Exact $1/N$ and Optimized Perturbative Evaluation of $\mu_c$
for Homogeneous Interacting Bose Gases

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In the framework of the $O(N)$ three-dimensional effective scalar field model for homogeneous dilute weakly interacting Bose gases we use the $1/N$ expansion, within the large $N$ limit, to evaluate the parameter $r_c$ which is directly related to the critical chemical potential $\mu_c$. This quantity enters the order-$a^2n^{2/3}$ coefficient contributing to the critical temperature shift $\Delta T_c$ where $a$ represents the $s$-wave scattering length and $n$ represents the density. Compared to the recent precise numerical lattice simulation results, our calculation suggests that the large $N$ approximation performs rather well even for the physical case $N = 2$. We then calculate the same quantity but using different forms of the optimized perturbative (variational) method, showing that these produce excellent results both for the finite $N$ and large-$N$ cases.

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I. INTRODUCTION

An issue that has attracted considerable attention recently and that is associated to the perturbation theory breakdown problem is the study of how interactions alter the critical temperature ($T_c$) of Bose-Einstein condensation (BEC). Due to its nonperturbative nature, this is clearly a non-trivial problem. The authors of Ref. [1] were able to show that the transition temperature for a dilute, homogeneous, three dimensional Bose gas can be expressed at next to leading order as

$$\frac{\Delta T_c}{T_0} = c_1 n^{1/3} + c_2' \ln(n^{1/3}) + c_2'' a^2 n^{2/3} + \mathcal{O}(a^3n),$$

where $T_0$ is the ideal gas critical temperature, $a$ is the $s$-wave scattering length and $n$ the number density. A similar structure is also discussed in Ref. [2]. As far as the numerical coefficients are concerned, the exact value, $c_1 = 1.303 \pm 0.006$ [1], the $1/N$ expansion at leading order ($c_1 = 2.33$) [4] and at next to leading order ($c_1 = 1.71$) [5] and also the linear $\delta$ expansion (LDE) at second order ($c_1 = 3.06$) [6]. Today the most precise values predict $c_1 = 1.30$ as shown by the lattice (Monte Carlo) simulations which produce $c_1 = 1.29 \pm 0.05$ [7] and $c_1 = 1.32 \pm 0.02$ [8] whereas very recent analytical studies, up to seven loop, predict $c_1 = 1.27 \pm 0.05$ (Variational Perturbation Theory, [10]) and $c_1 = 1.30$ (improved LDE, [11]). A clear review of all how those methods apply to the problem can be found in Ref. [12]. At the same time, the order-$a^2n^{2/3}$ non-perturbative coefficient, $c_2''$, has not been considered by most authors working on the BEC $\Delta T_c$ problem. At this point it is worth recalling that while $c_1$ is a function only of $\kappa$, which is related to the number density, $c_2'$ is a function of $\kappa$ as well as $R$, which is related to the critical chemical potential. This coefficient was first evaluated with lattice simulations which predict $c_2' = 75.7 \pm 0.4$ for $N = 2$ [10] and a previous analytical evaluation made use of the standard LDE [13, 14] as well as its recently improved version [11]. The best previous analytical results (in comparison to the lattice ones) were produced with the improved LDE which obtains $c_2' \approx 73.5$ at order-$\delta^5$ [11]. Very recently, the highest presently available (seven-loop order) perturbative contributions and a corresponding new evaluation of $R$ within the framework of the above mentioned variational perturbation theory, has been performed by Kastening [15]. The purpose of the present work is twofold. First, we

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use the traditional $1/N$ expansion to evaluate analytically, in the large-$N$ limit, the above mentioned quantity $\mathcal{R}$. This large $N$ result is a useful cross-check of a similar evaluation also presented very recently in Ref. [13], since our independent calculation is performed in a quite different way, as discussed in detail below. Using this result in conjunction with the results of Ref. [4] for the density number on $\mu_e$ may finally establish the analytical large-$N$ prediction for $c_{\mu e}^2$ at the next-to-leading $1/N$ order. Comparing our large-$N$ results for $\mu_e$ with the ones produced by the lattice simulations we show how the $1/N$ approximation performs very well, already at the first non trivial order, even for the homogeneous Bose gas case where $N = 2$. Second, by considering our $1/N$ result as being “exact” we perform standard as well as improved (LDE) optimized perturbative evaluations of $\mathcal{R}$ (or equivalently $\mu_e$), also at the $1/N$ order, so that the convergence and reliability of such a method in the BEC context can be further tested. As we recall below, the LDE method essentially introduces a mass term within the effective scalar field model, therefore avoiding the above mentioned infrared divergence problems of ordinary perturbation theory. Then, the interest of considering LDE in the large-$N$ limit is that at the relevant (next-to-leading) $1/N$ order we can define and calculate the LDE perturbative series to arbitrary orders for the quantity $\mu_e$, and thus check the convergence of the LDE method in principle to arbitrary orders. Our optimized perturbation analysis, in the large-$N$ limit, shows how the improved versions of this method also perform very well in connection with the $\Delta T_c$ problem. This work is organized as follows. In the next section we briefly recall the effective model which is considered throughout the paper. In Section III we perform the exact analytical evaluation at $1/N$ order of the critical chemical potential. Section IV is devoted to the optimized perturbation (LDE) applications both for the large-$N$ as well as the finite $N$ case. We pay special attention to the convergence problem. Our conclusions are presented in Section V.

II. THE EFFECTIVE SCALAR THREE DIMENSIONAL MODEL

The studies concerning the equilibrium properties of BEC can be addressed by means of a non-relativistic effective theory described by a complex scalar field. In the dilute limit, which is the regime involved in those experiments, only two-body interactions are important \[1\] and one may then consider the following $U(1)$ invariant finite temperature Euclidean action

$$
S_E = \int_0^\beta d\tau \int d^3 x \left\{ \psi^*(x, \tau) \left( \frac{d}{d\tau} - \frac{1}{2m_A} \nabla^2 \right) \psi(x, \tau) - \mu \psi^*(x, \tau) \psi(x, \tau) + \frac{2\pi a}{m_A} [\psi^*(x, \tau) \psi(x, \tau)]^2 \right\},
$$

where, in natural units, $\beta$ is the inverse of the temperature, $\mu$ is the chemical potential and $m_A$ the mass of the atoms. At the relevant low temperatures involved in BEC the internal degrees of freedom are unimportant and this can be taken as an effective model of hard core spheres with local interactions for which $a$ represents the s-wave scattering length.

The field $\psi$ can be decomposed into imaginary-time frequency modes $\psi_j(x, \omega_j)$, with discrete bosonic Matsubara frequencies $\omega_j = 2\pi j/\beta$, where $j$ is an integer. Near the transition the chemical potential becomes very small as compared to the temperature ($|\mu| << T$) and, since the correlation length tends to infinity, the distances are large compared to the thermal wavelength $\lambda = \sqrt{2\pi \beta/m_A}$. Therefore, only the zero modes matter and one may integrate out all other modes to obtain an effective three-dimensional scalar action with an $O(2)$ symmetry, given by \[3\]

$$
S_\phi = \int d^3 x \left[ \frac{1}{2} \nabla \phi \right]^2 + \frac{1}{2} r \phi^2 + \frac{u}{4!} (\phi^2)^2 + \frac{A}{2} \phi^2,
$$

where $r = -2m_A \mu$ and $u = 48\pi a m_A T$. In the large-$N$ limit mainly considered in this work, and also in Refs. [4, 11, 14, 10], the field $\phi$ in Eq. (3) is formally considered as having $N$ components ($\phi_i$, $i = 1, \ldots, N$). In this case, the Bose-Einstein condensate effective action Eq. (3) is the $N = 2$ special case of the general $O(N)$ invariant action. The three-dimensional effective theory described by Eq. (3) is super-renormalizable and requires only a mass counterterm ($A$) to eliminate any ultraviolet divergence. Using this action one may use any non perturbative technique to evaluate the physical quantities related to the number density

$$
\kappa \equiv \frac{\Delta \langle \phi^2 \rangle_c}{u} = \frac{\langle \phi^2 \rangle_u - \langle \phi^2 \rangle_0}{u},
$$

and to the critical chemical potential

$$
\mathcal{R} \equiv \frac{r_c}{u^2} = -\frac{\Sigma(0)}{u^2},
$$
where the subscripts $u$ and 0 in Eq. (4) mean that the density is to be evaluated in the presence and in the absence of interactions, respectively, and $\Sigma(0)$ is the self-energy with zero external momentum.

The actual relations between the two nonperturbative coefficients appearing in Eq. (1) and these physical quantities are given by

$$c_1 = -128\pi^3 [\zeta(3/2)]^{-4/3} \kappa,$$

and

$$c' = -\frac{2}{3} [\zeta(3/2)]^{-5/3} b'^2 + \frac{7}{9} [\zeta(3/2)]^{-8/3} (192\pi^3 \kappa)^2 + \frac{64\pi}{9} \zeta(1/2) [\zeta(3/2)]^{-5/3} \ln \zeta(3/2),$$

where $b'^2$, in Eq. (4), is

$$b'^2 = \frac{\pi}{3} \left\{ \left[ \frac{1}{2} \ln(128\pi^3) + \frac{1}{2} - 72\pi^2 R - 96\pi^2 \kappa \right] \zeta(1/2) + \frac{\sqrt{\pi}}{2} - K_2 - \frac{\ln 2}{2\sqrt{\pi}} [\zeta(1/2)]^2 \right\},$$

with $K_2 = -0.13508335373$.

As discussed below, the relation between $r_c$ and $\Sigma(0)$ comes from the Hugenholtz-Pines (HP) theorem at the critical point. As a matter of fact, when quantum corrections are taken into account, the full propagator for the effective three-dimensional theory reads

$$G(p) = [p^2 + r + \Sigma_{\text{ren}}(p)]^{-1},$$

where $p^2$ represents the three-momentum and $\Sigma_{\text{ren}}(p)$ represents the renormalized self-energy. At the transition point ($p^2 = 0$), the system must have infinite correlation length and one then has

$$[G(0)]^{-1} = [r_c + \Sigma_{\text{ren}}(0)] = 0.$$

This requirement leads to the Hugenholtz-Pines theorem result $r_c = -\Sigma_{\text{ren}}(0)$. Since $r_c$ is at least of order $u$, it would be treated as a vertex in a standard perturbation type of calculation in which $G(p) = 1/p^2$ represents the bare propagator. This shows that perturbation theory is clearly inadequate to treat the BEC problem at the transition due to the presence of infrared divergences. One must then recur to nonperturbative methods like the numerical lattice Monte Carlo simulations (LS), the analytical $1/N$ technique or the LDE [17] (see for instance Refs. [18] for different types of applications). The problem is highly nontrivial since the Hugenholtz-Pines theorem automatically washes out all momentum independent contributions, such as the one-loop tadpole diagrams, which constitute the leading order of most approximations. In practice, this means that the first nontrivial contributions start with two-loop momentum dependent self-energy terms. However, having reduced the original model, Eq. (2), to the effective three-dimensional one, Eq. (3), makes it easier to tackle those contributions since one no longer has the problem of summing over the Matsubara’s frequencies, which is a hard task when the number of loops increases.

Here, as stated in the introduction, our aim is to apply the $1/N$ expansion and the LDE to Eq. (3) to explicitly evaluate the quantity $r_c = -2m_A \mu_c$ ($m_A$ being the relevant atom mass), comparing our results with the lattice simulation prediction which, for $N = 2$, is $r_c^{L_S} = (0.0019201 \pm 0.0000021) u^2$ [1, 4].

III. STANDARD LARGE-$N$ CALCULATION

The large-$N$ evaluation is carried out, as usual, by assigning a factor of $N$ to any closed loop, and considering $u \sim 1/N$. Then for the massless critical theory given by Eq. (3) the first non trivial contributions are at the next-to-leading order $1/N$, and come from the tadpole (with bubble insertions) type of graphs (see Figure 1):

$$r_c^{(N, \text{tad})} = -\frac{N}{6} u \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2 + \Sigma(p) - \Sigma(0)},$$

where

$$\Sigma(p) = \frac{2}{N} \int \frac{d^3 k}{(2\pi)^3} F(k) \frac{1}{(k + p)^2},$$

and
FIG. 1: Feynman graphs for the tadpole contributions to $r_c$ at $1/N$ order, with the resummed propagator (dotted lines).

FIG. 2: Feynman graph for the chain contribution to $r_c$ at $1/N$ order.

with the “dressed” (resummed) scalar propagator

$$ F(k) = \left( \frac{6}{Nu} + B(k) \right)^{-1}, \quad (13) $$

where

$$ B(k, D) = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 (k + q)^2} \to \frac{1}{8k} \text{ for } D \to 3, \quad (14) $$

represents the basic one-loop $D$-dimensional (massless) integral. The evaluation of Eq. (11) can be done exactly and most conveniently in dimensional regularization. Actually, up to a trivial factor of $-u/6$, it is exactly the same two-loop integral entering the exact evaluation of $\langle \phi^2 \rangle$ at the $1/N$ order, thus already performed in Ref. 4 (see also Appendix A of Ref. 14 for details). The result (which is finite in three dimensions) is

$$ r_c^{(N, \text{tad})} = \frac{N}{6} \frac{u^2}{96\pi^2}. \quad (15) $$

The other contributions to $r_c$ at the next-to-leading $1/N$ order come from the chain exchange type of graphs (see Figure 2):

$$ r_c^{(N, \text{exc})} = - \Sigma^{(N, \text{exc})}(0) = -\frac{2}{N} \int \frac{d^3 k}{(2\pi)^3} F(k) \frac{1}{k^2}, \quad (16) $$

which can also be calculated exactly in dimensional regularization in a very similar way. The result, however, is not finite in three dimensions and has therefore to be regularized and renormalized. Although it turns out to be a technically rather standard two-loop calculation in dimensional regularization, we shall give it in some details below since, to the best of our knowledge, it has not been done before.

Due to the non-finiteness of the final result one should use the arbitrary $D$-dimensional result for the propagator in Eq. (14) which reads

$$ B(k, D) = \frac{2^{3-D} \sqrt{\pi}}{(4\pi)^{D/2}} \frac{\Gamma(D/2 - 1) \Gamma(2 - D/2)}{\Gamma(D/2 - 1/2)} (k^2)^{D/2-2} \equiv b(D) (k^2)^{D/2-2}. \quad (17) $$
Taking this result in the (D-dimensional) integral, Eq. (16), and making a convenient change of variable, \( k = P^2/(D-4) \), one obtains the following integral:

\[
\Sigma^{(\text{N,exc})}(0) = -\frac{4}{N b(D)} \frac{1}{4-D} \int \frac{d^D P}{(2\pi)^D} \left( \frac{P^2}{P^2 + \frac{6}{N b(D)}} \right)^{(1-D/2)},
\]

which we wrote in a form convenient for straightforward dimensional regularization integration. The explicit result is

\[
\Sigma^{(\text{N,exc})}(0) = -\frac{4 \Gamma(\frac{D-4}{2}) \Gamma(1 + \frac{D-4}{2})}{(4\pi)^{D/2} (4-D)} \frac{6}{N b(D)^{D/2}} \frac{1}{N u} \left( 1 - \frac{P^2}{b(D)^2} \right).
\]

Taking now the limit \( D \to 3 \) and taking into account the standard \( \overline{\text{MS}} \) scheme conversion factor:

\[
u_u \to \nu_u \left( \frac{e^{\gamma_E/2} M}{2\sqrt{\pi}} \right)^{3-D},
\]

where \( M \) is the arbitrary renormalization scale and \( \gamma_E \approx 0.57776 \) the Euler constant one obtains, after renormalization by the appropriate counterterm (A) which removes the \(-N u^2/(6 \times 96\pi^2(3 - D)) \) divergence, the final result:

\[
r_{\text{c}}^{(\text{N,exc})}(0) = \frac{N}{6} \frac{u^2}{96\pi^2} \left[ 1 + \ln \left( \frac{48 M}{N u} \right) \right].
\]

Numerically, for the standard lattice simulation choice of scale, \( M = u/3 \), and the physically relevant number of field components, \( N = 2 \), this gives \( r_{\text{c}}^{(\text{N,exc})} \approx 0.001814945 u^2 \).

Then, adding the two contributions, Eqs. (16) and (21), one gets

\[
r_{\text{c}}^{(\text{N})} = \frac{N}{6} \frac{u^2}{48\pi^2} \left[ 1 + \ln \left( \frac{48 M}{N u} \right) \right],
\]

which is completely equivalent to the result very recently obtained in Ref. [13] (Eq. (25)), the latter being performed in a different independent way. Eq. (22) is directly comparable with the available lattice simulation results, since the latter were also obtained [1, 9] in the \( \text{MS} \) scheme.

As an interesting consistency check, note that the logarithmic scale dependence in Eqs. (22) is just identical to the one already obtained [11] as the coefficient \( c_2^N \) in Eq. (11): more precisely, putting the appropriate factors defined in Eq. (16) to pass from \( r_{\text{c}} \) to the relevant coefficients of the expression (11) (and also putting back the arbitrary \( N \) dependence of the relevant graph instead of its large \( N \) value in Eq. (21)), we have:

\[c_2^N(\ln \text{part}) = 16(N + 2\pi) \frac{1}{\Gamma(1/2)} \frac{1}{\Gamma(3/2)} \frac{1}{\Gamma(5/2)} \ln[48M/(N u)]\]

This is of course expected, since the divergence has a perturbative origin and arises solely from the setting sun graph, namely the order \( u^2 \) in a formal perturbative expansion of expression (10) (see also Fig. 2). In that sense, the novelty of our result resides more in the non-logarithmic piece of Eq. (22), as well as the explicitly non-perturbative dependence on the coupling \( u \) therein.

For \( N = 2 \) and \( M = u/3 \) we obtain numerically the exact large-\( N \) result \( r_{\text{c}}^{(\text{N})} \approx 0.002166755 u^2 \) which, when used in conjunction with the value \( \kappa = -Nu/(96\pi^2) \) obtained by Baym, Blaizot and Zinn-Justin [4] gives from Eq. (10):

\[c_2^N(1/\text{N order}, N = 2) \approx 95.7941\]

**IV. OPTIMIZED PERTURBATION EVALUATION**

In this Section our aim is to test the eventual convergence of the optimized perturbation (LDE method) towards the “exact” large-\( N \) result for \( r_{\text{c}} \) given by Eq. (22). This supplements a similar analysis performed in Ref. [11] for the quantity \( c_1 \) in the large-\( N \) limit, but for the independent quantity \( c_2^N \), so as to establish the reliability of the finite \( N \) LDE results recently obtained in Ref. [11]. According to the usual prescription the LDE can be implemented in the BEC case by considering the interpolated action

\[S_{\text{LDE}}^{\text{exc}} = \int d^3 x \left[ \frac{1}{2} \left( \nabla \phi \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\delta}{2} (r - m^2) \phi^2 + \frac{\delta u}{4!} (\phi^2)^2 + \frac{\delta A_5}{2} \phi^2 \right],\]

(24)
which interpolates between a free theory ($\delta = 0$) and the original interacting theory ($\delta = 1$) \cite{13,14}. Any physical quantity ($\Phi$) is then evaluated and renormalized exactly at some perturbative order by treating the dummy parameter ($\delta$) as a small number and $m$ as an arbitrary mass parameter to be fixed later. At the end, one sets $\delta = 1$. While the $\delta m^2$ is treated as a new quadratic vertex, the other part is absorbed by the propagator which is now naturally infrared regulated by the mass term. Accordingly, the LDE amounts in a first stage to perform an order by order perturbative calculation of the relevant Green’s functions (here the self-energy contributions $\Sigma(0)$) but in the massive theory as defined by Eq. (21). Thus, any finite order perturbative calculation will produce $m$-dependent physical quantities. To get rid of this unwanted dependence one usually demands that $\Phi$ has its optimum value at the point where it is less sensitive to $m$:

$$\frac{d\Phi}{dm} = 0,$$  \hspace{1cm} (25)

according to the Principle of Minimal Sensitivity (PMS) \cite{13} to which we shall refer as ordinary PMS (OPMS). However, we shall also consider below some specific variations of this method that had been more recently developed \cite{11,21,22}, which in practice essentially introduce some additional arbitrary variational parameters. As explicitly shown in Refs. \cite{21,22}, the whole procedure is compatible with the renormalization program which is implemented exactly as in the usual perturbative case. Finally note that, according to the original prescription \cite{6}, we have treated $r$ ($r_c$ at the critical point) as an interaction, since this quantity has a critical value which is at least of order $\delta$. The Feynman rules for the interpolated theory, in Euclidean space, are $-\delta r$, $\delta m^2$ and $-\delta A_0$, for the quadratic vertices and $-\delta u$ for the quartic vertex while the propagator is given by

$$G(p) = [p^2 + m^2]^{-1}.$$  \hspace{1cm} (26)

A. Reviewing the perturbative LDE calculation

To make this paper self contained let us briefly recall how the LDE is employed in the evaluation of $r_c$ by redoing the step by step evaluation performed in Ref. \cite{13}. Here, we shall stay within the large-$N$ limit only, while the evaluation performed in Ref. \cite{13} was carried out for any value of $N$. According to the HP theorem, applied in conjunction with the LDE \cite{4,13}, $\delta r_c^{(\delta)}$ is obtained from the perturbative evaluation of self energy diagrams, $\Sigma^{(\delta)}(0)$. To order-\(\delta\) one has only the plain tadpole contribution and a direct application of the Feynman rules for the interpolated theory and dimensional regularization (see Ref. \cite{13} for details) gives, in the large-$N$ limit, the finite contribution

$$\delta r_c^{(1)} = -\Sigma^{(1)}(0) = \delta u N \frac{m}{3 \pi}.$$

Carrying on to order-$\delta^2$, still within the large-$N$ limit, one must consider \cite{13}

$$\delta r_c^{(2)} = -\Sigma^{(2)}(0) = \delta u N \frac{m^*}{3 \pi} + \delta^2 N \frac{u}{16 \pi} r_c - \delta^2 \left( \frac{N}{3} \right)^2 \frac{u^2}{128 \pi},$$  \hspace{1cm} (28)

where $m^* = m(1 - \delta)^{1/2}$ must be expanded accordingly. Now, one replaces $\delta r_c$ appearing in the right hand side of the above equation, with the value $\delta r_c^{(1)}$ obtained at the previous order so that the right hand side remains of order-$\delta^2$. Then, one can see how the “double-scoop” contribution (third term on the RHS of Eq. (28)) is exactly canceled due to the HP condition applied to $r_c$ at first order. It is easy to convince oneself that at any higher order ($n$) all large-$N$ contributions of order-$(N \delta u)^k$, with $1 < k \leq n$ cancel. Therefore to get non trivial results with the LDE one must add the next-to-leading order, $O(N^k (\delta u)^{k+1})$ terms, shown in Figure 1, to Eq. (27). The first comes from the “setting-sun” contribution (see left-hand-side of equality in Figure 2), which has been explicitly evaluated by Braaten and Nieto \cite{23} and also in Ref. \cite{13} giving

$$-\Sigma_{ssun}(0) = \delta^2 \frac{Nu^2}{18(4\pi)^2} \left[ \frac{1}{4\epsilon} + \ln \left( \frac{M}{m} \right) + \frac{1}{2} + \ln(1/3) \right].$$  \hspace{1cm} (29)

As already mentioned at the end of section III, this contribution displays an ultraviolet pole as $\epsilon \equiv (3 - D)/2 \to 0$ ($D \to 3$) which is, within dimensional regularization, the only primitive ultraviolet divergence associated with the
effective super-renormalizable three-dimensional theory. The pole produced by this primitive divergence fixes the mass counterterm coefficient in the \( \overline{\text{MS}} \) scheme

\[
\delta A_\delta = -\delta^2 \frac{N}{18} \frac{u^2}{(8\pi)^2} \frac{1}{c} .
\]  

(30)

As usual, this “vertex” must be considered also at higher orders so diagrams whose divergences arise from “setting-sun” sub-diagrams maybe rendered finite. One then gets the finite second order result

\[
\delta r_c^{(2)} = -\Sigma^{(2)}(0) = \delta u \frac{N m^*}{3} \frac{8\pi}{\delta^2} + \frac{N u^2}{18(4\pi)^2} \left[ \ln \left( \frac{M}{m} \right) + \frac{1}{2} + \ln(1/3) \right] .
\]  

(31)

Note that contrary to the original massless case, the massive LDE interpolated action allows for the unique plain tadpole contribution, given by Eq. (27), to survive at any order. This, more precisely, modifies the simplest LDE substitution formula

\[
\text{Eq. (33)}
\]

by introducing consistently into the substitution the relevant critical exponent:

\[
\delta r_c(M_1) = r_c(M_2) + \frac{N u^2}{18(4\pi)^2} \ln \left( \frac{M_1}{M_2} \right) .
\]  

(32)

It is not too difficult to see that this relation will be verified at any order in \( \delta \). At order-\( \delta^2 \) the only diagram which is scale dependent is the “setting sun”. At a higher order \( \mu \geq 3 \) this order-\( \delta^2 \) type of contribution can appear only as a subdiagram \( \Sigma_{\text{ssun}}(p) \). At the same order a similar graph appears, but this time \( \delta r_c \) replaces the setting sun insertion. However, the “vertex” \( \delta r_c \) is always replaced by its expansion in \( \delta \) which contains, at order-\( \delta^2 \), exactly the same scale dependent term as given by the setting sun with zero external momentum \( \Sigma_{\text{ssun}}(0) \), with a reversed sign producing a net combination proportional to \( \Sigma_{\text{ssun}}(p) - \Sigma_{\text{ssun}}(0) \) which, in three dimensions, turns out to be scale independent. This means that, apart from the order-\( \delta^2 \) setting sun, all contributions to \( \delta r_c^{(n)} \) are automatically scale independent. This perturbative procedure, to order-\( \delta^4 \) is clearly illustrated in Ref. [12]. In the present work we are interested in the large-\( N \) limit only so it is easier to add up the whole series (represented by the diagrams of Figures 1 and 2) in a closed form and then develop it to the desired order in \( \delta \) as illustrated in the next subsection.

### B. Resumming the large-\( N \) series

As shown in the previous subsection, a main difference between the LDE and the exact large-\( N \) above calculation is that with the former one considers a massive theory, which effectively introduces a mass term in the (dressed) propagator. So, for practical reasons the LDE evaluation can be performed as discussed in the previous sub-section or alternatively as we do now. Having defined the relevant perturbative series in powers of \( u \) for the massive case, we can proceed to the usual ordinary PMS (OPMS) or its improved version (IPMS). Accordingly, we apply on the latter series the substitutions:

\[
m \to m' = m (1 - \delta)^{1/2} ; \quad u \to u \delta ,
\]  

(33)

in the standard OPMS-LDE [17], or

\[
m \to m (1 - \delta)^{1/2}[1 + (1 - a) \delta]^{1/2} ; \quad u \to u \delta ,
\]  

(34)

and subsequent generalizations in the IPMS-LDE [11]. The difference between the OPMS and the IPMS is basically that within the latter one introduces more arbitrary interpolation parameters at each successive perturbative order such that the PMS criterion is generalized by requiring \( \partial \Phi / \partial m = 0, \partial^2 \Phi / \partial m^2 = 0 \), etc. It turns out [11] that the IPMS essentially reduces a basic problem concerning the OPMS. In fact, as discussed in many BEC applications [13, 14, 16], at high orders the OPMS generates many possible (complex) optimum values. On the other hand, the IPMS produces, in general, a unique (real) optimum. In the following we will compare results obtained with both methods.

We shall also consider for completeness another variation of the LDE-PMS introduced in Ref. [21], that we may call a “renormalization group inspired” interpolation. This, more precisely, modifies the simplest LDE substitution formula Eq. (33) by introducing consistently into the substitution the relevant critical exponent:

\[
m \to m (1 - \delta)^{1/\omega'} ,
\]  

(35)
where \( \omega' = 2 \Omega/(2 - \eta) \), \( \eta \) being the anomalous dimension of the critical propagator \( \sim 1/p^{2-\eta} \) and \( \Omega \equiv \beta'(g_c) \), where \( g_c \) is the critical coupling and \( \beta(g) \) the renormalization group beta function (see e.g. Ref. [24] for a review and definitions of the critical exponents). The benefit of the latter method in the present case is that the value of the critical exponent is known exactly for large \( N \): \( \omega'(N \to \infty) = 1 \). (Incidentally, our more empirical IPMS approach introducing more parameters is in practice equivalent to the latter at second order in \( \delta \), Eq. [34], where the only additional parameter \( a \) plays a role similar to the critical exponent \( \omega' \)). In our numerical applications below, we shall thus consider those three different forms of the LDE optimized perturbative expansion.

In order to resum the large-\( N \) LDE type of series one should now evaluate, in place of expression Eq. (11), the equivalent contributions coming from the graphs shown in Figure 1 now including a mass term in all propagators, including the plain tadpole term discussed in the previous section

\[
\begin{align*}
\rho_{c}^{\text{tad}}(m) &= -\frac{Nu}{6} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2 + m^2} \\
&\quad + \frac{uN}{18} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{(k^2 + m^2)^2} \left[ 1 + \frac{uN}{6} B(k,m) \right]^{-1} \left[ \frac{1}{(k+p)^2 + m^2} - \frac{1}{k^2 + m^2} \right],
\end{align*}
\]

(36)

where the first term on the right-hand side represents the plain one-loop tadpole while the series represents the tadpole with bubble insertions shown in Figure 1. However, note carefully that at the end the first term of the series will not be considered since it does not correspond to any real physical contribution being an artifact of the resummation as discussed, for an equivalent series, in Ref. [14]. Now, the massive dressed propagator reads

\[
B(k,m) = \int \frac{d^3 q}{(2\pi)^3} \frac{1}{[q^2 + m^2] \left[ (k+q)^2 + m^2 \right]} = \frac{1}{4\pi k} \arctan \left( \frac{k}{2m} \right),
\]

(37)

with \( k = |k| \) and similarly for \( p, q \) in Euclidean space. Contrary to the corresponding expression in the \( m = 0 \) case one cannot integrate exactly Eq. (36) due to the non-trivial dependence in \( k \) and \( m \) of \( B(k,m) \). However, the coefficients of the resulting perturbative series at arbitrary orders can be calculated numerically by simple integrals [14 16]. After discarding the spurious term discussed above the result can be written as

\[
\rho_{c}^{\text{tad}}(m) = \frac{Nu}{24\pi} - Nu^2 \sum_{i=1}^{\infty} C_i \left( -\frac{uN}{m} \right)^i,
\]

(38)

where the perturbative coefficients are given by

\[
C_i = \frac{1}{96\pi^3} \left( \frac{1}{48\pi} \right)^i \int_0^\infty \frac{dz}{(z^2 + 1)(z^2 + 4)} [A(z)]^i,
\]

(39)

with

\[
A(z) = \frac{2}{z} \arctan \frac{z}{2},
\]

(40)

and \( z = k/m \) (see e.g. Ref. [14] for more details). Similarly, the contributions of the exchange-type, shown in Figure 2, are now to be evaluated with the massive propagator \( B(k,m) \) in Eq. (37)

\[
\rho_{c}^{\text{exc}}(m) = -\Sigma(0)^{\text{exc}} = -\int \frac{d^3 k}{(2\pi)^3} \frac{1}{|k^2 + m^2|} \left( \frac{u}{3} \right) \left[ 1 + \frac{NuB(k,m)}{6} \right]^{-1}.
\]

(41)

Note that the first term of this series is nothing but and additional plain tadpole contribution, however with no overall \( N \) factor. This is consistent, since we should in fact consider for our purpose all contributions at the next-to-leading \( 1/N \) order. Explicitly this terms gives a contribution \( m u/(12\pi) \) to \( \rho_c \), which combines with the first term of Eq. (38) so that the final one-loop tadpole contribution to \( \rho_c \) is \( (N+2) m u/(24\pi) \), which is consistent with the usual result for arbitrary \( N \) (see e.g. [13]) . Next, the second term corresponds to the only divergent and scale dependent contribution (”setting-sum”) discussed above, which may be more conveniently evaluated separately as shown in Refs. [13 23]. All other higher order terms are finite and scale independent being easily resummed. Once more, the integral cannot be done analytically but arbitrary order coefficients of the corresponding perturbative series are evaluated numerically and the final series can be cast into the form:

\[
\rho_{c}^{\text{exc}}(m) = \frac{Nu^2}{18(4\pi)^2} \left[ \ln \left( \frac{M}{m} \right) + \frac{1}{2} + \ln(1/3) \right] - m u \sum_{i=2}^{\infty} D_i \left( -\frac{uN}{m} \right)^i,
\]

(42)
As expected, the scale dependence \((\ln(M))\) within the perturbative result Eq. (12) above again has the very same coefficient than the corresponding scale dependence of the non-perturbative result, Eq. (21). Note also that our selection of graphs takes automatically care of the HP theorem with \(r_c\) being appropriately replaced. It is easy to see that the order-\(\delta^2\) result obtained in the previous sub-section can be immediately reproduced by our resummed type of calculation. The final total \(O(N^{-1})\) contribution is then obtained by adding Eqs. (38) and (22) (with however the consistent replacement \(N \to N + 2\) in the first one-loop tadpole term of Eq. (38), as discussed above).

C. The large-\(N\) case and convergence of the LDE

Let us now perform the OPMS-LDE and the IPMS-LDE substitutions discussed above in order to obtain optimized results also discussing the convergence properties of each method. Actually, as discussed above in section IV.B, the first series given by Eq. (38) is equivalent, up to a trivial overall factor \(-u/6\), to the series defining the quantity \(k\) relevant to the \(\Delta T_c/T_c\) shift in the critical temperature. For the latter series, a good numerical convergence towards the exact result has already been obtained in the case of the IPMS [11]. On the other hand, the exact analytical result for the chain-exchange type of graphs, Fig. 2, is unambiguously defined separately as Eq. (21). We can thus for numerical analysis simplicity discard in a first stage the part of the perturbative contributions Eq. (38) to concentrate on the other non-trivial series Eq. (22) in order to examine its eventual convergence properties in the framework of the OPMS and/or IPMS. Since we shall also consider in a second stage the complete series, it provides in this way two numerically independent analysis of the (eventual) convergence properties of the different PMS methods. Accordingly the exact result to which the resummation of Eq. (22) alone should be compared with, is the second term in Eq. (22), which numerically is equal to \(\sim 0.00181497\ u^2\).

We give the results of IPMS and standard OPMS in Table I, second and fourth column respectively. The third column indicates the corresponding value of the IPMS leading parameter \(a\), cf. Eq. (34). Next, we give in the last right column the results obtained when considering, following Refs. [10, 21], the renormalization group inspired interpolation method (that we shall dub RGPMS for comparison), as defined in Eq. (35). Notice that at second order \((k = 2)\), the IPMS and OPMS result coincide simply because it is not possible to implement an (independent) new interpolation parameter, so that the IPMS actually starts at order three. (The empty RGPMS result entry at second order is simply due to the fact that for \(\omega' = 1\) there is an accidental exact cancellation of the linear in \(m\) (tadpole) term of Eq. (35), such that the RGPMS non-trivial optimization solutions also start at order three).

One can see that at first rough sight, all different PMS approaches seem to perform very well, all exhibiting a numerically convergence behaviour towards the exact result as the order increases. Moreover, all the results are already remarkably good at third order. In practice the IPMS method, introducing one more parameter at each successive order, becomes quite cumbersome at large orders, so that we somewhat arbitrarily stopped our analysis at order twelve, beyond which it became unreasonably time consuming for our calculation (performed with Mathematica [23]). When examining in more details the behavior of each different method, we first observe that the (real part of) the standard OPMS method results get very close to the exact result at larger and larger orders, letting aside the embarrassing non-zero but small imaginary parts. However, in Table I we have selected among several (many at large orders) real and complex optimal results those closest to the known exact result. Nevertheless, if selecting instead only the real solutions we find a slightly less convergent series of approximations, but still quite satisfactory (the real solutions are provided in Table I, whenever they exist). Alternatively, the IPMS has the advantage that it gives at any order a unique real optimal solution, and seemingly converging somewhat more slowly but regularly towards the expected result. As far as the RGPMS method is concerned, some of the optimization results appear at large orders to be even better than the two previous ones, being impressively close to the exact one as the perturbative order increases. This may be due to the fact that the known large-\(N\) exact value \(\omega' = 1\) enters this interpolation. However, the RGPMS suffers a priori from the same drawback than the ordinary PMS, namely that the real optimal solutions are not unique, as explicitly illustrated in the table, and we have also selected the best among many more (mainly complex) optimized solutions. It is interesting to remark finally that, as observed similarly in ref. [11], as the order increases the leading IPMS parameter \(a\) seems to approach (though very slowly) a critical value \(a = 2\) (which, upon

\[ D_i = \frac{1}{6\pi^2} \left( \frac{1}{48\pi} \right)^i \int_0^{\infty} dz \frac{z^2}{(z^2 + 1)} |A(z)|^i. \]  

(43)
neglecting other higher order IPMS interpolation parameters, would correspond to $\omega' = 1$ in the RGPMS approach. 
Next, as a second stage independent numerical analysis using the same OPMS, IPMS or RGPMS methods, we consider

| $k$ | $r_{cOPMS}/u^2$ | $r_{cIPMS}/u^2$ | $r_{cIPMS}/u^2$ | $r_{cIPMS}/u^2$ |
|-----|----------------|----------------|----------------|----------------|
| 2   | 0.00269        | 0.00206        | 0.00206        | 0.00206        |
| 3   | 0.001845       | 0.001859       | 0.001859       | 0.001859       |
| 4   | 0.001834       | 0.001809 + 3.6 \times 10^{-5} I, 0.001857 | 0.0018199 | 0.0018199 |
| 5   | 0.001830       | 0.001809 + 6 \times 10^{-5} I | 0.001821 | 0.001821 |
| 6   | 0.001825       | 9.1 \times 10^{-6} I, 0.001837 | 0.0018195 | 0.0018195 |
| 7   | 0.001824       | 1.6 \times 10^{-6} I, 0.001830 | 0.001815 ± 2 \times 10^{-5} I, 0.001819 | 0.0018185 |
| 8   | 0.001823       | 1.63 \times 10^{-6} I, 0.001830 | 0.001815 ± 2 \times 10^{-5} I, 0.001819 | 0.0018185 |
| 9   | 0.001822       | 1.65 \times 10^{-6} I, 0.001827 | 0.0018153, 0.0018180 |
| 10  | 0.0018213      | 1.67 \times 10^{-6} I, 0.0018199 ± 9.1 \times 10^{-6} I | 0.0018152 ± 5 \times 10^{-6} I, 0.0018179 |
| 11  | 0.0018208      | 1.69 \times 10^{-6} I, 0.0018252 | 0.00181513, 0.0018176 |
| 12  | 0.0018178      | ± 1.1 \times 10^{-5} I, 0.0018219 | 0.00181509, 0.0018173 |

the full perturbative series ($r_{cIPMS}^{\text{tot}}$) consisting of the sum of expressions defined in Eq. 38 and 52, respectively. The results of the optimization are shown in Table II. The numerical behaviour appears to be similar to the one in Table I, namely with all different PMS methods appearing to converge quite well. However, we stress once more the uniqueness of the real IPMS optimized solutions at each orders, exhibiting thus a relatively slower but completely unambiguous apparent convergence for this method. Finally, if neglecting the imaginary parts of the optimized solutions (which are however very small in this case), the RGPMS approach appears again to give the best results at large orders, though real solutions do not always exist at arbitrary orders, just like in the OPMS case.

| $k$ | $r_{cOPMS}/u^2$ | $r_{cIPMS}/u^2$ | $r_{cIPMS}/u^2$ | $r_{cIPMS}/u^2$ |
|-----|----------------|----------------|----------------|----------------|
| 2   | 0.002063       | 0.002063       | 0.002063       | 0.002063       |
| 3   | 0.001973       | 1.14 ± 0.001960 ± 7.9 \times 10^{-6} I | 0.002060 | 0.002060 |
| 4   | 0.002041       | 1.70 ± 0.001999 | 0.002010 ± 2.1 \times 10^{-5} I |
| 5   | 0.002065       | 1.73 ± 0.002020, 0.001995 | 0.0021254 | 0.0021254 |
| 6   | 0.002081       | 1.75 ± 0.002033 ± 3.4 \times 10^{-5} I, 0.001999 | 0.002135 ± 7.5 \times 10^{-6} I |
| 7   | 0.002093       | 1.76 ± 0.002046, 0.002001 | 0.0021422 | 0.0021422 |
| 8   | 0.002101       | 1.78 ± 0.002055 ± 1.6 \times 10^{-4} I, 0.002002 | 0.0021463 ± 3.7 \times 10^{-6} I |
| 9   | 0.002108       | 1.79 ± 0.002067 ± 1.7 \times 10^{-4} I, 0.002061 | 0.0021495 | 0.0021495 |
| 10  | 0.002114       | 1.80 ± 0.002076 ± 1.7 \times 10^{-4} I, 0.002061 | 0.0021517 ± 2.2 \times 10^{-6} I |
| 11  | 0.002118       | 1.81 ± 0.002083 ± 1.7 \times 10^{-4} I, 0.002071 | 0.0021535 | 0.0021535 |
| 12  | 0.002122       | 1.82 ± 0.002089 ± 1.8 \times 10^{-4} I, 0.002006 | 0.0021549 ± 1.5 \times 10^{-6} I |
| 13  | 0.002121       | ± 1.3 \times 10^{-4} I, 0.002008 | 0.0021605 ± 7 \times 10^{-7} I |

D. The finite $N = 2$ case

Here, for completeness, we extend some of the results of Ref. 11 for $r_{c}$ at arbitrary $N$ by incorporating the very recently available improved six-loop and seven-loop relevant perturbative contributions performed in Refs. 10, 17, 26. One obtains to order-$u^{7}$ and for $N = 2$
\[ r_c^{(5)} = -\Sigma_{\text{ren}}(0) = \frac{um}{6\pi} + u^2 A_2 \left[ \ln \left( \frac{M}{m} \right) + \frac{1}{2} + \ln \frac{1}{3} \right] + m^2 \sum_{i=3}^{7} \left( \frac{u}{m} \right)^i A_i , \]  

where in our normalization, \( A_2 = 1.40724 \times 10^{-3} \), \( A_3 = 8.50891 \times 10^{-5} \), \( A_4 = 3.46941 \times 10^{-6} \), \( A_5 = 2.23296 \times 10^{-7} \), \( A_6 = 1.81417 \times 10^{-8} \), and \( A_7 = 1.71627 \times 10^{-9} \). Performing the IPMS-LDE and OPMS-LDE substitutions, already discussed, one obtains the optimized LDE results shown in Table III below \(^2\) for \( M = u/3 \). As one can see, the results of our alternative IPMS method show a remarkable agreement with the numerical lattice result. Concerning the results of the standard OPMS method, as shown in the right column of Table III, they approach closely the lattice result at third order while the fourth order result departs from the LS value. Then, the fifth and higher orders prediction approach the LS result again, considering again here the real parts of complex optimal solutions (although the imaginary parts are very small in this case).

### TABLE III: IPMS versus standard OPMS results for the physical \( N = 2 \) \( \text{BEC} \) \( r_c \), and correspondingly \( c_2' \) at different orders \( k \).

| \( k \) | \( r_c^{\text{IPMS}}/u^2 \) | \( c_2'(\text{IPMS}) \) | \( r_c^{\text{OPMS}}/u^2 \) | \( c_2'(\text{OPMS}) \) |
|---|---|---|---|---|
| 2 | 0.00315 | 101.2 | 0.00315 | 101.2 |
| 3 | 0.001828 | 69.85 | 1.77 | 0.00215 \( \pm \) 7.8 \( \times \) 10\(^{-4}\)I |
| 4 | 0.0018988 | 78.31 | 2.02 | 0.00247 |
| 5 | 0.0018992 | 74.44 | 2.116 | 0.00226 \( \pm \) 3.3 \( \times \) 10\(^{-4}\)I |
| 6 | 0.0019034 | 75.51 | 2.183 | 0.00205 \( \pm \) 4.5 \( \times \) 10\(^{-4}\)I |
| 7 | 0.0019026 | 2.237 | 0.001887 \( \pm \) 2.0 \( \times \) 10\(^{-4}\)I |

V. CONCLUSIONS

Considering an effective three-dimensional model for homogeneous dilute weakly interacting Bose gases we have evaluated the physical quantity \( r_c \), which is directly related to the critical chemical potential. Our evaluations made use of the standard \( 1/N \) approximation as well as different variations of the optimized perturbation (linear \( \delta \) expansion) method. As discussed in the text, this quantity is important in the determination of a numerical coefficient \( (c_2') \) appearing at order-\( a^2n^{2/3} \) in the expansion of \( \Delta T \). Surprisingly, the large-\( N \) result \( r_c \approx 0.00216672u^2 \) compares well with the most recent lattice simulation result, \( r_c^{\text{LS}} = (0.0019201 \pm 0.0000021)u^2 \) \(^1\) \([^2\) even for the case \( N = 2 \). As a matter of fact the large-\( N \) approximation seems to work better for the determination of \( r_c \) than for the \( \langle \phi^2 \rangle \) prediction performed in Ref. \[^4\]. Then, having established the large-\( N \) result we have used the linear \( \delta \) expansion, in the large-\( N \) limit, to examine its convergence properties, in a way similar as it was done in a previous work \[^14\] for the quantity \( \langle \phi^2 \rangle \). Our analysis shows that the standard optimization procedure produces converging results which can be better controlled if one uses a recently improved version of the method \[^11\]. In this way one is able to obtain, at each order in \( \delta \), only one real optimized solution, contrary to what happens if one uses the standard optimization procedure. We also obtained that another variation \[^20\] of the LDE, introducing consistently the critical exponent \( \omega' = 1 \) performs very well in the large \( N \) limit. Apart from determining the large-\( N \) result our work shows the reliability of the LDE finite \( N \) results for \( r_c \) obtained in Ref. \[^11\]. Finally, concerning the large-\( N \) \[^4\] and IPMS-LDE \[^11\] results for \( \kappa \), one can establish the analytical values for \( c_2' \). For \( N = 2 \) they are \( c_2' \approx 95.79 \) (large-\( N \) approximation). Also, our numerical results in Table III for \( c_2' \) from the IPMS-LDE at the latest available orders \( (\delta^5, \delta^6) \) compare very well with the lattice simulation result \( c_2' = 75.7 \pm 0.4 \) \[^1\], \[^4\].

\(^2\) The slightly different results at orders 3-5 of the IPMS method for \( c_2' \) with respect to Table VIII of Ref. \[^11\], are simply due to our updated use of the most recent analytical and numerical improvements in the determination of the perturbative series coefficients given in Ref. \[^12\]. Note that \( c_2' \) cannot be determined at order \( \delta^5 \) since this requires the use of some eight-loop contributions to \( \langle \phi^2 \rangle \) which are not available at present.

Note added: After completing a first version of this paper (eprint archive: cond-mat/0408538 v1) we became aware...
of the very recent work by Kastening [15], where an analytical evaluation of \( r_c \) in the large \( N \) limit is also performed as well as the relevant perturbative contributions up to seven loop for arbitrary \( N \) (partly using some previously unpublished seven-loop material by B. Nickel), with a corresponding new evaluation of \( r_c \) in the framework of the “variational perturbation theory” (VPT). Our large \( N \) exact result for \( r_c \), Eq. [22], maybe be considered as an independent cross-check of his result, since ours is performed with a different method. Concerning his numerical VPT results for \( N = 2 \), at seven-loop order they lie in the range\(^3\): \( r_c/u^2 \sim 0.0019028 – 0.0019401 \), thus appearing very consistent with our results in Table III.

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\(^3\) Note for comparison a trivial factor of 4 (for \( N = 2 \)) difference in the normalization of \( r_c/u^2 \) in ref. 15 with respect to us or LS results