SEMIGROUP IDENTITIES IN THE MONOID OF
TRIANGULAR TROPICAL MATRICES

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Abstract. We show that the submonoid of all \( n \times n \) triangular tropical matrices satisfies a
nontrivial semigroup identity and provide a generic construction for classes of such identities.
The utilization of the Fibonacci number formula gives us an upper bound on the length of these
2-variable semigroup identities.

1. Introduction

Varieties in the classical theory are customarily determined as the solutions of systems of equa-
tions. The “weak” nature of semigroups, i.e., lack of inverses, forces the utilization of a different
approach than the familiar one, in which semigroup identities simulate the role of equations in
the classical theory. These semigroup identities are at the heart of the theory of semigroup vari-
eties \[10\], and have been intensively studied for many years. A new approach for studying these
semigroup identities has been provided by the use of tropical algebra, as introduced in \[7\].

Tropical algebra is carried out over the tropical semiring \( \mathbb{T} := \mathbb{R} \cup \{-\infty\} \) with the operations
of maximum and summation (written in the standard algebraic way),

\[
a + b := \max\{a, b\}, \quad ab := a + b,\]

serving respectively as addition and multiplication \[6 \ 8 \ 9\]. This semiring is an additively idem-
potent semiring, i.e., \( a + a = a \) for every \( a \in \mathbb{T} \), in which \( 0 := -\infty \) is the zero element and \( 1 := 0 \)
is the multiplicative unit.

As shown in \[7\] for the case of \( 2 \times 2 \) tropical matrices, linear representations of semigroups by
matrices over idempotent semirings, and in particular over the tropical semiring, establish a useful
machinery for identifying and proving semigroup identities. It has been also shown in \[7\] that the
monoid of \( 2 \times 2 \) tropical matrices admits a nontrivial semigroup identity. Therefore, aiming for
a generalization of the former results, it is natural to inquire about possible nontrivial identities
in the wider monoid of \( n \times n \) tropical matrices. These matrices essentially serve as the target for
representing semigroups, and thus enable representations and study of a larger range of monoids
and semigroups.

In the present paper we deal with \( n \times n \) triangular tropical matrices, improving the machinery
introduced in \[7\] by bringing in the perspective of graph theory. It is well known that graph
theory, especially the theory of direct graphs, is strongly related to tropical matrices \[2 \ 3\] and
provides a powerful computational tool in tropical matrix algebra; the correspondence between
tropical matrices and weighted digraphs is intensively used for proving the results of this paper. A
background on the interplay between digraphs and tropical matrices is given in \[2\]
Before approaching tropical matrices, we first address semigroup identities in general (cf. \cite{3}), providing a generic construction of classes of identities that preserve certain required properties (in particular, balancing), to be used in our further study. The following refinement property is very useful for this study, especially for proving existence of identities:

**Theorem 3.10.** A semigroup that satisfies an \( n \)-variable identity, also satisfies a refined 2-variable identity with exponent set \{1, 2\}.

This refinement of semigroup identities assists us to deal with tropical matrices by better utilizing their view as digraphs.

The monoid \( M_n(\mathbb{T}) \) of \( n \times n \) matrices over the tropical semiring plays, as one would expect, an important role both in theoretical algebraic study and in applications to combinatorics, as well as in semigroup representations and automata. In contrast to the case of matrices over a field, we identify nontrivial semigroup identities, satisfied by the submonoid \( U_n(\mathbb{T}) \) (resp. \( L_n(\mathbb{T}) \)) of all upper (resp. lower) \( n \times n \) triangular tropical matrices.

To simplify the exposition we open with a certain type of matrices, i.e., diagonally equivalent matrices, and have the following preliminary theorem:

**Theorem 4.8.** Any two triangular tropical matrices \( X, Y \in U_n(\mathbb{T}) \) having the same diagonal satisfy the (nontrivial) identities:

\[
\tilde{w}_{(C,P,n-1)} X \tilde{w}_{(C,P,n-1)} = \tilde{w}_{(C,P,n-1)} Y \tilde{w}_{(C,P,n-1)}, \tag{1.1}
\]

where \( \tilde{w}_{(C,P,n-1)} \) is any word having as its factors all the words of length \( n - 1 \) generated by \( C = \{X, Y\} \) of powers \( P = \{1, 2\} \), such that \( \tilde{w}_{(C,P,n-1)} X \tilde{w}_{(C,P,n-1)} \) and \( \tilde{w}_{(C,P,n-1)} Y \tilde{w}_{(C,P,n-1)} \) are generated by \( C \) and powers \( P \). (To be explained in the text below.)

Using this result, basically proved by combinatorial arguments on the associated (colored) weighted digraphs of products of tropical matrices, we obtain the main result of the paper.

**Theorem 4.10.** The submonoid \( U_n(\mathbb{T}) \subset M_n(\mathbb{T}) \) of upper triangular tropical matrices satisfies the nontrivial semigroup identities \( \eqref{1.1} \), with \( X = AB \) and \( Y = BA \), for any \( A, B \in U_n(\mathbb{T}) \).

This theorem generalizes the identity of the submonoid \( U_2(\mathbb{T}) \) of \( 2 \times 2 \) triangular tropical matrices, introduced in \cite{7} Theorem 3.6, which has been utilized to serve as the target of a faithful (linear) representation for the bicyclic monoid; the latter plays an important role in the study of semigroups. The use of this faithful representation, together with the identity admitted by \( U_2(\mathbb{T}) \), led in \cite{7} to an easy proof of Adjan’s identity of the bicyclic monoid (see \cite{1} for Adjan’s original work).

The well known Fibonacci number formula provides us an easy way to compute an upper bound for the length of the 2-variable semigroup identities discussed in this paper (cf. \cite{3}). This upper bound can be improved further, depending on the structure of the identities.

In the past years, most of the theory of semigroups and matrix representations has been developed for matrices built over fields or rings; the above results nicely demonstrate our new approach to represent semigroups, much along the line of group representations, attained by a “direct” use of matrices, realized as linear operators, but now taking place over semirings.

The results of this paper opens up the possibility of using representation theory over the tropical semiring to study wider classes of semigroups and monoids and to prove their possible (minimal) semigroup identities.

2. Background: Tropical Matrices and Weighted Digraphs

Recalling that \( \mathbb{T} \) is a semiring, then in the usual way, we have the semiring \( M_n(\mathbb{T}) \) of \( n \times n \) matrices with entries in \( \mathbb{T} \), whose addition and multiplication are induced from \( \mathbb{T} \) as in the familiar matrix construction. The **unit** element \( I \) of \( M_n(\mathbb{T}) \), is the matrix with all entries \( 1 \) on the main diagonal and whose off-diagonal entries are all \( 0 = -\infty \); the **zero** matrix is \( (0) = 0I \). Therefore, \( M_n(\mathbb{T}) \) is also a multiplicative monoid, and in the sequel it is always referred to as a monoid. Formally, for any nonzero matrix \( A \in M_n(\mathbb{T}) \) we set \( A^0 := I \). A given matrix \( A \in M_n(\mathbb{T}) \) with entries \( a_{i,j} \) is
written as \( A = (a_{i,j}) \), \( i, j = 1, \ldots, n \). We denote by \( U_n(\mathbb{T}) \) (resp. \( L_n(\mathbb{T}) \)) the submonoid of \( M_n(\mathbb{T}) \) of all upper (resp. lower) triangular matrices.

Given two matrices \( X = (x_{i,j}) \) and \( Y = (y_{i,j}) \) in \( M_n(\mathbb{T}) \), we write
\[
X \sim_{\text{diag}} Y \iff x_{i,i} = y_{i,i}, \quad \text{for all } i = 1, \ldots, n,
\]
and say that \( X \) and \( Y \) are diagonally equivalent if \( [2.1] \) holds.

**Remark 2.1.** It is readily checked that \( AB \sim_{\text{diag}} BA \) for any upper (or lower) triangular matrices \( A \) and \( B \).

The associated weighted digraph \( G_A := (\mathcal{V}, \mathcal{E}) \) of an \( n \times n \) tropical matrix \( A = (a_{i,j}) \) is defined to have vertex set \( \mathcal{V} := \{1, \ldots, n\} \), and edge set \( \mathcal{E} \) having a directed edge \( (i, j) \in \mathcal{E} \) from \( i \) to \( j \) (of weight \( a_{i,j} \)) whenever \( a_{i,j} \neq 0 \). A path \( \gamma \) is a sequence of edges \( (i_1, j_1), \ldots, (i_m, j_m) \), with \( j_k = i_{k+1} \) for every \( k = 1, \ldots, m - 1 \). We write \( \gamma := \gamma_{i,j} \) to indicate that \( \gamma \) is a path from \( i = i_1 \) to \( j = j_m \), and call \( \gamma_{i,s} \) (resp. \( \gamma_{s,j} \)), where \( s = i_k \) and \( 1 < k < m \), the prefix (resp. suffix) of \( \gamma_{i,j} \) if \( \gamma_{i,j} = \gamma_{i,s} \circ \gamma_{s,j} \).

The length \( \ell(\gamma) \) of a path \( \gamma \) is the number of its edges. Formally, we consider also paths of length 0, which we call empty paths. The weight \( w(\gamma) \) of a path \( \gamma \) is defined to be the tropical product of the weights of all the edges \( (i_k, j_k) \) composing \( \gamma \), counting repeated edges. The weight of an empty path is formally set to be 0.

A path is simple if each vertex appears at most once. (Accordingly, an empty path is considered also as simple.) A path that starts and ends at the same vertex is called a cycle; an edge \( \rho = (i, i) \) is called a self-loop, or loop for short. We write \((\rho)^k\) for the composition \( \rho \circ \cdots \circ \rho \) of a loop \( \rho \) repeated \( k \) times, and call it a multiloop. The notation \((\rho)^0\) is formal, and stands for an empty loop, which can be realized as a vertex.

**Remark 2.2.** When a matrix \( A \) is triangular, its associated digraph \( G_A \) is an acyclic digraph, possibly with loops. Since this paper concerns only with triangular matrices, in what follows we assume all graphs are acyclic digraphs.

Given a path \( \gamma_{i,j} \) from \( i \) to \( j \) in an acyclic digraph \( G_A \), it contains a unique simple path from \( i \) to \( j \), denoted \( \tilde{\gamma}_{i,j} \), where the remaining edges are all loops. Relabeling the vertices of \( G_A \), we may always assume that \( i < j \) and thus have \( \ell(\tilde{\gamma}_{i,j}) \leq j - i \).

It is well known that powers of a tropical matrix \( A = (a_{i,j}) \) correspond to paths of maximal weight in the associated digraph, i.e., the \((i,j)\)-entry of \( A^m \) corresponds to the highest weight of all the paths \( \gamma_{i,j} \) from \( i \) to \( j \) of length \( m \) in \( G_A \).

When dealing with product \( A_1 \cdots A_m \) of different \( n \times n \) matrices the situation becomes more complicated. Namely, we have to equip the weighted edges \( e_i \in \mathcal{E}_i \) of each digraph \( G_{A_i} \) with a unique color, say \( c_i \), and define the digraph
\[
G_{A_1 \cdots A_m} := \bigcup G_{A_i},
\]
whose vertex set is \( \{1, \ldots, n\} \) and its edge set is the union of edge sets \( \mathcal{E}_i \) of \( G_{A_i} = (\mathcal{V}, \mathcal{E}_i) \) colored by the \( c_i \)'s, where \( i = 1, \ldots, m \). (Thus, \( G_{A_1 \cdots A_m} \) could have multiple edges, but with different colors.) We called such a weighted digraph a colored digraph.

Then, having such coloring, the \((i,j)\)-entry of the matrix product \( B = A_1 \cdots A_m \) corresponds to the highest weight of all colored paths \( (i_1, j_1), \ldots, (i_m, j_m) \) of length \( m \) from \( i = i_1 \) to \( j = j_m \) in the digraph \( G_{A_1 \cdots A_m} \), where each edge \( (i_k, j_k) \) has color \( c_k \), \( k = 1, \ldots, m \), i.e., every edge is contributed uniquely by the associated digraph \( G_{A_k} \) of \( A_k \), respecting the color ordering. We call such a path a proper colored path.

In what follows when considering paths in colored digraphs \( G_{A_1 \cdots A_m} \), we always restrict to those colored paths that respect the sequence of coloring \( c_1, \ldots, c_m \), i.e., properly colored, determined by the product concatenation \( A_1 \cdots A_m \). (For this reason we often preserve the awkward notation \( G_{A_1 \cdots A_m} \), which records the product concatenation \( A_1 \cdots A_m \).)
Notation 2.3. Given a matrix product $B = A_1 \cdots A_m$, we write $\langle B \rangle$ to indicate that $B$ is realized as a word “restoring” the product concatenation $A_1 \cdots A_m$, and thus denote $G_{A_1 \cdots A_m}$ as $G_{\langle B \rangle}$, while $B$ denotes the result of the matrix product. We also write $\langle B \rangle = \langle B' \rangle \langle B'' \rangle$ for the product concatenation of $\langle B' \rangle = A_1 \cdots A_k$ and $\langle B'' \rangle = A_{k+1} \cdots A_m$, with $1 < k < m$.

3. Semigroup identities

3.1. Semigroup elements. Assuming that $S := (S, \cdot)$ is a multiplicative monoid with identity element $e_S$, we write $s^t$ for the $s \cdot s \cdots s$ with $s$ repeated $t$ times and formally identify $s^0$ with $e_S$.

Let $X$ be a countably infinite set of “variables” (or letters) $x_1, x_2, x_3, \ldots$, i.e., $X := \{x_i : i \in \mathbb{N}\}$. An element $w$ of the free semigroup $X^+$ generated by $X$ is called a word (over $X$), written uniquely as

$$w = x_1^{i_1} \cdots x_m^{i_m} \in X^+, \quad i_k \in \mathbb{N}, \ t_k \in \mathbb{N},$$

(3.1)

where $x_i \neq x_{i+1}$ for every $k$. We write $\kappa_{x_i}(w)$ for the number of occurrences of the variable $x_i \in X$ in the word $w$. Then

$$\text{cont}(w) := \{x_i \in X \mid \kappa_{x_i}(w) \geq 1\}$$

is called the content of $w$ and

$$\ell(w) := \sum_{x_i \in \text{cont}(w)} \kappa_{x_i}(w)$$

is the length of $w$. A word $w \in X^+$ is said to be finite if $\ell(w)$ is finite. We assume that the empty word, denoted $e$, belongs to $X^+$ and set $\ell(e) = 0$. A word $w$ is called $k$-uniform if each letter $x_i \in \text{cont}(w)$ appears in $w$ exactly $k$ times, i.e., $\kappa_{x_i}(w) = k$ for all $x_i \in \text{cont}(w)$. We say that $w$ is uniform if it is $k$-uniform for some $k$.

We say that $w_2 \in X^+$ is a factor of a word $w \in X^+$, written $w_2 | w$, if $w = w_1w_2w_3$ for some $w_1, w_3 \in X^+$. When $w = w_1w_2$, we call the factors $w_1$ and $w_2$ respectively the prefix and suffix of $w$, denoted as $\text{pre}(w)$ and $\text{suffix}(w)$. Given a word $w \in X^+$ we write $\text{pre}_{x_i}(w)$ (resp. $\text{suffix}_{x_i}(w)$) for the prefix (resp. suffix) of $w$ of maximal length that consists only the variable $x_i$, in particular, when $w \neq e$, $\text{pre}_{x_i}(w) = x_i^{j_i}$ for some $x_i$ and $j_i \in \mathbb{Z}_+$.

A word $u$ is a subword of $v$, written $u \subseteq v$, if $v$ can be written as $v = w_0u_1w_1u_2w_2 \cdots u_mw_m$ where $u_1$ and $u_i$ are words (possibly empty) such that $u = u_1u_2 \cdots u_m$, i.e., the $u_i$ are factors of $u$. Clearly, any factor of $v$ is also a subword, but not conversely.

Given a finite subset $P \subseteq \mathbb{N}$, we define the “down closure” of $P$ to be

$$P^\downarrow := \{t \in \mathbb{N} \mid t \leq p \text{ for some } p \in P\}.$$ 

The exponent set $\text{exp}(w)$ of a word $w$ of the Form 3.1 is defined as

$$\text{exp}(w) := \{t_k \mid t_k > 0\}.$$ 

In this paper we always assume all words are finite; thus $|\text{exp}(w)|$ is finite for any word $w$. When $\text{exp}(w) = \text{exp}(w) = \{1, \ldots, m\}$ we say that $w$ is a word of exponent $\langle m \rangle$.

Henceforth, we always assume that $P = P^\downarrow \subseteq \mathbb{N}$ is a nonempty subset of the form

$$P := \{1, \ldots, m\} \quad \text{and that} \quad n \geq m = \max\{p \mid p \in P\}.$$ 

Given finite nonempty subsets $C \subseteq X$ and $P \subseteq \mathbb{N}$, for any $n \geq m$, $n \in \mathbb{N}$, we define

$$\mathcal{W}_n[C, P] := \{w \in X^+ \mid \text{cont}(w) \subseteq C, \ \text{exp}(w) \subseteq P, \ \ell(w) = n\},$$

in particular $\mathcal{W}_n[C, P] \subseteq C^+$.

We denote by $\tilde{w}_{(C, P, n)}$ a word in $C^+$ for which every member of $\mathcal{W}_n[C, P]$ is a factor, i.e.,

$$\tilde{w}_{(C, P, n)} \in C^+ \text{ such that } u|\tilde{w}_{(C, P, n)} \text{ for every } u \in \mathcal{W}_n[C, P].$$

(3.2)
We call \( \tilde{w}_{(C,P,n)} \) an \( n \)-power word of \( C \) and \( P \). We say that \( \tilde{w}_{(C,P,n)} \) is faithful if
\[
\text{cont}(\tilde{w}_{(C,P,n)}) = C \quad \text{and} \quad \exp(\tilde{w}_{(C,P,n)}) = P.
\]
Note that \( \ell(\tilde{w}_{(C,P,n)}) \geq n \), while for \( |C| > 1 \) we have \( \ell(\tilde{w}_{(C,P,n)}) > n \). When \( |C| = 1 \), say \( C = \{x_i\} \), then \( x_i^t \) is an \( n \)-power word for any \( t \geq n \), and \( x_i^t \) is faithful only if \( t = n = \max\{p \mid p \in P\} \).

**Example 3.1.** Suppose \( C = \{x, y\}, P = \{1, 2\} \).

(i) When \( n = 2 \) we have the set
\[
W_2(C, P) = \{x^2, xy, yx, y^2\} \subset C^+,
\]
for which
\[
\tilde{w}_{(C,P,2)} = x^2y^2x
\]
is a faithful 2-power (nonuniform) word of \( C \) and \( P \) of length 5.

(ii) If \( n = 3 \) we get the set
\[
W_3(C, P) = \{x^3, x^2y, xy^2, y^2x, yxy, y^2x\} \subset C^+,
\]
for which
\[
\tilde{w}_{(C,P,3)} = l3
\]
is a faithful 3-power uniform word of \( C \) and \( P \) of length 8.

**Remark 3.2.** Given an \( n \)-power word \( \tilde{w}_{(C,P,n)} \) of \( C \) and \( P \), it is easy to see that for any \( w_1, w_2 \in C^+ \), the word of the form \( \tilde{w}_{(C,P,n)}^0 = w_1 \tilde{w}_{(C,P,n)} w_2 \) is also an \( n \)-power word of \( C \) and \( P \). Therefore, taking appropriate \( w_1 \) and \( w_2 \), \( \tilde{w}_{(C,P,n)} \) can be extended to a uniform \( n \)-power word. Similar extension can be performed for faithful \( n \)-power words, preserving their faithfulness.

In what follows, we always work with \( n \)-power words \( \tilde{w}_{(C,P,n)} \) which are faithful and with \( |C| > 1 \).

An \( n \)-power word \( \tilde{w}_{(C,P,n)} \) of \( C \) and \( P \) is called a minimal \( n \)-power word if \( \ell(\tilde{w}_{(C,P,n)}) \leq \ell(\tilde{w}_{(C,P,n)}^0) \) for any \( n \)-power word \( \tilde{w}_{(C,P,n)}^0 \in C^+ \).

**Example 3.3.** The power words in Example 3.1 are minimal power words.

**Remark 3.4.** Given a word \( w \in \mathcal{X}^+ \), we may consider \( \mathcal{X} \) to be a set of generic matrices \( A_1, A_2, \ldots \).

Using Notation 2.3 the word \( w = A_{i_1} \cdots A_{i_m} \) can be realized as a product concatenation of matrices and thus can be written equivalently as \( \langle B \rangle = A_{i_1} \cdots A_{i_m} \), where the result of the matrix product is \( B = (b_{i,j}) \). Then every entry \( b_{i,j} \) of \( B \) corresponds to a proper colored path \( \gamma_{i,j} \) from \( i \) to \( j \) of length \( m \) in the digraph \( G_{ \langle B \rangle } \), cf. [3]. Thus, the proper coloring of all paths of length \( m \) is uniquely determined by \( w \). Conversely, given a properly colored path \( \gamma_{i,j} \) of length \( \ell(w) \) in \( G_{ \langle B \rangle } \), one can recover the word \( w \) from the coloring of the edges consisting \( \gamma_{i,j} \).

3.2. **Semigroup identities.** A (nontrivial) semigroup identity is a formal equality of the form
\[
\Pi : u = v, \quad (3.3)
\]
where \( u \) and \( v \) are two different (finite) words of the Form (3.1) in the free semigroup \( \mathcal{X}^+ \). For a monoid identity, we allow \( u \) and \( v \) to be the empty word as well. We discuss, for simplicity, semigroup identities, but minor changes apply to monoid identities as well.

A semigroup \( S := (S, \cdot) \) satisfies the semigroup identity (3.3) if for every morphism \( \phi : \mathcal{X}^+ \to S \) one has \( \phi(u) = \phi(v) \).

**Remark 3.5.** Semigroup identities can be thought of as special case of polynomial identities (PI’s), namely as monomial identities.
We say that an identity \( \Pi : u = v \) is an \textit{n-variable identity} if \( |\text{cont}(u) \cup \text{cont}(v)| = n \). The \textit{exponent set}, denoted \( \exp(\Pi) \), of \( \Pi \) is defined to be \( \exp(u) \cup \exp(v) \). An identity \( \Pi \) is said to be \textit{balanced} if \( \kappa_{x_i}(u) = \kappa_{x_i}(v) \) for every \( x_i \in X \), and it is called \textit{uniformly balanced} if furthermore \( u \) and \( v \) are \( k \)-uniform for some \( k \). We define the \textit{length} \( \ell(\Pi) \) of \( \Pi \) to be \( \ell(\Pi) := \max\{\ell(u), \ell(v)\} \). It is readily checked that if \( \Pi \) is balanced, then \( \ell(u) = \ell(v) \).

**Example 3.6.** Let us give some very elementary examples of semigroup identities.

(i) A commutative semigroup admits the 2-variable uniformly balanced identity \( \Pi : xy = yx \), whose exponent set is \( \exp(\Pi) = \{1\} \).

(ii) An idempotent semigroup admits the (non-balanced) 1-variable identity \( \Pi : x^2 = x \), whose content is \( \text{cont}(\Pi) = \{x\} \) and its exponent set is \( \exp(\Pi) = \{1, 2\} \).

(iii) A virtually abelian semigroup admits the 2-variable uniformly balanced identity \( \Pi : x^n y^n = y^n x^n \), whose exponent set is \( \exp(\Pi) = \{n\} \).

(iv) The 2-variable identity \( \Pi : x^i y^j = y^j x^i \), whose exponent set is \( \exp(\Pi) = \{i, j\} \), is balanced but not uniformly balanced for nonzero \( i \neq j \).

An identity \( \Pi : u = v \) is called a \textit{minimal identity} of the semigoup \( S \) if \( \ell(\Pi) \leq \ell(\Pi') \) for any nontrivial identity \( \Pi' : u' = v' \) of \( S \).

**Remark 3.7.** Let \( \mathcal{I} \) be a set of semigroup identities. The set of all semigroups satisfying every identity in \( \mathcal{I} \) is denoted by \( V[\mathcal{I}] \) and is called the \textit{variety of semigroups} defined by \( \mathcal{I} \). It is easy to see that \( V[\mathcal{I}] \) is closed under subsemigroups, homomorphic images, and direct products of its members. The famous Theorem of Birkhoff says that conversely, any class of semigroups closed under these three operations is of the form \( V[\mathcal{I}] \) for some set of identities \( \mathcal{I} \).

3.3. Construction of semigroup identities. Although the main part of this paper utilizes 2-variable identities of exponent \( \langle 2 \rangle \), for future study, we present the construction of identities that are of interest in full generality.

Given an \( n \)-power word \( \widetilde{w}(C, P, n) \) as in (3.2), with \( C = \{x_1, \ldots, x_m\} \) and \( P = \{t_1, \ldots, t_j\} \), we aim to build a nontrivial balanced identity \( \Pi : u = v \) of content \( C \) and exponent set \( P \). To preserve the exponent set \( P \) for \( \Pi \), if necessary, we first extend \( \widetilde{w}(C, P, n) \) to \( \widetilde{w}'(C, P, n) \) (cf. Remark 3.2), and construct the words \( u \) and \( v \) such that \( \widetilde{w}'(C, P, n) \) is their prefix and suffix.

Let \( t_{\max} := \max\{t_1, \ldots, t_j\} \), \( t_{\min} := \min\{t_1, \ldots, t_j\} \), and let \( d := t_{\max} - t_{\min} \). We assume the following:

\[
|C| > 1, \quad |P| > 1, \quad t_{\max} \geq 2t_{\min}.
\]

Letting

\[
z_1 := x_1^{t_{\min}} \cdots x_m^{t_{\min}}, \quad z_2 := x_1^{t_{\min}} \cdots x_1^{t_{\min}},
\]

we define the identity

\[
\Pi(C, P, n) : \quad \widetilde{w}'(C, P, n) z_1 \widetilde{w}'(C, P, n) = \widetilde{w}'(C, P, n) z_2 \widetilde{w}'(C, P, n),
\]

where \( \widetilde{w}'(C, P, n) \) is defined as

\[
\widetilde{w}'(C, P, n) := w_1 \widetilde{w}(C, P, n) w_2,
\]

with \( w_1 \) and \( w_2 \) given as follows (letting \( \widetilde{w} := \widetilde{w}(C, P, n) \), for short):

\[
w_1 := \begin{cases} 
  x_m & \text{if } \text{pre}_{x_1}(\widetilde{w}) > d \\
  x_1 & \text{if } \text{pre}_{x_m}(\widetilde{w}) > d \\
  e & \text{otherwise}
\end{cases} \quad \text{and} \quad w_2 := \begin{cases} 
  x_m & \text{if } \text{suf}_{x_1}(\widetilde{w}) > d \\
  x_1 & \text{if } \text{suf}_{x_m}(\widetilde{w}) > d \\
  e & \text{otherwise}
\end{cases}
\]

Clearly, by this construction, \( \Pi(C, P, n) \) is a balanced identity.
In view of Remark 3.2, \( \tilde{w}'_{(C,P,n)} \) can be extended further to be uniform, which then makes the identity (3.4) uniformly balanced. In particular when the given \( n \)-power word \( \tilde{w}_{(C,P,n)} \) is uniform, we can instead explicitly define \( w_1 \) and \( w_2 \) in (3.5) as

\[
w_1 := \begin{cases} 
  z_1 & \text{if } \text{pre}_{x_1}(\tilde{w}) > d \\
  z_2 & \text{if } \text{pre}_{x_m}(\tilde{w}) > d \\
  e & \text{otherwise}
\end{cases} \quad w_2 := \begin{cases} 
  z_2 & \text{if } \text{sup}_{x_1}(\tilde{w}) > d \\
  z_1 & \text{if } \text{sup}_{x_m}(\tilde{w}) > d \\
  e & \text{otherwise}
\end{cases}
\]

(3.7)

to obtain a uniformly balanced identity.

**Notation 3.8.** Throughout this paper we use the notation \( x \) and \( y \) to mark specific instances of the variables \( x \) and \( y \) in a given expression, although these notations stand for the same variables \( x \) and \( y \), respectively.

**Example 3.9.** Let \( C = \{x, y\} \) and \( P = \{1, 2\} \), and set \( z_1 = xy, z_2 = yx \).

(i) Starting with the \( 2 \)-power word \( \tilde{w}_{(C,P,2)} = x^2y^2x \) of \( C \) and \( P \) given in Example 3.1 (i), by the rule of (3.6) we extend it to \( \tilde{w}'_{(C,P,2)} = yx^2y^2x \), a uniform word, and define the identity

\[
\Pi_{(C,P,2)} : \ yx^2y^2x yx^2y^2x = yx^2y^2x yx yx^2y^2x.
\]

(3.8)

This identity is uniformly balanced.

(ii) Taking the uniform \( 3 \)-power word \( \tilde{w}_{(C,P,3)} = 13 \) of \( C \) and \( P \) as in Example 3.1 (ii), we get the uniformly balanced identity

\[
\Pi_{(C,P,3)} : \ 13 \ x \ y \ 13 = 13 \ y \ x \ 13.
\]

(3.9)

(In this case, by (3.6) there is no need for extension.)

For both identities, \( \Pi_{(C,P,2)} \) and \( \Pi_{(C,P,3)} \), we have \( \text{cont}(\Pi_{(C,P,n)}) = \text{cont}(\tilde{w}'_{(C,P,n)}) = \text{cont}(\tilde{w}_{(C,P,n)}) \) and \( \text{exp}(\Pi_{(C,P,n)}) = \text{exp}(\tilde{w}'_{(C,P,n)}) = \text{exp}(\tilde{w}_{(C,P,n)}) \), for \( n = 2, 3 \).

**Theorem 3.10.** A semigroup \( S := (S, \cdot) \) that satisfies an \( n \)-variable identity \( \Pi : u = v \), for \( n \geq 2 \), also satisfies a refined \( 2 \)-variable identity \( \tilde{\Pi} : \tilde{u} = \tilde{v} \) of exponent \( \langle 2 \rangle \).

**Proof.** Since \( S \) satisfies the \( n \)-variable identity \( \Pi : u = v \), then by definition \( \phi(u) = \phi(v) \) for every morphism \( \phi : A^+ \to S \). Suppose \( C := \text{cont}(\Pi) = \{x_1, \ldots, x_n\} \), and write \( C \) as the disjoint union \( C = C_1 \cup C_2 \), for nonempty subsets \( C_1 \) and \( C_2 \). Pick two variables, say \( y_1, y_2 \), and consider the words \( \tilde{u} \) and \( \tilde{v} \), obtained respectively from \( u \) and \( v \) by substituting \( y_1y_2 \) for every \( x_i \in C_1 \) and \( y_2y_1 \) for every \( x_j \in C_2 \).

It is easy to verify that \( \tilde{\Pi} : \tilde{u} = \tilde{v} \) is a \( 2 \)-variable identity, with set of exponents \( \text{exp}(\tilde{\Pi}) \subseteq \{1, 2\} \). We claim that \( S \) satisfies the identity \( \tilde{\Pi} : \tilde{u} = \tilde{v} \). Indeed, assume \( \phi : A^+ \to S \) sends \( \phi : y_1 \mapsto s_1 \) and \( \phi : y_2 \mapsto s_2 \), then \( \phi : y_1y_2 \mapsto s_1s_2 = a \) and \( \phi : y_2y_1 \mapsto s_2s_1 = b \). But, since \( a \) and \( b \) satisfy \( \Pi \) by hypothesis, and \( \tilde{a} \) and \( \tilde{b} \) can be decomposed as concatenation of the terms \( y_1y_2 \) and \( y_2y_1 \), then \( s_1 \) and \( s_2 \) satisfy \( \tilde{\Pi} \).

In the sequel, in view of Theorem 3.10 we focus on \( 2 \)-variable identities \( \Pi_{(C,P,n)} \) of exponent \( \langle 2 \rangle \) of the Form (3.2), with \( C := \{x, y\} \) and \( P := \{1, 2\} \). For ease of exposition, for a given \( n \)-power word \( \tilde{w}_{(C,P,n)} \) we begin with the \( 2 \)-variables identity having exponent set \( \langle 2 \rangle \) of the form

\[
\Pi_{(C,P,n)} : \ \tilde{w}'_{(C,P,n)} = \tilde{w}_{(C,P,n)} \ \text{by} \ \tilde{w}'_{(C,P,n)},
\]

(3.10)

where here, using Remark 3.2, \( \tilde{w}' := \tilde{w}'_{(C,P,n)} \) is an extended \( n \)-power word, obtained by the rule of Formula (3.6); in particular, \( \text{pre}_x(\tilde{w}') \), \( \text{pre}_y(\tilde{w}') \), \( \text{sup}_x(\tilde{w}') \), \( \text{sup}_y(\tilde{w}') \) are all \( \leq 1 \), which preserve the exponent \( \langle 2 \rangle \) property of \( \Pi_{(C,P,n)} \). (Note that in comparison to (3.4), the intermediate terms \( z_i \) are now consisting of only one letter.)
Clearly the identity (3.10) is not balanced, however it can be easily refined by substituting
\[ x := \tilde{x} \tilde{y} \quad \text{and} \quad y := \tilde{y} \tilde{x} \]  
(3.11)
to receive back balanced identity (of exponent 2)
\[ \tilde{I}_{\tilde{C},P,n} : \tilde{w}'_{\tilde{C},P,n} \tilde{x}\tilde{y} \tilde{w}'_{\tilde{C},P,n} = \tilde{w}'_{\tilde{C},P,n} \tilde{y}\tilde{x} \tilde{w}'_{\tilde{C},P,n}, \]  
(3.12)
with \( \tilde{C} = \{\tilde{x}, \tilde{y}\}, P = \{1,2\} \), and \( \tilde{w}'_{\tilde{C},P,n} \) is the word obtained from \( \tilde{w}'_{\tilde{C},P,n} \) substitution (3.10).
(Note that now \( \tilde{w}'_{\tilde{C},P,n} \) need not be an \( n \)-power word of \( \tilde{C} = \{\tilde{x}, \tilde{y}\} \) and \( P = \{1,2\} \).)

4. IDENTITIES OF TRIANGULAR TROPICAL MATRICES

Aiming to prove the existence of a semigroup identity for the monoid \( U_n(\mathbb{T}) \) of \( n \times n \) triangular matrices, we start with the case of diagonally equivalent matrices, which is easier to deal with; then we generalize the results to the whole monoid \( U_n(\mathbb{T}) \).

Remark 4.1. Suppose \( S = M_n(\mathbb{T}) \) is the monoid of all \( n \times n \) tropical matrices, then any semigroup identity \( \Pi : u = v \) that \( S \) admits is balanced. Indeed, otherwise assume \( \kappa_{x_i}(u) \neq \kappa_{x_i}(v) \) for some \( x_i \) and take morphism \( \phi : x_j \mapsto I \) for each \( j \neq i \) (recall that \( I \) is the identity matrix) and \( \phi : x_i \mapsto \alpha I \) for some fixed \( \alpha \neq 1 \) to reach a contradiction.

However, for an easy exploration, for the certain class of diagonally equivalent matrices, we first work with unbalanced identities for the Form (3.10) and then refine them to balanced identities as in (3.11).

When dealing with matrix identities, we sometimes denote generic matrices (standing for variables \( x, y, \ldots \)) by capital letters \( X, Y, \ldots \), as well as the words they generate. To demonstrate the main idea of our approach for proving existence of semigroups identities for tropical matrices, we first prove the existence of a semigroups identity admitted by the monoid of \( 2 \times 2 \) triangular tropical matrices.

4.1. The monoid of \( 2 \times 2 \) triangular tropical matrices. An explicit semigroup identity of the case of \( 2 \times 2 \) tropical matrices has been proven in [2] by using the machinery of Newton’s polytope, applied to generic matrices, which allows the identification of different tropical polynomials describing the same function. Essentially, this machinery transfers the identification problem to the realm of convex sets, reducing it to a comparison of convex hulls. However, computing the convex-hulls becomes difficult in high dimensional cases, and it is not easily applicable.

To demonstrate our new approach, we bring a simpler proof of the above case, based now on the colored paths in associated digraphs. This proof is given for an explicit identity of the form (3.10).

Theorem 4.2. Any two matrices \( X, Y \in U_2(\mathbb{T}) \) such that \( X \sim_{\text{diag}} Y \) satisfy the identity
\[ U := XY \underline{X} \underline{XY} = XY \underline{Y} \underline{XY} =: V \]  
(4.1)
of the Form (3.10).

Proof. Write \( U = (u_{ij}) \) and \( V = (v_{ij}) \). The equality \( u_{ii} = v_{ii} \) is obvious for the diagonal entries. Consider the \((1,2)\)-entry, say of the matrix product \( U \), this entry corresponds to a colored path \( \gamma_{1,2} \) from 1 to 2 of maximal weight in the digraph \( G_{XY\underline{X}XY} = G_{U} \). Clearly, \( \gamma_{1,2} \) is of length 5 and it contains a simple path \( \tilde{\gamma}_{1,2} \subseteq \gamma_{1,2} \) of length 1; namely the simple path \( \tilde{\gamma}_{1,2} \) is an edge. Let \( e_X \) be the edge in \( \tilde{\gamma}_{1,2} \) contributed by \( G_X \) (or equivalently by \( \underline{X} \)). If \( e_X \) is a loop, we are done since \( e_X \) can be replaced by \( e_Y \) in \( G_{XY\underline{X}XY} = G_{V} \), since \( \underline{X} \sim_{\text{diag}} \underline{Y} \), yielding a path of the same weight. Otherwise, by the same argument, it is enough to show that \( G_{XY\underline{X}XY} \) has another path of the same length and weight in which the contribution of \( G_X \) is a loop.

Assume \( e_X \) is not a loop, i.e., \( \tilde{\gamma}_{1,2} = e_X \), then \( \gamma_{1,2} \) has the form
\[ \gamma_{1,2} = (\rho_1)^2 \circ e_X \circ (\rho_2)^2, \]
where \( \rho_1 \) and \( \rho_2 \) are loops. If \( w(\rho_1) > w(\rho_2) \) (resp. \( w(\rho_2) > w(\rho_1) \)) then the path \((\rho_1)^3 \circ e_X \circ \rho_2 \) (resp. \( e_X \circ (\rho_2)^4 \)) would have a higher weight than \( \gamma_{1,2} \) has — a contradiction. (Note that a loop can be contributed equivalently either by \( G_X \) or \( G_Y \).) Thus \( w(\rho_1) = w(\rho_2) \), and hence \( e_X \circ (\rho_2)^4 \) and \((\rho_1)^3 \circ e_X \circ \rho_2 \) are paths of the same weight as \( \gamma_{1,2} \) in which \( G_X \) contributes a loop. \( \square \)

**Remark 4.3.** Although we have proven Theorem 4.2 for the explicit identity \( (4.1) \) of diagonally equivalent matrices, the same proof also holds for any identity of \( 2 \times 2 \) triangular tropical matrices given in the general form as in \( (3.10) \).

**Corollary 4.4** ([7, Theorem 3.6]). Any matrices \( A, B \in U_2(\mathbb{T}) \) satisfy the semigroup identity
\[
AB^2A AB AB^2A = AB^2A BA AB^2A. \tag{4.2}
\]

**Proof.** Apply Theorem 4.2 for \( X := AB \) and \( Y := BA \) as in \( (3.11) \). \( \square \)

One easily sees that \( (4.2) \) is a 2-variable uniformly balanced identity of length 10. Moreover, by [7], we know that this is an identity of minimal length that \( U_2(\mathbb{T}) \) satisfies.

### 4.2. The general case.

We now turn to prove the existence of a semigroup identity in the general case of \( n \times n \) triangular tropical matrices, first generalizing Theorem 4.2 to identities of the Form \( (3.10) \) for \( n \times n \) diagonally equivalent triangular matrices, which later provides the proof for all \( n \times n \) triangular matrices.

**Remark 4.5.** Viewing the entries of a product of \( n \times n \) triangular tropical matrices as colored paths in the associated digraph, cf. [2], one gets only paths containing simple subpaths of length \(< n \). Therefore, concerning identities of the Form \( (3.10) \), applied to \( n \times n \) triangular matrices, it is enough to implement an identity \( \Pi_{(C,P,n-1)} \) constructed by using \((n-1)\)-power words.

Given two \( n \times n \) triangular matrices \( X \sim_{\text{diag}} Y \), let \( Z \in U_n(\mathbb{T}) \) be the product concatenation
\[
\langle Z \rangle := \langle L \rangle \underline{\text{x}} \langle R \rangle, \quad \langle L \rangle = \langle R \rangle = \hat{w}_{(C,P,n-1)}, \tag{4.3}
\]
with \( C := \{X,Y\} \) and \( P := \{1,2\} \). Recall that the notation \( \underline{\text{x}} \) is used to mark the specific instance of the matrix \( X \) in the expression, although it is just the same matrix as \( X \), and that \( \langle Z \rangle \) stands for the product concatenation (i.e., a formal word) whose product result is the matrix \( Z = (z_{i,j}) \) given in \( (4.3) \).

In the view of [2] the \((i,j)\)-entry \( z_{i,j} \) of \( Z = (z_{i,j}) \) corresponds to a colored path \( \gamma_{i,j} \) of maximal weight and length \( \ell(\gamma_{i,j}) = \ell(\langle Z \rangle) \) from \( i \) to \( j \) in the associated digraph \( G(\langle Z \rangle) \) of \( \langle Z \rangle \). Then, for \( j > i \), cf. Remark 2.2, the simple colored subpath \( \hat{\gamma}_{i,j} \subset \gamma_{i,j} \) from \( i \) to \( j \) is of length \( \leq j - i \leq n - 1 \), and thus \( \hat{\gamma}_{i,j} \) contains exactly \( \ell(\langle Z \rangle) - \ell(\hat{\gamma}_{i,j}) \) loops.

**Remark 4.6.** The matrix product concatenations \( \langle L \rangle \) and \( \langle R \rangle \) in \( (4.3) \) have been taken to be \( \hat{w}_{(C,P,n-1)} \) — the \((n-1)\)-power word of \( C \) and \( P \) for which every member of \( W_{n-1}[C,P] \) is a factor. This allows us to deal also with cases in which the involved matrices have diagonal entries 0 which means that some vertices in the associated digraphs are not adjacent to a loop.

In the sequel exposition, when working with matrices whose diagonal entries are all nonzero, we may replace \( \hat{w}_{(C,P,n-1)} \) by a word for which every member of \( W_{n-1}[C,P] \) is a subword (and not necessarily a factor), to possibly obtain shorter semigroup identities.

Given a simple path \( \hat{\gamma}_{i,j} \) from \( i \) to \( j \), we write \( W(\hat{\gamma}_{i,j}) \) for the word recorded uniquely by the coloring of \( \hat{\gamma}_{i,j} \). In particular \( \ell(W(\hat{\gamma}_{i,j})) = \ell(\hat{\gamma}_{i,j}) \).

The next lemma plays a central role in this paper.

**Lemma 4.7.** Suppose \( \langle Z \rangle \) is as in \( (4.3) \), where \( X, Y \in U_n(\mathbb{T}) \). Let \( \gamma_{i,j} \), where \( i < j \), be a colored path of maximal weight in \( G(\langle Z \rangle) \) for which the contribution of \( G_X \) is a non-loop edge \( e_X \). Then
$G_X$ has another colored path of the same length and weight in which the contribution of $G_X$ is a loop.

The proof of the lemma is quite technical, thus, before proving it formally, let us outline its major idea. Given a path $\gamma_{i,j}$ from $i$ to $j$, it contains a unique simple subpath $\tilde{\gamma}_{i,j}$ from $i$ to $j$ which corresponds to a subword $W$ of $\langle Z \rangle$; all the other edges of $\gamma_{i,j}$ are loops. We want to show that if the edge $e_X$ contributed by $G_X$ appears in $\tilde{\gamma}_{i,j}$, then $e_X$ can be excluded from $\gamma_{i,j}$ by “shifting” $\tilde{\gamma}_{i,j}$ in one of three ways without changing its weight: either shifting $\tilde{\gamma}_{i,j}$ to the left or to the right such that $W$ becomes a factor of $\langle L \rangle$ or $\langle R \rangle$ respectively, or by writing $\gamma_{i,j}$ as a composition $\tilde{\gamma}_{i,k} \circ \tilde{\gamma}_{k,j}$, and $W = W_1 W_2$ correspondingly, and shifting $\tilde{\gamma}_{i,k}$ to the left and $\tilde{\gamma}_{k,j}$ to right such that both $W_1$ and $W_2$ are factors of $\langle L \rangle$ and $\langle R \rangle$ respectively. As a consequence of these “shifts”, which are possible since on each vertex $G_X$ and $G_Y$ have the same loops of the same weight, the contribution of $G_X$ to $\gamma_{i,j}$ becomes a loop, while the edge $e_X$ is replaced by $e_X$, contributed by another $G_X$.

Proof of Lemma 4.7. Let $m = \ell(\langle R \rangle) = \ell(\langle L \rangle)$, since $\langle L \rangle = \langle R \rangle = \tilde{w}(G_{1:n-1})$ an $(n - 1)$-power word, then $m \gg n$, and thus $\ell(\gamma_{i,j}) = 2m + 1$. Let $\tilde{m} = \ell(\tilde{\gamma}_{i,j})$. In particular, $\tilde{m} < n$ since $\gamma_{i,j}$ is a simple path in $G_\langle Z \rangle$ an acyclic digraph on $n$ vertices, and thus $\tilde{m} < m$. Recall that by hypothesis $e_X \in \tilde{\gamma}_{i,j}$. The proof is delivered by cases, determined by the structure of the path $\gamma_{i,j}$.

Write

$$\gamma_{i,j} = \tilde{\gamma}_{i,s} \circ (\rho_s)^{p_s} \circ \tilde{\gamma}_{s,t} \circ (\rho_t)^{p_t} \circ \tilde{\gamma}_{t,j}, \quad i \leq s < t \leq j,$$

(4.4)

where $\tilde{\gamma}_{i,s}$ (resp. $\tilde{\gamma}_{t,j}$) is the maximal simple path (could be empty) appearing as the prefix (resp. suffix) of $\gamma_{i,j}$, $\rho_s$ and $\rho_t$ are loops which must exist due to length considerations, and $\tilde{\gamma}_{s,t}$ is a subpath (needs not be simple).

Therefore, $e_X$ does not belong to $\tilde{\gamma}_{i,s}$ or $\tilde{\gamma}_{t,j}$, by length considerations, and thus $\ell(\tilde{\gamma}_{i,s}), \ell(\tilde{\gamma}_{t,j}) < \tilde{m} - 1$ and hence $\rho_s, \rho_t > 0$.

Define $\langle F \rangle := W(\tilde{\gamma}_{i,s})$ and $\langle G \rangle := W(\tilde{\gamma}_{t,j})$ to be the words (whose terms are generic matrices) determined by the coloring of the simple paths $\tilde{\gamma}_{i,s}$ and $\tilde{\gamma}_{t,j}$, and set $\langle L' \rangle := \langle L \rangle_{\text{pre}}(\langle F \rangle)$ and $\langle R' \rangle := \langle R \rangle_{\text{post}}(\langle G \rangle)$. In other words $\langle L' \rangle$ and $\langle R' \rangle$ are the words obtained from $\langle L \rangle$ and $\langle R \rangle$ after removing respectively the initial and the terminal words (which in this case are factors) corresponding to the simple paths $\tilde{\gamma}_{i,s}$ and $\tilde{\gamma}_{t,j}$.

Let

$$\mu_{s,t} = (\rho_s)^{p_s} \circ \tilde{\mu}_{s,t} \circ (\rho_t)^{p_t}, \quad s < t,$$

(4.5)

be the subpath of $\gamma_{i,j}$, given by its intermediate non-simple part according to (4.4), and let $\tilde{\mu}_{s,t}$ be the simple subpath contained in $\mu_{s,t}$. Define $\langle H \rangle := W(\tilde{\mu}_{s,t})$ – the matrix product concatenation, realized as a subword, corresponding to the coloring of the path $\tilde{\mu}_{s,t}$.

We claim that $G_{\langle L' \rangle}$ contains a path similar to $\tilde{\mu}_{s,t}$ and $G_{\langle R' \rangle}$ contains a path similar $\tilde{\mu}_{s,t}$. It is enough to show that $\langle H \rangle$ is a subword of $\langle L' \rangle$. Indeed, $\langle F \times H \rangle \subseteq \langle L \rangle$ by word construction, where $\langle F \rangle$ is the prefix of $\langle L \rangle$ by hypotheses, thus $\langle H \rangle \subseteq \langle L' \rangle$ is a subword of $\langle L' \rangle$. The case of $\tilde{\mu}_{s,t} \subseteq G_{\langle R' \rangle}$ is dual.

Let $\rho_{\text{max}}$ denote the loop of maximal weight in $\rho_{s,t}$, then we have the following possible cases:

I. $\rho_s = \rho_{\text{max}}$: Then, there is a path $\mu'_{s,t} = (\rho_{\text{max}})^{q_s} \circ \tilde{\mu}_{s,t} \circ (\rho_t)^{q_t}$ with $\tilde{\mu}_{s,t} \subseteq G_{\langle R' \rangle}$ and $q_s > \ell(\langle L' \rangle)$, such that $w(\mu'_{s,t}) = w(\mu_{s,t})$, since otherwise we would get a contradiction to the maximality of weight of $\mu_{s,t}$. Thus $e_X' = \rho_s = \rho_{\text{max}} - a$ loop.

II. $\rho_t = \rho_{\text{max}}$: Then, there is a path $\mu'_{s,t} = (\rho_s)^{q_s} \circ \tilde{\mu}_{s,t} \circ (\rho_{\text{max}})^{q_t}$, with $\tilde{\mu}_{s,t} \subseteq G_{\langle L' \rangle}$ and $q_t > \ell(\langle R' \rangle)$, such that $w(\mu'_{s,t}) = w(\mu_{s,t})$, since otherwise we would get a contradiction to the maximality of weight of $\mu_{s,t}$. Thus $e_X' = \rho_t = \rho_{\text{max}} - a$ loop.

III. $\rho_{\text{max}} \in \tilde{\gamma}_{s,t}$: Then, there is a path

$$\mu'_{s,t} = (\rho_s)^{q_s} \circ \tilde{\mu}_{s,k} \circ (\rho_{\text{max}})^{q_k} \circ \tilde{\mu}_{k,t} \circ (\rho_{q_t})^{q_t}, \quad s < k < t,$$
with $\tilde{\mu}_{s,k} \subset G_{L'}$ and $\tilde{\mu}_{k,t} \subset G_{R'}$ such that $\tilde{\mu}_{s,t} = \tilde{\mu}_{s,k} \circ \tilde{\mu}_{k,t}$. Thus $w(\mu'_{s,t}) = w(\mu_{s,t})$, since otherwise we would get a contradiction to the maximality of weight of $\mu_{s,t}$. Hence $\rho_k = \rho_{\text{max}}$ and $\rho_Y' = \rho_Y - \text{a loop}$.

Therefore, in all the above cases we get that $e_Y'$ is a loop in $\mu'_{s,t}$ - a path in $G_{L'} \times (R')$. Then, concatenate the simple paths $\tilde{\gamma}_{i,s}$ and $\tilde{\gamma}_{t,j}$

$$\gamma'_{i,j} = \tilde{\gamma}_{i,s} \circ \mu'_{s,t} \circ \tilde{\gamma}_{t,j}, \quad i < s < t < j,$$

to obtain another path in $G_{L'} \times (R')$ for which the contribution of $G_{Y'}$ is a loop, as desired. \hfill $\Box$

**Theorem 4.8.** Any two diagonally equivalent matrices $X, Y \in U_n(\mathbb{T})$, i.e., $X \sim_{\text{diag}} Y$, satisfy the identities $\Pi_{(C,P,n-1)}$ of the Form (3.10), with $x = X$ and $y = Y$.

**Proof.** Write $X = (x_{i,j})$ and $Y = (y_{i,j})$. Let $U = (u_{i,j})$ and $V = (v_{i,j})$ be the matrix products determined respectively by the left and the right words of the identity (3.10). We need to show that $u_{i,j} = v_{i,j}$ for every $i \leq j$. (The case of $j > i$ is trivial since $x_{i,j} = y_{i,j} = -\infty$, for any $j > i$, and hence $u_{i,j} = v_{i,j} = -\infty$.)

It is easy to see that $u_{i,i} = v_{i,i}$ for every $i = 1, \ldots, n$. Assume now that $i < j$, and consider the associated colored digraphs $G_{U}$ and $G_{V}$ with matrix products $U$ and $V$, realized as words $\langle U \rangle$ and $\langle V \rangle$ as given by (3.10). Assume first that $u_{i,j} \neq -\infty$, then the value of the entry $u_{i,j}$ corresponds to a colored path $\gamma_{i,j}$ from $i$ to $j$ of maximal weight and of length $\ell(U)$ in the digraph $G_{U}$. By Lemma 4.7 we may assume that the contribution of $G_{X_{\gamma_{i,j}}}$ is a loop $\rho_{X_{\gamma_{i,j}}}$, but then $G_{V_{\gamma_{i,j}}}$ also contains a similar colored path $\gamma'_{i,j}$ in which the contribution of $G_{Y_{\gamma_{i,j}}}$ is also a loop $\rho_{Y_{\gamma_{i,j}}}$, replacing $\rho_{X_{\gamma_{i,j}}}$, which by hypothesis have the same weight, i.e., $w(\rho_{X_{\gamma_{i,j}}}) = w(\rho_{Y_{\gamma_{i,j}}})$. Dually, the same argument also holds for a path in $G_{V_{\gamma_{i,j}}}$ in which the contribution of $G_{Y_{\gamma_{i,j}}}$ is a loop.

Suppose now that $u_{i,j} = -\infty$, and assume that $v_{i,j} \neq -\infty$. This means that there exists a colored path $\gamma'_{i,j}$ from $i$ to $j$ in $G_{Y_{\gamma_{i,j}}}$, and in particular, by Lemma 4.7 a path in which the contribution of $G_{Y_{\gamma_{i,j}}}$ is a loop $\rho_{Y_{\gamma_{i,j}}}$, and thus $u_{i,j} \neq -\infty$ - a contradiction.

Putting all together, we have $u_{i,j} = v_{i,j}$ for every $i, j$. \hfill $\Box$

**Example 4.9.** Assume $X \sim_{\text{diag}} Y$, and set $x = X$ and $y = Y$.

(i) If $X, Y \in U_3(\mathbb{T})$ then they satisfy the identity

$$\Pi_{(C,P,2)}: \quad xy^2y^2x + yxy^2y^2 = yx^2y^2x + yxy^2y^2x.$$  \hfill (4.6)

(ii) If $X, Y \in U_4(\mathbb{T})$ then they satisfy the identity

$$\Pi_{(C,P,3)}: \quad 13 x 13 = 13 y 13.$$  \hfill (4.7)

(To preserve the exponent (2) of the identities, we use the power words of Example 3.7 with an additional instance, denoted as $\gamma$, of $y$ given by the rule of (3.4).)

**Theorem 4.10.** The submonoid $U_n(\mathbb{T}) \subset M_n(\mathbb{T})$ of upper triangular tropical matrices satisfies the identities $\Pi_{(C,P,n-1)}$ of the Form (3.12), which we recall is (3.10) with $x = X = AB$ and $y = Y = BA$, for $n \times n$ generic matrices $A, B$.

**Proof.** It easy to verify that for any triangular matrices $A, B \in U_n(\mathbb{T})$ the matrix products $X = AB$ and $Y = BA$ are diagonally equivalent. The proof is then completed by Theorem 4.8. \hfill $\Box$

**Example 4.11.** Set $x = AB$ and $y = BA$.

(i) The monoid $U_3(\mathbb{T})$ of $3 \times 3$ triangular tropical matrices satisfies the identity (4.6).

(ii) The monoid $U_4(\mathbb{T})$ of $4 \times 4$ triangular tropical matrices admits the identity (4.7).
5. Identity length: An upper bound

In the previous section we proved the existence of semigroup identities of the Form (3.12), satisfied by $U_n(\mathbb{T})$, we now discuss the length of this identity, providing a very naive upper bound.

The very well known Fibonacci sequence $F_n$ is defined by the recursive relation

$$F_n := F_{n-1} + F_{n-2}, \quad \text{for every } n \geq 2,$$

where $F_0 = 0$ and $F_1 = 1$, and has the closed formula (known as Binet’s Fibonacci number formula):

$$F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}.$$

Therefore, we see that the Fibonacci number $2F_n$ gives the number of elements in $W_n[C, P]$, i.e., all possible factors of length $n$, for the case of $C := \{x, y\}$ and $P := \{1, 2\}$. (The multiplier 2 stands for the two possibilities for starting a sequence, either with $x$ or $y$.)

A naive construction of an $n$-power word $\tilde{w}_{(C, P, n)}$, i.e., by concatenating all factors in $W_n[C, P]$, gives us the upper bound

$$\ell(\tilde{w}_{(C, P, n)}) \leq 2(n + 1) F_n.$$

(The multiplier of $n + 1$, stands for the length of a factor with a possible additional letter between sequential factors, aiming to preserve the exponent $\langle 2 \rangle$ property.) Thus, considering the refinement given by $x = \tilde{x}\tilde{y}$ and $y = \tilde{y}\tilde{x}$, we have

$$\ell(\Pi_{(C, P, n)}) \leq 8(n + 1) F_n + 2.$$

Obtusely, this rough upper bound assumes that the factors $W_n[C, P]$ do not overlap in $\tilde{w}_{(C, P, n)}$. Dealing with possible overlap factors, one can reduce the multiplier $n + 1$ in (5.3), and hence can improve further this upper bound.

Recall that, in view of Remark 4.5, for $n \times n$ triangular matrices it is enough to consider identities $\Pi_{(C, P, n-1)}$, for which the upper bound is then smaller, that is

$$\ell(\Pi_{(C, P, n-1)}) \leq 8n F_{n-1} + 2.$$

6. Remarks and open problems

A natural question arisen from our identity construction in §3.2 is about the minimality of semigroup identities admitted by $U_n(\mathbb{T})$.

**Conjecture 6.1.** When $\tilde{w}_{(C, P, n-1)}$ is a minimal $n$-power word of $P$ and $C$, then the identity $\Pi_{(C, P, n-1)}$ in (3.12) is a minimal semigroup identity admitted by $U_n(\mathbb{T})$.

The results of this paper, and those of [7], lead us to the conjecture, which has already been conjectured earlier in [7], that

**Conjecture 6.2.** Also the monoid $M_n(\mathbb{T})$ of $n \times n$ tropical matrices satisfies a nontrivial semigroup identity for all $n$.

(The conjecture has been proven in [7] Theorem 3.9 for the case of $n = 2$.)

Another reason for conjecturing this is that every finite subsemigroup of $M_n(\mathbb{T})$ has polynomial growth [4, 12]. In particular, the free semigroup on 2 generators is not isomorphic to a subsemigroup of $M_n(\mathbb{T})$. While Shneerson [14] has given examples of polynomial growth semigroups that do not satisfy any nontrivial identity (no such example exists for groups by Gromov’s Theorem [5]), we conjecture that this is not the case for $M_n(\mathbb{T})$. 


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