Distributed Optimization for Problems with Variational Inequality Constraints

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Abstract

We consider a class of constrained multi-agent optimization problems where the goal is to cooperatively minimize a sum of agent-specific nondifferentiable merely convex functions. The constraint set is characterized as a variational inequality (VI) problem where each agent is associated with a local monotone mapping. We first discuss how the proposed formulation can be employed in addressing several important classes of distributed optimization problems, including those subject to complementarity constraints, nonlinear equality constraints, and equilibrium constraints. In addressing the model of interest, our contributions are as follows: (i) We develop an iteratively regularized incremental gradient method where at each iteration, agents communicate over a cycle graph to update their solution iterates using their local information about the objective and the mapping. The proposed method is single-timescale in the sense that it does not involve any excessive hard-to-project computation per iteration. (ii) We derive non-asymptotic agent-wise convergence rates for the suboptimality of the global objective function and infeasibility of the VI constraints measured by a suitably defined dual gap function. (iii) To analyze the convergence rate in the solution space, assuming the objective function is strongly convex and smooth, we derive non-asymptotic agent-wise rates on an error metric that relates the generated iterates with the Tikhonov trajectory. The proposed method appears to be the first fully iterative scheme equipped with iteration complexity that can address distributed optimization problems with VI constraints. We provide preliminary numerical experiments for computing the best equilibrium in a transportation network problem. We also compare the performance of the proposed scheme with that of the existing incremental gradient methods in addressing constrained finite-sum problems.

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I. Introduction

Distributed optimization over networks has been an appealing area of research in recent years. In this regime, a collection of agents (e.g., processing units, sensors) communicate their local information with their neighboring agents to cooperatively optimize a global objective. It is through this cooperation that learning from massive datasets can be made possible. Moreover, the decentralized storage of the data over the network may allow for preserving the privacy of the agents. It is for these reasons that distributed optimization has found a wide range of applications in wireless sensor networks, machine learning, and signal processing [1]. Despite the significant advances in the design and analysis of the optimization methods over networks, the existing models and algorithms are still less satisfactory in some regimes than those of centralized optimization. For example, there is still much left to be understood about how to tackle the presence of nonlinearity and uncertainty in the functional constraints, while requiring a low number of communications and enforcing weak assumptions on the network topology. The goal in this work is to tackle some of the shortcomings in distributed constrained optimization through considering a new unifying mathematical framework described as follows. Consider a system with \( m \) agents where the \( i \)th agent is associated with a component function \( f_i : \mathbb{R}^n \to \mathbb{R} \) and a mapping \( F_i : \mathbb{R}^n \to \mathbb{R}^n \). Our goal is to solve the following distributed constrained optimization problem

\[
\min \sum_{i=1}^{m} f_i(x) \quad (P)
\]

subject to \( x \in \text{SOL}(X, \sum_{i=1}^{m} F_i) \),

where \( X \subseteq \mathbb{R}^n \) is a set and \( \text{SOL}(X, \sum_{i=1}^{m} F_i) \) denotes the solution set of the variational inequality \( \text{VI}(X, \sum_{i=1}^{m} F_i) \) defined as follows: \( x \in X \) solves \( \text{VI}(X, \sum_{i=1}^{m} F_i) \) if we have \((y - x)^T \sum_{i=1}^{m} F_i(x) \geq 0 \) for all \( y \in X \). Problem \((P)\) represents a distributed optimization framework in the sense that the information about \( f_i \) and \( F_i \) is locally known to the \( i \)th agent, while the set \( X \) is globally known to all the agents. In this work, we consider the case where the local functions \( f_i \) are nondifferentiable and merely convex, and mappings \( F_i \) are single-valued, continuous, and merely monotone. Model \((P)\) captures the canonical formulation of distributed optimization

\[
\min \sum_{i=1}^{m} f_i(x) \quad (1)
\]

subject to \( x \in X \),
that has been extensively studied in the literature. Indeed, by choosing $F_i(x) := 0_n$ for all $i$, model $[P]$ is equivalent to model $[\Pi]$. In what follows, we show how the proposed model offers a framework for extending $[\Pi]$ to capture more challenging distributed constrained optimization problems.

(i) **Distributed optimization problems with complementarity constraints:** Nonlinear complementarity problems (NCP) have been employed to formulate diverse applications in engineering and economics. The celebrated Wardrop’s principle of equilibrium in traffic networks and also, the Walras’s law of competitive equilibrium in economics are among important examples that can be represented using NCP (cf. [2]). Formally, NCP is defined as follows. Given a mapping $F : \mathbb{R}_+^n \to \mathbb{R}^n$, $x \in \mathbb{R}^n$ solves NCP$(F)$ if $0 \leq x \perp F(x) \geq 0$, where $\perp$ denotes the perpendicularity operator between two vectors. It is known that NCP$(F)$ can be cast as VI$(\mathbb{R}_+^n, F)$ (see Proposition 1.1.3 in [3]). In many applications where $F$ is merely monotone, NCP$(F)$ may admit multiple equilibria. In such cases, one may consider finding the best equilibrium with respect to a global metric $f : \mathbb{R}^n \to \mathbb{R}$. For example in traffic networks, the total travel time of the network users can be considered as the objective $f$. In fact, the problem of computing the best equilibrium of an NCP is important to be addressed particularly in the design of transportation networks where there is a need to estimate the efficiency of the equilibrium [4], [5]. In this regime, the goal is to minimize $f(x)$ where $x$ solves NCP$(F)$. Motivated by applications in traffic equilibrium problems, stochastic variants of NCP have been considered more recently [6], [7]. Consider a stochastic NCP given by

$$x \geq 0, \quad \mathbb{E}[F(x, \xi(\omega))] \geq 0, \quad x^T \mathbb{E}[F(x, \xi(\omega))] = 0,$$

where $\xi : \Omega \to \mathbb{R}^d$ is a random variable associated with the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $F : \mathbb{R}_+^n \times \Omega \to \mathbb{R}^n$ is a stochastic single-valued mapping. Let $\mathcal{S}_i$ denote a local index set of independent and identically distributed samples from the random variable $\xi$. Employing a sample average approximation scheme, one can consider a distributed NCP given by

$$x \geq 0, \quad \sum_{i=1}^m \sum_{\ell \in \mathcal{S}_i} F(x, \xi_\ell) \geq 0, \quad x^T \left( \sum_{i=1}^m \sum_{\ell \in \mathcal{S}_i} F(x, \xi_\ell) \right) = 0.$$

Let $f : \mathbb{R}_+^n \times \Omega \to \mathbb{R}^n$ denote a stochastic objective function that measures the performance of a given equilibrium at a realization of $\xi$. Then, the problem of distributed computation of
the best equilibrium of the preceding NCP is formulated as

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} \sum_{\ell \in S_i} f(x, \xi_{\ell}) \\
\text{subject to} & \quad x \in \text{SOL} \left( \mathbb{R}_+^n, \sum_{i=1}^{m} \sum_{\ell \in S_i} F(\bullet, \xi_{\ell}) \right).
\end{align*}
\]  

Importantly, the proposed model (P) captures problem (2) by defining

\[ X \triangleq \mathbb{R}_+^n, \quad f_i(x) \triangleq \sum_{\ell \in S_i} f(x, \xi_{\ell}), \quad F_i(x) \triangleq \sum_{\ell \in S_i} F(x, \xi_{\ell}). \]

In section V, we present preliminary numerical experiments where we solve problem (2) for a given transportation network.

(ii) Distributed optimization with local nonlinear inequality and linear equality constraints: Another class of problems that can be captured by problem (P) is given as

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} f_i(x) \\
\text{subject to} & \quad g_{i,1}(x) \leq 0, \ldots, g_{i,n_i}(x) \leq 0, \quad \text{for } i \in [m], \\
& \quad A_i x = b_i, \quad \text{for } i \in [m], \\
& \quad x \in X,
\end{align*}
\]  

where agent \( i \) is associated with function \( f_i : \mathbb{R}^n \to \mathbb{R} \), functions \( g_{i,j} : \mathbb{R}^n \to \mathbb{R} \) for \( j \in [n_i] \), and parameters \( A_i \in \mathbb{R}^{d_i \times n} \) and \( b_i \in \mathbb{R}^{d_i} \). The notation \([m]\) is used to abbreviate \( \{1, \ldots, m\}\).

The set \( X \) is globally known to all the agents while \( f_i(x), g_{i,j}(x), A_i, \) and \( b_i \) are locally known to agent \( i \). In the following, we show that problem (3) can be represented as model (P).

Lemma 1. Consider problem (3). Let functions \( g_{i,j}(x) \) be continuously differentiable and convex for all \( i \in [m] \) and \( j \in [n_i] \). Assume that the feasible region of problem (3) is nonempty and the set \( X \) is closed and convex. Then, problem (3) is equivalent to (P) where we define

\[ F_i : \mathbb{R}^n \to \mathbb{R}^n \text{ as } F_i(x) \triangleq A_i^T (A_i x - b_i) + \sum_{j=1}^{n_i} \max\{0, g_{i,j}(x)\} \nabla g_{i,j}(x). \]

Proof. See Appendix A.

(iii) Distributed optimization with coupling nonlinear equality constraints: Another class of problems that can be reformulated as (P) is given as

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} f_i(x) \\
\text{subject to} & \quad \sum_{i=1}^{m} F_i(x) = 0_n,
\end{align*}
\]  

where agent \( i \) is associated with a local mapping \( F_i : \mathbb{R}^n \to \mathbb{R}^n \) and a local objective \( f_i : \mathbb{R}^n \to \mathbb{R} \). This model finds relevance to the multi-user optimization problems in network resource allocation applications where the objective and constraints are not separable by each
### TABLE I: Comparison of incremental gradient schemes for solving finite-sum problems

| Reference | Method          | Problem class              | Problem formulation                                                                 | Convergence rate(s) |
|-----------|-----------------|----------------------------|-------------------------------------------------------------------------------------|---------------------|
| [11]      | Projected IG    | $f_i \in C_{0,L}^0$       | min$_{x \in X} \sum_{i=1}^{m} f_i(x)$                                                | $O\left(\frac{1}{\sqrt{k}}\right)$ |
| [12, 13]  | IAG             | $f_i \in C_{0,L}^1$       | min$_{x \in \mathbb{R}^n} \sum_{i=1}^{m} f_i(x)$                                    | $O\left(\rho^k\right)$ |
| [14]      | SAGA            | $f_i \in C_{0,L}^{1,1}$   | min$_{x \in X} \sum_{i=1}^{m} f_i(x)$                                                | $O\left(\frac{1}{k}\right), O\left(\rho^k\right)$ |
| [15]      | Proximal IAG    | $f_i \in C_{0,L}^{1,1}$   | min$_{x \in X} \sum_{i=1}^{m} f_i(x)$                                                | $O\left(\frac{1}{k}\right)$ |
| [16]      | IG              | $f_i \in C_{0,L}^{2,1}$   | min$_{x \in X} \sum_{i=1}^{m} f_i(x)$                                                | $O\left(\rho^k\right)$ |
| [17]      | Primal-Dual IG  | $f_i \in C_{0,L}^{0,0}$   | $Ax - b \in -K$                                                                     | $O\left(\frac{1}{k}\right)$ |

This work

| pair-IG    | $f_i \in C_{0,L}^{0,0}$, $F_i$ is monotone | min$_{x \in X} \sum_{i=1}^{m} f_i(x)$ | $x \in \text{SOL}(X, \sum_{i=1}^{m} F_i)$ | $O\left(k^{b-0.5}\right)$ | $O\left(k^{-b}\right)$ |

where $0 < b < 0.5$

user [8], [9], [10]. Note that the feasible solution set of problem (4) is equal to SOL($\mathbb{R}^n, \sum_{i=1}^{m} F_i$), implying that (4) is captured by (P).

**Scope and literature review:** In addressing the proposed formulation (P), our focus in this paper lies in the development of an incremental gradient (IG) method. IG methods are among popular avenues for addressing the classical distributed optimization model [1] and they have received an increasing attention in recent years in addressing applications arising in sensor networks and machine learning [11], [18], [13], [16]. In these schemes, utilizing the additive structure of the problem, the algorithm cycles through the data blocks and updates the local estimates of the optimal solution in a sequential manner [19]. While the first variants of IG schemes find their roots in addressing neural networks as early as in the 1980s [20], the complexity analysis of these schemes has been a trending research topic in the fields of control and machine learning in the past two decades. In addressing the constrained problems with easy-to-project constraint sets, the projected incremental gradient (P-IG) method and its subgradient variant were developed [21]. Considering the smooth case, the P-IG scheme is described as follows. Given an initial point $x_{0,1} \in X$ where $X \subseteq \mathbb{R}^n$ denotes the constraint set, for each $k \geq 0$, consider the update rules given by

$$x_{k,i+1} := \mathcal{P}_X\left(x_{k,i} - \gamma_k \nabla f_i(x_{k,i})\right), \quad \text{for } i \in [m],$$

$$x_{k+1,1} := x_{k,m+1},$$

where $x_{k,i} \in \mathbb{R}^n$ denotes agent $i$’s local copy of the decision variables at iteration $k$, $\mathcal{P}$ denotes the Euclidean projection operator defined as $\mathcal{P}_X(z) \triangleq \arg\min_{x \in X} \|x - z\|_2$, and $\gamma_k > 0$ denotes...
the stepsize parameter. To motivate our research, we provide an overview of the different variants of existing IG schemes and then, highlight some of the shortcomings of these methods in the constrained regime, in particular, in addressing VI constraints in [P]. Recently, under the assumption of strong convexity and twice continuous differentiability of the objective function, and also, boundedness of the generated iterates, the standard IG method was proved to converge with the rate $O(1/k)$ in the unconstrained case [16]. This is an improvement to the previously known rate of $O(1/\sqrt{k})$ for the merely convex case. Accelerated variants of IG schemes with provable convergence speeds were also developed, including the incremental aggregated gradient method (IAG) [12], [13], SAG [22], and SAGA [14]. While addressing the merely convex case, SAGA using averaging achieves a sublinear convergence rate, assuming strong convexity and smoothness, this is improved for non-averaging variants of SAGA and IAG to a linear rate. Table I presents a summary of the standard IG schemes in addressing unconstrained and constrained finite-sum problems. As evidenced, most of the past research efforts on the design and analysis of algorithms for distributed constrained optimization problems have focused on addressing easy-to-project sets or sets with linear functional inequalities. This has been done through employing duality theory, projection, or penalty methods (see [23], [24], [25], [26]). Also, a celebrated variant of the dual based schemes is the alternating direction method of multipliers (ADMM) (e.g., see [27], [28]). Other related papers that have utilized duality theory in distributed constrained regimes include [29], [24], [30]. Despite the extensive work in the area of constrained optimization, no provably convergent iterative method exits in the literature that can be employed to solve distributed optimization problems with VI constraints. In fact, we are unaware of any IG methods with complexity guarantees that can be employed for addressing any of the three individual subclass problems (i), (ii), and (iii) mentioned earlier.

Contributions: Motivated by the aforementioned research gap, we make the following main contributions.

(i) Complexity guarantees for addressing model [P]: The main contribution of this work lies in the development of a distributed iterative method equipped with agent-specific iteration complexity guarantees for solving distributed optimization problems with VI constraints of the form [P]. To this end, employing a regularization-based relaxation technique, we propose a projected averaging iteratively regularized incremental gradient method (pair-IG) presented by Algorithm 1. In Theorem 1, under merely convexity of the global objective
function and merely monotonicity of the global mapping, we derive new non-asymptotic suboptimality and infeasibility convergence rates for each agent’s generated iterates. This implies a total iteration complexity of $O((C_f + C_F)^4 \epsilon^{-4})$ for obtaining an $\epsilon$-approximate solution where $C_f$ and $C_F$ denote the bounds on the global objective function’s subgradients and the global mapping over a compact convex set $X$, respectively. Iterative regularization (IR) has been recently employed as a constraint-relaxation technique in a class of bilevel optimization problems [31], [32] and also in regimes where the duality theory may not be directly applied [33], [34]. Of these, in our recent work [34] we employed the IR technique to derive a provably convergent method for solving problem (P) in a centralized framework, where the information of the objective function is globally known by the agents. Unlike in [34], here we assume that the agents have access only to local information about both the objective function and the mapping. It is worth emphasizing that this lack of centralized access to information introduces a major challenge in both the design and the complexity analysis of the new algorithmic framework in addressing the distributed model (P). Motivated by the need for distributed implementations, in a preliminary version of this work, in [35], we addressed a subclass of the distributed model (P) formulated as (3). Extending [35], here we show that the convergence and rate analysis in [35] can be extended to the much broader class of problems of the form (P) without any degradation of the speed of the algorithm. In addressing hierarchical optimization problems, there have been other iterative methods proposed in works such as [36], [37], [38], [39]. We, however, note that all of these schemes can only address a subclass of (P) in centralized regimes and under more restrictive assumptions such as strong convexity of the global objective function. We also note that compared to the existing methods that address VI problems (e.g., see [40], [41], [33], [18]), our work provides an avenue for addressing a significantly more general class of problems where VI is employed as a tool to characterize the constraints in distributed optimization.

(ii) Distributed averaging scheme: In pair-IG, we employ a distributed averaging scheme where agents can choose their initial averaged iterate arbitrarily and independent from each other. This relaxation in the proposed IG method appears to be novel, even for the classical IG schemes in addressing (1).

(iii) Rate analysis in the solution space: Motivated by the recent developments of iterative methods for MPECs [42], it is important to characterize the speed of the proposed scheme in the solution space. To this end, under strong convexity of the global objective function, in
Theorem 2 we derive agent-specific rate statements that compare the generated sequence of each agent with the so-called Tikhonov trajectory, that is defined as the trajectory of the unique solutions to a family of regularized optimization problems.

(iv) Preliminary numerical results: To validate the theoretical results, we provide preliminary numerical experiments for computing the best equilibrium in a multi-agent traffic network problem. We also compare the performance of pair-IG with that of the existing IG methods in addressing constrained finite-sum problems.

Outline: The remainder of the paper is organized as follows. In section II, we provide the algorithm outline, main assumptions, and preliminaries. Section III includes the convergence and rate analysis of the proposed scheme. In section IV we study the solution quality and derive bounds on the distance between the iterates from Algorithm I and the Tikhonov trajectory. Section V contains numerical implementations, and concluding remarks follow in section VI.

Notation: A function \( f : \mathbb{R}^n \to \mathbb{R} \) is said to be in the class \( C^{k,r}_{\mu,L} \) if \( f \) is \( \mu \)-strongly convex in \( \mathbb{R}^n \), \( k \) times continuously differentiable, and its \( r \)th derivative is Lipschitz continuous with constant \( L \). Throughout, a vector \( x \in \mathbb{R}^n \) is assumed to be a column vector and \( x^T \) denotes its transpose. For any vector \( x \in \mathbb{R}^n \), we use \( \|x\| \) to denote the \( \ell_2 \)-norm. For a convex function \( f : \mathbb{R}^n \to \mathbb{R} \) with the domain \( \text{dom}(f) \) and any \( x \in \text{dom}(f) \), a vector \( \tilde{\nabla} f(x) \in \mathbb{R}^n \) is called a subgradient of \( f \) at \( x \) if \( f(x) + \tilde{\nabla} f(x)^T (y - x) \leq f(y) \) for all \( y \in \text{dom}(f) \). We let \( \partial f(x) \) denote the subdifferential set of function \( f \) at \( x \). The Euclidean projection of vector \( x \) onto set \( X \) is denoted by \( P_X(x) \). We let \( \mathbb{R}_+^n \) and \( \mathbb{R}_{++}^n \) denote the set \( \{x \in \mathbb{R}^n \mid x \geq 0\} \) and \( \{x \in \mathbb{R}^n \mid x > 0\} \), respectively. Given a set \( S \subseteq \mathbb{R}^n \), we let \( \text{int}(S) \) denote the interior of \( S \). Given an integer \( m \), we let \([m]\) abbreviate the set \( \{1, \ldots, m\} \).

II. Algorithm outline

In this section we present the main assumptions on problem \([P]\), the outline of the proposed algorithm, and a few preliminary results that will be applied later in the rate analysis. Throughout the paper, we let \( f(x) \triangleq \sum_{i=1}^m f_i(x) \) and \( F(x) \triangleq \sum_{i=1}^m F_i(x) \) denote the global objective and global mapping in problem \([P]\), respectively.

Assumption 1 (Properties of problem \([P]\)). Suppose the following conditions hold.

(a) Function \( f_i : \mathbb{R}^n \to \mathbb{R} \) is real-valued and merely convex (possibly nondifferentiable) on its domain for all \( i \in [m] \).
(b) Mapping \( F_i : \mathbb{R}^n \to \mathbb{R}^n \) is real-valued, continuous, and merely monotone on its domain for all \( i \in [m] \).

(c) The set \( X \subseteq \text{int} ( \text{dom}(f) \cap \text{dom}(F) ) \) is nonempty, convex, and compact.

Remark 1. Under Assumption \([\text{I}]\) we have the following immediate results. From Theorem 2.3.5 and Corollary 2.2.5 in \([3]\), the set \( \text{SOL}(X, F) \) is nonempty, convex, and compact. For all \( i \), the nonemptiness of the subdifferential set \( \partial f_i(x) \) for any \( x \in \text{int}(\text{dom}(f_i)) \) is implied from Theorem 3.14 in \([43]\). Also, Theorem 3.16 in \([43]\) implies that \( f_i \) has bounded subgradients over the compact set \( X \). Further, mapping \( F_i \) is bounded over the set \( X \).

In view of compactness of the set \( X \) and continuity of \( f \), throughout the paper we let positive scalars \( M_X < \infty \) and \( M_f < \infty \) be defined as \( M_X \triangleq \sup_{x \in X} \|x\| \) and \( M_f \triangleq \sup_{x \in X} |f(x)| \), respectively. We also let \( f^* \in \mathbb{R} \) denote the optimal objective value of problem \((P)\). In view of Remark \([\text{I}]\), throughout we let scalars \( C_F > 0 \) and \( C_f > 0 \) be defined such that for all \( i \in [m] \) and for all \( x \in X \) we have \( \|F_i(x)\| \leq \frac{C_F}{m} \), and \( \|\nabla f_i(x)\| \leq \frac{C_f}{m} \) for all \( \nabla f_i(x) \in \partial f_i(x) \).

This implies that \( |f(x) - f(y)| \leq C_f \|x - y\| \) for all \( x, y \in X \).

Remark 2. Under Assumption \([\text{I}]\) and from Theorem 3.61 in \([43]\), function \( f_i \) is Lipschitz continuous with the parameter \( \frac{C_f}{m} \) over the set \( X \), i.e., for all \( i \in [m] \) we have \( |f_i(x) - f_i(y)| \leq \frac{C_f}{m} \|x - y\| \) for all \( x, y \in X \). We also have \( \|\nabla f(x)\| \leq C_f \) for all \( x \in X \) and all \( \nabla f(x) \in \partial f(x) \).

This implies that \( |f(x) - f(y)| \leq C_f \|x - y\| \) for all \( x, y \in X \).

We now present an overview of the proposed method given by Algorithm \([\text{I}]\). We use vector \( x_{k,i} \) to denote the local copy of the global decision vector maintained by agent \( i \) at iteration \( k \). At each iteration, agents update their iterates in a cyclic manner. Each agent \( i \in [m] \) uses only its local information including the subgradient of the function \( f_i \) and mapping \( F_i \) and evaluates the regularized mapping \( F_i + \eta \nabla f_i \) at \( x_{k,i} \). Here, \( \gamma_k \) and \( \eta_k \) denote the stepsize and the regularization parameter at iteration \( k \), respectively. Importantly, through employing an iterative regularization technique, we let both of these parameters be updated iteratively at suitable prescribed rates (cf. Theorem \([\text{I}]\)). Each agent computes and returns a weighted averaging iterative denoted by \( \tilde{x}_{k,i} \) where the weights are characterized in terms of the stepsize \( \gamma_k \) and an arbitrary scalar \( r \in [0, 1) \). Notably, this averaging technique is carried out in a distributed fashion in the sense that agents do not require to start from the
Algorithm 1 projected averaging iteratively regularized Incremental subGradient (pair-IG)

**input:** Agent 1 arbitrarily chooses an initial vector $x_{0,1} \in X$. Agent $i$ arbitrarily chooses $\bar{x}_{0,i} \in X$, for all $i \in [m]$. Let $S_0 := \gamma_0^r$ with an arbitrary $0 \leq r < 1$.

**for** $k = 0, 1, \ldots, N - 1$ **do**

Update $S_{k+1} := S_k + \gamma_k^r$

**for** $i = 1, \ldots, m$ **do**

$$x_{k+1,i} := P_X \left( x_{k,i} - \gamma_k \left( F_i(x_{k,i}) + \eta_k \nabla f_i(x_{k,i}) \right) \right)$$  \hfill (5)

$$\bar{x}_{k+1,i} := \left( \frac{S_k}{S_{k+1}} \right) \bar{x}_{k,i} + \left( \frac{\gamma_k^r}{S_{k+1}} \right) x_{k+1,i}$$  \hfill (6)

**end for**

Set $x_{k+1,1} := x_{k,m+1}$

**end for**

**return:** $\bar{x}_{N,i}$ for all $i \in [m]$

same initialized averaging iterate. This is in contrast with the standard incremental gradient schemes where the averaging scheme is limited to a centralized initialization. Next we show that for any $i \in [m]$, $\bar{x}_{N,i}$ is indeed a well-defined weighted average of $\bar{x}_{0,i}$ and the iterates $x_{k-1,i+1}$ for $k \in [N]$.

**Lemma 2.** Consider the sequence $\{ \bar{x}_{k,i} \}$ generated by agent $i \in [m]$ in Algorithm 1. For $k \in \{0, \ldots, N\}$, let us define the weights $\lambda_{k,N} \triangleq \frac{\gamma_k^r}{\sum_{j=0}^{N} \gamma_j^r}$. Then for all $i \in [m]$ we have

$$\bar{x}_{N,i} = \lambda_{0,N} \bar{x}_{0,i} + \sum_{k=1}^{N} \lambda_{k,N} x_{k-1,i+1}.$$  \hfill (7)

Further, for a convex set $X$ we have $\bar{x}_{N,i} \in X$.

**Proof.** We use induction on $N \geq 0$ to show the equation. For $N = 0$, from $\lambda_{0,0} = 1$ we have $\bar{x}_{0,i} = \lambda_{0,0} \bar{x}_{0,i}$. Now, assume that the equation holds for some $N \geq 0$. This implies

$$\bar{x}_{N,i} = \lambda_{0,N} \bar{x}_{0,i} + \sum_{k=1}^{N} \lambda_{k,N} x_{k-1,i+1} = \frac{\gamma_0^r \bar{x}_{0,i} + \sum_{k=1}^{N} \gamma_k^r x_{k-1,i+1}}{\sum_{j=0}^{N} \gamma_j^r}.$$  \hfill (7)

Using equation (7), we now show that the hypothesis statement holds for any $N + 1$. From equation (6) we have $\bar{x}_{N+1,i} = \left( \frac{S_N}{S_{N+1}} \right) \bar{x}_{N,i} + \left( \frac{\gamma_N^r + 1}{S_{N+1}} \right) x_{N,i+1}$. Note that from equation (6) in
Algorithm 1 we have $S_k = \sum_{t=0}^{k} \gamma_t^r$ for all $k \geq 0$. From this and using equation (7) we obtain

$$
\bar{x}_{N+1,i} = \left( \sum_{t=0}^{N} \gamma_t^r \right) \bar{x}_{N,i} + \left( \sum_{t=0}^{N+1} \gamma_t^r \right) x_{N,i+1} = \frac{\gamma_r^0 \bar{x}_{0,i} + \sum_{k=1}^{N} \gamma_k^r x_{k-1,i+1} + \gamma_{N+1}^r x_{N,i+1}}{\sum_{t=0}^{N+1} \gamma_t^r}.
$$

From the definition of $\lambda_{k,N}$ we conclude that the hypothesis holds for $N + 1$ and thus, the result holds for all $N \geq 0$. To show the second part, note that from the initialization in Algorithm 1 and the projection in equation (5), we have $\bar{x}_{0,i}, x_{k-1,i+1} \in X$ for all $i$ and $k \geq 1$. From the first part, $\bar{x}_{N,i}$ is a convex combination of $\bar{x}_{0,i}, x_{0,i+1}, \ldots, x_{N-1,i+1}$. Therefore, from the convexity of the set $X$ we conclude that $\bar{x}_{N,i} \in X$. \hfill \Box

For the ease of presentation throughout the analysis, we define a sequence $\{x_k\}$ as follows.

**Definition 1.** Consider Algorithm 1. Let the sequence $\{x_k\}$ be defined as $x_k \triangleq x_{k-1,m+1} = x_{k,1}$, for all $k \geq 1$, with $x_0 \triangleq x_{0,1}$.

In the following result, we characterize the distance between the local variable of any arbitrary agent with that of the first and the last agent at any given iteration. This result will be utilized in the analysis.

**Lemma 3.** Consider Algorithm 1. Let Assumption 1 hold. Then the following inequalities hold for all $i \in [m]$ and $k \geq 0$

(a) $\|x_k - x_{k,i}\| \leq \frac{(i-1)\gamma_k (C_F + \eta_k C_f)}{m}$.

(b) $\|x_{k,i+1} - x_{k+1}\| \leq \frac{(m-i)\gamma_k (C_F + \eta_k C_f)}{m}$.

**Proof.** (a) Let $k \geq 0$ be an arbitrary integer. We use induction on $i$ to show this result. From Definition 1 for $i = 1$ and $k \geq 0$ we have $\|x_k - x_{k,1}\| = 0$, implying that the result holds for $i = 1$. Now suppose the hypothesis statement holds for some $i \in [m]$. We have

$$
\|x_k - x_{k,i+1}\| = \left\| \mathcal{P}_X (x_k) - \mathcal{P}_X \left( x_{k,i} - \gamma_k \left( F_i(x_{k,i}) + \eta_k \nabla f_i (x_{k,i}) \right) \right) \right\|
\leq \|x_k - x_{k,i}\| + \gamma_k \left\| F_i(x_{k,i}) + \eta_k \nabla f_i (x_{k,i}) \right\|
\leq \|x_k - x_{k,i}\| + \frac{\gamma_k (C_F + \eta_k C_f)}{m} \leq \frac{i\gamma_k (C_F + \eta_k C_f)}{m},
$$

where the first inequality is obtained from the nonexpansivity property of the projection. Therefore the hypothesis statement holds for any $i$ and the proof of part (a) is completed.
(b) To show this result, we use downward induction on $i \in [m]$. Note that the relation trivially holds for the base case $i = m$. Suppose it holds for some $i \in \{2, \ldots, m\}$. We show that it holds for $i - 1$ as well. From Definition 1 we have

$$\|x_k,i - x_k+1\| = \|x_k,i - x_{k,i+1} + x_{k,i+1} - x_{k,m+1}\| \leq \|x_k,i - x_{k,i+1}\| + \|x_{k,i+1} - x_{k,m+1}\|.$$ 

From equation (5), the hypothesis statement, and the nonexpansivity property of the projection, we obtain

$$\|x_k,i - x_k+1\| \leq \left\| P_X(x_{k,i}) - P_X\left(x_{k,i} - \gamma_k \left(F_i(x_{k,i}) + \eta_k \nabla f_i(x_{k,i})\right)\right)\right\| + \frac{(m - i)\gamma_k(C_F + \eta_k C_f)}{m} \leq \frac{(m - i + 1)\gamma_k(C_F + \eta_k C_f)}{m}.$$ 

This completes the proof of part (b).

We note that the generated agent-wise iterates $\bar{x}_{k,i}$ in Algorithm 1, as the scheme proceeds, may not be solutions to VI$(X, F)$ and so, they may not necessarily be feasible to problem (P). To quantify the infeasibility of these iterates, we employ a dual gap function (cf. Chapter 1 in [3]) defined as follows.

**Definition 2** (The dual gap function). Consider a closed convex set $X \subseteq \mathbb{R}^n$ and a continuous mapping $F : X \to \mathbb{R}^n$. The dual gap function at $x \in X$ is defined as $\text{GAP}(x) \triangleq \sup_{y \in X} F(y)^T(x - y)$.

Note that under Assumption 1 the dual gap function is well-defined. This is because for any $x \in X$, we have $\text{GAP}(x) \geq 0$. Further, it is known that when the mapping $F$ is continuous and monotone, $x \in \text{SOL}(X, F)$ if and only if $\text{GAP}(x) = 0$ (cf. [4]). We conclude this section by presenting the following result that will be utilized in the rate analysis.

**Lemma 4.** Let $\beta \in [0, 1)$ and $\Gamma \geq 1$ be given scalars and $K$ be an integer. Then for all $K \geq \left(2^{1/\beta} - 1\right)\Gamma$, we have

$$\frac{(K + \Gamma)^{1-\beta}}{2(1 - \beta)} \leq \sum_{k=0}^{K} (k + \Gamma)^{-\beta} \leq \frac{(K + \Gamma)^{1-\beta}}{1 - \beta}.$$ 

**Proof.** See Appendix B.

III. Rate and Complexity Analysis

In this section we present the convergence and rate analysis of the proposed method under Assumption 1. After obtaining a preliminary inequality in Lemma 5 in terms of the sequence
generated by the last agent, in Lemma 5 we derive inequalities that relate the global objective and the dual gap function at the iterate of other agents with those of the last agent. Utilizing these results, in Proposition 1 we obtain agent-specific bounds on the objective function value and the dual gap function. Consequently, in Theorem 1 we derive convergence rate statements under suitably chosen sequences for the stepsize and the regularization parameter.

**Lemma 5.** Consider Algorithm 1. Let Assumption 1 hold. Let \( \{\gamma_k\} \) and \( \{\eta_k\} \) be nonincreasing and strictly positive sequences. For any arbitrary \( y \in X \), for all \( k \geq 0 \) we have

\[
2\gamma_k^r \left( \eta_k (f(x_k) - f(y)) + F(y)^T (x_k - y) \right) \leq \gamma_k^{r-1} \|x_k - y\|^2 - \gamma_k^{r-1} \|x_{k+1} - y\|^2 + \gamma_k^{r+1} (C_F + \eta_k C_f)^2.
\]

**Proof.** Let \( y \in X \) be an arbitrary vector and \( k \geq 0 \) be fixed. From the update rule (5), for \( i \in [m] \) we have

\[
\|x_{k,i+1} - y\|^2 = \|P_X \left( x_{k,i} - \gamma_k \left( F_i(x_{k,i}) + \eta_k \nabla f_i(x_{k,i}) \right) \right) - P_X(y)\|^2.
\]

Employing the nonexpansivity of the projection we have

\[
\|x_{k,i+1} - y\|^2 \leq \|x_{k,i} - \gamma_k \left( F_i(x_{k,i}) + \eta_k \nabla f_i(x_{k,i}) \right) - y\|^2
\]

\[
= \|x_{k,i} - y\|^2 + \gamma_k^2 \left\| F_i(x_{k,i}) + \eta_k \nabla f_i(x_{k,i}) \right\|^2 - 2\gamma_k \left( F_i(x_{k,i}) + \eta_k \nabla f_i(x_{k,i}) \right)^T (x_{k,i} - y).
\]

From the triangle inequality and recalling the bounds on \( \nabla f_i(x) \) and \( F_i(x) \), we obtain

\[
\|x_{k,i+1} - y\|^2 \leq \|x_{k,i} - y\|^2 + \gamma_k^2 \left( \frac{C_F + \eta_k C_f}{m} \right)^2 + 2\gamma_k \left( F_i(x_{k,i}) + \eta_k \nabla f_i(x_{k,i}) \right)^T (y - x_{k,i}).
\]

The last term in the preceding relation is bounded as follows

\[
2\gamma_k \left( F_i(x_{k,i}) + \eta_k \nabla f_i(x_{k,i}) \right)^T (y - x_{k,i}) = 2\gamma_k F_i(x_{k,i})^T (y - x_{k,i}) + 2\gamma_k \eta_k \nabla f_i(x_{k,i})^T (y - x_{k,i})
\]

\[
\leq 2\gamma_k F_i(y)^T (y - x_{k,i}) + 2\gamma_k \eta_k (f_i(y) - f_i(x_{k,i})),
\]

where the last inequality is implied from the monotonicity of \( F_i \) and convexity of \( f_i \). Combining with equation (9) we have

\[
\|x_{k,i+1} - y\|^2 \leq \|x_{k,i} - y\|^2 + \gamma_k^2 \left( \frac{C_F + \eta_k C_f}{m} \right)^2 + 2\gamma_k F_i(y)^T (y - x_{k,i})
\]

\[
+ 2\gamma_k \eta_k (f_i(y) - f_i(x_{k,i})).
\]
Adding and subtracting $2\gamma_k F_i(y)^T x_k + 2\gamma_k \eta_k f_i(x_k)$ we get

$$
\|x_{k,i+1} - y\|^2 \leq \|x_{k,i} - y\|^2 + \gamma_k^2 \left( \frac{C_F + \eta_k C_f}{m} \right)^2 + 2\gamma_k F_i(y)^T (y - x_k) + 2\gamma_k \eta_k (f_i(y) - f_i(x_k)) \\
+ 2\gamma_k \left( |F_i(y)^T (x_k - x_{k,i})| + \eta_k |f_i(x_k) - f_i(x_{k,i})| \right).
$$

Using the Cauchy-Schwarz inequality and Remark 2 we obtain

$$
\|x_{k,i+1} - y\|^2 \leq \|x_{k,i} - y\|^2 + \gamma_k^2 \left( \frac{C_F + \eta_k C_f}{m} \right)^2 + 2\gamma_k F_i(y)^T (y - x_k) + 2\gamma_k \eta_k (f_i(y) - f_i(x_k)) \\
+ 2\gamma_k \left( \frac{C_F}{m} \|x_k - x_{k,i}\| + \frac{\eta_k C_f}{m} \|x_k - x_{k,i}\| \right).
$$

Summing over $i \in [m]$ and considering Definition 1 we have

$$
\|x_{k+1} - y\|^2 \leq \|x_k - y\|^2 + \gamma_k^2 \left( \frac{C_F + \eta_k C_f}{m} \right)^2 + 2\gamma_k F_i(y)^T (y - x_k) + 2\gamma_k \eta_k (f_i(y) - f_i(x_k)) \\
+ \frac{2\gamma_k (C_F + \eta_k C_f)}{m} \sum_{i=1}^{m} \|x_k - x_{k,i}\|.
$$

From Lemma 3 we obtain

$$
\|x_{k+1} - y\|^2 \leq \|x_k - y\|^2 + \gamma_k^2 \left( \frac{C_F + \eta_k C_f}{m} \right)^2 + 2\gamma_k F_i(y)^T (y - x_k) + 2\gamma_k \eta_k (f_i(y) - f_i(x_k)) \\
+ \frac{2\gamma_k (C_F + \eta_k C_f)}{m} \sum_{i=1}^{m} \frac{(i - 1) \gamma_k (C_F + \eta_k C_f)}{m} \\
= \|x_k - y\|^2 + \gamma_k^2 \left( \frac{C_F + \eta_k C_f}{m} \right)^2 + 2\gamma_k F_i(y)^T (y - x_k) + 2\gamma_k \eta_k (f_i(y) - f_i(x_k)).
$$

Multiplying the both sides by $\gamma_k^{r-1}$ we can obtain the result.

In the next result we provide inequalities that relate the objective function and the dual gap function at the generated averaged iterate of the last agent with that of any other agent, respectively. This result will be utilized in Proposition 1.

**Lemma 6.** Consider problem (P) and the sequences $\{\bar{x}_{N,i}\}$ generated in Algorithm 1 for $i \in [m]$ for some $N \geq 1$. Let Assumption 1 hold and let $\{\gamma_k\}$ and $\{\eta_k\}$ be strictly positive and nonincreasing sequences. Then for any $i \in [m]$ we have

$$
f(\bar{x}_{N,i}) - f(\bar{x}_{N,m}) \leq C_F \lambda_{0,N} \|\bar{x}_{0,i} - \bar{x}_{0,m}\| + \frac{(m - i) C_F (C_F + \eta_0 C_f)}{m} \sum_{k=0}^{N} \lambda_{k,N} \gamma_k,
$$

$$
\text{GAP}(\bar{x}_{N,i}) - \text{GAP}(\bar{x}_{N,m}) \leq C_F \lambda_{0,N} \|\bar{x}_{0,i} - \bar{x}_{0,m}\| + \frac{(m - i) C_F (C_F + \eta_0 C_f)}{m} \sum_{k=0}^{N} \lambda_{k,N} \gamma_k,
$$

where $\lambda_{k,N} \triangleq \frac{\gamma_k^i}{\sum_{j=0}^{N} \gamma_j^i}$ for $k \in \{0, \ldots, N\}$.
Proof. Note that the results are trivial when \( m = 1 \). Throughout, we assume that \( m \geq 2 \).

From the Lipschitz continuity of function \( f \) from Remark 2 and invoking Lemma 2, we can write the following for all \( i \in [m] \).

\[
f(\bar{x}_{N,i}) - f(\bar{x}_{N,m}) \leq C_f \lambda_{0,N} \|\bar{x}_{0,i} - \bar{x}_{0,m}\| + C_f \sum_{k=1}^{N} \lambda_{k,N} \|x_{k-1,i+1} - x_{k-1,m+1}\|. \tag{11}
\]

Next, using Lemma 3(b) for any \( k \geq 1 \) and \( i \in [m] \) we have

\[
\|x_{k-1,i+1} - x_{k-1,m+1}\| \leq \frac{(m-i)\gamma_{k-1} (C_F + \eta_{k-1} C_f)}{m}. \tag{12}
\]

From (11), (12), and the nonincreasing sequence \( \{\eta_k\} \), we have

\[
f(\bar{x}_{N,i}) - f(\bar{x}_{N,m}) \leq \frac{(m-i)C_f (C_F + \eta_0 C_f)}{m} \sum_{k=1}^{N} \lambda_{k,N} \gamma_{k-1} + C_f \lambda_{0,N} \|\bar{x}_{0,i} - \bar{x}_{0,m}\|.
\]

Since \( \{\gamma_k\} \) is nonincreasing and \( 0 \leq r < 1 \), we obtain

\[
\sum_{k=1}^{N} \lambda_{k,N} \gamma_{k-1} \leq \frac{1}{\sum_{j=0}^{N} \gamma_j} \sum_{k=1}^{N} \gamma_j^{r+1} \leq \frac{1}{\sum_{j=0}^{N} \gamma_j} \sum_{k=0}^{N-1} \gamma_j^{r+1} \leq \frac{1}{\sum_{j=0}^{N} \gamma_j} \sum_{k=0}^{N} \gamma_j^{r+1} = \sum_{k=0}^{N} \lambda_{k,N} \gamma_k.
\]

From the last two relations we obtain equation (10a). Next we show (10b). From Definition 2 we have

\[
\text{GAP}(\bar{x}_{N,i}) = \sup_{y \in X} F(y)^T (\bar{x}_{N,i} - y) = \sup_{y \in X} F(y)^T (\bar{x}_{N,i} + \bar{x}_{N,m} - \bar{x}_{N,m} - y)
\]

\[
\leq \sup_{y \in X} F(y)^T (\bar{x}_{N,i} - \bar{x}_{N,m}) + \sup_{y \in X} F(y)^T (\bar{x}_{N,m} - y) \leq C_F \|\bar{x}_{N,i} - \bar{x}_{N,m}\| + \text{GAP}(\bar{x}_{N,m}),
\]

Rearranging the terms we obtain \( \text{GAP}(\bar{x}_{N,i}) - \text{GAP}(\bar{x}_{N,m}) \leq C_F \|\bar{x}_{N,i} - \bar{x}_{N,m}\| \). The rest of the proof can be done in a similar fashion to the proof of (10a). \( \square \)

Next we construct agent-wise error bounds in terms of the objective function value and the dual gap function at the averaged iterates generated in Algorithm 1.

**Proposition 1** (Agent-wise error bounds). Consider problem \( [P] \) and the averaged sequence \( \{\bar{x}_{k,i}\} \) generated by agent \( i \) in Algorithm 1 for \( i \in [m] \). Let Assumption 7 hold and \( \{\gamma_k\} \) and \( \{\eta_k\} \) be nonincreasing and strictly positive sequences. Then we have for \( i \in [m] \), \( N \geq 1 \), and \( r \in [0,1) \):

\[
(a) \ f(\bar{x}_{N,i}) - f^* \leq \left( \sum_{k=0}^{N} \gamma_k \right)^{-1} \left( \frac{2M_f^2 \gamma_{N-1}^r}{\gamma_N} + \frac{\left( C_F + \eta_0 C_f \right)^2}{2} \sum_{k=0}^{N} \gamma_k^{r+1} \right)
\]

\[
+ \frac{(m-i)C_f (C_F + \eta_0 C_f)}{m} \sum_{k=0}^{N} \gamma_k^{r+1} + \gamma_0^r f(\bar{x}_{0,m}) - \gamma_0^r f(x_{0,1}) + C_f \gamma_0 \|\bar{x}_{0,i} - \bar{x}_{0,m}\|.
\]
Proof. (a) Let \( x^* \in X \) denote an arbitrary optimal solution to problem (F). From feasibility of \( x^* \) we have \( F(x^*) \leq 0 \). Substituting \( y \) by \( x^* \) in relation (8) and using the preceding relation we have

\[
2\gamma_k^r \eta_k \left( f(x_k) - f^* \right) \leq \gamma_k^{r-1} \left( ||x_k - x^*||^2 - ||x_{k+1} - x^*||^2 \right) + \gamma_k^{r+1} \left( C_F + \eta_k C_f \right)^2.
\]

Dividing both sides by \( 2\eta_k \) we have

\[
\gamma_k^r \left( f(x_k) - f^* \right) \leq \frac{\gamma_k^{r-1}}{2\eta_k} \left( ||x_k - x^*||^2 - ||x_{k+1} - x^*||^2 \right) + \frac{\gamma_k^{r+1}}{2\eta_k} \left( C_F + \eta_k C_f \right)^2. \tag{13}
\]

Adding and subtracting the term \( \frac{\gamma_k^{r-1}}{2\eta_k} ||x_k - x^*||^2 \) we have

\[
\gamma_k^r \left( f(x_k) - f^* \right) \leq \frac{\gamma_k^{r-1}}{2\eta_k} ||x_k - x^*||^2 - \frac{\gamma_k^{r-1}}{2\eta_k} ||x_{k+1} - x^*||^2 + \left( \frac{\gamma_k^{r-1}}{2\eta_k} - \frac{\gamma_k^{r-1}}{2\eta_k} \right) ||x_k - x^*||^2 + \frac{\gamma_k^{r+1}}{2\eta_k} \left( C_F + \eta_k C_f \right)^2. \tag{14}
\]

Recalling the definition of scalar \( M_X \) we have

\[
||x_k - x^*||^2 \leq 2||x_k||^2 + 2||x^*||^2 \leq 4M_X^2. \tag{15}
\]

Taking into account \( r < 1 \), the nonincreasing property of the sequences \( \{\gamma_k\} \) and \( \{\eta_k\} \), we have term \( 1 \geq 0 \). Using (15) and taking summation from (14) over \( k \in [N] \), we obtain

\[
\sum_{k=1}^{N} \gamma_k^r \left( f(x_k) - f^* \right) \leq \frac{\gamma_0^{r-1}}{2\eta_0} ||x_1 - x^*||^2 - \frac{\gamma_N^{r-1}}{2\eta_N} ||x_{N+1} - x^*||^2 + \left( \frac{\gamma_N^{r-1}}{2\eta_N} - \frac{\gamma_0^{r-1}}{2\eta_0} \right) 4M_X^2 + \frac{(C_F + \eta_0 C_f)^2}{2} \sum_{k=1}^{N} \frac{\gamma_k^{r+1}}{\eta_k}. \tag{16}
\]

Rewriting equation (13) for \( k = 0 \) and then, adding and subtracting \( f(\bar{x}_{0,m}) \), we have

\[
\gamma_0^r \left( f(\bar{x}_{0,m}) - f^* + f(x_0) - f(\bar{x}_{0,m}) \right) \leq \frac{\gamma_0^{r-1}}{2\eta_0} ||x_0 - x^*||^2 - \frac{\gamma_0^{r-1}}{2\eta_0} ||x_1 - x^*||^2 + (C_F + \eta_0 C_f)^2 \frac{\gamma_0^{r+1}}{2\eta_0}.
\]
Adding the preceding equation with (16) we obtain
\[
\gamma_0^r (\bar{x}_{0,m} - f^*) + \sum_{k=1}^{N} \gamma_k^r (f(x_k) - f^*) \leq \frac{\gamma_0^{r-1} \|x_0 - x^*\|^2}{2\eta_0} + 2M_X^2 \left( \frac{\gamma_N^{r-1}}{\eta_N} - \frac{\gamma_0^{r-1}}{\eta_0} \right) - \frac{\gamma_N^{r-1}}{2\eta_N} \|x_{N+1} - x^*\|^2 + \frac{(C_F + \eta_0 C_f)^2}{2} \sum_{k=0}^{N} \gamma_k^{r+1} \eta_k + \gamma_0^r (f(\bar{x}_{0,m}) - f(x_0)).
\]

From (15) and neglecting the nonpositive term we obtain
\[
\gamma_0^r (\bar{x}_{0,m} - f^*) + \sum_{k=1}^{N} \gamma_k^r (f(x_k) - f^*) \leq \frac{2M_X^2 \gamma_N^{r-1}}{\eta_N} + \frac{(C_F + \eta_0 C_f)^2}{2} \sum_{k=0}^{N} \gamma_k^{r+1} \eta_k + \gamma_0^r (f(\bar{x}_{0,m}) - f(x_0)).
\]

Next, dividing both sides by \(\sum_{k=0}^{N} \gamma_k^r\) we have
\[
\frac{\gamma_0^r f(\bar{x}_{0,m}) + \sum_{k=1}^{N} \gamma_k^r f(x_k)}{\sum_{k=0}^{N} \gamma_k^r} - f^* \leq \left( \sum_{k=0}^{N} \gamma_k^r \right)^{-1} \left( \frac{2M_X^2 \gamma_N^{r-1}}{\eta_N} + \frac{(C_F + \eta_0 C_f)^2}{2} \sum_{k=0}^{N} \gamma_k^{r+1} \eta_k \right) + \gamma_0^r (f(\bar{x}_{0,m}) - f(x_0)).
\]

Taking into account the convexity of \(f\) we have
\[
f \left( \frac{\gamma_0^r x_{0,m} + \sum_{k=1}^{N} \gamma_k^r x_{k-1,m+1}}{\sum_{k=0}^{N} \gamma_k^r} \right) \leq \gamma_0^r f(\bar{x}_{0,m}) + \sum_{k=1}^{N} \gamma_k^r f(x_{k-1,m+1}).
\]

Invoking Lemma 2 from the preceding two relations we obtain
\[
f(\bar{x}_{N,m}) - f^* \leq \left( \sum_{k=0}^{N} \gamma_k^r \right)^{-1} \left( \frac{2M_X^2 \gamma_N^{r-1}}{\eta_N} + \frac{(C_F + \eta_0 C_f)^2}{2} \sum_{k=0}^{N} \gamma_k^{r+1} \eta_k \right) + \gamma_0^r f(\bar{x}_{0,m}) - \gamma_0^r f(x_{0,1}).
\]

Adding equation (10a) with the preceding inequality we obtain the desired result.

(b) From equation (8), for an arbitrary \(y \in X\) we have
\[
2\gamma_k^r F(y)^T (x_k - y) \leq \gamma_k^{r-1} \left( \|x_k - y\|^2 - \|x_{k+1} - y\|^2 \right) + 2\gamma_k^r \eta_k (f(y) - f(x_k)) + \gamma_k^{r+1} (C_F + \eta_k C_f)^2.
\]

From the triangle inequality and definition of \(M_f\) we have \(|f(y) - f(x_k)| \leq 2M_f\). We obtain:
\[
2\gamma_k^r F(y)^T (x_k - y) \leq \gamma_k^{r-1} \left( \|x_k - y\|^2 - \|x_{k+1} - y\|^2 \right) + 4\gamma_k^r \eta_k M_f + \gamma_k^{r+1} (C_F + \eta_k C_f)^2. 
\] (17)

Adding and subtracting \(\gamma_k^{r-1} \|x_k - y\|^2\), we have:
\[
2\gamma_k^r F(y)^T (x_k - y) \leq \gamma_k^{r-1} \|x_k - y\|^2 - \gamma_k^{r-1} \|x_{k+1} - y\|^2 + 4\gamma_k^r \eta_k M_f + \left( \gamma_k^{r-1} - \gamma_k^{r-1} \right) \|x_k - y\|^2 + \gamma_k^{r+1} (C_F + \eta_k C_f)^2. 
\] (18)
Using the nonincreasing property of \( \{\gamma_k\} \) and recalling \( 0 \leq r < 1 \), we have \( \gamma^r_k - \gamma_k^r \geq 0 \). Thus, we can write: term 2 \( \leq (\gamma^r_k - \gamma_k^r) \cdot 4M^2_X \). Taking summation over \( k \in [N] \) in equation (18) and dropping a nonpositive term we obtain

\[
2 \sum_{k=1}^{N} \gamma^r_k F(y)^T (x_k - y) \leq \gamma^r_0 \|x_1 - y\|^2 + 4M_f \sum_{k=1}^{N} \gamma^r_k \eta_k + 4M^2_X \left( \gamma^r_N - \gamma^r_0 \right) + (C_F + \eta_0 C_f)^2 \sum_{k=1}^{N} \gamma^r_{k+1}. \tag{19}
\]

Writing equation (17) for \( k = 0 \) and adding and subtracting \( 2\gamma^r_0 F(y)^T x_{0,m} \), we have

\[
2\gamma^r_0 F(y)^T (x_{0,m} - y + x_0 - x_{0,m}) \leq 4\gamma^r_0 \eta_0 M_f + \gamma^r_0 \left( \|x_0 - y\|^2 - \|x_1 - y\|^2 \right) + \gamma^r_{r+1} (C_F + \eta_0 C_f)^2.
\]

Adding the preceding relation with equation (19) we have

\[
2\gamma^r_0 F(y)^T (z_{0,m} - y) + 2 \sum_{k=0}^{N} \gamma^r_k F(y)^T (x_k - y) \leq 4M^2_X \left( \gamma^r_N - \gamma^r_0 \right) + (C_F + \eta_0 C_f)^2 \sum_{k=0}^{N} \gamma^r_{k+1} + \gamma^r_{r+1} \|x_0 - y\|^2 + 4M_f \sum_{k=0}^{N} \gamma^r_k \eta_k + 2\gamma^r_0 F(y)^T (x_{0,m} - x_0).
\]

Using the Cauchy-Schwarz inequality we have: term 3 \( \leq 2\gamma^r_0 C_F \|x_{0,m} - x_{0,1}\| \). We also have \( \|x_0 - y\|^2 \leq 4M^2_X \). Dividing the both sides of the preceding inequality by \( 2 \sum_{k=0}^{N} \gamma^r_k \) and invoking Lemma 2, we have

\[
F(y)^T (\bar{x}_{N,m} - y) \leq \left( \sum_{k=0}^{N} \gamma^r_k \right)^{-1} \left( 2M^2_X \gamma^r_N - 1 \right) + 2M_f \sum_{k=0}^{N} \gamma^r_k \eta_k + \frac{(C_F + \eta_0 C_f)^2}{2} \sum_{k=0}^{N} \gamma^r_{k+1} + \gamma^r_0 C_F \|\bar{x}_{0,m} - x_{0,1}\|.
\]

Taking the supremum on both sides with respect to \( y \) over the set \( \mathbb{X} \) and recalling Definition 2, we have

\[
\text{GAP} (\bar{x}_{N,m}) \leq \left( \sum_{k=0}^{N} \gamma^r_k \right)^{-1} \left( \frac{(C_F + \eta_0 C_f)^2}{2} \sum_{k=0}^{N} \gamma^r_{k+1} + 2M_f \sum_{k=0}^{N} \gamma^r_k \eta_k + 2M^2_X \gamma^r_{N-1} \right) + \gamma^r_0 C_F \|\bar{x}_{0,m} - x_{0,1}\|.
\]

Adding equation (10b) with the preceding inequality we obtain the desired inequality. \( \square \)

In the following we present the main result of this section. We provide non-asymptotic rate statements for each agent \( i \in [m] \) in terms of suboptimality measured by the global objective function, and infeasibility characterized by the dual gap function. We note that
Theorem 1 (Agent-wise rate statements for Algorithm 1). Consider problem (P). Let the averaged sequence \( \{\bar{x}_{k,i}\} \) be generated by agent \( i \in [m] \) using Algorithm 1. Let Assumption 1 hold. Let the stepsize sequence \( \{\gamma_k\} \) and the regularization sequence \( \{\eta_k\} \) be updated using \( \gamma_k := \frac{\gamma_0}{\sqrt{k+1}}, \eta_k := \frac{\eta_0}{(k+1)^r} \), respectively, where \( \gamma_0, \eta_0 > 0 \) and \( 0 < b < 0.5 \). Then the following inequalities hold for all \( i \in [m] \), all \( N \geq 2^\frac{2}{1-r} - 1 \), and all \( r \in [0,1) \):

(a) \( f(\bar{x}_{N,i}) - f^* \leq \frac{2 - r}{(N + 1)^{0.5 - b}} \left( \frac{2M^2}{\gamma_0} + \frac{\gamma_0 (C_F + \eta_0 C_f)^2}{\eta_0 (1 - r + 2b)} \right) + f(\bar{x}_{0,m}) - f(x_{0,1}) + C_F \|\bar{x}_{0,i} - \bar{x}_{0,m}\| + \frac{2(m - i)\gamma_0 C_f (C_F + \eta_0 C_f)}{m(1 - r)} \right).\) 

(b) GAP(\( \bar{x}_{N,i}\)) \leq \frac{2 - r}{(N + 1)^b} \left( \frac{2M^2}{\gamma_0} + \frac{2M_F \eta_0}{1 - 0.5r - b} \right) + C_F \|\bar{x}_{0,m} - x_{0,1}\| + C_F \|\bar{x}_{0,i} - \bar{x}_{0,m}\| + \frac{(C_F + \eta_0 C_f)^2 \gamma_0}{1 - r} + \frac{2(m - i)C_F (C_F + \eta_0 C_f) \gamma_0}{m(1 - r)}.

Proof. (a) Consider the inequality in Proposition 1(a). Substituting \( \gamma_k \) and \( \eta_k \) by their update rules we obtain

\[ f(\bar{x}_{N,i}) - f^* \leq \left( \sum_{k=0}^{N} \frac{\gamma_0^r}{(k + 1)^{0.5b}} \right)^{-1} \left( \frac{2M^2 (N + 1)^{0.5(1-r)+b}}{\gamma_0} \right) \gamma_0^{-r} + \frac{(C_F + \eta_0 C_f)^2}{2} \left( \sum_{k=0}^{N} \frac{\gamma_0^{1+r}}{\eta_0 (k + 1)^{0.5(1-r)+b}} \right) + \frac{(m - i)C_f (C_F + \eta_0 C_f)}{m} \left( \sum_{k=0}^{N} \frac{\gamma_0^{r+1}}{(k + 1)^{0.5(1+r)}} \right) + \frac{\gamma_0^r f(\bar{x}_{0,m}) - \gamma_0^r f(x_{0,1}) + C_f \gamma_0^r \|\bar{x}_{0,i} - \bar{x}_{0,m}\|}{\eta_0} \].

In the next step, to apply Lemma 4 we need to ensure that the conditions in that result are met. From \( 0 \leq r < 1 \) and \( 0 < b < 0.5 \), we have \( 0 \leq 0.5r < 1, \leq 0.5(1 + r) - b < 1, \leq 0.5r + b < 1, \) and \( 0 \leq 0.5(1 + r) < 1 \). Further, from \( N \geq 2^\frac{2}{1-r} - 1 \), \( 0 < b < 0.5 \), and \( 0 \leq r < 1 \) we have that \( N \geq \max \left\{ 2^{1/(1-0.5r)}, 2^{1/(1-0.5(1+r)+b)}, 2^{1/(1-0.5(1+r))} \right\} - 1 \).

Therefore, all the necessary conditions of Lemma 4 are met.

\[ f(\bar{x}_{N,i}) - f^* \leq \left( \frac{\gamma_0^r (N + 1)^{1-0.5b}}{2(1 - 0.5r)} \right)^{-1} \left( \frac{2M^2 (N + 1)^{0.5(1-r)+b}}{\gamma_0} \right) \gamma_0^{-r} + \frac{\gamma_0^{1+r} (C_F + \eta_0 C_f)^2 (N + 1)^{1-0.5(1+r)+b}}{2\eta_0 (1 - 0.5(1 + r) + b)} + \frac{(m - i)C_f (C_F + \eta_0 C_f) \gamma_0^{r+1} (N + 1)^{1-0.5(1+r)}}{m(1 - 0.5(1 + r))} + \frac{\gamma_0^r f(\bar{x}_{0,m}) - \gamma_0^r f(x_{0,1}) + C_f \gamma_0^r \|\bar{x}_{0,i} - \bar{x}_{0,m}\|}{\eta_0} \].
From the preceding relation we obtain
\[ f(\bar{x}_{N,i}) - f^* \leq (2 - r) \left( \frac{2M^2}{\eta_0 \gamma_0 (N + 1)^{0.5 - b}} + \frac{\gamma_0 (C_F + \eta_0 C_f)^2}{2\eta_0 (1 - 0.5(1 + r) + b)(N + 1)^{0.5 - b}} \right) \]
\[ + \frac{(m - i)C_f}{m} \frac{(C_F + \eta_0 C_f) \gamma_0}{m(1 - 0.5(1 + r))(N + 1)^{0.5}} \left( \frac{f(\bar{x}_{0,m}) - f(x_{0,1})}{N + 1})^r + \frac{f(\bar{x}_{0,i}) - f(x_{0,1})}{N + 1})^r - \gamma_0 N, i \right) \frac{R}{\eta_0} \gamma_0 C_F \| \bar{x}_{0,m} - x_{0,1} \| \]
\[ + \frac{(C_F + \eta_0 C_f)^2}{2} \left( \frac{f(\bar{x}_{0,m}) - f(x_{0,1})}{N + 1})^r + \frac{f(\bar{x}_{0,i}) - f(x_{0,1})}{N + 1})^r \right) \frac{R}{\eta_0} \gamma_0 C_F \| \bar{x}_{0,i} - \bar{x}_{0,m} \| \]
\[ + \frac{\gamma_0^r C_F}{m} \| \bar{x}_{0,i} - \bar{x}_{0,m} \| + \left( \frac{(C_F + \eta_0 C_f)^2}{2} + \frac{(m - i)C_f}{m} \right) \frac{R}{\eta_0} \gamma_0 C_F \| \bar{x}_{0,m} - x_{0,1} \| \]
\[ + \frac{\gamma_0^r C_F}{m} \| \bar{x}_{0,i} - \bar{x}_{0,m} \| \frac{R}{\eta_0} \gamma_0 C_F \| \bar{x}_{0,m} - x_{0,1} \| \]
Utilizing the bounds in Lemma \[4\] we obtain
\[ \text{GAP}(\bar{x}_{N,i}) \leq \left( \sum_{k=0}^{N} \frac{\gamma_0^r}{(k + 1)^{0.5}} \right)^{-1} \left( \frac{2M^2}{\eta_0 \gamma_0 (N + 1)^{0.5(1 - r)}} \gamma_0^{1-r} \right) \]
\[ + \frac{2M^2}{\eta_0 \gamma_0 (N + 1)^{0.5(1 - r)}} \gamma_0^{1-r} \left( \frac{R}{\eta_0} \gamma_0 C_F \| \bar{x}_{0,m} - x_{0,1} \| + \| \bar{x}_{0,i} - \bar{x}_{0,m} \| \right) \]
\[ + \frac{(C_F + \eta_0 C_f)^2}{2} \left( \frac{R}{\eta_0} \gamma_0 C_F \| \bar{x}_{0,m} - x_{0,1} \| \right) + \frac{\gamma_0^r C_F}{m} \| \bar{x}_{0,i} - \bar{x}_{0,m} \| \left( \frac{R}{\eta_0} \gamma_0 C_F \| \bar{x}_{0,m} - x_{0,1} \| \right) \]
\[ + \frac{\gamma_0^{1-r}}{m} \left( \frac{R}{\eta_0} \gamma_0 C_F \| \bar{x}_{0,m} - x_{0,1} \| \right) \frac{R}{\eta_0} \gamma_0 C_F \| \bar{x}_{0,i} - \bar{x}_{0,m} \| \]
Note that from $b < 0.5$ and $0 ≤ r < 1$ we have $1 - 0.5r ≥ b$. Hence equation (21) holds.

**Remark 3** (Iteration complexity of Algorithm 1). Consider the rate results presented by relations (20) and (21). Let us choose $r := 0$ and suppose $γ_k := \frac{(C_F + C_f)^{-1}}{\sqrt{k+1}}$ and $η_k := \frac{1}{\sqrt{k+1}}$ for $k ≥ 0$. Let $ε > 0$ be an arbitrary small scalar such that $f(\bar{x}_{N,i}) - f^* + \text{GAP} (\bar{x}_{N,i}) < ε$ for all $i ∈ [m]$. Then, we obtain the iteration complexity of $N_ε = O((C_F + C_f)^4ε^{-4})$ for each agent. Interestingly, this iteration complexity matches the complexity of the proposed method in our earlier work [34] for addressing formulation (P) in a centralized regime where the information of the objective function $f$ is globally known. Importantly, this indicates that there is no sacrifice in the iteration complexity in addressing the distributed formulation (P). Another important observation to make is that the iteration complexity of the proposed distributed method is independent of the number of agents $m$.

**IV. Rate analysis in the solution space**

In this section we study the convergence rate properties of the proposed method in the solution space. To this end, we compare the sequences generated from Algorithm 1 with the Tikhonov trajectory (formally introduced in Definition 3). Throughout this section, we make the following additional assumption.

**Assumption 2.** Consider problem (P). For all $i ∈ [m]$ let the component function $f_i : \mathbb{R}^n → \mathbb{R}$ be continuously differentiable and $μ_{f_i}$—strongly convex over the set $X$.

Note that under this assumption, the equation (5) in Algorithm 1 can be written as

$$x_{k,i+1} := \mathcal{P}_X (x_{k,i} - γ_k (F_i(x_{k,i}) + η_k \nabla f_i(x_{k,i}))).$$

(22)

Next, we comment on the strong convexity parameter of the global objective function.

**Remark 4.** Under Assumption 2 we note that any function $f_i$ is strongly convex with a parameter $μ_{\text{min}} \triangleq \min_{i∈[m]} μ_{f_i}$. This also implies that the global function $f(x) ≜ \sum_{i=1}^{m} f_i(x)$ is $mμ_{\text{min}}$—strongly convex. Another implication is that under Assumption (b), (c), and Assumption 2 problem (P) has a unique optimal solution. Throughout this section, we denote the unique optimal solution of (P) by $x^*$. 

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A. Preliminaries

Here we provide some preliminary results that will be used later. We first introduce the notion of Tikhonov trajectory.

**Definition 3** (Tikhonov trajectory). Let \( \{\eta_k\} \subseteq \mathbb{R}^n_{++} \) be a given sequence. Consider a class of regularized VI problems for \( k \geq 0 \) given by

\[
\text{VI} \left( X, \sum_{i=1}^{m} (F_i + \eta_k \nabla f_i) \right).
\]

Let \( x_{\eta_k}^* \) denote the unique solution to (23). The sequence \( \{x_{\eta_k}^*\} \) is called the Tikhonov trajectory associated with problem (P).

In the following, we establish the convergence of the Tikhonov trajectory to the unique optimal solution of problem (P). This result will be used later in Theorem 2.

**Lemma 7** (Properties of the Tikhonov trajectory). Consider problem (P) and Definition 3.

Let Assumption 1(b), (c), and Assumption 2 hold. Then:

(a) Let \( \{\eta_k\} \) be a strictly positive sequence such that \( \lim_{k \to \infty} \eta_k = 0 \). Then, \( \lim_{k \to \infty} x_{\eta_k}^* \) exists and is equal to \( x^* \).

(b) For any two nonnegative integers \( k_1 \) and \( k_2 \), we have

\[
\|x_{\eta_{k_2}}^* - x_{\eta_{k_1}}^*\| \leq C_{f} \mu_{\min} \left(1 - \frac{\eta_{k_2}}{\eta_{k_1}}\right).
\]

Proof. The proof can be done in a similar fashion to the proof of Lemma 4.5 in [34].

Next we obtain a recursive bound on an error metric that is characterized by the sequence \( \{x_k\} \) in Definition 1 and the Tikhonov trajectory. This result will be utilized in Theorem 2.

**Lemma 8.** Let \( \{x_k\} \) be given by Definition 4. Let Assumption 1(b), (c), and Assumption 2 hold. Suppose \( \{\gamma_k\} \) and \( \{\eta_k\} \) are strictly positive and nonincreasing such that \( \gamma_0 \eta_0 \mu_{\min} \leq 0.5 \). Then for any \( k \geq 1 \) we have

\[
\|x_{k+1}^* - x_{\eta_k}^*\|^2 \leq (1 - \gamma_k \eta_k \mu_{\min}) \|x_k - x_{\eta_{k-1}}^*\|^2 + \frac{1.5 C_f^2}{m^2 \gamma_k \eta_k \mu_{\min}^3} \left(1 - \frac{\eta_k}{\eta_{k-1}}\right)^2 + \gamma_k^2 \left(C_F + \eta_k C_f\right)^2.
\]

Proof. From Algorithm 1, the nonexpansivity of the projection, and \( x_{\eta_k}^* \in X \), for any \( i \in [m] \) and \( k \geq 1 \) we have

\[
\|x_{k+1,j} - x_{\eta_k}^*\|^2 \leq \|x_{k,j} - x_{\eta_k}^*\|^2 + \gamma_k^2 \|F_i(x_{k,j}) + \eta_k \nabla f_i(x_{k,j})\|^2 - 2 \gamma_k (F_i(x_{k,j}) + \eta_k \nabla f_i(x_{k,j}))^T (x_{k,j} - x_{\eta_k}^*).
\]
From the definition of $C_F$ and $C_f$ we have
\[
\|x_{k,i+1} - x^*_{k,i}\|^2 \leq \|x_{k,i} - x^*_{k,i}\|^2 + \gamma_k^2 \left( \frac{C_F + \eta_k C_f}{m} \right)^2 + 2\gamma_k \left( F_i(x_{k,i}) + \eta_k \nabla f_i(x_{k,i}) \right)^T \left( x^*_{k,i} - x_{k,i} \right).
\]

From the strong monotonicity of $\nabla f_i$ and the monotonicity of $F_i$ we can write
\[
2\gamma_k \left( F_i(x_{k,i}) + \eta_k \nabla f_i(x_{k,i}) \right)^T \left( x^*_{k,i} - x_{k,i} \right) \leq 2\gamma_k F_i \left( x^*_{k,i} \right)^T \left( x^*_{k,i} - x_{k,i} \right)
+ 2\gamma_k \eta_k \nabla f_i \left( x^*_{k,i} \right)^T \left( x^*_{k,i} - x_{k,i} \right) - \mu_{\text{min}} \| x^*_{k,i} - x_{k,i} \|^2.
\]

From the preceding two relations we obtain
\[
\|x_{k,i+1} - x^*_{k,i}\|^2 \leq (1 - 2\gamma_k \eta_k \mu_{\text{min}}) \|x_{k,i} - x^*_{k,i}\|^2 + \gamma_k^2 \left( \frac{C_F + \eta_k C_f}{m} \right)^2 + 2\gamma_k F_i \left( x^*_{k,i} \right)^T \left( x^*_{k,i} - x_{k,i} \right)
+ 2\gamma_k \eta_k \nabla f_i \left( x^*_{k,i} \right)^T \left( x^*_{k,i} - x_{k,i} \right).
\]

Adding and subtracting $2\gamma_k F_i \left( x^*_{k,i} \right)^T x_k + 2\gamma_k \eta_k \nabla f_i \left( x^*_{k,i} \right)^T x_k$ in the previous relation we get
\[
\|x_{k,i+1} - x^*_{k,i}\|^2 \leq (1 - 2\gamma_k \eta_k \mu_{\text{min}}) \|x_{k,i} - x^*_{k,i}\|^2 + \gamma_k^2 \left( \frac{C_F + \eta_k C_f}{m} \right)^2 + 2\gamma_k F_i \left( x^*_{k,i} \right)^T \left( x^*_{k,i} - x_{k,i} \right)
+ 2\gamma_k \left( \eta_k \nabla f_i \left( x^*_{k,i} \right)^T \left( x^*_{k,i} - x_{k,i} \right) + \left| F_i \left( x^*_{k,i} \right)^T \left( x_k - x_{k,i} \right) \right| \right)
+ 2\gamma_k \eta_k \nabla f_i \left( x^*_{k,i} \right)^T \left( x_k - x_{k,i} \right).
\]

Employing the Cauchy-Schwarz inequality we obtain
\[
\|x_{k,i+1} - x^*_{k,i}\|^2 \leq (1 - 2\gamma_k \eta_k \mu_{\text{min}}) \|x_{k,i} - x^*_{k,i}\|^2 + \gamma_k^2 \left( \frac{C_F + \eta_k C_f}{m} \right)^2 + 2\gamma_k F_i \left( x^*_{k,i} \right)^T \left( x^*_{k,i} - x_{k,i} \right)
+ 2\gamma_k \eta_k \nabla f_i \left( x^*_{k,i} \right)^T \left( x^*_{k,i} - x_{k,i} \right) + \frac{2\gamma_k (C_F + \eta_k C_f)}{m} \| x_k - x_{k,i} \|^2.
\]

Next we take summations over $i \in [m]$ from both sides. Recall $f(x) \triangleq \sum_{i=1}^m f_i(x)$ and $F(x) \triangleq \sum_{i=1}^m F_i(x)$. Using Definition 1 for $x_{k,1}$ and recalling $1 - 2\gamma_k \eta_k \mu_{\text{min}} < 1$ we have
\[
\sum_{i=1}^m \|x_{k,i+1} - x^*_{k,i}\|^2 \leq (1 - 2\gamma_k \eta_k \mu_{\text{min}}) \|x_k - x^*_{k,i}\|^2 + \sum_{i=2}^m \|x_{k,i} - x^*_{k,i}\|^2 + \gamma_k^2 \left( \frac{C_F + \eta_k C_f}{m} \right)^2
+ \frac{2\gamma_k (C_F + \eta_k C_f)}{m} \sum_{i=1}^m \| x_k - x_{k,i} \| + 2\gamma_k F \left( x^*_{k,i} \right)^T \left( x^*_{k,i} - x_k \right) + 2\gamma_k \eta_k \nabla f \left( x^*_{k,i} \right)^T \left( x^*_{k,i} - x_k \right).
\]

(25)

From Lemma 3, $\|x_k - x_{k,i}\| \leq (i - 1) \gamma_k \left( C_F + \eta_k C_f \right) / m$ for all $i \in [m]$. Invoking this relation and Definition 1 we obtain
\[
\|x_{k+1} - x^*_{k}\|^2 \leq (1 - 2\gamma_k \eta_k \mu_{\text{min}}) \|x_k - x^*_{k}\|^2 + \gamma_k^2 \left( C_F + \eta_k C_f \right)^2
+ 2\gamma_k \left( F \left( x^*_{k} \right) + \eta_k \nabla f \left( x^*_{k} \right) \right)^T \left( x^*_{k} - x_k \right).
\]
From Definition $3$, $x^*_{\eta_k}$ is the solution to problem $(23)$. Recalling $x_k \in X$, we have: term 1 $\leq 0$. We obtain
\[
\|x_{k+1} - x^*_{\eta_k}\|^2 \leq (1 - 2\gamma_k \eta_k \mu_{\text{min}}) \|x_k - x^*_{\eta_k}\|^2 + \gamma_k^2 (C_F + \eta_k C_f)^2. \tag{26}
\]
Next, consider the term $\|x_k - x^*_{\eta_k}\|^2$ as follows.
\[
\|x_k - x^*_{\eta_k}\|^2 = \|x_k - x^*_{\eta_{k-1}}\|^2 + \|x^*_{\eta_{k-1}} - x^*_{\eta_k}\|^2 + 2 \left( x_k - x^*_{\eta_{k-1}} \right)^T \left( x^*_{\eta_{k-1}} - x^*_{\eta_k} \right). \tag{27}
\]
Next, we bound term 2 by recalling $2a^T b \leq \|a\|^2/\alpha + a \|b\|^2$ where $a, b \in \mathbb{R}^n$ and $\alpha > 0$. For $\alpha := 1/\gamma_k \eta_k \mu_{\text{min}}$, bounding term 2 in the preceding inequality we obtain
\[
\|x_k - x^*_{\eta_k}\|^2 = (1 + \gamma_k \eta_k \mu_{\text{min}}) \|x_k - x^*_{\eta_{k-1}}\|^2 + \left( 1 + \frac{1}{\gamma_k \eta_k \mu_{\text{min}}} \right) \|x^*_{\eta_{k-1}} - x^*_{\eta_k}\|^2.
\]
From Lemma $7(b)$ we obtain
\[
\|x_k - x^*_{\eta_k}\|^2 \leq (1 + \gamma_k \eta_k \mu_{\text{min}}) \|x_k - x^*_{\eta_{k-1}}\|^2 + \left( 1 + \frac{1}{\gamma_k \eta_k \mu_{\text{min}}} \right) \frac{C_f^2}{\mu_{\text{min}}^2} \left( 1 - \frac{\eta_k}{\eta_{k-1}} \right)^2. \tag{27}
\]
From equations $(26)$ and $(27)$ we have
\[
\|x_{k+1} - x^*_{\eta_k}\|^2 \leq (1 - 2\gamma_k \eta_k \mu_{\text{min}}) (1 + \gamma_k \eta_k \mu_{\text{min}}) \|x_k - x^*_{\eta_{k-1}}\|^2 + \left( 1 + \frac{1}{\gamma_k \eta_k \mu_{\text{min}}} \right) \frac{C_f^2}{\mu_{\text{min}}^2} \left( 1 - \frac{\eta_k}{\eta_{k-1}} \right)^2 + \gamma_k^2 (C_F + \eta_k C_f)^2.
\]
Using $0 < \gamma_k \eta_k \mu_{\text{min}} \leq 0.5$ we have the desired result.

\[\square\]

B. Convergence analysis

In this section, our goal is to derive a non-asymptotic convergence rate statement that relates the generated sequences by Algorithm $1$ to the Tikhonov trajectory. We begin with providing a class of sequences for the stepsize and the regularization parameter and prove some properties for them that will be used in the analysis.

Definition 4 (Stepsize and regularization parameter). Let $\gamma_k := \frac{\gamma}{(k+1)^{\alpha}}$ and $\eta_k := \frac{\eta}{(k+1)^{\beta}}$ for all $k \geq 0$ where $\gamma, \eta, \Gamma, a$ and $b$ are strictly positive scalars. Let $a > b$, $a + b < 1$, and $3a + b < 2$. Assume that $\Gamma \geq 1$ and it is sufficiently large such that $\Gamma^a + b \geq 2 \gamma \eta \mu_{\text{min}}$ and $\Gamma^{1-a-b} \geq \frac{4}{\gamma \eta \mu_{\text{min}}}$.

Lemma 9. Consider Definition $4$. The following results hold.

(i) $\{\gamma_k\}$ and $\{\eta_k\}$ are strictly positive and nonincreasing such that $\gamma_0 \eta_0 \mu_{\text{min}} \leq 0.5$. 

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(ii) For all integers \( k_1 \) and \( k_2 \) such that \( k_2 \geq k_1 \geq 0 \) we have \( 1 - \frac{\eta_{k_2}}{\eta_{k_1}} \leq \frac{k_2 - k_1}{k_2 + 1} \).

(iii) For all \( k \geq 1 \) we have \( \frac{1}{\tau k} \left( 1 - \frac{\eta_{k}}{\eta_{k-1}} \right)^2 \leq \frac{1}{\gamma \eta (2 - 3a - \beta)} \).

(iv) For all \( k \geq 1 \) we have \( \frac{\eta_{k-1}}{\eta_{k}} \leq \frac{\eta_{k}}{\eta_{k}} (1 + 0.5 \gamma \eta \mu_{\min}) \).

Proof. See Appendix C.

The main contribution in this section is presented by Theorem 2 where we derive agent-specific rates relating the sequences generated by Algorithm 1 with the Tikhonov trajectory.

**Theorem 2** (Comparison with the Tikhonov trajectory). Consider problem \( \mathbb{P} \). Let Assumptions 1(b), (c), and Assumption 2 hold. Consider \( \{x_k\} \) and \( \{x^*_k\} \) given in Definitions 1 and 3 respectively. Let the stepsize sequence \( \{\gamma_k\} \) and the regularization sequence \( \{\eta_k\} \) be given by Definition 4. Then for all \( k \geq 0 \) and all \( i \in [m] \) we have

\[
\|x_{k+1,i} - x^*_i\|^2 \leq \frac{2(i - 1)^2 (C_F + \eta_0 C_f)^2 \gamma^2}{m^2 (k + 1)^2} + \frac{2\tau B_0 \gamma}{\mu_{\min} \eta (k + 1)^{a - 5}},
\]

where \( \tau \triangleq \max \left\{ \mu_{\min} \eta \gamma^{-1} B_0^{-1} \Gamma^{a-b} \|x_1 - x^*_0\|^2, 2 \right\} \) and \( B_0 \triangleq \frac{1.5 C_f^2 \gamma_k^2}{m^2 \mu_{\min}^2 \eta (2 - 3a - \beta)} + (C_F + \eta_0 C_f)^2 \).

Proof. Consider (24). From Lemma 9 for \( k \geq 1 \) we have

\[
\|x_{k+1} - x^*_k\|^2 \leq (1 - \gamma \eta \mu_{\min}) \|x_k - x^*_k\|^2 + \frac{1.5 C_f^2 \gamma_k^2}{m^2 \mu_{\min}^2 \eta (2 - 3a - \beta)} + \gamma_k^2 (C_F + \eta_0 C_f)^2.
\]

Let us define the terms \( v_k \triangleq \|x_k - x^*_k\|^2, \alpha_k \triangleq \gamma \eta \mu_{\min}, \) and \( \beta_k \triangleq B_0 \gamma_k^2 \) for \( k \geq 1 \). Therefore, for all \( k \geq 1 \) we have

\[
v_{k+1} \leq (1 - \alpha_k) v_k + \beta_k. \tag{28}
\]

From Lemma 9(iii), for all \( k \geq 1 \) we have

\[
\frac{\beta_{k-1}}{\alpha_{k-1}} \leq \frac{B_0 \gamma_k}{\mu_{\min} \eta_k} (1 + 0.5 \gamma \eta \mu_{\min}) = \frac{\beta_k}{\alpha_k} (1 + 0.5 \alpha_k). \tag{29}
\]

Next, we show that \( v_{k+1} \leq \tau \frac{\beta_k}{\alpha_k} \) for all \( k \geq 0 \). We apply induction on \( k \geq 0 \). Note that this relation holds for \( k := 0 \) as an implication of the definition of \( \tau \). Suppose \( v_k \leq \tau \frac{\beta_{k-1}}{\alpha_{k-1}} \) holds for some \( k \geq 1 \). From (28) we obtain \( v_{k+1} \leq (1 - \alpha_k) \frac{\beta_{k-1}}{\alpha_{k-1}} + \beta_k \). Using the upper bound for the right-hand side given by (29) we have

\[
v_{k+1} \leq \tau (1 - \alpha_k)(1 + 0.5 \alpha_k) \frac{\beta_k}{\alpha_k} + \beta_k = \tau (1 - \alpha_k + 0.5 \alpha_k - 0.5 \alpha_k^2) \frac{\beta_k}{\alpha_k} + \beta_k
\]

\[
= \tau \frac{\beta_k}{\alpha_k} - \tau (1 - 0.5) \beta_k - 0.5 \tau \alpha_k \beta_k + \beta_k \leq \tau \frac{\beta_k}{\alpha_k} + (1 - 0.5 \tau) \beta_k.
\]
From the definition of \( \tau \), \( \tau \geq 2 \) implying that \( 1 - 0.5\tau \leq 0 \). This completes the proof of induction. Recall from Lemma 3 that we have \( \|x_k - x_{k,i}\| \leq (i - 1)\gamma_k (C_F + \eta_k C_f) / m \) for any \( i \in [m] \). For all \( k \geq 0 \) and \( i \in [m] \) we have

\[
\|x_{k+1,i} - x^*_{\eta_k}\|^2 \leq 2 \|x_{k+1,i} - x_{k+1}\|^2 + 2 \|x_{k+1} - x^*_{\eta_k}\|^2 \\
\leq \frac{2(i - 1)^2 (C_F + \eta_0 C_f)^2 \gamma_k^2}{m^2} + \frac{2\tau B_0 \gamma_k}{\mu_{\min} \eta_k} = \frac{2(i - 1)^2 (C_F + \eta_0 C_f)^2 \gamma^2}{m^2 (k + \Gamma + 1)^2a} + \frac{2\tau B_0 \gamma}{\mu_{\min} \eta (k + \Gamma)^a - b}.
\]

Hence the proof is completed.

\[\square\]

V. Numerical results

In this section we present the implementation results of Algorithm 1 in addressing two of the motivating examples discussed earlier in section I. These include a traffic equilibrium problem and a soft-margin support vector classification problem. We also provide comparisons between the performance of the proposed method with that of other standard IG methods.

(i) A traffic equilibrium problem: For an illustrative example, we consider the transportation network in [6]. We first describe the network and present the NCP formulation. Then, we implement Algorithm 1 to solve model (2) and compute the best equilibrium.

Consider a transportation network with the set of nodes \( \{n_1, n_2\} \) and the set of directed arcs \( \{a_1, a_2, a_3, a_4, a_5\} \). As shown in Figure 1 arcs \( a_1, a_2, \) and \( a_3 \) are directed from node \( n_1 \) to node \( n_2 \), and arcs \( a_4 \) and \( a_5 \) are directed in the reverse way. Note that \( a_1 \) and \( a_4 \) construct a two-way road. The same holds for \( a_2 \) and \( a_5 \). We let \( d \triangleq [d_1, d_2]^T \) denote the expected travel demand vector where \( d_1 \) and \( d_2 \) correspond to the demand from \( n_1 \) to \( n_2 \), and from \( n_2 \) to \( n_1 \), respectively. Let the vector \( h \triangleq [h_1, \ldots, h_5]^T \) denote the traffic flow on the arcs. The travel cost on each arc is assumed to be a linear function in terms of \( h \). More precisely, the travel cost on arc \( i \) is equal to \( [Ch + q]_i \) where we let the cost matrix \( C \in \mathbb{R}^{5 \times 5} \) and vector \( q \in \mathbb{R}^5 \) be given by

![Fig. 1: A transportation network with 2 nodes and 5 arcs](image-url)
We note that the matrix $C$ is positive semidefinite. The diagonal values of $C$ are rounded by two decimal places for the ease of presentation. Intuitively speaking, the structure of $C$ implies that the cost of each arc in a two-way road depends on the flows on the both directions. Let $u \triangleq [u_1, u_2]^T$ denote the (unknown) vector of minimum travel costs between the origin-destination (OD) pairs, i.e., $u_1$ denotes the minimum travel cost from $n_1$ to $n_2$, and $u_2$ denotes the minimum travel cost from $n_2$ to $n_1$. The Wardrop user equilibrium principle represents the path choice behavior of the users based on the following rationale: (i) For any OD pair among all possible arcs, users tend to choose the arc(s) with the minimum cost. (ii) For any OD pair, the arc(s) that have the minimum cost will have positive flows and will have equal costs. (iii) For any OD pair, arcs with higher costs than the minimum value will have no flows. Mathematically the Wardrop’s principle can be characterized as

$$0 \leq Ch + q - B^T u \perp h \geq 0, \quad 0 \leq Bh - d \perp u \geq 0,$$

(31)
where \( B \in \mathbb{R}^{2 \times 5} \) denotes the (OD pair, arc)-incidence matrix given as

\[
B := \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}.
\]

Throughout, we assume that the demand vector \( d \) and the cost vector \( q \) are subject to uncertainties. Let us define decision vector \( x \in \mathbb{R}^7 \), random variable \( \xi \in \mathbb{R}^{10} \), and stochastic mapping \( F(\bullet, \xi) : \mathbb{R}^7 \to \mathbb{R}^7 \) as

\[
x \triangleq \begin{bmatrix} h \\ u \end{bmatrix}, \quad \xi \triangleq \begin{bmatrix} \tilde{d} \\ \tilde{q} \end{bmatrix}, \quad F(x, \xi) \triangleq \begin{bmatrix} C & -B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix} + \begin{bmatrix} \tilde{q} \\ -\tilde{d} \end{bmatrix}.
\]

Then from section I, the Wardrop equation (31) can be characterized as

\[
\text{VI}(\mathbb{R}^7_+, \mathbb{E}[F(\bullet, \xi)])
\]

Notably due to positive semidefinite property of \( C \), the mapping \( \mathbb{E}[F(\bullet, \xi)] \) is merely monotone. Consequently, the aforementioned VI may have multiple equilibria. Among them, we seek to find the best equilibrium with respect to a welfare function \( f \) defined as the expected total travel time over the network by all users, i.e., \( f(x) \triangleq \mathbb{E}\left[(Ch + \tilde{q})^T1_5\right] \) where \( 1_5 \in \mathbb{R}^n \) denotes a vector with all unit elements.

**Set-up:** For this experiment, we assume that \( \tilde{d}_1 \sim \mathcal{N}(210, 10) \), \( \tilde{d}_2 \sim \mathcal{N}(120, 10) \). Also for \( i = 1, \ldots, 5 \) we let \( \tilde{q}_i \) be normally distributed with the mean equal to \( q_i \) and the standard deviation of 300, where the vector \( q \) is given by (30). Following the formulation (2) we generate 1000 samples for each parameter and distribute the data equally among 10 agents. We let \( \gamma_k := \frac{\gamma_0}{\sqrt{k+1}} \) and \( \eta_k := \frac{\eta_0}{(k+1)^{\alpha_2}} \) and consider different values for the initial stepsize \( \gamma_0 \) and the initial regularization parameter \( \eta_0 \). The results are as shown in Figure 2. We use standard averaging by assuming that \( r = 0 \). Notably for quantifying the infeasibility, we consider the metric

\[
\phi(x) \triangleq \|\max\{0, -x\}\|^2 + \|\max\{0, -F(x)\}\|^2 + |x^T F(x)|,
\]

where \( F(x) \triangleq \sum_{i=1}^m F_i(x) \) and \( F_i(x) \triangleq \sum_{\ell \in S_i} F(x, \xi_\ell) \). Note that \( \phi(x) = 0 \) if and only if \( 0 \leq x \perp F(x) \geq 0 \). We choose this metric over the dual gap function employed earlier in the analysis because in this particular example, the dual gap function becomes infinity at some of the evaluations of the generated iterates. This is due to the unboundedness of the set \( X := \mathbb{R}_+^n \). Unlike the dual gap function, \( \phi(x) \) stays bounded and is more suitable to plot.

**Insights:** In Figure 2 we observe that in all four different settings the infeasibility metric decreases as the algorithm proceeds. This indeed implies that the generated iterates by the agents tend to satisfy the NCP constraints with an increasing accuracy. In terms of the suboptimality metric we observe that the each agent’s objective value becomes more and more stable over time. Intuitively this implies that the agents asymptotically reach to an equilibrium. We should note that although the function \( f \) is minimized, it is minimized only over the set of equilibria. The fact that the objective values in Figure 2 are not necessarily
The decreasing is mainly because of the impact of feasibility violation of the iterates with respect to the NCP constraints throughout the implementations. As evidenced, pair-IG performs with much robustness to the choice of the initial values of $\gamma_0$ and $\eta_0$.

### (ii) Distributed support vector classification

Consider a distributed soft-margin SVM problem described as follows. Let a dataset be given as $D \triangleq \{(u_j, v_j) \mid u_j \in \mathbb{R}^n, v_j \in \{-1, +1\}, \text{ for } j \in S\}$ where $u_j$ and $v_j$ denote the attributes and the binary label of the $j^{th}$ data point, respectively, and $S$ denotes the index set. Let the data be distributed among $m$ agents by defining $D_i$ such that $\cup_{i=1}^{m} D_i = D$. Let $S_i$ denote the index set corresponding to agent $i$ such that $\sum_{i=1}^{m} |S_i| = |S|$. Then given $\lambda > 0$, the distributed SVM problem is given as

$$\begin{align*}
\text{minimize}_{w, b, z} & \quad \sum_{i=1}^{m} \left( \frac{1}{2m} \|w\|^2 + \frac{1}{\lambda} \sum_{j \in S_i} z_j \right) \\
\text{subject to} & \quad v_j(w^T u_j + b) \geq 1 - z_j, \text{ for } j \in S_i \text{ and } i \in [m], \\
& \quad z_j \geq 0, \quad \text{for } j \in S_i \text{ and } i \in [m].
\end{align*}$$

(32)

To solve problem (32) using Algorithm 1, we apply Lemma 1 by casting (32) as model (3). We also implement some of the well-known existing IG schemes, namely projected IG, proximal IAG, and SAGA. Unlike Algorithm 1, these schemes require a projection step onto the constraint set. To compute the projections we use the Gurobi-Python solver.

**Set-up:** We consider 20 agents and assume that $\lambda := 10$. We let $\gamma_k := \frac{1}{\sqrt{k+1}}$ and $\eta_k := \frac{1}{(k+1)^{0.25}}$ in Algorithm 1. We use identical initial stepsizes in all the methods. We provide the comparisons with respect to the runtime and report the performance of each scheme over 200 seconds. We
use a synthetic dataset with different values for $n$ and $|S|$. The suboptimality is characterized in terms of the global objective and the infeasibility is measured by quantifying the violation of constraints of problem (32) aligned with ideas in Lemma 1.

**Insights:** In Figure 3, we observe that with an increase in the dimension of the solution space, i.e., $n$, or the size of the training dataset, i.e., $|S|$, the projection evaluations in the standard IG schemes take longer and consequently, the performance of the IG methods is deteriorated in large scale settings. However, utilizing the reformulation in Lemma 1, the proposed method does not require any projection operations for addressing problem (32). As such, the performance of Algorithm 1 does not get affected severely with the increase in $n$ or $|S|$. Note that in Figure 3, the reason that the IG schemes do not show any updates for $|S| = 200$ and $|S| = 500$ beyond a time threshold is because of the interruption in their last update when the method reaches the 200 seconds time limit.

**VI. Conclusion**

We introduce a new unifying formulation for distributed constrained optimization where the constraint set is characterized as the solution set of a merely monotone variational inequality problem. We show that the proposed formulation can capture several important classes of multi-agent constrained optimization problems including those subject to complementarity, nonlinear equality, or equilibrium constraints. We develop an iteratively regularized incremental gradient method where at each iteration agents communicate over a cycle graph to update their iterates using their local information about the objective function and the mapping. We derive new iteration complexity bounds in terms of the global objective function and a suitably defined infeasibility metric. To analyze the convergence in the solution space, we also provide non-asymptotic agent-wise convergence rate statements that relate the iterate of each agent with that of the Tikhonov trajectory. We validate the theoretical results on an illustrative transportation network problem and also, on a distributed support vector classification problem. A potential, yet challenging, future direction to our research is to extend the current algorithm to address distributed optimization problems with VI constraints requiring weaker assumptions on the network and with possibly asynchronous communications.
Appendix

A. Proof of Lemma 1

For each $i \in [m]$, let us define function $\Theta_i : \mathbb{R}^n \to \mathbb{R}$ as $\Theta_i(x) \triangleq \frac{1}{2} \| A_i x - b_i \|^2 + \frac{1}{2} \sum_{j=1}^{n} (\max\{0, g_{i,j}(x)\})^2$. Note that $\frac{1}{2} \| A_i x - b_i \|^2$ is a continuously differentiable and convex function for all $i$. Also, for all $i$ and $j$, the function $\frac{1}{2} (\max\{0, g_{i,j}(x)\})^2$ is continuously differentiable with the gradient map of $\max\{0, g_{i,j}(x)\} \nabla g_{i,j}(x)$ (see page 380 in [19]). Thus, we have that $\nabla \Theta_i(x) = F_i(x)$ where $F_i(x)$ is given in the statement of Lemma 1. Next, we show that $F_i$ is a monotone mapping. From the convexity of $g_{i,j}(x)$, the function $\max\{0, g_{i,j}(x)\}$ is convex. Now, note that the function $\frac{1}{2} (\max\{0, g_{i,j}(x)\})^2$ can be viewed as the composition of the nondecreasing function $h(y) \triangleq \frac{1}{2} y^2$ for $y \in \mathbb{R}_+$ and the convex function $\max\{0, g_{i,j}(x)\}$. Thus, $\frac{1}{2} (\max\{0, g_{i,j}(x)\})^2$ is convex. This implies that $\Theta_i$ is a convex function and consequently, its gradient mapping that is $F_i(x)$, is monotone. Recalling the first-order optimality conditions for the convex optimization problems and taking into account the definition of $\text{SOL}(X, \sum_{i=1}^{m} F_i)$, we have that $\text{SOL}(X, \sum_{i=1}^{m} F_i) = \arg\min_{x \in X} \sum_{i=1}^{m} \Theta_i(x)$. Let $\mathcal{Z}$ denote the feasible set of problem (3). To complete the proof, it suffices to show that $\mathcal{Z} = \{ x \in X \ \mid \ \sum_{i=1}^{m} \Theta_i(x) = 0 \}$ is nonempty and that $\sum_{i=1}^{m} \Theta_i(x) \geq 0$ for all $x \in X$, we conclude that $x \in \arg\min_{x \in X} \sum_{i=1}^{m} \Theta_i(x)$. Thus, we showed that $\mathcal{Z} \subseteq \arg\min_{x \in X} \sum_{i=1}^{m} \Theta_i(x)$. Now, consider an arbitrary $\bar{x} \in \arg\min_{x \in X} \sum_{i=1}^{m} \Theta_i(x)$. Thus, $\bar{x} \in X$. Also, the assumption that the feasible set of problem (3) is nonempty implies that there exits $x_0 \in X$ such that $A_i x_0 = b_i$, $g_{i,j}(x_0) \leq 0$ for all $i \in [m]$ and $j \in [n]$. Thus, we have $\sum_{i=1}^{m} \Theta_i(x_0) = 0$. From the nonnegativity of the function $\sum_{i=1}^{m} \Theta_i(x)$ and that $\bar{x} \in \arg\min_{x \in X} \sum_{i=1}^{m} \Theta_i(x)$, we must have $\sum_{i=1}^{m} \Theta_i(\bar{x}) = 0$. Therefore, we obtain $A_i \bar{x} = b_i$, $g_{i,j}(\bar{x}) \leq 0$ for all $i \in [m]$ and $j \in [n]$. Thus, we have shown that $\arg\min_{x \in X} \sum_{i=1}^{m} \Theta_i(x) \subseteq \mathcal{Z}$. Hence, we have $\mathcal{Z} = \text{SOL}(X, \sum_{i=1}^{m} F_i)$ and the proof is completed.

B. Proof of Lemma 2

Using $\beta \in [0,1)$ and $\Gamma \geq 1$ we can write

$$
\sum_{k=0}^{K} \frac{1}{(k + \Gamma)\beta} \leq \int_{-1}^{K} \frac{dx}{(x + \Gamma)^\beta} = \frac{(K + \Gamma)^\beta - (\Gamma - 1)^\beta}{1 - \beta} \leq \frac{(K + \Gamma)^{1-\beta}}{1 - \beta}.
$$
We can also write
\[
\sum_{k=0}^{K} \frac{1}{(k + \Gamma)\beta} \geq \int_{0}^{K+1} \frac{dx}{(x + \Gamma)\beta} = \frac{(K + 1 + \Gamma)^{1-\beta} - \Gamma^{1-\beta}}{1 - \beta} \geq \frac{(K + \Gamma)^{1-\beta} - 0.5(K + \Gamma)^{1-\beta}}{1 - \beta},
\]
where the last inequality is implied from \( K \geq \left(2^{1-\beta} - 1\right)\Gamma \). From the preceding relations we observe that the desired relation holds.

C. Proof of Lemma 9

(i) This result holds directly from Definition 4 and that \( \gamma_0 = \frac{\eta}{\Gamma} \) and \( \eta_0 = \frac{\eta}{\Gamma} \).

(ii) For any \( k \geq 1 \) we have
\[
1 - \eta \frac{k_2}{\eta_1} = 1 - \eta \frac{(k_2 + \Gamma)^{-b}}{(k_1 + \Gamma)^{-b}} = 1 - \left(\frac{k_1 + \Gamma}{k_2 + \Gamma}\right)^b \leq 1 - \frac{k_1 + \Gamma}{k_2 + \Gamma},
\]
where the last inequality is due to \( b < 0 \) and that \( k_1 \leq k_2 \). We obtain
\[
1 - \eta \frac{k_2}{\eta_1} \leq \frac{1 - \frac{k_1 + \Gamma}{k_2 + \Gamma}}{1 + \frac{k_1 + \Gamma}{k_2 + \Gamma}} \leq k_2 - k_1.
\]

(iii) Let us use (ii) for \( k_1 := k - 1 \) and \( k_2 := k \). For all \( k \geq 1 \) we have
\[
\frac{1}{\gamma_k} \left(1 - \eta_k \frac{k_2}{\eta_1}ight)^2 \leq \frac{(k + \Gamma)^a(k + \Gamma)^b}{\gamma^3 \eta(k + \Gamma)^2} = \frac{1}{\gamma^3 \gamma \eta(k + \Gamma)^2 - 3a - b} \leq \frac{1}{\gamma^3 \eta \Gamma^2 - 3a - b},
\]
where the last relation is implied by \( 3a + b < 2, \Gamma > 0 \), and that \( k \geq 1 \).

(iv) For all \( k \geq 1 \) we can write
\[
\frac{1}{\gamma_k \eta_k \mu_{\min}} \left(\eta \gamma_k^{\gamma_k-1} - 1\right) = \frac{(k + \Gamma)^{a+b}}{\gamma \eta \mu_{\min}} \left(\left(1 + \frac{1}{k + \Gamma - 1}\right)^{a-b} - 1\right) \leq \frac{(k + \Gamma)^{a+b}}{\gamma \eta \mu_{\min}(k + \Gamma - 1)},
\]
where the last relation is implied by \( a - b < 1, k \geq 1, \) and \( \Gamma \geq 1 \). We obtain
\[
\frac{1}{\gamma_k \eta_k \mu_{\min}} \left(\eta \gamma_k^{\gamma_k-1} - 1\right) \leq \frac{1}{\gamma \eta \mu_{\min}(k + \Gamma)^{a-b}} \left(1 + \frac{1}{\Gamma}\right) \leq \frac{2}{\gamma \eta \mu_{\min} \Gamma^{a-b}} \leq 0.5,
\]
where the last two relations are implied by \( \Gamma \geq 1 \) and \( \Gamma^{1-a-b} \geq \frac{4}{\gamma \eta \mu_{\min}} \), respectively. This implies the relation in part (iv).

References

[1] A. Nedić and J. Liu, “Distributed optimization for control,” Annual Review of Control, Robotics, and Autonomous Systems, vol. 1, pp. 77–103, 2018.
[2] M. C. Ferris and J.-S. Pang, “Engineering and economic applications of complementarity problems,” SIAM Review, vol. 39, no. 4, pp. 669–713, 1997.
[3] F. Facchinei and J.-S. Pang, Finite-dimensional Variational Inequalities and Complementarity Problems. Vols. I,II. Springer Series in Operations Research, New York: Springer-Verlag, 2003.
[4] N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani, *Algorithmic Game Theory*. New York, NY, USA: Cambridge University Press, 2007.

[5] R. Johari, *Efficiency Loss in Market Mechanisms for Resource Allocation*. PhD thesis, MIT, 2004.

[6] X. Chen, C. Zhang, and M. Fukushima, “Robust solution of monotone stochastic linear complementarity problems,” *Mathematical Programming*, vol. 117, pp. 51–80, 2009.

[7] C. Zhang and X. Chen, “Stochastic nonlinear complementarity problem and applications to traffic equilibrium under uncertainty,” *Journal of Optimization Theory and Applications*, vol. 137, no. 277, pp. 51–80, 2008.

[8] J. Koshal, A. Nedić, and U. V. Shanbhag, “Multiuser optimization: Distributed algorithms and error analysis,” *SIAM Journal on Optimization*, vol. 21, no. 3, pp. 1046–1081, 2011.

[9] B. Turan, C. A. Uribe, H. T. Wai, and M. Alizadeh, “Resilient primal–dual optimization algorithms for distributed resource allocation,” *IEEE Transactions on Control of Network Systems*, vol. 8, no. 1, pp. 282–294, 2021.

[10] T. Chen, A. Mokhtari, X. Wang, A. Ribeiro, and G. B. Giannakis, “Stochastic averaging for constrained optimization with application to online resource allocation,” *IEEE Transactions on Signal Processing*, vol. 65, no. 12, pp. 3078–3093, 2017.

[11] A. Nedić, *Subgradient Methods for Convex Minimization*. PhD thesis, MIT, 2002.

[12] D. Blatt, A. O. Hero, and H. Gauchman, “A convergent incremental gradient method with a constant step size,” *SIAM Journal on Optimization*, vol. 18, no. 1, pp. 29–51, 2007.

[13] M. Gürbüzbalaban, A. Ozdaglar, and P. A. Parrilo, “On the convergence rate of incremental aggregated gradient algorithms,” *SIAM Journal on Optimization*, vol. 27, no. 2, pp. 1035–1048, 2017.

[14] A. Defazio, F. Bach, and S. Lacoste-Julien, “SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives,” *Advances in Neural Information Processing Systems*, pp. 1646–1654, 2014.

[15] N. D. Vanli, M. Gürbüzbalaban, and A. Ozdaglar, “Global convergence rate of proximal incremental aggregated gradient methods,” *SIAM Journal on Optimization*, vol. 28, no. 2, pp. 1282–1300, 2018.

[16] M. Gürbüzbalaban, A. Ozdaglar, and P. A. Parrilo, “Convergence rate of incremental gradient and incremental Newton methods,” *SIAM Journal on Optimization*, vol. 29, no. 4, pp. 2542–2565, 2019.

[17] A. Jalilzadeh, “Primal-dual incremental gradient method for nonsmooth and convex optimization problems,” [https://arxiv.org/abs/2011.02059](https://arxiv.org/abs/2011.02059).

[18] M. Wang and D. P. Bertsekas, “Incremental constraint projection methods for variational inequalities,” *Mathematical Programming*, vol. 150, no. 2, pp. 321–363, 2015.

[19] D. P. Bertsekas, *Nonlinear Programming*. Belmont, MA: Athena Scientific, 3th ed., 2016.

[20] D. P. Bertsekas, “Incremental gradient, subgradient, and proximal methods for convex optimization: A survey,” 2017. [https://arxiv.org/abs/1507.01030v2](https://arxiv.org/abs/1507.01030v2).

[21] A. Nedić and D. P. Bertsekas, “Incremental subgradient methods for nondifferentiable optimization,” *SIAM Journal on Optimization*, vol. 12, no. 1, pp. 109–138, 2001.

[22] N. L. Roux, M. Schmidt, and F. Bach, “A stochastic gradient method with an exponential convergence rate for finite training sets,” *Advances in Neural Information Processing Systems*, pp. 2663–2671, 2012.

[23] T.-H. Chang, A. Nedić, and A. Scaglione, “Distributed constrained optimization by consensus-based primal-dual perturbation method,” *IEEE Transactions on Automatic Control*, vol. 59, no. 6, pp. 1524–1538, 2014.

[24] N. S. Aybat and E. Y. Hamedani, “A primal-dual method for conic constrained distributed optimization problems,” *Advances in Neural Information Processing Systems*, pp. 5049–5057, 2016.
[25] G. Scutari and Y. Sun, “Distributed nonconvex constrained optimization over time-varying digraphs,” *Mathematical Programming*, vol. 176, pp. 497–544, 2019.

[26] A. Nedić and T. Tatarenko, “Convergence rate of a penalty method for strongly convex problems with linear constraints,” in *59th IEEE Conference on Decision and Control (CDC)*, pp. 372–377, 2020.

[27] A. Makhdoumi and A. Ozdaglar, “Convergence rate of distributed ADMM over networks,” *IEEE Transactions on Automatic Control*, vol. 62, no. 10, pp. 5082 – 5095, 2017.

[28] W. Tang and P. Daoutidis, “Distributed nonlinear model predictive control through accelerated parallel ADMM,” in *American Control Conference (ACC)*, pp. 1406–1411, 2019.

[29] D. P. Bertsekas, “Incremental aggregated proximal and augmented Lagrangian algorithms,” 2015. https://arxiv.org/abs/1509.09257.

[30] E. Y. Hamedani and N. S. Aybat, “A primal-dual algorithm for general convex-concave saddle point problems,” https://arxiv.org/abs/1803.01401.

[31] F. Yousefian, “Bilevel distributed optimization in directed networks,” in *Proceedings of the American Control Conference (accepted)*, 2021. https://arxiv.org/abs/2006.07564.

[32] M. Amini and F. Yousefian, “An iterative regularized mirror descent method for ill-posed nondifferentiable stochastic optimization,” 2019. https://arxiv.org/abs/1901.00506.

[33] F. Yousefian, A. Nedić, and U. V. Shanbhag, “On smoothing, regularization, and averaging in stochastic approximation methods for stochastic variational inequality problems,” *Mathematical Programming*, vol. 165, no. 1, pp. 391–431, 2017.

[34] H. D. Kaushik and F. Yousefian, “A method with convergence rates for optimization problems with variational inequality constraints,” *SIAM Journal on Optimization (under minor revision)*, 2021. https://arxiv.org/abs/2007.15845.

[35] H. D. Kaushik and F. Yousefian, “An incremental gradient method for large-scale distributed nonlinearly constrained optimization,” in *Proceedings of the American Control Conference (accepted)*, 2021. https://arxiv.org/abs/2006.07956.

[36] A. Beck and S. Sabach, “A first order method for finding minimal norm-like solutions of convex optimization problems,” *Mathematical Programming*, vol. 147, pp. 25–46, 2014.

[37] S. Sabach and S. Shtern, “A first order method for solving convex bilevel optimization problems,” *SIAM Journal on Optimization*, vol. 27, no. 2, pp. 640–660, 2017.

[38] G. Garrigos, L. Rosasco, and S. Villa, “Iterative regularization via dual diagonal descent,” *Journal of Mathematical Imaging and Vision*, vol. 60, no. 2, pp. 189–215, 2018.

[39] H.-K. Xu, “Viscosity approximation methods for nonexpansive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[40] A. Juditsky, A. Nemirovski, and C. Tauvel, “Solving variational inequalities with stochastic mirror-prox algorithm,” *Stochastic Systems*, vol. 1, no. 1, pp. 17–58, 2011.

[41] A. N. Iusem, A. Jofré, and P. Thompson, “Incremental constraint projection methods for monotone stochastic variational inequalities,” *Mathematics of Operations Research*, vol. 44, no. 1, pp. 236–263, 2019.

[42] S. Cui, U. V. Shanbhag, and F. Yousefian, “Complexity guarantees for an implicit smoothing-enabled method for stochastic MPECs,” 2021. https://arxiv.org/abs/2104.08406.

[43] A. Beck, *First-Order Methods in Optimization*. Philadelphia, PA: MOS-SIAM Series on Optimization, 2017.

[44] P. Marcotte and D. Zhu, “Weak sharp solutions of variational inequalities,” *SIAM Journal on Optimization*, vol. 9, no. 1, pp. 179–189, 1998.