UNIQUENESS OF WEAK SOLUTIONS FOR THE SEMILINEAR WAVE EQUATIONS WITH SUPERCritical BOUNDARY/INtERIOR SOURCES AND DAMPING

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Abstract. We consider finite energy solutions of a wave equation with supercritical nonlinear sources and nonlinear damping. A distinct feature of the model under consideration is the presence of nonlinear sources on the boundary driven by Neumann boundary conditions. Since Lopatinski condition fails to hold (unless the dim(Ω) = 1), the analysis of the nonlinearities supported on the boundary, within the framework of weak solutions, is a rather subtle issue and involves the strong interaction between the source and the damping. Thus, it is not surprising that existence theory for this class of problems has been established only recently. However, the uniqueness of weak solutions was declared an open problem. The main result in this work is uniqueness of weak solutions. This result is proved for the same (even larger) class of data for which existence theory holds. In addition, we prove that weak solutions are continuously depending on initial data and that the flow corresponding to weak and global solutions is a dynamical system on the finite energy space.

1. Introduction. Let Ω ⊂ ℝ^n be a bounded domain with sufficiently smooth boundary Γ. We consider the following model of semilinear wave equation with nonlinear boundary/interior monotone dissipation and nonlinear boundary/interior sources:

\[
\begin{aligned}
u_{tt} + a g_0(u_t) &= \Delta u + f(u) \quad \text{in } \Omega \times [0,T] \equiv Q_T, \\
\partial_\nu u + g(u_t) &= h(u) \quad \text{in } \Gamma \times [0,T] \equiv \Sigma_T, \\
u(0) &= u_0 \in H^1(\Omega) \quad \text{and} \quad u_t(0) = u_1 \in L^2(\Omega)
\end{aligned}
\] (1)

Here the functions \( g_0 \in C(\mathbb{R}) \), \( g_1 \in C(\mathbb{R}) \) are assumed monotone increasing. Equation (1) belongs to a class of problems characterized by the competing nature of the sources and the dampings. While the damping term is usually considered in the context of stability (in time) of solutions, in the case of strong nonlinearities present in the system the damping plays a critical role in establishing existence of solutions. It is the interplay between the damping and the source that lies in the heart of the wellposedness analysis. Our interest is in wellposedness of finite energy (i.e. \( H^1(\Omega) \times L^2(\Omega) \)) solutions associated with the equation (1).

In order to put things into perspective, we note that in the case of linear sources, existence and uniqueness of finite energy solutions are well known and follow from application of monotone operator theory [2, 20]. In fact, monotone operator theory
can still be applied and yields a desirable solution in the case when the interior source \( f \) is assumed \textit{locally Lipschitz} from \( H^1(\Omega) \) \textit{into} \( L_2(\Omega) \). In that case, the problem under consideration can be treated as a locally Lipschitz perturbation of a monotone operator. This yields local (in time) existence and uniqueness of finite energy solutions. Thus, the wellposedness problem with locally Lipschitz interior source and \textit{without any} boundary sources has been well understood.

Instead, problems of great physical interest and of mathematical novelty which escape the established mathematical technology are the following: (1) an interior source which may be supercritical with respect to Sobolev’s embeddings (hence does not satisfy the locally Lipschitz condition), and (2) the appearance of a \textit{nonlinear} boundary source \( h(u) \).

In both cases, the presence of the damping in the system becomes essential. This has been recognized in the literature first in the case of \( \text{interior sources} \) \( [15, 28, 32, 3] \) and, more recently, in the case of \textit{boundary sources} \( [20, 31, 5] \). In fact, the treatment of boundary sources is much more subtle for the reasons described below. The main difficulty in studying the wellposedness of system (1) on the finite energy space \( H = H^1(\Omega) \times L_2(\Omega) \) with nonlinear boundary term \( h(u) \) is due to the fact that the Lopatinski condition does not hold for the Neumann problem, i.e. the linear map \( h \rightarrow U(t) = (u(t), u_t(t)) \) where \( U(t) \) solves

\[
\begin{align*}
  u_{tt} &= \Delta u \text{ in } Q_T \\
  \partial\nu u &= h \text{ on } \Sigma_T \\
  u(0) &= u_0 \in H^1(\Omega) \text{ and } u_t(0) = u_1 \in L_2(\Omega)
\end{align*}
\]

is not bounded from \( L_2(\Sigma) \rightarrow H^1(\Omega) \times L_2(\Omega) \), unless the dimension of \( \Omega \) is equal to one or initial data are compactly supported \( [29, 21] \). In fact, the maximal amount of regularity that one obtains is in general \( H^{2/3}(\Omega) \times H^{-1/3}(\Omega) \) \( [22, 30] \) - hence there is a “loss” of \( 1/3 \) derivative. The \( H^{2/3}(\Omega) \) regularity of the wave-Neumann map was exploited in \( [19] \) in proving that locally Lipschitz sources supported on the boundary (without any damping) generate local (in time) solutions but in the topology \textit{above} finite energy levels. When dealing with finite energy solutions, one way to cope with this issue is to start with initial data suitably small. The corresponding theory, developed within the framework of potential well theory \( [24, 32] \), provides existence results for undamped equation (1) with \( g_0 = g = 0, f = 0 \). In such case, the issue of Lopatinski condition does not enter the picture, since the candidate solutions remain invariant within the well. However, this approach is inadequate for studying local or global existence of solutions \textit{without any restrictions on the size of initial data}.

For this reason, the problem of wellposedness of finite energy solutions with \textit{initial data of an arbitrary size} has attracted considerable attention in the field. With motivations coming from boundary control theory, it became clear that the boundary damping, and its interaction with the source, will have to play a major role. The idea is to exploit the boundary dissipation as a sort of “regularization” \( [23] \). In fact, this philosophy was pursued in \( [20] \), where it was shown that finite energy solutions do exist locally for locally Lipschitz functions \( f, h \) and for any dissipation \( g \) that is continuous, monotone and bounded linearly (from above and below) at infinity. The linear bound imposed on the boundary dissipation was dictated by energy decay considerations which necessitate such bound.

A beneficial role of \textit{superlinear interior} damping on extending the life span of solutions was for the first time exhibited in \( [15] \). In \( [15] \), the wave equation with \textit{interior} polynomial damping and source was considered. The sources were assumed
locally Lipschitz, thus local (in time) wellposedness (existence and uniqueness) of solutions is standard. However, such solutions are typically not global. It turns out that if the damping is strong enough ($m \geq p$ in Assumption 1), then solutions live forever. The result in [15] is optimal (for the class of locally Lipschitz sources considered), since for $m < p$, the authors [15] exhibit finite time blow up of solutions. Results of related nature are presented also in [28], where the authors consider $\Omega = \mathbb{R}^n$ with interior sources that are no longer assumed locally Lipschitz. In this latter case, even local existence of solutions necessitates the presence of the damping (more on this later).

For purely boundary sources, [31] provides a full analysis of existence of finite energy solutions. As already mentioned, boundary sources are more subtle, since there is no linear estimate on the corresponding wave operator which would allow one to control inhomogeneity on the boundary at the finite energy level. Thus, once again, the damping placed on the boundary is instrumental for the local existence theory, not only in the case of supercritical sources, but also with sources that are locally Lipschitz. In fact, [31] proves existence of finite energy solutions to (1) with $g_0 = 0, f = 0$, and with both boundary damping $g(u_t)$ and source $h(u)$ of a polynomial structure (this result will be elaborated on in (4)).

While the existing literature contains a number of more recent results pertaining to existence theories for the mixed boundary value problem (this also includes the authors’ work), the uniqueness of solutions with nonlinear boundary sources has been an open problem [31], except for the case with strongly $L_2$ monotone damping, i.e. $g$ satisfies $g'(s) > m_0 > 0, s \in \mathbb{R}$, and locally Lipschitz boundary sources (see [20, 12]). In addition, the same uniqueness question has been unresolved [14, 28] in the case of interior but supercritical sources, which fail to satisfy the locally Lipschitz condition.

It is the uniqueness of weak, local (in time) solutions that is of prime interest to this work. The main contribution of this paper is to provide an affirmative answer to the open question asked, both in the case of interior (supercritical) and boundary sources. In fact, we shall show (Theorem 2.2) that under a mild regularity condition imposed on $h$ and $f$ (i.e. $h, f \in C^2$), and sufficiently strong interaction between the source and the damping, every weak solution (defined below) is unique with respect to the topology governed by finite energy. This puts to rest the open problem of uniqueness of finite energy solutions with boundary sources and/or supercritical interior sources. In addition, as a byproduct of our method, we will also show (Corollary 2) that weak solutions depend continuously with respect to the initial data in the norm of the finite energy space. This completes the picture for Hadamard wellposedness of weak solutions to the problem considered in (1), hence also in [31, 14]. A consequence of this is that under appropriate a-priori bounds guaranteed either by the structure of the source, or the strength of the damping, the semiflow generated by such solutions forms a dynamical system on the finite energy space - see Corollary 3.

2. Main results. In order to focus our exposition and streamline the notation, we shall consider the most representative case when $n = 3$. It goes without saying that the analysis can be easily adapted to other values of $n$ ($n = 2$ being the least interesting, since the concept of criticality of Sobolev’s embeddings is much less pronounced).
Assumption 1. With reference to system (1), assume

1. Damping : \( g, g_0 \) are monotone increasing and continuous functions such that \( g(0) = g_0(0) = 0 \). In addition, the following growth conditions at infinity hold:

\[
\text{There exist positive constants } m_q, M_q, l_m, L_m \text{ such that for } |s| > 1, m_q s^{q+1} \leq g(s) s \leq M_q s^q \text{ and } l_m s^{m+1} \leq g_0(s) s \leq L_m s^m \text{ with } q \geq 1, m \geq 0 \text{ positive. When } 0 < q < 1, \text{ it is assumed that } m_q s^{q-1} \leq g'(s) \leq M_q s^{q-1} \text{ for } s \neq 0.
\]

2. Sources

- **Internal source** : \( f \in C^1(\mathbb{R}) \) and the following growth conditions are imposed on \( f(s) \) for \( |s| > 1 \): \( |f'(s)| \leq C|s|^{p-1} \), where \( 1 \leq p \leq 3 \). If \( p > 3 \), we additionally assume that \( f \in C^2(\mathbb{R}) \), \( |f''(s)| \leq C|s|^{p-2} \) with \( 3 < p \leq \frac{6m}{m+1} \) and \( a > 0 \).

- **Boundary source** Here we distinguish two cases: sublinear and superlinear damping. In both cases the interaction between the power of the source and the damping is controlled by the relation \( 1 \leq k \leq \frac{4q}{q+1} \).

   In the case of sublinear damping \( q < 1 \) we assume:

   \[ h \in C^1(\mathbb{R}), |h'(s)| \leq C|s|^{k-1} + 1, \quad 0 < q < 1, 1 \leq k \leq \frac{4q}{q+1} \]

   In the superlinear case \( q \geq 1 \) we assume:

   \[ h \in C^2(\mathbb{R}) \text{ and } |h''(s)| \leq C[1 + |s|^{k-2}], \quad 2 \leq k \leq \frac{4q}{q+1} \]

Remark 1. **Comment 1** : In line with Sobolev’s embeddings, the internal source \( f(u) \), with \( p \leq 3 \), is locally Lipschitz from \( H^1(\Omega) \to L_2(\Omega) \). Such sources are often referred to as subcritical \((p < 3)\) and critical when \( p = 3 \). When \( 3 < p < 5 \), the local Lipschitz property \( H^1 \to L_2 \) is no longer valid and the corresponding sources are defined in the literature as supercritical. For \( 5 \leq p < 6 \), Sobolev embeddings still yield \( L_1(\Omega) \) integrability of the source, however the potential energy associated with this source : \( \int_{\Omega} f(u)dx \), \( f \) being the antiderivative, may not be defined on the finite energy space. In this case, the sources are no longer within the framework of potential well theory [32, 24]. We shall refer to these sources as super-supercritical and note that Assumption 1 allows for both types of supercriticality.

**Comment 2** : Classification of the boundary sources with respect to the “criticality” of Sobolev’s embedding \( H^{1/2}(\Gamma) \to L_4(\Gamma) \) may not be that relevant, since even in the subcritical case \( k \leq 2 \), when the boundary source is locally Lipschitz \( H^1(\Omega) \to L_2(\Gamma) \), this Lipschitz property, due to the loss of 1/3 derivatives for the Neumann - wave map, does not translate into Lipschitz behavior of the corresponding wave map. So, even in the subcritical case, the analysis is more subtle, requiring special treatment that involves an interaction with the damping (unlike the interior case). Sources with \( 2 < k < 3 \), where the local Lipschitz property \( H^1(\Omega) \to L_2(\Gamma) \) is lost, are referred to as supercritical. Sources with \( 3 \leq k < 4 \) are super-supercritical, where again potential well theory does not apply. Assumption 1 covers critical and both types of supercritical boundary sources.

**Comment 3** : The relations describing interaction between the source and the damping (i.e. between \( m \) (resp. \( q \)), and \( p \) (resp. \( k \))) were introduced in [28] for the interior case and in [31] for the boundary case.

We introduce next the definition of a weak solution:

**Definition 2.1** (Weak solution). By a weak solution of (1), defined on some interval \((0, T)\), we mean a function \( u \in C_0(0, T; H^1(\Omega)), u_t \in C_0(0, T; L_2(\Omega)) \) such that
(a) \( u_t \in L_{m+1}(0, T; \Omega), \ u_t|_\Gamma \in L_{q+1}(0, T; \Gamma) \)

(b) For all \( \phi \in C(0, T; H^1(\Omega)) \cap C^1(0, T; L_2(\Omega)) \cap L_{m+1}(0, T; \Omega), \phi|_\Gamma \in L_{q+1}(0, T; \Gamma) \)

\[
\int_0^T \int_\Omega (-u_t \phi_t + \nabla u \nabla \phi) \, d\Omega dt + \int_0^T \int_\Omega a g_0(u_t) \phi \, d\Omega dt + \int_0^T \int_\Gamma g(u_t) \phi \, d\Gamma dt \\
= -\int_\Omega u_t \phi d\Omega|_0^T + \int_0^T \int_\Gamma h(u) \phi \, d\Gamma dt + \int_0^T \int_\Omega f(u) \phi \, d\Omega dt \tag{3}
\]

(c) \( \lim_{t \to 0} (u(t) - u_0, \phi)_{H^1(\Omega)} = 0 \) and \( \lim_{t \to 0} (u(t) - u_1, \phi)_{L_2(\Omega)} = 0 \) for all \( \phi \in C(0, T, H^1(\Omega)) \cap C^1(0, T; L_2(\Omega)) \cap L_{m+1}(0, T; \Omega), \phi|_\Gamma \in L_{q+1}(0, T; \Gamma) \).

Here \( C_w(0, T, \Omega) \) denotes the space of weakly continuous functions with values in a Banach space \( Y \).

The main contribution of this paper is a uniqueness result, and its consequences, for local weak solutions. However, in order to set the ground for our main findings, we proceed with a brief account of existence results. As mentioned before, the question of existence of weak solutions has received considerable attention in the literature. In fact, several results are presently available, and some of those most relevant to this work are reported below.

2.1. Known results. We begin by recalling results which are already known and provide the foundation for our analysis.

1. Existence of weak solutions with boundary sources and boundary damping: [31] Assume \( g_0 = 0, f = 0 \), with boundary damping \( g(u_t) \) and source \( h(u) \) of polynomial structures. More specifically, \( h(u) = |u|^{k-1}u, \ g(u) = |u|^{q-1}u \), under the restrictions:

\[
1 \leq k < 3, \ k < -\frac{4q}{q + 1} \tag{4}
\]

[31] proves existence of weak solutions which are local in time. If, in addition, \( 1 \leq k \leq q \), weak solutions are global in time. Locally Lipschitz source \( h \) was treated earlier in [20] (damping bounded linearly at infinity) and later extended (damping \( g \) subjected to a quadratic growth at infinity) in [12]. References [20] and [9] provide additionally an analysis of asymptotic decay rates to zero equilibrium of finite energy solutions. Reference [12] deals with a more general long time behavior problem that involves existence and properties of global attractors.

2. Existence of weak solutions with supercritical interior sources: With \( g = 0, h = 0, 1 < p < 5 \) and \( p < \frac{6m}{m+1} \), existence of weak solutions for a bounded domain \( \Omega \) has been known: see for instance [14] and also [3] where a more general degenerate damping was treated. For \( \Omega = \mathbb{R}^3 \) and compactly supported initial data, existence of weak solutions with the same restrictions as above were established in [28, 26], [26] shows additionally that in the case of no damping or linear interior damping, the exponent \( p \) may be supercritical, i.e. \( p < 5 \). Uniqueness of weak solutions was shown in [28] for the locally Lipschitz case only (i.e. \( p \leq 3 \)).

3. Existence and uniqueness with boundary sources: [20] establishes both existence and uniqueness of weak solutions in the locally Lipschitz case \( k \leq 2 \), with the strongly monotone damping, i.e. \( g'(s) > 0, s \in \mathbb{R} \). Existence and
uniqueness of weak solutions along with long time behavior analysis is given in [12] for locally Lipschitz, dissipative boundary/interior sources and coercive at infinity damping.

4. **Existence of weak solutions in the super-supercritical case**: i.e. $5 \leq p < 6, 3 \leq k < 4$ in Assumption 1. Existence of weak solutions in the super-supercritical case with boundary source-damping has been treated in [6, 9] and recently extended in [5] to both interior and boundary sources-damping subject to Assumption 1.

5. **Global existence and finite time blow up of weak solutions**: The solutions referred to above are global in time under the additional assumptions $1 \leq k < 3, k \leq q, 1 \leq p \leq 5, p \leq m$. When $k > q$ or $p > m$ the solutions blow up in a finite time [5]. When $h = 0, g = 0$, this result was first proved in [15] for the subcritical case $p \leq 3$, with global existence when $m \geq p$ and finite time blow up when $m < p$. Extension to the supercritical case $3 < p < 5$ in the case of $\mathbb{R}^n$ with compactly supported initial data (respectively bounded domain) are in [28] (respectively [33]). Blow-up results involving source-damping supported on the boundary are also given in [34] and [6, 9]. In fact, there are several more results in the literature pertaining to various types of blow-up of energy function. Since the phenomenon of blow-up of energy function is not essential to our present work, we do not report here many of other important results.

**Remark 2.** The proof of existence of weak solutions established in [31] relies on Schauder’s Fixed Point. This method depends on certain compactness properties exhibited by weak solutions. Instead, the proof of existence given in [5] and inspired by [20] relies on the theory of nonlinear semigroups generated by maximal monotone operators. Though the problem is not monotone, monotonicity of a subcomponent of nonlinear dynamics is exploited. Careful passage with the limit (where, again, monotonicity is critical) avoids the use of compactness and allows for substantial enlarging of parameters (to super-supercritical) characterizing the growth of the sources.

2.2. **New results.** The main goal of this paper is to resolve the open question of uniqueness of finite energy solutions. Our main result reads:

**Theorem 2.2.** Under the Assumption 1 and $p \leq 5, k \leq 3$ weak solutions are unique. For $p > 5$ and $k > 3$ the same conclusion holds assuming additionally that $u_0 \in L_r(\Omega) \cap L_s(\Gamma)$, where $r = \frac{3}{2}(p-1)$ and $s = 2(k-1)$ and $L_s(\Gamma) \equiv \{ u \in H^1(\Omega), u|_\Gamma \in L_s(\Gamma) \}$

This result has been known only in the case of strongly monotone damping, i.e. $g'(s) > m > 0$ on the boundary and locally Lipschitz sources [20]. In the absence of these requirements the problem of uniqueness, both in the interior and boundary cases, has been an open question [14, 31]. To our best knowledge, Theorem 2.2 is the first one which provides a positive answer to this open problem.

**Remark 3.** Uniqueness result in Theorem 2.2 applies to all weak solutions constructed in [31]. In the super-supercritical case (where weak solutions are constructed in [5]), uniqueness result requires slightly higher integrability of the initial displacement.
Figure 1. The picture on the left shows the range of exponents $m, p$ for the case of interior damping/source interaction for which local existence has been known. Uniqueness results were known up to $p \leq 3$. The picture on the right shows that we proved uniqueness for the entire range of local existence obtained before and moreover, we extended that range to include the boundaries of their region and the super-supercritical case of $p \in [5, 6)$. In addition, Corollary 2 stated below shows that weak solutions in the lighter shaded area on Fig 1 are continuous with respect to initial data taken from finite energy space. As a consequence, weak solutions corresponding to all parameters in a lighter shaded area (supercritical) are (locally) Hadamard well-posed.

Figure 2. Left picture presents the range of exponents $q, k$ for the boundary damping/source case for which local existence was obtained in ([31]). Right picture shows that we obtained uniqueness for the entire range of local existence and extended it to include the boundaries and the extra shaded super-supercritical region $k \in [3, 4)$. Corollary 2 also shows that solutions in the lighter shaded area are continuous with respect to the initial data. This implies (local) Hadamard well-posedness of weak solutions in both sub and supercritical regions. In particular, weak solutions considered in [31] are locally Hadamard well-posed.

The very first step in proving Theorem 2.2 is establishing an energy identity valid for all weak solutions. This is accomplished in section 2, but for the sake of reference, we state this result below: With the energy function defined as

$$E(t) = \frac{1}{2} \int_{\Omega} [\|\nabla u(t)\|^2 + |u_t(t)|^2] dt$$
Lemma 2.3. The energy identity given in (8) holds for all weak solutions.

It is well known that existence of weak solutions combined with uniqueness and an energy identity valid for all weak solutions leads to (1) Strong continuity (rather than continuity in weak topology) of solutions, and (2) Continuous dependence on initial data. In fact, we will derive these important properties as a byproduct of Theorem 2.2 and Lemma 3.1.

Corollary 1. With reference to weak solutions asserted by Theorem (2.2) we obtain:

\[ u \in C([0, T]; H^{1}(\Omega)), \quad u_t \in C([0, T]; L^{2}(\Omega)) \]  

(5)

Proof. The energy identity along with the fact that the functions \( \int_{\Omega} g_0(u_t(t))u_t(t)d\Omega \), \( \int_{\Gamma} g(u_t(t))u_t(t)d\Gamma \), \( \int_{\Omega} f(u(t))u_t(t)d\Omega \), \( \int_{\Gamma} h(u(t))u_t(t)d\Gamma \), are in \( L^1(0, T) \) imply that the energy \( E(t) \) corresponding to each solution is continuous. This means that weak solutions are norm–continuous. Combining this with weak continuity, we obtain strong continuity of weak solutions, as desired.

The proof of the uniqueness result in Theorem 2.2 in general does not imply Hadamard wellposedness, i.e. continuous dependence with respect to finite energy initial data. However, in the supercritical case, when \( p < 5, k < 3 \), we will be able to show that weak solutions are continuous with respect to the initial data taken from finite energy space. In order to formulate the corresponding result it will be convenient to introduce the following notation:

\[ H \equiv H^1(\Omega) \times L^2(\Omega), \quad U(t) \equiv (u(t), u_t(t)) \]

Corollary 2 (Continuous Dependence on Initial Data). We consider system (1) with the same assumptions as Theorem 1 and we take \( p < 5, k < 3 \). Then weak solutions to (1) depend continuously on the initial data in finite energy norm. That is to say, for all \( T < T_{\text{max}} \) and all sequences of initial data such that \( U_n(0) \to U_0 \) in \( H \) the corresponding weak solutions \( U_n(t), U(t) \in H \) satisfy \( U_n \to U \) in \( C(0, T; H) \).

Corollary 3. Under the Assumptions of Corollary 2 and with \( k \leq q, p \leq m \), \( p < 5, k < 3 \), the semi-flow generated by equation (1) is a dynamical system with respect to the strong topology of \( H \).

Remark 4. We note that in the case of boundary damping, solutions defined by (1) generate semi-flow and not a flow (unlike the case of interior damping). This is due to the fact that the backward problem may not be well-posed.

Remark 5. Finally we remark that uniqueness of solutions in “above critical” cases can be sometimes resolved by considering the dynamics in the topology below the energy level [16, 13, 7]. However, this method proves successful when the damping is linear and supercriticality is only incremental, see [16, 13].

The remainder of this paper is devoted to the proof of Theorem 1 and Corollaries 2, 3.

3. Energy identity. The first step in the proof of our results is to derive the energy identity for weak solutions of system (1). The validity of this equality depends on the presence of the damping in the system.
Lemma 3.1. Let $u$ be a weak solution of system (1) with the following a priori regularity:

$$(a) \ u \in B([0, T]; H^{1}(\Omega)); \ u_t \in B([0, T]; L_2(\Omega))$$

where $B([0, T]; X)$ is the space of $X$-valued functions which are bounded on $[0, T]$, endowed with the usual norm $|x|_{B([0, T]; X)} = \sup_{t \in [0, T]} |x(t)|_X$ and

$$(b) \ u_t \in L_{m+1}(0, T; \Omega) \cap \tilde{L}_{q+1}(0, T; \Gamma)$$

Then the following energy identity takes place for all $0 \leq s \leq t \leq T$

$$E(t) = -\int_s^t \int_{\Omega} a_0(u_t)u_t \ d\Omega dz + \int_s^t \int_{\Gamma} (g(u_t))u_t \ d\Gamma dz$$

$$+ E(s) + \int_s^t \int_{\Omega} f(u)u_t \ d\Omega dz + \int_s^t \int_{\Gamma} h(u)u_t \ d\Gamma dz$$

(8)

where $E(t) \equiv \frac{1}{2}|u_t(t)|^2_{\Omega} + \frac{1}{2}|
\nabla u(t)|^2_{\Omega}$

Notice that the result of Lemma 3.1 can be formally obtained by integrating by parts in time and using Green’s formula. However, the procedure is only formal. Its justification requires additional smoothness of solutions, an information that is not available. In order to overcome the problem, we shall use a finite-difference approximation of time derivatives combined with weak continuity methods (see [16]).

3.1. Proof of Lemma 3.1. We already know that weak solutions enjoy the following regularity

$$u \in C_w([0, T]; H^{1}(\Omega)); \ u_t \in C_w([0, T]; L_2(\Omega))$$

In order to derive the energy identity, we will use finite difference approximation of time derivatives (following [16]). We define the following three finite difference operators depending on the parameter $h$:

$$u^+_h(t) \equiv u(t + h) - u(t)$$

$$u^-_h(t) \equiv u(t) - u(t - h)$$

$$D_h u(t) \equiv \frac{1}{2h}[u(t + h) - u(t - h)]$$

with the extensions: $u(t) = u(0)$ for $t \leq 0$ and $u(t) = u(T)$ for $t \geq T$.

In (3), we use the variational form with the test function $\phi(t) = D_h u(t) \in H^{1}(\Omega)$ (which satisfies for all $h$ all the properties listed in part (b) of the definition for a weak solution) and obtain:

$$- \int_0^T \int_{\Omega} u_t(t) D_h u_t(t) \ d\Omega dt + \int_0^T \int_{\Omega} \nabla u(t) \nabla D_h u(t) \ d\Omega dt$$

$$+ \int_0^T \int_{\Omega} g_0(u(t))D_h(u(t)) \ d\Omega dt + \int_0^T \int_{\Gamma} g(u(t))D_h(u(t)) \ d\Gamma dt$$

$$+ \int_{\Omega} u_t D_h u d\Omega |_{t=0}^T = \int_0^T \int_{\Omega} f(u(t))D_h u(t) \ d\Omega dt + \int_0^T \int_{\Gamma} h(u(t))D_h u(t) \ d\Gamma dt$$

(10)

Next, we shall use the following proposition, which is a slight extension of the corresponding result used in [16].
Proposition 1. • Let \( u \) be weakly continuous with values in a Hilbert space \( X \). Then

\[
\lim_{h \to 0} \int_0^T (u(t), D_h u(t)) dt = \frac{1}{2}||u(T)||_X^2 - ||u(0)||_X^2 
\]  

(11)

• For \( u \in W^{1,p}(0, T; Y) \), the following limits are well defined in \( L^p(0, T; Y) \), for all \( 1 < p < \infty \) and any Banach space \( Y \):

\[
\lim_{h \to 0} D_h u = u_t; \lim_{h \to 0} \frac{1}{h} u^+_h = u_t; \lim_{h \to 0} \frac{1}{h} u^-_h = u_t 
\]  

(12)

Moreover, if \( u_t \) is weakly continuous with the values in \( Y \), then for every \( t \in (0, T) \), \( D_h u(t) \to u_t(t) \) weakly in \( Y \) and

\[
\frac{1}{h} u^+_h(T) \to u_t(T), \frac{1}{h} u^-_h(0) \to u_t(0); \text{ weakly in } Y 
\]  

(13)

• In addition to previous assumptions, let \( V \subset X \subset V' \), \( u_{tt} \in L^2(0, T; V') \) and \( u \in L^2(0, T; V) \). Then

\[
\lim_{h \to 0} \int_0^T (u_{tt}(t), D_h u(t)) dt = \frac{1}{2}||u_t(T)||_X^2 - ||u_t(0)||_X^2 
\]  

(14)

In (10), we take limit as \( h \to 0 \) and we use Proposition 1 parts (1) and (2) to obtain (after some algebra):

\[
E(T) = -\lim_{h \to 0} \int_Q a g_0(u_t(t)) D_h u(t) dQ + \lim_{h \to 0} \int_{\Sigma} g(u_t(t)) D_h u(t) d\Sigma 
\]

\[
+ E(0) + \lim_{h \to 0} \int_Q f(u(t)) D_h u(t) dQ + \lim_{h \to 0} \int_{\Sigma} h(u(t)) D_h u(t) d\Sigma 
\]  

(15)

In Proposition 1, part (2) we showed that \( D_h u(t) \in L^p(0, T; Y) \), for any \( p \geq 1 \) and any Banach space \( Y \), provided that \( u_t(t) \in L^p(0, T; Y) \). Since we have \( u_t(t) \in L^{m+1}(0, T; \Omega) \), we know that \( D_h u(t) \in L^{m+1}(0, T; \Omega) \), uniformly as \( h \to 0 \).

Now notice that since \( u_t \in L^{m+1}(0, T; \Omega) \), by Assumption 1

\[
g_0(u_t) \in L^\infty(0, T; \Omega) 
\]

So we have \( g_0(u_t) \in L^{\infty(0, T; \Omega)} \), \( D_h u \in L^{m+1}(0, T; \Omega) \), for any \( m \geq 0 \) and from Proposition 1 part (2) we know that \( \lim_{h \to 0} D_h u = u_t \) in \( L^{m+1}(0, T; \Omega) \). Therefore we obtain:

\[
\lim_{h \to 0} \int_0^T \int_{\Omega} g_0(u_t(t)) D_h u(t) \ d\Omega dt = \int_0^T \int_{\Omega} g_0(u_t(t)) u_t(t) \ d\Omega dt 
\]  

(16)

Similarly, from Proposition 1, part (2) we know that \( u_t(t) \in L^{q+1}(0, T; \Gamma) \) implies \( D_h(u_t(t)) \in L^{q+1}(0, T; \Gamma) \), for any \( q \geq 0 \) and also \( \lim_{h \to 0} D_h u = u_t \) in \( L^{q+1}(0, T; \Gamma) \). Moreover, since \( u_t \mid \Gamma \in L^{q+1}(0, T; \Gamma) \), then \( g(u_t) \in L^{\infty}(0, T; \Gamma) \). Combining all these results along with Proposition 1 we obtain

\[
\lim_{h \to 0} \int_0^T \int_{\Gamma} g(u_t(t)) D_h u(t) \ d\Gamma dt = \int_0^T \int_{\Gamma} g(u_t(t)) u_t(t) \ d\Gamma dt 
\]  

(17)
We know that \( u(t) \in H^1(\Omega) \) for all \( t \in [0, T] \), thus \( |u(t)|_{L^q(\Omega)} \leq C_0 \) for any \( t \leq T \). Therefore by using the growth condition imposed on \( f \) we infer \( |f(u(t))|_{L^p(\Omega)} \leq C(C_0) \) for all \( t \leq T \), i.e.

\[
f(u) \in L_\infty(0, T; L^p_\#(\Omega))
\]

Since \( \frac{\theta}{p} \geq \frac{m+1}{m} \) and \( D_h u \to u_t \) in \( L_{m+1}(0, T; \Omega) \), on the strength of part (2) of Proposition 1 we infer

\[
\lim_{h \to 0} \int_0^T \int_\Omega f(u(t))D_h(u(t)) \, d\Omega dt = \int_0^T \int_\Omega f(u(t))u_t(t) \, d\Omega dt
\]

Similarly, by the growth condition imposed on \( h \), trace theory, the embedding \( H^{1/2}(\Gamma) \subset L_2(\Gamma) \) (meaning \( |h(u(t))|_{L^2(\Gamma)} \leq C'(|u(t)|_{H^{1}(\Omega)}) \)), and the inequality \( \frac{\theta}{p} \geq \frac{m+1}{q} \), we have

\[
h(u) \in L_\infty(0, T; L^{m+1}(\Gamma)) \subset L^{\frac{m+1}{q}}(0, T; \Gamma)
\]

and thus by the second part of Proposition 1 we obtain:

\[
\lim_{h \to 0} \int_0^T \int_\Omega h(u(t))D_h(u(t)) \, d\Gamma dt = \int_0^T \int_\Gamma h(u(t))u_t(t) \, d\Gamma dt
\]

Combining (15), (16), (17), (19) and (21), we obtain the desired energy identity for weak solutions (8).

4. Proof of Theorem 2.2. The main idea behind the proof is to exploit the differentiability of the sources \( h(u), f(u) \) along with the damping in order to “compensate” for the missing regularity of the potential energy. In the process of doing this we shall exploit some techniques from dynamical system theory that allow to infer regularity of attractors from the dissipation present in the system. While this phenomenon is well understood in parabolic dynamics, it is much more subtle in hyperbolic problems, where there is no inherent global smoothing.

**Step 1:** Our first step is to apply Lemma (2.3) to the equation satisfied by the difference of two weak solutions.

Let \( u(t) \neq v(t) \) be two different weak (local in time) solutions of system (1). Thus \( \tilde{u} \equiv u - v \) satisfies the following equation

\[
\begin{aligned}
&\tilde{u}_{tt} + a g_0(u_t) - a g_0(v_t) = \Delta \tilde{u} + f(u) - f(v) \quad \text{in } \Omega \times [0, \infty) \\
&\partial_\nu \tilde{u} + g(u_t) - g(v_t) = h(u) - h(v) \quad \text{in } \Gamma \times [0, \infty) \\
&\tilde{u}(0) = 0 \in H^1(\Omega) \text{ and } \tilde{u}_t(0) = 0 \in L^2(\Omega)
\end{aligned}
\]

From the definition of weak solutions, \( u, v \in C_w(0, T_{\text{max}}; H^1(\Omega)) \), \( u_t, v_t \in C_w(0, T_{\text{max}}; L^2(\Omega)) \) and \( u_t, v_t \in L_{m+1}(0, T; \Omega) \cap L_{q+1}(0, T; \Gamma) \). Let \( T < T_{\text{max}} \) and consider the two solutions \( u(t) \) and \( v(t) \) such that, for \( 0 \leq t < T \)

\[
|u(t)|_{H^{1}(\Omega)} \leq R , \quad |u_t(t)|_{L^{m+1}(\Omega)} \leq R , \quad |v(t)|_{H^{1}(\Omega)} \leq R \quad \text{and} \quad |v_t(t)|_{L^{q+1}(\Omega)} \leq R
\]

and with the following interior and boundary regularities:

\[
\int_0^T \int_\Omega |u_t|^{m+1} + |v_t|^{m+1} \, d\Omega dt < R, \quad \text{for } T < T_{\text{max}}
\]

\[
\int_0^T \int_\Gamma |u_t|^q + |v_t|^{q+1} \, d\Gamma dt \leq R, \quad \text{for } T < T_{\text{max}}
\]

Our goal is to show that \( \tilde{u}(T) \equiv 0 \) for all \( T \leq T_{\text{max}} \).
In what follows we shall denote generic constants (depending on the geometry of domain etc, but not on the particular solution) by $C$. Constants depending on the bounds in (23) - (25) are denoted by $C(R)$.

Using the energy identity (8) from Lemma (2.3) for the difference of two weak solutions we obtain the following identity:

$$
\frac{1}{2}|\tilde{u}(T)|_{\Omega}^2 + \frac{1}{2}\int \nabla \tilde{u}(T)_{\Omega}^2 = - \int_Q (g_0(u(t)) - g_0(v_1)) \tilde{u} dQ - \int \left( g(u) - g(v_1) \right) \tilde{u} d\Sigma + \int_Q (f(u) - f(v)) \tilde{u} dQ + \int \left( h(u) - h(v) \right) \tilde{u} d\Sigma \tag{26}
$$

Let $\tilde{E}(t) \equiv \frac{1}{2}|\tilde{u}(t)|_{\Omega}^2 + \frac{1}{2}\int \nabla \tilde{u}(t)^2 + |\tilde{u}(t)|^2_{\Omega}$ be the energy of the system. Then, using the monotonicity property of $g_0$ and $g$, we obtain the following inequality for the energy:

$$
\tilde{E}(T) \leq \left| \int_Q (f(u) - f(v)) \tilde{u} \ dQ \right| + \left| \int \left( h(u) - h(v) \right) \tilde{u} d\Sigma \right| + \int_0^T \tilde{E}(t) dt \tag{27}
$$

Next, we will estimate the first two terms on the right hand side of (27). Here we note that in the case when $f$ is locally Lipschitz $H^1 \rightarrow L_2$, such estimate for the first term is standard. However, when $f$ is supercritical (i.e. $p > 3$), the estimate must exploit “hidden” regularity due to the damping. The situation with the second term on the right side of (27) is even more problematic, since even with $h$ locally Lipschitz, this term will not be estimated by the “energy”, since the energy does not control the boundary traces $\tilde{u}|_{\Gamma}$. The remaining sections in this manuscript will deal with these issues.

Before going into the details of the estimates, we state a few preliminary facts that will be used repeatedly throughout the rest of the proof without further mention.

1. Sobolev’s Embeddings. For all $0 < \varepsilon \leq 1$

$$
H^1(\Omega) \rightarrow L_6(\Omega) , \quad H^{1-\varepsilon}(\Omega) \rightarrow L^{\frac{6}{1-\varepsilon}}(\Omega)
$$

$$
H^{1/2}(\Gamma) \rightarrow L_4(\Gamma) \text{ and } H^{1/2-\varepsilon}(\Gamma) \rightarrow L^{\frac{4}{1-\varepsilon}}(\Gamma) \tag{28}
$$

2. From the definition of $\tilde{E}(t)$ we have that

$$
|\tilde{u}(t)|^2_{L_2(\Omega)} + |\tilde{u}(t)|^2_{H^1(\Omega)} \leq C\tilde{E}(t) , \forall t \leq T_{\max} \tag{29}
$$

3. Interpolation of Sobolev’s scale, combined with Young’s inequality, gives the following estimate valid for all $0 \leq \theta < 1$ and $\varepsilon > 0$:

$$
|u(t)|^2_{H^{\theta}(\Omega)} \leq \varepsilon |u(t)|^2_{H^1(\Omega)} + C_{\varepsilon, \theta} |u(t)|^2_{L_2(\Omega)} \tag{30}
$$

4. By [4] we have $|u|^2_{L_2(\Gamma)} \leq C|u|_{H^1(\Omega)}|u|_{L_2(\Omega)}$. This trace estimate, followed by Young’s inequality gives:

$$
|u(t)|^2_{L_2(\Gamma)} \leq \varepsilon |u(t)|^2_{H^1(\Omega)} + C_{\varepsilon} |u(t)|^2_{L_2(\Omega)} \tag{31}
$$

**Step 2:** Estimate for $R_f = \left| \int_0^T \int_Q (f(u) - f(v)) \tilde{u} \ dQ \right|$
1. If \( f \) is locally Lipschitz \( H^1(\Omega) \to L_2(\Omega) \) (when \( p \leq 3 \)), then we can take \( a \geq 0 \).

In this case the estimate is straightforward

\[
R_f \leq \int_0^T |f(u) - f(v)|_{L_2(\Omega)} \cdot |\tilde{u}_t(t)|_{L_2(\Omega)} \, dt
\]

\[
\leq C(R) \int_0^T \tilde{E}(t) \, dt
\]  

(32)

(33)

2. When \( p > 3 \), we have \( a > 0 \), \( f \in C^2(\mathbb{R}) \) and \( |f''(u)| \leq C||u|^{p-2} + 1| \).

The growth assumption imposed on \( f'' \) implies the following estimates

\[
|f'(u)| \leq C|u|^{p-1} \text{ and } |f(u)| \leq C|u|^p, \text{ for } |u| > 1
\]

(34)

\[
|f'(u) - f'(v)| \leq C|u - v||u|^{p-2} + |v|^{p-2} + 1
\]

(35)

\[
|f(u) - f(v)| \leq C|u - v||u|^{p-1} + |v|^{p-1} + 1
\]

(36)

Using integration by parts in time and (34), (36) and (35), we obtain the following preliminary inequality for \( R_f \):

\[
R_f \leq \left| \int_\Omega (f(u(T)) - f(v(T))) \tilde{u}(T) \, d\Omega \right| + \int_0^T \int_\Omega (f'(u)u_t(t) - f'(v)v_t(t))\tilde{u}(t) \, dQ \, dt
\]

\[
\leq C \int_\Omega \left| u(T) \right|^{p-1} + |v(T)|^{p-1} + 1 \tilde{u}^2(T) \, d\Omega + \frac{1}{2} \int_\Omega f'(u(T))\tilde{u}^2(T) \, d\Omega
\]

\[
+ \int_0^T \int_\Omega (f'(u) - f'(v))\tilde{u}(t)v_t(t) \, dQ + \frac{1}{2} \int_0^T \int_\Omega f''(u(t))\tilde{u}^2(t)u_t(t) \, dQ
\]

\[
\leq C \int_\Omega \tilde{u}^2(t) |u_t(t)| + |v_t(t)| \, dQ + C \int_\Omega \left| u(T) \right|^{p-1} + |v(T)|^{p-1} + 1 \tilde{u}^2(T) \, d\Omega
\]

\[
+ C \int_0^T \int_\Omega \left( |u(t)|^{p-2} + |v(t)|^{p-2} \right) \tilde{u}^2(t) |u_t(t)| + |v_t(t)| \, dQ + \int_\Omega \tilde{u}^2(t) \, d\Omega
\]

(37)

Now we need to estimate all the terms on the right side of (37). The terms that do not depend on the exponent \( p \) are simpler.

1. Estimate for \( |\tilde{u}(T)|^2 \). Since \( \tilde{u} \) is differentiable w.r.t \( t \) and \( \tilde{u}(0) = 0 \) and from (29), we write:

\[
|\tilde{u}(T)|^2 \leq \int_\Omega |\tilde{u}_t(t)|^2 \, d\Omega \leq T_{\max} \int_0^T \tilde{u}_t(t)^2 \, dt \cdot T \, d\Omega
\]

\[
\leq T_{\max} \int_0^T |\tilde{u}_t(t)|^2 \, dt \leq 2T_{\max} \int_0^T \tilde{E}(t) \, dt
\]

(38)

2. Estimate for \( \int_0^T \int_\Omega \tilde{u}^2(t) |u_t(t)| \, dQ \). We use Holder’s Inequality with \( p = 3 \) and \( q = 3/2 \) and obtain:

\[
\int_0^T \int_\Omega \tilde{u}^2(t) |u_t(t)| \, dQ \leq \int_0^T |\tilde{u}(t)|^2 \, dt \cdot |u_t(t)|_{L_{3/2}(\Omega)} \, dt
\]

(39)

From facts (28) and (29), we have \( |\tilde{u}(t)|_{L_{3/2}(\Omega)} \leq C\tilde{E}(t) \). From the definition of weak solutions, we know that \( |u_t(t)|_{L_{3/2}(\Omega)} \) is bounded for any \( t \in [0, T] \).

Therefore, (39) becomes

\[
\int_0^T \int_\Omega \tilde{u}^2(t) |u_t(t)| \, dQ \leq C(R) \int_0^T \tilde{E}(t) \, dt
\]

(40)
In the same manner we obtain the estimate for \( \int_0^T \int_{\Omega} \tilde{u}^2(t)|v_r(t)| \, dQ \)
\[
\int_0^T \int_{\Omega} \tilde{u}^2(t)|v_r(t)| \, dQ \leq C(R) \int_0^T \tilde{E}(t) \, dt \tag{41}
\]

3. Estimate for \( \int_{\Omega} |u(T)|^{p-1}\tilde{u}^2(T) \, d\Omega \). First, we write
\[
\int_{\Omega} |u(T)|^{p-1}\tilde{u}^2(T) \, d\Omega \leq |\tilde{u}(T)|^2_{L^p(\Omega)} + \int_{\Omega \cap \{|u(T)|>1\}} |u(T)|^{p-1}\tilde{u}^2(T) \, d\Omega \tag{42}
\]

The first term on the right side of (42) was estimated in part 1. For the second term, we consider the following two cases for the exponent \( p \):

**Case 1: supercritical.** If \( 3 < p < 5 \), then \( 2 < p-1 < 4 \). Since \( |u(T)| > 1 \), there exists \( \varepsilon_0 > 0 \) such that \( |u(T)|^{p-1} \leq |u(T)|^{4-\varepsilon_0} \). We choose \( \varepsilon < \frac{\varepsilon_0}{4} \) and apply Holder’s inequality with \( q = \frac{3}{1-\varepsilon} \) and \( \bar{q} = \frac{3}{2(1-\varepsilon)} \). Then we use (23), the fact that \( (4-\varepsilon_0) \frac{3}{2(1-\varepsilon)} \leq 6 \), (30) and (29) to obtain
\[
\int_{\Omega \cap \{|u(T)|>1\}} |u(T)|^{p-1}\tilde{u}^2(T) \, d\Omega \leq \left( \int_{\Omega} |\tilde{u}(T)|^{\frac{q}{1-\varepsilon}} \, d\Omega \right)^{\frac{1}{q}} \left( \int_{\Omega} |u(T)|^{(4-\varepsilon_0)(\frac{3}{2(1-\varepsilon)})} \, d\Omega \right)^{\frac{2(1-\varepsilon)}{q}} \tag{43}
\]
\[
\leq C|\tilde{u}(T)|^2_{H^{1-\varepsilon}}|u(T)|^{(4-\varepsilon_0)(\frac{3}{2(1-\varepsilon)})}(\Omega) \\
\leq \varepsilon \tilde{E}(T) + C_{\varepsilon}(R)|\tilde{u}(T)|^2_{L^2(\Omega)}
\]

Combining (42) with (38) and (43), we obtain the final estimate in this case:
\[
\int_{\Omega} |u(T)|^{p-1}\tilde{u}^2(T) \, d\Omega \leq \varepsilon \tilde{E}(T) + C_{\varepsilon}(R)T_{\text{max}} \int_0^T \tilde{E}(t) \, dt \tag{44}
\]

**Case 2: super-supercritical.** If \( p \geq 5 \), then \( p \leq \frac{6m}{m+1} \Rightarrow m \geq 5 \). In this case, we take advantage of the regularity of initial data \( u_0 \). Let \( u_0 \in L_r \), where \( r = \frac{3(p-1)}{2} \). Note that for \( p = 5 \Rightarrow r = 6 \), this is not an extra condition.

By density of \( H^2(\Omega) \) in \( L_r(\Omega) \), we can choose \( \{u_\eta\} \in H^2(\Omega) \) such that \( u_\eta \rightarrow u(0) \) in \( L_r(\Omega) \) as \( n \rightarrow \infty \). For any \( \eta > 0 \), \( \exists N \) such that \( |u_\eta - u_n|_{L_r(\Omega)} \leq \eta \frac{1}{p-1} \), for \( n \geq N \).
\[
\int_{\Omega} |u(T)|^{p-1}\tilde{u}^2(T) \, d\Omega \leq C \int_{\Omega} |u(T) - u_\eta|^{p-1}\tilde{u}^2(T) \, d\Omega \\
\quad + \int_{\Omega} |u_\eta - u_N|^{p-1}\tilde{u}^2(T) \, d\Omega \\
\quad + \int_{\Omega} |u_N|^{p-1}\tilde{u}^2(T) \, d\Omega \tag{45}
\]

For the first two terms on the right side of (45), we use Holder’s Inequality with \( q = 3 \) and \( \bar{q} = 3/2, (28) \), and (29). For the third term, we use Holder
with \( q = \frac{3}{1+2} \) and \( \overline{q} = \frac{3}{2(1-s)} \) and (28)

\[
\int_{\Omega} |u(T)|^{p-1} \tilde{u}^2(T) \, d\Omega \leq |\tilde{u}(T)|^2_{H^{1-s}(\Omega)} |u_N|^{p-1}_{L^{\frac{m}{m+1}}(\Omega)} \\
+ C \tilde{E}(T) \left( \int_{\Omega} \int_0^T u_t(t) \, dt \right) \frac{3(p-1)}{2(m+1)} ^{\frac{1}{2}} \\
+ C \tilde{E}(T) |u_0 - u_N|^{p-1}_{L^p(\Omega)}
\]

(46)

To deal with the double integral, we apply Holder’s inequality again, with \( q = m + 1 \) and \( \overline{q} = (m+1)/m \) and since \( p \leq \frac{6m}{m+1} \) and \( p \geq 5 \Rightarrow \frac{3(p-1)}{2(m+1)} \leq 1 \)

\[
\left( \int_{\Omega} \left( \int_0^T u_t(t) \, dt \right)^{\frac{3(m-1)}{3m}} \right)^{\frac{2}{3}} \leq C(R) \left( \int_{\Omega} \left( \int_0^T |u_t(t)|^{m+1} \, dt \right)^{\frac{3(m-1)}{3m}} \right)^{\frac{2}{3}} \leq C(R) \left( \int_0^T \tilde{E}(t) \, dt \right)
\]

(47)

The first term on the right side of (46) is estimated as follows: from (30),

\[
|\tilde{u}(T)|^2_{H^{1-s}(\Omega)} |u_N|^{p-1}_{L^{\frac{m}{m+1}}(\Omega)} \leq \varepsilon_1 |u_N|^{p-1}_{L^{\frac{m}{m+1}}(\Omega)} |\tilde{u}(T)|^2_{H^{1}(\Omega)} \\
+ C\varepsilon_1 |u_N|^{p-1}_{L^{\frac{m}{m+1}}(\Omega)} |\tilde{u}(T)|^2_{L^2(\Omega)}
\]

where \( \varepsilon_1 \) is arbitrary. Taking \( \varepsilon_1 = \varepsilon \) and recalling (38)

\[
|\tilde{u}(T)|^2_{H^{1-s}(\Omega)} |u_N|^{p-1}_{L^{\frac{m}{m+1}}(\Omega)} \leq \varepsilon \tilde{E}(T) |u_N|^{p-1}_{L^{\frac{m}{m+1}}(\Omega)} \\
+ C\varepsilon |u_N|^{p-1}_{L^{\frac{m}{m+1}}(\Omega)} \int_0^T \tilde{E}(t) \, dt
\]

(48)

Combining (46) with (47) and (48), and using (38) and the regularity of \( u_N \) (since \( H^2(\Omega) \subset L^{\frac{m}{m+1}}(\Omega) \)) and the fact that \( |u_N|^{p-1}_{L^{\frac{m}{m+1}}(\Omega)} \leq C_{\eta,u_0} < \infty \) for \( u_0 \in L_r(\Omega) \), we obtain the final estimate in this case:

\forall \varepsilon > 0 \ , \forall \eta > 0 \ and \ \forall u_0 \in L_r(\Omega), \ \exists \ C_\varepsilon > 0, \ C_{\eta,u_0} > 0 \ such \ that \n
\[\int_{\Omega} |u(T)|^{p-1} \tilde{u}^2(T) \, d\Omega \leq C_\varepsilon C_{\eta,u_0} \int_0^T \tilde{E}(t) \, dt \]

\[+ \left[ C(R) T^{\frac{m(p-1)}{3m}} + \eta + \varepsilon C_{\eta,u_0} \right] \tilde{E}(T) \]

(49)

The estimate for \( \int_{\Omega} |v(T)|^{p-1} \tilde{v}^2(T) \, d\Omega \) follows the same strategy as the one used above, since \( v(T) \) satisfies the same assumptions as \( u(T) \).

Remark 6. We note that under slightly stronger assumption on initial condition \( u_0 \in L^{\frac{m}{m+1}}(\Omega) \), the constant \( C_{\eta,u_0} \) is uniform with respect to the norm of \( u_0 \), i.e. \( C_{\eta,u_0} = C_{\eta,|u_0|_{L^{\frac{m}{m+1}}(\Omega)}} \). However, this incrementally stronger assumption would prevent to obtain the conclusion for the endpoint of supercritical case, \( p = 5 \). On the other hand the “loss” of uniformity has no impact in the context of uniqueness of solutions. It does impact, however, continuous dependence on the initial data.

4. Estimate for \( \int_0^T \int_{\Omega} |u(t)|^{p-2} u_t(t) \tilde{u}^2(t) \, dQ \). It is sufficient to look at \( \int_0^T \int_{\Omega \setminus \{|u(t)| > 1\}} |u(t)|^{p-2} u_t(t) \tilde{u}^2(t) \, dQ \), since for \( |u(t)| \leq 1 \), we obtain the
term estimated in part 2. Let \( \tilde{\Omega} = \Omega \triangle \{ |u(t)| > 1 \} \). We start by applying Holder’s Inequality with \( q = 3 \) and \( \overline{q} = 3/2 \)

\[
\int_0^T \int_\Omega |u(t)|^{p-2} |u_t(t)| \tilde{u}^2(t) \leq \int_0^T |\tilde{u}(t)|^2 \|u_t(t)\|_{L^2(\Omega)} \left( \int_\Omega |u(t)|^{\frac{3(q-2)}{q-3}} \|u_t(t)\|_{L^2(\Omega)}^2 \right)^{\frac{3}{2}} \tag{50}
\]

We use Holder’s inequality again, with \( q = \frac{6}{p-2} \) and \( \overline{q} = \frac{4}{6-p} \), and notice that \( \frac{6}{p-2} \leq m + 1 \iff p \leq \frac{6m}{m+1} \). Hence (50) becomes:

\[
\int_0^T \int_\tilde{\Omega} |u(t)|^{p-2} |u_t(t)| \tilde{u}^2(t) \ dt \leq \int_0^T C \tilde{E}(t) |u(t)|_{L^{p-2}(\Omega)}^2 |u_t(t)|_{L^\frac{6}{p-2}(\Omega)} \leq \int_0^T C \tilde{E}(t) |u(t)|_{L^{p-2}(\Omega)}^2 |u_t(t)|_{L^{m+1}(\Omega)} \tag{51}
\]

In (51), we know from (23) that \( |u(t)|_{H^1(\Omega)} \leq R \) and thus we obtain:

\[
\int_0^T \int_\tilde{\Omega} |u(t)|^{p-2} |u_t(t)| \tilde{u}^2(t) \ dt \leq C(R) \int_0^T \tilde{E}(t) |u(t)|_{L^{m+1}(\Omega)} \ dt \tag{52}
\]

5. Similarly, following the strategy used in the previous step, we obtain the estimates for the last two terms on the right side of (37)

\[
\int_0^T \int_{\Omega \triangle \{ |u(t)| > 1 \}} |u(t)|^{p-2} |v_t(t)| \tilde{v}^2(t) \ dt \leq C(R) \int_0^T \tilde{E}(t) |v_t(t)|_{L^{m+1}(\Omega)} \ dt
\]

\[
\int_0^T \int_{\Omega \triangle \{ |v(t)| > 1 \}} |v(t)|^{p-2} |v_t(t)| \tilde{v}^2(t) \ dt \leq C(R) \int_0^T \tilde{E}(t) |v_t(t)|_{L^{m+1}(\Omega)} \ dt \tag{53}
\]

Combining the results obtained in steps 1 through 5 back into (37), we obtain our final estimate for \( R_f \), which we will state as a lemma:

**Lemma 4.1.** \( \forall \varepsilon > 0, \forall \eta > 0 \), and, \( \forall u_0 \in L_r(\Omega), v_0 \in L_r(\Omega) \) there exist constants \( 0 < C_r < \infty, 0 < C_{q, u_0, v_0} < \infty \) such that

\[
R_f \leq C(R)C_rC_{q, u_0, v_0} \int_0^T (|u_t(t)|_{L^{m+1}(\Omega)} + |v_t(t)|_{L^{m+1}(\Omega)} + 1) \tilde{E}(t) \ dt
\]

\[
+ \tilde{E}(T)(C(R)T^\lambda + \eta + \varepsilon C_{q, u_0, v_0}) \tag{54}
\]

where \( \lambda \equiv \frac{m(p+1)}{m+1} > 0 \)

We note that Remark 6 applies as well to the estimate in the Lemma (4.1).

**Step 3:** Estimate for \( R_n = \left| \int_0^T \int_\Gamma (h(u(t)) - h(v(t))) \tilde{u}_t(t) \ d\Sigma \right| \)

We first consider *superlinear damping*, where \( q \geq 1 \). The growth assumption imposed on the boundary source \( h: h \in C^2(R) \) and \( |h''(u)| \leq C|u|^{k-2} \), where \( 2 \leq k < \frac{q}{q+1} \) and \( |s| \geq 1 \) implies the following estimates

\[
|h''(u)| \leq C||u|^{k-1} + 1|, \ |h(u)| \leq C||u|^{k+1}
\]

\[
|h'(u) - h'(v)| \leq C|u - v|||u|^{k-2} + |v|^{k-2} + 1|
\]

\[
|h(u) - h(v)| \leq C|u - v|||u|^{k-1} + |v|^{k-1} + 1| \tag{55}
\]
In order to estimate \( R_h \), we first integrate by parts and use (55)
\[
R_h \leq \left| \int_{\Gamma} \left[ h(u(T)) - h(v(T)) \right] \tilde{u}(T) \, d\Gamma \right|
\]
\[
+ \left| \int_{\Gamma} \int_0^T \left( h'(u(t)) u_t(t) - h'(v(t)) v_t(t) \right) \tilde{u}(t) \, dt \, d\Gamma \right|
\]
\[
\leq \int_{\Gamma} \left[ h(u(T)) - h(v(T)) \right] \tilde{u}(T) \, d\Gamma \left| + \frac{1}{2} \int_{\Gamma} \int_0^T h''(u(t)) \tilde{u}^2(t) u_t(t) \, dt \, d\Gamma \right|
\]
\[
+ \left| \int_{\Gamma} \int_0^T \left( h'(u(t)) - h'(v(t)) \right) \tilde{u}(t) v_t(t) \, dt \, d\Gamma \right| + \left| \frac{1}{2} \int_{\Gamma} h'(u(T)) \tilde{u}^2(T) \, d\Gamma \right|
\]
\[
\leq C |\tilde{u}(T)|^2_{L_2(\Gamma)} + C \int_{\Gamma} \tilde{u}^2(T) \left[ |u(T)|^{k-1} + |v(T)|^{k-1} \right] \, d\Gamma
\]
\[
+ C \int_0^T \int_{\Gamma} \tilde{u}^2(t) \left( |u_t(t)| + |v_t(t)| \right) \left[ |u(t)|^{k-2} + |v(t)|^{k-2} + 1 \right] \, d\Sigma
\]  (56)

Now we need to estimate the terms on the right side of (56):

- **Estimate for \( |\tilde{u}(T)|^2_{L_2(\Gamma)} \):

  Combining facts (31), (23) and (38), we see that
  \[
  |\tilde{u}(T)|^2_{L_2(\Gamma)} \leq \varepsilon \tilde{E}(T) + C_\varepsilon |\tilde{u}(T)|^2_{L_2(\Omega)}
  \]
  and
  \[
  |\tilde{u}(T)|^2_{L_2(\Gamma)} \leq \varepsilon \tilde{E}(T) + C_\varepsilon \int_0^T \varepsilon \tilde{E}(t) \, dt
  \]  (57)

- **Estimate for \( \int_0^T \int_{\Gamma} \tilde{u}^2(t) |u_t(t)| \, d\Gamma \, dt \): We use Holder's Inequality with \( p = \overline{p} = 2 \) and obtain:
  \[
  \int_0^T \int_{\Gamma} \tilde{u}^2(t) |u_t(t)| \, d\Sigma \leq \int_0^T |\tilde{u}(t)|^2_{L_4(\Gamma)} |u_t(t)|_{L_{q+1}(\Gamma)} \, dt
  \]  (58)
  Since \( q \geq 1 \), then \( |u_t(t)|_{L_{q+1}(\Gamma)} \leq |u_t(t)|_{L_{q+1}(\Gamma)} \). This fact and (28) gives us
  \[
  \int_0^T \int_{\Gamma} \tilde{u}^2(t) |u_t(t)| \, d\Sigma \leq C \int_0^T \tilde{E}(t) |u_t(t)|_{L_{q+1}(\Gamma)} \, dt
  \]  (59)

  In the same manner we obtain the estimate for \( \int_0^T \int_{\Gamma} \tilde{u}^2(t) |v_t(t)| \, d\Sigma \)
  \[
  \int_0^T \int_{\Gamma} \tilde{u}^2(t) |v_t(t)| \, d\Sigma \leq C \int_0^T \tilde{E}(t) |v_t(t)|_{L_{q+1}(\Gamma)} \, dt
  \]  (60)

- **Estimate for \( \tilde{u}^2(T) |u(T)|^{k-1} \) \( d\Gamma \):

  Note that it is enough to look at \( \int_{\Gamma \cap \{|u(T)| > 1\}} \tilde{u}^2(T) |u(T)|^{k-1} \, d\Gamma \), since for \( |u(T)| \leq 1 \) we obtain \( |\tilde{u}(T)|^2_{L_2(\Gamma)} \); which was estimated above.

  In order to proceed, we consider the following two cases for \( k \):

  **Case 1: Supercritical** If \( 2 \leq k < 3 \), then \( k - 1 < 2 \). Thus, \( \exists \varepsilon_0 > 0 \) such that \( |u(T)|^{k-1} \leq |u(T)|^{2-\varepsilon_0} \). We choose \( \varepsilon < \frac{2}{k-2} \), we apply Holder’s Inequality with \( p = \frac{2}{1+2\varepsilon} \) and \( \overline{p} = \frac{2}{1-2\varepsilon} \) and we use (28) and the fact that \( \frac{2(2-\varepsilon_0)}{1-2\varepsilon} \leq 4 \)
  \[
  \int_{\Gamma \cap \{|u(T)| > 1\}} \tilde{u}^2(T) |u(T)|^{k-1} \, d\Gamma \leq |\tilde{u}(T)|^2_{L_{(2-\varepsilon_0)/(1-2\varepsilon)}(\Gamma)} |u(T)|^{2-\varepsilon_0}_{L_{2-\varepsilon_0/(1-2\varepsilon)}(\Gamma)}
  \]
  \[
  \leq C |\tilde{u}(T)|^2_{H^{1-\varepsilon}(\Omega)} |u(T)|^{2-\varepsilon_0}_{H^{1/(1-2\varepsilon)}(\Gamma)}
  \]  (61)
In (61), we use (23), (30) and (38) to obtain
\[ \int_{\Gamma \cap \{ |u(T)| > 1 \}} \hat{u}^2(T) |u(T)|^{k-1} \, d\Gamma \leq \varepsilon \hat{E}(T) + C_{\varepsilon}(R) T_{\max} \int_0^T \hat{E}(t) \, dt \] (62)

**Case 2: super-supercritical** If \( k \geq 3 \), then \( k \leq \frac{4q}{q+1} \) implies \( q \geq 3 \). In this case, we take advantage again of the regularity of initial data \( u_0 \). Let \( u_0 \in \dot{L}_s(\Gamma) \), where \( s = 2(k - 1) \). Note that for \( k = 3 \Rightarrow s = 4 \), this is not an extra assumption on \( u_0 \).

By density of \( H^1(\Gamma) \) into \( L_s(\Gamma) \), we can choose \( \{ u_n \} \in H^1(\Gamma) \subset L_{\frac{s}{k-2r}}(\Gamma) \) such that \( u_n \to u_0 \) in \( L_s(\Gamma) \) as \( n \to \infty \). Thus, for any \( \eta > 0 \), \( \exists N \) such that \( |u_0 - u_n|_{L_s(\Gamma)} \leq \eta \), \forall \, n \geq N.

\[ \int_{\Gamma} \hat{u}^2(T) |u(T)|^{k-1} \, d\Gamma \leq \int_{\Gamma} |u(T) - u_0|^{k-1} \hat{u}^2(T) \, d\Gamma \]
\[ + \int_{\Gamma} |u_0 - u_N|^{k-1} \hat{u}^2(T) \, d\Gamma \]
\[ + \int_{\Gamma} |u_N|^{k-1} \hat{u}^2(T) \, d\Gamma \]

For the first two terms on the right side of (63), we use Holder's Inequality with \( p = \frac{2}{1-2r} \) and \( \overline{p} = \frac{2}{1-2r} \) and (28). For the third term, we use Holder with \( p = \frac{2}{1-2r} \) and \( \overline{p} = \frac{2}{1-2r} \) and (28)

\[ \int_{\Gamma} \hat{u}^2(T) |u(T)|^{k-1} \, d\Gamma \leq |\hat{u}(T)|^2_{H^{1-\varepsilon}(\Omega)} |u_N|_{L^{\frac{1}{k-2r}}(\Gamma)}^{k-1} + \]
\[ + C \hat{E}(T) \left( \int_{\Gamma} \left| \int_0^T u_t(t) \, dt \right|^\varepsilon \, d\Gamma \right)^{1/2} \]
\[ + C \hat{E}(T)|u_0 - u_N|^{k-1}_{L_s(\Gamma)} \]

The first term on the right side of (64) is estimated as follows: from (30), we have
\[ |\hat{u}(T)|^2_{H^{1-\varepsilon}(\Omega)} |u_N|_{L^{\frac{1}{k-2r}}(\Gamma)}^{k-1} \leq \varepsilon_2 |u_N|_{L^{\frac{1}{k-2r}}(\Gamma)}^{k-1} |\hat{u}(T)|^2_{H^1(\Omega)} \]
\[ + C_{\varepsilon_2} |u_N|_{L^{\frac{1}{k-2r}}(\Gamma)}^{k-1} |\hat{u}(T)|^2_{L^2(\Omega)} \]
where \( \varepsilon_2 \) is arbitrary. Taking \( \varepsilon_2 = \varepsilon \) and recalling (38), we obtain
\[ |\hat{u}(T)|^2_{H^{1-\varepsilon}(\Omega)} |u_N|_{L^{\frac{1}{k-2r}}(\Gamma)}^{k-1} \leq \varepsilon \hat{E}(T)|u_N|_{L^{\frac{1}{k-2r}}(\Gamma)}^{k-1} \]
\[ + C_{\varepsilon} |u_N|_{L^{\frac{1}{k-2r}}(\Gamma)}^{k-1} \int_0^T \hat{E}(t) \, dt \]

To deal with the double integral in (64), we apply Holder's inequality with \( p = q + 1 \) and \( \overline{p} = (q + 1)/q \) and since \( k \leq \frac{4q}{q+1} \leq \frac{q+3}{2} \Leftrightarrow \frac{2(k-1)}{q+1} \leq 1 \)
\[ \left( \int_{\Gamma} \left| \int_0^T u_t(t) \, dt \right|^\varepsilon \, d\Gamma \right)^{1/2} \leq T^{\frac{k-1}{q+1}} \int_{\Gamma} \left( \int_0^T |u_t(t)|^{q+1} \, dt \right)^{\frac{k-1}{q+1}} \, d\Gamma \]
\[ \leq C |R| T^{\frac{4(k-1)}{q+1}} \]
Combining (64) with (65) and (66), and using (38) and the regularity of \( u_N \) (since \( H^1\Gamma \subset L^{\infty} \)), we obtain our final estimate in this case: \( \forall \varepsilon > 0, \forall \eta > 0 \) and \( \forall u_0 \in \tilde{L}_s(\Gamma), \exists C_\varepsilon > 0, C_{\eta,u_0} > 0 \) such that

\[
\int_{\Gamma} |u(T)|^{k-1} \tilde{u}^2(T) \, d\Gamma \leq [C(R)T^{\frac{q(k-1)}{q+1}}] + \eta + \varepsilon C_{\eta,u_0} \tilde{E}(T) + C_\varepsilon C_{\eta,u_0} \int_0^T \tilde{E}(t) \, dt \quad (67)
\]

- Estimate for \( \int_0^T \int_{\Gamma} |u(t)|^k |u(t)|^2 \, d\Sigma \). It is sufficient to consider this term with \( |u(t)| > 1 \), since otherwise, we obtain the term that we already estimated in step 2. Let \( \Gamma = \tilde{\Gamma} \cap \{|u(t)| > 1\} \). We apply Holder’s Inequality with \( p = \overline{p} = 2 \) and obtain

\[
\int_0^T \int_{\Gamma} |u(t)|^{k-2} |u(t)| \tilde{u}^2(t) \leq \int_0^T \tilde{u}(t)^2 L_{L^q(\Gamma)} \left( \int_{\Gamma} |u(t)|^2 L^2(\Gamma) \right)^{\frac{1}{2}} \quad (68)
\]

In (68), we use (28), (29) and Holder’s Inequality, this time with \( p = \frac{2}{q-1} \) and \( \overline{p} = \frac{2}{q+1} \) and obtain

\[
\int_0^T \int_{\Gamma} |u(t)|^{k-2} |u_t(t)| \tilde{u}^2(t) \, d\Sigma \leq \int_0^T \tilde{E}(t) |u(t)|^k L_{L^2(\Gamma)} |u_t(t)| L_{\frac{4}{q+1}(\Gamma)} \leq C(R) \int_0^T \tilde{E}(t) |u_t(t)| L_{q+1}(\Gamma) \, dt \quad (69)
\]

In the last inequality of (69) we used the fact that \( \frac{1}{q-1} \leq q + 1 \iff k \leq \frac{4q}{q+1} \) and assumption (23).

We can obtain the estimates for \( \int_0^T \int_{\Gamma} \tilde{u}^2(t) |v_t(t)| u(t) |v(t)|^2 \, d\Sigma \) and \( \int_0^T \int_{\Gamma} \tilde{u}^2(t) |u_t(t)| v(t) |v(t)|^2 \, d\Sigma \), similarly.

Combining the results obtained in steps 1 through 5 back into (56), we obtain our final estimate for \( R_h \), which is presented in the next lemma.

**Lemma 4.2.** \( \forall \varepsilon > 0, \forall \eta > 0 \) and \( \forall u_0, v_0 \in \tilde{L}_s(\Gamma) \), \( \exists \) constants \( 0 < C_\varepsilon < \infty, 0 < C_{\eta,u_0, v_0} < \infty \) such that

\[
R_h \leq C(R)C_\varepsilon C_{\eta,u_0,v_0} \int_0^T \left( |u_t(t)| L_{q+1}(\Gamma) + |v_t(t)| L_{q+1}(\Gamma) + 1 \right) \tilde{E}(t) \, dt + \tilde{E}(T)(C(R)T^\gamma + \eta + \varepsilon C_{\eta,u_0,v_0}) \quad (70)
\]

where \( \gamma = \frac{q(k-1)}{q+1} > 0 \).

**Remark 7.** If we assume a slightly stronger assumption on initial condition \( u_0 \in L^{\infty} \), then the constant \( C_{\eta,u_0} = C_{\eta,u_0} L^{\infty}(\Gamma) \) becomes uniform with respect to the norm of \( u_0 \). Nevertheless, this stronger assumption would prevent us from obtaining the conclusion for the endpoint of super-supercritical case, \( k = 3 \). While the “loss” of uniformity does not affect the uniqueness argument, it will impact the continuous dependance on initial data.
In the case of sublinear damping, i.e. \( 1 \leq k \leq \frac{4q}{q+1} \), and a posteriori, \( q \geq 1/3 \), the argument is different and less involved. In order to estimate \( R_h \), we will need to account for the boundary damping term occurring in the inequality for the energy of the difference of two solutions. That is to say,

\[
\dot{E}(T) = - \int_0^T \int_G (g(u_t) - g(v_t)) \, \dot{u}_t \, d\Sigma + \int_0^T \int_\Omega (f(u) - f(v)) \, \dot{u}_t \, dQ \\
+ \int_0^T \int_G (h(u) - h(v)) \, \dot{u}_t \, d\Sigma + 2 \int_0^T \int_\Omega \dot{u}\dot{u}_t \, d\Omega \, dt
\]

It is now critical to obtain a sharp estimate for the loss of coercivity in the damping term. Since \( 0 \leq g'(s) \leq M_q |s|^{q-1} \), \( s \neq 0 \), \( g' \in L_1(R) \) then

\[
\int_0^T \int_G (g(u_t) - g(v_t)) \, \dot{u}_t \, d\Sigma = \int_0^T \int_\Omega \dot{u}(t) \int_0^1 g'(s u_t(t) + (1-s)v_t(t)) \, ds \, d\Sigma \tag{72}
\]

Define \( \hat{g}(u,v) \equiv \int_0^1 g'(s u(t) + (1-s)v(t)) \, ds \geq 0 \). With this notation, (72) becomes

\[
\int_0^T \int_G (g(u_t) - g(v_t)) \, \dot{u}_t \, d\Sigma = \int_0^T \int_G \hat{g}(u_t,v_t) \dot{u}_t^2(t) \, d\Sigma \tag{73}
\]

We proceed with estimating term \( R_h \) by using the weight \( \hat{g}(u_t,v_t) \).

\[
R_h \leq \int_0^T \int_G \frac{|h(u(t)) - h(v(t))|}{\sqrt{\hat{g}(u_t,v_t)}} |\dot{u}(t)| \sqrt{\hat{g}(u_t,v_t)} \, d\Sigma \leq \varepsilon \int_0^T \int_G \hat{g}(u_t,v_t) \dot{u}_t^2(t) \, d\Sigma + C \varepsilon \int_0^T \int_G \frac{(h(u(t)) - h(v(t)))^2}{\hat{g}(u_t,v_t)} \, d\Sigma \tag{74}
\]

The first term on the right side of (74) will be absorbed by the damping term in (73). For the second term (call it \( N_h \)), we need a sharp estimate from below for \( \hat{g}(u_t,v_t) \), which leads to the evaluation of the integral: \( \int_0^1 g'(s u_t(t) + (1-s)v_t(t)) \, ds \). Using the assumption on the derivative of \( g \), we obtain

\[
\int_0^1 g'(s u(t) + (1-s)v(t)) \, ds \geq m_q \int_0^1 |s u(t) + (1-s)v(t)|^{q-1} \, ds \tag{75}
\]

We make a change of variable \( z = su + (1-s)v \) and then the integral becomes

\[
\int_0^1 g'(s u(t) + (1-s)v(t)) \, ds \geq m_q \int_v^u |z|^{q-1} \frac{dz}{u-v} \tag{76}
\]

In order to estimate from below this integral, we consider the following cases:

First, if \( 0 < v \leq u \). Then

\[
\hat{g}(u_t,v_t) \geq m_q \int_v^u |z|^{q-1} \frac{dz}{u-v} = m_q \frac{u^q - v^q}{q(u-v)} = \frac{u^{q-1}}{q} \frac{1 - (\frac{v}{u})^q}{1 - \frac{v}{u}} \tag{77}
\]

Let \( x = \frac{v}{u} \). Analyzing the function \( f(x) = \frac{1-x^q}{1-x} \), for \( 0 \leq x \leq 1 \) and \( 0 < q < 1 \), we see that \( f \) is decreasing and \( \lim_{x \rightarrow 1^-} f(x) = q \). Thus \( f(x) \geq q \) for all \( x \in [0,1] \).

Using this fact in (77), we obtain that

\[
\hat{g}(u,v) \geq m_q u^{q-1} \Rightarrow \frac{1}{\hat{g}(u,v)} \leq \frac{u^{1-q}}{m_q} \tag{78}
\]

In the case when \( v \leq 0 \leq u \) similar calculations lead to

\[
\frac{1}{\hat{g}(u_t,v_t)} \leq \frac{g(u_t(t) - v_t(t))}{m_q |u_t(t)|^q + |v_t(t)|^q} \leq C_q [u_t(t)]^{1-q} + |v_t(t)|^{1-q}
\]
The analysis in the remaining sectors for \( u, v \) can be reduced by symmetry to the two cases discussed above. Hence for all \( u, v \in R \) we have
\[
\dot{g}(u, v) \geq m_q |u|^{q-1} + |v|^{q-1}
\]
(79)
or equivalently \( \dot{g}^{-1}(u, v) \leq M_q |u|^{1-q} + |v|^{1-q} \)

Continuing the estimate for \( N_h \) term, we arrive at
\[
N_h \leq \frac{C_e}{m_q} \int_0^T \int_{\Gamma} (h(u(t)) - h(v(t)))^2 |u_t(t)|^{1-q} \, d\Sigma
\]
\[
\leq C_N \int_0^T \int_{\Gamma} \tilde{u}^2(t) |u(t)|^{2(k-1)} + |v(t)|^{2(k-1)} + 1|(|u_t(t)|^{1-q} + |v_t(t)|^{1-q}) \, d\Sigma
\]
\[
\tag{80}
\]
It is enough to estimate the term \( \int_0^T \int_{\Gamma} \tilde{u}^2(t) |u(t)|^{2(k-1)} |u_t(t)|^{1-q} \, d\Sigma \). We use Holder’s inequality in space with \( p = \frac{2}{q-1} \)
\[
\int_0^T \int_{\Gamma} \tilde{u}^2(t) |u(t)|^{2(k-1)} |u_t(t)|^{1-q} \, d\Sigma \leq \int_0^T |u(t)|^2 \|L_{k+1}(\Gamma)\| \left( \int_0^T |u(t)|^{4(k-1)} |u_t(t)|^{2(q-1)} \, d\Gamma \right)^{\frac{1}{2}}
\]
\[
\tag{81}
\]
From (28), (29), and Holder’s inequality, with \( p = 1/(k-1) \) and \( \overline{p} = 1/(2-k) \) we get
\[
\int_0^T \int_{\Gamma} \tilde{u}^2(t) |u(t)|^{2(k-1)} |u_t(t)|^{1-q} \, d\Sigma \leq C \int_0^T \tilde{E}(t) |u(t)|^{2(k-1)} |u_t(t)|^{1-q} \, d\Sigma
\]
\[
\tag{82}
\]
Using (28) and (23), we know that \( |u(t)|^{2(k-1)} \leq C |u(t)|^{2(k-1)} \leq CR^{2(k-2)} \). We also have that \( k \leq \frac{4q}{q+1} \Longleftrightarrow \frac{2-2q}{2-k} \leq q + 1 \). Thus \( |u_t(t)|^{1-q} \leq |u_t(t)|^{1-q} \), and (81) becomes
\[
\int_0^T \int_{\Gamma} \tilde{u}^2(t) |u(t)|^{2(k-1)} |u_t(t)|^{1-q} \, d\Sigma \leq C \int_0^T \tilde{E}(t) \|u_t(t)\|_{L^{q+1}(\Gamma)} + 1 \, d\Sigma
\]
\[
\tag{82}
\]

The estimates involving the terms \( |v_t(t)|^{1-q} \) and \( |v(t)|^{2(k-1)} \) are the same. Therefore we have proved the following lemma.

**Lemma 4.3.** Let \( q < 1 \), \( 1 \leq k \leq \frac{4q}{q+1} \), \( g'(s) \geq m_q |s|^{q-1} \). Then for every \( \epsilon > 0 \) there exists a constant \( C_\epsilon > 0 \) such that
\[
R_h \leq C_\epsilon \int_0^T (|u_t(t)|_{L^{q+1}(\Gamma)} + |v_t(t)|_{L^{q+1}(\Gamma)} + 1) \tilde{E}(t) \, dt
\]
\[
+ \epsilon \int_0^T \int_{\Gamma} |g(u_t) - g(v_t)| \, d\Gamma \, dt
\]
\[
\tag{83}
\]

**Step 4: Completion of the Proof:**

In (27), we use Lemma (4.1)-(4.3) to obtain: \( \forall \epsilon > 0 \), \( \forall \eta > 0 \) and \( \forall \, u_0, \, v_0 \in L_r(\Omega) \cap L_s(\Gamma), \exists \) constants \( 0 < C_\epsilon < \infty \), \( 0 < C_{q,u_0,v_0} < \infty \) such that
\[
\tilde{E}(t) \leq 2\tilde{E}(T)(C(R)(T^\lambda + T^\gamma) + \eta + \epsilon C_{q,u_0,v_0})
\]
\[
+ C(R)C_\epsilon C_{q,u_0,v_0} \int_0^T (|u_t(t)|_{L^{q+1}(\Omega)} + |v_t(t)|_{L^{q+1}(\Omega)} + |u_t(t)|_{L^{q+1}(\Gamma)} + |v_t(t)|_{L^{q+1}(\Gamma)} + 2) \tilde{E}(t) \, dt
\]
\[
\tag{84}
\]
where \( \lambda = \frac{m(p+1)}{n+1} > 0 \) and \( \gamma = \frac{a(k-1)}{q+1} > 0 \). \( C(R) \) in (84) denotes a function bounded for bounded arguments of \( R \) where \( R \) denotes an upper bound for the a-priori regularity assumed to hold for weak solutions in (23), (24), (25).

In (84), we choose \( \varepsilon, \eta \) and \( T \) such that \( C(R)(T^\lambda + T^\gamma) + \eta + \varepsilon C_{\eta,u_0,v_0} < 1 \) (it is enough to look at a small interval for \( T \), since the process can be reiterated), we apply Gronwall’s inequality with \( L_1 \) kernel (\( u_t, v_t \in L_{m+1}(0,T;\Omega) \cap L_{q+1}(0,T;\Gamma) \)) and obtain that \( \tilde{E}(T) = 0 \), \( \forall T \leq T_{\text{max}} \Rightarrow \tilde{u}(T) = 0 \), \( \forall T \leq T_{\text{max}} \).

5. Proofs of the Corollaries 2, 3. Corollary 2 is a consequence of the uniqueness Theorem 2.2 and the energy identity.

Given \( U_0 = (u(t = 0), u_n(t = 0)) \in H \), let \( U_n^0 \to U_0 \) in \( H \). The uniquely determined weak solutions corresponding to these initial data are denoted, respectively, by \( U(t) \) and \( U_n(t) \). From the definition of weak solutions

\[
\begin{align*}
&u, u^0 \in C(0,T_{\text{max}}; H^1(\Omega)),\ u_t, u^n_0 \in C(0,T_{\text{max}}; L_2(\Omega)), \\
&u_t, u^n_t \in L_{m+1}(0,T_{\text{max}}; \Omega),\ u_t, u^n_t \in L_{q+1}(0,T_{\text{max}}; \Gamma),
\end{align*}
\]

where the bounds on the respective norms are bounded by some constant \( R \) uniformly in \( n \). Our aim is to prove that \( U_n \to U \) in \( C(0,T; H) \). In fact, our proof provides the estimate for the convergence.

For the energy of the difference of two solutions we shall adopt the same notation as before \( \tilde{E}(T) \equiv \int_\Omega |\nabla u_n(t) - \nabla u(t)|^2 + |u_n(t) - u(t)|^2 + |u_{nt}(t) - u_t(t)|^2|d\Omega \)

Using the energy identity from Lemma 1, applied to the difference of two solutions, calculations performed in the previous sections, after accounting for the non-zero initial condition \( \tilde{E}(0) \), give: \( \forall \varepsilon > 0, \eta > 0 \) there exists a constant \( C_\varepsilon < \infty \) and \( C_{\eta,u_0,u_n(0)} < \infty \) such that

\[
\begin{align*}
\tilde{E}(T) \leq & [\eta + C(R)(T^\lambda + T^\gamma) + \varepsilon C_{\eta,u_0(0),u(0)}]|\tilde{E}(T)| + \tilde{E}(0)) \\
+ & C_\varepsilon(R)C_{\eta,u_0(0),u(0)} \int_0^T \tilde{E}(t)|u_t(t)|_{L_{m+1}(\Omega)} + |u_{nt}(t)|_{L_{m+1}(\Omega)} \\
+ & |u_t(t)|_{L_{q+1}(\Gamma)} + |u_{nt}(t)|_{L_{q+1}(\Gamma)} + 1|dt
\end{align*}
\]

(86)

where the constants \( C_{\eta,u_0,u_n(0)} \) depend on \( \eta \) and the norms: \( |u_n(0)|_{L_\infty(\Omega)}, |u_n(0)|_{L_{q+1}(\Gamma)} \) (see Remark 6), and \( \lambda = \frac{m(p+1)}{n+1} > 0 \) and \( \gamma = \frac{a(k-1)}{q+1} > 0 \).

Since \( p < 5, k < 3 \), these norms are controlled, for \( \varepsilon \) sufficiently small, by \( H^1(\Omega) \) regularity of the initial data. Indeed, strict inequalities \( r = 3/(p-1) < 6 \) and \( s = 2(k-1) < 4 \) combined with Sobolev’s embeddings yield the conclusion. Thus, in the course of the proof of Theorem 2.2 there is no need for introducing approximation \( u_N \), along with the parameter \( \eta \) measuring the error of that approximation. As a consequence, we can take \( \eta = 0 \) and replace the constants \( C_{\eta,u_0,u_n(0)} \) by \( C(R) \). This observation, along with (86), leads to the following inequality:

\[
\begin{align*}
\tilde{E}(T) \leq & (\tilde{E}(T) + \tilde{E}(0))|\varepsilon C(R) + C(R)(T^\lambda + T^\gamma)| \\
+ & C_\varepsilon(R) \int_0^T \tilde{E}(t)|u_t(t)|_{L_{m+1}(\Omega)} + |u_{nt}(t)|_{L_{m+1}(\Omega)} \\
+ & |u_t(t)|_{L_{q+1}(\Gamma)} + |u_{nt}(t)|_{L_{q+1}(\Gamma)} + 1|dt
\end{align*}
\]

(87)
Selecting \( T \) small enough (depending on \( R \)), so \( T \leq T_0 \) and then taking \( \varepsilon \) sufficiently small gives

\[
\hat{E}(T) \leq C(R)\hat{E}(0) + C(R) \int_0^T \hat{E}(t)s(t)dt
\]

where \( T \leq T_0 \) and

\[s(t) = [|u_t(t)|_{L_{m+1}(\Omega)} + |u_{nt}(t)|_{L_{m+1}(\Omega)} + |u(t)|_{L_{q+1}(\Gamma)} + |u_{nt}(t)|_{L_{q+1}(\Gamma)} + 1] \in L_1(R)\]

Gronwall’s inequality with \( L_1 \) kernel yields:

\[
\hat{E}(T) \leq C(R)\hat{E}(0)e^{\int_0^T s(t)dt} \leq C(T, R)\hat{E}(0), \quad T \leq T_0
\]

Since the choice of \( T_0 \) depends only on \( R \) the above relation can be reiterated for any \( T \) when \( T < T_{\max} \), which proves Hadamard wellposedness when \( p < 5, k < 3 \).

**Remark 8.** When \( p > 5 \), or \( k > 3 \), we recall that uniqueness of weak solutions was shown to hold for the initial data taken from \( H_{r,s} = H^1(\Omega) \cap L_r(\Omega) \cap \tilde{L}_s(\Gamma) \times L_2(\Omega) \), where \( r = 3/2(p-1) \), \( s = 2(k-1) \). For this range of parameters, one can still prove Hadamard wellposedness, but in the space that is strictly contained in \( H_{r,s} \). More specifically, for \( p \geq 5, k \geq 3 \) the solutions are continuous with respect to initial data taken from the space

\[H_{r,s,\delta} = H^1(\Omega) \cap L_{r+\delta}(\Omega) \cap \tilde{L}_{s+\delta}(\Gamma) \times L_2(\Omega) \subset H_{r,s}\]

where \( \delta \) can be taken sufficiently small constant. Indeed, when \( p \geq 5, k \geq 3 \), the dependence of the constants \( C_{u_n(0),u(0)} \) is not uniform with respect to \( n \), unless \( u(0), u_n(0) \in L_{r+\delta}(\Omega) \cap \tilde{L}_{s+\delta}(\Gamma) \) for some \( \delta > 0 \) (see Remark 6). The important observation is that this new space is still invariant under the dynamics. This last statement follows from

\[
|u(t)|_{L_{r+\delta}(\Omega)} \leq |u(0)|_{L_{r+\delta}(\Omega)}C \int_0^t |u_t(s)|_{L_{m+1}(\Omega)}ds
\]

\[
|u(t)|_{L_{s+\delta}(\Gamma)} \leq |u(0)|_{L_{s+\delta}(\Gamma)} + C \int_0^t |u_t(s)|_{L_{q+1}(\Gamma)}ds
\]

where the following (strict) relations between the parameters were used.

\[
s = 2(k-1) < (q+1), \text{ when } 3 < k \leq \frac{4q}{q+1}
\]

\[
r = 3/2(p-1) < m + 1, \text{ when } 5 < p \leq \frac{6m}{m + 1}
\]

This proves the invariance of \( H_{r,s,\delta} \) under the dynamics for a sufficiently small \( \delta > 0 \). Having established a “right” topological framework for the super-supercritical case, Hadamard wellposedness follows now along the same argument as before. However, the obtained result is slightly sub-optimal with respect to the “uniqueness” statement in Theorem 2.2 - see (89), where in the latter case one can take \( \delta = 0 \).

**Proof of Corollary 3.** Since weak solutions are unique and also global in time \( (p \leq m, k \leq q) \), they generate dynamic flow - say a semigroup \( S(t) \) - on \( H \). This semigroup is strongly continuous in \( H \) with respect to time (see Corollary 1) and also, on the strength of Corollary (2), with respect to the initial data. This implies that \( (S(t), H) \) is a dynamical system.
Appendix: Proof of Proposition 1. The proof is elementary and only for reader’s convenience we outline the details. The first and third part of Proposition 1 follow from the proof presented in [16]. For the second part of Proposition 1, we need to show that \( \lim_{h \to 0} D_h u = u_t \) in \( L_p(0, T; Y) \), for all \( u \in W^{1,p}(0, T; Y) \), where \( 1 < p < \infty \). For this it suffices to analyze \( \lim_{h \to 0} \frac{1}{h} u_h^+ = u_t \) in \( L_p(0, T; Y) \) with the same conclusion being valid for \( \lim_{h \to 0} \frac{1}{h} u_h^- = u_t \) in \( L_p(0, T; Y) \), and the operator \( D_h \), we obtain \( \lim_{h \to 0} D_h u = u_t \) in \( L_p(0, T; Y) \).

Step 1. The following inequality is proved by standard calculus argument (note that \( u(t) = u(T), t > T \)) and interpolation theorem.

\[
\frac{1}{h} u_h^+(t)_{L_p(0,T;Y)} \leq C |u_t(t)|_{L_p(0,T;Y)} \text{ for any } p \geq 1 \tag{92}
\]

Indeed, (92) follows from two extreme ends \( p = 1 \) and \( p = \infty \) estimates followed by the application of Marcinkiewicz Interpolation Theorem.

Step 2. We will apply the well-known “stability-consistency” theorem which states that stability plus consistency implies convergence.

Theorem 6.1. Let \( Y \) be a Banach space, \( T \) a linear operator and \( \{T_n\}_{n=1}^\infty \) : \( Y \to Y \) a sequence of linear operators such that \( |T_n y|_Y \leq C |T y|_Y \), for any \( y \in Y \). If there exists a dense set \( D \in Y \) such that \( |T_n y - T y|_Y \leq d_n |y|_D \), where \( d_n \to 0 \) for all \( y \in D \), then \( T_n y \to T y \) for all \( y \in Y \), as \( n \to \infty \).

We apply the previous theorem to the following family of operators:

\[
T_h : L_p(0, T; Y) \to L_p(0, T; Y) \text{ defined by } T_h u = \frac{1}{h} u_h^+ \tag{93}
\]

\[
T : L_p(0, T; Y) \to L_p(0, T; Y) \text{ defined by } T u = u_t \tag{94}
\]

From Step 1, \( |T_h u|_{L_p(0,T;Y)} \leq C |T u|_{L_p(0,T;Y)} \) for any \( u \in L_p(0,T;Y) \). What is left to show is that there exists a dense set \( D \in L_p(0, T; Y) \) such that \( |T_h u - T u|_{L_p(0,T;Y)} \leq c_h |u_D|, c_h \to 0, \) for all \( u \in D \). But here it suffices to take \( D = W^{2,p}(0,T;Y) \), a dense subset of \( L_p(0,T;Y) \), when \( 1 < p < \infty \).

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