Spectral statistics of chaotic systems with a point-like scatterer

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Abstract

The statistical properties of a Hamiltonian $H_0$ perturbed by a localized scatterer are considered. We prove that when $H_0$ describes a bounded chaotic motion, the universal part of the spectral statistics are not changed by the perturbation. This is done first within the random matrix model. Then it is shown by semiclassical techniques that the result is due to a cancellation between diagonal diffractive and off-diagonal periodic-diffractive contributions. The compensation is a very general phenomenon encoding the semiclassical content of the optical theorem.
In quantum systems, the chaotic or disordered nature of the classical motion is reflected in the statistical properties of the high lying eigenvalues and eigenvectors. For example, the spectral statistics of ballistic cavities are universal for energy ranges that are small compared to the inverse time of flight through the system. These universal properties are well described by random matrix theory (RMT) \cite{1,2}.

Consider now a perturbation imposed to a chaotic system. On the classical side, the dynamics is structurally stable, in the sense that a generic smooth perturbation leaves the dynamics chaotic. We are interested in the quantum mechanical effects of a particular class of perturbations that are non-classical. If the unperturbed motion is described by the Hamiltonian \( H_0 \) acting in an \( N \)-dimensional Hilbert space, we consider Hamiltonians of the form

\[
H = H_0 + \lambda N |v\rangle \langle v|,
\]

where \( |v\rangle \) is a fixed vector. \( N \) is included in the perturbation for future convenience. The eigenvalues \( \{\omega_i\} \) of \( H \) satisfy the equation

\[
\sum_k \frac{|v_k|^2}{\omega - \epsilon_k} = \frac{1}{\lambda N},
\]

with \( \{\epsilon_k\} \) the eigenvalues of \( H_0 \) and \( v_k = \langle \varphi_k | v \rangle \) the amplitudes of \( |v\rangle \) in the eigenbasis of \( H_0 \).

Rank-one perturbations like in Eq. (1) appear in several contexts. The most common one is when a local short-range impurity or point scatterer is added to the system \cite{3}. The physical consequences of such a perturbation were studied for Fermi gases \cite{4,5}, in the context of RMT \cite{6} and for ballistic motion of particles in regular \cite{7} and chaotic \cite{8} cavities. Another context is in the physics of many body problems. In the basis of \( H_0 \), the perturbation simply reads \( v_k v_l^* \) and is said to be separable. In a mean field approach, it is the simplest model leading to collective modes via the residual interaction \cite{9}. Although our results are general, we adopt for simplicity the language of the localized point scatterer.

A local perturbation is purely wave-mechanical. For a system with \( f \) degrees of freedom, it represents a modification of the dynamics in a volume \( \propto (2\pi\hbar)^f \) in phase space, that
tends to zero in the semiclassical limit. For example, the addition of a point scatterer in a ballistic cavity leaves invariant the classical motion while quantum mechanically it induces wave effects like diffraction. The modifications of the eigenvalues by the perturbation are described by Eq.(2). A statistical analysis of the perturbed spectrum was done for the energy levels of a regular integrable cavity with a point scatterer in [7,10]. It was demonstrated that a short range repulsion between the eigenvalues, different from RMT, is induced by the perturbation, thus considerably modifying the initial Poisson distribution. More recently, M. Sieber [8] has studied, using semiclassical techniques, the modifications by a point scatterer of the spectral statistics of chaotic systems. He showed that diffractive orbits produce finite contributions which may induce deviations with respect to the random matrix model. Whether this deviation really exists for chaotic systems, or on the contrary if there are other (non-diagonal) semiclassical contributions that cancel the purely diffractive terms is the question we answer here.

We prove by two different approaches, namely a purely statistical model and a semiclassical calculation, that a local perturbation produces no deviations with respect to RMT. At the first place, assuming that the unperturbed eigenvalues and eigenvectors components in Eq.(2) are distributed according to RMT, i.e. their joint probability density are given by the standard formulae [1,2] 

\[ P(\{\epsilon_k\}) \propto \prod_{i>j} |\epsilon_i - \epsilon_j|^\beta, \]  

and 

\[ P(\{v_k\}) = \prod_i \left( \frac{\beta N}{2\pi} \right)^{1-\beta/2} \exp(-\beta N|v_i|^2/2), \]  

we show that the joint probability density for the perturbed eigenvalues is exactly the same as the distribution of the unperturbed ones 

\[ P(\{\omega_k\}) \propto \prod_{i>j} |\omega_i - \omega_j|^\beta. \]  

Here, \( \beta = 1 \) (resp. 2) for systems with (resp. without) time-reversal symmetry. In the second place, and to complete the analysis, a semiclassical calculation of the spectral form
factor is considered. It is expressed as a double sum over all the periodic and diffractive orbits of the system. The latter are orbits that are scattered by the perturbation. In [8] the diagonal contribution of the diffractive orbits was obtained (cf Eq.(13) below). We compute the off-diagonal contribution coming from the interference of periodic and diffractive orbits, and find that this contribution exactly cancels the diagonal diffractive term. We thus recover the statistics of RMT. The basic physical ingredient at the basis of this cancellation is the unitarity of quantum scattering processes, i.e. conservation of the flux scattered by the impurity. Although our semiclassical result is less general than Eq.(5) – it is only valid for the short time behavior of a two-point function –, it applies to systems with arbitrary diffraction coefficient not expressible as a Hamiltonian of the form (I).

In chaotic and disordered system the local universal fluctuations of the spectrum are essentially related to the properties of the Jacobian (3). We ignore here problems related to the confinement of the eigenvalues, that are of minor importance for our purpose. We start the proof of Eq.(5) from the joint distribution function of both the old and new eigenvalues, obtained in Ref. [6]

\[ P(\{\epsilon_i\}, \{\omega_j\}) \propto \prod_{i>j}(\epsilon_i - \epsilon_j)(\omega_i - \omega_j)^{1-\beta/2} e^{-\rho \sum_i (\omega_i - \epsilon_i)^2}, \]

with \( \rho = \beta/2\lambda \). We restrict for simplicity to \( \lambda > 0 \) (\( \lambda < 0 \) is treated in the same manner). Eq.(\( \beta \)) imposes the restrictions \( \epsilon_i \leq \omega_i \leq \epsilon_i + 1 \) (trapping). The distribution for the perturbed eigenvalues, \( \omega_i \), is then defined as

\[ P(\{\omega_i\}) = \int_{-\infty}^{\omega_1} d\epsilon_1 \int_{\omega_1}^{\omega_2} d\epsilon_2 \ldots \int_{\omega_N-1}^{\omega_N} d\epsilon_N P(\{\epsilon_i\}, \{\omega_j\}) \]
\[ \propto e^{-\rho \sum_i \omega_i} \prod_{i>j}(\omega_i - \omega_j) W(\beta, \rho), \]

with

\[ W(\beta, \rho) = \int_{-\infty}^{\omega_1} \frac{e^{\rho\epsilon_1} d\epsilon_1}{F(\epsilon_1)} \ldots \int_{\omega_N-1}^{\omega_N} \frac{e^{\rho\epsilon_N} d\epsilon_N}{F(\epsilon_N)} \prod_{i>j}(\epsilon_i - \epsilon_j), \]

and \( F(\epsilon) = \prod_{j} |\epsilon - \omega_j|^{1-\beta/2} \). Using the standard expression for the Vandermonde determinant in \( W \) and integrating term by term one obtains
\[ W(\beta, \rho) = \det \left[ I_j^{(i-1)} \right]_{i,j=1,\ldots,N}, \] (7)

where \( I_j^{(i)} = \partial_\rho^i I_j \) is the i-th derivative with respect to \( \rho \) of

\[ I_j = I_j^{(0)} = \int_{\omega_{j-1}}^{\omega_j} e^{\rho \epsilon} d\epsilon F(\epsilon). \] (8)

For \( j = 1 \), \( \omega_{j-1} = -\infty \).

It is straightforward to check that the \( I_j \)'s satisfy, for any \( j \), the following differential equation \[11\]

\[ \left[ \prod_i (\partial_\rho - \omega_i) + \frac{\beta}{2 \rho} \sum_i \prod_{j \neq i} (\partial_\rho - \omega_j) \right] I_j = 0. \] (9)

This differential equation allows to write

\[ I_j^{(N)} = \sum_{i=0}^{N-1} a_i I_j^{(i)}, \]

with some coefficients \( a_i \). \( W(\beta, \rho) \) as defined in Eq.(7) is the Wronskian of this equation. It then follows that

\[ \partial_\rho W = a_{N-1} W, \] (10)

where in our case \( a_{N-1} = \sum_i \omega_i - \beta N/2 \rho \). Integration of Eq.(10) leads to

\[ W(\beta, \rho) = \frac{W_0}{\rho^{\beta N/2}} \exp(\rho \sum_i \omega_i). \]

When this result is replaced in Eq.(8) one gets

\[ P(\{\omega_i\}) \propto W_0 \prod_{i>j} (\omega_i - \omega_j). \] (11)

\( W_0 \) is an integration factor, independent of \( \rho \). We compute it from the asymptotic behavior of \( I_j^{(i)} \) when \( \rho \to \infty \). In this limit the integral in Eq.(8) may be evaluated explicitly

\[ \lim_{\rho \to \infty} I_j \propto \frac{e^{\rho \omega_j}}{\rho^{\beta/2} \prod_{i \neq j} |\omega_i - \omega_j|^{1-\beta/2}}. \]

To leading order, \( I_j^{(i)} = \omega_j^i I_j \) and from Eq.(7) one gets

\[ W_0 \propto \prod_{i>j} |\omega_i - \omega_j|^\beta \] (11)
From this equation and Eq. (11) we recover the random matrix distribution function Eq. (5).

A chaotic system coupled to the environment through a one-channel antenna has been considered in [12]. The model is equivalent to Eq. (2) but with imaginary $\lambda$. For $\lambda \to \infty$ the imaginary part of the perturbed energies is small and Eq. (5) is obtained. Our method, which takes explicit care of the trapping problem, allows to prove this result for arbitrary $\lambda$.

In real physical systems agreement with random matrix theory is observed in a limited range. This universal behavior concerns correlations over energy ranges that are small compared to $h/T_{\text{min}}$, with $T_{\text{min}}$ the typical period of the shortest periodic orbit. The above random matrix calculation establishes that the universal part of the spectrum is not changed by the presence of the scatterer. On the other hand, the non-universal behavior of the correlation functions occurring at scales of the order of, or larger than, $h/T_{\text{min}}$ are modified by the scattering center, since new diffractive orbits are introduced [13, 14].

Let us now turn to a semiclassical treatment of the spectral correlations. These are based on trace formulae expansions of the density of states $d(\omega) = \sum_k \delta(\omega - \omega_k)$, written as a sum of smoothed plus oscillatory terms $d = \bar{d} + d^{(\text{osc})}$. We characterize the correlations by the spectral form factor defined as

$$K(\tau) = \int_{-\infty}^{\infty} \frac{d\eta}{\bar{d}} \left\langle d^{(\text{osc})} \left( E + \frac{\eta}{2} \right) d^{(\text{osc})} \left( E - \frac{\eta}{2} \right) \right\rangle \exp \left( 2\pi i \eta \tau \bar{d} \right). \tag{12}$$

The average indicated by brackets is taken over an energy window containing many quantum levels but whose size is small compared to $E$. We again consider a fully chaotic system with a point-like scatterer. In the geometrical theory of diffraction $d^{(\text{osc})} = d^{(\text{osc})}_p + d^{(\text{osc})}_d$, where $d^{(\text{osc})}_p$ and $d^{(\text{osc})}_d$ are expressed as interferent sums over periodic and diffractive orbits, respectively [13, 14].

$$d^{(\text{osc})}_{p,d}(E) = \sum_{p,d} A_{p,d} \exp \left( i \frac{S_{p,d}(E)}{\hbar} - i \frac{\pi}{2} \mu_{p,d} \right), \tag{13}$$

with

$$A_p = \frac{T_p}{2\pi \hbar |\det(M_p - 1)|^{1/2}}$$

and

$$A_d = \frac{T_d \mathcal{D}(\vec{n}, \vec{n}') e^{-i\sigma(f+1)/4} |\det N|^{1/2}}{4\pi \hbar \kappa(2\pi \hbar)^{(f-1)/2}}.$$
$S_{p,d}(E)$ is the action of the periodic (resp. diffractive) orbits, $T_{p,d}$ denotes their period, $M_p$ is the monodromy matrix of the periodic orbit, $N$ is the matrix $N_{ij} = \partial^2 S_d / \partial y_i \partial y_j$ (where $\vec{y}$ are coordinates orthogonal to the diffractive trajectory). $D(\vec{n}, \vec{n}')$ is the scattering amplitude of the scattering center located at $\vec{x}_0$ with incoming $\vec{n}$ and outgoing $\vec{n}'$ directions, defined in terms of the perturbed ($G$) and unperturbed ($G_0$) Green’s functions by the relation

$$G(\vec{x}, \vec{x}') = G_0(\vec{x}, \vec{x}') + \frac{\hbar^2}{2m} G_0(\vec{x}, \vec{x}_0) D(\vec{n}, \vec{n}') G_0(\vec{x}_0, \vec{x}') .$$

Using the properties of the periodic orbits of chaotic systems, the diagonal contribution of $d^{(osc)}_p$ in Eq.(12) gives the short-time random matrix result $K_p(\tau) = (2/\beta) \tau$ [15]. The one-scattering contribution of the diffractive orbits in the same approximation is [8]

$$K_d(\tau) = \tau \frac{\hbar^2}{8\beta \pi^2} \left( \frac{k}{2\pi} \right)^{2j-4} \sigma , \quad (14)$$

with $k$ the modulus of the wave vector at the impurity and $\sigma$ its total cross section

$$\sigma = \int |D(\vec{n}, \vec{n}')|^2 d\Omega \ d\Omega' . \quad (15)$$

($d\Omega$ is the solid angle element). For simplicity, we restrict the calculations to one scattering event (multiple scattering may be considered likewise).

Our purpose is to compute the off-diagonal cross-term coming from the product of $d^{(osc)}_p$ and $d^{(osc)}_d$ in Eq.(12). The semiclassical expression for this contribution is

$$K_{pd}(\tau) = \frac{2\pi \hbar}{d} \left( \sum_{p,d} A_p A_d^* \exp(i(S_p - S_d)/\hbar) \right) \left( T - \frac{T_p + T_d}{2} \right) + c.c.) . \quad (16)$$

After energy smoothing, $K_{pd}$ has significant contributions only from orbits with close actions $S_p \approx S_d$ (having therefore approximately the same period). Pairs of orbits satisfying this condition may be constructed by considering the neighborhood of the forward scattering orbits. To each periodic orbit passing nearby the scatterer $\mathcal{O}$ we associate an “almost periodic” diffractive orbit that is similar to the periodic orbit but comes back to $\mathcal{O}$ with a slightly different momentum. In Eq.(16) the double sum now involves all the possible pairs of trajectories constructed this way. Consider a surface of section that includes $\mathcal{O}$ and is
transversal to the momentum of the periodic orbit when it comes near \( \mathcal{O} \). Let coordinates measured from \( \mathcal{O} \) and momenta in the plane be denoted by \((\vec{q}, \vec{p})\). Consider all the periodic orbits of period \( T \) that cut the section through a differential element \( d^{f-1}q \, d^{f-1}p \) located at a distance \( \vec{q} \) from \( \mathcal{O} \). The difference of action between these periodic orbits and the diffractive orbits associated to them as mentioned above is

\[
S_p - S_d = -(1/2) \, Q_{ij} \, q_i \, q_j ,
\]

with

\[
Q_{ij} = \partial^2 S / \partial q_i \partial q_j + \partial^2 S / \partial q'_i \partial q_j + \partial^2 S / \partial q'_i \partial q'_j,
\]

and \( \vec{q} \) (\( \vec{q}' \)) are initial (resp. final) coordinates on the surface of section. Moreover, one can show that

\[
| \det Q | = | \det (M_p - 1) \det N | \cos^2 \theta ,
\]

where \( \theta \) is the angle between the normal to the surface of section and the momentum of the diffractive orbit.

By generalizing arguments used in the derivation of the Hannay - Ozorio de Almeida sum rule \[16\] one can prove the following sum rule

\[
\sum_p \delta(T - T_p) \, \chi(\vec{q}_p, \vec{p}_p) \left| \det(M_p - 1) \right| = \frac{\int d^{f-1}q \, d^{f-1}p \, \chi(\vec{q}, \vec{p})}{\Sigma} ,
\]

where \( \chi(\vec{q}, \vec{p}) \) is a test function defined on the surface of section and \((\vec{q}_p, \vec{p}_p)\) are the coordinates of the points at which the periodic orbit \( p \) crosses the surface of section. \( \Sigma = \int d^f x \, d^f p \, \delta(E - H(x, p)) \) is the total phase-space volume at energy \( E \). From Eq. \[16\], using Eqs.\[17\] and \[19\], we have

\[
K_{pd} = \frac{\tilde{d} \, \tau^2 e^{i\pi(f+1)/4}}{\beta k (2\pi\hbar)^{(f-3)/2} \Sigma} \int \sqrt{| \det(M_p - 1) | | \det N |} \, \times \mathcal{D}^*(\vec{n}, \vec{m}) e^{-(i/2\hbar)Q_{ij}q_i q_j} \, d^{f-1}q \, d^{f-1}p + c.c.
\]

Integrating the quadratic form in the exponent, taking into account Eq.\[18\], using the semiclassical density of states \( \tilde{d} = \Sigma / (2\pi\hbar)^f \) and the fact that the differential element for the momenta may be written \( d^{f-1}p = (\hbar k)^{f-1} \cos \theta \, d\Omega \), one obtains the final expression
\[
K_{pd}(\tau) = \frac{\tau^2}{2\pi^3} \left( \frac{k}{2\pi} \right)^{f-2} \int i \left[ D^*(\vec{n}, \vec{n}) - D(\vec{n}, \vec{n}) \right] d\Omega.
\] (20)

This is the result for the cross-term contribution. Note that it depends only on \( D(\vec{n}, \vec{n}) \), which translates the fact that in general interference terms between periodic and diffractive orbits can be large only for the diffraction in the forward direction.

To make contact with Eq. (14) we use a general relation valid for the elastic scattering on a finite range potential. The conservation of the flux scattered by the scattering center imposes a relation between the imaginary part of the scattering amplitude and the scattering cross section. This is the well known optical theorem \([17]\), that in \( f \) dimensions takes the form

\[
i \left[ D^*(\vec{n}, \vec{n}) - D(\vec{n}, \vec{n}) \right] = -\frac{1}{4\pi} \left( \frac{k}{2\pi} \right)^{f-2} \int |D(\vec{n}, \vec{n}')|^2 d\Omega'.
\]

Combining this relation with Eq. (20) one gets our final result

\[
K_{pd}(\tau) = -K_d(\tau).
\] (21)

The interference between periodic and diffractive orbits exactly cancels the diagonal contribution of the diffractive orbits, Eq.(14). We recover from semiclassical methods, at least for a two-point function and short times, the RMT result.

The two basic elements producing the cancellation are the sum rule (19) and the optical theorem. Only the former is characteristic of chaotic systems, the latter being very general. The present semiclassical results may be extended by similar methods to multiple scattering events. In a wider context, it should be mentioned that this is one of the rare cases in which a calculation of off-diagonal contributions (whose role is essential in producing the correct result) is done explicitly for chaotic systems.

We have concentrated on the fluctuation properties of eigenvalues of chaotic systems, and have demonstrated that they are unchanged by a local perturbation. This applies to high lying states, were the statistical hypotheses hold. On the opposite extreme, a local perturbation may lead to important modifications of the properties of the ground state of the system. Take for example a negative \( \lambda \). According to Eq.(2), each perturbed eigenvalue
remains trapped by two unperturbed ones, except the ground state. The energy of the ground state may diminish arbitrarily with increasing $|\lambda|$ and, as can easily be shown, the associated wavefunction becomes more and more localized at the impurity. In our considerations we have ignored the presence of this “collective” mode.

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