Gallai's path decomposition conjecture for graphs of small maximum degree

Bonamy, Marthe; Perrett, Thomas J.

Published in:
Discrete Mathematics

Link to article, DOI:
10.1016/j.disc.2019.01.005

Publication date:
2019

Document Version
Peer reviewed version

Citation (APA):
Bonamy, M., & Perrett, T. J. (2019). Gallai's path decomposition conjecture for graphs of small maximum degree. Discrete Mathematics, 342(5), 1293-1299. https://doi.org/10.1016/j.disc.2019.01.005

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
Gallai’s path decomposition conjecture
for graphs of small maximum degree

Marthe Bonamy\textsuperscript{*} and Thomas J. Perrett\textsuperscript{†}

September 2016

Abstract
Gallai’s path decomposition conjecture states that the edges of any connected graph on $n$ vertices can be decomposed into at most $\frac{n+1}{2}$ paths. We confirm that conjecture for all graphs with maximum degree at most five.

1 Introduction

A decomposition $\mathcal{D}$ of a graph $G$ is a collection of subgraphs of $G$ such that each edge belongs to precisely one graph in $\mathcal{D}$. A path decomposition is a decomposition $\mathcal{D}$ such that every subgraph in $\mathcal{D}$ is a path. If $G$ has a path decomposition $\mathcal{D}$ such that $|\mathcal{D}| = k$, then we say that $G$ can be decomposed into $k$ paths. In answer to a question of Erdős, Gallai conjectured the following, see [4].

Conjecture 1.1. [4] Every connected graph on $n$ vertices can be decomposed into $\left\lceil \frac{n}{2} \right\rceil$ paths.

Gallai’s conjecture is easily seen to be sharp: If $G$ is a graph in which every vertex has odd degree, then in any path decomposition of $G$ each vertex must be the endpoint of some path, and so at least $\left\lfloor \frac{n}{2} \right\rfloor$ paths are required. Lovász [4] proved that every graph on $n$ vertices has a decomposition $\mathcal{D}$ consisting of paths and cycles, and such that $|\mathcal{D}| = \left\lfloor \frac{n}{2} \right\rfloor$. By an argument similar to the above, it follows that in a graph with at most one vertex of even degree, such a decomposition must be a path decomposition. Thus, Gallai’s conjecture holds for all graphs with at most one vertex of even degree.

Let $G_E$ denote the subgraph of $G$ induced by the vertices of even degree. Building on Lovász’s result, Conjecture 1.1 has been proved for several classes of graphs defined by imposing some structure on $G_E$. The first result of this kind was obtained by Pyber.

Theorem 1.1. [1] If $G$ is a graph on $n$ vertices such that $G_E$ is a forest, then $G$ can be decomposed into $\left\lfloor \frac{n}{2} \right\rfloor$ paths.

Later, Theorem 1.1 was strengthened by Fan, who proved the following.

Theorem 1.2. [2] If $G$ is a graph on $n$ vertices such that each block of $G_E$ is a triangle free graph of maximum degree at most 3, then $G$ can be decomposed into $\left\lfloor \frac{n}{2} \right\rfloor$ paths.

\textsuperscript{*}CNRS, LaBRI, Université de Bordeaux, France
\textsuperscript{†} Technical University of Denmark, Denmark
Gallai’s conjecture is also known to hold for a variety of other graph classes. In 1988, Favaron and Koudier [6] proved that the conjecture holds for graphs where the degree of every vertex is either 2 or 4. More recently, Botler and Jiménez [3] proved that the conjecture holds for 2k-regular graphs of large girth and admitting a pair of disjoint perfect matchings. Jiménez and Wakabayashi [7] showed that the conjecture holds for a subclass of planar, triangle-free graphs satisfying a distance condition on the vertices of odd degree. Finally, it was shown by Geng, Fang and Li [5], that the conjecture holds for maximal outerplanar graphs. In this article, we prove that Gallai’s conjecture holds for the class of graphs with maximum degree at most 5.

**Theorem 1.3.** Let $G$ be a connected graph on $n$ vertices. If $\Delta(G) \leq 5$, then $G$ admits a path decomposition into $\lceil \frac{n}{2} \rceil$ paths.

To prove Theorem 1.3, we show that if $G$ is a smallest counterexample, then $G$ cannot contain one of 5 configurations. This restriction is enough to show that $G_E$ is a forest, whence the result follows by Theorem 1.1. It seems that proving Theorem 1.3 for graphs of maximum degree 6 will require some new ideas. However, we think the approach of considering graphs of bounded maximum degree allows step-by-step improvements which could eventually lead to a general solution.

In proving special cases of Conjecture 1.1, the presence of a ceiling in the bound brings with it a number of technical complications. It is therefore tempting to explore ways of proving a stronger, ceiling-free version except in a few special cases. We say a graph is an odd semi-clique if it is obtained from a clique on $2^k + 1$ vertices by deleting at most $k - 1$ edges. By a simple counting argument, we can see that an odd semi-clique on $2^k + 1$ vertices does not admit a path decomposition into $k$ paths. It is natural to ask if these are the only obstructions:

**Question 1.1.** Does every connected graph $G$ that is not an odd semi-clique admit a path decomposition into $\lfloor \frac{|V(G)|}{2} \rfloor$ paths?

## 2 Definitions and notation

All graphs in this article are finite and simple, that is they contain no loops or multiple edges. We say that a path decomposition $\mathcal{D}$ of a graph $G$ is good if $|\mathcal{D}| \leq \lceil \frac{|V(G)|}{2} \rceil$.

In figures we make use of the following conventions: Solid black circles denote vertices for which all incident edges are depicted. White hollow circles denote vertices which may have other, undepicted incident edges. Vertices containing a number indicate a vertex of that specific degree. A dotted line between two vertices indicates that those vertices are non-adjacent.

We will often modify a path decomposition of a graph $G$ to give a path decomposition of another graph $G'$. To describe these modifications we use a number of fixed expressions, which we formally define here. Let $\mathcal{D}$ be a path decomposition of $G$. Let $P \in \mathcal{D}$ be a path and $Q$ be a subpath of $P$. If $R$ is a path in $G'$ with the same end vertices as $P$, we say that we replace $Q$ with $R$ to mean that we define a new path $P' = P - Q + R$ and redefine $\mathcal{D}$ to be the collection $\mathcal{D} - P + P'$. If $R$ is a path in $G'$ with an endpoint in common with $P$, we say that we extend $P$ with $R$ to mean that we define a new path $P' = P + R$ and redefine $\mathcal{D}$ to be the collection $\mathcal{D} - P + P'$. For a vertex $u$ on $P$, we say that we split $P$ at $u$ to mean that we define paths $P_1$ and $P_2$ such that $P_1 \cup P_2 = P$ and
$P_1 \cap P_2 = u$, and redefine $D$ to be the collection $D - P + P_1 + P_2$. Finally, for a path $R$ in $G'$, we say that we add the path $R$ to mean that we redefine $D$ to be the collection $D + R$.

**Proposition 2.1.** Let $G$ and $G'$ be two graphs such that $|V(G)| \geq |V(G')| + 2$, and let $D$ be a path decomposition of $G$. If there is a good path decomposition $D'$ of $G'$ and $|D| \leq |D'| + 1$ then $D$ is a good path decomposition of $G$.

Proof. Let $|V(G)| = n$. We have $|D| \leq |D'| + 1 \leq \lceil \frac{n-2}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$. Thus $D$ is a good path decomposition of $G$. \(\square\)

**Proposition 2.2.** Let $G, G_1$ and $G_2$ be three graphs such that $|V(G)| \geq |V(G_1)| + |V(G_2)|$, and let $D$ be a path decomposition of $G$. If there are good path decompositions $D_1$ and $D_2$ of $G_1$ and $G_2$ (respectively) and $|D| \leq |D_1| + |D_2| - 1$, then $D$ is a good path decomposition of $G$.

Proof. Let $G, G_1$ and $G_2$ have $n, n_1$ and $n_2$ vertices respectively. We have $|D| \leq |D_1| + |D_2| - 1 \leq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 1 \leq \lceil \frac{n}{2} \rceil$. Thus $D$ is a good path decomposition of $G$. \(\square\)

**Proposition 2.3.** Let $G, G_1$ and $G_2$ be three graphs such that $|V(G)| \geq |V(G_1)| + |V(G_2)| + 1$, and let $D$ be a path decomposition of $G$. If there are good path decompositions $D_1$ and $D_2$ of $G_1$ and $G_2$ (respectively) and $|D| \leq |D_1| + |D_2|$, then $D$ is a good path decomposition of $G$.

Proof. Let $G, G_1$ and $G_2$ have $n, n_1$ and $n_2$ vertices respectively. We have $|D| \leq |D_1| + |D_2| \leq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil \leq \lceil \frac{n}{2} \rceil$. Thus $D$ is a good path decomposition of $G$. \(\square\)

## 3 Main Result

Let $G$ be a graph with $\Delta(G) \leq k$. We first prove that a number of configurations are reducible in $G$, if Gallai’s conjecture holds for all smaller graphs of maximum degree $k$.

**Lemma 3.1.** Let $k \in \mathbb{N}$. Let $G$ be a connected graph with maximum degree $\Delta(G) \leq k$, and suppose that $G$ does not admit a good path decomposition. If $G$ is vertex minimal with these properties, then $G$ does not contain any of the following configurations (see Figure 1):

- $C_1$: A vertex of degree 2 whose neighbors are not adjacent.
- $C_2$: A cut-edge $uv$ such that $d(u)$ and $d(v)$ are even.
- $C_3$: An edge $uv$ such that $u$ and $v$ have precisely 2 common neighbors, and $d(u) = d(v) = 4$.
- $C_4$: An edge $uv$ such that $d(u) = d(v) = 4$, and for $t_1, t_2, t_3$ (resp. $w_1, w_2, w_3$) the three other neighbors of $u$ (resp. $v$), the pairs $t_1t_2$ and $w_1w_2$ are not edges and $t_3 \neq w_3$.
- $C_5$: A triangle $uvw$ such that $d(u) = 4$ and $d(v), d(w) \in \{2, 4\}$.

Proof.
Claim 1. $G$ does not contain the configuration $C_1$.

Proof. Suppose that the claim is false. Let $u$ be the vertex of degree 2 with $N(u) = \{v, w\}$, and let $G'$ be the graph $G - u + vw$. Since $v$ and $w$ are non-adjacent, $G'$ is a simple graph. By the minimality of $G$, we have that $G'$ admits a good path decomposition $D'$. By Proposition 2.3, we obtain a good path decomposition of $G$ by replacing the edge $vw$ with the path $vw$ (see Figure 2).

This contradicts the assumption that $G$ has no such decomposition.

Claim 2. $G$ does not contain the configuration $C_2$.

Proof. Suppose that the claim is false. Deleting $uv$ results in two connected graphs $G_1$ and $G_2$, containing $u$ and $v$ respectively. By the minimality of $G$, both $G_1$ and $G_2$ admit good path decompositions $D_1$ and $D_2$. To obtain a path decomposition of $G$, note that, since $u$ has odd degree in $G_1$, there is a path $P_u \in D_1$ ending at $u$. Similarly, there is a path $P_v \in D_2$ ending at $v$. Now let $D$ be the path decomposition of $G$ formed by taking the union $D_1 \cup D_2$, deleting $P_u$ and $P_v$, and adding a new path $P = P_u + uv + P_v$ (see Figure 3). By Proposition 2.2, $D$ is a good path decomposition of $G$, a contradiction.

Claim 3. $G$ does not contain the configuration $C_3$.

Proof. Suppose that the claim is false. Let $x$ and $y$ be the common neighbours of $u$ and $v$. Since $d(u) = d(v) = 4$, both $u$ and $v$ both have precisely one other neighbour. Let these vertices be $u'$ and $v'$ respectively. Since $u$ and $v$ have precisely two common neighbours, we have that $u' \neq v'$. Suppose
first that at most one of the edges $xu’, u’y, yv’, v’x$ is present in $G$, say $xu’$. If $G - u - v$ is connected, then let $G’$ be the graph $G - u - v + u’y + v’x$. Otherwise let $G’ = G - u - v + u’y + v’x + xy$. Note that $G’$ is connected, and so by the minimality of $G$, it admits a good path decomposition. Now, replace $v’x$ by $xv’$ and replace $u’y$ by $u’uy$. Furthermore, if $xy \in E(G’) \setminus E(G)$, then replace $xy$ by $xuvy$. Otherwise add a new path $xuvy$ to the decomposition. By Proposition 2.1, and since we add at most one new path, the resulting decomposition is a good path decomposition of $G$. This contradicts the assumption that $G$ has no such decomposition.

Next, suppose that $xu’, u’y, yv’, v’x \in E(G)$, so the graph $G’ = G - u - v$ is connected. By the minimality of $G$, the graph $G’$ has a good path decomposition. Now replace the edge $xu’$ with the path $xvuu’$, and add a new path $u’xuvv’$ to the decomposition. By Proposition 2.1, and since we add at most one new path, the resulting decomposition is a good path decomposition of $G$, contradicting the assumption.

Finally, suppose that precisely two or three of the edges $xu’, u’y, yv’, v’x$ are present in $G$. As a consequence, from the set $\{xu’, u’y, yv’, v’x\} \setminus E(G)$, we may choose an edge, $xu’$ say, such that the graph $G’ = G - u - v + xu’$ is connected. By the minimality of $G$, the graph $G’$ has a good path decomposition. Now replace $xu’$ by $xuu’$, and add a new path $xuvv’$ to the decomposition. Again, by Proposition 2.1, and since we add at most one new path, the resulting decomposition is a good path decomposition of $G$, contradicting the assumption.

\textbf{Claim 4.} \textit{G does not contain the configuration $C_4$.}

\textbf{Proof.} Suppose that the claim is false. Since $G$ does not contain Configuration $C_3$, the vertices $u$ and $v$ do not have precisely two common neighbours. First suppose that $u$ and $v$ have 3 common neighbours $x, y$ and $z$. In this case, since there is a pair of non-adjacent vertices amongst $N(u) \setminus \{v\}$, we may assume $xy \notin E(G)$. Furthermore, by the definition of Configuration $C_4$, the third vertex $z$ is non-adjacent to at least one of $x$ or $y$. We conclude that there are two non-edges amongst $x, y$ and $z$, say these are $xy$ and $yz$. Let $G’$ be the graph $G - u - v + xy + yz$. It is easy to see that $G’$ is connected. By the minimality of $G$, the graph $G’$ has a good path decomposition. In this decomposition, replace $xy$ by $xuy$ and replace $yz$ by $yvz$. Finally, add a new path $xuvz$ (see Figure 4). This gives a good path decomposition of $G$, a contradiction. We may thus assume that $u$ and $v$ have at most one common neighbour.

![Figure 4: The reduction of $C_4$ in the case where $u$ and $v$ have three common neighbors.](image)

We now consider three cases depending on the structure of $G - \{u, v\}$. In each case we assume the previous ones do not apply (up to symmetry).

1. \textit{Assume that $G - u$ has at least three connected components.} Because $uv$ is not a cut-edge, the component of $G - u$ containing $v$ contains at least one other neighbor of $u$. Thus $G - u$
has precisely three components, and $t_1$ and $t_2$ lie in different components of $G - u$. Let $G'$ be the graph formed from $G - u$ by adding the edge $t_1t_2$. Thus $G'$ has two components $G_1$ and $G_2$, and by the minimality of $G$, both have good path decompositions $D_1$ and $D_2$. Without loss of generality we suppose $G_2$ contains $v$. Let $P \in D_1$ be the path containing the edge $t_1t_2$. Furthermore, let $P_1$ and $P_2$ be the possibly empty subpaths of $P - t_1t_2$ containing $t_1$ and $t_2$ respectively. Note that since $v$ has degree 3 in $G'$, there is some path $Q \in D_2$ which ends at $v$. We construct a path decomposition of $G$ by taking the union $D_1 \cup D_2$ and replacing $P$ and $Q$ with the paths $P_1 + t_1uv + Q$ and $P_2 + t_2ut_3$. By Proposition 2.3 and since we introduced no new paths, the resulting path decomposition is good, a contradiction.

2. Assume that $G - \{u, v\}$ has at least four connected components. Since both $G - u$ and $G - v$ have at most two connected components, there are precisely four connected components $C_1, C_2, C_3$ and $C_4$. Furthermore, two of these components contain both a neighbour of $u$ and a neighbour of $v$, one component contains only a neighbour of $u$, and one component contains only a neighbour of $v$. Relabelling if necessary, we may suppose that $t_1, w_1 \in C_1$, $t_2, w_2 \in C_2$, $t_3 \in C_3$ and $w_3 \in C_4$. This relabelling preserves the fact that $t_1t_2, w_1w_2 \not\in E(G)$ and $t_3 \neq w_3$. Consider the graph $G_1$ obtained from $C_1$ and $C_2$ by adding the edges $t_1t_2$ and $w_1w_2$. Similarly, consider the graph $G_2$ obtained from $C_3$ and $C_4$ by adding the edge $t_3w_3$. By the minimality of $G$, we obtain good path decompositions of $G_1$ and $G_2$, which we merge in the obvious way. The edge $t_1t_2$ is replaced with $t_1ut_3$, $w_1w_2$ with $w_1w_3$, and $t_3w_3$ with $t_3w_3$ to obtain a path decomposition of $G$. By Proposition 2.3 this yields a good path decomposition of $G$.

3. Now $G - \{u, v\}$ has at most three connected components, and each of $G - u$ and $G - v$ has at most two connected components. Let $T = \{t_1, t_2, t_3\}$ and $W = \{w_1, w_2, w_3\}$. We claim that we can relabel the vertices in $T$ and $W$ such that the graph $G - u - v + t_1t_2 + w_1w_2$ is connected and the properties that $t_1t_2, w_1w_2 \not\in E(G)$ and $t_3 \neq w_3$ are preserved. Indeed if $u$ and $v$ have a common neighbour, let $t \in T$ and $w \in W$ be such that $t = w$. Otherwise let $t = t_1$ and $w = w_1$. Suppose first that $t$ and $w$ lie in the same component of $G - u - v$. Since $G - u - v$ has at most 3 components, and $G - u$ and $G - v$ have at most 2 components, there are non edges $tt'$ and $wu'$ for some $t' \in T$ and $w' \in W$ such that $G - u - v + tt' + uu'$ is connected. Furthermore, since $t$ and $w$ are the only possible common neighbours of $u$ and $v$, we have that the single vertices in $T \setminus \{t, t'\}$ and $W \setminus \{w, w'\}$ are not equal. Thus, letting $t_1 = t, t_2 = t', w_1 = w, w_2 = w'$ and setting $t_3$ and $w_3$ to be the remaining vertices gives the desired relabeling.

Suppose now that $t$ and $w$ lie in different components of $G - u - v$. In particular this implies that $T \cap W = \emptyset$. Again, since $G - u - v$ has at most 3 components, and $G - u$ and $G - v$ have at most 2 components, there are non-edges $e_T$ and $e_W$ amongst the vertices of $T$ and $W$ respectively, such that $G - u - v + e_T + e_W$ is connected. We relabel the vertices in $T$ and $W$ such that $t_1$ and $t_2$ are the endpoints of $e_T$, $w_1$ and $w_2$ are the endpoints of $e_W$, and $t_3$ and $w_3$ are the remaining vertices. Since $T \cap W = \emptyset$, we have that $t_3 \neq w_3$ are required.

Let $G'$ be the graph obtained from $G - \{u, v\}$ by adding the edges $t_1t_2$ and $w_1w_2$. By the argument above, $G'$ is connected, and so by the minimality of $G$, there is a good path decomposition of $G'$. We obtain a path decomposition of $G$ by replacing $t_1t_2$ with $t_1ut_2$ and
$w_1w_2$ with $w_1vw_2$, and adding the path $t_3uw_3$. Note that since $t_3 \neq w_3$ the latter is really a path. By Proposition [2.1] and since we add at most one new path, this yields a good path decomposition of $G$.

Figure 5: The reduction of $C_4$ in the connected case.

Claim 5. $G$ does not contain the configuration $C_5$.

Proof. We first consider the case where a pair in $\{u, v, w\}$, say $\{u, v\}$, has three common neighbors. Let $x$ and $y$ be the two neighbors of $\{u, v\}$ besides $w$. We argue that $wxy$ induces a triangle. Indeed, first assume there are at least two edges missing, say $xw, wy \notin E(G)$. Consider the graph $G'' = G + xw + wy$, note that it is connected, and consider a good path decomposition of it. We obtain a path decomposition of $G$ by replacing the edge $xw$ with $xuw$, replacing the edge $wy$ with $wvy$, and adding the path $xvuy$, see Figure 6. By Proposition [2.1] this yields a good path decomposition of $G$.

Figure 6: The reduction of $C_5$ when $u$ and $v$ have three common neighbors that induce at least two non-edges.

Assume now that there is precisely one edge missing, say the edge $xy$. Consider $G'$, the graph obtained from $G - \{u, v\}$ by adding the edge $xy$. If $G'$ is connected, then by the minimality of $G$, it has a good path decomposition. From this, we obtain a path decomposition of $G$ by replacing the edge $xy$ with $xuvy$ and adding the path $xvwuy$, see Figure 7. By Proposition [2.1] this yields a good path decomposition of $G$.

Figure 7: The reduction of $C_5$ when $u$ and $v$ have three common neighbors that induce precisely one non-edge.

Therefore $x, y$ and $w$ induce a triangle. Let $G' = G - \{u, v\}$, and note that $G'$ is connected. Thus, by the minimality of $G$, the graph $G'$ admits a good path decomposition $D'$. We obtain
a path decomposition of $G$ as follows: First assume without loss of generality that $xy$ and $wy$ do not belong to the same path of $D'$. Let $Q'$ be the path of $D'$ containing the edge $xw$, and $Q = Q' - xw + xu$. We consider $D'' = D' - Q' + Q$. Let $P''$ be the path of $D''$ containing the edge $xy$. We write $P'' = P'_1 + xy + P'_2$, where $P'_1$ and $P'_2$ may be empty paths. Set $P_1 = P'_1 + xyuw$ and $P_2 = P'_2 + yuwvxw$. Note that $P_1$ and $P_2$ are paths even if $P'_1$ also contains the edge $xu$ or in other words $P'_1 = Q$. Finally, let $R'$ be the path of $D''$ containing the edge $yw$, and set $R = R' + wu$. We note that $D = D'' - P'' - R' + P_1 + P_2 + R$ is a path decomposition of $G$, with precisely one more path than $D'$, see Figure 8. Thus $D$ is a good path decomposition by Proposition 2.1, a contradiction. Therefore no pair in $\{u, v, w\}$ has three common neighbors.

![Figure 8: The reduction of $C_5$ when $u$ and $v$ have three common neighbors that induce a triangle. We assume $P'$ and $R'$ are distinct, though $Q'$ might be the same as $R'$ or $P'$ or be altogether distinct from both.](image)

Now, since $d(u), d(v), d(w) \leq 4$ and by Claim 3, we conclude that no pair of vertices in $\{u, v, w\}$ has a common neighbor other than the third vertex. If they exist, let $\{x_1, x_2\}, \{y_1, y_2\}$ and $\{z_1, z_2\}$ be the two other neighbors of $u, v$ and $w$ respectively. We consider three cases.

1. **Assume first that one of $v$ and $w$ has degree 2**, say $d(v) = 2$. Let $G'$ be the graph obtained from $G - v$ by contracting the edge $uw$. Note that $G'$ is connected and $|V(G')| = |V(G)| - 2$. By the minimality of $G$, there is a good path decomposition $D'$ of $G'$. To obtain a path decomposition of $G$, we consider two cases depending on whether $ux_1$ and $ux_2$ belong to the same path in $D'$, see Figures 9 and 10. If they do not, then replace $ux_1$ with the path $wux_1$, and replace $ux_2$ with the path $wux_2$. However, if $ux_1$ and $ux_2$ belong to the same path $P \in D'$, then split $P$ at $u$ into two paths $P_1$ and $P_2$. Extend $P_1$ with the edge $uw$ and extend $P_2$ with the path $uwv$. Note that no edge incident to $w$ is in $P_1$ or $P_2$. By Proposition 2.1 and since we created at most one new path, this yields a good path decomposition of $G$.

2. **Assume that one of the edges $ux_1, ux_2, vy_1, vy_2, wz_1, wz_2$ is not a cut-edge**. Assume without loss of generality that $ux_1$ is such an edge. Let $G'$ be the graph obtained from $G - u$ by contracting the edge $vw$ to a vertex $s$, and adding the edge $sx_2$. Note that $G'$ is connected and $|V(G')| = |V(G)| - 2$, so by the minimality of $G$, there is a good path decomposition $D'$ of $G'$.

We obtain a path decomposition of $G$ as follows. We first replace any subpath of the form $yz$, $y \in \{y_1, y_2\}$, $z \in \{z_1, z_2\}$ with $ywz$ (preferably) or with $ywuz$ (if there are two such subpaths). We then replace any subpath of the form $x_2st$, $t \in \{y_1, y_2, z_1, z_2\}$, with $x_2urt$ where $r$ is the vertex of $\{v, w\}$ adjacent to $t$. We replace any remaining edge of the form $ts$, $t \in \{x_2, y_1, y_2, z_1, z_2\}$ with $tr$, where $r$ is the vertex of $\{u, v, w\}$ adjacent to $t$. Let $D''$ be the
resulting collection of disjoint paths in $G$. Note that since $d(s) = 5$, there is a path $P$ in $D'$ that ends in $s$, thus a path $P'$ in $D''$ that ends in $r \in \{u, v, w\}$. We consider the set of edges of $G$ that do not belong to a path in $D''$. If that set does not induce a path, then we extend $P'$ to $wu$ or $wv$. Note that this guarantees the only remaining edges induce a path $Q$, which we add to the path collection. By Proposition 2.1 and since we added at most one new path, this yields a good path decomposition of $G$.

![Figure 9](image1.png)

Figure 9: The reduction of $C_5$ when $u$ and $w$ have precisely one common neighbor and $d(v) = 2$.

3. Now $d(u) = d(v) = d(w) = 4$ and every edge with precisely one endpoint in $\{u, v, w\}$ is a cut-edge. Consider the graph $G'$ obtained from $G - \{u, v, w\}$ by adding the three edges $x_1y_1, x_2y_2$, and $z_1z_2$. Note that $G'$ has precisely three connected components $G_1, G_2$, and $G_3$. By the minimality of $G$, there are good path decompositions of $G_1$, $G_2$ and $G_3$. We obtain a path decomposition of $G$ by replacing $x_1y_1$ with the path $x_1uwv_1$, replacing $x_2y_2$ with $x_2uvy_2$, and replacing $z_1z_2$ with the path $z_1wz_2$ (see Figure 10). These paths are all distinct since the edges $x_1y_1, x_2y_2$, and $z_1z_2$ belong to different components of $G'$. Note that the total number of paths involved in the resulting path decomposition of $G$ is at most $|V(G_1)| + |V(G_2)| + |V(G_3)| = |V(G)|$, thus it is a good path decomposition.

![Figure 10](image2.png)

Figure 10: An example of the reduction of $C_5$ when $d(u) = d(v) = d(w) = 4$, the triangle $(u, v, w)$ is adjacent to no other triangle and some edge in $E(\{u, v, w\}, \{x_1, x_2, y_1, y_2, z_1, z_2\})$ is not a cut-edge.

By Claims 1, 2, 3, 4 and 5, the lemma statement holds. □

Recall that $G_E$ denotes the graph induced on the vertices of even degree in $G$. 

9
clique, which again gives a contradiction since
with this extra information, the argument above shows that, in fact, if any edge amongst
that \( w \) is not in \( E \)
configuration \( G \). But now \( G \) and \( w \) graph
Proof of Theorem 1.3. Let \( G \) be a connected graph such that \( G \notin \{K_3, K_5\} \). If \( \Delta(G) \leq 5 \) and \( G \) does not contain configurations \( C_1, \ldots, C_5 \), then the graph \( G_E \) is a forest.

Proof. Let \( H = G_E \) and suppose for a contradiction that \( H \) contains a cycle \( C \). Suppose further that there is \( v \in V(C) \) with \( d(v) = 2 \), and let \( N(v) = \{u, w\} \). Since \( C \) is a cycle in \( H \), we have that \( d(u), d(w) \in \{2, 4\} \). Furthermore, since \( G \) does not contain configuration \( C_1 \), we have that \( uw \in E(G) \). Now \( G \neq K_3 \), so at least one of \( u \) and \( w \) has degree 4. It follows that \( u, v \) and \( w \) form configuration \( C_5 \), a contradiction. Thus, if \( C \) is a cycle in \( H \), then \( d_G(v) = 4 \) for all vertices \( v \in V(C) \). Since \( G \) does not contain configuration \( C_5 \), it immediately follows that \( |C| > 3 \).

Let \( uv \) be an edge of \( C \). Let \( t_1, t_2, t_3 \) be the neighbours of \( u \) apart from \( v \) and let \( w_1, w_2, w_3 \) be the neighbours of \( v \) apart from \( u \). Note that, since \( uv \) is an edge of \( C \), at least one of \( t_1, t_2, t_3 \) has degree 4. Similarly, at least one of \( w_1, w_2, w_3 \) has degree 4. Now, \( u \) and \( v \) do not have 3 common neighbours, since otherwise \( G \) contains configuration \( C_3 \), a contradiction. Furthermore, since \( G \) does not contain configuration \( C_3 \), the vertices \( u \) and \( v \) have at most one common neighbour. Thus, in what follows, we allow the possibility that \( t_1 = w_1 \), but always assume that \( t_2, t_3 \notin \{w_1, w_2, w_3\} \) and \( w_2, w_3 \notin \{t_1, t_2, t_3\} \).

Suppose first that \( t_1 t_2 \notin E(G) \). Since \( G \) does not contain configuration \( C_4 \), we must have that \( w_1 w_2, w_2 w_3, w_1 w_3 \in E(G) \). Otherwise, since \( t_3 \notin \{w_1, w_2, w_3\} \), we have that \( G \) contains configuration \( C_4 \), a contradiction. But now the vertices \( w_1, w_2, w_3 \) form a clique, and at least one of them has degree 4. It follows that \( G \) contains configuration \( C_3 \), a contradiction.

It follows that all of the edges \( t_1 t_2, t_1 t_3, w_1 w_2, w_1 w_3 \in E(G) \). As a consequence, \( t_1 \neq w_1 \), otherwise this vertex would have degree 6, which is larger than \( \Delta(G) \). Thus \( \{t_1, t_2, t_3\} \cap \{w_1, w_2, w_3\} = \emptyset \).

With this extra information, the argument above shows that, in fact, if any edge amongst \( t_1, t_2, t_3 \) is not in \( E(G) \), then \( w_1, w_2, w_3 \) induce a clique. Thus, either \( \{t_1, t_2, t_3\} \) or \( \{w_1, w_2, w_3\} \) induce a clique, which again gives a contradiction since \( G \) does not contain configuration \( C_3 \).

The proof of Theorem 1.3 now follows easily.

Proof of Theorem 1.3. Let \( G \) be a smallest counterexample to the theorem. By Lemma 3.1, the graph \( G \) does not contain configurations \( C_1, \ldots, C_5 \). Thus, by Lemma 3.2, the graph \( G_E \) is a forest. But now \( G \) admits a good path decomposition by Theorem 1.1, a contradiction.
4 Acknowledgements

The authors wish to thank François Dross for useful discussions. The second author was supported by ERC Advanced Grant GRACOL, project number 320812.

References

[1] L. Pyber, Covering the edges of a connected graph by paths, J. Combin. Theory Ser. B, 66 (1996) 152–159.

[2] G. Fan, Path decompositions and Gallai’s conjecture, J. Combin. Theory Ser. B, 93 (2005) 117–125.

[3] F. Botler and A. Jiménez, On path decompositions of 2k-regular graphs, Electron. Notes Discrete Math., 50 (2015) 163–168.

[4] L. Lovász, On covering of graphs, in: Theory of Graphs (ed. P. Erdős, G. Katona), Akad. Kiadó, Budapest (1968) 231–236.

[5] X. Geng, M. Fang and D. Li, Gallai’s conjecture for outerplanar graphs, J. Interdiscip. Math. 18 (2015) 593–598.

[6] O. Favaron and M. Kouider, Path partitions and cycle partitions of Eulerian graphs of maximum degree 4, Studia Sci. Math. Hungar., 23 (1988) 237-244.

[7] A. Jiménez and Y. Wakabayashi, On path-cycle decompositions of triangle-free graphs. Preprint.