BOST-CONNES-MARCOLLI SYSTEM FOR THE SIEGEL MODULAR VARIETY

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ABSTRACT. We present a generalization of the Connes-Marcolli $GL_{Q,2}$-system by constructing a quantum statistical mechanical system. Specifically, we introduce the Connes-Marcolli system associated with the Siegel modular variety of degree 2. We investigate the system’s KMS$_\beta$-states for various inverse temperatures $\beta > 0$. Our results reveal a spontaneous phase transition occurring at $\beta = 3$. We demonstrate that the system lacks a KMS$_\beta$ state for $\beta < 3$ with $\beta \neq 1$, identify the explicit extremal Gibbs states for $\beta > 4$, and prove that a unique KMS$_\beta$ state exists for every $\beta > 0$ with $3 < \beta \leq 4$.

1. INTRODUCTION

By the end of last century, Bost and Connes [1], motivated by the ideas introduced by B. Julia [14], constructed a quantum statistical mechanical system $(\mathcal{A}, (\sigma_t)_{t \in \mathbb{R}})$ with unexpected connection between class field theory of $\mathbb{Q}$ and the theory of $C^*$-dynamical systems. Bost and Connes defined a dense rational subalgebra $\mathcal{A}_Q \subset \mathcal{A}$ such that the evaluation of equilibrium states at low temperatures on $\mathcal{A}_Q$ generate the maximal abelian extension $\mathbb{Q}^{ab}$ of $\mathbb{Q}$. As a quantum statistical $C^*$-dynamical system, the Bost-Connes system has also the following interesting thermodynamical property: it exhibits a phase transition at inverse temperature $\beta = 1$. Moreover, this transition happens to be spontaneous in the sense that the symmetry of the system $(\mathcal{A}, (\sigma_t)_{t \in \mathbb{R}})$ changes radically with small changes of temperature. For $\beta > 1$, the system admits $\zeta(\beta)$ as its partition function and the extremal KMS$_\beta$ states have type I while for $0 < \beta < 1$ it admits a unique KMS$_\beta$ states of type $\text{III}_1$.

Over the past several years, several generalizations of the BC system have been studied. The construction of Bost and Connes was first generalized by Connes and Marcolli [8] to quadratic number fields. They introduced the so called the $GL_{2,Q}$-system and showed that maximal abelian extensions of quadratic number fields are generated by evaluating the ground states on a dense rational arithmetic subalgebra. In a subsequent work by Connes, Marcolli and Ramachandran [9], the connection between the $GL_{2}$-system and CM-fields was formally studied. The construction of Connes and Marcolli was further generalized to arbitrary number fields by Ha and Paugam [12]. The former authors reformulated the $GL_{2,Q}$ system in the adelic language, making explicit its relation to the Shimura datum $(GL_2, \mathbb{H}^\pm)$ over any given number field. They generalized the construction of Bost and Marcolli to an arbitrary Shimura datum $(G, X)$ and introduced a formal definition of the abstract Bost-Connes-Marcolli system associated to the pair $(G, X)$. It was shown that these systems admit the Dedekind zeta function as the partition function and the group of connected components of the idèle class group acts as the symmetry group.

In [18], Laca, Larsen and Neshveyev gave yet another reformulation of the $GL_{2,Q}$-system in terms of groupoid $C^*$-algebras. They recovered the classification results of KMS$_\beta$ states obtained by Connes and Marcolli and proved uniqueness in the critical range $1 < \beta \leq 2$. A by-product of the author’s work is the development of a general framework for analyzing dynamical systems of the
that in the region \( \beta > 3 \) the group \( H \) can not easily resolve this issue by excluding the subset \( \mathbb{H}_2^+ \times \{0_2, \} \). Our approach consists of first replacing the homogeneous space \( \mathbb{H}_2^+ \) by the quotient \( K \setminus PGSp_4^+(\mathbb{R}) \) (where \( K \) is a compact subgroup of \( PGSp_4(\mathbb{R})^+ \)) and then prove a one-to-one correspondence of the KMS\(_{\beta} \) states between the two systems. We first establish the correspondence between the set of KMS\(_{\beta} \) states and Borel measures, which allows us to study the properties of those measures instead of working directly with the KMS\(_{\beta} \) states. The second difficulty arises from the structure of the Hecke pair \( (GS^p_{2n}(\mathbb{Q}), Sp_{2n}(\mathbb{Z})) \). As we show in this paper, the case \( n = 2 \) is already computationally demanding and even in this case it is not always possible to directly apply some techniques used in [18] (especially in the critical interval \( 3 < \beta \leq 4 \)). As a first result we show in Theorem 3.8 that the \( GSp_4 \)-system does admit any KMS\(_{\beta} \) state for \( 0 < \beta < 3 \) and \( \beta \notin \{1, 2\} \). We next show that the extremal states in the region \( \beta > 4 \) correspond to Gibbs states and give an explicit construction of these states in Theorem 3.9. The final main result (Theorem 3.17) is uniqueness theorem: we show that in the region \( 3 < \beta \leq 4 \), the \( GSp_4 \)-system admits a unique KMS\(_{\beta} \) state. To show this, we split the proof into two parts. The main ingredient of the first is the convergence of Dirichlet L-functions for non-trivial characters. The second part relies on a variant of the technique used in [18]. As stated above, the structure of the Hecke pair \( (GS^p_{2n}(\mathbb{Q}), Sp_{2n}(\mathbb{Z})) \) becomes less explicit for \( n \geq 2 \) and in order to compute the number of right representatives in a double coset one has to work with upper bounds instead of explicit formulas. We achieve this by using the root datum of the group \( GS^p_{2n} \) and use the equidistribution of Hecke points for the group \( GS^p_{2n} \) to establish the second ergodicity result.

**Acknowledgment.** The author would like to thank his advisor Matilde Marcolli for her guidance throughout this project. The author is also grateful to George Elliott, Serguey Neshveyev and Jean Renault for their valuable comments on this work.

This work has been partially supported by the NSERC Postgraduate Scholarship PGSD 2-535022-2019.

**Notations and conventions.**

We use the common notations \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \) together with \( \mathbb{R}_+^* = (0, +\infty) \) and \( \mathbb{Z}^+ = R_+^* \cap \mathbb{Z}; \)

If \( R \) is a ring, we denote its group of multiplicative units by \( R^\times; \)

We use the notation Mat\(_n(R) \) for the ring of square matrices with entries in \( R \). We denote by \( E_{i,j} \) the usual elementary matrix with 1 in the \( (i, j) \) position and 0 elsewhere;

The group of units in the ring Mat\(_n(R) \) is denoted by \( GL_n(R) \). If \( A \) is a square matrix, then \( A^t \) stands for its transpose. If \( A_1, \ldots, A_n \) are square matrices we denote by diag\((A_1, \ldots, A_n)\) the square matrix with \( A_1, \ldots, A_n \) as diagonal blocks and 0’s otherwise. We use \( I_n \) and \( 0_n \) to denote the \( n \times n \) identity matrix and the a rectangular zero matrix;

\(|F|\) denotes the cardinality of a finite set \( F; \)

the set of prime numbers is denoted by \( \mathbb{P}; \)

given a nonempty finite set of prime numbers \( F \subset \mathbb{P}, \) we denote by \( \mathbb{N}(F) \) the unital multiplicative subsemigroup of \( \mathbb{N} \) generated by \( p \in F; \)

1.8

3.17

2
we write \( f(x) = O(g(x)) \) if \( |f(x)|/g(x) \) is bounded at \(+\infty\); for two sequences \( \{a_n\} \) and \( \{b_n\} \), we write \( a_n \sim b_n \) if \( \lim_n (a_n/b_n) = 1 \) and
\[
\sum_n a_n \sim \sum_n b_n
\]
if the two series are simultaneously divergent or convergent;
if \( Y \) is subset of \( X \), we denote \( Y^c = X \setminus Y = \{ a \in X : a \notin Y \} \);
for a number field \( K \), we denote by \( \mathbb{A}_K = \mathbb{A}_{K,f} \times \mathbb{A}_{K,\infty} \) the adèle ring of \( K \), where \( \mathbb{A}_{K,f} \) is the ring of finite adèles and \( \mathbb{A}_{K,\infty} \) the infinite adèles of \( K \). The ring of integers of \( K \) is denoted by \( \mathcal{O}_K \).

**BACKGROUND**

1.1. **Operator algebraic formulation of quantum statistical mechanics.** We briefly review the operator algebraic formulation of quantum statistical mechanics. For a more comprehensive treatment of this material, we refer the reader to [2], [3].

Given an abstract \( C^* \)-algebra \( \mathcal{A} \), we know from Gelfand–Naimark theorem that \( \mathcal{A} \) is \(*\)-isomorphic to a \(*\)-subalgebra of the algebra of bounded operators on a Hilbert space. This result, together with the axioms of quantum mechanics, motivates the following operator algebraic formulation of quantum statistical mechanics.

**Definition 1.** A quantum statistical mechanical system \((\mathcal{A}, (\sigma_t)_{t \in \mathbb{R}})\) is a \( C^* \)-algebra \( \mathcal{A} \) together with a strongly continuous one-parameter group of automorphisms \((\sigma_t)_{t \in \mathbb{R}}\); that is, the map
\[
t \rightarrow \sigma_t(a)
\]
is norm continuous for every \( a \in \mathcal{A} \).

We also say that the pair \((\mathcal{A}, (\sigma_t)_{t \in \mathbb{R}})\) is a \( C^* \)-dynamical system. One should view \( \mathcal{A} \) as the algebra of observables of a quantum system with time evolution implemented by the one-parameter group \((\sigma_t)_{t \in \mathbb{R}}\). If the algebra \( \mathcal{A} \) is unital (with unit element \( e \)), a state on \( \mathcal{A} \) is a linear functional \( \phi : \mathcal{A} \rightarrow \mathbb{C} \) satisfying the following normalization and positivity conditions:
\[
\phi(e) = 1, \quad \phi(a^*a) \geq 0
\]
If \( a \in \mathcal{A} \) is self adjoint, we have a decomposition of the form \( a = a^+ - a^- \) where \( a^+, a^- \in \mathcal{A}^+ \) and one should think of \( \phi(a) \) as the expectation value of the observable \( a \) in the physical state \( \phi \).

When the algebra \( \mathcal{A} \) is non-unital we shall always work with weights (which we will define shortly) first and then replace the normalization condition \( \phi(e) = 1 \) by
\[
\|\phi\| := \sup_{x \in \mathcal{A}, \|x\| \leq 1} |\phi(x)| = 1
\]
(1)
to get a state. A weight on \( \mathcal{A} \) is a function \( \phi : \mathcal{A}^+ \rightarrow [0, \infty) \) (here \( \mathcal{A}^+ \) is the convex cone of positive elements in \( \mathcal{A} \)) such that \( \phi(\lambda a + b) = \lambda \phi(a) \) and \( \phi(\lambda a) = \lambda \phi(a) \) for \( \lambda \in \mathbb{R}^+ \) and all \( a, b \in \mathcal{A}^+ \).

If the algebra \( \mathcal{A} \) is unital, any weight can be written as \( \phi = \lambda \omega \), where \( \lambda > 0 \) and \( \omega \) is a state. Hence the two notions of states and weights essentially coincide in the unital case. In general, this is not true in the nonunital case. As noted in [5], one should think of states as probability Borel measures on noncommutative spaces while weights correspond to regular Borel measures in the commutative case.
In statistical mechanics we are interested in the so-called thermal equilibrium states at different temperatures. The Kubo-Martin-Shwinger (KMS) \([13, 29]\) condition at inverse temperature \( \beta \) was
proposed in 1961 by Haag, Winnik and Hugenholtz as an equilibrium condition in the $C^*$-algebraic setting of statistical mechanics. We recall the notion of a KMS$_\beta$-weight from [7]. See and [2] [3] for a more detailed discussion on KMS$_\beta$-weights.

**Definition 2.** Let $A$ be a $C^*$-algebra, $\phi$ a weight on $A$ and $\sigma_t$ a strongly continuous one-parameter group of automorphisms of $A$. We set $N_\phi = \{a \in A \mid \phi(a^*a) < \infty\}$ and let $\beta > 0$. We say that $\phi$ is a KMS$_\beta$-weight if:

1. $\phi \circ \sigma_t = \phi$ for every $t \in \mathbb{R}$
2. For every $a, b \in N_\phi \cap N_\phi^$, there exists a bounded continuous function $F$ on the closed strip $\Omega = \{x \in \mathbb{C} \mid 0 \leq \text{Im} \leq \beta\}$ and holomorphic on $\Omega^0$ such that

$$F(t) = \phi(a\sigma_t(b)), \quad F(t + i\beta) = \phi(\sigma_t(b)a)$$

Although this was the original definition of the KMS$_\beta$ condition, in practice the following equivalent characterization (See [17, Theorem 6.36]) is often used.

**Proposition 1.1.** Let $\phi$ be a weight on a $C^*$-dynamical system $(A, (\sigma_t)_{t \in \mathbb{R}})$. Then $\phi$ is a KMS$_\beta$ weight if and only if

1. $\phi \circ \sigma_t = \phi$ for every $t \in \mathbb{R}$
2. For every $\sigma$-analytic element $a$ in $A$, we have

$$\phi(aa^*) = \phi(\sigma_{i\beta}(a)^*\sigma_{i\beta}(a)).$$

1.2. **Groupoid algebras and Hecke pairs.** An important class of $C^*$-dynamical systems arises as the algebra of compactly supported functions on (locally compact) topological groupoids. We review, without proofs, the general results related to these systems and derived in the first two sections of [18]. For general information about groupoids and groupoid $C^*$-algebras, we refer the reader to [28].

Consider a countable group $G$ acting on a locally compact second countable topological space $X$. The transformation groupoid is the space $G \times X$ with unit space $X$ with the source and target maps given by $s(g, x) = x$ and $t(g, x) = gx$

and the composition is defined by

$$(g, x)(h, y) = (gh, y) \quad \text{if} \quad x = hy.$$ 

If $\Gamma$ is a subgroup of $G$ and the action of $\Gamma$ is free and proper, we introduce a new groupoid $\Gamma \backslash G \times \Gamma X$ by taking the quotient of $G \times X$ by the following action of $\Gamma \times \Gamma$:

$$(\gamma_1, \gamma_2)(g, x) := (\gamma_1g\gamma_2^{-1}, \gamma_2x).$$

(2)

In all our settings, the main motivation for taking the quotient by this action is physical. In fact, to obtain a well behaved partition function of the $C^*$-dynamical system we will introduce shortly, one should necessarily take the quotient by the group $\Gamma \times \Gamma$ (See also [8] for another motivation based on the $\mathbb{Q}$-lattice picture).

The groupoid $G = \Gamma \backslash G \times \Gamma X$ is an étale groupoid, i.e. the source map $s$ (and hence the target map) is a local homeomorphism. In particular, it has discrete fibers [28, Prop. 2.8]. In this case, we introduce the algebra $C_c(G)$ of continuous compactly supported functions on the quotient space $\Gamma \backslash G \times \Gamma X$ and we define the convolution of two such functions by
\[(f_1 * f_2)(\omega) = \sum_{\omega_1 \omega_2 = \omega} f_1(\omega_1) f_2(\omega_2) = \sum_{\omega_1 \in G(\omega)} f_1(\omega_1) f_2(\omega_1^{-1} \omega). \quad (3)\]

Notice that this is a finite sum since \(f_1\) has compact support and the fibers are discrete. If the action of \(\Gamma\) is proper but not free, the quotient space \(\Gamma \backslash G \times \Gamma \times X\) is no longer a groupoid (cf. Proposition 2.11). Under the assumption that \(X\) is a homogeneous space of the form \(\tilde{X} / H\), where now the action of \(G\) on \(\tilde{X}\) is free and proper, we can still define a natural convolution algebra from the groupoid algebra \(C_c(\Gamma \backslash G \times \Gamma \times \tilde{X})\) as was done in the \(GL_2\)-case in [8]. More specifically, viewing the elements of \(C_c(\Gamma \backslash G \times \Gamma \times \tilde{X})\) as \(\Gamma \times \Gamma\)-invariant functions, we rewrite the convolution product \((3)\) as

\[(f_1 * f_2)(g, x) = \sum_{s \in G \backslash G} f_1(gs^{-1}, sx) f_2(s, x). \quad (4)\]

We then define a convolution algebra on the quotient \(\Gamma \backslash G \times \Gamma \times X\) by restricting the convolution product \((4)\) to weight zero functions on \(C_c(\Gamma \backslash G \times \Gamma \times \tilde{X})\), namely functions satisfying

\[f(g, x\alpha) = f(g, x), \quad \forall \alpha \in H.\]

Define an involution on \(C_c(\Gamma \backslash G \times \Gamma \times X)\) by

\[f^*(\omega) = \overline{f(\omega^{-1})}.\]

For each \(x \in X\), we have a \(*\)-representation \(\pi_x\) of \(C_c(\Gamma \backslash G \times X)\) on the Hilbert space \(L^2(\Gamma \backslash G)\) defined by

\[(\pi_x)(f)\delta_{\Gamma h} = \sum_{g \in \Gamma \backslash G} f(gh^{-1}, hx)\delta_{\Gamma g}, \quad f \in C_c(\Gamma \backslash G \times \Gamma \times X).\]

One can show that the operators \(\pi_x(f)\) are uniformly bounded [12, 18] and we denote by \(\mathcal{B} = C^*_r(\Gamma \backslash G \times X)\) the completion of \(C_c(\Gamma \backslash G \times X)\) in the reduced norm

\[\|f\| = \sup_{x \in X} \|\pi_x(f)\|. \quad (5)\]

In fact, it is easy to verify that \((5)\) defines a \(C^*\)-seminorm. The fact that we get a norm follows from the identity

\[\langle \pi_x f\delta_{\Gamma g_1}, \delta_{\Gamma g_2} \rangle = f(g_2g_1^{-1}, g_1x).\]

Let \(Y\) be any clopen \(\Gamma\)-invariant subset of \(X\) and denote by \(\Gamma \backslash G \rtimes \Gamma Y\) the quotient of the space

\[\{(g, y) \mid g \in G, \ y \in Y, \ gy \in Y\},\]

by the action of \(\Gamma \times \Gamma\) defined in \((2)\). We denote by \(C_c(\Gamma \backslash G \rtimes \Gamma Y)\) the algebra of compactly supported functions on \(\Gamma \backslash G \rtimes \Gamma Y\) with the convolution product given by

\[(f_1 * f_2)(g, y) = \sum_{s \in \Gamma \backslash G} f_1(gs^{-1}, sy) f_2(s, y),\]

and involution

\[f^*(g, y) = \overline{f(g^{-1}, gy)}.\]
We let \( A = C^*_\Gamma (\Gamma \setminus G \boxtimes Y) \) be the corner algebra \( eB e \), where \( e \) is the \( \Gamma \times \Gamma \)-invariant function on \( G \times X \) defined by

\[
e(g, x) = \begin{cases} 
1 & \text{if } (g, x) \in \Gamma \times Y \\
0 & \text{otherwise.}
\end{cases}
\]

Given \( x \in X \), we put \( G_x = \{ g \in G \mid gx \in Y \} \).

Then we have a representation of \( C_c(\Gamma \setminus G \boxtimes Y) \) on the Hilbert space \( H_x = l^2(\Gamma \setminus G_x) \) given by

\[
\pi_x(f) \delta_{\Gamma y} = \sum_{g \in \Gamma \setminus G_x} f(gh^{-1}, hx) \delta_{\Gamma y}, \quad f \in C_c(\Gamma \setminus G \boxtimes Y),
\]

and the algebra \( A \) coincides ([18]) with the completion of \( C_c(\Gamma \setminus G \boxtimes Y) \) in the norm defined by

\[
\|f\| = \sup_{y \in Y} \|\pi_y(f)\|.
\]

Assume that we are given a homomorphism

\[
N : G \longrightarrow \mathbb{R}_+^*,
\]

such that \( \Gamma \subseteq \ker(N) \). We then define a one-parameter group of automorphisms of \( \mathcal{B} \) by

\[
\sigma_t(f)(g, x) = N(g)^{it} f(g, x), \quad \text{for } f \in C_c(\Gamma \setminus G \times X).
\]

The operator on \( l^2(\Gamma \setminus G) \) given by

\[
H_x \delta_{\Gamma y} = \log N(g) \cdot \delta_{\Gamma y}
\]

is the Hamiltonian and the dynamics \( \sigma_t \) is then spatially implemented as

\[
\pi_x(\sigma_t(a)) = e^{itH_x} \pi_x(a) e^{-itH_x}, \quad \forall x \in X, \forall a \in \mathcal{B}.
\]

The following result will be the starting point of our KMS\( _\beta \)-analysis of the dynamical system \((\mathcal{A}, \sigma_t)\).

**Proposition 1.2.** Let \( G, X \) and \( Y \) as described earlier and suppose \( \Gamma \) acts freely on \( X \). Then for \( \beta > 0 \) there exists a one-to-one correspondence between KMS\( _\beta \) weights \( \phi \) on \( \mathcal{A} \) with domain of definition containing \( C_c(\Gamma \setminus Y) \) and Radon measures \( \mu \) on \( Y \) such that

\[
\mu(gB) = N(g)^{-\beta} \mu(B)
\]

for every \( g \in G \) and every Borel compact subset \( B \subseteq Y \) such that \( gB \subseteq Y \). If \( \nu \) denotes the induced measure on \( \Gamma \setminus Y \), then the corresponding weight \( \phi \) is given by

\[
\phi(f) = \int_{\Gamma \setminus Y} f(e, y) d\nu(y), \quad f \in \mathcal{A}.
\]

**Proof.** See [18, Proposition 2.1] \( \Box \)
Recall that if \( G \) be a group and \( \Gamma \) a subgroup, the pair \((G, \Gamma)\) is called is called a Hecke pair if for any \( a \in G \)
\[
[\Gamma : \Gamma \cap a^{-1}G] < \infty.
\]
If \((G, \Gamma)\) is a Hecke pair then every double coset of \( \Gamma \) contains finitely many right and left cosets of \( \Gamma \):
\[
\Gamma a \Gamma = \bigsqcup_{\gamma \in \Gamma \setminus (T\gamma a^{-1}G)} \gamma a \Gamma = \bigsqcup_{\gamma \in \Gamma \setminus (T\gamma a^{-1}G)} \Gamma g \gamma,
\]
so \(|\Gamma \setminus \Gamma a\Gamma| = [\Gamma : \Gamma \cap a^{-1}G]\). We denote the cardinality of this set by \(\degG(a)\).

Let \( \beta \in \mathbb{R} \) and \( S \) is a semisubgroup of \( G \) containing \( \Gamma \). Then we define
\[
\zeta_{S,\Gamma}(\beta) := \sum_{s \in \Gamma \cap S} N(s)^{-\beta} = \sum_{s \in \Gamma \cap S / \Gamma} N(s)^{-\beta} \degG(s). \tag{8}
\]

Let \( G \) be a group acting on a set \( X \) and suppose \((G, \Gamma)\) is a Hecke pair. The Hecke operator associated to \( g \in G \) is the operator \( T_g \) on \( \Gamma \)-invariant functions on \( X \) defined by
\[
(T_g f)(x) = \frac{1}{\degG(g)} \sum_{h \in \Gamma \setminus \Gamma g \Gamma} f(hx). \tag{9}
\]

### 1.3. Abstract Bost-Connes-Marcolli systems.

In this section, we briefly recall, without proofs, the general properties of abstract Bost-Connes-Marcolli systems introduced in [12].

A BCM datum is a tuple \( (G, X, V, M) \) with \((G, X)\) a Shimura datum, \((V, \psi)\) a faithful representations of \( G \) and \( M \) an enveloping semigroup (see definition 3) for \( G \) contained in \( \text{End}(V) \). A level structure on \( D \) is a triple \( \mathcal{L} = (L, K, K_M) \) with \( L \subseteq V \) a lattice, \( K \subseteq G(M) \) a compact subgroup and \( K_M \subseteq M(M) \) a compact open subsemigroup such that

- \( K_M \) stabilizes \( L \otimes \mathbb{Z} \mathcal{Z} \)
- \( \psi(K) \) is contained in \( K_M \).

The pair \((D, \mathcal{L})\) is called a BCM pair. We let
\[
Y_{D,\mathcal{L}} = K_M \times \text{Sh}(G, X),
\]
and we denote the points of \( Y_{D,\mathcal{L}} \) by \( y = (\rho, [z, l]) \). We let \( Y_{D,\mathcal{L}}^X \) be the set of invertible elements \( y = (\rho, [z, l]) \) in \( Y_{D,\mathcal{L}} \). We have a partially defined action of \( G(M) \) on \( Y_{D,\mathcal{L}} \):
\[
g \cdot y = (gy, [zl^{-1}]) \quad \text{for} \quad y = (\rho, [z, l]).
\]

We consider the subspace
\[
\mathcal{U}_{D,\mathcal{L}} \subseteq G(M) \times Y_{D,\mathcal{L}}
\]
of pairs \((g, y)\) such that \( g \cdot y \in Y_{D,\mathcal{L}} \). This space is a groupoid with source and target maps \( s : \mathcal{U}_{D,\mathcal{L}} \to Y_{D,\mathcal{L}} \) and \( t : \mathcal{U}_{D,\mathcal{L}} \to Y_{D,\mathcal{L}} \) given by \( s(g, y) = y \) and \( t(g, y) = gy \). The unit space is \( Y_{D,\mathcal{L}} \) and composition is defined as
\[
(g_1, y_1) \circ (g_2, y_2) = (g_1 g_2, y_2) \quad \text{if} \quad y_1 = g_2 y_2.
\]

There is an action of \( K^2 \) on the groupoid \( \mathcal{U}_{D,\mathcal{L}} \) given by
\[
(\gamma_1, \gamma_2) \cdot (g, y) := (\gamma_1 g \gamma_2^{-1}, \gamma_2 y)
\]
and the quotient stack $\mathcal{Z}_{D,L} = [K^2/U_{D,L}]$ has the structure of a stack-groupoid (see [12, Appendix A]).

From now on we suppose that that $(D,L)$ is a BCM pair such that the Shimura datum $(G,X)$ is classical, i.e

$$\text{Sh}(G,X) = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \times X.$$ 

We let $\Gamma = G(\mathbb{Q}) \cap K$ and

$$\mathcal{U}^{\text{princ}} := \{(g, \rho, z) \in G(\mathbb{Q}) \times K_M \times X \mid g\rho \in K_M\}.$$ 

We let $X^+$ be a connected component of $X$, $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$ (where $G(\mathbb{R})^+$ is the identity component of $G(\mathbb{R})$) and $\Gamma_+ = G(\mathbb{Q})^+ \cap K$. Consider the groupoid

$$\mathcal{U}^+ := \{(g, \rho, z) \in G(\mathbb{Q})^+ \times K_M \times X^+ \mid g\rho \in K_M\},$$

with the composition given by

$$(g_1, \rho_1, z_1) \circ (g_2, \rho_2, z_2) = (g_1g_2, \rho_2, z_2) \text{ if } (\rho_1, z_1) = (g_2\rho_2, \rho_2z_2)$$

(10)

There is a natural action of $\Gamma^2$ (resp. $\Gamma_+^2$) on $\mathcal{U}^{\text{princ}}$ (resp. $\mathcal{U}^+$) given by

$$(\gamma_1, \gamma_2) \cdot (g, \rho, z) := (\gamma_1g\gamma_2^{-1}, \gamma_2\rho, \gamma_2z)$$

(11)

and the quotient stack $\mathcal{Z}^{\text{princ}}_{D,L}$ (resp. $\mathcal{Z}^+_{D,L}$) of $\mathcal{U}^{\text{princ}}$ (resp. $\mathcal{U}^+$) by $\Gamma^2$ (resp. $\Gamma_+^2$) has again the structure of a stack-groupoid.

For an arbitrary BCM pair $(D,L)$, the relation between the three groupoids $\mathcal{Z}_{D,L}$ and $\mathcal{Z}^{\text{princ}}_{D,L}, \mathcal{Z}^+_{D,L}$ is unclear a priori. The following important result obtained in [11, Propositions 5.2, Proposition 5.3] provides sufficient conditions for these three stack-groupoids to coincide.

**Proposition 1.3.** We denote by $h(G,K)$ the cardinality of the finite set $G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K$. Assume that $h(G,K) = 1$ and the natural map $\Gamma \to G(\mathbb{Q})/G(\mathbb{Q})^+$ is surjective. Then the natural maps

$$\mathcal{Z}^+_{D,L} \to \mathcal{Z}^{\text{princ}}_{D,L}, \quad \mathcal{Z}^{\text{princ}}_{D,L} \to \mathcal{Z}_{D,L}$$

are isomorphisms.

Hence when the condition of Proposition 1.3 are satisfied, it is enough to work with the BCM system associated to one of the three stack-groupoids. Let $\mathcal{H}(D,L) = C_c(Z_{D,L})$ be the algebra of continuous compactly supported functions on the coarse quotient $Z_{D,L}$ of $U$ by the action of $K^2$. We view its elements as functions on $U_{D,L}$ satisfying the following properties:

$$f(\gamma g, y) = f(g, y), \quad f(g\gamma, y) = f(g, \gamma y), \quad \forall \gamma \in K, \quad g \in G(\mathbb{A}_f), \quad y \in Y_{D,L}$$

The convolution product on $\mathcal{H}(D,L)$ is defined by the expression

$$(f_1 * f_2)(g, y) := \sum_{h \in K \backslash G(\mathbb{A}_f)} f_1(gh^{-1}, hy) f_2(h, y),$$

(12)

and the involution is given by

$$f^*(g, y) = f(g, y^{-1})$$
\[ f^*(g, y) := \overline{f(g^{-1}, gy)}. \]

Let \( y = (\rho, [z, l]) \in Y_{\mathcal{D},\mathcal{L}} \) and we put \( G_y = \{ g \in G(\mathcal{A}_f) \mid g\rho \in K_M \} \). We define a \(^*\)-representation \( \pi_y : \mathcal{H}(\mathcal{D}, \mathcal{L}) \to \mathcal{B}(l^2(K \setminus G_y)) \) by

\[
(\pi_y(f)\xi)(g) := \sum_{h \in K \setminus G_y} f(gh^{-1}, hy)\xi(h), \quad f \in \mathcal{H}(\mathcal{D}, \mathcal{L})
\]

where \( \xi \) is the standard basis of \( l^2(K \setminus G_y) \). The operators \( \pi_y(f) \), for \( y \in Y_{\mathcal{D},\mathcal{L}} \) and \( f \in \mathcal{H}(\mathcal{D}, \mathcal{L}) \) are uniformly bounded [12, Lemma 4.16] and we obtain a \( C^* \)-algebra \( \mathcal{A} \) after completing \( \mathcal{H}(\mathcal{D}, \mathcal{L}) \) in the norm

\[
\|f\| = \sup_{y \in Y_{\mathcal{D},\mathcal{L}}} \|\pi_y(f)\|.
\]

Given a homomorphism

\[
N : \text{GL}(V) \to \mathbb{R}^*_+,\n\]

we define a time evolution on \( \mathcal{H}(\mathcal{D}, \mathcal{L}) \) by

\[
\sigma_t(f)(g, y) = N(\psi(g))^{it} f(g, y),
\]

so that the operator on \( l^2(K \setminus G_y) \) given by

\[
(H_y\zeta)(g) = \log N(\psi(g))\zeta(g)
\]

is the Hamiltonian. The resulting \( C^* \)-dynamical system \( (\mathcal{A}, \sigma_t) \) is the Bost-Connes-Marcolli system associated to the BCM pair \( (\mathcal{D}, \mathcal{L}) \).

The zeta function associated to the BCM pair \( (\mathcal{D}, \mathcal{L}) \) plays an important role in the KMS\(_\beta\) analysis of the system \( (\mathcal{A}, \sigma_t) \). It is defined as the complex-valued series

\[
\zeta_{\mathcal{D},\mathcal{L}}(\beta) := \sum_{g \in \text{Sym}_f^\times \setminus \text{Sym}_f} N(\psi(g))^{-\beta}, \quad \beta \in \mathbb{C}
\]

where \( \text{Sym}_f(\mathcal{D}, \mathcal{L}) := \psi^{-1}(K_M) \) and \( \text{Sym}_f^\times(\mathcal{D}, \mathcal{L}) \) denotes the group of invertible elements in \( \text{Sym}_f(\mathcal{D}, \mathcal{L}) \). The pair \( (\mathcal{D}, \mathcal{L}) \) is called summable if there exists \( \beta_0 \in \mathbb{R} \) such that \( \zeta_{\mathcal{D},\mathcal{L}}(\beta) \) converges in the right plane \( \{ \beta \in \mathbb{C} \mid \text{Re}(\beta) > \beta_0 \} \) and extends to a meromorphic function on the full complex plane. By [12, Proposition 4.19] we know that if \( y \in Y_{\mathcal{D},\mathcal{L}}^\times \), then \( G_y = \text{Sym}_f(\mathcal{D}, \mathcal{L}) \) and the zeta function \( \zeta_{\mathcal{D},\mathcal{L}}(\beta) \) coincides with the partition function

\[
\zeta_0(\beta) = \frac{1}{|K \setminus K_0|} \text{Tr}(e^{-\beta H_y}), \quad K_0 = \psi^{-1}(K_M^\times)
\]

of the system \( (\mathcal{A}, \sigma_t) \).
2. Bost-Connes-Marcolli System for the Siegel Modular Variety

2.1. The Symplectic Group. Let \( n \in \mathbb{N} \) and \( R \) be a commutative unital ring. The symplectic group of similitudes of degree \( n \) is defined by

\[
GSp_{2n}(R) = \{ g \in GL_{2n}(R) : \exists \lambda(g) \in R^\times \mid g^t \Omega g = \lambda(g) \Omega \},
\]

where

\[
\Omega = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \quad 1_n \text{ is the } n \times n \text{ identity matrix.}
\]

The function \( \lambda : GSp_{2n}(R) \rightarrow R^\times \) is called the multiplier homomorphism. Its kernel is the symplectic group \( Sp_{2n}(R) \) and there is an exact sequence

\[
1 \rightarrow Sp_{2n}(R) \rightarrow GSp_{2n}(R) \rightarrow R^\times \rightarrow 1. \tag{14}
\]

If \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_{2n}(R) \) then the following assertions are equivalent:

(i) \( \lambda(g) = \lambda(g^t) \)

(ii) The inverse of the matrix \( g \) is given by:

\[
g^{-1} = \lambda(g)^{-1} \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} \tag{15}
\]

(iii) The blocks \( A, B, C, D \) satisfy the conditions

\[
A^tC = C^tA, \quad B^tD = D^tB, \quad A^tD - C^tB = \lambda(g)1_n \tag{16}
\]

(iv) The blocks \( A, B, C, D \) satisfy the conditions

\[
A^tB = B^tA, \quad C^tD = D^tC, \quad A^tD - B^tC = \lambda(g)1_n \tag{17}
\]

For \( r \in R^\times \), we put

\[
S_n(r) := \{ g \in GSp_{2n}(R) \mid \lambda(g) = r \}.
\]

We then obtain an embedding of symplectic groups of different degrees as follows. Given \( r \in \mathbb{N} \) and \( 0 < j < n \), define the map

\[
S_n(q) \times S_{n-j}(q) \rightarrow S_n(q)
\]

\[
(M_1, M_2) \mapsto M_1 \odot M_2,
\]

where

\[
M_1 \odot M_2 := \begin{pmatrix} A_1 & 0_{j \times (n-j)} & B_1 & 0_{j \times (n-j)} \\ 0_{(n-j) \times j} & A_2 & 0_{(n-j) \times j} & B_2 \\ C_1 & 0_{j \times (n-j)} & D_1 & 0_{j \times (n-j)} \\ 0_{(n-j) \times j} & C_2 & 0_{(n-j) \times j} & D_2 \end{pmatrix}, \quad M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.
\]

Note that

\[
(M_1 \odot M_2) \cdot (N_1 \odot N_2) = (M_1N_1) \odot (M_2N_2) \tag{18}
\]
Consider the following elements of $Sp_{2n}(R)$:
\[
\begin{cases}
(1_n \quad \alpha_1 E_{ii}) ,
(1_n \quad \alpha_2 E_{ii}) ,
(1_n \quad \alpha_3 (E_{ij} + E_{ji})) ,
(1_n \quad \alpha_4 (E_{ij} + E_{ji})) ,
(1_n \quad \alpha_5 E_{ij}) ;
\end{cases}
\]
where $\alpha_1, \ldots, \alpha_5 \in R$. If $F$ is a field, then the group $Sp_{2n}(F)$ is generated \cite{24} by the matrices given in (19) with $\alpha_1, \ldots, \alpha_5 \in F$.

As a connected reductive algebraic group, the center $Z$ of $G = GSp_{2n}$ consists of scalar matrices and the standard maximal torus is
\[
T = \{ \text{diag}(u_1, \ldots, u_n, v_1, \ldots, v_n) : u_1 v_1 = \cdots = u_n v_n \neq 0 \}.
\]
If $t \in T$, we often write
\[
t = \text{diag}(u_1, \ldots, u_n, u_1^{-1} \lambda(t), \ldots, u_n^{-1} \lambda(t)),
\]
We fix the following characters $e_i \in \text{Hom}(T, G_m)$:
\[
e_i(t) = u_i, \quad i = 0, 1, \ldots, n \quad \text{where} \quad u_0 := \lambda(t).
\]
and cocharacters $f_i \in \text{Hom}(G_m, T)$:
\[
\begin{aligned}
f_0(u) &= \text{diag}(1_n, \ldots, 1_n, u, \ldots, u), \\
f_1(u) &= (1_n, \ldots, 1_n, u, \ldots, 1_n), \\
&\vdots \\
f_n(u) &= (1_n, \ldots, 1_n, u, \ldots, 1_n, u^{-1}).
\end{aligned}
\]

**Proposition 2.1.** The root datum of $GSp_{2n}$ is described as follows. We set
\[
X = \mathbb{Z} e_0 \oplus \mathbb{Z} e_1 \oplus \ldots \mathbb{Z} e_n,
\]
\[
X^\vee = \mathbb{Z} f_0 \oplus \mathbb{Z} f_1 \oplus \ldots \mathbb{Z} f_n.
\]
and let $\langle \cdot, \cdot \rangle$ the natural pairing on $X \times X^\vee$:
\[
\langle e_i, f_j \rangle = \delta_{ij}.
\]
Then we have the following set of simple roots:
\[
\alpha_1(t) = u_{n-1}^{-1} u_n, \quad \ldots \quad \alpha_{n-1}(t) = u_1^{-1} u_2, \quad \alpha_n(t) = u_1^2 u_0^{-1},
\]
where $t$ has the form in (20). In terms of the basis $e_i, i = 0, 1, \ldots, n$, we have
\[
\alpha_1 = e_n - e_{n-1}, \quad \ldots \quad \alpha_{n-1} = e_2 - e_1, \quad \alpha_n = 2e_1 - e_0.
\]
The corresponding coroots are
\[
\alpha_i^\vee = f_n - f_{n-1}, \quad \ldots \quad \alpha_{n-1}^\vee = f_2 - f_1, \quad \alpha_n^\vee = f_1.
\]
Let $R = \{ \alpha_1, \ldots, \alpha_n \}$ and $R^\vee = \{ \alpha_1^\vee, \ldots, \alpha_n^\vee \}$. Then
\[
(X, R, X^\vee, R^\vee)
\]
is the root datum of $GSp_{2n}$. The Cartan matrix is given by

$$\langle \alpha_i, \alpha_j^\vee \rangle = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & \vdots \\ -1 & 2 & -1 & \vdots \\ \vdots & \vdots & \vdots & \ddots \\ -1 & 2 & -1 & -1 \\ -2 & 2 \end{pmatrix}.$$ 

**Proof.** See [31, page 134-136]

2.2. **The Symplectic envelopping semigroup.** As noted in section 1.3, the abstract definition of the Bost-Connes-Marcolli system associated to a general Shimura datum $(G, X)$ requires the notion of an enveloping semigroup which plays the role of Mat$_{2, \mathbb{Q}}$ in the the case of the $GL_{2, \mathbb{Q}}$-system.

**Definition 3.** Let $G$ be a reductive group over a field $F$. An enveloping semigroup for $G$ is a multiplicative semigroup $M$ which is irreducible and normal and such that $M^\times = G$.

It is always possible to construct enveloping semigroup (see [12, Appendix B.2]). For the case $G = GSp_{2n}$, we are considering in the paper, we have the following explicit description of $M$. Given a commutative $\mathbb{Q}$-algebra $R$ we have

$$M(R) := MSp_{2n}(R) = \{ m \in \text{Mat}_{2n}(R) \mid \exists \lambda(m) \in R, m_0 \Omega m = \lambda(m) \Omega \},$$

since $m \in MSp_{2n}(R)^\times$ if and only if $\lambda(m) \in R^\times$.

2.3. **The Siegel modular group.** The group $\Gamma_n = Sp_{2n}(\mathbb{Z})$ is called the Siegel modular group of degree $n$. For $m \in \mathbb{N}$, we denote by $GL_m(\mathbb{Z})$ the unimodular group of degree $m$ and note that $\Gamma_n \subseteq U_{2n}$ with equality if $n = 1$. We let $MSp_{2n}^+(\mathbb{Z}) = \{ M \in MSp_{2n}(\mathbb{Z}) \mid \lambda(M) > 0 \}$. Then given any $M \in MSp_{2n}^+(\mathbb{Z})$, we denote by $\Gamma_n M \Gamma_n$ the double coset generated by $M$ and put

$$D(\Gamma_n M \Gamma_n) := \{ D \in \text{Mat}_n(\mathbb{Z}) \text{ such that there exists } \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_n M \Gamma_n \}.$$ 

For each $D \in D(\Gamma M \Gamma)$, we set

$$B(D, \Gamma_n M \Gamma_n) := \{ B \in \text{Mat}_n(\mathbb{Z}) \text{ such that there exists } \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_n M \Gamma_n \}.$$ 

We define the following equivalence relation on $B(D)$:

$$B \sim B' \Leftrightarrow (B - B')D^{-1} \in \text{Sym}_n(\mathbb{Z}),$$

and we write $B \equiv B' \mod D$ if $B \sim B'$. For $r \in \mathbb{N}$, we put

$$S_n(r) := \{ g \in GSp_{2n}(\mathbb{Z}) \mid \lambda(g) = r \}.$$ 

Let $M \in \text{Mat}_n(\mathbb{Z})$ with $\det(M) > 0$ and $N \in S_n(r)$. Then by the Elementary Divisor Theorems (See [16, Theorem 2.2, Chapter VI]) the double cosets $U_m M U_m$ and $\Gamma_n N \Gamma_n$ contain unique representative of the form

$$\text{Elm}(M) = \text{diag}(a_1, a_2, \ldots, a_m), \quad a_1, a_2, \ldots, a_n \in \mathbb{N},$$

(23)

(24)
with \(a_1 \mid a_2 \mid \cdots \mid a_m\) and

\[
\text{Elm}(N) = \text{diag}(a_1, \ldots, a_n, d_1, \ldots, d_n), \quad a_1, \ldots, a_n, d_1, \ldots, d_n \in \mathbb{N},
\]

such that \(a_i d_i = r, i = 1, \ldots, n\) and \(a_1 \mid a_2 \mid \cdots \mid a_n \mid d_n \mid \cdots \mid d_{n-1} \mid d_1\).

**Theorem 2.2.** Let \(M \in \text{MSp}_{2n}^+(\mathbb{Z})\). Then a set of representatives of the right cosets relative to \(\Gamma_n\) in \(\Gamma_n \text{MSp}_{2n}^+(\mathbb{Z})\) is given by the matrices

\[
\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad A = \lambda(M)(D^t)^{-1}
\]

where

1. \(D\) runs through a set of representatives of \(\text{GL}_n(\mathbb{Z}) \setminus \text{Mat}_n(\mathbb{Z})\);
2. \(B\) runs through a set of representatives of \(\mod D\) incongruent matrices in \(\text{B}(D, \text{MSp}_{2n}^+(\mathbb{Z}))\).

**Proof.** See [16, Theorem 3.4, Chapter VI] \(\Box\)

**Proposition 2.3.** Let \(p\) be a prime number and \(l \in \mathbb{N}\). Then the set \(S_n(p^l)\) decomposes into finitely many right cosets relative to \(\Gamma_n\). A set of representatives is given by

\[
\begin{pmatrix} p^l(D^t)^{-1} & B \\ 0 & D \end{pmatrix}
\]

where \(D\) runs through a set of representatives of

\[
\text{GL}_n(\mathbb{Z}) \setminus \left\{ D \in \text{Mat}_n(\mathbb{Z}) \mid \text{Elm}(D) = \text{diag}(d_1, d_2, \ldots, d_n) \text{ and } d_i \mid p^l \text{ for all } i = 1, 2, \ldots, n \right\}
\]

and \(B\) runs through a set of representatives of \(\mod D\) incongruent matrices in

\[
\text{B}(D) := \left\{ B \in \text{Mat}_n(\mathbb{Z}) \mid B^t D = D^t B \right\}
\]

**Proof.** Every right coset \(\Gamma_n M\) contains a representatives of the form in (26). Suppose we have two representatives \(N\) and \(M\) of this form:

\[
N = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad M = \begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix},
\]

with \(\Gamma_n N = \Gamma_n M\). Then since

\[
\left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_n \right\} = \left\{ \begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix} \begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix} \mid U \in \text{GL}_n(\mathbb{Z}), S \in \text{Sym}_n(\mathbb{Z}) \right\},
\]

we obtain from the conditions in (16)-(17) that there exists \(U \in \text{GL}_n(\mathbb{Z})\) such that \(D(A')^t = \lambda(M)U^{-1}\) and hence

\[
D(D')^{-1} = U^{-1}.
\]

This shows that \(D = D'\) and consequently \(A = A'\). Moreover from the equality (28) we know that there exists \(S \in \text{Sym}_n(\mathbb{Z})\) such that

\[-A(B')^t + B(A')^t = -A(B')^t + AB',
\]

\[= \lambda(M)S.
\]

Since \(A^t D = \lambda(M) = \lambda(N)\) we obtain that \(B - B' = SD\), i.e \(B \equiv B' \mod D\). \(\Box\)

**Lemma 2.4.** Let \(p\) be a prime number, \(l \in \mathbb{N}\), \(D \in \text{Mat}_n(\mathbb{Z})\) such that \(\text{Elm}(D) = \text{diag}(d_1, d_2, \ldots, d_n)\) with \(d_i \mid p^l\) for \(i = 1, 2, \ldots, n\) and \(B(D)\) is as in equation (27). Given \(U, V \in \text{GL}_n(\mathbb{Z})\), then
(1) \(|B(D) \mod D| = |B(UDV) \mod UDV|,
(2) \(|B(D) \mod D| = d_1^n d_2^{n-1} \ldots d_n|.

Proof. Since U, V \in \mathcal{U}_n(\mathbb{Z})$, we have the following bijection

\[
B(D) \to B(UDV)
\]

\[
M \mapsto (U^t)^{-1}MV
\]

This proves the first claim. Hence we can suppose that \(D = \text{Elm}(D)\) to prove the second assertion. Since \(d_1| \ldots |d_n\), we can write

\[
B(D) = \{M = (b_{jk}) \mid b_{jk} \in \mathbb{Z}, b_{jk} = b_{kj} \frac{d_k}{d_l} \text{ for } j \leq k\}.
\]

By definition of the relation in (22) the entries \(b_{jk}\) may be reduced \(\mod d_k\). This shows that \(B(D)\) consists exactly of \(d_1^n d_2^{n-1} \ldots d_n\) equivalence classes \(\mod D\). \(\square\)

Let \(r \in \mathbb{N}\) and define

\[
R_{\Gamma_n}(r) := \sum_{g \in \Gamma_n \backslash S_n(r)/\Gamma_n} \deg_{\Gamma_n}(g).
\]

**Proposition 2.5.** The function \(R_{\Gamma_n} : \mathbb{N} \to \mathbb{N}\) is a multiplicative function, i.e. given relatively prime numbers \(q, r \in \mathbb{N}\), we have

\[
R_{\Gamma_n}(qr) = R_{\Gamma_n}(q)R_{\Gamma_n}(r).
\]

Proof. Observe first that if \(g \in S_n(qr)\), then \(\text{Elm}(g) = \text{Elm}(g_1)\text{Elm}(g_2)\) for some \(g_1 \in S_n(q)\) and \(g_2 \in S_n(r)\). Hence

\[
R_{\Gamma_n}(qr) = \sum_{g \in \Gamma_n \backslash S_n(qr)/\Gamma_n} \deg_{\Gamma_n}(g) = \sum_{g_2 \in \Gamma_n \backslash S_n(q)/\Gamma_n} \sum_{g_1 \in \Gamma_n \backslash S_n(r)/\Gamma_n} \deg_{\Gamma_n}(g_1g_2),
\]

so it is enough to show that \(\deg(g_1g_2) = \deg(g_1)\deg(g_2)\). We decompose the double cosets \(\Gamma_n \backslash g_1/\Gamma_n\) and \(\Gamma_n \backslash g_2/\Gamma_n\) into finitely many right cosets \(\Gamma_n Q_i, i = 1, \ldots, \deg(g_1)\) and \(\Gamma_n R_i, i = 1, \ldots, \deg(g_2)\) and consider the right cosets given by \(\Gamma Q_i R_j\). Suppose that \(\Gamma Q_i R_j = \Gamma Q_k R_l\). Then there exists some \(\gamma \in \Gamma_n\) and a matrix \(M \in GSp_{2n}(\mathbb{Q})\) such that

\[
M = Q_k^{-1}\gamma Q_i = R_l R_j^{-1}, \quad \lambda(Q_k) = \lambda(Q_i) = q, \quad \lambda(R_k) = \lambda(R_i) = r.
\]

Recall from (15) that \(Q_k = \lambda(Q_k)^{-1} Q_k^{-1} Q_k^t\Omega\) and so after writing \(M = \{\frac{m_{ij}}{n_{ij}}\}_{ij}\) where \((n_{ij}, m_{ij}) = 1\), we see that the integers \(m_{ij}\) divide \(\lambda(Q_k) = q\) and similarly \(m_{ij}\) divide \(\lambda(R_j) = r\). By assumption \((q, r) = 1\) so \(M \in \Gamma_n\) since \(\lambda(A) = 1\). This shows that \(\Gamma_n Q_k = \Gamma_n Q_i\) and \(\Gamma_n R_l = \Gamma_n R_k\), in other words \(i = k\) and \(l = j\). To conclude we simply observe that the cosets \(\Gamma Q_i R_j\) form a partition of \(\Gamma_n \backslash g_1g_2/\Gamma_n\). \(\square\)

Let \(p\) be a prime and \(l \in \mathbb{N}\). For arbitrary \(n\), it is in general not possible to obtain a closed formula for \(\deg(a)\) if \(a \in S_{np}\) is given in its elementary form. On the other hand, an upper bound of \(\deg(a)\) will be enough in most of our calculations. We first suppose that \(a\) is given by

\[
a = \text{diag}(p^{k_1}, p^{k_2}, \ldots, p^{k_n}, p^{l-k_1}, \ldots, p^{l-k_n}), \quad \lfloor l/2 \rfloor \leq k_1 \leq k_2 \leq \cdots \leq k_n
\]
Note that this is not the elementary symplectic form of a since \( k_1 \geq [l/2] \). We shall use the root datum of \( GSp_{2n} \) given in Proposition 2.1. The set \( \Phi^+ \) of positive roots is given by (see [31, page 167])

\[
    e_j - e_i, \quad 1 \leq i < j \leq n \quad e_j + e_i - e_0, \quad 1 \leq i < j \leq n \quad 2e_i - e_0, \quad 1 \leq i \leq n.
\]

where we have used our choice of the basis \( e_i, i = 1, \ldots, n \) (the choice of the basis used in [31] is different but the computations are essentially the same). Hence

\[
    2\rho = \sum_{\alpha \in \Phi^+} \alpha = 2 \sum_{i=0}^{n-1} (n-i)e_{n-i} - \frac{1}{2}n(n+1)e_0.
\]

We set

\[
    \lambda = \sum_{i=1}^{i=n} k_i f_i + lf_0.
\]

Observe that \( \langle \lambda, \alpha \rangle \geq 0 \) for all \( \alpha \in \Phi^+ \). Then using the degree formula in Proposition 7.4 in [10] (see also the proof of Corollary 1.9 in [6] for a similar result in the case of \( GL_n \)) we obtain

\[
    \deg(a) = p^{(\sum_{i=0}^{i=n-1} 2(n-i)k_{n-i}) - \frac{1}{2}n(n+1)t}(1 + O(p^{-1})) \tag{29}
\]

where the big \( O \) depends only on \( n \). If \( a \in S_{g_0} \) is given in its elementary symplectic form (23), we apply left and right (symplectic) permutations matrices to \( a \) (which leaves invariant the degree) and use the formula (29).

2.4. Structure theorems of the symplectic group. As seen in the previous section, the class number \( h(G, K) \) plays a detrimental role in the definition of the abstract BCM system. The aim of this section is to show that for \( G = GSp_{2n} \) we have that \( h(G, K) = 1 \) where \( K \) is any open compact subgroup of \( GSp_{2n}(\bar{\mathbb{Z}}) \). The proof relies on the notion of strong approximation in algebraic groups, which we briefly review. Given a linear reductive group \( G \) over a global field \( K \) and a nonempty finite set \( S \) of places of \( K \), we denote by \( \mathbb{A}_S \) the ring of \( S \)-adèles and let \( G(\mathbb{A}_S) \) be the ring of \( S \)-adèles of \( G \)

\[
    G(\mathbb{A}_S) := \{ g = (g_v) \in \prod_{v \notin S} G(K_v) \mid g_v \in G(O_v) \text{ for almost all } v \notin S \}.
\]

For any given \( S \), there is a canonical embedding \( G(K) \hookrightarrow G(\mathbb{A}_S) \). An algebraic group \( G \) over a global field \( K \) has the strong approximation with respect to \( S \) if \( G(K) \) is dense in \( G(\mathbb{A}_S) \). It is well known that strong approximation does not hold in general (for example one can take the group \( G = GL_n \) with \( S = \{ \infty \} \)). The following theorem provides a necessary and sufficient condition for the strong approximation theorem to hold for algebraic groups.

**Theorem 2.6.** (See [15], [26] in characteristic zero and [27],[20] [19] in positive characteristic), Let \( G \) be an absolutely almost simple simply connected algebraic group over a field \( K \) and \( S \) a finite nonempty set of places of \( K \). Then \( G \) has strong approximation with respect to \( S \) if and only if the group \( G_S = \prod_{v \in S} G(K_v) \) is noncompact.
We can conclude from Theorem 2.6 that the symplectic group \( G = Sp_{2n} \) (over \( \mathbb{Q} \)) has the strong approximation with respect to \( S = \{ \infty \} \). In the rest of this section, we provide an elementary proof of this result using matrix factorization. For this, we first put

\[
\Gamma_n(N) = \{ \gamma \in \Gamma_n | \gamma^t \Omega \gamma \equiv \Omega \mod N \},
\]

(30)

for every positive integer \( N \)

**Lemma 2.7.** Let \( \pi_N \) be the projection \( \pi_N : Sp_{2n}(\mathbb{Z}) \to Sp_{2n}(\mathbb{Z}/N\mathbb{Z}) \) defined by \( \pi_N(\gamma) = \gamma \mod N \). Then the sequence

\[
1 \longrightarrow \Gamma_n(N) \longrightarrow \Gamma_n \overset{\pi_N}{\longrightarrow} Sp_{2n}(\mathbb{Z}/N\mathbb{Z}) \longrightarrow 1
\]

is exact.

**Proof.** The only nontrivial part is the surjectivity of the map \( \pi_N \). This follows directly from [23, Theorem 1] \( \square \)

We denote by \( \hat{\mathbb{Z}} = \lim_{\leftarrow N > 1} \mathbb{Z}/N\mathbb{Z} \) the ring of profinite integers and we write \( M_{2n}(\hat{\mathbb{Z}}) \) for the ring of \( 2n \times 2n \)-matrices with coefficients in \( \hat{\mathbb{Z}} \). The profinite compact group \( GL_{2n}(\hat{\mathbb{Z}}) = \lim_{\leftarrow N > 1} GL_{2n}(\mathbb{Z}/N\mathbb{Z}) \) is a subset of \( M_{2n}(\hat{\mathbb{Z}}) \) and consists of invertible matrices. The subgroup \( Sp_{2n}(\hat{\mathbb{Z}}) \) is defined by exactness of the sequence (14).

**Proposition 2.8.** \( Sp_{2n}(\mathbb{Z}) \) is dense in \( Sp_{2n}(\hat{\mathbb{Z}}) \).

**Proof.** Since \( M_{2n}(\hat{\mathbb{Z}}) = \lim_{\leftarrow N > 1} M_{2n}(\mathbb{Z}/N\mathbb{Z}) \) is a profinite ring, a system of neighborhoods of the zero matrix is given by \( \{ NM_{2n}(\hat{\mathbb{Z}}) | N \in \mathbb{N} \} \) and thus a system of neighborhood of \( 1_{2n} \) in \( Sp_{2n}(\hat{\mathbb{Z}}) \) is given by \( \{ U_N | N \in \mathbb{N} \} \) where \( U_N = \{ 1 + NM_{2n}(\hat{\mathbb{Z}}) \} \cap Sp_{2n}(\hat{\mathbb{Z}}) \}. \) Given \( N \in \mathbb{N} \), consider the projection map \( \pi_N : Sp_{2n}(\hat{\mathbb{Z}}) \to Sp_{2n}(\mathbb{Z}/N\mathbb{Z}) \) and note that \( \ker \pi_N = U_N \). By Lemma 2.7 the projection \( \pi_N \) restricted to \( Sp_{2n}(\mathbb{Z}) \) is surjective. Hence for any given \( x \in Sp_{2n}(\hat{\mathbb{Z}}) \) and \( N \in \mathbb{N} \), we choose \( \gamma_N \in Sp_{2n}(\mathbb{Z}) \) such that \( \pi_N(\gamma_N) = \pi_N(x) \). Then \( x^{-1}\gamma_N \in U_N \), that is \( \gamma_N \in xU_N \). Taking \( N \) large enough shows that \( Sp_{2n}(\mathbb{Z}) \) is dense in \( Sp_{2n}(\hat{\mathbb{Z}}) \). \( \square \)

For a prime \( p \), we denote by \( \mathbb{Q}_p \) the field of \( p \)-adic numbers and \( \mathbb{Z}_p \) its compact subring of \( p \)-adic integers. We consider \( \mathbb{A}_{\mathbb{Q},f} \) the ring of finite adèles of \( \mathbb{Q} \), that is, the restricted product of the fields \( \mathbb{Q}_p \) with respect to \( \mathbb{Z}_p \) and we denote by \( I_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q},f}^t \) the idèle group.

**Theorem 2.9.** The algebraic group \( G = Sp_{2n} \) (over \( \mathbb{Q} \)) has the strong approximation with respect to the infinite place \( S = \{ \infty \} \).

**Proof.** We denote by \( H \) the closure of \( Sp_{2n}(\mathbb{Q}) \) in \( Sp_{2n}(\mathbb{A}_{\mathbb{Q},f}) \). We have a dense diagonal embedding of \( \mathbb{Q} \) inside \( \mathbb{A}_{\mathbb{Q},f} \), whence the subgroup \( H \) contains the group generated by the matrices of the form (19) with \( \alpha_1, \ldots, \alpha_5 \in \mathbb{A}_{\mathbb{Q},f} \). In particular, given any prime number \( p \), the subgroup \( H \) contains the set of matrices of the form (19) with \( \alpha_i = 1 \) for \( q \neq p \) and \( i = 1, \ldots, 5 \). Since \( Sp_4(\mathbb{Q}_p) \) is generated by these type of matrices, we see that \( H \) contains the elements \( M = (M_p)_p \in Sp_{2n}(\mathbb{A}_{\mathbb{Q},f}) \), with \( M_p \in Sp_{2n}(\mathbb{Q}_p) \) and \( M_q = 1 \) for \( q \neq p \). Hence for any finite set of primes \( F \), we have the inclusion

\[
\{(x)_q \in Sp_{2n}(\mathbb{A}_{\mathbb{Q},f}) \mid \forall q \in F, x_q \in Sp_{2n}(\mathbb{Q}_p) \text{ and } x_q = 1 \text{ if } q \not\in F \} \subseteq H.
\]

The result follows since the union of these subsets over all finite set of primes \( F \) is dense in \( Sp_4(\mathbb{A}_{\mathbb{Q},f}) \). \( \square \)
Corollary 2.10. Let $K$ be an open compact subgroup of $GSp_{2n}(\mathbb{A}_f)$. Then $\lambda(K) \subseteq \hat{\mathbb{Z}}^\times$ and if $\lambda(K) = \hat{\mathbb{Z}}^\times$, we have
\[ GSp_{2n}(\mathbb{A}_f) = K \cdot GSp_{2n}^+(\mathbb{Q}) = GSp_{2n}^+(\mathbb{Q}) \cdot K. \] (31)
In particular we get that $h(G, K) = 1$ for the maximal open compact subgroup $K = GSp_{2n}(\hat{\mathbb{Z}})$.

Proof. The first assertion follows from the fact that the map $\lambda : GSp_{2n}(\mathbb{A}_f) \to \mathbb{A}_f^\times$ is continuous and $\hat{\mathbb{Z}}^\times$ is the unique maximal subgroup of $\mathbb{Q}_p^\times$. To show (31), let $g \in GSp_{2n}(\mathbb{A}_f)$ so that $\lambda(g) \in \mathbb{A}_f^\times$. Since $\mathbb{A}_Q,f = \mathbb{Q}^\times \hat{\mathbb{Z}}^\times$, we can write
\[ \lambda(g) = \alpha \cdot x, \]
for some $\alpha \in \mathbb{Q}^\times$ (we choose $\alpha > 0$ if necessary) and $x \in \hat{\mathbb{Z}}^\times = \lambda(K)$. Thus we can choose $k \in K$ such that $\lambda(k) = x$. Consider the matrix
\[ g' = \text{diag}(\alpha^{-1}, 1, \ldots, 1, 1, \alpha^{-1}, 1, \alpha^{-1}) g^{-1}. \]
Observe that $g' \in S_p_{2n}(\mathbb{A}_f)$ and by Theorem 2.9 the open set $g' \cdot S_p_{2n}(\hat{\mathbb{Z}}) \subseteq S_p_{2n}(\mathbb{A}_f)$ contains $\eta \in S_p_{2n}(\mathbb{Q})$ such that $\eta = g' \cdot h$ for some $h \in S_p_{2n}(\hat{\mathbb{Z}})$. Moreover by Proposition 2.8 the group $S_p_{2n}(\mathbb{Z})$ is dense in $S_p_{2n}(\hat{\mathbb{Z}})$, hence we can find $\gamma \in S_p_{2n}(\hat{\mathbb{Z}})$ such that $\gamma \in g'^{-1} \eta U$, where $U = K \cap S_p_{2n}(\hat{\mathbb{Z}})$. This shows that $g \in K \cdot GSp_{2n}^+(\mathbb{Q})$ as desired. Considering the automorphism $x \mapsto x^{-1}$ for $x \in GSp_{2n}(\mathbb{A}_f)$, we see that $GSp_{2n}(\mathbb{A}_f) = GSp_{2n}^+(\mathbb{Q}) \cdot K$. \[ \square \]

2.5. Siegel upper half plane.

Definition 4. The Siegel upper half plane of degree $n$ consists of all symmetric complex $n \times n$-matrices whose imaginary part is positive definite:
\[ \mathbb{H}_n^+ = \{ \tau = \tau_1 + i\tau_2 \in \text{Mat}_n(\mathbb{C}) \mid \tau^t = \tau, \quad \tau_2 > 0 \} \] (32)
Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_{2n}^+(\mathbb{R})$ and $\tau \in \mathbb{H}_n^+$. Then the matrix $C\tau + D$ is invertible and if we define
\[ g \cdot \tau := (A\tau + B)(C\tau + D)^{-1}, \]
then the map
\[ \tau \mapsto g \cdot \tau \]
is an action of $GSp_{2n}^+(\mathbb{R})$ on $\mathbb{H}_n^+$ [25]. If we write $\tau = \tau_1 + i\tau_2 \in \mathbb{H}_n^+$ and $d\tau = d\tau_1 d\tau_2$ is the Euclidean measure, then the element of volume on $\mathbb{H}_n^+$ given by
\[ d^*\tau := \det(\tau_2)^{-(n+1)}d\tau, \] (33)
is invariant under all transformations of the group $GSp_{2n}^+(\mathbb{R})$, i.e
\[ d^*(g \cdot \tau) = d^*\tau, \quad \text{for all } g \in GSp_{2n}^+(\mathbb{R}). \]
Given an element $\tau = \tau_2 + i\tau_2 \in \mathbb{H}_n^+$, the relation
\[ \begin{pmatrix} 1_n & \tau_1 \\ 0_n & 1_n \end{pmatrix} \begin{pmatrix} \tau_2^{1/2} & 0_n \\ 0_n & \tau_2^{-1/2} \end{pmatrix} \cdot i1_n = \tau \]
shows that the action of $GSp_{2n}(\mathbb{R})$ is transitive. The stabilizer of $i1_n$ is the subgroup

$$S = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in GL_{2n}(\mathbb{R}) \right\} \cap GSp_{2n}(\mathbb{R}),$$

Hence the group $Z(\mathbb{R}) \backslash GSp_{2n}(\mathbb{R})$ (where $Z(\mathbb{R})$ denotes the center of $GSp_{2n}(\mathbb{R})$) acts transitively on $H^+_n$ and we have the following identification

$$H^+_n = PGSp_{2n}(\mathbb{R})/K,$$

(34)

where $K$ is the compact group $K = Z(\mathbb{R}) \backslash S \simeq \mathbb{U}^n/\{\pm 1_{2n}\}$ and $\mathbb{U}^n = S \cap S_{2n}(\mathbb{R})$ is isomorphic to the unitary group of order $n$ through the map

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + iB.$$ 

The action of the group $GSp_{2n}(\mathbb{R})$ is the Riemann zeta function.

We consider the connected Shimura datum $(GSp_{2n}^+, \mathbb{H}^+_n)$ together with the BCM pair

$$G^+ = GSp_{2n}^+, \quad X^+ = \mathbb{H}^+_n, \quad V = \mathbb{Q}^{2n}, \quad M = MSp_{2n}(\hat{\mathbb{Z}}),$$

$$L = \mathbb{Z}^{2n}, \quad K = GSp_{2n}(\hat{\mathbb{Z}}), \quad K_M = MSp_{2n}(\hat{\mathbb{Z}}).$$

Let $M \in GSp_{2n}^+(\mathbb{Q}) \cap GSp_{2n}(\hat{\mathbb{Z}})$. Since $\hat{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$ it is clear that $M \in MSp_{2n}(\mathbb{Z})$ and there exists $M' \in MSp_{2n}(\mathbb{Z})^+$ such that $MM' = 1$ so that $\lambda(M) = 1$, that is $\Gamma_+ = GSp_{2n}^+(\mathbb{Q}) \cap GSp_{2n}(\hat{\mathbb{Z}}) = \Gamma_n$. Corollary 2.10 tells us that $h(G, K) = 1$ so by Proposition 1.3 it is enough to work with the space $\mathfrak{z}_{D,L}$. As in the case of the $GL_2$ system, the first difficulty that arises is the presence of points in $H^+_n$ with nontrivial stabilizers:

$$\lambda$$

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Proposition 2.11. The groupoid structure on $\Gamma_{P,L}$ does not pass to the quotient by the action of $\Gamma_n \times \Gamma_n$.

Proof. Let $g = \begin{pmatrix} 1_n & 0_n \\ 0_n & 1_l \end{pmatrix} \in GSp_{2n}^+(\mathbb{Q})$ and assume the groupoid composition is defined when we pass to the quotient. Since $g \cdot \frac{1}{2} i_{12n} = i_{12n}$ we obtain that

$$(g, 0, i_{12n})(g, 0, \frac{1}{2} i_{12n}) = (g^2, 0, \frac{1}{2} i_{12n}),$$

where the equality holds in the quotient. On the other hand, let $\gamma = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$ so that $g \gamma^{-1} g = -\frac{1}{2} \gamma$ and $\gamma \cdot i_{12n} = g \cdot \frac{1}{2} i_{12n}$. We then have the following equality in the quotient:

$$(g^2, 0, \frac{1}{2} i_{12n}) = (g \gamma^{-1}, 0, \gamma \cdot i_{12n})(g, 0, \frac{1}{2} i_{12n}) = (g \gamma^{-1} g, 0, \frac{1}{2} i_{12n}).$$

Hence there exist $\gamma_1, \gamma_2 \in \Gamma_n = Sp_{2n}(\mathbb{Z})$ satisfying the following two conditions:

$$\gamma_2 \cdot i_{12n} = i_{12n},$$

$$\gamma_1 g^2 \gamma_2^{-1} = \frac{1}{2} \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}.$$

The first condition implies that $\gamma_2$ is of the form $\gamma_2 = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ while the second condition gives

$$\gamma_1 = \begin{pmatrix} \frac{1}{2} B & -2A \\ 2A & 2B \end{pmatrix}.$$

Note that since $\gamma_1 \in Sp_{2n}(\mathbb{Z})$ we get

$$\begin{pmatrix} \frac{1}{4}(BA - AB) & A^2 + B^2 \\ -B^2 - A^2 & 4(BA - AB) \end{pmatrix} = \Omega,$$

that is $I = 4(A^2 + B^2)$ for some $A', B' \in \text{Mat}_n(\mathbb{Z})$, which is a contradiction. \qed

2.7. The $GSp_{4,\mathbb{Q}}$-system. We restrict our attention to the case $n = 2$ and fix the following notation. We let $Y = \mathbb{H}_2^+ \times MSp_4(\mathbb{Z})$ so that $X = GSp_4(\mathbb{Q})^+ Y = \mathbb{H}_2^+ \times MSp_4(\mathbb{A}_f, \mathbb{Q})$ and $\Gamma_2 = Sp_4(\mathbb{Z})$. As observed above the action of $\Gamma_2$ on $Y$ is not free and it turns out that the set of points in $Y$ with non-trivial stabilizers strictly contains $\mathbb{H}_2^+ \times \{0_1\}$. Let

$$F_Y = \{ h \in MSp_4(\mathbb{Z}) \mid \text{rank}_{\mathbb{Q}_p}(h_p) \leq 2 \quad \text{for all primes } p \}.$$  

Then the action of $\Gamma_2$ on $\tilde{Y} = Y \setminus (\mathbb{H}_2^+ \times F_Y)$ is free. To see this, we suppose that for some $\gamma \in \Gamma_2$ we have $\gamma \cdot \tau = \tau$ and $\gamma h_p = h_p$, for some $\tau \in \mathbb{H}_2^+$ and $h_p \in MSp_4(\mathbb{Q}_p)$ with $\text{rank}_{\mathbb{Q}_p}(h_p) > 2$ for some prime $p$. Then we can find $T \in GL_4(\mathbb{Q}_p)$ such that

$$T \gamma T^{-1} = \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & x_4 \end{pmatrix},$$

for some $x_1, x_2, x_3, x_4 \in \mathbb{Q}_p$. The rest of the proof is similar to the case of the $GSp_{2,\mathbb{Q}}$-system.
for some $x_1, \ldots, x_4 \in \mathbb{Q}_p$. Since the entries of $\gamma$ are in $\mathbb{Z}$ we see that $x_4 \in \mathbb{Q}$ and thus $C_{Q,\gamma}(x) = (x - 1)^3(x - x_4)$. On the other hand, since $\gamma$ fixes a point in $\mathbb{H}^+_4$, then there exists $P \in Sp_4(\mathbb{R})$ such that 

$$P \gamma P^{-1} = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \circ \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix}, \quad a_i, b_i \in \mathbb{R}, \quad a_1^2 + b_1^2 = 1, a_2^2 + b_2^2 = 1,$$

So $C_{L,\gamma}(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4)$ where $\lambda_1 = a_1 + ib_1$ and $\lambda_2 = a_2 + ib_2$. Hence $\lambda_1 = \lambda_2 = 1$ and $\gamma = 1$. It is easy to see that the set of points in $Y$ with non-trivial stabilizers is strictly larger than $\mathbb{H}^+_2 \times \{0\}$.

The fact that $F_Y$ is not invariant under scalar matrices in $GSp^+_4(\mathbb{Q})$ creates a new difficulty that was not present in the $GL_2$-system. For this reason, and for the purpose of KMS$_\beta$ analysis, instead of working with the quotient $\mathbb{H}^+_2 = GSp^+_4(\mathbb{R})/K$, we consider first the quotient $PGSp^+_4(\mathbb{R}) = GSp^+_4(\mathbb{R})/Z(\mathbb{R})$, where $Z(\mathbb{R})$ is the center of the group $GSp^+_4(\mathbb{R})$. From now on we refer to this system as the $GSp_4$-system and we call the original dynamical system (corresponding to the Shimura datum $(GSp^+_4, \mathbb{H}^+_2)$) the Connes-Marcolli $GSp_4$-system. We will show later that the two systems have the same thermodynamical properties. Since now $PGSp^+_4(\mathbb{R})$ is a group, we get the following:

**Proposition 2.12.** For $\beta \neq 0$, there exists a correspondence between KMS$_\beta$ states on the $GSp_4$-system and $\Gamma_2$-invariant measures $\mu$ on $PGSp^+_4(\mathbb{R}) \times MSp_4(\mathbb{A}_{f,q})$ such that

$$\nu(\Gamma_2 \backslash PGSp^+_4(\mathbb{R}) \times MSp_4(\mathbb{Z})) = 1, \quad \mu(gB) = \lambda(g)^{-\beta} \mu(B)$$

for any $g \in GSp^+_4(\mathbb{Q})$ and Borel compact subset $B \subset PGSp^+_4(\mathbb{R}) \times MSp_4(\mathbb{A}_{f,q})$. Here $\nu$ denotes the measure on $\Gamma_2 \backslash PGSp^+_4(\mathbb{R}) \times MSp_4(\mathbb{A}_{f,q})$ corresponding to $\mu$.

**Proof.** Since the action of $\Gamma_2$ on $PGSp^+_4(\mathbb{R})$ is free, Proposition 1.2 applied to the group $G = GSp^+_4(\mathbb{Q})$ and the spaces $X = PGSp^+_4(\mathbb{R}) \times MSp_4(\mathbb{A}_{f,q})$ and $Y = PGSp^+_4(\mathbb{R}) \times MSp_4(\mathbb{Z})$ gives a one-to-one correspondence between the set of KMS$_\beta$ states on $(\mathcal{A}, \sigma)$ and $\Gamma_2$-invariant measures $\mu$ on $Y$ such that $\mu(gB) = \lambda(g)^{-\beta} \mu(B)$ if $gZ$ and $Z$ are measurable subsets of $Y$. The equality $MSp_4(\mathbb{A}_f) = GSp^+_4(\mathbb{Q})MSp_4(\mathbb{Z})$ allows us to extend ([18, Lemma 2.2]) this measure to a Radon measure on $X$ such that $\mu(gB) = \lambda(g)^{-\beta} \mu(B)$ for every Borel subset $B \subseteq X$. Since the algebra $\mathcal{A}$ is not unital, from the normalization condition (1) and equation (7) we obtain that $\nu$ is a probability measure on $\Gamma_2 \backslash Y$. \qed

From now on, we let $Y = PGSp^+_4(\mathbb{R}) \times MSp_4(\mathbb{Z})$ and $X = PGSp^+_4(\mathbb{R}) \times MSp_4(\mathbb{A}_{f,q})$. For $\beta > 0$, we denote by $\mathcal{E}_\beta$ the set of Radon measures on $X$ satisfying the properties in Proposition 2.12. Note that the extremal KMS$_\beta$ states correspond to point mass measures.

3. KMS$_\beta$ states analysis

3.1. High temperature region.

We begin the KMS$_\beta$ of the $GSp_4$-system analysis by first considering the high temperature region $0 < \beta < 3$. Our first goal is to show that the $GSp_4$-system constructed above does not admit a KMS$_\beta$ state for $0 < \beta < 3$ with $\beta \notin \{1, 2\}$. We first show some useful lemmas.

**Lemma 3.1.** Let $F$ be a finite set of prime numbers and $g = (g_p)_{p \in F} \in \prod_{p \in F} MSp_4(\mathbb{Z}_p) \subset \prod_{p \in F} MSp_4(\mathbb{Q}_p)$ with $\lambda(g_p) \neq 0$ for all $p \in F$. Then there exist $g_1 \in S_F$ and $g_2 \in \prod_{p \in F} GSp_4(\mathbb{Z}_p)$ such that $g = g_1g_2$. 

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Proof. It follows from Corollary 2.10 that we can find \( g_1 \in GSp_4^+(\mathbb{Q}) \) and \( g_2 \in GSp_4(\mathbb{Z}_p) \) such that \( g = g_1 g_2 \) with \( g_1 \in GSp_4(\mathbb{Q}) \), \( q \neq p \) and \( g_1 \in MSp_4(\mathbb{Z}_p) \) and \( \lambda(g_1) \in \mathbb{N}_F \), that is \( g_1 \in S_F \). \( \square \)

For \( k_0, k_1, k_2 \in \mathbb{Z} \), we set

\[
P_{k_0} := \begin{pmatrix} 0 & 0 \\ 0 & p^{k_0} \end{pmatrix} \otimes 0_2, \quad P_{k_1,k_2} := \begin{pmatrix} 0 & 0 \\ 0 & p^{k_1} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & p^{k_2} \end{pmatrix}.
\]

\[
Z_{k_0}^{(0)} := Sp_4(\mathbb{Z}) P_{k_0} GSp_4(\mathbb{Z}_p), \quad Z_{k_1,k_2}^{(1)} := Sp_4(\mathbb{Z}_p) P_{k_1,k_2} GSp_4(\mathbb{Z}_p).
\] (36)

Lemma 3.2. The sets \( Z_{k_0}^{(0)} \) and \( Z_{k_1,k_2}^{(1)} \), \( k_0, k_1, k_2 \in \mathbb{Z} \) are pairwise disjoint. Moreover, given any nonzero matrix \( a \in MSp_4(\mathbb{Q}_p) \) with \( \lambda(a) = 0 \), then \( a \in Z_{k_0}^{(0)} \cup Z_{k_1,k_2}^{(1)} \) for some \( k_0, k_1, k_2 \in \mathbb{Z} \).

Proof. We first fix some notations. For \( 1 \leq i, j \leq 4 \), let \( E_{ij} \) be the elementary matrix with coefficient 1 at the position \((i, j)\) and 0 otherwise. For \( U \in GL_2(\mathbb{Z}_p) \) and \( S \in Sym_2(\mathbb{Z}_p) \), we put

\[
J(U) = \begin{pmatrix} U^t & 0_2 \\ 0_2 & U^{-1} \end{pmatrix}, \quad J(S) = \begin{pmatrix} 1_2 & S \\ 0_2 & 1_2 \end{pmatrix}
\]

Consider \( g \in MSp_4(\mathbb{Q}_p) \) with \( \mu(g) = 0 \). Let \( g_0 \) be any entry of \( g \) with maximal \( p \)-adic valuation and we write \( g_0 = a_0 p^{k_0} \), where \( a_0 \in \mathbb{Z}_p^\times \). Using the matrices \( \Omega \) and \( J_1(P) \) (where \( P \) is a permutation matrix), we may assume that \( g_{11} = g_0 \). If \( a \) is an entry of the matrix \( g \), we set

\[
U_a = 1_2 - g_0^{-1} a E_{21}, \quad S_a = -g_0^{-1} a E_{11}, \quad \bar{S}_a = -g_0^{-1} a (E_{12} + E_{21})
\]

Observe that by maximality, these matrices are in \( Sp_4(\mathbb{Z}_p) \). We multiply \( g \) from the right by

\[
J(U_{g_{12}}) J(S_{g_{13} + g_{1} g_{12} g_{14}}) J(\bar{S}_{g_{14}})
\]

to obtain a matrix whose first row is \( g_0 e_1 \). Taking the transpose and repeating this process, we obtain a matrix whose first column is equal to \( g_0 e_1 \). The symplectic relations 16 and 17 imply that this matrix has the following form:

\[
\begin{pmatrix} g_0 & 0 \\ 0 & 0 \end{pmatrix} \otimes M, \quad M \in Mat_2(\mathbb{Q}_p), \quad \det(M) = 0.
\]

If \( M = 0 \), then one has

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes 1_2 \cdot \begin{pmatrix} g_0 & 0 \\ 0 & 0 \end{pmatrix} \otimes M \cdot \begin{pmatrix} 0 & a_0^{-1} \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a_0^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & p^{k_0} \end{pmatrix} \otimes 0_2 \in P_{k_0}.
\]

Otherwise, we can use right and left multiplication to find \( \gamma_1 \in SL_2(\mathbb{Z}_p) \) and \( \gamma_2 \in GL_2(\mathbb{Z}_p) \) such that \( \gamma_1 M \gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & p^{k_1} \end{pmatrix} \), for some \( k_1 \in \mathbb{Z} \). Then the matrix

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \gamma_1 \cdot \begin{pmatrix} g_0 & 0 \\ 0 & 0 \end{pmatrix} \otimes M \cdot \begin{pmatrix} 0 & 0 \\ -a_0 \det(g_2) & a_0^{-1} \end{pmatrix} \otimes \gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & p^{k_0} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & p^{k_1} \end{pmatrix} \in P_{k_0,k_1}
\]

has the desired form. One can easily check that this decomposition is unique. \( \square \)

Lemma 3.3. Let \( p \) be a prime number and \( g \in MSp_4(\mathbb{Z}_p) \) such that \( |\lambda(g)|_p = p^{-k}, k \in \mathbb{N} \). Then there exist \( \gamma_1, \gamma_2 \in Sp_4(\mathbb{Z}_p) \) such that \( \gamma_1 g \gamma_2 \) is of the form

\[
diag(a_1, a_2, d_1, d_2), \quad |a_1|_p \geq |a_2|_p \geq |d_1|_p \geq |d_2|_p.
\]
Proof. The proof is similar to [16, Theorem 2.2, Chapter V].

**Lemma 3.4.** Let \( p \) be a prime and we put

\[
g_{1,p} := \text{diag}(1, 1, p, p), \quad g_{2,p} := \text{diag}(p, p, p, p), \quad g_{3,p} := \text{diag}(1, p, p^2, p).
\]

A set of representatives of the right cosets relative to \( \Gamma_2 \) in \( \Gamma_2 g_{1,p} \Gamma_2 \) is given by the matrices

\[
\begin{pmatrix}
p & 0 & 0 & 0 \\
p & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
p & 0 & 0 & 0 \\
0 & 1 & 0 & k_1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & -k_2 & k_3 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & p & 0 \\
0 & 0 & k_2 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & k_4 & k_5 \\
0 & 1 & k_5 & k_6 \\
0 & 0 & p & 0 \\
0 & 0 & 0 & p
\end{pmatrix},
\]

where \( 0 \leq k_1, k_2, \ldots, k_6 < p \).

A set of representatives of the right cosets relative to \( \Gamma_2 \) in \( \Gamma_2 g_{3,p} \Gamma_2 \) is given by the matrices

\[
\begin{pmatrix}
p^2 & 0 & 0 & 0 \\
p & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & -r_4 & r_5 r_4 + r_6 & r_5
\end{pmatrix}, \quad \begin{pmatrix}
p & -pr_1 & 0 & 0 \\
p & p^2 & 0 & 0 \\
0 & 0 & p & 0 \\
1 & 0 & r_1 & 1
\end{pmatrix}, \quad \begin{pmatrix}
p & 0 & r_8 & r_9 \\
0 & p & r_9 & r_{10} \\
0 & 0 & p & 0 \\
0 & 0 & 0 & p
\end{pmatrix}, \quad \begin{pmatrix}
p & 0 & r_8 & r_9 \\
0 & p & r_9 & r_{10} \\
0 & 0 & p & 0 \\
0 & 0 & 0 & p
\end{pmatrix},
\]

where \( 1 \leq r_1, r_2, r_4, r_5 < p, \ 1 \leq r_3, r_6 < p^2, \ \text{and} \ 0 \leq r_8, r_9, r_{10} < p \) are such that \( r_p \begin{pmatrix} r_8 & r_9 \\ r_9 & r_{10} \end{pmatrix} = 1 \), where \( r_p(B) \) denotes the rank of the matrix \( B \in \text{Mat}_2(\mathbb{Z}) \) over \( \mathbb{Z}/p\mathbb{Z} \). In particular we have:

\[
\deg_{\Gamma_2}(g_{1,p}) = (1 + p)(1 + p^2), \quad \deg_{\Gamma_2}(g_{2,p}) = 1, \quad \deg_{\Gamma_2}(g_{3,p}) = p + p^2 + p^3 + p^4.
\]

Proof. Recall that \( S_4(p) \) denotes the set of matrices \( g \in GSp_4(\mathbb{Z}) \) such that \( \mu(g) = p \). This set consists of a single coset:

\[
S_4(p) = \Gamma g_{1,p} \Gamma.
\]

From this we can see that

\[
\mathcal{D}(\Gamma_2 g_{1,p} \Gamma_2) = \{ \Gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Gamma_1, \Gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1, \Gamma_1 \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} \Gamma_1 \}
\]

The decomposition of \( \Gamma g_{1,p} \Gamma \) into right cosets follows then from applying from applying Theorem 28 with \( n = 2 \) and \( n = 1 \) (notice that we are using the convention that for \( n \) impair, the matrices \( B \) are under the diagonal).

The decomposition of \( \Gamma g_{2,p} \Gamma \) is trivial. To decompose the double cosets \( \Gamma g_{3,p} \Gamma \), we use the following simple criterion:

\[
M \in \Gamma_2 g_{3,p} \Gamma_2 \iff r_p(M) = 1 \text{ and } M \in GSp_4(\mathbb{Z}).
\]

We then obtain

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\[ \mathcal{D}(\Gamma_2 g_{3,p} \Gamma_2) = \{ \Gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1, \Gamma_1 \begin{pmatrix} p & 0 \\ 0 & p^2 \end{pmatrix} \Gamma_1 \}. \]

For each \( D \in U_2 \setminus \mathcal{D}(\Gamma_2 g_{3,p} \Gamma_2) \), the set \( \mathcal{B}(D, \Gamma_2 g_{3,p} \Gamma_2) \) is then obtained by applying again Theorem 28 and using the relations

\[
A^tD = p^2 \mathbb{1}_2, \quad B^tD = D^tB, \quad AB^t = BA^t.
\]

\[ \square \]

**Lemma 3.5.** Let \( p \) be a prime and denote by \( G_p \) the subgroup of \( GSp^+_4(\mathbb{Q}) \) generated by \( \Gamma_2 \) and the matrices \( g_{1,p}, g_{2,p} \) and \( g_{3,p} \) defined in Lemma 3.4. Suppose that \( \mu_p \) is a \( \Gamma_2 \)-invariant measure on \( PGS_{p_4}^+(\mathbb{R}) \times MSp_4(\mathbb{Q}_p) \) such that

1. \( \mu_p(PGS_{p_4}^+(\mathbb{R}) \times MSp_4(\mathbb{Z}_p)) < \infty \)
2. \( \mu_p(g Z) = \lambda(g)^{-1} \mu_p(Z) \) for all \( g \in G_p \) and Borel \( Z \subseteq PGS_{p_4}^+(\mathbb{R}) \times MSp_4(\mathbb{Q}_p) \).

If \( \beta \notin \{1, 2, 3\} \) then \( PGS_{p_4}^+(\mathbb{R}) \times GSp_4(\mathbb{Q}_p) \) is subset of full measure in \( PGS_{p_4}^+(\mathbb{R}) \times MSp_4(\mathbb{Q}_p) \).

**Proof.** We define a measure \( \tilde{\mu}_p \) on \( MSp_4(\mathbb{Q}_p) \) by \( \tilde{\mu}_p(Z) = \mu_p(PGS_{p_4}^+(\mathbb{R}) \times Z) \). Note that by assumption we have \( \tilde{\mu}_p(MSp_4(\mathbb{Q}_p)) < \infty \). For any \( g \in G_p \) and any positive integrable \( \Gamma_2 \)-invariant function on \( MSp_4(\mathbb{Q}_p) \), we get from the second condition that

\[
\int_{MSp_4(\mathbb{Q}_p)} T_g f \, d\tilde{\mu}_p = \lambda(g)^{-1} \int_{MSp_4(\mathbb{Q}_p)} f \, d\mu_p. \tag{37}
\]

For \( k_0, k_1, k_2 \in \mathbb{Z} \), consider the functions \( f_{k_0}^{(0)} = I_{Z_{k_0}^{(0)}} \) and \( f_{k_1+k_2}^{(1)} = I_{Z_{k_1+k_2}^{(1)}} \), where the sets \( Z_{k_0}^{(0)} \) and \( Z_{k_1+k_2}^{(1)} \) are as in equation (36). Given \( g \in GSp_{p_4}^+(\mathbb{Q}) \), we have that the function \( T_g f_{0,0}^{(1)} \) is continuous and \( \Gamma \)-invariant. By Proposition 2.8 the group \( \Gamma \) is dense in \( Sp_4(\mathbb{Z}_p) \), whence \( T_g f_{0,0}^{(1)} \) is left \( Sp_4(\mathbb{Z}_p) \)-invariant. For \( k_1, k_2 \in \mathbb{Z} \), we can expand the expression \( T_{g_{2,p} g_{1,p}} f_{0,0}^{(1)}(P_{k_1, k_2}) \) using the explicit representatives given in Lemma 3.4:

\[
\begin{aligned}
&\frac{1}{\deg(g_{2,p}^{-1} g_{1,p})} \left( f_{0,0}^{(1)} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p^{k_1-1} \\ 0 & 0 & 0 & p^{k_2-1} \end{array} \right) \right) + \sum_{k=0}^{p-1} f_{0,0}^{(1)} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p^{k_1-1} \\ 0 & 0 & 0 & p^{k_2} \end{array} \right) \\
+ &\sum_{0 \leq k', a \leq p-1} f_{0,0}^{(1)} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & k' p^{k_1-1} & 0 & 0 \\ 0 & 0 & p^{k_1-1} & p^{k_2-1} \end{array} \right) + \sum_{0 \leq a, b, n \leq p-1} f_{0,0}^{(1)} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p^{k_1} \\ 0 & 0 & 0 & p^{k_2} \end{array} \right) \right)
\end{aligned}
\]

Since \( T_{g_{2,p} g_{1,p}} f_{0,0}^{(1)}(\gamma_1 P_{k_1, k_2} \gamma_2) = T_{g_{2,p} g_{1,p}} f_{0,0}^{(1)}(P_{k_1, k_2}) \) for \( \gamma_1 \in Sp_4(\mathbb{Z}_p) \) and \( \gamma_2 \in GSp_4(\mathbb{Z}_p) \), it follows that
\[
\deg(g_{2,p}^{-1}g_{1,p}) T_{g_{2,p} g_{2,p}} f_{0,0}^{(1)} = f_{1,1}^{(1)} + f_{1,0}^{(1)} + (p - 1)f_{1,1}^{(1)} + f_{0,1}^{(1)} + (p^2 - 1)f_{1,1}^{(1)} + (p - 1)f_{1,0}^{(1)} + (p - 1)^2 f_{1,1}^{(1)} + f_{0,0}^{(1)} + (p - 1)f_{1,1}^{(1)} + f_{0,1}^{(1)} + (p - 1)^2 f_{1,1}^{(1)} \\
+ (p - 1)^2 f_{1,1}^{(1)} + (p - 1)^3 f_{1,1}^{(1)}.
\]

Similarly, the expansion of \( T_{g_{2,p} g_{3,p}} f_{0,1}(P_{k_1,k_2}) \) is given by

\[
\frac{1}{\deg(g_{2,p}^{-2}g_{3,p})} f_{0,1} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & p^{k_1-2} \\ 0 & 0 & p^{k_2-1} \end{array} \right) + \sum_{0 \leq a \leq p-1} f_{0,1} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & p^{k_1-1} \\ 0 & 0 & ap^{k_1-2} + p^{k_2-2} \end{array} \right) \\
+ \sum_{0 \leq b \leq p-1 \atop 0 \leq c \leq p^2-1} f_{0,1} \left( \begin{array}{ccc} 0 & 0 & bp^{k_1-2} \\ 0 & 0 & p^{k_1-1} \\ 0 & 0 & p^{k_2-1} \end{array} \right) + \sum_{r_p(b_1 b_2 b_3) = 1} f_{0,1} \left( \begin{array}{ccc} 0 & 0 & b_1 p^{k_1-2} + b_2 p^{k_2-2} \\ 0 & 0 & b_2 p^{k_1-2} \\ 0 & 0 & p^{k_2-1} \end{array} \right) \\
+ \sum_{0 \leq l \leq p^2-1 \atop 0 \leq k,d \leq p-1} f_{0,1} \left( \begin{array}{ccc} 0 & 0 & (kd + l)p^{k_1-2} + kp^{k_2-2} \\ 0 & 0 & kp^{k_1-1} \\ 0 & 0 & dp^{k_1-1} + p^{k_2-1} \end{array} \right).
\]

Note that there are \( p - 1 \) positive integers divisible by \( p \) between 0 and \( p^2 - 1 \). Similarly there are \( p(p - 1)^2 \) positive integers divisible by \( p \) of the form \( kd + l \) where \( 0 < k, d < p \) and \( 1 < l < p^2 \). Hence

\[
\deg(g_{2,p}^{-2}g_{3,p}) T_{g_{2,p} g_{3,p}} f_{0,0}^{(1)} = f_{2,1}^{(1)} + f_{1,2}^{(1)} + (p - 1)f_{2,2}^{(1)} + f_{1,0}^{(1)} + (p - 1)f_{1,1}^{(1)} + (p^2 - p)f_{1,2}^{(1)} + (p - 1)f_{2,2}^{(1)} + (p - 1)^2 f_{2,2}^{(1)} + (p^2 - 1) - (p - 1)^2 f_{2,1}^{(1)} \\
+ (p - 1)^2 f_{1,2}^{(1)} + (p - 1)^3 f_{1,1}^{(1)} \\
+ (p^2 - p)(p - 1)f_{2,2}^{(1)} + (p - 1)^2 f_{1,1}^{(1)} \\
+ (p - 1)^2 f_{2,2}^{(1)} + (p^2 - p - 1) - (p - 1)^2 f_{2,2}^{(1)} \\
= f_{0,1}^{(1)} + f_{1,0}^{(1)} + (p^2 + p - 2)f_{1,1}^{(1)} + p^3 f_{1,2}^{(1)} + p^2 f_{2,1}^{(1)} + (p^4 - p^3)f_{2,2}^{(1)}.\]
Similar computations lead to the following identities:
\[
\deg(g_{2,p}^{-1}T_{g_{2,p}^{-1}}f_{0,0}^{(1)} = f_{1,1}^{(1)},
\]
\[
\deg(g_{2,p}^{-2}g_{1,p}T_{g_{2,p}^{-2}g_{1,p}}f_{0,0}^{(1)} = f_{1,1}^{(1)} + pf_{2,1}^{(1)} + p f_{1,2}^{(1)} + (p^3 + p^2 - p) f_{2,2}^{(1)},
\]
\[
\deg(g_{2,p}^{-2}T_{g_{2,p}^{-2}}f_{0,0}^{(1)} = f_{2,2}^{(1)}.
\]

We then have
\[
f_{0,0}^{(1)} = \deg(g_{2,p}^{-1}g_{1,p}T_{g_{2,p}^{-1}g_{1,p}}f_{0,0}^{(1)} - p \deg(g_{2,p}^{-1}g_{3,p}T_{g_{2,p}^{-1}g_{3,p}}f_{0,0}^{(1)} - (p + p^3) \deg(g_{2,p}^{-1}T_{g_{2,p}^{-1}}f_{0,0}^{(1)}
\]
\[
+ p^3 \deg(g_{2,p}^{-2}g_{1,p}T_{g_{2,p}^{-2}g_{1,p}}f_{0,0}^{(1)} - p^6 \deg(g_{2,p}^{-2}T_{g_{2,p}^{-2}}f_{0,0}^{(1)}.
\]

Since \(\deg(g_{2,p}^{-i}g_k) = \deg(g_k), i \in \mathbb{N}, k = 1, 2, 3,\) it follows from equation (37) that
\[
\tilde{\mu}_p(Z_{0,0}^{(1)}) = R(p, \beta)\tilde{\mu}_p(Z_{0,0}^{(1)}),
\]
where
\[
R(p, \beta) = p^{1-\beta} + p^{2-\beta} + p^{3-\beta} + p^{-\beta} - p^{1-2\beta} - p^{2-2\beta} - 2p^{3-2\beta} - p^{4-2\beta} - p^{5-2\beta} + p^{3-3\beta}
\]
\[
+ p^4-3\beta + p^5-3\beta + p^6-3\beta - p^{6-4\beta}
\]
\[
= 1 - p^6 (p^{-\beta} - 1) (p^{-\beta} - p^{-1}) (p^{-\beta} - p^{-2}) (p^{-\beta} - p^{-3}).
\]

We can repeat the same computations with the functions \(T_g f_{0,0}^{(0)}, g \in GSp_4^+(\mathbb{Q})\) instead. As a summary we get
\[
\deg(g_{2,p}^{-1}g_{1,p}T_{g_{2,p}^{-1}g_{1,p}}f_0^{(0)} = (1 + p)f_0^{(0)} + (p^3 + p^2)f_1^{(0)}
\]
\[
\deg(g_{2,p}^{-1}g_{3,p}T_{g_{2,p}^{-1}g_{3,p}}f_0^{(0)} = f_0^{(0)} + (p^2 - 1 + p^3 + p)f_1^{(0)} + p^4f_2^{(0)}
\]
\[
\deg(g_{2,p}^{-1}T_{g_{2,p}^{-1}}f_0^{(0)} = f_1^{(0)}
\]
\[
\deg(g_{2,p}^{-2}g_{1,p}T_{g_{2,p}^{-2}g_{1,p}}f_0^{(0)} = (1 + p)f_1^{(0)} + (p^3 + p^2)f_2^{(0)}
\]
\[
\deg(g_{2,p}^{-2}T_{g_{2,p}^{-2}}f_0^{0} = f_2^{(0)}
\]
and
\[
\tilde{\mu}_p(Z_{0,0}^{(0)}) = R(p, \beta)\tilde{\mu}_p(Z_{0,0}^{(0)}).
\]

Suppose that \(\mu_p(Z_{0,0}^{(1)}) \neq 0.\) Since \(\beta \notin \{1, 2, 3\},\) by equation (38) we get that \(\beta = 0.\) Hence
\[
\tilde{\mu}_p(Z_{k_1,k_2}^{(1)}) = \tilde{\mu}_p(Z_{k_1+2,k_2+2}^{(1)}), \quad k_1, k_2 \in \mathbb{Z}.
\]

This is a contradiction since \(\tilde{\mu}_p(MSp_4(\mathbb{Z})) < \infty.\) This shows that \(\tilde{\mu}_p(Z_{0,0}^{(1)}) = 0\) and by induction we see that \(\tilde{\mu}_p(Z_{k_1,k_2}^{(1)}) = 0\) for \(k_1, k_2 \in \mathbb{N}.\) By equation (40) we see that this indeed holds for all \(k_1, k_2 \in \mathbb{Z}.\) The same argument shows that \(\tilde{\mu}_p(Z_{k}^{(0)}) = 0\) for \(k \in \mathbb{Z}.\) It follows from Lemma 3.2 that the \(\tilde{\mu}_p\)-measure of the set of nonzero matrices \(g \in MSp_4(\mathbb{Q})\) with \(\lambda(g) = 0\) is zero. \(\square\)
Corollary 3.6. We denote by \( MS_p(A_f)^* \) the set of elements \( h \in MS_p(A_f) \) such that \( \lambda(m_p) \neq 0 \) for all primes \( p \). Let \( \mu_\beta \) be a measure in \( E_\beta \) and \( \beta \notin \{0, 1, 2, 3\} \). Then \( PGSp_4^+(\mathbb{R}) \times MS_p(A_f)^* \) is subset of full measure in \( PGSp_4^+(\mathbb{R}) \times MS_p(A_f) \).

Proof. Given a prime \( p \), consider the restriction of \( \mu_\beta \) to the set

\[
PGSp_4^+(\mathbb{R}) \times MS_p(\mathbb{Q}_p) \times \prod_{p \neq q} MS_p(\mathbb{Z}_q),
\]

and the measure \( \mu_{\beta,p} \) on \( PGSp_4^+(\mathbb{R}) \times MS_p(\mathbb{Q}_p) \) obtained from the projection on the first two coordinates. Since \( \mu_\beta \in K_\beta \) we have that \( \mu_{\beta,p}(PGSp_4^+(\mathbb{R}) \times MS_p(\mathbb{Z}_p)) < \infty \). Given any \( g \in G_p \) and Borel \( Z \in PGSp_4(\mathbb{R})^+ \times MS_p(\mathbb{Q}_p) \) we get

\[
\mu_{\beta,p}(gZ) = \mu_\beta(g(Z \times \prod_{q \neq p} MS_p(\mathbb{Z}_q))) = \lambda(g)^{-\beta} \mu_{\beta,p}(Z).
\]

Thus the measure \( \mu_{\beta,p} \) satisfies the conditions of Lemma 3.5, whence \( PGSp_4^+(\mathbb{R}) \times GSp_4(\mathbb{Q}_p) \) is a subset of full \( \mu_{\beta,p} \)-measure. This shows that the \( \mu_\beta \)-measure of the set

\[
\{PGSp_4^+(\mathbb{R}) \times MS_p(\mathbb{Q}_p) \times \prod_{p \neq q} MS_p(\mathbb{Z}_q) \mid \lambda(h_p) = 0\}
\]

is zero. Finally observe that the complement of \( \mathbb{A}_f^+ \times MS_p^+(\mathbb{A}_f) \) is equal to

\[
\bigcup_{p \in \mathcal{P}} GSp_4^+(\mathbb{Q})\{PGSp_4^+(\mathbb{R}) \times MS_p(\mathbb{Q}_p) \times \prod_{p \neq q} MS_p(\mathbb{Z}_q) \mid \lambda(h_p) = 0\},
\]

which completes the proof. \( \square \)

Given a prime number \( p \in \mathcal{P} \) and \( \beta \in \mathbb{R}_{>0} \), we consider the subsemigroup of \( GSp_4(\mathbb{Q})^+ \) given by

\[
S_{2,p} = \bigcup_{l \geq 0} S_2(p^l).
\]

Note that \( \Gamma_2 \subseteq S_{2,p} \) for any prime number \( p \) and the corresponding Dirichlet series (cf. Definition 1.2) is

\[
\zeta_{S_{2,p},\Gamma_2}(\beta) = \sum_{g \in \Gamma_2 \backslash S_{2,p} / \Gamma_2} \lambda(g)^{-\beta} \deg_{\Gamma_2}(g) = \sum_{l=0}^{\infty} p^{-\beta l} R_{\Gamma_2}(p^l).
\]

Proposition 3.7. Suppose \( \beta \in \mathbb{R}_{>0} \). Then \( \zeta_{S_{2,p},\Gamma_2}(\beta) < \infty \) if and only if \( \beta > 3 \). In this case we have that

\[
\zeta_{S_{2,p},\Gamma_2}(\beta) = \frac{1 - p^{2-2\beta}}{(1 - p^{3-\beta})(1 - p^{2-\beta})(1 - p^{1-\beta})(1 - p^{-\beta})}.
\]

Proof. We combine the results from Proposition 2.3 and Lemma 2.4 to first compute \( R_{\Gamma_2}(p^l) \):
\[
R_{\Gamma_2}(p^l) = \sum_{d_1|d_2|p^l} \deg_{\Gamma_1}(\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}) d_1^2 d_2 \\
= \sum_{l_1 \leq l_2 \leq l} \deg_{\Gamma_1}(\begin{pmatrix} p^{l_1} & 0 \\ 0 & p^{l_2} \end{pmatrix}) p^{2l_1+l_2} \\
= \sum_{i=0}^{l} \sum_{k=0}^{l-i} \deg_{\Gamma_1}(\begin{pmatrix} p^i & 0 \\ 0 & p^{k+i} \end{pmatrix}) p^{2i} p^{i+k} \\
= \sum_{i=0}^{l} p^{3i} + \sum_{i=0}^{l} \sum_{k=0}^{l-i} p^{k-1}(1+p)p^{3i+k} \\
= \frac{1}{(1-p)^2(1+p+p^2)} \left(1 - p^{2l}(p+p^2+p^3) + p^{3l}(p^2+p^4)\right), \\
\]

since \(\deg_{\Gamma_1}(\begin{pmatrix} p^{l_1} & 0 \\ 0 & p^{l_2} \end{pmatrix}) = p^{2l_2-l_1-1}\) for \(l_2 \neq l_1\) \([16,\ \text{Theorem 4.1, Chapter IV}]\). It is then clear that the series \(\zeta_{S_2,\Gamma_2}(\beta)\) converges if and only if \(\beta > 3\) and

\[
\zeta_{S_2,\Gamma_2}(\beta) = \frac{1}{(1-p)^2(1+p+p^2)} \left(\frac{(1-p)^2(1+p+p^2)p^{-\beta}(p+p^\beta)}{(1-p^{3-\beta})(1-p^{-\beta})(1-p^{2-\beta})}\right) \\
= \frac{1-p^{2-2\beta}}{(1-p^{3-\beta})(1-p^{2-\beta})(1-p^{1-\beta})(1-p^{-\beta})},
\]

for \(\beta > 3\) as desired. \(\square\)

We are now ready to prove the main theorem of this section.

**Theorem 3.8.** The \(GSp_4\)-system \((\mathcal{A}, (\sigma_t)_{t \in \mathbb{R}^+})\) does not admit a KMS\(_\beta\) state for \(0 < \beta < 3\) with \(\beta \notin \{1, 2\}\).

**Proof.** We put

\[
Y_p := P GSp_4^+(\mathbb{R}) \times GSp_4(\mathbb{Z}_p) \times \prod_{q \neq p} MSp_4(\mathbb{Z}_q), \quad \text{(42)}
\]

and note that

\[
PGSp_4^+(\mathbb{R}) \times (MSp_4(\mathbb{Z}_p) \cap GSp_4(\mathbb{Q}_p)) \times \prod_{q \neq p} MSp_4(\mathbb{Z}_q) = \bigcup_{s \in \Gamma_2 \setminus S_2,\Gamma_2} \Gamma_2 s Y_p. \quad \text{(43)}
\]

It is easy to see that the sets \(\Gamma_2 s Y_p\) are disjoint for \(s \in \Gamma_2 \setminus S_2,\Gamma_2/\Gamma_2\) and the complement of their union in \(\mathbb{H}_2^+ \times MSp_4(\mathbb{Z})\) is a subset of \((\mathbb{H}_2^+ \times MSp_4(\mathbb{A}_f))^c\), whence by Lemma 3.5 it has full measure for \(\beta \notin \{1, 2, 3\}\). Let \(\mu_\beta \in \mathcal{E}(\mathcal{K}_\beta)\) and \(\nu_\beta\) the measure on \(\Gamma_2 \setminus \overline{Y}\). Note that if \(g \in G_p \cap GSp_4(\mathbb{Z}_p)\) then necessarily \(\lambda(g) = 1\) and \(g \in MSp_4(\mathbb{Z})\), hence \(G_p \cap GSp_4(\mathbb{Z}_p) = \Gamma\). We can then apply \([18,\ \text{Lemma 2.7}]\) to the group \(G_p\) (a simple calculation shows that any elementary matrix in \(S_2,\Gamma_2\) is generated by \(g_{1,p}, g_{2,p}\) and \(g_{3,p}\)) and the spaces \(\overline{X}\) and \(Y_0 = Y_p\). We obtain that for any \(g \in \Gamma_2 \setminus S_2,\Gamma_2/\Gamma_2\), we have

\[27\]
\[ \nu_\beta(\Gamma_2 \setminus \Gamma_2 gY_p) = \lambda(g)^{-\beta} \deg_{\Gamma_2}(g) \nu_\beta(\Gamma_2 \setminus Y_p). \]

Observe that

\[ \nu_\beta(\Gamma_2 \setminus PGSp_4^+(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}})) = \sum_{g \in \Gamma_2 \setminus S_{2,p}/\Gamma_2} \nu_\beta(\Gamma_2 \setminus \Gamma_2 gY_p) \]

\[ = \sum_{g \in \Gamma_2 \setminus S_{2,p}/\Gamma_2} \lambda(g)^{-\beta} \deg_{\Gamma_2}(g) \nu_\beta(\Gamma_2 \setminus Y_p). \]

Hence

\[ 1 = \zeta_{S_{2,p},\Gamma_2}(\beta) \nu_\beta(\Gamma_2 \setminus Y_p), \tag{44} \]

which is not possible for \( \beta < 3 \) by Proposition 3.7. This shows that there are no KMS_\beta-states for \( \beta < 3 \) and \( \beta \notin \{0, 1, 2\} \).

\[ \square \]

3.2. Low temperature region and Gibbs states.

In this section, we study the equilibrium states of the \( GSp_4 \)-system within the low temperature region. More specifically, we give an explicit construction of the extremal KMS_\beta states for \( \beta > 4 \), which of course provides a complete description of the set of KMS_\beta states.

**Theorem 3.9.** For \( \beta > 4 \), the extremal KMS states of the \( GSp_4 \)-system are given by the Gibbs states

\[ \phi_\beta(f) = \frac{\zeta(2\beta - 2) \text{Tr}(\pi_y(f) e^{-\beta H_y})}{\zeta(\beta) \zeta(\beta - 1) \zeta(\beta - 2) \zeta(\beta - 3)}, \tag{45} \]

where \( y \in PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}) \).

**Proof.** Let \( F \) be an arbitrary finite set of primes and denote by \( G_F \) the group generated by \( G_p \) for \( p \in F \). We denote by \( MSp_4^+(\mathbb{Z}) = GSp_4^+(\mathbb{Q}) \cap MSp_4(\mathbb{Z}) \) and we put

\[ S_F := \{ m \in MSp_4^+(\mathbb{Z}) \mid \lambda(m) \in \mathbb{N}(F) \}, \]

and

\[ Y_F = PGSp_4^+(\mathbb{R}) \times \prod_{p \in F} GSp_4(\mathbb{Z}_p) \times \prod_{q \notin F} MSp_4(\mathbb{Z}_q). \]

Similarly to the proof of Theorem 3.8 (we replace \( Y_p \) by \( Y_F \) and \( S_{2,p} \) by \( S_F \)), we get

\[ 1 = \nu_\beta(\Gamma_2 \setminus Y_F) \zeta_{S_F,\Gamma_2}(\beta) = \nu_\beta(\Gamma_2 \setminus Y_F) \prod_{p \in F} \zeta_{S_{2,p},\Gamma_2} \]

Note that \( Y_F \subseteq Y_{F'} \) for \( F' \subseteq F \) and the intersection of \( Y_F \) over all finite primes is the set \( PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}) \). Hence for \( \beta > 4 \), we get

\[ \nu_\beta(\Gamma_2 \setminus (PGSp_4^+(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}))) = \zeta_{MSp_4(\mathbb{Z}),\Gamma_2}(\beta)^{-1}. \]
where
\[ \zeta_{MSp^+_4(z),\Gamma_2}(\beta) = \prod_{p \in \mathcal{P}} \zeta_{S_{2,p},\Gamma_2} = \frac{\zeta(\beta)\zeta(\beta-1)\zeta(\beta-2)\zeta(\beta-3)}{\zeta(2\beta-2)}. \]

On the other hand, the sets \( \Gamma_2s(PGSp^+_4(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})) \) are disjoint for \( s \in \Gamma_2 \backslash MSp^+_4(\mathbb{Z})/\Gamma_2 \). We thus obtain
\[
\nu_\beta(\Gamma_2 \backslash MSp^+_4(\mathbb{Z})(PGSp^+_4(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}))) = \sum_{s \in \Gamma_2 \backslash MSp^+_4(\mathbb{Z})/\Gamma_2} \nu_\beta(\Gamma_2 \backslash GSp_4(\hat{\mathbb{Z}})) = 1,
\]

Hence \( MSp^+_4(\mathbb{Z})(PGSp^+_4(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}})) \) has full measure in \( PGSp^+_4(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}}) \) and by Corollary 2.9 the subset \( PGSp^+_4(\mathbb{R}) \times GSp_4(\mathcal{A}_f) \) has full measure in \( PGSp^+_4(\mathbb{R}) \times MSp_4(\mathcal{A}_f) \). Conversely, any probability \( \Gamma_2 \)-invariant measure on \( PGSp^+_4(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}) \) extends [18, Lemma 2.4] uniquely to a measure on \( PGSp^+_4(\mathbb{R}) \times GSp_4(\mathcal{A}_f) \) satisfying condition 35.

Suppose now that \( \mu_\beta \in \mathcal{E}_\beta \) is a Dirac measure centered on \( y \in PGSp^+_4(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}) \). Then
\[
\phi(f) = \sum_{s \in \Gamma_2 \backslash MSp^+_4(\mathbb{Z})/\Gamma_2} \int_{\Gamma_2 \backslash \Gamma_2s(PGSp^+_4(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}))} f(1, \omega)d\nu_\beta(\omega)
\]
\[
= \zeta_{MSp^+_4(z),\Gamma_2}(\beta)^{-1} \sum_{h \in \Gamma_2 \backslash MSp^+_4(\mathbb{Z})} \lambda(h)^{-\beta} f(1, hy)
\]
\[
= \frac{\text{Tr}(\pi_y(f)e^{-\beta H_y})}{\text{Tr}(e^{-\beta H_y})},
\]

since the operator \( H_y \) is positive and \( G_y = MSp^+_4(\mathbb{Z}) \) for \( y \in GSp^+_4(\mathbb{R}) \times GSp_4(\hat{\mathbb{Z}}) \).

\[\square\]

3.3. The critical region.

We denote by \( \hat{\mathcal{E}}_\beta \) the subset of right \( GSp_4(\hat{\mathbb{Z}}) \)-invariant measures in \( \mathcal{E}_\beta \). The next proposition shows that this set is not empty for \( 3 \leq \beta < 4 \).

**Proposition 3.10.** For each \( \beta \in (3, 4] \), the \( GSp_4 \)-system \( \{A, (\sigma_t)_{t \in \mathbb{R}}\} \) admits at least one KMS_\beta-state.

**Proof.** We generalize the construction in [18]. By the correspondence in Proposition 2.12, it is enough to construct a measure \( \mu_\beta \) on \( H_2 \times MSp_4(\mathcal{A}_f) \) such that \( \mu_\beta \in \mathcal{E}_\beta \). For each prime \( p \) and \( 3 < \beta \leq 4 \), we consider the normalized Haar measure on \( GSp_4(\mathbb{Z}_p) \) so that the total volume is \( \zeta_{S_{2,p},\Gamma_2}(\beta)^{-1} \) (we denote this measure by \( \text{meas}_{\beta,p} \)). Observe that \( GSp_4(\mathbb{Q}_p) = G_p GSp_4(\mathbb{Z}_p) \) and hence by [18, lemma 2.4] we can uniquely extend this measure to a measure \( \mu_{\beta,p} \) on \( GSp_4(\mathcal{Q}_p) \) such that if \( Z \) is a compact measurable subset in \( GSp_4(\mathcal{Q}_p) \), then
\[
\mu_{\beta,p}(Z) = \sum_{g \in G_p} |\lambda(g)|_p^{-\beta} \text{meas}_{\beta,p}(gZ \cap GSp_4(\mathbb{Z}_p)),
\]

where
\[
\zeta_{MSp^+_4(z),\Gamma_2}(\beta) = \prod_{p \in \mathcal{P}} \zeta_{S_{2,p},\Gamma_2} = \frac{\zeta(\beta)\zeta(\beta-1)\zeta(\beta-2)\zeta(\beta-3)}{\zeta(2\beta-2)}. \]
where \( |a|_p \) denotes the \( p \)-adic valuation of \( a \). Since \( \mu_{\beta,p}(hZ) = |\lambda(h)|_p^{\beta} \mu_{\beta,p}(Z) \) for \( g \in GSp_4(\mathbb{Q}_p) \), it is clear that \( \mu_{\beta,p} \) is left \( GSp_4(\mathbb{Z}_p) \)-invariant. It is also right \( GSp_4(\mathbb{Z}_p) \)-invariant since the Haar measure \( \text{meas}_{\beta,p} \) is right translation invariant. We extend the measure \( \mu_{\beta,p}(Z) \) to a measure on \( MSp_4(\mathbb{Q}_p) \) by setting \( \mu_{\beta,p}(Z) := \mu_{\beta,p}(Z \cap GSp_4(\mathbb{Q}_p)) \) for Borel \( Z \subseteq MSp_4(\mathbb{Q}_p) \). To extend this measure to \( MSp_4(\mathbb{A}_f) \) we first check that \( \mu_{\beta,p}(MSp_4(\mathbb{Z}_p)) = 1 \). The proof of Lemma (3.5) applied to the space \( MSp_4(\mathbb{Q}_p) \) shows that the set \( MSp_4(\mathbb{Z}_p) \cap GSp_4(\mathbb{Q}_p) \) has full measure. Since this set is precisely \( S_{2,p}GSp_4(\mathbb{Z}_p) \), then similarly to the calculation in the proof of Theorem 3.8 we get

\[
\mu_{\beta,p}(MSp_4(\mathbb{Z}_p)) = \sum_{g \in \Gamma_2 \setminus S_{2,p}/\Gamma_2} \lambda(g)^{-\beta} \deg_{\Gamma_2}(g) \mu_{\beta,p}(GSp_4(\mathbb{Z}_p)) = 1.
\]

We thus define a measure on \( MSp_4(\mathbb{A}_f) \) by \( \mu_{\beta,f} = \prod_p \mu_{\beta,p} \). Then for \( g \in GSp_4(\mathbb{Q}) \) and measurable subset \( Z \subseteq MSp_4(\mathbb{A}_f) \), we get

\[
\mu_{\beta,f}(gZ) = \left( \prod_{p \in \mathcal{P}} |\lambda(g)|_p^{\beta} \right) \mu_{\beta,f}(Z) = \lambda(g)^{-\beta} \mu_{\beta,f}(Z).
\]

If we denote by \( \mu_{\beta,PGSp_4^+(\mathbb{R})} \) the normalized Haar measure on \( PGSp_4^+(\mathbb{R}) \) such that \( \nu_{\beta,PGSp_4^+(\mathbb{R})} \) is a probability measure on \( \Gamma_2 \setminus PGSp_4^+(\mathbb{R}) \), then it is clear that the measure defined by \( \mu_{\beta} := \mu_{\beta,PGSp_4^+(\mathbb{R})} \times \mu_{\beta,f} \) is an element of \( \mathcal{K}_\beta \). By construction, \( \mu_{\beta} \) is also right \( GSp_4(\mathbb{Z}) \)-invariant; that is \( \mu_{\beta} \in \hat{\mathcal{E}}_\beta \).

Our next goal is to show that for \( 3 < \beta \leq 4 \), the KMS\(_\beta \) constructed in Proposition 3.10 is the unique equilibrium state. We first recall the definition of an ergodic action.

**Definition 5.** If \( \mu \in \mathcal{E}_\beta \), the action of \( G \) on the measure space \((X, \mu)\) is ergodic if the following holds: If \( A \) is any \( G \)-invariant Borel subset of \( X \), then \( \mu(A) = 0 \) or \( \mu(A^c) = 0 \).

Recall that if \( W \) be a locally compact group then a character of \( W \) is a continuous homomorphisms \( \chi : W \to \mathbb{T} \).

**Lemma 3.11.** For \( n \in \mathbb{N} \), we let \( G_n = 1 + p^n \mathbb{Z}_p^\times \subseteq \mathbb{Z}_p \) and \( \chi \) be any character of \( \mathbb{Z}_p^\times \). Then \( G_k \subseteq \ker(\chi) \) for some \( k \in \mathbb{N} \).

**Proof.** Consider the open subset of \( \mathbb{T} \) given by

\[
V = \{ z \in \mathbb{T} \mid \text{Re}(z) > 0 \}.
\]

Observe first that the only subgroup of \( V \) is \( \{1\} \) and a fundamental system of neighborhood of the neutral element of \( \mathbb{Z}_p^\times \) is given by the subgroups

\[
G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \ldots
\]

Consider now the open subset of \( \mathbb{T} \) given in (46). Since the character \( \chi \) is continuous, there exists an integer \( k \geq 1 \) such that \( \chi(G_k) \subseteq V \). Now \( \chi \) is homomorphism and therefore the subset \( \chi(G_k) \) is a subgroup of \( V \), that is \( \chi(G_k) = 1 \). \( \square \)

**Lemma 3.12.** Let \( m \) be an integer and \( B \) a finite set of prime numbers. Then the set

\[
\{(n, \ldots, n) \in \prod_{p \in B} \mathbb{Z}_p^\times \mid n \in \mathbb{Z} \text{ and } (n, p) = 1 \text{ } \forall p \in B\}
\]

is dense in \( \prod_{p \in B} \mathbb{Z}_p^\times \).
Proof. Any $a \in \mathbb{Z}_p^\times$ admits a $p$-adic expansion of the form

$$a = \sum_{i \geq 0} c_i p^i, \quad 0 < c_i < p.$$ 

Hence, given $\epsilon > 0$ and $x = (x_1, \ldots, x_{|B|}) \in \prod_{p \in B} \mathbb{Z}_p^\times$, we choose $a_k \in \mathbb{Z}$ and $e_k \in \mathbb{N}$ such that

$$|x_k - a_k|_{p_k} < \frac{\epsilon}{2}, \quad (a_k, p_k) = 1, \quad p_k^{-e_k} < \frac{\epsilon}{2}$$

for $k = 1, \ldots, |B|$. By the Chinese remainder theorem, the congruence system

$$n \equiv a_k \mod p_k^k, \quad 1 \leq k \leq |B|$$

has a solution $n \in \mathbb{Z}$. The condition $(a_k, p_k)$ implies that $(n, p_k) = 1$ for all $1 \leq k \leq |B|$. Hence

$$|n - x_k|_{p_k} \leq |n - a_k|_{p_k} + |a_k - x_k|_{p_k} \leq p_k^{-e_k} + \frac{\epsilon}{2} \leq \epsilon,$$

as desired. 

\[\square\]

Given $m \in \mathbb{N}$, a Dirichlet character modulo $m$ is a function $\chi_m : \mathbb{Z} \to \mathbb{C}$ obtained by extending a character of $(\mathbb{Z}/m\mathbb{Z})^\times$ to 0 on $\mathbb{Z}/m\mathbb{Z}$ and lifted to $\mathbb{Z}$ by composition. The corresponding Dirichlet $L$-function is defined by

$$L(s, \chi_m) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad s \in \mathbb{C}.$$ 

**Theorem 3.13.** Let $\hat{\mu}_\beta \in \hat{\mathcal{E}}_\beta$ and $A = L^\infty(PGSp_4^+(\mathbb{R}) \times MS\text{Sp}_4(\mathbb{A}_f), \hat{\mu}_\beta)$. We have that

$$A^{PGSp_4^+(\mathbb{R}) \times MS\text{Sp}_4^+(\mathbb{Q})} = \mathbb{C}. \quad (48)$$

**Proof.** It is enough to show that the action of $GSp_4^+(\mathbb{Q})$ on $(MS\text{Sp}_4(\mathbb{A}_f), \hat{\mu}_\beta,f)$ (where $\hat{\mu}_\beta,f$ is the measure on $MS\text{Sp}_4(\mathbb{A}_f)$ obtained by projecting onto the second factor) is ergodic. The strategy is similar to [22] and [18]. Since every $GSp_4^+(\mathbb{Q})$-invariant subset is completely determined by its intersection with $MS\text{Sp}_4(\hat{\mathbb{Z}})$, it is enough to show that the closed subspace

$$H = \{ f \in L^2(MS\text{Sp}_4(\hat{\mathbb{Z}})), \, d\hat{\mu}_{\beta,f} \mid V_m f = f, \forall m \in MS\text{Sp}_4^+(\mathbb{Z}) \}, \quad (V_m f)(x) := f(mx),$$

consists of constant functions. Denote by $P$ the orthogonal projection onto $H$. Since every function in $H$ is $\Gamma_2$-invariant, it is enough to show that $P$ maps $\Gamma_2$-invariants functions to constants.

Consider the subspace

$$H_F = \{ f \in L^2(MS\text{Sp}_4(\hat{\mathbb{Z}})), \, d\hat{\mu}_{\beta,f} \mid V_s f = f, \forall s \in S_F \},$$

and denote by $P_F$ the orthogonal projection onto $H_F$. Consider the subset

$$W_F = \prod_{p \in F} GSp_4(\mathbb{Z}_p) \times \prod_{q \not\in F} MS\text{Sp}_4(\mathbb{Z}_q).$$

We obtain by Lemma 3.1 and Corollary 3.12 that the disjoint union $\bigcup_{s \in \Gamma_2 \setminus S_F/\Gamma_2} s W_F$ is a subset of full measure. Hence given any $\Gamma_2$-invariant function $f \in L^2(MS\text{Sp}_4(\hat{\mathbb{Z}})), \, d\hat{\mu}_{\beta,f}$, we deduce from [18, Lemma 2.9] that
\[ P_{F}f = \zeta_{S_{F}, \Gamma_{2}}(\beta)^{-1} \sum_{s \in \Gamma_{2} \setminus S_{F} / \Gamma_{2}} \lambda(s)^{-\beta} R_{\Gamma_{2}}(s) T_{sf}. \] (49)

We fix a finite set of primes \( B \) such that \( B \cap F = \emptyset \) and consider the functions in \( L^{2}(M \beta_{p}(\hat{Z}), \, d\mu_{\beta,f}) \) of the form

\[ \chi_{B}(x) = \begin{cases} \chi(\lambda((x_{p})_{p \in B})) & \text{if } x \in W_{B} \\ 0 & \text{otherwise} \end{cases}, \]

where \( \chi \) is a character of the compact abelian group \( \prod_{p \in B} \mathbb{Z}_{p}^{\times} \).

We will first show that \( P\chi_{B}(x) \) is constant a.e. This is easy to prove if the character \( \chi \) is trivial. Indeed, in this case \( \chi_{B} = 1_{B} \) and the projection formula (49) we get that \( P_{B}\chi_{B} = \zeta_{S_{B}, \Gamma_{2}}(\beta)^{-1} \).

Since \( P = PP_{F} \) the result follows. We now consider the case where \( \chi \) is non-trivial. We first write \( \chi = \prod_{p \in B} \chi_{p} \), where \( \chi_{p} \) is a character of \( \mathbb{Z}_{p}^{\times} \) given by

\[ \chi_{p}(a) = \chi(1, \ldots, a, \ldots, 1). \]

Then by Lemma 3.11 for each prime \( p \in B \) there exists an integer \( k_{p} \in \mathbb{N} \) such that \( G_{k_{p}} \subseteq \ker(\chi_{p}) \).

Let \( m = \prod_{p \in B}^{k_{p}} \) and define \( \chi_{m} : \mathbb{Z} \to \mathbb{C} \) by

\[ \chi_{m}(n) = \begin{cases} \chi((n, \ldots, n)) & \text{if } (n, m) = 1 \\ 0 & \text{otherwise} \end{cases} \]

It is clear that \( \chi_{m} \) is multiplicative and \( \chi_{m}(n+m) = \chi_{m}(n) \) if \( (n, m) \neq 1 \). Suppose that \( (n, m) = 1 \) so that \( n \in \mathbb{Z}_{p}^{\times} \). Hence

\[ \chi_{m}(n+m) = \prod_{p \in B} \chi_{p}(n+m) = \prod_{p \in B} \chi_{p}(n) \chi_{p}(1+mn^{-1}) = \prod_{p \in B} \chi_{p}(n) = \chi_{m}(n), \]

since \( p_{p}^{k_{p}} \) divides \( m \). Hence \( \chi_{m} \) is a Dirichlet character modulo \( m \). We claim that \( \chi_{m} \) is nontrivial. Indeed, let \( a \in \prod_{p \in B}^{\mathbb{Z}_{p}^{\times}} \) such that \( \chi(a) \neq 1 \). By Lemma 3.6 the set

\[ \{(n, \ldots, n) \in \prod_{p \in B}^{\mathbb{Z}_{p}^{\times}} : n \in \mathbb{Z} \text{ and } (n, m) = 1 \} \]

is dense in \( \prod_{p \in B}^{\mathbb{Z}_{p}^{\times}} \). Since \( \chi \) is continuous there exists \( n_{0} \in \mathbb{Z} \) with \( (n_{0}, m) = 1 \) and \( 1 \neq \chi(n_{0}, \ldots, n_{0}) = \chi_{m}(n_{0}) \) as desired.

Since \( F \cap B = \emptyset \), for \( s \in S_{F} \) we can write

\[ (T_{s}\chi_{B})(x) = \begin{cases} \chi(\lambda((x_{p})_{p \in B}))\chi_{m}(\lambda(s)) & \text{if } x \in W_{B} \\ 0 & \text{otherwise}. \end{cases} \]

This allows us to obtain an explicit upper bound of the \( L^{2} \)-norm of \( P\chi_{B} \) as follows. Since \( P = PP_{F} \), from the projection formula (49) we get
\[ \|P\chi_B\| = \|PP_F\chi_B\| \leq \|P_F\chi_B\| \leq |\zeta_{S_F,\Gamma_2}(\beta)^{-1} - \sum_{s \in F / \Gamma} \lambda(s)^{-\beta} R_{\Gamma_2}(s) \chi_m(\lambda(s))| \]
\[ = |\zeta_{S_F,\Gamma_2}(\beta)^{-1} \sum_{n \in \mathbb{N}(F)} n^{-\beta} R_{\Gamma_2}(n) \chi_m(n)| \]
\[ = |\zeta_{S_F,\Gamma_2}(\beta)^{-1} \prod_{p \in F} \sum_{l=0}^{\infty} p^{-l\beta} R_{\Gamma_2}(p^l) \chi_m(p^l)|, \]

since the function \(R_{\Gamma_2}(n)\) is multiplicative by Lemma 2.5. As in the proof of Proposition 3.7 we get

\[ \|P\chi_B\| \leq \zeta_{S_F,\Gamma_2}(\beta)^{-1} \prod_{p \in F} \left( \frac{1}{1 - \chi_m(p)p^{-\beta}} - \frac{p + p^2 + p^3}{1 - \chi_m(p)p^{2-\beta}} + \frac{p^2 + p^4}{1 - \chi_m(p)p^{3-\beta}} \right) \]
\[ = \prod_{p \in F} \left( \frac{(1 + p^{1-\beta}(\chi_m(p) - 1)) - p^{2-\beta} \chi_m(p)(1 - p^{2-\beta})(1 - p^{-\beta})}{(1 - \chi_m(p)p^{3-\beta})(1 - \chi_m(p)p^{2-\beta})(1 - \chi_m(p)p^{1-\beta})(1 - \chi_m(p)p^{-\beta})(1 - p^{-2-\beta})} \right) \]
\[ = \left( \frac{\sum_{n \in \mathbb{N}(F)} \frac{\chi_m(n)}{n^{2-\beta}}} {\zeta_{\mathbb{N}(F)}(\beta - 3)\zeta_{\mathbb{N}(F)}(\beta - 2)\zeta_{\mathbb{N}(F)}(\beta - 1)} \right) \prod_{p \in F} \left( \frac{1 + \chi_m(p)p^{1-\beta}}{1 + p^{1-\beta}} \right) \]
\[ \leq \frac{L(\beta - 3, \chi_m)L(\beta - 2, \chi_m)L(\beta - 1)}{\zeta_{\mathbb{N}(F)}(\beta - 3)\zeta_{\mathbb{N}(F)}(\beta - 2)\zeta_{\mathbb{N}(F)}(\beta - 1)}. \]

Since the character \(\chi_m\) is non-trivial and \(3 < \beta \leq 4\), it follows from [30, Proposition 12, Chapter VI] that the right hand side above can be made arbitrary small as \(F \not\subseteq P\) (with \(F \cap B = \emptyset\)). This shows that \(P\chi_B = 0\), in particular \(P\chi_B\) is again a constant function when \(\chi\) is a nontrivial character.

Consider now the functions \(F_B \in L^2(MSp_4(\mathbb{Z}), d\mu_{\beta,f})\) of the form

\[ F_B(x) = \begin{cases} 
  f((x_p)_{p \in B}) & \text{if } x \in W_B \\
  0 & \text{otherwise},
\end{cases} \]

where \(f \in L^2(\prod_{p \in B} GSp_4(\mathbb{Z}_p), d\mu_B)\) and \(\mu_B = (\pi_B)_*(\mu_{\beta,f})\), where \(\pi_B\) is the projection \(\pi_B : MSp_4(\mathbb{Z}) \to \prod_{p \in B} MSp_4(\mathbb{Z}_p)\). We first show that it is enough to assume that \(f\) is \(\Gamma_2\)-invariant. Indeed, when this is not the case, we denote by \(Q\) the projection onto the space of \(Sp_4(\mathbb{Z})\)-invariant functions. Since \(\hat{\mu}_{\beta,f}\) is \(Sp_4(\mathbb{Z})\)-invariant we have that

\[ QF_B(x) = \int_{Sp_4(\mathbb{Z})} F_B(gx) dg = \begin{cases} 
  \int_{\prod_{p \in B} Sp_4(\mathbb{Z}_p)} f(gx) dg_B & \text{if } x \in W_B \\
  0 & \text{otherwise},
\end{cases} \]
Since the Haar measure on $Sp_4(\hat{\mathbb{Z}})$ is left-translation-invariant and $PQ = P$ (since $\Gamma_2$ is dense in $Sp_4(\hat{\mathbb{Z}})$), this shows that WLOG we can assume that $f$ is $\Gamma_2$-invariant. Then by the density of $\Gamma_2$ in $\prod_{p \in B} Sp_4(\hat{\mathbb{Z}}_p)$, we see that the function $f$ depends only on $\lambda((x_p)_{p \in B})$, i.e

$$F_B(x) = \begin{cases} f'(\lambda(x_p)_{p \in B}) & \text{if } x \in W_B \\ 0 & \text{otherwise,} \end{cases}$$

where $f'$ is a square integrable function in $\prod_{p \in B} Z_p^*$ (with the natural pushforward measure). The linear span of $\chi$ (where $\chi$ is a character of $\prod_{p \in B} Z_p^*$) form a dense subspace of such square integrable functions and we have shown that $P\chi$ is constant. It follows that $PF_B$ is constant.

We can easily verify that the adjoint of the the operator $V_s$ (where $s \in S_B$) is given by

$$(V_s^* h)(x) = \begin{cases} \lambda(s)^{\beta} h(s^{-1}x) & \text{if } x \in sMSp_4(\hat{\mathbb{Z}}) \\ 0 & \text{otherwise.} \end{cases}$$

Hence the map $\lambda(s)^{-\beta/2}V_s^*$ maps isometrically the functions of the form $F_B$ to functions in $L^2(MSp_4(\hat{\mathbb{Z}}), d\mu_{\beta,f})$ of the form

$$G_B(x) = \begin{cases} g((x_p)_{p \in B}) & \text{if } x \in sW_B \\ 0 & \text{otherwise,} \end{cases}$$

where $g \in L^2(s \prod_{p \in B} GSp_4(\mathbb{Z}_p), d\mu_B)$. It follows then from $P = PV_s^*$ that the projection $P$ maps every function of the form (50) to a constant function. The set $\cup_{s \in \Gamma_2 \backslash S_B/\Gamma_2} sW_B$ has full measure in $MSp_4(\hat{\mathbb{Z}})$ and thus $P$ maps functions depending only on $(x_p)_{p \in B}$ to constants. Finally observe that as $B \not\supseteq \mathcal{P}$, the union of $L^2(\prod_{p \in B} MSp_4(\mathbb{Z}_p), d\mu_B)$ over all finite sets of primes is a dense subspace of $L^2(\prod_{p \in B} MSp_4(\mathbb{Z}_p), d\hat{\mu}_{\beta,f})$ (this follows from weak convergence of measures). Since $P$ maps this dense subspace to constant functions, this finishes the proof.

\[\blacksquare\]

**Lemma 3.14.** Let $f$ be a smooth function on $\Gamma \backslash PGSp^+_4(\mathbb{R})$ with a compact support and let $\Omega$ be any compact subset of $\Gamma \backslash PGSp^+_4(\mathbb{R})$. If we denote by $d\mu$ the normalized Haar measure on $\Gamma \backslash PGSp^+_4(\mathbb{R})$, then for all $\epsilon > 0$ there exist $\kappa_1(\epsilon) > 0$, $\kappa_2 > 0$ and $M_2 > 0$ (depending on $f$) such that the inequality

$$\frac{1}{R_{\Gamma_2}(m)} \sum_{a \in \Gamma_2 \backslash S_m/\Gamma_2} \left| (T_a f)(\tau) - \int_{\Gamma_2 \backslash PGSp^+_4(\mathbb{R})} f \, d\mu \right| \deg(a) \leq \kappa_1 m^{(2\epsilon-1)} \prod_{i=1}^{l} (1+\kappa_2 p_i^{-1}) \frac{1}{(1-p^{-2\epsilon-1})^2},$$

holds for all $\tau \in \Omega$ and every integer of the form $m = \prod_{i=1}^{l} p_i^{t_i}$ where $\min\{p_1, \ldots, p_l\} > M_2$.

**Proof.** Let $p$ be a given prime and $l \in \mathbb{N}$. The formula of $R_{\Gamma_2}(p^l)$ in the proof of Proposition 3.7 gives

$$(R_{\Gamma_2}(p^l) - p^{2l})(1-p)^2(1+p+p^2) = 1 - p^{2l} - p^{1+2l} - p^{2+2l} - p^{2+2l} + p^{1+3l} + p^{2+3l} \geq 1 + 3p^{3l} - p^{2+2l} \geq p^{3l}(1-p^{2-1}) \geq 0, \quad \forall l \geq 2$$

A simple verification for the case $l = 1$ shows that
Proposition 3.15. Let \( J \) be any nonempty finite set of prime numbers, \( 3 < \beta \leq 4 \), \( f \) a smooth function on \( \Gamma \setminus GSp^+_4(\mathbb{R}) \) with a compact support and \( \Omega \) any compact subset of \( \Gamma \setminus GSp^+_4(\mathbb{R}) \). Then for any \( \delta > 0 \), there exists a sequence \( \{F_n\}_{n \geq 1} \) consisting of finite set of prime numbers that are disjoint from \( J \) and such that for all \( \tau \in \Omega \) we have

\[
\left| (T_{\eta_n}f)(\tau) - \int_{\Gamma \setminus GSp^+_4(\mathbb{R})} f d\mu \right| < \delta.
\]

Proof. Given \( 3 < \beta \leq 4 \), we fix \( 0 < \epsilon < \frac{\beta - 3}{2} \leq \frac{1}{2} \) and choose \( \kappa_1, \kappa_2 \) and \( M_2 \) as in Lemma 3.14. Let \( M_1 > 0 \) be such that

\[
x^\beta(1 - x^{2\epsilon - 1} - \kappa_2 x^{2\epsilon - 2}) > \kappa_2 \quad \forall x > M_1,
\]

and we set \( M := \max\{M_1, M_2\} \). Let \( F \) be a finite set of prime numbers with \( F \cap J = \emptyset \) and \( \min\{p \in F\} > M \). Then by Lemma 3.14, we have

\[
\frac{1}{R_{\Gamma_2}(m)} \sum_{a \in \Gamma_2 \setminus S_m / \Gamma_2} \left| (T_a f)(\tau) - \int_{\Gamma_2 \setminus GSp^+_4(\mathbb{R})} f d\mu \right| \deg(a)
\]

\[
\leq \frac{\kappa_1}{R_{\Gamma_2}(m)} \sum_{k_{ij} \leq m_{ij} \leq \lfloor l/2 \rfloor} \prod_{i=1}^{i=r} p_{2i,2k_{ij} - 2m_{ij}} \left( 1 + O(p_i^{-1}) \right)
\]

\[
\leq \frac{\kappa_1}{R_{\Gamma_2}(m) p^{-3l}} \prod_{i=1}^{i=l} \sum_{k_{ij} \leq m_{ij} \leq \lfloor l/2 \rfloor} p_{i,2(2\epsilon - 1)k_{ij} + (-3 - 2\epsilon) m_{ij}} \left( 1 + \kappa_2 p_i^{-1} \right)
\]

\[
\leq \kappa_1 m^{2\epsilon - 1} \prod_{i=1}^{i=l} \sum_{k_{ij} \leq m_{ij} \leq \lfloor l/2 \rfloor} p^{-(3 + 2\epsilon) k_{ij} - (1 + 2\epsilon) m_{ij}} \left( 1 + \kappa_2 p_i^{-1} \right)
\]

\[
\leq \kappa_1 m^{2\epsilon - 1} \prod_{i=1}^{i=l} \left( 1 + \kappa_2 p_i^{-1} \right) \sum_{m_{ij} = 0}^{\infty} \sum_{k_{ij} = 0}^{\infty} p^{-(3 + 2\epsilon) k_{ij} - (1 + 2\epsilon) m_{ij}}
\]

\[
\leq \kappa_1 m^{2\epsilon - 1} \prod_{i=1}^{i=l} \left( 1 + \kappa_2 p_i^{-1} \right) \frac{1}{(1 - p^{-1 - 2\epsilon})^2},
\]

holds for \( \min\{p_1, \ldots, p_l\} > M_2 \) as desired.

\( \square \)
\[
\left| (T_F f)(\tau) - \int_{\Gamma_2 \backslash GSp^+_4(\mathbb{R})} f \, d\mu \right| \\
\leq \kappa_1 \xi_{S_F, F}(\beta)^{-1} \left( \sum_{m \in \mathbb{N}(F)} m^{(2\epsilon - 1)} \prod_{i=1}^\infty \frac{1}{(1 + \kappa_2 p_i^{-1})(1 - p^{2\epsilon - 1})} R_F(m) m^{-\beta} \right) \\
\leq \kappa_1 \xi_{S_F, F}(\beta)^{-1} \left( \prod_{p \in F} p^{(2\epsilon - 1)} R_F(p) p^{-\beta}(1 + \kappa_2 p^{-1})(1 - p^{2\epsilon - 1}) \right) \\
\leq \kappa_1 \xi_{S_F, F}(\beta)^{-1} \left( \prod_{p \in F} \frac{1}{1 + p^{2\epsilon - 1} p^{-\beta}} \right) \xi_{\mathbb{N}(F)}(\beta - 2 - 2\epsilon) \xi_{\mathbb{N}(F)}(\beta - 1 - 2\epsilon) \xi_{\mathbb{N}(F)}(\beta - 2) \xi_{\mathbb{N}(F)}(\beta - 1).
\]

Notice that by our choice of \( M \) and \( F \), Equation (52) gives

\[
(1 + \kappa_2 p^{-1}) \frac{1 + p^{2\epsilon - 1} p^{-\beta}}{1 + p^{2\epsilon - 1}} < 1, \quad \forall p \in F.
\]

Hence

\[
\left| (T_F f)(\tau) - \int_{\Gamma_2 \backslash GSp^+_4(\mathbb{R})} f \, d\mu \right| \leq \kappa_1 \prod_{p \in F} \frac{\xi_{\mathbb{N}(F)}(\beta - 2 - 2\epsilon) \xi_{\mathbb{N}(F)}(\beta - 1 - 2\epsilon) \xi_{\mathbb{N}(F)}(\beta - 2)}{(1 - p^{2\epsilon - 1}) \xi_{\mathbb{N}(F)}(\beta - 3) \xi_{\mathbb{N}(F)}(\beta - 2) \xi_{\mathbb{N}(F)}(\beta - 1)}.
\]

As \( F \not\supseteq \mathcal{P} \) with \( F \cap J = \emptyset \), the right hand side can be made arbitrary small since \( 3 < \beta \leq 4, \beta - 2 - 2\epsilon > 1 \) and \( \epsilon > 0 \).

\textbf{Theorem 3.16.} Let \( \hat{\mu}_\beta \in \hat{\mathcal{C}}_\beta \) and \( A = L^\infty(GSp^+_4(\mathbb{R}) \times MSp_4(\mathbb{A}_F), \hat{\mu}_\beta) \). We have that

\[
A^{GSp^+_4(\mathbb{Q}) \times GSp_4(\hat{\mathbb{Z}})} = \mathbb{C}.
\]

\textbf{Proof.} Consider the space \( \mathcal{H} = L^2(GSp^+_4(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}}), d\hat{\mu}_\beta) \). Observe that any \( GSp^+_4(\mathbb{Q}) \times GSp_4(\hat{\mathbb{Z}}) \)-invariant subset of \( GSp^+_4(\mathbb{R}) \times MSp_4(\mathbb{A}_F) \) is completely determined by its intersection with \( GSp^+_4(\mathbb{R}) \times MSp_4(\hat{\mathbb{Z}}) \), hence it is enough to show that any \( MSp^+_4(\mathbb{Z}) \times GSp_4(\hat{\mathbb{Z}}) \)-invariant function in \( \mathcal{H} \) is constant. We denote by \( H \) the closed subspace of \( MSp^+_4(\mathbb{Z}) \times GSp_4(\hat{\mathbb{Z}}) \)-invariant functions in \( \mathcal{H} \) and denote by \( P \) the orthogonal projection onto \( H \). We will show that the image under \( P \) of a dense subspace consists of constant functions. Given any non-empty finite sets of primes \( F \) and \( J \), we denote by \( H_F \) the closed subspace of \( S_F \)-invariant functions in \( \mathcal{H} \) and by \( P_F \) the orthogonal projection onto \( H_F \). Let \( \mathcal{H}_J \) be the subspace of \( \Gamma_2 \times \prod_{p \in J} MSp_4(\mathbb{Z}_p) \). Recall that \( S_J Y_J \) is a subset of full measure, whence by [18, Lemma 2.9] given any function \( f \) in \( \mathcal{H}_J \) we get

\[
P_J f = \zeta_{S_F, F}(\beta) \sum_{s \in S_J / \Gamma_2} \lambda(s)^{-\beta} \deg_{\Gamma_2}(s) T_s f.
\]
It follows that the value \( P_J f(\tau, m) \) depends only on \( \tau \in GSp_4^+(\mathbb{R}) \). We can then write

\[
P_J f(x) = \begin{cases} 
\tilde{f}(\tau) & \text{if } x = (\tau, m) \in Y_J \\
0 & \text{otherwise}
\end{cases}
\]

We put \( \tilde{f}_J := P_J f \). Observe that since \( \tilde{f}_J \) is \( S_J \)-invariant we get that \( \tilde{f} \) is \( \Gamma_2 \)-invariant and therefore can view it as a square integrable function on \( \Gamma_2 \setminus PSp_4^+(\mathbb{R}) \). We first suppose that \( \tilde{f} \) is smooth with compact support \( \Omega \). For \( F \cap J = \emptyset \), the projection formula (54) gives

\[
P_F \tilde{f}_J(x) = \begin{cases} 
T_F \tilde{f}(\tau) & \text{if } x = (\tau, m) \in Y_J \\
0 & \text{otherwise.}
\end{cases}
\]

We put \( (T_F \tilde{f})_J := P_F \tilde{f}_J(x) \). Given \( \epsilon > 0 \), by Proposition 3.15 there exists some finite set of primes \( F \) disjoint from \( J \) such that

\[
\left| T_F \tilde{f}(\tau) - \int_{\Gamma_2 \setminus PSp_4^+(\mathbb{R})} \tilde{f} \, d\mu \right| < \epsilon, \quad \forall \tau \in \Omega.
\]

Since \( PP_J = PP_F = P \), we get

\[
\left\| Pf - P1_{Y_J} \int_{\Gamma_2 \setminus PSp_4^+(\mathbb{R})} \tilde{f} \, d\mu \right\|_2 = \left\| PP_J f - P1_{Y_J} \int_{\Gamma_2 \setminus PSp_4^+(\mathbb{R})} \tilde{f} \, d\mu \right\|_2 = \left\| PFP_J f - P1_{Y_J} \int_{\Gamma_2 \setminus PSp_4^+(\mathbb{R})} \tilde{f} \, d\mu \right\|_2 \\
\leq \left\| PFP_J f - 1_{Y_J} \int_{\Gamma_2 \setminus PSp_4^+(\mathbb{R})} \tilde{f} \, d\mu \right\|_2 \\
\leq \left\| (T_F \tilde{f})_J - 1_{Y_J} \int_{\Gamma_2 \setminus PSp_4^+(\mathbb{R})} \tilde{f} \, d\mu \right\|_2 < \epsilon.
\]

Hence using the projection formula 54 with \( f = 1_{Y_J} \) we get

\[
P f = P1_{Y_J} \int_{\Gamma_2 \setminus PSp_4^+(\mathbb{R})} \tilde{f} \, d\mu = P1_{Y_J} \int_{\Gamma_2 \setminus PSp_4^+(\mathbb{R})} \tilde{f} \, d\mu = \zeta_{S_J, \Gamma_2}(\beta)^{-1} \int_{\Gamma_2 \setminus PSp_4^+(\mathbb{R})} \tilde{f} \, d\mu,
\]

which is constant. Any integrable functions on \( \Gamma_2 \setminus PSp_4^+(\mathbb{R}) \) can be approximated by a compactly supported smooth function and hence \( P f \) is constant for all \( \Gamma_2 \)-invariant functions in \( H_J \). The results follows since the union of \( H_J \) over all finite set of primes is dense in the space of square integrable \( \Gamma_2 \times GSp_4(\mathbb{Z}) \)-invariant functions.

\[
\square
\]

**Theorem 3.17.** For \( 3 < \beta \leq 4 \), The \( GSp_4 \)-system admits a unique KMS\( \beta \) state.

**Proof.** We will show that the set \( \mathcal{E}_\beta \) consists of a single point. We first use [18, Proposition 4.6] together with Theorem 3.13 and Theorem 3.16 to conclude that \( A^{GSp_4^+(\mathbb{Q})} = \mathbb{C} \) where \( A = L^\infty(PSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{R}, \mathbb{Q}), \hat{\mu}_\beta) \) for any \( \hat{\mu}_\beta \in \mathcal{E}_\beta \) and \( 3 < \beta \leq 4 \), in other words there exists a
unique right $GSp_4(\hat{\mathbb{Z}})$-invariant measure $d\hat{\mu}_\beta$ in $\mathcal{E}_\beta$. Suppose now that $v_\beta$ is any other point of $\mathcal{E}_\beta$. Then the measure defined by

$$\omega = \int_{GSp_4(\hat{\mathbb{Z}})} g \cdot v_\beta \, dg$$

is an element of $\mathcal{E}_\beta$ and by unicity we get that $\omega = \hat{\mu}_{\beta,f}$. Since the point $\hat{\mu}_{\beta,f}$ is extremal we conclude that $\mu_{\beta,f} = v_\beta$. This completes the proof. \qedsymbol

**Remark 1.** We have studied the $GSp_4$-system in the region $\beta > 0$ with $\beta \notin \{1, 2, 3\}$. Let us now consider the cases where the inverse temperature is a pole of the Dirichlet series (41). If $\beta = 2$, it is possible to construct explicit measures $\mu_2 \in \mathcal{E}_2$. We consider the normalized Haar measure on $A_{f,Q}$ such that $\text{meas}(\hat{\mathbb{Z}}) = 1$ and

$$\text{meas}(aE) = \prod_p |a_p|_p \text{meas}(E). \quad (55)$$

for any $a \in A_{f,Q}^2$ and measurable subset $E \subseteq A_{f,Q}$. Let $\mu_f$ be the product measure on $A_{f,Q}^2$. Since

\[
\begin{pmatrix}
0 & 0 & 0 & x_1 \\
0 & 0 & 0 & x_2 \\
0 & 0 & 0 & x_3 \\
0 & 0 & 0 & x_4
\end{pmatrix} \in \text{MSp}_4(A_{f,Q}), \quad x_1, \ldots, x_4 \in A_{f,Q}
\]

we may consider $\mu_f$ as a measure on $\text{MSp}_4(A_{f,Q})$ such that $\mu_f(\text{MSp}_4(\hat{\mathbb{Z}})) = 1$. We claim that $\mu_2 = \mu_\infty \times \mu_f \in \mathcal{E}_2$. By construction it is enough to show that $\mu_2$ satisfies the scaling condition (35). Given $g \in GSp_4^+(\mathbb{Q})$, we can find $\gamma_1, \gamma_2 \in \Gamma_2$ and a diagonal matrix $D \in GSp_4^+(\mathbb{Q})$ such that $g = \gamma_1 D \gamma_2$. Since $\gamma \hat{\mathbb{Z}}^4 = \hat{\mathbb{Z}}^4$ for any $\gamma \in \Gamma_2$ and the Haar measure is translation invariant we conclude that $\mu_1(gB) = \lambda^{-2} \mu(B)$ for any Borel subset of $\text{MSp}_4(A_{f,Q})$.

One would expect to use a similar construction for $\beta = 1$ and $\beta = 3$. However, since the only subspace of $A_{f,Q}^2$ stable under the action of $GSp_4(\mathbb{Q})^+$ is $A_{f,Q}^4$ itself, this argument fails in the case $\beta = 1$ or $\beta = 3$. We conjecture that the $GSp_4$-system does not admit any KMS state in these two cases.

**Remark 2.** The results we prove in this paper completely classify the KMS$_\beta$ states on the Bost-Connes-Marcolli $GSp_4$-system. In fact, we will show that given $\beta > 0$ with $\beta \notin \{1, 2, 3\}$, there exists a one-to-one correspondence between KMS$_\beta$ on the Connes-Marcolli $GSp_4$-system and the $GSp_4$ system $(\mathcal{A}, \sigma_t)$. Recall the set

$$F_Y = \{ h \in \text{MSp}_4(\hat{\mathbb{Z}}) \mid \text{rank}_{\mathcal{O}}(h_p) \leq 2 \text{ for all } p \in \mathcal{P} \},$$

and consider the dynamical system $I = C^*_r(\Gamma_2 \backslash GSp_4^+(\mathbb{Q}) \times \mathbb{R}^*_+ \times F_Y)$, we claim that $I$ cannot have any KMS$_\beta$ states. To see this, observe that any KMS$_\beta$ state on $I$ gives rise to a regular $\Gamma_2$-invariant measure $\mu_\beta$ on the space $\mathbb{H}_2^+ \times F_Y$ (note that unlike the case where the underlying space is an $r$-discreet principal groupoid, the support of this measure is not necessarily contained in $\mathbb{H}_2^+ \times F_Y$). By the KMS$_\beta$ condition, this measure still satisfies the scaling property 2.12. Now since $\mathbb{H}_2^+ = (U^2/\{\pm 1_4\}) \backslash PGSp_4^+(\mathbb{R})$, we can define a measure on $PGSp_4^+(\mathbb{R}) \times F_Y$ by the formula

$$\int_{PGSp_4^+(\mathbb{R}) \times F_Y} f(x) d\hat{\mu}_\beta(x) = \int_{\mathbb{H}_2^+ \times F_Y} \left( \int_{U^2/\{\pm 1_4\}} f(xg) \, dg \right) d\mu_\beta(x).$$
The measure $\mu_\beta$ satisfies the condition 2.12. There is a canonical extension of this measure to a $\Gamma_2$-invariant measure $\mu_\beta \in E_\beta$ on the space $P GSp_4^+(\mathbb{R}) \times MSp_4(\mathbb{A}_f)$. This leads to a contradiction since the set $P GSp_4^+(\mathbb{R}) \times F_Y$ has measure zero by Corollary 3.6. This shows that the set $\mathbb{H}_2^+ \times F_Y$ can be ignored in the analysis of KMS $\beta$ states for $\beta > 0$ and $\beta \notin \{1, 2, 3\}$ and if we let $Y = Y \setminus F_Y$, it is clear that different KMS $\beta$ state on $C^*_r(\Gamma_2 \setminus GSp_4(\mathbb{Q}) \boxtimes_{\Gamma_2} \tilde{Y})$ give rise to different KMS $\beta$ state on the $GSp_4$-system $(\mathcal{A}, \sigma_t)$.

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