Admissible topologies on $C(Y, Z)$ and $O_Z(Y)$

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Abstract

Let $Y$ and $Z$ be two given topological spaces, $O(Y)$ (respectively, $O(Z)$) the set of all open subsets of $Y$ (respectively, $Z$), and $C(Y, Z)$ the set of all continuous maps from $Y$ to $Z$. We study Scott type topologies on $O(Y)$ and we construct admissible topologies on $C(Y, Z)$ and $O_Z(Y) = \{f^{-1}(U) \in O(Y) : f \in C(Y, Z) \text{ and } U \in O(Z)\}$, introducing new problems in the field.

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1 Introduction and Preliminaries

We denote by $Y$ and $Z$ two fixed topological spaces and by $C(Y, Z)$ the set of all continuous maps from $Y$ to $Z$. If $t$ is a topology on $C(Y, Z)$, then the corresponding topological space is denoted by $C_t(Y, Z)$.

Let $X$ be a topological space and $Q$ a subset of $X$. By $\text{Cl}(Q)$ we denote the closure of $Q$ in $X$.

By $O(X)$ we denote the family of all open subsets of $X$ under a given topology and by $O_Z(Y)$ the family $\{f^{-1}(U) : f \in C(Y, Z) \text{ and } U \in O(Z)\}$.

Let $X$ be a space and let $F : X \times Y \to Z$ be a continuous map. By $\widehat{F}$ we denote the map from $X$ to the set $C(Y, Z)$, such that $\widehat{F}(x)(y) = F(x, y)$, for every $x \in X$ and $y \in Y$. Let $G$ be a map from $X$ to $C(Y, Z)$. By $\widehat{G}$ we denote the map from $X \times Y$ to $Z$, such that $\widehat{G}(x, y) = G(x)(y)$, for every $(x, y) \in X \times Y$. 
Definition 1 (see [1] and [2]) A topology $t$ on $C(Y, Z)$ is called admissible, if for every space $X$, the continuity of the map $G: X \to C_t(Y, Z)$ implies the continuity of the map $\tilde{G}: X \times Y \to Z$ or equivalently the evaluation map $e: C_t(Y, Z) \times Y \to Z$, defined by $e(f, y) = f(y)$ for every $(f, y) \in C(Y, Z) \times Y$, is continuous.

Definition 2 (see, for example, [8]) The Scott topology $\Omega(Y)$ on $O(Y)$ is defined as follows: a subset $\mathcal{I} \in O(Y)$ belongs to $\Omega(Y)$ if:

$(\alpha)$ $U \in \mathcal{I}$, $V \in O(Y)$, and $U \subseteq V$ imply $V \in \mathcal{I}$ and

$(\beta)$ for every collection of open sets of $Y$, whose union belongs to $\mathcal{I}$, there are finitely many elements of this collection whose union also belongs to $\mathcal{I}$.

Definition 3 (see [11]) The strong Scott topology $\Omega_1(Y)$ on $O(Y)$ is defined as follows: a subset $\mathcal{I} \in O(Y)$ belongs to $\Omega_1(Y)$ if:

$(\alpha)$ $U \in \mathcal{I}$, $V \in O(Y)$, and $U \subseteq V$ imply $V \in \mathcal{I}$ and

$(\beta)$ for every open cover of $Y$, there are finitely many elements of this cover whose union belongs to $\mathcal{I}$.

Definition 4 (see, for example, [11] and [13]) The Isbell topology on $C(Y, Z)$, denoted here by $t_{Is}$, is the topology which has as a subbasis the family of all sets of the form:

$$(\mathcal{I}, U) = \{ f \in C(Y, Z) : f^{-1}(U) \in \mathcal{I} \},$$

where $\mathcal{I} \in \Omega(Y)$ and $U \in O(Z)$.

Definition 5 (see, for example, [11] and [13]) The strong Isbell topology on $C(Y, Z)$, denoted here by $t_{sIs}$, is the topology which has as a subbasis the family of all sets of the form:

$$(\mathcal{I}, U) = \{ f \in C(Y, Z) : f^{-1}(U) \in \mathcal{I} \},$$

where $\mathcal{I} \in \Omega_1(Y)$ and $U \in O(Z)$.

Definition 6 (see [4]) The compact open topology on $C(Y, Z)$, denoted here by $t_{co}$, is the topology which has as a subbasis the family of all sets of the form:

$$(K, U) = \{ f \in C(Y, Z) : f(K) \subseteq U \},$$

where $K$ is a compact subset of $Y$ and $U \in O(Z)$.

It is known that $t_{co} \subseteq t_{Is}$ (see, for example, [13]).

Definition 7 (see, for example, [11]) A subset $K$ of a space $X$ is said to be bounded, if every open cover of $X$ has a finite subcover for $K$.  

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Definition 8 (see, for example, [8]) A space $X$ is called corecompact, if for every $x \in X$ and for every open neighborhood $U$ of $x$ there exists an open neighborhood $V$ of $x$ such that $V \subseteq U$ and the subset $V$ is bounded in the space $U$.

Definition 9 (see, for example, [11]) A space $X$ is called locally bounded, if for every $x \in X$ and for every open neighborhood $U$ of $x$ there exists a bounded open neighborhood $V$ of $x$ such that $V \subseteq U$.

Below, we give some well known results on admissible topologies:

1. The compact open topology on $C(Y, Z)$ is admissible, if $Y$ is a regular locally compact space (see [1]).

2. The Isbell topology on $C(Y, Z)$ is admissible, if $Y$ is a corecompact space (see, for example, [11] and [15]).

3. The strong Isbell topology on $C(Y, Z)$ is admissible, if $Y$ is a locally bounded space (see, for example, [11]).

4. A topology which is larger than an admissible topology is also admissible (see [1]).

For a summary of all the above results and some open problems on function spaces see [5].

In this paper, we give Scott type topologies on the set $O(Y)$ and define using these topologies, in a standard way, new admissible topologies on the sets $C(Y, Z)$ and $O_Z(Y)$. We finally introduce questions on the field.

## 2 Scott type topologies on the set $O(Y)$

Throughout the text, by $\tau_X$ we will denote the corresponding topology on $X$, where $X$ is a topological space.

Definition 10 Let $Y$ and $Z$ be two topological spaces. The topology on $Y$, denoted here by $\tau^Z_Y$, which has as a subbasis the family:

$$O_Z(Y) = \{ f^{-1}(U) \in O(Y) : f \in C(Y, Z) \text{ and } U \in O(Z) \}$$

is called the $Z$-topology corresponding to the topology $\tau_Y$ of $Y$.

Clearly, $\tau^Z_Y \subseteq \tau_Y$.

Example 2.1 (1) Let $\mathbb{R}$ be the set of real numbers equipped with its usual topology $\tau_\mathbb{R}$ and let $Z$ be any set together with its trivial topology that is, the indiscrete topology. Then, $O_Z(\mathbb{R}) = \{ \emptyset, \mathbb{R} \}$ and, therefore, $\tau^Z_\mathbb{R} \neq \tau_\mathbb{R}$. 

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(2) It is well known that (see, for example, [16] (Example 92), [9], and [17]) for a fixed space $Z$ there exists a space $Y$ such that every continuous map from $Y$ to $Z$ is constant. This means that $\mathcal{O}_Z(Y) = \{\emptyset, Y\}$ and, therefore, $\tau_Y^Z \neq \tau_Y$.

(3) Let $Z = S$ be the Sierpiński space, that is $Z = \{0, 1\}$ and $\tau_Z = \{\emptyset, \{1\}, \{0, 1\}\}$. If $Y$ is another topological space, then $C(Y, S) = \{\mathcal{X}_V : V \in \mathcal{O}(Y)\}$, where $\mathcal{X}_V : Y \to S$ denotes the characteristic function of $V$.

$$\mathcal{X}_V(y) = \begin{cases} 1 & \text{if } y \in V, \\ 0 & \text{if } y \notin V. \end{cases}$$

In this case we observe that:

$$\mathcal{O}_Z(Y) = \{\mathcal{X}_V^{-1}(\{1\}) : \mathcal{X}_V \in C(Y, S)\}$$
$$= \{V : V \in \mathcal{O}(Y)\}$$
$$= \mathcal{O}(Y).$$

Thus, $\tau_Y^S = \tau_Y$.

**Definition 11** Let $Y$ and $Z$ be two topological spaces. A subset $K$ of $Y$ is called $Z$-compact, if $K$ is compact in the space $(Y, \tau_Y^Z)$, where $\tau_Y^Z$ is the $Z$-topology corresponding to the topology $\tau_Y$ of $Y$.

**Example 2.2** Every compact subset of a space $Y$ is $Z$-compact, but the converse statement is not true. Indeed, let $\mathbb{R}$ be the set of real numbers with the usual topology and $Z$ a set equipped with the trivial topology. Then, every subset of $\mathbb{R}$ is $Z$-compact, while subsets of the set of real numbers are not necessarily compact, in general.

**Definition 12** Let $Y$ and $Z$ be two topological spaces. The $Z$-compact open topology on $C(Y, Z)$, denoted here by $t_{co}^Z$, is the topology, which has as a subbasis the family of all sets of the form:

$$(K, U) = \{f \in C(Y, Z) : f(K) \subseteq U\},$$

where $K$ is a $Z$-compact subset of $Y$ and $U \in \mathcal{O}(Z)$.

**Remark 2.1** For the topologies $t_{co}$ and $t_{co}^Z$ we have that:

$$t_{co} \subseteq t_{co}^Z.$$
Example 2.3 Let $Y$ be an arbitrary topological space and $S$ the Sierpiński space. By Example 2.1(3), we have that a subset $K$ of $Y$ is $S$-compact if and only if $K$ is compact in $(Y, \tau_Y)$. In this case, we have $t_{Sco}^S = t_{co}$.

Furthermore, we observe that, whenever we consider the topology on $O(Y)$ which has as subbasis the family:

$$\{< K >: K \text{ is compact in } Y\},$$

where $< K > = \{U \in O(Y) : K \subset U\}$, then the topological spaces $C_{\text{co}}(Y, S)$ and $O(Y)$ are homeomorphic. Indeed, it would be enough to consider the homeomorphism $T : C(Y, S) \to O(Y)$ for which $T(\mathcal{X}_V) = V$, for every $\mathcal{X}_V \in C(Y, S)$.

So, in the case where $Z = S$, it would be that $t_{Sco}^S = t_{co}$ on $C(Y, S)$ and this topology coincides with the topology on $O(Y)$, which has as subbasis the set $\{< K >: K \text{ is compact in } Y\}$.

Definition 13 By $\tau_1^Z$ we denote the family of all subsets of $O(Y)$ that are defined as follows: a subset $I_H$ of $O(Y)$ belongs to $\tau_1^Z$ if:

(α) $f^{-1}(U) \in I_H \cap O(Z(Y), V \in O(Y)$ and $f^{-1}(U) \subseteq V$ imply $V \in I_H$ and

(β) for every collection $\{f^{-1}_\lambda(U_\lambda) : \lambda \in \Lambda\}$ of elements of $O(Z(Y)$, whose union belongs to $I_H$, there are finitely many elements $f^{-1}_\lambda(U_\lambda)$, $i = 1, 2, \ldots, n$ of this collection, such that:

$$\bigcup\{f^{-1}_\lambda(U_\lambda) : i = 1, 2, \ldots, n\} \in I_H.$$

Proposition 2.1 The family $\tau_1^Z$ defines a topology on $O(Y)$, called the Z-Scott topology.

Proof. The proof follows trivially by Definition 13. □

Definition 14 By $\tau_1^{Z,s}$ we denote the family of all subsets of $O(Y)$ that are defined as follows: a subset $I_H$ of $O(Y)$ belongs to $\tau_1^{Z,s}$ if:

(α) $f^{-1}(U) \in I_H \cap O(Z(Y), V \in O(Y)$ and $f^{-1}(U) \subseteq V$ imply $V \in I_H$ and

(β) for every collection $\{f^{-1}_\lambda(U_\lambda) : \lambda \in \Lambda\}$ of elements of $O(Z(Y)$ whose union is equal to $Y$, there are finitely many elements $f^{-1}_\lambda(U_\lambda)$, $i = 1, 2, \ldots, n$ of this collection, such that:

$$\bigcup\{f^{-1}_\lambda(U_\lambda) : i = 1, 2, \ldots, n\} \in I_H.$$

Proposition 2.2 The family $\tau_1^{Z,s}$ defines a topology on $O(Y)$, called the strong Z-Scott topology.

Proof. The proof follows trivially by Definition 14. □
Example 2.4  (1) Let $Y = \{0, 1\}$ be equipped with the topology $\tau_Y = \{\emptyset, \{0\}, Y\}$ and let $Z$ be a set equipped with the trivial topology. Then,

$$\mathcal{O}_Z(Y) = \{\emptyset, Y\},$$
$$\Omega(Y) = \Omega_1(Y) = \{\emptyset, \{0\}, \{0, Y\}, \{0\}, \emptyset\}$$

and

$$\tau^Z_1 = \tau^Z_{1,s} = \{\emptyset, \{\{0\}\}, \{0\}, \{0, Y\}, \{\{0\}, Y\}\}.$$ 

(2) If $Y$ is an arbitrary topological space and $S$ is the Sierpiński space, then a subset $H$ of $\mathcal{O}_Z(Y)$, will belong to $\tau^Z_s$, if:

1. $X^{-1}_i V(\{1\}) \in H \cap \mathcal{O}_S(Y)$, $V \in \mathcal{O}(Y)$, and $X^{-1}_i V(\{1\}) \subset W$ implies that $W \in \mathcal{H}$. Equivalently, $V \in \mathcal{H} \cap \mathcal{O}(Y) = \mathcal{H}$, $V \in \mathcal{O}(Y)$ and $V \subset W$ implies that $W \in H$.

2. For every collection $\{X^{-1}_i V(\{1\}) : V \in \mathcal{O}(Y)\}$ of elements of $\mathcal{O}_S(Y) = \mathcal{O}(Y)$, whose union belongs to $H$, there exist finitely many elements $\{X^{-1}_i V(\{1\}) : i = 1, 2, \cdots, n\}$, such that:

$$\bigcup \{X^{-1}_i V(\{1\}) : i = 1, 2, \cdots, n\} \in \mathcal{H}.$$ 

Equivalently, for every collection $\{V_\lambda \in \mathcal{O}(Y) : \lambda \in \Lambda\}$, such that $\bigcup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{H}$, there are finitely many elements $\{V_\lambda : i = 1, 2, \cdots, n\}$, such that $\bigcup\{V_\lambda : i = 1, 2, \cdots, n\} \in \mathcal{H}$. So, $\tau^Z_1$ is the Scott topology on $\mathcal{O}(Y)$.

Remark 2.2 For the topologies $\Omega(Y), \Omega_1(Y), \tau^Z_1,$ and $\tau^Z_{1,s}$ we have the following comparison:

$$\Omega_1(Y) \subseteq \tau^Z_{1,s} \subseteq \tau^Z_1 \subseteq \Omega(Y)$$

Definition 15 The $t^Z_1$ topology on $C(Y, Z)$ is the topology which has as a subbasis the family of all sets of the form:

$$\{(\mathcal{H}, U) = \{f \in C(Y, Z) : f^{-1}(U) \in \mathcal{H}\},$$

where $\mathcal{H}$ is open in the topology $\tau^Z_1$ on $\mathcal{O}(Y)$ and $U \in \mathcal{O}(Z)$.

Definition 16 The $t^Z_{1,s}$ topology on $C(Y, Z)$ is the topology which has as a subbasis the family of all sets of the form:

$$\{(\mathcal{H}, U) = \{f \in C(Y, Z) : f^{-1}(U) \in \mathcal{H}\},$$

where $\mathcal{H}$ is open in the topology $\tau^Z_{1,s}$ on $\mathcal{O}(Y)$ and $U \in \mathcal{O}(Z)$. 
Proposition 2.3 Let $Y$ and $Z$ be two topological spaces and let $K$ be a $Z$-compact subset of $Y$. Then, the set:

$$IH_K = \{ U \in \mathcal{O}(Y) : K \subseteq U \}$$

is open in $\mathcal{O}(Y)$ with the topology $\tau^Z_1$.

Proof. Let $f^{-1}(U) \in IH_K \cap \mathcal{O}_Z(Y)$, $V \in \mathcal{O}(Y)$ and let $f^{-1}(U) \subseteq V$. Then, $K \subseteq f^{-1}(U) \subseteq V$ and, therefore, $V \in IH_K$.

Now, let $\{ f^{-1}_\lambda(U_\lambda) : \lambda \in \Lambda \}$ be a collection of sets of $\mathcal{O}_Z(Y)$, whose union belongs to $IH_K$. Then:

$$K \subseteq \bigcup \{ f^{-1}_\lambda(U_\lambda) : \lambda \in \Lambda \}.$$

Since $K$ is $Z$-compact, there are finitely many elements $f^{-1}_\lambda(U_\lambda)$, $i = 1, 2, \ldots , n$ of this collection such that:

$$K \subseteq \bigcup \{ f^{-1}_\lambda(U_\lambda) : i = 1, 2, \ldots , n \}$$

and, therefore,

$$\bigcup \{ f^{-1}_\lambda(U_\lambda) : i = 1, 2, \ldots , n \} \in IH_K.$$

Thus, the set $IH_K$ is open in $\mathcal{O}(Y)$ with the topology $\tau^Z_1$. □

Remark 2.3 (1) We observe that for every $Z$-compact subset $K$ of $Y$ we have:

$$(IH_K, U) = \{ f \in C(Y, Z) : f^{-1}(U) \in IH_K \} = \{ f \in C(Y, Z) : K \subseteq f^{-1}(U) \} = \{ f \in C(Y, Z) : f(K) \subseteq U \}.$$

This says that $t^Z_{co} \subseteq t^Z_1$. Thus, by Remarks 2.1 and 2.2 we get the following comparison between the topologies $t_{co}, t^Z_{co}, t_{Is}, t^Z_{Is}, t^Z_1$, and $t^Z_{1,s}$:

$$t^Z_{co} \subseteq t^Z_1 \subseteq t^Z_{1,s}$$

(2) Let

$$\mathcal{T} = \{ \tau^Z_Y : Z \text{ is an arbitrary topological space} \}.$$

We immediately see that $(\mathcal{T}, \subseteq)$ has an upper bound, namely $\tau^Z_Y$, which is also the maximal element for $\mathcal{T}$ because, if $Z = S$, then $\tau^Z_Y = \tau^S_Y$. 

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In a similar way, we can prove that the set:

\[ T_{co} = \{ t_{co}^Z : Z \text{ is an arbitrary topological space} \} \]

has a lower bound, namely \( t_{co} \), which is a minimal element and, if \( Z = S \), then \( t_{co} = t_{co}^S \).

Also, in a similar manner the set:

\[ T_1 = \{ t_1^Z : Z \text{ is an arbitrary topological space} \} \]

has a lower bound, namely \( t_{Is} \), which is a minimal element and, if \( Z = S \), then \( t_{Is} = t_{Is}^S \).

Finally, the set:

\[ T_{1,s} = \{ t_{1,s}^Z : Z \text{ is an arbitrary topological space} \} \]

has a lower bound, namely \( t_{sIs} \), which is a minimal element and, if \( Z = S \), then \( t_{sIs} = t_{sIs}^S \).

**Theorem 2.4** Let \( Z \) be a \( T_i \)-space, where \( i = 0, 1, 2 \). Then, the topological spaces \( C_{t_{co}}^Z(Y, Z) \), \( C_{t_1}^Z(Y, Z) \), and \( C_{t_{1,s}}^Z(Y, Z) \) are also \( T_i \)-spaces.

**Proof.** Since \( Z \) is a \( T_i \)-space, where \( i = 0, 1, 2 \), the space \( C_{t_{co}}^Z(Y, Z) \) will also be a \( T_i \)-space (see, for example, [2]). Thus, by Remark 2.3, the spaces \( C_{t_{co}}^Z(Y, Z) \), \( C_{t_1}^Z(Y, Z) \) and \( C_{t_{1,s}}^Z(Y, Z) \) are \( T_i \)-spaces. □

**Theorem 2.5** The following statements are true:

1. If \( Y \) is a regular locally compact space, then the topologies \( t_{co}^Z \), \( t_1^Z \) and \( t_{1,s}^Z \) are admissible.
2. If \( Y \) is a corecompact space, then the topologies \( t_1^Z \) and \( t_{1,s}^Z \) are admissible.
3. If \( Y \) is a locally bounded space, then the topology \( t_{1,s}^Z \) is admissible.

**Proof.** The proof of this theorem follows from Remark 2.3 and from the fact that a topology larger than an admissible topology is also admissible. □

**Definition 17** Let \( Y \) and \( Z \) be two topological spaces. The space \( Y \) is called **locally \( Z \)-compact**, if the space \((Y, \tau_Y^Z)\), where \( \tau_Y^Z \) is the \( Z \)-topology corresponding to the topology \( \tau_Y \) of \( Y \), is locally compact.

**Remark 2.4** We observe that, if a space \( Y \) is regular locally \( Z \)-compact, then for every \( y \in Y \) and for every neighborhood \( f^{-1}(W) \in \tau_Y^Z \) of \( y \), where \( f \in C(Y, Z) \) and \( W \in \mathcal{O}(Z) \), there exists \( g^{-1}(V) \in \tau_Y^Z \) such that the closure of the set \( g^{-1}(V) \) is compact in the space \((Y, \tau_Y^Z)\) and \( y \in g^{-1}(V) \subseteq \text{Cl}(g^{-1}(V)) \subseteq f^{-1}(W) \).
**Theorem 2.6** Let $Z$ be a space and $Y$ a regular locally $Z$-compact space. Then, the $t^Z_{co}$ topology on $C(Y, Z)$ is admissible.

**Proof.** It is sufficient to prove that the evaluation map:

$$ e : C_{t^Z_{co}}(Y, Z) \times Y \to Z $$

is continuous.

Let $(f, y) \in C(Y, Z) \times Y$ and let also $W \in \mathcal{O}(Z)$, such that:

$$ e(f, y) = f(y) \in W. $$

Then, we have that:

$$ y \in f^{-1}(W). $$

Since the space $Y$ is regular locally $Z$-compact, there exists $g^{-1}(V) \in \tau^Z_Y$, such that the set $\text{Cl}(g^{-1}(V))$ is compact in the space $(Y, \tau^Z_Y)$ and

$$ y \in g^{-1}(V) \subseteq \text{Cl}(g^{-1}(V)) \subseteq f^{-1}(W). $$

Since $\text{Cl}(g^{-1}(V)) \subseteq f^{-1}(W)$, we have that $f \in (\text{Cl}(g^{-1}(V)), W)$. Thus,

$$(\text{Cl}(g^{-1}(V)), W) \times g^{-1}(V)$$

is an open neighborhood of $(f, y)$ in $C_{t^Z_{co}}(Y, Z) \times Y$.

We finally prove that:

$$ e((\text{Cl}(g^{-1}(V)), W) \times g^{-1}(V)) \subseteq W. $$

Let $(h, z) \in (\text{Cl}(g^{-1}(V)), W) \times g^{-1}(V)$. Then:

$$ h \in (\text{Cl}(g^{-1}(V)), W) \text{ and } z \in g^{-1}(V). $$

Therefore, $h(g^{-1}(V)) \subseteq h(\text{Cl}(g^{-1}(V))) \subseteq W$ and $e(h, z) = h(z) \in W$.

Thus, the evaluation map $e$ is continuous and, therefore, the $t^Z_{co}$ topology on $C(Y, Z)$ is admissible too. □

**Theorem 2.7** Let $X$, $Y$, and $Z$ be three topological spaces. If the space $Y$ is locally $Z$-compact, then the map:

$$ T : C_{t^Z_{co}}(X, Y) \times C_{t^Z_{co}}(Y, Z) \to C_{t^Z_{co}}(X, Z), $$

with $T(f, g) = g \circ f$ for every $(f, g) \in C(X, Y) \times C(Y, Z)$, is continuous.
Proof. Let \((f, g) \in C(X, Y) \times C(Y, Z)\) and let:

\[
T(f, g) = g \circ f \in (K, U) \in t^Z_{co},
\]

where \(K\) is a compact subset of the space \((X, \tau^Z_X)\) and \(U\) an open subset of \(Z\). Then, we have that:

\[
(g \circ f)(K) \subseteq U
\]

or, equivalently:

\[
K \subseteq f^{-1}(g^{-1}(U)).
\]

Now, since the space \(Y\) is locally \(Z\)-compact, the space \((Y, \tau^Z_Y)\) is locally compact. Thus, for an arbitrary \(y \in g^{-1}(U)\) there exists \(W_y \in \mathcal{O}(Z)\) such that \(\text{Cl}(W_y)\) is compact in \((X, \tau^Z_X)\) and

\[
y \in W_y \subseteq \text{Cl}(W_y) \subseteq g^{-1}(U). \tag{1}
\]

So, we have that:

\[
g^{-1}(U) = \bigcup \{W_y : y \in g^{-1}(U)\}
\]

and, therefore,

\[
K \subseteq f^{-1}(g^{-1}(U)) = \bigcup \{f^{-1}(W_y) : y \in g^{-1}(U)\}.
\]

Since \(K\) is a compact subset of the space \((X, \tau^Z_X)\), there are finitely many elements \(y_1, \ldots, y_n \in g^{-1}(U)\), such that:

\[
K \subseteq \bigcup \{f^{-1}(W_{y_i}) : i \in \{1, \ldots, n\}\}.
\]

So, we have

\[
f(K) \subseteq \bigcup \{W_{y_i} : i \in \{1, \ldots, n\}\}
\]

or, equivalently:

\[
f \in (K, \bigcup \{W_{y_i} : i \in \{1, \ldots, n\}\}).
\]

Also, by relation (1) we have

\[
g\left(\bigcup \{\text{Cl}(W_{y_i}) : i \in \{1, \ldots, n\}\}\right) \subseteq U
\]

and, therefore,

\[
g \in \left(\bigcup \{\text{Cl}(W_{y_i}) : i \in \{1, \ldots, n\}\}, U\right).
\]

We observe that:

\[
(K, \bigcup \{W_{y_i} : i \in \{1, \ldots, n\}\}) \in t^Y_{co}
\]
and that
\[
(\bigcup\{\text{Cl}(W_{y_i}) : i \in \{1, \cdots, n\}\}, U) \in \mathcal{T}_{\mathcal{C}}^{Z}.
\]
By all the above it suffices to prove that:
\[
T\left((K, \bigcup\{W_{y_i} : i \in \{1, \cdots, n\}\}) \times (\bigcup\{\text{Cl}(W_{y_i}) : i \in \{1, \cdots, n\}\}, U)\right) \subseteq (K, U).
\]
Let
\[
(h_1, h_2) \in (K, \bigcup\{W_{y_i} : i \in \{1, \cdots, n\}\}) \times (\bigcup\{\text{Cl}(W_{y_i}) : i \in \{1, \cdots, n\}\}, U).
\]
Then,
\[
h_1(K) \subseteq \bigcup\{W_{y_i} : i \in \{1, \cdots, n\}\}
\]
and
\[
h_2(\bigcup\{\text{Cl}(W_{y_i}) : i \in \{1, \cdots, n\}\}) \subseteq U.
\]
Therefore,
\[
(h_2 \circ h_1)(K) = h_2(h_1(K)) \subseteq h_2(\bigcup\{W_{y_i} : i \in \{1, \cdots, n\}\}) \subseteq h_2(\bigcup\{\text{Cl}(W_{y_i}) : i \in \{1, \cdots, n\}\}) \subseteq U,
\]
so that \(T(h_1, h_2) = h_2 \circ h_1 \in (K, U)\). Thus, the map \(T\) is continuous. \(\square\)

**Definition 18** Let \(Y\) and \(Z\) be two topological spaces. A subset \(B\) of \(Y\) is called \textit{Z-bounded}, if \(B\) is bounded in the space \((Y, \tau^{Z}_Y)\), where \(\tau^{Z}_Y\) is the \(Z\)-topology corresponding to the topology \(\tau_Y\) of \(Y\).

**Definition 19** Let \(Z\) be a space. A space \(Y\) is called \textit{locally Z-bounded}, if for every \(y \in Y\) and for every open neighborhood \(U\) of \(y\), there exists a \(Z\)-bounded neighborhood \(g^{-1}(V) \in \mathcal{O}_Z(Y)\) of \(y\), such that \(g^{-1}(V) \subseteq U\).

**Theorem 2.8** Let \(Z\) be a space and let \(Y\) be a locally \(Z\)-bounded space. Then, the \(i_{1, s}^{Z}\) topology on \(C(Y, Z)\) is admissible.

**Proof.** It is sufficient to prove that the evaluation map:
\[
e : C_{i_{1, s}^{Z}}(Y, Z) \times Y \to Z
\]
is continuous.

For this, we let \((f, y) \in C(Y, Z) \times Y\) and let \(W \in \mathcal{O}(Z)\), such that:
\[
e(f, y) = f(y) \in W.
\]
Then:
\[ y \in f^{-1}(W). \]

Also, since \( Y \) is locally \( Z \)-bounded, there exists a \( Z \)-bounded neighborhood \( g^{-1}(V) \in \mathcal{O}_Z(Y) \) of \( y \), such that:
\[ y \in g^{-1}(V) \subseteq f^{-1}(W). \]

Consider the set:
\[ \mathcal{H}_{g^{-1}(V)} = \{ U \in \mathcal{O}(Y) : g^{-1}(V) \subseteq U \}. \]

We prove that \( \mathcal{H}_{g^{-1}(V)} \) belongs to the \( \tau_{1,s}^Z \) topology.

Indeed, let \( h^{-1}(U) \in \mathcal{H}_{g^{-1}(V)} \cap \mathcal{O}_Z(Y) \), \( U_1 \in \mathcal{O}(Y) \) and \( h^{-1}(U) \subseteq U_1 \). Then:
\[ g^{-1}(V) \subseteq h^{-1}(U) \subseteq U_1 \]

and, therefore, \( U_1 \in \mathcal{H}_{g^{-1}(V)} \). Now, let \( \{ f^{-1}_{i_y}(U_i) : i \in I \} \) be a collection of elements of \( \mathcal{O}_Z(Y) \), whose union is equal to the set \( Y \). Then, for every \( y \in Y \), there exists \( i_y \in I \), such that \( y \in f^{-1}_{i_y}(U_{i_y}) \). Since \( Y \) is locally \( Z \)-bounded, there exists a \( Z \)-bounded neighborhood \( g^{-1}_{i_y}(V_{i_y}) \in \mathcal{O}_Z(Y) \) of \( y \), such that:
\[ y \in g^{-1}_{i_y}(V_{i_y}) \subseteq f^{-1}_{i_y}(U_{i_y}). \]

We can easily deduce that the set \( \{ g^{-1}_{i_y}(V_{i_y}) : i_y \in I \} \) is an open cover of \( (Y, \tau_Y^Z) \).

Since \( g^{-1}(V) \) is \( Z \)-bounded, there exist finitely many sets
\[ g^{-1}_{i_{y_1}}(V_{i_{y_1}}), \cdots, g^{-1}_{i_{y_n}}(V_{i_{y_n}}) \]
such that:
\[ g^{-1}(V) \subseteq \bigcup\{ g^{-1}_{i_{y_k}}(V_{i_{y_k}}) : k = 1, 2, \cdots, n \}. \quad (2) \]

We now consider the sets \( f^{-1}_{i_{y_1}}(U_{i_{y_1}}), \cdots, f^{-1}_{i_{y_n}}(U_{i_{y_n}}) \) of \( \mathcal{O}_Z(Y) \), for which:
\[ g^{-1}_{i_{y_k}}(V_{i_{y_k}}) \subseteq f^{-1}_{i_{y_k}}(U_{i_{y_k}}), \quad k = 1, 2, \cdots, n. \]

But relation (2) gives:
\[ g^{-1}(V) \subseteq \bigcup\{ g^{-1}_{i_{y_k}}(V_{i_{y_k}}) \} \subseteq \bigcup\{ f^{-1}_{i_{y_k}}(U_{i_{y_k}}) : k = 1, 2, \cdots, n \} \]

and, therefore:
\[ \bigcup\{ f^{-1}_{i_{y_k}}(U_{i_{y_k}}) : k = 1, 2, \cdots, n \} \in \mathcal{H}_{g^{-1}(V)}. \]

So, the set \( \mathcal{H}_{g^{-1}(V)} \) is open in the \( \tau_{1,s}^Z \) topology.
Also, since \( g^{-1}(V) \subseteq f^{-1}(W) \), we have that \( f \in (f^{-1}(V), W) \). Thus, the set \((f^{-1}(V), W) \times g^{-1}(V)\) is an open neighborhood of \((f, y)\) in \( C_{t_1Z}(Y, Z) \times Y\).

We finally prove that:

\[
e((f^{-1}(V), W) \times g^{-1}(V)) \subseteq W.
\]

Let \((h, z) \in (f^{-1}(V), W) \times g^{-1}(V)\). Then:

\[
h \in (f^{-1}(V), W) \text{ and } z \in g^{-1}(V).
\]

Therefore, \( z \in g^{-1}(V) \subseteq h^{-1}(W) \) and \( e(h, z) = h(z) \in W\).

Thus, the evaluation map \( e \) is continuous and, therefore, the topology \( t_1Z \) is admissible. \( \square \)

**Definition 20** Let \( Z \) be a space. A space \( Y \) is called \( Z \)-corecompact, if for every \( y \in Y \) and for every open neighborhood \( U \) of \( y \) in \( Y \), there exists a neighborhood \( f^{-1}(V) \in \mathcal{O}_Z(Y) \) of \( y \) such that \( f^{-1}(V) \subseteq U \) and the subset \( f^{-1}(V) \) is \( Z \)-bounded in the space \( U \) (in symbols we write \( f^{-1}(V) \ll U \)).

**Remark 2.5** (1) In Definition 20 we considered the space \( U \) to be a subspace of the space \((Y, \tau_Y)\), that is \( U \) is the space which is equipped with the topology:

\[
(\tau_Y)_U = \{ U \cap V : V \in \tau_Y \}.
\]

(2) Let \( Y \) and \( Z \) be two topological spaces, \( U, W, U_i \in \mathcal{O}(Y) \), and \( f^{-1}(V), f_i^{-1}(V_i) \in \mathcal{O}_Z(Y) \), where \( i = 1, \ldots , n \). We observe that:

(i) If \( f^{-1}(V) \subseteq U \ll W \), then \( f^{-1}(V) \ll W \).

(ii) If \( f^{-1}(V) \ll U \subseteq W \), then \( f^{-1}(V) \ll W \).

(iii) If \( f_i^{-1}(V_i) \ll U_i \), for every \( i = 1, \ldots , n \), then

\[
\bigcup \{ f_i^{-1}(V_i) : i = 1, \ldots , n \} \ll \bigcup \{ U_i : i = 1, \ldots , n \}.
\]

**Theorem 2.9** Let \( Z \) be a space and let \( Y \) be a \( Z \)-corecompact space. Then, the \( t_1Z \) topology on \( C(Y, Z) \) is admissible.

**Proof.** It is sufficient to prove that the evaluation map:

\[
e : C_{t_1Z}(Y, Z) \times Y \to Z
\]

is continuous.
For this, let \((f, y) \in C(Y, Z) \times Y\) and let \(W \in \mathcal{O}(Z)\) be such that:

\[ e(f, y) = f(y) \in W. \]

Then, we have that:

\[ y \in f^{-1}(W). \]

Since the space \(Y\) is \(Z\)-corecompact, there exists a neighborhood \(g^{-1}(V) \in \mathcal{O}_Z(Y)\) of \(y\), such that:

\[ g^{-1}(V) \ll f^{-1}(W). \]

We now consider the set:

\[ \mathbb{H}^g_{g^{-1}(V)} = \{ U \in \mathcal{O}(Y) : g^{-1}(V) \ll U \} \]

and we prove that the set \(\mathbb{H}^g_{g^{-1}(V)}\) is open in the \(\tau^Z_Y\) topology.

Indeed, let \(f^{-1}(U) \in \mathbb{H}^g_{g^{-1}(V)} \cap \mathcal{O}_Z(Y)\), \(V_1 \in \mathcal{O}(Y)\) and \(f^{-1}(U) \subseteq V_1\). Then, we have:

\[ g^{-1}(V) \ll f^{-1}(U) \subseteq V_1 \]

and, therefore, by Remark 2.5, \(g^{-1}(V) \ll V_1\). Thus, \(V_1 \in \mathbb{H}^g_{g^{-1}(V)}\). Now, let \(\{f_i^{-1}(U_i) : i \in I\}\) be a collection of sets of \(\mathcal{O}_Z(Y)\), such that:

\[ \bigcup\{f_i^{-1}(U_i) : i \in I\} \in \mathbb{H}^g_{g^{-1}(V)} \]

or equivalently:

\[ g^{-1}(V) \ll \bigcup\{f_i^{-1}(U_i) : i \in I\}. \]

Clearly, for every \(y \in \bigcup\{f_i^{-1}(U_i) : i \in I\}\), there exists \(i_y \in I\) such that \(y \in f_{i_{y}}^{-1}(U_{i_{y}})\). Since \(Y\) is \(Z\)-corecompact, there exists a neighborhood \(g^{-1}_{i_{y}}(V_{i_{y}}) \in \mathcal{O}_Z(Y)\) of \(y\), such that:

\[ y \in g^{-1}_{i_{y}}(V_{i_{y}}) \ll f_{i_{y}}^{-1}(U_{i_{y}}). \]

By all the above we get:

\[ \bigcup\{g^{-1}_{i_{y}}(V_{i_{y}}) : i_{y} \in I\} = \bigcup\{f_{i_{y}}^{-1}(U_{i_{y}}) : i_{y} \in I\} = \bigcup\{f_i^{-1}(U_i) : i \in I\}. \]

Since

\[ g^{-1}(V) \ll \bigcup\{f_i^{-1}(U_i) : i \in I\} \]

and since \(\{g^{-1}_{i_{y}}(V_{i_{y}}) : i_{y} \in I\}\) is an open cover of \(\bigcup\{f_i^{-1}(U_i) : i \in I\}\) with respect to the topology \((\tau^Z_Y)_{\bigcup\{f_i^{-1}(U_i) : i \in I\}}\), there exist finitely many sets \(g^{-1}_{i_{y_1}}(V_{i_{y_1}}), \ldots, g^{-1}_{i_{y_n}}(V_{i_{y_n}})\) of this collection such that:

\[ g^{-1}(V) \subseteq \bigcup\{g^{-1}_{i_{y_k}}(V_{i_{y_k}}) : k = 1, 2, \ldots, n\}. \]  

(3)
We now consider the sets $f_{i_{y_k}}^{-1}(U_{i_{y_k}}), \cdots, f_{i_{y_n}}^{-1}(U_{i_{y_n}})$ of $\mathcal{O}_Z(Y)$, for which:

$$g_{i_{y_k}}^{-1}(V_{i_{y_k}}) << f_{i_{y_k}}^{-1}(U_{i_{y_k}}), \ k = 1, 2, \ldots, n.$$ 

By Remark 2.5 and by (3) above, we get:

$$g^{-1}(V) \subseteq \bigcup \{g_{i_{y_k}}^{-1}(V_{i_{y_k}}) : k = 1, 2, \ldots, n\} << \bigcup \{f_{i_{y_k}}^{-1}(U_{i_{y_k}}) : k = 1, 2, \ldots, n\}$$

and, therefore:

$$g^{-1}(V) << \bigcup \{f_{i_{y_k}}^{-1}(U_{i_{y_k}}) : k = 1, 2, \ldots, n\}.$$ 

So, $\bigcup \{f_{i_{y_k}}^{-1}(U_{i_{y_k}}) : k = 1, 2, \ldots, n\} \in I_{g^{-1}(V)}$. Thus, the set $I_{g^{-1}(V)}$ is open in the $\tau^Z_{1}$ topology.

Also, since $g^{-1}(V) << f^{-1}(W)$, we have that $f \in (I_{g^{-1}(V)}, W)$. Thus, the set $(H_{g^{-1}(V)}, W) \times g^{-1}(V)$ is an open neighborhood of $(f, y)$ in $C_{\tau^Z_{1}}(Y, Z) \times Y$.

We finally prove that:

$$e((H_{g^{-1}(V)}, W) \times g^{-1}(V)) \subseteq W.$$ 

For this, let $(h, z) \in (H_{g^{-1}(V)}, W) \times g^{-1}(V)$. Then, $h \in (H_{g^{-1}(V)}, W)$ and $z \in g^{-1}(V)$. Therefore, $z \in g^{-1}(V) << h^{-1}(W)$ and $e(h, z) = h(z) \in W$.

Thus, the evaluation map $e$ is continuous and, consequently, the $t^Z_{1}$ topology is admissible. □

By Theorem 2.9 and Remark 2.3 we obtain the following corollary.

**Corollary 2.10** Let $Z$ be any space and let $Y$ be a $Z$-corecompact space. Then, the $t^Z_{1}$ topology on $C(Y, Z)$ is admissible.

### 3 Admissible topologies on $\mathcal{O}_Z(Y)$

Let $I \subseteq \mathcal{O}_Z(Y)$, $\mathcal{H} \subseteq C(Y, Z)$ and let $U \in \mathcal{O}(Z)$. We set:

$$(I, U) = \{f \in C(Y, Z) : f^{-1}(U) \in I\}$$

and

$$(\mathcal{H}, U) = \{f^{-1}(U) : f \in \mathcal{H}\}.$$
Definition 21 (see [7]) (1) Let $\tau$ be a topology on $\mathcal{O}_Z(Y)$. The topology on $C(Y,Z)$, for which the set
\[ \{(H,U) : H \in \tau, U \in \mathcal{O}(Z)\} \]
is a subbasis, is called the dual topology to $\tau$ and is denoted by $t(\tau)$.
(2) Let $t$ be a topology on $C(Y,Z)$. The topology on $\mathcal{O}_Z(Y)$, for which the set:
\[ \{(H,U) : H \in t, U \in \mathcal{O}(Z)\} \]
is a subbasis, is called the dual topology to $t$ and is denoted by $\tau(t)$.

Let $X$ be a space and let $G : X \rightarrow C(Y,Z)$ be a map. By $\overline{G}$ we denote the map from $X \times \mathcal{O}(Z)$ to $\mathcal{O}_Z(Y)$, for which $\overline{G}(x,U) = (G(x))^{-1}(U)$ for every $x \in X$ and $U \in \mathcal{O}(Z)$.

Let $\tau$ be a topology on $\mathcal{O}_Z(Y)$. We say that a map $M$ from $X \times \mathcal{O}(Z)$ to $\mathcal{O}_Z(Y)$ is continuous with respect to the first variable if for every fixed element $U$ of $\mathcal{O}(Z)$, the map $M_U : X \rightarrow (\mathcal{O}_Z(Y), \tau)$, for which $M_U(x) = M(x,U)$ for every $x \in X$, is continuous.

Definition 22 (see [7]) A topology $\tau$ on $\mathcal{O}_Z(Y)$ is called admissible, if for every space $X$ and for every map $G : X \rightarrow C(Y,Z)$ the continuity with respect to the first variable of the map $\overline{G} : X \times \mathcal{O}(Z) \rightarrow (\mathcal{O}_Z(Y), \tau)$ implies the continuity of the map $\overline{G} : X \times Y \rightarrow Z$

It is known that (see [7]):
(1) A topology $t$ on $C(Y,Z)$ is admissible if and only if the topology $\tau(t)$ on $\mathcal{O}_Z(Y)$ is admissible.
(2) A topology $\tau$ on $\mathcal{O}_Z(Y)$ is admissible if and only if the topology $t(\tau)$ on $C(Y,Z)$ is admissible.

Theorem 3.1 The following statements hold:
(1) If $Y$ is a regular locally compact space (or a regular locally $Z$-compact space), then the topologies $\tau(t^Z_{\text{co}})$, $\tau(t^Z_1)$, and $\tau(t^Z_{1,s})$ on $\mathcal{O}_Z(Y)$ are admissible.
(2) If $Y$ is a corecompact space (or a $Z$-corecompact space), then the topologies $\tau(t^Z_1)$ and $\tau(t^Z_{1,s})$ on $\mathcal{O}_Z(Y)$ are admissible.
(3) If $Y$ is a locally bounded space (or a locally $Z$-bounded space), then the topology $\tau(t^Z_{1,s})$ on $\mathcal{O}_Z(Y)$ is admissible.

Proof. The proof of this theorem follows immediately from Theorems 2.5, 2.6, 2.7 and 2.8. □
Corollary 3.2 The following statements are true:

(1) If \( Y \) is a regular locally compact space (or a regular locally \( Z \)-compact space), then the topologies \( t(\tau(t_{co})) \), \( t(\tau(t_1)) \), and \( t(\tau(t_{1,s})) \) on \( C(Y, Z) \) are admissible.

(2) If \( Y \) is a corecompact space (or a \( Z \)-corecompact space), then the topologies \( t(\tau(t_1)) \) and \( t(\tau(t_{1,s})) \) on \( C(Y, Z) \) are admissible.

(3) If \( Y \) is a locally bounded space (or a locally \( Z \)-bounded space), then the topology \( t(\tau(t_{1,s})) \) on \( C(Y, Z) \) is admissible.

Corollary 3.3 The following propositions are true:

(1) If \( Y \) is a regular locally compact space (or a regular locally \( Z \)-compact space), then the topologies \( \tau(t(\tau(t_{co}))), \tau(t(\tau(t_1))), \) and \( \tau(t(\tau(t_{1,s}))) \) on \( \mathcal{O}_Z(Y) \) are admissible.

(2) If \( Y \) is a corecompact space (or a \( Z \)-corecompact space), then the topologies \( \tau(t(\tau(t_1))) \) and \( \tau(t(\tau(t_{1,s}))) \) on \( \mathcal{O}_Z(Y) \) are admissible.

(3) If \( Y \) is a locally bounded space (or a locally \( Z \)-bounded space), then the topology \( \tau(t(\tau(t_{1,s}))) \) on \( \mathcal{O}_Z(Y) \) is admissible.

4 Some open questions

In this section we give some interesting in our opinion open questions applied to the topologies \( t_{co}, t_1, \) and \( t_{1,s}. \)

**Question 1.** Let \( Y \) and \( Z \) be two topological spaces. Is the topology \( t_1 \) on \( C(Y, Z) \) regular in the case where \( Z \) is regular?

**Question 2.** Let \( Y \) and \( Z \) be two topological spaces. Is the topology \( t_1 \) on \( C(Y, Z) \) completely regular in the case where \( Z \) is completely regular?

**Question 3.** Find two topological spaces \( Y \) and \( Z \) such that

3.1. \( t_{co} \neq t_{co}. \)
3.2. \( t_1 \neq t_{1,s}. \)
3.3. \( t_{1,s} \neq t_{s1s}. \)

**Question 4.** Do the topologies \( t_{co} \) and \( t_{co} \) coincide on the set \( C(\mathbb{R}^\omega, \mathbb{R}) \) where \( \mathbb{R} \) is the set of real numbers with the usual topology and \( \omega \) is the first infinite cardinal?

**Question 5.** Do the topologies \( t_{co} \) and \( t_{co} \) coincide on the set \( C(\mathbb{N}^\omega, \mathbb{N}) \), where \( \mathbb{N} \) is the set of natural numbers with its usual topology?

**Question 6.** Let \( X, Y, \) and \( Z \) be three topological spaces. Is the map:

\[
T : C_{t_1,s}(X, Y) \times C_{t_1,s}(Y, Z) \to C_{t_1,s}(X, Z),
\]

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with \( T(f, g) = g \circ f \), for every \((f, g) \in C(X, Y) \times C(Y, Z)\), continuous in the case where \( Y \) is locally \( Z \)-bounded?

**Question 7.** Let \( X, Y, \) and \( Z \) be three topological spaces. Is the map:

\[
T : C_{t_1}(X, Y) \times C_{t_1}(Y, Z) \to C_{t_1}(X, Z),
\]

with \( T(f, g) = g \circ f \), for every \((f, g) \in C(X, Y) \times C(Y, Z)\), continuous in the case where \( Y \) is \( Z \)-corecompact?

**Notation.** Let \( X \) be a space and \( F : X \times Y \to Z \) be a continuous map. By \( F_x \), where \( x \in X \), we denote the continuous map of \( Y \) into \( Z \) such that \( F_x(y) = F(x, y) \), \( y \in Y \). By \( \hat{F} \) we denote the map of \( X \) into the set \( C(Y, Z) \) such that \( \hat{F}(x) = F_x \), \( x \in X \).

We recall that a topology \( t \) on \( C(Y, Z) \) is called *splitting* if for every space \( X \), the continuity of a map \( F : X \times Y \to Z \) implies that of the map \( \hat{F} : X \to C_t(Y, Z) \) (see [1] and [2]).

It is known that:

1. The compact open topology \( t_{co} \) is always splitting (see [11] and [14]).
2. The Isbell topology is always splitting (see [10], [12], [13], and [15]).
3. If \( Z \) is the Sierpiński space and \( Y \) is an arbitrary space, then the Isbell topology coincides with the greatest splitting topology (see [14] and [15]).

By Remark 2.3(2) if \( Z \) is the Sierpiński space, then the topology \( t_{Z}^1 \) coincides with the Isbell topology on \( C(Y, Z) \) and, therefore, this topology is a splitting topology. In this case the topology \( t_{co}^1 \) is also splitting.

**Question 8.** Find necessary and sufficient conditions for the space \( Z \) such that the topology \( t_{co}^1 \) on \( C(Y, Z) \) to be splitting.

**Question 9.** Find necessary and sufficient conditions for the space \( Z \) such that the topology \( t_{co}^1 \) on \( C(Y, Z) \) to be splitting.

**Question 10.** Let \( Y \) be an arbitrary topological space and \( Z = \{0, 1\} \) with the discrete topology. Which of the following relations is true or false on \( C(Y, Z) \)?

(i) \( t_{co} \neq t_{co}^1 \),
(ii) \( t_{Is} \neq t_{Is}^1 \), and
(iii) \( t_{sIs} \neq t_{sIs}^1 \).

**Question 11.** Let \( Y \) be an arbitrary topological space and \( Z = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \) with the usual topology of the real line. Which of the following relations is true or false on \( C(Y, Z) \)?

(i) \( t_{co} \neq t_{co}^1 \),
(ii) \( t_{Is} \neq t_{Is}^1 \), and
(iii) \( t_{sIs} \neq t_{sIs}^1 \).

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Let $Y$ be a corecompact space which is not basic locally compact. (A space is basic locally compact if for every point there exists a basis of compact neighbourhoods). Then, the Isbell topology $t_{Is} \equiv t^S_1$, in $C(Y,S)$ does not coincide to the compact open topology $t_{co} \equiv t^{S}_{co}$ (see [14] and [15]).

**Question 12.** For what spaces $Y$ and $Z$ does the equality $t^Z_1 = t^Z_{co}$ hold?

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