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On the Unitary Representations of the Braid Group $B_6$

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Abstract: We consider a non-abelian leakage-free qudit system that consists of two qubits each composed of three anyons. For this system, we need to have a non-abelian four dimensional unitary representation of the braid group $B_6$ to obtain a totally leakage-free braiding. The obtained representation is denoted by $\rho$. We first prove that $\rho$ is irreducible. Next, we find the points $y \in \mathbb{C}^*$ at which the representation $\rho$ is equivalent to the tensor product of a one dimensional representation $\chi(y)$ and $\hat{\mu}_6(\pm i)$, an irreducible four dimensional representation of the braid group $B_6$. The representation $\hat{\mu}_6(\pm i)$ was constructed by E. Formanek to classify the irreducible representations of the braid group $B_n$ of low degree. Finally, we prove that the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ is a unitary relative to a hermitian positive definite matrix.

Keywords: braid group; unitarity; equivalence; irreducibility

1. Introduction

Due to Artin, the braid group $B_n$ is represented in the group $\text{Aut}(F_n)$ of automorphisms of the free group $F_n$ generated by $x_1, \ldots, x_n$. The matrix representation of $B_n$ was published by W. Burau in 1936. This representation was known as a Burau representation. Since then, other matrix representations of $B_n$ have been constructed. For more details, see [1].

Braid group unitary representations have been essential in topological quantum computations. To understand the $d$-dimensional systems in which anyons are exchanged, a lot of work has been made. The exchange of $n$ anyons inside the qudit system, the $d$-dimensional analogues of qubits, has been governed by the braid group $B_n$ which has $n-1$ generators $\tau_1, \ldots, \tau_{n-1}$. Here, $\tau_i$ exchanges the particle $i$ with its neighbor, particle $i+1$.

When the topological charge of the qudits changes due to the braiding of the anyons from different qudits, a leakage of some of the information will occur in the computational Hilbert space, the fusion space of the anyons.

The leakage-free braiding of anyons has been under investigation for a while. To perform universal quantum computation without any leakage, the requirement would be to consider two-qubit gates. This would be very restrictive and this property can only be realized for two-qubit systems related to the Ising-like anyons model [2].

R. Ainsworth and J.K. Slingerland showed that a non-abelian, leakage-free qudit of dimension $d$ involving $n$ anyons is equivalent to a non-abelian $d$-dimensional representation of the braid group $B_n$. Here, $n$ is the sum of the number of anyons $n_1$ inside the first qudit and the number of anyons $n_2$ inside the second qudit. As for the dimension $d$ of the representation of $B_n$, it is the product of the dimensions $d_1$ and $d_2$ of the Hilbert spaces of the individual qudits.

Moreover, it was proved in [2] that the number of anyons per qubit is either 3 or 4. Thus, there are mainly 3 different types of two-qubit systems and a 4-dimensional representation of the corresponding braid group is constructed for each. Taking into account E. Formanek’s result that there
is no $d$-dimensional representation of $B_n$ with $d < n - 2$, it was verified in [2] that the only possible type of two-qubit system is having 2 qubits of which each is composed of 3 anyons.

This system is a non-abelian leakage-free qudit system of dimension 4 involving 6 anyons. It is equivalent to a non-abelian 4-dimensional representation of the braid group $B_6$. This representation is denoted by $\rho$. Since the number of anyons is 6, there are 5 elementary exchanges $\tau_1, \ldots, \tau_5$. The exchanges $\tau_1, \tau_2, \tau_4$, and $\tau_5$ satisfy the following relations:

$$\rho(\tau_i) = \rho_1(\tau_i) \otimes I_{d_2} \quad (1 \leq i \leq n_1 - 1)$$

and

$$\rho(\tau_i) = I_{d_1} \otimes \rho_2(\tau_i) \quad (n_1 + 1 \leq i \leq n - 1),$$

where $\rho_1$ and $\rho_2$ are the $d_1$ and $d_2$-dimensional representations of $B_{n_1}$ and $B_{n_2}$ on the Hilbert spaces of the first and second qudit respectively. $I_{d_1}$ and $I_{d_2}$ are the $d_1$ and $d_2$-dimensional identity matrices respectively. Here, $n_1 = n_2 = 3$ and $d_1 = d_2 = 2$.

The matrix $\rho(\tau_3)$ is constructed by imposing braid group relations. For more details, see [2].

In our work, we consider the unitary representation $\rho$ and the irreducible representation $\mu_6(\pm i)$ which is defined by E. Formanek in [3]. Both representations are 4-dimensional representations of the braid group $B_6$.

First, we prove that the unitary representation $\rho : B_6 \rightarrow GL_4(\mathbb{C})$ is irreducible.

As the representation $\rho$ is proved to be irreducible, it follows that it is equivalent to the tensor product of a one-dimensional representation $\chi(y)$ and the irreducible 4-dimensional representation $\mu_6(\pm i)$, where $y \in \mathbb{C}^*$. For more details, see [3].

We then determine the points $y \in \mathbb{C}^*$ at which the two representations $\rho$ and $\chi(y) \otimes \mu_6(\pm i)$ are equivalent.

Finally, we show that the representation $\chi(y) \otimes \mu_6(\pm i)$ is a unitary relative to a hermitian positive definite matrix.

2. Preliminaries

Definition 1 (See [4]). The braid group on $n$ strings, $B_n$, is the abstract group with presentation $B_n = \{\sigma_1, \ldots, \sigma_{n-1}|\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_i \text{ for } i = 1, \ldots, n-2\}$.

The Hecke algebra representation of $B_6$ was constructed by V.F.R. Jones in [5]. E. Formanek obtained a low-degree representation of $B_6$ by conjugating the representation constructed by V.F.R. Jones by a certain permutation matrix. For more details, see [3].

Definition 2 (See [3]). The representation $\mu_6 : B_6 \rightarrow GL_5(\mathbb{Z}[t^{\pm 1}])$ is given by:

$$\mu_6(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & -t \\ 0 & -t & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -t \end{pmatrix}, \quad \mu_6(\sigma_2) = \begin{pmatrix} -t & 0 & 0 & 0 & 0 \\ 0 & 1 & -t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mu_6(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & -t & 0 \\ 0 & -t & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}, \quad \mu_6(\sigma_4) = \begin{pmatrix} -t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -t \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -t \end{pmatrix},$$

and
Theorem 2 (See [3]). The system is equivalent to a 4-dimensional representation of the braid group $B_3$. If $z$ is a root of $t$, this result is obtained from $\mu_6(\alpha_5)$.

Definition 3 (See [3]). A representation $\rho : B_n \to GL_r(\mathbb{C})$ is of Burau type if $r \geq 2$ and it is equivalent to an irreducible representation $\chi(y) \otimes \beta_n(z) : B_n \to GL_{n+1}(\mathbb{C})$ or $\chi(y) \otimes \hat{\beta}_n(z) : B_n \to GL_{n+2}(\mathbb{C})$ where $\chi(y)$ is a one-dimensional representation, $\beta_n(z)$ is the reduced Burau representation, and $\hat{\beta}_n(z)$ is the composition factor of the reduced Burau representation.

Definition 4 (See [3]). Let $y \in \mathbb{C}^*$. The representation $\chi(y) : B_n \to \mathbb{C}^*$ is the one-dimensional representation defined by $\chi(y)(\alpha_i) = y$.

Theorem 1 (See [3]). 1. For $z \in \mathbb{C}^*$, $\mu_5(z)$ and $\mu_6(z)$ are irreducible unless $z$ is a root of $(t^2 + t + 1)(t^2 + 1)$; 2. If $z$ is a root of $t^2 + t + 1$, then the composition factors of $\mu_6(z)$ are $\chi(-z)$ and $\hat{\beta}_6(z)$ and the composition factors of $\mu_5(z)$ are $\chi(z)$; 3. If $z$ is a root of $t^2 + 1$, then the composition factors of $\mu_6(z)$ are $\chi(1)$ and an irreducible representation $\mu_6(z) : B_6 \to GL_4(\mathbb{C})$, where $\mu_6(z)(\alpha_i)$ has eigen values $1, 1, -z, -z$. The composition factors of $\mu_5(z)$ are $\chi(1)$ and an irreducible representation $\mu_5(z)$, which is the restriction of $\mu_6(z)$ to $B_5$.

Theorem 2 (See [3]). Let $\rho : B_6 \to GL_r(\mathbb{C})$ be an irreducible representation, where $2 \leq r \leq 5$. Then one of the following is true.

1. The representation $\rho$ is of a Burau type;
2. For some $y \in \mathbb{C}^*$, $\rho$ is equivalent to $\chi(y) \otimes \mu_6(\pm i) : B_6 \to GL_4(\mathbb{C})$. Distinct pairs $(y, \pm i)$ give inequivalent representations;
3. For some $y, z \in \mathbb{C}^*$, $\rho$ is equivalent to $\chi(y) \otimes \mu_6(z) : B_6 \to GL_5(\mathbb{C})$, where $z$ is not a root of $(t^2 + t + 1)(t^2 + 1)$.

3. Irreducibility of $\rho : B_6 \to GL_4(\mathbb{C})$

The construction of a two-qubit system with a minimum amount of leakage has been of great interest. The only two-qubit system that can be realized without leakage is the system of two 3-anyon qubits. This system is equivalent to a 4-dimensional representation of the braid group $B_6$. This representation which was constructed in [2] is denoted by $\rho$.

In this section, we prove that $\rho : B_6 \to GL_4(\mathbb{C})$ is irreducible. We denote $\tau_i$, the exchange of the $i^{th}$ and $(i + 1)^{th}$ anyon, by $\alpha_i$, where $1 \leq i \leq 5$.

Definition 5 (See [2]). The representation $\rho : B_6 \to GL_4(\mathbb{C})$ is defined as follows:

$$
\rho(\alpha_1) = \begin{pmatrix}
 a & 0 & 0 & 0 \\
 0 & a & 0 & 0 \\
 0 & 0 & a & 0 \\
 0 & 0 & 0 & a
\end{pmatrix},
\rho(\alpha_2) = \begin{pmatrix}
 1 & 0 & 0 & 0 \\
 \frac{1}{a - a^2} & c & 0 & 0 \\
 -c & 0 & 1 & 0 \\
 0 & -c & 0 & \frac{1}{a - a^2}
\end{pmatrix},
$$
As the system considered in our work is leakage-free, the eigenvalues of the generators $\sigma_a$ are equal.

Thus, there are no non-trivial proper invariant subspaces of dimension 1.

This implies that $c = e = a$ or $f = \bar{a}$ and $e = c = \sqrt{1 - \frac{1}{2 - a^2 - \bar{a}^2}}$.

By simple computations, the relation $\rho(\sigma_2)\rho(\sigma_3)\rho(\sigma_2) = \rho(\sigma_3)\rho(\sigma_2)\rho(\sigma_3)$ yields the equation $a^2 = -\bar{a}^2$. But, $a = e^{i\theta}$. Therefore, $e^{2i\theta} = -e^{-2i\theta}$. This implies that $e^{4i\theta} = -1$. Consequently, $\beta^{4\theta} = 1$. That is, $a$ must be a primitive eighth root of unity. Furthermore, $c = \sqrt{1 - \frac{1}{2 - a^2 - \bar{a}^2}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$.

Note that since $a$ is a primitive eighth root of unity, $a^8 = 1$ and $a^2 \neq 1$. Then, $\bar{a}^3 \neq a$. Consequently, $\bar{a}^3 \neq a, f \neq f^3$, and $f \neq f^3$. This emphasizes that the defined matrices $\rho(\sigma_i), 1 \leq i \leq 5$, are well-defined.

Now we study the irreducibility of $\rho$. For simplicity, we denote $\rho(\sigma_i)$ by $\sigma_i$ for $1 \leq i \leq 5$.

**Lemma 1.** The representation $\rho : B_6 \to GL_4(\mathbb{C})$ has no non-trivial proper invariant subspaces of dimension 1.

**Proof.** Let $S$ be a proper invariant subspace of dimension 1. We consider all the possible cases.

**Case 1:** $S = \langle e_i \rangle, i = 1, 2, 3, 4$.

For simplicity, we take $i = 1$. Since $S$ is invariant, it follows that $\sigma_2(e_1) = \begin{pmatrix} \frac{1}{a-e} & 0 \\ 0 & -c \\ -c & 0 \end{pmatrix} \in S$.

This implies that $c = 0$, a contradiction.

**Case 2:** $S = \langle e_i + \mu e_{i+1} \rangle, i = 1, 3, \mu \in \mathbb{C}^*$.

For simplicity, we take $i = 1$. Since $S$ is invariant, it follows that $\sigma_2(e_1 + \mu e_2) = \begin{pmatrix} \frac{1}{a^2-\mu} & 0 \\ \frac{1}{\mu} & -c \\ -c & 0 \end{pmatrix} \in S$.

This implies that $c = 0$, a contradiction.

Thus, there are no non-trivial proper invariant subspaces of dimension 1. □

**Lemma 2.** The representation $\rho : B_6 \to GL_4(\mathbb{C})$ has no non-trivial proper invariant subspace of dimension 2.

**Proof.** Let $S$ be a proper invariant subspaces of dimension 2. We consider all the possible cases.

**Case 1:** $S = \langle e_i, e_{i+1} \rangle, i = 1, 3$.
For simplicity, we take \( i = 1 \). Since \( S \) is invariant, it follows that \( \sigma_2(e_1) = \begin{pmatrix} \frac{1}{a-p^2} \\ 0 \\ -c \\ 0 \end{pmatrix} \in S \).

This implies that \( c = 0 \), a contradiction.

**Case 2:** \( S = \langle e_{i}, e_{j} \rangle, i = 1, 2, j = 3, 4 \).

For simplicity, we take \( i = 1 \). Since \( S \) is invariant, it follows that \( \sigma_2(e_1) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{f-j^2} \\ -e \end{pmatrix} \in S \).

This implies that \( e = 0 \). But, \( e = c \). Thus, \( c = 0 \), a contradiction.

**Case 3:** \( S = \langle e_{i}, e_{1} + ue_{2} \rangle, i = 3, 4, u \in \mathbb{C}^* \).

For simplicity, we take \( i = 3 \). Since \( S \) is invariant, it follows that \( \sigma_3(e_3) = \begin{pmatrix} 0 \\ x \\ 0 \\ 0 \end{pmatrix} \in S \).

This implies that \( e = 0 \). But, \( e = c \). Thus, \( c = 0 \), a contradiction.

**Case 4:** \( S = \langle e_{i}, e_{3} + ve_{4} \rangle, i = 1, 2, v \in \mathbb{C}^* \).

For simplicity, we take \( i = 1 \). Since \( S \) is invariant, it follows that \( \sigma_4(e_1) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{f-j^2} \\ -e \end{pmatrix} \in S \).

This implies that \( e = 0 \). But, \( e = c \). Thus, \( c = 0 \), a contradiction.

**Case 5:** \( S = \langle e_{1} + ue_{2}, e_{3} + ve_{4} \rangle, u, v \in \mathbb{C}^* \).

Since \( S \) is invariant, it follows that \( \sigma_3(e_1 + ue_2) = \begin{pmatrix} x \\ xu \\ 0 \\ 0 \end{pmatrix} \in S \). This implies that \( a = \bar{a} \). That is, \( a^2 - \bar{a}^2 = 0 \).

But, from the equation \( \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \), we have \( a^2 + \bar{a}^2 = 0 \). Thus, \( 2a^2 = 0 \) which gives a contradiction.

Thus, there are no non trivial proper invariant subspaces of dimension 2. 

Now, we state the theorem of irreducibility.

**Theorem 3.** The representation \( \rho : B_6 \to GL_4(\mathbb{C}) \) is irreducible.

**Proof.** By Lemma 1 and Lemma 2, there are no proper invariant subspaces of dimensions 1 and 2. Clearly, the representation \( \rho \) is unitary, that is \( \sigma_i^\rho \sigma_i^\rho = I_4 \) for \( 1 \leq i \leq 5 \).

We note that if the representation is unitary, then the orthogonal complement of a proper invariant subspace is again a proper invariant subspace. As there is no proper invariant subspace of dimension 1, there is no proper invariant subspace of dimension 3.

As a result, all the possible proper subspaces are not invariant. Consequently, \( \rho \) is irreducible.

**4. The Representations \( \rho \) and \( \chi(y) \otimes \hat{\mu}_0( \pm i) \) Are Equivalent**

By Theorem 3, the representation \( \rho \) is irreducible. The eigen values of \( \rho(\sigma_i) \) for \( 1 \leq i \leq 5 \) are different from those of \( \hat{\beta}_4(\hat{z}) \), the composition factor of the reduced Burau representation. Therefore,
the representation $\rho$ is not equivalent to the tensor product of a one dimensional representation $\chi(y)$ and $\hat{\mu}_4(z)$. That is, $\rho$ is not of a Burau type.

Moreover, $\rho$ is a 4-dimensional representation. Consequently, Theorem 2 implies that the representation $\rho$ is equivalent to the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ for some $y \in \mathbb{C}^*$.

Note that, by Theorem 1, the representation $\hat{\mu}_6(z)$ is irreducible for $z = \pm i$ since the roots of the polynomial $t^2 + 1$ are clearly $\pm i$.

In this section, we determine the points $y \in \mathbb{C}^*$ at which the representations $\rho$ and $\chi(y) \otimes \hat{\mu}_6(\pm i)$ are equivalent.

Since the two representations are equivalent, the determinants of the matrices $(\chi(y) \otimes \hat{\mu}_6(\pm i))(\sigma_i)$ and $\rho(\sigma_i)$ are equal for $1 \leq i \leq 5$.

By simple computations, the determinant $\text{Det}[(\chi(y) \otimes \hat{\mu}_6(\pm i))(\sigma_i)] = -y^4$ and $\text{Det}[\rho(\sigma_i)] = 1$ for all $i = 1, 2, 3, 4, 5$. Thus, $-y^4 = 1$. This implies that $y = \pm \sqrt{i}$.

As a result, the two considered representations are equivalent at the following points: $y_1 = \sqrt{i}$, $y_2 = -\sqrt{i}$, $y_3 = -\sqrt{-i}$, and $y_4 = \sqrt{-i}$, where $i$ is the complex number such that $i^2 = -1$.

5. Unitarity of $\chi(y) \otimes \hat{\mu}_6(\pm i) : B_6 \to GL_4(\mathbb{C})$

As the representation $\rho$ is proved to be unitary and equivalent to the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ for some $y \in \mathbb{C}^*$, the representation $\chi(y) \otimes \hat{\mu}_6(\pm i)$ is a unitary relative to a matrix $M$.

In this section, we find the matrix $M$ and we prove that $M$ is a hermitian and positive definite.

**Definition 6** (See [3]). The representation $\hat{\mu}_6(\pm i) : B_6 \to GL_4(\mathbb{C})$ is defined as follows:

$$
\begin{align*}
\sigma_1 & \mapsto \begin{pmatrix} \pm i & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & \pm i & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \\
\sigma_2 & \mapsto \begin{pmatrix} 1 & \pm i & 0 & 0 \\ 0 & \pm i & 0 & 0 \\ 0 & 0 & 1 & \pm i \\ 0 & 0 & 0 & \pm i \end{pmatrix}, \\
\sigma_3 & \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \pm i & 0 & 0 \\ 0 & 0 & \pm i & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \\
\sigma_4 & \mapsto (1/2) \begin{pmatrix} 1 & \pm i & 0 & -1 \pm i & 0 \\ 0 & 1 & \pm i & 0 & -1 \pm i \\ -1 \pm i & 0 & 1 & \pm i & 0 \\ 0 & -1 \pm i & 0 & 1 & \pm i \end{pmatrix}, \\
\sigma_5 & \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & \pm i \end{pmatrix}.
\end{align*}
$$

Now, we state the following theorem:

**Theorem 4.** The images of the generators of $B_6$ under $\hat{\mu}_6(\pm i)$ are unitary relative to a hermitian positive definite matrix $M$.

**Proof.** Let,

$$
M = \begin{pmatrix} 2 & 1 - i & 0 & 0 \\ 1 + i & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 - i \\ 0 & 0 & 1 + i & 2 \end{pmatrix}.
$$

Here, $i$ is the complex number such that $i^2 = -1$ and $M$ is an invertible matrix whose determinant equals 4.

For simplicity, we denote $(\hat{\mu}_6(\pm i))(\sigma_i)$ by $\sigma_i$ for $1 \leq i \leq 5$. 


By direct computations, \( \sigma_1 M \sigma_1^* = \sigma_2 M \sigma_2^* = \sigma_3 M \sigma_3^* = \sigma_4 M \sigma_4^* = \sigma_5 M \sigma_5^* = M \), where \( \sigma_i^* \) is the complex conjugate transpose of \( \sigma_i \) for \( 1 \leq i \leq 5 \). Therefore, the representation \( \hat{\mu}_6(\pm i) \) is a unitary relative to the matrix \( M \).

Let \( M^* \) be the complex conjugate transpose of \( M \). Clearly, \( M^* = M \). This implies that the invertible matrix \( M \) is hermitian.

By computations, the eigen values of the matrix \( M \) are \( 2 + \sqrt{2} \) and \( 2 - \sqrt{2} \). Clearly, both values are positive. Consequently, \( M \) is a positive definite matrix.

As a result, the representation \( \hat{\mu}_6(\pm i) \) is a unitary relative to an invertible hermitian positive definite matrix \( M \). \( \square \)

Note that the unitarity of the representation \( \hat{\mu}_6(\pm i) \) relative to the matrix \( M \) clearly implies that the representation \( \chi(y) \otimes \hat{\mu}_6(\pm i) \) is also a unitary relative to the same matrix \( M \).

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