Bell’s nonlocality in a general nonsignaling case: quantitatively and conceptually

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Abstract
Quantum violation of Bell inequalities is now used in many quantum information applications and it is important to analyze it both quantitatively and conceptually. In the present paper, we analyze violation of multipartite Bell inequalities via the local probability model – the LqHV (local quasi hidden variable) model [Loubenets, J. Math. Phys. 53, 022201 (2012)], incorporating the LHV model only as a particular case and correctly reproducing the probabilistic description of every quantum correlation scenario, more generally, every nonsignaling scenario. The LqHV probability framework allows us to construct nonsignaling analogs of Bell inequalities and to specify parameters quantifying violation of Bell inequalities – Bell’s nonlocality – in a general nonsignaling case. For quantum correlation scenarios on an N-qudit state, we evaluate these nonlocality parameters analytically in terms of dilation characteristics of an N-qudit state and also, numerically – in d and N. In view of our rigorous mathematical description of Bell’s nonlocality in a general nonsignaling case via the local probability model, we argue that violation of Bell inequalities in a quantum case is not due to violation of the Einstein-Podolsky-Rosen (EPR) locality conjectured by Bell but due to the improper HV modelling of ”quantum realism”.

Keywords: Nonsignaling – Bell’s nonlocality – The LqHV modelling – Quantum realism
1 Introduction

In more than 50 years since the seminal paper [1] of Bell, there is still no a unique conceptual view\(^1\) on quantum nonlocality conjectured by Bell [2, 3] for explaining quantum violation of the local hidden variable (LHV) statistical constraints. However, in quantum information, nonlocality of a multipartite quantum state is defined purely mathematically – via violation by this state of a Bell inequality and it is specifically in this sense quantum nonlocality is now used in all experimental quantum information processing tasks.

Moreover, from the practical point of view, it is also important to know violation or nonviolation by an \(N\)-partite quantum state of Bell inequalities of some specific class, hence, nonlocality or locality of an \(N\)-partite state under the corresponding class of correlation scenarios, for example, under correlation scenarios with some specific numbers \(S_1, \ldots, S_N\) of settings at \(N\) sites. The latter type of partial locality of an \(N\)-partite quantum state, \(S_1 \times \cdots \times S_N\)-setting locality, was analyzed in [5, 6, 16, 17, 18].

However, in all cases, quantifying nonlocality of an \(N\)-partite quantum state, full or partial, is associated with finding the maximal violation by this state of the corresponding class of Bell inequalities. In the literature, the well-known attainable upper bounds [19, 20, 21, 22] on quantum violation of specific Bell inequalities concern the Clauser-Horne-Shimony-Holt (CHSH) inequality and the Mermin-Klyshko inequality. It is also well known that the maximal quantum violation of bipartite Bell inequalities on correlation functions cannot exceed the real Grothendieck’s constant \(K_G^{(\mathbb{R})} \in [1.676, 1.783]\) independently of a dimension of a bipartite state and numbers of measurement settings and outcomes per site. But the latter is not already the case for quantum violation of bipartite Bell inequalities on joint probabilities and last years bounds on the maximal quantum violation of general\(^2\) Bell inequalities were intensively discussed in the literature, see [6, 8, 23, 24, 25, 26, 27, 28] and references therein.

In the sense of violation of a Bell inequality, nonlocality is also inherent to a general nonsignaling\(^3\) correlation scenario and, in this case, we refer [5] to it as Bell’s nonlocality. As we analyzed this mathematically in [5], for an arbitrary correlation scenario, Bell’s locality (in the sense of nonviolation of all general Bell inequalities) implies the EPR (Einstein–Podolsky–Rosen) locality 29 and the EPR locality implies

\(^1\)See Introductions in [4, 5, 6, 7, 8] and discussions in [9, 10, 11, 12, 13, 14].

\(^2\)That is, Bell inequalities of an arbitrary type – either on correlation functions or on joint probabilities or of a more complicated form.

\(^3\)On this notion, see section 3 in [5], also, section 2 below.
nonsignaling – but not vice versa. Therefore, Bell’s nonlocality does not necessarily lead to violation of the EPR locality.

In the present paper, we analyze Bell’s nonlocality via the local probability model, the LqHV (local quasi hidden variable), introduced in [6, 30, 31] and incorporating the LHV probability model only as a particular case. The LqHV model correctly reproduces the probabilistic description of every nonsignaling correlation scenario (in particular, every quantum scenario) and this allows us to construct nonsignaling analogs of Bell inequalities and to specify parameters quantifying Bell’s nonlocality, partial and full, in a general nonsignaling case. For quantum correlation scenarios on an N-partite state, we evaluate these nonlocality parameters analytically via dilation characteristics of an N-partite state. For an N-qudit state, we also evaluate them numerically – in d, S and N.

In view of our rigorous mathematical description of Bell’s nonlocality in a general nonsignaling case via the local probability model, we argue that violation of Bell inequalities in a quantum case is not due to violation of the EPR locality conjectured by Bell [2, 3] but due to the improper HV modelling of “quantum realism”.

The paper is organized as follows.

In Section 2, we introduce the functional approach [15] to constructing general multipartite Bell inequalities for arbitrary numbers of settings and outcomes at each site and present the single general representation incorporating in a unique manner all Bell inequalities. This representation allows us to specify in section 4 violation of a Bell inequality in an arbitrary nonsignaling case.

In Section 3, we introduce the notion of a LqHV (local quasi hidden variable) probability model [6, 30, 31] and discuss its validity for a general correlation scenario.

In Section 4, we find nonsignaling analogs of Bell inequalities and specify parameters quantifying Bell’s nonlocality, partial and full, in a general nonsignaling case.

In Section 5, for quantum correlation scenarios on an N-qudit state, we evaluate these nonlocality parameters analytically and numerically. As an example, we specify the quantum analog of the bipartite Bell inequality presented in [32].

In Section 6, we discuss the conceptual issues of Bell’s nonlocality.
2 Multipartite Bell inequalities for arbitrary numbers of settings and outcomes per site

In this section, we present the functional approach [15] to constructing general multipartite Bell inequalities. In contrast to the polytope approach [19,21,33], which is valuable for finding Bell inequalities on correlation functions and joint probabilities in case of small numbers of settings and outcomes per site, the functional approach leads to a single general representation for Bell inequalities of any type with arbitrary numbers of settings and outcomes at each site. This allows us further easily to analyze in section 4 a modification of this general representation for an arbitrary nonsignaling case.

Consider an $N$-partite correlation scenario, where each $n$-th of $N \geq 2$ parties (players) performs $S_n \geq 1$ measurements with outcomes $\lambda_n \in \Lambda_n$ of any nature and an arbitrary spectral type. We label each measurement at $n$-th site by a positive integer $s_n = 1,\ldots,S_n$ and each $N$-partite joint measurement, induced by this correlation scenario and with outcomes

$$(\lambda_1,\ldots,\lambda_N) \in \Lambda = \Lambda_1 \times \cdots \times \Lambda_N,$$

by an $N$-tuple $(s_1,\ldots,s_N)$, where $n$-th component specifies a measurement at $n$-th site.

For concreteness, we further denote by $\mathcal{E}_{S,\Lambda}$, $S = S_1 \times \cdots \times S_N$, an $S_1 \times \cdots \times S_N$-setting correlation scenario with outcomes in $\Lambda$ and by $P^{(S,\Lambda)}(E_{S,\Lambda})$ - a joint probability distribution of outcomes under an $N$-partite joint measurement $(s_1,\ldots,s_N)$ of a scenario $E_{S,\Lambda}$.

The superscript $E_{S,\Lambda}$ at notation $P^{(S,\Lambda)}(E_{S,\Lambda})$ indicates that, under a scenario $E_{S,\Lambda}$, this joint probability distribution may depend not only on parties’ settings $(s_1,\ldots,s_N)$, specifying this joint measurement, but also on settings of all (or some) other measurements of this scenario. This is, for example, the case for scenarios with two-sided memory [34].

A correlation scenario $E_{S,\Lambda}$ is called nonsignaling if, for any two joint measurements $(s_1,\ldots,s_N)$ and $(s'_1,\ldots,s'_N)$ with common settings $s_{n_1},\ldots,s_{n_M}$ at some $1 \leq n_1 < \ldots < n_M \leq N$ sites, the marginal probability distributions of $P^{(S,\Lambda)}_n(E_{S,\Lambda})$ and $P^{(S,\Lambda)}_n(E_{S,\Lambda})$, describing measurements at these sites, coincide. For details on the mathematical specification of nonsignaling and the EPR locality, see section 3 in [5].

\footnote{For the general framework on the probabilistic description of multipartite correlation scenarios, see [3].}
For a correlation scenario $\mathcal{E}_{S,A}$, consider a linear combination

$$B^{(\mathcal{E}_{S,A})}_{\Phi_{S,A}} = \sum_{s_1, \ldots, s_N} \langle f(s_1, \ldots, s_N)(\lambda_1, \ldots, \lambda_N) \rangle_{\mathcal{E}_{S,A}},$$

$$\Phi_{S,A} = \{ f(s_1, \ldots, s_N) : \Lambda \to \mathbb{R} \mid s_n = 1, \ldots, S_n, \ n = 1, \ldots, N \},$$

of averages

$$\langle f(s_1, \ldots, s_N)(\lambda_1, \ldots, \lambda_N) \rangle_{\mathcal{E}_{S,A}}$$

for the most general form, specified for each joint measurement $(s_1, \ldots, s_N)$ by a bounded real-valued function $f(s_1, \ldots, s_N)(\lambda_1, \ldots, \lambda_N)$ of measurement outcomes $(\lambda_1, \ldots, \lambda_N) \in \Lambda$ at all $N$ sites.

Depending on a choice of a function $f(s_1, \ldots, s_N)$, an average (3) may refer either to the joint probability of events at $M \leq N$ sites or, for example, in case of real-valued outcomes at each $n$-th site, to the mean value

$$\langle \lambda_1^{(s_1)} \cdot \ldots \cdot \lambda_n^{(s_M)} \rangle = \int_{\Lambda} \lambda_1 \cdot \ldots \cdot \lambda_n P(s_1, \ldots, s_N) \ (d\lambda_1 \times \cdots \times d\lambda_N)$$

of the product of outcomes observed at $M \leq N$ sites under a joint measurement $(s_1, \ldots, s_N)$. In quantum information, the average (4) is referred to as a correlation function. For $M = N$, a correlation function is called full.

The probabilistic description of an arbitrary correlation scenario $\mathcal{E}_{S,A}$ admits a \textit{LHV} (local hidden variable) probability model if each of its joint probability distributions

$$\{ P^{(\mathcal{E}_{S,A})}_{(s_1, \ldots, s_N)}, \ s_1 = 1, \ldots, S_n, \ldots, s_N = 1, \ldots, S_N \}$$

admits the representation

$$P^{(\mathcal{E}_{S,A})}_{(s_1, \ldots, s_N)} \ (d\lambda_1 \times \cdots \times d\lambda_N) = \int_{\Omega} P_{s_1,1}(d\lambda_1|\omega) \cdot \ldots \cdot P_{s_N,1}(d\lambda_N|\omega) \ \nu_{\mathcal{E}_{S,A}}^{lhv} (d\omega)$$

via a single probability distribution $\nu_{\mathcal{E}_{S,A}}^{lhv}$ of some variables $\omega \in \Omega$ and conditional probability distributions $P_n, s_n(\cdot|\omega)$ of outcomes $\lambda_n$ at $n$-th sites for the main statements on the LHV modelling of a general multipartite correlation scenario, see section 4 in [5].
site, referred to as "local" in the sense that each $P_{n,s_n}$ depends only on a measurement setting $s_n$ at $n$-th site.

In quantum theory, variables $\omega \in \Omega$ are generally referred to as "hidden variables" (HV) – this and "locality" of distributions $P_{n,s_n}$ explains the title "LHV" of this model.

Note that, though, in the general LHV representation \[ (6) \], each distribution $P_{n,s_n}(\cdot|\omega)$ depends only on a measurement setting $s_n$ at $n$-th site, a probability distribution $\nu_{lhv}$ of variables $\omega$, which has a simulation character, may, in general, depend via the subscript $E_{S,\Lambda}$ on measurement settings at all (or some) sites. This is, for example, the case in quantum LHV models considered in section 5 of [5].

If, in addition to representation \[ (6) \], some distributions $P_{n,s_n}(\cdot|\omega)$ corresponding to different sites are correlated (via $\omega$), then we refer to such an LHV model as conditional. The LHV model considered by Bell in [1] represents an example of a conditional LHV model.

From representation \[ (6) \] it follows that each LHV correlation scenario is nonsignaling, though not vice versa.

Let a correlation scenario $E_{S,\Lambda}$ admit a LHV model. Then a linear combination \[ (2) \] of its averages satisfies the tight LHV constraints \[ (7) \] with the LHV constants

$$B_{\Phi_{S,\Lambda}}^{\inf} = \inf_{\lambda_n^{(s_n)} \in \Lambda_n, \forall s_n, \forall n} \sum_{s_1, \ldots, s_N} f_{(s_1, \ldots, s_N)}(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)})$$ (8)

$$B_{\Phi_{S,\Lambda}}^{\sup} = \sup_{\lambda_n^{(s_n)} \in \Lambda_n, \forall s_n, \forall n} \sum_{s_1, \ldots, s_N} f_{(s_1, \ldots, s_N)}(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)})$$ (9)

If a correlation scenario $E_{S,\Lambda}$ admits a conditional LHV model, then a linear combination $B_{\Phi_{S,\Lambda}}^{(E_{S,\Lambda})}$ of its averages satisfies not only unconditional LHV constraints \[ (7) \] but also their conditional version where the LHV constants $B_{\Phi_{S,\Lambda}}^{\sup}_{\text{cond}}$ and $B_{\Phi_{S,\Lambda}}^{\inf}_{\text{cond}}$ are defined similarly to \[ (8), (9) \] but via conditional supremum and infimum.

Some of the LHV constraints \[ (7) \] may be fulfilled for a wider (than LHV) class of correlation scenarios. This is, for example, the case for the LHV constraints on joint probabilities following explicitly from nonsignaling of probability distributions. Moreover, some of constraints \[ (7) \] may be simply trivial, i.e. fulfilled for all correlation scenarios, not necessarily nonsignaling.

Each of the tight linear LHV constraints \[ (7) \] that may be violated under a non-LHV scenario is referred to as a Bell (or Bell-type) inequality.

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6This terminology formed historically, see Introduction in [5].
If outcomes at each $n$-th site are real-valued, let $\lambda_n \in \tilde{\Lambda}_n = [-1, 1]$, and we specify $\Phi_{\tilde{S}, \tilde{\Lambda}}$ for the full correlation functions, that is, for the collection $\Phi_{\tilde{S}, \tilde{\Lambda}}$ of functions $f(s_1, \ldots, s_N)(\lambda_1 \cdot \ldots \cdot \lambda_N) = \alpha(s_1, \ldots, s_N)\lambda_1 \cdot \ldots \cdot \lambda_N$, then $B_{\Phi_{\tilde{S}, \tilde{\Lambda}}}^{\sup} = -B_{\Phi_{\tilde{S}, \tilde{\Lambda}}}^{\inf}$ and a Bell inequality (7) on the full correlation functions takes the form [15]:

$$\left| \sum_{s_1, \ldots, s_N} \alpha(s_1, \ldots, s_N) \left\langle \lambda^{(s_1)}_1 \cdot \ldots \cdot \lambda^{(s_N)}_N \right\rangle_{lhv} \right| \leq \max_{\eta_n \in [-1, 1]^{S_n}, \forall n} |F_\alpha(\eta_1, \ldots, \eta_N)| = \max_{\eta_n \in [-1, 1]^{S_n}, \forall n} |F_\alpha(\eta_1, \ldots, \eta_N)|,$$

where

$$F_\alpha(\eta_1, \ldots, \eta_N) = \sum_{s_1, \ldots, s_N} \alpha(s_1, \ldots, s_N)\eta^{(s_1)}_1 \cdot \ldots \cdot \eta^{(s_N)}_N \quad (11)$$

is the $N$-linear form of $S_n$-dimensional real-valued vectors $\eta_n = (\eta^{(1)}_n, \ldots, \eta^{(S_n)}_n) \in [-1, 1]^{S_n}$.

Note that, the value of the maximum in the right-hand side of (10) does not depend on a number of measurement outcomes at each site and is determined only by the extreme values $\pm 1$ of these outcomes. Therefore, the form of each correlation Bell inequality (7) does not depend on a spectral type of outcomes at each site, in particular, on their number. This observation is rather essential since, in the polytope approach, the classification of correlation Bell inequalities essentially depends on a number of measurement outcomes at each site, see, for example, in [21].

The specification of the general representation (7) for Bell inequalities on joint probabilities is derived quite similarly and is presented by Eq. (39) in [15], incorporating in a unique manner all Bell inequalities on joint probabilities derived in the literature via the other approaches, for example, the Bell inequalities in [35].

### 3 The LqHV Modelling

As it is well known since the seminal paper [1] of Bell, the probabilistic description of an arbitrary quantum correlation scenario cannot be reproduced via a LHV model.

However, as we proved in [6, 30], the probabilistic description of every quantum correlation scenario, more generally, every nonsignaling scenario, can be correctly reproduced via a LqHV (local quasi hidden variable) probability model – the notion introduced for an arbitrary correlation scenario in [6]. Moreover, all quantum correlation scenarios on an $N$-partite state with projective quantum measurements at each site admit [8, 31] a single LqHV model.
In a LqHV model, all scenario joint probability distributions (5) admit the representation

\[ P_{(E_{S},\Lambda)}(s_{1},...,s_{N}) = \int_{\Omega} P_{1,s_{1}}(d\lambda_{1}|\omega) \cdot \ldots \cdot P_{N,s_{N}}(d\lambda_{N}|\omega) \mu_{\text{lqhv}}^{E_{S},\Lambda}(d\omega), \] (12)

which is quite similar by its form to the LHV representation (6) with only one difference – in (12), a normalized distribution \( \mu_{E_{S},\Lambda}^{\text{lqhv}} \) of variables \( \omega \in \Omega \) is real-valued and does not need to be positive.

Therefore, a LHV model (6) constitutes a particular case of a LqHV probability model (12) whenever a distribution \( \mu_{E_{S},\Lambda}^{\text{lqhv}} \) is positive. Also, the affine model [36] for a family of nonsignaling probability distributions constitutes a LqHV model of a particular type.

Clearly, a LqHV model (12) is “local” in the same sense as it was meant by Bell [1] for the LHV representation (6). The term ”quasi” in its title ”LqHV” refers only to ”hidden variables” (HV), specifically, to a possible nonpositivity of a distribution \( \mu_{E_{S},\Lambda}^{\text{lqhv}} \) of these variables.

For an arbitrary correlation scenario \( E_{S},\Lambda \), the following statements are equivalent [30]:

- the probabilistic description of a correlation scenario admits a LqHV model (12);
- a correlation scenario is nonsignaling;
- there is a real-valued distribution

\[ \tau_{E_{S},\Lambda}^{\text{lqhv}} \left( d\lambda_{1}^{(1)} \times \ldots \times d\lambda_{1}^{(s_{1})} \times \ldots \times d\lambda_{N}^{(1)} \times \ldots \times d\lambda_{N}^{(s_{N})} \right) \] (13)

of all scenario outcomes, returning each scenario joint probability distribution \( P_{(E_{S},\Lambda)}^{(E_{S},\Lambda)}(s_{1},...,s_{N}) \) as the corresponding marginal.

Note that a nonsignaling correlation scenario, which we further specify by \( E_{S,n}^{\text{ns}} \), may admit a variety of LqHV models (12).

### 4 Nonsignaling analogs and Bell’s nonlocality

Let us now construct analogs of the LHV constraints (7) for a general nonsignaling case. Substituting into averages (3) of a linear combination (2) the LqHV representation (12) for joint probability distributions \( P_{(E_{S},\Lambda)}^{(E_{S},\Lambda)}(s_{1},...,s_{N}) \), recalling for a normalized real-valued distribution \( \mu \)
the Jordan decomposition via positive distributions $\mu^{(\pm)}$:

$$\mu = \mu^{(+)} - \mu^{(-)}, \quad \mu^{(+)}(\Omega) - \mu^{(-)}(\Omega) = 1,$$

(14)

and minimizing over all possible LqHV models for $\mathcal{E}_{S,A}^{ns}$, we come to the following analogs of constraints (7) for a nonsignaling scenario $\mathcal{E}_{S,A}^{ns}$:

$$
B_{\Phi_{S,A}}^{\inf} - \frac{\gamma_{\mathcal{E}_{S,A}^{ns}} - 1}{2}(B_{\Phi_{S,A}}^{\sup} - B_{\Phi_{S,A}}^{\inf}) 
\leq B_{\Phi_{S,A}}^{\sup} + \frac{\gamma_{\mathcal{E}_{S,A}^{ns}} - 1}{2}(B_{\Phi_{S,A}}^{\sup} - B_{\Phi_{S,A}}^{\inf}),
$$

(15)

where the parameter $\gamma_{\mathcal{E}_{S,A}^{ns}}$ has the form

$$
\gamma_{\mathcal{E}_{S,A}^{ns}} = \inf_{\mu_{lhv}^{\mathcal{E}_{S,A}^{ns}}} \frac{\|\mu_{lhv}^{\mathcal{E}_{S,A}^{ns}}\|_{var}}{\var} \geq 1,
$$

(16)

with infimum taken over all possible LqHV models (12) for a scenario $\mathcal{E}_{S,A}^{ns}$ and notation $\|\mu\|_{var}$ meaning the total variation norm of a real-valued distribution $\mu$. For a normalized real-valued distribution $\mu$, this norm $\|\mu\|_{var} \geq 1$, with $\|\mu\|_{var} = 1$ if and only if $\mu$ is a probability distribution. For a discrete distribution, the total variation norm reduces to the sum of all its absolute values.

From (16) it follows that the parameter $\gamma_{\mathcal{E}_{S,A}^{ns}}$, specifying in (15) violation of a Bell inequality under a nonsignaling scenario $\mathcal{E}_{S,A}^{ns}$, does not depend on a form of this inequality and $\gamma_{\mathcal{E}_{S,A}^{ns}} = 1$ (no violation) if and only if a nonsignaling scenario $\mathcal{E}_{S,A}^{ns}$ is an LHV one. Moreover, it is easy to prove, quite similarly to our proof of Lemma 3 in [6], that the parameter $\gamma_{\mathcal{E}_{S,A}^{ns}}$, given by (16), is otherwise expressed as $\gamma$

$$
\gamma_{\mathcal{E}_{S,A}^{ns}} = \sup_{\Phi_{S,A}} \frac{1}{B_{\Phi_{S,A}}^{lhv}} \left| B_{\Phi_{S,A}}^{(\mathcal{E}_{S,A}^{ns})}\right|,
$$

(17)

that is, constitutes the maximal violation under a nonsignaling scenario $\mathcal{E}_{S,A}^{ns}$ of all general Bell inequalities for $S_n$ settings and outcomes $\lambda_n \in \Lambda_n$ at each $n$-th site.

In order to construct analogs of Bell inequalities for an arbitrary class $\mathfrak{S}_{ns}$ of nonsignaling scenarios $\mathcal{E}_{S,A}^{ns}$ with $S_1, \ldots, S_N$ settings and

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7For this decomposition see, for example, section 3 in [6].
outcome sets $\Lambda_1, \ldots, \Lambda_N$ at $N$ sites, for example, for all quantum correlation scenarios $E^o_{S,A}$ on a state $\rho$ or for all possible nonsignaling scenarios $E^{ns}_{S,A}$, we maximize (15) over all scenarios $E^{ns}_{S,A}$ and come to the following analogs of Bell inequalities (7) in a general nonsignaling case:

\[
\begin{align*}
B_{\inf}^{\Phi_{S,A}} & \leq B_{\Phi_{S,A}}^{(E^{ns}_{S,A})} = B_{\sup}^{\Phi_{S,A}} - B_{\inf}^{\Phi_{S,A}} \\
\gamma & \leq B_{\sup}^{\Phi_{S,A}} + \frac{B_{\sup}^{\Phi_{S,A}} - B_{\inf}^{\Phi_{S,A}}}{2}.
\end{align*}
\]

In (18), the LqHV parameter $\gamma_{\Phi_{S,A}}^{\Phi_{S,A}}$ is given by

\[
\gamma_{\Phi_{S,A}}^{\Phi_{S,A}} = \sup_{E^{ns}_{S,A} \in \Phi_{ns}} \gamma_{E^{ns}_{S,A}} = \sup_{E^{ns}_{S,A} \in \Phi_{ns}} \inf_{E^{ns}_{S,A} \in \Phi_{ns}} \left| \frac{\mu_{\Phi_{S,A}}^{lhv}}{E^{ns}_{S,A}} \right| \geq 1
\]

and, in view of (17), constitutes the maximal violation

\[
\gamma_{\Phi_{S,A}}^{\Phi_{S,A}} = \sup_{E^{ns}_{S,A} \in \Phi_{ns}} \frac{1}{B_{\Phi_{S,A}}^{lhv}} \left| \frac{\mu_{\Phi_{S,A}}^{lhv}}{E^{ns}_{S,A}} \right| \geq 1
\]

under nonsignaling scenarios $E^{ns}_{S,A} \in \Phi_{ns}$ of general Bell inequalities for $S_n$ settings and outcomes $\lambda_n \in \Lambda_n$ at each $n$-th site. Clearly, $\gamma_{\Phi_{S,A}}^{\Phi_{S,A}} = 1$ if and only if $\Phi_{ns}$ is a class of nonsignaling scenarios, each admitting a LHV model.

As an example, consider the nonsignaling analogs (15) for the Clauser-Horne (CH) inequalities\(^8\) on joint probabilities

\[
-1 \leq B_{CH|lh} \leq 0.
\]

These Bell inequalities correspond to the bipartite case with two settings and two outcomes per site – in notation of [35], this is the (2222) case.

From (15) and (21) it follows that, for an arbitrary class $\Phi_{ns}$ of nonsignaling scenarios (2222), the analogs of the CH inequalities take the form:

\[
-\gamma_{2222}^{\Phi_{ns}} + \frac{1}{2} \leq B_{CH|\Phi_{ns}} \leq \gamma_{2222}^{\Phi_{ns}} - \frac{1}{2}.
\]

For example, for all quantum correlation scenarios (2222), the maximal quantum Bell violation $\gamma_{2222}^{\Phi_{ns}} = \sqrt{2}$ and inequalities (22) reduce to the well-known quantum analogs of the CH inequalities.

\(^8\)For these Bell inequalities, see, for example, subsection 3.2 in [15].
From (18), (19), (20) it follows that, for a $G_{ns}$-class of nonsignaling scenarios with $S_1, \ldots, S_N$ settings but arbitrary outcome sets $\Lambda_1, \ldots, \Lambda_N$ at $N$ sites, the parameter

$$\Upsilon_{G_{ns}S_1 \times \cdots \times S_N} = \sup_{\Lambda, \mathcal{E}_{S_1, \ldots, S_N} \in \mathcal{G}_{ns}} \left( \inf_{\mathcal{E}^\text{LHV}_{S_1, \ldots, S_N}} \left\| \mu^\text{LHV}_{\mathcal{E}_{S_1, \ldots, S_N}} \right\|_{\text{var}} \right) \geq 1 \tag{23}$$

quantifies the $S_1 \times \cdots \times S_N$-setting Bell’s nonlocality whereas the parameter

$$\Upsilon_{\mathcal{E}_{ns}S_1 \times \cdots \times S_N} = \sup_{S_1, \ldots, S_N} \Upsilon_{\mathcal{G}_{ns}S_1 \times \cdots \times S_N} \geq 1 \tag{24}$$

– the full Bell’s nonlocality. Locality, the $S_1 \times \cdots \times S_N$-setting and full, corresponds to $\Upsilon_{\mathcal{G}_{ns}S_1 \times \cdots \times S_N} = 1 \text{ and } \Upsilon_{\mathcal{E}_{ns}} = 1$, respectively.

Note that if $\Upsilon_{\mathcal{G}_{ns}S_1 \times \cdots \times S_N} = 1$ for some integers $S_1, \ldots, S_N \geq 1$, then $\Upsilon_{\mathcal{G}_{ns}S'_1 \times \cdots \times S'_N} = 1$ for all positive integers $S'_1 \leq S_1, \ldots, S'_N \leq S_N$.

5 Quantum nonlocality

Consider now the nonlocality parameters $\Upsilon_{\mathcal{E}_{ns}S_1 \times \cdots \times S_N}$ and $\Upsilon_{\mathcal{E}_{ns}}$ for a particular class $\mathcal{G}_{ns}$ of nonsignaling scenarios – quantum correlation scenarios performed on a state $\rho$ on a Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$.

For this class of nonsignaling scenarios, we shortly denote

$$\Upsilon_{\mathcal{E}_{ns}S_1 \times \cdots \times S_N} \to \Upsilon_{(\rho)S_1 \times \cdots \times S_N}, \quad \Upsilon_{\mathcal{E}_{ns}} \to \Upsilon_{\rho}, \tag{25}$$

and, as it is generally accepted, refer to Bell’s nonlocality as quantum nonlocality.

According to (23), (24), an $N$-partite quantum state $\rho$ is:

- the $S_1 \times \cdots \times S_N$-setting nonlocal if and only if $\Upsilon_{(\rho)S_1 \times \cdots \times S_N} > 1$;
- fully nonlocal if and only if $\Upsilon_{\rho} > 1$.

Let us now evaluate the quantum nonlocality parameters $\Upsilon_{(\rho)S_1 \times \cdots \times S_N}$ and $\Upsilon_{\rho}$.

We recall that, for every quantum state $\rho$ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and arbitrary positive integers $S_1, \ldots, S_N \geq 1$, there exists [6, 17, 18] an $S_1 \times \cdots \times S_N$-setting source operator $T_{(\rho)S_1 \times \cdots \times S_N}$ – that is, a self-adjoint trace class dilation of a state $\rho$ to the space

$$(\mathcal{H}_1) \otimes S_1 \otimes \cdots \otimes (\mathcal{H}_N) \otimes S_N. \tag{26}$$

Clearly, $T_{(\rho)S_1 \times \cdots \times S_N} = \rho$ and $\text{tr}[T_{(\rho)S_1 \times \cdots \times S_N}] = 1$. 

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In view of the analytical upper bound (53) in [6], we have:

\[ \Upsilon_{\rho} \leq \sup_{S_1, \ldots, S_N} \left( \inf_{T_{S_1 \times \cdots \times S_N}} \| T_{S_1 \times \cdots \times S_N}^{(\rho)} \|_{\text{cov}}, \right. \]

where infimum is taken over all source operators \( T_{S_1 \times \cdots \times S_N}^{(\rho)} \) with only one setting at some \( n \)-th site and over all sites \( n = 1, \ldots, N \), and notation \( \| \cdot \|_{\text{cov}} \) means the covering norm – a new norm introduced for self-adjoint trace class operators by relation (11) in [6].

For every self-adjoint trace class operator \( W \) on a tensor product Hilbert space \( G_1 \otimes \cdots \otimes G_m \), the covering norm satisfies [6] the relation

\[ |\text{tr} [W]| \leq \| W \|_{\text{cov}} \leq \| W \|_1, \]

where \( \| \cdot \|_1 \) is the trace norm and the equality \( \| W \|_{\text{cov}} = |\text{tr} [W]| \) is true if a self-adjoint trace class operator \( W \) is tensor positive [6], that is,

\[ \text{tr} [W \{ X_1 \otimes \cdots \otimes X_m \}] \geq 0 \]

for all positive bounded operators \( X_j \) on \( G_j \), \( j = 1, \ldots, m \).

Since, for every source operator, \( \text{tr} [T_{S_1 \times \cdots \times S_N}^{(\rho)}] = 1 \), from (28) it follows that \( \| T_{S_1 \times \cdots \times S_N}^{(\rho)} \|_{\text{cov}} \geq 1, \forall T_{S_1 \times \cdots \times S_N}^{(\rho)}, \) and \( \| T_{S_1 \times \cdots \times S_N}^{(\rho)} \|_{\text{cov}} = 1 \) if \( T_{S_1 \times \cdots \times S_N}^{(\rho)} \) is tensor positive. This and the analytical upper bounds imply:

If, for an \( N \)-partite quantum state \( \rho \) and arbitrary integers \( S_1, \ldots, S_N \geq 1 \), there exists a tensor positive source operator \( T_{S_1 \times \cdots \times S_N}^{(\rho)} \) for some \( n = 1, \ldots, N \), then \( \Upsilon_{S_1 \times \cdots \times S_N}^{(\rho)} = 1 \) and this \( N \)-partite state is the \( S_1 \times \cdots \times S_N \)-setting local, that is, satisfies all general \( S'_1 \times \cdots \times S'_N \)-setting Bell inequalities with \( S'_1 \leq S_1, \ldots, S'_N \leq S_N \) measurement settings at \( N \) sites.

If, for an \( N \)-partite quantum state \( \rho \), tensor positive source operators \( T_{S_1 \times \cdots \times S_N}^{(\rho)} \) for some arbitrary \( n = 1, \ldots, N \), exist for all integers \( S_1, \ldots, S_N \geq 1 \), then \( \Upsilon_{\rho} = 1 \) and this \( N \)-partite state \( \rho \) is (fully) local, that is, satisfies all general Bell inequalities.

For an \( N \)-qudit quantum state \( \rho_{d,N} \) on \( (\mathcal{H})^{\otimes N}, \) \( d = \dim \mathcal{H} < \infty \), let us now evaluate the analytical upper bounds [27] in \( d, S \) and \( N \).
From (20) and our results in [6, 8, 28] it follows that the quantum nonlocality parameters $\Upsilon_{N}$ and $\Upsilon_{N}^{(\rho_{d,N})}$ admit the upper bounds

$$\Upsilon_{N}^{(\rho_{d,N})} \leq (2 \min\{d, S\} - 1)^{N-1},$$
$$\Upsilon_{N} \leq (2d - 1)^{N-1},$$

under all generalized $N$-partite quantum measurements. These upper bounds are attainable. For $N = d = S = 2$, the upper bound (31) is attained on the CHSH inequality. For $d = S = 2$, $N \geq 3$, it is attained on the Mermin–Klyshko inequality.

For the analysis of attainability of (31) beyond the two-qubit case, let us consider the quantum analogs of the Zohren-Gill (ZG) inequalities [32] on joint probabilities:

$$1 \leq B_{\text{ZG}} |_{\rho_{d,N}} \leq 2.$$ (32)

These Bell inequalities correspond to the bipartite case with two settings ($N = S = 2$) and $d$ outcomes at each site. For this case, the critical value in the upper bound (31) is equal to $\min\{\sqrt{d}, 3\}$ and by (18) this critical value leads to the following quantum analogs of the ZG inequalities under projective bipartite quantum measurements:

$$\frac{3 - \sqrt{d}}{2} \leq B_{\text{ZG}} |_{\rho_{d,N}} \leq \frac{3 + \sqrt{d}}{2},$$
$$0 \leq B_{\text{ZG}} |_{\rho_{d,N}} \leq 3,$$ for $d \leq 9$ (33)

$$0 \leq B_{\text{ZG}} |_{\rho_{d,N}} \leq 3,$$ for $d \geq 9$. (34)

The left-hand side inequality in (34) just coincides with inequality (8) in [32], which was conjectured by Zohren and Gill (in view of their numerical results) as the quantum analog of the Bell inequality [32] for an infinite dimensional case.

6 Conceptual issues

In the present paper, we have analyzed violation of Bell inequalities via the local probability model, the LqHV probability model [6, 30], incorporating the LHV model only as a particular case. An arbitrary correlation scenario admits a LqHV model if and only if it is nonsignaling.
The LqHV modelling framework allowed us to construct the nonsignaling analogs of Bell inequalities and to specify parameters quantifying Bell’s nonlocality, partial and full, in a general nonsignaling case. For quantum correlation scenarios on an arbitrary $N$-partite quantum state, we evaluate these nonlocality parameters analytically in terms of dilation characteristics of an $N$-partite state. For an $N$-qudit state, we also evaluate these parameters numerically, in $d, S$ and $N$.

We stress that, in a LqHV model, locality is introduced quite similarly as in a LHV model and the only difference between these two local probability models is nonpositivity of a normalized real-valued distribution $\mu_{\text{LqHV}}^{E,\Lambda}$ in the general LqHV representation. Therefore, a LHV model constitutes a particular case of a LqHV model whenever a distribution $\mu_{\text{LqHV}}^{E,\Lambda}$ is positive.

From the mathematical modelling point of view, a LqHV model reproduces correctly all scenario joint probability distributions, so that it is not important that a simulation distribution $\mu_{\text{LqHV}}^{E,\Lambda}$ may have negative values – for observed events, there are no negative probabilities.

However, from the conceptual point of view, it is important to understand a reason of nonvalidity of an LHV model in a general nonsignaling case, specifically, in a quantum case.

Bell argued that violation of the LHV statistical constraints under space-like separated quantum measurements points to violation of the EPR locality under these measurements. However, as we analyzed mathematically in section 3 of [5], for an arbitrary correlation scenario, Bell’s nonlocality does not imply violation of the EPR locality. Moreover, under space-like separated quantum measurements, the EPR locality is not violated.

Furthermore, in view of the existence of the local probability model, the LqHV model, correctly reproducing the Hilbert space description of every quantum correlation scenario, we argued in [7] that quantum violation of Bell inequalities can be explained not by violation of the EPR locality but by nonclassicality leading to violation of "classical realism" embedded into a HV model via a probability distribution. In a LqHV model, the locality is preserved but "classical realism" is replaced by "quantum realism" modeled by a real-valued distribution.

The results of the present paper on the rigorous mathematical description of Bell’s nonlocality in a general nonsignaling case via the local probability model confirm our opinion gradually formed in [4, 5, 6, 7, 8] – violation of Bell inequalities in a quantum case is not due to violation of the EPR locality conjectured by Bell [2, 3] but due to the improper HV modelling of "quantum realism".
In conclusion, we stress that the Kolmogorov probability axioms refer to the probabilistic description of a single measurement and are true for a measurement of any type, classical or quantum. But a correlation scenario is described by a family of joint measurements and, though, for each joint measurement, the Kolmogorov probability axioms are fulfilled, the probabilistic description of the whole correlation scenario does not need to be described in terms of a single probability space inherent to the Kolmogorov probability model. In quantum theory, the Kolmogorov probability model is referred to as the HV model.

However, as we proved in [6, 30], the probabilistic description of every nonsignaling correlation scenario, in particular, every quantum correlation scenario, does admit the LqHV probability model. By taking off the specification on locality in the LqHV model and generalizing it for the probabilistic description of an arbitrary measurement situation, we introduced in [30] the notion of the qHV (quasi hidden variable) model – equivalently, the quasi-classical probability model – incorporating the Kolmogorov probability model only as a particular case. We proved [31] that, in its context-invariant form, the qHV model reproduces the probabilistic description of all joint von Neumann measurements on an arbitrary Hilbert space.

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