THE POINCARÉ SERIES OF MULTIPLIER IDEALS OF A SIMPLE COMPLETE IDEAL IN A LOCAL RING OF A SMOOTH SURFACE

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Abstract. For a simple complete ideal \( \mathcal{P} \) of a local ring at a closed point on a smooth complex algebraic surface, we introduce an algebraic object, named Poincaré series \( P_\mathcal{P} \), that gathers in an unified way the jumping numbers and the dimensions of the vector space quotients given by consecutive multiplier ideals attached to \( \mathcal{P} \). This paper is devoted to prove that \( P_\mathcal{P} \) is a rational function giving an explicit expression for it.

1. Introduction

Multiplier ideals are a recent and important tool in singularity theory and in birational geometry. They have the virtue of giving information on the type of singularity corresponding to an ideal, divisor or metric and of accomplishing several vanishing theorems which made them very useful. As a reference, including historic development, for this concept we refer to [12, Ch. 9, 10, 11]. In spite of the utility of multiplier ideals, which is due to that many of their properties and applications are known, to compute these ideals is very hard because it involves facts as either to calculate resolution of singularities or to obtain very difficult integrals. As a consequence, very few explicit computations are known. The most remarkable is the one of multiplier ideals of arbitrary monomial ideals [10].

Intimately related to multiplier ideals are the jumping numbers (see [6], where one can also read about the antecedents of these numbers). Jumping numbers are a sequence of rational numbers that provide a sequence of invariants for the singularity in question, extending in a natural way the information given by the log-canonical threshold since this is the smallest jumping number.

In the line of looking for explicit computations related to multiplier ideals, we shall consider the local ring \( R \) at a closed point on a smooth complex algebraic surface. Our aim consists of studying the sequence of multiplier ideals of a simple complete ideal \( \mathcal{P} \) of \( R \). It is well known that the class of simple complete ideals plays a crucial role in the so-called Zariski theory of complete ideals [27, 28]. This theory was inspired by the work of Enriques and Chisini [7, L. IV, Ch. II, Sect. 17] and it has had further developments due mainly to Lipman (see [14]) who also gave a concept preceding the one of multiplier ideal, the so-called adjoint ideal [15].

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Very recently, Järvilehto [11] obtained an explicit description of the jumping numbers attached to simple complete ideals $\mathcal{P}$ as above. He gives a formula where the set of jumping numbers $\mathcal{H}$ can be seen as a union of finitely many sets

$$\mathcal{H} = \bigcup_{i=1}^{g^*+1} \mathcal{H}_i$$

and each $\mathcal{H}_i$ is determined by the maximal contact values of the divisorial valuation defined by $\varphi$ [28]. Furthermore, the jumping numbers of an ideal in the local ring at a rational singularity on a complex algebraic surface can also be obtained by an algorithm provided by Tucker in [24].

In this paper, we consider the family (ordered by inclusion) of multiplier ideals defined by $\mathcal{P}$ and taking into account that the vector space given by the quotient between two consecutive multiplier ideals is finitely generated (a consequence of Nakayama Lemma), we attach to $\mathcal{P}$ a Poincaré series whose coefficients are the dimensions of the above vector spaces (see Definition 2.3). With the help of the explicit description of the jumping numbers in [11], we give in Theorem 2.2 a characterization of the jumping numbers belonging to each set $\mathcal{H}_i$, that depends on the fact that certain irreducible exceptional divisors of a log-resolution of $\varphi$ contribute these jumping numbers. This concept was introduced in [20] by Smith and Thompson, where the set of irreducible exceptional divisors which contribute jumping numbers associated with a singular curve on a smooth surface is described. A similar result was obtained by Favre and Jonsson using different techniques (see Proposition 2.4, Lemma 2.11 and Fact 2 in the proof of Theorem 6.1 of [8]). These contributing exceptional divisors are essential for our development and allow to prove our main result (Theorem 2.1) which states that the mentioned Poincaré series is a rational function and provides an explicit computation for it. This series is an algebraic object that involves jumping numbers and the dimensions of its above mentioned corresponding vector spaces. The explicit description we give allows to get information that multiplier ideals add to the jumping numbers. In fact, we prove that the coefficient corresponding to each jumping number $\iota$ is the sum of the dimensions of certain vector spaces attached to the indices $i$ such that $\iota \in \mathcal{H}_i$. These dimensions are always one except for the last index $g^* + 1$, in which case they can be calculated from the expression of $\iota$ described in [11]. An important aid to compute our Poincaré series is the description we show in Theorem 2.3 of the previous multiplier ideal to a given one.

To make easier the reading of this paper, in the next section we state the necessary notations and our main results while the proofs are relegated to the last section.

2. Results

We fix, along the paper, a local ring $R$ at a closed point of a smooth complex algebraic surface. Denote by $K$ the quotient field of $R$. Consider a simple complete ideal $\varphi$ of $R$ and set $\nu$ its corresponding valuation (of $K$ centered at $R$). $\nu$ is defined by a divisor $E_n$ obtained from a finite simple sequence of point blowing ups

$$\pi : X = X_n \xrightarrow{\pi_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = \text{Spec}(R),$$

determined by the centers of $\nu$ at the spaces $X_j$ [28]. We shall denote by $E_j$ the prime exceptional divisor created by $\pi_j$ (and, abusing of notation, also its strict transform on $X$). Associated with the above objects, there exists a rooted tree, $\Gamma$, usually named the dual graph of $\nu$ (or of $\varphi$), where each vertex represents an exceptional divisor $E_j$ (on $X$) and two vertices are joined by an edge whenever the corresponding divisors intersect (see
Figure 1: its root is the vertex corresponding to the first exceptional divisor, $E_1$. The star vertices of the dual graph (labelled with $st_i$ in Figure 1) will be those whose associated exceptional divisors $E_{st_i}$ meet three distinct prime exceptional divisors. From now on, we shall denote by $g^*$ the number of star vertices. A vertex of $\Gamma$ will be called a dead vertex if it has only one adjacent vertex.

An exceptional divisor $E_{j_0}$ precedes another one $E_{j_1}$ if $j_0 < j_1$. Also, $E_{j_1}$ is named proximate to $E_{j_0}$ whenever $E_{j_0}$ precedes $E_{j_1}$ and the point to be blown-up in $X_{j_1-1}$ to create $E_{j_1}$ is in the strict transform of $E_{j_0}$. If $E_{j_1}$ is proximate to, at most, one prime exceptional divisor, then we shall say that $E_{j_1}$ is a free divisor; otherwise $E_{j_1}$ will be a satellite divisor. Notice that the divisors corresponding to star vertices, $E_{st_i}$, are characterized by the fact that $E_{st_i}$ is satellite and $E_{st_i+1}$ is free.

We associate to each star vertex $st_i$, inductively, a rooted subtree $\Gamma_i$ of $\Gamma$ in the following manner: $\Gamma_1$ is the subgraph of $\Gamma$ whose vertices are those corresponding to the divisors $E_j$ such that $j \leq st_1$ and, for $1 < i \leq g^*$, $\Gamma_i$ is the subgraph of $\Gamma$ whose vertices correspond to divisors $E_j$ such that $j \leq st_i$ but they are not vertices of $\Gamma_k$, $1 \leq k < i - 1$; the root of $\Gamma_1$ is the one of $\Gamma$ and, for each $i > 1$, the root of $\Gamma_i$ is the vertex adjacent to $st_{i-1}$. Also we define $\Gamma_{g^*+1}$ to be the rooted subtree of $\Gamma$ whose vertices are those which are not in $\Gamma_k$, $1 \leq k \leq g^*$; its root is the vertex adjacent to $st_{g^*}$.

Along this paper we stand $\{\beta_i\}_{i=0}^{g^*+1}$ for the maximal contact values (or Zariski exponents) of the valuation $\nu$ [21, Sect. 6]. Also, set $e_i := \gcd(\beta_0, \beta_1, \ldots, \beta_i)$, $0 \leq i \leq g$ and $n_i := e_{i-1}/e_i$, for $1 \leq i \leq g$. If the last prime exceptional divisor $E_n$ is free (as in Figure 1) then $g = g^*$ and, otherwise, $g = g^* + 1$ (in this case there is no subgraph $\Gamma_{g+1}$). Also we shall denote by $F_i$ ($1 \leq i \leq g^*$) the divisor $E_{st_i}$ corresponding to a star vertex of $\Gamma$ and we stand $F_{g^*+1}$ for the last obtained exceptional divisor $E_{n}$.

A concept that we shall use often in this paper is given in the following

**Definition 2.1.** Given a prime exceptional divisor $E_j$, an $E_j$-general element for the valuation $\nu$ will be an element $\varphi \in R$ giving an equation of an analytically irreducible germ of curve whose strict transform on $X_j$ is smooth and intersects $E_j$ transversally at a non-singular point of the exceptional locus. The $E_n$-general elements are usually named general elements of the valuation $\nu$.

A remarkable fact is the description of the maximal contact values of $\nu$ as values of certain $E_j$-general elements. Specifically, if $\varphi_j \in R$ denotes any $E_j$-general element for $\nu$,
whenever \( 1 \leq C \), GALINDO AND F. MONSERRAT

**Definition 2.2.** Following definition:

Concerning the multiplier ideals attached to the ideal \( \mathcal{P} \) and its corresponding log-resolution \( \pi \), \( \rho \) of the ideal \( \mathcal{P} \) gives a log-resolution of the ideal \( \mathcal{P} \) such that \( \mathcal{P} \mathcal{O}_X = \mathcal{O}_X(-D) \). Notice that if \( \varphi_j \in R \) is an \( E_j \)-general element for \( \nu \) then \( a_j = \nu(\varphi_j) \). Thus, for any positive rational number \( \iota \), the multiplier ideal of \( \varphi \) and \( \iota \) can be defined as \( \mathcal{J}(\varphi^\iota) := \pi_* \mathcal{O}_X(K_{X|X_0} - [\iota D]) \), where \( K_{X|X_0} \) is the relative canonical divisor and \([\cdot]\) represents the round-down or the integral part of the corresponding divisor. The family of multiplier ideals is totally ordered by inclusion, parameterized by non-negative rational numbers. Furthermore, there is an increasing sequence \( t_0 < t_1 < \cdots \) of positive rational numbers, called jumping numbers, such that \( \mathcal{J}(\varphi^{t_i}) = \mathcal{J}(\varphi^{t_{i+1}}) \) for \( t_i \leq \iota < t_{i+1} \) and \( \mathcal{J}(\varphi^{t_{i+1}}) \nsubseteq \mathcal{J}(\varphi^{t_i}) \) for each \( t \geq 0; t_0 \), usually named log-canonical threshold of \( \varphi \), is the least positive rational number with such a property.

Denote, as above, by \( g^* \) the number of star vertices in \( \Gamma \) and set

\[
\mathcal{H}_i := \left\{ \iota(i, p, q, r) := \frac{p}{e_{i-1}} + \frac{q}{\beta_i} + \frac{r}{e_i} \mid \frac{p}{e_{i-1}} + \frac{q}{\beta_i} \leq 1; p, q \geq 1, r \geq 0 \right\}
\]

whenever \( 1 \leq i \leq g^* \), and

\[
\mathcal{H}_{g^*+1} := \left\{ \iota(g^* + 1, p, q) := \frac{p}{e_{g^*}} + \frac{q}{\beta_{g^*+1}} \mid p, q \geq 1 \right\},
\]

\( p, q \) and \( r \) being integer numbers. In [11], it is proved that the set \( \mathcal{H} \) of jumping numbers of the ideal \( \varphi \) can be computed as \( \mathcal{H} = \cup_{i=1}^{g^*+1} \mathcal{H}_i \).

Now, inspired by the terminology introduced in [20], for a simple complete ideal \( \varphi \) of \( R \) and its corresponding log-resolution \( \pi \) (see [11]) and divisor \( D = \sum_{j=1}^n a_j E_j \), we give the following definition:

**Definition 2.3.** A candidate jumping number from a prime exceptional divisor \( E_j \) given by \( \pi \) is a positive rational number \( \iota \) such that \( a_j \) is an integer number. Also, we shall say that \( E_j \) contributes \( \iota \) whenever \( \iota \) is a candidate jumping number from \( E_j \) and \( \mathcal{J}(\varphi^\iota) \nsubseteq \pi_* \mathcal{O}_X(-[\iota D] + K_{X|X_0} + E_j) \).

Assume \( \iota \in \mathcal{H} \) and \( \iota \neq t_0 = \min \mathcal{H} \). We denote by \( \iota^{<} \) the largest jumping number which is less than \( \iota \). By convention we set \( \mathcal{J}(\varphi^{\iota^{<}}) = R \). Nakayama Lemma proves that, for any \( \iota \in \mathcal{H} \), \( \mathcal{J}(\varphi^{\iota^{<}})/\mathcal{J}(\varphi^{\iota}) \) is a finitely generated \( \mathbb{C} \)-vector space, \( \mathbb{C} \) being the field of complex numbers. Thus, we can define the object to be studied as

**Definition 2.3.** Let \( \varphi \) a simple complete ideal of \( R \). The Poincaré series of multiplier ideals of \( \varphi \) is defined to be the following fractional series:

\[
P_{\varphi}(t) := \sum_{\iota \in \mathcal{H}} \dim_{\mathbb{C}} \left( \frac{\mathcal{J}(\varphi^{\iota^{<}})}{\mathcal{J}(\varphi^{\iota})} \right) t^{\iota},
\]

\( t \) being an indeterminate.

Our main result is
Theorem 2.1. The Poincaré series $P_\Psi(t)$ can be expressed as

$$
P_\Psi(t) = \frac{1}{1-t} \left( \sum_{i=1}^{g^*} \sum_{\iota \in \mathcal{H}, \iota < 1} t^\iota + \left( \frac{1}{1-t} + \frac{t}{(1-t)^2} \right) \sum_{\iota \in \Omega} t^{\iota} \right),$$

where

$$\Omega := \{ \iota \in \mathcal{H}_{g^*+1} \mid \iota \leq 2 \text{ and } \iota - 1 \not\in \mathcal{H}_{g^*+1} \}.$$

We must clarify that $P_\Psi(t)$ is not an element in $\mathbb{C}(t)$ but there exist finitely many (exactly $g^* + 1$) “roots of $t$” which could be considered as another indeterminates, say $z_1, z_2, \ldots, z_{g^*+1}$, such that $P_\Psi(t) \in \mathbb{C}(z_1, z_2, \ldots, z_{g^*+1})$. To prove this theorem we shall use the following results:

Theorem 2.2. A jumping number $\iota$ of a simple complete ideal $\Psi$ belongs to the set $\mathcal{H}_i$ ($1 \leq i \leq g^* + 1$) if and only if the prime exceptional divisor $F_i := E_{\pi i}$ contributes $\iota$.

Theorem 2.3. Let $\iota$ be a jumping number of a simple complete ideal $\Psi$. Then

$$\pi_* \mathcal{O}_X \left( -[\iota D] + K_{X|X_0} + \sum_{l=1}^{s} F_{i_l} \right) = \mathcal{J} \left( \Psi^{\iota} \right),$$

where $\{i_1, i_2, \ldots, i_s\}$ is the set of indexes $i$, $1 \leq i \leq g^* + 1$, such that $\iota \in \mathcal{H}_i$.

As a direct consequence of Theorem 2.2 and Clause (c) of the forthcoming Proposition 3.1 we get the following result that tells us which are the prime exceptional divisors contributing a given jumping number.

Proposition 2.1. The prime exceptional divisors that contribute a jumping number $\iota$ of a simple complete ideal $\Psi$ are those divisors $F_i$ such that $\iota \in \mathcal{H}_i$.

Remark. In [24] it is announced that, in a future work of the author, a similar result to Proposition 2.1 for jumping numbers that are less than one will be provided.

3. Proofs

Along this section we shall use the above notations. We start by proving Theorem 2.2.

3.1. Proof of Theorem 2.2. It will be useful the following result, whose proof can be deduced from Section 3 of [20] and, therefore, we omit it.

Proposition 3.1. Let $\iota$ be a positive rational number and $E_j$ a prime exceptional divisor given by the sequence $\pi$ of $\Gamma$. Then

(a) $\pi_* \mathcal{O}_X (-[\iota D] + K_{X|X_0}) \neq \pi_* \mathcal{O}_X (-[\iota D] + K_{X|X_0} + E_j)$ if and only if $-[\iota D] \cdot E_j \geq 2$.

(b) Assume that $\iota$ is a jumping number. Then $E_j$ contributes $\iota$ if and only if $\iota$ is a candidate jumping number from $E_j$ and $-[\iota D] \cdot E_j \geq 2$.

(c) If $E_j$ contributes a jumping number $\iota$ then $E_j = F_i$ for some $i \in \{1, 2, \ldots, g^* + 1\}$.

We shall divide the proof of the direct implication of the theorem in two parts, 1 and 2.

1. Assume that $g^* = g$, that is $\Gamma$ contains a subgraph $\Gamma_{g+1}$, and consider three subcases.
a. Let us prove that the divisor $F_g$ contributes any $\iota \in \mathcal{H}_g$.

Consider the subgraph of $\Gamma$ whose vertices are those corresponding to $F_g$ and the prime exceptional divisors that meet $F_g$, that we denote by $F'_{g-2}$, $F'_{g-1}$ and $F'_{g+1}$ (see Figure 2). We suppose that $F'_{g-1}$ is the exceptional divisor created immediately before that $F_g$ (that is, $E_{stg-1}$). Notice that, in Figure 2 $F'_{g-2}$ and $F'_{g-1}$ can appear interchanged but our reasoning in that case will be similar.

![Figure 2. $F_g$ contributes $\mathcal{H}_g$](image)

Recall that $E_j \cdot E_j = -\text{card}(P_j)$, $P_j$ being the set of prime exceptional divisors which are proximate to $E_j$ and card meaning cardinality, $E_k \cdot E_j = 1$ whenever $E_k \cap E_j \neq \emptyset$ and $k \neq j$, and $E_k \cdot E_j = 0$ otherwise, $j, k \in \{1, 2, \ldots, n\}$. By Proposition 3.1 we see that the inequality we have to prove is

\[
(2) \quad - \lfloor \nu(\varphi_2) \rfloor - \lfloor \nu(\varphi_1) \rfloor + 2\lfloor \nu(\varphi) \rfloor \geq 2,
\]

$\varphi = \varphi_0$ being an $F_g$-general element for $\nu$ and $\varphi_l$ an $F'_{g+l}$-general element for $\nu$, $l \in \{-2, -1, 1\}$. In fact, we shall prove that the equality holds in (2).

Set, as above, $\{\beta_i\}_{i=0}^{g+1}$ the sequence of maximal contact values of $\nu$ and denote by $\bar{\beta}_{g+l}^{\varphi_l}$, $l \in \{-2, -1, 1\}$, the $i$-th maximal contact value of the divisorial valuation $\nu_{\varphi_l}$ defined by $F'_{g+l}$ (notice that $\varphi_l$ is a general element of this valuation). Also, remind that $e_i := \gcd(\bar{\beta}_0, \ldots, \bar{\beta}_i)$ and $n_i := e_{i-1}/e_i$, and denote with a super-index $\varphi_l$ the analogous values attached to $\nu_{\varphi_l}$.

By [3] and taking into account that for $h \in R$,

\[
\nu(h) = \min\{(h, \psi) | \psi \text{ is a general element of } \nu\},
\]

$(h, \psi)$ being the intersection multiplicity of the germs given by $h$ and $\psi$ [21], we get the following equalities:

\[
\nu(\varphi_2) = e_{g-1} \bar{\beta}_g^{\varphi_2}; \\
\nu(\varphi_1) = e_{g-1} \bar{\beta}_g^{\varphi_1}; \\
\nu(\varphi_1) = e_{g-1} \bar{\beta}_g + 1; \\
\nu(\varphi) = e_{g-1} \bar{\beta}_g = n_g \bar{\beta}_g.
\]

Now, we state a result which will be useful in the proof.

**Lemma 3.1.** $n_g^{\varphi_2} \bar{\beta}_g - n_g \bar{\beta}_g^{\varphi_2} = 1$. 
Proof. Set $\beta_g'$ and $\beta_g^{\varphi_2}$ the Puiseux exponents of the valuations $\nu$ and $\nu_{\varphi_2}$ (see [21] for the definition). Lemma 1.8 in [4] proves that

\[ \frac{\bar{\beta}_g}{n_g} = \beta_g' + \frac{n_g-1}{n_g} \bar{\beta}_{g-1} - 1 \]

and, analogously,

\[ \frac{\bar{\beta}_g^{\varphi_2}}{n_g^{\varphi_2}} = \beta_g'_{\varphi_2} + \frac{n_g^{\varphi_2}-1}{n_g^{\varphi_2}} \bar{\beta}_{g-1} - 1. \]

Since $\bar{\beta}_i^{\varphi_2} = \kappa \bar{\beta}_i$, for $i < g$, where $\kappa = \bar{\beta}_0^{\varphi_2} / \bar{\beta}_0$ (see [2], for example), one gets

\[ \frac{n_g-1}{n_g} \bar{\beta}_{g-1} = \frac{n_g^{\varphi_2}-1}{n_g^{\varphi_2}} \bar{\beta}_{g-1} \]

since $e_g = e_g^{\varphi_2} = 1$. So

\[ \frac{\bar{\beta}_g}{n_g} - \frac{\bar{\beta}_g^{\varphi_2}}{n_g^{\varphi_2}} = \beta_g' - \beta_g'_{\varphi_2}. \]

Bearing in mind that $\beta_g'$ and $\beta_g'_{\varphi_2}$ are consecutive convergents of a finite continued fraction whose denominators are respectively $n_g$ and $n_g^{\varphi_2}$, by [17] Th. 7.5 the equality

\[ \beta_g' - \beta_g'_{\varphi_2} = \frac{1}{n_g n_g^{\varphi_2}} \]

holds. The statement follows from (3), (4) and (5). □

Returning to the proof of our theorem, we recall that in [11] it is proved that $\iota \in \mathcal{H}_g$ has the form $\iota(g, p, q, r)$ (see page 4 in this paper). Since $e_g = 1$ we get

\[ \iota = \frac{(p + n_g g) \bar{\beta}_g + q n_g}{n_g \bar{\beta}_g}. \]

For simplicity’s sake we set $s := p + n_g$. Now stand $\alpha$ and $\beta$ for $\alpha := \bar{\beta}_g^{\varphi_2} / \bar{\beta}_g$ and $\beta := e_g^{\varphi_2-1} / n_g$. Inequality (2) to be proved can be expressed

\[ - [(q n_g + s \bar{\beta}_g) \alpha] - [(q n_g + s \bar{\beta}_g) \beta] - [(q n_g + s \bar{\beta}_g)(1 + \frac{1}{n_g \bar{\beta}_g})] + 2(q n_g + s \bar{\beta}_g) \geq 2. \]

For $\iota \in \mathcal{H}_g$, it is necessary that $(p/n_g) + (q/\bar{\beta}_g) \leq 1$, which is true if and only if $p \bar{\beta}_g + q n_g \leq n_g \bar{\beta}_g$. However, equality can not happen in this case because $p \geq 1$, $q \geq 1$ and $\text{gcd}(\bar{\beta}_g, n_g) = 1$.

Let $C$ be a germ of curve given by a general element of $\nu$. It holds that $\pi^* C = \tilde{C} + D$, where $\tilde{C}$ denotes the strict transform of $C$ by $\pi$ and $D$ the attached to $\varphi$ above mentioned divisor. Then

\[ (\pi^* C) \cdot F_g = \tilde{C} \cdot F_g + D \cdot F_g = 0. \]

Since $(-D) \cdot F_g = -\nu(\varphi_{-2}) - \nu(\varphi_{-1}) - \nu(\varphi_1) + 2\nu(\varphi)$, we obtain

\[ \alpha + \beta = 1 - \frac{1}{n_g \bar{\beta}_g}. \]
Set $\alpha_1 := \alpha + (1/n_g \bar{\beta}_g)$ and $\beta_1 := \beta$. By Lemma 3.1 one has
\[
(qn_g + s \bar{\beta}_g)\alpha = \frac{qn_g \bar{\beta}_g^{\varphi-2}}{\beta_g^{\varphi}} + s \bar{\beta}_g^{\varphi-2} = qn_g^{\varphi-2} + s \bar{\beta}_g^{\varphi-2} - \frac{q}{\beta_g^{\varphi}}.
\]
Thus,
\[
(qn_g + s \bar{\beta}_g)\alpha_1 = qn_g^{\varphi-2} + s \bar{\beta}_g^{\varphi-2} - \frac{q}{\beta_g^{\varphi}} + \frac{qn_g + s \bar{\beta}_g}{n_g \beta_g} = qn_g^{\varphi-2} + s \bar{\beta}_g^{\varphi-2} + \frac{s}{n_g}.
\]
Recall that $s = p + rn_g$, $p < e_{g-1}$, $q < \bar{\beta}_g$ and $p \bar{\beta}_g + qn_g < n_g \bar{\beta}_g$. Then
\[
[(qn_g + s \bar{\beta}_g)\alpha_1] + [(qn_g + s \bar{\beta}_g)\beta_1] = qn_g + s \bar{\beta}_g - 1, \tag{8}
\]
\[
[(qn_g + s \bar{\beta}_g)\alpha_1] = qn_g^{\varphi-2} + s \bar{\beta}_g^{\varphi-2} + r, \tag{9}
\]
\[
[(qn_g + s \bar{\beta}_g)\alpha] = qn_g^{\varphi-2} + s \bar{\beta}_g^{\varphi-2} - 1 \tag{10}
\]
and
\[
[(qn_g + s \bar{\beta}_g)(1 + \frac{1}{n_g \beta_g})] = [qn_g + s \bar{\beta}_g + \frac{qn_g + s \bar{\beta}_g}{n_g \beta_g}] = [qn_g + s \bar{\beta}_g + \frac{q}{\beta_g} + \frac{p}{n_g} + r] = qn_g + s \bar{\beta}_g + r. \tag{11}
\]
From (8), (9) and (10), we get
\[
-(qn_g + s \bar{\beta}_g)\alpha - (qn_g + s \bar{\beta}_g)\beta = -(qn_g + s \bar{\beta}_g) + r + 2.
\]
This concludes the proof of this case since now (11) proves equality in (6).

b. Now, we are going to prove that for any $i < g$ the divisor $F_i$ contributes any $i \in \mathcal{H}_i$.

With the same conventions and notations as above, consider the subgraph of $\Gamma$ of vertices corresponding to $F_i$ and those divisors which meet it (see Figure 3).

![Figure 3. $F_i$ contributes $\mathcal{H}_i$](image)

b1. Firstly, we assume that the divisor $F_{i+1}'$ is free.

Set $\nu$ the valuation given by the last free divisor represented in $\Gamma_{i+1}$. Notice that for any analytically irreducible element $h$ in $R$ whose strict transform (the associated germ of curve) cuts transversally some of the divisors created to define $\nu$, the equality $\nu(h) =$
\((\overline{\beta}_0'/\beta_{01}')ν(h)\) holds. Along this proof, we set \(κ := \overline{\beta}_0'/\beta_{01}'\) and any jumping number \(ν' = ν(i, p, q, r)\) in the set \(H_i\) satisfies:

\[
ν' = \frac{(p + rn_0^ν)i_1^ν + qe_{i-1}^ν}{e_{i-1}^νi_1^ν} = \frac{(p + rn_0^ν)κi_1^ν + qke_{i-1}^ν}{κ^2e_{i-1}^νi_1^ν} = \frac{1}{κ}ν',
\]

where the super-indices ν and ν make reference to the valuation that corresponds to the free divisor represented in the graph \(Γ\), the reasoning in 1a shows that, by the conditions given in [11], is less than or equal to \(\overline{\beta}_0\). Again, set \(ν\) into account that \(iν\) is the jumping number \([q + \overline{n}_2^ν]i_1^ν + qe_{i-1}^ν]/(e_{i-1}^ν\overline{β}_1^ν)\) of the simple complete ideal defined by \(ν\), which belongs to the corresponding set \(H_{i1}\).

Now, our result is proved because inequality (2) in our case coincides with the same inequality for \(ν\), which accomplishes it since the valuation \(ν\) is in the situation 1a.

b2. Now suppose that \(F_{g+1}'\) is a satellite divisor.

We can assume that \(i = g - 1\), because when \(i < g - 1\) a reasoning as in 1b1 would finish the proof.

Again, set \(ν\) the valuation defined by the free divisor represented in the graph \(Γ_g\). Keeping our notation, we must prove

\[-[ν(ϕ_{-2})] - [ν(ϕ_{-1})] - [ν(ϕ_{1})] + (t + 1)ν(ϕ) \geq 2,\]

where \(t\) is the cardinality of the set of prime exceptional divisors in [11] proximate to \(F_{g-1}'\).

The worst case happens when \(t = 2\), so we assume it. By using the valuation \(ν\) and taking into account that \(ν' = \frac{1}{κ}ν\), the reasoning in 1a shows that

\[ν(ϕ) - [ν(ϕ_{-2})] - [ν(ϕ_{-1})] \geq 2 + r.\]

Looking at the sequence of values for \(ν\) (see [3, 1.5.1]), set \(a \geq a\) (b ≥ a, respectively) for the value at the divisor where the strict transform of the germ given by \(ϕ_1\) intersects transversally (for the value at the defining divisor of \(ν\), respectively). Then

\[ν(ϕ) = qe_{g-2}' + s\overline{β}_{g-1}' \]

where \(s\) is obtained as above. Now, since

\[(12) \quad ν(ϕ_1) = 2ν(ϕ) + a + b,\]

because \(t = 2\), we get

\[(13) \quad ν(ϕ_1) = 2(qe_{g-2}' + s\overline{β}_{g-1}') + ta + tb = 2(qe_{g-2}' + s\overline{β}_{g-1}') \left[1 + \frac{a + b}{2ke_{g-2}'\overline{β}_{g-1}'}\right].\]

It is clear that \((a + b)\overline{β}_{01}' = \overline{β}_0'\) and then the right hand side in (13) equals

\[2(qe_{g-2}' + s\overline{β}_{g-1}') \left[1 + \frac{a + b}{2(a + b)e_{g-2}'\overline{β}_{g-1}'}\right] = 2(qe_{g-2}' + s\overline{β}_{g-1}') \left[1 + \frac{1}{2ke_{g-2}'\overline{β}_{g-1}'}\right].\]

So, \([ν(ϕ_1)] ≤ 2(qe_{g-2}' + s\overline{β}_{g-1}') + r\) since

\[\frac{(qe_{g-2}' + s\overline{β}_{g-1}')}{e_{g-2}'\overline{β}_{g-1}'}\]

is the jumping number \(ν\) that, by the conditions given in [11], is less than or equal to \(r\).

It only remains to prove that

\(c. \quad F_{g+1}'\) contributes any jumping number \(ν \in H_{g+1}\).
Indeed, \( \iota \) has the form 
\[
\iota = \left( \frac{p}{e_g} \right) + \left( \frac{q}{\beta_g + 1} \right),
\]
where \( p, q > 1 \) and \( e_g = 1 \), so 
\[
\iota = \left( q + p\beta_g + 1 \right)/\beta_g + 1.
\]
Keeping the above notations, the subgraph we are interested in is the one depicted in Figure 4. As above set \( \varphi \) and \( \varphi - 1 \) analytically irreducible elements of type \( \varphi_i \) attached to the divisors \( F_{g+1} \) and \( F'_g \) respectively. Then, we must prove 
\[
-\lfloor \nu(\varphi - 1) \rfloor + \nu(\varphi) \geq 2.
\]
And this happens since 
\[
\nu(\varphi - 1) = \frac{q + p\beta_g + 1}{\beta_g + 1}(\beta_g + 1 - 1),
\]
\[
\nu(\varphi) = q + p\beta_g + 1 \text{ and } p \geq 1.
\]

2. To end the proof of the direct implication, let us assume that \( g^* = g - 1 \). We shall only prove that the divisor \( F_g \) contributes any \( \iota \in \mathcal{H}_g \). A proof for the case of the remaining divisors \( F_i, i < g \), works similarly to the analogous case in 1.

Here, the subgraph of \( \Gamma \) of vertices corresponding to \( F_g \) and divisors meeting it will be the one in Figure 5 and with the same notations of 1a, we must prove 
\[
-\lfloor \nu(\varphi - 2) \rfloor - \lfloor \nu(\varphi - 1) \rfloor + \nu(\varphi) \geq 2,
\]
that is 
\[
-[(qn_g + s\beta_g)\alpha] - [(qn_g + s\beta_g)\beta] + qn_g + s\beta_g \geq 2.
\]
Finally, since 
\[
1 = (-D) \cdot F_g = -\nu(\varphi - 2) - \nu(\varphi - 1) + \nu(\varphi),
\]
the same reasoning we did to show (7) allows to set \( n_g \beta_g (\alpha - \beta + 1) = 1 \), where \( \alpha \) and \( \beta \) are as above, and so we also get \( \alpha + \beta = 1 - (1/n_g \beta_g) \), i.e., \( n_g \beta_g^{\varphi - 2} + \beta_g e_{g-1}^{\varphi - 1} = n_g \beta_g - 1 \).

Then,

\[
[(q g_n + s \beta_g) \alpha] + [(q g_n + s \beta_g) \beta] - (q g_n + s \beta_g) =
\]

\[
= \frac{q \beta_g^{\varphi - 2} n_g}{\beta_g} + s \beta_g^{\varphi - 2} + l \left( \frac{q (n_g \beta_g - 1)}{\beta_g} - q e_{g-1}^{\varphi - 1} \right) + \left( \frac{s (n_g \beta_g - 1)}{n_g} - s \beta_g^{\varphi - 2} \right) - (q g_n + s \beta_g) =
\]

\[
= [q g_n - \frac{q}{\beta_g}] + [s \beta_g - \frac{s}{n_g}] - (q g_n + s \beta_g) \leq (q g_n - 1) + (s \beta_g - 1) - (q g_n + s \beta_g) = -2.
\]

In order to prove the converse implication we shall consider two previous lemmas.

**Lemma 3.2.** Consider an \( n \)-tuple of non-negative integers \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) and let \( D(\alpha) \) be the divisor \(- \sum_{j=1}^{n} \alpha_j E_j \). For every nonempty subset \( \{F_{i_1}, F_{i_2}, \ldots, F_{i_t}\} \) of the set of divisors \( \{F_i\}_{i=1}^{t+1} \), the following equality holds:

\[
\dim_{\mathbb{C}} \frac{\pi_* O_X(D(\alpha) + \sum_{i=1}^{t} F_{i})}{\pi_* O_X(D(\alpha))} = \sum_{i=1}^{t} \dim_{\mathbb{C}} \frac{\pi_* O_X(D(\alpha) + F_{i})}{\pi_* O_X(D(\alpha))}.
\]

**Proof.** Without loss of generality we can assume that \( i_1 < i_2 < \cdots < i_t \). In a first step, we prove (14) for \( t = 2 \).

Consider the following commutative diagram of ideals in \( R \) and injective maps, where, for any sum \( G \) of divisors \( E_j \), we stand \( \mathcal{E}_\alpha(G) \) for \( \pi_* O_X(D(\alpha) + G) \) and an expression like \( A \xrightarrow{p} B \) means that the dimension of the vector space quotient \( B/A \) equals \( p \).

\[
\begin{array}{ccc}
\mathcal{E}_\alpha :\mathcal{E}_\alpha(\emptyset) & \xrightarrow{[p_{i_1}]} & \mathcal{E}_\alpha(F_{i_1}) \\
& \downarrow & \downarrow & \downarrow \\
& [q_{i_2}] & \mathcal{E}_\alpha(F_{i_2}) & \xrightarrow{[q_{i_2}]} \mathcal{E}_\alpha(F_{i_1} + F_{i_2}).
\end{array}
\]

We only need to prove that

\[
\tag{15} p_{i_1} = q_{i_1},
\]

because it holds the following vector space isomorphism

\[
\frac{\mathcal{E}_\alpha(F_{i_1} + F_{i_2})}{\mathcal{E}_\alpha} \cong \frac{\mathcal{E}_\alpha(F_{i_1} + F_{i_2})}{\mathcal{E}_\alpha(F_{i_2})} \oplus \frac{\mathcal{E}_\alpha(F_{i_2})}{\mathcal{E}_\alpha}.
\]

The symmetric isomorphism given by the diagram proves that \( p_{i_2} = q_{i_2} \) is also true.

(15) holds when either \( \alpha_{st_{i_1}} = 0 \) or \( \alpha_{st_{i_2}} = 0 \), therefore we can assume that both values are positive. Set \( \nu_{i_1} \) and \( \nu_{i_2} \) the divisorial valuations defined by \( F_{i_1} \) and \( F_{i_2} \), respectively.
By the proof of [9, Th. 1], a basis of the vector space $\mathcal{E}_\alpha(F_{i_1})/\mathcal{E}_\alpha$ is given by classes defined by “monomials” of the type $\prod_{k=0}^{g'+1} \varphi_{l_k}^{a_k}$, $a_k \geq 0$, where $l_k$ and $\varphi_{l_k}$, $0 \leq k \leq g' + 1$, are elements as in the paragraph after Definition 2.1 and associated with the valuation $\nu_1$ (in particular $\varphi_{l'_{g'+1}}$ denotes a general element of $\nu_1$). Clearly $\nu_1(\prod_{k=0}^{g'+1} \varphi_{l_k}^{a_k}) = \alpha_{st_{i_1}} - 1$.

Taking into account that $F_{i_2}$ corresponds either to a star vertex (as $F_{i_1}$) or to the last vertex of the dual graph of $\nu$ (always denoted by $st_{i_2}$) and that $i_1 < i_2$, it holds that all generator “monomials” $\prod_{k=0}^{g'+1} \varphi_{l_k}^{a_k}$ as above have the same valuation $\nu_{i_2}$, that is a multiple of $\alpha_{st_{i_1}} - 1$ and larger than or equal to $\alpha_{st_{i_2}}$. This shows that $p_{i_1} = q_{i_1}$ because the classes in $\mathcal{E}_\alpha(F_{i_1} + F_{i_2})/\mathcal{E}_\alpha$ of the “monomials” spanning $\mathcal{E}_\alpha(F_{i_1})/\mathcal{E}_\alpha$ are linearly independent and one cannot find any element $\prod_{k=0}^{g'+1} \varphi_{l_k}^{a_k}$ such that $\nu_{i_1}(\prod_{k=0}^{g'+1} \varphi_{l_k}^{a_k}) = \alpha_{st_{i_1}} - 1$ and $\nu_{i_2}(\prod_{k=0}^{g'+1} \varphi_{l_k}^{a_k}) = \alpha_{st_{i_2}} - 1$.

Notice that in the above reasoning, the specific values $\alpha_j$ for those indices not corresponding to the divisors defining the valuations $\nu_{i_1}$ and $\nu_{i_2}$ are not relevant.

When $t > 2$, we can reduce the proof to the above situation. Indeed, if we consider the diagram in Figure 6 we can do the above reasoning for the sub-diagrams including three consecutive ideals of the type described for $t = 2$ (changing $D(\alpha)$ by sums of the type $D(\alpha) + \sum_{l \in L} F_l$, with a suitable set of indices $L$). We have set in boldface letter the dimensions we know and without this type of letter those that we compute in an iterative manner by using the reasoning for $t = 2$. This implies that we can compute the dimensions.
appearing in the arrows of the top line of the diagram. The sum of those dimensions is the dimension we desire to compute and it is \( p_i + p_{i+1} + \cdots + p_t \).

We shall use the above lemma to prove the following one, which shows that Theorem 2.2 holds for less than 1 jumping numbers. Before stating it, we recall that the less than 1 jumping numbers of \( \mathcal{P} \) (and their multiplier ideals) coincide with those associated with the curve in \( \text{Spec}(R) \) defined by a general element of the valuation \( \nu, \varphi, [11, \text{Prop. 9.3}] \). Furthermore, Varchenko [25] (see also [1]) proved that these jumping numbers are exactly the exponents \( \alpha \) in the interval \([0,1]\) corresponding to non-zero terms of the Hodge spectrum of \( \varphi \), \( \text{Sp}(\varphi) = \sum_{\alpha \in \mathbb{Q}} n_{\alpha}(\varphi)t^\alpha \), which is a fractional Laurent polynomial with integer coefficients that can be defined using the mixed Hodge structure and the monodromy on the cohomology of the Milnor fiber of \( \varphi \). We must mention here that, although the Hodge spectrum of a hypersurface singularity was first defined in [22] and [23], we consider the definition used in [19, 18, 1].

**Lemma 3.3.** If \( \iota \) is a jumping number of a simple complete ideal \( \mathcal{P} \) such that \( 0 < \iota < 1 \) and a divisor \( F_i \) (\( 1 \leq i \leq g^*+1 \)) contributes \( \iota \), then \( \iota \in H_i \).

**Proof.** Assume that \( F_i \) contributes \( \iota \). Consider the set of indices \( \Pi := \{k \mid 1 \leq k \leq g^*+1 \text{ and } \iota \in H_k \} \). Taking into account that \( F_k \) contributes \( \iota \) for all \( k \in \Pi \) (by the direct implication of Theorem 2.2) one has

\[
\mathcal{J}(\varphi^\iota) \subseteq \pi_*\mathcal{O}_X \left(-[\iota D] + K_{X|X_0} + \sum_{k \in \Pi} F_k \right) \subseteq \mathcal{J}(\varphi^{\iota^\leq}).
\]

By [1], it happens that

\[
n_{\iota}(\varphi) = \dim_{\mathbb{C}} \mathcal{J}(\varphi^{\iota^\leq})/\mathcal{J}(\varphi^{\iota}),
\]

and, by [18, 1.5], \( n_{\iota}(\varphi) \) is the cardinality of \( \Pi \). Therefore, by Lemma 3.2, the equality \( \pi_*\mathcal{O}_X \left(-[\iota D] + K_{X|X_0} + \sum_{k \in \Pi} F_k \right) = \mathcal{J}(\varphi^{\iota^\leq}) \) holds. As a consequence if \( \iota \) would not be in \( \Pi \), then \( \mathcal{J}(\varphi^{\iota^\leq}) = \pi_*\mathcal{O}_X \left(-[\iota D] + K_{X|X_0} + \sum_{k \in \Pi} F_k + F_i \right) \), which is a contradiction with Lemma 3.2 and (16). \( \square \)

**Remark.** Notice that, from the above proof, it follows that the dimension of the quotient \( \pi_*\mathcal{O}_X \left(-[\iota D] + K_{X|X_0} + F_i \right)/\mathcal{J}(\varphi^{\iota}) \) is 1 whenever \( \iota \in H_i, 1 \leq i \leq g^*+1 \), and \( 0 < \iota < 1 \).

Now we shall take advantage of Lemma 3.3 to prove that its statement is true for whichever jumping number \( \iota \) of \( \varphi \). Notice that \( \iota \neq 1 \) by [11, Prop. 8.9]. Consider a divisor \( F_i, 1 \leq i \leq g^*+1 \), such that \( F_i \) contributes \( \iota > 1 \) and let us prove that \( \iota \in H_i \).

If \( \iota \) is an integer then \( \iota \in H_{g^*+1} \) (by [11, Prop. 8.11]). As \( \iota D \cdot F_i = [\iota D] \cdot F_i \leq -2 \) (by Proposition 3.1), \( \iota \) must be \( g^*+1 \) since otherwise the above equality would not happen because \( D \cdot E_j = 0 \) whenever \( E_j \neq F_{g^*+1} \).

If \( \iota = g^*+1 \) then \( \nu(\varphi) \) is a positive integer and, therefore, \( \iota \in H_{g^*+1} \) (again by [11, Prop. 8.11]).
Hence we can assume from now on that \( i \) is not an integer and \( i \neq g^* + 1 \). Set \( a := \lfloor i \rfloor \) and \( \beta := i - a \). We shall distinguish two cases, proving that the second one cannot hold.

**Case 1.** \( \beta \) is a jumping number of \( \varphi \). Obviously \( \beta \) is a candidate jumping number from \( F_i \). We have \( \lfloor \beta D \rfloor \cdot F_i = \lfloor iD \rfloor \cdot F_i - aD \cdot F_i = \lfloor iD \rfloor \cdot F_i \leq -2 \) (by Proposition 3.1) and, thus, \( F_i \) contributes \( \beta \). Since \( 0 < \beta < 1 \) we apply Lemma 3.3 concluding that \( i = a + \beta \in \mathcal{H}_i \).

**Case 2.** \( \beta \) is not a jumping number. \( F_i \) contributes \( i \) so \( -\lfloor \beta D \rfloor \cdot F_i = -\lfloor iD \rfloor \cdot F_i + aD \cdot F_i = -\lfloor iD \rfloor \cdot F_i \geq 2 \). Now, \( \pi_* \mathcal{O}_X(-\lfloor \beta D \rfloor + K_{X|X_0} + F_i) \neq \pi_* \mathcal{O}_X(-\lfloor \beta D \rfloor + K_{X|X_0}) \) by Proposition 3.1(a). Since \( \beta \nu(\varphi_i) \) is an integer number it holds that \( \pi_* \mathcal{O}_X(\lfloor (\beta - \epsilon)D \rfloor + K_{X|X_0}) \neq \pi_* \mathcal{O}_X(\lfloor \beta D \rfloor + K_{X|X_0}) \) for all \( \epsilon > 0 \). Then \( \beta \) should be a jumping number, which is a contradiction.

This concludes the proof of Theorem 2.2. \( \square \)

3.2. **Proof of Theorem 2.3.** Next we prove Theorem 2.3 which states that if \( i \) is a jumping number of a simple complete ideal \( \varphi \), then

\[
\pi_* \mathcal{O}_X \left( -\lfloor iD \rfloor + K_{X|X_0} + \sum_{l=1}^s F_{i_l} \right) = J(\varphi^{t^<})
\]

where \( \{i_1, i_2, \ldots, i_s\} \) is the set of indexes \( i, 1 \leq i \leq g^* + 1 \), such that \( i \in \mathcal{H}_i \).

To begin with, we shall prove some technical lemmas which keep the above notation. The first one is related to Enriques’ “principle of discharge” [26, pag. 28]:

**Lemma 3.4.** Let \( F \) be a divisor of \( X \) with exceptional support and let \( E_k \) be a prime exceptional divisor such that \( F \cdot E_k > 0 \). Then \( \pi_* \mathcal{O}_X(-F) = \pi_* \mathcal{O}_X(-F - E_k) \).

**Proof.** It suffices to take global sections on the natural exact sequence

\[
0 \rightarrow \mathcal{O}_X(-F - E_k) \rightarrow \mathcal{O}_X(-F) \rightarrow \mathcal{O}_X(-F) \otimes \mathcal{O}_{E_k} \rightarrow 0
\]

\( \square \)

Our next lemma follows easily from the fact that the equality \( D \cdot E_k = 0 \) happens whenever \( k \neq n \).

**Lemma 3.5.** Let \( E_k \) be a prime exceptional divisor such that \( i \) is a candidate jumping number from \( E_k \). If \( E_k \neq F_i \) for all \( i \in \{1, 2, \ldots, g^* + 1\} \), then either \( i \) is a candidate jumping number from all prime exceptional divisors meeting \( E_k \) or \( i \) is not a candidate jumping number from none of them. \( \square \)

**Lemma 3.6.** Let \( i \in \{1, 2, \ldots, g^*\} \) and let \( F'_{i-2}, F'_{i-1} \) and \( F'_{i+1} \) be the three prime exceptional divisors meeting \( F_i \), whose corresponding vertices in \( \Gamma \) are depicted in Figure 5. If \( i \) is a candidate jumping number from \( F_i \) and \( F'_{i+1} \), then \( i \) is also so from \( F'_{i-2} \) and \( F'_{i-1} \).

**Proof.** Notice that, due to the equality \( D \cdot F_i = 0 \) and the fact that \( i \) is a candidate jumping number from \( F_i \) and \( F'_{i+1} \), it is enough to prove that \( i \) is a candidate jumping number from \( F'_{i-1} \). With the same notation of the proof of Theorem 2.2 we notice that \( \nu(\varphi_1) = t \nu(\varphi) + e_i \), where \( t \) is the number of proximate to \( F_i \) prime exceptional divisors.
Therefore \( i.e. \) is a positive integer because \( i \) is a candidate jumping number from \( F_i \) and \( F'_i \). Since \( \nu(\varphi) = e_i^{i-1} \beta_i \) and \( e_i \) divides \( \beta_i \) one has that \( \nu(\varphi) \) is also a positive integer and, hence, \( i \) is a candidate jumping number from \( F'_{i-1} \). \( \square \)

From now on, denote by \( \Delta \) the set of prime exceptional divisors (provided by the sequence \( \pi \) of \( \{1\} \)) from which \( i \) is a candidate jumping number.

**Lemma 3.7.** Let \( E_k \) be a divisor in \( \Delta \) such that \( E_k \neq F_i \) for all \( i \in \{1, 2, \ldots, g^* + 1\} \) and consider a subset \( S \) of \( \Delta \) such that \( E_k \in S \) and the cardinality of the set \( \mathcal{S}_k := \{E_j \in S \mid E_j \cap E_k \neq \emptyset\} \) is less than or equal to 1. Then

\[
\pi_* \mathcal{O}_X \left( -[tD] + K_{X|X_0} + \sum_{E_j \in S} E_j \right) = \pi_* \mathcal{O}_X \left( -[tD] + K_{X|X_0} + \sum_{E_j \in S \setminus \{E_k\}} E_j \right).
\]

**Proof.** Set \( G \) the divisor \( \lfloor tD\rfloor - K_{X|X_0} - \sum_{E_j \in S} E_j \). By Lemma 3.4 it suffices to prove that \( G \cdot E_k > 0 \). It is clear that \( G \cdot E_k = \lfloor tD\rfloor \cdot E_k-K_{X|X_0} \cdot E_k-E_k^2-\epsilon \), where \( \epsilon = 0 (\epsilon = 1, \text{ respectively}) \) whenever the cardinality of the set \( \mathcal{S}_k \) equals 0 (1, respectively). Taking into account that \( K_{X|X_0} \cdot E_k = -E_k^2 - 2 \), we get

\[
G \cdot E_k = \lfloor tD\rfloor \cdot E_k + 2 - \epsilon.
\]

When \( \mathcal{S}_k \) is empty, the condition \( G \cdot E_k > 0 \) is equivalent to \( \lfloor tD\rfloor \cdot E_k \geq -1 \), which is true by Proposition 3.1. Otherwise we must prove that \( \lfloor tD\rfloor \cdot E_k \geq 0 \). Indeed, by Lemma 3.5, \( i \) is a candidate jumping number from all prime exceptional divisors meeting \( E_k \) and, then, \( \lfloor tD\rfloor \cdot E_k = tD \cdot E_k = 0 \). \( \square \)

Given two prime exceptional divisors \( E_k \) and \( E_j \), we shall denote by \( [E_k, E_j] \) \((\lfloor E_k, E_j \rfloor, \text{ respectively})\) \( \lfloor E_k, E_j \rfloor \), \((\lfloor E_k, E_j \rfloor, \text{ respectively})\) the set of prime exceptional divisors corresponding to the vertices of the dual graph \( \Gamma \) of \( \varphi \) which are on the shortest path that joins the vertices associated with \( E_k \) and \( E_j \) (the set \( [E_k, E_j] \setminus \{E_j\} \), \((\lfloor E_k, E_j \rfloor \setminus \{E_k, E_j\} \), \((\lfloor E_k, E_j \rfloor \setminus \{E_k, E_j\} \), respectively). The length of \( [E_k, E_j] \) will be the length of the mentioned path (that is, its number of edges).

**Lemma 3.8.** Let \( S \) be a subset of \( \Delta \) and \( E_k \) a prime exceptional divisor given by \( \pi \) that is different from \( F_i \) for all \( i \in \{1, 2, \ldots, g^* + 1\} \). Assume that either \( E_k \) does not belong to \( S \), or \( E_k \) is in \( S \) and either the associated with \( E_k \) vertex in the dual graph \( \Gamma \) of \( \pi \) is a dead one or it is adjacent to a vertex whose associated divisor does not belong to \( S \). Consider any divisor \( F_r \) such that there is no divisor \( F_i \) satisfying \( F_i \in \lfloor E_k, F_r \rfloor \). Then

\[
\pi_* \mathcal{O}_X \left( -[tD] + K_{X|X_0} + \sum_{E_j \in S} E_j \right) = \pi_* \mathcal{O}_X \left( -[tD] + K_{X|X_0} + \sum_{E_j \in S \setminus \{E_k, F_r\}} E_j \right).
\]

**Proof.** It follows by applying Lemma 3.7 and making induction on the length of \( [E_k, F_r] \). \( \square \)
We conclude this subsection proving Theorem 2.3 with the help of the above lemmas. It is clear that
\[ J\left(\varphi^{<}\right) = \pi_sO_X \left(-[tD] + K_{X|X_0} + \sum_{E_i \in \Delta} E_i \right). \]
For each \( i = 1, 2, \ldots, g^* + 1 \), define \( \Delta_i' \) as the set of divisors in \( \Delta \) associated with some vertex of the subgraph \( \Gamma_i \) of \( \Gamma \) such that they do not contribute \( \iota \). Then, by Theorem 2.2
\[ \Delta \setminus \bigcup_{i=1}^{g^*+1} \Delta_i' = \{ F_{i_1}, F_{i_2}, \ldots, F_{i_g} \}. \]

To prove Theorem 2.3 it will be enough to show that
\[ J\left(\varphi^{<}\right) = \pi_sO_X \left(-[tD] + K_{X|X_0} + \sum_{E_j \in \Delta \setminus \Delta_i} E_j \right) \]
for any \( i \in \{1, 2, \ldots, g^* + 1\} \), where \( S_i := \bigcup_{k=1}^i \Delta_k' \). We shall assume that \( g^* > 0 \) and we shall apply induction on \( i \) (the result for \( g^* = 0 \) can be easily proved using reasonings of the forthcoming induction procedure).

Let us show (18) for \( i = 1 \). Set \( E_1 \) and \( E_t \) the divisors corresponding to the two dead vertices of the subgraph \( \Gamma_1 \). Applying Lemma 3.8 with \( E_k = E_1 \) and \( S = \Delta \), it happens
\[ J\left(\varphi^{<}\right) = \pi_sO_X \left(-[tD] + K_{X|X_0} + \sum_{E_j \in \Delta \setminus E_1} E_j \right) \]
Again by Lemma 3.8 but taking \( E_k = E_t \) and \( S = \Delta \setminus [E_1, F_1] \), we obtain
\[ J\left(\varphi^{<}\right) = \pi_sO_X \left(-[tD] + K_{X|X_0} + \sum_{E_j \in \Delta \setminus (E_1 \cup F_1)} E_j \right) \]
If either \( F_1 \not\in \Delta \) or \( F_1 \) contributes \( \iota \), then (18) holds for \( i = 1 \) because \( S_1 \subseteq [E_1, F_1] \). If \( F_1 \in \Delta_1' \) and \( F_2 \in \Delta \), using Lemma 3.6 we get
\[ \left([tD] - K_{X|X_0} - \sum_{E_j \in \Delta \setminus \{F_1\}} E_j \right) \cdot F_1 = tD \cdot F_1 - K_{X|X_0} \cdot F_1 - F_1^2 - 1 > 0 \]
taking into account that \( K_{X|X_0} \cdot F_1 = -F_1^2 - 2 \) and \( D \cdot F_1 = 0 \). If, otherwise, \( F_1 \in \Delta_1' \) and \( F_2 \not\in \Delta \), the fact that the left hand side of the above equality is also positive follows easily from Proposition 3.1. Thus, applying Lemma 3.4 we conclude the proof of Equality (18) for \( i = 1 \).

Assume now that (18) is true for \( i, 1 \leq i \leq g^* \), and let us show it for \( i + 1 \). With the notation of Lemma 3.6 either whether \( \iota \) is a candidate jumping number from the divisors \( F_i \) and \( F_{i+1}' \) (in which case Lemma 3.6 shows that \( [tD] \cdot F_{i-1} = tD \cdot F_i = 0 \)), what implies that \( F_i \) does not contribute \( \iota \) (by Proposition 3.1) and, therefore, \( F_i \not\in \Delta \setminus S_i \), or whether
\( i \) is not a candidate jumping number either from \( F_{i+1}' \) or \( F_i \), one can apply the induction hypothesis and Lemma 3.8 (with \( E_k = F_{i+1}' \) and \( S = \Delta \setminus S_i \)) getting the equality

\[
J \left( \wp^{<} \right) = \pi_s O_X \left( -|tD| + K_{X|X_0} + \sum_{E_j \in \Delta \setminus (S_i \cup [F_{i+1}', F_{i+1}])} E_j \right).
\]

In the cases \( i < g^* \) or \( i = g^* = g - 1 \) we consider the divisor \( E_q \) associated with the dead vertex of \( \Gamma \) in the subgraph \( \Gamma_{i+1} \). Applying again Lemma 3.8, taking \( E_k = E_q \) and \( S = \Delta \setminus (S_i \cup [F_{i+1}', F_{i+1}]) \), we have

\[
J \left( \wp^{<} \right) = \pi_s O_X \left( -|tD| + K_{X|X_0} + \sum E_j \right),
\]

where \( E_j \) runs over the set \( \Delta \setminus (S_i \cup \{F_{i+1}\}) \).

As a consequence, Equality (18) for \( i + 1 \) happens whenever \( F_{i+1} \notin S_{i+1} \). So we can assume that \( F_{i+1} \in S_{i+1} \) (that is, \( i \) is a candidate jumping number from \( F_{i+1} \) and \( F_{i+1} \) does not contribute \( i \)). To conclude our proof we study the three, a priori, existing possibilities.

**Case a.** \( F_{i+1} \) meets a divisor in \( \Delta \setminus S_{i+1} \) associated with a vertex of the graph \( \Gamma_{i} \cup \Gamma_{i+1} \). This case cannot happen because then the divisor \( F_i \) must contribute \( i \) and meet \( F_{i+1} \); but then \( F_{i-2}, F_{i-1} \in \Delta \) (by Lemma 3.6) and this implies that \( |tD| \cdot F_i = D \cdot F_i = 0 \), which is a contradiction by Proposition 3.1.

**Case b.** \( F_{i+1} \) meets a divisor in \( \Delta \setminus S_{i+1} \) associated with a vertex of the graph \( \Gamma_{i+2} \) (this cannot happen if \( i = g^* \); so we assume here that \( i < g^* \)). Applying Lemma 3.6, every divisor meeting \( F_{i+1} \) belongs to \( \Delta \) and, therefore, \( |tD| \cdot F_{i+1} = iD \cdot F_{i+1} = 0 \). Hence

\[
\left( |tD| - K_{X|X_0} - \sum_{E_j \in \Delta \setminus (S_i \cup \{F_{i+1}\})} E_j \right) \cdot F_{i+1} = 1
\]

since \( K_{X|X_0} \cdot F_{i+1} = -F^2_{i+1} - 2 \). Thus, in this case, by Lemma 3.4, (18) holds for \( i + 1 \).

**Case c.** \( F_{i+1} \) does not meet any divisor in \( \Delta \setminus (S_{i+1} \setminus \{F_{i+1}\}) \). Then

\[
\left( |tD| - K_{X|X_0} - \sum_{E_j \in \Delta \setminus (S_{i+1} \setminus \{F_{i+1}\})} E_j \right) \cdot F_{i+1} = \left( |tD| - K_{X|X_0} - F_{i+1} \right) \cdot F_{i+1} = |tD| \cdot F_{i+1} + 2 > 0,
\]

where the inequality holds since, by Proposition 3.4, \( |tD| \cdot F_{i+1} \geq -1 \). This finishes the proof after applying Lemma 3.4.

### 3.3. Proof of Theorem [2.1]

We end this paper by proving our main result, Theorem 2.1, that provides an explicit expression for the series \( P_g(t) \).

For a start we state the following result, proved in [5] Lem. 4] in a more general framework, that will be useful.
Theorem 3.1. Let $V = \{\nu_1, \nu_2, \ldots, \nu_s\}$ be a finite family of divisorial valuations of $K$ centered at $R$. Set $S_V := \{(\nu_1(h), \nu_2(h), \ldots, \nu_s(h)) \in \mathbb{Z}^s | h \in R \setminus \{0\}\}$, $\mathbb{Z}$ denoting the integer numbers and

$$B^l := (\nu_1(\psi_1), \nu_2(\psi_1), \ldots, \nu_s(\psi_1)) = (B^l_1, B^l_2, \ldots, B^l_s),$$

where $1 \leq l \leq s$ and $\psi_l$ is a general element for $\nu_l$. Then the following statements hold:

1. Suppose $s \geq 2$, fix and index $l$ and consider another one $1 \leq k \leq s$, $k \neq l$; if $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_s) \in S_V$, then

$$d_k(\alpha) := \dim \mathbb{C} \frac{\pi_s \mathcal{O}_X(\sum_{j=1}^s \alpha_j E_j)}{\pi_s \mathcal{O}_X(-\sum_{j=1}^s \alpha_j E_j - E_k)} = \dim \mathbb{C} \frac{\pi_s \mathcal{O}_X(-\sum_{j=1}^s (\alpha_j + B^l_j) E_j)}{\pi_s \mathcal{O}_X(-\sum_{j=1}^s (\alpha_j + B^l_j) E_j - E_k)}.$$

2. Assume that $d_l(\alpha) \neq 0$ for some index $l$, then

$$\dim \mathbb{C} \frac{\pi_s \mathcal{O}_X(-\sum_{j=1}^s (\alpha_j + B^l_j) E_j)}{\pi_s \mathcal{O}_X(-\sum_{j=1}^s (\alpha_j + B^l_j) E_j - E_l)} = 1 + \dim \mathbb{C} \frac{\pi_s \mathcal{O}_X(\sum_{j=1}^s \alpha_j E_j)}{\pi_s \mathcal{O}_X(\sum_{j=1}^s \alpha_j E_j - E_l)}. \quad \square$$

Now we return to the situation and notations of this paper, and recall that a divisor with exceptional support, $E$, is named antinef whenever $E \cdot E_j \leq 0$ for all $j = 1, 2, \ldots, n$. We shall use the following well-known result (see [16, Lem. 1.2] for instance): If $E$ is a divisor on $X$ with exceptional support, then there is an antinef divisor $E^-$ (called antinef closure of $E$) such that $E^- \geq E$ and $\pi_s \mathcal{O}_X(-E) = \pi_s \mathcal{O}_X(-E^-)$ (in fact, $E^-$ is the least antinef divisor $\geq E$). It can be computed by means of the following procedure: If $E \cdot E_j \leq 0$ for all $j$ then $E^- = E$; otherwise set $E^0 := E + E_j$, where $E_j$ is such that $E \cdot E_j > 0$, and repeat the procedure replacing $E$ by $E^0$. Due to Lemma 3.4 the antinef closure of $E$ will be obtained after finitely many steps.

For each $i \in H_i$, $1 \leq i \leq g^* + 1$, we denote by $d_i^t$ the following dimension that we shall use in the proof.

$$d_i^t := \dim \mathbb{C} \left( \frac{\pi_s \mathcal{O}_X(K_{X|X_0} - [\ell D] + F_i)}{J(\varphi^t)} \right).$$

By Theorem 2.23 and Lemma 3.2 we have that $p_i(t) = \sum_{i=1}^{g^*+1} P_i(t)$, where $P_i(t) = \sum_{i \in H_i} d_i^t t^i$. So, we only need to compute $P_i(t)$ for any $i$. Write $z_i = t^{1/(e_i - \beta_i)}$.

Firstly assume that $1 \leq i \leq g^*$. To get $P_i(t)$, we shall use the following

Lemma 3.9. Let $i$ be an index as above. Then,

1. The map $(p, q, s) \mapsto \nu(i, p, q, s)$ gives a bijection between the sets $B := \{(p, q, s) \in \mathbb{Z}^3 | p, q \geq 1, p\beta_i + qe_i - 1 \leq n_i\beta_i \text{ and } 0 \leq s \leq e_i - 1\}$ and $\mathcal{H}_i \cap [0, 1]$.

2. If $i \in H_i$ and $i > 1$, then there exist a unique $(p, q, s) \in B$ and a unique positive integer $r$ such that $i = \nu(i, p, q, s) + r = \nu(i, p, q, s + re_i)$.

Proof. (1) follows from the fact that $e_i - 1$ and $\beta_i$ are relatively prime and (2) from (1) and the arithmetical expressions of the jumping numbers in $\mathcal{H}_i$. \qed

As a consequence of the above result we obtain the equality

\[ P_t(t) = \sum_{p,q \geq 1, r \geq 0, p \beta_i + qe_i - 1 + rn_i \beta_i} d_i^{n_j(i,p,q,r)} z_i^{p \beta_i + qe_i - 1 + rn_i \beta_i}. \]

Fix any triple of non-negative integers \((p, q, s)\) in \( B \) and write

\[ \sigma(i, p, q, s) := z_i^{p \beta_i + qe_i - 1 + sm_i \beta_i} \sum_{r \geq 1} d_i^{n_j(i,p,q,s+r e_i)} z_i^{re_i n_i \beta_i}. \]

Then one gets

\[ P_t(t) = \sum_{(p,q,s) \in B} \sigma(i, p, q, s). \]

So, we are going to compute the expressions \( \sigma(i, p, q, s) \). To do it we shall prove the following

**Lemma 3.10.** With the above notations and assumptions, it holds that

\[ d_i^{n_j(i,p,q,s+r e_i)} = d_i^{n_j(i,p,q,s)} \]

for all non-negative integer \( r \).

**Proof.** We shall reason by induction on \( r \). Since the equality is evident for \( r = 0 \), we assume that \( d_i^{n_j(i,p,q,s+r e_i)} = d_i^{n_j(i,p,q,s+(r-1)e_i)} \). Set \( J \left( \varphi^{n_j(i,p,q,s+r e_i)} \right) = \pi_\ast \mathcal{O}_X \left( -\sum_{j=1}^n \alpha_j E_j \right) \), where \( (\alpha_1, \ldots, \alpha_n) = (\lfloor t(i, p, q, s + re_i) \nu(\varphi_1) \rfloor - \kappa_1, \ldots, \lfloor t(i, p, q, s + re_i) \nu(\varphi_n) \rfloor - \kappa_n) \), \( \sum_{j=1}^n \kappa_j E_j \) being \( K_X | X_0 \). Thus

\[ J \left( \varphi^{n_j(i,p,q,s+(r+1)e_i)} \right) = \pi_\ast \mathcal{O}_X \left( -\sum_{j=1}^n (\alpha_j + \nu(\varphi_j)) E_j \right). \]

Let \( \sum_{j=1}^n \beta_j E_j \left( \sum_{j=1}^n (\beta_j + \nu(\varphi_j)) E_j \right) \), be the antiflne closure of the divisor \( \sum_{j=1}^n \alpha_j E_j - F_i \left( \sum_{j=1}^n (\alpha_j + \nu(\varphi_j)) E_j - F_i \right) \). As \( D \cdot E_j = 0 \) \( (1 \leq j \leq n-1) \) and \( D \cdot E_n = -1 \), \( D \) being the associated to \( \varphi \) divisor \( D = \sum_{j=1}^n \nu(\varphi_j) E_j \), it is easy to deduce from the above described procedure for computing antiflne closures that \( \beta_j = 0 \) whenever \( j < n \) and \( \beta_n \leq \beta_n \). Moreover one has that \( \beta_{st_1} = \beta'_{st_1} = \alpha_{st_1} - 1 \) (since \( F_i \) contributes \( \nu \)).

Now, consider the commutative diagram

\[
\begin{array}{ccc}
\pi_\ast \mathcal{O}_X(\alpha + F_i) & \xrightarrow{g} & \pi_\ast \mathcal{O}_X(\alpha) \\
\pi_\ast \mathcal{O}_X(\alpha + F_i) & \xrightarrow{h} & \pi_\ast \mathcal{O}_X(\beta' + F_i) \\
\end{array}
\]

\[
\begin{array}{ccc}
\pi_\ast \mathcal{O}_X(\alpha + F_i) & \xrightarrow{f} & \pi_\ast \mathcal{O}_X(\alpha + F_i) \\
\pi_\ast \mathcal{O}_X(\alpha + F_i) & \xrightarrow{i} & \pi_\ast \mathcal{O}_X(\beta' + F_i) \\
\end{array}
\]

where \( D(\alpha) := -\sum_{j=1}^n \alpha_j E_j \), \( D(\beta) := -\sum_{j=1}^n \beta_j E_j \), \( D(\alpha + \varphi) := -\sum_{j=1}^n (\alpha_j + \nu(\varphi_j)) E_j \), \( D(\beta + \varphi) := -\sum_{j=1}^n (\beta_j + \nu(\varphi_j)) E_j \), \( f \) and \( h \) are the identity homomorphisms (notice that \( \pi_\ast \mathcal{O}_X(D(\alpha)) = \pi_\ast \mathcal{O}_X(D(\beta) - F_i) \) and...
\[ \pi_*(D(\alpha + \varphi)) = \pi_* O_X(D(\beta' + \varphi) - F_i) \], \( i \) is defined by the product by \( \varphi_n^\beta_n - \beta_n' \) and \( g \) and \( m \) are given by the product by \( \varphi_n \) (notice that \( \nu(\varphi_j) = \nu_{E_j}(\varphi_n) \), \( \nu_{E_j} \) being the divisorial valuation provided by the exceptional divisor \( E_j \)). This proves \( d_{i(i,p,q,s+(r+1)e_i)} = d_{i(i,p,q,s+re_i)} \) and thus our lemma, since \( g \) is an isomorphism because \( m \) is also an isomorphism by the proof in [5, Lem. 4] of Statement (1) in Theorem 3.11. \( \square \)

Notice that \( d_{i(i,p,q,s)}^e = 1 \) taking into account that \( \iota(i,p,q,s) < 1 \) (by the remark after Lemma 3.3) and the remark after Lemma 3.9. Therefore

\[ \sigma(i,p,q,s) = z_1^{\beta_1 + qe_{i-1} + sn_i} \sum_{r \geq 1} r e_{i,n_i} \beta_i = z_1^{\beta_1 + qe_{i-1} + (s+e_i)n_i} \left( \beta_i + 1 \right) \]

and this implies that

\[ P_i(t) = \frac{1}{1-t} \sum_{i\in \mathcal{H}_i, i < 1} t^i. \]

Now we shall obtain an expression for the series \( P_{g^*+1}(t) \). We shall use a similar result to Lemma 3.9 whose proof follows from the fact that \( e_{g^*} \) and \( \beta_{g^*+1} \) are relatively prime.

**Lemma 3.11.** The following statements hold:

1. The map \( (s,q) \mapsto \iota(g^*+1,s,q) \) gives a bijection between the sets \( T := \{(s,q) \in \mathbb{Z}^2 \mid 1 \leq s \leq e_{g^*} \text{ and } 1 \leq q \leq \beta_{g^*+1} \} \) and \( \Omega := \{i \in \mathcal{H}_{g^*+1} \mid i \leq 2 \text{ and } i - 1 \notin \mathcal{H}_{g^*+1} \} \).
2. If \( \iota \in \mathcal{H}_{g^*+1} \) and \( \iota > 2 \) then there exist a unique \( (s,q) \in T \) and a unique positive integer \( r \) such that \( \iota = \iota(g^*+1,s,q) + r = \iota(g^*+1,s+re_{g^*},q) \). \( \square \)

Set \( T \) as in Lemma 3.11. As a consequence of that lemma, one can see that

\[ P_{g^*+1}(t) = \sum_{(s,q) \in T} \tau(s,q), \]

where

\[ \tau(s,q) := \sum_{r \geq 0} d_{\iota(g^*+1,s+re_{g^*},q)}^{\beta_{g^*+1} + qe_{g^*}} z_{g^*+1}. \]

Applying (2) in Theorem 3.11 for any fixed pair \( (s,q) \in T \), it happens that

\[ d_{\iota(g^*+1,s+re_{g^*},q)}^{\beta_{g^*+1}} = d_{\iota(g^*+1,s,q)}^{\beta_{g^*+1}} + r \text{ for all } r \geq 0. \]

From this fact, one can deduce the equality

\[ \tau(s,q) = z_{g^*+1}^{\beta_{g^*+1} + qe_{g^*}} \left[ d_{\iota(g^*+1,s,q)}^{\beta_{g^*+1}} + \frac{z_{g^*+1}}{1 - z_{g^*+1}^{\beta_{g^*+1}}} \right], \]

therefore

\[ P_{g^*+1}(t) = \sum_{\iota \in \Omega} t^\iota \left( d_{\iota(g^*+1,s,q)}^{\beta_{g^*+1}} + \frac{t}{1-t} \right), \]

\( \Omega \) being as in Lemma 3.11.

Since \( d_{\iota(g^*+1)}^{2} = 1 \) whenever \( \iota < 1 \) (by the remark after Lemma 3.8), it only remains to prove that \( d_{\iota(g^*+1)}^{2} = 1 \) for all \( \iota \in \mathcal{H}_{g^*+1} \) such that \( 1 < \iota \leq 2 \) and \( \iota - 1 \) is not a
jumping number (recall that 1 is not a jumping number \[11\), Prop. 8.9\]). Thus, consider a jumping number \( \iota \) satisfying these conditions and, reasoning by contradiction, assume that \( \pi^{\iota+1} \geq 2 \). Let \( \alpha_j \) be the coefficient of \( E_j \) in the divisor \([\iota D] - K_{X|X_0}, 1 \leq j \leq n\). The natural monomorphism of vector spaces

\[
\frac{\pi_* \mathcal{O}_X(-[\iota D] + K_{X|X_0} + E_n)}{\mathcal{J}(\psi^\iota)} \xrightarrow{f_{\psi^\iota}} \frac{\pi_* \mathcal{O}_X(-[\alpha_n - 1]E_n)}{\mathcal{O}_X(-\alpha_n E_n)}
\]

and \( [9] \) Th. 1 show that the vector space on the left is generated by classes of elements of the type \( \prod_{k=0}^{g+1} \varphi_k^{b_k} \), where \( b_k, 0 \leq k \leq g + 1, \) are nonnegative integers and \( t_0, \ldots, t_{g+1} \) are as in the paragraph after Definition 2.1. Since two elements of this type satisfying the condition \( b_{g+1} = 0 \) are linearly dependent (see the proof of \( [9] \) Th. 1) there exists \( f \in R \) such that the class of \( f \varphi_n \) is a non-zero element of the vector space \( \pi_* \mathcal{O}_X(-[\iota D] + K_{X|X_0} + E_n)/\mathcal{J}(\psi^\iota) \) and therefore \( \nu_{E_n}(f \varphi_n) = \nu(f \varphi_n) = \alpha_n - 1 \) and \( \nu_{E_j}(f \varphi_n) \geq \alpha_j, 1 \leq j < n \). Thus \( \nu_{E_n}(f) = \alpha_n - 1 - \nu_{E_n}(\varphi_n) \) and \( \nu_{E_j}(f) \geq \alpha_j - \nu_{E_j}(\varphi_n), 1 \leq j < n \). This means that \( f \) is in \( \pi_* \mathcal{O}_X(-[(\iota - 1)D] + K_{X|X_0} + E_n) \) but it is not in \( \pi_* \mathcal{O}_X(-[\iota D] + K_{X|X_0}) \) what implies that \( \iota - 1 \) is a jumping number, which contradicts our assumptions. This concludes the proof of Theorem 2.1. \( \square \)

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