A Newton Algorithm for Semi-Discrete Optimal Transport with Storage Fees and Quantitative Convergence of Cells

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Abstract. In this paper we will continue analysis of the semi-discrete optimal transport problem with storage fees, previously introduced by the authors, by proving convergence of a damped Newton algorithm for a specific choice of storage fee function, along with quantitative convergence of the associated Laguerre cells under limits of various parameters associated with the problem. A convergence result for cells in measure is proven without the additional assumption of a Poincaré-Wirtinger inequality on the source measure, while convergence in Hausdorff metric is shown when assuming such an inequality. These convergence results also yield approximations to the classical semi-discrete optimal transport problem.

Contents

1. Introduction
   1.1. Semi-discrete optimal transport with storage fees
2. Setup
   2.1. Notation and conventions
   2.2. Classical optimal transport
   2.3. Outline of the paper
3. Properties of the mapping $w_{h,\epsilon}$
   3.1. Solutions of the approximating problem with $F_{w,h,\epsilon}$
   3.2. Estimates on $w_{h,\epsilon}$
4. Convergence of Algorithm 1
5. $\mu$-symmetric convergence of Laguerre cells
   5.1. The Exchange Digraph
   5.2. Proof of Theorem 2.11
6. Estimates on Hausdorff Distance
7. Injectivity of $G$
8. Quantitative Hausdorff Convergence
   8.1. Alternative spectral estimates on $DG$
   8.2. Quantitative invertibility of $G$
A. Strong Convexity of $C$
References

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1. Introduction

1.1. Semi-discrete optimal transport with storage fees. In this paper we deal with the following problem. Let $X \subset \mathbb{R}^n$, $n \geq 2$ be compact and $Y := \{y_i\}_{i=1}^N \subset \mathbb{R}^n$ a fixed collection of finite points, along with a cost function $c : X \times Y \to \mathbb{R}$ and a storage fee function $F : \mathbb{R}^N \to \mathbb{R}$. We also fix a Borel probability measure $\mu$ with $\text{spt} \, \mu \subset X$, and assume $\mu$ is absolutely continuous with respect to Lebesgue measure. The classical (semi-discrete) optimal transport problem would be to fix a discrete probability measure $\nu$ supported on $Y$, and to find a measurable mapping $T : X \to Y$ such that $T_\# \mu(E) := \mu(T^{-1}(E)) = \nu(E)$ for any measurable $E \subset Y$, and $T$ satisfies

\[
\int_X c(x, T(x)) d\mu = \min_{T_\# \mu = \nu} \int_X c(x, \tilde{T}(x)) d\mu.
\]

(1.1)

The variation introduced in [BK19] is instead, to find a pair $(T, \lambda)$ with $\lambda = (\lambda^1, \ldots, \lambda^N) \in \mathbb{R}^N$ and $T : X \to Y$ measurable satisfying

\[
T_\# \mu = \sum_{i=1}^N \lambda^i \delta_{y_i},
\]

such that

\[
\int_X c(x, T(x)) d\mu + F(\lambda) = \min_{\lambda \in \mathbb{R}^N, \ T_\# \mu = \sum_{i=1}^N \lambda^i \delta_{y_i}} \int_X c(x, \tilde{T}(x)) d\mu + F(\tilde{\lambda}).
\]

(1.2)

In [BK19] the authors have shown under appropriate conditions, existence of solutions to the above variant, along with a dual problem with strong duality, and a characterization of dual maximizers and primal minimizers.

In this paper, our goal is to propose and show convergence of a damped Newton algorithm, in the vein of [KMT19] for the classical semi-discrete transport problem, in the case of storage function given by

\[
F(\lambda) = F_w(\lambda) := \begin{cases} 
0, & \lambda \in \prod_{i=1}^N [0, w^i], \\
+\infty, & \text{else},
\end{cases}
\]

(1.3)

where $w = (w^1, \ldots, w^N) \in \mathbb{R}^N$ is some fixed vector with nonnegative components. The minimization (1.2) with this choice of $F$ can be thought of as a problem where the $i$th target point has a hard capacity constraint given by $w^i$, with no other associated cost of storage. In particular, if $w$ is chosen in such a way that $\sum_{i=1}^N w^i = 1$, it is easily seen this recovers the classical optimal transport problem with target measure $\nu = \sum_{i=1}^N w^i \delta_{y_i}$.

There are a number of difficulties that prevent a direct translation of the damped Newton algorithm from [KMT19] to the above problem. First, in the classical case one fixes a discrete target measure $\nu = \sum_{i=1}^N \lambda^i \delta_{y_i}$, and the Newton algorithm is used to approximate the weight vector $\lambda = (\lambda^1, \ldots, \lambda^N)$. However, in our problem with storage fees, the weight vector $\lambda$ itself must be chosen as part of the minimization and hence is not fixed, thus it is not even a priori clear what quantity to approximate with a Newton algorithm. Additionally, unlike the classical problem, it is possible that $\lambda^i = 0$ for one or more of the entries in an optimal choice for the weight vector, but the algorithm from [KMT19] uses the assumption that all $\lambda^i$ have strictly positive lower bounds in a crucial way to obtain the convergence. To remedy these issues, we will first attempt to approximate the storage function $F_w$ instead:
we will use the characterization for solutions found in \cite{BK19} to find approximating storage functions $\tilde{F}_w$, along with minimizers of the problem \eqref{eq:1.2} with $F = \tilde{F}_w$. However, a second difficulty arises as the functions of the form $F_w$ have both highly singular behavior in their subdifferentials at the boundary of their effective domains, while being nonstrictly convex everywhere. Thus, we will further replace functions of this form with uniformly convex, smooth approximations. Finally, in order to show this further approximation does not take us too far afield of the original problem, we prove that the “cells” arising in the problem, i.e., the sets $T^{-1}(y_i)$ for each $i$, for the optimal maps $T$ that are constructed, will converge to those of the original problem as the approximating storage functions converge to the original $F_w$. This convergence is in the $L^1(\mu)$ sense, and for optimal cells which are nonempty for the original problem we actually obtain Hausdorff convergence as well. The convergence result for the damped Newton algorithm and the $L^1(\mu)$ convergence of cells will be proved under the same conditions on the cost function $c$ and the source measure $\mu$ as in \cite{KMT19}, but without the assumption of a Poincaré-Wirtinger inequality. For the result on Hausdorff convergence, in order to obtain a quantitative result we will assume a slightly stronger version of the Poincaré-Wirtinger inequality than in \cite{KMT19}.

**Remark 1.1** (Data clustering). One application of the problem we consider here is to data clustering. Suppose there is some data set that is so large, it can be viewed as being distributed according to an absolutely continuous measure $\mu$. The goal is then to partition the data into $N$ clusters, where for each cluster a “representative element” $y_i$ is given and the affinity of the data is measured by the cost function $c$. Then, solving the optimal transport problem with storage fee $F_w$ will yield such a clustering, with the additional constraint that the $i$th cluster can be no larger than $w_i$.

In Section 2 below we introduce some preliminary notions in order to be able to state our damped Newton algorithm. As such, we also defer the statement of our main theorems to Section 2 along with the outline for the remainder of the paper.

1.1.1. **Prior results.** There are a number of existing numerical methods which apply a Newton’s algorithm to semi-discrete optimal transport and Monge-Ampère equations. The original idea of approximating a solution to the semi-discrete Monge-Ampère equation via constructing an envelope of affine functions goes back to Aleksandrov and Pogorelov (in the geometric setting of Minkowski’s theorem, \cite{Ale05}). The authors of \cite{OP88} apply a Newton method to solve a semi-discrete Monge-Ampère equation with Dirichlet boundary condition, and prove local convergence of their algorithm, and global convergence was established in \cite{Mir15}; their setting is for weak solutions of Aleksandrov type which differ slightly from optimal transport solutions.

In the context of solutions to the classical optimal transport problem, the variational method of extremizing the so-called Kantorovich functional goes back to \cite{AHA98}, and \cite{BFO14} observe good empirical behavior of Newton type methods for this problem (but without convergence proofs). The case of optimal transport with the quadratic cost on the torus is solved with a damped Newton method in \cite{LR05} with a proof of convergence based on regularity theory of the Monge-Ampère equation due to Caffarelli (\cite{Caf92}), and is refined in \cite{SAK15}. A joint work of the second author, \cite{KMT19} gives a damped Newton algorithm that applies to a wider class of cost functions and proves global linear and local superlinear
convergence for Hölder continuous source measures satisfying a Poincaré-Wirtinger inequality. Finally, [MMT18] shows convergence for a damped Newton algorithm when the source is a singular measure supported on unions of simplices.

2. Setup

2.1. Notation and conventions. Here we gather notation and conventions to be used in the remainder of the paper. As mentioned above, we fix positive integers $N$ and $n$ and a collection $Y := \{y_i\}_{i=1}^N \subset \mathbb{R}^n$. The standard $N$-simplex will be denoted

$$\Lambda := \{\lambda \in \mathbb{R}^N \mid \sum_{i=1}^N \lambda^i = 1, \lambda^i \geq 0\},$$

and to any vector $\lambda \in \Lambda$ we associate the discrete measure $\nu_\lambda := \sum_{i=1}^N \lambda^i \delta_{y_i}$. The notation $\mathbf{1}$ will refer to the vector in $\mathbb{R}^N$ whose components are all 1. We also reserve the notation $\|V\| := \sqrt{\sum_{i=1}^N |V^i|^2}$ for the Euclidean ($\ell^2$) norm of a vector $V \in \mathbb{R}^N$, while $\|V\|_1 := \sum_{i=1}^N |V^i|$ and $\|V\|_\infty := \max_{i \in \{1, \ldots, N\}} |V^i|$ will respectively stand for the $\ell^1$ and $\ell^\infty$ norms. We also write $\|M\|$ for the operator norm of a matrix $M$, the distinction from the Euclidean norm of a vector should be clear from context.

Given any set $A$, we write $\delta(x \mid A) := \begin{cases} 0, & x \in A, \\ +\infty, & x \notin A, \end{cases}$ for the indicator function of the set $A$, and for any vector $w \in \mathbb{R}^N$ with nonnegative entries, we denote $F_w := \sum_{i=1}^N \delta(\cdot \mid [0, w^i]) = \delta(\cdot \mid \prod_{i=1}^N [0, w^i])$. We will also use $\mathcal{L}$ to denote the $n$-dimensional Lebesgue measure and $\mathcal{H}^k$ for the $k$-dimensional Hausdorff measure.

Regarding the cost function $c$, we will generally assume the following standard conditions from optimal transport theory:

\begin{align*}
(\text{Reg}) & \quad c(\cdot, y_i) \in C^2(X), \forall i \in \{1, \ldots, N\}, \\
(\text{Twist}) & \quad \nabla_x c(x, y_i) \neq \nabla_x c(x, y_k), \forall x \in X, i \neq k.
\end{align*}

We also assume the following condition, originally studied by Loeper in [Loe09].

**Definition 2.1.** We say $c$ satisfies *Loeper’s condition* if for each $i \in \{1, \ldots, N\}$ there exists a convex set $X_i \subset \mathbb{R}^n$ and a $C^2$ diffeomorphism $\exp_i^t(\cdot) : X_i \to X$ such that

$$\forall t \in \mathbb{R}, 1 \leq k, i \leq N, \{p \in X_i \mid -c(\exp_i^t(p), y_k) + c(\exp_i^t(p), y_i) \leq t\} \text{ is convex.}$$

See Remark 2.4 below for a discussion of these conditions.

We also say that a set $\bar{X} \subset X$ is $c$-convex with respect to $Y$ if $(\exp_i^{-1}(\bar{X}))$ is a convex set for every $i \in \{1, \ldots, N\}$.

It will be convenient to also introduce $c$-convex functions and the $c$ and $c^*$-transforms. In the semi-discrete case the $c^*$-transform of a function defined on $X$ will be a vector in $\mathbb{R}^N$, while the $c$-transform of a vector in $\mathbb{R}^N$ will be a function whose domain is $X$.

**Definition 2.2.** If $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ (not identically $+\infty$) and $\psi \in \mathbb{R}^N$, their $c$- and $c^*$-transforms are a vector $\varphi^c \in \mathbb{R}^N$ and a function $\psi^{c^*} : X \to \mathbb{R} \cup \{+\infty\}$ respectively, defined
by
\[(\varphi^c)^i := \sup_{x \in X} (-c(x, y_i) - \varphi(x)), \quad (\psi^c^\star)(x) := \max_{i \in \{1, \ldots, N\}} (-c(x, y_i) - \psi^i).\]

If \(\varphi : X \to \mathbb{R} \cup \{+\infty\}\) is the \(c^\star\)-transform of some vector in \(\mathbb{R}^N\), we say \(\varphi\) is a \(c^\star\)-convex function. A pair \((\varphi, \psi)\) with \(\varphi : X \to \mathbb{R} \cup \{+\infty\}\) and \(\psi \in \mathbb{R}^N\) is a \(c^\star\)-conjugate pair if \(\varphi = \psi^c^\star\) and \(\psi = \psi^c^\star\).

**Definition 2.3.** For any \(\psi \in \mathbb{R}^N\) and \(i \in \{1, \ldots, N\}\), we define the \(i\)th Laguerre cell associated to \(\psi\) as the set
\[\text{Lag}_i(\psi) := \{x \in X \mid -c(x, y_i) - \psi^i = \psi^c^\star(x)\}.
\]

We also define the function \(G : \mathbb{R}^N \to \Lambda\) by
\[G(\psi) := (G_1^i(\psi), \ldots, G_N^i(\psi)) = (\mu(\text{Lag}_1(\psi)), \ldots, \mu(\text{Lag}_N(\psi))\),
\]and denote for any \(\epsilon \geq 0\),
\[\mathcal{K}^\epsilon := \{\psi \in \mathbb{R}^N \mid G^i(\psi) > \epsilon, \forall i \in \{1, \ldots, N\}\}.
\]

**Remark 2.4.** The above conditions \((\text{Reg}), (\text{Twist}), (\text{QC})\) are the same ones assumed in \cite{KMT19}. As is also mentioned there, the conditions \((\text{Reg})\) and \((\text{Twist})\) are standard in the existence theory of optimal transport, while \((\text{QC})\) holds if \(Y\) is a finite set sampled from from a continuous space, and \(c\) is a \(C^4\) cost function satisfying what is known as the Ma-Trudinger-Wang condition (along with an additional convexity assumption on the domain of \(c\), which we do not detail here). The Ma-Trudinger-Wang condition was first introduced in a strong form in \cite{MTW05}, and in \cite{TW09} in a weaker form. This is also known to be a necessary condition for the regularity theory of the Monge-Ampère type equation arising in optimal transport, see \cite{Loc09}.

If \(\mu\) is absolutely continuous with respect to Lebesgue measure, under \((\text{Twist})\) the Laguerre cells associated to different indices are disjoint up to sets of \(\mu\)-measure zero. Then by the generalized Brenier’s theorem \cite[Theorem 10.28]{Vil09}, for any vector \(\psi \in \mathbb{R}^N\) it is known that the \(\mu\text{-a.e. single valued map} T_\psi : X \to Y\) defined by \(T_\psi(x) = y_i\) whenever \(x \in \text{Lag}_i(\psi)\), is a minimizer in the classical optimal transport problem \((\mathbf{1.1})\), where the source measure is \(\mu\) and the target measure is defined by \(\nu = \nu_G(\psi)\).

In order to introduce the damped Newton algorithm we will analyze for our problem \((\mathbf{1.2})\), we must introduce a few more pieces of notation. The motivation for these will be explained in detail in the following section.

**Definition 2.5.** For \(h > 0\) and \(\epsilon \geq 0\) define the functions \(g : \mathbb{R} \to \mathbb{R}\) and \(w_{h,\epsilon} : \mathbb{R}^N \to \mathbb{R}^N\) by
\[g(t) := 2 \left( 1 + t^2 - t\sqrt{1 + t^2} \right),
\]
\[w^i_{h,\epsilon}(\psi) := (G^i(\psi) - \epsilon)g \left( \frac{\psi^i}{h} \right).
\]
Also, we write for any \(\epsilon_0 > 0\)
\[\mathcal{W}^{\epsilon_0} := \{\psi \in \mathbb{R}^N \mid w^i_{h,\epsilon}(\psi) \geq \epsilon_0, \forall i \in \{1, \ldots, N\}\}.
\]
and given any \( w \in \mathbb{R}^N \) with nonnegative entries, we define the set
\[
\Sigma_{w,h,\epsilon} := \{ \psi \in \mathcal{K}^\epsilon \mid \sum_{i=1}^N w^i = \sum_{i=1}^N w_{h,\epsilon}^i(\psi) \}.
\]

We now use the above notation to propose the following damped Newton algorithm to approximate solutions of (1.2). Note below, we do not lose any generality in assuming \( w^i \leq 1 \) for each \( i \), as \( \mu \) is a probability measure.

**Parameters:** Fix \( h, \epsilon > 0 \), and \( w \in \mathbb{R}^N \) such that \( \sum_{i=1}^N w^i \geq 1 \), \( w^i \in [0, 1] \).

**Input:** A tolerance \( \zeta > 0 \) and an initial \( \psi_0 \in \mathbb{R}^N \) such that
\[
\epsilon_0 := \frac{1}{2} \min \left[ \min_i w_{h,\epsilon}^i(\psi_0), \min_i w^i \right] > 0.
\]

**While:** \( \|w_{h,\epsilon}(\psi_k) - w\| \geq \zeta \)

**Step 1:** Compute \( \tilde{d}_k = -[Dw_{h,\epsilon}(\psi_k)]^{-1}(w_{h,\epsilon}(\psi_k) - w) \)

**Step 2:** For each \( \ell \in \mathbb{N} \) let \( r_\ell \in \mathbb{R} \) be such that \( \psi_{k+1,\ell} := \psi_k + 2^{-\ell} \tilde{d}_k + r_\ell 1 \) satisfies \( \psi_{k+1,\ell} \in \Sigma_{w,h,\epsilon} \).

**Step 3:** Determine the minimum \( \ell \in \mathbb{N} \) such that \( \psi_{k+1,\ell} \) satisfies
\[
\begin{align*}
\min_i w_{h,\epsilon}^i(\psi_{k+1,\ell}) &\geq \epsilon_0 \\
\|w_{h,\epsilon}(\psi_{k+1,\ell}) - w\| &\leq (1 - 2^{-\ell+1})\|w_{h,\epsilon}(\psi_k) - w\|
\end{align*}
\]

**Step 4:** Set \( \psi_{k+1} = \psi_k + 2^{-\ell} \tilde{d}_k + r_\ell 1 \) and \( k \leftarrow k + 1 \).

**Algorithm 1:** Damped Newton’s algorithm

We pause to provide some explanation of this algorithm. For \( h, \epsilon \geq 0 \) fixed, define for any \( t_0 \geq 0 \), the function \( \sigma_{t_0,h} : \mathbb{R} \to \mathbb{R} \) by
\[
(2.2) \quad \sigma_{t_0,h}(t) = \begin{cases} 
-h \sqrt{t(t_0-t)} & \text{if } t \in [0,t_0] \\
+\infty & \text{else}
\end{cases},
\]
and for any \( w \in \mathbb{R}^N \), \( w^i \geq 0 \), the function \( F_{w,h,\epsilon} : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \) by
\[
F_{w,h,\epsilon}(\lambda) = \sum_{i=1}^N \sigma_{w^i,h}(\lambda^i - \epsilon) + \delta(\lambda \mid \Lambda)
\]
\[
= \begin{cases} 
-h \sum_{i=1}^N \sqrt{(\lambda^i - \epsilon)(w^i - \lambda^i + \epsilon)} , & \lambda \in \Lambda \cap \prod_{i=1}^N [\epsilon, w^i + \epsilon] \\
+\infty , & \text{else}
\end{cases}
\]

It can be seen that \( F_{w,h,\epsilon} \) is a uniformly convex approximation to \( F_w = F_{w,0,0} \) when \( h, \epsilon > 0 \). Detailed calculations will be deferred to Proposition 3.2 in the following section, but if \( \psi \in \mathbb{R}^N \) is a vector such that \( w_{h,\epsilon}(\psi) = w \), using the results of [BK19] it can be seen for the map \( T_\psi \) defined as in Remark 2.4 the pair \( (T_\psi, G(\psi)) \) is the unique solution to the minimization problem (1.2) with storage fee function given by \( F_{w,h,\epsilon} \). Thus the algorithm generates a vector \( \psi \) and a storage fee function \( \tilde{F} \) approximating the original \( F_w \), such that \( (T_\psi, G(\psi)) \) solves the optimal transport problem with storage fee \( \tilde{F} \). The normalization
ψ ∈ Σ_{w,h,ϵ} at each step in Algorithm 1 is necessary in order to ensure that the magnitude of the error vector \( w_{h,ϵ}(ψ_k) - w \) will actually go to zero.

The first theorem of our paper is the following on convergence of the above algorithm. We remark that in contrast to the main result of [KMT19], we do not require a Poincaré-Wirtinger inequality on the measure \( μ \) (see Remark 2.13 below). Also, see Definition 2.9 below for the notion of a universal constant.

**Theorem 2.6.** Suppose \( c \) satisfies (Reg), (Twist), and (QC). Also suppose \( X \) is a bounded set that is \( c \)-convex with respect to \( Y \), \( μ = ρdx \) for some density \( ρ \in C^{0,α}(X) \) for some \( α ∈ (0,1] \), and \( \text{spt} \, μ ⊂ X \). Then if \( h ∈ (0,1) \), \( ϵ ∈ (0,1/2N) \), and \( \sum_{i=1}^{N} w_i ≥ 1 \), Algorithm 1 converges globally with linear rate, and locally with superlinear rate \( 1 + α^2 \).

Specifically, the iterates of Algorithm 1 satisfy

\[
\|w_{h,ϵ}(ψ_{k+1}) - w\| \leq (1 - \frac{1}{2})\|w_{h,ϵ}(ψ_k) - w\|
\]

where

\[
τ_k := \min \left( \frac{1}{8L\tilde{L}1+α√N}\|w_{h,ϵ}(ψ_k) - w\|, 1 \right).
\]

where \( L \) and \( κ \) are as in Proposition 3.3 and \( \tilde{L} ≤ \frac{C}{h^{18}ε^9} \) for some universal constant \( C \).

In addition as soon as \( τ_k = 1 \) we have

\[
\|w_{h,ϵ}(ψ_{k+1}) - w\| ≤ \frac{2LL1+α√N}\|w_{h,ϵ}(ψ_k) - w\|^{1+α^2}.
\]

**Remark 2.7.** In [KMT19], the goal is to find a root of the mapping \( G - β \) which is in fact the gradient of the concave dual functional in the Kantorovich problem. However, in our case the mapping \( w_{n,ϵ} - w \) is not the gradient of any scalar function (seen easily as \( Dw_{h,ϵ} \) is not symmetric).

Since Algorithm 1 only produces solutions to an approximating problem, we are concerned with how close these solutions might be to the solutions of our original problem. The second and third theorems of our paper show that solutions of (1.2) with the choice \( F = F_{\tilde{w},h,ϵ} \) are in fact close to the solution of the problem with \( F_w \), if \( \tilde{w} \) is close to \( w \).

**Definition 2.8.** If \( A, B ⊂ \mathbb{R}^n \) are Borel sets, then their \( μ \)-symmetric distance will be denoted by

\[
\Delta_μ(A, B) := μ(AΔB) = μ((A \setminus B) \cup (B \setminus A)).
\]

(2.4)

In what follows, it will be possible in theory to obtain the exact dependence of constants on various quantities involving the storage fee function, cost function, domain, and the density of the source measure by tracing these bounds through the results of [KMT19]. However, we are most interested in the dependencies on the parameters \( h \) and \( ϵ \), thus in the interest of brevity we will introduce the following terminology. The constants below are the same as those introduced in [KMT19] Remark 4.1.

**Definition 2.9.** Suppose \( c \) satisfies (Reg) and (Twist). \( X \) is a bounded set, \( c \)-convex with respect to \( Y \), \( μ = ρdx \) for some density \( ρ \in C^{0,α}(X) \) for some \( α ∈ (0,1] \), and \( \text{spt} \, μ ⊂ X \). Then we will say that a positive, finite constant is **universal** if it has bounds away from
Suppose Theorem 2.11. proximating problems to those of the original problem, in terms of the

\[ \epsilon \]

the paper (apart from that of \( N \)) in particular it would require careful book-keeping of exactly what norms are being used. We comment that if the collection \( \{ y_1, \ldots, y_N \} \) is constructed by sampling from a continuous domain \( Y \), and \( c \) is a cost function on \( X \times Y \) satisfying \( (\text{Reg}), (\text{Twist}) \), and the Ma-Trudinger-Wang condition (along with appropriate convexity conditions on \( X \) and \( Y \), which we will not detail here), then of the constants introduced in Definition 2.9 only \( \epsilon_{tw} \) will depend on \( N \). In particular, if this is the case, the dependencies of all universal constants that arise in the paper (apart from that of \( \epsilon_{tw} \)) can be seen to be polynomial in \( N \).

The following theorem gives a quantified measure of closeness for Laguerre cells of the approximating problems to those of the original problem, in terms of the \( \mu \)-symmetric distance.

**Theorem 2.11.** Suppose \( c \) satisfies \( (\text{Reg}) \) and \( (\text{Twist}) \), and \( \mu \) is absolutely continuous. Also suppose \( h > 0 \), \( \epsilon \in (0, \frac{1}{2N}) \), and \( w \in \mathbb{R}^N \) with \( \sum_{i=1}^{N} w^i \geq 1 \), \( w^i > 0 \). Then if \( \psi_{h,\epsilon} \in \mathcal{K}^\epsilon \) and \((T, \lambda)\) is a pair minimizing (1.2) with the storage fee function \( F_w \),

\[
\|G(\psi_{h,\epsilon}) - \lambda\|_1 \leq 2(N\epsilon + \|w_{h,\epsilon}(\psi_{h,\epsilon}) - w\|_1) + 2N\sqrt{2C_L h}
\]

and

\[
\sum_{i=1}^{N} \Delta_\mu(\text{Lag}_i(\psi_{h,\epsilon}), T^{-1}(\{y_i\})) \leq 8N(N\epsilon + \|w_{h,\epsilon}(\psi_{h,\epsilon}) - w\|_1) + 2N\sqrt{2C_L h},
\]

where \( C_L > 0 \) is the universal constant from Lemma A.1.

In view of Proposition 3.2 below, the above Theorem 2.11 implies the following. Suppose \( w, \tilde{w} \in \mathbb{R}^N \), and \((T_{h,\epsilon}, \lambda_{h,\epsilon})\) and \((T, \lambda)\) are minimizers of (1.2) with storage functions \( F_{\tilde{w},h,\epsilon} \) and \( F_w \) respectively. By [BK19] Proposition 3.5 and Theorem 4.7, there exists a vector \( \psi_{h,\epsilon} \) such that \( T_{h,\epsilon}^{-1}(\{y_i\}) = \text{Lag}_i(\psi_{h,\epsilon}) \) up to sets of zero \( \mu \) measure. By the uniqueness statement of Proposition 3.2, we see that \( w_{h,\epsilon}(\psi_{h,\epsilon}) = \tilde{w} \), hence the above theorem shows the \( \mu \)-symmetric distance between \( T_{h,\epsilon}^{-1}(\{y_i\}) \) and \( T^{-1}(\{y_i\}) \) is controlled by \( h, \epsilon \), and \( \|w_{h,\epsilon}(\psi_{h,\epsilon}) - w\|_1 \) (recall this last term is the error term from Algorithm 1).

The final theorem below shows that when the Laguerre cell associated to the problem with \( h = 0 = \epsilon \) has nonzero Lebesgue measure, the above closeness can be measured in the Hausdorff distance. Before stating this result, we recall the following definition.
**Remark 2.13.** A probability measure $\mu$ on $X$ satisfies a $(q,1)$-Poincaré-Wirtinger inequality for some $1 \leq q \leq \infty$ if there exists a constant $C_{pw} > 0$ such that for any $f \in C^1(X)$,

$$\|f - \int_X f\,d\mu\|_{L^q(\mu)} \leq C_{pw}\|\nabla f\|_{L^1(\mu)}.$$ 

For brevity, we will write this as “$\mu$ satisfies a $(q,1)$-PW inequality”.

**Remark 2.14.** Recall that some kind of connectedness condition on spt $\mu$ is necessary in order to obtain invertibility of the derivative of the map $G$ in nontrivial directions (see the discussion immediately preceding [KMT19, Definition 1.3]), and a Poincaré-Wirtinger inequality can be viewed as a quantitatively strengthened version of connectivity which is sufficient for our purposes.

It is classical that if $\rho$ is bounded away from zero on its support, it will satisfy a $(\frac{n}{n-1}, 1)$-PW inequality, and due to scaling $q = \frac{n}{n-1}$ is the largest possible value of $q$. We will only use the case of $q > 1$ in order to obtain quantitative bounds on the Hausdorff convergence of Laguerre cells, namely for Theorem 2.14. We also remark that in Theorem 2.14, we can make do with $q = 1$ if all of the Laguerre cells of the limit problem have nonzero measure.

**Theorem 2.14.** Suppose $c$ and $\mu$ satisfy the same conditions as Theorem 2.10 and $\mu$ satisfies a $(q,1)$-PW inequality for some $q \geq 1$. Also suppose $h > 0$, $\epsilon \in (0, \frac{1}{2N})$, and $w \in \mathbb{R}^N$ with $\sum_{i=1}^N w_i > 1$, $w_i \geq 0$, and $(T, \lambda)$ is a pair minimizing (1.2) with the storage fee function $F_w$.

1. If $\{h_k\}_{k=1}^\infty, \{\epsilon_k\}_{k=1}^\infty \subset \mathbb{R}_{>0}$, $\{\psi_k\}_{k=1}^\infty \in \mathcal{K}^c$, are sequences such that $w_{h_k, \epsilon_k}(\psi_k) \rightarrow w$, $h_k \rightarrow 0$, $\epsilon_k \rightarrow 0$ as $k \rightarrow 0$, and $\mathcal{L}(T^{-1}(\{y_i\})) > 0$, then

$$\lim_{k \rightarrow 0} d_H(\text{Lag}_c(\psi_k), T^{-1}(\{y_i\})) = 0.$$ 

2. If $q > 1$, $\psi_{h, \epsilon} \in \mathcal{K}^c$, there are universal constants $C_1, C_2 > 0$ such that,

$$d_H(\text{Lag}_c(\psi_{h, \epsilon}), T^{-1}(\{y_i\})) \leq \frac{C_1 C_{pw} N^{\frac{1}{q}} q(N\epsilon + \|w_{h, \epsilon}(\psi_{h, \epsilon}) - w\|_1 + 2N\sqrt{2C_L h})}{\epsilon^{1/(q-1)}(\arccos(1 - C_2 \mathcal{L}(T^{-1}(\{y_i\}))^2))^{n-1}},$$

as long as

$$\frac{N^{\frac{1}{q}} C_\Delta C_T C_{pw} q(N\epsilon + \|w_{h, \epsilon}(\psi_{h, \epsilon}) - w\|_1 + 2N\sqrt{2C_L h})}{\epsilon^{1/q(q-1)}(\arccos(1 - C_2 \mathcal{L}(T^{-1}(\{y_i\}))^2))^{n-1}} < \mathcal{L}(T^{-1}(\{y_i\}))$$

where $C_\Delta$ and $C_T$ are the universal constants defined in Lemma 6.5 and Lemma A.1 respectively.

**Remark 2.15.** The proof of Theorem 2.14 involves a bound on the Lebesgue measure of the symmetric difference of Laguerre cells which could in theory be used to prove the $\mu$-symmetric convergence of the Laguerre cells (as the density of $\mu$ is bounded). However, we opt to present a completely different proof for Theorem 2.11 as the method we present here can be applied under less stringent hypotheses. More specifically, in order to exploit the bound on the Lebesgue measure of symmetric difference of cells (Lemma 6.5), we would require a $(1, 1)$-PW inequality to obtain convergence, and a $(q, 1)$-PW inequality with $q > 1$ to obtain a quantitative rate of convergence of the $\mu$-symmetric difference, while our proof of Theorem 2.11 does not require any kind of PW inequality.
2.2. Classical optimal transport. Our results can also be applied to the classical optimal transport problem of transporting $\mu$ to a discrete measure $\nu = \sum_{i=1}^{N} \beta_i \delta_{y_i}$, subject to cost function $c$. Essentially, our results allow for a linearly convergent algorithm for a regularized version of the classical optimal transport problem without the assumption of any PW inequality, and under a $(q, 1)$-PW inequality with $q > 1$ gives quantitative uniform convergence of the Laguerre cells to that of the true transport map. To the best of the authors’ knowledge there is no other result currently on the Hausdorff convergence of Laguerre cells available in the literature.

As mentioned above, solving the optimal transport problem with storage fee given by $F_w$ for the choice $w = \beta$ is equivalent to solving the classical optimal transport problem. For $\zeta > 0$, take $h < \frac{\zeta^2}{2 c^2 N^2 c_L}$ and $\epsilon < \frac{\zeta}{8 N}$, then run Algorithm 1 with error tolerance $\sqrt[2]{\frac{\zeta}{c N}}$ to obtain some $\psi$. The estimate (2.5) in Theorem 2.11 yields $\|G(\psi) - \beta\| \leq \zeta$, which is exactly the form of the error in the Newton algorithm of [KMT19] (without assuming any PW inequality), along with the estimate on the $\mu$-symmetric difference of Laguerre cells (which can be seen to be strictly stronger than the estimate on $\|G(\psi) - \beta\|$). Under the additional PW inequality assumption, Theorem 2.14 gives uniform closeness of the Laguerre cells.

2.3. Outline of the paper. In Section 3 which follows, we give some useful properties of the mapping $w_{h,\epsilon}$ defined above. In Section 4 we prove Theorem 2.6 on the convergence rate of our Algorithm 1. We also provide a crude estimate on the number of iterations necessary to get within a desired error in terms of the parameters $h$, $\epsilon$, and $N$. In Section 5 we use the theory of directed graphs to prove Theorem 2.11 on the $\mu$-symmetric convergence of Laguerre cells. The remainder of the paper starting with Section 6 is devoted to the proof of Theorem 2.14 on the Hausdorff convergence of Laguerre cells. In Section 6 we gather some estimates on the Hausdorff measure of differences of Laguerre cells, mostly using convex geometry; the aforementioned bound on the Lebesgue measure of the difference of Laguerre cells is also found in this section. In Section 7 we establish invertibility properties of the mapping $G$ which will be necessary to convert the estimates from the previous section into the desired form of our main theorem. Finally in Section 8 we give the actual proof of Theorem 2.14. There, we first show some alternative spectral estimates of the transformation $DG$ which will be necessary in the proof of Theorem 2.14, followed by a quantitative strengthening of the invertibility of the mapping $G$ from the previous section. Appendix A contains a short result on strong convexity of the transport cost as a function of the dual variables $\psi$ which we need for the proof of Theorem 2.14.

3. Properties of the mapping $w_{h,\epsilon}$

In this section, we gather some properties and estimates on the mapping $w_{h,\epsilon}$ which will be crucial in the proofs of all of our main theorems. For the remainder of the paper, we assume that $c$ satisfies (Reg), (Twist), and $\mu$ is absolutely continuous. For this section and the following, we also assume $c$ satisfies (QC), $\mu = \rho dx$ for some density $\rho \in C^{0,\alpha}(X)$, for some $\alpha \in (0, 1]$, and $X$ is a bounded set, $c$-convex with respect to $Y$ such that $\text{spt} \mu \subset X$.

3.1. Solutions of the approximating problem with $F_{w,h,\epsilon}$. We will begin by justifying the remarks following Algorithm 1.
Definition 3.1. The subdifferential of a convex function $F : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ at any point $x$ is defined by the set
\[
\partial F(x) := \{ p \in \mathbb{R}^N \mid F(y) \geq F(x) + \langle p, y - x \rangle, \forall y \in \mathbb{R}^N\}.
\]

Proposition 3.2. Fix $h, \epsilon > 0$ and $w \in \mathbb{R}^N$ with $w_i \geq 0$, $\sum_{i=1}^N w_i \geq 1$. Then if $\psi \in \mathbb{R}^N$ is such that $w_{h,\epsilon}(\psi) = w$, the pair $(T_\psi, G(\psi))$ is the unique solution to the minimization problem (1.2) with storage fee function given by $F_{w,h,\epsilon}$ (with $T_\psi$ defined as in Remark 2.4).

Proof. We first calculate for any $t_0 \geq 0$ and $t \in (\epsilon, t_0 + \epsilon)$,
\[
\frac{d}{dt} \sigma_{t_0,h}(t - \epsilon) = h \frac{2(t - \epsilon) - t_0}{2 \sqrt{(t - \epsilon)(t_0 - t + \epsilon)}}.
\]
Thus for any $t$ and $t_1 \geq 0$ if we take the choice
\[
t_0 = 2(t - \epsilon) \left(1 + \left(\frac{t_1}{h}\right)^2 - \frac{t_1}{h} \sqrt{1 + \left(\frac{t_1}{h}\right)^2}\right) = (t - \epsilon)g\left(\frac{t_1}{h}\right)
\]
we obtain
\[
\frac{d}{dt} \sigma_{t_0,h}(t - \epsilon) = h \frac{2(t - \epsilon) - (2(t - \epsilon)(1 + \left(\frac{t_1}{h}\right)^2 - \frac{t_1}{h} \sqrt{1 + \left(\frac{t_1}{h}\right)^2}))}{2 \sqrt{(t - \epsilon)(2(t - \epsilon)(1 + \left(\frac{t_1}{h}\right)^2 - \frac{t_1}{h} \sqrt{1 + \left(\frac{t_1}{h}\right)^2})) - (t - \epsilon)}}
\]
\[
= h \frac{2(t - \epsilon)\left(-\left(\frac{t_1}{h}\right)^2 + \frac{t_1}{h} \sqrt{1 + \left(\frac{t_1}{h}\right)^2}\right)}{2(t - \epsilon)\sqrt{2(1 + \left(\frac{t_1}{h}\right)^2 - \frac{t_1}{h} \sqrt{1 + \left(\frac{t_1}{h}\right)^2}) - 1}}
\]
\[
= h \frac{\left(\frac{t_1}{h}\right)^2 - \frac{t_1}{h} \sqrt{1 + \left(\frac{t_1}{h}\right)^2}}{\sqrt{1 + \left(\frac{t_1}{h}\right)^2 - 2\frac{t_1}{h} \sqrt{1 + \left(\frac{t_1}{h}\right)^2} + \left(\frac{t_1}{h}\right)^2}}
\]
\[
= t_1 \sqrt{\frac{1 + \left(\frac{t_1}{h}\right)^2 - \frac{t_1}{h} \sqrt{1 + \left(\frac{t_1}{h}\right)^2}}{(\sqrt{1 + \left(\frac{t_1}{h}\right)^2 - \frac{t_1}{h} \sqrt{1 + \left(\frac{t_1}{h}\right)^2})^2}}} = t_1.
\]

Thus, taking $t = G(\psi)$ and $t_1 = \psi^i$, $t_0 = (w_{h,\epsilon}(\psi))^i$ for each $i$ in the calculation above, we see that if $w_{h,\epsilon}(\psi) = w$, we will have $\psi \in \partial F_{w,h,\epsilon}(G(\psi))$. Since $F_{w,h,\epsilon}$ is a proper, convex function that is $+\infty$ outside the set $\Lambda$, by [BK19 Theorem 4.7] we obtain that the pair $(T_\psi, G(\psi))$ is the unique minimizing pair in the problem (1.2) with storage fee function $F_{w,h,\epsilon}$. \qed

3.2. Estimates on $w_{h,\epsilon}$. Next we will obtain invertibility of $Dw_{h,\epsilon}$ on the set $\Sigma_{w,h,\epsilon}$. This normalization will be critical in obtaining the necessary estimates to justify convergence of our Newton algorithm. For the remainder of this section and the following Section 4, we will not be as explicit in terms of the dependence of various quantities on $N$. Related to this, for any vector valued map $\Phi : \Omega \to \mathbb{R}^N$ on any domain $\Omega \subset \mathbb{R}^N$, we will write associated
Moreover, if \[ \sum_{G}G + \ldots \] Next we show boundedness of \( \Sigma w, h, \epsilon \). In particular, \( \psi \), whose value may change from line to line.

Throughout the proof, \( \psi \) is differentiable on \( \psi \). Since adding a multiple of \( \psi \) does not change the value of \( G(\psi) \) and \( \sum_{i=1}^{N} (G^{i}(\psi) - \epsilon) < 1 \leq \sum_{i=1}^{N} w^{i} \), we can see there exists some \( r \in \mathbb{R} \) such that \( \sum_{i=1}^{N} w^{i}_{h, \epsilon}(\psi + r \mathbf{1}) = \sum_{i=1}^{N} (G^{i}(\psi + r \mathbf{1}) - \epsilon) g^{i}_{h} \). Hence \( \psi \) is nonempty.

Next we show boundedness of \( \Sigma_{w, h, \epsilon} \). If \( \psi \in \Sigma_{w, h, \epsilon} \), we calculate

\[
\sum_{i=1}^{N} w^{i} = \sum_{i=1}^{N} (G^{i}(\psi) - \epsilon) g^{i}_{h} \leq \sum_{i=1}^{N} (G^{i}(\psi) - \epsilon) \max_{j} g^{j}_{h} = \max_{j} g^{j}_{h}(1 - N \epsilon).
\]

\[
\text{Hence } \max_{j} g^{j}_{h} \geq \sum_{i=1}^{N} w^{i} \geq \frac{1}{1-N \epsilon} > 1. \]

In particular we must have an upper bound on some component \( \psi^{k} \), i.e. \( \psi^{k} \leq M_{1} \) where \( M_{1} := h g^{-1}(\frac{1}{1-N \epsilon}) < +\infty \). Now since \( X \) is compact,
there exist constants $M_1$ and $m_1$ such that $m_1 < c(\cdot, y_i) < M_1$ for all $i \in \{1, \ldots, N\}$. If, for any $i$, $\psi_i > \tilde{M}_1 + M_1 - m_1$ then we would have $\text{Lag}_i(\psi) = \emptyset$, contradicting $\psi \in \mathcal{K}^e$.

A similar calculation yields the bound

$$\min_j g(\psi_j^i / h) \leq \sum_{i=1}^N w^i \geq \frac{N}{1 - N\epsilon} \leq 2N$$

thus by an analogous argument we obtain the uniform bounds

$$\tilde{m} \leq \psi_i \leq \tilde{M}, \quad \forall \psi \in \Sigma_{w,h,\epsilon}, \ i \in \{1, \ldots, N\},$$

$$\tilde{M} := \tilde{M}_1 + M_1 - m_1 = h g^{-1}(\frac{1}{1 - N\epsilon}) + M_1 - m_1 > 0$$

(3.4)

$$\tilde{m} := \tilde{M}_2 - M_1 + m_1 := h g^{-1}(2N) - M_1 + m_1 < 0.$$  

We now calculate bounds on $\tilde{M}$ and $\tilde{m}$ in terms of $N$ and $\epsilon$. If $g(t) = a$ for some value $a > 1$, we find

$$\frac{a}{2} = 1 + t^2 - t\sqrt{1 + t^2} = 1 + t(t - \sqrt{1 + t^2}) = 1 + t \left( \frac{-1}{t + \sqrt{1 + t^2}} \right) = \frac{\sqrt{1 + t^2}}{t + \sqrt{1 + t^2}}$$

hence

(3.5) \quad (1 - \frac{a}{2})\sqrt{1 + t^2} = \frac{at}{2} \implies (1 - \frac{a}{2})^2 = t^2(\frac{a^2}{4} - (1 - \frac{a}{2})^2) \implies t^2 = \frac{(2 - a)^2}{4a - 4}.

Now if $a = \frac{1}{1 - N\epsilon} < 2$, we have $t = g^{-1}(a) > 0$, hence by (3.5) above,

(3.6) \quad 0 < \tilde{M} \leq C \left( 1 + h \frac{2 - \frac{1}{1 - N\epsilon}}{2\sqrt{\frac{1}{1 - N\epsilon} - 1}} \right) = C \left( 1 + h \frac{1}{2\sqrt{N\epsilon(1 - N\epsilon)}} \right) \leq \frac{C}{\sqrt{2N\epsilon}},

where we have used that $\epsilon < \frac{1}{2N}$. Similarly, for $a = 2N > 2$, $t = g^{-1}(a) < 0$ hence using (3.5) again yields

(3.7) \quad 0 > \tilde{m} = -C(1 + h) \frac{2N - 2}{2\sqrt{2N - 1}} \geq -C \left( 1 + \frac{hN}{\sqrt{N}} \right) = -C\sqrt{N}.

Combining this with (3.6) immediately gives (3.1).

We will also use for some estimates on $g$ and $g'$. We calculate,

$$g'(\tilde{M} / h) = -\frac{2(\tilde{M} / h - \sqrt{1 + (\tilde{M} / h)^2})^2}{\sqrt{1 + (\tilde{M} / h)^2}} = -\frac{2(\tilde{M} - \sqrt{h^2 + \tilde{M}^2})}{h\sqrt{h^2 + \tilde{M}^2}}$$

$$= -\frac{2h^3}{\sqrt{h^2 + \tilde{M}^2}(\tilde{M} + \sqrt{h^2 + \tilde{M}^2})^2} \leq -\frac{h^3}{2(h^2 + \tilde{M}^2)^{3/2}} \leq -\frac{h^3}{2(2Nh^2 + C)^{3/2}} \leq -Ch^3N^{3/2}\epsilon^{3/2}$$

where we have used (3.6) in the last line. At the same time,

$$g'(\tilde{m} / h) = -\frac{2(\tilde{m} - \sqrt{h^2 + \tilde{m}^2})}{h\sqrt{h^2 + \tilde{m}^2}} \geq -\frac{CN}{h^2},$$
since $g'$ is a decreasing, negative function, we have that for any $\psi \in \Sigma_{w,h,\epsilon}$ and index $i$, the estimates

\begin{equation}
Ch^3N^3 \leq \left| g'(\frac{\psi_i}{h}) \right| \leq \frac{CN}{h^2}.
\end{equation}

Additionally using (3.7) and that $h \leq 1$, for any $\psi \in \Sigma_{w,h,\epsilon}$ and index $i$ we have (recall $\bar{m}$ could be negative here)

\begin{equation}
1 \leq g(\frac{\psi_i}{h}) \leq g(\frac{\bar{m}}{h}) = 2\left(1 + \left(\frac{\bar{m}}{h}\right)^2 - \frac{\bar{m}}{h} \sqrt{1 + \left(\frac{\bar{m}}{h}\right)^2}\right)
\end{equation}

\begin{equation}
= \frac{2\sqrt{h^2 + m^2}}{h^2} \left(\sqrt{h^2 + m^2} - \bar{m}\right) \leq \frac{\sqrt{h^2 + m^2}}{h^2} \leq \frac{CN}{h^2}.
\end{equation}

Under the current assumptions, we see by [KMT19, Theorem 4.1] that $G$ is uniformly $C^{1,\alpha}$ on $\Sigma_{w,h,\epsilon} \subset \mathcal{K}^\epsilon$. We then calculate the derivative of $w_{h,\epsilon}$ as

\begin{equation}
Dw_{h,\epsilon}(\psi) = \text{diag}(g(\frac{\psi_i}{h}))DG(\psi) + \frac{1}{h} \text{diag}((G^i(\psi) - \epsilon)g'(\frac{\psi_i}{h}))
\end{equation}

\begin{equation}
= \text{diag}(g(\frac{\psi_i}{h})) \left(\frac{1}{h} \text{diag} \left(\frac{(G^i(\psi) - \epsilon)g'(\frac{\psi_i}{h})}{g(\frac{\psi_i}{h})}\right) + DG(\psi)\right)
\end{equation}

where diag of a vector in $\mathbb{R}^N$ is the $N \times N$ diagonal matrix with the entries of the vector on the diagonal. Since $g \geq 1$ on $\mathbb{R}$, we see $\text{diag}(g(\frac{\psi_i}{h}))$ is invertible with all eigenvalues larger than 1. For any unit vector $V \in \mathbb{R}^N$ we have

\[
\left\langle \frac{1}{h} \text{diag} \left(\frac{(G^i(\psi) - \epsilon)g'(\frac{\psi_i}{h})}{g(\frac{\psi_i}{h})}\right) V, V \right\rangle + \langle DG(\psi)V, V \rangle
\]

\[
= \frac{1}{h} \sum_{i=1}^{N} \frac{(G^i(\psi) - \epsilon)g'(\frac{\psi_i}{h})}{g(\frac{\psi_i}{h})} (V^i)^2 + \langle DG(\psi)V, V \rangle =: A + B.
\]

By [KMT19, Theorem 1.1 and 1.3], $DG$ is symmetric, every off diagonal entry is nonnegative, and each row sums to zero, hence $B \leq 0$. We also calculate

\[
A \leq \frac{1}{h} \max_j \left(\frac{(G^j(\psi) - \epsilon)g'(\frac{\psi_j}{h})}{g(\frac{\psi_j}{h})}\right) \sum_{i=1}^{N} (V^i)^2 = \frac{1}{h} \max_j \left(\frac{(G^j(\psi) - \epsilon)g'(\frac{\psi_j}{h})}{g(\frac{\psi_j}{h})}\right)
\]

\[
= \frac{1}{h} \max_j \left(\frac{w_{h,\epsilon}(\psi)g'(\frac{\psi_j}{h})}{g(\frac{\psi_j}{h})}\right) \leq \frac{-m_{h,\epsilon}\epsilon_0}{h \epsilon_0} \leq -C\epsilon_0h^6N^{-\frac{1}{2}}\epsilon_3^2
\]

where we have used (3.9) and that $\psi \in \mathcal{W}^{\epsilon_0}$, hence $Dw_{h,\epsilon}(\psi)$ is invertible and we obtain (3.3).

Finally, since $\Sigma_{w,h,\epsilon}$ is bounded by above and $g'$ is clearly a $C^1$ function on $\mathbb{R}$, we can again use [KMT19, Theorem 4.1] to conclude that $w_{h,\epsilon}$ is actually $C^{1,\alpha}$ on $\Sigma_{w,h,\epsilon}$. The only thing left is to verify the dependencies of $L \geq 0$ from (3.2). Since $g$ is decreasing on $\mathbb{R}$, by (3.9) we immediately see that $\|w_{h,\epsilon}\|_{L^\infty(\Sigma_{w,h,\epsilon})} \leq \frac{CN}{h^2}$. Also calculating using (3.8), (3.10), (3.9), and that $\|G\|_{C^1(\mathcal{K}^\epsilon)} \leq \frac{CN}{h^2}$ from [KMT19, Theorem 1.3], we see that $\|w_{h,\epsilon}\|_{C^1(\Sigma_{w,h,\epsilon})} \leq C(N^2h^{-2} + Nh^{-2}) \leq \frac{CN^2}{h^2}$. 

For the remainder of the proof, we will not keep explicit track of the dependencies on $N$. Finally, note that

$$[Dw_{h,\epsilon}]_{C^0, \alpha} \leq C \left( \|g(\frac{\cdot}{h})\|_{L^\infty} [DG]_{C^0, \alpha} + [g(\frac{\cdot}{h})]_{C^0, \alpha} \|DG\|_{L^\infty} \right)$$

$$+ \frac{1}{h} \left( [G - \epsilon \mathbf{1}]_{C^0, \alpha} \|g(\frac{\cdot}{h})\|_{L^\infty} + [G - \epsilon \mathbf{1}]_{L^\infty} [g(\frac{\cdot}{h})]_{C^0, \alpha} \right)$$

$$\leq C \left( \|g(\frac{\cdot}{h})\|_{L^\infty} [DG]_{C^0, \alpha} + \text{diam}(\Sigma_{w,h,\epsilon}) \|g(\frac{\cdot}{h})\|_{L^\infty} \|DG\|_{L^\infty} \right)$$

$$+ \frac{\text{diam}(\Sigma_{w,h,\epsilon})}{h} \left( [DG]_{L^\infty} \|g(\frac{\cdot}{h})\|_{L^\infty} + [G - \epsilon \mathbf{1}]_{L^\infty} [g(\frac{\cdot}{h})]_{C^0, \alpha} \right)$$

(3.11)

where all norms and seminorms of $g$ and $g'$ are taken over $[\tilde{m}, \tilde{M}]$ and the remainder over $\Sigma_{w,h,\epsilon}$.

Fixing an index $i$, for any $\psi_1 \neq \psi_2 \in \Sigma_{w,h,\epsilon}$ we have

$$\left| g'(\frac{\psi_1}{h}) - g'(\frac{\psi_2}{h}) \right| \leq \sup_{t \in [\tilde{m}, \tilde{M}]} \left| g''(\frac{t}{h}) \right| \left| \psi_1 - \psi_2 \right| \leq \frac{C \|\psi_1 - \psi_2\|}{h}$$

(3.12)

since by direct computation we see

$$g''(t) = \frac{-4t^3 + 4(1 + t^2)^{3/2} - 6t}{(1 + t^2)^{3/2}} = 4 - 2 \frac{2t^3 + 3t}{(1 + t^2)^{3/2}} = 4 - 4 \frac{t}{(1 + t^2)^{1/2}} - 2 \frac{t}{(1 + t^2)^{3/2}}$$

and so

$$|g''(t)| \leq 4 + 4 \left| \frac{t}{(1 + t^2)^{1/2}} \right| + 2 \left| \frac{t}{(1 + t^2)^{3/2}} \right| \leq 4 + 4 + 2 \min(|t|, |t|^{-2}) \leq 10.$$ 

At the same time using (3.8),

$$\left| g(\frac{\psi_1}{h}) - g(\frac{\psi_2}{h}) \right| \leq \sup_{t \in [\tilde{m}, \tilde{M}]} \left| g'(\frac{t}{h}) \right|^2 \left| \psi_1 - \psi_2 \right| \leq \frac{C \|\psi_1 - \psi_2\|}{h^5}$$

(3.13)

Finally, carefully tracing through the proofs leading to [KMT19] Theorem 4.1 yields that

$$[DG]_{C^0, \alpha(\bar{K})} \leq \frac{C}{\epsilon^2},$$

(3.14)

thus we can combine this with (3.8), (3.9), (3.13), (3.12), and the fact that $\|G\|_{C^1(\bar{K})} \leq CN$ in (3.11) to obtain

$$[Dw_{h,\epsilon}]_{C^0, \alpha(\Sigma_{w,h,\epsilon})} \leq C \max \left(h^{-2} \epsilon^{-2}, h^{-3} \epsilon^{-\frac{1}{2}} \right)$$

as desired.

\[ \square \]

4. Convergence of Algorithm

Here we provide the proof of our first main theorem, on global linear and locally superlinear convergence of Algorithm. We remark that the proof below also shows that $\Sigma_{w,h,\epsilon}$ is locally a $C^1$ manifold of codimension 1 in $\mathbb{R}^n$. Again in this section, we will not track the explicit dependencies on $N$. 
Proposition 4.1. There is a function \( r \in C^{1,\alpha}(\mathbb{K}^{\varepsilon}) \) such that for any \( \psi \in \mathbb{R}^{N}, r(\psi) \) is the unique number such that \( \pi(\psi) := \psi - r(\psi)1 \in \Sigma_{w,h,\epsilon} \). Moreover,
\[
\|D\pi\|_{C^{0,\alpha}(\mathbb{R}^{\varepsilon})} \leq \frac{C}{h^{18} \epsilon^{9}}
\]
for some universal \( C > 0 \).

Proof. First we carry out some preliminary analysis. Again, \( C > 0 \) will denote a suitable universal constant throughout the proof. Define \( \mathbb{R}^{N} \times \mathbb{R} \ni (\psi, r) \rightarrow \Phi(\psi, r) \in \mathbb{R} \) by
\[
\Phi(\psi, r) = \sum_{i=1}^{N} w_{h,i}^{i}(\psi - r1) - w^{i} = \sum_{i=1}^{N} (G^{i}(\psi - r1) - \epsilon) g(\frac{\psi^{i} - r}{h}) - \sum_{i=1}^{N} w^{i}
\]
\[
= \sum_{i=1}^{N} (G^{i}(\psi) - \epsilon) g(\frac{\psi^{i} - r}{h}) - \sum_{i=1}^{N} w^{i}.
\]
Note for any \( \psi \in \mathbb{R}^{N} \) such that \( w_{h,i}^{i}(\psi) \geq 0 \) for all \( i \in \{1, \ldots, N\} \), we must have \( G^{i}(\psi) \geq \epsilon \), hence \( \psi \in \mathbb{K}^{\varepsilon} \) for such \( \psi \). A quick calculation yields that if \( (\psi, r) \) are such that \( \psi \in \mathbb{K}^{\varepsilon} \) and \( \psi - r1 \in \Sigma_{w,h,\epsilon} \), we have using the calculation immediately preceding (3.8),
\[
\frac{\partial}{\partial r} \Phi(\psi, r) = -\frac{1}{h} \sum_{i=1}^{N} (G^{i}(\psi) - \epsilon) g'(\frac{\psi^{i} - r}{h}) \geq C N \frac{1}{h^{3}} (1 - N \epsilon) > 0.
\]
At the same time, the strict monotonicity of \( g \) along with the fact that \( \sum_{i=1}^{N} w^{i} \geq 1 > \sum_{i=1}^{N} (G^{i}(\psi) - \epsilon) \) and \( g(\mathbb{R}) = (1, \infty) \) implies that for any \( \psi \in \mathbb{R}^{N} \), there exists a unique \( r(\psi) \in \mathbb{R} \) such that \( \Phi(\psi, r(\psi)) = 0 \), thus the function \( \psi \mapsto r(\psi) \) is well-defined. By the above calculation and the implicit function theorem we have that this function \( r \) is differentiable near any \( \psi \in \mathbb{K}^{\varepsilon} \). Differentiating the expression \( \Phi(\psi, r(\psi)) = 0 \) with respect to \( \psi^{j} \) at such a \( \psi \), we find that
\[
0 = \sum_{i=1}^{N} \left( D_{j} G^{i}(\psi) g(\frac{\psi^{j} - r(\psi)}{h}) + (G^{i}(\psi) - \epsilon) g'(\frac{\psi^{j} - r(\psi)}{h}) \delta^{i}_{j} - D_{j} r(\psi) \right)
\]
\[
\Rightarrow D_{j} r(\psi) = \frac{\sum_{i=1}^{N} h D_{j} G^{i}(\psi) g(\frac{\psi^{i} - r(\psi)}{h}) + \delta^{i}_{j} (G^{i}(\psi) - \epsilon) g'(\frac{\psi^{i} - r(\psi)}{h})}{\sum_{i=1}^{N} (G^{i}(\psi) - \epsilon) g'(\frac{\psi^{i} - r(\psi)}{h})}.
\]
(4.1)
We can see \( \|Dr\| \) is uniformly bounded on \( \mathbb{K}^{\varepsilon} \): we calculate
\[
\|D_{j} r\|_{L^{\infty}(\mathbb{K}^{\varepsilon})} \leq 1 + \left| \frac{\sum_{i=1}^{N} D_{j} G^{i}(\psi) g(\frac{\psi^{i} - r(\psi)}{h})}{\frac{1}{h} \sum_{i=1}^{N} (G^{i}(\psi) - \epsilon) g'(\frac{\psi^{i} - r(\psi)}{h})} \right|
\]
\[
\leq 1 + \frac{g(\frac{N}{h}) \sum_{i=1}^{N} |D_{j} G^{i}(\psi)|}{h^{3/2} \epsilon^{3/2} (1 - N \epsilon)}
\]
(4.2)
where we have used $\|G\|_{C^1(\mathcal{K}^\epsilon)} \leq C$ from \cite[Theorem 1.3]{KMT19}, \eqref{eq:3.8}, \eqref{eq:3.9}, and that $\epsilon < \frac{1}{2N^2}$.

Since $\mathcal{K}^\epsilon = \bigcap_{i=1}^N (G^i)^{-1}(\mathcal{K}^\epsilon)$, the implicit function theorem combined with \cite[Theorem 5.1]{KMT19} along with the fact that $\partial X$ is locally Lipschitz shows that $\partial \mathcal{K}^\epsilon$ is locally Lipschitz. Thus $W^{1,\infty}(\mathcal{K}^\epsilon) = C^{0,1}(\mathcal{K}^\epsilon)$, hence $r$ is uniformly Lipschitz continuous on $\mathcal{K}^\epsilon$.

We will now show a Hölder bound on $Dr$. Note that for each $j$, we can write $D_j r = \frac{H_i}{h}$ where $H_j(\psi) := \frac{1}{h}(G^j(\psi) - \psi)g'(\frac{\psi - r(\psi)}{h}) + \sum_{i=1}^N D_j G^j(\psi)g'(\frac{\psi - r(\psi)}{h})$ belongs to $C^{0,\alpha}(\mathcal{K}^\epsilon)$ (using \cite[Theorem 4.1]{KMT19}) and $H_2(\psi) := \frac{1}{h} \sum_{i=1}^N (G^i(\psi) - r(\psi))g'(\frac{\psi - r(\psi)}{h})$ belongs to $C^{0,1}(\mathcal{K}^\epsilon)$, with $H_2 \leq -\frac{C h^3 N^{3/2}}{h} (1 - N\epsilon) < 0$ uniformly. Note that

$$H_2(\pi(\psi)) = \frac{1}{h} \sum_{i=1}^N (G^i(\psi - r(\psi))\mathbf{1} - \psi)g'(\frac{\psi - r(\psi)}{h})$$

Thus for $\psi_1 \neq \psi_2 \in \mathcal{K}^\epsilon$, using \eqref{eq:3.8},

$$\|D_j r(\psi_1) - D_j r(\psi_2)\| = \left| \frac{H_1(\psi_1) - H_1(\psi_2)}{H_2(\psi_1) - H_2(\psi_2)} \right| \leq \frac{H_1(\psi_1) - H_1(\psi_2)}{H_2(\psi_1) - H_2(\psi_2)}$$

$$\leq \frac{|H_1(\psi_1) - H_1(\psi_2)|}{H_2(\psi_1) - H_2(\psi_2)} + \frac{\|H_1\|_{\mathcal{L}^\infty(\mathcal{K}^\epsilon)} H_2 \|\pi(\psi_2) - \pi(\psi_1)\|}{(\frac{C h^3 N^{3/2}}{h} (1 - N\epsilon))^2}$$

$$\leq C \left( \frac{|H_1(\psi_1) - H_1(\psi_2)|}{h^2 N^{3/2} (1 - N\epsilon)} + \frac{\|H_1\|_{\mathcal{L}^\infty(\mathcal{K}^\epsilon)} H_2 \|\pi(\psi_2) - \pi(\psi_1)\|^{1-\alpha} [\alpha \pi(\psi_1)]}{(h^2 N^{3/2} (1 - N\epsilon))^2} \right) \left\| \psi_1 - \psi_2 \right\|^\alpha,$$

hence $D_j r$ is uniformly $C^{0,\alpha}$ on $\mathcal{K}^\epsilon$. Our next task will be to estimate $[Dr]_{C^{0,\alpha}(\mathcal{K}^\epsilon)}$. In order to do this we estimate each of the terms in the above expression.

A quick calculation yields

$$\|H_1\|_{\mathcal{L}^\infty(\mathcal{K}^\epsilon)} \leq C \left( \frac{1}{h^3} + \frac{1}{h^2} \right) \leq \frac{C}{h^3},$$

and since $\pi(\psi) \in \Sigma_{w,h,\epsilon}$, by \eqref{eq:3.1} we have

$$\|\pi(\psi_2) - \pi(\psi_1)\| \leq \text{diam}(\Sigma_{w,h,\epsilon}) \leq \frac{C}{\epsilon^2}. $$

To estimate $[H_2]_{C^{0,1}(\mathcal{K}^\epsilon)}$, let $H_3(\psi) := (G^i(\psi) - \psi)g'(\frac{\psi}{h})$ so that $H_2(\psi) = \frac{1}{h} \sum_{i} H_3(\pi(\psi))$. Just as we estimated the final two terms in \eqref{eq:3.11}, we see that $[H_3]_{C^{0,1}(\Sigma_{w,h,\epsilon})} \leq \frac{C}{h^2}$ by using the bound $\|G\|_{C^1(\mathcal{K}^\epsilon)} \leq C$ with \eqref{eq:3.8} and \eqref{eq:3.12}. Furthermore since $\pi(\psi) = \psi - r(\psi)\mathbf{1}$, we see that

$$[\pi]_{C^{0,1}(\mathcal{K}^\epsilon)} \leq 1 + N^{1/2}[r]_{C^{0,1}(\mathcal{K}^\epsilon)} \leq \frac{C}{h^{3/2}}$$
Finally we bound $[H_1]_{C^{0,\alpha}((\kappa^\varepsilon))}$. Let $H_{4,i}(\psi) := D_J G^i(\psi) g(\frac{\psi}{h})$ so that $H_1(\psi) = (G^i(\psi) - \epsilon) g' \left( \frac{\psi - r(\psi)}{h} \right) + \sum_i H_{4,i}(\pi(\psi))$. For $\psi_1, \psi_2 \in \overline{\kappa^\varepsilon}$ we have

$$
|H_{4,i}(\pi(\psi_1)) - H_{4,i}(\pi(\psi_2))| = \left| D_J G^i(\psi_1) g \left( \frac{\pi(\psi_1)}{h} \right) - D_J G^i(\psi_2) g \left( \frac{\pi(\psi_2)}{h} \right) \right|
$$

$$
\leq \left[ DG \right]_{C^{0,\alpha}((\kappa^\varepsilon))} g \left( \frac{\bar{m}}{h} \right) \| \psi_1 - \psi_2 \|^\alpha + \| G \|_{C^1(\kappa^\varepsilon)} \sup_{s \in [\bar{m}, \bar{M}]} g' \left( \frac{s}{h} \right) \frac{\| \pi(\psi_1) - \pi(\psi_2) \|}{h} \| \psi_1 - \psi_2 \|^\alpha
$$

$$
\leq C \left( \frac{1}{h^2} + \frac{1}{h^{3+4\alpha}} \right) \| \psi_1 - \psi_2 \|^\alpha
$$

where we have used (3.8) to estimate $g'$, (3.9) to estimate $g \left( \frac{\bar{m}}{h} \right)$, [KMT19, Theorem 1.3] to estimate $\| G \|_{C^1(\kappa^\varepsilon)}$, (3.14) for $[DG]_{C^{0,\alpha}((\kappa^\varepsilon))} \leq C \frac{1}{h^2}$, and (4.5). Hence we see, using (4.6),

$$
[H_1]_{C^{0,\alpha}((\kappa^\varepsilon))} \leq C \left[ \left[ G^i \right]_{C^{0,\alpha}((\Sigma_{w,h,e,i}))} \right] g' \left( \frac{\bar{m}}{h} \right) \| G \|_{L^\infty((\bar{m}, \bar{M}))} + \| G \|_{L^\infty((\kappa^\varepsilon))} [DG]_{C^{0,\alpha}((\bar{m}, \bar{M}))} + \sum_i [H_{4,i}]_{C^{0,\alpha}((\kappa^\varepsilon))} \left[ \pi \right]_{C^{0,1}((\kappa^\varepsilon))}
$$

$$
\leq C \left( \left[ \left[ G^i \right]_{C^{0,\alpha}((\Sigma_{w,h,e,i}))} \right] \| G \|_{L^\infty((\bar{m}, \bar{M}))} + \| G \|_{L^\infty((\kappa^\varepsilon))} [DG]_{C^{0,\alpha}((\bar{m}, \bar{M}))} + \sum_i [H_{4,i}]_{C^{0,\alpha}((\Sigma_{w,h,e,i}))} \left[ \pi \right]_{C^{0,1}((\kappa^\varepsilon))} \right)
$$

Putting the above together with (4.3), (4.4), (4.5), and (4.7) we get

$$
[Dr]_{C^{0,\alpha}((\kappa^\varepsilon))} \leq C \left( \frac{[H_1]_{C^{0,\alpha}((\kappa^\varepsilon))}}{h^2 N^{\frac{3}{2}} (1 - N \epsilon)} + \frac{\| H_1 \|_{L^\infty((\kappa^\varepsilon))}}{h^2 N^{\frac{3}{2}} (1 - N \epsilon)} \right) \frac{\| \pi(\psi_2) - \pi(\psi_1) \|^{1-\alpha} \left[ \pi \right]_{C^{0,1}((\kappa^\varepsilon))}^{\alpha}}{(h^2 N^{\frac{3}{2}} (1 - N \epsilon))^2}
$$

$$
\leq C \left( \frac{1}{h^{11} \epsilon^{\frac{5}{2}}} + \frac{1}{h^{15} \epsilon^{\frac{1}{2}}} \right) \left( \frac{1}{h^{14} \epsilon^{\frac{1}{2}}} + \frac{1}{h^{14} \epsilon^{\frac{1}{2}}} \right) = C \frac{1}{h^{13} \epsilon^{\frac{5}{2}}}
$$
Finally,
\[
\|D\pi\|_{C^{0,\alpha}(\mathcal{K};\mathbb{R}^N)} \leq C(1 + \|Dr\|_{L^\infty(\mathcal{K})} + [Dr]_{C^{0,\alpha}(\mathcal{K})}) \leq \frac{C}{h^{18} \epsilon^9}
\]
by the calculation above combined with (4.2).

With the above estimate, we can now prove linear convergence and locally superlinear convergence of our algorithm. This is done essentially as in [KMT19].

**Proof of Theorem 2.6.** Let \( \bar{\psi} := \psi_k \) be the vector chosen at the \( k \)th step of Algorithm 1, \( \bar{v} := (Dw_{h,\epsilon}(\bar{\psi}))^{-1}(w_{h,\epsilon}(\bar{\psi}) - w) \), and define the curve \( \bar{\psi}(t) := \pi(\bar{\psi} - t\bar{v}) \) (where \( \pi \) is defined in Proposition 4.1). We also take \( \tilde{L} := \|D\pi\|_{C^{0,\alpha}(\mathcal{K};\mathbb{R}^N)} \), which has the bound claimed in the statement of the theorem by Proposition 4.1. As noted above \( \bar{\psi} \in \mathcal{K}^t \cap \mathcal{W}^{\alpha_0} \), hence by Proposition 3.3 we have the estimates (3.2) and (3.3). Let \( \tau_1 := \inf\{t \geq 0 \mid \bar{\psi}(t) \notin \mathcal{W}^{\alpha_0} \} \), then \( w^j_{h,\epsilon}(\bar{\psi}(\tau_1)) = \frac{\epsilon_0}{2} \) for some \( 1 \leq j \leq N \), thus (using that \( \bar{\psi} \in \Sigma_{w,h,\epsilon} \) so \( \pi(\bar{\psi}) = \bar{\psi} \) and \( \|\bar{v}\| \leq \frac{\|w_{h,\epsilon}(\bar{\psi}) - w\|}{\kappa} \)) we calculate

\[
\frac{\epsilon_0}{2} \leq \|w_{h,\epsilon}(\bar{\psi}(\tau_1)) - w_{h,\epsilon}(\bar{\psi})\| \leq L\|\bar{\psi}(\tau_1) - \bar{\psi}\|
\]

\[
= L\|\pi(\bar{\psi} - \tau_1 \bar{v}) - \pi(\bar{\psi})\| \leq \tilde{L}\tau_1\|\bar{\psi}(\tau_1) - \bar{\psi}\| \leq \frac{\tilde{L}\tau_1\|w_{h,\epsilon}(\bar{\psi}) - w\|}{\kappa}.
\]

The above gives a lower bound of \( \frac{\epsilon_0}{2\tilde{L}\tau_1\|w_{h,\epsilon}(\bar{\psi}) - w\|} \) on the first exit time \( \tau_1 \), and \( w \) is uniformly \( C^{1,\alpha} \) on the image \( \bar{\psi}([0, \tau_1]) \) while \( \pi \) remains uniformly \( C^{1,\alpha} \) on the segment \( [\bar{\psi}, \bar{\psi} - \tau_1 \bar{v}] \). We will now Taylor expand in \( t \). Note that

\[
\frac{d}{dt} \bigg|_{t=0} w_{h,\epsilon}(\bar{\psi}(t)) = -Dw_{h,\epsilon}(\bar{\psi}(t))\bar{v} + \langle Dr(\bar{\psi}(t)), \bar{v} \rangle Dw_{h,\epsilon}(\bar{\psi}(t))\mathbf{1}\bigg|_{t=0}
\]

\[
= -\langle w_{h,\epsilon}(\bar{\psi}) - w, \bar{v} \rangle + \langle Dr(\bar{\psi}), \bar{v} \rangle Dw_{h,\epsilon}(\bar{\psi})\mathbf{1}.
\]

Using (4.1) and that \( \bar{\psi} \in \Sigma_{w,h,\epsilon} \), we obtain

\[
\langle Dr(\bar{\psi}), \bar{v} \rangle = \frac{\langle Dw_{h,\epsilon}(\bar{\psi})^T \mathbf{1}, Dw_{h,\epsilon}(\bar{\psi})^{-1}(w_{h,\epsilon}(\bar{\psi}) - w) \rangle}{\langle Dw_{h,\epsilon}(\bar{\psi}) \mathbf{1}, \mathbf{1} \rangle} = 0.
\]

Now Taylor expanding we obtain

\[
w_{h,\epsilon}(\bar{\psi}(t)) = w_{h,\epsilon}(\bar{\psi}(0)) + \left( \frac{d}{du} \bigg|_{u=0} w_{h,\epsilon}(\bar{\psi}(u)) \right) t + \int_0^t \left( \frac{d}{du} \bigg|_{u=s} w_{h,\epsilon}(\bar{\psi}(u)) - \frac{d}{du} \bigg|_{u=0} w_{h,\epsilon}(\bar{\psi}(u)) \right) ds
\]

\[
= (1 - t)w_{h,\epsilon}(\bar{\psi}) + tw + R(t).
\]

We see that

\[
R(t) = \int_0^t \left( \langle \nabla w^i_{h,\epsilon}(\bar{\psi}(s)), \hat{\psi}(s) \rangle - \langle \nabla w^i_{h,\epsilon}(\bar{\psi}(0)), \hat{\psi}(0) \rangle \right) ds
\]

\[
= \int_0^t \left( \langle \nabla w^i_{h,\epsilon}(\bar{\psi}(s)) - \nabla w^i_{h,\epsilon}(\bar{\psi}(0)), \hat{\psi}(s) \rangle + \langle \nabla w^i_{h,\epsilon}(\bar{\psi}(0)), \hat{\psi}(s) - \hat{\psi}(0) \rangle \right) ds.
\]
We will examine the two inner products separately. For \( t \in [0, \tau_1] \) we have
\[
\int_0^t \langle \nabla w_{h,\epsilon}^i(\bar{\psi}(s)) - \nabla w_{h,\epsilon}^i(\bar{\psi}(0)), \dot{\bar{\psi}}(s) \rangle ds \\
\leq \int_0^t \| \nabla w_{h,\epsilon}^i(\bar{\psi}(s)) - \nabla w_{h,\epsilon}^i(\bar{\psi}(0)) \| \| \dot{\bar{\psi}}(s) \| ds \\
\leq \int_0^t ([Dw_{h,\epsilon}]_{C^{0,\alpha}(\Sigma_{w,\epsilon})} \| \bar{\psi}(s) - \bar{\psi}(0) \| \| D\pi(\bar{\psi} - s\bar{v}) \| \| \bar{v} \| ) ds \\
\leq \int_0^t ([Dw_{h,\epsilon}]_{C^{0,\alpha}(\Sigma_{w,\epsilon})} \| D\pi \|_{C^{0,\alpha}(\mathcal{K}^T)} \| s\bar{v} \| \alpha^2 \| D\pi(\bar{\psi} - s\bar{v}) \| \| \bar{v} \| ) ds \\
\leq \frac{L\tilde{L}^{1+\alpha} \| \bar{v} \| \alpha^2 + 1}{\alpha^2 + 1} t^{\alpha^2 + 1}
\]
and
\[
\int_0^t \langle \nabla w_{h,\epsilon}^i(\bar{\psi}(0)), \dot{\bar{\psi}}(s) - \bar{\psi}(0) \rangle ds \leq \int_0^t \| \nabla w_{h,\epsilon}^i(\bar{\psi}(0)) \| \| \dot{\bar{\psi}}(s) - \bar{\psi}(0) \| ds \\
\leq \int_0^t \| Dw_{h,\epsilon}(\bar{\psi}(0)) \| \| (D\pi(\bar{\psi}(0)) - D\pi(\bar{\psi}(s)))\bar{v} \| ds \\
\leq \int_0^t \| Dw_{h,\epsilon}(\bar{\psi}(0)) \| \| D\pi \|_{C^{0,\alpha}(\mathcal{K}^T)} \| \bar{\psi}(s) - \bar{\psi}(0) \| \| \bar{v} \| ds \\
\leq \int_0^t \| Dw_{h,\epsilon}(\bar{\psi}(0)) \| \| D\pi \|_{C^{0,\alpha}(\mathcal{K}^T)} \| D\pi \|_{C^{0,\alpha}(\mathcal{K}^T)} \| s\bar{v} \| \alpha^2 \| \bar{v} \| ds \\
= \frac{L\tilde{L}^{1+\alpha} \| \bar{v} \| \alpha^2 + 1}{\alpha^2 + 1} t^{\alpha^2 + 1}
\]
where we have used \( \dot{\bar{\psi}}(s) = -(D\pi(\bar{\psi} - s\bar{v}))(\bar{v}) \). Hence for \( t \in [0, \tau_1] \) we obtain the bound on the remainder term \( R \) above as
\[
\| R(t) \| \leq \frac{2L\tilde{L}^{1+\alpha} \sqrt{N} \| \bar{v} \| \alpha^2 + 1}{\alpha^2 + 1} t^{\alpha^2 + 1} \leq \frac{2L\tilde{L}^{1+\alpha} \sqrt{N} \| w_{h,\epsilon}(\bar{\psi}) - w \| \alpha^2 + 1}{\kappa^{1+\alpha^2}} t^{\alpha^2 + 1}.
\]
At this point, the remainder of the proof proceeds exactly as that of \[KMT19\] Proposition 6.1] following equation (6.3) there, with \( w_{h,\epsilon} \) replacing the map \( G \) and \( \alpha^2 \) instead of \( \alpha \). For the convenience of the reader we give the analogous expressions for \( \tau_i \) which are
\[
\tau_1 \geq \frac{\kappa\varepsilon_0}{2L\tilde{L} \| w_{h,\epsilon}(\bar{\psi}) - w \|}, \\
\tau_2 = \min(\tau_1, \frac{\kappa^{1+\alpha^2} \sqrt{\varepsilon_0}}{2L\tilde{L}^{1+\alpha} \sqrt{N} \| w_{h,\epsilon}(\bar{\psi}) - w \|}), \\
\tau_3 = \min(\tau_2, \frac{\kappa^{1+\alpha^2}}{4L\tilde{L}^{1+\alpha} \sqrt{N} \| w_{h,\epsilon}(\bar{\psi}) - w \|}, 1).
\]
Finally, note that since \( \sum_{i=1}^N w_{h,\epsilon}(\bar{\psi})^i = \sum_{i=1}^N w^i \), we have the bound
\[
\| w_{h,\epsilon}(\bar{\psi}) - w \| \leq 2 \sum_{i=1}^N w^i \leq 2N.
\]
With these expressions, we can calculate

$$\bar{\tau}_k \leq \frac{\epsilon_0^{-\alpha} \kappa^{1+\frac{1}{\alpha^2}}}{(4L\tilde{L}^{1+\alpha}\sqrt{N}) \frac{1}{\alpha^2} \| w_{h,\epsilon}(\bar{\psi}_k) - w \|^{1+\frac{1}{\alpha^2}}} \leq \tau_3,$$

hence the global linear and local superlinear convergence can be obtained just as in [KMT19, Proposition 6.1].

We now use the above estimate Proposition 4.1 to give a crude estimate on the number of iterations necessary to obtain an approximation of a solution to within an error of $\xi$. As the bounds in our convergence Theorems 2.11 and 2.14 involve the quantity $\| w_{h,\epsilon}(\psi) - w \|$, the estimate below can be used to tune the parameters $h$ and $\epsilon$ effectively when implementing Algorithm 1. Note that Corollary 4.2 is far from tight, as it does not take into account that our rate derived in Proposition 4.1 goes to zero or that we have locally $1 + \alpha^2$-superlinear convergence, but still serves as a starting point.

**Corollary 4.2.** There exists a universal constant $C > 0$ so that for every $\zeta > 0$, and $\epsilon_0$, $h$, $\epsilon$ sufficiently small depending on universal quantities, Algorithm 1 terminates in at most $\frac{\log \frac{\tau_k}{\tau_0}}{\log(1-\eta)}$ steps where $\eta = C\epsilon_0^{1+\frac{2}{\alpha^2}} h^{6+\frac{18}{\alpha} + \frac{27}{\alpha^2}} \epsilon^{\frac{3}{2} + \frac{9}{\alpha} + \frac{25}{2\alpha^2}}$.

**Proof.** If $\bar{\tau}_k \neq 1$, we have

$$\bar{\tau}_k = \frac{\epsilon_0^{-\alpha} \kappa^{1+\frac{1}{\alpha^2}}}{(4L\tilde{L}^{1+\alpha}\sqrt{N}) \frac{1}{\alpha^2} \| w_{h,\epsilon}(\bar{\psi}_k) - w \|^{1+\frac{1}{\alpha^2}}}$$

$$\geq C\frac{\epsilon_0^{-\alpha} \kappa^{1+\frac{1}{\alpha^2}}}{(h^{-18\epsilon^{-9}})^{1+\alpha} \max(h^{-2\epsilon^{-2}}, h^{-3\epsilon^{-2}})}$$

$$\geq C\frac{\epsilon_0^{-\alpha} \kappa^{1+\frac{1}{\alpha^2}}}{(h^{-18\epsilon^{-9}})^{1+\alpha} (h^{-3\epsilon^{-2}})}$$

$$= C\epsilon_0^{1+\frac{2}{\alpha^2}} h^{6+\frac{18}{\alpha} + \frac{27}{\alpha^2}} \epsilon^{\frac{3}{2} + \frac{9}{\alpha} + \frac{25}{2\alpha^2}},$$

and we may assume $h$, $\epsilon_0$, $\epsilon$ are sufficiently small so that $1 - \frac{C\epsilon_0^{1+\frac{2}{\alpha^2}} h^{6+\frac{18}{\alpha} + \frac{27}{\alpha^2}} \epsilon^{\frac{3}{2} + \frac{9}{\alpha} + \frac{25}{2\alpha^2}}}{2} \geq \frac{1}{2}$. Hence regardless of which value $\bar{\tau}_k$ takes at each iteration, after $\ell$ iterations we have

$$\| w(\psi_{\ell}) - w \| \leq (1 - \eta)^\ell \| w(\psi_0) - w \| \leq 2N(1 - \eta)^\ell$$

where $\eta = C\epsilon_0^{1+\frac{2}{\alpha^2}} h^{6+\frac{18}{\alpha} + \frac{27}{\alpha^2}} \epsilon^{\frac{3}{2} + \frac{9}{\alpha} + \frac{25}{2\alpha^2}}$. Solving $(1 - \eta)^\ell \| w(\psi_0) - w \| \leq 2N(1 - \eta)^\ell \leq \xi$ for $\ell$, we see that

$$\ell \geq \frac{\log \frac{\xi}{2N}}{\log(1-\eta)}$$

suffices.

5. $\mu$-symmetric convergence of Laguerre cells

5.1. The Exchange Digraph. We now work toward proving Theorems 2.11 and 2.14 on convergence of the Laguerre cells in Algorithm 1 as $h$ and $\epsilon$ approach 0. We also note that for the results in this section, the only conditions that are used are that the cost function $c$
satisfies \( \text{(Reg)} \) and \( \text{(Twist)} \), and the source measure \( \mu \) is absolutely continuous with respect to Lebesgue measure: we do not assume \( \text{(QC)} \) or any regularity on the density of \( \mu \).

For this section, suppose \( F_1, F_2 : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \) are two proper convex functions equal to \(+\infty\) outside of \( \Lambda \). By [BK19, Theorem 2.3 and Proposition 3.5] there exist pairs \((T_1, \lambda_1)\) and \((T_2, \lambda_2)\) minimizing \([1.2]\) with storage fee functions equal to \( F_1 \) and \( F_2 \) respectively, along with (see [BK19, Theorem 4.7]) vectors \( \psi_1, \psi_2 \in \mathbb{R}^N \) such that \( G(\psi_1) = \lambda_1, G(\psi_2) = \lambda_2 \). As mentioned before, up to sets of \( \mu \) measure zero, we have \( T_1^{-1}(y_i) = \text{Lag}_i(\psi_1) \) and \( T_2^{-1}(y_i) = \text{Lag}_i(\psi_2) \).

We now define a weighted directed graph (digraph), \( D \), as follows. The vertex set is \( y_1, \ldots, y_N \). When \( i \neq j \), there is a directed edge from \( y_i \) to \( y_j \) if \( \mu(\text{Lag}_i(\psi_1) \cap \text{Lag}_j(\psi_2)) > 0 \), and in this case that edge is assigned weight \( \mu(\text{Lag}_i(\psi_1) \cap \text{Lag}_j(\psi_2)) \). We denote the weight of an edge \( e \) by \( w(e) \).

Essentially this digraph keeps track of how much mass is shifted from one Laguerre cell to a different one under a change of the storage fee function. Indeed note that \( \lambda_2^i = \lambda_1^i - \text{deg}^+(y_i) + \text{deg}^-(y_i) \) where

\[
\text{deg}^+(y_i) := \sum_{\{e \text{ is directed out from } y_i\}} w(e),
\]

\[
\text{deg}^-(y_i) := \sum_{\{e \text{ is directed into } y_i\}} w(e),
\]

denote outdegree and indegree respectively.

First we use an argument reminiscent of the \( c \)-cyclical monotonicity of optimal transport plans to prove the following lemma.

**Lemma 5.1.** \( D \) is acyclic

**Proof.** Suppose for sake of contradiction there exists a cycle \( y_{i_1}, e_1, y_{i_2}, \ldots, y_{i_l}, e_l, y_{i_1} \) where \( i_{j+1} = i_j \) and \( e_j \) is a directed edge from \( y_{i_j} \) to \( y_{i_{j+1}} \). Let \( m_0 := \min_{1 \leq j \leq l} w(e_j) > 0 \), then for each \( 1 \leq j \leq l \) there exists a measurable set \( A_j \subset \text{Lag}_{i_j}(\psi_1) \cap \text{Lag}_{i_{j+1}}(\psi_2) \) with \( \mu(A_j) = m_0 \), and we define \( A_{l+1} = A_1 \).

Now define the sets \( \{\bar{C}_k\}_{k=1}^N \) by

\[
\bar{C}_k = \begin{cases} 
(\text{Lag}_{i_j}(\psi_2) \cup A_{j+1}) \setminus A_j, & \text{if } k = i_{j+1}, 1 \leq j \leq l, \\
\text{Lag}_k(\psi_2), & \text{if } k \not\in \{i_1, \ldots, i_l\},
\end{cases}
\]

and the map \( \bar{T} : X \to Y \) defined by \( \bar{T}(x) = \sum_{k=1}^N y_k \mathbb{1}_{\bar{C}_k}(x) \). Since \( \text{Lag}_i(\psi_1) \) and \( \text{Lag}_j(\psi_1) \) are disjoint up to sets of \( \mu \) measure zero for \( i \neq j \), we must have that the sets \( A_j \) are mutually disjoint up to \( \mu \) measure zero sets, thus \( \bar{T}_\# \mu = \sum_{k=1}^N m_k (\bar{C}_k) \delta_{y_k} = \sum_{k=1}^N \lambda^k_2 \delta_{y_k} \) but \( \bar{T} \neq T_2 \) on a set of positive \( \mu \) measure. By [BK19, Corollary 4.5], \( (T_2, \lambda_2) \) is the unique minimizer of \([1.2]\) with storage fee function \( F_2 \), thus we must have

\[
\sum_{k=1}^N \int_{\bar{C}_k} c(x, y_k) d\mu(x) + F_2(\lambda_2) > \sum_{k=1}^N \int_{\text{Lag}_k(\psi_2)} c(x, y_k) d\mu(x) + F_2(\lambda_2).
\]
Thus,

\[
0 < \sum_{k=1}^{N} \int \hat{c}_k c(x, y_k) d\mu(x) - \sum_{k=1}^{N} \int \hat{\lambda}_k(x, y_k) d\mu(x) \\
= \sum_{k=1}^{N} \int_{\text{Lag}_k(\psi_2)} c(x, y_k) d\mu(x) - \sum_{k=1}^{N} \int_{\text{Lag}_k(\psi_2)} c(x, y_k) d\mu(x) \\
+ \sum_{j=1}^{l-1} \left( \int_{A_{j+1}} c(x, y_{i_{j+1}}) d\mu(x) - \int_{A_j} c(x, y_{i_j}) d\mu(x) \right) \\
= \sum_{j=1}^{l-1} \left( \int_{A_{j+1}} c(x, y_{i_{j+1}}) d\mu(x) - \int_{A_j} c(x, y_{i_j}) d\mu(x) \right). \\
\tag{5.2}
\]

On the other hand, defining the sets \( \{\tilde{D}_k\}_{k=1}^{N} \) by

\[
\tilde{D}_k = \begin{cases} 
(\text{Lag}_{i_{j+1}}(\psi_1) \cup A_j) \setminus A_{j+1}, & k = i_{j+1}, \ 1 \leq j \leq l, \\
\text{Lag}_k(\psi_1), & k \not\in \{i_1, \ldots, i_l\},
\end{cases}
\tag{5.3}
\]

and taking the map \( \tilde{T}(x) = \sum_{k=1}^{N} y_k \mathbb{1}_{\tilde{D}_k}(x) \), we can make an analogous calculation which yields the opposite inequality as \( (5.2) \), giving a contradiction. \( \square \)

For the next three Lemmas \( 5.2, 5.3 \) and \( 5.5 \) we shall be concerned about the case where

\[
F_1(\lambda) = \sum_{i=1}^{N} \delta(\lambda^i | [a^i, b^i]),
\]

\[
F_2(\lambda) = \delta(\lambda^1 | [a^1, b^1 + \eta]) + \sum_{i=2}^{N} \delta(\lambda^i | [a^i, b^i]) \\
\tag{5.4}
\]

where \( a^i \leq b^i \) and \( \sum a^i \leq 1 \leq \sum b^i \). Recall that \( (T_1, \lambda_1), (T_2, \lambda_2) \) are the minimizers in \( [1,2] \) associated with \( F_1, F_2 \) respectively; in particular we must have \( a^i \leq \lambda_1^i \leq b^i \) all \( i \in \{1, \ldots, N\} \), \( a^1 \leq \lambda_2^1 \leq b^1 + \eta \), and \( a^i \leq \lambda_2^i \leq b^i \) for all \( 2 \leq i \leq N \).

**Lemma 5.2.** Suppose we take \( F_1 \) and \( F_2 \) as in \( (5.4) \) and there exists some vertex \( y_m \) of \( D \) with an incoming edge. Then \( \lambda_1^m = b^m \).

**Proof.** Let \( i_1 = m \). Suppose the incoming edge, which we denote \( e_1 \), goes from \( y_{r+1} \) to \( y_{i_1} \). We claim that there is a path \( P = (y_{i_1}, e_1, y_{i_2}, \ldots, y_{i_{r-1}}, e_{r-1}, y_{i_r}) \), where \( e_j \) is an edge from \( y_{i_{j+1}} \) to \( y_{i_j} \), such that the last vertex \( y_i \) has no incoming edges.

We construct such a path recursively. Let \( P_1 = (y_{i_1}, e_1, y_{i_2}) \) and suppose that \( P_r = (y_{i_1}, e_1, y_{i_2}, \ldots, y_{i_r}, e_r, y_{i_{r+1}}) \) has been constructed. If \( y_{i_{r+1}} \) has no incoming edges then \( P_r \) is the desired path and we are done. If not \( y_{i_{r+1}} \) has an incoming edge which we denote \( e_{r+1} \). Let \( y_{i_{r+2}} \) be the originating vertex of \( e_{r+1} \) and let \( P_{r+1} = (y_{i_1}, e_1, y_{i_2}, \ldots, y_{i_{r+1}}, e_{r+1}, y_{i_{r+2}}) \). If the above process does not terminate then since we only have finitely many vertices we must eventually repeat a vertex, i.e. there is \( r > j \) so that \( i_j = i_r \). However this means that \( P_r \) contains a cycle which contradicts Lemma \( 5.1 \) above.
Now let \( m_0 = \min(b^{m} - \lambda^m_i, w(e_1, \ldots, w(e_{l-1}))) \). Suppose for sake of contradiction that \( \lambda^m_i < b^m \), then \( m_0 > 0 \). Note that
\[
\lambda^m_i = \lambda^m_i - \deg^+(y_i) + \deg^-(y_i) \leq b^i - w(e_{l-1}) + 0 \leq b^i - m_0.
\]

Now just as in the proof of Lemma 5.1 for \( j \in \{2, \ldots, l\} \) there exist sets \( A_j \) so that \( A_j \subset \text{Lag}_{i_j}(\psi_1) \cap \text{Lag}_{i_j-1}(\psi_2) \), and \( \mu(A_j) = m_0 \). We define \( A_1 = A_{l+1} = \emptyset \). Now define the sets \( \{\tilde{C}_k\}_{k=1}^N \) by
\[
\tilde{C}_k = \begin{cases} (\text{Lag}_{i_j}(\psi_2) \cup A_j) \setminus A_{j+1}, & k = i_j, j \in \{1, \ldots, l\}, \\
\text{Lag}_k(\psi_2), & k \notin \{i_1, \ldots, i_l\}.
\end{cases}
\]

and the map \( \tilde{T} : X \to Y \) defined by \( \tilde{T}(x) = \sum_{k=1}^N y_k \chi_{\tilde{C}_k}(x) \). Just as in the proof of Lemma 5.2 above, we have \( \tilde{T}_\#\mu = \sum_{k=1}^N \mu(\tilde{C}_k) \delta_{y_k} \) and \( \tilde{T} \neq T_2 \) on a set of positive \( \mu \) measure (however, note that we do not have \( \mu(\tilde{C}_k) = \lambda^m_k \) for \( k = i_1, i_l \)). Since \( (T_2, \lambda^m_2) \) is the unique minimizer of \( 1.2 \) with storage fee function \( F_2 \) by [BK19, Corollary 4.5], we must have
\[
\sum_{k=1}^N \int_{\tilde{C}_k} c(x, y_k) d\mu(x) + F_2(\mu(\tilde{C}_1), \ldots, \mu(\tilde{C}_N))) > \sum_{k=1}^N \int_{\text{Lag}_k(\psi_2)} c(x, y_k) d\mu(x) + F_2(\lambda^m_2).
\]

However now note that
\[
\mu(\tilde{C}_k) = \begin{cases} \lambda^m_i - m_0, & k = i_1, \\
\lambda^m_i + m_0, & k = i_l, \\
\lambda^m_2, & \text{else}.
\end{cases}
\]

By (5.5), we have that \( \mu(\tilde{C}_i) = \lambda^m_i + m_0 \leq b^i \). Also for \( k \neq i_l \) we have \( \mu(\tilde{C}_k) \leq \lambda^m_2 \leq b^k \), hence \( F_2(\mu(\tilde{C}_1), \ldots, \mu(\tilde{C}_N))) = 0 \). Thus the above becomes
\[
\sum_{k=1}^N \int_{\tilde{C}_k} c(x, y_k) d\mu(x) > \sum_{k=1}^N \int_{\text{Lag}_k(\psi_2)} c(x, y_k) d\mu(x),
\]

and by a calculation identical to the one leading to (5.2), we have
\[
0 < \sum_{j=1}^l \left( \int_{A_j} c(x, y_j) d\mu(x) - \int_{A_{j+1}} c(x, y_j) d\mu(x) \right).
\]

On the other hand, define the sets \( \{\tilde{D}_k\}_{k=1}^N \) by
\[
\tilde{D}_k = \begin{cases} (\text{Lag}_{i_j}(\psi_1) \cup A_j) \setminus A_{j+1}, & k = i_j, j \in \{1, \ldots, l\}, \\
\text{Lag}_k(\psi_1), & k \notin \{i_1, \ldots, i_l\}.
\end{cases}
\]

Note that
\[
\mu(\tilde{D}_k) = \begin{cases} \lambda^m_i + m_0, & k = i_1, \\
\lambda^m_i - m_0, & k = i_l, \\
\lambda^m_1, & \text{else}.
\end{cases}
\]

By definition of \( m_0 \) we have \( m_0 \leq b^m - \lambda^m_i = b^i - \lambda^m_i \), hence we have \( \mu(\tilde{D}_{i_1}) \leq b^i \). Thus as above, \( F_2(\mu(\tilde{D}_1), \ldots, \mu(\tilde{D}_N))) = 0 \) and a similar argument yields the opposite inequality of (5.7) to obtain a contradiction. \( \square \)
Lemma 5.3. Suppose we take \( F_1 \) and \( F_2 \) as in (5.4). Then for \( i \neq 1 \), \( \lambda_2^i \leq \lambda_1^i \). Furthermore, if \( y_i \) has an incoming edge it must have an outgoing edge. Finally, \( y_1 \) has no outgoing edges.

Proof. Recall that \( \lambda_1^i = \lambda_1^i - \deg^+(y_i) + \deg^-(y_i) \).
Suppose \( i \neq 1 \). If \( y_i \) has no incoming edges then \( \deg^-(y_i) = 0 \) so \( \lambda_2^i = \lambda_1^i - \deg^+(y_i) \leq \lambda_1^i \).
If \( y_i \) has at least one incoming edge then \( \lambda_1^i = b_i^j \) by Lemma 5.2 above. Since \( i \neq 1 \) and \( F_2(\lambda_2) < +\infty \), we must have \( \lambda_2^i \leq b_i^j \). In either case \( \lambda_2^i \leq \lambda_1^i \).
Now if \( y_i \) has an incoming edge then
\[
\deg^+(y_i) = \lambda_1^i - \lambda_2^i + \deg^-(y_i) \geq \deg^-(y_i) > 0,
\]
so there must be an outgoing edge.
Finally suppose for sake of contradiction that \( y_1 \) has an outgoing edge. We recursively construct a path similar to that in the proof of Lemma 5.2. Set \( i_1 = 1 \), \( P_1 = (y_{i_1}, e_1, y_{i_2}) \) and suppose that \( P_l = (y_{i_l}, e_1, y_{i_2}, \ldots, e_1, y_{i_{l+1}}) \) has been constructed where \( e_j \) is an edge directed from \( y_{i_j} \) to \( y_{i_{j+1}} \). If \( y_{i_{l+1}} = y_{i_1} \) then we have constructed a cycle which contradicts Lemma 5.1. If \( y_{i_{l+1}} \neq y_{i_1} = y_1 \), then \( y_{i_{l+1}} \) has an outgoing edge which we denote \( e_{l+1} \). Set \( y_{i_{l+2}} \) to be the tail of \( e_{l+1} \) and let \( P_{l+1} = (y_{i_{l+1}}, e_1, y_{i_2}, \ldots, y_{i_l}, e_{l+1}, y_{i_{l+2}}) \). Since we only have finitely many vertices the above process must repeat a vertex which will produce a cycle. This contradicts Lemma 5.1 hence \( y_1 \) cannot have any outgoing edges. \( \square \)

Remark 5.4. Recall that in an directed acyclic graph the vertices can be given an ordering, called a topological ordering, so that every edge goes from a vertex with smaller index to a vertex with larger index. See [BJG09, Proposition 2.1.3] and the associated footnote for more details.

Lemma 5.5. Suppose again we take \( F_1 \) and \( F_2 \) as in (5.4). Then every edge has outdegree at most \( \eta \), in particular every vertex has weight at most \( \eta \). In this case we have \( \| \lambda_1 - \lambda_2 \|_1 \leq 2\eta \) and \( \sum_{i=1}^N \Delta_\mu(\text{Lag}_i(\psi_1), \text{Lag}_i(\psi_2)) \leq 2N\eta \).

Proof. Let \( y_{i_1}, \ldots, y_{i_N} \) be a topological ordering. By Lemma 5.3 we may assume \( i_N = 1 \). Consider the function
\[
f(k) = \sum_{j=1}^k \deg^+(y_{i_j}) - \deg^-(y_{i_j}) = \sum_{j=1}^k \lambda_1^{i_j} - \lambda_2^{i_j}
\]
for \( k \leq N - 1 \).
By Lemma 5.3 \( f \) is increasing. Let \( E_k \) be the collection of edges directed from one of the vertices \( y_{i_1}, \ldots, y_{i_k} \) and into one of the vertices \( y_{i_{k+1}}, \ldots, y_{i_N} \). Then we have
\[
f(k) = \sum_{e \in E_k} w(e);
\]
as we have imposed a topological ordering, there is no edge directed from one of the vertices \( y_{i_{k+1}}, \ldots, y_{i_N} \) to one of the vertices \( y_{i_1}, \ldots, y_{i_k} \). In particular \( f(k) \geq \deg^+(y_{i_k}) \), thus \( f(N - 1) \geq \deg^+(y_{i_k}) \) for all \( k \leq N - 1 \). Note that \( E_{N-1} \) is the collection of all edges directed to \( y_{i_N} = y_1 \). Hence
\[
\deg^+(y_{i_k}) \leq f(N - 1) = \sum_{e \in E_{N-1}} w(e) = \deg^-(y_1).
\]
If $y_1$ has no incoming edges then this gives us $\deg^+(y_1) = 0$. Otherwise by Lemma 5.2
\[
\deg^-(y_1) = \lambda_2^1 - \lambda_1^1 + \deg^+(y_1) = \lambda_2^1 - b^1
\]
where $\deg^+(y_1) = 0$ by Lemma 5.3. Since $F_2(\lambda_2) < +\infty$, we must have $\lambda_2^1 \leq b^1 + \eta$ hence each vertex has outdegree at most $\eta$.
Next by Lemma 5.3, $\lambda_2^i \leq \lambda_1^i$ for $i \neq 1$, since $\lambda_1, \lambda_2 \in \Lambda$ this implies $\lambda_2^1 \geq \lambda_1^1$. Hence
\[
\|\lambda_1 - \lambda_2\|_1 = \sum_{i=1}^{N} |\lambda_2^i - \lambda_1^i| = \lambda_2^1 - \lambda_1^1 + \sum_{i=2}^{N} (\lambda_1^i - \lambda_2^i)
\]
\[
= \lambda_2^1 - \lambda_1^1 + (1 - \lambda_1^1) - (1 - \lambda_2^1)
\]
\[
= 2(\lambda_2^1 - \lambda_1^1) = 2(\deg^- (y_1) - \deg^+(y_1)) \leq 2\eta
\]
where we have used $\sum_{i=1}^{N} \lambda_1^i = \sum_{i=1}^{N} \lambda_2^i = 1$.
Next we have
\[
\mu(\text{Lag}_i(\psi_1) \setminus \text{Lag}_i(\psi_2)) = \mu(\text{Lag}_i(\psi_1) \cap (\text{Lag}_i(\psi_2))^c)
\]
\[
= \mu(\text{Lag}_i(\psi_1) \cap \bigcup_{j \neq i} \text{Lag}_j(\psi_2)) = \sum_{j \neq i} \mu(\text{Lag}_i(\psi_1) \cap \text{Lag}_j(\psi_2)) = \deg^+(y_i) \leq \eta
\]
and so $\sum_{i=1}^{N} \mu(\text{Lag}_i(\psi_1) \setminus \text{Lag}_i(\psi_2)) \leq N\eta$. A similar argument gives
\[
\sum_{i=1}^{N} \mu(\text{Lag}_i(\psi_1) \setminus \text{Lag}_i(\psi_2)) = \sum_{i=1}^{N} \deg^-(y_i) = \sum_{i=1}^{N} \deg^+(y_i) \leq N\eta
\]
where the final equality comes from
\[
\sum_{i=1}^{N} \deg^-(y_i) = \sum_{i=1}^{N} (\deg^+(y_i) + \lambda_2^i - \lambda_1^i) = \sum_{i=1}^{N} \deg^+(y_i),
\]
finishing the proof. \qed

By perturbing each of the coordinates separately, we can now analyze the digraph $D$ when $F_1$ and $F_2$ are characteristic functions of two different hyperrectangles.

**Theorem 5.6.** Suppose we have
\[
F_1(\lambda) = \sum_{i=1}^{N} \delta(\lambda^i | [a_1^i, b_1^i]),
\]
\[
F_2(\lambda) = \sum_{i=1}^{N} \delta(\lambda^i | [a_2^i, b_2^i]).
\]
Then $\|\lambda_1 - \lambda_2\|_1 \leq 2(\|a_1 - a_2\|_1 + \|b_1 - b_2\|_1)$ and $\sum_{i=1}^{N} \Delta_{\mu}(\text{Lag}_i(\psi_1), \text{Lag}_i(\psi_2)) \leq 2N(\|a_1 - a_2\|_1 + \|b_1 - b_2\|_1)$. 

Proof. If \( a_1 = a_2 \) then this follows from induction on the number of equal terms in \( b_1, b_2 \), repeatedly applying Lemma 5.5 and the triangle inequality. The case \( a_1 \neq a_2 \) is handled with a symmetric argument and the triangle inequality. \( \square \)

**Corollary 5.7.** Suppose that \( F_1, F_2 : \mathbb{R}^N \to \mathbb{R} \cup \{ \pm \infty \} \) are two proper convex functions equal to \( +\infty \) outside of \( \Lambda \). Then

\[
\sum_{i=1}^{N} \Delta_{\mu}(\text{Lag}_i(\psi_1), \text{Lag}_i(\psi_2)) \leq 4N\|\lambda_1 - \lambda_2\|_1.
\]

*Proof.* Define

\[
\tilde{F}_1(\lambda) = \sum_{i=1}^{N} \delta(\lambda^i \mid [a^i_1, b^i_1]),
\]

\[
\tilde{F}_2(\lambda) = \sum_{i=1}^{N} \delta(\lambda^i \mid [a^i_2, b^i_2]),
\]

where \( a^i_1 = b^i_1 = \lambda^i_1 \) and \( a^i_2 = b^i_2 = \lambda^i_2 \). We see that if \((\tilde{T}_1, \tilde{\lambda}_1), (\tilde{T}_2, \tilde{\lambda}_2)\) are minimizers for \(1.2\) with storage fee functions \(\tilde{F}_1\) and \(\tilde{F}_2\), then up to sets of \(\mu\) measure zero \(\tilde{T}_1^{-1}(\{y_i\}) = \text{Lag}_i(\psi_1)\) and \(\tilde{T}_2^{-1}(\{y_i\}) = \text{Lag}_i(\psi_2)\) for each \(i \in \{1, \ldots, N\}\). Hence the result follows from applying Theorem 5.6 to \(\tilde{F}_1, \tilde{F}_2\).

\( \square \)

**Remark 5.8.** By taking \( F_1, F_2 \) to be the indicator functions for two points in \( \Lambda \), the above corollary gives us quantitative stability for the \( \mu\)-symmetric difference of the Laguerre cells in the original semi-discrete optimal transport problem (without storage fees).

### 5.2. Proof of Theorem 2.11

We are now ready to prove the first quantitative convergence theorem of our Laguerre cells.

**Proof of Theorem 2.11.** Let \( w \in \mathbb{R}^N \) with \( \sum_{i=1}^{N} w^i \geq 1 \), \( w^i \geq 0 \) and \( \psi_{h, \epsilon} \in \mathcal{K}^e \), and let \((T, \lambda)\) be a pair minimizing \(1.2\) with the storage fee function \( F_w \). Then if we define \( \lambda_{h, \epsilon} := G(\psi_{h, \epsilon}) \) and \( \overline{w} := w_{h, \epsilon}(\psi_{h, \epsilon}) \), by Proposition 3.2, the pair \((T_{w, \epsilon}, \lambda_{h, \epsilon})\) minimizes \(1.2\) with storage fee equal to \( F_{\overline{w}, \epsilon} \). By [BK19, Theorem 4.7], there also exists a pair \((T_{\overline{w}, \epsilon}, \lambda_{\overline{w}, \epsilon})\) which minimizes \(1.2\) with storage fee \( F_{\overline{w}, 0, \epsilon} \). Let

\[
\mathcal{C}(\tilde{\lambda}) = \min_{S_{\mu} \in \mathcal{K}^c} \int c(x, S(x))d\mu = \sup_{\psi \in \mathbb{R}^N} \left( -\int \psi\psi^c d\mu - \langle \psi, \overline{\lambda} \rangle \right).
\]

We have

\[
\mathcal{C}(\lambda_{h, \epsilon}) + F_{\overline{w}, h, \epsilon}(\lambda_{h, \epsilon}) = \min_{\lambda \in \Lambda} \left( \mathcal{C}(\tilde{\lambda}) + F_{\overline{w}, h, \epsilon}(\lambda) \right) \leq \mathcal{C}(\lambda_{\overline{w}, \epsilon}) + F_{\overline{w}, h, \epsilon}(\lambda_{\overline{w}, \epsilon}),
\]

thus

\[
\mathcal{C}(\lambda_{h, \epsilon}) - \mathcal{C}(\lambda_{\overline{w}, \epsilon}) \leq F_{\overline{w}, h, \epsilon}(\lambda_{h, \epsilon}) - F_{\overline{w}, h, \epsilon}(\lambda_{\overline{w}, \epsilon}) \leq -F_{\overline{w}, h, \epsilon}(\lambda_{h, \epsilon}) \leq h.
\]

Next by Corollary A.2 from the appendix, we have \( \frac{1}{32C_L} \|\lambda_{h, \epsilon} - \lambda_{\overline{w}, \epsilon}\|^2 \leq \mathcal{C}(\lambda_{h, \epsilon}) - \mathcal{C}(\lambda_{\overline{w}, \epsilon}) \leq h \) as \( \lambda_{\overline{w}, \epsilon} \) is the minimizer of \( \mathcal{C} \) on the convex set \( \prod_{i=1}^{N}[\epsilon, \overline{w}^i + \epsilon] \), which can be seen from \( F_{\overline{w}, 0, \epsilon} = \delta(\cdot \mid \prod_{i=1}^{N}[\epsilon, \overline{w}^i + \epsilon]) \). Since the \( l^1 \) and \( l^2 \) norms on \( \mathbb{R}^N \) are comparable,

\[
\|\lambda_{h, \epsilon} - \lambda_{\overline{w}, \epsilon}\|_1 \leq \sqrt{N}\|\lambda_{h, \epsilon} - \lambda_{\overline{w}, \epsilon}\| \leq 4N\sqrt{2C_L}h.
\]
Now by Theorem 5.6, \( \| \lambda w, \epsilon - \lambda w \|_1 \leq 2N\epsilon + 2\| w - w \|_1 \) and so the triangle inequality gives
\[
\| G(\psi_{h,\epsilon}) - \lambda \|_1 = \| \lambda h, \epsilon - \lambda \|_1 \leq 2(N\epsilon + \| w - w \|_1 + 2N\sqrt{2C_L h})
\]
proving (2.5), and then Corollary 5.7 gives
\[
\sum_{i=1}^N \Delta \mu(\text{Lag}_i(\psi_{h,\epsilon}), T^{-1}(\{y_i\})) \leq 8N(N\epsilon + \| w - w \|_1 + 2N\sqrt{2C_L h})
\]
proving (2.6) \( \Box \)

6. ESTIMATES ON HAUSDORFF DISTANCE

We will now work towards proving Theorem 2.14 which is a quantitative rate of convergence of Laguerre cells in the Hausdorff distance of sets. Recall the following definition of Hausdorff distance.

**Definition 6.1.** If \( x \in \mathbb{R}^n \) and \( A \subset \mathbb{R}^n \), we define
\[
d(x, A) := \inf_{y \in A} \| x - y \|.
\]

Then for two nonempty sets \( A \) and \( B \subset \mathbb{R}^n \), the **Hausdorff distance** between \( A \) and \( B \) is defined by
\[
d_H(A, B) := \max \left( \sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A) \right).
\]

**Definition 6.2.** Let us denote
\[
\omega_j = \frac{\pi^{j/2}}{\Gamma(\frac{j}{2} + 1)}
\]
for the volume of the unit ball in \( \mathbb{R}^j \).

We start with a simple lemma in convex geometry.

**Lemma 6.3.** If \( A \) is a bounded convex set with \( \mathcal{L}(A) > 0 \) then \( A \) contains a ball of radius \( R_A \mathcal{L}(A) \) where
\[
R_A := \frac{2^{n-1}}{\omega_n (n + 2)^n \text{diam}(A)^{n-1}}.
\]

**Proof.** Let \( S \) be a simplex in \( A \) with volume at least \( \frac{1}{(n+2)^n} \mathcal{L}(A) \) as given by the main theorem of [Las11]. Since \( S \) is convex and is contained in a ball of radius \( \frac{\text{diam}(A)}{2} \), we have \( \mathcal{H}^{n-1}(\partial S) \leq n\omega_n \left( \frac{\text{diam}(A)}{2} \right)^{n-1} \) (see [Sch93, p. 211]). Then it is standard that \( S \) contains a ball of radius \( r \), where
\[
r = \frac{n \text{vol}(S)}{\mathcal{H}^{n-1}(\partial S)} \geq \frac{2^{n-1} \mathcal{L}(A)}{\omega_n (n + 2)^n \text{diam}(A)^{n-1}},
\]
see for example the last formula in the proof of [VG67, Corollary 3] and the discussion following it. \( \Box \)
In the next proposition, we estimate the term $\sup_{x \in B} d(x, A)$ from the definition of Hausdorff distance by the Lebesgue measure of the difference of the two sets, when they are convex. We opt to take a different approach from the proof of Theorem 2.11, ultimately we will control the Lebesgue measure of the symmetric difference of Laguerre cells directly by the dual variables $\psi$, then attempt to quantitatively invert the map $G$, allowing us to invoke the first estimate in Theorem 2.11.

**Proposition 6.4.** Let $A \subset B$ be bounded convex sets with $\mathcal{L}(A) > 0$. Then

$$\mathcal{L}(B \setminus A) \geq \omega_n \left( \frac{\sup_{x \in B} d(x, A)^n}{(2\pi)^{n-1}} \left( \arccos \left( 1 - \frac{2R_A^2 \mathcal{L}(A)^2}{\text{diam}(B)^2} \right) \right) \right)^{n-1}.$$

**Proof.** Let $D_A = 2R_A \mathcal{L}(A)$ be the diameter of the ball contained in $A$ from Lemma 6.3. Let $x \in B \setminus A$ be arbitrary. We shall first consider the case where $n = 2$.

![Figure 1](image)

First $P$, $Q$ are points chosen on the boundary of the disk contained in $A$ so that $R_1 = R_2$ where $R_1$ and $R_2$ are the lengths of the segments $xP$ and $xQ$ (such $P$, $Q$ exist by a continuity argument, see Figure 1). Set $r := d(x, A)$. Next let $S$ be the shaded circular sector, i.e. $S := B_r(x) \cap \Delta(P, Q, x)$ where $\Delta(P, Q, x)$ is the triangle with vertices $P$, $Q$, $x$. Let $\theta$ be the measure of the angle $\angle PxQ$ and set $R := R_1 = R_2$.

Note that $S \subset B \setminus A$. Then by the law of cosines

$$2R^2 - 2R^2 \cos \theta = R_1^2 + R_2^2 - 2R_1 R_2 \cos \theta = D_A^2$$

$$\implies \cos \theta = 1 - \frac{D_A^2}{2R^2} \leq 1 - \frac{D_A^2}{2 \text{diam}(B)^2}.$$
Thus we estimate the area of $S$ as
\[
\pi r^2 \frac{\theta}{2\pi} \geq \frac{r^2}{2} \arccos(1 - \frac{D_A^2}{2 \operatorname{diam}(B)^2}) = \frac{1}{2} d(x, A)^2 \arccos(1 - \frac{D_A^2}{2 \operatorname{diam}(B)^2}).
\]

Since $x \in B$ was arbitrary we obtain
\[
\mathcal{L}(B \setminus A) \geq \frac{1}{2} \sup_{x \in B} d(x, A)^2 \arccos(1 - \frac{D_A^2}{2 \operatorname{diam}(B)^2})
\]
as desired.

Now in higher dimensions the construction above yields a spherical sector instead of the circular sector, $S$. By slicing with planes through $x$ and the center of the ball and applying the argument used when $n = 2$ we see that this spherical sector has angle $\theta$ in all directions.

Hence we calculate that the volume of our spherical sector is estimated as
\[
\omega_n r^n \left(\frac{\theta}{2\pi}\right)^{n-1} \geq \omega_n r^n \left(\frac{\arccos(1 - \frac{D_A^2}{2 \operatorname{diam}(B)^2})}{(2\pi)^{n-1}}\right)^{n-1} = \omega_n d(x, A)^n \left(\frac{\arccos(1 - \frac{D_A^2}{2 \operatorname{diam}(B)^2})}{(2\pi)^{n-1}}\right)^{n-1}.
\]

Hence we have
\[
\mathcal{L}(B \setminus A) \geq \frac{\omega_n \sup_{x \in B} d(x, A)^n}{(2\pi)^{n-1}} \left(\frac{\arccos(1 - \frac{D_A^2}{2 \operatorname{diam}(B)^2})}{(2\pi)^{n-1}}\right)^{n-1}
\]
as desired. \hfill \Box

For any index $i \in \{1, \ldots, N\}$ and a set $E \subset \mathbb{R}^n$, we will use the notation
\[
[E]_i := (\exp_i)^{-1}(E).
\]

The following lemma is a simple use of the coarea formula to control the Lebesgue measure of the difference of Laguerre cells corresponding to different dual variables $\psi_1$ and $\psi_2$, in terms of the difference $\|\psi_1 - \psi_2\|_\infty$.

**Lemma 6.5.** Let $\psi_1, \psi_2 \in \mathbb{R}^n$. Then for some universal $C_\Delta > 0$, 
\[
\mathcal{L}(\text{Lag}_i(\psi_1) \setminus \text{Lag}_i(\psi_2)) \leq C_\Delta N \|\psi_1 - \psi_2\|_\infty.
\]

**Proof.** Suppose $x \in \text{Lag}_i(\psi_1) \setminus \text{Lag}_i(\psi_2)$, then there is a $k \neq i$ so that $c(x, y_k) + \psi^i_k \leq c(x, y_i) + \psi^i_i$ while $c(x, y_i) + \psi^i_i \leq c(x, y_k) + \psi^k_i$, combining these yields
\[
\psi^k_k - \psi^i_i \leq c(x, y_k) - c(x, y_k) \leq \psi^k_i - \psi^i_i.
\]

Hence writing $f(x) = c(x, y_k) - c(x, y_k)$,
\begin{equation}
(6.1) \quad \text{Lag}_i(\psi_1) \setminus \text{Lag}_i(\psi_2) \subset \bigcup_{k \neq i} (f^{-1}(\{\psi^k_k - \psi^i_i, \psi^k_i - \psi^i_i\})).
\end{equation}

We proceed to bound $\mathcal{L}(f^{-1}(\{\psi^k_k - \psi^i_i, \psi^k_i - \psi^i_i\}))$ using the coarea formula. We have
\[
\mathcal{L}(f^{-1}([a, b])) = \int_{f^{-1}([a, b])} d\mathcal{L}(x) = \int_a^b \int_{f^{-1}(\{t\})} \frac{1}{\|\nabla f(x)\|} d\mathcal{H}^{n-1}(x) dt \leq \frac{b - a}{\epsilon_{tw}} (\sup_{t \in [a, b]} \mathcal{H}^{n-1}(f^{-1}(\{t\})))
\]
where we recall $\epsilon_{tw}$ is from Definition 2.9.
Next we bound \( \sup_{t \in (a,b)} H^{n-1}(f^{-1}({t})) \). Let \( A_t := \{ x \mid f(x) \leq t \} \). We claim that \( f^{-1}({t}) \subset \partial A_t \). Clearly \( f^{-1}({t}) \subset A_t \subset \overline{A}_t \). Suppose for sake of contradiction that there is \( x \in f^{-1}({t}) \cap \text{int} A_t \). Then \( x \) has an open neighborhood \( U \) so that for every \( y \in U \), \( f(y) \leq t = f(x) \). In particular \( f(x) \) is a local maximum and so \( \nabla f(x) = 0 \), which contradicts (Twist).

By (QC), \([A]_i \) is convex and contained in \([X] \). Hence \( H^{n-1}([\partial A]_i) = H^{n-1}(\partial [A]_i) \leq H^{n-1}(\partial [X]_i) = H^{n-1}([\partial X]_i) \) (again see [Sch93, p. 211]). Hence up to some universal constant \( C_1 \) (depending of the Lipschitz constant of the map \( \exp_i \)) we have \( H^{n-1}(f^{-1}({t})) \leq H^{n-1}(\partial [A]_i) \leq C_1 H^{n-1}(\partial X) \). Putting the above together gives

\[
\mathcal{L}(f^{-1}(a, b)) \leq \frac{b - a}{\epsilon_{tw}} (\sup_{t \in [a, b]} H^{n-1}(f^{-1}({t}))) \leq \frac{C_1 H^{n-1}(\partial X)}{\epsilon_{tw}} (b - a).
\]

Since \( \psi^k_1 - \psi^i_1 - (\psi^k_2 - \psi^i_2) \leq 2\| \psi_1 - \psi_2 \|_{\infty} \), by combining the above with (6.1) we have

\[
\mathcal{L}(\text{Lag}_i(\psi_1) \setminus \text{Lag}_i(\psi_2)) \leq \sum_{k \neq i} \mathcal{L}(f^{-1}([\psi^k_2 - \psi^i_2, \psi^k_1 - \psi^i_1])) \leq \frac{2NC_1H^{n-1}(X)}{\epsilon_{tw}} \| \psi_1 - \psi_2 \|_{\infty}
\]

as desired.

Finally, we apply the bound Proposition 6.4 to the images of Laguerre cells under the coordinates induced by the maps \((\exp_i^c)^{-1}(\cdot)\), which are convex by (QC). Combining with Lemma 6.5 above allows us to control the Hausdorff distance between Laguerre cells by the difference of the dual variables defining the cells.

**Theorem 6.6.** Suppose that \( \| \psi_1 - \psi_2 \|_{\infty} < \frac{\mathcal{L}(\text{Lag}_i(\psi_1))}{2C_\Delta N} \) where \( C_\Delta \) is the constant from Lemma 6.5. Then for some universal constants \( C_1 > 0 \) and \( C_2 > 0 \),

\[
d_H(\text{Lag}_i(\psi_1), \text{Lag}_n(\psi_2))^n \leq \frac{C_1N\| \psi_1 - \psi_2 \|_{\infty}}{(\arccos(1 - C_2\mathcal{L}(\text{Lag}_i(\psi_1)))^2)^{n-1}}.
\]

**Proof.** By (QC), we see that \([\text{Lag}_i(\psi)]_i \) is a convex set for any \( i \).

Applying Proposition 6.4 with \( A = [\text{Lag}_i(\psi_1)]_i \cap [\text{Lag}_i(\psi_2)]_i \) and \( B = [\text{Lag}_i(\psi_1)]_i \) we obtain

\[
\mathcal{L}([\text{Lag}_i(\psi_1)]_i \setminus [\text{Lag}_i(\psi_2)]_i) \geq \frac{\omega_n(\sup_{x \in \text{Lag}_i(\psi_1)}(d((\exp_i^c)^{-1}(x), A)))^n}{(2\pi)^{n-1}} \left( \arccos(1 - \frac{2R_A^2\mathcal{L}(A)^2}{\text{diam}([\text{Lag}_i(\psi_1)]_i)^2}) \right)^{-1} \geq \frac{\omega_n(\sup_{x \in \text{Lag}_i(\psi_1)}(d((\exp_i^c)^{-1}(x), [\text{Lag}_i(\psi_2)]_i)))^n}{(2\pi)^{n-1}} \left( \arccos(1 - \frac{2R_A^2\mathcal{L}(A)^2}{\text{diam}([\text{Lag}_i(\psi_1)]_i)^2}) \right)^{-1}
\]

as \( [\text{Lag}_i(\psi_1)]_i \setminus ([\text{Lag}_i(\psi_1)]_i \cap [\text{Lag}_i(\psi_2)]_i) = [\text{Lag}_i(\psi_1)]_i \setminus [\text{Lag}_i(\psi_2)]_i \). Similarly, we also see

\[
\mathcal{L}([\text{Lag}_i(\psi_2)]_i \setminus [\text{Lag}_i(\psi_1)]_i) \geq \frac{\omega_n(\sup_{x \in \text{Lag}_i(\psi_2)}(d((\exp_i^c)^{-1}(x), [\text{Lag}_i(\psi_1)]_i)))^n}{(2\pi)^{n-1}} \left( \arccos(1 - \frac{2R_A^2\mathcal{L}(A)^2}{\text{diam}([\text{Lag}_i(\psi_2)]_i)^2}) \right)^{-1}
\]
and so
\[
\max(\mathcal{L}([\text{Lag}_i(\psi_2)]_i, \mathcal{L}([\text{Lag}_i(\psi_1)]_i, \mathcal{L}([\text{Lag}_i(\psi_1)]_i, \mathcal{L}([\text{Lag}_i(\psi_2)]_i))
\geq \frac{\omega_n d_H([\text{Lag}_i(\psi_1)]_i, [\text{Lag}_i(\psi_2)]_i)^n}{(2\pi)^{n-1}} \min_{j=1,2} \left( \arccos(1 - \frac{2R^2_\lambda(\mathcal{A})^2}{\text{diam}([\text{Lag}_i(\psi_j)]_i)^2}) \right)^{n-1}.
\]

Now we find that, using Lemma 6.5 and the assumption on \( \|\psi_1 - \psi_2\|_\infty \), for both \( j = 1, 2 \),
\[
\frac{2R^2_\lambda(\mathcal{A})^2}{\text{diam}([\text{Lag}_i(\psi_j)]_i)^2} = \frac{2^{2n-1}\mathcal{L}(\mathcal{A})^2}{\omega_n^2(n+2)^{2n} \text{diam}(\mathcal{X})^{2n}} \geq \frac{2^{2n-1}\mathcal{L}([\text{Lag}_i(\psi_1)]_i) - \mathcal{L}([\text{Lag}_i(\psi_1)]_i, [\text{Lag}_i(\psi_2)]_i)^2}{\omega_n^2(n+2)^{2n} \text{diam}(\mathcal{X})^{2n}} \geq \frac{2^{2n-1}\mathcal{L}([\text{Lag}_i(\psi_1)]_i)^2}{4\omega_n^2(n+2)^{2n} \text{diam}(\mathcal{X})^{2n}}.
\]
Combining the above estimate with Lemma 6.5 and (6.2),
\[
CN\|\psi_1 - \psi_2\|_\infty \geq \max(\mathcal{L}([\text{Lag}_i(\psi_2)]_i, [\text{Lag}_i(\psi_1)]_i, \mathcal{L}([\text{Lag}_i(\psi_1)]_i, [\text{Lag}_i(\psi_2)]_i))
\geq \frac{\omega_n d_H([\text{Lag}_i(\psi_1)]_i, [\text{Lag}_i(\psi_2)]_i)^n}{(2\pi)^{n-1}} \left( \arccos(1 - C_2\mathcal{L}([\text{Lag}_i(\psi_1)]_i)^2) \right)^{n-1}.
\]
Since the map \((\exp^c)^{-1}(\cdot)\) is bi-Lipschitz with universal Lipschitz constants, there is some universal \( C > 0 \) such that
\[
Cd_H([\text{Lag}_i(\psi_1)]. [\text{Lag}_i(\psi_2))]_i^n \leq d_H([\text{Lag}_i(\psi_1)]_i, [\text{Lag}_i(\psi_2)]_i)^n,
\]
finishing the proof. \( \square \)

7. Injectivity of \( G \)

In Theorem 6.6 above, we obtain quantitative control of the Hausdorff distance between Laguerre cells associated to different dual variables, but this control is in terms of the dual variables themselves. For Theorem 2.14 we want to obtain this estimate in terms of parameters that we have control over, namely \( h, \epsilon \), and the difference \( \|w_{h,\epsilon}(\psi) - \tilde{u}\| \). We begin working toward this goal, first by showing invertibility of the map \( G \) on the set of dual variables we are concerned with.

Definition 7.1. If \( \varphi : X \to \mathbb{R} \cup \{ +\infty \} \) (not identically \( +\infty \)), its pseudo-\( c \)-transform is a vector \( \varphi^{c!} \in \mathbb{R}^N \), defined by
\[
(\varphi^{c!})^i := \sup_{x \in \exp^c(\mu)} (-c(x, y_i) - \varphi(x)).
\]
Also let \( \Psi_c = \{ \psi \in \mathbb{R}^n : \psi = \psi^{c!} \} \).

Lemma 7.2. Suppose \( \psi_1, \psi_2 \in \mathbb{R}^N \) are such that \( \lambda := G(\psi_1) = G(\psi_2) \), and suppose that \( \lambda^i > 0 \) for some index \( i \). If \( x \in \text{Lag}_i(\psi_1) \) and \( \rho(x) > 0 \) then \( x \in \text{Lag}_i(\psi_2) \).

Proof. Suppose by contradiction, for such an \( x \) we have \( x \notin \text{Lag}_i(\psi_2) \). As the zero set of a continuous function \( \text{Lag}_i(\psi_2) \) is closed, hence there is a neighborhood of \( x \), say \( U \), so that \( U \cap \text{Lag}_i(\psi_2) = \emptyset \). Next since \( \rho(x) > 0 \), by continuity of \( \rho \) there is an open neighborhood of \( x \), say \( V \subset U \) so that \( \rho > 0 \) on \( V \).
Now we claim that $V \cap \text{int} (\text{Lag}_i (\psi_1)) \neq \emptyset$. By (QC), we see that $\text{Lag}_i (\psi_1)$ is diffeomorphic to a convex set, and furthermore $\text{Lag}_i (\psi_1)$ is compact. Hence $\text{Lag}_i (\psi_1)$ is homeomorphic to a closed unit ball of some dimension. Since $\mu(\text{Lag}_i (\psi_1)) = \lambda^i > 0$, we see that it must be homeomorphic to $B_n$, the closed unit ball in $\mathbb{R}^n$. This implies any open neighborhood of any point in $\text{Lag}_i (\psi_1)$ has nontrivial intersection with $\text{int} (\text{Lag}_i (\psi_1))$, and we obtain that $V \cap \text{int} (\text{Lag}_i (\psi_1)) \neq \emptyset$.

Since $\rho > 0$ on $V \cap \text{int} (\text{Lag}_i (\psi_1))$, which is open and non-empty, we have $\mu(V \cap \text{int} (\text{Lag}_i (\psi_1))) > 0$ while $V \cap \text{int} (\text{Lag}_i (\psi_1)) \subset \text{Lag}_i (\psi_1) \setminus \text{Lag}_i (\psi_2)$. However this contradicts [Vil09, Remark 10.29], as we must have $T_{\psi_1} = T_{\psi_2} \mu$-a.e. (where $T_{\psi_i}$ are defined as in Remark 2.4).

We now show the consequences of assuming a $(q,1)$-PW inequality on the source density $\rho$, the improvement afforded by taking $q > 1$ over the case of a $(1,1)$-PW inequality will be essential in the following section. In order to remain consistent with [KMT19], we recall some notation.

**Definition 7.3.** We will write $\text{int}(X)$ to denote the interior of the set $X$. Given an absolutely continuous measure $\mu = \rho dx$ and a set $A \subset X$ with Lipschitz boundary, we will write

$$|\partial A|_\rho := \int_{\partial A \cap \text{int}(X)} \rho d\mathcal{H}^{n-1}(x), \quad |A|_\rho := \mu(A).$$

**Lemma 7.4.** Suppose that $\mu = \rho dx$ satisfies a $(q,1)$-PW inequality where $q \geq 1$. Then

$$\inf_{A \subset \text{int}(X)} \min(|A|_\rho, |X \setminus A|_\rho)^{1/q} \geq \frac{1}{2^{\frac{1}{q}} C_{pw}},$$

where the infimum is over $A \subset \text{int}(X)$ whose boundary is Lipschitz with finite $\mathcal{H}^{n-1}$-measure, and $\min(|A|_\rho, |X \setminus A|_\rho) > 0$.

**Proof.** Let $A \subset \text{int}(X)$ be a Lipschitz domain as in the statement above, recall that we must have $q \leq \frac{n}{n-1} \leq 2$. Since we have a $(q,1)$-PW inequality instead of a $(1,1)$ inequality, by following the same method as [KMT19] Lemma 5.3 we obtain the inequality

$$C_{pw} |\partial A|_\rho \geq \|1_A - \int_X 1_A d\mu\|_{L^q(\mu)}\]

$$

$$= \left( \int_A \left|1 - |A|_\rho\right|^q d\mu + \int_{X \setminus A} \left| |A|_\rho\right|^q d\mu \right)^{\frac{1}{q}}$$

$$= \left( |A|_\rho |X \setminus A|_\rho^q + |A|_\rho^q |X \setminus A|_\rho \right)^{\frac{1}{q}}$$

$$= |A|_\rho^\frac{1}{q} |X \setminus A|_\rho^\frac{1}{q} (|X \setminus A|_\rho^{q-1} + |A|_\rho^{q-1})^{\frac{1}{q}}$$

$$\geq |A|_\rho^\frac{1}{q} |X \setminus A|_\rho^\frac{1}{q}$$

$$\geq 2^{-\frac{1}{q}} \min(|A|_\rho, |X \setminus A|_\rho) \frac{1}{q},$$

hence taking an infimum gives the claim.

**Lemma 7.5.** Suppose $\mu = \rho dx$ satisfies a $(1,1)$-PW inequality and $\psi_1, \psi_2 \in \Psi_c$. Then $\psi_1 - \psi_2 \in \text{span}(1)$ if and only if $G(\psi_1) = G(\psi_2)$.
Proof. It is easy to see from Definition 2.2 of \( c \)-transforms that \( \psi_1 - \psi_2 \in \text{span}(1) \) implies \( G(\psi_1) = G(\psi_2) \), so we only show the opposite implication.

Suppose \( \lambda := G(\psi_1) = G(\psi_2) \) and let \( \varphi_1 := \psi_1^\ast \), \( \varphi_2 := \psi_2^\ast \). Also, write \( T := T_{\psi_1} = T_{\psi_2} \) (up to \( \mu \)-a.e.), which is the Monge solution to problem (1.1) pushing \( \mu \) forward to the discrete measure \( \nu_\lambda \). Finally, without loss of generality we may assume that \( \lambda^1 > 0 \) and (by subtracting a multiple of 1) \( \psi_1^1 = \psi_2^1 \), and define \( S := \{ i \in \{1, \ldots, N \} \mid \psi_1^i = \psi_2^i \text{ and } \lambda^i > 0 \} \).

If we define the set

\[
A := \bigcup_{i \in S} \text{Lag}_i(\psi_1),
\]

then \( \mu(A) \geq \lambda^1 > 0 \), and since it is a union of Laguerre cells, by (QC) we see \( A \) has Lipschitz boundary. If \( \mu(A) < 1 \), by Lemma 7.4 we can conclude that \( \partial A \rho > 0 \). Then by [KMT19, (5.3)], we see there exist \( i \in S \), \( j \not\in S \) and a point \( x \in \text{Lag}_i(\psi_1) \cap \partial A \cap \text{int}(X) \) where \( \rho(x) > 0 \). Then \( x \in \text{Lag}_{i,j}(\psi_1) \subset \text{Lag}_i(\psi_1) \) so by Lemma 7.2 above we must also have \( x \in \text{Lag}_i(\psi_2) \).

Then we can calculate

\[
(7.1) \quad \varphi_1(x) + \psi_1^i = -c(x, y_i) = \varphi_2(x) + \psi_2^i \implies \varphi_1(x) = \varphi_2(x).
\]

Arguing as in the proof of Lemma 7.2 above, since \( x \in \text{Lag}_i(\psi_1) \cap \text{int}(X) \) and \( \rho(x) > 0 \), we see that \( \lambda^j = \mu(\text{Lag}_j(\psi_1)) > 0 \). Since \( x \in \text{Lag}_{i,j}(\psi_1) \subset \text{Lag}_j(\psi_1) \), we can apply Lemma 7.2 again to see \( x \in \text{Lag}_j(\psi_2) \).

Hence

\[
\varphi_1(x) + \psi_1^j = -c(x, y_j) = \varphi_2(x) + \psi_2^j \implies \psi_1^j = \psi_2^j,
\]

but this would imply \( j \in S \), a contradiction.

Now since \( \mu(A) = 1 \), the set \( A \cap \rho^{-1}((0, \infty)) \) must be dense in \( \rho^{-1}((0, \infty)) \). Then we can make the same calculation leading to (7.1) above to find that \( \varphi_1 = \varphi_2 \) on this dense set.

Since \( \varphi_1 \) and \( \varphi_2 \) are \( c \)-transforms of vectors they are continuous on \( \mathbb{R}^n \), thus they must actually be equal everywhere on \( \rho^{-1}(0, \infty) \), hence on its closure \( \text{spt } \mu \).

With the above, we then see that

\[
\psi_1 = \psi_1^c = \psi_2^c = \psi_2
\]

as desired. \[ \Box \]

We are finally ready to prove the desired invertibility result.

**Proposition 7.6.** Suppose \( \mu = p \text{d}x \) satisfies a \((1, 1)\)-PW inequality. Then \( G : \overline{\mathcal{K}^0}/1 \to \Lambda \) is a homeomorphism.

**Proof.** First let \( f(\psi) = \psi^c - \psi \). Note that directly from Definition 2.2, for an arbitrary \( x \in X \) we have \( |\psi_1^c(x) - \psi_2^c(x)| \leq \|\psi_1 - \psi_2\|_\infty \). A similar calculation then yields

\[
\|\psi_1^c - \psi_2^c\|_\infty \leq \sup_{x \in \text{spt } \mu} |\psi_1^c(x) - \psi_2^c(x)| \leq \|\psi_1 - \psi_2\|_\infty,
\]

hence by the triangle inequality, \( f \) is continuous, in particular \( \Psi_c = f^{-1}(\{0\}) \) is closed.

Now for any \( \psi \in \mathcal{K}^0 \) it is clear there for each index \( i \) must exist a point \( x_i \in \text{spt } \mu \cap \text{Lag}_i(\psi) \), while just as in the proof of [BK19, Proposition 4.1] we see that \( \psi = \psi^c e^c \). Then for any \( x \in X \), we would have

\[
-c(x_i, y_i) - \psi^c(x_i) = \psi_i = (\psi^c e^c)^i \geq -c(x, y_i) - \psi^c(x),
\]
hence for such a \( \psi \) we have
\[
\psi = \psi^{c^*c} = \psi^{c^*c^*},
\]
in particular \( K^0 \subset \Psi_c \), thus \( \overline{K^0} \subset \Psi_c \). Then by Lemma 7.5, \( G(\psi_1) = G(\psi_2) \) if and only if \( \psi_1 - \psi_2 \in \text{span}(1) \) for \( \psi_1, \psi_2 \in \overline{K^0} \), and we obtain that the induced map (which we also call \( G \)) \( G : \overline{K^0}/1 \rightarrow \Lambda \) is well-defined and injective.

Next note that \( \overline{K^0}/1 \) is closed and bounded and hence compact. Hence, \( \Lambda = G(K^0) \subset G(\overline{K^0}/1) = G(\overline{K^0}/1) \). Finally, since \( G \) is a continuous bijection with compact domain it follows by [GG99, Theorem 2.6.7] that \( G \) is a homeomorphism.

8. Quantitative Hausdorff Convergence

8.1. Alternative spectral estimates on \( DG \). We now obtain an estimate away from zero on the first nonzero eigenvalue of the mapping \( DG \) over the set \( K^\epsilon \) of a different nature than that of [KMT19, Theorem 5.1]. The estimate there is of order \( \epsilon^3 \) under the assumption of a \((1,1)\)-PW inequality, however we will show a estimate which is of order \( N^{-4}\epsilon^2 \) under the assumption of a \((q,1)\)-PW inequality. As can be seen, in the case of \( q = 1 \) we have traded two factors of \( \epsilon \) for factors of \( N^{-2} \), we desire to make this modification in order to be able to obtain a quantitative rate of convergence in the Hausdorff metric of the Laguerre cells (i.e., for Theorem 2.14). This is namely because we will first obtain the convergence rate in terms of the dual variables \( \psi \), thus we will have a need to estimate the Lipschitz norm of the inverse of \( G \), but as the parameter \( \epsilon \rightarrow 0 \) in order to obtain a finite bound, we will be forced to use this new spectral estimate, along with taking \( q > 1 \) in the Poincaré-Wirtinger inequality.

Recall \( DG \) is negative semidefinite on \( K^\epsilon \) by [KMT19, Theorem 5.1]. We work toward the following estimate.

**Theorem 8.1.** Fix \( \epsilon > 0 \) and assume \( \mu = \rho dx \) satisfies a \((q,1)\)-PW inequality where \( q \geq 1 \), then the second eigenvalue of \( DG \) on \( K^\epsilon \) is bounded above by
\[
-\frac{2^{1-\frac{1}{q}}}{C\sqrt{N}^4C_{pw}} \epsilon^{1/q} < 0.
\]

At this point, fix \( \epsilon > 0 \) and some \( \psi \in K^\epsilon \) and let \( W \) be the (undirected) weighted graph constructed in [KMT19, Section 5.3]: the vertices of \( W \) consist of the collection \( Y \), and for any \( y_i \) and \( y_j \), \( i \neq j \) there exists an edge which is given weight \( w_{ij} \) defined by
\[
w_{ij} := D_i G^i(\psi) = D_j G^j(\psi) = \int_{\text{Lag}_{i,j}(\psi)} \frac{\rho(x)}{\|
abla_x c(x, y_i) - \nabla_x c(x, y_j)\|} d\mathcal{H}^{n-1}(x),
\]
where we have used the notation
\[
\text{Lag}_{i,j}(\psi) := \text{Lag}_i(\psi) \cap \text{Lag}_j(\psi)
\]
for \( i, j \in \{1, \ldots, N\} \).

**Proposition 8.2.** If \( \mu = \rho dx \) satisfies a \((q,1)\)-PW inequality where \( q \geq 1 \) then \( W \) is connected by edges of weight at least \( \frac{2^{1-\frac{1}{q}}}{C\sqrt{N}^2C_{pw}} \epsilon^{1/q} \), that is: the weighted graph consisting of all vertices of \( W \) and only those edges of weight greater than or equal to \( \frac{2^{1-\frac{1}{q}}}{C\sqrt{N}^2C_{pw}} \epsilon^{1/q} \) is connected.
Proof. Suppose by contradiction that the proposition is false. This implies that removing all edges with weight strictly less than \( \frac{2^{1 - \frac{1}{q}}}{C_{1} N^{2} C_{pw}} \epsilon^{1/q} \) yields a disconnected graph. In other words, we can write \( W = W_{1} \cup W_{2} \) where \( W_{1}, W_{2} \neq \emptyset \) and are disjoint, such that every edge connecting a vertex in \( W_{1} \) to a vertex in \( W_{2} \) has weight strictly less than \( \frac{2^{2 - \frac{1}{q}}}{N^{2} C_{pw}} \epsilon^{1/q} \). Letting \( A := \bigcup_{y_{i} \in W_{1}} \operatorname{Lag}_{i}(\psi) \) we see that

\[
|\partial A|_{\rho} \leq 2 C_{V} \sum_{\{(i,j) : y_{i} \in W_{1}, y_{j} \in W_{2}\}} w_{ij} < \frac{2^{2 - \frac{1}{q}}}{N^{2} C_{pw}} \epsilon^{1/q} |W_{1}| |W_{2}| \leq \frac{2^{2 - \frac{1}{q}}}{N^{2} C_{pw}} \epsilon^{1/q} \frac{N^{2}}{4} = \frac{1}{2^{\frac{3}{2}} C_{pw}} \epsilon^{1/q}.
\]

On the other hand since both \( W_{1} \) and \( W_{2} \) are nonempty we have \( |A|_{\rho}, |X \setminus A|_{\rho} \geq \epsilon \). Hence

\[
\frac{|\partial A|_{\rho}}{\min(|A|_{\rho}, |X \setminus A|_{\rho})^{1/q}} \leq \frac{\epsilon^{1/q}}{2^{\frac{3}{2}} C_{pw} \epsilon^{1/q}} = \frac{1}{2^{\frac{3}{2}} C_{pw}}
\]

which contradicts Lemma 7.4. \( \square \)

Recall that given a weighted graph \( W \), the \textit{weighted graph Laplacian} is the \( N \times N \) matrix with entries

\[
L_{ij} := \begin{cases} -w_{ij}, & i \neq j, \\ \sum_{k \in \{1, \ldots, N\} \setminus \{i\}} w_{ik}, & i = j. \end{cases}
\]

If \( W \) is the graph we have defined above and \( L \) its weighted graph Laplacian, then by [KMT19, Theorem 1.3] we can see that \( L = -DG(\psi) \).

**Proof of Theorem 8.1** Let \( \hat{W} \) be the graph formed by dividing all of the edge weights in \( W \) by \( \frac{\epsilon^{1/q}}{C_{V} N^{2} C_{pw}} \). If \( L \) and \( \hat{L} \) are the weighted graph Laplacians of the graphs \( W \) and \( \hat{W} \) respectively, clearly \( \hat{L} = \frac{C_{V} N^{2} C_{pw}}{2^{1 - \frac{1}{q}} \epsilon^{1/q}} L \).

Now construct the graph \( \hat{W} \) from \( \hat{W} \) by the following procedure: if an edge connecting \( y_{i} \) and \( y_{j} \) has weight \( w_{ij} < 1 \), we remove the edge, and if \( w_{ij} \geq 1 \), we set the weight of the edge equal to 1. By Proposition 8.2 we see that \( \hat{W} \) is a connected graph whose edge weights are all 1 over \( N \) vertices, and in particular it has diameter

\[
\operatorname{diam}(\hat{W}) = \sup \sum_{i,j} w_{i,j} \leq N,
\]

here the supremum is taken over all pairs of vertices in \( \hat{W} \) and collections of edges forming a path between those two vertices, and the sum runs over all edges in such a collection. Let us write \( \hat{L} \) for the graph Laplacian of \( \hat{W} \) and use \( \lambda_{2} \) to denote the second eigenvalue of a positive semidefinite matrix. Then, using [Fie75, Lemma 3.2] to obtain the first inequality below and then [Moh91, Theorem 4.2] to obtain the second to final inequality, we find that

\[
\lambda_{2}(-DG(\psi)) = \lambda_{2}(L) = \frac{2^{1 - \frac{1}{q}} \epsilon^{1/q}}{C_{V} N^{2} C_{pw}} \lambda_{2}(\hat{L}) \geq \frac{2^{1 - \frac{1}{q}} \epsilon^{1/q}}{C_{V} N^{2} C_{pw}} \lambda_{2}(\hat{L}) \geq \frac{2^{1 - \frac{1}{q}} \epsilon^{1/q}}{C_{V} N^{2} C_{pw}} \cdot \frac{4}{N \operatorname{diam}(\hat{W})} \geq \frac{2^{3 - \frac{1}{q}} \epsilon^{1/q}}{C_{V} N^{4} C_{pw}},
\]

finishing the proof. \( \square \)
8.2. Quantitative invertibility of $G$.

**Proposition 8.3.** Suppose that $\mu = \rho dx$ satisfies a $(q,1)$-PW inequality with $q > 1$. Then for any $\psi_1 \in K^*$, $\psi_2 \in \mathbb{R}^N$ such that $\langle \psi_1 - \psi_2, 1 \rangle = 0$,

$$\|\psi_1 - \psi_2\| \leq \frac{N^2 \nabla C_{pw} q \|G(\psi_1) - G(\psi_2)\|}{4e^{1/q}(q-1)}.$$  

**Proof.** By Proposition 7.6, the restriction of $\psi$ to $K^0 \cap \{ \psi \mid \langle \psi - \psi_1, 1 \rangle = 0 \}$ is invertible, let $H$ denote this inverse; by Theorem 8.1 we see that

$$\|DH(\lambda)\| \leq \frac{C_{\nabla} N^4 C_{pw}}{4(\min \lambda)^{1/q}}.$$  

We calculate,

$$\|\psi_1 - \psi_2\| = \sqrt{\sum_i \left( \int_0^1 \langle \nabla H^i(tG(\psi_1) + (1-t)G(\psi_2)), G(\psi_1) - G(\psi_2) \rangle dt \right)^2}$$

$$\leq \sqrt{\sum_i \left( \|G(\psi_1) - G(\psi_2)\| \int_0^1 \|\nabla H^i(tG(\psi_1) + (1-t)G(\psi_2))\| dt \right)^2}$$

$$\leq \sqrt{N} \|G(\psi_1) - G(\psi_2)\| \int_0^1 \|DH(tG(\psi_1) + (1-t)G(\psi_2))\| dt$$

$$\leq \sqrt{N} \|G(\psi_1) - G(\psi_2)\| \int_0^1 \frac{C_{\nabla} N^4 C_{pw}}{4(\min_i (tG^i(\psi_1) + (1-t)G^i(\psi_2)))^{1/q}} dt$$

$$\leq \sqrt{N} \|G(\psi_1) - G(\psi_2)\| \int_0^1 \frac{C_{\nabla} N^4 C_{pw}}{4(\epsilon)^{1/q}} dt$$

$$= \frac{N^4 \sqrt{N} C_{\nabla} C_{pw} q \|G(\psi_1) - G(\psi_2)\|}{4(q-1)e^{1/q}},$$

here it is crucial that $q > 1$ in order to obtain the final line. \hfill \Box

With this quantitative invertibility of the map $G$ in hand, we are ready to prove Theorem 2.14.

**Proof of Theorem 2.14.** We begin with statement (1). By [BK19] Proposition 3.5, Proposition 4.4, Corollary 4.5, there exists some $\psi \in \mathbb{R}^N$ such that $T^i = T_\psi$ $\mu$-a.e. and $\lambda = G(\psi)$. Under the hypotheses of (1), by Theorem 2.11 (2.5), we see that $\|G(\psi_k) - \lambda\| \to 0$ as $k \to \infty$. Then by Proposition 7.6 we must have $\psi_k \to \psi$. Combining this with Theorem 5.6 gives the claim in (1).

Now we turn to claim (2), assume $q > 1$. Then combining (2.5) from Theorem 2.11 and Proposition 8.3 gives

$$\|\psi_{h,\epsilon} - \psi\| \leq \frac{N^2 \nabla C_{pw} q \|G(\psi_{h,\epsilon}) - \lambda\|}{4e^{1/q}(q-1)}$$

$$\leq \frac{N^2 \nabla C_{pw} q \|G(\psi_{h,\epsilon}) - \lambda\|}{4e^{1/q}(q-1)} \leq \frac{N^2 \nabla C_{pw} q (N \epsilon + \|w_{h,\epsilon}(\psi_{h,\epsilon}) - w\|_1 + 2N \sqrt{2C_L h})}{2e^{1/q}(q-1)}.$$
Finally, since the $\ell^\infty$ norm is bounded by the Euclidean norm, by (2.7) we can apply Theorem 6.6 finishing the proof.

A. Strong Convexity of $C$

Lemma A.1. $C$ is strongly convex. In particular

$$tC(x) + (1-t)C(y) \geq C(tx + (1-t)y) + \frac{1}{8C_LN}t(1-t)\|y-x\|^2,$$

where $[G]_{C^{0,1}(\mathbb{R}^N)} \leq C_LN$, and $C_L > 0$ is universal.

Proof. Let

$$B(\psi) = \int \psi^* d\mu.$$

We see that $C(\lambda) = B^*(-\lambda)$; also by [AG17] $B$ is $C^{1,1}$, $\nabla B = -G$, and $B$ is convex (see [KMT9] Theorem 1.1). By [AG17] Theorem 5.1 we see the Lipschitz constant of $G$ is bounded from above by $C_LN$ where $C_L > 0$ is some universal constant. Now

$$0 \leq tB(x) + (1-t)B(y) - B(tx + (1-t)y)$$

$$= tB(x) + (1-t)\left( B(x) + \langle y-x, \nabla B(x) \rangle + \int_0^1 \langle \nabla B((1-s)x + sy) - \nabla B(x), y-x \rangle ds \right)$$

$$- \left( B(x) + \langle tx + (1-t)y - x, \nabla B(x) \rangle \right)$$

$$+ (1-t) \int_0^1 \langle \nabla B((1-s(1-t))x + s(1-t)y) - \nabla B(x), y-x \rangle ds \right)$$

$$\leq (1-t) \int_0^1 \| \nabla B((1-s)x + sy) - \nabla B(x) \| \| y-x \| ds$$

$$+ (1-t) \int_0^1 \| \nabla B((1-s(1-t))x + s(1-t)y) - \nabla B(x) \| \| y-x \| ds$$

$$\leq C_LN(1-t) \left( \int_0^1 \| y-x \|^2 ds + (1-t) \int_0^1 \| y-x \|^2 ds \right)$$

$$\leq (1-t)C_LN\| y-x \|^2.$$

By repeating a similar argument we get $tB(x) + (1-t)B(y) - B(tx + (1-t)y) \leq tC_LN\| y-x \|^2$. Hence $tB(x) + (1-t)B(y) - B(tx + (1-t)y) \leq 2C_LNt(1-t)\| y-x \|^2$.

In the terminology of [AP95] Definition 1, we have shown that $B$ is $\sigma$-smooth where $\sigma(x) := 2C_LNx^2$. Since it is well-known that $\sigma^*(z) = \frac{1}{8C_LN}z^2$, by [AP95] Proposition 2.6 we see that $C$ is $\sigma^*$-convex, i.e.

$$tC(x) + (1-t)C(y) \geq C(tx + (1-t)y) + \frac{1}{8C_LN}t(1-t)\| y-x \|^2,$$

finishing the proof. □

Corollary A.2. Let $K$ be a convex subset of the domain of $C$. Let $\lambda_{\text{min}}$ be the minimizer of $C$ on $K$ and $\lambda \in K$ be arbitrary. Then

$$C(\lambda) - C(\lambda_{\text{min}}) \geq \frac{1}{32C_LN}\| \lambda - \lambda_{\text{min}} \|^2.$$
Proof. By choice of \( \lambda_{\text{min}} \), we have \( \frac{1}{2} C(\lambda) \geq \frac{1}{2} C(\lambda_{\text{min}}) \) and \( -C(\lambda_{\text{min}}) \geq -C(\frac{1}{2}(\lambda + \lambda_{\text{min}})) \). Hence by the above lemma we have

\[
C(\lambda) - C(\lambda_{\text{min}}) \geq \frac{1}{2}(C(\lambda) + C(\lambda_{\text{min}})) - C(\frac{1}{2}(\lambda + \lambda_{\text{min}})) \geq \frac{1}{32C_LN} \| \lambda - \lambda_{\text{min}} \|^2.
\]

\( \square \)

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