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Research Article

Razumikhin-Type Theorems on Exponential Stability of SDDEs Containing Singularly Perturbed Random Processes

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This paper concerns Razumikhin-type theorems on exponential stability of stochastic differential delay equations with Markovian switching, where the modulating Markov chain involves small parameters. The smaller the parameter is, the rapid switching the system will experience. In order to reduce the complexity, we will “replace” the original systems by limit systems with a simple structure. Under Razumikhin-type conditions, we establish theorems that if the limit systems are $\mu$-th-moment exponentially stable; then, the original systems are $\mu$-th-moment exponentially stable in an appropriate sense.

1. Introduction

The stability of time delay systems is a field of intense research [1, 2]. In [2], the global uniform exponential stability independent of time delay linear and time invariant systems subjected to point and distributed delays was studied. Moreover, noise and time delay are often the sources of instability, and they may destabilize the systems if they exceed their limits [3].

Hybrid delay systems driven by continuous-time Markov chains have been used to model many practical systems in which abrupt changes may be experienced in the structure and parameters caused by phenomena such as component failures or repairs. An area of particular interest has been the automatic control of the underlying systems, with consequent emphasis on the analysis of stability of the stochastic models. For systems with time delay, there are two approaches to proving stability that correspond to the conventional Lyapunov stability theory. The first is based on Lyapunov-Krasovskii functionals, the second on Lyapunov-Razumikhin functions. The latter one originated with Razumikhin [4] for the ordinary differential delay equation which is called Razumikhin-type theorem and was developed by several people [5]. In his paper, Mao [6] was the first who established a Razumikhin-type theorem for stochastic functional differential equations (SFDEs). Roughly speaking, a Razumikhin-type theorem states that if the derivative of a Lyapunov function along trajectories is negative whenever the current value of the function dominates other values over the interval of time delay; then, the Lyapunov function along trajectories will converge to zero. The Razumikhin methods have been widely used in the study of stability for functional and differential-delay systems. In this work, we shall investigate stochastic differential delay equations with Markovian switching (SDDEwMSs). The switching we shall use will be a finite-state Markov chain, which incorporates various considerations into the models and often results in the underlying Markov chain having a large state space. To overcome the difficulties and to reduce the computational complexity, much effort has been devoted to the modeling and analysis of such systems, in which one of the main ideas is to split a large-scale system into several classes and lumping the states in each class into one state; see [7–9]. Starting from the work [10], by introducing a small parameter $\varepsilon > 0$, a number of asymptotic properties of the Markov chain $\varepsilon^2(\cdot)$ have been established. One of the main results in [9] is that a complicated system can be replaced by the corresponding limit system having a much simpler structure. In [11, 12], long-term behavior of SDEwMSs and SDDEwMSs was investigated, respectively, while in [13, 14] the stability of random
delay system with two-time-scale Markovian switching was studied. Using the stability of the limit system as a bridge, the desired asymptotic properties of the original system is obtained using perturbed Lyapunov function methods. In this work, we shall establish a Razumikhin-type theorem for SDDEs driven by pure jumps is discussed in SDDEs driven by Brownian motion. The exponential stability for all $\ell \geq 0$ where

\[
i = \phi_1(0), \quad \Phi_1 = \phi_1(0) + \xi,
\]

Let $\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Throughout the paper, we let $B(t) = (B_1(t), \ldots, B_m(t))^T$ be an $m$-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. Let $| \cdot |$ denote the Euclidean norm in $\mathbb{R}^n$ as well as the trace norm of a matrix. For $r > 0$, $C([-r, 0]; \mathbb{R}^n)$ denotes the family of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\| = \sup_{0 < t < r} \|\phi(t)\|$. Denote by $C^b_r([-r, 0]; \mathbb{R}^n)$ the family of all $\mathbb{P}$ measurable and bounded $C([-r, 0]; \mathbb{R}^n)$-valued random variable. We will denote the indicator function of a set $G$ by $\mathbb{1}_G$. Let $r(t) \geq 0$ be a right-continuous Markov chain on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$ with the generator $\mathbb{G}$ given by

\[
\mathbb{P} \{r(t + \delta) = j | r(t) = i\} = \begin{cases}
\eta_{ij} \delta + o(\delta), & \text{if } i \neq j, \\
1 + \gamma_i \delta + o(\delta), & \text{if } i = j,
\end{cases}
\]

where $\delta > 0$ and $\gamma_i$ is the transition rate from $i$ to $j$ satisfying $\gamma_{ij} > 0$ if $i \neq j$ and $\gamma_i = -\sum_{j \neq i} \gamma_{ij}$. We assume the Markov $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost every sample path $r(\cdot)$ is a right-continuous step function with finite number of simple jumps in any finite subinterval of $[0, \infty)$. As a standing assumption, we assume that the Markov chain is irreducible. This is equivalent to the condition that for any $i, j \in \mathbb{S}$, we can find $i_1, i_2, \ldots, i_k \in \mathbb{S}$ such that

\[
y_{ij} y_{i_1j}, y_{i_2j}, \ldots, y_{i_kj} > 0.
\]

Thus, $\Gamma$ always has an eigenvalue $0$. The algebraic interpretation of irreducibility is $\text{rank}(\Gamma) = N - 1$. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = 0$, subject to $\sum_{j=1}^N \pi_j = 1$ and $\pi_j > 0$ for all $j \in \mathbb{S}$. For a real valued function $\sigma(\cdot)$ defined on $\mathbb{S}$, we define

\[
\Gamma(\cdot)(\cdot) := \sum_{\ell \in \mathbb{S}} \psi_{\ell}(\cdot)\sigma(\ell),
\]

for each $\psi \in \mathbb{S}$. Consider the following stochastic delay system with Markovian switching:

\[
dx(t) = f(x(t), x(t - r), r(t)) dt + g(x(t), x(t - r), r(t)) d\mathbb{B}(t),
\]

\[
x_0 = \xi \in C([-\tau, 0]; \mathbb{R}^n), \quad r(0) \in \mathbb{S},
\]

where $f: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^n$ and $g: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^m$. To highlight the fast and slow motions, we introduce a parameter $\epsilon > 0$ and rewrite the Markov chain $r(t)$ as $r^\epsilon(t)$ and the generator $\Gamma$ as $\Gamma^\epsilon$. $\Gamma^\epsilon$ is given by

\[
\Gamma^\epsilon = \frac{1}{\epsilon} \Gamma + \tilde{\Gamma},
\]

where $\tilde{\Gamma}$ represents the fast varying motions, and $\tilde{\Gamma}$ represents the slowly changing motions. We denote $\Gamma^\epsilon = (\eta_{ij}^{(\epsilon)})_{N \times N}$, $\tilde{\Gamma} = (\tilde{\eta}_{ij})_{N \times N}$, and $\Gamma = (\eta_{ij})_{N \times N}$. To the reduction of complexity, $\tilde{\Gamma}$ needs to have a certain structure. Suppose that

\[
\mathbb{S} = \mathbb{S}^1 \cup \mathbb{S}^2 \cup \cdots \cup \mathbb{S}^l
\]

with $\mathbb{S}^i = \{s_{i1}, \ldots, s_{iN}\}$ and $N = N_1 + N_2 + \cdots + N_l$, and that

\[
\tilde{\Gamma} = \text{diag}(\tilde{\Gamma}^1, \ldots, \tilde{\Gamma}^l),
\]

where for each $k \in \{1, \ldots, l\}$ and $\tilde{\Gamma}^k$ is a generator of a Markov chain taking values in $\mathbb{S}^k$. We impose the following hypothesis:

(H1) For each $k \in \{1, \ldots, l\}$ and $\tilde{\Gamma}^k$ is irreducible. To highlight the effect of the fast switching, we rewrite the system (4) as

\[
dx^\epsilon(t) = f(x^\epsilon(t), x^\epsilon(t - r), r^\epsilon(t)) dt + g(x^\epsilon(t), x^\epsilon(t - r), r^\epsilon(t)) d\mathbb{B}(t),
\]

\[
x^0_0 = \xi \in C([-\tau, 0]; \mathbb{R}^n), \quad r^\epsilon(0) = r_0.
\]

To assure the existence and uniqueness of the solution, we give the following standard assumptions.

(H2) For any integer $R$, there is constant $h_R > 0$, such that

\[
|f(x, y, \kappa) - f(x_1, y_1, \kappa)| \leq h_R |x - x_1| + |y - y_1|,
\]

for all $\kappa \in \mathbb{S}$ and those $x, x_1, y, y_1 \in \mathbb{R}^n$ with $\|x\| \vee |x_1| \vee |y\| \vee |y_1| \leq R$.

(H3) There is an $h > 0$, such that for any $x, y \in \mathbb{R}^n$, $\kappa \in \mathbb{S}$, \[f(x, y, \kappa) \leq h (1 + |x| + |y|), \quad f(0, 0, \kappa) \equiv 0, \quad g(0, 0, \kappa) \equiv 0.\] Under the assumptions (H2) and (H3), system (8) has a unique solution denoted by $x^\epsilon(t_\ell,f(t))$ on $t \geq -\tau$, where the notation $x^\epsilon(t_\ell,f(t))$ emphasizes the dependence on the initial data.
Theorem 1. Let (H1)–(H3) hold; there is a function \( V(x, i) \in C^p(\mathbb{R}^n \times \mathcal{S}; \mathbb{R}) \) satisfying (H4), and there are positive constants \( \lambda, c_1, c_2, \) and \( q > 1 \) such that

\[
\begin{align*}
&\text{(i)} \; c_1|x|^p \leq V(x, i) \leq c_2|x|^p, \\
&\text{(ii)} \; \mathbb{E}[\max_{\xi \in \mathcal{S}^p} \mathcal{L}V(x(t), x(t-r), i)] \leq -\lambda \mathbb{E}[\max_{\xi \in \mathcal{S}^p} V(x(t), i)] + q \mathbb{E}[\max_{\xi \in \mathcal{S}^p} V(x(t), i)], \quad -r \leq \theta \leq 0,
\end{align*}
\]

where

\[
\mathcal{L}V(x, y, i) = V_x(x, i) \overline{g}(x, y, i) \overline{g}^T(x, y, i) + \frac{1}{2} \text{trace} [V_{xx}(x, i) \overline{g}(x, y, i) \overline{g}^T(x, y, i)] + \sum_{j=1}^{\ell} \overline{r}_{ij} V(x, j).
\]

Then, for all \( \xi \in C([-r, 0]; \mathbb{R}^n) \),

\[
\limsup_{t \to -\infty} \mathbb{E}[|\mathcal{c}^t(\xi)|^p] \leq \nu_2 e^{-\nu_1 t},
\]

where

\[
\nu_1 = \min \left\{ \rho \lambda, \frac{\log q}{r} \right\},
\]

\( \nu_2 \) is a fixed constant such that

\[
\nu_2 = \frac{c_2}{c_1 - r} \sup_{\xi \in \mathcal{S}^i} \mathbb{E}[|\xi|^p].
\]

Remark 2. Note that the conditions of Theorem 1 are sufficient conditions for the average system (16) \( \mathcal{X}(t) \) (or the limit process \( \mathcal{X}(t) \)). However the conclusion of Theorem 1 is about the process \( \mathcal{X}(t) \). Since the structure of the average system (16) is much simpler than that of \( \mathcal{X}(t) \), this theorem has reduced the computational complexity for the system (8).

Remark 3. lim sup_{t \to -\infty} \mathbb{E}[|\mathcal{c}^t(\xi)|^p] does exist by (11).

Proof of Theorem 1. Define

\[
\nabla (x, \xi) = \sum_{i=1}^{I} V(x, i) I_{\{\xi \in \mathcal{S}^i\}} = V(x, i), \quad \text{if } \xi \in \mathcal{S}^i.
\]

Note that

\[
\sum_{\kappa=1}^{N} \mathbb{V}(x, \kappa) = \sum_{\kappa=1}^{N} \mathbb{V}(x, i) I_{\{\xi \in \mathcal{S}^i\}} = 0.
\]
We extend $r(t)$ to $[-\tau, 0]$ by setting $r(t) = r(0)$; then, $E\overline{y}(x(t), r'(t))$ is right continuous on $t \geq -\tau$.

Let $\overline{\nu} \in (0, v_1)$ be arbitrary, and define

$$U(t) := e^{\overline{\nu} t} \lim_{\varepsilon \to 0} \sup E\overline{y}(x^\varepsilon(t), \overline{\nu} (t))$$

$$= e^{\overline{\nu} t} \lim_{\varepsilon \to 0} \sup E\overline{y}(x^\varepsilon(t), r'(t)).$$

If we can show that $U(t) \leq c_1 v_2$, then the proof is completed.

If $t \in [-\tau, 0]$, by condition (i),

$$U(t) \leq \lim_{\varepsilon \to 0} \sup E\overline{y}(x^\varepsilon(t), r'(t)) = \sup E\overline{y}(0, 0) \leq c_2 \sup E\overline{y}(x^\varepsilon(0))^p = c_1 v_2.$$  (22)

If $t \geq 0$, we will prove that $U(t) \leq c_1 v_2$. Otherwise, there exists a $\rho \in (0, \infty)$ such that all $t \in [-\tau, \rho)$, $U(t) \leq c_1 v_2$ and $U(\rho) \geq c_1 v_2$ as well as $U(\rho + \overline{\delta}) > U(\rho)$ for all sufficiently small $\overline{\delta}$.

For $t \in [\rho - \tau, \rho)$,

$$\lim_{\varepsilon \to 0} \sup E\overline{y}(x^\varepsilon(t), \overline{\nu} (t))$$

$$= e^{\overline{\nu} t} U(t)$$

$$\leq e^{\overline{\nu} t} U(\rho) = e^{\overline{\nu} t} \lim_{\varepsilon \to 0} \sup E\overline{y}(x^\varepsilon(\rho), \overline{\nu} (\rho))$$

$$\leq e^{\overline{\nu} t} \lim_{\varepsilon \to 0} \sup E\overline{y}(x^\varepsilon(\rho), \overline{\nu} (\rho)).$$

Then $\lim_{\varepsilon \to 0} E\overline{y}(x^\varepsilon(\rho), \overline{\nu} (\rho)) = 0$, and hence, we have

$$\lim_{\varepsilon \to 0} \sup E\overline{y}(x^\varepsilon(t), \overline{\nu} (t)) = 0, \quad t > 0.$$  (23)

Then $U(\rho) = 0$, which is a contradiction. Hence we see that $\lim_{\varepsilon \to 0} E\overline{y}(x^\varepsilon(\rho), \overline{\nu} (\rho)) \neq 0$. For $t \in [\rho - \tau, \rho)$, there exists a $q > 1$ such that

$$\lim_{\varepsilon \to 0} \sup E\overline{y}(x^\varepsilon(t), \overline{\nu} (t))$$

$$\leq q \lim_{\varepsilon \to 0} \sup E\overline{y}(x^\varepsilon(\rho), \overline{\nu} (\rho)), \quad \overline{\nu} < \frac{\log q}{\tau}.\quad (24)$$

Consequently, there exists a sufficiently small $\varepsilon_0 > 0$, such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$E \left[ \min_{i \in \mathbb{N}} \max_{\varepsilon \in [0,\infty)} \left( \frac{x^\varepsilon(t)}{\varepsilon}, \frac{r'(t)}{\varepsilon} \right) \right]$$

$$\leq q E \left[ \max_{i \in \mathbb{N}} \left( \frac{x^\varepsilon(t)}{\varepsilon}, \frac{r'(t)}{\varepsilon} \right) \right], \quad \theta \in [-\tau, 0].$$

By condition (ii),

$$E \left[ \max_{i \in \mathbb{N}} \overline{\nu} \overline{y}(x^\varepsilon(t), x^\varepsilon(t-\tau), \overline{\nu} (t)) \right] \leq -\lambda E \left[ \max_{i \in \mathbb{N}} \overline{y}(x^\varepsilon(t), \overline{\nu} (t)) \right];$$

then,

$$E \left[ \overline{\nu} \overline{y}(x^\varepsilon(t), x^\varepsilon(t-\tau), \overline{\nu} (t)) \right] \leq -\lambda E \left[ \overline{y}(x^\varepsilon(t), \overline{\nu} (t)) \right].$$

Noting that $\overline{\nu} < \nu \leq \lambda$, we have

$$E \left[ \overline{\nu} \overline{y}(x^\varepsilon(t), x^\varepsilon(t-\tau), \overline{\nu} (t)) \right] \leq -\lambda E \left[ \overline{y}(x^\varepsilon(t), \overline{\nu} (t)) \right].$$

We now consider

$$U(\rho + \overline{\delta}) - U(\rho)$$

$$= \lim_{\varepsilon \to 0} \sup E\overline{y}(x^\varepsilon(\rho + \overline{\delta}), \overline{\nu} (\rho + \overline{\delta}))))$$

$$- e^{\overline{\nu} t} \sup E\overline{y}(x^\varepsilon(\rho), \overline{\nu} (\rho)))$$

$$= \lim_{\varepsilon \to 0} \sup \int_{\rho}^{\rho + \overline{\delta}} e^{\overline{\nu} t} \left[ \overline{\nu} \overline{y}(x^\varepsilon(t), x^\varepsilon(t-\tau), \overline{\nu} (t)) \right. dt$$

$$\left. + \overline{\nu} \overline{y}(x^\varepsilon(t), \overline{\nu} (t)) \right]$$

$$= \lim_{\varepsilon \to 0} \sup \int_{\rho}^{\rho + \overline{\delta}} e^{\overline{\nu} t} \left[ \overline{\nu} \overline{y}(x^\varepsilon(t), x^\varepsilon(t-\tau), \overline{\nu} (t)) \right.$$

$$\left. + \overline{\nu} \overline{y}(x^\varepsilon(t), \overline{\nu} (t)) \right] dt.$$  (32)

By the definition of operator $\overline{L}$, we have

$$\overline{L}(x^\varepsilon(t), x^\varepsilon(t-\tau), r'(t))$$

$$= \overline{L}_x (x^\varepsilon(t), r'(t)) f(x^\varepsilon(t), x^\varepsilon(t-\tau), r'(t))$$

$$+ \frac{1}{2} \text{trace} \left[ \nabla_{xx} (x^\varepsilon(t), r'(t)) \right].$$
\[
\begin{align*}
&\times g(x^e(t), x^e(t-\tau), r^e(t)) \\
&\times g^T(x^e(t), x^e(t-\tau), r^e(t)) \\
&+ \sum_{k=1}^{N} \bar{Y}_{r^e(t)c} \nabla (x^e(t), \kappa) \\
&= \nabla_x (x^e(t), r^e(t)) f(x^e(t), x^e(t-\tau), r^e(t)) \\
&+ \frac{1}{2} \text{trace} \left[ \nabla_{xx} (x^e(t), r^e(t)) \\
&\times g(x^e(t), x^e(t-\tau), r^e(t)) \\
&\times g^T(x^e(t), x^e(t-\tau), r^e(t)) \right] \\
&+ \sum_{k=1}^{N} \bar{Y}_{r^e(t)c} \nabla (x^e(t), \kappa) \\
&= V_x (x^e(t), \bar{r}^e(t)) \bar{f} (x^e(t), x^e(t-\tau), \bar{r}^e(t)) \\
&+ \frac{1}{2} \text{trace} \left[ V_{xx} (x^e(t), \bar{r}^e(t)) \\
&\times g(x^e(t), x^e(t-\tau), r^e(t)) \\
&\times g^T(x^e(t), x^e(t-\tau), r^e(t)) \right] \\
&+ \sum_{j=1}^{l} \bar{Y}_{r^e(t)c} \nabla (x^e(t), j) \\
&+ V_x (x^e(t), \bar{r}^e(t)) \left[ f(x^e(t), x^e(t-\tau), r^e(t)) \\
&- \bar{f} (x^e(t), x^e(t-\tau), \bar{r}^e(t)) \right] \\
&+ \frac{1}{2} \text{trace} \left[ V_{xx} (x^e(t), \bar{r}^e(t)) \\
&\times g(x^e(t), x^e(t-\tau), r^e(t)) \\
&\times g^T(x^e(t), x^e(t-\tau), r^e(t)) \right] \\
&- \sum_{j=1}^{l} \bar{Y}_{r^e(t)c} \nabla (x^e(t), j) \\
&= \mathcal{L}V (x^e(t), x^e(t-\tau), \bar{r}^e(t)) \\
&+ V_x (x^e(t), \bar{r}^e(t)) \\
&\times \left[ f(x^e(t), x^e(t-\tau(t)), r^e(t)) - \bar{f} (x^e(t), x^e(t-\tau(t)), \bar{r}^e(t)) \right] \\
&+ \frac{1}{2} \text{trace} \left[ V_{xx} (x^e(t), \bar{r}^e(t)) \\
&\times g(x^e(t), x^e(t-\tau), r^e(t)) \\
&\times g^T(x^e(t), x^e(t-\tau), r^e(t)) \right] \\
&+ \sum_{k=1}^{N} \bar{Y}_{r^e(t)c} \nabla (x^e(t), \kappa) \\
&= \mathcal{L}V (x^e(t), x^e(t-\tau), \bar{r}^e(t)) \\
&+ \sum_{j=1}^{l} \bar{Y}_{r^e(t)c} \nabla (x^e(t), j) \\
&= \text{trace} \left[ \left( \sum_{k=1}^{N} \bar{Y}_{r^e(t)c} \nabla (x^e(t), \kappa) \right) \\
&- \sum_{j=1}^{l} \bar{Y}_{r^e(t)c} \nabla (x^e(t), j) \right] dt \\
&=: I_1 + I_2 + I_3 + I_4.
\end{align*}
\]
By the definition of $\overline{\Gamma}$,

$$f(x^e(t), x^e(t - \tau), r^e(t)) - \overline{\Gamma}(x^e(t), x^e(t - \tau), r^e(t))$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} f(x(t), x(t - \tau), s_{ij}) \times \left[ I_{[r(t) = s_{ij}]} - \mu_j I_{[\Gamma(t) = i]} \right].$$

(35)

This, together with assumption (H2), implies

$$\lim_{\varepsilon \to 0} \mathbb{E} \int_{0}^{\rho \varepsilon} e^{\overline{\Gamma} V_x(x^e(t), r^e(t))} \times [f(x^e(t), x^e(t - \tau), r^e(t)) - \overline{\Gamma}(x^e(t), x^e(t - \tau), r^e(t)) - \overline{\Gamma}(x^e(t), x^e(t - \tau), r^e(t)) ] \, dt$$

$$\leq \lim_{\varepsilon \to 0} \left[ \mathbb{E} \int_{0}^{\rho \varepsilon} e^{\overline{\Gamma} V_x(x^e(t), r^e(t))} \times [f(x^e(t), x^e(t - \tau), r^e(t)) - \overline{\Gamma}(x^e(t), x^e(t - \tau), r^e(t))] \, dt \right].$$

(36)

By the argument of Lemma 7.14 in [9], we have

$$\sum_{k=1}^{N} \overline{\Gamma}_{r(t) \neq \kappa} V(x^e(t), \kappa) = \overline{\Gamma} V(x^e(t), \cdot) (r^e(t)),$$

$$\sum_{j=1}^{l} \overline{\Gamma}_{X(t), j} V(x^e(t), j) = \overline{\Gamma} V(x^e(t), j) (r^e(t)),$$

hence

$$I_4 = \lim_{\varepsilon \to 0} \mathbb{E} \int_{0}^{\rho \varepsilon} e^{\overline{\Gamma} V(x^e(t), \cdot) (r^e(t))} \times \overline{\Gamma} V(x^e(t), \cdot) (r^e(t)) \, dt$$

$$= \lim_{\varepsilon \to 0} \left[ \mathbb{E} \int_{0}^{\rho \varepsilon} e^{\overline{\Gamma} V(x^e(t), \cdot) (r^e(t))} \times \overline{\Gamma} V(x^e(t), \cdot) (r^e(t)) \, dt \right].$$

(37)

By assumption (H4) and the argument of Lemma 7.14 in [9], we have the right side of above inequality is equivalent to 0, that is, $I_4 = 0$.

Therefore by the condition (ii)

$$U(\rho + \varepsilon) - U(\rho)$$

$$= \lim_{\varepsilon \to 0} \mathbb{E} \int_{0}^{\rho \varepsilon} e^{\overline{\Gamma} V(x^e(t), x^e(t - \tau), r^e(t)) + \overline{\Gamma} V(x^e(t), r^e(t))} \, dt \leq 0;$$

this is

$$U(\rho + \varepsilon) \leq U(\rho).$$

(41)
This contradicts the definition of $\rho$. The proof is now completed. \hfill \Box

Example 4. Let $r^*(\cdot)$ be a Markov chain generated by $\Gamma^r$ given in (5) with
\[
\Gamma = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\]
\[
\Gamma = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix},
\]
(42)

The generator $\Gamma$ consists of two irreducible blocks. The stationary distributions are $\mu^1 = (0.5, 0.5), \mu^2 = (1/7, 2/7, 4/7)$, and
\[
\Gamma = \begin{pmatrix} -1 & 1 \\ 6 & 6 \\ 7 & -7 \end{pmatrix}.
\]
(43)

Consider a one-dimensional equation
\[
dx^*(t) = f(x^*(t), r^*(t)) dt + g(x^*(t), r^*(t)) dw(t)
\]
(44) with
\[
f(x, s_{11}) = \frac{x}{8}, \quad f(x, s_{12}) = \frac{x}{8},
\]
\[
g(x, s_{11}) = \frac{x \cos x}{8 \sqrt{2}}, \quad g(x, s_{12}) = \frac{x \sin x}{8 \sqrt{2}},
\]
\[
f(x, s_{21}) = -28 (x + \sin x),
\]
\[
f(x, s_{22}) = 7x + 14 \sin x,
\]
\[
f(x, s_{23}) = -\frac{7}{4} x,
\]
\[
g(x, s_{21}) = \frac{\sqrt{7}}{4} \sin x,
\]
\[
g(x, s_{22}) = -\frac{\sqrt{7}}{4} \cos x,
\]
\[
g(x, s_{23}) = \frac{\sqrt{7}}{4}.
\]
Then the limit equation is
\[
d\bar{x}^*(t) = f(\bar{x}(t), \bar{r}(t)) dt + g(\bar{x}(t), \bar{r}(t)) dw(t)
\]
(46) where $\bar{r}$ is the Markov chain generated by $\bar{\Gamma}$ and
\[
\bar{\Gamma}(x, 1) = \frac{x}{8}, \quad \bar{\Gamma}(x, 2) = -3x,
\]
\[
\bar{\Gamma}(x, 1) = \frac{x}{16}, \quad \bar{\Gamma}(x, 2) = \frac{x}{4}.
\]
(47)

Let $V(x, 1) = 2x^2, V(x, 2) = x^2$; then,
\[
\mathcal{L}V(x, y, 1) \leq -\frac{1}{2} |x|^2 + \frac{|y|^2}{128},
\]
\[
\mathcal{L}V(x, y, 2) \leq -\frac{36}{7} |x|^2 + \frac{|y|^2}{16}.
\]
(48)

Consequently
\[
\max_{i=1,2} \mathcal{L}V(x, y, i) \leq -\frac{1}{2} |x|^2 + \frac{1}{16} |y|^2
\]
(49)
\[
= -\frac{1}{4} \left[ \max_{i=1,2} \mathcal{L}V(x, i) \right] + \frac{1}{16} \left[ \min_{i=1,2} \mathcal{L}V(y, i) \right].
\]
(50)

It is easy to see that we can find a $q > 1$ such that $(1/4) - (q/16) > 0$. Therefore, for any $\phi \in L^2_{\mathbb{P}}([-\tau, 0]; \mathbb{R}^n)$ satisfying $E[\min_{t \in \mathbb{R}} \phi(t)] \leq q E[\max_{t \in \mathbb{R}} \phi(0)]$ on $-\tau \leq t \leq 0$, (49) yields
\[
E\left[ \max_{i=1,2} \mathcal{L}V(x, y, i) \right] \leq -\left( \frac{1}{4} - \frac{q}{16} \right) E\left[ \max_{i=1,2} \mathcal{L}V(x, i) \right].
\]
(51)

Hence, by Theorem 1, the solution $x^*(t)$ is mean square stable when $\epsilon$ is sufficient small.

4. Stochastic Delay System with Pure Jumps

In this section we discuss the stability of the following stochastic delay system with pure jumps:
\[
dx^*(t) = f(x^*(t), x^*(t-\tau), r^*(t)) dt
\]
\[
+ \int_{\mathbb{R}^m} b(x^*(t-\tau), x^*(t-\tau), r^*(t), z) \tilde{N}(dt, dz),
\]
\[
x_0 = \xi \in C([-\tau, 0], \mathbb{R}^m), \quad r(0) \in \mathbb{S},
\]
(52)

where $x^*(t-\tau) = \lim_{\tau \to 0} x^*(s)$, $b : \mathbb{R}^n \times \mathbb{N} \times \mathbb{S} \to \mathbb{R}^{n \times m}$. We assume that the each column $b^{(k)}$ of the $n \times m$ matrix $b = [b_{ij}]$ depends on $z$ only through the $i$th coordinate $z_i$, that is,
\[
b^{(k)}(x, y, k, z) = b^{(k)}(x, y, k, z_k);
\]
\[
z = (z_1, \ldots, z_m) \in \mathbb{R}^m, \quad k \in \mathbb{S},
\]
(53)

$N(t, z)$ is a $m$-dimensional Poisson process, and the compensated Poisson, process is defined by
\[
\tilde{N}(dt, dz) = \left( \tilde{N}_1(dt, dz_1), \ldots, \tilde{N}_d(dt, dz_m) \right)
\]
\[
= (N_1(dt, dz_1) - \lambda_1(dz_1)) dt, \ldots,
\]
\[
N_m(dt, dz_m) - \lambda_m(dz_m) dt,
\]
(54)

where $[N_{j}, j = 1, \ldots, m]$ are independent one-dimensional Poisson random measures with characteristic measure $[\lambda_j, j = 1, \ldots, m]$ coming from $m$ independent one-dimensional Poisson point processes.

The averaged system of (18) is defined as follows:
\[
dx = \bar{f}(\bar{x}(t), \bar{x}(t-\tau), \bar{r}(t)) dt
\]
\[
+ \int_{\mathbb{R}^m} \bar{b}(\bar{x}(t-\tau), \bar{x}(t-\tau), \bar{r}(t), z) \tilde{N}(dt, dz),
\]
\[
\bar{x}_0 = \xi \in C([-\tau, 0], \mathbb{R}^n), \quad \bar{r}(0) \in \mathbb{S},
\]
(55)
where \( \bar{x}(t) = \lim_{\tau \to 0} x(s), \bar{b} : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^{m \times n} \). Similar to the definition of \( \bar{f} \), we define
\[
\bar{b}(x, y, i, z) = \sum_{j=1}^{N} \mu^{i}_j b(x, y, s_{ij}, z).
\] (56)
For each \( s_{ij} \in S \) with \( i \in \{1, \ldots, l\} \) and \( j \in \{1, \ldots, N \} \).

To assure the existence and uniqueness of the solution of (52), we also give the following standard assumptions.

(H2') For any integer \( R \), there is a constant \( h_R > 0 \), such that
\[
|f(x, y, i) - f(x, y, i)| + \sum_{k=1}^{m} \int_{\mathbb{R}} [b^{(k)}(x, y, \kappa, z_k) - b^{(k)}(x, y, \kappa, z_k)] \lambda_k(dz_k)
\leq h_R(|x_2 - x_1| + |y_2 - y_1|)
\] (57)
for all \( i \in S \) and those \( x_1, x_2, y_1, y_2 \in \mathbb{R}^n \) with \( |x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq R \).

(H3') There is an \( h > 0 \), such that for any \( x, y \in \mathbb{R}^n \), \( i \in S \),
\[
|f(x, y, i)| + \sum_{k=1}^{m} \int_{\mathbb{R}} [b^{(k)}(x, y, \kappa, z_k)] \lambda_k(dz_k)
\leq h(1 + |x| + |y|), \ f(0, 0, 0) \equiv 0, \ b(0, 0, \kappa, z) \equiv 0.
\] (58)
Given \( V \in C^p(\mathbb{R}^n \times S, \mathbb{R}_+ \) \), we define the operator \( LV \) by
\[
LV(x, y, i) = V_x(x, i) f(x, y, i) + \sum_{j=1}^{N} y_j V(x, j)
+ \sum_{k=1}^{m} \int_{\mathbb{R}} \{V(x + b^{(k)}(x, y, \kappa, z_k), x) - V(x, i)
+ V_x(x, i) b^{(k)}(x, y, \kappa, z_k) \} \lambda_k(dz_k),
\] (59)
where
\[
V_x(x, i) = \left( \frac{\partial V(x, i)}{\partial x_1}, \ldots, \frac{\partial V(x, i)}{\partial x_m} \right).
\] (60)
We need the following lemma, for details see [16].

**Lemma 5.** Let (H1) and (H2'), (H3') hold, as \( \varepsilon \to 0 \); then, \( (x(\cdot), \bar{f}(\cdot)) \) converges weakly to \( (\bar{x}(\cdot), \bar{f}(\cdot)) \) in \( D([0, \infty), \mathbb{R}^n \times S) \), where \( D([0, \infty), \mathbb{R}^n \times S) \) is the space of functions defined on \([0, \infty)\) that are right continuous and have left limits taking values in \( \mathbb{R}^n \times S \) and endowed with the Skorohod topology.

We now state our main result in this section.

**Theorem 6.** Let (H1) and (H2'), (H3') hold; there is a function \( V(x, i) \in C^p(\mathbb{R}^n \times S, \mathbb{R}_+) \) satisfying (H4), and there are positive constants \( \lambda, c_1, c_2 \) and \( q > 1 \) such that

(i) \( c_1|x|^p \leq V(x, i) \leq c_2|x|^p \),
(ii) \( \mathbb{E}[\max_{i \in S} \mathbb{E}[V(x(t), x(t-\tau), i)]] \leq -\lambda \mathbb{E}[\max_{i \in S} \mathbb{E}[V(x(t), i)]] \) provided \( \mathbb{E}[\min_{i \in S} \mathbb{E}[V(x(t+\theta, i)] - \mathbb{E}[\max_{i \in S} \mathbb{E}[V(x(t), i)] \), \( -\tau < \theta \leq 0 \).

Then, for all \( \xi \in C([-\tau, 0]; \mathbb{R}^n) \),
\[
\limsup_{\varepsilon \to 0} \mathbb{E}[|x^\varepsilon(t)|^p] \leq v_4 e^{-\gamma t},
\] (61)
where
\[
v_4 = \frac{c_2}{c_1 - \varepsilon \log \frac{q}{\tau}} \text{ and } v_4 \text{ is a fixed constant such that } v_4 \leq \frac{c_2}{c_1 - \varepsilon \log \frac{q}{\tau}}.
\] (62)

**Proof.** As the proof of Theorem 1, define
\[
\bar{V}(x, \xi) = \int_{-\tau}^{0} \mathbb{E}[V(x, i) l_{i \in S} \mathbb{E}[V(x, i)]] = V(x, i) \text{ if } \xi \in S.
\] (63)
We extend \( r(t) \) to \([-\tau, 0] \) by setting \( r(t) = r(0) \). Then, \( \mathbb{E}[V(x(t), r(t))] \) is right continuous on \( t \geq -\tau \).

Let \( \bar{V} \in \mathbb{V} \), arbitrary, and define
\[
U(t) := e^{-\gamma t} \limsup_{\varepsilon \to 0} \mathbb{E}[V(x^\varepsilon(t), \bar{f}^\varepsilon(t))]
= e^{-\gamma t} \limsup_{\varepsilon \to 0} \mathbb{E}[V(x^\varepsilon(t), r^\varepsilon(t))].
\] (64)
If we can show that \( U(t) \leq c_1 v_4 \), then the proof is completed.

If \( t \in [-\tau, 0] \), by condition (i), is the same as the proof of Theorem 1, we have \( U(t) \leq c_1 v_4 \).

In the following we shall prove that \( U(t) \leq c_1 v_4 \) if \( t \geq 0 \). Otherwise, there exists the smallest \( p \in (0, \infty) \) such that all \( t \in [-\tau, \rho) \), \( U(t) \leq c_1 v_4 \), and \( U(p) \geq c_1 v_4 \) as well as \( U(p + \delta) > U(p) \) for all sufficiently small \( \delta \).

As the same in the proof of Theorem 1 we can have that
\[
\lim_{\varepsilon \to 0} \mathbb{E}[V(x^\varepsilon(t), \bar{f}^\varepsilon(t))] \neq 0. \text{ Hence for } t \in [p - \tau, \rho), \text{ there exists a } q \text{ such that }
\]
\[
\limsup_{\varepsilon \to 0} \mathbb{E}[V(x^\varepsilon(t), \bar{f}^\varepsilon(t))]
< q \limsup_{\varepsilon \to 0} \mathbb{E}[V(x^\varepsilon(t), \bar{f}^\varepsilon(t))], \bar{V} < \frac{\log q}{\tau}.
\] (65)
Consequently, there exists a sufficiently small \( \varepsilon_0 > 0 \), such that for any \( \varepsilon \in (0, \varepsilon_0) \),
\[
\mathbb{E}[\min_{i \in S} V(x^\varepsilon(t+\theta), i)] \leq q \mathbb{E}[\max_{i \in S} V(x^\varepsilon(t), i)] \text{ for } \theta \in [-\tau, 0).
\] (66)

By condition (ii),
\[
\mathbb{E}[\max_{i \in S} V(x^\varepsilon(t), x^\varepsilon(t-\tau), i)] \leq -\lambda \mathbb{E}[\max_{i \in S} V(x^\varepsilon(t), i)],
\] (67)
we then have for $\bar{v} < v \leq \lambda$,
\[
E \left[ LV (x^\varepsilon (t), x^\varepsilon (t - \tau), \bar{r}^\varepsilon (t)) \right] \leq -\mathbb{V} E \left[ V (x^\varepsilon (t), \bar{r} (t)) \right].
\] (68)

We now consider
\[
U (\rho + \bar{d}) - U (\rho) = \lim_{\varepsilon \to 0} \left[ e^{\varepsilon (\rho + \bar{d})} E \left[ V (x^\varepsilon (\rho + \bar{d}), \bar{r} (\rho + \bar{d})) \right] - e^{\varepsilon \rho} E \left[ V \left( x^\varepsilon (\rho), \bar{r} (\rho) \right) \right] \right]
\]
\[
= \lim_{\varepsilon \to 0} \sup \int_{\rho}^{\rho + \bar{d}} e^{\varepsilon \rho} \left[ LV (x^\varepsilon (t), x^\varepsilon (t - \tau), \bar{r}^\varepsilon (t)) \right] dt.
\] (69)
\[
= \lim_{\varepsilon \to 0} \sup \int_{\rho}^{\rho + \bar{d}} e^{\varepsilon \rho} \left[ LV (x^\varepsilon (t), x^\varepsilon (t - \tau), \bar{r}^\varepsilon (t)) \right] dt.
\]

This implies
\[
U (\rho + \bar{d}) - U (\rho) = \lim_{\varepsilon \to 0} \sup \int_{\rho}^{\rho + \bar{d}} e^{\varepsilon \rho} \left[ LV (x^\varepsilon (t), x^\varepsilon (t - \tau), \bar{r}^\varepsilon (t)) \right] dt.
\] (70)

By the definition of the operator $L$ similar to that of the proof of Theorem 1, we have
\[
L V (x^\varepsilon (t), x^\varepsilon (t - \tau), r^\varepsilon (t)) = \nabla_x (x^\varepsilon (t), r^\varepsilon (t)) f (x^\varepsilon (t), x^\varepsilon (t - \tau), r^\varepsilon (t))
\]
\[
+ \sum_{k=1}^{m} \int_{\mathbb{R}} \left\{ V \left( x^\varepsilon (t) + b^{(k)} (x^\varepsilon (t - \tau), x^\varepsilon ((t - \tau) - r^\varepsilon (t), z_k) \right)
\]
\[
- V \left( x^\varepsilon (t), r^\varepsilon (t) \right) - \nabla_x \left( x^\varepsilon (t), r^\varepsilon (t) \right) b^{(k)}
\]
\[
\times \left( x^\varepsilon (t - \tau), x^\varepsilon ((t - \tau) - r^\varepsilon (t), z_k) \right) \right\} \lambda_k (dz_k)
\]
\[
+ \sum_{j=1}^{N} \bar{V} \left( x^\varepsilon (t), j \right)
\]
\[
= L V (x^\varepsilon (t), x^\varepsilon (t - \tau), \bar{r}^\varepsilon (t))
\]
\[
+ V_\varepsilon \left( x^\varepsilon (t), \bar{r}^\varepsilon (t) \right)
\]
\[
\times \left[ f (x^\varepsilon (t), x^\varepsilon (t - \tau), r^\varepsilon (t)) - \bar{f} (x^\varepsilon (t), x^\varepsilon (t - \tau), \bar{r}^\varepsilon (t)) \right]
\]
\[
+ \sum_{k=1}^{m} \int_{\mathbb{R}} \left\{ V \left( x^\varepsilon (t) + b^{(k)} \right)
\]
\[
\times \left( x^\varepsilon (t - \tau), x^\varepsilon ((t - \tau) - r^\varepsilon (t), z_k) \right) \right\} \lambda_k (dz_k)
\]
\[
- \lim_{\varepsilon \to 0} \sup \int_{\rho}^{\rho + \bar{d}} e^{\varepsilon \rho} \left[ LV (x^\varepsilon (t), x^\varepsilon (t - \tau), r^\varepsilon (t)) \right] dt.
\]
\[ \sum_{j=1}^{l_k} b^{(k)}(x^x(t), z_j) \]

By assumption (H2'),

\[ b^{(k)}(x^x(t), x^x((t-\tau)-), r^x(t), z_k) \]
\[ -\tilde{b}^{(k)}(x^x(t)-, x^x((t-\tau)-), \tilde{r}^x(t), z_k) \]
\[ = \sum_{j=1}^{l_k} b^{(k)}(x^x(t)-, x^x((t-\tau)-), s_{ij}, z_k) \]
\[ \times I_{t^* (t)=s_{ij}} - \mu_j I_{r^x (t)=0} \]

By the definition of \( \tilde{b}^{(k)} \),

\[ b^{(k)}(x^x(t), x^x((t-\tau)-), r^x(t), z_k) \]
\[ = \sum_{j=1}^{l_k} b^{(k)}(x^x(t)-, x^x((t-\tau)-), s_{ij}, z_k) \]
\[ \times I_{t^* (t)=s_{ij}} - \mu_j I_{r^x (t)=0} \]

By assumption (H2'), we have

\[ J_4 = \lim_{\varepsilon \to 0} \sum_{k=1}^{m} \sum_{i=1}^{l_k} \int_{\rho}^{\rho+\delta} e^{\eta} V_x(x^x(t), r^x(t)) \]
\[ \times \int_{R} b^{(k)}(x^x(t)-, x^x((t-\tau)-), r^x(t), z_k) \]
\[ \times \frac{1}{2} \lambda_k(dz_k) dt \]

\[ \leq \lim_{\varepsilon \to 0} \sum_{k=1}^{m} \sum_{i=1}^{l_k} \int_{\rho}^{\rho+\delta} e^{\eta} V_x(x^x(t), r^x(t)) \]
\[ \times \int_{R} b^{(k)}(x^x(t)-, x^x((t-\tau)-), r^x(t), z_k) \]
\[ \times \frac{1}{2} \lambda_k(dz_k) dt \]

By the argument of Lemma 7.14 in [9], the right side of the inequality above is equivalent to 0, that is, \( I_4 = 0 \). Similarly, by mean-value theorem, we can show that there exists \( \eta^{(k)}(t) \) which is between \( x^x(t)+b^{(k)}(x^x(t)-), x^x((t-\tau)-), r^x(t), z_k \) and \( x^x(t)+\tilde{b}^{(k)}(x^x(t)-, x^x((t-\tau)-), r^x(t), z_k) \) such that

\[ J_5 = \lim_{\varepsilon \to 0} \sum_{k=1}^{m} \sum_{i=1}^{l_k} \int_{\rho}^{\rho+\delta} e^{\eta} V_x(x^x(t), r^x(t)) \]
\[ \times \int_{R} b^{(k)}(x^x(t)-, x^x((t-\tau)-), r^x(t), z_k) \]
\[ \times \frac{1}{2} \lambda_k(dz_k) dt \]

\[ \leq \lim_{\varepsilon \to 0} \sum_{k=1}^{m} \sum_{i=1}^{l_k} \int_{\rho}^{\rho+\delta} e^{\eta} V_x(x^x(t), r^x(t)) \]
\[ \times \int_{R} b^{(k)}(x^x(t)-, x^x((t-\tau)-), r^x(t), z_k) \]
\[ \times \frac{1}{2} \lambda_k(dz_k) dt \]

\[ \leq \lim_{\varepsilon \to 0} \sum_{k=1}^{m} \sum_{i=1}^{l_k} \int_{\rho}^{\rho+\delta} e^{\eta} V_x(x^x(t), 0) \]
\[ \times \int_{R} b^{(k)}(x^x(t)-, x^x((t-\tau)-), r^x(t), z_k) \]
\[ \times \frac{1}{2} \lambda_k(dz_k) dt \]
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\[ x^* (t - \tau - \cdot), s_{ij}, z_k \]
\[ \times \left[ I_{[\tau^*(\cdot) = s_{ij}]} - \mu \int_{[\tau^*(\cdot) = i}] \right] \]
\[ \times \lambda_k (dz_k) \ln \left[ \frac{1}{2} \right]. \]

(74)

By the argument of Lemma 7.14 in [9], we have \( J_3 = 0 \). Similar to the proof of Theorem 1, we can derive \( J_2 = 0, J_5 = 0 \). Therefore we arrive at

\[ U (\rho + \delta) - U (\rho) \]
\[ = \lim_{\epsilon \to 0} \mathbb{E} \int_{[\rho]} e^{\epsilon \mathbb{L} V (x^*(t), x^*(t - \tau), \bar{x}^*(t))}
\[ + \mathbb{E} \int_{[\rho]} e^{\epsilon \mathbb{L} V (x^*(t), \bar{x}^*(t))} dt \leq 0; \]

then,

\[ U (\rho + \delta) \leq U (\rho). \]

(76)

This contradicts the definition of \( \rho \). The proof is therefore completed. \( \square \)

We shall give an example to illustrate our theory:

Example 7. Let \( r^*(\cdot) \) be a Markov chain generated by

\[ T^* = \frac{1}{\varepsilon} I^* + \frac{1}{\varepsilon} \left( \begin{array}{ccc}
-1 & 0 & 0 \\
1 & -1 & 0 \\
0 & 1 & 1 \\
\end{array} \right) \]

(77)

here we set \( \bar{\Gamma} = 0 \). The stationary distribution is \( \mu = (4/19, 8/19, 3/19, 4/19) \). Consider a one-dimensional equation

\[ dx^* (t) = f (x^*(t), r^*(t)) dt 
\[ + \int_0^{\infty} \sigma (r^*(t), z) x^* ((t - \tau) - \cdot) \bar{N} (dt, dz) \]

(78)

with

\[ f (x, 1) = 2 \sin x, \quad f (x, 2) = -\frac{19}{8} x, \]
\[ f (x, 3) = -\frac{19}{6} x, \quad f (x, 4) = -2 \sin x. \]

(79)

Let

\[ \beta (z) = \frac{4}{19} \sigma (1, z) + \frac{8}{19} \sigma (2, z) + \frac{3}{19} \sigma (3, z) + \frac{4}{19} \sigma (4, z), \]
\[ \int_0^{\infty} \beta^2 (z) \lambda (dz) < 2. \]

(80)

Then the limit equation is

\[ d\bar{x} (t) = -\frac{3}{2} \bar{x} (t) dt + \int_0^{\infty} \beta (z) \bar{x} ((t - \tau) - \cdot) \bar{N} (dt, dz). \]

(81)

Let \( V (x) = x^2; \) then,

\[ \| V (x, y) \| \leq 3 |x|^2 + \int_0^{\infty} \beta^2 (z) \lambda (dz) |y|^2. \]

(82)

We can find a \( q > 1 \) such that \( 3 - 2q > 0 \). Therefore, for any \( \phi \in L^2 ([-\tau, 0]; \mathbb{R}^n) \) satisfying \( \mathbb{E} \left[ \min_{\theta \in \mathbb{R}} \phi (\theta) \right] \leq q \mathbb{E} \left[ \max_{\theta \in \mathbb{R}} \phi (0) \right] \) on \(-\tau \leq \theta \leq 0, (49)\) yields

\[ \mathbb{E} \left[ \max_{\theta \in \mathbb{R}} \phi ((x, y, i) \right] \leq (3 - 2q) \mathbb{E} \left[ \max_{\theta \in \mathbb{R}} V (x, i) \right]. \]

(83)

Hence, by Theorem 6, the solution \( x^*(t) \) is mean square stable.

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