Quantum rings in a space with topological defects

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Abstract. Topological defects generally they occur in a system when phase transitions take place. We have examples of arrangements with these defects both in Gravitation and Condensed Matter. Depending on how they are generated, this kind of defects make the space present curvature or torsion. There is a wide literature about this matter, much from a gauge field point of view [1, 2, 3], and probing different aspects [6] as the existence of Landau levels [4, 5] or quantum phases [7], for example. However, there is not an unique fundamental theory for describing defects in solids. In our work we follow the Geometric Theory of Defects (TGD) which has the work by Katanaev [8] as a good way of describing continuum media with topological defects. This last theoretical framework is our compass. Besides, we are interested in mesoscopic systems, an important subject right before the role played by nanodevices. In our work we study quantum rings in a space with a screw dislocation. This kind of space presents torsion. The confinement is modelled by a potential that describes different kinds of mesoscopic systems (quantum dots, antidots and quantum wires). We found some results showing the changes in the behaviour of properties as energy eigenvalues and magnetization.

1. Introduction
The study of rings of mesoscopic dimension in the presence of external fields exhibits a series of interesting phenomena in physics, for example Aharonov-Bohm (AB) [9, 10], quantum hall [11], persistent currents [12, 13] and Berry’s phase [14]. This results has been exploited in various experiments [15, 16]. The simplest model for a ring-like device is a one-dimensional (1D) ring, in which the applied magnetic field can be represented by an AB flux tube. Such 1D models have played an important role in the understanding of quantum interference effects in rings.

Defects in solids modify properties as their melting point, stress and so on. However there is not a fundamental theory despite of several articles have been published about this matter. One of the best theories describing defects in solids is based upon Riemann-Cartan geometry in the seminal article by Katanaev [8]. In this formalism non-Euclidian metrics account for the boundary conditions imposed by defects in elastic media. The theory, in the continuum limit, describes the solid by a Riemann-Cartan manifold where curvature and torsion are associated to disclinations and dislocations, respectively, in the medium [17]. Besides, in this theoretical framework, the elastic deformations are incorporated in the metric of the manifold. We can assume that a medium without deformation is invariant by translations (spatial homogeneity) and by rotations (spatial isotropy) in any coordinates framework. In this preferential coordinates
framework the medium is described for a euclidian metric as

\[ ds^2 = \sum_{i=1}^{3} (dx^i)^2. \]  

(1)

The metric in (1) is named cartesian. For this kind of space, if we move a vector along a closed path, the vector can return to its initial position. This movement along a closed path is named parallel transport [18]. If there is a defect, the parallel transport the vectors moved cannot be back to the same initial point from where they had left.

In this work we deal with dislocations which can be viewed using the Volterra process [19]. The existence of dislocations is related to the presence of torsion in the medium. Torsion tensor and Burgers vector are related by Burgers

\[ b^i = \int \int_S dx^\mu \wedge dx^{\nu} T_{\mu\nu}^i, \]  

(2)

where \( dx^\mu \wedge dx^{\nu} \) is the surface element and \( T_{\mu\nu}^i \) is the torsion tensor. The integration region when Burgers vector points out in a certain direction \( x^3 \) corresponds to the surface generated by Volterra process, which before the dislocation appearing was formed by directions \( x^1 \) e \( x^2 \). By the expression (2) we see torsion tensor represents physically a superficial density of the Burgers vector [20]. In this work we study the quantum rings in the presence of topological defects. We use geometric theory of defects for describing two-dimensional (2D) quantum rings in the presence of a defect. We investigate situations where the defect is related to torsion. In addition we consider a potential that can describe different kinds of mesoscopic systems (quantum dots, wires or antidots) [16, 21, 22]. We apply this potential in a 2D ring and calculate their energy spectrum. Different kinds of insulators can be obtained from the values for effective mass through our data. It is shown that the energy spectrum depends on geometric parameters related to the topological defect.

2. Confining potential

Let be a cylindrical wire with a topological defect through his center. Our space has a screw dislocation, mathematically represented for this metric

\[ ds^2 = d\rho^2 + (dz + \beta d\varphi)^2 + \rho^2 d\varphi^2; \]  

(3)

There is a coupling between the \( z \) and \( \varphi \) coordinates, something intrinsic to how screw dislocations are constructed. The strength of the Burgers vector and \( \beta \) are related by \( b^z = 2\pi\beta \). In our approach the modelling of quantum rings is made with the potential \( V_{TI}(\rho) \) proposed by Tan and Inkson, which describe some kinds of mesoscopic systems, e.g., dots, antidots and wires [16].

\[ V_{TI}(\rho) = \frac{a_1}{\rho^2} + a_2 \rho^2 - V_0, \]  

(4)

with

\[ \omega_0 \equiv \left( \frac{8a_2}{\mu} \right)^{1/2}, \quad V_0 \equiv 2(a_1a_2)^{1/2}. \]  

(5)

From the metric we obtain the hamiltonian and reach the following time independent Schrödinger equation (TISE)

\[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Psi}{\partial \varphi^2} - \frac{2\beta}{\rho^2} \left( \frac{\partial}{\partial \varphi} \frac{\partial}{\partial z} \right) \Psi + \left( 1 + \frac{\rho^2}{\beta^2} \right) \frac{\partial^2 \Psi}{\partial z^2} - \frac{2\mu a_1}{\hbar^2} \frac{\Psi}{\rho^2} - \frac{2\mu a_2}{\hbar^2} \rho^2 \Psi + 2\mu \frac{E - V_0}{\hbar^2} \Psi = 0 \]  

(6)
Assuming the ansatz $\Psi(\rho, \varphi, z) = R(\rho)e^{im\varphi}e^{ikz}$, we obtain the radial equation

$$\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{dR}{d\rho} \right) - \frac{M^2}{\rho^2} R + \left( \frac{2\mu E - V_0}{\hbar^2} - k^2 - \frac{2\mu a^2}{\hbar^2} \rho^2 \right) R = 0,$$

where $M \equiv (m - k\beta)^2 + \frac{2\mu a^2}{\hbar^2}$.

By the change $\xi = \sqrt{\frac{2\mu a^2}{\hbar^2}} \rho^2$, we also obtain

$$\frac{\xi}{4} \frac{d^2 R}{d\xi^2} + \frac{dR}{d\xi} - \frac{M^2}{4\xi} R + \beta' R - \frac{\xi}{4} R = 0,$$

which has the solution

$$R(\xi) = e^{-\frac{\xi}{2} \sqrt{\frac{|M|}{2}}} F \left( -\beta' + \frac{|M| + 1}{2}, |M| + 1, \xi \right)$$

Normalization condition impose the hypergeometric function $F \left( -\beta' + \frac{|M| + 1}{2}, |M| + 1, \xi \right)$ must be a $n$-degree polynomial [23] and this way

$$-\beta' + \frac{|M| + 1}{2} = -n,$$

what give us the following energy eigenvalues

$$E = \hbar \omega_0 \left( n + \frac{|M|}{2} + \frac{1}{2} \right) + \frac{k^2 \hbar^2}{2\mu} + V_0$$

In the first parcel of (14) we see that, even without any external magnetic field, appears an amount similar to those when Landau levels are present. Here, we could say torsion (because of $\beta$) played a role like an effective magnetic (or Landau-like) contribution.

3. External Fields

The last result, where torsion plays a role similar to magnetic fields, motivate us to investigate how is the influence upon system here studied if an external field plus potential $V_{TI}(\rho)$ are present. Initilly we turn on an external magnetic field without confinement and find the following TISE

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Psi}{\partial \rho} \right) + \left[ \frac{1}{\rho} \left( \frac{\partial}{\partial \varphi} - \beta \frac{\partial}{\partial z} \right) - \frac{ie B\rho}{\hbar} \right]^2 \Psi + \frac{\partial^2 \Psi}{\partial z^2} + \frac{2\mu E}{\hbar^2} \Psi = 0,$$

Its radial solution and energy eigenvalues are, respectively, given by

$$R(\xi) = e^{-\frac{\xi}{2} \sqrt{\frac{|m-k\beta|}{2}}} F \left( -\beta' + \frac{|m-k\beta| + 1}{2}, |m-k\beta| + 1, \xi \right),$$

$$E = \hbar \omega_c \left[ n + \frac{|m-k\beta|}{2} - \frac{m-k\beta}{2} + \frac{1}{2} \right] + \frac{k^2 \hbar^2}{2\mu}.$$
Next, if we add the potential $V_{TI}(\rho)$ to the external magnetic field, we find the following TISE

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Psi}{\partial \rho} \right) + \left[ \frac{1}{\rho} \left( \frac{\partial}{\partial \varphi} - \beta \frac{\partial}{\partial z} \right) - \frac{ie B \rho}{\hbar c} \right]^2 \Psi + \frac{\partial^2 \Psi}{\partial z^2} - \left( \frac{2\mu a_1}{\hbar^2} + \frac{2\mu a_2}{\hbar^2} \rho^2 + \frac{2\mu V_0}{\hbar^2} \right) \Psi + \frac{2\mu E}{\hbar^2} \Psi = 0 \quad .$$

(16)

Using the ansatz $\Psi(\rho, \varphi, z) = R(\rho) e^{im\varphi} e^{ikz}$ again and with the variable change $\xi = \lambda \rho^2$, we obtain

$$\xi \frac{d^2 R}{d\xi^2} + \frac{dR}{d\xi} - \frac{M^{'2}}{4\xi} R + \beta^{'} R - \frac{\xi}{4} R = 0 \quad ,$$

(17)

where $M^{'2} = (m - k\beta)^2 - \frac{2m a_1}{\hbar^2}$, $\lambda^2 = \frac{e^2 B^2}{4\epsilon_0 c^2} + \frac{2\mu a_2}{\hbar^2}$ and $\beta^{'} = \frac{1}{4\lambda} \left( \frac{2\mu E}{\hbar^2} + \frac{eB}{\hbar c} (m - k\beta) - k^2 - \frac{2aV_0}{\hbar^2} \right)$.

So, the radial solution and eigenvalues are, respectively, given by

$$R(\xi) = e^{-\frac{\xi}{2} \sqrt{\frac{|M'|}{2}}} F \left( -\beta^{'} + \frac{|M'|}{2} + 1, \frac{|M'|}{2} + 1, \xi \right) \quad ,$$

(18)

$$E = \hbar \omega \left[ n + \frac{|M'|}{2} + \frac{1}{2} \right] - \frac{(m - k\beta)}{2\hbar \omega_c} + \frac{k^2 h^2}{2\mu} + V_0 \quad .$$

(19)

Here $\omega = \omega_0 + \omega_c$. We see there are two contributions to the Landau-like frequency $\omega$: one because the potential $V_{TI}$ and another related to the external magnetic field.

From (19) we find the following for magnetization

$$\mathcal{M} = -\frac{\epsilon h}{\mu} \left[ n + \frac{|M'|}{2} + \frac{1}{2} \right] + \frac{\epsilon h (m - k\beta)}{\mu} \quad .$$

(20)

Looking at the last expressions, we see the torsion causes a shift in the quantum number $m$, what modifies the energy spectrum and how the magnetic field influence the physical properties we have approached.

4. Concluding remarks

Through this work we have found the torsion influences physical properties, namely, energy spectrum and magnetization. There is a shift on quantum number $m$ and, mathematically, torsion has a behaviour as if it would be a kind of magnetic field. We can apply this formalism to different insulators, simply substituting the known values for effective mass in the literature. The torsion influence occurs by the presence of Burgers vector contribution which is related to $\beta$ in our data. We also find that our system presents a quantization as if would be a Landau system: here, $\omega_0$ could play the role of cyclotron frequency.

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