Long time existence for a two-dimensional strongly dispersive Boussinesq system

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\section*{ABSTRACT}
We prove a long time existence result for the solutions of a two-dimensional Boussinesq system modeling the propagation of long, weakly nonlinear water waves. This system is exceptional in the sense that it is the only linearly well-posed system in the (abcd) family of Boussinesq systems whose eigenvalues of the linearized system have nontrivial zeroes. This new difficulty is solved by the use of “good unknowns” and of normal form techniques.

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\section*{1. Introduction}
\subsection*{1.1. The general setting}

The four-parameter (abcd) Boussinesq systems for long wavelength, small amplitude gravity-capillary surface water waves introduced in \cite{1, 2} couple the elevation of the wave \( \zeta = \zeta(x, t) \) to a measure of the horizontal velocity \( v = v(x, t), x \in \mathbb{R}^N, N = 1, 2, t \in \mathbb{R} \) and read as follows:

\begin{equation}
\begin{aligned}
\partial_t \zeta + \nabla \cdot v + \epsilon \nabla \cdot (\zeta v) + \epsilon (a \nabla \Delta v - b \Delta \partial_t \zeta) &= 0, \\
\partial_t v + \nabla \zeta + \frac{\epsilon}{2} \nabla (|v|^2) + \epsilon (c \nabla \Delta \zeta - d \Delta \partial_t v) &= 0.
\end{aligned}
\end{equation}

Here \( a, b, c, d \) are modeling parameters which satisfy the constraint \( a + b + c + d = \frac{1}{3} - \tau \) where \( \tau \geq 0 \) is a measure of surface tension effects, \( \tau = 0 \) for pure gravity waves.

In (1.1), the small parameter \( \epsilon \) is defined by

\[ \epsilon = a/h \sim (h/\lambda)^2, \]

where \( h \) denotes the mean depth of the fluid, \( a \) a typical amplitude of the wave, and \( \lambda \) a typical horizontal wavelength.

It was established in \cite{1} that, in suitable Sobolev classes, the error between the solutions of the full water waves system and their approximation given by (1.1) is of order \( O(\epsilon^2 t) \). Since the corresponding solutions of the full water wave system have been proven in \cite{3, 4} to exist on time scales of order \( O(1/\epsilon) \), one needs to establish long time
existence results for the Boussinesq systems, the “optimal” existence time scale being \(O(1/\epsilon)\). Note that the “dispersive” methods used to prove the local well-posedness of the corresponding Cauchy problems in low order Sobolev spaces lead to time scales of order \(O(1/\sqrt{\epsilon})\) (see for instance [5]).

The existence of solutions of the Boussinesq systems on time scales of order \(O(1/\epsilon)\) has been established in [6–10] for all the locally well posed Boussinesq systems except the case \(b = d = 0, a = c > 0\) which is in some sense special since the “generic” case \(b = d = 0, a, c > 0, a \neq c\) is linearly ill-posed. We also refer to [11] for the case of Full-Dispersion Boussinesq systems.

Remark 1.1. The global well-posedness of Boussinesq systems has been only established in a few cases, including the one-dimensional case \(a = c = b = 0, d > 0\) that can be viewed as a dispersive perturbation of the hyperbolic Saint-Venant (shallow water) system, see [12–14], and the Hamiltonian cases \(b = d > 0, a \leq 0, c < 0\), see [15] for the one-dimensional case and [16] for the two-dimensional case. We also refer to [17, 18] for scattering results in the energy space for those one-dimensional Hamiltonian systems when \(b = d > 0\).

Recall that the linearization of (1.1) around the null solution is well-posed (see [2]) provided that
\[
a \leq 0, \quad c \leq 0, \quad b \geq 0, \quad d \geq 0, \quad (1.2)
\]
or
\[
a = c > 0, \quad b \geq 0, \quad d \geq 0. \quad (1.3)
\]

Actually the linear well-posedness occurs when the nonzero eigenvalues of the linearization of (1.1) at \((0, 0)\)
\[
\lambda_{\pm}(\xi) = \pm i|\xi| \left(\frac{(1 - \epsilon a |\xi|^2)(1 - \epsilon c |\xi|^2)}{(1 + \epsilon d |\xi|^2)(1 + \epsilon b |\xi|^2)}\right)^{1/2}
\]
are purely imaginary.

This article will focus on the exceptional case (1.3) with \(b = d = 0, a = c = 1\) which is the only linearly well-posed case with eigenvalues having nontrivial zeroes, leading to difficulties not present in the other well-posed systems.

The one-dimensional (1D) case was considered in [19] and we will restrict to the two-dimensional (2D) still open case under the physical condition \(\text{curl } \mathbf{v} = 0, N = 2\).

In this article, we shall thus establish the long time existence theory for the following strongly dispersive (2D) Boussinesq system
\[
\begin{cases}
\partial_t \zeta + (1 + \epsilon \Delta) \nabla \cdot \mathbf{v} + \epsilon \nabla \cdot (\zeta \mathbf{v}) = 0, \\
\partial_t \mathbf{v} + (1 + \epsilon \Delta) \nabla \zeta + \frac{\epsilon}{2} \nabla (|\mathbf{v}|^2) = 0,
\end{cases}
(1.4)
\]
with the initial data
\[
\zeta|_{t=0} = \zeta_0, \quad \mathbf{v}|_{t=0} = \mathbf{v}_0.
(1.5)
\]
We shall assume that \(\mathbf{v}\) is curl-free, i.e.,
\[
\partial_1 v^2 = \partial_2 v^1.
(1.6)
\]
If (1.6) holds for \(t = 0\), then it holds for all time \(t > 0\) in the lifespan of the solutions to the system (1.4).
It was established in [5], using various dispersive properties of the underlying linear group (see [20–22] in the context of the KdV equation) that the Cauchy problem for (1.4) is locally well-posed for initial data in $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$, $s > 3/2$. However, the corresponding lifespan of the solution is $O(1/\sqrt{\epsilon})$, smaller though than the expected $O(1/\epsilon)$. The fact that purely dispersive methods do not yield the correct lifespan is understandable since this kind of methods are stable by perturbations which destroy the nonlinear structure of the system (essentially that of the shallow-water system) which of course plays a crucial role in the long time behavior of the solution.

In [19], by introducing good unknowns (in the sense of Alinhac in [23–25]), the authors symmetrized the 1D version of (1.4) to avoid the loss of derivatives. Using normal formal techniques on the set away from the spatial resonance set, the authors established the well-posedness of the 1D Boussinesq over the time scalar $\epsilon^{-\frac{3}{2}}$.

The goal of the present article is to extend the lifespan of the local solution for the 2D Boussinesq (1.4).

1.2. The main result

We now state the main result of this article as follows

**Theorem 1.1.** Assume that $((\zeta_0, v_0)) \in H^{N_0}(\mathbb{R}^2)$ for some $N_0 \geq 5$ satisfying $\hat{\zeta}_0(0) = 0$, $\hat{v}_0(0) = 0$ and

$$\partial_1 v_0^2 = \partial_2 v_0^1, \quad (1.7)$$

Then there exist a small $\epsilon_0 > 0$ and a constant $T_0 = T_0(\|\zeta_0\|_{H^{N_0}} + \|v_0\|_{H^{N_0}})$ such that for any $\epsilon \in (0, \epsilon_0]$, there exists a unique solution $(\zeta, v) \in C(0, T_0\epsilon^{-\frac{3}{2}}; H^{N_0}(\mathbb{R}^2))$ of system (1.4)–(1.5) such that

$$\sup_{t \in [0, T_0\epsilon^{-\frac{3}{2}}]} (\|\zeta(t)\|_{H^{N_0}} + \|v(t)\|_{H^{N_0}}) \leq C(\|\zeta_0\|_{H^{N_0}} + \|v_0\|_{H^{N_0}}), \quad (1.8)$$

where $C > 0$ is a universal constant.

**Remark 1.2.** If $\hat{\zeta}_0(0) = 0$, $\hat{v}_0(0) = 0$, (1.4) shows that $\hat{\zeta}(t, 0) = 0$, $\hat{v}(t, 0) = 0$ holds for all time $t > 0$ in the lifespan of the solutions to (1.4). Therefore, throughout the whole article, we shall use the condition $\hat{\zeta}(t, 0) = 0$, $\hat{v}(t, 0) = 0$ so that we could use the homogenous Littlewood–Paley decompositions.

**Remark 1.3.** The curl-free condition (1.6) of the velocity $v$ guarantees that the system (1.4) could be symmetrized so that there is no loss of derivatives. With (1.6), the principal part of (1.4) is similar to the 1D case of (1.4) in [19].

**Remark 1.4.** Reaching the expected time scale $O(1/\epsilon)$ for the solutions of (1.4) is still an open problem.

1.3. Main ideas of the proof

The main ideas of the proof rely heavily on symmetrization techniques and normal form techniques.
First, to avoid losing derivatives, we introduce a good unknown
\[ \mathbf{u} = \mathbf{v} + \epsilon \mathbf{B}'(\zeta, \mathbf{v}), \]
where \( \mathbf{B}'(\zeta, \mathbf{v}) \) is a nonlocal bilinear term which is defined at the beginning of Section 3.1. Here \( \mathbf{u} \) is called the good unknown of Alinhac in [23–25]. With \( (\zeta, \mathbf{u}) \), we symmetrize (1.4) to the following dispersive equation
\[ \partial_t \mathbf{V} - i\Lambda_{\epsilon} \mathbf{V} = S_{\epsilon, \nu}^\tau + Q_{\epsilon, \nu}^\tau + O(\epsilon), \]
where \( \Lambda_{\epsilon} = |D|(1 + \epsilon\Delta), \) and \( \mathbf{V} = \zeta + i|D|^{-1} \text{div} \mathbf{u} \) is the unknown which satisfies
\[ \|\mathbf{V}\|^2_{H^0_{\omega_0}} \sim \|\zeta\|^2_{H^0_{\omega_0}} + \|\mathbf{v}\|^2_{H^0_{\omega_0}}. \]
In (1.9), \( S_{\epsilon, \nu}^\tau \) is the symmetric quadratic term which is of order \( O(\epsilon) \), \( Q_{\epsilon, \nu}^\tau \) is the quadratic term of order \( O(\sqrt{\epsilon}) \), and term \( O(\epsilon) \) contains all the remained nonlinear terms of order \( O(\epsilon) \). The argument in this step is similar to the 1D Boussinesq in [19]. A standard energy estimate leads to the well-posedness over time scalar \( \frac{1}{\sqrt{\epsilon}} \).

The difference from the 1D Boussinesq is that there are two extra terms in \( Q_{\epsilon, \nu}^\tau \), i.e.,
\[ Q_{\epsilon, +}^\tau(V^+, V^+), \quad Q_{\epsilon, -}^\tau(V^-, V^+), \]
where \( V^+ = \mathbf{V} \) and \( V^- = \tilde{\mathbf{V}} \). Fortunately, the delicate derivation shows that the symbols of these two terms are bounded by \( \epsilon|\zeta - \eta| \) which is the Fourier multiplier for the low frequency quantity. Then these two terms are also of order \( O(\epsilon) \). The readers could refer to Section 3 for details.

To improve the bounds on the quadratic terms \( Q_{\epsilon, +}^\tau(V^+, V^-) \) and \( Q_{\epsilon, -}^\tau(V^-, V^-) \), we shall employ normal form techniques. To sketch the main idea, we only consider the simple model
\[ \partial_t \mathbf{V} - i\Lambda_{\epsilon} \mathbf{V} = Q_{\epsilon, +}^\tau(V^+, V^-), \]
where \( Q_{\epsilon, +}^\tau(V^+, V^-) \) is the quadratic term whose symbol \( q_{\epsilon, +}^\tau(\zeta, \eta) \) satisfies
\[ |q_{\epsilon, +}^\tau(\zeta, \eta)| \leq \epsilon|\zeta| \varphi_{\leq 5}(\sqrt{\epsilon}|\eta|) \varphi_{\leq 6}\left(\frac{|\zeta - \eta|}{\eta}\right), \]
and \( \text{supp}(q_{\epsilon, +}^\tau) \subset \left\{(\zeta, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \sqrt{\epsilon}|\eta| \leq 64, \frac{31}{32}|\eta| \leq |\zeta| \leq \frac{33}{32}|\eta| \right\} \).

The definition of the symbol \( q_{\epsilon, +}^\tau(\zeta, \eta) \) and the notation \( \varphi_{\leq} \) can be found in Sections 2.1 and 2.2. (1.11) yields that the rough bound of \( Q_{\epsilon, +}^\tau(V^+, V^-) \) is of order \( O(\sqrt{\epsilon}) \) which directly leads to the existence time of (1.10) being of order \( O\left(\frac{1}{\sqrt{\epsilon}}\right)\).

Since nontrivial zeroes of the phase occurs in the set of moderate frequencies, instead of employing the normal form transformation directly for \( Q_{\epsilon, +}^\tau(V^+, V^-) \), we use suitably modified normal form techniques when the integral regime is far away from the zero sets of the phase.

Assuming that
\[ \|\mathbf{V}(t)\|_{H^0_{\omega_0}} = O(1), \quad \text{for any} \ t \in [0, T_\epsilon], \quad T_\epsilon = O(\epsilon^{-\frac{3}{2}}), \]
we only need to show
\[ \| V(t) \|_{H^5_0}^2 \leq 1 + \epsilon^2 t. \] (1.12)

Then we could obtain the existence time of order \( O(\epsilon^{-3}) \) by a standard continuity argument. Indeed, an energy estimate for (1.10) yields
\[ \| V(t) \|_{H^5_0}^2 \leq \| V(0) \|_{H^5_0}^2 + \frac{1}{(4\pi)^2} \left| \int_{0}^{t} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle \xi \rangle^{2N_0} q_{+, -}^{\epsilon}(\xi, \eta) \overline{V^+(\xi - \eta)} (\eta) \overline{V^+(\xi)} d\eta d\xi dt \right|. \]

Defining the profiles \( f \) and \( g \) of \( V \) and \( \langle \nabla \rangle^{N_0} V \) as follows
\[ f = e^{-it\Lambda_c} V \quad \text{and} \quad g = \langle \nabla \rangle^{N_0} f, \]
we have
\[ A = \int_{0}^{t} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{it\Phi_{+, -}(\xi, \eta)} q_{+, -}^{\epsilon}(\xi, \eta) \overline{f^+(\xi - \eta)} \cdot g^+(\eta) \cdot \overline{g^-(\xi)} \ \eta d\xi dt, \]
where
\[ \Phi_{+, -}(\xi, \eta) = -\Lambda_c(\xi) + \Lambda_c(\xi - \eta) - \Lambda_c(\eta), \quad q_{+, -}^{\epsilon}(\xi, \eta) = \langle \eta \rangle^{-N_0}(\xi)^{N_0} q_{+, -}^{\epsilon}(\xi, \eta). \]

By the definitions of profiles, we have
\[ \| V \|_{H^5_0} \sim \| f \|_{H^5_0} \sim \| g \|_{L^2} \sim 1. \] (1.13)

Thanks to Lemma 2.1 and (1.11), we have
\[ \Phi_{+, -}(\xi, \eta) \sim |\eta| \phi_{+, -}(\xi, \eta), \quad \text{with} \ \phi_{+, -}(\xi, \eta) \ \text{defined in (2.5),} \]
\[ |q_{+, -}^{\epsilon}(\xi, \eta)| \leq \epsilon |\xi| \varphi_{\leq 5}(\sqrt{\epsilon |\eta|}) \varphi_{\leq -6}(\frac{|\xi - \eta|}{|\eta|}) \leq \sqrt{\epsilon}. \] (1.14)

Moreover, since
\[ \partial f = e^{-it\Lambda} Q_{+, -}(V^+, V^-), \]
using (1.13) and (1.14), we have
\[ \| D^{-1} \partial f \|_{H^5_0} = \| D^{-1} \partial g \|_{L^2} \leq \epsilon. \] (1.15)

According to the expressions of the phase \( \Phi_{+, -}(\xi, \eta) \), generally, we shall divide the integral regime into two cases: phase far away from the spatial resonance set and phase near the spatial resonance set. For the former case, we could use the normal formal techniques, that is, integrating by parts with respect to (w.r.t.) time \( t \). While for the latter case, we shall use the smallness of the volume of the integral regime. However, after similar arguments as that for the 1D Boussinesq system in [19], we could not improve the existence time scale \( \frac{1}{\sqrt{t}} \). This is because of the rough estimates over the latter regime, which is caused by the high dimension of the space. Therefore, to improve the existence time scale, we balance the size of symbol \( q_{+, -}^{\epsilon}(\xi, \eta) \) and the volume of the
integral regime when the phase near the spatial resonance set. To do so, we compare the sizes of $|\xi - \eta|$ and $|\eta|$.

Precisely, we divide the integral regime into the following three parts:

1. For low frequencies $\sqrt{\varepsilon} |\eta| \leq \frac{1}{2}$, there holds

$$|\phi^\varepsilon_{+,-}(\xi, \eta)| \sim 1 \quad \text{and} \quad \left| \frac{\tilde{q}^\varepsilon_{+,-}(\xi, \eta)}{ij\Phi^\varepsilon_{+,-}(\xi, \eta)} \right| \leq \frac{\varepsilon}{|\phi^\varepsilon_{+,-}(\xi, \eta)|} \leq \varepsilon.$$ Integrating by parts w.r.t. $t$ and using (1.13) and (1.15), we have

$$\left| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \Sigma^\varepsilon(\xi, \eta) \varphi_{\leq -2}(\sqrt{\varepsilon} |\eta|) |d\eta d\xi dt| \right| \leq \varepsilon + \varepsilon^2 t. \quad (1.16)$$

2. For moderate frequencies with phase far away from the spatial resonance set, i.e.,

$$\frac{1}{4} \leq \sqrt{\varepsilon} |\eta| \leq 64, \quad |\phi^\varepsilon_{+,-}(\xi, \eta)| \geq 2^{-D-1},$$

there holds

$$\left| \frac{\tilde{q}^\varepsilon_{+,-}(\xi, \eta)}{ij\Phi^\varepsilon_{+,-}(\xi, \eta)} \right| \leq \frac{\varepsilon}{|\phi^\varepsilon_{+,-}(\xi, \eta)|} \leq 2^D \varepsilon.$$ Here $D \in \mathbb{N}$ is a large number which will be determined later on. Integrating by parts w.r.t. $t$ and using (1.13) and (1.15), we obtain

$$\left| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \Sigma^\varepsilon(\xi, \eta) \varphi_{[-1, 0]}(\sqrt{\varepsilon} |\eta|) \varphi_{\leq -D}(\phi^\varepsilon_{+,-}(\xi, \eta)) |d\eta d\xi dt| \right| \leq 2^D \varepsilon + 2^D \varepsilon^2 t. \quad (1.17)$$

3. For moderate frequencies with phase near the spatial resonance set, i.e.,

$$\frac{1}{4} \leq \sqrt{\varepsilon} |\eta| \leq 64, \quad |\phi^\varepsilon_{+,-}(\xi, \eta)| \leq 2^{-D},$$

we shall split the integral regime into the following two parts

$$\frac{|\xi - \eta|}{|\eta|} \leq 2^{-K+1} \quad \text{and} \quad \frac{|\xi - \eta|}{|\eta|} \in [2^{-K}, 2^{-\frac{3}{2}}],$$

where $K \in \mathbb{N}$ is a large number will be determined later on.

(i) For case $\frac{|\xi - \eta|}{|\eta|} \leq 2^{-K+1}$, (1.14) gives rise to

$$|\tilde{q}^\varepsilon_{+,-}(\xi, \eta)| \approx \sqrt{\varepsilon},$$

and sine theorem yields

$$\angle(\xi, \eta) \approx \sin(\angle(\xi, \eta)) = \sin(\angle(\xi - \eta, \tilde{\xi})) \frac{|\xi - \eta|}{|\eta|} \leq 2^{-K+1},$$

where $\angle(\xi, \eta)$ is the angle between vectors $\xi$ and $\eta$. Since the bound of $|\tilde{q}^\varepsilon_{+,-}(\xi, \eta)|$ is not small enough, we shall use the smallness of the volume of integral regime whose size is determined by the size of $\angle(\xi, \eta)$.
By localizing the angular $\angle(\xi, \eta)$, we could obtain
\[
\left| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathcal{S}^c(\xi, \eta) \varphi_{[-1,5]}(\sqrt{\epsilon}|\eta|) \cdot \varphi_{\leq -D-1}(\phi_{+, -}(\xi, \eta)) \cdot \varphi_{\geq -K+1} \left( \frac{|\xi - \eta|}{|\eta|} \right) d\eta d\xi dt \right| \leq \sqrt{\epsilon} 2^{-\frac{\xi}{2}} t.
\]
(1.18)

In (1.18), we obtained a small factor $2^{-\frac{\xi}{2}}$ which is the contribution of the size of $\angle(\xi, \eta)$.

(ii) For case $\frac{|\xi - \eta|}{|\eta|} \in [2^{-K}, 2^{-5}]$, (1.14) yields
\[
|\tilde{q}_{+, -}^\epsilon(\xi, \eta)| \leq \epsilon 2^K |\xi - \eta|.
\]
(1.19)

Since $|\xi - \eta|$ is a good Fourier multiplier for the low frequency quantity, the bound of $|\tilde{q}_{+, -}^\epsilon(\xi, \eta)|$ has a small coefficient $\epsilon 2^K$. Then using the volume of the integral regime whose size is determined by the size of the phase, we have
\[
\left| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathcal{S}^c(\xi, \eta) \varphi_{[-1,5]}(\sqrt{\epsilon}|\eta|) \cdot \varphi_{\leq -D-1}(\phi_{+, -}(\xi, \eta)) \cdot \varphi_{\geq -K+1} \left( \frac{|\xi - \eta|}{|\eta|} \right) d\eta d\xi dt \right| 
\approx \epsilon 2^{\frac{\xi}{2}} 2^{-\frac{D}{2}} t.
\]
(1.20)

Combining (1.16), (1.17), (1.18), and (1.20), taking optimal $K$ and $D$, we obtain
\[
|A| \leq \epsilon t.
\]

Then we arrive at the energy estimate (1.12). The details of the proof are given in Section 4.

**Remark 1.5.** If we did not split the regime (3) into two parts: $\frac{|\xi - \eta|}{|\eta|} \leq 2^{-K+1}$ and $\frac{|\xi - \eta|}{|\eta|} \in [2^{-K}, 2^{-5}]$, the integral over the regime (3) is bounded by $2^{-\frac{D}{2}} t$. By the optimal choice $2^D \sim \epsilon^{-1}$, one only have
\[
|A| \leq \sqrt{\epsilon} t.
\]

That is to say, the normal formal technique does not improve the energy estimates. Thus, we have to decompose the integral regime in a more flexible way.

**2. Preliminary**

**2.1. Definitions and notations**

The notation $f \sim g$ means that there exists a constant $C$ such that $\frac{1}{C} f \leq g \leq C f$. Notations $f \leq g$ and $g \geq f$ mean that there exists a constant $C$ such that $f \leq C g$. We shall use $C$ to denote a universal constant which may changes from line to line. For any $s \in \mathbb{R}, H^s(\mathbb{R}^2)$ denotes the classical $L^2$ based Sobolev spaces with the norm $\| \cdot \|_{H^s}$. The notation $\| \cdot \|_{L^p}$ stands for the $L^p(\mathbb{R}^2)$ norm for $1 \leq p \leq \infty$.

For vectors $\xi, \eta \in \mathbb{R}^2$, the notation $\angle(\xi, \eta)$ represents the angle between $\xi$ and $\eta$.

The $L^2(\mathbb{R}^2)$ scalar product is denoted by $(u \mid v)_{L^2} \equiv \int_{\mathbb{R}^2} u \overline{v} dx$.

If $A, B$ are two operators, $[A, B] = AB - BA$ denotes their commutator.
The Fourier transform of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^2)$ is denoted by $\hat{u}$, which is defined as follows

$$\hat{u}(\xi) \overset{\text{def}}{=} \mathcal{F}(u)(\xi) = \int_{\mathbb{R}^2} e^{-ix \cdot \xi} u(x) dx.$$ 

We use $\mathcal{F}^{-1}(f)$ to denote the inverse Fourier transform of $f(\xi)$.

If $f$ and $u$ are two functions defined on $\mathbb{R}^2$, the Fourier multiplier $f(D)u$ is defined in term of Fourier transform, i.e.,

$$\widehat{f(D)u}(\xi) = f(\xi)\hat{u}(\xi).$$

We shall use notations

$$\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}, \quad \langle \nabla \rangle = (1 + |\nabla|^2)^{\frac{1}{2}}.$$ 

For two well-defined functions $f(x), g(x)$, and their bilinear form $Q(f, g)$, we use the convection that the symbol $q(\xi, \eta)$ of $Q(f, g)$ is defined in the following sense

$$\mathcal{F}(Q(f, g))(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} q(\xi, \eta)\widehat{f}(\xi - \eta)\widehat{g}(\eta) d\eta.$$ 

### 2.2. Para-differential decomposition theory

Our proof of the main result relies on suitable energy estimates for the solutions of (1.4). To do so, we introduce para-differential formulations (see, e.g., [26]) to symmetrize the system (1.4).

We fix an even smooth function $\phi : \mathbb{R} \to [0, 1]$ supported in $[-\frac{3}{4}, \frac{3}{4}]$ and equals to 1 in $[-\frac{5}{8}, \frac{5}{8}]$. For any $k \in \mathbb{Z}$, we define

$$\phi_k(x) \overset{\text{def}}{=} \phi\left(\frac{x}{2^k}\right) - \phi\left(\frac{x}{2^{k-1}}\right), \quad \phi_{\leq k}(x) \overset{\text{def}}{=} \sum_{l \leq k} \phi_l(x), \quad \phi_{\geq k}(x) \overset{\text{def}}{=} 1 - \phi_{\leq k-1}(x).$$

While for any interval $I$ of $\mathbb{R}$, we define

$$\phi_I(x) \overset{\text{def}}{=} \sum_{k \in I} \phi_k(x) = \sum_{k \in I \cap \mathbb{Z}} \phi_k(x).$$

Then for any $x \in \mathbb{R}$,

$$\sum_{k \in \mathbb{Z}} \phi_k(x) = 1 \quad \text{and} \quad \text{supp} \phi_k(\cdot) \subset \left\{x \in \mathbb{R} \mid |x| \in \left[\frac{5}{8} 2^k, \frac{3}{2} 2^k\right]\right\}. \tag{2.1}$$

We use $P_k$, $P_{\leq k}$, $P_{\geq k}$, and $P_I$ to denote the Littlewood–Paley projection operators of the Fourier multiplier $\phi_k$, $\phi_{\leq k}$, $\phi_{\geq k}$, and $\phi_I$, respectively.

We shall use the following para-differential decomposition: for any functions $f, g \in \mathcal{S}'(\mathbb{R}^2)$,

$$fg = T_f g + T_g f + R(f, g), \tag{2.2}$$

with the para-differential operators being defined as follows

$$T_f g = \sum_{j \in \mathbb{Z}} P_{\leq j} f \cdot P_j g, \quad R(f, g) = \sum_{j \in \mathbb{Z}} P_j f \cdot P_{|j-6, j+6]} g.$$
2.3. Analysis of the phases

In this subsection, we shall discuss the quadratic phase function $\Phi_{\mu, \nu}(\xi, \eta)$ which is defined as follows:

$$\Phi_{\mu, \nu}(\xi, \eta) = -\Lambda_\epsilon(\chi) + \mu \Lambda_\epsilon(\chi - \eta) + \nu \Lambda_\epsilon(\eta), \quad \mu, \nu \in \{+, -\}, \quad (2.3)$$

where $\Lambda_\epsilon(\chi)$ is defined by

$$\Lambda_\epsilon(\chi) = (1 - \epsilon|\chi|^2)|\chi| = |\chi| - \epsilon|\chi|^3.$$  

A direct calculation shows that

$$\Phi_{\mu, \nu}(\xi, \eta) = (|\xi| - \mu|\xi - \eta| - \nu|\eta|)[\epsilon(|\xi|^2 + |\xi - \eta|^2 + |\eta|^2 - \mu\nu|\xi - \eta||\eta|]$$

$$+ \mu|\xi||\xi - \eta| + \nu|\xi||\eta| - 1 + 3\mu\nu\epsilon|\xi - \eta||\eta|]. \quad (2.4)$$

Then we have the following lemma.

**Lemma 2.1.** Assuming that $(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$ satisfy $\xi \neq \eta$, $\xi \neq 0$, $\eta \neq 0$ and $\angle(\xi, \eta) \neq \pi$, we have

$$\Phi_{+, -}(\xi, \eta) = \frac{|\xi||\eta|}{|\xi| + |\xi - \eta| + |\eta|} \phi_{+, -}(\xi, \eta),$$

with

$$\phi_{+, -}(\xi, \eta) = 4 \cos^2 \left(\frac{1}{2} \angle(\xi, \eta)\right) \left[\epsilon(|\xi|^2 + |\eta|^2) - |\xi||\eta| - 1\right]$$

$$+ \epsilon \left(4 \cos^2 \left(\frac{1}{2} \angle(\xi, \eta)\right) - 3 \right)|\xi - \eta|(|\xi| + |\xi - \eta| + |\eta|), \quad (2.5)$$

and

$$\Phi_{-, -}(\xi, \eta) = (|\xi| + |\xi - \eta| + |\eta|) \phi_{-, -}(\xi, \eta),$$

with

$$\phi_{-, -}(\xi, \eta) = \epsilon(|\xi|^2 + |\eta|^2) - |\xi||\eta| - 1$$

$$+ \epsilon \left(\frac{3}{4 \cos^2 \left(\frac{1}{2} \angle(\xi, \eta)\right)} - 1 \right)|\xi - \eta|(|\xi| - |\xi - \eta| + |\eta|). \quad (2.6)$$

**Proof.** Since

$$|\xi| - |\xi - \eta| + |\eta| = \frac{(1 + \cos \angle(\xi, \eta))}{|\xi| + |\xi - \eta| + |\eta|}$$

$$= \frac{4|\xi||\eta|\cos^2 \left(\frac{1}{2} \angle(\xi, \eta)\right)}{|\xi| + |\xi - \eta| + |\eta|},$$

$$\frac{|\xi||\eta||\xi - \eta|}{|\xi| + |\xi - \eta| + |\eta|} = \frac{|\xi||\eta||\xi - \eta|(|\xi| - |\xi - \eta| + |\eta|)}{(1 + \cos \angle(\xi, \eta))^2}$$

$$= \frac{|\xi - \eta|(|\xi| - |\xi - \eta| + |\eta|)}{4 \cos^2 \left(\frac{1}{2} \angle(\xi, \eta)\right)} ,$$

(2.5) and (2.6) follow from (2.4).
2.4. Bilinear estimates with the angle localized

In order to improve the energy estimates near the spatial resonance set, we shall localize the angle \( \angle(\xi, \eta) \) between \( \xi \) and \( \eta \) whose small size makes a crucial contribution in the energy estimates. To catch the contribution caused by the localized angle, we need the following bilinear estimate.

**Lemma 2.2.** Let \( l, k, k_1, k_2 \in \mathbb{Z}, |k - k_2| \leq 2, l \leq -2, \) and \( m(\xi, \eta) \) satisfy

\[
\|m\|_{L^{\infty}_{k, k_1, k_2}} \overset{\text{def}}{=} \|m(\xi, \eta)\|_{L^{\infty}_{\xi, \eta}(|\xi| < 2^l, |\xi - \eta| < 2^{k_2})} < +\infty.
\]

For any \( f, g \in L^2(\mathbb{R}^2) \), defining a bilinear form as follows

\[
T_k(f, g)(\xi) = \int_{\mathbb{R}^2} m(\xi, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta) \varphi_k(\xi) \varphi_k(\xi - \eta) \varphi_l(\angle(\xi, \eta)) d\eta,
\]

we have

\[
\|T_k(f, g)(\xi)\|_{L^2} \lesssim 2^l 2^k \|m\|_{L^{\infty}_{k, k_1, k_2}} \cdot \|f_k\|_{L^2} \cdot \|g_k\|_{L^2},
\]

where \( f_k = \mathcal{F}^{-1}[\hat{f}(\xi) \varphi_k(\xi)], \) \( g_k = \mathcal{F}^{-1}[\hat{g}(\xi) \varphi_k(\xi)]. \)

**Remark 2.3.** The main contribution of the bilinear estimate (2.7) is to gain the factor \( 2^l. \)

**Proof.** First, for a given small number \( 2^n \in (0, 8)(n \in \mathbb{Z}) \), we decompose the unit circle \( S^1 \) into the union of \( N_n \) angular sectors, and each sector has angular size \( 2^n \), where \( N_n = O(2^{-n}) \). The number of the overlaps is bounded by a universal number \( n_0 \in \mathbb{N} \) that is independent of \( n \). We use notation \( \omega \in S^1 \) as the angular vector and \( \Delta \omega \) as the size of each angular sector. Then there exists a partition of unity \( \{b_n'(\omega)\} = \{b_n(\omega)\}_{j=1, \ldots, N_n} \) corresponding to the decomposition of \( S^1 \).

We define the angular vectors of \( \xi, \eta, \xi - \eta \in \mathbb{R}^2 \) as follows

\[
\omega_\xi \overset{\text{def}}{=} \frac{\xi}{|\xi|} \in S^1, \quad \omega_\eta \overset{\text{def}}{=} \frac{\eta}{|\eta|} \in S^1, \quad \omega_{\xi - \eta} \overset{\text{def}}{=} \frac{\xi - \eta}{|\xi - \eta|} \in S^1.
\]

By virtue of the sine theorem, we have the angular partition of unity \( \{b_{k_2-k_1+1}^l(\omega_\xi)\}, \{b_{k_2-k_1+1}^l(\omega_\eta)\} \) for \( \xi, \xi - \eta, \eta, \) where

\[
|\omega_\xi - \omega_\eta| \sim \angle(\xi, \eta) \sim 2^l, \quad |\omega_{\xi - \eta} - \omega_\eta| \sim \angle(\xi - \eta, \eta) \sim 2^{k_2-k_1+l} \text{ or } |\omega_{\xi - \eta} + \omega_\eta| \sim \angle(\xi - \eta, -\eta) \sim 2^{k_2-k_1+l},
\]

\[
\Delta \omega_{\xi - \eta} \sim 2^l, \quad \Delta \omega_\xi \sim 2^{k_2-k_1+l}, \quad \Delta \omega_\eta \sim 2^{k_2-k_1+l}.
\]

Since \( |\xi - \eta| \geq |\eta| \sin(\angle(\xi, \eta)) \approx |\eta| \angle(\xi, \eta) \), there holds \( k_2 - k_1 + l \leq 4 \) so that the decompositions in (2.8) are reasonable. Then we have
We remark that \( j_{\xi}, j_{\xi - \eta}, j_{\eta} \) in (2.9) are restricted by (2.8).

Thanks to the \( L^2 \)-orthogonality of \( \{ b_{k_2 - k_1 + l}(\omega_{\xi}) \} \), we have

\[
\| T_k(f, g)(\xi) \|_{L^2_{\xi}}^2 \leq \sum_{j_1, j_2} \left( \int_{\mathbb{R}^2} m(\xi, \eta) f_{k_1}(\xi - \eta) \tilde{g}_{k_2}(\eta) \varphi_k(\xi) \right) \leq \sum_{j_1, j_2} \left( \int_{\mathbb{R}^2} m(\xi, \eta) f_{k_1}(\xi - \eta) \tilde{g}_{k_2}(\eta) \varphi_k(\xi) \right)
\]

\[
\leq \| m \|_{L^\infty_{k_1, k_2}}^2 \sum_{j_1, j_2} \| f_{k_1}(\xi - \eta) b_{l_1}^{j_1}(\omega_{\xi - \eta}) \|_{L^2_{\xi - \eta}}^2 \| \tilde{g}_{k_2}(\eta) b_{l_2}^{j_2}(\omega_{\eta}) \|_{L^2_{\eta}}^2 \leq 2^{2k_1 + 1} \| m \|_{L^\infty_{k_1, k_2}}^2 \| f_{k_1} \|_{L^2} \| g_{k_2} \|_{L^2}.
\]

In the last inequality of (2.10), we used the volume of the integral regime.

Due to (2.8), for fixed \( j_{\xi - \eta} \), there are finite \( j_{\eta} \) such that (2.8) holds. While for fixed \( j_{\eta} \), there are finite \( j_{\xi} \) such that (2.8) holds. Using the \( L^2 \)-orthogonality of decompositions for the angle of \( \xi - \eta \) and \( \eta \), we deduce from (2.10) that

\[
\| T_k(f, g) \|_{L^2_{\xi}} \leq 2^{k_1 + 1} \| m \|_{L^\infty_{k_1, k_2}}^2 \| f_{k_1} \|_{L^2} \| g_{k_2} \|_{L^2}.
\]

This is exactly (2.7). The lemma is proved.

We end up this section with the following commutator estimate.

**Lemma 2.4.** Let \( s > -1 \), \( \nabla a \in L^\infty(\mathbb{R}^2) \) and \( b \in H^{s-1} \). There holds

\[
\| [ [D]^{-1} \div, T_a ] b \|_{H^s} \leq \| \nabla a \|_{L^\infty} \| b \|_{H^{s-1}}.
\]

The proof of Lemma 2.4 follows from the definition of \( T_{ab} \) and Theorem 3 of [27].

### 3. Symmetrization of system (1.4)

In this section, we shall symmetrize system (1.4) by using para-differential decomposition and introducing good unknowns. Then we state a main proposition on the symmetric system.

#### 3.1. Symmetrization of system (1.4)

First, we introduce a good unknown \( u \) with

\[
u = v + eB'(\xi, v),
\]

where \( B'(\cdot, \cdot) \) is a bilinear operator defined as
Using (1.4), (3.1), and (3.3), we have
\[ \partial_t \zeta + (1 + \epsilon \Delta) \nabla \cdot u = -\epsilon \nabla \cdot (\zeta v) + \epsilon (1 + \epsilon \Delta) \nabla \cdot B' (\zeta, v) \]
which implies
\[ \partial_t u + (1 + \epsilon \Delta) \nabla \zeta + \epsilon T_v \cdot \nabla u + \frac{\epsilon}{2} \nabla (T_v \zeta) = \frac{\epsilon}{2} \nabla (T_v |D|) \zeta + N'_u, \]
where
\[ N'_u = \frac{\epsilon}{2} T_v |D| \zeta - \epsilon T_v \cdot v - \epsilon R(v \cdot , \nabla v) - \epsilon B' ((1 + \epsilon \Delta) \nabla \cdot v, v) + \frac{\epsilon}{2} T_v \cdot \nabla B' - \frac{\epsilon}{2} B' (\nabla \cdot (|v|^2)), \]
we deduce from (3.2) and (3.4) that
\[
\partial_t V - i\Lambda_e V + \epsilon \nabla \cdot (T_v V) - \frac{i\epsilon}{2} |D|(T_v V) = -\frac{\epsilon}{2} \nabla \cdot (T_v \varphi \leq 5(\sqrt{\epsilon} |D|) V) - \frac{i\epsilon}{2} |D|(T_v \varphi \leq 5(\sqrt{\epsilon} |D|) \zeta) + N_V^e,
\]
where $\Lambda_e = |D|(1 + \epsilon \Delta)$ and
\[
N_V^e = -ie[D]^{-1} \nabla \cdot T_v \nabla u + ieT_{\text{div}}(|D|^{-1} \nabla u) - \frac{\epsilon}{2} |D|(|D|^{-1} \nabla, T_v) u + N_V^e + i|D|^{-1} \text{div} N_u^e.
\]

**Remark 3.1.** The left-hand side (L.H.S.) of (3.6) is the quasi-linear part of system (1.4). The first two terms in the right-hand side (R.H.S.) of (3.6) are the worst quadratic terms which are of order $O(\sqrt{\epsilon})$. The nonlinear terms in $N_V$ are of order $O(\epsilon)$.

We also remark that we could deduce from the evolution equations of $\zeta$ and $u$ that
\[
\partial_t V - i\Lambda_e V = -\epsilon \nabla \cdot (\zeta v) + \epsilon(1 + \epsilon \Delta) \nabla \cdot B' + i\epsilon |D|^{-1} \text{div} \left[-\frac{1}{2} \nabla(|v|^2) + B'(\partial_t \zeta, v) + B'(\zeta, \partial_t v)\right].
\]

Denoting by
\[
V^+ = V, \quad V^- = \bar{V},
\]
we shall rewrite the quadratic terms of (3.6) in terms of $V^+$ and $V^-$. Whereas we keep the remaining nonlinear terms in $N_V^e$ in terms of $\zeta$ and $v$.

By the definition of $V$, we have
\[
\zeta = \frac{1}{2}(V^+ + V^-) = \frac{1}{2} \sum_{\mu \in \{+,-\}} V^\mu, \quad \text{div } u = -\frac{i}{2} |D|(V^+ - V^-) = -\frac{i}{2} |D| \sum_{\nu \in \{+,-\}} \nu V^\nu.
\]
Since
\[
\text{div } u = \text{div } v + \epsilon \text{div } B', \quad \text{curl } v = 0,
\]
we have
\[
v = \Delta^{-1} \nabla (\text{div } u - \epsilon \text{div } B') = \frac{i}{2} \sum_{\nu \in \{+,-\}} \nu |D|^{-1} \nabla V^\nu - \epsilon \Delta^{-1} \nabla \text{div } B'.
\]
Then we have
\[
\zeta = \frac{1}{2} \sum_{\mu \in \{+,-\}} V^\mu, \quad \tilde{v} = \frac{i}{2} \sum_{\nu \in \{+,-\}} \nu |D|^{-1} \nabla V^\nu,
\]
\[
v = \tilde{v} - \epsilon \Delta^{-1} \nabla \text{div } B', \quad u = \tilde{v} - \epsilon \Delta^{-1} \nabla \text{div } B' + \epsilon B'.
\]

Before ending this subsection, we provide a lemma involving the bilinear operator $B'(\cdot, \cdot)$.

**Lemma 3.2.** Assume that the real-valued function $f \in L^\infty(\mathbb{R}^2)$ and vector function $g \in H^s(\mathbb{R}^2)$ for $s \geq -2$. There hold
\[ \mathcal{F}(B'(f, g))(\xi) = \mathcal{F}(B'(f, g))(-\xi) \]  
(3.9)

and for \( k = 0, 1, 2, \)
\[ \|B'(f, g)\|_{H^{k+1}} \leq C_B \epsilon^{-\frac{3}{2}}\|f\|_{L^{\infty}}\|g\|_{H^k}, \]  
(3.10)

where \( C_B > 0 \) is a universal constant.

The proof of the lemma is similar to that of Lemma 3.1 in [19]. We omit the proof here.

### 3.2. Main proposition for the symmetric system (3.6)

Now, we state the main proposition of this subsection.

**Proposition 3.3.** Assume that \( (f, v) \in H^{N_0}(\mathbb{R}^2) \) with \( N_0 \geq 5 \) solves (1.4). Then \( V \) defined in (3.5) satisfies the following system
\[ \partial_t V - i\Lambda V = S'_V + Q'_V + C'_V + N'_V, \]  
(3.11)

where

- The quadratic term \( S'_V \) is of the form
  \[ S'_V = \sum_{\mu \in \{-, +\}} S'_{\mu,+}(V^\mu, V^+), \]
  with the symbol \( s'_{\mu,+}(\xi, \eta) \) of \( S'_{\mu,+} \) satisfying
  \[ s'_{\mu,+}(\xi, \eta) = -s'_{\mu,+}(\xi, \eta), \]  
(3.12)
  \[ |\langle \xi \rangle^{-N_0} \langle \eta \rangle^{-N_0} (\langle \xi \rangle^{2N_0} s'_{\mu,+}(\xi, \eta) - \langle \eta \rangle^{2N_0} s'_{\mu,-}(\eta, \xi))| \leq \epsilon|\xi - \eta| \cdot \varphi_{\leq 6}(\frac{|\xi - \eta|}{\max\{|\xi|, |\eta|\}}). \]  
(3.13)

- The quadratic term \( Q'_V \) is of the form
  \[ Q'_V = \sum_{\mu, \nu \in \{-, +\}} Q'_{\mu,\nu}(V^\mu, V^\nu), \]
  with the symbol \( q'_{\mu,\nu}(\xi, \eta) \) of \( Q'_{\mu,\nu} \) satisfying
  \[ |q'_{\mu,-}(\xi, \eta)| \leq \epsilon|\xi| \cdot \varphi_{\leq 5}(\sqrt{|\xi|}|\eta|) \cdot \varphi_{\leq 6}(\frac{|\xi - \eta|}{|\eta|}), \]  
(3.14)
  \[ |q'_{\mu,+}(\xi, \eta)| \leq \epsilon|\xi - \eta| \cdot \varphi_{\leq 5}(\sqrt{|\xi|}|\eta|) \cdot \varphi_{\leq 6}(\frac{|\xi - \eta|}{|\eta|}). \]

- The cubic term \( C'_V = \epsilon^2 \nabla \cdot (T_{\Delta^{-1} \operatorname{div} B} V) \) satisfies
  \[ |\text{Re} \left\{ \left( \langle \nabla \rangle^{N_0} C'_V \right) \langle \nabla \rangle^{N_0} V \right\} | \leq \epsilon^2 \|\xi\|_{H^{N_0}}\|v\|_{H^{N_0}}\|V\|_{H^{N_0}}^2. \]  
(3.15)

- The nonlinear term \( N'_V \) satisfies
  \[ \|N'_V\|_{H^{N_0}} \leq \epsilon(1 + \|\xi\|_{H^{N_0}})(\|\xi\|_{H^{N_0}}^2 + \|V\|_{H^{N_0}}^2 + \|u\|_{H^{N_0}}^2). \]  
(3.16)
Remark 3.4. Proposition 3.3 suggests that there is no loss of derivative for the nonlinear terms of (3.6). Indeed, in the energy estimates, we shall use the symmetric structure of the quadratic terms $S^c_V$ to avoid the loss of derivatives (see (3.13)). We also use the symmetric structure of $C^c_V$ to avoid losing derivative (see the proof of (3.15)). The quadratic terms $S^c_V$, $Q^c_{\mu,+}$ and the nonlinear term $N^c_V$ are of order $O(\epsilon)$. Whereas the quadratic term $Q^c_{\mu,-}$ is of order $O(\sqrt{\epsilon})$ which directly leads to the existence time of order $O\left(\frac{1}{\sqrt{\epsilon}}\right)$. To enlarge the existence time scalar $\frac{1}{\sqrt{\epsilon}}$, we shall use norm formal techniques to deal with the worst quadratic term $Q^c_{\mu,-}$.

Proof of Proposition 3.3. The nonlinear terms in the R.H.S. of (3.11) stem from the nonlinear terms in (3.6). When we rewrite (3.6) to (3.11), using (3.8), the nonlinear terms of (3.11) first read as

$$S^c_V = -\epsilon \nabla \cdot (T_\nu V) + \frac{ie}{2} |D|(T_\xi V),$$

$$Q^c_V = -\frac{\epsilon}{2} \nabla \cdot (T_\xi \varphi_{\leq 5}(\sqrt{\epsilon}|D|)\tilde{V}) - \frac{ie}{2} |D|(T_\xi \varphi_{\leq 5}(\sqrt{\epsilon}|D|)\zeta),$$

$$C^c_V = \epsilon^2 \nabla \cdot (T_{\Delta^{-1}\nabla \text{div} B'} V),$$

$$N^c_V = \epsilon^2 \nabla \cdot (T_\xi \varphi_{\leq 5}(\sqrt{\epsilon}|D|)\Delta^{-1}\nabla \text{div} B') + N^c_V.$$

(1) For the quadratic term $S^c_V$, by virtue of (3.8), we rewrite it in terms of $V^+$ and $V^-$ as

$$S^c_V = \sum_{\mu \in \{+, -\}} S^c_{\mu,+}(V^\mu, V^+)$$

with

$$S^c_{\mu,+}(V^\mu, V^+) = -\mu \frac{ie}{2} \nabla \cdot (T_{|D|^{-1}\nabla \nu^\mu} V^+) + \frac{ie}{4} |D|(T_{V^\nu} V^+).$$

By the definition of the para-differential operator, we have

$$\mathcal{F}\left(S^c_{\mu,+}(V^\mu, V^+)(\xi)\right) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} s^c_{\mu,+}(\xi, \eta) \tilde{V}^\mu(\xi - \eta) \tilde{V}^+(\eta) d\eta$$

with the symbol $s^c_{\mu,+}(\xi, \eta)$ of $S^c_{\mu,+}(V^\mu, V^+)$ being as

$$s^c_{\mu,+}(\xi, \eta) = ie \left(\mu \frac{1}{2} \xi \cdot (\xi - \eta)|\xi - \eta|^{-1} + \frac{1}{4} |\xi|\right) \sum_{j \in \mathbb{Z}} \varphi_{\leq j-7}(|\xi - \eta|) \varphi_j(|\eta|). \quad (3.17)$$

Then (3.17) yields $\hat{s}^c_{\mu,+}(\xi, \eta) = -\hat{s}^c_{\mu,+}(\xi, \eta)$ which is exactly (3.12). By a direct derivation, it is easy to check that

$$\sum_{j \in \mathbb{Z}} \varphi_{\leq j-7}(|\xi - \eta|) \varphi_j(|\eta|) \lesssim \varphi_{\leq -6} \left(\frac{|\xi - \eta|}{|\eta|}\right). \quad (3.18)$$

Since
\[
\langle \xi \rangle^{2N_0} s_{\mu,+}^0(\xi, \eta) = \langle \eta \rangle^{2N_0} s_{-\mu,+}^0(\eta, \xi)
\]

\[
= i\epsilon \langle \xi \rangle^{2N_0} \left( \mu \frac{1}{2} \xi : (\xi - \eta)|\xi - \eta|^{-1} + \frac{1}{4} |\xi| \right) \sum_{j \in \mathbb{Z}} \varphi_{\leq J^{-1}}(|\xi - \eta|) \varphi_j(|\eta|)
\]

\[
- i\epsilon \langle \eta \rangle^{2N_0} \left( \mu \frac{1}{2} \eta : (\xi - \eta)|\xi - \eta|^{-1} + \frac{1}{4} |\eta| \right) \sum_{j \in \mathbb{Z}} \varphi_{\leq J^{-1}}(|\xi - \eta|) \varphi_j(|\xi|),
\]

and

\[ |\varphi_j(|\xi|) - \varphi_j(|\eta|)| \leq \frac{1}{\min\{|\xi|, |\eta|\}} |\xi - \eta|, \]

using (3.18), we obtain

\[ |\langle \xi \rangle^{2N_0} \langle \eta \rangle^{2N_0} (\langle \xi \rangle^{2N_0} s_{\mu,+}^0(\xi, \eta) - \langle \eta \rangle^{2N_0} s_{-\mu,+}^0(\eta, \xi))| \leq \epsilon |\xi - \eta| \cdot \varphi_{\leq -6} \left( \frac{|\xi - \eta|}{\max\{|\xi|, |\eta|\}} \right). \]

This is exactly (3.13).

(2) For the quadratic term \( \mathcal{Q}^r_V \), by virtue of (3.8), we rewrite it in terms of \( V^+ \) and \( V^- \) as

\[
\mathcal{Q}^r_V = \sum_{\mu, \nu \in \{+,-\}} Q^r_{\mu,\nu}(V^\mu, V^\nu),
\]

where

\[
Q^r_{\mu,\nu}(V^\mu, V^\nu) = -\nu \frac{i\epsilon}{8} \nabla \cdot (T_{V^\mu} \varphi_{\leq 5} (\sqrt{\epsilon}|D|)|D|^{-1} \nabla V^\nu) - \frac{i\epsilon}{8} |D|(T_{V^\mu} \varphi_{\leq 5} (\sqrt{\epsilon}|D|) V^\nu).
\]

Applying Fourier transformation to \( Q^r_{\mu,\nu}(V^\mu, V^\nu) \), we have

\[
\mathcal{F}(Q^r_{\mu,\nu}(V^\mu, V^\nu))(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} q^r_{\mu,\nu}(\xi, \eta) \tilde{V}^\mu(\xi - \eta) \tilde{V}^\nu(\eta) d\eta,
\]

with the symbol \( q^r_{\mu,\nu}(\xi, \eta) \) of \( Q^r_{\mu,\nu}(V^\mu, V^\nu) \) being as follows

\[
q^r_{\mu,\nu}(\xi, \eta) = \frac{i\epsilon}{8} \xi : \left( \frac{\nu}{|\eta|} - \frac{\xi}{|\xi|} \right) \varphi_{\leq 5} (\sqrt{\epsilon}|\eta|) \sum_{j \in \mathbb{Z}} \varphi_{\leq J^{-1}}(|\xi - \eta|) \varphi_j(|\eta|). \tag{3.19}
\]

Thanks to (3.18) and (3.19), we obtain

\[
|q^r_{\mu,-}(\xi, \eta)| \approx \epsilon |\xi| \varphi_{\leq 5} (\sqrt{\epsilon}|\eta|) \varphi_{\leq -6} \left( \frac{|\xi - \eta|}{|\eta|} \right),
\]

which is the first part of (3.14). And we also have

\[
|q^r_{\mu,+}(\xi, \eta)| \approx \epsilon |\xi| \varphi_{\leq 5} (\sqrt{\epsilon}|\eta|) \varphi_{\leq -6} \left( \frac{|\xi - \eta|}{|\eta|} \right), \tag{3.20}
\]

where we used the fact that \( \frac{\eta}{|\eta|} - \frac{\xi}{|\xi|} \sim \angle(\xi, \eta) \). Moreover, we have

\[
\text{supp } (q^r_{\mu,\nu}) \subset \left\{ (\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \sqrt{\epsilon}|\eta| \leq 64, \ |\xi - \eta| \leq \frac{1}{32} |\eta| \right\}
\]

\[
\subset \left\{ (\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \sqrt{\epsilon}|\eta| \leq 64, \ \frac{31}{32} |\eta| \leq |\xi| \leq \frac{33}{32} |\eta| \right\}. \tag{3.21}
\]
For $q_{\mu,+}(\xi, \eta)$, we only consider $\angle(\xi, \eta) \neq 0$. Thanks to sine theorem, we have
\[
\frac{|q_{\mu,+}(\xi, \eta)|}{\sin(\angle(\xi, \eta))} = \frac{|\eta|}{\sin(\angle(\xi, \eta) - \eta)} = \frac{|\xi|}{\sin(\angle(\xi - \eta, \eta))}.
\]
By virtue of (3.21), we see that $\angle(\xi, \eta) < \frac{\pi}{2}$ so that $\sin(\angle(\xi, \eta)) \sim \angle(\xi, \eta)$. Then we have
\[
|\xi| \angle(\xi, \eta) \sim |\xi - \eta| \sin(\angle(\xi - \eta, \eta)),
\]
which along with (3.20) implies
\[
|q_{\mu,+}(\xi, \eta)| \leq \epsilon |\xi - \eta| \cdot \varphi_{\leq \epsilon}(\sqrt{\epsilon}|\eta|) \varphi_{\leq \epsilon}(\sqrt{\frac{\epsilon}{|\eta|}}).
\]
This is the second part of (3.14).

(3) For the cubic term $C_\nu$, we first have
\[
\twidetilde{C}_\nu(\xi) = \frac{ie^2}{4\pi^2} \int_{\mathbb{R}^2} \left( \xi \cdot \frac{\xi - \eta}{|\xi - \eta|^2} \right) \cdot (\xi - \eta) \cdot \twidetilde{B}(\xi - \eta) \sum_{j \in \mathbb{Z}} \varphi_{\leq j-7}(\frac{|\xi - \eta|}{|\eta|}) \varphi_j(|\eta|) \widetilde{V}(\eta) d\eta.
\]
Then there holds
\[
((\nabla)^N_{\nu} C_\nu | (\nabla)^N_{\nu} V)_2 = \frac{ie^2}{2(2\pi)^4} \int_{\mathbb{R}^4} \langle \xi, \eta \rangle^{2N_0} \left( \frac{\xi - \eta}{|\xi - \eta|^2} \right) \cdot ((\xi - \eta) \cdot \twidetilde{B}(\xi - \eta)) \sum_{j \in \mathbb{Z}} \varphi_{\leq j-7}(\frac{|\xi - \eta|}{|\eta|}) \varphi_j(|\eta|) \widetilde{V}(\eta) \widetilde{V}(\xi) d\eta d\xi
\]
and
\[
((\nabla)^N_{\nu} C_\nu | (\nabla)^N_{\nu} V)_2
= -\frac{ie^2}{(2\pi)^4} \int_{\mathbb{R}^4} \langle \xi, \eta \rangle^{2N_0} \left( \frac{\xi - \eta}{|\xi - \eta|^2} \right) \cdot ((\xi - \eta) \cdot \twidetilde{B}(\xi - \eta)) \sum_{j \in \mathbb{Z}} \varphi_{\leq j-7}(\frac{|\xi - \eta|}{|\eta|}) \varphi_j(|\eta|) \widetilde{V}(\xi) \widetilde{V}(\eta) d\eta d\xi
= -\frac{ie^2}{(2\pi)^4} \int_{\mathbb{R}^4} \langle \eta, \xi \rangle^{2N_0} \left( \frac{\xi - \eta}{|\xi - \eta|^2} \right) \cdot ((\xi - \eta) \cdot \twidetilde{B}(\eta - \xi)) \sum_{j \in \mathbb{Z}} \varphi_{\leq j-7}(\frac{|\xi - \eta|}{|\eta|}) \varphi_j(|\eta|) \widetilde{V}(\eta) \widetilde{V}(\xi) d\eta d\xi.
\]
Thanks to (3.9), we have
\[
\twidetilde{B}(\eta - \xi) = \twidetilde{B}(\xi - \eta)
\]
which leads to
\[
\text{Re}\left\{((\nabla)^N_{\nu} C_\nu | (\nabla)^N_{\nu} V)_2\right\} = \frac{ie^2}{2(2\pi)^4} \int_{\mathbb{R}^4} \langle \xi, \eta \rangle^{N_0} \langle \eta \rangle^{N_0} \widetilde{V}(\eta) \cdot \langle \xi \rangle^{N_0} \widetilde{V}(\xi) d\eta d\xi,
\]
where
\[
c'(\xi, \eta) = \langle \xi \rangle^{-N_0} \langle \eta \rangle^{-N_0} \sum_{j \in \mathbb{Z}} \varphi_{\leq j-7}(\frac{|\xi - \eta|}{|\eta|}) \langle \xi \rangle^{2N_0} (\xi \cdot (\xi - \eta)) \varphi_j(|\eta|)
= -\langle \eta \rangle^{2N_0} (\eta \cdot (\xi - \eta)) \varphi_j(|\xi|) \frac{\xi - \eta}{|\xi - \eta|^2}.
\]
Since for fixed \((\xi, \eta)\),
\[
|\varphi_j(|\xi|) - \varphi_j(|\eta|)| \leq \frac{1}{\min\{|\xi|, |\eta|\}} |\xi - \eta|,
\]
and the summation in \(c'(\xi, \eta)\) is finite, using (3.18), we have \(|\xi| \sim |\eta|\) and
\[
|c'(\xi, \eta)| \leq |\xi - \eta|.
\]
Then we obtain
\[
\left| \text{Re} \left\{ \left( (\nabla)^{N_0}_V c_{V}^\epsilon \right) \cdot \langle \nabla \rangle^{N_0}_V \right\} \right| \leq \varepsilon^2 \int_{\mathbb{R}^4} |\xi - \eta| \| \widehat{\mathbf{B}}'(\xi - \eta) \| \cdot \langle \eta \rangle^{N_0}_V |\widehat{\mathbf{v}}(\eta)| \cdot \langle \xi \rangle^{N_0}_V |\widehat{\mathbf{v}}(\xi)| d\eta d\xi
\]
\[
\approx \varepsilon^2 \| \xi \| \| \mathbf{v} \| \| \mathbf{v} \|^{2}_{H^N}
\]
which along with (3.10) implies
\[
\left| \text{Re} \left\{ \left( (\nabla)^{N_0}_V c_{V}^\epsilon \right) \cdot \langle \nabla \rangle^{N_0}_V \right\} \right| \leq \varepsilon^2 \| \xi \| \| \mathbf{v} \| \| \mathbf{v} \|^{2}_{H^N}.
\]
This is (3.15).

(4) For the nonlinear term \(N'_{V}\), we estimate it term by term.

First, by definition of para-differential operators and (3.10), we have
\[
\varepsilon^2 \| \nabla \cdot (T_{\xi} \varphi \leq \sigma (\sqrt{\varepsilon} D)^{-1} \nabla \text{div} \mathbf{B}') \|_{H^N}^2
\]
\[
\approx \varepsilon^2 \| \xi \| \| \mathbf{v} \| \| \mathbf{v} \|^{2}_{H^N} \leq \varepsilon^2 \| \xi \| \| \mathbf{v} \| \| \mathbf{v} \|^{2}_{H^N}.
\]
Using (2.11), we have
\[
\varepsilon \| \|D\|^{-1} \text{div} (T_{\xi} \mathbf{v}) \|_{H^N} \approx \varepsilon \| \nabla \mathbf{v} \|_{L^\infty} \| \mathbf{v} \|_{H^N} \approx \varepsilon \| \mathbf{v} \|_{H^N} \| \mathbf{v} \|_{H^N}.
\]
By the definition of para-differential operators, we have
\[
\varepsilon \| T_{\text{div} \mathbf{v}} (|D|^{-1} \text{div} \mathbf{u}) \|_{H^N} \approx \varepsilon \| \nabla \mathbf{v} \|_{L^\infty} \| \mathbf{u} \|_{H^N} \approx \varepsilon \| \mathbf{v} \|_{H^N} \| \mathbf{u} \|_{H^N}.
\]
Then we obtain
\[
\| N'_{V} \|_{H^N} \approx \varepsilon (\| \mathbf{v} \|_{H^N} + \| \xi \|_{H^N}) \| \mathbf{u} \|_{H^N} + \varepsilon^2 \| \xi \|_{H^N} \| \mathbf{v} \|_{H^N} + \| N'_{V} \|_{H^N} + \| N'_{u} \|_{H^N}.
\]
For \(N'_{v}\), due to the definition of para-differential operators and (3.10), we have
\[
\varepsilon^2 \| \nabla \cdot (R(\xi, \mathbf{v})) + \frac{\varepsilon^2}{2} \nabla \cdot (\nabla \mathbf{B}') \|_{H^N} \approx \varepsilon \| \nabla \mathbf{v} \|_{L^\infty} \| \mathbf{v} \|_{H^N} + \varepsilon^2 \| \xi \|_{H^N} \| \mathbf{v} \|_{H^N} + \| N'_{v} \|_{H^N} + \| N'_{u} \|_{H^N}.
\]
and
\[
\varepsilon^2 \| \nabla \cdot (\nabla D (1 + \varepsilon \Delta^{-1} \varphi \geq \sigma (\sqrt{\varepsilon} D)^{-1} \mathbf{v}) \|_{H^N} \approx \varepsilon \| \nabla \mathbf{v} \|_{L^\infty} \| \mathbf{v} \|_{H^N} \approx \varepsilon \| \xi \|_{H^N} \| \mathbf{v} \|_{H^N},
\]
which imply
\[
\| N'_{v} \|_{H^N} \approx \varepsilon (1 + \| \xi \|_{H^N}) \| \xi \|_{H^N} \| \mathbf{v} \|_{H^N}.
\]
For \(N'_{u}\), by virtue of the definition of para-differential operators, we have
\[
\| \frac{\varepsilon}{2} T_{\nabla; \varphi_{\geq 6}} (\sqrt{\varepsilon} |D|) \zeta - \varepsilon T_{\nabla} \cdot v - \varepsilon R(\varepsilon \cdot v, \nabla v) \|_{H^0} \\
\leq \varepsilon (\| \nabla \zeta \|_{L^\infty} \| \zeta \|_{H^0} + \| \nabla v \|_{L^\infty} \| v \|_{H^0}) \leq \varepsilon (\| \zeta \|_{H^0}^2 + \| v \|_{H^0}^2),
\]

By the definition of \( B'(\cdot, \cdot) \), we have
\[
\varepsilon \| B'((1 + \varepsilon \Delta) \nabla \cdot v, v) \|_{H^0} = \frac{\varepsilon}{2} \| T_{(1 + \varepsilon \Delta) \nabla} v ((1 + \varepsilon \Delta)^{-1} \varphi_{\geq 6} (\sqrt{\varepsilon} |D|) v) \|_{H^0} \\
\leq \varepsilon \| \nabla v \|_{L^\infty} \| v \|_{H^0} \leq \varepsilon \| v \|_{H^0}^2.
\]

Using the definition of \( B'(\cdot, \cdot) \) again and (3.10), we have
\[
\varepsilon^2 \| T_v \cdot \nabla B'((\zeta, v)) \|_{H^0} \leq \varepsilon^2 \| v \|_{L^\infty} \| B'((\zeta, v)) \|_{H^0} \leq \varepsilon^2 \| v \|_{L^\infty} \| \zeta \|_{H^0} \leq \varepsilon^2 \| \zeta \|_{H^0} \| v \|_{H^0}^2,
\]
\[
\varepsilon^2 \| B'(\nabla \cdot (\zeta v), v) \|_{H^0} \leq \varepsilon^2 \| \nabla \cdot (\zeta v) \|_{L^\infty} \| v \|_{H^0} \leq \varepsilon^2 \| \zeta \|_{H^0} \| v \|_{H^0}^2,
\]
\[
\frac{\varepsilon^2}{2} \left\| B'((\zeta, \nabla(|v|^2)) \right\|_{H^0} \leq \varepsilon^2 \| \zeta \|_{L^\infty} \| \nabla(|v|^2) \|_{H^0} \leq \varepsilon^2 \| \zeta \|_{H^0} \| v \|_{H^0}^2.
\]

Then we obtain
\[
\| N_{u}^c \|_{H^0} \leq \varepsilon \| \zeta \|_{H^0}^2 + \varepsilon (1 + \| \zeta \|_{H^0}) \| v \|_{H^0}^2. \tag{3.24}
\]

Combining (3.22), (3.23), and (3.24), we have
\[
\| N_{v}^c \|_{H^0} \leq \varepsilon (1 + \| \zeta \|_{H^0})(\| \zeta \|_{H^0}^2 + \| v \|_{H^0}^2 + \| u \|_{H^0}^2).
\]

This is exactly (3.16).

Therefore, we complete the proof of the proposition. \( \square \)

### 4. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1 via the symmetric formulation (3.6) and the normal form techniques. The proof relies on the standard continuity argument and the main priori estimates for (1.4) which are stated in the following subsections.

#### 4.1. Ansatz for the continuity arguments

The first ansatz for the continuity arguments involves the amplitude \( \zeta \) as follows
\[
\varepsilon \| \zeta \|_{L^\infty} \leq \frac{1}{2C_B}, \quad \text{for any } t \in [0, T_0 \varepsilon^{-\frac{1}{2}}], \tag{4.1}
\]
where \( C_B \) is a constant stated in Lemma 3.2.

Let us define the energy functional for (1.4) as
\[
E_{N_0}(t) \overset{\text{def}}{=} \| \zeta(t, \cdot) \|_{H^0}^2 + \| v(t, \cdot) \|_{H^0}^2.
\]

For simplicity and without loss of generality, we assume that
\[
E_{N_0}(0) \overset{\text{def}}{=} \| \zeta_0 \|_{H^0}^2 + \| v_0 \|_{H^0}^2 = 1. \tag{4.2}
\]

The second ansatz is about the energy which reads
\[ E_{N_0}(t) \leq 2C_0, \quad \text{for any } t \in \left[0, T_0\epsilon^{-\frac{3}{2}}\right], \]  

(4.3)

where \( C_0 > 1 \) is an universal constant that will be determined at the end of the proof. We take
\[ T_0 = \frac{C_1}{C_2}, \quad C_0 = 2C_1, \]

where \( C_1 \) and \( C_2 \) are the constants stated in Proposition 4.1.

The standard continuity argument shows that: since for sufficiently small \( \epsilon > 0, \)
\[ E_{N_0}(0) = 1 \leq 2C_0, \quad \epsilon\|\zeta_0\|_{L^\infty} \leq \frac{1}{4C_B} \leq \frac{1}{2C_B}, \]
ansatz (4.1) and (4.3) hold on a short time interval \([0, t^*] \), where \( t^* \) is the maximal possible time on which (4.1) and (4.3) are correct. Without loss of generality, we assume that \( t^* = T_0\epsilon^{-\frac{3}{2}} \).

To close the continuity argument, we need to verify that: there exists a sufficiently small \( \epsilon_0 > 0 \) such that for any \( \epsilon \in (0, \epsilon_0) \), we improve the ansatz (4.1) and (4.3) as follows
\[ \epsilon\|\zeta\|_{L^\infty} \leq \frac{1}{4C_B}, \quad \text{for any } t \in \left[0, T_0\epsilon^{-\frac{3}{2}}\right], \]  

(4.4)

\[ E_{N_0}(t) \leq C_0, \quad \text{for any } t \in \left[0, T_0\epsilon^{-\frac{3}{2}}\right]. \]  

(4.5)

Theorem 1.1 follows from the above continuity argument and the local regularity theorem. To complete the last step of the continuity argument that improves the constants in the ansatz, we need Proposition 4.1 that concerns the \textit{a priori} energy estimates on (1.4).

### 4.2. The \textit{a priori} energy estimates

In this subsection, we shall derive the \textit{a priori} energy estimates on the solutions of (1.4)–(1.5). The main result is stated in the following proposition.

**Proposition 4.1.** Under the ansatz (4.1) and (4.3), the solution \((\zeta, v)\) to (1.4)–(1.5) satisfies
\[ E_{N_0}(t) \leq C_1 + C_2\epsilon^3 t, \quad \text{for any } t \in (0, T_0\epsilon^{-\frac{3}{2}}], \]  

(4.6)

where \( C_1 \) and \( C_2 \) are universal constants, and \( T_0 = \frac{C_1}{C_2}. \)

**Proof.** We divide the proof into several steps.

**Step 1.** Energy estimates. First, thanks to (3.10) and (4.1), we have
\[ \epsilon\|B'(\zeta, v)\|_{H^{\frac{1}{2}}} \leq C_B\epsilon\|\zeta\|_{L^\infty}\|v\|_{H^1} \leq \frac{1}{2}\|v\|_{H^1}, \]

which along with (3.1) and (3.5) implies
\[ E_{N_0}(t) \sim \|\zeta\|_{H^0}^2 + \|u\|_{H^0}^2 \sim \|V\|_{H^0}^2. \]  

(4.7)

With (4.7), we start to derive the energy estimates on the dispersive formulation (3.11) as follows
\[
\frac{1}{2} \frac{d}{dt} \|V(t)\|_{H^0}^2 = \text{Re} \left\{ \langle \nabla \rangle^{N_0} \mathcal{S}_V \ | \ \langle \nabla \rangle^{N_0} V \rangle_2 + \langle \nabla \rangle^{N_0} \mathcal{Q}_V \ | \ \langle \nabla \rangle^{N_0} V \rangle_2 \right. \\
+ \left. \langle \nabla \rangle^{N_0} \mathcal{C}_V^c \ | \ \langle \nabla \rangle^{N_0} V \rangle_2 + \langle \nabla \rangle^{N_0} \mathcal{N}_V^c \ | \ \langle \nabla \rangle^{N_0} V \rangle_2 \right\}.
\]

Thanks to the estimates (3.15) and (3.16) in Proposition 3.3, using (4.2), (4.3), and (4.7), we obtain

\[
\mathcal{E}_{N_0}(t) \leq 1 + |\text{Re}(I)| + |\text{Re}(II)| + |\text{Re}(III)| + et,
\]

where

\[
I \overset{\text{def}}{=} \sum_{\mu \in \{+, -\}} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle \xi \rangle^{2N_0} \hat{\eta} \mu_\mu + (\xi, \eta) \nabla^\mu (\xi - \eta) \nabla^+ (\eta) \nabla^+ (\xi) d\eta d\xi dt,
\]

\[
II \overset{\text{def}}{=} \sum_{\mu \in \{+, -\}} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle \xi \rangle^{2N_0} \hat{q} \mu_\mu - (\xi, \eta) \nabla^\mu (\xi - \eta) \nabla^- (\eta) \nabla^- (\xi) d\eta d\xi dt,
\]

\[
III \overset{\text{def}}{=} \sum_{\mu \in \{+, -\}} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle \xi \rangle^{2N_0} \hat{q} \mu_\mu + (\xi, \eta) \nabla^\mu (\xi - \eta) \nabla^+ (\eta) \nabla^+ (\xi) d\eta d\xi dt.
\]

By virtue of (3.13) and (3.14), it is easy to get the energy estimate

\[
\mathcal{E}_{N_0}(t) \leq 1 + \sqrt{et},
\]

which gives rise to the local existence of (1.4)–(1.5) on a finite time interval of scalar \(\frac{1}{\sqrt{et}}\). To enlarge the time scalar \(\frac{1}{\sqrt{et}}\), we shall use normal formal techniques to deal with the worst term III that is involving the quadratic term \(Q_{\mu, -}(\nu^+, \nu^-)\).

**Step 2. Estimate for Re(I) and Re(II).** In this step, we shall derive the following estimate

\[
|\text{Re}(I)| + |\text{Re}(II)| \leq et.
\]

For I, by the expression of I, we have

\[
\bar{I} = \sum_{\mu \in \{+, -\}} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle \xi \rangle^{2N_0} \hat{\eta} \mu_\mu + (\xi, \eta) \nabla^\mu (\xi - \eta) \nabla^- (\eta) \nabla^- (\xi) d\eta d\xi dt,
\]

which along with (3.12) and the fact that

\[
\nabla^\mu (\xi) = \nabla^- (\xi)
\]

shows

\[
\bar{I} = -\sum_{\mu \in \{+, -\}} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle \eta \rangle^{2N_0} \hat{q} \mu_\mu + (\eta, \xi) \nabla^- (\xi - \eta) \nabla^+ (\eta) \nabla^+ (\xi) d\eta d\xi dt
\]

\[
= -\sum_{\mu \in \{+, -\}} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle \eta \rangle^{2N_0} \hat{q} \mu_\mu + (\eta, \xi) \nabla^- (\xi - \eta) \nabla^+ (\eta) \nabla^+ (\xi) d\eta d\xi dt.
\]

Since \(\text{Re}(I) = \frac{1}{2} (I + \bar{I})\), we have

\[
\text{Re}(I) = \sum_{\mu \in \{+, -\}} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \hat{q} \mu_\mu + (\xi, \eta) \nabla^\mu (\xi - \eta) \cdot \langle \eta \rangle^{N_0} \nabla^+ (\eta) \cdot \langle \xi \rangle^{N_0} \nabla^+ (\xi) d\eta d\xi dt,
\]
where
\[ z_{\mu,+}^{\epsilon}(\xi, \eta) \overset{\text{def}}{=} \frac{1}{2} \langle \xi \rangle^{-N_0} \langle \eta \rangle^{-N_0} \langle \xi \rangle^{2N_0} z_{\mu,+}^{\epsilon}(\xi, \eta) - \langle \eta \rangle^{2N_0} z_{-\mu,+}^{\epsilon}(\xi, \eta). \]

Thanks to (3.13), we have
\[ |z_{\mu,+}^{\epsilon}(\xi, \eta)| \leq \epsilon |\xi - \eta| \cdot \phi_{\xi} \leq 6 \left( \max \left\{ |\xi|, |\eta| \right\} \right). \]

Then we have
\[
|\text{Re}(I)| \leq \epsilon \sum_{\mu \in \{+, -\}} \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\xi - \eta| |\hat{V}(\xi - \eta)| \cdot \langle \eta \rangle^{N_0} |\hat{V}^+(\xi)| \cdot \langle \xi \rangle^{N_0} |\hat{V}^+(\xi)| d\eta d\xi dt
\]
\[
\leq \epsilon t \sup_{\tau \in (0, t)} \left( \|\hat{V}(\tau, \xi)\|_{L^1} \left\| \hat{V}^+(\tau, \xi) \right\|_{L^1} \right) \]
\[
\leq \epsilon t \sup_{\tau \in (0, t)} \left( \|V(\tau)\|_{H^{N_0}} \|V^+(\tau)\|_{H^{N_0}} \right) \leq \epsilon t \sup_{\tau \in (0, t)} \|V(\tau)\|_{H^{N_0}}^3.
\]
By virtue of (4.7) and (4.3), we obtain
\[
|\text{Re}(I)| \leq \epsilon t. \tag{4.11}
\]

Following a similar derivation as for (4.11), using the second inequality of (3.14), we have
\[
|\text{Re}(II)| \leq |II| \leq \epsilon t. \tag{4.12}
\]

Due to (4.11) and (4.12), we obtain (4.10).

**Step 3.** Estimate for Re(III). Due to the first inequality of (3.14), III is of order $O(\sqrt{\epsilon})$. In order to improve the bound of III, we shall apply a normal form technique to III. The aim of this step is to derive the following estimate
\[
|III| \leq 1 + \epsilon^3 t. \tag{4.13}
\]

**Step 3.1.** The evolution equation and estimates of the profile. We introduce the profiles $f$ and $g$ of $V$ and $\langle \nabla \rangle^{N_0} V$ as follows
\[ f = e^{-itA} V \quad \text{and} \quad g = \langle \nabla \rangle^{N_0} f. \]

Due to (4.7), we have
\[
E_{N_0}(t) \sim \|V\|_{H^{N_0}}^2 \sim \|f\|_{H^{N_0}}^2 = \|g\|_{L^2}^2. \tag{4.14}
\]

By virtue of (3.7), we have
\[
\partial_t f = \epsilon e^{-itA} \left\{ -\nabla \cdot (\hat{\nu} + (1 + \epsilon \Delta) \nabla \cdot B') + i |D|^{-1} \text{div} \left[ -\frac{1}{2} \nabla (|\nu|^2) + B' (\partial_\xi \nu, \nu) + B' (\xi, \nu) \right] \right\} \tag{4.15}
\]

Then we have
\[
|\|D|^{-1} \partial_t f\|_{H^{N_0}} \leq \epsilon (\|\xi \nu\|_{H^{N_0}} + \|(1 + \epsilon \Delta) B'\|_{H^{N_0}} + \|\nu^2\|_{H^{N_0}}
\]
\[
+ \|D|^{-1} (B' (\partial_\xi \nu, \nu))\|_{H^{N_0}} + \|D|^{-1} (B' (\xi, \nu))\|_{H^{N_0}}. \]
By virtue of the definition of $B^\prime(\cdot, \cdot)$, we have

$$\text{supp } B^\prime(\cdot, \cdot)(\xi) \subset \{ \xi \in \mathbb{R}^2 \mid \sqrt{\epsilon} |\xi| \geq 2^5 \},$$

which along with (1.4), (3.10) and the definition of $B^\prime(\cdot, \cdot)$ implies

$$\| (1 + \epsilon A) B^\prime \|_{H^0} \leq \| \xi \|_{L^\infty} \| v \|_{H^0},$$

$$\| D^{-1} (B^\prime (\partial_1 \xi, v)) \|_{H^0} \leq \sqrt{\epsilon} \| \partial_1 \xi \|_{L^\infty} \| v \|_{H^0}$$

$$\leq \sqrt{\epsilon} (\| v \|_{H^2} + \epsilon \| \xi \|_{H^2} \| v \|_{H^0}) \| v \|_{H^0} \leq (1 + \epsilon \| \xi \|_{H^0}) \| v \|_{H^0}^2,$$

$$\| D^{-1} (B^\prime (\xi, \partial_1 v)) \|_{H^0} \leq \| \xi \|_{L^\infty} \| D^{-1} \partial_1 v \|_{H^0}$$

$$\leq \| \xi \|_{H^2} (\| v \|_{H^0} + \epsilon \| v \|_{H^0}^2) \leq \| \xi \|_{H^0}^2 + \epsilon \| \xi \|_{H^0} \| v \|_{H^0}^2).$$

Thanks to (4.3), we have

$$\| D^{-1} \partial_1 f \|_{H^0} \leq \varepsilon. \quad (4.16)$$

**Step 3.2.** The profile formulation of III. We denote

$$\Sigma^\prime_\mu(\xi, \eta) = (\xi)^{2N_0} q^\mu_{\mu^-}(\xi, \eta) \overline{V^\mu(\xi - \eta)} \overline{V^\prime}(-\xi).$$

Now we rewrite $\Sigma^\prime_\mu(\xi, \eta)$ in terms of the profiles $f$ and $g$ as follows

$$\Sigma^\prime_\mu(\xi, \eta) = e^{iP^\mu_{\mu^-}(\xi, \eta)} q^\mu_{\mu^-}(\xi, \eta) \hat{f}^\mu(\xi - \eta) \cdot \hat{g}(-\xi),$$

where

$$P^\mu_{\mu^-}(\xi, \eta) = -\Lambda_\mu(\xi) + \mu \Lambda_\mu(\xi - \eta) - \Lambda_\mu(\eta), \quad \hat{q}^\mu_{\mu^-}(\xi, \eta) = \langle \eta \rangle^{-N_0} (\xi)^{N_0} q^\mu_{\mu^-}(\xi, \eta).$$

Thanks to (3.14), we have $\text{supp } (\hat{q}^\mu_{\mu^-}) = \text{supp } (q^\mu_{\mu^-})$ and

$$|\hat{q}^\mu_{\mu^-}(\xi, \eta)| \leq \epsilon |\xi| \cdot \varphi_{\leq 5}(\sqrt{\epsilon} |\eta|) \cdot \varphi_{\leq 6}(\frac{|\xi - \eta|}{|\eta|}) \quad (4.17)$$

Due to Lemma 2.1 and (3.21), for any $(\xi, \eta) \in \text{supp } (q^\mu_{\mu^-})$, we have

$$P^\mu_{\mu^-}(\xi, \eta) \sim |\eta| P^\mu_{\mu^-}(\xi, \eta), \quad (4.18)$$

where $P^\mu_{\mu^-}(\xi, \eta)$ is defined in (2.5) and (2.6).

**Step 3.3.** Estimate for $\int_0^1 \int_{\mathbb{R}^2 \times \mathbb{R}^2} \Sigma^\prime_\mu(\xi, \eta) d\eta d\xi dt$. First, due to (3.21), for any $(\xi, \eta) \in \text{supp } (q^\mu_{\mu^-})$,

$$\sin (\angle(\xi, \eta)) \leq \frac{|\xi - \eta|}{|\eta|} \leq \frac{1}{32},$$

which gives rise to

$$\sin (\angle(\xi, \eta)) \approx \angle(\xi, \eta) \leq \frac{1}{31}, \quad 1 \approx \cos \left(\frac{1}{2} \angle(\xi, \eta) \right) \geq \frac{31}{32}. \quad (4.19)$$

We divide the integral regime into the following three parts:

1. For low frequencies $\sqrt{\epsilon} |\eta| \leq \frac{1}{2}$, using (2.5), (4.19), (4.17), and (4.18), we have

$$P^\mu_{\mu^-}(\xi, \eta) \approx 4(\epsilon |\eta|^2 - 1),$$

so that
where we used the facts that

\[ \frac{\dot{q}^\varepsilon_{+, -}(\xi, \eta)}{i \phi^\varepsilon_{+, -}(\xi, \eta)} \leq \frac{\varepsilon}{|\phi^\varepsilon_{+, -}(\xi, \eta)|} \leq \varepsilon. \]  

(4.20)

Integrating by parts w.r.t. \( t \), we have

\[
\int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathcal{L}^\varepsilon_+ (\xi, \eta) \varphi_{\leq -2}(\sqrt{\varepsilon} |\eta|) \ dn \eta \, d \xi \, dt
\]

\[
= \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\dot{q}^\varepsilon_{+, -}(\xi, \eta)}{i \phi^\varepsilon_{+, -}(\xi, \eta)} e^{i \phi^\varepsilon_{+, -}(\xi, \eta)} \tilde{f}^\varepsilon_+(\tau, \xi - \eta) \cdot \tilde{g}^\varepsilon_-(\tau, \eta) \cdot \tilde{g}^\varepsilon_-(\tau, -\xi) \varphi_{\leq -2}(\sqrt{\varepsilon} |\eta|) \ dn \eta \, d \xi \, dt
\]

\[
- \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\dot{q}^\varepsilon_{+, -}(\xi, \eta)}{i \phi^\varepsilon_{+, -}(\xi, \eta)} e^{i \phi^\varepsilon_{+, -}(\xi, \eta)} \partial_t (\tilde{f}^\varepsilon_+(\xi - \eta) \cdot \tilde{g}^\varepsilon_-(\eta) \cdot \tilde{g}^\varepsilon_-(\xi)) \varphi_{\leq -2}(\sqrt{\varepsilon} |\eta|) \ dn \eta \, d \xi \, dt.
\]

For \( A_1 \), by virtue of (4.20), we have

\[
|A_1| \leq \varepsilon \sup_{\tau \in [0, t]} \left( \|\tilde{f}^\varepsilon_+(\tau, \cdot)\|_{L^2} \cdot \|\tilde{g}^\varepsilon_-(\tau, \cdot)\|_{L^2} \right) \leq \varepsilon \sup_{\tau \in [0, t]} \left( \|f(\tau, \cdot)\|_{L^2} \cdot \|g(\tau, \cdot)\|_{L^2} \right),
\]

which along with (4.3) and (4.14) implies

\[
|A_1| \leq \varepsilon. \tag{4.21}
\]

For \( A_2 \), using (4.20), we have

\[
|A_2| \leq \varepsilon t \sup_{\tau \in [0, t]} \left( \|\tilde{f}^\varepsilon_+(\tau, \cdot)\|_{L^1} \cdot \|\tilde{g}^\varepsilon_-(\tau, \cdot)\|_{L^1} \right) \leq \varepsilon t \sup_{\tau \in [0, t]} \left( \|f(\tau, \cdot)\|_{L^2} \cdot \|g(\tau, \cdot)\|_{L^2} \right),
\]

where we used the facts that \( \sqrt{\varepsilon} \leq \frac{1}{2} \), \( |\xi - \eta| \leq \frac{1}{2} |\eta| \) and \( |\xi| \sim |\eta| \) in the second inequality. By virtue of (4.3), (4.14), and (4.16), we obtain

\[
|A_2| \leq \varepsilon^2 t,
\]

which along with (4.21) shows

\[
\left| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathcal{L}^\varepsilon_+ (\xi, \eta) \varphi_{\leq -2}(\sqrt{\varepsilon} |\eta|) \ dn \eta \, d \xi \, dt \right| \leq \varepsilon + \varepsilon^2 t. \tag{4.22}
\]
(2) For moderate frequencies with phase far away from the spatial resonance set, i.e.,

$$\frac{1}{4} \leq \sqrt{\epsilon}|\eta| \leq 64, \quad |\phi_{+,-}^\epsilon(\xi, \eta)| \geq 2^{-D-1},$$

we have

$$\left| \frac{\tilde{q}_{+,-}^\epsilon(\xi, \eta)}{i\Phi_{+,-}^\epsilon(\xi, \eta)} \right| \approx \frac{\epsilon}{|\phi_{+,-}^\epsilon(\xi, \eta)|} \leq 2^D \epsilon.$$

Here $D \in \mathbb{N}$ is a large number which will be determined later on.

Following similar a derivation as in (4.22), we obtain

$$\left| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \Sigma^\epsilon_+(\xi, \eta) \varphi_{[-1,5]}(\sqrt{\epsilon}|\eta|) \varphi_{\leq -D}(\phi_{+,-}^\epsilon(\xi, \eta)) d\eta d\xi dt \right| \leq 2^D \epsilon + 2^D \epsilon^2 t. \quad (4.23)$$

(3) For moderate frequencies with phase near the spatial resonance set, i.e.,

$$\frac{1}{4} \leq \sqrt{\epsilon}|\eta| \leq 64, \quad |\phi_{+,-}^\epsilon(\xi, \eta)| \leq 2^{-D},$$

we shall split the integral regime into the following two parts

$$\frac{|\xi - \eta|}{|\eta|} \leq 2^{-K+1} \quad \text{and} \quad \frac{|\xi - \eta|}{|\eta|} \in [2^{-K}, 2^{-5}],$$

where $K \in \mathbb{N}$ is a large number which will be determined later on. We use the smallness of $\angle(\xi, \eta)$ for the former part, while we use the smallness of the symbol $\tilde{q}_{+,-}$ and the volume of the integral regime for the latter part.

(i) For case $\frac{|\xi - \eta|}{|\eta|} \leq 2^{-K+1}$, using sine theorem, we have

$$\sin(\angle(\xi, \eta)) = \sin(\angle(\xi - \eta, \xi)) \frac{|\xi - \eta|}{|\eta|} \leq 2^{-K+1},$$

which along with (4.19) gives rise to

$$2^l \sim \angle(\xi, \eta) \leq 2^{-K+2}, \quad \text{i.e.,} \quad l \leq -K + 3. \quad (4.24)$$

Now localizing the angle $\angle(\xi, \eta)$, we have

$$\left| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \Sigma^\epsilon_+(\xi, \eta) \varphi_{[-1,5]}(\sqrt{\epsilon}|\eta|) \cdot \varphi_{\leq -D-1}(\phi_{+,-}^\epsilon(\xi, \eta)) \cdot \varphi_{\leq -K} \left( \frac{|\xi - \eta|}{|\eta|} \right)$$

$$\cdot \varphi_k(|\xi|) \varphi_k(|\xi - \eta|) \varphi_k(|\eta|) \varphi_l(\angle(\xi, \eta)) d\eta d\xi dt \right| \leq t \sup_{t \in [0, t]} \left( \|\hat{g}_k\|_{L^2} \|T_k(f, g)\|_{L^2} \right), \quad (4.25)$$

where

$$T_k(f, g) \overset{\text{def}}{=} \int_{\mathbb{R}^2} e^{i\Phi_{+,-}^\epsilon(\xi, \eta)} \tilde{q}_{+,-}^\epsilon(\xi, \eta) \varphi_{[-1,5]}(\sqrt{\epsilon}|\eta|) \cdot \varphi_{\leq -D-1}(\phi_{+,-}^\epsilon(\xi, \eta)) \cdot \varphi_{\leq -K} \left( \frac{|\xi - \eta|}{|\eta|} \right)$$

$$\cdot f^+ (\xi - \eta) g^-(\eta) \varphi_k(|\xi|) \varphi_k(|\xi - \eta|) \varphi_k(|\eta|) \varphi_l(\angle(\xi, \eta)) d\eta$$
Due to (4.17), there holds $|\tilde{q}_{+, -}(\xi, \eta)| \leq \sqrt{\epsilon}$ and $|k - k_2| \leq 2$. Then using the bilinear estimate (2.7), we have

$$
\|T_k(f, g)\|_{L^2} \leq \sqrt{\epsilon} 2^k \cdot 2^{1 + \frac{1}{2}} \|f_k\|_{L^2} \|g_k\|_{L^2}.
$$

(4.26)

Thanks to (4.24), (4.25), and (4.26), we obtain

$$
\left| \int_0^t \int_{\mathbb{R}^2} \mathcal{S}_+^c(\xi, \eta) \varphi_{[-1, 5]}(\sqrt{\epsilon} \eta) \cdot \varphi_{[-D-1]}(\phi_{+, -}^c(\xi, \eta)) \cdot \varphi_{-K} \left( \frac{|\xi - \eta|}{|\eta|} \right) \cdot \varphi_k \left( \frac{|\xi - \eta|}{|\eta|} \right) \varphi_k \left( \frac{|\eta|}{\eta} \right) d\eta d\xi dt \right|
$$

$$
\leq \sqrt{\epsilon} 2^{1 - \frac{K}{2}} t \sup_{\tau \in [0, t]} (\|\nabla f\|_{L^2} \|g\|_{L^2} \|g_k\|_{L^2}),
$$

which along with (4.3) and (4.14) implies

$$
\left| \int_0^t \int_{\mathbb{R}^2} \mathcal{S}_+^c(\xi, \eta) \varphi_{[-1, 5]}(\sqrt{\epsilon} \eta) \cdot \varphi_{[-D-1]}(\phi_{+, -}^c(\xi, \eta)) \cdot \varphi_{-K} \left( \frac{|\xi - \eta|}{|\eta|} \right) \cdot \varphi_k \left( \frac{|\xi - \eta|}{|\eta|} \right) d\eta d\xi dt \right|
$$

$$
\leq \sqrt{\epsilon} 2^{1 - \frac{K}{2}} t \sup_{\tau \in [0, t]} (\|\nabla f\|_{L^2} \|g\|_{L^2} \|g_k\|_{L^2}) \leq \sqrt{\epsilon} 2^{1 - \frac{K}{2}} t.
$$

(4.27)

(ii) For case $\frac{|\xi - \eta|}{|\eta|} \in [2^{-K}, 2^{-5}]$, using (4.17), we have

$$
|\tilde{q}_{+, -}(\xi, \eta)| \leq \epsilon 2^K |\xi - \eta|.
$$

(4.28)

We denote the integral regime in this case by

$$
\mathcal{S}_+^c = \left\{ (\xi, \eta) \in \text{supp}(q_{+, -}^c) \mid |\phi_{+, -}^c(\xi, \eta)| \leq 2^{-D}, \frac{|\xi - \eta|}{|\eta|} \in [2^{-K}, 2^{-5}] \right\}.
$$

(4.29)

First, transforming the variables $(\xi, \eta)$ in polar variables $(r, \theta, r_\eta, \theta_\eta)$, using (4.28), we have

$$
\left| \int_{\mathcal{S}_+^c} \mathcal{S}_+^c(\xi, \eta) \varphi_{[-1, 5]}(\sqrt{\epsilon} \eta) \cdot \varphi_{[-D-1]}(\phi_{+, -}^c(\xi, \eta)) \cdot \varphi_{-K+1} \left( \frac{|\xi - \eta|}{|\eta|} \right) \varphi_k \left( \frac{|\xi - \eta|}{|\eta|} \right) d\eta d\xi dt \right|
$$

$$
\leq \epsilon 2^K t \sup_{\tau \in [0, t]} \int_{\mathcal{S}_+^c} |\xi - \eta||f^+(\tau, \xi - \eta)| \cdot |\tilde{g}^-(\tau, \eta)| \cdot |\tilde{g}^-(\tau, - \xi)| \cdot r \tau r_\eta d\theta \eta r_\eta d\theta_\eta d\xi dr_\eta.
$$

(4.30)

Here and in what follows, we use an abuse of notations for the functions in different coordinates.

In order to use the volume of $\mathcal{S}_+^c$, we introduce the coordinates transformation on $\mathcal{S}_+^c$ as follows:

$$
\Psi_+ : \mathcal{S}_+^c \to \tilde{\mathcal{S}}_+^c \subset \mathbb{R}^2 \times \mathbb{R}^2,
$$

$$(r, \theta, r_\eta, \theta_\eta) \mapsto (r, \theta, \bar{r}_\eta, \theta_\eta) = (r, \theta, \phi_{+, -}^c(\xi, \eta), \theta_\eta),$$

we have
\[
\det \left( \frac{\partial \Psi_+ (r_\zeta, \theta_\zeta, r_\theta, \theta_\theta)}{\partial (r_\zeta, \theta_\zeta, r_\theta, \theta_\theta)} \right) = \frac{\partial \phi^\xi_+ - (\xi, \eta)}{\partial r_\eta}.
\]

Thanks to the expression of \(\phi^\xi_+ - (\xi, \eta)\) in (2.5), we have

\[
\frac{\partial \phi^\xi_+ - (\xi, \eta)}{\partial r_\eta} = 4 \epsilon \cos^2 \left( \frac{1}{2} \angle (\xi, \eta) \right) (2r_\eta - r_\zeta)
\]

\[
+ \epsilon \left( 4 \cos^2 \left( \frac{1}{2} \angle (\xi, \eta) \right) - 3 \right) \left( (r_\zeta + r_\eta) \partial_\eta |\xi - \eta| + \partial_\eta |\xi - \eta|^2 + |\xi - \eta| \right).
\]

Since

\[
|\xi - \eta|^2 = r_\zeta^2 + r_\eta^2 - 2r_\zeta r_\eta \cos (\angle (\xi, \eta)),
\]

we have

\[
\partial_\eta |\xi - \eta|^2 = 2 (r_\eta - r_\zeta \cos (\angle (\xi, \eta))) \quad \partial_\eta |\xi - \eta| = \frac{1}{|\xi - \eta|} (r_\eta - r_\zeta \cos (\angle (\xi, \eta))).
\]

Without loss of generality, we only consider \(\angle (\xi, \eta) \neq 0\). Since

\[
r_\eta - r_\zeta \cos (\angle (\xi, \eta)) = - |\xi - \eta| \cos (\angle (\xi - \eta, \eta)),
\]

we have

\[
\partial_\eta |\xi - \eta|^2 = -2 |\xi - \eta| \cos (\angle (\xi - \eta, \eta)), \quad \partial_\eta |\xi - \eta| = - \cos (\angle (\xi - \eta, \eta)), \quad (4.31)
\]

which implies that

\[
\frac{\partial \phi^\xi_+ - (\xi, \eta)}{\partial r_\eta} = 4 \epsilon \cos^2 \left( \frac{1}{2} \angle (\xi, \eta) \right) (2r_\eta - r_\zeta)
\]

\[
- \epsilon \left( 4 \cos^2 \left( \frac{1}{2} \angle (\xi, \eta) \right) - 3 \right) \left( \cos (\angle (\xi - \eta, \eta)) (r_\zeta + r_\eta + 2 |\xi - \eta|) - |\xi - \eta| \right).
\]

Using (4.19), the facts that \(|\xi - \eta| \leq \frac{1}{32} r_\eta\) and \(r_\zeta \approx r_\eta\), we have

\[
\det \left( \frac{\partial \Psi_+ (r_\zeta, \theta_\zeta, r_\theta, \theta_\theta)}{\partial (r_\zeta, \theta_\zeta, r_\theta, \theta_\theta)} \right) = \frac{\partial \phi^\xi_+ - (\xi, \eta)}{\partial r_\eta} \approx 4 \epsilon r_\eta - 2 \epsilon r_\eta \cos (\angle (\xi - \eta, \eta)) \sim \epsilon r_\eta, \quad (4.32)
\]

which along with the fact that \(\sqrt{\epsilon} r_\eta \sim 1\) yields

\[
\det \left( \frac{\partial \Psi_+ (r_\zeta, \theta_\zeta, r_\theta, \theta_\theta)}{\partial (r_\zeta, \theta_\zeta, r_\theta, \theta_\theta)} \right) \sim \sqrt{\epsilon}. \quad (4.33)
\]

Changing the variables \((r_\zeta, \theta_\zeta, r_\theta, \theta_\theta)\) to \((r_\zeta, \theta_\zeta, \tilde{r}_\eta, \theta_\theta)\), using (4.33), we deduce from (4.30) that

\[
\left| \int_0^t \int_{S^*_+} \Sigma_+ (\xi, \eta) \phi_{[-1, 1]} (\sqrt{\epsilon} |\eta|) \phi_{[-D - 1]} (\phi^\xi_+ - (\xi, \eta)) \cdot \phi_{\geq -k + 1} \left( \frac{|\xi - \eta|}{|\eta|} \right) d\eta d\xi dt \right|
\]

\[
\leq \sqrt{\epsilon} 2^{K_1} t \sup_{\tau \in [0, t]} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} |\xi - \eta| |\tilde{f}^+ (\tau, \xi - \eta)|
\]

\[
\cdot |\tilde{g}^- (\tau, \eta)| \cdot |\tilde{g}^- (\tau, - \xi)| \cdot 1_{S^*_+} (\xi, \eta) \cdot r_\zeta r_\eta d\theta_\eta d\tilde{r}_\eta d\theta_\zeta d\xi
\]
we have
\[ \int_0^t \int_{S_h^+} \mathcal{X}^e_+ (\xi, \eta) \varphi_{[-1,5]} (\sqrt{c}|\eta|) \cdot \varphi_{\lesssim -D-1} (\phi^e_{+, -} (\xi, \eta)) \, d\eta d\xi dt \]
\[ \leq \varepsilon^{3/2} 2^{2-D} t \sup_{t \in [0, t]} \left\{ \|g(t, \cdot)\|_{L^2} \cdot \left( \int_{S_h^+}^{3/2} \left( \int_0^t \int_0^{2-D} \int_0^{2-D} \left| \mathcal{g}^- (\xi, \eta) \cdot r_\eta d\theta_\eta d\theta_\xi d\eta \cdot r_\xi d\theta_\xi dr_\xi \right)^{1/2} \right)^2 \right\} \]
which along with (4.3) and (4.14) implies that
\[ \int_0^t \int_{S_h^+} \mathcal{X}^e_+ (\xi, \eta) \varphi_{[-1,5]} (\sqrt{c}|\eta|) \cdot \varphi_{\lesssim -D-1} (\phi^e_{+, -} (\xi, \eta)) \cdot \varphi_{\gtrsim -K+1} \left( \frac{|\xi|}{|\eta|} \right) \, d\eta d\xi dt \]
\[ \leq \varepsilon^{3/2} 2^{2-D} t. \]  

Changing variables \((r_\xi, \theta_\xi, \bar{r}_\eta, \theta_\eta)\) to \((\bar{r}_\xi, \theta_\xi, r_\eta, \theta_\eta)\), using (4.33) and the fact that \(r_\eta = |\eta| \leq 2^K |\xi - \eta|\), we have
\[ \left| \int_0^t \int_{S_h^+} \mathcal{X}^e_+ (\xi, \eta) \varphi_{[-1,5]} (\sqrt{c}|\eta|) \varphi_{\lesssim -D-1} (\phi^e_{+, -} (\xi, \eta)) \, d\eta d\xi dt \right| \leq \varepsilon^{3/2} 2^{2-D} t. \]

Due to (4.27) and (4.34), taking
\[ \sqrt{c} 2^{-\frac{5}{4}} \sim \varepsilon^{3/2} 2^{2-D} 2^{-\frac{D}{4}}, \quad \text{i.e.,} \quad 2^K \sim 2^{D} \varepsilon^{-\frac{1}{8}}, \]
we have
\[ \int_0^t \int_{S_h^+} \mathcal{X}^e_+ (\xi, \eta) \varphi_{[-1,5]} (\sqrt{c}|\eta|) \varphi_{\lesssim -D-1} (\phi^e_{+, -} (\xi, \eta)) \, d\eta d\xi dt \leq 2^{-\frac{D}{8}} \varepsilon n t. \]  

Thanks to (4.23) and (4.35), taking
\[ 2^D \varepsilon^{3/2} \sim 2^{-\frac{D}{8}} \varepsilon^{\frac{3}{4}}, \quad \text{i.e.,} \quad 2^D \sim \varepsilon^{-\frac{1}{8}}, \]
we have
\[ \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathcal{X}^e_+ (\xi, \eta) \varphi_{[-1,5]} (\sqrt{c}|\eta|) \, d\eta d\xi dt \leq 1 + \varepsilon^2 t, \]
which along with (4.22) implies
Step 3.4. Estimate for \( \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathcal{Q}_e^-(\xi, \eta) d\eta d\xi dt \). Thanks to (2.6) and (4.17), we could derive the estimate for \( \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathcal{Q}_e^-(\xi, \eta) d\eta d\xi dt \) in a similar way as that for \( \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathcal{Q}_e^+ (\xi, \eta) d\eta d\xi dt \). That is
\[
\left| \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathcal{Q}_e^-(\xi, \eta) d\eta d\xi dt \right| \leq 1 + e^\hat{t} t. \tag{4.37}
\]

To archive (4.37), we only need to check that on
\[ S'_- = \left\{ (\xi, \eta) \in \text{supp} (q^e_{-,-}) \mid |\phi_{-,-}^e (\xi, \eta)| \leq 2^{-D}, \quad \frac{|\xi - \eta|}{|\eta|} \in [2^{-K}, 2^{-3}] \right\}, \]
there exists a coordinates transformation \( \Psi_- \) such that
\[
\det \left( \frac{\partial \Psi_- (r_z, \theta_z, r_\eta, \theta_\eta)}{\partial (r_z, \theta_z, r_\eta, \theta_\eta)} \right) \sim \sqrt{\varepsilon}. \tag{4.38}
\]

Indeed, introducing the coordinates transformation \( \Psi_- \) as follows
\[
\Psi_- : S'_- \rightarrow \tilde{S}'_- \subset \mathbb{R}^2 \times \mathbb{R}^2,
\]
\[
(r_z, \theta_z, r_\eta, \theta_\eta) \mapsto (\tilde{r}_z, \tilde{\theta}_z, \tilde{r}_\eta, \tilde{\theta}_\eta) = (r_z, \theta_z, \phi_{-,-}^e (\xi, \eta), \theta_\eta),
\]
we have
\[
\det \left( \frac{\partial \Psi_- (r_z, \theta_z, r_\eta, \theta_\eta)}{\partial (r_z, \theta_z, r_\eta, \theta_\eta)} \right) = \partial_{r_\eta} \phi_{-,-}^e (\xi, \eta).
\]

Using (2.6), we have
\[
\partial_{r_\eta} \phi_{-,-}^e (\xi, \eta) = \varepsilon (2r_\eta - r_z) + \varepsilon \left( \frac{3}{4 \cos^2 \left( \frac{1}{2} \angle (\xi, \eta) \right)} - 1 \right) \frac{(r_z + r_\eta) \partial_{r_\eta} |\xi - \eta| + |\xi - \eta| - \partial_{r_\eta} |\xi - \eta|^2}{|\xi - \eta|^2},
\]
which along with (4.31) implies
\[
\partial_{r_\eta} \phi_{-,-}^e (\xi, \eta) = \varepsilon (2r_\eta - r_z) + \varepsilon \left( 1 - \frac{3}{4 \cos^2 \left( \frac{1}{2} \angle (\xi, \eta) \right)} \right) \times \left[ (r_z + r_\eta - 2 |\xi - \eta|) \cos (\angle (\xi - \eta, \eta)) - |\xi - \eta| \right].
\]

By virtue of (4.19), using the facts that \( |\xi - \eta| \leq \frac{1}{32} |\eta| \) and \( |\xi| \approx |\eta| \), we have
\[
\partial_{r_\eta} \phi_{-,-}^e (\xi, \eta) \approx \varepsilon r_\eta \left[ 1 + \frac{1}{2} \cos (\angle (\xi - \eta, \eta)) \right] \sim \varepsilon r_\eta,
\]
which along with the fact that \( \sqrt{\varepsilon} |\eta| \sim 1 \) implies
\[
\det \left( \frac{\partial \Psi_- (r_z, \theta_z, r_\eta, \theta_\eta)}{\partial (r_z, \theta_z, r_\eta, \theta_\eta)} \right) = \partial_{r_\eta} \phi_{-,-}^e (\xi, \eta) \sim \sqrt{\varepsilon}, \quad \text{for any } (\xi, \eta) \in S'_-.
\]
This is exactly (4.38).
Step 3.5. Estimate for III. Thanks to (4.36) and (4.37), we obtain (4.13).

Step 4. The \textit{a priori} energy estimate. Combining the estimates (4.8), (4.10), (4.13), we arrive at the final energy estimate (4.6). This completes the proof of the proposition. □

Remark 4.2. A variant of the long time issue considered in this article would be to look for the lifespan of solutions of

\[
\begin{aligned}
\partial_t \zeta + (1 + \Delta) \nabla \cdot \mathbf{v} + \nabla \cdot (\zeta \mathbf{v}) &= 0, \\
\partial_t \mathbf{v} + (1 + \Delta) \nabla \zeta + \frac{1}{2} \nabla (|\mathbf{v}|^2) &= 0,
\end{aligned}
\]  

(4.39)

with the initial data

\[
\zeta|_{t=0} = \zeta_0 = O(\epsilon), \quad \mathbf{v}|_{t=0} = \mathbf{v}_0 = O(\epsilon).
\]

This issue was studied in [19] in the one-dimensional case and the lifespan was proven to be \(O(1/\epsilon^{4/3})\), improving the \(O(1/\epsilon)\) result obtained by pure dispersive methods. By adapting the method in [19] to the two-dimensional case one can obtain the same \(O(1/\epsilon^{4/3})\) result for (4.39).

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