ON THE KODAIRA DIMENSION OF ORTHOGONAL MODULAR VARIETIES

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Abstract. We prove that up to scaling there are only finitely many integral lattices $L$ of signature $(2, n)$ with $n \geq 21$ or $n = 17$ such that the modular variety defined by the orthogonal group of $L$ is not of general type. In particular, when $n \geq 108$, every modular variety defined by an arithmetic group for a rational quadratic form of signature $(2, n)$ is of general type. We also obtain similar finiteness in $n \geq 9$ for the stable orthogonal groups. As a byproduct we derive finiteness of lattices admitting reflective modular form of bounded vanishing order, which proves a conjecture of Gritsenko and Nikulin.

Contents

1. Main results 1
2. Convention 7
3. Construction of cusp form 8
4. Reflective obstruction 14
5. Single volume estimate 18
6. Volume sum 29
7. Effective computation 34

Appendix A. Singularity over 0-dimensional cusp 38
References 45

1. Main results

It is one of classical problems in the theory of modular forms of several variables to determine the birational type of arithmetic quotients of Hermitian symmetric domains. Tai [37], Freitag [9] and Mumford [26] proved that the Siegel modular variety $\mathcal{A}_g$ is of general type in $g \geq 7$, which first revealed the phenomenon that in higher dimension, modular varieties would be often of general type even for basic class of arithmetic groups, hence unirational case should be rare. Our purpose is to address this problem for modular varieties of orthogonal type.

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Let $L$ be an integral lattice of signature $(2, n)$ and $O(L)$ be its orthogonal group. The Hermitian symmetric domain $\mathcal{D}_L$ of type IV attached to $L$ is defined as one of the two components of the space

$$\{ \mathbb{C} \omega \in \mathbb{P}(L \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \}.$$ 

Let $O^+(L)$ be the subgroup of $O(L)$ preserving $\mathcal{D}_L$. The quotient space

$$\mathcal{F}_L = O^+(L) \backslash \mathcal{D}_L$$

has the structure of a quasi-projective variety of dimension $n$. It is invariant under scaling of $L$.

**Theorem 1.1.** Up to scaling there are only finitely many integral lattices $L$ of signature $(2, n)$ with $n \geq 21$ or $n = 17$ such that $\mathcal{F}_L$ is not of general type. In particular, when $n \geq 108$, $\mathcal{F}_L$ is always of general type.

The proof is effective: we will derive an explicit bound $D(n)$ determined by $n$ such that for primitive lattices $L$ of signature $(2, n)$, $\mathcal{F}_L$ is of general type whenever the exponent $D(L)$ of its discriminant group $A_L$ satisfies $\sqrt{D(L)} \geq D(n)$. (Recall that the exponent of a finite abelian group is the maximal order of its elements.) Asymptotically,

$$D(n) \sim \frac{3^2 \cdot 2^{2n+1} \cdot \pi^{n/2+1} \cdot e^2}{\Gamma(n/2 + 1)}.$$

The absence of non-general type case in large $n$ is a consequence of the convergence $D(n) \to 0$. The bound $n \geq 108$ is obtained by computing a variant of this estimate, rather than itself (§7.1). In this way, the logic to deduce finiteness is to show, in a quantitative manner, that $\mathcal{F}_L$ must be of general type if the primitive lattice $L$ is “large”, measuring the size of $L$ by $n$ and $D(L)$.

As for the non-existence in higher dimension, the case of full orthogonal group covers that of general arithmetic group.

**Corollary 1.2.** Let $V$ be a rational quadratic space of signature $(2, n)$ with $n \geq 108$ and $\Gamma$ be an arithmetic subgroup of $O^+(V)$. The quotient space $\Gamma \backslash \mathcal{D}_V$ is always of general type.

This holds because we can find a lattice $L \subset V$ that is stable under the action of $\Gamma$ and hence $\Gamma \backslash \mathcal{D}_V$ dominates $\mathcal{F}_L$, the latter being of general type.

Another class of arithmetic groups that are often studied is the stable orthogonal groups $\tilde{O}^+(L)$ for $L$ even, which is the kernel of $O^+(L) \to O(A_L)$. The quotient $\tilde{O}^+(L) \backslash \mathcal{D}_L$ is a covering of $\mathcal{F}_L$ (and changes under scaling). For them we obtain finiteness result in $n \geq 9$.

**Theorem 1.3.** There are only finitely many even lattices $L$ of signature $(2, n)$ with $n \geq 9$ such that $\tilde{O}^+(L) \backslash \mathcal{D}_L$ is not of general type.
The study of Kodaira dimension of orthogonal modular varieties has been pioneered in the nineties by Kondō [21], [22] and Gritsenko [11], whose main object was the moduli spaces of polarized $K3$ surfaces. They created several techniques for constructing pluricanonical forms, which were subsequently developed by Gritsenko-Hulek-Sankaran in the series of fundamental work [12], [13], [14]. In particular, in [12] they almost completed the $K3$ case by using quasi-pullback of the Borcherds $\Phi_{12}$ function [4]. This method gives a fairly nice bound (see also [15], [16], [38]), but can be applied only in dimension $n < 26$. On the other hand, their second paper [14] (originally designed for the $K3$ case before [12]) used the Gritsenko lifting [11] and estimate of Hirzebruch-Mumford volume [13], and studied for the first time a series of higher dimensional orthogonal modular varieties. In contrast to the quasi-pullback of $\Phi_{12}$, the method of [14] gives coarser bound in lower dimension but instead can be applied in any dimension. The proof of Theorem 1.1 is based on a generalization of the method of [14].

In algebraic geometry, orthogonal modular varieties also appear as the period spaces of (lattice-)polarized holomorphic symplectic manifolds. Theorem 1.1 says that the moduli spaces of polarized symplectic manifolds must be of general type when the second Betti number is sufficiently large. Informally, one cannot have explicit parametrization of generic such varieties. For known examples, Theorems 1.1 and 1.3 cover the O’Grady’s 10-dimensional case and the $K3^{[N]}$-type case, proving finiteness of polarization types with non-general type moduli space. In particular, when $N >> 0$, moduli space for $K3^{[N]}$-type is of general type for any polarization type. This extends the results of [15], [16]. A natural question is whether there are only finitely many deformation types of polarized symplectic manifolds with non-general type moduli space. In view of Huybrechts’ theorem [18], the gap between this problem and results as above rests on the possibility of Fujiki constant.

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1.1. Structure of the proof. We now give a coherent account of the proof. Let $L$ be an integral lattice of signature $(2, n)$. A standard approach for proving that $\mathcal{F}_L$ is of general type is to produce pluricanonical forms on a toroidal compactification of $\mathcal{F}_L$ via modular forms. When $n \geq 9$, Gritsenko-Hulek-Sankaran [12] showed that there exists a projective toroidal compactification $\overline{\mathcal{F}}_L$ of $\mathcal{F}_L$ that has only canonical quotient singularity and has no brach divisor in the boundary. (In the Appendix we supplement their proof for the 0-dimensional cusp case.) Furthermore, they showed that when $n \geq 3$, every component of the ramification divisor of the projection $\mathcal{D}_L \to \mathcal{F}_L$
is defined by a reflection of $L$, in particular has ramification index 2. The canonical divisor of $\overline{F}_L$ is then $\mathbb{Q}$-linearly equivalent to

$$K_{\overline{F}_L} \sim_{\mathbb{Q}} n\mathcal{L} - \Delta - B/2,$$

where $\mathcal{L}$ is the $\mathbb{Q}$-line bundle of modular forms of weight 1 (the Hodge bundle), $\Delta \subset \overline{F}_L$ the boundary divisor, and $B \subset \overline{F}_L$ the branch divisor of $\mathcal{O}_L \to \mathcal{F}_L$. The bundle $\mathcal{L}$ is big, and this is the source for proving that $K_{\overline{F}_L}$ is big. We view $\Delta$ and $B/2$ as obstruction for $K_{\overline{F}_L}$ to be big, and deal with them separately by dividing the canonical weight $n$.

**Theorem 1.4.** (1) Let $n \geq 21$ or $n = 17$. For every lattice $L$ of signature $(2, n)$ there exists a nonzero cusp form of weight $< n$ with respect to $O^+(L)$.

(2) Let $4|n$ with $n \geq 16$. For every lattice $L$ of signature $(2, n)$ there exists a nonzero cusp form of weight $n$ with respect to $O^+(L)$.

**Theorem 1.5.** Fix a rational number $a > 0$. Up to scaling there are only finitely many lattices $L$ of signature $(2, n)$ such that the $\mathbb{Q}$-divisor $a\mathcal{L} - B/2$ of $\mathcal{F}_L$ is not big.

Theorem 1.4 (2) is not used here. In Theorem 1.5, sections of $a\mathcal{L}$ over $\mathcal{F}_L$ always extend over $\overline{F}_L$ by the Koecher principle, so we may replace $\mathcal{F}_L$ by $\overline{F}_L$.

It is straightforward to derive Theorem 1.1 from these two sub-theorems. Let $n' < n$ be the weight of cusp form in Theorem 1.4 (1), and we apply Theorem 1.5 with $a = 1$. This tells that in the range $n \geq 21$ or $n = 17$, for all but finitely many lattices (up to scaling), we can find a division

$$K_{\overline{F}_L} \sim_{\mathbb{Q}} (n'\mathcal{L} - \Delta) + (n''\mathcal{L} - B/2)$$

such that $n'\mathcal{L} - \Delta$ is effective and $n''\mathcal{L} - B/2$ is big. Therefore $K_{\overline{F}_L}$ is big for those lattices $L$. Since $\overline{F}_L$ has canonical singularity, its desingularization is of general type. This proves Theorem 1.1.

Theorems 1.4 and 1.5 are independent, and both effective. In Theorem 1.4 (1), the weight of cusp form can be taken to be $n/2 + l + 5$ where $l \leq 6$ is as defined in Table 1. In particular, it does not exceed $n/2 + 11$. In Theorem 1.5, finiteness up to scaling for integral lattices is equivalent to finiteness for primitive lattices. Then, for primitive $L$, we show that $a\mathcal{L} - B/2$ is big if the exponent $D(L)$ of $A_L$ exceeds the explicit bound (6.7):

$$\sqrt{D(L)} \geq g(n) \cdot (1 + a^{-1})^{n-1} \cdot (n/2a)$$

$$\sim \frac{3^2 \cdot 2^{2n+11} \cdot \pi^{n/2+1}}{\Gamma(n/2+1)} \cdot (1 + a^{-1})^{n-1} \cdot (n/2a).$$

The asymptotic (1.1) is obtained by putting $a = n/2 - 11$ in this bound.

For Theorem 1.5 it suffices to prove finiteness for fixed $n$, in view of Theorem 1.1. We use in place of Theorem 1.4 (1) the following.
Theorem 1.6. For all but finitely many even lattices $L$ of signature $(2, n)$ with $n \geq 5$ and containing $2U$, we can find a nonzero cusp form of weight $< n$ with respect to $\tilde{\mathcal{O}}^+(L)$.

Combined with Theorem 1.5 (note that $U$ is primitive and that the ramification divisor of $\tilde{\mathcal{O}}^+(L)$ is contained in that of $\mathcal{O}^+(L)$), this proves finiteness of even lattices $L$ with $n \geq 9$ and containing $2U$ such that $\tilde{\mathcal{O}}^+(L) \backslash D_L$ is not of general type. In order to extend this to general even lattices, we use overlattice construction. If $L'$ is a (finite-index) overlattice of a lattice $L$, we have $\tilde{\mathcal{O}}^+(L) \subset \tilde{\mathcal{O}}^+(L')$ inside $\mathcal{O}^+(L_\mathbb{Q}) = \mathcal{O}^+(L'_\mathbb{Q})$, hence $\tilde{\mathcal{O}}^+(L) \backslash D_L$ dominates $\tilde{\mathcal{O}}^+(L') \backslash D_{L'}$.

Lemma 1.7. Let $L$ be an even lattice of signature $(2, n)$ with $n \geq 8$. There exists an even overlattice $L'$ of $L$ containing $2U$ such that $D(L') = D(L)$.

Proof. Recall that even overlattice $L'$ of $L$ corresponds to isotropic subgroup $G = L'/L$ of $A_L$ and $A_{L'} \simeq G^+/G$. By Nikulin [27], $L'$ contains $2U$ if $G^+/G$ has length $\leq n - 3$. Let $A_L = \oplus_p A_p$ be the decomposition into $p$-parts. By Wall’s classification [40], there exists a nondegenerate subgroup $A'_p$ of $A_p$ of the same exponent as $A_p$ and length $\leq 2$. We have $A_p = A'_p \oplus (A'_p)^\perp$. If $G_p$ is a maximal isotropic subgroup of $(A'_p)^\perp$, $G_p^+ \cap (A'_p)^\perp / G_p$ is anisotropic and so has length $\leq 3$. We then put $G = \oplus_p G_p$. $\square$

By this lemma, we see that for even lattices $L$ at each $n \geq 9$, $\tilde{\mathcal{O}}^+(L) \backslash D_L$ must be of general type if $D(L)$ exceeds some bound. Since $|A_L| \leq D(L)^{n/2}$, Theorem 1.3 follows from finiteness of class number. (For $\tilde{\mathcal{O}}^+(L)$ the bound of $|A_L|$ and $n$ can be improved: see [24] for detail.)

Theorems 1.1 and 1.3 are thus reduced to Theorems 1.4, 1.5, and 1.6. Theorems 1.4 and 1.6 are proven in §3 via the Gritsenko-Borcherds additive lifting [11], [2]. For Theorem 1.4 we use an explicit combination of Eisenstein series, and for Theorem 1.6 we apply a recent result of Bruinier-Ehlen-Freitag [5]. The proof of Theorem 1.5 occupies §4 – §6. In §4 we relate the problem to the comparison of Hirzebruch-Mumford volume between $\mathcal{F}_L$ and its branch divisors, generalizing an argument of [14]. This volume ratio will be estimated in §5 and §6 for primitive $L$. In §5 we give an estimate for each component of the branch divisor, and in §6 we take their sum over all components. The proof of Theorems 1.1 and 1.3 will be thus completed at the end of §6 except the bound $n \geq 108$.

§7 is devoted to some explicit calculation. In §7.1 we derive the bound $n \geq 108$ by refining the bound (1.1) for a particular class of lattices. In §7.2 we work out the odd unimodular lattices as a typical example of transition of Kodaira dimension. In the Appendix we prove that toroidal compactification has canonical singularity over the 0-dimensional cusps when the
fans are chosen regular. This result was first found by Gritsenko-Hulek-
Sankaran [12] and is one of the basis of the present article, but their proof
needs to be modified.

In the rest of the introduction, we explain another direct consequences of
Theorems 1.4 and 1.5

1.2. Special orthogonal group. Let $\text{SO}^+(L)$ be the subgroup of $\text{O}^+(L)$ con-
sisting of isometries of determinant 1. When $n$ is odd, $\text{O}^+(L)$ is generated
by $\text{SO}^+(L)$ and $-1$, so the quotient $\text{SO}^+(L)\backslash \mathcal{D}_L$ is the same as $\mathcal{F}_L$. On the
other hand, when $n$ is even, $\text{SO}^+(L)$ contains no reflection nor its composi-
tion with $-1$, so the projection $\mathcal{D}_L \to \text{SO}^+(L)\backslash \mathcal{D}_L$ is unramified in codi-
mension 1. Furthermore, canonical forms on smooth projective models of
$\text{SO}^+(L)\backslash \mathcal{D}_L$ correspond to cusp forms of weight $n$ with respect to $\text{SO}^+(L)$
(cf. [12], [9]). Theorem 1.4 implies the following.

Corollary 1.8. (1) Let $n \geq 22$ be even. Then $\text{SO}^+(L)\backslash \mathcal{D}_L$ is of general type
for every lattice $L$ of signature $(2, n)$.

(2) Let $4|n$ with $n \geq 16$. For every lattice $L$ of signature $(2, n)$, smooth
projective models of $\text{SO}^+(L)\backslash \mathcal{D}_L$ have positive geometric genus. In particu-
lar, $\text{SO}^+(L)\backslash \mathcal{D}_L$ has nonnegative Kodaira dimension for $n = 16, 20$.

1.3. Reflective modular forms. Let $n \geq 3$. A modular form $F$ on $\mathcal{D}_L$ with
respect to some $\Gamma < \text{O}^+(L)$ and a character is said to be reflective if $\text{div}(F)$
is set-theoretically contained in the ramification divisor of $\mathcal{D}_L \to \mathcal{F}_L$. If
$F$ has weight $\alpha$ and every component of $\text{div}(F)$ has multiplicity $\leq \beta$, we
say (temporarily) that $F$ has slope $\leq \beta/\alpha$. In that case, taking the average
product of $F$ over $\Gamma\backslash \text{O}^+(L)$, we see that the $\mathbb{Q}$-divisor $\beta B/2 - \alpha \mathcal{L}$ of $\mathcal{F}_L$ is
$\mathbb{Q}$-effective. Hence $(\alpha/\beta)\mathcal{L} - B/2$ cannot be big by the Koecher principle.
For every $r \geq \beta/\alpha$, $r^{-1}\mathcal{L} - B/2$ is not big too. Theorem 1.5 implies the
following.

Corollary 1.9. Let $r > 0$ be a fixed rational number. Then up to scaling
there are only finitely many lattices $L$ of signature $(2, n)$ with $n \geq 4$ which
carries a reflective modular form of slope $\leq r$. In particular, for a fixed
natural number $\beta$, there are up to scaling only finitely many lattices $L$ with
$n \geq 4$ which carries a reflective modular form of vanishing order $\leq \beta$.

Gritsenko and Nikulin [17] defined Lie reflective modular forms as re-
flexive modular forms of vanishing order $\leq 1$ with some conditions on the
Fourier coefficients. Their motivation comes from the theory of generalized
Kac-Moody algebras. They conjectured that the set of lattices possessing
such a modular form is finite up to scaling ([17] Conjecture 2.5.5). Corol-
lary 1.9 gives a positive answer in $n \geq 4$: 
Corollary 1.10. Up to scaling there are only finitely many lattices \( L \) of signature \((2, n)\) with \( n \geq 4 \) which carries a Lie reflective modular form.

In the singular weight case, reflective modular forms are classified in [33], [8], [34] for a certain class of simple lattices.

2. Conventions

We summarize basic definitions. By an (integral) lattice \( L \) we mean a free \( \mathbb{Z} \)-module of finite rank equipped with a nondegenerate symmetric bilinear form \( (\ , \) : \( L \times L \to \mathbb{Z} \). The lattice \( L \) is said to be even if \((l, l) \in 2\mathbb{Z}\) for every \( l \in L \). The scaling \( L(a) \) of a lattice \( L \) by a natural number \( a \geq 1 \) has the same underlying \( \mathbb{Z} \)-module as \( L \), with the pairing multiplied by \( a \). A lattice \( L \) is said to be primitive if it is not isometric to a scaling of any other lattice. A vector \( l \in L \) is said to be primitive if \( L/\mathbb{Z}l \) is free. For such \( l \), the positive generator of the ideal \((l, L)\) of \( \mathbb{Z} \) is denoted by \( \text{div}(l) \). When \((l, l) \neq 0 \), the orthogonal splitting \( L = \mathbb{Z}l \oplus (l^⊥ \cap L) \) holds if and only if \( \text{div}(l) = |(l, l)| \). The rank 2 hyperbolic even unimodular lattice is called the hyperbolic plane and will be denoted by \( U \).

The dual lattice of a lattice \( L \) is written as \( L^\vee \). The quotient group \( A_L = L^\vee / L \) is called the discriminant group. Its length is denoted by \( l(A_L) \). \( A_L \) is equipped with a natural \( \mathbb{Q}/\mathbb{Z} \)-valued symmetric bilinear form. When \( L \) is even, this symmetric form comes from the \( \mathbb{Q}/2\mathbb{Z} \)-valued quadratic form \( A_L \to \mathbb{Q}/2\mathbb{Z}, l + L \mapsto (l, l) + 2\mathbb{Z} \), which we call the discriminant form of \( L \). In some literatures, scaling of this form by \( 1/2 \) is called the discriminant form. The kernel of the natural map \( \text{O}^+(L) \to \text{O}(A_L) \) is denoted by \( \text{O}^+(L) \) and called the stable orthogonal group.

The genus of a lattice \( L \) is the set of lattices \( L' \) of the same signature as \( L \) such that \( L \otimes \mathbb{Z}_p \simeq L' \otimes \mathbb{Z}_p \) for every \( p \). By the Hasse-Minkowski theorem, there is no loss of generality in assuming that \( L' \) is contained in \( L_\mathbb{Q} \). By Nikulin [27], two even lattices of the same signature are in the same genus if and only if their discriminant forms are isometric. Two lattices \( L', L'' \) on \( L_\mathbb{Q} \) are said to be properly equivalent if \( \gamma(L') = L'' \) for some \( \gamma \in \text{SO}(L_\mathbb{Q}) \). If we require only \( \gamma \in \text{O}(L_\mathbb{Q}) \), this is equivalent to \( L' \simeq L'' \) (abstractly isometric).

Let \( L \) be a lattice of signature \((2, n)\) with \( n \geq 3 \). Let \( \mathcal{O}(-1) \to \mathcal{D}_L \) be the restriction of the tautological bundle over \( \mathbb{P}(L_\mathbb{C}) \). The complement of the zero section in \( \mathcal{O}(-1) \) is identified with the affine cone \( \mathcal{D}^*_L \) over \( \mathcal{D}_L \) (the vertex removed). A modular form of weight \( k \) with respect to a finite-index subgroup \( \Gamma \) of \( \text{O}^+(L) \) is a \( \Gamma \)-invariant holomorphic section of \( \mathcal{O}(-k) \). It corresponds to a \( \Gamma \)-invariant holomorphic function on \( \mathcal{D}^*_L \) that is homogeneous of degree \(-k\) on each fiber of \( \mathcal{D}^*_L \to \mathcal{D}_L \). We write \( M_k(\Gamma) \) for the space of modular forms of weight \( k \) with respect to \( \Gamma \). When \( \Gamma \) contains \(-1\), we
will consider only even weight $k$ because in that case modular forms of odd weight must be identically zero.

Let $l \in L$ be a primitive isotropic vector, which corresponds to the 0-dimensional rational boundary component $\mathbb{C}l$ of $\mathcal{D}_L$. Let $M = l^\perp \cap L/\mathbb{Z}l$. Choose a vector $l' \in L_\mathbb{Q}$ with $(l, l') = 1$, and identify $M_\mathbb{Q}$ with $\langle l, l' \rangle^\perp \cap L_\mathbb{Q}$.

Let $M_\mathbb{R}^+$ be the positive cone in $M_\mathbb{R}$, i.e., one of the two components of $\{ m \in M_\mathbb{R} | (m, m) > 0 \}$, and $\mathcal{D}_l = M_\mathbb{R} + iM_\mathbb{R}^+$ be the associated tube domain.

We have an embedding depending on $l'$

$$\mathcal{D}_l \hookrightarrow \mathcal{D}_l^*, \quad v \mapsto l' + v - \frac{1}{2}((v, v) + (l', l'))l,$$

whose image is $\{ \omega \in \mathcal{D}_l^* | (\omega, l) = 1 \}$ which gives a nowhere vanishing section of $\mathcal{O}(-1)$. This also induces an isomorphism $\mathcal{D}_l \simeq \mathcal{D}_l$ (tube domain realization). In this way, depending on the choice of $l'$, modular forms on $\mathcal{D}_l$ are translated to holomorphic functions $F(Z)$ on $\mathcal{D}_l$. It is invariant under translation by a lattice $U(l)_\mathbb{Z}$ on $M_\mathbb{Q}$ (see the Appendix), hence admits a Fourier expansion of the form

$$F(Z) = \sum_{m \in U(0)_\mathbb{Z}} c(m) \chi^m, \quad \chi^m = e^{2\pi i (m, Z)}.$$

(This is expansion by characters on the torus $M_\mathbb{C}/U(l)_\mathbb{Z}$.) By the Koecher principle, we have $c(m) = 0$ when $m \notin M_\mathbb{R}^+$. If $c(m) = 0$ for all $m$ with $(m, m) = 0$ at all primitive isotropic $l \in L$, this modular form is called a cusp form. The space of cusp forms is denoted by $S_k(\Gamma) \subset M_k(\Gamma)$.

3. Construction of cusp form

In this section we prove Theorems 1.4 and 1.6. We construct a desired cusp form via the Gritsenko-Borcherds lifting [11], [2]. For Theorem 1.4 we first make a reduction of lattice, and then construct the source cusp form explicitly using Eisenstein series. For Theorem 1.6 we resort to Bruinier-Ehlen-Freitag’s result [5].

3.1. Reduction of lattice. For the proof of Theorem 1.4 we first simplify the given lattice using a classical reduction trick (cf. [10], [39]).

**Lemma 3.1.** Let $L$ be a lattice of signature $(2, n)$. There exists a lattice $L'$ on $L_\mathbb{Q}$ such that

1. $O^+(L) \subset O^+(L')$ inside $O^+(L_\mathbb{Q})$ and
2. $L'$ is a scaling of a lattice $L''$ for which the $p$-component of $A_{L''}$ is $p$-elementary of length $\leq n/2 + 1$ for every $p$.

**Proof.** This is described in [39] §8.5 (see also [10] p.198–199). It is useful to observe that $L'$ is obtained by inductively taking $L_{i+1} = L_i + p_i^{-1}L_i \cap p_iL_i'$ from $L_1 = L$, and finally taking $L' = L_N \cap aL_N'$. $\square$
**Corollary 3.2.** Let $L$ be a lattice of signature $(2, n)$ with $n \geq 11$. There exists a lattice $L_1$ on $L \mathbb{Q}$ such that $O^+(L) \subset O^+(L_1)$ and that $L_1$ is a scaling of an even lattice $L_2$ containing $2U$.

**Proof.** Let $L'$ and $L''$ be as in the lemma. Let $L_2 \subset L''$ be the maximal even sublattice of $L''$ and $L_1 \subset L'$ be the corresponding sublattice of $L'$. Since $O^+(L'') \subset O^+(L_2)$, we have $O^+(L') \subset O^+(L_1)$ and hence $O^+(L) \subset O^+(L_1)$. When $L''$ is even, we have $L_2 = L''$; when $L''$ is odd, $A_{L''}$ is an index 2 quotient of an index 2 subgroup of $A_{L_2}$. Hence $l(A_{L_2}) \leq l(A_{L''}) + 2 \leq n/2 + 3$. Then $\text{rk}(L_2) - l(A_{L_2}) \geq 5$ by our assumption $n \geq 11$. By Nikulin’s theory ([27] Corollary 1.10.2), $L_2$ contains $2U$. \hfill \Box

Note that we did not make full use of the property (2) in Lemma 3.1. This will be used in §7.1.

We have a natural isomorphism

$$ (3.1) \quad D_L^* = D_{L_1}^* \cong D_{L_2}^*,$$

where the first comes from the equality $L_\mathbb{Q} = (L_1)_\mathbb{Q}$ and the second from the identification $L_1 = L_2$ as $\mathbb{Z}$-modules. The inclusion $O^+(L) \subset O^+(L_1) \cong O^+(L_2)$ is compatible with this isomorphism. Note that the induced isomorphism $D_L \cong D_{L_2}$ preserves the rational boundary components.

**Lemma 3.3.** Let $F$ be a cusp form on $D_{L_2}$ with respect to $O^+(L_2)$. Via (3.1), $F$ gives a cusp form on $D_L$ of the same weight with respect to $O^+(L)$.

**Proof.** We check that $F$ is still a cusp form for $O^+(L_1)$. Let $l, l', M$ be as in the last paragraph of §2 for $L_2$. For $L_1 = L_2(a)$ we use $l'/a \in (L_1)_\mathbb{Q}$ in place of $l' \in (L_2)_\mathbb{Q}$. Then the tube domain realization of $D_{L_1}$ differs from that of $D_{L_2}$ by scalar multiplication by $a$, both on $M_\mathbb{C}$ and $D_{L_2}^*$. Hence if we view $U(l')_\mathbb{Z} \subset M(a)_\mathbb{Q}$ naturally, the Fourier expansion of $F$ for $l, l'/a, L_1$ is multiplication by $a^k$ of the one for $l, l', L_2$. \hfill \Box

In this way, for the proof of Theorem 1.4 we may (and do) assume in the rest of this section that $L$ is even and contains $2U$.

### 3.2. Lifting

Gritsenko-Borcherds additive lifting [11], [2], essentially equivalent to that of Oda [28] and Rallis-Schiffmann [30] in a common situation, is a lifting from modular forms of one variable to orthogonal modular forms. We assume throughout that $L$ is an even lattice of signature $(2, n)$ with $n \geq 3$ and contains $2U$. We fix an embedding $2U \hookrightarrow L$ and write $L$ in the form $L = 2U \oplus K$ with $K$ negative-definite of rank $n - 2$. We put $M = U \oplus K$. As explained in §2, via the splitting $L = U \oplus M$ we can identify $O^+(L)$-modular forms with holomorphic functions $F$ on the tube domain $M_\mathbb{R} + iM_\mathbb{R}^+$. The lattice of parallel translation coincides to $M$, so the
Fourier expansion has the form \( F(Z) = \sum_m c(m)\chi^m \) where \( m \in M^\vee \cap M^+_\mathbb{R} \) (see \([11]\) §2).

Let \( \text{Mp}_2(\mathbb{Z}) \) be the metaplectic double cover of \( \text{SL}_2(\mathbb{Z}) \). It is well-known that \( \text{Mp}_2(\mathbb{Z}) \) is generated by the two elements

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let \( \mathbb{C}[A_L] \) be the group ring over \( A_L \). If \( \lambda \in A_L \), we write \( e_\lambda \in \mathbb{C}[A_L] \) for the corresponding basis vector. The Weil representation is a unitary representation \( \rho_L : \text{Mp}_2(\mathbb{Z}) \to \text{GL}(\mathbb{C}[A_L]) \)
defined by

\[
\rho_L(T)(e_\lambda) = e((\lambda, \lambda)/2)e_\lambda, \quad \rho_L(S)(e_\lambda) = \frac{\sqrt{-1^{n/2-1}}}{\sqrt{|A_L|}} \sum_{\mu \in A_L} e(-\lambda_\mu)e_\mu.
\]

Here \( e(x) = \exp(2\pi ix) \) for \( x \in \mathbb{Q}/\mathbb{Z} \). The orthogonal group \( O(A_L) \) of \( A_L \) acts on \( \mathbb{C}[A_L] \) by permuting the standard basis vectors \( e_\lambda \).

**Lemma 3.4.** The permutation representation of \( O(A_L) \) on \( \mathbb{C}[A_L] \) commutes with the Weil representation.

**Proof.** It suffices to check that

\[
\rho_L(T) \circ \gamma = \gamma \circ \rho_L(T), \quad \rho_L(S) \circ \gamma = \gamma \circ \rho_L(S)
\]

for every \( \gamma \in O(A_L) \). The first equality follows from

\[
\rho_L(T)(e_{\gamma\lambda}) = e((\gamma\lambda, \gamma\lambda)/2)e_{\gamma\lambda} = e((\lambda, \lambda)/2)e_{\gamma\lambda} = \gamma(\rho_L(T)(e_\lambda)).
\]

The second follows from

\[
\sqrt{|A_L|} \sqrt{-1^{n/2}} \rho_L(S)(e_{\gamma\lambda}) = \sum_{\mu \in A_L} e(-\lambda_\mu)e_\mu = \sum_{\mu \in A_L} e(-\lambda, \gamma^{-1}\mu)e_\mu = \sum_{\mu' \in A_L} e(-\lambda, \mu')e_{\gamma\mu'} = \sqrt{|A_L|} \sqrt{-1^{n/2}} \gamma(\rho_L(S)(e_\lambda))
\]

where we put \( \mu' = \gamma^{-1}\mu \). \( \square \)

Modular forms of type \( \rho_L \) with respect to \( \text{Mp}_2(\mathbb{Z}) \) have Fourier expansion of the form

\[
f(\tau) = \sum_{\lambda \in A_L} \sum_{n \geq 0} c_\lambda(n)q^n e_\lambda, \quad q = e^{2\pi i \tau}.
\]

If \( l \) is an integral or half-integral weight such that \( l \equiv n/2 \mod \mathbb{Z} \), we write \( M_l(\rho_L) \) for the space of modular forms of weight \( l \) and type \( \rho_L \), and \( S_l(\rho_L) \)
the subspace of cusp forms. By Lemma 3.4, the group \( O(A_L) \) acts on \( M_l(\rho_L) \).

Explicitly, if \( f \) has Fourier expansion as above, then
\[
(\gamma \cdot f)(\tau) = \sum_{\lambda, n} c_{\lambda}(n)q^n e_{\gamma \lambda} = \sum_{\lambda, n} c_{\gamma^{-1} \lambda}(n)q^n e_{\lambda}.
\]

It is clear that this action preserves \( S_l(\rho_L) \).

We have a natural isomorphism \( O(A_L) \cong O^+(L)/\tilde{O}^+(L) \) by Nikulin [27]. Via this \( O(A_L) \) also acts on \( S_k(\tilde{O}^+(L)) \) by the Petersson slash operator. Basic properties of the Gritsenko-Borcherds lifting, in a form we need, are summarized as follows.

**Theorem 3.5** (Gritsenko [11], Borcherds [2]). Let \( L \) be an even lattice of signature \( (2, n) \) with \( n \geq 3 \) containing \( 2U \). Write \( L = 2U \oplus K = U \oplus M \). Let \( l \) be an integral or half-integral weight with \( l \equiv n/2 \mod \mathbb{Z} \). Then there exists an injective, \( O(A_L) \)-equivariant linear map
\[
(3.3) \quad S_l(\rho_L) \to S_k(\tilde{O}^+(L)), \quad k = l + n/2 - 1.
\]

If \( F = \sum c(m)x^m \) is the lifting of \( f = \sum c_{\lambda}(n)q^n e_{\lambda} \), its Fourier coefficients are given by \( c(0) = 0 \) and for \( m \neq 0 \in M^+ \cap M^+_R \)
\[
(3.4) \quad c(m) = \sum_{m/a \in M^+} a^{-1} c_{[m/a]}((m/a, m/a)/2),
\]
where \([m/a]\) denotes the class in \( A_M \cong A_L \).

Let us add a few comments, because some of the properties stated above are scattered or only implicit in the literatures.

(1) In [11] Theorem 3.1, Gritsenko constructed the lifting in the form of Jacobi lifting, namely a lifting from Jacobi forms of weight \( k \) and index \( 1 \) for \( K(-1) \) to \( O^+(L) \)-modular forms of the same weight. Since those Jacobi forms canonically correspond to modular forms of type \( \rho_L \) and weight \( l = k - n/2 + 1 \) (see [11] p.1187–1188), his lifting can be interpreted as a lifting from modular forms of type \( \rho_L \). Borcherds ([2] Theorem 14.3) extended the lifting in this second form to general even lattices \( L \) which does not necessarily contain \( 2U \). The formula (3.4) is obtained by combining explicit forms of the Jacobi lifting ([11] p.1193) and that of the correspondence between Jacobi forms and modular forms of type \( \rho_L \) ([11] Lemma 2.3). This coincides with Borcherds’ calculation of Fourier expansion of his lifting (loc. cit. item 5: his notation \( M, K, n, \lambda, n\lambda, \delta, m^+ \) is read \( L, M, a, l/a, m, [m/a], k \) here and \( z, z' \) are the standard basis of \( U \)), so the two liftings indeed agree.

(2) Injectivity: in Gritsenko’s construction, the Jacobi form corresponding to a cusp form \( f \in S_l(\rho_L) \) is recovered as the 1st Fourier-Jacobi coefficient of the lifting of \( f \) at the 1-dimensional cusp associated to the chosen
embedding $2U \subset L$. Thus the lifting map (3.3) is injective in the present case. (This can also be checked directly by looking the Fourier coefficients at $(1, \mathbb{Z}, K^\vee)$.) It is not known whether injectivity holds in general when $L$ does not contain $2U$.

(3) Cusp condition: the property that the lifting of a cusp form is a cusp form is established in [11] for maximal lattices $L$. Indeed, the Fourier expansion (3.4) shows that $F$ vanishes at 1-dimensional cusps adjacent to the standard 0-dimensional cusp, and when $L$ is maximal, every 1-dimensional cusp is $\widetilde{O}^+(L)$-equivalent to such a cusp. (In [12] this was extended to a wider class of lattices.) Borcherds [2], in his formulation, calculated the Fourier expansion of $F$ at every 0-dimensional cusp not necessarily coming from $U$. From his general formula one observes that the lifting of a cusp form is a cusp form. (In his notation: if $m = n\lambda \in K^\vee$ is isotropic, then $c_\gamma(\lambda^2/2) = c_\gamma(0)$ is zero for all possible $(n, \lambda, \delta)$, so the coefficient of $\chi^m = e((m, Z))$ is zero.) We note that for the Oda lifting this property was proved in [28] §6, Corollary 2.

(4) $O(AL)$-equivariance: the equivariance of the lifting with respect to $O(AL)$ is implicit in [2] but not stated explicitly. For completeness let us supplement a self-contained proof in case $L$ contains $2U$. Let $f = \sum c_\lambda(n)q^n e_\lambda$ be a cusp form of type $\rho_L$ and $F = \sum c(m)\chi^m$ be its lifting. Let $\gamma \in O(AL)$ be an isometry of $A_L$. By (3.2) and (3.4) the lifting of $\gamma^{-1} \cdot f$ has Fourier expansion $\sum c_\gamma(m)\chi^m$ where

$$c_\gamma(m) = \sum_{a|m} a^{k-1} c_{\gamma[m/a]}((m/a, m/a)/2).$$

Since $O^+(M) \to O(AL)$ is surjective by [27], we can lift $\gamma$ to an isometry of the lattice $M$, say $\hat{\gamma} \in O^+(M)$. We have $m/a \in M^\vee$ if and only if $\hat{\gamma}m/a \in M^\vee$. Therefore

$$c_\gamma(m) = \sum_{a|m} a^{k-1} c_{\hat{\gamma}[m/a]}((\hat{\gamma}m/a, \hat{\gamma}m/a)/2) = c(\hat{\gamma}m).$$

On the other hand, since the factor of automorphy on $O^+(M) \subset O^+(L)$ is constantly 1, the Petersson slash operator by $\hat{\gamma}$ is just the ordinary pullback of functions on $M_{+} = iM_{R}^+$. Thus the lifting of $\gamma^{-1} \cdot f$ is equal to the Petersson slash of the lifting of $f$ by $\gamma$.

3.3. **Proof of Theorem 1.4** Let us record a consequence of Theorem 3.5 in a ready-to-use form.

**Corollary 3.6.** Let $L$ be an even lattice of signature $(2, n)$ with $n \geq 3$ and containing $2U$. If there exists a nonzero, $O(AL)$-invariant cusp form of type $\rho_L$ and weight $l$, we have a nonzero cusp form of weight $l + n/2 - 1$ with respect to $O^+(L)$.
We are thus reduced to constructing a cusp form of type $\rho_L$ invariant under $O(A_L)$. We use Eisenstein series of Bruinier-Kuss [6].

Let $l > 2$ be a weight with $l + n/2 - 1 \in 2\mathbb{Z}$. The Eisenstein series $E_L^l(\tau)$ of weight $l$ and type $\rho_L$ is defined by ([6] §4)

$$E_L^l(\tau) = \frac{1}{2} \sum_{(M,\phi)} \phi(\tau)^{-2l} \cdot \rho_L(M,\phi)^{-1}(e_0),$$

where $(M,\phi)$ runs over the coset $\langle T \rangle \backslash M_{p_2}(\mathbb{Z})$. This series converges normally on $\mathbb{H}$ and gives a modular form of type $\rho_L$ and weight $l$ whose constant term is $2e_0$. It is $O(A_L)$-invariant because $e_0$ is fixed by $O(A_L)$ and the $O(A_L)$-action commutes with $\rho_L$ by Lemma 3.4. If $E_L^l(\tau) = \sum c_{1,l}(n)q^n$, it is shown in [6] Theorem 7 that the coefficients $c_{1,l}(n)$ in $n > 0$ are given by

$$(-1)^{(2l-2+n)/4} \times \text{(nonnegative rational number)}.$$

Note that the Eisenstein series in [6] are rather for the dual representation of $\rho_L$. But the conversion is immediate because $\rho_L^\vee = \rho_{L(-1)}$ under the natural identification $\mathbb{C}[A_L]^\vee = \mathbb{C}[A_{L(-1)}]$ induced by the basis $e_\lambda$. So our $E_L^l$ is $E_l$ for $L(-1)$ in the notation of [6].

Let $E_6(\tau) = 1 - 504q - \cdots$ be the classical scalar-valued Eisenstein series of weight 6.

**Lemma 3.7.** Choose a weight $l > 2$ satisfying $l + n/2 \equiv 3 \mod 4$. Then

$$f = E_L^l \cdot E_6 - E_L^{l+6}$$

is a nonzero, $O(A_L)$-invariant cusp form of weight $l + 6$ and type $\rho_L$.

**Proof.** The constant term of $f$ is equal to $1 \cdot 2e_0 - 2e_0 = 0$, so $f$ is a cusp form. Since $E_L^l$ and $E_L^{l+6}$ are $O(A_L)$-invariant, so is $f$. To see the nonvanishing of $f$, we observe that the Fourier coefficient of $f$ at $qe_0$ is calculated as

$$1 \cdot c_{0,l}(1) - 504 \cdot 2 - c_{0,l+6}(1).$$

By our choice of $l$, we have $c_{0,l}(1) \leq 0$ and $c_{0,l+6}(1) \geq 0$. Therefore (3.6) is nonzero, whence $f$ does not vanish. \qed

According to the congruence of $n$ modulo 8, the minimal weight $l > 2$ satisfying $l + n/2 \equiv 3 \mod 4$ is as in Table 1. In particular, $l \leq 6$.

| $n \mod 8$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------|---|---|---|---|---|---|---|---|
| $l$       | 3 | 5/2 | 6 | 11/2 | 5 | 9/2 | 4 | 7/2 |

If $n \geq 21$ or $n = 17$, we have $l + 6 < n/2 + 1$ for this value of $l$. Thus for every even lattice $L$ in this range, the cusp form $f$ defined by (3.5) has
weight $< n/2 + 1$. By Corollary 3.6, when $L$ contains $2U$, the lifting of $f$ is a nonzero cusp form for $O^+(L)$ of weight $< n$. This proves Theorem 1.4 (1).

When $4|n$ with $n \geq 16$, $l = n/2 - 5$ satisfies the congruence $l + n/2 \equiv 3 \mod 4$ and $l > 2$. Then $f$ has weight $n/2 + 1$, so its lifting is a cusp form of weight $n$ for $O^+(L)$. This proves Theorem 1.4 (2).

Remark 3.8. One may also try other combinations such as $E_l^4 - E_{l+4}$, but their nonvanishing seems nontrivial. There are lattices $L$ for which $E_l^4 = E_{l+4}$ for the minimal weight $l$, e.g., $II_{2,18}$, $II_{2,18} \oplus A_1$, $II_{2,18} \oplus A_2$.

3.4. Proof of Theorem 1.6. In view of Theorem 1.4, it is sufficient to see the finiteness for each $5 \leq n \leq 20$. Let $n$ be fixed. Bruinier-Ehlen-Freitag [5] recently estimated the dimension formula for $\rho_L$-valued cusp forms in [3], [36]. By [5] Corollary 4.7, there are only finitely many finite quadratic forms $A$ of length $\leq n - 2$ such that $S_l(\rho_A) = 0$ for any $l \leq 3$. By Nikulin [27], even lattices $L$ of signature $(2, n)$ containing $2U$ are determined by its discriminant form $A = A_L$. Hence for all but finitely many such lattices $L$ we have $S_l(\rho_L) \neq 0$ for some $l \leq 3 < n/2 + 1$. By taking the lifting, this proves Theorem 1.6.

Remark 3.9. The dimension formula for $O(A)$-invariant cusp forms is more complicated, partly involving an equivariant version of Gauss sum. This Gauss sum will be studied in a future paper.

4. Reflective obstruction

This section is the start of the proof of Theorem 1.5. In §4.1 we classify the branch divisors of $\mathcal{F}_L$. In §4.2 we show that the $\mathbb{Q}$-divisor $a\mathcal{L} - B/2$ of $\mathcal{F}_L$ is big if a certain inequality involving Hirzebruch-Mumford volumes holds. These volumes (or rather their ratio) will be estimated in §5 and §6. The proof of Theorem 1.5 will be completed at §6.3.

4.1. The branch divisor. Let $L$ be a lattice of signature $(2, n)$ with $n \geq 3$. Recall that the reflection $\sigma_l$ with respect to a primitive vector $l \in L$ with $(l, l) \neq 0$ is defined by

$$\sigma_l : L_\mathbb{Q} \to L_\mathbb{Q}, \quad v \mapsto v - \frac{2(v, l)}{(l, l)}l.$$

When $\sigma_l \in O^+(L)$, namely $\sigma_l$ preserves $L$ and $(l, l) < 0$, the vector $l$ is called a reflective vector. According to [12] Corollary 2.13, every irreducible component of the ramification divisor of $\mathcal{D}_L \to \mathcal{F}_L$ is the fixed divisor of a reflection $\sigma_l \in O^+(L)$, that is, the hyperplane section

$$\mathbb{P}(K_C) \cap \mathcal{D}_L = \mathcal{D}_K \quad \text{where} \quad K = l^\perp \cap L.$$
Hence classification of the branch divisors of \( F_L \) is equivalent to that of \( O^+(L) \)-equivalence classes of reflective vectors. The starting point is the following well-known property.

**Lemma 4.1.** Let \( l \in L \) be a primitive vector with \( (l, l) < 0 \) and \( K = l^\perp \cap L \) be its orthogonal complement. Then \( l \) is reflective if and only if either we have the splitting \( L = \mathbb{Z}l \oplus K \) or \( L \) contains \( \mathbb{Z}l \oplus K \) with index 2. In the first case we have \((l, l) = -\text{div}(l)\), and in the second case \((l, l) = -2\text{div}(l)\).

**Proof.** The sublattice \( \mathbb{Z}l \oplus K \) of \( L \) consists of vectors \( l' \) such that \((l, l)(l, l')\). If we choose a vector \( l_0 \in L \) such that \((l, l_0) = \text{div}(l)\), the quotient group \( L/(\mathbb{Z}l \oplus K) \) is cyclic of order \(-\text{div}(l)/\text{div}(l)\), generated by \( l_0 \). Suppose that the reflection \( \sigma_l \) preserves \( L \). Then the vector

\[
(l_0 - \sigma_l(l_0)) = (2(l_0, l)/l)l = (2\text{div}(l)/l)l
\]

is contained in \( L \). The primitivity of \( l \) implies \( 2\text{div}(l)/l \in \mathbb{Z} \), so that \(-\text{div}(l)/\text{div}(l) = 1 \) or 2. Conversely, suppose that \( L \) contains \( \mathbb{Z}l \oplus K \) with index \( \leq 2 \). By the above calculation \( \sigma_l(l_0) \) is contained in \( L \). Since \( \mathbb{Z}l \oplus K \) is clearly preserved by \( \sigma_l \), so is \( L \). \( \square \)

According to this lemma, we shall say that a reflective vector \( l \) is of **split type** when \( L = \mathbb{Z}l \oplus K \), and **non-split type** when \( \mathbb{Z}l \oplus K \) is of index 2 in \( L \). We denote by \( R_1, R_2 \) the sets of \( O^+(L) \)-equivalence classes of reflective vectors of split type, non-split type respectively. The union \( R_1 \cup R_2 \) corresponds to the set of irreducible components of the total branch divisor \( B \) of \( F_L \).

Each component is described as follows. Let \( l \in L \) be a reflective vector and \( B_l \) be the component of \( B \) defined by \( l \). Let \( \Gamma_l < O^+(L) \) be the stabilizer of the vector \( l \). We view \( \Gamma_l \) as a subgroup of \( O^+(K) \) naturally where \( K = l^\perp \cap L \). Note that \( \Gamma_l < O^+(K) \) contains \(-1 \) because \( -\sigma_l \) fixes \( l \) and restricts to \(-1 \) on \( K \). The projection \( D_K \to B_l \) from the ramification divisor descends to a birational morphism \( \Gamma_l/D_K \to B_l \). This gives the normalization of \( B_l \).

**Lemma 4.2.** The subgroup \( \Gamma_l < O^+(K) \) is described as follows.

1. When \( l \) is of split type, we have \( \Gamma_l = O^+(K) \).
2. When \( l \) is of non-split type, \( \Gamma_l \) is equal to the stabilizer of an order 2 element of \( A_K \). In particular, \( [O^+(K) : \Gamma_l] < 2^r \) where \( r = l((A_K)_2) \).

**Proof.** The split case is obvious. When \( l \) is of non-split type, we choose a vector \( l_0 \in L \) generating \( L/(\mathbb{Z}l \oplus K) \cong \mathbb{Z}/2 \) and let \( k_0 \in K' \) be its orthogonal projection to \( K_0 \). The element \( x = [k_0] \in A_K \) is of order 2. For \( \gamma \in O^+(K) \) the isometry \((\text{id}, \gamma)\) of \( \mathbb{Z}l \oplus K \) preserves \( L \) if and only if it fixes the element \( [l_0] = ([l]/2, x) \) of \( A_{\mathbb{Z}l \oplus K} \). Hence \( \Gamma_l < O^+(K) \) coincides with the stabilizer of \( x \), and \( [O^+(K) : \Gamma_l] = |O^+(K) \cdot x| \). The orbit \( O^+(K) \cdot x \) is contained in the set of order 2 elements of \( A_K \). \( \square \)
4.2. Hirzebruch-Mumford volume. Let \( L \) be a lattice of signature \((2, n)\) with \( n > 0 \). (This will be both \( L \) and \( K = l^\perp \cap L \) in \( \S 11 \)) Let \( \Gamma < \text{O}^+(L) \) be a finite-index subgroup. Gritsenko-Hulek-Sankaran [13] introduced the Hirzebruch-Mumford volume \( \text{vol}^\text{HM}_\Gamma(G) \) of \( \Gamma \) following the proportionality principle of Hirzebruch and Mumford [25]. It determines the growth of the dimension of \( \text{M}_k(\Gamma) \) by ([13] Proposition 1.2)

\[
(4.1) \quad \dim \text{M}_k(\Gamma) = \frac{2}{n!} \text{vol}^\text{HM}_\Gamma(\Gamma)k^n + O(k^{n-1}).
\]

We may adopt this as an equivalent definition of \( \text{vol}^\text{HM}_\Gamma(\Gamma) \). If \( \Gamma' < \Gamma \) is a finite-index subgroup, we have

\[
(4.2) \quad \text{vol}^\text{HM}_\Gamma(\Gamma') = [\langle \Gamma, -1 \rangle : \langle \Gamma', -1 \rangle] \cdot \text{vol}^\text{HM}_\Gamma(\Gamma).
\]

Proposition 4.3. Let \( L \) be a lattice of signature \((2, n)\) with \( n \geq 3 \) for which we are studying whether the \( \mathbb{Q} \)-divisor \( a\mathcal{L} - B/2 \) of \( \mathcal{F}_L \) is big where \( a \in \mathbb{Q}_{>0} \). We relate this problem to the comparison of the Hirzebruch-Mumford volumes between \( \text{O}^+(L) \) and the branch divisors. If \( l \in L \) is a reflective vector with orthogonal complement \( K = l^\perp \cap L \), we consider the volume ratio

\[
\text{vol}^\text{HM}_\Gamma(L, K) := \frac{\text{vol}^\text{HM}_\Gamma(\text{O}^+(K))}{\text{vol}^\text{HM}_\Gamma(\text{O}^+(L))}.
\]

Now let \( L \) be a lattice of signature \((2, n)\) with \( n \geq 3 \) for which we are studying whether the \( \mathbb{Q} \)-divisor \( a\mathcal{L} - B/2 \) of \( \mathcal{F}_L \) is big where \( a \in \mathbb{Q}_{>0} \). We relate this problem to the comparison of the Hirzebruch-Mumford volumes between \( \text{O}^+(L) \) and the branch divisors. If \( l \in L \) is a reflective vector with orthogonal complement \( K = l^\perp \cap L \), we consider the volume ratio

\[
\text{vol}^\text{HM}_\Gamma(L, K) := \frac{\text{vol}^\text{HM}_\Gamma(\text{O}^+(K))}{\text{vol}^\text{HM}_\Gamma(\text{O}^+(L))}.
\]

Proposition 4.3. Let \( L \) be a lattice of signature \((2, n)\) with \( n \geq 3 \). Let \( a > 0 \) be a rational number. The \( \mathbb{Q} \)-divisor \( a\mathcal{L} - B/2 \) of \( \mathcal{F}_L \) is big if we have

\[
(4.3) \quad \sum_{[l] \in \mathbb{R}_l} \text{vol}^\text{HM}_\Gamma(L, K) + 2^{n+1} \cdot \sum_{[l] \in \mathbb{R}_l} \text{vol}^\text{HM}_\Gamma(L, K) < \left(1 + \frac{1}{a}\right)^{-n} \cdot \frac{2a}{n}.
\]

Proof. By definition, \( a\mathcal{L} - B/2 \) is big if we could show that an estimate

\[
(4.4) \quad h^0(ka\mathcal{L} - (k/2)B) > c \cdot k^n
\]

holds for some \( c > 0 \) in \( k \gg 0 \), where \( k \) runs so that both \( k \) and \( ka \) are even numbers. We shall bound the left-hand side from below. Choose representatives \( l_1, \ldots, l_r \in L \) for \( \mathcal{R}_1 \cup \mathcal{R}_2 \). Let \( K_i = l_i^\perp \cap L \) and \( \Gamma_i < \text{O}^+(K_i) \) be the stabilizer of \( l_i \). The following is essentially proved in [14] Proposition 4.1.

Lemma 4.4. When both \( k \) and \( ka \) are even numbers, we have

\[
(4.5) \quad h^0(ka\mathcal{L} - (k/2)B) \geq \dim \text{M}_{ka}(\text{O}^+(L)) - \sum_{i=1}^{r} \sum_{j=0}^{k/2-1} \dim \text{M}_{ka+2j}(\Gamma_i).
\]

Proof. For a nonnegative integer \( j \geq 0 \), \( H^0(ka\mathcal{L} - jB) \) is the space of \( \text{O}^+(L) \)-modular forms of weight \( ka \) which have zero of order \( \geq 2j \) along every \( \mathcal{D}_K \). The quasi-pullback of such modular forms to \( \mathcal{D}_K \) is defined by ([14], [14])

\[
(4.6) \quad H^0(ka\mathcal{L} - jB) \rightarrow \text{M}_{ka+2j}(\Gamma_i), \quad F \mapsto (F/(\cdot, l_i)^2)|_{\mathcal{D}_K}.
\]
Note that the vanishing order of $F$ along $D_{K_i}$ must be even because $\Gamma_i$ contains $-1$. We obtain from (4.6) the exact sequence

$$0 \to H^0(kaL - (j + 1)B) \to H^0(kaL - jB) \to \bigoplus_{i=1}^r M_{ka+2j}(\Gamma_i).$$

Iteration of this for $j = 0, \cdots, k/2 - 1$ gives the desired inequality. □

We study asymptotic behavior of the right-hand side of (4.5) with respect to $k$. For the first term, we have by (4.1)

$$\dim_{M_{ka}}(O + (L)) = \left(\frac{2}{(n-1)!}\right) \cdot \text{vol}_{HM}(O + (L)) \cdot a^n \cdot k^n + O(k^{n-1}).$$

The second term is estimated as

$$\sum_{i=1}^r \sum_{j=0}^{k/2-1} \dim M_{ka+2j}(\Gamma_i)$$

$$= \sum_{i=1}^r \sum_{j=0}^{k/2-1} \left\{ \frac{2}{(n-1)!} \cdot \text{vol}_{HM}(\Gamma_i) \cdot (ka + 2j)^{n-1} + O(k^{n-2}) \right\}$$

$$\leq \sum_{i=1}^r \frac{k}{2} \cdot \left\{ \frac{2}{(n-1)!} \cdot \text{vol}_{HM}(\Gamma_i) \cdot (a + 1)^{n-1} \cdot k^{n-1} + O(k^{n-2}) \right\}.$$  

$$= \frac{1}{(n-1)!} \left( \sum_{i=1}^r \text{vol}_{HM}(\Gamma_i) \right) \cdot (a + 1)^{n-1} \cdot k^n + O(k^{n-1}).$$

Comparing the coefficients of $k^n$ in these two asymptotics, we see that (4.4) holds if

$$\sum_{i=1}^r \frac{\text{vol}_{HM}(\Gamma_i)}{\text{vol}_{HM}(O^+(L))} < \left(1 + \frac{1}{a}\right)^{1-n} \cdot \frac{2a}{n}.$$  

It remains to classify $l_1, \cdots, l_r$ by split/non-split type. We have $\Gamma_i = O^+(K_i)$ if $l_i$ is of split type. When $l_i$ is of non-split type, we have

$$\text{vol}_{HM}(\Gamma_i) = [O^+(K_i) : \Gamma_i] \cdot \text{vol}_{HM}(O^+(K_i)) < 2^{n+1} \cdot \text{vol}_{HM}(O^+(K_i))$$

by (4.2) and Lemma 4.2. □

We use the relation (4.2) to extend the definition formally to $O(L)$

$$\text{vol}_{HM}(O(L)) := \text{vol}_{HM}(O^+(L))/[O(L) : O^+(L)].$$

It is often convenient to consider the following variant of $\text{vol}_{HM}(L, K)$

$$\text{vol}_{HM}(L, K) := \frac{\text{vol}_{HM}(O(K))}{\text{vol}_{HM}(O(L))}.$$  

The quotient

$$\frac{\text{vol}_{HM}(L, K)}{\text{vol}_{HM}^+(L, K)} = \frac{[O(L) : O^+(L)]}{[O(K) : O^+(K)]}$$

(4.7)
is equal to 1 or 2 or 1/2.

5. SINGLE VOLUME ESTIMATE

By Proposition 4.3, to show that $a L - B / 2$ is big is reduced to estimating the sum of the volume ratios $\nu_{\text{HM}}^+ (L, K)$. In order to deduce the finiteness as in Theorem 1.5, we want to estimate it for primitive lattices $L$ in a way that reflects the “size” of $L$. This is the task of §5 and §6. In this §5 we estimate $\nu_{\text{HM}}^+ (L, K)$ for each reflective vector, and in the next §6 we take their sum over all components of the branch divisor. The final result is Propositions 6.4, 6.6 and (6.6), where the dimension $n$ and the exponent $D(L)$ of $A_L$ play the role of measuring the size of $L$. Derivation of Theorem 1.5 from these estimates is done in §6.3, which we encourage the reader to read before going to the technical detail of the estimate.

The central idea of §5 and §6 is to reserve the reflection of $n$ and $D(L)$ through the whole process of estimate. Some step in §5 might seem indirect, but they are designed so that we can finally obtain a reasonable bound in §6.

A word on primitivity assumption: in each subsection (except §6.3) we will not assume that the given lattice $L$ is primitive until the final step. This is not for the sake of generality, but rather is an indispensable piece in the proof for the non-split case.

Throughout we write $D(L)$ for the exponent of the discriminant group $A_L$ of a lattice $L$. Clearly $D(L)$ divides $|A_L|$, and the set of prime divisors of $D(L)$ equals that of $|A_L|$.

5.1. VOLUME FORMULA. In [13], Gritsenko-Hulek-Sankaran derived an exact formula for the Hirzebruch-Mumford volume by carefully comparing various volume formulae related to orthogonal groups. Let $L$ be a lattice of signature $(2, n)$ with $n > 0$. We write $g^+_p (L)$ for the number of proper spinor genera in the genus of $L$. Since $L$ is indefinite of rank $\geq 3$, proper spinor genus coincides with proper equivalence class ([20] Theorem 6.3.2). For each prime $p$ we write $\alpha_p (L)$ for the local density of the $\mathbb{Z}_p$-lattice $L \otimes \mathbb{Z}_p$. This is also denoted as $\alpha_p (L, L)$ in literatures (cf. [20] p.98).

**Theorem 5.1** ([13] Theorem 2.1). Let $L$ be a lattice of signature $(2, n)$ with $n > 0$. Then

$$\nu_{\text{HM}}^+ (O(L)) = \frac{2}{g^+_p (L)} \cdot |A_L|^{(n+3)/2} \cdot \prod_{k=1}^{n+2} \pi^{-k/2} \Gamma (k/2) \prod_p \alpha_p (L)^{-1},$$

where $\Gamma (m)$ is the Gamma function.
Computation of the formula (5.1) amounts to that of the spinor class number $g^{+}_{sp}(L)$ and the local densities $\alpha_{p}(L)$. Below we use the notation

$$L \otimes \mathbb{Z}_p = \bigoplus_{j \geq 0} L_{p,j}(p^j), \quad \text{rk}(L_{p,j}) = n_{p,j}(L)$$

for a Jordan decomposition of $L \otimes \mathbb{Z}_p$. Each $L_{p,j}$ is a unimodular $\mathbb{Z}_p$-lattice. When $p > 2$, Jordan decomposition is unique up to isometry. For $p = 2$, $n_{2,j}(L)$ and whether $L_{2,j}$ is even or odd are uniquely determined. See [20] §5.3 and [10] §8.3.

Let $P$ be the set of odd prime divisors $p$ of $D(L)$ for which $n_{p,j}(L) \leq 1$ for all $j$. We will later use the following estimate of $g^{+}_{sp}(L)$.

**Lemma 5.2.** We have

$$g^{+}_{sp}(L) \leq 4 \cdot 2^{|P|}.$$

**Proof.** This can be seen from [7] Chapter 11.3. If $p \notin P \cup \{2\}$, then $n_{p,j}(L) \geq 2$ for some $j$. By Lemma 3.3 loc. cit, the group $\theta(\text{SO}(L \otimes \mathbb{Z}_p))$ of spinor norms of $\text{SO}(L \otimes \mathbb{Z}_p)$ contains

$$\theta(\text{SO}(L_{p,j}(p^j))) = \theta(\text{SO}(L_{p,j})) = \mathbb{Z}_p^{\times} \cdot (\mathbb{Q}_p^{\times})^2$$

for such $p$. By Theorem 3.1 Note 2, equality (3.35) and Lemma 3.6 (i) loc. cit., we then have

$$g^{+}_{sp}(L) \leq \prod_{p \in D(L)} [\mathbb{Z}_p^{\times} : \mathbb{Z}_p^{\times} \cap \theta(\text{SO}(L \otimes \mathbb{Z}_p))]$$

$$\leq \prod_{p \in P \cup \{2\}} [\mathbb{Z}_p^{\times} : (\mathbb{Z}_p^{\times})^2]$$

$$= 4 \cdot 2^{|P|}.$$

□

Next we recall the formula of $\alpha_{p}(L)$ given in [20] §5.6 (see especially p.98 and Theorem 5.6.3). We write $s_{p}(L)$ for the number of indices $j$ with $L_{p,j} \neq 0$, and set

$$w_{p}(L) = \sum_{j} j \cdot n_{p,j}(L) \cdot \left(\frac{n_{p,j}(L) + 1}{2} \right) \cdot \sum_{k} n_{p,k}(L).$$

For an even unimodular $\mathbb{Z}_p$-lattice $N$ of rank $r \geq 0$, we define $\chi(N)$ by $\chi(N) = 0$ if $r$ is odd, $\chi(N) = 1$ if $N \cong (r/2)U \otimes \mathbb{Z}_p$, and $\chi(N) = -1$ otherwise. For a natural number $m$ we put

$$P_p(m) = \prod_{k=1}^{m}(1 - p^{-2k})$$
when \( m > 0 \), and \( P_p(0) = 1 \). Then for \( p \neq 2 \), we have
\[
\alpha_p(L) = 2^{w(L)-1} \cdot p^{w_p(L)} \cdot \prod_j P_p([n_{p,j}(L)/2]) \cdot \prod_j (1 + \chi(L_{p,j})p^{-n_{p,j}(L)/2} - 1),
\]
where \( j \) ranges over indices with \( L_{p,j} \neq 0 \).

The 2-adic density is more complicated. Consider a decomposition \( L_{2,j} = \mathbb{Z}_2^+ \oplus \mathbb{Z}_2^- \) such that \( L_{2,j}^+ \) is even and \( L_{2,j}^- \) is either 0 or odd of rank \( \leq 2 \). Put \( n_{2,j}^+(L) = \text{rk}(L_{2,j}^+) \). We also set \( q(L) = \sum_{j \geq 0} q_j(L) \), where \( q_j(L) = 0 \) if \( L_{2,j} \) is even, \( q_j(L) = n_{2,j}(L) \) if \( L_{2,j} \) is odd and \( L_{2,j+1} \) is even, and \( q_j(L) = n_{2,j}(L) + 1 \) if both \( L_{2,j} \) and \( L_{2,j+1} \) are odd. Here zero-lattice is counted as an even lattice. For an index \( j \) with \( L_{2,j} \neq 0 \), we define \( E_{2,j}(L) \) by \( E_{2,j}(L) = 1 + \chi(L_{2,j})2^{-n_{2,j}(L)/2} \) if both \( L_{2,j-1} \) and \( L_{2,j+1} \) are even and \( L_{2,j} \neq \langle \epsilon_1, \epsilon_2 \rangle \) with \( \epsilon_1 \equiv \epsilon_2 \mod 4 \), and \( E_{2,j}(L) = 1 \) otherwise. We also let \( s_2'(L) \) be the number of indices \( j \geq -1 \) such that \( L_{2,j} = 0 \) and either \( L_{2,j-1} \) or \( L_{2,j+1} \) is odd. Then we have
\[
\alpha_2(L) = 2^{n+1+w_2(L)-q(L)+s_2(L)+s_2'(L)} \cdot \prod_j P_2(n_{2,j}(L)/2) \cdot \prod_j E_{2,j}(L)^{-1},
\]
where \( j \) ranges over indices with \( L_{2,j} \neq 0 \).

5.2. Split case. We now begin the estimate of \( \text{vol}_{HM}(L, K) \). We first consider the split case. For later purpose (5.3) we will not assume until Proposition 5.8 that the lattice \( L \) is primitive. So our initial setting is: \( L \) is a lattice of signature \((2, n)\) with \( n \geq 2 \), and \( l \in L \) is a primitive vector of norm \((l, l) = -D \) such that we have the orthogonal splitting
\[
L = \mathbb{Z}l \oplus K \simeq \langle -D \rangle \oplus K, \quad K = l^\perp \cap L.
\]
We denote the prime decompositions of \( D, D(L), |A_L| \) respectively by
\[
D = \prod_p p^{\nu_p}, \quad D(L) = \prod_p p^{\mu_p}, \quad |A_L| = \prod_p |A_L|_p.
\]
It is clear that \( \nu_p \leq \mu_p \). We use the Jordan decomposition of \( L \otimes \mathbb{Z}_p \) that is induced from a Jordan decomposition of \( K \otimes \mathbb{Z}_p \). Then
\[
K_{p,j} \simeq L_{p,j} \quad (j \neq \nu_p),
\]
\[
n_{p,\nu_p}(K) = n_{p,\nu_p}(L) - 1.
\]
Substituting \( L \) and \( K \) into the formula (5.1), we obtain
\[
\text{vol}_{HM}(L, K) = \frac{g_{s_p}^+(L)}{g_{s_p}^+(K)} \cdot \frac{\pi^{n/2+1}}{\Gamma(n/2 + 1)} \cdot \left( \frac{1}{D} \right)^{n/2+1} \cdot |A_L|^{-1/2} \cdot \prod_p \frac{\alpha_p(L)}{\alpha_p(K)}.
\]
If we put for each prime \( p \)
\[
a_p(L, K) := p^{-\nu_p(n/2+1)} \cdot |A_L|^{-1/2} \cdot \frac{\alpha_p(L)}{\alpha_p(K)},
\]
this can be rewritten as

\[
\text{vol}_{HM}(L, K) = \frac{g_{Lp}^+(L)}{g_{Lp}^+(K)} \cdot \frac{\pi^{n/2+1}}{\Gamma(n/2+1)} \cdot \prod_p a_p(L, K).
\]

Below we shall estimate \(a_p(L, K)\) for each \(p\). The case \(p \nmid 2D(L)\) is easy (Lemma 5.5(1)). When \(p | D(L)\), we rearrange \(a_p(L, K)\) as follows.

**Lemma 5.3.** Let \(p\) be a prime. For an index \(j\) with \(L_{p,j} \neq 0\) we put

\[
m_{p,j}(L) := \sum_{k \geq 0} |k - j| \cdot n_{p,k}(L) - \mu(p).
\]

Then

\[
a_p(L, K) = p^{-m_{p,\nu(p)}(L)/2} \cdot \frac{\alpha_p(L) \cdot p^{-\nu_p(L)}}{\alpha_p(K) \cdot p^{-\nu_p(K)}} \cdot p^{-\mu(p)/2}.
\]

**Proof.** It suffices to check that

\[
\log_p |A_L|_p + \nu(p)(n + 2) = 2w_p(L) - 2w_p(K) + m_{p,\nu(p)}(L) + \mu(p).
\]

We have

\[
\log_p |A_L|_p + \nu(p)(n + 2) = \sum_{k \geq 0} k \cdot n_{p,k}(L) + \sum_{k \geq 0} \nu(p) \cdot n_{p,k}(L).
\]

Using the relation of \(n_{p,k}(L)\) and \(n_{p,k}(K)\), we can calculate

\[
w_p(L) - w_p(K) = \sum_{k < \nu(p)} k \cdot n_{p,k}(L) + \nu(p) \cdot \sum_{k \geq \nu(p)} n_{p,k}(L).
\]

Therefore

\[
\log_p |A_L|_p + \nu(p)(n + 2) = 2w_p(L) + 2w_p(K)
\]

\[
= \sum_{k < \nu(p)} (\nu(p) - k)n_{p,k}(L) + \sum_{k \geq \nu(p)} (k - \nu(p))n_{p,k}(L).
\]

\(\square\)

**Figure 1.** \(m_{p,j}(L)\) (when \(L \otimes \mathbb{Z}_p\) is primitive)
Proof.

of Jordan decomposition. This will be reserved through the rest of this section. The number $m_{p,v(p)}(L)$ will be central in our estimate. When $L \otimes \mathbb{Z}_p$ is primitive, i.e., $n_{p,0}(L) > 0$, one can understand $m_{p,j}(L)$ as the area of the slanted region in Figure 1. Let us first bound the middle term of (5.3)

$$\frac{\alpha_p(L) \cdot p^{-w_p(L)}}{\alpha_p(K) \cdot p^{-w_p(K)}}$$

in the next Lemma 5.5. The result is to be reflected in the following definition of $\varepsilon_{p,j}(L)$.

**Definition 5.4.** Let $L$ be a lattice of signature $(2, n)$. Let $p$ be a prime divisor of $2D(L)$ and $j$ be an index with $L_{p,j} \neq 0$. We set

$$\varepsilon_{p,j}(L) = \begin{cases} p^{-m_{p,j}(L)/2} \cdot (1 + p^{-[n, j]/2}), & p \notin P \cup \{2\}, \\ 4 \cdot p^{-m_{p,j}(L)/2}, & p \in P, \\ 2^{-m_{p,j}(L)/2}, & p = 2. \end{cases}$$

Note that when $2 \nmid D(L)$, namely $L \otimes \mathbb{Z}_2$ is unimodular, we have $m_{2,0}(L) = 0$ and hence $\varepsilon_{2,0}(L) = 1$. Note also that $\varepsilon_{p,j}(L)$ does not depend on the choice of Jordan decomposition.

**Lemma 5.5.** The following inequalities hold.

1. When $p \nmid 2D(L)$, we have

$$a_p(L, K) \leq 1 + p^{-[n/2]-1}.$$

2. When $p | D(L)$ with $p \notin P \cup \{2\}$, we have

$$a_p(L, K) \leq \varepsilon_{p,v(p)}(L) \cdot p^{-\mu(p)/2}.$$

3. For $p \in P$ we have

$$g^+_p(L) \cdot \prod_{p \in P} a_p(L, K) \leq 4 \cdot \prod_{p \in P} \varepsilon_{p,v(p)}(L) \cdot p^{-\mu(p)/2}.$$

4. For $p = 2$ we have

$$a_2(L, K) \leq 2^5 \cdot \varepsilon_{2,v(2)}(L) \cdot 2^{-\mu(2)/2}.$$

**Proof.** (1) Let $p \nmid D(L)$ with $p > 2$. In this case $a_p(L, K)$ reduces to $\alpha_p(L)/\alpha_p(K)$. Since both $L \otimes \mathbb{Z}_p$ and $K \otimes \mathbb{Z}_p$ are unimodular, we have $s_p(L) = s_p(K) = 1$ and $w_p(L) = w_p(K) = 0$. Then

$$\frac{\alpha_p(L)}{\alpha_p(K)} = \frac{P_p(((n + 2)/2))}{P_p(((n + 1)/2))} \cdot \frac{1 + \chi(K_{p,0})p^{-(n+1)/2}}{1 + \chi(L_{p,0})p^{-(n+2)/2}}$$

$$= \begin{cases} 1 - \chi(L_{p,0})p^{-(n+2)/2}, & n : \text{even}, \\ 1 + \chi(K_{p,0})p^{-(n+1)/2}, & n : \text{odd}, \\ \leq 1 + p^{-[n/2]-1}. \end{cases}$$
(2) Next we consider the case \( p | D(L) \) with \( p > 2 \). When \( n_{p,v(p)}(L) > 1 \), we have \( s_p(L) = s_p(K) \). Then
\[
\frac{\alpha_p(L) \cdot p^{-w_p(L)}}{\alpha_p(K) \cdot p^{-w_p(K)}} = \frac{P_p([n_{p,v(p)}(L)/2]) \cdot 1 + \chi(K_{p,v(p)}) p^{-n_{p,v(p)}(K)/2}}{P_p([n_{p,v(p)}(K)/2]) \cdot 1 + \chi(L_{p,v(p)}) p^{-n_{p,v(p)}(L)/2}} \leq 1 + p^{-\lceil n_{p,v(p)}(L)/2 \rceil}
\]
by the same calculation as in case (1). On the other hand, if \( n_{p,v(p)}(L) = 1 \), we have \( s_p(L) = s_p(K) + 1 \) so that
\[
\frac{\alpha_p(L) \cdot p^{-w_p(L)}}{\alpha_p(K) \cdot p^{-w_p(K)}} = 2.
\]
By (5.3), this gives the desired inequality in case \( p \not\in P \).

(3) When \( p \in P \), the equality (5.4) is still valid. This, combined with (5.3) and Lemma 5.2 gives the desired inequality.

(4) Finally let \( p = 2 \). Note that \( L_{2,v(2)} \) is odd. It is easy to check that
\[
s_2(L) - s_2(K) \leq 1,
\]
\[
s'_2(L) - s'_2(K) \leq 2,
\]
\[
q(K) - q(L) = q_{v(2)}(K) - q_{v(2)}(L) + q_{v(2)-1}(K) - q_{v(2)-1}(L) \leq -1 + 0 = -1,
\]
\[
\prod_j \frac{P_2(n_{2,j}^+(L)/2)}{P_2(n_{2,j}^+(K)/2)} = \frac{P_2(n_{2,v(2)}^+(L)/2)}{P_2(n_{2,v(2)}^+(K)/2)} \leq 1,
\]
\[
\prod_j \frac{E_{2,j}(L)}{E_{2,j}(K)} = \prod_{j=v(2)-1}^{v(2)} \frac{E_{2,j}(K)}{E_{2,j}(L)} \leq \frac{1 + 1}{1 - 2^{-1}} \cdot \frac{1 + 1}{1} = 2^4.
\]
Actually, examining the cases when \( s'_2(L) > s'_2(K) \) holds, we can see
\[
2^{s'_2(L)-s'_2(K)} \cdot \prod_j E_{2,j}(K)/E_{2,j}(L) \leq 2^4.
\]
This gives
\[
\frac{\alpha_2(L) \cdot 2^{-w_2(L)}}{\alpha_2(K) \cdot 2^{-w_2(K)}} \leq 2^5.
\]

By this lemma we obtain
\[
g_{2,p}^+(L) \cdot \prod_p \alpha_p(L, K) < 2^7 \cdot \zeta([n/2] + 1) \cdot \prod_{p | D(L)} \epsilon_{p,v(p)}(L) \cdot D(L)^{-1/2}
\]
regardless of whether \( D(L) \) is even or odd. Substituting this into (5.2) gives the following intermediate estimate of \( \text{vol}_{HM}(L, K) \).
Proposition 5.6. Let $L$ be a lattice of signature $(2,n)$ with $n \geq 2$, and $K = l^\perp \cap L$ be the orthogonal complement of a reflective vector $l \in L$ of split type of norm $(l,l) = -D = -\prod_p p^{\nu(p)}$. Then we have

$$\text{vol}_H(L, K) < \frac{1}{g_{sp}(K)} \cdot \frac{2^7 \cdot \pi^{n/2+1} \cdot \zeta([n/2]+1)}{\Gamma(n/2+1)} \cdot D(L)^{-1/2} \cdot \prod_{p|D(L)} \varepsilon_{p,\nu(p)}(L).$$

The point here is that the right-hand side reserves $D(L)$ which measures the size of $L$, and that except $g_{sp}(K)^{-1}$ it depends only on $L$ and $D$ but not on $K$.

The estimate of $\text{vol}_H(L, K)$ is thus shifted to that of $\prod_p \varepsilon_{p,\nu(p)}(L)$. Recall that what we finally need to estimate is not single $\text{vol}_H(L, K)$ but rather their following combination which will arise in the summation process (§6.1).

**Definition 5.7.** Let $L$ be a lattice of signature $(2,n)$. For $p|2D(L)$ we put

$$\varepsilon_p(L) = \sum_{j, L_{p,j} \neq 0} \varepsilon_{p,j}(L).$$

Then we set

$$\varepsilon(L) = \prod_{p|D(L)} \varepsilon_p(L) = \sum_J \left( \prod_{p|D(L)} \varepsilon_{p,j(p)}(L) \right),$$

where $J = (j(p))_{p|D(L)}$ runs through multi-indices such that $L_{p,j(p)} \neq 0$ for every $p$. Note that when $2 \nmid D(L)$, we have $\varepsilon_2(L) = 1$.

*From now on we assume that $L$ is primitive.* The main step in the proof of Theorem 1.5 is the following.

**Proposition 5.8.** For primitive lattices $L$ the numbers $\varepsilon(L)$ are bounded in $n \geq 4$: there exists a constant $\varepsilon < \infty$ independent of $L$ and $n$ such that $\varepsilon(L) \leq \varepsilon$ for every primitive lattice $L$ of signature $(2,n)$ with $n \geq 4$.

This proposition will not be used until Proposition 6.4 but we want to give the proof here because it would not be easy to remember $\varepsilon(L)$. In the proof the following easy estimate of $m_{p,j}(L)$ will be used several times.

**Lemma 5.9.** If $L$ is primitive, we have

$$m_{p,j}(L) \geq \max(0, n - n_{p,j}(L)).$$
Proof. (See also Figure [1]) Note that \( L_{p,0} \neq 0 \) by the primitivity of \( L \), and \( L_{p,\mu(p)} \neq 0 \) by the definition of \( \mu(p) \). We have

\[
m_{p,j}(L) = j(n_{p,0}(L) - 1) + (\mu(p) - j)(n_{p,\mu(p)}(L) - 1) + \sum_{k \neq j, j \mu(p)} |k - j| n_{p,k}(L)
\]

\[
\geq \sum_{k \neq j} n_{p,k}(L) - 2
\]

\[
= n - n_{p,j}(L).
\]

The inequality \( m_{p,j}(L) \geq 0 \) is clear from the second line. \( \square \)

(Proof of Proposition [5.8].) Since we will not change the lattice \( L \) through the argument, let us abbreviate \( n_{p,j}(L) = n_{p,j}, m_{p,j}(L) = m_{p,j} \) and \( \varepsilon_{p,j}(L) = \varepsilon_{p,j} \). We divide the set of prime divisors of \( D(L) \) into the following six sets, some of which could be empty:

\[
P_1 = \{2\},
\]

\[
P_2 = P,
\]

\[
P_3 = \{ p > 2 \mid \exists j \ n_{p,j} = n + 1 \},
\]

\[
P_4 = \{ p > 2 \mid \exists j \ n_{p,j} = n \},
\]

\[
P_5 = \{ p > 2 \mid \forall j \ n_{p,j} < n \text{ and } \exists j \ n_{p,j} > n/2 + 1 \},
\]

\[
P_6 = \{ p \notin P \cup \{2\} \mid \forall j \ n_{p,j} \leq n/2 + 1 \}.
\]

We will show that for each \( P_i \), there exists a constant \( \varepsilon(i) < \infty \) independent of \( L \) and \( n \) such that \( \prod_{p \in P_i} \varepsilon_{p}(L) \leq \varepsilon(i) \). Then our assertion follows by putting \( \varepsilon = \prod_{i=1}^{6} \varepsilon(i) \).

\( (P_1) \) There exists at most one index \( j \) such that \( n_{2,j} > n/2 + 1 \). We have \( \varepsilon_{2,j} \leq 1 \) for this index. For the remaining indices \( j \) we have \( n_{2,j} \leq n/2 + 1 \), so \( m_{2,j} \geq n/2 - 1 \) by Lemma [5.9] hence \( \varepsilon_{2,j} \leq 2^{(2-n)/4} \). Since there are at most \( n + 2 \) indices \( j \) with \( L_{2,j} \neq 0 \), we obtain

\[
\varepsilon_{2}(L) < 1 + (n + 2)2^{(2-n)/4}.
\]

Since \( (n + 2)2^{(2-n)/4} \) converges to 0 as \( n \to \infty \), the number

\[
\varepsilon(1) = \max_{n \geq 3} (1 + (n + 2)2^{(2-n)/4})
\]

is finite, and we have \( \varepsilon_{2}(L) < \varepsilon(1) \).

\( (P_2) \) If \( p \in P \), we have \( m_{p,j} \geq (n^2 - 1)/4 \) by calculating the definition of \( m_{p,j} \), and thus \( \varepsilon_{p}(L) \leq 4(n + 2)p^{(1-n^2)/8} \). It follows that

\[
\prod_{p \in P} \varepsilon_{p}(L) \leq \prod_{p > 2} \max(4(n + 2)p^{(1-n^2)/8}, 1).
\]
For fixed $n$ there are only finitely many $p$ such that $4(n + 2)p^{(1-n^2)/8} > 1$, so the right-hand side is actually a finite product. When $n \geq 6$ we have $4(n + 2)p^{(1-n^2)/8} < 1$ for any $p > 2$, so this product gets equal to 1. Therefore

$$
\varepsilon(2) = \max_{n \geq 3} \left( \prod_{p > 2} \max(4(n + 2)p^{(1-n^2)/8}, 1) \right)
$$

is finite, and we have $\prod_{p \in \mathbb{P}} \varepsilon_p(L) \leq \varepsilon(2)$.

(P3) For primes $p$ in $P_3$, we have $(n_{p,0}, n_{p,\mu(p)}) = (1, n + 1)$ or $(n + 1, 1)$, and $n_{p,j} = 0$ for other indices $j$. We have $(m_{p,0}, m_{p,\mu(p)}) = (n\mu(p), 0)$ and $(0, n\mu(p))$ in the respective cases, so

$$
\varepsilon_p(L) = (1 + p^{-[(n+1)/2]}) + 2p^{-n\mu(p)/2} \leq 1 + 3p^{-2}.
$$

If we put

$$
\varepsilon(3) = \prod_{p > 2}(1 + 3p^{-2}),
$$

we have $\prod_{p \in \mathbb{P}} \varepsilon_p(L) < \varepsilon(3)$ because every factor of $\varepsilon(3)$ is larger than 1. When $p \geq 11$, we have $1 + 3p^{-2} < 1 + p^{-3/2}$, so $\varepsilon(3)$ is dominated by some multiple of $\zeta(3/2)$, hence finite.

(P4) There are three possibilities:

1. $(n_{p,0}, n_{p,\mu(p)}) = (2, n)$ or $(n, 2)$, and $n_{p,j} = 0$ for all other $j$;
2. $(n_{p,0}, n_{p,\mu(p)}) = (1, n)$ or $(n, 1)$, and $n_{p,j} = 1$ for some $0 < j < \mu(p)$.
3. $(n_{p,0}, n_{p,\mu(p)}) = (1, 1)$, and $n_{p,j} = n$ for some $0 < j < \mu(p)$;

In case (1), we have

$$
\varepsilon_p(L) = p^{-\mu(p)/2}(1 + p^{-[n/2]}) + p^{(1-n)\mu(p)/2}(1 + p^{-1})
\leq p^{-1/2}(1 + p^{-2}) + p^{-3/2}(1 + p^{-1}).
$$

(5.6)

In case (2), we have $m_{p,k} \geq 1$ for $k$ with $n_{p,k} = n$, and $m_{p,k} \geq n - 1$ for $k$ with $n_{p,k} = 1$. Hence

$$
\varepsilon_p(L) \leq p^{-1/2}(1 + p^{-[n/2]}) + 4p^{(1-n)/2} \leq p^{-1/2}(1 + p^{-2}) + 4p^{-3/2}.
$$

(5.7)

In case (3), we have $m_{p,j} = 0$ for $j$ with $n_{p,j} = n$, and $m_{p,0}, m_{p,\mu(p)} \geq n$. Therefore

$$
\varepsilon_p(L) \leq (1 + p^{-[n/2]}) + 4p^{-n/2} \leq 1 + 5p^{-2}.
$$

(5.8)

We have the bounds (5.6), (5.7), (5.8) in the respective cases, but actually $1 + 5p^{-2}$ is greater than other two bounds. Therefore

$$
\varepsilon_p(L) \leq 1 + 5p^{-2}
$$

in any case. If we put

$$
\varepsilon(4) = \prod_{p > 2}(1 + 5p^{-2}),
$$

we have $\prod_{p \in \mathbb{P}} \varepsilon_p(L) < \varepsilon(4)$ because every factor of $\varepsilon(4)$ is larger than 1. When $p \geq 11$, we have $1 + 5p^{-2} < 1 + p^{-3/2}$, so $\varepsilon(4)$ is dominated by some multiple of $\zeta(3/2)$, hence finite.
we have $\prod_{p_i} \varepsilon_p(L) < \varepsilon(4)$. Since $1 + 5p^{-2} < 1 + p^{-3/2}$ in $p \geq 29$, $\varepsilon(4)$ is dominated by a multiple of $\zeta(3/2)$ and hence finite.

$(P_5)$ We must have $n \geq 5$ in this case. There exists only one index $j$ with $n_{p,j} > n/2 + 1$, for which we have $m_{p,j} \geq 1$ by Lemma 5.9 and hence $\varepsilon_{p,j} \leq p^{-1/2}(1 + p^{-2})$. There remain at most $(n + 1)/2$ indices $j$ with $L_{p,j} \neq 0$. For them we have $m_{p,j} > n/2$, so $\varepsilon_{p,j} < 2p^{-n/4}$. It follows that

$$\varepsilon_p(L) < p^{-1/2}(1 + p^{-2}) + (n + 1)p^{-n/4}.$$  

As in the $(P_2)$ case, there are only finitely many pairs $(n, p)$ such that the right-hand side is greater than 1. Therefore

$$\varepsilon(5) = \max_{n \geq 5} \left( \prod_{p \geq 2} \max(p^{-1/2}(1 + p^{-2}) + (n + 1)p^{-n/4}, 1) \right)$$

is finite, and we have $\prod_{p} \varepsilon_p(L) < \varepsilon(5)$.

$(P_6)$ By Lemma 5.9 we have $m_{p,j} \geq n/2 - 1$ and so $\varepsilon_{p,j} \leq 2p^{(2-n)/4}$ for every index $j$ with $L_{p,j} \neq 0$. Thus $\varepsilon_p(L) \leq 2(n + 1)p^{(2-n)/4}$. As before

$$\varepsilon(6) = \max_{n \geq 4} \left( \prod_{p \geq 2} \max(2(n + 1)p^{(2-n)/4}, 1) \right)$$

is finite, and we have $\prod_{p} \varepsilon_p(L) \leq \varepsilon(6)$. The proof of Proposition 5.8 is now finished. □

Remark 5.10. (1) We needed the condition $n \geq 4$ only in the $(P_4)$-(3) case. In other cases the boundedness can be easily extended to $n = 3$.

(2) In the proof we actually gave a bound at each $n$, say $\varepsilon(i, n)$, and $\varepsilon(i)$ was defined as $\max_n(\varepsilon(i, n))$. It would be useful to record the explicit form of $\varepsilon(i, n)$. Avoiding small $n$ and sharpening the estimate for $p = 2$, we may take the bound as follows.

$$\varepsilon(1, n) = 1 + 2^{-n/2+1} \quad (n \geq 14),$$
$$\varepsilon(2, n) = 1 \quad (n \geq 6),$$
$$\varepsilon(3, n) = \prod_{p > 2} (1 + 3p^{-n/2}) < \prod_{p > 2} (1 + p^{-n/2+1}),$$
$$\varepsilon(4, n) = \prod_{p > 2} (1 + 5p^{-n/2}) < \zeta([n/2] - 2),$$
$$\varepsilon(5, n) = 1 \quad (n \geq 14),$$
$$\varepsilon(6, n) = 1 \quad (n \geq 16).$$

In particular, the total bound satisfies

$$\prod_{i=1}^{6} \varepsilon(i, n) < \zeta([n/2] - 2)^2$$
in \( n \geq 16 \), so \( \varepsilon \) can be taken to be asymptotically 1. There is still room of improvement (by refining the classification by \( \max_j (n_{p,j}) \) and the number of \( j \) with \( n_{p,j} \neq 0 \)), but we stop here.

(3) By a similar argument as in case (\( P_1 \)), we can see that \( \varepsilon_p(L) \leq 1 + 2(n+2)p^{(2-n)/4} \) for \( p \notin P \cup \{2\} \). The product \( \prod_p (1+2(n+2)p^{(2-n)/4}) \) converges at each \( n \geq 7 \) and is bounded with respect to \( n \). This gives a simpler proof in \( n \geq 7 \).

5.3. Non-split case. Next we consider the non-split case. Let \( L \) be a lattice of signature \((2,n)\) with \( n \geq 2 \). Let \( l \in L \) be a reflective vector of non-split type. The sublattice

\[ L' = \mathbb{Z}l \oplus K \quad \text{where} \quad K = l^\perp \cap L, \]

is of index 2 in \( L \). The vector \( l \) is reflective of split type in \( L' \). Hence the definitions and results in \S 5.2 before Proposition 5.8 are valid for \((L',K)\). Our approach is to reduce the estimate of the sum of \( \text{vol}^+_{\text{HM}}(L,K) \) of non-split type for \( L \) to that of \( \text{vol}^+_{\text{HM}}(L',K) \) of split type for \( L' \) over various \( L' \subset L \). This reduction step will be done in \S 6.2. Here we prepare in advance the counterpart of Proposition 5.8.

We assume that \( L \) is primitive and estimate \( \varepsilon(L') = \prod_p \varepsilon_p(L') \). (In many cases \( L' \) remains primitive, but not always.) When \( p > 2 \), we have \( L \otimes \mathbb{Z}_p = L' \otimes \mathbb{Z}_p \) and hence \( L' \otimes \mathbb{Z}_p \) is primitive.

**Lemma 5.11.** Assume that \( L \) is primitive and write \( L' = L''(2^\nu) \) with \( L'' \) primitive. Then \( \rho \leq 2 \) and \( \varepsilon_2(L') = 2^{\nu/2} \varepsilon_2(L'') \).

**Proof.** We have \( n_{2,k}(L'') = n_{2,k+\rho}(L') \) for every \( k \). In particular, if we write \( D(L'')_2 = 2^{\mu(2')^\nu} \) and \( D(L')_2 = 2^{\mu(2')^\nu} \), then \( \mu(2') = \mu(2''') + \rho \). By the definition of \( m_{2,j} \) we see that

\[ m_{2,j}(L'') + \mu(2''') = m_{2,k+p}(L') + \mu(2'). \]

Hence \( m_{2,j}(L'') = m_{2,j+p}(L') + \rho \), and so \( 2^{\nu/2} \varepsilon_2,j(L'') = \varepsilon_2,j+\rho(L') \). This implies \( \varepsilon_2(L') = 2^{\nu/2} \varepsilon_2(L'') \).

We next check \( \rho \leq 2 \). By the primitivity of \( L \otimes \mathbb{Z}_2 \), there exist vectors \( l,m \in L \otimes \mathbb{Z}_2 \) such that \( (l,m) \in \mathbb{Z}_2^\times \). Since \( L' \otimes \mathbb{Z}_2 \subset L \otimes \mathbb{Z}_2 \) is of index 2, \( 2l \) and \( 2m \) are contained in \( L' \otimes \mathbb{Z}_2 \), and satisfies \( (2l,2m) \in 4\mathbb{Z}_2^\times \). On the other hand, we must have \( (l',m') \in 2^\nu \mathbb{Z}_2 \) for all \( l',m' \in L' \otimes \mathbb{Z}_2 \). Therefore \( \rho \leq 2 \).

**Proposition 5.12.** Let \( L \) be a primitive lattice of signature \((2,n)\) with \( n \geq 4 \), and let \( L' = \mathbb{Z}l \oplus K \) for a reflective vector \( l \in L \) of non-split type. Then

\[ \varepsilon(L') \leq 2\varepsilon \]

where \( \varepsilon \) is the constant introduced in Proposition 5.8.
Proof. For $p > 2$ we have $n_{p,j}(L') = n_{p,j}(L'')$ for every $j$, so $\varepsilon_p(L') = \varepsilon_p(L'')$. By Lemma 5.11 we have $\prod_{p | D(L')} \varepsilon_p(L') \leq 2 \prod_{p | D(L'')} \varepsilon_p(L'')$. Then we can apply Proposition 5.8 to the primitive lattice $L''$. □

6. Volume sum

Single volume ratios have been estimated in §5. Next we take their sum over the sets $R_I$, $R_{II}$ of branch divisors of each type. The proof of Theorem 1.5 will be completed at the end of this section.

6.1. Split case. We first deal with reflective vectors of split type. Let $L$ be a lattice of signature $(2, n)$ with $n \geq 3$. We will not assume primitivity of $L$ until Proposition 6.4. For each natural number $D$ dividing $D(L)$, we write $R_I^+(D)$ for the set of $O(L)$-equivalence classes of reflective vectors of split type of norm $-D$. Note that if we have a splitting $L \simeq \langle -D \rangle \oplus K$, then $D$ must divide $D(L)$. We thus have the division

$$R_I = \bigsqcup_{D | D(L)} R_I^+(D).$$

We also denote by $R_I(D)$ the set of $O(L)$-equivalence classes of reflective vectors of split type of norm $-D$. It is more convenient to work with $O(L)$ than with $O^+(L)$.

Lemma 6.1. We have

$$\sum_{[l] \in R_I^+(D)} \text{vol}_{HM}(L, K) = \sum_{[l] \in R_I(D)} \text{vol}_{HM}(L, K),$$

where $K = l^\perp \cap L$ for $[l] \in R_I^+(D)$ or $R_I(D)$.

Proof. We have a natural projection $R_I^+(D) \to R_I(D)$. The cardinality of the fiber over $[l] \in R_I(D)$ is at most 2 and equal to

$$[O(L) : O^+(L)]/[O(K) : O^+(K)].$$

Indeed, when $O(L) = O^+(L)$, we have $R_I^+(D) = R_I(D)$ and also $O(K) = O^+(K)$; when $O(L) \neq O^+(L)$, the fiber consists of one element if and only if $O(L) \cdot l = O^+(L) \cdot l$, namely $\gamma(l) = l$ for some $\gamma \in O(L)\setminus O^+(L)$. This is equivalent to $O(K) \neq O^+(K)$. Now the claim follows by comparison with (4.7). □

We first estimate $\sum_{R_I(D)} \text{vol}_{HM}(L, K)$ for each $D$, and next take their sum over all possible $D$. Two reflective vectors of split type are $O(L)$-equivalent if and only if their orthogonal complements are isometric. Thus $R_I(D)$ is
canonically identified with the set of isometry classes of lattices $K$ such that $K \oplus \langle -D \rangle \cong L$. We consider division into genera:

$$\mathcal{R}_l(D) = \bigsqcup_{a=1}^{\kappa} \mathcal{R}_l(D)_a.$$ 

Each $\mathcal{R}_l(D)_a$ consists of isometry classes of lattices $K$ in the same genus.

**Lemma 6.2.** The number $\kappa$ of possible genera of $K$ is at most 9.

**Proof.** Scaling $L$ if necessary, we may assume that $L$ (and hence $K$) is even. By Nikulin’s theory [27], it suffices to show that, with the discriminant forms $A_L$ and $A_{\langle -D \rangle}$ fixed, the number of isometry classes of finite quadratic forms $A$ such that

$$(6.1) \quad A_L \cong A_{\langle -D \rangle} \oplus A$$

is at most 9.

For $p > 2$, the $p$-component $A_p$ of $A$ is uniquely determined by this relation, as can be seen from Wall’s canonical form for quadratic forms on $p$-groups ([40]). Alternatively, one can also directly resort to the Witt cancellation for $\mathbb{Z}_p$-lattices in $p > 2$ (see [20] Corollary 5.3.1).

For $p = 2$ we use Kawauchi-Kojima’s invariants $\sigma_r$ ([19]) of quadratic forms on 2-groups. (Here we identify, as in [40] Theorem 5, quadratic forms and symmetric bilinear forms with no direct summand of order 2.) These invariants are defined for each positive integer $r \geq 1$, and take values in the semigroup $(\mathbb{Z}/8) \cup \{\infty\}$. They have the properties that for two such forms $B, B'$, (i) $\sigma_r(B \oplus B') = \sigma_r(B) + \sigma_r(B')$, and (ii) $B$ and $B'$ are isometric if and only if their underlying abelian groups are isomorphic and $\sigma_r(B) = \sigma_r(B')$ for every $r \geq 1$. Furthermore, (iii) when the abelian group underlying $B$ is isomorphic to $\mathbb{Z}/2^k$, we have $\sigma_r(B) < \infty$ for $r \neq k + 1$.

Now, with $(A_L)_2$ and $(A_{\langle -D \rangle})_2$ fixed in (6.1), the abelian group underlying $A_2$ is uniquely determined. We have $\sigma_r((A_{\langle -D \rangle})_2) < \infty$ except for one value of $r$. At these $r$, $\sigma_r(A_2)$ is uniquely determined by $\sigma_r(A_2) = \sigma_r((A_L)_2) - \sigma_r((A_{\langle -D \rangle})_2)$. Hence the isometry class of $A_2$ is determined by the value of $\sigma_r(A_2)$ at the remaining one $r$. $\square$

Since $\text{vol}_{HM}(O(K))$ depends only on the genus of $K$, we see that

$$\sum_{\mathcal{R}_l(D)} \text{vol}_{HM}(L, K) = \sum_{a=1}^{\kappa} |\mathcal{R}_l(D)_a| \cdot \text{vol}_{HM}(L, K).$$

If $K \in \mathcal{R}_l(D)_a$, we have

$$|\mathcal{R}_l(D)_a| \leq g_{sp}^{+}(K)$$
because proper spinor genus coincides with proper equivalence class, which is finer than isometry class. We now substitute Proposition 5.6. We set

\[ f(n) = \frac{2^7 \cdot 9 \cdot \pi^{n/2+1} \cdot \zeta([n/2] + 1)}{\Gamma(n/2 + 1)}. \]

Then

\[ \sum_{R \in \mathcal{R}_1(D)} \text{vol}_{HM}(L, K) < f(n) \cdot D(L)^{-1/2} \cdot \prod_{p|D(L)} \epsilon_{p, \nu(p)}(L), \]

where the indices \( \nu(p) \) are defined by \( D = \prod_p p^{\nu(p)} \).

We finally take the sum over the set of possible norms \(-D\). We can identify \( D = \prod_p p^{\nu(p)} \) with the multi-index \((\nu(p))_{p|D(L)}\). If \( R_1(D) \neq \emptyset \), then \( L_{p, \nu(p)} \neq 0 \) at each \( p \). Thus the set of possible norms \(-D\) can be regarded as a subset of the set of multi-indices \( J = (j(p))_{p|D(L)} \) such that \( L_{p, j(p)} \neq 0 \) at each \( p \). Since \( \epsilon_{p, j}(L) > 0 \) for all \((p, j)\) with \( p|D(L) \) and \( L_{p, j} \neq 0 \), we obtain by adding redundant \( J \)

\[
\sum_{D} \sum_{R \in \mathcal{R}_1(D)} \text{vol}_{HM}(L, K) < \sum_{D} f(n) \cdot D(L)^{-1/2} \cdot \prod_{p|D(L)} \epsilon_{p, \nu(p)}(L)
\leq f(n) \cdot D(L)^{-1/2} \cdot \sum_{J} \prod_{p|D(L)} \epsilon_{p, j(p)}(L)
= f(n) \cdot D(L)^{-1/2} \cdot \epsilon(L)
\]

where \( \epsilon(L) \) is as defined in Definition 5.7.

Let us summarize the argument so far, which worked without assuming \( L \) primitive. This will be used again in the next section.

**Lemma 6.3.** Let \( L \) be a lattice of signature \((2, n)\) with \( n \geq 3 \). Then

\[ \sum_{[\mathcal{I}] \in \mathcal{R}_1} \text{vol}^+_{HM}(L, K) < f(n) \cdot \epsilon(L) \cdot D(L)^{-1/2}. \]

Now assuming primitivity of \( L \) and that \( n \geq 4 \), we obtain from Proposition 5.8 the final estimate in the split case.

**Proposition 6.4.** For a primitive lattice \( L \) of signature \((2, n)\) with \( n \geq 4 \) we have

\[ \sum_{[\mathcal{I}] \in \mathcal{R}_1} \text{vol}^+_{HM}(L, K) < f(n) \cdot \epsilon \cdot D(L)^{-1/2} \]

where \( \epsilon \) is the constant introduced in Proposition 5.8 and \( f(n) \) is the function defined by (6.2).
6.2. Non-split case. We next consider the non-split case. Let $L$ be a lattice of signature $(2, n)$ with $n \geq 3$. Recall from [5,3] that for a reflective vector $l \in L$ of non-split type, our approach is to reduce the calculation of $\text{vol}_{\text{HM}}^+(L, K)$ to that of $\text{vol}_{\text{HM}}^+(L', K)$ where $K = l^\perp \cap L$ and $L' = \mathbb{Z}l \oplus K$. Let us denote

$$\Gamma_L' = \mathcal{O}^+(L) \cap \mathcal{O}^+(L'),$$

the intersection considered inside $\mathcal{O}(L_{\mathbb{Q}}) = \mathcal{O}(L'_{\mathbb{Q}})$. If we abuse notation to write

$$[\mathcal{O}^+(L) : \mathcal{O}^+(L')] = [\mathcal{O}^+(L) : \Gamma_L']/[\mathcal{O}^+(L') : \Gamma_L'],$$

we have by the relation (4.2)

$$\text{vol}_{\text{HM}}^+(L, K) = [\mathcal{O}^+(L) : \mathcal{O}^+(L')] \cdot \text{vol}_{\text{HM}}^+(L', K).$$

Let $T$ be the set of index 2 sublattices $L'$ of $L$ for which there exists a reflective vector $l$ of $L$ of non-split type such that $L' = \mathbb{Z}l \oplus (l^\perp \cap L)$. We write $\mathcal{T} = T/\mathcal{O}^+(L)$. For each $L' \in T$ let $R[L']$ be the set of vectors $l \in L'$ which is primitive in $L'$ and splits $L'$, namely $L' = \mathbb{Z}l \oplus (l^\perp \cap L')$. We put $\mathcal{R}[L'] = R[L']/\mathcal{O}^+(L')$. In other words, $\mathcal{R}[L']$ is $\mathcal{R}_I$ for $L'$.

Lemma 6.5. We have

$$\sum_{l \in \mathcal{R}_I} \text{vol}_{\text{HM}}^+(L, K) \leq \sum_{[L'] \in \mathcal{T}} [\mathcal{O}^+(L) : \Gamma_{L'}] \left( \sum_{l \in R[L']} \text{vol}_{\text{HM}}^+(L', K) \right).$$

Here $K = l^\perp \cap L$ for $[l] \in \mathcal{R}_I$ in the left-hand side, while $K = l'^\perp \cap L'$ for $[l] \in \mathcal{R}[L']$ in the right-hand side.

Proof. For each $L' \in T$, let $R'[L'] \subset R[L']$ be the subset consisting of splitting vectors $l$ of $L'$ such that $l$ is still primitive in $L$ and that $l'^\perp \cap L' = l^\perp \cap L'$. This is equal to the set of reflective vectors $l$ of $L$ of non-split type such that $L' = \mathbb{Z}l \oplus (l^\perp \cap L)$. Thus the set of reflective vectors of $L$ of non-split type is divided as $\bigsqcup_{L' \in \mathcal{T}} R'[L']$, according to which index 2 sublattice is $\mathbb{Z}l \oplus (l^\perp \cap L)$. Taking quotient by $\mathcal{O}^+(L)$, we obtain

$$\mathcal{R}_I = \bigsqcup_{[L'] \in \mathcal{T}} \mathcal{R}[L']/\Gamma_{L'}$$

because $\Gamma_{L'} < \mathcal{O}^+(L)$ is the stabilizer of $L'$ in the $\mathcal{O}^+(L)$-action on $T$. Hence $\mathcal{R}_I$ can be embedded into the formal disjoint union

$$\bigsqcup_{[L'] \in \mathcal{T}} \mathcal{R}[L']/\Gamma_{L'}.$$
(Note that when considered as sets of vectors of \( L \), the sets \( R[L'] \) may have overlap with each other.) By (6.4) we have
\[
\sum_{[l] \in R} \text{vol}_{HM}^+(L, K) = \sum_{[l] \in R} [O^+(L) : O^+(L')] \cdot \text{vol}_{HM}^+(L', K)
\]
\[
\leq \sum_{[l'] \in T} [O^+(L) : O^+(L')] \left( \sum_{[l] \in R[L']} \text{vol}_{HM}^+(L', K) \right).
\]
Here \( K = l^\perp \cap L \) in the first line, while \( K = l^\perp \cap L' \) in the second line. Consider the projection \( R[L']/\Gamma_{L'} \rightarrow R[L'] \). Its fibers have at most \( [O^+(L') : \Gamma_{L'}] \) elements, so we have
\[
\sum_{[l] \in R[L']/\Gamma_{L'}} \text{vol}_{HM}^+(L', K) \leq [O^+(L') : \Gamma_{L'}] \cdot \sum_{[l] \in R[L']} \text{vol}_{HM}^+(L', K)
\]
Then our assertion follows by recalling (6.3). \( \Box \)

We estimate the right-hand side of (6.5). Recall that Lemma 6.3 is still valid for \( L' \). This gives for each \( [L'] \in T \)
\[
\sum_{R[L']} \text{vol}_{HM}^+(L', K) < f(n) \cdot \varepsilon(L') \cdot D(L')^{-1/2} \leq f(n) \cdot \varepsilon(L') \cdot D(L)^{-1/2}.
\]
In the second inequality we have \( D(L') \geq D(L) \) because \( A_L \) is an index 2 quotient of an index 2 subgroup of \( A_{L'} \).

We now assume primitivity of \( L \) and \( n \geq 4 \). By Proposition 5.12 we have
\[
\sum_{R[L']} \text{vol}_{HM}^+(L', K) < f(n) \cdot 2\varepsilon \cdot D(L)^{-1/2}.
\]
Since the right-hand side does not depend on \( L' \), we obtain
\[
\sum_{[l] \in R} \text{vol}_{HM}^+(L, K) < \left( \sum_{[L'] \in T} [O^+(L) : \Gamma_{L'}] \right) \cdot f(n) \cdot 2\varepsilon \cdot D(L)^{-1/2}.
\]
Since \( \Gamma_{L'} < O^+(L) \) is the stabilizer of \( L' \in T \) in the \( O^+(L) \)-action on \( T \), then \( [O^+(L) : \Gamma_{L'}] \) equals to the cardinality of the \( O^+(L) \)-orbit of \( L' \) in \( T \). Therefore
\[
\sum_{[L'] \in T} [O^+(L) : \Gamma_{L'}] = |T| < 2^{n^2}.
\]
We arrive at the final estimate in the non-split case.

**Proposition 6.6.** For a primitive lattice \( L \) of signature \((2, n)\) with \( n \geq 4 \) we have
\[
\sum_{[l] \in R} \text{vol}_{HM}^+(L, K) < 2^{n+3} \cdot f(n) \cdot \varepsilon \cdot D(L)^{-1/2}
\]
where \( \varepsilon \) is the constant introduced in Proposition 5.8 and \( f(n) \) is the function defined by (6.2).
The above method can be used to give estimate of more general sum
\[ \sum_l \text{vol}^+_H(L, K) \] where \( l \) runs over (up to \( O^+(L) \)) primitive vectors such that \( \mathbb{Z}l \oplus (l^\perp \cap L) \) is of a fixed index in \( L \).

6.3. Proof of Theorem 1.5. We can now prove Theorem 1.5 by combining the estimates obtained so far. Let \( L \) be a primitive lattice of signature \((2, n)\) with \( n \geq 4 \). We put
\[ g(n) = f(n) \cdot (1 + 4^{a+2}) \cdot \varepsilon \]
where \( f(n) \) and \( \varepsilon \) are as introduced in (6.2) and Proposition 5.8 respectively. By Propositions 6.4 and 6.6, the left-hand side of (4.3) is bounded as
\[ \sum_{\mathcal{R}_1} \text{vol}^+_H(L, K) + 2^{n+1} \cdot \sum_{\mathcal{R}_1} \text{vol}^+_H(L, K) < g(n) \cdot D(L)^{-1/2}. \]

By Proposition 4.3, the \( \mathbb{Q} \)-divisor \( aL - B/2 \) is big if the inequality
\[ (6.7) \quad g(n) \cdot (1 + a^{-1})^{n-1} \cdot (n/2a) \leq \sqrt{D(L)} \]
holds.

If we fix \( n \), there are only finitely many primitive lattices \( L \) whose \( D(L) \) does not exceed this bound. Indeed, the discriminant is bounded by \( |A_L| \leq D(L)^{n+1} \), and there are only finitely many lattices of fixed signature with bounded discriminant. Thus we obtain the finiteness at each fixed \( n \). Next, when \( n \) grows, the left-hand side of (6.7) converges to 0 due to the rapid decay of the Gamma factor \( \Gamma(n/2 + 1)^{-1} \) in \( f(n) \). Therefore the inequality (6.7) holds for every primitive lattice \( L \) when \( n \) is sufficiently large. This completes the proof of Theorem 1.5.

7. Effective computation

7.1. Bound of \( n \). In this subsection we explicitly compute a bound of \( n \) above which all \( \mathcal{F}_L \) is of general type. By §3, we always have a nonzero \( O^+(L) \)-cusp form of weight \( \leq n/2 + 11 \). So we may take \( a = n/2 - 11 \) in (6.7). Since \( \varepsilon \to 1 \) (Remark 5.10 (2)) and \( (1 + a^{-1})^{n-1} \to e^2 \) for this value of \( a \), the resulting bound is asymptotically given by (1.1). This is smaller than 1 at least in \( n \geq 300 \), which gives a first bound.

We can improve this using Lemma 3.1. In the following we assume that \( L \) is a lattice of signature \((2, n)\) such that \((A_L)_p \cong (\mathbb{Z}/p)^{l_p} \) with \( l_p \leq n/2 + 1 \) for every \( p \). It suffices to compute a bound of \( n \) for such lattices. For them we can improve some part of §4–6 as follows.

First, if \( l \in L \) is reflective of non-split type, then \( \text{div}(l) = 2^b \mathbb{Z} \) with \( b \) odd and \( a \leq 1 \). When \( a = 0 \), we have \((A_K)_2 \cong \mathbb{Z}/2 \oplus (A_L)_2, (A_L)_2 \cong \mathbb{Z}/2 \oplus (A_K)_2 \) and \([O^+(K) : \Gamma_l] \leq 2^{l/2} \) by Lemma 4.2. When \( a = 1 \), we have \((A_K)_2 \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{l/2-2} \) and \((A_L)_2 \cong (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/2)^{l/2-2} \). The gluing element \( x \) in
(A_K)_2 satisfies \(x = 2y\) for every element \(y\) of order 4, so is \(O(A_K)\)-invariant. Hence \(\Gamma_l = O^+(K)\). Thus the left-hand side of (4.3) can be replaced by

\[
\sum_{R_l} \text{vol}^+_M(L, K) + \sum_{R_l, a = 1} \text{vol}^+_M(L, K) + 2^{l_2} \cdot \sum_{R_l, a = 0} \text{vol}^+_M(L, K).
\]

(The spinor genera \(g^+_{sp}(L), g^+_{sp}(L'), g^+_{sp}(K)\) are always equal to 1 by [7] Theorem 11.1.5. Also the set \(P\) is empty (for \(L\) and also for \(L'\)). We will not touch on the estimates in Lemma [5.5] (1), (2). On the other hand, the bound (5.5) can be improved to \(\leq 4\) for \(l\) of split type. For non-split type \(l\), replacing \(L\) by \(L'\), the bound (5.5) can be sharpened to \(\leq 1\). Finally, we have

\[
\varepsilon_2(L') = 2^{-(l_2+1)/2} + 2^{(l_2+1-n)/2}
\]

in the non-split case with \(a = 0\). In other cases we do not improve the estimate of \(\varepsilon_2(L), \varepsilon_2(L')\) in Remark [5.10] (2). (Note that \(L'\) is primitive.) To sum up, writing

\[
h(n) = 9 \cdot \pi^{n/2+1} \cdot \zeta([n/2] - 2)^3 / \Gamma(n/2 + 1),
\]

we have

\[
\sum_{R_l} \text{vol}^+_M(L, K) < 4 \cdot h(n) \cdot D(L)^{-1/2},
\]

\[
\sum_{R[L']} \text{vol}^+_M(L', K) < h(n) \cdot D(L)^{-1/2} \quad (a = 1),
\]

and when \(a = 0\),

\[
2^{l_2} \sum_{R[L']} \text{vol}^+_M(L', K) < \left(2^{(l_2-1)/2} + 2^{(3l_2+1-n)/2}\right) \cdot h(n) \cdot D(L)^{-1/2}
\]

\[
\leq \left(2^{n/4} + 2^{n/4+2}\right) \cdot h(n) \cdot D(L)^{-1/2}.
\]

Repeating the process in §6.2 we obtain

\[
(7.1) < \tilde{h}(n) \cdot D(L)^{-1/2}
\]

where

\[
\tilde{h}(n) = (4 + 2^{n+2} + 2^{5n/4+2} + 2^{5n/4+4}) \cdot h(n).
\]

Thus every \(\mathcal{F}_L\) is of general type when

\[
\tilde{h}(n) \cdot (1 + a^{-1})^{n-1} \cdot (n/2a) \leq 1, \quad a = n/2 - 11.
\]

This holds in \(n \geq 109\). When \(n = 108\), the left-hand side is still smaller \(\sqrt{2}\), and the unimodular case is of general type by the next §7.2. We thus obtain the bound stated in Theorem [1.1].

It would be possible to improve the bound of \(n\) by doing case-by-case refined estimate for lattices whose \(D(L)\) is smaller than the uniform bound above.
7.2. Example: odd unimodular lattice. As an explicit example we work out the odd unimodular lattices $I_{2,n} = 2(1) \oplus n(-1)$. The even unimodular case $II_{2,2+8m}$ is studied by Gritsenko-Hulek-Sankaran [14], who proved that $\mathcal{F}_{II_{2,n}}$ is of general type in $n \geq 42$.

**Proposition 7.1.** The variety $\mathcal{F}_{I_{2,n}}$ is of general type when $n \geq 39$.

**Proof.** We work with the maximal even sublattice $L$ of $I_{2,n}$, which is isometric to

$$L = 2U \oplus D_{n-2} = 2U \oplus mE_8 \oplus D_N, \quad 1 \leq N \leq 8.$$

By convention, $D_1 = \langle -4 \rangle$ and $D_2 = 2A_1$. The case $N = 1$ is treated in [14], where $\mathcal{F}_L$ is shown to be of general type in $m \geq 5$. We consider the remaining case $N \geq 2$. The discriminant form $A = A_{D_N}$ is as follows. We write $\langle \varepsilon/2^\mu \rangle$ for the quadratic form on $\mathbb{Z}/2^\mu$ for which the standard generator has norm $\varepsilon/2^\mu$ modulo $2\mathbb{Z}$.

- If $N$ is odd, $A \cong \langle -N/4 \rangle$;
- if $N = \pm 2 \ (8)$, $A \cong \langle \mp 1/2 \rangle \oplus \langle \mp 1/2 \rangle$;
- if $N = 4$, $A = (\mathbb{Z}/2)_{\mp} = \langle x_1, x_2 \rangle$ with $(x_1, x_2) = 1$ and $(x_1, x_2) = 1/2$;
- if $N = 8$, $A \cong A_{U(2)}$.

Hence $O^+(I_{2,n}) = O^+(L)$ when $N \neq 4$ and $[O^+(L) : O^+(I_{2,n})] = 3$ for $N = 4$.

One can work out the general dimension formula in [36], [3] for $\rho_A^{O(A)}$-valued cusp forms. This gives for $l > 2$ with $l + N/2 \in 2\mathbb{Z}$

$$\dim S_A^{(\rho_A)^{O(A)}} = \dim S_A^{(\rho_A)^{O(A)}} = \begin{cases} 
[(2l + N)/8] - 1 & N : \text{odd}, \\
[(l - 2)/4] & N = 2, \\
[(l - 2)/6] & N = 4, \\
[l/4] & N = 6, \\
[l/4] - 1 & N = 8.
\end{cases}$$

The minimal weight $l$ of $O(A)$-invariant cusp forms is as in Table 2.

| $N$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|
| $l$ | 7 | 13/2 | 8 | 11/2 | 5 | 9/2 | 8 |

Next we calculate the branch obstruction. Let $e, f$ be the hyperbolic basis of $U$ and $\delta_1, \cdots, \delta_N$ the root basis of $D_N$ with $(\delta_1, \delta_2) = 0, (\delta_1, \delta_3) = 1$ and $(\delta_1, \delta_{i+1}) = 1$ for $i \geq 2$. Then $l_1 = e - f$ and $l_2 = \delta_1 - \delta_2$ are reflective vectors of non-split type of norm $-2, -4$ respectively. When $N = 2$, we also have the splitting $(-2)$-vector $l_3 = \delta_1$. If we write $K_l = l_1^\perp \cap L$, then

$$K_1 \cong \langle 2 \rangle \oplus U \oplus D_N \oplus mE_8,$$
\[ K_2 \simeq 2U \oplus D_{N-1} \oplus mE_8, \]
\[ K_3 \simeq 2U \oplus A_1 \oplus mE_8. \]

By the Eichler criterion ([32]), every reflective vector of \( L \) is \( O^+(L) \)-equivalent to one of \( l_1, l_2, l_3 \). The stabilizer \( \Gamma_i \) of \( l_i \) coincides to \( O^+(K_i) \) when \( (i, N) \neq (1, 6), (2, 5) \). In those exceptional cases, \([O^+(K_i) : \Gamma_i]\) = 3. The volume ratio \( \text{vol}_{\tilde{H}_M}^i(L, K_i) \) is calculated as follows:

\[
\begin{align*}
N = 2 & \quad \frac{\pi \cdot (2\pi)^{4m+2} \cdot (1 - 2^{-8m-4})}{(4m + 2)! \cdot L(4m + 3, \chi_{-4})} & \quad i = 1 \\
N = 3 & \quad \frac{2^{4m+9/2} \cdot (4m + 3)! \cdot L(4m + 3, \chi_{-8})}{\pi^{4m+3} \cdot (1 - 2^{-4m-3}) \cdot B_{8m+6}} & \quad \frac{\pi^{4m+3} \cdot (1 + 2^{-4m-2})}{(4m + 2)! \cdot L(4m + 3, \chi_{-4})} \\
N = 4 & \quad \frac{(1 + 2^{-4m-3}) \cdot (4m + 4)}{(1 - 2^{-4m-4}) \cdot |B_{4m+4}|} & \quad \frac{2 \cdot (4m + 3)! \cdot L(4m + 3, \chi_{-4})}{\pi^{4m+3} \cdot (1 - 2^{-4m-3}) \cdot B_{8m+6}} \\
N = 5 & \quad \frac{2^{4m+11/2} \cdot (4m + 4)! \cdot L(4m + 4, \chi_{8})}{\pi^{4m+4} \cdot (1 - 2^{-4m-4}) \cdot |B_{4m+4}|} & \quad \frac{3 \cdot |B_{4m+4}|}{2^{4m+1} \cdot (1 - 2^{-4m-4})} \\
N = 6 & \quad \frac{\pi \cdot (2\pi)^{4m+4} \cdot (1 - 2^{-8m-8})}{3 \cdot (4m + 4)! \cdot L(4m + 5, \chi_{-4})} & \quad \frac{\pi^{4m+5} \cdot (1 - 2^{-4m-4})}{(4m + 4)! \cdot L(4m + 5, \chi_{-4})} \\
N = 7 & \quad \frac{2^{4m+13/2} \cdot (4m + 5)! \cdot L(4m + 5, \chi_{-8})}{\pi^{4m+5} \cdot (1 + 2^{-4m-5}) \cdot B_{8m+10}} & \quad \frac{2 \cdot (4m + 5)! \cdot L(4m + 5, \chi_{-4})}{\pi^{4m+5} \cdot (1 + 2^{-4m-5}) \cdot B_{8m+10}} \\
N = 8 & \quad \frac{(1 - 2^{-4m-5}) \cdot (4m + 6)}{(1 - 2^{-4m-6}) \cdot B_{4m+6}} & \quad \frac{2m + 3}{2^{4m+4} \cdot (1 - 2^{-4m-6}) \cdot B_{4m+6}} \\
\end{align*}
\]

and

\[ \text{vol}_{\tilde{H}_M}^i(L, K_i) = \frac{\pi^{4m+3}}{2^{4m+1} \cdot (4m + 2)! \cdot L(4m + 3, \chi_{-4})}. \]

Here \( \chi_D(\cdot) = \left( \frac{D}{\cdot} \right) \) is the quadratic Kronecker symbol and \( B_{2k} \) is the Bernoulli number. We insert these datum and \( a = n/2 + 1 - l \) into

\[ \sum_i \text{vol}_{\tilde{H}_M}^i(\Gamma_i)/\text{vol}_{\tilde{H}_M}^i(O^+(L)) < (1 + a^{-1})^{1-n}(2a/n). \]

The resulting inequality holds when \( n \geq 39 \). \( \square \)

Using quasi-pullback of Borcherds’ \( \Phi_{12} \) as in [12], [15], we can see that \( \mathcal{F}_L \) is of general type also in \( n = 23, 24 \) (embed \( D_N \) in \( E_8 \) with \( D_N^* \cong D_8 \)). On the other hand, \( \mathcal{F}_L \) is rational in \( n \leq 16 \) and unirational in \( n \leq 20 \). See
for \(n \leq 18\); \(L\) is the period lattice of quartic \(K\)3 surfaces in \(n = 19\), and of double EPW sextics in \(n = 20\) (\([29, 15]\)).

**Appendix A. Singularity over 0-dimensional cusp**

Let \(L\) be a lattice of signature \((2, n)\). Let \(\Gamma\) be a finite-index subgroup of \(O^+(L)\) and \(F(\Gamma) = \Gamma \backslash D_L\) the associated modular variety. For simplicity we assume \(-1 \in \Gamma\), which does not affect \(F(\Gamma)\).

0-dimensional cusps of the Baily-Borel compactification of \(F(\Gamma)\) correspond to primitive isotropic vectors \(l\) in \(L\) up to the \(\Gamma\)-action. We write \(M_I = l^\perp \cap L/\mathbb{Z}l\). Let \(N(l)_\mathbb{Q}\) be the stabilizer of \(l\) in \(O^+(L_\mathbb{Q})\). The unipotent radical \(U(l)_\mathbb{Q}\) of \(N(l)_\mathbb{Q}\) consists of the Eichler transvections \(E_{l,m}, m \in (M_I)_\mathbb{Q}\), which does not affect \(U(l)_\mathbb{Q}\) and the associated modular variety. For simplicity we write \((M_I)_\mathbb{Q}\) for \((M_I)_\mathbb{R}\), which does not affect \(U(l)_\mathbb{Q}\) and the associated modular variety. For simplicity we write \((M_I)_\mathbb{Q}\) for \((M_I)_\mathbb{R}\), which does not affect \(U(l)_\mathbb{Q}\) and the associated modular variety. For simplicity we write \((M_I)_\mathbb{Q}\) for \((M_I)_\mathbb{R}\), which does not affect \(U(l)_\mathbb{Q}\) and the associated modular variety.

Choose representatives \(l_1, \ldots, l_N \in L\) of primitive isotropic vectors modulo \(\Gamma\). We put a \(\mathbb{Z}\)-structure on \((M_I)_\mathbb{R} = (M_I)_\mathbb{Q}\) by \(U(l_i)_\mathbb{Z}\). Let \(C_i\) be the union of the positive cone \((M_I)_\mathbb{R}^+\) of \((M_I)_\mathbb{R}\) and the rays \(\mathbb{R}_{\geq 0}m\) for \(m \in (M_I)_\mathbb{Q}\) in the boundary of \((M_I)_\mathbb{R}^+\). According to \([1]\), toroidal compactification of \(F(\Gamma)\) can be constructed by choosing for each \(i\) an \(N(l_i)_\mathbb{Z}\)-admissible fan \(\Sigma_i\) in \((M_I)_\mathbb{R}\) with \(|\Sigma_i| = C_i\). (There is no ambiguity of choice at the 1-dimensional cusps, and the choices of fan at each \(i\) are independent.) By \([1]\), we can choose \(\Sigma_i\) to be regular with respect to \(U(l_i)_\mathbb{Z}\).

Our purpose in this appendix is to supplement a proof of the following

**Theorem A.1** (\([12]\)). When the fans \(\Sigma_i\) are regular, the toroidal compactification \(F(\Gamma)^{\Sigma}\) associated to \(\Sigma = (\Sigma_i)\) has canonical singularity at the points lying over the 0-dimensional cusps.

This theorem was first found by Gritsenko-Hulek-Sankaran (\([12]\) §2.2), but as we explain later (Remark A.8), their proof needs to be modified.

Since Tai (\([37]\)), proof of such a statement consists of the following steps:

1. find a finite linear quotient model \(V/G\) of the singularity;
(2) the Reid–Shepherd-Barron–Tai criterion [31], [37] tells whether 
$V/G$ has canonical singularity in terms of the eigenvalues of each 
element $g$ of $G$;
(3) so we are reduced to analyze $V$ as a representation of the cyclic 
group $(g)$ for each $g \in G$.

In §A.1 we first present a certain class of representations $V$ of the cyclic 
groups $\mathbb{Z}/m$ and show that $V/(\mathbb{Z}/m)$ has canonical singularity by the RST criterion. This part is elementary linear algebra and independent of modular 
varieties. We then study local model $V/G$ of the toroidal compactification 
and show (§A.3) that for each $g \in G$, $V|_{(g)}$ belongs to the class of representations we have studied in advance.

### A.1. Some cyclic quotients

Let $G = \mathbb{Z}/m$ be the standard cyclic group of 
order $m > 1$. By a representation of $G$ we always mean a finite-dimensional complex representation. For $\mu \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}$ we denote by $\chi_{\mu}$ the character $G \to \mathbb{C}^\times$ that sends $\bar{1} \in G$ to $e(\mu)$. For $d|m$ we write

$$V_d = \bigoplus_{k \in \mathbb{Z} / (d)^\times} \chi_{k/d}.$$ 

It is classical that a representation of $G$ defined over $\mathbb{Q}$ is isomorphic to 
$\oplus_{m} V_{d_m}$ for some $d|m$ (see [35] §13.1). When $m = m'm''$, we can view $\mathbb{Z}/m'$ 
as a subgroup of $\mathbb{Z}/m$ of index $m''$ by multiplication by $m''$:

$$\mathbb{Z}/m' \simeq m''\mathbb{Z}/m \subset \mathbb{Z}/m.$$ 

If we put $d'' = (d, m'')$ and $d' = d/d''$, the restriction of $V_d$ to $\mathbb{Z}/m' \subset \mathbb{Z}/m$ 
is isomorphic to a direct sum of copies of $V_{d'}$.

If $d|m$ and $\mu \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}$, we write $W_{d,\mu}$ for the $G$-representation

$$W_{d,\mu} = \mathbb{C}[\mathbb{Z}/d] \otimes \chi_{\mu} = \bigoplus_{k \in \mathbb{Z}/d} \chi_{k/d} \otimes \chi_{\mu}.$$ 

Eigenvalues of $\bar{1} \in G$ on $W_{d,\mu}$ are the $e(\mu)$-shift of the $d$-th roots of 1. Restriction rule is as follows.

**Lemma A.2.** Let $m = m'm''$. We put $\mu' = m''\mu$, $d'' = (d, m'')$ and $d' = d/d''$.
The restriction of $W_{d,\mu}$ to $\mathbb{Z}/m' \subset \mathbb{Z}/m$ is isomorphic to $(W_{d',\mu'})^{\otimes d''}$.

**Proof.** We have $\chi_{\mu}|_{\mathbb{Z}/m'} = \chi_{\mu}'$. The image of $\mathbb{Z}/m'$ by the reduction map 
$\mathbb{Z}/m \to \mathbb{Z}/d$ is $d''\mathbb{Z}/d \simeq \mathbb{Z}/d'$, and $\mathbb{C}[\mathbb{Z}/d]|_{\mathbb{Z}/d'} \simeq \mathbb{C}[\mathbb{Z}/d''^\times]^{\otimes d''}$.

**Example A.3.** Let $g \in \text{GL}_d(\mathbb{C})$ be the linear transformation

$$g = \text{diag}(e(\alpha_1), \cdots, e(\alpha_d)) \circ (2, 3, \cdots, d, 1)$$ 

where $\alpha_i \in \mathbb{C}/\mathbb{Z}$. Let $m = \text{ord}(g) < \infty$. The eigenpolynomial of $g$ is

$$x^d - e(\sum_i \alpha_i).$$

If $\mu \in \mathbb{Q}/\mathbb{Z}$ is an element with $d\mu = \sum_i \alpha_i$, it follows that $\mathbb{C}^d \simeq$
$W_{d,\mu}$ as a representation of $\langle g \rangle \simeq \mathbb{Z}/m$. When $m = m'm''$, the restriction of the cyclic permutation $(2, \cdots, d, 1)$ to $\langle g^{m''} \rangle \simeq \mathbb{Z}/m'$ splits into $d''$ copies of cyclic permutation of length $d'$. In §A.3, $W_{d,\mu}$ and Lemma A.2 will appear in this form.

Based on Lemma A.2, we make the following definition.

**Definition A.4.** Let $U$ be a representation of $G$ defined over $\mathbb{Q}$. Let $\{(d_i, \mu_i)\}_i$ be a finite set of pairs $(d_i, \mu_i)$ with $d_i|m$ and $\mu_i \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}$. We say that $\theta = (U, (d_i, \mu_i))$ is an admissible data for $G$ if for every nontrivial subgroup $G' \simeq \mathbb{Z}/m'$ of $G$, either $U|_{G'}$ is nontrivial or $d'_i := d_i/(d_i, m'') > 1$ for some $i$.

To such a data $\theta$ we associate the $G$-representation

$$V_\theta = U \oplus \bigoplus_i W_{d_i, \mu_i}.$$  

If we put

(A.2)  

$$\theta|_{G'} = (U|_{G'}, ((d'_i, \mu'_i)^{x_{d''}})_i)$$

for a subgroup $G' \simeq \mathbb{Z}/m'$ of $G$, Lemma A.2 shows that $V_{\theta|_{G'}} \simeq V_{\theta|_{G''}}$ as $G'$-representation. We have $(\theta|_{G'})|_{G''} = \theta|_{G''}$ for $G'' \subset G'.G'.G'$. Hence admissibility of $\theta$ for $G$ implies that of $(\theta|_{G'})$ for $G'$.  

Recall that a linear transformation of finite order is called quasi-reflection (or pseudo-reflection) if all but one of its eigenvalues are 1.

**Lemma A.5.** Let $\theta = (U, (d_i, \mu_i))$ be an admissible data for $G = \mathbb{Z}/m$. Suppose that $G$ contains an element $g$ acting by quasi-reflection on $V_\theta$. Let $m' = \text{ord}(g)$ and $m'' = m'/m'$. Then $g$ acts on $V_\theta$ by reflection, so $m' = 2$, and $m''$ is odd. The reflective vector $\delta \in V_\theta$ of $g$ is also an eigenvector of $G$, and contained in either $U$ or $W_{d_i, \delta}$ for some $i$. When $\delta \in U$, we have $\mathbb{C}\delta \simeq V_2$ as $G$-representation. When $\delta \in W_{d_i, \mu_i}$, we have $d_i = 2$.

**Proof.** We can write $g = g_0^{m''}$ for a generator $g_0$ of $G$. There is only one eigenvalue $\lambda$ of $g_0$ such that $\lambda^{m''} \neq 1$, and the remaining eigenvalues of $g_0$ are $m''$-th root of 1. In particular, $\lambda$ has multiplicity 1. Let $\delta$ be a generator of the 1-dimensional $\lambda$-eigenspace of $g_0$. Since every eigenvalue of $g_0$ occurs in $U$ or one of $W_{d_i, \mu_i}$, the multiplicity one property implies that $\delta \in U$ or $\delta \in W_{d_i, \mu_i}$ for some $i$.

First consider the case $\delta \in U$. Again by the multiplicity one, $\delta$ is contained in a sub $G$-representation isomorphic to $V_d$ for some $d|m$. Since $V_d|_{\langle g \rangle} \simeq (V_d)^{a\alpha}$ for $d' = d/(d, m'')$ while $g$ acts on this space by quasi-reflection, we must have $d' = 2$ and $a = 1$. Hence $d = 2$, namely $\mathbb{C}\delta \simeq V_2$ as $G$-representation. Since $(-1)^{m''} = -1$, $m''$ is odd.

Next consider the case $\delta \in W_{d_i, \mu_i}$. Since $g$ acts trivially on $U$ and $W_{d_i, \mu_i}$ for $j \neq i$, the admissibility condition says that we must have $d'_i > 1$ in
\( W_{d_i,\mu_i}(g) \simeq (W_{d_i,\mu_i'})^{\otimes m'} \). On the other hand, \( g \) has only one \( \neq 1 \) eigenvalue on \( W_{d_i,\mu_i} \), so \( d_i' = 2 \), \( d_i'' = 1 \) and \( \mu_i' = 0 \) or \( 1/2 \). Hence \( d_i = 2 \) and \( g \) acts by reflection. Since \( W_{2,\mu_i}(g) \simeq W_{2,\mu_i'} \), \( m'' \) is odd. \( \square \)

We can now present the main result of this subsection.

**Proposition A.6.** Let \( \theta = (U,(d_i,\mu_i)_i) \) be an admissible data for \( G = \mathbb{Z}/m \). Then \( V_\theta/G \) has canonical singularity.

**Proof.** If \( V \) is a representation of \( G \) and \( g \in G \) has eigenvalues \( e(\alpha_1),\ldots,e(\alpha_n) \) with \( 0 \leq \alpha_i < 1 \), the Reid-Tai sum of \( g \) is defined by

\[
\Sigma_V(g) = \sum_{i=1}^{n} \alpha_i.
\]

(Similar invariant appears in the dimension formula for modular forms: see \[36\], \[31\].) The Reid–Shepherd-Barron–Tai criterion \[31\], \[37\] says that when \( G \) contains no quasi-reflection, \( V/G \) has canonical singularity if and only if \( \Sigma_V(g) \geq 1 \) for every \( g \neq \text{id} \in G \). We apply this to \( V = V_\theta \) or its variation.

We first consider the case \( G \) contains no reflection on \( V_\theta \).

**Lemma A.7.** Let \( \theta = (U,(d_i,\mu_i)_i) \) be an admissible data for \( G = \langle g \rangle = \mathbb{Z}/m \). Assume that \( g \) does not act as reflection on \( V_\theta \). Then \( \Sigma_{V_\theta}(g) \geq 1 \).

**Proof.** Let \( W = \bigoplus_i W_{d_i,\mu_i} \). It is clear that \( \Sigma_{V_\theta}(g) \geq 1 \) in the following cases:

- \( U \) contains \( V_\theta \) with \( d \geq 3 \) or \( (V_2)^{\otimes 2} \);
- \( W \) contains \( W_{d,\mu} \) with \( d \geq 3 \) or \( W_{2,\mu} \oplus W_{2,\mu'} \);
- \( U \) contains \( V_2 \) and \( W \) contains \( W_{2,\mu} \).

The remaining cases are

1. \( U = V_2 \oplus (V_1)^{\otimes 2} \) and \( W = \bigoplus_i W_{1,\mu_i} \);
2. \( U \) is trivial and \( W = W_{2,\mu} \oplus \bigoplus_i W_{1,\mu_i} \).

In both cases \( m \) must be even, say \( m = 2m' \). If \( m' = 1 \), the eigenvalue \(-1\) has multiplicity at least 2 because \( g \) is not reflection. Then \( \Sigma_{V_\theta}(g) \geq 1 \). We show that the case \( m' > 1 \) does not occur. Consider the restriction to the subgroup \( G' = \langle g^2 \rangle \simeq \mathbb{Z}/m' \). Then \( U|_{G'} \) is trivial. On the other hand, \( W|_{G'} \simeq \bigoplus_i W_{1,2\mu_i} \) in case (1) and \( W|_{G'} \simeq (W_{1,2\mu})^{\otimes 2} \oplus \bigoplus_i W_{1,2\mu_i} \) in case (2) (in the sense of restriction in \( \langle A.2\rangle \)). By admissibility, we must have \( m' = 1 \). \( \square \)

When \( G \) contains no reflection, we can apply this lemma to all subgroups \( G' \) of \( G \) and their generators because \( \theta|_{G'} \) is admissible for \( G' \). By the RST criterion we obtain Proposition \[A.6\] in this case.

We next consider the case \( G \) contains an element \( g \) acting as reflection on \( V_\theta \). We may assume \( G \neq \langle g \rangle \). Let \( m'' = m/2 > 1 \) be the index of \( \langle g \rangle \)
in $G$, and $\delta$ a reflective vector of $g$. By Lemma A.5 $m''$ is odd, and $\delta$ is an eigenvector for $G$ contained in $U$ or some $W_{d,\mu}$. We write $\bar{G} < G$ for the subgroup of order $m''$. We have the decomposition $G = \bar{G} \oplus \langle g \rangle$ and $\bar{G}$ is canonically identified with $G/\langle g \rangle$. We set $\bar{V} = V_\theta/\langle g \rangle$, which is a $\bar{G}$-representation. We have $V_\theta/G \cong \bar{V}/\bar{G}$, and we want to apply the previous step to $(\bar{V}, \bar{G})$. Note that $\bar{G}$ cannot contain reflection because its order $m''$ is odd.

When $\delta \in U$, consider the $G$-decomposition $V_\theta = V' \oplus \mathbb{C}\delta$. By Lemma A.5 $\mathbb{C}\delta \cong V_2$ as $G$-representation. Then as $\bar{G}$-representation

$$\bar{V} = V' \oplus (\mathbb{C}\delta)^{\otimes 2} \cong V' \oplus V_1 \cong V_\theta.$$ Since $\theta|_{\mathbb{C}}$ is admissible for $\bar{G}$, $\bar{V}/\bar{G} \cong V_\theta/\bar{G}$ has canonical singularity by the previous step.

When $\delta \in W_{d,\mu}$, we have $d_i = 2$ by Lemma A.5 Since $W_{d,\mu}|_{\bar{G}} \cong (W_{1,2})^{\otimes 2}$, then $\eta = (U, (d_j, \mu_j)_{j \neq i})|_{\bar{G}}$ must be admissible for $\bar{G}$. Hence $\Sigma_{\eta}(h) \geq 1$ for every $h \neq id \in \bar{G}$ by Lemma A.7. Since $V_\eta$ is a direct summand of $\bar{V}$, we have $\Sigma\bar{V}(h) \geq 1$. Hence $\bar{V}/\bar{G}$ has canonical singularity. This finishes the proof of Proposition A.6 \hfill \Box

### A.2. Toroidal compactification

We go back to modular varieties and explain toroidal compactification over 0-dimensional cusp. We keep the notation in the beginning of this appendix. Let $l \in L$ be a primitive isotropic vector and $D_l = (M_l)_{\mathbb{R}} + i(M_l)_{\mathbb{Z}}$ the tube domain associated to $l$. We choose a vector $l' \in L_{\mathbb{Q}}$ with $(l, l') = 1$ and identify $(M_l)_{\mathbb{Q}}$ with $(l, l')_{\mathbb{Q}} = L_{\mathbb{Q}}$. As explained in §2 this induces the tube domain realization

$$\iota_l : D_l \cong \mathbb{D}_L, \quad \nu \mapsto \mathbb{C}(l' + \nu - \frac{1}{2}(\nu, \nu) + (l', l')) l,$$

which depends on $l'$. Via this, $U(l)_{\mathbb{Q}} \cong (M_l)_{\mathbb{Q}}$ acts on $D_l$ by parallel transformation. If we form the torus $T_l = (M_l)_{\mathbb{C}}/U(l)_{\mathbb{Z}}$, then $\iota_l^{-1}$ maps $X_l = D_l/U(l)_{\mathbb{Z}}$ isomorphically to the open set $D_l/U(l)_{\mathbb{Z}} = \text{ord}^{-1}((M_l)^+)_{\mathbb{R}}$ of $T_l$. The group $N(l)_{\mathbb{Z}}$ acts on $X_l$ through the $N(l)_{\mathbb{Z}}$-action on $D_l$.

The action of $N(l)_{\mathbb{Z}}$ on $U(l)_{\mathbb{Q}} \cong (M_l)_{\mathbb{Q}}$. Hence if $\pi : N(l)_{\mathbb{Q}} \to O^+(M_l)_{\mathbb{Q}}$ is the natural map, $N(l)_{\mathbb{Z}}$ is contained in $\pi^{-1}(O^+(U(l)_{\mathbb{Z}}))$, of which $U(l)_{\mathbb{Z}}$ is a normal subgroup. Thus $N(l)_{\mathbb{Z}}$ is canonically a subgroup of $\pi^{-1}(O^+(U(l)_{\mathbb{Z}}))/U(l)_{\mathbb{Z}}$. By (A.1), the splitting $L_{\mathbb{Q}} = \langle l, l' \rangle_{\mathbb{Q}} \oplus (M_l)_{\mathbb{Q}}$ given by $l'$ induces an isomorphism

$$\varphi_l : \pi^{-1}(O^+(U(l)_{\mathbb{Z}}))/U(l)_{\mathbb{Z}} \to O^+(U(l)_{\mathbb{Z}}) \cong (U(l)_{\mathbb{Q}}/U(l)_{\mathbb{Z}}).$$

The right side group is canonically a subgroup of

$$\text{GL}(U(l)_{\mathbb{Z}}) \rtimes (U(l)_{\mathbb{Q}}/U(l)_{\mathbb{Z}}) = \text{Aut}(T_l) \rtimes (T_l)_{\text{tor}} \subset \text{Aut}(T_l) \rtimes T_l.$$
We thus obtain an embedding depending on $l$

$$\varphi_r : \overline{N(l)}_\mathbb{Z} \hookrightarrow \text{Aut}(T_i) \ltimes T_i.$$ 

By the definition of $\overline{N(l)}_\mathbb{Z}$, the projection $\varphi_r(\overline{N(l)}_\mathbb{Z}) \rightarrow \text{Aut}(T_i)$ is injective. If we express $\varphi_r(g) = (\gamma, a) \in \text{Aut}(T_i) \ltimes T_i$ for $g \in \overline{N(l)}_\mathbb{Z}$, then $\gamma = \pi(\tilde{g})$ and $a = [\tilde{g}(l') - l']$ where $\tilde{g} \in N(l)_\mathbb{Z}$ is a lift of $g$.

The affine group $\text{Aut}(T_i) \ltimes T_i$ acts on $T_i$ naturally: $\text{Aut}(T_i)$ by torus automorphisms (fixing the identity), and $T_i$ by translation. The $N(l)_\mathbb{Z}$-action on $X_i$ is the restriction of the action of $\text{Aut}(T_i) \ltimes T_i$ on $T_i$ through $\varphi_r$ and $\iota_r$.

**Remark A.8.** In [12] p. 534, Gritsenko-Hulek-Sankaran implicitly assume that $\varphi_r(\overline{N(l)}_\mathbb{Z})$ is contained in $\text{Aut}(T_i)$ for some $l' \in L_C$ so that the translation component $a = a_g$ is trivial for every $g$. If this holds, $N(l)_\mathbb{Z}$ will decompose into $N(l)_\mathbb{Z} \ltimes U(l)_\mathbb{Z}$. However, this assumption seems to be too strong in general. For each $g$, $a_g$ varies holomorphically with $l'$ so that it is not $1$ for generic $l'$, and it seems highly nontrivial or even impossible for general $\Gamma$ that one can find a specific $l'$ such that $a_g = 1$ for all $g$. (Note that the isomorphism $D_{L_0}(F) \cong U(F)_C$ in loc. cit. depends on the choice of a base point $C \omega$ of $D_{L_0}(F)$. This isomorphism is the extension of $\iota_r$, and $C \omega$ is another intersection point of $\mathbb{P}(l, l')_C$ with the isotropic quadric.)

On the other hand, in the important example $\Gamma = \tilde{O}^+(L)$ with $L$ even, $\varphi_r(\overline{N(l)}_\mathbb{Z})$ is indeed contained in $\text{Aut}(T_i)$ if $l'$ is taken from $L'$. Hence in this case the proof of [12] works.

Now let $\Sigma_i$ be the $\overline{N(l)}_\mathbb{Z}$-admissible regular fan in $(M_l)_\mathbb{Z}$ we have chosen for $l$. This defines a torus embedding $T_i \hookrightarrow T_{\Sigma_i}$. The partial compactification $X_{\Sigma_i}$ of $X_i$ in the direction of $l$ is by definition the interior of the closure of $X_i$ in $T_{\Sigma_i}$. The group $\overline{N(l)}_\mathbb{Z}$ acts on $X_{\Sigma_i}$ properly discontinuously. We have a natural map

$$X_{\Sigma_i}/\overline{N(l)}_\mathbb{Z} \rightarrow \mathcal{F}(\Gamma)^{\Sigma_i},$$

which is locally isomorphic at the points lying over the 0-dimensional cusp $\mathbb{C} \cap (1)$ (11 p. 175). Hence Theorem [A.1] reduces to the following assertion (cp. [12] Theorem 2.17).

**Theorem A.9.** Let $N$ be a free abelian group of finite rank and $T = T_N$ be the associated torus. Let $G$ be a finite subgroup of $\text{Aut}(T) \ltimes T$ such that $G \rightarrow \text{Aut}(T)$ is injective. Let $\Sigma$ be a regular fan in $N_\mathbb{R}$ preserved by $G$, and $T_{\Sigma} = T_{N\Sigma}$ the torus embedding defined by $\Sigma$. Then $T_{\Sigma}/G$ has canonical singularity.

In the next subsection we prove this by reducing it to Proposition [A.6]. Note that the injectivity condition on $G \rightarrow \text{Aut}(T)$ is essential: consider the extreme situation $G \subset T$, where one loses control of the Reid-Taı sum.
A.3. **Proof of Theorem A.9** Let \( x \) be a point of \( T_\Sigma \) and \( G_x \subset G \) be the stabilizer of \( x \). It suffices to prove that \( T_x T_\Sigma / G_x \) has canonical singularity. By the well-known cyclic reduction ([31], [37]), this reduces to showing that \( T_x T_\Sigma / \langle g \rangle \) has canonical singularity for every \( g \in G_x \). We write \( m \) for the order of \( g \). Let \( \text{orb}(\sigma) \) be the \( T \)-orbit \( x \) belongs to, where \( \sigma \) is a regular cone in \( \Sigma \). Write \( g = (\gamma, a) \in \text{Aut}(T) \ltimes T \). Since \( g \) preserves \( \text{orb}(\sigma) \), \( \gamma \) preserves the cone \( \sigma \), permuting its rays. The open embedding \( T_\sigma \hookrightarrow T_\Sigma \) is \( g \)-equivariant, hence \( T_x T_\Sigma = T_x T_\sigma \) as \( \langle g \rangle \)-representation. We are thus reduced to showing that \( T_x T_\sigma / \langle g \rangle \) has canonical singularity.

Since \( g \) has finite order, we have the \( g \)-decomposition

\[
T_x T_\sigma = T_x (\text{orb}(\sigma)) \oplus N_x (\text{orb}(\sigma)).
\]

Let \( N_0 = \mathbb{Z}(\sigma \cap N) \) and \( N_1 = N / N_0 \), which are free \( \gamma \)-modules. We have a natural isomorphism \( \text{orb}(\sigma) \simeq N_1 \), so that \( T_x (\text{orb}(\sigma)) \simeq (N_1)_\mathbb{C} \). The rays of \( \sigma \) define a basis of \( N_0 \), and \( \gamma \) acts on \( N_0 \) by permuting these basis vectors. Let \((d_1, \cdots, d_N)\) be the cyclic type of this permutation (\( \sum_i d_i = \text{rk}(N_0) \)).

**Proposition A.10.** (1) Via the isomorphism \( T_x (\text{orb}(\sigma)) \simeq (N_1)_\mathbb{C} \), the \( g \)-action on \( T_x (\text{orb}(\sigma)) \) is identified with the \( \gamma \)-action on \( (N_1)_\mathbb{C} \). In particular, it is defined over \( \mathbb{Q} \).

(2) As a representation of \( \langle g \rangle \simeq \mathbb{Z}/m \), the normal space \( N_x (\text{orb}(\sigma)) \) is isomorphic to \( \bigoplus_{i=1}^N W_{d_i, \mu_i} \) for some \( \mu_1, \cdots, \mu_N \in \mathbb{Q}/\mathbb{Z} \).

(3) The data \((T_x (\text{orb}(\sigma)), (d_i, \mu_i))\) for \( \langle g \rangle \simeq \mathbb{Z}/m \) is admissible in the sense of Definition A.4.

Theorem A.9 follows from the assertion (3) and Proposition A.10.

**Proof.** We first show that (3) follows from (1) and (2). Suppose we have a factorization \( m = m' m'' \) with \( m' \neq 1 \) and consider the restriction of \((N_1)_\mathbb{C}, (d_i, \mu_i))\) to the subgroup \( \langle g^{m''} \rangle \simeq \mathbb{Z}/m' \) of \( \langle g \rangle \simeq \mathbb{Z}/m \). As explained in Example A.3, the restriction of the cyclic permutation \((2, \cdots, d_i, 1)\) to \( \mathbb{Z}/m' \subset \mathbb{Z}/m \) splits into copies of \((2, \cdots, d'_i, 1)\) where \( d'_i = d_i / (d_i, m'') \). Therefore, if \( d'_i = 1 \) for all \( 1 \leq i \leq N \), the \( \gamma^{m''} \)-action on \( N_0 \) must be trivial. If furthermore \( \gamma^{m''} \) acts on \( N_1 \) trivially, then \( \gamma^{m''} = \text{id} \). By the injectivity of \( \langle g \rangle \rightarrow \text{GL}(N) \), we have \( g^{m''} = \text{id} \), so \( m' = 1 \). This shows that \((N_1)_\mathbb{C}, (d_i, \mu_i))\) is admissible.

We check (1). We write \( T_1 = T_{N_1} \). We have a canonical isomorphism \( T_1 T_1 \simeq (N_1)_\mathbb{C} \) for every \( y \in T_{N_1} \). Via this \( \gamma : T_1 T_1 \rightarrow T_{\gamma y} T_1 \) is identified with \( \gamma : (N_1)_\mathbb{C} \rightarrow (N_1)_\mathbb{C} \), and the translation \( t_a : T_{\gamma y} T_1 \rightarrow T_x T_1 \) with the identity of \( (N_1)_\mathbb{C} \).

We verify (2). We write \( T_0 = T_{N_0} \). Via the generators of the rays of \( \sigma \), \( T_0 \subset (T_0)_\mathbb{C} \) is isomorphic to \((\mathbb{C}^r)^\gamma \subset \mathbb{C}^r \), and \( \gamma \) acts on \((T_0)_\mathbb{C} \simeq \mathbb{C}^r \) by permuting the basis vectors. We have a canonical isomorphism \( T_{\sigma \cdot} \simeq \mathbb{C}^r \) in the sense of Definition A.4, since \( \delta \cdot \simeq \delta \cdot 1 \).


$T \times T_0 \circ T_0$ which makes $T_\sigma$ a vector bundle over $T_1$ with zero section orb$(\sigma)$. Let $\pi: T_\sigma \to T_1 \simeq orb(\sigma)$ be the projection. If $y \in T$, the $\pi$-fiber through $y$ gets isomorphic to $(T_0)_\sigma$ by

$$\varphi_y : \pi^{-1}(\pi(y)) \to (T_0)_\sigma, \quad [(y, z)] \mapsto z.$$ 

This trivialization depends on $y$: if we replace $y$ by $y' = b^{-1}y$ where $b \in T_0$, then $\varphi_y \circ \varphi^{-1}$ acts on $(T_0)_\sigma$ by the torus action by $b$.

Now take a point $y \in T$ with $\pi(y) = x$, the fixed point of $g = t_a \circ \gamma$ in question. Via $\varphi_y$ and $\varphi_{yy'}$ the map $\gamma: \pi^{-1}(x) \to \pi^{-1}(\gamma x)$ is identified with the permuting action of $\gamma$ on $(T_0)_\sigma$, and via $\varphi_{yy'}$ and $\varphi_y$ the map $t_a: \pi^{-1}(\gamma x) \to \pi^{-1}(x)$ with the torus action of an element of $T_0$ on $(T_0)_\sigma$.

Via the trivialization $(T_0)_\sigma \simeq \mathbb{C}^r$, the last action is expressed by a diagonal matrix. Hence via $\varphi_y$ and $(T_0)_\sigma \simeq \mathbb{C}^r$, the map $g: \pi^{-1}(x) \to \pi^{-1}(x)$ is expressed by a direct sum of linear transformations of the form

$$\text{diag}(e(\alpha_1), \ldots, e(\alpha_i)) \circ (2, 3, \ldots, d_i, 1)$$

over $i = 1, \ldots, N$. In view of Example A.3 this proves our assertion. \qed

### A.4. No ramifying boundary divisor

We keep the notation in §A.2. In [12], Gritsenko-Hulek-Sankaran also proved the following.

**Proposition A.11.** The natural projection $X_{\Sigma_l} \to F(\Gamma)^\Sigma$ has no ramification divisor at the boundary.

This is equivalent to saying that no nontrivial element of $\overline{N(l)}_\Sigma$ fixes a boundary divisor of $X_{\Sigma_l}$. By the same reason the proof of this assertion also needs to be modified, but this is easier than Theorem A.1. It suffices to check the following.

**Lemma A.12.** Let $N$ and $T$ be as in Theorem A.9. Let $g = (\gamma, a)$ be a finite order element of $\text{Aut}(T) \times T$ such that $\gamma \neq \text{id}$. Let $\sigma \subset N$, be a ray fixed by $\gamma$. Then the $g$-action on $T_\sigma$ does not fix the boundary divisor orb$(\sigma)$.

**Proof.** Let $N_0 = \mathbb{Z}(1 \cap N)$ and $N_1 = N/N_0$. Via the natural isomorphism orb$(\sigma) \simeq T_{N_1}$, $g$ acts on orb$(\sigma)$ by $t_a \circ \gamma$ where $\tilde{a} \in T_{N_1}$ is the image of $a$ and $\tilde{\gamma}$ is the $\gamma$-action on $N_1$. If this was identity, then $\tilde{a} = 1$ and $\tilde{\gamma} = \text{id}$. Hence $\gamma$ acts on both $N_0$ and $N_1$ trivially, so $\gamma = \text{id}$. \qed

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