Non-triviality Conditions for
Integer-valued Polynomial Rings on Algebras

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Abstract
Let $D$ be a commutative domain with field of fractions $K$ and let $A$ be a torsion-free $D$-algebra such that $A \cap K = D$. The ring of integer-valued polynomials on $A$ with coefficients in $K$ is $\text{Int}_K(A) = \{ f \in K[X] \mid f(A) \subseteq A \}$, which generalizes the classic ring $\text{Int}(D) = \{ f \in K[X] \mid f(D) \subseteq D \}$ of integer-valued polynomials on $D$.

The condition on $A \cap K$ implies that $D[X] \subseteq \text{Int}_K(A) \subseteq \text{Int}(D)$, and we say that $\text{Int}_K(A)$ is nontrivial if $\text{Int}_K(A) \neq D[X]$. For any integral domain $D$, we prove that if $A$ is finitely generated as a $D$-module, then $\text{Int}_K(A)$ is nontrivial if and only if $\text{Int}(D)$ is nontrivial. When $A$ is not necessarily finitely generated but $D$ is Dedekind, we provide necessary and sufficient conditions for $\text{Int}_K(A)$ to be nontrivial. These conditions also allow us to prove that, for $D$ Dedekind, the domain $\text{Int}_K(A)$ has Krull dimension 2.

Keywords: Integer-valued polynomial, Algebraic algebra of bounded degree, Maximal subalgebra, Krull dimension

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1 Introduction
Given a (commutative) integral domain $D$ with fraction field $K$, we define $\text{Int}(D) := \{ f \in K[X] \mid f(D) \subseteq D \}$, which is the ring of integer-valued polynomials on $D$. Integer-valued polynomials and the properties of $\text{Int}(D)$ have been well studied; the book [4] covers the major theory in this area and provides an extensive bibliography. In recent years, researchers have begun to study a generalization of $\text{Int}(D)$ to polynomials that act on a $D$-algebra rather than on $D$ itself [7, 8, 9, 10, 11, 16, 18, 19, 20, 22, 23, 27]. For this generalization, we let $A$ be a torsion-free $D$-algebra such that $A \cap K = D$, and let $B = K \otimes_D A$, which is the extension of $A$ to a $K$-algebra. By identifying $K$ and $A$ with their images under the injections $k \mapsto 1$ and $a \mapsto 1 \otimes a$, we can evaluate polynomials in $K[X]$ at elements of $A$. This allows us to define $\text{Int}_K(A) := \{ f \in K[X] \mid f(A) \subseteq A \}$, which is the ring of integer-valued polynomials on $A$ with coefficients in $K$. With notation as above, the condition $A \cap K = D$ ensures that $D[X] \subseteq \text{Int}_K(A) \subseteq \text{Int}(D)$.

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**Definition 1.1.** We say that $\text{Int}_K(A)$ is *nontrivial* if $\text{Int}_K(A) \neq D[X]$.

The goal of this paper is to determine when $\text{Int}_K(A)$ is nontrivial. Some results in this direction were proved by Frisch in [11, Lem. 4.1] and [11, Thm. 4.3]; these are restated below in Proposition 2.5. In the traditional case, necessary and sufficient conditions for $\text{Int}(D)$ to be nontrivial were given by Rush in [20]. Using Rush’s criteria, we prove (Theorem 2.12) that when $D$ is any integral domain and $A$ is finitely generated as a $D$-module, $\text{Int}_K(A)$ is nontrivial if and only if $\text{Int}(D)$ is nontrivial. Part of this work involves conditions under which we have $D[X] \subseteq \text{Int}_K(M_n(D)) \subseteq \text{Int}_K(A)$ for some $n$, where $M_n(D)$ is the algebra of $n \times n$ matrices with entries in $D$. This led us to investigate whether having $\text{Int}_K(M_n(D)) = \text{Int}_K(A)$ implies that $A \cong M_n(D)$. While this is not true in general, the result does hold if $D$ is a Dedekind domain and $A$ can be embedded in $M_n(D)$ (Theorem 2.18).

If we drop the assumption that $A$ is finitely generated as a $D$-module, determining whether $\text{Int}_K(A)$ is nontrivial becomes more complicated. However, when $D$ is Dedekind, we are able to give necessary and sufficient conditions for $\text{Int}_K(A)$ to be nontrivial (Theorem 3.4). Our work on this topic also allows us to prove that if $D$ is Dedekind, then $\text{Int}_K(A)$ has Krull dimension 2 (Corollary 3.10). This generalizes another theorem of Frisch [9, Thm. 5.4] where it was assumed that $A$ was finitely generated as a $D$-module.

## 2 Integral Algebras of Bounded Degree

Throughout, $D$ denotes an integral domain with field of fractions $K$, and $A$ denotes a $D$-algebra. We will always assume that $A$ satisfies certain conditions, which we call our *standard assumptions*.

**Definition 2.1.** When $A$ is a torsion-free $D$-algebra such that $A \cap K = D$, we say that $A$ is a $D$-algebra with *standard assumptions*. When $A$ is finitely generated as a $D$-module, we say that $A$ is of *finite type*.

As mentioned in the introduction, the condition that $A \cap K = D$ implies that

$$D[X] \subseteq \text{Int}_K(A) \subseteq \text{Int}(D)$$

and it is natural to consider when $D[X] = \text{Int}_K(A)$ or $\text{Int}_K(A) = \text{Int}(D)$. This latter equality is investigated in [21], where the following theorem is proved. Unless stated otherwise, all isomorphisms are ring isomorphisms.

**Theorem 2.2.** [21, Thms. 2.10, 3.10] Let $D$ be a Dedekind domain with finite residue rings. Let $A$ be a $D$-algebra of finite type with standard assumptions. For each maximal ideal $P$ of $D$, let $\hat{A}_P$ and $\hat{D}_P$ be the $P$-adic completions of $A$ and $D$, respectively. Then, the following are equivalent.

1. $\text{Int}_K(A) = \text{Int}(D)$.
2. For each nonzero prime $P$ of $D$, there exists $t \in \mathbb{N}$ such that $A/PA \cong \bigoplus_{i=1}^{t} D/P$.
3. For each nonzero prime $P$ of $D$, there exists $t \in \mathbb{N}$ such that $\hat{A}_P \cong \bigoplus_{i=1}^{t} \hat{D}_P$.

In this paper, we examine the containment $D[X] \subseteq \text{Int}_K(A)$. In the traditional setting of integer-valued polynomials, the ring $\text{Int}(D)$ is said to be *trivial* if $\text{Int}(D) = D[X]$, and we adopt the same terminology for $\text{Int}_K(A)$. Clearly, for $\text{Int}_K(A)$ to be nontrivial it is necessary that $\text{Int}(D)$ be nontrivial, so we begin by reviewing the situation for $\text{Int}(D)$. Section 1.3 of [4] and a paper by Rush [20] give several results regarding the triviality or non-triviality of $\text{Int}(D)$. We will summarize these theorems after recalling several definitions.
Definition 2.3. An ideal \( a \) of \( D \) is said to be the colon ideal or conductor ideal of \( q \in K \) if
\[
a = (D :_D q) = \{ d \in D \mid dq \in D \}.
\]
For a commutative ring \( R \), we denote by \( \text{nil}(R) \) the nilradical of \( R \), which is the set of all nilpotent elements of \( R \), or, equivalently, the intersection of all nonzero prime ideals of \( R \). For \( x \in \text{nil}(R) \), we let \( \nu(x) \) equal the nilpotency of \( x \), i.e., the smallest positive integer \( n \) such that \( x^n = 0 \). If \( I \subseteq R \) is an ideal, let \( V(I) = \{ P \in \text{Spec}(R) \mid P \supseteq I \} \).

The following proposition summarizes several sufficient and necessary conditions on \( D \) in order for \( \text{Int}(D) \) to be nontrivial.

Proposition 2.4.

1. [4, Cor. I.3.7] If \( D \) is a domain with all residue fields infinite, then \( \text{Int}(D) \) is trivial.

2. [4, Prop. I.3.10] Let \( D \) be a domain. If there is a proper conductor ideal \( a \) of \( D \) such that \( D/a \) is finite, then \( \text{Int}(D) \) is nontrivial.

3. [4, Thm. I.3.14] Let \( D \) be a Noetherian domain. Then, \( \text{Int}(D) \) is nontrivial if and only if there is a prime conductor ideal of \( D \) with finite residue field.

4. [26, Cor. 1.7] Let \( D \) be an integral domain. Then, the following are equivalent:
   
   (i) \( \text{Int}(D) \) is nontrivial.
   
   (ii) There exist \( a, b \in D \) with \( b \notin aD \) such that the two sets \( \{ |D/P| \mid P \in V((aD : b)) \} \) and \( \{ \nu(x) \mid x \in \text{nil}(D/(aD : b)) \} \) are bounded.

If \( A \) is finitely generated as a \( D \)-module, Frisch has shown that the analogs of the above conditions in Proposition 2.4 hold for \( \text{Int}_K(A) \):

Proposition 2.5. Let \( D \) be a domain. Let \( A \) be a \( D \)-algebra of finite type with standard assumptions.

1. [11, Lem. 4.1] Assume there is a proper conductor ideal \( a \) of \( D \) such that \( D/a \) is finite. Then, \( \text{Int}_K(A) \) is nontrivial.

2. [11, Thm. 4.3] Assume that \( D \) is Noetherian. Then, \( \text{Int}_K(A) \) is nontrivial if and only if there is a prime conductor ideal of \( D \) with finite residue field.

In particular, [11, Thm. 4.3] shows that for a Noetherian domain \( D \) and a finitely generated algebra \( A \), \( \text{Int}_K(A) \) is nontrivial if and only if \( \text{Int}(D) \) is nontrivial. In Theorem 2.12, we will show that this holds even if \( D \) is not Noetherian. Additionally, we can weaken our assumptions on \( A \). Recall the following definition, which can be found in [14] or [15], among other sources.

Definition 2.6. Let \( R \) be a commutative ring and \( A \) an \( R \)-algebra. We say that \( A \) is an algebraic algebra (over \( R \)) if every element of \( A \) satisfies a polynomial equation with coefficients in \( R \). We say that \( A \) is an algebraic algebra of bounded degree if there exists \( n \in \mathbb{N} \) such that the degree of the minimal polynomial equation of each of its elements is bounded by \( n \). If we insist that each element of \( A \) satisfy a monic polynomial with coefficients in \( R \), then we say that \( A \) is an integral algebra over \( R \).
Algebraic algebras are usually discussed over fields, in which case an algebraic algebra is also an integral algebra. Over a domain however, the two structures are not equivalent. For example, \( A = \mathbb{Z} \left[ \frac{1}{2} \right] \) is an algebraic algebra over \( \mathbb{Z} \) that is not an integral algebra. In this case, \( A \) does not satisfy our standard assumption that \( A \cap \mathbb{Q} \) should equal \( \mathbb{Z} \). However, if we instead take \( A = \mathbb{Z} \oplus \mathbb{Z} \left[ \frac{1}{2} \right] \) (so that \( B = \mathbb{Q} \otimes \mathbb{Z} \), \( A \cong \mathbb{Q} \oplus \mathbb{Q} \), \( D \) is the diagonal copy of \( \mathbb{Z} \) in \( B \), and \( K \) is the diagonal copy of \( \mathbb{Q} \) in \( B \)), then \( A \) is an algebraic algebra over \( D \), \( A \) is not an integral algebra over \( D \), and \( A \cap K = D \).

Note also that if \( A \) is finitely generated as a \( D \)-module, then \( A \) is an integral algebra of bounded degree, with the bound given by the number of generators (see [2] Thm. 1, Chap. V) or [1] Prop. 2.4). However, the converse does not hold. For instance, \( A = \mathbb{D} \langle X_1, X_2, \ldots \rangle / \langle \{ X_i, X_j \mid i, j \geq 1 \} \rangle \) is not finitely generated, but if \( f \in A \) with constant term \( d \in D \), then \( f \) satisfies the polynomial \((X - d)^2\). Thus, this \( A \) is an integral algebra of bounded degree.

For our purposes, the importance of having a bounding degree \( n \), is that it guarantees that \( \text{Int}_K(A) \) contains \( \text{Int}_K(M_n(D)) \), where \( M_n(D) \) denotes the algebra of \( n \times n \) matrices with entries in \( D \).

**Lemma 2.7.** Let \( D \) be a domain and let \( A \) be a \( D \)-algebra with standard assumptions. Assume that \( A \) is an integral \( D \)-algebra of bounded degree \( n \). Then, \( \text{Int}_K(M_n(D)) \subseteq \text{Int}_K(A) \).

*Proof.* Let \( a \in A \) and let \( \mu_a \in D[X] \) be monic of degree \( n \) such that \( \mu_a(a) = 0 \). Let \( f(x) = g(X) / d \in \text{Int}_K(M_n(D)) \), where \( g \in D[X] \) and \( d \in D \setminus \{0\} \). By [12] Lem. 3.4, \( g \) is divisible modulo \( dD[X] \) by every monic polynomial in \( D[X] \) of degree \( n \); hence, \( \mu_a \) divides \( g \) modulo \( d \). It follows that \( g(a) \in dA \) and \( f(a) \in A \). Since \( a \) was arbitrary, \( f \in \text{Int}_K(A) \). \( \square \)

**Remark 2.8.** The converse of Lemma 2.7 does not hold, even in the case when \( \text{Int}_K(M_n(D)) \) is nontrivial, as Example 2.11 below will show.

Thus, in the case of an integral algebra of bounded degree \( n \), to prove that \( \text{Int}_K(A) \) is nontrivial it suffices to show that \( \text{Int}_K(M_n(D)) \) is nontrivial. This task is more tractable, because the polynomials given in the next definition can be used to map \( M_n(D) \) into \( M_n(P) \), where \( P \) is a maximal ideal of \( D \) with a finite residue field.

**Definition 2.9.** For each prime power \( q \) and each \( n > 0 \), let

\[
\phi_{q,n}(X) = (X^q^n - X)(X^{q^{n-1}} - X) \cdots (X^q - X).
\]

**Lemma 2.10.** [3] Thm. 3] Let \( \mathbb{F}_q \) be the finite field with \( q \) elements. Then, \( \phi_{q,n} \) sends each matrix in \( M_n(\mathbb{F}_q) \) to the zero matrix. Consequently, if \( P \subset D \) is a maximal ideal of \( D \) with residue field \( D/P \cong \mathbb{F}_q \), then \( \phi_{q,n} \) maps \( M_n(D) \) into \( M_n(P) \).

**Proposition 2.11.** Let \( D \) be a domain. If \( \text{Int}(D) \) is nontrivial, then \( \text{Int}_K(M_n(D)) \) is nontrivial, for all \( n \geq 1 \).

*Proof.* Let \( n \geq 1 \) be fixed. Since \( \text{Int}(D) \) is nontrivial, by [20] Cor. 1.7 there exist \( a, b \in D \) with \( b \notin aD \) such that \( \{ |D/P| \mid P \in V((aD : b)) \} \) and \( \{ \nu(x) \mid x \in \text{nil}(D/(aD : b)) \} \) are bounded. Let \( I = (aD : b) \). Note that, because the former condition holds, each prime ideal containing \( I \) is maximal, so the nilradical of \( D/I \) is equal to the Jacobson radical of \( D/I \).

Let \( \{ q_1, \ldots, q_s \} = \{ |D/P| \mid P \in V(I) \} \). By Lemma 2.10 we have \( \phi_{q,n}(M_n(D)) \subseteq M_n(P) \) for each maximal ideal \( P \subset D \) whose residue field has cardinality \( q \). Then

\[
g(X) = \prod_{i=1,\ldots,s} \phi_{q_i,n}(X)
\]
is a monic polynomial such that \( g(M_n(D)) \subseteq \prod_i M_n(P_i) \subseteq M_n(J) \), where \( J = \sqrt{T} \). Considering everything modulo \( I \), we have \( \mathcal{J}(M_n(D/I)) \subseteq M_n(J/I) \).

Now, since \( \{ u(x) \mid x \in \text{nil}(D/I) \} \) is bounded, the nilpotency of every element in \( J/I \) is bounded by some positive integer \( t \). It is a standard exercise that a matrix over a commutative ring with nilpotent entries is a nilpotent matrix. Moreover, it easily follows that the nilpotency of every matrix in \( M_n(J/I) \) is bounded by some \( m \in \mathbb{N} \), depending only on \( t \) and \( n \). Hence, \( g(X)^m \) maps every matrix \( M_n(D/I) \) to 0, so that \( g(X)^m \) maps \( M_n(D) \) into \( M_n(J) \). Finally, it is now easy to see that the polynomial \( \frac{1}{m} \cdot g(X)^m \) is in \( \text{Int}_K(M_n(D)) \) but not in \( D[X] \).

Combining Lemma 2.7 with Proposition 2.11 we obtain our desired theorem.

**Theorem 2.12.** Let \( D \) be a domain and let \( A \) be \( D \)-algebra with standard assumptions. Assume that \( A \) is an integral \( D \)-algebra of bounded degree. Then, \( \text{Int}_K(A) \) is nontrivial if and only if \( \text{Int}(D) \) is nontrivial. In particular, if \( A \) is finitely generated as a \( D \)-module, then \( \text{Int}_K(A) \) is nontrivial if and only if \( \text{Int}(D) \) is nontrivial.

Lemma 2.7 shows that, for an integral algebra \( A \) of bounded degree \( n \), the following containments hold:

\[
D[X] \subseteq \text{Int}_K(M_n(D)) \subseteq \text{Int}_K(A) \subseteq \text{Int}(D).
\]

While our focus has been on whether \( \text{Int}_K(A) \) equals \( D[X] \), for the remainder of this section we will consider the containment \( \text{Int}_K(M_n(D)) \subseteq \text{Int}_K(A) \). In particular, we will examine to what extent \( \text{Int}_K(M_n(D)) \) is unique among rings of integer-valued polynomials. That is, if \( \text{Int}_K(M_n(D)) = \text{Int}_K(A) \), then can we conclude that \( A \cong M_n(D) \)? In general, the answer is no, as we show below in Example 2.15. However, in Theorem 2.13 we will prove that for \( D \) Dedekind, if \( A \) can be embedded in \( M_n(D) \), then having \( \text{Int}_K(M_n(D)) = \text{Int}_K(A) \) implies that \( A \cong M_n(D) \).

We first recall the definition of a null ideal of an algebra.

**Definition 2.13.** Let \( R \) be a commutative ring and \( A \) an \( R \)-algebra. The null ideal of \( A \) with respect to \( R \), denoted \( N_R(A) \), is the set of polynomials in \( R[X] \) that kill \( A \). That is, \( N_R(A) = \{ f \in R[X] \mid f(A) = 0 \} \). In particular, \( N_{D/P}(A/PA) = \{ f \in (D/P)[X] \mid f(A/PA) = 0 \} \) denotes the null ideal of \( A/PA \) with respect to \( D/P \).

There is a close relationship between polynomials in rings of integer-valued polynomials and polynomials in null ideals.

**Lemma 2.14.** Let \( D \) be a domain and let \( A \) and \( A' \) be \( D \)-algebras with standard assumptions.

1. Let \( g(X)/d \in K[X] \), where \( g \in D[X] \) and \( d \neq 0 \). Then, \( g(X)/d \in \text{Int}_K(A) \) if and only if the residue of \( g \) (mod \( d \)) is in \( N_{D/dD}(A/dA) \).

2. \( \text{Int}_K(A) = \text{Int}_K(A') \) if and only if \( N_{D/dD}(A/dA) = N_{D/dD}(A'/dA') \) for all \( d \in D \). 

**Proof.** Notice that \( g \in \text{Int}_K(A) \) if and only if \( g(A) \subseteq dA \) if and only if \( g(A/dA) = 0 \mod d \). This proves (1), and (2) follows easily.

**Example 2.15.** Let \( D = \mathbb{Z}_p \) be the localization of \( \mathbb{Z} \) at an odd prime \( p \). Take \( A \) to be the quaternion algebra \( A = D \oplus Di \oplus D j \oplus D k \), where \( i, j, \) and \( k \) are the imaginary quaternion units satisfying \( i^2 = j^2 = -1 \) and \( ij = k = -ji \). It is well known (cf. [13, Exercise 3A] or [9, Sec. 2.5]) that \( A/p^k A \cong M_2(\mathbb{Z}/p^k \mathbb{Z}) \cong M_2(D/p^k D) \) for all \( k > 0 \). By Lemma 2.14, \( \text{Int}_Q(A) = \text{Int}_Q(M_2(D)) \). However, \( A \) contains no nonzero nilpotent elements (and is in fact contained in the division ring \( \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}k \)) and so cannot be isomorphic to \( M_2(D) \).
Thus, in general, $\text{Int}_K(A) = \text{Int}_K(M_n(D))$ does not imply that $A \cong M_n(D)$. However, as mentioned above, we do have such an isomorphism if $A$ can be embedded in $M_n(D)$. Proving this theorem involves some results of Racine [24, 25] about maximal subalgebras of matrix rings, which we now summarize.

**Proposition 2.16.**

(1) ([24] Thm. 1) Let $\overline{A}$ be a maximal $\mathbb{F}_q$-subalgebra of $M_n(\mathbb{F}_q)$. Let $V$ be an $\mathbb{F}_q$-vector space of dimension $n$, so that $M_n(\mathbb{F}_q) \cong \text{End}_{\mathbb{F}_q}(V)$. Then, $\overline{A}$ is one of the following two types.

(I) The stabilizer of a proper nonzero subspace of $V$. That is, $\overline{A} = S(W) = \{ \varphi \in \text{End}_{\mathbb{F}_q}(V) \mid \varphi(W) \subseteq W \}$, where $W$ is a proper nonzero $\mathbb{F}_q$-subspace of $V$.

(II) The centralizer of a minimal field extension of $\mathbb{F}_q$. That is, $\overline{A} = C_{\text{End}_{\mathbb{F}_q}(V)}(\mathbb{F}_q^l) = \{ \varphi \in \text{End}_{\mathbb{F}_q}(V) \mid \varphi x = x\varphi, \forall x \in \mathbb{F}_q^l \}$, where $l \in \mathbb{Z}$ is a prime dividing $n$.

(2) ([25] Theorem p. 12) Let $D$ be a Dedekind domain and let $A$ be a maximal $D$-subalgebra of $M_n(D)$. Then, there exists a maximal ideal $P$ of $D$ such that $A/PA$ is a maximal subalgebra of $M_n(D/P)$.

Racine’s classification allows us to establish a partial uniqueness result for the null ideal of $M_n(\mathbb{F}_q)$, and hence for $\text{Int}_K(M_n(D))$.

**Lemma 2.17.** Let $\overline{A}$ be an $\mathbb{F}_q$-subalgebra of $M_n(\mathbb{F}_q)$ such that $N_{\mathbb{F}_q}(\overline{A}) = N_{\mathbb{F}_q}(M_n(\mathbb{F}_q))$. Then $\overline{A} = M_n(\mathbb{F}_q)$.

**Proof.** Suppose the claim is not true, so that $\overline{A}$ is contained in a maximal $\mathbb{F}_q$-subalgebra of $M_n(\mathbb{F}_q)$; hence, without loss of generality, we may assume that $\overline{A} \subsetneq M_n(\mathbb{F}_q)$ is a maximal $\mathbb{F}_q$-subalgebra.

We will show that $N_{\mathbb{F}_q}(\overline{A})$ properly contains $N_{\mathbb{F}_q}(M_n(\mathbb{F}_q))$. Note that $N_{\mathbb{F}_q}(M_n(\mathbb{F}_q)) = (\phi_{q,n}(X))$ by [3] Thm. 3, where $\phi_{q,n}$ is the polynomial from Definition 2.4.

Let $V$ be an $\mathbb{F}_q$-vector space of dimension $n$, so that $M_n(\mathbb{F}_q) \cong \text{End}_{\mathbb{F}_q}(V)$. Assume first that $\overline{A} = S(W)$ is of Type I as in Proposition 2.16 and let $\dim_{\mathbb{F}_q}(W)$. Note that conjugating $\overline{A}$ by an element of $GL(n, q)$ will change the matrices in $\overline{A}$, but not the polynomials in the null ideal $N_{\mathbb{F}_q}(\overline{A})$. Moreover, up to conjugacy by an element in $GL(n, q)$, we may assume that $W$ has basis $e_1, e_2, \ldots, e_m$, where $e_i$ is the standard basis vector with 1 in the $i$th component and 0 elsewhere. Under this basis, the matrices in $\overline{A}$ are block matrices of the form $\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ where $A_1$ is $m \times m$ and $A_3$ is $(n - m) \times (n - m)$.

One consequence of this representation is that every matrix in $S(W)$ has a reducible characteristic polynomial. As shown in the proof of [3] Thm. 3, $\phi_{q,n}$ is the least common multiple of all monic polynomials in $\mathbb{F}_q[X]$ of degree $n$. Hence, $\phi_{q,n} \in N_{\mathbb{F}_q}(\overline{A})$, because the characteristic polynomial of each matrix in $\overline{A}$ divides $\phi_{q,n}$. However, if $\phi$ is the quotient of $\phi_{q,n}$ by an irreducible polynomial in $\mathbb{F}_q[X]$ of degree $n$, then $\phi \notin N_{\mathbb{F}_q}(M_n(\mathbb{F}_q))$. Thus, $N_{\mathbb{F}_q}(\overline{A})$ properly contains $N_{\mathbb{F}_q}(M_n(\mathbb{F}_q))$.

Now, assume that $\overline{A}$ is of Type II of Proposition 2.16 so that $\overline{A} = C_{\text{End}_{\mathbb{F}_q}(V)}(\mathbb{F}_q^l)$ for some prime $l$ dividing $n$. Then, by [24] Thm. VIII.10, we have $\overline{A} \cong M_{n/l}(\mathbb{F}_q^l)$, and so

\[ N_{\mathbb{F}_q}(\overline{A}) = (\phi_{q,l/n}(X)) \supseteq (\phi_{q,n}(X)) = N_{\mathbb{F}_q}(M_n(\mathbb{F}_q)). \]

As before, the null ideal of $\overline{A}$ strictly contains the null ideal of $M_n(\mathbb{F}_q)$.

\[ \square \]
Theorem 2.18. Let $D$ be a Dedekind domain with finite residue fields. Let $A$ be a $D$-algebra of finite type with standard assumptions. Assume that $n \geq 1$ is such that $A$ can be embedded in $M_n(D)$. Then, $\text{Int}_K(A) = \text{Int}_K(M_n(D))$ if and only if $A \cong M_n(D)$.

Proof. Clearly, $A \cong M_n(D)$ implies that $\text{Int}_K(A) = \text{Int}_K(M_n(D))$. So, assume that $\text{Int}_K(M_n(D)) = \text{Int}_K(A)$. As we will prove shortly in Lemma 2.17, $\text{Int}_K(A)$ (and likewise $\text{Int}_K(M_n(D))$) is well-behaved with respect to localization at primes of $D$: for each prime $P$ of $D$, we have $\text{Int}_K(A)_P = \text{Int}_K(A_P)$. Hence, $\text{Int}_K(M_n(D_P)) = \text{Int}_K(A_P)$ for each $P$. Since $D$ is Dedekind, $D_P$ is a discrete valuation ring, so there exists $\pi \in D_P$ such that $PD_P = \pi D_P$. Moreover, we have $D_P/\pi D_P \cong D/P$ and $A_P/\pi A_P \cong A/PA$, so that $N_{D_P/\pi D_P}(A_P/\pi A_P) = N_{D/P}(A/PA)$ (and likewise for $M_n(D)$).

By Lemma 2.14 (2), we conclude that the null ideals $N_{D/P}(M_n(D/P))$ and $N_{D/P}(A/PA)$ are equal for all maximal ideals $P$ of $D$.

Now, suppose by way of contradiction that the image of $A$ in $M_n(D)$ does not equal $M_n(D)$. As in Lemma 2.17 we may assume that the image of $A$ in $M_n(D)$ is a maximal $D$-subalgebra of $M_n(D)$.

By Proposition 2.16 there exists a maximal ideal $P$ of $D$ such that $A/PA$ is isomorphic to a maximal subalgebra of $M_n(D/P)$. By Lemma 2.17 the null ideals $N_{D/P}(A/PA)$ and $N_{D/P}(M_n(D/P))$ are not equal. This is a contradiction. Therefore, $A \cong M_n(D)$. 

3 General Case

We return now to the study of when $\text{Int}_K(A)$ is nontrivial. Because of Theorem 2.12, $A$ being an integral $D$-algebra of bounded degree can be sufficient for $\text{Int}_K(A)$ to be nontrivial, but it is not necessary. There exist $D$-algebras $A$ that are neither finitely generated, nor algebraic over $D$ (let alone integral or of bounded degree), but for which $\text{Int}_K(A)$ is nontrivial, as the next example shows.

Example 3.1. Let $D = \mathbb{Z}$ and let $A = \prod_{i \in \mathbb{N}} \mathbb{Z}$ be an infinite direct product of copies of $\mathbb{Z}$. Then, the element $(1, 2, 3, \ldots)$ cannot be killed by any polynomial in $\mathbb{Z}[X]$, so $A$ is not algebraic over $\mathbb{Z}$. However, since operations in $A$ are done component-wise, any polynomial in $\text{Int}(\mathbb{Z})$ is also in $\text{Int}_\mathbb{Q}(A)$. Hence, $\text{Int}_\mathbb{Q}(A) = \text{Int}(\mathbb{Z})$, so in particular $\text{Int}_\mathbb{Q}(A)$ is nontrivial.

Ultimately, the previous example works because for each prime $p$ there exists a polynomial that sends each element of $A/pA$ to 0. More explicitly, each element of $\prod_{i \in \mathbb{N}} F_p$ is killed by the polynomial $X^p - X$. This suggests that for $\text{Int}_K(A)$ to be nontrivial, it may be enough that there exists a finite index prime $P$ of $D$ with $A/PA$ algebraic of bounded degree over $D/P$ (since $D/P$ is a field in this case, this is equivalent to having $A/PA$ be integral of bounded degree over $D/P$). We will prove below in Theorem 3.4 that if $D$ is a Dedekind domain, then this condition is necessary and sufficient for $\text{Int}_K(A)$ to be nontrivial.

Our work will involve localizing $D$, $A$, and $\text{Int}_K(A)$ at $P$, and exploiting properties of $D_P$. In [27 Prop. 3.2], it is shown that if $D$ is Noetherian and $A$ is a free $D$-module of finite rank, then $\text{Int}_K(A)_P = \text{Int}_K(A_P)$ (in fact, [27 Prop. 3.2] will hold if $A$ is merely finitely generated as a $D$-module). The next lemma shows that we can drop this finiteness assumption if $D$ is Dedekind.

Lemma 3.2. Let $D$ be a Dedekind domain and $A$ a $D$-algebra with standard assumptions. Let $P$ be a prime ideal of $D$. Then $\text{Int}_K(A_P) = \text{Int}_K(A)_P$.

Proof. The containment $\text{Int}_K(A)_P \subseteq \text{Int}_K(A_P)$ follows from the proof of [27 Prop. 3.2], which itself is an adaptation of a technique of Rush involving induction on the degrees of the relevant polynomials (see [4 Thm. I.2.1] or [26 Prop. 1.4]).
For the other inclusion, let \( f \in \text{Int}_K(A_P) \) and write \( f(X) = \frac{g(X)}{d} \) for some \( g \in D[X] \) and \( d \in D \setminus \{0\} \). Since \( D \) is Dedekind, we may write \( dD = P^\alpha I \), where \( \alpha \geq 0 \) and \( I \) is an ideal of \( D \) coprime with \( P \) (possibly equal to \( D \) itself). If \( \alpha = 0 \) then \( f \in D_P[X] \subseteq \text{Int}_K(A_P) \). If \( \alpha \geq 1 \), let \( c \in I \setminus P \). We claim that \( cf \in \text{Int}_K(A) \), from which the statement follows since \( c \in D \setminus P \).

If \( Q \subset D \) is a prime ideal different from \( P \), then \( cf \in D_Q[X] \subseteq \text{Int}_K(A_Q) \); that is, \( cf(A_Q) \subset A_Q \). Now, \( f(A) \subseteq f(A_P) \subseteq A_P \) by assumption, so \( cf(A) \subset cA_P = A_P \), since \( c \notin P \). Since \( A = \bigcap_{Q \in \text{Spec}(D)} A_Q \), it follows that \( cf(A) \subset A \), and we are done. \( \square \)

Recall (Definition 2.13) that the null ideal of \( A \) in \( R \) is \( N_R(A) = \{ f \in R[X] | f(A) = 0 \} \).

**Proposition 3.3.** Let \( D \) be a Dedekind domain and \( A \) a \( D \)-algebra with standard assumptions. Let \( P \) be a prime ideal of \( D \). Then, the following are equivalent.

1. \( N_{D/P}(A/PA) \supseteq (0) \).
2. \( D_P[X] \subset \text{Int}_K(A_P) \).
3. \( D/P \) is finite and \( A/PA \) is a \( D/P \)-algebraic algebra of bounded degree.

**Proof.**

(1) \( \Rightarrow \) (2) Let \( g \in D[X] \) be a monic pullback of a nontrivial element \( \overline{g} \in N_{D/P}(A/PA) \) and let \( \pi \in P \setminus P^2 \). Then, \( g(A_P) \subseteq PAP = \pi A_P \), so \( \frac{g(X)}{\pi} \in \text{Int}_K(A_P) \setminus D_P[X] \).

(2) \( \Rightarrow \) (1) Let \( f(X) = \frac{g(X)}{d} \in \text{Int}_K(A_P) \setminus D_P[X] \), with \( g \in D[X] \setminus P[X] \) and \( d \in P \). Let \( v_P \) denote the canonical valuation on \( D_P \). If \( v_P(d) = e > 1 \) and \( \pi \in P \setminus P^2 \), then \( \pi^{e-1} f(X) \) is still an element of \( \text{Int}_K(A_P) \) which is not in \( D_P[X] \). So, \( g(A_P) \subseteq \frac{d}{\pi^{e-1}} A_P \subseteq \pi A_P \). Hence, \( \overline{g} \in (D_P/PD_P)[X] \cong (D/P)[X] \) is a nontrivial element of \( N_{D/P}(A/PA) \).

(1) \( \Leftrightarrow \) (3) Note that

\[
N_{D/P}(A/PA) = \bigcap_{\pi \in A/PA} N_{D/P}(\overline{\pi}) = \bigcap_{\pi \in A/PA} (\mu_{\overline{\pi}}(X))
\]

where, for each \( \pi \in A/PA \), \( \mu_{\overline{\pi}} \in (D/P)[X] \) is the minimal polynomial of \( \overline{\pi} \) over the field \( D/P \).

If \( N_{D/P}(A/PA) \) is nonzero, then it is equal to a principal ideal generated by a monic non-constant polynomial \( \overline{g} \in (D/P)[X] \). Since \( N_{D/P}(A/PA) \subseteq N_{D/P}(D/P) \), it follows that \( D/P \) is finite (if not, then \( N_{D/P}(D/P) = (0) \), because the only polynomial which is identically zero on an infinite field is the zero polynomial). Moreover, each element \( \overline{g} \in A/PA \) is algebraic over \( D/P \) (otherwise the corresponding \( N_{D/P}(\overline{g}) \) is zero) and its degree over \( D/P \) is bounded by \( \deg(\overline{g}) \).

Conversely, assume \( D/P \) is finite and \( A/PA \) is a \( D/P \)-algebraic algebra of bounded degree \( n \). Then, there are finitely many polynomials over \( D/P \) of degree at most \( n \), and the product of all such polynomials is a nontrivial element of \( N_{D/P}(A/PA) \). \( \square \)

We can now establish the promised criterion for \( \text{Int}_K(A) \) to be nontrivial.

**Theorem 3.4.** Let \( D \) be a Dedekind domain and let \( A \) be a \( D \)-algebra with standard assumptions. Then \( \text{Int}_K(A) \) is nontrivial if and only if there exists a prime ideal \( P \) of \( D \) of finite index such that \( A/PA \) is a \( D/P \)-algebraic algebra of bounded degree.

**Proof.** Clearly, \( D[X] \subsetneq \text{Int}_K(A) \) if and only if there exists a prime ideal \( P \subset D \) such that the two \( D \)-modules \( D[X] \) and \( \text{Int}_K(A) \) are not equal locally at \( P \), that is, \( D_P[X] \subsetneq \text{Int}_K(A_P) \). Since \( \text{Int}_K(A)_P = \text{Int}_K(A_P) \) by Lemma 3.2, we can apply Proposition 3.3 and we are done. \( \square \)
**Example 3.5.** Theorem 3.4 applies to the following examples.

1. Let $D = \mathbb{Z}$ and $A = \mathbb{Z}$, the absolute integral closure of $\mathbb{Z}$. Then, for each $n \in \mathbb{N}$, there exists $\alpha \in \mathbb{Z}$ of degree $d > n$ such that $O_{\alpha} = \mathbb{Z}[\alpha]$. It follows that for each prime $p \in \mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z}$ is an algebraic $\mathbb{Z}/p\mathbb{Z}$-algebra of unbounded degree. Thus, $\text{Int}_\mathbb{Q}(\mathbb{Z}) = \mathbb{Z}[X]$.

2. Let $D = \mathbb{Z}_{(p)}$ and $A = \mathbb{Z}_p$. Then, $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$, so $\mathbb{Z}_{(p)}[X] \subsetneq \text{Int}_\mathbb{Q}(\mathbb{Z}_p)$.

3. Let $D = \mathbb{Z}$ and $A = \hat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$, the profinite completion of $\mathbb{Z}$, where $\mathbb{P}$ denotes the set of all prime numbers. For each prime $p \in \mathbb{Z}$, we have $\hat{p}\mathbb{Z} = \prod_{p' \neq p} \mathbb{Z}_{p'} \times p\mathbb{Z}_p$, so $\hat{\mathbb{Z}}/\hat{p}\mathbb{Z} \cong \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$. Thus, $\mathbb{Z}[X] \subsetneq \text{Int}_\mathbb{Q}(\hat{\mathbb{Z}})$.

If $\hat{A}$ is the $P$-adic completion of a $D$-algebra $A$, then we can say more about $\text{Int}_K(\hat{A})$. The following lemma also appears in [21]. We include it in its entirety since the proof is quite short.

**Lemma 3.6.** Let $D$ be a discrete valuation ring (DVR) with maximal ideal $P = \pi D$. Let $A$ be a $D$-algebra with standard assumptions, and let $\hat{A}$ be the $P$-adic completion of $A$. Then, $\text{Int}_K(\hat{A}) = \text{Int}_K(A)$.

**Proof.** The containment $\text{Int}_K(\hat{A}) \subseteq \text{Int}_K(A)$ is clear, since $A$ embeds in $\hat{A}$. Conversely, let $f \in \text{Int}_K(A)$ and $\alpha \in \hat{A}$. Suppose $f(X) = g(X)/\pi^k$, where $g \in D[X]$ and $k \in \mathbb{N}$. If $k = 0$, then the conclusion is clear, so assume that $k > 1$.

Via the canonical projection $\hat{A} \to A/\pi^kA$, we see that there exists $a \in A$ such that $\alpha \equiv a \pmod{\pi^k\hat{A}}$. Since the coefficients of $g$ are central in $A$, we get $g(\alpha) \equiv g(a) \pmod{\pi^k\hat{A}}$. Thus, $f(\alpha) = f(a) + \lambda/\pi^k$, where $\lambda \in \pi^k\hat{A}$, so that $f(\alpha) \in \hat{A}$. Hence, $f \in \text{Int}_K(\hat{A})$ and $\text{Int}_K(\hat{A}) = \text{Int}_K(A)$.

Thus, in Example 3.5 (2), we have $\text{Int}_\mathbb{Q}(A) = \text{Int}(\mathbb{Z}_{(p)})$. Moreover, in Example 3.5 (3) we have $\text{Int}_\mathbb{Q}(A) = \text{Int}(\mathbb{Z})$ (see also [3] where the profinite completion of $\mathbb{Z}$ was considered in order to study the polynomial overrings of $\text{Int}(\mathbb{Z})$). A more general example, which results in proper containments among all of $D[X]$, $\text{Int}_K(A)$, and $\text{Int}(D)$, is the following.

**Example 3.7.** Let $D$ be a DVR with maximal ideal $P = \pi D$ and finite residue field. Let $A$ be a $D$-algebra of finite type with standard assumptions and such that $\text{Int}_K(A) \subsetneq \text{Int}(D)$. Let $\hat{A}$ be the $P$-adic completion of $A$. Then, $P$ satisfies the conditions of Theorem 3.4 with respect to $A$, so $D[X] \subsetneq \text{Int}_K(A)$; and $\text{Int}_K(\hat{A}) = \text{Int}_K(A)$ by Lemma 3.6. Thus,

$$D[X] \subsetneq \text{Int}_K(\hat{A}) = \text{Int}_K(A) \subsetneq \text{Int}(D).$$

In general, $\hat{A}$ is not finitely generated as a $D$-module (this is the case, for instance, when $A$ is countable but $\hat{A}$ is uncountable). So, $\hat{A}$ can provide an example of a $D$-algebra that is not finitely generated and for which the integer-valued polynomial ring is properly contained between $D[X]$ and $\text{Int}(D)$.

**Remark 3.8.** Lemma 3.6 also gives us another approach to Example 2.15. With notation as in that example, we have $\hat{A} \cong M_2(\mathbb{Z}_p)$ (indeed, this follows from the fact that $A/p^kA \cong M_2(\mathbb{Z}/p^k\mathbb{Z})$ for all $k > 0$). Thus, $\text{Int}_\mathbb{Q}(A) = \text{Int}_\mathbb{Q}(M_2(\mathbb{Z}_p)) = \text{Int}_\mathbb{Q}(M_2(\mathbb{Z}_{(p)}))$ even though $A \not\cong M_2(\mathbb{Z}_{(p)})$. 

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We close this paper by using the conditions of Proposition 3.3 to prove that when $D$ is Dedekind, $\text{Int}_K(A)$ has Krull dimension 2. This result was shown by Frisch [9, Thm. 5.4] in the case where $A$ is of finite type. Our work does not require $A$ to be finitely generated, and somewhat surprisingly does not require a full classification of the prime ideals of $\text{Int}_K(A)$.

Recall that a nonzero prime ideal $\mathfrak{P}$ of $\text{Int}_K(A)$ is called unitary if $\mathfrak{P} \cap D \neq (0)$, and is called non-unitary if $\mathfrak{P} \cap D = (0)$.

**Theorem 3.9.** Let $D$ be a Dedekind domain and let $A$ be a $D$-algebra with standard assumptions. Let $\mathfrak{P}$ be a nonzero prime ideal of $\text{Int}_K(A)$.

1. If $\mathfrak{P}$ is non-unitary, then $\mathfrak{P}$ has height 1.
2. If $\mathfrak{P}$ is unitary, then let $P = \mathfrak{P} \cap D$.
   
   i. If $P$ does not satisfy any of the conditions of Proposition 3.3, then $\mathfrak{P}$ has height 2.
   
   ii. If $P$ satisfies one of the conditions of Proposition 3.3, then $\mathfrak{P}$ is maximal and has height at most 2.

**Proof.** (1) Following [9, Lem. 5.3], the non-unitary prime ideals of $\text{Int}_K(A)$ are in one-to-one correspondence with the prime ideals of $K[X]$. Since $K[X]$ has dimension 1, the non-unitary primes of $\text{Int}_K(A)$ are all of height 1.

(2) Let $P$ be a nonzero prime of $D$. Assume first that $P$ does not satisfy any of the conditions of Proposition 3.3. Then, $D_P[X] = \text{Int}_K(AP) = \text{Int}_K(A)_P$. It follows that the unitary primes of $\text{Int}_K(A)$ are in one-to-one correspondence with the primes of $D_P[X]$. Since $D$ is Dedekind, we know that $D_P[X]$ has dimension 2; hence, all the primes of $\text{Int}_K(A)$ under consideration have height 2.

For the remainder of the proof, assume that $P = \mathfrak{P} \cap D$ does satisfy the conditions of Proposition 3.3. Since $\mathfrak{P} \cap D = P$, the prime ideal $\mathfrak{P}$ survives in $\text{Int}_K(A)_P = \text{Int}_K(AP)$ and clearly its extension $\mathfrak{P}^e$ is still a prime unitary ideal (so, $\mathfrak{P}^e \cap D_P = PD_P$). It is sufficient to show that $\mathfrak{P}^e$ is maximal, so we may work over the localizations. Thus, without loss of generality we will assume that $D$ is a DVR. In particular, this means that $P = \pi D$, for some $\pi \in D$.

Let $f \in N_{D/P}(A/PA)$, $f \neq 0$, and let $g \in D[X]$ be a pullback of $g(X)$. Then $g(A) \subseteq PA = \pi A$. Consequently, for each $f \in \text{Int}_K(A)$ we have $(g \circ f)(A) \subseteq \pi A$. Consider the ideal $\mathfrak{A} = \{F \in \text{Int}_K(A) \mid F(A) \subseteq PA\}$ of $\text{Int}_K(A)$.

Because $P = \pi D$ is principal, we have $\mathfrak{A} = \pi \text{Int}_K(A)$, which is contained in $\mathfrak{P}$. Hence, for each $f \in \text{Int}_K(A)$, $g \circ f \in \mathfrak{P}$.

Now, if we consider the $D/P$-algebra $\text{Int}_K(A) / \mathfrak{P}$, we see that each element of $\text{Int}_K(A)/\mathfrak{P}$ is annihilated by $\mathfrak{P}(X)$. But $\text{Int}_K(A)/\mathfrak{P}$ is a domain, and for it to be annihilated by a nonzero polynomial, it must be finite. Thus, in fact $\text{Int}_K(A)/\mathfrak{P}$ is a finite field, and so $\mathfrak{P}$ is maximal.

Finally, to show that $\mathfrak{P}$ has height at most 2, let $\mathfrak{Q}$ be a prime of $\text{Int}_K(A)$ such that $(0) \subseteq \mathfrak{Q} \subseteq \mathfrak{P}$. If $\mathfrak{Q}$ is unitary, then we have $\mathfrak{Q} \cap D = P$, and by our work above $\mathfrak{Q}$ is maximal, hence equal to $\mathfrak{P}$. If $\mathfrak{Q}$ is non-unitary, then it has height 1 by part (1) of the theorem. It follows that $\mathfrak{P}$ has height at most 2.

**Corollary 3.10.** Let $D$ be a Dedekind domain with quotient field $K$. Let $A$ be a $D$-algebra with standard assumptions. Then, $\text{Int}_K(A)$ has Krull dimension 2.

**Proof.** If $\text{Int}_K(A) = D[X]$, then its dimension equals that of $D[X]$, which is 2. So, assume that $\text{Int}_K(A)$ is nontrivial. By Theorem 3.3 there exists a prime $P$ of $D$ that satisfies the conditions of Proposition 3.3.
Let \( \mathfrak{P} = \{ f \in \text{Int}_K(A) \mid f(0) \in P \} \). Since \( \text{Int}_K(A) \subseteq \text{Int}(D) \), \( \mathfrak{P} \) is an ideal of \( \text{Int}_K(A) \), and it is easily seen to be prime and unitary, with \( \mathfrak{P} \cap D = P \). Moreover, it contains the non-unitary ideal \( XK[X] \cap \text{Int}_K(A) \). Hence, \( \mathfrak{P} \) has height at least 2, and so \( \dim(\text{Int}_K(A)) \geq 2 \). However, \( \dim(\text{Int}_K(A)) \leq 2 \) by Theorem 3.9 so we conclude that \( \dim(\text{Int}_K(A)) = 2 \). □

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