THE ALGEBRA OF $SL_3(\mathbb{C})$ CONFORMAL BLOCKS

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Abstract. We construct and study a family of toric degenerations of the algebra of conformal blocks for a stable marked curve $(C, \vec{p})$ with structure group $SL_3(\mathbb{C})$. We find that this algebra is Gorenstein. For the genus 0,1 cases we find the level of conformal blocks necessary to generate the algebra. In the genus 0 case we also find bounds on the degrees of relations required to present the algebra. Along the way we recover polyhedral rules for counting conformal blocks originally due to Senechal, Mathieu, Kirillov, and Walton.

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1. INTRODUCTION

Let $(C, \vec{p})$ be a smooth, complex, projective curve of genus $g$ with $n$ marked points. In this paper we study the projective coordinate algebras of the moduli stack $\mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C}))$ of quasi-parabolic principal $SL_3(\mathbb{C})$-bundles on $(C, \vec{p})$. In particular, we construct toric degenerations of these algebras out of polyhedra designed to compute the dimensions of the spaces $V_{C,\vec{p}}(\lambda, L)$ of $SL_3(\mathbb{C})$ conformal blocks.

For any trivalent graph $\Gamma$ with $n$ leaves and first Betti number equal to $g$, we will define a lattice polytope $\mathcal{C}B_{\Gamma}^*$, see Section 3. Additionally we will define for any $n$-tuple $\lambda$ of dominant $SL_3(\mathbb{C})$ weights, and non-negative integer $L$ a lattice polytope $CB_{\Gamma}^*(\lambda, L)$. We prove the following.

**Theorem 1.1.** Let $C$ be an $n$-marked curve of genus $g$, and let $\Gamma$ be a trivalent graph with first Betti number $g$ and $n$ leaves. The Cox ring (total coordinate ring) of $\mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C}))$ flatly degenerates to the semigroup algebra associated $CB_{\Gamma}^*$. 

This work was supported by the NSF fellowship DMS-0902710.
For the definition of the Cox ring of \( \mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C})) \), see below. We also obtain the following corollary.

**Corollary 1.2.** Let \( C, \vec{p}, \Gamma \) be as above, and let \( \mathcal{L}(\vec{\lambda}, L) \) be the effective line bundle on \( \mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C})) \) corresponding to the \( n \)-tuple \( \vec{\lambda} \) of dominant \( SL_3(\mathbb{C}) \) weights and the non-negative integer \( L \). The projective coordinate algebra \( R_{C,\vec{p}}(\vec{\lambda}, L) \) corresponding to \( \mathcal{L}(\vec{\lambda}, L) \) flatly degenerates to the semigroup algebra \( \mathbb{C}[CB^*_\Gamma(\vec{\lambda}, L)] \).

We use these semigroups to study the algebra Cox(\( \mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C})) \)) and \( R_{C,\vec{p}}(\vec{\lambda}, L) \). In particular we find that the Cox(\( \mathcal{M}_{C,\vec{p}}(SL_3(\mathbb{C})) \)) is always Gorenstein (Theorem 3.1), and we find a generating set and bounds on degrees of necessary relations when the genus of \( C \) is 0 (Theorem 3.2), and a generating set when the genus is 1 (Theorem 3.4).

### 1.1. Preliminaries.

Vector spaces of conformal blocks \( V_{C,\vec{p}}(\vec{\lambda}, L) \) for a simple Lie algebra \( g \) originate from the Wess-Zumino-Witten model of conformal field theory in \( 1+1 \) dimensions. Here \( (C, \vec{p}) \) is the marked curve, \( L \) is a non-negative integer known as the *level* and \( \vec{\lambda} \) is an \( n \)-tuple of dominant weights for the algebra \( g \). See [TUY], [U], [Be], and [L], or our abridged account below for the construction of these spaces.

Conformal blocks for \( g \) began receiving attention from algebraic geometers because of their relation to the moduli stack \( \mathcal{M}_{C,\vec{p}}(G) \) of quasi-parabolic principal \( G \) bundles on the curve \( (C, \vec{p}) \), for \( \text{Lie}(G) = g \). For a Borel subgroup \( B \subset G \), the Picard group of \( \mathcal{M}_{C,\vec{p}}(G) \), calculated in [LS], is a product of \( n \) copies of the character group of \( B \) with a copy of \( \mathbb{Z} \).

**Remark 1.3.** Throughout we take \( \mathcal{M}_{C,\vec{p}}(G) \) to be the moduli stack of quasi-parabolic bundles with parabolic structure at each \( p_i \in \vec{p} \) coming from a Borel subgroup \( B \subset G \). This is actually not a restriction for our purposes, as the projective coordinate rings of the moduli stacks for larger parabolic groups all appear as projective coordinate rings of this stack. This situation is analogous to (and indeed is a consequence of) the fact that the projective coordinate rings of the flag varieties \( G/P \) of \( G \), all appear as the projective coordinate rings of the full flag variety \( G/B \).

Conformal blocks make a related appearance in the geometry of the moduli stacks \( \mathcal{M}_{g,n} \) of semi-stable curves of genus \( g \) with \( n \) marked points. The spaces \( V_{C,\vec{p}}(\vec{\lambda}, L) \) fit together into a vector bundle \( V(\vec{\lambda}, L) \) with projective connection over
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$\mathcal{M}_{g,n}$. In this way, conformal blocks provide a rich source of vector bundles for studying the geometry of the moduli of curves. Special attention has been given to the dimension $\mathcal{V}_{g,n}(\vec{\lambda}, L) = \dim(V_{\text{C,}\vec{p}}(\vec{\lambda}, L))$ of spaces of conformal blocks due to their appearance in formulas for the Chern classes of the bundles $\mathcal{V}(\vec{\lambda}, L)$. Formulas for Chern classes, and the degrees of these bundles have been explored in many recent papers, see [Far], [AGS], [AGSS], [GG].

There are several rules for computing the dimensions of conformal blocks in the literature, some special, some general, starting with the Verlinde Formula.

(4) $\mathcal{V}_{g,n}(\vec{\lambda}, L) = |T_L|^{-1} \sum_{\mu \in \Delta_L} \text{Tr}_\vec{\lambda}(\exp(2\pi i \frac{(\mu + \rho)}{L + h^\vee})) \prod_\alpha [2\sin(\pi \frac{\alpha(\mu + \rho)}{L + h^\vee})]^{2-2g}$

Here $h^\vee$ is the dual Coxeter number of $g$, $\rho$ is the half sum of the positive roots, $\text{Tr}_\vec{\lambda}$ is a character associated to the tensor product of irreducible representations $V(\vec{\lambda})$, and $|T_L|$ is the cardinality of a certain subgroup of a maximal torus of $G$.

This expression was first written down by Verlinde in [Ver], owing much to the conformal field theoretic origins of conformal blocks. It was rigorously proved by Tsuchiya-Ueno-Yamada [TUY], and Faltings [Fal], using a combinatorial structure known as a fusion algebra, see also Beauville [Be]. The utility of this structure is owed to Tsuchiya-Ueno-Yamada’s proof of the fusion rules for conformal blocks in [TUY]. These rules allow one to express the dimension of space $V_{\text{C,}\vec{p}}(\vec{\lambda}, L)$ in terms of the spaces of conformal blocks on $\mathbb{P}^1$ with three marked points, $V_{0,3}(\alpha, \beta, \gamma, L)$.

The $g = 0, n = 3$ case of these spaces can be described purely with representation theoretic data for the algebra $g$, see [TUY], [U]. This has lead several authors to discover positive polyhedral counting rules for their dimensions. By this we mean that the dimensions of the spaces $V_{\text{C,}\vec{p}}(\vec{\lambda}, L)$ can be computed by counting the lattice points in a convex polytope. Tsuchiya, Ueno, and Yamada noticed that the fact that the spaces $V_{0,3}(r_1, r_2, r_3, L)$ are multiplicity free for $G = \text{SL}_2(\mathbb{C})$ can be used to create a polyhedral counting rule in this case. Several rules for $\text{SL}_3(\mathbb{C})$ have been found by Walton, Mathieu, Senechal, and Kirillov, [KMSW], (see section 0 below), along with partial results for $\text{SL}_n(\mathbb{C})$, $n > 3$.

1.2. Relating counting to commutative algebra. We are interested in relating these polyhedral counting rules and other combinatorial structures which inform the spaces $V_{\text{C,}\vec{p}}(\vec{\lambda}, L)$ to the algebraic geometry and commutative algebra of the moduli stack $\mathcal{M}_{g,\vec{p}}(G)$. In [M4] we defined the algebra of conformal blocks $V_{\text{C,}\vec{p}}(G)$. This is the fiber of a sheaf of commutative algebras $\mathcal{V}(G)$ on $\mathcal{M}_{g,\vec{p}}$, which is isomorphic to a direct sum of all the vector bundles $\mathcal{V}(\vec{\lambda}, L)$. When the curve $(C, \vec{p})$ is smooth, the fiber of this sheaf agrees with the Cox ring (total coordinate ring) of $\mathcal{M}_{g,\vec{p}}(G)$.

(5) $V_{\text{C,}\vec{p}}(G) = \text{Cox}(\mathcal{M}_{\text{C,}\vec{p}}(G)) = \bigoplus_{\vec{\lambda}, L} H^0(\mathcal{M}_{\text{C,}\vec{p}}(G), \mathcal{L}(\vec{\lambda}, L)) = \bigoplus_{\vec{\lambda}, L} V_{\text{C,}\vec{p}}(\vec{\lambda}, L)$

For fixed $\vec{\lambda}, L$ we also obtain the flat subsheaf $\mathcal{R}(\vec{\lambda}, L) \subset \mathcal{V}(G)$, where the fiber over $(C, \vec{p})$ smooth is the projective coordinate ring of $\mathcal{M}_{\text{C,}\vec{p}}(G)$ defined by $\mathcal{L}(\vec{\lambda}, L)$.

(6) $\mathcal{R}_{\text{C,}\vec{p}}(\vec{\lambda}, L) = \bigoplus_{N=0}^\infty H^0(\mathcal{M}_{\text{C,}\vec{p}}(G), \mathcal{L}(\vec{\lambda}, L)^\otimes N)$
In [M4] we give partial degenerations of $V_{C,p}(G)$ to simpler, but not necessarily toric, algebras for all $G$. We do this by first passing to nodal curves $(C,\vec{p}) \in \hat{M}_{g,n}$, then using a filtration on the algebra of conformal blocks built from the aforementioned factorization rules. This effectively reduces the problem of constructing a toric degeneration of $V_{C,p}(G)$ to constructing such a degeneration for the algebra of conformal blocks $V_{0,3}(G)$ on a three pointed genus 0 curve, see Section 3. Any polyhedral counting formulas developed for the conformal blocks on this pointed curve are then prime suspects for a toric degeneration of $V_{0,3}(G)$. We show that counting formulas of Walton, Mathieu, Senechal, and Kirillov, [KMSW] for $SL_3(\mathbb{C})$ conformal blocks are shadows of such a degeneration.

In the case $G = SL_2(\mathbb{C})$ the multiplicity-free property of $SL_2(\mathbb{C})$ tensor products is enough to establish a large family of toric degenerations that are useful in describing the algebras $V_{C,p}(SL_2(\mathbb{C}))$ and $R_{C,p}(\vec{r}, L)$. In [M4] degenerations of the algebra $V_{C,p}(SL_2(\mathbb{C}))$ are constructed and shown to be isomorphic to members of a class of semigroup algebras $\mathbb{C}[P_T]$ originating from mathematical biology. Here, $P_T$ is a semigroup which depends on a trivalent graph $\Gamma$ of genus $g$ with $n$ leaves. In [BW], Buczynska and Wieszewski prove that members of this class of semigroup algebras are generated in degree 1, with quadratic relations, when $\Gamma$ is a tree. In a pair of papers, [Bu], [BBKM], Buczynska, Buczynski, Kubjas and Michalek showed that $\mathbb{C}[P_T]$ is in general generated in degree $g + 1$.

In [M7], we showed that $\mathbb{C}[2oP_T]$ is generated in degree 1 with quadratic relations when $n = 0$, and $g > 2$, where $2oP_T$ is the second Minkowski sum. In addition, Abe [A] has shown that $V_{C}(SL_2(\mathbb{C}))$ is generated in degree 1. The algebras $R_{C,p}(\vec{r}, L)$ are also studied in [M7], where they are shown to degenerate to semigroup algebras $\mathbb{C}[P_T(\vec{r}, L)]$. It was proved that $\mathbb{C}[P_T(\vec{r}, L)]$ is generated in degree 1 with quadratic relations when each entry of $\vec{r}$ and $L$ are even.

Degenerations which correspond combinatorially to the factorization rules have also been constructed in the $g = 0 \ SL_2(\mathbb{C})$ case by Sturmfels and Xu, see [SX]. The algebra $V_{2^1,p}(SL_2(\mathbb{C}))$ was studied in [StV], where it was presented as a quotient of the projective coordinate ring of an even spinor variety $S^+$.  

2. SPACES OF CONFORMAL BLOCKS

For what follow see Beauville, [Be], Looijenga, [L], Ueno, [U], and Tsuchiya-Ueno-Yamada, [TUY]. Conformal blocks for a simple Lie algebra $\mathfrak{g}$ are constructed from the representation theory of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$. This algebra is defined as the $\mathbb{C}$–central extension of $\mathbb{C}((z)) \otimes \mathfrak{g}$ defined by the Killing form $<,>$ of $\mathfrak{g}$.

$$\hat{\mathfrak{g}} = \mathbb{C}((z)) \otimes \mathfrak{g} \oplus \mathbb{C}c$$

The element $c$ commutes with the $\mathbb{C}((z)) \otimes \mathfrak{g}$ component. Elsewhere the bracket is computed as follows.

$$[X \otimes f, Y \otimes g] = fg[X,Y] + c < X, Y > Res_{z=0}(gdf)$$

We take $\mathcal{H}(\lambda, L)$ to be the simple integrable module of $\hat{\mathfrak{g}}$ defined by a dominant $\mathfrak{g}$–weight of level $L$. We let $\mathcal{H}(\vec{X}, L)$ be the tensor product $\mathcal{H}(\lambda_1, L) \otimes \ldots \otimes \mathcal{H}(\lambda_n, L)$.

Fix a stable curve $(C, \vec{p}) \in \hat{M}_{g,n}$. At each point $p_i$ we choose a generator $t_i$ of the maximal ideal of $\mathbb{C}[C]_{p_i}$. Laurent expansion in $t_i$ defines a map of Lie algebras
The residue formula \( \sum_i \text{Res}(f_{id_i}; p_i) = 0 \) implies that the Lie algebra map \( \sum \eta_i \) defines a representation of \( \mathcal{O}(C \setminus \tilde{p}) \otimes \mathfrak{g} \) on the module \( \mathcal{H}(\tilde{\lambda}, L) \). This allows us to define the space of conformal blocks as follows.

\[
V_{C, \tilde{p}}(\tilde{\lambda}, L) = \text{Hom}_C(\mathcal{H}(\tilde{\lambda}, L)/[\mathcal{O}(C \setminus \tilde{p}) \otimes \mathfrak{g}]\text{Hom}_C(\mathcal{H}(\tilde{\lambda}, L), \mathbb{C})
\]

When the genus of \( C \) is 0, this is a subspace of the space of tensor product invariants \( V(\lambda_1^*) \otimes \ldots \otimes V(\lambda_n^*) \), see [U]. Later we will use this fact in the case \( n = 3 \).

There is an alternative formulation of conformal blocks as the following vector space,

\[
V_{C, \tilde{p}}(\tilde{\lambda}, L) = \text{Hom}_C(\mathcal{H}(0, L) \otimes V(\tilde{\lambda})/[\mathcal{O}(C \setminus \tilde{p}) \otimes \mathfrak{g}]\text{Hom}_C(\mathcal{H}(0, L) \otimes V(\tilde{\lambda}), \mathbb{C})
\]

for details. This formulation makes the identification of conformal blocks with global sections of the line bundle \( \mathcal{L}(\tilde{\lambda}, L) \) on \( \mathcal{M}_{C, \tilde{p}}(G) \) more clear. This moduli is constructed as a quotient stack,

\[
\mathcal{M}_{C, \tilde{p}}(G) = L_{C, \tilde{p}}(G)/\mathbb{Q} \times G/B^n
\]

where \( \mathbb{Q} \) is the affine Grassmanian \( \text{ind-variety} \), \( G/B \) is the full flag variety of the associated simple algebra group, and \( L_{C, \tilde{p}}(G) \) is an \( \text{ind-group} \) which corresponds to \( \mathcal{O}(C \setminus \tilde{p}) \otimes \mathfrak{g} \). It is well known that the variety \( G/B \) carries a line bundle \( \mathcal{L}(\lambda) \) with global sections \( V(\lambda^*) \) for any dominant weight \( \lambda \). Furthermore, the \( \text{ind-variety} \) \( \mathbb{Q} \) carries an ample line bundle \( \mathcal{L}(L) \) with global sections \( \mathcal{H}(0, L)^* \).

In this way, since the groups and spaces involved are sufficiently nice (see [LS], [KNR]), the space of global sections \( H^0(\mathcal{M}_{C, \tilde{p}}(G), \mathcal{L}(\tilde{\lambda}, L)) \) can be constructed by taking \( L_{C, \tilde{p}}(G) \)-invariants in the global sections \( \mathcal{H}(0, L)^* \otimes V(\tilde{\lambda}) \) of the line bundle \( \mathcal{L}(\lambda_1) \otimes \ldots \otimes \mathcal{L}(\lambda_n) \otimes \mathcal{L}(L) \), which then agrees with \( V_{C, \tilde{p}}(\tilde{\lambda}, L) \).

\section{Description of the Degenerations}

To prove Theorems [1.1] and [1.2] we construct a series of degenerations, starting with those constructed in [M4], we will describe this below. This has the effect of reducing questions on the commutative algebra of \( V_{C, \tilde{p}}(SL_3(\mathbb{C})) \) to questions about convex geometry. We analyze the polytope \( CB_3^* \) to prove the following theorem.

\textbf{Theorem 3.1.} The algebra \( V_{C, \tilde{p}}(SL_3(\mathbb{C})) \) is Gorenstein.

In the \( g = 0, 1 \) cases we get more.

\textbf{Theorem 3.2.} For generic \((\mathbb{P}^1, \tilde{p}) \in \mathcal{M}_{0, n}, \) The algebra \( V_{\mathbb{P}^1, \tilde{p}}(SL_3(\mathbb{C})) \) is generated by conformal blocks of level 1, and is presented by a homogenous ideal generated by forms of degree 2 and 3.

In [M4], it was shown that \( V_{\mathbb{P}^1, \tilde{p}}(SL_3(\mathbb{C})) \) is a sub-algebra of a polynomial ring over the algebra of \( SL_3(\mathbb{C}) \)-invariant \( n \)-tensors, \( R^n( SL_3(\mathbb{C})) \otimes \mathbb{C}[X] \). With this in mind, we can combine this theorem with Proposition 3.5.1 of [U], which characterizes conformal blocks in the genus 0 case.
Corollary 3.3. For generic $p_i$, the algebra $V_{p_1, \tilde{p}}(SL_3(\mathbb{C}))$ is generated by the tensors $f \otimes X \in [V(\lambda_1^*) \otimes \cdots \otimes V(\lambda_n^*)]^{sl_3(\mathbb{C})} \otimes \mathbb{C}\{X\}$, with $\lambda_i \in \{(1,0),(0,1),(0,0)\}$ which satisfy the following condition. For any element $\phi_k = v_1 \otimes \cdots \otimes v_{k-1} \otimes b_{\lambda_k} \otimes v_{k+1} \cdots \otimes v_n \in V(\lambda_1) \otimes \cdots \otimes V(\lambda_n)$, where $b_{\lambda_k}$ is a highest weight vector, we have the following equalities.

$$\sum_{\tilde{m}_k : |\tilde{m}_k| = \ell_k \ j \neq k} \prod_{j \neq k} \left( \ell_j \ m_j \right) (p_j - p_k)^{-m_j} < f \prod_{j \neq k} (p_j(e_\theta)^m_j)|\phi_k > = 0$$

where $\ell_k = 2 - \lambda_k(\theta)$, $\tilde{m}_k = (m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_n) \in \{0,1,2\}^{n-1}$, $|\tilde{m}_k| = \sum_{j \neq k} m_j$, and $e_\theta \in sl_3(\mathbb{C})$ is the raising operator for the longest root $\theta = L_1 - L_3$. The symbol $\rho_j$ indicates that the element is to act on the $j$-th component of the tensor $\phi_k$.

Here we have specialized Proposition 3.5.1 of [U] to the case $g = sl_3(\mathbb{C})$, with $L = 1$.

We also combine Theorem 3.2 with an analysis of a special semigroup in Section 9 to obtain a generating set in the genus 1 case.

Theorem 3.4. For generic $(C, \tilde{p}) \in \mathcal{M}_{1,n}$, the algebra $V_{C, \tilde{p}}(SL_3(\mathbb{C}))$ is generated by conformal blocks of levels 1, 2, and 3.

These results are proved by following the program described in [M4]. First, we degenerate the algebra of conformal blocks $V_{C, \tilde{p}}(SL_3(\mathbb{C}))$ to the invariants of a torus in a tensor product of many copies of $V_{0,3}(SL_3(\mathbb{C}))$. We begin by choosing a trivalent graph $\Gamma$ of genus $g$ with $n$ leaves labeled $\{1, \ldots, n\}$.

![Figure 1. A trivalent graph $\Gamma$ with associated forest $\hat{\Gamma}$.](image)

We consider the forest $\hat{\Gamma}$ of trinodes obtained by splitting every edge of $\Gamma$ not connected to a leaf. To each trinode $v \in V(\Gamma)$ we associate a copy of $V_{0,3}(G)$ and make the tensor product $\bigotimes_{v \in V(\Gamma)} V_{0,3}(G)$. This algebra carries the action of a large torus $T^\Gamma$, built by associating to each edge $e \in E(\hat{\Gamma})$ a product $T \times \mathbb{C}^*$ of a maximal torus $T \subset G$ with the non-zero complex numbers. The torus $T^\Gamma$ acts on the component $\bigotimes_{v \in V(\Gamma)} V_{0,3}(\lambda_v, \eta_v, L_v) \subset \bigotimes_{v \in V(\Gamma)} V_{0,3}(G)$ with the character obtained as the sum of differences $\sum_{v, v' \in V(\Gamma)} (\lambda_v^*, L_v) - (\lambda_{v'}^*, L_{v'})$ where $v$ and $v'$
The algebra of invariants \( \bigotimes_{v \in V(\Gamma)} V_{0,3}(G) \) is the direct sum of the components \( \bigotimes_{v \in V(\Gamma)} V_{0,3}(\lambda_v, \gamma_v, \eta_v, L_v) \) such that all the levels \( L_v \) agree and \( \lambda_v^* = \lambda_v \) for any two \( v, v' \) related as above.

This algebra is multigraded by networks of dominant \( G \)-weights, as depicted in Figure 2. The edge labeled "\( \lambda \)" is oriented to indicate that the "input" trinode receives the weight \( \lambda^* \) and the "output" trinode receives the weight \( \lambda \).

**Theorem 3.6.** [M4] For every trivalent graph \( \Gamma \) of genus \( g \) with \( n \) labeled leaves, there is a flat degeneration

\[
V_{C, \vec{p}}(G) \Rightarrow \left[ \bigotimes_{v \in V(\Gamma)} V_{0,3}(G) \right]^{\mathbb{T}^e}
\]

The two vector spaces in this theorem are isomorphic by Tsuchiya-Ueno-Yamada’s proof of the factorization rules. The content of the theorem is that the multiplication operation in the algebra \( \left[ \bigotimes_{v \in V(\Gamma)} V_{0,3}(G) \right]^{\mathbb{T}^e} \) is the "highest part" of the multiplication operation in \( V_{C, \vec{p}}(G) \) with respect to a natural term order.

This theorem reduces the problem to finding toric degenerations of \( V_{0,3}(SL_3(\mathbb{C})) \) which are \((T \times \mathbb{C}^*)^3\)-invariant. We find three such degenerations, each producing the same polytope \( CB_3^* \), with three different "gluing" projections to the associated triples of dominant weights in \( (\mathbb{R}_{\geq 0})^3 \). As a result we actually obtain \( 3|V(\Gamma)| \) possible semigroup degenerations of \( V_{C, \vec{p}}(SL_3(\mathbb{C})) \) per graph \( \Gamma \), each one a potential tool of study.

We will see that the algebra \( V_{0,3}(SL_3(\mathbb{C})) \) is generated by the components multigraded by the simple weights \((0,1)\) and \((1,0)\). These components are all multiplicity free, and so have a unique invariant associated to them. Furthermore, the weights \((0,1)\) and \((1,0)\) are dual to each other, so our network graphical device above serves to give a presentation of \( V_{0,3}(SL_3(\mathbb{C})) \), see Figure 3.
We label these generators from left to right, top down, $X, S, T, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13}$. These come with a map $\partial$ to triples of dominant weights of $SL_3(\mathbb{C})$.

$$\partial(X) = [(0,0),(0,0),(0,0)]$$

$$\partial(T) = [(0,1),(0,1),(0,1)], \quad \partial(S) = [(1,0),(1,0),(1,0)],$$

$$\partial(P_{32}) = [(0,0),(0,1),(1,0)], \quad \partial(P_{12}) = [(1,0),(0,1),(0,0)],$$

$$\partial(P_{13}) = [(1,0),(0,0),(1,1)], \quad \partial(P_{23}) = [(0,0),(1,0),(0,1)],$$

$$\partial(P_{21}) = [(0,1),(1,0),(0,0)], \quad \partial(P_{31}) = [(0,1),(0,0),(1,0)],$$

We let $\partial_i$ denote the $i-$the component of the map $\partial$. Note that $\partial_i(M) \in \{(1,0),(0,1),(0,0)\}$ for any $M$ in the collection above. The three degenerations of $V_{0,3}(SL_3(\mathbb{C}))$ are then given by the following three binomial relations.

(15) \quad \quad \quad \quad \quad XST = P_{12}P_{23}P_{31}

(16) \quad \quad \quad \quad \quad XST = -P_{21}P_{32}P_{13}

(17) \quad \quad \quad \quad \quad P_{12}P_{23}P_{31} = P_{21}P_{32}P_{13}

Note that these relations define isomorphic semigroups, but non-isomorphic semigroups-with-map $\partial$ to $\mathbb{R}_{\geq 0}^2$. This is important as the semigroups we obtain as degenerations of $V_{C,\mu}(SL_3(\mathbb{C}))$ are constructed as fiber products of these "local" semigroups with respect to the map $\partial$ over the structure of the graph $\Gamma$. 

Figure 3.
We let $\text{CB}_*^3$ denote the semigroup defined by the relation $P_{12}P_{23}P_{31} = P_{21}P_{32}P_{13}$ and $\text{BZ}_*^3$ denote the semigroup defined by the relation $XST = P_{12}P_{23}P_{31}$. We let $\text{CB}_*^3(\alpha, \beta, \gamma, L)$ and $\text{BZ}_*^3(\alpha, \beta, \gamma, L)$ denote the $\partial$–fibers over $(\alpha, \beta, \gamma)$ in the $L$–th Minkowski sums $L \circ \text{CB}_*^3$, $L \circ \text{BZ}_*^3$, respectively. Both of these semigroups have appeared in the work of Walton, Senechal, Mathieu and Kirillov on counting $\text{SL}_3(\mathbb{C})$ conformal blocks, [KMSW]. In particular, we recover their counting results.

**Corollary 3.7.** The dimension of the space $V_{0,3}(\alpha, \beta, \gamma, L)$ is equal to the number of lattice points in the polytope $\text{CB}_*^3(\alpha, \beta, \gamma, L)$.

Our approach owes much to the results in their papers, in particular Theorem 1.1 is a product of our attempt to prove their counting formulas with techniques more familiar to us. This implies that the dimension of a general space of $\text{SL}_3(\mathbb{C})$ conformal blocks can be counted by enumerating the lattice points in a convex polytope. We will build a polytope $\text{CB}_*^\Gamma$, and let $\text{CB}_*^\Gamma(\vec{\lambda}, L)$ be the fiber over $\vec{\lambda}$ in $L \circ \text{CB}_*^\Gamma$.

**Corollary 3.8.** The dimension of the space $V_{\text{C},\vec{p}}(\vec{\lambda}, L)$ is equal to the number of lattice points in the polytope $\text{CB}_*^\Gamma(\vec{\lambda}, L)$.

We construct the polytope $\text{CB}_*^\Gamma$ in a manner similar to $[\bigotimes_{v \in V(\Gamma)} V_{0,3}(G)]^T$. Once again, we consider the forest $\hat{\Gamma}$ and assign each trinode a copy of $\text{CB}_*^3$. Next, we take the fiber product polytope by requiring that the $\partial$ values assigned at edges identified by $\hat{\Gamma} \rightarrow \Gamma$ be dual to each other. The lattice points of $\text{CB}_*^\Gamma$ can be visualized as oriented multigraphs as above.

In general, we would like to have presentation data for the semigroup algebras $\mathbb{C}[\text{CB}_*^\Gamma]$, and $\mathbb{C}[\text{CB}_*^\Gamma(\vec{\lambda}, L)]$. Theorem 3.2 addresses this question when the genus of $\Gamma$ is 0, but it would be nice to have analogous results for $g > 0$.

**Problem 3.9.** Find a minimal generating set of the semigroup algebras $\mathbb{C}[\text{CB}_*^\Gamma]$, and $\mathbb{C}[\text{CB}_*^\Gamma(\vec{\lambda}, L)]$.

Having $3|V(\Gamma)|$ semigroups for each graph $\Gamma$ is also a bit of an embarrassment of riches. The number of lattice points in these polytopes, indeed the multigraded
Hilbert functions of $\mathbb{C}[CB^+_3]$ do not depend on the graph $\Gamma$, or even the choice of a semigroup from our three possibilities at each trinode. It would be fruitful to describe how these polytopes are related.

**Problem 3.10.** For two trivalent graphs $\Gamma, \Gamma'$ with first Betti number $g$ and $n$ leaves, find a piecewise linear map $p_{\Gamma, \Gamma'} : CB^+_3 \to CB^+_3$, which is a bijection on lattice points.

It is not difficult to write such a map down in the $SL_2(\mathbb{C})$ case. See [Bu], [BBKM] for further discussion on problems like these.

3.1. **Remarks on generalizations.** Our approach, inspired by standard monomial theory, and the theory of branching algebras (see [HTW], [M5]), is to treat elements of the theory of conformal blocks as combinatorial commutative algebra objects. To make these techniques work for general simple $\mathfrak{g}$, the approach should be as follows, see Section 5 for relevant definitions. First we find a filtration $\mathfrak{v}$ on $R_3(\mathfrak{g})$ with associated graded algebra an affine semigroup algebra $\mathbb{C}[P_3]$. The filtration should have the property that it is refined by the filtration defined by assigning the conformal blocks level $v_\mathfrak{v}$. That is, if $v_\mathfrak{v}(f) < v_\mathfrak{v}(g)$, then $v(f) < v(g)$. It then follows that the level filtration $v_\mathfrak{v}$ induces a grading on $\mathbb{C}[P_3]$, and the dimension of the space of conformal blocks $V_{0,3}(\lambda, \eta, \mu, L)$ with weights can be computed as the points in $P_3$ with these weights, such that the level grading is less than or equal to $L$. This would then allow us to define a toric degeneration of the algebra of conformal blocks, see Section 5.

4. **Commutative algebras associated to $sl_3(\mathbb{C})$**

We fix once and for the choice of roots $L_1 - L_2, L_2 - L_3, L_1 - L_3$ for $sl_3(\mathbb{C})$ and identify the set of dominant $sl_3(\mathbb{C})$ weights with respect to this choice with $\mathbb{Z}_{\geq 0}^2$. Recall that the fundamental irreducible representations of $sl_3(\mathbb{C})$ are $V(1,0) \cong \mathbb{C}^3$ and $V(0,1) \cong \Lambda^2(\mathbb{C}^3)$. We give these spaces the bases $x_1, x_2, x_3$ and $y_1, y_2, y_3$, respectively, with $y_i = x_j \wedge x_k$.

We let $R$ denote the Cox ring of the full flag variety $SL_3(\mathbb{C})/B$. This is a $\mathbb{Z}_{\geq 0}^2$ graded, $SL_3(\mathbb{C})$ algebra, with a full, multiplicity free decomposition into irreducible representations.

\begin{equation}
R = \bigoplus_{\alpha \in \mathbb{Z}_{\geq 0}^2} V(\alpha)
\end{equation}

The algebra $R$ is a natural byproduct of the embedding of $SL_3(\mathbb{C})/B$ into the variety $\mathbb{P}(\mathbb{C}^3) \times \mathbb{P}(\Lambda^2(\mathbb{C}^3))$ given by sending a flag $\mathcal{F} = \{ \ell \subset w \}$ to its Plücker coordinates. The pullbacks of the two generators of $Pic(\mathbb{P}(\mathbb{C}^3) \times \mathbb{P}(\Lambda^2(\mathbb{C}^3)))$ generate the Picard group of $SL_3(\mathbb{C})/B$ and give a multigraded surjection,

\begin{equation}
\text{Cox}(\mathbb{P}(\mathbb{C}^3) \times \mathbb{P}(\Lambda^2(\mathbb{C}^3))) = \mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3] \to R.
\end{equation}

The kernel of this surjection is generated by the form $x_1y_1 + x_2y_2 + x_3y_3$, which can be seen as the condition that the normal of $w \in \mathcal{F}$ vanish on the coordinates of $\ell \in \mathcal{F}$. 


We let \( R_3(sl_3(\mathbb{C})) \) be the algebra of invariants \([R \otimes R \otimes R]^{sl_3(\mathbb{C})}\). As a vector space, this is the direct sum of all spaces of invariants in triple tensor products of irreducible \( sl_3(\mathbb{C}) \) representations.

\[
R_3(sl_3(\mathbb{C})) = \bigoplus_{\alpha, \beta, \gamma} (V(\alpha) \otimes V(\beta) \otimes V(\gamma))^{sl_3(\mathbb{C})}
\]

Our analysis of conformal blocks involves the algebra \( R_3(sl_3(\mathbb{C})) \) and the following semigroup.

**Definition 4.1.** We let \( BZ_3 \) denote the semigroup of non-negative integer weight-ings of the diagram in Figure 5 such that the sums of pairs of weights opposite each other across the hexagon are equal.

\[
\begin{align*}
\mathbf{a}_1 & = c_4 \\
\mathbf{a}_2 & \\
\mathbf{a}_3 & \\
\mathbf{c}_3 & \\
\mathbf{b}_1 & \\
\mathbf{b}_2 & \\
\mathbf{b}_3 & \\
\mathbf{b}_4 & = \\
\mathbf{c}_1 & = \mathbf{c}_2
\end{align*}
\]

**Figure 5.**

\[
\begin{align*}
a_2 + a_3 & = b_3 + c_2, \\
b_2 + b_3 & = c_3 + a_2, \\
c_2 + c_3 & = a_3 + b_2.
\end{align*}
\]

We orient the triangle counter-clockwise. For \( a_1, a_2, a_3, a_4 \) the consecutive weights along an edge, we obtain a map to the dominant weights of \( sl_3(\mathbb{C}) \), \((a_1 + a_2, a_3 + a_4) \in \mathbb{Z}_{\geq 0}^2\). For a triangle \( X \in BZ_3 \), define \( \partial(X) = (\alpha, \beta, \gamma) \in (\mathbb{Z}_{\geq 0}^2)^3 \) to be the vector of the three dominant weights obtained this way from the three sides of \( X \).

**Theorem 4.2** (Berenstein, Zelevinksy, [BZ1]). The dimension \( \dim((V(\alpha) \otimes V(\beta) \otimes V(\gamma))^{sl_3(\mathbb{C})}) \) is equal to the number of BZ triangles in \( \{X \in BZ_3| \partial(X) = (\alpha, \beta, \gamma)\} = BZ_3(\alpha, \beta, \gamma) \).

Triangles for a fixed \((\alpha, \beta, \gamma)\) are all related to each other by adding or subtracting multiples of the triangle in Figure 6.

**Remark 4.3.** For two lattice cones \( P \subset \mathbb{R}^n, Q \subset \mathbb{R}^m \), and a map \( \pi : P \rightarrow Q \), a collection of lattice points \( \mathcal{M} \subset \mathbb{Z}^n \) is known as a Markov basis for \( \pi \) if the lattice points in any fiber \( \pi^{-1}(b), b \in Q \) can all be connected by elements of \( \mathcal{M} \). The triangle above and its inverse therefore constitute a Markov basis for \( \partial \).
Definition 4.4. Define $Q_{\min}(\alpha, \beta, \gamma) \in BZ_3(\alpha, \beta, \gamma)$ to be the unique triangle with some corner entry equal to 0.

It follows easily that all other members of the set $BZ_3(\alpha, \beta, \gamma)$ are obtained from $Q_{\min}(\alpha, \beta, \gamma)$ by successive applications of the triangle in Figure 6. This element can be applied until some entry in the internal hexagon is 0.

Remark 4.5. It follows that the triples $\alpha, \beta, \gamma$ for which $|BZ_3(\alpha, \beta, \gamma)| = 1$ are those which admit a triangle with a 0 corner and a 0 hexagon weight.

The semigroup $BZ_3$ is generated by the eight elements depicted in Figure 7; these elements have been represented by their honeycomb graphs. Honeycombs are a graphical device invented by Knutson, Tao, and Woodward [KTW] to make the combinatorics of BZ triangles more transparent. Each weight $a$ along the interior hexagon is replaced with an edge connected to the center of the triangle weighted by $a$.

From left to right, top down these are $S, T, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13}$. These elements are subject to the familiar relation

$$ST = P_{12}P_{23}P_{31}.$$
We factor $Q_{\min}(\alpha, \beta, \gamma)$ in the unique way which prefers $ST$ to $P_{12}P_{23}P_{31}$.

\begin{equation}
Q_{\min}(\alpha, \beta, \gamma) = S^{s(\min)}T^{t(\min)} \prod P_{ij}^{p_{ij}(\min)}
\end{equation}

Letting $k = \min\{t(\min), s(\min)\}$, the number of triangles in $BZ_3(\alpha, \beta, \gamma)$ is then equal to $k + 1$.

We now relate these two algebraic objects by using $BZ_3$ to construct a presentation of $R_3(sl_3(\mathbb{C}))$. The following appears in [M5] and [MZ].

**Theorem 4.6.** The algebra $R_3(sl_3(\mathbb{C}))$ carries a $T^3$-invariant flat degeneration to the semigroup algebra $\mathbb{C}[BZ_3]$.

Recall that we have fixed the bases of $V(1,0)$ and $V(0,1)$ to be $x_1, x_2, x_3$ and $y_1, y_2, y_3$ respectively. Let $S$ be the unique invariant in $V(1,0)^{\otimes 3}$, and $T$ be the unique invariant in $V(0,1)^{\otimes 3}$. These are the determinants of the matrices $X$ and $Y$ below, respectively. Let $P_{ij}$ be the the inner product of the $i-j$ column of $X$ with the $j-i$ column of $Y$, this represents the invariant in $V(1,0) \otimes V(0,1)$ where these representations are in the $i,j$ positions in the tensor product, respectively.

$$X = \begin{vmatrix}
x_1^1 & x_1^2 & x_1^3 \\
x_2^1 & x_2^2 & x_2^3 \\
x_3^1 & x_3^2 & x_3^3 \\
\end{vmatrix} \quad Y = \begin{vmatrix}
y_1^1 & y_1^2 & y_1^3 \\
y_2^1 & y_2^2 & y_2^3 \\
y_3^1 & y_3^2 & y_3^3 \\
\end{vmatrix}$$

These are the unique invariant forms in their corresponding tensor products, and therefore they must correspond to the eight generators of $BZ_3$.

Next we make use of the concept of a subduction basis for a valuation on the commutative algebra $R_3(sl_3(\mathbb{C}))$.

**Definition 4.7.** Let $A$ be a commutative algebra over a field $\mathfrak{k}$, with a valuation $v : A \to \mathbb{R} \cup \{-\infty\}$ which restricts to the trivial valuation on $\mathfrak{k}$. This defines a filtration on $A$, let $in_v(A)$ be the associated graded algebra. We say a set $X \subset A$ is a subduction basis for $(A, v)$ if the initial forms $in_v(X)$ generate $in_v(A)$.

If the valuation $v$ is sufficiently nice, for example if the dimension of the space of elements of $A_{<-r}$, with value less than or equal to a fixed $r$ is finite, then the existence of finite subduction bases can be a powerful tool for studying $A$ (this happens for all algebras and valuations we consider). In particular, $X$ generates $A$ as a $\mathfrak{k}$ algebra, and it is possible to construct a presentation $gr_v(A) = \mathfrak{k}[X]/in_v(I)$, where $I$ is the ideal of presentation of $A$ by $X \subset A$, see [M5], [Ka]. This is essentially the subduction algorithm and the “syzygy lifting” property of SAGBI bases, see [St], Chapter 11.

It follows that $S, T$ and the $P_{ij}$ are a subduction basis for $R_3(sl_3(\mathbb{C}))$ with respect to the filtration defined in [M5]. This implies that $S, T, P_{ij}$ generate $R_3(sl_3(\mathbb{C}))$, and that the binomial relation defining $BZ_3$ lifts to generate the presentation ideal of $R_3(sl_3(\mathbb{C}))$ by $S, T, P_{ij}$.

\begin{equation}
R_3(sl_3(\mathbb{C})) = \mathbb{C}[S,T,P_{12},P_{13},P_{23},P_{21},P_{31},P_{32}] / \langle ST - P_{12}P_{23}P_{31} + P_{21}P_{32}P_{13} \rangle
\end{equation}

This relation can be constructed by writing the determinant of the product of two $3 \times 3$ matrices, $X, Y$ as either $ST = \det(X)\det(Y')$ or $\det(XY^t) = P_{12}P_{23}P_{31} - P_{21}P_{32}P_{13} + h$, where $h$ involves the terms $P_{ii}$, $i \in \{1, 2, 3\}$. Recall that all of these
terms vanish as the defining relation of $R$. Notice that the generator of this ideal has two binomial degenerations up to isomorphism, one gives a presentation of $\mathbb{C}[BZ_3]$, and the other defines a semigroup algebra for a semigroup $CB_3$.

$$(24) \quad \mathbb{C}[CB_3] = \mathbb{C}[S, T, P_{12}, P_{13}, P_{23}, P_{21}, P_{31}, P_{32}] / < P_{12}P_{23}P_{31} - P_{21}P_{32}P_{13} >$$

Elements of the semigroup $CB_3$ can be represented by the following bird feet diagrams. The relation can be seen as the fact that the product of the elements in the second column equals the product of the elements in the third column.

![Bird feet diagrams](image)

**Figure 8.**

Later we will relate $CB_3$ and $BZ_3$ to their "quantum" analogues $CB_3^* \cong BZ_3^*$, both obtained by introducing an extra generator $X$.

5. **Valuations on the algebra $R_3(sl_3(\mathbb{C}))$**

In this section we define a valuation on the branching algebra $R_3(sl_3(\mathbb{C}))$ which will help us construct and study the algebra of $SL_3(\mathbb{C})$ conformal blocks. This function is built out of representation data for the subalgebra $sl_2(\mathbb{C}) \subset sl_3(\mathbb{C})$ defined by the longest root $\theta = L_1 - L_3$. First we decompose each irreducible representation of $sl_3(\mathbb{C})$ into its $sl_2(\mathbb{C})$ isotypical components with respect to this root.

$$(25) \quad V(\alpha) = \bigoplus W_{\alpha,i}$$

$$(26) \quad R^{\otimes 3} = \bigoplus W_{\alpha,i} \otimes W_{\beta,j} \otimes W_{\gamma,k}$$

This gives a *branching decomposition* of the algebra $R^{\otimes 3}$. For any map of groups $\phi : \mathfrak{h} \rightarrow \mathfrak{g}$, there is a filtration of the analogue of $R$ for the algebra $\mathfrak{g}$, $R_\mathfrak{g} = \bigoplus V(\lambda)$ by its $\mathfrak{h}$—isotypical decomposition, see [M5] and [Gr]. It follows that multiplication in the algebra $R^{\otimes 3}$ is lower-triangular with respect to the indices $i, j, k$. This can be understood by considering the product of two such isotypical components. Since the product map is $sl_2(\mathbb{C})$—linear the image of the product must decompose as follows.

$$(27) \quad [W_{\alpha_1,i_1} \otimes W_{\beta_1,j_1} \otimes W_{\gamma_1,k_1}] \times [W_{\alpha_2,i_2} \otimes W_{\beta_2,j_2} \otimes W_{\gamma_2,k_2}] \subset$$
Furthermore, a product $f \times g$ always has a component with indices $i = i_1 + i_2, j = j_1 + j_2, k = k_1 + k_2$, \cite{M5}. This implies that the function $v_\theta$, which assigns \[ \sum f_{\alpha, \beta, \gamma, i, j, k} = f \in R \] the number $\frac{1}{2} \max \{ i + j, k \}$, is a valuation on $R$. Another construction of this type of filtration by dominant weight data appears in \cite{Gr}, 3.15.2. From these observations we get the following proposition.

**Proposition 5.1.** The function $v_\theta$ defines a valuation on the algebra $R_3(sl_3(\mathbb{C})) \subset R^{ \otimes 3}$.

We note that the method of filtering an algebra by dominant weight data is used to prove Theorem 3.6 in \cite{M4}. As a consequence of the above proposition, it follows that $v_\theta(fg) = v_\theta(f) + v_\theta(g)$, i.e., if $f$ has highest component of weight $(i_1, j_1, k_1)$, and $g$ has highest component of weight $(i_2, j_2, k_2)$ then $fg$ has highest component of weight $(i_1 + i_2, j_1 + j_2, k_1 + k_2)$. This is useful for studying conformal blocks because of the following (reformulated) theorem of Ueno.

**Theorem 5.2** (Ueno, \cite{U}). The space of $sl_3(\mathbb{C})$ conformal blocks of level $L$ and weights $\alpha^*, \beta^*, \gamma^*$ can be identified with $V_L(\alpha^*, \beta^*, \gamma^*) = \{ f \in (V(\alpha) \otimes V(\beta) \otimes V(\gamma))^{sl_3(\mathbb{C})} \mid v_\theta(f) \leq L \}$

**Proof.** We translate from the result of Ueno, consider $V(\alpha) \otimes V(\beta) \otimes V(\gamma)$ as linear functions on the representation $V(\alpha^*) \otimes V(\beta^*) \otimes V(\gamma^*)$, and consider the isotypical decomposition of this representation,

\[ V(\alpha^*) \otimes V(\beta^*) \otimes V(\gamma^*) = \bigoplus W_i \otimes W_j \otimes W_k, \]

then by \cite{U}, Corollary 3.5.2 we have

\[ \{ f \mid i + j + k > 2L \Rightarrow f|_{W_i \otimes W_j \otimes W_k} = 0 \} \]

Consider the $sl_2(\mathbb{C})$ isotypical decomposition of $[V(\alpha^*) \otimes V(\beta^*) \otimes V(\gamma^*)]^*.$

\[ \sum W_i \otimes W_j \otimes W_k \]

This follows from the self-duality of $sl_2(\mathbb{C})$ representations. As a consequence, the dual functions of the $W_i \otimes W_j \otimes W_k$ component of $V(\alpha^*) \otimes V(\beta^*) \otimes V(\gamma^*)$ are given by the $W_i \otimes W_j \otimes W_k$ component of $V(\alpha) \otimes V(\beta) \otimes V(\gamma).$ A function $f$ vanishes on all $W_i \otimes W_j \otimes W_k,$ $i + j + k > 2L$ if and only if it is in the span of the components with $i + j + k \leq 2L.$

**Lemma 5.3.** The invariants $P_{ij}, S, T$ each have $v_\theta$ value 1.

**Proof.** We compute the decompositions of $V(0, 1)$ and $V(1, 0)$ as $\theta(sl_2(\mathbb{C})) \subset sl_3(\mathbb{C})$ representations. These are $V(1, 0) = V(1) \oplus V(0)$ with $V(0)$ the span of $x_2,$ and $V(0, 1) = V(1) \oplus V(0)$ with $V(0)$ the span of $y_2.$ Each monomial in the expressions for $S, T,$ and $P_{ij}$ is then in a component with at most two copies of $V(1)$ in the tensor product.
Proposition 5.4. For $\alpha, \beta, \gamma$ fixed, the invariants in $V(\alpha) \otimes V(\beta) \otimes V(\gamma)$ take on distinct, consecutive $v_\theta$ values.

To prove this proposition we use the BZ triangles to keep track of the dimension of the space of invariants $(V(\alpha) \otimes V(\beta) \otimes V(\gamma))^{sl_3(\mathbb{C})}$. To the minimum triangle $Q_{\min}(\alpha, \beta, \gamma)$ above we associate the following monomial in $R_3(sl_3(\mathbb{C}))$.

\begin{equation}
M_{\min} = S^{(\min)} T^{(\min)} \prod P_{ij}^{p_{ij}(\min)}
\end{equation}

This monomial has weight $(\alpha, \beta, \gamma)$ by construction, and therefore defines an invariant in $(V(\alpha) \otimes V(\beta) \otimes V(\gamma))^{sl_3(\mathbb{C})}$. Furthermore, we can compute the $v_\theta$-value of this monomial,

\begin{equation}
v_\theta(M_{\min}) = s(\min) + t(\min) + \sum p_{ij}(\min)
\end{equation}

Recall $k = \min\{s(\min), t(\min)\}$, we produce $k$ more monomials in $(V(\alpha) \otimes V(\beta) \otimes V(\gamma))^{sl_3(\mathbb{C})}$, one associated to each $1 \leq \ell \leq k$ as follows.

\begin{equation}
M_\ell = (ST)^{-\ell}(P_{12} P_{23} P_{31})^\ell M_{\min}
\end{equation}

Each application of the Laurent monomial $(ST)^{-1}(P_{12} P_{23} P_{31})^1$ has the effect of raising the value of $v_\theta$ by 1 while keeping the monomial in $(V(\alpha) \otimes V(\beta) \otimes V(\gamma))^{sl_3(\mathbb{C})}$. In this way we get $k + 1$ monomials

\begin{equation}
M_{\min} = M_0, M_1, \ldots, M_k
\end{equation}

with $k + 1$ distinct $v_\theta$ values $v_\theta(M_{\min}), v_\theta(M_{\min}) + 1, \ldots, v_\theta(M_{\min}) + k$, where

\begin{equation}
v_\theta(M_k) = v_\theta(M_{\min}) + \ell
\end{equation}

It follows from general properties of valuations that these monomials must be linearly independent, and we have already established that the invariant space is of dimension exactly $k + 1$, this proves the proposition. We can extend this proposition to the following theorem.

Theorem 5.5. The associated graded algebra of $R_3(sl_3(\mathbb{C}))$ with respect to the valuation $v_\theta$ is isomorphic to $\mathbb{C}[S, T, P_{12}, P_{13}, P_{23}, P_{21}, P_{31}, P_{32}] / < P_{12} P_{23} P_{31} - P_{21} P_{32} P_{13} >$.

Proof. The proposition above shows that the initial forms of the monomial generators $S, T, P_{ij}$ generate the associated graded algebra. This implies that they are a subduction basis for $R_3(sl_3(\mathbb{C}))$ with respect to the valuation $v_\theta$. From this it follows that

\begin{equation}
gr_{v_\theta} (R_3(sl_3(\mathbb{C}))) \cong \mathbb{C}[X] / \text{in}_{v_\theta}(I)
\end{equation}

For any set of generators $X \subset R_3(sl_3(\mathbb{C}))$ containing our subduction basis, and $I$ the associated ideal of presentation. In this case we have

\begin{equation}
I = < P_{12} P_{23} P_{31} - P_{21} P_{32} P_{13} + ST >,
\end{equation}
and the initial form of the principal generator of this ideal is $P_{12}P_{23}P_{31} - P_{21}P_{32}P_{13}$.

It also follows that the monomials $M_0, M_1, \ldots, M_\ell$ form a basis of the space $V_{v_\theta(M, m, n)} + \epsilon(\alpha^*, \beta^*, \gamma^*) \subset (V(\alpha) \otimes V(\beta) \otimes V(\gamma))^{sl_3(\mathbb{C})}$.

6. The algebra of conformal blocks $V_{0,3}(SL_3(\mathbb{C}))$

In this section we translate our results on $R_3(sl_3(\mathbb{C}))$ directly over to $V_{0,3}(SL_3(\mathbb{C}))$. In [M4], the algebra of conformal blocks $V_{0,3}(SL_3(\mathbb{C}))$ was shown to be isomorphic to the following subalgebra of $R_3(sl_3(\mathbb{C})) \otimes \mathbb{C}[X]$, with $X$ a variable.

\begin{equation}
V_{0,3}(SL_3(\mathbb{C})) = \bigoplus_{\alpha, \beta, \gamma, k \leq L} V_k(\alpha, \beta, \gamma) \otimes CX^L
\end{equation}

From the results of the last section we get the following theorem.

**Theorem 6.1.** The algebra $V_{0,3}(SL_3(\mathbb{C}))$ is generated by the elements $S \otimes X, T \otimes X, P_{ij} \otimes X$, and $1 \otimes X \subset R_3(sl_3(\mathbb{C})) \otimes \mathbb{C}[X]$, subject to the relation

\begin{equation}
[S \otimes X][T \otimes X][1 \otimes X] - [P_{12} \otimes X][P_{23} \otimes X][P_{31} \otimes X] + [P_{21} \otimes X][P_{32} \otimes X][P_{13} \otimes X]
\end{equation}

Furthermore, we can extend the valuation $v_\theta$ to a valuation on $V_{0,3}(SL_3(\mathbb{C}))$. The associated graded algebra of this valuation has the following presentation.

\begin{equation}
gr_{v_\theta}(V_{0,3}(SL_3(\mathbb{C}))) = \mathbb{C}[S, T, X, P_{12}, P_{23}, P_{31}, P_{21}, P_{32}, P_{13}] / < P_{12}P_{23}P_{31} - P_{21}P_{32}P_{13} >
\end{equation}

This can be seen to be isomorphic to the semigroup algebra of $CB_3^*$.

7. The algebra of conformal blocks $V_{C, \tilde{p}}(SL_3(\mathbb{C}))$

In this section we use results from [M4] and the previous section to obtain toric degenerations of the algebra of $SL_3(\mathbb{C})$ conformal blocks over a general marked stable curve $(C, \tilde{p})$. We also use these degenerations to show that $V_{C, \tilde{p}}(SL_3(\mathbb{C}))$ is a Gorenstein algebra.

**Definition 7.1.** Let $\mathfrak{k}$ be a field, then a $\mathbb{Z}$-graded $\mathfrak{k}$-algebra $R$ is said to be Gorenstein if the Matlis Dual

\begin{equation}H_m^{\dim(R)}(R)^* = \text{Hom}_R(H_m^{\dim(R)}(R), \mathfrak{k}),\end{equation}

is isomorphic to grade-shifted copy $R(-a)$ of $R$. Here $m$ is the maximal ideal generated by elements in $R$ of positive degree, and $\text{Hom}_R(-, -)$ is the functor of graded $\mathfrak{k}$-morphisms. The number $a$ is called the $a$-invariant of the graded Gorenstein algebra $R$, see [BH].

For an affine semigroup algebra this property is expressed as follows.

**Proposition 7.2.** Let $S_P$ be the semigroup given by the lattice points in a polyhedral cone $P$. Then the algebra $\mathbb{C}[S_P]$ is Gorenstein if and only if there is a lattice point $\omega \in \text{int}(P)$ with $\text{int}(P) \cap S_P = \omega + S_P$. Furthermore, in the presence of a grading, we have $a(\mathbb{C}[S_P]) = \deg(\omega)$. 

This is Corollary 6.3.8 in [BH]. By a theorem of Stanley (see [BH], Corollary 4.4.6), the Gorenstein property on a graded domain depends only on details of its Hilbert function. Since this is unchanged by flat degeneration, we reduce establishing this property for $V_{C,P}(SL_3(\mathbb{C}))$ to the case $\mathbb{C}[BZ_4^*]$, given that we can establish Theorem 1.1.

To prove Theorem 1.1, we can combine the degeneration described in Theorem 3.4 with the one defined on each component $V_{0,3}(SL_3(\mathbb{C}))$ in the previous section, it remains only to translate operations on semigroups to operations on their corresponding semigroup algebras. We have $\mathbb{C}[(CB_3^*)_{v \in V(\Gamma)}] = \bigotimes_{v \in V(\Gamma)} \mathbb{C}[CB_3^*]$. This algebra carries an action of the torus $T^3$, which appeared in the statement of Theorem 3.6. The algebra $\mathbb{C}[CB_3^*]$ can be constructed as the invariant subalgebra of $\bigotimes_{v \in V(\Gamma)} \mathbb{C}[CB_3^*]$ with respect to the torus $T^3$.

$\mathbb{C}[CB_3^*] = (\bigotimes_{v \in V(\Gamma)} \mathbb{C}[CB_3^*])^T$.

All degenerations we have constructed are $T^3$-invariant, this proves Theorem 1.1. Degeneration arguments also allow us to prove that deeper structural properties hold for $V_{C,P}(SL_3(\mathbb{C}))$.  

**Theorem 7.3.** The algebra $V_{C,P}(SL_3(\mathbb{C}))$ has the structure of a graded Gorenstein algebra, with $a$—invariant equal to 6.

**Proof.** This is a consequence of the following Lemmas. $\square$

**Lemma 7.4.** If $P$ is polytope with $C[P]$ a Gorenstein semigroup algebra with $\omega \in a \circ P$, and if $Q = P \cap H$ is polytope obtained from $P$ by intersecting with a subspace $H$ in a way such that $\omega \in a \circ Q$ then $C[Q]$ is a Gorenstein semigroup algebra with $*-$canonical generator $\omega$.

**Proof.** The proper facets of $Q$ are all obtained as intersections of proper facets of $P$ with $H$, therefore a point in the relative interior of $P$ which is also in $H$ is also in the relative interior of $Q$. Since all interior lattice points in the Minkowski sum $k \circ Q$ are of the form $kX$ for $X$ in the relative interior of $Q$, this shows that any lattice point $X$ in the interior must be divisible by $\omega$. $\square$

**Lemma 7.5.** $BZ_3^*$ is Gorenstein with $\omega_3 = P_{12}P_{23}P_{31}XST$.

**Proof.** This follows from Lemma 7.4 above. The polytope $BZ_3^*$ can be obtained from the simplex in $\mathbb{R}^9$ given as the convex hull of the origin and the points $(0, \ldots, 1, \ldots, 0)$. One takes the 3rd Minkowski sum of this simplex and intersects it with the hyperplanes defined by the hexagon relations. The generator of the $*-$canonical module of this simplex is the point $(1, 1, 1, 1, 1, 1, 1, 1, 1)$, which agrees with $\omega_3 \in BZ_3^*$. $\square$

**Lemma 7.6.** If $P_1$ and $P_2$ are normal polytopes which give Gorenstein semigroup algebras with $*-$canonical generators $\omega_1$ and $\omega_2$ in the same degree, then $\mathbb{C}[P_1 \times P_2]$ is Gorenstein with $*-$canonical generator $\omega_1 \times \omega_2$.

**Proof.** This follows from the fact that any point $(X, Y)$ in the interior of $k \circ (P_1 \times P_2)$ must have $X$ and $Y$ in the interiors of $P_1$ and $P_2$, respectively. $\square$
Since all $BZ^*_T$ are obtained by intersecting a product $(BZ^*_T)^{V(\Gamma)}$ with a subspace, this proves the theorem. The generator $\omega_T$ of the $*$-canonical module is the multi-graded component $V_{C,\rho}(2,2),\ldots,(2,2),6)$, hence we get the following corollary by the same argument.

**Corollary 7.7.** The projective coordinate algebra $R_{C,\rho}((1,1),\ldots,(1,1),3)$ is Gorenstein with $*$-canonical generator in degree 2.

## 8. The case $g = 0$

In this section we discuss Theorem 3.2. The semigroup $CB^*_T$ is highly dependent on the topology of the graph $\Gamma$. However, when the graph $\Gamma$ has genus 0, the situation simplifies considerably. In this section we outline the proofs of the following theorems. From now on let $T$ denote a trivalent tree with $n$ leaves. In both propositions below the active ingredient is that properties of $CB^*_T$ are preserved by fiber product over one of the three components $\partial_1, \partial_2, \partial_3$ of $\partial$, because the values of these components always take the values $(0,1),(1,0), (0,0)$ on a point of $CB^*_T$.

The following Proposition is a consequence of the argument used in [MZ], Theorem 1.2. The gist is that if we take a fiber product $S \times \mathbb{Z}_{\geq 0} CB^*_T$, then any factorization of an element $w \in S$ which is the image of $w' \in S \times \mathbb{Z}_{\geq 0} CB^*_T$, can be extended to a factorization of $w'$.

**Proposition 8.1.** The polytope $CB^*_T$ is normal.

The restriction of any element $\omega \in CB^*_T$ to a trinode is a member of $CB^*_T$. This allows us to describe the lattice points in $CB^*_T$ as follows. First, we define an element of $CB^*_T$ to be complete if its associated network on $T$ assigns every edge an arrow. Figure 9. A complete weighting on a tree.

Note that there are exactly two complete weightings of any tree $T$, determined by the direction of the arrow on a leaf edge. Recall now that a disjoint union of trees is called a forest. We say a forest $F \subset T$ is proper if it has at least two leaves, and every leaf of $F$ is a leaf of $T$. The following lemma is immediate.

**Lemma 8.2.** Every element $\omega \in CB^*_T$ with some non-trivial edge uniquely defines a complete weightings on the components of a proper forest of $T$. Furthermore, complete weightings on the components of a proper forest in $T$ define unique elements of $CB^*_T$.

The fact that any component of a proper forest in $T$ can be given two complete weightings, coupled with Lemma 8.2 give a complete description of the generators of $CB^*_T$. The following is a consequence of the argument for Theorem 1.2 in [MZ].
Figure 10. A complete weighting on a proper subtree.

**Proposition 8.3.** The ideal \( I_T \) which presents the toric algebra \( \mathbb{C}[CB_T^+] \) by its degree 1 component is generated by forms of degree 2 and 3.

As consequence we obtain Theorem 5.2 by a flat degeneration argument. In a reduced family of graded algebras, the generic maximum degrees of required generators and relations are bounded by the degrees at a closed point. For an application of this argument over the stack \( \overline{M}_{g,n} \), see [M7].

This same type of argument works with any of the semigroups we have constructed in the \( g = 0 \) case, in particular we could substitute \( BZ^*_3 \) with \( X + S + T = P_{12} + P_{23} + P_{31} \) as the cubic relation.

9. **The case \( g > 0 \)**

In this section we show that the degree of generation of \( V_{C,\vec{p}}(SL_3(\mathbb{C})) \) for \( (C, \vec{p}) \in \mathcal{M}_{g,n} \) is essentially controlled by the behavior of a specific polytope. The wealth of polytopes \( BZ^*_T \) at our disposal allows us the advantage of picking one with favorable properties. We let \( BZ^*_{g,n} \) be the polytope obtained by fiber-product, dual to the graph \( \Gamma(g, n) \) depicted in Figure 11.

For \( g \geq 1 \), the polytope \( BZ^*_{g,n} \) itself is a fiber product of of the polytope \( BZ^*_T \) corresponding to the tree \( T_0 \) and the polytope \( BZ^*_{g,1} \). As a result, we can control the behavior of \( BZ^*_{g,n} \) to a certain degree.

**Proposition 9.1.** The semigroup algebra \( \mathbb{C}[BZ^*_{g,n}] \) is generated in degree bounded by the degree necessary to generate \( \mathbb{C}[BZ^*_{g,1}] \).

**Proof.** We split the graph \( \Gamma(g, n) \) at the edge \( e \) which separates the tree \( T_0 \) component from the rest of the graph.

\[
\Gamma(g, n) = \Gamma(g, 1) \cup_e T_0
\]

Consider an element \( w \in L \circ BZ^*_{g,n} \), and its restrictions \( w|_{T_0} \) and \( w|_{\Gamma(g,1)} \). We once again use the argument from Theorem 1.2 of [MZ] to show that any factorization of \( w|_{\Gamma(g,1)} \) can be extended to a factorization of all of \( w \). \( \square \)
Using this proposition, we can prove Theorem 3.4 with the following lemma. The semigroup $BZ_{1,1}^*$ is composed of those triangles which are dual to the graph composed of a single loop with an edge. These are triangles such that the boundary values for two chosen edges are dual.

**Lemma 9.2.** *The semigroup algebra* $\mathbb{C}[BZ_{1,1}^*]$ *is generated by elements of level 1, 2, and 3.*

*Proof.* We leave it to the reader to verify that the elements depicted in Figure 12 suffice to generate $BZ_{1,1}^*$.

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**Figure 11.** The graph $\Gamma(g, n)$, which has $g$ loops and $n$ edges.

**Figure 12.** Generators of $BZ_{1,1}^*$.
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