On Axially Symmetric Space-Times Admitting Homothetic Vector Fields in Lyra’s Geometry

Ragab M. Gad\textsuperscript{1,2} * and A. E. Al Mazrooei\textsuperscript{1†}

\textsuperscript{1} Mathematics Department, Faculty of Science, University of Jeddah, 21589 Jeddah, KSA
\textsuperscript{2} Mathematics Department, Faculty of Science, Minia University, 61915 El-Minia, Egypt

Abstract

This paper investigates axially symmetric space-times which admit a homothetic vector field based on Lyra’s geometry. The cases when the displacement vector is function of $t$ and when it is constant are studied. In the context of this geometry, we find and classify the solutions of the Einstein’s field equations (EFE) for the space-time under consideration which display a homothetic symmetry.

Keywords: Axially symmetric space-times; homothetic vector field; Lyra’s geometry; Einstein’s field equations

PACS Nos.: 04.20.Cv, 04.20.jb

1 Introduction

Space-time symmetries play an important role in the features of space-time that can be described as exhibiting some form of symmetry. The most important of symmetries is in simplifying Einstein field equations and provide a classification of the space-times according to the structure of the corresponding Lie algebra.

\*E-mail: ragab2gad@hotmail.com
\†E-mail:abo.mhmd@hotmail.com
Symmetries have been studied in the theory of General Relativity (GR) based on Riemannian geometry and in the theory of teleparallel gravity (TPG) based on the Weitzenbock geometry. In GR different kinds of symmetries like isometry, homothetic, conformal, Ricci collineations and matter collineations have been extensively studied [1]-[18]. Also some of them have been studied in TPG, over the past few years [19]-[26].

In a series of paper, Gad and his collaborator [27]-[29] studied the homothetic symmetry based on Lyra’s geometry. They classified the space-times according to admitting such symmetry. For zero displacement vector field, they obtained results agree with those obtained previously in GR, based on Riemannian geometry. They shown also that in the case when the displacement vector field is constant, it is not possible to compare the results obtained in the context of Lyra’s geometry with that obtained in GR, using Riemannian geometry. This means that in Lyra’s geometry if the displacement vector field is considered as a constant this does not give meaningful results.

In the framework of Lyra’s geometry, many authors have made attempts to find solutions of EFE [30]-[36]. These attempts were by imposing certain conditions upon the scale factors of the space-time in addition to the conditions on the matter which represents such space-time and some restrictions on its physical properties. In the same context, we will obtaining solutions to EFE for an axially symmetric space-time by assuming only that such space-time displays a homothetic symmetry, that is, exhibits a self-similarity of the first kind (cf. [6, 37, 38]).

The paper is organized as follows: In the next Section, we will study the homothetic symmetry of axially symmetric space-times based on Lyra’s geometry. Section 3 deals with the solutions of EFE and their classifications of the space-times under considerations. Finally, in Section 4, concluding remarks are given.

2 The version of model and homothetic vector field in Lyra’s geometry

In Lyra’s geometry the metric or the measure of length of displacement vector $\zeta^\mu = x^\sigma dx^\mu$ between two points $p(x^\mu)_{\mu=1}$ and $q(x^\mu + d x^\mu)_{\mu=1}$ is given by absolute invariant under both gauge function, $x^o = x^o(x^\nu)_{\nu=1}$, and
coordinate system, \( \{x^\mu\}_{\mu=1}^n \), as follows

\[
    ds^2 = g_{\mu\nu}x^\alpha dx^\mu x^\alpha dx^\nu,
\]

where \( g_{\mu\nu} \) is a metric tensor as in Riemannian geometry.

In Lyra’s geometry, a generalized affine connection characterized not only by the Riemannian connection, \( \{ \alpha_{\mu\nu} \} \), but also by a function \( \phi_\mu \), which arises through gauge transformation, and it is given by

\[
    \Gamma^\alpha_{\mu\nu} = (x^\alpha)^{-1} \{ \alpha_{\mu\nu} \} + \frac{1}{2} (\delta^\alpha_\mu \phi_\nu + \delta^\alpha_\nu \phi_\mu - g_{\mu\nu} \phi^\alpha),
\]

(2.1)

where \( \phi \) is called a displacement vector field and satisfies \( \phi^\alpha = g^{\alpha\beta} \phi_\beta \). We consider \( \phi \) to be a timelike vector, where

\[
    \phi_\mu = (\beta(t), 0, 0, 0).
\]

Throughout the paper \( M \) will denoted a 4-dimensional Lyra manifold with Lorentz metric \( g \) which is a generalization to the 4-dimensional Riemannian manifold [39, 40]. As in Riemannian geometry, a global vector field \( \zeta = \zeta^\mu(t,x,y,z) \) on \( M \) is called homothetic vector field if the following condition

\[
    \mathcal{L}_\zeta g_{\mu\nu} = g_{\mu\rho} \nabla_\rho \zeta^\alpha + g_{\nu\rho} \nabla_\rho \zeta^\alpha = 2\psi g_{\mu\nu},
\]

(2.3)

holds where \( \psi \) is a constant (the homothetic constant) on \( M \), \( \mathcal{L} \) denotes a Lie derivative and \( \nabla \) is the covariant derivative, such that

\[
    \nabla_\mu \zeta^\nu = \frac{1}{x^\alpha} \partial_\mu \zeta^\nu + \Gamma^\alpha_{\mu\nu} \zeta^\alpha,
\]

\[
    \nabla_\mu \zeta^\nu = -\frac{1}{x^\alpha} \partial_\mu \zeta^\nu - \Gamma^\alpha_{\mu\nu} \zeta^\alpha.
\]

(2.4)

Here \( \Gamma^\alpha_{\mu\nu} \) is a Lyra connection given by equation (2.1).

In equation (2.3), if \( \psi \neq 0 \), \( \zeta \) is called proper homothetic vector field and the this equation is called homothetic equation. It is worth mention here if \( \psi = 0 \), then equation (2.3) is called Killing equation and \( \zeta \) is called a Killing vector field on \( M \).

Consider the axially symmetric metric in the form [41]

\[
    ds^2 = dt^2 - A^2(t)(d\chi^2 + f^2(\chi)d\phi^2) - B^2(t)dz^2,
\]

(2.5)

with the convention \( x^0 = t, x^1 = \chi, x^2 = \phi, x^3 = z \) and \( A \) and \( B \) are functions of \( t \) only while \( f \) is a function of the coordinate \( \chi \) only.
The study of homothetic vector fields, $\zeta = \zeta^\mu (t, x, y, z)^{\mu=1}$, on axially symmetric space-times (2.5) is based on an examination of the ten equations obtained from (2.3). For the model (2.5), using (2.1) and apart from the factor $1/x^0$, i.e., we choose the normal gauge $x^0 = 1$, equations (2.3) are reduced to the following system of equations:

\begin{align*}
\zeta_1^1 + \left(\frac{A_t}{A} + \frac{1}{2}\beta\right)\zeta^0 &= \psi, \quad (2.6) \\
\zeta_1^1 + f^2 \zeta_1^2 &= 0, \quad (2.7) \\
A^2 \zeta_1^1 + B^2 \zeta_3^3 &= 0, \quad (2.8) \\
\zeta_0^0 - A^2 \zeta_1^1 &= 0, \quad (2.9) \\
\zeta_2^0 + \frac{f_1}{f} \zeta_0^0 + \left(\frac{A_t}{A} + \frac{1}{2}\beta\right)\zeta^0 &= \psi, \quad (2.10) \\
A^2 f^2 \zeta_2^2 + B^2 \zeta_3^3 &= 0, \quad (2.11) \\
\zeta_0^0 - A^2 \zeta_2^2 &= 0, \quad (2.12) \\
\zeta_3^3 + \left(\frac{B_t}{B} + \frac{1}{2}\beta\right)\zeta^0 &= \psi, \quad (2.13) \\
\zeta_0^0 - B^2 \zeta_3^3 &= 0, \quad (2.14) \\
\zeta_0^0 + \frac{1}{2}\beta \zeta^0 &= \psi. \quad (2.15)
\end{align*}

Solving equation (2.15) and using the result back into equations (2.9), (2.12) and (2.14), we get

\begin{align*}
\zeta^0 &= \left[\psi \int e^{\frac{A_t}{A} \int \beta dt} dt + c_0\right]e^{-\frac{A_t}{A} \int \beta dt}, \quad (2.16) \\
\zeta^1 &= F_1(x, y, z), \\
\zeta^2 &= F_2(x, y, z), \\
\zeta^3 &= F_3(x, y, z),
\end{align*}

where $c_0$ is a constant of integration and $F_1(x, y, z)$, $F_2(x, y, z)$ and $F_3(x, y, z)$ are arbitrary functions which are to be determined.

Differentiating equations (2.6) and (2.13) with respect to $t$ and using (2.15) and (2.16), we get respectively

\begin{align*}
\frac{A_t}{A} + \frac{1}{2}\beta &= \frac{a}{\left[\psi \int e^{\frac{A_t}{A} \int \beta dt} dt + c_0\right]e^{-\frac{A_t}{A} \int \beta dt}}. \quad (2.17)
\end{align*}
\[
\frac{B_t}{B} + \frac{1}{2} \beta = \frac{c}{[\psi \int e^{\frac{1}{2} \int \beta dt} dt + c_0]e^{-\frac{1}{2} \int \beta dt}},
\]

(2.18)

where \(a\) and \(c\) are constants of integration. Without loss of generality, we assume that \(a = c\). Substituting these results back into (2.6) and (2.13), using the obtained results in (2.7), (2.8), and (2.11), we get

\[
\zeta^1 = (\psi - a)x + c_1,
\]

(2.19)

\[
\zeta^3 = (\psi - a)z + c_3,
\]

(2.20)

where \(c_1\) and \(c_3\) are constants of integration. Consequently, \(\zeta^2\) depends on \(y\) only.

Integrating equations (2.17) and (2.18), using \(a = c\), we get the following relation

\[
B(t) = nA(t),
\]

(2.21)

where \(n\) is a constant of integration.

From equation (2.10), we have the following two cases

1. **Case i:** If we assume that

\[
\frac{f_x}{f} \zeta^1 = c_4 \neq 0.
\]

(2.22)

Integrating this equation, we get

\[
f(\chi) = c_5 \chi^{\frac{c_4}{n-2}},
\]

(2.23)

where \(c_5\) is a constant of integration. Inserting the above results into (2.10), we get

\[
\zeta^2 = (\psi - b)y + c_2,
\]

(2.24)

where \(c_2\) is a constant of integration and \(b = a + c_4\).

Without loss of generality, we assume that \(c_0 = c_1 = c_2 = c_3 = 0\), therefore, from equations (2.16), (2.19), (2.20) and (2.24) we obtain the following homothetic vector field

\[
\zeta = ([\psi \int e^{\frac{1}{2} \int \beta dt} dt]e^{-\frac{1}{2} \int \beta dt}) \partial_t + (\psi - a)\chi \partial_\chi + (\psi - b) y \partial_y + (\psi - a) z \partial_z.
\]

(2.25)

It is of interested to note that the function (2.23) does not satisfy that \(\frac{f_x}{f}\) equals constant or zero. Consequently, the metric (2.5) with
\( g_{22} = c_5 \chi^{\frac{\psi - a}{c_4}} \) does not a solution to EFE, as we see in the next section. Therefore, this function will withdraw from consideration inspected the special case, when \( c_4 = \psi - a \). The latter case gives

\[
\zeta^2 = c_2
\]

and

\[
f(\chi) = c_5 \chi.
\] (2.26)

In this case, the homothetic vector becomes

\[
\zeta = ([\psi \int e^{\frac{1}{2} \int \beta dt} e^{-\frac{1}{2} \int \beta dt} \partial_t + (\psi - a) \chi \partial_\chi + c_2 y \partial_y + (\psi - a) z \partial_z].
\] (2.27)

2. Case ii: If

\[
\frac{f_\chi}{f} \zeta^1 = 0.
\] (2.28)

- When \( \zeta^1 \neq 0 \), this implies

\[
f(\chi) = c_6,
\] (2.29)

where \( c_6 \) is a constant of integration.

In this case the homothetic vector is the same as given by equation (2.25).

- When \( \zeta^1 = 0 \), this implies

\[
f(\chi) = c_8 e^{c_7 \chi},
\] (2.30)

where \( c_7 \) and \( c_8 \) are constants of integration.

In this case the homothetic vector is given by

\[
\zeta = ([\psi \int e^{\frac{1}{2} \int \beta dt} e^{-\frac{1}{2} \int \beta dt} \partial_t + (\psi - b) y \partial_y + (\psi - c) z \partial_z].
\] (2.31)

All the above considerations lead to the following theorem:

**Theorem 2.1** In Lyra’s geometry, if a displacement vector is function of \( t \), that is, \( \beta = \beta(t) \), axially symmetric space-time described by metric (2.5) admits the homothetic vector field if

\[
B(t) = nA(t).
\]

Such HVF is given by equation (2.25) if \( f(\chi) \) is given by equation (2.29) and it is given by equation (2.27) or (2.31), if \( f(\chi) \) is given by equation (2.26) or (2.30), respectively.
Now we will investigate the situation when the displacement vector is constant, that is, $\beta = \text{constant}$.

In this case, from equation (2.16), the time-like component of the homothetic vector $\zeta^0$, the only component depends on $\beta$, becomes

$$\zeta^0 = \left(\frac{2\psi}{\beta} e^{\frac{1}{2} \beta t} + c_0\right) e^{-\frac{1}{2} \beta t}.$$ 

While the space-like components are given as the case when $\beta = \beta(t)$ according to the values of $f(\chi)$.

Now we conclude that the homothetic vector field given by (2.25) and with the above constrains on the scale factors is similar to the vector as obtained in the theory of General Relativity (see for instance, [42, 43]).

It is worthy to note that, in the case $\beta = \text{const.}$, we cannot compare the results with that obtained in the theory of General Relativity, using Riemannian geometry, because in this case the component $\zeta^0$ tends to infinity when $\beta = 0$. This means that in Lyra's geometry if the displacement vector field is considered as a constant this does not give meaningful results.

### 3. Field equations and their solutions

In this section, we shall determine the exact solutions of EFE by assuming that the space-time under consideration admits a homothetic vector field (self-similarity).

The field equations in normal gauge for Lyra's geometry as obtained by Sen [44] (in gravitational units $c = 8\pi G = 1$) read as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -T_{\mu\nu} - \frac{3}{2} \phi_\mu \phi_\nu + \frac{3}{4} g_{\mu\nu} \phi_\alpha \phi^\alpha,$$

(3.1)

the left hand side is the usual Einstein tensor as in Riemannian geometry, whereas $\phi_\mu$ is a time-like displacement field vector defined by (2.2) and $T_{\mu\nu}$ is the energy momentum tensor corresponding to perfect fluid. It is of interesting to assume that the matter of field in space-time under consideration is represented by a perfect fluid, that is, the energy-momentum tensor is defined by

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - pg_{\mu\nu},$$

(3.2)

Here $p$ is the pressure, $\rho$ the energy density and $u_\mu$ the four velocity vector, it must verify $\mathcal{L}_{\zeta_\mu} u_\nu = 0$. For the space-time (2.5) the 4-velocity vector can be
defined by $u^\mu = u_\mu = (1, 0, 0, 0)$ and it is verified $g_{\mu\nu}u^\mu u^\nu = 1$. In view of the metric (2.5), the volume element, the four-acceleration vector, the rotation, the expansion scalar and the shear scalar can be written, respectively, in a comoving coordinates system as (see [45])

\[ V = \sqrt{-g} = A^2 f B, \]  
\[ \dot{u}_i = 0, \]
\[ \omega_{ij} = 0, \]
\[ \Theta = \frac{2A_t}{A} + \frac{B_t}{B}, \]
\[ \sigma^2 = \frac{1}{g}(11(A_t)^2 + 5(B_t)^2 + \frac{2A_t B_t}{AB}). \]

The non vanishing components of the shear tensor $\sigma_{ij}$ are

\[ \sigma_{11} = A(\frac{1}{3}\Theta A - A_t), \]
\[ \sigma_{22} = Af^2(\frac{1}{3}\Theta A - A_t), \]
\[ \sigma_{33} = B(\frac{1}{3}\Theta B - B_t), \]
\[ \sigma_{44} = -\frac{2}{3}\Theta. \]

For the line element (2.5), the field equations (3.1) with equation (??) lead to the following system of equations (see for instance Gad [45])

\[ \frac{A_{tt}}{A} + \frac{B_{tt}}{B} + \frac{A_t B_t}{AB} + \frac{3}{4}\beta^2 = -p, \]  
\[ \frac{2A_{tt}}{A} + (\frac{A_t}{A})^2 - \frac{f_{,xx}}{f A^2} + \frac{3}{4}\beta^2 = -p, \]  
\[ (\frac{A_t}{A})^2 + \frac{2A_t B_t}{AB} - \frac{f_{,xx}}{f A^2} - \frac{3}{4}\beta^2 = \rho, \]  
\[ \rho, t + (\rho + p)(\frac{2A_t}{A} + \frac{B_t}{B}) = 0, \]

In GR, Cahill and Taub [6] and Bicknell and Henriksen [15] (see also [16]) pointed out, if the matter field is a perfect fluid, then the only barotropic equation of state which is compatible with self-similarity (characterized by the existence of homothetic vector field) is of the form

\[ p = k\rho, \]

where $\rho$ is the total energy density, $p$ is the pressure and $k$ is a constant in the range $0 \leq k \leq 1$. This equation of state is nevertheless physically consistent
in the whole range of \( k \). When \( k = 0 \), the above equation describes dust, \( k = 1/3 \) gives the equation of state for radiation and \( k = 1 \) considers the effective "stiff fluid" distribution. The latter was apparently first proposed by Zeldovich [46]. It should have applied in the early Universe, with this case, the velocity of sound equals the velocity of light, so no material in this Universe could be more stiff.

Exact solutions for the Einstein field equations (3.6)-(3.9) can be found under the assumption that the space-time (2.5) admits homothetic vector field. The obtained solutions can be classified according to the value of the scale factor \( f(\chi) \) and the constant \( k \).

Using (2.21) and (3.10), the Einstein field equations (3.6)-(3.9) reduce to the following equations

\[
\frac{2A_{tt}}{A} + \left(\frac{A_{t}}{A}\right)^2 + \frac{3}{4}\beta^2 = -k\rho, \tag{3.11}
\]

\[
3\left(\frac{A_{t}}{A}\right)^2 - \frac{3}{4}\beta^2 = \rho, \tag{3.12}
\]

\[
\rho = \frac{m}{A^{3(1+k)}}, \tag{3.13}
\]

where \( m \) is a constant of integration.

From equations (3.11)-(3.13), we get

\[
\frac{2A_{tt}}{A} + 4\left(\frac{A_{t}}{A}\right)^2 = \frac{m(1-k)}{A^{3(1+k)}}, \tag{3.14}
\]

In the following we discuss two cases, when \( k = 1 \) and \( k = 0 \):

**Class I: \( k = 1 \)**

In this case

\[
A(t) = (\alpha t + \alpha_1)^{\frac{1}{4}}, \tag{3.15}
\]

where \( \alpha \) and \( \alpha_1 \) are constants of integration.

According to the values of \( f(\chi) \), given by equations (2.26), (2.29) and (2.30), we have, respectively, the following solutions

\[
ds^2 = dt^2 - (\alpha t + \alpha_1)^{\frac{2}{3}}(d\chi^2 + c_5\chi d\phi^2 + ndz^2), \tag{3.16}
\]

\[
ds^2 = dt^2 - (\alpha t + \alpha_1)^{\frac{2}{3}}(d\chi^2 + c_6d\phi^2 + ndz^2), \tag{3.17}
\]

\[
ds^2 = dt^2 - (\alpha t + \alpha_1)^{\frac{2}{3}}(d\chi^2 + c_8e^{c_7\chi}d\phi^2 + ndz^2), \tag{3.18}
\]
**Class II:** $k = 0$

From equation (3.14), we have the following equation

\[ A^2 A_{tt} + 2AA_t = \frac{m}{2}. \]  
(3.19)

Solving the above equation, we get

\[ A(t) = \frac{1}{2}(6mt^2 - \alpha_2 t + \alpha_3)^{\frac{1}{3}}. \]  
(3.20)

where $\alpha_2$ and $\alpha_3$ are constants of integration.

For the values of $f(\chi)$, as given in theorem (2.1), we have the following class of solutions

\[
\begin{align*}
    ds^2 &= dt^2 - \frac{1}{4}(6mt^2 - \alpha_2 t + \alpha_3)^{\frac{2}{3}}(d\chi^2 + c_5idxd\phi^2 + ndz^2), \\
    ds^2 &= dt^2 - \frac{1}{4}(6mt^2 - \alpha_2 t + \alpha_3)^{\frac{2}{3}}(d\chi^2 + c_6idxd\phi^2 + ndz^2), \\
    ds^2 &= dt^2 - \frac{1}{4}(6mt^2 - \alpha_2 t + \alpha_3)^{\frac{2}{3}}(d\chi^2 + c_8e^{c_7}\chi d\phi^2 + ndz^2).
\end{align*}
\]  
(3.21 - 3.23)

For the class I of solutions the expressions for density $\rho$, pressure $p$ and displacement field $\beta$ are given by

\[ p = \rho = \frac{m}{(at + \alpha_1)^2}, \]

which shows that $\rho$ and $p$ are not singular,

\[ \beta^2 = \frac{4}{9}\left(\frac{\alpha^2 - 3m}{(at + \alpha_1)^2}\right). \]

It is observed, from equations (3.15) and (2.21) that $A(t)$ and $B(t)$ can be singular only for $t \to \infty$. Thus the line element (2.5) is singular free even at $t = 0$.

For the same class of solutions, using equations (3.3) - (3.5), we have the following physical properties:

The volume element is

\[ V = nf(\chi)(at + \alpha_1). \]

Here $f(\chi)$ takes one of the values given by equations (2.26), (2.29) and (2.30). For all values of $f(\chi)$, we see that the volume element increases as
the time increases. This shows that the solutions (3.16)-(3.18) are expanding with time.

The expansion scalar, which determines the volume behavior of the fluid, is given by

$$\Theta = \frac{\alpha}{\alpha t + \alpha_1}.$$ 

The only non-vanishing component of the shear tensor, \(\sigma_{ij}\), is

$$\sigma_{44} = -\frac{2\alpha}{3(\alpha t + \alpha_1)}.$$  

Hence, the shear scalar \(\sigma\) is given by

$$\sigma^2 = 2\left(\frac{\alpha}{9(\alpha t + \alpha_1)}\right)^2.$$  

Since \(\lim_{t \to \infty} \left(\frac{\sigma}{\Theta}\right) \neq 0\), then the solutions (3.16)-(3.18) do not approach isotropy for large values of \(t\) and do not admit acceleration and rotation, since \(\dot{u}_i = 0\) and \(\omega_{ij} = 0\).

As the above discussion, the class II of solutions (3.21)-(3.23) has the following physical properties

\(p = 0\),

\(\rho = \frac{8m}{6mt^2 - \alpha_2 t + \alpha_3}\),

\(\beta^2 = \frac{4}{9} \left(\frac{\alpha_2^2 - 24m\alpha_3}{(6mt^2 - \alpha_2 t + \alpha_3)^2}\right)\),

\(V = \frac{n}{4} f(\chi)(6mt^2 - \alpha_2 t + \alpha_3)\),

\(\Theta = \frac{12mt - \alpha_2}{6mt^2 - \alpha_2 t + \alpha_3}\),

\(\sigma_{44} = -\frac{2}{3} \left(\frac{12mt - \alpha_2}{6mt^2 - \alpha_2 t + \alpha_3}\right)\),

\(\sigma^2 = \frac{2}{9} \left(\frac{12mt - \alpha_2}{6mt^2 - \alpha_2 t + \alpha_3}\right)^2\),

and

\(\lim_{t \to \infty} \left(\frac{\sigma}{\Theta}\right) \neq 0\).

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4 Discussion and conclusion

Homothetic symmetry is one of the most important types of symmetries, because it plays a dominant role in the dynamics of cosmological models [47, 48].

In this paper, we studied this symmetry of axially symmetric space-times within the framework of Lyra’s geometry. When the displacement vector, $\phi$, is function of $t$, we obtained HVF and found that its expression depends on the scale factor $f(\chi)$. In the case when $\phi$ is constant the time component of the HVF tends to infinity when $\beta = 0$. This means that we can’t compare the obtained results with those obtained in GR, using Riemannian geometry.

The second aim of this paper was to use the homothetic symmetry to simplify EFE. We assumed that the space-time under consideration admits this symmetry and found the exact solutions without more assumptions on the space-time as made in the most of literatures. We classified the obtained solutions according to the values of $f(\chi)$ and the constant $k$ into two classes. We found that the two classes of solutions are singularity-free at the initial epoch $t = 0$ and have vanishing accelerations. For these solutions $\lim_{t \to \infty} (\sigma) \neq 0$, that is, they are not approach isotropy for large time $t$. The all obtained solutions are expanding with time because their volume element increases as the time increases.

Acknowledgments

This project was funded by the Deanship of Scientific Research (DSR), University of Jeddah, Jeddah, under grant No. (G-1436-965-348). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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