Lifespan of solutions to a damped fourth-order wave equation with logarithmic nonlinearity

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Abstract This paper is devoted to the lifespan of solutions to a damped fourth-order wave equation with logarithmic nonlinearity

\[ u_{tt} + \Delta^2 u - \Delta u - \omega \Delta u_t + \alpha(t) u_t = |u|^{p-2} u \ln |u|. \]

Finite time blow-up criteria for solutions at both lower and high initial energy levels are established, and an upper bound for the blow-up time is given for each case. Moreover, by constructing a new auxiliary functional and making full use of the strong damping term, a lower bound for the blow-up time is also derived.

Keywords Lifespan; Damped; Fourth-order wave equation; Logarithmic nonlinearity; Initial energy.

AMS Mathematics Subject Classification 2010: 35L35, 35L76.

1 Introduction

In this paper, we are concerned with the following initial boundary value problem for a damped fourth-order wave equation with logarithmic nonlinearity

\[
\begin{aligned}
&u_{tt} + \Delta^2 u - \Delta u - \omega \Delta u_t + \alpha(t) u_t = |u|^{p-2} u \ln |u|, \quad (x,t) \in \Omega \times (0,T), \\
&u(x,t) = \Delta u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,
\end{aligned}
\]  

(1.1)

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary \( \partial\Omega \), \( T \in (0, +\infty) \) is the maximal existence time of the solution \( u(x,t) \), \( \omega > 0 \), \( \alpha(t) : [0, \infty) \to [0, \infty) \) is a nonincreasing bounded differentiable function, and the exponent \( p \) satisfies

\[ (A) \quad 2 < p < 2_*, \]

where \( 2_* = +\infty \) if \( n \leq 4 \) and \( 2_* = \frac{2n}{n-4} \) if \( n \geq 5 \).

Problems like (1.1) have their roots in many branches of physics such as nuclear physics, optics and geophysics. They may also be used to describe some phenomena of granular materials such as the longitudinal motion of an elastic-plastic bar. Interested reader may refer to [1, 2, 3, 4, 10] for more background of problems like (1.1). It is well known that the damping terms
(both strong $\Delta u_t$ and weak $u_t$) prevent solutions from blowing up while the nonlinear terms force solutions to blow up. So it is of great interest to investigate how one dominates the other, and much effort has been devoted to this direction during the past few years. For example, Gazzola et al. \[9\] investigated the following damped wave equation

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = |u|^{p-2}u$$

(1.2)

in a bounded domain of $\mathbb{R}^n$, where $\omega \geq 0$, $\mu > -\omega \lambda_1$ and $p > 2$. By using the potential well method first proposed by Sattinger et al. \[23, 24\], they obtained the existence of global and finite time blow-up solutions to (1.2) for initial data at different energy levels. As for the damped fourth-order wave equations, Lin et al. \[20\] considered the following hyperbolic equation with strong damping

$$u_{tt} + \Delta^2 u - \Delta u - \omega \Delta u_t = f(u)$$

(1.3)

in a bounded domain of $\mathbb{R}^n$ with $\omega > 0$. Under certain conditions on the initial data and on the nonlinearity $f$, they proved the existence of global weak solutions and global strong solutions by using the classical potential well method. When the nonlinearity $f(u)$ grows super-linearly with respect to $u$ as $u$ tends to infinity, the solutions to (1.3) may blow up in finite time. In 2018, Wu \[25\] considered the following initial boundary value problem

$$\begin{cases}
u_{tt} + \Delta^2 u - \Delta u - \omega \Delta u_t + \alpha(t)u_t = |u|^{p-2}u, & (x, t) \in \Omega \times (0, T), \\
u(x, t) = \Delta u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\
u(x, 0) = u_0(x), & x \in \Omega,
end{cases}$$

(1.4)

where $\omega$ and $\alpha(t)$ fulfill the same conditions as that of problem (1.1) and $p$ satisfies the so-called subcritical condition, i.e.,

$$p \in (2, \infty) \quad \text{if} \quad n \leq 4; \quad p \in \left(\frac{2n}{n-4}\right) \quad \text{if} \quad n \geq 5.$$  

After showing that the unstable set is invariant under the flow of (1.4), he proved a blow-up result for problem (1.4) with initial energy smaller than the depth of the potential well, by applying concavity argument. Moreover, a lower bound for the blow-up time is derived. Later, problem (1.4) was reconsidered by Guo et al. \[10\] and the results of \[25\] were extended in two aspects. The first is that they obtained a blow-up result for high initial energy, and the second is that lower bound for the blow-up time is also derived for some supercritical $p$, with the help of inverse Hölder’s inequality and interpolation inequality.

On the other hand, evolution equations with logarithmic nonlinearity have also attracted more and more attention in recent years, due to their wide applications to quantum field theory and other applied sciences. Among the huge amount of interesting literature, we only refer the interested reader to \[5, 6, 7, 8, 11, 12, 14, 15, 16, 19, 22\], where qualitative properties of solutions to hyperbolic or parabolic equations with logarithmic nonlinearities were studied. In particular, Di et al. \[8\] considered the following initial boundary value problem for a semilinear wave equation with strong damping and logarithmic nonlinearity

$$\begin{cases}u_{tt} - \Delta u - \Delta u_t = |u|^{p-2}u, & (x, t) \in \Omega \times (0, T), \\
u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\
u(x, 0) = u_0(x), & x \in \Omega,
end{cases}$$

(1.5)

when $\Omega \subset \mathbb{R}^n(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $2 < p < +\infty$ if $n = 1, 2$ and $2 < p < \frac{2n}{n-2}$ if $n \geq 3$. The existence of global or finite time blow-up solutions to problem
with initial energy less than or equal to the depth of the potential well was investigated by using the potential well method. Moreover, the decay rate of the energy functional was obtained for global solutions and upper and lower bounds for the blow-up time were also derived for blow-up solutions. However, the case that the initial energy is larger than the depth of the potential well was not considered in [8], and we do not know whether or not problem (1.5) admits finite time blow-up solutions for this case. In addition, the lower bound for the blow-up time was obtained when \( p \) is subcritical, i.e., \( p < \frac{2n-2}{n-2} \). When \( p \in \left( \frac{2n-2}{n-2}, \frac{2n}{n-2} \right) \) for \( n \geq 3 \), whether a lower bound for the blow-up time can be obtained is still open.

Motivated mainly by [8, 10, 25], we will consider problem (1.1) and investigate how the damping terms and logarithmic nonlinearity determine the blow-up conditions and blow-up time of the solutions. More precisely, we shall present some sufficient conditions for the solutions to problem (1.1) to blow up in finite time with both lower and high initial energy and derive an upper bound for the blow-up time for each case. Moreover, we also estimate a lower bound for the blow-up time, which, thanks to the strong damping term, also includes some supercritical case. For simplicity, we only consider (1.1) for the case \( \omega = 1 \) and \( \alpha(t) \equiv 1 \). The main results can be extended to the general case with little difficulty.

The organization of this paper is as follows. In Section 2, as preliminaries, some notations, definitions and lemmas that will be used in the sequel are introduced. Finite time blow-up of solutions and upper bound for the blow-up time with lower and high initial energy will be considered in Section 3 and Section 4, respectively. In Section 5 we derive a lower bound for the blow-up time.

2 Preliminaries

In this section, we introduce some notations and lemmas which will be used in the sequel. In what follows, we denote by \( \| \cdot \|_r \) the \( L^r(\Omega) \)-norm \((1 \leq r \leq \infty)\), by \((\cdot, \cdot)\) the \( L^2(\Omega) \)-inner product and by \( \lambda_1 > 0 \) the first eigenvalue of \(-\Delta\) in \( \Omega \) under homogeneous Dirichlet boundary condition. Set

\[
H = \left\{ u \in H^2(\Omega) \cap H_0^1(\Omega) \colon u = \Delta u = 0 \text{ on } \partial \Omega \right\},
\]

and equip it with the norm

\[
\| u \|_H = \sqrt{\| \Delta u \|_2^2 + \| \nabla u \|_2^2}.
\]

For simplicity, we also denote the \( H^1(\Omega) \)-norm by

\[
\| u \| = \sqrt{\| u \|_2^2 + \| \nabla u \|_2^2}.
\]

Obviously, for any \( u \in H \), we have

\[
\lambda_1 \| u \|^2 \leq \| u \|_H^2. \tag{2.1}
\]

For any \( u \in H \), define

\[
J(u) = \frac{1}{2} \| u \|_H^2 - \frac{1}{p} \int_{\Omega} |u|^p \ln |u| \, dx + \frac{1}{p^2} \| u \|_p^p, \tag{2.2}
\]

\[
I(u) = \| u \|_H^2 - \int_{\Omega} |u|^p \ln |u| \, dx, \tag{2.3}
\]

\[
N = \left\{ u \in H \setminus \{0\} \colon I(u) = 0 \right\}, \tag{2.4}
\]

\[
d = \inf_{u \in H \setminus \{0\}} \sup_{\lambda > 0} J(\lambda u) = \inf_{u \in N} J(u), \tag{2.5}
\]
where $\mathcal{N}$ is called the Nehari manifold and $d$ is the depth of the potential well (also called mountain pass level). In what follows, we shall show that $\mathcal{N}$ is non-empty and $d$ is positive.

The following lemma gives some properties of the so-called fibering map $J(\lambda u)$. Since the proof is more or less standard (see [8] for example), we omit it here.

**Lemma 2.1.** Let $p$ satisfy (A). Then for any $u \in H \setminus \{0\}$, we have

(i) $\lim_{\lambda \to 0^+} J(\lambda u) = 0$, $\lim_{\lambda \to +\infty} J(\lambda u) = -\infty$.

(ii) there exists a unique $\lambda^* = \lambda^*(u) > 0$ such that $\frac{d}{d\lambda} J(\lambda u)|_{\lambda = \lambda^*} = 0$. $J(\lambda u)$ is increasing on $0 < \lambda < \lambda^*$, decreasing on $\lambda^* < \lambda < +\infty$ and takes its maximum at $\lambda = \lambda^*$.

(iii) $I(\lambda u) > 0$ on $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ on $\lambda^* < \lambda < +\infty$ and $I(\lambda^* u) = 0$.

Let $\sigma$ be any positive number such that $p+\sigma < 2^*$. Then it is well known that the embedding from $H$ to $L^{p+\sigma}(\Omega)$ is compact and there is a positive constant $B_{\sigma}$ such that

$$\|u\|_{p+\sigma} \leq B_{\sigma}\|u\|_H, \quad \forall u \in H.$$  \hspace{1cm} (2.6)

**Lemma 2.2.** There is a positive constant $C_*$ such that $\|u\|_H \geq C_*$ for any $u \in \mathcal{N}$.

**Proof.** First, it follows from Lemma 2.1 (iii) that $\mathcal{N}$ is non-empty. For any $u \in \mathcal{N}$, using (2.6) and the basic inequality $\ln s \leq \frac{1}{\epsilon^2}\sigma^2$ for $s \geq 1$ and $\epsilon > 0$, we have

$$\|u\|_H^2 = \int_{\Omega} |u|^p \ln |u| dx = \int_{\Omega_1} |u|^p \ln |u| dx + \int_{\Omega_2} |u|^p \ln |u| dx$$

$$\leq \int_{\Omega_2} |u|^p \ln |u| dx \leq \frac{1}{\epsilon^2} \int_{\Omega_2} |u|^{p+\sigma} dx$$

$$\leq \frac{1}{\epsilon^2} \|u\|_{p+\sigma} \leq \frac{B_{p+\sigma}^{p+\sigma}}{\epsilon^2} \|u\|_H^{p+\sigma},$$

where $\Omega_1 = \{x \in \Omega : |u(x)| < 1\}$ and $\Omega_2 = \{x \in \Omega : |u(x)| \geq 1\}$. Recalling that $p > 2$, we obtain from (2.6) that $\|u\|_H \geq \left(\frac{\epsilon^2}{B_{p+\sigma}}\right)^{1/(p+\sigma-2)} \geq C_*$. The proof is complete. \hfill $\Box$

**Lemma 2.3.** The depth $d$ of the potential well is positive and there is a nonnegative function $v_0 \in \mathcal{N}$ such that $J(v_0) = d$.

**Proof.** By (2.2) and (2.3) we have

$$J(u) = \frac{p-2}{2p} \|u\|_H^2 + \frac{1}{p} I(u) + \frac{1}{p^2} \|u\|_p^p, \quad u \in H.$$  \hspace{1cm} (2.8)

Therefore, for any $u \in \mathcal{N}$, by combining Lemma 2.4 with (2.8) we obtain

$$J(u) \geq \frac{p-2}{2p} \|u\|_H^2 \geq \frac{p-2}{2p} C_*^2 \geq d_0 > 0.$$  \hspace{1cm} (2.9)

By the definition of $d$ one sees that $d \geq d_0$, i.e., $d$ is positive.

To show that $d$ can be attained, let $\{v_k\}_{k=1}^\infty \subset \mathcal{N}$ be a minimizing sequence of $J$. It is easy to check that $\{|v_k|\}_{k=1}^\infty \subset \mathcal{N}$ is also a minimizing sequence of $J$. Therefore, without loss of generality, we may assume that $v_k \geq 0$ a.e. in $\Omega$ for all $k \in \mathbb{N}$. Then $J(v_k)$ is bounded, which, together with (2.3), implies that $\{v_k\}_{k=1}^\infty \subset \mathcal{N}$ is bounded in $H$. Noticing that the embedding
from $H$ to $L^{p+\sigma}(\Omega)$ is compact, we see that there is a subsequence of \(\{v_k\}_{k=1}^{\infty} \subset N\), which we still denote by \(\{v_k\}_{k=1}^{\infty} \subset N\), and a $v_0 \in H$ such that

\[
v_k \rightharpoonup v_0 \text{ weakly in } H \text{ as } k \to \infty,
\]

\[
v_k \to v_0 \text{ strongly in } L^{p+\sigma}(\Omega) \text{ as } k \to \infty,
\]

\[
v_k \to v_0 \text{ a.e. in } \Omega \text{ as } k \to \infty.
\]

Hence, $v_0 \geq 0$ a.e. in $\Omega$. Furthermore, using the dominated convergence theorem, we obtain

\[
\int_{\Omega} |v_0|^p \ln |v_0| dx = \lim_{k \to \infty} \int_{\Omega} |v_k|^p \ln |v_k| dx,
\] (2.10)

\[
\int_{\Omega} |v_0|^p dx = \lim_{k \to \infty} \int_{\Omega} |v_k|^p dx.
\] (2.11)

Moreover, by the weak lower semicontinuity of $\| \cdot \|_{H}$, we have

\[
\|v_0\|_H \leq \lim inf_{k \to \infty} \|v_k\|_H.
\] (2.12)

Therefore, it follows from (2.10)-(2.12) that

\[
J(v_0) \leq \lim inf_{k \to \infty} J(v_k) = d,
\] (2.13)

and

\[
I(v_0) \leq \lim inf_{k \to \infty} I(v_k) = 0.
\] (2.14)

It remains to show that $v_0 \not\equiv 0$ and $I(v_0) = 0$ to complete the proof. By (2.10) and Lemma 4.3 we know

\[
\int_{\Omega} |v_0|^p \ln |v_0| dx = \lim_{k \to \infty} \int_{\Omega} |v_k|^p \ln |v_k| dx
\]

\[
= \lim_{k \to \infty} \|v_k\|_H^2 \geq C^2_*,
\]

which implies that $v_0 \not\equiv 0$.

If $I(v_0) < 0$, then by Lemma 2.1 (iii) we know that there exists a $\lambda^* \in (0, 1)$ such that $I(\lambda^* v_0) = 0$, i.e., $\lambda^* v_0 \in N$. By the definition of $d$, we see

\[
d \leq J(\lambda^* v_0) = \frac{(p-2)\lambda^2}{2p} \|v_0\|_H^2 + \frac{\lambda^p}{p^2} \|v_0\|_p^p
\]

\[
= \lambda^2 \left[ \frac{p-2}{2p} \|v_0\|_H^2 + \frac{\lambda^{p-2}}{p^2} \|v_0\|_p^p \right]
\]

\[
< \lambda^2 \left[ \frac{p-2}{2p} \|v_0\|_H^2 + \frac{1}{p^2} \|v_0\|_p^p \right]
\]

\[
\leq \lambda^2 \lim inf_{k \to \infty} \left[ \frac{p-2}{2p} \|v_k\|_H^2 + \frac{1}{p^2} \|v_k\|_p^p \right]
\]

\[
= \lambda^2 \lim inf_{k \to \infty} J(v_k) = \lambda^2 d,
\]

a contradiction. Therefore, $I(v_0) = 0$ and $v_0 \not\equiv 0$, which means that $v_0 \in N$. Recalling (2.13) and the definition of $d$ again one sees that $J(v_0) = d$. The proof is complete. □
In this paper, we consider weak solutions to problem (1.1). For completeness, we state, without proof, the local existence theorem which can be established by slightly modifying the argument in [21]. Sometimes \( u(x,t) \) will be simply written as \( u(t) \) if no confusion arises.

**Theorem 2.1.** (See [13, 17]) Let \( u_0 \in H \) and \( u_1 \in L^2(\Omega) \). Then the problem (1.1) admits a unique weak solution \( u \in L^\infty(0, T_0; H) \), \( u_1 \in L^\infty(0, T_0; L^2(\Omega)) \cap L^2(0, T_0; H^1(\Omega)) \), for \( T_0 > 0 \) suitably small. Moreover, the energy functional satisfies
\[
E'(t) = -\int_\Omega (u_t^2 + |\nabla u|^2) \, dx \leq 0, \tag{2.15}
\]
where
\[
E(t) = E(u(t)) = \frac{1}{2} \|u_t\|_2^2 + J(u(t)). \tag{2.16}
\]

At the end of this section, we present the well-known concavity lemma which will play essential role in proving the blow-up result.

**Lemma 2.4.** (See [13, 17]) Suppose that a positive, twice-differentiable function \( \psi(t) \) satisfies the inequality
\[
\psi''(t)\psi(t) - (1 + \theta)(\psi'(t))^2 \geq 0,
\]
where \( \theta > 0 \). If \( \psi(0) > 0 \), \( \psi'(0) > 0 \), then \( \psi(t) \to \infty \) as \( t \to t_* \leq t^* = \frac{\psi(0)}{\psi'(0)} \).

## 3 Blow-up for lower initial energy

In this section, we will investigate the blow-up phenomena of solutions to problem (1.1) with lower initial energy. We first show that the unstable set \( \mathcal{U} \) is invariant under the flow of problem (1.1), where
\[
\mathcal{U} = \{ u \in H : I(u) < 0, \ J(u) < d \}, \tag{3.1}
\]
and \( d \) is the depth of the potential well defined in (2.5).

**Lemma 3.1.** Let \( u_0 \in \mathcal{U} \) and \( u_1 \in L^2(\Omega) \) such that \( E(0) < d \). Then \( u(t) \in \mathcal{U} \) for all \( t \in [0, T) \) and
\[
\frac{p - 2}{2p} \|u(t)\|_H^2 + \frac{1}{p^2} \|u(t)\|_p^p > d, \quad \forall \ t \in [0, T). \tag{3.2}
\]

**Proof.** First, it follows from (2.2), (2.16) and (2.15) that
\[
J(u(t)) \leq E(t) \leq E(0) < d, \quad \forall \ t \in [0, T).
\]

Therefore, in order to prove \( u(t) \in \mathcal{U} \) for all \( t \in [0, T) \), it suffices to show that \( I(u(t)) < 0 \) for all \( t \in [0, T) \). Assume by contradiction that there exists a \( t_1 \in (0, T) \) such that \( u(t_1) \in \mathcal{N} \). Then by the variational definition of \( d \), we obtain
\[
d \leq J(u(t_1)) \leq E(t_1) \leq E(0) < d, \tag{3.3}
\]
a contradiction.

For any \( t \in [0, T) \), since \( I(u(t)) < 0 \), it follows from Lemma 2.1 (iii) that there exists a \( \lambda(t) \in (0, 1) \) such that \( I(\lambda(t)u(t)) = 0 \), i.e., \( \lambda(t)u(t) \in \mathcal{N} \). By the definition of \( d \) and (2.8), we have
\[
\frac{p - 2}{2p} \|u(t)\|_H^2 + \frac{1}{p^2} \|u(t)\|_p^p \geq \frac{(p - 2)\lambda^2(t)}{2p} \|u(t)\|_H^2 + \frac{\lambda^p(t)}{p^2} \|u(t)\|_p^p = J(\lambda^2(t)u(t)) \geq d.
\]

The proof is complete.
With the preliminaries given above, we can show the first blow-up results for problem (1.1) with lower initial energy.

**Theorem 3.1.** Let $p$ satisfy $(A)$, $u_0 \in U$ and $u_1 \in L^2(\Omega)$ such that $E(0) < d$. Then the solution $u(x,t)$ to problem (1.1) blows up at a finite time $T$ in the sense that

$$
\lim_{t \to T^-} \left( \|u(t)\|_2^2 + \int_0^t \|u(s)\|^2 \, ds \right) = \infty.
$$

(3.3)

Moreover, the blow-up time $T$ can be estimated from above as follows

$$
T \leq \frac{4 \left( (a^2 + (p - 2)b s \|u_0\|_2^2)^{1/2} + a \right)}{(p - 2)b},
$$

(3.4)

where $a, b$ are constants that will be fixed in the proof.

**Proof.** Assume by contradiction that the solution $u$ exists globally. As was done in [25], fix $T^* > 0$ and define the functional

$$
G(t) = \|u(t)\|_2^2 + \int_0^t \|u(s)\|^2 \, ds + (T^* - t) \|u_0\|^2 + b(t + \tau)^2, \quad t \in [0, T^*],
$$

where $T^*$, $b$, and $\tau$ are positive constants to be fixed later. Taking derivative we have

$$
G'(t) = 2(u, u_t) + 2 \int_0^t (uu_s + \nabla u \cdot \nabla u_s) \, dx + 2b(t + \tau).
$$

(3.6)

Taking derivative again and using (2.15), (2.16) and Lemma 3.1, we obtain

$$
G''(t) = 2\|u_t\|_2^2 + 2(u, u_{tt}) + 2 \int_\Omega (uu_t + \nabla u \cdot \nabla u_t) \, dx + 2b
$$

$$
= (p + 2)\|u_t\|_2^2 + (p - 2)\|u\|_H^2 - 2pE(t) + 2b + \frac{2}{p^2}\|u\|_p^p
$$

$$
\geq (p + 2)\|u_t\|_2^2 + 2pd - 2pE(t) + 2b
$$

$$
= (p + 2)\|u_t\|_2^2 + 2p(d - E(0)) + 2p \int_0^t \|u_s(s)\|^2 \, ds + 2b.
$$

Choosing $b = d - E(0) > 0$ and noticing $p > 2$ we get

$$
G''(t) \geq (p + 2)(\|u_t\|_2^2 + \int_0^t \|u_s(s)\|^2 \, ds + b) > 0.
$$

(3.8)

Combining (4.15), (3.6) with (3.8) we know, for any $t \in [0, T^*]$, that

$$
G(t)G''(t) - \frac{p + 2}{4}(G'(t))^2
$$

$$
\geq (p + 2) \left[ \|u(t)\|_2^2 + \int_0^t \|u(s)\|^2 \, ds + b(t + \tau)^2 \right] \cdot \left[ \|u_t\|_2^2 + \int_0^t \|u_s(s)\|^2 \, ds + b \right]
$$

$$
- (p + 2) \left[ (u, u_t) + \int_\Omega (uu_s + \nabla u \cdot \nabla u_s) \, dx + b(t + \tau) \right]^2.
$$

Since

$$
(u, u_t) \leq \|u(t)\|_2 \|u_t\|_2,
$$

(3.9)
\[
\int_0^t \int_\Omega u_0 \, dx \, ds \leq \left( \int_0^t \int_\Omega u^2 \, dx \, ds \right)^{1/2} \left( \int_0^t \int_\Omega u_s^2 \, dx \, ds \right)^{1/2}, \\
\int_0^t \int_\Omega \nabla u \cdot \nabla u_s \, dx \, ds \leq \left( \int_0^t \int_\Omega |\nabla u|^2 \, dx \, ds \right)^{1/2} \left( \int_0^t \int_\Omega |\nabla u_s|^2 \, dx \, ds \right)^{1/2},
\]

where we require that \( G \) is finite by the above argument and let \( T = \max \{ 0, 2\| u_0 \|^2 - (p - 2)(u_0, u_1) \} \), then

\[
G(0) = \| u_0 \|^2 + T^* \| u_0 \|^2 + br^2 > 0, \\
G'(0) = 2(u_0, u_1) + 2br > 0,
\]

and

\[
\frac{4G(0)}{(p - 2)G'(0)} = \frac{2[\| u_0 \|^2 + T^* \| u_0 \|^2 + br^2]}{(p - 2)(u_0, u_1) + br} \leq T^*,
\]

for suitably large \( T^* \). According to Lemma 2.4, there exists a \( T_0 > 0 \) satisfying

\[
T_0 \leq \frac{4G(0)}{(p - 2)G'(0)}
\]

such that

\[
G(t) \to \infty \text{ as } t \to T_*^-.
\]

This contradicts with the assumption that \( G(t) \) is well defined on the closed \([0, T^*] \) for any \( T^* > 0 \).

To derive an upper bound for the blow-up time, we proceed as follows: Let \( T \) be the maximal existence time of \( u(x, t) \) (which is finite by the above argument) and let \( G(t) \) be given in (4.4), with the exception that \( T^* \) is replaced by \( T \) and \( t \in [0, T] \), where \( T \leq 0, T \). Similarly to the foregoing arguments, one can show that

\[
T \leq 2[\| u_0 \|^2 + T |u_0|^2 + br^2]
\]

which guarantees

\[
T \leq T(\tau) \leq \frac{2[\| u_0 \|^2 + br^2]}{(p - 2)(u_0, u_1) + br} - 2\| u_0 \|^2.
\]

Set \( a = 2\| u_0 \|^2 - (p - 2)(u_0, u_1) \) and \( \tau_0 = \frac{\sqrt{a^2 + (p - 2)^2 b^2 u_0^2}}{(p - 2)b} \). Then it is an easy matter to verify that \( \tau_0 \) satisfies (3.12), \( T(\tau) \) attains its minimum at \( \tau_0 \) and

\[
T(\tau_0) = \frac{4[\sqrt{a^2 + (p - 2)^2 b^2 u_0^2}]^{1/2} + a}{(p - 2)b}.
\]
Therefore,
\[ T \leq \frac{4\left[(a^2 + (p - 2)^2b\|u_0\|_2^2)^{1/2} + a\right]}{(p - 2)^2b}. \]
The proof is complete.

**Remark 3.1.** By (2.10) and (2.8) and recalling that \( p > 2 \) one sees \( E(0) < 0 \) implies \( I(u_0) < 0 \). Therefore, Theorem 3.4 implies that the solution \( u(x, t) \) to problem (1.1) blows up in finite time for negative initial energy.

## 4 Blow-up for high initial energy

In this section we shall build a blow-up criterion for problem (1.1) at high initial energy level. Some ideas used in this section are borrowed from [10] and [18]. As a preliminary, we first establish a lemma that will play a fundamental role.

**Lemma 4.1.** Let \( p \) satisfy (A). Assume that \( u_0 \in H \) and \( u_1 \in L^2(\Omega) \) such that
\[ 0 < E(0) < \frac{C_0}{p}(u_0, u_1). \] (4.1)
Then the solution \( u(x, t) \) to problem (1.1) satisfies
\[ (u, u_t) - \frac{p}{C_0}E(t) \geq \left[(u_0, u_1) - \frac{p}{C_0}E(0)\right]e^{C_0 t}, \quad t \in [0, T). \] (4.2)
Here
\[ C_0 = \min \left\{ p + 2, p(p - 2)\lambda_1, \frac{(p - 2)(\lambda_1 + \lambda_2^2)}{2} \right\} > 0. \] (4.3)

**Proof.** Set \( F(t) = (u, u_t) \). By direct calculations and recalling (2.10) we have
\[ F'(t) = \|u_t\|^2_2 + (u, u_{tt}) \]
\[ = \|u_t\|^2_2 + (u, -\Delta^2 u + \Delta u + \Delta u_t - u_t + |u|^{p-2}u \ln |u|) \]
\[ = \|u_t\|^2_2 - \|u_t\|^2_H - \int_{\Omega} \nabla u \cdot \nabla u_t dx - (u, u_t) + \int_{\Omega} |u|^p \ln |u| dx \]
\[ \leq \frac{p + 2}{2} \|u_t\|^2_2 + \frac{p - 2}{2} \|u\|^2_H - \int_{\Omega} \nabla u_t \cdot \nabla u_t dx - (u, u_t) - pE(t) + \frac{1}{p} \|u\|_p^p \]
\[ \geq \frac{p + 2}{2} \|u_t\|^2_2 + \frac{p - 2}{2} \|u\|^2_H - \int_{\Omega} \nabla u_t \cdot \nabla u_t dx - (u, u_t) - pE(t). \] (4.4)

By using Cauchy inequality, we can estimate the third and fourth terms in the last inequality as follows
\[ |\int_{\Omega} \nabla u \cdot \nabla u_t dx| \leq \frac{C_0}{4p} \|\nabla u\|_2^2 + \frac{p}{C_0} \|\nabla u_t\|_2^2, \] (4.5)
\[ |(u, u_t)| \leq \frac{C_0}{4p} \|u\|_2^2 + \frac{p}{C_0} \|u_t\|_2^2. \] (4.6)
Substituting (4.5) and (4.6) into (4.4) we arrive at
\[ F'(t) \geq \frac{p + 2}{2} \|u_t\|^2_2 + \frac{p - 2}{2} \|u\|^2_H - \frac{C_0}{4p} \|u\|^2 - \frac{p}{C_0} \|u_t\|^2 - pE(t). \] (4.7)
Set $H(t) = F(t) - \frac{p}{C_0} E(t)$. Then in view of (2.1), (2.15), (4.3) and (4.7) we obtain

$$H'(t) = F'(t) - \frac{p}{C_0} E'(t) = F'(t) + \frac{p}{C_0} \|u_t\|^2$$

$$\geq \frac{p}{2} \|u_t\|^2 + \frac{p-2}{2} \|u\|_\infty^2 - H - \frac{\lambda_1}{2} \|u\|^2 - pE(t)$$

$$\geq \frac{p}{2} \|u_t\|^2 + \left[ \frac{(p-2)\lambda_1}{2} - \frac{p(p-2)\lambda_1}{4p} \right] \|u\|^2 - pE(t)$$

$$= \frac{p}{2} \|u_t\|^2 + \frac{(p-2)(\lambda_1 + \lambda_2^2)}{4} \|u\|^2 - pE(t)$$

$$\geq C_0 \left[ \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u\|^2 - \frac{p}{C_0} E(t) \right] \geq C_0 H(t).$$

Since $H(0) > 0$ by (4.11) and (4.2) follows after an application of Gronwall’s inequality to $H(t)$. The proof is complete. \hfill \Box

With Lemma 4.1 at hand, we are now in the position to prove high initial energy blow-up and estimate an upper bound for the blow-up time for problem (1.1).

**Theorem 4.1.** Let all the assumptions in Lemma 4.1 hold. Then the solution $u(x,t)$ to problem (1.1) blows up at some finite time $T$ in the sense of (3.3). Moreover, if

$$E(0) < \frac{C_0}{p} \|u_0\|^2,$$

then

$$T \leq \frac{2(\|u_0\|^2 + \beta t_0^2)}{(p-2)(\|u_0\|^2 + \beta t_0) - 2\|u_0\|^2}.$$  \hspace{1cm} (4.10)

Here $C_0$ is the positive constant given in (4.3),

$$\beta = 2 \left[ \frac{C_0}{p} \|u_0\|^2 - E(0) \right] > 0,$$  \hspace{1cm} (4.11)

to is suitably large such that

$$(p-2) \left[ (u_0, u_1) + \beta t_0 \right] > 2\|u_0\|^2.$$  \hspace{1cm} (4.12)

**Proof.** We divide the proof into two steps.

**Step I: Finite time blow-up.** Suppose by contradiction that (3.3) will not happen for any finite $T$. Then $\|u(\cdot, t)\|_2$ is well-defined for all $t \geq 0$. Without loss of generality, we may assume that $E(t) \geq 0$ for all $t \geq 0$. Otherwise by Remark 3.1 we know that $u(x,t)$ blows up in finite time.

On one hand, it follows from Lemma 4.1 that

$$\frac{d}{dt} \|u(t)\|^2 = 2(u, u_t) \geq 2H(0)e^{C_0t} + \frac{2p}{C_0} E(t) \geq 2H(0)e^{C_0t}.$$  \hspace{1cm} (4.13)

Integration of (4.13) over $[0, t]$ yields

$$\|u(t)\|^2 = \|u_0\|^2 + 2 \int_0^t \int u u_x dx d\tau \geq \|u_0\|^2 + 2 \int_0^t H(0)e^{C_0\tau} d\tau$$

$$= \|u_0\|^2 + \frac{2H(0)}{C_0}(e^{C_0t} - 1).$$  \hspace{1cm} (4.14)
On the other hand, by virtue of Minkowski inequality, Hölder inequality, \[\text{and}\] the definition of \(\lambda_1\) and the fact \(E(t) \geq 0\) one gets
\[
\|u(t)\|_2 \leq \|u_0\|_2 + \|u(t) - u_0\|_2 = \|u_0\|_2 + \int_0^t u_\tau d\tau
\]
\[
\leq \|u_0\|_2 + \int_0^t \|u_\tau\| d\tau \leq \|u_0\|_2 + \frac{1}{\sqrt{1 + \lambda_1}} \int_0^t \|u_\tau\| d\tau
\]
\[
\leq \|u_0\|_2 + \frac{\sqrt{T}}{\sqrt{1 + \lambda_1}} \left( \int_0^T \|u_\tau\|^2 d\tau \right)^{1/2} = \|u_0\|_2 + \frac{\sqrt{T}}{\sqrt{1 + \lambda_1}} (E(0) - E(t))^{1/2}
\]
\[
\leq \|u_0\|_2 + \sqrt{\frac{E(0)}{1 + \lambda_1}} t^{1/2},
\]
which contradicts (4.14) when \(t\) is sufficiently large. Therefore, \(u(x, t)\) blows up in finite time.

**Step II: Upper bound for the blow-up time.** From now on, we assume that \(T > 0\) is the blow-up time of \(u(x, t)\), which is finite by Step I. According to Lemma \[\text{and}\] the assumption that \(E(t) \geq 0\), we see that
\[
\frac{d}{dt} \|u(t)\|_2^2 = 2(u, u_t) \geq 2H(0)e^{C_0 t} + \frac{2p}{C_0} E(t) > 0, \quad t \in [0, T),
\]
which implies \(\|u(t)\|_2^2\) is increasing with respect to \(t\). To estimate \(T\) from above, as was done in the proof of Theorem \[\text{we define}\]
\[
K(t) = \|u(t)\|_2^2 + \int_0^t \|u(s)\|^2 ds + (T - t)\|u_0\|^2 + \beta(t + t_0)^2, \quad t \in [0, T),
\]
where \(\beta\) and \(t_0\) are given in (4.11) and (4.12), respectively. By applying similar argument to that in the proof of the first part of Theorem \[\text{we obtain}\]
\[
K(t)K''(t) - \frac{p + 2}{4} (K'(t))^2
\]
\[
= 2K(\|u_t\|_2^2 - \|u\|_2^2 + \int_{\Omega} |u|^p \ln |u| dx + \beta)
\]
\[
- (p + 2) \left( (u, u_t) + \int_0^t \int_{\Omega} uu_s + \nabla u \cdot \nabla u_s dx ds + \beta(t + t_0) \right)^2
\]
\[
= 2K(\|u_t\|_2^2 - \|u\|_2^2 + \int_{\Omega} |u|^p \ln |u| dx + \beta)
\]
\[
+ (p - 2) \left[ \eta(t) - (K(t) - (T - t)\|u_0\|^2) (\|u_t\|_2^2 + \int_0^t \|u_s\|^2 ds + \beta) \right],
\]
where
\[
\eta(t) = \left[ \|u(t)\|_2^2 + \int_0^t \|u(s)\|^2 ds + \beta(t + t_0)^2 \right] \left[ \|u_t\|_2^2 + \int_0^t \|u_s\|^2 ds + \beta \right]
\]
\[
- \left( (u, u_t) + \int_0^t \int_{\Omega} uu_s + \nabla u \cdot \nabla u_s dx ds + \beta(t + t_0) \right)^2.
\]
Using \[\text{and}\] Cauchy-Schwarz inequality we can show that \(\eta(t) \geq 0\) on \([0, T)\). There-
As was done in deriving (2.15), (2.16), (4.11), and the monotonicity of \( \|u(t)\|_2^2 \), we have

\[
K(t)K''(t) - \frac{p+2}{4}(K'(t))^2
\]

\[
\geq 2K(t)\left[\|u_t\|_2^2 - \|u\|_H^2 + \int_{\Omega} |u|^p \ln |u| dx + \beta\right] - (p+2)K(t)\left[\|u_t\|_2^2 + \int_0^t \|u_s\|^2 ds + \beta\right]
\]

\[
= K(t)\left[(p-2)\|u\|_H^2 - 2pE(0) + (p-2)\int_0^t \|u_s\|^2 ds + \frac{2}{p}\|u\|_p^p - p\beta\right]
\]

\[
\geq K(t)\left[(p-2)(\lambda_1 + \lambda_1^2)\|u\|_2^2 - 2pE(0) - p\beta\right]
\]

\[
\geq 2pK(t)\left[C_0 \|u_0\|_2^2 - E(0) - \beta/2\right] \geq 0, \quad t \in [0, T).
\]

Besides, \( K(0) = \|u_0\|_2^2 + T\|u_0\|^2 + \beta t_0^2 > 0 \) and \( K'(0) = 2(u_0, u_1) + 2\beta t_0 > 0 \) by (4.12). Applying Lemma 2.4 to \( K(t) \) yields

\[
T \leq \frac{4K(0)}{(p-2)K'(0)} = \frac{2(\|u_0\|_2^2 + T\|u_0\|^2 + \beta t_0^2)}{(p-2)((u_0, u_1) + \beta t_0)}.
\]

Since \( \frac{2\|u_0\|^2}{(p-2)((u_0, u_1) + \beta t_0)} < 1 \) by (4.12), we further obtain

\[
T \leq \frac{2(\|u_0\|_2^2 + \beta t_0^2)}{(p-2)((u_0, u_1) + \beta t_0) - 2\|u_0\|^2}.
\]

The proof of Theorem 4.1 is complete. \( \square \)

**Remark 4.1.** As was done in deriving (3.4), one can also minimize the right-hand side term of (4.10) for \( t_0 \) satisfying (4.12) to obtain a more accurate upper bound for \( T \). Interested reader may check it.

## 5 Lower bound for the blow-up time

Since the lower bound for the blow-up time provides a safe time interval for the system under consideration, it is more important in practice to estimate \( T \) from below. In this section, our aim is to determine a lower bound for the blow-up time of problem \( (1.1) \) by constructing a new auxiliary functional. Throughout this section we shall use \( C, C_1, C_2, \cdots \) to denote generic positive constants which may depend on \( \Omega, p, n \), but are independent of the solution \( u(x, t) \).

**Theorem 5.1.** Assume that \( p \) satisfies

\[
\frac{2n}{n+2} < \frac{2n(p-1)}{n+2} < 2s, \quad \text{i.e.,}
\]

\[
p \in (2, \infty) \quad \text{if} \quad n \leq 4, \quad p \in \left(2, \frac{2n-2}{n-4}\right) \quad \text{if} \quad n \geq 5.
\]

Let \( u(x, t) \) be a weak solution to problem \( (1.1) \) that blows up at \( T \) in the sense of (3.3). Then

\[
T \geq \int_{N(0)}^{\infty} \frac{ds}{C_4 + C_5 s^{p-1+\mu}},
\]

where \( N(0) = \|u_1\|_2^2 + \|u_0\|_H^2 \).
Proof. For simplicity, we only prove this theorem for \( n \geq 3 \). The case for \( n = 1, 2 \) is similar (and simpler). We aim to determine a time interval \((0, T_0)\) on which the quantity \( \|u(t)\|_H^2 \) is bounded. Clearly \( T_0 \) is a lower bound for \( T \) since both \( \|u(t)\|_2^2 \) and \( \|u(t)\|^2 \) can be bounded by \( \|u(t)\|_H^2 \).

Define

\[
N(t) = \|u_t(t)\|_2^2 + \|u(t)\|_H^2, \quad t \in [0, T_0).
\] (5.2)

Then

\[
\lim_{t \to T_0^-} N(t) = +\infty. \tag{5.3}
\]

Differentiating (5.2) and making use of Green’s second identity, we obtain

\[
N'(t) = 2([u_t, u_{tt}] + (\Delta u, \Delta u_t) + (\nabla u, \nabla u_t))
\]

\[
= 2(u_t, u_{tt} + \Delta^2 u - \Delta u)
\]

\[
= 2(u_t, \Delta u_t - u_t + |u|^{p-2} \ln |u|)
\]

\[
= -2\|u_t\|^2 + 2 \int_{\Omega} u_t |u|^{p-2} \ln |u| \, dx. \tag{5.4}
\]

Set \( \Omega_1 = \Omega_1(t) = \{ x \in \Omega : |u(x, t)| < 1 \} \) and \( \Omega_2 = \Omega_2(t) = \{ x \in \Omega : |u(x, t)| \geq 1 \} \). Since \( p \) satisfies (5.1), we can choose \( \mu > 0 \) suitably small such that \( \frac{2(n(p-1)+\mu)}{n+2} < 2 \), which implies that \( H \) can be embedded into \( L^{\frac{2(n(p-1)+\mu)}{n+2}}(\Omega) \) continuously. We use \( B_\mu \) to denote the embedding constant from \( H \) to \( L^{\frac{2(n(p-1)+\mu)}{n+2}}(\Omega) \), i.e.,

\[
\|v\|_H \leq B_\mu \|v\|_H, \quad \forall \, v \in H. \tag{5.5}
\]

Using Hölder’s inequality, Cauchy inequality, (5.5) and the basic inequalities \( |s^{p-2} \ln s| \leq (e(p-1))^{-1} \) for \( 0 < s < 1 \) and \( \ln s \leq \frac{1}{e} s^\mu \) for \( s \geq 1 \), we can estimate the second term on the right-hand side of (5.4) as follows

\[
\int_{\Omega} u_t |u|^{p-2} \ln |u| \, dx = \int_{\Omega_1} u_t |u|^{p-2} \ln |u| \, dx + \int_{\Omega_2} u_t |u|^{p-2} \ln |u| \, dx
\]

\[
\leq \left( \int_{\Omega_1} |u_t|^{\frac{2n}{n+2}} \, dx \right)^{\frac{n+2}{2n}} \left( \int_{\Omega_1} |u|^{p-2} \ln |u| \right)^{\frac{2n}{n+2}} \frac{n+2}{2n}
\]

\[
+ \left( \int_{\Omega_2} |u_t|^{\frac{2n}{n+2}} \, dx \right)^{\frac{n+2}{2n}} \left( \int_{\Omega_2} |u|^{p-2} \ln |u| \right)^{\frac{2n}{n+2}} \frac{n+2}{2n}
\]

\[
\leq \|u_t\|_{\frac{2n}{n+2}} \left[ (e(p-1))^{-1}|\Omega_1|^{\frac{n+2}{2n}} + (e\mu)^{-1} \left( \int_{\Omega} |u|^{\frac{2(n(p-1)+\mu)}{n+2}} \, dx \right)^{\frac{n+2}{2n}} \right]
\]

\[
\leq C \|u_t\| \left[ (e(p-1))^{-1}|\Omega_1|^{\frac{n+2}{2n}} + (e\mu)^{-1} B_\mu^{p-1+\mu} \|u\|_H^{p-1+\mu} \right]
\]

\[
\leq \varepsilon \|u_t\|^2 + C(\varepsilon) \left[ C_1 + C_2 \|u\|_H^{p-1+\mu} \right]
\]

\[
\leq \varepsilon \|u_t\|^2 + C(\varepsilon) \left[ C_1 + C_3 N^{p-1+\mu}(t) \right]. \tag{5.6}
\]

Therefore, it follows by taking \( \varepsilon \leq 1 \) and substituting (5.6) into (5.4) that

\[
N'(t) \leq C_4 + C_5 N^{p-1+\mu}(t). \tag{5.7}
\]
Integrating (5.7) over $[0, t]$, we have

$$\int_{0}^{t} \frac{N'(\tau)}{C_4 + C_5 N^{p-1+\mu}(\tau)} \, d\tau \leq t. \quad (5.8)$$

Letting $t \to T_0^-$ and recalling (5.3), we obtain

$$\int_{N(0)}^{\infty} \frac{ds}{C_4 + C_5 s^{p-1+\mu}} \leq T_0 \leq T. \quad (5.9)$$

Recalling that $p-1+\mu > 1$, the left-hand side term in (5.9) is finite. The proof is complete. \(\square\)

**Remark 5.1.** By making full use of the damping term, we obtain the lower bound for the blow-up time not only for subcritical exponent $p$, but also for some supercritical ones. We point out that this observation can also be applied to problem (1.5) considered in [8].

**Acknowledgement**

The authors would like to express their sincere gratitude to Professor Wenjie Gao for his enthusiastic guidance and constant encouragement.

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