Node Placement and Distributed Magnetic Beamforming Optimization for Wireless Power Transfer

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Abstract—In multiple-input single-output (MISO) wireless power transfer (WPT) via magnetic resonant coupling, multiple transmitters are deployed to enhance the efficiency of power transfer to a single receiver by jointly adapting their source currents/voltages so as to constructively combine the induced magnetic fields at the receiver, a technique known as magnetic beamforming. In practice, since the transmitters (power chargers) are usually at fixed locations and the receiver (e.g., mobile phone) is desired to be freely located in a target region for wireless charging, its received power can fluctuate significantly over locations even with adaptive magnetic beamforming applied. To achieve uniform power coverage, the transmitters need to be optimally placed in the region such that a minimum charging power can be achieved for the receiver regardless of its location, which motivates this paper. First, we derive the optimal magnetic beamforming solution in closed-form for a distributed MISO WPT system with fixed locations of the transmitters and receiver to maximize the deliverable power to the receiver subject to a given sum-power constraint at all transmitters. By applying adaptive magnetic beamforming based on this optimal solution, we then jointly optimize the locations of all transmitters to maximize the minimum power deliverable to the receiver over a given one-dimensional (1D) region. Although the formulated problem is non-convex, we propose an iterative algorithm for solving it efficiently. Extensive simulation results are provided which show the significant performance gains by the proposed design with optimized transmitter locations and magnetic beamforming as compared to other benchmark schemes with non-adaptive and/or heuristic transmitter current allocation and node placement. Last, we extend the node placement problem to the case of two-dimensional (2D) region, and propose efficient designs for this case.

Index Terms—Near-field wireless power transfer, distributed wireless charging, magnetic resonant coupling, magnetic beamforming, node placement optimization, uniform power coverage.

I. INTRODUCTION

Near-field wireless power transfer (WPT) has drawn significant interests recently due to its high efficiency for delivering power to electric loads without the need for any wire. Inductive coupling (IC) [1]–[3] is the conventional method to realize near-field WPT for short-range applications typically within centimeters. The wireless power consortium (WPC) that developed the “Qi” standard [4] is the main industrial organization for commercializing wireless charging based on IC. Recently, magnetic resonant coupling (MRC) [5]–[9] has been applied to significantly enhance the efficiency and range of WPT compared to IC, thus opening up a broader avenue for practical applications such as biomedical device charging [10], [11], electric vehicle charging [12], [13], etc. In MRC-enabled WPT (termed MRC-WPT), compensators each being a capacitor of variable capacity are embedded in the electric circuits of individual power transmitters and receivers to tune their oscillating frequencies to be the same as the operating frequency adopted by all input voltage/current sources so as to achieve resonance. Alternatively, resonators each of which constitutes a simple RLC circuit resonating at the source frequency can be employed in close proximity of the coils of any off-resonance transmitters/receivers to help efficiently transfer power between them. With MRC, the total reactive power consumption in the system is effectively reduced due to resonance and thus high power transfer efficiency is achieved over longer distance than the conventional IC. The preliminary experiments in [5] show that MRC enables a single transmitter to transfer 60 watts of power wirelessly with 40%–50% efficiency to a single receiver at a distance of about 2 meters. More recent experiment and simulation results on the controlling, scalability analysis, and performance characterization of MRC-WPT systems have been reported in the literature (see e.g. [14]–[16]). Formed after the merging between the alliance for wireless power (AW4P) that developed the “Rezence” specification and the power matters alliance (PMA), the new AirFuel alliance is currently the main industrial organization for developing wireless charging based on MRC [17]. The Rezence specification advocates a superior charging range, the capability of charging multiple devices simultaneously, and the use of two-way communication via e.g. Bluetooth between the charger unit and devices for real-time charging control. These features make Rezence and its future extensions a promising technology for high-performance near-field wireless charging.

In the current Rezence specification, one transmitter with a single coil is used in the power transmitting unit, i.e., only the single-input multiple-output (SIMO) MRC-WPT is considered, as shown in Fig. 1(a). Although this centralized WPT system performs well when the receivers are all sufficiently close to the transmitter, the power delivered to each receiver decays significantly as it moves away from the transmitter. This thus motivates distributed WPT where the single centralized transmitter coil is divided into multiple coils (i.e., separate transmitters) each with a smaller size (coil radius), which are then placed in different locations to cover a given target region, as shown in Fig. 1(b). By coordinating the transmissions of distributed coils via jointly allocating their
source currents/voltages, in [18] it is shown that their induced magnetic fields at one or more receivers can be constructively combined, thus achieving a magnetic beamforming gain in a manner analogous to multi-antenna beamforming in far-field wireless communication or WPT [19]–[21]. Besides, distributed WPT shortens the maximum distance from each receiver to its nearest transmitter(s) in a given region compared to centralized WPT, thus achieves more uniform charging performance over the region.

Generally speaking, there are similarities as well as differences between magnetic beamforming in near-field WPT and its far-field counterpart. For instance, as shown in Fig. 2(a), the objective of the beamforming design in far-field multiple-input single-output (MISO) WPT is to ensure that the electromagnetic (EM) waves propagated from different transmitter (TX) antennas are aligned at the single receiver antenna, i.e., their arriving phases are all identical so that they can be coherently added to maximize the received signal amplitude/power. This is practically realized by adjusting the phase and amplitude of the transmitted signal at each antenna. In contrast, as shown in Fig. 2(b), magnetic beamforming in near-field MISO WPT aims to achieve that the polarities of the magnetic fields generated by different transmitters are all identical (i.e., upward or downward) at a given receiver location and hence these magnetic fields can be constructively added to maximize the magnetic flux intensity at the receiver coil. This can be realized by adjusting the direction and magnitude of the current flowing in each transmitter coil. For example, in Fig. 2(b), the current at coil 2 should flow in the opposite direction of that in coil 1 in order for their generated magnetic fields to be both downward at the receiver coil. However, different from far-field WPT in which the receivers are passive and have no effect on the power radiated by the transmitters, the cross-coupling effect among the transmitters and receivers in near-field WPT is generally strong. As a result, changing the input current/voltage (load resistance) at one transmitter (receiver) not only affects its own transmitted (received) power, but also influences the power of all other transmitters and receivers in general. Therefore, the modeling and design of magnetic beamforming in near-field WPT are generally different from its far-field counterpart based on EM wave propagation.

The optimal magnetic beamforming design in MISO MRC-WPT systems is investigated in [22], [23]. Specifically, [22] formulates a convex optimization problem to jointly design the source currents at all transmitters to maximize the WPT efficiency subject to a given minimum power deliverable to the load at a single receiver. On the other hand, [23] jointly optimizes the transmitter currents to maximize the deliverable power to a single receiver by considering a sum-power constraint for all transmitters as well as practical peak voltage and current constraints at individual transmitters. Recently, the selective WPT technique is proposed for MISO MRC-WPT systems where one transmitter is selected at each time (i.e., transmitter selection) to deliver wireless power to a single receiver, and a simple control mechanism is devised for performance optimization [24]. Selective WPT is also proposed for SIMO MRC-WPT systems in [25], [26]. This technique delivers power to only one selected receiver (i.e., receiver selection) at each time to eliminate the magnetic cross-coupling effect among different receivers and hence achieve more balanced power transfer to them, assuming that their natural frequencies are set well separated from each other. Alternatively, [27] proposes to jointly optimize the load resistance of all receivers in a SIMO MRC-WPT system to exploit the magnetic cross-coupling effect to alleviate the near-far issue by delivering balanced power to individual receivers regardless of their distances to the power transmitter. In general, selective WPT requires a simpler control mechanism than magnetic beamforming, while its performance is also limited since only one pair of transmitter and receiver is allowed for power transfer at each time. In contrast, magnetic beamforming enables multiple transmitters to send power to one or more receivers simultaneously by properly assigning the source currents (load resistance values) at individual transmitters (receivers), thus in general achieving better performance than the simple transmitter/receiver selection.

The studies in [18], [22]–[27] show promising directions to improve the efficiency as well as performance fairness in MRC-WPT systems, but all of them assume that the transmitters and receivers are at fixed locations in a target region. In practice, each wireless device (e.g., mobile phone) is desired to be freely located in any position in the region (e.g., on a charging table) when it is being charged, for the convenience of its user. In this case, if the locations of the transmitters are not appropriately designed in a MISO MRC-WPT system, the deliverable power to the receiver can fluctuate significantly over different locations. Such power fluctuation degrades the quality of service, since the power requirement of the receiver load may not be satisfied at all locations in the region, even when magnetic beamforming is...
applied to adapt to the location of the receiver. To achieve uniform power coverage, one possible method is to allow the transmitters to track the location of the receiver and move toward it in real time, which however may not be feasible as in practice transmitters like power chargers are usually at fixed locations. Alternatively, the transmitters can be optimally placed at their initial deployment such that a minimum charging power is ensured to be achievable for the receiver regardless of its location in the region. This thus motivates our work in this paper to optimize the transmitter locations in a MISO MRC-WPT system jointly with receiver location-adaptive magnetic beamforming to maximize the minimum power deliverable to the receiver over a target region.

The main results of this paper are summarized as follows:

- First, we formulate the magnetic beamforming problem for a MISO MRC-WPT system with distributed transmitters to maximize the deliverable power to the load at a single receiver subject to a given transmitters’ sum-power constraint, by assuming that the power transmitters and receiver are all at fixed locations. We derive the closed-form solution to the magnetic beamforming problem in terms of the mutual inductances between the transmitters and the receiver. Our solution shows that the optimal current allocated to each transmitter is proportional to the mutual inductance between its coil and that of the receiver. For the special case when the transmitters are sufficiently separated from each other, we show that the optimal magnetic beamforming reduces to the simple transmitter selection scheme \[24\] where all power is allocated to one single transmitter that has the highest mutual inductance value with the receiver.

- To demonstrate the performance gain of magnetic beamforming, we compare it to an uncoordinated MISO WPT system with equal current allocation over all transmitters \[23\], as well as the transmitter selection scheme \[24\]. We also compare the performance of distributed WPT with magnetic beamforming versus centralized WPT subject to the same total size of transmitter coils.

- Next, by applying adaptive magnetic beamforming with the optimal solution derived, we formulate the node placement problem to jointly optimize the transmitter locations to maximize the minimum power deliverable to the receiver in a given one-dimensional (1D) region, i.e., a line of finite length where the receiver can be located in any point in the line. Although simplified, the 1D case is studied first for the purpose of exposition as well as drawing useful insights. The formulated problem is non-convex, while we propose an iterative algorithm for solving it approximately by utilizing the fact that the transmitters should be symmetrically located over the mid-point of the target line to maximize the minimum deliverable power. We present extensive simulation results to verify the effectiveness of our proposed transmitter location optimization algorithm in improving both the minimum and average deliverable power over the target line as compared to a heuristic design that uniformly locates the transmitters.

- At last, we extend the node placement problem to the more general two-dimensional (2D) target region case, i.e., a disk in 2D with a finite radius. A practical scenario for this setup could be a round table with built-in wireless chargers mounted below its surface, and a receiver that can be freely placed on its surface. Using an example of five transmitters, we show that the design approach for the 1D case can be similarly applied to obtain the optimal locations of the transmitters under the 2D setup with magnetic beamforming to maximize the minimum deliverable power over the target disk region, by exploiting the property of rotational symmetry in 2D.

The rest of this paper is organized as follows. Section II introduces the system model. Section III formulates the magnetic beamforming problem and presents its optimal solution. Section IV formulates the node placement problem for the 1D target region case, and presents an iterative algorithm for solving it. Section V presents simulation results for the 1D case. Section VI extends the node placement problem to the 2D target region case with an example of five transmitters. Finally, we conclude the paper in Section VII.

II. SYSTEM MODEL

In this paper, we consider a MISO MRC-WPT system with \(N \geq 1\) identical single-coil transmitters, indexed by \(n, n \in \{1, \ldots, N\}\), and a single-coil receiver, indexed by 0 for convenience. It is assumed that all the transmitters and receiver are each equipped with a Bluetooth communication module to enable information exchange among them to achieve coordinated WPT \[17\]. Each transmitter \(n\) is connected to a stable energy source supplying sinusoidal voltage over time given by \(\tilde{v}_n(t) = \Re\{v_ne^{jwt}\}\), where \(v_n\) is a complex number denoting the steady state voltage in phasor form and \(w > 0\) denotes its angular frequency. Note that \(\Re\{\cdot\}\) represents the real part of a complex number. On the other hand, the receiver is connected to an electric load, e.g., battery of a mobile phone. Let \(i_n(t) = \Re\{i_ne^{jwt}\}\), where \(i_n = i_n^r + j i_n^i\), with \(j^2 = -1\), denotes the steady state current at transmitter \(n\). This current produces a time-varying magnetic flux in the transmitter coil, which passes through the receiver coil and induces time-varying current in it. We denote \(\tilde{i}_0(t) = \Re\{i_0e^{jwt}\}\), with \(i_0 = i_0^r + j i_0^i\), as the steady state current at the receiver.

First, we consider the case of WPT in 1D. As shown in Fig. 3, we assume that the receiver can move horizontally along a line that lies in the \((x, y)\) plane with its \((x, y)\)-coordinates satisfying \(|x| \leq d\), with \(d > 0\), and \(y = 0\) while it has a fixed height of \(z = z_0, z_0 > 0\). We denote this line as the 1D target line. Note that \(|\cdot|\) denotes the absolute value of a real/complex number. The transmitters are also installed horizontally at fixed locations along a line that is in parallel with and below the target line, at a fixed height of \(z = 0\). Let \(x_n\) with \(|x_n| \leq d\) denote the location of transmitter \(n\) (receiver) over the \(x\)-axis. In this paper, we consider that \(x_n\)’s are symmetric over \(x = 0\).\(^1\) Hence, we consider the following two cases for the symmetric deployment of the transmitters.

\(^1\)We will show later in Section IV that such symmetric structure of the transmitters maximizes the minimum deliverable power over the target line.
Denote \( c_{tx} \) and the capacity of the compensator in each transmitter (receiver), respectively. As shown in Fig. 3(a), we consider that the transmitters and receiver are all located horizontally in parallel \((x, y)\) planes at heights \(z = 0\) and \(z = z_0\), respectively, hence their orientations are identical. In this case, from the so-called Conway’s mutual inductance formula \([30]\), we have
\[
 h_{nk} = \mu \pi b_{tx}^2 c_{coil,tx}^2 \int_0^\infty J_0 (d_{nk} u) \left( J_1 (e_{coil,tx} u) \right)^2 du,
\]
\[
 h_{n0} = \mu \pi b_{tx} b_{rx} e_{coil,tx} e_{coil,rx} \int_0^\infty J_0 (d_{n0} u) J_1 (e_{coil,tx} u) J_1 (e_{coil,rx} u) e^{-z_0 u} du,
\]
where \( d_{nk} = |x_n - x_k|,\) \( d_{n0} = |x_n - x_0|,\) and \( J_n (u) = \sum_{m=0}^\infty (-1)^m (u/2)^{2m+n}/(m! (m+n)!))\) is the Bessel function of the first kind of order \( \alpha \in \{0, 1\} \). \( d_{nk} \) denotes the factorial of a positive integer. The integration terms in \([1]\) and \([2]\) can be computed numerically, while there are no closed-form analytical expressions for them. In practice, the transmitters and receiver commonly use small coils for WPT; therefore, \( h_{n0} \) in \([2]\) can be simplified as follows.

**Lemma 2.1:** If \( e_{coil,tx}, e_{coil,rx} \ll z_0 \), we have
\[
 h_{n0} \approx \beta - \frac{2 \pi^2 - d_{n0}^2}{\sqrt{(x_0^2 + n_0^2)^2}},\]
where \( \beta = \mu \pi b_{tx} b_{rx} e_{coil,tx}^2 e_{coil,rx}^2 / 4 \) is a constant with the given coil parameters.

**Proof:** Please see Appendix A.

To validate the accuracy of the proposed approximation in \([3]\), we consider Case 2 in Fig. 3(b) with \( N = 5 \) identical transmitters, \( d = 1 \text{m}, \) and variable \( z_0, \) where the physical and electrical characteristics of the coils in the transmitters and receiver are given in Tables I and II (see Section III-B), respectively. We assume that the transmitters are uniformly located over \(|x| \leq 1 \text{m},\) with \( x_1 = x_3 = 0.5 \text{m}, x_2 = x_4 = 1 \text{m},\) and \( x_0 = 0.\)
and $x_5 = 0$. Figs. 4a) and 4b) compare the actual and approximated values of the mutual inductance between transmitter 1 and the receiver, $h_{10}$, versus the receiver’s x-coordinate $x_0$ under heights of $z_0 = 0.2$ m and $z_0 = 0.4$ m, respectively. It is observed that the approximation is tight in general; whereas there are discrepancies at $x_0 = 0$. It is also observed that the discrepancies decrease when $z_0$ increases. The similar result can be obtained for the mutual inductance between other transmitters and the receiver, while the peak value of the mutual inductance shifts over the x-axis accordingly, i.e., it moves from $x_0 = x_1 = 0$ to $x_0 = x_n$ when transmitter $n$ is considered instead of transmitter 1. In this paper, we use the approximation in (3) to formulate the node placement optimization problems in Sections IV and VI while the actual values given by (2) are used for all simulations to achieve the best accuracy. The approximation in (3) is acceptable in our case, since the design objective is to maximize the minimum transferable power to the receiver load, while such minimum occurs when the receiver is sufficiently away from all transmitters. In this case, from Fig. 4, it is observed that the approximation is indeed much tighter when $x_0$ deviates from zero.

By applying Kirchhoff’s circuit laws to the electric circuits of the transmitters and receiver in our considered MRC-WPT system shown in Fig. 5, we obtain

$$\begin{align*}
r_{tx}i_n - jw h_{n0}i_0 + jw \sum_{k=1, k \neq n}^N h_{nk}i_k &= v_n, \quad n = 1, \ldots, N, \quad (4) \\
r_{tx}i_0 - jw \sum_{n=1}^N h_{n0}i_n &= 0. \quad (5)
\end{align*}$$

Let $p_n$ and $p_0$ denote the power drawn from the energy source at transmitter $n$ and that delivered to the load at the single receiver. In practice, we have $p_n = \text{Re}\{v_n i_n^*\}$ and $p_0 = r_{tx} |i_0|^2$, where $(\cdot)^*$ denotes the conjugate of a complex number. With the results in (4) and (5), we thus obtain

$$\begin{align*}
p_n &= \left( r_{tx} + \frac{w^2}{r_{tx}} h_{n0}^2 \right) |i_n|^2 + \frac{w^2}{r_{tx}} \sum_{k=1, k \neq n}^N h_{nk} h_{k0} \left( i_n \hat{\gamma}_k + \hat{i}_n \gamma_k \right) \\
&\quad + w \sum_{k=1, k \neq n}^N h_{nk} \left( i_n \hat{\gamma}_k - \hat{i}_n \gamma_k \right), \quad (6) \\
p_0 &= \frac{w^2 r_{tx}}{r_{tx}} \left( \sum_{n=1}^N h_{n0} i_n \right)^2 + \left( \sum_{n=1}^N h_{n0} \hat{i}_n \right)^2. \quad (7)
\end{align*}$$

Accordingly, the sum-power drawn from all transmitters’ sources can be derived as

$$\begin{align*}
p_{\text{sum}} &= \sum_{n=1}^N p_n = \\
r_{tx} \sum_{n=1}^N |i_n|^2 + \frac{w^2}{r_{tx}} \left( \sum_{n=1}^N h_{n0} i_n \right)^2 + \left( \sum_{n=1}^N h_{n0} \hat{i}_n \right)^2. \quad (8)
\end{align*}$$

From (6) and (8), it follows that with fixed $i_n$’s, the power consumption of each individual transmitter depends on all the mutual inductances between the transmitters and the receiver, $h_{n0}$’s, as well as those between any pair of transmitters, $h_{kn}$’s, while their sum-power consumed depends on $h_{n0}$’s only. From (7) and (8), it also follows that the real-part currents $i_n$’s and the imaginary-part currents $\hat{i}_n$’s contribute in the same way to $p_0$ or $p_{\text{sum}}$. Therefore, in this paper, we set $\hat{i}_n = 0$, $n = 1, \ldots, N$, and focus on designing $i_n$’s. Moreover, since each

\text{Note that in this paper, we only consider the sum-power constraint for all transmitters. In the case that the peak power constraint for each individual transmitter is considered, both $i_n$’s and $\hat{i}_n$’s should be optimized jointly.}
\[ h_{n0} \text{ is a function of both } x_0 \text{ and } x_n \text{ with given } z_0 \text{ (see, e.g., (3)), we re-express } p_0 \text{ and } p_{\text{sum}} \text{ in (7) and (8) as functions of } x_0, x_n \text{’s, and } \tilde{t}_n \text{’s as}
\]
\[ p_0(x_0, \{x_n\}, \{\tilde{t}_n\}) = \frac{w^2 r_{\text{tx},1}}{r_{\text{tx}}^2} \left( \sum_{n=1}^{N} h_{n0}\tilde{t}_n \right)^2, \]
\[ p_{\text{sum}}(x_0, \{x_n\}, \{\tilde{t}_n\}) = r_{\text{tx}} \sum_{n=1}^{N} \tilde{t}_n^2 + \frac{w^2}{r_{\text{tx}}} \left( \sum_{n=1}^{N} h_{n0}\tilde{t}_n \right)^2. \]

Next, we introduce four metrics to evaluate the performance of the MRC-WPT system, which are the average value, the minimum value, the maximum value, and the min-max ratio of the deliverable power to the receiver load over the target line (or target region in general), defined as
\[ p_{\text{avg}}(\{x_n\}, \{\tilde{t}_n\}) = \int_{d}^{d} p_0(x_0, \{x_n\}, \{\tilde{t}_n\}) dx_0, \]
\[ p_{\text{min}}(\{x_n\}, \{\tilde{t}_n\}) = \min_{|x_0|\leq d} p_0(x_0, \{x_n\}, \{\tilde{t}_n\}), \]
\[ p_{\text{max}}(\{x_n\}, \{\tilde{t}_n\}) = \max_{|x_0|\leq d} p_0(x_0, \{x_n\}, \{\tilde{t}_n\}), \]
\[ \xi(\{x_n\}, \{\tilde{t}_n\}) = \frac{p_{\text{min}}(\{x_n\}, \{\tilde{t}_n\})}{p_{\text{max}}(\{x_n\}, \{\tilde{t}_n\})}. \]

In practice, it is desirable to have both large \( p_{\text{avg}} \) and \( p_{\text{max}} \) to maximize the WPT efficiency, and yet have acceptably high \( p_{\text{min}} \) to achieve uniform performance over the target region. However, in general, there are trade-offs in achieving these objectives at the same time, e.g., maximizing \( p_{\text{max}} \) versus \( p_{\text{min}} \). In the rest of this paper, we first design the magnetic beamforming via adjusting \( \tilde{t}_n \)’s by assuming fixed locations of the transmitters and receiver \( (x_n \text{’s and } x_0) \) to maximize the deliverable power subject to a given sum-power constraint for all transmitters. Next, with the obtained optimal magnetic beamforming solution, we optimize the transmitter locations \( x_n \text{’s} \) to maximize the minimum power deliverable to the receiver over the target region, i.e., \( p_{\text{min}} \) given in (12), for both the cases of 1D and 2D target regions, respectively.

### III. Optimal Magnetic Beamforming

In this section, we first present the results on the magnetic beamforming optimization. Then we use a numerical example to demonstrate the performance advantages of optimal distributed magnetic beamforming.

#### A. Problem Formulation and Solution

Assume that \( x_n \text{’s} \) and \( x_0 \) are given, and hence the mutual inductance values \( h_{n0} \text{’s} \) are known. We formulate the magnetic beamforming problem for designing the transmitter currents \( \tilde{t}_n \)’s to maximize the deliverable power to the receiver load, \( p_0 \) given in (9), subject to a maximum sum-power constraint at all transmitters, denoted by \( p_{\text{max}} > 0 \), as follows.

\[ (P1): \max_{\{\tilde{t}_n\}} \frac{w^2 r_{\text{tx},1}}{r_{\text{tx}}^2} \left( \sum_{n=1}^{N} h_{n0}\tilde{t}_n \right)^2 \]
\[ \text{s.t. } r_{\text{tx}} \sum_{n=1}^{N} \tilde{t}_n^2 + \frac{w^2}{r_{\text{tx}}} \left( \sum_{n=1}^{N} h_{n0}\tilde{t}_n \right)^2 \leq p_{\text{max}}. \]

(P1) is a non-convex quadratically constrained quadratic programming (QCQP) problem [33], since its objective is to maximize a convex quadratic function in (13). However, we obtain the optimal solution to (P1) in the following proposition.

**Proposition 3.1:** The optimal solution to (P1) is given by \( \tilde{t}_n^* \) and \( r_{\text{tx}}^* \) as
\[ \tilde{t}_n^* = \frac{h_{n0}}{\sqrt{p_{\text{max}}} \left( \sum_{k=1}^{N} h_{k0}^2 \right)^{1/2}}, \]
\[ r_{\text{tx}}^* = \frac{\sum_{k=1}^{N} h_{k0}^2}{\sum_{k=1}^{N} h_{k0}^2}. \]

**Proof:** Please see Appendix B.

**Remark:** The magnetic beamforming for the case of general complex-valued mutual inductance and/or unequal transmitter parameters is also recently investigated in [22], [23]. However, in Proposition 6.1, the beamforming solution is derived for the special case of identical transmitters with real-valued mutual inductance, which has a simpler structure compared to that in [22], [23] and thus facilitates our node placement design to be discussed later in Sections VII and VIII.

From (17), it follows that the current allocated to each transmitter \( n \) is proportional to the mutual inductance between its coil and that of the receiver, \( h_{n0} \). This is due to the fact that in (17), the denominator is the same for all transmitters, while only the numerator changes linearly with \( h_{n0} \). Moreover, it can be seen that when there exists an \( n \) such that \( |h_{n0}| \geq |h_{k0}| \), \( \forall k \neq n \), then \( \tilde{t}_k^* \approx 0 \). This means that all transmit power is allocated to transmitter \( n \) which has the dominant mutual inductance magnitude with the receiver (e.g., when the receiver is directly above transmitter \( n \) and more far apart from its adjacent transmitters), i.e., the transmitter selection technique [24] is optimal. To implement the optimal magnetic beamforming solution in practice, each transmitter \( n \) needs to estimate the mutual inductance between its coil and that of the receiver, \( h_{n0} \), in real time [27], and send it to a central controller via e.g. the Bluetooth communication considered in the Rezence specification [17]. Given the information received from all transmitters, the central controller computes the optimal transmitter currents \( \tilde{t}_n \)’s and sends them to the individual transmitters for implementing distributed magnetic beamforming. As shown in Fig. 3, [2] it is more convenient to use voltage source than current source at each transmitter in practice. With \( \tilde{t}_n \)’s derived, the optimal receiver current \( i_0^* \) can be obtained from (5) as
\[ i_0^* = \frac{\left( \sum_{n=1}^{N} h_{n0}\tilde{t}_n \right)^2}{r_{\text{rx}}}. \]

By substituting \( \tilde{t}_n^* \) and \( i_0^* \) into (4), one can compute the optimal source voltages \( v_n^* \)’s that generate the optimal currents \( \tilde{t}_n^* \) and \( i_0^* \) at the transmitters and receiver, respectively, for practical implementation. Specifically, for each transmitter \( n \), we have
\[ v_n^* = r_{\text{tx}} \tilde{t}_n^* + \frac{w^2 h_{n0}}{r_{\text{rx}}} \sum_{k=1}^{N} h_{k0} \tilde{t}_k^* + jw \sum_{k=1, k\neq n}^{N} h_{nk} \tilde{t}_k^*. \]
Note that the obtained $\xi_n^*$'s, $v_n^*$'s, and $i_0^*$ always satisfy the Kirchhoff’s circuit laws given in (4) and (5).

Next, by substituting $i_n = i_n^*$, $n = 1, \ldots, N$, in (6), the power delivered to the load with optimal magnetic beamforming is given by

$$p_0^* (x_0, \{x_n\}) = p_0 (x_0, \{x_n\}; \{i_n^*\}) = \frac{r_{tx,l}}{r_{tx}} \left(1 - \frac{1}{1 + \frac{w^2}{r_{tx} l_{tx}} \sum_{n=1}^{N} h_{w0}^2} \right) p_{\text{max}}. \quad (19)$$

From (19), it follows that the deliverable power with optimal magnetic beamforming is a function of $h_{w0}^2$'s, hence invariant to the signs of individual $h_{w0}$'s. This is expected, since magnetic beamforming ensures that the magnetic fields generated by different transmitters are constructively added at the receiver.

### B. Numerical Example

We consider an MRC-WPT system with $N = 5$ identical transmitters and a single receiver that is connected a load with resistance $r_{tx,l} = 100\Omega$. The physical and electrical characteristics of coils in the transmitters and receiver are given in Tables I and II respectively. The material of wire used for manufacturing coils is assumed to be copper. We set $x_0 = 0.2\text{m}$, $d = 1\text{m}$ (i.e., the line length is 2m in total), $w = 42.6 \times 10^6\text{rad/sec}$ [51], and $p_{\text{max}} = 30\text{W}$. In this example, we assume that transmitters are uniformly located over $|x| \leq 1\text{m}$, with $x_1 = -x_3 = 0.5\text{m}, x_2 = -x_4 = 1\text{m}$, and $x_5 = 0$. For performance comparison, we consider the uncoordinated WPT with equal current allocation over all transmitters, as well as the transmitter selection technique which only selects the transmitter with the largest mutual inductance (squared) value with the receiver for WPT with the full transmit power, $p_{\text{max}}$.

Fig. 6 compares the deliverable load power $p_0$ given in (2) versus the receiver location $x_0$ by three schemes: equal (transmitter) current with uniform (transmitter) location (ECUL), optimal (transmitter) current with uniform (transmitter) location (OCUL), and transmitter selection with uniform (transmitter) location (TSUL). It is observed that ECUL in general delivers higher power to the load than OCUL and TSUL, and also achieves a larger minimum power over the receiver location $x_0$. It is also observed that the three schemes all tend to deliver more power to the load when the receiver is in close proximity of one of the transmitters at $x_0 = 0, x_0 = \pm 0.5\text{m}$, and $x_0 = \pm 1\text{m}$. Furthermore, it is observed that TSUL performs quite close to OCUL except in the middle areas between any two adjacent transmitters, where the minimum deliverable power occurs. This observation is expected since when the receiver is in the middle of any two adjacent transmitters, optimal magnetic beamforming with both transmitters delivering power to the receiver load achieves a more pronounced combining gain as compared to the transmitter selection with only one of the two transmitters selected for WPT.

Next, we show the performance of centralized WPT, where a single transmitter is located at $x_1 = 0$ which sends wireless power to a receiver moving along the target line. For this centralized transmitter case, we set $b_h = 400$ turns and $\epsilon_{\text{coil,tx}} = 250\text{mm}$, where the radius of its coil is $N = 5$.

### TABLE I

| Coil   | Radius $r_{\text{coil,tx}}$ ($\text{mm}$) | Number of turns $b_h$ | Wire size $r_{\text{wire,tx}}$ ($\text{mm}$) | Wire resistivity $\sigma_{\text{tx}}$ ($\Omega/m$) |
|--------|----------------------------------------|-----------------------|---------------------------------------------|---------------------------------------------|
| Transmitter | 50                                      | 400                   | 0.1                                         | $1.68 \times 10^{-8}$                        |
| Receiver | 25                                      | 200                   | 0.1                                         | $1.68 \times 10^{-8}$                        |

### TABLE II

| Coil   | Internal resistance $r_{\text{tx}}$ ($\Omega$) | Self-inductance $l_{\text{tx}}$ ($\mu\text{H}$) | Series compensator $c_{\text{tx}}$ ($\mu\text{F}$) |
|--------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| Transmitter | 67.20                                        | 63.27                                         | 8.71                                          |
| Receiver | 16.80                                        | 7.04                                          | 78.29                                         |

### Fig. 6. The load power profile by distributed WPT.

### Fig. 7. The load power profile by centralized WPT.

### TABLE III

| Schemes   | $P_{0,\text{avg}}$ ($\text{W}$) | $P_{0,\text{min}}$ ($\text{W}$) | $P_{0,\text{max}}$ ($\text{W}$) | $\xi$ ($\%$) |
|-----------|---------------------------------|---------------------------------|---------------------------------|--------------|
| Distributed | OCUL                             | 21.54                           | 5.91                            | 25.54        | 23.14        |
|           | ECUL                             | 16.79                           | 3.13                            | 24.94        | 5.41         |
|           | TSUL                             | 21.41                           | 3.23                            | 25.54        | 12.69        |
| Centralized | 18.47                           | 0.00                            | 25.67                           | 0.00         |

This table shows the performance comparison between distributed versus centralized WPT.
times larger than that of each transmitter in the case of distributed WPT (i.e., 50mm) for fair comparison. Fig. 7 plots \( p_0 \) for centralized WPT versus \( x_0 \). It is observed that the deliverable power to the load is zero at \( x_0 = \pm 0.389 \), while its global and local maximums occur at \( x_0 = 0 \) and \( x_0 = 0.514 \), respectively. Note that from (2), it follows \( h_10 = 0 \) at \( x_0 = 0.389 \); as a result, by setting \( h_10 = 0 \) in (7), we have \( p_0 \) = 0, regardless of the transmit current.

The details of performance comparison between distributed versus centralized WPT in terms of the four metrics introduced in Section II (see (11)–(14)) are given in Table III. It is observed that distributed WPT with OCUL and TSUL achieves the global and local maximums occur at \( x_0 = 0 \) and \( x_0 = 0.514 \), respectively. We leave other possible node placement locations in general. We leave other possible node placement problem formulations to our future work.

IV. NODE PLACEMENT OPTIMIZATION IN 1D

In this section, we first present the node placement optimization problem for the 1D target line, and then propose an iterative algorithm to solve it.

A. Problem Formulation

Let \( \tau \) denote the minimum deliverable power to the load over the target line (see Fig. 3). The node placement problem is thus formulated as

\[
(P2): \max_{\tau, \{x_n\}} \tau
\]

\[
s.t. \quad p_0^*(x_0, \{x_n\}) \geq \tau, \quad |x_0| \leq d,
\]

\[
|x_n| \leq d, \quad n = 1, \ldots, N,
\]

with \( p_0^* \) given in (19). First, it can be easily shown by contradiction that the optimal solution \( x_n \)’s to (P2) must be symmetric over \( x = 0 \), as shown in Fig. 3 With symmetric transmitter locations, then it follows that the load power distribution over the target line is also symmetric over \( x = 0 \); as a result, the constraint (21) only needs to be considered over \( 0 \leq x_0 \leq d \). With the above observations, we simplify (P2) for the cases of even and odd \( N \), respectively, as follows. When \( N \) is even, we have

\[
(P2 - EvenN) : \max_{\tau, \{d_n\}} \tau
\]

\[
s.t. \quad \sum_{n=1}^{M} f_{z_0}(d_n, x_0) = \frac{2z_0^2 - (d_n + x_0)^2}{z_0^2 + (d_n + x_0)^2}^2, \quad \tau < d_n \leq d, \quad n = 1, \ldots, M,
\]

with

\[
f_{z_0}(d_n, x_0) = \frac{2z_0^2 - (d_n - x_0)^2}{z_0^2 + (d_n - x_0)^2}^2 + \frac{2z_0^2 - (d_n + x_0)^2}{z_0^2 + (d_n + x_0)^2}^2
\]

\[
\tau
\]

\[
\text{if } \tau < \frac{r_{\text{tx}, l}}{r_{\text{tx}}} p_{\text{max}} \text{ otherwise.}
\]

Note that since it can be easily verified that the constraint in (21) is infeasible regardless of \( x_0 \) when \( \tau \geq \frac{r_{\text{tx}, l} p_{\text{max}}}{r_{\text{tx}}} \), we define \( g(\tau) = \infty \) for \( \tau \geq \frac{r_{\text{tx}, l} p_{\text{max}}}{r_{\text{tx}}} \) in (24) for convenience. On the other hand, when \( N \) is odd, we have

\[
(P2 - OddN) : \max_{\tau, \{d_n\}} \tau
\]

\[
s.t. \quad \frac{f_{z_0}(0, x_0)}{2} + \sum_{n=1}^{M} f_{z_0}(d_n, x_0) \geq g(\tau), \quad 0 \leq x_0 \leq d,
\]

\[
0 \leq d_n \leq d, \quad n = 1, \ldots, M.
\]

B. Proposed Iterative Algorithm

In this subsection, we focus on the problem (P2–EvenN) for the even \( N \) case, while the proposed algorithm can be similarly applied for (P2–OddN) in the odd \( N \) case. In (P2–EvenN), we need to find the largest \( \tau = 0 \leq \tau \leq p_{\text{max}} \) under which the problem is feasible over all possible transmitter (one-sided) locations \( d_n \)’s. To this end, we apply the bisection method to find the largest \( \tau \) by using the fact that if (P2–EvenN) is not feasible for a certain \( \hat{\tau} = 0 \leq \hat{\tau} \leq p_{\text{max}} \), then it cannot be feasible for \( \hat{\tau} < \tau \leq p_{\text{max}} \). Similarly, if (P2–EvenN) is feasible for \( \hat{\tau} \), then it must be feasible for \( 0 \leq \tau < \hat{\tau} \). The detail of our proposed algorithm is given in the following.

Initialize \( \tau = 0 \) and \( \hat{\tau} = p_{\text{max}} \). At each iteration, we first set \( \tau = (\tau + \hat{\tau})/2 \), and test the feasibility of (P2–EvenN) given \( \tau \) by considering the following feasibility problem.

\[
(P2F - EvenN) : \text{Find} \{0 \leq d_n \leq d, \quad n = 1, \ldots, M\}
\]

\[
s.t. \quad \sum_{n=1}^{M} f_{z_0}(d_n, x_0) \geq g(\tau), \quad 0 \leq x_0 \leq d,
\]

\[
0 \leq d_n \leq d, \quad n = 1, \ldots, M
\]

If (P2F–EvenN) is feasible, we save its solution as \( d_n^* \), \( n = 1, \ldots, M \), and update \( \tau = \tau \) to search for larger values of \( \tau \) in the next iteration. Otherwise, if (P2F–EvenN) is infeasible, we update \( \tau = \tau \) to search for smaller values of \( \tau \) in the next iteration. We stop the search when \( \tau - \hat{\tau} \leq \epsilon \), where \( \epsilon > 0 \) is a small constant controlling the algorithm accuracy. It can be easily shown that the algorithm converges after about \( \log_2(p_{\text{max}}/\epsilon) \) iterations. After convergence, we return \( d_n^* \)’s as the solution to (P2–EvenN), and set \( x_n^* = -x_n^*_{M+n} = d_n^* \), \( n = 1, \ldots, M \), as the solution to (P2) for the even \( N \) case.
TABLE IV

| Algorithm 1                      |
|---------------------------------|
| a) Initialize $\varepsilon > 0$, $\delta > 0$, $itr_{\text{max}} \geq 1$, $\text{f}_{\text{max}} \geq 1$, $\tau = 0$, and $\tau_{\text{max}} = \text{f}_{\text{max}}$. |
| b) While $\tau - \tau_{\text{max}} > \varepsilon$ do: |
|   1) Set $\tau = (\tau - \tau_{\text{max}})/2$. |
|   2) Set $\text{Flag} = 0$, $itr = 1$, $\text{rpt} = 1$, and $d_n = nd/M$, $n = 1, \ldots, M$. |
|   • While $\text{Flag} = 0$, $itr < itr_{\text{max}}$, and $\text{rpt} \leq \text{rpt}_{\text{max}}$: |
|     ◦ Given $d_n$'s, check the constraint (24). If it holds, then set $\text{Flag} = 1$ and go to step 3; otherwise, find the derivatives $\partial f_{\text{min}}/\partial d_n$'s as in (31) and set $d_n = \max\{0, d_n - \delta\}$ if $\partial f_{\text{min}}/\partial d_n < 0$, or $d_n = \min\{d, d_n + \delta\}$ otherwise, $n = 1, \ldots, M$. |
|     • Set $itr = itr + 1$. |
|   3) If $\text{Flag} = 1$, then set $d_n = d_n$, $n = 1, \ldots, M$, and $\tau = \tau$; otherwise set $\tau = \tau_{\text{max}}$. |
| c) Return $d_n$'s as the solution to (P2–EvenN). |

Now, we focus on solving the feasibility problem (P2F–EvenN) at each iteration. Since (P2F–EvenN) is non-convex, we use the following gradient-based method to search for a feasible solution to this problem in an iterative manner. Initialize $d_n = (2n - 1)d/(N - 1)$, $n = 1, \ldots, M$. At each iteration $itr = 1, 2, \ldots$, given $d_n$'s, we check whether the constraint (24) holds or not. If the constraint holds, we return $d_n$’s as a feasible solution to (P2F–EvenN) and stop the search; otherwise, we update $d_n$’s as follows. First, we find $\dot{x}_0 = \arg \min_{0 \leq x_0 \leq d} \sum_{n=1}^{M} f_n(d_n, x_0)$, which can be numerically obtained with given $d_n$’s. Define $f_{\text{min}} = \sum_{n=1}^{M} f_n(d_n, \dot{x}_0)$, which represents the minimum value of the summation term on the left hand side (LHS) of the constraint in (24) over $0 \leq x_0 \leq d$, with the given $d_n$’s. Next, we have

\[
\frac{\partial f_{\text{min}}}{\partial d_n} = \frac{\partial f_n}{\partial d_n} (d_n, \dot{x}_0) = 
\begin{align*}
&-6 \left( 8 \dot{x}_0^4 + (d_n - \dot{x}_0)^4 - 6 \dot{x}_0^2 (d_n - \dot{x}_0)^2 \right) (d_n - \dot{x}_0) \\
&\quad \cdot \left( \dot{x}_0^2 + (d_n - \dot{x}_0)^2 \right)^6 \\
&-6 \left( 8 \dot{x}_0^4 + (d_n + \dot{x}_0)^4 - 6 \dot{x}_0^2 (d_n + \dot{x}_0)^2 \right) (d_n + \dot{x}_0) \\
&\quad \cdot \left( \dot{x}_0^2 + (d_n + \dot{x}_0)^2 \right)^6.
\end{align*}
\]

Accordingly, we set $d_n = \max\{0, d_n - \delta\}$ if $\partial f_{\text{min}}/\partial d_n < 0$, or $d_n = \min\{d, d_n + \delta\}$ otherwise, $n = 1, \ldots, M$, with $\delta > 0$ denoting a small step size. It can be easily verified that the above update helps increase $f_{\text{min}}$ if $\delta$ is chosen to be sufficiently small. We repeat the above procedure for a maximum number of iterations, denoted by $itr_{\text{max}} \geq 1$, after which we stop the search and return that (P2F–EvenN) is infeasible since the constraint (24) still does not hold with all $d_n$’s derived. In practice, the performance of the gradient-based search for the feasible solution to (P2F–EvenN) depends on the initial values of $d_n$’s as the search in general converges to a local maximum of the LHS function of (24).

To improve the accuracy of the search, if it fails to find a feasible solution to (P2F–EvenN) after $itr_{\text{max}}$ iterations, then we repeat the search with a new initial point given by $d_n = \min\{d, \max\{0, (2n - 1)d/(N - 1) + \Delta d_n\}\}$, $n = 1, \ldots, M$, with randomly generated $\Delta d_n$ which is uniformly distributed over $[-d/(N - 1), d/(N - 1)]$. The maximum number for the set of randomly generated initial points is limited by $\text{rpt}_{\text{max}} \geq 1$, and we decide (P2F–EvenN) is infeasible if we fail to find a feasible solution to (P2F–EvenN) with all $\text{rpt}_{\text{max}}$ sets of initial points generated. In general, a larger $\text{rpt}_{\text{max}}$ can help improve the overall accuracy of the bisection search, but at the cost of more computational complexity.

To summarize, the complete algorithm to solve (P2–EvenN) is given in Table IV denoted by Algorithm 1.

V. Simulation Results

In this section, we present further simulation results to evaluate the performance of our proposed transmitter node placement algorithm, i.e., Algorithm 1. We consider the same
It is observed that OCOL consistently achieves better performance than the other two schemes, although the gain decreases when \( d \) is small or large. This can be explained as follows. When \( d \) is small, the mutual inductance between the receiver and different transmitters is less sensitive to their locations, which implies that the gain of transmitter placement optimization is small. In this case, from (17), it follows that the transmitter currents tend to be all equal, hence the magnetic beamforming gain over the equal current allocation is also negligible. Similarly, when \( d \) is large, the distance between transmitters is large for both UL and OL designs, since there are only five transmitters available to cover the target line. In this case, the magnetic coupling between the transmitters is small, hence they can be treated as independent transmitters. As shown in Fig. 7 using a single transmitter for WPT cannot provide any magnetic beamforming gain. As a result, both transmitter location and current optimization do not yield notable performance gains.

Last, we consider the practical problem of finding the optimal number of transmitters, \( N \), to cover a given target line most efficiently. In this example, it is assumed that the total length of coil wires for manufacturing all \( N \) transmitters is fixed as \( 200\pi \) in meter, and thus the radius of each individual transmitter coil shrinks as \( N \) increases. Specifically, we set the transmitter coil radius as \( \epsilon_{\text{coil,tx}} = 250/N \) in millimeter and keep the number of the turns fixed as \( b_{\text{tx}} = 400 \) regardless of \( N \). The other parameters of the coils are assumed to be the same as in Section III-B. Fig. 11 plots the minimum load power \( p_{0,\text{min}} \) over the number of transmitters \( N \), under the aforementioned schemes of OCOL, ECOL, and OCUL. It is observed that for all three schemes, \( p_{0,\text{min}} \) first increases and then decreases with \( N \). This implies that using either a small number of transmitters each with larger coil or a large number of transmitters each with smaller coil is both inefficient in maximizing the minimum deliverable power. Note that for the case of \( N = 1 \), i.e., centralized WPT, \( p_{0,\text{min}} = 0 \), which is in accordance with the result in Fig. 7.

VI. NODE PLACEMENT OPTIMIZATION IN 2D

In this section, we extend the node placement optimization to the 2D region case. As shown in Fig. 12, we assume that the receiver can move horizontally within a disk of radius \( \rho > 0 \)
that lies in the \((x, y)\) plane at a fixed height of \(z = z_0\) with its center at the origin \((x = 0, y = 0, z = z_0)\). We denote this disk region as the 2D target disk. On the other hand, it is assumed that transmitters are all horizontally placed in a disk in parallel to and below the target disk, which has the same radius of \(\rho\), a fixed height of \(z = 0\), and its center at the origin. Let \((x_n, y_n)\), with \(\sqrt{x_n^2 + y_n^2} \leq \rho\), \((x_0, y_0)\), with \(\sqrt{x_0^2 + y_0^2} \leq \rho\) denote the \((x, y)\)-coordinates of transmitter \(n\) (receiver). In this case, the mutual inductance expressions given in (1) and (2) as well as the approximation given in (3) can be modified by setting \(d_{nk} = \sqrt{(x_n - x_k)^2 + (y_n - y_k)^2}\) and \(d_{n0} = \sqrt{(x_n - x_0)^2 + (y_n - y_0)^2}\). Accordingly, the transmitters’ sum power and the deliverable power to the receiver load given in (8) and (7) can be re-expressed as functions of \((x_0, y_0), (x_n, y_n)\)’s and \(\tau_n\)’s as \(p_0((x_0, y_0), \{(x_n, y_n)\}, \{\tau_n\})\) and \(p_n((x_0, y_0), \{(x_n, y_n)\}, \{\tau_n\})\), respectively. Define \(\mathcal{R} = \{(x, y) \mid \sqrt{x^2 + y^2} \leq \rho\}\), which is a convex set over \(x\) and \(y\). In general, \(\mathcal{R}\) represents a generic 2D disk with a radius of \(\rho\) that lies in the \((x, y)\) plane with an arbitrary fixed height of \(z = \tilde{z}\) and its center at the origin. In the rest of this section, when we refer to \(\mathcal{R}\) as the target disk, we implicitly assume that the height is set as \(\tilde{z} = z_0\); otherwise, the height is \(\tilde{z} = 0\) and \(\mathcal{R}\) is used to refer to the disk region where the transmitters are all located. The four performance metrics introduced for the 1D case, i.e., the average value, the minimum value, the maximum value, and the min-max ratio of the load power given in (11)–(14), can be similarly re-defined for the 2D case. Specifically, each metric is a function of \((x_n, y_n)\)’s and \(\tilde{\tau}_n\)’s in the 2D case. For brevity, the details are omitted. With the optimal transmitter currents given in (17) for magnetic beamforming, the deliverable power to the load in (19) can be then rewritten as \(p^*_m((x_0, y_0), \{(x_n, y_n)\})\).

Last, note that a practical example of our considered 2D setup could be a round non-metallic table with built-in wireless chargers mounted below its surface where the receiver can be freely placed on the table for wireless charging. In this case, \(2\rho\) denotes the diameter of the table, and \(z_0\) is the thickness of its surface.

### A. Problem Formulation and Solution

Similar to (P2) for the 1D case, we formulate the node placement problem to maximize the minimum deliverable power to the receiver over the target disk \(\mathcal{R}\) in 2D as

\[
\text{(P3):} \quad \max_{\tau, \{(x_n, y_n)\}} \tau \quad \text{s.t.} \quad p^*_m((x_0, y_0), \{(x_n, y_n)\}) \geq \tau, (x_0, y_0) \in \mathcal{R}, \quad \tau > 0, \quad n = 1, \ldots, N. \tag{32}
\]

Similar to the 1D case, it can be verified that the optimal transmitter locations in (P3) must be rotationally symmetric over \(\mathcal{R}\). In general, as shown in Fig. 13, a rotationally symmetric structure for the transmitters’ locations in \(\mathcal{R}\) needs to place them either at the origin and/or over one or more concentric rings, where each ring has the same center at the origin, an arbitrary radius that is less than or equal to \(\rho\), consists of at least two transmitters that are equally spaced over the ring, and has an arbitrary rotation angle with respect to the \(x\)-axis. For \(N = 1\), only one rotationally symmetric structure exists by placing the single transmitter at the origin.

For \(N \geq 2\), in the following we first present a sufficient and necessary condition to ensure that a structure consisting of \(Q\) transmitter rings, with \(1 \leq Q \leq \lfloor N/2 \rfloor\) and \(\lfloor \cdot \rfloor\) denoting the largest integer that is no greater than a given real number, is rotationally symmetric over \(\mathcal{R}\). Based on this condition, we then specify the total number of distinct rotationally symmetric structures that exist for a given \(N\), denoted by \(S_N\). For each ring \(q, q \in \{1, \ldots, Q\}\), we denote \(N_q\) as the number of its located transmitters, \(\rho_q\), with \(0 < \rho_q \leq \rho\), as its radius, and \(\phi_q\), with \(0 \leq \phi_q \leq 2\pi/N_q\), as its rotation from the \(x\)-axis. By default, we have \(\sum_{q=1}^Q N_q \leq N\), where \(N - \sum_{q=1}^Q N_q\) remaining transmitters (if any) are all placed at \((x = 0, y = 0, z = 0)\). Without loss of generality, we also set \(\phi_1 = 0\). Then, we have the following lemma.

**Lemma 6.1:** A structure with \(Q \geq 1\) transmitter rings is rotationally symmetric over \(\mathcal{R}\) if and only if (iff) there exists a common divider \(u \geq 2\) such that \(N_q \mod u = 0, \forall q = 1, \ldots, Q\).

**Proof:** Please see Appendix C.

With Lemma 6.1, the following proposition thus follows.

**Proposition 6.1:** For \(N \geq 2\), we have \(S_N = |\mathbb{P}_N|\), where \(\mathbb{P}_N\) is the set consisting of all prime numbers less than or equal to \(N\) and \(| \cdot |\) denotes the cardinality of a set.

**Proof:** Please see Appendix D.

Notice that \(|\mathbb{P}_N| < N\), for \(N \geq 2\), and hence the total number of rotationally symmetric structures for each given \(N\) is less than \(N\). For example, when \(N = 5\), from Proposition 6.1 it follows that \(\mathbb{P}_N = \{2, 3, 5\}\), thus \(S_N = 3\) and in total only three distinct rotationally symmetric structures exist, as shown in Figs. 13(a)–(c), respectively. Moreover, different from the magnetic beamforming optimization which needs to be computed in real time according to the receiver’s location, the node placement optimization can be solved offline before the transmitters are initially deployed. Thus, the complexity of optimizing over \(S_N\) structures to achieve the optimal transmitter placement is practically affordable for a given \(N\). Last, as \(N\) increases, based on the so-called prime number theorem [32], we have asymptotically \(|\mathbb{P}_N| \approx N/\ln(N)\).

Next, for each rotationally symmetric structure \(s, s = 1, \ldots, S_N\), derived from Proposition 6.1, we first simplify (P3)
by exploiting the symmetry in the structure, and then solve it using a similar algorithm like Algorithm 1 for the 1D case. Let \( \{(x^*_n,s^*_n)\} \) and \( \tau^*_s \) denote the optimized transmitter locations and the resulting minimum load power for structure \( s \), respectively. The optimal solution to (P3) is thus given by \( \{(x^*_n,s^*_n)\} \), where \( s = \arg \max_{s \in \{1, \ldots, s_N\}} \tau^*_s \). Note that the optimal structures for different \( N \) are in general not identical. Even for a fixed \( N \), the optimal structure may vary depending on the system parameters (e.g., \( \rho \) as shown later in Table VII).

Now, we illustrate the above procedure for the case of \( N = 5 \) transmitters, while the approach is general and can be applied to the cases with other \( N \) values. For Structure 1 shown in Fig. 14(a), (P3) is simplified as

\[
(P3 - 5TX - S1): \max_{\tau, \rho_1} \tau
\]  

subject to

\[
\sum_{n=1}^{N} \int_{z_{0,n}}^{
frac{\pi}{2}} \rho_1 (x_0, y_0) \geq g(\tau), (x_0, y_0) \in \mathcal{R},
\]

\[
0 \leq \rho_1 \leq \rho,
\]

where

\[
\int_{z_{0,n}}^{
frac{\pi}{2}} \rho_1 (x_0, y_0) = \frac{2 \rho_1 \cos(\theta_n) - (\rho_1 \cos(\theta_n) - x_0)^2 - (\rho_1 \sin(\theta_n) - y_0)^2}{z_{0,n}^2 + (\rho_1 \cos(\theta_n) - x_0)^2 + (\rho_1 \sin(\theta_n) - y_0)^2}.
\]

Moreover, we have \( \theta_n = 2\pi(n - 1)/5 \), \( n = 1, \ldots, 5 \), and \( \mathcal{R} = \{(x, y) | \sqrt{x^2 + y^2} \leq \rho, 0 \leq \cos^{-1}(x/\sqrt{x^2 + y^2}) \leq 2\pi/5, \} \), with \( \mathcal{R} \subset \mathcal{R} \) (the regions of \( \mathcal{R} \) for Structures 2 and 3 are shown in Figs. 14(b) and 14(c), respectively). In (P3–5TX–S1), \( \rho_1 \), with \( 0 \leq \rho_1 \leq \rho \), is the single decision variable (with \( \tau \) as an auxiliary variable), hence Algorithm 1 can be easily modified to solve this problem. Let \( \rho^*_1 \) denote the obtained solution to (P3–5TX–S1). Accordingly, we set \( \{(x^*_n, y^*_n, 1) = (\rho^*_1 \cos(\theta_n), \rho^*_1 \sin(\theta_n))\}, n = 1, \ldots, 5 \), for Structure 1. Similarly, we can simplify (P3) for Structure 2 shown in Fig. 14(b), for which two transmitters are placed at the origin. For Structure 3 shown in Fig. 14(c), we need to jointly optimize three decision variables \( \rho_1, \rho_2 \), and \( \phi_2 \), with \( 0 \leq \rho_1, \rho_2 \leq \rho \) and \( 0 \leq \phi_2 \leq \pi \) (\( \phi_1 = 0 \) by default). The details are omitted for brevity. Last, note that if Structures 1–3 are rotated around the origin, their optimal solutions remain unchanged, as explained in the following. For example, let us rotate Structure 1 shown in Fig. 14(a) around the origin by setting \( \phi_1 = \Delta \phi \), with \( 0 < \Delta \phi < 2\pi \) in rad. If \( \Delta \phi = 2m\pi/5 \), with \( m = 1, \ldots, 4 \), then it follows that the load power distribution of the rotated Structure 1 over the target region \( \mathcal{R} \) is the same as that for the reference Structure 1 with \( \phi_1 = 0 \), and hence \( \rho^*_1 \) is the optimal solution to the rotated version of Structure 1 as well. Otherwise, if \( \Delta \phi \neq \{2m\pi/5 | m = 1, \ldots, 4\} \), then the load power distribution of the rotated Structure 1 can be simply obtained by rotating the load power distribution of the reference Structure 1 around the origin by \( \Delta \phi \) radians. Obviously, the minimum, maximum, and average values for the deliverable load power all remain invariant. Hence, \( \rho^*_1 \) is still the optimal solution to the rotated version of Structure 1. The similar argument is valid for Structures 2 and 3. As a result, in the rest of this paper we do not consider the rotated versions of Structures 1–3 in our analysis/simulations.

B. Numerical Example

To illustrate the performance of joint magnetic beamforming and transmitter location optimization in the 2D disk region case, we consider the same system parameters as in Section III-B for the 1D target line, which is now replaced by a disk of radius \( \rho = 0.35m \) (i.e., with 0.7m in diameter which is the standard size for a round table with 2–3 seats). Hence, the target region area (0.385m²) is about ten times larger than the sum-area of all transmitter coils (0.0393m²).

\footnote{In practice, the transmitter deployment shown in Fig. 14(b) can be implemented by replacing transmitters 4 and 5 (which are co-located at the origin) by an aggregate transmitter with the same coil radius \( \rho_{\text{coil,tx}} \), but \( 2h_0 \) turns of wire. Specifically, from (17), it follows that the optimal currents allocated to transmitters 4 and 5, i.e., \( i_n ^* \) and \( i_n ^* \), respectively, are always identical, since \( b_{40} = b_{50} \) for any receiver location. Hence, the two transmitters can be aggregated to a single transmitter with the aforementioned specification, without change of performance.}
As shown in Figs. [14]a)–(c), three rotationally symmetric structures exist for the system of $N = 5$ transmitters. After obtaining the optimized transmitter locations for the three rotationally symmetric structures, we have $\rho_1^* = 0.228$m with $\tau_1^* = 17.17$ for Structure 1. For Structure 2, we obtain $\rho_1^* = 0.13$m and $\tau_2^* = 2.85$. For Structure 3, we obtain $\rho_1^* = 0.241$m, $\phi_2^* = \pi/2$, and $\tau_3^* = 6.89$. Since $\tau_1^* > \tau_2^* > \tau_3^*$, it follows that Structure 1 has the best performance in terms of maximizing the minimum deliverable power to the receiver load over the given target disk region $\mathcal{R}$, with the system setup considered above. For benchmark performance, we also consider centralize WPT (see Fig. [14]a) where the five transmitters are all placed at the origin $(x = 0, y = 0, z = 0)$. Note that this benchmark structure is a special case of Structures 1–3.

Fig. 15 shows the power deliverable to the receiver load versus its location $(x_0, y_0)$ in $\mathcal{R}$, under distributed WPT including the three rotationally symmetric structures as well as centralize WPT (benchmark structure), with the optimized transmitter locations in each of the three structures in distributed WPT and the optimal magnetic beamforming adaptive to the receiver location applied. The detailed performance comparison among the four structures is summarized in Table VI from which it is observed that the minimum deliverable power achieved by Structure 1 is indeed much larger than those of the other structures based on the actual simulation results. Note that $p_{\text{min}}$ reported in Table VI for each of Structures 1, 2, and 3 slightly differs from $\tau_3^*$ obtained previously by solving its corresponding optimization problem. This is due to the fact that the approximation in (3) is used to compute $h_{n0}$'s in the node placement optimization problem, but the actual mutual inductance expression in (2) is used in all simulations to achieve the best accuracy.

Next, Table VII shows the impact of changing the target disk radius $\rho$ on the performance of WPT in 2D. First, it is observed that when $\rho$ increases, the minimum deliverable power $p_{\text{min}}$ decreases for all structures. It is also observed that when $\rho = 0.1$m, the four structures perform the same, since the optimal solution is to place all the transmitters at the center $(x = 0, y = 0, z = 0)$. It is further observed that when $\rho = 0.3$m, Structure 1 outperforms the other structures, while Structure 3 achieves slightly larger $p_{\text{min}}$ over the other cases when $\rho = 0.6$m. Last, it is observed that the minimum deliverable power of centralize WPT (benchmark structure) significantly drops when $\rho > 0.1$m, which shows the inefficacy of centralize WPT in the 2D case.

### Table VI

**Performance comparison between different transmitter placement schemes in 2D.**

| Scheme          | $p_{\text{avg}}$ (W) | $p_{\text{min}}$ (W) | $p_{\text{max}}$ (W) | $\xi$ (%) |
|-----------------|-----------------------|-----------------------|-----------------------|------------|
| Distributed     | Structure 1: 24.02     | 18.24                 | 25.54                 | 71.42      |
| Distributed     | Structure 2: 22.31     | 2.70                  | 25.62                 | 10.54      |
| Distributed     | Structure 3: 24.03     | 8.15                  | 25.34                 | 51.91      |
| Centralized     | 17.93                 | 0                     | 25.65                 | 0          |

### Table VII

**Impact of region radius $\rho$ on the minimum deliverable power $p_{\text{min}}$ under different transmitter placement schemes in 2D.**

| $\rho$ (m) | Centralized $p_{\text{min}}$ (W) | Distributed $p_{\text{min}}$ (W) |
|-----------|----------------------------------|----------------------------------|
| 0.1       | 25.59                            | 25.59                            |
| 0.3       | 0                                | 25.11                            |
| 0.6       | 4.29                             | 2.69                             |

---

Fig. 15. The load power distribution under different transmitter placement schemes in 2D.

Next, Table VII shows the impact of changing the target disk radius $\rho$ on the performance of WPT in 2D. First, it is observed that when $\rho$ increases, the minimum deliverable power $p_{\text{min}}$ decreases for all structures. It is also observed that when $\rho = 0.1$m, the four structures perform the same, since the optimal solution is to place all the transmitters at the center $(x = 0, y = 0, z = 0)$. It is further observed that when $\rho = 0.3$m, Structure 1 outperforms the other structures, while Structure 3 achieves slightly larger $p_{\text{min}}$ over the other cases when $\rho = 0.6$m. Last, it is observed that the minimum deliverable power of centralize WPT (benchmark structure) significantly drops when $\rho > 0.1$m, which shows the inefficacy of centralize WPT in the 2D case.

#### VII. Conclusion

In this paper, we study the node placement optimization for a MISO MRC-WPT system with distributed magnetic beamforming. First, we propose the optimal magnetic beamforming solution to jointly assign the currents at different transmitters subject to their sum-power constraint with given locations of the transmitters and receiver. We show that although distributed WPT with optimal magnetic beamforming achieves better performance than centralize WPT, the resulting load power profile still fluctuates over a given target region considerably. This motivates us to formulate a node placement problem to jointly optimize the transmitter locations with adaptive magnetic beamforming to maximize the minimum power delivered to the load over a 1D line region. We propose an efficient algorithm for solving this problem based on bisection method and gradient-based search, which is shown by simulation to be able to improve the load power distribution significantly. Finally, we extend our design approach to the case of 2D disk region and show that significant performance gain can also be achieved in this case. In this paper, for simplicity we assume identical transmitter coils of equal size, while the performance of WPT may be further improved if the sizes of transmitter coils can be optimized jointly with the...
transmitter locations, which is an interesting problem worthy of further investigation. Moreover, in this paper we assume that the transmitters and receiver are placed in two parallel planes, and thus their coils all have the same orientation. Reformulating and solving the node placement problem for the general scenarios where the coils of transmitters and receivers can have arbitrary locations/orientations is also interesting to investigate in future work.

**APPENDIX**

**A. Proof of Lemma 2.7**

In (2), we can express $J_1(e_{\text{coil}},u)e_{\text{coil}}u^2/4 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n+1} J_{m,n}^e(u)$, with $J_{m,n}^e(u) = (e_{\text{coil}},u/2)^{2m+1}/(e_{\text{coil}},u/2)^{2m+1}/(m+1)^2(m+1)!$. Given $e_{\text{coil}},e_{\text{coil}}u \ll n_0$, we have $J_{m,n}^e(u)e^{-n_0u} \approx 0$ over $u \geq 0$, since its maximum value over $u$ is $\beta_{m,n}^e = (m+1)^2/(m+1)!$, which decreases to zero as $(e_{\text{coil}},u/2)^{2m+1} \rightarrow 0$ and $(e_{\text{coil}},u/2)^{2m+1} \rightarrow 0$ for $m_1, m_2 \geq 1$. Hence, we can simplify (2) as

$$h_{n_0} \approx \beta \int_0^{\infty} J_0(d_{n_0}u)u^2e^{-n_0u}du,$$

where $\beta = \pi n_0 h_{r_{\text{tx}}} e_{\text{coil}}^2/4$ is defined for convenience.

Next, let $J_{0,\gamma}(\psi) = L\{J_0(\psi)\}$, where $\gamma$ denotes a real number and $L\{\cdot\}$ represents the Laplace transform. Specifically, we have

$$J_{0,\gamma}(\psi) = \int_0^{\infty} J_0(\gamma u)e^{\psi u}du = \frac{1}{\sqrt{\gamma^2 + \psi^2}}.$$  

It is known that for any real function $o(u)$, with $O(\psi)$ denoting its Laplace transform, we have $L\{u^n o(u)\} = (-1)^n o^n(\psi)/\partial^\psi \psi_0$, $n = 1, 2$, and so on.

From (39) and (40), it then follows that $h_{n_0} \approx \beta_0^2 J_{0,\gamma}(\psi)/\partial^\psi \psi_0 = \beta_0(2\psi^2 - \gamma^2)/\gamma_0^2 + \psi^2$, with $\psi = n_0$ and $\gamma = d_{n_0}$. The proof is thus completed.

**B. Proof of Proposition 2.7**

For (P1), the optimal current solution $i_n$’s to (P1) can be obtained by leveraging the Karush-Kuhn-Tucker (KKT) conditions of the optimization problem [13]. Let $\lambda \geq 0$ denote the dual variable corresponding to the constraint (16). The Lagrangian of (P1) is given by

$$L = \frac{\psi^2}{\rho_{r_{\text{tx}}}}\left(\sum_{n=1}^N h_{n_0}i_n\right)^2 - \lambda \left(\sum_{n=1}^N \sum_{n=1}^N h_{n_0}i_n - p_{\text{max}}\right).$$

The KKT conditions of (P1) are also given by

$$\sum_{n=1}^N i_n^2 + \frac{\psi^2}{\rho_{r_{\text{tx}}}}\left(\sum_{n=1}^N h_{n_0}i_n\right)^2 \leq p_{\text{max}},$$

$$\lambda \geq 0,$$

$$\frac{\psi^2}{\rho_{r_{\text{tx}}}}\left(\sum_{n=1}^N h_{n_0}i_n\right) - \lambda \rho_{r_{\text{tx}}}i_n = 0, \forall n,$$

$$\lambda \left(\sum_{n=1}^N i_n^2 + \frac{\psi^2}{\rho_{r_{\text{tx}}}}\left(\sum_{n=1}^N h_{n_0}i_n\right)^2 - p_{\text{max}}\right) = 0.$$  

where (42) and (43) are the feasibility conditions for the primal and dual solutions, respectively. (44) is due to the fact that the gradient of the Lagrangian with respect to the optimal primal solution $i_n$’s must vanish, and (45) stands for the complimentary slackness. To solve the set of equations in (42)-(45), we consider two possible cases as follows.

- Case 1: $\lambda = 0$. It can be verified that any set of $\tilde{i}_n$’s satisfying $\sum_{n=1}^N h_{n_0}i_n = 0$ and $\rho_{r_{\text{tx}}}\sum_{n=1}^N i_n \leq p_{\text{max}}$ can satisfy the KKT conditions (42)-(45) in this case. However, the resulting $\tilde{i}_n$’s will make the objective function of (P1) in (15) equal to zero, which cannot be the optimal value of (P1); therefore, this case cannot lead to the optimal solution to (P1).

- Case 2: $\lambda > 0$. From (44), it follows that $i_n = \left(h_{k_0}/h_{n_0}\right)\gamma_k \neq n$. Moreover, from (45), it follows that $\rho_{r_{\text{tx}}}\sum_{n=1}^N i_n \leq p_{\text{max}}$. Accordingly, we obtain $\tilde{i}_n = \left(h_{k_0}/h_{n_0}\right)\gamma_k = 0, n = 1, \ldots, N$, and $\lambda = (u^r_{r_{\text{tx}}} h_{n_0})/(\rho_{r_{\text{tx}}}^2 + u^2_{r_{\text{tx}}} \sum_{n=1}^N h_{n_0}^2)$, where $\kappa$ is given by

$$\kappa = \frac{\sqrt{p_{\text{max}}}}{\sqrt{\sum_{n=1}^N h_{n_0}^2}}.$$  

The obtained $\tilde{i}_n$’s and $\lambda$ satisfy the KKT conditions (42)-(45).

Note that except the above set of primal and dual solutions to (P1), $\tilde{i}_n$’s and $\lambda$, given in Case 2, there is no other solution that satisfies the KKT conditions (42)-(45). Thus, we can conclude that the solution obtained in Case 2 is indeed the optimal solution to (P1) because the KKT conditions are necessary (albeit not necessarily sufficient) for the optimality of a non-convex optimization problem, which is the case of (P1). The proof is thus completed.

**C. Proof of Lemma 6.7**

By definition, rotational symmetry refers to the property of an object if it looks the same after a certain turn around its center. Based on this definition, a structure with $Q$ transmitter rings, shown in Fig. 13 is rotationally symmetric iff there exists a common rotation angle $\phi$, with $0 < \phi < 2\pi$, such that by setting $\phi_0 = \phi$, the ring looks the same after the rotation. Without loss of generality, we set $k_q = 1$. By considering all $Q$ transmitter rings, we thus have $\phi = 2\pi k/\psi$, with $k = 1, \ldots, \min_{q\in\{1,\ldots,Q\}} N_q/\psi$, where $u$ is a common divider such that $N_q = \min_{q\in\{1,\ldots,Q\}} N_q/\psi$, and $u$ is a common divider such that $N_q = \min_{q\in\{1,\ldots,Q\}} N_q/\psi$. In this case, the condition $0 < \phi < 2\pi$ (which is necessary and sufficient for rotational symmetry) holds iff there exists at least one common divider $u$ no smaller than 2, i.e., $u \geq 2$. The proof is thus completed.

**D. Proof of Proposition 6.1**

First, we show that a structure with $Q$ transmitter rings is rotationally symmetric and distinct over $\mathcal{R}$ iff $N_1 = \ldots =
transmitter rings is rotationally symmetric and distinct. The proof of the ‘if’ part is thus completed.

Secondly, we show that if the considered structure with \( Q \) transmitter rings is rotationally symmetric and distinct over \( R \), then \( N_1 = \ldots = N_Q = u \) and \( u \in \mathbb{P}_N \), must hold. In this case, since the structure is assumed to be rotationally symmetric over \( R \), from Lemma 6.1 it follows that a common divider \( u \geq 2 \) exists such that \( N_q \mod u = 0, q = 1, \ldots, Q \). Hence, we can represent \( N_q = a_q u \), where \( a_q = N_q / u \) is a positive integer. Moreover, from the distinction of the structure, it follows that \( a_q = 1, q = 1, \ldots, Q \); otherwise, each ring with \( a_q > 1 \) can be divided into \( a_q \) rings each with \( u \) transmitters. In this case, each ring can be divided into different radii and rotation angles in general, while in our case their radii are all set the same. This means that the considered structure is a special case of a more general structure with more rings each consisting of a prime number \( (u) \) of transmitters, and thus cannot be distinct. Thus, \( u \in \mathbb{P}_N \) must hold. The proof of the ‘only if’ part is thus completed.

To sum up, we have shown that a structure with \( Q \) transmitter rings is rotationally symmetric and distinct \( \iff \) \( N_1 = \ldots = N_Q = u \) and \( u \in \mathbb{P}_N \). With this result, it immediately follows that \( S_N = |\mathbb{P}_N| \). The proof of this proposition is thus completed.

REFERENCES

[1] J. G. Bolger, F. A. Kirsten, and L. S. Ng, “Inductive power coupling for an electric highway system,” in Proc. 28th IEEE Veh. Technol. Conf. (VTC), pp. 137-144, Mar. 1978.

[2] C. Kim, D. Seo, J. You, J. Park, and B. H. Cho, “Design of a contactless battery charger for cellular phone,” IEEE Trans. Ind. Electron., vol. 44, no. 6, pp. 1238-1247, Dec. 2001.

[3] W. Chiuei-Sen, O. H. Stielau, and C. A. Covic, “Design considerations for a contactless electric vehicle battery charger,” IEEE Trans. Ind. Electron., vol. 52, no. 5, pp. 1308-1314, Oct. 2005.

[4] Available online at http://www.wirelesspowercounsell.com.

[5] A. Kurs, A. Karalis, R. Moffatt, J. D. Joannopoulos, P. Fisher, and M. Soljacic, “Wireless power transfer via strongly coupled magnetic resonators,” Science, vol. 317, no. 83, pp. 83-86, July 2007.

[6] Y. Zhang and Z. Zhao, “Frequency splitting analysis of two-coil resonant wireless power transfer,” IEEE Antennas Wireless Propagat. Lett., vol. 13, pp. 400-402, Feb. 2014.

[7] J. Shin, S. Shin, Y. Kim, S. Ahn, S. Lee, G. Jung, S. Jeon, and D. Cho, “Design and implementation of shaped magnetic-resonance-based wireless power transfer system for roadway-powered moving electric vehicles,” IEEE Trans. Ind. Electron., vol. 61, no. 3, pp. 1179-1192, Mar. 2014.

[8] E. Bou-Balust, R. Sedwick, A. P. Hu, and E. Alarcon, “Advances in non-Radiative resonant inductive coupling wireless power transfer: a comparison of alternative circuit and system models driven by emergent applications,” in Proc. IEEE Int. Symposium Circuits and Systems (ISCAS), pp. 1-4, June 2014.

[9] Y. Li, J. Li, K. Wang, W. Chen, and X. Yang, “A maximum efficiency point tracking control scheme for wireless power transfer systems using magnetic resonant coupling,” IEEE Trans. Power Electron., vol. 30, no. 7, pp. 3998-4008, July 2015.

[10] K. Na, H. Jang, H. Ma, and F. Bien, “Tracking optimal efficiency of magnetic resonance wireless power transfer system for biomedical capsule endoscopy,” IEEE Trans. Microw. Theory Techn., vol. 63, no. 1, pp. 295-303, Jan. 2015.

[11] Q. Xu, H. Wang, Z. Gao, Z.-H. Mao, J. He, and M. Sun, “A novel magnetic-based system for position-varying wireless power transfer to biomedical implants,” IEEE Trans. Magn., vol. 49, no. 8, pp. 4774-4779, Feb. 2013.

[12] S. Li and C. C. Mi, “Wireless power transfer for electric vehicle applications,” IEEE Trans. Emerg. Sel. Topics Power Electron., vol. 30, no. 1, pp. 4-17, Mar. 2015.

[13] M. Ibrahim, L. Bernard, P. Pichon, E. Laboure, A. Razez, O. Cayol, D. Ladas, and J. Irving, “Inductive charger for electric vehicle: advanced modeling and interoperability analysis,” IEEE Trans. Power Electron., vol. 31, no. 12, pp. 8096-8114, Jan. 2016.

[14] W. X. Zhong, C. Zhang, X. Liu, and S. Y. R. Hui, “A methodology for making a three-coil wireless power transfer system more energy efficient than a two-coil counterpart for extended transfer distance,” IEEE Trans. Power Electron., vol. 30, no. 2, pp. 933-942, Feb. 2015.

[15] J. Yin, D. Lin, C.-K. Lee, and S. Y. R. Hui, “A systematic approach for load monitoring and control in wireless power transfer systems without any direct output measurement,” IEEE Trans. Power Electron., vol. 30, no. 3, pp. 1657-1667, Mar. 2015.

[16] E. Bou-Balust, A. P. Hu, and E. Alarcon, “Scalability analysis of SIMO non-radiative resonant wireless power transfer systems based on circuit models,” IEEE Trans. Circuits Syst. I, Reg. Papers, vol. 62, no. 10, pp. 2574-2583, Oct. 2015.

[17] Available online at http://www.airfuel.org.

[18] J. Jadidian and D. Katabi, “Magnetic MIMO: how to charge your phone in your pocket,” in Proc. 20th Int. Conf. Mobile computing and networking (ACM), pp. 495-506, Sept. 2014.

[19] R. Zhang and C. K. Ho, “MIMO broadcasting for simultaneous wireless information and power transfer,” IEEE Trans. Wireless Commun., vol. 12, no. 5, pp. 1989-2001, May 2013.

[20] S. L. Xiao, N. Dusit, P. Wang, D. I. Kim, and Z. Han, “Wireless networks with RF energy harvesting: a contemporary survey,” IEEE Commun. Surveys Tuts., vol. 17, no. 2, pp. 757-789, Second quarter 2015.

[21] S. Bi, Y. Zeng, and R. Zhang, “Wireless powered communication networks: an overview,” IEEE Wireless Communications, vol. 23, no. 4, pp. 10-18, April 2016.

[22] H.-D. Lange, A. Ludwig, and C. D. Sariss, “Convex optimization of wireless power transfer systems with multiple transmitters,” IEEE Trans. Antennas Propag., vol. 62, no. 9, pp. 4623-4636, Sept. 2014.

[23] G. Yang, M. R. V. Moghadam, R. Zhang, “Magnetic beamforming for wireless power transfer,” in Proc. IEEE Int. Conf. Acoustics, Speech, and Signal Processing (ICASSP), pp. 3936-3940, Mar. 2016.

[24] B.-H. Choi, B.-C. Park, and J.-H. Lee, “Near-field beamforming loop array for selective wireless power transfer,” IEEE Microw. Compon. Lett., vol. 25, no. 11, pp. 748-750, Nov. 2015.

[25] Y. Zhang, T. Lu, Z. Zhao, F. He, K. Chen, and L. Yuan, “Selective wireless power transfer to multiple loads using receivers of different resonant frequencies,” IEEE Trans. Power Electron., vol. 30, no. 11, pp. 6001-6005, Nov. 2015.

[26] Y.-J. Kim, D. Ha, W. J. Chappell, and P. P. Irazoqui, “Selective wireless power transfer for smart power distribution in a miniature-sized multiple-receiver system,” IEEE Trans. Ind. Electron., vol. 63, no. 3, pp. 1853-1862, Mar. 2016.

[27] M. R. V. Moghadam and R. Zhang, “Multituser wireless power transfer via magnetic resonant coupling: performance analysis, charging control, and power region characterization,” IEEE Trans. Signal Inf. Process. Netw., vol. 2, no. 1, pp. 72-83, Mar. 2016.
[29] S. Kisseleff, I. F. Akyildiz, and W. Gerstacker, “Beamforming for magnetic induction based wireless power transfer systems with multiple receivers,” in Proc. IEEE Global Communications Conference (GLOBECOM), pp. 1-7, Dec. 2015.

[30] J. T. Conway, “Inductance calculations for noncoaxial coils using Bessel functions,” IEEE Trans. Magn., vol. 43, no. 3, pp. 1023-1034, Mar. 2007.

[31] R. Tseng, B. Novak, S. Shevde, and K. Grajski, “Introduction to the alliance for wireless power loosely-coupled wireless power transfer system specification version 1.0,” in Proc. IEEE Int. Conf. Wireless Power Transfer (WPT), pp. 79-83, May 2013.

[32] T. M. Apostol, Introduction to Analytic Number Theory, Springer, New York, 1976.

[33] S. Boyd and L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.