Methods for summing general Kapteyn series

R C Tautz\(^1\), I Lerche\(^2\) and D Dominici\(^3\)

\(^1\) Zentrum für Astronomie und Astrophysik, Technische Universität Berlin, Hardenbergstraße 36, D-10623 Berlin, Germany
\(^2\) Institut für Geowissenschaften, Naturwissenschaftliche Fakultät III, Martin-Luther-Universität Halle, D-06099 Halle, Germany
\(^3\) Department of Mathematics, State University of New York at New Paltz, 1 Hawk Dr, New Paltz, NY 12561-2443, USA

E-mail: rct@gmx.eu, lercheian@yahoo.com and dominicd@newpaltz.edu

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Abstract
The general features and characteristics of Kapteyn series, which are a special type of series involving the Bessel function, are investigated. For many applications in physics, astrophysics and mathematics, it is crucial to have closed-form expressions in order to determine their functional structure and parametric behavior. The closed-form expressions of Kapteyn series have mostly been limited to special cases, even though there are often similarities in the approaches used to reduce the series to analytically tractable forms. The goal of this paper is to review the previous work in the area and to show that Kapteyn series can be expressed as trigonometric or gamma function series, which can be evaluated in a closed form for specific parameters. Two examples with a similar structure are given, showing the complexity of Kapteyn series.

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1. Introduction
Kapteyn series of the first and second kinds have arisen in a large variety of physics problems since their discovery by Kapteyn (1893). Series of the first kind are of the form

\[ K_1 = K_1(\{a_n\}, \alpha, \beta, c, b) = \sum_{n=-\infty}^{\infty} a_n J_{\alpha n + \beta}(cn + b), \] (1)

with \(\alpha, \beta, c\) and \(b\) fixed but possibly complex and \(\{a_n\}\) is a sequence of complex coefficients. Series of the second kind are of the form

\[ K_2 = K_2(\{a_n\}, \alpha, \beta, \gamma, \epsilon, c, b, f, g) \]
\[ = \sum_{n=-\infty}^{\infty} a_n J_{\alpha n + \beta}(cn + b) J_{\gamma n + \epsilon}(fn + g), \] (2)
with $\alpha, \beta, \gamma, \epsilon, c, b, f$ and $g$ fixed but possibly complex and $\{a_n\}$ is a sequence of complex coefficients. Over the years, the subject of summing Kapteyn series has attracted both physics (see below) and mathematics researchers (e.g. Erdelyi 1981, Watson 1966).

One of the most important aspects of such Kapteyn series is to provide closed-form expressions for the series in particular problems of physical interest. This aspect is often crucial, as one attempts to determine the functional structure and parametric behavior for the problem at hand. For example, for a system of $N$ equally charged particles uniformly spaced on a rotating ring, Budden (1924) used Kapteyn series to show how the far-field radiation distribution varies as $N \to \infty$ (see also Lerche and Tautz 2008). Earlier, Schott (1912) had discussed radiation from a single relativistic particle moving in a circle. Both radiation problems involved Kapteyn series of the second kind with $\alpha = \gamma = 1, \beta = \epsilon = 0, c = f$ and $b = g = 0$ but with different coefficients $a_n$ for the two problems.

To name a few illustrations from a broad range of physical applications (see Tautz and Lerche 2009, for a review), the large variety of physics problems involving various Kapteyn series includes (i) Kepler’s problem (Dattoli et al 1995), (ii) pulsars (Harrison and Tademaru 1975, Lerche and Tautz 2007), (iii) side-band spectra of terahertz electromagnetic waves (Citrin 1999, Lerche et al 2009), (iv) high-intensity Compton scattering (Harvey et al 2009, Lerche and Tautz 2010), (v) queueing theory (Dominici 2007b) and (vi) cosmic ray transport theories (Tautz and Dominici 2010, Tautz and Lerche 2010).

In Dominici (2007a), Kapteyn series of the form

$$K_1(z, t) = \sum_{n=1}^{\infty} r^n J_n (nz)$$  \hspace{1cm} (3)

were studied and a series representation was derived in powers of $z$. Furthermore, the radius of convergence was analyzed. Also, general Kapteyn series of the first kind were considered. In Dominici (2010), an asymptotic approximation in terms of a Kapteyn series was obtained for the zeros of the Hermite polynomials.

As far as can be determined, physical applications of Kapteyn series to date seem to involve only structural behaviors of the form

$$\sum a_n J_{\nu+n} (cn)$$  \hspace{1cm} (4)

for the Kapteyn series of the first kind and

$$\sum a_n J_{\nu+n} (cn) J_{-\nu+(n+\epsilon)} (cn)$$  \hspace{1cm} (5)

for the Kapteyn series of the second kind. What would be of value is to determine broad ranges of the parameters and coefficients $a_n$ so that rather general closed-form representations of the Kapteyn series $K_1$ and $K_2$ are available. Such knowledge would then obviate having to evaluate each application of Kapteyn series de novo.

Much effort in this general direction has been devoted by Nielsen (1901, 1904) who summed particular Kapteyn series of the second type. Curiously, in respect of Nielsen’s work, Watson (1966) remarks ‘series of the type

$$\sum \beta_n J_{\mu+n} \left[ \left( \frac{\mu + v}{2} + n \right) z \right] J_{\mu+n} \left[ \left( \frac{\mu + v}{2} + n \right) z \right]$$

have been studied in some detail by Nielsen (1901). But the only series of this type which have, as yet, proved to be of practical importance are some special series with $\mu = v$ and with simple coefficients.’ However, Watson also goes on to say that ‘Schott (1912) has shown that

$$\sum_{n=1}^{\infty} J_n (nz)^2 = \frac{1}{2} \{(1 - z^2)^{-1/2} - 1\}.$$
but the direct inspection of equation (31) from Nielsen (1901) shows that the formula ascribed to Schott was already available. Indeed, many further direct summations of Kapteyn series of the second kind are to be found in Nielsen’s (1901) work such as (his equation (31a))

$$2 \sum_{n=1}^{\infty} J_n[(2n+1)x]J'_n[(2n+1)x] = \frac{1}{2x}[(1-4x^2)^{-1/2} - 1].$$

(6)

The difference in philosophy between Nielsen and Schott is that Nielsen treated the summation as a pure mathematics problem requiring summation, while Schott worked out the physics problem of synchrotron radiation involving the series. Thus, physics applications of the Kapteyn series arose a decade (or so) after the series was originally summed in the closed form.

It would seem that to prejudge the ability to provide summations of as many as possible Kapteyn series of the second kind as not of practical use is basically not appropriate, for applications often follow much later than the basic mathematical results.

For these reasons, it seems relevant to consider de novo the series $K_2$ and to attempt to determine procedures for evaluating such series for as broad a range of parameters and coefficients as possible. In sections 2 and 3, general methods and two specific examples for the summation of Kapteyn series of the second kind will be presented, respectively. Section 4 provides a short summary and a discussion of the results.

2. Methods for summing $K_2$ series

In this section, different approaches are investigated that have proven useful for summing various Kapteyn series of the second kind.

2.1. An integral representation procedure

One of the main techniques for summing Kapteyn series of the second kind is (Erdelyi 1981, Gradshteyn and Ryzhik 2000)

$$J_\mu(z)J_\nu(z) = \frac{2}{\pi} \int_0^{\pi/2} d\theta \ J_{\mu+\nu}(2z \cos \theta) \cos(\mu - \nu)\theta,$$

(7)

which is valid for $\mu, \nu$ and any integer values, and is otherwise valid for $\Re(\mu + \nu) > -1$. Note here the requirement that $J_\mu$ and $J_\nu$ have the same argument.

Thus, when considering

$$K_2 \equiv \sum_{n=-\infty}^{\infty} a_n J_{\alpha n+\beta} (cn + b) J_{\gamma n+\epsilon} (fn + g),$$

(8)

one restricts the evaluation using equation (7) to $c = f$ and $b = g$. Then, for $\Re(\mu + \nu) > -1$, one requires $\Re[(\alpha + \gamma)n + \beta + \epsilon] > -1$ for all integers $n$, with $a_n \neq 0$. If $a_n \neq 0$ for some integer $n$, then one requires $\Re(\alpha + \gamma) = 0$ and $\Re(\beta + \epsilon) > -1$.

Hence, one has to evaluate

$$K_2 = \sum_{n=-\infty}^{\infty} a_n J_{\alpha n+\beta} (cn + b) J_{-\alpha n+\gamma n+\epsilon} (cn + b),$$

(9)

where $\gamma = \gamma_R + i\gamma_I$ with $\gamma_R = -\alpha R$ and $\alpha = \alpha_R + i\alpha_I$, and $\Re(\beta + \epsilon) > -1$.

Starting with the case $\mu = -\nu$ in equation (7), one has for the Kapteyn series of the second kind, $K_2([a_n], \alpha, \beta, -\alpha, -\beta, c, b, c, b)$, i.e.
\[ K_2 = \sum_{n=-\infty}^{\infty} a_n J_{\alpha n + \beta} (cn + b) J_{-(\alpha n + \beta)} (cn + b) \]  
(10a)

\[ = \frac{2}{\pi} \int_{0}^{\pi/2} \sum_{n=-\infty}^{\infty} a_n J_0 [2 \cos \theta (cn + b) \cos [2 \theta (\alpha n + \beta)]] \cos \theta (cn + b). \]  
(10b)

Using the fact that
\[ J_0(z) = \frac{1}{\pi} \int_{0}^{\pi} \cos(z \sin \psi) \, d\psi, \]  
(11)

one obtains
\[ K_2 = \frac{2}{\pi^2} \int_{0}^{\pi/2} \int_{0}^{\pi} \sum_{n=-\infty}^{\infty} a_n \cos [2 \theta (\alpha n + \beta)] \cos [2 \cos \theta \sin \psi (cn + b)] \, d\theta \int_{0}^{\pi} \psi \sum_{n=-\infty}^{\infty} a_n \sum_{j=\pm 1} j \{ \cos [2 \theta (\alpha n + \beta) - 2j \cos \theta \sin \psi (cn + b)] \}. \]  
(12a)

\[ = \frac{1}{\pi^2} \int_{0}^{\pi/2} \int_{0}^{\pi} \psi \sum_{n=-\infty}^{\infty} \sum_{j=\pm 1} j \{ \cos [2 \theta (\alpha n + \beta) - 2j \cos \theta \sin \psi (cn + b)] \}. \]  
(12b)

To the extent that one can sum expressions such as
\[ S = \sum_{n=-\infty}^{\infty} a_n \cos (nA + B) \]  
(13)

in the closed form, \( K_2 \) can be, at worst, reduced to a double integral and, at best, can be evaluated in the closed form. The reduction depends precisely on the functional forms chosen for \( a_n \) and on the convergence of the terms in the series \( S \) (normally, but not necessarily, by taking \( A \) to be real).

For example, if \( a_n = (n + p)^{-1} \), where \( p \) is not an integer, then one can write
\[ S = -p \psi B + \frac{\partial S'}{\partial A} \sin B, \]  
(14)

where
\[ S' = \sum_{n=-\infty}^{\infty} \frac{\cos nA}{n^2 - p^2}. \]  
(15)

Because \( S' \) can be given in the closed form, it is possible to sum a variety of Kapteyn series of the second kind with this procedure.

Thus, for sideband spectra in the terahertz regime, one can show (Lerche et al 2009) that
\[ \sum_{v=0}^{v=n} J_{v+n}(av) J_{v-n}(av) \frac{1}{\psi^2 - b^2} = (-1)^{n-1} \frac{\pi}{2b} \csc(\pi b) J_{n+b}(ab) J_{n-b}(ab) \]  
(16)

for \( n \) integer and \( n \geq 1 \), with \( 0 < a < 1 \) and \( 0 < b < 1 \) together with
\[ \sum_{v=0}^{\infty} (-1)^{n-v} J_{v+n+1} \left( a(v+\frac{1}{2}) \right) J_{v-n} \left( a(v+\frac{1}{2}) \right) \]
\[ = (-1)^{n} \frac{\pi}{4b} \sec(\pi b) J_{n+\frac{1}{2}+b}(ab) J_{n+\frac{1}{2}-b}(ab) \]  
(17)

for \( n \) integer and \( n \geq 1 \), with \( 0 < a < 1 \) and \( 0 < b < \frac{1}{2} \).
Similarly, for high-intensity Compton scattering, series arise of the form
\[ \sum_{n=1}^{\infty} \frac{n^p}{(a+n)^q} J_n^2(na) \]  
with \( p \) and \( q \) positive integers, which can be reduced to an analytic closed form apart from a single elliptic integral that has to be added to the rest of the closed-form expressions (Lerche and Tautz 2010).

Besides such evaluations where \( \mu = -\nu \), the more general case with \( \Re(\mu + \nu) > -1 \) needs to be considered. Writing equation (7) with \( \mu = \alpha n + \beta \) and \( \nu = -\alpha_R n + i\gamma n + \epsilon \), one obtains
\[ \mu + \nu = i(\gamma + \alpha)n + \beta + \epsilon \]  
\[ \mu - \nu = 2\alpha_R n + i(\alpha - \gamma)n + \beta - \epsilon. \]

Thus, one can write
\[ K_2 = \frac{2}{\pi} \int_{0}^{\pi/2} d\theta \sum_{n=-\infty}^{\infty} a_n J_{2\Re(\alpha)n+\beta+\epsilon}(2 \cos \theta (cn + b)) \cos[\theta (2\alpha_R n + i(\alpha - \gamma)n + \beta - \epsilon)]. \]  
with \( \Re(\beta + \epsilon) > -1 \). The cosine factors in equation (21) converge if and only if \( \alpha_1 = \gamma_1 \), so that
\[ K_2 = \frac{2}{\pi} \int_{0}^{\pi/2} d\theta \sum_{n=-\infty}^{\infty} a_n J_{2\Re(\alpha)n+\beta+\epsilon}(2 \cos \theta (cn + b)) \cos[\theta (2\alpha_R n + \beta - \epsilon)]. \]  

Therefore, \( J_{\nu}(x) \equiv \frac{2}{\pi} \int_{0}^{x} dt \sin \left( x \cosh t - \frac{\pi \nu}{2} \right) \cosh t \) for \( x \in \mathbb{R} \), so that \( c \) and \( b \) are real. But if \( \nu \) contains an imaginary part proportional to \( n \), as in equation (22), then the series in equation (22) diverges exponentially unless \( a_n \) converges fast enough (e.g. \( a_n \) proportional to \( \exp(-bn^2) \)). Thus, one requires \( \alpha_1 = 0 \).

Under such conditions, one can write
\[ K_2 = \left( \frac{2}{\pi} \right)^2 \int_{0}^{\infty} dt \int_{0}^{\pi/2} d\theta \sum_{n=-\infty}^{\infty} a_n \cos[\nu (2\alpha_R n + \beta - \epsilon)] \times \cosh[(\beta + \epsilon)t] \sin[2 \cos \theta (cn + b) \cosh t - \frac{\pi}{2} (\beta + \epsilon)]. \]  
Again, one sees that for choices of \( a_n \) such that the series in equation (24) can be summed in the closed form, then \( K_2 \) is reduced, at worst, to a double integral and, at best, can be evaluated explicitly.

All of these procedures for summing the general second-order Kapteyn series represented by \( K_2 \) are dependent on the integral representation from equation (7) for \( J_{\mu}(z)J_{\nu}(z) \), valid for \( \Re(\mu + \nu) > -1 \) when \( \mu, \nu \) are not integers and otherwise generally valid.

But just because there are values of \( \Re(\mu + \nu) \leq -1 \), for which equation (7) cannot be used, does not mean that other Kapteyn series of the \( K_2 \) form cannot be summed. One needs other procedures to effect the summations when \( \Re(\mu + \nu) \leq -1 \).
2.2. Series representation procedures

The use of series representations to sum $K_2$ types of Kapteyn series was already known to Nielsen (1901) and later the same procedure was given by Watson (1966). Following Nielsen (1901), the sense of the argument is as follows: one considers first integrals of the form

$$I_{\nu}(a) = \int_0^{\pi/2} dx \cos^{\nu-1} x \cos ax \tag{25}$$

and, for $\Re(\nu) > 0$, expresses the result as

$$I_{\nu}(a) = \frac{\pi}{2^{\nu}} \cos \frac{a\pi}{2} \frac{\Gamma(v + 1)}{\Gamma((v + 1 + a)/2))\Gamma((v + 1 - a)/2)} \tag{26}$$

Then one considers equation (7) with $\mu = n - a$ and $\nu = n + a$. According to Nielsen, one then uses the series expansion definition of the Bessel function under the integral sign as

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} k! \Gamma(\alpha + k + 1)} \tag{27}$$

and so one integrates equation (7) term by term with $\mu$ and $\nu$ as defined above by using equation (27). Effectively, one trades a sum over Bessel functions of the Kapteyn kind for a power series. The resulting power series can often, but not universally, be either summed in the closed form or can be evaluated for specific parameter values. In this way, Nielsen argued that

$$\sin \frac{v\pi}{\sqrt{\pi}} + 2 \sum_{n=1}^{\infty} J_{n+\frac{1}{2}}(2nx)J_{n-\frac{1}{2}}(2nx) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{n!\Gamma(n + \frac{1}{2})2^{2n}}{\Gamma(n + 1 + v)\Gamma(n + 1 - v)} \tag{28}$$

and

$$2 \sum_{n=0}^{\infty} J_{n+\frac{1}{2}}[(2n + 1)x]J_{n+\frac{1}{2}+\frac{1}{2}}[(2n + 1)x] = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{n!\Gamma(n + \frac{3}{2})2^{2n+1}}{\Gamma(n + 1 + v)\Gamma(n + 2 - v)} \tag{29}$$

which are illustrated in figures 1 and 2, respectively, for varying $\nu$ and for $x = 1/2$. Numerically, the evaluation of infinite series is carried out as follows: first, a number of terms (usually 100) is summed directly; to accelerate the convergence of the sum, then for example Wynn’s epsilon method (e.g., Brezinski 2000, Hamming 1986) can be used, which samples a number of additional terms (usually 100) in the sum and tries to fit a polynomial multiplied by a decaying
exponential. Thus, the series are well approximated and the required computer time is kept moderate.

Note that both equations (28) and (29) are valid for arbitrary \( \nu \in \mathbb{C} \) and for complex \( x \in \mathbb{C} \) with \( |x| < 1/2 \). However, for \( \Im(\nu) > 1 \), both series attain extremely large values, depending on \( \Re(\nu) \) and \( x \).

Nielsen then commented that for \( \nu = 0 \), one has the particular cases

\[
1 + 2 \sum_{n=1}^{\infty} J_n(2nx)^2 = (1 - 4x^2)^{-1/2}
\]

and

\[
2 \sum_{n=0}^{\infty} J_n[(2n + 1)x]J_{n+1}[(2n + 1)x] = \frac{1}{2x}[(1 - 4x^2)^{-1/2} - 1].
\]

Note that the results shown here correct two misprints in Nielsen’s results (his equations (30a) and (31)), which are (i) Nielsen wrote \( J_{n-\nu}[(2n + 1)x] \) on the left-hand side of equation (29) instead of \( J_{n+1-\nu}[(2n + 1)x] \) and (ii) he included the term \( n = 0 \) when summing the left-hand side of equation (30).

In fact, however, a more general representation is possible if, instead of using Nielsen’s series expansion of the Bessel function under the integral sign of equation (7), one were to write

\[
J_{2n}(2nx \cos \theta) = \frac{1}{\pi} \int_{0}^{\pi} d\psi \cos 2n\psi \cos(2nx \cos \psi)
\]

then one sees immediately that the ability to evaluate a Kapteyn series is again reduced to the question of whether one can sum

\[
\sum_{n=1}^{\infty} \alpha_n \cos 2n\psi \cos(2nx \cos \psi)
\]

for particular choices of \( \alpha_n \). Nielsen’s choice of \( \alpha_0 = 1 \) is just one example where the summation can be achieved.

Perhaps of more general interest is to ask how reducible equation (2) is for arbitrary parameter values. Then, using equation (27) for each of the Bessel functions in
\( K_2 \), one has

\[
K_2 = \sum_{n=-\infty}^{\infty} a_n \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \left( \frac{cn+b}{2} \right)^{an+\beta} \frac{(-1)^k (cn+b)^{2k}}{2^{2k}k! \Gamma(an+\beta+k+1)} \times \left( \frac{fn+g}{2} \right)^{\gamma r+\epsilon} \frac{(-1)^r (fn+g)^{2r}}{2^{2r}r! \Gamma(\gamma n+\epsilon+r+1)}.
\]

(33)

Unless \( fn + g = \Lambda(cn + b) \), where \( \Lambda \) is a constant, it is difficult to make further headway with equation (33). But when such is the case then one has

\[
K_2 = \sum_{n=-\infty}^{\infty} B_n \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \left( \frac{cn+b}{2} \right)^{(a+\gamma)n+\epsilon+\beta} \times \frac{(-1)^k (-1)^r}{2^{2k}2^{2r}k!r!} \frac{(cn+b)^{2(k+r)} \Lambda^{2r}}{\Gamma(an+\beta+k+1) \Gamma(\gamma n+\epsilon+r+1)}.
\]

(34)

where \( B_n = a_n \Lambda^{\gamma n+\epsilon} \). Now, setting \( k + r = m \), one obtains

\[
K_2 = \sum_{n=-\infty}^{\infty} B_n \left( \frac{cn+b}{2} \right)^{(a+\gamma)m+\epsilon+\beta} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \left( \frac{cn+b}{2} \right)^{2m} \frac{(-1)^m \Lambda^{2m} \Lambda^{-2k}}{k!(m-k)! \Gamma(an+\beta+k+1) \Gamma(\gamma n+\epsilon+1+m-k)}.
\]

(35)

The representation of \( K_2 \) in the closed form (or at worst as an integral) then rests on the extent to which one can sum the various component sums occurring in equation (35).

One can write

\[
K_2 = \sum_{n=-\infty}^{\infty} B_n \left( \frac{cn+b}{2} \right)^{(a+\gamma)m+\epsilon+\beta} Q_n.
\]

(36)

where

\[
Q_n = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \left( \frac{cn+b}{2} \right)^{2m} \frac{(-1)^m \Lambda^{2m} \Lambda^{-2k}}{k!(m-k)! \Gamma(an+\beta+k+1) \Gamma(\gamma n+\epsilon+1+m-k)}.
\]

(37)

Note that the replacement of \( r \) by \( (m-k) \) causes \( Q_n \) to be a single power series in \((cn+b)/2\) which can be written as

\[
Q_n = \sum_{m=0}^{\infty} \left( \frac{cn+b}{2} \right)^{2m} (-1)^m \Lambda^{2m} R_m
\]

(38)

with

\[
R_m = \sum_{k=0}^{\infty} \frac{\Lambda^{2k}}{k!(m-k)!} \left[ \Gamma(an+\beta+k+1) \Gamma(\gamma n+\epsilon+(m-k)+1) \right]^{-1}.
\]

(39)

The basic question is: under what conditions is \( R_m \) expressible in a closed form? If it is, then one can determine the conditions under which \( Q_n \) is expressible in the closed form and so arrange values of \( B_n \) so that \( K_2 \) is in the closed form. It would seem that only for particular values of the parameters it is possible to effect closed-form results, as those for instance given by Nielsen (his equation (8))

\[
\left( \frac{x}{\mu+v} \right)^{\mu+v} = (\mu+v) \Gamma(1+\mu) \Gamma(1+v) \sum_{n=0}^{\infty} \left( \frac{\mu+v+n-1}{n} \right) (\mu+v+2n)^{-(\mu+v+1)} \times J_{\mu+n}[((\mu+v+2n)x)]
\]

(40)
3. Examples for $K_2$ series

There are special cases of Kapteyn series of the second kind, which are often needed and which rely on other methods than those described above. One example is a Kapteyn series, which consists of $J_2(nz)$ for $z \in \mathbb{C}$ with $|z| < 1$, combined with a power of $n$ in the form

$$K_2 \equiv K_2(n^{2q}, 1, 0, 1, 0, z, 0, 0) = \sum_{n=1}^{\infty} n^{2q} J_2(nz). \quad (41)$$

Two different distinctions can be made: (i) $q < 0$ and (ii) $q > 0$, each for (a) integer $q \in \mathbb{N}$ and (b) arbitrary $q \in \mathbb{R}$.

3.1. The case $q < 0$

First, write $p = -q$ so that

$$K_2 = \sum_{n=1}^{\infty} J_2(nz) n^{2p}. \quad (42)$$

The general procedure is the following: use Bessel’s equation $J_n(z)$, to show that (cf Watson 1966, equation 17.33) for consecutive indices $p$, the following two Kapteyn series of the first kind are related through the equation

$$\left( \frac{d}{dz} \right)^2 \sum_{n=1}^{\infty} J_{2n}(2nz) n^{2p} = 4(1 - z^2) \sum_{n=1}^{\infty} J_{2n}(2nz) n^{2(p-1)}. \quad (43)$$

The second initial condition, i.e. the sum for $p = 1$ (cf Watson 1966, section 17.23),

$$\sum_{n=1}^{\infty} J_{2n}(2nz) n^{2p} = \frac{z^2}{2}, \quad (44)$$

together with Meissel’s (1892)) investigation suggests that one should write the Kapteyn series of the first kind as a polynomial in $z^2$ (see Watson 1966, section 17.23). By evaluating the recurrence relation, equation (43), it has been shown (Tautz and Dominici 2010) that the Kapteyn series of the first kind can be expressed as

$$\sum_{n=1}^{\infty} J_{2n}(2nz) n^{2p} = \sum_{k=1}^{p} \frac{z^{2k}}{k^2} \sum_{j=1}^{k} \left( -1 \right)^{j+k} j^{2(k-p)} (k-j)! (k+j)! . \quad (45)$$

To obtain the corresponding Kapteyn series of the second kind, equation (41), one employs equation (7) for $\mu = \nu = n$. Then one evaluates equation (45) with the argument $2nz \cos \theta$ and integrates over $\theta$, noting that (cf Gradshteyn and Ryzhik 2000, section 3.621)

$$\frac{2}{\pi} \int_{0}^{\pi/2} \cos^{2n} \theta \, d\theta = \frac{(2n-1)!!}{(2n)!!} = \frac{\Gamma(n + \frac{1}{2})}{n! \sqrt{\pi}} , \quad n \in \mathbb{N}, \quad (46)$$

where $(\cdot)!!$ is the double factorial. Hence, the result is

$$K_2(z) = \sum_{k=1}^{\Theta} \frac{z^{2k}}{k! \sqrt{\pi}} \sum_{j=0}^{k-1} \frac{(-1)^{j+k} (k-j)^{2(k-p)}}{j! (2k-j)!} , \quad (47)$$

with

$$\Theta = \begin{cases} p, & p \in \mathbb{N} \\ \infty, & p \in \mathbb{R} \setminus \mathbb{N} \end{cases} \quad (48)$$

and is valid for arbitrary $p \in \mathbb{R} < 0$. 

9
Figure 3. The functions $\xi_k(p)$ for varying $p$ with $k \in \{2, 3, 4, 5\}$ as given through equation (47) and, in the explicit form, equation (49). The vertical axis should merely illustrate the zeros of the functions $\xi_k(p)$, thus explaining why, for $p$ integer, a finite power series is obtained in equation (47).

Equation (47) has some important features. For integer $p$, a sum with $p(p+1)/2$ terms is obtained, whereas, for non-integer $p$, an infinite power series occurs. But even in that case, it is advantageous to exchange one infinite series (the Kapteyn series) by another infinite series (a power series), because the convergence behavior of a power series is better understood, thus allowing for a more reliable estimate of the number of terms needed to obtain a desired accuracy.

The reason for the distinction between a finite/infinite power series is that, when expanding the coefficients of the powers $z^{2k}$, one finds non-algebraic functions $\xi_k(p)$ of the form

\[
\begin{align*}
\xi_2(p) &= -\frac{1}{16} + 4^{-(1+p)} \\
\xi_3(p) &= \frac{1}{768} (5 - 2^{7-2p} + 3^{5-2p}) \\
\xi_4(p) &= \frac{1}{18432} (-7 + 2^{13-4p} + 7 \times 2^{7-2p} - 9^{4-p}) \\
\xi_5(p) &= \frac{1}{541920} (42 - 2^{21-4p} - 3 \times 2^{13-2p} + 5^{9-2p} + 9^{6-p}),
\end{align*}
\]

(49)

each of which has zeros at the first integers, i.e. $\xi_k = 0$ for $p = 1, \ldots, k - 1$. The functions $\xi_k(p)$ are illustrated in figure 3 for varying $p$ and for $k \in \{2, 3, 4, 5\}$.

3.2. The case $q > 0$

Even though structurally similar, the opposite case with $q > 0$ results in a completely different behavior when evaluating the Kapteyn series in terms of a power series. In Dominici (2011), Kapteyn series of the form

\[
K_2(z, q) = K_2(n^{2q}, 1, 0, 1, 0, 2z, 0, 2z, 0) = \sum_{n=1}^{\infty} n^{2q} f_n^2(2nz),
\]

(50)

were investigated and it was found that

\[
K_2(z, q) = \sum_{n=1}^{\infty} \left[ \frac{1}{(n!)^2} \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2(n+q)} \right] z^{2n},
\]

(51)
for $q \geq 0$ and $|z| < 1/2$. The result in equation (51) allows one to compute $K_2(z, q)$ numerically for any $q \geq 0$; however, sometimes it is difficult to use it to get closed-form expressions. Here, a different approach will be introduced using differential operators.

From Watson (1966, equation 5.4 (4)), it is known that the function $y_n(t) = f_n^2(e^t)$ satisfies the differential equation

$$\frac{d^3 y_n}{dt^3} + 4(e^{2t} - n^2) \frac{dy_n}{dt} + 4 e^{2t} y_n = 0.$$  \hspace{1cm} (52)

By changing variables to $2nz = e^t$ in equation (52), one obtains

$$\left(\frac{d^3}{dz^3} + 3z \frac{d^2}{dz^2} + \frac{d}{dz} \right) J_n^2(2nz) = 0,$$  \hspace{1cm} (53)

or

$$\left(\frac{d^3}{dz^3} + 3z \frac{d^2}{dz^2} + \frac{d}{dz} \right) J_n^2(2nz) = \left[4(1 - 4z^2) \frac{d}{dz} - 16\right] n^2 J_n^2(2nz).$$  \hspace{1cm} (54)

Introducing the function

$$g_q(z, n) = n^2 J_n^2(2nz),$$  \hspace{1cm} (55)

one has

$$\left(\frac{d^3}{dz^3} + 3z \frac{d^2}{dz^2} + \frac{d}{dz} \right) g_q = \left[4(1 - 4z^2) \frac{d}{dz} - 16\right] g_{q+1},$$  \hspace{1cm} (56)

for all $n \in \mathbb{N}^+$, i.e. all positive integers.

Thus, it follows that the function $K_2(z, q)$ satisfies

$$\left(\frac{d^3}{dz^3} + 3z \frac{d^2}{dz^2} + \frac{d}{dz} \right) K_2(z, q) = \left[4(1 - 4z^2) \frac{d}{dz} - 16\right] K_2(z, q + 1),$$  \hspace{1cm} (57)

while equation (2) gives for the initial condition

$$K_2(z, 0) = -\frac{1}{2} + \frac{1}{2\sqrt{1 - 4z^2}}.$$  \hspace{1cm} (58)

Since $J_0(0) = 0$ for all $n = 1, 2, \ldots$, one has $K_2(0, q) = 0$. Solving equation (57) for $K_2(z, q + 1)$, one obtains

$$K_2(z, q + 1) = \frac{1}{4\sqrt{1 - 4z^2}} \int_0^z \frac{dw}{\sqrt{1 - 4w^2}} \left(w^2 \frac{d}{dw} + 3w \frac{d^2}{dw} + \frac{d}{dw} \right) K_2(w, q),$$  \hspace{1cm} (59)

where one must be careful that $|z| < 1/2$ to ensure that the integral is convergent.

Integrating equation (59) by parts, one obtains the final result for the series $K_2$ in the form of a the recurrence relation, which reads

$$K_2(z, q + 1) = \left[\frac{z^2}{4(1 - 4z^2)} \frac{d^2}{dz^2} + \frac{z(1 - 8z^2)}{4(1 - 4z^2)^2} \frac{d}{dz} + \frac{2z^2(1 + 2z^2)}{(1 - 4z^2)^3} \right] K_2(z, q)$$

$$- \frac{4}{\sqrt{1 - 4z^2}} \int_0^z \frac{dw}{(1 - 4w^2)^{3/2}} K_2(w, q).$$  \hspace{1cm} (60)

Using equations (60) and (58), it is straightforward (albeit tedious) to compute the explicit expressions for the first orders of the series $K_2$, yielding the relations

$$K_2(z, 1) = \frac{z^2(1 + z^2)}{(1 - 4z^2)^{5/2}},$$

$$K_2(z, 2) = \frac{z^2(1 + 37z^2 + 118z^4 + 27z^6)}{(1 - 4z^2)^{13/2}},$$

$$K_2(z, 3) = \frac{z^2(1 + 217z^2 + 5036z^4 + 23630z^6 + 22910z^8 + 2250z^{10})}{(1 - 4z^2)^{19/2}}.$$
Figure 4. The first functions $K_2(z, q)$ for varying $z \in [0, 1/2]$ with $q \in \{1, 2, 3, 4\}$ as given through equation (62) and, in the explicit form, equation (61). The order $q$ varies from right to left, i.e. the solid line shows $K_2(1, z)$ while the dotted line shows $K_2(4, z)$.

The first functions $K_2(z, q)$ for varying $z \in [0, 1/2]$ with $q \in \{1, 2, 3, 4\}$ as given through equation (62) and, in the explicit form, equation (61). The order $q$ varies from right to left, i.e. the solid line shows $K_2(1, z)$ while the dotted line shows $K_2(4, z)$.

In general, one has

$$K_2(z, 4) = z^2 \left(385\,875z^{14} + 7\,119\,756z^{12} + 15\,359\,862z^{10} + 8\,635\,578z^8 + 1\,515\,705z^6 + 8\,0130z^4 + 973z^2 + 1\right) \left(1 - 4z^2\right)^{-25/2},$$

which are illustrated in figure 4.

In general, one has

$$K_2(z, q) = \frac{z^2 P_q(z^2)}{(1 - 4z^2)^{q+1/2}}, \quad |z| < \frac{1}{2},$$

where $P_q(z)$ is a polynomial of degree $2q - 1$. The structure of the polynomials $P_q(z)$ is quite complicated and will be analyzed in a forthcoming paper.

4. Discussion and conclusion

In this paper, the general features and characteristics of Bessel function series were investigated. Special emphasis was focused on Kapteyn series, which appear in many applications of theoretical physics and mathematics, such as radiation and optimization problems. In their original form, with the index of summation appearing in both the index and the argument of the Bessel function(s) involved, the convergence of such series is, in general, unclear. Therefore, it is appropriate and necessary to undertake every effort of rewriting such sums in terms of, at worst, infinite power series or (double) integrals, the convergence of which can be estimated more reliably. More importantly, in many cases it has proven possible to find closed analytical expressions for Kapteyn series of both the first and the second kind. This is indispensable for cases where Kapteyn series constitute only part of large mathematical expressions that have to be dealt with numerically.

However, the quest for closed-form expressions of Kapteyn series has mostly been limited to special cases that have arisen in specific problems such as those listed in section 1. Often one notes that a similar procedure proves useful for different forms of Kapteyn series, even in such cases where the summation coefficients are considerably diverse. But no general answer has been found to date to the following problem: for which parameter regimes of the coefficients do the Kapteyn series have a closed-form expression?
It was the aim of this paper to shed some light on that question. Starting from the most general form of Kapteyn series of the second kind, i.e. involving the product of two Bessel functions, the Kapteyn series was decomposed into series over trigonometric functions or, more generally, algebraic expressions involving gamma functions. The ability to sum such series depends on the precise choice of the parameters. However, it has been shown that the likelihood for any analytical tractability is increased if both Bessel functions are of equal order, i.e. \((αn + β) = (γn + ϵ)\) in equation (2).

Two specific examples with applications, for instance, in cosmic ray diffusion theories (Tautz and Dominici 2010, Shalchi and Schlickeiser 2004, Tautz et al 2006) were illustrated, where the summation coefficients are simply powers of the summation index. It has been shown that, depending very sensitively on the parameters chosen for the summation coefficients, power series with a finite/infinite number of terms are obtained, where the relative magnitude of each term can now easily be estimated.

Future work should, presumably, concentrate on the application of one or more of the above methods (or indeed combinations of the methods) to determine to the maximum extent possible the broadest range of conditions for summability of the general Kapteyn series. Of course, new methods for attempting such summations are welcome and it would be highly interesting to see any such ideas that would add to the capability to effect Kapteyn series summations. Another interesting question is to decide if for every Kapteyn series there exists an annihilator differential operator that can be used to obtain structural relations between series of different orders.

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References

Brezinski C 2000 J. Comput. Appl. Math. 122 1–21
Budden R F 1924 Proc. Lond. Math. Soc. 2 471–8
Citrin D S 1999 Phys. Rev. B 60 5659–63
Dattoli G, Torre A, Lorenzutta S and Maino G 1998 Comput. Math. Appl. 35 117–25
Dominici D 2007a Integral Transforms Spec. Funct. 18 409–18
Dominici D 2007b Proc. Appl. Math. Mech. 7 2050005–6
Dominici D 2010 Progress in Industrial Mathematics at ECMI 2008 ed A D Fitt, J Norbury, H Ockendon and E Wilson (Berlin: Springer) pp 99–103
Dominici D 2011 J. Comput. Appl. Math. 236 39–48
Erdelyi A 1981 Higher Transcendental Functions (Malaba, FL: Krieger)
Gradshteyn I S and Ryzhik I N 2000 Table of Integrals, Series, and Products (London: Academic)
Hamming R W 1986 Numerical Methods for Scientists and Engineers 2nd edn (New York: Dover)
Harrison R E and Tademaru E 1975 Astrophys. J. 201 447–61
Harvey C, Heinzl T and Ilderton A 2009 Phys. Rev. A 79 063407
Kapteyn W 1893 Ann. Sci. de l’Ecole Norm. Sup. Ser. 3 10 91–122
Lerche I and Tautz R C 2007 Astrophys. J. 665 1288–91
Lerche I and Tautz R C 2008 J. Phys. A: Math. Theor. 41 035202
Lerche I and Tautz R C 2010 J. Phys. A: Math. Theor. 43 115207
Lerche I, Tautz R C and Citrin D 2009 J. Phys. A: Math. Theor. 42 365206
Meissel E 1892 Astron. Nachr. 130 363–8
Nielsen N 1901 Ann. Sci. Normale Sup. 18 37–75
Nielsen N 1904 Handbuch Theorie Cylinderfunktionen (Leipzig: Teubner)
Schott G A 1912 Electromagnetic Radiation (Cambridge: Cambridge University Press)
Shalchi A and Schlickeiser R 2004 Astron. Astrophys. 420 799–808
Tautz R C and Dominici D 2010 Phys. Lett. A 374 1414–9
Tautz R C and Lerche I 2009 Adv. Math. Phys. 2009 425164
Tautz R C and Lerche I 2010 Phys. Lett. A 374 4573–80
Tautz R C, Shalchi A and Schlickeiser R 2006 J. Phys. G: Nucl. Part. Phys. 32 809–33
Watson G N 1966 A Treatise on the Theory of Bessel Functions (Cambridge: Cambridge University Press)