ABSTRACT. We extend the quandle cocycle invariant to oriented singular knots and links using algebraic structures called oriented singquandles and assigning weight functions at both regular and singular crossings. This invariant coincides with the classical cocycle invariant for classical knots but provides extra information about singular knots and links. The new invariant distinguishes the singular granny knot from the singular square knot.

1. Introduction

Quandles are algebraic structures whose axioms model the three Reidemeister moves in knot theory. Around 1982, the notion of a quandle was introduced independently by Joyce [13] and Matveev [15] and they used it to construct representations of the braid groups. Joyce and Matveev associated to each knot a quandle that determines the knot up to isotopy and mirror image. Since then quandles and racks have been investigated by topologists in order to construct knot and link invariants and their higher analogues (see for example [9] and references therein).

Singular knot theory was introduced in 1990 by V. A. Vassiliev in [19] as an extension of classical knot theory. Since then many classical knot invariants have been extended to singular knots. Braid theory for singular knots was given by Birman in [2]. More precisely, Birman introduced singular braids and conjectured that the monoid of singular braids maps injectively into the group algebra of the braid group. A proof of this conjecture was given by Paris in [18]. Fiedler extended the Kauffman state models of the Jones and Alexander polynomials to the context of singular knots [11]. In [12] Gemein investigated extensions of the Artin representation and the Burau representation to the singular braid monoid and the relations between them. Juyumaya and Lambropoulou constructed a Jones-type invariant for singular links using a Markov trace on a variation of the Hecke algebra [14]. Recently, quandles have been used to study singular knots [1,8,16].

The authors of [6] introduced a cohomology theory for quandles, defined state-sum invariants using quandle cocycles as weights and computed the invariants for some families of classical knots and knotted surfaces. Since then quandle cocycle invariants were generalized, using quandle modules and non-abelian 2-cocycles in [3], extended to the biquandle case in [5], and to virtual knots in [7]. In this article, We extend the quandle cocycle invariant to oriented singular knots and links using algebraic structures called oriented singquandles and assigning weight functions at both classical and singular crossings. This invariant coincides with the classical cocycle invariant for classical knots but provides extra information about singular knots and links. The new invariant distinguishes the singular granny knot from the singular square knot.

This article is organized as follows. In Section 2 we review the necessary ingredients of quandles and give examples. In Sections 3 and 4 the diagrammatics of the generating set of

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singualar Reidemeister moves are presented and are used to motivate the definition of singu-
delles. In Section 5 the conditions on both the weights assigned at regular crossings and singu-
crossings are derived from the diagrammatic. The cocycle invariant is extended to singular
knots in Section 6 and used in Section 7 to distinguish singular knots and links with the same
number of colorings.

2. Review of Quandles

In this section we review the basics of quandles. More details can be found for example in
[9,13,15]. A set \((X,\cdot)\) is called a quandle if the following three identities are satisfied.

1. For all \(x \in X\), \(x \cdot x = x\).
2. For all \(y, z \in X\), there is a unique \(x \in X\) such that \(x \cdot y = z\).
3. For all \(x, y, z \in X\), we have \((x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z)\).

A quandle homomorphism between two quandles \((X,\cdot)\) and \((Y,\triangleright)\) is a map \(f: X \to Y\) such
that \(f(x \cdot y) = f(x) \triangleright f(y)\), where \(
\cdot\) and \(\triangleright\) denote respectively the quandle operations of \(X\) and \(Y\). If furthermore \(f\) is a bijection then it is called a quandle isomorphism between \(X\) and \(Y\). The set of quandle automorphisms of a quandle \(X\) is a group denoted by \(\text{Aut}(X)\).

The following are some typical examples:

- Any non-empty set \(X\) with the operation \(x \cdot y = x\) for all \(x, y \in X\) is a quandle called a trivial quandle.
- Any group \(X = G\) with conjugation \(x \cdot y = y^{-1}xy\) is a quandle.
- Let \(G\) be an abelian group. For elements \(x, y \in G\), define \(x \cdot y = 2y - x\). Then \(\cdot\) defines a quandle structure on \(G\) called Takasaki quandle. In case \(G = \mathbb{Z}_n\) (integers mod \(n\)) the quandle is called dihedral quandle. This quandle can be identified with the set of reflections of a regular \(n\)-gon with conjugation as the quandle operation.
- Any \(\Lambda = (\mathbb{Z}[T^{\pm 1}])\)-module \(M\) is a quandle with \(x \cdot y = Tx + (1 - T)y\), \(x, y \in M\), called an Alexander quandle.
- A generalized Alexander quandle is defined by a pair \((G, f)\) where \(G\) is a group and \(f \in \text{Aut}(G)\), and the quandle operation is defined by \(x \cdot y = f(xy^{-1})y\).

The axioms of a quandle correspond respectively to the three Reidemeister moves of types
I, II and III (see [9] for example). In fact, one of the motivations of defining quandles came
from knot diagrammatic. Given a quandle \((X,\cdot)\), axiom (2) states that right multiplication by
\(y \in X\), given by by \(R_y(x) = x \cdot y\) for \(x \in X\) is a permutation of \(X\). The subgroup of \(\text{Aut}(X)\),
generated by the permutations \(R_y\), \(y \in X\), is called the inner automorphism group of \(X\), and
is denoted by \(\text{Inn}(X)\). A quandle is connected if \(\text{Inn}(X)\) acts transitively on \(X\). The operation
\(\cdot\) on \(X\) defined by \(x \cdot y = R_y^{-1}(x)\) is a quandle operation.

3. Oriented Singular Knots and Quandles

Similarly to quandles, singular quandles are algebraic structures that can be derived from
the generating set of singular Reidemeister moves given in Figure 2. Generating sets of oriented
singular Reidemeister moves were studied in [1]. The generating set obtained in [1] is illustrated
in Figure 1.

Now that we have a generating set of singular Reidemeister moves, the axioms of singquandle
can be derived easily. The first step in this process is to define four binary functions that are
indicated in Figure 2.
The axioms of singular quandles are then derived by considering the singular Reidemeister moves and writing the equations obtained using the above binary operations. See Figures 3, 4 and 5.

Figure 3. The Reidemeister move Ω4a and colorings
Definition 4.1. \[1\] Let \((X, \ast)\) be a quandle. Let \(R_1\) and \(R_2\) be two maps from \(X \times X\) to \(X\). The triple \((X, \ast, R_1, R_2)\) is called an oriented singquandle if the following axioms are satisfied

\[
\begin{align*}
R_1(x, y) \ast R_2(x, y) &= R_2(y, x \ast y) \quad \text{coming from } \Omega_5a (4.4) \\
R_1(x, y) \ast R_2(x, y) &= R_2(y, x \ast y) \quad \text{coming from } \Omega_5a (4.5)
\end{align*}
\]

The following lemma gives a family of examples of singquandles over \(\mathbb{Z}_n\).

Lemma 4.2. Let \(n\) be a positive integer, let \(a\) be an invertible element in \(\mathbb{Z}_n\) and let \(b, c \in \mathbb{Z}_n\), then the binary operations \(x \ast y = ax + (1 - a)y\), \(R_1(x, y) = bx + cy\) and \(R_2(x, y) = acx + [b + c(1 - a)]y\) make the triple \((\mathbb{Z}_n, \ast, R_1, R_2)\) into an oriented singquandle.

Proof. First we have the inverse operation: \(x \ast y = a^{-1}x + (1 - a^{-1})y\). Now let \(R_1(x, y) = bx + cy\), then by axiom (4.4) of Definition 4.1 we have \(R_2(x, y) = acx + (c(1 - a) + b)y\). Now using axiom (4.5) of Definition 4.1 we obtain that \((1 - a)(1 - b - c) = 0 \in \mathbb{Z}_n\). Substituting, we find that the following is an oriented singquandle for any invertible \(a\), any \(b\) and \(c\) in \(\mathbb{Z}_n\) such that
the condition \((1 - a)(1 - b - c) = 0\) holds in \(\mathbb{Z}_n\). We thus have:

\[
x \ast y = ax + (1 - a)y
\]

\[
R_1(x, y) = bx + cy
\]

\[
R_2(x, y) = acx + [b + c(1 - a)]y
\]

\[
□
\]

5. DERIVING THE AXIOMS OF 2-COCYCLES FROM SINGULAR KNOTS

Let \(A\) be abelian group and let \((X, \ast)\) be a quandle. A \(2\)-cocycle of \(X\) on \(A\) is a function \(\phi : X \times X \to A\) that satisfies some conditions coming from Reidemeister moves. For \(x, y \in X\) we think of the value \(\phi(x, y)\) as a weight associated to a positive crossing and the weight \(-\phi(x, y)\) associated to a negative crossing in a knot diagram as shown in Figure 6. The weight function \(\phi\) must respect the following three conditions. First, the function \(\phi\) satisfies the so called \(2\)-cocycle condition for all \(x, y, z \in X\)

\[
\phi(x, y) + \phi(x \ast y, z) = \phi(x, z) + \phi(x \ast z, y \ast z).
\]

The above condition is imposed by Reidemeister move III. Moreover, Reidemeister move I imposes the condition \(\phi(x, x) = 0\) on a \(2\)-cocycle. Finally, Reidemeister move II imposes that the total Boltzmann weight should be zero; this condition is automatically satisfied due to the weight at a positive and negative crossing canceling each other (see Figure 6). For \(x, y \in X\) the values of the function \(\phi(x, y)\) on a given crossing are usually called the Boltzmann weights.

Now, we will extend this definition to singular knots and links. To this end, we define a function \(\phi' : X \times X \to A\) that represents the Boltzmann weight at a singular crossing. This is shown in the right figure in Figure 6.

![Figure 6. Weight functions at classical and singular crossings.](image)

![Figure 7. The Reidemeister move Ω4α and colorings.](image)
Naturally, we now want these two functions to satisfy the conditions under singular Reidemeister moves. Using Figure 7 we deduce the following condition:

\[-\phi(x \ast y, y) + \phi'(x \ast y, z) + \phi(R_1(x \ast y, z), y) = \phi(z, y) + \phi'(x, z \ast y) - \phi(R_2(x, z \ast y), y).
\] (5.1)

Figure 8. The Reidemeister move \(\Omega_4\) and colorings

Figure 8 implies the following equation:

\[\phi(y \ast R_1(x, z), x) - \phi(y \ast R_1(x, z), R_1(x, z)) = -\phi((x \ast R_2(x, z)) \ast z, z) + \phi(y, R_2(x, z)).\] (5.2)

Notice that in this equation we got rid of the term \(\phi'(x, z)\) which appeared on both sides of the equation before simplification.

Figure 9. The Reidemeister move \(\Omega_5\) and colorings

Figure 9 implies that we have the following condition:

\[\phi'(x, y) + \phi(R_1(x, y), R_2(x, y)) = \phi(x, y) + \phi'(y, x \ast y).
\] (5.3)

6. Cocycle Invariants of Singular knots and links

In this section we extend the definition of the 2-cocycle invariant to singular knots and prove that it is indeed an invariant for singular knots. First, recall that given an abelian group \(A\) and a quandle \((X, \ast)\), a 2-cocycle is a function \(\phi: X \times X \to A\) that satisfies the following for all \(x, y, z \in X\):

\[\phi(x, y) + \phi(x \ast y, z) = \phi(x, z) + \phi(x \ast z, y \ast z) \text{ and } \phi(x, x) = 0.
\] (6.1)

In order to construct our invariant we need to solve the system made of equations (5.1), (5.2), (5.3) and (6.1). First we need to recall the notion of coloring of a knot diagram by an oriented singquandle. Let \(D\) be a diagram of a singular knot and let \((X, \ast, R_1, R_2)\) be an oriented singquandle, then a coloring of \(D\) by \((X, \ast, R_1, R_2)\) is defined in a similar way to the
case of colorings of classical knots by quandles. To be precise, the colorings at positive, negative and singular crossings are given by Figure 2. As in the case of classical knots we will assign the weights $\phi(x, y)$ and $-\phi(x, y)$ at a positive and negative crossing respectively as shown Figure 6. Furthermore, we associate the weight $\phi'(x, y)$ at a singular crossing as shown in Figure 6. Now, given an abelian group $A$ (denoted multiplicatively) and cocycles $\phi, \phi' : X \times X \rightarrow A$, we define the state sum in exactly the same manner as state sum for classical knots (see [4] page 3953).

**Definition 6.1.** Let $(X, *, R_1, R_2)$ be an oriented singquandle and let $A$ be an abelian group. Assume that $\phi, \phi' : X \times X \rightarrow A$ satisfy the equations (5.1), (5.2), (5.3) and (6.1). Then the state sum for a diagram of a singular knot $K$ is given by $\Phi_{\phi, \phi'}(K) = \sum_\mathcal{C} \prod_\tau \psi(x, y)$, where the product is taken over all classical and singular crossings $\tau$ of the digaram of $K$, the sum is taken over all possible colorings $\mathcal{C}$ of the knot $K$.

Notice that in this definition $\psi(x, y) = \phi(x, y)^{\pm 1}$ at classical positive or negative crossing and $\psi(x, y) = \phi'(x, y)$ at a singular crossing as in Figure 6. Now we have the following theorem stating that $\Phi_{\phi, \phi'}(K)$ is an invariant of singular knots.

**Theorem 6.2.** Let $\phi, \phi' : X \times X \rightarrow A$ be maps satisfying the conditions of Definition 6.1. The state sum associated with $\phi$ and $\phi'$ is invariant under the moves listed in the generating set of singular Reidemester moves in Figure 1 so it defines an invariant of singular knots and links.

**Proof.** As in classical knot theory, there is one-to-one correspondence between colorings before and after each of the generating set of singular Reidemester moves. The invariance follows automatically from equations (5.1), (5.2), (5.3) and (6.1) involving $\phi$ and $\phi'$. ■

## 7. Distinguishing singular knots and links using the cocycle invariant

In this section, we first provide examples of singquandles and their 2-cocycles. We then use the cocycle invariant introduced in Definition 6.1 to distinguish pairs of singular knots and links. Notice that since the target groups of the maps $\phi$ and $\phi'$ will be finite cyclic groups $A = \langle u, u^n = 1 \rangle$, then the quandle cocycle invariants will have the form $\Phi_{\phi, \phi'}(K) = \sum_{i=0}^{n-1} a_i u^i$.

The following examples were computed independently using Mathematica and Maple. They were also checked by hand computations.

**Example 7.1.** Now we give a list of four singquandles and their 2-cocycles. Precisely, in each of the following four items $(X, *, R_1, R_2)$ is a singquandle and the maps $\phi$ and $\phi'$ satisfy the equations (5.1), (5.2), (5.3) and (6.1), thus allowing for the computation of the cocycle invariants.

1. Let $X$ be the singquandle $\mathbb{Z}_6$ with the operations $*$ and $\ast$ and the functions $R_1$ and $R_2$ given by, $\forall x, y \in X$, $x * y = -x + 2y = x \ast y$ and $R_1(x, y) = 3 + 2x - y$ and $R_2(x, y) = 3 + x$. Let the coefficient group be $A = \mathbb{Z}_6$ and let the classical 2-cocycle be defined by $\phi(x, y) = 2x + 3x^2 - 2y - xy - 2y^2$ and the weight at singular crossings be defined by $\phi'(x, y) = 3 + x + x^2 + 2y - xy$.

2. Let $X$ be the singquandle $\mathbb{Z}_6$ with the operations $*$ and $\ast$ and the functions $R_1$ and $R_2$ given by, $\forall x, y \in X$, $x * y = -x + 2y = x \ast y$ and $R_1(x, y) = 3 + x$ and $R_2(x, y) = 3 + 3x + 3x^2 + y$. Let the coefficient group be $A = \mathbb{Z}_2$ and let the classical 2-cocycle be defined by $\phi(x, y) = y(x + 1)$ and the weight at singular crossings be defined by $\phi'(x, y) = 1 + x + xy$. 
(3) Let $X$ be the singquandle $\mathbb{Z}_{10}$ with the operations $\ast$ and $\bar{\ast}$ and the functions $R_1$ and $R_2$ given by, $\forall x, y \in X$, $x \ast y = 3x - 2y$, $x \bar{\ast}y = -3x + 4y$ with $R_1(x, y) = x$ and $R_2(x, y) = 5x + 5x^2 + y$. Let the coefficient group be $A = \mathbb{Z}_4$ and let the classical 2-cocycle be defined by $\phi(x, y) = (x - y)(y - x)$ and the weight at singular crossings be defined by $\phi'(x, y) = 1 + 2x + 3x^2 + y^2$.

(4) Let $X$ be the singquandle $\mathbb{Z}_{10}$ with the operations $\ast$ and $\bar{\ast}$ and the functions $R_1$ and $R_2$ given by, $\forall x, y \in X$, $x \ast y = 3x - 2y$, $x \bar{\ast}y = -3x + 4y$ with $R_1(x, y) = 3x + 8y$ and $R_2(x, y) = 4x + 7y$. Let the coefficient group be $A = \mathbb{Z}_4$ and let the classical 2-cocycle be defined by $\phi(x, y) = 2(2 + 3x + 3y)^3$, and the weight at singular crossings be defined by $\phi'(x, y) = (1 + x + y)^5$.

Now we give examples of pairs of singular knots and links and we distinguish them using the list of singquandles and their 2-cocycles given in Example 7.1.

**Example 7.2.** In this example we show that the cocycle invariant distinguishes two singular knots with 5 classical crossings and a singular crossing. Consider the oriented singquandle $(\mathbb{Z}_6, \ast, R_1, R_2)$ with cocycles $\phi(x, y) = 2x + 3x^2 - 2y - xy - 2y^2$ and $\phi'(x, y) = 3 + x + x^2 + 2y - xy$ from item (1) of Example 7.1. We will use the 2-cocycle invariant to distinguish the following singular knots listed as $5^k_1$ and $5^k_8$ in [17].

![Figure 10. Diagram for 5^k_1.](image)

The coloring of $5^k_1$ by the above signquandle gives the set of equations

\[
\begin{align*}
x &= w \bar{\ast}R_2(x, y) = -w + 2x \\
y &= z \bar{\ast}(R_1(x, y) \ast x) = 2y - z \\
z &= R_2(x, y) \bar{\ast}w = 3 + 2w - x \\
w &= (R_1(x, y) \ast x) \bar{\ast}z = 3 - y + 2z.
\end{align*}
\]

Coloring the singular knot $5^k_8$ by the above singquandle gives the set of equations

\[
\begin{align*}
x &= z \ast (R_1(x, y) \ast z) = 2x + 2y + 3z \\
y &= R_2(x, y) \ast z = 3 - x + 2z \\
z &= w \ast y = -w + 2y \\
w &= (R_1(x, y) \ast z) \ast x = 3 + 4x - y + 4z.
\end{align*}
\]
These two systems give that the two singular knots have 6 colorings by this singquandle. Now we compute the 2-cocycle invariant using $\phi, \phi': X \times X \to \mathbb{Z}_6$ as defined above. The singular knot $5^k_1$ has $\Phi_{\phi,\phi'}(5^k_1) = 6u^3$ and the second knot $5^k_8$ has $\Phi_{\phi,\phi'}(5^k_8) = 6u^0 = 6$, thus distinct.

**Example 7.3.** In this example we show that the cocycle invariant detects the change in the singular crossing. Consider the oriented singquandle $(\mathbb{Z}_6, *, R_1, R_2)$ with cocycles $\phi(x, y) = 2x + 3x^2 - 2y - xy - 2y^2$ and $\phi'(x, y) = 3 + x + x^2 + 2y - xy$ from item (1) of Example 7.1. We will use the 2-cocycle invariant to distinguish the following singular knots listed as $5^k_6$ and $5^k_7$ in [17].

Coloring of $5^k_6$ by the above singquandle gives the set of equations

$$\begin{align*}
x &= (R_1(x, y)*y)*w = 3 + 2w + 2x + 3y \\
y &= w*z = -w + 2z \\
z &= R_2(x, y)*R_1(x, y) = 3 + 3x - 2y \\
w &= z*(R_1(x, y)*y) = 2x - z.
\end{align*}$$
Figure 13. Diagram for $5^k_7$.

Coloring the singular knot $5^k_7$ by the above singquandle gives the set of equations

$$
\begin{align*}
&x = R_1(x, y)\bar{z} = 3 - 2x + y + 2z \\
y = z \ast (R_2(x, y) \ast w) = 4w - 2x - z \\
z = w \ast R_1(x, y) = -w + 4x - 2y \\
w = (R_2(x, y) \ast w) \ast y = 3 - 2w + x + 2y.
\end{align*}
$$

These two systems give that the two singular knots have 6 colorings by this singquandle. Now we compute the 2-cocycle invariant using $\phi, \phi' : X \times X \to \mathbb{Z}_6$ as defined above. The singular knot $5^k_6$ has $\Phi_{\phi, \phi'}(5^k_6) = 6u^3$ and the second knot $5^k_7$ has $\Phi_{\phi, \phi'}(5^k_7) = 6u^0 = 6$, thus distinct.

Example 7.4. Consider the oriented singquandle $(\mathbb{Z}_6, \ast, R_1, R_2)$ with cocycles $\phi(x, y) = y(x+1)$ and $\phi'(x, y) = 1 + x + xy$ from item (2) of Example 7.1. Now we color the two singular links $K_1$ and $K_2$ by the oriented singquandle $(\mathbb{Z}_6, \ast, R_1, R_2)$ as shown in the figure below.

Figure 14. Diagram $K_1$.

The coloring of the first singular link $K_1$ by this singquandle gives the set of equations

$$
\begin{align*}
x &= R_2(x, y) = 3 + 3x + 3x^2 + y \\
y &= z \ast R_2(x, y) = 2y - z \\
z &= R_2(R_1(x, y), w) \bar{w} = 3 + w + 3x + 3x^2 \\
w &= R_1(R_1(x, y), w) = x.
\end{align*}
$$
Figure 15. Diagram $K_2$.

Coloring the second singular link $K_2$ by this singquandle gives the set of equations

$$\begin{align*}
x &= R_2(x, y) = 3 + 3x + 3x^2 + y \\
y &= z \ast R_2(x, y) = 2y + 5z \\
z &= R_2(R_1(x, y), w) = 3 + w + 3x + 3x^2 \\
w &= R_1(R_1(x, y), w) \ast z = -x + 2z.
\end{align*}$$

These two systems give that the two singular links $K_1$ and $K_2$ have 6 colorings by this singquandle. We now compute the 2-cocycle invariant using $\phi, \phi' : X \times X \rightarrow \mathbb{Z}_2$ as defined above. The first singular link $K_1$ has $\Phi_{\phi, \phi'}(K_1) = 6u$ and the second link $K_2$ has $\Phi_{\phi, \phi'}(K_2) = 6u^0 = 6$. Thus the two links are distinct.

**Example 7.5.** Consider the oriented singquandle $(\mathbb{Z}_{10}, \ast, R_1, R_2)$ with cocycles $\phi(x, y) = (x - y)(y - x)$ and $\phi'(x, y) = 1 + 2x + 3x^2 + y^2$ from item (3) of Example 7.1. Now we color the two singular knots $K_1$ and $K_2$ by the oriented singquandle $(\mathbb{Z}_{10}, \ast, R_1, R_2)$ as shown in the figure below.

Figure 16. Diagram $K_3$. 

The coloring the first singular knot $K_3$ by this singquandle gives the set of equations

\[
\begin{align*}
  x &= k \ast (R_2(x, y) \ast R_1(x, y)) = 7k + 6x + 8y \\
  y &= (R_2(x, y) \ast R_1(x, y)) \ast k = 4k + 3x + 5x^2 + 9y \\
  t &= R_1(x, y) \ast y = 7x + 4y \\
  z &= (R_2(w, z) \ast t) \ast R_1(w, z) = 8t + 9w + 5w^2 + 9z \\
  w &= t \ast z = 7t + 4z \\
  k &= R_1(w, z) \ast (R_2(w, z) \ast t) = 6t + 7w + 8z.
\end{align*}
\]

\[\text{Figure 17. Diagram } K_4.\]

Coloring the second singular knot $K_4$ by this singquandle gives the set of equations

\[
\begin{align*}
  x &= k \ast (R_2(x, y) \ast R_1(x, y)) = 7k + 6x + 8y \\
  y &= (R_2(x, y) \ast R_1(x, y)) \ast k = 4k + 3x + 5x^2 + 9y \\
  t &= R_1(x, y) \ast y = 7x + 4y \\
  w &= R_1(w, z) \ast s = 4s + 7w \\
  z &= s \ast t = 7s + 4t \\
  s &= t \ast R_1(w, z) = 7t + 4w \\
  k &= R_2(w, z) \ast z = 5w + 5w^2 + z.
\end{align*}
\]

These two systems give that the two singular knots $K_3$ and $K_4$ have 40 colorings by this singquandle. We now compute the 2-cocycle invariant using $\phi, \phi'$ : $X \times X \rightarrow \mathbb{Z}_4$ as defined above. The first singular knot $K_3$ has $\Phi_{\phi, \phi'}(K_3) = 10 + 10u + 10u^2 + 10u^3$ and the second knot $K_4$ has $\Phi_{\phi, \phi'}(K_4) = 10u + 20u^2 + 10u^3$.

**Example 7.6.** In this example we use the 2-cocycle invariant to distinguish the singular square knot and granny knot each with two singular crossings. Consider the oriented singquandle $(\mathbb{Z}_{10}, \ast, R_1, R_2)$ with cocycles $\phi(x, y) = 2(2 + 3x + 3y)^3$ and $\phi'(x, y) = (1 + x + y)^5$ from item (4) of Example 7.1. Recall that $x \ast y = 3x - 2y$, $x \ast y = -3x + 4y$, $R_1(x, y) = 3x + 8y$ and $R_2(x, y) = 4x + 7y$.  

\[\text{Example 7.1} \]
The coloring the singular square knot by the above singquandle gives the set of equations
\[
\begin{align*}
x &= R_1(z, w) = 8w + 3z \\
y &= R_2(x, y) \ast x = y \\
z &= (R_1(x, y) \ast R_2(x, y)) \ast w = 4w + 7x \\
w &= R_2(z, w) \ast R_1(z, w) = w.
\end{align*}
\]
This system of equations has three free variables giving that the number of colorings of the singular square knot is $10^3$.

Coloring the singular granny knot by the above singquandle gives the set of equations
\[
\begin{align*}
x &= R_1(z, w) \ast R_2(z, w) = z \\
y &= R_2(x, y) \ast x = y \\
z &= R_1(x, y) \ast R_2(x, y) = x \\
w &= R_2(z, w) \ast (R_1(x, y) \ast R_2(x, y)) = w.
\end{align*}
\]
This system of equations has also three free variables giving that the number of colorings of the singular granny knot is $10^3$. We now compute the 2-cocycle invariant using $\phi, \phi' : X \times X \to \mathbb{Z}_4$. 
as defined above. The singular square knot has $\Phi_{\phi,\phi'} = 378 + 250u + 122u^2 + 250u^3$ and the singular granny knot has $\Phi_{\phi,\phi'} = 370 + 250u + 130u^2 + 250u^3$. Thus this invariant distinguish them.

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