Robustness properties of dimensionality reduction with Gaussian random matrices

HAN Bin\textsuperscript{1} & XU ZhiQiang\textsuperscript{2,3,*}

\textsuperscript{1}Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada; \textsuperscript{2}LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; \textsuperscript{3}School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

Email: bhan@ualberta.ca, xuzq@lsec.cc.ac.cn

Received September 3, 2016; accepted December 4, 2016; published online June 5, 2017

Abstract In this paper, motivated by the results in compressive phase retrieval, we study the robustness properties of dimensionality reduction with Gaussian random matrices having arbitrarily erased rows. We first study the robustness property against erasure for the almost norm preservation property of Gaussian random matrices by obtaining the optimal estimate of the erasure ratio for a small given norm distortion rate. As a consequence, we establish the robustness property of Johnson-Lindenstrauss lemma and the robustness property of restricted isometry property with corruption for Gaussian random matrices. Secondly, we obtain a sharp estimate for the optimal lower and upper bounds of norm distortion rates of Gaussian random matrices under a given erasure ratio. This allows us to establish the strong restricted isometry property with the almost optimal restricted isometry property (RIP) constants, which plays a central role in the study of phaseless compressed sensing. As a byproduct of our results, we also establish the robustness property of Gaussian random finite frames under erasure.

Keywords phase retrieval, finite frames, sparse approximation, restricted isometry property, Johnson-Lindenstrauss lemma

MSC(2010) 94A12, 94A20, 41A99, 60B20

Citation: Han B, Xu Z Q. Robustness properties of dimensionality reduction with Gaussian random matrices. Sci China Math, 2017, 60: 1753–1778, doi: 10.1007/s11425-016-9018-x

1 Introduction and motivations

1.1 Problem setup

In this paper, we are interested in investigating various robustness properties of dimensionality reduction with Gaussian random matrices having arbitrarily erased rows. Then we shall use the results to study the robustness properties of the Johnson-Lindenstrauss lemma and restricted isometry property (RIP).

Throughout the paper, $A = (a_{j,k})_{1 \leq j \leq m, 1 \leq k \leq n} \in \mathbb{R}^{m \times n}$ will be a Gaussian random matrix such that each entry $a_{j,k}$ is an independent identically distributed (i.i.d.) random variable under the standard Gaussian/normal distribution $N(0, 1)$ with zero mean and unit standard deviation. For $T \subseteq \{1, \ldots, m\}$, we shall adopt the notation $A_T \in \mathbb{R}^{|T| \times n}$ to denote the $|T| \times n$ sub-matrix of $A$ by keeping the rows of $A$
with the row indices from $T$, where $|T|$ is the cardinality of the set $T$. Let $x_0 \in \mathbb{R}^n$ be a fixed vector with $\|x_0\| = 1$, where $\|x_0\|$ is the Euclidean norm of the vector $x_0$. For $\epsilon > 0$ and $0 \leq \beta < 1$, we define

$$
\Omega_{\epsilon, \beta} := \Omega_{\epsilon, \beta}(A, x_0) := \left\{ \frac{1}{|T|} \| A_T x_0 \|^2 - 1 \leq \epsilon \text{ for all } T \subseteq \{1, \ldots, m\} \text{ satisfying } |T^c| \leq \beta m \right\},
$$

(1.1)

where $T^c := \{1, \ldots, m\} \setminus T$. For every fixed $\epsilon > 0$, it follows from the definition in (1.1) that $P(\Omega_{\epsilon, \beta})$ is a decreasing function in terms of $\beta$, where the probability is taken over the Gaussian random matrix $A$. Define $y = (y_1, \ldots, y_m)^T := Ax_0$. Since all entries of $A$ are i.i.d. and $\|x_0\| = 1$, we see that all the entries of $y$ are i.i.d. Gaussian/normal distribution $\mathcal{N}(0, 1)$. Note that $A_T x_0 = y_T$ for any $T \subseteq \{1, \ldots, m\}$. Thus, all our estimates on $P(\Omega_{\epsilon, \beta})$ in this paper are independent of $n$ and $x_0$.

It is well known in the literature by standard tail-bounds for the chi-squared distribution (see [1, Lemma 4.1]) that

$$
P\left\{ \frac{1}{m} \| A x_0 \|^2 > 1 + \epsilon \right\} < e^{-(\epsilon^2/\epsilon^3)/6} m,
$$

$$
P\left\{ \frac{1}{m} \| A x_0 \|^2 < 1 - \epsilon \right\} < e^{-(\epsilon^2/\epsilon^3)/6} m,
$$

(1.2)

Consequently, with high probability, a normalized Gaussian random matrix $\frac{1}{\sqrt{m}} A$ has the following almost norm preservation property:

$$
P(\Omega_{\epsilon, 0}) = P\left\{ \frac{1}{m} \| A x_0 \|^2 - 1 \leq \epsilon \right\} \geq 1 - 2e^{-(\epsilon^2/\epsilon^3)/6} m, \quad \forall m \in \mathbb{N}, \quad 0 < \epsilon < 1.
$$

(1.3)

The inequality in (1.3) also implies the Johnson-Lindenstrauss lemma (see [1,18]). For $N$ points $p_1, \ldots, p_N \in \mathbb{R}^n$ and for $0 < \epsilon < 1$, the Johnson-Lindenstrauss lemma says that for $m = O\left(\frac{\ln N}{\epsilon^2} \right)$, there exists a projection matrix $A \in \mathbb{R}^{m \times n}$ such that the following almost norm/distance preservation property holds:

$$
(1 - \epsilon) \| p_j - p_k \|^2 \leq \| A p_j - A p_k \|^2 \leq (1 + \epsilon) \| p_j - p_k \|^2, \quad \forall 1 \leq j, k \leq N.
$$

(1.4)

To establish the above almost norm preservation property in (1.4), the projection matrix $A$ is often taken to be a random matrix so that the almost norm preservation property in (1.3) holds with high probability. The Johnson-Lindenstrauss lemma is a fundamental technique to reduce the dimensionality of the data and has many applications in information theory, machine learning and algorithms (see [7,12] and the references therein).

In the compressed sensing literature, the restricted isometry property of a measurement matrix is also related to (1.3). For $x \in \mathbb{R}^n$ and $s \in \mathbb{N}$, we say that $x$ is $s$-sparse if $x$ has no more than $s$ nonzero entries. Under a measurement matrix $A \in \mathbb{R}^{m \times n}$, we have $y := (y_1, \ldots, y_m)^T := Ax$ with $m$ measurements $y_1, \ldots, y_m$. To successfully recover the unknown sparse signal $x$ from the measurement vector $y$, it is important for the matrix $A$ to satisfy the restricted isometry property with a small positive RIP constant $0 < \epsilon_s < 1$, i.e.,

$$
(1 - \epsilon_s) \| v \|^2 \leq \| A v \|^2 \leq (1 + \epsilon_s) \| v \|^2, \quad \text{for all } s\text{-sparse vectors } v \in \mathbb{R}^n.
$$

(1.5)

The above restricted isometry property with a small positive RIP constant $\epsilon_s$ is often established by considering $A$ to be a random matrix such as a normalized Gaussian random matrix so that the almost norm preservation property in (1.3) holds with high probability for $\epsilon = \epsilon_s$ (see [6,9,19,21]). In particular, (1.5) with $s = n$ simply becomes the definition of a near-tight frame with distortion $\epsilon_n$ for the finite dimensional space $\mathbb{R}^n$ (see [24]).

When $\beta > 0$, we suppose that at most $\beta m$ rows of the Gaussian random matrix $A$ are arbitrarily erased. It is of interest in both theory and application to study how large is the erasure ratio $\beta$ so that a normalized Gaussian random matrix $\frac{1}{\sqrt{m}} A$ with any arbitrarily erased $\beta m$ rows still has the almost norm preservation property with high probability. Particularly, we are interested in the following two problems.
Problem 1. Given $0 < \epsilon < 1$ and $0 < \alpha < 1$, what is the maximum $\beta$ so that
\[
P(\Omega_{\epsilon, \beta}) \geq 1 - 3e^{-\alpha(\epsilon^2/4 - \epsilon^2/6)m}, \quad \text{for all } m \in \mathbb{N}.
\] (1.6)

Problem 2. Given $0 < \beta < 1$ and $\alpha > 0$, what is the minimum $\epsilon$ so that
\[
P(\Omega_{\epsilon, \beta}) \geq 1 - 2e^{-\alpha m}, \quad \text{for all } m \in \mathbb{N}.
\] (1.7)

In these two problems, we require that $P(\Omega_{\epsilon, \beta})$ tend to 1 with the exponential order as $m$ goes to infinity. The order plays a key role in the analysis of strong RIP and the robustness property of Johnson-Lindenstrauss lemma. Hence, it is natural to require that $P(\Omega_{\epsilon, \beta})$ take the form of (1.6) and (1.7).

1.2 Applications

Let us briefly explain our motivation for considering $P(\Omega_{\epsilon, \beta})$ with $\beta > 0$ in the setting of Johnson-Lindenstrauss lemma and of compressed sensing.

(i) Phase retrieval is an active topic recently (see [2–4, 8]). In compressive phase retrieval, one can only obtain the magnitude of measurements, i.e., $|Ax_0| := (|a_1, x_0|, \ldots, |a_m, x_0|)$, where $x_0 \in \mathbb{R}^n$ is $s$-sparse. A simple observation is that if $|Ax| = |Ax_0|$ then there exists $T \subseteq \{1, \ldots, m\}$ with $|T| \geq m/2$ so that $A_T x = A_T x_0$ or $A_T x = -A_T x_0$. This implies that one can use $\ell_1$ minimization subject to $|Ax| = |Ax_0|$ to recover $x_0$ provided that $A_T$ satisfies RIP property for all $T \subseteq \{1, \ldots, m\}$ with $|T| \geq m/2$ (see [22, Theorem 2.2]). Hence, one introduces the concept of strong RIP requiring that the matrices $A_T$ satisfy the RIP property for all subsets $T \subseteq \{1, \ldots, m\}$ with $|T^c| \leq \beta m$ (see [22]). For example, Bandeira and Mixon [5] investigated the case where $\beta$ is small enough while in [22], Voroninski and Xu considered the case $\beta = 1/2$. To achieve strong RIP, it is natural to first establish the almost norm preservation property with high probability by replacing $\Omega_{\epsilon, \beta}$ and $\epsilon$ in (1.3) with $\Omega_{\epsilon, \beta}$ and $\epsilon_s$, respectively for $\beta > 0$. The results presented in this paper show that strong RIP holds with high probability for any $\beta \in [0, 1)$ (see Corollary 1.6). Our results also improve the strong RIP constant for the case where $\beta = 1/2$.

(ii) In Johnson-Lindenstrauss lemma and compressed sensing, note that each projected vector $Ax$ has $m$ entries. The projected vectors are often transmitted through network by $m$ parallel channels, i.e., each entry of $Ax$ is transmitted through an independent channel in a parallel manner (see [16]). In practice, it is possible that the channels suffer from erasure and we only receive the corrupted projected vectors $A_T x$ instead of $Ax$ for $j = 1, \ldots, N$, where $T \subseteq \{1, \ldots, m\}$ is an unknown subset with $|T^c| \leq \beta m$ for some given corruption/erasure ratio $0 < \beta < 1$. Consequently, it is important to first establish the almost norm preservation property with high probability for $\Omega_{\epsilon, \beta}$ with $\beta > 0$ which leads to the robustness of the Johnson-Lindenstrauss lemma and of the RIP.

(iii) As said before, in data communication, the erasure can often occur. In finite frame theory, the robustness of a finite frame against erasures attracts much attention recently (see [13, 15, 17]). In [13], the term numerically erasure-robust frames (NERFs) is introduced which uses the max norm $|T^c| \leq \beta m$ $\sigma_{\alpha}(A_T)$ to measure the robustness against $\beta m$-erasures, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ are singular values of $A_T$. The robustness of random frames is investigated in both [13, 24]. Note (1.5) with $s = n$ simply becomes the definition of a near-tight frame with distortion $\epsilon_s$ for the finite dimensional space $\mathbb{R}^n$. The robust RIP is also naturally linked to the robustness property of random finite frames in $\mathbb{R}^n$. The investigation of $P(\Omega_{\epsilon, \beta})$, and hence of the robustness of RIP, will lead to results on NERFs.

1.3 Main results

1.3.1 Results for Problem 1

We first consider Problem 1. To study how large is the erasure ratio $\beta$ so that a normalized Gaussian random matrix $A$ with arbitrarily erased $\beta m$ rows still has the almost norm preservation property with high probability, we introduce a quantity $\beta_{\epsilon, \alpha}^{\text{max}}$ to characterize the largest possible such erasure ratio $\beta$ with a given fixed high probability rate $\alpha > 0$. For $\epsilon > 0$ and $\alpha > 0$, we define
\[
\beta_{\epsilon, \alpha}^{\text{max}} := \sup\{0 \leq \beta < 1 : P(\Omega_{\epsilon, \beta}) \geq 1 - 3e^{-\alpha(\epsilon^2/4 - \epsilon^2/6)m} \text{ for all } m \in \mathbb{N}\}.
\] (1.8)
Theorem 1.1. Let \( \Omega_{\epsilon,0} \) be a \( m \times n \) random matrix with i.i.d. entries obeying \( \mathcal{N}(0,1) \). For any given \( N \) points \( p_1, \ldots, p_N \in \mathbb{R}^n \), with probability at least \( 1 - \frac{2}{N(N-1)} e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} \) > 0,
\[
(1 - \epsilon) ||p_j - p_k||^2 \leq \frac{1}{m} ||A_T p_j - A_T p_k||^2 \leq \frac{1}{|T|} ||A_T p_j - A_T p_k||^2 \\
\leq (1 + \epsilon) ||p_j - p_k||^2, \quad \forall 1 \leq j, k \leq N \quad \text{and} \quad T \in T_{\epsilon,\alpha},
\]
where \( T_{\epsilon,\alpha} \) is defined to be
\[
T_{\epsilon,\alpha} := \left\{ T \subseteq \{1, \ldots, m\} : |T^c| \leq m \left( \frac{1 - \sqrt{\alpha}}{32} \right) \frac{\epsilon}{\ln \frac{1}{\epsilon}} \right\}. \tag{1.14}
\]

Another consequence of Theorem 3.5 is the following result on the robust restricted isometry property.

Corollary 1.3. Let \( 0 < \alpha < 1 \) and \( 0 < \epsilon < \frac{1 - \sqrt{\alpha}}{4} \). Let \( s, m, n \in \mathbb{N} \) satisfy
\[
s \ln \frac{24en}{\epsilon s} < \alpha(\epsilon^2/16 - \epsilon^3/24)m - \ln 6. \tag{1.15}
\]
Let \( A \) be an \( m \times n \) random matrix with i.i.d. entries obeying \( \mathcal{N}(0,1) \). With probability at least
\[
1 - 6 \left( \frac{24en}{\epsilon s} \right)^s e^{-\alpha(\epsilon^2/16 - \epsilon^3/24)m} > 0,
\]
we have
\[
(1 - \epsilon)||v||^2 \leq \frac{1}{m} ||A_T v||^2 \leq \frac{1}{|T|} ||A_T v||^2 \leq (1 + \epsilon)||v||^2, \quad \forall \text{s-sparse} \ v \in \mathbb{R}^n \text{ and } T \in T_{\epsilon/2,\alpha}. \tag{1.16}
\]
where \( T_{\epsilon/2,\alpha} \) is defined in (1.14).
Corollary 1.3 also establishes the robustness property for Gaussian random finite frames in $\mathbb{R}^n$. If
\[
m > \frac{n \ln \frac{24e}{\epsilon} + \ln 6}{(\alpha^2/16 - \epsilon^2/24)}
\] (1.17)
then we can directly check that (1.15) holds with $s = n$. Consequently, under the condition in (1.17), with probability at least $1 - 6(\frac{24e}{\epsilon})^n e^{-\alpha^2(16 - \epsilon^2/24)n} > 0$, the following robustness property for near-tight frames with distortion $\epsilon$ holds:
\[
(1 - \epsilon)||v||^2 \leq \frac{1}{m}||A_Tv||^2 \leq \frac{1}{|T|}||A_Tv||^2 \leq (1 + \epsilon)||v||^2, \quad \forall v \in \mathbb{R}^n \text{ and } T \in T_{\epsilon/2, \alpha}.
\]
Note that for a finite frame with $m$ elements from the space $\mathbb{R}^n$, we must have $m \geq n$ and the ratio $m/n$ is called the redundancy rate of the finite frame. By regarding the rows of $A \in \mathbb{R}^{m \times n}$ as elements in $\mathbb{R}^n$, Corollary 1.3 with $s = n$ shows that robustness of Gaussian random finite near-tight frames, up to erasure ratio no more than $(1 - \sqrt{32})^{e/2(\ln 24)}$, can be achieved with the redundancy rate $\frac{n \ln 24\epsilon}{(\alpha^2/16 - \epsilon^2/24)}$.

1.3.2 Results for Problem 2

We now turn to Problem 2, which is also related to erasure robust frames (see [13, 24]). For a given $0 < \beta < 1$, we determine the minimum $\epsilon$ so that with high probability $\frac{1}{|T|}||A_Tx_0||^2 \in [1 - \epsilon, 1 + \epsilon]$ for all $T \subseteq \{1, \ldots, m\}$ satisfying $|T| \leq \beta m$. For this purpose, we consider the most general case instead of the particular subsets $\Omega_{s, \beta}$ in (1.1). Recall that $x_0 \in \mathbb{R}^n$ with $||x_0|| = 1$. For $0 \leq \beta < 1$ and $0 \leq \theta \leq \omega \leq \infty$, we define
\[
\Omega_{\theta, \omega, \beta} := \Omega_{\theta, \omega, \beta}(A, x_0) := \left\{ \frac{1}{|T|}||A_Tx_0||^2 \in [\theta, \omega], \forall T \subseteq \{1, \ldots, m\} \text{ satisfying } |T| \leq \beta m \right\}.
\] (1.18)
Obviously, $\Omega_{s, \beta}$ in (1.1) simply becomes $\Omega_{s, \beta} = \Omega_{1 - \epsilon, 1 + \epsilon, \beta}$. For $0 < \beta < 1$ and $\alpha > 0$, we define
\[
\theta_{\beta}^{\max} := \theta_{\beta}^{\max}(\alpha) := \sup\{0 \leq \theta < \infty : P(\Omega_{\theta, \infty, \beta}) \geq 1 - \exp(-\alpha m) \text{ for all } m \in \mathbb{N}\},
\omega_{\beta}^{\min} := \omega_{\beta}^{\min}(\alpha) := \inf\{0 \leq \omega \leq \infty : P(\Omega_{0, \omega, \beta}) \geq 1 - \exp(-\alpha m) \text{ for all } m \in \mathbb{N}\}.
\] (1.19)
A simple observation from the above definitions is
\[
P(\Omega_{\theta_{\beta}^{\max}, \omega_{\beta}^{\min}, \beta}) \geq 1 - 2 \exp(-\alpha m), \quad \forall m \in \mathbb{N}.
\] (1.20)
If $0 < \theta_{\beta}^{\max} \leq \omega_{\beta}^{\min} < 2$, then Problem 2 is solved with $\epsilon = \max(1 - \theta_{\beta}^{\max}, \omega_{\beta}^{\min} - 1) > 0$. Similar to (1.18), we define
\[
\hat{\Omega}_{\theta, \omega, \beta} := \left\{ \frac{1}{m}||A_Tx_0||^2 \in [\theta, \omega], \forall T \subseteq \{1, \ldots, m\} \text{ satisfying } |T| \leq \beta m \right\}
\] (1.21)
and we can define $\hat{\theta}_{\beta}^{\max}$ and $\hat{\omega}_{\beta}^{\min}$ similar to $\theta_{\beta}^{\max}$ and $\omega_{\beta}^{\min}$, respectively by replacing $\Omega$ with $\hat{\Omega}$.

We now briefly explain why we are interested in $\Omega_{\theta, \omega, \beta}$. An $m \times n$ matrix $A$ is said to have the strong restricted isometry property of sparse order $s \in \mathbb{N}$ and level $[\theta, \omega, \beta]$ with $0 < \theta \leq \omega < 2, 0 \leq \beta < 1$ if
\[
\theta ||v||^2 \leq \frac{1}{m}||A_Tv||^2 \leq \omega ||v||^2, \quad \forall s\text{-sparse } v \in \mathbb{R}^n \text{ and } T \subseteq \{1, \ldots, m\} \text{ with } |T| \leq \beta m.
\] (1.22)
The strong restricted isometry property plays a critical role in the study of phaseless compressed sensing in [5, 22]. In [22], Voronin and Xu investigated the case where $\beta = 1/2$ and showed that the normalized Gaussian matrix has the strong RIP of order $s$ and level $[\theta_0, \omega_0, 1/2]$ with high probability provided $m = O(s \log en)$. Here, $\theta_0$ and $\omega_0$ are constants satisfying $0 \leq \theta_0 \leq \omega_0 < 2$. The original motivation for this work is to extend the result in [22] to the arbitrary erasure/corruption ratio $\beta \in (0, 1)$. To show that there indeed exists a measurement matrix $A$ having the strong restricted isometry property of sparse order $s$ and level $[\theta, \omega, \beta]$ in (1.22), the matrix $A$ is often constructed by an $m \times n$ Gaussian random matrix with i.i.d. entries obeying $\mathcal{N}(0, 1)$ and one would like to have $P(\hat{\Omega}_{\theta, \omega, \beta}) > 0$ for $0 < \theta \leq \omega < 2$ with the largest possible $\theta$ and the smallest possible $\omega$, i.e., if we can prove the inequalities $0 < \theta_{\beta}^{\max} \leq \hat{\omega}_{\beta}^{\min} < 2$.
for some $\alpha > 0$, as we shall see in Corollary 1.6, (1.22) holds with high probability. Thus, the desired inequalities $0 < \hat{\theta}_\beta^{\max}(\alpha), \omega_\beta^{\min}(\alpha)$ and $\hat{\theta}_\beta^{\max}(\alpha), \omega_\beta^{\min}(\alpha)$.

Theorem 1.4. For all $0 < \beta < 1$ and $0 < \alpha < \pi/12(1-\beta)^2 h_\beta$ with $h_\beta := \min\left(\frac{3}{4} - \frac{1}{2} \beta, 1 - \beta\right)$,

\[0 < \frac{\pi}{6} (1 - \beta) h_\beta + \frac{2\alpha}{1 - \beta} - 2 \sqrt{\pi a h_\beta/3} \leq \theta_\beta^{\max}(\alpha) \leq \min\left(\frac{\pi}{2} \left(\frac{\ln 1}{\beta}\right)^2, 1\right),\] \hspace{1cm} (1.23)

\[0 < \frac{\pi}{6} (1 - \beta)^2 h_\beta + 2\alpha - 2(1 - \beta) \sqrt{\pi a h_\beta/3} \leq \hat{\theta}_\beta^{\max}(\alpha) \leq (1 - \min\left(\frac{\pi}{2} \left(\frac{\ln 1}{\beta}\right)^2, 1\right).\] \hspace{1cm} (1.24)

Moreover, for all $0 < \beta < 1$ and $\alpha > 0$,

\[0 < \max\left(c_g^2 \cdot \ln \frac{2}{1 - \beta} \cdot \chi(1/2,1)(\beta) \cdot \frac{n}{2} \beta^2\right) \leq \omega_\beta^{\min}(\alpha) \leq 2 \ln \frac{e}{1 - \beta} + \frac{2\alpha}{1 - \beta} + 4 \sqrt{\frac{\alpha}{1 - \beta} \ln \frac{e}{1 - \beta}},\] \hspace{1cm} (1.25)

where the absolute constant $c_g$ is defined in (3.15), and

\[1 \leq \omega_\beta^{\min}(\alpha) \leq 1 + \sqrt{12\alpha}, \quad 0 \leq \beta < 1, \quad 0 < \alpha \leq \frac{1}{12}.\] (1.26)

Theorem 1.4 establishes the strong restricted isometry property for Gaussian random matrices for all $\beta \in [0,1)$, since $0 < \hat{\theta}_\beta^{\max}(\alpha) \leq \omega_\beta^{\min}(\alpha) < 2$ for all $0 < \alpha < \min\left(\frac{\pi}{12} (1 - \beta)^2 h_\beta, \frac{1}{12}\right)$. Note that $\theta_\beta^{\max}(\alpha)$ and $\hat{\theta}_\beta^{\max}(\alpha)$ are non-increasing functions of $\alpha$ while $\omega_\beta^{\min}(\alpha)$ and $\omega_\beta^{\min}(\alpha)$ are non-decreasing functions for $\alpha \in (0, \infty)$. As a direct consequence of Theorem 1.4, for all $0 < \beta < 1$, $\lim_{\alpha \to 0^+} \omega_\beta^{\min}(\alpha) = 1$ and

\[\frac{\pi}{8} (1 - \beta)^2 \leq \lim_{\alpha \to 0^+} \theta_\beta^{\max}(\alpha) \leq \frac{(1 - \beta)^2}{(1 - \sqrt{2\pi^2})^2}, \quad \frac{\pi}{8} (1 - \beta)^3 \leq \lim_{\alpha \to 0^+} \hat{\theta}_\beta^{\max}(\alpha) \leq \frac{(1 - \beta)^3}{(1 - e^{-\sqrt{2\pi^2}})^2}.\]

Moreover,

\[\frac{c_g^2 \ln 4}{1 + \ln 2} \ln \frac{e}{1 - \beta} \leq \lim_{\alpha \to 0^+} \omega_\beta^{\min}(\alpha) \leq 2 \ln \frac{e}{1 - \beta}, \quad \forall \frac{1}{2} < \beta < 1.\]

Thus, up to multiplicative constants, our estimates in Theorem 1.4 for $\theta_\beta^{\max}(\alpha)$, $\omega_\beta^{\min}(\alpha)$ and $\hat{\theta}_\beta^{\max}(\alpha)$, $\hat{\omega}_\beta^{\min}(\alpha)$ are optimal as $\beta \to 1^-$ and $\alpha \to 0^+$.

As an application of Theorem 1.4, we have the following robustness properties of Johnson-Lindenstrauss lemma and restricted isometry property with a given erasure ratio $0 < \beta < 1$.

Corollary 1.5. Let $0 < \beta < 1$ and $0 < \alpha < \min\left(\frac{1}{12} \frac{\pi}{12} (1-\beta)^2 h_\beta\right)$ with $h_\beta := \min\left(\frac{3}{4} - \frac{1}{2} \beta, 1 - \beta\right)$. Let $m, n, N \in \mathbb{N}$ be such that $m > \frac{1}{\alpha} \ln \frac{2}{N(N-1)}$. Let $A$ be an $m \times n$ Gaussian random matrix with i.i.d. entries obeying $N(0,1)$. For any given $N$ points $p_1, \ldots, p_N \in \mathbb{R}^n$, with probability at least $1 - 2N(N-1)e^{-am} > 0$,

\[\hat{\theta} \|p_j - p_k\|^2 \leq \frac{1}{m} \|A_T p_j - A_T p_k\|^2 \leq \omega \|p_j - p_k\|^2,\]

\[\frac{\hat{\theta}}{1 - \beta} \|p_j - p_k\|^2 \leq \frac{1}{|T|} \|A_T p_j - A_T p_k\|^2 \leq \omega \|p_j - p_k\|^2,\] \hspace{1cm} (1.27)

\[\forall 1 \leq j, k \leq N, \quad T \subseteq \{1, \ldots, m\}, \quad |T^c| \leq \beta m,\]

where $0 < \hat{\theta} \leq \omega < 2 \beta$ and $\omega > 0$ are given by

\[\hat{\theta} := \frac{\pi}{6} (1 - \beta)^2 h_\beta + 2\alpha - 2(1 - \beta) \sqrt{\pi a h_\beta/3}, \quad \omega := 1 + \sqrt{12\alpha},\] \hspace{1cm} (1.28)
Corollary 1.6. Let $0 < \beta < 1$ and $0 < \alpha < \min\left(\frac{1}{12}, \frac{5}{12}(1-\beta)^2 h_\beta\right)$ with $h_\beta := \min\left(\frac{3}{4} - \frac{1}{2}\beta, 1 - \beta\right)$. Let $m, n, s \in \mathbb{N}$ such that

$$\alpha m > s \ln \frac{24en}{\epsilon s} + \ln 4.$$  \hspace{1cm} (1.29)

Let $A$ be an $m \times n$ Gaussian random matrix with i.i.d. entries obeying $\mathcal{N}(0, 1)$. For any $0 < \epsilon < 1$, with probability at least $1 - 4(\frac{24n}{\epsilon s})^e \epsilon^{-\alpha m} > 0$,

$$\hat{\theta}(1 - \epsilon)\|v\|^2 \leq \frac{1}{m}\|Av\|^2 \leq \hat{\omega}(1 + \epsilon)\|v\|^2,$$

$$\frac{\hat{\theta}}{1 - \beta}(1 - \epsilon)\|v\|^2 \leq \frac{1}{|T|}\|Av\|^2 \leq \omega(1 + \epsilon)\|v\|^2,$$

\[\forall \text{s-sparse } v \in \mathbb{R}^n, \quad T \subseteq \{1, \ldots, m\}, \quad |T^c| \leq \beta m,\]

where the positive real numbers $\hat{\theta}, \hat{\omega}$ and $\omega$ are given in (1.28) with $0 < \hat{\theta} \leq \hat{\omega} < 2$.

Corollary 1.6 also establishes the robustness property for Gaussian random finite frames in $\mathbb{R}^n$ for a given erasure ratio $0 < \beta < 1$. If

$$m > \frac{n \ln \frac{24e}{\epsilon} + \ln 4}{\alpha},$$

(1.31)

then we can directly check that (1.29) holds with $s = n$. Consequently, under the condition in (1.31), with probability at least $1 - 4(\frac{24n}{\epsilon s})^e \epsilon^{-\alpha m} > 0$, the following robustness properties for Gaussian random finite frames hold:

$$\hat{\theta}(1 - \epsilon)\|v\|^2 \leq \frac{1}{m}\|Av\|^2 \leq \hat{\omega}(1 + \epsilon)\|v\|^2,$$

$$\frac{\hat{\theta}}{1 - \beta}(1 - \epsilon)\|v\|^2 \leq \frac{1}{|T|}\|Av\|^2 \leq \omega(1 + \epsilon)\|v\|^2,$$

where $\hat{\theta}, \hat{\omega}$ and $\omega$ are defined in (1.28). Let $\hat{\theta}$ and $\omega$ be defined in (1.28). The ratio between the upper and lower frame bounds in the above second inequality (called the condition number of a frame) satisfies

$$\lim_{\alpha \to 0^+} \frac{(1 + \epsilon)\omega}{(1 - \epsilon)\hat{\theta}/(1 - \beta)} \leq \frac{12(1 + \epsilon)}{\pi(1 - \epsilon)} \frac{\ln \frac{e}{\epsilon}}{\min\left(\frac{3}{4} - \frac{1}{2}\beta, 1 - \beta\right)} = \frac{12(1 + \epsilon)}{\pi(1 - \epsilon)} \frac{\ln \frac{e}{\epsilon}}{(1 - \beta)^2}, \quad \forall \frac{1}{2} \leq \beta < 1.$$

On the other hand, according to Theorem 1.4, as $\alpha \to 0^+$, the ratio between the optimal upper frame bound $\omega(1 - \epsilon)$ and optimal lower frame bound $(1 - \epsilon)\hat{\theta}/(1 - \beta)$ must be approximately no less than

$$\lim_{\alpha \to 0^+} \frac{(1 + \epsilon)\omega}{(1 - \epsilon)\hat{\theta}/(1 - \beta)} \geq \frac{(1 + \epsilon)c^2_\gamma \ln \frac{e}{\epsilon}}{(1 - \epsilon)\min\left(\frac{3}{4} - \frac{1}{2}\beta, 1 - \beta\right)}$$

$$= \frac{2c^2_\gamma(1 + \epsilon)}{\pi(1 - \epsilon)} \frac{\ln \frac{e}{\epsilon}}{(1 - \epsilon)^2} \geq \frac{2c^2_\gamma(1 + \epsilon)}{\pi(1)(ln 2 + 1)(1 - \epsilon) (1 - \beta)^2}$$

for all $1/2 \leq \beta < 1$. This shows that up to a multiplicative constant, our estimate for the robustness property of Gaussian random finite frames in $\mathbb{R}^n$ is asymptotically optimal as $\beta \to 1^-$.

Remark 1.7. If $\beta = 1/2$ in Corollary 1.6, then the Gaussian random matrix satisfies strong RIP with the lower RIP constant $\theta \approx 10^{-2}$ with high probability provided $m = O(s \log(en/s))$ which improves the lower bound $\theta \approx 10^{-5}$, which is presented in [22].

Remark 1.8. It is of interest to extend the main results in this paper from Gaussian random matrices to other random matrices such as sub-Gaussian matrices and circulant matrices (see [23]). If $A$ is the Bernoulli matrix, i.e., $P(a_{j,k} = 1) = P(a_{j,k} = -1) = 1/2$. Define 2-sparse vectors $v_1 := (1, 1, 0, \ldots, 0)^T \in \mathbb{R}^n$ and $v_2 := (1, -1, 0, \ldots, 0)^T \in \mathbb{R}^n$. Then either $\inf\{\frac{1}{m}\|Av_1\| : T \subseteq \{1, \ldots, m\}, |T^c| \leq m/2\} = 0$ or $\sup\{\frac{1}{m}\|Av_2\| : T \subseteq \{1, \ldots, m\}, |T^c| \leq m/2\} = 0$, i.e., for any $\theta > 0$, either $P(\Omega_{[\theta, \infty], 1/2}(A, v_1)) = 0$ or $P(\Omega_{[\theta, \infty], 1/2}(A, v_2)) = 0$ for all $m \in \mathbb{N}$. As a consequence, the strong restricted isometry property for $\beta = 1/2$ cannot hold for Bernoulli random matrices. This shows that the results and study for sub-Gaussian random matrices will be essentially different to Gaussian random matrices. We shall study random matrices other than Gaussian random matrices elsewhere.
1.4 Outline of the paper

The structure of the paper is as follows. In Section 2, we shall provide some auxiliary results for the proofs of the main results in later sections. In Section 3, we shall study the robustness properties of Gaussian random matrices with arbitrarily erased rows for small distortion rates \( \epsilon \to 0^+ \). In particular, we shall prove Theorem 1.1 in Section 3. In certain sense, we studied in Theorem 1.1 the quantities \( \beta_{c,\alpha}^{\max} \) and \( \beta_{c,\alpha}^{\min} \) for the case of small erasure ratios \( \epsilon \to 0^+ \). In Section 4, we shall study the robustness properties of Gaussian random matrices with arbitrarily erased rows for a given corruption/erasure ratio \( 0 < \beta < 1 \). In particular, we are interested in the behavior of \( \beta_{c,\alpha}^{\max} \), \( \beta_{c,\alpha}^{\min} \) and \( \beta_{c,\alpha}^{\max} \), \( \beta_{c,\alpha}^{\min} \) when \( \beta \to 1^- \). We shall prove Theorem 1.4 and other results related to Theorem 1.4 in Section 4. We shall also show that our result leads to the establishment of the strong restricted isometry property for Gaussian random matrices. As applications of the main results in this paper for dimensionality reduction, in Section 5 we shall prove Corollaries 1.2, 1.3, 1.5 and 1.6.

2 Auxiliary results

In this section, we provide some auxiliary results that will be used in later sections. For \( y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m \), we define \( y_1, \ldots, y_m \) to be the nonincreasing rearrangements of \( y_1, \ldots, y_m \) in terms of magnitudes such that \( |y_1| \geq \cdots \geq |y_m| \). Let \( m \in \mathbb{N} \). For any \( 0 \leq \gamma \leq 1 \) such that \( \gamma m \) is an integer, we define

\[
T_\gamma := \{ T \subseteq \{1, \ldots, m\} : |T^c| = \gamma m \}.
\]

The following simple observation will facilitate our discussion in later sections.

**Lemma 2.1.** For \( 0 \leq \gamma \leq \beta \leq 1 \) such that both \( \gamma m \) and \( \beta m \) are integers,

\[
\min_{T \in T_\beta} \frac{1}{|T|} \| A_T x_0 \|_2^2 \leq \min_{T \in T_\gamma} \frac{1}{|T|} \| A_T x_0 \|_2^2 \leq \max_{T \in T_\gamma} \frac{1}{|T|} \| A_T x_0 \|_2^2 \leq \max_{T \in T_\beta} \frac{1}{|T|} \| A_T x_0 \|_2^2
\]

and

\[
\max_{T \in T_\gamma} \left| \frac{1}{|T|} \| A_T x_0 \|_2^2 - 1 \right| \leq \max_{T \in T_\beta} \left| \frac{1}{|T|} \| A_T x_0 \|_2^2 - 1 \right|.
\]

**Proof.** Let \( k_\gamma := \gamma m \) and \( k_\beta := \beta m \). By \( 0 \leq \gamma \leq \beta \), we have \( k_\gamma \leq k_\beta \) and it follows from \( |y_1| \geq \cdots \geq |y_m| \) that

\[
\min_{T \in T_\beta} \frac{\| A_T x_0 \|_2^2}{|T|} = \frac{y_{\beta(k_\beta+1)}^2 + \cdots + y_{\beta m}^2}{m - k_\beta} \leq \frac{y_{\gamma(k_\gamma+1)}^2 + \cdots + y_{\gamma m}^2}{m - k_\gamma} = \min_{T \in T_\gamma} \frac{\| A_T x_0 \|_2^2}{|T|} \leq \max_{T \in T_\gamma} \frac{\| A_T x_0 \|_2^2}{|T|} = \frac{y_{\gamma(k_\gamma+1)}^2 + \cdots + y_{\gamma m}^2}{m - k_\gamma} \leq \frac{y_{\beta(k_\beta+1)}^2 + \cdots + y_{\beta m}^2}{m - k_\beta} = \max_{T \in T_\beta} \frac{\| A_T x_0 \|_2^2}{|T|}.
\]

This proves (2.2). The inequality

\[
\max_{T \in T_\gamma} \left| \frac{1}{|T|} \| A_T x_0 \|_2^2 - 1 \right| \leq \max_{T \in T_\beta} \left| \frac{1}{|T|} \| A_T x_0 \|_2^2 - 1 \right|
\]

follows directly from the above inequalities.

The following well-known concentration inequalities for the standard Gaussian/normal distribution (see [20]) will be used later.

**Theorem 2.2.** Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a Lipschitz function with Lipschitz constant 1 satisfying \( |f(x) - f(y)| \leq \|x - y\| \) for all \( x, y \in \mathbb{R}^m \). For i.i.d. standard Gaussian/normal random variables \( X_1, \ldots, X_m \sim \mathcal{N}(0, 1) \) and for all \( \delta \geq 0 \),

\[
P\{f(X_1, \ldots, X_m) < \delta + \mathbb{E}[f(X_1, \ldots, X_m)]\} \geq 1 - e^{-\delta^2/2}, \tag{2.3}
\]

\[
P\{f(X_1, \ldots, X_m) > -\delta + \mathbb{E}[f(X_1, \ldots, X_m)]\} \geq 1 - e^{-\delta^2/2}. \tag{2.4}
\]
Lemma 2.3. Let $y_1, \ldots, y_m$ be i.i.d. Gaussian/normal random variables obeying $N(0, 1)$. Then for all nonempty subsets $S \subseteq \{1, \ldots, m\}$ and $\delta > 0$,

\[
P\left\{ \sqrt{\frac{1}{|S|} \sum_{j \in S} y_{(j)}^2} < \delta + E \sqrt{\frac{1}{|S|} \sum_{j \in S} y_{(j)}^2} \right\} \geq 1 - e^{-\delta^2|S|/2} \tag{2.5}
\]

and

\[
P\left\{ \sqrt{\frac{1}{|S|} \sum_{j \in S} y_{(j)}^2} > -\delta + E \sqrt{\frac{1}{|S|} \sum_{j \in S} y_{(j)}^2} \right\} \geq 1 - e^{-\delta^2|S|/2}. \tag{2.6}
\]

Proof. Define $F_S(x) := \sqrt{\sum_{j \in S} y_{(j)}^2}$. Then it is easy to observe that

\[
|F_S(x) - F_S(y)|^2 \leq \sum_{j \in S} (|x_{(j)}| - |y_{(j)}|)^2 \leq \sum_{j=1}^m (|x_{(j)}| - |y_{(j)}|)^2 = \|x\|^2 + \|y\|^2 - 2 \sum_{j=1}^m |x_{(j)}y_{(j)}| \leq \|x - y\|^2,
\]

where in the last step we used the rearrangement inequality $\sum_{j=1}^m |x_jy_j| \leq \sum_{j=1}^m |x_{(j)}y_{(j)}|$. Therefore, $F_S$ is a Lipschitz function with Lipschitz constant 1. By Theorem 2.2, we have

\[
P\left\{ \sqrt{\frac{1}{|S|} \sum_{j \in S} y_{(j)}^2} < \delta + E \sqrt{\frac{1}{|S|} \sum_{j \in S} y_{(j)}^2} \right\} = P\left\{ \sum_{j \in S} y_{(j)}^2 < \delta |S| + E \sum_{j \in S} y_{(j)}^2 \right\} \geq 1 - e^{-\delta^2|S|/2}.
\]

The inequality in (2.6) can be proved similarly. \qed

The following result extends [14, Example 10]. We provide a proof here by modifying the proof of [14, Example 10].

Lemma 2.4. Let $y_1, \ldots, y_m \in \mathbb{R}^m$ be i.i.d. Gaussian/normal random variables obeying $N(0, 1)$ and define $y_{(1)}, \ldots, y_{(m)}$ to be the nonincreasing rearrangements of $y_1, \ldots, y_m$ in terms of magnitudes such that $|y_{(1)}| \geq \cdots \geq |y_{(m)}|$.

(i) For $1 \leq j \leq m$ and $1 \leq p \leq 2$,

\[
\sqrt{\frac{\pi m + 1 - j}{2}} \leq E|y_{(j)}| \quad \text{and} \quad E|y_{(j)}|^p \leq C_p \sum_{\ell=j}^m \frac{1}{\ell} \leq C_p \left( \frac{1}{j} + \ln \frac{m}{j} \right), \tag{2.7}
\]

where $C_p$ is a positive constant (e.g., $C_1 \leq \sqrt{\pi}, C_2 \leq 2$) depending only on $p$ and given by

\[
C_p := p \sup_{0 < t < \infty} t^{p-1} \int_0^\infty e^{(t^2-x^2)/2} \, dx \leq p \left( \frac{\pi}{2} \right)^{1/2}. \tag{2.8}
\]

(ii) For $1 \leq k \leq m$,

\[
E\left( \sqrt{\frac{1}{k} \sum_{j=1}^k y_{(j)}^2} \right) \leq \sqrt{2 \ln \frac{em}{k}}. \tag{2.9}
\]

(iii) Let $0 \leq \gamma < 1$ and $m \in \mathbb{N}$ such that $k := \gamma m \in \mathbb{N}$. Then

\[
\sqrt{\frac{\pi}{6}} \sqrt{(1-\gamma) \frac{1-\gamma + \frac{m}{2}}{1 + \frac{m}{2}}} \leq E\left( \sqrt{\frac{1}{m-k} \sum_{j=k+1}^m y_{(j)}^2} \right) \leq \sqrt{2 - \frac{2\gamma}{1-\gamma} \ln \frac{1 + \frac{m}{\gamma}}{1 + \frac{m}{\gamma}}}. \tag{2.10}
\]

Proof. Define $u(t) := \sqrt{\frac{2}{\pi}} \int_t^\infty e^{-x^2/2} \, dx$. As shown in [14, Example 10],

\[
E|y_{(j)}|^p = \int_0^\infty P\{ |y_{(j)}| > t^{1/p} \} \, dt = \sum_{\ell=0}^{m-j} \binom{m}{\ell} \int_0^{t^{1/p}} (u(t^{1/p}))^{m-\ell - (1 - u(t^{1/p}))^{\ell}} \, dt
\]
\[ E|y(j)| = \sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{m-j} \frac{m}{\ell} \int_0^\infty (u(t))^{m-\ell} (1-u(t))^{\ell} e^{pt-1} e^{\ell^2/2} (-du(t)). \]

(i) Since \( e^{\ell^2/2} \geq 1 \) for all \( t \in \mathbb{R} \), for \( p = 1 \), by a change of variable \( x = u(t) \), as proved in [14, Example 10],

\[ E|y(j)| \geq \sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{m-j} \frac{m}{\ell} \int_0^1 x^{m-\ell} (1-x)^{\ell} dx = \sqrt{\frac{\pi}{2}} \frac{m+1-j}{m+1}. \]

For \( 1 \leq p \leq 2 \), by a change of variable \( x = u(t) \), we deduce that

\[ E|y(j)|^p = \sum_{\ell=0}^{m-j} \int_0^\infty \left( \frac{m}{\ell} \right)^p (u(t))^{m-\ell} (1-u(t))^{\ell} (-du(t)) \]

\[ \leq C_p \sum_{\ell=0}^{m-j} \frac{1}{m-\ell} = C_p \left( \frac{1}{j} + \sum_{\ell=j+1}^{m} \frac{1}{\ell} \right) \leq C_p \left( \frac{1}{j} + \int_j^m \frac{1}{x} dx \right) = C_p \left( \frac{1}{j} + \ln \frac{m}{j} \right). \]

It is easy to prove that if \( 1 \leq p \leq 2 \), then \( C_p < \infty \). Indeed, define

\[ f(t) := t^{p-1} \int_t^\infty e^{(t^2-s^2)/2} ds = t^{p-1} \int_0^\infty e^{-ts-t^2/2} ds = t^{p-1} \int_0^\infty e^{ts-t^2/2} dt = \frac{\pi}{2} t^{p-1}. \]

We also have

\[ f(t) = t^{p-1} \int_0^t e^{-ts-t^2/2} ds \leq t^{p-1} \int_0^\infty e^{-ts} ds = t^{p-2}. \]

Therefore, \( C_p = p \sup_{0 < t < \infty} f(t) \leq p \sup_{0 < t < \infty} \min(\sqrt{\frac{\pi}{2}} t^{p-1}, t^{p-2}) = p(\frac{\pi}{2})^{1-\frac{p}{2}} < \infty. \)

(ii) By (2.8) with \( p = 2 \), we have \( C_2 \leq 2 \) and

\[ E\left( \frac{1}{k} \sum_{j=1}^k y_j^2 \right) \leq \frac{1}{k} \sum_{j=1}^k \frac{2}{2} \sum_{j=1}^m \frac{1}{\ell} \leq \sqrt{2 + 2 \ln \frac{m}{k}} = 2 \ln \frac{m}{k} \]

by \( \sum_{\ell=k+1}^{m} \frac{1}{\ell} \leq \int_k^m \frac{1}{x} dx = \ln \frac{m}{k} \). This proves (2.9).

(iii) By Cauchy-Schwarz inequality, we have

\[ \sum_{j=k+1}^m \frac{m+1-j}{m+1} |y(j)| \leq \sqrt{\sum_{j=k+1}^m \left( \frac{m+1-j}{m+1} \right)^2} \sqrt{\sum_{j=k+1}^m y_j^2}. \]

Therefore, it follows from the first inequality in (2.7) that

\[ E\left( \frac{1}{m-k} \sum_{j=k+1}^m y_j^2 \right) \geq \frac{1}{\sqrt{m-k}} \left( \sum_{j=k+1}^m \left( \frac{m+1-j}{m+1} \right)^2 \right)^{-1/2} \sum_{j=k+1}^m \frac{m+1-j}{m+1} E|y(j)| \]

\[ \geq \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{m-k}} \left( \sum_{j=k+1}^m \left( \frac{m+1-j}{m+1} \right)^2 \right)^{-1/2} \sum_{j=k+1}^m \frac{m+1-j}{m+1} \]

\[ = \sqrt{\frac{\pi}{2}} \left( \frac{1}{m-k} \sum_{j=k+1}^m \left( \frac{m+1-j}{m+1} \right)^2 \right) \sum_{j=1}^{m-k} j^2 \]
Lemma 2.5. For any positive real number $\epsilon > 0$,

$$\epsilon_g := \max_{x \geq 2} \frac{\epsilon_g^2 \ln(2x) - 1}{x - 1} = \frac{\epsilon_g^2 \ln 4 - 1}{\frac{1}{\beta_g} - 1} = \begin{cases} 
\epsilon_g^2 \ln 4 - 1 > 0, & \text{if } \epsilon_g^2 \ln \frac{4}{\sqrt{e}} > 1, \\
-\epsilon_g^2 W_0(-2e^{-1-1/\epsilon_g}) > 0, & \text{if } \epsilon_g^2 \ln \frac{4}{\sqrt{e}} \leq 1
\end{cases}$$

(2.11)

and

$$0 < \beta_g \leq \frac{1}{2} \text{ with } \beta_g := \begin{cases} 
\frac{1}{2}, & \text{if } \epsilon_g^2 \ln \frac{4}{\sqrt{e}} > 1, \\
-W_0(-2e^{-1-1/\epsilon_g}), & \text{if } \epsilon_g^2 \ln \frac{4}{\sqrt{e}} \leq 1.
\end{cases}$$

(2.12)

Proof. Let $t_g := 2e^{-1-1/\epsilon_g}$. If $0 < \epsilon_g^2 \ln \frac{4}{\sqrt{e}} \leq 1$, then $0 < t_g < e^{-1}$ and therefore, both $W_0(-t_g)$ and $\beta_g$ are well-defined. Since $W_0$ is an increasing function, it is also easy to prove that

$$-W_0(-t_g) \leq \frac{1}{2} \Leftrightarrow t_g \leq \frac{1}{2}e^{-1/2} \Leftrightarrow \epsilon_g^2 \ln \frac{4}{\sqrt{e}} \leq 1.$$  

(2.13)

To prove (2.11), we define $f(x) := \frac{\epsilon_g^2 \ln(2x) - 1}{x - 1}$ for $x > 1$. Then

$$f'(x) = \frac{g(x)}{x(x - 1)^2} \text{ with } g(x) := x + \epsilon_g^2(x - x \ln(2x) - 1).$$

By $g'(x) = 1 - \epsilon_g^2 \ln(2x)$, the function $g$ increases on $(0, \frac{1}{\epsilon_g})$ and decreases on $(\frac{1}{\epsilon_g}, \infty)$. Note that $f'$ has the same sign as $g$ on $(1, \infty)$. If $t_g \geq e^{-1}$, then $\epsilon_g^2 \ln 2 \geq 1$ and $g'(x) \leq 0$ for all $x > 1$. Hence, $f'(x) \leq 0$ for all $x > 1$ and $f$ decreases on $(1, \infty)$. Therefore, $\epsilon_g = \max_{x \geq 2} f(x) = f(2) = f(1/\beta_g)$, since $\beta_g = 1/2$ by $\epsilon_g^2 \ln \frac{4}{\sqrt{e}} > \epsilon_g^2 \ln 2 \geq 1$. Since $g'$ has only one real root on $(0, \infty)$, if $t_g < e^{-1}$, then $g$ has at most two real roots on $(0, \infty)$ given by $x_1 := -W_{-1}(-t_g) < 1 < -W_0(-t_g) := x_2$. Thus, $f$ decreases on $(x_2, \infty)$ and increases on $(1, x_2)$, from which we have $\epsilon_g = \max_{x \geq 2} f(x) = f(\max(2, x_2)) = f(1/\beta_g)$, since $\max(2, x_2) = 1/\beta_g$ by (2.13).

$$\Box$$

3 Gaussian random matrices under arbitrary erasure of rows for small $\epsilon$

In this section, we study the robustness property of (normalized) Gaussian random matrices under arbitrary erasure of rows for small $0 < \epsilon < 1$. At the end of this section, we shall prove Theorem 1.1.
3.1 A lower bound for $\beta_{e,\alpha}^{\max}$

To provide a lower bound for $\beta_{e,\alpha}^{\max}$ in (1.8), we first prove the following result.

**Lemma 3.1.** Let $A$ be an $m \times n$ random matrix with i.i.d. entries obeying $\mathcal{N}(0,1)$. For $0 < \alpha < 1$ and $0 < \epsilon \leq \min(1, \frac{1 - \sqrt{\alpha}}{\sqrt{\alpha}^2})$, if

$$
0 < \beta \leq \frac{1 - \sqrt{\alpha}}{1 + \epsilon} \quad \text{and} \quad 0 < \beta \ln \frac{e}{\beta} \leq \frac{\epsilon}{2} \left(1 - \sqrt{\alpha} - \frac{\alpha \epsilon}{2}\right)^2,
$$

then

$$
P(\Omega_{e,\beta}) \geq 1 - 3e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}, \quad \forall m \in \mathbb{N}. 
$$

**Proof.** Set $y := Ax_0$. Since each entry in $A$ is an i.i.d. random variable obeying $\mathcal{N}(0,1)$ and since $\|x_0\| = 1$, a simple calculation leads to $y_j \sim \mathcal{N}(0,1)$ for every $j = 1, \ldots, m$ and all $y_1, \ldots, y_m$ are independent.

Let $m \in \mathbb{N}$ be arbitrary but fixed at this moment. Define $\gamma := |\beta m|/m$. Then $0 \leq \gamma \leq \beta$ and $\gamma m \in \mathbb{N}$. If $\gamma = 0$, then $\beta m < 1$ and consequently $|T^c| \leq \beta m$ implies that $T^c$ is the empty set. Therefore, if $\gamma = 0$, then $\Omega_{e,\beta} = \Omega_{e,0}$ and the claim in (3.2) is trivially true by (1.3). Thus, in the following discussion, we assume $\gamma > 0$.

By Lemma 2.1, we conclude that

$$
P(\Omega_{e,\beta}) = P\left\{1 - \epsilon \leq \frac{\|y_T\|^2}{|T|} \leq 1 + \epsilon : \forall |T^c| \leq \beta m\right\} = P\left\{1 - \epsilon \leq \min_{T \in T_\gamma} \frac{\|y_T\|^2}{|T|} \leq \max_{T \in T_\gamma} \frac{\|y_T\|^2}{|T|} \leq 1 + \epsilon\right\},
$$

where $T_\gamma$ is defined in (2.1). Recall that $y_1, \ldots, y_m$ are nonincreasing rearrangements of $y_1, \ldots, y_m$ in terms of magnitude such that $|y_1| \geq |y_2| \geq \cdots \geq |y_m|$. Let $T \in T_\gamma$ and $k := |T^c| = \gamma m$. Then $|T| = m - k$. Since $\|y_T\|^2 = \|y\|^2 - \|y_{T^c}\|^2$, we observe that

$$
\|y\|^2 - (y_1^2 + \cdots + y_k^2) \leq \|y_T\|^2 \leq \|y\|^2 - (y_{(m-k+1)}^2 + \cdots + y_{(m)}^2).
$$

Using the above inequalities and noting that $k = \gamma m$, we can easily deduce that

$$
P\left\{1 - \epsilon \leq \min_{T \in T_\gamma} \frac{\|y_T\|^2}{|T|} \leq \max_{T \in T_\gamma} \frac{\|y_T\|^2}{|T|} \leq 1 + \epsilon\right\} = P\left\{1 - \epsilon \leq \frac{\|y\|^2 - (y_1^2 + \cdots + y_k^2)}{m - k} \leq \frac{\|y\|^2 - (y_{(m-k+1)}^2 + \cdots + y_{(m)}^2)}{m - k} \leq 1 - \epsilon\right\} = P(\gamma m - k + 1) \leq (1 - \gamma)(1 - \epsilon)
$$

and

$$
\frac{y_1^2 + \cdots + y_k^2}{k} \leq \frac{\|y\|^2}{k} \leq \frac{\|y\|^2}{k} - (1 - \gamma)(1 + \epsilon)
$$

$$
\geq P(E_0 \cap E_1 \cap E_2) = 1 - P(E_0^c \cup E_1^c \cup E_2^c) \geq 1 - P(E_0^c) - P(E_1^c) - P(E_2^c),
$$

where

$$
E_0 := \left\{1 - \sqrt{\alpha} \leq \frac{\|y\|^2}{m} \leq 1 + \sqrt{\alpha}\right\},
$$

$$
E_1 := \left\{\frac{y_1^2 + \cdots + y_k^2}{k} \leq 1 - \epsilon + \frac{1 - \sqrt{\alpha}}{\gamma}\right\},
$$

$$
E_2 := \left\{\frac{y_{(m-k+1)}^2 + \cdots + y_{(m)}^2}{k} \geq 1 + \epsilon - \frac{1 - \sqrt{\alpha}}{\gamma}\right\}.
$$
Since $E_0 = \{| \frac{1}{m} \| y \|^2 - 1 \| \leq \sqrt{\alpha} \epsilon \}$, it follows directly from (1.3) that $P(E_0) \geq 1 - 2e^{-(\alpha \epsilon^2/4 - \alpha^3/2 \epsilon^3/6)m}$. Thus, by $0 < \alpha < 1$, we have

$$P(E_0^c) \leq 2e^{-(\alpha \epsilon^2/4 - \alpha^3/2 \epsilon^3/6)m} = 2e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}e^{-(1-\sqrt{\alpha})\alpha^2m/6} \leq 2e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}.$$  

Next, we estimate $P(E_1)$ and $P(E_2)$. By (2.9) of Lemma 2.4 and noting that $k = \gamma m$, we have

$$\mathbb{E} \left[ \frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2 \right] \leq \sqrt{2 \ln \frac{e}{\gamma}}.$$  

For $\delta > 0$, it follows from (2.5) of Lemma 2.3 with $S = \{1, \ldots, k\}$ that

$$P \left\{ \frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2 \leq \delta + \sqrt{2 \ln \frac{e}{\gamma}} \right\} \geq P \left\{ \frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2 \leq \delta + \mathbb{E} \left[ \frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2 \right] \right\} \geq 1 - e^{-\delta^2 \gamma m/2}. \quad (3.7)$$

Take

$$\delta := \sqrt{1 - \epsilon - \frac{1 - \sqrt{\alpha}}{\gamma} \epsilon} - \sqrt{2 \ln \frac{e}{\gamma}}. \quad (3.8)$$

We claim that

$$\delta \geq \sqrt{\frac{\alpha \epsilon^2}{2 \gamma}} > 0. \quad (3.9)$$

Then it follows from (3.7) and the above inequality that

$$P(E_1^c) = P \left\{ \frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2 > \sqrt{1 - \epsilon - \frac{1 - \sqrt{\alpha}}{\gamma} \epsilon} \right\} \leq e^{-\delta^2 \gamma m/2} \leq e^{-\alpha \epsilon^2 m/4} \leq e^{-(\epsilon^2/4 - \epsilon^3/6)m}.$$  

On the other hand, by the first inequality in (3.1) and the fact that $0 < \gamma \leq \beta$, we have

$$1 + \epsilon - \frac{1 - \sqrt{\alpha}}{\beta} \epsilon \leq 1 + \epsilon - \frac{1 - \sqrt{\alpha}}{\beta} \epsilon \leq 1 + \epsilon - (1 + \epsilon) = 0,$$

which yields

$$P(E_2) = P \left\{ \frac{y^2_{(m)} + \cdots + y^2_{(m-k+1)}}{k} \geq 1 + \epsilon - \frac{1 - \sqrt{\alpha}}{\gamma} \epsilon \right\} \geq P \{ y^2_{(m)} + \cdots + y^2_{(m-k+1)} \geq 0 \} = 1,$$

i.e., $P(E_2^c) = 0$. Putting all estimates together, we complete the proof of (3.2).

We now prove (3.9). By our assumption $0 < \epsilon \leq \frac{1 - \sqrt{\alpha}}{\beta \epsilon^2}$, we observe that

$$f_{\epsilon, \alpha} \geq 0 \quad \text{with} \quad f_{\epsilon, \alpha} := \frac{\epsilon}{2} \left( \sqrt{1 - \sqrt{\alpha}} - \sqrt{\alpha} \right). \quad (3.10)$$

Now, it is straightforward to check that

$$2x^2 + 2\sqrt{\alpha \epsilon^2}x \leq 2f_{\epsilon, \alpha}^2 + 2\sqrt{\alpha \epsilon^2} f_{\epsilon, \alpha} = (1 - \sqrt{\alpha}) \epsilon - \frac{\alpha \epsilon^2}{2}, \quad \text{for all} \quad x \in [0, f_{\epsilon, \alpha}]. \quad (3.11)$$

Note that the function $x \ln \frac{e}{x}$ is an increasing function on the interval $(0, 1]$. Since $0 < \gamma \leq \beta < 1$, by the second inequality in (3.1), we have

$$0 < \gamma \ln \frac{e}{\gamma} \leq \sqrt{\beta \ln \frac{e}{\beta}} \leq f_{\epsilon, \alpha}.$$  

Consequently, plugging $x = \sqrt{\gamma \ln \frac{e}{\gamma}}$ into the inequality in (3.11), we deduce that

$$2\gamma \ln \frac{e}{\gamma} + 2\sqrt{\alpha \epsilon^2} \sqrt{\gamma \ln \frac{e}{\gamma}} \leq (1 - \sqrt{\alpha}) \epsilon - \frac{\alpha \epsilon^2}{2}.$$  

Since \( \gamma > 0 \), dividing \( \gamma \) on both sides of the above inequality, we obtain

\[
2 \ln \frac{e}{\gamma} + 2 \sqrt{\frac{\alpha \epsilon^2}{2 \gamma}} \sqrt{2 \ln \frac{e}{\gamma}} \leq \frac{1 - \sqrt{\alpha} \epsilon}{\gamma} - \frac{\alpha \epsilon^2}{2 \gamma}.
\]

Hence, by \( 1 - \epsilon \geq 0 \), we conclude that

\[
\left( 2 \ln \frac{e}{\gamma} + \sqrt{\frac{\alpha \epsilon^2}{2 \gamma}} \right)^2 = \frac{\alpha \epsilon^2}{2 \gamma} (\ln 2 + \gamma) + \frac{\alpha \epsilon^2}{2 \gamma} \leq \frac{1 - \sqrt{\alpha} \epsilon}{\gamma} \leq 1 - \epsilon + \frac{1 - \sqrt{\alpha} \epsilon}{\gamma},
\]
i.e., we proved

\[
\sqrt{2 \ln \frac{e}{\gamma} + \sqrt{\frac{\alpha \epsilon^2}{2 \gamma}}} \leq \sqrt{1 - \epsilon + \frac{1 - \sqrt{\alpha} \epsilon}{\gamma}},
\]
which is simply the inequality in (3.9).

The following result establishes a lower bound for \( \beta_{\epsilon, \alpha}^{\text{max}} \) in (1.8).

**Theorem 3.2.** Let \( A \) be an \( m \times n \) random matrix with i.i.d. entries obeying \( \mathcal{N}(0,1) \). For \( 0 < \alpha < 1 \) and \( 0 < \epsilon \leq \min(1, \frac{1 - \sqrt{\alpha}}{4 \alpha}) \), if

\[
0 < \beta \leq \frac{(1 - \sqrt{\alpha}) \epsilon}{16 \ln \frac{1}{(1 - \sqrt{\alpha}) \epsilon}},
\]
then (3.2) holds. Consequently,

\[
\beta_{\epsilon, \alpha}^{\text{max}} > \frac{\epsilon}{\ln \frac{1}{\epsilon}} \frac{1 - \sqrt{\alpha}}{32}, \quad \forall 0 < \epsilon \leq \frac{1 - \sqrt{\alpha}}{4}.
\]

**Proof.** We first show that (3.12) implies (3.1) in Lemma 3.1. Since \( 0 < \epsilon \leq 1 \) and \( 0 < \alpha < 1 \), we have \( 0 < (1 - \sqrt{\alpha}) \epsilon \leq 1 \). The first inequality in (3.1) follows from (3.12), since

\[
0 < \beta \leq t_{\epsilon, \alpha} := \frac{(1 - \sqrt{\alpha}) \epsilon}{16 \ln \frac{1}{(1 - \sqrt{\alpha}) \epsilon}} \leq \frac{(1 - \sqrt{\alpha}) \epsilon}{16 \ln 4} < \frac{(1 - \sqrt{\alpha}) \epsilon}{2 < (1 - \sqrt{\alpha}) \epsilon \leq (1 - \sqrt{\alpha}) \epsilon}.
\]

Let \( f_{\epsilon, \alpha} \) be defined in (3.10). By \( 0 < \epsilon \leq \frac{1 - \sqrt{\alpha}}{4 \alpha} \), we have \( \frac{\alpha \epsilon}{2} \leq \frac{1 - \sqrt{\alpha}}{4} \) and hence, \( f_{\epsilon, \alpha} > 0 \) and

\[
f_{\epsilon, \alpha}^2 = \frac{\epsilon}{2} \left( \sqrt{1 - \alpha} - \sqrt{\frac{\alpha \epsilon}{2}} \right)^2 \geq \frac{\epsilon}{2} \left( \sqrt{1 - \alpha} - \sqrt{\frac{1 - \sqrt{\alpha}}{8}} \right)^2 = \frac{9 - 4 \sqrt{2}}{16} (1 - \sqrt{\alpha}) \epsilon.
\]

A basic calculation shows that

\[
\ln(4ez) < (8 - 4 \sqrt{2}) z, \quad \forall z \geq \ln 4,
\]
from which it is straightforward to deduce, by setting \( z = \ln(1/x) \), that

\[
\frac{x}{4 \ln(1/x)} \ln \frac{4e \ln(1/x)}{x} < \frac{9 - 4 \sqrt{2}}{4} x, \quad \forall 0 < x \leq \frac{1}{4}.
\]

Plugging \( x := \frac{(1 - \sqrt{\alpha}) \epsilon}{4} \leq 1/4 \) into the above inequality, by (3.14), we conclude that

\[
t_{\epsilon, \alpha} \frac{\epsilon}{\ln(1/x)} = \frac{x}{4 \ln(1/x)} \ln \frac{4e \ln(1/x)}{x} < \frac{9 - 4 \sqrt{2}}{4} x = \frac{9 - 4 \sqrt{2}}{16} (1 - \sqrt{\alpha}) \epsilon \leq f_{\epsilon, \alpha}^2.
\]

Since \( \beta \ln \frac{e}{\epsilon} \) is an increasing function on \( (0, 1] \), the second inequality in (3.1) follows from (3.12) by noting that \( 0 < \beta \ln \frac{e}{\epsilon} \leq t_{\epsilon, \alpha} \ln \frac{e}{t_{\epsilon, \alpha}} < f_{\epsilon, \alpha}^2 \).

Since \( t_{\epsilon, \alpha} < \frac{(1 - \sqrt{\alpha}) \epsilon}{1 + \epsilon} \) and \( t_{\epsilon, \alpha} \ln \frac{e}{t_{\epsilon, \alpha}} < f_{\epsilon, \alpha}^2 \), there must exist \( \delta > 0 \) such that (3.1) holds for all \( 0 < \beta \leq t_{\epsilon, \alpha} + \delta \). Note that \( \epsilon \leq \frac{1 - \sqrt{\alpha}}{4} < \frac{1 - \sqrt{\alpha}}{4 \alpha} \) by \( 0 < \alpha < 1 \). We have \( \ln \frac{e}{\epsilon} \geq \ln \frac{1}{\epsilon} = \ln \frac{1}{\epsilon} + \ln \frac{4}{1 - \sqrt{\alpha}} \) and therefore,

\[
\beta_{\epsilon, \alpha}^{\text{max}} > t_{\epsilon, \alpha} + \delta > t_{\epsilon, \alpha} = \frac{(1 - \sqrt{\alpha}) \epsilon}{16 \ln \frac{1}{\epsilon} + \ln \frac{4}{1 - \sqrt{\alpha}}} \geq \frac{(1 - \sqrt{\alpha}) \epsilon}{32 \ln \frac{1}{\epsilon}}.
\]

This proves (3.13). \( \square \)
3.2 An upper bound for $\beta_{c,\alpha}^{\max}$

We now show that the order $\beta_{c,\alpha}^{\max} = O(\epsilon/\ln \frac{1}{\beta})$ for small $\epsilon$ given in Theorem 3.2 is optimal. To do so, we recall a well-known inequality on order statistics (see (ii) of [14, Example 10] or see [10, Lemma 3.3.1]): there exists an absolute positive constant $c_g$ (depending only on the Gaussian/normal distribution $\mathcal{N}(0, 1)$) such that

$$c_g \sqrt{\ln \frac{2m}{j}} \leq E|y(j)|, \quad \forall 1 \leq j \leq m/2. \quad (3.15)$$

We first estimate the quantity $P(\Omega_{c,\beta})$.

**Lemma 3.3.** Let $m \in \mathbb{N}$ and $0 < \beta \leq 1/2$ be such that $\beta m \in \mathbb{N}$. Let $\alpha > 0$ and $0 < \epsilon \leq 1$. If

$$\delta := c_g \sqrt{\beta \ln \frac{2}{\beta} - \sqrt{(1 - \epsilon)\beta + \epsilon} > 0}, \quad (3.16)$$

then

$$P(\Omega_{c,\beta}) \leq e^{-(\epsilon^2/4 - \epsilon^3/6)\beta m} + e^{-\delta^2 m/2}. \quad (3.17)$$

**Proof.** Let $y := Ax_0$ and $k := \beta m$. By Lemma 2.1 and (3.3), we have

$$P(\Omega_{c,\beta}) = P\left\{ \max_{T \in T_{a}} \left| \frac{1}{|T|} \sum_{T \in T_{a}} \|y_T\|^2 - 1 \right| \leq \epsilon \right\} \leq P\left\{ \min_{T \in T_{a}} \|y_T\|^2 \geq (1 - \epsilon)(1 - \beta)m \right\}$$

$$= P\left\{ \|y\|^2 - (y_{(1)}^2 + \cdots + y_{(k)}^2) \geq (1 - \epsilon)(1 - \beta)m \right\}$$

$$\leq P\left\{ \frac{1}{m} \|y\|^2 > 1 + \epsilon \right\} + P\left\{ y_{(1)}^2 + \cdots + y_{(k)}^2 \leq \|y\|^2 - (1 - \epsilon)(1 - \beta)m \text{ and } \frac{1}{m} \|y\|^2 \leq 1 + \epsilon \right\}$$

$$\leq P\left\{ \frac{1}{m} \|y\|^2 > 1 + \epsilon \right\} + P\left\{ y_{(1)}^2 + \cdots + y_{(k)}^2 \leq \eta m \right\},$$

where $\eta := (1 - \epsilon)\beta + \epsilon + \delta > 0$ since $0 < \epsilon \leq 1$. It follows directly from (1.2) that

$$P\left\{ \frac{1}{m} \|y\|^2 > 1 + \epsilon \right\} \leq e^{-(\epsilon^2/4 - \epsilon^3/6)\beta m}. \quad (3.18)$$

On the other hand,

$$P\left\{ y_{(1)}^2 + \cdots + y_{(k)}^2 \leq \eta m \right\} = P\left\{ \sqrt{\frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2} - \sqrt{\frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2} \leq \sqrt{\eta/\beta} - \sqrt{\frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2} \right\}. \quad (3.19)$$

Since $k = \beta m \leq m/2$, it follows from the inequality in (3.15) and $|y_{(1)}| \geq \cdots \geq |y_{(k)}|$ that

$$E\sqrt{\frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2} \geq E(|y_{(k)}|) \geq c_g \sqrt{\ln \frac{2m}{k}} = c_g \sqrt{2 \ln \frac{2m}{\beta}}.$$

Therefore,

$$P\left\{ y_{(1)}^2 + \cdots + y_{(k)}^2 \leq \eta m \right\} = P\left\{ \sqrt{\frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2} - \sqrt{\frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2} \leq \sqrt{\eta/\beta} - \sqrt{\frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2} \right\}$$

$$\leq P\left\{ \sqrt{\frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2} - \sqrt{\frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2} \leq \sqrt{\eta/\beta} - c_g \sqrt{2 \ln \frac{2m}{\beta}} \right\}$$

$$= P\left\{ \sqrt{\frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2} - \sqrt{\frac{1}{k} \sum_{j=1}^{k} y_{(j)}^2} \leq -\delta \right\}. \quad (3.20)$$
By our assumption $\delta > 0$, applying (2.6) in Lemma 2.3 with $S = \{1, \ldots, k\}$, we get
\[
P\{\sum_{j=1}^{k} y_j^2(1) + \cdots + y_j^2(k) \leq \eta m\} \leq P\left\{\sqrt{\frac{1}{k} \sum_{j=1}^{k} y_j^2} - E\left\{\sqrt{\frac{1}{k} \sum_{j=1}^{k} y_j^2}\right\} \leq -\frac{\delta}{\sqrt{\beta}}\right\} \leq e^{-\beta^2 k/(2\beta)} = e^{-\beta^2 m/2}.
\] (3.19)

Combining the above inequality with (3.18), we conclude that (3.17) holds.

To provide an upper bound for $\beta_{e,\alpha}^{\max}$ in (1.8), here we introduce a related quantity. For $\epsilon > 0$, $\alpha > 0$ and $C > 0$, we define
\[
\beta_{e,\alpha,C}^{\max} := \sup\{0 < \beta < 1 : P(\Omega_{e,\beta}) \geq 1 - Ce^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} \text{ for sufficiently large } m \in \mathbb{N}\}.
\] (3.20)

If the above set on the right-hand side of (1.8) is empty, then we simply define $\beta_{e,\alpha,C}^{\max} := 0$. Trivially, $\beta_{e,\alpha}^{\max} \leq \beta_{e,\alpha,C}^{\max}$ for all $\epsilon > 0$ and $\alpha > 0$.

**Theorem 3.4.** Let $c_g$ and $c_g$ be the absolute positive constants defined in (3.15) and (2.11). Then
\[
\beta_{e,\alpha,C}^{\max} < \left(\frac{1 + e^{-1}}{c_g \ln \frac{2\epsilon}{\epsilon}}\right), \quad \forall 0 < \epsilon < \min(1, c_g), \quad \alpha > 0, \quad C > 0.
\] (3.21)

**Proof.** If $\beta_{e,\alpha,C}^{\max} = 0$, the claim is trivially true. Hence, we assume $\beta_{e,\alpha,C}^{\max} > 0$. We first prove that
\[
f_{\epsilon}(\beta) := c_g \sqrt{\beta \ln \frac{2}{\beta}} - \sqrt{(1 - \epsilon)\beta + \epsilon} \leq 0, \quad \forall 0 < \beta < \min\left(\frac{1}{2}, \beta_{e,\alpha,C}^{\max}\right).
\] (3.22)

By the continuity of the function $f_{\epsilon}$, it suffices to prove (3.22) under the extra assumption that $\beta$ is a rational number. Suppose that (3.22) fails for some rational number $\beta$ such that $0 < \beta < \min(\frac{1}{2}, \beta_{e,\alpha,C}^{\max})$. Then there exists $\epsilon > 0$ such that
\[
\delta = c_g \sqrt{\beta \ln \frac{2}{\beta}} - \sqrt{(1 - \epsilon)\beta + \epsilon} > 0,
\]
where $\delta$ is also defined in (3.16). Consequently, by Lemma 3.3,
\[
P(\Omega_{e,\beta}) \leq e^{-(\epsilon^2/4 - \epsilon^3/6)m} + e^{-\delta^2 m/2}
\] (3.23)
provided $\beta m \in \mathbb{N}$. On the other hand, by the definition of $\beta_{e,\alpha,C}^{\max}$ and $0 < \beta < \beta_{e,\alpha,C}^{\max}$,
\[
P(\Omega_{e,\beta}) \geq 1 - Ce^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} \quad \text{for sufficiently large } m \in \mathbb{N}.
\] (3.24)

Consequently, combining (3.23) and (3.24), we have
\[
1 - Ce^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} \leq P(\Omega_{e,\beta}) \leq e^{-(\epsilon^2/4 - \epsilon^3/6)m} + e^{-\delta^2 m/2}
\]
for sufficiently large $m \in \mathbb{N}$ satisfying $\beta m \in \mathbb{N}$. Since $\beta$ is a rational number, there are infinitely many sufficiently large $m \in \mathbb{N}$ satisfying $\beta m \in \mathbb{N}$. Letting such $m$ go to $\infty$, we deduce from the above inequality that
\[
1 \leq Ce^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} + e^{-(\epsilon^2/4 - \epsilon^3/6)m} + e^{-\delta^2 m/2} \to 0,
\]
which is a contradiction. Therefore, (3.22) must hold.

Define a function
\[
F_{\epsilon}(\beta) := c_g^2 \beta \ln \frac{2}{\beta} - ((1 - \epsilon)\beta + \epsilon), \quad \beta > 0.
\]

Then it is trivial to see that $f_{\epsilon}(\beta)$ and $F_{\epsilon}(\beta)$ have the same sign on the interval $\beta \in (0, 1)$. As a direct consequence of (3.22), it is straightforward to see that
\[
\min\left(\frac{1}{2}, \beta_{e,\alpha,C}^{\max}\right) \leq \inf\{0 < \beta < 1/2 : F_{\epsilon}(\beta) > 0\} =: \beta_{e}.
\] (3.25)
If the above set on the right-hand side is empty, then we simply define \( \beta_\epsilon = 1/2 \). Let \( \beta_g \) and \( \epsilon_g \) be defined as in Lemma 2.5. Since \( 0 < \beta_g \leq 1/2 \) and \( \epsilon_g > 0 \), we deduce that
\[
F_\epsilon(\beta_g) = \epsilon_g^2 \beta_g \ln \frac{2}{\beta_g} - ((1 - \epsilon)\beta_g + \epsilon) = (1 - \beta_g)(\epsilon_g - \epsilon) > 0, \quad \forall 0 < \epsilon < \epsilon_g,
\]
where we used \( \epsilon_g^2 \beta_g \ln \frac{2}{\beta_g} - 1 = \epsilon_g(\frac{1}{\beta_g} - 1) \) by (2.11). Since \( \lim_{\beta_\epsilon \to 0^+} F_\epsilon(\beta) = -\epsilon < 0 \), \( F_\epsilon \) must have a real root inside the interval \((0, \beta_g)\). Hence, \( 0 < \beta_\epsilon < \beta_g \leq 1/2 \) and \( F_\epsilon(\beta_\epsilon) = 0 \). For \( 0 < \epsilon < \epsilon_g \),
\[
\epsilon = F_\epsilon(\beta_\epsilon) + \epsilon = \left( \epsilon_g^2 \beta_g \ln \frac{2}{\beta_g} - 1 + \epsilon \right) \beta_\epsilon \geq \left( \epsilon_g^2 \beta_g \ln \frac{2}{\beta_g} - 1 \right) \beta_\epsilon = \epsilon_g \left( \frac{1}{\beta_g} - 1 \right) \beta_\epsilon \geq \epsilon_g \beta_\epsilon,
\]
by \( 0 < \beta_\epsilon \leq 1/2 \). It follows from the above inequality that \( \beta_\epsilon \leq \frac{\epsilon}{\epsilon_g} \) for all \( 0 < \epsilon < \epsilon_g \). Hence, for \( 0 < \epsilon < \epsilon_g \), it follows from \( F_\epsilon(\beta_\epsilon) = 0 \) and \( 0 < \beta_\epsilon \leq \frac{\epsilon}{\epsilon_g} \) that
\[
\beta_\epsilon = \frac{\epsilon + (1 - \epsilon)\beta_\epsilon}{\epsilon_g^2 \beta_g \ln \frac{2}{\beta_g}} \leq \frac{\epsilon + (1 - \epsilon)\frac{\epsilon}{\epsilon_g}}{\epsilon_g^2 \beta_g \ln \frac{2}{\beta_g}} < \frac{(1 + \epsilon g - 1)\epsilon}{\epsilon g^2 \beta_g \ln \frac{2}{\beta_g}}.
\]
By \( 0 < \beta_\epsilon < \beta_g \leq 1/2 \) and \( \min(1, 2\beta_g \ln \frac{2}{\beta_g}) \leq 1/2 \), we conclude that
\[
\beta_{\epsilon,\alpha}^{\max} = \min \left( \frac{1}{2}, \beta_{\epsilon,\alpha,\alpha}^{\max} \right) \leq \beta_\epsilon < \left( \frac{1 + \epsilon g - 1}{\epsilon g^2 \beta_g \ln \frac{2}{\beta_g}} \right), \quad \forall 0 < \epsilon < \min(1, \epsilon_g).
\]
This proves (3.21).

3.3 An estimate for \( \hat{\beta}_{\epsilon,\alpha}^{\max} \)

We now study the relation of the quantity \( \hat{\beta}_{\epsilon,\alpha}^{\max} \) in (1.10) using \( \hat{\Omega}_{\epsilon,\beta} \) in (1.9) with a uniform normalization factor \( \frac{1}{m} \).

Similar to Lemma 3.3 and Theorem 3.2, we have the following result on the lower bound of \( \beta_{\epsilon,\alpha}^{\max} \).

**Theorem 3.5.** Let \( A \) be an \( m \times n \) random matrix with i.i.d. entries obeying \( \mathcal{N}(0,1) \). For \( 0 < \alpha < 1 \) and \( 0 < \epsilon \leq \min(1, \frac{1 - \sqrt{\alpha}}{\alpha/2}) \), if \( 0 < \beta < 1 \) satisfies the second inequality in (3.1), i.e.,
\[
0 < \beta \ln \frac{\epsilon}{\beta} \leq \frac{\epsilon}{2} \left( \sqrt{1 - \sqrt{\alpha}} - \frac{\sqrt{\alpha} \epsilon}{2} \right)^2,
\]
then
\[
P(\hat{\Omega}_{\epsilon,\beta}) \geq 1 - 3e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}, \quad \forall m \in \mathbb{N}.
\]
Consequently, under the same conditions as in Theorem 3.2, all the claims in Theorem 3.2 hold with \( \beta_{\epsilon,\alpha}^{\max} \) being replaced by \( \hat{\beta}_{\epsilon,\alpha}^{\max} \).

**Proof.** Let \( \gamma := [\beta m]/m \) and \( k := \gamma m \). Then \( P(\hat{\Omega}_{\epsilon,\beta}) = P(\hat{\Omega}_{\epsilon,\gamma}) \). As in Lemma 3.1, it suffices to consider \( \gamma > 0 \). By \( \|y_T\| \leq \|y\| \) and \( 0 < \alpha < 1 \), it follows from (3.3) that
\[
P(\hat{\Omega}_{\epsilon,\gamma}) = P\left\{ 1 - \epsilon \leq \min \left\{ \frac{1}{m}, \frac{1}{m} \|y_T\|^2 \leq \frac{1}{m} \|y\|^2 \leq 1 + \epsilon \right\} \right\}
\geq P\left\{ (1 - \epsilon)m \leq \|y\|^2 - (y_{(1)}^2 + \cdots + y_{(k)}^2) \right\} \frac{1}{m} \|y\|^2 \leq 1 + \sqrt{\alpha}\epsilon \geq \right\}
\geq P\left\{ (1 - \sqrt{\alpha})\epsilon m \right\} \frac{1}{m} \|y\|^2 \leq 1 \leq \sqrt{\alpha}\epsilon \right\}
\geq P(E_0 \cap E_3) \geq 1 - P(E_0^c) - P(E_3^c),
\]
where \( E_0 := \{ \frac{1}{m} \|y\|^2 - 1 \leq \sqrt{\alpha}\epsilon \} \) as in (3.4) and \( E_3 := \{ y_{(1)}^2 + \cdots + y_{(k)}^2 \leq (1 - \sqrt{\alpha})\epsilon m \} \). By (1.3) and \( 0 < \alpha < 1 \), we have
\[
P(E_0^c) \leq 2e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m} \leq 2e^{-\alpha(\epsilon^2/4 - \epsilon^3/6)m}.
\]
Recall from (3.7) that the following inequality holds for any $\delta > 0$:
\[
P\left\{ \sqrt{\frac{1}{k} \sum_{j=1}^{k} \gamma^2_{(j)}} \leq \delta + \sqrt{2 \ln \frac{e}{\gamma}} \right\} \geq P\left\{ \sqrt{\frac{1}{k} \sum_{j=1}^{k} \gamma^2_{(j)}} \leq \delta + \mathbb{E} \left( \sqrt{\frac{1}{k} \sum_{j=1}^{k} \gamma^2_{(j)}} \right) \right\} \geq 1 - e^{-\delta^2 \gamma^2 / 2}. \tag{3.29}
\]

Set
\[
\delta := \sqrt{\frac{(1 - \sqrt{\alpha}) \epsilon}{\gamma}} - \sqrt{2 \ln \frac{e}{\gamma}}.
\]

Since the function $\beta \ln \frac{e}{\beta}$ is an increasing function on $(0, 1]$, by $0 < \gamma \leq \beta < 1$, we deduce from (3.27) that
\[
0 < \gamma \ln \frac{e}{\gamma} \leq \beta \ln \frac{e}{\beta} \leq \frac{\epsilon}{2} \left( \sqrt{1 - \sqrt{\alpha}} - \sqrt{\frac{\alpha \epsilon}{2}} \right)^2.
\]

Since $\gamma > 0$, dividing $\frac{\gamma}{2}$ on both sides and then taking square root on the above inequality, we see that
\[
\sqrt{2 \ln \frac{e}{\gamma}} \leq \sqrt{\frac{\epsilon}{\gamma}} \left( \sqrt{1 - \sqrt{\alpha}} - \sqrt{\frac{\alpha \epsilon}{2}} \right) = \sqrt{\frac{(1 - \sqrt{\alpha}) \epsilon}{\gamma}} - \sqrt{\frac{\alpha \epsilon^2}{2 \gamma}},
\]
from which it is trivial to see that $\delta \geq \sqrt{\frac{\alpha \epsilon^2}{2 \gamma}} > 0$ holds. By the definition of the set $E_3$, it follows from (3.29) and $\delta \geq \sqrt{\frac{\alpha \epsilon^2}{2 \gamma}} > 0$ that
\[
P(E_3^\delta) = P\left\{ \sqrt{\frac{1}{k} \sum_{j=1}^{k} \gamma^2_{(j)}} > \sqrt{\frac{(1 - \sqrt{\alpha}) \epsilon}{\gamma}} \right\} = P\left\{ \sqrt{\frac{1}{k} \sum_{j=1}^{k} \gamma^2_{(j)}} > \delta + \sqrt{2 \ln \frac{e}{\gamma}} \right\}
\leq e^{-\delta^2 \gamma^2 / 2} \leq e^{-\alpha \epsilon^2 / 4} \leq e^{-\alpha \epsilon^2 (4 - e^2) / 6}.
\]

Therefore, $P(\hat{\Omega}_{\epsilon, \beta}) = P(\hat{\Omega}_{\epsilon, \gamma}) \geq 1 - P(E_3^\delta) = P(E_3^\delta) \geq 1 - 3e^{-\alpha \epsilon^2 (4 - e^2) / 6}$. This proves (3.28).

It has been proved in the proof of Theorem 3.2 that (3.12), combined with $0 < \alpha < 1$ and $0 < \epsilon \leq \min(1, \frac{1 - \sqrt{\alpha}}{4 \alpha})$, implies the conditions in (3.27) (i.e., the second inequality in (3.1)) with $\leq$ being replaced by $<$. Therefore, all the claims in Theorem 3.2 hold with $\beta_{\max}^\epsilon$ being replaced by $\hat{\beta}_{\max}^\epsilon$.

To provide an upper bound for $\hat{\beta}_{\max}^\epsilon$, we define $\hat{\beta}_{\max}^{\epsilon, c_{\max}}$, as in (3.20) with $\Omega_{\epsilon, \beta}$ being replaced by $\hat{\Omega}_{\epsilon, \beta}$. Trivially, $\hat{\beta}_{\max}^{\epsilon, c_{\max}} \leq \hat{\beta}_{\max}^{\epsilon, c_{\max}}$ for all $\epsilon > 0$ and $\alpha > 0$.

**Theorem 3.6.** Let $c_g$ be the absolute positive constant in (3.15). Then
\[
\hat{\beta}_{\epsilon, \alpha, C}^{\max} < \frac{\epsilon}{c_g^2 \ln \frac{2 \epsilon^2}{\epsilon}} \quad \forall 0 < \epsilon < \min(1, 2e^{-1} c_g^2), \quad \alpha > 0, \quad C > 0. \tag{3.30}
\]

**Proof.** If $\hat{\beta}_{\max}^{\epsilon, c_{\max}} = 0$, the claim is trivially true. Hence, we assume $\hat{\beta}_{\max}^{\epsilon, c_{\max}} > 0$. We first prove that
\[
g_{\epsilon}(\beta) := c_g \sqrt{\frac{\beta \ln \frac{2 \epsilon}{\beta} - \sqrt{2 \epsilon}}{\beta}} \leq 0, \quad \forall 0 < \beta < \hat{\beta}_{\max}^{\epsilon, c_{\max}}. \tag{3.31}
\]

By the continuity of $g_{\epsilon}$, it suffices to prove (3.31) for all rational numbers $\beta$. Suppose not. Then there exists $\hat{\epsilon} > 0$ such that $\delta := c_g \sqrt{\beta \ln \frac{2 \epsilon}{\beta} - \sqrt{2 \epsilon + \hat{\epsilon}}} > 0$. Let $y = Ax_0$ and $k := \beta m$ for $m \in \mathbb{N}$ satisfying $\beta m, \epsilon \in \mathbb{N}$. Then by Lemma 2.1 and $\min_{T \in T_\beta} \|y_T\|^2 = \|y\|^2 - (y_{(1)}^2 + \cdots + y_{(k)}^2)$,
\[
P(\hat{\Omega}_{\epsilon, \beta}) = P\left\{ 1 - \epsilon \leq \min_{T \in T_\beta} \frac{1}{m} \|y_T\|^2 \leq \max_{T \in T_\beta} \frac{1}{m} \|y_T\|^2 \leq 1 + \epsilon \right\}
\leq P\left\{ \min_{T \in T_\beta} \|y_T\|^2 \geq (1 - \epsilon)m \right\}
= P\left\{ y_{(1)}^2 + \cdots + y_{(k)}^2 \leq \|y\|^2 - (1 - \epsilon)m \right\}
\]

\[
\leq P \left\{ \frac{1}{m} \|y\|^2 > 1 + \bar{\epsilon} \right\} + P \left\{ y_{(1)}^2 + \cdots + y_{(k)}^2 \leq \eta m \right\}
\]

with \( \eta := \epsilon + \bar{\epsilon} \). Then the proof of Lemma 3.3 shows that (3.19) holds. By (3.18) and (3.19), we conclude

\[
P(\hat{\Omega}_{\epsilon, \beta}) \leq P \left\{ \frac{1}{m} \|y\|^2 > 1 + \bar{\epsilon} \right\} + P \left\{ y_{(1)}^2 + \cdots + y_{(k)}^2 \leq \eta m \right\} \leq e^{-\left(\epsilon^2/4 - \epsilon^3/6\right)m} + e^{-\delta^2 m/2} \to 0 \quad (3.32)
\]
as \( m \to \infty \). Since \( 0 < \beta < \beta_{\epsilon, \alpha, C}^{\max} \), the definition of \( \beta_{\epsilon, \alpha, C}^{\max} \) implies \( P(\hat{\Omega}_{\epsilon, \beta}) \geq 1 - Ce^{-\left(\epsilon^2/4 - \epsilon^3/6\right)m} \) for sufficiently large \( m \in \mathbb{N} \) with \( \beta m \in \mathbb{N} \) (note that \( \beta \) is assumed to be a rational number). This leads a contradiction to (3.32). Hence, (3.31) must hold.

Note that \( g_\epsilon \) is an increasing function on \((0, \frac{\epsilon}{2})\) and \( \epsilon < 2e^{-1}c_g^2 \) implies \( g_\epsilon(\frac{\epsilon}{2}) = c_g \sqrt{2c / \epsilon} - \sqrt{\epsilon} > 0 \). Define \( \beta_\epsilon := \inf\{0 < \beta < 1 : g_\epsilon(\beta) > 0\} \). By \( \lim_{\beta \to 0^+} g_\epsilon(\beta) = -\sqrt{\epsilon} < 0 \), we have \( 0 < \beta_\epsilon < \frac{2}{6} < 1 \) and \( g_\epsilon(\beta_\epsilon) = 0 \). Now it follows from \( g_\epsilon(\beta_\epsilon) = 0 \) and \( 0 < \beta_\epsilon < \frac{2}{6} < \frac{\epsilon}{c_g} \) that \( \epsilon = c_g^2 \beta_\epsilon \ln \frac{2\sqrt{\epsilon}}{\beta_\epsilon} > c_g^2 \beta_\epsilon \), which implies \( \beta_\epsilon < \frac{\epsilon}{c_g} \). Combining this with \( g_\epsilon(\beta_\epsilon) = 0 \), we have

\[
\beta_\epsilon = \frac{\epsilon}{c_g^2 \ln \frac{2\sqrt{\epsilon}}{\beta_\epsilon}} < \frac{\epsilon}{c_g^2 \ln \frac{2\sqrt{\epsilon}}{\epsilon}}.
\]

By (3.31), we must have \( \beta_{\epsilon, \alpha, C}^{\max} < \beta_\epsilon \). This proves (3.30).

### 3.4 Proof of Theorem 1.1

We are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** The left-hand inequality in (1.11) follows directly from (3.13) of Theorem 3.2. Since \( 0 < \epsilon < 4c_g^4 \), we have \( \sqrt{\epsilon} < 2c_g \) and therefore, \( \ln \left(\frac{2\sqrt{\epsilon}}{\epsilon}\right) \geq \ln \frac{2\sqrt{\epsilon}}{\epsilon} = \frac{1}{4} \ln \frac{2\sqrt{\epsilon}}{\epsilon} \). By Theorem 3.4, we have

\[
\beta_{\epsilon, \alpha}^{\max} \leq \beta_{\epsilon, \alpha}^{\max} < \frac{(1 + c_g^{-1})\epsilon}{c_g^2 \ln \frac{2\sqrt{\epsilon}}{\epsilon}} \leq \frac{(1 + c_g^{-1})\epsilon}{\frac{1}{2} c_g^2 \ln \frac{1}{\epsilon}} = \frac{2 + 2c_g}{c_g^2 \epsilon} \ln \frac{1}{\epsilon}.
\]

This proves the right-hand inequality in (1.11).

The left-hand inequality in (1.12) follows directly from Theorem 3.5 and (3.13) of Theorem 3.2. Since \( \epsilon < 4c_g^4 \), we have \( \sqrt{\epsilon} < 2c_g^2 \) and hence \( \ln \left(\frac{2\sqrt{\epsilon}}{\epsilon}\right) \geq \ln \frac{2\sqrt{\epsilon}}{\epsilon} = \frac{1}{4} \ln \frac{2\sqrt{\epsilon}}{\epsilon} \). Now the right-hand inequality in (1.12) follows directly from Theorem 3.6.

### 4 Gaussian random matrices under arbitrary erasure of rows for given \( 0 < \beta < 1 \)

In this section, we study the robustness property of Gaussian random matrices with arbitrarily erased rows for a given corruption/erasure ratio \( 0 < \beta < 1 \). We shall prove Theorem 1.4 by breaking its proof into several parts.

#### 4.1 Estimate \( \theta_{\beta}^{\max}(\alpha) \) and \( \omega_{\beta}^{\min}(\alpha) \)

To prove Theorem 1.4, we first estimate \( \theta_{\beta}^{\max}(\alpha) \) and \( \omega_{\beta}^{\min}(\alpha) \).

**Lemma 4.1.** For \( 0 < \beta < 1 \) and \( 0 < \alpha < \frac{\epsilon}{12}(1 - \beta)^2 h_{\beta} \) with \( h_{\beta} := \min\left\{ \frac{3}{4} - \frac{1}{2} \beta, 1 - \beta \right\} \), (1.23) holds.

**Proof.** Define \( \gamma := |\beta m|/m \) and \( k := \gamma m \). Let \( y := (y_1, \ldots, y_m)^T := Ax_0. \) Since \( \gamma m = |\beta m| \), by Lemma 2.1 and (3.3), for \( \theta \geq 0 \), we have

\[
P(\Omega_{\theta, \infty, \beta}) = P \left\{ \min_{T \in T_\epsilon} \frac{1}{T} \|y_T\|^2 \geq \theta \right\} = P \left\{ \left( \frac{1}{m - k} \sum_{j=k+1}^{m} y_{(j)}^2 \right) \geq \sqrt{\theta} \right\}.
\]
By the left-hand inequality in (2.10) and 0 ≤ γ ≤ β < 1, we have
\[
E \left[ \frac{1}{m - k} \sum_{j=k+1}^{m} y_{(j)}^2 \right] \geq \sqrt{\frac{\pi}{6}} \sqrt{(1 - \gamma) \frac{1 - \gamma + \frac{1}{2m}}{1 + \frac{1}{m}}} \geq \sqrt{\frac{\pi}{6}} \sqrt{(1 - \beta) \frac{1 - \beta + \frac{1}{2m}}{1 + \frac{1}{m}}}
\]
\[
\geq \sqrt{\frac{\pi}{6}} \sqrt{(1 - \beta)} \inf_{0 < x < 1} \frac{1 - \beta + \frac{1}{2x}}{1 + x} = \sqrt{\frac{\pi}{6}} (1 - \beta) h_\beta.
\]
Therefore, for 0 < β < 1, by (4.1) and (2.6) of Lemma 2.3 with \( \delta := \sqrt{\frac{\pi}{6}} (1 - \beta) h_\beta - \sqrt{\theta} \), if
\[
\delta \geq \sqrt{\frac{2\alpha}{1 - \beta}} > 0,
\]
then we have
\[
P(\Omega_{[\theta, \infty], \beta}) = P\left\{ \sqrt{\frac{\pi}{6}} \sqrt{(1 - \beta) h_\beta} \geq \sqrt{\theta} - \sqrt{\frac{1 - \beta}{2}} \right\} = 1 - \frac{e^{\delta^2(m-k)/2}}{1 - \beta} = 1 - e^{-\delta^2 m \alpha / \beta} \geq 1 - e^{-\alpha m},
\]
since \( \frac{1 - \beta}{2} \geq 1 \) by 0 ≤ γ ≤ β < 1. This shows that if \( \sqrt{\theta} \leq \sqrt{\frac{\pi}{6}} (1 - \beta) h_\beta - \sqrt{\frac{2\alpha}{1 - \beta}} \), then
\[
P(\Omega_{[\theta, \infty], \beta}) \geq 1 - e^{-\alpha m}
\]
for all \( m \in \mathbb{N} \). Since 0 < α ≤ \( \sqrt{\frac{\pi}{6}} (1 - \beta)^2 h_\beta \), we have \( \sqrt{\frac{\pi}{6}} (1 - \beta) h_\beta - \sqrt{\frac{2\alpha}{1 - \beta}} > 0 \). Consequently, by the definition of \( \theta_{max}(\alpha) \), we conclude that
\[
\theta_{max}(\alpha) \geq \left( \sqrt{\frac{\pi}{6}} (1 - \beta) h_\beta - \sqrt{\frac{2\alpha}{1 - \beta}} \right)^2 = \frac{\pi}{6} (1 - \beta) h_\beta + \frac{2\alpha}{1 - \beta} - 2 \sqrt{\pi \alpha h_\beta / 3} > 0.
\]
This proves the left-hand side of (1.23).

We now estimate the upper bound for \( \theta_{max}(\alpha) \). By the second inequality in (2.7) with \( p = 1 \), we have
\[
E \left[ \frac{1}{m - k} \sum_{j=k+1}^{m} y_{(j)}^2 \right] \leq E[|y_{(k+1)}|] \leq \frac{\pi}{2} \left( \frac{1}{k + 1} + \ln \frac{m}{k + 1} \right) \leq \frac{\pi}{2} \ln \frac{m}{k} = \frac{\pi}{2} \ln \frac{1}{\gamma},
\]
where we used the basic inequality \( \frac{1}{k+1} \leq \ln (1 + \frac{k}{2}) \) for all \( k > 0 \). Suppose that (4.2) holds for sufficiently large \( m \in \mathbb{N} \). For convenience, we only consider the case that \( \beta \) is rational, since the general result follows from the fact that the rational numbers are dense in \( \mathbb{R} \). We assume that \( m \in \mathbb{N} \) is sufficiently large and satisfies \( m\beta \in \mathbb{N} \), i.e., we have γ = β. Note that \( k = \gamma m = \beta m \). By (4.1) and (4.3), applying (2.5) of Lemma 2.3 with \( \delta := \sqrt{\theta} - \sqrt{\frac{1}{\beta}} \ln \frac{1}{\beta} > 0 \), we have
\[
P(\Omega_{[\theta, \infty], \beta}) = P\left\{ \sqrt{\frac{\pi}{6}} (1 - \beta) h_\beta - \sqrt{\frac{2\alpha}{1 - \beta}} \right\} \leq P\left\{ \sqrt{\frac{\pi}{6}} (1 - \beta) h_\beta - \sqrt{\frac{2\alpha}{1 - \beta}} \right\} \leq e^{-\delta^2 (m-k)/2} = e^{-\delta^2 (1-\beta) m/2}.
\]
Consequently, if $\sqrt{\theta} > \sqrt{\frac{2}{3}} \ln \frac{1}{B}$ and if (4.2) holds for sufficiently large $m \in \mathbb{N}$, then the above inequalities imply

$$1 - e^{-\alpha m} \leq P(\Omega_{[0, \omega], \beta}) \leq e^{-\delta^2(1-\beta)m/2},$$

which cannot be true for sufficiently large $m$ since $\alpha > 0$ and $\delta > 0$. This proves that $\theta_{\beta}^{\max}(\alpha) \leq \frac{2}{3} (\ln \frac{1}{B})^2$. Also, it is trivial to see that

$$\mathbb{E} \left[ \frac{1}{m - k} \sum_{j=k+1}^{m} y_j^2 \right] \leq \sqrt{\frac{1}{m - k} \sum_{j=k+1}^{m} \mathbb{E} y_j^2} \leq \sqrt{\frac{1}{m} \sum_{j=1}^{m} \mathbb{E} y_j^2} = 1.$$

The above same argument shows that $\theta_{\beta}^{\max}(\alpha) \leq 1$. This proves the upper bound in (1.23).

We next estimate $\omega_{\beta}^{\min}(\alpha)$.

**Lemma 4.2.** For $0 < \beta < 1$ and $\alpha > 0$, (1.25) holds.

**Proof.** Define $\gamma := |\beta m|/m$ and $k := \gamma m$. Let $y := (y_1, \ldots, y_m)^T := Ax_0$. Since $\gamma m = |\beta m|$, by Lemma 2.1 and (3.3), for $\omega \geq 0$, we have

$$P(\Omega_{[0, \omega], \beta}) = P \left\{ \max_{T \in T_j} \| y_T \|^2 \leq \omega \right\} = P \left\{ \sqrt{\frac{1}{m - k} \sum_{j=1}^{m-k} y_j^2} \leq \sqrt{\omega} \right\}. \quad (4.4)$$

By (2.9), we have

$$\mathbb{E} \left[ \frac{1}{m - k} \sum_{j=1}^{m-k} y_j^2 \right] \leq \sqrt{2 \ln \frac{em}{m-k}} = \sqrt{2 \ln \frac{e}{1-\gamma}}.$$

By (4.4) and the above estimate, applying (2.5) of Lemma 2.3 with $\delta := \sqrt{\omega} - \sqrt{2 \ln \frac{e}{1-\gamma}} \geq \sqrt{\frac{2a}{1-\gamma}} > 0$, we have

$$P(\Omega_{[0, \omega], \beta}) \geq P \left\{ \sqrt{\frac{1}{m - k} \sum_{j=1}^{m-k} y_j^2} - E \left[ \sqrt{\frac{1}{m - k} \sum_{j=1}^{m-k} y_j^2} \right] \leq \sqrt{\omega} - E \left[ \sqrt{\frac{1}{m - k} \sum_{j=1}^{m-k} y_j^2} \right] \right\}$$

$$\geq 1 - e^{-\delta^2(m-k)/2} \geq 1 - e^{-\alpha \frac{m-k}{1-\beta}} = 1 - e^{-\alpha m}. \quad (4.5)$$

If $\sqrt{\omega} \geq \sqrt{\frac{2a}{1-\beta}} + \sqrt{2 \ln \frac{e}{1-\beta}}$, then we have $\sqrt{\omega} \geq \sqrt{\frac{2a}{1-\beta}} + \sqrt{2 \ln \frac{e}{1-\beta}} \geq \sqrt{\frac{2a}{1-\beta}} + \sqrt{2 \ln \frac{e}{1-\beta}}$, by $0 \leq \gamma \leq \beta < 1$ and the above inequality shows that

$$P(\Omega_{[0, \omega], \beta}) \geq 1 - e^{-\alpha m} \quad (4.6)$$

holds for all $m \in \mathbb{N}$. Therefore, we proved

$$\omega_{\beta}^{\min}(\alpha) \leq \left( \sqrt{\frac{2a}{1-\beta}} + \sqrt{2 \ln \frac{e}{1-\beta}} \right)^2 = 2 \ln \frac{e}{1-\beta} + \frac{2a}{1-\beta} + 4 \sqrt{\frac{a}{1-\beta} \ln \frac{e}{1-\beta}}.$$

This proves the right-hand inequality in (1.25).

Without loss of generality, we assume that $\beta$ is a rational number and $m$ is sufficiently large satisfying $\beta m \in \mathbb{N}$. Thus, $\gamma = \beta$ and $k = \beta m$. By (2.7), we have

$$E[|y_{(m-k)}|] \geq \sqrt{\frac{\pi}{2}} \frac{k + 1}{m + 1} \geq \sqrt{\frac{\pi}{2}} \frac{k}{2m} = \sqrt{\frac{\pi}{2}}.$$

If in addition $1/2 \leq \beta < 1$, then $m-k = (1-\beta)m \leq m/2$ and by (3.15), we have

$$E[|y_{(m-k)}|] \geq c_2 \sqrt{\ln \frac{2m}{m-k}} = c_2 \sqrt{\ln \frac{2}{1-\beta}}.$$
i.e., we have
\[ E \left( \frac{1}{m-k} \sum_{j=1}^{m-k} y_{(j)}^2 \right) \geq E|y(m-k)| \geq \frac{\beta}{\sup_{u \in [\Omega]} E(u)} \leq \left( 1 + \frac{1}{m-k} \right)^{\frac{1}{2}} \left( 1 + \frac{1}{m-k} \right)^{\frac{3}{2}} \geq \left( \frac{2}{1-\beta} \right)^{\frac{1}{2}} \left( \frac{2}{1-\beta} \right)^{\frac{3}{2}}. \]

By (4.4) and the above inequality, applying (2.6) of Lemma 2.3 with \( \delta := t_{\beta} - \sqrt{\omega} > 0 \) and \( S = \{1, \ldots, m-k\} \), we have
\[
P(\Omega_{[\theta, \infty], \beta}) = P \left( \sqrt{\omega} - \ln \left( \frac{2}{1-\beta} \right) \leq \sqrt{\omega} - \ln \left( \frac{2}{1-\beta} \right) \right)
\leq P \left( \sqrt{\omega} - \ln \left( \frac{2}{1-\beta} \right) \leq \sqrt{\omega} - t_{\beta} \right)
= P \left( \sqrt{\omega} - \ln \left( \frac{2}{1-\beta} \right) \leq \sqrt{\omega} - t_{\beta} \right)
\leq e^{-\delta^2(m-k)/2} = e^{-\delta^2(1-\beta)m/2}.
\]

Consequently, if (4.6) holds for sufficiently large \( m \in \mathbb{N} \), then
\[ 1 - e^{-\alpha m} \leq P(\Omega_{[\theta, \infty], \beta}) \leq e^{-\delta^2(1-\beta)m/2}, \]
which cannot be true when \( m \) is sufficiently large. This proves that \( \delta = t_{\beta} - \sqrt{\omega} \leq 0 \) and hence \( \omega_{\beta}^{\min}(\alpha) \geq t_{\beta}^2 \). This proves the left-hand inequality in (1.25).

\subsection*{4.2 Estimate \( \hat{\theta}_{\beta}^{\max}(\alpha) \) and \( \omega_{\beta}^{\min}(\alpha) \)}

As a direct consequence of Lemma 4.1, we have the following corollary.

\textbf{Corollary 4.3.} For \( 0 < \beta < 1 \) and \( 0 < \alpha < \frac{\pi}{12}(1-\beta)^2h_{\beta} \) with \( h_{\beta} := \min(\frac{3}{4} - \frac{1}{2}\beta, 1-\beta) \), (1.24) holds.

\textbf{Proof.} Define \( \gamma := \lfloor \beta m \rfloor / m \) and \( k := \gamma m \). Let \( y := (y_1, \ldots, y_m)^T := Ax_0 \). Then \( 0 \leq \gamma \leq \beta < 1 \). By the definition of \( \hat{\Omega}_{\theta, \infty}, \beta \), we have
\[
P(\hat{\Omega}_{\theta, \infty}, \beta) = P \left( \frac{1}{m} \sum_{j=k+1}^{m} y_{(j)}^2 \geq \sqrt{\theta} \right) = P \left( \frac{1}{m} \sum_{j=k+1}^{m} y_{(j)}^2 \geq \sqrt{\theta} \right) \geq \frac{\theta}{1-\gamma} \geq P(\hat{\Omega}_{\theta, \infty}, \beta),
\]
where we used \( \frac{1}{1-\gamma} \leq \frac{1}{1-\beta} \) by \( 0 \leq \gamma \leq \beta < 1 \). Consequently, for all \( 0 \leq \theta < (1-\beta)\hat{\theta}_{\beta}^{\max}(\alpha) \), we have \( \frac{\theta}{1-\beta} \leq \hat{\theta}_{\beta}^{\max}(\alpha) \) and by the definition of \( \hat{\theta}_{\beta}^{\max}(\alpha) \), we have
\[ 1 - e^{-\alpha m} \leq P(h_{\beta}(\hat{\Omega}_{\theta, \infty}, \beta)) \leq P(\hat{\Omega}_{\theta, \infty}, \beta), \quad \forall m \in \mathbb{N}.
\]

For \( 0 < \beta < 1 \) and \( 0 < \alpha < \frac{\pi}{12}(1-\beta)^2h_{\beta} \), it follows from the above inequality and Lemma 4.1 that
\[
\hat{\theta}_{\beta}^{\max}(\alpha) \geq (1-\beta)\hat{\theta}_{\beta}^{\max}(\alpha) \geq \frac{\pi}{6}(1-\beta)^2h_{\beta} + 2\alpha - 2(1-\beta)\sqrt{\pi \alpha h_{\beta}/3} > 0.
\]
This proves the left-hand inequality in (1.24).

Note that we proved the upper bound of \( \hat{\theta}_{\beta}^{\max}(\alpha) \) in (1.23) of Lemma 4.1 by assuming that \( \beta \) is rational and \( m \) is sufficiently large with \( \beta m \in \mathbb{N} \). For such \( \beta \) and \( m \), we have \( \gamma = \beta \) and \( P(\hat{\Omega}_{\theta, \infty}, \beta) = P(\hat{\Omega}_{\theta, \infty}, \beta) \). Consequently, the same proof of Lemma 4.1 yields
\[
\hat{\theta}_{\beta}^{\max}(\alpha) = (1-\beta)\hat{\theta}_{\beta}^{\max}(\alpha) \leq (1-\beta) \min \left( \frac{\pi}{2} \left( \ln \frac{1}{\beta} \right)^2, 1 \right).
\]
This proves the right-hand inequality in (1.24). \( \square \)
4.3 Proof of Theorem 1.4

We are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** (1.23) has been proved in Lemma 4.1 and (1.25) has been proved in Lemma 4.2. The inequalities in (1.24) have been proved in Corollary 4.3.

We now prove (1.26) by estimating $\omega_{\beta}^{\min}(\alpha)$. By the definition of $\Omega_{[0,\omega],\beta}$ and $\frac{1}{m}\|A_T x_0\|^2 \leq \frac{1}{m}\|A x_0\|^2$ for all $T \subseteq \{1, \ldots, m\}$, we have

$$
\Omega_{[0,\omega],\beta} = \left\{ \frac{1}{m}\|A x_0\|^2 \leq \omega \right\} = \Omega_{[0,\omega],0}.
$$

By the second inequality in (1.2), it is trivial to conclude that $\omega_{\beta}^{\min}(\alpha) \geq 1$. Let $\epsilon := \sqrt{12}\alpha$. By $0 < \alpha \leq \frac{1}{12}$, we have $0 < \epsilon \leq 1$ and $e^2/4 - e^3/6 \geq e^2/12 = \alpha$. Thus, it follows from the first inequality in (1.2) that

$$
\Pr\left\{ \frac{1}{m}\|A x_0\|^2 \leq 1 + \epsilon \right\} \geq 1 - e^{-(e^2/4-e^3/6)m} \geq 1 - e^{-\alpha m}, \quad \forall m \in \mathbb{N}.
$$

Therefore, $\omega_{\beta}^{\min}(\alpha) \leq 1 + \epsilon = 1 + \sqrt{12}\alpha$ for all $0 < \alpha \leq \frac{1}{12}$. This completes the proof of Theorem 1.4. \qed

5 Proofs of corollaries

We now provide proofs of Corollaries 1.2 and 1.3 as well as the proofs of Corollaries 1.5 and 1.6.

**Proof of Corollary 1.2.** We assume that all the points $p_1, \ldots, p_N$ are distinct. For every $T \in T_{\epsilon,\alpha}$, we have $|T| \leq \beta m$ for all $0 \leq \beta \leq \frac{1 - \sqrt{\alpha}}{32\ln \frac{1}{\epsilon}}$. Let $M = m$ or $M = |T|$. By Theorem 1.1 (or Theorems 3.2 and 3.5), for $j \neq k$, we have

$$
\Pr\left\{ \left| \frac{\|A_T p_j - A_T p_k\|^2}{M\|p_j - p_k\|^2} - 1 \right| \leq \epsilon \right\} \geq 1 - 3e^{-\alpha(e^2/4-e^3/6)m}, \quad \forall m \in \mathbb{N}.
$$

Here, we used the inequality

$$
0 < \beta \leq \frac{1 - \sqrt{\alpha}}{32\ln \frac{1}{\epsilon}} \leq \frac{1 - \sqrt{\alpha}}{16\ln \frac{1}{(1-\sqrt{\alpha})\epsilon}},
$$

provided that $0 < \epsilon < \frac{1 - \sqrt{\alpha}}{32\ln \frac{1}{\epsilon}}$. Since there are $\binom{N}{2} = \frac{N(N-1)}{2}$ pairs $\{p_j, p_k\}$ with $j \neq k, j, k = 1, \ldots, N$, using union bounds, we conclude that

$$
\Pr\left\{ \left| \frac{\|A_T p_j - A_T p_k\|^2}{M\|p_j - p_k\|^2} - 1 \right| \leq \epsilon, \forall T \in T_{\epsilon,\beta}, j \neq k, j, k = 1, \ldots, N \right\} \geq 1 - \frac{3N(N-1)}{2}e^{-\alpha(e^2/4-e^3/6)m} > 0,
$$

where we used the assumption of $m > 2\ln(3)(N(N-1))\alpha(e^2/4-e^3/6)$ in the last inequality. This proves that (1.13) holds with probability at least $1 - \frac{3N(N-1)}{2}e^{-\alpha(e^2/4-e^3/6)m}$. \qed

**Proof of Corollary 1.3.** We slightly modify the argument in [6, Lemma 5.1]. Let $\Lambda \subseteq \{1, \ldots, n\}$ with $|\Lambda| = s$. Set $\mathbb{R}^\Lambda := \{ x \in \mathbb{R}^n : x \text{ is supported inside } \Lambda \}$ and $S^\Lambda := \{ x \in \mathbb{R}^\Lambda : \|x\| = 1 \}$. It is well known that there exists a subset $Q_{\Lambda,\epsilon} \subset S^\Lambda$ such that $|Q_{\Lambda,\epsilon}| \leq (24/\epsilon)^s$ and

$$
S^\Lambda \subseteq \bigcup_{\zeta \in Q_{\Lambda,\epsilon}} \{ x \in \mathbb{R}^n : \|x - \zeta\| \leq \epsilon/8 \}.
$$

Let $M = m$ or $M = |T|$. By Theorem 3.5, with probability at least $1 - 3(24/\epsilon)^s e^{-\alpha(e^2/16-e^3/24)m}$, we have

$$
\sqrt{1 - \epsilon/2\|v\|} \leq \frac{1}{\sqrt{M}}\|A_T v\| \leq \sqrt{1 + \epsilon/2}, \quad \forall T \in T_{\epsilon/2,\alpha} \quad \text{and} \quad v \in Q_{\Lambda,\epsilon}.
$$

(5.1)
We next consider the case where \( A \) satisfies (5.1). Define \( \lambda := \sup \{ \frac{1}{\sqrt{m}} \| A_T x \| : x \in S^\Lambda, T \in T_{e/2, \alpha} \} \). For every \( x \in S^\Lambda \), there exists \( v_x \in Q_{\Lambda, \epsilon} \) such that \( \| x - v_x \| \leq \epsilon/8 \) and hence,

\[
\frac{1}{\sqrt{m}} \| A_T x \| \leq \frac{1}{\sqrt{m}} \| A_T v_x \| + \frac{1}{\sqrt{m}} \| A(x - v_x) \| \leq \sqrt{1 + \epsilon/2 + \lambda \| x - v_x \|} \leq \sqrt{1 + \epsilon/2 + \lambda \epsilon/8}.
\]

By the definition of \( \lambda \), we must have \( \lambda \leq \sqrt{1 + \epsilon/2 + \lambda \epsilon/8} \), which implies that

\[
\lambda \leq \frac{\sqrt{1 + \epsilon/2}}{1 - \epsilon/8} \leq \sqrt{1 + \epsilon}
\]

for all \( 0 < \epsilon < 1 \). Therefore, for all \( x \in \mathbb{R}^\Lambda \) and \( T \in T_{e/2, \alpha} \),

\[
\frac{1}{\sqrt{m}} \| A_T x \| \leq \lambda \| x \| \leq \sqrt{1 + \epsilon} \| x \|
\]

and

\[
\frac{1}{\sqrt{m}} \| A_T x \| \geq \frac{1}{\sqrt{m}} \| A_T v_x \| - \frac{1}{\sqrt{m}} \| A_T (x - v_x) \| \geq \sqrt{1 - \frac{\epsilon}{2} - \frac{\lambda \epsilon}{8}} \geq \sqrt{1 - \frac{\epsilon}{2} - \frac{\epsilon}{8} \sqrt{1 + \epsilon}} \geq \sqrt{1 - \epsilon},
\]

where the last inequality holds for all \( 0 \leq \epsilon \leq 1 \). Thus, with probability at least

\[
1 - 3(24/\epsilon)^{e-\alpha(\epsilon^2/16 - \epsilon^3/24)m},
\]

we have

\[
(1 - \epsilon) \| x \|^2 \leq \frac{1}{\sqrt{m}} \| A_T x \|^2 \leq (1 + \epsilon) \| x \|^2, \quad \forall x \in \mathbb{R}^\Lambda, \quad T \in T_{e/2, \alpha}.
\]

Proof of Corollary 1.5. Since \( 0 < \alpha < \min (\frac{1}{12}, \frac{n}{10} (1 - \beta)^2 h_{\beta}) \), by the definition of \( \tilde{\theta} \) and \( \tilde{\omega} \) in (1.28) we have \( 0 < \tilde{\theta} \leq \tilde{\omega} < 2 \). By the left-hand inequality in (1.24) of Theorem 1.4, we have \( \mathbb{P}(\Omega_{[\theta, \infty], \beta}) \geq 1 - e^{-am} \). By (1.26) of Theorem 1.4, we deduce that \( \mathbb{P}(\Omega_{[\theta, \omega], \beta}) \geq 1 - e^{-am} \). Consequently, we have

\[
P\left\{ \tilde{\theta} \| p_j - p_k \|^2 \leq \frac{1}{m} \| A_T p_j - A_T p_k \|^2 \leq \tilde{\omega} \| p_j - p_k \|^2, \quad \forall T \subseteq \{1, \ldots, m\}, |T^c| \leq \beta m \right\}
\]

\[
\geq 1 - 2e^{-am}, \quad \forall m \in \mathbb{N}
\]

for every \( j, k = 1, \ldots, N \). Similarly, by the left-hand inequality of (1.23) and the right-hand inequality of (1.25) of Theorem 1.4, we have

\[
P\left\{ \frac{\tilde{\theta}}{1 - \beta} \| p_j - p_k \|^2 \leq \frac{1}{|T|} \| A_T p_j - A_T p_k \|^2 \leq \tilde{\omega} \| p_j - p_k \|^2, \quad \forall T \subseteq \{1, \ldots, m\}, |T^c| \leq \beta m \right\}
\]

\[
\geq 1 - 2e^{-am}, \quad \forall m \in \mathbb{N},
\]

for every \( j, k = 1, \ldots, N \). Since there are \( \binom{N}{2} = \frac{N(N-1)}{2} \) pairs \( \{p_j, p_k\} \) with \( j \neq k, j, k = 1, \ldots, N \), we conclude that each of two inequalities in (1.27) holds with probability at least \( 1 - N(N-1)e^{-am} > 0 \) for all \( m > \frac{1}{a} \ln N(N-1) \). The proof is completed by taking union bounds of these two cases. \( \square \)

Proof of Corollary 1.6. We use the same notation as in the proof of Corollary 1.3. Define \( T_{e/2} := \{ T \subseteq \{1, \ldots, m\} : |T^c| \leq \beta m \} \). Let \( T_{e/2} \) be a subset of \( \mathbb{N} \) with probability at least \( 1 - 2e^{-am} \),

\[
\sqrt{\tilde{\theta} (1 - \epsilon/2) \| v \|^2} \leq \frac{1}{\sqrt{m}} \| A_T v \| \leq \sqrt{\tilde{\omega} (1 + \epsilon/2)}, \quad \forall T \in T_{e/2},
\]

(3.5)

where \( v \in \mathbb{R}^n \) is a fixed vector. We next consider the case where \( A \) satisfies (3.5). Define \( \lambda := \sup \{ \frac{1}{\sqrt{m}} \| A_T x \| : x \in S^\Lambda, T \in T_{e/2} \} \). For every \( x \in S^\Lambda \), there exists \( v_x \in Q_{\Lambda, \epsilon} \) such that \( \| x - v_x \| \leq \epsilon/8 \) and hence,

\[
\frac{1}{\sqrt{m}} \| A_T x \| \leq \frac{1}{\sqrt{m}} \| A_T v_x \| + \frac{1}{\sqrt{m}} \| A(x - v_x) \| \leq \sqrt{\tilde{\omega} (1 + \epsilon/2) + \lambda \| x - v_x \|} \leq \sqrt{\tilde{\omega} (1 + \epsilon/2) + \lambda \epsilon/8}.
\]
By the definition of $\lambda$, we must have $\lambda \leq \sqrt{\omega(1+\epsilon/2)} + \lambda\epsilon/8$, from which we have $\lambda \leq \sqrt{\omega(1+\epsilon/2)/(1−\epsilon/8)} \leq \sqrt{\omega(1+\epsilon)}$ for all $0 < \epsilon < 1$. Therefore, for all $x \in \mathbb{R}^\Lambda$ and $T \in T_{\leq \beta}$, $\frac{1}{\sqrt{m}}\|A_T x\| \leq \lambda\|x\| \leq \sqrt{\omega(1+\epsilon)}\|x\|$ and
\[
\frac{1}{\sqrt{m}}\|A_T x\| \geq \frac{1}{\sqrt{m}}\|A_T v_x\| - \frac{1}{\sqrt{m}}\|A_T(x - v_x)\| \geq \sqrt{\theta}(1-\epsilon/2) - \lambda\epsilon/8
\]
\[
\geq \sqrt{\bar{\omega}}(1-\epsilon/2 - \sqrt{\omega/\theta}(1+\epsilon/8)) \geq \sqrt{\bar{\theta}}(1-\epsilon/2 - \sqrt{1+\epsilon/8}) \geq \sqrt{\bar{\theta}}\sqrt{1-\epsilon},
\]
where we used the fact that $\bar{\omega}/\bar{\theta} \geq 1$ by $0 < \bar{\theta} \leq \bar{\omega}$. Thus, with probability at least $1 - 2(24/\epsilon)^s e^{-\alpha m}$, we have
\[
\bar{\theta}(1-\epsilon)\|x\| \leq \frac{1}{m}\|A_T x\|^2 \leq \omega(1+\epsilon)\|x\|^2, \quad \forall x \in \mathbb{R}^\Lambda, \quad T \in T_{\leq \beta}.
\]
(5.4)

Note that there are total $\binom{n}{s}$ subsets $\Lambda$. Therefore, (5.4) holds for every such subset $\Lambda$. Hence, the first inequality in (1.16) holds with probability at least $1 - 2(24/\epsilon)^s e^{-\alpha m} > 0$ by $s\ln(24en)/\epsilon s < \alpha m - \ln 2$.

By Theorem 1.4, with probability at least $1 - 2e^{-\alpha m}$,
\[
\sqrt{\frac{\bar{\theta}}{1-\beta}}(1-\epsilon/2)\|v\| \leq \frac{1}{\sqrt{|T|}}\|A_T v\| \leq \sqrt{\omega(1+\epsilon/2)}, \quad \forall T \in T_{\leq \beta}.
\]
(5.5)

Now the above same proof shows that the second inequality in (1.16) holds with probability at least $1 - 2(24en)/\epsilon s e^{-\alpha m} > 0$ by $s\ln(24en)/\epsilon s < \alpha m - \ln 2$. The proof is completed by taking union bounds of these two cases.

Acknowledgements Bin Han was supported by Natural Sciences and Engineering Research Council of Canada (Grant No. 05865). Zhiqiang Xu was supported by National Natural Science Foundation of China (Grant Nos. 11422113, 91630203, 11021101 and 11331012) and National Basic Research Program of China (973 Program) (Grant No. 2015CB856000). The authors are thankful to Yang Wang for informing them the reference [24] and its connection with their work.

References
1 Achlioptas D. Database-friendly random projections: Johnson-Lindenstrauss with binary coins. J Comput System Sci, 2003, 66: 671–687
2 Balan R. Stability of phase retrievable frames. In: Proceedings of SPIE, vol. 8858. Los Angeles: SPIE, 2013, doi: 10.1117/12.2026135
3 Balan R, Casazza P, Edidin D. On signal reconstruction without phase. Appl Comput Harmon Anal, 2006, 20: 345–356
4 Balan R, Wang Y. Invertibility and robustness of phaseless reconstruction. Appl Comput Harmon Anal, 2015, 38: 469–488
5 Bandeira A S, Mixon D G. Near-optimal phase retrieval of sparse vectors. In: Proceedings of SPIE, vol. 8858. Los Angeles: SPIE, 2013, doi: 10.1117/12.2024355
6 Baraniuk R, Davenport M, DeVore R, et al. A simple proof of the restricted isometry property for random matrices. Constr Approx, 2008, 28: 253–263
7 Baraniuk R, Wakin M. Random projections of smooth manifolds. Found Comput Math, 2009, 9: 51–77
8 Candès E J, Strohmer T, Voroninski V. Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming. Comm Pure Appl Math, 2013, 66: 1241–1274
9 Candès E J, Tao T. Near-optimal signal recovery from random projections: Universal encoding strategies? IEEE Trans Inform Theory, 2006, 52: 5406–5425
10 Chafaï D, Guédon O, Lecué G, et al. Interactions Between Compressed Sensing Random Matrices and High Dimensional Geometry. Paris: Soc Math France, 2012
11 Corless R M, Gonnet G H, Hare D E G, et al. On the Lambert W function. Adv Comput Math, 1996, 5: 329–359
12 Dirksen S. Dimensionality reduction with subgaussian matrices: A unified theory. Found Comput Math, 2016, 16: 1367–1396
13 Fickus M, Mixon D G. Numerically erasure-robust frames. Linear Algebra Appl, 2012, 437: 1394–1407
14 Gordon Y, Litvak A E, Schutt C, et al. On the minimum of several random variables. Proc Amer Math Soc, 2006, 134: 3665–3675
15 Goyal V K, Kovacevic J, Kelner J A. Quantized frame expansions with erasures. Appl Comput Harmon Anal, 2001, 10: 203–233
16 Haupt J, Bajwa W, Rabbat M, et al. Compressed sensing for networked data. IEEE Signal Process Mag, 2008, 25: 92–101.
17 Holmes R B, Paulsen V I. Optimal frames for erasures. Linear Algebra Appl, 2004, 377: 31–51
18 Johnson W B, Lindenstrauss J. Extensions of Lipschitz mappings into a Hilbert space. Contemp Math, 1984, 26: 189–206
19 Krahmer F, Ward R. New and improved Johnson-Lindenstrauss embeddings via the restricted isometry property. SIAM J Math Anal, 2011, 43: 1269–1281
20 Ledoux M. The Concentration of Measure Phenomenon. Providence: Amer Math Soc, 2001
21 Vershynin R. Introduction to the non-asymptotic analysis of random matrices. In: Compressed Sensing: Theory and Applications. Cambridge: Cambridge University Press, 2012, 210–268
22 Voroninski V, Xu Z. A strong restricted isometry property, with an application to phaseless compressed sensing. Appl Comput Harmon Anal, 2016, 40: 386–395
23 Vybiral J. A variant of the Johnson-Lindenstrauss lemma for circulant matrices. J Funct Anal, 2011, 260: 1096–1105
24 Wang Y. Random matrices and erasure robust frames. ArXiv:1403.5969, 2014