On Axiomatization of Lewis’ Conditional Logics

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Abstract

This paper first shows that the popular axiomatic systems proposed by Nute for Lewis’ conditional logics are not equivalent to Lewis’ original systems. In particular, the axiom CA which is derivable in Lewis’ systems is not derivable in Nute’s systems. Then the paper proposes a new set of axiomatizations for Lewis’ conditional logics, without using CSO, or RCEA, or the rule of interchange of logical equivalents. Instead, the new axiomatizations adopt two axioms which correspond to cautious monotonicity and cautious cut in nonmonotonic logics, respectively. Finally, the paper gives a simple resolution to a puzzle about the controversial axiom of simplification of disjunctive antecedents, using a long neglected axiom in one of Lewis’ systems for conditional logics.

Keywords: conditional logic, axiomatization, simplification of disjunctive antecedents, nonmonotonic logic

1 Introduction

Lewis proposed two conditional logics, denoted by $V$ and $VC$, respectively. Each of them has three different axiomatizations in the literature. Two were proposed by Lewis himself, one in (Lewis, 1971), where $V$ and $VC$ were named $C0$ and $C1$, respectively, the other in (Lewis, 1973). A third formulation was offered by Nute (1980b, 1984; Nute & Cross (2001). Lewis’ formulations have less but some cumbersome axioms. Nute’s formulations have more but neater axioms, making them easier to compare with other systems. Thus, Nute’s axiomatizations are more popular in the literature now. When referring to Lewis’ conditional logics, often are Nute’s axiomatizations presented, for instance in
I will show in this paper, however, that Nute’s systems are not equivalent to Lewis’ original ones. In particular, the axiom CA derivable from Lewis’ systems is not derivable from Nute’s systems. By replacing MOD with CA in Nute’s systems, the defects can be amended.

Both Lewis’ systems in (Lewis, 1971) and Nute’s systems contain the axiom CSO, which says that bi-conditionally implied propositions can be interchanged with each other for antecedents. From CSO together with RCE (namely a conditional from \( \varphi \) to \( \psi \) can be derived if \( \varphi \) entails \( \psi \)), the rule of interchange of logical equivalents for antecedents (RCEA, henceforth) can be derived. Instead of CSO, Lewis’ systems in (Lewis, 1973) contain the rule of interchange of logical equivalents (RE, henceforth). I will propose some new axiomatizations for Lewis’ logics. They contain neither CSO, nor RCEA or RE, and hence may shed light on nonclassical conditional logics, where these axiom and rules are invalidated. The new systems I propose indicate that it is hard to abandon these axiom and rules in conditional logics, since they can be recovered from other intuitive axioms.

Finally, I will show that an axiom in one of Lewis’ systems can be used to solve a puzzle triggered by the controversial axiom of simplification of disjunctive antecedents (SDA, henceforth), which is intuitively valid but trivializes conditional implication to strict implication if added to any conditional logic with RCEA.

2 Preliminaries

For reference, I list all related axioms and rules for conditional logics in this paper as follows:

(PC) All tautologies and derivable rules in classical logic

(ID) \( \varphi > \varphi \)

(CM) \( (\varphi > \psi \land \chi) \rightarrow (\varphi > \psi) \land (\varphi > \chi) \)

(CC) \( (\varphi > \psi) \land (\varphi > \chi) \rightarrow (\varphi > \psi \land \chi) \)

(CV) \( (\varphi > \chi) \land \neg(\varphi > \neg \psi) \rightarrow (\varphi \land \psi > \chi) \)

(CA) \( (\varphi > \chi) \land (\psi > \chi) \rightarrow (\varphi \lor \psi > \chi) \)

(AC) \( (\varphi > \psi) \land (\varphi > \chi) \rightarrow (\varphi \land \psi > \chi) \)

\[1\text{In (Arló Costa, 2014), the author wrote: “...it is useful to see first that the system VC can be axiomatized via the axioms ID, MP, MOD, CSO, CV and CS with RCEC and RCK as rules of inference.”} \]
(RT) \[(\varphi > \psi) \land (\psi \land \varphi > \chi) \rightarrow (\varphi > \chi)\]

(CSO) \[(\varphi > \psi) \land (\psi > \varphi) \rightarrow ((\varphi > \chi) \leftrightarrow (\psi > \chi))\]

(MOD) \[(\varphi > \neg \varphi) \rightarrow (\psi > \neg \varphi)\]

(DAE) \[(\varphi \lor \psi > \varphi) \lor (\varphi \lor \psi > \psi) \lor ((\varphi \lor \psi > \chi) \leftrightarrow (\varphi > \chi) \land (\psi > \chi))\]

(PIE) \[(\varphi > \neg \psi) \lor ((\varphi \land \psi > \chi) \leftrightarrow (\varphi > (\psi \rightarrow \chi)))\]

(CMP) \[(\varphi > \psi) \rightarrow (\varphi \rightarrow \psi)\]

(CS) \[\varphi \land \psi \rightarrow (\varphi > \psi)\]

(SDA) \[(\varphi \lor \psi > \chi) \rightarrow ((\varphi > \chi) \land (\psi > \chi))\]

(RCM) \[
\begin{align*}
\psi & \rightarrow \psi \\
\chi > \varphi & \rightarrow (\chi > \psi)
\end{align*}
\]

(RCE) \[
\begin{align*}
\varphi & \rightarrow \psi \\
\varphi > \psi
\end{align*}
\]

(RCN) \[
\begin{align*}
\psi & \rightarrow \varphi \\
\varphi > \psi
\end{align*}
\]

(RCK) \[
\begin{align*}
\psi_1 \land \ldots \land \psi_n & \rightarrow \psi \\
(\varphi > \psi_1) \land \ldots \land (\varphi > \psi_n) & \rightarrow (\varphi > \psi)
\end{align*}
\]

(n ≥ 0)

(RCEA) \[
\begin{align*}
\varphi & \leftrightarrow \psi \\
(\varphi > \chi) & \leftrightarrow (\psi > \chi)
\end{align*}
\]

(RCEC) \[
\begin{align*}
\varphi & \leftrightarrow \psi \\
(\chi > \varphi) & \leftrightarrow (\chi > \psi)
\end{align*}
\]

(Re) \[
\begin{align*}
\psi & \leftrightarrow \psi' \\
\varphi & \leftrightarrow \varphi[\psi / \psi']
\end{align*}
\]

All the axioms and rules above had been discussed in the literature (e.g. Lewis, 1973; Nute, 1980b, 1984) before. Note that I slightly reformulate the axiom MOD here. The standard formulation of MOD in the literature (including Lewis’ works) is

MOD’ \[(\neg \varphi > \varphi) \rightarrow (\psi > \varphi).\]

There are two reasons why I reformulate it. One is that it is this reformulation rather than the standard one that corresponds directly to the associated model condition of worlds selection functions, normally formulated in the literature as follows:

(mod) \[f(w, \varphi) = \emptyset \implies f(w, \psi) \cap [\varphi] = \emptyset,\]
where \([\psi]\) denotes the truth set of \(\psi\), and \(f\) is the selection function, associating with a possible world \(w\) and a sentence \(\varphi\) a set of \(\varphi\)-worlds that are closest to \(w\). Rather, the standard formulation MOD’ corresponds to the following condition instead:

\[(\text{mod}') \quad f(w, \neg \varphi) = \emptyset \implies f(w, \psi) \cap [\neg \varphi] = \emptyset.\]

Of course, if the rule RCEA or RE is available, the difference between the two formulations is immaterial. But if one works on conditional logics without such rules admissible, the two formulations might turn out to be very different. This is related to my second reason for choosing the reformulation. In a proof of the derivation of CSO from MOD’ and PIE below, I find if the reformulation MOD is used then the rule RCEA or RE is dispensable; otherwise, such rules are required for the derivation.

### 3 Amendments of Nute’s Axiomatizations

Nute’s axiomatization for \(\mathbf{V}\) and \(\mathbf{VC}\) are as follows:

\[
\mathbf{Vn} = \langle \text{PC, ID, CM, CC, CV, MOD’, CSO; RCEC} \rangle \\
\mathbf{VCn} = \langle \text{PC, ID, CM, CC, CV, MOD’, CSO, CMP, CS; RCEC} \rangle.
\]

I will show that CA is not derivable in neither of these systems. Since \(\mathbf{VCn}\) is the stronger one, it suffices to prove that CA is not derivable in \(\mathbf{VCn}\).

**Proposition 1.** \(\not\vdash_{\mathbf{VCn}} \text{CA} \).

**Proof.** Let \(U = \{0, 1, 2, 3\}, A = \{1, 2\}\), and \(B = \{1, 3\}\). Define \(g : U \times \wp(U) \rightarrow \wp(U)\) as follows:

\[g(i, X) = \begin{cases} 
\{1\} & \text{if } X = A \text{ and } i = 0 \\
\{i\} & \text{if } i \in X \\
X & \text{otherwise}
\end{cases}\]

Now I verify that \(g\) satisfies the following conditions: for all \(i \in U\) and \(X, Y \in \wp(U)\)

- (id) \(g(i, X) \subseteq X\)
- (mod) \(g(i, X) = \emptyset \implies g(i, Y) \cap X = \emptyset\)

---

Nute’s original axiomatization used the rule RCK instead of the axioms CM and CC. But to reduce inference rules to the minimum, I prefer to use these two axioms instead of the rule RCK. It can be easily shown that they are equivalent as long as RCEC is provided.
(cv) \[ g(i, X) \cap Y \neq \emptyset \implies g(i, X \cap Y) \subseteq g(i, X) \]

(cso) \[ g(i, X) \subseteq Y \text{ and } g(i, Y) \subseteq X \implies g(i, X) = g(i, Y) \]

(cen) \[ i \in X \implies g(i, X) = \{i\} \]

(id) and (cen) are obvious. (mod) holds since \[ g(i, X) = \emptyset \] iff \[ X = \emptyset \]. It remains to verify (cv) and (cso). For (cv), suppose \[ g(i, X) \cap Y \neq \emptyset \]. Consider the following cases:

1. \[ X = A \text{ and } i = 0. \] Then \[ g(i, X) = \{1\} \]. Since \[ g(i, X) \cap Y \neq \emptyset \], we have \[ 1 \in Y \]. Hence \[ X \cap Y = X \text{ or } X \cap Y = \{1\} \]. In both cases, we have \[ g(i, X \cap Y) = g(i, X) \].

2. \[ i \in X. \] Then \[ g(i, X) = \{i\}. \] Since \[ g(i, X) \cap Y \neq \emptyset \], we have \[ i \in Y. \] Then \[ i \in X \cap Y \]. Hence \[ g(i, X \cap Y) = \{i\} = g(i, X) \].

3. \[ X \neq A \text{ or } i \neq 0, \text{ and } i \notin X. \] Then \[ g(i, X) = X. \] Since \[ i \notin X \], we have \[ i \notin X \cap Y. \] Then either \[ g(i, X \cap Y) = \{1\} \] or \[ g(i, X \cap Y) = X \cap Y. \] If \[ g(i, X \cap Y) = X \cap Y \], we have \[ g(i, X \cap Y) \subseteq X = g(i, X). \] If \[ g(i, X \cap Y) \neq X \cap Y \] and \[ g(i, X \cap Y) = \{1\} \], by the definition of \[ g \], we have \[ X \cap Y = A. \] Hence \[ 1 \in X \] and \[ g(i, X \cap Y) \subseteq g(i, X) \].

For (cso), suppose \[ g(i, X) \subseteq Y \text{ and } g(i, Y) \subseteq X. \] Consider the following cases:

1. \[ X = A \text{ and } i = 0. \] Then \[ g(i, X) = \{1\} \]. Since \[ g(i, X) \subseteq Y \], we have \[ 1 \in Y \]. Since \[ g(i, Y) \subseteq X \text{ and } i \notin X \], we have \[ Y = A \text{ or } g(i, Y) = Y \subseteq X. \] In the former case, we have \[ g(i, X) = g(i, Y). \] In the latter case, we have \[ Y = \{1\} \], and hence \[ g(i, Y) = \{1\} = g(i, X) \].

2. \[ i \in X. \] Then \[ g(i, X) = \{i\}. \] Since \[ g(i, X) \subseteq Y \], we have \[ i \in Y. \] Hence \[ g(i, Y) = \{i\} = g(i, X) \].

3. \[ X \neq A \text{ or } i \neq 0, \text{ and } i \notin X. \] Then \[ g(i, X) = X. \] Since \[ g(i, X) \subseteq Y \], we have \[ X \subseteq Y \]. If \[ Y = A \text{ and } i = 0 \], then \[ g(i, Y) = \{1\}. \] By \[ g(i, Y) \subseteq X \], we have \[ 1 \in X \]. Then by \[ X \subseteq Y = A \], we have \[ X = \{1\} \text{ or } X = Y. \] In both cases we have \[ g(i, X) = g(i, Y). \] If \[ i \in Y \], then \[ g(i, Y) = \{i\}. \] Since \[ g(i, Y) \subseteq X \], we have \[ i \in X \], contradicting that \[ i \notin X \]. In other cases, we have \[ g(i, Y) = Y. \] Since \[ g(i, Y) \subseteq X \], we have \[ Y \subseteq X. \] Hence \[ g(i, X) = g(i, Y) \].

Given a model \( \mathfrak{M} = (W, f, V) \), the truth set of \( \varphi \) in \( \mathfrak{M} \), denoted \([\varphi]_\mathfrak{M}\), is inductively defined as follows:

- \([p]_\mathfrak{M} = V(p) \) for \( p \in PV \)
\[\neg \varphi \equiv \overline{\varphi}, \quad \varphi \land \psi \equiv \overline{\varphi} \cap \overline{\psi}, \quad \varphi \supset \psi \equiv \{w \in W \mid f(w, \overline{\varphi}) \subseteq \overline{\psi}\}\]

We say that \(\varphi\) is valid in \(\mathfrak{F} = (W, f)\) if for all models \(M\) based on \(\mathfrak{F}\), \(\overline{\varphi}_M = W\).

Let \(\mathfrak{G} = (U, g)\). By the frame conditions that \(\mathfrak{G}\) satisfies, it can be easily verified that all axioms in \(\text{VC}_n\) are valid in \(\mathfrak{G}\), and \(\mathfrak{G}\) preserves validity for the rule \(\text{RCEC}\). But \(\text{CA}\) is not valid in \(\mathfrak{G}\), since \(g(0, A \cup B) = \{1, 2, 3\} \not\subseteq \{1, 3\} = g(0, A) \cup g(0, B)\). Therefore \(\not\vdash \text{VC}_n\text{CA}\).

\[\square\]

**Corollary 2.** \(\not\vdash \text{VC}_n\text{CA}\)

Now I show that by replacing \(\text{MOD'}\) with \(\text{CA}\) in the corresponding systems, Nute’s axiomatizations can be amended. Let

\[
\begin{align*}
\text{Va} &= (\text{PC, ID, CSO, DAE; RCK}) \\
\text{Vb} &= (\text{PC, ID, MOD', PIE; RCK, RE}) \\
\text{Vc} &= (\text{PC, ID, CM, CC, CV, CA, CSO; RCEC}) \\
\text{VCa} &= (\text{PC, ID, CSO, DAE, CMP, CS; RCK}) \\
\text{VCb} &= (\text{PC, ID, MOD', PIE, CMP, CS; RCK, RE}) \\
\text{VCc} &= (\text{PC, ID, CM, CC, CV, CA, CSO, CMP, CS; RCEC}),
\end{align*}
\]

where \(\text{Va}\) and \(\text{VCa}\) are Lewis’ first axiomatizations of \(V\) and \(\text{VC}\), respectively; \(\text{Vb}\) and \(\text{VCb}\) are his second axiomatizations; \(\text{Vc}\) and \(\text{VCc}\) are amendments of Nute’s systems for \(V\) and \(\text{VC}\), respectively. To prove that these amendments are equivalent to Lewis’ original systems, it suffices to prove that \(\text{Vc}\) is equivalent to both \(\text{Va}\) or \(\text{Vb}\). For completeness, I will show that \(\text{Vc}\) is equivalent to both \(\text{Va}\) and \(\text{Vb}\).

**Proposition 3.** \(\text{Vc} = \text{Va} = \text{Vb}\).

**Proof.** First, I show that \(\text{Vc} \supseteq \text{Va}\). For simplification of proofs, I will show first that \(\text{RCM, RCE, RCN, and RCEA}\) are derivable in \(\text{Vc}\).

For \(\text{RCM}:

\[
\begin{align*}
(1) \quad & \varphi \rightarrow \psi & \text{Assumption}
\end{align*}
\]

\[\footnote{Nute also gave the axiomatization \(\text{VW}_n = (\text{PC, ID, CV, MOD', CSO, CMP; RCEC, RCK})\) for Lewis’ system \(\text{VW}\). So neither is CA derivable from \(\text{VW}_n\). The reason why CA is missing from Nute’s axiomatization is not clear, since no explicit proof of completeness of these systems was given in his writings. I guess the reason may be that CA is derivable in his axiomatization of \(\text{C2}\).} \]
(2) \(\varphi \land \psi \leftrightarrow \varphi\)  \hspace{2cm} (1), PC

(3) \((\chi > \varphi \land \psi) \leftrightarrow (\chi > \varphi)\)  \hspace{2cm} (2), RCEC

(4) \((\chi > \varphi \land \psi) \rightarrow (\chi > \psi)\)  \hspace{2cm} CM, PC

(5) \((\chi > \varphi) \rightarrow (\chi > \psi)\)  \hspace{2cm} (3), (4), PC

For RCE:

(1) \(\varphi \rightarrow \psi\)  \hspace{2cm} Assumption

(2) \((\varphi > \varphi) \rightarrow (\varphi > \psi)\)  \hspace{2cm} (1), RCM

(3) \(\varphi > \varphi\)  \hspace{2cm} ID

(4) \(\varphi > \psi\) \hspace{2cm} (2), (3), PC

For RCN:

(1) \(\psi\)  \hspace{2cm} Assumption

(2) \(\varphi \rightarrow \psi\)  \hspace{2cm} (1), PC

(3) \(\varphi > \psi\) \hspace{2cm} (2), RCE

For RCEA:

(1) \(\varphi \leftrightarrow \psi\)  \hspace{2cm} Assumption

(2) \(\varphi \rightarrow \psi, \psi \rightarrow \varphi\) \hspace{2cm} (1), PC

(3) \(\varphi > \psi, \psi > \varphi\) \hspace{2cm} (2), RCE

(4) \((\varphi > \chi) \leftrightarrow (\psi > \chi)\) \hspace{2cm} (3), CSO, PC

Now I prove that RCK is derivable in \(Vc\). The case for \(n = 0\) is just RCN. The case for \(n = 1\) is just RCM. It remains to prove the case for \(n = 2\). The case for \(n > 2\) can be obtained similarly.

(1) \(\psi_1 \land \psi_2 \rightarrow \psi\) \hspace{2cm} Assumption

(2) \((\varphi > \psi_1 \land \psi_2) \rightarrow (\varphi > \psi)\) \hspace{2cm} (1), RCM

(3) \((\varphi > \psi_1) \land (\varphi > \psi_2) \rightarrow (\varphi > \psi_1 \land \psi_2)\) \hspace{2cm} CC

(4) \((\varphi > \psi_1) \land (\varphi > \psi_2) \rightarrow (\varphi > \psi)\) \hspace{2cm} (2), (3), PC

Next I prove that DAE is derivable in \(Vc\). By CA, it suffices to prove \((\varphi \lor \psi > \varphi) \lor (\varphi \lor \psi > \psi) \lor ((\varphi \lor \psi > \chi) \rightarrow (\varphi > \chi) \land (\psi > \chi))\).
This completes the proof of $V_c \supseteq V_a$.

Now I prove $V_a \supseteq V_b$. First I prove that RCE, RCEA, and RCEC are derivable in $V_a$.

For RCE:

1. $\varphi \rightarrow \psi$  
   Assumption
2. $(\varphi \rightarrow \varphi) \rightarrow (\varphi \rightarrow \psi)$  
   (1), RCK
3. $\varphi \rightarrow \varphi$  
   ID
4. $\varphi \rightarrow \psi$  
   (2), (3), PC

For RCEA:

1. $\varphi \leftrightarrow \psi$  
   Assumption
2. $\varphi \rightarrow \psi$, $\psi \rightarrow \varphi$  
   (1), PC
3. $(\varphi \rightarrow \varphi), (\psi \rightarrow \varphi)$  
   (2), RCK
(4) \( \varphi > \varphi, \psi > \psi \)  
(5) \( \varphi > \psi, \psi > \varphi \)  
(6) \( (\varphi > \chi) \leftrightarrow (\psi > \chi) \)

For RCEC:

(1) \( \varphi \leftrightarrow \psi \)  
(2) \( \varphi \rightarrow \psi, \psi \rightarrow \varphi \)  
(3) \( (\chi > \varphi) \rightarrow (\chi > \psi), (\chi > \psi) \rightarrow (\chi > \varphi) \)  
(4) \( (\chi > \varphi) \leftrightarrow (\chi > \psi) \)

From RCEA and RCEC, RE is obtained by a simple induction.

Then I prove that CA is derivable in Va. By DAE, it suffices to prove
\((\varphi \lor \psi > \varphi) \lor (\varphi \lor \psi > \psi) \rightarrow CA\)

(1) \( \varphi \rightarrow \varphi \lor \psi, \psi \rightarrow \varphi \lor \psi \)  
(2) \( \varphi > \varphi \lor \psi, \psi > \varphi \lor \psi \)  
(3) \( (\varphi \lor \psi > \varphi) \rightarrow ((\varphi > \chi) \rightarrow (\varphi \lor \psi > \chi)) \)  
(4) \( (\varphi \lor \psi > \psi) \rightarrow ((\psi > \chi) \rightarrow (\varphi \lor \psi > \chi)) \)  
(5) \( (\varphi \lor \psi > \varphi) \lor (\varphi \lor \psi > \psi) \rightarrow CA \)

Now I prove that MOD' is derivable in Va.

(1) \( \varphi \land \neg \varphi \rightarrow \varphi \land \psi \)  
(2) \( (\neg \varphi > \varphi) \land (\neg \varphi > \neg \varphi) \rightarrow (\neg \varphi > \neg \varphi \land \psi) \)  
(3) \( \neg \varphi > \neg \varphi \)  
(4) \( \neg \varphi \land \psi \rightarrow \neg \varphi \)  
(5) \( \neg \varphi \land \psi > \neg \varphi \)  
(6) \( (\neg \varphi > \varphi) \rightarrow (\neg \varphi \land \psi > \varphi) \)  
(7) \( \varphi \land \psi \rightarrow \varphi \)

\[\text{This is of course not a proper notation. It is used to abbreviate the rather long formula (scheme) } (\varphi \lor \psi > \varphi) \lor (\varphi \lor \psi > \psi) \rightarrow ((\varphi > \chi) \land (\psi > \chi) \rightarrow (\varphi \lor \psi > \chi)). \text{ I will use this kind of abbreviation occasionally below. The abbreviated formulas should be easily recovered from context.}\]
Next I prove that PIE is derivable in $Va$. It suffices to prove

(a) $(\varphi \land \psi > \chi) \rightarrow (\varphi > (\psi \rightarrow \chi))$, and

(b) $(\varphi > \neg \psi) \lor ((\varphi > (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi > \chi))$

For (a): Let $\alpha = (\varphi \land \psi) \lor (\varphi \land \neg \psi)$

(1) $\varphi \land \neg \psi \rightarrow \neg \psi \lor \chi$ PC

(2) $\varphi \land \neg \psi > \neg \psi \lor \chi$ (1), RCE

(3) $\chi \rightarrow \neg \psi \lor \chi$ PC

(4) $(\varphi \land \psi > \chi) \rightarrow (\varphi \land \psi > \neg \psi \lor \chi)$ (3), RCK

(5) $(\varphi \land \psi > \neg \psi \lor \chi) \land (\varphi \land \neg \psi > \neg \psi \lor \chi) \rightarrow (\alpha > \neg \psi \lor \chi)$ CA

(6) $(\varphi \land \psi > \chi) \rightarrow (\alpha > \neg \psi \lor \chi)$ (2), (4), (5), PC

(7) $\alpha \leftrightarrow \varphi, \neg \psi \lor \chi \leftrightarrow (\psi \rightarrow \chi)$ PC

(8) $(\varphi \land \psi > \chi) \rightarrow (\varphi > (\psi \rightarrow \chi))$ (6), (7), RE

For (b): Let $\alpha = (\varphi \land \neg \psi) \lor (\varphi \land \psi)$

(1) $(\alpha > \varphi \land \neg \psi) \lor (\alpha > \varphi \land \psi) \lor ((\alpha > (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi > (\psi \rightarrow \chi)))]$ DAE, PC

(2) $\alpha \leftrightarrow \varphi$ PC

(3) $(\varphi > \varphi \land \neg \psi) \lor (\varphi > \varphi \land \psi) \lor ((\varphi > (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi > (\psi \rightarrow \chi)))]$ (1), (2), RE

(4) $\psi \land (\psi \rightarrow \chi) \rightarrow \chi$ PC

(5) $(\varphi \land \psi > \psi) \land (\varphi \land \psi > (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi > \chi)$ (4), RCK

(6) $\varphi \land \psi > \psi$ PC, RCE

(7) $(\varphi > \varphi \land \neg \psi) \lor (\varphi > \varphi \land \psi) \lor ((\varphi > (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi > \chi))$ (3), (5), (6), PC
(8) \((\varphi \land \psi) \land (\psi \rightarrow \chi) \rightarrow \chi\) \hspace{1cm} \text{PC}

(9) \((\varphi > \varphi \land \psi) \land (\varphi > (\psi \rightarrow \chi)) \rightarrow (\varphi > \chi)\) \hspace{1cm} (8), \text{RCK}

(10) \((\varphi > \varphi \land \psi) \land (\varphi > (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi > \chi)\) \hspace{1cm} (6), (9), \text{CSO}

(11) \varphi \land \neg \psi \rightarrow \neg \psi \hspace{1cm} \text{PC}

(12) \((\varphi > \varphi \land \neg \psi) \rightarrow (\varphi > \neg \psi)\) \hspace{1cm} (11), \text{RCK}

(13) \neg (\varphi > \neg \psi) \rightarrow \neg (\varphi > \varphi \land \neg \psi) \hspace{1cm} (12), \text{PC}

(14) \neg (\varphi > \neg \psi) \land (\neg (\varphi > \varphi \land \psi) \land (\varphi > (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi > \chi)\) \hspace{1cm} (7), (13), \text{PC}

(15) \neg (\varphi > \neg \psi) \land (\varphi > (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi > \chi) \hspace{1cm} (10), (14), \text{PC}

(16) \((\varphi > \neg \psi) \lor ((\varphi > (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi > \chi))\) \hspace{1cm} (15), \text{PC}

This completes the proof of \(V_a \supseteq V_b\).

Now I prove \(V_b \supseteq V_c\). The derivation of CC and CM is straightforward using RCK. The rule RCEC is a special case of RE. It remains to show that CA, CV, and CSO are derivable in \(V_b\).

For CV:

(1) \((\varphi > \neg \psi) \land (\varphi > (\psi \rightarrow \chi)) \rightarrow (\varphi \land \psi > \chi)\) \hspace{1cm} \text{PIE, PC}

(2) \chi \rightarrow (\psi \rightarrow \chi) \hspace{1cm} \text{PC}

(3) \((\varphi > \chi) \rightarrow (\varphi > (\psi \rightarrow \chi))\) \hspace{1cm} (2), \text{RCK}

(4) \((\varphi > \chi) \land \neg (\varphi > \neg \psi) \rightarrow (\varphi \land \psi > \chi)\) \hspace{1cm} (1), (3), \text{PC}

For CSO: Let \(\alpha = \varphi > \neg \psi, \beta = \psi > \neg \varphi\). By PC, it suffices to prove

(c) \((\neg \alpha \land \neg \beta \land (\varphi > \psi) \land (\psi > \varphi) \rightarrow ((\varphi > \chi) \leftrightarrow (\psi > \chi)))\),

(d) \((\alpha \land (\varphi > \psi) \land (\psi > \varphi) \rightarrow ((\varphi > \chi) \leftrightarrow (\psi > \chi)))\), and

(e) \((\beta \land (\varphi > \psi) \land (\psi > \varphi) \rightarrow ((\varphi > \chi) \leftrightarrow (\psi > \chi)))\).

For (c):

(1) \((\neg \alpha \rightarrow ((\varphi \land \psi > \chi) \leftrightarrow (\varphi > (\psi \rightarrow \chi))))\) \hspace{1cm} \text{PIE}

(2) \psi \rightarrow (\psi \rightarrow \chi) \rightarrow \chi \hspace{1cm} \text{PC}

(3) \((\varphi > \psi) \land (\varphi > (\psi \rightarrow \chi)) \rightarrow (\varphi > \chi)\) \hspace{1cm} (2), \text{RCK}

(4) \chi \rightarrow (\psi \rightarrow \chi) \hspace{1cm} \text{PC}
(5) \((\varphi > \chi) \rightarrow (\varphi > (\psi \rightarrow \chi))\)  

(6) \(\neg \alpha \land (\varphi > \psi) \rightarrow ((\varphi > \chi) \leftrightarrow (\varphi \land \psi > \chi))\) \hspace{1cm} (1), (3), (5), PC

(7) \(\neg \beta \land (\psi > \varphi) \rightarrow ((\psi > \chi) \leftrightarrow (\varphi \land \psi > \chi))\) \hspace{1cm} analogous to (1)–(6)

(8) \(\neg \alpha \land \neg \beta \land (\varphi > \psi) \land (\psi > \varphi) \rightarrow ((\varphi > \chi) \leftrightarrow (\psi > \chi))\) \hspace{1cm} (6), (7), PC

For (d):

(1) \(\neg \psi \land \psi \rightarrow \chi\) \hspace{1cm} PC

(2) \(\alpha \land (\varphi > \psi) \rightarrow (\varphi > \chi)\) \hspace{1cm} (1), RCK

(3) \(\alpha \land (\varphi > \psi) \rightarrow (\varphi > \neg \varphi)\) \hspace{1cm} (2), PC

(4) \((\varphi > \neg \varphi) \rightarrow (\psi > \neg \varphi)\) \hspace{1cm} MOD

(5) \(\alpha \land (\varphi > \psi) \rightarrow (\psi > \neg \varphi)\) \hspace{1cm} (3), (4), PC

(6) \(\alpha \land (\varphi > \psi) \land (\psi > \varphi) \rightarrow (\psi > \varphi \land \neg \varphi)\) \hspace{1cm} (5), RCK, PC

(7) \(\varphi \land \neg \varphi \rightarrow \chi\) \hspace{1cm} PC

(8) \(\alpha \land (\varphi > \psi) \land (\psi > \varphi) \rightarrow (\psi > \chi)\) \hspace{1cm} (6), (7), RCK, PC

(9) \(\alpha \land (\varphi > \psi) \land (\psi > \varphi) \rightarrow ((\varphi > \chi) \leftrightarrow (\psi > \chi))\) \hspace{1cm} (2), (8), PC

Note that in the above derivation, I use MOD instead of MOD’, so that I can dispense with RE. If MOD’ is used instead, then the derivation is longer, with an additional line of transforming MOD’ to MOD, using RE.

(e) can be proved analogously to (d).

For CA: Let \(\alpha = \varphi \lor \psi > \neg \varphi, \beta = \varphi \lor \psi > \neg \psi\). It suffices to prove

(f) \(\neg \alpha \land \neg \beta \land (\varphi > \chi) \land (\psi > \chi) \rightarrow (\varphi \lor \psi > \chi)\),

(g) \(\alpha \land (\varphi > \chi) \land (\psi > \chi) \rightarrow (\varphi \lor \psi > \chi)\), and

(h) \(\beta \land (\varphi > \chi) \land (\psi > \chi) \rightarrow (\varphi \lor \psi > \chi)\).

For (f):

(1) \(\neg \alpha \land ((\varphi \lor \psi) \land \varphi > \chi) \rightarrow (\varphi \lor \psi > (\varphi \rightarrow \chi))\) \hspace{1cm} PIE, PC

(2) \((\varphi \lor \psi) \land \varphi \leftrightarrow \varphi\) \hspace{1cm} PC

(3) \(\neg \alpha \land (\varphi > \chi) \rightarrow (\varphi \lor \psi > (\varphi \rightarrow \chi))\) \hspace{1cm} (1), (2), RE

(4) \(\neg \beta \land (\psi > \chi) \rightarrow (\varphi \lor \psi > (\psi \rightarrow \chi))\) \hspace{1cm} analogous to (1)–(3)
(5) \( \varphi \lor \psi > \varphi \lor \psi \) 
(6) \((\varphi \lor \psi) \land (\varphi \rightarrow \chi) \land (\psi \rightarrow \chi) \rightarrow \chi \)
(7) \((\varphi \lor \psi > (\varphi \rightarrow \chi)) \land (\varphi \lor \chi > (\psi \rightarrow \chi)) \rightarrow (\varphi \lor \psi > \chi)\) (5), (6), RCK, PC
(8) \( \neg \alpha \land \neg \beta \land (\varphi > \chi) \land (\psi > \chi) \rightarrow (\varphi \lor \psi > \chi) \) (3), (4), (7), PC

For (g):

(1) \((\varphi \lor \psi) \land \neg \varphi \rightarrow \psi \)
(2) \(\varphi \lor \psi > \varphi \lor \psi \)
(3) \(\alpha \rightarrow (\varphi \lor \psi > \psi) \) (1), (2), RCK, PC
(4) \(\psi > \varphi \lor \psi \)
(5) \(\alpha \land (\psi > \chi) \rightarrow (\varphi \lor \psi > \chi) \) (3), (4), CSO
(6) \(\alpha \land (\varphi > \chi) \land (\psi > \chi) \rightarrow (\varphi \lor \psi > \chi) \) (5), PC

(h) can be prove analogously to (g).
This completes the proof of \( Vb \supseteq Vc \).

\[ \square \]

Corollary 4. \( V_{Cc} = V_{Ca} = V_{Cb} \).

4 New Axiomatizations of Lewis’ Conditional Logics

I propose the following new axiomatizations of Lewis’ conditional logics, which are denoted by \( V' \) and \( VC' \), respectively.

\[ V' = \{ PC, ID, CM, CA, CV, AC, RT; RCEC \} \]
\[ VC' = \{ PC, ID, CM, CA, CV, AC, RT, CMP, CS; RCEC \} \]

Both systems replace the axiom CSO by the axioms AC and RT in \( Vc \) and \( VCc \), respectively. Meanwhile, CC is omitted, since it is derivable from other axioms and rules. I will prove that the new axiomatizations are equivalent to Lewis’ original ones. By Proposition 3 and Corollary 4 it suffices to prove that \( V' \) is equivalent to \( Vc \).

Proposition 5. \( V' = Vc \).
Proof. First, I show that $Vc \supseteq V'$, i.e. AC and RT are derivable in $Vc$.

For AC:

(1) $(\varphi \vartriangleright \varphi) \land (\varphi \vartriangleright \psi) \rightarrow (\varphi \vartriangleright \varphi \land \psi)$ \hspace{1cm} CC
(2) $\varphi \vartriangleright \varphi$ \hspace{1cm} ID
(3) $(\varphi \vartriangleright \psi) \rightarrow (\varphi \vartriangleright \varphi \land \psi)$ \hspace{1cm} (1), (2), PC
(4) $\varphi \land \psi \rightarrow \varphi$ \hspace{1cm} PC
(5) $\varphi \land \psi \vartriangleright \varphi$ \hspace{1cm} (4), RCE
(6) $(\varphi \vartriangleright \psi) \rightarrow ((\varphi \vartriangleright \varphi \land \psi) \land (\varphi \land \psi \vartriangleright \varphi))$ \hspace{1cm} (3), (5), PC
(7) $(\varphi \vartriangleright \varphi \land \psi) \land (\varphi \land \psi \vartriangleright \varphi) \rightarrow ((\varphi \vartriangleright \chi) \leftrightarrow (\varphi \land \psi \vartriangleright \chi))$ \hspace{1cm} CSO
(8) $(\varphi \vartriangleright \psi) \land (\varphi \vartriangleright \chi) \rightarrow (\varphi \land \psi \vartriangleright \chi)$ \hspace{1cm} (6), (7), PC

For RT:

(1) $\psi \land \varphi \rightarrow \varphi$ \hspace{1cm} PC
(2) $\psi \land \varphi \vartriangleright \varphi$ \hspace{1cm} (1), RCE
(3) $\varphi \vartriangleright \varphi$ \hspace{1cm} ID
(4) $(\varphi \vartriangleright \psi) \rightarrow (\varphi \vartriangleright \psi \land \varphi)$ \hspace{1cm} (3), CC, PC
(5) $(\varphi \vartriangleright \psi) \rightarrow ((\varphi \vartriangleright \psi \land \varphi) \land (\psi \land \varphi \vartriangleright \varphi))$ \hspace{1cm} (2), (4), PC
(6) $(\varphi \vartriangleright \psi \land \varphi) \land (\psi \land \varphi \vartriangleright \varphi) \rightarrow ((\varphi \vartriangleright \chi) \leftrightarrow (\varphi \land \psi \vartriangleright \chi))$ \hspace{1cm} CSO
(7) $(\varphi \vartriangleright \psi) \land (\psi \land \varphi \vartriangleright \chi) \rightarrow (\varphi \vartriangleright \chi)$ \hspace{1cm} (5), (6), PC

Then I show that $V' \supseteq Vc$.

For CSO:

(1) $(\varphi \vartriangleright \psi) \land (\varphi \vartriangleright \chi) \rightarrow (\varphi \land \psi \vartriangleright \chi)$ \hspace{1cm} AC
(2) $(\psi \vartriangleright \varphi) \land (\varphi \land \psi \vartriangleright \chi) \rightarrow (\psi \vartriangleright \chi)$ \hspace{1cm} RT
(3) $(\varphi \vartriangleright \psi) \land (\psi \vartriangleright \varphi) \land (\varphi \vartriangleright \chi) \rightarrow (\psi \vartriangleright \chi)$ \hspace{1cm} (1), (2), PC
(4) $(\varphi \vartriangleright \psi) \land (\psi \vartriangleright \varphi) \land (\psi \vartriangleright \chi) \rightarrow (\varphi \vartriangleright \chi)$ \hspace{1cm} analogous to (1)–(3)
(5) $(\varphi \vartriangleright \psi) \land (\psi \vartriangleright \varphi) \rightarrow ((\varphi \vartriangleright \chi) \leftrightarrow (\psi \vartriangleright \chi))$ \hspace{1cm} (3), (4), PC
To prove CC, note that we have proved that RCE can be obtained from PC, ID, CM, and RCEC in the proof of Proposition 3. Since RCEA follows from RCE and CSO, we also have RCEA in V'. Now we have the following derivation for CC:

(1) $\varphi \land \psi \land \chi > \varphi \land \psi \land \chi$  \hspace{1cm} ID
(2) $\varphi \land \psi \land \chi > \psi \land \chi$  \hspace{1cm} (1), CM, PC
(3) $\varphi \land \psi \land \chi \leftrightarrow \chi \land \varphi \land \psi$  \hspace{1cm} PC
(4) $\chi \land \varphi \land \psi > \psi \land \chi$  \hspace{1cm} (2), (3), RCEA, PC
(5) $(\varphi > \psi) \land (\varphi > \chi) \rightarrow (\varphi \land \psi > \chi)$  \hspace{1cm} AC
(6) $(\varphi > \psi) \land (\varphi > \chi) \rightarrow (\varphi \land \psi > \psi \land \chi)$  \hspace{1cm} (4), (5), RT, PC
(7) $\varphi \land \psi \leftrightarrow \psi \land \varphi$  \hspace{1cm} PC
(8) $(\varphi \land \psi > \psi \land \chi) \rightarrow (\psi \land \varphi > \psi \land \chi)$  \hspace{1cm} (7), RCEA
(9) $(\varphi > \psi) \land (\psi \land \varphi > \psi \land \chi) \rightarrow (\varphi > \psi \land \chi)$  \hspace{1cm} RT
(10) $(\varphi > \psi) \land (\varphi > \chi) \rightarrow (\varphi > \psi \land \chi)$  \hspace{1cm} (6), (8), (9), PC

\[ \square \]

Corollary 6. VC' = VCc.

The axiom CSO was criticized by Gabbay (1972). One may be inclined to abandon it directly. However, the above new systems show that CSO can be recovered from AC and RT. It should be easy to notice that AC and RT correspond to cautious monotonicity and cautious cut (a.k.a. cumulative transitivity) in nonmonotonic logics. Both cautious monotonicity and cautious cut are regarded as the minimal requirements for nonmonotonic consequences. If AC and RT are also taken to be minimal for conditional logics, then the above proof shows that CSO is inevitable in conditional logics. If CSO is inevitable, then RCEA is also inevitable, since it follows from CSO and another very modest axiom CM. The new axiomatization indicates that it is difficult to construct nonclassical conditional logics for characterizing default conditionals. It also leads us to a puzzle about the controversial axiom SDA, which is the converse of CA.
5 A Resolution of a Puzzle about SDA

The axiom SDA suggests that conditionals with disjunctive antecedents have conjunctive reading. For example, when I say that if John or Mary comes to my party, I’ll be happy, it is reasonable to conclude that if John comes to my party I’ll be happy, and if Mary comes to my party I’ll be happy. But if SDA is contained in any conditional logic with the rule RCEA, the so called fallacy of strengthening the antecedent which is rejected in all conditional logics will be recovered. This can be shown by the following simple derivation:

\[
\begin{align*}
(1) & \quad \varphi \leftrightarrow (\varphi \lor (\varphi \land \psi)) \\
(2) & \quad (\varphi \lor (\varphi \land \psi) > \chi) \rightarrow \varphi \land \psi > \chi \quad \text{SDA, PC} \\
(3) & \quad (\varphi > \chi) \rightarrow (\varphi \land \psi > \chi) \quad \text{(1), (2), RCEA, PC}
\end{align*}
\]

There are mainly three approaches to solving this puzzle. The first approach, adopted in (Loewer, 1976; McKay & van Inwagen, 1977; Nute, 1980a; Lewis, 1977), is to abandon SDA and apply something other than logic such as translation lore to account for the intuitive validity of SDA. The second approach, adopted in (Nute, 1975, 1978, 1980b), is to keep SDA while giving up the rule RCEA by developing nonclassical conditional logics. As we have seen in Section 4, this means that some other intuitively reasonable axioms such as AC or RT have to be abandoned too. In (Fine, 1975), both the first two approaches were suggested. The third approach, adopted in (Alonso-Ovalle, 2006; Klinedinst, 2007; Paoli, 2012), is to give nonclassical interpretations for disjunction, so that the disjunctive antecedents in conditionals have conjunctive reading. All the approaches are somewhat ad hoc, in the sense that conditionals with disjunctive antecedents are treated as special and different from other conditionals.

It has been noticed that SDA has counterexamples in both counterfactual and indicative conditionals. The following is one for counterfactuals given in (McKay & van Inwagen, 1977):

(1) If Spain fought on the Axis side or fought on the Allied side, it would fight on the Axis side.

(2) If Spain fought on the Allied side, it would fight on the Axis side.

By SDA, (1) implies (2). But obviously (2) is false even if (1) is true. A similar counterexample for indicative conditionals was given in (Carlstrom & Hill, 1978):

(3) If Ivan is playing tennis or playing baseball, then he is playing baseball.
(4) If Ivan is playing tennis, then he is playing baseball.

By SDA, (3) implies (4). But we can have (3) true and (4) false. Both counterexamples have the following form: $\varphi \lor \psi > \varphi$ is true but $\psi > \varphi$ false. As far as I know, no other forms of counterexamples of SDA have been discovered. Considering that SDA has only counterexamples of such special forms, one can not resist keeping SDA while explaining away such counterexamples by attributing them as abnormal uses of conditionals with disjunctive antecedents. But we still face the conflict between SDA and RCEA. Remarkably, one of Lewis’ axioms for conditional logics, namely the old-fashioned axiom DAE, which has been neglected for a long time, can perfectly account for both the intuitive validity of SDA and its counterexamples! The axiom DAE says that either $\varphi \lor \psi > \varphi$ is true, or $\varphi \lor \psi > \psi$ is true, or $(\varphi \lor \psi > \chi)$ is logically equivalent to $(\varphi > \chi) \land (\psi > \chi)$. From DAE it follows that

$$-(\varphi \lor \psi > \varphi) \land -(\varphi \lor \psi > \psi) \to \text{SDA},$$

which is weaker than SDA. But it is not too weak, since as long as we exclude the cases when the disjunctive antecedent conditionally implies one of its disjuncts, which are exactly the counterexamples for SDA we have found, SDA is obtained. I think this resolution of the puzzle around SDA is better than previous ones, since we can dispense with any special treatments of the conditionals with disjunctive antecedents. It is a big surprise that Lewis himself did not discover this simple solution, even though he had published a note (Lewis, 1977) about SDA some years after he proposed the axiom DAE in (Lewis, 1971).

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