Superrenormalizable Gauge and Gravitational Theories

E. T. Tomboulis
Department of Physics
University of California, Los Angeles
Los Angeles, CA 90095-1547

Abstract

We investigate 4-dim gauge theories and gravitational theories with nonpolynomial actions containing an infinite series in covariant derivatives of the fields representing the expansion of a transcendental entire function. A class of entire functions is explicitly constructed such that: (i) the theory is perturbatively superrenormalizable; (ii) no (gauge-invariant) unphysical poles are introduced in the propagators. The nonpolynomial nature is essential; it is not possible to simultaneously satisfy (i) and (ii) with any polynomial series in derivatives. Cutting equations are derived verifying the absence of unphysical cuts and the Bogoliubov causality condition within the loop expansion. A generalized KL representation for the 2-point function is obtained exhibiting the consistency of physical positivity with the improved convergence of the propagators. Some physical effects, such as extended bound excitations in the spectrum, are briefly discussed.

\[1\] Research supported in part by NSF Grant PHY 95-310223, and Monbusho, Japan

\[2\] e-mail address: tomboul@physics.ucla.edu
1 Introduction

In this paper we investigate 4-dimensional gauge theories defined by non-polynomial actions with an infinite number of derivatives. Specifically, we consider Langrangians of the form:

$$-\frac{1}{2g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \text{tr} F_{\mu\nu} h\left(\frac{D^2}{\Lambda^2}\right) F^{\mu\nu},$$

where $h$ is a transcendental entire function having an infinite series expansion in the covariant D'Alembertian $D^2$, and $\Lambda$ some scale. Similarly, we consider gravitational theories with actions including terms of the form

$$R_{\mu\nu} h_2\left(-\frac{\nabla^2}{\Lambda^2}\right) R^{\mu\nu} \quad \text{and} \quad R h_0\left(-\frac{\nabla^2}{\Lambda^2}\right) R$$

where $h_2$ and $h_0$ are entire functions.\(^3\)

When the functions $h$, or $h_2, h_0$, are taken to be polynomials, these are the Lagrangians of the familiar higher-derivative (covariant Pauli-Villars) regularization of gauge theory \([1]\). As it is well-known, such regularization renders gauge theory superrenormalizable at the expense of introducing massive ghosts. It is easily shown that this will always be the case for any polynomial $h$. Here we consider the question whether it is possible to choose non-polynomial $h$ so as to obtain good UV behavior while avoiding the introduction of ghosts. Somewhat surprisingly, we find that there is a class of transcendental entire functions, which can be explicitly constructed, and give a superrenormalizable theory, while, at least formally within the perturbative loop expansion, maintaining unitarity and causality. Superrenormalizability and unitarity appear interconnected. The requirement that the function $h$ be entire, thus possessing no singularities anywhere in the finite complex plane, is absolutely crucial for this to be possible.

To avoid a potential confusion at the outset, let us stress that what is being considered here is not the expansion in derivatives (powers of momenta) of the nonlocal effective action resulting from integration over some of the fields of a local field theory. Note that such an effective action necessarily contains singularities corresponding to the thresholds for production of the integrated out degrees of freedom. By the same token it cannot define a unitary S-matrix solely in terms of the remaining fields appearing in it, since the integrated-out fields still can occur in the intermediate state cuts.

Though no nonlocal kernels are explicitly introduced in our actions, the dependence of the argument of the nonpolynomial $h$ on derivatives does introduce an effective nonlocality. Actions with general nonlocal kernels are, of course, known to lead to problems with causality. The nonlocality due to transcendental entire functions with

\(^3\)Additional structures involving higher than second powers of $F, R$ may be included, but will not be considered in this paper.
derivative-dependent argument is, on the other hand, of a rather mild sort sometimes termed 'localizable' in distribution theory. As we will see, many things work for our actions pretty much as for polynomial actions precisely because of the similar properties of polynomial and nonpolynomial entire functions.

The idea of nonpolynomial entire Langrangians as a natural extension of the usual polynomial ones is certainly not new. Efimov, in particular, pursued such investigations [3], mostly in the context of attempts to obtain finite scalar theories, some time ago.

In the context of relativistic particle mechanics, Kato [4] considered actions analogous to the field theory actions considered here. To the usual (gauge-fixed) particle action \((dx^\mu/d\tau)^2\), he adds a term \((dx^\mu/d\tau)f(d/d\tau)(dx_\mu/d\tau)\), with \(f\) some function. To this one must add Lagrange multiplier terms incorporating the constraints of reparametrization invariance. This is then analogous to the BRS action (2.1) below, with \(dx^\mu/d\tau\) corresponding to \(F_{\mu\nu}\). For appropriate choice of meromorphic function \(f\), he finds a class of theories which includes the open bosonic string. It might be that the theories considered here have some kind of underlying extended structure associated with them. In this paper, however, we study them as a field theory problem.

The contents of the paper are as follows. In section 2 the action for the gauge theory case is introduced. A brief review of some features of higher derivative regularization provides the motivation for the introduction of nonpolynomial entire functions. The structure of the resulting interaction vertices is quite complicated, and is examined in section 3, with technical details relegated to Appendix A. Provided that the function \(h\) satisfies appropriate asymptotic conditions, detailed power counting shows that only 1-loop divergences occur in the perturbative loop expansion. These asymptotic conditions are supplemented in section 4 by the requirement of the absence of unphysical poles at tree level. A class of entire functions \(h\) satisfying all the requirements is then explicitly constructed. After discussing the relation between Euclidean and Minkowski Feynman rules in section 5, we turn to the basic issues of unitarity and causality to any order in the loop expansion in section 6. The special nature of the vertices allows one to obtain a largest time equation and hence generalized Cutkosky rules, which, applied to physical amplitudes, give the unitarity condition equations. No gauge-invariant unphysical poles occur in the intermediate states, whereas the cancellation of longitudinal and FP ghost gauge dependent excitations occurs as in the standard gauge theories and is explicitly verified. Similarly, the Bogoliubov causality condition equation is shown to hold. Details concerning the derivations are relegated to Appendices B and C. In section 7 we consider the 2-point function and obtain a generalized Källen-Lehmann representation for it. This makes explicit how, in this type of theory, the absence of unphysical excitations can be consistent with the improved convergence of propagators. With slight modifications, the entire development can be repeated for gravitation, which, in fact, provides one of the main motivations for this study. This is done in section 8.

The coupling to matter is discussed, though not in explicit detail, in the concluding section 9. There is a variety of potentially rather interesting physical effects in these
gauge theories, such as the appearance of bound extended excitations, due to the modified short distance behavior. These matters are also briefly discussed in section 9.

2 Action

Consider the Lagrangian

\[ \mathcal{L} = -\frac{1}{2g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} - \frac{\alpha}{2} \text{tr} F_{\mu\nu} h(-\frac{D^2}{\Lambda^2}) F^{\mu\nu} - \frac{1}{2\xi} f_a[A] w(-\frac{\Box}{\Lambda^2}) f_a[A] + \overline{\tau}_a M_{ab} c_b. \]  

(2.1)

\(D^2 = D_\mu D^\mu\) and \(\Box = \partial_\mu \partial^\mu\) denote the covariant and ordinary D’Alembertian, respectively. \(f_a[A]\) is a gauge-fixing function, with \(w\) a gauge-fixing weighting function. Note that the corresponding FP ghost term \(M_{ab} c_b = \delta^c_a f_a[A, x]\), where \(\delta^c_a f_a\) is the infinitesimal transformation of \(f_a\) with gauge transformation parameter \(c^b\), does not depend on \(w\). \(h\) is a given function to be specified, and \(\Lambda\) an arbitrary mass. The coupling \(\alpha\) can, of course, be absorbed in the definition of the function \(h\), but is more convenient to keep it explicit.

(2.1) is invariant under the BRS transformation:

\[ \delta A^a_\mu = D^a_{\mu b} c_b \epsilon, \quad \delta c_a = -\frac{1}{2} f_{abc} c_b c_c \epsilon, \quad \delta \overline{\tau}_a = -\frac{1}{\xi} w(-\frac{\Box}{\Lambda^2}) f_a[A] \epsilon. \]  

(2.2)

With \(f^a = \partial^\mu A^a_\mu\), and the usual rescalings \(A \rightarrow gA\), \(\xi \rightarrow g^2 \xi\), the bare propagator is then given by

\[ D^{\mu\nu}_{ab}(k) = -\frac{i}{(2\pi)^4} \frac{\delta^{ab}}{k^2 + i\epsilon} \left( \frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{1 + g^2 \alpha h(k^2/\Lambda^2)} + \frac{\xi k_\mu k_\nu/k^2}{w(k^2/\Lambda^2)} \right). \]  

(2.3)

Further definition of (2.1) hinges on the specification of the function \(h\).

Polynomial \(h\) - Higher Derivative (HD) Regularization. In the HD regularization scheme \([1]\), the function \(h(x)\) is chosen to be a polynomial, \(h(x) = p_n(x)\), of degree \(n\). With the weight \(w(x)\) also a polynomial, and if \(n \geq 2\), and \(\text{deg } w \geq n\), straightforward power counting shows that, at finite \(\Lambda\), the only divergent diagrams are one-loop diagrams with 0, 2, 3 and 4 external gauge field legs and no external ghost legs. All other one-loop diagrams, and all IPI multi-loop diagrams are superficially convergent. Superficially convergent multi-loop diagrams may, of course, still contain subdivergences due to one-loop subdiagrams. The theory is thus rendered superrenormalizable.

To completely regulate the theory then, the remaining one-loop divergences must be regulated separately. Dimensional regularization is straightforward to implement.

---

\(D_{ab} = D_{\mu b} F^{\mu a}\), \(\{t_a, t_b\} = i f_{abc} t_c\).
and very convenient for this purpose. Alternatively, and perhaps more in the spirit of the original HD scheme, additional Pauli-Villars (PV) regulators may be used.

For discussion of renormalization, it is very convenient to note that by taking $\text{deg } w$ sufficiently high all gauge dependent divergences disappear; renormalization may then be performed by the addition of only gauge invariant counterterms. So at finite $\Lambda$, the remaining one-loop divergences are formally manifestly gauge-invariant for $\text{deg } w \geq n$, since FP ghost field and vertex renormalizations are finite. They may be removed by adding the one-loop $F_{\mu\nu}^2$ counterterm; it is important to note that the function $h(x) = p_n(x)$ does not get renormalized. These statements can, of course, be made rigorous only in the presence of appropriate one-loop regularization. Dimensional regularization works well. The introduction of additional PV regulators, on the other hand, requires considerable care to avoid conflicts with gauge or BRS invariance, and has been the subject of several recent investigations; for a review and discussion see Ref.[2], and references therein.

In HD regularization, where $h(x) = p_n(x)$, the theory (2.1) is rendered superrenormalizable at the expense of introducing ghosts. Indeed, as it is evident from (2.3), the transverse part of the propagator acquires $n$ additional poles from the $n$ zeroes of the polynomial $1 + g^2 \alpha p_n(k^2/\Lambda^2)$. Note that some of these will, in general, be complex. The residues of some of these will necessarily be negative (more generally, have a negative real part). This follows from the improved UV behavior of (2.3). Indeed, by the factorization theorem for polynomials and partial fraction decomposition one may write:

\[
\frac{1}{k^2(1 + g^2 \alpha p_n(k^2))} = \frac{r_0}{k^2} + \sum_{i=1}^{n} \frac{r_i}{k^2 - M_i^2}.
\]

The assertion that at least one $r_i$ must be negative follows immediately by multiplying (2.4) by $k^2$ and taking the large $k^2$ limit. More generally, the spectral function in the Källen-Lehmann representation for the dressed propagator must contain negative contributions and satisfy a superconvergence relation.

Entire transcendental $h$. The question we consider in this paper then is: is it possible to choose the function $h(x)$ in (2.1) so that no unphysical poles are introduced while at the same time maintaining the (super)renormalizability of the theory?

It is, of course, clear from the above argument that the answer is no as long as $h(x)$ is taken to be a polynomial of any finite order (fundamental theorem of algebra!). One, therefore, has to consider non-polynomial functions. Now a polynomial is an entire function, i.e. holomorphic anywhere in the finite complex plane. This property is necessary for the action to be well-defined everywhere (including the complex domain needed for analyticity and unitarity considerations). The natural generalization of a polynomial possessing this property is a transcendental (i.e. non-polynomial) entire function, which we will take $h(z)$ to be. This means that it can be represented by an
everywhere convergent power series about any point, in particular the origin:

\[ h(z) = \sum_{n=0}^{\infty} a_n z^n \]  

(2.5)

with \( a_n = \frac{1}{n!} h^{(n)}(0) \). Infinite radius of convergence (in fact, absolute convergence) implies \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = 0 \). The operator function \( h(-D^2/\Lambda^2) \) is then defined through (2.5) as a power series in the covariant D'Alembertian \(D^2\), and gives a well-defined non-polynomial action (2.1).

Recall [5] that the standard growth scale for entire functions is based on exponentials of powers as comparison functions: if \( f(z) \) is of order \( \rho \), then \( \exp (r(\rho - \epsilon)) < \max_{|z|=r} |f(z)| < \exp (r(\rho + \epsilon)) \) for arbitrary positive \( \epsilon \), and \( r \) sufficiently large. (Polynomials are of order zero.) It may thus at first appear that controllable UV behavior would not be possible. The overall growth scale provided by \( \rho \), however, ignores any dependence of growth on the direction in which \( z \) grows large. A more refined growth measure is obtained by defining the order \( \rho(\alpha, \beta) \) of \( f(z) \) in the angle \( \alpha \leq \text{arg} \ z \leq \beta \). It is a remarkable property of entire functions that, for appropriate \( f(z) \), \( \rho(\alpha, \beta) \) may range from zero to arbitrarily large values as \( \alpha, \beta \) vary. This property can be exploited in order to obtain controllable UV behavior [3]. Our basic requirement will be that \( h(z) \) in (2.1) exhibit at most polynomial behavior along the real axis.

3 Perturbative expansion and renormalization

With \( h \) a transcendental entire function, as in eq. (2.5), the action (2.1) now possesses, in addition to the usual YM vertices, an infinite set of interaction vertices. In an obvious notation, suppressing spacetime and group indices and with \( \delta^n/\delta A^n \equiv \delta^n/\delta A(x_1) \ldots \delta A(x_n) \), an \( N \)-point \( h \)-dependent vertex is given by:

\[
V^{(N)}(x; x_1, \ldots, x_N) \equiv \mathcal{V}^{(N)}(\{\partial x_i\}) \prod_{i=1}^{N} \delta(x - x_i)
\]

\[
= \text{tr} \left( \frac{\delta^n F[A]}{\delta A^{n''}} \right) \cdot v^{(n)}(x, \partial x; x_1, \ldots, x_n) \cdot \frac{\delta^n F[A]}{\delta A^{n'}} \bigg|_{A=0}.
\]

\[
\quad \equiv \text{tr} \ F^{(n'')} \cdot v^{(n)} \cdot F^{(n')} , \quad N = n' + n + n'' \quad (3.1)
\]

\(^{5}\text{Let}\quad M_f(r, \alpha, \beta) \equiv \max_{\alpha \leq \theta \leq \beta} |f(re^{i\theta})| , \quad (2.6)\)

and define the order \( \rho(\alpha, \beta) \) of \( f(z) \) in the angle \( \alpha \leq \text{arg} \ z \leq \beta \) by

\[
\lim_{r \to \infty} \frac{\ln \ln M_f(r, \alpha, \beta)}{\ln r} . \quad (2.7)
\]

A concise and fairly complete account of the theory of growth of entire functions is given in Chapter 1 of the second reference in [5].
where
\[ v^{(n)} = \sum_{r=0}^{\infty} a_r \left( -\frac{D^2[A]}{\Lambda^2} \right)^r \big|_{A=0} \]
\[ = \sum_{l=\frac{1}{2} \left( \frac{n+1}{2} + 1 \right)}^{n} \sum_{\sigma} \sum_{r=l}^{\infty} a_r S_{r,l,n}^\sigma \left( -\frac{\Box}{\Lambda^2} \right)^{(r-l)}, \left( \frac{1}{\Lambda^2} \delta^b \delta [\frac{\partial^2}{\Lambda^2} + \Box] |_{A=0} \right)^l \]
\[ \equiv \sum_{l=\frac{1}{2} \left( \frac{n+1}{2} + 1 \right)}^{n} \sum_{\sigma} v_{l,n}^\sigma (x, \partial x; x_1, \ldots, x_n) \right. \]
(3.2)

In (3.2), \( S_{r,l,n}^\sigma \) stands for the sum of all possible ways of distributing \((r - l)\) powers of \((-\Box)\) in the \(l + 1\) positions among an ordered sequence, indexed by \(\sigma\), of \(l\) factors of \(\delta^b \delta [\frac{\partial^2}{\Lambda^2} + \Box] |_{A=0}\), \(b = 1, 2\) (see A.1). The total number of \(\delta/\delta A\)'s among these \(l\) factors is \(n\), and \(b = 1\) or \(2\) since \([-D^2 + \Box]\) is at most bilinear in \(A\). The total number \(\Gamma\) of such ordered sequences is then
\[ \Gamma = (n - l)! \left( \frac{l}{(n - l)} \right)! = \frac{l!^2}{(2l - n)!} \].
(3.3)

The structure of \(v_{l,n}^\sigma\) is examined in Appendix A, where it is explicitly reexpressed in terms of the function \(h\) and its derivatives \(h^{(m)}\). (It is of course important that one be able to do this, so that the asymptotic behavior of (3.1) can be related to that of \(h(z)\).) For arbitrary configuration of momenta \(q_1, \ldots, q_N\) carried by the legs of the vertex (3.1), \(v^{(n)}\) is given through (A.9), (A.11)-(A.14) as a sum of products of rational functions of momenta and \(h\) or its derivatives. Our fundamental requirement is that \(h\) behaves asymptotically for real values of its argument as a polynomial. The assumptions of the power counting theorem [3] are then satisfied.

Let \(q_i = c_i k + p_i, i = 1, \ldots, M \leq N\) for some set of constants \(c_i\) and fixed finite momenta \(p_i\), and with \(k\) growing arbitrarily large. By choosing the \(c_i\)'s and \(M\) the growth of the vertex (3.1) along every hyperplane in the space of the vertex momenta can then be examined. As shown in Appendix A, in all cases the leading asymptotic behavior of \(v_{l,n}^\sigma\) is given by a sum of terms that grow at most either as
\[ h^{(s)}(k^2) k^{2s-n}, \quad 0 \leq s \leq l \],
(3.4)
or
\[ \frac{1}{k^2} k^{n_{\text{int}}}, \]
(3.5)
where \(n_{\text{int}}\) is the number among the legs of \(v_{l,n}^\sigma\) carrying momentum of order \(k\). Let
\[ \gamma \equiv \lim_{|z| \to \infty} \left( \frac{\ln|h(z)|}{\ln|z|} \right) \].
(3.6)

Footnotes:
6 The vertices (3.1), and hence integrands of graphs, are, in the terminology of reference [3], functions in the class \(A_n\).
7 In the following, to avoid cluttering the notation we often write \(h(k^2/\Lambda^2) = h(k^2)\).
Note that finiteness of the limit $\gamma$ defined in (3.6) implies the requirement that $h(z)$ exhibit at most polynomial behavior at infinity on the real axis. Detailed power counting (Appendix A) shows that UV divergences arise solely from terms with growth of type (3.4) provided $\gamma \geq 2$. Then the superficial degree of divergence of a 1PI graph $G$ with $L$ loops, $E$ external gauge boson lines and no external ghost lines is:

$$\delta_G = 4 - 2\gamma(L - 1) - E, \quad \gamma \geq 2. \quad (3.7)$$

Thus only 1-loop diagrams with $E = 2, 3, 4$ are superficially divergent. All other diagrams, i.e. 1-loop graphs with $E > 4$, and all $L > 1$ graphs with any number of external legs are superficially finite. Also, graphs with any number of external ghost legs are convergent for all $L$. The theory is then superrenormalizable by power counting, and the 1-loop divergences present are gauge-invariant. Renormalization of (2.1) is thus performed very simply by the addition of only the gauge-invariant 1-loop $F^2$ counterterm:

$$L_R = -\frac{1}{2g^2}\text{tr}F_{\mu\nu}F^{\mu\nu} - \frac{\alpha}{2}\text{tr}F_{\mu\nu}h(-\frac{D^2}{\Lambda^2})F^{\mu\nu} - \frac{1}{2g^2}(Z_3 - 1)\text{tr}F_{\mu\nu}F^{\mu\nu} , \quad (3.8)$$

where $g = g(\mu)$ is now the renormalized coupling fixed at some renormalization scale $\mu$. With the customary rescaling $A \to gA$, $\xi \to g^2\xi$, equations (2.3) and (3.1) then again give the propagator and vertices, now in terms of the renormalized coupling.

It is crucial for what follows that the function $h$ does not get renormalized, or, more precisely, the functional dependence on its argument is not altered under renormalization.

We still have to show that functions $h$ with the required properties can be found.

## 4 Construction of the entire function $h$

In view of the form of the denominator in (2.3), it is convenient to define

$$\tilde{h}(z) \equiv 1 + g^2\alpha h(z) . \quad (4.1)$$

We require that the function $h(z)$ be an entire transcendental function with the following properties:

(i) $\tilde{h}(z)$ is real and positive on the real axis, and has no zeroes anywhere in the complex plane, $|z| < \infty$.

(ii) $|h(z)|$ has the same asymptotic behavior along the real axis at $\pm\infty$. 

8
(iii) There exists $\Theta > 0$ such that

$$|h(z)| \xrightarrow{|z| \to \infty} |z|^\gamma, \quad \gamma \geq 2$$

for arguments in the cones:

$$\mathcal{C} = \{ z \mid -\Theta < \arg z < \Theta, \quad \pi - \Theta < \arg z < \pi + \Theta \} , \quad 0 < \Theta < \pi/2 .$$

(4.2)

Condition (i) is the requirement that no poles appear in the transverse bare propagator (2.3) other than the physical (positive residue) massless gauge boson pole. Reality of the action (2.1) is ensured. Condition (iii) ensures that the power counting requirements for superrenormalizability of the previous section are satisfied. The appropriate asymptotic behavior is imposed in compliance with (ii), and not only on the real axis but in conelike regions surrounding it. This is in fact necessary since amplitudes are defined as boundary values of complex functions on the real axis. Condition (ii) is not strictly necessary as far as power counting requirements go. Rigorous power counting is considered to be performed in Euclidean space. It is, however, necessary if we are to obtain the usual formal identity in (asymptotic behavior of) Feynman rules for Euclidean and Minkowski $k^2$. This is important in the derivation of unitarity cutting rules. (The relation between Euclidean and Minkowski amplitudes is discussed in the following section.)

The conditions (i) - (iii) lead directly to a general form of $h$. It is a basic result that an entire function with no zeroes anywhere in the complex plane can only be the exponential of an entire function. Thus, if (i) were to be satisfied, we must have:

$$h(z) = \exp H(z) , \quad \text{where } H \text{ is entire, and, from (iii), should exhibit logarithmic asymptotic behavior in the region } \mathcal{C}. \quad \text{Thus we arrive at the form:}$$

$$H(z) = \int_0^{p_\gamma(z)} \frac{1 - \zeta(w)}{w} \, dw ,$$

(4.3)

with:

(a) $p_\gamma(z)$ a real polynomial of degree $\gamma$, and $p_\gamma(0) = 0$,

(b) $\zeta(z)$ entire and real on the real axis, and $\zeta(0) = 1$,

(c) $|\zeta(z)| \to 0$ for $|z| \to \infty$, $z \in \mathcal{C}$.  \quad (4.4)

We take for $h$ in (2.1):

$$h(z) = \left( e^{H(z)} - 1 \right) ,$$

(4.5)

where $H(z)$ is given by (4.3). Since $\zeta(z)$ is bounded in domains extending to infinity, it must be of order $\rho > 1/2$ (Wiman’s theorem), and $h(z)$ is of infinite order.

\footnote{But it can actually also be done directly in Minkowski space using Zimmermann’s trick \cite{ref}.}
The absence of zeros requirement in (i) is now satisfied if we set \( g(\mu)^2 = \alpha^{-1} \). Since \( \alpha \) is an arbitrary parameter, this is always possible. We should, however, consider its meaning under changes of renormalization scale.

Assume that, at a given scale \( \mu \), we have \( g(\mu)^2 \neq \alpha^{-1} \). Now evolve \( g(\mu) \) to the value of \( \mu = \mu_0 \) where \( g(\mu_0)^2 = \alpha^{-1} \). (We make the ‘naturalness’ assumption that, with (4.3) in (2.3), one of the couplings \( g_{--2} \) and \( \alpha \) is not unnaturally small or large relative to the other.) By RG invariance, physical quantities are unchanged under this change of the parametrization of the theory in terms of \( g(\mu) \), \( \mu \) to one in terms of \( g(\mu_0) \), \( \mu_0 \).

Another way of looking at this is to note that a specification of \( g(\mu) = g_1 \) at one value of \( \mu \) is equivalent to a specification \( \bar{\Lambda} = \bar{\Lambda}_1 \) of a RG-invariant scale \( \bar{\Lambda} \) corresponding to the renormalization of \( g \). Suppose a different specification \( g(\mu) = g_2 \) is made, and evolving from \( g_2 \) one obtains \( g(\mu_0) = g_1 \). Then, by a familiar argument, \( \bar{\Lambda}_2 = \left( \frac{\mu_0}{\mu} \right) \bar{\Lambda}_1 \); and the scale \( \Lambda \) must be rescaled by the same amount to keep the same numerical value. So two versions of the theory specified by \( g(\mu)^2 = \alpha^{-1} \) and \( g(\mu_0)^2 = \alpha^{-1} \), respectively, for given \( \alpha \), differ only by a change in mass scale.

We may indeed always assume, with no loss of generality, that we work with a renormalization prescription such that \( g(\mu_0)^2 = \alpha^{-1} \) at some convenient renormalization scale \( \mu_0 \). In fact, note that any split between a renormalized \( 1/g^2 \) and a constant part (coefficient \( a_0 \) in (2.5)) in \( h \) is a renormalization prescription. Since the \( h \) modification to the action is relevant only in an UV regime set by the scale \( \Lambda \), it is natural to fix \( \mu_0 \) of order \( \Lambda \).

Choosing to renormalize at \( \mu_0 \) then, (i) is satisfied, and the bare transverse propagator has no additional poles. This will be extremely convenient in the following, in particular in deriving cutting rules and equations for unitarity and causality. It is clearly not essential, however, and we may choose a different renormalization point. The technical nuisance then would be that, if we want to avoid dealing with fictitious poles, we must work with cutting equations in terms of dressed propagators. Indeed, consider any other \( \mu \) where (2.3) will have additional poles at \( k^2 \) such that:

\[
\exp H(k^2/\Lambda^2) = (1 - \frac{1}{g(\mu)^2 \alpha}) .
\]

(4.6)

In fact, it will have an infinite number of (complex) poles since (4.6) must have an infinite number of roots (Picard’s little theorem). They are all, however, clearly unphysical since their position moves with \( \mu \) and is actually driven off to infinity as \( \mu \to \mu_0 \). We know, of course, that they must cancel, at any fixed \( \mu \), by RG invariance. Thus, if in analogy to \( \Pi \), occuring in the transverse bare 2-point function, we consider the dressed counterpart

\[
\Pi'(z) \equiv 1 + g^2(\mu) \alpha h(z) + g^2(\mu) \pi(z, \Lambda/\mu, g(\mu), \mu) ,
\]

(4.7)
occuring in the RG invariant transverse inverse full 2-point function, cp. eq. \((7.3)\) below\(^\text{10}\), we have:

\[
\frac{1}{g(\mu_0)^2} + \alpha h(k^2) + \pi(k^2, g(\mu_0), \mu_0) = \frac{1}{g(\mu_0)^2} + \alpha h(k^2) + \pi(k^2, g(\mu), \mu) + \Delta(g, \mu, \mu_0) = \frac{1}{g(\mu)^2} + \alpha h(k^2) + \pi(k^2, g(\mu), \mu). \tag{4.8}
\]

Here \(\Delta(g, \mu, \mu_0)\) is the finite difference between the counterterms renormalizing the self-energy at scales \(\mu\) and \(\mu_0\), respectively - it provides the finite renormalization between \(g(\mu)\) and \(g(\mu_0)\), and the shift responsible for the zeroes in the bare part, (eq. \((4.6)\)), at scale \(\mu\). The equality \((4.8)\) shows how these zeros in the bare part at scale \(\mu\) cancel against the self energy contribution to reproduce their absence at \(\mu_0\). In short, the requirement (i) above may be replaced by the RG invariant statement that no zeros \((4.6)\) survive in \(\tilde{h}\). But this is automatically satisfied once (i) is fulfilled at some renormalization scale \(\mu = \mu_0\). From now on we will assume that the coupling has been renormalized at scale \(\mu_0\).

Clearly, many examples of functions that satisfy the stated conditions on \(\zeta(z)\) can be given. An obvious choice is to assume exponential fall-off and take:

\[
\zeta(z) = \exp - \left( C_n z^{2n} + \cdots + C_1 z^2 \right) \quad (C_n > 0, \ n \geq 1). \tag{4.9}
\]

Define \(4n\) equal angular sectors \(\omega^\epsilon_j\) with common vertex at the origin:

\[
\omega^\epsilon_j : \quad (2j - 3) \frac{\pi}{4n} + \frac{\epsilon}{2n} < \theta < (2j - 1) \frac{\pi}{4n} - \frac{\epsilon}{2n} \tag{4.10}
\]

with \(0 < \epsilon \ll 1\). The sectors \(\omega_j \equiv \omega^\epsilon_j=0\) then divide the plane in \(4n\) sectors. Now elementary estimates show that, for any (arbitrarily) small \(\epsilon > 0\), and \(|z|\) larger than some number \(Z(\epsilon)\), one has:

\[
|\zeta(z)| > \exp \left[ C_n|z|^{2n}(1-\epsilon)\epsilon \right] \quad \begin{array}{l}
|z| \to \infty \quad \infty, \quad z \in \omega^\epsilon_j, \quad j \text{ even} \\
|z| \to \infty \quad 0, \quad z \in \omega^\epsilon_j, \quad j \text{ odd}.
\end{array} \tag{4.11}
\]

The cones \(C\) in \((4.2)\) then are the sectors \(\omega_1\) and \(\omega_{1+2n}\), with \(\Theta = \pi/4n\). Note that as \(n\) increases \(\Theta\) decreases, but the total area of the angular sectors in which \(\lim_{|z|\to\infty} \zeta(z) \to 0\) always occupies half of the total plane area.

The simplest choice \(n = 1\). Then \(h\) is given by \((4.3)\) with

\[
H(z) = \left[ \sum_{m=1}^{\infty} (-1)^{(m-1)} \frac{C^m}{m!} \frac{p_n(z)^{2m}}{2m} \right], \tag{4.12}
\]

and \(\Theta = \pi/4\).

\(^{10}\)An explicit factor of \((2\pi)^4 g^2\) has been factored out in the definition of the self-energy \(\pi\) in \((4.7)\) compared to that in \((7.4)\), \((7.3)\).
5 Relation between Minkowski and Euclidean formulation

By construction, the function $h(z)$ exhibits polynomial asymptotic behavior in certain directions in the complex plane, in particular the cones $\mathcal{C}$, eq. (4.2), surrounding the real axis. Being an entire function of infinite order, however, it must grow doubly exponentially in some other directions, such as the even-numbered sectors (4.10) for $\zeta(z)$ as in (4.9). This raises the issue of the relation between Minkowski and Euclidean formulations.

We may follow the standard path of Euclidean field theory: the theory is defined through the Euclidean path integral for the action (2.1), correlation functions are computed, and finally continued to Minkowski space by analytic continuation in the external momenta.

For ordinary local (gauge) field theories ($h = 0$), this is sometimes purely formally justified as 'Wick rotation of the integration contour' of the functional integral. But, in fact, actual rigorous justification at the non-perturbative level has only been obtained for 'simple' theories. Within the perturbative loop expansion, however, the procedure is indeed justified on a graph by graph basis. It may at first appear that this is no longer the case when we allow $h \neq 0$. But this is not so. Analytic continuation in the external momenta of the result of computation with Euclidean Feynman rules again formally agrees with the result of using Minkowski Feynman rules and Wick-rotating integration contours according to the following procedure.

For convenience, we choose the gauge-fixing weight function $w(z) = \overline{h}(z)$, and adopt the Feynman gauge $\xi = 1$. The basic point is that the vertices (3.4) and the factor $\overline{h}^{-1}$ in the propagator (2.3) do not contain any singularities. In the computation of some arbitrary graph then we proceed in a standard fashion introducing Schwinger parameters for the scalar propagator $(k^2 + i\epsilon)^{-1}$ in each propagator (2.3). This allows one to successively perform the momentum integrals. The integral over internal momentum $k$ is of the form

$$ \int d^d k \mathcal{V} \left( \{ p^\mu \}, \{ l^\mu \}, \{ k^\lambda \} \right) \exp \left[ A(x) k^2 + B(p, l, x) \cdot k \right] . \quad (5.1) $$

Here $A$ depends only on the Schwinger parameters $x$, while $B$ depends on $x$ and also linearly on the other loop momenta $l^\mu$, and external momenta $p^\mu$. $\mathcal{V}$ stands for the product of all vertex and $\overline{h}^{-1}$ factors, and thus consists of sums of products of polynomials and entire functions. We may then reexpress (5.1) as

$$ \mathcal{V} \left( \{ p^\mu \}, \{ l^\mu \}, \left\{ \frac{\partial}{\partial B^\mu}, \frac{\partial}{\partial B^\nu}, \ldots, \frac{\partial}{\partial B^\lambda} \right\} \right) \int d^d k \exp \left[ A(x) k^2 + B(p, l, x) \cdot k \right] . \quad (5.2) $$

$^{11}$These are the 'reconstruction theorems' showing that Minkowski correlation functions obtained by analytic continuation will obey the Wightman axioms if the theory obeys the usual Euclidean axioms such as reflection positivity. There are, of course, no such rigorous results for 4-dim gauge theories.
The Gaussian integral can now be performed by translation $k^\mu \to k^\mu - B/A$, scaling $k^\mu \to k^\mu/A^{-1/2}$, and finally Wick rotation to obtain:

$$i\left(\frac{\pi}{A}\right)^{d/2}V\left(\{p^\mu\}, \{l^\mu\}, \{\partial/\partial B^\mu, \partial/\partial B^\nu, \cdots, \partial/\partial B^\lambda\}\right) \exp\left(-B^2/A\right). \quad (5.3)$$

The momenta again appear only quadratically and linearly in the exponent. Performing the differentiations and picking another loop momentum $l^\mu = k'^\mu$, we have the $k'$-integration in the form (5.1). Continuing in this way all momenta integrations are then performed. \[\text{[72x699]}\]

We stress that the above is nothing but the standard use of Schwinger parameters. It goes through in the present context because the usual polynomial vertices are generalized to entire functions which again do not introduce any singularities in the integrands. The steps (5.1) - (5.3) may in fact be viewed here as a definition of the computational rules relating Minkowski to Euclidean Feynman rules.

### 6 Unitarity and Causality

As discussed in section 4, the tree-level propagator is arranged to have, at the chosen renormalization scale, no gauge-invariant unphysical poles. We now turn to the issue of unitarity and causality to any order in the loop expansion. Again, with the convenient choice $w(z) = \hbar(z)$ for the gauge weight function, the propagator (2.3) in configuration space is:

$$D_{\mu\nu ab}(x-x') = -\delta_{ab} \left( g_{\mu\nu} - (1 - \xi) \frac{\partial_\mu \partial_\nu}{\square} \right) \hbar^{-1}(\square) D(x-x') \quad (6.1)$$

where $D(x)$ is the usual bare massless scalar propagator. $D(x)$ has the decomposition

$$D(x-x') = \theta(x^0-x'^0)D^+(x-x') + \theta(x'^0-x^0)D^-(x-x') \quad (6.2)$$

where

$$D^\pm(x) = \frac{1}{(2\pi)^4} \int d^4k e^{-ikx} D^\pm(k) = \frac{1}{(2\pi)^4} \int d^4k e^{-ikx} 2\pi\theta(\pm k^0)\delta(k^2) \quad (6.3)$$

are the usual $\pm$ energy functions. (6.2) implies that $D(x)$ obeys the KL representation.

Substituting (6.2) into (6.1) gives for $x^0 \neq x'^0$:

$$D_{\mu\nu ab}(x-x') = \theta(x^0-x'^0)\hbar^{-1}(\square)D^+_{\mu\nu ab}(x-x', \xi) + \theta(x'^0-x^0)\hbar^{-1}(\square)D^-_{\mu\nu ab}(x-x', \xi), \quad x^0 \neq x'^0 \quad (6.4)$$

\[\text{[81x115]}\]

Some regularization, e.g. dimensional, or cutting an $\epsilon > 0$ off the Schwinger parameter integration region, is, of course, implicitly used so that this series of steps be always well defined. Since in the UV regions all vertices behave as polynomials, any of a number of conventional schemes may be used.
with
\[ D_{\mu\nu}^\pm(x, \xi) \equiv -\delta_{\mu\nu} \left( g_{\mu\nu} - (1 - \xi) \frac{\partial_{\mu}\partial_{\nu}}{\Box} \right) D^\pm(x) . \] (6.5)

\( D_{\mu\nu}^a(x - x') \) no longer satisfies this decomposition at \( x^0 = x'^0 \) where the r.h.s. of (6.4) differs from \( D_{\mu\nu}^a \) by \( x^0 \)-contact terms, i.e. terms proportional to \( \delta(x^0 - x'^0) \), and its derivatives, resulting from the action of the derivatives in (6.1) on the \( \theta \)-functions in (6.2). Now the derivation of results such as unitarity and causality conditions via largest time equations rely on the decomposition (6.4). Equal times regions may then, in some cases, require special consideration as the contact terms present a technical, but for the most part innocuous, complication.\[\text{13}\]

Recall first the case of ordinary gauge theories, i.e. take \( h = 1 \). We write \( D_{\mu\nu}^a \) for (6.1) with \( h = 1 \). When the theory possesses gauge invariance, explicit consideration of equal time contact terms becomes unnecessary since the contribution of such terms must cancel in any physical amplitude, as indicated by the presence of gauges where they are absent. In particular, in the Feynman gauge, \( \xi = 1 \), all derivatives are eliminated, and one indeed has
\[ D_{\mu\nu}^a(x) = \theta(x^0)D_{\mu\nu}^+(x) + \theta(-x^0)D_{\mu\nu}^-(x) \] (6.6)
where \( D_{\mu\nu}^a(x) \equiv D_{\mu\nu}^a(x, \xi = 1) \), valid for all \( x^0 \). For the ghost propagator \( D_{ab}(x) \equiv \delta_{ab}D(x) \), the corresponding equation follows trivially from (6.2) with \( D_{ab}^\pm(x) \equiv \delta_{ab}D^\pm(x) \). Given (6.6), one may proceed to derive \[8\],[9] the largest time equation, and hence cutting rules leading to unitarity conditions for physical amplitudes; and then note that, by virtue of the WI, these equations continue to hold if one replaces \( D_{\mu\nu}^a \) with (6.5) for arbitrary \( \xi \).

To follow the same procedure in the case we are considering here, when \( h \neq 1 \), appears at first somewhat problematic. Since the action of \( \overline{h}(\Box) \) induces contact term in all parts of the propagator, these cannot be cancelled by gauge invariance alone. (This, of course, reflects the fact that \( h \neq 0 \) is an actual, gauge invariant modification of the usual gauge theory action.)

It is, however, not difficult to circumvent this problem. The trick is to use \( D_{\mu\nu}^a(x) \) as the bare propagator, and include the \( \overline{h}(\Box) \) factors in the vertices where the propagator line ends. More precisely, write (2.3) as
\[ D_{\mu\nu}^a(k) = \overline{h}^{-1/2}(k^2) D_{\mu\nu}^a(k) \overline{h}^{-1/2}(k^2) , \] (6.7)
\[\text{14}\]
and define vertices:

\[ \hat{V}_{\mu_1 a_1 \ldots \mu_n a_n}(k_1, \ldots, k_n) \equiv V_{\mu_1 a_1 \ldots \mu_n a_n}(k_1, \ldots, k_n) \prod_{i=1}^{n} \bar{h}^{-1/2}(k_i^2) \]

\[ \hat{V}_{\mu abc}(k, p, q) \equiv V_{\mu abc}(k, p, q) \bar{h}^{-1/2}(k^2) \]

\[ \hat{J}_{\mu a}(k) \equiv J_{\mu a}(k) \bar{h}^{-1/2}(k^2) , \quad (6.8) \]

Here \( V_{\mu_1 a_1 \ldots \mu_n a_n} \) are the perturbative vertices from the expansion of the action (1) (Section 3), with \( V_{\mu abc}(k, p, q) \) the ghost-ghost-gauge boson vertex, and we redefined external sources \( J \) by inserting \( \bar{h}^{-1/2} \) factors. For this to work, it is, of course, crucial that \( h(z) \neq 0 \) and singularity-free for all \(|z| < \infty\); we set \( \bar{h}^{-1/2} \equiv \exp(-1/2H) \). This assignment of \( \bar{h} \) factors to vertices is, of course not unique, but a convenient, symmetric choice.

Consider now diagrams constructed from propagators \( D \), and vertices \( \hat{V}, \hat{J} \), the FP propagator remaining unchanged. We will refer to these rules as the ‘alternative’ rules. It is immediately seen that for any diagram contributing to any \( n \)-point function between arbitrary sources the same result is obtained with these rules as with the original rules (propagator \( D \), vertices \( V, J \)):

\[ G_{\alpha_1 \ldots \alpha_n}(p_1, \ldots, p_n) \prod_{i=1}^{n} J_{\alpha_i}^{i}(p_i) = \hat{G}_{\alpha_1 \ldots \alpha_n}(p_1, \ldots, p_n) \prod_{i=1}^{n} \hat{J}_{\alpha_i}^{i}(p_i) . \quad (6.9) \]

(Here the label \( \alpha_i \) stands for all polarization, Lorentz and group indices pertaining to the \( i \)-th leg; and different sources \( J^i \) may be chosen for each leg.)

For \( S \)-matrix elements all legs, in addition to be truncated, must be put on mass-shell\(^{14} \) and all wave-functions appropriately chosen. Now

\[ \bar{h}^{1/2}(0) = \bar{h}^{-1/2}(0) = 1 , \quad (6.10) \]

so all \( \bar{h} \) factors on external on-shell legs are actually irrelevant; and the residue of the (dressed) gauge boson propagator at the pole:

\[ k^2 \tilde{D}_{\mu \nu ab}(k) |_{k^2=0^+} = k^2 \bar{h}^{-1/2}(k^2) \tilde{D}_{\mu \nu ab}(k) \bar{h}^{-1/2}(k^2) |_{k^2=0^+} \]

\[ = k^2 \tilde{D}_{\mu \nu ab} |_{k^2=0^+} \]

\[ = \left[ -g_{\mu \nu} - (Z_3 - 1) \left( g_{\mu \nu ab} - \frac{k_\mu k_\nu}{k^2} \right) \right] |_{k^2=0^+} \]

\[ (6.11) \]

is the same for both sets of rules. (In (6.11) \( \tilde{D} \) and \( \tilde{D} \) denote the full propagators in the two sets of rules.) The residue defines the wave-function renormalization constant

\(^{14}\)As usual with massless poles, to avoid on-shell IR divergences, the mass-shell condition is taken with an infinitesimal mass, denoted \( k^2 = 0^+ \).
Each physical external wave-function, in addition to being of physical polarization, must be normalized by a factor of \(1/\sqrt{Z_3}\) in order to correctly account for the contribution of self-energy insertions on external legs.\(^{15}\) Truncating and putting all legs on shell then, we have the equality for amplitudes:

\[
A(p_1, \ldots, p_n) \mid p_i^2=0^+ = \prod_{i=1}^n J^{\alpha_i}_i(p_i) \prod_{i=1}^n p_i^2 G_{\alpha_1, \ldots, \alpha_n}(p_1, \ldots, p_n) \mid p_i^2=0^+
\]

\[= \prod_{i=1}^n J^{\hat{\alpha}_i}_i(p_i) \prod_{i=1}^n \hat{p}_i^2 \hat{G}_{\hat{\alpha}_1, \ldots, \hat{\alpha}_n}(p_1, \ldots, p_n) \mid p_i^2=0^+, \quad (6.12)
\]

Note that (6.12) holds for arbitrary \(J\)'s of any polarization; and by the equality (6.11), it continues to hold for correctly normalized physical wave functions. Furthermore, (6.9) - (6.12) are also valid directly in the renormalized theory, since the counterterms in (6.8) are included in the set of vertices \(V\). Once more, the absence of singularities in \(h(k^2)\), and the fact that its functional form does not change under renormalization, are crucial for obtaining the simple relations (6.9) - (6.12). These relations allow one to view any diagram as constructed according to either set of rules. One may then proceed pretty much as in the case of ordinary gauge theories.

**Unitarity**

We construct amplitudes using the alternative rules in the Feynman gauge. The bare propagator \(D_{\mu\nu ab}\) then satisfies the decomposition (6.6). The vertices \(\hat{V}\), eq. (6.8), are real for real momenta, and given by entire functions possessing no singularities, in particular no poles or cuts anywhere in the finite complex plane. It follows (Appendix C) that the Veltman largest time equation [8] holds, which in turn implies the cutting equation (generalized Cutkosky rule):

\[
\includegraphics[width=0.8\textwidth]{cutting_equation.png} \quad \text{(6.13)}
\]

(6.13) is a general cutting rule that applies to a single diagram, or to any collection of diagrams represented by the blob, and for arbitrary external wave-functions and momenta. On the shaded side of the 'cuts', each explicit factor of \(i\) assigned to each vertex and propagator is changed to \(-i\), and each \(i\epsilon\) in each propagator to \(-i\epsilon\). The cut blob on the right hand side stands for the sum over all possible cuts of the diagram, or of each diagram in the collection of diagrams represented by the blob. A possible cut is obtained by cutting propagator lines so that the vertices on the shaded (unshaded) side of the cut form a connected region containing at least one outgoing (ingoing) external

\(^{15}\)Note that these factors are needed to make the \(S\)-matrix gauge invariant whether or not the \(Z_3\)'s are UV finite.
The rules for cut lines are given by:

\[ D^{\pm}_{\mu
u ab}(k) = -g_{\mu\nu}\delta_{ab} D^{\mp}(k) \]

(6.14)

for gauge field and ghost lines, respectively. Although conveniently derived in terms of the alternative rules, the result (6.13), once obtained, may be trivially viewed either in the alternative or the original rules by use of (6.9). Note, in particular, that cut legs are on shell, so, from (6.10)

\[ h^{-1/2}(k^2) D^{\pm}_{\mu
u ab}(k) h^{-1/2}(k^2) |_{k^2=0^+} = D^{\pm}_{\mu\nu ab}(k) |_{k^2=0^+} , \]

and there is no distinction between a cut \( D^{\pm}_{\mu\nu ab}(k) \) and a cut \( D^{\pm}_{\mu\nu ab}(k) \) propagator.

To establish unitarity equations one lets the blobs in (6.13) include all diagrams to a certain order that contribute to a given process, with all external legs truncated and on shell. The physical S-matrix is furthermore defined only between external physical gauge bosons:

\[ p \]

\[ \text{physical S-matrix} \]

(6.15)

The label \( P \) denotes physically polarized on-shell gauge bosons. Note that since the Lagrangian (2.1) is real (hermitean), the rules for diagrams on the shaded side in (6.13) are indeed those for \( S^\dagger \). In the sum over intermediate state cuts on the r.h.s. of (6.15), however, the cuts, as given by (6.14), include cuts over gauge bosons of unphysical polarizations, as well as ghosts. Therefore, physical unitarity will hold only if these unphysical contributions cancel leaving only a sum over physical transverse cuts given by:

\[ D^{\mu\nu \pm}_{\rho\sigma ab}(k) = -g^{\mu\nu}_{\rho\sigma} \delta_{ab} D^{\pm}(k) \]

(6.16)

\[ D^{\mu\nu \pm}_{\rho\sigma ab} \]

arises from summation over only the two physical polarizations satisfying \( e^I_{\mu} k^\mu = 0 \); \( e^I_{\mu} \eta^\mu = 0 \) (\( I = 1, 2 \)), and

\[ \sum_{I=1}^{2} e^I_{\mu} e^I_{\nu} = -g^{\mu\nu} \equiv -g_{\mu\nu} - \left[ \frac{1}{2} (k_{\mu} k_{\nu} + k_{\nu} k_{\mu}) + k^2 \eta_{\mu} \eta_{\nu} \right] \frac{1}{(k \cdot \eta)^2 - k^2} , \]

(6.17)
where \( \eta^{\mu} \) is a timelike unit vector used to fix a timelike polarization direction, and

\[
\bar{k}_\mu \equiv k_\mu - 2(k \cdot \eta)\eta_\mu.
\]  

(6.18)

Having established the cutting equation (6.15), as well as the Ward identities, eq. (B.1), however, the demonstration of the cancellation of unphysical cuts is actually no different than that in the case of ordinary local gauge theory. This is clear since this demonstration relies only on the gauge, or BRS, invariance of the action. It is usually stated at the formal level of the path integral independence of the gauge-fixing term, which, of course, holds here as well. Since, however, we derived the cutting equation by means of the split (6.7)-(6.8) on a graph by graph basis, one should, for completeness, check the cancellation also diagrammatically. We outline the derivation in Appendix B, which, starting from the WI (B.1), verifies explicitly (see (B.7)) that indeed the sum over cuts on the r.h.s. of (6.15) reduces to the sum over only physical cuts (6.16).

\textit{Causality}

Again, working in terms of the alternative rules, the validity of the largest time equation implies (Appendix C) that the Bogoliubov Causality Condition (BCC) [10], [8] is satisfied:

\[
\begin{alignment}
\begin{aligned}
\text{blob} & = x^p_1 \cdots x^p_n \quad y^p_1 \cdots y^p_n \\
& \quad - \theta(y^0 - x^0) x^p_1 \cdots x^p_n \quad - \theta(x^0 - y^0) y^p_1 \cdots y^p_n
\end{aligned}
\end{alignment}
\]  

(6.19)

The blob represents a diagram, or a collection of diagrams, contributing to the \( n + n_1 + n_2 \)-point Green’s function, with \( n \) external (truncated) legs, and with \( n_1 \) and \( n_2 \) legs joined at the space-time points \( x \) and \( y \) by \( n_1 \) and \( n_2 \)-point vertices \( V \), respectively.\(^{16}\) \( n_1 = 1 \) and/or \( n_2 = 1 \) is the case of external source vertices at \( x \) and/or \( y \). The cut blobs stand for the sum over all cuts with the positions of the two vertices at \( x \) and \( y \) as shown. (Again, though the equation is conveniently derived in the alternative rules, it is equally well viewed in terms of the original rules by (B.9) and (B.13).)

The physical meaning of (6.19) is as follows. The first term on the r.h.s. is a (set of) cut diagram(s) representing a contribution to the product \( SS^\dagger \), with the vertices at \( x \) and \( y \) both in the diagram(s) making up the \( S \) factor of the product. We may now apply equation (6.19) to this diagram(s) for the \( S \) factor. Iterating this procedure, the r.h.s. of (6.19) can be reduced entirely to a sum of two groups of terms: one group multiplied by \( \theta(y^0 - x^0) \) and containing only cuts forcing positive energy flow from \( x \) to \( y \), the other group involving the opposite combination.

The vertices at \( x \) and \( y \) act as (multileg) external sources and the legs emanating from them correspond in general to particles off-shell. This is actually what gives

\(^{16}\) (6.19) is actually shown with \( n_1 = n_2 = 2 \).
the obvious intuitive meaning to the above physical interpretation of future directed positive flow since, of course, space-time points cannot be precisely pinpointed by wave-packets representing particles near mass-shell.

Integrating (6.19) over \(x\) and \(y\) converts (6.19) to an equation (now entirely in momentum space) for a (set of) diagram(s) contributing to an \(n\)-point amplitude.\(^{17}\) The l.h.s. is precisely the (set of) diagram(s) for the amplitude in question and is expressed by the r.h.s. in terms of cut graphs in what is in fact a dispersion relation in non-covariant form.\(^{18}\) In the presence of derivative interactions, however, some care must be exercised in converting (6.19) into a dispersion relation by integration over \(x\) and \(y\). This is because (6.19) was strictly derived for \(x^0 \neq y^0\). The action of derivatives at \(x, y\) on the \(\theta\)-function factors, resulting into \(\delta\)-functions and \(\delta\)-function derivatives which give a finite measure contribution to the equal times integration region, must then be properly taken into account as explained in Appendix C. The important special case of the 2-point function is considered below.

## 7 The 2-point function

Consider (6.19) for \(n_1 = n_2 = 1, n = 0\), i.e. for the two point function between sources at \(x, y\). To lowest approximation, where the blob is a single bare propagator line joining \(x\) and \(y\), (6.19) is nothing but eq. (6.4), sandwiched between arbitrary sources, in Feynman gauge (\(\xi = 1\)). Consider next (6.19) for the 2-point function between sources at \(x\) and \(y\) including an arbitrary number of insertions of the self-energy

\[ \Pi_{\mu\nu a b}(x' - x') = i\delta_{a b} \int d^4 k \ e^{-ik(x' - x')} (k_\mu k_\nu - g_{\mu\nu} k^2) \pi(k^2) \quad . \quad (7.1) \]

Summing over all insertions between sources of physical polarization, so as to eliminate the physically inessential longitudinal terms at the outset, one arrives at the BCC equation:

\[
\begin{align*}
\theta(x_0 - y_0) & - \theta(y_0 - x_0) \quad (7.2)
\end{align*}
\]

---

\(^{17}\)The two vertices at \(x\) and \(y\) are now internal vertices. They may also be taken to be external vertices: if they are originally chosen as \(n_1 + n_1'\)- and \(n_2 + n_2'\)-point vertices, respectively, then multiplication by the appropriate external wave-functions and integration over \(x\) and \(y\) converts (6.19) to an equation for an \(n + n_1' + n_2'\)-point amplitude.

\(^{18}\)It is noteworthy that such a dispersion relation can be written for any individual diagram. In some cases this relation may be converted to a more conventional dispersion relation in some external Lorentz invariant as the dispersed variable. For scalar theories this is developed in \([11]\).
In (7.2) the crosses indicate the sources, and the shaded blob stands for the physical transverse full propagator given by

\[
\tilde{D}_{\mu
u ab}^{tr}(x - y) = \delta_{ab} \sum_{I=1,2} \int \frac{d^4k}{(2\pi)^4} e^{I}_\mu(k)e^{I}_\nu(k) \frac{1}{n(k^2)} \frac{i}{k^2 + i\epsilon} \left[ 1 + \frac{e^{-ik(x-y)}}{(2\pi)^4 n^{-1}(k^2)} \right] 
\]

\[
\equiv \delta_{ab} \sum_{I=1,2} e^{I}_\mu(i\partial_x) e^{I}_\nu(-i\partial_y) \tilde{D}(x - y) 
\]

\[
= \delta_{ab} \left(-g_{\mu\nu} - \frac{\partial_x \partial_y}{\partial^2} \right) (1 - \delta^0_\mu)(1 - \delta^0_\nu) \tilde{D}(x - y) 
\]

(7.3)

(the last equality valid in the frame where \(\eta^\mu = (1, 0, 0, 0)\), no sum over \(\mu, \nu\)). Explicitly,

\[
iJ^{\mu a}(i \partial_x) \tilde{D}_{\mu
u ab}^{tr}(x - y)iJ^{\nu b}(-i \partial_y) = -iJ^{\mu a}(i \partial_x) \left[ \theta(x^0 - y^0) \tilde{D}^{+}_{\mu
u ab}(x - y) + \theta(y^0 - x^0) \tilde{D}^{-}_{\mu
u ab}(x - y) \right] (-i)J^{\nu b}(-i \partial_y) , 
\]

\(x^0 \neq y^0\) 

(7.4)

where

\[
\tilde{D}^{\pm}_{\mu
u ab} = \delta_{ab} \sum_{I=1,2} \int \frac{d^4k}{(2\pi)^4} e^{I}_\mu(k)e^{I}_\nu(k) n^{-1}(k^2)\theta(\pm k^0)\rho(k^2, k^2)e^{-ik(x-y)} 
\]

\[
\equiv \delta_{ab} \sum_{I=1,2} e^{I}(i\partial_x)e^{I}(-i\partial_y)\tilde{D}^{\pm}(x - y) 
\]

(7.5)

with

\[
\rho(k^2, q^2) = 2\pi Z_3 \delta(k^2) + \theta(k^2)\sigma(k^2, q^2) 
\]

(7.6)

\[
Z_3 = \left[ 1 + \frac{\pi(0)}{(2\pi)^4} \right]^{-1} 
\]

(7.7)

\[
\sigma(k^2, q^2) = \frac{1}{(2\pi)^4} \frac{2\text{Im} \pi(k^2)}{1 + \frac{\pi(k^2)}{(2\pi)^4 n^{-1}(q^2)}} \frac{1}{n(k^2)} 
\]

(7.8)

(7.3) is the result of cutting the propagator (7.3):

\[
\begin{align*}
\text{crosses} & = Z_3 \text{crosses} + \text{shaded blob} 
\end{align*}
\]

(7.9)

The first term on the r.h.s. of (7.9) represents the \(\delta\)-function term in \(\rho(k^2, k^2)\), and is the contribution to the cut of the pole in (7.3). The second term on the r.h.s. of (7.9)
represents the substitution of the second term in (7.4) into (7.3). It is easily seen that this gives precisely the structure depicted graphically in the second term in (7.9) with the cut self-energy given by:

\[
\Pi_{\mu ab}^\pm(k) = \delta_{ab} \left( -g_{\mu\nu}k^2 + k_\mu k_\nu \right) \theta(k^2) \theta(\pm k^0) 2\text{Im} \pi(k^2) \quad (7.10)
\]

Noting that \( \rho(k^2, q^2) \) is non-vanishing only for \( k^2 \geq 0 \), one can write:

\[
\tilde{D}^\pm(x-y) = \frac{1}{2\pi} \mathcal{H}^{-1} \left( -\Box / \Lambda^2 \right) \int_0^\infty d\kappa \rho(\kappa, -\Box) \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} 2\pi \theta(\pm k^0) \delta(k^2 - \kappa)
\]

\[
= \frac{1}{2\pi} \mathcal{H}^{-1} \left( -\Box / \Lambda^2 \right) \int_0^\infty d\kappa \rho(\kappa, -\Box) \Delta^\pm(x-y, \kappa)
\]

(7.11)

where \( \Delta^\pm(x-y, \kappa) \) is the \( \pm \)-energy function of the free scalar field of mass \( m^2 = \kappa \).

Substituting in (7.4) one obtains

\[
\tilde{D}(x-y) = \frac{1}{2\pi} \mathcal{H}^{-1} \left( -\Box / \Lambda^2 \right) \int_0^\infty d\kappa \rho(\kappa, -\Box) \left[ \theta(x^0 - y^0) \Delta^+(x-y, \kappa) + \theta(y^0 - x^0) \Delta^-(x-y, \kappa) \right]
\]

\[
= \frac{1}{2\pi} \mathcal{H}^{-1} \left( -\Box / \Lambda^2 \right) \int_0^\infty d\kappa \rho(\kappa, -\Box) \Delta(x-y, \kappa)
\]

(7.12)

where \( \Delta(x-y, m^2) \) is the free scalar propagator. Again, this derivation from (7.2)-(7.4) strictly follows for \( x^0 \neq y^0 \), but the result as written in (7.12) correctly includes also the point \( x^0 = y^0 \) (Appendix C). (7.12) is the (generalized) KL representation for the invariant function \( \tilde{D}(x-y) \) in the decomposition (7.3). Reverting to momentum space, one obtains the dispersion relation for the physical transverse full propagator:

\[
\tilde{D}^{tr}_{ijab}(p) = \frac{i}{(2\pi)^4} \delta_{ab} \left( \delta_{ij} - \frac{p_i p_j}{|p|^2} \right) \mathcal{H}^{-1}(p^2 / \Lambda^2) \int_0^\infty d\kappa \frac{\rho(\kappa, p^2)}{p^2 - \kappa + i\epsilon}
\]

\[
= \frac{i}{(2\pi)^4} \delta_{ab} \left( \delta_{ij} - \frac{p_i p_j}{|p|^2} \right) \tilde{D}(p)
\]

(7.13)

The BCC-equation (7.2) for (7.3) was obtained by summing the BCC-equation for the 2-point function with \( n \) self-energy insertions over all \( n \). This summation is exhibited by expanding the denominators in (7.7), (7.8) to write:

\[
Z_3 = \sum_{n=0}^\infty Z_3^{(n)}, \quad \sigma(k^2, p^2) = \sum_{n=0}^\infty \sigma^{(n)}(k^2) \mathcal{H}^{-n}(p^2 / \Lambda^2)
\]

(7.14)

with \( Z_3^{(0)} \equiv 1 \), \( \sigma^{(0)}(k^2) \equiv 0 \), so that

\[
\tilde{D}(p) = \mathcal{H}^{-1}(p^2 / \Lambda^2) \sum_{n=0}^\infty \left\{ \frac{Z_3^{(n)}}{p^2 + i\epsilon} + \frac{1}{2\pi} \mathcal{H}^{-n}(p^2 / \Lambda^2) \int_0^\infty d\kappa \frac{\sigma^{(n)}(\kappa)}{p^2 - \kappa + i\epsilon} \right\}
\]

(7.15)
Applying the BCC-equation (6.19) also to the self-energy (7.1) itself, and proceeding as above, one derives the standard dispersion relation:

\[ \Pi_{\mu\nu ab}(k) = i\delta_{ab}(g_{\mu\nu} - g_{\mu\nu}k^2) \left( \pi(k^2) - \pi(0) \right) = i\delta_{ab}(g_{\mu\nu} - g_{\mu\nu}k^2) \frac{1}{2\pi} \int_0^\infty d\kappa \frac{k^2\sigma_0(\kappa)}{(k^2 - \kappa + i\epsilon)(\kappa - i\epsilon)}, \tag{7.16} \]

where we defined \( \sigma_0(\kappa) \equiv 2\text{Im} \pi(\kappa) \). The dispersion integral over \( \sigma_0(\kappa) \) does not converge, hence requiring the subtracted dispersion (7.16), i.e. renormalization, the subtraction of \( \pi(0) \) being one particular convenient choice of counterterm (giving unit residue to the renormalized dressed propagator (7.3)). Note that the self-energy subtraction does not affect the imaginary part of \( \pi(k^2) \).

\( (7.2), (7.4) \) hold with the self-energy (7.1) computed in any approximation. When an appropriate (gauge invariant) set of graphs is included, the imaginary part \( \sigma_0(k^2) \), and hence the spectral density \( \rho(k^2, p^2) \) in (7.13) is positive. Indeed, from the demonstration of physical unitarity above, only physical positive residue particles propagate then in the cut (7.10). Eqs. (7.12), (7.13), (7.15), (7.16) now make explicit the fact that this positivity is indeed consistent with the improved asymptotic behavior of the propagator in this superrenormalizable theory.

Recall that in the case of the bare propagator, represented by the \( n = 0 \) term in (7.15), the improved asymptotic behavior is due to the \( \tilde{h}(k^2/\Lambda^2) \) factor, which does not contribute to the absorptive part - the only contribution is the usual \( \delta \)-function due to the pole at \( k^2 = 0 \). (7.13), (7.15) show that this continues to hold when radiative corrections are included: the external momentum dependence in the spectral density \( \rho(k^2, p^2) \) in (7.13) is positive. Indeed, from the demonstration of physical unitarity above, only physical positive residue particles propagate then in the cut (7.10). Eqs. (7.12), (7.13), (7.15), (7.16) now make explicit the fact that this positivity is indeed consistent with the improved asymptotic behavior of the propagator in this superrenormalizable theory.

Recall that in the case of the bare propagator, represented by the \( n = 0 \) term in (7.15), the improved asymptotic behavior is due to the \( \tilde{h}(k^2/\Lambda^2) \) factor, which does not contribute to the absorptive part - the only contribution is the usual \( \delta \)-function due to the pole at \( k^2 = 0 \). (7.13), (7.15) show that this continues to hold when radiative corrections are included: the external momentum dependence in the spectral density \( \rho(k^2, p^2) \) in (7.13) is positive. Indeed, from the demonstration of physical unitarity above, only physical positive residue particles propagate then in the cut (7.10). Eqs. (7.12), (7.13), (7.15), (7.16) now make explicit the fact that this positivity is indeed consistent with the improved asymptotic behavior of the propagator in this superrenormalizable theory.

19 This is, of course, the basic point of renormalizability by local counterterms.

20 The fact that the leading asymptotic behavior of the dressed propagator is entirely due to its bare
In summary, eqs. (7.13) - (7.16) show explicitly that the relation
\[ \lim_{p^2 \to \infty} p^2 \tilde{D}^{tr}(p) = 0 \] (7.17)
is indeed consistent with the result that \( Z_3 \geq 0, \sigma_0 > 0 \), i.e. that only positive residue physical particles propagate in the intermediate state cuts.

8 Gravity

In analogy to (2.1), we consider the gravitational action:
\[ \mathcal{L} = \sqrt{-g} \left\{ \frac{\beta}{\kappa^2} R - \beta_2 (R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2) + \beta_0 R^2 + \lambda \\
+ \left( R_{\mu\nu} h_2 (-\frac{\nabla^2}{\Lambda^2}) R^{\mu\nu} - \frac{1}{3} R h_2 (-\frac{\nabla^2}{\Lambda^2}) R \right) - R h_0 (-\frac{\nabla^2}{\Lambda^2}) R \\
- \frac{1}{2\xi} f^\mu[g] \nabla^\mu \nabla \left(-\frac{\nabla^2}{\Lambda^2}\right) f_\mu[g] + \bar{c}^\mu M_{\mu\nu} c^\nu \right\} \] (8.1)
where \( \nabla^2 = \nabla^\mu \nabla_\mu \) and \( \Box \) denote the covariant and ordinary D' Alembertian, respectively, and \( f_\mu[g] \) is the gauge-fixing function with gauge-term weight \( w \). In (8.1) two terms introducing the transcendental entire functions \( h_2 \) and \( h_0 \) have been added to the general 4-th order renormalizable gravitational Langrangian. Expanding, for convenience, about flat space:
\[ \sqrt{-g} g^{\mu\nu} = \eta^{\mu\nu} + \kappa \phi^{\mu\nu} \] (8.2)
and taking \( f_\mu = \partial_\mu \phi^{\mu\nu} \), the bare propagator is of the form:
\[ D_{\mu\nu\kappa\lambda}(k) = \frac{i}{(2\pi)^4 k^2 + i\epsilon} \left( \frac{2P^{(2)}_{\mu\nu\kappa\lambda}(k)}{\beta - \beta_2 \kappa^2 k^2 + \kappa^2 k^2 h_2 (k^2/\Lambda^2)} \right) \\
+ \frac{4P^{(0)}_{\mu\nu\kappa\lambda}(k)}{\beta + 6\beta_0 \kappa^2 k^2 - 6\kappa^2 k^2 h_0 (k^2/\Lambda^2)} \right) \]
\[ + (\xi - \text{proportional trace and longitudinal parts}) \] (8.3)
where \( P^{(2)} \) and \( P^{(0)} \) denote spin-2 and spin-0 projections, respectively.

The structure of the vertices involving \( h_2 \) or \( h_0 \) is completely analogous to (3.1) - (3.2). Again, in obvious matrix notation, we have:
\[ V^{(N)}(x; x_1, \ldots, x_N) = \text{tr} \left( \frac{\delta^{\nu'}}{\delta \phi^{\nu'\mu'}} \bigg|_{\phi=0} \right) \cdot v^{(n)}(x, \partial_x; x_1, \ldots, x_n) \cdot \frac{\delta^{\nu'}}{\delta \phi^{\nu'}} \bigg|_{\phi=0} \) \\
\[ \equiv \text{tr} \left( \delta^{\mu'} R^{(\mu')} \cdot v^{(n)} \cdot R^{(\mu')} \right) , \quad N = n' + n + n'' \] (8.4)
propagator part, i.e. self-energy insertions do not contribute to it, appears to be a general feature of superrenormalizable theories - cp. \( \phi^3 \)-theory in 4 dimensions.
\[
\begin{align*}
\psi^{(n)} &= \frac{\delta^n}{\delta \phi^n} \sum_{r=0}^{\infty} a_r \left( -\nabla^2 [g] \right)^r_{\phi=0} \\
&= \sum_{l=1}^{n} \sum_{\sigma} \sum_{r=t}^{\infty} a_r S_{\sigma r, l, n}^\sigma \left( \left( -\frac{\Box}{\Lambda^2} \right)^{r-l} \left( \frac{1}{\Lambda^2} \frac{\delta^b}{\delta \phi^b} \left[ -\nabla^2 + \Box \right]_{\phi=0} \right)^l \right) \\
&\equiv \sum_{l=1}^{n} \sum_{\sigma} \psi_{l, n}^\sigma(x, \partial x; x_1, \ldots, x_n).
\end{align*}
\]

In (8.5), \(\{a_r\}\) are the coefficients in the expansion (2.5) of either \(h_2\) or \(h_0\). \(S_{\sigma r, l, n}^\sigma\) stands for the sum of all possible ways of distributing \((r - l)\) powers of \((-\Box)\) in the \(l + 1\) positions among an ordered sequence, indexed by \(\sigma\), of \(l\) factors of \(\frac{\delta^b}{\delta \phi^b} \left[ -\nabla^2 + \Box \right]_{\phi=0}\).

Note that \([-\nabla^2 + \Box]\) can generate any number of \(\phi\) legs, so now \(b \in [1, n - l + 1]\), and the number of ordered sequences labeled by \(\sigma\) is

\[
\Gamma = l^{(n-l)} l!.
\]

This structure \(\psi_{l, n}^\sigma\) is again given by (A.1), and the analysis in Appendix A applied to (8.5) gives its behavior for arbitrary leg momenta configuration. In particular, if any subset of the leg momenta is of order \(k\) with \(k\) growing arbitrarily large, the leading asymptotic behavior of \(\psi_{l, n}^\sigma\) is given by a sum of terms that grow at most either as

\[
h_m^{(s)}(k^2) k^{2s}, \quad 0 \leq s \leq l, \quad m = 0, 2,
\]

or

\[
\frac{1}{k^2} k^{2n_{\text{int}}},
\]

We impose the requirement that both \(h_2\) and \(h_0\) exhibit the same, at most polynomial asymptotic behavior in a neighborhood of the real axis, so that

\[
\gamma \equiv \lim_{|z| \to \infty} \left( \frac{\ln |h_2(z)|}{\ln |z|} \right) = \lim_{|z| \to \infty} \left( \frac{\ln |h_0(z)|}{\ln |z|} \right)
\]

exists. Power counting (Appendix A) then shows that, provided \(\gamma \geq 2\), UV divergences arise solely due to (8.7), and the superficial degree of divergence of any 1PI graph without external ghost legs is bounded by

\[
\delta_G \leq 4 - 2\gamma(L - 1).
\]

Thus, if \(\gamma \geq 3\), only 1-loop diagrams are superficially divergent. Graphs with any external ghost lines are superficially convergent for all \(L\). Thus only gauge-invariant 1-loop divergences occur - the theory is superrenormalizable and is renormalized by the addition of all gauge-invariant 4-th order 1-loop counterterms:

\[
L_R = L + \sqrt{-g} \left\{ \frac{\beta(Z - 1)}{\kappa^2} R - \beta_2(Z_2 - 1)(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3} R^2) + \beta_0(Z_0 - 1)R^2 + \lambda(Z_{\lambda} - 1) \right\},
\]

(8.11)
where all the couplings in (8.11) now signify renormalized couplings at some scale $\mu$. Again, let us note that $h_2$ and $h_0$ are not altered by renormalization.

We now define

$$h_2(z) \equiv \frac{1}{\kappa^2 \Lambda^2} \left( \frac{\alpha (e^{H(z)} - 1) + \alpha_2 z}{z} \right), \quad h_0(z) \equiv \frac{1}{\kappa^2 \Lambda^2} \left( \frac{\alpha (e^{H(z)} - 1) + \alpha_0 z}{6z} \right). \quad (8.13)$$

An explicit form of the functions $h_2$, $h_0$ follows, as before, by imposing conditions (i) - (iii) of section 4 on each of them. We take

$$h_2(z) = \frac{1}{\kappa^2 \Lambda^2} \left( \frac{\alpha (e^{H(z)} - 1) + \alpha_2 z}{z} \right), \quad h_0(z) = \frac{1}{\kappa^2 \Lambda^2} \left( \frac{\alpha (e^{H(z)} - 1) + \alpha_0 z}{6z} \right). \quad (8.13)$$

Here $H(z)$ is again given by (4.3) but with the replacement $\gamma \rightarrow (\gamma + 1)$. (4.9), (4.12) again provide an explicit realization. $\alpha$, $\alpha_2$, $\alpha_0$ are parameters. Assume that the theory has been renormalized at some scale $\mu_0$. Then the bare propagator (8.3) will possess no gauge-invariant pole other than the transverse massless physical graviton pole if we set

$$\alpha = \beta(\mu_0), \quad \alpha_2 \frac{1}{\kappa^2 \Lambda^2} = \beta_2(\mu_0), \quad \alpha_0 \frac{1}{\kappa^2 \Lambda^2} = 6\beta_0(\mu_0). \quad (8.14)$$

This may be viewed as fixing the scale $\Lambda^2$ in terms of the (Planck) scale $1/\kappa^2$ present in the theory:

$$\Lambda^2 = \left( \frac{\alpha_2}{\beta_2(\mu_0)} \right) \frac{1}{\kappa^2}, \quad (8.15)$$

and fixing

$$\beta(\mu_0) = \alpha, \quad \frac{\beta_2(\mu_0)}{\alpha_2} = \frac{6\beta_0(\mu_0)}{\alpha_0}. \quad (8.16)$$

Note that any split between $\beta_2$, resp. $\beta_0$, and a constant part in the entire function $h_2$, resp. $h_0$ (i.e. a coefficient $a_0$ in the expansion (2.5)) is actually a renormalization convention. Thus, again, there is no loss in generality in assuming that we work with a renormalization prescription such that (8.16) holds at some conveniently chosen scale $\mu_0$. If now $\mu_0$ is taken as the renormalization point, $h_2(z) = h_0(z) = \beta(\mu_0) \exp H(z)$, and only the physical massless spin-2 pole occurs in the bare propagator. If another renormalization scale $\mu$ is chosen, the previous discussion (section 4) applies: (8.3) will acquire poles which, however, will cancel in the dressed physical propagator since, by RG invariance, the shift in the bare part leading to their appearance will cancel against a corresponding shift in the self-energy.

The discussion of unitarity and causality, and the 2-point function of the previous two sections goes through here as well in a closely analogous manner.

---

21Cp. the discussion in the vector gauge theory case above.

22As already pointed out, the crucial point is that there are no counterterms that renormalize $\{ a_n | n \geq 1 \}$ altering the non-trivial dependence of $h_2$ and $h_0$ on their argument.
9 Discussion

We have constructed a class of vector gauge and gravitational theories by including in the action a series of derivative terms representing the expansion of a transcendental entire function. The original Lagrangian to which these terms are added is renormalizable. The new terms then render it superrenormalizable, provided the entire function(s) determining the new vertices possess appropriate asymptotic behavior. Additional constraints follow from the requirement that no (gauge- and RG-invariant) unphysical poles are introduced in the propagators. These conditions then fix the class of allowed entire functions. Cutting equations may then be derived within the loop expansion, which allow one to verify that no unphysical cuts occur in the intermediate states to any order. The unitarity and superrenormalizability properties are closely related as discussed in section 7: the self-energy insertions are irrelevant to the improved UV behavior of the propagator.

The arbitrariness within the class of allowed entire functions is that of a theory admitting a class of potentials. This is nothing unusual in field theory. The interesting point here is that this potential depends on derivatives of the fields. The study in this paper was within the perturbative loop expansion. Harder to investigate issues outside perturbation theory, such as global stability, may lead to further constraints on the allowed class of functions, and/or the inclusion of additional terms in the action.

It is straightforward to introduce minimally coupled non-self-interacting matter, i.e. fermions in gauge theories, and any gravitating matter with action bilinear in the matter fields in the gravitational case. It is immediately seen that again only 1-loop graphs can be superficially divergent. In particular, any matter loop embellished by internal gauge boson or graviton propagators is superficially convergent; and so is any graph with any external matter field legs. The superrenormalizability properties of the theory are thus not altered by the matter coupling. Note in this connection that anomalies still occur as usual, since the relevant 1-loop graphs, such as fermion triangle graphs, will be divergent.

Self-interacting matter fields, such as scalars, do introduce multi-loop divergences through subgraphs of multiloops of these matter lines. Thus the theory is rendered merely renormalizable. Whether this introduces any inconsistency remains to be investigated. In this connection one may, of course, consider modifying also the matter Langrangian in a manner analogous to (2.1). For actions only bilinear in the matter fields, such a modification actually introduces nothing new since it may be absorbed in a redefinition of the matter fields. Only interaction terms trilinear or higher in matter fields, if present, will be affected. In this manner one may perhaps obtain

\[ P(\phi) = \nabla^2 \phi \] models and supergravity potentials are familiar examples.

The abelian version of (2.1) minimally coupled to fermions is a particularly interesting case. Since there are no photon self-interaction vertices, the only modification introduced by the \( h \)-term in (2.1) is in the photon propagator. This is then a version of QED with smooth UV behavior and apparently no UV Landau pole.
superrenormalizable theories also in the presence of Higgs scalars. In fact, the troublesome quadratic scalar divergences might be completely eliminated. This question will be addressed elsewhere.

Perhaps more interesting is the fact that the theories introduced here can be expected to have a spectrum of bound states, thus creating their own 'matter': these may be bound states of gauge bosons or gravitons, as well as of externally introduced matter. Indeed, at length scales of order $1/\Lambda$ or smaller, the inverse bare gauge boson or graviton propagator is given by a polynomial of degree $2 + 2\gamma$ or $4 + 2\gamma$, respectively, corresponding to a tree-level confining or ultraconfining potential between particles. This is, of course, reflected in the smooth UV behavior of the theory. At scales larger than $1/\Lambda$, (2.1) and (8.1) reduce to the usual theories, and revert to $1/k^2$ tree-level propagators corresponding to Coulomb potentials. The exact shape of the potential barrier between the two regimes is determined by the particular choice of the function $H$ in (4.3). For sufficiently large $\Lambda$, and exponential fall-offs as in (4.9), it can be made extremely steep. Systems bound by the short distance confining potential will thus be effectively permanently confined, and forming string-like excitations with almost linearly rising spectrum. Note that the formation mechanism for such states is perturbative, indeed classical, since, as we saw, the short distance behavior is completely dominated by the bare propagator. It is, therefore, not to be confused with any long-distance nonperturbative confining interaction arising from the nonperturbative IR dynamics in nonabelian gauge theories (2.1).

Such tightly bound states, e.g. scalars as gauge boson bound states, may have some interesting applications. In the gravitational case, the very smooth UV behavior of power-like approach to vanishing short-distance interaction, and the related nature of the excitation spectrum, have some obvious relevance for the problems of the avoidance of space-time singularities and of entropy in gravitational collapse.

It is plausible that appropriate supersymmetrization of the actions (2.1) and (8.1) may lead to cancellation of the remaining 1-loop divergences, and thus perturbatively finite theories.

**Acknowledgements**

Part of this work was done while the author was at the Yukawa Institute of Theoretical Physics. The hospitality and support of the Institute is gratefully acknowledged. I would also like to thank the theory group at the Physics Department, University of Tokyo (Komaba), for their hospitality and discussions, and Z. Bern and H. Sonoda for discussions.
A Appendix - Vertices and power counting

The general structure of each term in the sum \( S_{\alpha r,l,n} \cdot \mathbf{F}^{(n')} \) appearing in (3.1)-(3.2) is of the form:\(^{25}\)

\[
\sum_{\{n_i\}} (-\Box)^{n_1} B_1 (-\Box)^{n_2} B_2 (-\Box)^{n_3} \cdots (-\Box)^{n_l} B_l (-\Box)^{n_{l+1}} \cdot \mathbf{F}^{(n')} ,
\]

where

\[
B_i \equiv \frac{\delta}{\delta A_{b_i}} [ -\nabla^2 + \Box ]_{A=0} , \quad \sum_{i=1}^{l} b_i = n , \quad b_i \in \{1, 2\} .
\]

(A.2)

Note that each \( B_i \) is the source of either one derivative- coupled vertex leg contributing a linear power of momentum (\( b_i = 1 \)), or two legs and no powers of momentum (\( b_i = 2 \)). Let \( \delta^{(l)} \) denote the number of derivative-coupled (single-legged) \( B_i \)'s in (A.1). The following relation then holds:

\[
\delta^{(l)} + n = 2l
\]

(A.3)

The \( B_i \)'s in (A.1) are in a fixed ordered sequence labeled \( \alpha \); the number of such sequences is given by (3.3).

Given the sequence of the \( l \) \( B_i \)'s, there are \( l + 1 \) positions for distributing \( (r - l) \) factors \( \Box \) between them. The sums in (A.1) then are over all sets \( \{n_i\} \) such that

\[
n_1 + n_2 + \cdots + n_{l+1} = (r - l)
\]

(A.4)

The number of solutions to (A.4) is equal to\(^{26}\)

\[
\binom{(l + 1) + (r - l) - 1}{r - l} = \binom{r}{l} .
\]

(A.5)

Going over to momentum space, let \( q_i \) denote the sum of momenta of the legs emanating from \( B_i, i = 1, \ldots, l \), and \( q_{l+1} \) the sum of momenta of the legs from \( \mathbf{F}^{(n')} \). Then \((-\square)^{n_i}\) generates a factor of \((Q_i^2)^{n_i}\), where \( Q_i \equiv \left( \sum_{i'=1}^{l+1} q_{i'} \right) \) is the sum of the vertex momenta incoming through the \( \mathbf{F}^{(n')} \) factor, and all \( B_{i'} \)'s, \( i' = i, \ldots, l \), to the right\(^{27}\) of \((-\square)^{n_i}\). (We set \( Q_{l+1} = q_{l+1} \).) In general, for some vertex momenta configurations, equalities among some of the linear combinations \( Q_i \) may occur because of cancellations in partial momenta sums. (To examine the convergence properties of graphs we will need, according to the proof of the power counting theorem, ascertain

\[^{25}\text{To avoid notational clutter, in (A.1)-(A.2) and the rest of this section division of all d’Alembertians by } A^2 \text{ is left implied.}\]

\[^{26}\text{This is the number of colourings of } (r - l) \text{ indistinguishable balls (} m_j \text{ balls in the case of (A.6) below), with } (l + 1) \text{ colours (} l_j \text{ colours in (A.6)), and repetitions of any colour allowed.}\]

\[^{27}\text{Note that ’left’ and ’right’ can be interchanged by overall vertex momentum conservation.}\]
the behavior of the vertices along every hyperplane in the space of the momenta.) Let the \((l + 1)\) possible positions of the \(\Box\)’s in \((A.1)\) be split into \(J\) disjoint sets, where the \(j\)-th subset \((j \in [1,J])\) consists of \(l_j\) positions all giving the same momentum factor \(Q_j^2\). Let \(n_i^{(j)},\ i=1,\ldots,l_j\) denote the exponents in \((A.1)\) in these \(l_j\) positions, and

\[ \sum_{i=1}^{l_j} n_i^{(j)} = m_j \quad . \]  

(6)

Then

\[ \sum_{j=1}^{J} l_j = l + 1 \quad , \quad \sum_{j=1}^{J} m_j = r - l \quad , \]

(7)

where \(1 \leq J \leq l + 1\). So for \(J = 1, l_1 = l + 1, m_1 = r - l\) and all \(l + 1\) positions give the same \(Q\); for \(J = l + 1\), each of the \(l + 1\) positions results in a different partial momentum sum \(Q_i\), and \(l_j = 1, j = 1, \ldots, l + 1\). There are \(\binom{m_j + (l_j - 1)}{m_j}\) solutions to the constraint \((A.6)\).

The vertex \(v_{l,n}^\alpha\) in momentum space is then of the form:

\[
\left[ \sum_{r=l}^{\infty} \sum_{\{m_i\}} \delta_{r-l, m_1+\ldots+m_J} a_r \prod_{j=1}^{J} \left( m_j + (l_j - 1) \right) Q_j^{2m_j} \right] \phi_{l,n}^\alpha(\{Q\}) 
\equiv S_{l,n} \phi_{l,n}^\alpha(\{Q\}) \quad .
\]

(A.8)

Here \(\phi_{l,n}^\alpha(\{Q\})\) denotes the product of the \(l\) factors \(B_i\) in \((A.1)\), and carries all spacetime and group indices, here suppressed, on \(v_{l,n}^\alpha\). Each \(B_i\) is a linear combination of \(Q_i\) and \(Q_{i+1}\) if one-legged, and has no momentum dependence if two-legged. Now

\[
S_{l,n} = \prod_{j=1}^{J} \frac{((l_j - 1)!)^{-1} d(l_j - 1)}{dQ_j^{2(l_j - 1)}} \left( \sum_{r-l=0}^{\infty} \sum_{\{m_i\}} \delta_{r-l, m_1+\ldots+m_J} a_r \prod_{i=1}^{J} Q_i^{2(m_i+(l_i-1))} \right) 
\]

\[
= \prod_{j=1}^{J} \frac{((l_j - 1)!)^{-1} d(l_j - 1)}{dQ_j^{2(l_j - 1)}} \left( \sum_{\{m_i\}=0}^{\infty} a_{J-1+\sum_i (m_i+l_i-1)} \prod_{i=1}^{J} Q_i^{2(m_i+l_i-1)} \right) 
\]

\[
= \prod_{j=1}^{J} \frac{((l_j - 1)!)^{-1} d(l_j - 1)}{dQ_j^{2(l_j - 1)}} \left( \sum_{\{m_i\}=0}^{\infty} a_{J-1+\sum_i m_i} \prod_{i=1}^{J} Q_i^{2m_i} \right) 
\]

\[
= \prod_{j=1}^{J} \frac{((l_j - 1)!)^{-1} d(l_j - 1)}{dQ_j^{2(l_j - 1)}} S_J \left( Q_1, \ldots, Q_J \right) \quad .
\]

(A.9)

To evaluate \(S_J\) we may assume that the \(Q_i\) are ordered, if necessary by relabeling, into descending order: \(Q_1^2 > Q_2^2 > \ldots > Q_J^2\). Note that \(Q_j^2 \neq Q_j^2\) by assumption. Then

\[
S_J \left( Q_1, Q_2, Q_3, \ldots, Q_J \right) 
= \sum_{\{m_i=0 \mid i \geq 3\}} \sum_{m_2=0}^{\infty} \sum_{m_1=m_2}^{\infty} a_{J-1+m_1+m_3+\ldots+m_J} Q_1^{2m_1} \left( \frac{Q_2^2}{Q_1^2} \right)^{m_2} \prod_{i=3}^{J} Q_i^{2m_i}
\]

29
From (A.14) one easily obtains
\[ C_J \]
We may now iterate (A.10) to perform the rest of the sums. After (\( J - 1 \)) iterations one obtains:
\[ S_J(Q_1, Q_2, \ldots, Q_J) = \sum_{k=1}^{J} C_k(\{Q^{1}_{ij}\}_{i < j \leq k}) \prod_{m > k} \left( 1 - \frac{Q_{m}^{2}}{Q_{k}^{2}} \right)^{-1} \left( \frac{Q_{k}^{2}}{Q_{i}^{2}} \right) \tilde{h}_J(Q^{2}_{k}) \]
where
\[ \tilde{h}_J(Q^{2}) \equiv \sum_{m=J-1}^{\infty} a_m Q^{2m} = h(Q^{2}) - (1 - \delta_{J1}) \left( \sum_{m=0}^{J-2} a_m Q^{2m} \right) , \]
and the \( C_k(\{Q^{1}_{ij}\}_{i < j \leq k}) \)'s are given through
\[ C_1 = 1 \]
\[ C_k = - k \prod_{j=s+1}^{k} \left( 1 - \frac{Q_{j}^{2}}{Q_{s}^{2}} \right)^{-1} , \quad k \geq 2 . \]

From (A.14) one easily obtains \( C_k, k \geq 2 \), as a sum of \( 2^{k-2} \) terms, each term a product of \( k - 1 \) factors and of the form \( \pm \prod_{(ij)} \left( 1 - Q_{j}^{2}/Q_{i}^{2} \right)^{-1} \) over sets of pairs \((ij)\), with \( 1 \leq i < j \leq k \); for example, for \( k = 3 \), \( \{(i j)\} = \{(12), (13)\} \) and \( \{(12), (23)\} \).

Eqs. (A.8), (A.9), and (A.11)-(A.14) allow one to examine the behavior of the vertices in any direction in the space of the vertex momenta. For general UV power counting, assume that some of the \( l + 1 \) \( Q_i \)'s grow as \( Q_i = s_i k + s'_i q \sim s_j k \), where \( s_i, s'_i \) constants, \( q \) some finite momentum and \( k \) grows arbitrarily large. Let then a subset \( K \) consisting of \( K \) of the above \( J \) sets of equal \( Q' \)'s grow as \( Q_j \sim s_j k, j \in K \); whereas the set \( L \) of the other \( L = J - K \) sets stays finite, \( Q_j = s_j q, j \in L \). Let
\[ \sum_{j \in K} l_j = l', \quad \sum_{j \in L} l_j = l'', \quad l' + l'' = l + 1 , \]
so \( 1 \leq K \leq l' \), \( 0 \leq L \leq l'' \). Splitting the summation in (A.14) as \( S_J = S_K + S_L \), one easily obtains from (A.11)-(A.14) the leading asymptotic behavior:

\[ \text{For considerations of asymptotic behavior, } Q \text{ and } Q' \text{ are considered to belong to the same set if } Q \sim Q' \sim c_j k, j \in K. \]
\[ \frac{d^{l''-L}}{dq^{2(l''-L)}} \frac{d^{l'-K}}{dk^{2(l'-K)}} S_K \sim \sum_{s=0}^{l'-K} \frac{h^{(2)}(k^2)}{k^{2(l-s)}} - \frac{a_{J-2}}{k^{2(l+2-J)}} , \quad J \geq 2 \quad (A.15) \]

\[ \sim h^{(1)}(k^2) , \quad J = K = 1 \quad (A.16) \]

\[ \frac{d^{l''-L}}{dq^{2(l''-L)}} \frac{d^{l'-K}}{dk^{2(l'-K)}} S_L \sim \frac{y(q^2)}{k^{2(l'-K+1)}} \quad (A.17) \]

where \( y(q^2) \) is a combination of \( h(q^2) \) and its derivatives, and powers of \( 1/q^2 \).

Since each \( B_i \) is at most linear in the momenta, we have

\[ \phi_{i,n}^0 \leq B k^{\tilde{l}} , \quad \tilde{l} \leq \delta^{(I)} \leq l , \quad (A.18) \]

for some constant \( B \). If all \( B_i \)'s are one-legged (derivative coupled, \( \delta^{(I)} = l \)), then for non-exceptional vertex momenta configurations \( \tilde{l}' = l' \) for \( l + 1 \notin K \), or \( \tilde{l}' = l' - 1 \) for \( l + 1 \in K \). In general, however, we may have \( \tilde{l}' \neq l', l' - 1 \).

Now, starting at the \( i = l + 1 \) position in \( (A.11) \) count the total number of transitions \( K \rightarrow K, L \rightarrow K, K \rightarrow L \) encountered as one proceeds to \( i = 1 \). (Only transitions between distinct elements count in \( K \rightarrow K \).) Note that among the first \( l \) positions of the \( Q_i \) in \( (A.11) \) there occur either all \( K \) of the sets in \( K \) if \( l + 1 \notin K \), or at least \( K - 1 \) of the sets in \( K \) if \( l + 1 \in K \); and only a change \( K \rightarrow L \) may contribute positively to the difference \( \tilde{l}' - l' \). It follows that the number of transitions is greater or equal to \( (K + (\tilde{l}' - l')) \). Now, for any pair \( Q_{i-1}, Q_i = q_i + Q_{i-1} \), only if \( q_i \sim k \), i.e if momentum of order \( k \) is injected into \( B_i \), can any of the following three occur: \( i \) and \( i - 1 \) belong to distinct elements of \( K \); or \( i \in K \) while \( i - 1 \notin L \); or \( i \in L \) while \( i - 1 \in K \). Let then \( n^{\text{int}} \) denote the number of legs among the \( B_i \) carrying momentum of order \( k \). Then

\[ n^{\text{int}} \geq K + (\tilde{l}' - l') \quad , \quad (A.19) \]

and

\[ \frac{1}{k^{2(l'-K+1)}} k^{\tilde{l}'} \leq \frac{1}{k^{2}} k^{K+(\tilde{l}' - l')} \leq \frac{1}{k^{2}} k^{n^{\text{int}}} . \quad (A.20) \]

Noting that \( 0 \leq l' - K \leq l \) and

\[ l + 2 - J = (l'' - L) + (l' - K + 1) \geq (l' - K + 1) \quad , \quad (A.21) \]

and combining \((A.8), (A.9), (A.15), (A.17), (A.18), (A.3)\) and \((A.20)\), we arrive at eqs. \((3.4) - (3.5)\) of the main text.

**Power Counting**

Let \( \delta^F \) denote the number of derivative-coupled lines emanating from the \( F^2 \) factors in any given vertex in \( (2.1) \). Note that \( 0 \leq \delta^F \leq 2 \), and:

\[ \delta^F + n^F = 4 \quad (A.21) \]

\(^{29}\text{Cancellations may occur in the partial sums } Q_i, \text{ so that for some } j, Q_j \sim q, \text{ but } B_j \sim k. \text{ An example would be } q_j = -sk, Q_{i-1} = sk + q. \text{ Similarly, it may be that } Q_j \sim k, \text{ but } B_j \sim q. \text{ Also, two-legged } B_i \text{'s contribute no powers of momentum. The first case contributes positively, the latter two negatively to } l' - l'. \)
where \( n^F, 1 \leq n^F \leq 4 \), denotes the number of legs emanating from the \( F \)-factors in the vertex. Let

\[
\gamma^{(s)} \equiv \lim_{|z| \to \infty} \lim_{\text{Im } z \to 0} \left( \frac{\ln |h^{(s)}(z) z^s|}{\ln |z|} \right).
\]

Consider any 1PI graph \( G \) constructed with the vertices (3.1). The superficial degree of divergence is:

\[
\delta_G \leq \sum_{i \in V_1} \left( \delta_i^F \int + 2\gamma^{(s_i)} - n_i \right) + \sum_{i \in V_2} \left( \delta_i^F \int + n_i^\text{int} - 2 \right)
- \sum_{\text{int}} (2 + 2\gamma) + 4(I - V + 1) \quad (A.23)
\]

The index \( i \) enumerates the vertices which are split into two sets \( V_1 \) and \( V_2 \) with asymptotic behavior (3.4) and (3.5), respectively. The suffix 'int' indicates restriction to internal lines only. \( V = V_1 + V_2 \) is the total number of vertices, and \( I \) the number of internal lines \( \{i^\text{int}\} \). Noting that \( \delta_i^F \int \leq \delta_i^F \), and \( \delta_i^F \int \leq n_i^F \int \), (A.23) may be rewritten as

\[
\delta_G \leq \sum_{i \in V_1} \left( \delta^F_i - n_i + 2\gamma^{(s_i)} - \frac{1}{2} \sum_{\text{int}} 2(\gamma - 1) - 4 \right)
+ \sum_{i \in V_2} \left( n_i^F \int + n_i^\text{int} - \frac{1}{2} \sum_{\text{int}} 2(\gamma - 1) - 6 \right) + 4 \quad (A.24)
\]

Using (A.21), and the fact that each vertex must have at least two internal legs, and provided \( \gamma \geq 1 \), we obtain

\[
\delta_G \leq \sum_{i \in V_1} \left( -N_i + 2(\gamma^{(s_i)} - \gamma) + 2 \right) + \sum_{i \in V_2} \left( N_i^{\text{int}} - (\gamma - 1)N_i^\text{int} - 6 \right) + 4
\leq \sum_{i \in V_1} (2 - N_i) - \sum_{i \in V_2} \left( N_i^\text{int}(\gamma - 2) + 6 \right) + 4 \quad (A.25)
\]

In the last step we used the important fact that the (by construction) polynomial asymptotic behavior of \( h(z) \) means that

\[
\gamma^{(s)} - \gamma \leq 0 \quad . \quad (A.26)
\]

We set \( N_i \equiv n_i^F + n_i \), and similarly for \( N_i^\text{int} \), for the total number of lines and internal lines, respectively, out of the \( i \)-th vertex. Now \( N_i \geq 2 \), so the sum over the \( V_1 \) vertices in (A.25) is always non-positive. In fact we have the topological relation

\[
\sum_{i \in V_1} (N_i - 2) = 2(L_1 - 1) + E_1 + I_{12} \quad (A.27)
\]
where $L_1$ is the number of loops in $G$ constructed entirely of vertices in $\mathcal{V}_1$, and $E_1$ is the number of external lines attached to $\mathcal{V}_1$ vertices, while $I_{12}$ is the number of internal lines in $G$ connecting a $\mathcal{V}_1$ and a $\mathcal{V}_2$ vertex. Thus

$$\delta_G \leq 4 - 6V_2 - \sum_{i \in \mathcal{V}_2} (\gamma - 2)N_i^{\text{int}},$$

(A.28)

and, provided $\gamma \geq 2$, we always have $\delta_G < 0$ if $\mathcal{V} \neq \emptyset$, i.e. $V_2 \geq 1$ - any graph containing one or more $\mathcal{V}_2$ vertices is superficially convergent.

The only superficially divergent graphs then are those containing solely $\mathcal{V}_1$ vertices, i.e. the parts in (3.2) with asymptotic behavior (3.4). (A.25) with $\mathcal{V}_2 = \emptyset$ and (A.27) with $I_{12} = 0$ give:

$$\delta_G \leq 4 - E - 2(L - 1)$$

(A.29)

A slightly better estimate can be obtained directly from (A.23) with $\mathcal{V}_2 = \emptyset$:

$$\delta_G \leq \sum_i \left( \delta_i^{\text{F}} + 2\gamma^{(m_i)} - n_i - 4 \right) - 2(\gamma - 1)I + 4$$

$$\leq \sum_i \left( -n_i^{\text{F}} - n_i + 2 + 2(\gamma^{(m_i)} - \gamma) \right) - 2(\gamma - 1)(L - 1) + 4$$

$$\leq \sum_i \left( 2 - N_i \right) - 2(\gamma - 1)(L - 1) + 4$$

$$= 4 - 2\gamma(L - 1) - E$$

(A.30)

which is eq. (3.7) in the main text.

In the same manner, it is straightforward to include in (A.23) the usual YM vertices from the $\frac{1}{g^2}F^2$ and ghost terms in (2.1), and with the same result (3.7). In particular, graphs with one or more $\frac{1}{g^2}F^2$ vertices, or any external ghost legs are superficially convergent. This is actually rather obvious since these vertices and the ghost propagator remain unaltered, but the gauge propagator has improved UV behavior.

**Gravity**

The general structure of each term in the sum $S_{\alpha,r,l,n}^\alpha \cdot R^{n'}$ is again of the form (A.1) where now

$$B_i \equiv \frac{\delta}{\delta \phi^{b_i}} \left[ -\nabla^2 + \square \right] \big|_{\phi=0}, \quad \sum_{i=1}^l b_i = n, \quad b_i \in [1, (n - l) + 1].$$

(A.31)

Each $B_i$ contributes two derivatives, so $\delta^{(l)} = 2l$ is the total number of powers of momenta contributed by the $l$ $B_i$’s.

Given a sequence of the $l$ $B_i$’s, the analysis of the distribution of the ($-\square$)’s among the $l + 1$ positions is as before. Thus starting from (A.4), one repeats the argument leading to the series forms (A.8)-(A.9), and the summation result (A.11)-(A.14), and asymptotic forms (A.15)-(A.17). We now have

$$\phi_{\alpha,r}^{\alpha} \leq B k^{2\tilde{l}}, \quad 2\tilde{l} \leq \delta^{(l)} = 2l,$$

(A.32)
while \((A.19)\) remains unchanged. Collecting results as before, we obtain eqs. \((8.7)-(8.8)\)

The superficial degree of divergence of a 1PI graph with vertices \((8.4)\) is now

\[
\delta_G \leq \sum_{i \in V_1} (\delta_{R, \text{int}}^i + 2\gamma(s_i)) + \sum_{i \in V_2} (\delta_{R, \text{int}}^i + 2n_i^{\text{int}} - 2) - \sum_{\text{int}} (4 + 2\gamma) + 4(I - V + 1) \tag{A.33}
\]

where \(V_1\) and \(V_2\) again denote the sets of vertices with asymptotic behavior \((8.7)\) and \((8.8)\) respectively, and \(h_2\) and \(h_0\) have, by construction, equal index \(\gamma^{(s)}\) defined in \((A.22)\). Using \(\delta_{R, \text{int}}^i \leq 4\), and \(\delta_{R, \text{int}}^i \leq 2n_i^{\text{int}}\), proceeding as above (cp eqs. \((A.23)-(A.25)\)), we now obtain

\[
\delta_G \leq 4 - \sum_{i \in V_2} ((\gamma - 2)N_i^{\text{int}} + 6) \tag{A.34}
\]

Hence, if \(\mathcal{V} \neq \emptyset\), we always have \(\delta_G < 0\), provided \(\gamma \geq 2\). Now, with \(\mathcal{V}_2 = \emptyset\), \((A.34)\) gives \(\delta_G \leq 4\). A better estimate, however, follows directly from \((A.33)\) with \(\mathcal{V} = \emptyset\):

\[
\delta_G \leq \sum_{i} (\delta_{R, \text{int}}^i + 2\gamma(s_i) - 2\gamma - 4) - 2\gamma(L - 1) + 4 \\
\leq 4 - 2\gamma(L - 1) \tag{A.35}
\]

which is eq. \((8.10)\) in the main text. Again, and for the same reason as in the vector gauge theory case, the same result is obtained if we include any of the original, i.e. \(h\)-independent, gravitational vertices, and ghost vertices.

\section*{B Appendix - Ward identities}

The BRS invariance \((2.2)\) of the action \((2.1)\) implies, in the standard fashion, the Ward identities (WI):

\[
\int [DA][d\bar{c}][dc] \frac{\exp \{ i \int (\mathcal{L} + J \cdot A + \bar{\eta} c + \bar{c} \eta) \}}{\text{vol}(\frac{-\Box}{\Lambda^2})} \left( \partial_\mu A^{a \mu} \right) \eta^a - \frac{1}{2} \bar{\eta} f_{abc} c^b c^c \right] = 0 \tag{B.1}
\]

where \(J, \bar{\eta}, \eta\) are external sources for the gauge and ghost fields.\(^\text{30}\) The form of these identities is, of course, the same as in ordinary gauge theory since they follow only from the gauge (BRS) invariance of the action. We can, therefore, be brief. Differentiating \((B.1)\) w.r.t. \(\eta(x)\) and then setting \(\eta = \bar{\eta} = 0\) gives the ST identities in their usual form,\(^\text{30}\)

\(^{30}\)It should noted that here the absence of gauge dependent divergences means that the identities \((B.1)\) hold directly in the renormalized theory in terms of the renormalized Lagrangian \((B.8)\) without further ado - there is no need for a definition of a renormalized gauge transformation.
from which follows, in particular, the absence of radiative corrections to the longitudinal part of the propagator. Similarly, the WI for on-shell gauge boson amplitudes, needed in the discussion of unitarity, also follow by familiar manipulations:

\[
\begin{align*}
\sum_{i=1}^{n} \bar{\eta} & \equiv \sum_{i=1}^{n} \bar{\eta}_i \\
\end{align*}
\]

The short double line denotes multiplication by \(-ip_\mu\) where \(p_\mu\) is the momentum flowing into the vertex it is attached to. The wavy lines indicate truncated external gauge boson legs of arbitrary polarization, and the label \(o\) that they are on-shell. If, say, the \(j\)-th leg carries physical transverse polarization \(e_\mu\), then \(p_\mu e^\mu = 0\), and the corresponding term is absent in the r.h.s. sum.

By further differentiation of (B.1) w.r.t. \(\bar{\eta}, \eta\), and repeating the route leading to (B.2), one obtains analogous WI involving any number of external ghost legs. Off-shell, such identities look progressively more complicated as the number of legs increases. On mass-shell, however, they simplify considerably - they essentially reduce to the form of (B.2), with the blob carrying the additional ghost legs and summations over appropriate permutations of external lines.

Having obtained the WI (B.2), the cancellation of unphysical polarizations and FP ghosts on the r.h.s. of (6.15) can now be explicitly verified. Introduce, following [9], the auxiliary cuts

\[
\begin{align*}
\frac{k_\mu}{a} & \rightarrow \mathcal{D}_{ab}^+(k) \frac{-i\bar{k}_\mu}{2(k \cdot \eta)^2} \\
\frac{k}{b} & \rightarrow \mathcal{D}_{ab}^+(k) \frac{i\bar{k}_\mu}{2(k \cdot \eta)^2} \\
\end{align*}
\]

(B.3) on shell, \(k^2 = 0\), then gives the relation

\[
\begin{align*}
\end{align*}
\]

(B.4) between cut bare propagators. Consider first two-particle intermediate states. Using the WI, eq. (B.2), repeatedly, and noting the relation

\[
\bar{k}_\mu k^\mu = -2(k \cdot \eta)^2 , \quad k^2 = 0 ,
\]

(B.5)
one straightforwardly derives the equality
and hence, using also (B.4),:

and (B.7) is the desired relation showing that, for two-particle intermediate states, the sum over cuts on the r.h.s. of (6.15) (properly including the minus sign for ghost loops) reduces to the sum over only physical cuts (6.16).

The generalization to N-particle intermediate states follows directly from the same argument by means of the extensions of the WI (B.2) to multiple external ghost legs alluded to above. Alternatively, the N-particle state can be treated by induction starting from the two-particle state (B.7). As, however, all this is the same as in the familiar local gauge theory case, there is no need to belabor the point.

C Appendix - Cutting equations

We remark here on some points pertaining to the derivation of the cutting equations for our actions. A Feynman graph with N vertices is represented in coordinate space by N vertex points \( x_i, i = 1, \ldots, N \) joined by lines. To the \( i \)-th \( m_i \)-point vertex factor

\( (2\pi)^4 i\hat{V}_{\mu\nu...ab...}(k_{i_1}, \ldots, k_{i_{m_i}})\delta(\sum_k k_{i_k}) \) in momentum space, there corresponds the matrix vertex factor

\[
i\hat{V}_{\mu\nu...ab...}(x_i; x_{i_1}, x_{i_2}, \ldots, x_{i_{m_i}}) \equiv i\mathcal{V}_{\mu\nu...ab...}(i\partial_{x_{i_1}}, \ldots, i\partial_{x_{i_{m_i}}}) \prod_{l=1}^{m_i}\delta(x_i - x_{i_l}). \tag{C.8}
\]

Arbitrary external source vertices are included here as 1-point vertices. To every gauge boson (ghost) line joining the \( l \)-th leg of the vertex at \( x_i \) to the \( k \)-th leg of the vertex at \( x_j \) there corresponds a propagator factor of \( D_{\mu\nu ab}(x_{ik} - x_{jl}) (D_{ab}(x_{ik} - x_{jl})) \). We
will use a compact notation suppressing inessential labels and letting the enumerative index $i$ also stand for all associated spacetime and group indices. Thus we denote by $D_{ikji}$ any propagator joining $x_{j}$ to $x_{ik}$, and by $\tilde{V}(i; i_{1} \ldots i_{m})$ the vertex factor at $x_{i}$.

Consider a graph with $N$ vertices. Associated with the graph is the generalized function $F(\{x_{i}\})$ defined as the product of a factor of $\tilde{V}(i; i_{1} \ldots i_{m})$ for each vertex point $x_{i}$, and a propagator $D_{ikji}$ for the line joining the $l$-th leg emanating from the vertex at $x_{j}$ to the $k$-th leg emanating at $x_{i}$. Note that no integration is included in the definition of $F(\{x_{i}\})$, and thus there is yet no distinction between internal and external vertices. The actual amplitude for the graph is obtained, apart from combinatorial factors, by multiplying $F(\{x_{i}\})$ by the appropriate external wave-functions, and integrating over $x_{ik}, x_{i}, k = 1, \ldots, m_{i}, i = 1, \ldots, N$.

Given $F(x_{1}, \ldots, x_{N})$, we define a set of related functions which will all be denoted by $F$ but with one or more of their arguments $x_{i}$ underlined \[8\]. The function $F(x_{1}, \ldots, x_{i}, \ldots, x_{j}, \ldots, x_{N})$ with any number of its arguments $x_{i}$ underlined is defined as the product of the following factors:

(a) $i\tilde{V}(i; i_{1} \ldots i_{m})$ for each $x_{i}$ that is not underlined; $-i\tilde{V}(i; i_{1} \ldots i_{m})$ for each $x_{i}$ that is underlined.

(b) For a line joining the $l$-th leg of the vertex at $x_{j}$ and the $k$-th leg of the vertex at $x_{i}$:

(i) $D_{ikji}$ if neither $x_{i}$ nor $x_{j}$ are underlined.

(ii) $D^{*}_{ikji}$ if both $x_{i}$ and $x_{j}$ are underlined.

(iii) $D^{+}_{ikji}$ if $x_{i}$ but not $x_{j}$ is underlined.

(iv) $D^{-}_{ikji}$ if $x_{j}$ but not $x_{i}$ is underlined.

Note that (iii) implies (iv), and vice-versa, because of the relation $D^{+}(-x) = D^{-}(x)$. This makes the inherent ambiguity in assigning a direction to propagators, which are symmetric in their arguments, irrelevant; the point is that positive energy always flows from the not underlined to the underlined vertex, whereas there is no restriction on the sign of energy flow for lines connecting two vertices which are both either underlined or not underlined. Underlining vertices is clearly related to complex conjugation provided all vertex factors $\tilde{V}(i; i_{1} \ldots i_{m})$ are real. Given a graph with some vertices underlined, consider the graph obtained by removing all present underlinings, and underlining all previously not underlined vertices. It is easily verified that the respective functions $F$ are complex conjugate of each other.

Each vertex $\tilde{V}_{\mu \nu \ldots ab \ldots}(k_{i_{1}}, \ldots, k_{i_{m}})$ can be represented by an everywhere absolutely convergent series. By standard theorems, the series of generalized functions obtained by Fourier transforming converges to the corresponding Fourier transform $[C.8]$. Orders of summations may be interchanged, and the essentially combinatorial argument \[8\] on the functions $F(\{x_{i}\})$ leading to the largest time and cutting equations may in
fact be applied term by term in the series representations. Infinite radius of convergence is crucial for this argument to be applicable in the present case. Finite radius corresponding to singularities in the vertices would mean that the argument either fails or must be modified in a manner resulting into cut contributions from vertices. The other necessary ingredients, also required in the usual polynomial action case, are: (a) the quasi-local nature of each (term in the expansion of) vertex \( \hat{V}(i; i_1 \ldots i_m) \) (i.e. proportional to delta functions of \( (x_i - x_{i_k}) \) and their derivatives); (b) the propagators \( D_{i_k i_l} \) obeying the decomposition (6.6).

Assuming then a \( x_j^0 \) to be the largest among the time components of the points \( \{x_i| i = 1, \ldots, N\} \), the argument of \( \{8\} \) may now be applied to give a largest time equation, and hence, by summation over orderings, the result:

\[
F(\{x_i\}) + F(\{x_i\})^* = -\sum_{\{\not 0, \text{all}\}} F(\{x_i\}) \tag{C.9}
\]

In (C.9) the sum is over the \((2^N - 2)\) possible underlinings other than no and all underlinings. Multiplying (C.9) by any appropriate external wave functions and integrating over all \( \{x_{i_k}\}, x_i \) gives (6.13), the cutting equation (in momentum space). By noting that many terms on the r.h.s. vanish due to conflicting energy \( \theta \)-functions, it is easily seen that the sum over underlinings indeed reduces to the sum over cuts as described in the main text.

The following point should be noted. Although (C.9) holds for any time ordering of the \( x_i \)'s (thus allowing one to integrate), it was established, for each given configuration \( \{x_i\} \), only for one of the time components \( x^0_i \) being the largest, i.e. not for the case of two or more of the time components being equal and largest than the rest. Such equal time regions are of lower dimensionality, hence measure zero, in the \( N \)-dimensional integration space of the \( x_i \)'s. They therefore give no finite contribution provided the (regulated or subtracted) Feynman integrands are sufficiently regular in coordinate space

Return to (C.9), and assume that \( x^0_k < x^0_l \). Then the equation holds separately for the terms with and without \( x_k \) being underlined since it is certain that \( x_k \) is not the largest time \( \{8\} \). So, in particular,

\[
\sum_{\{\not k\}} F(\{x_i\}) = 0 \quad , \quad x^0_k < x^0_l \quad , \tag{C.10}
\]

where the sum is over all underlinings except \( x_k \). Similarly, considering the case \( x^0_k > x^0_l \) gives equation (C.10) with \( x_k \) and \( x_l \) interchanged. Adding these two equations gives

\[31\] A similar proviso applies to tadpoles, formed by lines closing upon themselves, though these actually give no contribution to the sum over cuts. The use of dimensional (or equivalent) regularization which sets infinite constants such as \( \delta^{(n)}(0) \), \( n \geq 0 \), to zero is extremely convenient in automatically handling these subtleties.
the relation:

\[
F(\{x_i\}) = - \sum_{\{i \neq k,l\}} F(\{x_i\}) - \theta(x^0_l - x^0_k) \sum_{i \cup \{k,l\}} F(\{x_i\}) - \theta(x^0_k - x^0_l) \sum_{k \cup \{k,l\}} F(\{x_i\}), \quad x^0_k \neq x^0_l. \tag{C.11}
\]

In (C.11) the term with no underlining and the term with neither \(x_k\) nor \(x_l\) underlined have been separated out. The two terms multiplied by the \(\theta\)-functions contain the sums of all underlinings with \(x_l\) but not \(x_k\) underlined, and with \(x_k\) but not \(x_l\) underlined, respectively.

Multiplying (C.11) by any appropriate wave-functions attached to external vertices, and integrating over all \(x_{i_k}, k = 1, \ldots, m_i, i = 1, \ldots, N\), and all \(x_i\) except \(x_k, x_l\) gives the BCC equation as originally stated [10]. An equivalent and more convenient statement is obtained by performing the same operations on (C.11) which results in (6.19) in the main text with \(x_k = x, x_l = y\).

Finally, integrating also over \(x, y\) allows one to express the BCC equation (6.19), entirely in momentum space, in the form of a dispersion relation. In performing this step, however, one must note that (C.11) was derived strictly only for \(x^0 \neq y^0\). In the presence of derivative interactions, finite contributions can arise from the point \(x^0 = y^0\) due to the action of derivatives at \(x, y\) on the \(\theta\)-functions in (6.19). The general rule is as follows. Any derivatives which, upon Fourier-transforming, correspond to external momenta injected into the graph along external lines (truncated vertex legs) attached at \(x\) and/or \(y\), or along a propagator leg connecting a source vertex at \(x\) and/or \(y\) to the rest of the diagram, must also act on the \(\theta(\pm(x^0 - y^0))\) factors, i.e. must be commuted to the front on the r.h.s. in (6.19) (cp. [10]).

As an example, consider the important case of the 2-point function between external sources \(J_x, J_y\) at \(x\) and \(y\). To lowest order where the blob stands for a single bare propagator joining \(x\) and \(y\), (6.19) is nothing but (6.4), in Feynman gauge, sandwiched between \(J^\mu\) and \(J^\nu\). Comparison with (6.1) trivially shows that the correct extension to include the point \(x^0 = y^0\) is obtained by letting derivatives in \(\hat{J}_x = J_x \hbar^{-1/2}(-\Box x/\Lambda^2)\) and \(\hat{J}_y\) also act on \(\theta(\pm(x^0 - y^0))\). Similarly, in the presence of an arbitrary number \(n\) of self-energy insertions between \(x\) and \(y\), one has:

\[
\int \prod_{k=1}^n \frac{dz_k}{2\pi} \frac{d\pi_k}{2\pi} \frac{d\pi_{z_k}}{2\pi} \cdot \hat{J}_x \cdot \hbar^{-1/2}(-\Box x) \cdot D_{zz_1} \cdot \hbar^{-1/2}(-\Box z_1) \cdot D_{z_1z_2} \cdot \hbar^{-1/2}(-\Box z_2) \cdot D_{z_2z_3} \cdot \cdots \cdot \hbar^{-1/2}(-\Box z_{n-1}) \cdot D_{z_{n-1}z_n} \cdot \hbar^{-1/2}(-\Box y) \cdot J_y
= \hat{J}_x \cdot \hbar^{-(n+1)}(-\Box x) \cdot \left[ \prod_{k=1}^n \frac{dz_k}{2\pi} \frac{d\pi_k}{2\pi} \frac{d\pi_{z_k}}{2\pi} \cdot \hbar^{-1} D_{zz_1} \cdot D_{z_1z_2} \cdots \cdot D_{z_{n-1}z_n} \right] \cdot J_y \tag{C.12}
\]

by integration by parts and translation invariance of \(D_{zz'}, \Pi_{zz'}\). Again, in applying (6.19) to (C.12), all derivatives in sources and the \(n + 1\) \(\hbar^{-1}\) factors must also act on
the $\theta$-factors on the r.h.s. This clearly generalizes to the propagator leg connecting a
source vertex at $x$ and/or $y$ to any $(2+n)$-legged blobs in (6.13).

References

[1] A. A. Slavnov, Theor. Math. Phys. 13 (1972) 174; 33 (1977) 977.

[2] T. D. Bakeyev and A. A. Slavnov, preprint SMI-02-96, hep-th/9601092.

[3] G. V. Efimov, Commun. Math. Phys. 5 (1967) 42; V. A. Alebastrov and G. V.
Efimov, Commun. Math. Phys. 38 (1974) 11.

[4] M. Kato, Phys. Lett. 245B (1990) 43.

[5] E. C. Titchmarsh, *The theory of Functions* (Oxford University Press, 2nd corrected
edition, 1968); B. Ja. Levin, *Distribution of zeros of entire functions* (American
Mathematical Society, Providence, Rhode Island, 1964).

[6] S. Weinberg, Phys. Rev. 118, 838 (1960).

[7] W. Zimmermann, Comm. Math. Phys. 11 (1968) 1; Y. Hahn and W. Zimmermann,
Comm. Math. Phys. 10 (1968) 330.

[8] M. Veltman, Physica 29 (1963) 186; G. ’t Hooft and M. Veltman, *Diagrammar* in
*Particle interactions at very high energies*, D. Speiser, F. Halzen and J. Weyers,
eds. (Plenum Press London 1974).

[9] G. ’t Hooft, Nucl. Phys. B33 (1971) 173; G. ’t Hooft and M. Veltman, Nucl. Phys.
B50 (1972) 318.

[10] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized
Fields* (Wiley-Interscience, New York, 1959).

[11] E. Remiddi, Helvetica Phys. Acta, 54, 365 (1981).