On the polynomial vector fields on $S^2$

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(MS received 25 January 2009; accepted 9 December 2010)

Let $\mathcal{X}$ be a polynomial vector field of degree $n$ on $M$, $M = \mathbb{R}^m$. The dynamics and the algebraic–geometric properties of the vector fields $\mathcal{X}$ have been studied intensively, mainly for the case when $M = \mathbb{R}^2$, and especially when $n = 2$. Several papers have been dedicated to the study of the homogeneous polynomial vector field of degree $n$ on $S^2$, mainly for the case where $n = 2$ and $M = S^2$. But there are very few results on the non-homogeneous polynomial vector fields of degree $n$ on $S^2$. This paper attempts to rectify this slightly.

1. Introduction and statement of the main results

Let $\mathbb{R}[x, y, z]$ be the ring of all polynomials in the variables $x$, $y$ and $z$ with real coefficients. The vector field

$$\mathcal{X} = P(x, y, z) \frac{\partial}{\partial x} + Q(x, y, z) \frac{\partial}{\partial y} + R(x, y, z) \frac{\partial}{\partial z}$$

(1.1)

is called a polynomial vector field of degree $n$ in $\mathbb{R}^3$ if $P, Q, R \in \mathbb{R}[x, y, z]$ and $n = \max\{\deg P, \deg Q, \deg R\}$. For simplicity, sometimes we will write the vector field $\mathcal{X}$ simply as $\mathcal{X} = (P, Q, R)$.

The vector field (1.1) is a homogeneous polynomial vector field $\mathcal{X}$ of degree $n$ in $\mathbb{R}^3$ if $P, Q$ and $R$ are homogeneous polynomials of degree $n$.

Let $f \in \mathbb{R}[x, y, z]$. The algebraic surface $f(x, y, z) = 0$ is an invariant algebraic surface of $\mathcal{X}$ if there exists $K \in \mathbb{R}[x, y, z]$ such that

$$\mathcal{X}f = P(x, y, z) \frac{\partial f}{\partial x} + Q(x, y, z) \frac{\partial f}{\partial y} + R(x, y, z) \frac{\partial f}{\partial z} = K(x, y, z)f(x, y, z).$$

(1.2)

As usual, $S^2$ denotes the two-dimensional sphere $\{(x, y, z) \in \mathbb{R}^3: x^2 + y^2 + z^2 = 1\}$. A polynomial vector field $\mathcal{X}$ of degree $n$ on $S^2$ is a polynomial vector field in $\mathbb{R}^3$ of degree $n$ such that restricting to $S^2$ defines a vector field on $S^2$, i.e. it must satisfy the equality

$$xP(x, y, z) + yQ(x, y, z) + zR(x, y, z) = 0 \quad \text{for all } (x, y, z) \in S^2.$$  

(1.3)

In particular, $\mathcal{X}$ is called a quadratic vector field on $S^2$ if $n = 2$. 

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There are other definitions of polynomial vector fields on the sphere \( S^2 \). For example, if (1.3) holds, then there exist \( P', Q' \) and \( R' \), which consist purely of the terms of degree \( n \) and \( n - 1 \) of \( P, Q \) and \( R \) such that \( xP' + yQ' + zR' = 0 \), not only in the points of \((x, y, z) \in S^2\), but also in all the points of \((x, y, z) \in \mathbb{R}^3\). Therefore, we can give a second definition of polynomial vector fields on \( S^2 \) as follows. A polynomial vector field of degree \( n \) on \( S^2 \) is the polynomial vector field \( X' = (P', Q', R') \) in \( \mathbb{R}^3 \) of degree \( n \) such that \( xP' + yQ' + zR' = 0 \) for \((x, y, z) \in \mathbb{R}^3\).

It is a standard result that this then implies that \((P', Q', R') = (x, y, z) \times (A, B, C)\) for suitable polynomials \((A, B, C)\). One motivation of this paper is to study the polynomial vector fields on \( S^2 \) induced by homogeneous vector fields in \( \mathbb{R}^3\) [1, 2].

In the second definition, any sphere \( x^2 + y^2 + z^2 = r^2 \) is an invariant algebraic surface of \( X' \) in \( \mathbb{R}^3 \) for any \( r \in \mathbb{R} \). However, in general, this does not hold for the polynomial vector fields on \( S^2 \) induced by non-homogeneous vector fields in \( \mathbb{R}^3 \). Therefore, we do not consider this second definition here.

The algebraic surface \( f(x, y, z) = 0 \) defines an invariant algebraic curve \( \{f = 0\} \cap S^2 \) of the polynomial vector field \( X \) on \( S^2 \) if

(i) there exists \( K \in \mathbb{R}[x, y, z] \) such that (1.2) holds, and

(ii) the intersection of the two invariant surfaces \( f(x, y, z) = 0 \) and \( S^2 \) is a curve.

The algebraic curve \( \{ax + by + cz = 0\} \cap S^2 \) is called a great circle on \( S^2 \), where \( a, b, c \) are real constants. Moreover, a great circle is an invariant great circle of a polynomial vector field \( X \) of degree \( n \) on \( S^2 \) if it is an invariant algebraic surface of \( X \).

All of the above definitions can be found, for example, in [2, 9, 10, 13].

Llibre and Pessoa [9] studied homogeneous polynomial vector fields of degree 2 on \( S^2 \). They determined the maximum number of invariant circles when such a vector field has finitely many invariant circles. The same authors found the upper bound for the number of invariant circles and invariant great circles of homogeneous polynomial vector fields of degree \( n \) on \( S^2 \) [10]. It was proved by Llibre and Pessoa in [11] that if a homogeneous polynomial vector field \( X \) of degree 2 on \( S^2 \) has at least a non-hyperbolic singularity, then it has no limit cycle. They also gave necessary and sufficient conditions for determining whether a singularity of a homogeneous polynomial vector field \( X \) of degree 2 on \( S^2 \) is a centre, and characterized the global phase portrait of \( X \) modulo limit cycles.

Coleman [2] and Sharipov [13] studied homogeneous vector fields of degree \( n \) in \( \mathbb{R}^3 \). They showed that any such vector fields naturally induced a tangent vector field on \( S^2 \). Camacho [1] proved some properties of the vector fields on \( S^2 \) induced by homogeneous vector fields of degree 2 in \( \mathbb{R}^3 \) (see also [8, 14, 15]).

We note that the results cited above are concerned with vector fields on \( S^2 \) homogeneous, or vector fields that are induced by homogeneous polynomial vector fields of \( \mathbb{R}^3 \).

In what follows we shall use many basic definitions of the qualitative theory of differential equations, such as singular point, node, saddle, focus, centre, antisaddle, period annulus, topological index or simply index of a singular point, contact point, periodic orbit, limit cycle, homoclinic loop, transversal arc, topological equivalence, etc. All of the precise definitions of these notions can be found, for example, in [4]. A node is called \textit{proper} if its two real eigenvalues are different.
Our main results are presented in the next two theorems. In the first we present results for polynomial vector fields on $S^2$ of arbitrary degree, and in the second results only for degree 2.

**Theorem 1.1.** Let $\mathcal{X}$ be a polynomial vector field on $S^2$ of degree $n$ having a finite number $s(\mathcal{X})$ of singular points in $S^2$. Assume that $\mathcal{X}$ is given by (1.1).

(i) If one of the three polynomial systems

\[
P = Q = x^2 + y^2 + z^2 - 1 = 0, \\
Q = R = x^2 + y^2 + z^2 - 1 = 0, \\
R = P = x^2 + y^2 + z^2 - 1 = 0
\]

has finitely many solutions in $\mathbb{C}P^3$, then $s(\mathcal{X}) \leq 2n^2$.

(ii) If $n \geq 2$, then $s(\mathcal{X}) \leq (2n - 2)(2n - 1) + 1$; if $n = 1$, then $s(\mathcal{X}) \leq 2$.

(iii) Assume that all the singular points of $\mathcal{X}$ have topological index 1 or $-1$. If $n \geq 2$, then $s(\mathcal{X}) \leq (2n - 2)(2n - 1)$; if $n = 1$, then $s(\mathcal{X}) = 2$.

(iv) Every great circle is either an invariant great circle of $\mathcal{X}$, or has at most $2n$ contact points (including singular points) with the orbits of $\mathcal{X}$.

The singular points of the vector field $\mathcal{X}$ on $S^2$ are included in the solution of the polynomial system $P = Q = x^2 + y^2 + z^2 - 1 = 0$. By the multi-dimensional Bezout theorem [3,12], the solutions of this polynomial system in $\mathbb{C}P^3$ are either infinite or less than $2n^2$. This is the proof of theorem 1.1(i). We note that statements (ii) and (iii) of theorem 1.1 improve the result of theorem 1.1(i) in the case where $n = 2$.

On the other hand, it is important to note that there are polynomial vector fields $\mathcal{X} = (P, Q, R)$ on $S^2$ for which the three polynomial systems in theorem 1.1(i) have infinitely many solutions in $\mathbb{C}P^3$, and consequently we cannot apply the Bezout theorem for bounding its number of singular points, but such polynomial vector fields can have finitely many singular points. An example of this kind of polynomial vector field on $S^2$ is

\[
P = yz(1 - 3x^2), \quad Q = xz(1 - 3y^2), \quad R = xy(1 - 3z^2).
\]

Clearly, each of the three systems of theorem 1.1(i) contains as a solution one circle of the sphere, so the Bezout theorem does not provide any bound for the number of singular points. But the singular points of $\mathcal{X}$ on $S^2$ are finite, namely $(\pm 1, \pm 1, \pm 1)/\sqrt{3}$, $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$.

**Theorem 1.2.** Let $\mathcal{X}$ be a quadratic polynomial vector field on $S^2$ having a finite number $s(\mathcal{X})$ of singular points of $\mathcal{X}$.

(i) Assume that all the singular points of $\mathcal{X}$ have topological index 1 or $-1$.

(a) The exact upper bound $6$ for $s(\mathcal{X})$ is reached.

(b) $\mathcal{X}$ has at most four centres. This bound is reached for a convenient $\mathcal{X}$.

(c) $\mathcal{X}$ has at most four foci. This bound is reached for a convenient $\mathcal{X}$.

(d) $\mathcal{X}$ has at most two saddles. This bound is reached for a convenient $\mathcal{X}$.
Assuming that $X$ has an invariant great circle, without loss of generality we can assume that it is $S^1 = \{z = 0\} \cap S^2$.

(1) The vector field $X$ can be written as (1.1) with

$$\begin{align*}
P(x, y, z) &= a_0 + a_1 y - a_0 x^2 + a_3 y^2 + a_4 z^2 + a_5 x y + a_7 y z, \\
Q(x, y, z) &= b_0 - a_1 x - (a_5 + b_0)x^2 \\
&\quad - b_0 y^2 + b_2 z^2 - (a_0 + a_3) x y - a_7 x z, \\
R(x, y, z) &= c_0 - c_0 x^2 - c_0 y^2 - c_0 z^2 - (a_0 + a_4) x z - (b_0 + b_2) y z.
\end{align*}$$

(1.4)

(2) The great circle $S^1$ is a periodic orbit of $X$ if and only if

$$(a_0 + a_3)^2 + a_5^2 - a_1^2 < 0.$$  \hspace{1cm} (1.5)

(3) The great circle $S^1$ is a hyperbolic limit cycle if and only if (1.5) holds and

$$\frac{a_5(a_0 + a_4) + (a_0 + a_3)(b_0 + b_2)}{\sqrt{a_1^2 - a_5^2 - (a_0 + a_3)^2(|a_1| + \sqrt{a_1^2 - a_5^2 - (a_0 + a_3)^2})}} \neq 0.$$  \hspace{1cm} (1.6)

(4) If (1.5) holds, if the right-hand side of (1.6) is equal to 0 and if

$$(a_0 + a_3)^2 + (b_0 + b_2)^2 \neq 0,$$

then the real parts of the two eigenvalues at every singular point of $X$ equal 0.

(iii) A homoclinic loop of $X$ does not contain any arc of a great circle.

(iv) There exist non-homogeneous $X$ having a great circle as a limit cycle.

(v) If $X$ is homogeneous, then any great circle cannot be a periodic orbit of $X$.

(vi) There exist $X$ having the equator of $S^2$ as a limit cycle and surrounding three singular points in one hemisphere and one singular point in the other.

(vii) If $X$ is homogeneous and has a hyperbolic saddle or a hyperbolic proper node, then there exists a great circle $C$ such that either all orbits of $X$ are transversal to $C$, or $C$ is an invariant great circle.

(viii) There exist $X$ having two centres, and all the other orbits are periodic.

(ix) There exist $X$ having two invariant great circles $C_1$ and $C_2$. Each of the four connected components of $S^2 \setminus \{C_1 \cup C_2\}$ is completely filled by a centre and its period annulus.

The paper is organized as follows. In §2 we introduce the notion of stereographic projection that will be used to prove many of our results. Theorems 1.1 and 1.2 will be proved in §§3 and 4, respectively.

We recall that Écalle [5] and Ilyashenko [7] proved independently that any analytic vector field on the sphere $S^2$ has finitely many limit cycles. So, in particular, any polynomial vector field on $S^2$ has a finite number of limit cycles.
2. Stereographic projection

The stereographic projection has been used frequently in the study of polynomial vector fields on $\mathbb{S}^2$. Here we will follow [9] with some small changes.

We identify $\mathbb{R}^2$ with the plane $ax + by + cz = 0$, where $a^2 + b^2 + c^2 = 1$. Suppose $c \neq 0$. Then the points in $\mathbb{R}^2$ are denoted by $(u, v, -(au + bv)/c)$. Let

$$\pi: \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{(a, b, c)\}$$

be the diffeomorphism given by

$$\pi(u, v, -\frac{au + bv}{c}) = (x, y, z) = \left(\frac{a\lambda - 2c^2(a - u)}{\lambda}, \frac{b\lambda - 2c^2(b - v)}{\lambda}, \frac{c\lambda - 2c(c^2 + au + bv)}{\lambda}\right),$$

where $\lambda = c^2(1 + u^2 + v^2) + (au + bv)^2$. Therefore, the stereographic projection

$$\pi^{-1}: \mathbb{S}^2 \setminus \{(a, b, c)\} \to \mathbb{R}^2$$

is defined by

$$\pi^{-1}(x, y, z) = \left(u, v, -\frac{au + bv}{c}\right) = \left(\frac{x - a(ax + by + cz)}{1 - (ax + by + cz)}, \frac{y - b(ax + by + cz)}{1 - (ax + by + cz)}, \frac{z - c(ax + by + cz)}{1 - (ax + by + cz)}\right).$$

Through the stereographic projection $\pi^{-1}$ the polynomial system $\mathcal{X}$ on $\mathbb{S}^2$ becomes the differential system

$$\begin{align*}
\dot{u} &= \bar{P} + (u - a)(\bar{aP} + \bar{bQ} + \bar{cR}), \\
\dot{v} &= \bar{Q} + (v - b)(\bar{aP} + \bar{bQ} + \bar{cR}).
\end{align*}$$

(2.1)

(2.2)

on the plane $\mathbb{R}^2$, where $\bar{S}(u, v) = \lambda^n S(\pi(u, v, -(au + bv)/c))$ and $S \in \{P, Q, R\}$. Let $t$ be the independent variable in the above differential system. Of course, the dot denotes the derivative with respect to $t$. Introducing the new independent variable $ds = dt/(2c^2\lambda^{n-1})$, we get the polynomial differential system

$$\begin{align*}
\dot{u} &= \bar{P} + (u - a)(\bar{aP} + \bar{bQ} + \bar{cR}), \\
\dot{v} &= \bar{Q} + (v - b)(\bar{aP} + \bar{bQ} + \bar{cR}).
\end{align*}$$

(2.2)

Now the dot denotes the derivative with respect to $s$. The planar vector field (2.2) will be called the stereographic projection of vector field $\mathcal{X} = (P, Q, R)$ at the point $(a, b, c) \in \mathbb{S}^2$.

In general we consider the stereographic projections at the points $(a, b, c) = (0, 0, 1)$ and $(a, b, c) = (0, 0, -1)$, and we denote their corresponding stereographic projections by $\pi_1$ and $\pi_{-1}$. These two projections form an atlas for the sphere $\mathbb{S}^2$. 

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On the polynomial vector fields on $\mathbb{S}^2$
Let
\[
P(x, y, z) = \sum_{i+j+k \leq n} p_{ijk} x^i y^j z^k,
\]
\[
Q(x, y, z) = \sum_{i+j+k \leq n} q_{ijk} x^i y^j z^k,
\]
\[
R(x, y, z) = \sum_{i+j+k \leq n} r_{ijk} x^i y^j z^k.
\]
(2.3)

If \((a, b, c) = (0, 0, 1)\), then system (2.2) becomes
\[
\begin{align*}
\dot{u} &= \tilde{P}(u, v) + u\tilde{R}(u, v) = \tilde{P}(u, v), \\
\dot{v} &= Q(u, v) + v\tilde{R}(u, v) = \tilde{Q}(u, v),
\end{align*}
\]
(2.4)

where
\[
\begin{align*}
\tilde{P}(u, v) &= \sum_{i+j+k \leq n} p_{ijk}(2u)^i(2v)^j(u^2 + v^2 - 1)^k(u^2 + v^2 + 1)^{n-i-j-k}, \\
\tilde{Q}(u, v) &= \sum_{i+j+k \leq n} q_{ijk}(2u)^i(2v)^j(u^2 + v^2 - 1)^k(u^2 + v^2 + 1)^{n-i-j-k}, \\
\tilde{R}(u, v) &= \sum_{i+j+k \leq n} r_{ijk}(2u)^i(2v)^j(u^2 + v^2 - 1)^k(u^2 + v^2 + 1)^{n-i-j-k}.
\end{align*}
\]
(2.5)

If \((a, b, c) = (0, 0, -1)\), then system (2.2) may be written
\[
\begin{align*}
\dot{u} &= \tilde{P}(u, v) - u\tilde{R}(u, v), \\
\dot{v} &= \tilde{Q}(u, v) - v\tilde{R}(u, v),
\end{align*}
\]
(2.6)

with
\[
\begin{align*}
\tilde{P}(u, v) &= \sum_{i+j+k \leq n} p_{ijk}(2u)^i(2v)^j(1 - u^2 - v^2)^k(u^2 + v^2 + 1)^{n-i-j-k}, \\
\tilde{Q}(u, v) &= \sum_{i+j+k \leq n} q_{ijk}(2u)^i(2v)^j(1 - u^2 - v^2)^k(u^2 + v^2 + 1)^{n-i-j-k}, \\
\tilde{R}(u, v) &= \sum_{i+j+k \leq n} r_{ijk}(2u)^i(2v)^j(1 - u^2 - v^2)^k(u^2 + v^2 + 1)^{n-i-j-k}.
\end{align*}
\]
(2.7)

In what follows, we study the properties of the polynomial vector fields \(X\) of degree \(n\) on \(S^2\) by using the stereographic projections (2.4) and (2.6).

3. Proof of theorem 1.1

We shall use the following result.

**Lemma 3.1.** Let \(X\) be the polynomial vector field of degree \(n\) on \(S^2\) given (1.1). Then the polynomial differential system (2.4) (respectively, (2.6)) has degree at most \(2n\). Moreover, it has degree at most \(2n - 1\) if \((0, 0, 1)\) (respectively, \((0, 0, -1)\)) is a singular point of \(X\).
Proof. We only prove this lemma for system (2.4). Similar arguments prove it for system (2.6).

We obtain from (2.5) that \( \deg \bar{P}(u, v) \leq 2n, \ \deg \bar{Q}(u, v) \leq 2n \). It follows from (1.3) that

\[
2u\bar{P}(u, v) + 2v\bar{Q}(u, v) + (u^2 + v^2 - 1)\bar{R}(u, v) = 0, \tag{3.1}
\]

which implies that \( \bar{R}(u, v) \) is a polynomial of degree at most \( 2n - 1 \). Hence, system (2.4) is a polynomial system of degree at most \( 2n \).

It follows from (2.5) that the homogeneous parts of \( \bar{P}(u, v) \) and \( \bar{Q}(u, v) \) of degree \( 2n \) are

\[
\left( \sum_{k=0}^{n} p_{00k} \right)(u^2 + v^2)^n, \quad \left( \sum_{k=0}^{n} q_{00k} \right)(u^2 + v^2)^n, \tag{3.2}
\]

respectively.

By the Poincaré–Hopf theorem, every vector field on \( S^2 \) has singular points (see, for example, [4]). Suppose that \((0, 0, 1)\) is a singular point of \( X \). Then we have

\[
\sum_{k=0}^{n} p_{00k} = 0, \quad \sum_{k=0}^{n} q_{00k} = 0. \tag{3.3}
\]

From (3.2) and (3.3) we get that \( \bar{P}(u, v) \) and \( \bar{Q}(u, v) \) are polynomials of degree at most \( 2n - 1 \). Equation (3.1) implies that \( \bar{R}(u, v) \) is a polynomial of degree at most \( 2n - 2 \). Therefore, system (2.4) has degree at most \( 2n - 1 \) if \((0, 0, 1)\) is a singular point of \( X \) on \( S^2 \).

Now we shall prove theorem 1.1. We only need to prove statements (ii)–(iv).

Proof of theorem 1.1(ii). Let \( p \) be a singular point of \( X \). Doing a convenient rotation of \( SO(3) \) we can assume that \( p = (0, 0, 1) \). Let \((u, v)\) be a singular point of system (2.4). Then \( \bar{P}(u, v) = -u\bar{R}(u, v), \ \bar{Q}(u, v) = -v\bar{R}(u, v) \). This, together with equation (3.1), shows that \( \bar{R}(u, v) = 0 \). Therefore, the coordinates \((u, v)\) of singular points are determined by the equations \( \bar{P}(u, v) = \bar{Q}(u, v) = \bar{R}(u, v) = 0 \). Since we have shown in the proof of lemma 3.1 that \( \deg \bar{P}(u, v) \leq 2n - 1, \ \deg \bar{Q}(u, v) \leq 2n - 1, \ \deg \bar{R}(u, v) \leq 2n - 2 \) by the Bezout theorem (see, for example, [6]), system (2.4) has at most \((2n - 2)(2n - 1) + 1\) isolated singular points. Noting that \((0, 0, 1)\) is also a singular point, the polynomial vector field \( X \) of degree \( n \) on \( S^2 \) has at most \((2n - 2)(2n - 1) + 1\) isolated singular points.

If \( n = 1 \) and \((0, 0, 1)\) is a singular point of \( X \), then \( \bar{R}(x, y, z) \equiv 0 \). By direct computation we know that \( X \) has at most 2 singular points if \( s(X) \) is finite.

Proof of theorem 1.1(iii). By the Poincaré–Hopf theorem, a vector field on \( S^2 \) with a finite number of singular points satisfies the condition that the sum of the indices of its singular points is +2 (see, for example, [4]).

Under the assumptions all the indices are 1 or –1. Suppose that \( X \) has \( s \) singular points with the index +1. Then the number of the singular points with index –1 is \( s - 2 \). This yields that the vector field \( X \) on \( S^2 \) has at most \( 2s - 2 \) isolated singular points. Now, using part (ii) of theorem 1.1, part (iii) follows.
Proof of theorem 1.1(iv). Without loss of generality we shall work with the great circle \( \{ z = 0 \} \cap S^2 \). Under the stereographic projection at the point \((0, 0, 1)\), the phase portrait of the vector field \( \mathcal{X} \) on \( S^2 \setminus (0, 0, 1) \) is topologically equivalent to that of system (2.4) with (2.5) in \( \mathbb{R}^2 \), and the great circle goes over the unite circle \( S^1 = \{ u^2 + v^2 = 1 \} \).

Suppose that the great circle \( S^1 \) touches the orbits of the polynomial differential system (2.4) at \( (u, v) \). Then it follows from (3.1) that

\[
\dot{u} + v = u \tilde{P}(u, v) + v \tilde{Q}(u, v) + (u^2 + v^2) \tilde{R}(u, v) = \frac{1}{2}(u^2 + v^2 + 1) \tilde{R}(u, v). \tag{3.4}
\]

So every contact point to \( S^1 \) satisfies

\[
\tilde{R}(u, v) = 0, \quad u^2 + v^2 = 1.
\]

Since

\[
\tilde{R}(u, v)|_{(u,v)\in S^1} = \sum_{i+j=0}^{n} 2^n r_{ij} u^i v^j \tag{3.5}
\]

is a polynomial of degree \( n \), part (iv) of theorem 1.1 follows using the Bezout theorem.

4. Proof of theorem 1.2

In this section we study quadratic polynomial vector fields on \( S^2 \). The next result will be very useful.

**Lemma 4.1.** Let \( \mathcal{X} \) be a quadratic polynomial vector field in \( \mathbb{R}^3 \) given by (1.1). Then \( \mathcal{X} \) is a quadratic polynomial vector field on \( S^2 \) if and only if the differential system associated to \( \mathcal{X} \) can be written as

\[
\begin{align*}
\dot{x} &= P(x, y, z) \\
&= a_0 + a_1 y + a_2 z - a_0 x^2 + a_3 y^2 + a_4 z^2 + a_5 xy + a_6 xz + a_7 yz, \\
\dot{y} &= Q(x, y, z) \\
&= b_0 - a_1 x + b_1 z - (a_5 + b_0) x^2 - b_0 y^2 + b_2 z^2 - (a_0 + a_3) xy + b_3 xz + b_4 yz, \\
\dot{z} &= R(x, y, z) \\
&= c_0 - a_2 x - b_1 y - (a_6 + c_0) x^2 - (b_4 + c_0) y^2 - c_0 z^2 - (a_7 + b_3) xy - (a_0 + a_4) xz - (b_0 + b_2) yz. \tag{4.1}
\end{align*}
\]

**Proof.** Let

\[
x P(x, y, z) + y Q(x, y, z) + z R(x, y, z) - (x^2 + y^2 + z^2 - 1) K(x, y, z) \equiv 0,
\]

where \( P, Q, R \) are defined as (2.3) with

\[
n = 2 \quad \text{and} \quad K(x, y, z) = \sum_{i+j+k=0}^{1} k_{ijk} x^i y^j z^k.
\]

Solving this equation and renaming the coefficients of (1.1), the lemma follows. \( \square \)
For system (4.1), we have
\[
\begin{align*}
\ddot{P}(u, v) &= a_0 - a_2 + a_4 - 2a_6u + 2(a_1 - a_7)v - 2(a_0 + a_4)u^2 + 4a_5uv \\
&\quad + 2(a_0 + 2a_3 - a_4)v^2 + 2a_6u^3 + 2(a_1 + a_7)uv + 2a_6uv^2 \\
&\quad + 2(a_1 + a_7)v^3 + (a_0 + a_2 + a_4)(u^2 + v^2)^2, \\
\ddot{Q}(u, v) &= b_0 - b_1 + b_2 - 2(a_1 + b_5)u - 2b_4v - 2(2a_5 + b_5 + b_2)u^2 \\
&\quad - 4(a_0 + a_3)uv - 2(b_0 + b_4)v^2 - 2(a_1 - b_3)u^3 + 2b_3u^2v \\
&\quad - 2(a_1 - b_3)uv^2 + 2b_4v^3 + (b_0 + b_1 + b_2)(u^2 + v^2)^2, \\
\ddot{R}(u, v) &= 2(a_0 - a_2 + a_4)u + 2(b_0 - b_1 + b_2)v - 4a_6u^2 - 4(a_7 + b_3)uv \\
&\quad - 4b_4v^2 - 2(a_0 + a_2 + a_4)u^3 - 2(b_0 + b_1 + b_2)u^2v \\
&\quad - 2(a_0 + a_2 + a_4)uv^2 - 2(b_0 + b_1 + b_2)v^3.
\end{align*}
\]
\begin{equation}
(4.2)
\end{equation}

We shall give the proof of part (i) of theorem 1.2 at the end of this section.

Proof of theorem 1.2(ii). By assumptions, \( S^1 = \{ z = 0 \} \cap S^2 \) is an invariant great circle of system \( \mathcal{X} \). By lemma 4.1, the vector field \( \mathcal{X} \) can be written as (4.1). Under the stereographic projection at the point \( (0, 0, 1) \), the vector field \( \mathcal{X} \) becomes system (2.4) and the great circle \( \{ z = 0 \} \cap S^2 \) is reduced to the unit circle \( S^1 : u^2 + v^2 = 1 \) of \( \mathbb{R}^2 \). Note that we denote by \( S^1 \) the great circle \( \{ z = 0 \} \cap S^2 \) and the unit circle \( u^2 + v^2 = 1 \); of course they are different objects, but they are diffeomorphic through the stereographic projection at the point \( (0, 0, 1) \).

It follows from (3.4) and (4.2) that \( S^1 \) is an invariant circle of (2.4) if and only if \( \ddot{R}(u, v) \) is divisible by \( u^2 + v^2 - 1 \), i.e. if and only if
\[
a_2 = a_6 = b_1 = b_4 = 0 \quad \text{and} \quad b_3 = -a_7
\]
\begin{equation}
(4.3)
\end{equation}
hold. Putting conditions (4.3) into (4.1) we obtain part (1) of theorem 1.2(ii).

The invariant great circle \( S^1 \) is a periodic orbit of \( \mathcal{X} \) if and only if there is no singular point on it. It follows from (1.4), (2.4) and (4.2) that if \( (\cos \theta, \sin \theta) \in S^1 \), then
\[
\begin{align*}
\mathcal{P}(\cos \theta, \sin \theta) &= 4 \sin \theta(a_1 + a_5 \cos \theta + (a_0 + a_3) \sin \theta), \\
\mathcal{Q}(\cos \theta, \sin \theta) &= -4 \cos \theta(a_1 + a_5 \cos \theta + (a_0 + a_3) \sin \theta).
\end{align*}
\]

Therefore, \( S^1 \) is a periodic orbit if and only if the three systems
\[
\begin{align*}
v &= 0, \quad a_1 + a_5u + (a_0 + a_3)v = 0, \quad u^2 + v^2 = 1, \\
u &= 0, \quad a_1 + a_5u + (a_0 + a_3)v = 0, \quad u^2 + v^2 = 1, \\
a_1 + a_5u + (a_0 + a_3)v &= 0, \quad u^2 + v^2 = 1
\end{align*}
\]
\begin{equation}
(4.4)
\end{equation}
\begin{equation}
(4.5)
\end{equation}
\begin{equation}
(4.6)
\end{equation}
have no solution. This is equivalent to (1.5). Statement (2) follows.

It is well known that a periodic orbit \( (u(t), v(t)) \) of period \( T \) is a hyperbolic limit cycle if and only if
\[
I = \int_0^T \left( \frac{\partial \mathcal{P}}{\partial u} + \frac{\partial \mathcal{Q}}{\partial v} \right)(u(t), v(t)) \, dt \neq 0.
\]
\begin{equation}
(4.7)
\end{equation}
Let \((u, v) = (\cos \theta, \sin \theta)\). Since \(dt = du/P(u, v)\), we have

\[
I_{|S^1} = \int_{S^1} \frac{P_u + Q_v}{P(u, v)} \, du
\]

\[
= -\int_0^{2\pi} \frac{\sin \theta(P_u(\cos \theta, \sin \theta) + Q_v(\cos \theta, \sin \theta))}{P(\cos \theta, \sin \theta)} \, d\theta
\]

\[
= \int_0^{2\pi} \frac{(2a_0 + a_3 + a_4) \cos \theta + (b_0 + b_2 - a_5) \sin \theta}{a_1 + a_5 \cos \theta + (a_0 + a_3) \sin \theta} \, d\theta
\]

\[
= \frac{-2\pi(a_5(a_0 + a_4) + (a_0 + a_3)(b_0 + b_2))}{\sqrt{a_1^2 - a_5^2 - (a_0 + a_3)^2(|a_1| + \sqrt{a_1^2 - a_5^2 - (a_0 + a_3)^2})}},
\]

where we use the fact that \(|a_1| > |a_5|\) (see (1.5)). Hence, \(S^1\) is a hyperbolic limit cycle of system (2.4) if and only if \(I_{|S^1} \neq 0\) and (1.5) holds. So statement (3) is proved.

Finally, we prove (4). We note that \(I_{|S^1} = 0\) if and only if

\[a_5(a_0 + a_4) + (a_0 + a_3)(b_0 + b_2) = 0.\]

Suppose that (1.5) holds and that \(I_{|S^1} = 0\). Then, by direct computation, we have

\[uP(u, v) + vQ(u, v) = -((a_0 + a_4)u + (b_0 + b_2)v)(u^2 + v^2 + 1)(u^2 + v^2 - 1),\]

(4.8)

which implies that the coordinates of the singular points of system (2.4) under the assumptions satisfy \((a_0 + a_4)u + (b_0 + b_2)v = 0\) if \(a_0 + a_4 \neq 0\), then

\[
\left. \frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v} \right|_{a_5=-(a_0+a_4)/(a_0+a_4)} = 0.
\]

If \(a_0 + a_4 = 0, b_0 + b_2 \neq 0\), then \(a_5(a_0 + a_4) + (a_0 + a_3)(b_0 + b_2) = 0\) implies \(a_0 + a_3 = 0\). We have

\[
\left. \frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v} \right|_{v=0, a_3=-a_0, a_4=-a_0} = 0.
\]

Since the coordinates of the singular points of system (2.4) under these assumptions satisfy \((a_0 + a_4)u + (b_0 + b_2)v = 0\), the above equation is also equal to zero if \((u^*, v^*)\) is a singular point of system (2.4). This proves (4), except perhaps if \((0, 0, 1)\) is a singular point of \(X\), but this is not the case because the origin of system (2.6) is a singular point if and only if \(a_0 + a_4 = b_0 + b_2 = 0\), which by assumption is not the case.

Under the assumptions of statement (4) of theorem 1.2(ii), if a singular point of \(X\) is a focus, then it is non-hyperbolic.

**Proof of theorem 1.2(iii).** Suppose that a homoclinic loop of \(X\) on \(S^2\) contains an arc of a great circle. Without loss of generality we assume that this great circle is \(\{z = 0\} \cap S^2\). It follows from statement (ii) of theorem 1.2 that if \(\{z = 0\} \cap S^2\) is an invariant great circle of \(X\), then \(X\) can be written as (1.4). Since the coordinates of the singular points on \(\{z = 0\} \cap S^2\) are determined by equations (4.4)–(4.6), the number of the singular points on the great circle is an even number, which implies that \(\{z = 0\} \cap S^2\) is not a homoclinic loop of \(X\).
Part (iii) of theorem 1.2 also holds for the vector fields on \( S^2 \) induced by homogeneous vector fields of degree two in \( \mathbb{R}^3 \) [8,14].

Parts (iv) and (v) of theorem 1.2 show some of the differences between homogeneous and non-homogeneous quadratic polynomial vector fields on \( S^2 \).

**Proof of theorem 1.2(iv).** Consider the vector field (1.4). If \( a_0 = a_3 = 0, a_1 = 2 \) and \( a_4 = a_5 = 1 \), then (1.6) becomes \( 2/\sqrt{3} - 1 \), and (1.5) holds. So part (iv) of theorem 1.2 follows from part (ii) of the same theorem.

**Proof of theorem 1.2(v).** Suppose that (1.4) is a quadratic homogeneous polynomial vector field (i.e. \( a_0 = a_1 = b_0 = b_4 = 0 \)) having \( S^1 \) is an invariant great circle. Since \((a_0 + a_3)^2 - a_1^2 + a_5^2 = a_3^2 + a_5^2 \geq 0\), (1.5) does not hold. Therefore, from (2) of theorem 1.2(ii), part (v) follows.

Part (v) of theorem 1.2 also holds for the vector fields on \( S^2 \) induced by a homogeneous vector field of degree 2 in \( \mathbb{R}^3 \) [8]. We also note that part (v) of theorem 1.2 was proved in [10] using different arguments.

**Proof of theorem 1.2(vi).** Consider system (4.1) with
\[
\begin{align*}
a_0 &= a_2 = a_3 = a_6 = 0, \\
a_1 &= 2, \\
a_4 &= a_5 = 1, \\
a_7 &= 10, \\
b_0 &= b_1 = b_2 = b_4 = 0, \\
b_3 &= -10.
\end{align*}
\]

It follows from system (2.4) that the stereographic projection of system (4.1) is given by
\[
\begin{align*}
\dot{u} &= 1 - 16v + 4uv - 2u^2 + 24u^2v + 24v^3 - u^4 + v^4 = \mathcal{P}(u,v), \\
\dot{v} &= -2u(-8 + 2u - v + 12u^2 + 12v^2 + u^2v + v^3) = \mathcal{Q}(u,v).
\end{align*}
\]

By direct computation, \( \mathcal{P}(0,v) = 1 - 16v - 2v^2 + 24v^3 + v^4 \) and \( \mathcal{P}(0,\pm\infty) = +\infty, \mathcal{P}(0,-1) = -8, \mathcal{P}(0,0) = 1, \mathcal{P}(0,1/2) = -71/54 \) and \( \mathcal{P}(0,1) = 8 \). This implies that system (4.9) has at least four singular points on the line \( u = 0 \) and three of them are inside the circle \( u^2 + v^2 = 1 \).

The system that we consider has the same form as system (1.4). Since the given coefficients of system (4.1) satisfy (1.5) and (1.6), it follows from part (ii) of theorem 1.2 that the great circle \( \{z = 0\} \cap S^2 \) is a limit cycle of system (4.1) with the given coefficients. On the other hand, the equality
\[
u\mathcal{P}(u,v) + v\mathcal{Q}(u,v) = -u(u^2 + v^2 - 1)(u^2 + v^2 + 1)
\]
shows that the coordinates of singular points of system (4.9) satisfy either \( u = 0 \) or \( u^2 + v^2 = 1 \). Since \( u^2 + v^2 = 1 \) is the stereographic projection of the great circle \( \{z = 0\} \cap S^2 \), which is a limit cycle, there is no singular point on \( u^2 + v^2 = 1 \). Therefore, system (4.1) with the given coefficients has only four singular points on \( S^2 \). One is the singular point \( (0,0,1) \) of the northern hemisphere and the other three are in the southern hemisphere.
Proof of theorem 1.2(vii). Here we use similar arguments to those of [8]. Without loss of generality we suppose that \( p = (0, 0, -1) \) is a singular point of \( \mathcal{X} \). Then \( \mathcal{X} \) becomes system (2.4) under the stereographic projection \( \pi^{-1} \) at the point \( p \). Since \( \mathcal{X} \) is a quadratic homogeneous polynomial vector field, it follows from lemma 4.1 that \( \mathcal{X} \) can be written as (4.1) with \( a_0 = a_1 = a_2 = a_4 = b_0 = b_1 = b_2 = c_0 = 0 \). A direct computation shows that \( (0, 0, 1) \) is also a singular point of \( \mathcal{X} \).

The origin \( q = (0, 0) \) is a singular point of the stereographic projection given by system (2.4). Assume that \( q \) is a hyperbolic saddle or a hyperbolic proper node of (2.4). Then the linear part of system (2.4) at \( q \),
\[
\begin{pmatrix}
-2a_6 & -2a_7 \\
-2b_3 & -2b_4
\end{pmatrix}
\]
has two different real eigenvalues. This implies
\[
(a_6 + b_4)^2 - 4(a_6b_4 - a_7b_3) = (b_4 - a_6)^2 + 4a_7b_3 > 0. \tag{4.10}
\]

Consider the straight line \( L = v - ku = 0 \). We have
\[
\frac{dL}{dt}\bigg|_{L=0} = 2(b_3 + (b_4 - a_6)k - a_7k^2)u((1 + k^2)u^2 - 1) - 4(a_5 + a_3k)(1 + k^2)u^2.
\]

Since (4.10) holds, the equation \( b_3 + (b_4 - a_6)k - a_7k^2 = 0 \) has two real zeros for \( k \) if \( a_7 \neq 0 \), and one real zero if \( a_7 = 0 \). Let \( k \) be one zero of the above equation. Then we have
\[
\frac{dL}{dt}\bigg|_{L=0} = -4(a_5 + a_3k)(1 + k^2)u^2.
\]

If \( \frac{dL}{dt}\big|_{L=0} \) does not change sign and \( \frac{dL}{dt}\big|_{L=0} \neq 0 \), then all orbits pass through \( L \) in the same direction. If \( \frac{dL}{dt}\big|_{L=0} \equiv 0 \), then \( L \) is an invariant straight line of system (2.4). Since \( \pi(L) \) is a great circle on \( \mathbb{S}^2 \), part (vii) of theorem 1.2 follows.

Part (vii) of theorem 1.2 shows that if a quadratic homogeneous polynomial vector field \( \mathcal{X} \) on \( \mathbb{S}^2 \) has either a hyperbolic saddle or a hyperbolic proper node, then a periodic orbit is entirely contained in a hemisphere. This assertion also holds for the vector fields on \( \mathbb{S}^2 \) induced by homogeneous quadratic polynomial vector fields of in \( \mathbb{R}^3 \) [1].

Proof of theorem 1.2(viii). There exists polynomial vector fields on \( \mathbb{S}^2 \) such that \( \mathbb{S}^2 \) is completely filled by centres and their families of period annulus. For example, the linear differential system of the form
\[
\dot{x} = -y, \quad \dot{y} = x, \quad \dot{z} = 0
\]
has the two poles of the sphere \( \mathbb{S}^2 \) as singular points and all the parallels as periodic orbits. The same occurs for the quadratic polynomial vector field
\[
\dot{x} = -y(z - 2), \quad \dot{y} = x(z - 2), \quad \dot{z} = 0.
\]

So part (viii) of theorem 1.2 is proved. But we provide a big family of quadratic polynomial vector fields on \( \mathbb{S}^2 \) with the same topological equivalent phase portrait.
In statement (4) of theorem 1.2 we suppose that \((a_0 + a_4)^2 + (b_0 + b_2)^2 \neq 0\). What happens if \((a_0 + a_4)^2 + (b_0 + b_2)^2 = 0\)? We will show that all of these quadratic polynomial vector fields on \(S^2\) have the phase portrait described in part (viii) of theorem 1.2.

Suppose that the vector field (1.4) satisfies (1.5) and that, additionally, \(a_4 = -a_0\) and \(b_2 = -b_0\), then (1.4) becomes

\[
\begin{align*}
\dot{x} &= P(x, y, z) = a_0 + a_1 y - a_0 x^2 + a_3 y^2 - a_0 z^2 + a_5 x y + a_7 y z, \\
\dot{y} &= Q(x, y, z) = b_0 - a_1 x - (a_5 + b_0) x^2 - b_0 y^2 - b_0 z^2 - (a_0 + a_3) x y - a_7 x z, \\
\dot{z} &= R(x, y, z) = -c_0 (x^2 + y^2 + z^2 - 1).
\end{align*}
\]

(4.11)

The stereographic projection of (4.11) at the point \((0, 0, 1)\) provides the differential system

\[
\begin{align*}
\dot{u} &= 2v(u - a_1 + a_7 + 2a_5 u + 2(a_0 + a_3) v + (a_1 + a_7)(u^2 + v^2)), \\
\dot{v} &= -2u(u - a_1 + a_7 + 2a_5 u + 2(a_0 + a_3) v + (a_1 + a_7)(u^2 + v^2)).
\end{align*}
\]

(4.12)

If \(a_1 + a_7 \neq 0\) and \(a_1^2 - a_7^2 - a_3^2 - (a_0 + a_3)^2 > 0\), we have

\[
\begin{align*}
& (a_1 - a_7 + 2a_5 u + 2(a_0 + a_3) v + (a_1 + a_7)(u^2 + v^2) \\
&= (a_1 + a_7) \left( \left( u + \frac{a_5}{a_1 + a_7} \right)^2 + \left( v + \frac{a_0 + a_3}{a_1 + a_7} \right)^2 + \frac{a_1^2 - a_7^2 - a_3^2 - (a_0 + a_3)^2}{(a_1 + a_7)^2} \right) \\
& \neq 0,
\end{align*}
\]

which implies that system (4.12) has the unique singular point at the origin which is a centre. Therefore, \((0, 0, -1)\) is a centre of system (4.11).

By the same arguments we conclude that the \((0, 0, 1)\) is also a centre of system (4.11). Hence \(S^2\) is completely filled by two centres at \((0, 0, \pm 1)\) and their periodic orbits.

**Proof of theorem 1.2(ix).** Suppose that a quadratic vector field \(\mathcal{X}\) on \(S^2\) has six isolated singular points at \((\pm 1, 0, 0)\), \((0, \pm 1, 0)\) and \((0, 0, \pm 1)\), respectively. It follows from lemma 4.1 that \(\mathcal{X}\) can be written as

\[
\begin{align*}
\dot{x} &= a_0 - a_0 x^2 - a_0 y^2 - a_0 z^2 + a_7 y z, \\
\dot{y} &= b_0 - b_0 x^2 - b_0 y^2 - b_0 z^2 + b_3 x z, \\
\dot{z} &= c_0 - c_0 x^2 - c_0 y^2 - c_0 z^2 - (a_7 + b_3) x y.
\end{align*}
\]

(4.13)

Making the change \(2t = \tau\), the stereographic projections of system (4.13) at the points \((0, 0, 1)\) and \((0, 0, -1)\) are given by

\[
\begin{align*}
\dot{u} &= v(a_7 - (a_7 + 2b_3) u^2 + a_7 v^2), \\
\dot{v} &= u(-b_3 + b_3 u^2 - (2a_7 + b_3) v^2),
\end{align*}
\]

(4.14)

and

\[
\begin{align*}
\dot{u} &= v(a_7 + (a_7 + 2b_3) u^2 - a_7 v^2), \\
\dot{v} &= u(b_3 - b_3 u^2 + (2a_7 + b_3) v^2),
\end{align*}
\]

(4.15)
respectively, where the dot now denotes the derivative with respect to $\tau$. If $a_7b_3 = 0$, then system (4.14), (4.15) has non-isolated singular points. Therefore, we suppose $a_7b_3 \neq 0$.

By direct computation we know that the linearized system of (4.14) and (4.15) at each singular point has two non-zero eigenvalues. So the topological index at every singular point is $\pm 1$ (for more details, see [4]). Since a vector field on $S^2$ with a finite number of singular points satisfies that the sum of their indices is 2 (see the Poincaré–Hopf theorem in, for example, [4]), system (4.13) on $S^2$ has two saddles whose indices are $-1$ and four antisaddles with indices 1. Without loss of generality we assume that $(0, 0, -1)$ is a saddle. This implies $a_7b_3 > 0$. Moreover, we take $a_7 = 1$ and $b_3 > 0$. By direct computations we check that system (4.15) has a saddle at $(0, 0)$, and that system (4.14) has the first integral

$$H(u, v) = \frac{-b_3u^2 + v^2}{(1 + u^2 + v^2)^2},$$

(4.16)

the four centres $(0, \pm 1)$ and $(\pm 1, 0)$, and the saddle $(0, 0)$.

It follows from (4.16) that system (4.14) has two invariant straight lines $v = \pm \sqrt{b_3}u$, which correspond to two invariant great circles $\{(x, y) \mid y = \pm \sqrt{b_3}x\} \cap S^2$ of system (4.13). The proof of part (ix) of theorem 1.2 is complete. 

Proof of theorem 1.2(i). From part (iii) of theorem 1.1 we know that $s(\mathcal{X}) \leq 6$. Since the quadratic polynomial vector field $\mathcal{X}$ on $S^2$ of part (ix) of theorem 1.2 has exactly six singular points, part (a) of theorem 1.2 is proved.

By part (a) of theorem 1.2, a quadratic polynomial vector field $\mathcal{X}$ such that all the indices of its singular points are $\pm 1$ has at most 6 singular points. Since centres have index 1 and the sum of all the indices is always 2 (by the Poincaré–Hopf theorem), the first part of theorem 1.2(b) follows. The bound of four centres is reached by part (ix) of theorem 1.2. Hence, statement (b) of theorem 1.2 is completely proved.

Of course, the proof of the first part of theorem 1.2(c) is the same as that of the proof of the first part of statement (b). Now we shall provide a quadratic polynomial vector field on $S^2$ with 4 foci.

Consider the quadratic polynomial vector field on $S^2$ given by

$$\begin{align*}
P(x, y, z) &= a_0 - a_0x^2 + a_3y^2 - a_0z^2 + a_7yz, \\
Q(x, y, z) &= b_0 - b_0x^2 - b_0y^2 + b_0z^2 - (a_0 + a_3)xy + b_3xz + b_4yz, \\
R(x, y, z) &= c_0 - c_0x^2 - (b_1 + c_0)y^2 - c_0z^2 - (a_7 + b_3)xy.
\end{align*}$$

(4.17)

Making the change $2t = \tau$, the stereographic projections of system (4.17) at the points $(0, 0, 1)$ and $(0, 0, -1)$ are given by

$$\begin{align*}
\dot{u} &= v(-a_7 + 2(a_0 + a_3)v - (a_7 + 2b_3)u^2 - 2b_4uv + a_7v^2), \\
\dot{v} &= -b_3u - b_3v - 2(a_0 + a_3)uv + b_3u^3 + b_4u^2v - (2a_7 + b_3)uv^2 - b_4v^3,
\end{align*}$$

(4.18)

and

$$\begin{align*}
\dot{u} &= v(a_7 + 2(a_0 + a_3)v + (a_7 + 2b_3)u^2 + 2b_4uv - a_7v^2), \\
\dot{v} &= b_3u + b_3v - 2(a_0 + a_3)uv - b_3u^3 - b_4u^2v + (2a_7 + b_3)uv^2 + b_4v^3.
\end{align*}$$

(4.19)
respectively. It follows from (4.18) and (4.19) that if

\[ b_4^2 + 4a_7b_3 < 0, \quad (a_0 + a_3)^2 - 4(a_7 + b_3)b_3 < 0, \quad b_4 \neq 0, \quad a_0 + a_3 \neq 0, \]

then \((-1, 0, 0)\) and \((0, 0, \pm 1)\) are foci of system (4.17). Hence, part (c) of theorem 1.2 is proved.

Since \(s(X) \leq 6\), saddles have index \(-1\) and the sum of all the indices is always 2, the first part of part (d) of theorem 1.2 follows. The bound of two saddles is reached by part (ix) of theorem 1.2. Hence, part (d) of theorem 1.2 is proved.

Acknowledgements

The authors thank the referee for comments that helped to improve the presentation of this paper. Y.Z. thanks the Departament de Matemàtiques of the Universitat Autònoma de Barcelona for its hospitality and support during the period in which this paper was started. J.L. was partly supported by DGICYT/FEDER Grant no. MTM2008-03437, by CICYT Grant no. 2009SGR 410 and ICREA Academia. Y.Z. was partly supported by the Spanish Grant SAB2005-0029, NSF of China (no. 10871214), the PhD Programs Foundation of the Ministry of Education of China (no. 20100171110040) and the Program for New Century Excellent Talents in University.

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(Issued 7 October 2011)