Quasinormal Modes of Charged Fermions and Phase Transition of Black Holes

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We study the quasinormal modes of massless charged fermions in a Reissner-Nordström-anti-de Sitter black hole spacetime. In the probe limit, we find that the imaginary part of quasinormal frequency will become positive when the temperature of the black hole is below a critical value. This indicates an instability of the black hole occurs and a phase transition happens. In the AdS/CFT correspondence, this transition can be viewed as a superconducting phase transition and the bulk fermion is regarded as the order parameter. When the coupling of the fermions and the background electric field becomes stronger, the critical temperature of the phase transition becomes higher. If the interaction between the fermion and the electric field can be ignored, namely in the case of a neutral fermion, the imaginary part of the quasinormal modes is always negative, which indicates that the black hole is stable and no phase transition occurs.

I. INTRODUCTION

Over the past years the holographic models of superconductors have attracted a lot of attentions since the works \cite{1,2}. For reviews see \cite{3}. In a simple model with the Einstein-Maxwell-complex scalar field system with a negative cosmological constant, the charged scalar field will condensate when the temperature of the anti-de Sitter (AdS) black hole is below a critical value. Above the critical temperature, the system has the Reissner-Norström (RN)-AdS black hole solution with a trivial scalar field, while below the critical temperature, a hairy black hole solution is more stable with a nontrivial scalar field. This indicates that a phase transition happens between the RN-AdS black hole and a hairy black hole when the temperature of the black hole arrives at the critical value. According to the AdS/CFT correspondence \cite{4–6}, the phase transition of the AdS black hole can be mapped to a superconducting phase transition on the boundary of the AdS space \cite{2}, the condensation of charged scalar field around the black hole corresponds to a condensation of the charged

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operators in the boundary field theory.

It is well known that a neutral scalar field perturbation of an asymptotically AdS spacetime is stable if the mass of the scalar field satisfies the Breitenlohner-Freedman (BF) bound \[ \frac{m^2}{L^2} < 1 \]. However, the perturbation of charged scalar field in an asymptotically AdS spacetime can cause the background to become unstable \[ 1-8,10 \]. In the holographic superconductor models, the presence of the instability of the perturbation of the charged scalar field just indicates the occurrence of the condensation in the boundary field theory.

To reveal the stability of a black hole, a useful method is to study the quasinormal modes of some perturbations in the black hole background, for a review see \[ 11 \]. Recently it was found that quasinormal modes of charged scalar field in some AdS black hole backgrounds can be used to disclose the relation to the superconductor transition \[ 12 \]. The occurrence of the unstable modes was observed in consistent with the superconducting phase transition.

Besides the charged scalar field, it is of great interest to examine whether some other fields can experience the condensation in the AdS black hole background, for example the charged fermionic field, its condensation might be related to the model of color superconductor in QCD theory. The perturbation of a charged fermionic field in an asymptotically flat spacetime was found always with decay modes \[ 13 \]. The fermionic field will either be absorbed by the horizon or go to the spatial infinity and cannot condensate near the black hole. It is well known that the AdS spacetime has different asymptotical behavior from that of the asymptotically flat spacetime and this difference plays crucial role on the dynamics of the inner geometry. For example, the late-time tail of the perturbations with a power law form in an asymptotically flat spacetime gives way to the exponential form in an asymptotically AdS spacetime \[ 14,15 \]. In this work we will concentrate on the study of the charged fermionic field perturbation in the background of an AdS black hole.

We employ a simple RN-AdS spacetime geometry while leaving fermionic fields as probes. We find that when the temperature is below a certain critical value, the imaginary part of the quasinormal modes will become positive, which indicates the occurrence of the instability of the black hole. This shows that at the critical temperature there appears a phase transition, the RN-AdS black hole background becomes unstable and a new stable hairy black hole solution is expected to emerge. We observe that the critical temperature grows with the increase of the charge coupling constant. This means that the stronger the fermions couple to the Maxwell field, the easier the instability occurs. Our result is consistent with the holographic superconducting phase transition discussed in \[ 3,12 \], however in our result the order parameter is a charged fermion instead of the charged scalar field. In this sense our model might be regarded as a simple model of color superconductor in QCD theory.

This paper is organized as follows. In Sec. II we briefly introduce the RN-AdS geometry and the action of the charged fermions in this background. Under an explicit representation, the Dirac equation is splitted in Sec. III to a concise form. In Sec. IV we use the Horowitz-Hubeny approach to Fourier expand the Dirac equations for the convenience to study the quasinormal modes of the fermionic perturbation numerically. We give the numerical results and analysis of the quasinormal modes in Sec. V. We conclude our paper in Sec. VI.
II. THE BACKGROUND GEOMETRY AND ACTION

For a general $d$-dimensional RN-AdS spacetime, the metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_{d-2,k}^2,$$

where

$$f(r) = k + \frac{r^2}{L^2} + \frac{Q^2}{4\pi L^{d-6}} - \left(\frac{r_0}{r}\right)^{d-3}. \quad (2)$$

$L$ is the AdS radius, $Q$ is the charge of the black hole and $r_0$ is related to the black hole mass $M$ via

$$r_0^{d-3} = \frac{16\pi G_d M}{(d-2) A_{d-2}}, \quad (3)$$

where $A_{d-2} = 2\pi^{d-1}/\Gamma\left(\frac{d-1}{2}\right)$ is the area of a unit $(d-2)$-sphere, $G_d$ is the Newton gravitational constant in the $d$-dimensional spacetime. $d\Omega_{d-2,k}$ in (1) is the metric of constant curvature. If $k = 0$, it is the metric of a flat Euclidean space $\mathbb{R}^{d-2}$; if $k > 0$, it is the line element of a $(d-2)$-sphere with radius $\frac{1}{\sqrt{k}}$; and if $k < 0$, it is the metric of a hyperbolic plane with radius of curvature $\frac{1}{\sqrt{-k}}$. Without loss of generality, one can take $k = 0$, and $\pm 1$.

We will focus on $k = 0$ case in this paper for simplicity and also for the relation to a superconductor in a plane. For the cases with $k \neq 0$, similar results can be obtained. In particular, we will consider a 4-dimensional RN-AdS spacetime and let $G_d = 1$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2), \quad (4)$$

The temperature of the black hole now is

$$T_H = \frac{3r_+}{4\pi L^2} - \frac{Q^2}{16\pi r_+^3}. \quad (5)$$

where $r_+$ is the horizon radius of the black hole.

The spin connection is defined as:

$$\omega_{\hat{a}\hat{b}\hat{c}} = e_{\hat{a}\hat{d}}\partial_{\hat{c}}e^\hat{d}_b + e_{\hat{a}\hat{d}}e^\hat{d}_e \Gamma^e_{\hat{b}\hat{c}}, \quad (6)$$

where $e^\hat{a}_b$ is the tetrad, $\Gamma^a_{bc}$ is the Christoffel connection and $\omega_{\hat{a}\hat{b}\hat{c}} = -\omega_{\hat{b}\hat{a}\hat{c}}$. The nonvanishing spin connections $\omega_{\hat{a}\hat{b}\hat{c}}$ for the metric (4) are:

$$\omega_{\hat{r}\hat{t}\hat{t}} = -\omega_{\hat{r}\hat{t}\hat{t}} = \frac{1}{2}f', \quad \omega_{\hat{r}\hat{x}\hat{x}} = -\omega_{\hat{x}\hat{x}\hat{x}} = \sqrt{f}, \quad \omega_{\hat{y}\hat{y}} = -\omega_{\hat{r}\hat{y}} = \sqrt{f}. \quad (7)$$

where a prime denotes the derivative with respect to $r$.

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1 The un-hatted letter $b$ is the index of the background spacetime while the hatted letter $\hat{a}$ denotes the index of the tangent space.
We consider the Einstein-Maxwell-fermion system with the action

\[ S_T = S_g + S_m, \]  
(8)

where

\[ S_g = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g} \left( \mathcal{R} - \frac{6}{L^2} \right), \]  
(9)

\[ S_m = \mathcal{N} \int d^4x \sqrt{-g} \left( -\frac{1}{4} F_{ab} F^{ab} + i \left( \bar{\Psi} \Gamma^a (D_a - i q A_a) \Psi - m \bar{\Psi} \Psi \right) \right), \]  
(10)

\( \kappa_4 \) is the 4-dimensional gravitational constant, \( \mathcal{R} \) is Ricci scalar, \( \mathcal{N} \) is a total coefficient of the action of matter, \( q \) is the coupling constant between the fermion field and Maxwell field. \( \Gamma^a \) is the Dirac gamma matrix and

\[ D_c = \partial_c + \frac{1}{2} \omega_{\hat{a}c} \Sigma^{\hat{a} \hat{b}}, \quad \Sigma^{\hat{a} \hat{b}} = \frac{1}{4} [\Gamma^\hat{a}, \Gamma^\hat{b}], \]

\[ \Gamma^\hat{b} = e^b_\hat{a} \Gamma^\hat{a}. \]  
(11)

The Dirac equation for the fermion is

\[ \Gamma^a (D_a - i q A_a) \Psi - m \Psi = 0. \]  
(12)

In the action (8), clearly we have a simple RN-AdS black hole solution with the electric potential \( A_t \) and a vanishing fermion \( \Psi \):

\[ A_t = Q \left( \frac{1}{r} - \frac{1}{r_+} \right), \quad \Psi = 0. \]  
(13)

In the following, we will work in the probe limit which means that the fermionic field does not backreact on the metric and Maxwell field.

### III. THE SPLIT OF THE DIRAC EQUATION

We can write the wave function \( \Psi(r, x_\mu) \) into the momentum space

\[ \Psi(r, x_\mu) = \psi(r)e^{-i\omega t + i\vec{k} \cdot \vec{x}}. \]  
(14)

where \( x_\mu \) denotes the coordinates in the boundary, \( x_\mu = (t, x, y) \), while \( \vec{k} = (k_x, k_y) \) and \( \vec{x} = (x, y) \). Under this transformation, we can write the Dirac equation (12) into

\[ \sqrt{f} \Gamma^\hat{\mu} \partial_\mu \psi - \frac{i \omega}{\sqrt{f}} \Gamma^{\hat{\mu}} \psi + \frac{i \vec{k} \cdot \Gamma^{\hat{\mu}}}{r} \psi + \frac{f'}{4} \Gamma^{\hat{\mu}} \psi - \left( iq \Gamma^a A_a + m \right) \psi = 0. \]  
(15)

where \( \vec{k} \cdot \Gamma^{\hat{\mu}} = k_x \Gamma^{\hat{x}} + k_y \Gamma^{\hat{y}}. \) We can choose the Dirac gamma matrices as [16]

\[ \Gamma^i = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}. \]  
(16)
where $I$ is the $2 \times 2$ unit matrix, $\sigma^\dagger$ is the Pauli matrix, explicitly,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (17)$$

It is easy to decompose the fermion field $\Psi$ into $\Psi_+$ and $\Psi_-$, which are the eigenvectors of $\Gamma^5$, i.e.

$$\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \quad P_\pm \Psi = \pm \Psi_\pm, \quad P_\pm = 1 \pm \Gamma^5, \quad \Gamma^5 = i\Gamma^t\Gamma^x\Gamma^y\Gamma^z. \quad (18)$$

Under this representation, the Dirac equation (15) can be decomposed into

$$\left(\sqrt{f}\partial_r + \frac{1}{4}\frac{f'}{\sqrt{f}} + \frac{\sqrt{f}}{r}\right)\sigma^r \psi_+ + \frac{i}{r}(\vec{k} \cdot \vec{\sigma})\psi_+ + \frac{i}{\sqrt{f}}(\omega + qA_t)\psi_+ - m\psi_+ = 0, \quad (19)$$

$$\left(\sqrt{f}\partial_r + \frac{1}{4}\frac{f'}{\sqrt{f}} + \frac{\sqrt{f}}{r}\right)\sigma^r \psi_- + \frac{i}{r}(\vec{k} \cdot \vec{\sigma})\psi_- - \frac{i}{\sqrt{f}}(\omega + qA_t)\psi_- - m\psi_- = 0. \quad (20)$$

It is easy to see from eqs. (19) and (20) that if $\omega \to -\omega$ and $q \to -q$, there is a permutation symmetry $\psi_+ \leftrightarrow \psi_-$ in (19) and (20). In the following, we will take $m = 0$, because the chiral modes of $\psi$ will decouple in this case, which can be easily seen from eqs. (19) and (20).

For the massless fermions, we will focus on the modes of $\psi_+$, because the modes of $\psi_-$ can be obtained accordingly if we change the sign of $\omega$ and $q$. Furthermore, for simplicity, we can set $k_y = 0$ because of the symmetry of $(x, y)$-plane [17]. We rewrite $\psi_+$ as

$$\psi_+ = r^{-1} f^{-1/4} \tilde{\psi}. \quad (21)$$

And then eq. (20) can be further simplified into

$$\sigma^r \tilde{\psi}' - \frac{i}{f}(\omega + qA_t - \sqrt{f} k_x \sigma^z )\tilde{\psi} = 0. \quad (22)$$

Here $\tilde{\psi}$ is a 2-component fermion wavefunction, we can decompose it as $\tilde{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ under the $\sigma^r$ matrix. Under this representation, eq. (22) can be further decomposed into

$$\psi_1' - \frac{i}{f}(\omega + qA_t)\psi_1 + \frac{i}{r\sqrt{f}} k_x \psi_2 = 0, \quad (23)$$

$$\psi_2' + \frac{i}{f}(\omega + qA_t)\psi_2 - \frac{i}{r\sqrt{f}} k_x \psi_1 = 0. \quad (24)$$

Like in the case of eqs. (19) and (20), if $\omega \to -\omega, q \to -q$ and $k_x \to -k_x$, there is also a permutation symmetry $\psi_1 \leftrightarrow \psi_2$ between eqs. (23) and (24). Therefore, we can only focus on the modes $\psi_1$ by extracting $\psi_2$ from eq. (23) and then inserting it into eq. (24).
IV. THE QUASINORMAL MODES

To calculate the quasinormal modes of the charged fermion, we introduce the tortoise coordinates \( r_* \) by

\[
dr_* = \frac{dr}{f}. \tag{25}
\]

Now, the black hole horizon is located at \( r_* \to -\infty \) and the spatial infinity is at a finite value of \( r_* \).

The ingoing boundary conditions at the horizon requires the wavefunction to have the form

\[
\psi_1(r) = e^{-i\omega r_*} u(r). \tag{26}
\]

For simplicity of the numerical calculation, we define \( x = 1/r \). Thus, the black hole horizon is at \( x_+ = 1/r_+ \) while the infinite boundary \( r \to \infty \) locates at \( x = 0 \). And the equation of \( u \) is given as

\[
S(x) \ddot{u}(x) + \frac{T(x)}{x - x_+} \dot{u}(x) + \frac{U(x)}{(x - x_+)^2} u(x) = 0, \tag{27}
\]

where the overdot denotes the derivative with respect to \( x \) and

\[
S(x) = \left( -L^2 Q^2 x^3 x_+^3 + 4x_+^2 + 4xx_+ + 4x^2 \right)^2, \tag{28}
\]

\[
T(x) = \frac{1}{2} \left( L^2 Q^2 x^3 x_+^3 - 4x_+^2 - 4xx_+ - 4x^2 \right) \left( 4L^2 Q^2 x^3 x_+^3 + 16iL^2 \omega x_+^3 - 3x^2 \left( L^2 Q^2 x_+^4 + 4 \right) \right), \tag{29}
\]

\[
U(x) = 2L^2 x_+^3 \left\{ 2\omega \left[ -8L^2 qQ x_+^4 - 4iL^2 Q^2 x^3 x_+^3 + 8L^2 qQ xx_+^3 + 3ix^2 \left( L^2 Q^2 x_+^4 + 4 \right) \right] \\
- (x - x_+) \left[ 2 \left( L^2 Q^2 x^3 x_+^3 - 4x_+^2 - 4xx_+ - 4x^2 \right) k_x^2 + iqQ \left( -8iL^2 qQ x_+^4 + 4 \right) \right] \\
+ 2L^2 Q^2 x^3 x_+^3 + 8x_+^2 + 8x \left( iL^2 qQ x_+^3 + x_+ \right) - x^2 \left( 3L^2 Q^2 x_+^4 + 4 \right) \right\}. \tag{30}
\]

Although the complex number \( i \) occurs in the potential \( U(x) \) here, it was argued in \([18, 22]\) and references therein that one still can have correct quasinormal modes in the cases with such a complex potential if some proper boundary conditions are imposed.

Next we employ the Horowitz-Hubeny method \([14]\) to calculate the quasinormal modes of this charged fermion. \( S(x), T(x) \) and \( U(x) \) can be Fourier expanded near \( x_+ \) to a finite term, \( i.e., S(x) = \sum_{n=0}^{6} s_n (x - x_+)^n, T(x) = \sum_{n=0}^{4} t_n (x - x_+)^n \) and \( U(x) = \sum_{n=0}^{4} u_n (x - x_+)^n \). We also expand the solution \( u(x) \) around \( x_+ \) as

\[
u(x) = (x - x_+)^\alpha \sum_{n=0}^{\infty} a_n (x - x_+)^n. \tag{31}\]

At the leading order, the equation has two solutions \( \alpha = 1/2 \) and \( \alpha = \frac{8iL^2 x_+}{12 - L^2 Q^2 x_+^2} \omega \). It is obvious that the solution \( \alpha = \frac{8iL^2 x_+}{12 - L^2 Q^2 x_+^2} \omega \) gives an outgoing mode at the horizon. As a
result, we choose $\alpha = 1/2$ in the expansion (31). Substituting eq. (31) into the eq. (27), we can obtain a recursion relation
\begin{equation}
    a_n = -\frac{1}{P_n} \sum_{k=0}^{n-1} [(k + \alpha)(k + \alpha - 1)s_{n-k} + (k + \alpha)t_{n-k} + u_{n-k}] a_k,
\end{equation}

where $P_n = (n + \alpha)(n + \alpha - 1)s_0 + (n + \alpha)t_0 + u_0$.

In the numerical calculations, we set the initial data $a_0 = 1$ whose scaling will not affect the final quasinormal modes $\omega$ because the equation of $\psi$ is linear. In practice, we can solve the expansion up to a large number $n = N$, then compare the results got from $N$ and the results from $n > N$, if the error of the two results is of the order $10^{-5}$, we will believe that the modes $\omega_N$ is acceptable.

V. NUMERICAL RESULTS

The boundary condition on the spatial infinity is that the fermion field vanishes there, which means
\begin{equation}
    u(0) = \sum_{n=0}^{N} a_n (0 - x_+)^{n+\alpha} = 0.
\end{equation}

This equation gives the eigenvalues of $\omega$. The quasinormal modes can be decomposed into real and imaginary parts:

\begin{equation}
    \omega = \text{Re}(\omega) + i\text{Im}(\omega).
\end{equation}

If $\text{Im}(\omega) < 0$, the perturbations always decay exponentially and then the background black hole is stable; if $\text{Im}(\omega) > 0$, however, the modes grow exponentially, which means the perturbations will make the background black hole unstable. This instability suggests a phase transition of black holes [8, 12].

Following [1], in the calculation we set $r_+ = Q = 1$. In this case, changing $L$ corresponds to the change of the temperature of the black hole. Furthermore, we set the x-direction momentum $k_x = 1$. For the calculation precision we Fourier expand the equation up to $N = 300$.

To find the marginally stable perturbation where $\Psi$ depends only on $r$ and is infinitesimally small, taking $\omega = 0$ in (27), we will have an equation on $q$ and $L$ from eq. (27). Requiring both the real and imaginary parts of the resulting equation on $q$ and $L$ to vanish, we have two constraints on the relation between $q$ and $L$, whose numerical results are shown in the left panel of Fig[1]. It indicates that only $q$ and $L$ at the crossing points of the two constraint curves can give the marginally stable solution with $\omega = 0$. The three crossing points $(q, L)$ shown in the left panel of Fig[1] are $(1.546, 1.951), (2.821, 1.915)$, and $(4.187, 1.887)$, respectively. When $q$ increases, we see that $L$ at the crossing points decreases. According to the temperature formula (5) of the black hole, the temperature increases as $L$ decreases. Thus the stronger the fermion field couples to the Maxwell field, the easier the instability
FIG. 1: (Left Panel) Eq. (27) with $\omega = 0$ gives the two constraints on $q$ and $L$: the real part (blue) and imaginary part (red dashed). (Right Panel) The critical temperature $T_{Hc}$ versus $q$.

occurs. This is physically reasonable. We show this relation of the critical temperature and the coupling $q$ in the right panel of Fig.1.

The marginally stable perturbation cannot guarantee the existence of the instability but just highly suggestive $[1]$. In examining the stability of the perturbation, one can release the condition $\omega = 0$, then in principle there does not exist any constraint relation between $q$ and $L$. In this paper we are interested in the quasinormal modes of the fermion perturbation and expect to see whether a positive imaginary part of quasinormal frequency which indicates the instability of the fermion perturbation will appear. Fig.2 shows the frequencies of the quasinormal modes in the case with a fixed $q = 1.546$ and different $L$. It can be seen that when $L > L_c = 1.951$ the imaginary part of the quasinormal frequency becomes positive, which means that the black hole is unstable under the perturbation when the temperature is low enough. $L = L_c$ corresponds to the critical temperature of the black hole $T_{Hc}$, when $T > T_{Hc}$ the RN-AdS black hole is stable under the fermionic perturbation, while the RN-AdS black hole becomes unstable when $T < T_{Hc}$ and the fermionic hair is expected to condensate and attach to the black hole. The RN-AdS black hole will give way to a fermion haired black hole.

In Table I we list the quasinormal modes shown in Fig.2. One can see that the imaginary part of the first node becomes positive when $L = 1.96$ and $L = 2$, while it is negative if $L = 1.94$. This manifests that the perturbation becomes unstable when $L$ changes from 1.94 to 1.96. The critical value is $L = 1.951$, where $\omega = 0$.

From the left panel of Fig.1 we can see that when $q = 0$, there are no critical values for $L$ which makes $\omega = 0$. This implies that for the neutral fermion perturbation, the black hole is always stable and the perturbation always decays exponentially. This is indeed observed in the quasinormal modes calculation as shown in Table II where we see that the imaginary part of the quasinormal modes of the fermion perturbation is always negative in the case of $q = 0$.

Fig. 3 shows the behavior of the quasinormal modes with respect to the black hole tem-
temperature. We can see from Fig. 3 that in the high temperature regime, both the real part and imaginary part of the quasinormal modes have a linear relation to the temperature of the black hole, which is similar to the result given in [22], where the quasinormal modes of a neutral massless fermion are numerically calculated in a RN-AdS black hole with a spherical horizon (namely the case with $k = 1$ in (1)). The linear behavior observed for the fermionic perturbation agrees to the finding for scalar perturbation [14, 15]. By numerical fitting, we find that the quasinormal frequencies behave like

$$Re(\omega) \approx 8.4193T_H, \quad Im(\omega) \approx -6.3981T_H,$$

(35)

TABLE II: Real and imaginary parts of the quasinormal modes in the case of $q = 0$ with different temperature shown in Fig. 3

| $T_H$ | 0.0537 | 0.0733 | 0.1019 | 0.2188 | 0.3531 | 0.6432 | 1.4721 | 2.6326 | 5.9484 | 10.5904 |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Re($\omega$) | 0.8253 | 0.9856 | 1.2209 | 2.2134 | 3.3206 | 5.7491 | 12.6862 | 22.3972 | 50.1418 | 88.9839 |
| Im($\omega$) | −0.5725 | −0.6992 | −0.8804 | −1.5724 | −2.4708 | −4.3155 | −9.5919 | −16.9810 | −38.0943 | −67.6533 |
where $T_H$ is the black hole temperature given in [5]. In the high temperature regime, Ref. [22] gives a relation (see eq. (8) in [22]) for the case of Schwarzschild-AdS black hole with a spherical horizon

$$Re(\omega) = 8.367T, \quad Im(\omega) = -6.371T. \quad (36)$$

We can see from (35) and (36) that their behaviors are quite similar to each other, except the differences in the numerical factors caused by the value of the nonzero black hole charge $Q$ we considered here.

In the low temperature regime, on the other hand, one can see that both the real and imaginary part obviously deviate from the linear behavior. This result is also consistent with the one of scalar fields in [14, 15].

FIG. 3: Quasinormal modes of the first node in the case of $q = 0$. The real part versus high temperature (Up-Left); the imaginary part versus high temperature(Up-Right); the real part versus low temperature (Down-Left); the imaginary part versus low temperature (Down-Right).

VI. CONCLUDING REMARKS

In this paper we considered the Einstein-Maxwell-Fermion system with a negative cosmological constant. Such a system has been intensively studied recently in the holographic fermion liquid model [16, 17, 23, 25]. We studied the stability of the RN-AdS black hole
under the perturbation of the charged fermion by numerically calculating the quasinormal modes of the perturbation in the probe limit. It is found that when the temperature of the black hole is below a critical value, the imaginary part of the quasinormal modes will become positive, which means that the perturbation of the fermion will grow exponentially which makes the black hole unstable; while when the temperature of the black hole is above the critical value, the perturbation will decay exponentially and the black hole is stable. Our result indicates that a phase transition occurs in the system from the high temperature phase, which is described by the RN-AdS black hole, to a low temperature phase, which should be described by some stable new black hole solution. Like the case with charged scalar field, we expect that the stable new black hole should be a one with nontrivial fermion condensation.

When the coupling constant $q$ increases, the critical temperature of the black hole becomes higher. This means that the stronger the coupling is, the much easier for the phase transition to occur. In the case of $q = 0$, namely for a neutral fermion, we found that the imaginary part of the quasinormal modes is always negative. This implies that the black hole is stable under the perturbation of the uncharged fermion and it will not cause any phase transition of the black hole. The behavior of the quasinormal modes agrees with that for the scalar perturbation [14, 15] and in consistent with fermionic perturbation [22] for spherical background. In high temperature regime, both the real and imaginary parts of the quasinormal modes are linearly proportional to the temperature of the black hole. In the AdS/CFT correspondence, $1/\text{Im}(\omega)$ is the time scale for the system approaching to thermal equilibrium of the boundary field theory. This means if we perturb the thermal field on the boundary, the time for it to approach a thermal equilibrium is proportional to the inverse of the temperature [14]. However the linearity breaks in the low temperature regime.

In the probe limit, we found the marginal stable mode exists with $\omega = 0$ of the charged fermion. This strongly indicates there should exist a stable charged black hole with fermion hair in AdS space. It would be of great interest to consider the backreaction of the fermion and to find such a black hole solution in our system, which is currently under investigation.

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