On Complex Harmonic Functions with a Two-Parameters Coefficient Condition defined by an Integral Operator

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Abstract. In this paper, We present some results concerning starlikeness and convexity on a class of complex harmonic function with a two-parameter coefficient condition defined by an integral operator.

1. Introduction

Let $f$ denote the class of analytic function of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \in \mathbb{C}$, if both $u$ and $v$ are real harmonic in $D$ expressed as

$$f = h + \overline{g}. \quad (1.2)$$

where $h$ is analytic and $g$ is co-analytic in $D$.

In 1984, Clunie and Sheil-Small [2] introduced the class $SH$ which is of the form (1.2) that are harmonic univalent and sense-preserving in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ expressed as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in U, |b_1| < 1. \quad (1.3)$$

In [4], the operator $I^\sigma$, the one-parameter Jung-Kim-Srivastava integral was defined as $I^\sigma f(z) = \frac{2\pi}{\Gamma(\sigma)} \int_0^1 (\log t)^{\sigma-1} f(t)dt$ and given as

$$I^\sigma f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\sigma a_n z^n.$$
The operator $I^\sigma f(z)$ was used to generalise the concepts of starlikeness and convexity of functions in the unit disk (see [3]).

Similarly, our aim here is to use this operator $I^\sigma f(z)$ to generalise starlikeness and convexity for harmonic functions with a two-parameter condition of the results obtained in [5, 6, 7]. Defining the operator on $f = h + g$, we have that

$$I^\sigma f(z) = I^\sigma h(z) + I^\sigma g(z).$$

(1.4)

where

$$I^\sigma h(z) = z + \sum_{n=2}^{\infty} \left( \frac{2}{n+1} \right)^\sigma a_n z^n, I^\sigma g(z) = \sum_{n=1}^{\infty} \left( \frac{2}{n+1} \right)^\sigma b_n z^n$$

Let $U = \{z \in \mathbb{C} : |z| < 1\}, P = (\beta, p) \in \mathbb{R}^2 : 0 \leq \beta \leq 1, p > 0$ and $U_{\sigma,n}(\beta, p) = \beta(2/(n+1))^{\sigma p} + (1-\beta)(2/(n+1))^{\sigma (p+1)}, n = 2, 3, \ldots, (\beta, p) \in P$.

For a fixed pair $(\beta, p) \in P$, we denote by $I_{\sigma,k}(\beta, p)$ the class of function $f$ of the form

$$f = h + g, h(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n, z \in U, |b_1| < 1,$$

and such that

$$|b_1| + \sum_{n=2}^{\infty} U_{\sigma,n}(\beta, p)(|a_n| + |b_n|) \leq 1, |b_1| < 1.$$

Moreover,

$$I_{\sigma,n}^0(\beta, p) = f \in I_{\sigma,k}(\beta, p) : b_1 = 0.$$

Clearly the functions in the class $I_{0,n}^0(1, 1)$ is starlike and the functions in the class $I_{0,n}^0(1, 2)$ is convex.

With respect to the following inequality

$$U_{0,n}(1, p) = n^p \leq U_{\sigma,n}(n, p) \leq n^{p+1} = U_{0,n}(0, p), n = 2, 3, \ldots, (\beta, p) \in P.$$

and by condition (1.4), we have the following inclusions:

$$I_{0,n}(0, p) \subset I_{\sigma,n}(\beta, p) \subset I_{0,n}(1, p), \quad I_{0,n}(0, p) \subset I_{\sigma,n}^0(\beta, p) \subset I_{0,n}^0(1, p).$$

2. Main Results

**Theorem 2.1** Let $(\beta, p) \in P$. If $f \in I_{\sigma,n}(\beta, p)$ ($I_{0,n}^0(\beta, p)$), then functions

$$z \rightarrow r^{-1} f(rz), z \rightarrow e^{-it} f(e^{-it} z), z \in U, r \in (0, 1), t \in \mathbb{R}.$$  

(2.1)

also belong to $I_{\sigma,n}(\beta, p)$ (resp. $I_{0,n}^0(\beta, p)$).

**Theorem 2.2** If $0 \leq \beta_1 \leq \beta_2 \leq 1, p > 0$, then

$$I_{\sigma,n}(\beta_1, p) \subset I_{\sigma,n}(\beta_2, p), I_{0,n}^0(\beta_1, p) \subset I_{0,n}^0(\beta_2, p).$$

(2.2)

If $\beta \in [0, 1]$ and $0 < p_1 \leq p_2$, then

$$I_{\sigma,n}(\beta, p_1) \supset I_{\sigma,n}(\beta, p_2), I_{0,n}^0(\beta, p_1) \supset I_{0,n}^0(\beta, p_2).$$

(2.3)
Theorem 2.3 Let \((\beta, p) \in P\). If \(p \geq 1\), then every function \(f \in I_{\sigma,n}^{0}(\beta, p)\) is univalent and maps the unit disk \(U\) onto a domain starlike with respect to the origin. If \(p \geq 2\), then every function \(I_{\sigma,n}^{0}(\beta, p)\) is univalent and maps the unit disk \(U\) onto a convex domain.

Proof. If \(p \geq 1\), then \(U_{\sigma,n}(\beta, p) \geq n\) form = 2, 3, \cdots, \(\beta \in [0, 1]\), so by the condition (1.4) we have that

\[
\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1.
\]

Hence \(f\) is univalent and starlike with respect to the origin. If \(p \geq 2\), then by (1.4) we obtain

\[
\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \leq 1.
\]

Therefore \(f\) is convex. Let \(\beta \in [0, 1]\) and set \(p_1(\beta) = 1 - \log_2(2 - \beta), p_2(\beta) = 1 - \log_2(2 - \beta), \log_21 = 0\) and let

\[
D_1 = (\beta, p) \in P : p \geq p_1(\beta), D_2 = (\beta, p) \in P : p \geq p_2(\beta).
\]

For \((\beta, p) \in D_1\) and \((\beta, p) \in D_2\), the following theorems shows starlikeness and convexity of functions of the \(I_{\sigma,n}^{0}\).

Theorem 2.4 If \((\beta, p) \in D_1\), then the functions of the class \(I_{\sigma,n}^{0}\), are starlike.

Proof. From the condition (1.4), it follows that \(f\) is a starlike function.

Corollary 2.5 [3] If \(f \in I_{\sigma,n}^{0}\), then \(f\) is a starlike function.

Theorem 2.6 Let \((\beta, p) \in P \setminus D_1\), if \(r_0(\beta, p) = 2^{p-1}(2 - \beta)\), then each function in \(I_{\sigma,n}^{0}\) maps the disk \(U_r\) onto a domain of starlike with respect to the origin where \(U_r = \{z \in \mathbb{C} : |z| < r\}, r > 0\) with \(U_0 = U\).

Proof. For \((\beta, p) \in P \setminus D_1\), we have \(r_0(\beta, p) < 1, \) let \(f \in I_{\sigma,n}(\beta, p), (\beta, p) \in P \setminus D_1\) and let \((r_0(\beta, p))\). By Theorem 2.1, the function \(f_r\) of the form \(f_r(z) = r^{-1}f(rx)\) belong to the class \(I_{\sigma,n}^{0}(\beta, p)\) and we have that

\[
\sum_{n=2}^{\infty} n \left(|a_n r^{n-1}| + |b_n r^{n-1}|\right) = \sum_{n=2}^{\infty} n r^{n-1}(|a_n| + |b_n|).
\]

(2.4)

By elementary inequality, we have that

\[
n r^{n-1} \leq n(r_0(\beta, p))^{n-1} \leq U_{\sigma,n}(\beta, p), n = 2, 3, \cdots.
\]

(2.5)

Corollary 2.7 [3] If \(f \in I_{\sigma,n}^{0}\), then \(f\) maps the unit disk onto a starlike domain with respect to the origin.

Theorem 2.8 Let \((\beta, p) \in P \setminus D_2\), if \(r_0^*(\beta, p) = 2^{p-1}(2 - \beta)\), then each function in \(I_{\sigma,n}^{0}\) maps the disk \(U_r\) onto a convex domain.

Proof. \((\beta, p) \in P \setminus D_2\), we have \(r_0^*(\beta, p) \leq 1\), from the condition (2.4) we have

\[
n^2 r^{n-1} \leq U_{\sigma,n}(\beta, p), n = 2, 3, \cdots.
\]

(2.6)

The next theorem presents the distortion for functions in \(I_{\sigma,n}\).
Theorem 2.9 Let \((\beta, p) \in P\). If \(f \in I_{\sigma,n}(\beta, p), z \in U, z \neq 0\), then
\[
|f(z)| \leq (1 + |b_1|)|z| + \frac{1 - |b_1|}{2p(2 - \beta)}|z|^2|f(z)| \geq (1 + |b_1|)|z| + \frac{1 - |b_1|}{2p(2 - \beta)}|z|^2.
\]

Proof. Let \(f \in I_{\sigma,n}(\beta, p), (\beta, p) \in P\), \(f\) of the form (1.3) and \(z \in U \setminus \{0\}\). Then by condition (1.4), we have that
\[
\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq \frac{1 - |b_1|}{U_{\sigma,n}(\beta, p)} - \sum_{n=3}^{\infty} \left( \frac{U_{\sigma,n}(\beta, p)}{U_{\sigma,n}(\beta, p)} - 1 \right) (|a_n| + |b_n|)
\]
Since \(U_{\sigma,n}(\beta, p) > U_{\sigma,2}(\beta, p), n = 2, 3, \cdots, (\beta, p) \in P\), we have that
\[
\sum_{n=2}^{\infty} (|a_n| + |b_n|) \leq \frac{1 - |b_1|}{U_{\sigma,n}(\beta, p)}.
\]
Therefore
\[
|f(z)| \leq \sum_{n=2}^{\infty} (|a_n| + |b_n|)|z|^n + (1 + |b_1|)|z| + \frac{1 - |b_1|}{U_{\sigma,n}(\beta, p)}|z|^2.
\]
which is the upper bounds. The lower bounds follows from (2.6).

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