GROUPOID CROSSED PRODUCTS OF CONTINUOUS-TRACE C*-ALGEBRAS

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ABSTRACT. We show that if \((A, G, \alpha)\) is a groupoid dynamical system with \(A\) continuous trace, then the crossed product \(A \rtimes_{\alpha} G\) is Morita equivalent to the C*-algebra \(C^*(G, E)\) of a twist \(E\) over a groupoid \(G\) equivalent to \(G\). This is a groupoid analogue of the well known result for the crossed product of a group acting on an elementary C*-algebra.

KEYWORDS: Groupoid C*-algebras, continuous-trace C*-algebras, crossed products

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1. Introduction

One of the basic results in the theory of crossed products of C*-algebras by groups is the result, due to Green [2, Theorem 18], computing the crossed product \(A \rtimes_{\alpha} G\) when \(A\) is elementary. The primary object of this note is to prove an analogue, up to Morita equivalence, of Green’s result for Groupoid crossed products.

For motivation, we recall some of the details of Green’s result. If \((A, G, \alpha)\) is a dynamical system with \(A = K(H)\) for a complex Hilbert space \(H\), then there is a short exact sequence of locally compact groups

\[
1 \longrightarrow T \overset{i}{\longrightarrow} E \overset{j}{\longrightarrow} G \longrightarrow 1
\]

that arises as follows. The unitary group \(U(H)\) acts on \(A = K(H)\) by automorphisms, via \(U \mapsto \text{Ad} U\), and by Wigner’s theorem every automorphism of \(A\) arises in this way. The kernel of \(U(H) \to \text{Aut} A\) is the center \(T : I_H \cong T\) of \(U(H)\). The algebraic isomorphism \(U(H)/T \cong \text{Aut} A\) is a homeomorphism if \(U(H)\) is given the strong operator topology. The action \(\alpha: G \to \text{Aut} A\) gives rise to the sequence \((1.1)\) via pullback,

\[
\begin{array}{c}
1 \longrightarrow T \overset{i}{\longrightarrow} E \overset{j}{\longrightarrow} G \longrightarrow 1 \\
\downarrow \quad \downarrow \quad \downarrow \alpha \\
1 \longrightarrow T \longrightarrow U(H) \overset{\text{Ad}}{\longrightarrow} \text{Aut} A \longrightarrow 1.
\end{array}
\]
In other words, $E$ is the fibered product

$$E := \{ (s, U) \in G \times U(H) : \alpha_s = \Ad U \},$$

and $E$ is a locally compact group if given the relative topology in $G \times U(H)$. (This is not trivial, because $U(H)$ is not locally compact. The construction is described in detail in [15 §7.3 & §D.3].)

Since $i(T)$ is central in $E$, every irreducible unitary representation $\pi$ of $E$ has a single $T$-type; i.e., for every $\pi$ there is an integer $k \in \mathbb{Z}$ such that $\pi(\alpha(z)) = z^k I_{\mathcal{H}}$ for $z \in T$. The unitary dual $\hat{E}$ decomposes as a disjoint union of closed subsets according to $T$-type. The twisted group $C^*$-algebra $C^*(G; E)$ is the quotient of $C^*(E)$ corresponding to $T$-type $\pi(i(z)) = z I_{\mathcal{H}_z}$.

Green’s result says that the crossed product $A \rtimes_{\alpha} G$ is isomorphic to the tensor product $C^*(G; E) \otimes A$. Since $A$ is elementary, $A \rtimes_{\alpha} G$ is Morita equivalent to $C^*(G; E)$.

We want to exhibit the analogous result for groupoid crossed products $\mathcal{A} \rtimes_{\alpha} G$ where $G$ is a second countable locally compact Hausdorff groupoid and $\mathcal{A}$ is an upper semicontinuous $C^*$-bundle of elementary $C^*$-algebras over the unit space $G(0)$. Our techniques require that the section algebra $A = \Gamma_0(G(0); \mathcal{A})$ be a separable continuous-trace $C^*$-algebra.

The groupoid analogue of central extensions of $G$ are called either $T$-groupoids over $G$ or twists over $G$. A twist over $G$ is a principal $T$-bundle $j : E \to G$ where $E$ has a groupoid structure making $j$ a groupoid homomorphism. The associated $C^*$-algebras $C^*(G; E)$ have been extensively studied [4, 7–9]. In the case where $A = \Gamma_0(G(0); \mathcal{A})$ has continuous trace with trivial Dixmier-Douady class $\delta(A)$, our main result (Theorem 4.1) says that there is a twist $E$ over $G$, analogous to (4.1), such that $\mathcal{A} \rtimes_{\alpha} G$ is Morita equivalent to $C^*(G; E)$. If $\delta(A)$ is nontrivial, we must replace $G$ by an equivalent groupoid $\tilde{G}$. Then we show that $\mathcal{A} \rtimes_{\tilde{\alpha}} \tilde{G}$ is Morita equivalent to $C^*(\tilde{G}; \tilde{E})$ for an appropriate twist $\tilde{E}$.

The converse also holds, and is much easier. In [5] we show that if $E$ is a twist over $G$, there is a Hilbert $C_0(G(0))$-module $H$ and an action $\alpha$ of $G$ on the generalized compacts $K(H)$ such that $C^*(G; E)$ is Morita equivalent to $K(H) \rtimes_{\tilde{\alpha}} \tilde{G}$.

Our result is closely related to the work on the Brauer group in [5]. The Brauer group $\text{Br}(G)$ consists of equivalence classes (appropriately defined) of $C^*$-dynamical systems $(\mathcal{A}, G, \alpha)$ with a continuous trace $C^*$-algebra $A = \Gamma_0(G(0); \mathcal{A})$. The main result of [5] is that $\text{Br}(G)$ is isomorphic to a group $\text{Ext}(G, \mathbf{T})$ whose elements are pairs $(\mathcal{G}, E)$ consisting of a groupoid $\mathcal{G}$ that is equivalent to $G$ and a twist $E$ of $\mathcal{G}$, where the pairs $(\mathcal{G}, E)$ are subject to a subtle equivalence relation.

The focus in [5] is on establishing a group isomorphism between $\text{Br}(G)$ and $\text{Ext}(G, \mathbf{T})$, and the difficulty resides in the precise equivalence relations that define the two groups. However, the relation between the (maximal or reduced) $C^*$-algebras $\mathcal{A} \rtimes_{\tilde{\alpha}} \tilde{G}$ and $C^*(\tilde{G}; \tilde{E})$ is not considered there.
Our main tool here is the Equivalence Theorem for Fell bundles from [10]. This has the advantage that explicit pre-imprimitivity bimodules for our Morita equivalences can be read off from the formulas in [10]. More significantly, it follows from [14, Theorem 14] that our results pass to the reduced algebras; that is, we also have a Morita equivalence of \( \mathcal{A} \rtimes_\alpha G \) and \( \mathcal{C}_r^*(G, E) \). However, because we use the Equivalence Theorem, all our results require separability, which is not the case for Green’s result.

In a different context, an equivalence class of groupoids represents a stack, and a twist over a groupoid represents an \( S^1 \)-gerbe. The twisted K-theory of a stack twisted by a gerbe is, by definition, the K-theory of the reduced C*-algebra \( \mathcal{C}_r^*(G, E) \), where \( G \) is a groupoid representing the stack, and \( E \) is a twist of \( G \) corresponding to the gerbe. In [15] Tu, Xu and Laurent-Gengoux consider the twisted K-theory of differentiable stacks, i.e., the case where \( G \) is a Lie groupoid and \( E \) is a smooth twist of \( G \). In the language of [15], the present paper deals with second countable locally compact stacks, and our results imply that, in that more general context, the K-theory of a crossed product \( \mathcal{A} \rtimes_\alpha G \) is naturally isomorphic to the twisted K-theory of the stack represented by \( G \) for an appropriate gerbe, and vice versa.

2. Preliminaries

For further references and results on upper semicontinuous C*-algebra bundles and upper semicontinuous Banach bundles we refer to [16, Appendix C] and [10, Appendix A], respectively; for groupoid crossed products, we refer to [11]; and for Fell bundles and their associated C*-algebras, to [10].

If \( p : \mathcal{B} \to X \) is a Banach bundle, we write \( B(x) \) for the Banach space that is the fibre of \( \mathcal{B} \) over \( x \). We write \( \Gamma(X; \mathcal{B}) \) for the continuous sections of \( \mathcal{B} \), and \( \Gamma_0(X; \mathcal{B}) \) and \( \Gamma_c(X; \mathcal{B}) \) for the continuous sections vanishing at infinity or with compact support, respectively.

We review the basic definitions for convenience. In all that follows, \( G \) will be a second countable locally compact Hausdorff groupoid with a Haar system \( \{ \lambda^u \}_{u \in G(0)} \).

2.1. Groupoid Crossed Products. A groupoid dynamical system \( (\mathcal{A}, G, \alpha) \) consists of an upper semicontinuous C*-bundle \( p : \mathcal{A} \to G(0) \) with a continuous left \( G \)-action

\[
\alpha : G \times_{s,p} \mathcal{A} := \{(x,a) \in G \times \mathcal{A} \mid s(x) = p(a) \} \to \mathcal{A}
\]

such that for every \( x \in G \) the map \( \alpha_x(a) := \alpha(x,a) \) is an isomorphism of C*-algebras

\[
\alpha_x : A(s(x)) \to A(r(x)).
\]
Then $\Gamma_c(G; r^*\mathcal{A})$ is a $*$-algebra with respect to
\[
f \ast g(x) := \int_G f(y) a_y(g(y^{-1}x)) A^r(y)(y) \quad \text{and} \quad f^*(x) = a_x(f(x^{-1})^*)\]
The crossed product $\mathcal{A} \rtimes_a G$ is the completion of $\Gamma_c(G; r^*\mathcal{A})$ with respect to all suitably bounded representations, and the reduced crossed product $\mathcal{A} \rtimes_a G$ is the completion of $\Gamma_c(G; r^*\mathcal{A})$ with respect to the regular representations. (See \cite{11}.)

**Remark 2.1 (Notation for Crossed Products).** A $C^*$-algebra $A$ can be given the structure of a $C_0(X)$-algebra if and only if there is an upper semicontinuous $C^*$-bundle $\mathcal{A}$ so that $A$ is $C_0(X)$-isomorphic to $I_0(X; \mathcal{A})$ \cite[Theorem C.26]{16}. If $(\mathcal{A}, \alpha, G)$ is a groupoid dynamical system and $A = I_0(G^{(0)}; \mathcal{A})$, then both $\mathcal{A} \rtimes_a G$ and $A \rtimes_a G$ are used to denote the crossed product. We usually prefer the bundle notation $\mathcal{A} \rtimes_a G$.

2.2. **Twists.** A twist $E$ over $G$, or alternatively, a $T$-groupoid over $G$, is a central groupoid extension
\[
G^{(0)} \times T \xrightarrow{i} E \xrightarrow{j} G.
\]
A central extension is one such that $i(r(e), z)e = e(i(s(e), z)e$ for all $e \in E$ and $z \in T$. In particular, $E$ admits a (left or right) $T$-action $z \cdot e := i(r(e), z)e$. Since (2.1) is meant to be an extension of topological groupoids, we are insisting that $i$ is a homeomorphism onto the kernel of $j$, and that $j$ is open and continuous. In particular, $E$ is also a principal $T$-bundle over $G$. Note that if $G$ is a group, then (2.1) is just a central extension of locally compact groups just as in (1.1).

As in \cite{13}, we associate a $C^*$-algebra to a twist (2.1) as follows. We let
\[
C_c(G; E) = \{ f \in C_c(E) : f(ze) = zf(e) \text{ for all } z \in T \text{ and } e \in E \}.
\]
Then $C_c(G; E)$ becomes a $*$-algebra with respect to
\[
f \ast g(e') = \int_G f(e) g(e^{-1}e') d\lambda^{r}(e')(j(e)) \quad \text{and} \quad f^*(e) = f(e^{-1}).
\]
The integral that defines $f \ast g$ makes sense because for fixed $e' \in E$ the expression $f(e)g(e^{-1}e')$ is a function of $j(e) \in G$. Its universal $C^*$-completion is denoted by $C^*_c(G; E)$ and its completion with respect to its regular representations is denoted by $C^*_r(G; E)$.

**Example 2.2 (Projective Representations).** Suppose that $E$ is a twist over a group $G$. Then we get a Haar measure on $E = T \times G$ as the product of Haar measures on $T$ and $G$ (normalized such that $T$ has measure 1). Then multiplication $f \ast g$ in $C_c(G; E)$ can be written as an integral over $E$,
\[
f \ast g(e') = \int_E f(e) g(e^{-1}e') d\lambda(e)
\]
In other words, \( C_c(G; E) \) is a sub \( * \)-algebra of the convolution algebra \( C_c(E) \). If \( \pi \) is a unitary representation of \( E \), then for \( f \in C_c(G; E) \) and \( z \in T \) we find:

\[
\pi(f) = \int_E f(e) \pi(e) d\lambda(e) = \int_E zf(ze) \pi(e) d\lambda(e) = z \int_E f(e) \pi(ze) d\lambda(e) = z \pi(z) \pi(f).
\]

In other words, if we decompose the Hilbert space \( \mathcal{H}_\pi \) according to \( T \)-type, then the restriction of \( \pi \) to \( C_c(G; E) \) is zero on all subspaces, except the one with \( T \)-type \( \pi(z) = z I_{\mathcal{H}_\pi} \). Thus, if \( G \) is a group then \( C^*(G; E) \) is the quotient of \( C^*(E) \) corresponding to those unitary representations \( \pi \) of \( E \) that satisfy \( \pi(i(z)) = z I_{\mathcal{H}_\pi} \).

Central extensions of \( G \) by \( T \) are classified (up to isomorphism) by \( H^2(G, T) \). If \( c \) is a Borel cross section for \( f : E \to G \) such that \( c(e) \) is the identity element of \( E \), then the corresponding Borel 2-cocycle \( \omega \in Z^2(G, T) \) is determined by \( \omega(s)c(r) = \omega(r,s)c(sr) \).

Recall that an \( \omega \)-multiplier representation \( \tilde{\pi} : G \to U(\mathcal{H}) \) such that \( \tilde{\pi}(s) \tilde{\pi}(r) = \omega(s,r) \tilde{\pi}(sr) \). Note that \( \omega \)-multiplier representations of \( G \) and are in one-to-one correspondence with unitary representations \( \pi \) of \( E \) that satisfy \( \pi(i(z)) = z I_{\mathcal{H}} \).

Therefore it is not surprising that \( C^*(G; E) \) is isomorphic to \( C^*(G, \tilde{\omega}) \) where \( C^*(G, \tilde{\omega}) \) is the universal \( C^* \)-algebra for \( \omega \)-multiplier representations of \( G \) and \( \tilde{\omega} \) is the complex conjugate of \( \omega \).

2.3. FELL BUNDLES AND THEIR \( C^* \)-ALGEBRAS. A Fell bundle over a locally compact Hausdorff groupoid \( G \) is an upper semicontinuous Banach bundle \( p : B \to G \) equipped with a continuous, bilinear, associative multiplication \( (a, b) \mapsto ab \) from \( B^{(2)} = \{ (a, b) \in B \times B : (p(a), p(b)) \in G^{(2)} \} \) to \( B \) such that the diagram

\[
\begin{array}{ccc}
B^{(2)} & \longrightarrow & B \\
p \downarrow & & \downarrow p \\
G^{(2)} & \longrightarrow & G
\end{array}
\]

commutes, and such that there is a continuous involution \( b \mapsto b^* \) from \( B \) to \( B \) such that

\[
\begin{array}{ccc}
B & \overset{b \mapsto b^*}{\longrightarrow} & B \\
p \downarrow & & \downarrow p \\
G & \overset{x \mapsto x^{-1}}{\longrightarrow} & G
\end{array}
\]

commutes, and such that, as usual,

\[(ab)^* = b^* a^* .\]

These axioms imply that for a unit \( u \in G^{(0)} \) the fiber \( B(u) \) is a Banach \( * \)-algebra with respect to the inherited operations, while an arbitrary fiber \( B(x) \) is a left \( B(r(x)) \) and right \( B(s(x)) \) bimodule. Finally, for \( B \) to be a Fell bundle
it is required that the $*$-algebra $B(u)$ is a $C^*$-algebra, while $B(x)$ must be a $B(r(x)) - B(s(x))$-imprimitivity bimodule when given inner products

$$\langle a, b \rangle_{\beta(s(x))} := a^* b$$

If $\mathcal{B}$ is a Fell bundle over $G$ we can make $\mathcal{B} = \mathcal{B}(G; \mathcal{B})$ into a $*$-algebra in a straightforward way (provided $G$ has a Haar system). That is, we define

$$f * g(x) := \int_G f(\eta)g(y^{-1}x) \, d\lambda^r(x)(y) \quad \text{and} \quad f^*(x) := f(x^{-1})^*.$$

We can then form the universal completion $C^*(G, \mathcal{B})$ as well as the reduced one $C^*_r(G, \mathcal{B})$.

Fell bundles and their associated $C^*$-completions include virtually all known $C^*$-algebras associated to dynamical systems (see [10, §2]). We include the examples that are relevant to our discussion here below.

**Example 2.3 (Twists Revisited).** Let $E$ be a twist over $G$ as in §2.2. Then $E$ is a principal $T$-bundle. If we let $\mathcal{B}$ be the associated complex line bundle; that is, let $\mathcal{B}$ be the quotient of $E \times \mathbb{C}$ by the diagonal $T$-action $z \cdot (e, \lambda) = (ze, \lambda)$. Then $\mathcal{B}$ is a line bundle over $G$ which we can treat as Fell bundle (see [10, Example 2.4]).

Note that the sections of $\mathcal{B}$ correspond to continuous functions on $E$ which satisfy

$$f(ze) = \bar{z}f(e).$$

Comparing the above with (2.2), it is not hard to see that $C^*(\mathcal{B})$ is isomorphic to $C^*(G, E^0)$ where $E^0$ is the conjugate $T$-bundle to $E$.

To recover $C^*(G; E)$, we work with the line bundle $\mathcal{C}$ associated to $E^0$. Note that $\mathcal{C}$ can be thought of as the quotient of $E \times \mathbb{C}$ with respect to the $T$-action $z \cdot (e, \lambda) = (ze, \lambda)$. Then $C^*(G, \mathcal{C}) \cong C^*(G; E)$, and it is not hard to see that $C^*_r(G, \mathcal{C}) \cong C^*_r(G; E)$.

**Example 2.4 (Groupoid Crossed Products).** Let $(\mathcal{A}, G, \alpha)$ be a groupoid dynamical system. Then we can make $\mathcal{B} := s^* \mathcal{A} = \{ (\gamma, a) : a \in \mathcal{A}(s(\gamma)) \}$ into a Fell bundle where $(\gamma, a)(\eta, b) = (\gamma \eta, a_{\gamma}^{-1}(1) b)$ and $(\gamma, a)^* = (\gamma^{-1}, a_{\gamma}^{-1}(1))$.

Since the map $(\gamma, a) \mapsto (a_{\gamma}(a), \gamma)$ is a Fell bundle isomorphism of $\mathcal{B}$ onto the bundle constructed in [10, Example 2.1], it follows as in [10, Example 2.8], that $C^*(G, \mathcal{B}) \cong \mathcal{A} \rtimes_{\alpha, r} G$. (We have used $\mathcal{B} = s^* \mathcal{A}$, rather than $r^* \mathcal{A}$ as in [10] as it makes some of the formulas in §3a bit tidier. This is also Muhly’s original formulation from [6, §3].) It follows as in [14, Example 11] that $C^*_r(G, \mathcal{B}) \cong \mathcal{A} \rtimes_{\alpha, r, s} G$.

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1This subtlety was overlooked in [10, Example 2.9] were it is erroneously claimed to be isomorphic to $C^*(G; E)$. As is often the case with mathematical constructs where a choice of sign or conjugate is involved, different authors make different choices. We have chosen to keep our notation consistent with the published literature and not redefine $C^*(G; E)$ to suit the present circumstances.
3. Building the $T$-Groupoid

In the next section we prove that, for continuous trace $A$ and locally compact second countable $G$, the crossed product $A \rtimes_{\alpha} G$ is Morita equivalent to $C^*(G; E)$. In this section we construct the pair $(G, E)$. The construction can be summarized as follows.

If the Dixmier-Douady invariant of $A$ is zero, then $A \cong K(H)$ for some $C_0(G^{(0)})$ Hilbert module $H$. In that case, define the $T$-groupoid

$$E := \{ (x, U) \mid U: H(s(x)) \rightarrow H(r(x)) \text{ is a unitary with } \alpha_x = \text{Ad} U \text{ for } x \in G \}$$

with $(x, U)(y, V) := (xy, UV)$. We show that $C^*(G; E)$ is Morita equivalent to $K(H) \rtimes_{\alpha} G$ for both the maximal and reduced $C^*$-algebras.

Even if the Dixmier-Douady invariant of $A$ is not zero, there always exists an open cover $\mathcal{U} = \{ U_i \}$ of $G^{(0)}$ by pre-compact open sets such that the restriction of $A$ to each $U_i$ has zero Dixmier-Douady invariant. Let $\mathcal{G} := G[\mathcal{U}]$ be the groupoid with unit space $G^{(0)} := \bigsqcup U_i$ obtained as the pullback of $G$ via $G^{(0)} \rightarrow G^{(0)}$. As a pullback groupoid, $\mathcal{G}$ is equivalent to $G$.

The convolution $C^*$-algebras (maximal or reduced) of equivalent groupoids are Morita equivalent. We show that the crossed products (maximal or reduced) $A \rtimes_{\alpha} G$ and $A \rtimes_{\alpha} \mathcal{G}$ are Morita equivalent, if we let $\mathcal{A}$ be the pullback of $A$ via $G^{(0)} \rightarrow G^{(0)}$.

By construction, $A$ has Dixmier-Douady invariant zero, and therefore there is a twist $E$ such that $A \rtimes_{\alpha} \mathcal{G}$ is Morita equivalent to $C^*(G; E)$.

A technical difficulty is to provide $E$ with the right topology, and to prove that it satisfies the axioms of a twist. To this end we introduce two auxiliary groupoids $\text{Aut}(\mathcal{A})$ and $\text{Iso}(\mathcal{H})$, whose construction may be of independent interest.

3.1. Two Useful Groupoids. In this section we introduce two groupoids $\text{Aut}(\mathcal{A})$ and $\text{Iso}(\mathcal{H})$ that are convenient in what follows. The constructions given here are valid and may be of use in other contexts, even if $\mathcal{A}$ is not continuous trace.

Let $p : \mathcal{A} \rightarrow G^{(0)}$ be an upper semicontinuous $C^*$-bundle over $G^{(0)}$. We define

$$\text{Aut}(\mathcal{A}) := \{ (u, a, v) \mid a : A(v) \rightarrow A(u) \text{ is a } *\text{-isomorphism} \}.$$ 

Then $\text{Aut}(\mathcal{A})$ is a groupoid with respect to the natural operations.

**Proposition 3.1.** If $p : \mathcal{A} \rightarrow G^{(0)}$ is an upper semicontinuous $C^*$-bundle over $G^{(0)}$, then $\text{Aut}(\mathcal{A})$ has a Hausdorff topology making it into a topological groupoid such that $\{ (u_i, \alpha_i, v_i) \}$ converges to $(u, \alpha, v)$ if and only if

- (a) $u_i \rightarrow u$,
- (b) $v_i \rightarrow v$ and
(c) if \( a_i \to a_0 \) in \( \mathcal{A} \) and \( p(a_i) = v \), then \( \alpha_i(a_i) \to \alpha(a_0) \) in \( \mathcal{A} \).

The proof of Proposition 3.1 is a bit tedious. In general, one can not specify a topology simply by specifying a criteria for nets to converge: the family of convergent nets must satisfy certain axioms (for example, see [3, Chap. 2]). Furthermore, it seems useful to exhibit a bona fide base for our topology. To avoid distractions, we’ll do this in [5].

**Proposition 3.2.** Let \( \{ \alpha_x \}_{x \in G} \) be a family of \(*\)-isomorphisms \( \alpha_x : A(s(x)) \to A(r(x)) \). Then \((\mathcal{A}, G, \alpha)\) is a groupoid dynamical system if and only if \( x \mapsto (r(x), \alpha_x, s(x)) \) is a continuous groupoid homomorphism \( G \to \text{Aut} \mathcal{A} \).

**Proof.** This is an immediate consequence of Proposition 3.1.

Now we suppose that \( H \) is a Hilbert \( C_0(G^{(0)}) \)-module. Then [1, Theorem II.13.18] implies that there is a topology on \( \mathcal{H} = \bigsqcup_{u \in G^{(0)}} H(u) \) making \( q : \mathcal{H} \to G^{(0)} \) into a continuous Banach bundle, or in this case a continuous Hilbert bundle, such that \( H \cong \Gamma_0(G^{(0)}; \mathcal{H}) \).

We let

\[
\text{Iso}(\mathcal{H}) := \{ (u, V, v) \mid V : H(v) \to H(u) \text{ is a unitary} \}
\]

be the groupoid with the obvious operations: \( (u, V, v)(v, W, w) = (u, VW, w) \) etc.

**Proposition 3.3.** If \( H = \Gamma_0(G^{(0)}; \mathcal{H}) \) is a Hilbert \( C_0(G^{(0)}) \)-module, then \( \text{Iso}(\mathcal{H}) \) has a Hausdorff topology making it into a topological groupoid such that a net \( \{ (u_i, V_i, v_i) \} \) converges to \( (u, V, v) \) if and only if

(a) \( u_i \to u \),
(b) \( v_i \to v \) and
(c) if \( h_i \to h_0 \) in \( \mathcal{H} \) and \( q(h_i) = v_i \), then \( V_i h_i \to V h_0 \) in \( \mathcal{H} \).

The proof is similar to that for Proposition 3.1 so we omit it. Note that the relative topology on \( U(H(u)) \) is the strong operator topology. In particular, \( \text{Iso}(\mathcal{H}) \) is not locally compact unless all the fibers \( H(u) \) are finite dimensional.

Now we restrict ourselves to the special case where \( H \) is an \( A - C_0(G^{(0)}) \)-imprimitive bimodule; i.e., \( H \) is a right Hilbert \( C_0(G^{(0)}) \)-module and \( A \) is isomorphic to the generalized compact operators \( \mathcal{K}(H) \) on \( H \) via the left action of \( A \). Then it is not hard to see that \( A \cong \Gamma_0(G^{(0)}; \mathcal{A}) \) with \( A(u) \) identified with the compact operators \( \mathcal{K}(H(u)) \) in such a way that \( \mathcal{A}(u)(h, k) \) is identified with the rank-one operator \( \theta_{h,k} \) where \( \theta_{h,k}(l) = \langle k, l \rangle_i h \). A unitary operator \( U : H(v) \to H(u) \) induces a \(*\)-isomorphism \( \alpha := \text{Ad } U : A(v) \to A(u) \) given by \( \alpha(a) \cdot h = U(a \cdot U^* h) \), with \( a \in A(v), h \in H(u) \).

**Proposition 3.4.** Let \( H = \Gamma_0(G^{(0)}; \mathcal{H}) \) and \( A = \mathcal{K}(H) = \Gamma_0(G^{(0)}; \mathcal{A}) \) be as above. Then there is a short exact sequence of topological groupoids

\[
\begin{array}{ccc}
G^{(0)} \times T & \overset{i}{\longrightarrow} & \text{Iso}(\mathcal{H}) & \overset{j}{\longrightarrow} & \text{Aut} \mathcal{A} \\
\end{array}
\]
where \( j(u, U, v) = (u, \text{Ad} U, v) \) is an continuous open surjection, and \( \iota(u, z) = (u, z \psi_j(u), u) \) is a homeomorphism onto the kernel of \( j \). In particular, given \((u, a, v) \in \text{Aut} \mathcal{A} \) with \( \alpha = \text{Ad} U \), there is a neighborhood \( N \) of \((u, a, v) \) and a continuous section \( \beta : N \to \text{Iso}(\mathcal{H}) \) for \( j \) such that \( \beta(u, a, v) = (u, U, v) \).

Since establishing some of the assertions in Proposition 3.3, such as the continuity and openness of \( j \), is a bit technical, we have moved the proof of the proposition to \( \S 7 \) so as not to distract from the matter at hand.

Note that \( \text{Iso}(\mathcal{H}) \) admits a \( T \)-action such that

\[
\zeta \cdot (u, U, v) = \iota(u, \zeta)(u, U, v) = (u, zU, v) = (u, U, v) \psi_j(v, \zeta).
\]

Thus, except for the fact that \( \text{Iso}(\mathcal{H}) \) and \( \text{Aut} \mathcal{A} \) are generally not locally compact, we can view \( \text{Iso}(\mathcal{H}) \) as a \( T \)-groupoid over \( \text{Aut} \mathcal{A} \).

### 3.2. The Construction of the Twist

Suppose that \((\mathcal{A}, G, \alpha)\) is a groupoid dynamical system with \( A = \Gamma_0(\mathcal{G}^{(0)}; \mathcal{A}) \) continuous trace. Then there is an open cover \( \mathcal{U} = \{ U_i \} \) of \( \mathcal{G}^{(0)} \) by pre-compact open sets such that \( A_{U_i} := \Gamma_0(U_i; \mathcal{A}) \) is Morita equivalent to \( C_0(U_i) \). (See, for example, [13, Proposition 5.5]) In particular, \( A_{U_i} \) can be identified with \( \mathcal{K}(H_i) \) for a Hilbert \( C_0(U_i) \)-module \( H_i = \Gamma_0(U_i; \mathcal{H}_i) \).

Let \( G := G[\mathcal{U}] \) be the pullback groupoid whose unit space is the disjoint union \( G^{(0)} := \bigsqcup U_i = \{ (i, u) : x \in U_i \} \); that is,

\[
G[\mathcal{U}] := \{ (i, x, j) : r(x) \in U_j \text{ and } s(x) \in U_i \}
\]

with \((i, x, j)(j, y, k) = (i, xy, k)\) etc. As is well-known, any groupoid of the form \( G[\mathcal{U}] \) for an open cover \( \mathcal{U} \) of \( G^{(0)} \) is equivalent to \( G \) via \( Z = \bigsqcup G_{U_i} \).

Let \( G \mathcal{A} \) be the pull-back via the obvious map \( \psi : G^{(0)} \to G^{(0)}; \)

\[
G \mathcal{A} = \{ (i, u, b) : u \in U_i \text{ and } b \in A(u) \}.
\]

Note that \( G^A \equiv \Gamma_0(\mathcal{G}^{(0)}; \mathcal{A}) \) by [12, Proposition 1.3]. It is not hard to check that \( G^A \) is Morita equivalent to \( C_0(G^{(0)}) \) via \( H = \Gamma_0(\mathcal{G}^{(0)}; \mathcal{H}) \) where \( \mathcal{H} = \bigsqcup H_i = \{ (i, u, h) : u \in U_i \text{ and } h \in H_i(u) \} \).

There is an action \( \alpha : G \to \text{Aut} \mathcal{A} \) of \( G \) on \( \mathcal{A} \) defined by

\[
\alpha(k, x, l) = (k, r(x), a_x, s(x), l),
\]

and it is straightforward to see that \( \alpha \) is continuous.

Let

\[
\mathcal{G}^{(0)} \times T \xrightarrow{j} \text{Iso}(\mathcal{H}) \xrightarrow{\iota} \text{Aut} \mathcal{A},
\]

be the short exact sequence coming from Proposition 3.4 applied to \( G^A \), and form the pull-back

\[
\mathcal{E} := \{ (h, \gamma) \in \text{Iso}(\mathcal{H}) \times \mathcal{G} : j(h) = \alpha(\gamma) \}
\]

\( ^2 \)In fact \((\mathcal{A}, G, \alpha)\) is a dynamical system whose class in the Brauer group \( Br(G) \) matches up with that of \((\mathcal{A}, G, \alpha)\) in \( Br(G) \) under the isomorphism of \( Br(G) \) with \( Br(\mathcal{G}) \) (see [5, Theorem 4.1]).
(equipped with the relative product topology) that completes the commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{j} & G \\
\text{pr}_1 & & \downarrow\alpha \\
\text{Iso}(\mathcal{H}) & \xrightarrow{\iota} & \text{Aut} \mathcal{A}.
\end{array}
\]

A description of \( E \) that is easier to work with is

\[
E = \{ (k, x, V, l) : x \in G, r(x) \in U_k, s(x) \in U_l, \\
\text{and } V : H(s(x)) \to H(r(x)) \text{ is a unitary with } \alpha_x = Ad V \}
\]

with multiplication \((k, x, V, l)(l, y, W, m) = (k, xy, VW, m)\).

Consideration of the commutative diagram that defines \( E \) shows that \( j \) is an open, continuous surjection with kernel identified with \( i(G(0) \times T) \) (for the obvious map \( i \)), and that \( E \) is a Hausdorff topological groupoid admitting a central \( T \)-action. In fact, \( E \) is a principal \( T \)-bundle over the locally compact space \( G \). Hence \( E \) is a locally compact \( T \)-groupoid over \( G \).

4. The Main Theorem

**Theorem 4.1.** Suppose that \( G \) is a second countable locally compact Hausdorff groupoid with Haar system \( \{ \lambda^u \}_{u \in G(0)} \) and that \( (\mathcal{A}, G, \alpha) \) is a dynamical system with \( A := \Gamma_0(G(0); \mathcal{A}) \) continuous trace. Let \( E \) be the \( T \)-groupoid over the equivalent groupoid \( G \) constructed in the previous section. Then \( \mathcal{A} \rtimes_\alpha G \) is Morita equivalent to \( C^*(G; E) \), and \( \mathcal{A} \rtimes_{\alpha, r} G \) is Morita equivalent to \( C^*_r(G, E) \).

We are going to realize both \( \mathcal{A} \rtimes_\alpha G \) and \( C^*(G; E) \) as Fell bundle \( C^* \)-algebras and then observe that the underlying Fell bundles are equivalent (as in [10, Definition 6.1]). Then Theorem 4.1 will follow from the Fell Bundle Equivalence Theorem [10, Theorem 6.4] and the observation that the Morita equivalence descends to the reduced algebras [13, Theorem 14].

Recall from Example 2.3 that \( C^*(G; E) \) is isomorphic to \( C^*(G, \mathcal{E}) \) where \( \mathcal{E} \) is the line bundle \( T \setminus (E, C) \) where \( z \cdot (g, \lambda) = (g, z \lambda) \). Thus

\[
\mathcal{E} = \{ [k, x, U, \lambda, l] : (k, x, U, l) \in E, \lambda \in C \text{ and } \text{Ad } U = \alpha_x \},
\]

and we have identified \((k, x, zU, \lambda, l)\) with \((i, x, U, z\lambda, j)\) for all \( z \in T \). In particular, the map \([k, x, U, \lambda, l] \mapsto \lambda U^*\) is well defined on \( \mathcal{E} \).

On the other hand, as in Example 2.3 \( \mathcal{A} \rtimes_\alpha G \) is isomorphic to \( C^*(G, \mathcal{B}) \) where \( \mathcal{B} = s^* \mathcal{A} \).

Let \( Z = \coprod_i G_{U_i} \) be the \((G, \mathcal{G})\)-equivalence from [33, We’ll show that

\[
\mathcal{E} = s^* \mathcal{H} = \{ (i, x, h) : s(x) \in U_i \text{ and } h \in H_i \},
\]
viewed as a bundle over \(Z\), is the desired equivalence between \(B\) and \(C\).

To start with, we need actions of \(B\) and \(C\) on \(E\) satisfying the properties (a), (b) and (c) laid out just prior to [10, Definition 6.1] \(^3\) the left \(B\)-action is given by

\[
(x,a) \cdot (i,y,h) := (i,xy,\alpha_y^{-1}(a) \cdot h),
\]

and the right \(C\)-action by

\[
(i,y,h) \cdot [i,z,U,\lambda,j] := (j,yz,\lambda U^*h).
\]

Continuity follows from the characterization of the topologies on \(\text{Aut}\ \mathcal{A}\) and \(\text{Iso}(\mathcal{H})\) in §3.1 and Lemma 7.4.

Next we check that the axioms of [10, Definition 6.1] hold. To see that the actions commute we observe that on the one hand,

\[
(x,a) \cdot ((i,y,h) \cdot [i,z,U,\lambda,j]) = (x,z) \cdot (j,yz,\lambda U^*h) = (j,xyz,\alpha_y^{-1}(a) \cdot \lambda U^*h).
\]

On the other hand,

\[
((x,a) \cdot (i,y,h)) \cdot [i,z,U,\lambda,j] = ((i,xy,\alpha_y^{-1}(a) \cdot h) \cdot [i,z,U,\lambda,j] = (j,xyz,\lambda U^*\alpha_y^{-1}(a) \cdot h) = (j,xyz,\lambda \alpha_z^{-1}(\alpha_y^{-1}(a)) \cdot U^*h).
\]

Hence the actions do commute.

To define inner products, we proceed as follows.

\[
\langle (i,x,h), (i,y,k) \rangle = \langle xy^{-1}, \alpha_y(\mathcal{A}(i)(h,k)) \rangle,
\]

and

\[
\langle (i,y,k), (j,z,l) \rangle = \langle i, y^{-1}z, U, [k,l] \rangle,
\]

where the definition of \(E\) allows us to choose any unitary \(U : H(i,s(z)) \to H(j,s(y))\) implementing \(\alpha_{y^{-1}z}\).

To see that the actions and inner products play nice together (as in [10, Definition 6.1(c)(iv)]), we proceed as follows. On the one hand,

\[
\langle (i,x,h), (i,y,k) \rangle \cdot (j,z,l) = \langle (xy^{-1}, \alpha_y(\mathcal{A}(i)(h,k))) \cdot (j,z,l) = (j, xy^{-1}z, \alpha_{z^{-1}}(\alpha_y(\mathcal{A}(i)(h,k))) \cdot l) = (j, xy^{-1}z, \alpha_{-1} \alpha_y^{-1}(\mathcal{A}(i)(h,k)) \cdot l).
\]

---

\(^3\)Sadly, condition (c) there should read: \(\|b \cdot e\| \leq \|b\|\|e\|\) (and not \(\|b \cdot e\| = \|b\|\|e\|\)).
On the other hand, 
\[ (i, x, h) \cdot \langle (i, y, k), (j, z, l) \rangle = (i, x, h) \cdot [i, y^{-1} z, U, (k, U l) \rangle_{c'} \]
\[ = (j, x y^{-1} z, (k, U l) \rangle_{c'} U^* (h)) \]
\[ = (j, x y^{-1} z, U^* ((k, U l) \rangle_{c'} h)) \]
\[ = ((j, x y^{-1} z, U^* \alpha_{(A(g) \rangle_{c'} (h, k)} \cdot U l)) \]
\[ = (j, x y^{-1} z, \alpha_{z^{-1} y} \cdot (A(g) \rangle_{c'} (h, k)) \cdot U l). \]

Thus
\[ \langle (i, x, h), (i, y, k) \rangle = (i, x, h) \cdot \langle (i, y, k), (j, z, l) \rangle \]
as required. Checking the rest of [10, Definition 6.1(c)] is more straightforward.

It remains only to check that \( E(i, x) \), equipped with the given actions and inner products, is a \( B(r(x)) - C(i, s(x)) \)-imprimitivity bimodule. But \( C(i, s(x)) = \{ [i, s(x), z I, \lambda, i] \} \) can be identified with \( C \) via the map \( [i, s(x), z I, \lambda, i] \mapsto z \lambda, \) and \( B(r(x)) \) with \( A(r(x)) \). Then \( E(i, x) \) is isomorphic to \( a_c H(i, x) \).

Thus axioms (a), (b) and (c) of [10, Definition 6.1] are satisfied. Thus \( \mathcal{B} \) is an equivalence between \( \mathcal{B} \) and \( \mathcal{C} \). This completes the proof of Theorem 4.4.

5. A Partial Converse

In this section, we briefly outline a proof of the following.

**Proposition 5.1.** Suppose that \( E \) is a \( T \)-groupoid over a second countable locally compact groupoid \( G \) with a Haar system \( \{ \lambda^u \}_{u \in G^{(0)}} \). Then there is a Hilbert \( C_0(G^{(0)}) \)-module \( H \) and a \( G \) action \( \alpha \) on \( A = K(H) \) so that \( A \triangleright_{\alpha} G \) is Morita equivalent to \( C^*(G; E) \), and \( A \triangleright_{\alpha,r} G \) is Morita equivalent to \( C^*_r(G; E) \).

Let \( \mathcal{C} \) be the Fell bundle associated to \( E \) as in Example 2.3. Recall that \( C_c(G; E) \) consists of functions in \( C_c(E) \) that satisfy \( f(ze) = z f(e) \). Then \( C_c(G; E) \) becomes a pre-Hilbert \( C_0(G^{(0)}) \)-module via the pre-inner product
\[ \langle f, g \rangle_{C_0(G^{(0)})}(u) = \int_G f(e) g(e) d\lambda_u(j(e)) = \int_G f(\overline{e^{-1}}) g(\overline{e^{-1}}) d\lambda_u(j(e)). \]
The completion, \( \mathcal{H} \), is the section algebra \( H = \Gamma_0(G^{(0)}; H) \) of a continuous Hilbert bundle \( \mathcal{H} \) over \( G^{(0)} \). The fibre \( H(u) \) over \( u \in G^{(0)} \) is the Hilbert space which is the completion of \( H_0(u) = \{ \phi \in C_c(E) : f(ze) = z f(e) \} \) with respect to the pre-inner product induced by \( \langle \cdot, \cdot \rangle_{C_0(G^{(0)})} \). (Of course, this is a Hilbert space with an

---

4Recall that if \( X \) is an \( A - B \)-imprimitivity bimodule, and if \( \theta : A \to C \) is an isomorphism, then the \( C \) - \( B \)-imprimitivity bimodule, where the left inner product is given by \( \langle x, y \rangle = \theta(\langle x, y \rangle) \) and the left \( C \)-action is given by \( c \cdot x := \theta^{-1}(c) \cdot x. \) (Of course the right Hilbert-\( B \) module structure is as before.)
inner product conjugate linear in the first variable.) The algebra of generalized compacts, \( \mathcal{K}(\mathcal{H}) \), is Morita equivalent to \( C_0(G^{(0)}) \). Thus \( \mathcal{K}(\mathcal{H}) \) is a continuous-trace C*-algebra with spectrum \( G^{(0)} \) and trivial Dixmier-Douady class \([13, \S 5.3]\). Furthermore, \( \mathcal{K}(\mathcal{H}) \cong \mathcal{I}_0(G^{(0)}; \mathscr{H}) \) for a suitable (continuous) C*-bundle \( \mathscr{H} \) over \( G^{(0)} \).

For each \( e \in E \), define a unitary \( u(e) : H(s(e)) \to H(r(e)) \) by
\[
(u(e))(f)(e') := f(e'e).
\]
Since \( u(ze) = zu(e) \), \( Ad u(e) : \mathcal{K}(\mathcal{H})(s(e)) \to \mathcal{K}(\mathcal{H})(r(e)) \) depends only on \( j(e) \in G \), we get an action \( a = \{ a_x \}_{x \in G} \) of \( G \) on \( \mathscr{H} \) by
\[
a_{ji(e)}(T) = \text{Ad} u(e) \circ T.
\]
(Just for the record, \( Ad u(e)(T) = u(e)Tu(e^{-1}) \).

It is not hard to see that this gives a continuous action of \( G \) on \( \mathscr{H} \). Hence \( (\mathscr{H}, G, a) \) is a groupoid dynamical system and we can form the groupoid crossed product: \( \mathscr{H} \rtimes_a G \). As in Example 2.4, this crossed product is isomorphic to the Fell bundle C*-algebra \( C^*(\mathcal{B}, \mathcal{E}) \) where \( \mathcal{B} = s^t \mathcal{H} = \{ (x, T) \in G \times \mathcal{H} : T \in \mathcal{K}(H(s(x))) \} \), with \((x, T)(y, S) = (xy, a^{-1}_y(T)S)\) and \((x, T)^* = (x^{-1}, a_x(T^*))\).

To prove Proposition 5.1 we will treat \( G \) as a \((G, G)\)-equivalence and show that \( \mathcal{E} = s^t \mathcal{H} = \{ (x, h) \in G \times \mathcal{H} : h \in H(s(x)) \} \), viewed as a bundle over \( G \), is an equivalence between \( \mathcal{B} \) and \( \mathcal{E} \). (This will suffice by the Equivalence Theorem \([10, \text{Theorem 6.4}] \) and \([14, \text{Theorem 14}] \).)

Now we proceed as in the previous section. We need actions of \( \mathcal{B} \) and \( \mathcal{E} \) on \( \mathcal{E} \) satisfying the properties (a), (b) and (c) laid out just prior to \([10, \text{Definition 6.1}] \). We define
\[
(x, T) \cdot (y, h) := (xy, a^{-1}_y(T)(h)) \quad \text{and} \quad (y, h) \cdot [e, \lambda] := (yji(e), \lambda u(e^{-1})(h))
\]
for \((x, T) \in \mathcal{B}, (y, h) \in \mathcal{E} \) and \([e, \lambda] \in \mathcal{E}\). The algebraic properties are easily checked, and continuity is not so hard to check.

Then we need to check that the actions commute. On the one hand,
\[
((x, T) \cdot (y, h)) \cdot [e, \lambda] = (xy, a^{-1}_y(T)(h)) \cdot [e, \lambda] = (xyji(e), \lambda u(e^{-1})(a^{-1}_y(T)(h))).
\]

On the other hand, keeping in mind that \( a_{ji(e)} = \text{Ad} u(e) \), we have
\[
(x, T) \cdot ((y, h) \cdot [e, \lambda]) = (x, T) \cdot (yji(e), \lambda u(e^{-1})(h)) = (xyji(e), \lambda a^{-1}_{ji(e)}(T)(u(e^{-1})(h))) = (xyji(e), \lambda u(e^{-1})a^{-1}_y(T)(h)).
\]

Thus we have verified Definition 6.1(a).

\footnote{To see that the \( \mathcal{E} \)-action is well-defined, keep in mind that \( u(z \cdot e) = zu(e) \).}
Next we define \( \langle \cdot, \cdot \rangle \) on \( \mathcal{E} \ast_s \mathcal{E} \) by

\[
\langle (x, h), (y, k) \rangle = (xy^{-1}, \alpha_y(\theta_{h,k}))
\]

where \( \theta_{h,k} : H(s(y)) \to H(s(y)) \) is the rank-one operator \( \theta_{h,k}(l) = (k, l)_c h \), where

\[
\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{E}0(\mathcal{G}(0))}(u).
\]

Note that if \( y = j(\varepsilon) \), then \( \alpha_y(\theta_{h,k}) = \theta_{u(\varepsilon)h,u(\varepsilon)k} \).

We define \( \langle \cdot, \cdot \rangle \) on \( \mathcal{E} \ast_r \mathcal{E} \) by

\[
\langle (j(f), k), (j(g), l) \rangle = [f^{-1}g, \langle u(f)k, u(g)l \rangle_{\mathcal{E}0(\mathcal{G}(0))}(r(f))].
\]

After checking that the above is actually well-defined, it is not too difficult to see that \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) satisfy (i)--(iv) of part (b) of the definition. For example, to verify (iv), consider on the one hand:

\[
\langle (x, h), (y, k) \rangle \cdot (z, l) = (xy^{-1}, \alpha_y(\theta_{h,k})) \cdot (z, l) = (xy^{-1}z, \alpha_{z^{-1}y}(\theta_{h,k})(l)).
\]

On the other hand, supposing that \( j(f) = y \) and \( j(g) = z \), we have

\[
(x, h) \cdot \langle (j(f), k), (j(g), l) \rangle = (x, h) \cdot [f^{-1}g, \langle u(f)k, u(g)l \rangle_{\mathcal{E}0(\mathcal{G}(0))}(r(f))]
= (xy^{-1}z, \langle u(g^{-1}f)k, \langle u(y^{-1}f)l \rangle_{\mathcal{E}0(\mathcal{G}(0))} s(g)u(g^{-1}f)(h) \rangle)
= (xy^{-1}z, \alpha_{u(g^{-1}f)k}(\theta_{h,k})(l))
= (xy^{-1}z, \alpha_{z^{-1}y}(\theta_{h,k})(l)).
\]

Thus (iv) holds.

We still need to verify part (c) of the Definition; that is, we need to see that

\[
E(x) = \{(x, h) : h \in H(s(x)) \}
\]

is a \( B(r(x)) - C(s(x)) \)-imprimitivity bimodule with respect to the inherited operations. First note that \( C(s(x)) = \{ [\langle i(s(x), z), \lambda \rangle] \} \), is easily identified with \( C \) via the map \( [\langle i(s(x), z), \lambda \rangle] \to \lambda \). On the other hand, \( B(r(x)) \) is easily identified with \( \mathcal{K}(H(r(x))) \). Then it is not hard to check that \( E(x) \) is the \( \mathcal{K}(H(r(x))) - C \)-imprimitivity bimodule given by \( a_s H(s(x)) \).

Thus axioms (a), (b) and (c) of [10] Definition 6.1 are satisfied and \( \mathcal{E} \) is the desired equivalence. This completes the proof of Proposition 5.1.

\footnote{See footnote 4 at the end of 44}
Recall that if $X$ is a set and $\rho$ is a collection of subsets of $X$ than cover $X$, then the collection of finite intersections of elements of $\rho$ form a basis for a topology $\tau$ on $X$. In this case, we say $\rho$ is a subbasis for $\tau$.

If $a, b \in A$, $U$ and $V$ are open sets in $G^{(0)}$ and $\epsilon > 0$, then let

$$W(U, a, b, V, \epsilon) = \{ (u, a, v) : u \in U, v \in V \text{ and } \|a(a(v)) - b(u)\| < \epsilon \}.$$ 

Of course, some of the $W(U, a, b, V, \epsilon)$ might be empty, but if $(u, a, v) \in \text{Aut} \mathcal{A}$, for any $a \in A$, there is a $b \in A$, such that $b(v) = a(u)$. Then $(u, a, v) \in W(U, a, b, V, \epsilon)$ for any $U$, $V$ and $\epsilon$. Hence the collection of all $W(U, a, b, V, \epsilon)$ cover $\text{Aut} \mathcal{A}$, and form a subbasis for a topology $\tau$ on $\text{Aut} \mathcal{A}$.

**Lemma 6.1.** We have $(u_i, a_i, v_i) \to (u, a, v)$ in $(\text{Aut} \mathcal{A}, \tau)$ if and only if (a), (b) and (c) of Proposition 5.1 hold.

**Proof.** Suppose that $(u_i, a_i, v_i) \to (u, a, v)$. Fix $a, b \in A$ such that $b(u) = a(a(v))$. Then for any $\epsilon > 0$ and any neighborhoods $U$ of $u$ and $V$ of $v$, we have that $W(U, a, b, V, \epsilon)$ is a neighborhood of $(u, a, v)$. Hence (a) and (b) hold.

There is an $i_0$ such that $i \geq i_0$ implies $(u_i, a_i, v_i) \in W(U, a, b, V, \epsilon)$. Hence if $i \geq i_0$, we have

$$\|a_i(a(v_i)) - b(u_i)\| < \epsilon.$$ 

Since $b(u_i) \to b(u)$ in $\mathcal{A}$ and $\epsilon > 0$ is arbitrary, [16] Proposition C.20 implies that $a_i(a(v_i)) \to a(a(v))$. Now if $a_i \to a$ in $\mathcal{A}$ with $p(a_i) = v_i$, let $a \in A$ be such that $a(v) = a_0$. Then by the above, $a_i(a(v_i)) \to a(a_0)$ in $\mathcal{A}$, while

$$\|a_i(a_i) - a_i(a(v_i))\| = \|a_i - a(v_i)\| \to 0,$$

since isomorphisms are isometric and $a$ is a section. It follows from another application of [16] Proposition C.20 that $a_i(a_i) \to a(a_0)$. This proves (c).

Conversely, suppose that $\{ (u_i, a_i, v_i) \}$ is a net in $\text{Aut} \mathcal{A}$ satisfying (a), (b) and (c). Let $W(U, a, b, V, \epsilon)$ be a subbasic open neighborhood of $(u, a, v)$. It will suffice to see that $\{ (u_i, a_i, v_i) \}$ is eventually in $W(U, a, b, V, \epsilon)$. Since (a) and (b) hold, we can assume that $u_i \in U$ and $v_i \in V$. Moreover,

$$\|a(a(v)) - b(u)\| < \epsilon.$$ 

But $\{ a \in \mathcal{A} : \|a\| < \epsilon \}$ is open and

$$a_i(a(v_i)) - b(u_i) \to a(a(v)) - b(u)$$

in view of assumption (c). Thus we eventually have

$$\|a_i(a(v_i)) - b(u_i)\| < \epsilon.$$ 

That is, $\{ (u_i, a_i, v_i) \}$ is eventually in $W(U, a, b, V, \epsilon)$ as required. 

**Remark 6.2.** It follows immediately that the relative topology on $\text{Aut} \ A(u)$ is the so-called “point-norm” topology (see [13] Lemma 7.18)].
More generally, we have the following immediate corollary.

**Corollary 6.3.** A net \( \{(u, \alpha_i, v)\} \) converges to \((u, \alpha, v)\) in \( \text{Aut} A \) if and only if \( \alpha_i(b) \to a(b) \) for all \( b \in A(v) \).

**Lemma 6.4.** The topology \( \tau \) on \( \text{Aut} A \) is Hausdorff.

**Remark 6.5.** At this point, we are not assuming that \( A \) is a continuous bundle. Hence the topology on \( A \) need not be Hausdorff in general [16, Example C.27].

**Proof.** The existence of neighborhoods \( W(U, a, b, V, e) \) of \((u, a, v)\) with \( U \) and \( V \) arbitrary implies that we only have to show that we can separate \((u, a, v)\) and \((u, \beta, v)\) with \( \alpha \neq \beta \). But this is not hard in view of Corollary 6.3.

**Proof of Proposition 3.4.** Since \( \text{Ad}_V \) is isometric, it follows that \( (u, a, v) \to \) \( (u, a, v) \) if and only if \( (v, a^{-1}, u) \to \) \( (v, a^{-1}, u) \). Hence, inversion is continuous.

We just need to see that \( (u_i, \alpha_i, v_i) \to (u, a, v) \) and \( (v_i, \beta_i, w_i) \to (v, \beta, w) \). But if \( a_i \to a_0 \) in \( A \) with \( p(a_i) = u_i \), then Lemma 6.1 implies that \( \beta_i(a_i) \to \beta(a_0) \) and hence that \( a_i(\beta_i(a_i)) \to a(\beta(a_0)) \). By Lemma 6.1, this suffices.

7. **Proof of Proposition 3.4.**

Recall that \( \mathcal{H}(u) \) is an \( A(u) \to \mathbb{C} \)-imprimitivity bimodule with
\[
\langle a(u), b(u) \rangle = \langle a, b \rangle(u) \quad \text{for all } a, b \in A.
\]

**Lemma 7.1.** The map \( j : \text{Iso}(\mathcal{H}) \to \text{Aut} A \) is continuous.

**Proof.** Suppose that \( (u_i, V_i, v_i) \to (u, V, v) \) in \( \text{Iso}(\mathcal{H}) \) and that \( a_i \to a_0 \) in \( A \) with \( p(a_i) = v_i \). It will suffice to see that \( (\text{Ad} V_i)(a_i) \to (\text{Ad} V)(a_0) \) in \( A \).

Given \( \epsilon > 0 \), there are elements \( c_j, d_j \in A \) such that
\[
\left\| \sum_{j=1}^{n} \langle c_j(v), d_j(v) \rangle - a_0 \right\| < \epsilon.
\]

Since \( \text{Ad} V(\langle c, d \rangle) = \langle Vc, Vd \rangle \) and since \( \text{Ad} V \) is isometric,
\[
\left\| \sum_{j=1}^{n} \langle Vc_j(v), Vd_j(v) \rangle - \text{Ad} V(a_0) \right\| < \epsilon.
\]
But \( \left( \sum_{j \in \mathfrak{A}(v)} \langle c_j(v_i) , d_j(v_i) \rangle - a_i \right) \) converges to \( \left( \sum_{j \in \mathfrak{A}(v)} \langle c_j(v) , d_j(v) \rangle - a_0 \right) \) in \( \mathfrak{A} \).

Hence we eventually have

\[
\left\| \sum_{j \in \mathfrak{A}(v)} \langle c_j(v_i) , d_j(v_i) \rangle - a_i \right\| < \epsilon.
\]

As above, this means we eventually have

\[
\left\| \sum_{j \in \mathfrak{A}(u)} \langle V_i c_j(v_i) , V_i d_j(v_i) \rangle - \text{Ad} V_i(a_i) \right\| < \epsilon.
\]

Since we certainly have \( \sum_{j \in \mathfrak{A}(u)} \langle V_i c_j(v_i) , V_i d_j(v_i) \rangle \rightarrow \sum_{j \in \mathfrak{A}(u)} \langle V c_j(v) , V d_j(v) \rangle \) by assumption, \cite{16} Proposition C.20] implies that \( \text{Ad} V_i(a_i) \rightarrow \text{Ad} V(a_0) \) as required.

The next result is classical. We sketch the proof as the construction will be required in the proof of the proposition.

**Lemma 7.2.** Given a \(*\)-isomorphism \( \alpha : A(v) \rightarrow A(u) \), there is a unitary \( U : H(v) \rightarrow H(u) \), determined up to a unimodular scalar, such that \( \alpha = \text{Ad} U \). Consequently, \( U \) is surjective with kernel the image of \( \mathfrak{L} \).

**Sketch of the Proof.** The only nontrivial bit is the construction of \( U \), and for this we follow the proof of \cite{13} Proposition 1.6.

Let \( e \) be a unit vector in \( H(v) \). Then \( p := \langle e , e \rangle \) is a minimal projection in \( A(v) \). Hence \( \alpha(p) \) is a minimal projection in \( A(u) \) and is of the form \( \sum_{j \in \mathfrak{A}(u)} \langle f_j , f_j \rangle \) for any unit vector \( f \) in the range of \( \alpha(p) \). Then we define \( U : H(v) \rightarrow H(u) \) by

\[
U h := \alpha(\langle v , e \rangle \cdot f).
\]

A computation show that \( U \) preserves inner products and if \( g \in H(u) \) we have \( U h = g \) with \( h = \alpha^{-1}(\langle A(u) \rangle g , f) e \). Hence \( U \) is a unitary. Another computation shows that \( \alpha(T) \cdot Uh = U(T \cdot h) \). Hence \( \alpha = \text{Ad} U \) as required.

By \cite{16} Proposition 1.15], the openness of \( j \) will follow from the cross section assertion. Since \( j \) is clearly a homeomorphism onto its range, we can complete the proof of Proposition 3.4 by proving the cross section result.

**Proposition 7.3.** Given \( (u_0 , a_0 , v_0) \in \text{Aut} \mathfrak{H} \) with \( a_0 = \text{Ad} V_0 \), there is an open neighborhood \( N \) of \( (u_0 , a_0 , v_0) \) and a continuous map \( \beta : N \rightarrow \text{Iso}(\mathfrak{H}) \) such that \( \beta \circ \gamma = \text{id}_N \) and such that \( \beta(u_0 , a_0 , v_0) = (u_0 , V_0 , v_0) \).

We’ll need the following lemma.

**Lemma 7.4.** The action of \( \mathfrak{H} \) on \( \mathfrak{H} \) is continuous in that if \( a_i \rightarrow a_0 \) in \( \mathfrak{H} \) and \( h_i \rightarrow h_0 \) in \( \mathfrak{H} \) with \( p(a_i) = v_i = q(h_i) \), then \( a_i \cdot h_i \rightarrow a_0 \cdot h_0 \) in \( \mathfrak{H} \).

**Proof.** Fix \( a \in A \) and \( h \in H \) such that \( a(v_0) = a_0 \) and \( h(v_0) = h_0 \). Since \( \|a_i - a(v_i)\| \rightarrow 0 \) and \( \|h_i - h(v_i)\| \rightarrow 0 \), it follows that \( \|a_i \cdot h_i - a(v_i) \cdot h(v_i)\| \rightarrow 0 \).
Since \(a(v_i) \cdot h(v_i) = a \cdot h(v_i)\), we have \(a(v_i) \cdot h(v_i) \to a_0 \cdot h_0\) in \(\mathcal{A}\). Hence \(a_i \cdot h_1 \to a_0 \cdot h_0\) by [16 Proposition C.20].

**Proof of Proposition 7.3** We follow the constructions in Lemma 7.2. Thus if \(e_0\) is a unit vector in \(H(v_0)\) and \(c_0 \in H(u_0)\) is a unit vector in the range of \(a_0(A(v_0)\langle e_0 , e_0 \rangle)\), then \(a_0 = \text{Ad} V\) for \(V\) defined by \(V(h) = a_0(A(v_0)\langle h , e_0 \rangle)c_0\). We can modify \(c_0\) by a unimodular scalar so that \(V = V_0\).

Let \(e \in H\) be such that \(e(v_0) = e_0\). Since \(v \mapsto \|e(v)\|\) is continuous and \(\phi \cdot e(v) = \phi(v)e(v)\) defines an element of \(H\) for any \(\phi \in C_c(G(0))\), we can assume that there is a neighborhood \(V\) of \(v_0\) such that \(\|e(v)\| = 1\) for all \(v \in V\). Similarly, choose \(c \in X\) such that \(c(u_0) = c_0\) such that there is a neighborhood \(U\) of \(u_0\) such that \(\|c(u)\| = 1\) for all \(u \in U\). Let \(a := \lambda(e , e)\) and \(b := \lambda(c , c)\) be the corresponding local rank-one projection fields. Then,

\[
N = \mathcal{W}(U, a, b, V, \frac{1}{4}) = \{ (u, a, v) : u \in U, v \in V \text{ and } \|a(a(v)) - b(u)\| < \frac{1}{4} \}
\]

is a basic open neighborhood of \((u_0, a_0, v_0)\) in \(\text{Aut} \mathcal{A}\) (see the proof of Proposition 3.1).

If \((u, a, v) \in N\), then

\[
\|a(a(v))c(u) - c(u)\| = \|a(a(v))c(u) - b(u)c(u)\| \leq \frac{1}{4}.
\]

In particular, \(a(a(v))c(u) \neq 0\) and

\[
f(u, a, v) := \|a(a(v))c(u)\|^{-1}a(a(v))c(u)
\]

is a unit vector in the range of the minimal projection \(a(a(v))\). Thus if \(n = (u, a, v) \in N\), then \(a = \text{Ad} V_n\) where

\[
V_n(h) = a(A(v_0)\langle h , e(v) \rangle)f(n).
\]

Hence we can define \(\beta : N \to \text{Iso}(\mathcal{A})\) by \(\beta(u, a, v) = (u, V_n(u, a, v), v)\). To complete the proof, we just need to see that \(\beta\) is continuous. Suppose that \(n_i = (u_i, a_i, v_i) \to n = (u, a, v)\). In view of Proposition 7.3, it will suffice to show that if \(h_i \to h\) in \(\mathcal{A}\) with \(h(i) = v_i\), then \(V_n(h_i) \to V_n(h)\).

But we claim

\[
A(v_0)\langle h_i , e(v_i) \rangle \to A(v_0)\langle h , e(v) \rangle
\]

in \(\mathcal{A}\). To see this, let \(f \in \mathcal{H}\) be such that \(f(v) = h\). Then \(\|f(v_i) - h_i\| \to 0\). Therefore

\[
\|A(v_0)\langle f(v_i) , e(v_i) \rangle - A(v_0)\langle h_i , e(v_i) \rangle\| \to 0.
\]

Since

\[
A(v_0)\langle f(v_i) , e(v_i) \rangle \to A(v_0)\langle f(v) , e(v) \rangle = A(v_0)\langle h , e(v) \rangle,
\]

the claim follows from [16 Proposition C.20]. Hence it follows from Proposition 3.1 that

\[
a_i(A(v_0)\langle f(v_i) , e(v_i) \rangle) \to a_i(A(v_0)\langle h , e(v) \rangle).
\]
Since \( f(n_i) \to f(n) \) in \( \mathcal{A} \), Lemma 7.4 implies that

\[
V_n(h_i) = \alpha_i \left( \langle A(v_i), e(v_i) \rangle \right) f(n_i) \to \alpha \left( \langle h_i, e(v) \rangle \right) f(n) = V_n(h).
\]

This completes the proof of Proposition 7.3 and also of Proposition 3.4.

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