Perspectives and completely positive maps

Frank Hansen

August 7, 2016

Abstract

We study the filtering of the perspective of a regular operator map of several variables through a completely positive linear map. By this method we are able to extend known operator inequalities of two variables to several variables; with applications in the theory of operator means of several variables. We also extend Lieb-Ruskai’s convexity theorem from two to \( n + 1 \) operator variables.

MSC2010 classification: 47A63

Key words and phrases: partial traces of operator means; Lieb-Ruskai’s convexity theorem for several variables.

1 Introduction

We study the filtering of a regular operator map through a completely positive linear map \( \Phi \). A main result is the inequality

\[
F(\Phi(A_1), \ldots, \Phi(A_k)) \leq \Phi(F(A_1, \ldots, A_k)),
\]

where \( A_1, \ldots, A_k \) are positive definite operators on a Hilbert space of finite dimensions, and \( F \) is a convex regular operator map of \( k \) variables. If \( G_k \) denotes any of the various geometric means of \( k \) variables studied in the literature we obtain as a special case the inequality

\[
\Phi(G_k(A_1, \ldots, A_k)) \leq G_k(\Phi(A_1), \ldots, \Phi(A_k)).
\]

This inequality extends result in the literature for \( k = 2 \), for geometric means of \( k \) variables that may be obtained inductively by the power mean of two
variables, and for means that are limits of such means, including the Karcher mean [3].

We extend Lieb-Ruskai’s convexity theorem from two to \( n + 1 \) operator variables. For \( n = 2 \) we obtain in particular that the map

\[
L(A, B, C) = C^* B^{-1/2} \left( B^{1/2} A^{-1} B^{1/2} \right)^{1/2} B^{-1/2} C
\]

is convex in arbitrary \( C \) and positive definite \( A \) and \( B \). In addition,

\[
L(\Phi(A), \Phi(B), \Phi(C)) \leq \Phi(L(A, B, C))
\]

for any completely positive linear map \( \Phi \) between operators acting on finite dimensional Hilbert spaces. In particular, this includes quantum channels and partial traces. For commuting \( A \) and \( B \) the generalised Lieb-Ruskai map reduces to

\[
L(A, B, C) = C^* A^{-1/2} B^{-1/2} C.
\]

In particular, \( L(A, A, C) = C^* A^{-1} C \).

### 2 Preliminaries

Let \( D \subseteq B(H) \times \cdots \times B(H) \) be a convex domain, where \( B(H) \) is the algebra of bounded linear operators on a Hilbert space \( H \).

We defined [10, Definition 2.1] the notion of a regular map \( F: D \rightarrow B(H) \), generalising the notion of a spectral function of Davis for functions of one variable, the notion of a regular matrix map of two variables by the author [7], and the notion of a regular operator map of two variables [5, Definition 2.1] by Effros and the author. Loosely speaking, a regular map is unitarily invariant and reduces block matrices in a simple and natural way. It retains regularity when compressed to a subspace.

Although we often restrict the study to finite dimensional spaces it is convenient to consider only such regular maps that may be defined also on an infinite dimensional Hilbert space \( H \). Since \( H \) in this case is isomorphic to \( H \oplus H \) this allows us to use block matrix techniques without imposing dimension conditions. Furthermore, it implies that a regular map is well-defined regardless of the underlying Hilbert space. We may thus port a regular map unambiguously from one Hilbert space to another. In this paper we only consider domains of the form,

\[
D^k(H) = \{(A_1, \ldots, A_k) \mid A_1, \ldots, A_k \geq 0\},
\]
of $k$-tuples of positive semi-definite operators, or domains,

$$\mathcal{D}^k_+(\mathcal{H}) = \{(A_1, \ldots, A_k) \mid A_1, \ldots, A_k > 0\},$$

of $k$-tuples of positive definite and invertible operators acting on a Hilbert space $\mathcal{H}$. The latter is the natural type of domain for perspectives.

### 2.1 Jensen’s inequality for regular operator maps

The following result was proved for $\mathcal{H} = \mathcal{K}$ in [10, Theorem 2.2 (i)]. It is just an exercise to generalise the statement and obtain the following:

**Lemma 2.1.** Let $F: \mathcal{D}^k(\mathcal{H}) \to B(\mathcal{H})$ be a convex regular map, and take a contraction $C: \mathcal{H} \to \mathcal{K}$ of $\mathcal{H}$ into a Hilbert space $\mathcal{K}$. If $F(0, \ldots, 0) \leq 0$ then the inequality

$$F(C^*A_1C, \ldots, C^*A_kC) \leq C^*F(A_1, \ldots, A_k)C$$

holds for $k$-tuples $(A_1, \ldots, A_k)$ in $\mathcal{D}^k(\mathcal{K})$.

The next result reduces to [10, Theorem 2.2 (ii)] for $\mathcal{H} = \mathcal{K}$ and $n = 2$. Since the generalisation is quite straightforward we leave the proof to the reader.

**Theorem 2.2** (Jensen’s inequality for regular operator maps).

Let $F: \mathcal{D}^k(\mathcal{H}) \to B(\mathcal{H})$ be a convex regular map and let $C_1, \ldots, C_n: \mathcal{H} \to \mathcal{K}$ be mappings of $\mathcal{H}$ into (possibly another) Hilbert space $\mathcal{K}$ such that

$$C_1^*C_1 + \cdots + C_n^*C_n = 1_{\mathcal{H}}.$$

Then the inequality

$$F\left(\sum_{i=1}^n C_i^*A_{i_1}C_i, \ldots, \sum_{i=1}^n C_i^*A_{i_k}C_i\right) \leq \sum_{i=1}^n C_i^*F(A_{i_1}, \ldots, A_{i_k})C_i$$

holds for $k$-tuples $(A_{i_1}, \ldots, A_{i_k})$ in $\mathcal{D}^k(\mathcal{K})$ for $i = 1, \ldots, n$.

**Corollary 2.3.** Let $\Phi: B(\mathcal{H}) \to B(\mathcal{K})$ be a completely positive unital linear map between operators on Hilbert spaces of finite dimensions, and let $F$ be a convex regular map. Then

$$F(\Phi(A_1), \ldots, \Phi(A_k)) \leq \Phi\left(F(A_1, \ldots, A_k)\right)$$

for $(A_1, \ldots, A_k) \in \mathcal{D}_k(\mathcal{H})$. 

3
Proof. By Choi’s decomposition theorem there exist operators $C_1, \ldots, C_n$ in $B(\mathcal{K}, \mathcal{H})$ with $C_1^* C_1 + \cdots + C_n^* C_n = 1_{\mathcal{K}}$ such that
$$ \Phi(A) = \sum_{i=1}^{n} C_i^* A C_i \quad \text{for} \quad A \in B(\mathcal{H}). $$

The statement now follows by the preceding theorem by choosing
$$(A_{i1}, \ldots, A_{ik}) = (A_1, \ldots, A_k)$$
for $i = 1, \ldots, n$. QED

Davis [4, Corollary] proved that $f(\Phi(A)) \leq \Phi(f(A))$ for an operator convex function $f$ with $f(0) = 0$ and a completely positive linear map $\Phi$ with $\Phi(1) \leq 1$. Jensen’s operator inequality is the slightly more general statement
$$ f\left(\sum_{i=1}^{n} C_i^* A_i C_i\right) \leq \sum_{i=1}^{n} C_i^* f(A_i) C_i $$
for tuples $(A_1, \ldots, A_n)$ and operators $C_1, \ldots, C_n$ with $C_1^* C_1 + \cdots + C_n^* C_n = 1$, see [2] Theorem 2.1 (iii)] and [8]. Jensen’s inequality for regular operator maps may in the same way be considered a generalisation of Corollary 2.3.

3 Perspectives

We introduced the perspective [10, Definition 3.1] of a regular operator map of $k$ variables as a generalisation of the operator perspective of a function of one variable defined by Effros [6]. A key result is that the perspective $\mathcal{P}_F$ of a convex regular operator map $F: \mathcal{D}^k_+(\mathcal{H}) \to B(\mathcal{H})$ of $k$ variables is a convex positively homogenous regular operator map of $k + 1$ variables [10] Theorem 3.2.

Theorem 3.1. Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a completely positive linear map between operators on Hilbert spaces of finite dimensions, and let $F : \mathcal{D}^k_+(\mathcal{H}) \to B(\mathcal{H})$ be a convex regular map. Then

$$ \mathcal{P}_F(\Phi(A_1), \ldots, \Phi(A_{k+1})) \leq \Phi(\mathcal{P}_F(A_1, \ldots, A_{k+1})), $$

for operators $(A_1, \ldots, A_{k+1})$ in $\mathcal{D}^k_+(\mathcal{H})$, where $\mathcal{P}_F$ is the perspective of $F$. 

Proof. We extend an idea of Ando \cite{Ando} from functions of one variable to regular operators maps. To a fixed positive definite $B \in B(H)$ we set

$$
\Psi(X) = \Phi(B)^{-1/2} \Phi(B^{1/2} X B^{1/2}) \Phi(B)^{-1/2}
$$

and notice that $\Psi: B(H) \rightarrow B(K)$ is a unital linear map. By the definition of complete positivity, we realise that also $\Psi$ is completely positive. Since $F$ is convex we may thus apply Corollary \cite{Luo, Ando} and obtain

$$
F\left(\Psi(B^{-1/2} A_1 B^{-1/2}), \ldots, \Psi(B^{-1/2} A_k B^{-1/2})\right) \leq \Psi\left(F\left(B^{-1/2} A_1 B^{-1/2}, \ldots, B^{-1/2} A_k B^{-1/2}\right)\right).
$$

Inserting $\Psi$ we obtain the inequality

$$
F\left(\Phi(B)^{-1/2} \Phi(A_1) \Phi(B)^{-1/2}, \ldots, \Phi(B)^{-1/2} \Phi(A_k) \Phi(B)^{-1/2}\right) \leq \Phi\left(F\left(B^{1/2} F(B^{-1/2} A_1 B^{-1/2}, \ldots, B^{-1/2} A_k B^{-1/2}) B^{1/2}\right) \Phi(B)^{-1/2}\right).
$$

By multiplying from the left and from the right with $\Phi(B)^{1/2}$ we obtain

$$
P_F(\Phi(A_1), \ldots, \Phi(A_k), \Phi(B)) = \Phi(B)^{1/2} F(\Phi(B)^{-1/2} \Phi(A_1) \Phi(B)^{-1/2}, \ldots, \Phi(B)^{-1/2} \Phi(A_k) \Phi(B)^{-1/2}) \Phi(B)^{1/2} \leq \Phi\left(F\left(B^{1/2} F(B^{-1/2} A_1 B^{-1/2}, \ldots, B^{-1/2} A_k B^{-1/2}) B^{1/2}\right)\right) = \Phi\left(P_F(A_1, \ldots, A_k, B)\right),
$$

which is the assertion. QED

Notice that we do not require $\Phi$ to be unital or trace preserving in the above theorem.

Theorem 3.2. Let $\Phi: B(H) \rightarrow B(K)$ be a completely positive linear map between operators on Hilbert spaces of finite dimensions, and let $F: D_{+}^{k+1}(H) \rightarrow B(H)$ be a convex and positively homogeneous regular map. Then

$$
F(\Phi(A_1), \ldots, \Phi(A_{k+1})) \leq \Phi\left(F(A_1, \ldots, A_{k+1})\right)
$$

for positive definite $A_1, \ldots, A_{k+1} \in B(H)$.
\textbf{Proof.} We proved elsewhere \cite[Proposition 3.3]{10} that a convex and positively homogeneous regular mapping $F$ of $k + 1$ variables is the perspective of its restriction
\[ G(A_1, \ldots, A_k) = F(A_1, \ldots, A_k, 1) \]
to $k$ variables. Since $G : \mathcal{D}_+^k(\mathcal{H}) \to B(\mathcal{H})$ is convex and regular the assertion follows from Theorem 3.1. QED

\textbf{Remark 3.3.} A geometric mean $G$ of several variables is an example of a concave positively homogeneous regular map. The inequality in Theorem 3.2 thus reduces to
\[ G(\Phi(A_1), \ldots, \Phi(A_k)) \geq \Phi(G(A_1, \ldots, A_k)). \]
This result was proved \cite[Theorem 4.1]{3} for all geometric means that may be obtained inductively by an application of the power mean of two variables. By a limiting argument this was then extended to the Karcher mean. However, there exist geometric means that cannot be obtained in this way, for example the means introduced in \cite[Section 4.2]{10}.

\section{Lieb-Ruskai’s convexity theorem}

Lieb and Ruskai \cite[Theorem 1]{12} proved convexity of the map
\[ L(A, K) = K^*A^{-1}K \]
in pairs $(A, K)$ of bounded linear operators on a Hilbert space, where $A$ is positive definite. Subsequently, Ando gave a very elegant proof of this result \cite[Theorem 1]{11}. If $K$ is positive definite we may write
\[ KA^{-1}K = K^{1/2}(K^{-1/2}AK^{-1/2})^{-1}K^{1/2} \]
as the perspective of the function $t \to t^{-1}$. Since this function is operator convex, we obtain convexity of the perspective $L(A, K)$, if $K$ is restricted to positive definite operators. This however is enough to obtain the general result. Indeed, the set of $(K, A)$ where $\|K\| < 1$ and $A \geq 1$ is convex, and the embedding
\begin{equation}
K \to \begin{pmatrix} A & K^* \\ K & A \end{pmatrix} > 0
\end{equation}
is affine into positive definite operators. It thus follows that

\[(K, A) \rightarrow \begin{pmatrix} A & K^* \\ K & A \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}^{-1} \begin{pmatrix} A & K^* \\ K & A \end{pmatrix}
\]

= \begin{pmatrix} A + K^* A^{-1} K & 2K^* \\ 2K & A + K A^{-1} K^* \end{pmatrix}

is convex in the specified set. In particular, \((K, A) \rightarrow K^* A^{-1} K\) is convex.

M.B. Ruskai kindly informed the author that Lieb and Ruskai obtained their much cited convexity result unaware that it was proved much earlier in another context by Kiefer [11].

Proposition 4.1. Let \(\Phi: B(H) \rightarrow B(K)\) be a completely positive linear map between operators on Hilbert spaces of finite dimensions. The inequality

\[\Phi(K)^* \Phi(A)^{-1} \Phi(K) \leq \Phi(K^* A^{-1} K)\]

is valid for positive definite \(A\) and arbitrary \(K\).

Proof. If we restrict \(K\) to positive definite operators the first inequality is already contained in Theorem 3.1. The same block matrix construction as in (1) applied to the completely positive linear map \(\Phi \otimes 1_2\) then leads to the inequality

\[\Phi(A) + \Phi(K^*) \Phi(A)^{-1} \Phi(K) \leq \Phi(A + K^* A^{-1} K)\]

for \(A \geq 1\) and \(\|K\| < 1\), and the statement follows. QED

Notice that the above inequality was obtained in [11, Corollary 3.1] if \(K\) is positive definite; cf. also [12, Theorem 2 and Theorem 3].

There is another way to consider Lieb-Ruskai’s convexity theorem which points to generalisations of the result to more than two operators. The geometric mean \(G_1\) of one positive definite operator is trivially given by \(G_1(A) = A\). It is a concave regular map and its inverse

\[A \rightarrow G_1(A)^{-1} = A^{-1}\]

is thus a convex regular map. The perspective

\[\mathcal{P}_{G_1^{-1}}(A, B) = B^{1/2} G_1(B^{-1/2} A B^{-1/2})^{-1/2} B^{1/2} = B A^{-1} B = L(A, B)\]

is therefore a convex regular map by [10, Theorem 3.2], and it is increasing when filtered through a completely positive linear map by Theorem 3.1. A similar construction may be carried out for any number of operator variables.
**Theorem 4.2.** Let $G_n$ be an extension of the function

$$(t_1, \ldots, t_n) \rightarrow t_1^{1/n} \cdots t_n^{1/n} \quad t_1, \ldots, t_n > 0$$

to an operator map defined in positive definite invertible operators on a Hilbert space $H$. Furthermore, suppose that $G_n$ is a positively homogeneous regular operator map which is concave, self-dual and congruence invariant, cf. the discussions in [2] and [10]. The operator map

$$L(A_1, \ldots, A_n, C) = C G_n(A_1, \ldots, A_n)^{-1} C,$$

is then convex in positive definite and invertible operators.

**Proof.** The (geometric) mean $G_n$ is a positive, concave, and regular map. The inverse

$$G_n(A_1, \ldots, A_n)^{-1} = G_n(A^{-1}, \ldots, A_n^{-1})$$

is therefore convex and regular. The perspective

$$P_{G_n^{-1}}(A_1, \ldots, A_n, C)$$

$$= C^{1/2} G_n(C^{-1/2} A_1 C^{-1/2}, \ldots, C^{-1/2} A_n C^{-1/2})^{-1/2}$$

$$= C^{1/2} G_n(C^{1/2} A_1^{-1} C^{1/2}, \ldots, C^{1/2} A_n^{-1} C^{1/2}) C^{1/2}$$

$$= C G_n(A_1^{-1}, \ldots, A_n^{-1}) C = C G_n(A_1, \ldots, A_n)^{-1} C$$

$$= L(A_1, \ldots, A_n, C),$$

where we used self-duality and congruence invariance of the geometric mean. It now follows, by [10, Theorem 3.2], that $L$ is a convex regular map. QED

**Remark 4.3.** It is interesting to notice that Theorem 4.2 alternatively may be obtained by adapting the arguments of Ando in [1, Theorem 1], and that this way of reasoning even imparts convexity of the map

$$L(A, B, C) = C^* G_2(A, B)^{-1} C,$$

where $C$ now is arbitrary and $A, B$ are positive definite and invertible. The argument uses the well-known fact that a block matrix of the form

$$\begin{pmatrix} A & C \\ C^* & B \end{pmatrix},$$
where \( A \) is positive definite and invertible, is positive semi-definite if and only if \( B \geq C^*A^{-1}C \). Indeed, by taking \( \lambda \in [0,1] \) and setting
\[
C = \lambda C_1 + (1 - \lambda)C_2 \\
T = \lambda C^*_1G_2(A_1, B_1)^{-1}C_1 + (1 - \lambda)C^*_2G_2(A_2, B_2)^{-1}C_2
\]
we obtain the equality
\[
X = \begin{pmatrix}
\lambda G_2(A_1, B_1) + (1 - \lambda)G_2(A_2, B_2) & C^* \\
C^* & T
\end{pmatrix}
= \lambda \begin{pmatrix}
G_2(A_1, B_1) & C_1 \\
C^*_1G_2(A_1, B_1)^{-1}C_1 & C^*_2G_2(A_2, B_2)^{-1}C_2
\end{pmatrix}
+ (1 - \lambda) \begin{pmatrix}
G_2(A_2, B_2) & C_2 \\
C^*_2G_2(A_2, B_2)^{-1}C_2 & C^*_1G_2(A_1, B_1)^{-1}C_1
\end{pmatrix}.
\]
Since the two last block matrices by construction are positive semi-definite, we obtain that the block matrix \( X \) is positive semi-definite. Therefore,
\[
T \geq C^*\left( \lambda G_2(A_1, B_1) + (1 - \lambda)G_2(A_2, B_2) \right)^{-1}C.
\]
We thus obtain
\[
\lambda L(A_1, B_1, C_1) + (1 - \lambda)L(A_2, B_2, C_2)
= \lambda C^*_1G_2(A_1, B_1)^{-1}C_1 + (1 - \lambda)C^*_2G_2(A_2, B_2)^{-1}C_2 = T
\geq C^*\left( \lambda G_2(A_1, B_1) + (1 - \lambda)G_2(A_2, B_2) \right)^{-1}C
\geq C^*G_2(\lambda A_1 + (1 - \lambda)A_2, \lambda B_1 + (1 - \lambda)B_2)^{-1}C
= L(\lambda A_1 + (1 - \lambda)A_2, \lambda B_1 + (1 - \lambda)B_2, \lambda C_1 + (1 - \lambda C_2),
\]
where we in the last inequality used concavity of the geometric mean and operator convexity of the inverse function.

It seems mysterious that we in the last proof only used concavity of \( G_2 \) while we in Theorem 4.2 used self-duality and congruence invariance in addition. However, if we want \( L(A, B, C) \) to be positively homogeneous, then \( G_2 \) must have the same property; and if we also want \( G_2 \) to be an extension of the geometric mean of positive numbers, then the geometric mean of two operators is the only solution satisfying all these requirements, cf. [10, Proposition 3.3]. This way of reasoning extends to any number of variables and we obtain:
Corollary 4.4. Let $G_n$ be any geometric mean of $n$ positive semi-definite and invertible operators. The operator function

$$L(A_1, \ldots, A_n, C) = C^*G_n(A_1, \ldots, A_n)^{-1}C$$

is convex in arbitrary $C$ and positive definite and invertible $A_1, \ldots, A_n$ acting on a Hilbert space.

Corollary 4.5. Let $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a completely positive linear map between operators on Hilbert spaces of finite dimensions. The inequality

$$L(\Phi(C), \Phi(A_1), \ldots, \Phi(A_n)) \leq \Phi(L(C, A_1, \ldots, A_n))$$

is valid for positive definite $A_1, \ldots, A_n$ and $C$.

It is known that the geometric mean of two variables is the unique extension of the function $(t, s) \to t^{1/2}s^{1/2}$ to a positively homogeneous, regular and concave operator map [5]. Therefore,

$$L(A, B, C) = CB^{-1/2}(B^{-1/2}A^{-1}B^{1/2})^{1/2}B^{-1/2}C$$

is the only sensible extension of Lieb-Ruskai’s map to three positive definite and invertible operators with symmetry condition $L(A, B, C) = L(B, A, B)$. Without the symmetry condition there are other solutions. The weighted geometric mean,

$$G_2(\alpha; A, B) = B^{1/2}(B^{-1/2}AB^{-1/2})^{\alpha}B^{1/2} \quad 0 \leq \alpha \leq 1,$$

is the perspective of the operator concave function $t \to t^\alpha$ and is therefore concave and congruent invariant [7, 5]. It is also manifestly self-dual. We can therefore apply a proof similar to the one used in the preceding theorem and obtain that the map

$$L(\alpha; A, B, C) = CB^{-1/2}(B^{1/2}A^{-1}B^{1/2})^{\alpha}B^{-1/2}C$$

is convex in positive semi-definite and invertible operators. Furthermore, it is positively homogeneous and therefore increasing when filtered through a completely positive linear map between operators on finite dimensional Hilbert spaces. It reduces to

$$L(\alpha; A, B, C) = CA^{-\alpha}B^{-(1-\alpha)}C$$
for commuting $A$ and $B$.

It is known that there for $n \geq 3$ exist many different extensions of the real function $(t_1, \ldots, t_n) \rightarrow t_1^{1/n} \cdots t_n^{1/n}$ to an operator mapping $G_n$ satisfying the conditions in the preceding theorem, cf. [10]. Notice that if $A_1, \ldots, A_n$ commute then

$$L(A_1, \ldots, A_n, C) = C^* A_1^{-1/n} \cdots A_n^{-1/n} C$$

and in particular $L(A, \ldots, A, C) = C^* A^{-1} C$.

Acknowledgments This work was supported by the Japanese Grant-in-Aid for scientific research 26400104.

References

[1] T. Ando. Concavity of certain maps of positive definite matrices and applications to Hadamard products. *Linear Algebra Appl.*, 26:203–241, 1979.

[2] T. Ando, C.-K. Li, and R. Mathias. Geometric means. *Linear Algebra Appl.*, 385:305–334, 2004.

[3] R. Bhatia and R.L. Karandikar. Monotonicity of the matrix geometric mean. *Math. Ann.*, 353:1453–1567, 2012.

[4] C. Davis. A Schwarz inequality for convex operator functions. *Proc. Amer. Math. Soc.*, 8:42–44, 1957.

[5] E. Effros and F. Hansen. Non-commutative perspectives. *Annals of Functional Analysis*, 5(2):74–79, 2014.

[6] E.G. Effros. A matrix convexity approach to some celebrated quantum inequalities. *Proc. Natl. Acad. Sci. USA*, 106:1006–1008, 2009.

[7] F. Hansen. Means and concave products of positive semi-definite matrices. *Math. Ann.*, 264:119–128, 1983.

[8] F. Hansen and Pedersen G.K. Jensen’s operator inequality. *Bull. London Math. Soc.*, 35:553–564, 2003.

[9] F. Hansen and G.K. Pedersen. Jensen’s inequality for operators and Löwner’s theorem. *Math. Ann.*, 258:229–241, 1982.

[10] Frank Hansen. Regular operator mappings and multivariate geometric means. *Linear Algebra and Its Applications*, 461:123–138, 2014.
[11] J. Kiefer. Optimum experimental designs. *J. Roy. Statist. Soc. Ser. B*, 21:272–310, 1959.

[12] E.H. Lieb and M.B. Ruskai. Some operator inequalities of the Schwarz type. *Adv. in Math.*, 12:269–273, 1974.

Frank Hansen: Institute for Excellence in Higher Education, Tohoku University, Sendai, Japan.
Email: frank.hansen@m.tohoku.ac.jp.