An Extension of Alzer’s Inequality by Convexity

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Abstract
In this article, we obtain two interesting general inequalities concerning Riemann sums of convex functions, which in particular, sharpen Alzer’s inequality and give a suitable converse for it.

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1 Introduction
In [1], H. Alzer proved the following inequality

\[
\frac{n}{n+1} \leq \left( (n+1) \sum_{i=1}^{n} i^r/n \sum_{i=1}^{n+1} i^r \right)^{1/r}
\]

(1)
where \( r \) is a positive real and \( n \) is a natural number. In other words, the Riemann sums \( \frac{1}{n} \sum_{i=1}^{n} \left( \frac{i}{n} \right)^r \) (\( n = 1, 2, \cdots \)) of the function \( x^r \) is a decreasing sequence. The proof of Alzer [1] is technical, but quite complicated. So, in several articles Alzer’s proof has been simplified, and also in many others, this inequality has been extended; see e.g. [2-4].

In this article, using some trivial facts about convex functions, we obtain some valuable results concerning special kinds of Riemann sums of convex functions, from which Alzer’s inequality with an its converse are handled at once.

2 Main Results

Throughout this section, we suppose that \( f : [a, b] \to \mathbb{R} \) is an arbitrary function on a closed interval \([a, b]\), and put

\[
A_n = \frac{b - a}{n} \sum_{i=1}^{n} f \left( x_i^{(n)} \right) \quad \text{and} \quad B_n = \frac{b - a}{n} \sum_{i=0}^{n-1} f \left( x_i^{(n)} \right),
\]

where

\[
x_i^{(n)} = a + i \frac{b - a}{n} \quad (i = 0, 1, \cdots, n; \ n = 1, 2, \cdots)
\]

When emphasizing, we write \( A_n(f) \) instead of \( A_n \), and so on. In the following theorem, we obtain some recursive inequalities concerning \( A_n \) and \( B_n \), which as a corollary, give Alzer’s inequality and a converse for it.

Theorem 2.1 With the above assumptions, if \( f \) is convex, then we have

\[
A_{n+1} + \frac{1}{n(n+2)} \left[ A_{n+1} - (b - a)f(a) \right] \leq A_n \leq A_{n+1} + \frac{1}{n^2} \left[ (b - a)f(b) - A_{n+1} \right] \tag{3}
\]

and

\[
B_{n+1} + \frac{1}{n(n+2)} \left[ B_{n+1} - (b - a)f(b) \right] \leq B_n \leq B_{n+1} + \frac{1}{n^2} \left[ (b - a)f(a) - B_{n+1} \right]. \tag{4}
\]

Moreover,

\[
A_{n+1} \leq (b - a) \left[ \frac{n}{2(n+1)} f(a) + \frac{n+2}{2(n+1)} f(b) \right] \tag{5}
\]

and

\[
B_{n+1} \leq (b - a) \left[ \frac{n+2}{2(n+1)} f(a) + \frac{n}{2(n+1)} f(b) \right]. \tag{6}
\]
If \( f \) is concave, all the above inequalities reverse. Moreover, all these inequalities are strict in the case of strict convexity or concavity.

**Proof.** Since \( x_i^{(n+1)} = \frac{i}{n+1}x_{i-1}^{(n)} + \frac{n+1-i}{n+1}x_i^{(n)} \) (1 ≤ \( i \) ≤ \( n \)), by Jensen’s inequality, we have

\[
f(x_i^{(n+1)}) \leq \frac{i}{n+1}f(x_{i-1}^{(n)}) + \frac{n+1-i}{n+1}f(x_i^{(n)}) \quad (1 \leq i \leq n),
\]

which by summing them up from \( i = 1 \) to \( i = n \), with some calculations, we get the left hand side of (3).

Similarly, since \( x_i^{(n)} = \frac{n+1-i}{n}x_{i-1}^{(n+1)} + \frac{i}{n}x_i^{(n+1)} \) (1 ≤ \( i \) ≤ \( n+1 \)), we have

\[
f(x_i^{(n)}) \leq \frac{n+1-i}{n}f(x_{i-1}^{(n+1)}) + \frac{i}{n}f(x_i^{(n+1)}) \quad (1 \leq i \leq n+1),
\]

which by summing them up from \( i = 1 \) to \( i = n+1 \), with some calculations, we get the right hand side of (4).

The right hand side of (3) and the left hand side of (4) are obtained from the right hand side of (4) and the left hand side of (3) respectively, by considering

\[
A_k - B_k = \frac{(b-a)}{k}[f(b) - f(a)] \quad (k = 1, 2, \cdots).
\]

The inequalities (5) and (6) are trivially obtained by comparing the left and right hand sides of (4) and (5) with each other respectively.

**Corollary 2.2.** If \( f \) is an increasing convex or concave function on \([a, b]\), then

\[
A_{n+1} \leq A_n \quad \text{and} \quad B_n \leq B_{n+1} \quad (n = 1, 2, \cdots). \quad (7)
\]

**Proof.** Since \( f \) is increasing, for each \( k \) we have

\[
(b-a)f(a) \leq A_k \leq (b-a)f(b) \quad (8)
\]

and

\[
(b-a)f(a) \leq B_k \leq (b-a)f(b). \quad (9)
\]

So, if \( f \) is convex, using (8), (9), the left hand side of (3) and the right hand side of (4), we get (7).

Now, if \( f \) is concave, then \(-f\) is convex, and (7) follows from the right hand side of (3) and the left hand side of (4) applying to \(-f\), by using (8) and (9) and taking into consideration that

\[
A_k(-f) = -A_k(f) \quad \text{and} \quad B_k(-f) = -B_k(f) \quad (k = 1, 2, \cdots).
\]
Corollary 2.3. If \( r \geq 1 \), then

\[
\frac{n}{n+1} \left( 1 + \frac{1}{n(n+2)} \right)^{1/r} \leq \left( (n+1) \sum_{i=1}^{n} i^r / n \sum_{i=1}^{n+1} i^r \right)^{1/r}
\]

(10)

\[
\leq \frac{n}{n+1} \left( 1 + \frac{(n+1)^{r+1} - \sum_{i=1}^{n+1} i^r}{n^2 \sum_{i=1}^{n+1} i^r} \right)^{1/r}.
\]

If \( 0 < r \leq 1 \) the inequalities in (10) reverse.

Obviously, these give us a refinement and a reverse of Alzer’s inequality (1).

**Proof.** If \( r \geq 1 \), the function \( f(x) = x^r \ (x \geq 0) \) is convex and so (10) follows from (3), by taking \( a = 0 \) and \( b = 1 \).

If \( 0 < r \leq 1 \) the function \( f \) is concave and so \( -f \) is convex. So, the inequalities in (3), and therefore the inequalities in (10) reverse.

**REFERENCES**

1. H. Alzer, On an inequality of H. Minc and L. Sathre, *J. Math. Anal. Appl.*, 179 (1993), 396-402.

2. F. Qi, Generalization of H. Alzer’s inequality, *J. Math. Anal. Appl.* 240 (1999), no. 1, 294–297.

3. Elezović, N.; Pečarić, J. On Alzer’s inequality. *J. Math. Anal. Appl.* 223 (1998), no. 1, 366–369.

4. J. Sandor, On an inequality of Alzer, *J. Math. Anal. Appl.* 192 (1995), no. 3, 1034-1035.