HEAT TRANSFER AND ENTROPY ANALYSIS OF MAXWELL HYBRID NANOFLIUID INCLUDING EFFECTS OF INCLINED MAGNETIC FIELD, JOULE HEATING AND THERMAL RADIATION

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Abstract. In this numerical study, researchers explore the flow, heat transfer and entropy of electrically conducting hybrid nanofluid over the horizontal penetrable stretching surface with velocity slip conditions at the interface. The non-Newtonian fluid models lead to better understanding of flow and heat transfer characteristics of nanofluids. Therefore, non-Newtonian Maxwell mathematical model is considered for the hybrid nanofluid and the uniform magnetic field is applied at an angle to the direction of the flow. The Joule heating and thermal radiation impact are also considered in the simplified model. The governing nonlinear partial differential equations for hybrid Maxwell nanofluid flow, heat transfer and entropy generation are simplified by taking boundary layer approximations and then reduced to ordinary differential equations using suitable similarity transformations. The Keller box scheme is then adopted to solve the system of ordinary differential equations. The Ethylene glycol based Copper Ethylene glycol (Cu-EG) nanofluid and Ferro-Copper Ethylene glycol (Fe$_3$O$_4$–Cu-EG) hybrid nanofluids are considered to produce the numerical results for velocity, temperature and entropy profiles as well as the skin friction factor and the local Nusselt number. The main findings indicate that hybrid Maxwell nanofluid is better thermal conductor when compared with the conventional nanofluid, the greater angle of inclination of magnetic field offers greater resistance to fluid motion within boundary layer and the heat transfer rate act as descending function of nanoparticles shape factor.

1. Introduction. The research of fluid flow and heat transfer generated by means of stretching medium has large number of industrial applications, for example,
manufacturing of composite materials, drying of porous solids, thermal insulation, oil recovery, underground species transport, manufacturing of glass and thermal solar systems. In all of these industrial processes, heat transfer and flow investigation are of significant importance because the final product quality is determined on the basis of coefficient of velocity gradient (skin friction) and the rate of convective heat exchange. To understand the fluid flow along with heat transfer characteristics over a moving surface, readers are recommended to study [13, 4, 15, 35, 6, 47, 5, 21, 48]. Recently the focus of researchers shifted towards study of nanofluids flow and the process of heat transfer due to enhanced thermal properties of nanofluids [46, 34, 57, 10, 8, 50]. Detailed survey of literature on nanofluids flow and heat transfer over a flat surface can be found in the review articles of Wang et al. [58] and Keblinski et al. [31] and the references therein.

In general, the addition of nanoparticles in Newtonian base fluid changes the behavior of fluids to non-Newtonian. However, it depends on the concentration of nanoparticles in the base fluid, nanoparticles shape, nanoparticles size and the interaction between them. Therefore it seems that theoretical non-Newtonian models will lead to better understanding of flow and heat transfer of nanofluids. Moreover, the viscosity and thermal conductivity are not constants with the sudden motion and the variation in temperature of the nanofluid. In light of this, research in nanofluids flow focused to the study of variable thermo-physical parameters and non-Newtonian models of nanofluids. Qayyum et al. [44] studied the flow of third grade nanofluid past a stretching surface with variable thickness and convective conditions. They exhibited that the velocity and temperature profile rises with the rising values of third grade fluid and thermal conjugate parameters. Khan et al. [43] used the shooting technique to obtain the numerical solutions of inclined magnetohydrodynamics Williamson fluid flow over a nonlinear stretching sheet with cumulative effects of variable viscosity in the presence of nanoparticles. They concluded that increasing values of angle of inclination, Hartmann number and variable viscosity shows the reduction in velocity profile. On the other hand velocity gradient rises with higher values of Harmann number. Eid et al. [19] analyzed two-dimensional MHD flow of a Carreau nanofluid passing over a permeable stretching surface with thermal radiation. They utilized the shooting scheme to obtain the numerical results. They found enhancement in thickness of thermal boundary layer with increasing concentration of nanoparticles and decreasing strength of applied magnetic field. Kho et al. [33] presented the Casson nanofluid flow induced by the stretching sheet with the impact of Lorentz forces, permeability and thermal radiation. They determined that the velocity gradient and heat transfer rate drops at the boundary surface as Casson and magnetic parameters increase in the fluid. Choi [14] observed that the heat transfer is maximized when nanoparticles are spherical in shape. The reason he described for greater heat transfer is the greater surface area of spherical shaped particles when compared with non-spherical particles. Tausif et al. [55] used the Brinkman nanofluid model to examine the impact of Lorentz forces on the Casson nanofluid with Zirconium dioxide (ZrO$_2$) nanoparticles. Their study presented analysis for cylindrical, platelet, brick and blade shaped nanoparticles. Shen et al. [51] introduced the Cattaneo heat flux model flow of Maxwell viscoelastic nanofluid over a vertical sheet with natural convection and considered the five different types of nanoparticle shapes containing sphere, hexahedron, tetrahedron, column and lamina. They adopted the numerical finite difference scheme with $L_1$-algorithm to get the solution. The results clearly showed that the sphere
shape nanoparticle has the greatest rate of heat transfer and the lowest convective heat exchange rate. Qing et al. [45] examined the entropy analysis on MHD Casson nanofluid flow passing through a permeable stretching surface. They used the numerical successive linearization method and found that the increase in Hartmann number, permeability parameter, Reynolds and Brinkman numbers causes an increase in the entropy generation. Moreover, for increasing values of thermal radiation parameter the temperature distribution increases and for growing values of Brownian motion the concentration profile decreases. Bhatti et al. [11] used the numerical scheme of Chebyshev spectral collocation and investigated the Powell-Eyring nanofluid over a permeable stretching surface with magnetic fields. They concluded that the entropy distribution rises by raising values of Hartmann number and the radiation parameter. Recently, Sithole et al. [52] examined the entropy analysis and the impact of MHD, chemical reactions on second grade nanofluid over a heated stretching sheet with non-linear thermal radiation using the spectral linearization method. The results showed the entropy of the system increased for the higher values of Hartmann, Reynolds and Brinkmann numbers and decreases with increasing values of temperature difference ratio parameter. Readers are recommended to study [41, 24, 22, 30, 7] for further insight into the effects of variable nanofluid properties, non-Newtonian nanofluid models and entropy analysis.

Keeping above in view Suresh [53] introduced the concept of hybrid nanofluids to further enhance the positive features of ordinary nanofluids. Hybrid nanofluids are constructed by mixing two different types of nanoparticles and are the focus of recent research in the field of nanofluids. Devi and Anjali [17] used RK-Fehlberg integration method to study the three-dimensional flow of Copper-Alumina/water (Cu−Al2O3/water) hybrid nanofluid. The flow is induced by the unidirectional linear stretching of flat surface with consideration of Lorentz forces. The numerical results gave the impression that heat exchange rate of Cu−Al2O3/water hybrid nanofluid is greater than the Cu−water nanofluid. Afrand et al. [3] examined the impact of temperature distribution and nanoparticles concentration on rheological behavior of magnetite Ferrofluid-Silver/Ethylene glycol (Fe3O4−Ag/EG) hybrid nanofluid. Hayat and Nadeem [23] considered the three dimensional Brinkman hybrid nanofluid model to investigate the heat transfer characteristics of Copper-oxide/water (CuO/water) and Silver-Copper-oxide/water (Ag−CuO/water) nanofluids over a linearly stretching, rotating surface with thermal radiation and homogeneous-heterogeneous reactive flow. Their results deducted that the hybridity enhanced the temperature profile along with the rate of heat transfer at the boundary of the surface. Ghadikolaei et al. [20] scrutinized the thermo-physical properties of Titanium-Copper/water (TiO2−Cu/H2O) hybrid nanofluid with shape factor in the presence of Lorentz forces. Hussian et al. [25] pondered at the flow of a hybrid nanofluid containing Alumina-Copper/water (Al2O3−Cu/water) flowing through an open cavity with an adiabatic square obstacle inside the cavity. They used the finite element method for the numerical solutions and discussed the impact of different physical parameters on hybrid nanofluids. Mehrali et al. [37] synthesized reduced graphene Oxide-Magnetite-Ferrofluid (GO−Fe3O4) hybrid nanofluid using graphene oxide, iron salts and tannic acid for the process of redundancy and stabilization. It was observed that the use of this hybrid nanofluid increases the overall thermal conductivity of the system by 11%. They also observed that heat transfer performance of (GO−Fe3O4) hybrid nanofluid improve with the application of magnetic field and the entropy is reduced.
by up to 41% on the use of graphene instead of distilled water. Further details regarding the flow and heat transfer characteristics of hybrid nanofluids can be found in [54, 38, 42, 9, 29].

The present research deals with the flow and heat transfer and entropy analysis of electrically conducting hybrid Maxwell nanofluid over a stretching surface. The focus is to analyze the combined effect of applied magnetic field, Joule’s heating and thermal radiation on flow and temperature profiles, heat transfer rate and the overall entropy of the system. The flow is induced due to the stretching of flat surface and velocity slip conditions are assumed at the boundary. The magnetic field is applied at an angle to the direction of the flow.

2. Mathematical formulation. Assume an incompressible, electrically conducting, two-dimensional laminar slip flow of non-Newtonian Maxwell hybrid nanofluid which covers the space over an infinite stretching flat porous plate. The hybrid nanofluid is manufactured by adding Copper (Cu) nanoparticle into the Ethylene glycol (base fluid) at a contact volume friction ($\phi_1$) then Ferro-Copper ($Fe_3O_4-Cu$) nanoparticles are dispersed into the Copper-Ethylene glycol nanofluid to make it a hybrid nanofluid at volume friction ($\phi_2$). A uniform magnetic field is applied at an angle $\Gamma$ about the $x$–axis, where the disk is placed along the $x$–axis. The induced magnetic field is considered negligible with respect to the applied magnetic field and the Maxwell hybrid nanofluid is considered optically thick. Therefore the Rosseland approximations can be considered for the radiation effects. The stretching velocity, $U_w(x,t)$ applied magnetic field strength $B(t)$ and the temperature at the surface of the disk, $T_w(x,t)$ are

$$U_w(x,t) = \frac{cx}{1 - \varpi t}, \quad B(t) = \frac{B_0}{\sqrt{1 - \varpi t}}, \quad T_w(x,t) = T_\infty + \frac{cx}{1 - \varpi t}, \quad (1)$$

respectively. In the above equations $c$ and $\frac{1}{T_\infty^2}$ (with $\varpi t < 1$) are the initial and effective stretching rate, $t$ is the time and $T_\infty$ is the ambient temperature. The geometry of the flow model can be seen from Figure (1) given below.

**Figure 1.** Flow geometry.
The constitutive equations for conservation of mass, momentum and energy under boundary layer assumptions along with suitable boundary conditions for the hybrid Maxwell nanofluid are given as (for details see, Mukhopadhyay [39])

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, 
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \lambda \left[ u^2 \frac{\partial^2 u}{\partial x^2} + v^2 \frac{\partial^2 u}{\partial y^2} + 2uv \frac{\partial^2 u}{\partial x \partial y} \right] = \frac{\mu_{hnf}}{\rho_{hnf}} \left( \frac{\partial^2 u}{\partial y^2} \right) - \frac{\sigma_{hnf} B^2(t) u}{\rho_{hnf}} \sin^2(\Gamma),
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k_{hnf}}{(\rho C_p)_{hnf}} \left( \frac{\partial^2 T}{\partial y^2} \right) - \frac{1}{(\rho C_p)_{hnf}} \left( \frac{\partial q_r}{\partial y} \right) + \frac{\sigma_{hnf} B^2 u^2}{(\rho C_p)_{hnf}} \sin^2(\Gamma).
\]

\[
u(x, 0) = U_w + W_1 \mu_{hnf} \left( \frac{\partial u}{\partial y} \right), \quad \psi(x, 0) = V_w, \quad -k_f \left( \frac{\partial T}{\partial y} \right) = h_f (T_w - T),
\]

\[
u \to 0, \quad T \to T_\infty \text{ as } y \to \infty.
\]

Here, \( u \) and \( v \) are the velocities in the horizontal and vertical directions respectively. The thermal relaxation is represented by the relation \( \lambda = \lambda_0 (1 - \varpi t) \), where \( \lambda_0 \) is the initial relaxation rate. \( \mu_{hnf}, \rho_{hnf}, \sigma_{hnf}, \kappa_{hnf}, (\rho C_p)_{hnf} \) and \( q_r \) are the viscosity, density, electrical conductivity, thermal conductivity, specific heat capacity and thermal radiation factor of hybrid nanofluid. \( V_w \) is the porosity, \( W_1 = W_0 \sqrt{1 - \varpi t} \) is the velocity slip factor and \( W_0 \) is the initial slip parameter. The other parameter such as \( \kappa_f \) and \( h_f \) are the thermal conductivity and heat transfer coefficient of the base fluid. The relationships between thermo-physical parameters of hybrid nanofluid, conventional nanofluid, base fluid and nanoparticles shape factor are given in Table (1) and Table (2) of [29].

Using Rosseland approximations (see details in Brewster [12]) the thermal radiation \( q_r \) can be written as

\[
q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial y},
\]

where, \( \sigma^* \) and \( k^* \) express the Stefan-Boltzmann constant and absorption coefficient respectively.

### 3. Solution of the problem

In this section initially, constitutive system of partial differential equations (2)-(6) is reduced to a system of ordinary differential equation by using the following stream functions \( \psi \) and \( \theta \) and the similarity variable \( \eta \), in the form

\[
\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial x}, \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \psi}{\partial y},
\]

\[
\eta(x, y) = \sqrt{\frac{c}{\nu f(1 - \varpi t)}} y, \quad \psi(x, y) = \sqrt{\frac{\nu f c}{(1 - \varpi t)}} x f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}.
\]

Using above transformations and making use of relations given in Table (1) of [29] and equation (7), equation (2) is identically satisfied and equations (3)-(6) are reduced to

\[
A \left( \frac{\eta}{2} f'' + f' \right) + f'^2 - f f'' = \frac{f'''}{\phi_a \phi_b} + \beta \left( f^2 f''' - 2f f' f'' \right) + \frac{\phi_e}{\phi_b} M \sin^2(\Gamma) f' = 0,
\]
\[ \theta'' \left(1 + \frac{1}{\phi_d} P_R N_R \right) + \frac{\phi_c}{\phi_d} \left[ f\theta' - f' \theta - A \left( \theta + \frac{\eta}{2} \theta' \right) + \frac{\phi_c}{\phi_e} M E^* c \sin^2(\Gamma) f'^2 \right] = 0, \] (11)

\[ f(0) = S, \quad f'(0) = 1 + \frac{A}{(1 - \phi_1)^{2.5}(1 - \phi_2)^{2.5}} f''(0), \quad \theta'(0) = -B_1(1 - \theta(0)), \] (12)

where

\[ \phi_a = \frac{\mu_{hnf}}{\rho_f} = (1 - \phi_1)^{2.5-1} (1 - \phi_2)^{2.5-1}, \] (14)

\[ \phi_b = \frac{\rho_{hnf}}{\rho_f} = (1 - \phi_2) \left\{ (1 - \phi_1) + \phi_1 \frac{\rho_{p_1}}{\rho_f} + \phi_2 \frac{\rho_{p_2}}{\rho_f} \right\}, \] (15)

\[ \phi_c = \frac{(\rho C_p)_{hnf}}{(\rho C_p)_f} = (1 - \phi_2) \left\{ (1 - \phi_1) + \phi_1 \frac{(\rho C_p)_{p_1}}{(\rho C_p)_f} + \phi_2 \frac{(\rho C_p)_{p_2}}{(\rho C_p)_f} \right\}, \] (16)

\[ \phi_d = \frac{\kappa_{hnf}}{\kappa_f} = \frac{[(\kappa_{p_2} + (m - 1) \kappa_{nf}) - (m - 1) \phi_2 (\kappa_{nf} - \kappa_{p_2})]}{(\kappa_{p_2} + (m - 1) \kappa_{nf}) + \phi_2 (\kappa_{nf} - \kappa_{p_2})}, \] (17)

\[ \phi_e = \frac{\sigma_{hnf}}{\sigma_f} = \frac{3(\phi_1 \sigma_{p_1} + \phi_2 \sigma_{p_2})}{\sigma_f} - \phi_1 \phi_2. \] (18)

Here, primes stand for differentiation of function with respect to \( \eta \), \( A = \Xi \) is the unsteadiness parameter, \( \beta = c \lambda_0 \) is the Maxwell parameter, \( M = \frac{\sigma_f T^3}{\rho_f} \) is the magnetic parameter/Harman number, \( P_R = \frac{v}{\alpha_f} \) is the Prandtl number, \( \alpha_f = \frac{\kappa_f (\rho C_p)_f}{\rho f} \) is the thermal diffusivity parameter, \( N_R = \frac{16}{3} \frac{\sigma^* T^3 w}{\kappa_f (\rho C_p)_f} \) is the radiation parameter, \( E^* c = \frac{U_w^2}{(C_p)_f (T_w - T_\infty)} \) is the local Eckert number, \( S = -V_w \sqrt{\frac{1 - \omega T}{\kappa_f c}} \) is the suction/injection parameter, \( \Lambda = W_0 \sqrt{\frac{1 - \omega T}{\kappa_f c}} \) is the velocity slip parameter and \( B_1 = \frac{h_f}{c} \sqrt{\frac{\nu_f (1 - \omega T) \rho f}{c}} \) is the Biot number. In equations (14)-(18) \( \phi_1 \) is the nanoparticle volume concentration of Copper nanoparticle into the Ethylene glycol base fluid and \( \phi_2 \) is the Ferro solid nanoparticles volume concentration in the mixture. \( \phi_{hnf} = \phi_1 + \phi_2 \) is the nanoparticle volume concentration coefficient of the hybrid nanofluid. \( \kappa_{nf} \) is the thermal conductivity of the nanofluid and \( \rho_f \), \( (C_p)_f \), \( \kappa_f \) and \( \sigma_f \) are density, effective heat capacity, thermal conductivity and electrical conductivity of the base fluid respectively. Here, \( p_1 \) represents the Copper solid nanoparticle and \( p_2 \) represents the Ferro solid nanoparticles. The other properties like \( \rho_{p_1}, \rho_{p_2}, (C_p)_{p_1}, (C_p)_{p_2}, \kappa_{p_1}, \kappa_{p_2}, \sigma_{p_1}, \) and \( \sigma_{p_2} \) are the density, specific effective capacity, thermal conductivity and electrical conductivity of the Copper and Ferro solid nanoparticles respectively [18, 26]. It is also observed that the local Eckert number in equation (11) is dependent of variable \( x \), therefore \( x \) cannot be eliminated from the energy equation. To overcome this and to avoid the non-similar solutions for our problem, we should take Eckert number as \( Ec = \frac{E^* c}{(C_p)_f (1 - \omega T)} \).

The boundary value problem defined by equations (10)-(13) is difficult to solve analytically. Different numerical techniques are available in literature that can accurately approximate solutions for the given nonlinear boundary value problem (see for example, Thorley and Tiley [56]). The techniques include the characteristics
method, shooting technique, explicit and implicit finite difference method, finite volume method and the implicit box scheme. The box scheme adopted by the Keller [32] is employed here to find the approximate solutions. The main advantage of this method over the other difference methods is that the mesh spacing need not be uniform so that a finer mesh can be used in regions where we expect rapid changes in the solution. Secondly being implicit, the method is unconditionally stable and has second order accuracy even with non-uniform mesh. An important step in this scheme is to reformulate the problem as a system of first order ODEs with differences centered at the center of the rectangular mesh box or at the boundary. The flow chart in Figure 2 explain the implementation of Keller box numerical scheme.

Figure 2. Flow Sheet of Keller Box method

The first step of the method is to convert the boundary value problem defined in (10)-(13) to a system of first order ODEs, that is,

\begin{align*}
u &= f', \\
v &= u', \\
t &= \theta',
\end{align*}

(19)\hspace{1cm} (20)\hspace{1cm} (21)

\begin{align*}
A \left( \frac{\eta}{2} v + u \right) + u^2 - f v + \theta' \phi_a \phi_b &+ \beta \left( f^2 v' - 2 f u v \right) + \phi_c M \sin^2(\Gamma) u = 0, \\
\left( 1 + \frac{1}{\phi_c} Pr Nr \right) t' + Pr \phi_c \phi_d \left[ ft - u \theta - A(\theta + \eta t) + \phi_e M E c \sin^2(\Gamma) u^2 \right] &= 0, \\
f(0) &= S, \quad u(0) = 1 + \frac{\Lambda}{\phi_u} v(0), \quad t(0) = -Bi(1 - \theta(0)), \quad u(\infty) \to 0, \quad \theta(\infty) \to 0.
\end{align*}

(22)\hspace{1cm} (23)\hspace{1cm} (24)

The system of first order ODEs are then transformed to difference equations by replacing functions with their mean averages i.e. \( \left( \ast \right)_{j-\frac{1}{2}} = \frac{\left( \ast \right)_j + \left( \ast \right)_{j-1}}{2} \) and their derivatives by backward differences, \( \left( \ast \right)'_{j-\frac{1}{2}} = \frac{\left( \ast \right)_j - \left( \ast \right)_{j-1}}{h} \). This gives

\begin{align*}
\frac{u_j + u_{j-1}}{2} &= \frac{f_j - f_{j-1}}{h}, \\
\frac{v_j + v_{j-1}}{2} &= \frac{u_j - u_{j-1}}{h}, \\
\frac{t_j + t_{j-1}}{2} &= \frac{\theta_j - \theta_{j-1}}{h},
\end{align*}

(25)\hspace{1cm} (26)\hspace{1cm} (27)
To deal with the nonlinearity in the difference equations the Newton iteration is adopted, that is

\[ j^{(i+1)} = j^{(i)} + \delta j^{(i)} \]  

(30)

The initial guess in Newton method is chosen in a way that a second order accuracy is achieved in a single Newton iteration. Substitution of Eq. (30) in Eqs. (25)-(29) and dropping higher order terms of \( \delta j^i \), we obtain a linear tridiagonal system of the form

\[
A \left\{ \left( \frac{u_j + u_{j-1}}{2} \right) + \frac{\eta}{2} \left( \frac{v_j + v_{j-1}}{2} \right) \right\} + \left( \frac{u_j + u_{j-1}}{2} \right)^2 \\
- \left( \frac{f_j + f_{j-1}}{2} \right) \left( \frac{v_j + v_{j-1}}{2} \right) \frac{1}{\phi_a \phi_b} \left( \frac{v_j + v_{j-1}}{h} \right) \\
+ \beta \left( \frac{f_j + f_{j-1}}{2} \right)^2 \left( \frac{v_j + v_{j-1}}{h} \right) \\
- 2\beta \left( \frac{f_j + f_{j-1}}{2} \right) \left( \frac{u_j + u_{j-1}}{2} \right) \left( \frac{v_j + v_{j-1}}{2} \right) \\
+ \frac{\phi_c}{\phi_b} M \sin^2(\Gamma) \left( \frac{u_j + u_{j-1}}{2} \right) = 0,
\]

(28)

\[
+ Pr \frac{\phi_c}{\phi_d} \left[ - \left( \frac{u_j + u_{j-1}}{2} \right) \left( \frac{\theta_j + \theta_{j-1}}{2} \right) \right] - A \left\{ \left( \frac{\theta_j + \theta_{j-1}}{2} \right) + \frac{\eta}{2} \left( \frac{t_j + t_{j-1}}{2} \right) \right\} \\
+ Pr \frac{\phi_c}{\phi_d} \left[ \left( \frac{\phi_c ECM \sin^2(\Gamma)}{2} \right) \left( \frac{u_j + u_{j-1}}{2} \right)^2 \right] = 0
\]

(29)

To deal with the nonlinearity in the difference equations the Newton iteration is adopted, that is

\[ j^{(i+1)} = j^{(i)} + \delta j^{(i)} \]  

(30)

The initial guess in Newton method is chosen in a way that a second order accuracy is achieved in a single Newton iteration. Substitution of Eq. (30) in Eqs. (25)-(29) and dropping higher order terms of \( \delta j^i \), we obtain a linear tridiagonal system of the form

\[
\delta f_j - \delta f_{j-1} - \frac{1}{2} h (\delta u_j + \delta u_{j-1}) = (r_1)_{j-\frac{1}{2}},
\]

(31)

\[
\delta u_j - \delta u_{j-1} - \frac{1}{2} h (\delta v_j + \delta v_{j-1}) = (r_2)_{j-\frac{1}{2}},
\]

(32)

\[
\delta \theta_j - \delta \theta_{j-1} - \frac{1}{2} h (\delta t_j + \delta t_{j-1}) = (r_3)_{j-\frac{1}{2}},
\]

(33)

\[
(a_1) \delta f_j + (a_2) \delta f_{j-1} + (a_3) \delta u_j + (a_4) \delta u_{j-1} + (a_5) \delta v_j + (a_6) \delta v_{j-1} + (a_7) \delta \theta_j + (a_8) \delta \theta_{j-1} + (a_9) \delta t_j + (a_{10}) \delta t_{j-1} = (r_4)_{j-\frac{1}{2}},
\]

(34)

\[
(b_1) \delta f_j + (b_2) \delta f_{j-1} + (b_3) \delta u_j + (b_4) \delta u_{j-1} + (b_5) \delta v_j + (b_6) \delta v_{j-1} + (b_7) \delta \theta_j + (b_8) \delta \theta_{j-1} + (b_9) \delta t_j + (b_{10}) \delta t_{j-1} = (r_5)_{j-\frac{1}{2}},
\]

(35)

where

\[
(r_1)_{j-\frac{1}{2}} = -f_j + f_{j-1} + \frac{h}{2} (u_j + u_{j-1}),
\]

(36)

\[
(r_2)_{j-\frac{1}{2}} = -u_j + u_{j-1} + \frac{h}{2} (v_j + v_{j-1}),
\]

(37)

\[
(r_3)_{j-\frac{1}{2}} = -\theta_j + \theta_{j-1} + \frac{h}{2} (t_j + t_{j-1}),
\]

(38)
(r_4)_{j - \frac{1}{2}} = -h \left[ -A \left( \frac{u_j + u_{j-1}}{2} + \frac{v_j + v_{j-1}}{4} \right) + \left( \frac{u_j + u_{j-1}}{2} \right)^2 \right] \\
- h \left[ - \frac{1}{2} \left( \frac{f_j + f_{j-1}}{2} \right) \left( \frac{v_j + v_{j-1}}{2} \right) - \frac{1}{\phi_a \phi_b} \left( \frac{v_j + v_{j-1}}{h} \right) \right] \\
- h \left[ \beta \left( \frac{f_j + f_{j-1}}{2} \right)^2 \left( \frac{v_j + v_{j-1}}{h} \right) \right] \\
- h \beta \left[ 2 \left( \frac{f_j + f_{j-1}}{2} \right) \left( \frac{u_j + u_{j-1}}{2} \right) \left( \frac{v_j + v_{j-1}}{2} \right) \right] \\
- h \left[ \frac{\phi_a}{\phi_b} M \sin^2(\Gamma) \left( \frac{u_j + u_{j-1}}{2} \right) \right], \quad \text{(39)}

(r_5)_{j - \frac{1}{2}} = -h \left[ \left( \frac{t_j - t_{j-1}}{h} \right) \left( 1 + \frac{1}{\phi_d} Pr Nr \right) \right] \\
- \frac{\phi_c}{\phi_d} h^* Pr \left[ \left( \frac{(f_j + f_{j-1})(t_j + t_{j-1})}{4} \right) - \left( \frac{(t_j + t_{j-1})(u_j + u_{j-1})}{4} \right) \right] \\
+ \frac{\phi_c}{\phi_d} Pr h A \left[ \left( \frac{\theta_j + \theta_{j-1}}{2} + \eta \frac{(t_j + t_{j-1})}{4} \right) \left( \frac{\theta_j + \theta_{j-1}}{2} \right) \right] \\
- \frac{\phi_c}{\phi_d} Pr h \left[ \frac{\phi_c}{\phi_d} ECM \sin^2(\Gamma) \left( \frac{u_j + u_{j-1}}{2} \right)^2 \right], \quad \text{(40)}

subject to boundary conditions
\[ \delta f_0 = 0, \quad \delta u_0 = 0, \quad \delta t_0 = 0, \quad \delta u_J = 0, \quad \delta \theta_J = 0, \quad \text{(41)} \]

The system if equations (30)-(35) can be written in a matrix form:
\[ A \delta = b, \quad \text{(42)} \]

where
\[ A = \begin{bmatrix} A_1 & C_1 \\ B_2 & A_2 & C_2 \\ \vdots & \ddots & \ddots \\ B_{J-1} & A_{J-1} & C_{J-1} \\ B_J & A_J \end{bmatrix}, \quad \delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_{J-1} \\ \delta_J \end{bmatrix}, \quad b = \begin{bmatrix} (r_1)_{j - \frac{1}{2}} \\ \vdots \\ (r_{J-1})_{j - \frac{1}{2}} \\ (r_J)_{j - \frac{1}{2}} \end{bmatrix}. \quad \text{(43)} \]

In the above $A$ is a $J \times J$ block tridiagonal matrix with each block size of $5 \times 5$, $\delta$ and $b$ are column matrices with $J$ rows. The numerical scheme adopted here is now reduced to the solution of linear systems of algebraic equations. The stability or convergence of solutions imply that the system in (42) is non-singular and have unique solutions. In general method of LU-factorization of block tri-diagonal matrices is adopted to the linear system. To check the validity of numerical scheme the results are compared with those already available in the literature [27, 28, 2, 16]. The comparison is presented for the calculation of heat transfer rate at the boundary. Table 1 presents an excellent agreement between the results obtained from the present scheme and those already available in literature.
Hartman number $M$ are carried out to present effects of Maxwell parameter.

In the above equations, $Re$ parameters.

Applying the dimensional less similarity transformations (9), we obtain

$$C_f = \frac{\tau_w}{\rho_f U_w^2}, \quad Nu_x = \frac{x q_w}{k_f (T_w - T_\infty)}, \quad N_G = \frac{T_\infty^2 c^2 E_G}{k_f (T_w - T_\infty)^2},$$

where $\tau_w$, $q_w$ and $E_G$ are

$$\tau_w = \mu_{hnf} (\frac{\partial u}{\partial y})_{y=0}, \quad q_w = -k_{hnf} \left(1 + \frac{16}{\kappa} \kappa_{fj}(\rho C_p f) \right) \left(\frac{\partial T}{\partial y}\right)_{y=0},$$

$$E_G = \frac{k_{hnf}}{T_\infty^2} \left\{ \left(\frac{\partial T}{\partial y}\right)^2 + \frac{16}{\kappa} \kappa_{fj}(\rho C_p f) \left(\frac{\partial T}{\partial y}\right)^2 \right\}$$

$$+ \frac{\mu_{hnf}}{T_\infty^2} \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma_{hnf} a^2(t) u^2}{T_\infty^2} \sin^2(\Gamma).$$

Applying the dimensional less similarity transformations (9), we obtain

$$C_f Re_x^{\frac{1}{2}} = \frac{f''(0)}{(1 - \phi_1)^{2.5}(1 - \phi_2)^{2.5}}, \quad Nu_x Re_x^{-\frac{1}{2}} = -\frac{k_{hnf}}{k_f} (1 + Nr) \theta'(0),$$

$$N_G = Re \left(\phi_d (1 + Nr) \theta'^2 + \frac{1}{\phi_a} \frac{B_r}{\Omega} \left( f''^2 + \phi_a \phi_v M \sin^2(\Gamma) f''^2 \right) \right).$$

In the above equations, $Re_x = U_w(x)/\nu_f$ is the local Reynolds number, $Re$ is the Reynolds number, $Br$ and Brinkman number $Br$ on the behavior of non-dimensional velocity, temperature and entropy generation profiles. The graphical representation of numerical results are presented in Figures (3)-(24). The nanoparticle volume concentration parameter $\phi$ is fixed at 0.18 (which is not changed in the given problem). The results are produced for the Copper Ethylene glycol (Cu-EG) Maxwell nanofluid and Ferro-Copper/Ethylene glycol (Fe$_3$O$_4$ - Cu/EG) Maxwell hybrid nanofluid. In addition to these Table 3 contains effects of governing parameters on skin friction factor and temperature gradient. Thermo-physical properties of
Cu-EG and Fe₃O₄ - Cu/EG nanofluids are tabulated in Table 2 (See for example [40, 49]).

| Thermo-physical | ρ (kg/m³) | cₚ (J/kgK) | k (W/mK) | σ (S/m) |
|-----------------|-----------|-------------|-----------|---------|
| Ethylene glycol (EG) | 1114 | 2415 | 0.252 | 5.5 × 10⁻⁶ |
| Pure water (H₂O) | 997.1 | 4179 | 0.613 | 0.05 |
| Copper (Cu) | 8933 | 385.0 | 401.00 | 5.96 × 10⁷ |
| Ferro (Fe₃O₄) | 5180 | 670 | 9.7 | 0.74 × 10⁶ |
| Copper oxide (CuO) | 6510 | 540 | 18 | 5.96 × 10⁷ |
| Alumina (Al₂O₃) | 3970 | 765.0 | 40.000 | 3.5 × 10⁷ |
| Titanium oxide (TiO₂) | 4250 | 686.2 | 8.9538 | 2.38 × 10⁶ |

Table 2. Thermo-physical properties of Base Fluid and Nanoparticles

The effects of Maxwell parameter β on velocity, temperature and entropy profiles for Cu-EG Maxwell nanofluid and Fe₃O₄ - Cu/EG Maxwell hybrid nanofluid are displayed in Figures (3)-(5).

Computations are performed for β = 0.01, 0.1, 0.3 at nanoparticles volume fric-
tion of φ₁ = 0.09 and φ₂ = 0.09. The velocity profiles in Figure (3) declines with growing values of β and therefore reduces the thickness of momentum boundary layer. The obvious explanation for the behavior is the reduction in fluid yield stress in the boundary layer with increasing values of parameter β. This also presents the decreasing trend for the skin friction factor at the boundary. Figure (3) for the fixed value of Maxwell parameter β can be used to compare the velocity behavior for the conventional and hybrid nanofluids. It is observed (for β = 0.3) the thickness of momentum boundary layer for hybrid nanofluid is more than the conventional nanofluid. In Figure (4) the rising trend is observed in temperature profiles with increasing values of parameter β. This behavior reveals boost in the thermal boundary layer thickness and fall in the heat transfer rate at the boundary. Figure (5) presents the graphs for entropy profile against the variation in parameter β. Initially the entropy profiles increases with positive variation in β, representing more energy entered into the system. At later times entropy reduces with increase
in parameter $\beta$ and asymptotically approaching to some constant value. This is due to the reduction in the temperature of the system and the reduction in the overall energy of the system. It is also noticed from Table (3) that the local Nusselt number declines for both conventional and hybrid nanofluids.

![Figure 4. Temperature distribution against $\beta$.](image)

![Figure 5. Entropy generation against $\beta$.](image)

Figures (6)-(8) captured the impact of variation in magnetic parameter/Hartman number $M$ on the velocity, temperature and entropy profiles when plotted against the similarity variable $\eta$. Figure (6) depicts that enhancing the strength of transverse applied magnetic field (Hartman number $M$) the fluid flow diminished progressively for both Cu-EG nanofluid and Fe$_3$O$_4$-Cu/EG non-Newtonian Maxwell hybrid nanofluid. This is due to the production of resistive Lorentz force whose strength increases with increasing strength of magnetic parameter $M$. Moreover the increase in strength of applied magnetic field present an inverse effect to the density of nanofluid, therefore the increasing values of parameter $M$ reduces the overall density which raised the temperature of nanofluid within boundary layer (see Figure (7)). The effect of $M$ on the entropy profile is discussed in Figure (8) and it is observed the magnetic parameter or the Hartman number tends
Figure 6. Velocity distribution against $M$.

Figure 7. Temperature distribution against $M$.

Figure 8. Entropy generation against $M$. 
to reduce the entropy generation in the system. Finally the appreciation in $M$ maximize the velocity gradient and minimize the temperature gradient at the boundary. Figures (9)-(11) present the effect of angle of inclination of magnetic field on the velocity, temperature and entropy profiles respectively. An inclined magnetic field offer greater resistance to the flow when compared with the transverse magnetic field. Therefore the velocity of nanofluids decreases with increase in the angle of inclination. Similarly the density of nanofluids reduces more when magnetic field is applied at an angle with the horizontal. Therefore the temperature of nanofluids rises with a greater inclination of parameter $\Gamma$. The effect of $\Gamma$ on the entropy of the system is discussed Figure in (11). It is observed that a more traverse magnetic field tends to raise the entropy of the system.

Figures (12)-(14) present the effect of nanoparticle volume friction parameter $\phi$ of conventional Maxwell nanofluid and $\phi_{hnf}$ of hybrid Maxwell nanofluid on the velocity, temperature and entropy generation profiles. In numerical computations $\phi_1$ is fixed at 0.09 and the analysis is carried out by considering different volume friction $\phi_2$. It is noted that the nanofluid velocity falls and the temperature rises with increasing volume of nanoparticles in the base fluid. The is due to higher thermal conductivity and density of solid nanoparticles which raises the overall
thermal conductivity and density of nanofluid. This fact is very much evident in Figure (13), that is, the increase in thermal conductivity of nanofluid boosts the temperature and thermal boundary layer thickness of nanofluid. It is also observed that the thermal conductivity for hybrid Maxwell nanofluid is higher when compared to the thermal conductivity of conventional Maxwell nanofluid (see Table 3). The graphical representation of entropy generation in Figure (14) depicts the entropy of the system increases for both conventional and hybrid nanofluids the increasing nanoparticles volume friction parameter $\phi$ and $\phi_{hnf}$.

There are no effects of thermal radiation parameter $N_r$, Eckert number $Ec$ and Biot number $Bi$ on the velocity profiles. Figures (15)-(20) exhibit graphs of temperature profile for $Cu$-EG and $Fe_3O_4 - Cu/EG$ hybrid nanofluids with variation in thermal radiation parameter $N_r$, Eckert number $Ec$, and Biot number $Bi$ respectively.

In Figure (15) an increase in radiation parameter $N_r$ shows significant enhancement in the fluid temperature function. Physically speaking, by strengthening $N_r$ yields more heat into the system, as a consequence the thickness of associated the thermal boundary layer and the rate of heat transfer increases in the boundary layer. Moreover, the Eckert number $Ec$ is the ratio of the kinetic energy to the
enthalpy (or the dynamic temperature to the temperature) and the greater values of $Ec$ means greater kinetic energy for nanoparticles those act as a driving force for higher rate of heat transfer. The frictional heating take place at the surface giving rise to the temperature of the corresponding nanofluid (see Figure 17). Biot number or the sheet convection parameter shows the ratio of conduction inside the fluid to the convection at its surface. Increasing sheet convection parameter means that the heat transfer through conduction dominates the convection coefficient at the surface of the fluid. This fact is evident through in with rising temperature profiles in Figure (19).

Figures (16),(18) and (20) exhibit effects of $Nr$, $Ec$ and $Bi$ on entropy profiles. The crossover point is observed in entropy profiles, that is, the opposite behavior in entropy profiles are observed before and after this point. Initially entropy increases with increasing values of parameters $Nr$, $Ec$ and $Bi$. Whereas the entropy started decreasing at some later times as a direct consequence of decreasing trend in temperature profiles.

Figure. (21) depicts the effect of different nanoparticle shapes on the temperature and entropy profiles. The computations are carried for spherical, hexahedron,
Figure 15. Temperature distribution against $Nr$.

Figure 16. Entropy generation against $Nr$.

Figure 17. Temperature distribution against $Ec$. 
Figure 18. Entropy generation against $Ec$.

Figure 19. Temperature distribution against $Bi$.

Figure 20. Entropy generation against $Bi$. 
tetrahedron, column and lamina shaped nanoparticles in the nanofluid. The graphical view shows that the dimensionless temperature of the nanofluid increases as the shape factor $m$ increases. It is noteworthy that spherical shaped particle has lowest temperature on the boundary due to its greater surface area which pull higher heat from the boundary layer. In addition Figure (22) shows the entropy of the system increases at slowest rate for the spherical-shaped nanoparticles.

In the end, the effects of Reynolds number $Re$ and Brinkam number $Br$ on the entropy generation profiles are presented in Figure (23)-(24). The numerical simulation shows that the higher value of $Re$ raises the overall entropy. This showed the dominance of inertial forces over the viscous forces. Moreover, it is found that the Brinkman number $Br$ augmentation increases the entropy generation rate.

5. **Conclusion.** A computational analysis of boundary layer flow of Maxwell hybrid nanofluid over a porous stretching disk with consideration of inclined magnetic field, Joule heating boundary slip and nanoparticles shape and thermal radiation is carried out for Copper Ethylene glycol ($Cu$-EG) and Ferro-Copper Ethylene glycol ($Fe_3O_4 - Cu/EG$) nanofluids in the present research. The main findings of the present work are:
1. The hybrid Maxwell nanofluid \( \text{Fe}_3\text{O}_4 - \text{Cu}/\text{EG} \) is observed to be a better thermal conductor than the conventional Maxwell nanofluid \( \text{Cu}-\text{EG} \).
2. The more inclined magnetic field has a greater effect on the motion of fluid particles in the boundary layer when compared with the transverse magnetic field.
3. The heat transfer rate increases with the higher concentration of nanoparticles.
4. The heat transfer rate is greater for the smaller shape factor \( m \) in the boundary layer.

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\[
\begin{array}{cccccccccc}
\beta & A & M & \Gamma & \phi & \phi_2 & \Lambda & Nr & Ec & Bi & C_f \frac{Re_x}{Cu-EG} & \frac{C_f}{C_{Cu/EG}} & \frac{Nu}{Re_x} & \frac{Nu}{Cu/EG} \\
0.01 & 0.6 & 0.6 & \pi/4 & 0.18 & 0.09 & 0.1 & 0.2 & 0.2 & 0.1 & 1.1918 & 1.8311 & 0.0818 & 0.0824 \\
0.1 & & & & & & & & & & 2.0011 & 2.0673 & 0.0813 & 0.0819 \\
0.3 & & & & & & & & & & 2.1039 & 2.0894 & 0.0804 & 0.0808 \\
0.6 & & & & & & & & & & 1.9181 & 1.8311 & 0.0818 & 0.0824 \\
1.6 & & & & & & & & & & 2.2368 & 2.1736 & 0.0839 & 0.0879 \\
2.6 & & & & & & & & & & 2.4621 & 2.3211 & 0.0896 & 0.0899 \\
\pi/4 & & & & & & & & & & 1.9181 & 1.8311 & 0.0818 & 0.0824 \\
\pi/3 & & & & & & & & & & 2.1000 & 2.0138 & 0.0810 & 0.0821 \\
\pi/2 & & & & & & & & & & 1.8610 & 2.0771 & 0.0802 & 0.0820 \\
0.09 & & & & & & & & & & 2.1314 & - & 0.0801 & - \\
0.15 & & & & & & & & & & 2.0229 & - & 0.0810 & - \\
0.18 & & & & & & & & & & 1.9112 & - & 0.0818 & - \\
0.0 & & & & & & & & & & 2.0121 & 2.0065 & 0.0852 & 0.0858 \\
0.06 & & & & & & & & & & 1.9118 & 1.8311 & 0.0818 & 0.0824 \\
0.09 & & & & & & & & & & 1.9118 & 1.8311 & 0.0818 & 0.0824 \\
0.1 & & & & & & & & & & 1.9118 & 1.8311 & 0.0921 & 0.1269 \\
0.2 & & & & & & & & & & 1.7228 & 1.6216 & 0.0810 & 0.0820 \\
0.4 & & & & & & & & & & 1.9118 & 1.8311 & 0.0996 & 0.2106 \\
0.2 & & & & & & & & & & 1.9118 & 1.8311 & 0.0706 & 0.0580 \\
0.4 & & & & & & & & & & 1.9118 & 1.8311 & 0.0818 & 0.0824 \\
0.6 & & & & & & & & & & 1.9118 & 1.8311 & 0.0881 & 0.0829 \\
0.1 & & & & & & & & & & 1.9118 & 1.8311 & 0.0828 & 0.0824 \\
0.2 & & & & & & & & & & 1.9118 & 1.8311 & 0.013 & 0.0821 \\
0.6 & & & & & & & & & & 1.9118 & 1.8311 & 0.0806 & 0.0815 \\
\end{array}
\]

Table 3. Values of Skin Friction = \(C_f \frac{Re_x}{Cu-EG}\) and Nusselt Number = \(\frac{Nu}{Re_x}\) for \(Pr = 6.2\) and \(m = 3\).

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