Quasi-energy spectra of a charged particle in planar honeycomb lattices

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Abstract. The low-energy spectrum of a particle in planar honeycomb lattices is conical, which leads to the unusual electronic properties of graphene. In this paper, we calculate the quasi-energy spectra of a charged particle in honeycomb lattices driven by a strong ac field, which is of fundamental importance for its time-dependent dynamics. We find that depending on the amplitude, direction and frequency of the external field, many interesting phenomena may occur, including band collapse, renormalization of the velocity of ‘light’, gap opening etc. Under suitable conditions, by increasing the magnitude of the ac field, a series of phase transitions from gapless phases to gapped phases appear alternately. At the same time, the Dirac points may disappear or change to a line. We suggest possible realization of the system in honeycomb optical lattices.

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1. Introduction

Lattice structure has an important impact on the dynamics of a system through related band structure. For many materials with a square/cubic lattice, their energy spectra near the bottom of the conduction band are parabolic and the low-energy excitation is a quasi-particle with an effective mass. However, graphene, a two-dimensional (2D) monolayer of carbon atoms, has a linear dispersion relation near so-called Dirac points due to its special honeycomb lattice [1]. Therefore in the long distance limit, the basic excitation is a massless Dirac fermion. There are extensive ongoing experimental and theoretical studies on graphene and related systems [2]. On the one hand graphene provides an excellent condensed-matter analogue of (2 + 1)-dimensional quantum electrodynamics, where the effective velocity of ‘light’ is 1/300 of the real velocity of light. On the other hand the unusual energy spectrum also leads to novel transport properties, such as the anomalous quantum Hall effect [3]. Besides graphene, the optical honeycomb lattice has also been realized in experiments [4]. The flexible tunability of the optical lattices gives people much more opportunity to explore interesting physics in 2D systems with honeycomb lattices. Recently much attention has been drawn in this direction, for instance Haldane’s quantum Hall effect was proposed in ultracold atoms in the optical honeycomb lattice [5], P-orbital physics in the honeycomb optical lattice was also addressed [6], Bose–Einstein condensation in a honeycomb optical lattice was considered in [7], to name a few.

An important issue here is the adjustment of band structure. One convenient and effective way is through an external time-dependent field. For a system driven by a time-periodic electric field, band suppression [8] and band collapse [9] appear. There are also many other interesting phenomena, including coherent destruction of tunneling [10], dynamical localization [11], photovoltaic effects [12], and photon-assisted Fano resonance [13]. Also the electronic dynamics in graphene in the presence of time-dependent fields was studied in [14]. Many of the mentioned phenomena have been observed in the electronic system and/or optical lattices. As the energy spectrum is essential for studying the optical and electronic properties of the relevant system, the quasi-energy spectrum plays a key role in understanding the time-dependent phenomena.

Thus, it is of great importance and interest to study the quasi-energy spectra of a particle in a honeycomb lattice in the presence of an ac field. This issue is addressed in detail in this paper. We show that the quasi-energy spectrum of the honeycomb lattice has quite a rich structure. A series of phase transitions (from gapless phase to gapped phase) appear by changing the amplitude, direction and frequency of the external electric field, which is quite easy in experiments. In particular, under special conditions the particle can be localized in one direction, or two directions, thus the dimension of the system reduces effectively from 2D to 1D and to 0D. The changing of the external field may also lead to effective mass generation and renormalization of the velocity of ‘light’.

2. Quasi-energy spectra

The planar honeycomb lattice (figure 1(a)) consists of two equivalent sublattices A (black dots) and B (white dots). We consider the charged particle hopping between the nearest neighbor sites from different sublattices. In addition, the system under consideration is subject to a
time-periodic electric field \( \mathbf{E}(t) = E_0 \cos(\omega t) \). The tight-binding Hamiltonian has the form

\[
H = \sum_{\langle i,j \rangle} V (c_i^* c_j + c_j^* c_i) + \sum_i e \mathbf{E}(t) \cdot \mathbf{r}_i \left( c_i^+ c_i \right),
\]

where \( \langle i,j \rangle \) refers to the nearest neighbor lattice sites and \( \mathbf{r}_i \) is the position of site \( i \). The probability amplitude \( \alpha(\mathbf{r}_i) \) for a particle at site \( i \) of sublattice A (or \( \beta(\mathbf{r}_i) \) at site of sublattice B) can be calculated by solving the corresponding Schrödinger equation. It is very helpful to make the unitary transformation

\[
\alpha(\mathbf{r}_i) \rightarrow \alpha(\mathbf{r}_i) e^{-i A(t) \cdot \mathbf{r}_i}, \quad \beta(\mathbf{r}_i) \rightarrow \beta(\mathbf{r}_i) e^{i A(t) \cdot \mathbf{r}_i},
\]

where \( A(t) = (eE_0 a/\omega) \sin(\omega t) \), with \( a \) the lattice constant. The subsequent Fourier transformation leads to

\[
i \frac{d}{dt} \begin{pmatrix} \alpha(\mathbf{k}) \\ \beta(\mathbf{k}) \end{pmatrix} = H(\mathbf{k}, t) \begin{pmatrix} \alpha(\mathbf{k}) \\ \beta(\mathbf{k}) \end{pmatrix},
\]

\[
H = \begin{pmatrix} 0 & V \sum_{j=1,2,3} e^{i(a k - A(t)) \cdot \mathbf{u}_j} \\ V \sum_{j=1,2,3} e^{-i(a k - A(t)) \cdot \mathbf{u}_j} & 0 \end{pmatrix}.
\]

From the Floquet theorem [15], there exists a complete set of states of the form \( \psi_\varepsilon(\mathbf{r}, t) = e^{-i t \phi_\varepsilon(\mathbf{r}, t)} \), where \( \varepsilon \) is the quasi-energy and \( \phi_\varepsilon(\mathbf{r}, t + T) = \phi_\varepsilon(\mathbf{r}, t), T = 2\pi/\omega \) (the period of the ac field), just like the Bloch wavefunction in a spatial periodic system. The quasi-energy can be obtained by diagonalizing the time evolution operator \( \hat{T} \int_0^T dt e^{-i H(\mathbf{k}, t)} \), where \( \hat{T} \) refers to time ordering.

2.1. Electric field along the Y-direction

We first consider the case of an electric field along the \( Y \)-direction. In the high-frequency or small hopping limit \( V/\omega \ll 1 \), we can obtain the following analytical result for the quasi-energy spectrum

\[
\varepsilon = \pm V \sqrt{A^2 + 4B^2 \left( \cos \frac{\sqrt{3}}{2} k_y a \right)^2 + 4AB \cos \frac{\sqrt{3}}{2} k_x a \cos \frac{3}{2} k_y a},
\]

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Figure 2. (a) The red solid curve: $|A|/2|B| = |J_0(x)/2J_0(x/2)|$ versus $x$; the blue dotted curve: $|1/2J_0(x)|$ versus $x$, where $x$ is $eE_0 a/\omega$ (for the red solid curve for the case $E \parallel Y$), or $(\sqrt{3}/2)eE_0 a/\omega$ (for the blue dotted curve for the case $E \parallel X$). The shadowed regimes II and IV are for the gapped phases when $E \parallel Y$.

where $A = J_0(eE_0 a/\omega)$ and $B = J_0(eE_0 a/2\omega)$ ($J_0$ is the zeroth-order Bessel function). When $E_0 \rightarrow 0$, $A, B \rightarrow 1$, we reproduce the results for the energy spectrum of a system with a honeycomb lattice without an ac electric field.

2.1.1. Quantum phase transition. The conduction band and the valence band may form conically shaped valleys that touch at some specific points (called conical points or Dirac points). These Dirac points are determined as $k_y a = 2\pi/3$, $\cos((\sqrt{3}/2)k_x a) = A/2B$. It is clear that there is no solution when $|A| > 2|B|$. In this situation, an energy gap is generated. In figures 2(a) and (b) (red solid curve), we show $|A|/2|B| = |J_0(x)/2J_0(x/2)|$ and $\Delta \epsilon / \nu$ versus $x = eE_0 a/\omega$, where $\Delta \epsilon$ is the difference between the minimum of the top band and the maximum of the bottom band. From figure 2 it can be seen that gap opening happens in the regimes (for $eE_0 a/\omega$) II = $[4.1, 5.1]$, IV = $[10.3, 11.3]$, ... . Figures 3(a)–(f) (corresponding to regimes I–V) show the quasi-energy spectra around the minimum of conduction band/maximum of valence band for different values of $eE_0 a/\omega$, which are obtained by exact numerical calculation based on equations (2) and (3).

We can see from figure 3 that as $eE_0 a/\omega$ increases, the system undergoes a series of phases corresponding to the regimes I–V, i.e. gapless phase (regime I) $\rightarrow$ gapped phase (regime II) $\rightarrow$ gapless phase (regime III) $\rightarrow$ gapped phase (regime IV) $\rightarrow$ gapless phase (regime V). The gapped phases II (figure 3(b)) and IV (figure 3(e)) [gapless phases I (figure 3(a)) and V (figure 3(f))] have similar properties as seen in figure 3.

To have more understanding of the nontrivial quantum phase transitions, we expand the quasi-energy spectrum near the bottom (top) of the conduction (valence) band, which has the form

$$\epsilon = \pm \sqrt{\left(\frac{G}{2}\right)^2 + C_x (\delta k_x a)^2 + C_y (\delta k_y a)^2}.$$

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Figure 3. Quasi-energy spectra for $eE_0\alpha/\omega = 3.5$ ((a), in regime I), $eE_0\alpha/\omega = 4.60$ ((b), in regime II), $eE_0\alpha/\omega = 4.81$ ((c), band collapse condition), $eE_0\alpha/\omega = 5.52$ ((d), in regime III), $eE_0\alpha/\omega = 10.8$ ((e), in regime IV), and $eE_0\alpha/\omega = 12.5$ ((f), in regime V) are obtained by exact numerical calculation.

Here, $k_x = \bar{k}_x + \delta k_x, k_y = \bar{k}_y + \delta k_y, \{\bar{k}_x, \bar{k}_y\}$ is the Dirac point. In the gapless phases $G = 0$, and $C_x = \frac{3}{8}V^2(4B^2 - A^2), C_y = \frac{9}{8}V^2A^2$ are related to the effective velocities of ‘light’. In the gapped phases, $G = 2V(|A| - 2|B|)$ is the quasi-energy gap, and $C_x = \frac{3}{8}V^2B(A - 2B), C_y = \frac{9}{8}V^2AB$ are related to the effective mass. The gap opening introduces a new energy scale and may also lead to the generation of an effective mass as seen in figures 3(b) and (e) and equation (5), since the quasi-energy spectrum near the bottom of the conduction band is parabolic. We find the gap $G = 2V(|A| - 2|B|)\theta(|A| - 2|B|)$, where $\theta$ is the step function. Thus $G$ is zero in the gapless phases and nonzero in the gapped phases. As one can see from figure 3, by tuning the ac field, the Dirac points may disappear in the gapped phase, or change from discrete points to a line as shown in figure 3(d). Near $\epsilon = 0$, the density of quasi-energy states $D(\epsilon)$ is zero for gapped phases, $D(\epsilon) \sim |\epsilon|$ near discrete Dirac points, and $D(\epsilon) \sim$ constant near the line. According to the generalized Kubo–Greenwood formula for time-dependent system [16], the conductivity depends on the density of quasi-energy states. Therefore by simply tuning the strength of the external ac field, we can realize the transitions from ‘metal’ phase to ‘semimetal’ phase and to ‘semiconductor’/‘insulator’ phase.

2.1.2. Band collapse. A typical character of figure 2 is that $|A|/2|B|$ is divergent at $eE_0\alpha/\omega = 4.81, 11.0, \ldots$. By checking equation (4), it is easy to see that band collapse happens and a gap opens under this condition $B = 0$ or $J_0(eE_0\alpha/2\omega) = 0$. Figure 3(c) shows the quasi-energy spectrum for $eE_0\alpha/\omega = 4.81$, twice 2.405, the first zero point of $J_0$. As expected, we see the flat conduction and valence bands, which are drastically different from the remaining figures of figure 3, and the case without an ac field. Moreover, we see that a gap between the conduction and valence bands opens. From equation (4), we can also see that the quasi-energy band collapses in the $Y$-direction when $A = 0$, i.e. $eE_0\alpha/\omega = 2.405, \ldots$. 

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2.1.3. Time-dependent dynamics. The collapse of the quasi-energy spectrum implies dynamical localization. To verify this picture, we calculate the time evolution of a particle initially at the origin. By the fourth-order Runge–Kutta method, we obtain the time evolution of the wavefunction and the mean square displacement of the particle, which is defined as 
\[
\langle m^2 \rangle = \langle m_x^2 \rangle + \langle m_y^2 \rangle,
\]
where 
\[
\langle m_x^2 \rangle = \sum_i |\alpha(r_i)|^2 x_i^2 + \sum_j |\beta(r_j)|^2 y_j^2,
\]
\[
\langle m_y^2 \rangle = \sum_i |\alpha(r_i)|^2 y_i^2 + \sum_j |\beta(r_j)|^2 x_j^2.
\]
In our calculation, the system size is $101 \times 101$, time step = 0.02 (in units of $1/\omega$), and the hopping constant is $V/\omega = 0.02$ as in the calculation of quasi-energy. In the long time regime, the mean square displacement of the particle shows the time-dependent behavior as $t^\alpha$. In figure 4(a), we show the time dependence of the mean square displacement. We can see that when the system is away from the band collapse condition ($eE_0a/\omega = 4.81$), $\alpha = 2$, the dynamics is a typical ballistic motion. Under the band collapse condition, the mean square displacement is finite ($<1$) for all time, i.e. $\alpha = 0$, indicating dynamical localization of the particle due to the presence of an ac field. In figure 4(b), we show the time dependence of the mean square displacement in the $X$- and $Y$-directions for $eE_0a/\omega = 2.405$, under which the quasi-energy spectrum is flat in the $Y$-direction. It is clear that the particle is localized in the $Y$-direction ($\langle m_y^2 \rangle < 0.18$), indicating a dimension reduction from 2D to 1D.

2.1.4. Renormalization of the velocity of ‘light’. For a system in gapless phases, we may obtain the velocity of ‘light’ by expanding the quasi-energy spectrum around Dirac points
\[
v_x = \sqrt{3} V B \sin \left( \frac{\sqrt{3}}{2} \vec{k}_x a \right) = \frac{\sqrt{3}}{2} V \sqrt{4 B^2 - A^2},
\]
\[
v_y = 3 V B \cos \left( \frac{\sqrt{3}}{2} \vec{k}_y a \right) = \frac{3}{2} V A,
\]
where $\{\vec{k}_x, \vec{k}_y\}$ refer to the Dirac points. We can see that the velocity of ‘light’ can be tuned by the amplitude and frequency of an external electric field. From figure 5(a), we can see that the velocity of ‘light’ for $eE_0a/\omega = 3.5$ reduces to $\sim \frac{1}{3}$ of that without an external electric field. The external field may also lead to anisotropy of the velocity of ‘light’, see figure 5(b) and

![Figure 4](http://www.njp.org/)
Figure 5. Renormalization of the velocity of ‘light’. (a) Quasi-energy spectrum for $eE_0a/\omega = 0.0$ and 3.5. (b) Anisotropy of the velocity of ‘light’. Quasi-energy spectrum for $eE_0a/\omega = 12.5$. For the purpose of comparison, we have shifted the coordinates of Dirac points.

An extremal example is that the velocity in one direction is finite, while in the other direction it becomes zero under the condition $A = 0$, i.e. $\cos((\sqrt{3}/2)\bar{k}_x a) = 0$, as shown in figure 3(d). The tunability of the velocity of ‘light’ gives us more opportunities to explore relativistic effects in electronic systems or optical lattices.

2.2. Electric field along the other directions

The quasi-energy spectrum is dependent on the direction of the electric field. For the case of an electric field along the $X$-direction, the quasi-energy spectrum still takes the form of equation (4) with $A = 1$ and $B = J_0((\sqrt{3}/2)eE_0a/\omega)$. In this case, the condition for the existence of Dirac points is $|2J_0((\sqrt{3}/2)eE_0a/\omega)| \geq 1$. From figures 2(a) and (b) (dotted blue curve), we see that when $(\sqrt{3}/2)eE_0a/\omega > 1.52$ or $eE_0a/\omega > 1.76$, a gap always opens, which is in contrast to the case with an electric field along the $Y$-direction discussed above. The band collapse appears under the new condition $J_0((\sqrt{3}/2)eE_0a/\omega) = 0$.

In the more general situation, the quasi-energy spectrum has the form $\epsilon = \pm V \sqrt{\sum_j |a_j e^{-i(k \cdot u_j)}|^2}$, where $a_j = J_0(eaE_0 \cdot u_j/\omega)$. The general condition for the existence of Dirac points or the gapless phase is that $|a_j|$ can form a triangle, or $|a_i| + |a_j| \geq |a_k|$, $\{i, j, k\} = \{1, 2, 3\}$. The conditions for the band collapse/dynamical localization become $a_i = a_j = 0$, $i, j \in \{1, 2, 3\}$. These conditions are two equations with two variables, which can be solved in the general situation.

Now we give a simple picture of the phenomena we have found. For an electric field $\mathbf{E}(t)$ in an arbitrary direction, the projection of this field in three directions along the bond (i.e. directions $\mathbf{u}_i$, $i = 1, 2, 3$ (see figure 1)) is $E_i$. Then the effective hopping constants along the bonds $\mathbf{u}_i$ are $V_{i}^{\text{eff}} = V J_0(eE_i a/\omega)$ due to the band suppression/coherent destruction of tunneling [8, 10]. Thus we may map our system to a system with honeycomb lattices and effective hopping constants $V_{i}^{\text{eff}}$ along bond $\mathbf{u}_i$. The dynamical localization phenomena we have found are quite clear in this picture. In fact, for the case $\mathbf{E}||Y$ and $eE_0a/\omega = 4.81$, $V_{1}^{\text{eff}} = V_{2}^{\text{eff}} = 0$, the particle initially staying at site $c$ can only oscillate between site $c$ and site $f$ (see figure 1(b)), which is the dynamical localization behavior shown in figure 4(a). For the
situation $eE_0a/\omega = 2.405$, $V_1^{\text{eff}} = 0$, the particle can only move through the path ‘abcde’ shown in figure 1(b), which is the dynamical localization in the $Y$-direction shown in figure 4(b). The asymmetry between the tunneling amplitudes (or the ‘bond ordering’), the dimerization pattern (see figure 1(b)) results in the moving of Dirac points. At the critical asymmetry ($|a_i| + |a_j| = |a_k|$, $(i, j, k) = (1, 2, 3)$), the Dirac points merge [17]. When the asymmetry increases further, the Dirac points vanish and a gap is opened. The larger asymmetry leads to a larger gap, as seen in figure 2(b) (where the larger asymmetry of effective tunneling amplitudes for $E \parallel X$ leads to a larger gap compared with the case of $E \parallel Y$). In the continuum limit, the fluctuation of the hopping constant maps to the axial vector potential [18]. It is different from the honeycomb lattices with ‘Kekule pattern’, in which the fluctuation of the hopping constant maps to the complex-valued Higgs field in the continuum limit. In that case the opening of the energy gap is related to the spontaneous breaking of an effective axial $U(1)$ symmetry [19], and was proposed for the split of zero-energy Landau levels in graphene in the presence of a magnetic field [20]. By tuning the external electric field, one may effectively obtain different bond patterns.

3. Realization in optical lattices

The honeycomb optical lattice can be realized in experiments [4] by using three pairs of laser beams with co-planar propagating wavevectors $\pm q_i$, $i = 1, 2, 3$. The magnitudes of the three pairs of wavevectors are the same and their directions form an angle of 120° with each other. The optical potential has the form

$$U = \sum_i U_i = U_0 \sum_i \cos(p_i \cdot r),$$

where $p_1 = q_2 - q_3$, $p_2 = q_3 - q_1$, and $p_3 = q_1 - q_2$. $U_0$ depends on the detuning between laser frequency and atomic transition frequency, and the resonant Rabi frequency proportional to the square root of laser intensity [21]. Thus $U_0$ can be tuned quite easily in experiments. For sufficiently large detuning, we may neglect spontaneous scattering and the high-frequency condition $U_0/\omega \ll 1$ can also be easily met. One may let the coordinate of the mirror in each direction $p_i$, reflecting the incoming traveling light wave, oscillate around its average with $d_i \cos(\omega t)$. In the moving frame of the potential, the atoms feel an effective force proportional to $d_i \cos(\omega t)$, which is proportional to the electrical field component in the $p_i$ direction [22]. One may observe the field-induced dynamical localization from the time-averaged occupation probability [21]. Bragg spectroscopy can tell the difference between the gapless phase (with the linear dispersion relation) and the gapped phase (with the parabolic dispersion relation). In Bragg spectroscopy, one may shine two laser beams with frequencies $\omega_1$, $\omega_2$ and wavevectors $k_1, k_2$ [23]. According to the generalized Fermi–Golden rule [24], the structure factor $S(q, \omega_0)$ is proportional to $\sum_n \delta(\epsilon_i - \epsilon_n + n\hbar\omega - \hbar\omega_0)$, where $\omega_0 = \omega_2 - \omega_1$, $q = k_2 - k_1$, and $\epsilon_i, \epsilon_n$ are the quasi-energies of the initial and final states. In the high-frequency regime, the $n = 0$ term dominates. In the gapless phase with the isotropic linear dispersion relation $\epsilon = v_q$, the structure factor has the form $S(q, \omega_0) \sim (2q^2 - q^2)/(|\sqrt{q^2} - q^2|\theta(\omega_0 - q_v/\hbar))$, with $q_v = |q|$; in the gapped phase, $S(q, \omega_0) \sim G/(v_x v_y)\theta(\omega_0 - G - q^2 v_x^2/2G - q^2 v_y^2/2G)$, where $G$ is the gap and $v_x, v_y$ are the velocities near the bottom of the parabolic band. One may check our theoretical predictions by another type of experiment with Bose–Einstein condensate in the optical lattices. One may measure the expansion rate $dw/dt$ of an initially trapped Bose–Einstein condensate of cold atoms in a honeycomb optical lattice,

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where $w$ is the width of the matter wave packet. If the initial wave packet is narrowly centered in $k$-space around some point $\bar{k}$, the time evolution of the wave packet is determined by the quasi-energy spectrum near $\bar{k}$, $\varepsilon(k) = \varepsilon(\bar{k}) + v \cdot \delta k + \gamma_x \delta k_x^2 + \gamma_y \delta k_y^2$. Here, $\delta k = k - \bar{k}$, $v$ is the velocity of the center of the wave packet, and $\gamma_x, \gamma_y$ are proportional to the expansion rate in the $x$- and $y$-directions [25]. Firstly, one may observe the anisotropic expansion due to the anisotropy of the quasi-energy spectrum. Secondly, at the Dirac point (in the gapless phase), the expansion rate vanishes, which is quite different from the gapped phase with finite expansion rate. Thirdly, at the band suppression/collapse point, $v_x = 0, v_y = 0$ and $\gamma_x = 0, \gamma_y = 0$, i.e. there is no moving of the center of the wave packet and no expansion in the $x/y$-direction. Thus one may see more interesting dynamic localization effects: effective dimensional reduction $2D \rightarrow 1D \rightarrow 0D$.

In summary, we have studied the quasi-energy spectra and the related dynamics of a particle in honeycomb lattices driven by an ac field. By tuning the amplitude, frequency and direction of the electric field, we can obtain quite rich phases, including gapped and gapless phases. Moreover, we may tune the velocity of the ‘light’, and realize band collapse and the dimensional reduction from $2D$ to $1D$ and to $0D$ (dynamical localization in one direction and two directions). In the current paper, we have focused on the high-frequency or small hopping regime, i.e. $V/\omega \ll 1$. In graphene with $V$ around 2.8 eV and the lattice constant $a$ 1.4 Å, our predictions may not be verified easily. However, they should be easier to observe in optical lattices. We leave the low-frequency regime for future study.

Acknowledgments

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References

[1] Wallace P R 1947 Phys. Rev. 71 622
[2] Geim A K and Novoselov K S 2007 Nat. Mater. 6 183
Castro Neto A H, Guinea F, Peres N M R, Novoselov K S and Geim A K 2009 Rev. Mod. Phys. 81 109
Osipov V A and Kochetov E A 2001 JETP Lett. 73 562
Morpurgo A F and Guinea F 2006 Phys. Rev. Lett. 97 196804
[3] Novoselov K S, Geim A K, Morozov S V, Jiang D, Katsnelson M I, Grigorieva I V, Dubonos S V and Firsov A A 2005 Nature 438 197
Zhang Y, Tan Y-W, Stormer H L and Kim P 2005 Nature 438 201
[4] Grynberg G, Lounis B, Verkerk P, Courtois J-Y and Salomon C 1993 Phys. Rev. Lett. 70 2249
[5] Shao L B, Zhu S-L, Shen L, Xing D Y and Wang Z D 2008 arXiv:0804.1850
[6] Wu C, Bergman D, Balents L and Sarma S D 2007 Phys. Rev. Lett. 99 070401
Wu C and Sarma S D 2008 Phys. Rev. B 77 235107
[7] Haddad L H and Carr L D 2008 arXiv:0803.3039v1
[8] Madison K W, Fischer M C, Diener R B and Qian Niu Raizen M G 1998 Phys. Rev. Lett. 81 5093
Diener R and Niu Q 2000 J. Opt. B: Quantum Semiclass. Opt. 2 618
[9] Holthaus M 1992 Phys. Rev. Lett. 69 351
[10] Grossmann F, Dittrich T, Jung P and Hänggi P 1991 Phys. Rev. Lett. 67 516

New Journal of Physics 11 (2009) 063032 (http://www.njp.org/)
[11] Dunlap D H and Kenkre V M 1986 *Phys. Rev. B* **34** 3625
Zhao X-G 1991 *Phys. Lett. A* **155** 299
Zhao X-G 1992 *Phys. Lett. A* **167** 291
Keay B J, Zeuner S, Allen S J Jr, Maranowski K D, Gossard A C, Bhattacharya U and Rodwell M J W 1995 *Phys. Rev. Lett.* **75** 4102
Longhi S, Marangoni M, Lobino M, Ramponi R, Laporta P, Cianci E and Foglietti V 2006 *Phys. Rev. Lett.* **96** 243901
Iyer R, Aitchison J S, Wan J, Dignam M M and de Sterke C M 2007 *Opt. Express* **15** 3212
[12] Vavilov M G, Ambegaokar V and Aleiner I L 2001 *Phys. Rev. B* **63** 195313
Vavilov M G, DiCarlo L and Marcus C M 2005 *Phys. Rev. B* **71** 241309
[13] Xie W, Chu W, Zhang W and Duan S 2008 *J. Phys.: Condens. Matter* **20** 325223
Duan S, Zhang W, Xie W, Ma Y and Chu W 2009 *New J. Phys.* **11** 013037
[14] Oka T and Aoki H 2009 *Phys. Rev. B* **79** 081406
Gusynin V P, Sharapov S G and Carbotte J P 2007 *Int. J. Mod. Phys. B* **21** 4611
[15] Shirley J H 1965 *Phys. Rev. B* **138** 979
[16] Shi J and Xie X C 2003 *Phys. Rev. Lett.* **91** 086801
[17] Wunsch B B, Guinea F and Sols F 2008 *New J. Phys.* **10** 103027
[18] Jackiw R and Pi S-Y 2007 *Phys. Rev. Lett.* **98** 266402
Chamon C, Hou C-Y, Jackiw R, Mudry C, Pi S-Y and Schnyder A P 2008 *Phys. Rev. Lett.* **100** 110405
[19] Hou C-Y, Chamon C and Mudry C 2007 *Phys. Rev. Lett.* **98** 186809
[20] Hatsugai Y, Fukui T and Aoki H 2008 *Physica E* **40** 1530
[21] Graham R, Schlautmann M and Zoller P 1992 *Phys. Rev. A* **45** R19
[22] Niu Q, Zhao X-G, Georgakis G A and Raizen M G 1996 *Phys. Rev. Lett.* **76** 4504
[23] Stamper-Kurn D M, Chikkatur A P, Gorlitz A, Inouye S, Gupta S, Prichard D E and Ketterle W 1999 *Phys. Rev. Lett.* **83** 2876
[24] Holthaus M 1992 *Z. Phys.* **89** 251
[25] Lignier H, Sias C, Ciampini D, Singh Y, Zenesini A, Morsch O and Arimondo E 2007 *Phys. Rev. Lett.* **99** 220403
Eckardt A, Holthaus M, Lignier H, Zenesini A, Ciampini D, Morsch O and Arimondo E 2008 *arXiv:0812.1997*