ON THE GSOR ITERATION METHOD FOR IMAGE RESTORATION

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Abstract. In this study, we present a generalization of the successive overrelaxation (GSOR) iteration method to find the solution of the image restoration problem. Moreover, an improved version of the GSOR (IGSOR) method is also given to solve the proposed problem. Convergence of the GSOR and IGSOR methods are investigated. Three numerical examples are given to illustrate the effectiveness and accuracy of the methods.

1. Introduction. Image restoration problem is an important area of image processing which attempts to restore degraded images. In almost all applications, the degraded image is a noisy and blurred version of the true image. The image restoration is widely used in various domains of applied science such as medical imaging [9, 10], film making and archival [7], astronomical imaging [22], engineering and optical systems [2], and many other areas [13]. The proposed problem can be modelled as the following linear and shift-invariant process [9]

\[ g = h \ast f + \eta, \]  

where \( \ast \) is convolution process, \( g \) is the observed image, \( h \) is the blurring kernel, \( f \) is the true image and \( \eta \) is the additive noise. The discretization of (1) leads to the following matrix-vector equation

\[ g = Af + \eta, \]  

where \( f, g \) and \( \eta \) are \( n^2 \)-dimensional vectors and \( A \) is \( n^2 \times n^2 \) blurring matrix which is given by the point spread function (PSF). Since the finite length (and width) of the observed image \( g \), a finite section of true image \( f \) can be restored. Therefore, from the convolution process, the true image can not be completely determined by observed (degraded) image where is defined in limited field of view (FOV). From this restrictions, some assumptions are necessary outside the FOV which are called boundary conditions (BCs).
Some known BCs which are widely used for image restoration problem in literature are zero, periodic, reflexive and antireflective. For zero BCs, it is supposed that the outside of FOV is black (zero). It can be seen that considering zeros for the outside of FOV leads to block Toeplitz with Toeplitz blocks structure for the blurring matrix $A$. Periodic BCs is derived from periodically extending of data in outside of FOV which leads to the block circulant with circulant blocks structure for the matrix $A$. It is shown that the matrix-vector multiplications can be efficiently implemented for zero and periodic BCs by fast Fourier transforms [23]. The reflexive BCs is given by reflecting the FOV to outside of domain. In this BC, the matrix $A$ has block Toeplitz-plus-Hankel with Toeplitz-plus-Hankel blocks structure. If the PSF is symmetric, the blurring matrix can be diagonalized with the two-dimensional fast cosine transform [18]. Capazzino proposed the antireflective BCs which is given by anti-reflection of the FOV data to outside [4]. This BC leads to block Toeplitz-plus-Hankel-plus-rank-2-correction structure for the blurring matrix. It is shown that for symmetric PSF, discrete sine transform can be used to diagonalize the blurring matrix $A$. Nagy et al. [17] and Perrone [19] presented the Kronecker product approximations to implement the reflexive and antireflective BCs. To apply Kronecker product approximations, being a symmetric property of PSF is not necessary. Hence, in this study, the proposed approximation is applied to implement image restoration problem with antireflective BCs.

Several papers have been presented to solve the image restoration problem in the literature. Chan et al. presented a high-order total variation method to restore images [5]. Landi [14] proposed a fast truncated Lagrange method to find the solution of problem (2). Sylvester Tikhonov-regularization method has been applied to solve image restoration problem by Bouhamidi and Jbilou [3]. Deng et al. presented a wavelet-based two-level method to restore images [6]. A special Hermitian and skew-Hermitian splitting (SHSS) method has been given by Lv et al. in 2013 [15]. To implement SHSS method, the following equivalent form of problem (2) is presented:

$$\begin{pmatrix}
I & A \\
-A^T & \mu^2 I
\end{pmatrix}
\begin{pmatrix}
e \\
x
\end{pmatrix} =
\begin{pmatrix}
g \\
0
\end{pmatrix},$$

where $e$ is an auxiliary variable and $0 < \mu < 1$ is a regularization parameter. In the sequel, we consider the system $Kx = b$ to approximate the solution of image restoration problem. Salkuyeh et al. proposed a generalization of the successive overrelaxation (GSOR) iteration method to solve a class of complex symmetric linear system of equations [21]. In this study, we apply the proposed method to solve problem (3). Convergence of this method is investigated and the optimal value of unknown parameter is given for the proposed method. Furthermore, an improved version of the GSOR (IGSOR) iteration method and its convergence properties are also presented.

This paper is organized as follows. In Section 2, the GSOR method has been applied to solve image restoration problem (3). Some convergence remarks has also been presented. The IGSOR method and its convergence properties have been given in Section 3. In Section 4, three numerical examples have been proposed to show the effectiveness of the GSOR and IGSOR methods. Finally, some concluding remarks have been drawn in Section 5.
2. Description of the method. In this section, we apply the GSOR iteration method to solve the image restoration problem (3).

The matrix $K$ in (3) can be split as $K = D - E - F$, where $D$, $E$ and $F$ are defined as follows:

$$
D = \begin{bmatrix} I & O \\ O & \mu^2 I \end{bmatrix}, \quad E = \begin{bmatrix} O & O \\ A^T & O \end{bmatrix}, \quad F = \begin{bmatrix} O & -A \\ O & O \end{bmatrix}.
$$

The above splittings leads to the following GSOR iteration method to solve the image restoration problem:

$$
x_{k+1} = (D - \omega E)^{-1}(\omega F + (1 - \omega)D)x_k + \omega(D - \omega E)^{-1}b,
$$

(k = 0, 1, 2, ...) where $0 \neq \omega \in \mathbb{R}$. Note that in Eq. (4), we have

$$
D - \omega E = \begin{bmatrix} I & O \\ -\omega A^T & \mu^2 I \end{bmatrix},
$$

$$
\omega F + (1 - \omega)D = \begin{bmatrix} (1 - \omega)I & -\omega A \\ O & (1 - \omega)\mu^2 I \end{bmatrix}.
$$

Therefore, from Eqs. (4)-(6), the GSOR iteration method is given as

$$
\begin{bmatrix} I & O \\ -\omega A^T & \mu^2 I \end{bmatrix} \begin{bmatrix} e_{k+1} \\ f_{k+1} \end{bmatrix} = \begin{bmatrix} (1 - \omega)I & -\omega A \\ O & (1 - \omega)\mu^2 I \end{bmatrix} \begin{bmatrix} e_k \\ f_k \end{bmatrix} + \omega \begin{bmatrix} g \\ 0 \end{bmatrix},
$$

for $k = 0, 1, 2, \ldots$. It can be easily seen that Eq. (7) can be written in the matrix-vector form as:

$$
\begin{bmatrix} e_{k+1} \\ f_{k+1} \end{bmatrix} = G_\omega \begin{bmatrix} e_k \\ f_k \end{bmatrix} + \omega(D - \omega E)^{-1} \begin{bmatrix} g \\ 0 \end{bmatrix},
$$

where

$$
G_\omega = (D - \omega E)^{-1}(\omega F + (1 - \omega)D).
$$

Note that we have

$$
(D - \omega E)^{-1} = \begin{bmatrix} I & O \\ \omega A^T & \frac{1}{\mu^2} I \end{bmatrix},
$$

and hence, the GSOR iteration method can also be summarized as Algorithm 1.

**Algorithm 1 (GSOR):** Let $f_0 = g$ and $e_0 = g - Af_0$ be the initial guesses. $e_{k+1}$ and $f_{k+1}$ for $k = 0, 1, 2, \ldots$ are computed by using the following iteration scheme until $\begin{bmatrix} e_{k+1} \\ f_{k+1} \end{bmatrix}$ satisfies the stopping criterion:

$$
\begin{cases}
  e_{k+1} := (1 - \omega)e_k - \omega Af_k + \omega g, \\
  f_{k+1} := \frac{\omega(1 - \omega)}{\mu^2} A^T e_k - \frac{\omega^2}{\mu^2} A^T Af_k + (1 - \omega)f_k + \frac{\omega^2}{\mu^2} A^T g.
\end{cases}
$$

In the sequel, the convergence of the GSOR iteration method (8) is investigated.

To this end, the following lemmas are required.

**Lemma 2.1.** ([1]) Consider the second-degree equation $x^2 - rx + s = 0$, where $r$ and $s$ are real. Both roots of the proposed quadratic equation are less than one in modules if and only if $|s| < 1$ and $|r| < 1 + s$. 
Lemma 2.2. ([20]) The iterative method $x_{k+1} = Gx_k + c$ converges for any $c$ and $x_0$ if and only if $\rho(G) < 1$ where $\rho(G)$ stands for the spectral radius of the matrix $G$.

In the next theorem, a necessary and sufficient condition is presented for convergence of the GSOR iteration method.

Theorem 2.3. Let $\sigma_i, i = 1, \ldots, n^2$, be the singular values of $A$ and $\sigma = \max_{1 \leq i \leq n^2} \{ \sigma_i \}$. Then, the GSOR method (8) is convergent if and only if

$$0 < \omega < \frac{2\mu}{\mu + \sigma}.$$ 

Furthermore, if

$$\rho(G_{\omega^*}) = \min_{0 < \omega < \frac{2\mu}{\mu + \sigma}} \rho(G_{\omega}),$$

then

$$\omega^* = \frac{2\mu}{\sqrt{\mu^2 + \sigma^2 + \mu}},$$

and $\rho(G_{\omega^*}) = 1 - \omega^*$.

Proof. Let $(\lambda, [x; y])$ be an eigenpair of the matrix $G_\omega$. In other words,

$$G_\omega \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$ 

Then we have,

$$\begin{bmatrix} (1 - \omega)I - \omega A \\ O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} I \\ -\omega A^T \mu^2 I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{9}$$

Hence, from the Eq. (9), the following system can be given:

$$\begin{cases} (1 - \omega - \lambda)x = \omega Ay, \\ (1 - \omega - \lambda)\mu^2 y = -\lambda \omega A^T x. \end{cases} \tag{10}$$

Now, if $\lambda = 1 - \omega$, for convergence of the GSOR method we must have $|1 - \omega| < 1$. In other words, for $0 < \omega < 2$, we have $|\lambda| < 1$.

Now, suppose that $\lambda \neq 1 - \omega$. In this case, we have $x \neq 0$, because otherwise from the second equation in (10) we deduce that $y = 0$, and hence $[x; y] = 0$, which contradicts with $[x; y]$ being an eigenvector of the matrix $G_\omega$. Furthermore, by straightforward computation, it can be seen that

$$\mu^2(1 - \omega - \lambda)^2 x = -\lambda \omega^2 A A^T x.$$

Note that the eigenvalues of the matrix $AA^T$ are real and nonnegative which are square of singular values of the matrix $A^T$. Therefore we have

$$(1 - \omega - \lambda)^2 = -\lambda \omega^2 \left( \frac{\sigma_i}{\mu} \right)^2, \quad i = 1, \ldots, n^2. \tag{11}$$

This equation can be written as:

$$\lambda^2 + \left( \omega^2 \left( \frac{\sigma_i}{\mu} \right)^2 + 2\omega - 2 \right) \lambda + (\omega - 1)^2 = 0, \quad i = 1, \ldots, n^2. \tag{12}$$
Now, from Lemmas 2.1-2.2 and similar to the proof of Theorem 1 in [21], the roots of Eq. (12) are less than one in modulus if and only if
\[
\begin{cases} 
|\omega - 1|^2 < 1, \\
\omega^2 \left(\frac{\sigma_i}{\mu}\right)^2 + 2\omega - 2 < 1 + (\omega - 1)^2, \quad i = 1, \ldots, n^2. 
\end{cases}
\] (13)
The first inequality in (13) is equivalent to \(0 < \omega < 2\) and the second one to
\[
0 < \omega < \frac{2}{1 + (\sigma_i/\mu)} = \frac{2\mu}{\mu + \sigma_i}, \quad i = 1, \ldots, n^2.
\]
Clearly, the latter inequalities hold if and only if
\[
0 < \omega < \frac{2\mu}{\mu + \sigma}.
\]
On the other hand, by letting
\[
\gamma_i = \frac{\sigma_i}{\mu}, \quad i = 1, \ldots, n^2,
\]
Eq. (11) can be written as \((1 - \omega - \lambda)^2 = -\lambda\omega^2\gamma_i^2, \quad i = 1, \ldots, n^2\). Now similar to the proof of Theorem 2 in [21], we deduce that
\[
\omega^* = \frac{2}{\sqrt{1 + (\sigma/\mu)^2} + 1} = \frac{2\mu}{\sqrt{\mu^2 + \sigma^2} + \mu},
\] (14)
and \(\rho(G_{\omega^*}) = 1 - \omega^*\), which completes the proof.

3. The improved GSOR method. From the previous section, it can be seen that the coefficient matrix \((D - \omega E)\) is near to a singular matrix for small values of \(\mu\). We present an improvement of the GSOR (IGSOR) method to overcome this problem. To this end, the following splitting is given for the matrix \(K\),
\[
\tilde{D} = \begin{bmatrix} (1 + \alpha)I & O \\ O & (\mu^2 + \alpha)I \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} O & O \\ A^T & O \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} \alpha I & -A \\ O & \alpha I \end{bmatrix},
\]
where \(\alpha\) is a positive parameter. Therefore, we have
\[
\tilde{D} - \omega \tilde{E} = \begin{bmatrix} (1 + \alpha)I & O \\ -\omega A^T & (\mu^2 + \alpha)I \end{bmatrix}, \\
\omega \tilde{F} + (1 - \omega)\tilde{D} = \begin{bmatrix} (1 + \alpha - \omega)I & -\omega A \\ O & (\mu^2 + \alpha - \mu^2\omega)I \end{bmatrix}.
\] (15)
Now, based on the new splitting (15), the IGSOR iteration method is presented for \(k = 0, 1, 2, \ldots\) as follows:
\[
x_{k+1} = \hat{G}_{\omega,\alpha}x_k + \omega(\tilde{D} - \omega \tilde{E})^{-1}b,
\] (16)
where
\[
\hat{G}_{\omega,\alpha} = (\tilde{D} - \omega \tilde{E})^{-1}(\omega \tilde{F} + (1 - \omega)\tilde{D}).
\] (17)
From Eqs. (15)-(16), to solve the image restoration problem, the IGSOR method can be written as follows:

\[
\begin{pmatrix}
(1 + \alpha)I & O \\
-\omega A^T & (\mu^2 + \alpha)I
\end{pmatrix}
\begin{pmatrix}
e_{k+1} \\
f_{k+1}
\end{pmatrix}
= \begin{pmatrix}
(1 + \alpha - \omega)I & -\omega A \\
\omega A & (\mu^2 + \alpha - \mu^2 \omega)I
\end{pmatrix}
\begin{pmatrix}
e_k \\
f_k
\end{pmatrix}
+ \omega \begin{pmatrix} g \\ 0 \end{pmatrix}.
\]

It can be easily seen that

\[
(\tilde{D} - \omega \tilde{E})^{-1} = \begin{pmatrix}
\frac{1}{1+\alpha} I & O \\
\frac{1}{(1+\alpha)(\mu^2 + \alpha)} A^T & \frac{1}{\mu^2 + \alpha} I
\end{pmatrix},
\]

hence, similar to the GSOR method, the IGSOR method can be summarized in the following algorithm:

**Algorithm 2 (IGSOR):** Let \( f_0 = g \) and \( e_0 = g - Af_0 \) be the initial guesses. \( e_{k+1} \) and \( f_{k+1} \) for \( k = 0, 1, 2, \ldots \) are computed by using the following iteration scheme until \( \begin{pmatrix} e_{k+1} \\ f_{k+1} \end{pmatrix} \) satisfies the stopping criterion:

\[
\begin{aligned}
e_{k+1} &= (1 - \frac{\omega}{1+\alpha})e_k - \frac{\omega}{1+\alpha}Af_k + \frac{\omega}{1+\alpha}g, \\
f_{k+1} &= \frac{\omega(1 + \alpha - \omega)}{(1 + \alpha)(\mu^2 + \alpha)} A^T e_k - \frac{\omega^2}{(1 + \alpha)(\mu^2 + \alpha)} A^T Af_k \\
&+ \frac{\mu^2 + \alpha - \mu^2 \omega}{(1 + \alpha)(\mu^2 + \alpha)} f_k + \frac{\mu^2 + \alpha - \mu^2 \omega}{(1 + \alpha)(\mu^2 + \alpha)} A^T g.
\end{aligned}
\]

Now, the next theorem gives some necessary and sufficient conditions for the convergence of the IGSOR iteration method.

**Theorem 3.1.** Let \( 0 < \mu < 1, \alpha > 0 \) and \( \sigma = \max_{1 \leq i \leq n^2} \{\sigma_i\} \), where \( \sigma_i, i = 1, \ldots, n^2 \) are the singular values of \( A \). Let also

\[
\hat{\alpha} = \frac{2\mu^4 + 2\mu^2 + 2\sqrt{2}\mu^2(1 - \mu^2)}{(\mu^2 - 3)^2 - 8},
\]

\[
\hat{\alpha}_1 = \frac{-(2\alpha \mu^2 + 4\mu^2 + 2\alpha) - \sqrt{\Delta_2}}{2(\sigma^2 - \mu^2)},
\]

\[
\hat{\alpha}_2 = \frac{-(2\alpha \mu^2 + 4\mu^2 + 2\alpha) + \sqrt{\Delta_2}}{2(\sigma^2 - \mu^2)},
\]

where

\[
\Delta_2 = (4\mu^4 - 8\mu^2 + 16\sigma^2 + 4)\alpha^2 + (16\mu^2 \sigma^2 + 16\sigma^2)\alpha + 16\mu^2 \sigma^2,
\]

and

\[
0 < \omega < 2(1 + \alpha).
\]

Then, the IGSOR method (16) is convergent if each of the following conditions is satisfied

(i): \( \sqrt{2} - 1 \leq \mu < 1, \sigma < \mu \) and

\[
0 < \alpha < \hat{\alpha}_2 \quad \text{or} \quad \alpha > \hat{\alpha}_1.
\]

(ii): \( \sqrt{2} - 1 \leq \mu < \min\{\sigma, 1\} \), and

\[
0 < \alpha < \hat{\alpha}_2.
\]
(iii): $0 < \mu < \sqrt{2} - 1$, $\sigma < \mu$, and

$$0 < \alpha < \min\{\hat{\alpha}, \hat{\alpha}_2\} \text{ or } \hat{\alpha}_1 < \alpha < \hat{\alpha}.$$  

(iv): $0 < \mu < \sqrt{2} - 1$, $\sigma > \mu$ and

$$0 < \alpha < \min\{\hat{\alpha}, \hat{\alpha}_2\}.$$  

Proof. Let $(\lambda, [x; y])$ be an eigenpair of the matrix $\tilde{G}_\omega$. Hence, we have

$$\tilde{G}_\omega \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix},$$

which can be written as the following form:

$$\begin{bmatrix} (1 + \alpha - \omega)I & -\omega A \\ O & (\mu^2 + \alpha - \mu^2)I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} (1 + \alpha)I & O \\ -\omega A^T & (\mu^2 + \alpha)I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$  

Hence, we have

$$\begin{cases} (\lambda(\alpha + 1) + \omega - \alpha - 1)x = -\omega Ay, \\ (\lambda(\mu^2 + \alpha) + \mu^2\omega - \alpha - \mu^2)y = \lambda \omega A^T x. \end{cases}$$  

From the above system, if $\lambda = 1 - \frac{\omega}{\alpha + 1}$, for the convergence of the method we must have $0 < \omega < 2(1 + \alpha)$. Similarly, if $\lambda = 1 - \frac{\mu^2\omega}{\mu^2 + \alpha}$, the condition $\omega < 2(1 + \frac{\alpha}{\mu^2})$ must be satisfied for the convergence of the IGSOR method.

Now suppose that $\lambda \neq 1 - \frac{\omega}{\alpha + 1}$ and $\lambda \neq 1 - \frac{\mu^2\omega}{\mu^2 + \alpha}$. From (18) we have

$$(\lambda(\alpha + 1) + \omega - \alpha - 1)(\lambda(\mu^2 + \alpha) + \mu^2\omega - \alpha - \mu^2)x = -\lambda \omega^2 AA^T x.$$  

Since the square of singular values of the matrix $\mu^T$ are equal to eigenvalues of the matrix $AA^T$, we have

$$(\lambda(\alpha + 1) + \omega - \alpha - 1)(\lambda(\mu^2 + \alpha) + \mu^2\omega - \alpha - \mu^2) = -\lambda \omega^2 \sigma_i^2,$$

$$i = 1, \ldots, n^2.$$  

The equation (19) can be written as:

$$\lambda^2 + \bar{r} \lambda + \bar{s} = 0,$$

where

$$\bar{r} = \frac{(\omega - \alpha - 1)(\alpha + \mu^2) + (\mu^2\omega - \alpha - \mu^2)(\alpha + 1) + \omega^2 \sigma_i^2}{(\alpha + 1)(\alpha + \mu^2)}$$

and

$$\bar{s} = \frac{(\mu^2\omega - \alpha - \mu^2)(\omega - \alpha - 1)}{(\alpha + 1)(\alpha + \mu^2)}.$$  

From Lemma 2.1, we know that the roots of the equation (20) are less than one in modulus if and only if

$$\begin{cases} |\bar{s}| < 1, \\ |\bar{r}| < 1 + \bar{s}. \end{cases}$$  

For $0 < \omega < \alpha + 2 + \frac{\mu^2}{\alpha + 1}$, it can be seen that $\bar{s} < 1$. Now, we consider the case $\bar{s} > -1$. Note that this case can be equivalently written as:

$$\mu^2\omega^2 + (-\alpha\mu^2 - 2\mu^2 - \alpha)\omega + 2\alpha \mu^2 + 2\alpha^2 + 2\mu^2 + 2\alpha > 0.$$  

Now let,

$$\Delta_1 = ((\mu^2 - 3)^2 - 8)\alpha^2 - (4\mu^4 + 4\mu^2)\alpha - 4\mu^4.$$
We consider two cases \((\mu^2 - 3)^2 - 8 \leq 0\) and \((\mu^2 - 3)^2 - 8 > 0\). The inequality \((\mu^2 - 3)^2 - 8 \leq 0\) along with \(0 < \mu < 1\) is equivalent to \(\sqrt{2} - 1 \leq \mu < 1\). In this case, we have \(\Delta_1 < 0\) and as a result we deduce that the equation (22) holds true. On the other hand, the case \((\mu^2 - 3)^2 - 8 > 0\) together with \(0 < \mu < 1\) is equivalent to \(0 < \mu < \sqrt{2} - 1\). In this case
\[
\alpha_{\pm} = \frac{2\mu^4 + 2\mu^2 \pm 2\sqrt{2} \mu^2 (1 - \mu^2)}{(\mu^2 - 3)^2 - 8},
\]
are the zeros of \(\Delta_1\). We have
\[
\alpha_- = \frac{2\mu^2 \left((1 + \sqrt{2})\mu^2 - (\sqrt{2} - 1)\right)}{(\mu^2 - 3)^2 - 8} < 0.
\]
Therefore, from \(\alpha > 0\) and \(\alpha_- < 0\), we deduce that \(\Delta_1 < 0\) for \(0 < \alpha < \alpha_+ = \hat{\alpha}\), and hence \(\hat{s} > 1\).

Now, we consider the case \(-1 - \hat{s} < \hat{r} < 1 + \hat{s}\). It is easy to see that
\[
\hat{r} + \hat{s} + 1 = \frac{(\mu^2 + \sigma_i^2)\omega^2}{(\alpha + 1)(\alpha + \mu^2)} > 0, \quad i = 1, \ldots, n^2,
\]
therefore we consider the second case that \(\hat{r} - \hat{s} - 1 < 0\). Since \(\sigma = \max_{1 \leq i \leq n^2} \{\sigma_i\}\), it is enough to find the conditions that the following inequality holds:
\[
\hat{r} - \hat{s} - 1 = \frac{(-\mu^2 + \sigma^2)\omega^2 + (2\alpha \mu^2 + 4\mu^2 + 2\sigma)\omega - 4\alpha \mu^2 - 4\alpha^2 - 4\mu^2 - 4\alpha}{(\alpha + 1)(\alpha + \mu^2)} < 0.
\]
Since \((\alpha + 1)(\alpha + \mu^2)\) is positive, the above condition is equivalent to
\[
c(\omega) = (\sigma^2 - \mu^2)\omega^2 + (2\alpha \mu^2 + 4\mu^2 + 2\sigma)\omega - 4\alpha \mu^2 - 4\alpha^2 - 4\mu^2 - 4\alpha < 0.
\]
It can be easily seen that the discriminant of \(c(\omega)\) is positive (\(\Delta_2 > 0\)) for all \(0 < \mu < 1\), \(\sigma_i\)’s and \(\alpha > 0\), and hence the roots of \(c\) are given by \(\hat{\alpha}_1\) and \(\hat{\alpha}_2\). Now, if \(\sigma < \mu\), then \(\hat{\alpha}_1 > \hat{\alpha}_2\) and hence for \(0 < \alpha < \hat{\alpha}_2\) or \(\alpha > \hat{\alpha}_1\), we have \(\hat{r} < 1 + \hat{s}\). Moreover, if \(\sigma > \mu\), then it follows from \(\sigma^2 - \mu^2 > 0\) and \(c(0) < 0\) that \(\hat{\alpha}_1 < 0 < \hat{\alpha}_2\). Therefore, the proposed inequality holds for \(0 < \alpha < \hat{\alpha}_2\), and this completes the proof. \(\square\)

4. Illustrative examples. In this section, we present three examples to demonstrate the accuracy and effectiveness of the GSOR and IGSOR iteration methods. Furthermore, the numerical results are compared with a recently proposed method. All the methods have been implemented in MATLAB 8.2 software on a PC with an Intel(R) Core(TM) i7-620M 2.67 GHz processor with 4 GB memory.

To compare the original image with the restored one, peak signal-to-noise ratio (PSNR) and relative error are defined as follows:
\[
\text{PSNR} = 10 \log_{10} \frac{255^2 \times n^2}{\|f_{\text{res}} - f_{\text{true}}\|_2^2}, \quad \text{relative error} = \frac{\|f_{\text{res}} - f_{\text{true}}\|_2}{\|f_{\text{true}}\|_2},
\]
where the size of the image is \(n \times n\) and \(f_{\text{true}}, f_{\text{res}}\) are original and restored images, respectively. In all the examples, the GSOR and IGSOR methods are compared with the SHSS iteration method. The outer iterations and stop tolerance are considered to 15 and \(1e - 06\), respectively. Furthermore, to apply the SHSS method, we used the restarted GMRES method in MATLAB, where the stop tolerance is set to \(10^{-6}\) and \(\text{restart} = 15\) \([20]\).
The values of unknown parameters in SHSS and GSOR methods are approximated by estimating the singular values of matrix $A$ and Tikhonov regularization parameter. To this end, the two matrices $B_k$ and $C_k$ have been found that minimized $\|A - \sum_k B_k \otimes C_k\|$. Thus, the singular values of $A$ have been approximated by the small matrices $B_k$ and $C_k$. There are some known methods to approximate the regularization parameters such as the discrepancy principle method [16], the L-curve criterion method [11], and generalized cross validation (GCV) method [8]. In our examples, the GCV method is applied to approximate the regularization parameter. In the GCV method, the regularization parameter is a value which minimizes the GCV function

$$G(\mu) = \frac{\|A(A^TA + \mu^2I)^{-1}A^Tg - g\|_2^2}{(\text{trace}(I - A(A^TA + \mu^2I)^{-1}A^T))^2}. \quad (23)$$

The Kronecker product approximation [19] of matrix $A$ is applied to find an approximation of proposed argument in (23). The values of $\omega$ and $\alpha$ in IGSOR method have also been experimentally determined in all examples.

**Example 1.** In this example, the $256 \times 256$ cameraman grayscale image is blurred by the Moffat function which is defined as follows:

$$h_{ij} = c \left(1 + \frac{i^2}{s_1^2} + \frac{j^2}{s_2^2}\right)^{-\beta},$$

where $c$ is the normalization constant, $s_1 = s_2 = 5$, $\beta = 8$ and $i, j = 1, 2, \ldots, 60$. The degraded image is given by adding 1% Gaussian white noise to blurred image. In this test, we crop the degraded image to lower dimension and apply the iterative methods to the cropped images. The PSNR of the degraded image is 21.54. The true image, PSF function and the observed image domain which is characterized by white lines are shown in Figure 1. The SHSS, GSOR and IGSOR iteration methods have been applied to image restoration problem with given values for unknown parameters in Table 1. The PSNR, relative error and CPU time of the proposed methods have been given in Tables 2-4. Furthermore, restored images with GSOR and IGSOR methods for various BCs have been drawn in Figures 2-3. As the numerical results show, the GSOR and IGSOR methods are effective methods to solve this problem.

![True Image](image1.png) ![The Moffat function PSF](image2.png) ![Blurred and noisy image](image3.png)

**Figure 1.** True image, PSF and degraded image in Example 1.
Table 1. Values of $(\alpha, \omega)$ in Example 1.

| Method   | Zero       | Periodic | Reflexive | Antireflective |
|----------|------------|----------|-----------|----------------|
| SHSS     | (0.3283, 0) | (0.3333, 0) | (0.3290, 0) | (0.4650, 0)    |
| GSOR     | (0.22, -)  | (-0.20, -) | (-0.14, -) | (-0.19, -)    |
| IGSOR    | (0.27, 0.36) | (0.31, 0.22) | (0.02, 0.34) | (0.01, 0.28)  |

Table 2. PSNR values of various methods in Example 1.

| Method   | Zero   | Periodic | Reflexive | Antireflective |
|----------|--------|----------|-----------|----------------|
| SHSS     | 21.08  | 22.09    | 23.98     | 24.17          |
| GSOR     | 21.12  | 22.18    | 24.13     | 24.40          |
| IGSOR    | 21.23  | 22.25    | 24.23     | 24.46          |

Table 3. Relative error of various methods in Example 1.

| Method   | Zero   | Periodic | Reflexive | Antireflective |
|----------|--------|----------|-----------|----------------|
| SHSS     | 0.1796 | 0.1592   | 0.1287    | 0.1259         |
| GSOR     | 0.1790 | 0.1582   | 0.1265    | 0.1224         |
| IGSOR    | 0.1767 | 0.1571   | 0.1252    | 0.1218         |

Table 4. CPU times of various methods in Example 1.

| Method   | Zero   | Periodic | Reflexive | Antireflective |
|----------|--------|----------|-----------|----------------|
| SHSS     | 5.26   | 5.05     | 5.74      | 5.31           |
| GSOR     | 0.49   | 0.52     | 0.49      | 0.51           |
| IGSOR    | 0.48   | 0.52     | 0.49      | 0.51           |

Example 2. In this example, we consider $128 \times 128 \times 15$ MRI image from Matlab image processing toolbox. The out-of-focus PSF has been applied to blur this image. The proposed PSF is implemented by using the function $psfDefocus$ with $dim = 9$ and $R = 4$ which is defined in [12]. The degraded image is given by adding 2% white Gaussian noise to blurred image. The PSNR of the degraded image is 26.37. The true and degraded images have been drawn in Figure 4. The optimal values of unknown parameters have been given in Table 5. Using the proposed values, the SHSS, GSOR and IGSOR methods have been applied to solve image restoration problem. PSNR, relative error and CPU time of these methods are presented in Tables 6-8. The restored image with the GSOR and IGSOR methods are drawn for various BCs in Figures 5-6. As can be seen from numerical results, the GSOR and IGSOR methods are reliable and effective methods to solve image restoration problem.

Example 3. In this example, we consider a $256 \times 256$ grayscale image to check the applicability of the GSOR and IGSOR methods in higher dimensions. To this end, the proposed grayscale image is blurred by the following truncated Gaussian PSF
Figure 2. Restored images with GSOR method for various BCs in Example 1.

Table 5. Values of \((\alpha, \omega)\) in Example 2.

| Method | Zero       | Periodic   | Reflexive  | Antireflective |
|--------|------------|------------|------------|----------------|
| SHSS   | (0.3277, -) | (0.3333, -) | (0.3339, -) | (0.6039, -)    |
| GSOR   | (-, 0.32)  | (-, 0.30)  | (-, 0.17)  | (-, 0.18)     |
| IGSOR  | (0.01, 0.09)| (0.001, 0.15)| (0.005, 0.22)| (0.01, 0.25)  |

Table 6. PSNR values of various methods in Example 2.

| Method | Zero | Periodic | Reflexive | Antireflective |
|--------|------|----------|-----------|----------------|
| SHSS   | 27.50| 27.71    | 28.81     | 28.39          |
| GSOR   | 27.53| 27.76    | 29.41     | 29.14          |
| IGSOR  | 27.61| 27.83    | 29.43     | 29.16          |

function:

\[ h_{ij} = \begin{cases} \ce^{-0.1(i^2+j^2)}, & \text{if } |i - j| \leq 8, \\ 0, & \text{otherwise,} \end{cases} \]

where \(c\) is the normalization parameter. Degraded image is given by adding 1% Gaussian white noise to the blurred image. The PSNR of the degraded image is 24.25. True image, PSF function and degraded image are shown in Figure 7. To
compare the SHSS, GSOR and IGSOR iteration methods, using the given values of unknown parameters in Table 9, the PSNR, relative error and CPU time of these methods are presented in Tables 10-12. Furthermore, the restored images with proposed methods for various BCs are drawn in Figures 8-9. Based on the results, comparison between SHSS, GSOR and IGSOR methods shows that the GSOR and IGSOR methods are more effective and applicable methods.

5. Conclusion. In this study, a generalization of the successive overrelaxation (G-SOR) iteration method is applied to solve the image restoration problem. To this
Table 7. Relative error of various methods in Example 2.

| Method    | Zero  | Periodic | Reflexive | Antireflective |
|-----------|-------|----------|-----------|----------------|
| SHSS      | 0.2771| 0.2704   | 0.2384    | 0.2502         |
| GSOR      | 0.2764| 0.2690   | 0.2224    | 0.2296         |
| IGSOR     | 0.2740| 0.2671   | 0.2222    | 0.2292         |

Table 8. CPU times of various methods in Example 2.

| Method    | Zero  | Periodic | Reflexive | Antireflective |
|-----------|-------|----------|-----------|----------------|
| SHSS      | 4.98  | 5.08     | 5.21      | 5.40           |
| GSOR      | 0.49  | 0.52     | 0.51      | 0.52           |
| IGSOR     | 0.49  | 0.51     | 0.53      | 0.50           |

Figure 5. Restored images with GSOR method for various BCs in Example 2.

Table 9. Values of $(\alpha, \omega)$ in Example 3

| Method    | Zero       | Periodic  | Reflexive | Antireflective |
|-----------|------------|-----------|-----------|----------------|
| SHSS      | (0.3330, -)| (0.3333, -)| (0.3333, -)| (0.5894, -)    |
| GSOR      | (-, 0.26)  | (-, 0.25) | (-, 0.25) | (-, 0.29)      |
| IGSOR     | (0.003, 0.19) | (0.008, 0.13) | (0.006, 0.13) | (0.05, 0.09)   |
Figure 6. Restored images with IGSOR method for various BCs in Example 2.

Figure 7. True image, PSF and degraded image in Example 3.

Table 10. PSNR values of various methods in Example 3

| Method | Zero  | Periodic | Reflexive | Antireflective |
|--------|-------|----------|-----------|----------------|
Table 11. Relative error of various methods in Example 3

| Method   | Zero       | Periodic   | Reflexive  | Antireflective |
|----------|------------|------------|------------|----------------|
| SHSS     | 0.1287     | 0.0726     | 0.0857     | 0.0811         |
| GSOR     | 0.1281     | 0.0706     | 0.0825     | 0.0801         |
| IGSOR    | 0.1276     | 0.0664     | 0.0795     | 0.0704         |

Table 12. CPU times of various methods in Example 3

| Method   | Zero        | Periodic    | Reflexive   | Antireflective |
|----------|-------------|-------------|-------------|----------------|
| SHSS     | 24.62       | 22.81       | 22.95       | 24.65          |
| GSOR     | 2.27        | 2.30        | 2.41        | 2.35           |
| IGSOR    | 2.26        | 2.28        | 2.29        | 2.33           |

Figure 8. Restored images with GSOR method for various BCs in Example 3.

end, the proposed problem is equivalently considered as a non-Hermitian positive definite system. The convergence of the GSOR is investigated and the optimal value of unknown parameter is given. Furthermore, an improved version of the GSOR (IGSOR) method is also presented. It is shown that the IGSOR method is convergent under some conditions. Three numerical examples are given to illustrate the effectiveness of the new methods. As the numerical results show, the GSOR and IGSOR methods are accurate and applicable methods to restore images.
Figure 9. Restored images with IGSOR method for various BCs in Example 3.

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