Studying the Properties of Compressible Magnetohydrodynamic Turbulence Using Synchrotron Fluctuation Statistics

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Abstract

We study the observable properties of compressible magnetohydrodynamic (MHD) turbulence covering different turbulence regimes, based on synthetic synchrotron observations arising from 3D MHD numerical simulations. Using the synchrotron emissivity and intensity, we first explore how the cosmic-ray spectral indices affect the measurements of the turbulence properties by employing normalized correlation functions. We then study how the anisotropy of the synchrotron total and polarization intensities arising from the three fundamental MHD modes varies with the viewing angle, i.e., the angle between the mean magnetic field and the line of sight. We employ the ratio of the quadrupole moment to the monopole moment (QM) for this purpose. Our numerical results demonstrate that: (1) the two-point correlation function of synchrotron statistics for the arbitrary cosmic-ray spectral index is related to the special case of the magnetic field index $\gamma = 2$, in agreement with the analytical formulae provided by Lazarian & Pogosyan; (2) the anisotropy of the synchrotron total and polarization intensities arising from the Alfvén and slow modes increases with the increase of the viewing angle, while that of fast mode remains almost unchanged with the viewing angle; and (3) the analytical formulae of the synchrotron intensities for studying turbulence can be applied to describe the statistics of the polarization intensities, and the QM can be successfully used to recover the turbulence anisotropy. This study validates the analytical approach of Lazarian & Pogosyan and opens up a way of studying turbulence from observations.

Unified Astronomy Thesaurus concepts: Interplanetary turbulence (830); Interstellar synchrotron emission (856); Magnetohydrodynamics (1964); Interstellar medium (847)

1. Introduction

Magnetohydrodynamic (MHD) turbulence, resulting from the interaction of fluids and magnetic fields, is ubiquitous in astrophysical environments. The most significant evidence is the spectral distribution of electron density fluctuations in the Milky Way, termed the big power law in the sky (Armstrong et al. 1995; Chepurnov & Lazarian 2010). MHD turbulence has significant impacts on fundamental astrophysical processes, such as star formation (Mac Low & Klessen 2004; McKee & Ostriker 2007; Crutcher 2012), the propagation and acceleration of cosmic rays (Yan & Lazarian 2008; Xu & Lazarian 2018), heat conduction (Narayan & Medvedev 2001; Lazarian 2006), and turbulent magnetic reconnection (Lazarian & Vishniac 1999; hereafter LV99; Lazarian et al. 2020). Therefore, a comprehensive understanding of the properties of MHD turbulence is indispensable to describing astrophysical processes.

The theory of MHD turbulence has been developing over several decades (see Biskamp 2003 for a book and Beresnyak & Lazarian 2019 for a more recent book). A key turning point in the construction of the modern MHD turbulence theory dates back to the study of Goldreich & Sridhar (1995; hereafter, GS95), which focuses on incompressible MHD turbulence. The most important contribution of the GS95 theory is the prediction of the scale-dependent anisotropy of the turbulence cascade. Further insight into the nature of the MHD turbulence cascade was provided by the theory of turbulent reconnection in Lazarian & Vishniac (1999; hereafter, LV99). Here, it was shown that MHD turbulence can be presented as eddies, whose rotation is aligned with the direction of the magnetic field in their vicinity. Indeed, the prediction of LV99 was that turbulent reconnection takes place within the eddy turnover time, and therefore the motions of the magnetic field lines that are perpendicular to the local magnetic field are not constrained by magnetic field tension. The turbulent energy therefore cascades along the path of least resistance, which involves mixing magnetic field lines rather than bending them. In this picture of MHD turbulence, it is worth noting that the anisotropic relation was originally proposed by GS95 in the frame of the mean magnetic field, which should in fact only be relevant in the local frame of reference (LV99; Cho & Vishniac 2000; Maron & Goldreich 2001). The decomposition of compressible MHD turbulence into three modes has been an important topic in the development of MHD turbulence theory (Cho & Lazarian 2002—hereafter, CL02; 2003—hereafter, CL03; Kowal & Lazarian 2010; Hernández-Padilla et al. 2020; Hu et al. 2021; Wang et al. 2021; Zhang & Xiang 2021). The research results have become important components of modern MHD turbulence theory. In particular, MHD turbulence has been numerically decomposed into Alfvén, slow, and fast modes in Fourier space (CL02; CL03), providing new perspectives for understanding MHD turbulence. Later, the Fourier decomposition was confirmed by wavelet decomposition, and generalized to these cases of no mean magnetic field and super-Alfvénic turbulence (Kowal & Lazarian 2010).
Although direct numerical simulations have achieved many valuable results relating to the properties of MHD turbulence, the current simulations—up to Reynolds number \( R_e \approx 10^3 \)—still cannot simulate realistic astrophysical environments (Beresnyak 2019), given the inherent high–Reynolds number characteristics of astrophysics, such as the interstellar medium (ISM) with \( R_e > 10^{16} \). Therefore, a new observation-based research perspective has been adopted, in an attempt to avoid the difficulties of direct numerical simulation.

Currently, a number of statistical techniques from observational perspectives have been developed to explore the properties of MHD turbulence. These can be divided into two major categories, in terms of the information obtained. One involves Doppler-shifted spectroscopic data (see Lazarian 2009 for a review, as well as Lazarian & Pogosyan 2000; Chepurnov & Lazarian 2009; Kandel et al. 2016, 2017). The other important branch of research relates to synchrotron radiation (e.g., Lazarian & Pogosyan 2012—hereafter, LP12; 2016—hereafter, LP16; Herron et al. 2018a; Zhang & Wang 2022), with the most typical approaches being those of LP12 and LP16.

When the relativistic electrons spiraling around the magnetic field are accelerated, a synchrotron radiation signal is emitted. Analyzing synchrotron radiation fluctuations provides a powerful way of studying the properties of astrophysical magnetic fields (Rickett 1990; Heiles & Haverkorn 2012; Haverkorn et al. 2019; Thomson et al. 2019; Wolleben et al. 2021; Erceg et al. 2022). LP12 first proposed a theoretical description of the synchrotron intensity fluctuations arising from magnetic turbulence. They provided analytical formulae that relate the correlation of synchrotron fluctuations for an arbitrary index of relativistic electrons to the correlation for a particular magnetic field index \( \gamma = 2 \), and predicted the anisotropy of the synchrotron intensity using the ratio between the quadrupole and monopole parts, being sensitive to the compressibility of the underlying turbulence. Furthermore, based on a statistical analysis of the synchrotron polarization intensity, LP16 suggested that polarization spatial and polarization frequency analysis techniques could reveal the properties of MHD turbulence, such as the spectrum, anisotropy, and correlation scales.

The polarization frequency analysis technique was studied by Zhang et al. (2016), who confirmed the theoretical predictions of LP16 and achieved the measurement of the spectral index of the underlying magnetic turbulence. Subsequently, the technique was applied to polarization observations of optical/infrared blazars (Guo et al. 2017). Based on synthetic and real simulation data of MHD turbulence, the polarization spatial analysis technique was successfully tested in the spatially coincident synchrotron emission and Faraday regions (Lee et al. 2016) as well as the spatially separated regions (Zhang et al. 2018). These efforts focused on studying how to recover the spectral index of the magnetic turbulence.

Using the quadrupole ratio method that was proposed in the synchronized intensity fluctuation study (LP12), Lee et al. (2019) numerically studied the statistical description of the anisotropy of the polarized synchrotron intensity arising from one spatial region and two spatially separated regions, respectively. Using the same method, Wang et al. (2020) explored the compressibility and anisotropy of MHD turbulence, and found that the anisotropic features of Alfven, slow, and fast modes are in agreement with the earlier direct numerical simulations of MHD turbulence (CL02; CL03).

In addition, synchrotron intensity and polarization gradient techniques (Lazarian et al. 2017; Lazarian & Yuen 2018) have been developed to trace the direction of the magnetic field (Zhang et al. 2019a, 2019b, 2020; Wang et al. 2021). It is noticed that the synchrotron polarization gradient was suggested by Gaensler et al. (2011) to constrain the sonic Mach number of the warm ionized ISM. Later, various polarization diagnostic quantities were derived in Herron et al. (2018a), then applied to simulation and observational data, to distinguish between backlit and internal emission (Herron et al. 2018b). Applying synchrotron diagnostic gradients to the archive data from the Canadian Galactic Plane Survey (Taylor et al. 2003; Zhang et al. 2019b) obtained consistent predictions for the gradient directions and the Galactic magnetic field directions. More recently, Wang et al. (2021) have explored the capabilities of gradient techniques in the case of the spatial separation of synchrotron polarization and Faraday rotation regions.

The purpose of this paper is to advance the study of the compressibility of MHD turbulence. We want to know whether the analytical descriptions provided in LP12 can be used to reveal the anisotropy of the magnetic turbulence. For example, does the change in the angle between the mean magnetic field and the line of sight (LOS) hinder the application of synchrotron statistics techniques? And does the spectral index of the relativistic electrons affect the measurement of the compressible turbulence properties, compared to the analytical descriptions of LP12? At the same time, studies of synchrotron radiation intensity are also generalized to the synchrotron polarization intensity.

This paper is organized as follows. In Section 2, we provide theoretical descriptions, including the fundamental theory of MHD turbulence, the calculation of synchrotron radiation, the analytical expressions of LP12, and the statistical methods. In Section 3, we introduce the procedure for the numerical simulation of MHD turbulence. The results are presented in Section 4. Finally, we offer a discussion and a summary in Sections 5 and 6, respectively.

2. Theoretical Descriptions

2.1. Fundamentals of MHD Turbulence Theory

The theory of GS95 is the starting point for understanding modern MHD turbulence. This focused on incompressible trans-Alfvénic turbulence with Mach number \( M_A = V_L / V_A = 1 \), where \( V_A \) and \( V_L \) are the Alfvénic velocity and the injection velocity \( V_L \) at the injection scale \( L_{inj} \), respectively. With an assumption of the critical balance, \( v_L k_L = V_A k_L \), GS95 predicted the anisotropic scaling relation with regard to the parallel and perpendicular wavenumbers, namely \( k_\parallel \propto k_\perp^{2/3} \), where \( v_L \) is the velocity at the scale \( k^{-1} \).

Later, the theory of GS95 was generalized to cases for both \( M_A < 1 \) (LV99) and \( M_A > 1 \) (Lazarian 2006), respectively. For the former, the injection velocity \( V_L \) is smaller than the Alfvénic velocity \( V_A \), which is called sub-Alfvénic turbulence.

4 The mean magnetic field is consistent with the large-scale or ordered magnetic field (Fletcher et al. 2011). The presence of the magnetic field makes MHD turbulence anisotropic (Montgomery & Turner 1981; Shobal et al. 1983; Higdon 1984; GS95), and the larger the mean magnetic field, the more pronounced the anisotropy. In the system of reference of the mean magnetic field, the anisotropy of the eddies is scale-independent (different from the anisotropy of GS95), and the degree of anisotropy is determined by the largest eddies (Cho et al. 2002; Esquivel & Lazarian 2005).
When driving at an injection scale $L_{\text{inj}}$, the turbulence is weak from the injection scale $L_{\text{inj}}$ to the transition scale

$$L_{\text{trans}} = L_{\text{inj}} M_A^2. \quad (1)$$

The turbulence becomes strong when the scale is smaller than the transition scale $L_{\text{trans}}$, but larger than the dissipation scale $L_{\text{diss}}$. In this case, the relationship between the parallel and perpendicular scales for an eddy is described as

$$l_1 \approx L_{\text{inj}}^{1/3} L_{\text{trans}}^{2/3} M_A^{-4/3}. \quad (2)$$

In terms of perpendicular motion, the spectrum of Alfvén modes for $l_1 < L_{\text{trans}}$ is Kolmogorov, which is self-evident from the eddy description of Alfvénic turbulence provided above. Compared to the trans-Alfvénic case, the eddies for sub-Alfvénic turbulence are more elongated. When the spectrum is measured in the global system—i.e., the system of reference regarding the mean magnetic field—the effects of perpendicular motions are dominated by the case both parallel to the mean magnetic field and perpendicular to it.

If $M_A > 1$, the turbulence is super-Alfvénic. Such turbulence at the scale close to $L_{\text{inj}}$ has an essentially hydrodynamic Kolmogorov property. The cascade becomes fully magnetic, from hydrodynamic turbulence to MHD turbulence, at the scale (Lazarian 2006)

$$L_A = L_{\text{inj}} M_A^3, \quad (3)$$

which corresponds to the scale at which the turbulent velocity becomes equal to the Alfvén velocity. In the range of $[L_A, L_{\text{diss}}]$, the super-Alfvénic turbulence is similar to trans-Alfvénic MHD turbulence, with an injection scale equal to $L_A$.

The theory of GS95 provided insights into the expected properties of compressible MHD turbulent (see also Lithwick & Goldreich 2001). The incompressible Alfvén and compressible slow modes have the same scale-dependent anisotropy as the Alfvén modes in incompressible turbulence (see above). The Alfvén modes also show the Kolmogorov property as a simple formula $E(k) \propto k^{-5/3}$, where $E$ is the power spectrum of the turbulent motions, while the scaling of the compressible fast mode is less clear. It has been suggested that for subsonic driving, the spectrum follows $E(k) \propto k^{-3/2}$ (CL03), while for supersonic driving, the spectrum steepens to $k^{-2}$. In addition, the Alfvén and slow modes show GS95 scale-dependent anisotropy, while the fast mode with isotropy is different from the above two modes.

2.2. Synchrotron Radiative Processes in Magnetic Turbulence

The magnetic field and the relativistic electron energy distribution are two major factors that determine synchrotron radiation. In this paper, we assume that the relativistic electron population has an isotropic pitch-angle distribution and a power-law energy distribution, described by

$$N(E)dE = N_0 E^{\alpha - 1}dE, \quad (4)$$

where $N(E)$ is the number density of the relativistic electron with energy between $E$ and $E + dE$, $N_0$ is a normalization constant, and $\alpha$ is the spectral index.

The synchrotron radiation intensity at a fixed frequency $\nu$ is calculated by Ginzburg & Syrovatskii (1965) and Ginzburg (1981):

$$I(\nu) = \frac{e^3}{4\pi m_e c^3} \int_0^{\sqrt{3\frac{2 - 2\alpha}{2 - 6\alpha}}} \sqrt{\frac{2 - 6\alpha}{12}} \Gamma \left(\frac{26 - 6\alpha}{12}\right) \times \left(\frac{3e}{2\pi m_e^2 c^5}\right)^{-\alpha} N_0 B(-\nu)(X, z) \nu^\alpha d\nu,$$  \quad (5)

where $X = (x, y)$ represents a 2D vector in the plane of the sky (POS), $\Gamma$ is the gamma function, $\gamma = 1 - \alpha$ is the index of the magnetic field, $B_\perp$ is the magnetic field component perpendicular to the LOS, and $L$ is the size of the emitting region. The other symbols have their usual meanings.

When the radiation is linearly polarized, the intrinsic polarization intensity is calculated by

$$P_0(X) = \Pi L I(\nu), \quad (6)$$

where $\Pi L = \frac{3 - 3\alpha}{5 - 3\alpha}$ is the fraction polarization degree. The observable Stokes parameters $Q$ and $U$ relating to the polarization angle $\Psi$ can be expressed as $Q(X) = P_0(X) \cos 2\Psi$ and $U(X) = P_0(X) \sin 2\Psi$, respectively. When the polarized emission does not experience the Faraday rotation effect, the polarization angle corresponds to the intrinsic polarization angle, namely $\Psi = \Psi_0 = \pi/2 + \arctan(B_y/B_x)$. When encountering the Faraday rotation effect, the polarization angle is written as $\Psi = \Psi_0 + \text{RM} \lambda^2$, with the Faraday rotation measure

$$\text{RM}(X, z) = 0.81 \int_0^z n_e(X, z') B_y(X, z') dz' \text{ rad m}^{-2}, \quad (7)$$

where $n_e$ is the number density of the thermal electrons, $B_y$ is the magnetic field component along the LOS, and $z$ is a variable. The integration of Equation $(7)$ is along the LOS from the observer to the position of the source at $z' = z$. Moreover, the observable polarization intensity can be expressed as

$$P(X) = \sqrt{Q^2(X) + U^2(X)}. \quad (8)$$

2.3. Analytical Expressions of Correlation for Anisotropic Turbulence

LP12 provides the expressions for the correlation function of the synchrotron intensity, which relates to the change in the relativistic electron spectral index for the isotropic and anisotropic turbulence, respectively. The formula for isotropic turbulence has been tested by numerical simulation (Herron et al. 2016). The anisotropic turbulence will be explored in this paper. In this case, the normalized correlation function (NCF) of the synchrotron emissivity is defined by

$$C_{\text{e}}^2 = \frac{\langle B_y^2(x)B_y^2(x + r) \rangle - \langle B_y^2(x) \rangle^2}{\langle B_y^2(x) \rangle^2 - \langle B_y^2(x) \rangle^2}, \quad (9)$$

where $\langle \ldots \rangle$ indicates an average through the whole volume space, $B_y(x) = \sqrt{\langle B_x^2 \rangle + B_y^2(x)}$ is the component of the magnetic field perpendicular to the LOS, $x = (x, y, z)$ represents...
the position vector of any spatial point in the emitting region, \( \mathbf{r} \) is the separation vector between two points, and \( \langle B_x \rangle \) indicates the mean magnetic field.

The index \( \gamma \) constituted a large problem in constructing a statistical theory for the synchrotron fluctuations. This problem was resolved by LP12, which obtained an analytical relation between the correlations of synchrotron fluctuations for an arbitrary \( \gamma \) and those for \( \gamma = 2 \). The relations obtained in LP12 demonstrated that the two types of statistics differed by a factor that was independent of the lag over which the correlations were measured. This opened up a way to study the synchrotron statistics for \( \gamma = 2 \), simplifying the problem significantly.

Setting \( \gamma = 2 \), Equation (9) can be expressed as (see LP12)

\[
\hat{C}(\mathbf{r}) = c(\mathbf{r})^2,
\]

where the symbol \( c(\mathbf{r}) \) on the right-hand side is

\[
c(\mathbf{r}) = \frac{\langle B_x(\mathbf{x})B_x(\mathbf{x} + \mathbf{r}) \rangle - \langle B_x(\mathbf{x}) \rangle^2}{\langle B_x(\mathbf{x}) \rangle^2 - \langle B_x(\mathbf{x}) \rangle^2}.
\]

Since the magnetic field is assumed to be far from the fluctuations along the mean magnetic field direction, i.e., the x-axis direction, Equation (10) is only associated with the component \( B_x \).

2.4. Analytical Expressions of Anisotropy for the Three Modes

Here, we briefly summarize the analytical expressions for the anisotropy of the three modes, as follows (see LP12 for more details).

**Alfvén mode.** The multipole of the synchrotron (polarization) intensity is expressed as

\[
\hat{D}_m(R) \approx C_m(2/3) \left[ W_l \left( \hat{P}_m - \frac{1}{2} \hat{P}_m + \hat{P}_m - 2 \right) \right] + W_l \sin^2 2 \int_{n=-\infty}^{\infty} \left[ \hat{P}_n - \frac{1}{2} \hat{P}_n + \hat{P}_n - 2 \right] A_{m-n}^A(\theta) \text{ } R^{5/3},
\]

where the multipole includes two parts that are associated with the weight functions \( W_l \) and \( W_t \). The relevant parameters in Equation (12) are introduced as follows.

1. The parameter \( C_m(2/3) \), depending on the scaling of the synchrotron correlation \( (\mu) \), can be determined by

\[
C_m(\mu) = -\frac{i^{m} \Gamma[\frac{1}{2}(|m| - \mu - 1)]}{2^{2+\mu} \Gamma[\frac{1}{2}(|m| + \mu + 3)]}.
\]

2. The isotropic and local weight functions, written as (Lazarian et al. 2022)

\[
W_l \approx \frac{M_k^2}{2 + 2M_k^2},
\]

\[
W_l + W_t \approx \frac{1 + M_k^2}{1 + 2M_k^2},
\]

characterize the relative proportions of the degree of isotropy and anisotropy, respectively.

3. The parameter \( \epsilon \), representing the level of anisotropy, is defined by

\[
\epsilon = \frac{\sigma_{xx} - \sigma_{yy}}{\sigma_{xx} + \sigma_{yy}},
\]

where \( \sigma_{xx} = \sigma_{y}^2 \sin^2 2 \theta + \frac{1}{2} \sigma_{\perp}^2 \cos^2 2 \theta, \sigma_{yy} = \frac{1}{2} \sigma_{\perp}^2 \). Here, \( \theta \) is the angle between the mean magnetic field and the LOS, \( \sigma_{\parallel} \) and \( \sigma_{\perp} \) denote the fluctuations parallel and perpendicular to the mean magnetic field, respectively.

4. The harmonic decomposition of the 2D spectra is defined by

\[
\hat{P}_m = \frac{1}{2\pi} \int_{0}^{2\pi} d\psi e^{-i\psi} \exp \left[ -M_{\perp}^2 \frac{\cos \psi \sin^2 2 \theta}{(1 - \cos^2 \psi \sin^2 2 \theta)^2} \right],
\]

where \( \hat{P}_m \) has the same expression as \( \hat{P}_m^A \), with the uncertain subscripts \( n \).

5. The geometrical function for the Alfvén mode is given by

\[
A_{m-n}^A(\theta) = \frac{1}{2\pi} \int_{0}^{2\pi} d\psi e^{-i(m-n)\psi} \frac{\cos^2 \theta}{1 - \cos^2 \psi \sin^2 2 \theta}.
\]

**Slow mode.** The multipole of the synchrotron fluctuations is defined by

\[
\hat{D}_m(R) \approx C_m(2/3) \left[ W_l \left( \hat{P}_m - \frac{1}{2} \hat{P}_m + \hat{P}_m - 2 \right) \right] + W_l \sin^2 2 \int_{n=-\infty}^{\infty} \left[ \hat{P}_n - \frac{1}{2} \hat{P}_n + \hat{P}_n - 2 \right] A_{m-n}^S(\theta) \text{ } R^{5/3},
\]

in the case of high \( \beta \), and by

\[
\hat{D}_m(R) \approx C_m(2/3) \left[ W_l \left( \hat{P}_m - \frac{1}{2} \hat{P}_m + \hat{P}_m - 2 \right) \right] + W_l \sin^2 2 \int_{n=-\infty}^{\infty} \left[ \hat{P}_n - \frac{1}{2} \hat{P}_n + \hat{P}_n - 2 \right] A_{m-n}^S(0) \text{ } R^{5/3},
\]

in the case of low \( \beta \). Here, the geometrical function for the slow mode is

\[
A_{m-n}^S(\theta) = \frac{1}{2\pi} \int_{0}^{2\pi} d\psi e^{-i(m-n)\psi} \frac{\sin^2 \psi}{1 - \cos^2 \psi \sin^2 2 \theta},
\]

with the other coefficients being the same as those of the Alfvén mode.

**Fast mode.** The multipole of the synchrotron fluctuations is

\[
\hat{D}_m(R) \approx C_m(1/2) \left[ W_l \left( \hat{\delta}_{0,0} - \hat{\delta}_{0,1} \right) + W_l \sin^2 \theta \right] \left( A_m^F(0) - \frac{1}{2} \left[ A_{m-2}^F(0) + A_{m+2}^F(0) \right] \right) R^{3/2},
\]

in the case of high \( \beta \), and

\[
\hat{D}_m(R) \approx C_m(1/2) \left[ W_l \left( \hat{\delta}_{0,0} - \hat{\delta}_{0,1} \right) + W_l \sin^2 \theta \right] \left( A_m^F(\theta) - \frac{1}{2} \left[ A_{m-2}^F(\theta) + A_{m+2}^F(\theta) \right] \right) R^{3/2},
\]

in the case of low \( \beta \). Here, the geometrical function of the fast mode, which is identical to the slow mode, is expressed by

\[
A_{m-n}^F(\theta) = \frac{1}{2\pi} \int_{0}^{2\pi} d\psi e^{-i(m-n)\psi} \frac{\sin^2 \psi}{1 - \cos^2 \psi \sin^2 2 \theta}.
\]
In the following, we study the anisotropy using the ratio of the quadrupole moment \((m = 2)\) to the monopole moment \((m = 0)\).

2.5. Statistical Measures

To numerically measure the anisotropy of the MHD turbulence, we utilize the ratio of the quadrupole moment to the monopole one \((QM)\):

\[
QM(R) = \frac{D_2}{D_0} = \frac{\int_0^{2\pi} e^{-2\xi} \hat{D}(R, \varphi) d\varphi}{\int_0^{2\pi} \hat{D}(R, \varphi) d\varphi},
\]

where \(R\) is a radial separation between two points and \(\varphi\) is the polar angle. The absolute value of QM characterizes the level of the anisotropy. And the larger the absolute value, the more pronounced the anisotropy. The normalized structure function of the radiation intensity \((Y = P \text{ or } I)\) is expressed as

\[
\hat{D} = 2(1 - \xi),
\]

which is related to the NCF:

\[
\xi(R) = \frac{\langle Y(X)Y(X + R) \rangle - \langle Y(X) \rangle^2}{\langle Y(X) \rangle^2 - \langle Y \rangle^2},
\]

where \(X = (x, y)\) is a 2D position vector and \(R\) denotes a separation vector between any two points on the POS.

3. MHD Turbulence Simulations and Analyses

3.1. Numerical Scheme

The evolution of the MHD turbulence is governed by the following set of equations:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
\]

\[
\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] + \nabla p - \mathbf{J} \times \mathbf{B} / 4\pi = \mathbf{f},
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) = 0,
\]

with the zero-divergence condition \(\nabla \cdot \mathbf{B}\) and an isothermal equation of state \(p = \rho c_s^2\). Here, \(t\) is the evolution time of the fluids, \(\mathbf{J} = \nabla \times \mathbf{B}\) is the current density, and \(\mathbf{f}\) is a random driving force acting on a large scale.

Numerically, we use a third-order-accurate hybrid, essentially nonoscillatory (ENO) scheme \((CL02)\), to solve the ideal isothermal MHD equations above in a periodic box of size \(2\pi\). More specifically, we combined two ENO finite-difference schemes to mitigate spurious oscillations near shocks. When the variables are sufficiently smooth, we use the third-order weighted ENO scheme \((Jiang \& Wu 1999)\), without characteristic mode decomposition. In the opposite case, we use the third-order convex ENO scheme \((Liu \& Osher 1998)\). The ENO schemes are generalizations of the total variation diminishing schemes \((Harten 1983)\). The latter typically degenerate to first-order accuracy at locations with smooth extrema, while the former maintain high-order accuracy there, even in multispatial dimensions.

To maintain \(\nabla \cdot \mathbf{B} = 0\), we first solve for the potential \(\varphi\) for the Poisson equation, \(\nabla^2 \varphi + \nabla \cdot \mathbf{B} = 0\), with the updated magnetic field \(\mathbf{B}\) obtained by the ENO scheme, then we compute the corrected magnetic field \(\mathbf{B}_c = \mathbf{B} + \nabla \varphi\), for which \(\nabla \cdot \mathbf{B}_c = 0\). Furthermore, we use a three-stage Runge–Kutta method for time integration, in units of the large eddy turnover time of \(\sim L/V\). The magnetic field can be presented as \(\mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B}\), i.e., a superposition of a regular magnetic field \(\mathbf{B}_0\) and a random/fluctuation magnetic field \(\delta \mathbf{B}\).

3.2. Generation of Data Cubes

In our simulations, we set the mean magnetic field to be 1 or 0.1 along the x-axis. The turbulence is driven by a solenoidal driving force at wavenumber \(k = 2.5\) (corresponding to the scale of \(\sim 0.4L\)) in Fourier space, with a continuous energy injection, and it is then transferred to the real volume space. When the simulation with a numerical resolution of 512 \(^3\) reaches a statistically steady state at \(t \sim 20\) in code units, we set the output to the primitive physical quantities that we need, such as density\(^8\) as well as the individual components of the magnetic field and velocity.

Based on the output 3D physical quantities and the parameters set at the initial moment, we obtain the typical parameters—e.g., the Alfvénic Mach number \(M_a = \frac{V_A}{V_c}\) and the sonic Mach number \(M_s = \frac{V_s}{V_c}\)—listed in Table 1, to characterize each run. The former characterizes the strength of the magnetic field, while the latter reflects the compressibility, where \(V_A = |\mathbf{B}|/\sqrt{4\pi \rho}\) is the Alfvénic speed and \(c_s = \sqrt{p/(\rho)}\) is the sound speed. The plasma parameter is described by \(\beta = p_i/p_m = 2M_a^2/M_s^2\), where \(\beta > 1\) represents gas pressure-dominated turbulence and \(\beta < 1\) represents magnetic pressure-dominated turbulence.

3.3. Analyses of Data Cubes

The inertial range and the strong turbulence range\(^10\) can be shown by the correlation function, the structure function, and the power spectrum. Since the calculations of the correlation and structure functions of the 3D magnetic field and velocity are time-consuming, we only use the power spectrum to determine these ranges. In Figure 1, we present power spectra of the magnetic fields and velocities calculated in different turbulence regimes, where the vertical dotted lines represent the injection, transition, and dissipation scales, respectively. The determination of these scales helps the calculation of the average QM. From this figure, we find that the power spectra of the magnetic fields and velocities satisfy \(k^{5/3}\) in four different turbulence regimes, but have different inertial ranges. Moreover, we find that the velocity spectra do not show obvious flattening in the vicinity of the dissipation range, so there are no bottleneck effects in the velocity spectra \((Falkovich 1994)\).

3.4. Decomposition of MHD Modes

We input 3D data cubes of the magnetic fields and decompose the data using the decomposition method. The decomposition of the MHD turbulence is related to the following unit vectors for the Alfvén, slow, and fast modes.

\(8\) In this paper, we do not consider density stratification and self-gravity effects, which are not a part of the theoretical predictions that we test and not essential for most of the applications of the technique.

\(9\) Typically, the synchrotron emission regions correspond to the subsonic regime, but for the sake of completeness, we consider both the subsonic and supersonic cases.

\(10\) When the assumption of the critical balance is satisfied, it corresponds to strong turbulence (see Section 2.1 for more details).
Table 1

Data Cubes with a Numerical Resolution of 512^3 Arising from the Compressible MHD Turbulence Simulations

| Run | $B_0$ | $M_s$ | $M_A$ | $\beta$ | $\delta B_{\text{rms}}/(\langle B \rangle)$ | $L_{\text{inj}}$ | $l_h$ | $l_v$ | $L_{\text{trans}}(L_A)$ | $l_{\text{diss}}$ |
|-----|------|------|------|------|-------------------------------|----------------|------|------|----------------|---------------|
| 1   | 1.0  | 9.92 | 0.50 | 0.005 | 0.465                         | $\sim 0.4L$  | $\sim 0.33L$ | $\sim 0.33L$ | $\sim 51.20$ | $\sim 0.025L$ |
| 2   | 1.0  | 4.46 | 0.55 | 0.030 | 0.467                         | $\sim 0.4L$  | $\sim 0.33L$ | $\sim 0.25L$ | $\sim 61.95$ | $\sim 0.025L$ |
| 3   | 1.0  | 3.16 | 0.58 | 0.067 | 0.506                         | $\sim 0.4L$  | $\sim 0.33L$ | $\sim 0.25L$ | $\sim 68.89$ | $\sim 0.025L$ |
| 4   | 1.0  | 0.87 | 0.70 | 1.295 | 0.579                         | $\sim 0.4L$  | $\sim 0.33L$ | $\sim 0.33L$ | $\sim 100.35$ | $\sim 0.025L$ |
| 5   | 1.0  | 0.48 | 0.65 | 3.668 | 0.614                         | $\sim 0.4L$  | $\sim 0.33L$ | $\sim 0.20L$ | $\sim 86.52$ | $\sim 0.025L$ |
| 6   | 0.1  | 3.11 | 1.69 | 0.591 | 5.254                         | $\sim 0.4L$  | $\sim 0.20L$ | $\sim 0.20L$ | $\sim 42.43$ | $\sim 0.031L$ |
| 7   | 0.1  | 0.45 | 1.72 | 29.219 | 6.345                      | $\sim 0.4L$  | $\sim 0.20L$ | $\sim 0.20L$ | $\sim 40.25$ | $\sim 0.031L$ |

Note. The different turbulence regimes are mainly characterized by the following parameters: the Alfvénic Mach number $M_A$, the sonic Mach number $M_s$, and the plasma parameter $\beta = 2M_s^2/M_A^2$. The other parameters are listed as follows: $\delta B_{\text{rms}}$—the rms of the random magnetic field; $\langle B \rangle$—the regular magnetic field; $L_{\text{inj}}$—the injection scale; $l_h$—the correlation length of the magnetic field; $l_v$—the correlation length of the velocity; $L_{\text{trans}}$—the transition scale for $M_A < 1$; $L_A$—the transition scale for $M_A > 1$; and $l_{\text{diss}}$—the dissipation scale.

Figure 1. The power spectra of the magnetic fields in different turbulence regimes. The vertical green, magenta, and cyan dotted lines denote the injection, transition, and dissipation scales of the turbulence, respectively.

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\begin{align}
\hat{\zeta}_r & \propto (1 + \alpha + \sqrt{D})(k_i \hat{k}_l) + (-1 + \alpha + \sqrt{D})(k_l \hat{k}_i), \\
\hat{\zeta}_s & \propto (1 + \alpha - \sqrt{D})(k_i \hat{k}_l) + (-1 + \alpha - \sqrt{D})(k_l \hat{k}_i),
\end{align}

\begin{equation}
\hat{\zeta}_A \propto -\hat{k}_l \times \hat{k}_i,
\end{equation}

where $D = (1 + \alpha)^2 - 4\alpha \cos^2 \theta$ and $\cos \theta = \hat{k}_l \cdot \hat{B}$. When projecting the magnetic field into $\hat{\zeta}_r$, $\hat{\zeta}_s$, and $\hat{\zeta}_A$, we can obtain the magnetic field component for each mode.
4. Results

To generate synthetic observations, we calculate the synchrotron radiation intensity using Equation (5), with the magnetic field from the simulation data and $\alpha = -1.0$. As for the calculation of the synchrotron radiation, we ignore the fluctuation of the electrons and only consider the fluctuation of the magnetic field. For the synchrotron polarized radiation, its intensity is calculated using Equation (8), with the assumption of a thermal electron density proportional to $\rho$, and we use $n_e = \rho$ in the actual calculation of the Faraday measure.

4.1. Correlation of MHD Turbulence

The effects of the electron spectral index of the cosmic rays on the NCFs of the synchrotron emissivity or intensity have been studied in Herron et al. (2016) using numerical simulations. However, it was not right to adopt an expression applicable to isotropic turbulence (see Equation (7) of Herron et al. 2016) to study the correlation property of the anisotropic turbulence. Here, we use the general formula, i.e., Equation (9), to explore the dependence of the NCFs of the synchrotron emissivity on the spectral index in the case of anisotropic turbulence. At the same time, the dependence of the spectral index is explored for different plasma modes in the sub-Alfvénic and subsonic turbulence regimes.

We first focus on the special case of the NCFs of the synchrotron emissivity, namely $\gamma = 2$. This is explored in four turbulence regimes: sub-Alfvénic and subsonic; sub-Alfvénic and supersonic; super-Alfvénic and subsonic; and super-Alfvénic and supersonic. The results obtained using Equation (10) are shown in Figure 2, where the upper and lower panels correspond to strong and weak magnetic field simulations, respectively. At the scale smaller than $L_{\text{trans}}$, the turbulence becomes strong and the statistical relationship can be well represented. This figure demonstrates that there is no expected consistency between the left-hand and right-hand sides of Equation (10), except for the presence of slight deviations in the case of sub-Alfvénic and subsonic turbulence (see the upper left panel). Notice that Equation (10) is derived under the assumption of only involving the fluctuation component perpendicular to the mean magnetic field, i.e., without the fluctuation along the mean magnetic field direction $B_x$. Therefore, it is not surprising that the discrepancy becomes more pronounced in the case of a weak magnetic field (the

Figure 2. Numerical tests of the correlation’s analytical formula in different turbulence regimes. The red dashed lines represent the results calculated by Equation (9) for the magnetic field index $\gamma = 2$, and the solid blue lines those on the right-hand side of Equation (10), i.e., the square of Equation (11). The vertical green and magenta dotted lines denote the injection and transition scales of the turbulence, respectively. The vertical black dotted line denotes the correlation length of the magnetic field.
lower panels) than in the strong one (the upper panels). In addition, there is a larger deviation for the high-$M_s$ regime, which may be due to the formation of shocks. In the range from the injection scale $L_{\text{inj}}$ (the green line) to the transition scale $L_{\text{trans}}$ (the magenta line), the turbulence is weak (LV99; Galtier et al. 2000), and therefore deviations from the LP12 results are expected (for more details about the difference of the turbulence in the weak and strong regimes, see Beresnyak & Lazarian 2019).

We then use the general expression, Equation (9), to explore whether the varied spectral indices affect the statistics of the NCFs of the synchrotron emissivity in the case of anisotropic turbulence. Figure 3 shows the NCFs of the synchrotron emissivity as a function of radial separation at different spectral indices. We find that changes in the spectral index hardly affect the scaling index measured, except in the super-Alfvénic and supersonic turbulence regimes. Comparing the subsonic turbulence with the supersonic turbulence, we find that the NCFs of the synchrotron emissivity have a marginal dependence on the spectral index for the latter. It is not difficult to understand this phenomenon, because the possible shock formation in the high-$M_s$ regime causes a deviation from the synchrotron statistics. In short, our study confirms that the correlation of the synchrotron emissivity for an arbitrary index $\gamma$ of the magnetic field is linked to those for $\gamma = 2$, namely

$$\text{CF}_\gamma(R) \approx \xi(\gamma) \text{CF}_{\gamma = 2}(R),$$

where $\xi(\gamma) = ((B_x^2) - \langle B_x^2 \rangle^2 / \langle B_x^2 \rangle - \langle B_x^2 \rangle)$ is a factor that changes only with $\gamma$ and not with the separation $R$. This is a practically important result, in agreement with the theoretical prediction of LP12. However, it is important to stress that the above result does not hold in the super-Alfvénic and supersonic turbulence regimes. Our numerical confirmation of the theory opens the way for quantitative studies of the synchrotron statistics for an arbitrary index of the cosmic-ray distribution.

Based on Run5, as listed in Table 1, we study the influence of the electron indices on the NCF of the synchrotron intensity arising from different MHD modes. The results of the synchrotron statistics are shown in Figure 4, where the range of spectral indices that we consider corresponds to various possible astrophysical environments. As shown in this figure, the electron spectral index has less effect on the results for the Alfvén mode, which is similar to the properties of predecomposition MHD turbulence and reflects the Alfvén mode dominating the properties of the MHD turbulence. However, the NCFs of the synchrotron intensity for the

![Figure 3](image_url)

**Figure 3.** The NCFs of the synchrotron emissivity for varying electron indices in different turbulence regimes. The vertical green and magenta dotted lines denote the injection and transition scales of the turbulence, respectively. The vertical black dotted line denotes the correlation length of the magnetic field.
4.2. Synchrotron Intensity Anisotropy Arising from Basic MHD Modes

The compressible nature of these two modes. The compressible slow and fast modes are completely dependent on the distribution of the electron index. This may be caused by the compressible nature of these two modes.

4.2. Synchrotron Intensity Anisotropy Arising from Basic MHD Modes

4.2.1. Qualitative Analysis of Anisotropy

To explore the effect of the angle between the mean magnetic field and the LOS on the anisotropy of the MHD turbulence, we rotate the direction of the mean magnetic field in the xoz plane along the γ-axis. The angle \(0^\circ\) denotes the LOS aligned with the x-axis, and the angle \(90^\circ\) denotes the LOS along the z-axis. Using statistics of the synchrotron polarization intensity, Wang et al. (2020) explored the anisotropy of the compressible MHD turbulence. Notice that they only considered the special case of the mean magnetic field perpendicular to the LOS. Here, we focus on a more general case; that is, we investigate the dependence of the anisotropy for three modes on the angle between the mean magnetic field and the LOS, based on the synchrotron radiation intensity.

We provide contour maps of the normalized structure function of the synchrotron radiation intensity for the three modes in Figure 5, which are based on the decomposed data of Run5, as listed in Table 1. From top to bottom, we display the contour maps for the Alfvén, slow, and fast modes, respectively. From left to right, the contour maps correspond to the angles \(90^\circ\left(\frac{\pi}{6}\right), 60^\circ\left(\frac{\pi}{3}\right), 30^\circ\left(\frac{\pi}{6}\right), \) and \(0^\circ\), respectively.

The angle of \(90^\circ\) represents the mean magnetic field parallel to the LOS, while the angle of \(0^\circ\) is the opposite case. As is shown in the upper and middle rows, the structures of the contour maps are almost isotropic at small angles for the three modes. As the angle increases, the Alfvén and slow modes exhibit significant anisotropic features, while the fast mode remains almost isotropic. In the range of the large angles, the structures of the contour maps for the three modes are similar to the earlier direct numerical simulations (CL02; CL03). It can be seen that synchrotron radiation statistics can efficiently reveal the information of the mean magnetic fields of the POS.

4.2.2. Contribution of Geometric Function Order to QM

In the sub-Alfvénic and subsonic turbulence regimes, the numerical simulations of the average QM are compared with analytical predictions. Given that the anisotropy part of the multipole involves different terms of the series expansion of the geometrical function for the Alfvén and slow modes, we first explore which terms of the geometrical function can match the anisotropy from the simulation observations. For our purpose of studying QM, the expansion orders \(m = 0\) and \(2\) can be fixed for the monopole moment and the quadrupole moment (see Equation (25)). The contribution of the monopole moment \((m = 0)\) only comes from the zeroth order of the expansion of the geometrical function \(A^{A, S}_{0, n}\), with \(m − n = 0\), while the contribution of the quadrupole moment \((m = 2)\) is related to the zeroth, second, and fourth orders of the series expansion of the geometrical function \(A^{A, S}_{0, n} ,\) with \(m − n = 0, 2,\) and \(4\). Furthermore, once the expansion order of the geometric function is fixed, the expansion of the power spectra will be determined via the unchanged \(n\) (see Equations (12) and (19) for details).

Figure 6 shows the simulation results of the average QM as a function of the angle \(\theta\) between the mean magnetic field and the LOS, in comparison with the analytical results arising from different expansion orders of the geometrical function. For the simulation results, we calculate the average QM by considering almost all the scales from 10 pixels to the transition scale (corresponding to the pixels listed in Table 1), namely in the strong turbulence range. As for the analytical results, we consider the individual contributions from the zeroth, second, and fourth orders of the geometric function and the summation of the three orders, corresponding to the labels \(m − n = 0, 2,\) and \(4\), and “sum.” in Figure 6, respectively. Note that there are no selections of parameter variations in the case of the fast mode.

From the left to the right panels, we plot the average QMs for the Alfvén, slow, and fast modes, respectively. It can be seen that the average QMs for the Alfvén and slow modes increase with the increase of the angle, while those of fast mode remain almost unchanged with the increase of the angle. We find that for the Alfvén and slow modes, the analytical results of the zeroth \(m − n = 0\) can match the simulations, and apart from some small deviations in the average QM amplitude, the overall trends of the numerical and analytical results are also
consistent for fast mode. The observed deviations as we add additional terms may mean that the expansion series in the LP12 analytics requires the use of higher-order terms, rather than limiting the expansion over the first two terms, as has been done in the present study. In addition, we show the error bars corresponding to the variability with scale (taken from the standard deviation of the QMs with the range of scales used for averaging). We find that the error bars for the QMs in the
numerical simulation are consistent with the variations in the analytical calculations. Therefore, we believe that the error bars provide an adequate depiction of the reliability of the data.

Comparing with the analytical curves of the QMs provided in LP12, we find that at the angle $\theta = 0^\circ$, the current analytical QMs show nonzero values, which are caused by stochastic deviations of the mean magnetic field from its original direction, due to an external driving setting. Note that the analytical formulae of the quadrupole moment include an important parameter, $\epsilon$, characterizing the level of the turbulence anisotropy in the analytical formula (see Equation (16)). In this regard, we have conducted a self-consistent treatment between the analytical theory and the simulations; that is, the new nonzero $\epsilon$ value from the simulation is substituted into the analytical formulae, resulting in the nonzero QM values.

### 4.2.3. Application to Subsonic and Supersonic Turbulence Regimes

In this section, we explore the relationship between the average QM and the angle for the three modes in two different turbulence regimes, namely the subsonic and supersonic turbulence regimes, which are shown in the top and bottom rows of Figure 7, respectively. For the former, the analytical results are obtained by considering the contribution of the zeroth order of the series expansion of the geometrical function, while for the latter, this is done by considering the contribution of the fourth order. The numerical results are calculated using the data cubes in Table 1.

In Figure 7, it is shown that the anisotropy of the Alfvén and slow modes (the left and middle columns) increases with the angle in the different turbulence regimes, while the fast mode remains almost isotropic for all the viewing angles, due to the QM value approaching zero. As is expected, the highest level of anisotropy appears in the case of the mean magnetic field perpendicular to the LOS. As is shown in the left and middle columns, the amplitude of the average QM for the Alfvén mode is slightly larger than that of the slow mode at all angles. This reflects the fact that the Alfvén mode dominates the properties of the slow mode, while the fast mode with smaller amplitude, is independent of these two modes. From the top to bottom rows, it is shown that the absolute values of the average QMs for the Alfvén and slow modes in the subsonic turbulence regime are larger than those in the supersonic case, especially for the slow mode. The reason for this may be due to the formation of shocks in the supersonic turbulence regime decreasing the anisotropy of the turbulence. Our simulations are in good agreement with the analytical predictions of LP12 for the Alfvén and slow modes in different cases. The deviations that we observe for the fast mode may be due to their relatively small amplitudes, which influence our decomposition. In particular, the effects of the “leakage” of the modes—i.e., the contamination of one type of mode by the other modes—is considered.

Here, we would like to mention to interested readers that the LP12 theoretical predictions did not consider the influence of the sonic Mach number $M_s$ on the synchrotron fluctuation statistics. Our studies in this section promote the application of the LP12 predictions in the different turbulence environments. They show that the LP12 theory can be applied to a variety of astrophysical conditions with different $M_s$ for the slow and Alfvén modes, but for fast modes, its accuracy drops with the increase of $M_s$. Fortunately, this is not important for practical applications, as most synchrotron-emitting media in astrophysical settings are warm and hot media with low $M_s$.

### 4.3. Extension to Intensities of Polarized Emission

Another important way of studying MHD turbulence is through the analysis of synchrotron polarization information (see Section 2.2 for the calculation of the Stokes parameters). Note that the numerical simulation studies mentioned above are based on the theoretical basis of the synchrotron radiation fluctuations from LP12. Here, we explore whether the relevant conclusions from the study of the synchrotron radiation fluctuation hold in the case of synchrotron polarized radiation. Moreover, we want to know to what extent the study of polarization statistics has an impact on the results in the presence of the Faraday rotation effect. In particular, we study whether the measurements of the anisotropy are affected by the Faraday rotation effect in the different frequency regimes. We assume that the emission region is extended to 1 kpc along the LOS and that the thermal electron density and magnetic field strength are set as $n_e \approx 0.01$ cm$^{-3}$ (Gaensler et al. 2008; Nota & Katgert 2010; Lee et al. 2019) and $B_z \approx 1.23$ $\mu$G, respectively.

First, we show the distributions of the Faraday measure for the three modes at different angles between the mean magnetic field and the LOS in Figure 8, from which we can clearly see that the Faraday measure for the three modes follows a non-Gaussian distribution (e.g., Kierdorf et al. 2020; Seta & Federrath 2021). As shown, the value of the Faraday measure for the three modes lies at different ranges: $\sim -2$ to $1$ rad m$^{-2}$ for the Alfvén mode, $\sim -2$ to $8$ rad m$^{-2}$ for the slow mode, and $\sim -3$ to $3$ rad m$^{-2}$ for the fast mode. This reflects the dispersion of the Faraday measure being different for the three modes, with the dispersion value of the slow mode being the largest. We find that the dispersion of the Faraday measure for the Alfvén and fast modes increases with the increasing angle, while the opposite is true for that of the slow mode. In addition, the mean of the Faraday measure for the three modes at different angles is close to 0.

Figure 9 shows the degree of polarization (top row) and the average QM (bottom row) arising from the synchrotron polarization intensities as a function of the viewing angle, at different frequencies, i.e., $\nu = 0.1$, 0.2, and 10 GHz. We can see that the degree of polarization increases with both the increase of the angle and the frequency. This is because the increase of the frequency and the increased viewing angle, which decreases the $B_\perp$ component of the projected magnetic fields, both reduce the Faraday rotation effect. At the same viewing angle and frequency, we find that the fast mode has a lower degree of polarization than the Alfvén and slow modes, due to the isotropy of the fast mode. The bottom row shows that the absolute values of the average QMs of the synchrotron polarization intensities for the Alfvén and slow modes increase with the viewing angle $\theta$, while that of the fast mode presents small changes around the average QM equal to 0. The overall shapes of the average QM distributions for the three modes have some changes at different frequencies, especially for the Alfvén and fast modes. These results reveal that in the wide frequency range considered, the Faraday rotation effect hardly hinders the measurement of the anisotropy for the high-frequency case.
5. Discussion

5.1. Numerical Tests of the LP12 Theory

Our numerical study uses synthetic observations to test the predictions of LP12. We focus on the effects of the relativistic electron spectral index and turbulence anisotropy on the statistics of synchrotron intensity and polarization. This study is essential for measuring the variations of the synchrotron parameters, e.g., the Faraday rotation (see Haverkorn et al. 2008; Waelkens et al. 2009; Xu & Zhang 2016), the Faraday tomography (Burn 1966; Haverkorn 2018), as well as the promising new techniques that involve tracing the magnetic field with synchrotron intensity (Lazarian et al. 2017) and synchrotron polarization gradients (Lazarian & Yuen 2018; Carmo et al. 2020).

The prediction of LP12—regarding whether the correlation function of the synchrotron emissivity depends on the spectral index in the case of the isotropic model—has been explored by Herron et al. (2016), but they, unfortunately, used the results that were obtained for the anisotropic turbulence to compare...
with the expression obtained for the toy model of the isotropic turbulence. In this paper, we test the general case, which utilizes the anisotropic model to explore the dependence of the NCF of the synchrotron emissivity on the spectral index. Our results are not consistent with LP12 for the super-Alfvénic and supersonic turbulence case. The observed deviations are mainly caused by supersonic effects, e.g., shocks, that are not considered within the LP12 theoretical model. Importantly, we explore the dependence of the NCF of the synchrotron intensity for all three basic MHD modes on the spectral index and obtain expected results.

Wang et al. (2021) demonstrated that the anisotropy of the three modes can be better distinguished by QM versus R when only considering the mean magnetic field perpendicular to the LOS. In this paper, we explore the anisotropy of the three modes from different angles. By using the average QM versus \( \theta \), we can see that the overall trends of the three modes vary with angle, from which we can distinguish the three modes well. However, it is difficult to judge the three modes from the magnitude of the average QM, because the overall variation for all the three modes is around \( \pm 0.1 \) in terms of the average QM. This depends primarily on the mean magnetic field, and is generally weakly dependent on the separation.

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The degree of polarization (upper panels) and the average QMs (lower panels) of the synchrotron polarization intensity as a function of the angle \( \theta \) for the Alfvén, slow, and fast modes, at different frequencies. The simulation results are based on Run4, as listed in Table 1, and the red dashed lines represent analytical predictions.

Figure 9. The degree of polarization (upper panels) and the average QMs (lower panels) of the synchrotron polarization intensity as a function of the angle \( \theta \) for the Alfvén, slow, and fast modes, at different frequencies. The simulation results are based on Run4, as listed in Table 1, and the red dashed lines represent analytical predictions.

5.2. Ways of Studying MHD Turbulence

The anisotropy of three modes has been explored in several studies. Initially, the anisotropy of the three modes was obtained from the magnetic field, the density, and the velocity information of the MHD turbulence (CL02; CL03). But this information is not directly available in real observations. Later, the study of the anisotropy of the three modes was extended to synchrotron polarized radiation (Wang et al. 2020) and to velocity centroids (Hernández-Padilla et al. 2020). In addition to the anisotropy, the identification and relative contributions of the different modes are important in different astrophysical environments. The former has been explored in Galactic turbulence (Zhang et al. 2020), while the latter have been studied in interstellar turbulence (Hernández-Padilla et al. 2020).

The structure function map serves as an indicator for preferentially predicting the anisotropic structure. The anisotropy direction is related to the mean field over the POS, and thus provides a way of tracing the magnetic field. The structure function map of the Alfvén mode in the 90° case shown in Figure 5 is the best proof, where the mean magnetic field is
along the $x$-axis direction. Compared with the structure function, the QM is more precise in measurements of the anisotropy and the magnetic field tracing. Its magnitude reflects the level of anisotropy, and positive and negative values represent the directions of the anisotropic structures. The two methods are synergistic for the study of the properties of MHD turbulence, particularly in terms of the anisotropy and the measurement of the magnetic field.

5.3. Synchrotron Statistics

The radio synchrotron emission is primarily from the hot/warm ionized diffuse medium, where the turbulence has a relatively low sonic Mach number $M_s \ll 2$ (Gaensler et al. 2011). However, some environments, such as the regions of active galactic nuclei and supernova remnants interacting with the surrounding cold molecular cloud, could still have a large $M_s$. For this case, the spectral lines related to velocity channel analysis and velocity correlation spectrum have been considered the main probe (Lazaraj & Pogosyan 2000, 2004; González-Casanova & Lazarian 2017; Yuen & Lazarian 2017; Yang et al. 2021).

The development of synchrotron radiation techniques for measuring MHD turbulence has been motivated by the massive amount of radio observational data that is currently available or will be available in the future, such as that from the Low-Frequency Array and Square Kilometer Array. The turbulence injection scale in spiral galaxies is expected to be of the order of the disk scale height, which is 100 pc or more (see Chepurnov et al. 2010). If the observations resolve smaller scales, our technique is applicable. The resolutions of new instruments constantly increase, and thus we expect our technique to apply to more distant galaxies. At the same time, the limitations in terms of the required resolution are less strict for the study of Milky Way magnetic fields. Thus, this technique has a lot of present-day as well as future applications.

6. Summary

In this paper, the statistics of the synchrotron intensity and synchrotron polarization fluctuation arising from compressible MHD turbulence have been studied numerically, with the results being compared to the predictions of the analytical theory in LP12. Using the QM and correlation function, we have explored the anisotropy of the MHD turbulence and the influence of the cosmic-ray electron spectral index on synchrotron statistics, respectively. The main results are briefly summarized as follows.

1. Our simulations show the good correspondence of the LP12 analytical expressions of correlation statistics for a variety of magnetic field indices $\gamma$. In particular, we find that the approximation corresponding to $\gamma = 2$ is adequate for describing the spatial variations of the fluctuations.

2. The degree of the anisotropy for the Alfvén and slow modes is larger in subsonic simulations compared to supersonic simulations. The numerical simulations are in good agreement with the analytical results of LP12.

3. As for the Alfvén and slow modes, the degree of anisotropy increases with the angle between the mean magnetic field and the LOS. The synchrotron statistics arising from the fast mode are almost isotropic. The LP12 theoretical predictions regarding the anisotropy of the plasma modes have been numerically confirmed.

4. Analytical expressions relating to synchrotron intensities have been generalized to the synchrotron polarization intensities. The results demonstrate that the relevant expressions are still applicable, while the Faraday depolarization effect impedes the measurement of the turbulence anisotropy.

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