On a class of Kirchhoff-Choquard equations involving variable-order fractional $p(\cdot)$—Laplacian and without Ambrosetti-Rabinowitz type condition

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Abstract
In this article we study the existence of weak solution, existence of ground state solution using Nehari manifold and existence of infinitely many solutions using Fountain theorem and Dual fountain theorem for a class of doubly nonlocal Kirchhoff-Choquard type equations involving the variable-order fractional $p(\cdot)$—Laplacian operator. Here the nonlinearity does not satisfy the well known Ambrosetti-Rabinowitz type condition.

Keywords: Kirchhoff-Choquard equation; Variable order fractional $p(\cdot)$—Laplacian; Fountain theorem; Dual fountain theorem; Nehari manifold; Ambrosetti-Rabinowitz type condition

Mathematics subject classification: 35J60; 35R11; 35A15; 46E35

1 Introduction
The purpose of this article is to study the following Kirchhoff-Choquard type problem:

\[
\begin{aligned}
&\frac{m}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}} \, dx \, dy}
\quad \left[(-\Delta)^{s(\cdot)} u + V(x)|u|^{p(x)-2}u\right]
\quad =
\quad \left(\int_{\Omega} F(y, u(y)) \, dy\right) f(x,u),
\quad x \in \Omega, \\
&\quad u = 0, \quad x \in \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]

where \( m : \mathbb{R}_+^+ \to \mathbb{R}_+^+ \), \( V : \Omega \to \mathbb{R}_+^+ \), \( \mu : \mathbb{R}^N \times \mathbb{R}^N \to (0, N) \), \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) \( p : \mathbb{R}^N \times \mathbb{R}^N \to (1, \infty) \) and \( s : \mathbb{R}^N \times \mathbb{R}^N \to (0, 1) \) are continuous functions, \( p(x) := p(x, x) \), \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain and \( F \) is the primitive of \( f \). The nonlocal operator \((-\Delta)^{s(\cdot)}\) is defined as

\[
(-\Delta)^{s(\cdot)} u(x) := P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y))}{|x-y|^{N+s(x,y)p(x,y)}} \, dy, \quad x \in \mathbb{R}^N,
\]

where P.V. denotes the Cauchy’s principal value. Note that such kind of operators are non-homogeneous in nature. If \( p(x,y) = p \), \( s(x,y) = s \) are constants then \((-\Delta)^{s(\cdot)}\) is reduced to nonlocal fractional \( p \)–Laplacian. The fractional Sobolev spaces and the corresponding nonlocal equations involving nonlocal operator have major applications to various nonlinear problems, including phase transitions, thin obstacle problem, crystal dislocation, soft thin films, minimal surfaces, material science, etc. (see for e.g., [1] and the references there in for more details). We also refer to the monograph [2] and [3] for

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the detailed study of the nonlocal fractional $p$–Laplacian with $1 < p < \infty$, its properties and problems involving it.

Motivated by the application of variable exponent Lebesgue and Sobolev spaces (see [4, 5, 6]) in the study of the flow of non-Newtonian fluid (for example electrorheological fluids) or the image restoration (see for e.g., [7, 8]), Kaufmann et al. [2] introduced fractional Sobolev spaces with variable exponents. Then Rădulescu, Bahrouni, Ho, Kim (11, 11, 12) studied the extensive properties of such spaces and associated problems involving the operator (1.1) when $s(\cdot, \cdot) = s$, constant. For $p(\cdot, \cdot) = 2$, Zhang et al. (13) discussed existence results for problems involving variable-order fractional Laplacian operator $(-\Delta)_0^s$. Recently, Biswas and Tiwari (14) studied problems involving the nonlocal operator (1.1) in fractional Sobolev spaces with variable order and variable exponents, which involve not only the variable exponents but also variable order.

The Choquard type of nonlinearity in (1.1) is motivated by the work of Pekar [15], studying the following nonlinear Schrödinger-Newton equation:

$$-\Delta u + V(x)u = (\mathcal{K}_\mu * u^2)u + \lambda f(x, u), \quad (1.2)$$

where $\mathcal{K}_\mu$ denotes the Riesz potential. This type of nonlinearity describes the self gravitational collapse of a quantum mechanical wave function (see [16]) and also plays a key role in the Bose–Einstein condensation (see [17]). For $V(x) = 1, \lambda = 0$, the equations of type (1.2) have extensively been studied in (18, 19, 20, 21). We cite [22, 23] for some recent works related to critical Choquard type problems involving local Laplacian. In the fractional Laplacian set up, Wu (24) investigated existence and stability of solutions for the equations

$$(-\Delta)^s u + \omega u = (\mathcal{K}_\mu * |u|^q)|u|^{q-2}u + \lambda f(x, u) \quad \text{in } \mathbb{R}^N,$$

(1.3)

where $q = 2$, $\omega = 0$ and $\mu \in (N - 2s, N)$. In the critical case, i.e. $q = 2\mu/s := (2N - 2\mu)/(N - 2s)$, Mukherjee and Sreenadh (25) obtained existence and multiplicity results for solutions of (1.3) in a smooth bounded domain for $w = 0$ and $f(x, u) = u$.

Recently Gao et al. (26) investigated the existence of ground state solution of Pohozaev-type for the following problem:

$$(-\Delta)^s u + V(x)u = (\mathcal{K}_\mu * F(u))F'(u) \quad \text{in } \mathbb{R}^N,$$

(1.4)

where $V \in C^1(\mathbb{R}^N, [0, \infty))$ and $F$ satisfies general Berestycki–Lions-type assumptions.

Very recently, Alves et al. (27) introduced a new kernel $A(x, y) := 1/(x^2 + y^2)$ for $x, y \in \mathbb{R}^N$ and then using the properties of that kernel the authors established Hardy-Littlewood-Sobolev-type inequality (27, Proposition 2.4) for variable exponents. Analogous to this, in the nonlocal setup involving variable order and variable exponents, Biswas and Tiwari (14) established a Hardy-Littlewood-Sobolev-type inequality result with some appropriate assumptions and studied the Choquard problem of type (1.1) for $n(\cdot) = 1, V \equiv 0$ and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ satisfying Ambrosetti-Rabinowitz type condition.

The study of Kirchhoff-type problems arise in various models of physical and biological systems and hence have received more attentions in recent years. Precisely, Kirchhoff established a model given by the following equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{L^2} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial t}\right|^2 dx\right) \frac{\partial^2 u}{\partial t^2} = 0,$$

which extends the classical D’Alembert wave equation by taking into account the effects of the changes in the length of the strings during the vibrations, where the constants $\rho, p_0, h, E, L$ represent physical parameters of the string. Subsequently, using the Nehari manifold and the concentration compactness principle in 28, Lü studied the Kirchhoff-Chaouard-type equation

$$\left(-a + b \int_{\mathbb{R}^n} |\nabla u|^2 dx\right) \Delta u + V_\lambda(x)u = (\mathcal{K}_\mu * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^n,$$

where $a \in \mathbb{R}^+, b \in \mathbb{R}^+_0$, $V_\lambda(x) = 1 + \lambda g(x), \lambda > 0$ and $g$ is a continuous potential function, $q \in (2, 6 - \mu)$. Fiscella and Valdinoci in 29 first proposed a stationary Kirchhoff model involving fractional Laplacian.
by considering the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, \((29)\) Appendix A). Recently, Pucci et al. \((30)\) extensively studied the existence and asymptotic behavior of entire solutions of the following superlinear Kirchhoff-Schrödinger-Choquard equation involving fractional \(p\)-Laplacian:

\[
M\left(\|u\|_p\right)(-\Delta)_p u + V(x)|u|^{p-2}u = \lambda f(x, u) + (K_m + |u|^{p_{\mu,s}^\gamma - 2})u \quad \text{in} \quad \mathbb{R}^N,
\]

where \(M : \mathbb{R}_+^+ \to \mathbb{R}_0^+\) is degenerate-type Kirchhoff function, \(V : \mathbb{R}^N \to \mathbb{R}_+^+\) is a scalar potential, \(p_{\mu,s}^\gamma = (pN - p\mu/2)/(N - ps)\) is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality, \(f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function with superlinear growth. Further Liang and Rădulescu \((31)\) studied the existence of infinitely many solutions of the problem \((1.6)\) using symmetric mountain pass lemma under some appropriate assumptions on \(f\).

Motivated by all the above works, in problem \((1.1)\), we consider the the study of nonlocal Kirchhoff-Choquard type problems with variable order and variable exponents. Now we fix some notations. For any domain \(D\) and any function \(\Phi : D \to \mathbb{R}\), we set

\[
\Phi^- := \inf_{D} \Phi(x) \quad \text{and} \quad \Phi^+ := \sup_{D} \Phi(x).
\]

We define the function space

\[
\mathcal{C}^+_\mu(D) := \{\Phi \in C(D, \mathbb{R}) : 1 < \Phi^- \leq \Phi^+ < \infty\}.
\]

Concerning the variable order \(s\) and the variable exponents \(p, \mu\) and the potential function \(V\) we assume the followings:

\((S1)\) \(s : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}\) is a continuous and symmetric function, i.e., \(s(x, y) = s(y, x)\) for all \((x, y) \in \mathbb{R}^N \times \mathbb{R}^N\) with \(0 < s^- \leq s^+ < 1\).

\((P1)\) \(p \in \mathcal{C}^+_{\mu}((\mathbb{R}^N \times \mathbb{R}^N)\) is a continuous and symmetric function, i.e., \(p(x, y) = p(y, x)\) for all \((x, y) \in \mathbb{R}^N \times \mathbb{R}^N\) such that \(s^+ p^+ < N\).

\((\mu1)\) \(\mu : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}\) is a continuous and symmetric function, i.e., \(\mu(x, y) = \mu(y, x)\) for all \((x, y) \in \mathbb{R}^N \times \mathbb{R}^N\) with \(0 < \mu^- \leq \mu^+ < N\).

\((V1)\) \(V \in C(\Omega, \mathbb{R})\) such that \(V(x) \geq 0\) for all \(x \in \Omega\).

Through out this article, \(p_{\mu,s}^\gamma(x) := \frac{\mathfrak{N}(x)}{\mathfrak{N}(x)(\mathfrak{N}(x))}\) denotes the Sobolev-type critical exponent, where \(\mathfrak{N}(x) := s(x, x)\). Next, we consider the following assumption on the Kirchhoff function \(m\) in \((1.1)\).

\((M1)\) \(m : \mathbb{R}_0^+ \to \mathbb{R}_0^+\) is continuous and defined as \(m(t) = a + bt^{\theta - 1}, a \geq 0, b > 0\) such that \(\theta \in [1, 2, \frac{p_{\mu,s}^\gamma}{p_{\mu,s}^\gamma - q}])\), where \(p^+ \geq 2\) and \(\mu\) satisfies \((1.7)\) and \(q \in \mathcal{C}_+((\mathbb{R}^N \times \mathbb{R}^N)\) verifies

\[
\frac{2}{q(x, y)} + \frac{\mu(x, y)}{N} = 2, \quad \forall x, y \in \mathbb{R}^N.
\]

Here \(a = 0\) represents the degenerate Kirchhoff equation and \(a > 0\) represents non-degenerate Kirchhoff equation. In case of the degenerate Kirchhoff problems, the transverse oscillations of a stretched string, with nonlocal flexural rigidity, depends continuously on the Sobolev deflection norm of \(u\) via \(m(\cdot)\). The fact \(m(0) = 0\) means that the base tension of the string is zero, a very realistic model from a physical point of view. We would like to mention that in this article we study problem \((1.1)\) for both the degenerate and non-degenerate cases. Let \(M(t) := \int_0^t m(s) ds\) denote the primitive of \(m\). Then we have next two remarks as consequences of \((M1)\).

**Remark 1.1.** When \(a = 0\), we have the following observations:

- For any \(\tau > 0\), there exists \(m_0 := m_0(\tau) > 0\) such that \(m(t) \geq m_0\) whenever \(t \geq \tau\).
- \(\theta M(t) - m(t) t\) is non decreasing for \(t > 0\) and \(\theta M(t) - m(t) t = 0\) for all \(t \geq 0\).
Remark 1.2. When $a > 0$, we have the following observations:

- $m(t) = a + bt^{\theta - 1}$, $a > 0$ and $m(t) > \inf_{t \geq 0} m(t) = a > 0$.
- $\theta M(t) - m(t)t$ is non decreasing for $t > 0$ and $\theta M(t) - m(t)t \geq 0$ for all $t \geq 0$.
- For each $t \geq 0$

$$
\begin{array}{l}
M(t) \geq M(1)t^\theta \quad \text{for all } t \in [0, 1], \\
\leq M(1)t^\theta \quad \text{for all } t \geq 1, \\
\leq M(1)(1 + t^\theta) \quad \text{for all } t \geq 0,
\end{array}
$$

(1.8)

where $M(1) = a + \frac{b}{\theta}$.

The assumptions that we consider for the nonlinearity $f(x, t)$ in (1.1) are as follows:

- $(f1)$ $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ such that $|f(x, t)| \leq C (1 + |t|^q)$, where $C > 0$ is a constant, $r \in C_+(\mathbb{R}^N)$ satisfies $1 < r(x)q^- \leq r(x)q^+ < p_*(x)$, $r^- > \frac{q^+}{p_*}$ and $q \in C_+(\mathbb{R}^N \times \mathbb{R}^N)$ verifies (1.7).

- $(f2)$ $f(x, t) = o\left(\frac{|t|^{\theta^+ - 2}}{|t|}\right)$ as $|t| \to 0$, uniformly in $x \in \Omega$.

- $(f3)$ $\lim_{|t| \to \infty} \frac{F(x, t)}{|t|^{\theta^+ - 2}} = +\infty$ uniformly in $x \in \Omega$, where $F(x, t) := \int_0^t f(x, s)ds$ is the primitive of $f$.

- $(f4)$ There exists $\theta > 1$ such that $\theta F(x, t) \geq F(x, s)$ for $(x, t) \in \Omega \times \mathbb{R}$ and $s \in [0, 1]$, where $F(x, t) = 2tF(x, t) - \theta p^+ F(x, t)$.

The condition $(f4)$ is originally due to Jeanjean [32] in the case $p(x) = 2$, and then was used in [33] for $p-$Laplacian equations in bounded domain. It is worthy to note that the assumptions $(f2) - (f4)$ allow us to consider the nonlinearities which do not satisfy following standard Ambrosetti–Rabinowitz (AR) type condition:

- $(AR)$ There exists $\omega > \theta p^+$ such that

$$0 < \omega F(x, t) \leq 2tF(x, t), \quad t \neq 0, \quad \text{for all } x \in \Omega.$$

An example of such function is $f(x, t) = t|t|^{\theta^+ - 1} \log(1 + |t|)$. Ambrosetti–Rabinowitz condition ensures that an Euler–Lagrangian functional has the mountain pass geometry structure and also plays a pivotal role in establishing the boundedness of the Palais-Smale sequence of the functional. Therefore, relaxing (AR) condition not only includes a larger class of nonlinearities but also calls for delicate analysis to establish the compactness results and hence interests many studies (see for e.g., [34, 35, 36] and references there in). We make the following remarks about $f(x, t)$.

Remark 1.3. Since $f(x, 0) = 0 = F(x, 0)$, thanks to $(f2)$, from $(f4)$ we get $F(x, t) \geq 0$, that is,

$$2tF(x, t) - \theta p^+ F(x, t) \geq 0 \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}. \quad (1.9)$$

The following crucial remark was studied in [36] for local $p-$Laplacian set up.

Remark 1.4. $F(x, t) \geq 0$ for all $(x, t) \in \Omega \times \mathbb{R}$.

Proof. For $t > 0$, using (1.9), we have

$$
\frac{d}{dt} \frac{F(x, t)}{t^{2^{p^*}-2}} = \frac{t^{\theta^+} f(x, t) \cdot \theta p^+ t^{\theta^+ - 1} F(x, t)}{t^{2^{p^*}+1}} = \frac{F(x, t)}{2t^{p^*+1}} \geq 0.
$$

Also from $(f2)$, we can easily deduce that $\lim_{t \to 0^+} \frac{F(x, t)}{t^{2^{p^*}-2}} = 0$. Using the above two facts, it follows that $F(x, t) \geq 0$ for all $(x, t) \in \Omega \times \mathbb{R}, \quad t \geq 0$.

Similarly, for $t < 0$, proceeding as above, we get $\lim_{t \to 0^-} \frac{F(x, t)}{(-t)^{2^{p^*}-2}} = 0$, and therefore $F(x, t) \geq 0$ for all $(x, t) \in \Omega \times \mathbb{R}, \quad t \leq 0$. Thus the result follows.
In view of the above remark, we have the following assertion:

**Remark 1.5.** From (1.9) and Remark 1.4, we obtain \( f(x, t) \geq 0 \) for all \((x, t) \in \Omega \times \mathbb{R}, \ t \geq 0 \) and \( f(x, t) \leq 0 \) for all \((x, t) \in \Omega \times \mathbb{R}, \ t \leq 0 \). Therefore for all \( x \in \Omega \), we have \( F(x, t) \) is non decreasing in \( t \geq 0 \) and is non increasing in \( t \leq 0 \).

**Definition 1.6** (Weak solution). A function \( u \in E \) (defined in section 2) is said to be weak solution of (1.1), if for all \( w \in E \)

\[
m(\sigma(u)) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p(x,y) - 2 (u(x) - u(y))(w(x) - w(y))}{|x - y|^{N+\sigma(x,y)p(x,y)}} \, dx \, dy \\
+ \int_{\Omega} V(x)|u(x)|^{p(x) - 2} u(x)w(x) \, dx = \int_{\Omega} \int_{\Omega} F(y, u(y)) f(x, u(x))w(x) \, dx \\
\quad |x - y|^{\mu(x,y)}
\]

The weak solutions are characterized as the critical point of the associated energy functional \( J \) (see Definition 2.11). Now we state the main results to be proved in this article.

**Theorem 1.7.** Let (S1), (P1), (µ1), (V1) and (M1) hold. Also let \( f \) satisfy (f1) – (f4). Then (1.1) admits a nontrivial weak solution.

The next theorem deals with the ground state solution of the problem (1.1), which is a solution that minimizes the functional \( J \) among all nontrivial solutions. To study ground state solution for (1.1) without (AR) condition, we consider the assumption:

\[
(f4)' \quad \frac{f(x,t)}{|x|^\frac{N+\sigma}{2}} \text{ is increasing in } t > 0 \text{ and decreasing in } t < 0 \text{ for all } x \in \Omega.
\]

Again we can see that the conditions (f2), (f3), (f4)' are weaker than (AR).

**Theorem 1.8.** Let (S1), (P1), (µ1), (V1) and (M1) hold. Also let \( f \) satisfy (f1) – (f3) and (f4)'. Then the problem (1.1) admits a nontrivial ground state solution.

Next, for the odd nonlinearity \( f(x, t) \), we state the existence results of infinitely many solutions using the Fountain theorem and the Dual fountain theorem.

**Theorem 1.9.** Let (S1), (P1), (µ1), (V1) and (M1) hold. Also let \( f \) satisfy (f1) – (f4) with \( f(x, -t) = -f(x, t) \). Then the problem (1.1) has a sequence of nontrivial weak solutions with unbounded energy.

**Theorem 1.10.** Let (S1), (P1), (µ1), (V1) and (M1) hold. Also let \( f \) satisfy (f1) – (f4) with \( f(x, -t) = -f(x, t) \). Then the problem (1.1) has a sequence of nontrivial weak solutions with negative critical values converging to zero.

In this article we first prove the existence of nontrivial solution using Mountain pass theorem with Cerami condition, then using Nehari manifold we study existence of nontrivial ground state solution and finally existence of infinitely many solutions are achieved by applying Fountain theorem and Dual fountain theorem for the problem (1.1) which covers both degenerate and non-degenerate cases. The main feature of the problem (1.1) is the presence of both the nonlocal Kirchhoff and Choquard terms together for which (1.1) remains no longer a point wise identity and hence it is categorized as a doubly nonlocal problem. Moreover, due to the involvement of the variable order and variable exponents, the problem possess non-homogeneous nature. These facts induce some further mathematical difficulties in the use of classical methods of nonlinear analysis. According to best of our knowledge the equation of type (1.1) involving the non-homogeneous operator \((-\Delta)^{s_c} \) is studied for the first time in this article. Also we want to mention that the same results will hold in case of local \( p(x) \)–Lapacian, which are also new, as per best of our knowledge, in the literature. The main novelty of this work is that unlike as in [21, 14] (also as in most of the studies regarding Choquard-type problems existed in the literature) we relax the well known Ambrosetti-Rabinowitz type condition on our nonlinearity \( f \) and hence we need to carry out some extra careful and delicate analysis to overcome the difficulties and establish the Cerami condition so that we can prove the desired results in this article.
2 Preliminary results and functional settings

In this section first we briefly discuss some basic properties of the variable exponent Lebesgue spaces, which will be used as tools to prove our main results.

For $\Theta \in C_+^0$ define the variable exponent Lebesgue space $L^{\Theta}(\Omega)$ as

$$L^{\Theta}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ is measurable : } \int_\Omega |u(x)|^{\Theta(x)} \, dx < \infty \right\}$$

which is a separable, reflexive, uniformly convex Banach space (see [4, 37, 6]) with respect to the Luxemburg norm

$$\|u\|_{L^{\Theta}(\Omega)} := \inf \left\{ \eta > 0 \colon \int_\Omega \frac{|u(x)|^{\Theta(x)}}{\eta} \, dx \leq 1 \right\}.$$  

Define the modular $\rho : L^{\Theta}(\Omega) \to \mathbb{R}$ as $\rho(u) := \int_\Omega |u|^{\Theta(x)} \, dx$, for all $u \in L^{\Theta}(\Omega)$.

**Proposition 2.1.** (37) Let $u_n, u \in L^{\Theta}(\Omega) \setminus \{0\}$, then the following properties hold:

(i) $\eta = \|u\|_{L^{\Theta}(\Omega)}$ if and only if $\rho(u) = 1$.

(ii) $\rho(u) > 1$ if and only if $\|u\|_{L^{\Theta}(\Omega)} > 1$.

(iii) If $\|u\|_{L^{\Theta}(\Omega)} > 1$, then $\|u\|_{L^{\Theta^{-}}(\Omega)} \leq \rho(u) \leq \|u\|_{L^{\Theta^{+}}(\Omega)}$.

(iv) If $\|u\|_{L^{\Theta}(\Omega)} < 1$, then $\|u\|_{L^{\Theta^{-}}(\Omega)} \leq \rho(u) \leq \|u\|_{L^{\Theta^{+}}(\Omega)}$.

(v) $\lim_{n \to \infty} \|u_n - u\|_{L^{\Theta}(\Omega)} = 0 \iff \lim_{n \to \infty} \rho(u_n - u) = 0$.

Let $\Theta'$ be conjugate function of $\Theta$, that is, $1/\Theta(x) + 1/\Theta'(x) = 1$.

**Proposition 2.2.** (Hölder inequality) (37) For any $u \in L^{\Theta}(\Omega)$ and $v \in L^{\Theta'}(\Omega)$, we have

$$\left| \int_\Omega uv \, dx \right| \leq 2\|u\|_{L^{\Theta}(\Omega)} \|v\|_{L^{\Theta'}(\Omega)}.$$

The above result is also valid for $\Omega = \mathbb{R}^N$, which is called generalized Hölder inequality.

**Lemma 2.3.** (38) Let $\vartheta_1(x) \in L^\infty(\Omega)$ such that $\vartheta_1 \geq 0$, $\vartheta_1 \not= 0$. Let $\vartheta_2 : \Omega \to \mathbb{R}$ be a measurable function such that $\vartheta_1(x)\vartheta_2(x) \geq 1$ a.e. in $\Omega$. Then for every $u \in L^{\vartheta_1(x)\vartheta_2(x)}(\Omega)$,

$$\|u\|_{L^{\vartheta_1(x)\vartheta_2(x)}(\Omega)} \leq \left\| \vartheta_1^{-} \right\|_{L^{\vartheta_1(x)}(\Omega)} \left\| u \right\|_{L^{\vartheta_2(x)}(\Omega)} + \left\| \vartheta_1^{+} \right\|_{L^{\vartheta_1(x)}(\Omega)} \left\| u \right\|_{L^{\vartheta_2(x)}(\Omega)}.$$

Note that the above lemma also holds if we replace $\Omega$ by $\mathbb{R}^N$.

Next we recall the fractional Sobolev spaces with variable order and variable exponents (see [14]). First denote $W^{s(\cdot),\varphi(\cdot),p(\cdot)}(\Omega) = W$ and then define

$$W := \left\{ u \in L^{\varphi}(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x - y|^{N+s(x,y)p(x,y)}} \, dx \, dy < \infty, \text{ for some } \eta > 0 \right\}$$

endowed with the norm

$$\|u\|_W := \inf \left\{ \eta > 0 : \rho_W\left( \frac{u}{\eta} \right) < 1 \right\},$$

where

$$\rho_W(u) := \int_\Omega |u|^{\varphi(x)} \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x - y|^{N+s(x,y)p(x,y)}} \, dx \, dy$$

is a modular on $W$. Then, $(W, \|\cdot\|_W)$ is a separable reflexive Banach space (see [14]). On $W$ we also make use of the following norm

$$|u|_W := \|u\|_{L^{\varphi}(\Omega)} + \|u\|_W.$$
Theorem 2.5. An embedding result which is studied in [14].

Banach space $W_0^{s,p}(\Omega)$ is defined as follows:

$$||u||_W := \inf \left\{ \eta > 0 : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\eta^{p(x,y)}} dx dy < 1 \right\}.$$ 

Note that $||\cdot||_W$ and $|\cdot|_W$ are equivalent norms on $W$ with the relation

$$\frac{1}{2} ||u||_W \leq |u|_W \leq 2 ||u||_W,$$ 

for all $u \in W$.

Remark 2.4. If we substitute $\Omega$, a smooth bounded domain, by $\mathbb{R}^N$, all the above results regarding the fractional Sobolev spaces with variable order and variable exponents hold.

We define the subspace $X_0$ of $W^{s,\frac{p_0}{p_0}}(\mathbb{R}^N)$ as

$$X_0 = X_0^{s,\frac{p_0}{p_0}}(\mathbb{R}^N) := \{ u \in W^{s,\frac{p_0}{p_0}}(\mathbb{R}^N) : u = 0 \ a.e. \ in \ \Omega \}.$$ 

which is endowed with the following norm:

$$||u||_{X_0} := \inf \left\{ \eta > 0 : \rho_{X_0} \left( \frac{u}{\eta} \right) < 1 \right\},$$

where $\rho_{X_0}(u) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+\gamma(x,y)}} dx dy$ is a convex modular on $X_0$. We have the following embedding result which is studied in [14].

Theorem 2.5. Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$ and $s(\cdot, \cdot)$ and $p(\cdot, \cdot)$ satisfy (S1) and (P1), respectively. Then for any $\gamma \in C_+\left(\mathbb{R}_+^N\right)$ with $1 < \gamma(x) < p_0^*(x)$ for all $x \in \Omega$, there exits a constant $C = C(N, s, p, \gamma, \Omega) > 0$ such that for every $u \in X_0$,

$$||u||_{L^{\gamma(x)}(\Omega)} \leq C ||u||_{X_0}.$$ 

Moreover, this embedding is compact.

Remark 2.6. For $u \in X_0$, from the proof of the Theorem 2.5, we get

$$||u||_{X_0} \leq ||u||_{W^{s,\frac{p_0}{p_0}}(\mathbb{R}^N)} \leq C ||u||_{X_0},$$

that is, these two norms are equivalent on $X_0$. Since $X_0$ is closed subspace of the separable reflexive Banach space $W^{s,\frac{p_0}{p_0}}(\mathbb{R}^N)$ with respect to $||\cdot||_{W^{s,\frac{p_0}{p_0}}(\mathbb{R}^N)}$, we have that $(X_0, ||\cdot||_{X_0})$ is separable reflexive Banach space.

If (V1) holds true, we define the following space

$$E = \left\{ u \in X_0 : \int_{\Omega} \frac{V(x)|u(x)|}{\eta^{p(x)}} dx < +\infty, \ for \ some \ \eta > 0 \right\}$$

equipped with the norm

$$||u||_E := \inf \left\{ \eta > 0 : \rho_E \left( \frac{u}{\eta} \right) < 1 \right\},$$

where

$$\rho_E(u) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+\gamma(x,y)}} dx dy + \int_{\Omega} V(x)|u(x)|^{p(x)} dx$$

defines a convex modular in $E$. On $E$ we also can make use of the the following norm

$$|u|_E := |u|_V + ||u||_{X_0},$$

where

$$|u|_V := \inf \left\{ \eta > 0 : \int_{\Omega} \frac{V(x)|u(x)|^{p(x)}}{\eta^{p(x)}} dx < 1 \right\}.$$
One can easily verify that \( \| \cdot \|_E \) and \( |\cdot|_E \) are equivalent norms on \( E \) with the relation
\[
\frac{1}{2} \| u \|_E \leq |u|_E \leq 2 \| u \|_E, \quad \text{for all } u \in E. \tag{2.2}
\]

Next we can prove the following results similarly as in [37].

**Proposition 2.7.** For \( u \in E \setminus \{0\} \), we have
(i) \( \eta = \| u \|_E \) if and only if \( \rho_E(\eta) = 1 \);
(ii) \( \rho_E(u) > 1 \) (\( = 1 \); \( < 1 \)) if and only if \( \| u \|_E > 1 \) (\( = 1 \); \( < 1 \)), respectively;
(iii) if \( \| u \|_E \geq 1 \), then \( \| u \|_E^p \leq \rho_E(u) \leq \| u \|_E^+ \);
(iv) if \( \| u \|_E < 1 \), then \( \| u \|_E^p \leq \rho_E(u) \leq \| u \|_E^+ \).

As a consequence of the above proposition, we can derive the following result.

**Proposition 2.8.** Let \( u, u_n \in E, n \in \mathbb{N} \). Then the following statements are equivalent:
(i) \( \lim_{n \to \infty} \| u_n - u \|_E = 0 \).
(ii) \( \lim_{n \to \infty} \rho_E(u_n - u) = 0 \).

**Lemma 2.9.** \( (E, \| \cdot \|_E) \) is a separable reflexive Banach space.

**Proof.** First we show that \( (E, \| \cdot \|_E) \) is a Banach space. For that, let \( \{u_n\} \) be a Cauchy sequence in \( E \). Therefore for any \( \epsilon > 0 \) there exists \( N_\epsilon \in \mathbb{N} \) such that if \( n, k \geq N_\epsilon \)
\[
\| u_n - u_k \|_E \leq \epsilon. \tag{2.3}
\]

Since \( \| u \|_E \geq \| u \|_{X_0} \) and \( (X_0, \| \cdot \|_{X_0}) \) is a Banach space, there exists \( u \in X_0 \) such that \( u_n \to u \) in \( X_0 \) strongly as \( n \to \infty \). So, there exists a subsequence \( \{u_{n_j}\} \) such that \( u_{n_j}(x) \to u(x) \) a.e. \( x \in \mathbb{R}^N \). Now using Fatou’s lemma and (2.3) with \( \epsilon = 1 \), we have
\[
\int_\Omega V(x)|u(x)|^{p(x)}dx \leq \liminf_{n \to \infty} \int_\Omega V(x)|u_n(x)|^{p(x)}dx
\]
\[
\leq \liminf_{n \to \infty} \int_\Omega V(x)|u_n(x) - u_{N_1}(x) - u_{N_1}(x)|^{p(x)}dx
\]
\[
\leq 2^{p^+} \liminf_{n \to \infty} \left[ \int_\Omega V(x)|u_n(x) - u_{N_1}(x)|^{p(x)}dx + \int_\Omega V(x)|u_{N_1}(x)|^{p(x)}dx \right]
\]
\[
\leq 2^{p^+} \left[ 1 + \int_\Omega V(x)|u_{N_1}(x)|^{p(x)}dx \right] < \infty. \tag{2.4}
\]

Therefore \( u \in E \). Now again by Fatou’s lemma and (2.3), we get for all \( n, n_j \geq N_\epsilon \)
\[
\rho_E(u_n - u) \leq \liminf_{j \to \infty} \rho_E(u_n - u_{n_j}) \leq \epsilon \tag{2.5}
\]

and then Proposition 2.8 infers \( u_{n_j} \to u \). Hence \( (E, \| \cdot \|_E) \) is a Banach space. For proving reflexivity of \( E \) we define the map \( T : E \to L^{p^+}(\Omega) \times L^{p^+}((\mathbb{R}^N \times \mathbb{R}^N)^N) \)
\[
T(u) = \left( V^1, x/\Omega u, \frac{|u(x) - u(y)|}{\beta(x,y)} \right).
\]

The norm on \( L^{p^+}(\Omega) \times L^{p^+}((\mathbb{R}^N \times \mathbb{R}^N)^N) \) is given as
\[
\| u \| = \| u \|_{L^{p^+}(\Omega)} + \| u \|_{L^{p^+}((\mathbb{R}^N \times \mathbb{R}^N)^N)}.
\]

Clearly \( T \) is an isometry. Hence \( T(E) \) is reflexive being a closed subspace of the reflexive Banach space \( L^{p^+}(\Omega) \times L^{p^+}((\mathbb{R}^N \times \mathbb{R}^N)^N) \) (see [39, Proposition 3.20]) and consequently \( E \) is reflexive. Arguing similarly, we get \( E \) is separable (see [39, Proposition 3.25]).
Using Theorem 2.5 and the fact \( \|u\|_E \geq \|u\|_{X_0} \), we have the following embedding theorem.

**Theorem 2.10.** Let \( \Omega \subset \mathbb{R}^N, N \geq 2 \) be a smooth bounded domain and let (S1), (P1) and (V1) hold. Then for any \( \gamma \in C_+(\overline{\Omega}) \) with \( 1 < \gamma(x) < p^*_s(x) \) for all \( x \in \overline{\Omega} \), there exists a constant \( C = C(N, s, p, \gamma, \Omega) > 0 \) such that for every \( u \in E \),

\[
\|u\|_{L^{\gamma(x)}(\Omega)} \leq C\|u\|_E.
\]

Moreover, this embedding is compact.

The function space \( E \) is the solution space for the problem (1.1). The energy functional \( J \) associated to (1.1) is defined as follows. For \( u \in E \), we set

\[
\sigma(u) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{p(x, y)} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+s(x,y)p(x,y)}} \, dx \, dy + \int_{\Omega} V(x) \frac{|u(x)|^{\overline{p}(x)}}{\overline{p}(x)} \, dx.
\]

\[
\Psi(u) := \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{F(x, u(x))F(y, u(y))}{|x - y|^{\overline{p}(x,y)}} \, dx \, dy.
\]

\[
I(u) := \int_{\Omega} \int_{\Omega} \left( \frac{F(y, u(y))}{|x - y|^{\overline{p}(x,y)}} \right) f(x, u(x))u(x) \, dx.
\]

**Definition 2.11.** The energy functional \( J : E \to \mathbb{R} \) associated with problem (1.1) is defined as

\[
J(u) = M(\sigma(u)) - \Psi(u).
\]

To establish the smoothness of the energy functional \( J \) we first recall the following Hardy-Littlewood-Sobolev type result ([14, Proposition 4.1]).

**Proposition 2.12.** Let (\( \mu_1 \)) hold and let \( q \in C_+(\mathbb{R}^N \times \mathbb{R}^N) \) be a continuous function satisfying

\[
\frac{2}{q(x,y)} + \frac{\mu(x,y)}{N} = 2, \quad \text{for all } x, y \in \mathbb{R}^N.
\]

If \( h, g \in L^{\mu}((\mathbb{R}^N)^N) \cap L^q(\mathbb{R}^N) \) then

\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{h(x)g(y)}{|x - y|^{\mu(x,y)}} \, dx \, dy \right| \leq C(N, q, \mu) \left( \|h\|_{L^\mu((\mathbb{R}^N)^N)} \|g\|_{L^q(\mathbb{R}^N)} + \|h\|_{L^{\mu}(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)} \right).
\]

**Corollary 2.13.** In particular for \( h(x) = g(x) = |u(x)|^{\beta(x)} \in L^{\mu}(\Omega) \cap L^q(\Omega) \), \( u \in E \), we have

\[
\left| \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{\beta(x)} |u(y)|^{\beta(y)}}{|x - y|^{\mu(x,y)}} \, dx \, dy \right| \leq C(N, q, \mu, \beta) \left( \|u\|_{L^\beta(\Omega)}^2 + \|u\|_{L^{\mu}(\Omega)}^2 \right),
\]

where \( \beta \in C_+(\overline{\Omega}) \) such that \( 1 < \beta_q^- \leq \beta(x) \beta_q^- \leq \beta(x) \beta_q^+ < p^*_s(x) \), for all \( x \in \overline{\Omega} \).

**Remark 2.14.** From Proposition 2.13 we can define the variable order and variable exponent Hardy-Littlewood-Sobolev critical exponent as

\[
p^*_s,\mu(x) := p^*_s(x) = \frac{\overline{p}(x)}{\overline{\mu}(x)} = \frac{2N - \overline{\mu}(x)}{N - \overline{\overline{\mu}}(x)},
\]

where \( \overline{\mu}(x) = q(x, x) \) and \( \overline{\overline{\mu}}(x) = \mu(x, x) \).

**Lemma 2.15.** The functional \( J \) as defined in the Definition 2.11 is of class \( C^1 \) and for all \( u, w \in E \),

\[
\langle J'(u), w \rangle = m(\sigma(u)) \left[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y))(w(x) - w(y))}{|x - y|^{N+s(x,y)p(x,y)}} \, dx \, dy \right.
\]

\[
+ \int_{\Omega} V(x)|u(x)|^{\overline{p}(x)-2} u(x) w(x) \, dx - \int_{\Omega} \int_{\Omega} \frac{F(y, u(y)) f(x, u(x))w(x)}{|x - y|^{\overline{p}(x,y)}} \, dx \, dy.
\]
Proof. Clearly, \( J \) is well defined. Also it is easy to see that \( M(\sigma(\cdot)) \) is Gateaux-differentiable in \( E \) and the derivative function at \( u \in \Omega \) is given as

\[
m(\sigma(u)) \left[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N+s(x,y)p(x,y)}} \, dx \, dy + \int_{\Omega} V|u|^p \, dw \right]
\]

for all \( w \in E \). Let \( \{u_n\} \) be a sequence in \( E \) such that \( u_n \to u \) strongly in \( E \) as \( n \to \infty \). Then \( u_n(x) \to u(x) \) a.e. in \( \mathbb{R}^N \). Let \( p' \) and \( \tilde{p} \) denote the conjugate of \( p \) and \( p' \), respectively. Then the sequences \( \left\{ \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))}{|x - y|^{N+s(x,y)p(x,y)/p'(x,y)}} \right\} \) and \( \left\{ V(x)^{1/p'}(x) |u_n(x)|^{p(x,y)-2}u_n(x) \right\} \) are bounded in \( L^{p'/(\cdot)}(\mathbb{R}^N \times \mathbb{R}^N) \) and in \( L^{p/(\cdot)}(\Omega) \), respectively and as \( n \to \infty \)

\[
\mathcal{U}_n(x,y) := \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))}{|x - y|^{(N+s(x,y)p(x,y))/p'(x,y)}} \to \mathcal{U}(x,y) := \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{(N+s(x,y)p(x,y))/p'(x,y)}}
\]

for a.e. \( x, y \in \mathbb{R}^N \), and

\[
V(x)^{1/p'}(x) |u_n(x)|^{p(x,y)-2}u_n(x) \to V(x)^{1/p'}(x) |u(x)|^{p(x,y)-2}u(x)
\]

for a.e. \( x \in \Omega \). Thus by [10], Proposition 5.4.7, we get as \( n \to \infty \)

\[
\mathcal{U}_n \rightharpoonup \mathcal{U} \text{ weakly in } L^{p'/(\cdot)}(\mathbb{R}^N \times \mathbb{R}^N)
\]

and

\[
V^{1/p'}(|u_n(\cdot)|^{p(\cdot)-2}u_n(\cdot)) \rightharpoonup V^{1/p'}(|u(\cdot)|^{p(\cdot)-2}u(\cdot)) \text{ weakly in } L^{p/(\cdot)}(\Omega).
\]

Hence for any \( w \in E \), by Theorem [2.10] and definition of weak convergence,

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))(w(x) - w(y))}{|x - y|^{(N+s(x,y)p(x,y))/p'(x,y)}} \, dx \, dy
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(w(x) - w(y))}{|x - y|^{(N+s(x,y)p(x,y))/p'(x,y)}} \, dx \, dy,
\]

\[
\lim_{n \to \infty} \int_{\Omega} V(x)|u_n(x)|^{p(x,y)-2}u_n(x)w(x) \, dx = \int_{\Omega} V(x)|u(x)|^{p(x,y)-2}u(x)w(x) \, dx. \tag{2.6}
\]

Next, using Proposition [2.12] and arguing similarly as in [27, Section 3], one can see that \( \Psi \) is of class \( C^1 \) such that Gateaux-derivative of \( \Psi \) is given as

\[
\Psi'(u)w = \int_{\Omega} \int_{\Omega} \frac{F(y,u)}{|x - y|^{\mu(x,y)}} f(x,u)w(x) \, dx \, dy
\]

for all \( w \in E \). Moreover

\[
\int_{\Omega} \int_{\Omega} \frac{F(y,u_n)}{|x - y|^{\mu(x,y)}} f(x,u_n)w(x) \, dx \, dy \to \int_{\Omega} \int_{\Omega} \frac{F(y,u)}{|x - y|^{\mu(x,y)}} f(x,u)w(x) \, dx \, dy \text{ as } n \to \infty. \tag{2.7}
\]

Finally combining (2.6)-(2.7), we obtain as \( n \to \infty \)

\[
\|J'(u_n) - J'(u)\|_{E^*} = \sup_{w \in E, ||w||_{E^*}=1} |(J'(u_n) - J'(u))w| \to 0.
\]

This completes the proof. \( \square \)

3 Proof of the main results

Here we give the proofs of the main theorems in this article. From now on wards \( C \) is treated as a generic positive constant which may vary from line to line. The notation \( o_n(1) \) implies \( \lim_{n \to +\infty} o_n(1) = 0 \).
3.1 Proof of Theorem 1.7

To prove the Theorem 1.7 we need some Lemmas and results presented below.

Lemma 3.1 (Mountain Pass Geometry 1). Assume that (S1), (P1), (μ1), (V1), (M1) and (f1) − (f4) hold. Then there exist $R > 0$ and $\delta > 0$ such that $J(u) > R$ for all $u \in E$ with $\|u\|_E = \delta$.

Proof. First we estimate $\Psi(u)$. For that, using (f1) and Theorem 2.10, one can easily check that $F(\cdot, u) \in L^{r^-}(\Omega) \cap L^{r^+}(\Omega)$. Hence by Proposition 2.12, we get

$$\Psi(u) = \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{F(x, u(x))F(y, u(y))}{|x - y|^\mu(x, y)} dxdy \leq C(N, q, \mu) \left[ \|F(\cdot, u)\|_{L^{r^+}(\Omega)}^2 + \|F(\cdot, u)\|_{L^{r^-}(\Omega)}^2 \right]. \quad (3.1)$$

From (f1) and (f2) we deduce that for any $\epsilon > 0$, there exist some constants $\overline{C}(\epsilon), \underline{C}(\epsilon) > 0$ such that $|f(x, t)| \leq \epsilon t^{\frac{p_+}{2}} + C(\epsilon) t^{r(x) - 1}$ for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$ and

$$|F(x, t)| \leq \epsilon |t|^{\frac{p_+}{2}} + C(\epsilon) |t|^{r(x)} \quad \text{for a.e. } x \in \Omega \text{ and for all } t \in \mathbb{R}. \quad (3.2)$$

Using (3.2) and Lemma 2.3, we have

$$\|F(\cdot, u)\|_{L^{r^+}(\Omega)} \leq \left[ \int_{\Omega} \left( \epsilon \|u(x)\|^{\frac{p_+}{2}} + C(\epsilon) \|u(x)\|^{r(x)} \right)^{\frac{1}{q^+}} \right]^{1/q^+} \leq 2 \left[ \epsilon \left( \int_{\Omega} \|u(x)\|^{\frac{p_+}{2}} \right)^{\frac{1}{q^+}} + C(\epsilon) \|u\|_{L^{r^+}(\Omega)} \right] \leq 2 \left[ \epsilon \|u\|^{\frac{p_+}{2}} + C(\epsilon) \left\{ \|u\|_{L^{r^+}(\Omega)} + \|u\|_{L^{r^-}(\Omega)} \right\} \right]. \quad (3.3)$$

Similarly, we deduce

$$\|F(\cdot, u)\|_{L^{r^-}(\Omega)} \leq 2 \left[ \epsilon \|u\|^{\frac{p_+}{2}} + C(\epsilon) \left\{ \|u\|_{L^{r^-}(\Omega)} + \|u\|_{L^{r^+}(\Omega)} \right\} \right] \quad (3.4)$$

Plugging (3.3) and (3.4) into (3.1), we derive

$$\Psi(u) \leq C_h \left[ \epsilon^2 \|u\|^{\frac{p_+}{2}} + C(\epsilon) \left\{ \|u\|_{L^{r^+}(\Omega)}^2 + \|u\|_{L^{r^-}(\Omega)}^2 \right\} \right] + \left\{ \epsilon^2 \|u\|^{\frac{p_+}{2}} + C(\epsilon) \left\{ \|u\|_{L^{r^+(\Omega)}} + \|u\|_{L^{r^-}(\Omega)} \right\} \right\}, \quad (3.5)$$

where $C_h > 0$ is a constant. Now by applying Theorem 2.10 in (3.5), we have

$$\Psi(u) \leq C \left[ \epsilon^2 \|u\|^{\frac{p_+}{2}} + C(\epsilon) \left\{ \|u\|_{L^p(\Omega)}^2 + \|u\|_{L^p(\Omega)}^{-2} \right\} \right]. \quad (3.6)$$

Let $u \in E$, $\|u\|_E < 1$. Therefore using (3.5), Remark 1.1 or Remark 1.2, and Proposition 2.4, we get

$$J(u) \geq M(1) \{\sigma(u)\}^\theta - C \left[ \epsilon^2 \|u\|^{\frac{p_+}{2}} + C(\epsilon) \left\{ \|u\|_{L^p(\Omega)}^2 + \|u\|_{L^p(\Omega)}^{-2} \right\} \right] \geq M(1) \left\{ \frac{\rho_E(u)}{\epsilon} \right\}^\theta - C \epsilon^2 \|u\|^{\frac{p_+}{2}} - 2CC(\epsilon)^2 \|u\|_{L^p(\Omega)}^{-2} \geq M(1) \left\{ \frac{\rho_E(u)}{\epsilon} \right\}^\theta - C \epsilon^2 \|u\|^{\frac{p_+}{2}} - 2CC(\epsilon)^2 \|u\|_{L^p(\Omega)}^{-2} \quad (3.7)$$

By taking $0 < \epsilon < \left[ M(1)/(2C(p^+)^\theta) \right]^{1/2}$ in (3.7), since $\|u\|_E < 1$ and $\frac{p_+}{2} < r^-$ we can choose $0 < \delta < 1$ sufficiently small such that (3.7) infers that there exists some $R > 0$ such that $J(u) > R > 0$ for $\|u\|_E = \delta$. \qed
Lemma 3.2 (Mountain Pass Geometry 2). Assume that (S1), (P1), (μ1), (V1), (M1) and (f1)−(f4) hold. Then there exists $e \in X_0$ with $\|e\|_{X_0} > \delta$ such that $J(e) < 0$, where $\delta$ is given by Lemma 3.4.

Proof. Choose $u \in E, u > 0$ such that $\|u\|_E = 1$ and $\int_\Omega \int_\Omega \frac{|u(x)|}{|x-y|^{N+\sigma}} \frac{|u(y)|}{|x-y|^{N+\sigma}} \, dx \, dy > 0$. Now for $t > 1$ large, using Remark 1.1 (or Remark 1.2) and Proposition 2.7, we get

$$J(tu) \leq \frac{M(1)}{(p^-)^{\theta}} (\rho_{E}(tu))^\theta - \Psi(tu) \leq \frac{M(1)}{(p^-)^{\theta}} (\rho_{E}(u))^\theta - \Psi(tu) = \frac{M(1)}{(p^-)^{\theta}} (\rho_{E}(u))^\theta - \Psi(tu).$$

(3.8)

It follows from (f3) that for any $l > 0$ there exists $C_l > 0$ such that $F(x, tu(x)) > l \|tu(x)\|^{\frac{4}{2}}$, whenever $|tu(x)| > C_l$ for a.e. $x \in \Omega$. Therefore, using the above inequality in (3.8), we deduce that

$$J(tu) \leq \frac{M(1)}{(p^-)^{\theta}} (\rho_{E}(u))^\theta - \frac{\mu}{2} \rho_{E}(u)^2 - \frac{\mu}{2} \rho_{E}(u)^2 \int_\Omega \int_\Omega \frac{|u(x)|^{\frac{4}{2}} |u(y)|^{\frac{4}{2}}}{|x-y|^{N-\sigma}} \, dx \, dy.$$  

(3.9)

After taking $0 < l < \frac{4M(1)}{(p^-)^{\theta}} \left( \int_\Omega \int_\Omega \frac{|u(x)|^{\frac{4}{2}} |u(y)|^{\frac{4}{2}}}{|x-y|^{N-\sigma}} \, dx \, dy \right)^{-1/2}$ in (3.9) we can choose $t_u > 0$ large enough so that $|t_u u(x)| > C_l$ for a.e. $x \in \Omega$ with $\|t_u u\|_E > \delta$ such that $J(t_u u) < 0$. Thus by fixing $e = t_u u$, the result follows. \(\square\)

Definition 3.3 (Cerami condition). J is said to be satisfying Cerami condition $(C)_c$ for any $c \in \mathbb{R}$, if for any sequence $\{u_n\}$ in $E$

$$(C)_c \quad J(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|_E) \|J'(u_n)\|_{E^*} \to 0 \quad \text{as} \quad n \to \infty,$$

then $u_n \to u$ strongly in $E$ as $n \to \infty$.

Lemma 3.4. Let (S1), (P1), (μ1), (V1), (M1) and (f1)−(f4) hold. Then the functional $J$ satisfies the Cerami condition $(C)_c$ for any $c \in \mathbb{R}$.

Proof. Let $\{u_n\} \subset E$ be a Cerami sequence for $J$ at level $c \in \mathbb{R}$. Then by Definition 3.3

$$J(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|_E) \|J'(u_n)\|_{E^*} \to 0 \quad \text{as} \quad n \to +\infty,$$

(3.10)

which implies that

$$\langle J'(u_n), u_n \rangle \to 0 \quad \text{as} \quad n \to +\infty,$$

(3.11)

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $E$ and its dual $E^*$.

First we discuss the degenerate case, i.e., $a = 0$. Hence we divide the proof into two parts.

Case: inf\limits_{n \in \mathbb{N}} \|u_n\|_E = d_* > 0. First we prove that the sequence $\{u_n\}$ is bounded in $E$. Indeed, arguing by contradiction, we assume that $\{u_n\}$ is unbounded in $E$, that is, \(\|u_n\|_E \to +\infty \quad \text{as} \quad n \to +\infty.\)

(3.12)

Without loss of generality, we assume $\|u_n\|_E > 1$ and $u_n \rightharpoonup w$ weakly in $E$ and $w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^N$ and thus by applying Theorem 2.10 it follows that

$$w_n \to w \quad \text{strongly in} \quad L^{\gamma}(\Omega) \quad \text{for any} \quad 1 < \gamma(x) < p_*^{\gamma}(x) \quad \text{as} \quad n \to +\infty.$$

(3.13)

Let $\Omega_0 := \{x \in \Omega : w(x) \neq 0\}$. Thus we have \(\|u_n(x)\| \to +\infty \quad \text{a.e.} \quad x \in \Omega_0 \quad \text{as} \quad n \to +\infty.\)

(3.14)

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When \( x \in \Omega_0 \), we have \( |w_n(x)| > 0 \) for large \( n \). Therefore using this fact together with \((f3)\) and \((3.14)\), for each \( x \in \Omega_0 \) and sufficiently large \( n \), we get

\[
\lim_{|u_n(x)| \to \infty} \frac{F(x, u_n(x))}{|w_n(x)|^{\frac{\rho^+}{2}}} = +\infty. \tag{3.15}
\]

Now using Remark 1.4 (3.15) and Fatou’s lemma, we derive

\[
\liminf_{n \to \infty} \int_{\Omega} \frac{F(y, u_n(y)) |w_n(y)|^{\frac{\rho^+}{2}}}{|x - y|^{\mu(x,y)}} \, dy \geq \liminf_{n \to \infty} \int_{\Omega} \frac{F(y, u_n(y)) |w_n(y)|^{\frac{\rho^+}{2}}}{|x - y|^{\mu(x,y)}} \, dy
\]

\[
\geq \int_{\Omega_0} \liminf_{n \to \infty} \frac{F(y, u_n(y)) |w_n(y)|^{\frac{\rho^+}{2}}}{|x - y|^{\mu(x,y)}} \, dy
\]

\[
= +\infty. \tag{3.16}
\]

Combining together \((3.15)\) and \((3.16)\) for each \( x \in \Omega_0 \), we obtain

\[
\left( \int_{\Omega} \frac{F(y, u_n(y)) |w_n(y)|^{\frac{\rho^+}{2}}}{|x - y|^{\mu(x,y)}} \, dy \right) \frac{F(x, u_n(x))}{|w_n(x)|^{\frac{\rho^+}{2}}} \to +\infty \text{ as } n \to \infty,
\]

that is,

\[
\frac{1}{\|u_n\|_{E^p}} \left( \int_{\Omega} \frac{F(y, u_n(y))}{|x - y|^{\mu(x,y)}} \, dy \right) F(x, u_n(x)) \to +\infty \text{ as } n \to \infty. \tag{3.17}
\]

We claim that \( \text{meas}(\Omega_0) = 0 \). Indeed, if not, then for large \( n \) taking \( \|u_n\|_{E^p} > 1 \) and using \((3.10)\), \((3.12)\), \((3.17)\) and Remark 1.4 Remark 1.4 with Fatou’s lemma, we get

\[
\frac{1}{(p^*)^q} \geq \liminf_{n \to \infty} \frac{1}{(p^*)^q} \frac{[\rho_E(u_n)]^q}{\|u_n\|_{E^p}^{\rho^+}}
\]

\[
\geq \liminf_{n \to \infty} \frac{[\sigma(u_n)]^q}{\|u_n\|_{E^p}^{\rho^+}}
\]

\[
= \liminf_{n \to \infty} \frac{1}{\|u_n\|_{E^p}^{\rho^+}} \left( \frac{J(u_n) + \Psi(u_n)}{M(1)} \right)
\]

\[
\geq \liminf_{n \to \infty} \frac{[\Psi(u_n)]}{M(1)\|u_n\|_{E^p}^{\rho^+} - 1}
\]

\[
\geq \frac{1}{2M(1)} \liminf_{n \to \infty} \int_{\Omega_0} \frac{1}{\|u_n\|_{E^p}^{\rho^+}} \left( \int_{\Omega} \frac{F(y, u_n(y))}{|x - y|^{\mu(x,y)}} \, dy \right) F(x, u_n(x)) \, dx - 1
\]

\[
\geq \frac{1}{2M(1)} \liminf_{n \to \infty} \int_{\Omega_0} \frac{1}{\|u_n\|_{E^p}^{\rho^+}} \left( \int_{\Omega} \frac{F(y, u_n(y))}{|x - y|^{\mu(x,y)}} \, dy \right) F(x, u_n(x)) \, dx - 1 = +\infty,
\]

which is a contradiction and hence \( \text{meas}(\Omega_0) = 0 \). Therefore

\[
w(x) = 0 \text{ a.e. } x \in \Omega. \tag{3.18}
\]

Given any real number \( \kappa > 1 \), by \((f1)\), it follows that

\[
F(x, \kappa t) \leq C \left( |\kappa t| + |\kappa t|^r(x) \right)
\]

for any \( x \in \Omega \) and for all \( t \in \mathbb{R} \),

which together with Theorem 2.10 yields that

\[
|F(x, \kappa w_n(x))|^{\rho^+} \leq \overline{K}(x), \quad |F(x, \kappa w_n(x))|^{\rho^+} \leq \overline{H}(x) \text{ a.e. } x \in \Omega \text{ for some } \overline{K}, \overline{H} \in L^1(\Omega).
\]
Note that from (3.18) we have $w_n \to 0$ strongly in $L^{\gamma}(\Omega)$, for all $1 < \gamma < p^*(x)$ and $w_n(x) \to 0$ a.e. in $\Omega$, hence using the continuity of $F$, we deduce
\[
\lim_{n \to +\infty} F(x, \kappa w_n(x)) = F(x,0) = 0 \text{ a.e. } x \in \Omega.
\]
Therefore by Lebesgue dominated convergence theorem, we have
\[
\lim_{n \to +\infty} \|F(\cdot, \kappa w_n(\cdot))\|_{L^p_+(\Omega)} = 0 \text{ and } \lim_{n \to +\infty} \|F(\cdot, \kappa w_n(\cdot))\|_{L^p_-(\Omega)} = 0. \tag{3.19}
\]
Using Proposition 2.12 and (3.19), we get as $\kappa w_n(\cdot) \to 0$.

Combining Remark 1.1, 1.4, 1.5 with (3.12), we have $\kappa w_n(\cdot) \to 0$.

Hence, using the continuity of $\Psi$, we deduce
\[
\Psi(\kappa w_n) \leq C(N,q,\mu) \left[\|F(\cdot, \kappa w_n(\cdot))\|_{L^p_+(\Omega)}^2 + \|F(\cdot, \kappa w_n(\cdot))\|_{L^p_-(\Omega)}^2\right] \to 0. \tag{3.20}
\]
As $J(tu_n)$ is continuous in $t \in [0,1]$, for each $n \in \mathbb{N}$, there exists $t_n \in [0,1]$ such that
\[
J(t_n u_n) = \max_{t \in [0,1]} J(t u_n). \tag{3.21}
\]
We claim that
\[
J(t_n u_n) \to +\infty \text{ as } n \to +\infty. \tag{3.22}
\]
For any real number $C > 1$, choose $\kappa = \left[C/(\min \{1, m \theta p^-\})\right]^{1/p^-}$. Using (3.12), we have $\kappa w_n(\cdot) \in (0,1)$ for $n$ sufficiently large. Thus by (M1), Remark 1.1, 3.20, 3.21 and Proposition 2.7, we get
\[
J(t_n u_n) \geq J\left(\frac{\kappa}{\|u_n\|_E} w_n\right) = J(\kappa w_n) = M(\sigma(\kappa w_n)) - \Psi(\kappa w_n)
\]
\[
= \frac{1}{\theta} \kappa \sigma(\kappa w_n) + o_n(1)
\]
\[
\geq \frac{m_0 \theta p^-}{\theta p^-} \kappa w_n + o_n(1)
\]
\[
\geq \frac{m_0 \theta p^-}{\theta p^-} (C + o_n(1))
\]
Hence (3.22) is proved. Because $J(0) = 0$ and $J(u_n) \to c$ as $n \to \infty$, we can see that
\[
t_n \in (0,1) \text{ and } J(t_n u_n) = \max_{t \in [0,1]} J(t u_n) = 0. \tag{3.23}
\]
Combining Remark 1.1, 1.4, 1.5 with (f4), (1.5), (3.10), (3.11) and (3.23), we obtain
\[
\frac{1}{\theta} J(t_n u_n) + o_n(1)
\]
\[
= \frac{1}{\theta} J(t_n u_n) - \int_{\mathbb{R}^N} \frac{J'(t_n u_n)}{\theta p^-} dx
\]
\[
= \frac{1}{\theta p^-} \left[ M(\sigma(t_n u_n)) - \int_{\mathbb{R}^N} \frac{\sigma(t_n u_n)}{\theta p^-} dx \right] + \int_{\mathbb{R}^N} \frac{F(x, t_n u_n)}{\theta p^-} dx
\]
\[
\leq \left[ M(\sigma(t_n u_n)) - \int_{\mathbb{R}^N} \frac{\sigma(t_n u_n)}{\theta p^-} dx \right] + \int_{\mathbb{R}^N} \frac{F(x, u_n(\cdot))}{\theta p^-} dx
\]
\[
= \left[ M(1) \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|t_n u_n(x) - t_n u_n(y)|^p}{|x - y|^{N + sp(x,y)} dx dy} \right)^{\theta^-1} \right. \times \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N + sp(x,y)} dx dy} \right)^{\theta^-1} \right]
\]
\[
= J(u_n) - \frac{1}{\theta p^-} J'(u_n, u_n) = c + o_n(1),
\]

which contradicts (3.22). Hence \( \{u_n\} \) is bounded in \( E \). Therefore passing to the limit \( n \to \infty \), if necessary, to a subsequence, thanks to Theorem 2.10, we have \( u_n \rightharpoonup u \) weakly in \( E \), \( u_n(x) \to u(x) \) a.e. in \( \Omega \) and \( u_n \to u \) strongly in \( L^{q^+}(\Omega) \) for all \( \gamma \in C_+^{1}(\Omega) \) with \( 1 < q(x) < p^*_s(x) \). Let us denote \( v_n := u_n - u \). Then clearly

\[
 v_n \to 0 \quad \text{strongly in } L^{q^+}(\Omega). \tag{3.24}
\]

In order to prove \( \{u_n\} \) converges strongly to \( u \) in \( E \) as \( n \to \infty \), we define the following functional. Let \( \phi \in E \) be fixed and let \( B_\phi \) denote the linear functional on \( E \) defined by

\[
 B_\phi(v) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)| p(x,y)^{-2}(\phi(x) - \phi(y))(v(x) - v(y))}{|x - y|^{n+q(x,y)p(x,y)}} dx dy + \int_\Omega |\phi|^{p(x)} v dx
\]

for all \( v \in E \). For \( (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \), let us denote

\[
 \Upsilon(x,y) := \frac{|\phi(x) - \phi(y)|}{|x - y|^{\frac{n}{p(x,y)} + q(x,y,p(x,y))}} \quad \text{and} \quad U(x,y) := \frac{|v(x) - v(y)|}{|x - y|^{\frac{n}{p(x,y)} + q(x,y,p(x,y))}}.
\]

Then from Hölder inequality and Lemma 2.28 it follows that

\[
 |B_\phi(v)| \leq C \left[ \|\Upsilon\|_{L^{p^*(\cdot)}(\mathbb{R}^N \times \mathbb{R}^N)} \|U\|_{L^{p^*(\cdot)}(\mathbb{R}^N \times \mathbb{R}^N)} \right] + \|V\|_{L^{p^*(\cdot)}(\Omega)} \|\phi\|_{L^{p^*(\cdot)}(\Omega)} \|v\|_{L^{p^*(\cdot)}(\Omega)}
\]

\[
 \leq C \left[ \|\phi\|_{L^{p^*(\cdot)}(\Omega)}^{p^*_0 - 1} \|v\|_{L^{q^*(\cdot)}(\Omega)} \right] + \left[ \|\phi\|_{L^{p^*(\cdot)}(\Omega)}^{p^*_0 - 1} \|v\|_{L^{q^*(\cdot)}(\Omega)} + \|\phi\|_{L^{p^*(\cdot)}(\Omega)}^{p^*_0 - 1} \|v\|_{L^{q^*(\cdot)}(\Omega)} \right]
\]

\[
 \leq C_1 \left[ \|\phi\|_{L^{p^*(\cdot)}(\Omega)}^{p^*_0 - 1} \|v\|_{L^{q^*(\cdot)}(\Omega)} \right]
\]

where \( C_1 > 0 \) is a constant. Thus for each \( \phi \in E \) the linear functional \( B_\phi \) is continuous on \( E \). Hence \( u_n \rightharpoonup u \) weakly in \( E \) implies that

\[
 \lim_{n \to +\infty} B_\phi(v_n) = 0. \tag{3.25}
\]

Since \( \{m(\sigma(u_n)) - m(\sigma(u))\} \) is bounded a sequence in \( \mathbb{R} \), from (3.25) we get

\[
 \lim_{n \to +\infty} [m(\sigma(u_n)) - m(\sigma(u))] B_\phi(v_n) = 0. \tag{3.26}
\]

Sine \( \{u_n\} \) is bounded in \( E \), using \( (f1) \), Theorem 2.10 for each \( n \in \mathbb{N} \), we obtain \( f(\cdot, u_n(\cdot)) v_n(\cdot) \in L^{q^+}(\Omega) \) and furthermore Hölder’s inequality and (3.24) infer

\[
 \|f(\cdot, u_n(\cdot)) v_n(\cdot)\|_{L^{r^+}(\Omega)} \leq 2C \left[ \left( \int_\Omega |v_n(x)|^{q^+} dx \right)^{\frac{1}{q^+}} + \left( \int_\Omega |u_n(x)|^{(r(x)-1)q^+} |v_n(x)|^{q^+} dx \right)^{\frac{1}{q^+}} \right]
\]

\[
 \leq 2C \left[ \|v_n\|_{L^{q^+}(\Omega)} + \|u_n\|_{L^{r^+}(\Omega)}^{(r(x)-1)q^+} \right] \left[ \|v_n\|_{L^{q^+}(\Omega)} + \|u_n\|_{L^{r^+}(\Omega)}^{(r(x)-1)q^+} \right]
\]

\[
 \leq 2C \left[ \|v_n\|_{L^{q^+}(\Omega)} + \|u_n\|_{L^{r^+}(\Omega)}^{(r(x)-1)q^+} \right] \left[ \|v_n\|_{L^{q^+}(\Omega)} + \|u_n\|_{L^{r^+}(\Omega)}^{(r(x)-1)q^+} \right]
\]

\[
 \leq C \left[ \|v_n\|_{L^{q^+}(\Omega)} + \|u_n\|_{L^{q^+}(\Omega)} \right] = o_n(1). \tag{3.27}
\]

Similarly we can deduce that \( f(\cdot, u_n(\cdot)) v_n(\cdot) \in L^{q^-}(\Omega) \) and

\[
 \|f(\cdot, u_n(\cdot)) v_n(\cdot)\|_{L^{q^-}(\Omega)} = o_n(1). \tag{3.28}
\]
Therefore using Proposition 2.12, Theorem 2.10, boundedness of \( \{u_n\} \) in \( E \) and (f1) with (3.27) and (3.28), we get

\[
\int_\Omega \int_\Omega \frac{F(y, u_n(y))f(x, u_n(x))v_n(x)}{|x - y|^\alpha} \, dx \, dy \\
\leq C(N, q, \mu) \left[ \left( \int_\Omega \left| f(x, u_n(x)) \right|^{q^+} \, dx \right)^{1/q^+} \| f(\cdot, u_n(\cdot))v_n(\cdot) \|_{L^{q^+}(\Omega)} \right. \\
\left. + \left( \int_\Omega \left| f(x, u_n(x)) \right|^{q^-} \, dx \right)^{1/q^-} \| f(\cdot, u_n(\cdot))v_n(\cdot) \|_{L^{q^-}(\Omega)} \right] \\
\leq C \left( \| u_n \|_{L^{q^+}(\Omega)} + \| u_n \|_{L^{q^-}(\Omega)} \right) \left\{ \left\| f(\cdot, u_n(\cdot))v_n(\cdot) \right\|_{L^{q^+}(\Omega)} \right. \\
\left. + \left\| f(\cdot, u_n(\cdot))v_n(\cdot) \right\|_{L^{q^-}(\Omega)} \right\} \\
\leq C \left[ \| u_n \|_{L^{q^+}(\Omega)} + \left\| u_n \right\|_{L^{q^-}(\Omega)} \right] \times \left\{ \left\| f(\cdot, u_n(\cdot))v_n(\cdot) \right\|_{L^{q^+}(\Omega)} + \left\| f(\cdot, u_n(\cdot))v_n(\cdot) \right\|_{L^{q^-}(\Omega)} \right\} \\
= o_n(1). \tag{3.29}
\]

Now again from (f1), it follows that \( f(\cdot, u(\cdot))v_n(\cdot) \in L^{q^+}(\Omega) \cap L^{q^-}(\Omega) \) and hence by arguing similarly as above, we obtain

\[
\int_\Omega \int_\Omega \frac{F(y, u(y))f(x, u(x))v_n(x)}{|x - y|^\alpha} \, dx \, dy \\
\leq C \left[ \left\{ \| u \|_{L^{q^+}(\Omega)} + \left\| u \right\|_{L^{q^-}(\Omega)} \right\} \left\{ \left\| f(\cdot, u(\cdot))v_n(\cdot) \right\|_{L^{q^+}(\Omega)} + \left\| f(\cdot, u(\cdot))v_n(\cdot) \right\|_{L^{q^-}(\Omega)} \right\} \right] \\
= o_n(1). \tag{3.30}
\]

Since \( \{u_n\} \) is bounded, combining (3.24), (3.26), (3.29) and (3.30) we get

\[
o_n(1) = \langle J'(u_n) - J'(u), v_n \rangle \\
= m(\sigma(u_n)) \mathcal{B}_{u_n}(v_n) - m(\sigma(u_n)) \mathcal{B}_u(v_n) + \left[ m(\sigma(u_n)) - m(\sigma(u)) \right] \mathcal{B}_u(v_n) \\
+ \int_\Omega \int_\Omega \frac{F(y, u_n(y))f(x, u_n(x))v_n(x)}{|x - y|^\alpha} \, dx \, dy - \int_\Omega \int_\Omega \frac{F(y, u(y))f(x, u(x))v_n(x)}{|x - y|^\alpha} \, dx \, dy \\
= m(\sigma(u_n)) \left[ \mathcal{B}_{u_n}(v_n) - \mathcal{B}_u(v_n) \right] + o_n(1), \tag{3.31}
\]

that is,

\[
\lim_{n \to +\infty} m(\sigma(u_n)) \left[ \mathcal{B}_{u_n}(v_n) - \mathcal{B}_u(v_n) \right] = 0. \tag{3.32}
\]

By using Remark 1.1, we have in particular

\[
\lim_{n \to +\infty} \left[ \mathcal{B}_{u_n}(v_n) - \mathcal{B}_u(v_n) \right] = 0. \tag{3.33}
\]

Let us now recall well-known Simon inequalities, for all \( \zeta, \xi \in \mathbb{R}^N \), the followings hold:

\[
\left\{ \begin{array}{ll}
|\zeta - \xi|^p \leq \frac{1}{p} \left[ \left( |\zeta|^{p-2}\zeta - |\xi|^{p-2}\xi \right) \cdot (\zeta - \xi) \right]^{\frac{p}{2}} \left( |\zeta|^p + |\xi|^p \right)^{\frac{2-p}{2}}, & 1 < p < 2, \\
|\zeta - \xi|^p \leq 2^p \left( |\zeta|^{p-2}\zeta - |\xi|^{p-2}\xi \right) \cdot (\zeta - \xi), & p \geq 2.
\end{array} \right. \tag{3.34}
\]
Let us set
\[
g^{(1)}_n(x, y) := \left[ \frac{|u_n(x) - u_n(y)|^{p(x,y) - 2}(u_n(x) - u_n(y))(v_n(x) - v_n(y))}{|x - y|^{N + s(x,y)p(x,y)}} - \frac{|u(x) - u(y)|^{p(x,y) - 2}(u(x) - u(y))(v_n(x) - v_n(y))}{|x - y|^{N + s(x,y)p(x,y)}} \right];
\]
\[
g^{(2)}_n(x, y) := \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N + s(x,y)p(x,y)}};
\]
\[
g^{(3)}_n(x, y) := \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + s(x,y)p(x,y)}};
\]
\[
g^{(4)}_n(x) := V(x) \left( |u_n(x)|^{2} - 2u_n - |u|^2 u \right) v_n(x).
\]

For all \( n \in \mathbb{N} \), using (3.33) we get \( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g^{(1)}_n(x, y) \, dx \, dy \geq 0 \) and \( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g^{(4)}_n(x, y) \, dx \, dy \geq 0 \) which together with (3.33) and Remark 1.1 infer that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g^{(1)}_n(x, y) \, dx \, dy = 0 \tag{3.35}
\]
and
\[
\lim_{n \to +\infty} \int_{\Omega} g^{(4)}_n(x) \, dx = 0. \tag{3.36}
\]

In order to prove strong convergence of \( \{u_n\} \) in \( E \) by applying Simon’s inequality, we divide our discussion into four cases. Let us set
\[
\Delta_1 := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : 1 < p(x, y) < 2 \}; \quad \Delta_2 := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : p(x, y) \geq 2 \};
\]
\[
\tilde{\Delta}_1 := \{x \in \Omega : 1 < \underline{p}(x) < 2 \}; \quad \tilde{\Delta}_2 := \{x \in \Omega : \underline{p}(x) \geq 2 \}.
\]

(i) Case \( (x, y) \in \Delta_1 \). Since \( \{u_n\} \) is bounded in \( E \), using (3.34), (3.35), generalized Hölder inequality and Lemma 2.5, we have
\[
I_1 := \int_{\Delta_1} \frac{|u_n(x) - v_n(y)|^{p(x,y)}}{|x - y|^{N + s(x,y)p(x,y)}} \, dx \, dy
\]
\[
\leq \frac{1}{(p^*-1)} \int_{\Delta_1} \left( g^{(1)}_n(x, y) \right)^{\frac{p(x,y)}{p^*(x,y)}} \left( g^{(2)}_n(x, y) + g^{(3)}_n(x, y) \right)^{\frac{2-p(x,y)}{2}} \, dx \, dy
\]
\[
\leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left[ \left( g^{(1)}_n \right)^{\frac{p(x,y)}{p^*(x,y)}} \cdot \left( g^{(2)}_n \right)^{\frac{2-p(x,y)}{2}} + \left( g^{(1)}_n \right)^{\frac{p(x,y)}{p^*(x,y)}} \cdot \left( g^{(3)}_n \right)^{\frac{2-p(x,y)}{2}} \right] \, dx \, dy
\]
\[
\leq C \left[ \left( \|g^{(1)}_n\|_{L^\frac{p^*}{p(x,y)}(\mathbb{R}^N \times \mathbb{R}^N)} \right)^{\frac{2-p(x,y)}{2}} \left( \|g^{(2)}_n\|_{L^{\frac{2-p(x,y)}{2}}(\mathbb{R}^N \times \mathbb{R}^N)} \right) + \left( \|g^{(3)}_n\|_{L^{\frac{2-p(x,y)}{2}}(\mathbb{R}^N \times \mathbb{R}^N)} \right) \right] \]
\[
\leq C \left[ \left( \|g^{(1)}_n\|_{L^\frac{p^*}{p(x,y)}(\mathbb{R}^N \times \mathbb{R}^N)} + \|g^{(1)}_n\|_{L^\frac{p^*}{p(x,y)}(\mathbb{R}^N \times \mathbb{R}^N)} \right) \times \left( \|g^{(2)}_n\|_{L^\frac{2-p(x,y)}{2}} + \|g^{(2)}_n\|_{L^\frac{2-p(x,y)}{2}} + \|g^{(3)}_n\|_{L^\frac{2-p(x,y)}{2}} + \|g^{(3)}_n\|_{L^\frac{2-p(x,y)}{2}} \right) \right] = o_n(1). \tag{3.37}
\]
ii) Case \((x, y) \in \Delta_1\). Since \(\{u_n\} \) is bounded, \((3.34)\) \((3.36)\), Hölder inequality and Lemma \(2.3\) imply

\[
I_2 := \int_{\Delta_1} V(x)|v_n(x)|^{p(x)}dx \leq \int_{\Delta_1} \frac{1}{(p-1)} \int_{\Delta_1} (g_n^{(4)}(x))^{p(x)/2} \times \left\{V(x)(|u_n(x)|^{p(x)} + |u(x)|^{p(x)})\right\}^{2-\frac{p(x)}{2}}dx
\]

\[
\leq C \left[\int_{\Omega} (g_n^{(4)}(x))^{p(x)}V(x)|u_n(x)|^{p(x)}dx + \int_{\Omega} (g_n^{(4)}(x))^{p(x)}V(x)|u(x)|^{p(x)}dx\right]^{2-\frac{p(x)}{2}}dx
\]

\[
\leq C\|(g_n^{(4)})^{2-\frac{p(x)}{2}}\|_{L^{\frac{1}{p(x)}}(\Omega)} \left[\|V|u_n|^{p(x)}\|_{L^{\frac{1}{p(x)}}(\Omega)} + \|V|u|^{p(x)}\|\right]_L^{\frac{1}{p(x)}}(\Omega)
\]

\[
\leq C\left[\left\|\|g_n^{(4)}\|_{L^{\frac{1}{p(x)}}(\Omega)}\right\| + \|g_n^{(4)}\|_{L^{\frac{1}{p(x)}}(\Omega)} + \|g_n^{(4)}\|_{L^{\frac{1}{p(x)}}(\Omega)} + \|g_n^{(4)}\|_{L^{\frac{1}{p(x)}}(\Omega)}\right] = o_n(1). \tag{3.38}
\]

(iii) Case \((x, y) \in \Delta_2\). Using \((3.34), (3.35)\) and generalized Hölder inequality, we obtain

\[
I_3 := \int_{\Delta_2} \frac{|v_n(x) - v_n(y)|^{p(x,y)}}{|x - y|^{\frac{N+s(x,y)p(x,y)}{2}}}dx \leq 2^{p^+} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g_n^{(1)}(x, y)dx \leq o_n(1). \tag{3.39}
\]

(iv) Case \((x, y) \in \Delta_3\). Using \((3.34), (3.36)\) and Hölder inequality, we deduce

\[
I_4 := \int_{\Delta_3} V(x)|u_n(x)|^{p(x)}dx \leq 2^{p^+} \int_{\Omega} g_n^{(4)}(x)dx = o_n(1). \tag{3.40}
\]

Now taking into account \((3.33)\) and \((3.37)-(3.40)\), we get

\[
\rho_E(v_n) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{p(x,y)}}{|x - y|^{\frac{N+s(x,y)p(x,y)}{2}}}dx + \int_{\Omega} V(x)|v_n(x)|^{p(x)}dx
\]

\[
= I_1 + I_2 + I_3 + I_4 = o_n(1).
\]

Hence finally Proposition \(2.8\) yields that

\[
u_n \rightarrow u \quad \text{in } E \quad \text{strongly as } n \rightarrow \infty.
\]

**Case:** \(\inf_{n \in \mathbb{N}} \|u_n\|_E = 0\). If 0 is an isolated point for the sequence \(\{\|u_n\|_E\}\) then there exists a subsequence \(\{u_{n_k}\}\) of \(\{u_n\}\) such that

\[
\inf_{n \in \mathbb{N}} \|u_n\|_E = d_* > 0
\]

and therefore we can proceed as before. Otherwise, 0 is an accumulation point of the sequence \(\{\|u_n\|_E\}\). Hence there is a subsequence \(\{u_{n_k}\}\) of \(\{u_n\}\) such that \(u_{n_k} \rightarrow 0 \) in \(E\) strongly.

Next we consider the non-degenerate case, i.e., \(a > 0\). Hence the above proof reduces to Case 1. Then making use of Remark \(1.2\) and \((1.3)\) in place of Remark \(1.4\) in the above and arguing in a similar fashion with some minute modifications, the result follows.

Now we state the following version of Mountain pass theorem.

**Theorem 3.5 (Mountain pass theorem).** Let \(E\) be a real Banach space and suppose that \(J \in C^1(E, \mathbb{R})\) satisfies the condition

\[
\max\{J(0), J(u_1)\} \leq i < j \leq \inf_{\|u\|_E = \varrho_0} J(u)
\]

for some \(i < j\), \(\varrho_0 > 0\) and \(u_1 \in E\) with \(\|u_1\|_E = \varrho_0\). Let \(c \geq j\) be characterized by

\[
c_* = \inf_{\nu \in \Gamma} \max_{0 \leq t \leq 1} J(\nu(t)),
\]

where \(\Gamma = \{\nu \in C([0,1], E), \nu(0) = 0, \nu(1) = u_1\}\) is the set of continuous paths joining 0 and \(u_1\). Then there exists a Cerami sequence \(\{u_n\} \subset E\) such that

\[
J(u_n) \rightarrow c \geq j \quad \text{and } (1 + \|u_n\|_E)\|J'(u_n)\|_E \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]
Proof of Theorem 1.7: Since $J$ satisfies Lemma 3.1 and Lemma 3.2, by Mountain Pass theorem (Theorem 3.9), there exists a Cerami sequence $\{u_n\}$ for $J$ in $E$ such that

$$J(u_n) \to c_* \quad \text{and} \quad (1 + \|u_n\|_E)\|J'(u_n)\|_{E^*} \to 0, \text{ as } n \to \infty,$$

where $c_* > 0$ is the mountain pass level defined by

$$c_* := \inf_{\nu \in \Gamma} \sup_{t \in [0,1]} J(\nu(t)).$$

Now Lemma 3.3 implies $u_n \to u_*$ strongly in $E$ and hence $J'(u_*) = 0$. This infers that $u_*$ is a critical point of $J$ and therefore a weak solution to (1.1). Also $J(u_*) = c_* > 0$ and since $J(0) = 0$, finally we conclude that $u_* \neq 0$.

3.2 Proof of Theorem 1.8

Proof of Theorem 1.8: First we note that the condition $(f4)$ is the consequence of $(f4)'$. Indeed, for $t_2 \geq t_1 > 0$, $(f4)'$ implies

$$\mathcal{F}(x,t_2) - \mathcal{F}(x,t_1) = \theta p^+[\frac{2}{\theta p^+} (f(x,t_2)t_2 - f(x,t_1)t_1) - (F(x,t_2) - F(x,t_1))]
\begin{align*}
&= \theta p^+ \left[ \int_0^{t_2} f(x,\tau) \frac{\nu^+}{\nu^+ - 1} d\tau - \int_0^{t_1} f(x,\tau) \frac{\nu^+}{\nu^+ - 1} d\tau - \int_{t_1}^{t_2} f(x,\tau) \frac{\nu^+}{\nu^+ - 1} d\tau \right] \\
&= \theta p^+ \left[ \int_0^{t_1} \left( f(x,\tau) - f(x,t_1) \right) \frac{\nu^+}{\nu^+ - 1} d\tau + \int_{t_1}^{t_2} \left( f(x,\tau) - f(x,t_1) \right) \frac{\nu^+}{\nu^+ - 1} d\tau \right] \geq 0.
\end{align*}
$$

Similarly for $0 > t_1 \geq t_2$, we can deduce $\mathcal{F}(x,t_2) - \mathcal{F}(x,t_1) \geq 0$, that is $\mathcal{F}(\cdot,t)$ is increasing for $t \geq 0$ and decreasing for $t \leq 0$. Hence $(f4)$ follows. Therefore there exists a weak solution $v_* \neq 0$ of (1.1), thanks to Theorem 1.7 with $J'(v_*) = 0$ and $J(v_*) = b_*$, where $b_*$ is given as

$$b_* := \inf_{\nu \in \Gamma} \max_{0 \leq t \leq 1} J(\nu(t)),$$

where $\Gamma = \{ u \in C([0,1], E) : \nu(0) = 0, J(\nu(1)) < 0 \}$. We claim that $v_*$ is a ground state solution to (1.1). The Nehari manifold associated with the functional $J$ is defined as,

$$\mathcal{N} := \{ u \in E \setminus \{ 0 \} : \langle J'(u), u \rangle = 0 \}.$$

Since $v_*$ is a critical point of $J$, we have $v_* \in \mathcal{N}$. Let

$$\alpha_* = \inf_{u \in \mathcal{N}} J(u).$$

Hence $\alpha_* \leq b_*$. Therefore it is left to show $b_* \leq \alpha_*$. For $u \in \mathcal{N}$ define the fibering map $H : [0, +\infty) \to \mathbb{R}$ by $H(t) = J(tu)$. Then $H$ is differentiable with respect to $t$ and

$$H'(t) = \langle J'(tu), u \rangle = m(\sigma(tu)) \left[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^{p(x,y)-1} \frac{\|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+s(x,y)p(x,y)}} \, dx \, dy \right]
\begin{align*}
&+ \int_{\Omega} |\nabla(x)|^{-1} V(x) |u|^{\nabla(x)} \, dx - \int_{\Omega} \int_{\Omega} \frac{F(y, tu)f(x, tu)u(x)}{|x - y|^{\mu(x,y)}} \, dx \, dy.
\end{align*}
$$

Also we have $\langle J'(u), u \rangle = 0$, that is,

$$m(\sigma(u))\rho_E(u) = I(u) := \int_{\Omega} \int_{\Omega} \frac{F(y, u)f(x, u)u(x)}{|x - y|^{\mu(x,y)}} \, dx \, dy.
$$

(3.41)

(3.42)
Case I: $a = 0$. From (M1), Remark 3.11, 3.41 and 3.42 for $t > 1$, it follows that
\[
\mathcal{H}'(t) = m(\sigma(tu)) \left[ \int_{\Omega} \int_{\mathbb{R}^N} f^{\sigma}(x,y) - \frac{\sigma p^*}{2} t^{p^* - 1} F(x,tu) dxdy - \int_{\Omega} \int_{\Omega} F(y,\sigma(tu)) u(x) \frac{\sigma p^*}{2} t^{p^* - 1} dxdy \right]
\]
\[
- \int_{\Omega} \int_{\Omega} F(y,\sigma(u)) u(x) \frac{\sigma p^*}{2} t^{p^* - 1} dxdy - \int_{\Omega} \int_{\Omega} F(y,\sigma(u)) u(x) \frac{\sigma p^*}{2} t^{p^* - 1} dxdy.
\]
Thus we obtain
\[
\mathcal{H}'(t) = m(\sigma(tu)) \left[ \int_{\Omega} \int_{\mathbb{R}^N} f^{\sigma}(x,y) - \frac{\sigma p^*}{2} t^{p^* - 1} F(x,tu) dxdy - \int_{\Omega} \int_{\Omega} F(y,\sigma(tu)) u(x) \frac{\sigma p^*}{2} t^{p^* - 1} dxdy \right]
\]
\[
- \int_{\Omega} \int_{\Omega} F(y,\sigma(u)) u(x) \frac{\sigma p^*}{2} t^{p^* - 1} dxdy - \int_{\Omega} \int_{\Omega} F(y,\sigma(u)) u(x) \frac{\sigma p^*}{2} t^{p^* - 1} dxdy.
\]
Since $\mathcal{F}(x,\tau) = 2\tau f(x,\tau) - \theta p^* F(x,\tau) \geq 0$ for all $x \in \mathbb{R}^N, \tau \in \mathbb{R}$, it follows that
\[
\frac{d}{dt} F(x,tu) = \frac{\theta p^*}{2} t^{p^* - 1} F(x,tu) = \frac{f(x,tu)u(x) - \theta p^* F(x,tu)}{t^{p^* - 1} - \theta p^*} \geq 0.
\]
Hence $F(x,tu)$ is an increasing function in $t > 0$ for all $u \in E$. (3.44)

Now for $t > 1$, using (f4)', 3.34 and Remark 3.14 we deduce
\[
\mathcal{G}(t) = t^{\theta p^* - 1} \left[ \int_{\Omega} \int_{\mathbb{R}^N} F(y,\sigma(tu)) u(x) \frac{\sigma p^*}{2} t^{p^* - 2} dxdy - \int_{\Omega} \int_{\Omega} F(y,\sigma(tu)) u(x) \frac{\sigma p^*}{2} t^{p^* - 2} dxdy \right]
\]
\[
- \int_{\Omega} \int_{\Omega} F(y,\sigma(u)) u(x) \frac{\sigma p^*}{2} t^{p^* - 2} dxdy - \int_{\Omega} \int_{\Omega} F(y,\sigma(u)) u(x) \frac{\sigma p^*}{2} t^{p^* - 2} dxdy.
\]
Combining (3.43) and (3.45), we get $\mathcal{H}'(t) \leq 0$ for $t > 1$. Arguing similarly as above we can deduce $\mathcal{H}'(t) \geq 0$ for $t \leq 1$. Therefore 1 is the maximum point for $\mathcal{H}$, that is $J(u) = \max_{t \geq 0} J(tu)$. Next we define the map $\nu : [0,1] \to E$ as $\nu(t) = (t_0 u)t$, where $t_0 > 1$ satisfies $J(t_0 u) < 0$. This map is well-defined due to Lemma 3.2. So $\nu \in \Gamma$. Hence
\[
b_* \leq \max_{0 \leq t \leq 1} J(\nu(t)) \leq \max_{0 \leq t \leq 1} J(tu) = J(u).
\]
Since $u \in \mathcal{N}$ is arbitrary, we get $b_* \leq \alpha_*$. Therefore
\[
\inf_{u \in \mathcal{N}} J(u) = \alpha_* = b_* = J(\nu_*).
\]

Case II: $a > 0$. By replacing Remark 3.11 with Remark 3.22 in (3.43) and arguing in a similar way as in Case I we can conclude the proof.

3.3 Proof of Theorem 1.9

To prove the Theorem 1.9 we need the Fountain theorem of Bartsch 41, Theorem 2.5; (see also 42, Theorem 3.6). The next lemma is due to 43.
Lemma 3.6. Let $X$ be a reflexive and separable Banach space. Then there are $\{e_n\} \subset X$ and $\{f^*_n\} \subset X^*$ such that

$$X = \text{span}\{e_n : n = 1, 2, 3, \ldots\}, \quad X^* = \text{span}\{f^*_n : n = 1, 2, 3, \ldots\},$$

and

$$\langle f^*_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us denote

$$X_n = \text{span}\{e_n\}, \quad Y_k = \bigoplus_{n=1}^{k} X_n \quad \text{and} \quad Z_k = \bigoplus_{n=k}^{\infty} X_n.$$  \hspace{1cm} (3.46)

Now we recall the following Fountain theorem from [34].

Theorem 3.7 (Fountain theorem). Assume that $J \in C^1(E, \mathbb{R})$ satisfies the Cerami condition $(C)_c$ and $J(-u) = J(u)$. If for each sufficiently large $k \in \mathbb{N}$, there exists $g_k > \delta_k > 0$ such that

(B1) $b_k := \inf \{J(u) : u \in Z_k, \|u\|_E = \delta_k\} \to +\infty$, as $k \to +\infty$,

(B2) $a_k := \max \{J(u) : u \in Y_k, \|u\|_E = g_k\} \leq 0$.

then $J$ has a sequence of critical points $\{u_k\}$ such that $J(u_k) \to +\infty$.

Proof of Theorem 3.7: For the reflexive, separable Banach space $E$, define $Y_k$ and $Z_k$ as in (3.46). Now $J$ satisfies Cerami condition $(C)_c$ for all $c \in \mathbb{R}$, thanks to Lemma 3.3 and $J$ is even. So to prove the Theorem 1.9 it remains to verify conditions (B1) – (B2).

Verification of (B1): For $k$ large enough let us denote

$$\alpha_k = \sup_{u \in Z_k, \|u\|_E = 1} \|u\|_{L^\gamma(\Omega)},$$  \hspace{1cm} (3.47)

where $\gamma \in C_+(\overline{\Omega})$ such that $1 < \gamma(x) < p^*_n(x)$ for all $x \in \overline{\Omega}$. Then we have

$$\lim_{k \to +\infty} \alpha_k = 0.$$  \hspace{1cm} (3.48)

If not, supposing to the contrary, there exist $\epsilon_0 > 0, k_0 \geq 0$ and a sequence $\{u_k\}$ in $Z_k$ such that

$$\|u_k\|_E = 1 \quad \text{and} \quad \|u_k\|_{L^\gamma(\Omega)} \geq \epsilon_0$$

for all $k \geq k_0$. Since $\{u_k\}$ is bounded in $E$, there exists $u_0 \in E$ such that up to a subsequence, still denoted by $\{u_k\}$, we have $u_k \rightharpoonup u_0$ in $E$ as $k \to +\infty$ and

$$\langle f^*_j, u_0 \rangle = \lim_{k \to +\infty} \langle f^*_j, u_k \rangle = 0$$

for $j = 1, 2, 3, \ldots$. Thus we have $u = 0$. Furthermore, using Theorem 2.10 we obtain

$$\epsilon_0 \leq \lim_{k \to +\infty} \|u_k\|_{L^\gamma(\Omega)} = \|u_0\|_{L^\gamma(\Omega)} = 0,$$

which is a contradiction to the fact $\epsilon_0 > 0$. Hence (3.48) holds true. Let $u \in Z_k$ with $\|u\|_E > 1$. By using Remark 1.1 (or Remark 1.2) and Proposition 2.7 we have

$$J(u) \geq \frac{M(1)}{(p^+)^{\theta}} \|\rho(u)\|^\theta - \Psi(u) \geq \frac{M(1)}{(p^+)^{\theta}} \|u\|_{L_{\gamma}(\Omega)}^{\theta p^{-}} - \Psi(u).$$  \hspace{1cm} (3.49)

Now we get

$$\Psi(u) \leq C \left( \left\|u\right\|^{2}_{L_\gamma^-(\Omega)} + \left( \left\|u\right\|^{2r^-}_{L^r(\Omega)} + \left\|u\right\|^{2r^+}_{L^r(\Omega)} \right) \right)$$

$$\quad + \left\|u\right\|^{2}_{L_{\gamma}^+(\Omega)} + \left( \left\|u\right\|^{2r^-}_{L^r(\Omega)} + \left\|u\right\|^{2r^+}_{L^r(\Omega)} \right) \right)$$

$$\leq 2C \left\{ \left\|u\right\|^{2}_{E} \alpha^-_k + \left( \left\|u\right\|^{2r^-}_{E} \alpha^-_k \right) + \left\|u\right\|^{2r^+}_{E} \alpha^+_k \right\} \leq \tilde{C} \alpha_k \|u\|_{L_{\gamma}^+(\Omega)}^{2r^+},$$  \hspace{1cm} (3.50)
where $\tilde{C}$ is a positive constant. Thus (3.49) and (3.50) give us

$$J(u) \geq \frac{M(1)}{(p^+)^\theta} \|u\|_{E}^{\theta p^+} - \tilde{C} \alpha_k \|u\|_{E}^{2 r^+}. \quad (3.51)$$

Consider the real function $G : \mathbb{R} \to \mathbb{R}$,

$$G(t) = \frac{M(1)}{(p^+)^\theta} t^{\theta p^+} - \tilde{C} \alpha_k t^{2 r^+}.$$ 

Then from elementary calculus it follows that $G$ attains its maximum at

$$\delta_k = \left( \frac{M(1) \theta p^-}{2 r^+(p^+)^\theta \tilde{C} \alpha_k} \right)^{1/(2 r^+ - \theta p^-)}.$$

Therefore the maximum value of $G$ is given by

$$G(\delta_k) = \frac{M(1)}{(p^+)^\theta} \left( \frac{M(1) \theta p^-}{2 r^+(p^+)^\theta \tilde{C} \alpha_k} \right)^{\theta p^-/(2 r^+ - \theta p^-)} - \tilde{C} \alpha_k \left( \frac{M(1) \theta p^-}{2 r^+(p^+)^\theta \tilde{C} \alpha_k} \right)^{2 r^+/(2 r^+ - \theta p^-)} \left( 1 - \frac{\theta p^-}{2 r^+} \right).$$

Since $\theta p^- < 2 r^+$ and $\alpha_k \to 0$ as $k \to +\infty$, we have

$$G(\delta_k) \to +\infty \text{ as } k \to +\infty. \quad (3.52)$$

Again using (3.48), we get $\delta_k \to +\infty$ as $k \to +\infty$. Thus for $u \in Z_k$ with $\|u\|_E = \delta_k$, combining (3.51) and (3.52), it readily follows that as $k \to +\infty$

$$b_k = \inf_{u \in Z_k, \|u\|_E = \delta_k} J(u) \to +\infty.$$

**Verification of (B2)**: Due to the presence of Choquard type nonlinearity here we will use an indirect argument. Suppose assertion (B2) of Fountain theorem does not hold true for some given $k$. Then there exists a sequence $\{u_n\} \subset Y_k$ such that

$$\|u_n\|_E \to +\infty, \quad J(u_n) \geq 0. \quad (3.53)$$

Let us take $w_n := \frac{u_n}{\|u_n\|_E}$, then $w_n \in E$ and $\|w_n\|_E = 1$. Since $Y_k$ is of finite dimension, there exists $w \in Y_k \setminus \{0\}$ such that up to a subsequence, still denoted by $\{w_n\}$, $w_n \to w$ strongly and $w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^N$ as $n \to +\infty$. If $u(x) \neq 0$ then $|u_n(x)| \to +\infty$ as $n \to +\infty$. Similar to (3.17), it follows that for each $x \in \Omega$

$$\left( \int_\Omega \frac{F(y, u_n(y))|w_n(y)|^{\theta p^+}}{|y|^{\mu(x,y)} |u_n(y)|^{\theta p^+}} dy \right) \frac{F(x, u_n(x))}{|u_n(x)|^{\theta p^+}} |w_n(x)|^{\theta p^+} \to +\infty. \quad (3.54)$$

Thus using (3.53), (3.54) with Remark 1.1 and applying Fatou’s lemma, as $n \to \infty$

$$\frac{\Psi(u_n)}{\|u_n\|_{E}^{\theta p^+}} = \frac{1}{2} \int_\Omega \left( \int_\Omega \frac{F(y, u_n(y))|w_n(y)|^{\theta p^+}}{|y|^{\mu(x,y)} |u_n(y)|^{\theta p^+}} dy \right) \frac{F(x, u_n(x))}{|u_n(x)|^{\theta p^+}} |w_n(x)|^{\theta p^+} \frac{dx}{|x|^\mu} \to +\infty. \quad (3.55)$$

Since $\|u_n\|_E > 1$ for large $n$, using Remark 1.1 (or Remark 1.2, Proposition 2.7 and (3.55), we deduce as $n \to +\infty$

$$J(u_n) \leq \frac{M(1)}{(p^+)^\theta} \|u_n\|_{E}^{\theta p^+} - \Psi(u_n) = \left( \frac{M(1)}{(p^+)^\theta} - \frac{1}{\|u_n\|_{E}^{\theta p^+}} \Psi(u_n) \right)\|u_n\|_{E}^{\theta p^+} \to -\infty,$$

which contradicts (3.53). Thus for sufficiently large $k$ we can have $\delta_k > b_k > 0$ such that for $u \in Y_k$ with $\|u\|_E = \delta_k$ the assertion (B2) follows.
3.4 Proof of Theorem 1.10

To prove the Theorem 1.10 we need the Bartsch-Willem Dual fountain theorem (see [42, Theorem 3.18]). Since $E$ is reflexive separable Banach space, using Lemma 3.6 we can define $Y_k$ and $Z_k$ appropriately.

**Definition 3.8.** We say that $J$ satisfies the $(C)^*_c$ condition (with respect to $Y_k$) if any sequence $\{u_k\}$ in $E$ with $u_k \in Y_k$ such that

$$J(u_k) \to c \quad \text{and} \quad \|J'_{|Y_k}(u_k)\|_{E^*}(1 + \|u_k\|_E) \to 0, \quad \text{as} \quad k \to +\infty$$

contains a subsequence converging to a critical point of $J$, where $E^*$ is the dual of $E$.

**Theorem 3.9** (Dual fountain Theorem). Let $J \in C^1(E, \mathbb{R})$ satisfy $J(-u) = J(u)$. If for each $k \geq k_0$ there exist $\varrho_k > \delta_k > 0$ such that

\begin{align*}
(A1) \quad & a_k = \inf \{J(u) : u \in Z_k, \|u\|_E = \varrho_k\} \geq 0; \\
(A2) \quad & b_k = \sup \{J(u) : u \in Y_k, \|u\|_E = \delta_k\} < 0; \\
(A3) \quad & d_k = \inf \{J(u) : u \in Z_k, \|u\|_E \leq \varrho_k\} \to 0 \quad \text{as} \quad k \to +\infty; \\
(A4) \quad & J \text{satisfies the (C)}^*_c \text{ condition for every} \ c \in [d_{k_0}, 0].
\end{align*}

Then $J$ has a sequence of negative critical values converging to 0.

**Remark 3.10.** Here we would like to mention that in [42], assuming that $J$ satisfies $(PS)^*_c$ condition the Dual fountain theorem is proved using Deformation theorem which is still valid under Cerami condition. Therefore we see that like many critical point theorems the Dual fountain theorem is true under $(C)^*_c$ condition.

**Lemma 3.11.** Suppose that the hypotheses in Theorem 1.10 hold, then $J$ satisfies the $(C)^*_c$ condition.

**Proof.** Let $c \in \mathbb{R}$ and the sequence $\{u_k\}$ in $E$ be such that $u_k \in Y_k$ for all $k \in \mathbb{N}$, $J(u_k) \to c$ and $\|J'_{|Y_k}(u_k)\|_{E^*}(1 + \|u_k\|_E) \to 0$, as $k \to +\infty$. Therefore, we have

$$c = J(u_k) + o_k(1) \quad \text{and} \quad \langle J'(u_k), u_k \rangle = o_k(1).$$

Analogously to the proof of Lemma 3.4 we can show that $\{u_k\}$ is bounded in $E$. Hence there exists a subsequence, still denoted by $\{u_k\}$, and $u \in E$ such that $u_k \rightharpoonup u$ weakly in $E$ as $k \to +\infty$. On the other hand, Lemma 3.7 implies $E = \bigcup_{k \geq 1} Y_k = \text{span}\{e_k : k \geq 1\}$ and thus we can choose $v_k \in Y_k$ such that $v_k \to u$ strongly in $E$ as $k \to +\infty$. Therefore, using the facts $J'_{|Y_k}(u_k) \to 0$ and $u_k - v_k \to 0$ in $Y_k$, (see [34, Proposition 3.5]), we achieve

$$\lim_{k \to +\infty} \langle J'(u_k), u_k - u \rangle = \lim_{k \to +\infty} \langle J'(u_k), u_k - v_k \rangle + \lim_{k \to +\infty} \langle J'(u_k), v_k - u \rangle = 0.$$  \hspace{1cm} (3.59)

Again recalling the proof of Lemma 3.4 we can deduce $u_k \rightharpoonup u$ strongly in $E$ as $k \to +\infty$. Then, we conclude that $J$ satisfies the $(C)^*_c$ condition. Thus, we obtain that $J'(u_k) \to J'(u)$ as $k \to +\infty$. Let us prove $J'(u) = 0$. Indeed, taking $\omega_j \in Y_j$, for $k \geq j$, we have

$$\langle J'(u), \omega_j \rangle = \lim_{k \to +\infty} \left[ \langle J'(u) - J'(u_k), \omega_j \rangle + \langle J'(u_k), \omega_j \rangle \right]$$

$$= \lim_{k \to +\infty} \left[ \langle J'(u) - J'(u_k), \omega_j \rangle + \langle J'_{|Y_k}(u_k), \omega_j \rangle \right] = 0.$$

Therefore, $J'(u) = 0$ in $E^*$ and hence $J$ satisfies the $(C)^*_c$ condition for every $c \in \mathbb{R}$. \hfill \Box

**Proof of Theorem 1.10** For the reflexive, separable Banach space $E$, define $Y_k$ and $Z_k$ as in 3.40. $J$ is even and Lemma 3.11 ensures that $J$ satisfies Cerami condition $(C)^*_c$ for all $c \in \mathbb{R}$. So to prove Theorem 1.10 it is enough to verify conditions (A1) – (A3).

**Verification of (A1):** For all $u \in Z_k$ with $\|u\|_E < 1$, arguing in a similar fashion as (3.50) we can derive

$$\Psi(u) \leq C_d \alpha_k \|u\|_E$$  \hspace{1cm} (3.56)
which together with Remark 1.1 (or Remark 1.2) and Proposition 2.7 imply

\[ J(u) \geq \frac{M(1)}{(p^+)^\theta} \|u\|_{E}^{\theta p^+} - C_4 \alpha_k \|u\|_{E}, \tag{3.57} \]

where \( C_4 > 0 \) is some constant. Let us choose \( \varrho_k = \left( (\theta p^+ \gamma C_4 \alpha_k / M(1))^{1/(\theta p^+ - 1)} \right) \). Since \( \theta p^+ > 1 \), \( \tag{3.47} \) yields that

\[ \varrho_k \to 0 \quad \text{as} \quad k \to +\infty. \tag{3.58} \]

Thus for \( u \in \mathcal{Z}_k \) with \( \|u\|_E = \varrho_k \) and for sufficiently large \( k \), from (3.57) we get \( J(u) \geq 0 \).

**Verification of (A2):** Suppose assertion (A2) of Dual fountain theorem does not hold true for some given \( k \). Then there exists a sequence \( \{u_n\} \subset Y_k \) such that

\[ \|u_n\|_E \to +\infty, \quad J(u_n) \geq 0. \tag{3.59} \]

Now arguing similarly as in the proof of assertion (B2) of Theorem 3.7, we get (3.54) and (3.55) which together with Remark 1.1 (or Remark 1.2) and Proposition 2.7 imply that as \( n \to \infty \)

\[ J(u_n) \leq \frac{M(1)}{(p^-)^\theta} \|u_n\|_{E}^{\theta p^-} - \frac{1}{\|u_n\|_{E}^{\theta p^-}} \Psi(u_n) \|u_n\|_{E}^{\theta p^-} \to -\infty. \]

Hence we get a contradiction to (3.59). Thus there exists \( k_0 \in \mathbb{N} \) such that for all \( k \geq k_0 \) we have

\[ 1 > \varrho_k > \delta_k > 0 \quad \text{such that for} \quad u \in Y_k \quad \text{with} \quad \|u\|_E = \delta_k \quad \text{the assertion (A2) follows.} \]

**Verification of (A3):** Since \( Y_k \cap \mathcal{Z}_k \neq \emptyset \), we get \( d_k \leq b_k < 0 \). Now for \( u \in \mathcal{Z}_k \), \( \|u\|_E \leq \varrho_k \) using (3.56), we have

\[ J(u) \geq -C_4 \alpha_k \|u\|_E \geq -C_4 \alpha_k \varrho_k. \]

Therefore using (3.47) and (3.58), we obtain

\[ d_k \geq -C_4 \alpha_k \varrho_k \to 0 \quad \text{as} \quad k \to \infty. \]

Since \( d_k < 0 \), we finally conclude \( \lim_{k \to \infty} d_k = 0 \). Thus the proof of the theorem is complete.

\[ \square \]

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**References**

[1] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (8) (2007) 1245–1260.

[2] G. M. Bisci, V. D. Rădulescu, R. Servadei, Variational methods for nonlocal fractional problems, vol. 162, Cambridge University Press, 2016.

[3] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012) 521–573.

[4] L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka, Lebesgue and Sobolev spaces with variable exponents, Springer-Verlag, Heidelberg, 2011.

[5] V. D. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, Nonlinear Anal. 121 (2015) 336–369.
[6] V. D. Rădulescu, D. Repovš, Partial differential equations with variable exponents: variational methods and qualitative analysis, vol. 9, CRC press, 2015.

[7] S. N. Antontsev, J. Rodrigues, On stationary thermo-rheological viscous flows, Ann. Univ. Ferrara Sez. VII Sci. Mat. 52 (1) (2006) 19–36.

[8] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math. 66 (4) (2006) 1383–1406.

[9] U. Kaufmann, J. D. Rossi, R. E. Vidal, Fractional Sobolev spaces with variable exponents and fractional \( p(x) \)-Laplacians, Electron. J. Qual. Theory Differ. Equ. 76 (2017) 1–10.

[10] A. Bahrouni, Comparison and sub-supersolution principles for the fractional \( p(x) \)-Laplacian, J. Math. Anal. Appl. 458 (2) (2018) 1363–1372.

[11] A. Bahrouni, V. D. Rădulescu, On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent, Discrete Contin. Dyn. Syst. Ser. S 11 (3) (2018) 379.

[12] K. Ho, Y. H. Kim, A-priori bounds and multiplicity of solutions for nonlinear elliptic problems involving the fractional \( p(\cdot) \)-Laplacian, Nonlinear Anal. 188 (2019) 179–201.

[13] M. Xiang, B. Zhang, D. Yang, Multiplicity results for variable-order fractional Laplacian equations with variable growth, Nonlinear Anal. 178 (2019) 190–204.

[14] R. Biswas, S. Tiwari, Variable order nonlocal Choquard problem with variable exponents, Complex Var. Elliptic Equ. DOI [10.1080/17476933.2020.1751136].

[15] S. Pekar, Untersuchungen über die Elektronentheorie der Kristalle, Akademie-Verlag, Berlin, 1954.

[16] R. Penrose, Quantum computation, entanglement and state reduction, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. 356 (1743) (1998) 1927–1939.

[17] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, S. Stringari, Theory of Bose-Einstein condensation in trapped gases, Rev. Modern Phys. 71 (3) (1999) 463.

[18] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, Studies in Appl. Math. 57 (2) (1977) 93–105.

[19] P. Lions, The Choquard equation and related questions, Nonlinear Anal. 4 (6) (1980) 1063–1072.

[20] V. Moroz, J. Van Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, Trans. Am. Math. Soc. 367 (2015) 6557–6579.

[21] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: Hardy–Littlewood–Sobolev critical exponent, Commun. Contemp. Math. 17 (05) (2015) 1550005.

[22] D. Goel, V. D. Rădulescu, K. Sreenadh, Coron Problem for Nonlocal Equations Involving Choquard Nonlinearity, Adv. Nonlinear Stud. 20 (1) (2020) 141 – 161.

[23] D. Goel, K. Sreenadh, Critical growth elliptic problems involving Hardy-Littlewood-Sobolev critical exponent in non-contractible domains, Adv. Nonlinear Anal. 9 (1) (2019) 803–835.

[24] D. Wu, Existence and stability of standing waves for nonlinear fractional Schrödinger equations with Hartree type nonlinearity, J. Math. Anal. Appl. 411 (2) (2014) 530–542.

[25] T. Mukherjee, K. Sreenadh, Fractional Choquard equation with critical nonlinearities, NoDEA Nonlinear Differential Equations Appl. 24 (6) (2017) 63.

[26] Z. Gao, X. Tang, S. Chen. Ground state solutions of fractional Choquard equations with general potentials and nonlinearities, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM 113 (3) (2019) 2037–2057.
[27] C. O. Alves, L. S. Tavares, A Hardy–Littlewood–Sobolev-Type Inequality for Variable Exponents and Applications to Quasilinear Choquard Equations Involving Variable Exponent, Mediterr. J. Math. 16 (2) (2019) 55.

[28] D. Lii, A note on Kirchhoff-type equations with Hartree-type nonlinearities, Nonlinear Anal. 99 (2014) 35–48.

[29] A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, Nonlinear Anal. 94 (2014) 156–170.

[30] P. Pucci, M. Xiang, B. Zhang, Existence results for Schrödinger–Choquard–Kirchhoff equations involving the fractional $p$–Laplacian, Adv. Calc. Var. 12 (3) (2019) 253–275.

[31] S. Liang, V. D. Rădulescu, Existence of infinitely many solutions for degenerate Kirchhoff-type Schrödinger–Choquard equations, Electron. J. Differential Equations 2017 (230) (2017) 1–17.

[32] L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer-type problem set on $\mathbb{R}^N$, Proc. R. Soc. Edinb. Sect. A. 129 (4) (1999) 787–809.

[33] S. Liu, S. J. Li, Infinitely many solutions for a superlinear elliptic equation, Acta Math. Sinica (Chin. Ser.) 46 (4) (2003) 625–630.

[34] C. O. Alves, On superlinear $p(x)$-Laplacian equations in $\mathbb{R}^N$, Nonlinear Anal. 73 (2010) 2566–2579.

[35] G. Li, V. D. Rădulescu, D. D. Repovš, Q. Zhang, Nonhomogeneous Dirichlet problems without the Ambrosetti-Rabinowitz condition, Topol. Methods Nonlinear Anal. 51 (1) (2018) 55–77.

[36] S. Liu, On ground states of superlinear $p$–Laplacian equations in $\mathbb{R}^N$, J. Math. Anal. Appl. 361 (1) (2010) 48–58.

[37] X. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. 263 (2) (2001) 424–446.

[38] J. Giacomoni, S. Tiwari, G. Warnault, Quasilinear parabolic problem with $p(x)$–Laplacian: existence, uniqueness of weak solutions and stabilization, NoDEA Nonlinear Differential Equations Appl. 23 (3) (2016) 24.

[39] H. Brézis, Functional analysis, Sobolev spaces and partial differential equations, Springer, New York, 2010.

[40] M. Willem, Functional Analysis: Fundamentals and Applications, vol. 14, Birkhäuser, Basel, 2013.

[41] T. Bartsch, Infinitely many solutions of a symmetric Dirichlet problem, Nonlinear Anal. 20 (10) (1993) 1205–1216.

[42] M. Willem, Minimax theorems, vol. 24, Birkhäuser, Boston, 1996.

[43] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, Banach Space Theory: the Basis for Linear and Nonlinear Analysis, Springer, New York, 2011.