GLOBAL STABILITY FOR A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS (APPLICATION TO NICHOLSON’S BLOWFLIES AND MACKEY-GLASS MODELS)

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ABSTRACT. Global asymptotic and exponential stability of equilibria for the following class of functional differential equations with distributed delay is investigated

\[ x'(t) = -f(x(t)) + \int_0^\tau h(a)g(x(t-a))da. \]

We make our analysis by introducing a new approach, combining a Lyapunov functional and monotone semiflow theory. The relevance of our results is illustrated by studying the well-known integro-differential Nicholson’s blowflies and Mackey-Glass equations, where some delay independent stability conditions are provided. Furthermore, new results related to exponential stability region of the positive equilibrium for these both models are established.

1. Introduction. In this paper, we study the following general class of functional differential equations with distributed delay

\[
\begin{cases}
  x'(t) = -f(x(t)) + \int_0^\tau h(a)g(x(t-a))da, & t > 0, \\
  x(t) = \phi(t), & -\tau \leq t \leq 0.
\end{cases}
\] (1)

Various forms of system (1) can be found in the literature, we invite the reader to see [3], [6], [7], [8], [12], [15], [20], [21], [38], [39] and references therein.

As known, the dynamics of the subfamily of quasi-monotone functional differential equations of (1) is well understood, see [26], [28]; it admits a comparison principle and therefore, the chaotic behavior of this family is not possible [26]. However, much is unknown about the global dynamics of non-monotone functional differential equations (1) see [11], [13], [28], [30]. These models need more attention to elucidate the complex dynamics of its solutions. In fact, despite the existence of various techniques for the global study of functional differential equations, the determination of the asymptotic behavior of its solutions even for the simplest-looking equations is difficult. For example, in the case where \( g \) is an unimodal feedback (when \( g \) has exactly one maximum and changes monotonicity at only one point) and \( f(x) = x \), the trajectory may lead to complicated dynamics, as chaotic behavior [22]. In this context, based on the idea stated in the works of Mallet-Paret

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and Nussbaum [24], [25], many authors emphasize that the proofs of the convergence to the positive equilibrium is strongly related to the linearity of $f$ and by the sequel to the convergence of the sequence defined by $x_{n+1} = g(x_n)$. For the most related works in this direction see [4], [9], [14], [16], [18], [17], [19], [27] and further references therein. We also mention the work of Kuang in his book [15], where he investigated the following general functional differential equation,

$$x'(t) = -f(x(t)) + g \left( \int_{-\tau}^{\sigma} x(t+s) d\mu(s) \right),$$

and under some conditions on $f$ and $g$, he found a set of sufficient conditions which lead to the global attractiveness of the positive equilibrium.

Recently, Yuan et al [39] analyzed the following model

$$x'(t) = -\alpha x(t) + f_1(x(t)) + \int_{0}^{\tau} k(s) f_2(x(t-s)) ds,$$

and obtained sufficient conditions under which the positive equilibrium is globally attractive, the approach used there is related to the fluctuation method.

When a spatial diffusion is considered; many models are studied in the literature, see for example [34], [35], [36], [37]; some of the techniques used there can be adapted for our problem to obtain the stability region.

The purpose of this paper is to provide a new approach on analytic study of asymptotic behavior of solutions to problem (1), which can be applied to the non-monotone feedback case. More precisely, we will present some results which will enable us to give general conditions on $f$ and $g$ that ensure global asymptotic and exponential stability of equilibria regardless of the length of the delay. To this end, fluctuation and monotone semi-flow methods combined with a Lyapunov functional are used.

The main idea of this work is to derive necessary and sufficient conditions on $f$ and $g$ which guarantee the existence of a nondecreasing function satisfying certain properties. From then, the monotone semi-flow theory is applied. For this, the first part of this paper is devoted to dealing with the monotone feedback i.e. ($g$ is a nondecreasing function), where the global asymptotic stability of equilibria to (1) is proved. Even though this first case is simple to prevent, it will play an important role in the more interesting case, when $g$ is a non-monotone feedback. This is the goal of the second part of the paper. In this framework, the first objective is to obtain some estimates of solutions. In order to do this, we will split our study into two cases: $x^* \leq M$ and $x^* > M$, ($M$ being the value for which $g$ in (1) reaches its maximum and $x^*$ being the positive equilibrium). When dealing with the most interesting situation, namely $x^* > M$, and as mentioned above, we will build nondecreasing functions (in most cases) less or equal than our principal feedback function $g$ in (1) and satisfying a suitable condition stated later, therefore, we can apply the obtained results to the monotone semi-flow. Using a comparison principle result will lead us to the required estimates of solutions. From then on, we will use these established results to get a series of criteria for global asymptotic stability of positive equilibrium in the case where the functions $g$ are nondecreasing over $(0,M)$ and non-increasing over $(M,B)$ (unimodal functions).

We point out here that our resulting criteria improve, among other things, some existing results by relaxing the conditions on $f$ in system (1). Indeed, in [15] (respectively [39]), the condition that $f$ in (2) ($\alpha x - f_1(x)$ in (3)) is strictly increasing over $(0,B)$ is necessary in both works. In addition, as part of our discussion, we will
present some extended results of global stability concerning the nonlinear equation mentioned in [1] and studied in [5], Theorem 3.1. Furthermore, at our knowledge there is no result on exponential stability of equilibria of system (1); we will give a condition for which the exponential stability of equilibria is reached.

Our main results will be applied to two well-known models: integro-differential Nicholson’s blowflies and Mackey-Glass equations. Our theorems allow us to prove the absolute global asymptotic stability of the positive equilibrium whenever it is absolutely locally stable. We will also give some new results concerning the exponential stability of the positive equilibrium for these two cited models.

The paper is organized as follows: In the next section, we establish existence and uniqueness of positive solution and present a suitable Lyapunov functional. In section 3, we prove the asymptotic stability of equilibria when the monotonic case is considered. In Section 4, we show the strong persistence and determine some useful estimates of solutions in the case of non-monotone feedback. Section 5 is devoted to presenting some theorems related to the global asymptotic stability of equilibria. The global exponential stability of equilibria is investigated in Section 6. We illustrate our results by studying the integro-differential Nicholson’s blowflies and Mackey-Glass equations in Section 7. A discussion is included in the last section.

Throughout this paper, we will make the following assumptions:

we suppose that the function $h$ is positive and

$$\int_0^\tau h(a)da = 1.$$  

(T1) $f$ and $g$ are Lipschitz continuous with $f(0) = g(0)$.

(T2) $g(s) > g(0)$ for all $s > 0$ and there exists a number $B > 0$ such that

$$\max_{v \in [0, s]} g(v) < f(s)$$

for all $s > B$.

We will also use the notation

$$K(s) := g(s) - f(s).$$

Let $C := C([-\tau, 0], \mathbb{R})$ be the Banach space of continuous functions defined in $[-\tau, 0]$ with $||\phi|| = \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|$, and $C_+ = \{\phi \in C; \phi(\theta) \geq 0, -\tau \leq \theta \leq 0\}$ is the positive cone of $C$. Then $(C, C_+)$ is a strongly ordered Banach space, that is: for all $\phi, \psi \in C$, we write $\phi \geq \psi$ if $\phi - \psi \in C_+$; $\phi > \psi$ if $\phi - \psi \in C_+ \setminus \{0\}$; $\phi >> \psi$ if $\phi - \psi \in \text{Int}(C_+)$. We define the ordered intervals

$$[\phi, \psi] := \{\xi \in C; \phi \leq \xi \leq \psi\},$$

and for any $\chi \in \mathbb{R}$, we write $\chi^\ast$ for the element of $C$ satisfying $\chi^\ast(\theta) = \chi$ for all $\theta \in [-\tau, 0]$. The segment $x_t \in C$ of a solution is defined by the relation $x_t(\theta) = x(t + \theta)$ where $\theta \in [-\tau, 0]$ and $t \geq 0$, in particular $x_0 = \phi$. The family of maps

$$\Phi : [0, \infty) \times C_+ \rightarrow C_+,$$

such that

$$(t, \phi) \rightarrow x_t(\phi)$$

defines a continuous semiflow on $C_+$, [31]. The map $\Phi(t, \cdot)$ is defined from $C_+$ to $C_+$ which is denoted by $\Phi_t$,

$$\Phi_t(\phi) = \Phi(t, \phi).$$
The set of equilibria of the semiflow generated by (1) is given by
\[ E = \{ \chi^* \in C_+; \chi \in \mathbb{R} \text{ and } g(\chi) = f(\chi) \}. \]

2. Preliminaries. In this section, we first provide existence, uniqueness and boundedness of solution to problem (1), then a Lyapunov functional candidate is presented, some estimates of solutions are also given.

We begin by recalling this useful theorem related to a comparison principle, see ([28], page 78, Theorem 1.1) and which is adapted, for the reader convenience, to the context of our discussion.

We consider the following system
\[
\begin{cases}
  x'(t) = F(x), & t > 0, \\
  x(t) = \phi(t), & -\tau \leq t \leq 0,
\end{cases}
\]
where \( F : \Omega \to \mathbb{R} \) is continuous on \( \Omega \), an open subset of \( C \). We denote \( x(t, \phi, F) \) the maximal defined solution of (4).

**Theorem 2.1.** Let \( f_1, f_2 : \Omega \to \mathbb{R} \) be continuous, Lipschitz on each compact subset of \( \Omega \), and assume that either \( f_1 \) or \( f_2 \) is a nondecreasing function, with \( f_1(\phi) \leq f_2(\phi) \) for all \( \phi \in \Omega \). Then
\[
x(t, \phi, f_1) \leq y(t, \phi, f_2),
\]
holds for all \( t \geq 0 \) for which both are defined.

The following lemma, states existence, uniqueness and boundedness of the positive solution to (1). See also [33], [39] for similar results.

**Lemma 2.2.** For any \( \phi \in C_+ \), the problem (1) has a unique positive solution \( x(t) := x(t, \phi) \) on \( [0, \infty) \) satisfying \( x_0 = \phi \), provided \( \phi(0) > 0 \). In addition we have the following estimate,
\[
\limsup_{t \to \infty} x(t) \leq B.
\]
Moreover, the semi-flow \( \Phi_t \) admits a compact global attractor which attracts every bounded set in \( C_+ \).

**Proof.** We refer to [11] for the proof of existence and uniqueness of solution. Now we focus on the positivity of solution. We first claim that \( x(t) \geq 0 \), for all \( t > 0 \). Otherwise, suppose (without loss of generality) that there exists \( T \geq \tau \) such that \( x(T) = 0 \) and \( x'(T) \leq 0 \), and \( x(t) > 0 \) for all \( t \in [0, T) \) (since \( \phi(0) > 0 \)) then, from (1) we have
\[
-f(0) + \int_0^T h(a)g(x(T - a))da \leq 0,
\]
using the fact that \( g(s) > g(0) \) for all \( s > 0 \) we get,
\[
K(0) < 0,
\]
which is a contradiction. The claim is proved. Concerning the boundedness of solutions. We set \( \bar{g}(s) := \max_{s \in [0,]} g(v) \) and we consider the following system
\[
\begin{cases}
  y'(t) = -f(y(t)) + \int_0^t h(a)g(y(t - a))da, & t > 0, \\
  y(t) = \phi(t), & -\tau \leq t \leq 0.
\end{cases}
\]
(5)

According to Theorem 2.1 observe that if \( \phi \in [0^*, l^*] \) then \( 0 \leq y(t, \phi) \leq l \) for all \( t \geq -\tau \) with \( l \) is any constant greater than \( B \). We also have \( x(t) \leq y(t) \) for all
$t \geq -\tau$. Hence, $0 \leq x(t) \leq l$ for all $t \geq 0$ and $l \geq B$. Therefore, since $l$ is arbitrarily large, then the solution is global and the orbits of bounded sets for the semi-flow $\Phi$ are bounded. Next we claim that \( \limsup_{t \to \infty} y(t) \leq B \). Suppose, on the contrary, that $y(t_n) \to l$ and $y'(t_n) \to 0$. Thus, substituting $y(t_n)$ in (1),

$$y'(t_n) = -f(y(t_n)) + \int_0^\tau h(a)\bar{g}(y(t_n - a))da,$$

by the definition of $\limsup_{t \to \infty} y(t)$ and the monotonicity of the continuous function $\bar{g}$ we have

$$\bar{g}(\lim_{n \to \infty} y(t_n - a)) \leq \bar{g}(\lim_{n \to \infty} y(t_n)) := \bar{g}(l), \quad \forall a \in [0, \tau].$$

Passing to the limit in (7) and combining with (8) we get,

$$0 \leq -f(l) + \bar{g}(l),$$

and this provides a contradiction with (6). Therefore the solution semiflow is point dissipative on $C_+$, see [31]. Further, let $F : C \to \mathbb{R}$ be defined as

$$F(\phi) = -f(\phi(0)) + \int_0^\tau h(a)g(\phi(-a))da,$$

in view of (T1), $F$ is continuous and satisfies a Lipschitz condition on each bounded subset of $C$. So we can easily show that $F$ is completely continuous. Moreover, since the orbits of bounded sets are bounded, then the Proposition 5.5, [29] shows that the semiflow $\Phi_t$ is completely continuous for any $t \geq 0$. Finally according to [10], Theorem 3.4.8, the existence of a compact global attractor is established and the result is reached.

Next we will show that this solution is strictly positive if $\phi(0) > 0$. In fact, due to (T1) and (T2),

$$x'(t) \geq -f(x(t)) + g(0),$$

$$= f(0) - f(x(t))$$

according to (T1) we obtain

$$x'(t) \geq -L_1x(t),$$

with $L_1$ is $f$-Lipschitz constant; this gives

$$x(t) \geq \phi(0)e^{-L_1t} > 0.$$
where
\[
\psi(a) = \int_a^\tau h(s)ds. \tag{11}
\]

We introduce the functional \( V_\gamma : C \to \mathbb{R} \) by
\[
V_\gamma(\phi) = W_\gamma(\phi(0)) + J(\phi). \tag{12}
\]

The derivative of \( V_\gamma \) along the solution of (1) is given by the following theorem.

**Theorem 2.3.** All solutions of problem (1) satisfy the following estimate
\[
\frac{dV_\gamma(x_t)}{dt} = L(t) + H'(g(x(t)))K(x(t)), \quad \forall t > 0, \tag{13}
\]
where
\[
L(t) = \int_0^\tau \left(H(g(x(t))) - H(g(x(t-a))) + H'(g(x(t)))(g(x(t-a)) - g(x(t)))\right)h(a)da \leq 0, \quad \forall t > 0.
\]

**Proof.** We first compute the derivative of \( W_\gamma \) along the solutions of (1), we have
\[
\frac{d}{dt}W_\gamma(x(t)) = W'_\gamma(x(t)) \{ -f(x(t)) + \int_0^\tau h(a)g(x(t-a))da \}. \tag{14}
\]
Adding and subtracting the term \( W'_\gamma(x(t))g(x(t)) \) in the above equation, we get
\[
\frac{d}{dt}W_\gamma(x(t)) = W'_\gamma(x(t)) \{ -f(x(t)) + g(x(t)) \}
+ W'_\gamma(x(t))\int_0^\tau h(a)(g(x(t-a)) - g(x(t)))da.
\]
On the other hand, a direct computation of the derivative of \( J \) along solutions of (1) leads to,
\[
\frac{d}{dt}J(x_t) = \psi(0)H(g(x(t))) + \int_0^\tau \psi'(a)H(g(x(t-a)))da.
\]
Summing these two above equalities, we obtain
\[
\frac{d}{dt}V_\gamma(x_t) = W'_\gamma(x(t))\{ -f(x(t)) + g(x(t)) \}
+ W'_\gamma(x(t))\int_0^\tau h(a)(g(x(t-a)) - g(x(t)))da + \psi(0)H(g(x(t)))
+ \int_0^\tau \psi'(a)H(g(x(t-a)))da.
\]
In view of (10) and (11), it yields
\[
\frac{d}{dt}V_\gamma(x_t) = L(t) + H'(g(x(t)))(g(x(t)) - f(x(t))), \tag{14}
\]
where
\[
L(t) = \int_0^\tau \left(H(g(x(t))) - H(g(x(t-a)))\right)
\]
\[ + H'(g(x(t))) (g(x(t-a)) - g(x(t))) \int_0^a h(a) \, da \leq 0, \quad \forall t > 0. \]

2.1. Global asymptotic stability of the trivial equilibrium. Now we focus on global asymptotic stability of the trivial equilibrium. We begin by stating the following hypothesis:

\[ g(s) < f(s), \quad \forall s > 0. \] (15)

We have the extinction result as follows

**Theorem 2.4.** Assume that (15) holds. Then the trivial equilibrium of problem (1) is globally asymptotically stable.

**Proof.** Applying Theorem 2.3, by choosing \( \gamma = 0 \) in (10) and \( H(s) = s - g(0) \), by (13) we obtain

\[ \frac{dV_0(x_t)}{dt} \leq K(x(t)), \quad \forall t > 0. \]

Further \( K(s) < 0 \) for \( s > 0 \), thus, there results that the trivial equilibrium is globally asymptotically stable, see for instance [11], [15], [23].

2.2. Global stability of the positive equilibrium when \( g \) is nondecreasing.

We first present some assumptions, that will be used in this subsection.

Assume that there exists a positive constant \( x^* \) such that,

\[ \begin{align*}
  g(s) &> f(s), \quad \text{for } 0 < s < x^*, \\
  g(s) &< f(s), \quad \text{for } x^* < s \leq B.
\end{align*} \] (16)

Suppose also that \( f'(0) \) and \( g'(0) \) exist and verify \( g'(0) > f'(0) > 0 \) with \( f(s) > f(0), \quad \forall s > 0. \)

\[ f'(0) \] and \( g'(0) \) exist and verify \( g'(0) > f'(0) > 0 \) with \( f(s) > f(0), \quad \forall s > 0. \)

There exists a small constant \( \bar{\theta} > 0 \), such that

the function \( f \) is strictly increasing over \( [x^* - \bar{\theta}, x^* + \bar{\theta}] \)

and \( f(x^* - \bar{\theta}) > f(\sigma) \) for all \( \sigma \in [0, x^* - \bar{\theta}] \) and

\( f'(0) \) and \( g'(0) \) exist and verify \( g'(0) > f'(0) > 0 \) with \( f(s) > f(0), \quad \forall s > 0. \)

\[ f'(0) \] and \( g'(0) \) exist and verify \( g'(0) > f'(0) > 0 \) with \( f(s) > f(0), \quad \forall s > 0. \)

The hypothesis (18) is stated in order to rule out the oscillation of \( f \) in the neighborhood of \( x^* \). However and as it will be established in the next proof, we will not need it, if \( g \) is assumed to be a strictly increasing function.

The following result concerns the strong persistence of solutions of (1), for more details on this notion see [31].

**Lemma 2.5.** Assume that (16) holds, then the solution of problem (1) is strongly persistent provided the corresponding initial data satisfies \( \phi(0) > 0 \).

**Proof.** Notice that \( x(t) > 0 \) for any \( t > 0 \) for \( \phi \in C_+ \) and \( \phi(0) > 0 \). We now suppose that there exists an increasing sequence \( \{t_n\}_n, \ t_n \to \infty, \ t_0 \geq \tau \) and a non-increasing sequence \( \{x(t_n)\}_n \) such that \( x(t_n) < x^*, \ x(t_n) \to 0 \) as \( t_n \to \infty \) and \( x(t) = \min_{t \in [0, t_n]} x(t) \), by substituting \( x(t_n) \) in the equation (1), we have

\[ x'(t_n) = -f(x(t_n)) + \int_0^\tau h(a) g(x(t_n - a)) \, da \leq 0, \]
since \( g \) is a nondecreasing function, so \( g(x(t_n)) \leq g(x(s)) \) for all \( 0 \leq s \leq t_n \), and hence we get

\[
0 \geq x'(t_n) \geq -f(x(t_n)) + g(x(t_n)),
\]

we reach a contradiction with \( 0 < x(t_n) < x^* \) and (16).

\[\square\]

**Theorem 2.6.** Assume that (16)-(18) hold, then the positive equilibrium \( x^* \) of problem (1) is globally asymptotically stable.

**Proof.** First we choose a strictly convex function \( H \) used in (10)-(12) satisfying \( H'(g(x^*)) = H(g(x^*)) = 0 \). In addition, let \( \gamma = x^* \) in (10). We divide the proof into two cases:

**The first case.** we suppose that \( g \) is a strictly increasing function. So, by (13) we obtain

\[
\frac{dV_{x^*}(x_t)}{dt} \leq -\Psi(x(t)),
\]

with \( \Psi(s) = -H'(g(s))K(s) \) and \( \Psi(x^*) = 0 \), furthermore \( \Psi(s) > 0 \), for \( s \neq x^* \). Finally, from Lemma 2.5, all solutions of (1) are strongly persistent. There results that \( x^* \) is globally asymptotically stable (see e.g., [11], [23]).

**The second case.** we assume that \( g \) is a nondecreasing function. Then from (16)-(18) and for \( \theta \in (0, \bar{\theta}) \) we can build two strictly increasing functions \( g_\theta \) and \( g^\theta \) (see Appendix) satisfying the following assertions

\[
\begin{align*}
&g_\theta(x) > f(x), \quad \text{for } 0 < x < x^* - \theta \\
&g_\theta(x) < f(x), \quad \text{for } x^* - \theta < x \leq B, \\
&g^\theta(x) > f(x), \quad \text{for } 0 \leq x < x^* + \theta \\
&g^\theta(x) < f(x), \quad \text{for } x^* + \theta < x \leq B,
\end{align*}
\]

and

\[
g_\theta(s) \leq g(s) \leq g^\theta(s), \quad \text{for all } 0 \leq s \leq B.
\]

Next, we introduce the following auxiliary problems

\[
\begin{align*}
\begin{cases}
  y_\theta'(t) = -f(y_\theta(t)) + \int_0^\tau h(a)g_\theta(y_\theta(t-a))da, & t > 0, \\
  y_\theta(t) = x(t), & -\tau \leq t \leq 0,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
  y^\theta'(t) = -f(y^\theta(t)) + \int_0^\tau h(a)g^\theta(y^\theta(t-a))da, & t > 0, \\
  y^\theta(t) = x(t), & -\tau \leq t \leq 0,
\end{cases}
\end{align*}
\]

then, Theorem 2.1, gives that

\[
y_\theta(t) \leq x(t) \leq y^\theta(t), \quad \text{for all } 0 \leq t \leq B.
\]

Hence, letting \( t \) tends to infinity and using the first part of the proof, we reach that

\[
x^* - \theta \leq \liminf_{t \to \infty} x(t) \leq \limsup_{t \to \infty} x(t) \leq x^* + \theta.
\]

Now, since \( \theta \) is arbitrary small, we prove that all solutions of (1) converge to the positive equilibrium. Further, in order to reach the result, it is sufficient to prove the local stability of the positive equilibrium, that is,

\[
\forall \varepsilon > 0, \quad \exists \eta > 0 \quad \text{s.t.} \quad \|\phi - x^*\| \leq \eta \Rightarrow \|x_t - x^*\| \leq \varepsilon \quad \forall t \geq 0,
\]
with $||\phi|| = \sup_{t \in [-\tau, 0]} |\phi(t)|$. Indeed, for $\theta = \varepsilon$ we have,

\[
\forall \varepsilon_1 > 0, \exists \eta_1 > 0 \text{ s.t. } ||\phi - x^*|| \leq \eta_1 \Rightarrow ||y_{\varepsilon}(t + \gamma) - (x^* - \varepsilon)|| \leq \varepsilon_1 \forall t \geq 0, \forall \gamma \in [-\tau, 0].
\]

In particular choosing $\varepsilon_1 = \varepsilon$ and using the fact that $y_{\varepsilon}(t) \leq x(t)$ for all $t > 0$ it comes

\[
-\varepsilon \leq y_{\varepsilon}(t + \gamma) - (x^* - \varepsilon) \leq x(t + \gamma) - (x^* - \varepsilon), \quad \forall \gamma \in [-\tau, 0]
\]

accordingly,

\[
-2\varepsilon \leq x(t + \gamma) - x^*, \quad \forall t \geq 0, \quad \forall \gamma \in [-\tau, 0]. \tag{19}
\]

On the other hand, we have

\[
\forall \varepsilon_2 > 0, \exists \eta_2 > 0 \text{ s.t. } ||\phi - x^*|| \leq \eta_2 \Rightarrow ||y_{\varepsilon}(t + \gamma) - (x^* + \varepsilon)|| \leq \varepsilon_2 \forall t \geq 0, \forall \gamma \in [-\tau, 0].
\]

As above, choosing $\varepsilon_2 = \varepsilon$, by the fact that $x(t) \leq y_{\varepsilon}(t)$ for all $t > 0$ we get

\[
x(t + \gamma) - x^* \leq y_{\varepsilon}(t + \gamma) - x^* \leq 2\varepsilon,
\]

finally, combining the above estimates, for $\eta = \max\{\eta_1, \eta_2\}$, if $||\phi - x^*|| \leq \eta$, then

\[
-2\varepsilon \leq x(t + \gamma) - x^* \leq 2\varepsilon, \quad \forall t \geq 0, \quad \forall \gamma \in [-\tau, 0]. \tag{20}
\]

Consequently the positive equilibrium is globally asymptotically stable. \qed

3. The case when $g$ is non-monotone.

3.1. Persistence and estimates of solutions. Our main objective in this section is to exhibit some fundamental results, including the strong persistence and some estimates of solutions to problem (1) in the case where $g$ is non-monotone. We make the following assumption, that will be used from now on.

There exists a positive constant $x^*$ such that,

\[
\begin{align*}
\min_{\sigma \in (s, x^*)} g(\sigma) > f(s), & \quad \text{for } 0 < s < x^*, \\
\max_{\sigma \in [x^*, s]} g(\sigma) < f(s), & \quad \text{for } x^* < s \leq B.
\end{align*}
\tag{21}
\]

It is easy to show that, $x^*$ is the unique value that satisfies $g(x^*) = f(x^*)$.

**Remark 1.** The positive equilibrium of (1), which is the constant function $x^*(t) = x^*$ for all $t \in [-\tau, 0]$ satisfying the equation of (1) exists and is unique.

With the aim to prove the strong persistence and to obtain some estimates of solutions of (1), we need to build a nondecreasing function having some properties in order to apply the results of the previous section; this will be done by the help of the next lemmas.

**Lemma 3.1.** Suppose that (17) and (21) hold, then there exists a positive constant $0 < m < x^*$ satisfying

\[
\begin{align*}
g(s) & > f(m) \quad \text{for } m \leq s \leq B, \\
f(s) & < f(m) \quad \text{for } 0 \leq s < m, \\
f(s) & > f(m) \quad \text{for } m < s \leq B,
\end{align*}
\tag{22}
\]

in addition $f$ and $g$ are strictly increasing over $[0, m]$. 
Proof. First, from (17) there exists $\sigma_0 > 0$ such that $f$ and $g$ are strictly increasing over $[0, \sigma_0]$. Let $\gamma > 0$ be defined as

$$\gamma := \min_{\sigma \in [\sigma_0, \sigma^*]} f(\sigma) = f(\sigma_1).$$

Next, since $f(\sigma_1) > f(0)$ then there exists $m_1 \in (0, \sigma_0]$ such that $f(m_1) = f(\sigma_1)$. Notice that for $\varepsilon > 0$ so small, the positive constant $(m_1 - \varepsilon)$ satisfies

$$f(s) < f(m_1 - \varepsilon) \text{ for } 0 \leq s < m_1 - \varepsilon,$$

$$f(s) > f(m_1 - \varepsilon) \text{ for } m_1 - \varepsilon < s \leq B.$$

Indeed, since $f$ is strictly increasing over $[0, \sigma_0]$ and $m_1 \leq \sigma_0$ then $f(s) < f(m_1 - \varepsilon)$ for all $s \in [0, m_1 - \varepsilon]$ and $f(s) > f(m_1 - \varepsilon)$ for all $s \in (m_1 - \varepsilon, \sigma_0]$.

For $\sigma_0 \leq s \leq \sigma^*$ and from (23) we get

$$f(s) \geq f(m_1) = f(\sigma_1),$$

thus,

$$f(s) > f(m_1 - \varepsilon).$$

Next, if $\sigma^* < s \leq B$ then in view of (21)

$$f(s) > \max_{\sigma \in [\sigma^*, s]} g(\sigma) \geq g(\sigma^*) = f(\sigma^*) > f(\sigma_1),$$

from (23), we have

$$f(s) > f(m_1) > f(m_1 - \varepsilon).$$

We define $\alpha := \min_{\sigma \in [m_1 - \varepsilon, B]} g(\sigma)$, hence, if $f(m_1 - \varepsilon) < \alpha$ then $m = (m_1 - \varepsilon)$ satisfies (22), otherwise, there exists $m_2 \in (0, m_1 - \varepsilon]$ such that $f(m_2) = \alpha$. Finally using the fact that $g$ is strictly increasing over $[m_2 - \varepsilon, m_1 - \varepsilon]$ and the first assertion of (21) we see that $m = (m_2 - \varepsilon)$ satisfies (22). This completes the proof.

Now let us consider the following function,

$$g^R_m(s) = \begin{cases} 
  g(s), & \text{for } 0 < s < m, \\
  f(m), & \text{for } m < s \leq B,
\end{cases}$$

with $\bar{m}$ is the constant satisfying $\bar{m} < m$ and $g(\bar{m}) = f(m)$ where $m$ is defined in (22).

The following result is easily checked.

**Lemma 3.2.** Assume that (17) and (21) hold. Then $g^R_m$ defined in (24) is a non-decreasing function over $(0, B)$ satisfying

$$
\begin{align*}
  g^R_m(s) & \leq g(s), & 0 \leq s \leq B, \\
  g^R_m(s) & > f(s), & 0 < s < m, \\
  g^R_m(s) & < f(s), & m < s \leq B.
\end{align*}
$$

Now we are in position to prove the strong persistence of solutions of (1).

**Lemma 3.3.** Assume that (17), (21) hold, then the solution of problem (1) is strongly persistent provided the corresponding initial data satisfies $\phi(0) > 0$. 
Proof. We consider the following problem
\[
\begin{align*}
\dot{y}(t) &= -f(y(t)) + \int_{0}^{t} h(a)g_{m}^{B}(y(t-a))da, \quad t > 0, \\
y(t) &= \phi(t), \quad -\tau \leq t \leq 0,
\end{align*}
\]
with \(g_{m}^{B}\) is defined in (24). In view of Theorem 2.1, we have \(y(t) \leq x(t)\) for all \(t > 0\). Consequently, since \(g_{m}^{B}\) is a nondecreasing function, then the result is reached by applying Lemma 2.5.

In the following, we focus on functions \(g\) having a maximum. More precisely, assume that the function \(g\) satisfies: there exists a positive constant \(M\) such that,
\[
g(M) = \max_{s \in \mathbb{R}^{+}} g(s). \tag{25}
\]
We will investigate two cases, namely, \(x^{\ast} \leq M\) and \(x^{\ast} > M\).

3.2. The Case \(x^{\ast} \leq M\). In order to state our next result we need the following lemma,

Lemma 3.4. Under the hypotheses (21), (25). Then for all solutions \(x\) of problem (1) there exists \(T > 0\) such that
\[
x(t) \leq M, \quad \forall t \geq T.
\]

Proof. First, suppose that \(x^{\ast} < M\) it yields from (21),
\[
g(M) < f(M). \tag{26}
\]
We begin by assuming that all solutions of (1) satisfy \(x(t) > M > x^{\ast}\) for all \(t > 0\) then, due to the second assertion of (21) we have,
\[
K(x(t)) := g(x(t)) - f(x(t)) < 0.
\]
Moreover, by integrating the equation (1) over \((0, r), r > 0\) we obtain
\[
x(r) - x(0) = - \int_{0}^{r} f(x(s))ds + \int_{0}^{r} h(a) \int_{-a}^{r-a} g(x(s))dsda,
\]
thus,
\[
x(r) - x(0) \leq - \int_{0}^{r} f(x(s))ds + \int_{0}^{r} g(x(s))ds + C,
\]
with
\[
C = \int_{0}^{r} h(a) \int_{-a}^{0} g(\phi(s))dsda.
\]
Rearranging these terms, we find
\[
- \int_{0}^{r} K(x(s))ds \leq x(0) + C,
\]
passing to the limit as \(r\) tends to infinity we get,
\[
- \int_{0}^{\infty} K(x(s))ds \leq x(0) + C.
\]
Further, notice that $K(x(\cdot))$ is uniformly continuous; indeed, as all solutions $x$ of (1) are uniformly bounded and from (T1) we have,

$$|K(x(t_1)) - K(x(t_2))| \leq |g(x(t_1)) - g(x(t_2))| + |f(x(t_1)) - f(x(t_2))|,$$

$$\leq L_2|x(t_1) - x(t_2)| + L_1|x(t_1) - x(t_2)|,$$

with $L_1$ and $L_2$ are Lipschitz constants associated to $f$ and $g$ respectively. Since the solutions of (1) are of uniformly bounded first derivative, then

$$|K(x(t_1)) - K(x(t_2))| \leq C(L_2 + L_1)|t_1 - t_2|,$$

with $C$ is the upper bound of the derivative of $x$, accordingly,

$$K(x(t)) \to 0, \text{ as } t \to \infty.$$

From this, all solutions $x$ of (1) converge either to zero or to the positive equilibrium $x^*$: in both cases we reach a contradiction with the assumption. Therefore there exists $T > 0$ such that $x(T) \leq M$. Next, we claim that $x(t) \leq M$ for all $t \geq T$.

Indeed, again by contradiction, we suppose that there exists a positive constant $\bar{t} > T$ such that $x(\bar{t}) = M$ and $x'(\bar{t}) \geq 0$, then

$$0 \leq -f(M) + \int_0^\tau h(a)g(x(\bar{t} - a))da,$$

consequently, we arrive at

$$f(M) \leq g(M),$$

this is a contradiction with (26). Now if $x^* = M$ then

$$\max_{v \in [0, s]} g(v) = g(x^*), \quad \forall s > x^*. \quad (27)$$

Observe that, from the second assertion of (21) we get $g(x^*) < f(s)$ for all $x^* < s \leq B$, thus combining this with (27) we conclude that

$$\max_{v \in [0, s]} g(v) < f(s), \quad \forall s > x^*. \quad (28)$$

According to Lemma 2.2 (substituting the hypothesis (T2) by (28)) we show that $\limsup_{t \to \infty} x(t) \leq x^*$ and the result is reached since the semiflow admits a global compact attractor. The lemma is proved.

Next let us turn our attention towards the more interesting case.

3.3. The case $x^* > M$. This situation is more delicate to study and we need to impose additional hypotheses on $f$ and $g$.

Assume that

$$\begin{cases} f(s) < f(M) & \text{for } 0 \leq s < M, \\ f(s) > f(M) & \text{for } M < s \leq B. \end{cases} \quad (29)$$

Now, to avoid any possibility of infinitely oscillations of $g$ around $f(M)$, we will assume that $g$ satisfies the next hypotheses, setting

$$\mathbb{D} \equiv \{s \in [0, M] : g(s) = f(M)\},$$

there exists $\bar{m} \in (0, M)$, such that $g(\bar{m}) = f(M)$ and $\bar{m} = \max \mathbb{D}$. \quad (30)

Observe that $\mathbb{D} \neq \emptyset$ since $g(0) = f(0) < f(M) < g(M)$. In the same way we define the set

$$\mathbb{D} \equiv \{s \in [M, B] : g(s) = f(M)\}.$$
The rest of this subsection is devoted to estimating the solutions of (1) into two different situations namely, either \( D = \emptyset \) or \( \min D \) exists. The following lemma deals with the first case.

**Lemma 3.5.** Assume that \( D = \emptyset \). We also suppose that (17), (21), (29), (30) hold, then for all solutions \( x \) of problem (1) there exists \( T > 0 \) such that

\[
M \leq x(t) \leq B, \quad \forall t \geq T.
\]

**Proof.** It is clear that \( D = \emptyset \) implies,

\[
g(s) > f(M) \quad \text{for all} \quad s \in [M, B]. \tag{31}
\]

Now, in view of (30) and the fact that \( g(M) > f(M) \), the value \( \bar{m} \) defined in (30) satisfies

\[
\min_{\sigma \in [\bar{m}, M]} g(\sigma) = f(M), \tag{32}
\]

otherwise \( f(\bar{\sigma}) := \min_{\sigma \in [\bar{m}, M]} g(\sigma) < f(M) \) with \( \bar{\sigma} \in (\bar{m}, M) \). Since \( g(M) > f(M) \) then there exists \( \bar{\sigma} \in (\bar{m}, M) \) such that \( g(\bar{\sigma}) = f(M) \), which contradicts (30).

Next, we introduce the following function,

\[
g^B_M(s) = \begin{cases} 
\min_{\sigma \in [s, M]} g(\sigma), & \text{for} \quad 0 < s < \bar{m}, \\
f(M), & \text{for} \quad \bar{m} < s \leq B,
\end{cases} \tag{33}
\]

we claim that the function \( g^B_M \) is nondecreasing and satisfies,

\[
\begin{cases}
\begin{aligned}
g^B_M(s) &\leq g(s), &\text{for} &\quad 0 \leq s \leq B, \\
g^B_M(s) &> f(s), &\text{for} &\quad 0 < s < M, \\
g^B_M(s) &< f(s), &\text{for} &\quad M < s \leq B.
\end{aligned}
\end{cases} \tag{34}
\]

Indeed, from (31), (32) and (33) it is easily checked that \( g^B_M \) is nondecreasing and \( g^B_M(s) \leq g(s) \) for all \( s \in [0, B] \). We first take \( 0 < s < \bar{m} \), then using the fact that \( M < x^* \), we have

\[
g^B_M(s) = \min_{\sigma \in [s, M]} g(\sigma),
\]

\[
\geq \min_{\sigma \in [s, x^*]} g(\sigma),
\]

in view of (21) we get

\[
g^B_M(s) > f(s).
\]

For \( \bar{m} \leq s < M \), the hypothesis (29) gives that,

\[
g^B_M(s) = f(M) > f(s).
\]

Finally for \( M < s \leq B \), using again (29) we obtain

\[
g^B_M(s) = f(M) < f(s),
\]

and the claim is proved.

Next, we consider the following problem,

\[
\begin{align*}
y'(t) &= -f(y(t)) + \int_0^t h(a)g^B_M(y(t - a))da, & t > 0, \\
y(t) &= \phi(t), & -\tau \leq t \leq 0,
\end{align*} \tag{35}
\]
using Theorem 2.1, we have \( y(t) \leq x(t) \) for all \( t \geq -\tau \). Moreover, since \( g_B^M \) is a nondecreasing function and the problem (35) admits only \( M \) as positive equilibrium, then from Theorem 2.6, \( y(t) \) goes to \( M \) as \( t \) tends to infinity. At this stage we have proved that
\[
\liminf_{t \to \infty} x(t) \geq M.
\]
Employing now the same arguments as at the end of the proof of Lemma 2.2, we show that \( \Phi \) admits a compact attractor \( J \subset [M^*, B^*] \) that attracts all bounded sets of \( C_+ \). The result is reached.

Now, we will give an interest to the case where \( \min \mathbb{D} \) exists. In this context, and throughout the rest of the paper, we define the constant \( A \) as
\[
A := \min \{ s \in (M, B), \; g(s) = f(M) \},
\]
and \( M \) is defined in (25). The following lemma gives some estimates of solutions of (1).

**Lemma 3.6.** Suppose that (17), (18), (21), (29) are fulfilled and assume that the following condition holds
\[
f(A) > g(M).
\]
Then for all solutions \( x \) of problem (1) there exists \( T > 0 \) such that
\[
M \leq x(t) \leq A, \quad \forall t \geq T.
\]

**Proof.** First, we claim that \( x^* < A \). Conversely, suppose \( x^* \geq A \), then if \( x^* > A \), due to the first assertion of (21) and \( M < A \), we obtain
\[
f(M) := g(A) \geq \min_{\sigma \in [M, x^*]} g(\sigma) > f(M),
\]
which is a contradiction. Necessarily \( x^* \neq A \), if not, the first assertion of (21) and (37) give
\[
g(M) > f(M) = g(A) = f(A) > g(M),
\]
the claim is established.

On the other hand, we prove that for all solutions \( x \) of (1) it is impossible to have \( T > 0 \) such that \( x(t) > A \) for all \( t \geq T \). Indeed, if the contrary is true, so we take \( t \geq T + \tau \) and let us consider the following problem
\[
\begin{cases}
y'(t) = -f(y(t)) + \int_0^\tau h(a)\bar{g}(y(t-a))da, & t > 0, \\
y(t) = x(t), & -\tau \leq t \leq 0,
\end{cases}
\]
with
\[
\bar{g}(s) = \begin{cases}
\min_{\sigma \in [x, x^*]} g(\sigma) & \text{for } 0 < s \leq x^*, \\
\max_{\sigma \in [x^*, s]} g(\sigma) & \text{for } x^* < s \leq B,
\end{cases}
\]
the contradiction comes from \( x^* < A < x(t) \leq y(t) \) and \( y(t) \) converges to \( x^* \), since \( \bar{g} \) is a nondecreasing function and consequently Theorem 2.6 can be applied. Then there exists \( t_0 > 0 \) such that \( x(t_0) \leq A \), we claim that \( x(t) \leq A \) for all \( t \geq t_0 \). Otherwise, there exists \( t_1 > 0 \) such that \( x(t_1) = A \) and \( x'(t_1) \geq 0 \) thus, from (1),
\[
0 \leq x'(t_1) = -f(x(t_1)) + \int_0^\tau h(a)g(x(t_1-a))da,
\]
\[
\leq -f(A) + g(M),
\]
hence,

\[ f(A) \leq g(M), \]

this is a contradiction with (37). The upper bound of solutions \( x \) to problem (1) is proved.

Next we focus on the lower bound of \( x \). From (29), (36) the function \( g_{M}^{A} \) defined in (33) (by substituting \( B \) by \( A \)) is nondecreasing and satisfies (34). So by introducing the following problem

\[
\begin{aligned}
\left\{ \begin{array}{ll}
y'(t) = -f(y(t)) + \int_{0}^{t} h(a)g_{M}^{A}(y(t-a))da, & t > 0, \\
y(t) = x(t), & -\tau \leq t \leq 0,
\end{array} \right.
\end{aligned}
\]

and using Theorem 2.1 we have \( y(t) \leq x(t) \) for all \( t > 0 \). Thus \( \liminf_{t \to \infty} x(t) \geq M \).

Employing now the same arguments as at the end of the proof of Lemma 2.2, we show that \( \Phi \) admits a compact attractor \( J \subset [M^{*}, A^{*}] \).

Corollary 1. Assume that the function \( f \) is strictly increasing over \([0, B]\) and \( g \) is strictly decreasing over \([M, B]\), then the condition (37) is equivalent to

\[ M < f^{-1}(g(f^{-1}(g(M)))) \]  

(40)

Proof. Suppose (37) is verified. First observe that \( f \) maps \([0, B]\) to \([f(0), f(B)]\) and \( g \) maps \([M, B]\) to \([g(B), g(M)]\). Now, in view of \( f(0) < f(M) < g(M) < f(A) \) we have

\[ M < f^{-1}(g(M)) < A. \]

From this and the monotonicity of \( g \) we get

\[ g(A) < g(f^{-1}(g(M))) < g(M) < f(A), \]

by the definition of \( A \) in (37),

\[ f(M) < g(f^{-1}(g(M))) < f(A), \]

the result is reached by the monotonicity of \( f \).

Now if (40) holds, we first show that,

\[ M < f^{-1}(g(f^{-1}(g(M)))) < B. \]  

(41)

Indeed, according to (T2) we have

\[ f(0) < f(M) < g(M) \leq f(B), \]

so,

\[ M < f^{-1}(g(M)) < B, \]

it yields,

\[ f(0) < g(B) < g(f^{-1}(g(M))) < g(M) \leq f(B), \]  

(42)

since \( f \) is strictly increasing, we find

\[ 0 < f^{-1}(g(f^{-1}(g(M)))) < B, \]  

(43)

combining (43) and (40) we get (41).

Next, the constant \( A \) satisfies \( g(A) = f(M) \). Thus, using the monotonicity of \( f \) in (41) and (42) we get,

\[ f(M) := g(A) < g(f^{-1}(g(M))) < f(B), \]

the monotonicity of \( g \) over \([M, B]\) gives,

\[ A > f^{-1}(g(M)), \]
therefore the result is obtained by the monotonicity of \( f \).

**Remark 2.** The particular case \( f(s) = \mu s \) gives,

\[
M < \frac{g(M)}{\mu}.
\]

This last inequality is the well-known condition \((L)\) stated in page 2661, [27].

### 3.4. Global asymptotic stability of the positive equilibrium.

In order to prove the global asymptotic stability of the positive equilibrium, from now on we need the following hypotheses and definitions: there exists a unique positive constant \( M \) such that,

\[
g(M) = \max_{s \in \mathbb{R}^+} g(s)
\]

the function \( g \) is nondecreasing over \((0, M)\) and nonincreasing over \((M, B)\).

We define \( \bar{f} = f|_{[M, A]} \), the restriction function of \( f \) over \([M, A]\) and \( G(s) := \bar{f}^{-1}(g(s)) \) for \( s \in [M, A] \) when \( \bar{f} \) is strictly increasing over \([M, A]\).

The following corollary deals with the case when \( x^* \leq M \) and its proof is a direct consequence of Theorem 2.6.

**Corollary 2.** Assume that (17), (18), (21) hold, then the positive equilibrium \( x^* \) of problem (1) is globally asymptotically stable provided that \( x^* \leq M \).

We now establish the main theorem of this section related to the case \( x^* > M \).

**Theorem 3.7.** Under the assumptions of Lemma 3.6, the positive equilibrium of problem (1) is globally asymptotically stable provided that one of the following conditions holds:

- **(H1)** \((f(s) - f(0))(g(s) - g(0))\) is a nondecreasing function over \([M, A]\).
- **(H2)** \(f + g\) is a nondecreasing function over \([M, A]\).
- **(H3)** \((f(s) - f(0))(g(s) - g(0)) + f + g\) is a nondecreasing function over \([M, A]\).
- **(H4)** \(\bar{f}\) is strictly increasing over \([M, x^*]\) and \(\frac{(GoG)(s)}{s}\) is a nonincreasing function over \([M, x^*]\).
- **(H5)** \(\bar{f}\) is strictly increasing over \([M, A]\) and \(\frac{(GoG)(s)}{s}\) is a nonincreasing function over \([x^*, A]\).

**Proof.** First of all, according to Lemma 3.6 we know that for every solution \( x \) of (1) there exists \( T > 0 \) such that \( M \leq x(t) \leq A \) for all \( t > T \). Now we split the proof into two possible cases, non-oscillatory, that is the trajectory does not intersect the positive equilibrium infinitely many times, and oscillatory, that is the trajectories oscillate infinitely around the positive equilibrium:

1/ **Non-oscillatory case.** We first suppose that for all solutions \( x \) of (1) there exists \( T > 0 \) such that \( x(t) \leq x^* \) for all \( t > T \). Then, we introduce the following function

\[
g(s) = \begin{cases} 
\min_{\sigma \in [s, x^*]} g(\sigma) & \text{for } 0 < s < x^*, \\
g(x^*) & \text{for } x^* \leq s \leq A,
\end{cases}
\]
and the associated problem
\[
    \begin{cases}
        y'(t) = -f(y(t)) + \int_0^\tau h(a)g(y(t-a))da, & t > 0, \\
        y(t) = x(t), & -\tau \leq t \leq 0,
    \end{cases}
\]
thus since \( g(s) \leq g(s) \) for all \( s \leq x^* \) then \( y(t) \leq x(t) \) for all \( t > 0 \) and \( y(t) \) converges to \( x^* \) as \( t \) tends to infinity. Now if \( x(t) \geq x^* \) for all \( t \geq T \), as above, let \( \tilde{g} \) be defined as
\[
    \tilde{g}(s) = \begin{cases}
        \min_{\sigma \in [s,x^*]} g(\sigma) & 0 < s \leq x^*, \\
        \max_{\sigma \in [x^*,s]} g(\sigma) & x^* < s \leq A,
    \end{cases}
\]
the solution of the following problem,
\[
    \begin{cases}
        y'(t) = -f(y(t)) + \int_0^\tau h(a)\tilde{g}(y(t-a))da, & t > 0, \\
        y(t) = x(t), & -\tau \leq t \leq 0,
    \end{cases}
\]
verify \( x(t) \leq y(t) \) for all \( t > 0 \) and \( y(t) \) converges to \( x^* \) as \( t \) goes to infinity. Concerning the local stability we use the same arguments as in the proof of Theorem 2.6.

\textbf{2/ Oscillatory case.} We claim that this situation is not possible. Assume by contradiction that the solution \( x \) oscillates infinitely around the positive equilibrium \( x^* \), we set
\[
    x^\infty := \limsup_{t \rightarrow \infty} x(t), \quad x_\infty := \liminf_{t \rightarrow \infty} x(t),
\]
and
\[
    A = x_\infty > x^\infty > M.
\]
Using the fluctuations method, see [31], [32], there exist two sequences \( t_n \rightarrow \infty \) and \( s_n \rightarrow \infty \) such that
\[
    \lim_{n \rightarrow \infty} x(t_n) = x^\infty, \quad x'(t_n) = 0, \quad \forall n \geq 1,
\]
and
\[
    \lim_{n \rightarrow \infty} x(s_n) = x_\infty, \quad x'(s_n) = 0, \quad \forall n \geq 1,
\]
then substituting \( x(t_n) \) in problem (1) it follows that,
\[
    0 = -f(x(t_n)) + \int_0^\tau h(a)g(x(t_n-a))da,
\]
now, using the fact that
\[
    \lim_{n \rightarrow \infty} x(t_n - a) \geq x^\infty, \quad \forall a \in [0,\tau],
\]
and
\[
    g(y) \text{ is nonincreasing over } [M,A],
\]
then
\[
    f(x^\infty) \leq g(x_\infty).
\]
Similarly by taking in consideration that
\[
    \lim_{n \rightarrow \infty} x(s_n - a) \leq x^\infty, \quad \forall a \in [0,\tau],
\]
we get,
\[
    f(x_\infty) \geq g(x^\infty),
\]
from \( g(0) = f(0) \) we have,
\[
    f(x^\infty) - f(0) \leq g(x^\infty) - g(0), \quad f(x^\infty) - f(0) \geq g(x^\infty) - g(0).
\]
Multiplying the expression (50) by \( g(x^\infty) - g(0) > 0 \) and combining with (51) we obtain
\[
(f(x^\infty) - f(0))(g(x^\infty) - g(0)) \leq (f(x^\infty) - f(0))(g(x^\infty) - g(0)),
\]
this fact together with the hypothesis \((H1)\) give \( x^\infty \leq x^\infty \), so we reach a contradiction. Arguing as before we may conclude the results for \((H2)\) and \((H3)\). Now suppose that \((H4)\) holds. First notice that \( G \) defined just before the present theorem make sense, that is, for all \( s \in [M, A] \) the range of \( g \) is contained in \([\bar{f}(M), \bar{f}(A)]\) since \( \bar{f} \) is strictly increasing over \([M, A]\). In fact, for all \( s \in [M, A] \) and since \( g \) is non increasing over \([M, A]\) then \( g(A) \leq g(s) \leq g(M) \). Now using (36) and (37) we show that
\[
\bar{f}(M) \leq g(s) \leq \bar{f}(A), \quad \text{for all} \quad s \in [M, A].
\]
Therefore the function \( G \) is nonincreasing and maps \([M, A]\) to \([M, A]\).

In view of (48), (49) and the monotonicity of \( \bar{f} \) we arrive at,
\[
\begin{align*}
\bar{f}(M) & \leq g(s) \leq \bar{f}(A), \\
x^\infty & \leq G(x^\infty),
\end{align*}
\]
and
\[
\begin{align*}
x^\infty & \geq G(x^\infty),
\end{align*}
\]
with \( G(s) := \bar{f}^{-1}(g(s)) \). Now suppose that \( x^\infty < x^* \leq x^\infty \), then, applying the function \( G \), the inequalities (52)-(53) become
\[
\begin{align*}
x^\infty & \geq G(x^\infty) \geq (GoG)(x^\infty),
\end{align*}
\]
this gives,
\[
\frac{(GoG)(x^\infty)}{x^\infty} \leq 1 = \frac{(GoG)(x^*)}{x^*},
\]
due to \((H4)\) it ensures that \( x^* \leq x^\infty \), which is a contradiction. Moreover, if \( x^\infty \leq x^* < x^\infty \) then from (54) we get
\[
x^\infty = x^*.
\]
In addition, according to (48) we have
\[
f(x^\infty) \leq g(x^\infty) = g(x^*) = f(x^*) < f(x^\infty),
\]
the contradiction is also reached. Using the same arguments as in (H4) we establish the result for \((H5)\). The Theorem is proved. \( \square \)

**Remark 3.** If the hypothesis \((37)\) is not satisfied, we also have the global asymptotic stability of the positive equilibrium if \( \mathbb{D} = \emptyset \) and if one of the conditions \((H1)-(H3)\) is satisfied over \([M, B]\).

Now we deal with the case where the product function \( H(.) := (f(.) - f(0))(g(.) - g(0)) \) satisfies
\[
\begin{align*}
\max_{s \in [M, A]} H(s) & = H(M), \quad \text{and} \quad H \text{ is nondecreasing over} \quad (M, \bar{M}).
\end{align*}
\]

**Corollary 3.** Under the assumptions of Lemma 3.6 and (55). The positive equilibrium of (1) is globally asymptotically stable provided that
\[
(f(M) - f(0))(f(M) - f(0)) > (g(M) - g(0))(g(M) - g(0)).
\]
Proof. In order to prove this corollary, it suffices to show that $x^* \leq \bar{M}$ and $x(t) \leq \bar{M}$ for all $t \geq T$. Then, the result follows from (H1) in Theorem 3.7. First, from (56) and the fact that $K(M) := g(M) - f(M) > 0$ we have

$$
(f(M) - f(0))(f(M) - f(0)) > (g(M) - g(0))(g(M) - g(0)),
$$

so we have

$$
(f(M) - f(0))(f(M) - f(0)) > (g(M) - g(0))(f(M) - f(0)), \tag{57}
$$

thus, $K(\bar{M}) < 0$ and therefore $x^* < \bar{M}$. Now we first suppose that there exists $T > 0$ such that $x(t) \geq \bar{M}$ for all $t > T$. Notice that $M < \bar{M}$, then, as done in the first part of the proof of Theorem 3.7, we define $\bar{g}$ and by the sequel $y$ as in (45)-(46). So we have $x^* < \bar{M} \leq x(t) \leq y(t)$ and $y(t)$ converges to $x^*$, which is a contradiction. As a conclusion, there exists $t_0 > 0$ such that $x(t_0) < \bar{M}$. Now we claim that $x(t) < \bar{M}$ for all $t > t_0$, otherwise, there exists $t_1 > 0$ such that $x(t_1) = \bar{M}$ and $x'(t_1) \geq 0$, substituting $x(t_1)$ in (1) we get,

$$
f(\bar{M}) \leq g(M). \tag{58}
$$

On the other hand, by multiplying the expression (56) by $f(\bar{M}) - f(0)$ we have,

$$
(f(M) - f(0))^2(f(M) - f(0)) > H(M)(g(M) - g(0)),
$$

since the function $H$ reaches its maximum in $\bar{M}$ it follows that,

$$
(f(\bar{M}) - f(0))^2(f(\bar{M}) - f(0)) > (g(M) - g(0))^2(f(M) - f(0)),
$$

therefore $f(\bar{M}) > g(M)$, which contradicts (58). \hfill \Box

4. Global exponential stability of equilibria. Now let us investigate the global exponential stability of equilibria. For this, we first establish the following lemma.

Lemma 4.1. Suppose that there exist two positive constants $A_1$, $A_2$ and a positive function $w$ such that

$$
w'(t) \leq -A_1w(t) + A_2\int_0^t h(a)w(t-a)da. \tag{59}
$$

If $A_1 > A_2$, then there exist two positive constants $C$ and $A_3$ with $A_2 < A_3 < A_1$ such that

$$
w(t) \leq Ce^{-(A_1 - A_3)t}, \forall t \geq -\tau. \tag{60}
$$

Proof. First, observe that (59) can be reformulated as

$$
z'(t) \leq A_2\int_0^t h(a)e^{A_1z(t-a)}da, \tag{61}
$$

with $z(t) = e^{A_1t}w(t)$. Now, we can choose $C > 0$ and $A_2 < A_3 < A_1$ such that $Ce^{A_3t}$ is a supersolution of (61), that is

$$
A_3 \geq A_2\int_0^t h(a)e^{(A_1-A_3)a}da.
$$

Indeed, since $A_1 > A_2$ there exists $\varepsilon$ so small such that

$$
A_1 - \varepsilon \geq A_2\int_0^t h(a)e^{\varepsilon a}da.
$$

Next, we set $A_3 = A_1 - \varepsilon$, it follows that

$$
A_3 \geq A_2\int_0^t h(a)e^{(A_1-A_3)a}da,
$$
therefore $z(t) \leq Ce^{A_3 t}$ and thus $w(t) \leq Ce^{-(A_1 - A_3) t}$. The lemma is proved.

Now we are in position to present our main result concerning the exponential stability of equilibria.

**Theorem 4.2.** Under the conditions of Theorem 3.7. Suppose also that $f, g$ are differential functions and $f$ is strictly increasing function on $[M, A]$. Then the equilibrium $x^*$ of (1) is globally exponentially stable provided that

$$\inf_{s \in [M, A]} f'(s) > \sup_{s \in [M, A]} |g'(s)|.$$  \hfill (62)

**Proof.** First of all, in view of the proof of Theorem 3.7, notice that the oscillatory case is not possible. Thus, we set $y(t) = x(t) - x^*$, and we suppose that there exists $T > 0$ such that $y(t) \geq 0$ for all $t \geq T$ (the proof will be the same if we assume that $y(t) \leq 0$ for all $t \geq T$). Then $y$ satisfies the following problem

$$y'(t) = -f'(\theta(t))y(t) + \int_0^T h(a)g'(\theta_1(t-a))y(t-a)da,$$

where $\theta(t)$ ($\theta_1(t-a)$ respectively) is a value between $x(t)$ and $x^*$ ($x(t-a)$ and $x^*$ respectively). Now since $x(t)$ and $x^*$ belong to $[M, A]$ we obtain,

$$y'(t) \leq -\inf_{s \in [M, A]} f'(s)y(t) + \sup_{s \in [M, A]} |g'(s)| \int_0^T h(a)y(t-a)da.$$  

The result is established from (62) and by applying Lemma 4.1 for $A_1 = \inf_{s \in [M, A]} f'(s)$ and $A_2 = \sup_{s \in [M, A]} |g'(s)|$.

5. **Application to Nicholson’s blowflies and Mackey-Glass models.** The goal of this section is to apply our results to two well-known models, namely Blowflies and Mackey-Glass distributed delay equation. For a good survey in this direction see [1] and references therein.

First, observe that the condition (21) is verified whenever the positive equilibrium exists. We set $\int_0^T h(a)da = 1$.

5.1. **Nicholson’s blowflies model.** The blowflies model with distributed delay is defined as follows

$$x'(t) = -\delta x(t) + \int_0^T h(a)x(t-a)e^{-x(t-a)}da, \quad t > 0,$$

where the equation of equilibria is given by

$$\delta x^* = x^* e^{-x^*}.$$  

Thus either $x^* = 0$ or $x^* = \ln(1/\delta)$ whenever $1/\delta > 1$. For more details and results concerning this equation see [1] and references therein.

**Theorem 5.1.** Suppose that $\delta \geq 1$, then the trivial equilibrium of (63) is globally asymptotically stable.

**Proof.** The condition (15) is equivalent to $\delta \geq 1$, so the result is obtained by applying Theorem 2.4.
Theorem 5.2. The positive equilibrium of (63) is globally asymptotically stable provided that
\[ 1 < \frac{1}{\delta} \leq e^2. \]

Moreover this equilibrium is globally exponentially stable if,
\[ e < \frac{1}{\delta} < e^2. \]  

Proof. First, for \( g(s) = se^{-s} \), notice that sup_{s \in \mathbb{R}^+} g(s) = g(1) = e^{-1}. Now, assume that \( 1 < \frac{1}{\delta} \leq e \), then we have \( x^* \leq M := 1 \), with \( M \) is defined in (25). Therefore all assumptions of Corollary 2 are satisfied, hence we conclude that \( x^* \) is globally asymptotically stable. Next, suppose that
\[ e < \frac{1}{\delta} \leq e^2, \]  

this case corresponds to \( x^* > M := 1 \), thus we can define the constant \( A \) as in (36), namely,
\[ Ae^{-A} = \delta, \quad \text{and} \quad A > 1. \]  

We claim that the condition (37) stated in Lemma 3.6 is always satisfied, i.e. \( A \geq \frac{1}{\delta} e^{-1} > 1 \). Since the function \( f \) is strictly increasing and \( g \) is strictly decreasing over \((1, \infty)\) then by using Corollary 1 and Remark 2, the condition (37) is equivalent to (40), that is, the new claim becomes,
\[ \frac{1}{\delta^2} e^{-1} e^{-\frac{1}{\delta} e^{-1}} > 1. \]

If it is not true, so
\[ \frac{1}{\delta} e^{-1} e^{-\frac{1}{\delta} e^{-1}} \leq \delta, \]
by a simple computation we obtain,
\[ 2ln\left(\frac{1}{\delta} e^{-1}\right) - \frac{1}{\delta} e^{-1} + 1 \leq 0. \]  

However, the function \( R(s) := (2ln(s) - s + 1) \) is positive if \( 1 < s < \theta \) with \( \theta \simeq 3.51 \) and satisfies \( f(\theta) = 0 \), thus for \( s = \frac{1}{\delta} e^{-1} \) we have \( e < \frac{1}{\delta} < 3.51e \), hence the hypothesis (65) contradicts (67). At this stage we conclude that the solution \( x \) of problem (63) satisfies \( 1 < x(t) \leq A \) for all \( t \geq T \) provided that
\[ e < \frac{1}{\delta} < 3.51e. \]

Finally, observe that the function \( \frac{G(G(s))}{s} \) with \( G(s) := \frac{1}{\delta} se^{-s} \) is decreasing over \((1, x^*)\) if the assertion (65) is fulfilled. Therefore the hypothesis \((H4)\) in Theorem 3.7 is satisfied. Consequently, the positive equilibrium is globally asymptotically stable provided \( 1 < \frac{1}{\delta} \leq e^2 \).

On the other hand, the condition (62) holds if
\[ e < \frac{1}{\delta} < e^2, \]
as a consequence, the exponential stability follows from Theorem 4.2. \( \square \)
5.2. Mackey-Glass model of hematopoiesis. To model a blood cell production and haematological diseases, different authors studied the following Mackey-Glass model with distributed delay,

\[ x'(t) = -\delta x(t) + \int_0^\tau h(a) \frac{x(t-a)}{1+x^n(t-a)} da. \]  

(68)

For more results of this type of equations, see [2] and the references therein.

The equilibria are given by \( x^* = 0 \) and \( x^* = (\frac{1}{\delta} - 1)^\frac{1}{n} \) whenever it exists. We set \( g(s) = \frac{s}{1+s^n} \) and \( K(s) = g(s) - \delta s \), notice that \( \sup_{s \in \mathbb{R}^+} g(s) = g(M) \) with \( M := \frac{1}{(n-1)^\frac{1}{n}} \) and \( n > 1 \).

By the application of our theorems stated in the previous sections, we obtain

**Theorem 5.3.** Suppose that \( \delta \geq 1 \), then the trivial equilibrium of (68) is globally asymptotically stable.

**Proof.** The condition (15) is equivalent to \( \delta \geq 1 \). We conclude by applying Theorem 2.4.

Before stating the global stability of the positive equilibrium, we need to verify the requirement (37) of Lemma 3.6. The other conditions are clearly verified. This is the objective of the following lemma.

**Lemma 5.4.** Suppose that \( x^* > M \), then the condition (37) holds if and only if we have

\[ F(\frac{n}{n-1}) < 0, \]  

(69)

with

\[ F(s) := s^n - (\frac{1}{\delta})^2 s^{n-1} + (\frac{1}{\delta})^n s - (\frac{1}{\delta})^n. \]  

(70)

**Proof.** In the context of Mackey-Glass model, the condition (37) reads,

\[ \delta A > \frac{M}{1+M^n}, \]  

(71)

and

\[
\begin{cases}
\delta M = \frac{A}{1+A^n}, \\
M < A < B.
\end{cases}
\]  

(72)

From Corollary 1 and Remark 2, the condition (71) is equivalent to

\[ \delta M < g(\frac{1}{\delta} \frac{M}{1+M^n}), \]

this reads,

\[ 1 + (\frac{1}{\delta} \frac{M}{1+M^n})^n < (\frac{1}{\delta})^2 \frac{1}{1+M^n}. \]

Multiplying this last inequality by \( (1+M^n)^n \) and setting

\[ a := 1 + M^n = \frac{n}{n-1}, \]

we observe that the expression (71) is equivalent to,

\[ F(a) < 0, \]
with

\[ F(s) := s^n - \left(\frac{1}{\delta}\right)^2 s^{n-1} + \left(\frac{1}{\delta}\right)^n s - \left(\frac{1}{\delta}\right)^n. \]

The lemma is proved. \(\square\)

We will now present our version of Theorem 4.5 in [35], that provides sufficient conditions for the global asymptotic stability of the positive equilibrium. Although the equation considered in [35] only has a discrete delay, but the dynamical system setting and method used there should also work for our problem.

**Theorem 5.5.** Suppose that \( \delta < 1 \), then the positive equilibrium of (68) is globally asymptotically stable if one of the following conditions is satisfied

\[ 0 < n \leq 2, \quad (73) \]
\[ n > 2 \quad \text{and} \quad \frac{1}{\delta} < \frac{n}{n-2}. \quad (74) \]

**Proof.**

**First case.** \( 0 < n \leq 2 \). We will divide this case into three subcases

**Subcase 1.** \( 0 < n \leq 1 \). Observe that this corresponds to \( g \) is increasing, therefore the result follows from Theorem 2.6.

**Subcase 2.** \( 1 < n \leq 2 \) and \( \frac{1}{\delta} \leq \frac{n}{n-1} \). This corresponds to \( x^* \leq M \), so the Corollary 2 is applied to reach the global asymptotic stability of the positive equilibrium.

**Subcase 3.** \( 1 < n \leq 2 \) and \( \frac{1}{\delta} > \frac{n}{n-1} \). This situation corresponds to \( x^* > M \), notice that (69) is easily proved. Indeed, we rewrite (70) as

\[ F(s) := s^n - \left(\frac{1}{\delta}\right)^n + \left(\frac{1}{\delta}\right)^2 s \left(\frac{1}{\delta}\right)^n - s^n, \]

and so, \( F\left(\frac{n}{n-1}\right) < 0 \), hence from Lemma 5.4 the assertion (37) is verified. Moreover, by a straightforward computation we show that the function \( \frac{G(G(s))}{s} \) is decreasing for all \( s > 0 \) and all \( 0 < n \leq 2 \), with \( G(s) = \frac{1}{\delta} \frac{s}{1 + s^n} \), thus the condition (H4) of Theorem 3.7 is satisfied. Thus the assertion (73) is proved.

**Second case.** \( n > 2 \) and \( \frac{1}{\delta} \leq \frac{n}{n-1} \). As Corollary 2 can be applied for \( n > 2 \), by the same reasoning of Subcase 2 we get the result.

**Third case.** \( n > 2 \) and \( \frac{n}{n-1} < \frac{1}{\delta} < \frac{n}{n-2} \). By a simple calculation we remark that \( \frac{G(G(s))}{s} \) is a decreasing function in this case. We conclude that (H4) of Theorem 3.7 is established. It remains to show that (69) is satisfied in order to reach the result. The computation of the second derivative of the function \( F \) gives,

\[ F''(s) := (n-1)s^{n-3}(ns^2 - (n-2)(\frac{1}{\delta})^2), \]

by setting

\[ s^* := \left(\frac{1}{\delta}\right)^2 \frac{n-2}{n}, \]
observe that the function $F$ is concave for $s < s^*$ and convex for $s > s^*$. In addition, in view of $n > 2$ and $\frac{n}{n-1} < \frac{1}{\delta} < \frac{n}{n-2}$, it is easy to show that $$F(1) < 0, \quad F\left(\frac{1}{\delta}\right) = 0, \quad F'(s^*) > 0 \quad \text{and} \quad F'(\frac{1}{\delta}) > 0,$$
as a consequence it is clear that $F\left(\frac{n}{n-1}\right) < 0$. The result follows from Theorem 3.7.

The case leading to the exponential stability of the positive equilibrium is proved by applying Theorem 4.2.

**Theorem 5.6.** Suppose that $\delta < 1$, then the positive equilibrium of (68) is globally exponentially stable if the following condition is satisfied

\[ n > 1 \quad \text{and} \quad \frac{n}{n-1} < \frac{1}{\delta} < \frac{4n}{(n-1)^2}. \]

6. **Discussion.** In this paper we have investigated the global behavior of solutions for a class of functional differential equations (1). We have presented some conditions to ensure global asymptotic and exponential stability of the unique positive equilibrium. We illustrated the obtained results in some well-known biologic models namely, the Nicholson and Mackey-Glass equations.

In addition, when dealing with the following equation, which is mentioned in [1],

\[ x'(t) = -\frac{\alpha x(t)}{b + x(t)} + q \int_0^\tau h(a)x(t-a)e^{x(t-a)}da, \quad (75) \]

in the particular case of non autonomous equations, our methods give more explicit conditions than those presented in [5]. In fact the stability of equilibria of equation (75) is carried out by applying some of the previous results of this paper. More precisely we use the Theorem 2.4 for the trivial equilibrium and the Corollary 2, Theorem 3.7 (H1) together with the Remark 3, in order to prove global asymptotic stability of the positive equilibrium. We set

\[ f(s) = \frac{\alpha s}{b + s} \quad \text{and} \quad g(s) = qse^{-s}, \]

then we have

**Lemma 6.1.** The trivial equilibrium of (75) is globally asymptotically stable if one of the following conditions holds:

a) \[ b \leq \frac{\alpha}{q} \quad \text{and} \quad b \geq 1, \]
b) \[ e^{b-1} < \frac{\alpha}{q} \quad \text{and} \quad b < 1. \]

**Lemma 6.2.** Assume that \( \frac{\alpha}{q} < b \), then the positive equilibrium of (75) exists and is unique. Moreover this one is globally asymptotically stable if one of the following conditions holds:

a) \[ (b + 1)e^{-1} \leq \frac{\alpha}{q}, \]
b) \[ e^{-1} < \frac{\alpha}{q} < (b + 1)e^{-1}. \]
c) $B < \bar{x}$ and $qBe^{-B} > \frac{\alpha}{b+1}$ with $\bar{x} = \frac{1 - b + \sqrt{b^2 + 6b + 1}}{2}$, and
\[ B = \frac{bqe^{-1}}{\alpha - qe^{-1}}. \]

Proof. First, by a straightforward calculation, observe that, the condition $\frac{\alpha}{q} < b$ guarantees existence and uniqueness of the positive equilibrium. Next suppose that the assertion a) is satisfied, or equivalently $f(1) \geq g(1) := qe^{-1}$, from this and the fact that $\max \limits_{\mathbb{R}^+} g(s) = g(1) = qe^{-1}$ and $f$ is an increasing function, we find
\[ \max_{\sigma \in [0,s]} g(\sigma) - f(s) < 0 \quad \text{for all} \quad s > 1. \quad (76) \]

In view of (76), the assertion (T2) is verified for $B = 1$, thus due to Lemma 2.2 we have, for all solutions $x$ of (75) there exists $T > 0$ such that $x(t) \leq 1$ for all $t \geq T$, we use Corollary 2 to reach the result. Next suppose that $e^{-1} < \frac{\alpha}{q} < (b+1)e^{-1}$.

First notice that $B > 1$, we define the constant $B$ such that $f(B) = g(1) := qe^{-1}$, this gives,
\[ B = \frac{bqe^{-1}}{\alpha - qe^{-1}}, \]
from this value of $B$ and the monotonicity of $f$ we have,
\[ \max_{\sigma \in [0,s]} g(\sigma) - f(s) < 0 \quad \text{for all} \quad s > B. \]

Then Lemma 2.2, says that for all solutions $x$ of (75) there exists $T > 0$ such that $x(t) \leq B$ for all $t \geq T$. Now the hypothesis
\[ qBe^{-B} > \frac{\alpha}{b+1}, \]
is equivalent to $g(B) > f(1)$, and thus
\[ g(s) > f(1) \quad \text{for all} \quad 1 \leq s \leq B. \quad (77) \]

We can apply Lemma 3.5 in order to prove that, for all solutions of (75) there exists $T > 0$ such that
\[ 1 \leq x(t) \leq B, \]
for all $t \geq T$. On the other hand, it is not difficult to show that the product function $f.g$ is increasing if
\[ 0 < s < \bar{x} := \frac{1}{2} (1 - b + \sqrt{b^2 + 6b + 1}). \]
Consequently, by the assertion $B < \bar{x}$ we have
\[ 1 \leq x(t) < \bar{x}, \]
for all $t \geq T$. The result is reached by applying Theorem 3.7 (H1) together with the Remark 3, the proof is completed. □

Concerning the exponential stability we have
Lemma 6.3. Assume that
\[ e^{-1} < \frac{a}{q} < \min\{(b+1)e^{-1},b\}. \] (78)
Then the positive equilibrium of the equation (75) is globally exponentially stable if the following condition holds:
\[ \left( \frac{a}{q} \right)^2 - e^{-1}(2 + be^{-1}) \frac{a}{q} + e^{-2} > 0. \]

Proof. Observe that the assertion (78) implies that \( 1 \leq x(t) \leq B \) for all \( t \geq T \), with \( B := \frac{bq e^{-1}}{\alpha - q e^{-1}} > 1 \), see the proof of Lemma 6.2. Hence the result follows by applying Theorem 4.2 with \( C_1 = 1 \) and \( C_2 = B \). To be more precise we look for the condition that satisfies,
\[ \min_{s \in [1,B]} f'(s) > \max_{s \in [1,B]} |g'(s)|. \] (79)
Indeed, we have
\[ f'(s) = \frac{ab}{(b+s)^2} \text{ and } g'(s) = q(1-s)e^{-s}, \]
so, (79) is satisfied if \( f'(B) > g'(2) \), that is
\[ \frac{ab}{(b+B)^2} > qe^{-2}, \]
by substituting the value of \( B \) in this last inequality, and by a straightforward computation we arrive at,
\[ \left( \frac{a}{q} \right)^2 - e^{-1}(2 + be^{-1}) \frac{a}{q} + e^{-2} > 0. \]
\[ \square \]

Appendix: Construction of strictly increasing functions between \( f \) and \( g \).
This section is devoted to build strictly increasing functions under the hypotheses (16), (17) and (18). More precisely, we fix \( \theta \in (0,\bar{\theta}) \), and the first aim is:
1-Construct a strictly increasing function \( g^\theta \) smaller than \( g \) over \([0,B]\) and that intersects the function \( f \) only at point \((x^* - \theta)\) and satisfying \( g^\theta(x) > f(x) \) for \( x \in [0,x^* - \theta) \) and \( g^\theta(x) < f(x) \) for \( x \in [x^* - \theta,B]\).

The second aim is as follows:
2-Construct a function \( g_\theta \) greater than \( g \) over \([0,B]\) that intersects the function \( f \) only at point \((x^* + \theta)\), in addition, this function has to satisfy \( g_\theta(x) > f(x) \) for \( x \in [0,x^* + \theta) \) and \( g_\theta(x) < f(x) \) for \( x \in (x^* + \theta,B]\).

We note that these constructions are technical and we need to set some notations and definitions. Without loss of generality we suppose that \( f(0) = g(0) = 0 \).

We first introduce the following nondecreasing function
\[ \hat{f}(x) = \max_{s \in [0,x]} f(s). \]
Since \( f'(0) > 0 \) then there exists a small positive constant \( \bar{a} \) such that \( f \) is strictly increasing over \([0,\bar{a}]\). From this and (18), observe that \( \hat{f}(x) = f(x) \) if either \( x \in [0,\bar{a}] \) or \( x \in [x^* - \theta,x^* - \bar{a}]. \)
For \( \theta \in (0,\bar{\theta}) \) we define the following constants and functions:
\[ \gamma_1 := \max_{x \in [x^* - \theta,x^* - \bar{a}]} (f(x^* - \theta) - f(x)), \]
and
\[ \gamma_2 := \min_{x \in [x^*-\theta, x^*]} (g(x) - f(x)), \]

further, for \( x \in [x^* - \theta, x^*] \) we note
\[ g^\theta(x) = (1 - c_0)f(x) + c_0f(x^* - \theta), \]

with \( c_0 = \frac{\gamma_2 - \varepsilon}{\gamma_1} \) and \( \varepsilon \) is a positive constant satisfying
\[ \max\{\gamma_2 - \gamma_1, 0\} < \varepsilon < \gamma_2. \]

In addition, for
\[ \bar{\alpha} := \frac{f(x^* - \theta) + g^\theta(x^* - \theta)}{2}, \]

we introduce,
\[ T_\alpha(g(x)) := \begin{cases} 
  g(x) & \text{for } g(x) < \bar{\alpha}, \\
  \bar{\alpha} & \text{for } g(x) \geq \bar{\alpha}.
\end{cases} \]

For \( \theta \in (0, \bar{\theta}) \) we build the function \( g_\theta \) as follows:
\[ g_\theta(x) = \begin{cases} 
  \beta x, & \text{for } 0 \leq x \leq \bar{a}, \\
  \frac{T_\alpha(g(x)) + \bar{f}(x)}{2} + \varepsilon_1 x^2, & \text{for } \bar{a} \leq x \leq x^* - \theta, \\
  g^\theta(x), & \text{for } x^* - \theta \leq x \leq x^*, \\
  \frac{g(x^*) - g^\theta(x^*)}{B - x^*}(x - B) + g(x^*), & \text{for } x^* \leq x \leq B.
\end{cases} \]  

(80)

with \((\varepsilon_1, \beta)\) verify
\[ (x^* - \theta)^2 = g^\theta(x^* - \theta) - \frac{T_\alpha(g(x^* - \theta)) + \bar{f}(x^* - \theta)}{2}, \]

and
\[ \beta \bar{a} = \frac{T_\alpha(g(\bar{a})) + \bar{f}(\bar{a})}{2} + \varepsilon_1 \bar{a}^2. \]

(82)

**Remark 4.**

1. Notice that \( \varepsilon_1 \) in (81) is well defined since \( g^\theta(x^* - \theta) > \bar{\alpha} \).
2. In view of \( f'(0) < g'(0) \), there exists a small positive constant \( \bar{a} \) such that the constant \( \beta \) defined in (82) satisfies \( f(x) \leq \beta x \leq g(x) \) for all \( x \in [0, \bar{a}] \).

Observe that the function \( g_\theta \) defined in (80) is strictly increasing, \( g_\theta(x) \leq g(x) \) for all \( x \in [0, B] \) and it satisfies
\[ \begin{cases} 
  g_\theta(x) > f(x), & \text{for } 0 < x < x^* - \theta, \\
  g_\theta(x) < f(x), & \text{for } x^* - \theta < x \leq B.
\end{cases} \]

Now we focus on the second aim. For \( \theta \in (0, \bar{\theta}) \), we define the function \( \bar{g}^\theta \) as follows:
for \( x \in [x^*, x^* + \bar{\theta}], \)
\[ \bar{g}^\theta(x) = (1 - c_1)f(x) + c_1f(x^* + \theta), \]

where \( c_1 = \frac{\mu_2 - \varepsilon}{\mu_1} \) and \( \varepsilon \) is a constant satisfying
\[ \max\{\mu_2 - \mu_1, 0\} < \varepsilon < \mu_2, \]
with
\[ \mu_1 = \min_{x \in [x^* + \theta, x^* + \theta]} (f(x) - g(x)), \]
and
\[ \mu_2 = \max_{x \in [x^* + \theta, x^* + \theta]} (f(x) - f(x + \theta)). \]

The strictly increasing function \( g^\theta \) is given by
\[
g^\theta(x) = \begin{cases} 
\tilde{g}^\theta(x^*) - g(x^*) + g(x^*), & \text{for } 0 \leq x \leq x^*, \\
\tilde{g}^\theta(x), & \text{for } x^* < x \leq x^* + \theta, \\
\frac{f(B) - \varepsilon_1 - \tilde{g}^\theta(x^* + \theta)}{B - x^* - \theta}(x - x^* - \theta) + \tilde{g}^\theta(x^* + \theta), & \text{for } x^* + \theta \leq x \leq B,
\end{cases}
\]

with \( \varepsilon_1 \) satisfies
\[ 0 < \varepsilon_1 < \min\{(f(B) - g(B)), (f(B) - \tilde{g}^\theta(x^* + \theta))\}. \]

From this, observe that \( g^\theta \) is strictly increasing with \( g \leq g^\theta \) and
\[
\begin{cases} 
g^\theta(x) > f(x), & \text{for } 0 \leq x < x^* + \theta, \\
g^\theta(x) < f(x), & \text{for } x^* + \theta < x \leq B.
\end{cases}
\]

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