Abstract: A map is called edge-regular if it is edge-transitive but not arc-transitive. In this paper, we show that a complete graph $K_n$ has an orientable edge-regular embedding if and only if $n = p^d > 3$ with $p$ an odd prime such that $p^d \equiv 3 \pmod{4}$. Furthermore, $K_{p^d}$ has $\frac{p^d-3}{4d} \phi_1\left(\frac{p^d-1}{2}\right)$ non-isomorphic orientable edge-regular embeddings.

Keywords: edge-regular maps, arc-transitive maps, complete graphs

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1 Introduction

An orientable map is a 2-cell embedding of a finite graph in an orientable surface, and thus a map is an incident triple of vertex set $V$, edge set $E$, and face set $F$, denoted by $M = (V, E, F)$. The finite graph is called the underlying graph of the map, and the surface is called the supporting surface of the map. In this paper, we focus on the orientable supporting surface.

An automorphism of a map $M = (V, E, F)$ is a permutation of $V \cup E \cup F$ that preserves the incidence relation between vertices, edges, and faces. A map $M$ is called edge-transitive or an edge-transitive embedding of its underlying graph or arc-transitive if the automorphism group $\text{Aut}M$ is transitive on the edge set or the arc set, respectively. Sometimes, the arc-transitive map is called a rotary map or a symmetrical map. For convenience, a map $M$ is called edge-regular if $M$ is edge-transitive but not arc-transitive.

The purpose of this paper is to give a classification of orientable edge-regular maps with underlying graphs being complete graphs. For convenience, a map with underlying graph being a complete graph is called a complete map.

In general, a flag of a map is an incident triple of vertex, edge, and face. A map $M$ is called flag-transitive if $\text{Aut}M$ is transitive on the flag set. Since the action of $\text{Aut}M$ on the flag set is always semi-regular, a flag-transitive map is flag-regular, and it is simply called regular. It is easy to see that if a subgroup $G \leq \text{Aut}M$ is transitive on the edge set of $M$, then the index of $G$ in $\text{Aut}M$ is not more than four. Thus, an edge-transitive map is highly symmetrical.

The literature studying “symmetrical” maps is rich, refer to [1]. Recent investigation began with Biggs’ study in [2,3] of arc-transitive complete maps. Among the arc-transitive embeddings of $K_n$, constructed by Biggs, the unique embeddings for $n = 2, 3$, and $4$ are flag-regular, others with $n \geq 5$ a prime power are all arc-regular. In the past four decades, plenty of results about symmetrical maps have been obtained, see [4–10] and references therein. In particular, arc-transitive complete maps are classified in [3,7,11]; vertex-transitive complete maps are characterized in [12]. Very recently, some special families of edge-transitive
maps with underlying complete bipartite graphs are classified in [13–16]. In this paper, we study the edge-
regular complete maps. The main result of the paper is stated in the following theorem.

**Theorem 1.1.** A complete graph \( K_n \) has an orientable edge-regular embedding if and only if \( n = p^d \), where \( p \) is an odd prime and \( d \) is a positive integer such that \( p^d \equiv 3 \pmod{4} \). Furthermore, \( K_{p^d} \) has \( \frac{p^d - 3}{4d} \phi(p^d - 1) \) non-isomorphic edge-regular embeddings, where \( \phi \) is Euler’s function, and each of the maps has full automorphism group isomorphic to \( \mathbb{Z}_p^d : \mathbb{Z}_{p^d - 1} \).

**Remarks on Theorem 1.1.**
1. The main results of Theorem 1.1 were obtained by James [17] and Jones [18], both of which used the methods of group theories. However, the research approach adopted in the present paper is more concise and easier.
2. A map \( M \) is called a Cayley map if \( \text{Aut} M \) contains a subgroup \( G \) which is regular on the vertices. Furthermore, if \( G \subset \text{Aut} M \), then \( M \) is called a normal map. Thus, all the maps in Theorem 1.1 are normal maps.
3. A Cayley map \( M \) is called balanced if each generated element \( s \) and its inverse element \( s^{-1} \) of \( s \) are placed on the antipodal points, see [10]. It is known that an arc-transitive Cayley map \( M \) is balanced if and only if \( M \) is a normal map. Thus, the maps in Theorem 1.1 are not necessary balanced.

We end this section in the following corollary, the result can also be deduced from [19, Corollary 2].

**Corollary 1.2.** A normal edge-regular Cayley map is not necessarily a balanced map.

## 2 Construction and enumeration

Let \( p \) be an odd prime and \( d \) a positive integer. Let \( F = \text{GF}(p^d) \) be a field of order \( p^d \). Let \( F^+ \) and \( F^\times \) be the additive group and the multiplicative group of \( F \), respectively. Then \( F^+ \cong \mathbb{Z}_p^d \) and \( F^\times \cong \mathbb{Z}_{p^d - 1} \). Furthermore, \( F^\times \) naturally acts on \( F^+ \) by multiplication. This defines a group

\[
X = F^+ : F^\times \cong \text{AGL}(1, p^d),
\]

which is an affine group. Let \( k = p^d - 1 \). Since \( p \) is odd, \( k \) is even.

Assume that \( p^d \equiv 3 \pmod{4} \). Then \( \frac{k}{2} \) is odd, and \( d \) is odd too. Let \( F^\times = \langle \xi \rangle \), and let \( \rho = \xi^2 = \sigma = \xi^\frac{k}{2} \). Then \( \langle \xi \rangle = \langle \rho \rangle \times \langle \sigma \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Let

\[
G = F^+ : \langle \rho \rangle \cong \mathbb{Z}_p^d : \mathbb{Z}_4.
\]

We now construct edge-regular embeddings of the complete graph \( \Gamma = K_{p^d} \). Let \( \mathbf{0} \) be the identity of \( F^+ \). Then \( \Gamma = \text{Cay}(F^+, F^+ \setminus \{ \mathbf{0} \}) \), and in the case where \( p^d \equiv 3 \pmod{4} \), \( G \) is edge-transitive but not arc-transitive on \( \Gamma \). Noting that the group \( \langle \rho \rangle \) partitions \( \Gamma(\mathbf{0}) = F^+ \setminus \{ \mathbf{0} \} \) into two orbits, say \( \Delta_0 \) and \( \Delta_1 \). We make labelings for the elements of \( \Delta_0 \) and \( \Delta_1 \) as follows.

Let \( v \in \Delta_0 \) be a non-identity element of \( F^+ \), and set

\[
\beta_0 = v \quad \text{and} \quad \beta_{2i+2} = \beta_{2i}^p, \quad \text{where} \quad 0 \leq i < \frac{k}{2} - 1;
\]

and

\[
\beta_i \in F^+ \setminus \{ \mathbf{0}, \beta_0, \beta_2, \ldots, \beta_{k-2} \} \quad \text{and} \quad \beta_{2i+1} = \beta_{2i-1}^p, \quad \text{where} \quad 1 \leq i < \frac{k}{2}.
\]

Then the orbits \( \Delta_0 = \beta_0^{(\rho)} \) and \( \Delta_1 = \beta_1^{(\rho)} \).

For a vertex \( a \), a cyclic permutation of the neighbor set \( \Gamma(a) \) of \( a \) is called a rotation at \( a \) and denoted by \( R_a \). A rotation system \( R(\Gamma) \) of a graph \( \Gamma \) is the product of rotations at all vertices, that is, \( R(\Gamma) = \prod_{a \in V} R_a \). Then each rotation system \( R(\Gamma) \) defines an orientable embedding of \( \Gamma \), refer to [1, pp. 104–108].
Construction 2.1
Use the aforementioned notations.
(1) Label the end points of the arcs emitting from the vertex \(0 \in F^+\) as
\[\beta_0, \beta_1, \beta_2, \beta_3, \ldots, \beta_{k-1}.\]
(2) For each vertex \(x \in F^+\), label the end points of the arcs emitting from \(x\) as
\[\beta_0 x, \beta_1 x, \beta_2 x, \beta_3 x, \ldots, \beta_{k-1} x.\]
(3) Define a rotation of the end points of arcs emitting from \(x\) by
\[R_x = (\beta_0 x, \beta_1 x, \beta_2 x, \beta_3 x, \ldots, \beta_{k-1} x),\]
where \(x \in V = F^+\), and let \(R(\Gamma) = \prod_{x \in F^+} R_x.\)
(4) Observing that the rotation system \(R(\Gamma)\) is uniquely determined by \(\beta_0, \beta_1, \) and \(\rho\), we denote the orientable map defined by the rotation system as follows:
\[M(\beta_0, \beta_1, \rho).\]

We next study the maps \(M(\beta_0, \beta_1, \rho)\). The first lemma determines the relation between \(\beta_i\)'s and their inverses.

**Lemma 2.2.** There exist positive integers \(t\) and \(m\) such that \(\beta_1 = (\beta_0^{-1})^\rho = \beta_0^{-1}\) and \(\beta_0 = \beta_2m+1\), where \(m = \frac{p^d - 1}{2} - t\).

**Proof.** Assume that \(\beta_i\) and \(\beta_i^{-1}\) lie in the same orbit of \(\langle \rho \rangle\), where \(0 \leq i < p^d - 1\). Then \(\beta_i^{-1} = \beta_j^t\) with \(1 \leq t < \frac{p^d - 1}{2}\), and \((\beta_i^t)^\rho = (\beta_i^{-1})^\rho = (\beta_i^{-1})^{-1} = \beta_i\). Thus, \((\rho^t)^2 = 1\). A contradiction occurred since \(|\rho| = \frac{p^d - 1}{2}\) is odd. So \(\beta_i\) and \(\beta_i^{-1}\) lie in different orbits of \(\langle \rho \rangle\), where \(0 \leq i < p^d - 1\). Hence, \(\beta_0^{-1}\) lies in the orbit of \(\beta_i\) under \(\langle \rho \rangle\). It follows that there exists a positive integer \(t\) such that \(\beta_1 = (\beta_0^{-1})^\rho = (\beta_0^{-1})^{-1} = \beta_2^{-1}\).

Furthermore, \(\beta_0^{-1} = \beta_1^\rho\) and \(\beta_2^{-1} = \beta_1^{\rho^{p^d - 1 - t}} = \beta_2^{(\rho^{p^d - 1 - t})^t}\). Thus, letting \(m = \frac{p^d - 1}{2} - t\), we have \(\beta_0^{-1} = \beta_2m+1\). \(\square\)

For a vertex \(v \in VT\), noting that the vertex rotations \(R_v\) can be regarded as permutations not only of the set \(\Gamma(v)\) but also of the generating set \(S\). So Cayley maps have another equivalent definitions, see [19]. A map with underlying graph being Cayley graph \(\Gamma = \text{Cay}(G, S)\) is a Cayley map if the induced local cyclic permutations of \(S\) are all the same. The next lemma shows the edge-transitivity of \(M(\beta_0, \beta_1, \rho)\).

**Lemma 2.3.** Let \(M = M(\beta_0, \beta_1, \rho)\) and \(G = F^+ \cdot \langle \rho \rangle\), as defined above. Then the following statements hold.
(i) \(G \leq \text{Aut}M\), and \(G\) is transitive on the edges of \(M\);
(ii) \(M\) is a Cayley map of \(F^+\).

**Proof.** By the definition of the rotation system \(\prod_{x \in F^+} R_x\), each element of \(F^+\) is an automorphism of \(M\). Since \(F^+\) acts regularly on the vertices of \(M\), \(M\) is a Cayley map of \(F^+\), and the underlying graph of \(M\) is a complete graph of order \(p^d\). Furthermore, for each element \(x \in F^+\), the conjugation \(\rho x\) is such that, for \(0 \leq i \leq k - 1,$
\[(\beta_i x)^\rho = (\beta_i x)^{s^{-1} j} = \beta_i^{s j} x = \beta_i x,\]
reading the subscripts modulo \(k\). Thus, \(\rho\) is an automorphism of the map \(M\), refer to [1, pp. 109–110]. Since \(G\) is transitive on the edges of the underlying graph \(\Gamma\), we have that the map \(M\) is \(G\)-edge-transitive. \(\square\)

The Cayley map \(M(\beta_0, \beta_1, \rho)\) is balanced if \(s\) and \(s^{-1} = -s\) are placed on the antipodal points for all elements \(s \in F^+ \setminus \{0\}\). Let \(\rho_0\) be the unique involution of \(\text{GL}(1, p^d)\). Then
\[ \rho_0 : x \mapsto -x, \text{ for all } x \in F^+, \]

which is an automorphism of \( M \).

**Lemma 2.4.** A map \( M(\beta_0, \beta_1, \rho) \) is balanced if and only if \( \rho_0^{-1} = \beta_0 \rho^0, \) and \( \beta_1^{-1} = \beta_0 \rho^{-1}. \)

**Proof.** Assume that \( M(\beta_0, \beta_1, \rho) \) is balanced. Then for every \( i \) with \( 0 \leq i < \frac{p^d - 1}{2} \), the vertex \( \beta_i^{-1} \) is placed at the antipodal position of the vertex \( \beta_i \). Thus, \( \beta_0^{-1} = \beta_0 \rho_0^0 \) and \( \beta_1^{-1} = \beta_0 \rho^{-1}. \)

Conversely, assume that \( \beta_0^{-1} = \beta_0 \rho_0^0 \) and \( \beta_1^{-1} = \beta_0 \rho^{-1}. \) Then we have

\[
\beta_0^{-1} = (\beta_0^{-1})^\rho = \left( \beta_0 \rho_0^0 \right)^\rho = \beta_0 \rho_0^0, \quad \beta_1^{-1} = (\beta_1^{-1})^\rho = \left( \beta_0 \rho^{-1} \right)^\rho = \beta_0 \rho^{-1}.
\]

So \( \beta_i^{-1} = \beta_0 \rho^{-1} i \) is at the antipodal position of \( \beta_i \) for all \( j \) with \( 0 \leq j < \frac{p^d - 1}{2} \), and therefore, \( M(\beta_0, \beta_1, \rho) \) is balanced. \( \square \)

The mirror-image \( \overline{M} \) of an orientable map \( M \) is given by reversing the cyclic order of the edges at each vertex. An orientable map \( M \) is called **reflexible** if it has an automorphism that reverses the orientation of the supporting surface, equivalently, \( M \equiv \overline{M} \), see [20, Chapter 16].

**Lemma 2.5.** The orientable map \( M(\beta_0, \beta_1, \rho) \) is not reflexible, and the automorphism group \( \text{Aut} M \) is a subgroup of \( \text{AGL}(1, p^d) \).

**Proof.** Let \( M = M(\beta_0, \beta_1, \rho) \), and let \( A = \text{Aut} M \). Let \( A' \) be the group of elements of \( A \) which preserves the orientation of the supporting surface of \( M \). Then \( G \) is of index at most 2 in \( A' \), and \( A' \) is of index at most 2 in \( A \). Thus, \( G \triangleleft A' \triangleleft A \), and the index \( |A : G| = 1, 2, \) or 4. Since \( G \) is of odd order, it follows that \( G \) is a characteristic subgroup of \( A \). As \( G \) is primitive on the vertex set \( V \), so is \( A \). Hence, \( C_\alpha(G) \leq G \), and \( A \leq \text{Aut}(G) = \text{AGL}(1, p^d) = (F^+ : F^+) \rtimes \mathbb{Z}_d \). By Frattini’s argument, \( A_a \leq F^+ \rtimes \mathbb{Z}_d \) for any \( a \in V = F^+ \). Since \( A_a \leq D_{2p^d-1} \) and \( d \) is odd, it follows that \( A_a \leq F^+ \rtimes \mathbb{Z}_{p^d-1} \) and \( A \leq \text{AGL}(1, p^d) \).

If \( M \) is reflexible, then there exists an involution \( \tau \in A \setminus A' \) which reverses \( \rho \). However, note that \( d \) is odd, then this is not possible. So \( M \) is not reflexible. \( \square \)

An orientable map is called **chiral** if it is arc-regular, and each of its automorphisms preserves the orientation of the supporting surface. However, chirality does not always require arc-regularity, sometimes chiral simply means “not allowing reflections.” The next lemma determines the chirality of the maps \( M(\beta_0, \beta_1, \rho) \).

**Lemma 2.6.** Let \( M = M(\beta_0, \beta_1, \rho) \), a map as constructed in Construction 2.1. Then either

(i) \( M \) is chiral (arc-regular), \( \beta_i = \beta_0^i \), and \( \text{Aut} M = \text{AGL}(1, p^d) \); or

(ii) \( M \) is edge-regular, and \( \text{Aut} M = G \).

**Proof.** Assume that \( M \) is arc-transitive. Then \( \text{Aut} M = \text{AGL}(1, p^d) = F^+ : F^+ = F^+ : < \xi > \) by Lemma 2.5. Since \( \beta_1 \) and \( \beta_0 \) lie in different orbits of \( < \rho > = < \xi > \), there exists an odd integer \( \ell \) with \( 1 \leq \ell < p^d - 1 \) such that \( \beta_0 = \beta_0^{\xi^\ell} \). Thus, \( \beta_0^{\xi^\ell} = \beta_0 = \beta_1 = \beta_1^{\xi^\ell} = \beta_0^{\xi^\ell} \). Since \( < \xi > \) is regular on \( F^+ \setminus \{0\} \), it follows that \( \ell = 1 \), and so \( \beta_1 = \beta_0^{\xi} \).

Assume that \( M \) is not arc-transitive. Then by Lemma 2.5, \( \text{Aut} M = G \), which is regular on the edge set of \( M \). \( \square \)
Noting that the rotation \( R_k = (\beta_0 x, \beta_1 x, \beta_2 x, \beta_3 x, \ldots, \beta_{k-1} x) \) for all \( x \in V = F^+ \) is a circular permutation, namely,

\[
(\beta_0 x, \beta_1 x, \beta_2 x, \beta_3 x, \ldots, \beta_{k-1} x) = R_k = (\beta_k x, \beta_{k+1} x, \beta_{k+2} x, \beta_{k+3} x, \ldots, \beta_{k+k-1} x),
\]

where \( 0 \leq i \leq k - 1 \) and the subscripts are modulo \( k \), we have

\[
\mathcal{M}(\beta_0, \beta_1, \rho) = \mathcal{M}(\beta_{1i}, \beta_1, \rho).
\]

Therefore, to enumerate different edge-transitive embeddings of \( K_{p^d} \), we may fix the first element \( \beta_0 \).

The following lemma gives a counting formula of the maps \( \mathcal{M}(\beta_0, \beta_1, \rho) \).

**Lemma 2.7.** A complete graph \( K_{p^d} \) with \( p^d \equiv 3 \) (mod 4) has precisely \( \frac{p^d-3}{4d} \phi \left( \frac{p^d-1}{2} \right) \) non-isomorphic orientable edge-regular embeddings, and \( \frac{1}{d} \phi \left( \frac{p^d-1}{2} \right) \) non-isomorphic orientable chiral embeddings.

**Proof.** As argued above, we may fix \( \beta_0 \). Then there are precisely \( \frac{p^d-1}{2} \) choices for \( \beta_1 \), and there are precisely \( \phi \left( \frac{p^d-1}{2} \right) \) choices for \( \rho \), where \( \phi \) is Euler’s function. Thus, there are exactly \( \frac{p^d-1}{2} \phi \left( \frac{p^d-1}{2} \right) \) different triples for \( (\beta_0, \beta_1, \rho) \), so the number of different maps of the form \( \mathcal{M}(\beta_0, \beta_1, \rho) \) is equal to \( \frac{p^d-1}{2} \phi \left( \frac{p^d-1}{2} \right) \).

By Lemma 2.6, a map \( \mathcal{M}(\beta_0, \beta_1, \rho) \) is arc-transitive if and only if \( \beta_1 = \beta_0^i \) and if and only if it is arc-regular. Thus, the number of different arc-transitive embeddings of the form \( \mathcal{M}(\beta_0, \beta_0^i, \rho) \) of \( K_{p^d} \) equals the number of choices of \( \xi \) with \( |\xi| = p^d - 1 \), which is equal to \( \phi \left( \frac{p^d-1}{2} \right) \) since \( \frac{p^d-1}{2} \) is odd and \( \phi(p^d-1) = \phi \left( \frac{p^d-1}{2} \right) \times \phi(2) = \phi \left( \frac{p^d-1}{2} \right) \). The others are all edge-regular, and the number of them equals

\[
\frac{p^d-1}{2} \phi \left( \frac{p^d-1}{2} \right) - \phi \left( \frac{p^d-1}{2} \right) = \frac{p^d-3}{2} \phi \left( \frac{p^d-1}{2} \right).
\]

Now, we determine isomorphism classes of the \( \frac{p^d-1}{2} \phi \left( \frac{p^d-1}{2} \right) \) different maps described above. Let

\[
S = \left\{ \mathcal{M}(\beta_0, \beta_1, \rho) \mid \beta_1 \in F^\times \setminus \{0, \beta_0, \beta_2, \ldots, \beta_{k-1}\}, 1 \leq i < \frac{p^d-1}{2} \text{ and } \left( i, \frac{p^d-1}{2} \right) = 1 \right\}.
\]

Let \( \mathcal{M}_1, \mathcal{M}_2 \in S \) which are isomorphic, and let \( \varphi \) be an isomorphism from \( \mathcal{M}_1 \) to \( \mathcal{M}_2 \). Then \( \varphi \) induces an isomorphism from \( \text{Aut}\mathcal{M}_1 \) to \( \text{Aut}\mathcal{M}_2 \).

**Case 1.** First, assume that \( \mathcal{M}_1 \) is edge-regular. Then so is \( \mathcal{M}_2 \). By Lemma 2.6, we have

\[
\text{Aut}\mathcal{M}_1 = \text{Aut}\mathcal{M}_2 = F^\times : \langle \rho \rangle = G.
\]

Hence, \( G^\rho = (\text{Aut}\mathcal{M}_1)^\rho = \text{Aut}\mathcal{M}_2 = G \), namely, \( \varphi \) is an automorphism of \( G \). Since \( F^\times \) is a normal Sylow \( p \)-subgroup of \( G \), it follows that \( F^\times \) is a characteristic subgroup of \( G \), and \( \varphi \) normalizes \( F^\times \equiv Z_{p^d}^\rho \). Thus, \( \varphi \in \text{Aut}(Z_{p^d}^\rho) = \text{GL}(d, p) \), and \( \varphi \in N_{\text{GL}(d, p)}(\langle \rho \rangle) \). We may choose \( \varphi \) to normalize \( \langle \rho \rangle \). By [21, Theorem 7.3 (a) of Chapter 2],

\[
\varphi \in N_{\text{GL}(d, p)}(\langle \rho \rangle) = \Gamma(1, p^d) = Z_{p^d-1} = \langle \xi \rangle : \langle \tau \rangle,
\]

where \( \tau : \xi \mapsto \xi^p \) is the Frobenius automorphism of \( GF(p^d) \).

Since \( \rho = \xi^2 \) is an automorphism of \( M \in S \), it fixes all maps in \( S \). Let \( S_i \) be the subset of \( S \) which consists of edge-regular maps. So \( \rho = \xi^2 \) fixes each element of \( S_i \). It follows that \( \langle \xi \rangle : \langle \tau \rangle / \langle \xi \rangle \cong \langle \sigma \rangle \times \langle \tau \rangle \),

where \( \sigma = \xi^{p^d-1} \). Thus, \( N_{\text{GL}(d, p)}(\langle \rho \rangle) = \langle \xi \rangle : \langle \tau \rangle \) acting on \( S_i \) is isomorphic to \( \langle \sigma \rangle \times \langle \tau \rangle \). As \( \text{Aut}\mathcal{M} = G \), the group \( \langle \sigma \rangle \times \langle \tau \rangle \) acts semiregularly on \( S_i \). Obviously, two maps \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are isomorphic if and only if they lie in the same orbit of \( \langle \sigma \rangle \times \langle \tau \rangle \) acting on \( S_i \). Since \( |S_i| = \frac{p^d-3}{2} \phi \left( \frac{p^d-1}{2} \right) \), it follows that there are exactly

\[
\frac{1}{d} \cdot \frac{p^d-3}{2} \phi \left( \frac{p^d-1}{2} \right) \text{ non-isomorphic edge-regular maps in } S.
\]
\textbf{Case 2.} Assume now that $M_1$ and $M_2$ are arc-regular. By Lemma 2.6, we have
\[
\Aut M_1 = \Aut M_2 = F^\ast : \langle \xi \rangle = G. \langle \sigma \rangle,
\]
where $\sigma = \xi^{p^d - 1}$. So $(G.\langle \sigma \rangle)^p = (\Aut M_1)^p = \Aut M_2 = G.\langle \sigma \rangle$, namely, $\varphi$ is an automorphism of $G$. Arguing as in the previous case shows that $\varphi \in \Gamma L(1, p^d) = \mathbb{Z}_{p^d - 1} : \mathbb{Z}_d = \langle \xi \rangle : \langle \tau \rangle$, where $\tau : \xi \mapsto \xi^p$. Since for any $M(\beta_0, \beta_1, \rho^i) \in S$,
\[
M(\beta_0, \beta_1, \rho^i) = M(\beta_0^p, \beta_1^p, (\rho^i)^p) = M(\beta_0^p, \beta_1^p, \rho^p),
\]
and $M(\beta_0, \beta_1, \rho^i) \in S$ and $M(\beta_0^p, \beta_1^p, \rho^p) \in S$, it follows that each element of $\langle \xi \rangle : \langle \tau \rangle$ is an isomorphism among maps in $S$. Let $S_2$ be the subset of $S$ consisting of arc-regular maps. As $\xi$ is an automorphism of each map in $S_2$, the induced action of $\langle \xi \rangle : \langle \tau \rangle$ is isomorphic to $\langle \tau \rangle$ is semiregular on $S$. Then $\rho^i$ is said to be 2-regular. Assume that $G^\ast \cong \mathbb{Z}_{p}^d$ is a cyclic or dihedral group. By Kantor’s lemma 3.1. Let $\rho^i = \phi_{\langle \xi^p \rangle^2}$, where $\rho^i = \phi_{\langle \xi^p \rangle^2}$ is a 2-transitive but not arc-regular. By Lemma 2.6, we have $\rho^i = \phi_{\langle \xi^p \rangle^2} < \AGL(1, p^d)$.

\textbf{Proof.} Since $G$ is edge-transitive, but not arc-regular, we have that $G \leq S_n$ is a 2-homogeneous, but not 2-transitive permutation group on $V$. Noting that $G_n$ is a cyclic group or a dihedral group, we need to prove that $M$ is a map as given in Construction 2.1.

By Lemma 3.1, there exist an odd prime $p$ and a positive integer $d$ such that $G = N : G_n = \mathbb{Z}_p^d : \mathbb{Z}_{p^d - 1}$. where $n = p^d \equiv 3 \pmod{4}$, and $N$ is regular on the vertex set $V$. Thus, we may identify the vertex set $V$ with $N$. Furthermore, we identify $N$ with the additive group $F^\ast$ of the field $F = GF(p^d)$. Let the multiplicative group $F^\ast$ of $F$ be generated by $\xi$, namely, $F^\ast = \langle \xi \rangle = \mathbb{Z}_{p^d - 1}$. Let $\rho = \xi^2$, and let $\alpha$ be the vertex of $M$ corresponding to $0 \in F^\ast$. Then $G_n = \langle \rho \rangle \cong \mathbb{Z}_{p^d - 1}$.

The action of $G_n = \langle \rho \rangle$ on $V \setminus \{a\}$ is semiregular and divides $V \setminus \{a\}$ into two orbits. Since $\rho$ fixes the vertex $0$ and preserves the supporting surface, the rotation of the end points of the arcs emanating from $0$ has the form
\[
(\beta_0, \beta_1, \beta_2, \ldots, \beta_{k-2}, \beta_{k-1}).
\]
where $k = p^d - 1$ such that

$$\beta_i^p = \beta_{i+2}, \text{ for } 0 \leq i \leq k - 1,$$

reading the subscripts $i + 2$ modulo $k$. Thus, the two orbits of $\langle \rho \rangle$ acting on $\Gamma(a)$ are $\{\beta_0, \beta_2, \ldots, \beta_{k-2}\}$ and $\{\beta_1, \beta_3, \ldots, \beta_{k-1}\}$. Therefore, the rotation system for $\mathcal{M}$ is the same as the one constructed in Construction 2.1, and so the map $\mathcal{M}$ is as constructed in Construction 2.1. □

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