Max-Semi-Selfdecomposable Laws and Related Processes

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Abstract: Methods of construction of Max-semi-selfdecompsable laws are given. Implications of this method in random time changed extremal processes are discussed. Max-autoregressive model is introduced and characterized using the max-semi-selfdecompsable laws and exponential max-semi-stable laws. Some comments regarding the selfdecomposability of max-semi-stable laws are given.

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1. Introduction.

This investigation is motivated by the discussion of selfdecomposable (SD) and semi-selfdecomposable (semi-SD) laws and its relation to autoregressive (AR(1)) series and subordination of Levy processes in Satheesh and Sandhya (2004b). The class of semi-SD laws was characterized therein using an additive AR(1) model. That there is a close relationship between SD laws and AR (1) series had been observed by Gaver and Lewis (1980). Parallel to SD and semi-SD laws, Pancheva (1994) and Becker-Kern (2001) have discussed their maximum versions viz. max-SD and max-semi-SD laws. Again, paralleling subordination of Levy processes Pancheva, et al. (2003) has discussed a random time changed or compound extremal processes (EP). However, in
the literature we have not come across a maximum version of the AR(1) model and so here we introduce such a model and characterize it using max-semi-SD laws.

EPs are processes with increasing right continuous sample paths and independent max-increments. The univariate marginals of an EP determine its finite dimensional distributions. In general EPs are discussed with state space in $\mathbb{R}^d$, $d>1$ integer. This is because their max-increments are max-infinitely divisible (max-ID) and a discussion of max-ID laws makes sense only for distribution functions (d.f) on $\mathbb{R}^d$, $d>1$ integer as all d.fs on $\mathbb{R}$ are max-ID. However, we can discuss max-stable, max-semistable, max-SD and max-semi-SD laws on $\mathbb{R}$. Selfsimilar (SS) processes are those that are invariant in distribution under suitable scaling of time and space. Pancheva (1998, 2000) has developed SS and semi-SS EPs and showed that EPs whose univariate marginals are strictly max-stable (strictly max-semistable) constitute a class of SS (semi-SS) EPs having homogeneous max-increments. She also considered EPs whose univariate marginals are max-SD (max-semi-SD). Pancheva (1994, 1998, 2000) has described these laws and related processes in more generality, in terms of the invariance of their d.fs w.r.t one-parameter groups and cyclic groups of time-space changes. Here we will restrict our discussion to d.fs on $\mathbb{R}$ and the invariance w.r.t linear normalization. In the maximum scheme we have the following definitions.

**Definition 1.1** (Megyesi, 2002). A non-degenerate d.f $F$ is max-semi-stable$(a,b)$ if

$$F(x) = \exp\{-x^{-\alpha} h(\ln(x))\}, \ x>0, \ \alpha>0,$$

where $h(x)$ is a positive bounded periodic function with period $\ln(b)$, $b>1$, and there exists an $a>1$ such that $ab^{-\alpha} = 1$. This class is denoted by $\Phi_{\alpha,a,b}$, and,

$$F(x) = \exp\{-|x|^\alpha h(\ln(|x|))\}, \ x<0, \ \alpha>0,$$

(2)
where \( h(x) \) is a positive bounded periodic function with period \( |\ln(b)|, b<1 \), and there exists an \( a>1 \) such that \( ab^a = 1 \). This class is denoted by \( \Psi_{a,a,b} \).

\( \Phi_{a,a,b} \) is the extended Frechet type and \( \Psi_{a,a,b} \) the extended Weibull. The extended Gumbel has the translational invariance property and is not considered here.

**Definition 1.2** (Becker-Kern, 2001). A non-degenerate d.f \( F \) is max-semi-SD(\( c \)) if for some \( c>1 \) and \( v \in \mathbb{R} \) there is a non-degenerate d.f \( H \) such that

\[
F(x) = F(c^v x + \beta) H(x) \quad \forall \ x \in \mathbb{R},
\]

where \( \beta = 0 \) if \( v \neq 0 \) and \( \beta = \ln(c) \) if \( v = 0 \). If (3) holds for every \( c>1 \), then \( F \) is max-SD.

Pancheva, et al. (2003) has discussed random time changed or compound EPs and their theorem 3.1 together with property 3.2 reads: Let \( \{Y(t), t \geq 0\} \) be an EP having homogeneous max-increments with d.f \( F_t(y) = \exp\{-t\mu(\ell, y)^c\} \), \( y \geq \ell \), \( \ell \) being the bottom of the rectangle \( \{F>0\} \) and \( \mu \) the exponential measure of \( Y(1) \), that is, \( \mu(\ell, y)^c) = -\ln(F(y)) \). Let \( \{T(t), t \geq 0\} \) be a non-negative process independent of \( Y(t) \) having stationary, independent and additive increments with Laplace transform (LT) \( \varphi^t \). If \( \{X(t), t \geq 0\} \) is the compound EP obtained by randomizing the time parameter of \( Y(t) \) by \( T(t) \), then \( X(t) = Y(T(t)) \) and its d.f is:

\[
P\{X(t)<x\} = \{ \varphi(\mu(\ell, y)^c))\}^t.
\]

Pancheva, et al. (2003) also showed that in the above setup \( Y(T(t)) \) is also an EP. For a description of the process \( \{T(t)\} \) in this paper we need the following notion also.

**Definition 1.3** (Maejima an Naito, 1998). A probability distribution with characteristic function (CF) \( f \) is semi-SD(\( c \)) if for some \( 0<c<1 \) there exists a CF \( f_o \) such that
\[ f(s) = f(cs) f_0(s), \quad \forall \, s \in \mathbb{R}. \]

If this relation holds for every \( 0 < c < 1 \) then \( f \) is SD.

A stochastically continuous process \( \{ X(t), \, t \geq 0 \} \) having stationary, independent and additive increments and \( X(0) = 0 \) is called a Levy process. A Levy process \( X(t) \) such that \( X(1) \) is SD (semi-SD) will be called a SD (semi-SD) process. Since SD and semi-SD laws are ID, above Levy processes are well defined.

**Note.1** For brevity the above EPs will be referred to as max-stable, max-semi-stable, max-SD, max-semi-SD EPs and the Levy process above as SD and semi-SD processes.

**Note.2** In this paper stable and semi-stable laws, selfsimilarity and semi-selfsimilarity are considered in the strict sense only.

With this background we give methods to construct max-semi-SD laws in section.2. Implications of these constructions in the context of compound EPs are then discussed. In section.3 we introduce a max-AR(1) series and characterize it using the max-semi-SD law. The max-AR(1) model is then modified to characterize the exponential max-semi-stable laws. Finally in section.4, certain selfdecomposability properties of max-semi-stable laws are given.

2. Max-semi-SD laws and compound EPs.

We begin with some remarks.

**Remark.2.1** If \( h(x) \) in (1) is periodic w.r.t \( \ln(b_1) \) and \( \ln(b_2) \) such that \( \ln(b_1)/ \ln(b_2) \) is irrational then \( h(x) = \lambda, \) a constant. When \( h(x) \) is a constant \( F \) is max-stable.

**Remark.2.2** The d.fs in \( \Phi_{\alpha, a, b} \) can be represented in the form \( \exp\{-\psi(x)\} \), where \( \psi(x) \) satisfies \( \psi(x) = a \psi(bx) \), for some \( a > 1, \, b > 1, \) and \( \alpha > 0 \) satisfying \( ab^{-\alpha} = 1 \). Similarly
those in $Ψ_{a,a,b}$ can be represented as $\exp\{-ψ(x)\}$, where $ψ(x) = aψ(bx)$, for some $a>1$, $b<1$, and $α>0$ satisfying $ab^α = 1$. On the other hand, the general solution to $ψ(x) = aψ(bx)$, $ψ(x) = x^α h(ln(x))$, $x>0$ or $|x|^α h(ln(|x|))$, $x<0$, where $h(x)$ is as in (1) which can be proved along the lines in Lin (1994) or Pillai and Anil (1996).

Remark.2.3 An EP $\{Y(t)\}$ is SS if for any $b>0$ there is an exponent $H>0$ such that

$$\{Y(bt)\} \overset{d}{=} \{b^HY(t)\}. \quad (5)$$

If (5) holds for some $b>0$ only then $\{Y(t)\}$ is semi-SS. Without loss of generality we may consider the range of $b$ as $0<b<1$ in the description of selfsimilarity because (5) is equivalent to $\{(b^-1)^HY(t)\} \overset{d}{=} \{Y(b^{-1}t)\}$ and thus the whole range $b>0$ is covered.

Remark.2.4 In definition.1.2 of max-semi-SD($c$) laws, the case $ν=0$ is not considered here. Thus the combination $c>1$ and $ν\in \mathbb{R}−\{0\}$ implies that we are considering

$$F(x) = F(cx) H(x) \forall x\in \mathbb{R} \text{ and for some } c\in (0,1) \cup (1,\infty). \quad (6)$$

Thus there is a one-to-one correspondence between the possible scale changes in max-semi-stable and max-semi-SD laws. $F$ is max-SD if (6) holds for every $c\in (0,1) \cup (1,\infty)$.

Satheesh (2002) and Satheesh and Sandhya (2004a) have discussed $ϕ$-max-stable and $ϕ$-max-semi-stable laws as follows. For a LT $ϕ$ the d.f $ϕ\{-ln(F(x))\}$ is $ϕ$-max-stable ($ϕ$-max-semi-stable) if the d.f $F(x)$ is max-stable (max-semi-stable). When $ϕ$ is exponential we call the corresponding d.f as exponential max-stable (max-semi-stable). Satheesh (2002) has discussed the connection among: mixtures of max-ID laws (theorem.2.1b), randomizing the time parameter of an EP (theorem.2.2b), construction
of SD laws (property 2.1) and the possibility of constructing its max-analogue using mixtures of max-stable laws. This analogue is:

**Theorem 2.1** \( \varphi \)-max-stable laws corresponding to (1) and (2) are max-SD if \( \varphi \) is SD.

Generalizing this Satheesh and Sandhya (2004a) proved the following:

**Theorem 2.2** \( \varphi \)-max-semi-stable(\( a, b \)) laws corresponding to (1) and (2) are max-semi-SD(\( b \)) if \( \varphi \) is SD.

We now generalize this as follows.

**Theorem 2.3** \( \varphi \)-max-semi-stable(\( a, b \)) laws corresponding to (1) (respectively (2)) are max-semi-SD(\( b \)) if \( \varphi \) is semi-SD(\( b^{-\alpha} \), \( b>1 \)) (respectively semi-SD(\( b^\alpha \), \( b<1 \)).

*Proof.* If \( \varphi \) is semi-SD(\( b^{-\alpha} \), \( b>1 \)), then there exists a LT \( \varphi_o \) and the d.f of a \( \varphi \)-max-semi-stable(\( a, b \)) law is;

\[
\varphi\{ \psi(x) \} = \varphi\{ b^{-\alpha} \psi(x) \} \varphi_o\{ \psi(x) \}, \ \forall x \in \mathbb{R},
\]

Since \( \psi(x) = a \psi(bx) \), for \( a>1, b>1 \), and \( \alpha>0 \) satisfying \( ab^{-\alpha} = 1 \), we have;

\[
\varphi\{ \psi(x) \} = \varphi\{ \psi(bx) \} \varphi_o\{ \psi(x) \} \text{ for some } b>1,
\]

which completes the proof corresponding to the case (1). For the case corresponding to (2) we can proceed along the above lines and we have d.f.s satisfying

\[
\varphi\{ \psi(x) \} = \varphi\{ \psi(bx) \} \varphi_o\{ \psi(x) \} \text{ for some } b<1.
\]

This completes the proof of the theorem.

**Example 1** Consider the d.f \( F(x) = \{1+\psi(x)\}^{-\beta} \), where \( \psi \) is as in remark 2.2 and \( \beta>0 \). Since the gamma(1,\( \beta \)) law is SD the above d.f is max-semi-SD(\( b \)) being that of a gamma-max-semi-stable(\( a, b \)) law. By a similar line of argument, Satheesh and Sandhya
(2004b) have shown that the CF \( \{1+\psi(t)\}^{-\beta} \) is semi-SD(\( b \)), \( b<1 \). Write \( b=\lambda^\alpha \) for \( \lambda<1 \) and \( \alpha\in(0,2] \). Setting \( \varphi \) to be this semi-SD(\( \lambda^\alpha \)) the corresponding \( \varphi \)-max-semi-stable(\( a,\lambda \)) laws are max-semi-SD(\( \lambda \)).

In the case of compound EPs, extending the discussion in Pancheva, et al. (2003) we now have the following results. These results follow from the equation (4) and respectively from theorems 2.1, 2.2 and 2.3.

**Theorem 2.4** The EP obtained by compounding a max-stable EP is max-SD if the compounding process is SD.

**Theorem 2.5** The EP obtained from a random time changed max-semi-stable(\( a,b \)) EP is max-semi-SD(\( b \)) if the compounding process is SD.

**Theorem 2.6** The EP obtained by compounding a max-semi-stable(\( a,b \)) EP corresponding to (1) (respectively (2)) is max-semi-SD(\( b \)) if the compounding process is semi-SD(\( b^{-\alpha} \)), \( b>1 \) (respectively semi-SD(\( b^\alpha \)), \( b<1 \)).

Now a condition for a semi-SS EP to be SS. The proof follows from remark.1.

**Theorem 2.7** If a max-semi-stable(\( a,b \)) EP (hence semi-SS) \{\( Y(t) \)\} satisfies (5) for two values \( b_1 \) and \( b_2 \) such that \( \ln(b_1)/\ln(b_2) \) is irrational then it is max-stable (hence SS).

### 3. Autoregressive models with a maximum structure.

We now develop AR(1) models with a maximum structure and give stationary solutions to it characterizing max-semi-SD and exponential max-semi-stable laws.

**Definition 3.1** A sequence \{\( X_n \)\} of r.vs generates a max-AR(1) series if for some \( \rho>0 \) there exists an innovation sequence \{\( \epsilon_n \)\} of i.i.d r.vs such that
\[ X_n = \rho X_{n-1} \lor \varepsilon_n, \quad \forall \ n > 0 \text{ integer.} \quad (7) \]

Since \( X_{n-1} \) is a function only of \( \varepsilon_j, j = 1, 2, \ldots, n-1 \), it is independent of \( \varepsilon_n \).

Hence in terms of d.fs this is equivalent to;

\[ F_n(x) = F_{n-1}(x/\rho) F_\varepsilon(x) \quad \forall \ x \in \mathbb{R}. \]

Assuming the series to be marginally stationary we have;

\[ F(x) = F(x/\rho) F_\varepsilon(x) \quad (8) \]

Now comparing (8), (6) and (3) the following theorem is clear.

**Theorem 3.1** A sequence \( \{X_n\} \) of r.v.s generates a max-AR(1) series that is marginally stationary if and only if the distribution of \( X_n \) is max-semi-SD(\( c \)), \( c = 1/\rho \).

**Remark 3.1** Unlike the additive AR(1) scheme where semi-SD(\( \rho \)) laws characterizes only the case of \( \rho < 1 \), max-semi-SD(\( c \)) laws characterizes the max-AR(1) scheme including the explosive situation of \( \rho > 1 \) as well, where \( c = 1/\rho \).

Now let us modify the max-AR(1) scheme (7) as follows.

\[ \begin{align*}
X_n &= \rho X_{n-1}, \quad \text{with probability } p \\
    &= \rho X_{n-1} \lor \varepsilon_n, \quad \text{with probability } (1-p).
\end{align*} \quad (9) \]

In terms of d.fs and assuming marginal stationarity of \( X_n \) this is equivalent to;

\[ F(x) = F(x/\rho) \{p + (1-p) F_\varepsilon(x)\}. \]

Further assuming \( X_n \) and \( \varepsilon_n \) to be identically distributed and setting \( c = 1/\rho \) we have;

\[ F(x) = p F(cx)/\{1-(1-p)F(cx)\}. \quad (10) \]
This means that $F(x)$ is a geometric (geometric with mean $1/p$) maximum of its own type. Writing $F(x) = 1/[1+\psi(x)]$, we have from (10),

$$\frac{1}{1+\psi(x)} = \frac{p}{1+(1-p)\psi(x)} = \frac{1}{1+\frac{1}{p}\psi(cx)}, \forall x \in \mathbb{R}.$$ 

Hence, $\psi(x) = \frac{1}{p}\psi(cx)$, and thus $\psi(x)$ is now as in remark.2.2 with $a = \frac{1}{p}$ and $b = c = 1/\rho$. Hence $F(x)$ is an exponential max-semi-stable$(a,b)$ law (since here $\varphi$ is the LT of an exponential law). Notice that here also both $\rho < 1$ and $\rho > 1$ are permitted. As the converse is clear we have proved;

**Theorem.3.2** A sequence $\{X_n\}$ of r.v.s generates a max-AR(1) series with the structure of (9) that is marginally stationary and $X_n \overset{d}{=} \varepsilon_n$ if and only if the distribution of $X_n$ is an exponential max-semi-stable$(a,b)$ law, $a = \frac{1}{p}$ and $b = c = 1/\rho$.

**4. Selfdecomposability properties of max-semi-stable laws.**

Satheesh and Sandhya (2004b) have shown that a semi-stable law is infinitely divisible and semi-SD. Here we show that similar results hold good for max-semi-stable laws as well.

**Theorem.4.1** Max-semi-stable$(a,b)$ laws corresponding to (1) and (2) are max-semi-SD$(b)$.

**Proof.** If $F(x)$ is a max-semi-stable$(a,b)$ d.f corresponding to (1) then we have;

$$F(x) = \{F(bx)\}^a, x > 0, \text{ for some } a > 1, b > 1, \text{ where } ab^{-\alpha} = 1 \text{ for some } \alpha > 0.$$ 

$$= F(bx)\{F(bx)\}^{a-1}.$$
Hence by definition.1.2 $F$ is max-semi-SD($b$). A similar line of argument shows the fact for the case corresponding to (2). Notice that in the one-dimensional case $F$ is always max-ID.

**Theorem.4.2** Max-stable laws corresponding to (1) and (2) are max-SD.

*Proof.* If $F$ is a max-stable d.f then for each $b \in (0,1) \cup (1,\infty)$ there exists an $a>1$ such that:

$$F(x) = \{F(bx)\}^a, x \in \mathbb{R}.$$

$$= F(bx)\{F(bx)\}^{a^{-1}}, \text{ for every } b \in (0,1) \cup (1,\infty).$$

Hence $F$ is max-SD.

Thus we know how to construct max-semi-SD laws, its implication in compound EPs, related divisibility properties and the structure of max-AR(1) series.

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