CONGRUENCES FOR FRACTIONAL PARTITION FUNCTIONS

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Abstract. The coefficients of the generating function \((q; q)_\infty\) produce \(p_\alpha(n)\) for \(\alpha \in \mathbb{Q}\). In particular, when \(\alpha = -1\), the partition function is obtained. Recently, Chan and Wang (4) identified and proved congruences of the form \(p_\alpha(\ell n + c) \equiv 0 \pmod{\ell}\) where \(\ell\) is a prime such that \(\ell | a - db\) for \(d \in \{4, 6, 8, 10, 14, 26\}\). Expanding upon their work, we use the representation of powers of the Dedekind-eta functions in linear sums of Hecke eigenforms and their lacunarity to raise the power of the modulus to higher powers of \(\ell\). In addition, we generate congruences for when \(d = 2\) employing Hecke algebra.

1. Introduction

A partition of a non-negative integer \(n\) is a non-increasing sequence of positive integers that sum to \(n\). Per usual, let \(p(n)\) denote the number of distinct ways to partition \(n\). Euler discovered the generating function of the partition function to be:

\[
P(q) := \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_\infty},
\]

where \((q; q)_\infty := \prod_{n=1}^{\infty}(1 - q^n)\) is the \(q\)-Pochhammer symbol.

Ramanujan observed congruences relations in \(p(n)\) for \(n\) in special arithmetic progressions. Of the following he proved the first two, and Atkin affirmed the third. (8)

\[
\begin{align*}
p(5n + 4) & \equiv 0 \pmod{5} \\
p(7n + 5) & \equiv 0 \pmod{7} \\
p(11n + 6) & \equiv 0 \pmod{11}.
\end{align*}
\]

In addition to his observations, Ramanujan conjectured that for all powers of \(\ell \in \{5, 7, 11\}\), there exists a congruence relation in which the common difference of the arithmetic progression and the modulus share the same power of \(\ell\). His conjecture was proven to be false in the 1930s when Chawla and Gupta discovered \(7^3\) to be a counterexample. (8) Nonetheless, a slight modification of the conjecture was proven to hold true by Atkin (3) and Watson (10): that for any \(k \in \mathbb{Z}^+\) and a prime \(\ell\), by letting \(\delta_{\ell,k}\) be the unique integer in the set \(\{0, 1, 2, \ldots, \ell^k - 1\}\) such that \(\delta_{\ell,k} \equiv 1/24 \pmod{\ell^k}\), we have that for every nonnegative integer \(n\),

\[
\begin{align*}
p(5^k n + \delta_{5,k}) & \equiv 0 \pmod{5^k}, \\
p(7^k n + \delta_{7,k}) & \equiv 0 \pmod{7^{\lfloor k/2 \rfloor + 1}}, \\
p(11^k n + \delta_{11,k}) & \equiv 0 \pmod{11^k}.
\end{align*}
\]

By relaxing the condition that the common difference of the arithmetic sequence and the modulus have to be powers of the same prime, many more congruences are present. For all primes \(\ell\) co-prime to 6, Ono proved that there exists a congruence relation of the form \(p(An + B) \equiv 0 \pmod{\ell}\) for some \(A, B \in \mathbb{Z}\). (7) Generalizing this result, Ono and Ahlgren proved (2) in the following year that for all integers \(L\) co-prime to 6, there exists \(A, B \in \mathbb{Z}\) such that for all non-negative integers \(n\), \(p(An + B) \equiv 0 \pmod{L}\).
The continued search for congruence relations in the partition function led to the study of congruence relations in the fractional partition functions. The fractional partition function, $P_\alpha(n)$, is the usual partition function raised to the power of $\alpha$ for $\alpha \in \mathbb{Q}$. Throughout the rest of the paper, we let $\alpha = \frac{a}{b}$ where $a$ is a fraction of lowest terms, as well as that $b \geq 1$.

$$P_\alpha(q) := (q;q)_\infty^\alpha =: \sum_{n=0}^\infty p_\alpha(n)q^n.$$  

Unlike $p(n)$ that are integral, $p_\alpha(n)$ is a non-integral rational number for most choices of $n$ and $\alpha$. Therefore, it is imperative to keep in mind that not all modulo are worth exploration. Chan and Wang address this issue in a recent paper (Theorem 1.1 of [4]) through proving that $p_\alpha(n)$ are $\ell$–integer for $\ell$ prime and $\ell \nmid b$. In the rest of this paper, we let $\ell$ always refer to a prime.

In addition to the aforementioned observation, Chan and Wang (Theorem 1.2 of [4]) display an infinite family of congruences for fractional partition functions. To arrive at their results, Chan and Wang took advantage of the well-known formulas that produce the coefficients of an infinite family of congruences for fractional partition functions. To arrive at their results, Chan and Wang can be characterized as the following where as usual, we let $(\cdot)$ denote the Jacobi symbol.

**Theorem 1.1.** Suppose $\ell \mid a - db$ and that integers $d, \ell, r$ satisfy any of the following conditions:

1. $d = 1$ and $(\frac{24r+1}{\ell}) = -1$;
2. $d = 3$ and $(\frac{8r+1}{\ell}) \neq 1$;
3. $d \in \{4, 8, 14\}$, $\ell \equiv 5 \pmod{6}$ and $\ell \mid 24r + d$;
4. $d \in \{6, 10\}$, $\ell \geq 5$, $\ell \equiv 3 \pmod{4}$ and $\ell \mid 24r + d$;
5. $d = 26$, $\ell \equiv 11 \pmod{12}$, and $\ell \mid 24r + d$.

Then, for all non-negative $n$, $p_\alpha(\ell n + r) \equiv 0 \pmod{\ell}$.

Notice that for each $d \in \{4, 6, 8, 10, 14, 26\}$, there are conditions imposed on $\ell$, independent of the choice of $r, r$, and $d$. For example, for $d \in \{6, 10\}$, it is required that $\ell \geq 5$ as well as that $\ell \equiv 3 \pmod{4}$. For each of these $d$, we define a prime $\ell$ to be $d$-satisfactory if $\ell$ satisfies the conditions on $\ell$ that are independent of the choices of $r, r$, and $d$ listed above in the Theorem 1.1.

It is natural to ask about the significance of the list of $d$ in Chan and Wang’s theorem which brings us to a result by Serre on the lacunarity of certain powers of the Dedekind eta-function. The Dedekind eta-function is defined as $\eta(\tau) := q^{1/24}(q;q)_\infty$ where $q := e^{2\pi i \tau}$. In addition, recall that a Fourier expansion $\sum_{n=0}^\infty a(n)q^n$ is lacunary if

$$\lim_{N \to \infty} \frac{\#\{n \leq N : a(n) = 0\}}{N} = 1.$$  

In 1958, Serre proved ([5]) that $\eta(\tau)^d$ is lacunary for even positive integers $d$ if and only if $d \in \{2, 4, 6, 8, 10, 14, 26\}$. In addition, for each of these eta-powers, Serre provided explicit formulas of $\eta(\tau)^d$ as linear combinations of Hecke eigenforms.

While many of the even integers $d$ in Chan-Wang and Serre’s list coincide, it is noticeable that $d = 2$ is missing from Chan-Wang’s list. We delve into this case in this paper. In doing so, we define a prime $\ell$ to be 2-satisfactory if $\ell \not\equiv 1 \pmod{12}$. In addition, we build off of Chan and Wang’s results by raising the modulo $\ell$ in their congruences to higher powers of $\ell$.

Although Chan and Wan assembled an infinite family of congruences for fixed $d$ and modulo $\ell$, they were not able to do so for modulo higher powers of $\ell$. Through Theorem 1.2, we prove congruence relations for higher powers of $\ell$ for $d \in \{4, 6, 8, 10, 14, 26\}$ using the lacunarity of such corresponding $\eta$ powers. In Theorem 1.3, we do the same for $d = 2$. The cases for $d \in \{1, 3\}$ and higher powers of $\ell$ has been discussed by Bevilacqua, Chandran, and the author ([5]), by making use of the well-known Euler and Jacobi identity on $\eta(24\tau)$ and $\eta(8\tau)^3$. 

**Theorem 1.2.** For \( d \in \{4, 6, 8, 10, 14, 26\} \), let \( \ell \) be a \( d \)-satisfactory prime. If \( ord_d(\frac{24}{gcd(d, 24)}+1) = 1 \), then, we have

\[
p_\alpha(\ell^2 n + \delta) \equiv 0 \pmod{\ell^{ord_d(\alpha-d)}}.
\]

A similar but a slightly weaker statement holds true for \( d = 2 \).

**Theorem 1.3.** For \( d = 2 \), let \( \ell \) be a 2-satisfactory prime. If \( ord_d(\frac{24}{gcd(d, 24)}+1) = 1 \), then, we have

\[
p_\alpha(\ell^2 n + \delta) \equiv 0 \pmod{\ell^{ord_d(\alpha-d)-1}}.
\]

Both statements above rely on the lacunarity of the corresponding \( \eta \) functions as the relevant coefficients of the \( \eta \) powers are uniformly 0. For the case of \( d = 2 \), however, more can be said. With an adequate choice of an arithmetic progression and a prime power \( \ell^v \), the coefficients of \( \eta(12\tau)^2 \) are not all uniformly 0 and are multiples of \( \ell^v \). This segways to our last theorem.

**Theorem 1.4.** For \( d = 2 \), fix \( \ell \) and \( v \in \mathbb{Z}^+ \). Then, there exists a finite \( r \) such that when \( ord_d(\alpha - 2) \geq r + v \), and when integer \( \delta \in [0, \ell^r - 1] \) is chosen such that \( ord_d(12\delta + 1) = r \), we have

\[
p_\alpha(\ell^{r+1} n + \delta) \equiv 0 \pmod{\ell^v}.
\]

2. **Preliminaries**

2.1. **Modular Forms.** We first discuss preliminaries regarding modular forms. First, we recall that all modular forms of \( SL_2(\mathbb{Z}) \) are generated by \( E_4 \) and \( E_6 \) where:

\[
E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \quad \text{and}
\]

\[
E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n.
\]

Next, we define the congruence subgroup of \( SL_2(\mathbb{Z}) \), \( \Gamma_0(N) \), of level \( N \).

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.
\]

In addition, let \( M_k(\Gamma_0(N)) \) denote the complex vector space of modular forms of weight \( k \) with respect to \( \Gamma_0 \). If \( \chi \) is a Dirichlet character modulo \( N \), we say that a modular function \( f(\tau) \in M_k(\Gamma_0(N)) \) has Nebentypus character \( \chi \) if for all \( \tau \in \mathbb{H} \) and for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \),

\[
f(\frac{a\tau + b}{c\tau + d}) = \chi(d)(c\tau + d)^k f(\tau).
\]

The space formed by such modular forms is referred to as \( M_k(\Gamma_0(N), \chi) \). In addition, we note that the \( m \)th Hecke operator for \( m \in \mathbb{Z}^+ \), \( T_{m,k,\chi} \), is an endomorphism on \( M_k \). Its action on a Fourier expansion \( f(\tau) = \sum_{n=0}^{\infty} a(n)q^n \) is illustrated by the formula

\[
f(\tau) | T_{m,k,\chi} = \sum_{n=0}^{\infty} \left( \sum_{\delta | (m,n)} \chi(\delta)\delta^{k-1} a(mn/\delta^2) \right) q^n.
\]

When \( m = \ell \) prime, the expression reduces to

\[
f(\tau) | T_{\ell,k,\chi} := \sum_{n=0}^{\infty} \left( a(\ell n) + \chi(\ell)\ell^{k-1} a(n/\ell) \right) q^n.
\]
Recall that a modular form $f(\tau) \in M_k(\Gamma_0(N), \chi)$ is a Hecke eigenform if it is an eigenvector of $T_{m,k,\chi}$ for all $m \geq 1$, i.e. there exist $\lambda(m) \in \mathbb{C}$ such that

$$f(\tau) \mid T_{m,k} = \lambda(m)f(\tau).$$

Suppose that $f = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ is a cuspidal Hecke eigenform. In addition, supposed that $f$ is normalized, i.e. that $a(1) = 1$. Then, it follows that $a(m) = \lambda(m)$ and

$$a(n)a(\ell) = a(n\ell) + \chi(\ell)\ell^{\ell-1}a(\frac{n}{\ell}),$$

where $a(\frac{n}{\ell}) = 0$ for $\ell \nmid n$.

2.2. On the Powers of the Dedekind Eta Function. The Dedekind eta function is defined as:

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{i=1}^{\infty} (1 - q^i).$$

It is known by Martin (6) that $\eta(\tau)^d$ for $d \in \{1, 2, 3, 4, 6, 8, 12, 24\}$ are Hecke eigenforms. In addition, the Nebentypus character, $\chi$, for each eta-power are known by Carney, Etropolksi, and Pitman (Lemma 2.2 of [1]) to be the following.

**Lemma 2.1.** The Nebentypus character for $\eta(\tau)^d$ for integer $d$ is

$$\chi(d) := \begin{cases} (\frac{-1}{d}) & \text{if} \ s \in \mathbb{Z} \\ (\frac{12}{d}) & \text{if} \ s \not\in \mathbb{Z} \cup 3\mathbb{Z} \\ (\frac{1}{d}) & \text{if} \ s \in 3\mathbb{Z} \setminus 2\mathbb{Z}. \end{cases}$$

In addition, Serre proved (9) that $\eta(\tau)^d$ for even integers $d$ is lacunary if and only if $d \in \{2, 4, 6, 8, 10, 14, 26\}$. For each such $d$, he provides explicit formulas of $\eta(\frac{24}{\gcd(d,24)}\tau)^d$ in linear combination of Hecke eigenforms. The expression $\eta(\frac{24}{\gcd(d,24)}\tau)^d$ is an expression of integral powers of $q$. These formula play an integral role in proving our main results, and for this reason, we list the formulas.

First, if $d \in \{2, 4, 6, 8, 12\}$, then $\eta(\tau)^d$ itself is a Hecke eigenform. For $d = 10$, $\eta(12\tau)^{10}$ can be written as a linear combination of two Hecke eigenforms, $E_4(q^{12})\eta(12\tau)^2 \pm 48\eta(12\tau)^{10}$.

$$\eta^{10}(12\tau) = \frac{1}{96}((E_4(q^{12})\eta(12\tau)^2 + 48\eta(12\tau)^{10}) - (E_4(q^{12})\eta(12\tau)^2 - 48\eta(12\tau)^{10}))$$

Similarly, $\eta(12\tau)^{14}$ is a linear combination of two Hecke eigenforms, namely, $E_6(q^{12})\eta(12\tau)^2 \pm 360\sqrt{-3}\eta(12\tau)^{14}$.

$$\eta(12z)^{14} = \frac{1}{720\sqrt{-3}}((E_6(q^{12})\eta(12\tau)^2 + 360\sqrt{-3}\eta(12\tau)^{14})$$

$$- (E_6(q^{12})\eta(12\tau)^2 - 360\sqrt{-3}\eta(12\tau)^{14}))$$

For $d = 26$, $\eta(12\tau)^{26}$ can be written as a sum of four Hecke eigenforms, specifically,

$$E_6^2(12\tau)\eta(12z)^2 + 9398592\eta(12\tau)^{26} \pm 102960\sqrt{-3}E_6\eta(12z)^{14}$$

and

$$E_8^2(12\tau)\eta(12z)^2 - 6910272\eta(12\tau)^{26} \pm 20592E_8\eta(12z)^{10}.$$
Proof of Theorem 1.2. We work out the case for (2.3)
Throughout the paper, we denote $\eta$ and the author, (5), making use of the Euler and Jacobi identities for the $2.3$. Preliminary Results. The first result identifies the modulus that are fruitful to study.

Theorem 2.2. When written in lowest terms,

$$\text{denom}(p_\alpha(n)) = b^n \prod_{p \mid b} p^{\text{ord}_p(n)}. $$

From this theorem, we conclude that for a given rational number $\alpha$, whenever $\text{gcd}(L, b) = 1$, congruences modulo $L$ are meaningful to study. The second result that we state is a technical lemma (Lemma 2.1 of 4) which results from Frobenius endomorphism. This lemma allows us to move exponents through $\eta$-Pochhammer symbols, a crucial step in the proof of our main results.

Lemma 2.3. Let $\ell$ be a prime not dividing $b$. Then, for any $r \geq 1$,

$$(q; q)_\ell^\alpha \equiv (q^\ell; q^{\ell^{r+1}})^{\alpha} \pmod{\ell^r}.$$ 

Congruences modulo higher powers of $\ell$ for $d = 1, 3$ have been studied by Bevilaqua, Chandran, and the author, (3), making use of the Euler and Jacobi identities for the $\eta(24\tau)$ and $\eta(8\tau)^3$.

Theorem 2.4. When $\ell \nmid b$, and $c$ an integer,

(i) if $\left(\frac{2d+1}{\ell}\right) = -1$, then for all $n$ we have

$$p_\alpha(\ell n + c) \equiv 0 \pmod{\ell^{\text{ord}_\ell(c)}};$$

(ii) if $\left(\frac{8c+1}{\ell}\right) \neq 1$, then for all $n$ we have

$$p_\alpha(\ell n + c) \equiv 0 \pmod{\ell^{\text{ord}_\ell(c)}}.$$

3. Proofs of the Main Results

Proof of Theorem 1.2. We work out the case for $d = 4$. Similar conclusions can be made about $d = 6, 8$ following the same steps. First, we write $v := \text{ord}_\ell(\alpha - 4)$ so that $\alpha - 4 = \ell^v u$ for some $u \in \mathbb{Z}(\ell)$. It follows that

$$\sum_{n=0}^{\infty} p_\alpha(n)q^{6n+1} = q(q^6; q^6)^\alpha = q(q^6; q^6)^{\ell^v u+4}$$

$$= q(q^6; q^6)^4(q^6; q^6)^{\ell^v u} = \eta(6z)^4(q^6; q^6)^{\ell^v u}. $$

Now, applying Lemma 2.3, we have

$$\sum_{n=0}^{\infty} p_\alpha(n)q^{6n+1} = \eta(6z)^4(q^6; q^6)^{\ell^v u} \equiv \eta(6z)^4(q^6\ell; q^6\ell)^{\ell^v u-1} \pmod{\ell^v}. $$

Then, we extract terms of the form $q^\ell$ from both sides and replace them with $q$. Recall that we denote $\eta(6z)^4 = \sum_{n=0}^{\infty} a_4(n)q^n$. Now, let $r \in [0, \ell - 1]$ such that $6r + 1 \equiv 0 \pmod{\ell}$. Such $r$ exists
since \( \ell \) is 4-satisfactory, which ensures that \( \ell \) is co-prime with 6. As a result, we have that
\[
\sum_{n=0}^{\infty} p_\alpha(\ell n + r)q^{6n+6r+1} \equiv \sum_{n=0}^{\infty} a_4(\ell n)q^n \cdot (q^6; q^6)_\infty^{\ell n-1}u \quad (\text{mod } \ell^v).
\]

Since \( \ell \) is \( d \)-satisfactory and because \( 6r + 1 \equiv 0 \pmod{\ell} \), it follows from Theorem 1.1 that \( p_\alpha(\ell n + r) \equiv 0 \pmod{\ell} \), allowing us to divide each side with \( \ell \). We now have
\[
\frac{1}{\ell} \cdot \sum_{n=0}^{\infty} p_\alpha(\ell n + r)q^{6n+6r+1} \equiv \frac{1}{\ell} \cdot \sum_{n=0}^{\infty} a_4(\ell n)q^n \cdot (q^6; q^6)_\infty^{\ell n-1}u \quad (\text{mod } \ell^{v-1}).
\]

Now, let \( \gamma \) be an integer such that \( \gamma \in [0, \ell - 1] \). Extracting terms of the form \( q^{\ell n + \gamma} \) from each side, we have that for \( \delta \equiv \frac{\ell - 1}{6} \pmod{\ell^2} \), (Note that \( \delta \equiv \frac{2\ell - 1}{6} \pmod{\ell^2} \) is analogous to the condition on \( \delta \) in the theorem statement, \( \text{ord}_\ell(6\delta + 1) = 1 \).)
\[
p_\alpha(\ell^2n + \delta) \equiv 0 \quad (\text{mod } \ell^v),
\]
which proves the statement of the theorem.

Next, we work out the case of \( d = 10 \). Similar arguments can be made about \( d = 14, 26 \) to arrive at the desired conclusion. Our first steps are nearly analogous to when \( d = 4 \). We start by writing \( r := \text{ord}_\ell(\alpha - 10) \) so that \( \alpha - 10 = \ell^u \cdot u \) for some \( u \in \mathbb{Z}_\ell \). After applying the same beginning steps as with \( d = 4 \), we eventually arrive at
\[
\sum_{n=0}^{\infty} p_\alpha(\ell n + r)q^{12n+12r+\frac{12\ell - 5}{12}} \equiv \sum_{n=0}^{\infty} a_{10}(\ell n)q^n \cdot (q^{12\ell}; q^{12\ell})_\infty^{\ell n-2}u \quad (\text{mod } \ell^v).
\]

Notice that \( (q^{12\ell}; q^{12\ell})_\infty^{\ell n-2}u \) is an expression in terms of \( q^\ell \). In addition, from eq. (2.1), we have
\[
\eta(12\tau)^{10} = \frac{1}{96}(E_4(q^{12})\eta(12\tau)^2 \pm 48\eta(12\tau)^{10}) - (E_4(q^{12})\eta(12\tau)^2 \pm 48\eta(12\tau)^{10})).
\]

First, we note that since \( 96 = 2^5 \cdot 3 \) and \( \ell \) is 10-satisfactory, which guarantees that \( \gcd(96, \ell) = 1 \), \( \frac{1}{96} \) in eq. (2.1) does not stand in our way of deriving the congruences we want. Now, we observe that in the two Hecke eigenforms that we are taking a linear sum of, \( E_4(q^{12})\eta(12\tau)^2 \pm 48\eta(12\tau)^{10} \), because \( E_4(q^{12}) \), \( \eta(12\tau)^2 \), \( \eta(12\tau)^2 \) are expressions in terms of \( q^{12} \), all non-zero coefficients will be the coefficients of \( q^{12i+1} \) for some \( i \) or \( q^{12j+5} \) for some \( j \). As a result, for 10-satisfactory primes \( \ell \), \( a_{10}(\ell) = 0 \). In addition, since \( \eta(12\tau)^{10} \) is a Hecke eigenform, it has multiplicative coefficients for co-prime integers. Therefore, for integers \( n \) co-prime to a 14-satisfactory primes \( \ell \), \( a_{10}(\ell n) = 0 \).

Let \( \gamma \) be an integer such that \( \gamma \in [0, \ell - 1] \). Extracting terms of the form \( q^{\ell n + \gamma} \) from each side, we have that for \( \delta \equiv \frac{\ell - 5}{12} \pmod{\ell^2} \),
\[
p_\alpha(\ell^2n + \delta) \equiv 0 \quad (\text{mod } \ell^v).
\]
Again, we end with a note that \( \delta \equiv \frac{\ell^2 - 5}{12} \) (mod \( \ell^2 \)) is analogous to the condition on \( \delta \) in the theorem statement, \( \text{ord}_d(12\ell + 5) = 1 \).

**Proof of Theorem 1.3.** \( d = 2 \): Write \( v + 1 := \text{ord}_x(\alpha - 2) \) so that \( \alpha - 2 = \ell^{v+1}u \) for some \( u \in \mathbb{Z}_x \). It follows that

\[
\sum_{n=0}^{\infty} p_{\alpha}(n)q^{12n+1} = q(q^{12}; q^{12})_{\infty}^{\alpha} = q(q^{12}; q^{12})_{\infty}^{v+1}u + 2
\]

\[
= q(q^{12}; q^{12})_{\infty}^{2}(q^{12}; q^{12})_{\infty}^{v+1}u = \eta(12z)^{2}(q^{12}; q^{12})_{\infty}^{v+1}u.
\]

Now, applying Lemma 2.3 twice, we have

\[
\sum_{n=0}^{\infty} p_{\alpha}(n)q^{12n+1} \equiv \eta(12z)^{2}(q^{12}; q^{12})_{\infty}^{v+1}u \equiv \eta(12z)^{2}(q^{12}; q^{12})_{\infty}^{v+1}u \pmod{\ell^v}.
\]

Since \( \frac{\eta(12z)^2}{q} \) is an expression of \( q^{12} \), \( a_{2}(\ell) = 0 \) for 2-satisfactory primes \( \ell \). In addition, since \( \eta(12z)^2 \) is a cuspidal Hecke eigenform, for integer \( \gamma \in [0, \ell - 1] \), \( a_{2}(\ell^2n + \ell\gamma) = a_{2}(\ell)a_{2}(\ell n + \gamma) = 0 \).

Now, we extract terms of the form \( q^{\ell^2n+\ell\gamma} \) from both sides. From this step, we conclude that for all non-negative \( n \) where \( \delta \equiv \frac{\ell^2 - 5}{12} \) (mod \( \ell^2 \)),

\[
p_{\alpha}(\ell^2n + \delta) \equiv 0 \pmod{\ell^v}.
\]

Before proving the statement of Theorem 1.4, we prove an auxiliary lemma.

**Lemma 3.1.** Given fixed \( \ell \) and \( v \), there exists a positive integer \( r \), at most \( \ell^{2v} \), such that

\[
a_{2}(\ell^r) \equiv 0 \pmod{\ell^v}.
\]

**Proof of Lemma 3.1.** We note that \( \eta(12\ell)^2 = \sum_{i=0}^{\infty} a(12i + 1)q^{12i+1} \). Therefore, when \( \ell \not\equiv 1 \) (mod 12), the statement of the lemma holds true for \( r = 1 \).

We now focus on primes \( \ell \equiv 1 \) (mod 12). By Lemma 2.1, we deduce that for such \( \ell \), \( \chi^2(\ell) = 1 \). In addition, since \( \eta(12z)^2 \) is a normalized cuspidal Hecke eigenform, for all non-negative integers \( n \), its coefficients satisfy:

\[
a_{2}(n\ell) = a_{2}(n)a_{2}(\ell) - a_{2}(n/\ell).
\]

By letting \( n = \ell^k \), for \( k \in \mathbb{Z}^+ \), in eq. (3.1), we have that

\[
a_{2}(\ell^{k+1}) = a_{2}(\ell^k)a(\ell) - a_{2}(\ell^{k-1}).
\]

Observing the sequence \( a_{2}(\ell^i) \) for \( i \geq 0 \), we note that the sequence is periodic with respect to \( \ell^{2v} \) due to the pigeon hold principle with the length of its period is at most \( \ell^{2v} \). Let \( s \) denote the length of this period.

I claim that the period begins at \( a_{2}(1) \). To prove this claim, assume for the sake of contradiction that the period does not begin at \( a_{2}(1) \). Let the first term of the period be \( a(\ell^c) \) for some \( c > 0 \) and rearrange eq. (3.2) to \( a(\ell^{k+1}) = a(\ell^k)a(\ell) - a(\ell^{k-1}) \). Then,

\[
a_{2}(\ell^{c-1}) \equiv a_{2}(\ell^c)a_{2}(\ell) - a_{2}(\ell^{c+1}) \equiv a_{2}(\ell^{c+s})a_{2}(\ell) - a_{2}(\ell^{c+s+1}) \equiv a_{2}(\ell^{c+s-1}) \pmod{\ell^v}.
\]

This statement is a contradiction to the setup that the period starts at \( a_{12}(\ell^c) \). Thus, the period begins at \( a_{2}(1) \). Now, notice that

\[
a_{2}(\ell^{s-1}) \equiv a_{2}(\ell^s)a_{2}(\ell) - a_{2}(\ell^{s+1}) \equiv a_{2}(\ell^{s-1})a_{12}(\ell) - a_{2}(\ell^s) \equiv 0 \pmod{\ell^v}.
\]

Letting \( r = s - 1 \) in the statement of the lemma completes the proof. \( \Box \)
Proof of Theorem 1.4. $d = 2$: Let $r$ such that $a_2(\ell^n) \equiv 0 \pmod{\ell^n}$ from Lemma 3.1. Write $r+u := \ord_\ell(\alpha-2)$ so that $\alpha-2 = \ell^{r+u}u$ for some $u \in \mathbb{Z}_\ell$. It follows that

$$
\sum_{n=0}^{\infty} p_\alpha(n)q^{12n+1} = q(q^{12}; q^{12})_\infty^{\alpha} = q(q^{12}; q^{12})_\infty^{\ell^{r+u}u+2} = q(q^{12}; q^{12})_\infty^{2}(q^{12}; q^{12})_\infty^{\ell^{r+u}u} = \eta(12z)^2(q^{12}; q^{12})_\infty^{\ell^{r+u}u}.
$$

Now, applying Lemma 2.3 $r+1$ times, we have

$$(3.3) \sum_{n=0}^{\infty} p_\alpha(n)q^{12n+1} = \eta(12z)^2(q^{12}; q^{12})_\infty^{\ell^{r+u}u} \equiv \eta(12z)^2(q^{12}; q^{12})_\infty^{\ell^{r+1}u} (mod \ell^v).$$

Since $\eta(12\tau)^2$ is a cuspidal Hecke eigenform, for integer $\gamma$ such that $\gamma \in [0, \ell - 1],$

$$a_2(\ell^{r+1}n + \ell^r \gamma) = a_2(\ell^r) \equiv 0 \pmod{\ell^v}.$$ 

Now, extracting terms of the form $q^{\ell^{r+1}n + \ell^r \gamma}$ from both sides of eq (3.2), we have that for $\delta \equiv \frac{\ell^{r+1} - 1}{12} (mod \ell^{r+1}),$

$$p_\alpha(\ell^{r+1}n + \delta) \equiv 0 \pmod{\ell^v}. \square$$

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