A convex geometry perspective to the (SM)EFT space

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We present a convex geometry perspective to the Effective Field Theory (EFT) parameter space. We show that the set of $s^2$ terms in the EFT amplitudes form a convex cone, whose extremal rays correspond to tree-level one-particle UV-completions. In addition, all the extremal rays are determined by the symmetries of the EFT and can be systematically identified via group theoretical considerations. The implications are twofold. First, geometric information encoded in the EFT space can help reconstruct the UV completion. This suggests that the dim-8 operators are important in reverse-engineering the UV physics from the Standard Model EFT, and thus deserve more theoretical and experimental investigations. Second, theoretical bounds on the Wilson coefficients can be obtained by identifying the boundaries of the cone and are in general stronger than the current positivity bounds. We show explicit examples of these new bounds, and demonstrate that they originate from the scattering amplitudes corresponding to entangled states.

INTRODUCTION

Effective field theory (EFT) is an important framework to systematically parameterize new high-scale phenomena. Absent any clear signature of new particles from the LHC data, the Standard Model EFT (SMEFT) \[1\] has become a standard tool for studying indirect signs of new physics. If EFT operators are detected and the corresponding Wilson coefficients measured, the next step is to pin down the underlying UV theory. While determining the Wilson coefficients from a given UV theory is a systematized procedure \[4\]-\[13\], this inverse problem can be highly nontrivial, as one set of coefficients can be UV-completed in many ways.

A geometric perspective provides hints to this problem. Consider the subspace of the EFT parameters, spanned by the operators that lead to $s^2$ dependence in the forward 2-to-2 scattering amplitude. The Wilson coefficients are subject to positivity bounds \[14\] (see \[15\]-\[23\] for earlier works and recent generalizations; also see the applications in SMEFT \[24\]-\[28\] and in other areas \[29\]-\[38\]) for the EFT to have a UV-completion that satisfies the axiomatic principles of quantum field theory. These bounds are a set of linear homogeneous inequalities of the coefficients. The solutions form a convex cone whose vertex is the origin of the (linear) space spanned by the coefficients. To see how the geometry of this cone connects to the UV physics, consider a simple example where the cone is polyhedral and the measured coefficients are aligned with one of its edges, which is an extremal ray (ER). Since an ER is, by definition, not a sum of any two different rays in the cone, this EFT cannot be UV-completed with more than one (multiplet-)particle, as each particle would generate a set of coefficients inside the cone, and hence their sum is not an ER. This almost uniquely determines the UV-completion as some “one-particle extension”.

In this letter, we establish a connection between the geometry of the $s^2$-subspace of EFT and the UV physics behind. On the geometry side, the physical space is a convex cone that can be generated as positively weighted sums of its ERs. On the physics side, all the ERs can be interpreted as tree-level one-particle UV-completions, from which all other UV-completions can be generated. This leads to a geometric view that can be used to determine the UV physics from measurements. By using the convex nature of the subspace, one can often draw striking conclusions about the existence of states including their quantum numbers and couplings. In SMEFT, this subspace corresponds to dim-8 operators \[49\]-\[51\], which have attracted increasing attention as the LHC have accumulated more and more data. Various motivations for going beyond dim-6 have been discussed, e.g. in Ref. \[26\], \[28\], \[39\], \[52\]-\[57\]. A number of coefficients can be tested at the TeV level at the LHC \[54\], \[57\]-\[60\], while better sensitivities are expected at future colliders \[58\], \[61\]. Here we show that the geometric connection to the UV physics (which is absent at dim-6) gives a particularly important motivation to study these operators, as their coefficients, once determined and disentangled from the dim-6 ones, contain vital information for a bottom-up reconstruction of the UV-completion.

To formulate this mapping between ERs and UV states, an accurate description of the EFT cone is mandatory. The current positivity bound approach is not sufficient. Instead, we will take a more efficient and physically intuitive approach that follows the extremal representa-
tion of convex cones. This approach indeed proves the current positivity bounds. Before proceeding, it is instructive to introduce some basic concepts and facts in convex geometry.

A convex cone is a subset of a linear space that is closed under additions and positive scalar multiplications. An extremal ray (ER) of a convex cone $C_0$ is an element $x \in C_0$ that cannot be split into two other elements in a nontrivial way, i.e. if we write $x = y_1 + y_2$ with $y_1, y_2 \in C_0$, we must have $x = \lambda y_1$ or $x = \lambda y_2$. For example, the ERs of a polyhedral cone are its edges. The dual cone $C_0^*$ of $C_0$ is the set $C_0^* \equiv \{ x | x \cdot y \geq 0, \forall y \in C_0 \}$. We have $(C_0^*)^* = C_0$, and $C_1 \subset C_2$ implies $C_1^* \supset C_2^*$. The full set of positive linear combinations of elements in some set $X$ form a convex cone, denoted by cone($X$). The set of rank-$n$ positive semi-definite (PSD) matrices $\mathcal{P}_n$ is a convex cone, whose ERs are all the rank-1 matrices of form $m^i m^j$. We have $\mathcal{P}_n = \text{cone} \{ m^i m^j | m^i \in \mathbb{R}^n \}$.

**EFT AMPLITUDES AS CONVEX CONES**

Consider the forward scattering amplitude $M_{ij \rightarrow kl}(s, t = 0)$, where $1 \leq i, j, k, l \leq n$ represent the low-energy modes. Using analyticity of $M_{ij \rightarrow kl}(s)$ and the generalized optical theorem, we have the following dispersion relation

$$M^{ijkl} = \int_{\epsilon \Lambda}^\infty \frac{d \mu}{2 \pi} \text{Disc} M_{ij \rightarrow kl}(\mu) \left( j \leftrightarrow l \right) + c.c. \quad (1)$$

$$= \sum_X \int_{\epsilon \Lambda}^\infty \frac{d \mu}{2 \pi} \frac{m_{K_X}^{ij} m_{K_X}^{kl}}{\mu - \frac{1}{2} M^2} + \left( j \leftrightarrow l \right), \quad (2)$$

where the l.h.s. is the second-order derivative of $M_{ij \rightarrow kl}(s)$, with the low-energy discontinuity subtracted up to $\epsilon \Lambda$, a scale smaller than the EFT cutoff (see Appendix A for details). $M^2$ is the sum of the four interacting particle mass squared. $\sum_X$ denotes the sum over possible $X$ states along with their phase spaces, and we have defined $M_{ij \rightarrow X} \equiv m_{K_X}^{ij} + i m_{ij}^{kl}$.

The elastic version of this relation ($j = k, i = l$) has been widely used to derive positivity bounds (because $m_{K_X}^{ij} m_{K_X}^{ij} \geq 0$, see e.g., [14]). One may also mix different polarizations [21, 25, 29, 33] and different particles (e.g. [24, 28, 49, 62]), to get more bounds by using $M^{ijkl} u^i v^j u^k v^l \geq 0$ (because $u^i v^j u^k v^l m_{K_X}^{ij} m_{K_X}^{kl} = (u^t m_{K_X} u^t, v^t m_{K_X} v^t) \geq 0$, where $u^t$ and $v^t$ enumerate the particles and polarizations [63]. This can be viewed as the positivity bound from superposed states $u^t |i\rangle$ and $v^t |j\rangle$. In any case, the $M^{ijkl}$ on the l.h.s. is a low-energy quantity and can be expressed in terms of the Wilson coefficients, be it tree-level or higher loop-level, and we will use it as a proxy of the EFT space. At the tree level, $M^{ijkl}$ can be linearly mapped to the dim-8 coefficient space [24, 28, 29], so in the SMEFT discussions we will not distinguish the two.

Our goal is a more accurate characterization of the set $C$ of all possible $M^{ijkl}$. The main observation is that Eq. (2) defines $C$ as a convex cone. To see this, note that Eq. (2) represents a positively weighted sum of $m_{K_X}^{ij} m_{K_X}^{kl} + (j \leftrightarrow l)$, with integration regarded as a limit of summation. For a model-independent EFT, $m_{K_X}^{ij}$ are allowed to take arbitrary values in $\mathbb{R}^2$. Thus the set $C$ can be viewed as a convex cone

$$C = \text{cone} \left\{ (m^i m^j | m^i \in \mathbb{R}^n) \right\}, \quad (3)$$

where $i(j|k;l)$ means $j, l$ indices are symmetrized. Furthermore, $C$ is a salient cone, i.e. if $c \in C$, $c \neq 0$, then $-c \notin C$. This is because any nonzero element of $C$, after contracted with $\delta^{ik} \delta^{jl}$, is always positive as $\text{Tr}(mm^T) > 0$. We also define $C'$ as the set of $M^{ijkl}$ but without the $j \leftrightarrow l$ term: $C' = \text{cone} \{ m^{ij} m^{kl} \}$, which is the PSD matrix cone, $\mathcal{P}_2$.

A convex cone can be represented in two ways. Positivity bounds are related to the inequality representation, which follows from the Hahn-Banach separation theorem. The theorem states that $C$ is an intersection of half-spaces, each described by a linear inequality. Alternatively, the extremal representation follows from the Krein-Milman theorem: a salient cone $C$ is a convex hull of its ERs. Our approach follows the latter representation.

Before moving forward, we briefly comment on the incompleteness of the elastic positivity bounds from superposed states. As they are derived using $M^{ijkl} u^i v^j u^k v^l \geq 0$, these bounds describe the dual cone of $Q \equiv \text{cone}(u^i v^j u^k v^l)$. If $Q = C^*$, then $Q^*$ is an accurate description of $C$. However, we will show with explicit examples that $Q \subsetneq C^*$, and thus $Q^* \supsetneq C$, i.e. these bounds are not tight. In this respect, the extremal representation is a better approach.

**ERS AND UV STATES**

Since $C$ is the convex hull of its ERs, we first identify the latter. Note that $c^{ijkl}$ being extremal in $C'$ is a necessary condition for $c^{ijkl}$ to be extremal in $C$ (not sufficient). We shall first find the ERs of $C'$, then symmetrize the $j, l$ to get the potential ERs (PERs) of $C$, and finally discard the non-extremal ones.

Since $C'$ is $\mathcal{P}_n^2$, its ERs are simply rank-1 matrices, or 1D projectors, of the form $m^{ij} m^{kl}$. The PERs of $C$ are then $m^{ij} m^{kl}$. For a physics amplitude $M^{ijkl}$ to be extremal, according to Eq. (4), $M_{ij \rightarrow X}$ can only have a factorized dependence on $s$ and the phase space, i.e. $M_{ij \rightarrow X}(s, \Pi_X) = m_X^{ij}(s, \Pi_X)$. Otherwise, by splitting the integration region, $M^{ijkl}$ can be written as a sum of two different elements, which is non-extremal. The simplest such case is a one-particle extension to the theory, i.e. $M^{ijkl}$ is generated from a tree-level exchange of
this particle, so that the integration over $s$ and $\Pi_X$ vanishes. In practice, not all 1D projectors are allowed by symmetries of the theory. This will be discussed momentarily.

Heuristically, one-particle extensions may be thought of as the basic building blocks of the UV-completion. This provides an intuitive understanding of the $\mathcal{C}$ cone: a more inner ray of the $\mathcal{C}$ cone corresponds to a more elaborate UV-completion (including loop-level UV-completions), as it is some positively weighted sum of the outer elements; Indeed, the most outer elements—the ERs—correspond to one-particle UV-completions, which are the most fundamental. UV physics is in some sense “ordered in complexity” in this geometric view, which is a reason why the latter can help UV reconstruction.

**Symmetries**

We have so far not considered the symmetries of the EFT. $M^{ijkl}$ must be invariant under the internal (such as gauge) symmetries of the EFT. Also, since we define $M^{ijkl}$ to be stripped off any kinematic factor, $M^{ijkl}$ is invariant under an $SO(2)$ rotation around the forward direction that acts on the transverse polarization modes.

To determine the ERs of $\mathcal{C}'$, note that with symmetries its element is a sum of $m_X^l m_X^l$, where $X_a$ are the components of $X$ in the direct product space $r_i \otimes r_j$. For $X_a$ in one irreducible representation (irrep), $m_X^l m_X^l$ cannot be further decomposed, meaning it is extremal. So, instead of 1D projectors, now the ERs of $\mathcal{C}'$ are the projectors of all irreps in $r_i \otimes r_j$, which can be readily obtained from the Clebsch-Gordan coefficients. The ERs obtained by symmetrizing $j,l$ are not always extremal, as Fierz identities create additional linear dependence among the ERs. However, this can be easily examined. The physical interpretation of a PER is, again, a tree-level one-particle extension, but this particle is a multiplet of some irrep in $r_i \otimes r_j$. Loop-level UV-completion is unlikely to generate an ER, as $X$ is a multi-particle state which often contains more than one irrep. When all particles being considered live in the same multiplet, the number of ERs is finite, and $\mathcal{C}$ is a polyhedral cone following a theorem by Minkowski and Weyl.

We remark that it is possible to focus on a subset of particles closed under all symmetries. The ERs continue to be projective in this subspace, so most results derived in this case (such as bounds) are valid in general. In the following we will illustrate our approach with three subsets of fields in SMEFT: scalars, vectors and fermions. It is worth emphasizing that this approach is generically applicable to any EFT.

**THE HIGGS TRIANGULAR CONE**

The SM Higgs boson lives in the 2 of $SU(2)_L$ and carries hypercharge $1/2$. To find the PERs, we work with real scalars, define

$$H = \left( \begin{array}{cc} \phi_2 + i \phi_1 \\ \phi_4 - i \phi_3 \end{array} \right), \quad C = \left( \begin{array}{cc} 0 & 1_{2 \times 2} \\ -1_{2 \times 2} & 0 \end{array} \right), \quad (4)$$

and use the $\gamma$ matrices defined in [61]. The projectors of the irreps from $2 \otimes 2$ define the following PERs:

$$E_1^{ijkl} = \frac{1}{2} \left[ C^{i[jl]} + (C \gamma_4)^{i[jl]} \right], \quad E_{1S}^{ijkl} = 1_{4 \times 4} 1_{4 \times 4}, \quad E_{1A} = \gamma_4^{i[jl]},$$

$$E_3^{ijkl} = \frac{1}{2} \left[ (C \gamma_4)^{i[jl]} + (C \gamma_4 \gamma_1)^{i[jl]} \right], \quad E_3^{ijkl} = (\gamma_4 \gamma_1)^{i[jl]}, \quad E_{3A}^{ijkl} = (\gamma_1)^{i[jl]}, \quad (5)$$

where the subscripts 1,3 denote the 1 and 3 respectively and $S,A$ denote the exchange symmetry of the irrep. $I$ runs from 1 to 3. $E_1$ and $E_3$ consist of two terms, as required by hypercharge conservation. The corresponding UV particle can be easily identified. For example, $E_1$ is generated by a hypercharge-1 $SU(2)_L$ singlet vector $V^\mu$ that couples as $g_1 (H^T C \mu \delta H) V^{\mu \dagger}$, where $x \delta \mu y \equiv x D_\mu y - (D_\mu x)y$ is required by Bose symmetry.

**FIG. 1:** A cross section of the Higgs triangular cone, taken to be perpendicular to the direction $E_1 + E_{1S} + E_{1A}$. The colored circles represent the potential ERs.

Only 3 of the 6 PERs are linearly independent, as there are only 3 independent $H^T D^4$-type operators, conventionally taken to be $O_{S,n}$, $n = 0,1,2$, defined in [65]. The convex hull of the PERs determines $\mathcal{C}$ as a 3D triangular cone, whose cross section is shown in Figure 1. There are 3 ERs: $E_1$, $E_{1S}$ and $E_{1A}$. What can we learn from this cone? First, any UV-completeable EFT must stay within this cone. Its boundary can be identified by connecting pairs of the ERs and, after matching to the Wilson coefficients, are given by 3 bounds: $C_{S,0} \geq 0$, $C_{S,0} + C_{S,2} \geq 0$ and $C_{S,0} + C_{S,1} + C_{S,2} \geq 0$, $C_{S,n}$ being the coefficients of $O_{S,n}$. These are precisely the positivity bounds obtained from elastic scatterings of superposed Higgs modes, albeit numerically [27]. Here we have proven that they are the strongest bounds, even going beyond elastic scatterings. (This however is not always
the case; see the $W$-boson case.) Second, the shape of the cone contains non-trivial information about the UV completion. Suppose the coefficients are experimentally measured and fall into the blue region. We can immediately deduce that a new particle (or a multi-particle state, for loop-level UV-completions), which is a $SU(2)_L$ singlet and has hypercharge 1, must exist and couple to $HH$, in order to generate $E_1$, because the convex hull of all other PERs does not contain this point. Similarly, if it falls in the red (green) or orange region, we know that a new particle that lives in the 1S (1A) representation must exist.

**THE $W$-BOSON POLYHEDRAL CONE**

Our second example is the $W$-boson, which has 2 polarization modes and is charged under the 3 of $SU(2)_L$. The projection operators for $3 \otimes 3 = 1 \oplus 3 \oplus 5$ of $SU(2)_L$ are:

\[
P^1_{\alpha \beta \gamma \sigma} = \frac{1}{N} \delta_{\alpha \beta} \delta_{\gamma \sigma}, \quad P^2_{\alpha \beta \gamma \sigma} = \frac{1}{2} (\delta_{\alpha \gamma} \delta_{\beta \sigma} - \delta_{\alpha \sigma} \delta_{\beta \gamma}),
\]

\[
P^3_{\alpha \beta \gamma \sigma} = \frac{1}{2} (\delta_{\alpha \gamma} \delta_{\beta \sigma} + \delta_{\alpha \sigma} \delta_{\beta \gamma}) - \frac{1}{N} \delta_{\alpha \beta} \delta_{\gamma \sigma}, \quad (6)
\]

where $N = 3$. For the $SO(2)$ rotation around the forward direction, the projectors for $2 \otimes 2 = 1 \oplus 1 \oplus 2$ are similar but with $N = 2$. With these we can construct 9 PERs, which we denote as $E_{m,n}$, from the tensor product of the $m$-th $SO(2)$ and the $n$-th $SU(2)_L$ projectors. Taking $E_{1,1}$, $E_{1,3}$, $E_{2,2}$, $E_{3,1}$ and $E_{3,3}$ as a linearly independent basis, the remaining 4 PERs can be expressed as 5-vectors:

\[
E_{1,2} = (-2, 1, -1, -2, 1), \quad E_{2,1} = (-1, -1, -1, 1, 1)
\]

\[
E_{2,3} = (-10, -1, 5, 10, 1), \quad E_{3,2} = (-2, 1, 1, 0, 0). \quad (7)
\]

All these PERs except for $E_{3,3}$ are indeed extremal. This immediately determines $C$ as a 5D polyhedral cone with 8 edges.

This example remarkably illustrates the efficiency of the extremal approach in constraining the physical EFT space. To compare with the positivity bound approach, we switch to the inequality representation and, after mapping to the operator coefficients, the following inequalities are obtained:

\[
C_{T,2} \geq 0, \quad 4C_{T,1} + C_{T,2} \geq 0, \quad (8)
\]

\[
C_{T,2} + 8C_{T,10} \geq 0, \quad 8C_{T,0} + 4C_{T,1} + 3C_{T,2} \geq 0, \quad (9)
\]

\[
12C_{T,0} + 4C_{T,1} + 5C_{T,2} + 4C_{T,10} \geq 0, \quad (10)
\]

\[
4C_{T,0} + 4C_{T,1} + 3C_{T,2} + 12C_{T,10} \geq 0. \quad (11)
\]

Again, the corresponding operators $O_{T,n}$ are defined in [65] [66]. All these bounds except for $C_{T,2} \geq 0$ have not appeared previously in the literature, and are indeed stronger than those presented in [25] [27]. These coefficients parameterize the anomalous quartic-gauge-boson couplings, currently being measured at the LHC [58] [60], so they alone are important results. The first four bounds can be identified as positivity bounds by scattering various superposed states of $|W_{i,j}^{1,2}\rangle$ (superscripts for $SU(2)_L$ and subscripts for polarization modes). The last two bounds, Eqs. (10), (11), deserve more attention: they cannot be derived from any elastic scattering between superposed states, so they are beyond elastic positivity.

**MORE THAN POSITIVITY**

As explained already, positivity bounds fail to give a complete description of $C$, because in general $C^* \setminus Q \neq \emptyset$. The two bounds in Eqs. (10), (11) are indeed coming from elements of $C^* \setminus Q$:

\[
T_1 = 6E_{1,1} + 3E_{2,1} + 6E_{2,2} + 3/2E_{3,1} + 3E_{3,3} \quad (12)
\]

\[
T_2 = 5/2E_{1,1} + 5E_{1,2} + E_{3,1} + 15/2E_{2,1} + 3E_{3,3}. \quad (13)
\]

One can show that $T_{i,j}^{kl} M_{j,i}^{kl} \geq 0$, which lead to Eqs. (10) and (11) respectively, and that $T_{1,2} \notin Q$, which implies that those bounds cannot be derived from scattering between superposed states. More details are given in Appendix B.

The fact that $T_{1,2} \notin Q$ suggests that the dispersion relation of scattering amplitudes with entangled states can provide additional information about the UV completion. Positivity bounds would not capture this information unless there is a systematic and efficient way to tackle all elements in $C^*$. Note that the $T_{1,2}$ tensors are independent of this specific problem, and may lead to new bounds also for other operators or EFTs, whenever the number of states $n \geq 6$. Our extremal approach naturally captures all such cases.

**THE FERMION CIRCULAR CONE**

Lastly, we consider SM-like chiral fermions, with left- and right-handed components carrying different hypercharges, but other symmetries neglected for simplicity. Defining $J^\mu_{L,R} \equiv \bar{f}_{L,R} \gamma^\mu f_{L,R}$, we use the following basis:

\[
O_1 = -\partial^\mu J^\mu_{L} \partial_\mu J_{LV}, \quad O_2 = -\partial^\mu J^\mu_{R} \partial_\mu J_{RV},
\]

\[
O_3 = \partial^\mu J^\mu_{L} \partial_\mu J_{RV}, \quad O_4 = D^\mu (\bar{f}_L f_R) D_\mu (\bar{f}_R f_L). \quad (14)
\]

We simply show the PERs, in terms of the coefficient vector $\vec{C} = (C_1, C_2, C_3, C_4)$:

\[
M_L : (1, 0, 0, 0), \quad D_S : (0, 0, 0, 1), \quad V : (1, r^2, -2r, 0),
\]

\[
M_R : (0, 1, 0, 0), \quad D_A : (0, 0, -1, 1), \quad V' : (0, 0, -1, 2).
\]
$M_{L,R}$ are from Majorana-type scalar couplings with two $f_L$'s or two $f_R$'s. $D$ is a from Dirac-type scalar coupling, with subscripts $S,A$ indicating the exchange symmetry. $V(V')$ is from the vector coupling formed by same(opposite)-chirality fermions. $r$ is the ratio between $R/L$ couplings. Since $V$ is continuously parameterized by $r$, $\mathcal{C}$ has a curved boundary. The physics origin is that $f_L$ and $f_R$ are allowed to form the same kind of coupling, with no symmetry fixing their ratio. $\mathcal{C}$ can be described as a convex hull of a circular cone ($\overline{V}$) and two rays ($D_{S,A}$). In Figure 2 we show a 3D slice of $\mathcal{C}$. The boundaries are given by $C_1, C_2, C_4 \geq 0$ and $2\sqrt{C_1C_2} \geq \max(C_3, -C_3 - C_4)$.

![Figure 2](image-url)  
**FIG. 2:** A slice of the 4D fermion cone, taken to be perpendicular to the direction $(1, 1, 0, 1)$. The three axes are taken to be along $(1, -1, 0, 0), (0, 0, 1, 0), (-1, -1, 0, 2)$.

## A GEOMETRIC VIEW FOR UV-DETERMINATION

Let us reiterate what the Higgs example tells us in the general case. Let $\mathcal{E}_{\alpha}$ be the convex hull of all PERs with one of them, $E^4_a$, removed. If the measured coefficients, denoted as $\tilde{C}_{\alpha}$, are not contained by $\mathcal{E}_\alpha$, then a tree-level UV-completion must contain a particle that couples with the $E^a_\alpha$ irrep. This feature extends to loop-generated cases and even non-perturbative cases, with a proper reinterpretation of the “one particle”. For example, in the blue region of Figure 1 there must exist some multi-particle state or strongly coupled state that couples to $HH$, carries total hypercharge 1, and contains a $SU(2)_L$ singlet.

Quantitative statements may also be made. For a measured $\tilde{C}_{\alpha}$ in the blue region, there is a minimum $\lambda$ such that the $\tilde{C}_{\alpha} - \lambda \tilde{E}^a_1 \in \mathcal{E}_\alpha$. This sets a lower bound on the strength of the UV coupling that generates $\tilde{E}^a_1$. Similarly, an upper bound can be set using $\tilde{C}_{\alpha} - \lambda \tilde{E}^a_1 \in \mathcal{C}$ for all $\tilde{E}^a_1$. As a second example, consider the fermion cone and assume $\tilde{C}_{\alpha} \propto (1, 0.8, 1.4, 1)$ is observed (see the black point in Figure 2). If a small arc on $V$ (shown in black) is removed, the convex hull of remaining PERs does not contain $\tilde{C}_{\alpha}$. It follows that a new state exists and couples to the fermions with $V/A$-type couplings, and an upper bound on the coupling ratio $|g_A/g_V| < 0.35$ can be set. There are many other interesting and phenomenologically relevant examples, where convex geometry can be used as a tool to infer UV states.

In SMEFT, dim-6 coefficients are expected to be more accurately measured, yet they alone are insufficient to uniquely determine UV models. For example, the analogues of fermion PERs at dim-6 do not form a salient cone, and there are an infinite number of “flat directions”, i.e. combinations of UV particles, with no net dim-6 effects. An UV reconstruction from dim-6 coefficients solely is thus difficult, as a solution can always move along these directions. The geometric feature of dim-8 space provides a partial solution because, as we have shown, more concrete information can be drawn and, once combined with dim-6 coefficients, helps to pinpoint the UV-completion.

As a final remark, we have shown that the representations of convex cones, the concepts of dual cones and ERs, and several theorems in convex geometry help to develop a deeper understanding of the EFT space, to improve the positivity bounds, and to determine the UV-completion [57]. We hope that through this geometric perspective, other results in convex geometry may find their applications in particle physics.

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[66] C T, 10 denotes the coefficient of Q^2_{W^4} of [27], multiplied by g_s^4/4.
[67] The EFT hderon of [27] also connects the EFT space to the UV states from a geometric point of view. In particular, convex objects such as cyclic polytopes are found to constrain sequences of operators with increasing dimensions. Here we consider EFTs endowed symmetries and focus on the operators with the lowest possible dimension.

**APPENDIX**

### A. The dispersion relation

Here we present more details about how to get the dispersion relation (1b). Let M_{ij→kl}(s) be the forward amplitude M_{ij→kl}(s) but with the pole contributions subtracted out. Using unitarity and analyticity of the amplitude and Cauchy’s integral formula, we can derive a dispersion relation (see e.g. [21, 25])

\[ \tilde{M}^{ijkl} \equiv \frac{1}{2} \frac{d^2}{ds^2} \tilde{M}_{ij\rightarrow kl}(s = M^2/2) + c.c., \]

\[ = \int_{iM_{th}}^{\infty} \frac{d\mu}{2\pi} \text{Disc} M_{ij\rightarrow kl}^{M_{th}}(\mu) \left( \frac{\mu}{\mu - M^2} \right)^3 + c.c., \]

where the discontinuity of a complex function is defined as DiscA(s) = A(s + iε) − A(s − iε) and M_{th} is the threshold scale of the process ij → kl. Now, for a valid EFT, since we should be able to compute the amplitude in the IR to a desired accuracy within the EFT, we can subtract out the low energy parts of the dispersive integrals

\[ M^{ijkl} \equiv \tilde{M}^{ijkl} - \int \frac{(\epsilon A)^2}{M_{th}^3} \frac{d\mu}{2\pi} \text{Disc} M^{ijkl}_{ij\rightarrow kl}(\mu) \left( \frac{\mu}{\mu - M^2} \right)^3, \]

\[ - \int \frac{(\epsilon A)^2}{M^2} \frac{d\mu}{2\pi} \text{Disc} M^{ijkl}_{ij\rightarrow kl}(\mu) \left( \frac{\mu}{\mu - M^2} \right)^3 + c.c. \]

\[ = \int \frac{(\epsilon A)^2}{M^2} \frac{d\mu}{2\pi} \text{Disc} M^{ijkl}_{ij\rightarrow kl}(\mu) \left( \frac{\mu}{\mu - M^2} \right)^3 + (j \leftrightarrow l) + c.c., \]

where εA is a scale much smaller than Λ, but still larger than M and M_{th}, so that the denominator of the integrand is positive. Using Hermitian analyticity M^{ijkl}_{kl→ij}(s + iε) = M^{ijkl}_{ij→kl}(s − iε) and the generalized optical theorem M^{ijkl}_{ij→kl} − M^{ijkl}_{kl→ij} = i \sum_{X} M^{ijkl}_{ij→X} M^{kl→X}_{kl→X}, we can then get Eq. (2).

We also want to mention that in this paper we have focused on the s^2 subspace, as this is most accessible experimentally for SMEFT. However, our analysis will...
be similar for the $s^{2n}$ EFT subspaces ($n = 2, 3, ...$), which corresponds to taking $s^{2n}$ derivatives in Eq. (15).

B. Proof of more than positivity

Here we present more details about how to get the bounds in Eqs. (10) and (11) from $T_{1,2}$ given in Eqs. (12) and (13), and why these bounds can not be derived from the positivity bounds of scattering between superposed states. By construction, the $j,l$ indices of $T^{ijkl}_{1,2}$ are symmetrized. Viewing $ij$ (and $kl$) as one index, $T^{ijkl}_{1,2}$ are both PSD as they have the same positive eigenvalues: 15, 10, 10, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 5, 2, 2, 2, 2, 2, plus 16 zero eigenvalues. Therefore, $T^{ijkl}_{1,2} m^{ij} m^{kl} = T^{ijkl}_{1,2} m^{ij} m^{kl} \geq 0$. It follows that $T^{ijkl}_{1,2} M^{ijkl} \geq 0$, which then leads to Eqs. (10) and (11).

Now we show that $T^{ijkl}_{1,2} / \notin Q$, i.e. the same bounds cannot be derived from the positivity bounds of the form $u^i v^j u^k v^l M^{ijkl} \geq 0$. To that end, we need to show that $T^{ijkl}_{1,2}$ cannot be written as $\sum \alpha_a u^i_a v^j_a u^k_a v^l_a$ with $\alpha_a > 0$. Suppose this can be done for $T^{ijkl}_{1,2}$. Notice that $T^{ijkl}_{1,2} E^{ijkl}_b = \sum \alpha_a u^i_a v^j_a u^k_a v^l_a E^{ijkl}_b = 0$ for $b = (1, 2), (1, 3), (3, 1), (3, 2)$. Since $E^{ijkl}_b$ are projection operators, $u^i_a v^j_a u^k_a v^l_a E^{ijkl}_b$ are sums of squares, so for these $b$ values we have $u^i_a v^j_a u^k_a v^l_a E^{ijkl}_b = 0$ for each $a$. Then $T^{ijkl}_{1,2} E^{ijkl}_b = 0$ reduces to a system of quadratic equations for $u, v$, and one can check explicitly that it has no non-zero solution. Similarly, we can prove that $T^{ijkl}_{2,2} \notin Q$, that $T^{ijkl}_{2,2}$ cannot be written as $\sum \alpha_a u^i_a v^j_a u^k_a v^l_a$ with $\alpha_a > 0$, using $E^{ijkl}_b$ for $b = (2, 2), (2, 3), (3, 1), (3, 2)$. So Eqs. (10) and (11) cannot be derived from positivity bounds of scattering between $u^i |i\rangle$ and $v^j |j\rangle$. 

