The charm quark mass from non-relativistic sum rules

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Abstract
We present an analysis to determine the charm quark mass from non-relativistic sum rules, using a combined approach taking into account fixed-order and effective-theory calculations. Non-perturbative corrections as well as higher-order perturbative corrections are under control. For the PS mass we find $m_{PS}(0.7 \text{ GeV}) = 1.50 \pm 0.04 \text{ GeV}$ which translates into a $\overline{\text{MS}}$ mass of $\overline{m} = 1.25 \pm 0.04 \text{ GeV}$. 
1 Introduction

In the sum rule approach [1] to determine the mass $m$ of heavy quarks, $Q$, the sensitivity of the cross section $\sigma(e^+e^- \rightarrow Q\bar{Q})$ near threshold $\sqrt{s} \simeq 2m$ is exploited by comparison of the experimental value of the $n$-th moment $M_n$ to the theoretical prediction. The moments are defined as

$$M_n \equiv \int_0^\infty \frac{ds}{s^{n+1}} R_{QQ}(s) = \frac{12\pi^2 e_Q^2}{n!} \left( \frac{d}{dq^2} \right)^n \Pi(q^2)|_{q^2=0}$$

where $\Pi(q^2)$ is the vacuum polarization, $e_Q$ the electric charge of the heavy quark and $R_{QQ}(s) \equiv \sigma(e^+e^- \rightarrow Q\bar{Q})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ the normalized cross section. Traditionally, there have been two complementary theoretical approaches to determine $M_n$. If $n$ is chosen to be small, $n \lesssim 4$ the moments are evaluated using a fixed-order approach. Sometimes this approach is referred to as relativistic or low-$n$ sum rules. Alternatively, if $n$ is large, fixed-order perturbation theory breaks down due to the presence of terms $\alpha_s \sqrt{n}$. In order to get a reliable theoretical prediction these terms have to be resummed, counting $v \sim 1/\sqrt{n} \sim \alpha_s$, where $v$ is the (small) velocity of the heavy quarks. This is usually done in an effective-theory approach (for a review see Ref. [2]), treating the heavy quarks in the non-relativistic approximation. Therefore, this approach is referred to as non-relativistic sum rules.

Relativistic sum rules have been used to determine the bottom and charm quark masses [3, 4, 5, 6]. The first two moments are known at four loop [5, 7, 8, 9], i.e. $\mathcal{O}(\alpha_s^4)$, higher moments are currently known at $\mathcal{O}(\alpha_s^2)$ [10, 11]. The extracted mass and its error depend crucially on how the experimentally poorly known continuum cross section is treated and how the theoretical error is estimated. Recently the charm quark mass has been determined using this approach but replacing experimental data for $\sigma(e^+e^- \rightarrow c\bar{c})$ by input from lattice QCD [12], which via tuning uses different experimental input such as the $\eta_c$ mass.

Applications of the non-relativistic sum rules have been restricted to the determination of the bottom quark mass [13, 14] so far. The moments are known to next-to-next-to leading order (NNLO), in the counting of the effective theory, and in the case of Ref. [14] also include the resummation of logarithms [15] of the form $(\alpha_s \ln \sqrt{n})^l$ at next-to-leading (NLL) and partially at next-to-next-to-leading logarithmic accuracy (NNLL). A complete NNNLO calculation is still missing but partial results are available [16, 17]. This method of extracting $m$ is virtually insensitive to the continuum cross section but suffers from large higher-order corrections.

The main reason why non-relativistic sum rules have not been used in the case of charm quarks is that the application of perturbation theory in this context is thought to be questionable. First, the typical non-relativistic momentum and
energy scales, \(2m/\sqrt{n}\) and \(m/n\), are very small for large \(n\). To some extent this is related to the large higher-order corrections mentioned above and in fact is already a problem for the case of bottom quarks. Second, the non-perturbative contributions from vacuum condensates increase with increasing \(n\) and decreasing \(m\) and potentially make a precise determination of \(M_n\) impossible.

For what values of \(n\) can sum rules be used in the charm case and when is perturbation theory not applicable any longer? In order to answer this question we will consider \(M_n\) for all \(n \leq 16\) in the charm case. We will use an approach that combines techniques for low-\(n\) and large-\(n\) sum rules. In the case of the bottom quark it has been shown [18] that even though these techniques are completely different, the results are remarkably consistent. Encouraged by this we perform an all \(n\) analysis in the case of the charm quark. We will see that using a combined approach helps to keep the size of higher-order corrections under control. Also, the non-perturbative corrections due to the gluon condensate turn out to be much smaller than expected, even for large \(n\), if a threshold mass definition [19, 20, 21] is used. This will lead us to conclude that, contrary to common belief, the charm quark mass can be extracted from non-relativistic sum rules in a reliable way.

In Section 2 we will describe how to obtain the experimental moments, the theoretical moments and the non-perturbative contributions in turn. We then combine these results in Section 3 and determine the charm quark mass.

2 Determination of the moments

2.1 The experimental moment

We start with the determination of the experimental moments. This is conveniently split into three regions: the resonance region including the bound states \(J/\Psi\) and \(\Psi(2S)\) below threshold, the threshold region \(2M_{D_0} = 3.73\ \text{GeV} \leq \sqrt{s} \leq 4.8\ \text{GeV}\) and the continuum region \(\sqrt{s} > 4.8\ \text{GeV}\).

The mass and leptonic width of \(J/\Psi\) and \(\Psi(2S)\) are known to a high accuracy which leads to a very precise determination of the resonance contribution. We use \(M_{J/\Psi} = 3096.916(11)\ \text{MeV}\), \(M_{\Psi(2S)} = 3686.09(4)\ \text{MeV}\), \(\Gamma_{J/\Psi} = 5.55(14)\ \text{keV}\) and \(\Gamma_{\Psi(2S)} = 2.38(4)\ \text{keV}\) [22]. The resonance contribution is then given by

\[
M_n^{(\text{res})} = \frac{9\pi}{\alpha^2} \sum_i \frac{\Gamma_i}{M_i^{2n+1}}
\]

where \(i \in \{J/\Psi, \Psi(2S)\}\) and \(\alpha = \alpha(M_i) = 1/134\).

The contribution from the threshold and continuum region are much more difficult to determine. However, for increasing \(n\) these contributions become less
and less important. In fact, for $n > 5$ the combined threshold and continuum contribution to the moments is smaller than the error from the resonance contribution. Since our analysis will be driven by large $n$ a rather crude determination of these contributions with a large error will not affect the final result. This is one of the big advantages of this approach compared to a fixed-order low-$n$ analysis. In particular, there is no need to replace experimental data above threshold by theoretical input to (artificially) decrease the experimental error.

For the continuum contribution we use data points above and below threshold $[23]$ to obtain a very crude parameterization $R_{\text{ce}}(s) = 1.4 \pm 0.5$ for $\sqrt{s} > 4.8$ GeV. In the threshold region, we include $\Psi(3770), \Psi(4040), \Psi(4160)$ and $\Psi(4415) [22]$ according to Eq. (2) in addition to an underlying contribution parameterized by $R_{\text{ce}}(s) = -4.88 + 1.31\sqrt{s}$. This corresponds to a linear in $\sqrt{s}$ extrapolation between $R_{\text{ce}}(\sqrt{s} = 3.73$ GeV) = 0 and $R_{\text{ce}}(\sqrt{s} = 4.8$ GeV) = 1.4. This is of course a very crude estimate. We take this into account by taking as the error the full size of the underlying contribution. With this procedure we obtain the results as given in Table 1 for the experimental moments. The total error has been obtained by adding the separate errors in quadrature.

| $n$ | $10^{n-1}M_{\text{res}}^n$ | $10^{n-1}M_{\text{thr}}^n$ | $10^{n-1}M_{\text{cont}}^n$ | $10^{n-1}M_{\text{exp}}^n$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| 1   | 0.1190(28)      | 0.0361(176)     | 0.0608(217)     | 0.2158(281)     |
| 2   | 0.1167(28)      | 0.0202(92)      | 0.0132(47)      | 0.1501(107)     |
| 3   | 0.1162(28)      | 0.0115(49)      | 0.0038(14)      | 0.1315(58)      |
| 4   | 0.1171(29)      | 0.0067(27)      | 0.0012(4)       | 0.1250(39)      |
| 5   | 0.1192(30)      | 0.0039(15)      | 0.0004(1)       | 0.1235(33)      |
| 6   | 0.1221(30)      | 0.0023(8)       | 0.0002(1)       | 0.1246(32)      |
| 7   | 0.1257(31)      | 0.0014(5)       | 0.0001(1)       | 0.1272(32)      |
| 8   | 0.1299(33)      | 0.0009(3)       | 0               | 0.1308(33)      |
| 9   | 0.1346(34)      | 0.0005(2)       | 0               | 0.1352(34)      |
| 10  | 0.1397(35)      | 0.0003(1)       | 0               | 0.1400(35)      |
| 11  | 0.1452(37)      | 0.0002(1)       | 0               | 0.1454(37)      |
| 12  | 0.1510(38)      | 0.0001(0)       | 0               | 0.1512(38)      |
| 13  | 0.1572(40)      | 0.0001(0)       | 0               | 0.1573(40)      |
| 14  | 0.1637(41)      | 0.0001(0)       | 0               | 0.1638(41)      |
| 15  | 0.1706(43)      | 0               | 0               | 0.1706(43)      |
| 16  | 0.1778(45)      | 0               | 0               | 0.1778(45)      |

Table 1: Values of experimental moments (in $\text{[GeV]}^{-2n}$) and their errors. Entries smaller than $5 \times 10^{-5}$ are given as 0.

We stress that it is of course possible to get more precise results for small $n$, but in our approach this is not required. In fact, in what follows we will consider
only \( n > 2 \) and for our final result only moments with \( n \gtrsim 8 \) are relevant.

2.2 The theoretical moment

The evaluation of the theoretical moments follows the discussion given in Ref. [18]. We consider three ways to evaluate the moments: fixed-order (FO) moments, moments evaluated using an effective-theory approach (ET) writing

\[
M_n = \int_{-\infty}^{\infty} \frac{2 \, dE}{(2m)^{2n+1}} \, e^{-\frac{nE}{m}} \left( 1 - \frac{E}{2m} + \frac{nE^2}{(2m)^2} + \ldots \right) R_{cc}(E)
\]

with \( E = \sqrt{s} - 2m \) and, finally, moments using a combined approach. The latter are obtained by adding the FO and ET moments and subtracting the doubly counted terms [18]. These moments should provide us with a reliable theoretical prescription for all \( n \) as long as non-perturbative corrections are under control.

Since we are dealing (at least partially) with large \( n \) where the moments are dominated by the lowest lying resonances, we have to use a mass definition adapted to the description of such resonances, i.e. a threshold mass [19, 20, 21]. We use the PS mass [20] with the associated factorization scale \( \mu_F = 0.7 \) GeV. For the strong coupling we set \( \alpha_s(M_Z) = 0.118 \) and use three-loop evolution.

The main issue for a reliable extraction of the charm mass will be a realistic estimate of the theoretical error due to missing higher-order corrections. This is a notorious problem and there is no generally applicable procedure. We will therefore use a combination of different methods and criteria, as described below.

Let us use the 10th moment as an example to illustrate our determination of the theoretical error. In Figure 1 we display \( M_{10} \) as a function of the scale \( \mu \) for \( m_{PS}(\mu) = 1.50 \) GeV. We consider a range of different theoretical predictions.

- **FO:** this is a fixed-order calculation including all terms up to \( O(\alpha_s^3) \), keeping in mind that the constant term of \( O(\alpha_s^3) \) is actually not yet known for \( n > 2 \). We have fixed this constant to be the one of the first moment. This has only a very weak influence on the result.

- **ET-NNLL:** this is a renormalization-group improved effective-theory calculation complete at NNLO. The resummation of logarithms is complete at NLL and partially done at NNLL. For details we refer to Ref. [14].

- **ET-N3LO:** this is an effective-theory calculation where the logarithms have been re-expanded and kept up to NNLO. This result contains all terms at NNLO and the logarithmically enhanced NNNLO terms. It is used to gauge the impact of corrections beyond NNLO and the importance of resumming the logarithms.
Figure 1: Scale dependence of various theoretical predictions for $M_{10}$ evaluated with $m_{PS}(0.7 \text{ GeV}) = 1.50 \text{ GeV}$. $M_{10}^{\text{exp}}$ is shown as thin grey band and the range of scale variation used for $\delta m_{\text{th}}$ is indicated by the light-grey region.

CB-NNLL: this is a combined FO-ET calculation [18]. The ET input is as for ET-NNLL. In addition all terms of $\mathcal{O}(\alpha_s^3)$ have been included. This is the “best” available theoretical prediction.

CB-N3LO: this is also a combined FO-ET calculation, but the ET input has been taken as in ET-N3LO.

CB-NLL: this is also a combined FO-ET calculation, but the ET input is modified to be only at NLO/NLL. The subtraction to avoid double counting when combining with the FO result has to be adapted accordingly. This result is used to consider the convergence of the perturbative series.

The following points related to Figure 1 will be of importance and in fact are valid for all moments that are relevant to us, i.e. with $n \gtrsim 5$:

- The ET and CB results are very close, indicating that the relativistic corrections to the ET result are small. This is not surprising for large $n$. What is surprising to some extent is that the relativistic corrections turn out to be small also for small $n$.

- The ET-NNLL and CB-NNLL results have a peak slightly above $\mu \simeq 1 \text{ GeV}$. For scales below the peak, the results become very soon unreliable indicating
a breakdown of perturbation theory. The peak is close to $\mu = 2m/\sqrt{n}$, the typical momentum scale in the non-relativistic region.

- The ET-N3LO and CB-N3LO results are very similar to the ET-NNLL and CB-NNLL results except for small $\mu$. It is in this region only, where the resummation of logarithms actually becomes important and, therefore, the ET-N3LO and CB-N3LO results cannot be used any longer.

- The FO prediction does remarkably well even for large moments, where it is supposed to be inapplicable. This hinges on the fact that a threshold mass has been used in the FO approach.

Taking into account these observations we proceed as follows to determine the mass and its theoretical error due to missing higher-order corrections. We start by taking our best prediction, CB-NNLL, and determine a band of $m$ values by varying $1 \text{ GeV} \leq \mu \leq 2 \text{ GeV}$. The standard prescription would be to vary the scale by a factor two around the typical value $\mu = 2m/\sqrt{n}$. This would result in scales below 1 GeV though, which according to Figure 1, are not acceptable. However, if the upper limit of the standard variation is larger than 2 GeV (i.e. for $n \leq 9$) we use the larger value instead. Note that in any case the peak of the CB-NNLL result is included in the band of scale variation. We now extract the mass as the central value of the band with symmetric errors. The results are shown in the first column of Table 2. As anticipated small moments have a large error in our approach and will not play a significant role.

Given the remarkable consistency and the small errors of these results it would be tempting to simply take them at face value. However, the scale dependence alone is a dubious way to determine the theoretical error. In order to get a more reliable estimate we extend the analysis. We repeat the same exercise for the CB-N3LO case. The results are given in the second column of Table 2. As can be seen from Figure 1 the CB-N3LO calculation leads to larger moments (for small $\mu$) and therefore somewhat larger values for $m$. Also, the error is dominated by small scales $\mu \simeq 1$ where the CB-N3LO starts to become unreliable due to the importance of resumming logarithms. Therefore, taking the upper end of the $m$ band of the CB-N3LO results in an overestimate of the theoretical error. We thus combine the CB-NNLL and CB-N3LO results by subtracting the CB-NNLL error from the (smaller) CB-NNLL result and adding it to the (larger) CB-N3LO result. From this range we determine $m$ (as the central value) and symmetric errors. In this way we take into account the CB-N3LO tendency to give larger values of $m$ while discarding the unreliable small scale region of the CB-N3LO results. The results are given in the third column of Table 2.

Finally we perform two further cross checks on our error. We determine the mass using the CB-NLL calculation and using the same scale variation as above. The central value of the results are shown in column 4 of Table 2. These values
Table 2: Extracted mass and theoretical error in MeV, using various approaches. The central column shows the combined result with error.

| n  | CB-NLL     | CB-N3LO    | m(δm^{th}) | CB-NLL | FO O(α_s^3) |
|----|------------|------------|------------|--------|-------------|
| 3  | 1436(156)  | 1583(182)  | 1509(229)  | 1451   | 1434        |
| 4  | 1464(102)  | 1553(136)  | 1508(147)  | 1439   | 1434        |
| 5  | 1478(72)   | 1539(107)  | 1509(103)  | 1438   | 1438        |
| 6  | 1483(57)   | 1531(88)   | 1507(81)   | 1444   | 1442        |
| 7  | 1488(45)   | 1526(74)   | 1507(64)   | 1448   | 1447        |
| 8  | 1493(36)   | 1524(63)   | 1508(52)   | 1452   | 1452        |
| 9  | 1494(30)   | 1521(55)   | 1508(44)   | 1456   | 1457        |
| 10 | 1494(28)   | 1518(50)   | 1506(40)   | 1463   | 1460        |
| 11 | 1494(26)   | 1516(46)   | 1505(37)   | 1466   | 1463        |
| 12 | 1494(25)   | 1514(43)   | 1504(35)   | 1470   | 1467        |
| 13 | 1494(23)   | 1513(40)   | 1503(32)   | 1473   | 1470        |
| 14 | 1495(22)   | 1511(38)   | 1503(31)   | 1479   | 1473        |
| 15 | 1495(21)   | 1511(36)   | 1503(29)   | 1481   | 1476        |
| 16 | 1496(20)   | 1510(33)   | 1503(27)   | 1484   | 1479        |

all lie within the error band which gives further confidence in our results. The same is even true for the mass values determined by a FO approach, listed in the last column of Table 2. This could be taken as an indication that the error has been overestimated. However, the corrections in the ET approach are very large and the NLL results lie within the NNLL error band only because they have been improved using the FO results in a combined analysis. Thus we prefer to keep the larger error, anticipating relatively large NNNLO corrections in the effective theory.

To visualize the consistency of our approach, in Figure 2 we plot the extracted mass with its error for all n. The left (dark blue) bands show m_{PS} as extracted using a FO approach with the central value indicated by a dot. The right (light green) bands show the corresponding values using a ET-NNLL and a ET-N3LO approach. The latter leads to slightly larger values and errors for m_{PS} and is depicted by the dashed band. The two dots in this band indicate the two central values for ET-NNLL (lower) and ET-N3LO (higher) respectively. Finally, the middle (red) bands show our combined result, as given in in the third column of Table 2 with the central value again indicated by a dot.

As expected, the central value of the combined result is close to the ET result for large n. For small n, the combined result is also consistent with the FO result, at the price of having a huge error. The combined result makes use of all available
Figure 2: Extracted values with theoretical errors for $m_{PS}(0.7 \text{ GeV})$ for all moments $n \leq 16$, using a FO (left, dark blue bands), an ET (right, light green bands) and a combined approach (middle, red bands).

information and gives the most reliable prediction for large $n$.

### 2.3 The non-perturbative contribution

One of the main reasons why non-relativistic sum rules so far have not been used to extract the charm quark mass is the common belief that non-perturbative corrections are not under control. As we will see this is actually not the case.

A first hint that the situation is in much better control than anticipated is the fact that the (central value of the) mass extracted, as indicated by the points in Figure 2, is remarkably consistent for all values of $n \geq 5$. If there were large non-perturbative effects they would with all likelihood affect results with increasing $n$ more dramatically.

In order to get a more quantitative picture, we will consider the effects of the gluon condensate [24] which gives us a handle for the leading non-perturbative correction. The corresponding contribution to the sum rule has been computed to two loops [25] and reads

$$
\delta M_n^{(np)} = \frac{12 \pi^2 e_Q^2}{(4m^2)^{n+2}} \left( \frac{\alpha_s}{\pi} G^2 \right) a_n \left( 1 + \frac{\alpha_s}{\pi} \left[ b_n - (2n + 4) \delta b_X \right] \right)
$$

(4)
where the one-loop coefficients are given by

\[ a_n = -\frac{(2n + 2) \Gamma(4 + n) \Gamma(7/2) \Gamma(7/2 + n)}{15 \Gamma(4) \Gamma(7/2)} \] (5)

and \( b_n \) are the two-loop coefficients in the pole scheme, as listed in Ref. [25]. The shifts \( \delta b_X \) take into account the change in the mass scheme. For the PS mass we have

\[ b_n^{\text{PS}} \equiv b_n - (2n + 4) \delta b_{\text{PS}} = b_n - (2n + 4) C_F \frac{\mu_F}{m} \] (6)

where \( C_F = 4/3 \) is a colour factor.

There are two issues we have to consider: how large are the contributions due to the gluon condensate and how reliable is the prediction in Eq. (4)? The answer to both questions crucially depends on the mass scheme used. Regarding the former, the key features of Eq. (4) are that the coefficients grow like \( a_n \sim n^{3/2} \) and that \( \delta M_n^{(np)} \) increases rapidly for decreasing mass. To assess the situation, we calculated for the case of the PS scheme the ratio of the non-perturbative contributions, as given in Eq. (4), to the experimental moment, using \( m_{\text{PS}} = \mu = 1.5 \text{ GeV} \) and \( \langle (\alpha_s/\pi) G^2 \rangle = 0.005 \text{ GeV}^4 \) [26]. The results are shown in Table 3. As can be seen, the non-perturbative contributions are well below 10\% for all \( n \leq 16 \) indicating that they are in fact not (yet) very important. Another satisfactory feature of the PS scheme is that the series in Eq. (4) is well behaved. To show this we also list the relative importance of the two-loop corrections. Clearly, they should be small compared to the leading term, for Eq. (4) to be applicable. For the first moment, the higher order correction is 75\% of the leading term which is uncomfortably large. However, for larger values of \( n \), where the contribution starts to become somewhat more relevant, the corrections are smaller.

| \( n \)   | \( n = 1 \) | \( n = 4 \) | \( n = 7 \) | \( n = 10 \) | \( n = 13 \) | \( n = 16 \) |
|--------|---------|---------|---------|---------|---------|---------|
| \( \delta M_n^{(np,PS)}/M_n^{\text{exp}} \times 10^{-2} \) | 0.1     | 0.7     | 1.6     | 2.9     | 4.3     | 5.9     |
| \( \alpha_s b_n^{\text{PS}}/\pi \) | 0.75    | 0.72    | 0.61    | 0.46    | 0.28    | 0.09    |

Table 3: Importance of gluon condensate contribution to moments (first row) and relative importance of two-loop corrections to gluon condensate contribution (second row) in the PS scheme.

From the results in Table 3 we conclude that the gluon condensate contributions are under control for all values of \( n \) considered here. This seems to be in contradiction with what is commonly stated. However, we stress that the picture is completely different if either the pole mass or the \( \overline{\text{MS}} \) mass is used. It is well known that there is a close interplay between vacuum condensates and mass definitions [27, 26]. In the case of the pole mass, the corrections are also relatively
small (mainly because \( m_{\text{OS}} > m_{\text{PS}} \)) but the series in \( \alpha_s \) in Eq. (4) is completely unreliable because the two-loop corrections exceed the one-loop corrections for all \( n \). In the case of the \( \overline{\text{MS}} \) mass, the contributions are huge for large \( n \) (mainly because \( m_{\overline{\text{MS}}} < m_{\text{PS}} \)) and the corrections are also very large, unless extremely small scales \( \mu \lesssim 1 \text{ GeV} \) are used. For other threshold masses, such as e.g. the RS mass [21] we checked that the main conclusions are the same as for the PS mass.

Of course one might wonder about the contribution of further suppressed condensates such as the dimension 6 operator \( \langle G^3 \rangle \). However, as we will see, the contribution and induced error due to the \( \langle G^2 \rangle \) operator is so small that we can simply take this into account by increasing the error. Thus we conclude that if a mass definition adapted for quark pairs near threshold is used, the non-perturbative corrections are under control even in the charm case. We remark that this is also in agreement with a recent analysis [12] where contributions from the gluon condensate were found to be much smaller than expected.

3 Results

In this section we extract the PS charm quark mass and determine the various errors. The dominant error will be the error \( \delta m_{\text{th}} \) due to missing higher-order corrections discussed in Section 2.2. We will consider all \( 3 \leq n \leq 16 \) even though from Figure 2 it is clear that values \( n \lesssim 5 \) are “useless” in the sense that their error is too large. The results are summarized in Table 4.

The first column shows the central value for the mass. These entries differ slightly from the corresponding entries of Table 2 because the effect of the gluon condensate, as discussed in Section 2.3, has been included. Apart from the theoretical error, taken directly from Table 2, we include three further sources of errors.

First we consider the experimental error, \( \delta m_{\text{exp}} \). We simply vary the experimental moments in the range given in Table 1 and consider the effect on the extracted mass. As expected, the error decreases rapidly for increasing \( n \) and becomes very soon negligible.

A more relevant source of error is the uncertainty in the strong coupling. We vary \( 0.116 \leq \alpha_s(M_Z) \leq 0.120 \) [22]. The resulting error, \( \delta m_{\alpha} \), is listed again in Table 4.

Finally we consider the contribution and induced error due to the gluon condensate, \( \delta m_{GG} \). As discussed in Section 2.3 this contribution is surprisingly small. To determine the error we vary \( \langle (\alpha_s/\pi)G^2 \rangle = 0.005 \pm 0.004 \text{ GeV}^4 \) [26] and determine the corresponding change in \( m \). We then double this error to
take into account higher-order corrections to Eq. (4), higher dimensional vacuum condensates and the fact that previous determinations of $\langle (\alpha_s/\pi)G^2 \rangle$ resulted in somewhat larger values. Even so, the error does virtually not affect the final result.

The total error, listed in the last column of Table 4 is obtained by adding the various errors in quadrature. We also checked that the higher-order QED contributions have a negligible effect. In fact, they change the mass by a few MeV at most.

| $n$ | $m$  | $\delta m^{th}$ | $\delta m^{exp}$ | $\delta m^\alpha$ | $\delta m^{QED}$ | $\delta m$ |
|-----|------|----------------|------------------|------------------|----------------|-------------|
| 3   | 1508 | 229           | 11               | 41               | 2              | 233         |
| 4   | 1507 | 147           | 6                | 34               | 3              | 151         |
| 5   | 1508 | 103           | 4                | 29               | 3              | 107         |
| 6   | 1506 | 81            | 3                | 27               | 3              | 85          |
| 7   | 1505 | 64            | 3                | 24               | 4              | 69          |
| 8   | 1506 | 52            | 2                | 22               | 4              | 57          |
| 9   | 1504 | 44            | 2                | 20               | 4              | 49          |
| 10  | 1503 | 40            | 2                | 19               | 5              | 45          |
| 11  | 1503 | 37            | 2                | 18               | 5              | 41          |
| 12  | 1501 | 35            | 2                | 17               | 5              | 39          |
| 13  | 1500 | 33            | 1                | 16               | 5              | 36          |
| 14  | 1500 | 31            | 1                | 15               | 6              | 35          |
| 15  | 1500 | 29            | 1                | 14               | 6              | 33          |
| 16  | 1500 | 27            | 1                | 14               | 6              | 31          |

Table 4: Extracted charm quark mass with separate and total errors. All entries are in MeV and for the PS mass with $\mu_F = 0.7$ GeV.

The extracted mass is virtually independent of $n$. Thus, the only issue regarding how to combine the results of Table 4 is the determination of the final error. Given the remarkable consistency between the various results we argue it is safe to take a single moment result with a rather large $n$. Therefore we take as our final result

$$m_{PS}(0.7 \text{ GeV}) = 1.50 \pm 0.04 \text{ GeV} \tag{7}$$

Since the determination of the dominant error, $\delta m^{th}$ is somewhat arbitrary, we think it is misleading to give more significant figures in the error.

Converting this to the $\overline{\text{MS}}$ mass we obtain $\overline{m} \equiv m_{\overline{\text{MS}}}(m_{\overline{\text{MS}}}) = 1.25$ GeV. The error of 40 MeV in Eq. (7) results in an error of 35 MeV for the $\overline{\text{MS}}$ mass. However, there is also an error in the conversion itself. As an indication of this error we take the size of the fourth order term in the conversion and obtain an
additional error of 15 MeV. Finally, there is an error in the conversion induced by the uncertainty in \( \alpha_s \). Varying \( 0.116 \leq \alpha_s(M_Z) \leq 0.120 \) in the conversion and taking into account the correlation of this with the corresponding variation in the determination of the PS mass, this results in an error of 15 MeV as well. These errors are relatively large because the coupling is large due to the small scale. Thus a reduction in the error in Eq. (7) would only partially impact on the error in the \( \overline{\text{MS}} \) mass. Combining in quadrature the three errors in the conversion we obtain for the \( \overline{\text{MS}} \) mass

\[
\overline{m} = 1.25 \pm 0.04 \, \text{GeV}
\]  

This value is in good agreement with the world average \( \overline{m} = 1.27^{+0.07}_{-0.11} \) \cite{22}, but has a larger error than recent determinations using low-\( n \) moments \cite{5,6,9}. However, we would argue that our estimate of the theoretical error is more conservative and that this determination of the charm quark mass is in many respects complementary to the low-\( n \) sum rules.

## 4 Conclusions

The main result of this work is that the non-relativistic sum rule can be used to obtain a precise and reliable determination of the charm quark mass. The non-perturbative corrections are under control even for \( n \simeq 8 - 16 \) as long as a suitable threshold mass definition is used. The situation with respect to the large corrections in the effective theory is much improved if a combined analysis is performed, including available fixed order results.

We are aware that these statements are to a certain extent in contradiction with what would naively be expected. However, looking at the situation more carefully, they are actually not that surprising. It has been shown previously \cite{18} that in the case of the bottom quark, the FO as well as the ET approach work much better than expected. In the charm case large moments were also found to give consistent results \cite{12}. The value of the quark mass does not seem to be the driving force for the large corrections in the effective theory. In fact, the corrections are also large in the top case. Thus, the reduction in mass from bottom to charm does not completely alter the question regarding the applicability of perturbation theory. Given that completely different theoretical approaches give comparable results and that the size of the corrections are reasonable in a combined approach, we argue that the situation regarding non-relativistic sum rules in the charm case is similar to the bottom case. In spite of large partial NNNLO corrections to the sum rules in the bottom case \cite{17} we expect that the total NNNLO correction is within our error estimate, implying similar cancellation between the various NNNLO contributions as for the top case. With a careful, conservative error estimate the quark mass can be determined reliably.
The by far largest contribution to the error in the present determination of the charm quark mass comes from unknown higher-order corrections. An estimate of this error is notoriously difficult and to a large extent arbitrary. It is for this reason that we deliberately refrained from pushing the error estimate to an extreme. In particular, to make our error estimate as reliable as possible, we do not take the considerably smaller errors of the CB-NNLL result, nor do we take the smallest error in Table 4.

It is clear that neither a FO nor a ET analysis alone can cover the whole range of \(n\) and only a combined analysis can make use of all available information. In this sense the present analysis can be considered as to a large extent complementary to low-\(n\) sum rules, since it is clearly dominated by a large-\(n\) approach. This approach uses a different theoretical input and the consistency of this result with other determinations provides useful information.

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