A Systematic Eigenspectral Investigation of Spatially Varying Effective Mass Schrödinger Equation for Squared Class Trigonometric Potentials

Metin Aktaş
Ankara Yıldırım Beyazıt University, Engineering and Natural Sciences Faculty, Energy System Engineering Department, 06010, Ankara, Turkey
metinaktas01@gmail.com

Abstract
In the present paper, one particular attempt is to acquire explicit and analytical solutions of the position-dependent effective mass (PDEM) Schrödinger equation for various types of the squared trigonometric potentials. The algebraic procedure entitled as the point canonical transformation (PCT) is implemented in the process. Three different spatially dependent mass configurations are taken into account the establishment of the target system. In the final step, performing the computational tasks lead to the explicit determination of both discrete energy spectra and their corresponding wavefunctions.

Keywords: Schrödinger equation, position-dependent effective mass, point canonical transformation trigonometric potential

1. Introduction
On the solution to non-stationary mass dependence formalism of Schrödinger, Klein-Gordon and Dirac equations there has been much particular attention in the past few decades since it presents scientific viewpoints to analyse the various physical mechanisms involving in small scale fields of physics. Besides the notable subject for providing quantum mechanical description of some physical structures at the micro and nano scales, position-dependent effective mass (PDEM) Schrödinger equation which is widely used for quantum semiconductor heterostructures [1-7], quantum dots and liquids [8], $^3$He and $^4$He structures [9] has been achieved. For example, this concept is applied especially in the determination of electronic transport properties of semiconductors, the introduction of the effective interaction pictures of nuclear particles [10] as well as the dynamical properties of neutron superfluids in the neutron stars [11].

It is pointed out that the exact solvability condition involved in both nonrelativistic and relativistic frameworks of quantum mechanical systems is very crucial factor as it can serve as a useful model to describe well these formations. Despite limited to achieving fully solvable quantum structures, the explicit solution of the Schrödinger equation with some spatially dependent mass distributions can be obtained. A great many alternative mathematical methods have been proposed in the literature to get exact and analytical solutions for equations having non-constant mass forms. For instance, these are point canonical transform, Lie algebraic and group theoretical approaches as well as alternative perspectives [12-39], parity-time (PT)-symmetric [40-44] and supersymmetric quantum mechanical (SUSYQM) formalisms etc [45-47]. In addition to the quantum mechanical problem of a variable mass particle under the concept of instantaneous Galilean invariance,
path integral approach [48, 49] and some other attempts to recognize various physical configurations [50-61] are also presented. Recently, several applications have been performed in the solution of Schrödinger [62-71], Klein-Gordon and Dirac equations etc [72-91], which are defined by changing masses.

The point canonical transform (PCT) approach can be applied to map the Schrödinger equation (SE) into the wave equation under some positional variable mass cases for investigating the target spectrum. It is important here to note that the final configuration includes not only the energy spectra of the target system, but also the corresponding wavefunctions in terms of reference (source) potential. The canonical map presents the original map and the associated target equivalent defined by the mass distribution function. In the study of this research article, exact solutions of the Schrödinger equation, which is characterized by non-stationary effective masses will be handled by the PCT technique. In this work, squared class trigonometric potentials such as tangent, cotangent [92, 93] and trigonometric Pöschl-Teller ones [94] will be examined respectively. By selecting three different mass distribution functions depending on the spatial coordinate like $m = m(x)$ [95], then explicit forms of the target potentials and eigenfunctions are constructed.

The outline of the article will be organized as follows: In section 2, the general description of the PCT method is presented in the mapping of the spatially dependent effective mass Schrödinger equation. The third, fourth and fifth sections cover the straightforward PCT calculations for the wave equation configuration, describing various forms of mass distributions with the fixed parameters $a$ and $b$. The section 6 comprises of the analysis and discussion of the results.

2. Mapping of the Spatially Variable Effective Mass Schrödinger Equation (SE) via the PCT Formalism

This section employs the formulation of the spatially-dependent effective mass Schrödinger equation through the PCT procedure. In order to do this, we first consider the Schrödinger’s Hamiltonian in the case of position dependent mass as

$$-\frac{1}{2} \left[ \nabla_x \frac{1}{M(x)} \nabla_x \right] \Psi(x) + [E - V(x)]\Psi(x) = 0, \quad (1)$$

where $M(x) = m_0 m(x)$, and constant $m_0$. Then, its explicit form will be

$$\Psi''(x) - \left( \frac{m_0^2}{m} \right) \Psi'(x) + 2m[E - V(x)]\Psi(x) = 0, \quad (2)$$

In the equation, the atomic units $\hbar = c = 1$ is chosen. Also, prime and double prime factors in the wave function for its first and second derivatives with respect to $x$. On the other hand, the one dimensional SE concerning the constant mass is written

$$\Psi''(y) + 2[E - V(y)]\Psi(y) = 0. \quad (3)$$

By making the transformation $y \rightarrow x$ through a mapping function $y = f(x)$, and the wavefunction is regarded as

$$\Psi(y) = g(x) \Phi(x), \quad (4)$$

then the stationary mass SE can be transformed into the new form

$$\Phi''(x) + 2 \left( \frac{g''}{g} - \frac{f''}{f} \right) \Phi'(x) + \left[ \left( \frac{g''}{g} - \frac{f''}{f} \right) + 2(f')^2[E - V(f(x)) \right] \Phi(x) = 0. \quad (5)$$

When comparing the left-hand sides of equations (2) and (5) term-by-term gives the conditions:

$$g(x) = \frac{f'(x)}{\sqrt{m(x)}}, \quad (6)$$

and

$$[E - V(x)] = \frac{(f')^2}{m} [E - V(f(x))] + \frac{1}{2m} \left( \frac{g''}{g} - \frac{f''}{f} \right) \quad (7)$$
By setting \( m = (f')^2 \) and substituting it in the final equation, it will bring about the target problem including the energy spectra of the bound states, potential and wavefunction respectively

\[
E_n = \mathcal{E}_n, \\
V(x) = V(f(x)) + \frac{1}{8m} \left[ \frac{m''}{m} - \frac{7}{4} \left( \frac{m'}{m} \right)^2 \right], \\
\Psi_n(x) = [m(x)]^{1/4} \Phi_n(f(x)).
\]  

This algebraic approach can be applied in some cases to determine their target equivalents and wavefunctions explicitly. It is also important to note that applying the well-known strategic process preserves the structure of the wavefunction in the target system having the same class as that of the reference problem.

In the following sections three different mass distribution functions with spatial dependence forms like \( m = m(x) \) and the reference (source) potentials including squared tangent, cotangent [92, 93] as well as trigonometric Pöschl-Teller ones [94] are going to be considered respectively. In addition, target spectra will be generated explicitly by applying the PCT method given in this section.

3. PCT Calculations for Mass Distribution-I

Let us first consider the asymptotically vanishing mass distribution form given [95]

\[
m = m_1 = \frac{a^2}{(b+x^2)^2},
\]

where \( a \) and \( b \) are constant parameters. In this case, its mapping function will be

\[
y = f(x) = \int \sqrt{m_1} \, dx = \frac{a}{\sqrt{b}} \arctan\left( \frac{x}{\sqrt{b}} \right),
\]

with \( x = \sqrt{b} \tan \left( \frac{\sqrt{B}}{a} \right) \).

3.1. Squared Tangent Potential (STP) Case

Now, we pay our attention to the potential

\[
V(y) = V_0 \tan^2 y,
\]

where \( V_0 = \mu(\mu - 1) \) is the potential coefficient including the parameter \( \mu \geq 1 \) and \( y \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \). This is the original (source) form with the energy spectra corresponding to the wavefunctions [92].

\[
E_{2n}(\mu) = [4n(n + \mu + \mu)], \quad n = 0, 1, 2, ...
\]

\[
\Phi_{2n}(y; \mu) = C_n(\mu) \cos^n y \ {}_2F_1(-n, n + \mu; \mu + 1/2; \cos^2 y)
\]

They are the symmetric eigenvalues and eigenfunctions regarding for even parity solutions. In the equation, \( C_n(\mu) \) is the normalizable coefficient, and \( {}_2F_1 \) stands for the Gauss hypergeometric function. In addition to this, anti-symmetric eigenspectrum and the corresponding eigenfunctions results for odd parity solutions are given [93]

\[
E_{2n+1}(\mu) = [(2n + 1)(2n + 2\mu + 1) + \mu], \quad n = 0, 1, 2, ...
\]
Φ_{2n+1} (y; \mu) = B_n (\mu) (\sin y) (\cos^\mu y) \; _2F_1(-n, \; n + \mu + 1; \; \mu + 1/2; \; \cos^2 y). \quad (13)

By substituting mass function (9) and the mapping function (10) into the equation (8), the constructed target system for the even and odd parity solutions of the STP case takes the form

\[ E_{2n}(\mu) = \mathcal{E}_{2n}(\tau) \]

\[ V(x) = \tilde{V}_0 x^2 - \left[ \frac{1}{2a^2} (b + 2x^2) \right] \]

\[ \Psi_{2n}(x) = \frac{a}{\sqrt{b + x^2}} \Phi_{2n}(f(x)). \quad (14) \]

and

\[ E_{2n+1}(\mu) = \mathcal{E}_{2n+1}(\tau) \]

\[ \Psi_{2n+1}(x) = \frac{a}{\sqrt{b + x^2}} \Phi_{2n+1}(f(x)). \quad (15) \]

with \( \tilde{V}_0 = V_0 \left( \frac{a}{b} \right)^2. \)

3.2. Squared Cotangent Potential (SCP) Case

It can be defined

\[ V(y) = V_0 \cot^2 y = \frac{V_0}{\tan^2 y}, \]

where \( y \in (0, \pi). \)

Similarly, this reference potential has also two identical solutions for both even and odd index parities, as in the case of STP. Following the similar algebraic procedure in solving of the SE for SCP, then they are written [92, 93]

\[ \Phi_{2n}(y + \pi/2; \mu) = (\sin^\mu y) \; _2F_1(-n, \; n + \mu; \; \mu + 1/2; \; \sin^2 y) \quad (17) \]

eigenfunctions for the symmetric states and

\[ \Phi_{2n+1} = \Gamma_n \Phi_{2n}(y + \pi/2; \mu = 3) = \Gamma_n \sum_{k=0}^{n} \frac{(-n)_k (n+3)_k}{k! (7/2)_k} (\sin^{2k+3} y) \quad (18) \]

the anti-symmetric eigenfunctions and the normalizable coefficient \( \Gamma_n, n = 0, 1, 2, \ldots \) It should be mentioned here that the energy eigenvalues, but not eigenfunctions, of the Schrödinger’s Hamiltonian for the STP on the symmetric interval \((- \pi/2, \; \pi/2)\) have precisely uniform as those of the Hamiltonian for the SCP case on the asymmetric interval \((0, \; \pi)\) [93].

By following the procedure as in the above section, the target system for (16)–(18) is obtained.

\[ \tilde{E}_{2n} = \tilde{\mathcal{E}}_{2n} \]

\[ \tilde{V}(x) = \tilde{V}_0 x^2 - \left[ \frac{1}{2a^2} (b + 2x^2) \right] \]

\[ \tilde{\Psi}_{2n}(x) = \frac{a}{\sqrt{b + x^2}} \tilde{\Phi}_{2n}(f(x)). \quad (19) \]

and

\[ \tilde{E}_{2n+1} = \tilde{\mathcal{E}}_{2n+1} \]
\[ \Psi_{2n+1}(x) = \sqrt{\frac{a}{b x^2}} \Phi_{2n+1}(f(x)), \]  

(20)

with the same target potential.

3.3. Trigonometric Pöschl – Teller Potential (PTP) Case

In the section, we will take into account the general form of the potential as

\[ V_{PT}(y) = \left( \frac{V_{01}}{\sin^2 y} + \frac{V_{02}}{\cos^2 y} \right) \]

\[ = V_{01} \csc^2 y + V_{02} \sec^2 y. \]  

(21)

Here, the constants \( V_{01} = V_0 \chi(\chi - 1)/2 \) and \( V_{02} = V_0 \lambda(\lambda - 1)/2; \chi \geq 1, \lambda \geq 1 \) [94]. It is bounded in the region between the values of 0 and \( \pi/2 \). Solving the Schrödinger equation for (21) gives the energy eigenvalues as

\[ \tilde{E}_n(\chi, \lambda) = \left[ \frac{V_0}{2} (2n + \chi + \lambda) \right]^2, \]  

(22)

and the wavefunctions

\[ \Phi_n(y; \chi, \lambda) = C_n(\sin^2 y)(\cos^2 y) \, _2F_1(-n, n + \chi + \lambda, \chi + 1/2; \sin^2 y), \]  

(23)

where \( C_n \) is normalization constant. Hence, energy spectra, target potential and wave functions can be developed by following equations (9) and (10)

\[ \tilde{E}_n(\chi, \lambda) = \tilde{E}_n(\tau_1, \tau_2) \]

\[ V(x) = -\left( \frac{V_{01}}{x^2} + V_{02}(1 + \kappa x^2) \right) \]

\[ - \left[ 1 + \frac{1}{2} \kappa^2 (b + 2 x^2) \right] \]

(24)

\[ \Psi_n(x) = \sqrt{\frac{a}{b + x^2}} \Phi_n(f(x)), \]

regarding the parameter \( \kappa = (a/b)^2 \).

4. PCT Calculations for Mass Distribution-II

Now, we consider the mass distribution [95]

\[ m = m_2 = \frac{1}{x(a + bx)^2}. \]  

(25)

Here \( a \) and \( b \) are constant parameters. Thus its map function takes the form as

\[ y = \bar{f}(x) = \int \sqrt{m_2} \, dx \]

\[ = \frac{2}{\sqrt{a}} \arctan \left( \frac{\sqrt{a} y}{x} \right), \]

(26)

with the function \( x = \left( \frac{a}{b} \right) \tan^2 \left( \frac{\sqrt{a} y}{2} \right) \).

4.1. STP Case
In this context, the target potential and the wave functions for the symmetric and anti-symmetric cases for the uniform energy spectra in (14) and (15) can be obtained in compact form

\[ V(x) = \tilde{V}_0 x^2 + \left[ \frac{(a^2/32) x - (3 b^2/32)}{x} - (5/16)ab \right] \]

\[ \Psi_{2n}(x) = \frac{1}{\sqrt{\sqrt[4]{a+bx}}} \Phi_{2n} \left( \bar{f}(x) \right) \]  

(27)

\[ \Psi_{2n+1}(x) = \frac{1}{\sqrt{\sqrt[4]{a+bx}}} \Phi_{2n+1} \left( \bar{f}(x) \right). \]

where \( \tilde{V}_0 = (4 V_0/a) \).

4.2. SCP Case

By means of the PCT procedure, resulting system will become

\[ \tilde{V}(x) = \tilde{V}_0 x^{-2} + \left[ \frac{(a^2/32) x - (3 b^2/32)}{x} - (5/16)ab \right] \]

\[ \Psi_2(x) = \frac{1}{\sqrt{\sqrt[4]{a+bx}}} \Phi_2 \left( \tilde{f}(x) \right) \]

(28)

\[ \Psi_3(x) = \frac{1}{\sqrt{\sqrt[4]{a+bx}}} \Phi_3 \left( \tilde{f}(x) \right). \]

It is noted that the spectrum of this system and that of its original counterpart are identical.

4.3. Trigonometric PTP Case

The mapping function gives rise to the following configuration having the same eigenspectra

\[ \tilde{V}(x) = -\frac{(V_0 + \gamma)}{x^2} + V_{02} (1 + \gamma x^2) + \left[ \frac{(a^2/32) x - (4 b^2/32)}{x} - (5/16)ab \right] \]

\[ \Psi_n(x) = \frac{1}{\sqrt{\sqrt[4]{a+bx}}} \Phi_n \left( \tilde{f}(x) \right), \]  

(29)

where the parameter \( \gamma = (4/a) \).

5. PCT Calculations for Mass Distribution-III

In the section, the new mass distribution form is taken into account with the parameters \( a \) and \( b \) as [95]

\[ m = m_3 = \left( \frac{a x}{b^2 + a^2} \right)^2. \]  

(30)

Thus it can be mapped by

\[ y = \tilde{f}(x) = \int \sqrt{m_3} \, dx = \frac{a}{2b^2} \arctan \left( \frac{x^2}{b^2} \right), \]

(31)

\[ x = b \left[ \tan \left( \frac{2b^4 y}{a} \right) \right]^{1/2}. \]
5.1. STP Case

The mapping function yields the following expressions

\[ V(x) = \bar{V}_0 x^4 - \left[ \frac{(21/8 \ a^2) \ x^4}{x^4} + \left( \frac{5/8 \ b^4 a^2}{a^4} \right) \right] \]

\[ \Psi_{2n}(x) = \sqrt{\frac{a x}{b^4 + x^4}} \ \Phi_{2n} \left( \tilde{f} (x) \right) \]

(32)

\[ \Psi_{2n+1}(x) = \sqrt{\frac{a x}{b^4 + x^4}} \ \Phi_{2n+1} \left( \tilde{f} (x) \right), \]

with the parameter \( \bar{V}_0 = V_0 \left( \frac{a^2}{4 b^4} \right) \) and the same eigenvalues.

5.2. SCP Case

By applying the PCT procedure, then the target system can be determined

\[ V(x) = \bar{V}_0 x^{-4} - \left[ \frac{(21/8 \ a^2) \ x^4}{x^4} + \left( \frac{5/8 \ b^4 a^2}{a^4} \right) \right] \]

\[ \Psi_{2n}(x) = \sqrt{\frac{a x}{b^4 + x^4}} \ \Phi_{2n} \left( \tilde{f} (x) \right) \]

(33)

\[ \Psi_{2n+1}(x) = \sqrt{\frac{a x}{b^4 + x^4}} \ \Phi_{2n+1} \left( \tilde{f} (x) \right) \]

5.3. Trigonometric PTP Case

It is possible to express the final configuration of the target system concerning the uniform energy spectrum of the bound states as

\[ \tilde{V}(x) = - \left( \frac{V_0 V_1}{x^4} \right) + V_0 (1 + a \ x^4) - \left[ \frac{(21/8 \ a^2) \ x^4}{x^4} + \left( \frac{5/8 \ b^4 a^2}{a^4} \right) \right] \]

\[ \tilde{\Psi}_n(x) = \sqrt{\frac{a x}{b^4 + x^4}} \ \tilde{\Phi}_n \left( \tilde{f} (x) \right), \]

(34)

where \( o = (a^2/4 b^6) \) is the potential parameter.

6. Concluding Remarks

This paper mainly deals with accurate and analytical solutions of the position-dependent effective mass (PDEM) Schrödinger equation for quadratic form of some trigonometric type potentials such as tangent, cotangent and Pöschl-Teller cases by implementing the mathematical procedure known as the point canonical transform (PCT). It also contains target (constructed) potentials for the original counterpart with mass distribution functions having some particular forms. Moreover, performing the straightforward PCT calculations allows us to determine both the energy spectra of the bound states and the corresponding eigenfunctions. It is possible to here note that these results can also be adapted for description of several quantum few- and many-body physical systems including position-dependent unstable masses. Furthermore, they are applicable for analysing some problems encountered particularly in the fields of atomic and molecular, solid-states and condensed matter physics as well as examining the various devices or mechanisms with spatially changing of the mass configurations in quantum optics and physical chemistry etc.
Use of the PCT method enables us to express the mapping function. It plays a distinctive role for determining the target system, which includes compact forms of eigenspectra and wavefunctions. It is important to note that the energy levels of the reference (source) potential and the resulting ones have similar properties. Especially, the former and the target equivalent ones share the same energy eigenvalues. However, it can be noticed that former eigenfunctions and their counterparts (target potentials) have different forms. As a final remark, both the forms of the original and the spatial dependence properties of the mass distribution functions $m = m(x)$ ensure the ultimate decision for the system whether it is fully solvable or not solvable.

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