2-D MT Inversion Using Genetic Algorithm

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Abstract. In this paper the inverse problem of magnetotelluric (MT) for 2-D are solved by using genetic algorithm (GA). Some quick and effective algorithm for solving forward problem is designed to reduce the computation time of GA. It is shown that as long as the genetic operators and some control parameters are chosen correctly, with some auxiliary techniques very good results can be obtained. Some difficulties encountered in other optimization methods can be overcome.

1. Introduction
The MT inverse problem can be formulated into an optimization problem with a cost function which depends on the geological property parameters to be determined. The complex nonlinear error function usually depends on the unknown parameters. The usual methods to solve this problem are Quasi-Newton method or Conjugate Gradient method[1]. Nevertheless these local search methods may come into local maximum and bring significant difficulties because of calculating derivatives. To overcome these difficulties we take genetic algorithm because it has the advantages of global optimization, robustness and inherent parallelism.

In 1960’s Holland[3] proposed the genetic algorithm according to Darwin’s principle: survival of the fittest. After suitably coding the elements in search domain, this algorithm obtains the optimal solution by use of the genetic operators such as select, cross and mutation generation by generation. We find that as long as the genetic operators and some control parameters are chosen correctly very good results can be obtained. We give some numerical examples in this paper to illustrate the application of the GA and its effectiveness.

The mathematical model of MT are described in Section.2; The improvement algorithm of forward problem is described in Section.3; The fundamental method of GA and the improvement are illustrated in Section.4. Some numerical examples are given in Section.5. Finally, Section.6 gives the conclusion and further work.

2. Mathematical model
After the TM polarization the inverse problem of 2-D MT can be deduced to obtain resistivity parameters in a rectangular domain, that is to find:

\[ \overrightarrow{\rho} = (\rho_1, \rho_2, \cdots, \rho_s)^T, \]

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such that

\[ ERR(\bar{\rho}) = w_1 \sum_{j=1}^{N} \left[ \left( \frac{1}{\omega \mu} \left| \frac{H^j_x}{(H^j_y)_y} \right| \right)^2 - \rho_{\text{app}}(\omega_j) \right]^2 + w_2 \sum_{j=1}^{N} (\phi_{aj} - \phi_{cj})^2 = \min. \]  

(1)

where \( \rho_{\text{app}}(\omega_j) \) denotes the apparent resistivity corresponding to frequency \( \omega_j \); \( \phi_{aj}, \phi_{cj} \) are the measured and calculated phase angles respectively; \( w_1, w_2 \) are the weights of resistivity and phase; and \( H^j_x \) represents a complex function which satisfies the BVP described as System (2).

\[
\begin{align*}
\frac{\partial}{\partial y} \left( \rho \frac{\partial H^j_x}{\partial y} \right) + \frac{\partial}{\partial z} \left( \rho \frac{\partial H^j_x}{\partial z} \right) &= -i \omega j \mu H^j_x, \quad (y, z) \in \Omega = [y_{\min}, y_{\max}] \times [z_{\min}, z_{\max}] \\
H^j_x &= 1, \quad z = z_{\min} \\
\rho \frac{\partial H^j_x}{\partial y} + Z_{HS} H^j_x &= 0, \quad z = z_{\max} \\
\frac{\partial H^j_x}{\partial y} &= 0, \quad y = y_{\min}, y_{\max}
\end{align*}
\]

(2)

where \( \omega_j \) represents the \( j \)th frequency, \( j = 1, 2, \ldots, N \). \( \rho \) denotes the resistivity and can be treated as piecewise constants. i.e. \( \rho(y, z) = \rho_i \), as \( (y, z) \in \Omega_i \subset \Omega, i = 1, 2, \ldots, S \); \( Z_{HS} \) is the impedance on lower boundary.

When \( \bar{\rho} = (\rho_1, \rho_2, \ldots, \rho_j)^T \) is given, the solution of the above BVP uniquely exist, so that we can calculate the theoretic apparent resistivity \( \frac{1}{\omega \mu} \left| \frac{H^j_x}{(H^j_y)_y} \right| \) and \( ERR \) can be defined. The problem goes to an unconstrained optimization problem.

### 3. The improvement of algorithm for forward problem

We use finite element method (FEM) to solve the forward problem. After rearrange the stiff matrix, A sparse block matrix with two same diagonal blocks which are positive defined, symmetric can be obtained:

\[
\begin{pmatrix}
K_r - K_i \\
K_i & K_r
\end{pmatrix}
\begin{pmatrix}
V_r \\
V_i
\end{pmatrix}
= \begin{pmatrix}
F_r \\
F_i
\end{pmatrix}.
\]

(3)

By noticing that \( K_r \) and \( K_i \) are both sparse, We take 1-D compact storage techniques to deposit their lower part. By use of some bandwidth optimization technique, we significantly reduce the bandwidth of \( K_r \) and \( K_i \) after re-coding the node points of FEM mesh. The block relaxation iteration method is used to solve system (3). The iteration scheme is illuminated by equation (4).

\[
\begin{align*}
V_{r}^{I+1} &= \lambda K_r^{-1}(F_r + K_i V_{r}^{I}) + (1 - \lambda)V_r^I \\
V_{i}^{I+1} &= \lambda K_i^{-1}(F_i + K_r V_{i}^{I+1}) + (1 - \lambda)V_i^I.
\end{align*}
\]

(4)

When \( 1 < \lambda < 2 \) is the over-relaxation factor and \( V_{r}^{I}, V_{i}^{I} \) are the \( i \)-th iteration values of \( V_{r}^{I}, V_{i}^{I} \) respectively. It is noticed that after we obtain \( K_r^{-1} \) in each iteration only some matrix multiplication should be taken. We compute \( K_r^{-1} \) by Crout decomposition with 1-D compact storage technique. In this way, our algorithm reaches optimal tradeoff between the storage and the time cost for the computer.

### 4. Solution of the inverse problem

Minimization problem of 2-D MT inversion can be transferred to a maximization problem through a proper transformation. Suppose that \( y = f(X) \) is the function to be maximized with variable, \( X = (x_1, x_2, \ldots, x_n) \).
Genetic algorithm (GA) is a search method modelling the evolution of life. Independent variable can be viewed as an individual in the search domain whose function value is its fitness. Then we can code every individual by some rules.

For example, Suppose the search domain is: \([x_{\text{min}}, x_{\text{max}}]\). The method that transform \(x_i\) to binary codes is: First transform \([x_{\text{min}}, x_{\text{max}}]\) to \([0, 2^l]\) linearly, and \(x_i\) is transformed to:

\[
\overline{x_i} = (x_i - x_{\text{min}}) \frac{2^l - 1}{x_{\text{max}} - x_{\text{min}}},
\]

then we transform it to a binary string \((\overline{x_i})_2\), which is called the binary coding of \(x_i\). Inversely, Suppose the \(l\) binary codes of \(x_i\) is known \((\overline{x_i})_2\). The decoding process is that first transform it to decimal codes: \(x_i\), then:

\[
x_i = x_{\text{min}} + \frac{x_i}{2^l - 1}(x_{\text{max}} - x_{\text{min}}).
\]

We arrange \((\overline{x_1})_2, (\overline{x_2})_2, \ldots, (\overline{x_n})_2\) as an \(n \times l\) binary string. This binary string is finally the binary codes of the point in search domain. It can mimic the chromosome of individuals while the elements of the binary string can be viewed as the genes.

\(N\) individuals are selected randomly to form the initial population. There are four main genetic operators such as select, copy, cross and mutation. The probability of select is decided by the fitness of individuals. The higher the fitness is, the higher the probability is for the individual to be chosen. By select, copy, cross and mutation, we can obtain the new generation. This iteration continues until the stop criteria is satisfied. This process is generally called simple genetic algorithm (SGA).

4.1. Real number coding

Real number coding has many advantages which are listed below:

(i) Overcome the disadvantage of Hamming cliff induced by binary coding. For example 01111 and 11110 represent decimal 15 and 16 respectively. In order to change 15 to 16, we must change all the bits. This defect decreases the search efficiency. It is almost impossible to change all bits simultaneously. This is called Hamming cliff.

(ii) Refine the genetic operators. For example, instead of one-point cross or multi-point cross, we may take uniform cross. Suppose two individuals are selected to cross, their real number codes are

\[
x_1 = (x^1_1, x^1_2, \ldots, x^1_m), \quad x_2 = (x^2_1, x^2_2, \ldots, x^2_m).
\]

First generate \(m\) random numbers: \(a_i \in (0, 1), i = 1, 2, \ldots, m,\) After cross we get two new individuals:

\[
x_a = (y^1_1, y^1_2, \ldots, y^1_m), \quad x_\beta = (y^2_1, y^2_2, \ldots, y^2_m)
\]

\[
y^1_i = a_i x^1_i + (1 - a_i) x^2_i
\]

\[
y^2_i = a_i x^2_i + (1 - a_i) x^1_i \quad i = 1, 2, \ldots, m
\]

(iii) Mutation has more choices. For example, Uniform mutation. First randomly select an element of an individual: \(x_k \in [a_k, b_k]\). Then randomly select a real number \(x'_k\) in \([a_k, b_k]\) to substitute \(x_k\), that is:

\[
x_i = \begin{cases} 
  x_i, & i \neq k \\
  x'_i, & i = k
\end{cases}
\]

(iv) We select genetic operators and corresponding probabilities and the population size to obtain good results. The Elitism technique is taken so that the GA can get to the optimal solution in probability 1.
4.2. Selection of the cost function
We select the cost function as:

\[
F(\bar{\rho}) = \exp \left\{ -w_1 \sum_{j=1}^{N} \left[ \ln \left( \frac{1}{\omega_j \mu} \left| \frac{H_x}{H_y} \right| \right) - \ln (\rho_{\text{app}}(\omega_j)) \right]^2 - w_2 \sum_{j=1}^{N} (\phi_{aj} - \phi_{cj})^2 \right\} + M \sum_{j=1}^{M} W_j \left( \rho_{\text{app}}(\omega_j) - \rho_i^\theta \right)^2
\]

(9)

where \(w_1, w_2\) are the weights of resistivity and phase, and \(\rho_i^\theta\) denotes the estimate value of the apparent resistivity. We mainly focus on the weights selection of cost function.

4.3. Multi-scale method
The unknown parameter of MT inversion is resistivity which may vary from several ohm-meters to several thousand ohm-meters. Since the complexity and diversity of the physical model, the search scope is very wide and the number of iteration times is very large so that the GA is time consuming. We use multi-scale search method [4] as follows. Initially we do GA search in a large scale. After several iterations we reduce and re-center the searching domain according to the variant and the mathematical expectation of the parameters, then search again in a smaller scale. This process can be repeated several times and the computation time is reduced significantly.

4.4. Parallelism
Due to the inherent parallelism of GA, We design the parallel algorithm on SGI with 16 CPUs. The computation time is further reduced since the parallel computation of the fitness function.

5. The numerical results of 2-D MT problem
After taking the improvement techniques of forward and inverse problem, Very good results can be obtained without pre-information.

5.1. The comparison between 1-D calculation and theoretical value
We validate the accuracy of our forward problem algorithm in 1-D geophysical model that the exact solution can be obtained. The solution can be described by equation (10).

\[
H_i (y, z) = U (z) + iV (z) ,
\]

(10)

where \(U\) and \(V\) are the horizontal and vertical part of magnetic vector respectively. In the case of \(\rho = 10000, \omega = 2, \mu = 4\pi \times 10^{-7}, z_n = 10000\), approximation of \(U\) and \(V\) obtained by our method and exact solution of \(U\) and \(V\) and their errors are illustrated by Figure.1 and Figure.2, respectively.

It is obvious that our algorithm has higher accuracy. The solution for forward problem is well done.

5.2. the impact of weights in cost function
It is demonstrated in Figure.3 that we can obtain the best result when the weights of resistivity and phase are in inverse proposition to their variants. We take seven layers geological model including a rift and an abnormality as an example. the probability of cross is 0.6, the probability of mutation is 0.01, the number of iteration times is 20 and the inversion search scope is 1000 to 4500.

Experiments show that if the variants of resistivity are ten times larger than the phase, the best result can be get when their weights are 1:10.
5.3. Multi-scale numerical example
The result of inverse problem taking multi-scale technique is shown in Table 1. The population and iteration number are 20, the cross probability is 0.6, and the mutation probability is 0.01. In the initial scale search, the search space is $[1, 1000]^4$, We reduce the search scale twice.

It can be seen that by use of multi-scale search only totally 60 times of iteration, the maximal relative error reaches 0.004. Without taking multi-scale search, to obtain the same accuracy the computation time of GA would be ten times longer.
Table 1. Inverse value of 4 parameters with multi-scale technique (unit: Ω·m)

| parameters      | ρ_1  | ρ_2  | ρ_3  | ρ_4  |
|-----------------|------|------|------|------|
| accurate value  | 300  | 80   | 900  | 100  |
| 1\textsuperscript{st} scale value | 300.87 | 80.78 | 906.22 | 130.21 |
| 2\textsuperscript{nd} scale value | 292.39 | 79.28 | 894.45 | 106.49 |
| 3\textsuperscript{rd} scale value | 299.91 | 80.02 | 904.90 | 100.40 |

5.4. The Impact of Parallel Algorithm

Table 2. The Impact of Parallel Algorithm

| cpu No. | Param1 | Param2 | Param3 | Param4 | initial population | Generation No. | time(s) |
|---------|--------|--------|--------|--------|-------------------|----------------|---------|
| 1       | 305.36 | 89.77  | 98.24  | 914.16 | 20                | 20             | 984     |
| 8       | 305.36 | 89.77  | 98.24  | 914.16 | 20                | 20             | 587     |
| 1       | 303.32 | 89.43  | 100.42 | 886.83 | 200               | 20             | 7508    |
| 8       | 303.32 | 89.43  | 100.42 | 886.83 | 200               | 20             | 3846    |

6. Conclusion and further work

It is shown that by improved GA and the forward problem algorithm suggested by us, the 2-D MT inversion can be ameliorated both in robustness and accuracy. We can also see its promising highly parallel properties.

In order to further reduce computation time, we will focus on the further improvement of the genetic operators and refine the parallel algorithm. Generalize to 3-D MT Inversion and realize the combined inversion of MT and seismic are also our future work.

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