Estimation of the instantaneous volatility

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April 5, 2010

Abstract

This paper is concerned with the estimation of the volatility process in a stochastic volatility model of the following form: \( dX_t = a_t dt + \sigma_t dW_t \), where \( X \) denotes the log-price and \( \sigma \) is a càdlàg semi-martingale. In the spirit of a series of recent works on the estimation of the cumulated volatility, we here focus on the instantaneous volatility for which we study estimators built as finite differences of the power variations of the log-price. We provide central limit theorems with an optimal rate depending on the local behavior of \( \sigma \). In particular, these theorems yield some confidence intervals for \( \sigma_t \).

Keywords: Central limit theorem, Power variation, semimartingale.

AMS Classification (2000): Primary 60F05; Secondary 91B70, 91B82.

1 Introduction

The financial market objects offering a great complexity of modelling, the development and the study of financial models has attracted a lot of attention in recent years. For such models, a key parameter is the volatility, which is of paramount importance. The fact that the volatility is not constant has been observed for a long time. Thus, since the famous but too stringent Black and Scholes model, many stochastic volatility models have been introduced. Among them, models where jumps occur are now widely spread in the literature (see e.g. [9] for a review and [10] for a list of recent studies on this topic), mainly because there are able to fit skews and smiles that can not be captured by continuous models.

In this paper, we deal with the following kind of model:

\[
X_t = x + \int_0^t a_s ds + \int_0^t \sigma_s dW_s \quad \forall t \geq 0,
\]

where \( W \) is a Brownian motion and \( \sigma \) is a càdlàg semi-martingale (assumptions will be made precise in the next section). At this stage, one can remark the main restriction of our model: jumps only occur in the volatility but not in the price. This restriction will be explained in the sequel.

For such a model, a (now) classical tool for the estimation of the volatility is to make use of the

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power variations of order $p$ (see next section for details) that have some convergence properties to the cumulated volatility process: \( \int_0^t |\sigma_s|^p ds \) (when $p = 2$, this type of result is only the convergence of the quadratic variations to the angle bracket of the continuous semi-martingale $X$). The study of such estimators of the integrated volatility and their use for the detection of jumps have been deeply studied in the last years (see for instance [1] [2] [13] [19] [21] [23] [9] [15] for the discontinuous setting and the more recent papers [16] [22] [20]).

Unlike these works, the aim of this paper is to estimate rather the instantaneous volatility. Then, the natural idea is to study estimators which are built as “derivatives” of the power variations. More precisely, the proposed estimator of the instantaneous volatility is a normalized relative increment of cumulative volatility estimator, this relative increment being taken on a smaller and smaller interval. We provide some central limit theorems for the $\sigma_t$-estimator and we exhibit an optimal rate depending on the local behavior of $\sigma$. More precisely, if a Brownian component exists in $\sigma$, the best rate is of order $n^{1/4}$ and otherwise, it depends on the intensity of jumps. In particular, when the jump component has finite-variation, the optimal rate is of order $n^{1/3}$. These central limits lead in particular to some confidence intervals for $\sigma_t$ and to an asymptotic control of the relative error between the estimator and $\sigma_t$.

When jumps occur in the log-price $X$, it seems that we could extend some of the previous announced results by exploiting the fact that convergence properties for the power variations to the cumulated volatility still hold when $p < 2$. However, this extension generates some technicalities which are out of ours objectives.

The paper is organized as follows. In Section 2 we introduce the model we deal with, we present the different assumptions for this study and we state our main theorems: Central Limit Theorems for the instantaneous volatility. Section 3 is the proof of these theorems. From Theorem 3 we easily deduce a confidence interval for the instantaneous volatility, this is shown in Section 4. Moreover, we stress the fact that the confidence interval length increases with $p$. Finally, the volatility estimator is tested on some simulations in Section 5.

2 Model

We consider a stochastic process $(X_t)_{t \geq 0}$ defined on a filtered probability space by:

$$dX_t = a_t \, dt + \sigma_t \, dW_t, \quad t \geq 0,$$

where $W$ is an $(\mathcal{F}_t)$-adapted Wiener process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, $a : \mathbb{R}_+ \to \mathbb{R}$ and $\sigma$ are some càdlàg $(\mathcal{F}_t)$-adapted processes. We also assume that $\sigma$ is non-negative.

In this paper, we want to estimate $(\sigma_t)_{t \geq 0}$ using the asymptotic properties of the observed discrete increments of $X$: let $T$ be a positive number and assume that $X$ is observed at times $i \Delta_n$ for all $i = 0, 1, \ldots, [\frac{T}{\Delta_n}]$. In the sequel, we will assume that $\Delta_n \xrightarrow{n \to +\infty} 0$.

Then, for $p > 0$, we denote by $\hat{B}(p, \Delta_n)$, the process of power variations of order $p$, i.e. the stochastic process defined by

$$\hat{B}(p, \Delta_n)_t := \sum_{i=1}^{[t/\Delta_n]} |\Delta^p_i X|^p, \quad t \in [0, T]$$

where $\Delta^p_i X := X_i \Delta_n - X_{(i-1)\Delta_n}$.

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Before stating our main result, we introduce the following assumptions depending on parameter $q \in [1,2]$ which is related to the behavior of the small jumps of $(\sigma_t)$:

\((H_1^q)\) : $\sigma$ is a positive càdlàg semimartingale such that $\sigma_t = |Y_t|$ where $(Y_t)$ satisfies:

$$
dY_t = b_s ds + \eta_1(s)dW_s + \eta_2(s)dW_s^2 + \int_{\mathbb{R}} y 1_{|y|\leq 1}(\mu(ds, dy) - \nu(ds, dy)) + \int_{\mathbb{R}} y 1_{|y|> 1}\mu(ds, dy),$$

where $b, \eta_1, \eta_2$ are adapted càdlàg processes, $\mu$ denotes a random measure on $\mathbb{R}_+ \times \mathbb{R}$ with predictable compensator $\nu$ satisfying: $\nu(dt, dy) = dtF_t(dy)$ and $(\int (1 \wedge |y|^q)F_t(dy))_{t \geq 0}$ is a locally bounded predictable process.

The above assumption on the predictable compensator imply that $(\sigma_t)$ is quasi-left continuous and that the jump component has locally-finite $q$-variation. In order to obtain our main results, we actually need a little more constraining control of the jump component:

\((H_2^q)\) : For every $T > 0$,

$$
\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \int_{\{ |y| \leq \varepsilon \}} |y|^qF_t(dy) = 0 \quad a.s.
$$

As an example, we get $(Y_t)$ a solution to the following SDE:

$$
dY_t = b(Y_{t^-})dt + \zeta(Y_{t^-})d\tilde{W}_t + \kappa(Y_{t^-})dZ_t, \quad (2)
$$

where $b : \mathbb{R} \to \mathbb{R}$, $\zeta : \mathbb{R} \to \mathbb{R}$ and $\kappa : \mathbb{R} \to \mathbb{R}$ are some continuous functions with sublinear growth, $(W_t)_{t \geq 0}$ is a Brownian motion and $(Z_t)_{t \geq 0}$ is a centered purely discontinuous Lévy process independent of $(W_t)_{t \geq 0}$ with Lévy measure $\pi$ satisfying $\int (1 \wedge |y|^q\wedge 1)\pi(dy) < \infty$, $q \in [1,2]$, then Hypotheses $(H_1^q)$ and $(H_2^q)$ hold.

Then, we recall two results about the asymptotic properties of the observed discrete increments of $X$, from Lépingle [17] and Aït-Sahalia and Jacod [3, Theorem 2] respectively. On the same topic, we can also quote [8-11].

**Proposition 1.** Let $p$ be a positive number and set $m_p := \mathbb{E}[|U|^p]$ where $U \sim \mathcal{N}(0,1)$. Then, locally uniformly in $t$,

$$
\Delta_n^{1-\frac{q}{2}} \hat{B}(p, \Delta_n)_t \xrightarrow{p}{n \to +\infty} m_p A(p)_t \quad \text{with} \quad A(p)_t = \int_0^t \sigma_s^p ds.
$$

**Proposition 2.** Let $p \geq 2$ and assume Assumption 1 of [3]. Then, the sequence of continuous processes $(Y(n,p))_{n \in \mathbb{N}}$ defined for any $n \in \mathbb{N}$ by

$$
Y(n,p)_t := \frac{1}{\sqrt{\Delta_n}} \left( \Delta_n^{1-\frac{q}{2}} \hat{B}(p, \Delta_n)_t - m_p A(p)_t \right), \quad t \geq 0,
$$

converges stably to a random variable $Y(p)$ on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ of the original filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t, \mathbb{P})$ such that, for any $t \geq 0$, conditionally on $\mathcal{F}$, $Y(p)_t$ is a centered Gaussian variable with variance $\mathbb{E}[Y(p)_t^2 | \mathcal{F}] = (m_{2p} - m_p^2)A(2p)_t$. 

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Looking at these results, it is natural to try to estimate $\sigma_t^p$ by the following statistic: \( (\Sigma(p, \Delta_n, h_n)_t) \) defined for every \( t \leq \bar{T} \) with \( \bar{T} = T - h_1 \) by:
\[
\Sigma(p, \Delta_n, h_n)_t := \frac{\Delta_n^{-\frac{2}{p}}(\hat{B}(p, \Delta_n)_t + h_n - \hat{B}(p, \Delta_n)_t)}{m_p h_n}.
\]
(3)

Actually, this estimator is the mean of \( p \)-variations in a window of length \( h_n \) where \( (h_n) \) is a non-increasing sequence of positive numbers such that \( h_n \) tends to 0.

Throughout this paper, we will denote by \( \mathcal{L} - s \) the stable convergence. We recall that a sequence of random variables \( (Y_n) \) converges stably to \( Y \) or \( Y_n \overset{\mathcal{L} - s}{\to} Y \) if there exists an extension \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\) of \((\Omega, \mathcal{F}, \mathbb{P})\) and a random variable \( \hat{Y} \) defined on \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\) such that for every bounded measurable random variable \( H \), for every bounded continuous function \( f \), \( \mathbb{E}[Hf(Y_n)] \to \mathbb{E}[Hf(Y)] \) when \( n \to +\infty \) where \( \mathbb{E} \) denotes the expectation on the extension. We are now able to state our main results.

**Theorem 3.** Let \( p = 2 \) or \( p \geq 3 \) and let \((X_t)\) be a stochastic process solution to (1). Assume \((H^2_2)\) and \((H^2_q)\). Assume that \( \Delta_n \equiv o(h_n) \). Then,

(i) If \( h_n/\sqrt{\Delta_n} \to 0 \), \( \forall t \in [0, \bar{T}] \),
\[
\frac{1}{h_n}((\Sigma(p, \Delta_n, h_n)_t - \sigma_t^p)) \xrightarrow{\mathcal{L} - s} \sqrt{\varphi_1(p, t, \sigma)} U,
\]
where, conditionally on \( \mathcal{F} \), \( U \) is a standard Gaussian random variable independent of \( \mathcal{F}_t \) and \( \varphi_1(p, t, \sigma) = \frac{m_2 - m_1^2}{m_p^2} \sigma_t^{2p} \).

(ii) If \( \sqrt{\Delta_n}/h_n \to \beta \in \mathbb{R}_+ \), \( \forall t \in [0, \bar{T}] \),
\[
\frac{1}{h_n}((\Sigma(p, \Delta_n, h_n)_t - \sigma_t^p)) \xrightarrow{\mathcal{L} - s} \sqrt{\beta^2 \varphi_1(p, t, \sigma) + \varphi_2(p, t)} U,
\]
where \( \varphi_1(p, t, \sigma) \) and \( U \) are defined as before and,
\[
\varphi_2(p, t) = \frac{m_2}{m_1^2}(\sigma_t)_{2p-2}(||\eta||^2(t)) \text{ with } ||\eta||^2(t) = \eta_1^2(t) + \eta_2^2(t).
\]

Note that when the drift term \( a \) is null, the result is valid even if \( 2 < p < 3 \). Otherwise, the drift contributes in a bias for the estimator that is not negligible in case \( 2 < p < 3 \).

Now, we state a second result when there is no Brownian component in the volatility, i.e. when \( \eta_1 = \eta_2 = 0 \).

**Theorem 4.** Let \( p = 2 \) or \( p \geq 3 \) and let \((X_t)\) be a stochastic process solution to (1). Assume \((H^2_q)\) and \((H^2_q)\) with \( q \in [1, 2] \) and suppose that \( \eta_1 = \eta_2 = 0 \). Assume that \( \Delta_n \equiv o(h_n) \). Then,

(i) If \( q \in (1, 2] \), if \( \limsup_{n \to +\infty} h_n^{1/2+1/q}/\sqrt{\Delta_n} < +\infty \), \( \forall t \in [0, \bar{T}] \),
\[
\frac{h_n}{\Delta_n}((\Sigma(p, \Delta_n, h_n)_t - \sigma_t^p)) \xrightarrow{\mathcal{L} - s} \sqrt{\varphi_1(p, t, \sigma)} U,
\]
where \( \varphi_1(p, t, \sigma) \) and \( U \) are defined as in Theorem 3.

(ii) Assume that \( q = 1 \). If \( \lim_{n \to +\infty} h_n^{1/2}/\Delta_n = 0 \), [3] holds.

If \( \lim_{n \to +\infty} h_n^{1/3}/\Delta_n = \beta \in \mathbb{R}_+^* \) and if \( \theta_t^p \) defined, for all \( t \in [0, T] \), by
\[
\theta_t^p := p^p \sigma_t^{p-1} b_t + \frac{p(p-1)}{2} \sigma_t^{p-2} ||\eta||^2(t) - p \sigma_t^{p-1} \int_{\{0 < |\eta| \leq 1\}} y F_t(dy),
\]

(7)
is càd, then, \( \forall t \in [0, T] \),

\[
\sqrt{\frac{h_n}{\Delta_n}}(\Sigma(p, \Delta_n, h_n)_t - \sigma^p_t) \xrightarrow{\mathcal{L}_{n \to +\infty}} \sqrt{\varphi_1(p, t, \sigma)} U + \frac{\beta}{2} \theta^0.
\] (8)

Following Tauchen and Todorov [21] and according to concrete data, pure jump volatility process could be a more convenient model. In such a case, it seems that the right theorem to be applied is Theorem 4. In cases (i) in both Theorems, we get for all \( t > 0 \),

\[
\sqrt{\frac{h_n}{\Delta_n}} \left( \frac{\Sigma(p, \Delta_n, h_n)_t}{\sigma^p_t} - 1 \right) \xrightarrow{\mathcal{L}_{n \to +\infty}} \sqrt{m_2 p - m_p^2} U, \tag{9}
\]

where \( U \sim \mathcal{N}(0,1) \) and \( U \) is independent of \( \mathcal{F}_t \). This result is enough to obtain an estimation of \( \sigma^p_t \) and to obtain a confidence interval for it, together the convergence rate.

**Remark 5.** It must be stressed here that the convergence rate depends on the balance between the frequency of observations and the length \( h_n \) of the window. Hence, the following considerations justify the choice of a “good pair” \( (h_n, \Delta_n) \): let \( p = 2 \) or \( p \geq 3 \) and assume \( \Delta_n = o(h_n) \). Considering the window width \( h_n, r_n := \frac{h_n}{\Delta_n} \) corresponds to the number of observations on the interval \([t, t+h_n]\). Suppose \( \Delta_n = \frac{1}{n^q} \) and \( r_n := n^\rho, \quad 0 < \rho < 1 \), then \( h_n = n^{\rho-1} \). In this scheme, assuming \((H^1_2), (H^2_2)\), Theorem 5 yields the following convergence rates:

(i) \( \rho < \frac{1}{2} \) yields a convergence rate of order \( n^{\rho/2} \),

(ii) \( \rho \geq \frac{1}{2} \) yields a convergence rate of order \( n^{(1-\rho)/2} \).

In case \( \eta_1 = \eta_2 = 0 \), under Hypotheses \((H^1_q)\) and \((H^2_q)\) with \( 1 \leq q \leq 2 \), Theorem 5 yields the following convergence rates:

(i) if \( 1 < q \leq 2, \rho \leq \frac{2}{2+q} \), yields a convergence rate of order \( n^{\rho/2} \),

(ii) the same convergence rate occurs in case \( q = 1, \rho \leq \frac{2}{3} \); the best convergence rate is of order \( n^{1/3} \), obtained for \( \rho = 2/3 \). As an example in such a case, let us choose \( r_n = n^{2/3} \sim 300 \). It means 300 data which can be the daily observations and globally \( n = 300^{3/2} \sim 5200 \). This may correspond to a realistic data set.

### 3 Proofs

In every proofs \( C \) or \( C_p \) are constants which can change from a line to another. In order to make the notations easier to handle, we will denote by:

\[
\mathcal{D}^n_t = \left\{ i \in \mathbb{N}, \left[ \frac{t}{\Delta_n} \right] + 1 \leq i \leq \left[ \frac{t + h_n}{\Delta_n} \right] \right\}, \quad \mathcal{C}^n_{t,i} = \left\{ u \in \mathbb{R}, \ (i-1)\Delta_n \leq u \leq i\Delta_n \right\}, \quad \mathcal{C}^n_{t,s} = \left\{ u \in \mathbb{R}, \ (i-1)\Delta_n \vee t \leq u \leq s \right\}.
\]
3.1 Decomposition of the error

Following [13], we first decompose \( \Sigma(p, \Delta_n, h_n)_t - \sigma_t^p \) as follows:

\[
\Sigma(p, \Delta_n, h_n)_t - \sigma_t^p = \frac{Z_{t+h_n}^{(n,p)} - Z_t^{(n,p)}}{m_p h_n} + \left( \frac{1}{r_n} \sum_{i \in \mathcal{D}_t} \sigma_t^p \Delta_i - \sigma_t^p \right),
\]

(10)

where \( r_n = h_n/\Delta_n \) and

\[
Z_t^{(n,p)} := \Delta_n^{1-p/2} \hat{B}(p, \Delta_n)_t - m_p \sum_{i=1}^{[t/\Delta_n]} \Delta_i \sigma_t^p \Delta_i.
\]

On the one hand, denoting by \( \mathbb{E}_{i-1}^n[\ast] \) the conditional expectation with respect to \( \mathcal{F}_{(i-1)\Delta_n} \), one can notice that \( \mathbb{E}_{i-1}^n \left[ |\sigma_{(i-1)\Delta_n} \frac{\Delta_n W}{\sqrt{\Delta_n}}|^p \right] = \sigma_{(i-1)\Delta_n}^p m_p \), and it is easy to check that

\[
\frac{Z_{t+h_n}^{(n,p)} - Z_t^{(n,p)}}{h_n} = \Lambda_1^p(t) + \Lambda_2^p(t) + \Lambda_3^p(t),
\]

with

\[
\Lambda_1^p(t) := \frac{\Delta_n}{h_n} \sum_{i \in \mathcal{D}_t^p} \left( \left| \frac{\Delta_n X}{\sqrt{\Delta_n}} \right|^p - |\sigma_{(i-1)\Delta_n} \frac{\Delta_n W}{\sqrt{\Delta_n}}|^p \right) - \mathbb{E}_{i-1}^n \left[ \left| \frac{\Delta_n X}{\sqrt{\Delta_n}} \right|^p - |\sigma_{(i-1)\Delta_n} \frac{\Delta_n W}{\sqrt{\Delta_n}}|^p \right],
\]

\[
\Lambda_2^p(t) := \frac{\Delta_n}{h_n} \sum_{i \in \mathcal{D}_t^p} \sigma_t^p \left( \left| \frac{\Delta_n W}{\sqrt{\Delta_n}} \right|^p - m_p \right),
\]

\[
\Lambda_3^p(t) := \frac{\Delta_n}{h_n} \sum_{i \in \mathcal{D}_t^p} \mathbb{E}_{i-1}^n \left[ \left| \frac{\Delta_n X}{\sqrt{\Delta_n}} \right|^p - m_p \sigma_{(i-1)\Delta_n}^p - \sigma_t^p \Delta_i \right] + \frac{\Delta_n}{h_n} m_p \left( \sigma_t^p - \sigma_{(i+h_n)/\Delta_n}^p - \sigma_{[t/\Delta_n]}^p \right).
\]

On the other hand, let us now decompose the second part of (10). Itô’s formula applied to \( x \to |x|^p \) with \( p \geq 1 \) yields for every \( i \geq [t/\Delta_n] + 1 \):

\[
|Y_{i\Delta_n}|^p = |Y_t|^p + A_t \Delta_n - A_t + M_{i\Delta_n} - M_t,
\]

with,

\[
M_t = \int_0^t p \, \text{sgn}(Y_s)|Y_s|^{p-1} \eta(s) dW_s + \int_0^t p \, \text{sgn}(Y_s)|Y_s|^{p-1} \eta_2(s) dW_s^2,
\]

\[
A_t = \int_0^t \theta_s ds + \int_0^t \int_{|y| \leq 1} p \, \text{sgn}(Y_s)|Y_s-\eta|^p (\mu - \nu)(ds, dy)
\]

\[
+ \sum_{0 < s \leq t} \left( |Y_{s-} + \Delta Y_s|^p - |Y_{s-}|^p - p \, \text{sgn}(Y_s)|Y_{s-}|^{p-1} \Delta Y_s 1_{|\Delta Y_s| \leq 1} \right),
\]

and \( \theta_s = p \, \text{sgn}(Y_s)|Y_s|^{p-1} b_s + \frac{p(p-1)}{2} |Y_s|^{p-2} ||\eta||^2(s) \). Then, it follows that

\[
\frac{1}{r_n} \sum_{i \in \mathcal{D}_t^p} \sigma_t^p \Delta_i - \sigma_t^p = \Lambda_4^p(t) + \Lambda_5^p(t),
\]

with,

\[
\Lambda_4^p(t) = \frac{1}{r_n} \sum_{i \in \mathcal{D}_t^p} \left( [(t+h_n)/\Delta_n] - i + 1) (M_{i\Delta_n} - M_{(i-1)\Delta_n}) \right),
\]

\[
\Lambda_5^p(t) = \frac{1}{r_n} \sum_{i \in \mathcal{D}_t^p} \left( [(t+h_n)/\Delta_n] - i + 1) (A_{i\Delta_n} - A_{(i-1)\Delta_n}) \right).
\]
3.2 Preliminary Lemmas

In this section, we establish a series of useful lemmas for the sequel of the proof. First, we show in Lemma 6 that it is enough to prove the main results under \((\mathbf{H}_q^2)\) and the following assumption:

\[(\mathbf{SH}_q)\ a, b, \eta_1, \eta_2, \text{ and } \int_0^1 \int (|y|^{q} \wedge 1) F_s(dy)ds \text{ are bounded and there exists } M > 0 \text{ such that } F_s([-M, M]^c) = 0 \text{ a.s. } \forall s \geq 0.\]

**Lemma 6.** Assume that the conclusions of Theorem 3 and 4 hold for every \((X, \sigma)\) satisfying \((\mathbf{H}_q^2)\) and \((\mathbf{SH}_q)\) (with \(q \in [1, 2]\) depending on the statement). Then, the conclusions hold for every \((X, \sigma)\) satisfying \((\mathbf{H}_q^2)\) and \((\mathbf{SH}_q)\) with \(q \in [1, 2]\).

The proof of this lemma is based on a classical localization procedure and is done in the Appendix (see Section 6).

As a consequence of the preceding lemma, we now work under \((\mathbf{SH}_q)\). In the following preliminary result, we state a series of useful properties on \(\sigma\) under this assumption.

**Lemma 7.** Assume \((\mathbf{SH}_q)\).

(i) For every \(T > 0\) and every \(r > 0\),

\[\mathbb{E} \left[ \sup_{0 \leq t \leq T} (\sigma_t)^r \right] < \infty.\]  \(\quad (11)\)

(ii) For every \(0 \leq s \leq t \leq T\) such that \(|t - s| \leq 1\), it exists a deterministic constant \(C_T > 0\) such that:

\[\mathbb{E} \left[ \left| \sigma_t - \sigma_s \right|^r \mid \mathcal{F}_s \right] \leq C_T |t - s|^1 r^q, \quad \forall r > 0.\]  \(\quad (12)\)

\[\mathbb{E} \left[ \left| \int_s^t \sigma_u dW_u \right|^q \right] \leq C_T |t - s|^q, \quad \forall q > 0.\]  \(\quad (13)\)

\[\mathbb{E} \left[ \left| \int_s^t (\sigma_u - \sigma_s) dW_u \right|^q \right] \leq C_T |t - s|^q (\frac{q}{2} + 1), \quad \forall q > 0.\]  \(\quad (14)\)

The proof of this lemma is based on standard tools and is also done in the appendix (see Section 6).

Finally, the last preliminary result is an extension of a result by [11] on the stable-CLT for martingale increments.

**Lemma 8.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) denote a probability space. Let \(\{\xi^n, n, i \in \{0, \ldots, k_n\}, n \geq 1\}\) denote a triangular array of real martingale increments (for every \(n \geq 1, \mathcal{F}_{n,1} \subset \cdots \subset \mathcal{F}_{n,k_n}\)). Denote by \(S_n = \sum_{i=1}^{k_n} \xi^n_i\). Assume the following conditions:

(i) \(n \to \mathcal{F}_{n,k_n}\) is non-increasing and

\[\sum_{i=1}^{k_n} \mathbb{E}[\xi^n_i^2/\mathcal{F}_{n,i-1}] \overset{n \to +\infty}{\to} \eta \quad \text{a.s.}\]

where \(\eta\) is a \(\mathcal{G}\)-measurable random variable with \(\mathcal{G} = \cap_{n \geq 1} \mathcal{F}_{n,k_n}\).
(ii) For every $\varepsilon > 0$,

$$\sum_{i=1}^{k_n} \mathbb{E}[(\xi_i^n)^21_{\xi_i^n^2 \geq \varepsilon / F_{n,i-1}}] \xrightarrow{p} n \to +\infty 0.$$ 

Then, $(S_n)$ converges stably to $S$ where $S$ is defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and such that conditionally on $\mathcal{G}$, the distribution of $S$ is a centered Gaussian law with variance $\eta$.

Proof. Owing to Corollary 2 of [11], there exists $S$ defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that for every $\mathcal{G}$-measurable random variable $Z$, for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[Zf(S_n)] \xrightarrow{n \to +\infty} \tilde{\mathbb{E}}[Zf(S)].$$

(15)

(where $\tilde{\mathbb{E}}$ denotes the expectation on the extended probability space), and such that conditionally on $\mathcal{G}$, the distribution of $S$ is a centered Gaussian distribution with variance $\eta$. Now, let $Y$ denote a $\mathcal{F}$-random variable. Since $Z = \mathbb{E}[Y/\mathcal{G}]$ satisfies (15), it is enough to show that: $\mathbb{E}[(Y - \mathbb{E}[Y/\mathcal{G}])f(S_n)] \to 0$ as $n \to +\infty$. Since $S_n$ is $\mathcal{F}_{n,k_n}$-measurable,

$$|\mathbb{E}[(Y - \mathbb{E}[Y/\mathcal{G}])f(S_n)]| = |\mathbb{E}[(\mathbb{E}[Y/\mathcal{F}_{n,k_n}] - \mathbb{E}[Y/\mathcal{G}])f(S_n)]| \leq C\mathbb{E}[|\mathbb{E}[Y/\mathcal{F}_{n,k_n}] - \mathbb{E}[Y/\mathcal{G}]|]$$

and this last term tends to 0 owing to the definition of $\mathcal{G}$ and the non-increasing assumption on $(\mathcal{F}_{n,k_n})_{n \geq 1}$.

3.3 The CLTs for the Brownian martingale terms

In this section, we focus on the main terms of the decomposition which satisfy a central limit theorem.

Proposition 9. Assume that $\Delta_n = o(h_n)$ and (SH)$_2$.

(i). Then,

$$\rho_n (\Lambda_2^n(t) + \Lambda_1^n(t)) \xrightarrow{L-s, n \to +\infty} f(t,p)U,$$

where $U \sim \mathcal{N}(0,1)$, $U$ is independent of $\mathcal{F}_t$ and

$$f^2(t,p), \rho_n = \begin{cases} (\varphi_1(p,t,\sigma), \sqrt{r_n}) & \text{if } h_n = o(\sqrt{\Delta_n}), \\ (\beta^2 \varphi_1(p,t,\sigma) + \varphi_2(p,t), \frac{1}{\sqrt{h_n}}) & \text{if } \frac{\sqrt{\Delta_n}}{h_n} \to \beta \in \mathbb{R}^+_*, \\ \left(\frac{1}{2}r^2(\sigma_t)2^{-2}\|\eta\|^2(t), \frac{1}{\sqrt{h_n}}\right) & \text{if } \frac{\sqrt{\Delta_n}}{h_n} \to 0. \end{cases}$$

(17)

(ii). In case of pure jump process, meaning we assume that $\eta_1 = \eta_2 = 0$, then, $\Lambda_4 = 0$ and, for every $t \in [0,T]$,

$$\sqrt{\frac{h_n}{\Delta_n}} \Lambda_2^n(t) \xrightarrow{n \to +\infty} f(t,p)U,$$

with $f^2(t,p) = \varphi_1(p,t,\sigma)$.
Proof. Actually, in case (ii), the proof is easier since it only deals with \( \Lambda^p \), and is more or less included in what follows.

Let \( t > 0 \). Let \( \{ (\xi^n_i) : i = [t/\Delta_n], \ldots, [(t + h_n)/\Delta_n], n \geq 1 \} \) be the sequence of triangular arrays of square-integrable martingale increments (with respect to the filtration \( (\mathcal{F}_{(i-1)\Delta_n})_{i \geq [t/\Delta_n]+1} \)) defined by: \( \xi_n = \xi^{n,1}_i + \xi^{n,2}_i \) with

\[
\xi^{n,1}_i := \frac{\rho_n}{r_n} \left( \sigma_{(i-1)\Delta_n} \frac{\Delta^n W}{\sqrt{\Delta_n}} - E_{i-1} \left[ \sigma_{(i-1)\Delta_n} \frac{\Delta^n W}{\sqrt{\Delta_n}} \right] \right),
\]

\[
\xi^{n,2}_i := \frac{\rho_n}{r_n} \left( (\Delta^{i-1}_n)_{t > s} (\sigma_{(i-1)\Delta_n})_t - 1 \right) (M_{(i-1)\Delta_n} - M_{(i-1)\Delta_n \wedge t}).
\]

We first notice that \( \sum_{i \in \mathcal{D}^n} (\xi^{n,1}_i + \xi^{n,2}_i) = \rho_n (\Lambda^p(t) + \Lambda^p_n(t)) \). The following lemma gives the asymptotic predictable bracket of this sum of martingale increments.

**Lemma 10.** Let \( f(t, p) \) defined by \( f(t, p) \), then

\[
\sum_{i \in \mathcal{D}^n_t} E_{i-1} \left[ (\xi^{n,1}_i + \xi^{n,2}_i)^2 \right] \xrightarrow{n \to +\infty} f^2(t, p).
\]

**Proof.** Three sums have to be computed:

\[
\sum_{i \in \mathcal{D}^n_t} E_{i-1} \left[ (\xi^{n,1}_i)^2 \right], \quad \sum_{i \in \mathcal{D}^n_t} E_{i-1} \left[ (\xi^{n,2}_i)^2 \right], \quad \sum_{i \in \mathcal{D}^n_t} E_{i-1} \left[ \xi^{n,1}_i \xi^{n,2}_i \right].
\]

(i) First

\[
E_{i-1} \left[ (\xi^{n,1}_i)^2 \right] = \left( \frac{\rho_n}{r_n} \right)^2 (m_{2p} - m_p^2) (\sigma_{(i-1)\Delta_n})_t^{2p},
\]

and since \( \sigma \) is càd,

\[
\frac{1}{r_n} \sum_{i \in \mathcal{D}^n_t} \sigma_{(i-1)\Delta_n} \xrightarrow{n \to +\infty} \sigma_t \quad \text{a.s.}
\]

Thus, by the definition of \( \rho_n \),

\[
\sum_{i \in \mathcal{D}^n_t} E_{i-1} \left[ (\xi^{n,1}_i)^2 \right] \xrightarrow{n \to +\infty} \begin{cases} 
(m_{2p} - m_p^2) \sigma_t^{2p} & \text{if } h_n = o(\sqrt{\Delta_n}), \\
\beta^2 (m_{2p} - m_p^2) \sigma_t^{2p} & \text{if } \sqrt{\Delta_n}/h_n \to \beta \in \mathbb{R}^*_+, \\
0 & \text{if } \sqrt{\Delta_n}/h_n \to 0.
\end{cases}
\]

(ii) Second,

\[
E_{i-1} \left[ (\xi^{n,2}_i)^2 \right] = \left( \frac{\rho_n}{r_n} (\Delta^{i-1}_n)_{t > s} (\sigma_{(i-1)\Delta_n})_t - 1 \right)^2 \int_{c_{i-1}^n} E_{i-1} [\psi_s] \, ds,
\]

with \( \psi_s = p^2 |Y_s|^{2p-2} [\eta_1^2(s) + \eta_2^2(s)] \). One observes that

\[
E \left[ \sum_{i \in \mathcal{D}^n_t} \left( \frac{\rho_n (\Delta^{i-1}_n)_{t > s} (\sigma_{(i-1)\Delta_n})_t - 1)}{r_n} \right)^2 \int_{c_{i-1}^n} (E_{i-1} [\psi_s] - \psi_t) \, ds \right] \leq \Delta_n \sum_{i \in \mathcal{D}^n_t} \left( \frac{\rho_n (\Delta^{i-1}_n)_{t > s} (\sigma_{(i-1)\Delta_n})_t - 1)}{r_n} \right)^2 E \left[ \sup_{s \in [t, t + h_n]} |\psi_s - \psi_t| \right] \leq C p^2 h_n E \left[ \sup_{s \in [t, t + h_n]} |\psi_s - \psi_t| \right]
\]

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The function $\psi$ is càd. Therefore, using (11) and the fact that $\eta_1$ and $\eta_2$ are bounded, we deduce from the dominated convergence theorem that for every $t \in [0, T]$,

$$
\mathbb{E} \left[ \sup_{s \in [t, t+h_n]} |\psi_s - \psi_t| \right] \xrightarrow{n \to +\infty} 0.
$$

(23)

It follows from the definition of $\rho_n$ that

$$
\sum_{i \in D_i^n} \mathbb{E}_{i}^{n-1} \left[ (\xi_{i,t}^{n,2})^2 \right] - \left( \rho_n \left( \frac{(t+h_n)/\Delta_n - i + 1}{r_n} \right)^2 \Delta_n \psi_t \right) \xrightarrow{P} 0.
$$

Thus, since

$$
\frac{1}{h_n} \sum_{i \in D_i^n} \left( \frac{(t+h_n)/\Delta_n - i + 1}{r_n} \right)^2 \Delta_n \psi_t \xrightarrow{n \to +\infty} \frac{1}{3},
$$

we obtain that the order of $\sum_{i \in D_i^n} \mathbb{E}_{i}^{n-1} \left[ (\xi_{i,t}^{n,2})^2 \right]$ is $\frac{1}{3} \rho_n^2 h_n \psi_t$ thus

$$
\sum_{i \in D_i^n} \mathbb{E}_{i}^{n-1} \left[ (\xi_{i,t}^{n,2})^2 \right] \xrightarrow{P} \begin{cases} 
0 & \text{if } h_n = o(\Delta_n) \\
\frac{\psi_t}{3} & \text{if } \Delta_n/h_n \to \beta \in \mathbb{R}_+.
\end{cases}
$$

(24)

(iii) Finally, we consider the cross products $\mathbb{E}_{i}^{n-1} \left[ (\xi_{i,t}^{n,1}\xi_{i,t}^{n,2}) \right]$. First of all, it is easily seen that, $W$ and $W^2$ being independent, only the term in $W$ of $M$ will play a role. Thus we have:

$$
\mathbb{E}_{i}^{n-1} \left[ \xi_{i,t}^{n,1}\xi_{i,t}^{n,2} \right] = \alpha_{i,n}(t) \sigma_{(i-1)\Delta_n}^{p} \mathbb{E}_{i}^{n-1} \left[ \int_{C_n,i} p\sigma_s^{p-1} \eta_1(s) dw_s \left( |\Delta_n^p W|^p - \mathbb{E}_{i-1}^{n-1} \left[ |\Delta_n^p W|^p \right] \right) \right]
$$

with $\alpha_{i,n}(t) = (\rho_n/r_n)^2 \Delta_n^{-p/2} \left( (t+h_n)/\Delta_n - i + 1 \right)$. Now, by Itô’s formula,

$$
|\Delta_n^p W|^p = p \int_{C_n,i} \sgn(W_s - W_{(i-1)\Delta_n}) |W_s - W_{(i-1)\Delta_n}|^{p-1} |dW_s + \frac{p(p-1)}{2} \int_{C_n,i} |W_s - W_{(i-1)\Delta_n}|^{p-2} ds.
$$

Then, we have $\mathbb{E}_{i}^{n-1} \left[ \xi_{i,t}^{n,1}\xi_{i,t}^{n,2} \right] = T_i^{n,1} + T_i^{n,2}$ with

$$
T_i^{n,1} = p^2 \alpha_{i,n}(t) \sigma_{(i-1)\Delta_n}^{p} \int_{C_n,i} \mathbb{E}_{i}^{n-1} \left[ \sigma_s^{p-1} \eta_1(s) \sgn(W_s - W_{(i-1)\Delta_n}) |W_s - W_{(i-1)\Delta_n}|^{p-1} \right] ds,
$$

$$
T_i^{n,2} = \frac{p^2(p-1)}{2} \alpha_{i,n}(t) \sigma_{(i-1)\Delta_n}^{p} \mathbb{E}_{i}^{n-1} \left[ \int_{C_n,i} |W_s - W_{(i-1)\Delta_n}|^{p-2} ds \int_{C_n,i} \sigma_s^{p-1} \eta_1(s) dw_s \right].
$$

First, let us focus on $T_i^{n,2}$. By an integration by parts, one obtains that:

$$
\int_{C_n,i} \mathbb{E}_{i}^{n-1} \left[ \int_{C_n,i} |W_s - W_{(i-1)\Delta_n} \vee t|^{p-2} ds \int_{C_n,i} \sigma_s^{p-1} \eta_1(s) dw_s \right]
$$

$$
= \int_{C_n,i} \mathbb{E}_{i}^{n-1} \left[ \left( \int_{C_n,i} \sigma_u^{p-1} \eta_1(u) dw_u \right) |W_s - W_{(i-1)\Delta_n} \vee t|^{p-2} \right] ds
$$

$$
= \int_{C_n,i} \mathbb{E}_{i}^{n-1} \left[ \int_{C_n,i} (\sigma_u^{p-1} \eta_1(u) - \sigma_t^{p-1} \eta_1(t)) dw_u |W_s - W_{(i-1)\Delta_n} \vee t|^{p-2} \right] ds,
$$

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where in the last line we used that for every \( s \in [(i-1)\Delta_n, i\Delta_n] \),
\[
E^n_{i-1} \left[ (W_s - W_{(i-1)\Delta_n}) | W_s - W_{(i-1)\Delta_n} |^{p-2} \right] = 0.
\]

Then, using Cauchy-Schwarz inequality
\[
E^n_{i-1} \left[ \int_{C_{t,i}} \left( \sigma^p u \eta_1(u) - \sigma^p \eta_1(t) \right) dW_u \right] |W_s - W_{(i-1)\Delta_n}|^{p-2}
\]
\[
\leq \sqrt{E^n_{i-1} \left[ \int_{C_{t,i}} \left( \sigma^p u \eta_1(u) - \sigma^p \eta_1(t) \right) dW_u \right]^2} \cdot \sqrt{E^n_{i-1} \left[ |W_s - W_{(i-1)\Delta_n}|^{2p-4} \right]},
\]
\[
\leq \int_{C_{t,i}} E^n_{i-1} \left[ \sup_{u \in [t, t+h_n]} \left| \sigma^p u \eta_1(u) - \sigma^p \eta_1(t) \right|^2 \right] du. \quad (s - (i-1)\Delta_n)^{(p-2)/2}.
\]

Then (23) yields:
\[
\mathbb{E} \left[ T_{i}^{n,2} \right] \leq C \alpha_{i,n}(t) \mathbb{E} \left[ \sup_{u \in [t, t+h_n]} \left| \sigma^p u \eta_1(u) - \sigma^p \eta_1(t) \right|^2 \right]^{\frac{1}{2}} \int_{C_{t,i}} \left( s - (i-1)\Delta_n \right)^{-\frac{1}{2}} ds,
\]
\[
\leq C \frac{\rho^2 \Delta_n}{r^2 n} ((t+h_n)/\Delta_n - i + 1) \mathbb{E} \left[ \sup_{u \in [t, t+h_n]} \left| \sigma^p u \eta_1(u) - \sigma^p \eta_1(t) \right|^2 \right]^{\frac{1}{2}}.
\]

Thus, an argument similar to (23) yields:
\[
\sum_{i \in D_t^n} T_{i}^{n,2} \xrightarrow{\mathbb{P}} n \rightarrow +\infty 0.
\] (25)

Second, we focus on \( T_{i}^{n,1} \). Using again that \( \sigma \) and \( \eta \) are càd, one obtains that
\[
\sum_{i \in D_t^n} \left[ T_{i}^{n,1} - p^2 \alpha_{i,n}(t) \sigma^p \eta_1(t) \int_{C_{t,i}} E^n_{i-1} \left[ \text{sgn}(W_s - W_{(i-1)\Delta_n}) |W_s - W_{(i-1)\Delta_n}|^{p-1} \right] ds \right] \xrightarrow{\mathbb{P}} n \rightarrow +\infty 0.
\]

Then, since \( E^n_{i-1} \left[ \text{sgn}(W_s - W_{(i-1)\Delta_n}) |W_s - W_{(i-1)\Delta_n}|^{p-1} \right] = 0 \) we deduce that
\[
\sum_{i \in D_t^n} T_{i}^{n,1} \xrightarrow{\mathbb{P}} n \rightarrow +\infty 0.
\]

then with (24) that
\[
\sum_{i \in D_t^n} E^n_{i-1} \left[ \xi_{i}^{n,1} \xi_{i}^{n,2} \right] \xrightarrow{\mathbb{P}} n \rightarrow +\infty 0.
\] (26)

Thus, by (22), (24) and (26), we obtain that,
\[
\sum_{i \in D_t^n} E^n_{i-1} \left[ (\xi_{i}^{n,1} + \xi_{i}^{n,2})^2 \right] \xrightarrow{\mathbb{P}} n \rightarrow +\infty f^2(t, p).
\]
On the other hand, using (20)

\[ \sigma \]

and since

\[ \sum h \]

If condition is fulfilled.

\[ \psi \]

Since

These two lemmas conclude the proof of Proposition 9.

Lemma 11. The following Lindeberg condition holds:

\[
\sum_{i \in D_t^n} E_{i-1}^n \left[ (\xi_i^{n,1} + \xi_i^{n,2})^2 1_{|\xi_i^{n,1} + \xi_i^{n,2}| \geq \varepsilon} \right] \xrightarrow{n \to +\infty} 0 \text{ a.s. \ \forall \varepsilon > 0.}
\]  

(27)

Proof. Let us prove (27). We derive from the Cauchy-Schwarz and Chebyshev inequalities that,

\[
E_{i-1}^n \left[ \left( \xi_i^{n,1} + \xi_i^{n,2} \right)^2 1_{|\xi_i^{n,1} + \xi_i^{n,2}| \geq \varepsilon} \right] \leq E_{i-1}^n \left[ \left( \xi_i^{n,1} + \xi_i^{n,2} \right)^4 \right]^{\frac{1}{2}} \mathbb{P} \left[ \left\{ |\xi_i^{n,1} + \xi_i^{n,2}| \geq \varepsilon \right\} \mid \mathcal{F}_{(i-1)\Delta_n} \right]^{\frac{1}{2}},
\]

\[ \leq \frac{8}{\varepsilon} \left( E_{i-1}^n \left[ \left( \xi_i^{n,1} \right)^4 \right] + E_{i-1}^n \left[ \left( \xi_i^{n,2} \right)^4 \right] \right). \]

On the one hand, using (19),

\[ E_{i-1}^n \left[ \left( \xi_i^{n,1} \right)^4 \right] = \frac{\rho_n^4}{r_n^3} \sigma_{(i-1)\Delta_n}^4 \mathbb{E} \left[ |U|^p - m_p \right] \]

and since \( \sigma \) is locally bounded, we obtain that there exists \( C(\omega) \) such that for all \( t \geq 0 \),

\[ \sum_{i \in D_t^n} E_{i-1}^n \left[ (\xi_i^{n,1})^4 \right] \leq \frac{C(\omega)}{\varepsilon} \sum_{i \in D_t^n} \frac{\rho_n^4}{r_n^3} = \frac{C(\omega) \rho_n^4}{\varepsilon r_n^3}. \]

If \( h_n = o(\sqrt{\Delta_n}) \) (resp. \( \sqrt{\Delta_n} = O(h_n) \)), \( \rho_n^4/r_n^3 = 1/r_n \) (resp. \( \rho_n^4/r_n^3 = \Delta_n^{3/2}/h_n^5 \)). Thus, \( \sum_{i \in D_t^n} E_{i-1}^n \left[ (\xi_i^{n,1})^4 \right] \xrightarrow{n \to +\infty} 0 \text{ a.s.} \)

On the other hand, using (20)

\[
\mathbb{E} \left[ \sum_{i \in D_t^n} E_{i-1}^n \left[ (\xi_i^{n,2})^4 \right] \right] = \left( \frac{\rho_n}{r_n} \right)^4 \left( (t + h_n)/\Delta_n \right) - i + 1 \right)^4 \mathbb{E} \left[ (M_i \Delta_n - M_{(i-1)\Delta_n \land t})^4 \right]
\]

\[
\leq \sum_{i \in D_t^n} \left( \frac{\rho_n}{r_n} \right)^4 \left( (t + h_n)/\Delta_n \right) - i + 1 \right)^4 \mathbb{E} \left[ \left( \int_{C_i^{n,i}} \psi(s) \, ds \right)^2 \right],
\]

\[
\leq \sum_{i \in D_t^n} \left( \frac{\rho_n}{r_n} \right)^4 \left( (t + h_n)/\Delta_n \right) - i + 1 \right)^4 \Delta_n \int_{C_i^{n,i}} \mathbb{E} \left[ \psi(s)^2 \right] \, ds.
\]

Since \( \psi(s) \leq C|\sigma_i|^{2p-2} \), it follows from (11) that \( \sup_{s \in [0, T]} \mathbb{E} \left[ \psi(s)^2 \right] < +\infty \). Now,

\[
\sum_{i \in D_t^n} \left( \frac{\rho_n}{r_n} \right)^4 \left( (t + h_n)/\Delta_n \right) - i + 1 \right)^4 \Delta_n^2 \leq C \rho_n^4 r_n \Delta_n^2,
\]

and one checks that this right-hand member tends to 0 in every cases. It follows that the Lindeberg condition is fulfilled. \( \square \)

These two lemmas conclude the proof of Proposition 9.
3.4 The remainder terms

We focus on $\Lambda^q_\delta(t)$, recalling:

$$
\Lambda^q_\delta(t) = \frac{1}{r_n} \sum_{i \in D^p_n} \left( [(t + h_n)/\Delta_n] - i + 1 \right) (A_i \Delta_n - A_{(i-1)\Delta_n \lor t}).
$$

We obtain the following results of convergence in probability.

**Proposition 12.** Assume $(\text{SH})_q$ and $(\text{H}^2_\delta)$ with $q \in [1, 2]$. Then, for every $t \in [0, T]$:

$$
\frac{1}{h_n^{1/q}} \Lambda^q_\delta(t) \xrightarrow{p}{\rightarrow} 0. \tag{28}
$$

If the previous assumptions hold with $q = 1$ and if $(\theta^0_t)$ defined by (4) is càd, then for every $t \in [0, T]$,

$$
\frac{1}{h_n} \Lambda^1_\delta(t) \xrightarrow{p}{\rightarrow} \frac{\theta^0_t}{2}. \tag{29}
$$

**Remark 13.** Note that Assumption $(\text{H}^2_\delta)$ is only necessary at this stage of the proof where a kind of regularity of the small jumps is needed.

**Proof.** It will be useful to notice that, for every $\epsilon > 0$, $A_t = \mathcal{L}'(t) + \mathcal{M}'(t) + \mathcal{N}'(t)$, with

$$
\mathcal{L}'(t) = \int_0^t \left( \theta_s - \int_{|y| \leq 1} p|Y_s|^p-1 y F_y(dy) \right) ds,
$$

$$
\mathcal{M}'(t) = \int_0^t \int_{|y| \leq \epsilon} p|Y_s-y|^p-1 (\mu - \nu)(ds, dy) + \sum_{0 < s \leq t} H_{0^y_0} (Y_s - \Delta Y_s) 1_{||\Delta Y_s| \leq \epsilon},
$$

$$
\mathcal{N}'(t) = \sum_{0 < s \leq t} (|Y_s - \Delta Y_s|^p - |Y_s|^p) 1_{||\Delta Y_s| > \epsilon},
$$

where $g_p(x) = |x|^p$ and for every $f : \mathbb{R} \to \mathbb{R}$,

$$
H^f(x, y) = f(x + y) - f(x) - f'(x)y. \tag{30}
$$

With these notations, the above proposition is a consequence of the following lemma.

**Lemma 14.** Assume $(\text{SH})_q$ with $q \in [1, 2]$. Then,

(i) For every $\epsilon > 0$, there exists a.s. $n_0(\omega)$ such that for every $n \geq n_0(\omega)$,

$$
\frac{1}{r_n} \sum_{i \in D^p_n} \left( \left[ \frac{t + h_n}{\Delta_n} \right] - i + 1 \right) (N^n_{i\Delta_n} - N^n_{(i-1)\Delta_n \lor t}) = 0.
$$

(ii) Assume moreover $(\text{H}^2_\delta)$ with $q \in [1, 2]$. For every $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that for every $\varepsilon \leq \varepsilon_\delta$:

$$
\mathbb{P} \left[ \frac{1}{r_n} \sum_{i \in D^p_n} \left( \left[ \frac{t + h_n}{\Delta_n} \right] - i + 1 \right) (\mathcal{M}^\varepsilon_{i\Delta_n} - \mathcal{M}^\varepsilon_{(i-1)\Delta_n \lor t}) > \delta h_n^{1/q} \right] \xrightarrow{n \to +\infty} 0.
$$

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(iii) For every $\varepsilon > 0$, we have almost surely

$$
\limsup_{n \to +\infty} \frac{1}{h_n} \left( \frac{1}{r_n} \sum_{i \in \mathcal{D}_n^i} \left( \frac{t + h_n}{\Delta_n} - i + 1 \right) \left( \mathcal{L}_t^\varepsilon - \mathcal{L}_{(i-1)\Delta_n \vee t}^\varepsilon \right) \right) < +\infty. \tag{31}
$$

Assume moreover that $(\text{SH})_1$ and $(\text{H}^2)$ hold and that $(\theta_i^q)$ is càd. Then, almost surely,

$$
\limsup_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{h_n} \left( \frac{1}{r_n} \sum_{i \in \mathcal{D}_n^i} \left( \frac{t + h_n}{\Delta_n} - i + 1 \right) \left( \mathcal{L}_t^\varepsilon - \mathcal{L}_{(i-1)\Delta_n \vee t}^\varepsilon \right) \right) = \frac{\theta_i^q}{2}. \tag{32}
$$

**Proof.** (i) Let $T_t^\varepsilon$ denote the random time defined by $T_t^\varepsilon(\omega) := \inf \{ s > t, |\Delta Y_s| \geq \varepsilon \}$. For every $\delta > 0$, we have

$$
P \{ t \leq T_t^\varepsilon \leq t + \delta \} \leq \mathbb{E} \left[ \sum_{t \leq s \leq t + \delta} 1_{\{|\Delta Y_s| \geq \varepsilon \}} \right] \leq \mathbb{E} \left[ \int_t^{t+\delta} \int_{\{|y| \geq \varepsilon \}} F_s(dy) ds \right].
$$

Under $(\text{SH})_2$,

$$
\int_{\{|y| \geq \varepsilon \}} F_s(dy) \leq \varepsilon^{-2} \int_{\{|y| \geq \varepsilon \}} |y|^2 F_s(dy) \leq \varepsilon^{-2} \sup_{s \in [0,T]} \int |y|^2 F_s(dy) \leq M/\varepsilon^2.
$$

It follows from the dominated convergence theorem that $\mathbb{P} \{ T_t^\varepsilon = t \} = 0$. Thus, a.s., there exists $n_0(\omega)$ such that $T_t(\omega) > t + h_n$ for every $n \geq n_0(\omega)$. The result follows.

(ii) On the one hand, by the Doob inequality for discrete martingales, we have for every $q \in (1,2]$,

$$
\mathbb{E} \left[ \sum_{i \in \mathcal{D}_n^i} \left( \frac{t + h_n}{\Delta_n} - i + 1 \right) \int_{C_{t,n}^i} \int_{\{|y| \leq \varepsilon \}} p(\sigma_{s-}^{-1}y(\mu - \nu))(ds,dy)^q \right] \leq \mathbb{E} \left[ \sum_{i \in \mathcal{D}_n^i} \left( \frac{t + h_n}{\Delta_n} - i + 1 \right)^2 \int_{C_{t,n}^i} \int_{\{|y| \leq \varepsilon \}} p^2 \sigma_{s-}^{-2} y^2 \nu(ds,dy)^q \right].
$$

Then, using that $(\sum |u_i|)^{q/2} \leq \sum |u_i|^{q/2}$ (since $q/2 \leq 1$) and Jensen’s inequality, we obtain:

$$
\mathbb{E} \left[ \sum_{i \in \mathcal{D}_n^i} \left( \frac{t + h_n}{\Delta_n} - i + 1 \right) \int_{C_{t,n}^i} \int_{\{|y| \leq \varepsilon \}} p(\sigma_{s-}^{-1}y(\mu - \nu))(ds,dy)^q \right] \leq C \sum_{i \in \mathcal{D}_n^i} \left( \frac{t + h_n}{\Delta_n} - i + 1 \right)^q \Delta_n^{q-1} \int_{C_{t,n}^i} \mathbb{E} \left[ \int_{\{|y| \leq \varepsilon \}} \sigma_{s-}^{(p-1)q} |y|^q F_s(dy) \right] ds.
$$

Using Assumption $(\text{SH})_q$, we derive from Cauchy-Schwarz’s inequality, $(\text{H}^1)$, Assumption $(\text{H}^2)_q$ and the dominated convergence Theorem that:

$$
\sup_{s \in [0,T]} \mathbb{E} \left[ \int_{\{|y| \leq \varepsilon \}} \sigma_{s-}^{(p-1)q} |y|^q F_s(dy) \right] \xrightarrow{\varepsilon \to 0} 0.
$$

Thus, using that

$$
\sum_{i \in \mathcal{D}_n^i} \left( \frac{t + h_n}{\Delta_n} - i + 1 \right)^q \leq C r_n^{q+1},
$$

we obtain

$$
\sum_{i \in \mathcal{D}_n^i} \left( \frac{t + h_n}{\Delta_n} - i + 1 \right)^q \leq C t_n^{q+1}.
$$
it follows that, for every $q \in [1, 2]$, for every $\eta > 0$, there exists $\varepsilon_\eta^1 > 0$ such that for every $\varepsilon \leq \varepsilon_\eta^1$,
\[
\mathbb{E}\left[ \frac{1}{r_n} \sum_{i \in D_t^q} \left( \left[ \frac{t + h_n}{\Delta_n} \right] - i + 1 \right) \int_{C_{t,i}^n} \int_{-\varepsilon}^{\varepsilon} p\sigma_{s-1}^p y(\mu - \nu)(ds, dy) \right] \leq C \eta h_n^{1/q}. \tag{33}
\]

On the other hand, by the Taylor formula, we have
\[
|H^{p_r}(x, y)| \leq C \left( |x|^{p-2}|y|^2 + |y|^{2p} \right),
\]
when $p \geq 2$. Then, using that $(H_2^q)$ implies $(H_2^2)$ and $(H_2^{2p})$, we obtain that for every $\eta > 0$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \leq \varepsilon_\eta^2$,
\[
\mathbb{E}\left[ \sum_{s \in C_{t,i}^n} |H^{p_r}(Y_s^- - \Delta Y_s)|1_{\{|\Delta Y_s| \leq \varepsilon\}} \right] \leq C\eta \Delta_n.
\]

Thus, for every $\eta > 0$, there exists $\varepsilon_\eta^2$ such that for every $\varepsilon \leq \varepsilon_\eta^2$,
\[
\mathbb{E}\left[ \frac{1}{r_n} \sum_{i \in D_t^q} \left( \left[ \frac{(t + h_n)/\Delta_n} \right] - i + 1 \right) H^{p_r}(Y_s^- - \Delta Y_s)1_{\{|\Delta Y_s| \leq \varepsilon\}} \right] \leq C \rho r_n \Delta_n. \tag{34}
\]

Therefore, (ii) follows from (33) and (34).

(iii) Since $\theta_s, Y_s$ and $\int_{|y| \leq 1} yF_s(dy)$ are locally bounded, there exists almost surely $C_T(\omega)$ such that for every $t \in [0, T]$, for every $n \geq 1$, $|\mathcal{L}_t^i \Delta_n - \mathcal{L}_t^{(i-1)\Delta_n} \mathcal{Y}_t| \leq C_T(\omega) \Delta_n$. Assertion (31) follows. If moreover $(\theta^0_t)$ is càdlàg, then,
\[
\mathcal{L}_t^i \Delta_n - \mathcal{L}_t^{(i-1)\Delta_n} \Delta_n = \theta^0_t \Delta_n + R_n^\varepsilon(\varepsilon, t) + a_{\varepsilon, t}(\Delta_n),
\]
with $R_n^\varepsilon(\varepsilon, t) = -p\sigma_{t-1}^{p-1} \int_{C_{t,i}^n} \int_{|y| \leq \varepsilon} yF_s(dy) ds$. Then, since
\[
\frac{1}{h_n} \left( \frac{1}{r_n} \sum_{i \in D_t^q} \left( \left[ \frac{(t + h_n)/\Delta_n} \right] - i + 1 \right) \Delta_n \right) \xrightarrow{n \to +\infty} \frac{1}{2},
\]
it follows that
\[
\limsup_{n \to +\infty} \frac{1}{h_n} \frac{1}{r_n} \sum_{i \in D_t^q} \left( \left[ \frac{(t + h_n)/\Delta_n} \right] - i + 1 \right) \left( \mathcal{L}_t^i \Delta_n - \mathcal{L}_t^{(i-1)\Delta_n} \Delta_n \right) - \frac{\theta^0_t}{2} = \limsup_{n \to +\infty} \frac{1}{h_n r_n} \sum_{i \in D_t^q} \left( \left[ \frac{(t + h_n)/\Delta_n} \right] - i + 1 \right) |R_n^\varepsilon(\varepsilon, t)|,
\]
\[
\leq C \sigma_t^{p-1} \sup_{s \in [0, T]} \int_{|y| \leq \varepsilon} |y|F_s(dy).
\]

Finally, we deduce (32) from $(H_2^q)$. \hfill \Box
Lemma 15. Assume (SH)$_2$. Then, there exists $C_p > 0$ such that for all $t$

$$\sup_{t \in [0, T]} \mathbb{E} \left[ (\Lambda^p_t(t))^2 \right] \leq C_p \frac{\Delta^2}{h_n^2}.$$ 

As a consequence, for every $t \in [0, T]$,

$$\sqrt{\frac{h_n}{\Delta_n}} \Lambda^p_n(t) \xrightarrow{n \to +\infty} 0.$$ 

Proof. Set $\sigma^n_i = \sigma_i \Delta_n$. Then, by a martingale argument, we have

$$\mathbb{E} \left[ (\Lambda^p_t(t))^2 \right] \leq \frac{\Delta^2}{h_n^2} \sum_{i \in D_n^t} \mathbb{E} \left[ \left( \frac{\Delta^n X}{\sqrt{\Delta_n}} \right)^p \right] \leq \frac{\Delta^2}{h_n^2} \sum_{i \in D_n^t} \mathbb{E} \left[ (\Delta^n W)^p \right].$$

As $dX_t = a(t)dt + \sigma_t dW_t$, we have $\Delta^n X = \sigma_{(i-1)\Delta_n} \Delta^n W + \chi^n_i$, with

$$\chi^n_i = \int_{C_{n,i}} (\sigma_{s} - \sigma_{(i-1)\Delta_n}) dW_s + \int_{C_{n,i}} a_s ds.$$ 

Using a Taylor expansion of $g(x) = |x|^p$ on the interval $[\sigma_{(i-1)\Delta_n} \Delta^n W; \Delta^n X]$, we have:

$$||\Delta^n X|^p - |\sigma_{(i-1)\Delta_n} \Delta^n W|^p| \leq \sup_{x \in [\sigma_{(i-1)\Delta_n} \Delta^n W; \Delta^n X]} |g'(x)| |\chi^n_i|.$$ 

But $|g'(x)| = O(|x|^{p-1})$ thus using the relation $|x+y|^p \leq C_p(|x|^p + |y|^p)$ with $C_p$ a constant, we have

$$\sup_{x \in [\sigma_{(i-1)\Delta_n} \Delta^n W; \Delta^n X]} |g'(x)| \leq C_p (|\sigma_{(i-1)\Delta_n} \Delta^n W|^{p-1} + |\chi^n_i|^{p-1}),$$

$$||\Delta^n X|^p - |\sigma_{(i-1)\Delta_n} \Delta^n W|^p| \leq C_p (|\sigma_{(i-1)\Delta_n} \Delta^n W|^{p-1} |\chi^n_i| + |\chi^n_i|^p).$$

Finally there is a constant $C_p$ such that, for all $t \geq 0$:

$$\mathbb{E} \left[ (\Lambda^p_t(t))^2 \right] \leq C_p \frac{\Delta^2}{h_n^2} \sum_{i \in D_n^t} \mathbb{E} \left[ (\chi^n_i)^2 |\sigma_{(i-1)\Delta_n} \Delta^n W|^{2p-2} + |\chi^n_i|^{2p} \right],$$

$$\leq C_p \frac{\Delta^2}{h_n^2} \sum_{i \in D_n^t} \left[ \mathbb{E} \left[ (\chi^n_i)^{2p} \right] \frac{\Delta^2}{h_n^2} \mathbb{E} \left[ |\sigma_{(i-1)\Delta_n} \Delta^n W|^{2p} \right] \right] \frac{\Delta^2}{h_n^2} + \mathbb{E} \left[ (\chi^n_i)^{2p} \right].$$ (35)

First of all, the independence between $\sigma_{(i-1)\Delta_n}$ and $\Delta^n W$ and $\text{(H)}$ yield:

$$\mathbb{E} \left[ |\sigma_{(i-1)\Delta_n} \Delta^n W|^{2p} \right] = \Delta^n W \mathbb{E} \left[ |\sigma_{(i-1)\Delta_n}|^{2p} \right] \leq C_p \Delta^n W.$$ 

So it remains to give a majoration of $\mathbb{E} \left[ (\chi^n_i)^{2p} \right]$. Since $a$ is bounded by $M$,

$$\mathbb{E} \left[ (\chi^n_i)^{2p} \right] \leq C_p \left( \mathbb{E} \left[ \left( \int_{\mathbb{C}_{n,i}} (\sigma_s - \sigma_{(i-1)\Delta_n}) dW_s \right)^{2p} \right] + (M \Delta_n)^{2p} \right).$$
Now, using inequality (13) and since $p \geq 1$, $E \left[ \left| \chi_n^2 \right|^p \right] \leq C_p \left( \Delta_n^{p+1} + \Delta_n^{2p} \right) \leq C \Delta_n^{p+1}$. Thus (35) becomes:

$$E \left[ (\Lambda_n^1(t))^2 \right] \leq C_p \frac{\Delta_n^{2-p}}{h_n^2} \sum_{i \in \mathcal{D}_i} \left[ (\Delta_n^{p+1})^{2} \Delta_n^{p-1} + \Delta_n^{p+1} \right] \leq C \frac{\Delta_n^{1+p}}{h_n} \left[ \Delta_n^{1+\frac{1}{p}} + \Delta_n^2 \right],$$

the constant $C_p$ does not depend on $t$ and as $p \geq 2$, we have,

$$\sup_{t \in [0,T]} E \left[ (\Lambda_n^1(t))^2 \right] \leq C_p \frac{\Delta_n^{1+\frac{1}{p}}}{h_n},$$

which ends the proofs.

**Proposition 16.** Assume (SH)$_2$. Then,

$$E \left[ \mathbb{E}_{i-1}^n \left[ \frac{\Delta_n^p X^p}{\sqrt{\Delta_n}} \right] - m_p |\sigma_{(i-1)\Delta_n}|^p \right] \leq \begin{cases} C \Delta_n^{\frac{1}{2}} & \text{if } p = 2 \\ C \Delta_n^{\frac{p-2}{2} \frac{1}{p}} & \text{if } p > 2. \end{cases} \quad (36)$$

As a consequence, if $p = 2$ or $p \geq 3$, $\|\Lambda_n^1(t)\|_1 \leq C \sqrt{n}$, and

$$\max \left( \sqrt{\frac{h_n}{\Delta_n}}, \sqrt{\frac{1}{h_n}} \right) \Lambda_n^1(t) \xrightarrow{n \to +\infty} 0.$$

**Proof.** We begin the proof by the following remark. Scaling and independence properties of the Brownian motion and the Ito’s formula yield

$$m_p = \frac{p(p-1)}{2\Delta_n^2} \int_{\mathcal{D}_n} \mathbb{E}_{i-1}^n \left[ \left| W_i - W_{(i-1)\Delta_n} \right|^{p-2} \right] ds.$$

Keeping in mind this representation of $m_p$, we decompose the integrand of (36) as follows:

$$\mathbb{E}_{i-1}^n \left[ \frac{\Delta_n^p X^p}{\sqrt{\Delta_n}} \right] - m_p |\sigma_{(i-1)\Delta_n}|^p = A_{1,i}^n + A_{2,i}^n$$

where

$$A_{1,i}^n = \mathbb{E}_{i-1}^n \left[ \frac{\Delta_n^p X^p}{\sqrt{\Delta_n}} \right] - \frac{p(p-1)}{2\Delta_n^2} \int_{\mathcal{D}_n} \mathbb{E}_{i-1}^n \left[ \left| \int_{(i-1)\Delta_n}^s \sigma_u dW_u \right|^{p-2} \right] ds,$$

$$A_{2,i}^n = \frac{p(p-1)}{2\Delta_n^2} \int_{\mathcal{D}_n} \mathbb{E}_{i-1}^n \left[ \left| \int_{(i-1)\Delta_n}^s \sigma_u dW_u \right|^{p-2} \right] ds.$$

Then, the result is a consequence of Lemmas 17 and 18 corresponding to $A_{1,i}^n$ and $A_{2,i}^n$ respectively.

**Lemma 17.** Assume (SH)$_2$. Then,

$$E \left[ |A_{1,i}^n| \right] \leq \begin{cases} C \Delta_n & \text{if } p = 2 \\ C \Delta_n^{(\frac{p-1}{2}) \frac{1}{p}} & \text{if } p > 2. \end{cases} \quad (37)$$
Proof. First, we use Itô's formula to develop $A^n_i$:

$$
\left| \frac{\Delta^n_i X}{\Delta_n} \right|^p = \int_{C_n,i} p \cdot \text{sgn}(X_s - X_{(i-1)\Delta_n}) \left| \frac{X_s - X_{(i-1)\Delta_n}}{\Delta_n} \right|^{p-1} a_s ds
$$

$$
+ \frac{1}{2} p(p-1) \int_{C_n,i} \left| \frac{X_s - X_{(i-1)\Delta_n}}{\Delta_n} \right|^{p-2} \sigma^2_s ds + M^n_i,
$$

with $\mathbb{E}^{n}_{i-1} [M^n_i] = 0$. It follows that:

$$
A^n_{1,i} = \mathbb{E}^{n}_{i-1} \left[ \int_{C_n,i} p \cdot \text{sgn}(X_s - X_{(i-1)\Delta_n}) \left| \frac{X_s - X_{(i-1)\Delta_n}}{\Delta_n} \right|^{p-1} a_s ds \right]
$$

$$
+ \frac{1}{2} p(p-1) \mathbb{E}^{n}_{i-1} \left[ \int_{C_n,i} R^n_i(s) \sigma^2_s ds \right],
$$

with $R^n_i(s) := \frac{|X_s - X_{(i-1)\Delta_n}|^{p-2}}{\Delta_n^2} - \frac{|f_s - X_{(i-1)\Delta_n}| \sigma u dW_u|^{p-2}}{\Delta_n^2}$. Now, using that $a$ is bounded, we have

$$
\mathbb{E} \left[ |X_s - X_{(i-1)\Delta_n}|^{p-1} |a_s| \right] \leq C \left( (s - (i-1)\Delta_n)^{p-1} + \mathbb{E} \left[ \int_{(i-1)\Delta_n}^s \sigma u dW_u \right]^{p-1} \right)
$$

$$
\leq C (s - (i-1)\Delta_n)^{p-1} + C (s - (i-1)\Delta_n)^{\frac{p-1}{2}},
$$

owing to Inequality (13). Hence, for every $p \geq 2$,

$$
\mathbb{E} \left[ \int_{C_n,i}^p \frac{|X_s - X_{(i-1)\Delta_n}|^{p-1}}{\Delta_n^2} |a_s| ds \right] \leq C \Delta_n^{\frac{1}{2}(2-\frac{2}{p})}.
$$

Now, we observe that $R^n_i(s) = 0$ when $p = 2$ so the proof is ended in this case.

When $p > 2$, recall that for every $\bar{q} > 0$ and $\forall (u, v) \in \mathbb{R}^2$,

$$
|u|^{\bar{q}} - |v|^{\bar{q}} \leq \begin{cases} |u - v|^{\bar{q}} & \text{if } \bar{q} \leq 1 \\
C_{\bar{q}} (|u - v||u|^{\bar{q} - 1} + |u - v|^{\bar{q}}) & \text{if } \bar{q} > 1,
\end{cases}
$$

applying it with $\bar{q} = p - 2$ yields

$$
|R^n_i(s)| \leq \begin{cases} \frac{1}{\Delta_n^2} |f_{(i-1)\Delta_n} a u du|^{p-2} & \text{if } p \leq 3 \\
C \frac{1}{\Delta_n^2} \left( |f_{(i-1)\Delta_n} a u du| |f_{(i-1)\Delta_n} \sigma u dW_u|^{p-3} + |f_{(i-1)\Delta_n} a u du|^{p-2} \right) & \text{if } p > 3.
\end{cases}
$$

(39)

First, let $p \in (2, 3)$. Since $a$ is uniformly bounded,

$$
|R^n_i(s)| \leq C \Delta_n^{-\frac{p}{2}} (s - (i-1)\Delta_n)^{p-2}.
$$

Then, $\mathbb{E} \left[ \sigma^2_s \right]$ uniformly bounded, Cauchy-Schwarz and inequality (13) yield

$$
\mathbb{E} \left[ \int_{C_n,i} |R^n_i(s)\sigma^2_s| ds \right] \leq C \Delta_n^{-1}.
$$

(40)
Assume now that \( p > 3 \). First, for all \( s \in [(i-1)\Delta_n, i\Delta_n] \), we derive from a bounded and Cauchy-Schwarz inequality that

\[
\mathbb{E} \left[ \left| \int_{(i-1)\Delta_n}^{s} a_u du \right| \left| \int_{(i-1)\Delta_n}^{s} \sigma_u dW_u \right|^{p-3} \sigma_s^2 \right] \leq C(s - (i - 1)\Delta_n) \mathbb{E} \left[ \left| \int_{(i-1)\Delta_n}^{s} \sigma_u dW_u \right|^{2(p-3)} \right]^{\frac{1}{2}} \mathbb{E} \left[ \left| \sigma_s^2 \right|^{\frac{2}{2}} \right].
\]

Therefore, using inequalities (13) and (11), we have:

\[
\mathbb{E} \left[ \left| \int_{(i-1)\Delta_n}^{s} a_u du \right| \sigma_s^2 \left| \int_{(i-1)\Delta_n}^{s} \sigma_u dW_u \right|^{p-3} \right] \leq C(s - (i - 1)\Delta_n)^{\frac{p-3}{2} + 1}.
\]

Thus, we derive from (59), the preceding inequality and (10) that when \( p > 3 \),

\[
\mathbb{E} \left[ \int_{C_{n,i}} |R^n_t(s)\sigma_s^2|ds \right] \leq \frac{C}{\Delta_n^{\frac{p}{2}}} \int_{C_{n,i}} (s - (i - 1)\Delta_n)^{p-2} + (s - (i - 1)\Delta_n)^{\frac{p-3}{2}} ds \leq C\Delta_n^{\frac{1}{2}}.
\]

We now focus on \( A^n_{2,i} \).

**Lemma 18.** Assume \((\text{SH}_2)\). Then,

\[
\mathbb{E} \left[ |A^n_{2,i}| \right] \leq \begin{cases} C\Delta_n^{\frac{1}{2}} & \text{if } p = 2, \\ C\Delta_n^{\frac{(\frac{p}{2}-1)\wedge \frac{1}{2}}{4}} & \text{if } p > 2. \end{cases} \tag{41}
\]

**Proof.** In case \( p = 2 \) we deal with \( \frac{1}{\Delta_n} \int_{C_{n,i}} (\sigma_s^2 - \sigma_{(i-1)\Delta_n}^2) ds \). Hence by Cauchy-Schwarz, (11) and (12), we deduce that,

\[
\mathbb{E} \left[ \frac{1}{\Delta_n} \int_{C_{n,i}} (\sigma_s^2 - \sigma_{(i-1)\Delta_n}^2) ds \right] \leq C \frac{1}{\Delta_n} \int_{C_{n,i}} \mathbb{E} \left[ \left| \sigma_s - \sigma_{(i-1)\Delta_n} \right|^2 \right]^{\frac{1}{2}} \leq C\Delta_n^{\frac{1}{2}}. \tag{42}
\]

When \( p > 2 \), first,

\[
\frac{1}{\Delta_n^{\frac{p}{2}}} \int_{C_{n,i}} \mathbb{E}^{n}_{i-1} \left[ \sigma_s^2 \left| \int_{(i-1)\Delta_n}^{s} \sigma_u dW_u \right|^{p-2} - \sigma_{(i-1)\Delta_n} \left| \int_{(i-1)\Delta_n}^{s} dW_u \right|^{p-2} \right] ds = B^n_{1,i} + B^n_{2,i} \quad \text{with,}
\]

\[
B^n_{1,i} = \frac{1}{\Delta_n^{\frac{p}{2}}} \int_{C_{n,i}} \mathbb{E}^{n}_{i-1} \left[ \left( \sigma_s^2 - \sigma_{(i-1)\Delta_n}^2 \right) \left| \int_{(i-1)\Delta_n}^{s} \sigma_u dW_u \right|^{p-2} \right] ds,
\]

\[
B^n_{2,i} = \frac{\sigma_{(i-1)\Delta_n}^2}{\Delta_n^{\frac{p}{2}}} \int_{C_{n,i}} \mathbb{E}^{n}_{i-1} \left[ \left| \int_{(i-1)\Delta_n}^{s} \sigma_u dW_u \right|^{p-2} - \left| \int_{(i-1)\Delta_n}^{s} \sigma_{(i-1)\Delta_n} dW_u \right|^{p-2} \right] ds.
\]

Let us focus on \( B^n_{1,i} \) and let \( \bar{q} > 1 \) and \( \bar{r} > 1 \) satisfying \( \frac{1}{q} + \frac{1}{p} = 1 \) and \( \bar{r} > 2 + \frac{2}{p-2} \). Using Hölder inequality, we have

\[
\mathbb{E} \left[ \left| \sigma_s^2 - \sigma_{(i-1)\Delta_n}^2 \right| \cdot \left| \int_{(i-1)\Delta_n}^{s} \sigma_u dW_u \right|^{p-2} \right] \leq \left( \mathbb{E} \left[ \left| \sigma_s^2 - \sigma_{(i-1)\Delta_n}^2 \right|^\bar{q} \right] \right)^\frac{1}{\bar{q}} \cdot \left( \mathbb{E} \left[ \left| \int_{(i-1)\Delta_n}^{s} \sigma_u dW_u \right|^{\bar{r}(p-2)} \right] \right)^\frac{1}{\bar{r}}.
\]
Then, on the one hand, applying again Holder’s inequality applied with \( p = 2/q(> 1) \) and \( q = \bar{q}/(\bar{q} - 2) \), we derive from (11) and (12),

\[
E \left[ \sigma^2_s - \sigma^2_{(i-1)\Delta n} \right]^{\frac{1}{q}} \leq C E \left[ |\sigma_s - \sigma_{(i-1)\Delta n}|^2 \right]^{\frac{1}{q}} \leq C(s - (i - 1)\Delta n)^{\frac{1}{q}}.
\]

On the other hand, using (13),

\[
\left( E \left[ \int_{(i-1)\Delta n}^s \sigma_u dW_u \right]^{\frac{p}{p-2}} \right)^{\frac{p-2}{p}} \leq C(s - (i - 1)\Delta n)^{\frac{p-2}{p}}.
\]

Thus,

\[
E \left[ \sigma^2_s - \sigma^2_{(i-1)\Delta n} \right] \left| \int_{(i-1)\Delta n}^s \sigma_u dW_u \right|^{p-2} \leq C(s - (i - 1)\Delta n)^{\frac{p-2}{p}} + \frac{1}{p}. \quad (43)
\]

Hence, we have

\[
E \left[ |B_{1,i}^n| \right] \leq C\Delta_n^{\frac{1}{p}}. \quad (44)
\]

We now study \( B_{2,i}^n \). Set \( M^n_s = \int_{(i-1)\Delta n}^s (\sigma_u - \sigma_{(i-1)\Delta n}) dW_u \). By (45),

\[
\left| \int_{(i-1)\Delta n}^s \sigma_u dW_u \right|^{p-2} - \left| \int_{(i-1)\Delta n}^s \sigma_{(i-1)\Delta n} dW_u \right|^{p-2} \leq \begin{cases} |M^n_s|^{p-2} \leq C |M^n_s| \left( |M^n_s|^{p-3} + |M^n_s|^{p-2} \right) & \text{if } p \leq 3 \\ C \left[ |M^n_s| \int_{(i-1)\Delta n}^s \sigma_{(i-1)\Delta n} dW_u \right]^{p-3} + |M^n_s|^{p-2} & \text{if } p > 3. \end{cases} \quad (45)
\]

Hence, if \( p \leq 3 \), it follows from (14) and Cauchy-Schwarz inequality that

\[
E \left[ |B_{2,i}^n| \right] \leq C \Delta_n^{\frac{1}{p}} \int_{(i-1)\Delta n}^{r\Delta_n} E \left[ |M^n_s|^{2(p-2)} \right]^{\frac{1}{p}} E \left[ |\sigma_{(i-1)\Delta n}|^4 \right]^{\frac{1}{p}} ds,
\]

\[
\leq C \Delta_n^{\frac{1}{p}} \int_{(i-1)\Delta n}^{r\Delta_n} [(s - (i - 1)\Delta n)^{2(p-2)}]^{\frac{1}{p}} ds \leq C\Delta_n^{\frac{1}{p} - 1}. \quad (46)
\]

Assume now that \( p > 3 \). According to (15), we have two terms to manage with. On the one hand, by Cauchy-Schwarz and (14), we have

\[
E \left[ \sigma^2_{(i-1)\Delta n} |M^n_s| \left| \int_{(i-1)\Delta n}^s \sigma_{(i-1)\Delta n} dW_u \right|^{p-3} \right] \leq (E \left[ |M^n_s|^{2} \right])^{\frac{1}{2}} (s - (i - 1)\Delta n)^{\frac{p-2}{2}} E \left[ |\sigma_{(i-1)\Delta n}|^{2(p-1)} \right]^{\frac{1}{2}} \leq C(s - (i - 1)\Delta n)^{\frac{p-1}{p}}.
\]

On the other hand, Cauchy-Schwarz and (14) applied with \( q = 2(p-2) \geq 2 \) yield

\[
E \left[ \sigma^2_{(i-1)\Delta n} |M^n_s|^{p-2} \right] \leq \left( E \left[ |\sigma_{(i-1)\Delta n}|^4 \right] \right)^{\frac{1}{2}} \left( E \left[ |M^n_s|^{2(p-2)} \right] \right)^{\frac{1}{2}} \leq C(s - (i - 1)\Delta n)^{\frac{p-1}{p}}.
\]

Thus, it follows that when \( p > 3 \),

\[
E \left[ |B_{2,i}^n| \right] \leq C\Delta_n^{\frac{1}{p}}. \quad (47)
\]

Finally, we derive the lemma from (14), (46), (17).
3.5 Proof of main Theorems, a synthesis

Gathering the previous steps, Theorems 3 and 4 are now consequences of the classical following lemma:

Lemma 19. Let \( (X_n) \) and \( (Y_n) \) be some sequences of random variables defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \) with values in a Polish space \( E \). Assume that \( (X_n) \) converges \( \mathcal{L} - s \) to \( X \) and that \( (Y_n) \) in probability to \( Y \). Then, the sequence of random variables \( (Z_n = X_n + Y_n) \) converges \( \mathcal{L} - s \) to \( X + Y \).

Indeed, focus for instance on statements (4) and (6). By Lemma 6, it is enough to prove these convergences under (5). Finally, applying Lemma 19 with \( Y \), therefore (4) and (6) follow from Lemma 19 applied with \( Y \).

On the other hand, under the assumptions of Theorems 3 and 4, allowing us to build a confidence region to estimate for all \( t \) parameter \( \sigma_t \). Since the variance limits in their second part depend on the unknown parameters \( \eta_1(t) \) and \( \eta_2(t) \), we focus on their first part (i) when \( h_n/\sqrt{\Delta_n} \) (respectively \( \limsup_{n \to +\infty} h_n^{1/2+1/q}/\sqrt{\Delta_n} < +\infty \)). These confidence regions could be defined as follows:

\[
\left\{ \sigma_t, \sqrt{\frac{n}{X}} \sum(p, \Delta_n, h_n)_{t} - \sigma_t^p \right\} \leq 1.96 \]

and according to (4) or (6) with, for instance, asymptotic probability 0.95, we get:

\[
\mathbb{P} \left[ \sqrt{\frac{n}{X}} \sum(p, \Delta_n, h_n)_{t} - \sigma_t^p \left| m_p \right| \leq 1.96 \right] \to 0.95.
\]
Thus, with 0.95 asymptotic confidence,

\[
\sigma_t^p \in \left[ \frac{m_p \sqrt{r_n} \Sigma(p, \Delta_n, h_n)_t}{m_p \sqrt{r_n} + 1.96 \sqrt{m_{2p} - m_p^2}}, \quad \frac{m_p \sqrt{r_n} \Sigma(p, \Delta_n, h_n)_t}{m_p \sqrt{r_n} - 1.96 \sqrt{m_{2p} - m_p^2}} \right].
\] (49)

The confidence interval length is about \( r_n^{\frac{1}{2}} \). Actually, the most interesting point is that we obtain an asymptotic confidence interval for the relative error:

\[
\mathbb{P} \left[ \frac{\Sigma(p, \Delta_n, h_n)_t}{\sigma_t^p} - 1 \leq \frac{1.96 \sqrt{m_{2p} - m_p^2}}{m_p \sqrt{r_n}} \right] \xrightarrow{n \to \infty} 0.95.
\]

**Remark 20.** Finally, to compare this result with respect to \( p \), we have to compare asymptotic confidence intervals of \( \sigma_t \), depending on \( p \), namely

\[
\sigma_t \in \left[ \left( \frac{m_p \sqrt{r_n} \Sigma(p, \Delta_n, h_n)_t}{m_p \sqrt{r_n} + 1.96 \sqrt{m_{2p} - m_p^2}} \right)^{\frac{1}{p}}, \quad \left( \frac{m_p \sqrt{r_n} \Sigma(p, \Delta_n, h_n)_t}{m_p \sqrt{r_n} - 1.96 \sqrt{m_{2p} - m_p^2}} \right)^{\frac{1}{p}} \right].
\]

This interval length is about \( r_n^{\frac{1}{2}} \sqrt{\frac{m_{2p} - m_p^2}{p m_p}} \), and this length order is unhappily increasing with \( p \), so it could be not so good to use \( p > 2 \).

## 5 Simulations

In this section, we want to test numerically the volatility estimator. In order to be able to compare the estimations with the true volatility, we do not use some real datas but get our observations from *quasi-exact* simulations of toy models (by quasi-exact, we mean simulations of the process using an Euler scheme with a very small time-discretization step).

### 5.1 A numerical test in a continuous stochastic volatility model

In this part, we consider the stochastic volatility model proposed in [13] where the volatility is an Ornstein-Uhlenbeck process. Denote the price by \((S_t)\) and by \((\sigma_t)\) the (non-negative) stochastic volatility. Set \( X_t := \log(S_t) \) and \( v_t := \sigma_t^2 \). The model is defined by:

\[
\begin{cases}
    dX_t = (r - \frac{1}{2} \sigma_t^2)dt + \sigma(t)dW_t^1 \\
    dv_t = a(m - v_t)dt + \beta(\rho dW_t^1 + \sqrt{1 - \rho^2}dW_t^2),
\end{cases}
\]

where \( r, a, \beta \) and \( m \) are some positive parameters, \( \rho \in [-1, 1] \) and the processes \( W^1 \) and \( W^2 \) are independent one-dimensional Brownian motions.

We set \( X_0 = \log(50), v_0 = m \) and simulate *quasi-exactly* \((X_t, v_t)\) at times \( 0, 1/n, 2/n, \ldots, 1 \) with the following parameters:

\[
r = 0.05, \quad \rho = 0, \quad a = 1, \quad m = 0.05, \quad \text{and} \quad \beta = 0.05.
\]

Using the simulated observations \( X_0, X_{1/n}, \ldots, X_1 \), we compute the estimator \( \Sigma(p, 1/n, h_n) \) on \([h_n, 1]\) and compare its value with the true volatility. In Figures 1 and 2 we represent the corresponding
graphics for $n = 1000$ and $n = 10000$ and $h_n = n^{-1/2}$. In all the figures, we choose $p = 2$ since as shown in the computation of the confidence interval length in Remark 20 to increase $p$ is not a good choice. The process $(\sigma_t)$ is plotted as continuous line whereas the estimator $\Sigma(2, 1/n, h_n)$ is plotted as discontinuous line.

By Remark 5, taking $r_n = n^\rho$ with $\rho \in (0, 1/2)$ and $p \in \{2\} \cap (5/2, +\infty)$ (or equivalently $h_n = n^{\rho-1}$), we obtain a rate of order $n^{\rho/2}$. In particular, we can derive that the best rate is obtained in the limit case $\rho = 1/2$. This theoretical result is confirmed in the following computation. Denote by $E_n(p, h_n)$ the mean relative error defined by:

$$E_n(p, h_n) := \frac{1}{n} \sum_{k=1}^{n} \left| \frac{\Sigma(p, n^{-1}, h_n)^{1/p}_k/n - \sigma(k/n)}{\sigma(k/n)} \right|.$$  

We obtain the following results:

|       | $E_n(2, n^{-0.4})$ | $E_n(2, n^{-0.5})$ | $E_n(2, n^{-0.6})$ |
|-------|--------------------|--------------------|--------------------|
| $n = 10^4$ | 18.9%            | 16.6%             | 18.6%             |
| $n = 10^5$ | 12.2%            | 11.0%             | 12.3%             |

|       | $E_n(4, n^{-0.4})$ | $E_n(4, n^{-0.5})$ | $E_n(4, n^{-0.6})$ |
|-------|--------------------|--------------------|--------------------|
| $n = 10^4$ | 20.3%            | 17.5%             | 19.2%             |
| $n = 10^5$ | 13.0%            | 11.9%             | 12.9%             |

Here, Remark 20 is confirmed by the fact: the estimations seem to be better with $p = 2$ than with $p = 4$.  

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5.2 A numerical test in a jump model

In this last part, we assume that the volatility is a jump process solution to a SDE driven by a tempered stable subordinator $(Z_t^{\lambda,\beta})$ with Lévy measure $\pi(dy) = 1_{y>0} \exp(-\lambda y)/y^{1+\beta}dy$. This model can be viewed as a particular case of the Barndorff-Nielsen and Shephard model \[5\] (for other jump volatility models see e.g. \[9, 12\]):

\[
\begin{align*}
    dX_t &= (r - \frac{1}{2} \sigma^2(t))dt + \sigma(t) dW^1_t \\
    dv_t &= -\mu v_t dt + dZ_t^{(\lambda,\beta)}
\end{align*}
\]

with the following choice of parameters:

\[ r = 0.05, \quad \mu = 1, \quad \lambda = 1 \quad \text{and} \quad \beta = 1/2. \]

This concerns Example (2) in Section 1 where Hypotheses $(H^1_q)$ and $(H^2_q)$ hold for any $q \geq \beta$ and Theorem 4 can be applied. As in the preceding example, we simulate $(X_t, v_t)$ on the interval $[0,1]$ with $X_0 = \log(50)$ and $v_0 = 0.05$. In order to compare the two types of models, we chose some similar parameters. The main difference between these two models comes from the variations which are stronger in the first case. We obtain a quasi-exact sequel $(X_{k/n}, v_{k/n})$ with $k \in \{0, \ldots, n\}$. In Figures 3 and 4 we represent the estimated and true volatilities for some different choices of $h_n = n^{-1/2}$, $n = 10^3$ and $n = 10^4$.

For these computations, we obtain the following mean relative errors:
It seems the best result is obtained with \( h_n = n^{-0.6} \), according to Remark 7 in case \( \eta_1 = \eta_2 = 0 \): the best convergence rate is obtained with \( \rho = 2/3 \).

6 Appendix

Proof of Lemma 3 Since the arguments are almost the same for each statement of the main results, we only prove (4) (with \( p \in \{2\} \cup \{3, +\infty\} \) and \( t \geq 0 \)).

Let \((X, \sigma)\) satisfy \((H_q^2)\) and \((H_q^3)\). Then, there exists a sequence \((T_m)_{m \geq 1}\) of stopping times increasing to \(\infty\) such that the processes \((a_t), (b_t), (\eta(t)), (\eta_2(t))\) and \(I(1 \wedge |y|^p)F_t(dy)\) are bounded on \([0, T_m]\).

Then, let \(X^M, \sigma^M\) be defined by \(X^M = X_{t\wedge T_m}\) and \(\sigma^M = |Y^M|\) where

\[
Y_t^M = \int_0^{t \wedge T_m} b_s ds + \int_0^{t \wedge T_m} \eta_1(s) dW^1_s + \int_0^{t \wedge T_m} \eta_2(s) dW^2_s + \int_0^{t \wedge T_m} (g(\mu - \nu) ds) + \int_0^{t \wedge T_m} \mu(ds, dy).
\]

By construction, \((X^M, Y^M)\) satisfies \((SH)_q\) and \((H^2_q)\) for every \(M \in \mathbb{N}\). It follows from the assumptions of the proposition that for every \(M \in \mathbb{N}\),

\[
\sqrt{\tau_n} \left( \Sigma^M(p, \Delta_n, h_n)_t - (\sigma^M_t)^p \right) \xrightarrow{L_2, n \to +\infty} \sqrt{\varphi_1(p, t, \sigma)U},
\]

where \(\Sigma^M\) is the statistic related to \((X^M, \sigma^M)\) as in (3). \(U \sim \mathcal{N}(0, 1)\) and \(U\) is independent of \(\mathcal{F}_t\). Let us now prove (4). Let \(g\) be a bounded continuous function on \(\mathbb{R}\) and let \(H\) be a bounded \(\mathcal{F}\)-measurable random variable. Then, for all \(M \in \mathbb{N}\),

\[
\mathbb{E}[Hg(\sqrt{\tau_n}(\Sigma(p, \Delta_n, h_n)_t - (\sigma_t)^p))] - \mathbb{E}[g(\sqrt{\varphi_1(p, t, \sigma)U})]
= \mathbb{E}[Hg(\sqrt{\tau_n}(\Sigma^M(p, \Delta_n, h_n)_t - (\sigma^M_t)^p))] - \mathbb{E}[Hg(\sqrt{\varphi_1(p, t, \sigma)U})]
+ \mathbb{E}[Hg(\sqrt{\tau_n}(\Sigma^M(p, \Delta_n, h_n)_t - (\sigma^M_t)^p))] - \mathbb{E}[Hg(\sqrt{\varphi_1(p, t, \sigma)U})]
+ \mathbb{E}[Hg(\sqrt{\varphi_1(p, t, \sigma)U})] - \mathbb{E}[Hg(\sqrt{\varphi_1(p, t, \sigma)U})].
\]

Set \(B_M = \{\omega, t + h_1 < T_m(\omega)\}\). By construction, on \(B_M\), \(\alpha^M_t = \alpha_t\) and \(\Sigma^M(p, \Delta_n, h_n)_t = \Sigma(p, \Delta_n, h_n)_t\) for every \(n \in \mathbb{N}\). Thus, uniformly in \(n\), the first and third right-hand side terms are bounded by \(2\|g\|_{\infty}\|H\|_{\infty}^p\mathbb{P}[B_M^c]\) for every \(M \in \mathbb{N}\). Then, since Assumption \((H^2_q)\) implies that \(T_M \to +\infty\) a.s., \(\mathbb{P}[B_M^c] \to 0\) as \(M \to +\infty\). Now, by (4), for every \(M \in \mathbb{N}\),

\[
\mathbb{E}[Hg(\sqrt{\tau_n}(\Sigma^M(p, \Delta_n, h_n)_t - (\sigma^M_t)^p))] - \mathbb{E}[Hg(\sqrt{\varphi_1(p, t, \sigma)U})] \xrightarrow{n \to +\infty} \mathbb{E}[Hg(\sqrt{\varphi_1(p, t, \sigma)U})].
\]
and the result follows.

**Proof of Lemma 7** (i). Let us prove (11). Thanks to Jensen’s inequality, we can only consider the case \( r \geq 2 \). Since the jumps of \( Y \) are bounded, we can compensate the big jumps and write

\[
Y_t = \int_0^t \tilde{b}_u du + \int_0^t \eta_1(s) dW_s + \int_0^t \eta_2(s) dW_s^2 + \int_0^t y(\mu - \nu)(ds, dy),
\]

where \( \tilde{b}_t = b_t + \int_{\{|u|>1\}} yF_t(dy) \). Then, using (SH)\(_2\) and Burkholder-Davis-Gundy inequality, we have for every \( r \geq 2 \):

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |\sigma_t|^r \right] \leq C \left( T^r + T^{r/2} + \mathbb{E} \left[ \sup_{t \in [0,T]} \left( \int_0^t \int_\mathbb{R} y^2 \mu(ds, dy) \right)^{r/2} \right] \right),
\]

where \( C \) is a deterministic constant. Let us focus on the last term of the right-hand side. We can write:

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} \left( \int_0^t \int_\mathbb{R} y^2 \mu(ds, dy) \right)^{r/2} \right] &\leq C \mathbb{E} \left[ \sup_{t \in [0,T]} \left( \int_0^t \int_\mathbb{R} y^2 (\mu - \nu)(ds, dy) \right)^{r/2} \right] + C \mathbb{E} \left[ \left( \int_0^T \int_\mathbb{R} y^2 \nu(ds, dy) \right)^{r/2} \right] \\
&\leq C \left( \mathbb{E} \left[ \left( \int_0^T \int_\mathbb{R} y^4 \mu(ds, dy) \right)^{r/4} \right] + T^{r/2} \right),
\end{align*}
\]

where in the last inequality, we again used Burkholder-Davis-Gundy inequality and (SH)\(_2\). Set \( k_0 = \min\{k \in \mathbb{N}, r \leq 2^k\} \). By an iteration, we obtain

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left( \int_0^t \int_\mathbb{R} y^2 \mu(ds, dy) \right)^{r/2} \right] \leq C \left( \mathbb{E} \left[ \left( \int_0^T \int_\mathbb{R} y^{2k_0} \mu(ds, dy) \right)^{r/2k_0} \right] + C \sum_{i=1}^{k_0} T^{r/2^k} \right).
\]

Using that \(|u + v|^\rho \leq |u|^\rho + |v|^\rho\) when \( \rho \leq 1 \), we have

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left( \int_0^t \int_\mathbb{R} y^{2k_0} \mu(ds, dy) \right)^{r/2k_0} \right] \leq \mathbb{E} \left[ \left( \int_0^T \int_\mathbb{R} y^r \mu(ds, dy) \right)^{r/2k_0} \right] \leq C_T,
\]

and the result follows.

(ii). For (12), when \( r \geq 2 \), we obtain by a similar approach:

\[
\mathbb{E} [\|\sigma_t - \sigma_s\|^r | \mathcal{F}_s] \leq C_T(|t-s|^r + |t-s|^{r/2} + |t-s|),
\]

where \( C_T \) is a deterministic constant. This yields the result when \( r \geq 2 \). When \( r < 2 \) the result follows from the Jensen inequality. Let us prove (13). If \( 0 < q < 2 \), using Jensen inequality and the concavity of the map \( x \mapsto x^{q/2} \), we have

\[
\mathbb{E} \left[ \left( \int_s^t \sigma_u dW_u \right)^q \right] \leq \mathbb{E} \left[ \int_s^t \sigma_u dW_u \right]^{q/2} \leq \left( \int_s^t \mathbb{E} \left[ \sup_{s \leq u \leq t} \sigma_u^2 \right] du \right)^{q/2} \leq C(t-s)^{q/2},
\]

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owing to (11). When \( q \geq 2 \), we first derive from Burkholder-Davis-Gundy inequality that

\[
E \left[ \left| \int_s^t \sigma_u dW_u \right|^q \right] \leq C E \left[ \left| \int_s^t \sigma_u^2 du \right|^{q/2} \right].
\]

Then, Jensen inequality and the convexity of the map \( x \mapsto x^{q/2} \) yield

\[
\left( \frac{1}{t-s} \int_s^t \sigma_u^2 du \right)^{q/2} \leq \frac{1}{t-s} \int_s^t \sigma_u^q du,
\]

Thus,

\[
E \left[ \left| \int_s^t \sigma_u^2 du \right|^{q/2} \right] \leq (t-s)^{q/2-1} \int_s^t E \left[ \sigma_u^q \right] du,
\]

and (13) again follows from (11).

Finally, let us prove (14). With similar arguments as previously, we obtain:

\[
E \left[ \left| \int_s^t (\sigma_u - \sigma_s) dW_u \right|^q \right] \leq C \begin{cases} 
\int_s^t E \left[ |\sigma_u - \sigma_s|^2 \right] du \left( t-s \right)^{q/2-1} \int_s^t E \left[ |\sigma_u - \sigma_s|^q \right] du & \text{if } q \in (0, 2), \\
\left( t-s \right)^{q/2} \int_s^t E \left[ |\sigma_u - \sigma_s|^q \right] du & \text{if } q \geq 2,
\end{cases}
\]

and (14) follows from (12).

**Acknowledgements.** We are deeply grateful to the listeners of our presentations and to Jean Jacod for their valuable advices.

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