On the existence of smooth Cauchy steep time functions

E Minguzzi

Dipartimento di Matematica e Informatica ‘U. Dini’, Università degli Studi di Firenze, Via S. Marta 3, I-50139 Firenze, Italy
E-mail: ettore.minguzzi@unifi.it

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Abstract
A simple proof (based on results in Chruściel et al 2015 Ann. Henri Poincaré arXiv:1301.2909) is given that every globally hyperbolic spacetime admits a smooth Cauchy steep time function. This result is useful in order to show that globally hyperbolic spacetimes can be isometrically embedded in Minkowski spacetimes and that they split as a product. The proof is based on a recent result on the differentiability of Geroch’s volume functions.

Keywords: Cauchy hypersurface, global hyperbolicity, time function

1. Introduction

Let \((M, g)\) be a spacetime, namely a Hausdorff, paracompact, connected, time-oriented Lorentzian manifold, endowed with a \((C^3)\) metric \(g\) of signature \((- , + ,\cdots , +)\). Since every \(C^r\) manifold is \(C^r\) diffeomorphic to a \(C^\infty\) manifold [6], we shall not bother with the the degree of differentiability of \(M\), as we can suppose to have been given a smooth atlas.

On a spacetime, a time function \(\tau: M \to \mathbb{R}\) is a continuous function that increases over every causal curve, that is, \(x < y \Rightarrow f(x) < f(y)\), where as usually, we write \(x < y\) if there is a future-directed causal curve connecting \(x\) to \(y\). A Cauchy time function is a time function for which the level sets are Cauchy hypersurfaces, namely closed acausal sets intersected exactly once by any inextendible causal curve. Time function \(\tau\) is steep if it is continuously differentiable and \(- g(\nabla \tau, \nabla \tau) \geq 1\). A smooth Cauchy time function need not have a timelike gradient, e.g. it could have a lightlike gradient at some point, however, if it is steep it necessarily has a timelike gradient everywhere.

The existence of smooth Cauchy time functions with timelike gradients in globally hyperbolic spacetimes has been proved through different approaches in [1–3]. This implies that these spacetimes admit a smooth splitting as a product \(\mathbb{R} \times S\), \(g = -\beta^2 dt^2 \oplus h_t\), \(\beta: M \to (0, +\infty)\), where \(t\) is the Cauchy time function and \(h_t\) is a time-dependent Riemannian metric over the Cauchy hypersurface \([t] \times S\). Recently, Müller and Sánchez [7]
proved that the Cauchy time function can be found to be steep, a fact that implies that \((\mathcal{M}, g)\) can be embedded in \(N\)-dimensional Minkowski spacetime for some \(N \geq 2\), and that \(\beta\) can be chosen such that \(\beta \leq 1\). This result can also be used to express a formula for the Lorentzian distance in terms of the family of time functions \([8]\). A recent study on the existence of steep time functions for general cone structures can be found in \([9]\).

In this work we provide a simple proof for the existence of smooth Cauchy steep time functions in globally hyperbolic spacetimes by using some recent results on Geroch’s volume functions.

Let \((\mathcal{M}, g)\) be a spacetime and let us consider a non-negative continuous function \(\varphi\). The Geroch’s volume function is

\[
\tau_\varphi^+(p) = \int_{\mathcal{J}_+(p)} \varphi \, d\mu_g,
\]

where \(d\mu_g\) is the volume element of \(g\).

In a recent joint work with Grant and Chruściel \([2]\), we proved the following lemma, which basically follows from the differentiability properties of the exponential map.

**Lemma 1.1.** In globally hyperbolic spacetimes, the functions \(\tau_\varphi^\pm\) are continuously differentiable for all continuous compactly supported non-negative functions \(\varphi\). The gradient at \(p \in \mathcal{M}\) is past-directed timelike or vanishing depending, respectively, on whether or not \(E^\pm(p)\) intersects the set \(\{q : \varphi(q) > 0\}\).

For completeness we mention that the gradient is given by

\[
\nabla_X \tau_\varphi^\pm = \int_{E^\pm(p)} \varphi \, J(X) \, d\mu_g,
\]

where \(J(X)\) is the Jacobi field obtained by solving the Jacobi equation over each generator \(\gamma(s)\) of the horismos \(E^\pm(p)\), with an initial condition \(J(0) = X, \left(\frac{d}{ds}\right)J(0) = 0\).

In the proof of the main theorem \(1.3\), it will be useful to keep in mind the next topological result.

**Lemma 1.2.** Let \(K\) be a compact subset of a topological space and let \(\{U_i\}_{i \in S}\) be a locally finite family of subsets. Then, \(K\) is intersected by at most finitely many elements of the family.

**Proof.** Let \(A_i\) be the open (in \(K\)) subset of \(K\) made of points that admit a neighborhood which intersects at most \(i\) elements of \(\{U_i\}_{i \in S}\). Clearly, \(A_i \subset A_{i+1}\) and \(\bigcup_i A_i = K\). However, this open covering of \(K\) admits a finite subcovering, which proves that \(K = A_j\) for some \(j\). Since every \(p \in K\) admits an open neighborhood \(N_p\), which intersect at most \(j\) elements of the family \(\{U_i\}_{i \in S}\), and since \(K\) admits a finite subcovering of, say, \(r\) elements of the form \(N_p, K\) is intersected at most by \(rj\) elements of the family \(\{U_i\}_{i \in S}\).

We are thus ready to prove the existence of smooth Cauchy steep time functions. The proof uses just the previous lemmas and Geroch’s topological splitting theorem.
Theorem 1.3. Let \((M, g)\) be a globally hyperbolic spacetime. There exists a smooth Cauchy time function \(\tau: M \to \mathbb{R}\) with timelike past-directed gradient \(\nabla \tau\), which is steep, namely \(-g(\nabla \tau, \nabla \tau) > 1\).

**Proof.** In this proof, we shall say that \(\tau\) is ‘steep’ if \(-g(\nabla \tau, \nabla \tau) > 1\) with the strict inequality. According to Geroch’s topological splitting [4, 5], there is a continuous Cauchy time function \(t: M \to \mathbb{R}\). The level sets \(S_t\) are Cauchy hypersurfaces. We are first going to construct a continuously differentiable function \(\tau^-\) (resp. \(\tau^+\)) over \(M\), with past-directed timelike or vanishing gradient, which is steep over \(J^+(S_t)\) (resp. \(J^-(S_t)\)) and such that \(\tau^- - t > 0\) over \(J^+(S_t)\) (resp. \(-\tau^+ < t\) over \(J^-(S_t)\)). Then, \(\tau' = \tau^- - \tau^+\) will clearly be continuously differentiable, steep, and Cauchy over \(M\).

Next, from [6, theorem 2.6], \(C^\infty(M, \mathbb{R})\) is dense in \(C^i(M, \mathbb{R})\) endowed with the Whitney strong topology [6, p. 35], thus we can find \(\tau \in C^\infty(M, \mathbb{R})\), which approximates \(\tau'\), up to the first derivative, as accurately as we want over \(M\). In particular, we can find \(\tau\) such that \(|\tau - \tau'| < 1\) and \(-g(\nabla \tau, \nabla \tau) > 1\), where the former inequality implies that \(\tau\) is Cauchy, and the latter inequality implies that \(\tau\) is steep.

Let us construct \(\tau^-\), the plus case being analogous. For every \(p \in J^+(S_0)\) there is an open neighborhood \(V_p\) with compact closure contained in \(I^+(S_{-1})\). Let \(V_0 = M \setminus J^+(S_0)\), by paracompactness, the open covering \(\{V_0\} \cup \{V_p : p \in J^+(S_0)\}\) admits a locally finite refinement (necessarily countable by lemma 1.2 and \(\sigma\)-compactness of \(M\)) and a corresponding partition of unity \(\{\varphi_i, i \geq 0\}\), where, setting \(U_j = \{q : \varphi_j(q) > 0\}\), we have for \(j \geq 1\), supp \(\varphi_j = \overline{U_j} \subset I^+(S_{-1})\).

Let \(K_1 \subset J^+(S_0)\) be a sequence of compact sets such that \(K_0 = \emptyset, \bigcup K_t = J^+(S_0)\), \(K_t = J^-(K_t) \cap J^+(S_0)\), and

\[
K_t \cup \bigcup_{j : U_j \cap K_t \neq \emptyset} \overline{U_j} \subset K_{t+1}.
\]

(The indices \(j\) entering the union are finite in number due to lemma 1.2.) Let \(\lambda_j\) be a sequence of positive numbers and let \(\varphi = \sum_{j=1}^\infty \lambda_j \varphi_j\). Since \(\text{supp} \varphi\) is locally finite, \(\varphi\) is finite as at any point only a finite number of terms give a non-vanishing contribution. Moreover, \(\text{supp} \varphi = \bigcup_{j=1}^\infty \overline{U_j} \subset I^+(S_{-1})\) it is useful to recall that the closure operator is additive if the union is a locally finite family [10, lemma 20.5], and that the closure of a locally finite family is locally finite.

We are going to find a sequence \(\lambda_j\) such that \(\tau^- = \tau^- \varphi\) is steep over \(J^+(S_0)\). Observe that

\[
\tau^- \varphi = \sum_j \lambda_j \tau^- \varphi_j, \quad \nabla \tau^- \varphi = \sum_j \lambda_j \nabla \tau^- \varphi_j.
\]

By induction, suppose that we can find a finite sequence \(\lambda_1, \lambda_2, \ldots, \lambda_{n_j}\), \((\lambda_j = 0\) for \(j > n_j)\) such that \(\tau^-, \text{defined as above, is steep over } K_t\) and satisfies the inequality \(\tau^- - t > 0\) over \(K_t\). Let \(A_1\) be the finite index set such that if \(r \in A_1\), then \(U_r \cap (K_{t+1} \setminus \text{Int}K_t) = \emptyset\). We have \(U_t \cap K_{t-1} = \emptyset\), as otherwise \(U_t \subset K_t \Rightarrow U_t \subset \text{Int}K_t\), which is a contradiction. Let \(p \in K_{t+1} \setminus \text{Int}K_t\), then \(p\) belongs to some \(U_r, r \in A_1\), and hence \(E^- (p)\) intersects \(U_r\).

We now replace the constants \(\lambda_r, r \in A_1, \lambda_r \to \lambda'_r\), with larger constants chosen in such a way that \(\tau^-\) and \(\nabla \tau^-\) get replaced by

\[
\tau^- \rightarrow \tau^- \varphi + \sum_{r \in A_1} (\lambda'_r - \lambda_r) \tau^- \varphi_j, \quad \nabla \tau^- \varphi \rightarrow \nabla \tau^- \varphi + \sum_{r \in A_1} (\lambda'_r - \lambda_r) \nabla \tau^- \varphi_j,
\]

and so that \(\tau^- \varphi\) becomes steep at \(p\), and \(\tau^- - t\) becomes positive at \(p\).
Since $K_{i+1} \setminus \text{Int}K_i$ is compact and $\tau^--t$ and $\nabla \tau^-$ depend continuously on each $\lambda_j$, the redefinition of $\lambda_j$ and consequently of $\tau^-$ can be done so as to obtain steepness and positivity of $\tau^--t$ all over $K_{i+1}$. Observe that this redefinition can only increase the value of $\tau^-$ over $K_i$ and that it certainly does not spoil steepness there. Furthermore, it does not change $\tau^-$ over $K_{i-1}$, thus the built inductive process leads to the desired function $\tau^-$.

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