Density Matrix Deformations in Quantum and Statistical Mechanics at Planck-Scale

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Abstract: In this work the Quantum and Statistical Mechanics of the Early Universe, i.e. at Planck scale, is considered as a deformation of the well-known theories. In so doing the primary object under deformation in both cases is the density matrix. It is demonstrated that in construction of the deformed quantum mechanical and statistical density matrices referred to as density pro-matrices there is a complete analog. The principal difference lies in a nature of the deformation parameter that is associated with the fundamental length in the first case and with a maximum temperature in the second case. Consideration is also given to some direct consequences, specifically the use of the explicitly specified exponential ansatz in the derivation from the basic principles of a semiclassical Bekenstein-Hawking formula for the black hole entropy and of the high-temperature complement to the canonical Gibbs distribution.

Keywords: statistical mechanics deformation, deformed density matrix, deformed Gibbs distribution.
1. Introduction

As is known, at Planck scale Quantum Mechanics undergoes variation: it should be subjected to deformation. This is realized due to the presence of the Generalized Uncertainty Relations (GUR) and hence the fundamental length \[1\], \[2\]. The deformation in Quantum Mechanics at Planck scale takes different paths: commutator deformation \[3\], \[4\], \[5\], \[6\] or density matrix deformation \[7\], \[8\]. In the present work the second approach is extended by the author to the Statistical Mechanics at Planck scale. And similar to the quantum mechanics, the primary object is the density matrix. A complete analog in construction of the deformed quantum mechanical and statistical density matrices is revealed. The principal difference is the deformation parameter that is associated with the fundamental length in the first case and with a maximum temperature in the second case. Consideration is given to some consequences including the use of the explicitly specified exponential ansatz in the derivation from the basic principles of a semiclassical Bekenstein-Hawking formula for the black hole entropy and a high-temperature complement to the canonical Gibbs distribution. In conclusion possible applications of the obtained results are discussed.
2. Density Matrix Deformation in Quantum Mechanics at Planck Scale

In this section the principal features of QMFL construction with the use of the density matrix deformation are briefly outlined [7], [8]. As mentioned above, for the fundamental deformation parameter we use \( \alpha = l_{\text{min}}^2/x^2 \) where \( x \) is the scale. In contrast with [7], [8], for the deformation parameter we use \( \alpha \) rather than \( \beta \) to avoid confusion, since quite a distinct value is denoted by \( \beta \) in Statistical Mechanics: \( \beta = 1/k_B T \).

**Definition 1. (Quantum Mechanics with Fundamental Length)**

Any system in QMFL is described by a density pro-matrix of the form

\[
\rho(\alpha) = \sum_i \omega_i(\alpha) |i><i|,
\]

where

1. \( 0 < \alpha \leq 1/4 \);
2. The vectors \(|i>\) form a full orthonormal system;
3. \( \omega_i(\alpha) \geq 0 \) and for all \( i \) the finite limit \( \lim_{\alpha \to 0} \omega_i(\alpha) = \omega_i \) exists;
4. \( Sp[\rho(\alpha)] = \sum_i \omega_i(\alpha) < 1, \sum_i \omega_i = 1 \);
5. For every operator \( B \) and any \( \alpha \) there is a mean operator \( B \) depending on \( \alpha \):

\[
<i|B|i> = \sum_i \omega_i(\alpha) <i|B|i>.
\]

Finally, in order that our definition 1 agree with the result of section 2, the following condition must be fulfilled:

\[
Sp[\rho(\alpha)] - Sp^2[\rho(\alpha)] \approx \alpha.
\] (2.1)

Hence we can find the value for \( Sp[\rho(\alpha)] \) satisfying the condition of definition 1:

\[
Sp[\rho(\alpha)] \approx \frac{1}{2} + \sqrt{\frac{1}{4} - \alpha}.
\] (2.2)

According to point 5), \( <i> = Sp[\rho(\alpha)] \). Therefore for any scalar quantity \( f \) we have \( <f> = fSp[\rho(\alpha)] \). In particular, the mean value \( [<x_\mu, p_\nu> = i\hbar \delta_{\mu,\nu}Sp[\rho(\alpha)]\]

We denote the limit \( \lim_{\alpha \to 0} \rho(\alpha) = \rho \) as the density matrix. Evidently, in the limit \( \alpha \to 0 \) we return to QM.
As follows from definition 1, \(< (j >< j) >_\alpha = \omega_j(\alpha)\), from whence the completeness condition by \(\alpha\) is
\(< (\sum_i |i > < i|)_\alpha = 1 >_\alpha = Sp[\rho(\alpha)]\). The norm of any vector \(|\psi >\) assigned to \(\alpha\) can be defined as
\(< \psi|\psi >_\alpha = < \psi|\sum_i |i > < i|\alpha|\psi > = < \psi|1 >_\alpha = Sp[\rho(\alpha)]\), where \(< \psi|\psi >\) is the norm in QM, i.e. for \(\alpha \to 0\). Similarly, the described theory may be interpreted using a probabilistic approach, however requiring replacement of \(\rho\) by \(\rho(\alpha)\) in all formulae.

It should be noted:

I. The above limit covers both Quantum and Classical Mechanics. Indeed, since \(\alpha \sim L^2_p/x^2 = G\hbar/c^3x^2\), we obtain:

a. \((\hbar \neq 0, x \to \infty) \Rightarrow (\alpha \to 0)\) for QM;

b. \((\hbar \to 0, x \to \infty) \Rightarrow (\alpha \to 0)\) for Classical Mechanics;

II. As a matter of fact, the deformation parameter \(\alpha\) should assume the value \(0 < \alpha \leq 1\). However, as seen from (2.2), \(Sp[\rho(\alpha)]\) is well defined only for \(0 < \alpha \leq 1/4\), i.e. for \(x = il_{\text{min}}\) and \(i \geq 2\) we have no problems at all. At the point, where \(x = l_{\text{min}}\), there is a singularity related to complex values assumed by \(Sp[\rho(\alpha)]\), i.e. to the impossibility of obtaining a diagonalized density matrix at this point over the field of real numbers. For this reason definition 1 has no sense at the point \(x = l_{\text{min}}\).

III. We consider possible solutions for (2.1). For instance, one of the solutions of (2.1), at least to the first order in \(\alpha\), is
\[
\rho^*(\alpha) = \sum_i \alpha_i exp(-\alpha)|i > < i|
\]
where all \(\alpha_i > 0\) are independent of \(\alpha\) and their sum is equal to 1. In this way \(Sp[\rho^*(\alpha)] = exp(-\alpha)\). Indeed, we can easily verify that
\[
Sp[\rho^*(\alpha)] - Sp^2[\rho^*(\alpha)] = \alpha + O(\alpha^2).
\]
Note that in the momentum representation \(\alpha = p^2/p^2_{pl}\), where \(p_{pl}\) is the Planck momentum. When present in matrix elements, \(exp(-\alpha)\) can damp the contribution of great momenta in a perturbation theory.

IV. It is clear that within the proposed description the states with a unit probability, i.e. pure states, can appear only in the limit \(\alpha \to 0\), when all \(\omega_i(\alpha)\) except for one are equal to zero or when they tend to zero at this limit. In our treatment pure state are states, which can be represented in the form \(|\psi > < \psi|\), where \(< \psi|\psi > = 1\).
V. We suppose that all the definitions concerning a density matrix can be transferred to the above-mentioned deformation of Quantum Mechanics (QMFL) through changing the density matrix \( \rho \) by the density pro-matrix \( \rho(\alpha) \) and subsequent passage to the low energy limit \( \alpha \to 0 \). Specifically, for statistical entropy we have

\[
S_\alpha = -Sp[\rho(\alpha) \ln(\rho(\alpha))].
\]  

(2.4)

The quantity of \( S_\alpha \) seems never to be equal to zero as \( \ln(\rho(\alpha)) \neq 0 \) and hence \( S_\alpha \) may be equal to zero at the limit \( \alpha \to 0 \) only.

Some Implications:

I. If we carry out measurement on the pre-determined scale, it is impossible to regard the density pro-matrix as a density matrix with an accuracy better than particular limit \( \sim 10^{-66+2n} \), where \( 10^{-n} \) is the measuring scale. In the majority of known cases this is sufficient to consider the density pro-matrix as a density matrix. But on Planck’s scale, where the quantum gravitational effects and Plank energy levels cannot be neglected, the difference between \( \rho(\alpha) \) and \( \rho \) should be taken into consideration.

II. Proceeding from the above, on Planck’s scale the notion of Wave Function of the Universe (as introduced in \([9]\)) has no sense, and quantum gravitation effects in this case should be described with the help of density pro-matrix \( \rho(\alpha) \) only.

III. Since density pro-matrix \( \rho(\alpha) \) depends on the measuring scale, evolution of the Universe within the inflation model paradigm \([10]\) is not a unitary process, or otherwise the probabilities \( p_i = \omega_i(\alpha) \) would be preserved.

3. Deformed Density Matrix in Statistical Mechanics at Planck Scale

To begin, we recall the generalized uncertainty relations "coordinate - momentum" \([4],[5],[6]\):

\[
\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' L_p^2 \frac{\Delta p}{\hbar}.
\]  

(3.1)

Using relations (3.1) it is easy to obtain a similar relation for the "energy - time" pair. Indeed (3.1) gives

\[
\frac{\Delta x}{c} \geq \frac{\hbar}{\Delta pc} + \alpha' L_p^2 \frac{\Delta p c}{\hbar c},
\]  

(3.2)

then

\[
\Delta t \geq \frac{\hbar}{\Delta E} + \alpha' L_p^2 \frac{\Delta pc}{\hbar c} = \frac{\hbar}{\Delta E} + \alpha' t_p^2 \frac{\Delta E}{\hbar}.
\]  

(3.3)
where the smallness of $L_p$ is taken into account so that the difference between $\Delta E$ and $\Delta(pc)$ can be neglected and $t_p$ is the Planck time $t_p = L_p/c = \sqrt{G\hbar/c^3} \approx 0.54 \times 10^{-43}$ sec. From whence it follows that we have a maximum energy of the order of Planck’s:

$$E_{\text{max}} \sim E_p$$

Proceeding to the Statistical Mechanics, we further assume that an internal energy of any ensemble $U$ could not be in excess of $E_{\text{max}}$ and hence temperature $T$ could not be in excess of $T_{\text{max}} = E_{\text{max}}/k_B \sim T_p$. Let us consider density matrix in Statistical Mechanics (see [1], Section 2, Paragraph 3):

$$\rho_{\text{stat}} = \sum_n \omega_n |\varphi_n><\varphi_n|,
$$

(3.4)

where the probabilities are given by

$$\omega_n = \frac{1}{Q} \exp(-\beta E_n)$$

and

$$Q = \sum_n \exp(-\beta E_n)$$

Then for a canonical Gibbs ensemble the value

$$\Delta(1/T)^2 = Sp[\rho_{\text{stat}}(1/T)^2] - Sp^2[\rho_{\text{stat}}(1/T)],
$$

(3.5)

is always equal to zero, and this follows from the fact that $Sp[\rho_{\text{stat}}] = 1$. However, for very high temperatures $T \gg 0$ we have $\Delta(1/T)^2 \approx 1/T^2 \geq 1/T_{\text{max}}^2$. Thus, for $T \gg 0$ a statistical density matrix $\rho_{\text{stat}}$ should be deformed so that in the general case

$$Sp[\rho_{\text{stat}}(1/T)^2] - Sp^2[\rho_{\text{stat}}(1/T)] \approx \frac{1}{T_{\text{max}}^2},
$$

(3.6)

or

$$Sp[\rho_{\text{stat}}] - Sp^2[\rho_{\text{stat}}] \approx \frac{T^2}{T_{\text{max}}^2},
$$

(3.7)

In this way $\rho_{\text{stat}}$ at very high $T \gg 0$ becomes dependent on the parameter $\tau = T^2/T_{\text{max}}^2$, i.e. in the most general case

$$\rho_{\text{stat}} = \rho_{\text{stat}}(\tau)$$

and

$$Sp[\rho_{\text{stat}}(\tau)] < 1$$
and for $\tau \ll 1$ we have $\rho_{\text{stat}}(\tau) \approx \rho_{\text{stat}}$ (formula (3.4)).

This situation is identical to the case associated with the deformation parameter $\alpha = l^2_{\text{min}}/x^2$ of QMFL given in section 2 [8]. That is the condition $Sp[\rho_{\text{stat}}(\tau)] < 1$ has an apparent physical meaning when:

I. At temperatures close to $T_{\text{max}}$ some portion of information about the ensemble is inaccessible in accordance with the probability that is less than unity, i.e. incomplete probability.

II. And vice versa, the longer is the distance from $T_{\text{max}}$ (i.e. when approximating the usual temperatures), the greater is the bulk of information and the closer is the complete probability to unity.

Therefore similar to the introduction of the deformed quantum-mechanics density matrix in section 3 [8] and previous section of this paper, we give the following

**Definition 2. (Deformation of Statistical Mechanics)**

Deformation of Gibbs distribution valid for temperatures on the order of the Planck’s $T_p$ is described by deformation of a statistical density matrix (statistical density pro-matrix) of the form

$$\rho_{\text{stat}}(\tau) = \sum_n \omega_n(\tau)\langle \varphi_n | \varphi_n \rangle$$

having the deformation parameter $\tau = T^2/T^2_{\text{max}}$, where

I. $0 < \tau \leq 1/4$;

II. The vectors $|\varphi_n\rangle$ form a full orthonormal system;

III. $\omega_n(\tau) \geq 0$ and for all $n$ at $\tau \ll 1$ we obtain $\omega_n(\tau) \approx \omega_n = \frac{1}{Q}\exp(-\beta E_n)$ In particular, $\lim_{T_{\text{max}} \to \infty (\tau \to 0)} \omega_n(\tau) = \omega_n$.

IV. $Sp[\rho_{\text{stat}}(\tau)] = \sum_n \omega_n(\tau) < 1$, $\sum_n \omega_n = 1$;

V. For every operator $B$ and any $\tau$ there is a mean operator $\bar{B}$ depending on $\tau$

$$< B >_\tau = \sum_n \omega_n(\tau) < n|B|n > .$$

Finally, in order that our Definition 2 agree with the formula (3.7), the following condition must be fulfilled:

$$Sp[\rho_{\text{stat}}(\tau)] - Sp^2[\rho_{\text{stat}}(\tau)] \approx \tau.$$ (3.8)
Hence we can find the value for $Sp[\rho_{\text{stat}}(\tau)]$ satisfying the condition of Definition 2 (similar to Definition 1):

$$Sp[\rho_{\text{stat}}(\tau)] \approx \frac{1}{2} + \sqrt{\frac{1}{4} - \tau}.$$  \hspace{1cm} (3.9)

It should be noted:

I. The condition $\tau \ll 1$ means that $T \ll T_{\text{max}}$ either $T_{\text{max}} = \infty$ or both in accordance with a normal Statistical Mechanics and canonical Gibbs distribution 

II. Similar to QMFL in Definition 1, where the deformation parameter $\alpha$ should assume the value $0 < \alpha \leq 1/4$. As seen from (3.9), here $Sp[\rho_{\text{stat}}(\tau)]$ is well defined only for $0 < \tau \leq 1/4$. This means that the feature occurring in QMFL at the point of the fundamental length $x = l_{\text{min}}$ in the case under consideration is associated with the fact that highest measurable temperature of the ensemble is always $T \leq \frac{1}{2}T_{\text{max}}$.

III. The constructed deformation contains all four fundamental constants: $G, \hbar, c, k_B$ as $T_{\text{max}} = \varsigma T_p$, where $\varsigma$ is the denumerable function of $\alpha'$ (3.1) and $T_p$, in its turn, contains all the above-mentioned constants.

IV. Again similar to QMFL, as a possible solution for (3.8) we have an exponential ansatz

$$\rho^*_{\text{stat}}(\tau) = \sum_n \omega_n(\tau)|n > n| = \sum_n \exp(-\tau)\omega_n|n > n|$$

$$Sp[\rho^*_{\text{stat}}(\tau)] - Sp^2[\rho^*_{\text{stat}}(\tau)] = \tau + O(\tau^2).$$  \hspace{1cm} (3.10)

In such a way with the use of an exponential ansatz (3.10) the deformation of a canonical Gibbs distribution at Planck scale (up to factor $1/Q$) takes an elegant and completed form:

$$\omega_n(\tau) = \exp(-\tau)\omega_n = \exp(-\frac{T^2}{T_{\text{max}}^2} - \beta E_n)$$  \hspace{1cm} (3.11)

where $T_{\text{max}} = \varsigma T_p$

4. Comments to the Bekenstein-Hawking Formula and Measuring Procedure

It should be noted that an exponential ansatz yielding a result in case of the statistical mechanics, in quantum mechanics may be used in the derivation of
Bekenstein-Hawking formula for the black hole entropy in semiclassical approximation from the basic principles [8]: In the process factor 1/4 is interpreted as a growing density of the entropy for the observer at the conventional scales when measuring is performed close to the singularity. Also note that (2.3) may be written as a series

\[ Sp[\rho(\alpha)] - Sp^2[\rho(\alpha)] = \alpha + a_0\alpha^2 + a_1\alpha^3 + \ldots \]  

(4.1)

As a result, a measurement procedure using the exponential ansatz may be understood as the calculation of factors \(a_0, a_1, \ldots\) or the definition of additional members in the exponent which destroy \(a_0, a_1, \ldots\). It is easy to check that the exponential ansatz gives \(a_0 = -3/2\), being coincident with the logarithmic correction factor for the black hole entropy [15].

5. Conclusion

Thus, it has been shown that between the deformations of quantum and statistical mechanics of the early Universe there is a complete analog. Some consequences have been demonstrated. Of interest are possible applications presented in the work. Of particular interest is also the problem of a rigorous proof for the generalized uncertainty relations (GUR) in thermodynamics [12], [13] as a complete analog of the corresponding relations in Quantum Mechanics [4], [3, 4, 5, 6]. The methods developed in this study may be interesting for the investigation of black hole thermodynamics [14] based on GUR (3.1). These problems will be considered by the author in subsequent papers.

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