Extremality, left-modularity and semidistributivity

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Abstract. In this article we study the relations between three classes of lattices each extending the class of distributive lattices in a different way. In particular, we consider join-semidistributive, join-extremal and left-modular lattices, respectively. Our main motivation is a recent result by Thomas and Williams proving that every semidistributive, extremal lattice is left modular. We prove the converse of this on a slightly more general level. Our main result asserts that every join-semidistributive, left-modular lattice is join extremal. We also relate these properties to the topological notion of lexicographic shellability.

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1. Introduction

One of the most fundamental (and at the same time most important) classes of lattices is the class of distributive lattices. These lattices are characterized by satisfying the well-known distributive laws with respect to meet and join operations. More precisely, a lattice is distributive if for any choice of elements $a, b, c$ it holds that

\[(a \lor b) \land (a \lor c) = a \lor (b \land c),\]  
\[(a \land b) \lor (a \land c) = a \land (b \lor c).\]  

A celebrated result of Birkhoff’s states that a finite lattice is distributive if and only if it is isomorphic to the lattice of order ideals of some (underlying) finite partially ordered set (or poset) [1].

From this perspective it is straightforward to describe the covering pairs in a distributive lattice. In particular, an order ideal $a$ is covered by some
other order ideal \( b \) if and only if \( a \) is obtained by removing a maximal element from \( b \). We can therefore realize \( a \) as the join of the order ideals generated by the maximal elements of \( a \). This representation is canonical in the sense that it is minimal in size and contains elements as close to the bottom as possible. Of course, this idea of representing lattice elements in terms of as few as possible elements which are as far down as possible, may be considered for arbitrary lattices. Such canonical join representations play an important role in the solution of the word problem for free lattices [16,17]. There is even a nice characterization of the finite lattices in which every element admits such a canonical join representation. These are precisely the finite join semidistributive lattices, i.e. lattices where every three elements \( a, b, c \) satisfy the following implication:

\[
\text{if } a \lor b = a \lor c \text{ then } a \lor b = a \lor (b \land c). \tag{1.3}
\]

See [6, Chapter II.5] for more background.

The order ideal-representation of distributive lattices exhibits yet another intriguing property. By the reasoning from the previous paragraph, we may conclude that the order ideals generated by single elements of the underlying poset are join irreducible, i.e. they cannot be expressed as a nontrivial join of other elements. In fact, all join-irreducible order ideals are of this form. Thus, the number of join-irreducible elements in a distributive lattice agrees with the size of the underlying poset. In a related fashion, we can use any linear extension of the underlying poset to construct a maximal chain in a distributive lattice. Since for distributive lattices, all maximal chains have the same size, this implies that the maximum size of a maximal chain equals the number of join-irreducible elements. Following [10], finite lattices with this property are called join extremal.

Another remarkable property of distributive lattices is the fact that any three elements \( a, b, c \), where \( b < c \), satisfy the modular equality:

\[
(b \lor a) \land c = b \lor (a \land c). \tag{1.4}
\]

The element \( a \) is then called left modular. Drawing inspiration from group theory, Stanley introduced the class of supersolvable lattices [12]. These are graded lattices which possess a maximal chain consisting entirely of left-modular elements. In particular, every distributive lattice is supersolvable. Several researchers have subsequently studied non-graded lattices which have a maximal chain of left-modular elements, see for instance [9,11,13]. Such lattices are themselves called left modular and have various intriguing combinatorial and topological properties.

The three properties of distributive lattices that we have just reviewed are illustrated in Figure 1. The purpose of this article is the study of the interactions between the three induced lattice classes: join-semidistributive lattices, join-extremal lattices and left-modular lattices. As described above, each of these families contains the class of distributive lattices, but none of them is contained in another. The main motivation for the research presented here is the recent result which states that every semidistributive extremal
lattice is left modular [15, Theorem 1.4]. Our main result is the converse of this claim on the level of join-semidistributive lattices: in Theorem 3.2 we prove that every join-semidistributive, left modular lattice is join-extremal. Moreover, we consider the interaction of these properties with the topological property of lexicographic shellability. We end with an open question asking whether any semidistributive, lexicographically shellable lattice is necessarily left modular (Question 4.4).

An overview of the relations between various combinations of the considered properties is given in Figure 2. In this figure, the gray boxes denote the classes of lattices having the properties stated in the corresponding row and column headers. For instance, the green arrow from second box in the bottom row to the third box in the bottom row represents the statement:

“Every semidistributive, join-extremal lattice is left modular.”

which is true by virtue of Theorem 3.1 and Lemma 3.3.
2. Lattice-theoretic preliminaries

In this note we consider only finite lattices, and refer the interested reader to [5, 6, 7] for further background information. Moreover, we usually view lattices from an order-theoretic perspective. More precisely, if $L = (L; \lor, \land, \hat{0}, \hat{1})$ is a (finite) lattice given as an algebraic structure with signature $(2, 2, 0, 0)$, then we may consider this as a partially ordered set by ordering its elements via $a \leq b$ if and only if $a \lor b = b$ (or equivalently $a \land b = a$). In this case, we write $L = (L, \leq)$ instead.

The dual lattice is $L^d \overset{def}{=} (L, \geq)$. For an integer $n \geq 1$, we define $[n] \overset{def}{=} \{1, 2, \ldots, n\}$.

2.1. Covering pairs, perspectivity and join-irreducibles

Two elements $a, b \in L$ form a covering pair if $a < b$ and there does not exist $c \in L$ such that $a < c < b$. In that event we write $a \lessdot b$. Then, $a$ is covered by $b$ and $b$ covers $a$. The cover relation of $L$ is the set of covering pairs, defined by

$$\text{Cov}(L) \overset{def}{=} \{(a, b) : a \lessdot b\} \subseteq L \times L.$$ 

Two covering pairs $(a_1, b_1)$ and $(a_2, b_2)$ are perspectivity if either $b_1 \lor a_2 = b_2$ and $b_1 \land a_2 = a_1$ or $a_1 \lor b_2 = b_1$ and $a_1 \land b_2 = a_2$. In that case we write $(a_1, b_1) \sim (a_2, b_2)$. This is illustrated in Figure 3.

For $a, b \in L$ with $a \leq b$, we define the associated interval by

$$[a, b] \overset{def}{=} \{c \in L : a \leq c \leq b\}.$$ 

A subset $C \subseteq L$ is a chain if it can be written as $C = \{c_0, c_1, \ldots, c_k\}$ such that $c_0 < c_1 < \cdots < c_k$. The length of such a chain is $k$. A chain is maximal if
For every Lemma 2.1. \[ \text{len} \] by Vol. 84 (2023) Extremality, left-modularity and semidistributivity Page 5 of 12

Since \( L \) is finite, there must exist \( j \) such that \( \text{len}(L) = \hat{0}, \hat{1} \), and \( \{0, 1\} \) is the map \( J(L) \rightarrow [k] \), denoted by \( \gamma(L) \) is the maximum length of a maximal chain of \( L \).

An element \( j \in L \setminus \{\hat{0}\} \) is join irreducible if \( j = a \lor b \) implies \( j \in \{a, b\} \). Since \( L \) is finite, \( j \in L \) is join irreducible if and only if \( j \) covers a unique element, denoted by \( j_* \). We denote the set of all join-irreducible elements of \( L \) by \( J(L) \).

Lemma 2.1. For every \((a, b) \in \text{Cov}(L)\), there exists \( j \in J(L) \) such that \((a, b) \subseteq (j_*, j)\).

Proof. We proceed by induction on \( \text{len}(L) \). If \( \text{len}(L) = 1 \), then \( L = \{\hat{0}, \hat{1}\} \), \( \text{Cov}(L) = \{(0, 1)\} \) and \( J(L) = \{\hat{1}\} \). The claim then holds trivially, because \((\hat{0}, \hat{1}) \subseteq (\hat{0}, \hat{1})\).

Now suppose that \( \text{len}(L) > 1 \) and that the claim holds for all lattices of length strictly smaller than \( \text{len}(L) \). Pick \((a, b) \in \text{Cov}(L)\).

(i) If \( b \neq \hat{1} \), then by induction we can find \( j \in J([0, b]) \) such that \((a, b) \subseteq (j_*, j)\), because \( \text{len}([0, b]) < \text{len}(L) \). But then, \( j \in J(L) \) because \([0, b]\) is an interval of \( L \) and it holds \((a, b) \subseteq (j_*, j)\) in \( L \) as well.

(ii) If \( b = \hat{1} \), then there are two options. Either \( \hat{1} \in J(L) \) or not. In the first case, we have \( a = \hat{1} \) and the claim holds trivially, because \((a, b) \subseteq (\hat{1}, \hat{1})\). Otherwise, there exists \( c \in L \) such that \( c \neq a \) and \( c < \hat{1} \). Let \( z = a \land c \). Since \( L \) is finite, there must exist \( d \in L \) such that \( z < d \leq c < b \) and \( d \leq a \). Since \( \text{len}([0, d]) < \text{len}(L) \), by induction we can find \( j \in J([0, d]) \) such that \((z, d) \subseteq (j_*, j)\). But then \( j \in J(L) \), because \([0, d]\) is an interval of \( L \) and it holds \((z, d) \subseteq (j_*, j)\) in \( L \) as well. By definition, we have \( j \leq d \leq c < b \). If \( j \leq a \), then we have \( j \leq a \land c = z \) which contradicts \( j \lor z = d \). Thus, \( j \not\leq a \) and we conclude \( j \lor a = b \). Moreover, \( j \land a \leq j_* \), because \( j \not\leq a \) and \( j \in J(L) \), and \( j_* = j \land z \leq j \land a \), because \( z \leq a \). It follows that \( j_* = j \land a \), and we conclude \((a, b) \subseteq (j_*, j)\). \( \square \)

Inspired by this property, we consider for any maximal chain \( C = \{c_0, c_1, \ldots, c_k\} \) the map

\[ \gamma_C: J(L) \rightarrow [k], \quad j \mapsto \min\{s: j \leq c_s\}. \]  \hspace{1cm} (2.1)
Note that it is not necessarily the case that every join-irreducible element is *perspective* to some covering pair in $C$ other than the one induced by itself; see for instance Figure 3. As a corollary, we obtain the following well-known relation between the length of $L$ and the number of join-irreducibles.

**Corollary 2.2.** For any maximal chain $C$, the map $\gamma_C$ is surjective. Consequently, $\text{len}(L) \leq |J(L)|$.

**Proof.** Let $C = \{c_0, c_1, \ldots, c_k\}$ be a maximal chain of $L$. Lemma 2.1 implies that for every $s \in [k]$ there exists $j_s \in J(L)$ with $(c_{s-1}, c_s) \npreceq (j_s, j_s)$. By definition, we have $j_s \leq c_s$ and $j_s \nleq c_{s-1}$ which implies $\gamma_C(j_s) = s$. This yields the first part of the statement. The second part of the statement follows if we take $k = \text{len}(L)$.

2.2. Some generalizations of distributive lattices

2.2.1. Join-extremal lattices. Following [10], we award the lattices which satisfy equality in Corollary 2.2 a special name. The lattice $L$ is *join extremal* if $\text{len}(L) = |J(L)|$. It is *extremal* if both $L$ and $L^d$ are join extremal.

**Corollary 2.3.** Let $C$ be a maximal chain of $L$ with $\text{len}(L) = |C| - 1$. The map $\gamma_C$ is a bijection if and only if $L$ is join extremal.

**Proof.** Let $C = \{c_0, c_1, \ldots, c_k\}$ with $k = \text{len}(L)$. If $L$ is join extremal, then $k = |J(L)|$, and $\gamma_C$ is a surjective map between equinumerous sets. Therefore it must be a bijection. If $L$ is not join extremal, then $k < |J(L)|$, and the pigeon-hole principle tells us that $\gamma_C$ cannot be injective.

2.2.2. Left-modular lattices. Let $a, b, c \in L$ with $b < c$. Then, these elements satisfy the *modular inequality*

$$ (b \lor a) \land c \geq b \lor (a \land c). \quad (2.2) $$

If this holds with equality for all $b < c$, then the element $a$ is *left modular*. Following [4], the lattice $L$ is *left modular* if it has a maximal chain of size $\text{len}(L) + 1$ which consists entirely of left-modular elements. See also [9,14,15] for more background.

2.2.3. Join-semidistributive lattices. Lastly, $L$ is *join semidistributive* if for all $a, b, c \in L$ it holds that

$$ a \lor b = a \lor c \quad \text{implies} \quad a \lor b = a \lor (b \land c). \quad (2.3) $$

If $L$ and $L^d$ are join semidistributive, then $L$ is *semidistributive*.

3. Proof of the main result

In general, there are no implications among the three lattice properties defined before. Figure 4a shows a join-semidistributive lattice that is neither join extremal nor left modular; Figure 4b shows a left-modular lattice that is neither join extremal nor join semidistributive; Figure 4c shows a join-extremal lattice that is neither join semidistributive nor left modular.

The main motivation for this article comes from the following result that relates extremality and left-modularity for semidistributive lattices.
Theorem 3.1 ([15, Theorem 1.4]). Every semidistributive, extremal lattice is left modular.

Our main contribution is the converse to Theorem 3.1 on the level of join-semidistributive lattices.

Theorem 3.2. Every join-semidistributive, left-modular lattice is join extremal.

Proof. Suppose that $L$ is join semidistributive and left modular with $\text{len}(L) = k$. Fix a chain $C = \{c_0, c_1, \ldots, c_k\}$ of left-modular elements. We proceed by way of contradiction and assume that $\gamma_C$ is not injective. Let $j_1, j_2 \in J(L)$ with $j_1 \neq j_2$ and $\gamma_C(j_1) = s = \gamma_C(j_2)$.

This implies, by construction, that $j_i \not\leq c_{s-1}$ for $i \in \{1, 2\}$. Since $c_{s-1} \leq c_s$ and $j_i \leq c_s$ it follows that $c_{s-1} < j_i$ if and only if $j_i = c_s$. Thus, without loss of generality, we may assume that $j_1$ and $c_{s-1}$ are incomparable. We conclude:

$$j_1 \lor c_{s-1} = c_s = j_2 \lor c_{s-1}. \quad (3.1)$$

Together with (2.3), we get

$$c_s = c_{s-1} \lor (j_1 \land j_2). \quad (3.2)$$

Now we show that $j_1$ and $j_2$ are incomparable, and assume that $j_1 < j_2$. Since $c_{s-1}$ is left modular, we obtain using (3.1):

$$j_2 = c_s \land j_2 = (j_1 \lor c_{s-1}) \land j_2 = j_1 \lor (c_{s-1} \land j_2) \leq j_1 \lor j_{2*} \leq j_{2*},$$

which is a contradiction. Switching the roles of $j_1$ and $j_2$ discards the case $j_2 < j_1$. Since $j_1 \neq j_2$ by assumption, these elements must be incomparable.

Using the left-modularity of $c_{s-1}$ once again, we obtain from (3.2) that

$$j_1 = c_s \land j_1 = ((j_1 \land j_2) \lor c_{s-1}) \land j_1 = (j_1 \land j_2) \lor (c_{s-1} \land j_1). \quad (3.3)$$

Since $j_1 \in J(L)$, it follows that $j_1 = j_1 \land j_2$ or $j_1 = c_{s-1} \land j_1$. However, we have already established that $j_1$ and $j_2$ are incomparable (which discards the first case) and that $j_1$ and $c_{s-1}$ are incomparable (which discards the second
case). This contradiction shows that our assumption must have been wrong, meaning that \( \gamma_C \) is injective.

Together with Corollary 2.2 we get that \( \gamma_C \) is bijective, so that Corollary 2.3 implies that \( L \) is join extremal. \( \square \)

Since Theorem 3.2 is stated for join-semidistributive lattices, it is natural to wonder if we can weaken any of the assumptions of Theorem 3.1. For keeping the notation simple, we define \( M(L) \overset{\text{def}}{=} J(L^d) \). First of all, we record the following lemma.

**Lemma 3.3** ([6, Corollary 2.55]). If \( L \) is semidistributive, then \( |J(L)| = |M(L)| \).

In particular, semidistributive, join-extremal lattices are extremal. This gives us the following corollary.

**Corollary 3.4.** For semidistributive lattices, left-modularity and extremality are equivalent.

For \( j \in J(L) \), let us consider the set
\[
K(j) \overset{\text{def}}{=} \{ a \in L : j_* \leq a \text{ but } j \nleq a \}.
\]
If \( K(j) \) has a unique maximal element, then we write \( \kappa(j) \) for this element. In other words \( \kappa(j) \overset{\text{def}}{=} \bigvee K(j) \) if this join exists.

**Lemma 3.5.** Every maximal element of \( K(j) \) is in \( M(L) \). Consequently, \( \kappa(j) \in M(L) \) if it exists.

**Proof.** Let \( j \in J(L) \). Since \( j_* \in K(j) \), the set \( K(j) \) is not empty. Let \( m \in K(j) \) be maximal. Assume that \( m \notin M(L) \). If \( m = \hat{1} \), then \( j \leq m \), which is a contradiction. Therefore, \( m \neq \hat{1} \). By construction, this means that there exist (at least) two distinct elements \( a_1, a_2 \in L \) such that \( m \prec a_1 \) and \( m \prec a_2 \). Since \( m \in K(j) \) we conclude that \( j_* \leq m \) but \( j \nleq m \). In particular \( j_* \leq a_1 \) and \( j_* \leq a_2 \). Since \( m \) is maximal, we conclude that \( a_1, a_2 \notin K(j) \) meaning that necessarily \( j \leq a_1 \) and \( j \leq a_2 \). But then \( j \leq a_1 \land a_2 = m \), a contradiction. \( \square \)

**Lemma 3.6.** Let \( L \) be join semidistributive. If \( j, j' \in J(L) \) are distinct, then there does not exist \( m \in L \) which is maximal in both \( K(j) \) and \( K(j') \).

**Proof.** Assume that there exists \( m \in L \) which is a maximal element of both \( K(j) \) and \( K(j') \). By Lemma 3.5, \( m \in M(L) \). Let \( m^* \) denote the unique element in \( L \) that covers \( m \).

By assumption, \( m^* \notin K(j) \cup K(j') \), which means that \( j \leq m^* \) and \( j' \leq m^* \). Then, \( j \lor m = m^* \) and \( j' \lor m = m^* \), because \( m \prec m^* \). If we set \( z = j \land j' \), then (2.3) implies that \( m^* = m \lor z \). Since \( j \in J(L) \) and \( j \neq j' \), it follows that \( z \leq j_* \leq m \), which implies \( m \lor z = m \). We thus obtain the contradiction \( m^* = m \). \( \square \)

**Lemma 3.7.** Every join-semidistributive lattice with \( |J(L)| = |M(L)| \) is semidistributive.
(a) A join-semidistributive, join-extremal, left-modular lattice that is not semidistributive.

(b) A join-semidistributive, join-extremal lattice that is not left modular.

**Figure 5. Some join-semidistributive lattices.**

**Proof.** For \( j \in J(L) \) we denote by \( M(j) \) the set of maximal elements of \( K(j) \).

By Lemma 3.5, \( M(j) \subseteq M(L) \). Moreover, by Lemma 3.6, if \( j, j' \in J(L) \) with \( j \neq j' \) we have \( M(j) \cap M(j') = \emptyset \). Let \( J(L) = \{j_1, j_2, \ldots, j_k\} \) and write \( m_\ell \triangleq |M(j_\ell)| \) for \( \ell \in [k] \). Since \( K(j) \neq \emptyset \) by construction it follows that 
\[ m_\ell \geq 1 \text{ for all } \ell \in [k]. \]

If \( |M(L)| = k \), then we obtain
\[ k \leq m_1 + m_2 + \cdots + m_k \leq |M(L)| = k, \]

which enforces \( m_\ell = 1 \) for all \( \ell \in [k] \). Consequently, \( \kappa(j) \) exists for all \( j \in J(L) \).

By [6, Theorem 2.56] it follows that \( L^d \) is join-semidistributive.

As a consequence, every join-semidistributive, extremal lattice is semi-distributive. Finally, Figure 5b shows a join-semidistributive, join-extremal lattice that is not left modular. Therefore, we cannot weaken the assumptions of Theorem 3.1 in order to guarantee left-modularity.

**4. Shellability**

We now briefly touch a topological aspect of the lattices that we consider. An edge labeling of \( L \) is any map

\[ \lambda : \text{Cov}(L) \to M \]

for some set \( M \). If \( C = \{c_0, c_1, \ldots, c_k\} \) is a maximal chain of \( L \), then

\[ \lambda(C) \triangleq (\lambda(c_0, c_1), \lambda(c_1, c_2), \ldots, \lambda(c_{k-1}, c_k)) \]

is the associated label vector. If \( M \subseteq \mathbb{N} \), then \( C \) is increasing if \( \lambda(C) \) is strictly increasing. Following [2,3], the edge labeling \( \lambda \) is an EL-labeling if every interval \([a, b] \) of \( L \) contains a unique increasing maximal chain, and this maximal chain has the lexicographically smallest label vector among all maximal chains in \([a, b] \). The lattice \( L \) is EL-shellable if it admits an EL-labeling. EL-shellable
lattices have remarkable properties, for instance the order complex of $L \setminus \{0, 1\}$ is a shellable, hence Cohen–Macaulay, complex.

If $L$ is left modular with left-modular chain $C = \{c_0, c_1, \ldots, c_k\}$, then the map $\gamma_C$ induces an edge labeling of $L$ by setting

$$\lambda_C : \text{Cov}(L) \to [k], \quad (a, b) \mapsto \min\{\gamma_C(j) : j \in J(L), a \lor j = b\}.$$ 

It was shown in [8] that $\lambda_C$ is an EL-labeling, which yields the following result.

**Proposition 4.1** ([8]). Every left-modular lattice is EL-shellable.

The converse of Proposition 4.1 is not true, see for instance the lattice in Figure 4c. Together with Theorem 3.1, we obtain the following.

**Corollary 4.2.** Every semidistributive, extremal lattice is EL-shellable.

Theorems 3.1 and 3.2 state that we cannot distinguish between extremality and left-modularity on the level of semidistributive lattices. What about shellability? Proposition 4.1 on the level of join-semidistributive lattices reads as follows.

**Corollary 4.3.** Every join-semidistributive, left-modular lattice is EL-shellable.

The converse does not hold by virtue of the lattice in Figure 5b. This lattice is join semidistributive and EL-shellable, but not left modular. But are we able to distinguish left-modularity and EL-shellability on the level of semidistributive lattices?

**Question 4.4.** Is every semidistributive, EL-shellable lattice necessarily left modular?

We note that in view of Theorem 3.2, if Question 4.4 is true, then any semidistributive, EL-shellable lattice must necessarily be extremal. Is that always the case?

**Question 4.5.** Is every semidistributive, EL-shellable lattice necessarily extremal?

Figure 4b shows an EL-shellable lattice that is not extremal (and it is also not semidistributive).

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