Accurate analysis of steady state response of fully clamped orthotropic functionally graded rectangular plates

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Abstract. A Hamiltonian-based method is introduced to find exact solutions for steady state forced vibration of a fully clamped orthotropic functionally graded (FG) rectangular plate. The proposed method has two major steps. Firstly, the steady state response for an FG rectangular plate with Navier-type boundary conditions is analytically derived by a wave-based symplectic method. Subsequently, exact solutions of the fully clamped FG rectangular plate are obtained by a symplectic superposition technique. The overall procedure is systematic with a clearly defined, step-by-step derivation. The accuracy of the obtained exact solutions is highlighted in comparison with reference data. Numerical results are presented to reveal the effects of the material properties.

1. Introduction

Functionally graded materials made of a mixture of metal and ceramic with smooth and gradual variation of material properties through the thickness have been widely used in many fields, such as thermal barrier materials for aerospace applications and fusion reactors \cite{1}. Functionally graded (FG) plate as one of the important FG structural components are usually subjected to complicated loads or in a severe environment. Hence, the dynamic analysis of FG plates is highly significant for the design of engineering structures subjected to external dynamic excitations such as earthquakes, impacts, nuclear blast and other conditions.

Numerous studies of the dynamic response of FG plates have been conducted with various solutions approaches. Zenkour and Sobhy \cite{2}, Mechab et al. \cite{3} and Jung et al. \cite{4} presented analytical solutions of the dynamic response of FG plates based on the Navier’s solution. Sun and Luo \cite{5} derived exact integral solutions for FG plates under impulse load by using the dispersion relation and integral transforms. Yang and Shen \cite{6} employed the modal superposition method to determine the transient response of FG plates. Wattanasakulpong et al. \cite{7} studied the forced vibration of FG plates using the Ritz method. Ansari et al. \cite{8} analyzed the forced vibration behavior of FG plates by a Galerkin-based numerical approach. Parandvar and Farid \cite{9} investigated the dynamic response of FG plates in the framework of the finite element method (FEM). Rezaei Mojdehi et al. \cite{10} used the meshless method to study the dynamic response of FG plates. Alijani and Amabili \cite{11} proposed a method based on the pseudo arc-length continuation and collocation scheme to study the forced
vibration of FG rectangular plates. Huang and Shen [12] solved the dynamic response of the FG plate by using an improved perturbation technique. Nguyen and Nguyen-Xuan [13] employed the isogeometric analysis (IGA) to solve the dynamic response of FG plates. Song et al. [14] investigated the dynamic response of the FG plates with a moving mass based on Rayleigh-Ritz solutions.

In view of the existing literature, the theoretical studies on deriving analytical solutions was very limited [2-4] and most of the obtained exact solutions depend on the pre-determined trial functions such as trigonometric functions [2-4]. Therefore, it is necessary to develop a rigorous method to derived analytical solutions for the dynamic analysis of FG plates, which could directly benefit the rapid design of such FG structural components. In this paper, a Hamiltonian-based analytical method is proposed to study the steady state response of orthotropic FG rectangular plates resting on an elastic foundation. The symplectic method was first introduced by Zhong and his collaborators [15] and has been applied in the vibration analysis of rectangular plates [16]. The solution procedure has two major steps. At the first step, analytical solutions for steady state forced vibration of Navier-type FG plates are obtained and expressed in terms of symplectic eigenfunctions. At the second step, analytical solutions of the fully clamped FG plates are obtained by a superposition technique [17, 18]. Unlike the classical inversed or semi-inverse methods, the present method a direct method with a clearly defined, step-by-step derivation.

2. Basic theory

2.1. FGM properties

The orthotropic FGM material is considered here. The volume fractions of constituents are assumed to vary through the thickness according to the power law distribution (P-FGM). The effective material properties $P$, such as Young’s modulus $E_x$, $E_y$, shear modulus $G$ and density $\rho$, are assumed to vary continuously through the thickness $h$, and expressed respectively as [19]

$$ P(z) = P_h + (P_t - P_h) \left( \frac{1}{2} \left( \frac{z}{h} \right) \right)^k, \quad (1) $$

where $z$ is coordinate along the thickness; $k$ is the volume fraction index; $P_t$ and $P_h$ respectively are effective material properties at $z = h/2$ and $z = -h/2$. The Poisson's ratio $\nu$ is constant.

2.2. Basic equation of orthotropic FG rectangular plate

Figure 1 depicts an orthotropic FG rectangular plate resting on an elastic Winkler foundation. The origin of the Cartesian coordinate system ($x, y, z$) is at one corner of the middle plane of the plate. The length, width and thickness of the plate is selected as $a, b$ and $h$, respectively. $K_w$ is the stiffness parameter of the elastic foundation. The distributed harmonic force $q_{ext}(x, y)$ is applied within the action area $\Omega$ ($0 \leq x_1 \leq x_2 \leq a$, $0 \leq y_1 \leq y_2 \leq b$). Since the harmonic vibration is considered, all
quantities vary harmonically with time as \( e^{i\omega t} \) and this explicit dependence will henceforth be suppressed [20]. The stress-strain relations of the plate are expressed as

\[
\sigma = C \varepsilon, \tag{2}
\]

where \( \sigma = \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}^T \) and \( \varepsilon = \{-z \partial^2 w/\partial x^2, -z \partial^2 w/\partial y^2, -2z \partial^2 w/(\partial x \partial y)\}^T \) are the stress and strain tensors, respectively; \( w \) is the displacement along \( z \)-axis; \( C' = C' + iC'' \) is the viscoelastic tensor in which \( C' \) and \( C'' \) are the storage and loss moduli, respectively [21]; \( i = \sqrt{-1} \) is the imaginary unit. In order to obtain a continuous response, the loss modulus is assumed to be \( \eta \) in which \( \eta \) is the damping loss factor. Therefore, the viscoelastic tensor \( C' \) in Eq. (2) can be further expressed as

\[
C' = (1 + i\eta)C' = (1 + \eta i) \begin{bmatrix} C_{xx} & C_{xy} & 0 \\ C_{yx} & C_{yy} & 0 \\ 0 & 0 & C_{zz} \end{bmatrix}, \tag{3}
\]

where \( C_{xx} = E_x(z)/(1 - \nu_x \nu_y) \), \( C_{yy} = E_y(z)/(1 - \nu_x \nu_y) \), \( C_{xy} = \nu_y E_y(z)/(1 - \nu_x \nu_y) \), \( C_{zz} = G(z) \), \( E_x(z) \) and \( E_y(z) \) are Young’s moduli, \( G(z) \) is the shear modulus, \( \nu_x \) and \( \nu_y \) are Poisson’s ratios, and \( \nu_y/E_y(z) = \nu_x/E_x(z) \).

The moments and shear force resultants are defined by

\[
\{M_x, M_y, M_{xy}\} = \int_{-h/2}^{h/2} \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\} dz \quad \text{and} \quad \{Q_x, Q_y\} = \int_{-h/2}^{h/2} \{\sigma_{xx}, \sigma_{yy}\} dz, \tag{4}
\]

respectively. By using Eq. (4) and D’Alembert’s principle, the equation of motion for an orthotropic FG rectangular thin plate are expressed as [22]

\[
\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_y}{\partial x \partial y} + \frac{\partial^2 M_{xy}}{\partial y^2} + \left(\bar{m} \omega^2 - K_w\right) w = 0, \tag{5}
\]

where \( \bar{m} = \int_{-h/2}^{h/2} \rho(z) dz \) is density of unit area, \( \omega \) is the circular frequency.

3. Hamiltonian system and governing equation

3.1. Hamiltonian system

Defining \( \Psi = \{w, \theta_x, V_x, M_{xy}\}^T \) as a total unknown vector, the governing equations in the Hamiltonian system can be expressed in the matrix form of [16]

\[
\frac{\partial \Psi}{\partial x} = H\Psi, \tag{6}
\]

where \( V_x = Q_x + \partial M_{xy}/\partial y \), \( V_y = Q_y + \partial M_{xy}/\partial x \) are equivalent shear forces; \( H \) is the Hamiltonian operator matrix, i.e.,

\[
H = \begin{bmatrix}
0 & -1 & 0 & 0 \\
D_y \frac{\partial^2}{\partial y^2} & 0 & 0 & \frac{1}{D_y} \\
\frac{D_y D_x - D_x^2}{D_y} \frac{\partial^2}{\partial x \partial y^2} - \left(\bar{m} \omega^2 - K_w\right) & 0 & 0 & -\frac{D_y}{D_x} \frac{\partial^2}{\partial y^2} \\
0 & -4D_x \frac{\partial^2}{\partial y^2} & 1 & 0
\end{bmatrix}. \tag{7}
\]
in which \[ \{D_x, D_y, D_{xy}, D_s\} = \left\{ \frac{\beta}{2\pi}, 2\pi \alpha - \frac{\gamma^2}{2}, 2\pi \beta + \frac{\gamma^2}{2}, 2\pi \gamma \right\} \] are the bending and torsion stiffnesses.

The associate boundary conditions are:
\[ \mathbf{g}_i \Psi = 0 \text{ at } x = 0, a; \quad \mathbf{h}_i \Psi = 0 \text{ at } y = 0, b, \] where “\( i = S \)” and “\( i = C \)” represent the simply supported and clamped boundary conditions, respectively; \( \mathbf{g}_i \) and \( \mathbf{h}_i \) are the boundary index matrices, i.e.,
\[ \mathbf{g}_S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{g}_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{h}_S = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{h}_C = \begin{bmatrix} 1 \\ \frac{\partial}{\partial y} \\ 0 \\ 0 \\ 0 \end{bmatrix}. \]

### 3.2. Symplectic eigenproblem and wave propagation

In the Hamiltonian system, it is natural to use the method of separation of variables to solve Eq. (6). Assuming \( \Psi(x, y) = \psi_j(y) e^{i\mu x} \), the eigen-equation is
\[ \mathbf{H} \psi_j(y) = \mu_j \psi_j(y), \] where \( \mu_j \) and \( \psi_j(y) \) are eigenpairs, which represent the wave propagation parameter and wave shape function, respectively; \( e^{i\mu x} \) represents the wave decrease in the propagation direction [23].

Using boundary conditions (8) (at \( y = 0, b \) and \( i = S \)), \( \mu_j \) and \( \psi_j(y) \) can be obtained as
\[ \mu_n^{(1)} = -\frac{1}{D_y} (D_y \alpha_n^2 + \beta_n + \gamma_n), \quad \mu_n^{(2)} = -\mu_n^{(1)}, \] \[ \mu_n^{(3)} = -\frac{1}{D_y} (D_y \alpha_n^2 + \beta_n - \gamma_n), \quad \mu_n^{(4)} = -\mu_n^{(3)} \]

and
\[ \psi_n^{(1)}(y) = \begin{bmatrix} 1 -\mu_n^{(1)} \mu_n^{(1)} \beta_n - \gamma_n \end{bmatrix}^T \sin(\alpha_n y), \]
\[ \psi_n^{(2)}(y) = \begin{bmatrix} 1 -\mu_n^{(2)} \mu_n^{(2)} \beta_n - \gamma_n \end{bmatrix}^T \sin(\alpha_n y), \]
\[ \psi_n^{(3)}(y) = \begin{bmatrix} 1 -\mu_n^{(3)} \mu_n^{(3)} \beta_n + \gamma_n \end{bmatrix}^T \sin(\alpha_n y), \]
\[ \psi_n^{(4)}(y) = \begin{bmatrix} 1 -\mu_n^{(4)} \mu_n^{(4)} \beta_n + \gamma_n \end{bmatrix}^T \sin(\alpha_n y), \]
respectively. Here, \( \alpha_n = n \pi / b \), \( \beta_n = 2D_y \alpha_n^2 \), \( \gamma_n = 0.5 \sqrt{\chi_2 - D_y \alpha_n^4} \), \( \chi_2 = D_y + 2D_x \), \( \chi_1 = D_y + 2D_x \).

In review of Eqs. (10) and (11), the symplectic eigenvalues can be arranged in an ascending order according to \( \text{Re}(\mu) \), i.e.,
\[ \mu^+ = \{ \mu_n^{(1)}, \mu_n^{(3)}, \mu_n^{(1)}, \mu_n^{(3)}, \ldots, \mu_n^{(1)}, \mu_n^{(3)}, \ldots \} \]
and
\[ \mu^- = \{ \mu_n^{(2)}, \mu_n^{(4)}, \mu_n^{(2)}, \mu_n^{(4)}, \ldots, \mu_n^{(2)}, \mu_n^{(4)}, \ldots \} \]
which represent the vectors of right traveling wave propagation parameter and left traveling wave propagation parameter, respectively. The eigenfunctions corresponding to the eigenvalues are mutually adjoint symplectic orthogonal.
\[ \int_0^b \psi_i^{(+)} J \psi_i^{(+)} dy = -\int_0^b \psi_i^{(-)} J \psi_i^{(-)} dy = \delta_y; \quad \int_0^b \psi_i^{(+)} J \psi_i^{(-)} dy = \int_0^b \psi_i^{(-)} J \psi_i^{(+)} dy = 0 \]  

(12)

where \( J \) is the symplectic identity matrix [15] and \( \delta_y \) is the Kronecker delta. The eigenfunctions \( \psi_i^{(+)} \) and \( \psi_i^{(-)} \) are arranged corresponding to the eigenvalues and constitute wave shape matrices of right traveling waves and left traveling waves, i.e., \( \Phi^+ \) and \( \Phi^- \).

Therefore, the steady state response of the plate can be described in the wave space, i.e.,

\[ \Psi(x, y) = \Phi(y) \mathbf{c}(x), \]  

(13)

where \( \Phi = \{ \Phi^+, \Phi^- \} \) and \( \mathbf{c} = \{ \mathbf{c}^+, \mathbf{c}^- \}^T \) are the wave shape matrix and amplitude vector, respectively.

In addition, we can also take the \( y \)-coordinate to simulate the time coordinate and obtain a solution which is equivalent to Eq. (13). The wave propagation parameter \( \bar{\mu} \) and wave shape function \( \bar{\psi}(x) \) are listed as follows:

\[
\begin{align*}
\bar{\mu}^{(1)} &= -\frac{1}{\sqrt{D_y}} (D_y \bar{\alpha}_n^2 + \bar{\beta}_n + \bar{\gamma}_n), & \bar{\mu}^{(2)} &= -\bar{\mu}^{(1)} \\
\bar{\mu}^{(3)} &= -\frac{1}{\sqrt{D_y}} (D_y \bar{\alpha}_n^2 + \bar{\beta}_n - \bar{\gamma}_n), & \bar{\mu}^{(4)} &= -\bar{\mu}^{(3)},
\end{align*}
\]  

(14)

\[
\begin{align*}
\bar{\psi}^{(1)}(x) &= \left[ 1 - \bar{\mu}^{(1)} \bar{\mu}^{(1)} (\bar{\beta}_n - \bar{\gamma}_n) - (\bar{\beta}_n + \bar{\gamma}_n) \right]^T \sin(\bar{\alpha}_n x) \\
\bar{\psi}^{(2)}(x) &= \left[ 1 - \bar{\mu}^{(3)} \bar{\mu}^{(3)} (\bar{\beta}_n - \bar{\gamma}_n) - (\bar{\beta}_n + \bar{\gamma}_n) \right]^T \sin(\bar{\alpha}_n x) \\
\bar{\psi}^{(3)}(x) &= \left[ 1 - \bar{\mu}^{(1)} \bar{\mu}^{(1)} (\bar{\beta}_n + \bar{\gamma}_n) - (\bar{\beta}_n - \bar{\gamma}_n) \right]^T \sin(\bar{\alpha}_n x) \\
\bar{\psi}^{(4)}(x) &= \left[ 1 - \bar{\mu}^{(3)} \bar{\mu}^{(3)} (\bar{\beta}_n + \bar{\gamma}_n) - (\bar{\beta}_n - \bar{\gamma}_n) \right]^T \sin(\bar{\alpha}_n x),
\end{align*}
\]  

(15)

where \( \bar{\alpha}_n = n \pi / a \), \( \bar{\beta}_n = 2D_x \bar{\alpha}_n^2 \), \( \bar{\gamma}_n = 0.5 \sqrt{\left(2 \chi_x \bar{\alpha}_n^2 \right)^2 + 4D_y \left(\chi_x - D_x \bar{\alpha}_n^4 \right)} \). The total unknown vector can be expressed as

\[ \bar{\Psi} = \bar{\Phi}(x) \bar{\mathbf{c}}(y), \]  

(16)

where \( \bar{\Phi} = \{ \bar{\Phi}^+, \bar{\Phi}^- \} \) and \( \bar{\mathbf{c}} = \{ \bar{\mathbf{c}}^+, \bar{\mathbf{c}}^- \}^T \) are the wave shape matrix and amplitude vector, respectively.

### 3.3. Steady state forced vibration of the FG plate with Naiver boundary conditions

The steady state forced vibration of a simply supported orthotropic FG plate can be obtained by the wave propagation analysis [20]. Based on Eq. (13), the total unknown vector can be expressed as

\[ \Psi(x, y) = \begin{cases} \Phi(y) \mathbf{c}_l(x), & 0 \leq x \leq x_1 \\ \Phi(y) \mathbf{e}_m(x), & x_1 < x \leq x_2 \\ \Phi(y) \mathbf{c}_r(x), & x_2 < x \leq a \end{cases}, \]  

(17)

where the subscripts “l”, “m” and “r” denote the left side of the action area, action area and right side of the action area, respectively; the wave amplitude vectors \( \mathbf{c}_i(x) \) (\( i = l, m \) and \( r \)) are

\[ \mathbf{c}_i(x) = \mathbf{A}_i \left\{ \begin{array}{c} \mathbf{A}_v^- + \int_{x_i}^a T(x_2) \mathbf{e}^- dx_2 \\ \mathbf{T}(a) \mathbf{A}_v^- \end{array} \right\}, \]  

(18a)
\[ \mathbf{e}_m(x) = \Lambda(x) \left\{ A^+ + \int_{x_l}^{x_u} \mathbf{T}(-x)e^x dx \right\}, \quad (18b) \]

\[ \mathbf{e}_r(x) = \Lambda(x) \left\{ A^+ + \int_{x_l}^{x_u} \mathbf{T}(-x)e^x dx \right\}, \quad (18c) \]

where \( \Lambda(x) = \text{diag}\{\mathbf{T}(x), \mathbf{T}(-x)\} \) is the wave propagation matrix in which \( \mathbf{T}(x) = \text{diag}\{e^{i\alpha x}, e^{i\alpha x}, \ldots, e^{i\alpha x}, e^{i\alpha x}, \ldots\} \); \( e^+ \) and \( e^- \) are the directly excited waves and can be obtained as

\[ \{e^+, -e^-\}^T = -\mathbf{J} \int_{x_l}^{x_u} \Phi^T \mathbf{J} \{0, 0, -q_{ext}(x_e, y_e), 0\}^T dy_e. \]

By means of boundary conditions (8) (at \( x = 0, a \) and \( i = S \)), undetermined coefficient vectors \( A^+ \) and \( A^- \) are obtained as

\[ \begin{bmatrix} A^+ \\ A^- \end{bmatrix} = [\mathbf{I} - \mathbf{R}^T(x)\mathbf{T}(x)\mathbf{R}(x)]^{-1} \left[ \mathbf{R}^T(x)\mathbf{R}\int_{x_l}^{x_u} \mathbf{T}(x)e^x dx + \int_{x_l}^{x_u} \mathbf{T}(x)e^x dx \right], \]

where \( \mathbf{R} = -\int_0^b \Phi^T \mathbf{J} \mathbf{f} \mathbf{J} \mathbf{r} \mathbf{J} \mathbf{f} \mathbf{r} dy \), \( \mathbf{f} = \text{diag} \{1, 0, 0, 1\} \).

By means of Eq. (16), the total unknown vector has an equivalent form, i.e.,

\[ \overline{\Phi} (x, y) = \begin{cases} \Phi(x) \overline{\xi}(y), & 0 \leq y \leq y_{e1} \\ \Phi(x) \overline{\mu}(y), & y_{e1} < y \leq y_{e2} \\ \Phi(x) \overline{\nu}(y), & y_{e2} < y \leq b \end{cases}, \quad (19) \]

where \( \overline{\xi}(y) , \overline{\mu}(y) \) and \( \overline{\nu}(y) \) can be obtained from Eq. (18) by replacing “\( x \)” with “\( y \)”.

4. **Steady state forced vibration of CCCC orthotropic FG plates**

The steady state forced vibration of a CCCC plate can be regarded as a superposition of three subproblems (figure 2): (i) a SSSS plate subjected to harmonic excitation \( q_{ext} \); (ii) a SSSS plate with the harmonic moments \( M_y \bigg|_{y=0} = \sum_{n} E_n \sin(\alpha_n y) \) and \( M_y \bigg|_{y=b} = \sum_{n} F_n \sin(\alpha_n y) \); (iii) a SSSS plate with the harmonic moments \( M_y \bigg|_{y=0} = \sum_{n} G_n \sin(\alpha_n y) \) and \( M_y \bigg|_{y=b} = \sum_{n} H_n \sin(\alpha_n y) \) where \( E_n, F_n, G_n \) and \( H_n \) are the coefficients of the Fourier series expansion.
4.1. Solutions of subproblems

The solution of subproblem (i) has been obtained in Eq. (17). To simplify the manipulation, the displacement response can be rewritten as

\[ w_1 = \sum_{n=1}^{\infty} \left[ d_n^{(1)} \sinh \left( \mu_n^{(1)} x \right) + d_n^{(2)} \cosh \left( \mu_n^{(1)} x \right) + d_n^{(3)} \sinh \left( \mu_n^{(1)} x \right) + d_n^{(4)} \cosh \left( \mu_n^{(1)} x \right) \right] \sin \left( \alpha_n y \right) \]  

where \( d_n^{(1)} \), \( d_n^{(2)} \), \( d_n^{(3)} \) and \( d_n^{(4)} \) are constants determined by Eq. (17). Also, Eq. (19) have an equivalent form by using the time-like \( y \)-coordinate, i.e.,

\[ w_1 = \sum_{n=1}^{\infty} \left[ \tilde{d}_n^{(1)} \sinh \left( \tilde{\mu}_n^{(1)} y \right) + \tilde{d}_n^{(2)} \cosh \left( \tilde{\mu}_n^{(1)} y \right) + \tilde{d}_n^{(3)} \sinh \left( \tilde{\mu}_n^{(1)} y \right) + \tilde{d}_n^{(4)} \cosh \left( \tilde{\mu}_n^{(1)} y \right) \right] \sin \left( \tilde{\alpha}_n x \right) \]  

where \( \tilde{d}_n^{(1)} \), \( \tilde{d}_n^{(2)} \), \( \tilde{d}_n^{(3)} \) and \( \tilde{d}_n^{(4)} \) are constants.

The solution of subproblem (ii) is obtained by Eq. (13). The boundary conditions at \( x = 0 \) and \( a \) are

\[ g_s \Psi = \left\{ 0, \sum_{n=1}^{\infty} E_n \sin \left( \alpha_n y \right) \right\}^T \text{ at } x = 0, \quad g_s \Psi = \left\{ 0, \sum_{n=1}^{\infty} F_n \sin \left( \alpha_n y \right) \right\}^T \text{ at } x = a \]  

The displacement response of subproblem (ii) is archived as

\[ w_2 = \sum_{n=1}^{\infty} \left[ \sinh \left( \mu_n^{(3)} a - \mu_n^{(1)} x \right) E_n + \sinh \left( \mu_n^{(1)} x \right) F_n \right] \csc \left( \mu_n^{(3)} a \right) \sin \left( \alpha_n y \right) \]  

\[ - \left[ \sinh \left( \mu_n^{(1)} a - \mu_n^{(1)} x \right) E_n + \sinh \left( \mu_n^{(1)} x \right) F_n \right] \csc \left( \mu_n^{(1)} a \right) \frac{\sin \left( \alpha_n y \right)}{2 \gamma_n}. \]  

Similar to procedure of subproblem (ii), the displacement response of subproblem (iii) is obtained by Eq. (16)

\[ w_3 = \sum_{n=1}^{\infty} \left[ \sinh \left( \tilde{\mu}_n^{(3)} b - \tilde{\mu}_n^{(1)} y \right) G_n + \sinh \left( \tilde{\mu}_n^{(1)} y \right) H_n \right] \csc \left( \tilde{\mu}_n^{(3)} b \right) \sin \left( \tilde{\alpha}_n x \right) \]  

\[ - \left[ \sinh \left( \tilde{\mu}_n^{(1)} b - \tilde{\mu}_n^{(1)} y \right) G_n + \sinh \left( \tilde{\mu}_n^{(1)} y \right) H_n \right] \csc \left( \tilde{\mu}_n^{(1)} b \right) \frac{\sin \left( \tilde{\alpha}_n x \right)}{2 \gamma_n}. \]  

4.2. Superposition of boundary conditions

The displacement response of the CCCC plates can be represented by

\[ w(x,y) = w_1(x,y) + w_2(x,y) + w_3(x,y). \]  

To satisfy the boundary conditions of the four clamped edges, the superposition of the rotation angles of the three subproblems must be vanished. Thus, four sets of simultaneous equations are obtained in the following.

At \( y = 0 \), we have
\[ \overline{d}_i^{(0)} \overline{\mu}_i^{(0)} + \overline{d}_i^{(1)} \overline{\mu}_i^{(1)} + \sum_{n=1}^{\infty} \frac{2i\pi\alpha_n a^2}{D_n \tau_n \lambda_n} \left[ E_n - \cos(i\pi) F_n \right] + \overline{\mu}_i^{(1)} \coth \left( \overline{\mu}_i^{(1)} b \right) \frac{G_i}{2\gamma_i} = 0; \]  

At \( y = b \), we have
\[ \overline{\gamma}_i^{(0)} \cos \left( \overline{\mu}_i^{(0)} b \right) + \overline{\gamma}_i^{(1)} \sin \left( \overline{\mu}_i^{(1)} b \right) + \overline{\gamma}_i^{(3)} \cos \left( \overline{\mu}_i^{(3)} b \right) + \overline{\gamma}_i^{(4)} \sin \left( \overline{\mu}_i^{(4)} b \right) \]
\[ + \sum_{n=1}^{\infty} \frac{2i\pi\alpha_n a^2 \cos(n\pi)}{D_n \tau_n \lambda_n} \left[ E_n - \cos(i\pi) F_n \right] + \overline{\mu}_i^{(1)} \frac{G_i}{2\gamma_i} \coth \left( \overline{\mu}_i^{(1)} b \right) \]
\[ - \overline{\mu}_i^{(3)} \frac{G_i}{2\gamma_i} \coth \left( \overline{\mu}_i^{(3)} b \right) + \overline{\mu}_i^{(4)} \frac{H_i}{2\gamma_i} \coth \left( \overline{\mu}_i^{(4)} b \right) = 0; \]  

At \( x = 0 \), we have
\[ d_i^{(0)} \mu_i^{(0)} + d_i^{(1)} \mu_i^{(1)} = \frac{\mu_i^{(1)} E_i}{2\gamma_i} \coth \left( \mu_i^{(1)} a \right) - \frac{\mu_i^{(3)} E_i}{2\gamma_i} \coth \left( \mu_i^{(3)} a \right) + \frac{\mu_i^{(4)} F_i}{2\gamma_i} \coth \left( \mu_i^{(4)} a \right) \]
\[ - \mu_i^{(3)} \frac{F_i}{2\gamma_i} \coth \left( \mu_i^{(3)} a \right) + \sum_{n=1}^{\infty} \frac{2i\pi\alpha_n b^2}{D_n \tau_n \lambda_n} \left[ G_n - \cos(i\pi) H_n \right] = 0; \]  

At \( x = a \), we have
\[ d_i^{(0)} \mu_i^{(0)} + d_i^{(1)} \mu_i^{(1)} = \frac{\mu_i^{(1)} E_i}{2\gamma_i} \coth \left( \mu_i^{(1)} a \right) + \frac{\mu_i^{(3)} E_i}{2\gamma_i} \coth \left( \mu_i^{(3)} a \right) + \frac{\mu_i^{(4)} F_i}{2\gamma_i} \coth \left( \mu_i^{(4)} a \right) \]
\[ + \sum_{n=1}^{\infty} \frac{2i\pi\alpha_n b^2 \cos(n\pi)}{D_n \tau_n \lambda_n} \left[ G_n - \cos(i\pi) H_n \right] = 0; \]  

where \( \tau_n = i^{2n} \pi^2 + a^2 \left( \mu_i^{(0)} \right)^2 \), \( \lambda_n = i^{2n} \pi^2 + a^2 \left( \mu_i^{(0)} \right)^2 \), \( \bar{\tau}_n = i^{2n} \pi^2 + b^2 \left( \mu_i^{(0)} \right)^2 \), \( \bar{\lambda}_n = i^{2n} \pi^2 + b^2 \left( \mu_i^{(0)} \right)^2 \), \( i = 1, 2, 3, \ldots \). The displacement response of the CCCC orthotropic FG plate can be obtained through the solution of above simultaneous equations.

5. Numerical results

Numerical examples are presented for steady state forced vibration of orthotropic FG rectangular plates using a computer program developed in Matlab. Firstly, the proposed solutions are validated by comparing the displacement response with existing results. Subsequently, the effects of the FG distribution and orthogonal elastic moduli on the steady state response are studied. The displacement amplitude is defined as \( |w(x, y)| \) and the damping loss factor is selected as \( \eta = 0.01 \).

5.1. Validation and convergence study

To validate the proposed solutions, a fully clamped orthotropic rectangular plate subjected to two types of harmonic forces is considered here. The dimensionless displacement amplitudes at the center of the plate subjected to a uniformly-distributed harmonic force and a concentrated harmonic force are tabulated in tables 1 and 2, respectively. \( \omega_{11} \) is the first natural frequency. Clearly, the present results compare well with those reported by Sakata and Hosokawa [24] and Li [25]. The maximum error is 0.59\%. A convergence study is carried out in figure 3 by comparing the present results with Ref. [24] \((b/a = (D_{00} + 2D_1)/D_i = 1.0, \omega = 0.5\omega_{11})\). It is observed from the curve that the dimensionless displacement amplitude converges to that of Sakata and Hosokawa [24] using about 15 terms of eigenfunctions. To ensure the accuracy of numerical results, 20 terms of symplectic eigenfunctions are
taken to compute the steady state response of orthotropic FG rectangular plates in the following examples.

**Table 1.** Dimensionless displacement amplitude \( \left| w \right| \left( \frac{q_0 a^2}{D_s} \right) \times 10^3 \) at the center of a fully clamped orthotropic rectangular plate subjected to a uniformly-distributed harmonic force \( q_0 \).

| \( \frac{b}{a} \) | \( \frac{D_y + 2D_s}{D_s} \) | \( \frac{D_y}{D_s} \) | \( 0.5\omega_{11} \) Ref. [24] Present | \( 0.8\omega_{11} \) Ref. [24] Present |
|---|---|---|---|---|
| 1.0 | 1.0 | 1.0 | 1.7070 | 1.7069 | 3.6243 | 3.6243 |
| 0.5 | 1.0 | 1.9352 | 1.9352 | 4.1053 | 4.1052 |
| 0.5 | 0.5 | 2.4604 | 2.4603 | 5.2294 | 5.2293 |
| 2.0 | 1.0 | 1.0 | 3.4532 | 3.4558 | 7.5008 | 7.5063 |
| 0.5 | 1.0 | 3.6488 | 3.6487 | 7.9616 | 7.9615 |
| 0.5 | 0.5 | 3.6241 | 3.6239 | 7.9778 | 7.9770 |

**Table 2.** Dimensionless displacement amplitude \( \left| w \right| \left( \frac{q_0 a^2}{D_s} \right) \times 10^3 \) at the center of a fully clamped orthotropic rectangular plates subjected to a concentrated harmonic force \( q_0 \) applied at the center of the plate.

| \( \frac{b}{a} \) | \( \frac{D_y + 2D_s}{D_s} \) | \( \frac{D_y}{D_s} \) | \( 0.5\omega_{11} \) Ref. [24] Ref. [25] Present | \( 0.8\omega_{11} \) Ref. [24] Ref. [25] Present |
|---|---|---|---|---|
| 1.0 | 1.0 | 1.0 | 7.1830 * | 7.1739 | 13.9611 * | 13.9519 |
| 0.5 | 1.0 | 8.2324 | 8.2208 | 8.2219 | 16.0300 | 16.018 | 16.0195 |
| 0.5 | 0.5 | 10.5421 * | 10.5254 | 10.5254 | 20.4199 | 20.4033 |
| 2.0 | 1.0 | 1.0 | 8.9696 * | 8.9330 | 16.2345 * | 16.1979 |
| 0.5 | 1.0 | 9.9689 | 9.9577 | 9.9266 | 17.9681 | 17.960 | 17.9258 |
| 0.5 | 0.5 | 11.1983 * | 11.1317 | 19.5268 | 19.4602 |

**Figure 3.** The convergence study.
5.2. Parametric study
After verifying the accuracy and convergence of the present method, the effects of the FG distribution and orthogonal elastic moduli are investigated in this section. Consider an orthotropic FG plate subjected to a uniformly-distributed harmonic force resting on the elastic foundation. The material properties are tabulated in Table 3 [26, 27]. The other computation parameters are taken as \( a = 0.4 \, \text{m}, \quad b = 0.4 \, \text{m}, \quad h = 0.004 \, \text{m}, \quad q_{\text{ext}} = 1 \, \text{N/m}^2 \) \((x_1 = y_1 = 0, \quad x_2 = a, \quad y_2 = b)\) and \( K_w = 70000 \, \text{N/m}^3 \).

Figure 4 depicts the frequency-response at the center of the plate for CCCC boundary condition and two different volume faction indexes \((k = 0.1 \text{ and } 1)\). It is interesting to find that the resonant amplitudes for \( k = 1 \) are greater than those for \( k = 0.1 \) while the greater volume fraction index lead to the lower resonant frequency. To illustrate the observations, variations of the first resonant frequency and the specific stiffness \( D_x / m \) versus various volume fraction indexes, and variations of the resonant amplitude at the center of the plate and the flexibility \( 1/D_x \) versus various volume fraction indexes are plotted in figures 5 and 6, respectively. Since the resonant frequency and its resonant amplitude directly depend on the specific stiffness and flexibility, the variation trends of the resonant frequency and its resonant amplitude are in accordance with those of the specific stiffness and flexibility, respectively. Furthermore, it is noted that the specific stiffness shows a decreasing trend with the increasing volume fraction index while the curve of the flexibility shows an opposite trend. These findings are similar to those in figure 4. Hence, it could explain the reasons for different variation trends of the resonant frequency and its resonant amplitude with the increasing volume fraction index. It also implies that the volume fraction index has a complicated influence on the steady state response of the FG plates. An appropriate volume fraction index may improve the dynamic behaviors of the FG plate and result in a better design.

| Table 3. Material properties. |
|-------------------------------|
| \( E_x \) \text{(GPa)} | \( E_y \) \text{(GPa)} | \( G \) \text{(GPa)} | \( v_x \) | \( v_y \) | \( \rho \) \text{(kg/m}^3) |
| Bottom material | 70 | 35 | 70/2.6 | 0.3 | 0.15 | 2700 |
| Top material | 200 | 100 | 200/2.6 | 0.3 | 0.15 | 5700 |

Figure 4. Frequency-response at the center of the plate.
Figure 5. First resonant frequency and specific stiffness versus various volume fraction indexes.

Figure 6. Resonant amplitude at the center of the plate and flexibility versus various volume fraction indexes.

Subsequently, the effect of the orthogonal elastic moduli on the steady state response of the plate is investigated in figure 7. \( E(z), \nu, \) and \( \rho(z) \) are selected as those in table 3. The remaining material constants are taken as \( G(z) = E(z)/2.6, \nu_x = \nu_y, E(z)/E(z) \) and \( k = 1 \). Frequency-response curves at the center of the plate for CCCC boundary condition are shown in figure 7. Six ratios of \( E(z)/E(z) = 0.5, 1, 2, 5, 10, 20 \) are considered in each figure. For a fixed \( E(z) \), the first resonant frequency increases with the increase of \( E(z)/E(z) \) while its resonant amplitude at the center of the plate shows a decreasing trend. It indicates that the orthogonal elastic moduli have an influence on the steady state response of the plate, which cannot be neglected in the design of engineering structures. In addition, the displacement responses \( \text{Re}(w(x, y)e^{j\omega t}) (\omega t = 0.5\pi) \) of the CCCC orthotropic FG plate with \( E(z)/E(z) = 0.5 \) are shown in figure 8. It is clear to see that the distribution of displacements is smooth and strictly satisfy the CCCC boundary condition.

Figure 7. Frequency-response at the center of the CCCC plate with different modulus ratio.
6. Conclusion

Closed form solutions for steady state forced vibration of a fully clamped orthotropic FG rectangular plate is found by a two-step Hamiltonian-based method. The elastic foundation is considered and modelled as a continuously distributed Winkler’s spring system. At the first step, a Hamiltonian system is established to obtain the steady state forced vibration solution of a Navier-type rectangular plate and expressed in terms of wave shape functions. At the second step, a superposition technique is employed to derive analytical solutions for the steady state response of CCCC plates. Compared to the other analytical methods, the present method is a simple and rigorous without any trial functions in the entire process. Numerical examples are presented to validate the accuracy and stability of the present method. Some new results are given also.

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