A REMARK ON THE SCHRÖDINGER OPERATOR ON WIENER AMALGAM SPACES

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Abstract. In this paper, we study the boundedness of the Schrödinger operator $e^{i\Delta}$ on Wiener amalgam spaces and determine its optimal condition.

1. Introduction

In this paper, we consider the Schrödinger operator $e^{i\Delta}$ which is naturally arose from the Cauchy problem for the free Schrödinger equation

$$
\begin{aligned}
\left\{ 
\begin{array}{ll}
    i\partial_t u + \Delta u = 0, & t \in \mathbb{R}, \ x \in \mathbb{R}^n, \\
    u(0, x) = u_0(x), & x \in \mathbb{R}^n.
\end{array}
\right.
\end{aligned}
$$

Here, the differential operator $\Delta$ is the usual Laplacian. The Schrödinger operator is a special case of Fourier multipliers and denoted by

$$
e^{i\Delta} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-i|\xi|^2} \hat{f}(\xi) d\xi.
$$

Wiener amalgam spaces $W_{p,q}^s$ are variation of modulation spaces $M_{p,q}^s$, which were introduced from the view point of time-frequency analysis (see Feichtinger [4] and Gröchenig [5]). The essence of modulation and Wiener amalgam spaces is to measure integrability (or decay property) of functions with respect to their space variable and frequency variable simultaneously. On the other hand, in $L^p$-Sobolev spaces $L_p^s$ and Besov spaces $B_{p,q}^s$, the space and frequency variables of functions must be considered separately. This idea is a major difference among these spaces. The exact definitions and basic properties of modulation and Wiener amalgam spaces will be given in Section 2.2. In the following, if $s = 0$ we simply write $M_{p,q}$ and $W_{p,q}$ instead of $M_{p,q}^0$ and $W_{p,q}^0$, respectively.

The boundedness problems of Fourier multipliers have been studied by many researchers. In particular, we here focus on the unimodular Fourier multiplier $e^{-i|\xi|^\alpha}$ for $\alpha \geq 0$, which is denoted by the expression (1.1) whose phase function $|\xi|^2$ is replaced by $|\xi|^\alpha$. In [7], Hörmander studied that $e^{-i|\xi|^2}$ (i.e., $e^{i\Delta}$) is bounded on $L^p$ if and only if $p = 2$. After that, it was proved that for $1 < p < \infty$, $e^{-i|\xi|^\alpha}$ is bounded from $L^p$ to $L^p$ if and only if $p \geq \alpha n/1/p - 1/2$ (see, e.g., Miyachi [9]). Next, we shall consider the boundedness results on modulation spaces. On modulation spaces, Gröchenig and Heil [6] invented that $e^{-i|\xi|^{1/\alpha}}$ is bounded on $M_{p,q}$ for all $1 \leq p, q \leq \infty$ (see also Toft [11] and Wang, Zhao and Guo [14]). Then, the boundedness of $e^{-i|\xi|^{1/\alpha}}$ on modulation spaces was studied for $0 \leq \alpha \leq 2$ by Bényi, Gröchenig, Okoudjou and Rogers [1], and for $\alpha > 2$ by Miyachi, Nicola, Rivetti, Tabacco and Tomita [10]. Optimality of these results can be found in, e.g., [10, Theorem 1.2] and [12, Remark 3.4]. Extracting the Schrödinger operator case, namely, $\alpha = 2$, we have the following.

Theorem A. Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Then, $e^{i\Delta}$ is bounded from $M_{p,q}^s$ to $M_{p,q}$ if and only if $s \geq 0$.

Comparing the results on $L^p$-Sobolev and modulation spaces, we see that the Schrödinger operator is bounded on modulation spaces without loss of regularity, while on $L^p$-Sobolev spaces loss of the order up to $2n[1/p - 1/2]$ occurs. This is one of advantages in using modulation spaces.

The objective of this paper is to study the boundedness problem of the Schrödinger operator $e^{i\Delta}$ on variation of modulation spaces, Wiener amalgam spaces. In [3, Theorem 1.1], Cunanan and Sugimoto proved that for $1 \leq p, q \leq \infty$, $e^{i\Delta}$ is bounded from $W_{p,q}^s$ to $W_{p,q}$ if $s > n[1/p - 1/q]$. Conversely, in Cordero

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and Nicola [2, Proposition 6.1], it was shown that $e^{i\Delta}$ is bounded from $W^p_{p,q}$ to $W^p_{p,q}$ only if $s \geq n|1/p - 1/q|$. Moreover, in [2, Proposition 6.1], it was also stated that if $p = \infty$ and $q < \infty$, the necessary condition for the boundedness holds with the strict inequality, namely, $e^{i\Delta}$ is bounded from $W^\infty_{\infty,q}$ to $W^\infty_{\infty,q}$ only if $s > n/q$. Therefore, except for the case $p = \infty$ and $q < \infty$, there is a gap between the sufficient and necessary conditions for this boundedness problem. That is, the critical case $s = n|1/p - 1/q|$ is not included in the sufficient condition, although it is included in the necessary condition. The goal of this paper is to determine whether this critical case is needed or not. The exact answer is provided by the following.

**Theorem 1.1.** Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$.

1. For $p = q$, $e^{i\Delta}$ is bounded from $W^p_{p,q}$ to $W^p_{p,q}$ if and only if $s \geq 0$.
2. For $p \neq q$, $e^{i\Delta}$ is bounded from $W^p_{p,q}$ to $W^p_{p,q}$ if and only if $s > n|1/p - 1/q|$. 

Theorem 1.1 mentions that unlike modulation spaces, $e^{i\Delta}$ is not bounded on $W^p_{p,q}$ if $p \neq q$ (Wiener amalgam spaces coincide with modulation spaces in the case $p = q$). However, the boundedness of unimodular Fourier multiplier $e^{-i|D|^\alpha}$ on $W^p_{p,q}$ for $0 \leq \alpha \leq 1$ was given by Cunanan and Sugimoto [3, Corollary 2.1]. Since $e^{-i|D|^\alpha} = e^{i\Delta}$ if $\alpha = 2$, we have yet to determine an upper bound of $\alpha$ to obtain the boundedness of $e^{-i|D|^\alpha}$ on $W^p_{p,q}$.

We end this section with mentioning the plan of this paper. In Section 2, we will state basic notations which will be used throughout this paper, and then introduce the definitions and some basic properties of modulation and Wiener amalgam spaces. After stating and proving some lemmas needed to show our main theorem in Section 3, we will actually prove it in Section 4.

## 2. Preliminaries

### 2.1. Basic notations.

We collect notations which will be used throughout this paper. We denote by $\mathbb{R}$ and $\mathbb{Z}$ the sets of reals and integers, respectively. The notation $a \lesssim b$ means $a \leq CB$ with a constant $C > 0$ which may be different in each occasion, and $a \sim b$ means $a \lesssim b$ and $b \lesssim a$. We write $<(x) = (1 + |x|^2)^{1/2}$.

We denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^n$ by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ and its dual, the space of tempered distributions, by $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$. The Fourier transform and the inverse Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ are given by

$$\mathcal{F}f(x) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx$$

and

$$\mathcal{F}^{-1}f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi,$$

respectively. For $g \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier multiplier is denoted by $g(D)f = \mathcal{F}^{-1}[g \cdot \mathcal{F}f]$, and for $s \in \mathbb{R}$, the Bessel potential by $(I - \Delta)^{s/2}f = \mathcal{F}^{-1}[(\xi)^s \cdot \mathcal{F}f]$ for $f \in \mathcal{S}(\mathbb{R}^n)$.

We will use some function spaces. The Lebesgue space $L^p = L^p(\mathbb{R}^n)$ is equipped with the norm

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}$$

for $1 \leq p < \infty$. If $p = \infty$, $\|f\|_{\infty} = \sup_{x \in \mathbb{R}^n} |f(x)|$. Moreover, we denote the $L^p$-Sobolev space (or the Bessel potential space) $L^p_s$ by $L^p_s(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \|(I - \Delta)^{s/2}f\|_{L^p} < \infty \}$ for $1 < p < \infty$ and $s \in \mathbb{R}$. For $1 \leq q \leq \infty$, we denote by $l^q$ the set of all complex number sequences $\{a_k\}_{k \in \mathbb{Z}^n}$ such that

$$\|a_k\|_{l^q} = \left(\sum_{k \in \mathbb{Z}^n} |a_k|^q \right)^{1/q} < \infty,$$

with the usual modification for $q = \infty$. For the sake of simplicity, we write $\|a_k\|_{l^q}$ instead of the more correct notation $\|\{a_k\}_{k \in \mathbb{Z}^n}\|_{l^q}$.

### 2.2. Modulation and Wiener amalgam spaces.

We give the definitions of modulation and Wiener amalgam spaces. They are based on Feichtinger [4] and Gröchenig [5]. We fix a function (called a window function) $g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and denote the short-time Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to $g$ by

$$V_g f(x, \xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot t} g(t - x)f(t) dt.$$
We will sometimes write \( V_g[f] \) when the function \( f \) is complicated. Now, for \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \), the modulation space \( M^s_{p,q} \) is denoted by

\[
M^s_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{M^s_{p,q}} = \left\| \langle \xi \rangle^s V_g f(x, \xi) \right\|_{L^p(\mathbb{R}^n)} \right\|_{L^q(\mathbb{R}^n)} < +\infty \}
\]

and the Wiener amalgam space \( W^s_{p,q} \) is denoted by

\[
W^s_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{W^s_{p,q}} = \left\| \langle \xi \rangle^s V_g f(x, \xi) \right\|_{L^p(\mathbb{R}^n)} \right\|_{L^q(\mathbb{R}^n)} < +\infty \}
\]

These definitions are independent of the choices of the window function \( g \). From the definitions, we see that \( M^s_{p,p} = W^s_{p,p} \). We also remark that \( M^{2,2}_{2,2} = L^2 \). In the following, if \( s = 0 \) we will simply write \( M_{p,q} \) and \( W_{p,q} \) instead of \( M^s_{p,q} \) and \( W^s_{p,q} \) respectively.

We write \( X^s_{p,q} \) instead of \( X^s_{p,p} \) or \( W^s_{p,q} \). We note that \( X^s_{p,q} \) are Banach spaces and \( \mathcal{S} \subset X^s_{p,q} \subset \mathcal{S}' \). In particular, \( \mathcal{S} \) is dense in \( X^s_{p,q} \) if \( 1 \leq p, q < \infty \). For \( 1 \leq p, q < \infty \), the dual space of \( X^s_{p,q} \) can be seen as \( (X^s_{p,q})' = X^{-s}_{p',q'} \). Moreover, we have the following complex interpolation theorem. If \( 0 < \theta < 1 \), \( s = (1 - \theta) a_1 + \theta a_2 \), \( 1/p = (1 - \theta)/p_1 + \theta/p_2 \) and \( 1/q = (1 - \theta)/q_1 + \theta/q_2 \), we have \( (X^{s_1}_{p_1,q_1}, X^{s_2}_{p_2,q_2})_\theta = X^s_{p,q} \).

We finally present the basic inclusion relations by the lifting property.

**Proposition 2.1.** Let \( s_1, s_2 \in \mathbb{R} \) and \( 1 \leq p_1, p_2, q_1, q_2 \leq \infty \).

1. \( X^s_{p_1,q_1} \subset X^s_{p_2,q_2} \) if \( p_1 \leq p_2 \), \( q_1 \leq q_2 \), and \( s_1 \geq s_2 \).
2. \( X^s_{p_1,q_1} \subset X^s_{p_2,q_2} \) if \( q_1 > q_2 \), and \( s_1 - s_2 > n(1/q_2 - 1/q_1) \).

**Proposition 2.2.** Let \( s, t \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \). Then, \( (I - \Delta)^{t/2} \) maps \( X^s_{p,q} \) isomorphically onto \( X^{s-t}_{p,q} \), and \( \|(I - \Delta)^{t/2} f\|_{X^{s-t}_{p,q}} \sim \|f\|_{X^s_{p,q}} \).

3. **Lemma**

In this section, we state a key lemma and its proof, which will be used for the proof of the “ONLY IF” part of Theorem 1.1 (2).

**Lemma 3.1.** Let \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \). If \( e^{i\Delta} \) or \( e^{-i\Delta} \) is bounded from \( W^s_{p,q} \) to \( W^s_{p,q} \), then

\[
\left( \sum_{k \in \mathbb{Z}^n} |c_k|^p \right)^{1/p} \lesssim \left( \sum_{k \in \mathbb{Z}^n} (k)^{s q}|c_k|^q \right)^{1/q}
\]

holds for all finitely supported sequences \( \{c_k\}_{k \in \mathbb{Z}^n} \) (that is, \( c_k = 0 \) except for a finite number of \( k \)'s).

**Proof of Lemma 3.1.** Before beginning with the proof, we give the following notations which will be used here:

\[
\|f\|_{M^{(s_1, r_2)}_{p, q}} = \left\| \left( \langle \xi \rangle^{s_1} V_g f(x, \xi) \right) \right\|_{L^p(\mathbb{R}^n)} \quad \text{and} \quad \|f\|_{W^{(s_1, r_2)}_{p, q}} = \left\| \left( \langle \xi \rangle^{s_1} V_g f(x, \xi) \right) \right\|_{L^p(\mathbb{R}^n)}
\]

Note that \( \|f\|_{M^{(s, r)}_{p, q}} = \|f\|_{M^{(0, r)}_{p, q}} \) and \( \|f\|_{W^{(s, r)}_{p, q}} = \|f\|_{W^{(0, r)}_{p, q}} \). From the fact \( |V_g f(x, \xi)| = (2\pi)^{-n} |\hat{V}_g \hat{f}(\xi, -x)| \) (see [6, Section 2.2] for the detail calculation), \( \|f\|_{W^{(s_1, r_2)}_{p, q}} \sim \|\hat{f}\|_{M^{(s_2 - s_1)}_{p, q}} \) for \( 1 \leq p, q \leq \infty \). Hence, the assumption \( \|e^{\pm i\Delta} f\|_{W_{p, q}} \lesssim \|f\|_{W_{p, q}} \) is equivalent to the inequality

\[
\|e^{\mp i\Delta} \|_{M^{(s, r)}_{p, q}} \lesssim \|f\|_{M^{(0, r)}_{p, q}}
\]

In the following statements, we will consider (3.2) and write \( f \) instead of \( \hat{f} \) for the sake of simplicity.
Step 1: We first consider the left hand side of (3.2). We compute the short-time Fourier transform of $e^{\pm i|t|^2} f$.

$$
\left| V_g[e^{\pm i|t|^2} f](x, \xi) \right| = \left| \int_{\mathbb{R}^n} e^{-ix \cdot t} g(t-x) \cdot e^{\pm i|t|^2} f(t) dt \right|
$$

$$
= \left| \int_{\mathbb{R}^n} e^{-i(\xi+2x) \cdot t} g(t-x) \cdot e^{\pm i|t-x|^2} f(t) dt \right|
$$

$$
= \left| V_{g \cdot e^{\pm i|t|^2}} f(x, \xi + 2x) \right|,
$$

where it should be remarked that $g \cdot e^{\pm i|t|^2} \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ since $e^{\pm i|t|^2}$ and all their derivatives are $C^\infty$ slowly increasing functions. Since the norm of modulation spaces is independent of the choice of window functions,

$$
\left\| e^{\pm i|t|^2} f \right\|_{M_{(0,0)}^{0,q}(\mathbb{R}^n)} \sim \left\| V_{\psi} e^{\pm i|t|^2} \left[ e^{\pm i|t|^2} f \right] (x, \xi) \right\|_{L^q(\mathbb{R}^n)} \left\| L^p(\mathbb{R}^n) \right\|
$$

(3.3)

for any $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$.

We choose $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying that

$$
\text{supp } \varphi \subset [-1/8, 1/8]^n \quad \text{and} \quad |\hat{\varphi}| \geq c > 0 \text{ on } [-3/8, 3/8]^n
$$

for some positive constant $c$, and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying that

$$
\text{supp } \psi \subset [-3/8, 3/8]^n \quad \text{and} \quad \psi \equiv 1 \text{ on } [-1/4, 1/4]^n
$$

(see [8, Lemma 4.3] for existence of such functions $\varphi$ and $\psi$). We put

$$
f(t) = \sum_{\ell \in \mathbb{Z}^n} c_{\ell} \cdot \varphi (t - \ell)
$$

for a finitely supported sequence $\{c_{\ell}\}_{\ell \in \mathbb{Z}^n}$.

Under the conditions above, we first consider the inner $L^q$ norm in the last quantity in (3.3) and have

$$
\left\| V_{\psi} f(x, \xi + 2x) \right\|_{L^q(\mathbb{R}^n)} \geq \left( \sum_{k \in \mathbb{Z}^n} \int_{Q_k} \left| \int_{\mathbb{R}^n} e^{-i(\xi+2x) \cdot t} \overline{\psi(t-x)} \sum_{\ell \in \mathbb{Z}^n} c_{\ell} \cdot \varphi(t-\ell) dt \right|^q dx \right)^{1/q}
$$

(3.4)

where we set $Q_k = k + [-1/8, 1/8]^n$. Considering the supports of the functions in the integral of (3.4), we have for all $x \in Q_k$

$$
\text{supp } \psi(-x) \subset x + [-3/8, 3/8]^n \subset k + [-1/2, 1/2]^n;
$$

$$
\text{supp } \varphi(-\ell) \subset \ell + [-1/8, 1/8]^n,
$$

so that we see that $\text{supp } \psi(-x) \cap \text{supp } \varphi(-\ell) = \emptyset$ if $k \neq \ell$ for all $x \in Q_k$. Hence, the right hand side of (3.4) is equal to

$$
\left( \sum_{k \in \mathbb{Z}^n} |c_k|^q \int_{Q_k} \left| \int_{\mathbb{R}^n} e^{-i(\xi+2x) \cdot t} \overline{\psi(t-x)} \cdot \varphi(t-k) dt \right|^q dx \right)^{1/q}
$$

(3.5)

Moreover, we realize that $\psi(-x) \equiv 1$ on $\text{supp } \varphi(-k)$, since we have the fact that $\text{supp } \varphi(-k) \subset k + [-1/8, 1/8]^n \subset x + [-1/4, 1/4]^n$ which is given from the equivalence $x \in Q_k \Leftrightarrow k \in Q_x$. Then, we have

$$
(3.5) = \left( \sum_{k \in \mathbb{Z}^n} |c_k|^q \int_{Q_k} \left| \int_{\mathbb{R}^n} e^{-i(\xi+2x) \cdot t} \varphi(t-k) dt \right|^q dx \right)^{1/q}
$$

$$
= \left( \sum_{k \in \mathbb{Z}^n} |c_k|^q \int_{Q_k} |\hat{\varphi}(\xi + 2x)|^q dx \right)^{1/q}.
$$
Substituting this conclusion into the last quantity of (3.3), we have
\[ \left\| V_{\psi} f(x, \xi \mp 2x) \right\|_{L^q(\mathbb{R}^n)} \geq \left\| \left( \sum_{k \in \mathbb{Z}^n} |c_k|^q \int_{Q_k} |\widehat{\varphi}(\xi \mp 2x)|^q \, dx \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \]

\[ \geq \left( \sum_{m \in \mathbb{Z}^n} |c_m|^p \int_{Q_{\pm 2m}} \left( \int_{Q_m} |\varphi(\xi \mp 2x)|^q \, dx \right)^{p/q} \, d\xi \right)^{1/p} \]

\[ \geq \left( \sum_{m \in \mathbb{Z}^n} |c_m|^p \int_{Q_{\pm 2m}} \left( \int_{Q_m} |\varphi(\xi \mp 2x)|^q \, dx \right)^{p/q} \, d\xi \right)^{1/p} , \]

where we set \( Q_{\pm 2m} = \pm 2m + [\pm 1/8, 1/8]^n \). The assumption \(|\widehat{\varphi}| \geq c > 0\) on \([-3/8, 3/8]^n\) and the fact \( \xi \mp 2x \in (\pm 2m + [\pm 1/8, 1/8]^n) \varsubsetneq 2(m + [-1/8, 1/8]^n) \subset [-3/8, 3/8]^n\) yield that the last expression in (3.6) is estimated from below by
\[ \left( \sum_{m \in \mathbb{Z}^n} |c_m|^p \int_{Q_{\pm 2m}} \left( \int_{Q_m} c^q \, dx \right)^{p/q} \, d\xi \right)^{1/p} , \]

which gives
\[ \left\| V_{\psi} f(x, \xi \mp 2x) \right\|_{L^q(\mathbb{R}^n)} \geq \left( \sum_{m \in \mathbb{Z}^n} |c_m|^p \right)^{1/p} . \]

Combining this result with (3.3), we obtain
\[ (3.7) \quad \left\| e^{\pm i |\cdot|} f \right\|_{H^q(\mathbb{R}^n)} \geq \left( \sum_{m \in \mathbb{Z}^n} |c_m|^p \right)^{1/p} . \]

**Step 2:** Next, we estimate the right hand side of (3.2). The statement of this step is essentially the same as that of [5, Theorem 12.2.4]. Recall that we put
\[ f(t) = \sum_{\ell \in \mathbb{Z}^n} c_\ell \cdot \varphi(t - \ell) \]

for a finitely supported sequence \( \{c_\ell\}_{\ell \in \mathbb{Z}^n} \). We calculate the short-time Fourier transform of this function \( f \) with the \( x \)-weight. Using the \( N \)-times integration by parts with respect to the \( t \)-variable, we have for any \( N \geq 0 \)
\[ \langle x \rangle^N |V_{\psi} f(x, \xi)| = \langle x \rangle^N \left| \sum_{\ell \in \mathbb{Z}^n} c_\ell \int_{\mathbb{R}^n} e^{-i \xi \cdot t} \frac{g(t-x)}{t-x} \cdot \varphi(t-\ell) \, dt \right| \]
\[ \leq \langle x \rangle^N \sum_{\ell \in \mathbb{Z}^n} |c_\ell| \sum_{|\beta| \leq n} \int_{\mathbb{R}^n} \left| (\partial^{\alpha - \beta} g)(t-x) \cdot (\partial^\beta \varphi)(t-\ell) \right| \, dt . \]

From the inequality \( \langle x - \ell \rangle \lesssim \langle t - x \rangle \cdot \langle t - \ell \rangle \) and the Schwarz inequality, it follows that for any \( M \geq 0 \)
\[ \int_{\mathbb{R}^n} \left| (\partial^{\alpha - \beta} g)(t-x) \cdot (\partial^\beta \varphi)(t-\ell) \right| \, dt \]
\[ \lesssim \langle x - \ell \rangle^{-M} \int_{\mathbb{R}^n} \langle t - x \rangle^M \left| (\partial^{\alpha - \beta} g)(t-x) \right| \langle t - \ell \rangle^M \left| (\partial^\beta \varphi)(t-\ell) \right| \, dt \]
\[ \lesssim \langle x - \ell \rangle^{-M} \left\| \langle \cdot \rangle^M (\partial^{\alpha - \beta} g) \right\|_{L^2} \left\| \langle \cdot \rangle^M (\partial^\beta \varphi) \right\|_{L^2} \]
\[ \sim \langle x - \ell \rangle^{-M} , \]
since \( g, \varphi \in \mathcal{S}(\mathbb{R}^n) \). Thus, by the Peetre inequality \( \langle x \rangle^s \lesssim (\langle \ell \rangle^s \cdot \langle x - \ell \rangle^{|s|} ) \), we have for any \( M \geq 0 \)
\[
\langle x \rangle^s |V_{\ell}f(x, \xi)| \lesssim \langle x \rangle^s \langle \xi \rangle^{-N} \sum_{\ell \in \mathbb{Z}^n} |c_{\ell}| \langle x - \ell \rangle^{-M} \\
\lesssim \langle \xi \rangle^{-N} \sum_{\ell \in \mathbb{Z}^n} \langle \ell \rangle^s |c_{\ell}| \langle x - \ell \rangle^{-M+|s|}.
\]
Substitute this estimate into the right hand side of (3.2) and choose \( N \geq 0 \) such that \( N > n/p \). Then, we have
\[
\|f\|_{M^{(s,0)}_{\ell,p}} = \left\| \langle x \rangle^s V_{\ell}f(x, \xi) \|_{L^p(\mathbb{R}^n)} \right\|_{L^p(\mathbb{R}^n)} \\
\lesssim \left\| \langle \xi \rangle^{-N} \sum_{\ell \in \mathbb{Z}^n} \langle \ell \rangle^s |c_{\ell}| \langle x - \ell \rangle^{-M+|s|} \right\|_{L^p(\mathbb{R}^n)} \\
\sim \left\| \sum_{\ell \in \mathbb{Z}^n} \langle \ell \rangle^s |c_{\ell}| \langle x - \ell \rangle^{-M+|s|} \right\|_{L^p(\mathbb{R}^n)}.
\]
Moreover, since \( \mathbb{R}^n \) is decomposed by \( \mathbb{R}^n = \bigcup_{m \in \mathbb{Z}^n} \tilde{Q}_m \) with \( \tilde{Q}_m = m + [-1/2, 1/2)^n \), the last quantity in (3.8) is expressed as
\[
\left\{ \sum_{m \in \mathbb{Z}^n} \int_{\tilde{Q}_m} \left( \sum_{\ell \in \mathbb{Z}^n} \langle \ell \rangle^s |c_{\ell}| \langle x - \ell \rangle^{-M+|s|} \right) d\mathcal{L}^n \right\}^{1/q}.
\]
Hence, by the fact that \( \langle x - \ell \rangle \sim \langle m - \ell \rangle \) for all \( x \in \tilde{Q}_m \), we have
\[
\|f\|_{M^{(s,0)}_{\ell,p}} \lesssim \left\{ \sum_{m \in \mathbb{Z}^n} \left( \sum_{\ell \in \mathbb{Z}^n} \langle \ell \rangle^s |c_{\ell}| \langle m - \ell \rangle^{-M+|s|} \right) \right\}^{1/q}.
\]
Choosing \( M \geq 0 \) such that \( M > n + |s| \) and using the convolution relation \( \ell^q * \ell^1 \hookrightarrow \ell^q \) for \( 1 \leq q \leq \infty \), we obtain
\[
\|f\|_{M^{(s,0)}_{\ell,p}} \lesssim \left( \sum_{\ell \in \mathbb{Z}^n} \langle \ell \rangle^{sq} |c_{\ell}|^q \right)^{1/q}.
\]
**Step 3:** Combining (3.7) and (3.9) with the assumption, we have
\[
\left( \sum_{m \in \mathbb{Z}^n} |c_m|^p \right)^{1/p} \lesssim \left\| e^{+i|\xi|^2/2} \mathcal{F}^n \right\|_{M^{(0,0)}_{\ell,p}} \sim \left\| e^{+i|\xi|^2/2} \mathcal{F}^n \right\|_{W^p_{\ell,q}} \\
\lesssim \left\| \mathcal{F} \right\|_{W^p_{\ell,q}} \sim \left\| \mathcal{F} \right\|_{M^{(s,0)}_{\ell,p}} \lesssim \left( \sum_{\ell \in \mathbb{Z}^n} \langle \ell \rangle^{sq} |c_{\ell}|^q \right)^{1/q},
\]
which is the desired result. \( \square \)

4. Proof of the main theorem

In this section, we prove Theorem 1.1. As mentioned in Section 1, the “IF” part in Theorem 1.1 was already proved by Cunanan and Sugimoto [3] and also the “ONLY IF” part for \( p = \infty \) and \( 1 \leq q < \infty \) in Theorem 1.1 was proved by Cordero and Nicola [2], so that the main contribution of this paper is to show the “ONLY IF” part for the remaining cases of \( p \) and \( q \). However, for the reader’s convenience, we will prove the “IF” and “ONLY IF” parts for the whole range \( 1 \leq p, q \leq \infty \).

**Proof of the “IF” part of Theorem 1.1.** We divide the proof into the following three cases:

(a) \( 1 \leq p = q \leq \infty \);  \( b) 1 \leq p < q \leq \infty \);  \( c) 1 \leq q < p \leq \infty \).
Case (b): Recall from Proposition 2.1 (1) that $W_{p,q} \hookrightarrow W_{p,p}$ for $p<q$, and from Proposition 2.1 (2) that $W_{p,q}^s \hookrightarrow W_{p,p}$ for $p<q$ and $s>n(1/p-1/q)$. Then, Theorem A combined with $M_{p,p} = W_{p,p}$ gives
\[ \|e^{i\Delta} f\|_{W_{p,q}} \lesssim \|e^{i\Delta} f\|_{W_{p,p}} \lesssim \|f\|_{W_{p,p}} \lesssim \|f\|_{W_{p,q}^s}. \]

Case (c): As above, using Proposition 2.1 and Theorem A, we have the desired conclusion. □

Proof of the “ONLY IF” part of Theorem 1.1. We prove the “ONLY IF” part of Theorem 1.1 by considering the following four cases:

(a) $1 \leq p = q \leq \infty; \quad$ (b) $1 \leq p < q < \infty; \quad$ (c) $1 < q/p < \infty; \quad$ (d) otherwise.

Case (a): The relation $W_{p,p}^s = M_{p,p}^s$ and Theorem A complete the proof for this case.

Case (b): Note that the condition $1 \leq p < q < \infty$ implies that $1 < q/p < \infty$. Using (3.1) with $c_k = \langle k \rangle^{-s}|d_k|^{1/p}$ for a given finitely supported sequence $\{d_k\}_{k \in \mathbb{Z}^n}$, we have
\[ \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-sp}|d_k| \lesssim \left( \sum_{k \in \mathbb{Z}^n} |d_k|^q/p \right)^{p/q}. \]

We take the supremum over $\{d_k\}_{k \in \mathbb{Z}^n}$ such that $\|d_k\|_{\ell^{q/p}} = 1$. Then, we have by (4.1)
\[ \|\langle k \rangle^{-sp}\|_{\ell^{q/p}} \lesssim \sup_{\|d_k\|_{\ell^{q/p}} = 1} \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-sp}d_k \right) \lesssim 1, \]
which yields that $(q/p)^{sp} > n$, namely, $s > n(1/p-1/q) = n[1/p-1/q]$.

Case (c): By duality, our assumption implies $e^{-s\Delta} : W_{p',q'} \rightarrow W_{p,q}'$, which is equivalent from Proposition 2.2 that $e^{-s\Delta} : W_{p,q}^s \rightarrow W_{p,q}'$. Note that $1 < q/p \leq \infty$ implies that $1 \leq q' < q' < \infty$ and thus $1 < q'/q' < \infty$. Then, repeating the proof above, we obtain $(q'/p')^{sp'} > n$, that is, $s > -n(1/p-1/q) = n[1/p-1/q]$.

Case (d): We note that the remaining parts are, in detail,

(d-1) $1 \leq p < \infty$ and $q = \infty; \quad$ (d-2) $1 < p \leq \infty$ and $q = 1$.

We first consider the case (d-1). We assume towards a contradiction that $s \leq n[1/p-1/q] = n/p$. Note that we have $e^{i\Delta} : W_{p,\infty}^s \rightarrow W_{p,\infty}$ by the assumption, which implies that $e^{i\Delta} : W_{p,\infty}^{s/n} \rightarrow W_{p,\infty}$, and $e^{i\Delta} : W_{p,p} \rightarrow W_{p,p}$ by Theorem 1.1 (1). Interpolation yields for arbitrary $0 < \theta < 1$ that $e^{i\Delta} : W_{\tilde{s},\tilde{q}}^{\tilde{s}} \rightarrow W_{\tilde{p},\tilde{q}}$, where $1/\tilde{q} = \theta/p$ and $\tilde{s} = (1-\theta)n/p$. Now, we remark that $\tilde{s} = n(1/p-1/\tilde{q})$ and $1/p < \tilde{q} < \infty$. However, this is a contradiction, since we already proved in Case (b) that, for $1 \leq p < q < \infty$, $e^{i\Delta} : W_{\tilde{s},\tilde{q}}^{\tilde{s}} \rightarrow W_{\tilde{p},\tilde{q}}$ holds only if $\tilde{s} > n(1/p-1/\tilde{q})$. Thus, we obtain for $1 \leq p < \infty$, $e^{i\Delta} : W_{p,\infty}^s \rightarrow W_{p,\infty}$ holds only if $s > n/p$. The case (d-2) can be also given by a similar argument, so that we omit the detail. □

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