Heavy tetraquark confining potential in Coulomb gauge QCD

Carina Popovici\(^1\),\(^2\) and Christian S. Fischer\(^1\)

\(^1\) Institut für Theoretische Physik, Justus-Liebig-Universität Giessen, 35392 Giessen, Germany
\(^2\) Institut für Physik, Karl-Franzens-Universität Graz, A-8010 Graz, Austria

(Dated: March 25, 2014)

We present an analytic nonperturbative solution of the Yakubovsky equation for tetraquark states in the case of equal separations and energies, and demonstrate a direct connection between the tetraquark confinement potential and the temporal gluon propagator. To this end we employ a leading-order heavy quark mass expansion of the Coulomb gauge QCD action, and use the dressed two-point functions of the Yang-Mills sector only. As a result we find a bound state energy that rises linear with distance and a string tension twice as large as in a \(q\bar{q}\)-system.

PACS numbers: 11.10.St,12.38.Aw

I. INTRODUCTION

Exotic states in the heavy quark sector are an increasingly fascinating subject to study. With the discovery and confirmation of many new XYZ-states at BaBar, Belle, LHC and BES, interpretations of some of these clearly favor states which go beyond the time-honored classification of hadrons into mesons and baryons \([1, 2]\), opening up the exciting possibility of the identification of tetraquark, meson molecular or hybrid states. The idea of tetraquarks is around for quite some time. For the light quark sector, Jaffe proposed that the light scalar nonet including exciting possibility of the identification of tetraquark, meson molecule or hybrid states. The idea of tetraquarks is states which go beyond the time-honored classification of hadrons into mesons and baryons \([1, 2]\), opening up the confirmation of many new XYZ-states at BaBar, Belle, LHC and BES, interpretations of some of these clearly favor and their cousins cannot be identified with ordinary quarkonia and therefore strongly suggest an interpretation in terms of tetraquarks. Theoretically, tetraquarks can be described by a generalized Bethe-Salpeter equation for four particle states, originally proposed by Yakubovsky \([3]\) (see also Refs. \([6, 7]\) for pedagogical introductions). In a covariant setting, this equation, rounded off to account for quantum-field theoretical effects \([8]\), has been solved under the approximation that the 4q state is described by a coupled system of two-body equations with meson and diquark constituents \([9]\). Based on previous investigations, which showed that a rainbow-ladder kernel is most robust in meson Bethe-Salpeter calculations \([10–12]\), results of tetraquark masses have been obtained by employing a phenomenologically validated one gluon-exchange interaction. A complete classification of tetraquark states in terms of spin-flavor, color and spatial degrees of freedom has been constructed in \([13]\). Other investigations of tetraquark states include large \(N\)-limit calculations \([14]\), effective theory studies \([15–17]\) and relativistic quark models \([18]\).

From a fundamental perspective, tetraquarks offer interesting insights into the underlying structure of the strong interaction. The relationship between the non-perturbative scale associated with confinement (the string tension) and the gluon sector is of crucial importance in understanding the low-energy properties of QCD. On the lattice, Wilson loops exhibit an area law at intermediate distances that corresponds to a linearly rising potential, whereas the corresponding coefficient, so-called Wilsonian string tension, can be explicitly related to a hadronic scale \([19]\). Within continuous functional approaches, investigations carried out in Coulomb gauge have shown that in the heavy quark sector (and at least under truncation) one can identify a direct connection between the temporal Yang-Mills Green’s function and the potential that confines quarks, both in the two- and three-body case \([20, 21]\). In the Hamiltonian formalism, the physical string tension can be related to both the temporal Wilson loop \([22]\) and non-abelian color Coulomb potential \([23]\).

While the potential that confines two and three quarks has been relatively extensively studied with continuous methods as well as on the lattice (see for example Ref. \([24]\) for a review), the interaction between quarks in a 4q system has received little attention. On the lattice, the problem of van der Waals forces has been investigated, and it has been shown that a flux tube recombination takes place, i.e., around a level-crossing point, the confining potential flips between the disconnected ‘two-meson’ Ansatz and the state where the quarks and antiquarks are connected by a double-Y shaped flux tube, and this implies that the van der Waals forces are absent at long distances \([25]\). Continuum studies that have investigated the absence of long range forces in tetraquarks include Refs. \([26, 27]\).

In this work we will study the nature of the confining force in tetraquarks using a framework gauge fixed to Coulomb gauge. The realization of confinement in Coulomb gauge centers around the Gribov-Zwanziger scenario, which conjectures that the confining potential is provided by the temporal gluon propagator, whereas the spatial propagator is suppressed at long distances \([28]\). In addition, this gauge possesses a number of features that recommend it as an appropriate tool to study the low-energy sector of QCD: within the first order formalism, the total charge
of the system is conserved and vanishing, the system reduces naturally to the physical degrees of freedom \[29\], and the problem of divergent energy integrals disappears \[30\]. In Coulomb gauge, the Dyson-Schwinger equations for both Yang-Mills and quark sectors have been derived, and perturbative results have been obtained \[31\]–\[34\]. On the lattice, one important result (which shall be extensively used in this work) is that the temporal gluon propagator is energy-independent, and it behaves like \(1/\bar{q}^2\) for vanishing \(\bar{q}\) \[35\], \[36\]. The lattice results agree with the analytical findings obtained from the Hamiltonian approach to Yang-Mills theory \[23\]–\[25\].

This paper is a natural continuation of previous works including one of the authors \[20\]–\[21\], where meson and baryon bound states have been investigated via Bethe-Salpeter and Faddeev equations, respectively. Based on a leading-order expansion in the heavy quark mass originally developed within heavy quark effective theory \[HQET\] \[39\], a direct connection between the temporal gluon propagator and the string tension has been derived.\(^1\) Here we follow the same approach and consider the Yakubovsky equation for four-quark states \[5\] in Coulomb gauge at leading order in the mass expansion, in the symmetric case (i.e., the separation between quarks are equal), at equal energies, and by including only 2PI contributions. We will employ lattice results for the temporal gluon propagator, and in addition, we will use previous findings, namely that the kernel of the Bethe-Salpeter equation reduces non-perturbatively to the ladder truncation. In this setting, we will provide an exact analytical solution to the Yakubovsky equation, which then naturally leads to the confining potential of a 4q system.

The organization of this paper is as follows. In Sec. II we briefly survey the results obtained for heavy quark systems. We review the main steps of the expansion of QCD action in powers of the inverse quark mass, and discuss the results obtained for the heavy quark propagator and the corresponding (temporal) quark-gluon vertex. In Sec. III we present the Yakubovsky equation for tetraquark states. Similar to the case of meson and baryon states, we establish (at least under truncation) a direct relation between the physical string tension and the temporal component of the gluon propagator. A short summary and conclusions are presented in Sec. IV.

II. EXPANSION IN THE HEAVY QUARK MASS

In this section we outline the results obtained within heavy quark limit that are relevant for this work, and direct the reader to Ref. \[20\] for a full account. We employ the standard notations and conventions: spatial indices are labeled with roman letters \(i, j, \ldots\), and the superscripts \(a, \ b, \ldots\) denote color indices in the adjoint representation; flavor, Dirac spinor and (fundamental) color indices are commonly denoted with an index \(\alpha, \beta \ldots\). We work in Minkowsky space, with the metric \(\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)\). The Dirac \(\gamma\)-matrices satisfy \(\{\gamma^\mu, \gamma^\nu\} = 2g^\mu\nu\), where the notation \(\gamma^i\) refers to the spatial component and the minus sign arising from the metric has been explicitly taken into account. \(f^{abc}\) are the structure constants of the \(SU(N)\) group, with the Hermitian generators \(T^a, T^b = if^{abc}T^c\) normalized via \(\text{Tr}(T^aT^b) = \delta^{ab}/2\), and the Casimir factor \(C_F = (N^2 - 1)/2N\).

The idea that underlies the heavy quark mass expansion is the so-called heavy quark decomposition, i.e., the (full) quark field \(q_i\) is separated into two components via the spinors \(h\) and \(H\) as follows:

\[
q_a(x) = e^{-imx_0} \left[ h(x) + H(x) \right]_a, \quad h_a(x) = e^{imx_0} \left\{ \frac{1}{2} + \gamma^0 \right\} q_a(x), \quad H_a(x) = e^{imx_0} \left\{ \frac{1}{2} - \gamma^0 \right\} q_a(x)
\]

(similarly for the antiquark field). In our Coulomb gauge functional approach, this particular type of heavy quark transform adapted from HQET \[39\] can be simply regarded as an arbitrary decomposition. This is then inserted into the QCD generating functional, and, after integrating out the \(H\)-fields, an expansion in the heavy quark mass is performed (throughout this work we shall use the established terminology “mass expansion”, instead of “expansion in the inverse quark mass”). We mention here that the quark and antiquark sources are kept in all steps of the calculation, such that the full gap and Yakubovsky equations can be employed, whereas the kernels, propagators and vertices are replaced with their expressions at leading order in the mass expansion. The decomposition Eq. (2.1) leads to the suppression of the spatial gluon propagator at leading order in the mass expansion, which in turn means that at LO the attached gluons couple to the constituent quarks of the four-body state via a temporal quark-gluon vertex. We refer the reader to Ref. \[34\] for a detailed discussion regarding source terms and the expansion of the QCD action in the parameter \(1/m\).

Before we provide our solution for the heavy quark propagator, it is appropriate to briefly discuss our truncation scheme, which has also been employed in \[20\]–\[21\]. In the context of the heavy mass expansion, we restrict ourselves to dressed two-point functions of the Yang-Mills sector (i.e., the nonperturbative gluon propagators) and set all the

\(^1\) The heavy quark limit has also been recovered under a (perturbative) leading order truncation of Dyson-Schwinger equations \[40\]–\[41\].
pure Yang-Mills vertices and higher n-point functions occurring in the quark equations to zero. This truncation is justified by the fact that the total number of loops containing Yang-Mills vertices is drastically reduced: on the one hand, these vertices only contribute at second order perturbatively, since the leading order perturbative corrections containing purely temporal vertices vanish (temporal Yang-Mills vertices are zero at tree-level \([32]\)), and on the other hand, loops containing spatial Yang-Mills vertices are suppressed by the mass expansion.

From the full Coulomb gauge gap equation (i.e., first order formalism without mass expansion \([34]\)), supplemented by the Slavnov-Taylor identity, we find the following solution for the heavy quark propagator,

\[
W_{q_0q_\alpha\beta}(k_0) = \frac{-i\delta_{\alpha\beta}}{|k_0 - m - \mathcal{I}_r + i\varepsilon|} + \mathcal{O}(1/m),
\tag{2.2}
\]

with

\[
\mathcal{I}_r = \frac{1}{2}g^2C_F \int_r \frac{d\omega}{\omega^2} \frac{\mathcal{D}_{\sigma\alpha}(\omega)}{\omega^2} + \mathcal{O}(1/m).
\tag{2.3}
\]

The constant \(\mathcal{I}_r\) is implicitly regularized, under the assumption that the order of the integration is set such that the temporal integral is performed first, and the spatial integral is regularized and finite. The non-perturbative temporal gluon propagator entering \(\mathcal{I}_r\) is given by:

\[
W_{\sigma\sigma}(\vec{k}) = \delta_{ab} \frac{i}{k^2} D_{\sigma\sigma}(\vec{k}^2).
\tag{2.4}
\]

Following lattice results \([30]\), and also continuum investigations \([35]\), we assume that the dressing function \(D_{\sigma\sigma}\) is energy independent and diverges like \(1/k^2\) in the infrared. From the Slavnov-Taylor identity, combined with the solution Eq. 2.2, one easily finds that the temporal quark-gluon vertex remains non-perturbatively bare,

\[
\Gamma^2_{q_0q_\sigma\alpha\beta}(k_1, k_2, k_3) = [gT^a]_{\alpha\beta} + \mathcal{O}(1/m),
\tag{2.5}
\]

whereas the spatial vertex is subleading in the heavy mass expansion \([20]\). The heavy quark propagator Eq. 2.2 has a few remarkable properties which we shall discuss here briefly. Firstly, as a result of the mass expansion, this propagator has a single pole in the complex \(k_0\)-plane, as opposed to the conventional Feynman quark propagator. Therefore, it is necessary to explicitly define the Feynman prescription. It then follows that the closed quark loops vanish due to energy integration,

\[
\int \frac{dk_0}{|k_0 - m - \mathcal{I}_r + i\varepsilon| |k_0 + p_0 - m - \mathcal{I}_r + i\varepsilon|} = 0,
\tag{2.6}
\]

meaning that the theory is quenched at lowest order in the heavy quark mass expansion. A further observation is that the propagator Eq. 2.2 is diagonal in the outer product of the fundamental color, flavor and spinor spaces, due to the decoupling of the spin degree of freedom in the heavy quark limit \([39]\). Finally, the position of the pole in the heavy quark propagator does not have a physical meaning since the quark cannot be on-shell. As soon as the regularization is removed, the poles are shifted to infinity, and this implies that only the relative energy plays a role in a hadronic system (or, if a single quark is considered, one needs infinite energy to create it from the vacuum). Indeed, it has long been known that the absolute energy does not have a physical meaning, and that only the relative energy, which in the case of tetraquarks is derived from the Yakubovsky equation, must be considered \([42]\).

Since the heavy quark mass expansion breaks the charge conjugation symmetry, the antiquark and quark propagators are not equivalent. The Feynman prescription for the antiquark propagator is derived from the observation that the Bethe-Salpeter equation must have a physical interpretation of bound states – in this case, the quark and the antiquark are not connected by a primitive vertex, and hence they do not create a virtual quark-antiquark pair (closed loop) but a system composed of two separate unphysical particles. For the antiquark propagator we obtain (the derivation is similar to the quark propagator):

\[
W_{\bar{q}_{\bar{q}_0q_\alpha\beta}}(k_0) = \frac{-i\delta_{\alpha\beta}}{|k_0 + m - \mathcal{I}_r + i\varepsilon|} + \mathcal{O}(1/m),
\tag{2.7}
\]

and the corresponding vertex is given by:

\[
\Gamma^a_{\bar{q}\bar{q}_\sigma\alpha\beta}(k_1, k_2, k_3) = -[gT^a]_{\beta\alpha} + \mathcal{O}(1/m).
\tag{2.8}
\]

A last important consequence of the heavy quark mass expansion is the reduction of the interaction kernels from both Bethe-Salpeter and Yakubovsky equations to ladder exchange, since the crossed box contributions cancel due to energy integration over multiple quark propagators with the same Feynman prescription (see Ref. \([20]\) for a detailed discussion and calculation).
Bethe-Salpeter equation for mesons and Faddeev equation for baryons, the integral equation (3.1) depends only on the four-momentum (total energy) of the bound tetraquark state. \( \Gamma_{\alpha\beta\gamma\delta} \) is the pole dependence from \( \Gamma \). In our convention, states \([9]\). The amplitudes \( S \) contain three types of kernels, corresponding to quark-quark, antiquark-antiquark and quark-antiquark pairs. As discussed in the previous section, in the limit of heavy quark mass these kernels reduce to ladder gluon exchange:

\[
\begin{align*}
S_{\alpha\beta\gamma\delta}^{(qq)}(p_1, p_2, k) & = g^2 T_{\alpha\alpha'} T_{\beta\beta'}^a W_{\sigma\sigma'}(k) W_{\bar{q}q}(p_1^0 + k_0) W_{\bar{q}q}(p_2^0 - k_0), \\
S_{\alpha\beta\gamma\delta}^{(aa)}(p_1, p_2, k) & = g^2 T_{\alpha\alpha'}^a T_{\beta\beta'} W_{\sigma\sigma'}(k) W_{\bar{q}q}(p_1^0 + k_0) W_{\bar{q}q}(p_2^0 - k_0), \\
S_{\alpha\beta\gamma\delta}^{(aq)}(p_1, p_2, k) & = -g^2 T_{\alpha\alpha'}^a T_{\beta\beta'} W_{\sigma\sigma'}(k) W_{\bar{q}q}(p_1^0 + k_0) W_{\bar{q}q}(p_2^0 - k_0).
\end{align*}
\]

III. YAKUBOVSKY EQUATION FOR TETRAQUARK STATES

The Yakubovsky equation is a four-body bound state equation which has been successfully used to describe tetraquark states. It embodies two-, three- and four-quark irreducible diagrams. Since the irreducible three- and four-body forces all involve pure Yang-Mills vertices which, as discussed in the previous section, are neglected in our approach, it follows that there are no 3PI and 4PI contributions at this level of approximation. Consequently, it is appropriate to employ only the (anti)diquark and meson kernels that also appear in the corresponding Bethe-Salpeter equations. In a covariant setting, a similar approximation has been successfully applied to study tetraquark bound states \([9]\].

In this approximation, and by using the formulation \([8]\), where the correct multiplicity is taken into account via the inclusion of two-pair kernels, the Yakubovsky equation reads:

\[
\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = \int d \vec{k} \left\{ S_{\beta\alpha'\beta'}^{(qq)}(p_1, p_2; k) \Gamma_{\alpha'\beta'\gamma\delta}(p_1 + k, p_2 - k, p_3, p_4) + S_{\delta\alpha'\delta'}(p_3, p_4; k) \Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3 + k, p_4 - k) + S_{\gamma\alpha'\gamma'}(p_2, p_3; k) \Gamma_{\alpha'\beta\gamma\delta}(p_1 + k, p_2 + k, p_3 - k, p_4 + k) + S_{\delta\alpha'\delta'}^{(qa)}(p_1, p_2; k) \Gamma_{\alpha'\beta\gamma\delta}(p_1 + k, p_2 - k, p_3 - k, p_4 + q) + S_{\delta\alpha'\delta'}^{(aq)}(p_1, p_2; k) \Gamma_{\alpha'\beta\gamma\delta}(p_1 + k, p_2 - q, p_3 - q, p_4 - k) \right\}.
\]

The amplitudes \( S \) contain three types of kernels, corresponding to quark-quark, antiquark-antiquark and quark-antiquark pairs. As discussed in the previous section, in the limit of heavy quark mass these kernels reduce to ladder gluon exchange:

\[
\begin{align*}
S_{\alpha\beta\gamma\delta}^{(qq)}(p_1, p_2, k) & = g^2 T_{\alpha\alpha'} T_{\beta\beta'}^a W_{\sigma\sigma'}(k) W_{\bar{q}q}(p_1^0 + k_0) W_{\bar{q}q}(p_2^0 - k_0), \\
S_{\alpha\beta\gamma\delta}^{(aa)}(p_1, p_2, k) & = g^2 T_{\alpha\alpha'}^a T_{\beta\beta'} W_{\sigma\sigma'}(k) W_{\bar{q}q}(p_1^0 + k_0) W_{\bar{q}q}(p_2^0 - k_0), \\
S_{\alpha\beta\gamma\delta}^{(aq)}(p_1, p_2, k) & = -g^2 T_{\alpha\alpha'}^a T_{\beta\beta'} W_{\sigma\sigma'}(k) W_{\bar{q}q}(p_1^0 + k_0) W_{\bar{q}q}(p_2^0 - k_0).
\end{align*}
\]

In the above, we have already replaced the temporal quark-gluon vertices by their expressions Eqs. (2.5) and (2.8). In our convention, \( p_1, p_2 \) denote the quark momenta, \( p_3, p_4 \) the antiquark momenta, and \( P_0 = \sum_{i=1}^{4} P_i^0 \) is the pole four-momentum (total energy) of the bound tetraquark state. \( \Gamma_{\alpha\beta\gamma\delta} \) represents the quark-tetraquark vertex for a particular bound state and its indices denote explicitly only its quark content. Just like the heavy quark propagator, \( \Gamma_{\alpha\beta\gamma\delta} \) becomes a Dirac scalar due to the decoupling of the spin in the heavy mass limit. Similar to the homogeneous Bethe-Salpeter equation for mesons and Faddeev equation for baryons, the integral equation (3.1) depends only parametrically on the total energy \( P_0 \) (for notational convenience we have dropped the \( P_0 \) dependence from \( \Gamma_{\alpha\beta\gamma\delta} \)). We also note that, as in the case of meson and baryon bound states, the energy independence of the temporal gluon propagator will play a key role in the derivation of the confining potential. The Yakubovsky equation is diagrammatically shown in Fig. 4.
Before we specify our Ansatz for the Yakubovsky vertex, let us shortly recall the energy behavior of the meson and baryon vertices. Whereas in the case of mesons it was straightforward to show that the Bethe-Salpeter vertex was energy-independent, the quark-baryon vertex did contain an energy component, similar in structure to the quark propagator [21]. In case of tetraquarks the relative energy does not cancel, hence it is reasonable to assume that the Yakubovsky vertex obeys a separable Ansatz (recall that the dependence on the total energy is implicit):

$$\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = \Psi_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) \Gamma_{\delta}(p_1, p_2, p_3, p_4).$$

(3.3)

\(\Gamma_t\) and \(\Gamma_s\) represent the temporal and spatial component, respectively, and \(\Psi_{\alpha\beta\gamma\delta}\) denotes the color component, with \(\alpha, \beta\) being quark, and \(\gamma, \delta\) antiquark indices.

The color structure of tetraquarks is nontrivial since a singlet can be obtained via two different representations [13]. Interestingly, diquark and antidiquark pairs also contribute to the color singlet tetraquark state, although themselves cannot exist as color singlets (at least for \(N = 3\) colors). By writing the color function \(\Psi\) as

$$\Psi_{\alpha\beta\gamma\delta} = \delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\alpha\gamma}\delta_{\beta\delta},$$

(3.4)

and with the Fierz identity for the generators

$$2|T^a_{\alpha\beta}|T^a_{\delta\gamma} = \delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{N}\delta_{\alpha\beta}\delta_{\gamma\delta},$$

(3.5)

we calculate the color factors corresponding to various channels:

$$T^a_{\alpha\alpha}T^a_{\beta\beta}\Psi_{\alpha\beta\gamma\delta} = \frac{1}{2} \left( 1 - \frac{1}{N} \right) \Psi_{\alpha\beta\gamma\delta}$$

diquark and antidiquark channel

$$T^a_{\alpha\alpha}T^a_{\gamma\gamma}\Psi_{\alpha\beta\gamma\delta} = \frac{1}{2} \left( 1 + N - \frac{2}{N} \right) \Psi_{\alpha\beta\gamma\delta}$$

diagram channel

$$T^a_{\alpha\alpha}T^b_{\beta\beta}T^b_{\delta\delta}\Psi_{\alpha\beta\gamma\delta} = \frac{1}{4} \left( 1 - \frac{2}{N} + \frac{1}{N^2} \right) \Psi_{\alpha\beta\gamma\delta}$$

diquark-antidiquark channel

$$T^a_{\alpha\alpha}T^b_{\beta\beta}T^b_{\delta\delta}\Psi_{\alpha\beta\gamma\delta} = \frac{1}{4} \left( N^2 + N - 2 - \frac{2}{N} + \frac{2}{N^2} \right) \Psi_{\alpha\beta\gamma\delta}$$

diagram-meson channel

(3.6)

Inspecting the equation Eq. (3.3), we notice that the energy and three-momentum integrations separate since the temporal gluon propagator is energy independent, whereas the heavy quark propagator is independent of the spatial momentum. Fourier transforming the spatial part of the vertex

$$\Gamma_{\delta}(p_1, p_2, p_3, p_4) = \int d\vec{x}_1 d\vec{x}_2 d\vec{x}_3 d\vec{x}_4 \exp(-i \sum_{i=1}^4 \vec{p}_i \cdot \vec{x}_i) \Gamma_{\delta}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4),$$

(3.7)

we find that the convolution product with the temporal gluon propagator is given by

$$\int d\vec{k} W_{\sigma\rho}(\vec{k}) \Gamma_{\delta}(p_1, p_2, p_3, p_4) = \int d\vec{x}_1 d\vec{x}_2 d\vec{x}_3 d\vec{x}_4 \exp(-i \sum_{i=1}^4 \vec{p}_i \cdot \vec{x}_i) W_{\sigma\rho}(\vec{x}_2 - \vec{x}_1) \Gamma_{\delta}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)$$

(3.8)

and hence the spatial component of the vertex completely drops from the calculation. Motivated by the symmetry of the system, we further restrict to the case of equal quark separations \(|\vec{r}| = |\vec{x}_i - \vec{x}_j|\) \((i, j = 1, 4; i > j)\). The original equation can be recast into an equation for the temporal component \(\Gamma_t\):

$$\Gamma_t(p_1^0, p_2^0, p_3^0, p_4^0) = g^2 W_{\sigma\rho}(\vec{r}) \int d\vec{k} \left\{ \frac{1}{3} W_{\bar{q}q}(p_1^0 + k_0) W_{\bar{q}q}(p_2^0 - k_0) \Gamma_t(p_1^0 + k_0, p_2^0 - k_0, p_3^0, p_4^0) + (3.4) \right\}$$

- \frac{5}{6} W_{\bar{q}q}(p_3^0 - k_0) W_{\bar{q}q}(p_2^0 + k_0) \Gamma_t(p_1^0 + k_0, p_2^0 - k_0, p_3^0, p_4^0) + (1, 4) + (2, 4) + (1, 3) \right\}$$

- g^2 W_{\sigma\rho}(\vec{r})^2 \left\{ \frac{1}{9} W_{\bar{q}q}(p_1^0 + k_0) W_{\bar{q}q}(p_2^0 - k_0) W_{\bar{q}q}(p_3^0 + q_0) W_{\bar{q}q}(p_4^0 - q_0) \Gamma_t(p_1^0 + k_0, p_2^0 - k_0, p_3^0 + q_0, p_4^0 - q_0) + \frac{43}{18} W_{\bar{q}q}(p_1^0 + k_0) W_{\bar{q}q}(p_3^0 - k_0) W_{\bar{q}q}(p_2^0 + q_0) W_{\bar{q}q}(p_4^0 - q_0) \Gamma_t(p_1^0 + k_0, p_2^0 - q_0, p_3^0 - k_0, p_4^0 + q_0) + (1, 4; 2, 3) \right\},$$

(3.9)
where \((i,j)\) represent the terms attached to the corresponding pairs of (anti)quarks, and can be explicitly read off from Eqs. \([3.1, 3.2]\).  

Now in order to identify the structure of the solution, it is useful to rewrite the energy integral as follows:

\[
\int d k_0 W_{qq}(\tilde{p}_0^0 - k_0 - m)W_{qq}(\tilde{p}_1^0 + k_0 - m)\tilde{\Gamma}_i(\tilde{p}_1^0 + k_0, \tilde{p}_2^0 - k_0, \tilde{p}_3^0, \tilde{p}_4^0) = - \frac{2}{\tilde{p}_1^0 + \tilde{p}_2^0 - 2\mathcal{I}_r + i\varepsilon} \int d k_0 \frac{\Gamma_i(\tilde{p}_1^0 + \tilde{p}_2^0 - k_0 - m)\tilde{\Gamma}_i(\tilde{p}_1^0 + k_0, \tilde{p}_2^0 - k_0, \tilde{p}_3^0, \tilde{p}_4^0)}{\tilde{p}_1^0 + \tilde{p}_2^0 - 2\mathcal{I}_r + i\varepsilon}, \tag{3.10}
\]

where we have introduced the shifted momenta \(\tilde{p}_{1,2}^0 = p_{1,2}^0 + m\) for notational convenience. The integration over antiquark propagators leads to an identical formula, except that the mass term has the opposite sign (in this case, \(\tilde{p}_{3,4}^0 = p_{3,4}^0 - m\)), whereas in the integral over a quark and an antiquark, the mass term completely vanishes. The double integrals can straightforwardly be rewritten in a similar form. Without loss of generality, we can further restrict to equal energies, i.e., \(\tilde{p}_i^0 = \tilde{P}_i/4\). Inspired by the energy integral Eq. (3.10), and noticing the similarities with our previous three-body calculation [21], we make the following Ansatz for the tetraquark vertex

\[
\Gamma_i(\tilde{P}_1^0, \tilde{P}_2^0, \tilde{P}_3^0, \tilde{P}_4^0) = \sum_{i,j=1,3 \atop i<j} \frac{1}{\tilde{p}_i^0 + \tilde{p}_j^0 - 2\mathcal{I}_r - A(P_0, \mathcal{I}_r) + i\varepsilon} \tag{3.11}
\]

where \(A(P_0, \mathcal{I}_r)\) is a function that needs to be determined. For equal energies, the Ansatz takes the simpler form:

\[
\Gamma_i(\tilde{P}_1^0, \tilde{P}_2^0, \tilde{P}_3^0, \tilde{P}_4^0)|_{\tilde{P}_i^0 = \tilde{P}} = \frac{12}{P_0 - 4\mathcal{I}_r - 2A(P_0, \mathcal{I}_r) + i\varepsilon}. \tag{3.12}
\]

Plugging this back into Eq. (3.3) and using the result Eq. (2.4), we are left with an algebraic equation for the function \(A(P_0, \mathcal{I}_r)\). As has been emphasized in [20], there are only two possibilities for the bound state energy once all regulators are removed: either it is finite and linear rising with distance (i.e. a confined state), or it is infinite and therefore unphysical. Since we are searching for a confining solution for our tetraquark, the following condition has to be satisfied:

\[
P_0 - 4\mathcal{I}_r = 2C_F g^2 W_{\sigma\sigma}(\mathcal{r}). \tag{3.13}
\]

This condition essentially requires that the Fourier transform integral is convergent, such that the bound state energy remains finite:

\[
\int \frac{d \omega}{2\omega}(1 - e^{i\omega\mathcal{r}}) = |\mathcal{r}| \frac{1}{8\pi}. \tag{3.14}
\]

Replacing \(\mathcal{I}_r\) and \(W_{\sigma\sigma}\) by their expressions Eq. (2.16) and Eq. (2.17), respectively, and the gluon dressing function with \(D_{\sigma\sigma} = X/\alpha_s^2\), we can rewrite Eq. (3.13) as

\[
P_0 = \sigma_{4q} |\mathcal{r}| = \frac{g^2 C_F X}{4\pi} |\mathcal{r}|. \tag{3.15}
\]

Inserting the Ansatz (3.12) into Eq. (3.9) we find, after a laborious but fairly straightforward calculation, 

\[
A(P_0, \mathcal{I}_r) = \frac{5}{4C_F}(P_0 - 4\mathcal{I}_r), \tag{3.16}
\]

which gives for the temporal component of the tetraquark vertex function:

\[
\Gamma_i(\tilde{P}_1^0, \tilde{P}_2^0, \tilde{P}_3^0, \tilde{P}_4^0) = \sum_{i,j=1,3 \atop i<j} \frac{1}{\tilde{p}_i^0 + \tilde{p}_j^0 - 15 P_0 + \frac{15}{16} \mathcal{I}_r + i\varepsilon}. \tag{3.17}
\]

These results are inline with our previous findings for \(\bar{q}q\) and \(3q\) systems. From Eq. (3.15) we find that the ‘string tension’ \(\sigma_{4q}\) corresponding to a tetraquark state, i.e. the coefficient of the four-body linear confining term, is two times bigger than the one of \(\bar{q}q\) system calculated in Ref. [20]:

\[
\sigma_{4q} = \frac{g^2 C_F X}{4\pi} = 2\sigma_{\bar{q}q}. \tag{3.18}
\]
For comparison, the string tension for three quark states has the value $\sigma_{3q} = \frac{3}{2} \sigma_{\bar{q}q}$. Just like in the case of meson and baryon states, our results show that in Coulomb gauge and at leading order in the mass expansion there is a direct connection between the string tension and the nonperturbative Yang-Mills sector of the theory, at least under the truncation considered here. Notice also that the total mass has disappeared (similar to mesons), since the two quarks and two antiquarks move with opposite (and equal) velocities such that the center of mass is stationary. In fact this is related to our original specification for the Feynman prescription – recall that we have assigned the reversed sign for antiquarks, which corresponds to a particle that moves with opposite velocity. For comparison, in the case of baryons, where the three quarks move in the same direction with equal velocities, the total bound state energy contains three times the quark mass.

IV. SUMMARY AND CONCLUSIONS

In this paper we have derived the four-quark confinement potential in the heavy quark limit of Coulomb gauge QCD. To this end, we have solved the Yakubovsky equation for tetraquark states in a symmetric configuration and for equal quark energies. We have expanded the QCD action by using a method adapted from HQET, and restricted to the leading order. Further, we have truncated the system such that only nonperturbative propagators of the Yang-Mills sector are included, and all pure Yang-Mills vertices and higher order functions are neglected.

As in the meson and baryon cases, a direct connection between the physical string tension and the Yang-Mills sector of Coulomb gauge QCD (the temporal gluon propagator) has been established. A bound state energy that raises linearly with the distance has been derived, and the coefficient of the linearly rising term is found to be two times that of a meson system. Since only symmetric configurations have been considered, no statement can be made regarding the shape of the string that confines the quarks. However, the restriction to equal energies does not alter the validity of our statements – clearly the confining potential should hold for any configuration, including that of equal energies.

A possible extension of this work is to include the next order in the mass expansion, and analyze the contribution of the spatial gluon propagator which so far has been neglected. A different line of research is the inclusion of vertex corrections – this should trigger the phenomenon of charge screening which is expected to alter the value of the string tension. Finally, our result serves as a basis for phenomenological descriptions of heavy tetraquarks in terms of potentials.

Acknowledgments

We are grateful to Peter Watson for useful discussions and a critical reading of the manuscript. This work was supported by BMBF under contract 06GI7121 and by the Helmholtz International Center for FAIR within the LOEWE program of the State of Hesse.
[20] C. Popovici, P. Watson, and H. Reinhardt, Phys. Rev. D81, 105011 (2010), 1003.3863.
[21] C. Popovici, P. Watson, and H. Reinhardt, Phys.Rev. D83, 025013 (2011), 1010.4254.
[22] M. Pak and H. Reinhardt, Phys. Rev. D80, 125022 (2009), 0910.2916.
[23] D. Eppele, H. Reinhardt, and W. Schleifenbaum, Phys. Rev. D75, 045011 (2007), hep-th/0612241.
[24] C. Popovici, Mod.Phys.Lett. A28, 1330006 (2013), 1302.5642.
[25] F. Okiharu, H. Suganuma, and T. T. Takahashi, Phys.Rev. D72, 014505 (2005), hep-lat/0412012.
[26] T. Appelquist and W. Fischler, Phys.Lett. B77, 405 (1978).
[27] G. Feinberg and J. Sucher, Phys.Rev. D20, 1717 (1979).
[28] V. N. Gribov, Nucl. Phys. B139, 1 (1978).
[29] D. Zwanziger, Nucl. Phys. B518, 237 (1998).
[30] H. Reinhardt and P. Watson, Phys. Rev. D79, 045013 (2009), 0808.2436.
[31] P. Watson and H. Reinhardt, Phys. Rev. D75, 045021 (2007), hep-th/0612114.
[32] P. Watson and H. Reinhardt, Phys. Rev. D77, 025030 (2008), 0709.3963.
[33] P. Watson and H. Reinhardt, Phys. Rev. D76, 125016 (2007), 0709.0140.
[34] C. Popovici, P. Watson, and H. Reinhardt, Phys. Rev. D79, 045006 (2009), 0810.4887.
[35] A. Cucchieri and D. Zwanziger, Phys. Rev. D65, 014002 (2001), hep-th/0008248.
[36] M. Quandt, G. Burgio, S. Chimchinda, and H. Reinhardt, PoS CONFINEMENT8, 066 (2008), 0812.3842.
[37] D. Eppele, H. Reinhardt, W. Schleifenbaum, and A. P. Szczepaniak, Phys. Rev. D77, 085007 (2008), 0712.3694.
[38] A. P. Szczepaniak, Phys. Rev. D69, 074031 (2004), hep-ph/0306030.
[39] M. Neubert, Phys. Rept. 245, 259 (1994), hep-ph/9306320.
[40] P. Watson and H. Reinhardt, Phys.Rev. D86, 125030 (2012), 1211.4507.
[41] P. Watson and H. Reinhardt, Phys.Rev. D85, 025014 (2012), 1111.6078.
[42] S. L. Adler and A. Davis, Nucl.Phys. B244, 469 (1984).