THE STABLE AND AUGMENTED BASE LOCUS UNDER FINITE MORPHISMS

TANUJ GOMEZ

ABSTRACT. We study the pullback of the stable and augmented base locus under a finite surjective morphism between normal varieties over a perfect field.

1. INTRODUCTION

The stable and augmented base locus, denoted $B(\cdot)$ and $B_+(\cdot)$, are fundamental invariants in Birational Geometry and in Complex Geometry, particularly when positivity and asymptotic behaviour of divisors, line bundles, or vector bundles are discussed [Nak00, BDPP13, ELM’06].

In this article, we study the behaviour of the stable and augmented base locus under the pullback via a finite surjective morphism. Previous results in this direction, [DCL21] work in the setting of complex manifolds. They, however, rely on powerful analytic methods such as currents and Lelong numbers [Bou04]. If one is willing to impose strong positivity assumptions on the divisors, using [DCDC19] and [Bir17], these results can be extended to projective schemes. Our main result, Theorem 1.1, generalises these results to normal varieties over a perfect field.

Theorem 1.1. Let $X$ and $Y$ be normal projective varieties over a perfect field $k$. Let $f : X \to Y$ be a finite surjective morphism over $k$. Then the following hold:

(1.1.1) For every $\mathbb{Q}$-divisor $D$ on $Y$, we have an equality of reduced schemes $B(f^*D) = f^{-1}B(D)$.

(1.1.2) For every $\mathbb{R}$-divisor $\Delta$ on $Y$, we have an equality of reduced schemes $B_+(f^*\Delta) = f^{-1}B_+(\Delta)$.

In Section 2, we collect the definitions of $\mathbb{Q}$-divisors, $\mathbb{R}$-divisors, the stable and augmented base locus. In Section 3, we prove Theorem 1.1 in the separable case by passing to the Galois closure. There, we use generalised Reynolds operators to produce Galois-invariant sections to study the stable base locus. In Section 4, we use foliations to tackle the inseparable case.

1.1. Acknowledgements. The author would like to thank Johan Commelin, Fabio Bernasconi, Andreas Demleitner, Luca Di Cerbo, Erwan Rousseau and Roberto Svaldi for taking their time to read earlier drafts of this paper and giving the author useful comments and insights.

Date: 19th September 2022.
2020 Mathematics Subject Classification. 14A10, 14E99.
Key words and phrases. augmented base locus, stable base locus, finite morphisms, foliation.
Supported by the DFG-Graduiertenkolleg GK1821 ”Cohomological Methods in Geometry” at the University of Freiburg.

1
Special thanks to Calum Spicer for the insight into tackling the inseparable case; and to the author’s supervisor Stefan Kebekus for making this article even remotely readable!

2. Preliminaries

This section contains the necessary background on the stable and augmented base locus and foliations.

Convention 2.1. Throughout we work with a perfect field $k$. A $k$-variety is an irreducible, separated and integral scheme of finite type over $k$. We will, in Section 4, come across morphisms between $k$-varieties that are not morphism over $k$. To emphasize this, we will say $k$-morphism to mean a morphism between $k$-varieties that is a morphism of schemes over the base field $k$, a morphism is a morphism of schemes that is not necessarily over $k$.

Remark 2.2. The perfect assumption on $k$ is crucial for Section 4.

2.1. Stable and augmented base locus. We will follow [Bir17] for the definition of augmented base locus for $\mathbb{R}$-divisors on a projective scheme. We will then prove a couple of small but key lemmas that will be used throughout the later sections.

Definition 2.3 ($\mathbb{Q}$-divisors, $\mathbb{R}$-divisors and $\mathbb{R}$-linear equivalence). Let $\mathbb{C}$ be a $\mathbb{C}$-variety and denote the group of Cartier divisors on $\mathbb{C}$ as $\text{Div}(\mathbb{C})$.

(2.3.1) A $\mathbb{Q}$-divisor on $\mathbb{C}$ is an element in $\text{Div}(\mathbb{C}) \otimes \mathbb{Q}$.

(2.3.2) An $\mathbb{R}$-divisor on $\mathbb{C}$ is an element in $\text{Div}(\mathbb{C}) \otimes \mathbb{R}$.

(2.3.3) Two $\mathbb{R}$-divisors $\Delta_1$ and $\Delta_2$ are $\mathbb{R}$-linearly equivalent if $\Delta_1 - \Delta_2 = \sum a_iL_i$ where $a_i \in \mathbb{R}$ and $L_i$ are Cartier divisors linearly equivalent to 0. We denote $\mathbb{R}$-linear equivalence as $\Delta_1 \sim_\mathbb{R} \Delta_2$.

Definition 2.4 (Stable base locus). Let $\mathbb{C}$ be a $\mathbb{C}$-variety.

(2.4.1) For a Cartier divisor $\mathcal{F}$, the stable base locus is defined as

$$ B(\mathcal{F}) := \bigcap_{n \in \mathbb{N}} \text{Bs}(n\mathcal{F}) $$

where $\text{Bs}(n\mathcal{F})$ is the base locus of the linear system $|n\mathcal{F}|$.

(2.4.2) For a $\mathbb{Q}$-divisor $\mathcal{D}$ and a positive integer $b$ such that $b\mathcal{D}$ is Cartier, the stable base locus is defined as

$$ B^b(\mathcal{D}) := B(b\mathcal{D}). $$

Remark 2.5. In the setting of Definition (2.4.2), the stable base locus for $\mathcal{D}$ does not depend on the choice of the integer $b$. This follows from the observation, for any Cartier divisor $L$ and any positive integer $m$, that $\text{Bs}(L) \supseteq \text{Bs}(mL)$. This implies that $B(L) \supseteq B(mL)$ and by Definition 2.4, we find $B(L) \subseteq B(mL)$. Hence we will simply denote the stable base locus as $B(\mathcal{D}) := B^b(\mathcal{D})$.

We will use the definition of the augmented base locus for $\mathbb{R}$-divisors from [Bir17, Definition 1.2].

Definition 2.6 (Augmented base locus). Let $\Delta$ be an $\mathbb{R}$-divisor on a projective $k$-variety $Y$ and $A$ an ample Cartier divisor on $Y$. Choose a representation $\Delta \sim_\mathbb{R} \sum t_iA_i$.
where \( t_i \in \mathbb{R} \) and \( A_i \) are very ample Cartier divisors. Then, the augmented base locus is

\[
\mathbb{B}^+_{\sum t_i A_i}(\Delta) := \left( \bigcap_{m \in \mathbb{N}} \mathbb{B}\left( \langle m\Delta \rangle - A \right) \right)_{\text{red}},
\]

where \( \langle m\Delta \rangle = \sum [mt_i]A_i \).

**Remark 2.7.** The augmented base locus does not depend on the choice of an ample Cartier divisor \( A \), or representation \( \Delta \sim_{\mathbb{R}} \sum t_iA_i \) [Bir17, Lemma 3.1]. For this reason we drop the superfluous superscripts to simply denote the augmented base locus as

\[
\mathbb{B}(\Delta) := \mathbb{B}^+_{\sum t_i A_i}(\Delta).
\]

By [Bir17, Lemma 3.1] this definition is consistent with the definitions found in [Laz04b, ELM*06, ELM*09].

**Notation 2.8.** (Pullback of the stable and augmented base locus). In the setting of Definition 2.4, let \( g : Z \to Y \) be any surjective finite morphism that is not necessarily a \( k \)-morphism. We abuse notation by denoting \( g^{-1}\mathbb{B}(D) \) as the reduced scheme associated to the scheme theoretic pullback of \( \mathbb{B}(D) \). We do the same with the augmented base locus.

**Remark 2.9.** The following Lemma 2.10 will allow us to reduce statement (1.1.2) to statement (1.1.1) of Theorem 1.1.

**Lemma 2.10.** Let \( X \) and \( Y \) be projective \( k \)-varieties and \( f : X \to Y \) be a finite morphism. Suppose that for all Cartier divisors \( L \) we have \( \mathbb{B}(f^*L) = f^{-1}\mathbb{B}(L) \). Then for all \( \mathbb{R} \)-divisors \( \Delta \), we have \( \mathbb{B}_+(f^*\Delta) = f^{-1}\mathbb{B}_+(\Delta) \).

**Proof.** To set up the proof, let \( A \) be an ample Cartier divisor on \( Y \). Let \( \Delta \) be an \( \mathbb{R} \)-divisor and choose a representation \( \Delta \sim_{\mathbb{R}} \sum t_iA_i \) where \( t_i \in \mathbb{R} \) and the \( A_i \) are very ample Cartier divisors such that \( f^*A_i \) are very ample. This is possible, as the pullback of an ample divisor under a finite morphism is ample [Laz04a, Proposition 1.2.13]. Observe then that \( f^*\Delta \sim_{\mathbb{R}} \sum t_i f^*A_i \). By this choice of representation, we find that for all \( m \), we have \( f^*\langle m\Delta \rangle = \langle mf^*\Delta \rangle \).

Finally we compute that

\[
f^{-1}\mathbb{B}_+(\Delta) = \bigcap_{m \in \mathbb{N}} f^{-1}\mathbb{B}(\langle m\Delta \rangle - A)
= \bigcap_{m \in \mathbb{N}} \mathbb{B}(f^*\langle m\Delta \rangle - f^*A) \quad \text{by hypothesis}
= \bigcap_{m \in \mathbb{N}} \mathbb{B}(\langle mf^*\Delta \rangle - f^*A) = \mathbb{B}_+(f^*\Delta).
\]

The following is a trivial statement. As we will use it several times, we formulate it as a lemma.

**Lemma 2.11.** Let \( X \) and \( Y \) be normal \( k \)-varieties and \( D \) a \( \mathbb{Q} \)-divisor on \( Y \). For any finite surjective morphism\(^1\) \( f : X \to Y \) we find that \( \mathbb{B}(f^*D) \subseteq f^{-1}\mathbb{B}(D) \).

**Corollary 2.12.** Let \( X, Y \) and \( Z \) be normal \( k \)-varieties, let \( Z \to X \xrightarrow{f} Y \) be a composition of finite surjective morphisms, and let \( D \) be a \( \mathbb{Q} \)-divisor on \( Y \). If we have \( \mathbb{B}(f \circ g)^*L) = (f \circ g)^{-1}\mathbb{B}(L) \) then \( \mathbb{B}(f^*L) = f^{-1}\mathbb{B}(L) \) holds.

\[^1\text{not necessarily a } k \text{-morphism}\]
Proof. There is a chain of inclusions \( B((f \circ g)^*D) \subseteq g^{-1}B(f^*D) \subseteq g^{-1}(f^{-1}B(L)) \).
By hypothesis \( B((f \circ g)^*D) = (f \circ g)^{-1}B(D) \), hence the chain of inclusions are indeed equalities. The result follows from applying \( g \) to the second inclusion. \( \square \)

2.2. Foliation. We recall the definition of a foliation.

**Definition 2.13** (Foliation). Let \( X \) be normal \( k \)-variety and denote the sheaf of Kähler differentials on \( X \) by \( \Omega^1_X \). A foliation on \( X \) is a saturated subsheaf of the tangent sheaf \( \mathcal{T}_X := (\Omega^1_X)^\vee \) that is closed under the Lie bracket.

**Definition 2.14** (p-closed Foliation). Let \( \mathcal{F} \) be a foliation on a normal \( k \)-variety \( X \). Suppose that the characteristic of the field \( k \) is \( p > 0 \). We say that \( \mathcal{F} \) is \( p \)-closed if for all \( U \subseteq X \) open and \( \forall \vartheta \in \mathcal{F}(U) \), the composition of \( \vartheta \) with itself \( p \)-times is in \( \mathcal{F}(U) \).

3. The case of separable morphisms

This section is devoted to proving the following result:

**Theorem 3.1.** Let \( f : X \longrightarrow Y \) be a finite, surjective and separable \( k \)-morphism of normal projective \( k \)-varieties. Then for every \( \mathbb{Q} \)-divisor \( D \) on \( Y \), we have \( B(f^*D) = f^{-1}B(D) \).

3.1. Symmetric Sections. To prove Theorem 3.1 we will first study the partial case when \( f : X \longrightarrow Y \) is a finite Galois morphism, that is when \( f \) is isomorphic to a quotient of \( X \) by a finite group \( G \). In this case we make use of the group action \( G \) to produce a plethora of sections by generalising the classical Reynolds operator.

Suppose that \( L \) is a Cartier divisor on \( Y \) and \( \sigma \in H^0(X, \mathcal{O}_X(f^*L)) \), we can apply the classical Reynolds operator on \( \sigma \) to obtain a section, which we denote by, \( R_1(\sigma) \in H^0(Y, \mathcal{O}_Y(L)) \). Even if \( \sigma(x) \neq 0 \) for some \( x \in X \), there is no guarantee that \( R_1(\sigma)(f(x)) \neq 0 \). Hence it is hard to control the stable base locus with the Reynolds operator. The following Construction 3.2 remedies this issue.

**Construction 3.2.** Let \( X \) be a projective \( k \)-variety and \( G < \text{Aut}_k(X) \) be a non trivial finite group. Denote by \( f : X \longrightarrow Y \) the quotient of \( X \) by \( G \). Take a Cartier divisor \( L \) on \( Y \). Write \( G \) as \( G = \{ g_1, g_2, \ldots, g_N \} \), such that \( g_1 \) is the identity element. For each \( 1 \leq i \leq N \) there is a linear map

\[
R_i : H^0(X, \mathcal{O}_X(f^*L)) \longrightarrow H^0(X, \mathcal{O}_X(i \cdot f^*L)) = H^0(X, \text{Sym}^i \mathcal{O}_X(f^*L))
\]

\[
\sigma \longmapsto \sum_{1 \leq j_1 < j_2 \cdots < j_N \leq N} g_{j_1}^i \sigma \otimes g_{j_2}^i \sigma \otimes \cdots \otimes g_{j_N}^i \sigma.
\]

It should be clear that \( R_i(\sigma) \) are \( G \)-invariant by construction. It will also be useful to define the following linear maps for \( i = 1, \ldots, N-1 \):

\[
\tau_i : H^0(X, \mathcal{O}_X(f^*L)) \longrightarrow H^0(X, \mathcal{O}_X(i \cdot f^*L))
\]

\[
\sigma \longmapsto \sum_{1 < j_1 < j_2 \cdots < j_N \leq N} g_{j_1}^i \sigma \otimes g_{j_2}^i \sigma \otimes \cdots \otimes g_{j_N}^i \sigma
\]

**Remark 3.3.** In Construction 3.2, the definitions of \( R_i(\sigma) \) and \( \tau_i(\sigma) \) do not depend on the choice of ordering of \( G \).
Lemma 3.4. In the setting of Construction 3.2, given a section $\sigma \in H^0(X, \mathcal{O}_X(f^*L))$ there are relations:

$$R_i(\sigma) = \begin{cases} 
\sigma + \tau_i(\sigma) & \text{for } i = 1 \\
\sigma \otimes \tau_{i-1}(\sigma) + \tau_i(\sigma) & \text{for } 1 < i < N \\
\sigma \otimes \tau_{N-1}(\sigma) & \text{for } i = N.
\end{cases}$$

Proof. The cases $i = 1$ and $i = N$ are clear. For $1 < i < N$, we can split $R_i(\sigma)$ as follows

$$R_i(\sigma) = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq N} g_{j_1}^* \sigma \otimes g_{j_2}^* \sigma \otimes \cdots \otimes g_{j_i}^* \sigma$$

$$= \sum_{1 = j_1 < j_2 < \ldots < j_i \leq N} g_{j_1}^* \sigma \otimes g_{j_2}^* \sigma \otimes \cdots \otimes g_{j_i}^* \sigma$$

$$+ \sum_{1 < j_1 < \ldots < j_i \leq N} g_{j_1}^* \sigma \otimes g_{j_2}^* \sigma \otimes \cdots \otimes g_{j_i}^* \sigma$$

$$= \sigma \otimes \tau_{i-1}(\sigma) + \tau_i(\sigma). \quad \Box$$

Lemma 3.5. In the setting of Construction 3.2, let $\sigma \in H^0(X, \mathcal{O}_X(f^*L))$ and $x \in X$ be a scheme theoretic point such that $\sigma(x) \neq 0$. Then, there exists an $1 \leq i \leq N$ such that $R_i(\sigma)(x) \neq 0$.

Proof. Suppose that $R_i(\sigma)(x) = 0$ for all $i = 1, \ldots, N - 1$. Then by the relations of Lemma 3.4, one finds that

$$R_N(\sigma)(x) = (-1)^{N-1} \sigma \otimes \cdots \otimes \sigma \neq 0.$$ 

This implies that $R_N(\sigma)(x)$ is non-zero as it is assumed that $\sigma(x) \neq 0$. Hence $R_i(\sigma)(x)$ cannot simultaneously vanish for all $1 \leq i \leq N$. \hfill \Box

3.2. Proof of Theorem 3.1. Lemma 2.12 allows us to pass to the Galois closure and, without loss of generality, assume that $f : X \to Y$ is the quotient of $X$ by a finite group $G$.

The inclusions $\mathcal{B}(f^*L) \subseteq f^{-1}\mathcal{B}(L)$ follows from Lemma 2.11.

Conversely, take $x \notin \mathcal{B}(f^*D)$. This means that there exists an $m \in \mathbb{N}$ and a section $\sigma \in H^0(X, \mathcal{O}_X(m \cdot f^*D))$ such that $\sigma(x) \neq 0$. To show that $x \notin f^{-1}\mathcal{B}(D)$, we need to produce a section in $H^0(Y, \mathcal{O}_Y(n \cdot D))$, for some $n > 0$, that does not vanish at $f(x)$. By Lemma 3.5, we find that there exists an $i$, such that $R_i(\sigma) \in f^*H^0(Y, \mathcal{O}_Y(mi \cdot D))$ satisfying $R_i(\sigma)(x) \neq 0$. \hfill \Box

4. The Case of inseparable morphisms

The goal of this section is to remove the separability hypothesis of Theorem 3.1, and hence proving Theorem 1.1. In characteristic 0, every finite morphism is automatically separable. In positive characteristic, a finite field extension splits into a separable extension and a purely inseparable extension. Hence it enough to study purely inseparable morphisms. In this case, the idea is to use Ekedahl’s theory of foliations and purely inseparable morphisms to factorise a purely inseparable $k$-morphism into purely inseparable morphisms of height one [Eke87]. We then study the pullback of the stable and augmented base locus via a purely inseparable morphism of height one.

Setting 4.1. Assume that the perfect field $k$ has characteristic $p > 0$. 

4.1. 

Ekedahl’s theory of foliations and purely inseparable morphisms.

The original paper [Eke87] covers the setting of smooth $k$-varieties. However, it is crucial to work in the setting of normal $k$-varieties. The excellent [PW22] gives a nice exposition of this more general setting in [PW22, Section 2.4], which we follow closely.

**Definition 4.2 (Absolute Frobenius).** In Setting 4.1, let $X$ be a normal $k$-variety. The absolute Frobenius morphism $F_X : X \rightarrow X$ is the endomorphism of schemes induced by the identity on the underlying topological space and the morphism between the sheaves of rings, $F_X^* : \mathcal{O}_X \rightarrow \mathcal{O}_X$, is given by sending $s \mapsto s^p$.

**Remark 4.3.** In general, the absolute Frobenius morphism is not a $k$-morphism. The pushforward $(F_X)_* \mathcal{O}_X$ is isomorphic to $\mathcal{O}_X$ as sheaves. However the $\mathcal{O}_X$-module structure on $(F_X)_* \mathcal{O}_X$ is given by the rule

$$(r, s) \mapsto r^p \cdot s.$$

We will label $(F_X)_* \mathcal{O}_X$ with this $\mathcal{O}_X$-module structure by $\mathcal{O}_X^{\text{Frob}}$.

**Definition 4.4 (Purely inseparable morphism).** In Setting 4.1, a finite surjective morphism $g : X \rightarrow Z$ between normal $k$-varieties is called purely inseparable if the field extension $k(Z) \subseteq k(X)$ is a purely inseparable extension.

**Definition 4.5 (Purely inseparable morphism of height one).** In Setting 4.1, a finite surjective morphism $g : X \rightarrow Z$ between normal $k$-varieties is called purely inseparable of height one if it is purely inseparable and there exists a morphism $h : Z \rightarrow X$ such that $h \circ g = F_X$, where $F_X$ is the absolute Frobenius of $X$.

4.1.1. Foliations and finite morphisms. Here we highlight how to obtain a foliation from a finite morphism and vice-versa.

**Construction 4.6 (Quotient by a foliation).** In Setting 4.1, let $X$ be a normal $k$-variety and $\mathcal{F}$ a foliation on $X$. The functions of $X$ that are annihilated by $\mathcal{F}$ define a sheaf $\mathcal{A}_\mathcal{F}$. More precisely, the assignment sending each open set $U \subseteq X$ to

$$\mathcal{A}_\mathcal{F}(U) := \{ s \in \mathcal{O}_X(U) \mid \forall V \subseteq U \text{ open}, \forall \partial \in \mathcal{F}(V) : \partial s = 0 \},$$

defines $\mathcal{A}_\mathcal{F}$. The $\mathcal{O}_X$-module structure of $\mathcal{A}_\mathcal{F}$ is defined by the multiplication rule

$$(r, s) \mapsto r^p \cdot s.$$

There is an inclusion of $\mathcal{O}_X$-modules $\mathcal{A}_\mathcal{F} \hookrightarrow \mathcal{O}_X^{\text{Frob}}$. Taking the relative spectrum of this inclusion we obtain

$$
\begin{array}{ccc}
X & \rightarrow & \text{Spec}_X \mathcal{A}_\mathcal{F} \\
\downarrow & & \downarrow \\
\text{Spec } k & & \text{Spec } k
\end{array}
$$

The $k$-morphism $X \rightarrow \text{Spec}_X \mathcal{A}_\mathcal{F}$ is called the quotient of $X$ by $\mathcal{F}$.

---

\[4\] not necessarily a $k$-morphism
Construction 4.7 (Foliation associated to a finite morphism). In Setting 4.1, let \( f : X \to Y \) be a finite surjective \( k \)-morphism between normal \( k \)-varieties. The sections of \( \mathcal{F}_X \) which annihilate \( f^{-1}\mathcal{O}_Y \subseteq \mathcal{O}_X \) define a subsheaf of \( \mathcal{F}_X \). More precisely, the assignment sending each open set \( U \subseteq X \) to
\[
\mathcal{F}_f(U) := \{ \partial \in \mathcal{F}_X(U) \mid \forall V \subseteq U \text{ open, } \forall s \in f^{-1}\mathcal{O}_Y(V) : \partial s = 0 \}
\]
defines a subsheaf of \( \mathcal{O}_X \)-modules of \( \mathcal{F}_X \).

Lemma 4.8. In Construction 4.7, the sheaf \( \mathcal{F}_f \) is a \( p \)-closed foliation.

Proof. To show that \( \mathcal{F}_f \) is closed under the Lie bracket, fix open sets \( V \subseteq U \subseteq X \) and a function \( s \in f^{-1}\mathcal{O}_Y(V) \). For all \( \partial, \partial' \in \mathcal{F}_f(U) \), compute the Lie bracket to find
\[
[\partial, \partial'](s) = \partial(\partial'(s)) - \partial'(\partial(s)) = 0.
\]
A similar computation shows that \( \mathcal{F}_f \) is \( p \)-closed.

Now we show that \( \mathcal{F}_f \) is a saturated in \( \mathcal{F}_X \). If \( \mathcal{F}_f = \mathcal{F}_X \) or \( \mathcal{F}_f = 0 \), then there is nothing to prove, so let us assume \( \mathcal{F}_f \) is a proper subsheaf. Fix an open set \( U \subseteq X \). Suppose for a contradiction that \( \tau \) is a non-zero torsion section in \( \mathcal{F}_X \mid \mathcal{F}_f(U) \). By shrinking \( U \) to a smaller open set, if necessary, we can choose a lift \( \partial \in \mathcal{F}_X(U) \) of \( \tau \). Since \( \tau \) is non-zero, this implies that \( \partial \notin \mathcal{F}_f(U) \). Since \( \tau \) is a torsion section, there exists a non zero function \( r \in \mathcal{O}_X(U) \) such that \( r \cdot \tau = 0 \). This means that \( r \cdot \partial \notin \mathcal{F}_f(U) \).

To arrive at the contradiction, pick a section \( t \in f^{-1}\mathcal{O}_Y(V) \) for some \( V \subseteq U \) open such that its evaluation \( \partial(t) \neq 0 \) however then \( (r \cdot \partial)(t) = 0 \). Since \( X \) is integral, as it is normal, this leads to contradiction as it suggests that \( r \) is a zero divisor. \( \Box \)

4.1.2. Ekedahl’s Theorem. Restricting the class of morphisms in Construction 4.7, we obtain a correspondence between \( p \)-closed foliations and purely inseparable \( k \)-morphisms of height one.

Remark 4.9. It is a necessary condition that \( k \) is a perfect field in Theorem 4.10 below.

Theorem 4.10 ([PW22, Proposition 2.9]). In Setting 4.1, let \( X \) be a normal \( k \)-variety. There is a 1-1 correspondence between

(4.10.1) \( p \)-closed foliations \( \mathcal{F} \subset \mathcal{F}_X \), and
(4.10.2) purely inseparable \( k \)-morphisms \( X \to Z \) of height one.

The correspondence is given by:

- sending the foliation \( \mathcal{F} \) to the quotient of \( X \) by \( \mathcal{F} \), and
- sending the morphism \( X \to Z \) to the \( p \)-closed foliation \( \mathcal{F}_g \).

Moreover the correspondence satisfies \( \gcd(k(Z) : k(X)) = p^{\text{rank } \mathcal{F}} \). \( \Box \)

Remark 4.11. Theorem 4.10 asserts that the quotient by a \( p \)-closed foliation is in particular a normal variety, and the quotient morphism is finite and surjective.
4.2. Factorising purely inseparable morphisms. Using Theorem 4.10, we will factorise a purely inseparable $k$-morphism into height one pieces.

**Proposition 4.12.** Let $f : X \rightarrow Y$ be a purely inseparable $k$-morphism between normal $k$-varieties. Then there exists a positive integer $l$ and normal $k$-varieties $Z_1, \ldots, Z_l$ that factorise $f$

$$X = Z_1 \rightarrow Z_2 \rightarrow \cdots \rightarrow Z_l = Y.$$  

The morphisms $Z_l \rightarrow Z_{l+1}$ are purely inseparable $k$-morphisms of height one.

To prove Proposition 4.12, we will need the following technical lemma.

**Lemma 4.13.** In Construction 4.7, the foliation $\mathcal{F}_f$ is zero if and only if $f$ is a separable morphism.

**Remark 4.14.** The consequence of Lemma 4.13 is that, the quotient by $\mathcal{F}_f$ is the identity morphism if and only if $f$ is a separable morphism.

We will prove Lemma 4.13 after we give a proof of Proposition 4.12.

**Proof of Proposition 4.12.** If the morphism $f : X \rightarrow Y$ is the identity, we are done. Supposing otherwise, we will show that then it is possible to factorise $f : X \rightarrow Y$ into

$$(4.14.1) \quad X \xrightarrow{g} Z_2 \xrightarrow{f_2} Y$$

where $g$ is a purely inseparable $k$-morphism of height one and $f_2$ is a purely inseparable $k$-morphism. The morphism $g$ is simply the quotient of the foliation associated to the finite morphism $f : X \rightarrow Y$, which by Theorem 4.10 is purely inseparable of height one and by Lemma 4.13 is not the identity morphism.

Using the fact that purely inseparable morphisms are universal homeomorphisms [GW10, Exercise 12.32], we find that $f$ and $g$ are universal homeomorphisms and that the underlying topological spaces of $Y$, $X$ and $Z_2$ are all homeomorphic. Hence the morphism $f_2 : |Z_2| \rightarrow |Y|$, between the underlying topological spaces is given by $g^{-1} \circ f$.

For the morphism of sheaves of rings, observe that there is an inclusion of sheaves $f^{-1} \mathcal{O}_Y \subseteq g^{-1} \mathcal{O}_{Z_2} \subseteq \mathcal{O}_X$. The sheaf $g^{-1} \mathcal{O}_{Z_2}$ is the sheaf of functions annihilated by the foliation $\mathcal{F}_f$, which contains $f^{-1} \mathcal{O}_Y$, as in Construction 4.6. For an open set $V \subseteq X$ we obtain a ring homomorphism

$$(4.14.2) \quad \mathcal{O}_Y (f(V)) \hookrightarrow \mathcal{O}_{Z_2} (g(V)).$$

For any open set $U \subseteq Z_2$, by setting $V = g^{-1}(U)$ in Equation (4.14.2), we obtain a well defined morphism of sheaves $f_2^* : f_2^{-1} \mathcal{O}_Y \rightarrow \mathcal{O}_{Z_2}$. Hence we get a morphism of schemes $f_2 : Z_2 \rightarrow Y$ factorising $f$ as found in Equation (4.14.1).

We proceed by applying the same argument to $f_2 : Z_2 \rightarrow Y$. Since $f : X \rightarrow Y$ a finite morphism, this process of factorising must stop after a finite number of steps. This proves the proposition. □

In preparation to prove Lemma 4.13, it will be useful to give an alternative description of $\mathcal{F}_f$ in terms of differentials, to give us a concrete connection between $\mathcal{F}_f$ and the separability of $f$. 


Construction 4.15. In the Setting of Construction 4.7, let $\mathcal{M}$ be the image of the map $f^*\Omega^1_Y \rightarrow \Omega^1_X$. Denote by $\text{Ann} \mathcal{M}$ the sections of $\mathcal{T}_X$ that are annihilated by $\mathcal{M}$. This is described in the following exact sequence:

$$(4.15.1) \quad 0 \rightarrow \text{Ann} \mathcal{M} \rightarrow \mathcal{T}_X \rightarrow \mathcal{M}^\vee.$$  

Moreover, since the quotient of $f^*\Omega^1_Y$ by $\mathcal{M}$ is $\Omega^1_{X/Y}$, we find that $\text{Ann} \mathcal{M} = (\Omega^1_{X/Y})^\vee$.

Lemma 4.16. The sheaves $\mathcal{F}_f$ and $\text{Ann} \mathcal{M}$ constructed in Construction 4.7 and Construction 4.15 are isomorphic.

Proof. Let $U \subseteq X$ be an open set and fix a section $s \in f^{-1}\mathcal{O}_Y(U)$ and a derivation $\partial \in \text{Ann} \mathcal{M}(U)$. Observe that $\partial ds = \partial(s) = 0$, which shows that $\partial \in \mathcal{F}_f(U)$.

Conversely, fix a derivation $\partial \in \mathcal{F}_f(U)$ for any open set $U \subseteq X$. We need to show that for any $\omega \in \mathcal{M}(U)$, that $\partial \omega = 0$. Since $\mathcal{M}$ is locally generated by sections of the form $f^*dy$, where $y$ is a local section of $\mathcal{O}_Y$, it is enough to check that $\partial(f^*dy) = 0$. This follows from the computation $\partial(f^*dy) = f^*(\partial y) = 0$, since $f^*y$ is a local section of $f^{-1}\mathcal{O}_Y$.

$\square$

Proof of Lemma 4.13. By Lemma 4.16 and Construction 4.15, we get that $\mathcal{F}_f = \text{Ann} \mathcal{M} = (\Omega^1_{X/Y})^\vee$. This allows us to work with relative differentials.

Suppose that $f : X \rightarrow Y$ is separable. Since $\Omega^1_{X/Y}$ is supported on the ramification locus of $f$, it is torsion. Therefore the dual $(\Omega^1_{X/Y})^\vee = \mathcal{F}_f = 0$.

Conversely, suppose that $f$ is not separable. Denote the generic point of $X$ by $\eta$. We compute that

$$(\text{Ann} \mathcal{M})_\eta = \left((\Omega^1_{X/Y})^\vee\right)_\eta = \left((\Omega^1_{X/Y})_\eta\right)^\vee \quad [\text{Har}77, \text{III Proposition 6.8]}

= (\Omega^1_{K(X)/K(Y)})^\vee \neq 0 \quad [\text{Har}77, \text{II Theorem 8.6A}].$$

Hence $\mathcal{F}_f \neq 0$. $\square$

4.3. Stable base locus under purely inseparable morphisms. In this subsection, we make use of the Frobenius morphism to understand the pullback of the stable base locus via a purely inseparable height one morphism.

Lemma 4.17. In Setting 4.1, let $F_X : X \rightarrow X$ be the absolute Frobenius morphism and let $D$ be a $\mathbb{Q}$-divisor on $X$. Then $\mathbb{B}(F_X^*D) = F_X^{-1}\mathbb{B}(D)$.

Proof. The left hand side of the expression is simply $\mathbb{B}(pD)$ since $F_X^*D = pD$. The right hand side is $\mathbb{B}(D)$ as the stable base locus is a closed subset of $X$ and the absolute Frobenius morphism leaves the points of $X$ fixed. The expression immediately follows from the observation that $\mathbb{B}(pD) = \mathbb{B}(D)$. $\square$

Lemma 4.18. In Setting 4.1, let $g : X \rightarrow Z$ be a purely inseparable height one morphism$^3$ and $D$ a $\mathbb{Q}$-divisor on $Z$. Then $\mathbb{B}(g^*D) = g^{-1}\mathbb{B}(D)$.

Proof. Since $g$ is purely inseparable of height one, there exists a morphism $Z \rightarrow i : X$ such that $g \circ i = F_X$. We claim that $i \circ g = F_Z$, the Frobenius endomorphism of $Z$. Then by Lemma 4.17 and Lemma 2.12 the result follows.

$^3$not necessarily a $k$-morphism
To prove the claim, observe that $F_X \circ g = \varphi F_Z$ by [Sta18, Tag 0CC7]. Since $F_X = g \circ i$, if we can show that $g$ is an epimorphism, or in other words left cancellative, then the claim follows. This follows from the observation that $g$ is surjective and the map of sheaves of rings $g^\#: \mathcal{O}_Z \to \mathcal{O}_X$ is injective. □

4.4. Proof of Theorem 1.1. We start with a series of reduction steps. It is enough to prove Statement (1.1.1) due to Lemma 2.10.

In characteristic 0, the morphism $f : X \to Y$ is necessarily separable, this case is covered by Theorem 3.1. We can then assume we are in Setting 4.1, namely in positive characteristic.

Since any finite field extension can be split into a separable extension followed by a purely inseparable extension [Sta18, Tag 030K], we can assume without loss of generality that $f : X \to Y$ is a purely inseparable extension.

By Proposition 4.12, we obtain a factorisation of $f$

$$X \to Z_1 \to \cdots \to Z_l \to Y.$$ Since all the morphisms in the factorisation are purely inseparable of height one, by Lemma 4.18 we get $\mathbb{B}(f^*D) = f^{-1}\mathbb{B}(D)$.

References

[BdPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell, The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension, J. Algebraic Geom. 22 (2013), no. 2, 201–248, DOI:10.1090/s0161-116x-2012-00257-0, preprint: arXiv:math/0405285, ↑ 1

[Bir17] Caucher Birkar, The augmented base locus of real divisors over arbitrary fields, Math. Ann. 368 (2017), no. 3–4, 905–921, DOI:10.1007/s00208-016-1441-y, preprint: arXiv:1312.0239, ↑ 1, 2, 3

[Bou04] Sébastien Boucksom, Divisorial Zariski decompositions on compact complex manifolds, Ann. Sci. École Norm. Sup. (4) 37 (2004), no. 1, 45–76, DOI:10.1016/j.ansens.2003.04.002, ↑ 1

[BDCL21] Luca Di Cerbo and Luca Di Cerbo, On Seshadri constants of varieties with large fundamental group, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 19 (2019), no. 1, 335–344, DOI:10.2422/2036-2145.201609_012, preprint: arXiv:1411.1033, ↑ 1

[Bou06] Sébastien Boucksom, Foliations and inseparable morphisms, Proc. Symp. Pure Math. 86 part 2 (1987), 139–149, DOI:10.1090/pspum/086.2/927978, ↑ 5, 6

[ELM*06] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa, Asymptotic invariants of base loci, Ann. Inst. Fourier 56 (2006), no. 6, 1701–1734, DOI:10.5802/aif.2225, preprint: arXiv:0308116, ↑ 1, 3

[ELM*09] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa, Restricted volumes and base loci of linear series, Amer. J. Math. 131 (2009), no. 3, 607–651, Available at: jstor.org/stable/40263793, ↑ 3

[ELM*08] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa, Foliations and inseparable morphisms, Proc. Symp. Pure Math. 86 part 2 (2008), 139–149, DOI:10.1090/pspum/086.2/927978, ↑ 5, 6

[ELM*10] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa, Asymptotic invariants of base loci, Ann. Inst. Fourier 56 (2006), no. 6, 1701–1734, DOI:10.5802/aif.2225, preprint: arXiv:0308116, ↑ 1, 3

[GW10] Ulrich Görtz and Torsten Wedhorn, Algebraic geometry I. Advanced Lectures in Mathematics, Vieweg + Teubner, Wiesbaden, 2010, Schemes with examples and exercises. ↑ 8

[Har77] Robin Hartshorne, Algebraic geometry, Graduate Texts in Mathematics: 52, Springer-Verlag, New York, 1977, DOI: 10.1007/978-1-4757-3849-0, ↑ 9

[Laz04a] Robert Lazarsfeld, Positivity in algebraic geometry. I. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 48, Springer-Verlag, Berlin, 2004, DOI:10.1007/978-3-642-18808-4, ↑ 3

[Laz04b] Robert Lazarsfeld, Positivity in algebraic geometry. II. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 49, Springer-Verlag, Berlin, 2004, DOI:10.1007/978-3-642-18810-7, ↑ 3
Michael Nakamaye, *Stable base loci of linear series*, Math. Ann. **318** (2000), no. 4, 837–847, DOI:10.1007/s002080000149.

Zsolt Patakfalvi and Joe Waldron, *Singularities of general fibers and the LMMP*, Amer. J. Math. **144** (2022), no. 2, 505–540, DOI:10.1353/ajm.2022.0009 preprint: arXiv:1708.04268.

The Stacks Project Authors, *Stacks Project*, https://stacks.math.columbia.edu, 2018.

Tanuj Gomez, Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Ernst-Zermelo-Straße 1, 79104 Freiburg im Breisgau, Germany

Email address: tanuj.gomez@hotmail.com