On Reducible Verma Modules over
Jacobi Algebra

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Abstract. With this paper we start the study of reducible representations of the Jacobi algebra with the ultimate goal of constructing differential operators invariant w.r.t. the Jacobi algebra. In this first paper we show examples of the low level singular vectors of Verma modules over the Jacobi algebra. According to our methodology these will produce the invariant differential operators.

Keywords: Jacobi algebra, Verma modules, singular vectors

1 Introduction

The role of nonrelativistic symmetries in theoretical physics was always important. Currently one of the most popular fields in theoretical physics - string theory, pretending to be a universal theory - encompasses together relativistic quantum field theory, classical gravity, and certainly, nonrelativistic quantum mechanics, in such a way that it is not even necessary to separate these components.

Since the cornerstone of quantum mechanics is the Schrödinger equation then it is not a surprise that the Schrödinger group - the group that is the maximal group of symmetry of the Schrödinger equation - was the first to play a prominent role in theoretical physics. The latter is natural since originally the Schrödinger group, actually the Schrödinger algebra, was introduced in [1, 2] as a nonrelativistic limit of the vector-field realization of the conformal algebra. For a review on these developments we refer to [3].

Another interesting non-relativistic example is the Jacobi algebra [4, 5] which is the semi-direct sum of the Heisenberg algebra and the $sp(n)$ algebra. Actually the lowest case of the Jacobi algebra coincides with the lowest case of the Schrödinger algebra which makes it interesting to apply to the Jacobi algebra the methods we applied to the Schrödinger algebra. This is a project we start in the present short paper. Actually here we give as examples the low level singular vectors of Verma modules over the Jacobi algebra.
2 Preliminaries

The Jacobi algebra is the semi-direct sum \( \mathcal{G}_n := H_n \triangleright sp(n, \mathbb{R}) \mathbb{C} \) [4, 5]. The Heisenberg algebra \( H_n \) is generated by the boson creation (respectively, annihilation) operators \( a_i^+ (a_i^-) \), \( i, j = 1, \ldots, n \), which verify the canonical commutation relations
\[
[a_i^+, a_j^-] = \delta_{ij}, \quad [a_i^-, a_j^+] = [a_i^+, a_j^+] = 0.
\]

\( H_n \) is an ideal in \( \mathcal{G}_n \), i.e., \( [H_n, \mathcal{G}_n] = H_n \), determined by the commutation relations (following the notation of [6]):
\[
[a_k^+, K_{ij}^+] = [a_k^-, K_{ij}^-] = 0,
\]
\[
[a_{i}^-, K_{kj}^+] = \frac{1}{2}\delta_{ik}a_{j}^+ + \frac{1}{2}\delta_{ij}a_{k}^+,
\]
\[
[K_{ij}^-, a_i^+] = \frac{1}{2}\delta_{ik}a_j^- + \frac{1}{2}\delta_{ij}a_k^-,
\]
\[
[K_{ij}^0, a_k^+] = \frac{1}{2}\delta_{jk}a_i^+,
\]
\[
[a_k^-, K_{ij}^-] = \frac{1}{2}\delta_{jk}a_i^-.
\]

\( K_{ij}^{\pm, 0} \) are the generators of the \( S_n \equiv sp(n, \mathbb{R}) \mathbb{C} \) algebra:
\[
[K_{ij}^+, K_{kl}^-] = [K_{ij}^-, K_{kl}^+] = 0, \quad 2[K_{ij}^-, K_{kl}^0] = K_{il}^0 \delta_{kj} + K_{jl}^0 \delta_{ki},
\]
\[
2[K_{ij}^-, K_{kl}^+] = K_{ij}^0 \delta_{ki} + K_{ij}^0 \delta_{ki} + K_{kl}^0 \delta_{ij} + K_{kl}^0 \delta_{ij},
\]
\[
2[K_{ij}^+, K_{kl}^-] = -K_{ij}^0 \delta_{ji} - K_{ij}^0 \delta_{ij}, \quad 2[K_{ij}^0, K_{kl}^0] = K_{ji}^0 \delta_{ki} - K_{ji}^0 \delta_{ij}.
\]

In order to implement our approach we introduce a triangular decomposition of \( \mathcal{G}_n \):
\[
\mathcal{G}_n = \mathcal{G}_n^+ \oplus \mathcal{K}_n \oplus \mathcal{G}_n^- \quad (4)
\]
using the triangular decomposition \( S_n = S_n^+ \oplus \mathcal{K}_n \oplus S_n^- \), where:
\[
\mathcal{G}_n^+ = H_n^+ \oplus S_n^+ \quad (5)
\]
\[
H_n^+ = \text{l.s.} \{ a_i^+ : i = 1, \ldots, n \},
\]
\[
S_n^+ = \text{l.s.} \{ K_{ij}^+ : 1 \leq i < j \leq n \} \oplus \text{l.s.} \{ K_{ij}^0 : 1 \leq i < j \leq n \}
\]
\[
S_n^- = \text{l.s.} \{ K_{ij}^- : 1 \leq i < j \leq n \} \oplus \text{l.s.} \{ K_{ij}^0 : 1 \leq j < i \leq n \}
\]
\[
\mathcal{K}_n = \text{l.s.} \{ K_{ij}^0 : 1 \leq i \leq n \}
\]
Note that the subalgebra \( \mathcal{K}_n \) is abelian and is a Cartan subalgebra of \( S_n \). Furthermore, not only \( S_n^ \pm \) are its eigenspaces:
\[
[\mathcal{K}_n, \mathcal{G}_n^\pm] = \mathcal{G}_n^\pm
\]
Thus, \( \mathcal{K}_n \) plays for \( \mathcal{G}_n \) the role that Cartan subalgebras are playing for semi-simple Lie algebras.

3 Case \( \mathcal{G}_2 \)

Note that the algebra \( \mathcal{G}_1 \) is isomorphic to the (1+1)-dimensional Schrödinger algebra (without central extension). The representations of the latter are well...
known, cf. [7–9, 3]. Thus, we study the first new case of the $G_n$ series, namely, $G_2$.

For simplicity, we introduce the following notations for the basis of $S_2^+$:

$$S^+ : b_i^+ \equiv K_{ii}^+ , \quad i = 1, 2; \quad c^+ \equiv K_{12}^- , \quad d^+ \equiv K_{21}^0 \quad (7a)$$

$$S^- : b_i^- \equiv K_{ii}^- , \quad i = 1, 2; \quad c^- \equiv K_{12}^+ , \quad d^- \equiv K_{21}^0 \quad (7b)$$

$$K : h_i \equiv K_{ii}^0 , \quad i = 1, 2. \quad (7c)$$

Next, using (2) and (3) we give the eigenvalues of the basis of $G^+_2$ w.r.t. $K$:

$$h_1 : (b_1^+, b_2^+, c^+, d^+, a_1^+, a_2^+) : (1, 0, \frac{1}{2}, \frac{1}{2}, 0) , \quad (8)$$

$$h_2 : (b_1^+, b_2^+, c^+, d^+, a_1^+, a_2^+) : (0, 1, \frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}) ,$$

(e.g., $[h_1, b_1^+] = b_1^+$, $[h_2, d^+] = -\frac{1}{2}d^+$, etc). Naturally, the eigenvalues of the basis of $G^-_2$ w.r.t. $K$ are obtained from (8) by multiplying every eigenvalue by (-1).

Next we introduce the following grading of the basis of $G^+_2$:

$$(b_1^+, b_2^+, c^+, d^+, a_1^+, a_2^+) : (2\delta_1, 2\delta_2, \delta_1 + \delta_2, \delta_1 - \delta_2, \delta_1, \delta_2) \quad (9)$$

The grading of the $S^+_2$ part of the basis follows from the root system of $S^+_2$, while the grading of the $H^+_2$ part of the basis is determined by consistency with commutation relations (2). It is consistent also with formulae (8).

Naturally, the grading of the basis of $G^-_2$ w.r.t. are obtained from (9) by multiplying every grading by (-1).

4 Verma modules and singular vectors

4.1 Definitions

We shall introduce Verma modules over the Jacobi algebra analogously to the case of semi-simple algebras. Thus, we define a lowest weight Verma module $V^\Lambda$ over $G_n$ as the lowest weight module over $G_n$ with lowest weight $\Lambda \in \mathcal{K}^*_n$ and lowest weight vector $v_0 \in V^\Lambda$, induced from the one-dimensional representation $V_0 \cong \mathbb{C}v_0$ of $U(B_n)$, (where $B_n = \mathcal{K}_n \oplus \mathcal{G}^-_n$ is a Borel subalgebra of $G_n$), such that:

$$X v_0 = 0, \quad \forall X \in G^-_n$$

$$H v_0 = \Lambda(H) v_0 , \quad \forall H \in \mathcal{K}_n \quad (10)$$

Pursuing the analogy with the semi-simple case and following our approach we are interested in the cases when the Verma modules are reducible. Namely, we are interested in the cases when a Verma module $V^\Lambda$ contains an invariant submodule which is also a Verma module $V^{\Lambda'}$, where $\Lambda' \neq \Lambda$, and holds the analog of

$$X v'_0 = 0, \quad \forall X \in G^-_n$$

$$H v'_0 = \Lambda'(H) v'_0 , \quad \forall H \in \mathcal{K}_n \quad (11b)$$
Since $V^{A'}$ is an invariant submodule then there should be a mapping such that $v_0'$ is mapped to a singular vector $v_s \in V^A$ fulfilling exactly (11). Thus, as in the semi-simple case there should be a polynomial $\mathcal{P}$ of $\mathcal{G}_n^+$ elements which is eigenvector of $K_n$: $[H, \mathcal{P}] = A'(H)\mathcal{P}$, ($\forall H \in K_n$), and then we would have: $v_s = \mathcal{P}v_0$.

### 4.2 Case $G_2$

We shall consider several examples of reducible Verma modules with different weights.

**Weight 2$\delta_1$** As first example we try to find a singular vector of weight $A' \sim 2\delta_1$. There are six possible terms in $U(G_2)$ with this weight, thus, we try:

$$v_s^{2\delta_1} = (\nu_1 b_1^+ + \nu_2 c^+(d^+) + \nu_3 b_2^+(d^+) + \nu_4(a_1^+)^2 + \nu_5 a_1^+ a_2^+ d^+ + \nu_6 (a_2^+) (d^+)^2)v_0$$

where $\nu_k$ are numerical coefficients which may be fixed when we impose (11a) on (12). (Note that (11b) is fulfilled by every term of (12).)

After we impose (11a) on (12) we find the solution:

$$\Lambda(H_1) = \frac{3}{4}, \quad \nu_3 = -2\nu_6,$$

$$\nu_1 = -\nu_6(\Lambda(H_2) - \Lambda(H_1))(2\Lambda(H_2) - 2\Lambda(H_1) - 1),$$

$$\nu_2 = 2\nu_6(2\Lambda(H_2) - 2\Lambda(H_1) - 1),$$

$$\nu_4 = \nu_6(\Lambda(H_2) - \Lambda(H_1))(\Lambda(H_2) - \Lambda(H_1) - \frac{1}{2}),$$

$$\nu_5 = -\nu_6(2\Lambda(H_2) - 2\Lambda(H_1) - 1).$$

Thus, the singular vector is:

$$v_s^{2\delta_1} = \nu_6((\Lambda(H_2) - \frac{3}{4})(\Lambda(H_2) - \frac{3}{4})(a_1^+)^2 - 2b_1^+)) +$$

$$+ 2(\Lambda(H_2) - \frac{3}{4})(2c^+ - a_1^+ a_2^+)d^+ +$$

$$+ ((a_2^+)^2 - 2b_2^+)(d^+)^2)v_0, \quad \Lambda(H_1) = \frac{3}{4}$$

**Weight 2$\delta_2$** As next example we try to find a singular vector of weight $A' \sim 2\delta_2$. The possible singular vector is:

$$v_s^{2\delta_2} = (\mu_1 b_2^+ + \mu_2 (a_2^+)^2)v_0$$

Imposing (11a) on (15) we obtain:

$$\Lambda(H_2) = \frac{1}{4}, \quad \mu_1 = -2\mu_2,$$

Thus, the singular vector is:

$$v_s^{2\delta_2} = \mu_2((a_2^+)^2 - 2b_2^+)v_0, \quad \Lambda(H_2) = \frac{1}{4}$$
**Weight $\delta_1 + \delta_2$** Next we try a singular vector of weight $\Lambda' \sim \delta_1 + \delta_2$. The possible singular vector is:

$$v^{\delta_1 + \delta_2}_s = \left( \kappa_1 c^+ + \kappa_2 b^+_2 d^+ + \kappa_3 a^+_1 a^+_2 + \kappa_4 (a^+_2)^2 d^+ \right) v_0$$  \hspace{1cm} (18)

Imposing (11a) on (18) we obtain:

$$\Lambda(H_2) = \frac{3}{2} - \Lambda(H_1), \quad \kappa_1 = (3 - 4h(1)) \kappa_4, \quad \kappa_2 = -2 \kappa_4, \quad \kappa_3 = (2h(1) - \frac{3}{2}) \kappa_4$$  \hspace{1cm} (19)

Thus, the singular vector is:

$$v^{\delta_1 + \delta_2}_s = \kappa_4 \left( \left( \frac{3}{2} - 2h(1) \right) (2c^+ - a^+_1 a^+_2) + ((a^+_2)^2 - 2b^+_2 d^+) \right) v_0$$  \hspace{1cm} (20)

**Weight $\delta_1 - \delta_2$** Next we try a singular vector of weight $\Lambda' \sim \delta_1 - \delta_2$. The only possible singular vector is:

$$v^{\delta_1 - \delta_2}_s = \lambda d^+ v_0$$  \hspace{1cm} (21)

Imposing (11a) on (21) we obtain that $v^{\delta_1 - \delta_2}_s$ is a singular vector iff:

$$\Lambda(H_2) = \Lambda(H_1)$$  \hspace{1cm} (22)

**Weight $\delta_1$** Next we try a singular vector of weight $\Lambda' \sim \delta_1$. The possible singular vector is:

$$v^{\delta_1}_s = \left( \lambda_1 a^+_1 + \lambda_2 a^+_2 d^+ \right) v_0$$  \hspace{1cm} (23)

Imposing (11a) on (23) we obtain:

$$\lambda_1 = \lambda_2 = 0$$  \hspace{1cm} (24)

Thus, there is no singular vector of weight $\delta_1$.

**Weight $\delta_2$** Finally, we try a singular vector of weight $\Lambda' \sim \delta_2$. The only possible singular vector is:

$$v^{\delta_2}_s = \mu a^+_2 v_0$$  \hspace{1cm} (25)

Imposing (11a) on (25) we obtain:

$$\mu = 0$$  \hspace{1cm} (26)

Thus, there is no singular vector of weight $\delta_2$.

**Weight 3$\delta_2$** The only possible singular vector is:

$$v^{3\delta_2}_s = \mu b^+_2 a^+_2 v_0 + \nu (a^+_2)^3 v_0$$  \hspace{1cm} (27)

Imposing (11a) on (27) we obtain:

$$\mu = \nu = 0$$  \hspace{1cm} (28)

Thus, there is no singular vector of weight $3\delta_2$. 
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