SCALE MATRIX ESTIMATION UNDER DATA-BASED LOSS IN HIGH AND LOW DIMENSIONS

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Résumé. Nous considérons le problème d’estimation de la matrice d’échelle Σ du modèle additif $Y_{p \times n} = M + E$, du point de vue de la théorie de la décision. Ici, $p$ est le nombre de variables, $n$ le nombre d’observations, $M$ une matrice de paramètres inconnus de rang $q < p$ et $E$ un bruit aléatoire de distribution symétrique elliptique, de matrice de covariance proportionnelle $I_n \otimes \Sigma$. Ce problème d’estimation est abord sous une représentation canonique où la matrice d’observation $Y$ est décomposée en deux matrices, savoir, $Z_{q \times p}$ qui résume l’information contenue dans $M$ et une matrice $U_{m \times p}$, où $m = n - q$, qui résume l’information suffisante pour l’estimation de $\Sigma$. Comme les estimateurs naturels de la forme $\hat{\Sigma}_a = aS$ ($o = U^T U$ et $a$ est une constante positive) se comportent mal lorsque $p > m$ ($S$ n’est pas inversible), nous proposons des estimateurs alternatifs de la forme $\hat{\Sigma}_{a,G} = a(S + S^+ G(Z, S))$ où $S^+$ est l’inverse de Moore-Penrose de $S$ (qui coïncide avec l’inverse $S^{-1}$ lorsque $S$ est inversible). Nous fournissons des conditions sur la matrice de correction $SS^+G(Z, S)$ telles que $\hat{\Sigma}_{a,G}$ améliore $\hat{\Sigma}_a$ sous le cot bas sur les données $L_S(\Sigma, \hat{\Sigma}) = \text{tr}(S^+ \Sigma (\hat{\Sigma}^{-1} - I_p)^2)$. Nous adoptons une approche unifiée des deux cas où $S$ est inversible ($p \leq m$) et $S$ est non inversible ($p > m$).

Mots-clés. Distribution symétrique elliptique, cot bas sur les données, identité de type Stein-Haff, matrice de covariance, matrice d’échelle.

Abstract. We consider the problem of estimating the scale matrix $\Sigma$ of the additif model $Y_{p \times n} = M + E$, under a theoretical decision point of view. Here, $p$ is the number of variables, $n$ is the number of observations, $M$ is a matrix of unknown parameters with rank $q < p$ and $E$ is a random noise, whose distribution is elliptically symmetric with covariance matrix proportional to $I_n \otimes \Sigma$. We deal with a canonical form of this model where $Y$ is decomposed in two matrices, namely, $Z_{q \times p}$ which summarizes the information
contained in $M$, and $U_{m \times p}$, where $m = n - q$, which summarizes the sufficient information to estimate $\Sigma$. As the natural estimators of the form $\hat{\Sigma}_a = a S$ (where $S = U^T U$ and $a$ is a positive constant) perform poorly when $p > m$ ($S$ non-invertible), we propose estimators of the form $\hat{\Sigma}_{a,G} = a (S + SS^+ G(Z, S))$ where $S^+$ is the Moore-Penrose inverse of $S$ (which coincides with $S^{-1}$ when $S$ is invertible). We provide conditions on the correction matrix $SS^+ G(Z, S)$ such that $\hat{\Sigma}_{a,G}$ improves over $\hat{\Sigma}_a$ under the data-based loss $L_S(\Sigma, \hat{\Sigma}) = \text{tr}(S^+ \Sigma (\hat{\Sigma} \Sigma^{-1} - I_p)^2)$. We adopt a unified approach of the two cases where $S$ is invertible ($p \leq m$) and $S$ is non-invertible ($p > m$).

**Keywords.** Eliptically symmetric distributions, data-based loss, Stein-Haff type identity, covariance matrix, scale matrix.

1 Introduction

Consider the following additive model

$$Y = M + \mathcal{E}, \quad \mathcal{E} \sim ES(0_{np}, I_n \otimes \Sigma),$$

where $Y$ is an observed $n \times p$ matrix, $M$ denotes an $n \times p$ matrix of unknown parameters and $\mathcal{E}$ is an $n \times p$ elliptically symmetric distributed noise with unknown covariance matrix proportional to $I_n \otimes \Sigma$, where $\Sigma$ is an unknown $p \times p$ invertible scale matrix and $I_n$ is the $n$-dimensional identity matrix. Note that, the class of elliptically symmetric distributions encompasses a large number of important distributions such as Gaussian, Cauchy, exponential, Student and Weibull distributions. Our main assumption is that $M$ is of low-rank, that is,

$$\text{rank}(M) = q < p \quad (2)$$

Note that Model (1) is a common alternative representation of the multivariate low-rank regression model $Y = X \beta + \mathcal{E}$, where $X$ is an $n \times q$ matrix of known constants of rank $q < p$ and $\beta$ is an $q \times p$ matrix of unknown parameters. In the Gaussian setting, Model (1) arises in many fields that require to estimate $M$ as in signal processing, image processing, collaborative filtering. Thus, it has been considered by various authors such as Candès and Recht (2009), Ji et al. (2010) and Candès et al. (2013). Recently, Canu and Fourdrinier (2017) introduced the extended elliptical setting in Model (1). It is worth noting that many estimation procedures of $M$ rely on an accurate estimation of the scale matrix $\Sigma$, which is the aim of this paper.

Thanks to the low-rank assumption in (2), there exists a $n \times n$ orthogonal matrix $Q = (Q_1 Q_2)$, with $Q_2^T M = 0$, so that the canonical form of Model (1) is given by

$$Q^T Y = \begin{pmatrix} Q_1^T \\ Q_2^T \end{pmatrix} Y = \begin{pmatrix} Z \\ U \end{pmatrix} = \begin{pmatrix} \theta \\ 0 \end{pmatrix} + Q^T \mathcal{E}, \quad (3)$$
where $Z$ and $U$ are respectively $q \times p$ and $m \times p$ matrices (cf. Fourdrinier and Canu (2017) for more details). Note that, the canonical form (3) separates information about the mean structure $Z$ and the information concerning the scale $U$, since $S = U^T U$ summarizes the information to estimate $\Sigma$. Now, we restrict our attention to the setting where the joint density of $Z$ and $U$ is of the form

$$(z, u) \mapsto |\Sigma|^{-n/2} f \left[ \text{tr}\{(z - \theta)\Sigma^{-1}(z - \theta)^T \} + \text{tr}\{\Sigma^{-1} u^T u\} \right],$$

for some function $f$.

In the following, $E_{\theta, \Sigma}$ will denote the expectation with respect to the density (4) and $E_{\theta, \Sigma}^*$ the expectation with respect to the density

$$(z, u) \mapsto \frac{1}{K^*} |\Sigma|^{-n/2} F^* \left[ \text{tr}\{(z - \theta)\Sigma^{-1}(z - \theta)^T \} + \text{tr}\{\Sigma^{-1} u^T u\} \right],$$

where $F^*(t) = \frac{1}{t} \int_t^\infty f(\nu) \, d\nu$ and the normalizing constant $K^*$ is assumed to be finite. Note that, in the setting of a multivariate normal distribution, since $F^* = f$, these two expectations coincide.

As mentioned by James and Stein (1961), the natural estimators of the form $\hat{\Sigma}_a = a S$ (where $a$ is a positive constant) perform poorly. Therefore, we consider alternative estimators of the form $\hat{\Sigma}_{a,G} = a (S + SS^+ G(Z, S))$ and we derive dominance results under the data-based loss function

$$L_S(\hat{\Sigma}, \Sigma) = \text{tr} \left( S^+ \Sigma \left( \hat{\Sigma} \Sigma^{-1} - I_p \right)^2 \right),$$

and its associated risk

$$R(\hat{\Sigma}, \Sigma) = E_{\theta, \Sigma}[L_S(\hat{\Sigma}, \Sigma)],$$

where $\hat{\Sigma}$ is an estimator of $\Sigma$ and $S^+$ is the Moore-Penrose inverse of $S$. It is worth noticing that this type of loss function, called data-based since it involves $S^+$, was introduced by Efron and Morris (1976). Since then, it was considered by various authors, in a Gaussian setting by Kubokawa and Srivastava (2008) and Tsukuma and Kubokawa (2015), and in a spherical setting by Fourdrinier and Strawderman (2015).

The two main features of our approach is that we consider the general elliptically symmetric distribution context and we unify the two cases where $S$ is non-invertible ($p > m$) and $S$ is invertible ($p \leq m$). The primary decision-theoretic results are presented in Section 2. More precisely, we derive a sufficient condition on the correction matrix function $SS^+ G(Z, S)$ for which $\hat{\Sigma}_{a,G}$ improves on $\hat{\Sigma}_a$ under the data-based loss in (5). In Section 3, we provide numerical results through simulations.
2 Main result

Among the usual estimators $\hat{\Sigma}_a = a S$, there exists $a_o > 0$ such that $\hat{\Sigma}_{a_o}$ is optimal (that is, the risk of $\hat{\Sigma}_{a_o}$ is less than or equal to the risk of $\hat{\Sigma}_a$, for any $a > 0$); this is

$$a_o = \frac{1}{K^*(p \lor m)},$$

where $p \lor m = \max\{p, m\}$ (cf. Haddouche (2019) for a proof). The improvement over the class of $a S$’s will be shown through the improvement of

$$\hat{\Sigma}_{a_o, G} = a_o (S + SS^+ G(Z, S)),$$

over $\hat{\Sigma}_{a_o} = a_o S$, where

$$G(Z, S) = \frac{t}{\text{tr}(S^+)} SS^+$$

and $t$ is a positive constant. Note that the choice of this specific form of $G(Z, S)$ is motivated by the estimator considered by Konno (2009) in the normal case. We give sufficient conditions on the corrected factor $SS^+ G(Z, S)$, that is on the constant $t$, such that the risk difference

$$\Delta(G) = R(\hat{\Sigma}_{a_o, G}, \Sigma) - R(\hat{\Sigma}_{a_o}, \Sigma)$$

between $\hat{\Sigma}_{a_o, G}$ and $\hat{\Sigma}_{a_o}$ is non-positive. Of course, $\Delta(G) \leq 0$ makes only sense if and only if $R(\hat{\Sigma}_{a_o, G}, \Sigma) < \infty$. It is shown in Haddouche (2019) that this occurs as soon as the expectations $E_{\theta, \Sigma} [\|S^+ G\|_F^2]$, $E_{\theta, \Sigma} [\|\Sigma^{-1} SS^+ G\|_F^2]$, $E_{\theta, \Sigma} [\text{tr}(\Sigma S^+)]$ and $E_{\theta, \Sigma} [\text{tr}(\Sigma^{-1} S)]$ are finite. In that case,

$$\Delta(G) = a_o^2 K^* E_{\theta, \Sigma} \left[ \text{tr} \left( \Sigma^{-1} S S^+ G \left\{ I_p + S^+ G + SS^+ \right\} \right) \right] - 2 a_o E_{\theta, \Sigma} \left[ \text{tr} (S^+ G) \right]. \quad (8)$$

The dependence of the risk difference in (8) on the unknown parameter $\Sigma^{-1}$ is problematic. As a remedy, we apply the Stein-Haff type identity in the framework of elliptically symmetric distribution given in Fourdrinier Haddouche and Mezoued (2019).

**Lemma 1** Let $G(z, s)$ be a $p \times p$ matrix function such that, for any fixed $z$, $G(z, s)$ is weakly differentiable with respect to $s$. Assume that $E_{\theta, \Sigma} [\|S^+ G\|_F] < \infty$. Then we have

$$E_{\theta, \Sigma} \left[ \text{tr} \left( \Sigma^{-1} S S^+ G \right) \right] = K^* E_{\theta, \Sigma} \left[ \text{tr} \left( 2 S S^+ D_s \{SS^+ G\}^T + (m - (p \land m) - 1) S^+ G \right) \right].$$

Thanks to this identity, sufficient conditions for improvement of $\hat{\Sigma}_{a_o, G}$ over $\hat{\Sigma}_{a_o}$, are given in the following theorem (cf. Haddouche (2019) for a proof) through an upper bound of the risk difference in (8).
Theorem 1 Consider a density of the form (4). Let

\[ \hat{\Sigma}_{a_o,G} = a_o \left( S + \frac{t}{\text{tr}(S^+)} SS^+ \right) \] (9)

where \( t \) is a positive constant. Then \( \hat{\Sigma}_{a_o,G} \) improves over \( \hat{\Sigma}_{a_o} \) as soon as

\[ 0 \leq t \leq \frac{2((p \land m) - 1)}{(p \lor m) - (p \land m) + 1}. \]

where \( p \land m = \min\{p, m\} \).

3 Numerical study

We deal here with the non-invertible case \((p > m)\) for a Gaussian distribution \((K^* = 1)\) where the scale matrix have an autoregressive structure of the form \((\Sigma)_{ij} = 0.9^{|i-j|}\). Note that simulation on the Student distributions are under study. We evaluate numerically the performance of the alternative estimator \( \hat{\Sigma}_{a_o,G} \) in (9) where \( a_o = 1/p \) and \( t = 2(m - 1)/(p - m + 1) \), through the percentage relative improvement in average loss PRIAL of \( \hat{\Sigma}_{a_o,G} \) over \( \hat{\Sigma}_{a_o} \) defined as

\[ \text{PRIAL}(\hat{\Sigma}_{a_o,G}) = \frac{\text{average loss of } \hat{\Sigma}_{a_o} - \text{average loss of } \hat{\Sigma}_{a_o,G}}{\text{average loss of } \hat{\Sigma}_{a_o}} \times 100, \]

which is reported in the following table.

| \( p \) | \( m \) | PRIAL (%) |
| --- | --- | --- |
| 20 | 4 | 15.00 |
| 20 | 8 | 18.56 |
| 20 | 12 | 25.56 |
| 20 | 16 | 47.034 |
| 100 | 20 | 3.39 |
| 100 | 40 | 4.19 |
| 100 | 60 | 5.76 |
| 100 | 80 | 10.42 |

Results of 1000 Monte Carlo simulation for \((\Sigma)_{ij} = 0.9^{|i-j|}\).

For \( p = 20 \) and \( p = 100 \), the PRIAL increases with the values of \( m \). Note that, when \( p = 20 \) and \( m = 16 \) the PRIAL is close de 50%. Note that the data-based Loss is much more discriminant then the usual quadratic loss for which the PRIAL is lower.
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