\textbf{\LARGE \textit{\partial}}-Neumann Problem and Kohn’s Algorithm

\textit{Eric Ng}

\textbf{Contents}

1 Introduction \hfill 1  
2 Formulation of the Problem \hfill 2  
3 Domains of Operators \hfill 2  
4 Basic Estimate \hfill 5  
5 Tangential Sobolev Spaces and Pseudo-Differential Operators \hfill 10  
6 Strongly Pseudoconvex: Subelliptic Estimate \hfill 13  
7 Weakly Pseudoconvex: Subelliptic Multipliers \hfill 14  
8 Kohn’s Algorithm \hfill 22  

1 Introduction

This article gives a rapid introduction to the $\partial$-Neumann problem and Kohn’s algorithm [Kohn 79] for generating subelliptic estimate.
2 Formulation of the Problem

Let \( D \) be a bounded domain in \( \mathbb{C}^n \) with smooth boundary \( \partial D \). Assume \( D = \{ r < 0 \} \), \( \partial D = \{ r = 0 \} \) such that \( |dr| = 1 \) on \( \partial D \), i.e., \( r \) is a defining function for \( D \).

Define \( H^{p,q} \subset L^2_{p,q}(D) \) by \( H^{p,q} = \{ f \in L^2_{p,q}(D) \cap \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) : \bar{\partial}f = \bar{\partial}^*f = 0 \} \).

The \( \bar{\partial} \)-Neumann problem for \((p,q)\)-forms can then be stated as follows:

Given \( \alpha \in L^2_{p,q}(D) \) and \( \alpha \perp H^{p,q} \), does there exists \( \phi \) such that

\[
(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\phi = \alpha
\]

Note that if a solution of the above exists, then there is a unique solution \( \psi \) such that \( \psi \perp H^{p,q} \). We will denote this unique solution by \( N\alpha \). If a solution to the above exists for all \( \alpha \perp H^{p,q} \), then we extend \( N \) to a linear operator on \( L^2_{p,q}(D) \) by setting it 0 on \( H^{p,q} \). Then \( N \) is bounded and self-adjoint. Furthermore, if \( \bar{\partial}\alpha = 0 \), then taking \( \bar{\partial} \) on both sides of the above gives \( \bar{\partial}\bar{\partial}^*\bar{\partial}\psi = 0 \). Taking inner product with \( \bar{\partial}\psi \) gives \( \bar{\partial}^*\bar{\partial}\psi = 0 \). Thus we see that if \( \bar{\partial}\alpha = 0 \) and \( \alpha \perp H^{p,q} \) then

\[
\alpha = \bar{\partial}(\bar{\partial}^*\psi) = \bar{\partial}(\bar{\partial}^*N\alpha)
\]

It then follows that \( u = \bar{\partial}^*N\alpha \) is the unique solution to the equation \( \bar{\partial}u = \alpha \) which is orthogonal to the space \( \{ f \in L^2_{p,q-1}(D) : \bar{\partial}f = 0 \} \).

3 Domains of Operators

**Proposition 3.1 (Domain of \( \bar{\partial}^* \)).** Let \( \phi \in \Lambda^{p,q}(\overline{D}) \), then \( \phi \in \mathcal{D}^{p,q} := \Lambda^{p,q}(\overline{D}) \cap \text{Dom}(\bar{\partial}^*) \) if and only if \( \sigma(\bar{\partial}^*, dr)\phi = 0 \) on \( \partial D \).

Here the symbol \( \sigma(L, \eta) \) is defined as follows: suppose \( L \) is a differential operator of order \( k \) and that at a point \( x \) we select a covector \( \eta \), then \( \sigma(L, \eta) \) is defined as

\[
L \left( \frac{\rho^k}{k!} \right) |_x
\]

where \( \rho \) is any function which satisfies \( \rho(x) = 0 \) and \( d\rho = \eta \) at \( x \).
Proof. Observe that if

$$L = \sum_{\mu_1 + \cdots + \mu_m = k} a_{\mu_1, \ldots, \mu_m}(x) \frac{\partial^k}{\partial x_1^{\mu_1} \cdots \partial x_m^{\mu_m}} + L_{k-1}$$

where $L_{k-1}$ is a differential operator of order at most $k - 1$. Then

$$\sigma(L, \eta) = \sum_{\mu_1 + \cdots + \mu_m = k} a_{\mu_1, \ldots, \mu_m}(x) \eta_1^{\mu_1} \cdots \eta_m^{\mu_m}$$

where $\eta = \sum \eta_i dx^i$.

Integration by parts implies

$$(\bar{\partial}^* \phi, \psi)_D = (\phi, \bar{\partial} \psi)_D + \int_{\partial D} \langle \sigma(\bar{\partial}^*, dr) \phi, \psi \rangle$$

for all $\psi \in \Lambda^p_q(D)$. □

Now let

$$Q(\phi, \psi) = (\bar{\partial} \phi, \bar{\partial} \psi) + (\bar{\partial}^* \phi, \bar{\partial}^* \psi) + (\phi, \psi)$$

first defined for $(p, q)$-forms $\phi, \psi$ smooth up to the boundary and then complete the space to $\mathcal{D}^{p,q}$ according to the norm defined by the inner product $Q(\cdot, \cdot)$. We now apply Riesz representation theorem to $\mathcal{D}^{p,q}$ with inner product $Q(\cdot, \cdot)$ and the functional

$$\phi \mapsto (\phi, \psi)$$

where $\psi \in \mathcal{D}^{p,q}$ is fixed. This functional is bounded, because

$$|((\phi, \psi)|^2 \leq (\phi, \phi) \cdot (\psi, \psi) \leq Q(\phi, \phi) \cdot (\psi, \psi)$$

Thus there exists a unique $T\psi \in \mathcal{D}$ such that

$$(\phi, \psi) = Q(\phi, T\psi)$$

with

$$Q(T\psi, T\psi) \leq (\psi, \psi)$$

When $T\psi = 0$, it follows from

$$(\phi, \psi) = Q(\phi, T\psi)$$
that $(\phi, \psi) = 0$ for all $\phi \in \mathcal{D}$ and so $\psi = 0$. We define $F = T^{-1}$ so that the domain of $F$ is contained in $\mathcal{D}^{p,q}$. The operator $F$ is self-adjoint because $T$ is:

$$(T\phi, \psi) = Q(T\phi, T\psi) = Q(T\psi, T\phi) = (T\psi, \phi) = (\phi, T\psi)$$

and hence we have $(F\phi, \psi) = (\phi, F\psi) = Q(\phi, \psi)$ for $\phi, \psi \in \text{Dom}(F)$.

**Proposition 3.2 (Domain of $([\square + I])$. Let $\phi \in \Lambda^{p,q}(\mathcal{D})$, then $\phi \in \text{Dom}(F)$ if and only if**

1. $\phi \in \mathcal{D}^{p,q}$ (i.e. $\sigma(L, \eta)\phi = 0$ on $\partial D$)

2. $\bar{\partial}\phi \in \mathcal{D}^{p,q+1}$ (i.e. $\sigma(L, \eta)\bar{\partial}\phi = 0$ on $\partial D$)

**In this case**

$$F\phi = ([\square + I])\phi := \bar{\partial}\bar{\partial}^*\phi + \bar{\partial}^*\bar{\partial}\phi + \phi$$

**Proof.** Let $\phi \in \mathcal{D}^{p,q}$. $(F\phi, \psi) = ([\square + I]\phi, \psi)$ holds for $\psi \in \Lambda^{p,q}(\mathcal{D})$. By density, $(F\phi, \psi) = ([\square + I]\phi, \psi)$ holds for $\psi \in \mathcal{D}^{p,q}$. Recall the following 2 identities obtained from integration by parts:

$$(\bar{\partial}^*\phi, \psi)_D = (\phi, \bar{\partial}\psi)_D + \int_{\partial D} \langle \sigma(\bar{\partial}^*, dr)\phi, \psi \rangle$$

$$(\bar{\partial}\phi, \psi)_D = (\phi, \bar{\partial}^*\psi)_D + \int_{\partial D} \langle \sigma(\bar{\partial}, dr)\phi, \psi \rangle$$

Suppose $\phi \in \text{Dom}(F)$, we have

$$Q(\phi, \psi) = (\bar{\partial}\phi, \bar{\partial}\psi) + (\bar{\partial}^*\phi, \bar{\partial}^*\psi) + (\phi, \psi)$$

$$= (\bar{\partial}\phi, \bar{\partial}\psi) - \int_{\partial D} \langle \sigma(\bar{\partial}^*, dr)\bar{\partial}\phi, \psi \rangle$$

$$+ (\bar{\partial}^*\phi, \psi) - \int_{\partial D} \langle \sigma(\bar{\partial}, dr)\bar{\partial}^*\phi, \psi \rangle + (\phi, \psi)$$

$$= (\bar{\partial}\phi, \bar{\partial}\psi) - \int_{\partial D} \langle \sigma(\bar{\partial}^*, dr)\bar{\partial}\phi, \psi \rangle$$

$$+ (\bar{\partial}\phi, \bar{\partial}^*\psi) - \int_{\partial D} \langle \bar{\partial}^*\phi, \sigma(\bar{\partial}^*, dr)\psi \rangle + (\phi, \psi)$$

For $\psi \in \mathcal{D}^{p,q}$, the above equals

$$(\bar{\partial}^*\bar{\partial}\phi, \psi) + (\bar{\partial}\bar{\partial}^*\phi, \psi) + (\phi, \psi) - \int_{\partial D} \langle \sigma(\bar{\partial}^*, dr)\bar{\partial}\phi, \psi \rangle$$
since \( \sigma(\bar{\partial}^* dr) \psi = 0 \). Next, observe that we can take \( \psi = \sigma(\bar{\partial}^* dr)\bar{\partial} \phi \in \mathcal{D}^{p,q} \). Therefore \( \sigma(\bar{\partial}^* dr)\bar{\partial} \phi = 0 \). The converse is trivial. \( \square \)

**Proposition 3.3.** For \( \phi \in \Lambda^{p,q}(\bar{D}) \), we have, in respective domains, the following identities:

1. \[
\bar{\partial} \phi = \sum_{I,J} \sum_{k=1}^n \frac{\partial \phi_{I,J}}{\partial z_k} d\overline{z}_k \wedge dz^I \wedge d\overline{z}^J
\]

2. \[
\bar{\partial}^* \phi = (-1)^{p-1} \sum_{I,K} \sum_{j=1}^n \frac{\partial \phi_{I,jK}}{\partial z_j} dz^I \wedge d\overline{z}^K
\]

3. \[
\bar{\partial} \bar{\partial} \phi = -\sum_{I,J} \sum_k \frac{\partial^2 \phi_{I,J}}{\partial z_k \partial \overline{z}_k} d\overline{z}^I \wedge d\overline{z}^J + (-1)^p \sum_{I,K} \sum_j \sum_k \frac{\partial^2 \phi_{I,jK}}{\partial z_j \partial \overline{z}_k} d\overline{z}_k \wedge dz^I \wedge d\overline{z}^K
\]

4. \[
\bar{\partial} \bar{\partial}^* \phi = (-1)^{p-1} \sum_{I,K} \sum_j \sum_k \frac{\partial^2 \phi_{I,jK}}{\partial z_j \partial \overline{z}_k} d\overline{z}_k \wedge dz^I \wedge d\overline{z}^K
\]

5. \[
\Box \phi = -\sum_{I,J} \sum_k \frac{\partial^2 \phi_{I,J}}{\partial z_k \partial \overline{z}_k} d\overline{z}^I \wedge d\overline{z}^J = -\frac{1}{4} \sum_{I,J} \Delta \phi_{I,J} dz^I \wedge d\overline{z}^J
\]

**Proof.** Straight forward computations. \( \square \)

### 4 Basic Estimate

**Lemma 4.1.** Let \( D \) be a bounded domain in \( \mathbb{C}^n \) with smooth boundary \( \partial D \) and defining function \( r \) such that \( |dr| = 1 \) on \( \partial D \), then

\[
\left( \frac{-1}{2} \right)^n \int_{\partial D} \left( fg \frac{\partial r}{\partial z_j} \right) d\sigma = \int_D \left( \frac{\partial f}{\partial z_j} g + \frac{\partial g}{\partial z_j} f \right) dV
\]

where \( d\sigma \) is the volume form of \( \partial D \).
Proof. First observe that
\[
\left( \frac{\sqrt{-1}}{2} \right)^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^j \wedge \cdots \wedge dz^n \wedge d\bar{z}^n = \frac{\partial r}{\partial z_j} (\ast dr)
\]
\[
= \frac{\partial r}{\partial z_j} d\sigma
\]
Then by Stokes’ theorem, the result follows. \qed

**Proposition 4.2.** For \( \phi \in \Lambda^{0,1}(\overline{D}) \), \( \phi \in \mathcal{D}^{0,1} \) if and only if
\[
\sum_j \phi_j \frac{\partial r}{\partial z_j} = 0
\]
on \( \partial D \). And
\[
Q(\phi, \phi) = \sum_{j,k} \left\| \frac{\partial \phi_j}{\partial z_k} \right\|^2 + \sum_{j,k} \int_{\partial D} \frac{\partial^2 r}{\partial z_j \partial z_k} \phi_j \bar{\phi}_k + \|\phi\|^2
\]

**Proof.** The first part is straightforward calculation. The second part goes as follows:
\[
\|\bar{\partial} \phi\|^2 = \frac{1}{2} \sum_{j,k=1}^n \left\| \frac{\partial \phi_j}{\partial z_k} - \frac{\partial \phi_k}{\partial z_j} \right\|^2
\]
\[
= \sum_{j,k} \left\| \frac{\partial \phi_j}{\partial z_k} \right\|^2 - \sum_{j,k} \left\langle \frac{\partial \phi_j}{\partial z_k}, \frac{\partial \phi_k}{\partial z_j} \right\rangle
\]
\[
= \sum_{j,k} \left\| \frac{\partial \phi_j}{\partial z_k} \right\|^2 - \sum_{j,k} \left\langle \frac{\partial \phi_j}{\partial z_j}, \frac{\partial \phi_k}{\partial z_k} \right\rangle + \sum_{j,k} \int_{\partial D} \frac{\phi_k}{\partial z_k} \frac{\partial r}{\partial z_j} \frac{\partial \phi_j}{\partial z_j} - \sum_{j,k} \int_{\partial D} \frac{\phi_k}{\partial z_k} \frac{\partial \phi_j}{\partial z_j} \frac{\partial r}{\partial z_j}
\]
The second term above is \(-\|\bar{\partial}^* \phi\|^2\).
The third term above vanishes since \( \phi \in \mathcal{D}^{0,1} \).
For the fourth term, notice that
\[
\sum_j \phi_j \frac{\partial r}{\partial z_j} = 0
\]
on \( \partial D \) implies that any tangential derivative of LHS is nonzero. It also implies that
\[
\sum_j \phi_j \frac{\partial}{\partial z_j}
\]
is perpendicular to $dr$ and hence is tangential. So, on $\partial D$,

$$
\sum_k \phi_k \frac{\partial}{\partial z_k} \left( \sum_j \phi_j \frac{\partial r}{\partial z_j} \right) = 0
$$

Therefore, the fourth term equals to

$$
\sum_{j,k} \int_{\partial D} \frac{\partial^2 r}{\partial z_j \partial z_k} \phi_j \phi_k
$$

We remark that in the case $\phi \in \Lambda^{p,q}(D)$, $\phi \in D^{p,q}$ if and only if

$$
\sum_k \phi_{I,k} \frac{\partial r}{\partial z_k} = 0
$$
on $\partial D$ for all $I, J$. And

$$Q(\phi, \phi) = \sum_{I,J} \sum_k \left\| \phi_{I,J} \right\|^2 + \sum_{I,K} \sum_{i,j} \int_{\partial D} \frac{\partial^2 r}{\partial z_i \partial z_j} \phi_{I,iK} \phi_{I,jK} + \|\phi\|^2
$$

Theorem 4.3. We define a norm $E(\cdot)$ on $\Lambda^{0,1}(D)$ by setting

$$E(\phi)^2 = \sum_{j,k=1}^n \left\| \frac{\partial \phi_j}{\partial z_k} \right\|^2 + \int_{\partial D} |\phi|^2 d\sigma + \|\phi\|^2
$$

then

1. there exists a constant $c > 0$ such that for all $\phi \in D^{0,1}$,

$$Q(\phi, \phi) \leq cE(\phi)^2$$

2. if $\partial D$ is strongly pseudoconvex, there exists a constant $c' > 0$ such that for all $\phi \in D^{0,1}$,

$$Q(\phi, \phi) \geq c'E(\phi)^2$$

Proof. Trivial
Let $U \subset \overline{D}$ be a relatively open set that has nontrivial intersection with $\partial D$. Coordinates $(t_1, ..., t_{2n-1}, r)$ constitute a special boundary chart for $U$ if $r$ is the defining function for $D$ and $(t_1, ..., t_{2n-1}, 0)$ give a coordinate for $\partial D$.

Let $\omega_1, ..., \omega_n$ be an orthonormal basis for $\Lambda^{1,0}(U \cap \overline{D})$ such that $\omega_n = \sqrt{2} \partial r$ on $U \cap \partial D$. Let $L_1, ..., L_n$ be its dual basis for $T^{1,0}(U \cap \overline{D})$. Note that on $U \cap \partial D$, we have

$$\langle L_i, \sqrt{2} \partial r \rangle = \sqrt{2} \cdot L_i(r) = \delta_m = \sqrt{2} \cdot \overline{L}_i(r) = \langle \overline{L}_i, \sqrt{2} \partial r \rangle$$

Hence $L_1, ..., L_{n-1}$ and $\overline{L}_1, ..., \overline{L}_{n-1}$ are local bases for $T^{1,0}(U \cap \partial D)$ and $T^{0,1}(U \cap \partial D)$ respectively. Finally we set $T = L_n - \overline{L}_n$, making $\{L_1, ..., L_{n-1}, \overline{L}_1, ..., \overline{L}_{n-1}, T\}$ local basis for $\mathcal{C}T(U \cap \partial D)$. If we denote

$$c_{ij}(x) = \langle \partial \overline{\partial} r, L_i \wedge L_j \rangle$$

then by the Cartan structure formula

$$\langle A \wedge B, d\theta \rangle = A\langle B, \theta \rangle - B\langle A, \theta \rangle - \langle [A, B], \theta \rangle$$

we have, on $\partial D$, for $i, j < n$,

$$[L_i, \overline{L}_j] = c_{ij}T + \sum_{k=1}^{n-1} a_{ij}^k L_k + \sum_{k=1}^{n-1} b_{ij}^k \overline{L}_k$$

Define $\mathcal{D}^{p,q}_U = \{ \varphi \in \mathcal{D}^{p,q} : \text{supp}(\varphi) \subset U \cap \overline{D} \}$.

If $\phi \in \Lambda^{p,q}(\overline{D})$, then in terms of the above local basis, if we write

$$\phi = \sum_{I,J} \phi_{IJ} \cdot \omega_I \wedge \omega_J$$

then $\phi \in \mathcal{D}^{p,q}_U$ if and only if

$$\phi_{IJ}(x) = 0$$

for $n \in J$, $x \in U \cap \partial D$. Also, we have

$$\overline{\partial} \phi = \sum_{I,J} \sum_{k=1}^n \overline{L}_k(\phi_{I,J}) \cdot \omega_k \wedge \omega_I \wedge \omega_J + O(||\phi||)$$

and

$$\overline{\partial}^* \phi = (-1)^{p-1} \sum_{I,K} \sum_{j=1}^n L_j(\phi_{I,J}) \cdot \omega_I \wedge \omega_K + O(||\phi||)$$
Theorem 4.4 (Basic Estimate). If \( x_0 \in \partial D \) and \( D \) is pseudo-convex then there exists a neighborhood \( U \) of \( x_0 \) and a constant \( C > 0 \) such that

\[
||\phi||_z^2 + \sum_{I,K} \sum_{i,j} c_{ij} \phi_{I,iK} \overline{\phi_{I,jK}} d\sigma \leq CQ(\phi,\phi)
\]

for all \( \phi \in D^{p,q}_U \), with \( q \geq 1 \). Here \( ||\phi||_z \) denotes the norm given by

\[
||\phi||_z^2 = \sum_{I,J} \sum_j ||L_j \phi_{IJ}||^2 + ||\phi||^2
\]

Observe that if \( u \in C^\infty(U \cap \overline{D}) \) with \( u(x) = 0 \) on \( U \cap \partial D \), then integration by parts gives

\[
\int_{U \cap \overline{D}} L_j u \cdot \overline{L_j u} = \int_{U \cap \overline{D}} \overline{L_j u} \cdot L_j u + O(||u||^2)
\]

and so

\[
\sum_j ||L_j u||^2 \leq \sum_j ||\overline{L_j u}||^2 + ||u||^2,
\]

where the constant is independent of \( u \). Hence, we have

\[
||u||^2_1 \leq ||u||^2_z \tag{1}
\]

for all \( u \in C^\infty(U \cap \overline{D}) \) with \( u(x) = 0 \) on \( U \cap \partial D \). Therefore

\[
\sum_{I,K} ||\phi_{I,nK}||_1^2 \leq \sum_{I,K} ||\phi_{I,nK}||_z^2 \leq Q(\phi,\phi)
\]

Next, notice that

\[
\sum_{I,K} \left( \sum_{j=1}^{n-1} L_j (\phi_{I,jK}) \right)^2
\]

is dominated by

\[
C(||\bar{\partial}^* \phi||^2 + \sum_{I,K} ||\phi_{I,nK}||_1^2 + ||\phi||^2)
\]

The reason is that

\[
\bar{\partial}^* \phi = (-1)^{p-1} \sum_{I,K} \sum_{j=1}^n (L_j \phi_{I,jK}) \cdot \omega^I \wedge \omega^K + O(||\phi||)
\]
and so
\[ \sum_{I,K} \left\| \sum_{j=1}^{n-1} L_j(\varphi_{I,jK}) \right\|^2 \leq \|\tilde{\partial}^r \varphi\|^2 + \sum_{I,K} \|\varphi_{I,nK}\|^2 + \|\varphi\|^2 \]

Also
\[ \|\tilde{\partial}^r \varphi\|^2 + \sum_{I,K} \|\varphi_{I,nK}\|^2 + \|\varphi\|^2 \]

is dominated by
\[ CQ(\varphi, \varphi) \]

Combining the above, we obtain
\[ \|\varphi\|^2 + \sum_{I,K} \sum_{i,j} \int_{\partial D} c_{ij} \varphi_{I,iK} \varphi_{I,jK} d\sigma + \sum_{I,K} \|\varphi_{I,nK}\|^2 + \sum_{I,K} \left\| \sum_{j=1}^{n-1} L_j(\varphi_{I,jK}) \right\|^2 \leq Q(\varphi, \varphi) \]

for all \( \varphi \in D^{p,q}_U \) with \( q \geq 1 \).

Notice that conversely we have
\[ Q(\varphi, \varphi) \leq \|\varphi\|^2 + \left| \sum_{I,K} \sum_{i,j} \int_{\partial D} c_{ij} \varphi_{I,iK} \varphi_{I,jK} d\sigma \right| \]

for all \( \varphi \in D^{p,q}_U \). This inequality is a consequence of the definitions and holds without the assumption of pseudo-convexity. The estimates that we will derive will be valid for \( (p, q) \) forms if and only if they are valid for \( (0, q) \) forms, by noticing that we have given above a norm which is equivalent to \( Q(\cdot, \cdot) \).

5 Tangential Sobolev Spaces and Pseudo-Differential Operators

Consider \( \mathbb{R}^{2n} \) with coordinates \( (t_1, \ldots, t_{2n-1}, r) \), \( r < 0 \). Define the tangential Fourier transform \( \tilde{u} \) of \( u \) by
\[ \tilde{u}(\tau, r) = \left( \frac{1}{\sqrt{2\pi}} \right)^{2n-1} \int_{\mathbb{R}^{2n-1}} e^{-i(t, \tau)} \cdot u(t, r) dt \]
Define the operator $\Lambda^s$ by
\[
(\Lambda^s u)(\tau, r) = (1 + |\tau|^2)^{s/2} \tilde{u}(\tau, r)
\]
or in other words
\[
(\Lambda^s u)(t, r) = \left(\frac{1}{\sqrt{2\pi}}\right)^{2n-1} \int_{\mathbb{R}^{2n-1}} e^{i(t, \tau) \cdot \tau} \cdot (1 + |\tau|^2)^{s/2} \cdot \tilde{u}(\tau, r) d\tau
\]
Define the **tangential Sobolev norm** $||| \cdot |||_s$ by
\[
||| u |||_s^2 := \|\Lambda^s u\|^2 = \|\Lambda^s \tilde{u}\|^2 = \int_{\tau \in \mathbb{R}^{2n-1}} \int_{r = -\infty}^0 (1 + |\tau|^2)^s |\tilde{u}(\tau, r)|^2 dr d\tau
\]
Observe that if $u$ is a differentiable function,
\[
Du = (\sigma(D, i\xi) \tilde{u})^\vee
\]
where $\xi$ is the variable of the phase space. Indeed, we have the following definition:

$P$ is a **tangential pseudo-differential operator** of order $m$ on $C_0^\infty(U \cap D)$ if it can be expressed by
\[
P u(t, r) = \int_{\mathbb{R}^{2n-1}} e^{-i(t, \tau)} p(t, r, \tau) \tilde{u}(\tau, r) d\tau
\]
Here $p \in C^\infty(\mathbb{R}^{2n} \times \mathbb{R}^{2n-1})$, with $r \leq 0$. The function $p$ is called the symbol of $P$ and satisfies the following inequalities, for multiindices $\alpha = (\alpha_1, ..., \alpha_{2n})$, $\beta = (\beta_1, ..., \beta_{2n-1})$ there exists a constant $C = C(\alpha, \beta)$ such that:
\[
|D^\alpha D^\beta_r p(t, r, \tau)| \leq C(1 + |\tau|)^{m-|\beta|}
\]
Both, tangential $s$-norms and tangential pseudo-differential operators have natural extensions to the space $S(\mathbb{R}^{2n})$, i.e., the space of $C^\infty$ functions all of whose derivatives are rapidly decreasing.

**Proposition 5.1.** If $P$ is a tangential pseudo-differential operator of order $m$ then for each $s \in \mathbb{R}$ there exists $C_s > 0$ such that:
\[
||| Pu |||_s \leq C_s ||| u |||_{s+m} \quad \text{for all} \ u \in S(\mathbb{R}^{2n})
\]
Furthermore, if \( P^* \) is the adjoint of \( P \) then \( P^* \) is a tangential pseudo-differential operator of order \( m \) and if \( p \) and \( p^* \) are the symbols of \( P \) and \( P^* \) then \( p - p^* \) is the symbol of an operator of order \( m - 1 \). If \( P' \) is a tangential pseudo-differential operator of order \( m' \) with symbol \( p' \), then \( PP' \) is a tangential pseudo-differential operator of order \( m + m' \); if \( q \) is the symbol of \( PP' \) then \( pp' - q \) is the symbol of an operator of order \( m + m' - 1 \). Hence, the commutator \([P, P'] = PP' - P'P\) has order \( m + m' - 1 \).

Proof. Omitted

**Proposition 5.2.** Suppose \( x_0 \in \partial D \). Then there exists a neighborhood \( U \) of \( x_0 \) and constant \( C > 0 \) such that

\[
||\varphi||^2 \leq C(||\bar{\partial}\varphi||^2 + ||\bar{\partial}^*\varphi||^2 + ||\varphi||^2)
\]

for all \( \varphi \in \mathcal{D}^{p,q}_U \) if and only if there exists a neighborhood \( U' \) of \( x_0 \) and constant \( C' > 0 \) such that

\[
|||\varphi||^2 \leq C'|||\bar{\partial}\varphi||^2 + ||\bar{\partial}^*\varphi||^2 + ||\varphi||^2)
\]

for all \( \varphi \in \mathcal{D}^{p,q}_U \).

Proof. We prove the special case \( \epsilon = 1/2 \). For \( \varphi \in \mathcal{D}^{p,q}_U \), we have

\[
||\varphi||^2 \leq ||\bar{\partial}\varphi||^2 + ||\bar{\partial}^*\varphi||^2 + ||\varphi||^2 + \int_{\partial D} |\varphi|^2 \quad \text{(will be shown in the next section)}
\]

For \( f \in C^\infty(U' \cap \overline{D}) \), we have

\[
\int_{\partial D} |f|^2 = \int_{\mathbb{R}^{2n-1}} |\tilde{f}(\tau, 0)|^2 d\tau
\]

\[
= \int_{\mathbb{R}^{2n-1}} \int_{-\infty}^0 \frac{\partial}{\partial r} |\tilde{f}(\tau, r)|^2 dr d\tau
\]

\[
= \int_{\mathbb{R}^{2n-1}} 2\text{Re} \int_{-\infty}^0 \frac{\partial \tilde{f}(\tau, r)}{\partial r} \cdot \overline{\tilde{f}(\tau, r)} dr d\tau
\]

\[
\leq \int_{\mathbb{R}^{2n-1}} \left( \frac{1}{C} \int_{-\infty}^0 \left| \frac{\partial \tilde{f}(\tau, r)}{\partial r} \right|^2 dr + C \int_{-\infty}^0 |\tilde{f}(\tau, r)|^2 dr \right) d\tau
\]

Choose \( C = (1 + |\tau|^2)^{1/2} \), then we have

\[
\int_{\partial D} |f|^2 \leq |||D_rf|||^2_{1/2} + |||f|||^2_{1/2}
\]

12
Finally, by the ellipticity of $\bar{\partial} \oplus \bar{\partial}^*$, we have
\[
|||D_r f|||_{-\frac{1}{2}} \leq |||\bar{\partial} \varphi|||_{-\frac{1}{2}} + |||\bar{\partial}^* \varphi|||_{-\frac{1}{2}} + |||\Lambda f|||_{-\frac{1}{2}} \leq ||\bar{\partial} \varphi|| + ||\bar{\partial}^* \varphi|| + ||\varphi||_{\frac{1}{2}}
\]
for each component function $f$ of $\varphi$. This finishes the proof. \qed

6 Strongly Pseudoconvex: Subelliptic Estimate

Theorem 6.1 (Subelliptic Estimate). Let $D$ be a bounded domain in $\mathbb{C}^n$ with smooth strongly pseudoconvex boundary $\partial D$, then
\[
||\phi||_{\frac{1}{2}}^2 \leq Q(\phi, \phi)
\]
holds for all $\phi \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$.

Proof. We are going to show that
\[
||\phi||_{\frac{1}{2}}^2 \leq ||\bar{\partial} \phi||^2 + ||\bar{\partial}^* \phi||^2 + ||\phi||^2 + \int_{\partial D} |\phi|^2
\]
for all $\phi \in D^{p,q}$. For the above we just require $D$ to be a bounded domain with smooth boundary. Then the result follows because
\[
\int_{\partial D} |\phi|^2 \leq ||\bar{\partial} \phi||^2 + ||\bar{\partial}^* \phi||^2
\]
by strong pseudo-convexity of $D$.

Indeed, when $f \in C^\infty(\overline{D})$,
\[
||f||_{\frac{1}{2}}^2 \leq \int_D |r||\nabla f|^2 + \int_D |f|^2
\]
For a proof, see [Chen-Shaw].

Also, we have
\[
\int_D |r||\nabla f|^2 = \int_D -r|\nabla f|^2 \leq \text{const.} \left( \int_{\partial D} |f|^2 + \int_D |f|^2 \right) + \text{Re} \int_D r(\Delta f) \bar{f}
\]
by applying Green’s identity \( \int_D u \Delta v - v \Delta u = \int_{\partial D} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \) to \( u = r, v = |f|^2 \). Applying the above to \( \phi \) componentwise, we need only establish that

\[
\left| \int_D r(\Delta f) \bar{f} \right| \leq Q(\phi, \phi)
\]

for every component function \( f \) of \( \phi \). For this, observe that

\[
\int_D r(\Delta f) \bar{f} = 4 \sum_i \int_D r \frac{\partial^2 f}{\partial z_i \partial \bar{z}_i} \bar{f} = -4 \sum_i \int_D \left( \frac{\partial r}{\partial z_i} \frac{\partial f}{\partial \bar{z}_i} + r \frac{\partial f}{\partial \bar{z}_i} \frac{\partial \bar{f}}{\partial z_i} \right)
\]

\[
\leq \sum_i \left| \frac{\partial f}{\partial \bar{z}_i} \right| ||f|| + \sum_i \left| \frac{\partial f}{\partial z_i} \right|^2
\]

\[\square\]

**Theorem 6.2 (Boundary Regularity of Kohn’s Solution).** Let \( D \) be a bounded domain in \( \mathbb{C}^n \) with smooth strongly pseudoconvex boundary \( \partial D \). If \( \bar{\partial} f = 0 \) and \( u = \bar{\partial}^* N f \), then \( f \in H^s \) implies \( u \in H^{s+\frac{1}{2}} \) for all \( s \geq 0 \).

**Proof.** We are going to establish

\[
||\bar{\partial}^* N f||^2_{s+\frac{1}{2}} \leq ||f||^2_s
\]

for all \( s \geq 0 \). The way to do that is first by induction and then by interpolation. See Chapter 5 of [Chen-Shaw]. \[\square\]

### 7 Weakly Pseudoconvex: Subelliptic Multipliers

Now let us consider the case when \( \partial D \) is weakly pseudoconvex.

For \( x_0 \in D \), define \( I^q(x_0) \) (called the **multiplier ideal sheaf** at \( x_0 \) for \( (0,q) \)-forms) as follows:

\( f \in I^q(x_0) \) if and only if \( f \) is a \( C^\infty \) function germ at \( x_0 \) and there exists an open neighborhood \( U \) of \( x_0 \) in \( \mathbb{C}^n \) and positive constants \( \epsilon \) and \( C \) such that

\[
|||f\varphi|||^2_\epsilon \leq CQ(\varphi, \varphi)
\]

(2)
for all \( \varphi \in \mathcal{D}_U^{0,q} \), where \( \mathcal{D}_U^{0,q} \) is the set of all \((0,q)\)-forms on \( D \) with support in \( U \cap \overline{D} \) which are \( C^\infty \) up to \( \partial D \) and which belong to \( \text{Dom}(\bar{\partial}^*) \).

For \( x_0 \in D \), define \( M^q(x_0) \) (called the multiplier ideal module at \( x_0 \) for \((0,q)\)-forms) as follows:

\[ \sigma \in M^q(x_0) \text{ if and only if } \sigma \text{ is a } C^\infty(1,0) \text{-form germ at } x_0 \text{ and there exists an open neighborhood } U \text{ of } x_0 \text{ in } \mathbb{C}^n \text{ and positive constants } \epsilon \text{ and } C \text{ such that} \]

\[ \left\| \text{int}(\bar{\sigma}) \cdot \varphi \right\|_2^2 \leq CQ(\varphi, \varphi) \]

for all \( \varphi \in \mathcal{D}_U^{(0,q)} \), where \( \text{int}(\bar{\sigma}) \cdot \varphi \) is the interior product of \( \bar{\sigma} \) with \( \varphi \) defined by contracting the index of \( \sigma \) with one of the indices of \( \varphi \).

**Theorem 7.1.** If \( D \) is pseudoconvex and if \( x_0 \in \overline{D} \), then \( I^q(x_0) \) and \( M^q(x_0) \) have the following properties:

1. \( x_0 \in D \Rightarrow 1 \in I^q(x_0) \) for all \( q \geq 1 \).
2. \( x_0 \in \partial D \Rightarrow r \in I^q(x_0) \).
3. \( I^q(x_0) \) is an ideal.
4. If \( f \in I^q(x_0) \) and if \( g \in C^\infty(x_0) \) with \( |g| \leq |f| \) in a neighborhood of \( x_0 \), then \( g \in I^q(x_0) \).
5. \( I^q(x_0) = \sqrt{I^q(x_0)} \), its radical ideal.
6. If \( x_0 \in \partial D \) and \( \theta \) is a \( C^\infty \) germ of \((0,1)\)-form at \( x_0 \) with \( \langle \theta, \bar{\partial}r \rangle = 0 \), then \( \text{int}(\theta)\partial\bar{\partial}r \), being a \((1,0)\)-form, lies in \( M^q(x_0) \).
7. \( \partial I^q(x_0) \subset M^q(x_0) \), where \( \partial I^q(x_0) = \{ \partial f : f \in I^q(x_0) \} \).
8. \( \det_{n-q+1} M^q(x_0) \subset I^q(x_0) \).

**Proof.**

1. If \( x_0 \in D \), choose \( U \) so that \( \overline{U} \cap \partial D = \emptyset \), then \( \text{supp}(\varphi) \subset U \) and therefore

\[ \left\| \varphi \right\|_1^2 \leq \left\| \varphi \right\|_2^2 \]

for all \( \varphi \in \mathcal{D}_U^{0,q} \). Hence Equation (2) holds with \( \epsilon = 1 \).
2. We choose \( U \) so that \( r \) is defined on \( U \), and we have, by Equation (1),
\[
\|r\varphi\|_1^2 \leq \|r\varphi\|_2^2
\]
and also
\[
\|r\varphi\|_2^2 \leq \|\varphi\|_2^2 \leq Q(\varphi, \varphi)
\]
where the last inequality follows from Theorem (4.4).

3. It follows from the following inequality: for \( g \in C^\infty(\overline{U}) \) there exists \( C > 0 \) so that:
\[
\|||g||_\epsilon \leq C|||u||_\epsilon
\]
for all \( u \in C^\infty(U \cap \overline{D}) \). Thus, if \( f \in I^q(x_0) \) and \( g \in C^\infty(x_0) \) we can conclude that \( fg \in I^q(x_0) \) by replacing \( u \) with \( f\varphi \) in the above, with \( \varphi \in D^q_0 \) and \( U \) suitably small.

4. It follows from the following lemma:

**Lemma 7.2.** If \( \epsilon \leq 1 \), \( f, g \in C^\infty(U) \) and if \( |g| \leq |f| \), then
\[
\|||gu||_\epsilon \leq ||fu||_\epsilon + \text{const.}||u||
\]
for all \( u \in C^\infty_0(U \cap \overline{D}) \).

**Proof.** The operators \([\Lambda^\epsilon, g] \) and \([f, \Lambda^\epsilon] \) are of order \( \epsilon - 1 \) and hence bounded in \( L^2 \) so that we have
\[
\|||gu||_\epsilon = ||\Lambda^\epsilon(gu)|| = ||g\Lambda^\epsilon u|| + O(||u||)
\]
and
\[
||g\Lambda^\epsilon u|| \leq ||f\Lambda^\epsilon u|| = ||\Lambda^\epsilon(fu)|| + O(||u||) = ||fu||_\epsilon + O(||u||)
\]

5. We first need the following lemma:

**Lemma 7.3.** If \( 0 < \delta \leq \frac{1}{m} \), then there exists \( C > 0 \) such that
\[
\|||gu||_\epsilon^2 \leq ||g^m u||_\epsilon^{2m\delta} + C||u||^2
\]
for all \( u \in C^\infty_0(U \cap \overline{D}) \).
Proof. Proceeding by induction we assume that the left hand side above is bounded by \(|||g^k u|||_{k\delta} + \text{const.}||u||\) for \(k < m\). Using the idea of proof in the previous lemma, for any \(j\), with \(0 \leq j \leq k\) and \((k + j)\delta \leq 1\), we have,

\[
|||g^k u|||_{k\delta}^2 = (\Lambda^j g^k u, \Lambda^{(k-j)} \Lambda^{(k-j)} g^k u) \\
= (\Lambda^{(k+j)} \Lambda^{(k-j)} g^k u) + C ||u|| \cdot ||g^{k-j} u||_{(k-j)\delta} \\
\leq (\Lambda^{(k+j)} \Lambda^{(k-j)} g^k u) + C ||u|| \cdot ||g^{k-j} u||_{(k-j)\delta} \\
\leq |||g^{k+j} u|||_{(k+j)\delta} \cdot |||g^{k-j} u|||_{(k-j)\delta} + C ||u|| \cdot ||g^{k-j} u||_{(k-j)\delta}
\]

For \(m\) even, set \(k = j = m/2\), then

\[
|||g u|||_{\delta}^2 \leq |||g^k u|||_{k\delta}^2 + C ||u||^2 \leq |||g^m u|||_{m\delta} \cdot ||u|| + C ||u||^2 \leq |||g^m u|||_{m\delta}^2 + C ||u||^2
\]

For \(m\) odd, set \(k = (m + 1)/2, j = (m - 1)/2\), then

\[
|||g u|||_{\delta}^2 \leq |||g^k u|||_{k\delta}^2 \leq |||g^m u|||_{m\delta} \cdot ||g u||_{\delta} + C ||u|| \cdot ||g u||_{\delta}
\]
which also yields the desired estimate. \(\square\)

Proof of Property (5): If \(g \in \sqrt{T^q(x_0)}\), then on some neighborhood \(U\) of \(x_0\) we have \(||g||^n \leq \|f\|\), where \(f\) satisfies Equation (2). Hence

\[
|||g \varphi|||_{/m}^2 \leq |||g^m \varphi|||_2^2 + C ||\varphi||^2 \leq |||f \varphi|||_2^2 + C ||\varphi||^2 \leq CQ(\varphi, \varphi)
\]

\(\square\)

6. Suppose \(\bar{\partial} \bar{r} = \sum_{i,j} c_{ij} \cdot w_i \wedge \bar{w}_j\). It suffices to prove the statement for \(\theta = \bar{\omega}_1, \ldots, \bar{\omega}_{n-1}\). That means, we have to show that \(\text{int}(\bar{\omega}_k) \bar{\partial} \bar{r} = \sum_i c_{ik} \cdot \omega_i\) lies in \(M^q(x_0)\) for \(k = 1, 2, \ldots, n - 1\). Therefore we have to show for \(\sigma_k = \sum_i c_{ik} \cdot \omega_i\), we have

\[
|||\text{int}(\sigma_k) \varphi|||_2^2 \leq CQ(\varphi, \varphi)
\]
for all \(\varphi \in \mathcal{D}^{b,q}_{/U}, k = 1, 2, \ldots, n - 1\). Integration by parts gives

\[
\text{int}(\sigma_k) \varphi = \sum_{K} \sum_i c_{ik} \varphi_i K \cdot \bar{w}_K
\]

17
Therefore, letting $\psi_{kK} = \sum_i c_{ik} \varphi_{iK}$, we have to estimate
\[
\left\| \sum_i c_{ik} \varphi_{iK} \right\|^2 = \|\psi_{kK}\|^2_2
\]
Using the method in estimating $\| \cdot \|^2_2$ for strongly pseudoconvex domains, we have
\[
\|\psi_{kK}\|^2_2 \leq \int_{\partial D} |\psi_{kK}|^2 + \int_D |\psi_{kK}|^2 + \sum_j \left\| \frac{\partial \psi_{kK}}{\partial z_j} \right\|^2
\]
\[
\leq \int_{\partial D} |\psi_{kK}|^2 + \|\varphi\|^2_\infty
\]
Therefore we have to show that
\[
\int_{\partial D} |\psi_{kK}|^2 \leq Q(\varphi, \varphi)
\]
For this we consider
\[
\sum_k |\psi_{kK}|^2 = \sum_k \left| \sum_i c_{ik} \varphi_{iK} \right|^2 = \sum_k \sum_i c_{ik} \varphi_{iK} \bar{\psi}_{kK}
\]
\[
\leq \left( \sum_{i,k} c_{ik} \varphi_{iK} \bar{\varphi}_{kK} \right)^{\frac{1}{2}} \left( \sum_{i,k} c_{ik} \psi_{iK} \bar{\psi}_{kK} \right)^{\frac{1}{2}}
\]
\[
\leq \frac{1}{2C} \sum_{i,k} c_{ik} \varphi_{iK} \bar{\varphi}_{kK} + \frac{C}{2} \sum_{i,k} c_{ik} \psi_{iK} \bar{\psi}_{kK}
\]
\[
\leq \frac{1}{2C} \sum_{i,k} c_{ik} \varphi_{iK} \bar{\varphi}_{kK} + \frac{CC'}{2} \sum_k |\psi_{kK}|^2
\]
Choose $C = 1/C'$, then we get
\[
\sum_k |\psi_{kK}|^2 \leq \frac{1}{C} \sum_{i,k} c_{ik} \varphi_{iK} \bar{\varphi}_{kK}
\]
In particular,
\[
|\psi_{kK}|^2 \leq \frac{1}{C} \sum_{i,k} c_{ik} \varphi_{iK} \bar{\varphi}_{kK}
\]
for each $k$. Integrating both sides over $\partial D$ gives the desired inequality.
7. We first state a useful lemma:

**Lemma 7.4.** Let $L_1, \ldots, L_n$ be the special local basis defined in a neighborhood $U$ of $x_0 \in \partial D$. Let $u, v \in C^\infty_0(U \cap \overline{D})$, then we have

$$(L_i u, v) = -(u, \overline{L_i v}) + \delta_m \int_{\partial D} u \overline{v} d\sigma + (u, g_i v)$$

where $g_i \in C^\infty(\overline{U} \cap \overline{D})$.

**Proof.** In terms of a boundary coordinate system we have

$$L_i u = \sum_k a_i^k \frac{\partial u}{\partial t_k} + b_i \frac{\partial u}{\partial r}$$

where $b_i = \delta_m$ on $\partial D$, hence

$$(L_i u, v) = -(u, \overline{L_i v}) + \delta_m \int_{\partial D} u \overline{v} d\sigma$$

where

$$L_i^* v = -\overline{L_i v} - \left( \sum_k \frac{\partial \overline{a_i^k}}{\partial t_k} + \frac{\partial \overline{b_i}}{\partial r} \right) := -\overline{L_i v} + g_i v$$

$\square$

**Proof of Property (7):** Let $\varphi \in D_U^{0,q}$ and $f \in I^q(x_0)$ such that its order of subellipticity is $\epsilon$. We have to show that $\|[\text{int}(\overline{\partial f}) \cdot \varphi]\|_\delta^2 \leq C Q(\varphi, \varphi)$ for some $\delta > 0$.

Indeed,

$$\text{int}(\overline{\partial f}) \cdot \varphi = \sum_K \sum_j' (L_j f) \varphi_{jK} \cdot \overline{\omega_K}$$

Hence,

$$\|[\text{int}(\overline{\partial f}) \cdot \varphi]\|_\delta^2 = \sum_K \|[\sum_j' (L_j f) \varphi_{jK}]\|_\delta^2$$
Let $\psi_K = \sum_j (L_j f) \varphi_{jK} = O(||\varphi||)$. By commuting tangential pseudo-differential operators (as usual), we have

\[ ||\sum_j (L_j f) \varphi_{jK}||^2 = \sum_j \langle \Lambda^\delta ((L_j f) \varphi_{jK}), \Lambda^\delta \psi_k \rangle \]
\[ = \sum_j \langle (L_j f) \Lambda^\delta (\varphi_{jK}), \Lambda^\delta \psi_k \rangle + O(\sum_j ||(L_j f)\varphi_{jK}||_{2\delta - 1} \cdot ||\varphi||) \]
\[ = -\sum_j \langle f \Lambda^\delta \varphi_{jK}, \Lambda^\delta \psi_K \rangle - \sum_j \langle f \Lambda^\delta \psi_K, \Lambda^\delta \varphi_{jK} \rangle + O(\sum_j ||(L_j f)\varphi_{jK}|| \cdot ||\varphi||^2) + O(\sum_j ||\varphi||_{2\delta - 1} \cdot ||\varphi||) \]

There is no boundary term because $\varphi_{nK} = 0$ on $U \cap \partial D$ and tangential pseudo-differential operators does not change the normal components.

Also, we have

\[ \sum_j \langle f L_j \Lambda^\delta \varphi_{jK}, \Lambda^\delta \psi_K \rangle = \sum_j \langle L_j \varphi_{jK}, \Lambda^\Delta (\bar{\psi}_K) \rangle + O\left(\sum_j ||L_j \varphi_{jK}|| \cdot ||\varphi||\right) \]
\[ + O\left(\sum_j ||\Lambda^\Delta (\bar{\psi}_K)|| \cdot ||\varphi||\right) + O\left(\sum_j ||\varphi||_{2\delta - 1} \cdot ||\varphi||\right) \]

and

\[ \sum_j \langle f \Lambda^\delta \varphi_{jK}, \Lambda^\delta \psi_K \rangle = \sum_j \langle \Lambda^\Delta (f \varphi_{jK}), \Lambda^\Delta \psi_K \rangle + O\left(\sum_j ||f \varphi||_{2\delta} \cdot ||\varphi||\right) \]
\[ + O\left(\sum_j ||\varphi||_{2\delta - 1} \cdot ||\Lambda^\Delta \psi_K||\right) + O\left(\sum_j ||\varphi||_{2\delta - 1} \cdot ||\varphi||\right) \]

Therefore it remains to estimate $||\sum_j L_j \varphi_{jK}||^2$, $||\Lambda^\Delta (f \psi_K)||^2$ and $||\Lambda^\Delta (f \psi_K)||^2$. From basic estimate,

\[ ||\sum_j L_j \varphi_{jK}||^2 \leq ||\bar{\partial}^\ast \varphi||^2 + C||\varphi||^2 \]

From definition,

\[ ||\Lambda^\Delta (f \psi_K)||^2 \leq \sum_j \Lambda^\Delta ((L_j f) \varphi_{jK}) \leq O\left(\sum_j ||f \varphi||^2_{2\delta} + ||\varphi||^2_{2\delta - 1}\right) \]

and

\[ ||\Lambda^\Delta (f \psi_K)||^2 \leq \sum_i (\Lambda^\Delta (f \varphi_{iK}))^2 + (L_i f) (\bar{\Lambda}^\Delta \psi_{iK}) \leq C||\varphi||^2 \leq C Q(\varphi, \varphi) \]
Setting $\delta = \epsilon/2$, we have

$$|||\int(\partial f) \cdot \varphi|||^2 \leq CQ(\varphi, \varphi)$$

8. Suppose $\sigma^1, ..., \sigma^{n-q+1} \in M^q(x_0)$. So there is an $\epsilon > 0$ such that

$$|||\int(\sigma^k) \cdot \varphi|||^2 \leq Q(\varphi, \varphi)$$

for each $k$. Integration by part gives

$$\int(\sigma^k) \cdot \varphi = \sum'_K \sum_i \sigma^k_i \varphi_{iK} \cdot \omega_K$$

where $\sigma^k = \sum_j \sigma^k_j \omega_j$. It suffices to show that $\langle \sigma^1 \wedge ... \wedge \sigma^{n-q+1}, \theta \rangle$ lies in $I^q(x_0)$ for $\theta = \omega_{h_1} \wedge ... \wedge \omega_{h_{n-q+1}} (h_1, ..., h_{n-q+1}$ are arbitrary). We then have

$$\langle \sigma^1 \wedge ... \wedge \sigma^{n-q+1}, \omega_{h_1} \wedge ... \wedge \omega_{h_{n-q+1}} \rangle = \det(\sigma^k_{h_j})$$

Let $K$ be the $(q-1)$-tuple that is disjoint from $h_1 < ..... < h_{n-q+1}$. Consider the system of linear equations

$$\sigma^1_{h_1} \varphi_{h_1K} + ... + \sigma^1_{h_{n-q+1}} \varphi_{h_{n-q+1}K} = (\int(\sigma^1) \cdot \varphi)_K$$

$$\vdots$$

$$\sigma^{n-q+1}_{h_1} \varphi_{h_1K} + ... + \sigma^{n-q+1}_{h_{n-q+1}} \varphi_{h_{n-q+1}K} = (\int(\sigma^{n-q+1}) \cdot \varphi)_K$$

where $\varphi_{h_1K}, ..., \varphi_{h_{n-q+1}K}$ are considered unknowns. By Crammer’s rule, we have

$$\det(\sigma^k_{h_j}) \cdot \begin{pmatrix} \varphi_{h_1K} \\ \vdots \\ \varphi_{h_{n-q+1}K} \end{pmatrix} = \text{Adj}(\sigma^k_{h_j}) \begin{pmatrix} (\int(\sigma^1) \cdot \varphi)_K \\ \vdots \\ (\int(\sigma^{n-q+1}) \cdot \varphi)_K \end{pmatrix}$$

21
Hence,
\[ ||| \det(\sigma_h^k) \cdot \varphi_{h^\nu K} |||^2 \leq \left( ||| \int(\overline{\sigma^1}) \cdot \varphi_K |||^2 + \ldots + ||| \int(\overline{\sigma^{n-q+1}}) \cdot \varphi_K |||^2 \right) \]
\[ \leq Q(\varphi, \varphi) \]
for every \( \nu = 1, \ldots, n - q + 1 \). And as \( h^\nu \) and \( K \) can be made arbitrary, we have
\[ ||| \det(\sigma_h^k) \varphi |||^2 \leq Q(\varphi, \varphi) \]

\[ \square \]

**Proposition 7.5.** Let \( x \) be in the zero-set of \( I^q_k(x_0) \). Then \( x \) is in the zero-set of \( I^{q+1}_k(x_0) \) if and only if \( \dim_C (Z_{x}^{1,0}(I^q_k(x_0) \cap N_x)) \) where
\[ Z_{x}^{1,0}(I) = \{ L \in T^{1,0}_{x} : L(I) = 0 \} \]
and
\[ N_x = \{ L \in T^{1,0}_{x}(\partial D) : (\partial \bar{\partial} L) \wedge (\bar{L})_x = 0 \} \]

**Proof.** \[ \square \]

**Corollary 7.6.** Suppose the boundary of a smoothly bounded weakly pseudoconvex domain \( D \) in \( \mathbb{C}^n \) is real-analytic. Then for \( q \geq 1 \), the subelliptic estimate
\[ ||| \varphi |||_\varepsilon^2 \leq C Q(\varphi, \varphi) \]
holds for all \((0,q)\)-forms in the domain of \( \bar{\partial} \) and \( \partial^* \) with some \( \varepsilon > 0 \) and some \( C > 0 \) independent of \( \varphi \).

**Proof.** \[ \square \]

## 8 Kohn’s Algorithm

We now describe Kohn’s algorithm and restrict our attention to \((0,1)\)-forms. This section follows the treatment in [Siu].

(A) Initial Membership.

(I) \( r \in I^1(x_0) \)

(II) \( \sigma_k := \int(\overline{\omega_k})\partial \bar{\partial} r = \sum_i c_{ik} \cdot \omega_i \in M^1(x_0) \) for \( 1 \leq k \leq n - 1 \).
(B) Generation of New Members.

(I) If $f \in I^1(x_0)$, then $\partial f \in M^1(x_0)$

(II) If $\theta_1, \ldots, \theta_{n-1} \in M^1(x_0)$, then the coefficient of $\theta_1 \wedge \cdots \wedge \theta_{n-1} \wedge \partial r$ lies in $I^1(x_0)$.

(C) Real Radical Property.

If $g \in I^1(x_0)$ and $|f|^m \leq |g|$ for some positive integer $m$, then $f \in I^1(x_0)$.

Definition 1 (Order of finite type). The order $m$ at a point $x_0$ of the boundary $\partial D$ of $D$ is the supremum of

$$\frac{\text{ord}_0(r \circ \varphi)}{\text{ord}_0 \varphi}$$

over all local holomorphic curves $\varphi : U \to \mathbb{C}^n$ with $\varphi(0) = x_0$, where $U$ is an open neighborhood of 0 in $\mathbb{C}$ and $\text{ord}_0$ is the vanishing order at the origin 0.

A point $P$ of the boundary $\partial D$ of $D$ is said to be of finite type if $m$ is finite.

Definition 2 (Assigned order of subellipticity). Here we give the rules for assigning order of subellipticity:

1. $r \in I^1(x_0)$ has order 1.

2. $\sigma_k := \int \bar{\omega}_k \partial \bar{\partial} r = \sum_i c_{ik} \cdot \omega_i \in M^1(x_0)$ for $1 \leq k \leq n-1$ has order $\frac{1}{2}$.

3. If $f \in I^1(x_0)$ has order $\epsilon$, then $\partial f \in M^1(x_0)$ has order $\epsilon/2$.

4. If $\theta_1, \ldots, \theta_{n-1} \in M^1(x_0)$ have minimum order $\epsilon$, then the coefficient of $\theta_1 \wedge \cdots \wedge \theta_{n-1} \wedge \partial r \in I^1(x_0)$ has order $\epsilon$.

Notice the above multiplicity is the minimal multiplicity for the subelliptic estimate to hold.

Let us consider the domain $D \subset \mathbb{C}^{n+1}$ whose defining function $r$ is given, near the origin, by:

$$r(z_1, \ldots, z_n, w) = \text{Re}(w) + \sum_{j=1}^N |F_j(z_1, \ldots, z_n)|^2$$
where $F_j$'s are holomorphic functions vanishing at the origin. Then

$$\partial \bar{\partial} r = \sum_{j=1}^{N} \partial F_j \wedge \bar{\partial} F_j = \sum_{j=1}^{N} dF_j \wedge \bar{dF}_j$$

and hence $\partial D$ is pseudoconvex. The boundary point under consideration is the origin of $\mathbb{C}^{n+1}$. Next, notice that $dr = dw$ at the origin, so

$$\text{int}(dz^j) \partial \bar{\partial} r = \partial \bar{\partial} z_j r = \sum_{k=1}^{N} \left( \frac{\partial F_k}{\partial z_j} \right) \cdot \partial F_k = \sum_{k=1}^{N} \left( \frac{\partial F_k}{\partial z_j} \right) \cdot dF_k$$

lies in $M^1(x_0)$ for $1 \leq j \leq n$ and they have assigned order of subellipticities $1/2$.

Next we introduce 3 effectively comparable constants:

1. Let $p$ be the smallest positive integer such that

$$|z|^p \leq C \sum_{j=1}^{N} |F_j(z)|$$

on some neighborhood of the origin for some positive constant $C$.

2. Let $q$ be the smallest positive integer such that

$$(m_{\mathbb{C}^n,0})^q \subset \langle F_1, ..., F_N \rangle := \mathcal{I}$$

where $m_{\mathbb{C}^n,0}$ is the maximal ideal of function germs vanishing at 0.

3. Let $s$ be

$$\dim_{\mathbb{C}} \left( \mathcal{O}_{\mathbb{C}^n,0} / \langle F_1, ..., F_N \rangle \right) = \dim_{\mathbb{C}} \left( \mathcal{O}_{\mathbb{C}^n,0} / \mathcal{I} \right)$$

Indeed we have the following inequalities:

**Proposition 8.1.**

$$m = 2p$$
$$p \leq q$$
$$q \leq (n + 2)p$$
$$q \leq s$$
$$s \leq \binom{n + q - 1}{q - 1}$$
Proof.

$m \leq 2p$: By definition there is a $\psi = (\psi_1, \psi_2) : U \to \mathbb{C} \times \mathbb{C}^n$ where the first argument is $w$ and the second argument is $(z_1, ..., z_n)$ such that

$$m = \frac{\text{ord}_0 (r \circ \psi)}{\text{ord}_0 \psi}$$

Now observe that

$$\frac{\text{ord}_0 (r \circ \psi)}{\text{ord}_0 \psi} = \frac{\min(\text{ord}_0 \psi_1^*(\text{Re}(w)), \text{ord}_0 \psi_2^* \left( \sum_{j=1}^{N} |F_j|^2 \right))}{\min(\text{ord}_0 \psi_1^*(w), \frac{1}{2} \text{ord}_0 \psi_2^* \left( \sum_{j=1}^{n} |z_j|^2 \right))}$$

Let

$$\alpha = \text{ord}_0 \psi_1^*(\text{Re}(w)) = \text{ord}_0 \psi_1^*(w)$$

$$\beta = \frac{1}{2} \text{ord}_0 \psi_2^* \left( \sum_{j=1}^{n} |z_j|^2 \right)$$

$$\gamma = \text{ord}_0 \psi_2^* \left( \sum_{j=1}^{N} |F_j|^2 \right)$$

Clearly, $\gamma \leq 2p\beta$.

When $\alpha \leq \beta$:

$$m = \frac{\alpha}{\alpha} \leq 2p$$

When $\beta < \alpha \leq 2p\beta$:

$$m \leq \frac{2p\beta}{\beta} = 2p$$

When $2p\beta < \alpha$:

$$m \leq \frac{2p\beta}{\beta} = 2p$$

$m \geq 2p$: We use a simultaneous resolution of embedded singularities

$$\pi : \tilde{W} \to W$$
for some open neighborhood $W$ of the origin of $\mathbb{C}^n$ with exceptional divisors $\{Y_j\}_{j=1}^J$ in $\tilde{W}$ in normal crossing so that

$$\pi^*(m_{\mathbb{C}^n,0}) = \sum_{j=1}^J \sigma_j Y_j$$

and

$$\pi^*\mathcal{I} = \sum_{j=1}^J \tau_j Y_j$$

By the definition of $p$, we know that $p\sigma_j \leq \tau_j$ for $1 \leq j \leq J$.

Take $j_0$ with $\sigma_{j_0} > 0$ and $0 \in \pi(Y_{j_0})$ such that there is a regular point $Q$ with the property that $\pi(Q) = 0$ and $Q$ does not belong to any other $Y_j(j \neq j_0)$.

Take a local regular curve $\tilde{C}$ in $\tilde{W}$ represented by a holomorphic map $\tilde{\varphi} : U \to \tilde{W}$ from some neighborhood $U$ of $0 \in \mathbb{C}$ to $\tilde{W}$ such that $\tilde{\varphi}(0) = Q$ and the local complex curve $\tilde{C}$ is transversal to $Y_{j_0}$ and disjoint from any other $Y_j(j \neq j_0)$.

Now we define a holomorphic map $\psi = (\psi_1, \psi_2) : U \to \mathbb{C} \times \mathbb{C}^n$ by $\psi_0 \equiv 0$ and $\psi_1 = \pi \circ \tilde{\varphi}$.

Then

$$\frac{\text{ord}_0(r \circ \psi)}{\text{ord}_0 \psi} = \frac{\text{min}(\text{ord}_0 \psi_1^* w, \text{ord}_0 \psi_1^* \sum_{j=1}^N |F_j|^2)}{\text{min}(\text{ord}_0 \psi_0^* w, \frac{1}{2} \text{ord}_0 \psi_1^* \sum_{j=1}^n |z_j|^2)}$$

$$= \frac{2\tau_{j_0}}{\sigma_{j_0}}$$

$$\geq 2p$$

$p \leq q$: Notice that $z_i^q \in (m_{\mathbb{C}^n,0})^q \subset \langle F_1, ..., F_N \rangle$ for $1 \leq i \leq n$. Hence

$$|z_i|^q \leq C \sum_{j=1}^N |F_j(z)|$$

And so we have

$$|z|^q = \left(\sum_{i=1}^n |z_i|^2\right)^{q/2} \leq n^{q/2} \cdot \max_{1 \leq i \leq n} |z_i|^q \leq C \sum_{j=1}^N |F_j(z)|$$

26
We first state the

**Theorem 8.2 (Skoda’s division theorem).** Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$, $g_1, ..., g_m$ be holomorphic functions on $D$, $\alpha > 1$, $l = \inf(n, m - 1)$. For any holomorphic function $F$ on $D$ such that

$$
\int_D \frac{|F|^2}{(|g|^2)^{\alpha l + 1}} < \infty
$$

there exists $f_1, ..., f_m$ on $D$ such that

$$
F = g_1 f_1 + \cdots g_m f_m
$$

and

$$
\int_D \frac{|f|^2}{(|g|^2)^{\alpha l}} \leq \frac{\alpha}{\alpha - 1} \int_D \frac{|F|^2}{(|g|^2)^{\alpha l + 1}}
$$

We now apply the above theorem to $F(z_1, ..., z_n) = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$ where $\gamma_1 + \cdots + \gamma_n = (n + 2)p$, $\{g_1, ..., g_m\} = \{F_1, ..., F_N, 0, ..., 0\}$, $l = n$, $\alpha = \frac{n + 1}{n}$, $D$ be an open ball neighborhood of $0 \in \mathbb{C}^n$, then we have

$$
z_1^{\gamma_1} \cdots z_n^{\gamma_n} \in \langle F_1, ..., F_N \rangle
$$

and therefore

$$
q \leq (n + 2)p
$$

$q \leq s$:

$$
\mathcal{O}_{\mathbb{C}^n, 0}/\mathcal{I} \supset m_{\mathbb{C}^n, 0}^{1} (\mathcal{O}_{\mathbb{C}^n, 0}/\mathcal{I}) \supset m_{\mathbb{C}^n, 0}^{2} (\mathcal{O}_{\mathbb{C}^n, 0}/\mathcal{I}) \supset \cdots
$$

is a nest of subspace of the vector space $\mathcal{O}_{\mathbb{C}^n, 0}/\mathcal{I}$. So there exists $1 \leq l \leq s$ such that

$$
m_{\mathbb{C}^n, 0}^{l} (\mathcal{O}_{\mathbb{C}^n, 0}/\mathcal{I}) = m_{\mathbb{C}^n, 0}^{l+1} (\mathcal{O}_{\mathbb{C}^n, 0}/\mathcal{I})
$$

By Nakayama’s lemma, we have

$$
0 = m_{\mathbb{C}^n, 0}^{l} (\mathcal{O}_{\mathbb{C}^n, 0}/\mathcal{I})
$$

or

$$
m_{\mathbb{C}^n, 0}^{l} \subset \mathcal{I}
$$

So

$$
q \leq l \leq s
$$

27
Lemma 8.3 (Effective Nullstellensatz). Let $I$ be an ideal in $\mathcal{O}_{\mathbb{C}^n,0}$ such that
\[
dim_{\mathbb{C}} (\mathcal{O}_{\mathbb{C}^n,0}/I) \leq d
\]
Let $f$ be a holomorphic function germ on $\mathbb{C}^n$ at the origin which vanishes at the origin. Then
\[
f^{d^2} \in I
\]

Proof. □

Next we look at 2 applications of the Skoda division theorem:

Proposition 8.4. Let $D$ be a bounded Stein open subset of $\mathbb{C}^n$. Let $g_1,\ldots,g_n, \rho$ be holomorphic functions on some open neighborhood $\overline{D}$ of $D$. Let $Z$ be the zero set of $\{g_1,\ldots,g_n\}$ and $\rho$ vanishes on $Z$. Let $J$ be the Jacobian determinant of $g_1,\ldots,g_n$, then there exist holomorphic functions $h_1,\ldots,h_n$ on $D$ such that
\[
\rho J = \sum_{j=1}^{n} h_j g_j
\]

Proof. First notice that as $\rho$ vanishes on $Z$, we can choose $0 < \eta < 1$ such that
\[
\frac{|\rho|^2}{(\sum_{j=1}^{n} |g_j|^2)^m}
\]
is bounded on a neighborhood $U$ of $\overline{D}$. Consider the finite integral
\[
\int_K \frac{\prod_{j=1}^{n} \sqrt{-1} dw_j \wedge d\overline{w_j}}{(\sum_{j=1}^{n} |w_j|^2)^{(1-\frac{d}{2})m}}
\]

28
and its pullback by the map

\[(z_1, \ldots, z_n) \mapsto (w_1, \ldots, w_n) = (g_1, \ldots, g_n)\]

we know that

\[
\int_D \frac{|J|^2}{(\sum_{j=1}^n |g_j|^2)^{1-n/2}} < \infty
\]

Together with the boundedness of

\[
\frac{|\rho|^2}{(\sum_{j=1}^n |g_j|^2)^{\eta n}}
\]

we conclude that

\[
\int_D \frac{|\rho J|^2}{(\sum_{j=1}^n |g_j|^2)^{(1+\frac{\eta}{2})n}} < \infty
\]

Applying Skoda’s theorem concludes the proof. \(\square\)

**Proposition 8.5.** Let \(f\) be a holomorphic function germ in \(\mathcal{O}_{\mathbb{C}^n,0}\) which vanishes at 0. Then \(f^{n+1}\) belongs to the ideal

\[
\left\langle \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right\rangle
\]

**Proof.** By considering the resolution of singularities of the neighborhood \(U\) of 0 on which the holomorphic function germ \(f\) is defined, we know that on a compact neighborhood of 0,

\[
\frac{|f|^2}{\sum_{j=1}^n |\frac{\partial f}{\partial z_j}|^2}
\]

is uniformly bounded and \(|f|^2\) has a slightly larger vanishing order than that of \(\left| \frac{\partial f}{\partial z_j} \right|^2\). Hence,

\[
\int_{U'} \frac{|f^{n+1}|^2}{\left( \sum_{j=1}^n |\frac{\partial f}{\partial z_j}|^2 \right)^{\alpha(n+1)}} < \infty
\]

for some \(\alpha > 1\). Applying Skoda’s theorem concludes the proof. \(\square\)
**Example.** Consider the following example in $\mathbb{C}^{2+1}$. Let

$$r(z_1, z_2, w) = \text{Re}(w) + |z_1^M|^2 + |z_2^N + z_2 z_1^K|^2$$

near the origin, where $K > M \geq 2$ and $N \geq 3$.

The order of finite type is

$$2 \cdot \max(M, N)$$

As for the ideal $\langle F_1, F_2 \rangle$,

$$\dim_{\mathbb{C}} \left( \mathcal{O}_{\mathbb{C}^2, 0} / \langle F_1, F_2 \rangle \right) = MN$$

where $F_1(z_1, z_2) = z_1^M$ and $F_2(z_1, z_2) = z_2^N + z_2 z_1^K$. As the first step in Kohn’s algorithm, we get

$$Mz_1^{M-1}dz_1^1 + 0 \cdot dz_2^1$$

and

$$dF_2 = \frac{\partial F_2}{\partial z_1} dz_1^1 + \frac{\partial F_2}{\partial z_2} dz_2^1$$

as generators of $M_1$, the first multiplier ideal module. Next, taking determinant gives

$$z_1^{M-1} \cdot \frac{\partial F_2}{\partial z_2}$$

as generator of $J_1$, the first multiplier ideal sheaf without taking radicals. Taking radicals give

$$z_1 \cdot \frac{\partial F_2}{\partial z_2}$$

as generator of $I_1$, the first multiplier ideal sheaf. Next we take $\partial$ to get a new generator

$$\left( \frac{\partial F_2}{\partial z_2} + z_1 \frac{\partial^2 F_2}{\partial z_2 \partial z_1} \right) dz_1^1 + \left( z_1 \frac{\partial^2 F_2}{\partial z_2 \partial z_1} \right) dz_2^1$$

for $M_2$. So $J_2$ is generated by

$$z_1 \cdot \frac{\partial F_2}{\partial z_2}$$
and the 2 new determinants
\[ z_1^M \cdot \frac{\partial^2 F_2}{\partial z_2 \partial z_2} \]
and
\[ z_1 \cdot \frac{\partial F_2}{\partial z_1} \cdot \frac{\partial^2 F_2}{\partial z_2^2} - z_1 \cdot \frac{\partial F_2}{\partial z_2} \cdot \frac{\partial^2 F_2}{\partial z_2 \partial z_1} - \left( \frac{\partial F_2}{\partial z_2} \right)^2 \]
And by writing out those partial derivatives explicitly, we have \( z_1^{2K+2} \) and \( z_2^{2N-2} \) in \( J_2 \). Therefore \( I_2 = m_{\mathbb{C}^2,0} = \langle z_1, z_2 \rangle \) and so \( M_2 \) contains \( dz_1 \) and \( dz_2 \). Hence \( I_3 = \langle 1 \rangle \).

Now suppose \( z^K_1 \in J_2 \). Then there exists \( a(z_1, z_2) \), \( b(z_1, z_2) \) and \( c(z_1, z_2) \) such that
\[
z^K_1 = a(z_1, z_2) \cdot \left( z_1 \cdot \frac{\partial F_2}{\partial z_2} \right) + b(z_1, z_2) \cdot \left( z^M_1 \cdot \frac{\partial^2 F_2}{\partial z_2 \partial z_2} \right) + c(z_1, z_2) \cdot \left( z_1 \cdot \frac{\partial F_2}{\partial z_1} \cdot \frac{\partial^2 F_2}{\partial z_2 \partial z_2} - z_1 \cdot \frac{\partial F_2}{\partial z_2} \cdot \frac{\partial^2 F_2}{\partial z_2 \partial z_1} - \left( \frac{\partial F_2}{\partial z_2} \right)^2 \right) \]
Putting \( z_2 = 0 \) implies
\[
z^K_1 = a(z_1, 0) z^{K+1}_1 + c(z_1, 0) \left[ -(K + 1) z_1^{2K} \right] \]
but that means \( a(z_1, 0) = 0 = c(z_1, 0) \) and therefore it is a contradiction. So, \( z^K_1 \notin J_2 \). That means as we go from \( J_2 \) to \( I_2 \), we need to take at least \((K + 1)\)-th roots. That shows in particular that Kohn’s algorithm is not effective in terms of the order of finite type and the multiplicity of the ideal.
References

[Catlin 84] D. Catlin, Boundary invariants of pseudoconvex domain. Annals of Mathematics, 120 (1984), 529-586.

[Catlin 87] D. Catlin, Subelliptic estimates for the $\bar{\partial}$-Neumann problem on pseudoconvex domains. Annals of Mathematics, 126 (1987), 131-191.

[Chen-Shaw] S.-C. Chen, M.-C. Shaw, Partial Differential Equations in Several Complex Variables. AMS/IP Studies in Advanced Mathematics, Volume 19, 2001.

[D’Angelo] J. P. D’Angelo, Several Complex Variables and the Geometry of Real Hypersurfaces. CRC Press, Boca Raton, 1992.

[Folland-Kohn] G. B. Folland, J. J. Kohn, The Neumann Problem for the Cauchy-Riemann Complex. (AM-75). Princeton University Press, 1972.

[Kohn 63] J. J. Kohn, Harmonic Integrals on Strongly Pseudo-Convex Manifolds. Annals of Mathematics. Second Series, Vol. 78, No. 1 (Jul., 1963), pp. 112-148.

[Kohn 79] J. J. Kohn, Subellipticity of the $\bar{\partial}$-Neumann problem on pseudoconvex domains: sufficient conditions. Acta Math. 142 (1979), 79-122.

[Siu] Y.-T. Siu, Effective Termination of Kohn’s Algorithm for Subelliptic Multipliers. Pure and applied mathematics quarterly (Impact Factor: 0.64). 07/2007; DOI: 10.4310/PAMQ.2010.v6.n4.a11

[Skoda] H. Skoda, Application des techniques $L^2$ à la théorie des idéaux d’une algèbre de fonctions holomorphes avec poids. Ann. Sci. École Norm. Sup. 5 (1972), 548-580.