Counting Water Cells in Pattern Restricted Compositions

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Abstract In this paper, we consider statistics on compositions of a positive integer represented geometrically as bargraphs that avoid certain classes of consecutive patterns. A unit square exterior to a bargraph that lies along a horizontal line between any two squares contained within its subtended area is called a water cell since it is a place where a liquid would collect if poured along the top part of the bargraph from above. The total number of water cells in the bargraph representation of a k-ary word then gives what is referred to as the capacity of w. Here, we determine the distribution of the capacity statistic on certain pattern-restricted compositions, regarded as k-ary words. Several general classes of patterns are considered, including 12, 122, 122 and 2122, where a is arbitrary. As a consequence of our results, we obtain all of the distinct distributions for the capacity statistic on avoidance classes of compositions corresponding to 3-letter patterns having at most two distinct letters. Finally, in the case of 12, some further enumerative results are given when a = 2, including algebraic and bijective proofs for the total capacity of all Carlitz partitions of a given size having a fixed number of blocks.

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1. Introduction

Bargraphs have been studied recently from several standpoints and have found various refined enumerations. Recall that a bargraph is a first quadrant self-avoiding random walk starting at the origin and ending upon its first return to the x-axis in which there are three types of steps—an up step u = (0,1), a down step d = (0,−1) and a horizontal step h = (1,0). Bargraphs are also known as wall polyominoes [1] and skylines [2], and have proven to be effective in visualizing compositions [3] and other discrete structures [4]. Prellberg and Brak [5] and Feretic [6] were among the first to study statistics on bargraphs and found a generating function with two variables x and y marking the number of horizontal and up steps, respectively. Blecher et al. later considered such statistics as levels [7], peaks [8], descents [3] and area [3]. Generating function formulas related to bargraphs have also arisen in statistical physics [9], where they are used to model various kinds of polymers.

Four points (x,y), (x+1,y), (x+1,y+1), (x,y+1) that lie along a bargraph B or within the area it subtends in the first quadrant determine what is called a cell of B. If B contains m horizontal steps, then it can be identified as a sequence of columns w = w1w2⋯wm such that the j-th column from the left contains exactly wj cells. A composition of n is a sequence of positive integers, called parts, whose sum is n. Let Cn,m denote the set of compositions of n having m parts. Then members of Cn,m may be represented as bargraphs having m horizontal steps and that subtend a first quadrant area of n, where the i-th column height (from the left) of a bargraph corresponds to the size of the i-th part of a composition.

Here, we consider counting certain classes of compositions with respect to a parameter defined on their bargraph representations. Let us call a unit square s in the first quadrant a water cell of the bargraph B if s lies outside of the area subtended by B but along a horizontal line connecting two cells of B. Thus, if one were to pour a large amount of a liquid along an impermeable boundary modeled by B from above and assume the usual rules of water flow, then a water cell is a square s that would be immersed by virtue of its location. That is, two columns of strictly greater height are present one to the left and another to the right of the column of B above which s lies, with the height of s (above the x-axis) less than or equal the minimum of these two greater heights. The total number of water cells within a bargraph B whose...
sequence of column heights is given by the \( k \)-ary word \( w = w_1w_2 \cdots w_m \) has been termed the capacity of \( B \) (also applied to the word \( w \)), see [10]. For example, if \( k = 6 \) and \( w = 63451245 \) is a \( k \)-ary word of length 8 whose bargraph representation is shown below, then the capacity of \( w \) is \( 2 + 1 + 4 + 3 + 1 = 11 \).

\[ \text{Figure 1. The capacity of } w = 63451245 \]

Let \( \lambda = \lambda_1 \lambda_2 \cdots \lambda_r \) be a member of \( C_{n,r} \), and let \( \tau = \tau_1 \cdots \tau_s \) where \( s \leq r \) be a word of length \( s \) containing all the letters in \([b] = \{1, 2, \ldots, b\}\) for some \( b \geq 1 \). An occurrence of \( \tau \) is a string \( \lambda_{i_1} \cdots \lambda_{i+s} \) of \( \lambda \) which is order isomorphic to \( \tau \), i.e., \( \lambda_{s+i} < \lambda_{s+i+1} \) if and only if \( \tau_{s+i} < \tau_{s+i+1} \) for all \( 1 \leq s_1 \leq s_2 \leq s \), and likewise with the inequalities reversed. Then \( \lambda \) is said to avoid \( \tau \) if it contains no occurrences of \( \tau \). In this context, \( \tau \) is usually referred to as a subword or consecutive pattern. For example, the composition \( \lambda = 11311223 \in C_{4,8} \) avoids the subword pattern 212, but contains three occurrences of 112, namely, 113, 112 and 223. Let \( C_{n,r}(\tau) \) denote the subset of \( C_{n,r} \) whose members avoid the subword \( \tau \).

Here, we wish to enumerate members of \( C_{n,r}(\tau) \) represented geometrically as bargraphs according to the capacity statistic for various \( \tau \). Specifically, we will compute the generating function (g.f.) for the distribution of the capacity statistic (marked by \( q \)) on \( C_{n,r}(\tau) \). When \( q = 1 \), one obtains avoidance results for compositions with respect to the various patterns (see, e.g., [11]), and hence our results may be viewed as generalizations of the avoidance case for these patterns. For a different generalization of the avoidance case wherein the number of occurrences of a given pattern is considered, see [12] and also [13] for comparable work on finite set partitions.

This paper is organized as follows. In the next section, we compute the g.f. which counts \( 1^m \)-avoiding members of \( C_{n,r} \) according to the joint distribution of the number of runs of parts and the capacity. Note that avoiding \( 1^m \) is the same as avoiding \( x \cdots x \) of length \( m \) for all possible \( x \), i.e., there are no runs of length \( m \) or more of the same part \( x \). Some further enumerative results are given in the case \( m = 2 \) wherein there are no two consecutive parts the same (which coincides with what are known as Carlitz compositions). Among our results is a bijective proof for the total capacity of all Carlitz set partitions represented sequentially having a fixed number of blocks. In the third section, we compute the g.f. counting members of \( C_{n,r}(\tau) \) according to the number of water cells in the cases when \( \tau = 21^m 2, 12^m, 1^m 2 \) or 121, where \( a \) is arbitrary. As a consequence of the preceding results and symmetry, one obtains all of the distinct distributions for the capacity statistic on the set of \( \tau \)-avoiding compositions of a given size in the case when \( \tau \) has length three and at most two distinct letters.

2. Distribution of Water Cells in Compositions Avoiding \( 1^m \)

We will refer to compositions having parts in \([k]\) as \( k \)-ary. Let \( C_{n,r,k}(\tau) \) for a subword pattern \( \tau \) denote the subset of \( C_{n,r}(\tau) \) whose members are \( k \)-ary. Note that taking \( k \geq n \) in \( C_{n,r,k}(\tau) \) gives \( C_{n,r}(\tau) \), and hence one may consider the former when computing the distribution of a statistic on the latter. A run within a composition is a string of parts of the same kind. Let \( f_k = f_k^m(x,y;p,q) \) count \( k \)-ary \( 1^m \)-avoiding compositions of \( n \) having \( r \) parts (marked by \( x \) and \( y \), respectively) according to the number of runs and the capacity (marked by \( p \) and \( q \)). That is,

\[
f_k^m(x,y;p,q) = \sum_{n \geq 20} \sum_{\lambda \in C_{n,r,k}(1^m)} \left( \sum p^{\text{run}(\lambda)} q^{\text{cap}(\lambda)} \right) x^n y^r,
\]

where \( \text{run}(\lambda) \) and \( \text{cap}(\lambda) \) denote the number of runs and the capacity of a composition \( \lambda \).

We first compute the case when \( q = 1 \) which we will need to establish the general formula. Let

\[
\ell_k = \ell_k^m(x,y;p) = f_k^m(x,y;p,1).
\]

**Lemma 1.** If \( k \geq 1 \), then

\[
\ell_k = \frac{1}{1 - p^k \sum_{i=1}^{k-1} (x^i y - (x^i y)^m)}.
\]

**Proof.** Let \( \ell_{k,i} = \ell_{k,i}^m(x,y;p) \) denote the restriction of \( \ell_k \) to those compositions starting with \( i \). Then considering the length of the initial run of \( i \) implies

\[
\ell_{k,i} = p \left( x^i y + (x^i y)^2 + \cdots + (x^i y)^{m-1} \right) (\ell_k - \ell_{k,i}),
\]

which gives

\[
\ell_{k,i} = p \frac{x^i y - (x^i y)^m}{1 - (1 - p)x^i y - p(x^i y)^m} \ell_k, 1 \leq i \leq k.
\]

Summing both sides of the last equation over \( i \), and noting \( \ell_k = \ell_1 + \sum_{i=1}^{k} \ell_{k,i} \), yields (1).
Let \( a_k = a_k^{(m)}(x, y; p, q) \) be the restriction of \( f_k \) to those compositions that start with \( k \). We have the following product formula for \( a_k \).

**Lemma 2.** If \( k \geq 1 \), then

\[
a_k = \frac{p(x^k y - (x^k y)^m)}{1 - x^k y} \prod_{j=2}^k \left( 1 - x^{j-1} y + px^{j-1} y (1 - (x^{j-1} y)^{m-1}) \right) (1 - \frac{1}{1 - p y_j^{(m)}}),
\]

where \( y_j^{(m)} = \sum_{i=1}^{j-1} \frac{x^i y^{j-i} - (x^i y^{j-i})^m}{1 - x^i y} \).

**Proof.** First suppose \( \lambda \) enumerated by \( a_k \) is of the form \( \lambda = k' \lambda' \), where \( 0 \leq i \leq m-1 \) and \( \lambda' \) is \((k-1)\) -ary (possibly empty) and not starting with \( k-1 \). To determine this case, let \( a_k^* \) denote the restriction of \( a_k \) to compositions starting with a single \( k \). Then \( \lambda = k' \lambda' \) is synonymous with a composition enumerated by \((x^{i-1} y)^{-1} a_{k-1}^* \) where there is an extra weight of \( x^i \) (which has the effect of changing the initial run of \( k-1 \) of length \( i \) to a run of \( k \)). Thus, we get a contribution towards \( a_k \) of

\[
\sum_{i=1}^{m-1} x^i (x^{k-1} y)^{i-1} a_{k-1}^* = x \frac{1 - (x^k y)^{m-1}}{1 - x^k y} a_{k-1}
\]

in this case. Note that \((1 - (x^k y)^{m-1}) a_{k}^* = (1 - x^k y) a_k \). To realize this, note that both sides of the equality give the difference of the g.f. that counts \( k\) -ary compositions avoiding \( 1 \) and starting with a single letter \( k \) with the g.f. that counts compositions of the form \( \sigma = k^m \beta \), where \( \beta \) is a composition enumerated by \( f_k \) but not \( a_k \).

If \( \lambda \) counted by \( a_k \) is of the form \( \lambda'(k-1) \lambda' \), where \( \lambda' \) is \((k-1)\) -ary, then considering all possible \( i \) gives

\[
p \frac{x^k y - (x^k y)^m}{1 - x^k y} a_{k-1}.
\]

Finally, if \( \lambda = k' \lambda' k \lambda'' \), where \( \lambda' \) is \((k-1)\) -ary and nonempty, then each part of size \( r \) in \( \lambda' \) contributes \( k-r \) towards the capacity for all \( r \) since there is a \( k \) both to the left and right. Thus, \( \lambda' \) has weight \( e_{k-1}(x/q, q, k^q ; p)-1 \), while \( k \lambda'' \) contributes \( a_k \).

Accounting for the initial run of \( k \) yields

\[
p \frac{x^k y - (x^k y)^m}{1 - x^k y} (e_{k-1}(x/q, q, k^q ; p) - 1) a_k,
\]

where \( e_{k-1}(x/q, q, k^q ; p) \) is determined by (1). Combining the three previous cases then implies for \( k \geq 2 \) the recurrence

\[
\left( 1 - p \frac{x^k y - (x^k y)^m}{1 - x^k y} (e_{k-1}(x/q, q, k^q ; p) - 1) \right) a_k
\]

\[
= \left( p \frac{x^k y - (x^k y)^m}{1 - x^k y} + x (1 - x^{k-1} y) (1 - (x^k y)^{m-1}) \right) a_{k-1},
\]

which may be rewritten as

\[
(1 - x^k y - p (x^k y - (x^k y)^m) (e_{k-1}(x/q, q, k^q ; p) - 1)) a_k
\]

\[
= x (1 - (x^k y)^{m-1}) \left( p x^{k-1} y (1 - (x^k y)^{m-1}) \right) a_{k-1}.
\]

Iterating this last equation implies

\[
a_k = \frac{1}{p} \left( \sum_{j=2}^k \left( 1 - x^j y + px^j y (1 - (x^j y)^{m-1}) \right) \left( 1 - \frac{1}{1 - p y_j^{(m)}} \right) \right)
\]

\[
= \frac{1}{p} \left( 1 - x^j y + px^j y (1 - (x^j y)^{m-1}) \right) \left( 1 - \frac{1}{1 - p y_j^{(m)}} \right).
\]

where \( y_j^{(m)} \) is as defined, which gives (2)

**Theorem 3.** If \( k \geq 1 \), then

\[
f_k^{(m)}(x, y; p, q) = 1 + p \frac{xy - (xy)^m}{1 - xy}
\]

\[
+ \sum_{j=2}^k \left( p x^j y - (x^j y)^m \right) a_{j+1} a_j,
\]

where \( a_j \) is given by (2) above.

**Proof.** Let \( b_k \) be the same as \( f_k \) except that we count the \( k\) -ary compositions \( \lambda \) according to the capacity of \( \lambda(k-1) \). Upon writing \( \lambda \) enumerated by \( f_k \) and containing at least one \( k \) as \( \lambda = \lambda' k \lambda'' \) where \( \lambda' \) is \((k-1)\) -ary, we have

\[
f_k = f_{k-1} + a_k b_{k-1}, \quad k \geq 2,
\]

which implies

\[
f_k = 1 + p \frac{xy - (xy)^m}{1 - xy} + \sum_{j=2}^k a_j b_{j-1}.
\]

We now determine \( b_k \). If the last part of \( \lambda \) counted by \( b_k \) is \( k \), then the contribution is \( a_k \), upon writing
compositions in reverse order. Otherwise, the final part of \(\lambda\) belongs to \([k-1]\) or \(\lambda\) is empty, in either case the \(k+1\) may be regarded as the initial \(k\) in a composition \(\sigma\) enumerated by \(a_k^*\), where we divide by \(x^k y^p\) since this letter actually does not contribute to the weight of \(\sigma\).

This case is then accounted for by

\[
\frac{a_k^*}{x^k y^p}
\]

and thus

\[
b_k = a_k + \frac{a_k^*}{x^k y^p} = 1 + \frac{(p-1)x^k y - p(x^k y)^m}{p(x^k y - (x^k y)^m)} a_k.
\]

Substituting the last expression into (4) implies (3).

**Remark:** Note that the generating function \(f_k^{(m)}(1,x;p,q)\) counts all \(1^m\)-avoiding \(k\)-ary words (where the length is marked by \(x\)) according to the number of runs and the capacity. A similar comment will apply to the pattern-avoiding compositions considered in the subsequent section.

Taking \(p=q=1\) in (3) recovers an earlier formula for the g.f. enumerating \(1^m\)-avoiding compositions according to the number of parts.

**Corollary 4** (Mansour and Sirhan [12]). If \(k \geq 1\), then

\[
f_k^{(m)}(x,y;1,1) = \frac{1}{1 - \sum_{i=1}^{k-1} x^i y - (x^i y)^m}.
\]

**Proof:** We first simplify \(a_k\). Taking \(p=q=1\) in (2) implies that the numerator of \(a_k^{(m)}\) is given by

\[
x^k y - (x^k y)^m = \frac{1}{1 - x y} \prod_{j=2}^{k} (1 - (x^{j-1} y)^m),
\]

while the \(j\)-th factor in the denominator may be written as

\[
1 - (x^j y)^m = \frac{x^j y - (x^j y)^m}{1 - \sum_{i=1}^{j} x^i y - (x^i y)^m}
\]

\[
(1 - (x^j y)^m) \left(1 - \sum_{i=1}^{j-1} x^i y - (x^i y)^m\right)
\]

\[
= \frac{1 - \sum_{i=1}^{j-1} x^i y - (x^i y)^m}{1 - (x^j y)^m}.
\]

Therefore, we have

\[
a_k^{(m)}(x,y;1,1) = \frac{x^k y - (x^k y)^m}{1 - x y} \prod_{j=2}^{k} \left(1 - (x^{j-1} y)^m\right) \left(1 - \sum_{i=1}^{j-1} x^i y - (x^i y)^m\right)
\]

\[
= \frac{x^k y - (x^k y)^m}{1 - x y} \prod_{j=2}^{k} \left(1 - (x^{j-1} y)^m\right) \left(1 - \sum_{i=1}^{j-1} x^i y - (x^i y)^m\right)
\]

by telescoping. Thus, by (3), we have

\[
f_k^{(m)}(x,y;1,1) = \frac{1 - (xy)^m}{1 - xy}
\]

\[
+ \sum_{j=2}^{k} \frac{1 - (xy)^m}{1 - (xy)^m} \omega^{(m)}(x,y;1,1) _{\lambda=\sigma}^{(j)}(x,y;1,1)
\]

\[
= \frac{1 - (xy)^m}{1 - xy} + \sum_{j=2}^{k} \left(1 - \sum_{i=1}^{j-1} x^i y - (x^i y)^m\right)
\]

\[
\times \left(1 - \sum_{i=1}^{j-1} x^i y - (x^i y)^m\right)
\]

\[
= \frac{1 - (xy)^m}{1 - xy} + \sum_{j=2}^{k} \left(1 - \sum_{i=1}^{j-1} x^i y - (x^i y)^m\right)
\]

\[
\times \left(1 - \sum_{i=1}^{j-1} x^i y - (x^i y)^m\right)
\]

\[
= \frac{1 - (xy)^m}{1 - xy} + \sum_{j=2}^{k} \left(1 - \sum_{i=1}^{j-1} x^i y - (x^i y)^m\right)
\]

\[
\times \left(1 - \sum_{i=1}^{j-1} x^i y - (x^i y)^m\right)
\]

\[
= \frac{1 - (xy)^m}{1 - xy} + \sum_{j=2}^{k} \left(1 - \sum_{i=1}^{j-1} x^i y - (x^i y)^m\right)
\]

\[
\times \left(1 - \sum_{i=1}^{j-1} x^i y - (x^i y)^m\right)
\]

by telescoping, as desired.

### 2.1. The Case \(m = 2\)

Carlitz compositions (see, e.g., [[11], Section 3.2.2]) are those with no two consecutive parts the same and are enumerated by the \(m = 2\) case of Theorem 3 above. Note that the number of runs (marked by \(p\)) in a Carlitz composition is always equal the number of parts (marked by \(y\)), hence one may take \(p = 1\). Let \(c_k(x,y;\lambda) = F_k^{(2)}(x,y;1,1,\lambda)\) for \(k \geq 1\). Letting \(m = 2\) in Theorem 3 and simplifying gives the following explicit formula.

**Corollary 5.** The generating function that counts Carlitz compositions with parts in \([k]\) where \(k \geq 1\) according to the capacity is given by

\[
c_k(x,y;\lambda) = 1 + xy + \sum_{j=1}^{k} \frac{\lambda_{j}^{(2)}}{x^j y} a_j^{(2)}(x,y;1,1)
\]

where
\[
d_k^{(2)} = d_k^{(2)}(x, y; 1, q) \\
= x^k \prod_{j=2}^{k} \frac{(1 + x^{-j}) \left(1 + \sum_{i=1}^{j-1} \frac{x^i y^{k-i}}{1 + x^i y^{j-i}}\right)}{1 - (1 + x^j) \sum_{i=1}^{j-1} \frac{x^i y^{k-i}}{1 + x^i y^{j-i}}}.
\] (5)

Recall that a partition of a set is a collection of non-empty, mutually disjoint subsets, called blocks, whose union is the set. A partition \( \pi \) of \([n]\) having \( k \) blocks is said to be in standard form if its blocks \( \pi_i \) are labeled so that \( \min(B_1) < \min(B_2) < \cdots < \min(B_k) \). A partition \( \pi \) in standard form can be expressed equivalently by the canonical sequential form \( w_\pi = \pi_1 \pi_2 \cdots \pi_n \) wherein \( i \in B_{\pi_i} \) for all \( i \) (see, e.g., [14]). This sequence is onto \([k]\) and satisfies the restricted growth property [15], meaning for all \( 1 \leq i \leq n - 1 \). For example, if \( n = 12 \) and \( \pi = [1, 4, 2, 7, 11, 3, 5, 9, 12, 6, 8, 10] \), then \( w_\pi = 123134253523 \). One can represent the restricted growth sequence associated with a partition \( \pi \) of \([n]\) as a bargraph whose sequence of column heights satisfy the restricted growth property. At times, the sequential form \( w_\pi \) as well as the associated bargraph will be used interchangeably with \( \pi \).

A Carlitz (set) partition \( \pi \) (see, e.g., [16]) is one in which no two consecutive elements belong to the same block of \( \pi \); i.e., there are no two adjacent letters the same in \( w_\pi \), see the example above. We will denote the set of Carlitz partitions of \([n]\) having \( k \) blocks by \( \mathcal{P}_{n,k} \). Define the capacity of \( \pi \in \mathcal{P}_{n,k} \) as the number of water cells in the bargraph representation of \( w_\pi \). For example, if \( \pi \in \mathcal{P}_{12,5} \) as given above, then the capacity is \( 2 + 2 + 2 + 1 = 7 \), as illustrated in Figure 2.

![Figure 2](image.png)

**Figure 2.** The capacity of the Carlitz partition \( \pi = [1, 4, 2, 7, 11, 3, 5, 9, 12, 6, 8, 10] \).

Given \( k \geq 1 \), let \( d_k(x; q) \) denote the g.f. enumerating members of \( \mathcal{P}_{n,k} \) according to the capacity.

**Theorem 6.** The generating function that counts Carlitz partitions of \([n]\) having \( k \) blocks where \( k \geq 1 \) according to the capacity is given by

\[
d_k(x; q) = \frac{x^k}{(1+x)^{k-1}} \times \prod_{j=1}^{k-1} \left[1 - \sum_{i=1}^{j} x^i q^{j-i} \left(1 - \frac{x / (1+x)}{1 - \sum_{i=1}^{j} x^i q^{j-i+1}}\right)\right]^{-1}.
\] (6)

**Proof.** The canonical sequential form of \( \pi \in \mathcal{P}_{n,k} \) may be represented as \( \pi = \lambda \pi_1^{(1)} \pi_2^{(2)} \cdots k \pi(k) \), where \( \pi(i) \) is \( i \)-ary such that \( \pi(i) \) has no equal adjacent letters for each \( i \). In the notation from the previous section, this implies

\[
d_k(x; q) = d_k^{(2)}(1, x; 1, q) \prod_{j=1}^{k-1} (1 / q, x^{j-1}, 1), k \geq 1.
\] (7)

By (2) and (1) when \( m = 2 \), we have

\[
d_k^{(2)}(1, x; 1, q) = \frac{x^k}{1-x^k} \prod_{j=2}^{k} \left[1 - \frac{x(1-x)}{1 - \sum_{i=1}^{j-1} x^i q^{j-i}}\right] \times
\]

and

\[
\ell_j^{(2)}(1/q, x^{j-1}; 1) = \frac{x}{1+x} \ell_j^{(2)}(1/q, x^{j-1}; 1) = \frac{x / (1+x)}{1 + \sum_{i=1}^{j-1} x^i q^{j-i}}.
\]

from which formula (6) follows from (7).

Substituting \( q = 1 \) into (6) gives

\[
d_k(x; 1) = \frac{x^k}{(1+x)^{k-1}} \prod_{j=1}^{k-1} \left[1 - \frac{x / (1+x)}{1 - \sum_{i=1}^{j} x^i q^{j-i+1}}\right]^{-1} \times \prod_{j=1}^{k-1} (1 - jx) = \sum_{n \geq k} S(n-1, k-1) x^n,
\]

where \( S(n,k) \) denotes the Stirling number of the second kind. This reaffirms the well-known fact (see, for example, [16]) that there are \( S(n-1, k-1) \) members of \( \mathcal{P}_{n,k} \).

We now determine an expression for the sum of the capacities of all members of \( \mathcal{P}_{n,k} \). To do so, we compute the derivative with respect to \( q \) of \( d_k(x; q) \) at \( q = 1 \). We
handle the two factors occurring in the product expression in (6) separately. First observe that
\[ \frac{\partial}{\partial q} \ell_{j,j}^{(2)}(1/q, xq^j;1)|_{q=1} \]
\[ = \frac{x}{1 + x} \sum_{i=1}^{j} \left( x - \frac{x^2}{1 + x} \right)^2 (j-i) \]
\[ = \frac{x(1+x)}{(1-(j-1)x)^2} \frac{\left( \frac{j}{2} \right)^x}{(1+x)^2} \]
\[ = \frac{\left( \frac{j}{2} \right)^x}{(1+x)(1-(j-1)x)^2}. \]

and thus
\[ \frac{\partial}{\partial q} \ell_{j,j}^{(2)}(1/q,xq^j;1)|_{q=1} = \frac{\left( \frac{j}{2} \right)^x}{1-(j-1)x^2} \]
\[ = \frac{x(1+x)}{1-(j-1)x^2} \frac{1+x}{1+(1-x) \frac{\left( \frac{j}{2} \right)^x}{(1+x)^2}} \]
\[ = \frac{x}{1-(j-1)x} \frac{\left( \frac{j}{2} \right)^x}{(1+x)(1-(j-1)x)^2}, 1 \leq j \leq k-1. \]

Also, we have
\[ \frac{\partial}{\partial q} d_k^{(2)}(1,x;1,q)|_{q=1} \]
\[ = \frac{x}{1-(k-1)x} \sum_{j=2}^{k} \left[ \frac{1-(j-1)x}{1-(j-2)x} \right] \left( 1 + x \frac{1+x}{1-(j-2)x} \right) \]
\[ \times \sum_{i=1}^{k-1} \left( x - \frac{x^2}{1 + x} \right)^2 (j-i) \]
\[ = \frac{x}{1-(k-1)x} \sum_{j=2}^{k} \left( 1-(j-1)x \right) \frac{x(1+x)}{1-(j-1)x^2} \frac{\left( \frac{j}{2} \right)^x}{(1+x)^2} \]
\[ = \frac{x^2}{(1+x)(1-(k-1)x)} \left( 1-(j-2)x \right) \frac{\left( \frac{j}{2} \right)^x}{(1-(j-1)x)^2}. \]

with \( d_k^{(2)}(1,x;1,q)|_{q=1} = \frac{x}{1-(k-1)x} \), so that
\[ \frac{\partial}{\partial q} d_k^{(2)}(1,x;1,q)|_{q=1} \]
\[ = \frac{x}{1+(1-x) \frac{\left( \frac{j}{2} \right)^x}{(1+x)^2}} \sum_{j=2}^{k} \left( \frac{j}{2} \right)^x \]
\[ = \frac{x}{1-(k-1)x} \sum_{j=2}^{k} \left( 1-(j-1)x \right) \frac{x(1+x)}{1-(j-1)x^2} \frac{\left( \frac{j}{2} \right)^x}{(1+x)^2} \]
\[ = \frac{x}{1-(k-1)x} \frac{x}{1-(k-1)x} \frac{\left( \frac{j}{2} \right)^x}{(1+x)(1-(j-1)x)^2}. \]

Therefore, by (7), we get
\[ \frac{\partial}{\partial q} d_k(x; q)|_{q=1} = \frac{\left( \frac{k}{2} \right)^x}{1-(k-1)x} + \sum_{j=1}^{k-1} \frac{\left( \frac{j}{2} \right)^x}{1-(j-1)x}. \]

which yields the following result.

**Theorem 7.** The generating function for the number of water cells in all Carlitz partitions of \([n]\) having \(k\) blocks is given by
\[ \frac{\partial}{\partial q} d_k(x; q)|_{q=1} \]
\[ = \frac{x^k}{1+(1-x) \prod_{i=1}^{k} \left( 1-ix \right)} \left[ \frac{x}{1-(k-1)x} + \sum_{j=1}^{k-1} \frac{\left( \frac{j}{2} \right)^x}{1-(j-1)x} \right]. \]

Let \( F_n(a,b) \) denote the generalized Fibonacci sequence (see, e.g., [17], Section 3.1) defined by the recurrence \( F_n(a,b) = a F_{n-1}(a,b) + b F_{n-2}(a,b) \) for \( n \geq 2 \), with \( F_1(a,b) = 1 \) and \( F_0(a,b) = 0 \). Note that
\[ \sum_{n=0}^{\infty} F_n(a,b)x^n = \frac{x}{1-ax-bx^2}. \]

Then extracting the coefficient of \( x^n \) in (8) gives the following explicit formula for the total capacity.

**Corollary 8.** The number of water cells in all Carlitz partitions of \([n]\) having \(k\) blocks where \(n \geq k \geq 2\) is given by
\[ \frac{x}{1-ax-bx^2}. \]
\[
\begin{aligned}
\binom{k}{2} \sum_{r=1}^{n-k} S(n-r-1,k-1)F_r(k-2,k-1) \\
+ \sum_{j=1}^{k-1} \sum_{r=1}^{n-k} (-1)^{j-r} \binom{j}{2} S(n-r-1,k-1)F_r(j-2,j-1).
\end{aligned}
\]

**Combinatorial proof of Corollary 8.**

We first describe a class of words enumerated by \(F_n(j-2,j-1)\) for \(n \geq 1\) and \(j \geq 2\) that will be needed in our combinatorial interpretation. Given a \(j\)-ary word \(w = w_1 \cdots w_n\), a (circular) *succession* is an occurrence of two consecutive letters \(i, i+1\) (where \(i+1\) corresponds to 1 if \(i = j\)). Consider underlining non-overlapping successions in \(w\) starting from the left in a greedy manner. That is, we underline the first succession encountered starting from the left having no letters in common with previously underlined successions. For example, for the \(j\)-ary word \(w\) where \(n = 11\) and \(j = 6\), we have \(w = 13453612451\).

**Definition 9.** Let \(W_{n,j}\) denote the set of all \(j\)-ary words \(w = w_1 \cdots w_n\) where \(w_1 = 1\) such that (I) no two adjacent letters of \(\omega\) are the same, and (II) when non-overlapping successions are underlined in a greedy manner starting from the left, the last letter of \(w\) is not part of an underlined succession.

Note that by symmetry the cardinality of \(W_{n,j}\) would not change if the \(w_1 = 1\) requirement is replaced by \(w_1 = \ell\) for any fixed \(\ell \in [j]\). We will refer to \(j\)-ary words satisfying properties (I) and (II) above as being *restricted*. For example, if \(n = 7\) and \(j = 5\), then the \(j\)-ary word \(w = 3523123\) is restricted, whereas \(w = 1452354\) and \(w = 4212351\) are not since the last two letters, 34 and 51, respectively, form successions that would be underlined in these words. In what follows, a set of restricted words will often be assumed to start with a specific letter. Let \(w_{n,j} = \mid W_{n,j}\mid\).

**Lemma 10.** If \(n \geq 1\) and \(j \geq 2\), then \(w_{n,j} = F_n(j-2,j-1)\).

**Proof.** First note that there is \(F_1(j-1,j-2) = 1\) word of length one and \(F_2(j-2,j-1) = j-2\) words of length two since the second letter cannot be 1 or 2. If \(n \geq 3\), then there are \((j-2)w_{n-1,j}\) members of \(W_{n,j}\) in which there are two or more letters to the right of the rightmost underlined succession, as seen upon appending a letter from \([j]-\{i,i+1\}\) to a member of \(W_{n-1,j}\) ending in \(i\) for some \(i\). There are \((j-1)w_{n-2,j}\) possibilities if the penultimate letter and its predecessor form an underlined succession, upon appending \(i+1, r\) where \(r \neq i+1\) to a member of \(W_{n-2,j}\) ending in \(i\). Combining this case with the previous implies \(w_{n,j} = F_n(j-2,j-1)\) for all \(n \geq 1\).

Let \(P_{n,k}\) denote the set of “marked” Carlitz partitions obtained from members of \(P_{n,k}\) by marking any one of the water cells. Note that \(|P_{n,k}^*|\), which we compute below, equals the total number of water cells in all the members of \(P_{n,k}\). Given \(\rho = \lambda \rho(1) (\rho(2) \cdots k \rho(k) \in P_{n,k}\) represented sequentially, we will refer to the subsequence \(j \rho(j)\) as the \(j\)-ary section of \(\rho\) (and also when discussing the corresponding portion of the bargraph representation of \(\rho\), which we denote by \(\bar{\rho}\)). We first count members of \(P_{n,k}^*\) where the marked water cell lies above a column in the \(j\)-ary section for some \(2 \leq j \leq k-1\). To do so, given \(2 \leq j \leq k-1\) and \(1 \leq r \leq n-k\), consider the set \(S_{j,r}\) of triples \((\alpha, \rho, \omega)\), where \(\alpha = \{u < v\}\) is a two-element subset of \([j]\), \(\rho \in P_{n-r,k}\) and \(\omega\) is a restricted \(j\)-ary word of length \(r\) with first letter \(u\).

Note that \(|S_{j,r}| = \binom{\frac{n-r-1}{2}}{j} S(n-r-1,k-1)F_r(j-2,j-1)\), by the previous lemma.

Given \(s = (\alpha, \rho, \omega) \in S_{j,r}\), we form \(\pi \in P_{n,k}^*\) as follows. If \(\rho = \lambda \rho(1) \cdots k \rho(k)\) and the last letter of \(j \rho(j)\) does not equal \(u\), then let \(\pi = \lambda \rho(1) \cdots (j-1) \rho(j) \omega(j+1) \rho(j+j) \cdots k \rho(k)\), that is, we append \(\omega\) just prior to the first occurrence of \(j+1\), and then mark the water cell at height \(v\) directly above the column of \(\bar{\rho}\) corresponding to the initial letter \(u\) of \(\omega\). This operation is seen to yield (uniquely) all members \(\pi\) of \(P_{n,k}^*\) in which the marked cell has height \(v\) above a column of height \(u\) in the \(j\)-ary section of \(\bar{\rho}\) such that the subword of \(\pi\) starting with the letter corresponding to the column of the marked cell of \(\bar{\rho}\) and ending with the final letter of the \(j\)-ary section of \(\pi\) is restricted and of length \(r\). On the other hand, if the last letter of \(j \rho(j)\) equals \(u\), then we delete this letter from \(j \rho(j)\) and add it back to the end of \(\omega\) as \(\ell+1\) where \(\ell\) denotes the final letter of \(\omega\) (with \(\ell+1\) representing 1 if \(\ell = j\)). Note that \(u < j\) implies \(\rho(j)\) is nonempty in this case. Let \(\tilde{\rho}(j)\) denote \(\rho(j)\) minus its last letter and let \(\tilde{\omega} = \omega(\ell+1)\). We then form the partition \(\pi = \lambda \rho(1) \cdots (j-1) \rho(j) \tilde{\rho}(j) \tilde{\omega}(j+1) \rho(j+j) \cdots k \rho(k)\), which is seen to satisfy all the same properties as before except that the subword starting with the letter corresponding to the marked cell and ending just prior to the first occurrence of \(j+1\) is no longer restricted and has length \(r+1\). The correspondence \(s \rightarrow \pi\) then defines a bijection between \(S_{j,r}\) and all members of \(P_{n,k}^*\) in which the marked water cell occurs in a column belonging to the \(j\)-ary section where there are exactly \(r-1\) positions between it and the first column of height \(j+1\). Considering all possible \(j\) and \(r\) implies that there are...
\[ \sum_{j=2}^{k-1} \sum_{r=1}^{n-k} \binom{k-j}{2} S(n-r-1,k-1)F_r(j-2,j-1) \]

members of \( T_{n,k}^* \) in which the marked cell lies above a column within a \( j \)-ary section for some \( 2 \leq j \leq k-1 \).

Similar reasoning applies to members \( \pi \in T_{n,k}^* \) in which the marked cell lies above a column in the \( k \)-ary section, which gives \( \binom{k}{2} \sum_{r=1}^{n-k} S(n-r-1,k-1)F_r(k-2,k-1) \). However, this would be assuming that there is a terminal letter \( k \) that follows \( \pi \). Thus, we must subtract the number of those cells which do not yield de facto members of \( T_{n,k}^* \) when marked. By a 'non-water cell', we will mean one of height at most \( k \) that lies above some column in the \( k \)-ary section of \( \pi \in T_{n,k} \) which is not itself a water cell. Let \( \hat{P}_{n,k} \) be obtained from \( T_{n,k} \) by marking one of the non-water cells. Then we must subtract \( |\hat{P}_{n,k}| \) from the current total to obtain the desired \( |T_{n,k}^*| \). We now show

\[ |\hat{P}_{n,k}| = \sum_{j=1}^{k-1} \sum_{r=1}^{n-k} jS(n-r-1,k-1)F_r(j-2,j-1), \]

from which Corollary 8 follows. Suppose that the marked non-water cell in \( \pi \in T_{n,k}^* \) has height \( j+1 \). First assume \( 2 \leq j \leq k-1 \). Given \( 1 \leq r \leq n-k \), let \( T_{j,r} \) denote the set of triples \((i, \rho, \omega)\), where \( t \in [j], \rho \in T_{n-r,k} \) and \( w \) is a restricted \( j \)-ary word of length \( r \) having first letter \( t \).

Then the bijection defined above on \( S_{j,r} \) applies here and shows that \( |T_{j,r}| \) gives the cardinality of all members of \( \hat{P}_{n,k} \) in which the marked cell has height \( j+1 \) with exactly \( r-1 \) columns to the right of the one containing this cell. Therefore, summing over \( 2 \leq j \leq k-1 \) and \( 1 \leq r \leq n-k \) gives all possible members of \( \hat{P}_{n,k} \) in which the marked cell has height at least three.

We now consider the case \( j=1 \) and count members of \( \hat{P}_{n,k} \) where the marked cell has height two. In order to have height two, the column in which this cell lies must correspond to the final letter of a partition, for otherwise it would actually be a water cell. Thus, counting the remaining members of \( \hat{P}_{n,k} \) is equivalent to counting members of \( T_{n,k} \) ending in 1. Since \( F_r(-1,0)=(-1)^{r-1} \) for \( r \geq 1 \), the remaining case when \( j=1 \) in the expression to be shown above for \( |\hat{P}_{n,k}| \) is given by

\[ \sum_{r=1}^{n-k} (-1)^{r-1}S(n-r-1,k-1), \]

and thus it suffices to show that this is the cardinality of the subset of \( T_{n,k} \) whose members end in 1. To do so, we first let \( A_{n,k}^{(r)} \) for

1 \( \leq r \leq n-k \) denote the set of words of length \( n \) obtained by appending exactly \( r \) 1’s to members \( \sigma \in T_{n-r,k} \) (represented sequentially) and circling the final letter of \( \sigma \) if it ends in a 1 and let \( A_{n,k}^{(r)} = \bigcup_{r=1}^{n-k} A_{n,k}^{(r)} \). Define the sign of members of \( A_{n,k}^{(r)} \) by \((-1)^{r-1}\). Note that the sum of the signs of members of \( A_{n,k} \) is given by

\[ \sum_{r=1}^{n-k} (-1)^{r-1}S(n-r-1,k-1). \]

Define an involution on \( A_{n,k} \) by either circling the first 1 in the terminal sequence of 1's if it is not circled or erasing the circle enclosing the first 1 if it is. Note that this pairs members of \( A_{n,k} \) of opposite sign and is not defined in the case when \( r=1 \) and \( w \in A_{n,k}^{(1)} \) ends in a single (uncircled) 1. Such words \( w \) are seen to be equivalent to members of \( T_{n,k} \) ending in 1, which establishes the desired formula for this subset of \( T_{n,k} \) and completes the proof.

**Remark:** It is also possible to find an expression for the exponential generating function for the total capacity of all members of \( T_{n,k} \). Let \( d_k(x) = \frac{\partial}{\partial q} d_k(x,q) \big|_{q=1} \).

(Tex translation failed) and \( e'(x,y) = \sum e(x) y^k \). Then transferring from the ordinary to the exponential generating function gives the following formula:

\[
\begin{bmatrix}
(2s-3)y^2e^{ys} - y + 2s \\
+4ye^{ys} - y + s \\
-4y^2e^{ys} - y + 4 \\
-4y^2e^{2ys} - ye^{ys} - y + 2r + 21drdt \\
4y^2e^{2ys} - ye^{ys} - y + 1 + 2r + 21drdt \\
+4s(1-e^{ys} - y) dt
\end{bmatrix}
\]

\[
e'(x,y) = e^{-x} \int_0^x \frac{ye^t}{4} ds.
\]

3. **Capacity of Other Pattern-restricted Compositions**

In this section, we enumerate various avoidance classes of compositions according to the number of water cells in their bargraph representations. As corollaries to our results, we obtain the distribution for this statistic on the 112, 121, 122 and 212 avoidance classes. Note that this covers all patterns of length three having two distinct letters since the capacity statistic is preserved by the reversal operation. On the other hand, while the patterns 112 and 212 (and also 121 and 212) are equivalent when enumerating compositions avoiding either pattern according to the number of parts, this is not the case when the capacity is considered since it is unclear how this statistic behaves with respect to the reverse-complement operation.
We will use the following notation. Let $\tau$ be a subword pattern of length $m \geq 3$ (where $\tau$ here will actually denote a particular member of an infinite class of patterns containing one representative for each $m \geq 3$). Given $k \geq 1$, let $f_k^{(m)}(x, y; q)$ denote the g.f. counting $\tau$-avoiding $k$-ary compositions $\lambda$ of $n$ having $r$ parts (marked by $x$ and $y$, respectively) according to the capacity $s$ (marked by $q$), i.e.,

$$f_k^{(m)}(x, y; q) = \sum_{n \geq r \geq 0} \sum_{\lambda \in \mathcal{C}_{s,r,k}(\tau)} q^{\text{cap}(\lambda)} x^n y^r.$$

Let $g_k^{(m)}(x, y; q)$ be the same as $f$ except now we are counting $k$-ary compositions $\lambda$ according to the capacity of $\lambda k$, where it is assumed that $\lambda k$ avoids $\tau$. Similarly, define $h_k^{(m)}(x, y; q)$ as the g.f. counting $k$-ary $\lambda$ according to the capacity of $(k+1)\lambda$, where it is assumed that $(k+1)\lambda$ avoids $\tau$.

If $\lambda$ is $k$-ary and contains at least one letter $k$, then one may write $\lambda = k'r\lambda''$, where $\lambda'$ is $k$-ary and $\lambda''$ is $(k-1)$-ary. Suppose $\tau$ is such that its largest letter occurs in the final position (and possibly elsewhere as well). Then the preceding decomposition of $\lambda$ may be expressed in terms of generating functions as

$$f_k = f_{k-1} + x^k y g_{k-1} h_{k-1}, \quad k \geq 2. \quad (9)$$

In the following subsections, we use $f_k$, $g_k$ and $h_k$ as they are defined above for the particular pattern $\tau$ in question. Also, we will frequently regard a composition as a $k$-ary word, referring to its parts as letters by a slight abuse of notation.

### 3.1. The Case of 21⋯12-avoiding Compositions

Let $\tau = 21\cdots12$ be of length $m \geq 3$. Let

$$a_k^{(m)} = x^{k+m-1} y^{m-1} q^{m-2} \left( \frac{x^{m-2} - q^{m-2}}{x^{m-2} - q^{m-2}} \right)$$

for $m \geq 3$ and $k \geq 0$. In this case for $\tau$, we have the following recursive formulas for $f_k$ and $g_k$.

**Lemma 11.** If $k \geq 2$, then

$$f_k = f_{k-1} + x^k y (1 + a_k^{(m)}) g_{k-1} h_{k-1}$$

and

$$(1 + a_k^{(m)}) g_k = (1 + a_k^{(m)}) g_{k-1} + x^k y f^{(m)}_{k-1}(x/q, y q^k; 1) g_k. \quad (10)$$

**Proof.** To show (10), we must account for the $(k-1)$-ary $\lambda''$ factor within $\lambda$ in the decomposition $k'k\lambda''$ stated above and enumerated by $h_{k-1}$. Note first that $\lambda''$ may assume the form $\rho^*$, where $\rho$ is any composition enumerated by $g_{k-1}$ and $\tau$ denotes the reversal operation, with the capacity of $\rho k$ and $k\rho^*$ equal. Note further that it is also possible for $\lambda''$ to be of the form $\lambda'' = j^{m-2}(k-1)\beta$, where $j \in \{k-2\}$ and $\beta$ is $(k-1)$-ary (the reversal in this case would not be accounted for by $g_{k-1}$ since the last $m$ letters of $(\lambda'')^*(k-1)$ would form an occurrence of $\tau$). Considering all $i$, such $\lambda''$ are seen to have weight

$$x^{k-1} y^{m-1} ((x q^{k-2})^m - 2 + (x^2 q^{k-3})^m - 2 + \cdots + (x^{k-2} q^{m-2})^m - 2)$$

Combining the previous cases implies that $\lambda''$ contributes $(1 + a_{k-1}^{(m)}) g_{k-1}$, from which formula (10) follows from (9).

For (11), first note that the prior reasoning shows also that the contribution of $\lambda$ towards $g_k$, in the case when $\lambda$ is $(k-1)$-ary, is given by $(1 + a_k^{(m)}) g_{k-1}$. If $\lambda$ ends in $k$, then this letter is extraneous concerning the avoidance of $\tau$ (and thus may be deleted) and we get a contribution of $x^k y g_k$. So assume $\lambda = k'k\lambda''$, where $\lambda'$ is $k$-ary and possibly empty and $\lambda''$ is $(k-1)$-ary and nonempty. In this case, any letter of size $r$ in $\lambda''$ produces $k-r$ water cells since there is a letter $k$ both to its left and to its right in $\lambda k$. The distribution for the number of water cells in $\lambda''$ may then be obtained by the transformation $(x, y, q) \rightarrow (x/q, y q^k, 1)$. Thus, the $\lambda''$ factor contributes

$$f_{k-1}^{(m)}(x/q, y q^k; 1) - 1 - y^{m-2} \frac{((xy q^{k-1})^m - 2)}{x^2}$$

where the second subtracted term accounts for strings of the same letter of length exactly $m-2$ (which would produce an occurrence of $\tau$ in $\lambda k$). Therefore, compositions $k'k\lambda''$ of the given form contribute

$$x^k y g_k \left( f_{k-1}^{(m)}(x/q, y q^k; 1) - 1 - \frac{a_{k-1}^{(m)}}{x^2} \right) \text{ towards } g_k.$$

Combining all of the previous cases, and rewriting the resulting equation, yields (11).

Iterating (11) gives

$$g_k = \prod_{j=0}^{k-2} \left( 1 + a_{j-1}^{(m)} - x^j y f_{j-1}^{(m)}(x/q, y q^j; 1) \right), \quad k \geq 1.$$
By [[12], Theorem 2.4], we have
\[ f_k^{(m)}(x, y; 1) = \frac{1}{1 - \sum_{j=1}^{k} x^j y^{-1}} \], where \( m \geq 3 \),

which implies
\[ g_k = \prod_{j=2}^{k} \left( 1 + \alpha^{(m)}_{j-2} \right) \]
and
\[ \alpha^{(m)}_{j-1} = \frac{x^j y^{-1} - 1}{1 - \sum_{i=1}^{\frac{m(m-1)}{2}} x^i y^{-i}} \alpha^{(m)}_{j-1} \]
and \( \alpha^{(m)}_0 = x^m - q^{-1} \).

Thus, we can state the following result.

**Theorem 12.** If \( k \geq 2 \), then the generating function \( f_k^{(m)}(x, y; q) \) is given by
\[ f_k^{(m)}(x, y; q) = \frac{1}{1 - x y} \sum_{j=2}^{k} x^j y \left( 1 + \alpha^{(m)}_{j-2} \right) g_{j-1} \]
where
\[ g_k = \prod_{j=2}^{k} \left( 1 + \alpha^{(m)}_{j-2} \right) \]

\[ \alpha^{(m)}_{j-1} = \frac{x^j y^{-1} - 1}{1 - \sum_{i=1}^{\frac{m(m-1)}{2}} x^i y^{-i}} \alpha^{(m)}_{j-1} \]
and \( \alpha^{(m)}_0 = x^m - q^{-1} \).

Taking \( m = 3 \) in the prior theorem implies that

\[ f_k^{(3)}(x, y; q) = \frac{1}{1 - x y} \sum_{j=2}^{k} x^j y \left( 1 + \alpha^{(3)}_{j-2} \right) g_{j-1} \]
where
\[ g_k = \prod_{j=2}^{k} \left( 1 + \alpha^{(3)}_{j-2} \right) \]

\[ \alpha^{(3)}_{j-1} = \frac{x^j y^{-1} - 1}{1 - \sum_{i=1}^{3} x^i y^{-i}} \alpha^{(3)}_{j-1} \]
and \( \alpha^{(3)}_0 = x^3 - q^{-1} \).

3.2. The Case 12 · · · 2.

Let \( \tau = 12 \cdots 2 \) be of length \( m \geq 3 \). Then in this case the following recurrences are satisfied by \( g_k \) and \( h_k \).

**Lemma 13.** If \( k \geq 2 \), then
\[ g_k = \frac{(x^{k-1} y - (x^k y - 1))}{(1 - x y)(1 - (x^{k-1} y)^{m-1})} g_k \]
\[ + \frac{1 - (x^k y - 1)}{(1 - x y)(1 - (x^{k-1} y)^{m-1})} g_{k-1} \]
\[ + \frac{1}{1 - x y} \left( f_{k-1}^{(m)}(x, y; q, y^k; 1) - 1 \right) g_k \]
and
\[ \frac{(1 - (x^k y - 1))}{(1 - x y)(1 - (x^{k-1} y)^{m-1})} f_{k-1}^{(m)}(x, y; q, y^k; 1) h_k = h_{k-1} \]

**Proof.** To show (12), first note that if \( \lambda \) enumerated by \( g_k \) consists of (a possibly empty) sequence of the letter \( k \), then the contribution is \( \frac{1}{1 - x^k y} \). So assume \( \lambda \) has the form \( \lambda = \beta k^i \), where \( 0 \leq i \leq m - 3 \) and \( \beta \) is nonempty with final letter in \([k - 1]\). First suppose \( \beta \) contains at least one letter \( k \). Let \( \beta = \beta' k^j \beta'' \), where \( \beta' \) is \( k \)-ary and \( \beta'' \) is \((k - 1)\)-ary. Then \( \beta'k \) and \( \beta'' \) are accounted for by \( x^k g_k \) and \( f_{k-1}^{(m)}(x, y; q, y^k; 1) \), respectively, since any letter \( r \) in \( \beta'' \) is seen to have \( k-r \) water cells above it. Considering all possible \( i \) implies \( \lambda \) contributes
\[ \frac{x^k y - (x^k y)^{m-1}}{1 - x^k y} \left( f_{k-1}^{(m)}(x, y; q, y^k; 1) - 1 \right) g_k \]
in this case.

Now suppose \( \beta \) is \((k - 1)\)-ary. Let \( j_k = j_k^{(m)}(x, y; q) \) count \( \tau \)-avoiding \( k \)-ary compositions \( \rho \) according to the number of water cells in \( \rho(k + 1) \). Then \( \lambda \) in this case contributes
\[ \frac{1}{1 - x^k y} \left( j_{k-1}^{(m)}(x, y; q, y^k; 1) - 1 \right) g_k \]
towards \( g_k \). We now determine \( j_k \). Note that in addition to counting all compositions \( \rho \) enumerated by \( g_k \), the g.f. \( j_k \) also enumerates \( \rho \) of the form \( \rho = \rho' k^m \), where \( \rho' \) does not end in \( k \) and \( \rho' \) is nonempty. Let \( g_k^* = g_k^*(x, y; q) \) be the g.f. that counts nonempty \( \tau \)-avoiding \( k \)-ary compositions \( \sigma \) whose final letter belongs to \([k - 1]\) according to the capacity of \( \sigma k \). Then \( \lambda \) enumerated by \( g_k \) and not consisting of a string of \( k \)'s may be obtained by appending up to \( m - 3 \) \( k \)'s to the end of \( \sigma \) having the stated form, which implies
\[ \left( 1 + x^k y + \cdots + (x^k y)^{m-3} \right) g_k^* = \frac{1}{1 - x^k y}. \]
i.e.,
\[ g_k = \frac{(1-x^k y)g_k - 1}{1-(x^k y)^{m-2}}, \quad k \geq 1. \]

Thus, we have
\[ j_k = g_k + (x^k y)^{m-2} g_k^* \]
\[ = \frac{(1-(x^k y)^{m-1})g_k - (x^k y)^{m-2}}{1-(x^k y)^{m-2}}, \quad k \geq 1. \]

Combining all of the previous cases then implies
\[ g_k = \frac{1}{1-x^k y} + \frac{1-(x^k y)^{m-2}}{1-x^k y} (j_{k-1} - 1) \]
\[ + \frac{x^k y-(x^k y)^{m-1}}{1-x^k y} \left( f_{k-1}^{(m)} (x/q, yq^k ; 1) - 1 \right) g_k, \quad k \geq 2, \]

where \( j_k \) is as given. The last equality may be rewritten as (12).

To show (13), first note that \( h_{k-1} \) accounts for \((k-1)\)-ary compositions \( \lambda \) and \( x^k y h_k \) for \( \lambda \) starting with \( k \). So assume \( \lambda = \lambda' k' \lambda'' \), where \( \lambda' \) is \((k-1)\)-ary and nonempty, \( \lambda'' \) is \( k \)-ary and does not start with \( k \) (possibly empty), and \( r \geq 1 \). Note that \( \lambda' \) nonempty implies \( r \leq m-2 \) and that \( \lambda'' \) contributes \((1-x^k y)h_k \), by subtraction. Thus \( \lambda \) in this case is accounted for by
\[ \sum_{r=1}^{m-2} (x^k y)^r \left( f_{k-1}^{(m)} (x/q, yq^k ; 1) - 1 \right) (1-x^k y)h_k \]
\[ = (x^k y-(x^k y)^{m-1}) \left( f_{k-1}^{(m)} (x/q, yq^k ; 1) - 1 \right) h_k. \]

Combining the previous cases leads to (13).

By [12], Theorem 2.3, we have
\[ f_k^{(m)} (x, y ; 1) = \frac{1}{1-x^k y} \sum_{j=1}^k \frac{x^j y}{k \prod_{i=j+1}^{m-1} (1-x^i y)^{m-1}}, \quad m \geq 3. \]

Define
\[ \alpha_{j,m} = (1-(x^j y)^{m-2}) \]
\[ \times \frac{x^j y-(x^j y)^{m-1} f_{j-1}^{(m)} (x/q, yq^j ; 1)}{-(x^j y-(x^j y)^{m-1}) f_{j-1}^{(m)} (x/q, yq^j ; 1)}. \]

From (12), we get
\[ g_k = \frac{(x^k y)^{m-2} (x^{m-2} - 1)}{\alpha_{k,m}} \]
\[ + \frac{(1-(x^k y)^{m-2})(1-(x^j y)^{m-1})}{\alpha_{k,m}} g_{k-1}. \]

Thus, by induction on \( k \) with \( g_1 = 1/(1-xy) \), we have
\[ g_k = \prod_{i=2}^{k} \frac{(1-(x^i y)^{m-2})(1-(x^j y)^{m-1})}{\alpha_{i,m}} \]
\[ \times \frac{1}{1-xy} \sum_{j=2}^{k} \frac{(x^j y)^{m-2}(x^{m-2} - 1) \prod_{i=2}^{j-1} \alpha_{i,m}}{(1-(x^j y)^{m-2})(1-(x^j y)^{m-1})}. \]

Moreover, (13) shows that \( h_k = \frac{1-(x^j y)^{m-2}}{\alpha_{k,m}} h_{k-1} \),

which implies
\[ h_k = \prod_{j=1}^{k} \frac{1-(x^j y)^{m-2}}{\alpha_{j,m}}. \]

Hence, by (9), we can state the following result.

**Theorem 14.** If \( k \geq 2 \), then the generating function \( f_k = f_k^{(m)} (x, y ; q) \) is given by
\[ f_k = \frac{1}{1-xy} + \sum_{i=2}^{k} \frac{x^i y}{1-xy} \prod_{j=i+1}^{m-2} \frac{\xi_{j-1,m-1} \xi_{j-2,m-2}}{\alpha_{j-1,m} \alpha_{j,m}} \]
\[ \times \prod_{j=2}^{i} \frac{(x^j y)^{m-2}(x^{m-2} - 1) \prod_{i=2}^{j-1} \alpha_{i,m}}{(1-(x^j y)^{m-2})(1-(x^j y)^{m-1})}. \]

where \( \xi_{i,a} = \xi_{i,a} (x, y) = 1-(x^i y)^a \).

### 3.3. The Case 1 \cdots 12

Let \( \tau = 1 \cdots 12 \) be of length \( m \geq 3 \). We refine \( f_k \) in this case as follows. Let \( f_{k,i}^{(m)} (x, y ; q) \) denote the restriction of \( f_k \) to those compositions whose last part is \( i \). We have the following formula for \( f_{k,i}^{(m)} (x, y ; 1) \).

**Lemma 15.** If \( k \geq 1 \) and \( 1 \leq i \leq k \), then
\[ f_{k,i}^{(m)} (x, y ; 1) = x^i y \prod_{j=1}^{i-1} (1-(x^j y)^{m-1}) f_k^{(m)} (x, y ; 1). \] (14)

One may clearly add \( i \) to any \( \tau \)-avoiding \( k \)-ary composition \( \rho \) ending in \( j \geq i \) without introducing an occurrence of \( \tau \). On the other hand if \( j < i \), then the final run of the letter \( j \) in \( \rho \) cannot have length greater than \( m-2 \). Thus \( \rho \) in this case is accounted for by \( f_{k,i}^{(m)} (1-(x^j y)^{m-2}) \), by subtraction, as appending \( m-2 \) letters \( j \) to a composition enumerated by \( f_{k,j} \) results in one ending in at least \( m-1 \) letters \( j \), where \( f_{k,j} = f_{k,j}^{(m)} (x, y ; 1) \). Combining the previous cases and considering the penultimate letter within a composition enumerated by \( f_{k,j} \) gives
\[ f_{k,i} = x^i y + x^i y \sum_{j=1}^{k} f_{k,j} \]
\[ + x^i y \sum_{j=1}^{k-1} f_{k,j} \left( 1 - (x^i y)^{m-2} \right), 1 \leq i \leq k, \tag{15} \]
where the \( x^i y \) term represents the composition consisting of a single part \( i \).

Replacing \( i \) with \( i+1 \) in (15), and subtracting, gives
\[ f_{k,i+1} = -x^{i+1} y f_{k,i} + x^{i+1} y \left( 1 - (x^i y)^{m-2} \right) f_{k,i}, \]
which may be rewritten as
\[ f_{k,i+1} = x \left( 1 - (x^i y)^{m-1} \right) f_{k,i}, \tag{16} \]
\[ 1 \leq i \leq k-1. \]

Iterating (16), and noting
\[ f_{k,1} = xy + xy \sum_{j=1}^{k} f_{k,j} = xy \sum_{j=1}^{m} (x, y; 1), \]
leads to (14).

By using (14) and the fact
\[ f_{k}^m(x, y; 1) = 1 + \sum_{i=1}^{k} f_{k,i}^m(x, y; 1), \]
we get for \( m \geq 3 \) the formulas
\[ f_{k}^m(x, y; 1) = \frac{1}{1 - \sum_{j=1}^{i-1} x^j y \prod_{j=1}^{i-1} \left( 1 - (x^j y)^{m-1} \right)}, \tag{17} \]
\[ f_{k,i}^m(x, y; 1) = \frac{x^i y \prod_{j=1}^{i-1} \left( 1 - (x^j y)^{m-1} \right)}{1 - \sum_{j=1}^{i-1} x^j y \prod_{j=1}^{i-1} \left( 1 - (x^j y)^{m-1} \right)}, \tag{18} \]
where (17) was shown in [11, Theorem 2.2].

**Lemma 16.** If \( k \geq 2 \), then
\[ \left( 1 - x^k y - x^k y \sum_{j=1}^{m-1} f_{k,j}^m(x, y, q; 1) \right) \]
\[ + \sum_{i=1}^{k-1} x^i y q^{i-1} \left( 1 - (x^i y q^{i-1})^{m-2} \right) g_k \]
\[ \times \prod_{j=1}^{i-1} \left( 1 - (x^j y q^{j-1})^{m-1} \right) \]
\[ = \left( \frac{1 - x^{k-1} y^{m-1}}{1 - x^{k-1} y} \right) g_{k-1} \tag{19} \]
and
\[ \left( 1 - x^k y \right) h_k = \left( 1 - x^k y \right) \sum_{i=1}^{k-1} x^i y q^{i-1} \left( 1 - (x^i y q^{i-1})^{m-2} \right) \]
\[ \times \prod_{j=1}^{i-1} \left( 1 - (x^j y q^{j-1})^{m-1} \right) \]
\[ \left( 1 - x^k y \right) h_{k-1} = \left( 1 - x^k y \right) \sum_{i=1}^{k-1} x^i y q^{i-1} \left( 1 - (x^i y q^{i-1})^{m-2} \right) \]
\[ \times \prod_{j=1}^{i-1} \left( 1 - (x^j y q^{j-1})^{m-1} \right) \]

**Proof.** To show (19), first suppose \( \lambda \) enumerated by \( g_k \) is \((k-1)\)-ary, which gives \( g_{k-1} \). However, we must subtract the possibility of \( \lambda \) ending in a string of the letter \( k-1 \) of length at most \( m-1 \). Thus, we get
\[ \left( 1 - \frac{(k-1)^2}{1 - x^{k-1} y} \right) g_{k-1} \] in the case that \( \lambda \) is \((k-1)\)-ary. If \( \lambda \) ends in \( k \), then we clearly get \( x^k y g_k \). So assume \( \lambda = \lambda' k \hat{\lambda''} \), where \( \lambda' \) is \( k \)-ary and \( \lambda'' \) is \((k-1)\)-ary and nonempty. If \( \lambda'' \) ends in \( i \in [k-1] \), then the final string of \( i \)'s must have length at most \( m-2 \) in order for \( \lambda k \) to avoid \( \tau \). Thus, by subtraction, such \( \lambda'' \) would contribute \( f_{k-1,i}(x, q, q^{k-1} ; 1) \left( 1 - (x^i y q^{-i})^{m-2} \right) \).

Considering all possible \( i \), and using (14), then gives a contribution of
\[ x^k y g_k \sum_{i=1}^{k-1} f_{k-1,i}(x, q, q^{k-1} ; 1) \left( 1 - (x^i y q^{-i})^{m-2} \right) \]
\[ = x^k y \sum_{i=1}^{m} (x, q, q^{k-1} ; 1) g_k \]
\[ \times \sum_{i=1}^{k-1} x^i y q^{i-1} \left( 1 - (x^i y q^{-i})^{m-2} \right) \]
\[ \times \prod_{j=1}^{i-1} \left( 1 - (x^j y q^{-j})^{m-1} \right) \]
where the \( x^k y g_k \) factor accounts for \( \lambda' k \). Combining this with the previous cases leads to (19).
where the $1/(1 - x^k y)$ factor accounts for a possibly empty terminal string of the letter $k$. Then $k \lambda''$ gives $x^k y h_{k-1}$, which now fully accounts for the second term on the right-hand side of (20) (when divided by $1 - x^k y$) and completes the proof.

By Lemma 16 and (17), we have

$$g_k = \frac{1 - (x^k y)^{m-1}}{1 - x^k y} g_{k-1}$$

and

$$h_k = \frac{1}{1 - x^k y} \left( 1 + x^k y \sum_{i=1}^{k-1} y x^k y^{k-i} (1 - (x^k y)^{k-i-2}) \prod_{j=1}^{i-1} (1 - (x^k y)^{k-j})^{m-1} \right) h_{k-1},$$

which, by $g_1 = h_1 = 1/(1 - x y)$, leads to

$$g_k = \frac{1}{1 - x y} \prod_{i=2}^{k} \frac{1 - (x^i y)^{m-1}}{1 - x^i y} g_{k-1}$$

and

$$h_k = \frac{1}{1 - x^k y} \left( 1 + x^k y \sum_{i=1}^{\ell-1} y x^k y^{\ell-i} (1 - (x^k y)^{\ell-i-2}) \prod_{j=1}^{i-1} (1 - (x^k y)^{\ell-j})^{m-1} \right) h_{k-1}.$$  \hspace{1cm} (21)

Hence, by (9), we can state the following result.

**Theorem 17.** If $k \geq 2$, then the generating function $f_k = f_k^{(m)}(x; y; q)$ is given by

$$f_k = \frac{1}{1 - x y} \sum_{j=1}^{k} x^j y^j h_{j-1}.$$

where $g_k$ and $h_k$ are given by (21) and (22).

### 3.4. The Case 121

While we were unable to find an expression for the g.f. that counts water cells in 121-avoiding compositions for patterns of general length, we did find a recursive procedure for generating the distribution of water cells in the 121 case, which we describe in this subsection. Let $A_{n,k}$ denote the set of 121-avoiding $k$-ary words of length $n$. Given $n,k \geq 1$, let $a_{n,k} = a_{n,k}(p,q)$ be the joint distribution polynomial on $A_{n,k}$ for the statistics recording the sum of the letters and the capacity (marked by $p$ and $q$, respectively). Note that $[p^n]a_{m,k}(p,q)$ gives the distribution for the capacity statistic on $k$-ary 121-avoiding compositions of $n$ with $m$ parts.

To determine $a_{n,k}$, we will need the following further definitions. Let $b_{n,k} = b_{n,k}(p,q)$ be the same as $a_{n,k}$ except now we consider the capacity of $kw$ for $w \in A_{n,k}$. 


Let $b_{n,k}(i)$ where $i \in [k]$ denote the restriction of $b_{n,k}$ to words whose first letter is $i$. By the definitions, we have

$$b_{n,k} = \sum_{i=1}^{k} b_{n,k}(i), \quad n, k \geq 1,$$

with $b_{n,k}(k) = p^k b_{n-1,k}$ for $n \geq 1$ (where $b_{0,k} = 1$ for all $k \geq 0$). Considering whether or not $w$ enumerated by $a_{n,k}$ contains $k$, and if so, the position of the leftmost occurrence of $k$, leads to the formula

$$a_{n,k} - a_{n,k-1} = p b_{n-1,k} + b_{n-1,k-1}$$

$$+ \sum_{\ell=2}^{n-1} \sum_{i=1}^{k-1} b_{\ell-1,k-1}(i) b_{n-\ell,k}(j), \quad n, k \geq 2,$$

with $a_{1,k} = b_{1,k} = \frac{p^n}{1-p}$ and $a_{n,1} = b_{n,1} = p^n$ for $n, k \geq 1$.

We now determine a recurrence for the $b_{n,k}(i)$ array.

To aid in doing so, let $c_{n,k}(i; j)$ for $n \geq 1$ be the joint distribution on $w \in A_{n,k}$ of the form $w = iw'$ for the sum of the letters of $w$ and the number of water cells in $kwk$ (again, marked by $p$ and $q$, respectively). Note that the sum of the letters in $w$ equals the difference of $nk$ with the number of water cells in $kwk$, so $c_{n,k}(i; j)$ is actually a distribution of a single statistic, but we will keep track of both variables since this array is used to find a recurrence as follows for the $b_{n,k}(i)$ where this is not the case.

**Proposition 18.** If $n, k \geq 2$ and $1 \leq i \leq k-1$, then

$$b_{n,k}(i) = b_{n,k-1}(i) + p^{k+i} q^{k-i} b_{n-2,k} + p^k \sum_{j=1}^{k-1} c_{n,k-1,i}(j; i)$$

$$+ p^k \sum_{m=2}^{n-2} \sum_{j=1}^{m-1} c_{m,k}(j; i) b_{n-m,k}(\ell), \quad (24)$$

where $c_{1,k}(i; j) = 0$, $b_{1,k}(1) = p^n$ and $b_{n,k}(k) = p^k b_{n-1,k}$.

**Proof.** Let $w$ be a word enumerated by $b_{n,k}(i)$. Then there are $b_{n,k-1}(i)$ possibilities for $w$, by definition, if $w$ is $(k-1)$-ary, so assume $w$ contains at least one $k$. If $w$ is of the form $w = ikw'$, then there are $p^{k+i} q^{k-i} b_{n-2,k}$ possibilities. If $n \geq 3$ and $k$ appears only once and as the final letter, then there are $p^k \sum_{j=1}^{k-1} c_{n-1,k,j}(i; j)$ words $w$, upon considering the penultimate letter. So assume $w = w'kw''$, where $w'$ is $(k-1)$-ary of length $m$ for some $2 \leq m \leq n-2$ and $w''$ is $k$-ary. Let $j$ denote the last letter of $w'$ and $\ell \neq j$ the first letter of $w''$. Then there are $p^k c_{m,k}(i; j)$ possibilities for the $w'k$ and $b_{n-m-1,k}(\ell)$ for $Sw''$. Considering all $m$, $j$ and $\ell$ gives the remaining words $w$ and implies (24).

Let $c_{n,k}(i; \ell; j)$ for $n \geq 3$ denote the restriction of $c_{n,k}(i; j)$ to $w$ of the form $i\ell w'j$. Note that $c_{n,k}(i; \ell; j) = \sum_{\ell=1}^{k-1} c_{n,k}(i; \ell; j)$ for $n \geq 3$, by the definitions, with $c_{2,k}(i; \ell; j) = q^{2k} (p/q)^{j+i}$. We have the following recurrence formula for $c_{n,k}(i; \ell; j)$.

**Proposition 19.** If $n \geq 4$ and $k \geq 2$ and $i, \ell, j \in [k-1]$, then

$$c_{n,k}(i; \ell; j) = q^{2k} (p/q)^{j+i} \sum_{r=1}^{\ell} c_{n-2,k}(r; j)$$

$$+ p^\ell q^{k-1} \sum_{r+i+1}^{k-1} c_{n-1,k}(r,r; j), \quad i < \ell,$$

where $c_{3,k}(i; i; j) = 0$ if $\ell > i = j$ and $c_{3,k}(i; i; j) = q^{2k} (p/q)^{j+i+j}$ otherwise.

The initial conditions when $n = 3$ follow from the definitions, so assume $n \geq 4$. First assume $i < \ell$ and consider the third letter $r$ of $w$ enumerated by $c_{n,k}(i;i; j)$. If $r \leq \ell$ and $r \neq i$, then both the $i$ and $\ell$ may be deleted which gives the first term on the right side of (25). If $r > \ell$, then only the $i$ may be deleted which gives the second term in (25). If $i \geq \ell$, then $i$ is extraneous concerning the avoidance of 121 and thus may be deleted, which implies (26).

Note that recurrences (25) and (26) determine the $c_{n,k}(i; \ell; j)$ array which in turn gives the $c_{n,k}(i; j)$. For example, one can use these recurrences to compute $c_{n,k}(i; j)$ for all $n$ and $k$ up to some fixed number. Once the $c_{n,k}(i; j)$ are known, then the $b_{n,k}(i)$ array is determined by (24), which gives $b_{n,k}$, and hence $a_{n,k}$, by (23).

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