LIPSCHITZ EXTENSION CONSTANTS EQUAL PROJECTION CONSTANTS

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Dedicated to the memory of Gert Kjægård Pedersen

Abstract. For a Banach space \( V \) we define its Lipschitz extension constant, \( \mathcal{L}E(V) \), to be the infimum of the constants \( c \) such that for every metric space \( (Z, \rho) \), every \( X \subset Z \), and every \( f : X \to V \), there is an extension, \( g \), of \( f \) to \( Z \) such that \( L(g) \leq cL(f) \), where \( L \) denotes the Lipschitz constant. The basic theorem is that when \( V \) is finite-dimensional we have \( \mathcal{L}E(V) = PC(V) \) where \( PC(V) \) is the well-known projection constant of \( V \). We obtain some direct consequences of this theorem, especially when \( V = M_n(\mathbb{C}) \). We then apply known techniques for calculating projection constants, involving averaging of projections, to calculate \( \mathcal{L}E(M_n(\mathbb{C}))^{sa} \). We also discuss what happens if we also require that \( \|g\|_\infty = \|f\|_\infty \).

In my exploration of the relationship between vector bundles and Gromov–Hausdorff distance [20] I need to be able to extend matrix-valued functions from a closed subset of a compact metric space to the whole metric space, with as little increase of the Lipschitz constant as possible. There is a substantial literature concerned with extending Lipschitz functions, but I have had difficulty finding there the facts which I need. The purpose of this largely expository paper is to describe and employ a very strong relationship between the Lipschitz extension problem and what is referred to as the “projection constant” for finite-dimensional Banach spaces. This permits us to bring to bear on the Lipschitz extension problem the quite substantial literature concerning projection constants. This then provides the facts that I need, as well as other interesting facts.

In Section 1 we introduce what we call the Lipschitz extension constant, \( \mathcal{L}E(V) \), of a Banach space \( V \). I have not found exactly this definition in the literature, although there are definitions very close to it. We also recall the well-known definition of the projection constant, \( PC(V) \), of a Banach space \( V \). The basic theorem is that if \( V \) is finite-dimensional, then \( \mathcal{L}E(V) = PC(V) \). I have not found this theorem stated in the literature, probably because \( \mathcal{L}E(V) \) is not defined in the literature, but I am told that this theorem is well-known to specialists on the geometry of Banach spaces.

In Section 2 we give the proof that \( \mathcal{L}E(V) \leq PC(V) \), while in Section 3 we give the proof that \( PC(V) \leq \mathcal{L}E(V) \), thus proving the basic theorem. We also show that restricting attention to compact metric space, or to finite metric spaces, does not change this relation with the projection constant.

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In Section 4 we give some consequences of the basic theorem that come directly from using facts about projection constants that are available in the literature. One of these consequences is that \( \mathcal{L}(C) = 4/\pi \). Another of these consequences is the formula for \( \mathcal{L}(M_n(C)) \). Neither of these consequences seems to have been recorded in the literature before.

However, what I specifically need for my exploration of vector bundles and Gromov–Hausdorff distance is \( \mathcal{L}(M_n(C)^{sa}) \), where \( M_n(C)^{sa} \) denotes the Banach space of self-adjoint matrices in \( M_n(C) \) with the operator norm. I have not seen how to obtain this directly from facts stated in the literature. But in Sections 5 and 6 we discuss known techniques for dealing with projection constants, involving subspaces \( V \) of \( C(M) \) for \( M \) compact, and averaging of projections when there is a sufficiently large group of isometries present. In Section 7 we then use these techniques to show that for any \( n \geq 1 \) we have

\[
\mathcal{L}(M_n(C)^{sa}) = 2n\left(\frac{n}{n+1}\right)^{n-1} - 1.
\]

Finally, in Section 8 we use radial retractions to discuss extending Lipschitz functions without increasing their supremum norm.

I am deeply indebted to Assaf Naor for patiently answering my occasional emailed questions about this topic over the course of a number of months. He gave me important suggestions and brought to my attention important facts in the literature. I am equally deeply indebted to William B. Johnson for his comments on the first version of this paper and for patiently answering my subsequent emailed questions. He gave me further important suggestions and brought to my attention further important facts in the literature. I give some specific acknowledgments of their help at various points later in the paper. Much of this paper consists of little more than putting together the items in the literature that they pointed out to me, and so this paper must be considered largely expository.

For the convenience of the reader I have included a number of the arguments that appear in the literature, and I have tried to formulate them in a relatively constructive form. The audience that I have had in mind when writing this paper consists of topologists and geometers who may read my paper [20] and would like to gain an understanding of the facts about extending Lipschitz functions that I use there. More generally, this paper can be considered to be an advertisement, detectable by MathSciNet and Google searches, that mathematicians who discover that they need to extend vector-valued Lipschitz functions can find a body of facts in the literature on the geometry of Banach spaces which may be quite useful to them.

1. The definitions and the main theorem

Throughout this paper \((Z, \rho)\) will denote a metric space, and \( X \) will denote a closed subset of \( Z \) with the metric from \( \rho \). Throughout this section we will assume that our Banach spaces are over \( \mathbb{R} \) unless the contrary is indicated, since for the Lipschitz extension problem it is irrelevant whether they are over \( \mathbb{R} \) or \( \mathbb{C} \). We will let \( V \) denote a Banach space, often finite-dimensional. Let \( f \) be a function from \( X \) to \( V \). Its Lipschitz constant, \( L(f) \), is defined by

\[
L(f) = \sup\{\|f(x) - f(y)\|/\rho(x, y) : x, y \in X, x \neq y\}.
\]
It can easily happen that $L(f) = +\infty$. We define the Lipschitz constant of a function from $Z$ to $V$ similarly. In much of the discussion below we do not actually need to assume that $X$ is closed, since if $L(f) < \infty$ then $f$ extends to the closure of $X$ with no increase in Lipschitz constant.

In general it is not possible to extend to $Z$ a function from $X$ to $V$ without increasing its Lipschitz constant. I have not found the following definition in the literature, although there are definitions very close to it, such as $e(Y, Z)$ defined in section 1 of [13].

**Definition 1.1.** For a Banach space $V$ we let $\mathcal{LE}(V)$ denote the infimum of the constants $c$ such that for any metric space $(Z, \rho)$ and any $X \subseteq Z$, and any function $f : X \to V$, there is an extension, $g$, of $f$ to $Z$ such that $L(g) \leq cL(f)$. If no such constant $c$ exists, then we set $\mathcal{LE}(V) = +\infty$. We call $\mathcal{LE}(V)$ the *Lipschitz extension constant* of $V$. We define $\mathcal{LE}_c(V)$ much as above but using only metric spaces $Z$ that are compact, and we define $\mathcal{LE}_f(V)$ much as above but using only metric spaces that are finite sets.

Of course $\mathcal{LE}_f(V) \leq \mathcal{LE}_c(V) \leq \mathcal{LE}(V)$.

**Proposition 1.2.** Let $V$ be a finite-dimensional Banach space. Then $\mathcal{LE}(V) < \infty$. Furthermore, if $Z$ is compact then the infimum in the above definition is actually achieved, that is, for all $f$ there is an extension $g$ such that $L(g) \leq (\mathcal{LE}(V))L(f)$.

**Proof.** We use McShane’s theorem [13] which states that every $\mathbb{R}$-valued function on $X$ can be extended to $Z$ with no increase in the Lipschitz constant. We review its proof in the next section. Let $\{b_j\}$ be a basis for $V$ with $\|b_j\| = 1$ for all $j$, and let $\{\varphi_j\}$ be the dual basis. Suppose that we are given $f : X \to V$. Set $f_j = \varphi_j \circ f$ for each $j$. For each $j$ choose, by McShane’s theorem, an extension $g_j$ of $f_j$ to $Z$ such that $L(g_j) = L(f_j)$. Define $g : Z \to V$ by $g(z) = \sum g_j(z)b_j$. Then $g$ is an extension of $f$ to $Z$, and

$$L(g) \leq \sum L(g_j) \leq (\sum \|\varphi_j\|)L(f).$$

Note that $\sum \|\varphi_j\|$ is independent of $Z, X$ and $f$. Thus $\mathcal{LE}(V) < \infty$. (In fact, by Auerbach’s lemma [28] $\{b_j\}$ can be chosen such that $\|\varphi_j\| = 1$ for all $j$, so that $L(g) \leq nL(f)$ where $n = \dim(V)$, and so $\mathcal{LE}(V) \leq n$.)

Suppose now that $Z$ is compact. Choose some $c_1$ with $\mathcal{LE}(V) < c_1 < \infty$. By the Arzela–Ascoli theorem one sees easily that, for a given $f : X \to V$ with $L(f) < \infty$, the set of its extensions $g : Z \to V$ for which $L(g) \leq c_1L(f)$ forms a sup-norm compact subset of the functions from $Z$ to $V$. Furthermore $L$, as a function on the set of Lipschitz functions, can easily be verified to be lower semi-continuous for the sup-norm. Thus there will be at least one extension $g$ for which $L(g)$ is minimal. It is then easily seen that $L(g) \leq (\mathcal{LE}(V))L(f)$. $\square$

We now turn to the topic of projection constants [6, 10, 24, 9].

**Definition 1.3.** For a Banach space $V$ we let $\mathcal{PC}(V)$ denote the infimum of the constants $c$ such that for any Banach space $W$ into which $V$ is isometrically embedded there is a (linear) projection $P$ from $W$ onto $V$ such that $\|P\| \leq c$. If no such constant $c$ exists then we set $\mathcal{PC}(V) = +\infty$.

**Proposition 1.4.** Let $V$ be a finite dimensional Banach space. Then $\mathcal{PC}(V) < \infty$. Furthermore, the infimum in the above definition is actually achieved, that is, for every $W$ containing $V$ there is a projection $P$ such that $\|P\| \leq \mathcal{PC}(V)$. 

Proof. Let \( \{b_j\} \) and \( \{\varphi_j\} \) be as in the proof of Proposition 1.2. Use the Hahn–Banach theorem to extend each \( \varphi_j \) to a linear functional, \( \tilde{\varphi}_j \), on \( W \) with \( \|\tilde{\varphi}_j\| = \|\varphi_j\| \). Define \( P : W \to V \) by \( P(w) = \sum \tilde{\varphi}_j(w)b_j \). Then \( P \) is a projection of \( W \) onto \( V \), and \( \|P\| \leq \sum \|\varphi_j\| \). Note that this bound is independent of \( W \). Thus \( PC(V) < \infty \). (From Auerbach’s lemma, stated above, we actually obtain \( PC(V) \leq n \).

For a given \( W \) it follows from the definition of \( PC(V) \) that there is a sequence, \( \{P_n\} \), of projections from \( W \) onto \( V \) such that \( \|P_n\| \leq PC(V) + 1/n \) for each \( n \). Because \( V \) is finite dimensional, the collection of operators \( T \) from \( W \) to \( V \) for which \( \|T\| \leq k \) for some fixed constant \( k \) is compact for the topology of pointwise convergence (by essentially the same proof as that of Alaoglu’s theorem [22], or by applying Alaoglu’s theorem). A limit, \( P \), of the sequence \( \{P_n\} \) is easily seen to be a projection of \( W \) onto \( V \) such that \( \|P\| \leq PC(V) \).

I thank Assaf Naor for encouraging me to expect that projection constants are relevant to the Lipschitz extension problem.

The basic theorem used in this paper is:

**Theorem 1.5.** For any finite-dimensional Banach space \( V \) we have

\[ LE(V) = PC(V) = LE_c(V) = LE_f(V). \]

Thus we see that one benefit of introducing \( LE_c(V) \) and \( LE_f(V) \) is to see that if one is working in a setting where one is only dealing with compact, or finite, metric spaces, there is nevertheless no reduction in the Lipschitz extension constant. (They are also useful for technical purposes. See Theorem 3.3.)

The above theorem is false in general for infinite-dimensional Banach spaces. In fact, for every infinite-dimensional separable Banach space \( V \) we have \( PC(V) = \infty \), for (we paraphrase some lines on page 32 of [1]) Grothendieck showed in [9] that every operator \( T \) from \( \ell^\infty(\Gamma) \) to a separable Banach space is weakly compact. (Here \( \Gamma \) is any discrete set.) If \( T \) is actually a projection onto a separable subspace of \( \ell^\infty(\Gamma) \), then from the Dunford-Pettis theorem it follows that \( T \) is actually compact, and so has finite-dimensional range. But every Banach space can be isometrically embedded into some \( \ell^\infty(\Gamma) \). On the other hand, if \( M \) is a compact and metrizable space then \( C(M) \) is separable, and we will see that \( LE_c(C(M)) = 1 \) whenever \( M \) is compact. See also Theorem 4.3 below.

In Section 8, we will consider the variation on \( LE(V) \) in which we require of the extension \( g \) not only that \( L(g) \leq cL(f) \) but also that \( \|g\|_\infty = \|f\|_\infty \), where \( \|\cdot\|_\infty \) denotes the supremum norm using the norm of \( V \).

Since \( LE_c(V) \leq LE(V) \), to prove Theorem 1.5 it suffices just to prove that \( LE(V) \leq PC(V) \) and \( PC(V) \leq LE_c(V) = LE_f(V) \).

2. The proof that \( LE(V) \leq PC(V) \)

The basic extension theorem for functions with values in \( \mathbb{R} \) goes back to McShane in 1934 [15]. We sketch its proof. It will be convenient here to denote \( \max\{r,s\} \) for \( r, s \in \mathbb{R} \) by \( r \vee s \), and to use \( \bigvee \) for the supremum of a bounded subset of \( \mathbb{R} \). We will also use these symbols for the max and supremum of a collection of \( \mathbb{R} \)-valued functions on a set.

**Theorem 2.1** (McShane). Let \( (Z, \rho) \) be a metric space, and let \( X \) be a subset of \( Z \). Let \( f \) be an \( \mathbb{R} \)-valued function on \( X \). Then there is a (non-unique) extension,
Let \( g \), of \( f \) to \( Z \) such that \( L(g) = L(f) \). If \( \|f\|_\infty < \infty \), then we can arrange that also \( \|g\|_\infty = \|f\|_\infty \).

**Proof.** (See also theorem 1.5.6 of [27].) We can assume that \( L(f) < \infty \). For each \( x \in X \) define \( h_x \) on \( Z \) by \( h_x(z) = f(x) - \rho(z, x)L(f) \). Then \( L(h_x) = L(f) \) and \( h_x|_X \leq f \), while \( h_x(x) = f(x) \). Let \( g = \sqrt{\{h_x : x \in X\}} \). If we pick some “base-point” \( x_0 \), then for any \( z \in Z \) we have

\[
f(x) - f(x_0) \leq L(f)\rho(x, x_0) \leq L(f)(\rho(z, x) + \rho(z, x_0)),
\]

so that

\[
h_x(z) = f(x) - L(f)\rho(z, x) \leq f(x_0) + L(f)\rho(z, x_0).
\]

Thus \( g(z) < \infty \). Furthermore, \( L(g) \leq L(f) \) (see, e.g., proposition 1.5.5 of [27]), while \( g|_X = f \), so that, in fact, \( L(g) = L(f) \). If \( \|f\|_\infty < \infty \), then \( h_x \leq f(x) \leq \|f\|_\infty \) for all \( x \), so that \( g \leq \|f\|_\infty \). We can then replace \( g \) by \( g \vee (-\|f\|_\infty) \) to obtain the desired extension of \( f \) such that \( \|g\|_\infty = \|f\|_\infty \). \( \square \)

The above theorem fails already for complex-valued functions. Actually, this has nothing to do with the product of complex numbers, but rather involves just the fact that, as a Banach space, \( \mathbb{C} \) is \( \mathbb{R}^2 \) with the Euclidean metric. A standard example (e.g., example 1.5.7 of [27]) consists of a 4-point space \( Z = \{\alpha, \beta, \gamma, \mu\} \), with \( \alpha, \beta \) and \( \gamma \) having distance 2 from each other and distance 1 to \( \mu \). Let \( X = \{\alpha, \beta, \gamma\} \), and let \( f : X \to \mathbb{C} \) have range exactly the 3 cube-roots of 1 (or the vertices of any equilateral triangle in \( \mathbb{R}^2 \)). Then it is easily checked that there is no extension \( g \) of \( f \) to \( Z \) such that \( L(g) = L(f) \). Basically this is due to the difference in curvature between the metric space \( Z \) and the Euclidean space \( \mathbb{R}^2 \) ([12]). (See also theorem 1.3 of [14].) Euclidean \( \mathbb{R}^2 \) is “flat”, while \( Z \) is hyperbolic-like.

However, there do exist other Banach spaces to which Theorem 2.1 generalizes, and these will be useful for the proof. Let \( \Gamma \) be a discrete set, possibly infinite, even uncountable. We let \( \ell^\infty(\Gamma) \) denote the Banach space of bounded real-valued functions on \( \Gamma \) with the supremum norm. The following is well-known. (See, e.g., lemma 1.1 of [1].)

**Proposition 2.2.** Let \( Z, \rho, X \) and \( \Gamma \) be as above. Any function, \( f \), from \( X \) to \( \ell^\infty(\Gamma) \) has an extension, \( g \), to \( Z \) such that \( L(g) = L(f) \). In particular, \( \mathcal{LE}(\ell^\infty(\Gamma)) = 1 \). If \( \|f\|_\infty < \infty \), then we can arrange that also \( \|g\|_\infty = \|f\|_\infty \).

**Proof.** For each \( \gamma \in \Gamma \) let \( f_\gamma \) denote the \( \mathbb{R} \)-valued function whose value at \( x \in X \) is \( f(x) \) evaluated at \( \gamma \). Then \( L(f_\gamma) \leq L(f) \). Let \( g_\gamma \) be an extension of \( f_\gamma \) to \( Z \) as per Theorem 2.1, so that \( L(g_\gamma) = L(f_\gamma) \). Define \( g : Z \to \ell^\infty(\Gamma) \) by \( g(z)(\gamma) = g_\gamma(z) \). Then it is easily verified that \( g \) is the desired extension. If \( \|f\|_\infty < \infty \), then \( \|f_\gamma\|_\infty \leq \|f\|_\infty \) for each \( \gamma \). Thus we can choose the above \( g_\gamma \)'s such that \( \|g_\gamma\|_\infty = \|f_\gamma\|_\infty \) for each \( \gamma \). The resulting \( g \) will then satisfy \( \|g\|_\infty = \|f\|_\infty \). \( \square \)

**Proof that** \( \mathcal{LE}(V) \leq \mathcal{P}\mathcal{C}(V) \). Let \( V \) be a finite-dimensional Banach space. Let \( \Gamma \) be a subset of the unit ball of the dual space \( V' \) such that for every \( v \in V \) we have \( \|v\| = \sup\{\|\langle v, \gamma \rangle\| : \gamma \in \Gamma\} \). For example, \( \Gamma \) can be all of the unit sphere, or a dense subset of the unit ball, or the set of extreme points of the unit ball. Then each element of \( V \) can be viewed as a function on \( \Gamma \) in the evident way, and this provides an isometric embedding of \( V \) into \( \ell^\infty(\Gamma) \). By the definition of \( \mathcal{P}\mathcal{C}(V) \) and Proposition 1.4 there is a projection \( P \) from \( \ell^\infty(\Gamma) \) onto \( V \) such that \( \|P\| \leq \mathcal{P}\mathcal{C}(V) \).
Let $Z$, $\rho$, $X$ be as earlier, and let $f : X \to V$. Through the above embedding we can view $f$ as having its values in $\ell^\infty(\Gamma)$ and this does not change $L(f)$ or $\|f\|_\infty$. Then according to Proposition 2.3 we can find a function $h : Z \to \ell^\infty(\Gamma)$ such that $h|_X = f$ while $L(h) = L(f)$ (and $\|h\|_\infty = \|f\|_\infty$ if $\|f\|_\infty < \infty$). Set $g = P \circ h$. Then $g : Z \to V$ and $g|_X = f$, while $L(g) \leq \|P\|L(h) \leq PC(V)L(f)$.

In principle, the above proof gives a constructive method for producing extensions $g$ for which $L(g) \leq LE(V)L(f)$, for a given $V$. We need only make one choice of an isometric embedding of $V$ into an $\ell^\infty(\Gamma)$, and then find one projection, $P : \ell^\infty(\Gamma) \to V$ with $\|P\| \leq PC(V)$. We can then proceed as in the second paragraph of the above proof. The basic theorem then shows that this gives $g$ with $L(g) \leq LE(V)L(f)$. In fact, the above proof shows that $LE(V) \leq \|P\|$, and so the basic theorem will imply the well-known fact that $\|P\| = PC(V)$.

Notice, however, that the above proof does not give $\|g\|_\infty = \|f\|_\infty$ when $\|f\|_\infty < \infty$. We can only conclude that $\|g\|_\infty \leq PC(V)\|f\|_\infty$. But we will see in Section 4 that we can arrange that $\|g\|_\infty = \|f\|_\infty$ at the cost of only knowing that $L(g) \leq 2PC(V)L(f)$.

3. The proof that $PC(V) \leq LE_c(V) = LE_f(V)$

The proof of the inequality is a very minor reworking of the second proof of theorem 7.2 of [11], which is descended from the proof of theorem 2 of [14]. I am indebted to Assaf Naor for telling me that the proof of theorem 7.2 of [11] was what I needed here. We give an outline of the proof. But first we remark that the fact that the inequality $PC(V) \leq LE(V)$ holds follows swiftly from corollary 1 to theorem 3 of [14]. If $V$ is embedded isometrically in a Banach space $W$, then the identity map from $V$ to itself will, by definition, have an extension, $g$, to all of $W$ such that $L(g) \leq LE(V)$. Then corollary 1 to theorem 3 of [14] implies that there is a projection, $P$, from $W$ onto $V$ such that $\|P\| \leq LE(V)$.

Here is the outline of the proof that $PC(V) \leq LE_c(V)$. We must show that whenever $V$ is isometrically embedded in some Banach space $W$, then there is a projection, $P$, from $W$ onto $V$ such that $\|P\| \leq LE_c(V)$.

Suppose first that $W$ is finite-dimensional, and that $W$ contains $V$ isometrically. For any $r \in \mathbb{R}^+$ let $B^V(r)$ denote the closed ball about 0 of radius $r$ in $V$, and similarly for $B^W(r)$. Then $B^W(3)$ is a compact metric space which contains $B^V(3)$ as a closed subset. Let $f$ be the identity map from $B^V(3)$ into $V$. By the definition of $LE_c(V)$ there is a function, $g$, from $B^W(3)$ into $V$ such that $g(v) = v$ for $v \in B^V(3)$, and $L(g) \leq (LE_c(V)L(f) = LE_c(V)$.

Next we smooth $g$ in the direction of $V$ by convolving it with a non-negative symmetric $C^\infty$ function on $V$ supported in the interior of $B^V(1)$ and of integral 1 (for some choice of translation-invariant measure on $V$). The resulting function, $h$, when viewed as defined on $B^W(2)$, satisfies $L(h) \leq L(g)$ and $h(v) = v$ for $v \in B^V(2)$. Furthermore, the derivatives of $h$ in $V$-directions exist in the interior of $B^W(2)$ and are continuous there. Now choose a subspace $U$ of $W$ which is complementary to $V$, and choose a sequence $\{\psi_n\}$ of $C^\infty$ functions on $U$ supported in the interior of $B^V(1)$, which forms an approximate $\delta$-function at 0 in $U$. Convolv $h$ by each $\psi_n$ to obtain a sequence $\{j_n\}$ of smooth $V$-valued functions, viewed as defined on $B^W(1)$, such that $L(j_n) \leq L(h) \leq LE_c(V)$ for each $n$. Let $D_0j_n$ denote the total derivative of $j_n$ at $0 \in W$, so that $D_0j_n$ is a linear operator from $W$ to
V such that $\|D_0\hat{h}_n\| \leq LE_c(V)$. Because the $\psi_n$’s form an approximate $\delta$-function, one finds that $(D_0\hat{h}_n)(v)$ converges to $v$ for each $v \in V$.

The sequence $D_0\hat{h}_n$ is contained in the ball of linear operators from $W$ to $V$ of norm no greater than $LE_c(V)$, which is compact. Thus there is a subsequence which converges to an operator $P$. It is clear from the remarks just above that $Pv = v$ for any $v \in V$, and that $\|P\| \leq LE_c(V)$. Thus $P$ is our desired projection from $W$ onto $V$.

Suppose now that $W$ is infinite-dimensional. For each finite-dimensional subspace $U$ of $W$ which contains $V$ one can choose as above a projection, $P_U$, from $U$ onto $V$ such that $\|P_U\| \leq LE_c(V)$. An argument similar to the proof of Alaoglu’s theorem, or of Proposition 1.3 above, then yields a projection, $P$, from $W$ onto $V$ such that $\|P\| \leq LE_c(V)$.

We now turn to proving that $LE_c(V) \leq LE_f(V)$, so that they are equal. Let $(Z, \rho)$ be a compact metric space, and let $X \subseteq Z$. Consider first the case in which $X$ is a finite subset. Let $f : X \to V$. Let $S$ be a countable dense subset of $Z$ containing $X$, and let $S$ be enumerated in such a way that $X = \{s_j : 1 \leq j \leq n\}$. For each positive integer $k$ let $S_k = \{s_j : 1 \leq j \leq k\}$. Then for each $k > n$ we can find, by the definition of $LE_f(V)$, a function $g_k$ from $S_k$ to $V$ that extends $f$ and is such that $L(g_k) \leq LE_f(V)L(f)$. It is easily seen that for each $k$ we have $\|g_k\|_\infty \leq r$ where

$$r = \|f\|_\infty + (\text{diameter}(Z))LE_f(V)L(f).$$

For each $k > n$ define $\hat{g}_k$ to be the extension of $g_k$ to all of $S$ which has value $0_V$ for each $s_j$ with $j > k$. Because $V$ is finite-dimensional, the set of all functions from $S$ into the $r$-ball in $V$ about $0_V$ is compact for the topology of pointwise convergence, by Tychonoff’s theorem. Since $S$ is countable this topology is metrizable. Thus there is a subsequence, say $\{\hat{g}_{k_{n}}\}$, which converges pointwise to a function, $\hat{g}$, from $S$ to $V$. It is easily seen that $L(\hat{g}) \leq LE_f(V)L(f)$. Then $\hat{g}$ extends to a function, $g$, from the completion, $Z$, of $S$, still with $L(g) \leq LE_f(V)L(f)$, and $g$ is an extension of $f$.

Suppose now that $X$ is a subset of $Z$ which is not finite. Let $T$ be a countable dense subset of $X$, and enumerate its elements as a sequence $\{t_j\}$. For each $n$ let $T_n = \{t_j : 1 \leq j \leq n\}$, and let $f_n$ be the restriction of $f$ to $T_n$. Note that $L(f_n) \leq L(f)$. As seen in the paragraph above, for each $n$ we can find an extension, $g_n$, of $f_n$ to $Z$ such that $L(g_n) \leq LE_f(V)L(f)$. Thus the sequence $\{g_n\}$ is equicontinuous. Also, $\|g_n\|_\infty \leq r$ for each $n$, where $r$ is as defined in the previous paragraph. Thus by the Arzela-Ascoli theorem the sequence is totally bounded, and so has a subsequence which converges uniformly to a $V$-valued function, $g$, on $Z$. (Actually, pointwise convergence would suffice.) It is easily seen that $g$ is an extension of $f$ such that $L(g) \leq LE_f(V)L(f)$ as desired.

4. Some Consequences

In his recent book [27], Weaver remarks just after example 1.5.7 that it seems not to be known what is the smallest constant $c$ such that any $\mathbb{C}$-valued function $f$ on a subset of a metric space can be extended to a function $g$ on $Z$ such that $L(g) \leq cL(f)$, that is, what is $LE(\mathbb{C})$? But B. Grunbaum showed in [16] that $\mathcal{P}\mathcal{C}(\mathbb{R}^2) = 4/\pi$ when $\mathbb{R}^2$ is equipped with the Euclidean norm. From Theorem 1.5 we then immediately obtain:

**Corollary 4.1.** $LE(\mathbb{C}) = 4/\pi$. 

The reader will find it an easy and entertaining exercise to apply the techniques
that we describe in Sections 5 and 6 to give a proof of this fact.

A non-obvious result of Kadec and Snobar (6; and see theorem 9.12 of [20] or
theorem III.B.10 of [28]) states that for any Banach space $V$ of real dimension $n$
we have $\mathcal{P}(V) \leq \sqrt{n}$. Thus:

**Corollary 4.2.** For any Banach space of real dimension $n$ we have $\mathcal{LE}(V) \leq \sqrt{n}$.

But the Kadec–Snobar theorem has been improved in [10], so that if, for example,
$n = 4$ then $\mathcal{P}(V) \leq (2 + 3\sqrt{6})/5 < 2$, so that the same holds for $\mathcal{LE}(V)$. We refer
the reader to that paper for upper bounds for other values of $n$.

My interest in this whole topic originated in my need for information about
$\mathcal{LE}((M_n(\mathbb{C}))^{sa})$. But one can first ask for $\mathcal{LE}(M_n(\mathbb{C}))$. I am indebted to William
B. Johnson for telling me that from theorem 5.6b of [6] one can deduce that
$\mathcal{P}(M_n(\mathbb{C})) = (\mathcal{P}(\mathbb{C}^n))^2$. Now Rutovitz [23] showed that an inequality for $\mathcal{P}(\mathbb{C}^n)$
obtained by Grunbaum [6] is actually an equality. A proof along the lines that we
will use in the next sections is given in corollary III.B.16 of [28]. One obtains:

$$\mathcal{P}(\mathbb{C}^n) = n \int_{S^n} |z_1|d\lambda(z) = \Gamma(3/2)\Gamma(n+1)/\Gamma(n+1/2) \geq (1/2)\sqrt{n\pi},$$

where $S_n$ is the unit sphere in $\mathbb{C}^n$ and $\lambda$ is the rotationally invariant measure of
mass 1 on $S^n$. We thus obtain:

**Corollary 4.3.** For each $n \geq 2$ we have

$$\mathcal{LE}(M_n(\mathbb{C})) = \left(\Gamma(3/2)\Gamma(n+1)/\Gamma(n+1/2)\right)^2 \geq (\pi/4)n.$$

It is interesting to note, in contrast, that if $D_n$ denotes the $*$-subalgebra of
diagonal matrices in $M_n(\mathbb{C})$, then $D_n$ is isometric to $\ell^2(\Gamma_n)$ where $\Gamma_n$ is a set with
$n$ points, and so $\mathcal{LE}(D_n) = 4/\pi$, as follows easily from Corollary 4.1 and the proof
of Proposition 2.2. Thus $\mathcal{LE}(D_n)$ is independent of $n$, in contrast to $\mathcal{LE}(M_n(\mathbb{C}))$.

Finally, we now give an elementary argument which gives the Kadec–Snobar upper bound for $\mathcal{LE}(M_n(\mathbb{C}))$. Specifically, for $A = M_n(\mathbb{C})$ and for each $m$ with
$0 \leq m \leq n-1$, let $A_m$ be the linear subspace of matrices $\{t_{ij}\}$ such that $t_{ij} = 0$
unless $i-j = m \mod n$. Thus $A_0$ is our earlier $D_n$, the algebra of diagonal
matrices. For each $m$, multiplication by an appropriate permutation matrix carries
$A_m$ isometrically onto $A_0$. Thus $\mathcal{LE}(A_m) = \mathcal{LE}(A_0) \leq \sqrt{2}$ by considering real and
imaginary parts (or $= 4/\pi$ by the non-elementary Corollary 4.1). But $A = \sum_{m=0}^{n-1} A_m$ (with the precise relation between the norms being obscure), and from
this we can at least obtain quickly that $\mathcal{LE}(M_n(\mathbb{C})) \leq n\sqrt{2}$ (or $\leq 4n/\pi$).

5. The usefulness of $V \subset C(M)$

We now begin our discussion of techniques that will permit us to compute
$\mathcal{LE}((M_n(\mathbb{C}))^{sa})$. A standard fact about projection constants is that if $M$ is a compact
metric space and if $V$ is a finite-dimensional subspace of $C(M)$, then $\mathcal{P}(V)$
is equal to the norm of any projection from $C(M)$ onto $V$ of minimal norm. (See,
e.g., theorem III.B.5 of [28].) We will see in the next sections how this can be used.
I am much indebted to William B. Johnson for pointing out this path to me.

Here we will give a proof of this standard fact by using $\mathcal{LE}$ and then Theorem
1.5. This gives a somewhat constructive way of producing Lipschitz extensions with
minimal increase of the Lipschitz constant. I have not seen the following theorem stated in the literature.

**Theorem 5.1.** Let $M$ be a compact space. For any compact metric space $(Z, \rho)$, any subset $X \subseteq Z$, and any function $f : X \to C(M)$ there is an extension, $g$, of $f$ from $Z$ to $C(M)$ with $L(g) = L(f)$ and $\|g\|_{\infty} = \|f\|_{\infty}$. In particular, $\mathcal{L} \mathcal{E}_c(C(M)) = 1$.

**Proof.** We can assume that $L(f) < \infty$ and that $X$ is closed. Let $M_{\text{dis}}$ denote $M$ with the discrete topology. We view $f$ as a function from $X$ to $\ell^{\infty}(M_{\text{dis}})$. Then we construct an extension, $\tilde{g}$, of $f$ from $Z$ to $\ell^{\infty}(M_{\text{dis}})$ in almost the same way as was used in the proof of Proposition 2.2 using Theorem 2.1. For each $x \in X$ we set

$$h_x(z, m) = f(x, m) - L(f)\rho(z, x)$$

for $z \in Z$ and $m \in M$. Define $\tilde{g}$, with values in $\ell^{\infty}(M_{\text{dis}})$, by $\tilde{g}(z, m) = \bigvee_x h_x(z, m)$. Then $\tilde{g}(z, m) \leq \|f\|_{\infty}$ for all $z$ and $m$, and $\tilde{g}$ is an extension of $f$ with $L(\tilde{g}) = L(f)$.

What we need to show is that $\tilde{g}(z) \in C(M)$ for all $z \in Z$. Since $X$ is compact, $\{f(x) : x \in X\}$ is a compact subset of $C(M)$, and so is equicontinuous by the Arzela-Ascoli theorem. Then $\{h_x\}_{x \in X}$ is easily seen to be an equicontinuous family of functions on $Z \times M$. It follows that $\tilde{g}$, as a function on $Z \times M$, is continuous, and thus uniformly continuous. Consequently $\tilde{g}$, as a function on $Z$, has values in $C(M)$, as needed. We can now define $g$ by $g(z) = \tilde{g}(z) \lor (-\|f\|_{\infty})$ to obtain the desired extension of $f$. (A similar idea to the above proof is indicated in remark 3.3 of [11].)

When we apply Theorem 5.1 in a way very similar to that in the second paragraph of the proof that $\mathcal{L} \mathcal{E}(V) \leq P \mathcal{C}(V)$, we quickly obtain:

**Theorem 5.2.** Let $M$ be a compact space, and let $V$ be a finite-dimensional subspace of $C(M)$. Let $P$ be a projection from $C(M)$ onto $V$. Then $\mathcal{L} \mathcal{E}_c(V) \leq \|P\|$. Thus if $P$ is a projection of minimal norm, then

$$P \mathcal{C}(V) = \mathcal{L} \mathcal{E}(V) = \mathcal{L} \mathcal{E}_c(V) = \|P\|.$$  

We remark that another approach to proving Theorem 5.2 is to use peaking partitions of the identity, as discussed in lemma 2.1 of [10], to produce in $C(M)$ isometric copies of $\ell^{\infty}(\Gamma)$ containing subspaces which converge to $V$ for Banach-Mazur distance, where the $\Gamma$’s are finite sets of increasing size.

We also remark that from theorem 6b of [11] and its proof we quickly obtain in much the same way as for Theorem 5.2:

**Theorem 5.3.** Let $(M, d)$ be any metric space, and let $C_u(M)$ be the Banach space of all bounded uniformly continuous real-valued functions on $M$. Then $\mathcal{L} \mathcal{E}(C_u(M)) \leq 37$.

See also theorem 1.6 of [11] and its proof.

6. **Averaging of projections**

Given enough symmetry and favorable circumstances, one can construct projections of minimal norm. The following proposition is due to Rudin [21], but has antecedents for the circle group. See also theorem III.B.13 of [28]. The proof is straightforward.
Proposition 6.1. Let $G$ be a compact group, and let $\alpha$ be a strongly-continuous representation of $G$ by isometries on a Banach space $W$. Let $V$ be a subspace of $W$ which is $\alpha$-invariant, and suppose that there is a projection $Q$ from $W$ onto $V$. Define an operator $P$ from $W$ to $V$ by

$$Pw = \int_G \alpha_g(Q(\alpha_g^{-1}w))dg,$$

where the Haar measure on $G$ gives $G$ measure 1. Then $P$ is a projection from $W$ onto $V$, and $\|P\| \leq \|Q\|$. Furthermore, $P$ is $\alpha$-invariant, in the sense that $\alpha_g \circ P \circ \alpha_g^{-1} = P$ for all $g \in G$.

Thus if we are in a situation in which the $\alpha$-invariant projection from $W$ onto $V$ is unique, then we will know that this projection is a projection of minimal norm. But this situation is easily described. Let $\tilde{G}$ denote the set of equivalence classes of real irreducible representations of $G$. They are all finite-dimensional. For any $\gamma \in \tilde{G}$ a suitable multiple of its character, which we denote by $p_\gamma,$ will be an idempotent in $L^1(G)$ for convolution. (See [24], especially the appendix to III.5.) For any strongly continuous representation $\alpha$ of $G$ on a Banach space $W$ the operator $\alpha_{p_\gamma}$ that is the integrated form of $p_\gamma$ will be a projection from $W$ onto a subspace, $W_\gamma$, of $W$. This subspace is the $\gamma$-isotypic component of $W$, in the sense that any irreducible $\alpha$-invariant subspace of $W$ on which the representation of $G$ gives a representation isomorphic to $\gamma$ will be contained in $W_\gamma$. The $W_\gamma$’s are disjoint, and their algebraic direct sum is dense in $W$. The kernel of the projection $\alpha_{p_\gamma}$ is the closure of the direct sum of all the other isotypic components.

Let $V$ be an $\alpha$-invariant subspace of $W$. Then it too has isotypic components, $V_\gamma,$ and $V_\gamma \subseteq W_\gamma$ for each $\gamma \in \tilde{G}$.

Definition 6.2. We will say that an $\alpha$-invariant subspace $V$ of $W$ is $\alpha$-full if $V_\gamma = W_\gamma$ for each $\gamma$ for which $V_\gamma \neq \{0\}$.

If the subspace $V$ is $\alpha$-full and if $P$ is an $\alpha$-invariant projection onto $V$, then the kernel of $P$ must contain all of the $W_\gamma$’s for which $V_\gamma \neq W_\gamma$, and the direct sum of these $W_\gamma$’s will be dense in the kernel of $P$. Thus $P$ will be the unique $\alpha$-invariant projection onto $V$. A few moments of contemplating actions by the one-element group, and then the general situation, shows that if $V$ is not $\alpha$-full then an $\alpha$-invariant projection onto $V$, if it exists, can not be unique. Putting all of this discussion together, we obtain:

Theorem 6.3. Let $\alpha$ be a strongly-continuous representation of the compact group $G$ by isometries on a Banach space $W$. Let $V$ be an $\alpha$-invariant subspace of $W$. If a projection from $W$ onto $V$ exists, then an $\alpha$-invariant projection exists. This $\alpha$-invariant projection is unique exactly if $V$ is $\alpha$-full. If $V$ is $\alpha$-full, then this $\alpha$-invariant projection is a projection of minimal norm from $W$ onto $V$.

When this theorem is combined with Theorem 5.1 we obtain:

Corollary 6.4. Let $\alpha$ be a continuous action of $G$ on a compact space $M$, and let $\alpha$ also denote the corresponding representation of $G$ on $C(M)$. Let $V$ be a finite-dimensional $\alpha$-invariant subspace of $C(M)$ which is $\alpha$-full, and let $P$ be the (unique) $\alpha$-invariant projection of $C(M)$ onto $V$. Then $\|\mathcal{L}^\alpha(V)\| = \|P\|$.

We will make good use of this corollary in the next section. From it we can also obtain a swift proof of Corollary 4.1 as follows. Let $G = T = M$ where $T$ is the...
circle group, acting on itself by translation, and so acting on $C(T)$. Embed $\mathbb{R}^2$ into $C(T)$ by sending $(a,b) \in \mathbb{R}^2$ to $f_{a,b} \in C(T)$ defined by

$$f_{a,b}(t) = a \cos t + b \sin t.$$

7. The Calculation of $\mathcal{LE}((M_n(\mathbb{C}))^{sa})$

Let $E = (M_n(\mathbb{C}))^{sa}$, with $n \geq 2$ fixed throughout this section. We equip $E$ with the operator norm. We will use the technique suggested by Corollary 6.3 to calculate $\mathcal{LE}(E)$. We let $\text{tr}$ denote the (un-normalized) trace, and define a real-valued inner product on $E$ by $\langle a,b \rangle_E = \text{tr}(ab)$. Thus every linear functional on $E$ can be represented by an element of $E$. We let $M$ denote the set of rank-one projections in $E$. It is a compact subset of $E$ (and it corresponds to the set of extreme points of the state space of $E$). We define a linear mapping, $\varphi$, from $E$ into $C(M)$ by $\varphi_a(p) = \text{tr}(ap)$ for $a \in A$ and $p \in M$. It is easily seen that $\varphi$ is isometric. (Here $C(M)$ denotes real-valued functions.) Thus by Theorem 6.2 we know that $\mathcal{LE}(E)$ is equal to the norm of a projection of minimal norm from $C(M)$ onto $\varphi(E)$.

Let $\alpha$ denote the action of $G = SU(n)$ on $E$ by conjugation, and let $\beta$ denote the representation of $G$ on $C(M)$ coming from the action of $G$ on $M$ by conjugating rank-one projections. Then for any $u \in G$, $a \in E$ and $p \in M$ we have

$$(\beta_u \varphi_a)(p) = \varphi_a(u^{-1}pu) = \text{tr}(au^{-1}pu) = \text{tr}(uau^{-1}p) = \varphi_{\alpha_u}(a)(p),$$

so that $\varphi$ is $G$-equivariant, and $\varphi(E)$ is a $G$-invariant subspace of $C(M)$.

Now as a $G$-space $E$ decomposes into the direct sum of the subspace $E_0$ of scalar multiples of the identity and the subspace $E_1$ of elements of trace 0. The latter is isomorphic as $G$-space, via multiplication by $i = \sqrt{-1}$, to $su(n)$, the Lie algebra of $G$, with the representation of $G$ on $su(n)$ being the adjoint representation (which is irreducible since $su(n)$ is a simple Lie algebra). Thus the representation of $G$ on $\varphi(E)$ is isomorphic to the direct sum of the trivial representation and the adjoint representation. Let $p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-1} \end{pmatrix}$. The stability subgroup, $H$, of $p_0$ consists of the $u$’s in $G$ of the form $u = \begin{pmatrix} \lambda & 0 \\ 0 & u' \end{pmatrix}$ with $u' \in U(n-1)$ and $\lambda = \det(u')$. Thus we can identify $H$ with $U(n-1)$. The action of $G$ on $M$ is transitive, and so we can identify $M$ with $G/H$.

The representation of $G$ on $C(M)$ can be viewed as the representation of $G$ obtained by inducing to $G$ the trivial representation of $H$ in the way described in section III.6 of [2]. The Frobenius reciprocity theorem, in the form given in proposition III.6.2 of [2], shows that the multiplicity in $C(M)$ of an irreducible representation of $G$ will equal the multiplicity of the trivial representation of $H$ in the restriction to $H$ of that irreducible representation of $G$. When the representation of $G$ on $E_1$ is restricted to $H$, its subspace of invariant vectors consists of the scalar multiples of $\begin{pmatrix} n-1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}$, and so is 1-dimensional. We then see that the adjoint representation and the trivial representation of $G$ in $C(M)$ each occur with multiplicity 1. Thus $\varphi(E)$ is the direct sum of two isotypic components of the action of $G$ on $C(M)$. Consequently $\varphi(E)$ is a full $G$-invariant subspace of $C(M)$.

From Corollary 6.3 we see that the $G$-invariant projection from $C(M)$ onto $\varphi(E)$ is
From the discussion of multiplicity given above it is clear that the vector space of 
$G$-invariant operators from $C(M)$ onto $\varphi(E)$ is 2-dimensional. The projection, 
$P_0$, of $C(M)$ onto the constant functions is given by 

$$(P_0 f)(p) = \int_M f(q) dq,$$

where we use the $G$-invariant measure on $M$ which gives $M$ measure 1. It is easily 
sen that the operator $T$ defined by 

$$(T f)(p) = \int_M \langle p, q \rangle_E f(q) dq = \langle p, \int_M q f(q) dq \rangle_E$$

has values in $\varphi(E)$ and is $G$-invariant. Thus $P$ must be a linear combination of $P_0$ 
and $T$. So we will seek $P$ in the form $(P f)(p) = \int_M K(p, q) f(q) dq$ where $K$ has the form 

$$K(p, q) = \mu + \nu \langle p, q \rangle_E,$$

where $\mu$ and $\nu$ are constants to be determined. We must of course have $P 1 = 1$. 
By considering $\int_M \varphi_n(p) dp$ and its $G$-invariance, it is easy to see that $\int_M q dq$ is 
$\alpha$-invariant. Thus it must be a multiple of $I_n$. On taking the trace we find that 
$\int q dq = n^{-1} I_n$. From $P 1 = 1$ it then follows that $\mu = 1 - n^{-1} \nu$.

To obtain a second equation for $\mu$ and $\nu$ we use the reproducing-kernel property 
of $K$. For $f, g \in C(M)$ we let $(f, g)_M$ denote their usual inner product for $L^2(M)$.
For each $p \in M$ define $K_p$ by $K_p(q) = K(p, q)$. Then the definition of $P$ can be 
rewritten as $(P f)(p) = (K_p, f)_M$. Let $\{e_j\}_{j=1}^{n^2}$ be an orthonormal basis for 
$\varphi(E) \subseteq L^2(M)$. Then $K_p = \sum (K_p, e_j)_M e_j$ for each $p \in M$. But 
$$(K_p, e_j)_M = (P e_j)(p) = e_j(p).$$

Thus 

$$K(p, q) = \sum e_j(p) e_j(q).$$

Consequently 

$$\int_M K(q, q) dq = \sum (e_j, e_j)_M = n^2.$$ 

But from the definition of $K$ it is clear that $K(q, q)$ has the constant value $\mu + \nu$, 
so that $\mu + \nu = n^2$. From this and the equation $\mu = 1 - n^{-1} \nu$ obtained earlier we 
find that $\mu = -n$ and $\nu = n(n + 1)$.

We must now determine $\|P\|$, where we now revert to the sup-norm on $C(M)$. 
Now $P$ is $G$-invariant and $G$ acts transitively on $M$, so it suffices to determine the 
norm of the linear functional $f \mapsto (P f)(p_0)$ on $C(M)$. In view of the form of $P$ 
this will be given by 

$$\int_M | - n + n(n + 1) \langle p_0, q \rangle_E | dq.$$ 

To evaluate this integral we use the following specialization to $\mathbb{C}$ of lemma 3"ii of 
[19] (or lemma 4.4 of [18]):

**Lemma 7.1.** For any continuous function $h$ on the interval $[-1, 1]$ we have 

$$\int_M h(2 \langle p, q \rangle_E - 1) dq = (n - 1) 2^{1-n} \int_{-1}^1 h(t)(1 - t)(n-2) dt.$$
But
\[-n + n(n + 1)(p, q)_E = 2^{-n}n((n - 1) + (n + 1)(2(p, q)_E - 1)),\]
and so from Lemma 7.1 we see that
\[\| P \| = 2^{-n}n(n - 1) \int_{-1}^{1} |(n - 1) + (n + 1)t|(1 - t)^{n-2}dt.\]
Straightforward calculation of this integral yields \(\| P \| = 2 - n \left( \frac{n}{n + 1} \right)^{n-1} - 1\). We thus obtain:

**Theorem 7.2.** \(\mathcal{LE}(\mathcal{M}_n(\mathbb{C})^{sa}) = 2n \left( \frac{n}{n + 1} \right)^{n-1} - 1\).

We can rewrite this formula as
\[2(n + 2 + n^{-1}) \left( 1 - \frac{1}{n + 1} \right)^{n+1} - 1\]
and notice that \(\left( 1 - \frac{1}{n + 1} \right)^{n+1}\) converges to \(e^{-1}\) as \(n \to \infty\). If we rewrite our formula in the form
\[\mathcal{LE}(\mathcal{M}_n(\mathbb{C})^{sa}) = n \omega(n),\]
we then find that \(\omega(n)\) converges to \(2e^{-1}\) as \(n \to \infty\). We also note that our formula gives the correct answer for \(n = 1\).

### 8. Preservation of the supremum norm

We mentioned earlier that the extensions of Lipschitz functions that we have discussed so far do not always preserve the supremum norms of the functions. We will now show that the norm can be preserved at the cost of no more than doubling the Lipschitz constant. In Theorems 2.1 and 5.1 we were able to use the lattice structure of \(\ell^\infty(\Gamma)\) and \(C(M)\) and its relation to the norm in order to arrange preservation of the norm, but this technique is not generally available.

The tool which we use is that of radial retractions (which are also called “radial projections”). Let \(V\) be any normed vector space over \(\mathbb{R}\) or \(\mathbb{C}\). For any \(r \in \mathbb{R}\) with \(r > 0\) define the radial retraction \(\Pi_r\), a non-linear map from \(V\) into itself, by
\[\Pi_r(v) = \begin{cases} v & \text{if } \|v\| \leq r \\ \frac{rv}{\|v\|} & \text{if } \|v\| > r. \end{cases}\]
Let \(L(\Pi_r)\) denote the Lipschitz constant of \(\Pi_r\). The first assertion of the following proposition is basically known. See [3] and [25]. I have not seen the next two assertions stated in the literature, though they can be obtained via theorem 2 of [25].

**Proposition 8.1.** For any normed vector space \(V\) we have \(L(\Pi_r) \leq 2\). If \(V = C(X)\) for any compact space \(X\) containing at least two points, then \(L(\Pi_r) = 2\). If \(V\) is a \(C^*\)-algebra of dimension at least 2, then \(L(\Pi_r) = 2\).

**Proof.** We have \(\Pi_r(v) = r\Pi_1(v^{-1}v)\) for \(v \in V\), and so \(L(\Pi_r) = L(\Pi_1)\). Thus it suffices to prove the first assertion for \(\Pi = \Pi_1\), which we now do. If \(\|v\| \leq 1\) and \(\|w\| \leq 1\) then clearly \(\|\Pi(v) - \Pi(w)\| = \|v - w\|\). Suppose that \(\|v\| \geq 1\) while \(\|w\| \leq 1\). Then
\[\|v/\|v\| - w\| \leq \|v/\|v\| - v\| + \|v - w\| = \|v\| - 1 + \|v - w\|.\]
But $\|v\| - 1 \leq \|v - w\| + \|w\| - 1 \leq \|v - w\|$, so that $\|\Pi(v) - \Pi(w)\| \leq 2\|v - w\|$. Finally, suppose that $\|v\| \geq 1$ and $\|w\| \geq 1$. By symmetry we can assume that $\|w\| \leq \|v\|$. Then $\|v/\|w\|\| \geq 1$ and $\|w/\|w\|\| \leq 1$, so that we can apply the previous case to obtain

$$\|(v/\|w\|)/\|v/\|w\|\| - w/\|w\|\| \leq 2\|v/\|w\|\| - w/\|w\|\|. $$

Upon simplifying, we obtain the desired inequality.

Suppose now that $V = C_0(X)$ for $X$ compact, and that $x$ and $y$ are distinct points of $X$. Choose $g \in V$ such that $-1 \leq g \leq 1$, $g(x) = 1$ and $g(y) = -1$. For any $\epsilon > 0$ set $f = g + \epsilon$. Thus $\|g\|_\infty = 1$, while $\|f\|_\infty = 1 + \epsilon$ and $\|f - g\|_\infty = \epsilon$. But

$$f/\|f\| - g = (g + \epsilon)/(1 + \epsilon) - g = \epsilon(1 - g)/(1 + \epsilon).$$

Now $\|1 - g\|_\infty = 2$, and so $\|f/\|f\| - g\|_\infty = 2\epsilon/(1 + \epsilon)$. If $k$ is a constant such that

$$\|f/\|f\| - g\|_\infty = \|\Pi(f) - \Pi(g)\|_\infty \leq k\|f - g\|_\infty,$$

then the calculation above shows that $2\epsilon/(1 + \epsilon) \leq k\epsilon$. On letting $\epsilon$ go toward 0, we see that $2 \leq k$. (We remark that the above argument works for any linear subspace $V$ of $C(X)$, or of $\ell^\infty(\Gamma)$, which contains the constant functions and at least one non-constant function, i.e. for order-unit spaces.)

Now any unital $C^*$-algebra, commutative or not, will, if it has dimension at least 2, contain a commutative $C^*$-algebra of dimension at least 2 and thus also a $C_0(X)$ for an $X$ with at least 2 points. Since $\Pi$ will carry linear subspaces into themselves, we can apply what we have found for $C_0(X)$ to conclude that $L(\Pi) = 2$ for unital $C^*$-algebras. A bit more arguing deals with non-unital $C^*$-algebras. \qed

We remark that for a Hilbert space we have $L(\Pi) = 1$.

Suppose now that we have a metric space $(Z, \rho)$, a subset $X$, and a bounded function $f$ from $X$ into a Banach space $V$. If $g$ is an extension of $f$ to $Z$, and if $r = \|f\|_\infty$, then $h = \Pi_r \circ g$ will be an extension of $f$ to $Z$ such that $\|h\|_\infty = \|f\|_\infty$ and $L(h) \leq L(\Pi_r) L(g) \leq 2L(g)$.

We can formalize the situation with the following definition.

**Definition 8.2.** For a Banach space $V$ we let $\mathcal{LN}(\mathcal{E}(V))$ denote the infimum of the constants $c$ such that for any metric space $(Z, \rho)$ and any $X \subseteq Z$, and any bounded function $f : X \to V$, there is an extension, $g$, of $f$ to $Z$ such that $L(g) \leq cL(f)$ and $\|g\|_\infty = \|f\|_\infty$. We define $\mathcal{LN}(\mathcal{E}_c(V))$ and $\mathcal{LN}(\mathcal{E}_f(V))$ similarly.

Then our discussion above gives:

**Proposition 8.3.** For any Banach space $V$ we have $\mathcal{LN}(\mathcal{E}(V)) \leq 2\mathcal{E}(V)$.

Note that Proposition 2.2 says that $\mathcal{LN}(\mathcal{E}(\ell^\infty(\Gamma))) = 1$ and Theorem 5.1 says that $\mathcal{LN}(\mathcal{E}(C(M))) = 1$. It would be interesting to know the exact value for $\mathcal{LN}(\mathcal{E}(V))$ for various choices of $V$, especially for $M_n(\mathbb{C})$ and $(M_n(\mathbb{C}))''$. I have not found a $V$ for which I could prove that $\mathcal{LN}(\mathcal{E}(V)) \neq \mathcal{E}(V)$. We could look for other retractions of $V$ onto its unit ball. Thus we seek information about $\mathcal{LR}(V)$, where $\mathcal{LR}(V)$ is the infimum of the constants $c$ such that there is a retraction, $R$, from $V$ onto its unit ball with $L(R) \leq c$. We are thus looking for Lipschitz extensions to the metric space $V$ of the function consisting of the inclusion of the unit ball of $V$ into $V$. Proposition 5.1 shows that we always have $\mathcal{LR}(V) \leq 2$.

Assume that $V$ is finite-dimensional. Then arguments very similar to those towards the end of Section 3 show that $\mathcal{LN}(\mathcal{E}_c(V)) = \mathcal{LN}(\mathcal{E}_c(V))$; and somewhat
similar arguments, involving the directed sets of all compact (or finite) subsets of \( Z \) and of \( X \) and Tychonoff’s theorem, show that \( LN_\xi(V) = LN_\xi_c(V) \).

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