Some Symmetric Orbits for N-body Type Difference Equations

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Abstract

This paper introduces a new difference scheme to the difference equations for N-body type problems. To find the non-collision periodic solutions and generalized periodic solutions in multi-radial symmetric constraint for the N-body type difference equations, the variational approach and the method of minimizing the Lagrangian action are adopted and the strong force condition is considered correspondingly, which is an efficient method in studying those with singular potentials. And the difference equation can also be taken into consideration of other periodic solutions with symmetric or choreographic constraint in further studies.

1 Introduction

The N-body problem is a classical and important problem in celestial mechanics and mathematics. It consists of determining the orbits of N bodies interacting in accordance with the gravitational law of Newton. That is two bodies attracting each other, the force of attraction being directed along the line joining them, proportional to the product of the masses, and inversely
proportional to the square of the distance between them. For given $N$ bodies, the preceding N-body problem can be described by the following nonlinear system of second order differential equations:

$$-m_i\ddot{q}_i = \sum_{l=1, l\neq i}^{N} \frac{Gm_l m_i(q_i - q_l)}{|q_i - q_l|^3}, \quad i = 1, \cdots, N,$$  \hspace{1cm} (1)$$

where $m_i$ is the mass of $i$th body, $q_i$ is the position of $i$th body in $R^d$, $N$ is the number of bodies, and $G$ is the universal gravitational constant, which is always taken to be 1 for convenience in mathematics. Hence we take $G = 1$ in this paper.

Poincaré showed that it is impossible for $N \geq 3$ to find an explicit expression for the general solution. So even for the three-body problem, it is impossible to describe all the solutions. Till now, for planar three-body problem, there are only three orbits are found clearly for three equal masses, including the existence and the shape of the orbits. They are Euler’s solution (see Fig.1), Lagrange’s solution (see Fig.2) and Figure-Eight solution (see Fig.3). The Euler’s solution was discovered by Euler in 1765. It described a collinear configuration, in which one body locates on the center of masses of the three bodies, and the other two rotate around the center of the masses on the two endpoints of the diameter of the circle orbit. The Lagrange solution was discovered by Lagrange in 1772. The three bodies form an equilateral triangle on a circle orbit and the center of masses is the origin all the time. The Figure-Eight solution is a new, surprisingly and remarkable solution after the two solutions, which was found numerically by Moore first in 1993 (see [5]). In 2000, Chenciner and Montgomery rediscovered it and proved its existence at the same time. It describes three equal masses chasing one by one on an Eight-shaped curve in the plane. We refer the reader to the following articles and references therein for more detailed description: Chenciner et al [2], Montgomery [6] and Simó [7, 8] etc.

There is no other specific result for $N \geq 3$. Hence the numerical solution is more and more important in finding and describing new solutions vividly. In recent years, some new periodic solutions of the N-body problem have been found by minimizing the action of cycles with some special symmetric or choreographic constraint (e.g. [3], [4] etc), in which Fourier approach is used by most authors for the numerical solutions of N-body problem. In this paper, we propose the N-body type difference equations, and from which we can discover some new periodic orbits by minimizing the corresponding
The N-body type problems have been studied by Bahri, Rabinowitz [1], Zelati [10] among others. In [10], Zelati proved the existence of periodic solutions under the radial symmetry assumption. In this paper we prove the existence of periodic solutions for the difference equations with multi-radial symmetric constraint, which includes the radial symmetry. In the section on numerical examples, we propose a new variational difference method for the periodic solutions numerically.

The periodic solutions for N-body type problems (including N-body problems) can be described by

$$-m_i \ddot{q}_i = \nabla U(q), \quad i = 1, \ldots, N,$$

$$q(0) = q(T), \quad \dot{q}(0) = \dot{q}(T),$$

where $N$, $m_i$ and $q_i$ are defined as above, $q = (q_1, \ldots, q_N)$ in $(\mathbb{R}^d)^N$, $T$ is the period, and generally the singular potential function $U$ is

$$U(q) = \frac{1}{2} \sum_{1 \leq i \neq l \leq N} U_{il}(q_i - q_l)$$

with assumptions: $\forall 1 \leq i \neq l \leq N$ and $\forall \xi \in \mathbb{R}^d \setminus \{0\}$, we have

$$U_{il} \in C^2(\mathbb{R}^d \setminus \{0\}, \mathbb{R}), \quad U_{il}(\xi) = U_{ji}(\xi), \quad (A1)$$

$$U_{il}(\xi) \to -\infty, \quad \text{as} \quad |\xi| \to 0, \quad (A2)$$

$$U(q_1, \ldots, q_N) \leq 0, \quad \forall (q_1, \ldots, q_N) \in (\mathbb{R}^d)^N, \quad (A3)$$

where $U_{il}$ is usually described by

$$U_{il} = \frac{m_i m_l}{|q_i - q_l|^\alpha}. \quad (2)$$
2 Difference Equations

In order to formulate the difference equations for N-body type problems (N)-(P), we discretize time within one period \([0, T]\) with \(k + 1\) points \(t_0, \ldots, t_k\) and denote the set of these discretized points as \(S\), i.e. \(S = \{t_0, \ldots, t_k\}\), and introduce a difference scheme by

\[
q(t) \doteq \hat{q}(t) := \frac{t_s - t}{h} q(s - 1) + \frac{t - t_{s-1}}{h} q(s), \quad s = 1, \ldots, k + 1, \quad (DS)
\]

where \(\hat{q}(s)\) is an abbreviation of \(\hat{q}(t_s)\), \(t_s\) is the \(s\)th discretized point of \([0, T]\), \(\hat{q} = (\hat{q}_1, \ldots, \hat{q}_N)\), and \(h\) is the discretized step. Moreover, the traditional forth difference scheme is used to approximate the derivative of \(q\), which is used in [9]. Then the first order difference of \(q\) is

\[
\Delta q(t_s) = q(s + 1) - q(s),
\]

and the second order difference of \(q\) is

\[
\Delta^2 q(t_s) = q(s + 1) - 2q(s) + q(s - 1).
\]

Hence the derivatives of \(q(t_s)\) can be approximated by

\[
\dot{q}(t_s) \doteq \frac{1}{h} \Delta q(t_s), \quad \ddot{q}(t_s) \doteq \frac{1}{h^2} \Delta^2 q(t_s),
\]

By the difference approach above, (N) can be transformed into

\[
-\frac{1}{h^2} m_i (q_i(s + 1) - 2q_i(s) + q_i(s - 1)) = \nabla_{\hat{q}_i(s)} \hat{U},
\]

where \(i = 1, \ldots, N\) and

\[
\hat{U} = \sum_{s=1}^{k} U(\hat{q}(s)) = \frac{1}{2} \sum_{s=1}^{k} \sum_{1 \leq i \neq l \leq N} U_{il}(\hat{q}_l(s) - \hat{q}_i(s)). \quad (3)
\]

For convenience, we set the following notations:

\[
M_k = \begin{pmatrix}
2 & -1 & 0 & \cdots & -1 \\
-1 & 2 & -1 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & -1 & 2 & -1 \\
-1 & \cdots & 0 & -1 & 2
\end{pmatrix},
\]
\[
\hat{q} = \begin{pmatrix}
\hat{q}_1^T \\
\hat{q}_2^T \\
\vdots \\
\hat{q}_N^T
\end{pmatrix},
\hat{q}_i^T = \begin{pmatrix}
\hat{q}_{i(1)}^T \\
\hat{q}_{i(2)}^T \\
\vdots \\
\hat{q}_{i(d)}^T
\end{pmatrix},
\hat{q}_{i(p)}^T = \begin{pmatrix}
\hat{q}_{i(p(1)}^T \\
\hat{q}_{i(p(2)}^T \\
\vdots \\
\hat{q}_{i(p(k))}^T
\end{pmatrix},
\]

and

\[
M = \begin{pmatrix}
M_1 \\
\vdots \\
M_N
\end{pmatrix},
\]

where

\[
M_i = \begin{pmatrix}
m_{iM_k} \\
\vdots \\
m_{iM_k}
\end{pmatrix},
i = 1, \cdots, N, \quad p = 1, \cdots, d.
\]

It is easy to verify that \( M \) is a positive definitive matrix. Then the difference equation of (N) becomes

\[
\frac{1}{h} M \cdot \hat{q} - \nabla \hat{q}_i(s) \hat{U} = 0,
\]

and it is obviously that the difference boundary conditions can be described by

\[
\hat{q}(j) = \hat{q}(k+j), \quad j \in \mathbb{Z}^+, \quad (P1)
\]

where \( \mathbb{Z}^+ \) is the set of positive integer. Hence the Lagrangian functional of (N1)-(P1) is the following

\[
J(q) = \frac{1}{2h} < M \hat{q}, \hat{q} > -\hat{U} h.
\]

**Theorem 1.** There exists a small number \( \delta > 0 \) for all \( i \neq j, i, j = 1, \cdots, N, \) \( s \in S, \) such that \( |\hat{q}_i(s) - \hat{q}_j(s)| > \delta. \) Then the solution \( \hat{q} \) of (N1)-(P1) approximates the solution \( q \) of (N)-(P) when \( k \to \infty. \)

**Proof.** Since \( |\hat{q}_i - \hat{q}_j| > \delta, \) we can deduce the following equation from (2) and

\[
\frac{1}{h} M \cdot \hat{q} - \nabla \hat{q}_i(s) \hat{U} = 0,
\]

and it is obviously that the difference boundary conditions can be described by

\[
\hat{q}(j) = \hat{q}(k+j), \quad j \in \mathbb{Z}^+, \quad (P1)
\]

where \( \mathbb{Z}^+ \) is the set of positive integer. Hence the Lagrangian functional of (N1)-(P1) is the following

\[
J(q) = \frac{1}{2h} < M \hat{q}, \hat{q} > -\hat{U} h.
\]
\[ \nabla \hat{q}_i(s) \hat{U} = \frac{1}{2} \sum_{s=1}^{k} \sum_{1 \leq i \neq l \leq N} \frac{1}{|\hat{q}_l(s) - \hat{q}_i(s)|^{(\alpha+1)}} \]

\[ < \frac{1}{2} \sum_{s=1}^{k} \sum_{1 \leq i \neq l \leq N} \frac{1}{\delta^{(\alpha+1)}} \]

\[ = C, \]  (5)

where \( C \) is a constant. Similar to (DS), we introduce the difference scheme to \( \dot{q}(t) \), i.e. we have

\[ \hat{q}(t) = \frac{t_s - t}{h} \dot{q}(t_{s-1}) + \frac{t - t_{s-1}}{h} \dot{q}(t_s), \]

where \( t \in [t_{s-1}, t_s] \) and \( t_{s-1}, t_s \in S \). Notice that

\[ \ddot{\dot{q}}(t) = \frac{1}{h} (\dot{q}(t_s) - \dot{q}(t_{s-1})) = \frac{1}{h} \Delta \dot{q}(t_{s-1}). \]

Then when we let

\[ x = (q, \dot{q}), \]

the system \((N)-(P)\) can be rewritten as follows:

\[ \dot{x} = F(x), \quad x(0) = x(T), \]

where \( x \) satisfies \((DS)\) and \( F \in C(R^{2dN}, R^{2dN}) \), that is

\[
F(x) = \begin{pmatrix}
\dot{q} \\
\dot{\dot{q}}
\end{pmatrix}
= \begin{pmatrix}
\vdots \\
\dddot{q}_i \\
\vdots \\
-\frac{1}{m_e} \nabla U(q)
\end{pmatrix}.
\]

From \((5)\), we have \(|F| \leq M_0\) for some constant \( M_0 \), and for \( \forall \varepsilon > 0 \), there exist \( \delta' > 0 \), such that

\[ |F(x_1) - F(x_2)| \leq \varepsilon, \]
when $|x_1 - x_2| \leq \delta'$. From (DS), for $t \in [t_{s-1}, t_s], s = 1, \ldots, k$, clearly we have

$$|\hat{x}_k(t) - x(t_{s-1})| = \left| \frac{t - t_{s-1}}{h} x(t_{s-1}) + \frac{t - t_{s-1}}{h} x(t_s) - x(t_{s-1}) \right|$$

$$= \left| \frac{t - t_{s-1}}{h} (x(t_s) - x(t_{s-1})) \right|$$

$$= \left| \frac{t - t_{s-1}}{h} F(x(t_{s-1}))(t_s - t_{s-1}) \right|$$

$$= \left| (t - t_{s-1}) F(x(t_{s-1})) \right|$$

$$\leq h M_0.$$  

Therefore, there exists a $K(\epsilon) > 0$ such that

$$|F(\hat{x}_k(t)) - F(x(t_{s-1}))| \leq \epsilon, \text{ whenever } k \geq K(\epsilon). \quad (6)$$

The error within one period can be estimated by

$$\Delta_k(t) = \hat{x}_k(t) - \left( x(0) + \int_{0}^{t} F(\hat{x}_k(\tau)) d\tau \right).$$

Hence

$$\Delta_k(T) = \sum_{s=1}^{k} \int_{t_{s-1}}^{t_s} (F(x(t_{s-1})) - F(\hat{x}_k(\tau))) d\tau.$$ 

From (6), noting that the discretized step is $h = 1$, we have

$$|\Delta_k(T)| \leq \sum_{s=1}^{k} \epsilon h = T \epsilon.$$ 

This means $\Delta_k \to 0$, $x_k \to x$, i.e. $\hat{q} \to q$ when $k \to \infty$. The theorem is proved.

Let

$$\Omega = \{ \hat{q} \mid \hat{q} \in (R^d)^N, \text{ s.t. } \hat{q}_i \neq \hat{q}_l, \forall i \neq l \}, \quad \Lambda = \{ \hat{q} \mid \hat{q} \in H^1(R/T; \Omega) \}, \quad R^d = \prod_{i=1}^{b_d} R^{b_i}.$$ 

Then the multi-radial symmetry can be described as follows:

$$\Lambda_0 = \{ \hat{q} \mid \hat{q} \in \Lambda, \quad \hat{q}(b_i)(s + \tilde{k}) = -\hat{q}(b_i)(s), i = 1, \ldots, b_d, \sum_{i=1}^{b_d} b_i = d \}, \quad (Ms)$$
where $\tilde{k}^i = A/A_i$, $A_i, A \in \mathbb{Z}^+$, $A$ is a common multiple of $A_i$, and $\hat{q}^{(b)}(s + \tilde{k}) = -\hat{q}^{(b)}(s)$ means

$$\hat{q}^{(b)}(t) = -\hat{q}^{(b)}(\tau), \quad \text{where} \quad t \in [t_{s+\tilde{k}^i-1}, t_{s+\tilde{k}^i}], \quad \tau \in [t_{s-1}, t_s].$$

3 The Existence of Periodic Solutions

Let $H^1$ is a Hilbert space, $\hat{q} \in H^1([0, T], (\mathbb{R}^d)^N)$, i.e. $\hat{q}_i \in H^1([0, T], \mathbb{R}^d)$, $i = 1, \cdots, N$, and we define the norm of $\hat{q}$ in $H^1$ by

$$\| \hat{q} \| = \sqrt{S_1^2 + S_2},$$

where

$$S_1 = \frac{1}{T} \sum_{s=1}^{k} \int_{t_{s-1}}^{t_s} |\hat{q}(t)| \, dt, \quad S_2 = \sum_{s=1}^{k} \int_{t_{s-1}}^{t_s} \left| \dot{\hat{q}}(t) \right|^2 \, dt.$$

The following lemma describes the relationship between the critical points of $J$ and the periodic solutions of $\{N1\}-\{P1\}$.

Lemma 1. $(\hat{q}(1), \cdots, \hat{q}(k))^T$ is a critical point of $J$ if and only if $(\hat{q}(0), \cdots, \hat{q}(k+1))^T$ is a solution of $\{N1\}-\{P1\}$.

Proof. Since the difference equations transform the infinite dimensional dynamics into finite dimensional dynamics, it is reasonable to view $J$ as a continuously differentiable functional on $H^1$.

Let $\hat{q} = (\hat{q}(1), \cdots, \hat{q}(k))^T$ is the critical point of $J$ if and only if the first order derivative of $J$ on $\hat{q}$ is equal to zero. That is

$$J'(\hat{q}) = M\hat{q} - \nabla_\hat{q} U = 0.$$ 

Notice that

$$(\hat{q}(0), \hat{q}(1), \cdots, \hat{q}(k), \hat{q}(k+1))^T = (\hat{q}(k), \hat{q}(1), \cdots, \hat{q}(k), \hat{q}(1))^T.$$ 

This satisfies $\{N1\}-\{P1\}$ obviously. Thus the proof is completed.

Remark 1. Actually, $\hat{q}_i$ is a $k$-polygon in $\mathbb{R}^d$ as a simple case with the difference scheme $\{DS\}$. Hence we can call $\{\hat{q}(s)\}_{s=0}^{k+1} = (\hat{q}(0), \cdots, \hat{q}(k+1))^T$ a ‘cycle’ in $(\mathbb{R}^d)^N$, which means obviously that $\{\hat{q}_i(s)\}_{s=0}^{k+1} = (\hat{q}_i(0), \cdots, \hat{q}_i(k+1))^T$, $i = 1, \cdots, N$, are ‘cycles’ in $\mathbb{R}^d$.
We denote
\[ \Delta = \bigcup_{s=1}^{k}\{t \in [t_{s-1}, t_{s}] \mid \hat{q}_i(t) = \hat{q}_l(t), \text{ for some } i \neq l\}. \]

**Definition 1.** We call \( \hat{q} \) is a non-collision solution of (N1) and (P1), if \( \Delta = \phi \).

**Definition 2.** We call \( \hat{q} \) is a generalized solution of (N1) and (P1), if
\[ a) \ \text{meas}(\Delta) = 0; \]
\[ b) \ \hat{q} \text{ satisfy (N1)}; \]
\[ c) \ \sum_{i=1}^{N} \frac{m_i}{2} \left| \dot{q}_i(t) \right|^2 - U(\hat{q}(t)) \equiv C, \]  
(7)
where \( t \in [0, T] \backslash \Delta \) and \( C \) is a constant.

**Remark 2.** It is easy to verify that the integral of (7) with respect to \( t \) approximates \( J(\hat{q}) \) since
\[ \int_{t_{s-1}}^{t_s} U_{il}(\hat{q}_i(t) - \hat{q}_i(t))dt = U_{il}(\hat{q}_i(s) - \hat{q}_i(s)). \]  
(8)
For convenience, we suppose \( \hat{U} \) in (N1) has only one singular point, and without loss of generality, let it be \( 0 \in R^d \). Hence we have

**Definition 3.** Let some \( V_{il} \in C^1(R^d \backslash \{0\}, R) \), \( V_{il} \rightarrow +\infty (\xi \rightarrow 0) \) and
\[ -U_{il}(\xi) \geq |\nabla V_{il}(\xi)|^2, \quad \forall \xi \in R^d \backslash \{0\}, \]  
(SF)
where \( |\xi| \) is small. Then we say \( U_{il} \) satisfies Strong Force (SF) condition.

The condition (SF) is very efficient to overcome the obstacle coming from singular potential. In [4], (SF) is used to bound \( q \) away from \( S \), where \( S \) is a closed nonempty set, and \( U \rightarrow -\infty \) as \( q \rightarrow S \). Here, we introduce Lemma 2 based on the discussions above for N-body type difference equations.

Let \( \Gamma \) be the family of homotopic ‘cycles’ in \( H^1([0, T], (R^d \backslash \{0\})^N) \), then we have
Lemma 2. Assume condition (SF) is satisfied, and there exists a constant $c$ such that
\[ \langle M \hat{q}, \hat{q} \rangle \leq c, \quad |\hat{U}| \leq c, \]
for all $\hat{q} \in \Gamma$, then $\hat{q}$ is bounded away from $\{0\}$. That is, there exists $\delta > 0$ such that no cycles in $\Gamma$ intersect the $\delta$-neighborhood of $\{0\}$.

Proof. From (SF), it is easy to verify that $U_{il}$ are all negative, then from (3) we have the following estimate for $\hat{U}$:
\[
|\hat{U}| = \frac{1}{2} \sum_{s=1}^{k} \sum_{1 \leq i \neq l \leq N} |U_{il}(\hat{q}(s) - \hat{q}(s))| \leq c.
\]

Let
\[
B = \begin{pmatrix}
m_1 & \cdots & m_1 \\
\vdots & \ddots & \vdots \\
m_N & \cdots & m_N
\end{pmatrix},
\]
\[
\tilde{m} = \min\{m_1, \ldots, m_N\}, \quad y = B\hat{q},
\]
and let
\[
G = \{ \rho = (y, z) \in (R^d)^N \times R \mid y, \hat{q} \in (R^d\{0\})^N, \quad z = V(\hat{q}) \}
\]
be the graph of $V$ in (SF). Suppose the binary relation is
\[
\rho = (y, V(\hat{q})),
\]
whose arc length is \( a = a(\rho) \). Then we have

\[
a = \int_0^T |\dot{a}|dt = \sum_{s=1}^k \int_{t_{s-1}}^{t_s} |\dot{a}|dt
\]

\[
= \sum_{s=1}^k \int_{t_{s-1}}^{t_s} \sqrt{|\dot{y}|^2 + \langle \dot{q}, \nabla V \rangle^2} dt
\]

\[
\leq \sum_{s=1}^k \int_{t_{s-1}}^{t_s} \left( |\dot{y}|^2 + |\dot{q}|^2 |\nabla V|^2 \right)^{\frac{1}{2}} dt
\]

\[
\leq \sum_{s=1}^k \int_{t_{s-1}}^{t_s} \left( \frac{1}{m^2} |\nabla V|^2 \right)^{\frac{1}{2}} dt
\]

\[
\leq \sum_{s=1}^k \left( \int_{t_{s-1}}^{t_s} |\dot{y}|^2 dt \right)^{\frac{1}{2}} \left( \int_{t_{s-1}}^{t_s} \left( 1 + \frac{1}{m^2} |\nabla V|^2 \right) dt \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_{s=1}^k \int_{t_{s-1}}^{t_s} |\dot{q}|^2 dt \right)^{\frac{1}{2}} \left( \sum_{s=1}^k \int_{t_{s-1}}^{t_s} \left( 1 + \frac{1}{m^2} |\nabla V|^2 \right) dt \right)^{\frac{1}{2}}
\]

\[
\leq \left( \sum_{s=1}^k \int_{t_{s-1}}^{t_s} |\dot{B}|^2 dt \right)^{\frac{1}{2}} \left( T + \frac{1}{m^2} \sum_{s=1}^k \int_{t_{s-1}}^{t_s} |U|dt \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_{s=1}^k \int_{t_{s-1}}^{t_s} |\dot{B}(s) - \dot{q}(s-1)|^2 dt \right)^{\frac{1}{2}} \left( T + \frac{1}{m^2} \sum_{s=1}^k \int_{t_{s-1}}^{t_s} |U(\dot{q}(t))|dt \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_{s=1}^k \sum_{i=1}^N m_i |\dot{q}_i(s) - \dot{q}_i(s-1)|^2 \right)^{\frac{1}{2}} \left( T + \frac{1}{m^2} \sum_{s=1}^k \int_{t_{s-1}}^{t_s} |U|dt \right)^{\frac{1}{2}}
\]

\[
= \sqrt{(\dot{M} \dot{q}, \dot{q})} \left( T + \frac{1}{m^2} |\dot{U}| \right)^{\frac{1}{2}}
\]

\[
\leq \sqrt{c} \left( T + \frac{1}{m^2} c \right) \equiv C \text{ (a constant)}.
\]

Now let \( \{\tilde{q}^n\}_{n=1}^\infty \) be a sequence in \( \Gamma \). If \( \tilde{q}^n \) are not bounded away from \( \{0\} \), for \( \forall \varepsilon > 0 \), the \( \varepsilon \)-neighborhood of \( \{0\} \) in \( \mathbb{R}^d \), \( B_\varepsilon \), must intersect with infinite many \( \tilde{q}^n \). By passing to a subsequence, we can suppose that

\[
\inf |\tilde{q}^n| \to 0, \quad \text{as} \quad n \to \infty.
\]

This implies that the corresponding \( \rho^n \) ascend infinitely far up to the skylight of \( G \) at \( \{0\} \) as \( n \to \infty \).
Suppose there exists $\delta > 0$, $B_\delta$ is $\delta$-neighborhood of $\{0\}$, such that portions of $\hat{q}^\nu$ fall outside the ball $B_\delta$ and the others (infinite many) $\hat{q}^\nu$ in $B_\delta$. Then the variation of $V(\hat{q}^\nu)$ would become infinite as $n \to \infty$, but $a(\rho)$ is greater than the variation of $V(\hat{q}^\nu)$, a contradiction to the fact $a \leq C$.

On the other hand, if there is no such $\delta$, $\hat{q}^\nu$ will collapse to the point $\{0\}$, which implies that

$$\hat{U} \to +\infty.$$ 

This contradicts to the hypotheses at the beginning. Hence $\hat{U}$ is bounded away from $\{0\}$.

Now, suppose $\hat{q}^\nu$ converge uniformly to $\hat{q}^*$ in the norm in $H^1$. By the above argument, if $\hat{q}^\nu$ are not bounded away from $\{0\}$, $\hat{q}^*$ will be attached to $\{0\}$. The proof is complete. \qed

**Remark 3.**

(a) For the case that the potential $U$ has more than one singular points has the same result in Lemma 2.

(b) Lemma 2 means that there is no $\hat{q} \in \Gamma$ intersects the $\delta$-neighborhood of $\{0\}$ under (SF) condition.

(c) The converse-negative proposition of Lemma 2 shows that if $\bar{q} \in \partial \Lambda$, $q^\mu \to \bar{q}$ weakly in $H^1$ and strongly in $C^0$, then $J(q^\mu) \to +\infty$.

With respect to the periodic solutions of (N1)-(P1), we have the following theorem:

**Theorem 2.** Let $U_{il}$ and $\hat{U}$ in (3) satisfy (A1)-(A3) and $\hat{q}$ satisfy (Ms). Then (N1)-(P1) has infinitely many generalized solutions; furthermore, if (SF) holds, then (N1)-(P1) has infinitely many non-collision solutions.

**Proof.** It is obvious that $\Lambda_0$ is a subset of $\Lambda$, and $\hat{U}$ is an even function on $\Lambda_0$ for $U_{il} = U_{il}$, so we can easily check that the critical points of (3) on $\Lambda_0$ are also the critical points on $\Lambda$.

Since $U_{il}$ does not satisfy (SF), we can modify it as

$$U_{il}^\delta = U_{il} - \frac{\phi(|\hat{q}_l - \hat{q}_i|)}{|\hat{q}_l - \hat{q}_i|^2},$$

where $\delta$ is small and

$$\phi = \begin{cases} 
0, & \text{when } |\hat{q}_l - \hat{q}_i| \leq \frac{\delta}{2}, \\
1, & \text{when } |\hat{q}_l - \hat{q}_i| \geq \delta.
\end{cases}$$
Then we get a new Lagrangian functional with condition (SF) from (4),

\[ J^\delta(\hat{q}) = \frac{1}{2} < M\hat{q}, \hat{q} > -\tilde{U}^\delta, \]  

(9)

where \( \tilde{U}^\delta = \sum_{s=1}^{k} \sum_{1 \leq i < l \leq N} U^\delta_{il}. \)

Let \( J^\delta \) have infimum

\[ c_\delta = \inf \{ J^\delta(\hat{q}) \mid \hat{q} \in \Lambda_0 \}. \]

Consider minimizing sequence \( \hat{q}^\mu(s) \in \Lambda_0 \) such that \( J^\delta(\hat{q}^\mu(s)) \to c_\delta \) when \( \mu \to +\infty \). Then, when \( \mu \) is large enough, we have

\[ J^\delta(\hat{q}^\mu(s)) \leq c_\delta + 1. \]

Since \( \hat{q} \in \Lambda_0 \), we have that

\[ |\hat{q}_i(s)| \leq d \times \max \{|\hat{q}^b_i(s)|, |\hat{q}^{b'}_i(s)|\} \]

\[ = d \times \max \{\frac{1}{2}|\hat{q}^b_i(s) - \hat{q}^b_i(s + \hat{k})|, \frac{1}{2}|\hat{q}^{b'}_i(s) - \hat{q}^{b'}_i(s + \hat{k})|\} \]

\[ \leq d \times \max \{\frac{1}{2} \sum_{j=0}^{k-1} |\hat{q}_i(s - j) - \hat{q}_i(s - j + 1)|, \]

\[ \frac{1}{2} \sum_{j=0}^{k-1} |\hat{q}_i(s - j) - \hat{q}_i(s - j + 1)|\} \]

\[ \leq \frac{d}{2} \sum_{j=0}^{k-1} |\hat{q}_i(s - j) - \hat{q}_i(s - j + 1)| \]

\[ = \frac{d}{2} \sum_{j=0}^{k-1} |\hat{q}_i(s - j) - \hat{q}_i(s - j + 1)| \]
\[
\sum_{j=0}^{k-1} \left( \sqrt{m_i} |\hat{q}_i(s-j) - \hat{q}_i(s-j+1)| \right) \leq \sum_{j=0}^{k-1} \left( \sum_{i=1}^{N} m_i \right) \frac{1}{2} \left( \sum_{j=0}^{k-1} \frac{1}{m_i} \right) \frac{1}{2}
\]

where \( \bar{m} = \min_{i=1, \cdots, N} \{m_i\} \). This illuminates that there exists a \( \bar{q} \in \Lambda_0 \) and a subsequence \( \{\hat{q}^\mu\}_\mu \) such that

\[
\hat{q}^\mu \to \bar{q}(s), \quad s = 1, \cdots, k,
\]

where \( \bar{m} = \min_{i=1, \cdots, N} \{m_i\} \). This illuminates that there exists a \( \bar{q} \in \Lambda_0 \) and a subsequence \( \{\hat{q}^\mu\}_\mu \) such that

\[
\hat{q}^\mu \to \bar{q}(s), \quad s = 1, \cdots, k,
\]

Then \( \bar{q} \) is a critical point of (3).

Now, from the remarks of Lemma 2 and Lemma 1 there exist a subsequence \( \{\hat{q}^\mu(s)\} \) and \( \hat{q}^\ast \) in \( H^1(\mathbb{R}/T, (\mathbb{R}^d)^N) \), such that \( \hat{q}^\ast \) is a non-collision periodic solution of N-body problem (N1)-(P1).

Along the line of Theorem 1.1 in [10], we can find a generalized solution \( \hat{q}^\ast \) of (N1)-(P1) by constructing compact nested interval cover of \( [0, T] \setminus \Delta \) to show \( U^\delta \to U \) and \( \hat{q}^\ast \to \hat{q}^\ast \) when \( \delta \to 0 \). We can also prove \( Q_{T/C} \) is a \( T \)-periodic solution of (N1)-(P1) if \( Q \) is, where \( C \) is a positive integer, then we can find infinitely many \( T \)-periodic solutions of (N1)-(P1). This completes the proof.

4 Numerical Examples for the N-Body Problem

In this section, we present some numerical solutions of N-body problem (N1)-(P) as an example of N-body type. In this case, \( \alpha = 1 \) in (2), which is usually called Newtonian Potential.
The method of steepest descent is adopted in finding the minimum of the functional $J$ since this method is simple, easy to apply, and each iteration is fast. If the minimum points exist, the method guarantees that we can locate them after at least an infinite number of iterations. We implement it in MATLAB7.0 for getting the following orbits. And from the process in deducing $J$, we named the method *variational difference method*. The following examples show that it is at least very valid for our problem.

When we set the multi-radial symmetry with

$$q^{(1,2)} \left( t + \frac{T}{2} \right) = -q(t)^{(1,2)}, \quad q^{(1,2)} \in \mathbb{R}^2,$$

which is a simple radial symmetric constraint in the plane. We get the following two figures with different initial values for three-body problems with equal masses.

In the following tables, $\hat{q}_i$ in the columns is the initial value of $i$th body, where $i = 1, 2, 3$. And the $j$th raw is corresponding to the initial value of the $j$th difference point, where $j = 1, \cdots, k$. We set the values at random by MATLAB.

| $\hat{q}_1$ | $\hat{q}_2$ | $\hat{q}_3$ | $\hat{q}_1$ | $\hat{q}_2$ | $\hat{q}_3$ |
|-------------|-------------|-------------|-------------|-------------|-------------|
| 4.8375      | -3.5004     | -1.3367     | 4.8375      | -3.5004     | -1.3367     |
| 4.3081      | -5.8243     | 1.5174      | 4.3081      | -5.8243     | 1.5174      |
| 5.0723      | -3.3872     | -1.6835     | 5.0723      | -3.3872     | -1.6835     |
| 3.9921      | -5.7708     | 1.7809      | 3.9921      | -5.7708     | 1.7809      |
| $\vdots$    | $\vdots$    | $\vdots$    | $\vdots$    | $\vdots$    | $\vdots$    |
| -4.2971     | 3.7003      | 0.5988      | -4.2971     | 3.7003      | 0.5988      |
| -4.8963     | 5.8687      | -0.9714     | -4.8963     | 5.8687      | -0.9714     |
| -4.5792     | 3.6051      | 0.9749      | -4.5792     | 3.6051      | 0.9749      |
| -4.6106     | 5.8569      | -1.2464     | -4.6106     | 5.8569      | -1.2464     |

*Table 1*: The initial value of Fig.4  
*Table 2*: The initial value of Fig.5
When the distribution of the masses are $m_1 = 1, m_2 = 0.5, m_3 = 1.5$, four different numerical solutions are found with different initial values of $J$ for three-body problem.

| $\hat{q}_1$ | $\hat{q}_2$ | $\hat{q}_3$ |
|-----------|-------------|-------------|
| 0.9977    | 0.7023      | 0.9878      |
| 0.3897    | 0.5013      | 0.5027      |
| 0.9878    | 0.3214      | 0.8118      |
| 0.5027    | 0.2917      | 0.2859      |
| ...       | ...         | ...         |
| 0.8906    | 0.0109      | 0.1405      |
| 0.4815    | 0.0134      | 0.1421      |
| 0.5134    | 0.0484      | 0.1205      |
| 0.4002    | 0.0046      | 0.3442      |

Table 3: The initial value of Fig. 6

| $\hat{q}_1$ | $\hat{q}_2$ | $\hat{q}_3$ |
|-----------|-------------|-------------|
| 0.2916    | 0.2886      | 0.5938      |
| 0.3066    | 0.3100      | 0.3580      |
| 0.0974    | 0.9711      | 0.1567      |
| 0.7207    | 0.1348      | 0.9382      |
| ...       | ...         | ...         |
| 0.6674    | 0.6226      | 0.3063      |
| 0.1623    | 0.6986      | 0.3450      |
| 0.3109    | 0.1326      | 0.6609      |
| 0.0311    | 0.8893      | 0.8840      |

Table 4: The initial value of Fig. 7
Figure 6: $m = [1, 0.5, 1.5], J = 0.805$

Figure 7: $m = [1, 0.5, 1.5], J = 0.670$

| $\dot{q}_1$ | $\dot{q}_2$ | $\dot{q}_3$ |
|-------------|-------------|-------------|
| 0.5499      | 0.8989      | 0.7706      |
| 0.6606      | 0.8988      | 0.2469      |
| 0.0424      | 0.0684      | 0.6264      |
| 0.2721      | 0.7092      | 0.1738      |

Table 5: The initial value of Fig.

| $\dot{q}_1$ | $\dot{q}_2$ | $\dot{q}_3$ |
|-------------|-------------|-------------|
| 0.9342      | 0.3759      | 0.1146      |
| 0.4225      | 0.6273      | 0.4319      |
| 0.2644      | 0.0099      | 0.6649      |
| 0.8560      | 0.6991      | 0.6343      |

Table 6: The initial value of Fig.

Figure 8: $m = [1, 0.5, 1.5], J = 0.795$

Figure 9: $m = [1, 0.5, 1.5], J = 0.832$
The following orbits are also found for the planar N-body problems, where $m$ shows the number and the masses of the bodies, $J$ is the corresponding value of Lagrangian functional action.

**Figure 10:** $m = [1, 1, 1, 1], J = 1.522$

**Figure 11:** $m = [1, 1, 1, 1, 1], J = 2.794$

**Figure 12:** $m = [1, 0.5, 1.5, 2.5], J = 2.747$

**Figure 13:** $m = [1, 0.5, 1.5, 2, 3], J = 5.541$
For spacial N-body problems,

\[ q^{(1,2,3)} \left( t + \frac{T}{2} \right) = -q(t)^{(1,2,3)}, \quad q^{(1,2,3)} \in \mathbb{R}^3, \]

i.e. \( A_1 = A_2 = A_3 = 2 \) in \( \text{[MS]} \), it is also a simple radial symmetric constraint but in space. We find the orbits in the following figures. In each figure, the picture on the top left corner is the projection of the orbits on \( x-y \) plane, the picture on the top right corner is the projection of the orbits on \( y-z \) plane, the picture on the bottom left corner is the projection of the orbits on \( z-x \) plane, and the picture on the bottom right corner is the orbits in space.

**Figure 14:** \( m = [1, 0.5, 1, 0.5], J = 1.086 \)  
**Figure 15:** \( m = [1, 0.5, 1, 0.5, 1], J = 1.827 \)

**Figure 16:** \( m = [1, 1, 1], J = 0.783 \)  
**Figure 17:** \( m = [1, 0.5, 1.5], J = 0.832 \)
Figure 18: \( m = [1, 1, 1], J = 1.665 \)

Figure 19: \( m = [1, 0.5, 1.5, 2], J = 2.721 \)

Figure 20: \( m = [1, 0.5, 1.0, 0.5, 1], J = 2.112 \)

Figure 21: \( m = [1, 0.5, 1.0, 0.5], J = 1.200 \)
When we set $A_1 = 4, A_2 = 2$ in $\text{[Misc]}$, $R^d = R^3 = R^2 \times R^1$, we have the following multi-radial symmetric constraint

$$q_i^{(1,2)}(t + \frac{T}{4}) = -q_i^{(1,2)}(t), \quad q_i^{(1,2)} \in R^2,$$

$$q_i^{(3)}(t + \frac{T}{2}) = -q_i^{(3)}(t), \quad q_i^{(3)} \in R^1.$$

In this case, the following orbits are found.
5 Conclusion

The N-body type difference equation is first introduced in this paper with a new difference scheme. To prove the existence of periodic solutions for N-body type problem, the variational approach and the method of minimizing the Lagrangian functional action, both of which are typical in studying N-body dynamical systems, are employed in this paper. In finding the generalized solutions and non-collision solutions numerically, we use the method of minimizing the functional action of N-body type difference equation. The presented numerical solutions of N-body problem illuminate that the variational difference method, which is combined with the traditional steepest method, is rather effective in finding the periodic solutions for N-body type problem numerically. With these numerical examples, we can conclude that there exist some interesting orbits under multi-radial symmetric constraint; and surprisingly, even under radial symmetric constraint we have obtained some interesting orbits. Furthermore, difference equations transform infinity dimension dynamical systems into finite dimension dynamical systems, and it can be used in the studies for other solutions under corresponding symmetric or choreographic constraints.

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