Quasiparticles in the multicomponent Zhang-Hansson-Kivelson model

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We study the vortex solutions in a multicomponent Zhang-Hansson-Kivelson model for the fractional quantum Hall effect, at the self-dual point. Vortices with minimal free energy represent Laughlin quasiholes. We find at least two classes of solutions, distinguished by their global invariance, or by the number of conserved charges.

I. INTRODUCTION

Studying properties of the quantum Hall systems requires to know the properties of the elementary excitations. There exist several approaches to the question of quasiparticles in quantum Hall effect. One has as starting point the Laughlin wave function [1]. On the other hand, low energy excitations located at the edges can be described by means of conformal field theory [2]. As for the excitations in the bulk, they can be obtained in field theories with Chern-Simons term. A particularly useful effective theory for the bulk, inspired by the Landau-Ginsburg theory of the superconductivity, is the so-called Zhang-Hansson-Kivelson (ZHK) model [3]. It consists of a complex scalar field coupled to a statistical gauge field, which performs the statistical transmutation from bosons to fermions. Similarly to the Landau-Ginsburg model for the superconductivity [4], the ZHK model shows vortex solutions, vortices being identified to the Laughlin quasiholes, since they correspond to a depletion of the electron density. The vortices carry (fractional) electric charge, as well as statistical flux. A pedagogical introduction to Chern-Simons theories, as well as to vortex solutions in field theories with Chern-Simons term can be found in ref. [5].

In this letter, we study the vortex solutions in the situation where the bosonic field has p components. The relevant filling fractions are of the type \( \nu = p/(p\beta + 1) \), where \( \beta = 2n \) is an even integer (the number of fluxes attached to the electrons in the Jain construction). Let us point out that these filling fractions do not completely characterize the quantum Hall fluid; for example, \( p = 2s \) even integer can correspond to the spin polarised fractional quantum Hall effect, or to the spin singlet fractional quantum Hall effect [6].

Pursuing the analogy with the Landau-Ginsburg theory for the superconductors [4,5], we minimize the free energy at the self-dual point, where vortices do not interact. This point has also the merit that the minimum of the free energy is solution of first order differential equations [6]. For superconductors, the dual point correspond to the border between type I and type II superconductors [6], and above the dual point vortices interact repulsively.

When \( p > 1 \), we find that the vortex solutions have several types of solutions, depending on the monodromy conditions around the vortex center imposed to the electron fields. Different monodromy conditions lead to different number of independent electron densities, which ultimately lead to different number of conserved charges. Therefore, the bulk action of the ZHK model does not uniquely characterize a quantum Hall fluid, and different types of solutions correspond to different quantum Hall fluids associated to the same filling fraction.

For \( p = 2 \), a similar study was done by Ichinose and Sekiguchi [6], who found two type of excitations: the ones which carry \( su(2) \) charge were called merons, while the singlet superposition of merons was called vortex. Our first type of solutions correspond to the merons. The second type is similar to the vortex, in the sense that it carries only a \( u(1) \) charge; however, it carries only a fraction of the charge and the flux of the vortices in [6]. As pointed out in [6], turning on an interlayer tunneling (which break the symmetry from \( su(p) \) to \( u(1) \)) renders the meron excitations unstable. We therefore expect that the vortices carrying fractional flux are good excitations for this kind of system.

II. ZHK MODEL WITH P COMPONENTS

Throughout this paper we are going to use units in which \( e = \hbar = 1 \). Also, for simplicity we set \( e = 2m = 1 \). The ZHK model [3] couples a complex scalar field to a statistical gauge field with Chern-Simons dynamics and it was originally proposed to describe filling fraction of the type \( \nu = 1/(2n + 1) \). More complicated filling fractions can be described by a complex scalar field with \( p \) components, \( \psi_k \), coupled to \( p \) statistical gauge fields. The action of the multicomponent ZHK model is then:

\[
S = \sum_{k=0}^{p-1} \int d^3x \left( -|\tilde{D}^{(k)} \psi_k(z)|^2 + i\psi_k^*(x)D^{(k)}_\mu \psi_k(x) \right) -V[\psi] + S_{CS}[a],
\]

where the covariant derivatives \( D^{(k)}_\mu \) involve the external electromagnetic potential \( A_\mu \) and the statistical gauge field.
potentials $a_{\mu}^{(k)}$:

$$D_{\mu}^{(k)} = \partial_\mu + i(A_\mu + a_{\mu}^{(k)}) ,$$

$V[\psi]$ is a two body potential whose form will be precised later, and $S_{CS}[a]$ is a Chern-Simons term for the statistical gauge potentials. The Chern-Simons term is supposed to be of the form [3]

$$S_{CS}[a] = -\frac{1}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} a_{\mu}^{(k)} (K^{-1})_{\nu\lambda} \partial_\mu a_{\lambda}^{(l)} ,$$

with $\epsilon^{\mu\nu\lambda}$ being the Levi-Civita tensor and $K$ a $p \times p$ symmetric matrix with integer entries. We are interested here in the case $K_{kl} = \beta + \delta_{kl}$, when the filling factor is $\nu = \sum_k (K^{-1})_{kl} = p/(p\beta + 1)$.

The time components of the gauge potentials play the role of Lagrange multipliers, insuring the attachment of the statistical flux to the charge

$$\rho_k \equiv |\psi_k|^2 = -\frac{1}{2\pi} (K^{-1})_{kl} \epsilon^{ij} \partial_j \theta(\vec{x} - \vec{y}) \rho_l ,$$

where $\theta(\vec{x}) = \arctan(x_2/x_1)$ is the angle of the vector $\vec{x}$ in the plane. For a collection of $N_k$ point particles of density $\rho_k(\vec{x},t) = \sum_{a \in N_k} \delta(\vec{x} - \vec{x}_a^{(k)}(t))$, the gauge potentials $a_i^{(k)}$ could be removed via a singular gauge transformation. Under this transformation, the wave function for the particles acquire a phase factor

$$\varphi^{(k)}(\vec{x}) = \sum_{a,l} K_{kl} \theta(\vec{x} - \vec{x}_a^{(l)}) .$$

Since the diagonal entries of the matrix $K$ are odd, this singular gauge transformation turns bosons into fermions.

**III. STATIC SELF-DUAL SOLUTIONS**

We consider a potential $V[\phi]$ which has a minimum at $\rho_k = n/p$, $n$ being the total electron density

$$V[\psi] = \int d^2x \ d^2y \ \delta \rho_k(x) \ \delta \rho_l(y) \ V_{kl}(x-y) ,$$

where $\delta \rho_k(x) = \rho_k(x) - n/p$ is the deviation of the partial density from its average value. For simplicity we use a hard core potential. A self dual point exists when

$$V_{kl}(x-y) = 2\pi K_{kl} \delta(x-y) .$$

For a constant external magnetic field, $\epsilon_{ij} \partial_i A_j = B_{ext}$ and $A_0 = 0$, the static free energy functional is

$$\mathcal{F} = \int d^2x \sum_k \left( |\partial_i + i(A_i + a_i^{(k)})| \psi_k |^2 + 2\pi K_{kl} \delta \rho_k \delta \rho_l \right) ,$$

whose minimum is realized by the uniform configuration $a_i^{(k)} = -A_i$, $a_0^{(k)} = 0$, $\psi_k = \sqrt{n/p}$. This solution is consistent with the attachment of the statistical flux to the charge [3] if

$$n = \sum_k \rho_k = \frac{\nu}{2\pi} B_{ext} ,$$

that is, if the filling factor defined as the occupation of the Landau level equals exactly $\nu = p/(p\beta + 1)$. Small deviations from this filling fractions can be accommodated by creating locally non-uniform configurations. Let us now search such non-uniform configurations with finite energy. We transform the expression [3] using the Bogomol’nyi identity

$$|\tilde{D}\psi|^2 = |D_{\pm} \psi|^2 \pm B_{\psi}^2 \pm \epsilon_{ij} \partial_i J_j ,$$

where $D_{\pm} = D_1 \pm iD_2$ and $J_j = (\psi^* D_j \psi - \psi (D_j \psi^*)/2i$. After some algebra, and dropping the boundary contribution, we obtain that

$$\mathcal{F} = \int d^2x \sum_k \left( |D^{(k)} \psi_k|^2 + \frac{\nu}{2\pi p} B_{ext} B^{(k)} \right) ,$$

where $B^{(k)} = B_{ext} + b^{(k)}$ is the total magnetic field (external and statistical) seen by the component $k$. Therefore, the free energy is bounded by a multiple of the total magnetic flux. This bound is saturated by configurations with

$$D^{(k)} \psi_k = 0 ,$$

$$B^{(k)} = B_{ext} - 2\pi K_{kl} \rho_l .$$

To solve these equations, we parametrize $\psi_k$ by the modulus and the phase

$$\psi_k = \rho_k^{1/2} e^{i\chi_k} .$$

Then, the first equation in [3] become

$$A_i + a_i^{(k)} = \partial_i \chi_k + \frac{1}{2} \epsilon_{ij} \partial_j \ln \rho_k ,$$

The two equations [3] can be combined into a system of $p$ coupled equations for the partial densities $\rho_k$

$$\nabla^2 \ln \rho_k = 4\pi K_{kl} \left( \rho_l - n/p \right) .$$

Localized (finite energy) solutions have $\rho_k \to n/p$ when $|x| \to \infty$, as well as critical points $x_a$, where $\rho_k(x_a) = 0$
for some $k$. Let us consider a radially symmetric configuration with a critical point at $s_0 = 0$, and switch to polar coordinates $(r, \theta)$. This configuration is characterized by the vorticities $s_k$
\[
\chi_k(\theta + 2\pi) - \chi_k(\theta) = 2\pi s_k \quad \text{and} \quad \rho_k^{1/2}(r) \sim r^{s_k} \quad \text{at} \quad r \to 0.
\]
The magnetic flux carried by such a configuration is $\phi_k = 2\pi s_k$ (the flux quantum is equal to $2\pi$). The numbers $s_k$ are constrained via the monodromy conditions around the vortex center imposed to the fields $\psi_k$. When there is only one component, $p = 1$, the phase can only have a jump of $2\pi \times \text{integer}$ around the center of the vortex, so that the field $\psi$ is single valued. When the electron field has several components, more complicated monodromy conditions can appear. The two typical situations which can arise are the following:

- **i)** each component is single valued around the vortex center

  \[
  \psi_k(\theta + 2\pi) = \psi_k(\theta).
  \]

  The minimal configuration is then of the type $s_l = 1$ and $s_k = 0$ for $k \neq l$, so that the $l^{th}$ component carries one flux quantum and the others carry no flux. Vortex charges are given by $q_k = K_{kl}^{-1} = \beta/(p\beta + 1) - \delta_{kl}$.

- **ii)** the phase is matched between two components after a tour around the vortex center

  \[
  \psi_k(\theta + 2\pi) = \psi_k + \mod p(\theta).
  \]

  Then, there is only one independent density, $\rho_k = \rho_0$, and it satisfies the differential equation

  \[
  \nabla^2 \ln \rho_0 = 4\pi(p\beta + 1)(\rho_0 - n/p).
  \]

  The vortex carries charge $(q_k =) q_0 = -1/(p\beta + 1)$.

  All $p$ components see the same flux, corresponding to $s_0 = \ldots = s_{p-1} = 1/p$ and the center of the vortex is a branch point singularity of order $p$.

  Both above configurations have free energy $F = n/2\pi p$. In both cases, removing an electron of charge 1 is equivalent to creating $p\beta + 1$ quasi-holes, and it requires free energy $F = n/2\pi \nu$.

**IV. CHARGE/NEUTRAL MODES FACTORIZATION**

In the definition of the ZHK model \[\text{(1)},\] we have used the potentials $\alpha^{(k)}_\mu$. They couple in a simple manner to the fields $\psi_k$, but are mutually coupled by the Chern-Simons part. Alternatively, we can choose to put the Chern-Simons action into a diagonal form, by defining

\[
\alpha^{(k)}_\mu = \frac{1}{p}\sum_{l=0}^{p-1} e^{2\pi ikl/p} a^{(l)}_\mu.
\]

Since $\alpha^{(k)}_\mu = \alpha^{(p-k)}_\mu$, we obtain

\[
\mathcal{S}_{CS} = -\frac{\mu \nu \lambda}{4\pi} \int d^3 x \left( \nu \alpha^{(0)}_\mu \partial_\nu \alpha^{(0)}_\lambda + p \sum_{l=1}^{p-1} \alpha^{(k)}_\mu \partial_\nu \alpha^{(k)}_\lambda \right),
\]

Also, we redefine densities

\[
\tilde{\rho}_k = \sum_{l=0}^{p-1} e^{2\pi ikl/p} \rho_l.
\]

so that the flux to charge attachment is expressed in a diagonal form

\[
\tilde{\rho}_k = -\frac{c_k}{2\pi} \tilde{\psi}^{(k)} \quad \text{with} \quad c_k = \begin{cases} \nu, & k = 0 \\ p, & k \neq 0 \end{cases}
\]

and $\tilde{\psi}^{(k)} = \epsilon_{ij} \partial_i \alpha^{(j)}_\chi$. This decoupling reveals the special role played by the mode, $\alpha^{(0)}_\mu = 1/p \sum_k \alpha^{(k)}_\mu$, which will be called charge mode. The other modes $\alpha^{(k)}_\mu$, $k = 1, \ldots, p-1$ will be called neutral. This charge/neutral mode separation is very similar to the one employed by Balatsky and Fradkin \[\text{[1]}\] in the context of the singlet spin quantum Hall effect – with the exception that here the neutral modes are described by an abelian Chern-Simons theory.

In this new basis, the charges carried by the two types of vortices are

- **i)** electric charge $\tilde{q}_0 = -1/(p\beta + 1)$ and $\tilde{q}_k = e^{2\pi i kl/p}$ for the neutral modes $k \neq 0$.

- **ii)** electric charge $\tilde{q}_0 = -1/(p\beta + 1)$ and $\tilde{q}_k = 0$ for the neutral modes $k \neq 0$.

The two types of vortices carry the same electric charge $\tilde{q}_0$, but they have different neutral sectors: the first type carry a "spin" index, while the second does not carry any other charge except the electric one. Also, in this basis, the second type of vortex carries only one type of flux, with value $1/p$ of flux quantum, which explains the branch point singularity introduced at the center of the vortex.

An alternative point of view is that of the global invariance of $\psi_k$. In the first case, there is a $u(1)^{\otimes p}$ global invariance, corresponding to independent shifts of the phases $\chi_k$ (in fact, the invariance is extended to $u(1)^{\otimes su(p)}$ \[\text{(1)}\]). In the second case, the $p$ phases are not independent, and the global invariance is reduced to $u(1)$. Intermediate cases are possible; for example, for $p = 2s$ components, the global symmetry can be arranged to be $u(1) \otimes su(2)$, as is the case in the spin singlet fractional quantum Hall effect \[\text{[3]}\].
V. CONCLUSION

We conclude that the two different types of vortices described in section III correspond to two different types of quantum Hall fluids, both with the same value of the Hall conductivity but with different global invariance.

The conformal field theory on the edge should have the same conserved charges as the theory in the bulk. Therefore, we expect that the first class of solutions correspond to an edge theory with the full $su(p)$ symmetry \[10\]. In the second case, the global $su(p)$ symmetry should be broken, and it has been shown \[12\] that one way this can be realized is by introducing twist operators. They are able to remove completely the extra charges of the electron operators, as well as the contribution of the neutral modes from the correlation functions. The two types of conformal field theories, with and without the neutral modes, predict different behavior for the electron Green function \[13,14,8\], testable via tunneling conductance experiments \[13\].

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