LOCAL WIENER’S THEOREM AND COHERENT SETS OF FREQUENCIES

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Abstract. Using a local analogue of the Wiener–Levi theorem, we investigate the class of measures on Euclidean space with discrete support and spectrum. Also, we find a new sufficient conditions for a discrete set in Euclidean space to be a coherent set of frequencies.

1. Introduction

In this article we show that a local version of the Wiener–Levi theorem allows us to strengthen some theorems in the theory of quasicrystals. To formulate corresponding results, we recall some definitions (see, for example, [21]).

1.1. Notations and preliminaries. Denote by $S(\mathbb{R}^d)$ the Schwartz space of test functions $\psi \in C^\infty(\mathbb{R}^d)$ with finite norms

$$p_m(\psi) = \sup_{x \in \mathbb{R}^d} (1 + |x|)^m \max_{k_1 + \cdots + k_d \leq m} |D^k(\psi(x))|, \quad m = 0, 1, 2, \ldots,$$

$k = (k_1, \ldots, k_d) \in (\mathbb{N} \cup \{0\})^d$, $D^k = \partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d}$. These norms generate a topology on $S(\mathbb{R}^d)$, and elements of the space $S^*(\mathbb{R}^d)$ of continuous linear functionals on $S(\mathbb{R}^d)$ are called tempered distributions.

The Fourier transform of a tempered distribution $f$ is defined by the equality

$$(1) \quad \hat{f}(\psi) = f(\hat{\psi}) \quad \text{for all } \psi \in S(\mathbb{R}^d),$$

where

$$\hat{\psi}(y) = \int_{\mathbb{R}^d} \psi(x)e^{-2\pi i \langle x, y \rangle} dx$$

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is the Fourier transform of the function $\psi$. Note that the Fourier transform of every tempered distribution is also a tempered distribution. But here we consider only the case when $f$ and $\hat{f}$ are complex Radon measures on $\mathbb{R}^d$. For example, if

$$f(x) = \sum_n c_n e^{2\pi i \langle x, \gamma_n \rangle}, \quad \sum_n |c_n| < \infty,$$

then the Fourier transform of the measure $f(x) \, dx$ is equal to $\sum_n c_n \delta_{\gamma_n}$, where $\delta_\gamma$ means the unit mass at $\gamma \in \mathbb{R}^d$.

Furthermore, denote $\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$ for a countable set $\Lambda \subset \mathbb{R}^d$ without finite limit points. By the Poisson formula,

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n), \quad f \in S(\mathbb{R}^d),$$

and we have $\hat{\delta_{\mathbb{Z}^d}} = \delta_{\mathbb{Z}^d}$. If $L = A(\mathbb{Z}^d)$ for some nonsingular linear operator $A$ in $\mathbb{R}^d$, or, shortly, $L$ is a lattice of rank $d$, and $a \in \mathbb{R}^d$, then

$$\hat{\delta_{L+a}} = (\det A)^{-1} \sum_{\lambda \in L^*} e^{-2\pi i \langle y, a \rangle} \delta_\lambda(dy),$$

(2)

where $L^* = \{ y \in \mathbb{R}^d : \langle \lambda, y \rangle \in \mathbb{Z} \text{ for all } \lambda \in L \}$ is the conjugate lattice.

Let $\nu$ be a Radon measure on $\mathbb{R}^d$. Denote by $\nu_t$ its shift on $t \in \mathbb{R}^d$, i.e.,

$$\int g(x) \nu_t(dx) = \int g(x + t) \nu(dx).$$

A measure $\nu$ is translation bounded if variations of its translations $|\nu_t|$ are bounded in the unit ball uniformly in $t \in \mathbb{R}^d$. Note that every translation bounded measure on $\mathbb{R}^d$ satisfies the condition

$$|\nu|(B(0, r)) = O(r^d), \quad r \to \infty,$$

(3)

therefore it belongs to $S^*(\mathbb{R}^d)$. Here $B(x, r) = \{ t \in \mathbb{R}^d : |t - x| < r \}$.

The measure $\nu$ on $\mathbb{R}^d$ is pure point, if it has the form

$$\nu = \sum_{\lambda \in \Lambda} c_\lambda \delta_\lambda, \quad c_\lambda \in \mathbb{C}, \quad \text{with countable } \Lambda \subset \mathbb{R}^d.$$

If this is the case, we will write $\nu(\lambda) := c_\lambda$. Also, we say that $\Lambda$ is the support of $\nu$.

The measure $\nu$ on $\mathbb{R}^d$ has a uniformly discrete support $\Lambda$, if

$$\inf\{|x - x'| : x, x' \in \Lambda, \ x \neq x'\} > 0.$$
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Such measures are the main object in the theory of Fourier quasicrystals (see [1–3,6–11,13–19]). This theory was developed in connection with the experimental discovery of non-periodic pure point structures with diffraction patterns consisting of spots, which was made in the mid ’80s. Remark also that some properties of tempered distributions with discrete and closed or pure point supports were considered in [4–7,12,20].

1.2. Meyer’s theorem on measures with discrete support.

**Theorem 1** [16]. Let \( \mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \) be a measure on \( \mathbb{R} \) with a discrete and closed support \( \Lambda \) and \( a_\lambda \in A \) for a finite set \( A \subset \mathbb{C} \). If \( \mu \in S^*(\mathbb{R}) \) and its Fourier transform \( \hat{\mu} \) is a translation bounded measure on \( \mathbb{R} \), then

\[
\Lambda = E \cup \left( \bigcup_{j=1}^N (\alpha_j \mathbb{Z} + \beta_j) \right) \setminus F, \quad \alpha_j > 0, \: \beta_j \in \mathbb{R},
\]

where the sets \( E, F \) are finite.

M. Kolountzakis [8] extended the above theorem to measures on \( \mathbb{R}^d \). Also, he replaced the condition “the measure \( \hat{\mu} \) is translation bounded” with the weaker one “the measure \( \hat{\mu} \) satisfies condition (3)”. He also found a condition for the support of \( \mu \) to be a finite union of several disjoint shifted lattices of rank \( d \). His result is very close to Cordoba’s one:

**Theorem 2** [1]. Let \( \mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \) be a measure on \( \mathbb{R}^d \) with a uniformly discrete support \( \Lambda \) and \( a_\lambda \in A \) for a finite set \( A \subset \mathbb{C} \). If \( \hat{\mu} \) is a pure point and translation bounded measure, then \( \Lambda \) is a finite union of several disjoint shifted lattices of rank \( d \).

In papers [3] and [4] we get a small improvement to Cordoba’s result. In particular, we replaced the conditions “\( a_\lambda \) from a finite set” by “\( |a_\lambda| \) from a finite set”. But Cordoba’s type theorems are not true for some uniformly discrete measures \( \mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \) with translation bounded \( \hat{\mu} \) and a countable set \( \{a_\lambda\}_{\lambda \in \Lambda} \) ([14]).

The local version of the Wiener–Levi theorem (see Theorem 9 below) allows us to obtain such a result:

**Theorem 3** [6]. Let \( \mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \) be a measure on \( \mathbb{R}^d \) with a uniformly discrete support \( \Lambda \) such that \( \inf_{\lambda} |a_\lambda| > 0 \), and \( \hat{\mu} \) be a pure point measure satisfying (3). Then \( \Lambda \) is a finite union of several disjoint shifted lattices of rank \( d \).

Note that even if both \( \text{supp} \mu \) and \( \text{supp} \hat{\mu} \) are uniformly discrete, they can be finite unions of shifted lattices with incommensurable sides ([3]).

We supplement Theorem 3 with a description of the measure \( \mu \).
Theorem 4. Under the conditions of Theorem 3, there exist an integer $N$, lattices $L_1$, ..., $L_N$ in $\mathbb{R}^d$ of rank $d$ (some of them may coincide), points $\lambda_1, \ldots, \lambda_N \in \Lambda$, and functions $F_j(y) = \sum_s b_s^j e^{2\pi i \langle y, \alpha_s^j \rangle}$ with $\sum_s |b_s^j| < \infty$, $j = 1, \ldots, N$, where the set $\{\alpha_s^j : s \in N\}$ is bounded. Then

$$
\mu = \sum_{j=1}^{N} F_j(y) \delta_{L_j + \lambda_j}.
$$

Moreover,

$$
\hat{\mu} = \sum_{j=1}^{N} e^{2\pi i \langle x, \lambda_j \rangle} \nu^j,
$$

where $\nu^j$ are $d$-periodic pure point measures with lattices $L^*_j$ of periods, and $\lambda_j \in \Lambda$.

1.3. Coherent sets of frequencies. Let us remember that a uniformly discrete set $\Upsilon \subset \mathbb{R}^d$ is a coherent set of frequencies (or satisfies Kahane’s property), if every limit of a sequence of finite sums

$$
\sum c_\lambda e^{2\pi i \langle x, \lambda \rangle}, \quad \lambda \in \Upsilon, \ c_\lambda \in \mathbb{C},
$$

with respect to the topology of uniform convergence on every compact subset of $\mathbb{R}^d$ is almost periodic in the sense of H. Bohr.

Theorem 5 (Y. Meyer, [17]). Let $\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda$ be a Radon measure from $S^*(\mathbb{R}^d)$ with a uniformly discrete support $\Lambda$ and $\hat{\mu}$ be a translation bounded Radon measure. Then the set $\Upsilon = \{\lambda : a_\lambda = 1\}$ is a coherent set of frequencies.

Using the local version of the Wiener–Levi theorem (see Theorem 9 below), we obtain the following result.

Theorem 6. Let $\mu = \sum_{\lambda \in \Lambda} \mu(\lambda) \delta_\lambda$ and $\hat{\mu}$ be pure point translation bounded measures, let $\Upsilon \subset \Lambda$ be such that for all $\lambda \in \Upsilon$ and some $\varepsilon > 0$

i) $|\mu(\lambda)| \geq \varepsilon$,

ii) $|\lambda - \lambda'| \geq \varepsilon$ for all $\lambda' \in \Lambda \setminus \{\lambda\}$.

Then the set $\Upsilon$ is a coherent set of frequencies.

Corollary 1. Let $\mu = \sum_{\lambda \in \Lambda} \mu(\lambda) \delta_\lambda$ be a measure from $S^*(\mathbb{R}^d)$ with a uniformly discrete support $\Lambda$ and $\hat{\mu}$ be a pure point translation bounded measure. Then for every $\varepsilon > 0$ the set $\Upsilon = \{\lambda : |\mu(\lambda)| \geq \varepsilon\}$ is a coherent set of frequencies.
2. Proofs

2.1. The Wiener–Levi theorem. The following theorem is known as the Wiener–Levi theorem (see, for example, [22, Ch. VI]):

**Theorem 7.** Let

\[ F(t) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n t} \]

be an absolutely convergent Fourier series, and \( h(z) \) be a holomorphic function on a neighborhood of the closure of the set \( \{ F(t) : t \in [0, 1] \} \). Then the function \( h(F(t)) \) admits an absolutely convergent Fourier series expansion as well.

For \( h(z) = 1/z \) the theorem is well-known as Wiener’s theorem.

Denote by \( W \) the algebra of absolutely convergent series

\[ f(x) = \sum_{n} c_n e^{2\pi i \langle x, \lambda_n \rangle}, \quad \lambda_n \in \mathbb{R}^d, \quad x \in \mathbb{R}^d, \]

with the norm \( \| f \|_W = \sum_{n} |c_n| \). In [6] we proved the following result.

**Theorem 8.** Let \( K \subset \mathbb{C} \) be an arbitrary compact set, \( h(z) \) be a holomorphic function on a neighborhood of \( K \), and \( f \in W \). Then there is a function \( g \in W \) such that, for every \( x \in \mathbb{R}^d \) for which \( f(x) \in K \), we have \( h(f(x)) = g(x) \). In other words, for every \( f \in W \), there is a function \( g \in W \) such that \( h \circ f \) is its restriction to \( K \).

For \( K = f(\mathbb{R}^d) \) we obtain the global Wiener–Levi theorem for functions from the class \( W \).

Note that exponents of \( g \) belong to the span over \( \mathbb{Z} \) of exponents of \( f \).

The main consequence of Theorem 8 is the following result:

**Theorem 9** [6]. For every \( f \in W \) and \( \varepsilon > 0 \) there is a function \( g \in W \) such that, for every \( x \in \mathbb{R}^d \) for which \( |f(x)| \geq \varepsilon \), we have \( g(x) = 1/f(x) \), and for every \( x \in \mathbb{R}^d \) for which \( |f(x)| \leq \varepsilon/2 \), we have \( g(x) = 0 \).

2.2. Auxiliary results. The following statements are implicitly contained in [6]. For the sake of completeness we present them here with proofs.

**Proposition 1.** If \( \nu \) is a translation bounded measure and \( \psi \in S(\mathbb{R}^d) \), then the total mass of the variation \( |\psi \nu_t| \) of the measure \( \psi \nu_t \) is bounded uniformly in \( t \in \mathbb{R}^d \). Moreover, for every \( \varepsilon > 0 \) there is \( r(\varepsilon) < \infty \) such that the mass of restriction of each measure \( |\psi \nu_t| \) on the set \( \{ x \in \mathbb{R}^d : |x| > r(\varepsilon) \} \) is less than \( \varepsilon \).

**Proposition 2** (see also [9] and [12]). If \( \nu \) is a measure from \( S^*(\mathbb{R}^d) \) with uniformly discrete support \( \Lambda \), and \( \hat{\nu} \) is a measure satisfying (3), then \( \sup_{\lambda \in \Lambda} |\nu(\lambda)| < \infty \), hence the measure \( \nu \) is translation bounded.
Proposition 3. If \( \nu \) is a measure from \( S^*(\mathbb{R}^d) \), \( \hat{\nu} \) is a pure point measure satisfying (3), and \( \psi \in S(\mathbb{R}^d) \), then the convolution \( (\psi \ast \nu)(t) = \int \psi(t-x) \nu(dx) \) belongs to \( W \), and its Fourier transform equals \( \hat{\psi} \hat{\nu} \).

Proposition 4. If \( \nu \) is a translation bounded measure, \( \hat{\nu} \) is a translation bounded pure point measure, and \( g \in W \), then the Fourier transform \( \hat{g}(\nu) \) of the product \( g\nu \) is a translation bounded pure point measure.

Proofs of propositions 1–4. To prove Proposition 1, fix \( t \in \mathbb{R}^d \). Denote by \( N(r) \) the variation \( |\nu_t| \) in the ball \( B(0,r) \). Since the measure \( \nu \) is translation bounded, we see that \( N(r) \leq C(1+r)^d \) with a constant \( C \) independent of \( t \). Also, \( |\psi(x)| \leq C'(1+|x|)^{-d-1} \) for all \( x \in \mathbb{R}^d \) with a constant \( C' \). Therefore, integrating by parts, we obtain the estimate
\[
\int_{\mathbb{R}^d} |\psi(x)| |\nu_t|(dx) \leq C' \int_0^\infty (1+r)^{-d-1} dN(r) \leq CC'(d+1) \int_0^\infty (1+r)^{-2} dr.
\]
Also, if \( r(\varepsilon) \) is sufficiently large, we get
\[
\int_{|x|>r(\varepsilon)} |\psi(x)| |\nu_t|(dx) \leq C' \int_{r(\varepsilon)}^\infty (1+r)^{-d-1} dN(r)
\leq CC'(d+1) \int_{r(\varepsilon)}^\infty (1+r)^{-2} dr < \varepsilon.
\]

To prove Proposition 2, put \( \eta < (1/2) \inf \{|\lambda - \lambda'| : \lambda, \lambda' \in \Lambda \} \), and let \( \varphi \) be an odd \( C^\infty \) function such that \( \varphi(0) = 1 \) and \( \text{supp} \varphi \subset B(0,\eta) \). Then (1) implies the equality
\[
\nu(\lambda) = \int \varphi(x-\lambda) \nu(dx) = \int \tilde{\varphi}(y) e^{2\pi i \langle \lambda, y \rangle} \hat{\nu}(dx),
\]
where \( \tilde{\varphi} \in S(\mathbb{R}^d) \) is the inverse Fourier transform of the function \( \varphi \). Arguing as in the proof of Proposition 1, we see that the module of the latter integral is bounded uniformly in \( \lambda \in \Lambda \).

To prove Proposition 3, we can repeat arguments from the proof of Proposition 1 and get that the total variation of the measure \( \hat{\psi} \hat{\nu} \) is finite. Since the measure \( \hat{\psi} \hat{\nu} \) is pure point, we see that it has the form
\[
\sum_{n=1}^\infty b_n \delta_{\gamma_n}, \quad \sum_{n} |b_n| < \infty,
\]
and its inverse Fourier transform is equal to
\[
\sum_{n=1}^\infty b_n e^{2\pi i \langle \gamma_n, x \rangle} \in W.
\]
On the other hand, for any fixed \( t \) the inverse Fourier transform of the function \( \psi(t-x) \) equals \( \hat{\psi}(y)e^{2\pi i \langle y, t \rangle} \). By formula (1), we get

\[
(\psi * \nu)(t) = \int \psi(t-x) \nu(dx) = \int e^{2\pi i \langle t, y \rangle} \hat{\psi}(y) \hat{\nu}(dy).
\]

Therefore, the inverse Fourier transform of the measure \( \hat{\psi} \hat{\nu} \) is equal to \( \psi * \nu \).

To prove Proposition 4, note that the function \( g \) is bounded, hence the measure \( g \nu \) is translation bounded. Set

\[
\nu^\gamma(x) = e^{2\pi i \langle x, \gamma \rangle} \nu(x), \quad \gamma \in \mathbb{R}^d.
\]

Let \( g(x) = \sum_n c_n e^{2\pi i \langle x, \gamma_n \rangle} \). We have

\[
g \nu = \sum_n c_n \nu^\gamma_n, \quad \hat{g} \nu = \sum_n c_n \hat{\nu}^\gamma_n.
\]

Note that \( \sum_n |c_n| < \infty \), and \( \hat{\nu}^\gamma_n \) are pure point measures, hence \( \hat{g} \nu \) is a pure point measure too. Then for each \( y \in \mathbb{R}^d \) we get \( |\hat{\nu}^\gamma|(B(y, 1)) = |\hat{\nu}|(B(y - \gamma, 1)) \). Therefore,

\[
|\hat{g} \nu|(B(y, 1)) \leq \sum_n |c_n| |\hat{\nu}^\gamma_n|(B(y, 1)) \leq \sup_{t \in \mathbb{R}^d} |\hat{\nu}|(B(t, 1)) \sum_n |c_n|.
\]

So, \( \hat{g} \nu \) is a translation bounded pure point measure. \( \square \)

**2.3. Proof of Theorem 4.** Let \( \eta < \frac{1}{2} \inf\{|\lambda - \lambda'| : \lambda, \lambda' \in \Gamma\} \), and \( \varphi \) be an odd \( C^\infty \) function such that \( \varphi(0) = 1 \) and \( \text{supp} \varphi \subset B(0, \eta) \).

Put \( g = \varphi * \mu \). Clearly, for \( \lambda \in \Lambda \), \( g(\lambda) = \mu(\lambda) \) and, according to Proposition 3, \( g \in W \). We can then write

\[
g(x) = \sum_n c_n e^{2\pi i \langle x, \gamma_n \rangle}.
\]

By Theorem 3, we have

\[
\mu = \sum_{j=1}^N g \delta_{L_j + \lambda_j}, \quad (6)
\]

For every fixed \( j \) and each \( \gamma \in \mathbb{R}^d \) there is \( \alpha \) inside the parallelepiped generated by corresponding \( L_j^* \) such that \( \gamma - \alpha \in L_j^* \), therefore, \( e^{2\pi i \langle x, \gamma \rangle} = e^{2\pi i \langle x, \alpha \rangle} \) for \( x \in L_j \). Collecting similar terms for this \( j \), we obtain (4).

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Next, by (2), \( \hat{\delta}_{L_j+\lambda_j} \) is a uniformly discrete and translation bounded pure point measure. Furthermore, the measure \( g\delta_{L_j+\lambda_j} \) coincides with the restriction of the measure \( \mu \) to \( \lambda_j + L_j \) and, by Proposition 4, its Fourier transform \( \hat{g}\delta_{L_j+\lambda_j} \) is a pure point translation bounded measure. Set
\[
\nu^j = e^{-2\pi i \langle \lambda_j, y \rangle} \hat{g}\delta_{L_j+\lambda_j}.
\]
The inverse Fourier transform of \( \nu^j \) is equal to
\[
(\nu^j)^* = \sum_{x \in L_j+\lambda_j} \mu(x) \delta_{x-\lambda_j} = \sum_{x \in L_j} \mu(x + \lambda_j) \delta_x,
\]
and the inverse Fourier transform of the measure \( \nu_a^j \) for \( a \in \mathbb{R}^d \) is equal to
\[
e^{2\pi i \langle a, x \rangle} \sum_{x \in L_j} \mu(x + \lambda_j) \delta_x,
\]
which coincides with \( (\nu^j)^* \) for each \( a \in L_j^* \). Therefore, \( \nu_a^j = \nu^j \). So, \( \nu^j \) is \( d \)-periodic with the lattice \( L_j^* \) of periods and, by (6),
\[
\hat{\mu} = \sum_{j=1}^{N} g\delta_{L_j+\lambda_j} = \sum_{j=1}^{N} e^{2\pi i \langle \lambda_j, y \rangle} \nu^j. \quad \square
\]

2.4. Proof of Theorem 6 and its Corollary. Let \( \eta < \varepsilon / 2 \), and \( \varphi \) be the same as in the proof of Theorem 4. By Proposition 3 we have \( g = \varphi * \mu \in W \). Then \( g(\lambda) = (\varphi * \mu)(\lambda) = \mu(\lambda) \) for \( \lambda \in \Upsilon \). By Theorem 9, there is \( h \in W \) such that \( g(\lambda)h(\lambda) = 1 \) under condition \( |g(\lambda)| \geq \varepsilon \), in particular, for all \( \lambda \in \Upsilon \). Fix a parameter \( t \in \mathbb{R}^d \). Let \( F(x) \) be a convolution of the function \( \varphi \) and the measure \( e^{2\pi i \langle x, t \rangle} h(x) \mu(x) \), i.e.,
\[
F(x) = \sum_{\lambda \in \Lambda} \varphi(x - \lambda) e^{2\pi i \langle \lambda, t \rangle} h(\lambda) \mu(\lambda).
\]
By Proposition 4, the measure \( \hat{h}\mu \) is translation bounded. Using Proposition 3, we see that \( \hat{F}(y) = \hat{\varphi}(\hat{h}\mu)_t(y) \). Applying Proposition 1, we get that the total mass of the measure \( \hat{F}(y) \) is bounded by some constant \( C < \infty \), and the mass of its restriction to the set \( \{ x \in \mathbb{R}^d : |x| > r \} \) is less than \( 1/2 \) for a suitable \( r < \infty \). Note that \( C \) and \( r \) are independent of \( t \). Taking into account that \( F(x) \) is the inverse Fourier transform of the measure \( \hat{F}(y) \) and obvious equality \( F(\lambda) = e^{2\pi i \langle \lambda, t \rangle} \) for all \( \lambda \in \Upsilon \), we get
\[
e^{2\pi i \langle \lambda, t \rangle} = \int_{\mathbb{R}^d} e^{2\pi i \langle y, \lambda \rangle} \hat{F}(dy).
\]
Now, let $\sum c_\lambda e^{2\pi i(t,\lambda)}$ be any finite sum of exponents with $\lambda \in \Upsilon$. We have

$$\left| \sum c_\lambda e^{2\pi i(t,\lambda)} \right| = \left| \int_{\mathbb{R}^d} \sum c_\lambda e^{2\pi i(y,\lambda)} \hat{F}(dy) \right|$$

$$\leq \left| \int_{|y| \leq r} \sum c_\lambda e^{2\pi i(y,\lambda)} \hat{F}(dy) \right| + \left| \int_{|y| > r} \sum c_\lambda e^{2\pi i(y,\lambda)} \hat{F}(dy) \right|.$$

The first integral in the right-hand side does not exceed

$$C \sup_{|y| \leq r} \left| \sum c_\lambda e^{2\pi i(y,\lambda)} \right|;$$

and the second one is bounded by

$$\frac{1}{2} \sup_{t \in \mathbb{R}^d} \left| \sum c_\lambda e^{2\pi i(t,\lambda)} \right|.$$

Thus,

$$\sup_{t \in \mathbb{R}^d} \left| \sum c_\lambda e^{2\pi i(t,\lambda)} \right| \leq 2C \sup_{|y| \leq r} \left| \sum c_\lambda e^{2\pi i(y,\lambda)} \right|.$$

Therefore, the uniform convergence of the sequence of exponential sums on the ball $B(0, r)$ implies the uniform convergence on $\mathbb{R}^d$, and the limit of the sequence is an almost periodic function in the sense of H. Bohr.

Finally, by Proposition 2, the measure $\mu$ under the conditions of the Corollary is translation bounded, therefore all the conditions of Theorem 6 are met.

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