Introduction:

Let $W$ be a 6 dimensional vector space over the complex numbers, equipped with a non-degenerate symplectic form $\omega$. Let $G_\omega$ be the Grassmannian of $\omega$-isotropic 3-dimensional vector subspaces of $W$. Considering the Plucker embedding, the intersection of $G_\omega$ with a generic codimension 2 linear subspace is the Mukai model of a smooth Fano manifold of dimension 4, genus 9 index 2 and Picard number 1. We will also note by $P_\omega$ the 13-dimensional projective space spanned by $G_\omega$ under its Plucker embedding.

Notations:

In all the paper, $B$ will be a general double hyperplane section of $G_\omega$. For a hyperplane $H$ of $P_\omega$, we define $\bar{H} = H \cap G_\omega$, and for any $u \in G_\omega$, the corresponding plane of $\mathbb{P}(W)$ will be noted $\pi_u$.

Abstract

On a genus 9 Fano variety, Mukai’s construction gives a natural rank 3 vector bundle, but curiously in dimension 4, another phenomena appears. In the first part of this article, we will explain how to construct on a Fano 4-fold of genus 9 (named $B$), a canonical set of four stable vector bundles of rank 2, and prove that they are rigid. Those bundles was already known to A. Iliev and K. Ranestad in [I-R] and the results of this section could be viewed as particular cases of the work of A. Kuznetsov (Cf [K]). We’d like here to show their consequences in the geometry of the 4-fold, and study Zak duality in this case.

Indeed, this “four-ality” (Cf [M]) is also present in the geometry of lines included in $B$, and also in the Chow ring of $B$. In section 2 we show that the variety of lines in $B$, is an hyperplane section of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$. This description is explicit and could also be interesting in terms of Freudenthal geometries. Then in section 3, we compute the Chow ring of $B$ which appears to have a rich structure in codimension 2.

The 4 bundles constructed can embed $B$ in a Grassmannian $G(2,6)$, and the link with the order one congruence discovered by E. Mezzetti and P de Poi in [M-dP] will be done in section 4. In particular we will prove that the generic fano variety of genus 9 and dimension 4 can be obtained by their construction, and explain the choices involved. We will also describe in this part the normalization of the non quadratically normal variety they constructed, and also its variety of plane cubics.
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1 Construction of rank 2 vector bundles on $B$

This part is devoted to the construction of a canonical set of 4 stable and rigid rank 2 vector bundles on $B$. Many results are done in a universal way in [K], but we detail this description to use it in next sections. Let’s first recall some classical geometric properties of $G_\omega$. (Cf [I]) The union of the tangent spaces to $G_\omega$ is a quartic hypersurface of $P_\omega$, so a general line of $P_\omega$ has naturally 4 marked points. Dually, as the variety $B$ is given by a pencil $L$ of hyperplane sections of $G_\omega$, there are in this pencil, 4 hyperplanes $H_1, \ldots, H_4$ tangent to $G_\omega$. Denoting by $u_i$ the contact point of $H_i$ with $G_\omega$, we will first construct a rank 2 sheaf on $H_i \cap G_\omega$ with singular locus $u_i$, and it’s restriction to $B$ will be the vector bundle.

1.1 Data associated to a tangent hyperplane section

Let $u \in G_\omega$, and $H$ be a general hyperplane tangent to $G_\omega$ at $u$. For any $v$ in $G_\omega$, denote by $\pi_v$ the corresponding projective subspace of $\mathbb{P}(W)$, and consider the hyperplane section of $G_\omega$:

$$\bar{H}_u = \{v \in G_\omega, \pi_v \cap \pi_u \neq \emptyset\}$$

It’s proved in [I] the following:

Lemma 1.1 There is a conic $C$ in $\pi_u$ such that $v \in H \cap \bar{H}_u \iff \pi_v \cap C \neq \emptyset$. For $H$ general containing the tangent space of $G_\omega$ at $u$, $C$ is smooth. Furthermore, $H \cap \bar{H}_u$ contains the tangent cone $T_u G_\omega \cap G_\omega = \{v \in G_\omega| \dim \pi_v \cap \pi_u \geq 1\}$ which is embedded in $P_\omega$ as a cone over a veronese surface.

Let $Z$ be the following incidence:

$$Z_H = \{(p, v) \in C \times (H \cap G_\omega)| p \in \pi_v\}$$

Identifying $C$ with $\mathbb{P}_1$, we denote by $q_1$ and $q_2$ the projections from $C \times G_\omega$ to $\mathbb{P}_1$ and to $G_\omega$, and by $L$ the $\text{SL}_2$-representation $H^0 \mathcal{O}_{\mathbb{P}_1}(1)$. Restricting the surjection $L \otimes \mathcal{O}_{\bar{H}} \to q_{2*} \mathcal{O}_{Z_H}(1,0)$ to the hyperplane section $\bar{H}$, we obtain:

Proposition 1.2 The sheaf $\mathcal{E}$ defined by the following exact sequence:

$$0 \to \mathcal{E} \to L \otimes \mathcal{O}_{\bar{H}} \to q_{2*} \mathcal{O}_{Z_H}(1,0) \to 0$$

is reflexive of rank 2, $c_1(\mathcal{E}) = -1$ and is locally free outside $u$. 

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Proof: From lemma [1.1] the support of $q_{2*}O_{Z_H}(1,0)$ is $H \cap \bar{H}_u$ so it is an hyperplane section of $H$, hence $E$ has rank 2 and $c_1(E) = -1$. Furthermore, for $v$ in $q_2(Z_H)$, the fiber of the restriction of $q_2$ to $Z_H$: $q_{2*}^{-1}Z_H(v)$ has length 1 if $v$ is not in $H_u$, it has length 2 if $v \in H_u - \{u\}$ and $q_{2*}^{-1}Z_H(u)$ is the curve $C$. As $Z_H$ and $\bar{H} - u$ are smooth, the sheaf $q_{2*}O_{Z_H}(1,0)$ has projective dimension 1 outside $u$, hence $E$ is locally free outside $u$. □

Denote by $S_iL$ the $SL_2$-representation $H^0(O_{\mathbb{P}_1}(i))$ and by $K$ and $Q$ the tautological bundles on $G_\omega$, such that the following sequence is exact.

$$0 \to K \to W \otimes O_{G_\omega} \to Q \to 0$$

**Proposition 1.3** For $i > 0$ we have $R^iq_{2*}O_{Z_H}(1,0) = 0$, and the resolution of $q_{2*}O_{Z_H}(1,0)$ in $G_\omega$ is given by the following exact sequence:

$$0 \to S_3L \otimes O_{G_\omega}(-1) \to L \otimes \bigwedge^2 Q^\vee \to L \otimes O_{G_\omega} \to q_{2*}O_{Z_H}(1,0) \to 0 \quad (1)$$

Proof: We consider the injection from $q_1^*(O_{\mathbb{P}_1}(-2))$ to $W \otimes O_{\mathbb{P}_1 \times G_\omega}$ given by the conic $C$. So the incidence $Z_H$ is the locus where the map from $q_1^*(O_{\mathbb{P}_1}(-2)) \oplus q_2^*K$ to $W \otimes O_{\mathbb{P}_1 \times G_\omega}$ is not injective, hence $Z_H$ is obtained in $\mathbb{P}_1 \times G_\omega$ as the zero locus of a section of the bundle $O_{\mathbb{P}_1}(2) \boxtimes Q$.

Let $\mathcal{K}$ be the Koszul complex $\bigwedge(O_{\mathbb{P}_1}(-2) \boxtimes Q^\vee)$ of this section. We obtain the proposition [1.3] from the Leray spectral sequence applied to $\mathcal{K}$ twisted by $O_{\mathbb{P}_1 \times G_\omega}(1,0)$. □

Furthermore, we deduce from Bott’s theorem on $G_\omega$ the following:

**Corollary 1.4** We have the following equality $L = H^0(O_{Z_H}(1,0)) = H^0(q_{2*}O_{Z_H}(1,0))$, and for $i > 0$, all the groups $H^i(O_{Z_H}(1,0))$ and $H^i(q_{2*}O_{Z_H}(1,0))$ are zero. For $i \geq 0$ all the groups $H^i(q_{2*}O_{Z_H}(1,-1))$ and $H^i(q_{2*}O_{Z_H}(1,-1))$ are zero.

Proof: We will prove that on the isotropic Grassmannian $G_\omega$, the bundles $\bigwedge Q^\vee$ and $(\bigwedge Q^\vee)(-1)$ are acyclic for $i \in \{1, 2, 3\}$. Indeed, with the notations of [3.1] 4.3.3 and 4.3.4, they correspond to the partitions $(0, 0, -1)$, $(0, -1, -1)$, $(-1, -1, -1)$, $(-1, -1, -2)$, $(-1, -2, -2)$, $(-2, -2, -2)$. Now recall that the half sum of positive roots is $\rho = (3, 2, 1)$, so $\alpha + \rho$ either contains a 0 or is $(2, 1, -1)$. So in all cases $\alpha + \rho$ is invariant by a signed permutation, and the sheaves are acyclic. The corollary is now a direct consequence of this acyclicity. □

**Corollary 1.5** The sheaves $\mathcal{E}$ and $\mathcal{E}(-1)$ are acyclic. The vector space $V = H^0(\mathcal{E}(1))$ has dimension 6 and $\forall i > 0, h^i(\mathcal{E}(1)) = 0$, and $\mathcal{E}(1)$ is generated by its global sections.

Proof: The acyclicity of $\mathcal{E}$ and $\mathcal{E}(-1)$ is a direct consequence of the definition of $\mathcal{E}$ and of the previous corollary.
To obtain the second assertion, we restrict the sequence \([\boxdot]\) to the hyperplane section \(\bar{H}\), so we obtain the following monad:\([\square]\)

\[
0 \to S_3L \otimes \mathcal{O}_{\bar{H}}(-1) \to L \otimes \bigwedge^2 \mathcal{Q}_{\bar{H}}^* \to \mathcal{E} \to 0
\]

whose cohomology is \(\text{Tor}^1(q_{2*}(\mathcal{O}_{Z_H}(1,0)), \mathcal{O}_{\bar{H}})\) which is equal to \(q_{2*}(\mathcal{O}_{Z_H}(1, -1))\) because \(Z_H \subset q_2^{-1}(H)\). Twisting this monad by \(\mathcal{O}_{\bar{H}}(1)\) we obtain that \(H^0(\mathcal{E}(1))\) is the quotient of \(L \otimes W\) by \(S_3L \oplus L\) because \(W = H^0(\mathcal{Q}_H)\). Furthermore, the right part of the monad gives a sujection from \(L \otimes \mathcal{Q}_{\bar{H}}\) to \(\mathcal{E}(1)\). But \(L \otimes \mathcal{Q}_{\bar{H}}\) is globally generated, so \(\mathcal{E}(1)\) is also generated by its local sections.

The vanishing of \(h^i(\mathcal{E}(1))\) for \(i > 0\) is a corollary of the vanishing of \(h^i(q_{2*}(\mathcal{O}_{Z_H}(1,0))\), \(h^i(\mathcal{Q}_{\bar{H}})\) and \(h^i(\mathcal{O}_{\bar{H}})\) for \(i > 0\). \(\square\)

We can remark that the two vector spaces \(V\) and \(W\) of dimension 6 have not the same role. More precisely, the conic \(C\) gives a marked subspace of \(W\) so that we have the following:

**Remark 1.6** The tangent hyperplane \(H\) gives canonically the \(SL_2\)-equivariant sequences:

\[
0 \to S_3L \to W \to S_2L \to 0 \text{ and } 0 \to L \to V \to S_3L \to 0
\]

### 1.2 The 4 rank 2 vector bundles on \(B\)

The pencil of hyperplanes defining \(B\) contains the 4 tangent hyperplanes \(H_i\), so we can apply the previous construction to construct a rank 2 sheaf \(\mathcal{E}_i\) on each of the \(\bar{H}_i\), and define by \(E_i\) the restriction of \(\mathcal{E}_i\) to \(B\). Because \(B\) is smooth, it doesn’t contain the contact points \(u_i\), so \(E_i\) is locally free on \(B\).

**Corollary 1.7** All the cohomolgy groups of the vector bundles \(E_i\) vanish. In particular, the rank 2 vector bundles \(E_i\) are stable. The vector space \(H^0(E_i(1))\) has dimension 6, and \(\forall j > 0, h^j(E_i(1)) = 0\). The bundles \(E_i(1)\) are generated by their global sections.

**Proof:** It’s a direct consequence of corollary \([\boxdot]\), because \(B\) is a hyperplane section of \(\bar{H}_i\). (Note that the stablility condition for a \(E_i\) is equivalent to \(h^0E_i = 0\)) \(\square\)

### 1.3 The restricted incidences

Now, for each of the 4 hyperplanes \(H_i\) containing \(B\) and tangent to \(G_\omega\) at some point \(u_i\), let \(C_i\) be the conic of the projective plane \(\pi_{u_i}\) constructed in \([\boxdot]\). Consider the restriction of the incidences \(Z_{H_i}\) to \(B\). In other words, let \(Z_i, Z'_i\) be:

\[
Z_i = \{(p, v) \in C_i \times B \mid p \in \pi_v\}, \quad Z'_i = \{(p, v) \in Z_i \mid \dim(\pi_v \cap \pi_{u_i}) > 0\}
\]

where \(q_1\) and \(q_2\) still denote the projections from \(C_i \times G_\omega\) to \(C_i\) and \(G_\omega\).

**Remark 1.8** Let \(p\) be a fixed point of \(C_i\). The scheme \(Z_{i,p} = q_2(q_1^{-1}(p) \cap Z_i)\) is a 2 dimensional irreducible quadric in \(P_\omega\). The restriction of \(q_2\) to \(Z'_i\) is a double cover of a veronese surface \(V_i = q_2(Z'_i)\).

\(^1\)A monad is complex exact at all terms different from the middle one.
Proof: In fact \( \{b \in G_\omega | p \in \pi_v \} \) is a smooth quadric of dimension 3 (Cf [I]), so it doesn’t contain planes. This scheme is included in \( H_i \), so \( Z_{i,p} \) is just an hyperplane section of this smooth quadric. It is also proved in [I] that \( \{v \in G_\omega | \dim \pi_{w_i} \cap \pi_v > 0 \} \) is a cone over a veronese surface of vertex \( u_i \). As \( u_i \notin B \), the surface \( V_i \) is the intersection of this cone with an hyperplane which doesn’t contain the vertex \( u_i \), so it’s a veronese surface. □

Notations:

We denote by \( \sigma_i \) the class of a point on \( C_i \), and \( h_3 \) the class of a hyperplane in \( \mathbb{P}_\omega \). (the plucker embedding of \( G_\omega \)).

Proposition 1.9 The incidence \( Z_i \) is a divisor of classe 2h in

\[
\Pi = \text{Proj}(\mathcal{O}_{\mathbb{P}_1}(2) \oplus S_2L \otimes \mathcal{O}_{\mathbb{P}_1})
\]

Furthermore we have \( h_3 \sim h + 2\sigma_i \), where \( h \sim \mathcal{O}_{\Pi}(1) \) and \( \sigma_i \) is also the class of a point on the base of the fibration \( \Pi \). The divisor \( Z_i' \) of \( Z_i \) is equivalent to \( h - 2\sigma_i \).

Proof: Denotes by \( e_i \) the image of the map from \( \mathcal{O}_{\mathbb{P}_1}(-2) \) to \( W \otimes \mathcal{O}_{\mathbb{P}_1} \) associated to \( C_i \). Choose an element \( \phi' \) of \( \wedge^3 W^\vee \) such that \( \ker \phi' \) gives an hyperplane section of \( G_\omega \) containing \( B \) and different from the \( \tilde{H}_i \). (i.e \( \phi'(u_i) \neq 0 \)). We can remark that the incidence \( Z_i \) is given over \( \mathbb{P}_1 \) by the isotropic 2-dimensional subspaces \( l \) of \( \frac{e_i^+}{e_i} \), such that \( \phi'(e_i \otimes \wedge^2 l) = 0 \), because the condition \( \phi_i(e_i \otimes \wedge^2 l) = 0 \) is already satisfied by the definition of \( C_i \) and lemma [I] (where \( \phi_i \) denotes a trilinear form of kernel \( H_i \)).

The bundle \( e_i^\perp \) is isomorphic to \( S_2L \otimes \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1) \) where the trivial factor \( S_2L \) correspond to the plane \( \pi_{u_i} \). So the bundle \( \frac{e_i^+}{e_i} \) is isomorphic to \( L(1) \oplus L(-1) \) where those factors are isotropic for the symplectic form induced by \( \omega \). We can take local basis \( s_0, s_1 \) and \( s_2, s_3 \) of each factors such that the form induced by \( \omega \) is \( p_{0,2} + p_{1,3} \) where \( p_{i,j} \) denotes the Plucker coordinates associated to the \( s_i \).

So the relative isotropic grassmannian \( G_\omega(2, \frac{e_i^+}{e_i}) \) is the intersection of \( G(2, \frac{e_i^+}{e_i}) \) with the subsheaf \( \mathcal{O}_{\mathbb{P}_1}(2) \oplus \mathcal{O}_{\mathbb{P}_1}(-2) \) and \( S_2L \otimes \mathcal{O}_{\mathbb{P}_1} \), where the factor \( \mathcal{O}_{\mathbb{P}_1}(2) \) still correspond to \( s_0 \wedge s_1 \).

Now we need to compute the kernel of the map \( e_i \otimes \wedge^2 (\frac{e_i^+}{e_i}) \otimes \mathcal{O}_{\mathbb{P}_1} \). But the assumption \( \phi'(u_i) \neq 0 \) proves that it is \( \mathcal{O}_{\mathbb{P}_1}(-2) \oplus L \otimes L \otimes \mathcal{O}_{\mathbb{P}_1} \).

So we have an exact sequence:

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}_1}(-2) \oplus S_2L \otimes \mathcal{O}_{\mathbb{P}_1} \rightarrow \wedge^3 (\frac{e_i^+}{e_i}) \otimes \mathcal{O}_{\mathbb{P}_1} \oplus e_i^\perp \rightarrow 0
\]

and \( Z_i \) is a divisor of class 2h in \( \text{Proj}(\mathcal{O}_{\mathbb{P}_1}(2) \oplus S_2L \otimes \mathcal{O}_{\mathbb{P}_1}) \). The relation \( h_3 \sim h + 2\sigma_i \) is given by the map \( e_i \otimes \wedge^2 (\frac{e_i^+}{e_i}) \rightarrow \wedge W \).

The divisor \( Z_i' \) of \( Z_i \) is locally given by the vanishing of the exterior product with \( s_0 \wedge s_1 \) so it equivalent to \( h - 2\sigma_i \).

We will now study the relation between the conormal bundle of \( Z_i \) in \( \mathbb{P}_1 \times B \) and the bundle \( E_i \).

\(^2\)As \( SL_2 \)-representation, we will identify \( L \) with its dual.
1.4 Deformations of $E_i$

Lemma 1.10 We have the following exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_1 \times B}(-\sigma_i - h_3) \rightarrow q_2^*E_i \rightarrow \mathcal{I}_{Z_i}(\sigma_i) \rightarrow q_2^*(R^1q_2_*\mathcal{I}_{Z_i})(-\sigma_i) \rightarrow 0$$

(where $q_1$ and $q_2$ denotes the projections from $\mathbb{P}_1 \times B$ to $\mathbb{P}_1$ and $B$)

Proof: From the resolution of the diagonal of $\mathbb{P}_1 \times \mathbb{P}_1$, we obtain the relative Beilinson’s spectral sequence:

$$E^1_{a,b} = \left(\bigwedge \omega_{q_2}(\sigma_i)\right) \otimes R^b q_2_*((1 + a)\sigma_i) \Longrightarrow \mathcal{I}_{Z_i}(\sigma_i)$$

By the definition of $E_i$ (Cf prop 1.2) we have $E_i = q_2^*(\mathcal{I}_{Z_i}(\sigma_i))$. Furthermore, the projection $q_2(Z_i)$ is an hyperplane section of $B$, so $q_2^*\mathcal{I}_{Z_i} = \mathcal{O}_B(-1)$. We can conclude, remarking that $R^1q_2_*((1 + a)\sigma_i) = 0$, because the restriction of $q_2$: $Z_i \xrightarrow{q_2|_{Z_i}} q_2(Z_i)$ has all its fibers of length at most 2.

NB: The support of $R^1q_2_*\mathcal{I}_{Z_i}$ is the natural scheme structure (Cf [G-P]) on the scheme of fibers of $q_2$ intersecting $Z_i$ in length 2 or more. It is the veronese surface $V_i = q_2(Z_i)$.

So the previous lemma can now be translated in the following:

Corollary 1.11 The scheme $q_2^{-1}(V_i) \cup Z_i$ is in $\mathbb{P}_1 \times B$ the zero locus of a section of the bundle $q_2^*E_i(1, 1)$.

This gives also a geometric description of the marked pencil of sections of $E_i(h_3)$ given by the natural inclusion $L \subset V$ found in remak 1.6. Indeed, if we fixe a point $p$ on $C_i$, the restriction to $q_1^{-1}(p)$ of the section obtained in corollary 1.11 gives with the notations of lemma 1.8 the following:

Corollary 1.12 For any point $p$ on the conic $C_i$, the vector bundle $E_i(h_3)$ has a section vanishing on $Z_{i,p} \cup V_i$.

We can now study the restriction of $E_i$ to $Z_i$.

Proposition 1.13 The restriction $E_i|_{Z_i}$ of the vector bundle $q_2^*E_i$ to $Z_i$ fits into the following exact sequence:

$$0 \rightarrow \mathcal{O}_{Z_i}(h_3 - 3\sigma_i) \rightarrow q_2^*E_i|_{Z_i}(h_3) \rightarrow \mathcal{O}_{Z_i}(3\sigma_i) \rightarrow 0$$

Proof: Fix a point $p$ on $C_i$, and consider the corresponding section of $E_i(h_3)$ constructed in corollary 1.12. Its pull back gives a section of $q_2^*E_i(h_3)$ vanishing on $q_2^{-1}(Z_{i,p} \cup V_i)$, so its restriction to $Z_i$ gives a section of $q_2^*E_i|_{Z_i}(h_3 - \sigma_i - Z_i')$. Now, using the computation of the class of $Z_i'$ in $Z_i$ made in proposition 1.9, namely that $\mathcal{O}_{Z_i}(Z_i') = \mathcal{O}_{Z_i}(h_3 - 4\sigma_i)$, it gives a section of $E_i|_{Z_i}(3\sigma_i)$. We have to prove that it is a non vanishing section. To obtain this, we compute the second Chern’s class of $E_i|_{Z_i}(3\sigma_i)$. We will show that its image in the Chow ring of $\mathbb{P}_1 \times B$ is zero. Denote by $\alpha_i$ the second Chern class of $E_i$. From the lemma 1.11, we obtain the class of $Z_i$ in $\mathbb{P}_1 \times B$: $[Z_i] = \alpha_i + h_3.\sigma_i - [V_i]$. So we can compute $[Z_i].c_2(E_i(3\sigma_i))$. It is $(\alpha_i + h_3.\sigma_i - [V_i]).(\alpha_i - 3h_3.\sigma_i)$, but we will compute in proposition 3.7 the Chow ring of $B$, and this class vanish.
Corollary 1.14 The vector bundles $E_i$ are rigid, in other words we have $\text{Ext}^1(E_i, E_i) = 0$.

Proof: From the corollary 1.11 we have an exact sequence on $\mathbb{P}^1 \times B$:

$$0 \to q^*_2 E_i(-\sigma_i) \to q^*_2 (E_i) \otimes q^*_2 (E_i)(h_3) \to q^*_2 E_i(h_3 + \sigma_i) \to (q^*_2 E_i(h_3 + \sigma_i))_{|Z_i \cup q^*_2(V_i)} \to 0$$

The bundle $q^*_2 E_i(-\sigma_i)$ is acyclic, and the corollary 1.7 gives $H^0(q^*_2 E_i(h_3 + \sigma_i)) = L \otimes V$ and $H^1(q^*_2 E_i(h_3 + \sigma_i)) = 0$. The liaison exact sequence twisted by $q^*_2(E_i(h_3 + \sigma))$ is:

$$0 \to q^*_2 E_i(h_3) \otimes O_{q^*_2(V_i)}(\sigma_i - Z'_i) \to q^*_2 E_i(h_3 + \sigma_i))_{|Z_i \cup q^*_2(V_i)} \to E_i|_{Z_i}(h_3 + \sigma_i) \to 0$$

As $\sigma_i - Z'_i$ have degree $-1$ along the fibers of $q_2 : q^*_2(V_i) \to V_i$, the bundle $q^*_2 E_i(h_3) \otimes O_{q^*_2(V_i)}(\sigma_i - Z'_i)$ is acyclic, so the cohomology of $q^*_2 E_i(h_3 + \sigma_i))_{|Z_i \cup q^*_2(V_i)}$ can be computed with its restriction to $Z_i$. The propositions 1.13 and 1.9 show that $H^0 E_i|_{Z_i}(h_3 + \sigma_i) = S_2 L \oplus S_2 L \oplus S_4 L$. In conclusion, we have the exact sequence:

$$0 \to \text{Hom}(E_i, E_i) \to L \otimes V \to S_2 L \oplus S_2 L \oplus S_4 L \to \text{Ext}^1(E_i, E_i) \to 0$$

By the corollary 1.7 the bundle $E_i$ is stable, so it is simple, in other words we have $\text{Hom}(E_i, E_i) = \mathbb{C}$, and the above exact sequence gives $\text{Ext}^1(E_i, E_i) = 0$. □

2 The variety of lines in $B$

Remark 2.1 Let $\delta$ be an isotropic line of $\mathbb{P}(W)$. The set of isotropic planes of $\delta^\perp$ containing $\delta$ form a line in $G_\omega(3, 6)$, and all the line in $G_\omega(3, 6)$ are of this type for a unique element of $G_\omega(2, 6)$. In other words, the variety of lines in $G_\omega(3, 6)$ is naturally $G_\omega(2, 6)$.

Notations:

A point of $G_\omega(2, 6)$ will be denoted by a minuscule letter, and the corresponding line in $G_\omega(3, 6)$ by the majuscule letter. The variety of lines included in $B$ will be noted $F_B$. Denote by $I$ the incidence point/line in $B$. In other words:

$$I = \text{Proj}((K_2^\perp)^{d \otimes h_2}) = \{(\delta, p) \in F_B \times B | p \in \Delta \} \subset G_\omega(2, 6) \times G_\omega(3, 6)$$

The projections from $I$ to $F_B$ and $B$ will be denoted by $p_1$ and $p_2$.

2.1 A morphism from $F_B$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Each of the 4 conics $C_i$ will enable us to construct a morphism from $F_B$ to $\mathbb{P}^1$. We have the following geometric hint to expect at least a rational map: A general element $\delta$ of $F_B$ gives an isotropic 2-dimensional subspace $L_\delta$ of $W$. In general, the projectivisation of $L_\delta^\perp$ meets the plane containing $C_i$ in a point $p$. There is at least an element $m$ of $\Delta$ such that $p \in \pi_m$, so $m \in \Delta \subset B \subset H_i$. Now, the definition of $H_i$ and lemma 1.1 prove that $p$ must be on $C_i$.

But to show that it is everywhere defined, we will use the vector bundle $E_i$. We start by constructing line bundles on $F_B$. 

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Remark 2.2 Any line $\Delta$ included in the hyperplane section $\tilde{H}_{u_i} = q_2(Z_i)$ of $B$ intersect the veronese surface $V_i$. Furthermore, any such line is in a quadric $Z_{i,p_i}$ for a unique point $p_i$ of $C_i$. So the set $v_i = \{ \delta \in F_B | \Delta \subset \tilde{H}_{u_i} \}$ is a divisor in $F_B$.

Proof: As the line $\Delta$ is in $\tilde{H}_{u_i}$, we have from lemma [1.1] that for any $b \in \Delta$, the plane $\pi_b$ intersect the conic $C_i$. The line $\mathbb{P}(L_0)$ must intersect $C_i$ in some point $p_i$. Indeed, if it was not the case, this line would be orthogonal to $C_i$, so it would be in the plane $\pi_{u_i}$, but any line in this plane intersect $C_i$.

So the line $\Delta$ is in the quadric $Z_{i,p_i}$. Note that $\mathbb{P}(L_0) \cap C_i$ can’t contain another point, because it would imply that $\Delta \subset V_i$. Furthermore, the proposition [1.9] implies that $Z_{i,p_i} \cap V_i$ is a plane section of the quadric $Z_{i,p_i}$, so $\Delta$ intersect $V_i$ in a single point. \(\square\)

Remark 2.3 For any point $p_i$ of $C_i$, the scheme $p_2^{-1}(Z_{i,p_i})$ has several irreducible components of dimension 2, but some of these component are contracted by $p_1$ to a curve. Denote by $A_{i,p_i}$ the 2 dimensional part of $p_1(p_2^{-1}(Z_{i,p_i}))$.

Proof: The components of $p_2^{-1}(Z_{i,p_i})$ corresponding to the lines included in $Z_{i,p_i}$ are contracted to curves. \(\square\)

Proposition 2.4 The sheaf $p_1* p^*_2 E_i$ is a line bundle on $F_B$. There is a natural map $f_i$ from $H^0(\mathcal{O}_{C_i}(\sigma_i))^\vee \otimes \mathcal{O}_{F_B}$ to the dual bundle of $(p_1*p^*_2 E_i)$. The image of $f_i$ is also a line bundle on $F_B$, we will denote it by $\mathcal{O}_{F_B}(\alpha_i)$. By construction, for any $p_i \in C_i$, the divisor $A_{i,p_i}$ will be in the linear system $|\mathcal{O}_{F_B}(\alpha_i)|$.

Proof: By the corollary [1.7] the bundle $E_i$ is a quotient of $6\mathcal{O}_B(-1)$ and by definition [1.2] it is a subsheaf of $2\mathcal{O}_B$. So its restriction to any line $\Delta$ included in $B$ must be $\mathcal{O}_\Delta \oplus \mathcal{O}_\Delta(-1)$. So $R^1p_1*p^*_2 E_i = 0$ and $p_1*p^*_2 E_i$ is a line bundle. Denote this line bundle by $\mathcal{O}_{F_B}(-\alpha_i')$. Dualising and twisting the exact sequence defining $E_i$, we have the following exact sequence:

$$0 \rightarrow L \otimes \mathcal{O}_B(-2h_3) \rightarrow E_i(-h_3) \rightarrow \mathcal{L}_i \rightarrow 0$$

(2)

where $\mathcal{L}_i$ is supported on the hyperplane section $\tilde{H}_{u_i}$, and is singular along the veronese surface $V_i$. As the incidence $I$ is $\text{Proj}((\frac{K^*_i}{K_2})^\vee (h_2))$ (where $K_2$ is the tautological sub-bundle of $W \otimes \mathcal{O}_{G_{2,2W}}$), the relative dualising sheaf $\omega_{p_1}$ is $\mathcal{O}_I(2h_2 - 2h_3)$. So we have $R^1p_1*(p^*_2 E_i(-h_3)) = \mathcal{O}_{F_B}(\alpha_i' - 2h_2)$. So the base locus of this pencil of sections of $\mathcal{O}_{F_B}(\alpha_i')$ is the support of $R^1p_1*p^*_2(\mathcal{L}_i)$. We will now prove that this sheaf is a line bundle on $v_i$.

The morphism $p_1$ is projective of relative dimension 1. So, for any point $\delta$ of $F_B$ and any coherent sheaf $F$ on $B$, the fiber $(R^1p_1* p^*_2 F)_\delta$ is $H^1(F \otimes \mathcal{O}_\Delta)$, where $\Delta$ is the line in $B$ corresponding to $\delta$. The restriction of the sequence [2] to $\Delta$ gives the surjection:

$$2\mathcal{O}_\Delta(-2) \rightarrow \mathcal{O}_\Delta(-1) \otimes \mathcal{O}_\Delta(-2) \rightarrow \mathcal{L}_i \otimes \mathcal{O}_\Delta \rightarrow 0$$

(3)

When the line $\Delta$ is not in the hyperplane section $\tilde{H}_{u_i}$, the sheaf $\mathcal{L}_i \otimes \mathcal{O}_\Delta$ is supported by the point $\tilde{H}_{u_i} \cap \Delta$, so in this case we have $h^1(\mathcal{L}_i \otimes \mathcal{O}_\Delta) = 0$. Now, when the line $\Delta$ is in $\tilde{H}_{u_i}$, the sheaf $\mathcal{L}_i \otimes \mathcal{O}_\Delta$ has generic rank 1 because the veronese surface $V_i$ can’t contain
the line $\Delta$. We have proved in lemma 2.2 that $\Delta$ intersect $V_i$, hence for any element of $L$, the corresponding section of $E_i(h_3)$ vanishes on $\Delta$, so the map $2\mathcal{O}_\Delta(-2) \to \mathcal{O}_\Delta(-2)$ induced by the sequence (3) is zero, and for any $\delta$ in $V_i$, we have $h^1(L_i \otimes \mathcal{O}_\Delta) = 1$, and $R^1p_1^*(\mathcal{O})$ is a line bundle on $v_i$.

So we have proved that the base locus of $(p_1^*(\mathcal{O}))$ is the divisor $v_i$. In other words, the image of $f_i$ is the line bundle $\mathcal{O}_{F_B}(\alpha_i) = (p_1^*(\mathcal{O}))$ which is by construction a quotient of $L \otimes \mathcal{O}_{F_B}$. By definition $A_{i,p_i}$ is the closure of $\{\delta \in F_B | \text{length}(\Delta \cap Z_{i,p_i}) = 1\}$ which was identified set theoretically with an element of the linear system $|\alpha_i|$, so we conclude the proof with lemma : $\Box$

**Lemma 2.5** For a generic choice of a point $p_i$ on $C_i$, the support of the sheaf $R^1p_1^*(\mathcal{O})$ represent the class $\alpha_i$, and all its irreducible components are reduced.

**Proof:** First notice that the point $p_i$ on $C_i$, gives a section of $E_i(h_3)$, so an exact sequence:

$$0 \to \mathcal{O}_B(-2h_2) \to E_i(-h_3) \to I_{Z_{i,p_i} \cup V_i}(-h_3) \to 0$$

which gives a section of $\mathcal{O}_{F_B}(\alpha_i)$ vanishing on the support of the sheaf $R^1p_1^*(\mathcal{O})$. But this is the definition in [G-P] of the scheme structure on the set of lines included in $B$ and intersecting $Z_{i,p_i} \cup V_i$. So to show that this scheme structure is reduced on each component, we have to prove that $Z_{i,p_i}$ and $V_i$ are not in the ramification of the morphism: $p_2 : I \to B$. But a general point $m$ on $Z_{i,p_i} \cap V_i$ is the intersection of $Z_{i,p_i}$ and another quadric $Z_{i',p_i'}$, so there pass 4 distinct lines through $m$, and from remark 2.2 there are no other lines in $B$ through $m$. So $m$ is not in the ramification of the morphism $p_2 : I \to B$ because it is of degree 4. (Cf lemma 3.1). $\Box$

### 2.2 Description of the morphism

In the previous section, we have contructed 4 morphism $f_i$ from $F_B$ to $\mathbb{P}_1$. We will now prove that the morphism from $F_B$ to $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ is an embedding for a generic $B$, and that its image is an hyperplane section.

**Notations:**

For a point $p_i$ in the conic $C_i$, we denote by $C_B$ the affine cone over $B$. We consider:

$$F_{p_i} = \{\delta \in G_\omega(2,W) | \delta \cap p_i \in C_B\}$$

Unfortunately, we have to remark that $f_i^{-1}(p_i)$ is not exactly $F_{p_i} \cap F_B$:

**Remark 2.6** The intersection $F_{p_i} \cap F_B$ is equal to $G_\omega(2,p_i^\perp) \cap F_B$. It contains $f_i^{-1}(p_i)$ and residual curves corresponding to the lines included in the quadric $Z_{i,p_i}$.

**Proof:** If $\delta$ is already an element of $F_B$, then any isotropic plane of $\mathbb{P}(W)$ containing the line $\mathbb{P}(L_\delta)$ is an element of $B$ (Cf remark 2.1). For $\delta \in G_\omega(2,p_i^\perp)$, the plane $\pi_{\delta/p_i}$ is isotropic, so we have $\delta \cap p_i \in C_B$ and $G_\omega(2,p_i^\perp) \cap F_B = F_{p_i} \cap F_B$. Now, remark that in
independent and we have proved that $F$ smooth) but $p \subset \mathcal{G}$ have to prove that the residual curve of isotropic planes, the conditions conic must be smooth. To conclude that this conic represents the class of $\Delta$.

Nevertheless, we have the following:

**Lemma 2.7** For the generic double hyperplane section $B$ of $G_\omega(3, W)$, we can find points $p_i$ (resp $p_j$) in the conics $C_i$ (resp $C_j$) such that $F_{p_i} \cap F_{p_j}$ is a smooth conic in $G_\omega(2, W)$. Furthermore, this conic is in $F_B$ and represent the class $\alpha_i, \alpha_j$.

**Proof:** We can choose $p_i$ and $p_j$ respectively in $C_i$ and $C_j$ such that, $p_i$ is not in $p_j^\perp$. Remark first that this implies that the intersection $F_{p_i} \cap F_{p_j}$ is automatically included in $F_B$ because $p_i$ and $p_j$ are never in the same isotropic plane. This also implies that the isotropic Grassmannian $G_\omega(2, p_i^\perp \cap p_j^\perp)$ is a smooth 3-dimensional quadric. The intersection $F_{p_i} \cap F_{p_j}$ is given in this Grassmannian by the 2 conditions: $d \wedge p_i \in H_i$ and $d \wedge p_j \in H_i$ for an element $d$ of $G_\omega(2, p_i^\perp \cap p_j^\perp)$. Indeed, as $d \wedge p_i$ and $d \wedge p_j$ represent isotropic planes, the conditions $d \wedge p_i \in H_i$ and $d \wedge p_j \in H_j$ are automatically satisfied because $p_i \in C_i$ and $p_j \in C_j$ (cf lemma 1.1). Let $\delta$ be the intersection of $p_j^\perp$ and the 3 dimensional vector space $U_i$ represented by the contact point $u_i$ of $H_i$. The space $\delta$ is an element of $G_\omega(2, p_i^\perp \cap p_j^\perp)$ such that $p_i \wedge \delta$ is not in $H_j$ (because it is $u_i$, and $B$ is smooth) but $p_j \wedge \delta \in H_i$. So the two hyperplane sections are independent and we have proved that $F_{p_i} \cap F_{p_j}$ is a (may be singular) conic.

Note that from the genericity of $B$ we could assume also that $\mathbb{P}(p_j^\perp) \cap C_j = \{a_1, a_2\}$ and $\mathbb{P}(p_i^\perp) \cap C_i = \{b_1, b_2\}$ are 4 distinct points. According to lemma 1.1 the lines $(p_i, a_1), (p_i, a_2), (p_j, b_1), (p_j, b_2)$ are in $F_{p_i} \cap F_{p_j}$, but no three of those lines are in the same plane, so the conic $F_{p_i} \cap F_{p_j}$ contains 4 points with no trisecant line. Hence the conic must be smooth. To conclude that this conic represent the class $\alpha_i, \alpha_j$, we just have to prove that the residual curve of $F_{p_i} \cap F_B$ don’t intersect $F_{p_j}$. But if $\delta$ is such that $\Delta \subset Z_{i,p_i}$, the line $\mathbb{P}(L_\delta)$ contains the point $p_i$ which is not orthogonal to $p_j$, so $\delta \notin F_{p_j}$.

**Lemma 2.8** When $i, j, k$ are distinct, the morphism from $F_B$ to $C_i \times C_j \times C_k$ is dominant.

**Proof:** According to lemma 2.7 for a generic choice of $p_i \in C_i$ and $p_j \in C_j$ the subvariety $F_{p_i} \cap F_{p_j}$ of $F_B$ is a smooth conic, and we can also assume that the line $(p_i, p_j)$ doesn’t intersect $C_k$. Assume that the induced map from $F_{p_i} \cap F_{p_j}$ to $C_k$ is not dominant, then there is a point $p_k \in C_k$ such that $F_{p_i} \cap F_{p_j} \subset F_{p_k}$. So for any element $d$ of $F_{p_i} \cap F_{p_j}$ the corresponding line $D$ is in the plane $\mathbb{P}(<p_i,p_j,p_k>)$, and $p_k \notin (p_i,p_j)$, the vector space $<p_i,p_j,p_k>^\perp$ has dimension 3, and this contradicts the fact that $F_{p_i} \cap F_{p_j}$ is a smooth conic.

At this point, we need more details about the embedding of $F_B$ in $G_\omega(2, W)$:

**Notations:**

Still denote by $K_2$ the tautological rank 2 subsheaf of $W \otimes \mathcal{O}_{G_\omega(2, W)}$, and by $-h_2$ and $c_2$ its first and second Chern classes. As $G_\omega(2, W)$ is an hyperplane section of $G(2, W)$, we will do the computations in $G(2, W)$.
Remark 2.9 The Chow ring of $G(2,6)$ is
\[ \mathcal{O}[h_2, c_2]/(h_2^5 + 3h_2c_2^2 - 4h_2^3c_2 - h_2^4c_2 + 3h_2^2c_2^2 - c_2^3) \]

We have:

Lemma 2.10 The variety $F_B$ is obtained in $G\omega(2,W)$ as the zero locus of a section of the bundle $(\frac{K^\perp}{K_2^\perp})^\vee(h_2) \oplus (\frac{K_2^\perp}{K_2^\perp})^\vee(h_2)$. For a general choice of $B$, $F_B$ is smooth with $\omega_{F_B} = \mathcal{O}_{F_B}(-h_2)$. The class of $F_B$ in the chow ring of $G_\omega(2,W)$ is $4(h_2^3 - c_2)^2$, and $F_B$ has degree 24 in $G_\omega(2,W)$. In the Chow ring of $F_B$, we have the extra relation: $h_2^2 = 3c_2$.

Proof: To compute Chern classes, we just have to remark that $K_2^\perp$ is isomorphic to the dual of the tautological quotient. The vanishing locus claim is a consequence of the fact that $B$ is a double hyperplane section of $G_\omega(3,W)$, and that we have from the definition of $I$ the equality: $p_1^*(p_2^*\mathcal{O}_{G_\omega(3,W)}(h_3)) = (\frac{K_2^\perp}{K_2^\perp})^\vee(h_2)$. (with the notations introduced in \[2\]). So the choice of a generic 2-dimensional subspace of $H^0(\mathcal{O}_{G_\omega(3,W)}(h_3)))$ correspond to the choice of a generic section of $(\frac{K_2^\perp}{K_2^\perp})^\vee(h_2) \oplus (\frac{K_2^\perp}{K_2^\perp})^\vee(h_2)$. Hence, for a general choice of $B$, $F_B$ will be smooth because $K_2^\perp(h_2)$ is globally generated, and so is $\frac{K_2^\perp}{K_2^\perp}(h_2) \simeq (\frac{K_2^\perp}{K_2^\perp})^\vee(h_2)$. We can now conlude with the computations using the remark 2.9

Lemma 2.11 In the Chow ring of $F_B$ we have $\alpha_i^2 = 0$, $\alpha_i, \alpha_j, h_2 = 2$ and $\alpha_i, \alpha_j, \alpha_k = 1$ when $i, j, k$ are distinct. (where $h_2$ is the hyperplane section of $F_B$)

Proof: In proposition 2.4 we already proved that $\alpha_i^2 = 0$. We have proved in the lemma 2.8 that for a generic choice of $p_i, p_j, p_k$ the intersection $F_{p_i} \cap F_{p_j} \cap F_{p_k}$ is not empty. Furthermore, this intersection is included in the smooth conic $F_{p_i} \cap F_{p_j}$ and the line $G_\omega(2,2p_i^\perp \cap p_j^\perp \cap p_k^\perp)$. So it must be a point because the intersection of a smooth conic and a line in the 3-dimensional smooth quadric $G_\omega(2,2p_i^\perp \cap p_j^\perp)$ must be empty or a point. So $\alpha_i, \alpha_j, \alpha_k = 1$.

Lemma 2.12 The hyperplane section $h_2$ of $F_B$ is linearly equivalent to $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$.

Proof: We deduce from the lemma 2.11 that the image of the morphism from $F_B$ to $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ is a divisor $\bar{F}$ of class $(1,1,1,1)$. Furthermore, the lemmas 2.8 and 2.11 imply that $\psi : F_B \rightarrow \bar{F}$ is a birational morphism.

Now we should notice that $\bar{F}$ is normal because the restriction of $\psi : F_B \rightarrow \bar{F}$ to a double hyperplane section of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ is not ramified. Indeed, from the lemma 2.11 we have: $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^3 = 24 = h_2((\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^2)$, so the restriction of $\psi$ to a general double hyperplane section of $\bar{F}$ is not ramified.

From the normality of $\bar{F}$, we obtain a map $\psi^*(\omega_{\bar{F}}) \rightarrow \omega_{F_B}$. This implies that the divisor $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - h_2$ of $F_B$ is effective in $\bar{F}$. To obtain the result, we will show that its intersection with $h_2$ is zero. Remarking from lemma 2.10 that on $F_B$ we have $h_2^2 = 24$, we will end the proof after the following lemma:
Lemma 2.13 The image of the class $\alpha_i$ in the Chow ring of $G_\omega(2, W)$ is $h_2.c_2.2(h_2^2-c_2)$, and in the Chow ring of $F_B$ we have $\alpha_i.h_2^2 = 6$.

Proof: First, chose an hyperplane $H'$ such that $B = G_\omega(3, W) \cap H' \cap H_i$. Let $F_{H'}$ and $F_{H_i}$ be the variety of lines included respectively in $H'$ and $H_i$, and denote by $Y$ the Grassmannian $G_\omega(2, p_i^+)$. We proved in remark 2.6 that $\alpha_i$ is reprented by the 2-dimensional part of $Y \cap F_B$ for some point $p_i$ on the conic $C_i$. From the definition of $B$, we have $F_B = F_{H'} \cap F_{H_i}$. The variety $F_{H'}$ and $F_{H_i}$ represent the class $c_2((\frac{K_+}{K_2})^v(h_2))$ in the Chow ring of $G_\omega(2, W)$, but the intersection $Y \cap F_{H_i}$ has codimension one in $Y$. Indeed, for any $l$ in $Y$ the point $p_i \wedge l$ is already in $H_i$. The restriction of the sheaf $(K_2)^\perp$ to $Y$ has a section given by $p_i$, and the intersection $G_\omega(2, p_i^+) \cap F_{H_i}$ is the vanishing locus of a section of $\frac{K_+^v}{p_i+K_2^v}(h_2)$. Remarking that $Y$ represent the class $c_2$ in $G_\omega(2, W)$, we conclude that the class $\alpha_i$ is equivalent to $h_2.c_2.2((\frac{K_+}{K_2})^v(h_2))$ in the Chow ring of $G_\omega(2, W)$.

Using the remark 2.9 we compute that $h_2^2.\alpha_i = 6$. This ends the proof of lemma 2.13 and also of lemma 2.12. □

In conclusion, we have the following:

Theorem 2.14 The variety of lines included in a generic double hyperplane section of $G_\omega(3, 6)$ is an hyperplane section of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$.

Proof: In lemma 2.12 we proved that $h_2 \sim \sum_{i=1}^4 \alpha_i$, but we have $h^0(\mathcal{O}_F(1, 1, 1, 1)) = 15$ and $h^0(\mathcal{O}_{G_\omega(2, W)}(h_2)) = 14$, so the morphism $\psi : F_B \to \bar{F}$ is an isomorphism. □

3 The Chow ring of $B$

We keep the notations of section 2. We will study here the Chow ring of $B$. The 4-dimensional variety $B$ is a generic double hyperplane section of $G_\omega(3, 6)$. Its Chow ring in codimension 2 will appear to be surprisingly bigger than the codimension 2 part of the Chow ring of $G_\omega(3, 6)$. To study it, we will first compare it to the Chow ring of $I$ via the projection $p_2 : I \to B$.

Lemma 3.1 The variety $B$ contains 16 quadric cones of dimension 2. For any point $m$ different from the 16 vertex of those cones, the fiber $p_2^{-1}(\{m\})$ has length 4.

Proof: Let $m$ be any point of $B$. The lines containing $m$ and included in $B$ are the lines containing $m$ and included in the tangent cone of $B$. The tangent cone to $G_\omega(3, W)$ is a cone over a veronese surface. The smoothness of $B$ at $m$ implies that the tangent cone of $B$ at $m$ is a cone over the intersection of a veronese surface by a $\mathbb{P}_3$. So it is either 4 lines or a 2 dimensional quadric cone with smooth basis.

Now, let $\Gamma$ be such a cone included in $B$, we can deduce from lemma 3.5 of [1] that there is a point $e$ of $\mathbb{P}(W)$ included in all the planes $\pi_u$ for $u$ in $\Gamma$. So $\Gamma$ is the intersection of the 3-dimensional quadric $Q_e = \{e \wedge l \mid l \in G_\omega(2, \frac{w}{u})\}$ with 2 hyperplanes containing $B$. This proves that one of them contains $Q_e$. We deduce from the proposition 3.3 of
that this hyperplane must be one of the $H_i$ defined in section 1 and that $e$ is on one of the four conics $C_i$. But the description of the incidences $Z_i$ in proposition 1.9 proves

that there are only 4 cones for each $C_i$. □

Notations:

In the sequel, we will denote the Chow ring of a variety $X$ by $A_X$. Let $c'_1 = h_3$, $c'_2$ and $c'_3$ be the Chern classes of the tautological quotient $Q_3$ of $W \otimes O_{G_w(3,W)}$. Denote by $h'_3$ the class $p^*_3 h_3$ in $A_I$, and let $a_i$ be the second Chern class of the bundle $E_i$.

First recall from the section 2 and the lemma 2.10 that the Chow ring of the incidence $I$ is:

**Lemma 3.2** The Chow ring of $I$ is

$$A_I = \frac{A_{F_B}[h'_3]}{(h'_3^2 - 2 h_2 h'_3 + \frac{1}{2} h_2^2)}$$

So we have to get more informations on $F_B$.

**Lemma 3.3** For each choice of $i$, the variety $F_B$ can be identified with the blow up of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ in an elliptic sextic curve, where $v_i$ is the exceptional divisor.

The choice of one of the $\mathbb{P}_1$ (ie: the marking of one of the conics $C_i$) in theorem 2.14 enable us to consider the variety $F_B$ as an incidence:

$$\{(x, h)| x \in \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1, h \in L, x \in h\}$$

where $L$ is a marked subspace of $H^0(O_{\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1}(1, 1, 1))$. In other words, for each $i$, the variety $F_B$ with le line bundle $h_2 - a_i$ is identified with the blow up of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ in an elliptic sextic curve. By construction $|a_i|$ is the marked pencil of hyperplanes.

To show that $v_i$ is the exceptional divisor, we need to prove that $2a_i + v_i = h_2$. Take any subset $\{i, j, k\}$ of $\{1, 2, 3, 4\}$ of cardinal 3, and chose a point $p_j$ on $C_j$ and another one $p_k$ on $C_k$ such that $\mathbb{P}(p_j^* \cap p_k^* C)$ is empty. The divisor $v_i$ was constructed in proposition 2.4 as the set of lines included in the hyperplane section $H_{u_i}$ of $B$. So $\delta \in v_i$ implies that $\Delta \subset H_{u_i}$, which implies that the line $\mathbb{P}(L_{\delta})$ intersect $C_i$, so it can’t be in $\mathbb{P}(p_j^* \cap p_k^*)$, and we have $\alpha_j, \alpha_k, v_i = 0$. So set theoretically, $v_i$ is the exceptional divisor, but we proved in lemma 2.5 that $v_i$ is reduced. □

**Lemma 3.4** The projection $p_{1*}(p^*_2 a_i)$ of the second Chern class of $p^*_2 E_i$ to $A_{F_B}$ is $h_2 - a_i$. Furthermore, we have in $A_I$ the equality: $p^*_2 a_i = h_3'(h_2 - a_i) + h_2(2a_i - h_2)$

**Proof:** We know from the lemma 3.3 and proposition 2.4 that the first Chern class of $p_{1*}(p^*_2 E_i)$ is $-a_i - v_i = a_i - h_2$. So the first assertion is a direct consequence of Riemman-Roch-Grothendieck’s theorem because $(p^*_2 E_i)$ have no higher direct images by $p_1$. Indeed, we can compute in $A_I$ using the equality: $\omega_m = \mathcal{O}_f(2h_2 - 2h_3)$ obtained in 2 and it gives that the first Chern class of $p_{1*}(p^*_2 E_i)$ is $-p_{1*} p^*_2 a_i$.  

\[\]
The second assertion is obtained by the evaluation map: \( p_i^*p_1^*p_2^*E_i \to p_2^*E_i \). We can compute its cokernel as we done in \( \text{lemma 1.10} \) but the vanishing of \( R^1p_1^*(p_2^*E_i) \) gives the following exact sequence:

\[
0 \to p_i^*p_1^*p_2^*E_i \to p_2^*E_i \to \mathcal{O}_I(-h_3') \otimes p_i^*R^1p_1^*(p_2^*(E_i(-h_3))) \to 0
\]

so we have by relative duality the extention:

\[
0 \to \mathcal{O}_I(p_i^*\alpha_i - p_i^*h_2) \to p_2^*E_i \to \mathcal{O}_I(p_1^*h_2 - p_i^*\alpha_i - h_3') \to 0
\]

which gives the computation of \( p_2^*a_i \).

**Lemma 3.5** Let \( \gamma \in A_{2B}^1 \), then the class \( p_2^*\gamma \) can be written in \( A_1 \) by \( h_3', \gamma_0 + \gamma_1 \) with \( \gamma_i \) in \( A_{2B}^{i+1} \), and where \( \gamma_i \) is in the vector space generated by \( h_2^2\alpha_1, \ldots, h_2^2\alpha_4 \). More precisely, we have: \( 2h_2, \gamma_0 + \gamma_1 \in \mathbb{Q}.h_2^2 \)

**Proof:** We can first find classes \( \gamma_i \) in \( A_{2B}^{i+1} \) such that \( p_2^*\gamma = h_3', \gamma_0 + \gamma_1 \). Now remark that \( A_{2B}^1 \) is one dimensional, so \( h_3, \gamma \) is proportional to \( h_3^2 \). But from the relation \( h_3^2 = 2h_2, h_3' - \frac{4}{3} h_2^2 \), the class \( p_1,h_3' \) is proportional to \( h_2^2 \). So the class \( p_1,p_2^*\gamma \) is also in \( \mathbb{Q}.h_2^2 \), and we have \( 2h_2, \gamma_0 + \gamma_1 \in \mathbb{Q}.h_2^2 \). We can now conclude with the description of \( F_B \) in \( \text{lemma 3.3} \) that \( \gamma_1 \) and \( h_2 \) are in the vector space generated by \( \alpha_1, \ldots, \alpha_4 \), which gives the lemma.

**Lemma 3.6** The classes \( (a_1, a_2, a_3, a_4) \) form a basis of the vector space \( A_{2B}^2 \). We have in \( A_B \) the relation \( 2(a_1 + a_2 + a_3 + a_4) = 3h_3^2 \).

**Proof:** From the lemma \( \text{3.1} \), the map \( p_2^* : A_{2B}^2 \to A_2^2 \) is injective, so from the lemma \( \text{3.5} \), \( A_{2B}^2 \) is generated by the set of classes \( \{p_2^*(h_3', \alpha_1), p_2^*(h_2^2\alpha_i)\}_{i=1,4} \). As the picard group of \( B \) is generated by \( h_3 \), all the classes \( p_2^*(h_3', \gamma_0) \) are proportional to \( h_3^2 \). Now, we use the relation \( \frac{1}{3}h_3^2 = 2h_2, h_3' - h_3'^2 \) to eliminate \( h_3^2 \) in the expression of \( p_2^*a_i \) found in lemma \( \text{3.4} \). So, we obtain that \( a_i \) is in the affine space \( \frac{1}{2} p_2^*(h_2^2\alpha_i) + \mathbb{Q}.h_3^2 \). So we have proved that \( A_{2B}^2 \) is generated by \( h_3^2, a_1, a_2, a_3, a_4 \). Furthermore, as we have \( p_1,p_2^*a_i = h_2 - \alpha_i \), which is a free family in \( A_{2B}^1 \), the family \( (a_1, \ldots, a_4) \) is free in \( A_{2B}^2 \).

To obtain the relation with \( h_3^2 \), we substitute the expression of \( p_2^*a_i \) of lemma \( \text{3.4} \) in the relation found in lemma \( \text{2.12} \). We eliminate \( h_3', h_2 \) with the relation \( \frac{4}{3}h_2^2 + h_3'^2 = 2h_2.h_3' \), and we obtain: \( \sum_{i=1}^4 p_2^*a_i = \frac{3}{4}h_3^2 \).

So we are now ready to describe the Chow ring of \( B \).

**Proposition 3.7** The Chow ring of \( B \) is \( \mathbb{Q}[h_3, a_1, a_2, a_3, a_4]/\mathcal{I} \) where \( \mathcal{I} \) is generated by \( 3h_3^2 - 2 \sum_{i=1}^4 a_i, (8h_3.a_i - 3h_3^3)i\in \{1, \ldots, 4\}, (8.a_i.a_j - h_3^4)i\neq j, (i,j)\in \{1, \ldots, 4\}^2 \). (The class of a point is \( \frac{a_i.a_j}{2} \), the class of the veronese \( V_i \) is \( 2a_i - \frac{1}{3}h_3^2 \), and \( [V_i]^2 \) is 4 points).

**Proof:** As \( A_{2B}^2 \) is known to be one dimensional for \( i \in \{0, 1, 3, 4\} \), we just need to compute the relations, and it can be done by calculating the degrees, because we have found the structure of \( A_{2B}^2 \) in \( \text{lemma 3.6} \). The relations \( (8h_3.a_i - 3h_3^3) \) are consequences of the degree of \( G_{\omega}(3, 6) \) (ie 16) and the fact that \( a_i \) can be represented by the union of a quadric \( Z_{i,p_i} \) and the veronese surface \( V_i \).
Now remark that the class of $V_i$ is in the vector space generated by $a_i$ and $h_3^2$. Indeed, we have $p^*_3[V_i] = h_3^2, v_i + \gamma_1$ and the lemmas 3.3 and 3.5 give $v_i = h_2 - 2a_i$ and $\gamma_1 \in h_3^2, Q + h_2a_i, Q$, so $[V_i]$ is in the vector space generated by $h_3^2$ and $a_i$, and then $[Z_{i,p_i}]$ also. Computing their degree, we have: $[V_i] = 2a_i - \frac{1}{2}h_3^2$ and $[Z_{i,p_i}] = \frac{1}{2}h_3^2 - a_i$. The last relations are then consequences of the fact that we have: $[V_i], [V_3] = 0$ for $i \neq j$. □

NB:

It could be useful to have the link with the ring of $G_\omega(3,6)$ and the Chern classes of $Q_3$, so we state the following:

**Remark 3.8** Still denote by $c'_i$ the Chern classes of the tautological quotient $Q_3$. The Chow ring of $G_\omega(3,6)$ is

\[ \mathbb{Q}[c'_1, c'_2, c'_3]/(c'_1^2, c'_2^2 - 2c'_1c'_3, c'_1^2 - 2c'_2) \]

On $B$ we have the additional relation $c'_1c'_2 = 4c'_3$. In particular the rank of $A^2_{G_\omega(3,6)}$ is only one.

**Proof:** The Chow ring of $G_\omega(3,6)$ is given by

\[ \mathbb{Q}[x, y, z]/((xyz)^2, x^2y^2 + y^2z^2 + z^2x^2, x^2 + y^2 + z^2) \]

where $c'_1 = x + y + z$, $c'_2 = xy + yz + zx$ and $c'_3 = xyz$. □

4 Application to quadratic normality

In this part, we explain the link with the congruence of lines found in [M-dP]. They started with the intersection $\Gamma$ of $G(2,6)$ by a very particular $\mathbb{P}_{11}$ (NB: it was proved in [M-MM], that the choice of this $\mathbb{P}_{11}$ is unique up to the action of $GL_6$). Then they chose a general quadric containing a fixed subscheme of $\Gamma$ (which contains the singular locus of $\Gamma$) to obtain a reducible intersection with $\Gamma$. Then they checked with Macaulay2 that one of these irreducible components is smooth of degree 16 and sectional genus 9.

Here we prove that the generic Fano 4-fold of genus 9 can be obtained by their construction, and that the choice we need to do is generically finite. More precisely, the choice of $\Gamma$ correspond to the choice of a tangent hyperplane $H$ to $G_\omega(3,6)$, and the choice of the quadric will correspond to the choice of a non zero element in $[O_H(h_3)]$.

**Proposition 4.1** For any choice of $i$, the bundle $E_i(h_3)$ gives an embedding of the Fano manifold $B$ in the Grassmannian $G(2,6)$ as the congruence of lines constructed in [M-dP] (Theorems 8,9,10).

**Proof:** Their description of the congruence is in terms of equations, so we need to make an adapted choice of coordinates to get the link. Choose a basis $w_0, \ldots, w_5$ of $W$ such that $\omega^i = w_0 \wedge w_3 + w_1 \wedge w_4 + w_2 \wedge w_5$. Let $A$ and $B$ be the vector spaces generated respectively by $w_0, w_1, w_2$ and $w_3, w_4, w_5$, in particular, the form $\omega$ gives an identification between $A^3$ and $B$. The decomposition $W = A \oplus B$ gives a decomposition of $\wedge^3 W$. So
we can represent an element of $\mathfrak{g}^3 W$ as in $[1]$ by $(a, X, Y, b)$, with $a \in \wedge^3 A$, $b \in \wedge^3 B$, $X \in \text{Hom}(A, B)$, $Y \in \text{Hom}(B, A)$. The equations of $G(3, 6)$ are as: $\wedge^2 X = aY$, $\wedge^2 Y = bX$, $YX = abI_3$, and to obtain $G_\omega(3, 6)$ we had the linear relations $X = tX$ and $Y = tY$, so we take $X = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_5 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$ and $Y = \begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_5 & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix}$. (Cf $[1]$).

Now we need to choose an hyperplane $H$ as in section 1.1. So we consider the hyperplane $y_5 = y_2$. It is tangent to $G_\omega(3, 6)$ in $w_0 \wedge w_1 \wedge w_2$ and contains $w_3 \wedge w_4 \wedge w_5$. Furthermore, the conic described in lemma 3.7 is parametrized by $\lambda^2 w_0 + \lambda \mu w_1 + \mu^2 w_2$, so the incidence $Z_H$ is given in $\mathbb{P}_1 \times H$ by the equations:

$$a = 0, X \left( \begin{array}{c} \lambda^2 \\ \mu^2 \end{array} \right) = \begin{pmatrix} 0 & \lambda \\ 0 & -\mu \end{pmatrix}.$$ 

The 6 dimensional vector space $H^0(\mathcal{I}_{Z_H}(1, 1))$ is generated by

$$(-a, \lambda, -a, \mu, \lambda y_1 - \mu y_0, \lambda y_2 - \mu y_1, \lambda y_3 - \mu y_2, \lambda y_4 - \mu y_3)$$

So, the sheaf $\mathcal{E}$ constructed in proposition 1.2 is the image of the map:

$$6\mathcal{O}_H(-1) \xrightarrow{\begin{pmatrix} a & 0 & y_0 & y_1 & y_2 & y_3 \\ 0 & a & -y_0 & -y_1 & -y_2 & -y_3 \end{pmatrix}} 2\mathcal{O}_H$$

which is exactly the map of $[M-dP]$ th 9. Now, a generic hyperplane section of $G_\omega(3, 6)$ gives a linear relation between $a, (x_i), (y_i), b$ which is exactly the relation 29 of $[M-dP]$, which gives the identification with their congruence of lines.

So we can remark that the computations they made in affine charts of $G(2, 6)$ can be globally done in $G_\omega(3, 6)$.

### 4.1 Geometry of the focal locus of Mezzetti-de Poi’s congruence

Here we chose one of the 4 bundles, say $E_1$. Denote by $h$ the bundle $\mathcal{O}_{\text{Proj}(E_1(h_3))}(1)$. We proved in lemma 1.7 that the linear system $|h|$ gives a morphism from $\text{Proj}(E_1(h_3))$ to $\mathbb{P}_5$.

**Notations:**

Denote by $r$ the above projection from $\text{Proj}(E_1(h_3))$ to $\mathbb{P}_5$, and by $p_B$ the projection from $\text{Proj}(E_1(h_3))$ to its basis $B$.

Following $[M-dP]$, the focal locus of Mezzetti-de Poi’s congruence is defined like this:

**Lemma 4.2** The morphism from $r : \text{Proj}(E_1(h_3)) \rightarrow \mathbb{P}_5$ is birational. The class of the exceptional divisor $\mathcal{R}$ in $\text{Proj}(E_1(h_3))$ is $4h - h_3$.

**Proof:** The degree of $r$ is given by the length of the degeneracy locus of a generic map $5\mathcal{O}_B \rightarrow E_1(h_3)$, so it is the fourth segre class of $E_1$, which is the class of one point from proposition 3.7. So $r$ is birational, and the class of the exceptional divisor follows. $\square$

The manifold $B$ gives the family $LB = \{\mathbb{P}(E_{1,u})|u \in B\}$ of lines in $\mathbb{P}_5$. From the lemma 4.2 those lines are quadrisecant to $X$. 

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Notations:

Denote by $X$ the focal locus of $B$, (ie $X = r(\mathcal{R})$). Recall from $[M-dP]$ that $X$ has dimension 3, degree 6, and is singular along a rational smooth cubic curve $C$.

From the lemma [4.2] any line of this family intersects $X$ in length 4 or is included in $X$. We can now describe easily the normalisation $\tilde{X}$ of $X$, and note that we have as in the Palatini case some kind of duality between $X$ and a family of plane cubics included in $X$, but here it breaks over the singular locus of $X$:

**Proposition 4.3** The focal locus $X$ of Mezzetti-de Poi’s congruence is a projection of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ from a line. The variety of pencils of lines belonging to $LB$ is $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ blown up in the elliptic curve of degree 6 which is the double cover of the cubic curve $C$.

**Proof:** We pullback the situation to the incidence point/lines of $B$ via the projection $p_2 : I \to B$. We still denote by $p_2$ the projection from $\text{Proj}(p_2^*(E_1(h_3)))$ to $\text{Proj}(E_1(h_3))$. In the proof of lemma [3.4] we noticed the extension (4), which gives a surjection:

$$p_2^*(E_1(h_3)) \to \mathcal{O}_I(p_1^*h_2 - p_1^*\alpha_1)$$

This surjection gives an embedding of $I$ in $\text{Proj}(p_2^*(E_1(h_3)))$. The restriction of $p_2^*(h)$ to $I$ is $p_1^*(h_2 - \alpha_1)$, but it is also $p_1^*(\alpha_2 + \alpha_3 + \alpha_4)$ by lemma [2.12] So the linear system $|p_2^*(h)|$ contracts the fibers of $p_1 : I \to F_B$, and the image of $I$ coincide with the image of $F_B$ by $\alpha_2 + \alpha_3 + \alpha_4$. In conclusion, the map $r$ contracts the divisor $p_2(I)$ to the projection of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ from a line. \qed

We can also obtain a description of the plane curves in $X$ related to this family of lines:

**Remark 4.4** Any point of $F_B$ gives a plane in $\mathbb{P}_5$ intersecting $X$ in a point and a plane cubic. Those plane cubics are all singular in a point of $C$. Only 12 lines included in $X$ are in $LB$.

**Proof:** A point $\delta \in F_B$ correspond to a line $\Delta$ included in $B$. We already proved in §2 that the restriction of $E_1$ to $\Delta$ is always $\mathcal{O}_\Delta \oplus \mathcal{O}_\Delta(-1)$, so the lemma [4.2] implies that the intersection of $\mathcal{R}$ with $p_2^{-1}(\Delta)$ is given by a section of $(S_4(E_1))(3h_3)$. Hence this intersection contains the exceptional divisor of $\mathbb{P}(\mathcal{O}_\Delta \oplus \mathcal{O}_\Delta(1))$, so the plane $r(p_2^{-1}(\Delta))$ intersect $X$ in a cubic curve and a unique residual point included in all the lines $r(\mathbb{P}(E_1,p))$ for any point $p$ of $\Delta$. Note that the exceptional divisor $\mathcal{R}$ is the image of the composition:

$$I \subset \text{Proj}(p_2^*(E_1(h_3))) \to \text{Proj}(E_1(h_3))$$

so from lemma [3.1] the projection $\mathcal{R} \to B$ is finite of degree 4 except over 16 points of $B$, but we will notice later that 4 of them are contracted to points on $X$. To understand why the plane cubic described above is singular, we first notice that it is the image of the following curve $T_\Delta$ in $F_B$: The closure in $F_B$ of the lines included in $B$ intersecting $\Delta$ and different from $\Delta$. A general $\Delta$, intersect the hyperplane section $\tilde{H}_{u_1}$ in a single point $b$ which is not on the veronese $V_1$. From the corollaries [1.12] and [1.13] there is a single quadric $Z_{1,p_1}$ containing $b$. The 2 lines containing $b$ and included in this quadric
are included in $\tilde{H}_{u_1}$. So, from remark 2.2, they correspond in $F_B$ to points on the exceptional divisor $v_1 \subset F_B$. In conclusion, for a general $\Delta$, the curve $T_\Delta$ intersects the exceptional divisor $v_1$ in 2 points. The image of those 2 points of $F_B$ must be a single point of $C$ because they are in the line $r(\mathbb{P}(p_B^{-1}(b)))$, and this line intersects $C$ in at most one point because $b$ is not in the veronese $V_1$. So the plane cubic is singular at this point of $C$.

For the same reason, if $b$ is one of the 4 vertex of the cones in $\tilde{H}_{u_1} \cap B$, $r(\mathbb{P}(p_B^{-1}(b)))$ is contracted to a point on the curve $C$, that’s why only 12 lines of $X$ belong to the family $LB$. □

As all lines in those planes are trisecant to $X$, we have from the above remark that:

**Remark 4.5** The triple locus of the projection of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ from a generic plane is reducible, because it contains the union of 4 lines with a common point.

We give now some details on the marked “virtual” section related to the anormality of $X$.

**Remark 4.6** The ramification of the morphism $p_2 : I \rightarrow B$ is $p_1^{-1}(\Sigma)$ for some surface $\Sigma$ in $F_B$. The canonical sheaf of $\Sigma$ is $\omega_\Sigma = \mathcal{O}_\Sigma$, and $\Sigma$ contains the 16 rational curves parameterizing the 16 cones of $B$. The image of $\Sigma$ into $X$ is a section of $\mathcal{O}_X(2)$ that is not in a quadric of $\mathbb{P}_5$.

**Proof:** The canonical divisor of $I$ can be computed from §2, and it gives that $R \sim p_1^* h_2$. The 16 contracted curves must be in $R$, so we have only to prove that the image of $\Sigma$ in $X$ is not in a quadric of $\mathbb{P}_5$. But the answer is general for those anormality constructions: As $R$ is a divisor in the projective bundle $Proj(E_1(h_3))$, we have the exact sequence:

$$0 \rightarrow \omega_{p_B} \rightarrow \omega_{p_B}(R) \rightarrow \omega_R \otimes \omega_B^\vee \rightarrow 0$$

where $\omega_{p_B}$ is the relative dualising sheaf of $p_B : Proj(E_1(h_3)) \rightarrow B$. The ramification of the restriction of $p_B$ to $R$ is the zero locus of a section of $\omega_R \otimes \omega_B^\vee$ which gives a non zero element of $H^1(\omega_{p_B}) = \mathfrak{C}$. But we saw in the proof of remark 4.4 that the map $I \rightarrow B$ factors through $R$ (Cf the composition §3), so we have the result because the above sequence is also:

$$0 \rightarrow \omega_{p_B} \rightarrow \mathcal{O}_{Proj(E_1(h_3))}(2h) \rightarrow \mathcal{O}_R(2h) \rightarrow 0$$

□

### 4.2 The rank 2 reflexive sheaf on $\mathbb{P}_5$

If we take a general double hyperplane section of $X$, we obtain a smooth elliptic curve of degree 6. So from Serre’s construction, it is the vanishing locus of a section of a rank 2 vector bundle on $\mathbb{P}_3$, unique up to isomorphism. Curiously, there a way to globalise this construction over $\mathbb{P}_5$. In this part we will construct an $SL_2$-equivariant rank 2 reflexive sheaf on $\mathbb{P}_5$ using classical techniques developed in mathematical instanton
studies. (Cf \cite{Ba}, \cite{Tj}) Still consider vector spaces \( L \) and \( V \) of respective dimension 2 and 6. This construction will be essentially unique up to the \( SL_6 \) action. Indeed, it could be constructed from a tangent hyperplane to \( G_\omega(3,6) \) or like this:

\[
\begin{pmatrix}
0 & u^2 & 2uv & v^2 & 0 & 0 \\
-u^2 & 0 & 0 & 0 & 0 & 0 \\
-2uv & 0 & 0 & 0 & 0 & \frac{1}{2} \\
-v^2 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & v^2 \\
0 & 0 & -u^2 & -2uv & -v^2 & 0
\end{pmatrix}
\]

Let’s recall from \cite{M-M} that \( S^2 L \otimes \Lambda^2 V \) have an \( SL_2 \times SL_6 \) orbit made of net of alternating forms of constant rank 4. So let’s \( \beta \in S^2 L \otimes \Lambda^2 V \) be the element of this orbit. This element was considered in \cite{M-dP} to construct a \( \mathbb{P}_{11} \) containing their congruence, for instance, take the following:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

We can remark that \( \beta \) viewed as an element of \( \Lambda^2 (L \otimes V) \) has rank 6 because it can be represented by:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Denote by \( W \) the six dimensional image of this map, we will identify \( W \) with its dual via the induced alternating form. The inclusion \( W \subset L \otimes V \) gives a map \( \beta' \) from \( W \) to \( L \otimes \mathcal{O}_{\mathbb{P}_5}(1) \), and we can construct a complex:

\[
L \otimes \mathcal{O}_{\mathbb{P}_5}(-1) \xrightarrow{i_{\beta'}} W \otimes \mathcal{O}_{\mathbb{P}_5} \xrightarrow{\beta'} L \otimes \mathcal{O}_{\mathbb{P}_5}(1)
\]

exact on the left, with middle cohomology a rank 2 reflexive sheaf \( K \), and with right cohomology a sheaf \( \mathcal{L} \) supported on a smooth rational cubic curve \( C \) isomorphic to \( \mathcal{O}_{\mathbb{P}_5}(4) \). So we can compute from the complex \eqref{eq:complex} some of its invariants:

**Corollary 4.7** The sheaf \( K \) has rank 2, \( c_1 K = 0, c_2 K = 2, c_3 K = 0, c_4 K = -15 \). Its singular locus is the cubic curve \( C \), and we have \( H^0 K(1) = 0 \) and \( H^0 K(2) = 13 \), and \( H^1 K = 1 \).

we have now a way to structurate the variation of the focal locus with respect to the choice of the quadric in the of \cite{M-dP}. They are the vanishing locus of sections of the same bundle: \( K(2) \).
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