Functoriality and $K$-theory for $\text{GL}_n(\mathbb{R})$

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Abstract

We investigate base change and automorphic induction $\mathbb{C}/\mathbb{R}$ at the level of $K$-theory for the general linear group $\text{GL}_n(\mathbb{R})$. In the course of this study, we compute in detail the $C^*$-algebra $K$-theory of this disconnected group. We investigate the interaction of base change with the Baum-Connes correspondence for $\text{GL}_n(\mathbb{R})$ and $\text{GL}_n(\mathbb{C})$. This article is the archimedean companion of our previous article [11].

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1 Introduction

In the general theory of automorphic forms, an important role is played by base change and automorphic induction, two examples of the principle of functoriality in the Langlands program [8]. Base change and automorphic induction have a global aspect and a local aspect [1][7]. In this article, we focus on the archimedean case of base change and automorphic induction for the general linear group $\text{GL}(n, \mathbb{R})$, and we investigate these aspects of functoriality at the level of $K$-theory.

For $\text{GL}_n(\mathbb{R})$ and $\text{GL}_n(\mathbb{C})$ we have the Langlands classification and the associated $L$-parameters [8]. We recall that the domain of an $L$-parameter of $\text{GL}_n(F)$ over an archimedean field $F$ is the Weil group $W_F$. The Weil groups are given by

$$W_\mathbb{C} = \mathbb{C}^\times$$

and

$$W_\mathbb{R} = \langle j \rangle \mathbb{C}^\times$$

where $j^2 = -1 \in \mathbb{C}^\times$, $jc = \overline{c}j$ for all $c \in \mathbb{C}^\times$. Base change is defined by restriction of $L$-parameter from $W_\mathbb{R}$ to $W_\mathbb{C}$.

An $L$-parameter $\phi$ is tempered if $\phi(W_F)$ is bounded. Base change therefore determines a map of tempered duals.
In this article, we investigate the interaction of base change with the Baum-Connes correspondence for $\text{GL}_n(\mathbb{R})$ and $\text{GL}_n(\mathbb{C})$.

Let $F$ denote $\mathbb{R}$ or $\mathbb{C}$ and let $G = G(F) = \text{GL}_n(F)$. Let $C^*_r(G)$ denote the reduced $C^*$-algebra of $G$. The Baum-Connes correspondence is a canonical isomorphism [2][5][9]

$$
\mu_F : K^*_G(F)(EG(F)) \to K^*_r(G(F))
$$

where $EG(F)$ is a universal example for the action of $G(F)$.

The noncommutative space $C^*_r(G(F))$ is strongly Morita equivalent to the commutative $C^*$-algebra $C_0(A^t_n(F))$ where $A^t_n(F)$ denotes the tempered dual of $G(F)$, see [12, §1.2][13]. As a consequence of this, we have

$$
K^*_r(G(F)) \cong K^* A^t_n(F).
$$

This leads to the following formulation of the Baum-Connes correspondence:

$$
K^*_G(F)(EG(F)) \cong K^* A^t_n(F).
$$

Base change and automorphic induction $\mathbb{C}/\mathbb{R}$ determine maps

$$
BC_{\mathbb{C}/\mathbb{R}} : A^t_n(\mathbb{R}) \to A^t_n(\mathbb{C})
$$

and

$$
\mathcal{A}^t_{\mathbb{C}/\mathbb{R}} : A^t_n(\mathbb{C}) \to A^t_{2n}(\mathbb{R}).
$$

This leads to the following diagrams

$$
\begin{align*}
&K^*_G(\mathbb{C})(EG(\mathbb{C})) \xrightarrow{\mu_C} K^* A^t_n(\mathbb{C}) \\
&\downarrow \hspace{1cm} \downarrow

&K^*_G(\mathbb{R})(EG(\mathbb{R})) \xrightarrow{\mu_R} K^* A^t_n(\mathbb{R})
\end{align*}
$$

and

$$
\begin{align*}
&K^*_G(\mathbb{R})(EG(\mathbb{R})) \xrightarrow{\mu_R} K^* A^t_{2n}(\mathbb{R}) \\
&\downarrow \hspace{1cm} \downarrow

&K^*_G(\mathbb{C})(EG(\mathbb{C})) \xrightarrow{\mu_C} K^* A^t_n(\mathbb{C})
\end{align*}
$$

where the left-hand vertical maps are the unique maps which make the diagrams commutative.

In section 2 we describe the tempered dual $A^t_n(F)$ as a locally compact Hausdorff space.
In section 3 we compute the \( K \)-theory for the reduced \( C^* \)-algebra of \( \text{GL}(n, \mathbb{R}) \). The real reductive Lie group \( \text{GL}(n, \mathbb{R}) \) is not connected. If \( n \) is even our formulas show that we always have non-trivial \( K_0 \) and \( K_1 \). We also recall the \( K \)-theory for the reduced \( C^* \)-algebra of the complex reductive group \( \text{GL}(n, \mathbb{C}) \), see [13]. In section 4 we recall the Langlands parameters for \( \text{GL}(n) \) over archimedean local fields, see [8]. In section 5 we compute the base change map \( \mathcal{BC} : \mathcal{A}_n^i(\mathbb{R}) \to \mathcal{A}_n^i(\mathbb{C}) \) and prove that \( \mathcal{BC} \) is a continuous proper map. At the level of \( K \)-theory, base change is the zero map for \( n > 1 \) (Theorem 5.3) and is nontrivial for \( n = 1 \) (Theorem 5.5). In section 6, we compute the automorphic induction map \( \mathcal{AI} : \mathcal{A}_n(\mathbb{C}) \to \mathcal{A}_{2n}(\mathbb{R}) \). Contrary to base change, at the level of \( K \)-theory, automorphic induction is nontrivial for every \( n \) (Theorem 6.5). In section 7, where we study the case \( n = 1 \), base change for \( K^1 \) creates a map

\[
\mathcal{R}(U(1)) \longrightarrow \mathcal{R}(\mathbb{Z}/2\mathbb{Z})
\]

where \( \mathcal{R}(U(1)) \) is the representation ring of the circle group \( U(1) \) and \( \mathcal{R}(\mathbb{Z}/2\mathbb{Z}) \) is the representation ring of the group \( \mathbb{Z}/2\mathbb{Z} \). This map sends the trivial character of \( U(1) \) to \( 1 \oplus \epsilon \), where \( \epsilon \) is the nontrivial character of \( \mathbb{Z}/2\mathbb{Z} \), and sends all the other characters of \( U(1) \) to zero.

This map has an interpretation in terms of \( K \)-cycles. The \( K \)-cycle

\[
(C_0(\mathbb{R}), L^2(\mathbb{R}), id/dx)
\]

is equivariant with respect to \( \mathbb{C}^\times \) and \( \mathbb{R}^\times \), and therefore determines a class \( \mathcal{C} \in K^1_{\mathbb{C}^\times}(\mathbb{R}^\times) \) and a class \( \mathcal{C} \in K^1_{\mathbb{R}^\times}(\mathbb{R}^\times) \). On the left-hand-side of the Baum-Connes correspondence, base change in dimension 1 admits the following description in terms of Dirac operators:

\[
\mathcal{C} \mapsto (\mathcal{C}, \mathcal{C})
\]

This extends the results of [11] to archimedean fields.

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2 On the tempered dual of \( \text{GL}(n) \)

Let \( F = \mathbb{R} \). In order to compute the \( K \)-theory of the reduced \( C^* \)-algebra of \( \text{GL}(n, F) \) we need to parametrize the tempered dual \( \mathcal{A}_n^i(F) \) of \( GL(n, F) \).

Let \( M \) be a standard Levi subgroup of \( \text{GL}(n, F) \), i.e. a block-diagonal subgroup. Let \( 0M \) be the subgroup of \( M \) such that the determinant of each block-diagonal is \( \pm 1 \). Denote by \( X(M) = \overline{M/0M} \) the group of unramified characters of \( M \), consisting of those characters which are trivial on \( 0M \).
Let $W(M) = N(M)/M$ denote the Weyl group of $M$. $W(M)$ acts on the
discrete series $E_2(0M)$ of $^0M$ by permutations.

Now, choose one element $\sigma \in E_2(0M)$ for each $W(M)$-orbit. The *isotropy subgroup* of $W(M)$ is defined to be

$$W_\sigma(M) = \{ \omega \in W(M) : \omega.\sigma = \sigma \}.$$  

Form the disjoint union

$$\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) = \bigsqcup_M \bigsqcup_{\sigma \in E_2(0M)} X(M)/W_\sigma(M). \tag{1}$$

The disjoint union has the structure of a locally compact, Hausdorff space
and is called the *Harish-Chandra parameter space*. The parametrization of
the tempered dual $\mathcal{A}_n^t(\mathbb{R})$ is due to Harish-Chandra, see [10].

**Proposition 2.1.** There exists a bijection

$$\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) \rightarrow \mathcal{A}_n^t(\mathbb{R})$$

$$\chi^\sigma \mapsto i_{GL(n),MN}(\chi^\sigma \otimes 1),$$

where $\chi^\sigma(x) := \chi(x)\sigma(x)$ for all $x \in M$.

In view of the above bijection, we will denote the Harish-Chandra param-
eter space by $\mathcal{A}_n^t(\mathbb{R})$.

We will see now the particular features of the archimedean case, starting
with $GL(n, \mathbb{R})$. Since the discrete series of $GL(n, \mathbb{R})$ is empty for $n \geq 3$, we
only need to consider partitions of $n$ into 1’s and 2’s. This allows us to to
decompose $n$ as $n = 2q + r$, where $q$ is the number of 2’s and $r$ is the number
of 1’s in the partition. To this decomposition we associate the partition

$$n = (\underbrace{2, \ldots, 2}_q, 1, \ldots, 1)_r,$$

which corresponds to the Levi subgroup

$$M \cong \underbrace{GL(2, \mathbb{R}) \times \ldots \times GL(2, \mathbb{R})}_{q} \times \underbrace{GL(1, \mathbb{R}) \times \ldots \times GL(1, \mathbb{R})}_{r}.$$  

Varying $q$ and $r$ we determine a representative in each equivalence class
of Levi subgroups. The subgroup $^0M$ of $M$ is given by

$$^0M \cong \underbrace{SL^+(2, \mathbb{R}) \times \ldots \times SL^+(2, \mathbb{R})}_{q} \times \underbrace{SL^+(1, \mathbb{R}) \times \ldots \times SL^+(1, \mathbb{R})}_{r}.$$
where 

\[ SL^\pm(m, \mathbb{R}) = \{ g \in GL(m, \mathbb{R}) : |\det(g)| = 1 \} \]

is the unimodular subgroup of \( GL(m, \mathbb{R}) \). In particular, \( SL^\pm(1, \mathbb{R}) = \{ \pm 1 \} \cong \mathbb{Z}/2\mathbb{Z} \).

The representations in the discrete series of \( GL(2, \mathbb{R}) \), denoted \( \mathcal{D}_\ell \) for \( \ell \in \mathbb{N} \) (\( \ell \geq 1 \)) are induced from \( SL(2, \mathbb{R}) \) [8, p.399]:

\[ \mathcal{D}_\ell = \text{ind}_{SL^\pm(2, \mathbb{R}), SL(2, \mathbb{R})} (\mathcal{D}^\pm_\ell), \]

where \( \mathcal{D}^\pm_\ell \) acts in the space

\[ \{ f : \mathcal{H} \to \mathbb{C} | f \text{ analytic}, \|f\|^2 = \int \int |f(z)|^2 y^{\ell-1} dx dy < \infty \}. \]

Here, \( \mathcal{H} \) denotes the Poincaré upper half plane. The action of \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is given by

\[ \mathcal{D}^\pm_\ell(g)(f(z)) = (bz + d)^{-\ell+1} f\left( \frac{az + c}{bz + d} \right). \]

More generally, an element \( \sigma \) from the discrete series \( E_2(0M) \) is given by

\[ \sigma = i_{G,MN}(\mathcal{D}^\pm_{\ell_1} \otimes \ldots \otimes \mathcal{D}^\pm_{\ell_q} \otimes \tau_1 \otimes \ldots \otimes \tau_r \otimes 1), \quad (2) \]

where \( \mathcal{D}^\pm_{\ell_i} (\ell_i \geq 1) \) are the discrete series representations of \( SL^\pm(2, \mathbb{R}) \) and \( \tau_j \) is a representation of \( SL^\pm(1, \mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z} \), i.e. \( id = (x \mapsto x) \) or \( sgn = (x \mapsto \frac{x}{|x|}) \).

Finally we will compute the unramified characters \( X(M) \), where \( M \) is the Levi subgroup associated to the partition \( n = 2q + r \).

Let \( x \in GL(2, \mathbb{R}) \). Any character of \( GL(2, \mathbb{R}) \) is given by

\[ \chi(\det(x)) = (\text{sgn}(\det(x)))^t |\det(x)|^i \]

(\( \varepsilon = 0, 1, t \in \mathbb{R} \)) and it is unramified provided that

\[ \chi(\det(g)) = \chi(\pm 1) = (\pm 1)^i = 1, \text{ for all } g \in SL^\pm(2, \mathbb{R}). \]

This implies \( \varepsilon = 0 \) and any unramified character of \( GL(2, \mathbb{R}) \) has the form

\[ \chi(x) = |\det(x)|^i, \text{ for some } t \in \mathbb{R}. \quad (3) \]

Similarly, any unramified character of \( GL(1, \mathbb{R}) = \mathbb{R}^\times \) has the form

\[ \xi(x) = |x|^i, \text{ for some } t \in \mathbb{R}. \quad (4) \]
Given a block diagonal matrix $diag(g_1, ..., g_q, \omega_1, ..., \omega_r) \in M$, where $g_i \in GL(2, \mathbb{R})$ and $\omega_j \in GL(1, \mathbb{R})$, we conclude from (3) and (4) that any unramified character $\chi \in X(M)$ is given by

$$\chi(diag(g_1, ..., g_q, \omega_1, ..., \omega_r)) = \prod_{i=1}^{q} |\det(g_i)|^{t_i_1} \times \prod_{j=1}^{r} |\det(\omega_j)|^{t_j_1} \times \prod_{j=1}^{r} |\det(\omega_j)|^{t_j_q},$$

for some $(t_1, ..., t_{q+r}) \in \mathbb{R}^{q+r}$. We can denote such element $\chi \in X(M)$ by $\chi(t_1, ..., t_{q+r})$. We have the following result.

**Proposition 2.2.** Let $M$ be a Levi subgroup of $GL(n, \mathbb{R})$, associated to the partition $n = 2q + r$. Then, there is a bijection

$$X(M) \rightarrow \mathbb{R}^{q+r}, \chi(t_1, ..., t_{q+r}) \mapsto (t_1, ..., t_{q+r}).$$

Let us consider now $GL(n, \mathbb{C})$. The tempered dual of $GL(n, \mathbb{C})$ comprises the unitary principal series in accordance with Harish-Chandra [6, p. 277]. The corresponding Levi subgroup is a maximal torus $T \approx (\mathbb{C}^\times)^n$. It follows that $^0T \approx \mathbb{T}^n$ the compact $n$-torus.

The principal series representations are given by

$$\pi_{\ell, it} = i_{G,T,U}(\sigma \otimes 1), \quad (5)$$

where $\sigma = \sigma_1 \otimes ... \otimes \sigma_n$ and $\sigma_j(z) = (\frac{z}{|z|})^{\ell_j} |z|^{t_j}$ ($\ell_j \in \mathbb{Z}$ and $t_j \in \mathbb{R}$).

An unramified character is given by

$$\chi \left( \begin{array}{c} z_1 \\ \vdots \\ z_n \end{array} \right) = |z_1|^{t_{1_1}} \times ... \times |z_n|^{t_{n_1}}$$

and we can represent $\chi$ as $\chi(t_1, ..., t_n)$. Therefore, we have the following result.

**Proposition 2.3.** Denote by $T$ the standard maximal torus in $GL(n, \mathbb{C})$. There is a bijection

$$X(T) \rightarrow \mathbb{R}^{n}, \chi(t_1, ..., t_{n}) \mapsto (t_1, ..., t_{n}).$$

### 3 K-theory for $GL(n)$

Using the Harish-Chandra parametrization of the tempered dual of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ (recall that the Harish-Chandra parameter space is a locally compact, Hausdorff topological space) we can compute the $K$-theory of the reduced $C^*$-algebras $C^*_rGL(n, \mathbb{R})$ and $C^*_rGL(n, \mathbb{C})$. 

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6
3.1 \textit{K}-theory for $\text{GL}(n, \mathbb{R})$

We exploit the strong Morita equivalence described in [12, §1.2]. We infer that
\[
K_j(C^*_r \text{GL}(n, \mathbb{R})) = K^j(\bigsqcup_{(\mathcal{M}, \sigma)} X(M) / W_\sigma(M)) = \bigoplus_{(\mathcal{M}, \sigma)} K^j(X(M) / W_\sigma(M)),
\]
where $n_M = q + r$ if $M$ is a representative of the equivalence class of Levi subgroup associated to the partition $n = 2q + r$. Hence the $K$-theory depends on $n$ and on each Levi subgroup.

To compute (6) we have to consider the following orbit spaces:

(i) $\mathbb{R}^n$, in which case $W_\sigma(M)$ is the trivial subgroup of the Weil group $W(M)$;

(ii) $\mathbb{R}^n / S_n$, where $W_\sigma(M) = W(M)$ (this is one of the possibilities for the partition of $n$ into 1’s);

(iii) $\mathbb{R}^n / (S_{n_1} \times ... \times S_{n_k})$, where $W_\sigma(M) = S_{n_1} \times ... \times S_{n_k} \subset W(M)$ (see the examples below).

Definition 3.1. An orbit space as indicated in (ii) and (iii) is called a closed cone.

The $K$-theory for $\mathbb{R}^n$ may be summarized as follows
\[
K^j(\mathbb{R}^n) = \begin{cases} 
\mathbb{Z} & \text{if } n = j \mod 2 \\
0 & \text{otherwise} \end{cases}.
\]

The next results show that the $K$-theory of a closed cone vanishes.

Lemma 3.2. $K^j(\mathbb{R}^n / S_n) = 0, j = 0, 1$.

Proof. We need the following definition. A point $(a_1, ..., a_n) \in \mathbb{R}^n$ is called normalized if $a_j \leq a_{j+1}$, for $j = 1, 2, ..., n - 1$. Therefore, in each orbit there is exactly one normalized point and $\mathbb{R}^n / S_n$ is homeomorphic to the subset of $\mathbb{R}^n$ consisting of all normalized points of $\mathbb{R}^n$. We denote the set of all normalized points of $\mathbb{R}^n$ by $N(\mathbb{R}^n)$.

In the case of $n = 2$, let $(a_1, a_2)$ be a normalized point of $\mathbb{R}^2$. Then, there is a unique $t \in [1, +\infty[$ such that $a_2 = ta_1$ and the map
\[
\mathbb{R} \times [1, +\infty[ \to N(\mathbb{R}^2), (a, t) \mapsto (a, ta)
\]
is a homeomorphism.
If \( n > 2 \) then the map
\[
N(\mathbb{R}^{n-1}) \times [1, +\infty[ \to N(\mathbb{R}^n), (a_1, \ldots, a_{n-1}, t) \mapsto (a_1, \ldots, a_{n-1}, ta_n)
\]
is a homeomorphism. Since \([1, +\infty[\) kills both the \(K\)-theory groups \(K^0\) and \(K^1\), the result follows by applying Künneth formula.

The symmetric group \(S_n\) acts on \(\mathbb{R}^n\) by permuting the components. This induces an action of any subgroup \(S_{n_1} \times \ldots \times S_{n_k}\) of \(S_n\) on \(\mathbb{R}^n\). Write
\[
\mathbb{R}^n \cong \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_k} \times \mathbb{R}^{n_1-\ldots-n_k}.
\]
If \( n = n_1 + \ldots + n_k \) then we simply have \(\mathbb{R}^n \cong \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k}\).

The group \(S_{n_1} \times \ldots \times S_{n_k}\) acts on \(\mathbb{R}^n\) as follows.
\(S_{n_1}\) permutes the components of \(\mathbb{R}^{n_1}\) leaving the remaining fixed;
\(S_{n_2}\) permutes the components of \(\mathbb{R}^{n_2}\) leaving the remaining fixed; and so on. If \( n > n_1 + \ldots + n_k \) the components of \(\mathbb{R}^{n_1-\ldots-n_k}\) remain fixed. This can be interpreted, of course, as the action of the trivial subgroup. As a consequence, one identifies the orbit spaces
\[
\mathbb{R}^n/(S_{n_1} \times \ldots \times S_{n_k}) \cong \mathbb{R}^{n_1}/S_{n_1} \times \ldots \times \mathbb{R}^{n_k}/S_{n_k} \times \mathbb{R}^{n_1-\ldots-n_k}.
\]

**Lemma 3.3.** \(K^j(\mathbb{R}^n/(S_{n_1} \times \ldots \times S_{n_k})) = 0, j = 0, 1,\) where \(S_{n_1} \times \ldots \times S_{n_k} \subset S_n\).

**Proof.** It suffices to prove for \(\mathbb{R}^n/(S_{n_1} \times S_{n_2})\). The general case follows by induction on \(k\).

Now, \(\mathbb{R}^n/(S_{n_1} \times S_{n_2}) \cong \mathbb{R}^{n_1}/S_{n_1} \times \mathbb{R}^{n_2-n_1}/S_{n_2}\). Applying the Künneth formula and Lemma 3.2, the result follows.

We give now some examples by computing \(K^j C^\ast_r G(n, \mathbb{R})\) for small \(n\).

**Example 3.4.** We start with the case of \(GL(1, \mathbb{R})\). We have:
\[
M = \mathbb{R}^\times, \quad 0M = \mathbb{Z}/2\mathbb{Z}, \quad W(M) = 1 \quad \text{and} \quad X(M) = \mathbb{R}.
\]

Hence,
\[
A^1_1(\mathbb{R}) \cong \bigsqcup_{\sigma \in (\mathbb{Z}/2\mathbb{Z})} \mathbb{R}/1 = \mathbb{R} \sqcup \mathbb{R}, \quad (\ast)
\]
and the \(K\)-theory is given by
\[
K^j C^r_1 GL(1, \mathbb{R}) \cong K^j(A^1_1(\mathbb{R})) = K^j(\mathbb{R} \sqcup \mathbb{R}) = K^j(\mathbb{R}) \oplus K^j(\mathbb{R}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & , j = 1 \\ 0 & , j = 0. \end{cases}
\]
Example 3.5. For $GL(2, \mathbb{R})$ we have two partitions of $n = 2$ and the following data

| Partition | $M$ | $^0M$ | $W(M)$ | $X(M)$ | $\sigma \in E_2(^0M)$ |
|-----------|-----|-------|---------|--------|---------------------|
| $2+0$     | $GL(2, \mathbb{R})$ | $SL^\pm(2, \mathbb{R})$ | $\mathbb{R}$ | $\mathbb{R}^2$ | $\sigma = i_{G, P}(D^\ell_\pm), \ell \in \mathbb{N}$ |
| $1+1$     | $(\mathbb{R}^\times)^2$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{R}^2$ | $\sigma = i_{G, P}(\text{id} \otimes \text{sgn})$ |

Then the tempered dual is parameterized as follows

$$A^t_2(\mathbb{R}) \cong \bigsqcup_{(M, \sigma)} X(M)/W_\sigma(M) = \bigsqcup_{\ell \in \mathbb{N}} \mathbb{R} \sqcup (\mathbb{R}^2/\mathbb{Z}_2) \sqcup (\mathbb{R}^2/\mathbb{Z}_2) \sqcup \mathbb{R}^2,$$

and the $K$-theory groups are given by

$$K^*_j C^*_r GL(2, \mathbb{R}) \cong K^j(A^t_2(\mathbb{R})) \cong \bigoplus_{\ell \in \mathbb{N}} K^j(\mathbb{R}) \oplus K^j(\mathbb{R}^2) = \left\{ \begin{array}{ll} \bigoplus_{\ell \in \mathbb{N}} \mathbb{Z} & , j = 1 \\ \mathbb{Z} & , j = 0. \end{array} \right.$$ 

Example 3.6. For $GL(3, \mathbb{R})$ there are two partitions for $n = 3$, to which correspond the following data

| Partition | $M$ | $^0M$ | $W(M)$ | $X(M)$ |
|-----------|-----|-------|---------|--------|
| $2+1$     | $GL(2, \mathbb{R}) \times \mathbb{R}^\times$ | $SL^\pm(2, \mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})$ | $\mathbb{R}^2$ | $\mathbb{R}^3$ |
| $1+1+1$   | $(\mathbb{R}^\times)^3$ | $(\mathbb{Z}/2\mathbb{Z})^3$ | $S_3$ | $\mathbb{R}^3$ |

For the partition $3 = 2 + 1$, an element $\sigma \in E_2(^0M)$ is given by

$$\sigma = i_{G, P}(D^\ell_\pm \otimes \tau), \ell \in \mathbb{N} \text{ and } \tau \in (\mathbb{Z}/2\mathbb{Z}).$$

For the partition $3 = 1 + 1 + 1$, an element $\sigma \in E_2(^0M)$ is given by

$$\sigma = i_{G, P}(\prod_{i=1}^3 \tau_i), \tau_i \in (\mathbb{Z}/2\mathbb{Z}).$$

The tempered dual is parameterized as follows

$$A^t_3(\mathbb{R}) \cong \bigsqcup_{(M, \sigma)} X(M)/W_\sigma(M) = \bigsqcup_{N \times (\mathbb{Z}/2\mathbb{Z})} (\mathbb{R}^2/1) \bigcup_{(\mathbb{Z}/2\mathbb{Z})^3} (\mathbb{R}^3/S_3).$$

The $K$-theory groups are given by

$$K^*_j C^*_r GL(3, \mathbb{R}) \cong K^j(A^t_3(\mathbb{R})) \cong \bigoplus_{N \times (\mathbb{Z}/2\mathbb{Z})} K^j(\mathbb{R}) \oplus 0 = \left\{ \begin{array}{ll} \bigoplus_{N \times (\mathbb{Z}/2\mathbb{Z})} \mathbb{Z} & , j = 0 \\ 0 & , j = 1. \end{array} \right.$$
The general case of $GL(n, \mathbb{R})$ will now be considered. It can be split in two cases: $n$ even and $n$ odd.

- $n = 2q$ even

Suppose $n$ is even. For every partition $n = 2q + r$, either $W_\sigma(M) = 1$ or $W_\sigma(M) \neq 1$. If $W_\sigma(M) \neq 1$ then $\mathbb{R}^{n_M}/W_\sigma(M)$ is a cone and the $K$-groups $K^0$ and $K^1$ both vanish. This happens precisely when $r > 2$ and therefore we have only two partitions, corresponding to the choices of $r = 0$ and $r = 2$, which contribute to the $K$-theory with non-zero $K$-groups

| Partition | $M$ | $^0M$ | $W(M)$ |
|-----------|-----|-------|--------|
| 2q        | $GL(2, \mathbb{R})^q$ | $SL^\pm(2, \mathbb{R})^q$ | $S_q$ |
| 2(q - 1) + 2 | $GL(2, \mathbb{R})^{q-1} \times (\mathbb{R}^\times)^2$ | $SL^\pm(2, \mathbb{R})^{q-1} \times (\mathbb{Z}/2\mathbb{Z})^2$ | $S_{q-1} \times (\mathbb{Z}/2\mathbb{Z})$ |

We also have $X(M) \cong \mathbb{R}^q$ for $n = 2q$, and $X(M) \cong \mathbb{R}^{q+1}$, for $n = 2(q - 1) + 2$.

For the partition $n = 2q$ ($r = 0$), an element $\sigma \in E_2(0M)$ is given by

$$\sigma = i_{G,P}(D_{\ell_1}^+ \otimes \ldots \otimes D_{\ell_q}^+) \text{, } (\ell_1, \ldots, \ell_q) \in \mathbb{N}^q \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$  

For the partition $n = 2(q - 1) + 2$ ($r = 2$), an element $\sigma \in E_2(0M)$ is given by

$$\sigma = i_{G,P}(D_{\ell_1}^+ \otimes \ldots \otimes D_{\ell_{q-1}}^+ \otimes id \otimes sgn) \text{, } (\ell_1, \ldots, \ell_{q-1}) \in \mathbb{N}^{q-1} \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$  

Therefore, the tempered dual has the following form

$$A_n^t(\mathbb{R}) = A_{2q}^t(\mathbb{R}) = \left( \bigcup_{\ell \in \mathbb{N}^q} \mathbb{R}^{\ell} \right) \sqcup \left( \bigcup_{\ell' \in \mathbb{N}^{q-1}} \mathbb{R}^{\ell'+1} \right) \sqcup C$$

where $C$ is a disjoint union of closed cones as in Definition 3.1.

**Theorem 3.7.** Suppose $n = 2q$ is even. Then the $K$-groups are

$$K_jC_r^*GL(n, \mathbb{R}) \cong \left\{ \begin{array}{ll}
\bigoplus_{\ell \in \mathbb{N}} \mathbb{Z} & \text{if } j \equiv q \text{(mod } 2) \\
\bigoplus_{\ell \in \mathbb{N}^{q-1}} \mathbb{Z} & \text{otherwise.}
\end{array} \right.$$  

If $q = 1$ then the direct sum $\bigoplus_{\ell \in \mathbb{N}^{q-1}} \mathbb{Z}$ will denote a single copy of $\mathbb{Z}$.

- $n = 2q + 1$ odd

If $n$ is odd only one partition contributes to the $K$-theory of $GL(n, \mathbb{R})$ with non-zero $K$-groups:
An element $\sigma \in E_2^{(0)M}$ is given by

$$\sigma = i_{G,P}(D^\dagger_{\ell_1} \otimes \cdots \otimes D^\dagger_{\ell_q} \otimes \tau), \ (\ell_1, \ldots, \ell_q, \tau) \in \mathbb{N}^q \times (\mathbb{Z}/2\mathbb{Z}) \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$  

The tempered dual is given by

$$A_{\mathfrak{t}}^n(R) = A_{\mathfrak{t}}^{2q+1}(R) = \bigoplus_{\ell \in (\mathbb{N}^q \times (\mathbb{Z}/2\mathbb{Z}))} \mathbb{R}^{q+1} \sqcup C \text{ where } C \text{ is a disjoint union of closed cones as in Definition 3.1.}$$

**Theorem 3.8.** Suppose $n = 2q + 1$ is odd. Then the $K$-groups are

$$K_j C^*_rGL(n, \mathbb{R}) \cong \bigoplus_{\ell \in (\mathbb{N}^q \times (\mathbb{Z}/2\mathbb{Z}))} \mathbb{Z} \text{ if } j \equiv q + 1 \text{ (mod 2)}, \ 0 \text{ otherwise.}$$

Here, we use the following convention: if $q = 0$ then the direct sum is $\bigoplus_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}.$

We conclude that the $K$-theory of $C^*_rGL(n, \mathbb{R})$ depends on essentially one parameter $q$ given by the maximum number of $2'$s in the partitions of $n$ into $1'$s and $2'$s. If $n$ is even then $q = \frac{n}{2}$ and if $n$ is odd then $q = \frac{n-1}{2}$.

### 3.2 $K$-theory for $GL(n, \mathbb{C})$

Let $T^\circ$ be the maximal compact subgroup of the maximal compact torus $T$ of $GL(n, \mathbb{C})$. Let $\sigma$ be a unitary character of $T^\circ$. We note that $W = W(T)$, $W_\sigma = W_\sigma(T)$. If $W_\sigma = 1$ then we say that the orbit $W \cdot \sigma$ is generic.

**Theorem 3.9.** The $K$-theory of $C^*_rGL(n, \mathbb{C})$ admits the following description. If $n = j \text{ mod 2}$ then $K_j$ is free abelian on countably many generators, one for each generic $W$-orbit in the unitary dual of $T^\circ$, and $K_{j+1} = 0$.

**Proof.** We exploit the strong Morita equivalence described in [13, Prop. 4.1]. We have a homeomorphism of locally compact Hausdorff spaces:

$$A^\ell_n(\mathbb{C}) \cong \bigsqcup X(T)/W_\sigma(T)$$

by the Harish-Chandra Plancherel Theorem for complex reductive groups [6], and the identification of the Jacobson topology on the left-hand-side with the natural topology on the right-hand-side, as in [13]. The result now follows from Lemma 4.3. \qed
4 Langlands parameters for $GL(n)$

The Weil group of $\mathbb{C}$ is simply

$$W_{\mathbb{C}} \cong \mathbb{C}^\times,$$

and the Weil group of $\mathbb{R}$ can be written as disjoint union

$$W_{\mathbb{R}} \cong \mathbb{C}^\times \sqcup j\mathbb{C}^\times,$$

where $j^2 = -1$ and $jcj^{-1} = \overline{c}$ ($\overline{c}$ denotes complex conjugation). We shall use this disjoint union to describe the representation theory of $W_{\mathbb{R}}$.

**Definition 4.1.** An $L$-parameter is a continuous homomorphism

$$\phi : W_F \to GL(n, \mathbb{C})$$

such that $\phi(w)$ is semisimple for all $w \in W_F$.

$L$-parameters are also called Langlands parameters. Two $L$-parameters are equivalent if they are conjugate under $GL(n, \mathbb{C})$. The set of equivalence classes of $L$-parameters is denoted by $\mathcal{G}_n$. And the set of equivalence classes of $L$-parameters whose image is bounded is denoted by $\mathcal{G}_{t,n}$.

Let $F$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{A}_n(F)$ (resp. $\mathcal{A}^{t}_{n}(F)$) denote the smooth dual (resp. tempered dual) of $GL(n, F)$. The local Langlands correspondence is a bijection

$$\mathcal{G}_n(F) \to \mathcal{A}_n(F).$$

In particular,

$$\mathcal{G}^{t}_{n}(F) \to \mathcal{A}^{t}_{n}(F)$$

is also a bijection.

We are only interested in $L$-parameters whose image is bounded. In the sequel we will refer to them, for simplicity, as $L$-parameters.

**$L$-parameters for $W_{\mathbb{C}}$**

A 1-dimensional $L$-parameter for $W_{\mathbb{C}}$ is simply a character of $\mathbb{C}^\times$ (i.e. a unitary quasicharacter):

$$\chi(z) = \left(\frac{z}{|z|}\right)^\ell \otimes |z|^t$$

where $|z| = |z|_{\mathbb{C}} = z\overline{z}$, $\ell \in \mathbb{Z}$ and $t \in \mathbb{R}$. To emphasize the dependence on parameters $(\ell, t)$ we write sometimes $\chi = \chi_{\ell, t}$. 

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An $n$-dimensional $L$-parameter can be written as a direct sum of $n$ $1$-dimensional characters of $\mathbb{C}^\times$:

$$\phi = \phi_1 \oplus \ldots \oplus \phi_n,$$

with $\phi_k(z) = (\frac{x}{|x|})^{\ell_k} \otimes |x|^{t_k}, \ell_k \in \mathbb{Z}, t_k \in \mathbb{R}, k = 1, \ldots, n$.

$L$-parameters for $W_\mathbb{R}$

The $1$-dimensional $L$-parameters for $W_\mathbb{R}$ are as follows

$$\begin{align*}
\phi_{\varepsilon,t}(z) &= |z|_\mathbb{R}^t, \varepsilon \in \{0, 1\}, t \in \mathbb{R}.
\phi_{\varepsilon,t}(j) &= (-1)^\varepsilon.
\end{align*}$$

We may now describe the local Langlands correspondence for $GL(1, \mathbb{R})$:

$$\begin{align*}
\phi_{0,t} &\mapsto 1 \otimes |.|^t_{\mathbb{R}}
\phi_{1,t} &\mapsto sgn \otimes |.|^t_{\mathbb{R}}
\end{align*}$$

Now, we consider $2$-dimensional $L$-parameters for $W_\mathbb{R}$:

$$\phi_{\ell,t}(z) = \begin{pmatrix}
\chi_{\ell,t}(z) & 0 \\
0 & \overline{\chi}_{\ell,t}(z)
\end{pmatrix}, \phi_{\ell,t}(j) = \begin{pmatrix}
0 & (-1)^\ell \\
1 & 0
\end{pmatrix},$$

with $\ell \in \mathbb{Z}$ and $t \in \mathbb{R}$.

and

$$\phi_{m,t,n,s}(z) = \begin{pmatrix}
\chi_{0,t}(z) & 0 \\
0 & \chi_{0,s}(z)
\end{pmatrix}, \phi_{m,t,n,s}(j) = \begin{pmatrix}
(-1)^m & 0 \\
0 & (-1)^n
\end{pmatrix},$$

with $m, n \in \{0, 1\}$ and $t, s \in \mathbb{R}$.

The local Langlands correspondence for $GL(2, \mathbb{R})$ may be described as follows.

The $L$-parameter $\phi_{m_1,t_1,m_2,t_2}$ corresponds, via Langlands correspondence, to the unitary principal series:

$$\phi_{m_1,t_1,m_2,t_2} \mapsto \pi(\mu_1, \mu_2),$$

where $\mu_i$ is the character of $\mathbb{R}^\times$ given by

$$\mu_i(x) = (\frac{x}{|x|})^{m_i}|x|^{t_i}, m_i \in \{0, 1\}, t_i \in \mathbb{R}.$$
The $L$-parameter $\phi_{\ell,t}$ corresponds, via the Langlands correspondence, to the discrete series:

$$\phi_{\ell,t} \mapsto D_\ell \otimes |\text{det}(\cdot)|^{it}, \quad \text{with} \quad \ell \in \mathbb{N}, t \in \mathbb{R}.$$  

**Proposition 4.2.** (i) $\phi_{\ell,t} \cong \phi_{-\ell,t}$;  
(ii) $\phi_{\ell,m,s,t} \cong \phi_{m,s,\ell,t}$;  
(iii) $\phi_{0,t} \cong \phi_{1,t,0,t} \cong \phi_{0,t,1,t}$; 

The proof is elementary. We now quote the following result.

**Lemma 4.3.** Every finite-dimensional semi-simple representation $\phi$ of $W_\mathbb{R}$ is fully reducible, and each irreducible representation has dimension one or two.

## 5 Base change

We may state the base change problem for archimedean fields in the following way. Consider the archimedean base change $\mathbb{C}/\mathbb{R}$. We have $W_\mathbb{C} \subset W_\mathbb{R}$ and there is a natural map

$$\text{Res}^{W_\mathbb{C}}_{W_\mathbb{R}} : \mathcal{G}_n(\mathbb{R}) \longrightarrow \mathcal{G}_n(\mathbb{C}) \quad (8)$$

called restriction. By the local Langlands correspondence for archimedean fields [3, Theorem 3.1, p.236][8], there is a base change map

$$\mathcal{B} \mathcal{C} : \mathcal{A}_n(\mathbb{R}) \longrightarrow \mathcal{A}_n(\mathbb{C}) \quad (9)$$

such that the following diagram commutes

$$\begin{array}{ccc} 
\mathcal{G}_n(\mathbb{R}) & \xrightarrow{\text{Res}^{W_\mathbb{C}}_{W_\mathbb{R}}} & \mathcal{G}_n(\mathbb{C}) \\
\downarrow_{\mathcal{L}_n} & & \downarrow_{\mathcal{L}_n} \\
\mathcal{A}_n(\mathbb{R}) & \xrightarrow{\mathcal{B} \mathcal{C}} & \mathcal{A}_n(\mathbb{C}) 
\end{array}$$

Arthur and Clozel’s book [1] gives a full treatment of base change for $GL(n)$. The case of archimedean base change can be captured in an elegant formula [1, p. 71]. We briefly review these results.

Given a partition $n = 2q + r$ let $\chi_i$ ($i = 1, \ldots, q$) be a ramified character of $\mathbb{C}^\times$ and let $\xi_j$ ($j = 1, \ldots, r$) be a ramified character of $\mathbb{R}^\times$. Since the $\chi_i$’s
are ramified, $\tau_i(z) \neq \tau_i(z) = \chi_i(\tau)$, where $\tau$ is a generator of $Gal(\mathbb{C}/\mathbb{R})$. By Langlands classification [3], each $\chi_i$ defines a discrete series representation $\pi(\chi_i)$ of $GL(2, \mathbb{R})$, with $\pi(\chi_i) = \pi(\chi_i)$. Denote by $\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$ the generalized principal series representation of $GL(n, \mathbb{R})$

$$\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r) = \iota_{GL(n, \mathbb{R}), MN}(\pi(\chi_1) \otimes ... \otimes \pi(\chi_q) \otimes \xi_1 \otimes ... \otimes \xi_r \otimes 1). \tag{10}$$

The base change map for the general principal series representation is given by induction from the Borel subgroup $B(\mathbb{C})$ [1, p. 71]:

$$BC(\pi) = \Pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r) = \iota_{GL(n, \mathbb{C}), B(\mathbb{C})}(\chi_1, \chi_1, ..., \chi_q, \chi_q, \xi_1 \circ N, ..., \xi_r \circ N), \tag{11}$$

where $N = N_{\mathbb{C}/\mathbb{R}} : \mathbb{C}^\times \to \mathbb{R}^\times$ is the norm map defined by $z \mapsto z \overline{z}$.

We illustrate the base change map with two simple examples.

**Example 5.1.** For $n = 1$, base change is simply composition with the norm map

$$BC : \mathcal{A}_1(\mathbb{R}) \to \mathcal{A}_1(\mathbb{C}) \, , \, BC(\chi) = \chi \circ N.$$

**Example 5.2.** For $n = 2$, there are two different kinds of representations, one for each partition of 2. According to (10), $\pi(\chi)$ corresponds to the partition $2 = 2 + 0$ and $\pi(\xi_1, \xi_2)$ corresponds to the partition $2 = 1 + 1$. Then the base change map is given, respectively, by

$$BC(\pi(\chi)) = \iota_{GL(2, \mathbb{C}), B(\mathbb{C})}(\chi, \chi^T),$$

and

$$BC(\pi(\xi_1, \xi_2)) = \iota_{GL(2, \mathbb{C}), B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N).$$

### 5.1 The base change map

Now, we define base change as a map of topological spaces and study the induced $K$-theory map.

**Proposition 5.3.** The base change map $BC : \mathcal{A}_n(\mathbb{R}) \to \mathcal{A}_n(\mathbb{C})$ is a continuous proper map.

**Proof.** First, we consider the case $n = 1$. As we have seen in Example [3], base change for $GL(1)$ is the map given by $BC(\chi) = \chi \circ N$, for all characters $\chi \in \mathcal{A}_1(\mathbb{R})$, where $N : \mathbb{C}^\times \to \mathbb{R}^\times$ is the norm map.

Let $z \in \mathbb{C}^\times$. We have

$$BC(\chi)(z) = \chi(|z|^2) = |z|^{2it}.$$  

(12)
A generic element from $\mathcal{A}_t^1(\mathbb{C})$ has the form
\[ \mu(z) = \left( \frac{z}{|z|} \right)^{\ell} |z|^t, \]
where $\ell \in \mathbb{Z}$ and $t \in S^1$, as stated before. Viewing the Pontryagin duals $\mathcal{A}_t^1(\mathbb{R})$ and $\mathcal{A}_t^1(\mathbb{C})$ as topological spaces by forgetting the group structure, and comparing (12) and (13), the base change map can be defined as the following continuous map
\[ \varphi : \mathcal{A}_t^1(\mathbb{R}) \cong \mathbb{R} \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathcal{A}_t^1(\mathbb{C}) \cong \mathbb{R} \times \mathbb{Z} \]
\[ \chi = (t, \varepsilon) \mapsto (2t, 0) \]
A compact subset of $\mathbb{R} \times \mathbb{Z}$ in the connected component $\{ \ell \}$ of $\mathbb{Z}$ has the form $K \times \{ \ell \} \subset \mathbb{R} \times \mathbb{Z}$, where $K \subset \mathbb{R}$ is compact. We have
\[ \varphi^{-1}(K \times \{ \ell \}) = \begin{cases} \emptyset, & \text{if } \ell \neq 0 \\ \frac{1}{2}K \times \{ \varepsilon \}, & \text{if } \ell = 0, \end{cases} \]
where $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Therefore $\varphi^{-1}(K \times \{ \ell \})$ is compact and $\varphi$ is proper.

The Case $n > 1$. Base change determines a map $BC : \mathcal{A}_t^n(\mathbb{R}) \rightarrow \mathcal{A}_t^n(\mathbb{C})$ of topological spaces. Let $X = X(M)/W(\sigma(M))$ be a connected component of $\mathcal{A}_t^n(\mathbb{R})$. Then, $X$ is mapped under $BC$ into a connected component $Y = Y(T)/W(\sigma(T))$ of $\mathcal{A}_t^n(\mathbb{C})$. Given a generalized principal series representation
\[ \pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r) \]
where the $\chi_i$’s are ramified characters of $\mathbb{C}^\times$ and the $\xi$’s are ramified characters of $\mathbb{R}^\times$, then
\[ BC(\pi) = i_{G,B}(\chi_1, \chi_1^\tau, ..., \chi_q, \chi_q^\tau, \xi_1 \circ N, ..., \xi_r \circ N). \]
Here, $N = N_{\mathbb{C}/\mathbb{R}}$ is the norm map and $\tau$ is the generator of $Gal(\mathbb{C}/\mathbb{R})$.

We associate to $\pi$ the usual parameters uniquely defined for each character $\chi$ and $\xi$. For simplicity, we write the set of parameters as a $(q + r)$-uple:
\[ (t, t') = (t_1, ..., t_q, t'_1, ..., t'_r) \in \mathbb{R}^{q+r} \cong X(M). \]
Now, if $\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$ lies in the connected component defined by the fixed parameters $(t, \varepsilon) \in \mathbb{Z}^q \times (\mathbb{Z}/2\mathbb{Z})^r$, then
\[ (t, t') \in X(M) \mapsto (t, t, 2t') \in Y(T) \]
is a continuous proper map.
It follows that
\[
\mathcal{BC} : X(M)/W_\sigma(M) \to Y(T)/W_\sigma(T)
\]
is continuous and proper since the orbit spaces are endowed with the quotient topology. \hfill \Box

**Theorem 5.4.** The functorial map induced by base change
\[
K_j(C_r^n GL(n, \mathbb{C})) \xrightarrow{K_j(\mathcal{BC})} K_j(C_r^n GL(n, \mathbb{R}))
\]
is zero for \( n > 1 \).

**Proof.** We start with the case \( n > 2 \). Let \( n = 2q + r \) be a partition and \( M \) the associated Levi subgroup of \( GL(n, \mathbb{R}) \). Denote by \( X_\mathbb{R}(M) \) the unramified characters of \( M \). As we have seen, \( X_\mathbb{R}(M) \) is parametrized by \( \mathbb{R}^{q+r} \). On the other hand, the only Levi subgroup of \( GL(n, \mathbb{C}) \) for \( n = 2q + r \) is the diagonal subgroup \( X_\mathbb{C}(M) = (\mathbb{C}^\times)^{2q+r} \).

If \( q = 0 \) then \( r = n \) and both \( X_\mathbb{R}(M) \) and \( X_\mathbb{C}(M) \) are parametrized by \( \mathbb{R}^n \). But then in the real case an element \( \sigma \in E_2(0^M) \) is given by
\[
\sigma = i_{GL(n, \mathbb{R})}(\chi_1 \otimes \cdots \otimes \chi_n),
\]
with \( \chi_i \in \hat{\mathbb{Z}}/2\mathbb{Z} \). Since \( n \geq 3 \) there is always repetition of the \( \chi_i \)'s. It follows that the isotropy subgroups \( W_\sigma(M) \) are all nontrivial and the quotient spaces \( \mathbb{R}^n/W_\sigma \) are closed cones. Therefore, the \( K \)-theory groups vanish.

If \( q \neq 0 \), then \( X_\mathbb{R}(M) \) is parametrized by \( \mathbb{R}^{q+r} \) and \( X_\mathbb{C}(M) \) is parametrized by \( \mathbb{R}^{2q+r} \) (see Propositions 2.2 and 2.3).

Base change creates a map
\[
\mathbb{R}^{q+r} \to \mathbb{R}^{2q+r}.
\]
Composing with the stereographic projections we obtain a map
\[
S^{q+r} \to S^{2q+r}
\]
between spheres. Any such map is nullhomotopic [4, Proposition 17.9]. Therefore, the induced \( K \)-theory map
\[
K^j(S^{2q+r}) \to K^j(S^{q+r})
\]
is the zero map.

The Case \( n = 2 \). For \( n = 2 \) there are two Levi subgroups of \( GL(2, \mathbb{R}) \), \( M_1 \cong GL(2, \mathbb{R}) \) and the diagonal subgroup \( M_2 \cong (\mathbb{R}^\times)^2 \). By Proposition 2.2
$X(M_1)$ is parametrized by $\mathbb{R}$ and $X(M_2)$ is parametrized by $\mathbb{R}^2$. The group $GL(2, \mathbb{C})$ has only one Levi subgroup, the diagonal subgroup $M \cong (\mathbb{C}^\times)^2$. From Proposition 2.3 it is parametrized by $\mathbb{R}^2$.

Since $K^1(\mathcal{A}_2^l(\mathbb{C})) = 0$ by Theorem 5.1, we only have to consider the $K^0$ functor. The only contribution to $K^0(\mathcal{A}_2^l(\mathbb{R}))$ comes from $M_2 \cong (\mathbb{R} \times \mathbb{Z})^2$ and we have (see Example 3.5)

$$K^0(\mathcal{A}_2^l(\mathbb{R})) \cong \mathbb{Z}.$$ 

For the Levi subgroup $M_2 \cong (\mathbb{R} \times \mathbb{Z})^2$, base change is

$$BC : \mathcal{A}_2^l(\mathbb{R}) \longrightarrow \mathcal{A}_2^l(\mathbb{C})$$

$$\pi(\xi_1, \xi_2) \mapsto i_{GL(2,\mathbb{C}),B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N),$$

Therefore, it maps a class $[t_1, t_2]$, which lies in the connected component $(\varepsilon_1, \varepsilon_2)$, into the class $[2t_1, 2t_2]$, which lies in the connect component $(0, 0)$. In other words, base change maps a generalized principal series $\pi(\xi_1, \xi_2)$ into a nongeneric point of $\mathcal{A}_2^l(\mathbb{C})$. It follows from Theorem 3.9 that

$$K^0(BC) : K^0(\mathcal{A}_2^l(\mathbb{R})) \rightarrow K^0(\mathcal{A}_2^l(\mathbb{C}))$$

is the zero map.

\[\square\]

### 5.2 Base change in one dimension

In this section we consider base change for $GL(1)$.

**Theorem 5.5.** The functorial map induced by base change

$$K_1(C_1^\ast GL(1, \mathbb{C})) \xrightarrow{K_1(BC)} K_1(C_1^\ast GL(1, \mathbb{R}))$$

is given by $K_1(BC) = \Delta \circ \text{Pr}$, where $\text{Pr}$ is the projection of the zero component of $K^1(\mathcal{A}_1^l(\mathbb{C}))$ into $\mathbb{Z}$ and $\Delta$ is the diagonal $\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$.

**Proof.** For $GL(1)$, base change

$$\chi \in \mathcal{A}_1^l(\mathbb{R}) \mapsto \chi \circ N_{\mathbb{C}/\mathbb{R}} \in \mathcal{A}_1^l(\mathbb{C})$$

induces a map

$$K^1(BC) : K^1(\mathcal{A}_1^l(\mathbb{C})) \rightarrow K^1(\mathcal{A}_1^l(\mathbb{R})).$$

Any character $\chi \in \mathcal{A}_1^l(\mathbb{R})$ is uniquely determined by a pair of parameters $(t, \varepsilon) \in \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$. Similarly, any character $\mu \in \mathcal{A}_1^l(\mathbb{C})$ is uniquely determined by a pair of parameters $(t, \ell) \in \mathbb{R} \times \mathbb{Z}$. The discrete parameter $\varepsilon$ (resp., $\ell$) labels each connected component of $\mathcal{A}_1^l(\mathbb{R}) = \mathbb{R} \sqcup \mathbb{R}$ (resp., $\mathcal{A}_1^l(\mathbb{C}) = \bigsqcup_{\mathbb{Z}} \mathbb{R}$).
Base change maps each component ε of \( A_t^1(R) \) into the component 0 of \( A_t^1(C) \), sending \( t \in R \) to \( 2t \in R \). The map \( t \mapsto 2t \) is homotopic to the identity. At the level of \( K^1 \), the base change map is given by \( K_1(BC) = \Delta \circ Pr \), where \( Pr \) is the projection of the zero component of \( K^1(A_t^1(C)) \) into \( \mathbb{Z} \) and \( \Delta \) is the diagonal \( \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \).

6 Automorphic induction

We begin this section by describing the action of \( \text{Gal}(C/R) \) on \( \hat{W}_C = \hat{C}^\times \). Take \( \chi = \chi_{\ell,t} \in \hat{C}^\times \) and let \( \tau \) denote the nontrivial element of \( \text{Gal}(C/R) \). Then, \( \text{Gal}(C/R) \) acts on \( \hat{C}^\times \) as follows:

\[
\chi^\tau(z) = \chi(z).
\]

Hence,

\[
\chi_{\ell,t}^\tau(z) = \left( \frac{\zeta}{|z|} \right)^\ell |z|^\ell |z|_C^t = \left( \frac{z}{|z|} \right)^{-\ell} |z|^\ell |z|_C^t
\]

and we conclude that

\[
\chi_{\ell,t}^\tau(z) = \chi_{-\ell,t}(z).
\]

In particular,

\[
\chi^\tau = \chi \iff \ell = 0 \iff \chi = |.|_C^t
\]

i.e, \( \chi \) is unramified.

Note that \( W_C \subset W_R \), with index \([W_R : W_C] = 2\). Therefore, there is a natural induction map

\[
\text{Ind}_{C/R} : \mathcal{G}_1^t(C) \to \mathcal{G}_2^t(R).
\]

By the local Langlands correspondence for archimedean fields [3, 8], there exists an automorphic induction map \( \mathcal{A}C_{/R} \) such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{A}_1^t(C) & \xrightarrow{\mathcal{A}C_{/R}} & \mathcal{A}_2^t(R) \\
\varepsilon \mathcal{L}_1 & \Uparrow \text{Ind}_{C/R} & \varepsilon \mathcal{L}_2 \\
\mathcal{G}_1^t(C) & \xrightarrow{\text{Ind}_{C/R}} & \mathcal{G}_2^t(R)
\end{array}
\]

The next result describes reducibility of induced representations.
Proposition 6.1. Let $\chi$ be a character of $W_C$. We have:

(i) If $\chi \neq \chi^\tau$ then $\text{Ind}_{C/R}(\chi)$ is irreducible;

(ii) If $\chi = \chi^\tau$ then $\text{Ind}_{C/R}(\chi)$ is reducible. Moreover, there exist $\rho \in \hat{W}_R$ such that

$$\text{Ind}_{C/R}(\chi) = \rho \oplus \rho^\tau = \rho \oplus \text{sgn.}\rho,$$

where $\rho|_{W_C} = \chi$;

(iii) $\text{Ind}_{C/R}(\chi_1) \sim \text{Ind}_{C/R}(\chi_2)$ if and only if $\chi_1 = \chi_2$ or $\chi_1 = \chi_2^\tau$.

Proof. Apply Frobenius reciprocity

$$\text{Hom}_{W_R}(\text{Ind}_{C/R}(\chi_1), \text{Ind}_{C/R}(\chi_2)) \cong \text{Hom}_{W_C}(\chi_1, \chi_2).$$

Now, $W_R = W_C \cup jW_C$. Therefore, the restriction of $\text{Ind}_{C/R}(\chi)$ to $W_C$ is $\chi \oplus \chi^\tau$. The result follow since $\text{Ind}_{C/R}(\chi)$ is semi-simple. □

Proposition 6.2. A finite dimensional continuous irreducible representation of $W_R$ is either a character or isomorphic to some $\text{Ind}_{C/R}(\chi)$, with $\chi \neq \chi^\tau$.

Proof. It follows immediately from Lemma 4.3 □

6.1 The automorphic induction map

In this section we describe automorphic induction map as a map of topological spaces. We begin by considering $n = 1$.

Let $\chi = \chi_{t,t}$ be a character of $W_C$. If $\chi \neq \chi^\tau$, by proposition 4.2, $\phi_{t,t} \simeq \phi_{-t,-t}$. Hence,

$$\mathcal{AI}_{C/R}(c \mathcal{L}_1(\chi_{t,t})) = D|_t \otimes |det.|^t.$$

On the other hand, if $\chi = \chi^\tau$ then $\chi = \chi_{0,t}$ and $\chi(z) = |z|_t^t$. Therefore,

$$\mathcal{AI}_{C/R}(c \mathcal{L}_1(|.|_t^t)) = \mathcal{L}_2(\rho \oplus \text{sgn.}\rho) = \pi(\rho, \rho^{-1}),$$

where $\pi(\rho, \rho^{-1})$ is a reducible principal series and $\rho$ is the character of $\mathbb{R}^\times \simeq W_R^{ab}$ associated with $\chi_{0,t} = |.|_t^t$ via class field theory, i.e. $\rho|_{W_C} = \chi$.

Recall that

$$\mathcal{A}_1^t(\mathbb{C}) \cong \bigsqcup_{t \in \mathbb{Z}} \mathbb{R}$$

and

$$\mathcal{A}_2(\mathbb{R}) \cong \left( \bigsqcup_{t \in \mathbb{N}} \mathbb{R} \right) \bigsqcup \left( \mathbb{R}/S_2 \right) \bigsqcup \left( \mathbb{R}^2/S_2 \right) \bigsqcup \mathbb{R}^2.$$
As a map of topological spaces, automorphic induction for $n = 1$ may be described as follows:

$$(t, \ell) \in \mathbb{R} \times \mathbb{Z} \mapsto (t, |\ell|) \in \mathbb{R} \times \mathbb{N}, \text{ if } \ell \neq 0$$  \hspace{1cm} (14)

$$(t, 0) \in \mathbb{R} \times \mathbb{Z} \mapsto (t, t) \mapsto \mathbb{R}^2, \text{ if } \ell = 0. \hspace{1cm} (15)$$

More generally, let $\chi_1 \oplus ... \oplus \chi_n$ be an $n$ dimensional $L$-parameter of $W_C$. Then, either $\chi_k \neq \chi_k^+$ for every $k$, in which case automorphic induction is

$$\mathcal{AI}_{C/R}(cL_n(\chi_1 \oplus \oplus \chi_n)) = D_{|\ell_1|} \otimes |\det(\cdot)|^{i\ell_1} \oplus ... \oplus D_{|\ell_n|} \otimes |\det(\cdot)|^{i\ell_n}$$ \hspace{1cm} (16)

or for some $k$ (possibly more than one), $\chi_k = \chi_k^+$, in which case we have

$$\mathcal{AI}_{C/R}(cL_n(\chi_1 \oplus \oplus \Gamma C) \chi_n)) = D_{|\ell_1|} \otimes |\det(\cdot)|^{i\ell_1} \oplus ... \oplus \pi(\rho_k, \rho_k^{-1}) \oplus ... \oplus D_{|\ell_n|} \otimes |\det(\cdot)|^{i\ell_n}. \hspace{1cm} (17)$$

In order to describe automorphic induction as a map of topological spaces, it is enough to consider components of $\mathcal{A}_n^t(C)$ with generic $W$-orbit. For convenience, we introduce the following notation:

if $(t_1, ..., t_n)$ is in the component of $\mathcal{A}_n^t(C)$ labeled by $(\ell_1, ..., \ell_n)$, i.e

$$(t_1, ..., t_n) \in (\mathbb{R} \times \{\ell_1\}) \times ... \times (\mathbb{R} \times \{\ell_n\})$$

we write simply

$$(t_1, ..., t_n) \in \mathbb{R}_{(\ell_1, ..., \ell_n)^t}, \hspace{1cm} \ell_i \in \mathbb{Z}.$$  

There are two cases:

**Case 1:** $\chi_k \neq \chi_k^+$, i.e. $\ell_k \neq 0$, for every $k$,

$$\mathcal{AI} : (t_1, ..., t_n) \in \mathbb{R}_{(\ell_1, ..., \ell_n)^t} \mapsto (t_1, ..., t_n) \in \mathbb{R}_{(\ell_1, ..., \ell_n)}.$$  \hspace{1cm} (18)

So, $(|\ell_1|, ..., |\ell_n|) \in \mathbb{N}^n$.

**Case 2:** if there are $0 < m < n$ characters such that $\chi_k = \chi_k^+$, then

$$\mathcal{AI} : (t_1, ..., t_k, ..., t_n) \in \mathbb{R}_{(\ell_1, ..., \ell_{m}, \ell_n)} \mapsto (t_1, ..., t_k, t_k, ..., t_n) \in (\mathbb{R}^n/W)_{(|\ell_1|, ..., |\ell_n|)^*}.$$  \hspace{1cm} (19)

where $(|\ell_1|, ..., |\ell_n|)^* \in \mathbb{N}^{n-m}$ means that we have deleted the $m$ labels corresponding to $\ell_k = 0$. Note that if $m > 1$, necessarily $W \neq 0$.

We have the following result

**Proposition 6.3.** The automorphic induction map

$$\mathcal{AI}_{C/R} : \mathcal{A}_n^t(C) \rightarrow \mathcal{A}_m^t(R)$$

is a continuous proper map.
The proof follows from the above discussion and is similar to that of proposition 5.3.

Example 6.4. Consider \( n = 3 \). Then,

\[ \mathcal{A}_3^t(\mathbb{C}) \simeq \bigsqcup_{\sigma} \mathbb{R}^3/W_{\sigma} \]

and

\[ \mathcal{A}_6^t(\mathbb{R}) \simeq \left( \bigsqcup_{\ell \in \mathbb{N}^3} \mathbb{R}^3 \right) \sqcup \left( \bigsqcup_{\ell' \in \mathbb{N}^2} \mathbb{R}^4 \right) \sqcup \mathcal{C}, \]

where \( \mathcal{C} \) is a disjoint union of cones.

Let \( \chi_1 \oplus \chi_2 \oplus \chi_3 \) denote a 3-dimensional \( L \)-parameter of \( W_{\mathbb{C}} \). We have the following description of \( \mathcal{AL}_{\mathbb{C}/\mathbb{R}} \) as a map of topological spaces:

- \( \chi_1 \neq \chi_1^\tau, \chi_2 \neq \chi_2^\tau, \chi_3 \neq \chi_3^\tau \)

\[ (t_1, t_2, t_3) \in \mathbb{R}^3_{(t_1, t_2, t_3)} \longmapsto (t_1, t_2, t_3) \in \mathbb{R}^3_{(|t_1|, |t_2|, |t_3|)} \]

with \( \ell_i \in \mathbb{Z}\setminus\{0\} \).

- \( \chi_1 = \chi_1^\tau, \chi_2 \neq \chi_2^\tau, \chi_3 \neq \chi_3^\tau \)

\[ (t_1, t_2, t_3) \in \mathbb{R}^3_{(0, t_2, t_3)} \longmapsto (t_1, t_1, t_2, t_3) \in (\mathbb{R}^4/W)_{(|t_1|, |t_2|, |t_3|)} \]

with \( \ell_i \in \mathbb{Z}\setminus\{0\} \). Similar for the cases \((\ell_1, 0, \ell_3)\) and \((\ell_1, \ell_2, 0)\).

- \( \chi_1 = \chi_1^\tau, \chi_2 = \chi_2^\tau, \chi_3 \neq \chi_3^\tau \)

\[ (t_1, t_2, t_3) \in \mathbb{R}^3_{(0, 0, t_3)} \longmapsto (t_1, t_1, t_2, t_2, t_3) \in (\mathbb{R}^5/W)_{(|t_1|, |t_2|, |t_3|)} \]

with \( \ell_3 \in \mathbb{Z}\setminus\{0\} \). Similar for the cases \((\ell_1, 0, 0)\) and \((0, \ell_2, 0)\).

- \( \chi_1 = \chi_1^\tau, \chi_2 = \chi_2^\tau, \chi_3 = \chi_3^\tau \)

\[ (t_1, t_2, t_3) \in \mathbb{R}^3_{(0, 0, 0)} \longmapsto (t_1, t_1, t_2, t_2, t_3, t_3) \in (\mathbb{R}^6/W) \]

6.2 Automorphic induction in one dimension

Automorphic induction \( \mathcal{AI} \) induces a \( K \)-theory map at the level of \( K \)-theory groups \( K^1 \):

\[ K^1(\mathcal{AI}) : K^1(\mathcal{A}_3^t(\mathbb{R})) \to K^1(\mathcal{A}_1^t(\mathbb{C})). \]
We have

\[ K^1(\mathcal{A}_2^l(\mathbb{R})) \cong \bigoplus_{\ell \in \mathbb{N}} \mathbb{Z}. \]

Each class of 1-dimension L-parameters of \( W_C \) (characters of \( \mathbb{C}^\times \))

\[ [\chi] = [\chi_{\ell,t}] = [\chi_{-\ell,t}] \ (\ell \neq 0) \]

contributes with one generator to \( K^1(\mathcal{A}_2^l(\mathbb{R})) \). Note that, under \( \mathcal{AI} \), this is precisely the parametrization given by the discrete series \( D_{|\ell|} \).

On the other hand,

\[ K^1(\mathcal{A}_1^l(\mathbb{C})) \cong \bigoplus_{\ell \in \mathbb{Z}} \mathbb{Z}. \]

Again, each class (of characters of \( \mathbb{C}^\times \)) \([\chi]\) contributes with a generator to \( K^1(\mathcal{A}_1^l(\mathbb{C})) \), only this time \([\chi_{\ell,t}] \neq [\chi_{-\ell,t}] \), i.e, \( \ell \) and \(-\ell\) belong to different classes.

Note that we may write

\[ K^1(\mathcal{A}_2^l(\mathbb{R})) \cong \bigoplus_{\text{Discrete series}} \mathbb{Z} = \bigoplus_{[D|\ell]} \mathbb{Z} = \bigoplus_{\ell \in \mathbb{N}} \mathbb{Z} \]

and

\[ K^1(\mathcal{A}_1^l(\mathbb{C})) \cong \bigoplus_{\mathbb{C}^\times} \mathbb{Z} = \bigoplus_{[\chi_\ell]} \mathbb{Z} = \bigoplus_{\ell \in \mathbb{Z}} \mathbb{Z}. \]

The automorphic induction map

\[ K^1(\mathcal{AI}) : K^1(\mathcal{A}_2^l(\mathbb{R})) \to K^1(\mathcal{A}_1^l(\mathbb{C})) \]

may be interpreted, at the level of \( K^1 \), as a kind of ”shift” map

\[ [D_{|\ell|}] \mapsto [\chi_{|\ell|}] \]

More explicitly, the ”shift” map is

\[ \bigoplus_{\ell \in \mathbb{N}} \mathbb{Z} \to \bigoplus_{\ell \in \mathbb{Z}} \mathbb{Z}, ([D_1], [D_2], \ldots) \mapsto (..., 0, 0, [\chi_1], [\chi_2], \ldots) \]

where the image under \( K^1(\mathcal{AI}) \) on each component of the right hand side with label \( \ell \leq 0 \) is zero (because \( K^1(\mathcal{AI}) \) is a group homomorphism so it must map zero into zero).
6.3 Automorphic induction in \( n \) dimensions

In this section we consider automorphic induction for \( \text{GL}(n) \). Contrary to base change (see theorems 5.4 and 5.5), the \( K \)-theory map of automorphic induction is nonzero for every \( n \).

**Theorem 6.5.** The functorial map induced by automorphic induction

\[
K_j(C^*_r\text{GL}(2n, \mathbb{R})) \xrightarrow{K_j(\mathcal{A}\mathcal{I})} K_j(C^*_r\text{GL}(n, \mathbb{C}))
\]

is given by

\[
[D_{|\ell_1|} \otimes \ldots \otimes D_{|\ell_n|}] \mapsto [\chi_{|\ell_1|} \oplus \ldots \oplus \chi_{|\ell_n|}]
\]

if \( n \equiv j \pmod{2} \) and \( \chi_k \neq \chi_k^\tau \) for every \( k \), and is zero otherwise.

Here, \([D_{|\ell_1|} \otimes \ldots \otimes D_{|\ell_n|}]\) denotes the generator of the component \( \mathbb{Z}_{(|\ell_1|, \ldots, |\ell_n|)} \) of \( K_j(C^*_r\text{GL}(2n, \mathbb{R})) \) and \([\chi_{|\ell_1|} \oplus \ldots \oplus \chi_{|\ell_n|}]\) is the generator of the component \( \mathbb{Z}_{(|\ell_1|, \ldots, |\ell_n|)} \) of \( K_j(C^*_r\text{GL}(n, \mathbb{C})) \).

**Proof.** Let \( 0 \leq m < n \) be the number of characters \( \chi_k \) with \( \chi_k = \chi_k^\tau \).

**Case 1:** \( m = 0 \)

In this case, \( \chi_k \neq \chi_k^\tau \) for every \( k \). Each character \( \chi_{\ell_k}, \ell_k \neq 0 \), is mapped via the local langlands correspondence into a discrete series \( D_{|\ell_k|} \). At the level of \( K \)-theory, a generator \([D_{|\ell_k|}]\) is mapped into \([\chi_{|\ell_k|}]\). The result follows from (16).

**Case 2:** \( m > 0 \) odd

Then, if \( n \equiv j \pmod{2} \), \( K^j(\mathbb{R}^{n+m}) = 0 \) and \( K_j(\mathcal{A}\mathcal{I}) \) is zero.

**Case 3:** \( m > 0 \) is even

In this case \( K^j(\mathbb{R}^n) = K^j(\mathbb{R}^{n+m}) \). However, \( X_\mathbb{R}(M) \simeq \mathbb{R}^{n+m} \) corresponds precisely to the partition of \( 2n \) into 1’s and 0’s given by

\[
2n = 2(n - m) + 2m
\]

Hence, the number of 1’s in the partition is \( 2m \geq 4 \). It follows that \((t_1, \ldots, t_n)\) is mapped into a cone and, as a consequence, \( K_j(\mathcal{A}\mathcal{I}) \) is zero.

This concludes the proof. \( \square \)

7 Connections with the Baum-Connes correspondence

The standard maximal compact subgroup of \( \text{GL}(1, \mathbb{C}) \) is the circle group \( U(1) \), and the maximal compact subgroup of \( \text{GL}(1, \mathbb{R}) \) is \( \mathbb{Z}/2\mathbb{Z} \). Base change for \( K^1 \) creates a map

\[
\mathcal{R}(U(1)) \rightarrow \mathcal{R}(\mathbb{Z}/2\mathbb{Z})
\]
where $\mathcal{R}(U(1))$ is the representation ring of the circle group $U(1)$ and $\mathcal{R}(\mathbb{Z}/2\mathbb{Z})$ is the representation ring of the group $\mathbb{Z}/2\mathbb{Z}$. This map sends the trivial character of $U(1)$ to $1 \oplus \varepsilon$, where $\varepsilon$ is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$, and sends all the other characters of $U(1)$ to zero.

This map has an interpretation in terms of $K$-cycles. The real line $\mathbb{R}$ is a universal example for the action of $\mathbb{R}^\times$ and $\mathbb{C}^\times$. The $K$-cycle

$$(C_0(\mathbb{R}), L^2(\mathbb{R}), id/dx)$$

is equivariant with respect to $\mathbb{C}^\times$ and $\mathbb{R}^\times$, and therefore determines a class $\hat{\phi}_C \in K_1^{\mathbb{C}^\times}(L^2(\mathbb{C}^\times))$ and a class $\hat{\phi}_R \in K_1^{\mathbb{R}^\times}(L^2(\mathbb{R}^\times))$. On the left-hand-side of the Baum-Connes correspondence, base change in dimension 1 admits the following description in terms of Dirac operators:

$$\hat{\phi}_C \mapsto (\hat{\phi}_R, \hat{\phi}_R)$$

It would be interesting to interpret the automorphic induction map at the level of representation rings:

$$\mathcal{A}I^* : \mathcal{R}(O(2n)) \longrightarrow \mathcal{R}(U(n)).$$

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