Some New Properties of Jacobi’s Theta Functions*

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Abstract

In this paper, a monotonicity property for the quotient of two Jacobi’s theta functions with respect to the modulus $k$ is proved.

Mathematics Subject Classification (2000): 33E05

Keywords: Jacobi’s elliptic functions, Jacobi’s theta functions

1 Introduction and Main Result

Let $k \in (0, 1)$ denote the modulus of Jacobi’s elliptic functions $\text{sn}(u)$, $\text{cn}(u)$, and $\text{dn}(u)$, of Jacobi’s theta functions $\Theta(u)$, $H(u)$, $H_1(u)$, and $\Theta_1(u)$, and, finally, of Jacobi’s zeta function, $\text{zn}(u)$. Here we follow the notation of Carlson and Todd [3], in other references, like [8], Jacobi’s zeta function is denoted by $Z(u)$. Let $k' := \sqrt{1-k^2} \in (0, 1)$ be the complementary modulus, let $K \equiv K(k)$ and $E \equiv E(k)$ be the complete elliptic integral of the first and second kind, respectively, and let $K' \equiv K'(k) := K(k')$ and $E' \equiv E'(k) := E(k')$.

If we want to point out the dependence of these functions on the modulus $k$, we will write $K(k)$, $K'(k)$, $\text{sn}(u,k)$, $\Theta(u,k)$, etc. In this paper, we follow the old notation of the theta functions given by $\Theta(u,k) = \vartheta_0(v,\tau) = \vartheta_1(v,\tau)$, $H(u,k) = \vartheta_1(v,\tau)$, $H_1(u,k) = \vartheta_2(v,\tau)$ and $\Theta_1(u,k) = \vartheta_3(v,\tau)$, where $\tau = iK'/K$ and $v = u/(2K)$ (note that in some references, like [2] and [4], $v = u\pi/(2K)$).

For the definitions and many important properties of these functions, see, e.g., [2], [4], [5], [7], and [1].

In [3], Carlson and Todd proved that, for each $\lambda \in (0, 1)$, the functions $\text{sn}(\lambda K, k)$ and $\text{zn}(\lambda K, k)$ are strictly monotone increasing with respect to the modulus $k$, $0 < k < 1$. In addition, they investigated the degenerating behaviour of these functions as $k \to 0$ and especially as $k \to 1$. Hence the question arises, whether analogous monotonicity properties hold for the theta functions. Unfortunately, $\Theta(\lambda K, k)$ does not have the same monotonicity behaviour with respect to the modulus $k$ for every $\lambda \in (0, 1)$. Numerical examples show that for small $\lambda$ ($\lambda \leq 0.5$), $\Theta(\lambda K)$ is strictly monotone decreasing, for large $\lambda$ ($\lambda \geq 0.6$), $\Theta(\lambda K)$ is strictly monotone increasing in $k$ and for some $\lambda \in (0.5, 0.6)$, $\Theta(\lambda K)$ is not monotone at all in the whole interval $(0, 1)$. However, we are able to prove that for each $\lambda, \mu \in \mathbb{R}$, the quotient $\Theta(\lambda K)/\Theta(\mu K)$ of two theta functions is strictly monotone. In addition, the degenerative behaviour of $\Theta(\lambda K)/\Theta(\mu K)$ as $k \to 0$ and $k \to 1$ is given.

*published in: Journal of Computational and Applied Mathematics 178 (2005), 419–424.
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Theorem 1.

(i) Let $\lambda, \mu \in \mathbb{R}$. If $\cos(\lambda \pi) > \cos(\mu \pi)$, then $\Theta(\lambda K)/\Theta(\mu K)$ is a positive, strictly monotone decreasing function of the modulus $k \in (0, 1)$. If $\cos(\lambda \pi) = \cos(\mu \pi)$, i.e., $\lambda = \mu + 2\nu$, $\nu \in \mathbb{Z}$, then $\Theta(\lambda K)/\Theta(\mu K) = 1$.

(ii) Let $\lambda, \mu \in (0, 1)$, then $\Theta(\lambda K)/\Theta(\mu K) \to 1$ as $k \to 0$ and

\[
\frac{\Theta(\lambda K)}{\Theta(\mu K)} \sim \left(\frac{k'}{4}\right)^{(\mu - \lambda)(1 - (\lambda + \mu)/2)} \quad \text{as } k \to 1.
\]

(iii) Let $\mu \in (0, 1)$ and $k \in (0, 1)$. Then $f(\lambda) := \Theta((\mu - \lambda) K)/\Theta((\mu + \lambda) K)$ is a convex function of $\lambda \in (0, 1)$ with $f(0) = f(1) = 1$.

Remark. By the relation $\Theta_1(u) = \Theta(u + K)$, one gets analogous monotonicity properties for the quotients $\Theta_1(\lambda K)/\Theta_1(\mu K)$, $\Theta(\lambda K)/\Theta_1(\mu K)$, and $\Theta(\lambda K)/\Theta(\mu K)$.

In [6], we considered polynomials, whose $[-1, 1]$ inverse image consists of two Jordan arcs, i.e., we characterized polynomials $P_n$, for which $P_n^{-1}([-1, 1])$ consists of two Jordan arcs (in general, $P_n^{-1}([-1, 1])$ consists of $n$ Jordan arcs). Since these polynomials $P_n$ can be given with the help of an elliptic integral, Jacobi’s elliptic and theta functions appear in a natural way. When describing the shape of the two Jordan arcs, we need the above theorem, see Theorem 22, Lemma 33 and Lemma 34 of [6].

2 Proof of the Main Result

First, we collect some derivation formulas, which are an immediate consequence of formula (710.00) of [2].

Lemma 2. Let $k \in (0, 1)$. Then

\[
\frac{dK}{dk} = \frac{E - k'^2 K}{kk'^2 K}, \quad \frac{dK'}{dk} = \frac{k^2 K' - E'}{kk'^2 K}, \quad \frac{d}{dk}\left\{\frac{1}{K}\right\} = \frac{k^2 K - E}{kk'^2 K^2}, \quad \frac{d}{dk}\left\{\frac{K'}{K}\right\} = \frac{-\pi}{2kk'^2 K^2}.
\]

Concerning the limiting behaviour of $\text{sn}(\lambda K)$, etc., as $k \to 1$, Carlson and Todd [3] have proved the following.

Lemma 3. Let $0 < \lambda < 1$, then, as $k \to 1$, we have

\[
K \sim \log\left(\frac{1}{\phi}\right), \quad \text{sn}(\lambda K) \sim 1 - 2(\vartheta_2' \lambda^2), \quad \text{cn}(\lambda K) \sim \text{dn}(\lambda K) \sim 2(\vartheta_2' \lambda^2), \quad \text{zn}(\lambda K) \sim 1 - \lambda - 2(\vartheta_2' \lambda^2).
\]

Moreover, we will need the following formulas for the derivatives of the theta functions, which are a direct consequence of (1053.01), (1052.02) and (731.01)–(731.03) of [2].
Lemma 4. The following relations hold:
\[
\frac{\partial}{\partial u}\{\Theta(u)\} = \Theta(u)\zeta(u),
\]
\[
\frac{\partial}{\partial u}\{H(u)\} = \sqrt{k} \Theta(u)(\text{cn}(u)\text{dn}(u) + \text{sn}(u)\zeta(u)),
\]
\[
\frac{\partial}{\partial u}\{H_1(u)\} = \sqrt{k} \Theta(u)(-\text{sn}(u)\text{dn}(u) + \text{cn}(u)\zeta(u)),
\]
\[
\frac{\partial}{\partial u}\{\Theta_1(u)\} = \frac{1}{\sqrt{k}} \Theta(u)(-k^2\text{sn}(u)\text{cn}(u) + \text{dn}(u)\zeta(u)),
\]
and
\[
\frac{\partial^2}{\partial u^2}\{\Theta(u)\} = \Theta(u)(\text{dn}^2(u) + \zeta^2(u) - E/K),
\]
\[
\frac{\partial^2}{\partial u^2}\{H(u)\} = \sqrt{k} \Theta(u)\left(-k^2\text{sn}(u)\text{cn}^2(u) + 2\text{cn}(u)\text{dn}(u)\zeta(u) + \text{sn}(u)\zeta^2(u) - E/K\right),
\]
\[
\frac{\partial^2}{\partial u^2}\{H_1(u)\} = \sqrt{k} \Theta(u)\left(-k^2\text{sn}^2(u)\text{cn}(u) - 2\text{sn}(u)\text{dn}(u)\zeta(u) + \text{cn}(u)\zeta^2(u) - E/K\right),
\]
\[
\frac{\partial^2}{\partial u^2}\{\Theta_1(u)\} = \frac{1}{\sqrt{k}} \Theta(u)\left(\text{dn}(u)\left(1 - k^2\text{cn}^2(u)\right) - 2k^2\text{sn}(u)\text{cn}(u)\zeta(u) + \text{dn}(u)\zeta^2(u) - E/K\right).
\]

Remark. By (123.01), (123.03) of [2] and Lemma 4 the relation
\[
\frac{\partial}{\partial u}\left\{\log\left(\frac{H(v-u)}{H(v+u)}\right)\right\} = \frac{2\text{sn}(v)\text{cn}(v)\text{dn}(v)}{\text{sn}^2(u) - \text{sn}^2(v)} - 2\zeta(v)
\]
holds which implies
\[
\log\left(\frac{H(v-u)}{H(v+u)}\right) = \int_0^u \frac{2\text{sn}(v)\text{cn}(v)\text{dn}(v)}{\text{sn}^2(u) - \text{sn}^2(v)} \, du - 2u\zeta(v).
\]

Analogous formulas can be obtained for other ratios of theta functions.

The next lemma gives the derivatives with respect to the modulus \(k\) for the four theta functions \(\Theta(u), H(u), H_1(u), \Theta_1(u)\), where \(u = \lambda K\). Note that \(\lambda K\) as well as the theta functions themselves depend on the modulus \(k\).

Lemma 5. For \(\lambda \in \mathbb{R}\), the derivatives with respect to \(k\) of the four theta functions \(\Theta(\lambda K), H(\lambda K), H_1(\lambda K), \Theta_1(\lambda K)\) are given by
\[
\frac{d}{dk}\{\Theta(\lambda K)\} = -\frac{1}{2kk^2} \frac{\partial^2}{\partial u^2}\{\Theta(u)\}\bigg|_{u=\lambda K},
\]
\[
\frac{d}{dk}\{H(\lambda K)\} = -\frac{1}{2kk^2} \frac{\partial^2}{\partial u^2}\{H(u)\}\bigg|_{u=\lambda K},
\]
\[
\frac{d}{dk}\{H_1(\lambda K)\} = -\frac{1}{2kk^2} \frac{\partial^2}{\partial u^2}\{H_1(u)\}\bigg|_{u=\lambda K},
\]
\[
\frac{d}{dk}\{\Theta_1(\lambda K)\} = -\frac{1}{2kk^2} \frac{\partial^2}{\partial u^2}\{\Theta_1(u)\}\bigg|_{u=\lambda K}.
\]
Proof. The four theta functions \( \vartheta_j(v, \tau), \ j = 1, 2, 3, 4, \) satisfy a differential equation of the form

\[
\frac{\partial^2}{\partial v^2} \{ \vartheta_j(v, \tau) \} = 4i\pi \frac{\partial}{\partial \tau} \{ \vartheta_j(v, \tau) \},
\]

see [5, p. 375]. By Lemma [2] and since \( \Theta(u, k) = \vartheta_4(\frac{u}{2K}, \tau) \), where \( \tau = iK'/K \),

\[
\frac{\partial}{\partial k} \{ \Theta(u, k) \} = \frac{\partial}{\partial v} \{ \vartheta_4(\frac{u}{2K}, \tau) \} \frac{d}{dk} \{ \vartheta_4(\frac{u}{2K}, \tau) \} + \frac{\partial}{\partial \tau} \{ \vartheta_4(\frac{u}{2K}, \tau) \} \frac{d\tau}{dk} = \frac{u(k^2K - E)}{kk^2K} \frac{\partial}{\partial u} \{ \Theta(u, k) \} - \frac{1}{2kk^2} \frac{\partial^2}{\partial u^2} \{ \Theta(u, k) \}.
\]

Thus, for the derivative with respect to \( k \), by Lemma [2] we get

\[
\frac{d}{dk} \{ \Theta(\lambda K, k) \} = \frac{d}{dk} \{ \lambda K \} \frac{\partial}{\partial u} \{ \Theta(u, k) \} \bigg|_{u=\lambda K} + \frac{\partial}{\partial k} \{ \Theta(u, k) \} \bigg|_{u=\lambda K} = \frac{\lambda(E - k^2K)}{kk^2} \frac{\partial}{\partial u} \{ \Theta(u, k) \} \bigg|_{u=\lambda K} + \frac{\lambda K(k^2K - E)}{kk^2K} \frac{\partial}{\partial u} \{ \Theta(u, k) \} \bigg|_{u=\lambda K}
\]

\[
- \frac{1}{2kk^2} \frac{\partial^2}{\partial u^2} \{ \Theta(u, k) \} \bigg|_{u=\lambda K} = - \frac{1}{2kk^2} \frac{\partial^2}{\partial u^2} \{ \Theta(u, k) \} \bigg|_{u=\lambda K}.
\]

Since the last identity holds for the other three theta functions as well, this gives the assertion.

Proof of Theorem 1.

(i) By Lemma [5]

\[
\frac{d}{dk} \left\{ \frac{\Theta(\lambda K)}{\Theta(\mu K)} \right\} = - \frac{1}{2kk^2} \frac{\Theta(\lambda K)}{\Theta(\mu K)} \times (dn^2(\lambda K) + zn^2(\lambda K) - dn^2(\mu K) - zn^2(\mu K)).
\]

Thus, it remains to be shown that \( g(u) := dn^2(u) + zn^2(u) \) satisfies the inequality \( g(\lambda K) - g(\mu K) > 0 \) for all \( \lambda, \mu \in \mathbb{R} \) with \( \cos(\lambda \pi) > |< \cos(\mu \pi) \). This property holds since \( g(-u) = g(u), g(u + 2K) = g(u) \) and \( g(u) \) is a positive, strictly monotone decreasing function in \((0, K)\). Concerning the monotonicity of \( g(u) \), note that

\[
g'(u) = -2dn^2(u) \left( \frac{k^2sn(u)cn(u)}{dn(u)} - zn(u) \right) = h(u)
\]

where \( h(0) = h(K) = 0 \) and

\[
h''(u) = -\frac{2k^2k^2sn(u)cn(u)}{dn^3(u)} < 0 \text{ for } u \in (0, K).
\]

Thus \( h(u) > 0 \) and therefore \( g'(u) < 0 \) for \( u \in (0, K) \).
(ii) Assume that \( \lambda < \mu \) and let \( \alpha := (\lambda + \mu)/2, \beta := (\mu - \lambda)/2 \), i.e. \( \lambda = \alpha - \beta, \mu = \alpha + \beta, \alpha, \beta \in (0, 1) \). By (1052.02) of [2] and (1), we get

\[
\log \left( \frac{\Theta(\lambda K)}{\Theta(\mu K)} \right) = \log \left( \frac{\text{sn}(\mu K) H(\lambda K)}{\text{sn}(\lambda K) H(\mu K)} \right) = \log(\text{sn}(\mu K)) - \log(\text{sn}(\lambda K))
\]

\[
+ 2K \text{sn}(\alpha K) \text{cn}(\alpha K) \text{dn}(\alpha K) \int_{0}^{\beta} \frac{d\nu}{\text{sn}^2(\nu K) - \text{sn}^2(\alpha K)} - 2\beta K \text{zn}(\alpha K).
\]

Thus, as \( k \to 0 \),

\[
\log \left( \frac{\Theta(\lambda K)}{\Theta(\mu K)} \right) \sim \log(\text{sn}(\mu \pi/2)) - \log(\text{sn}(\lambda \pi/2)) + \pi \sin(\alpha \pi/2) \cos(\alpha \pi/2) \int_{0}^{\beta} \frac{d\nu}{\text{sn}^2(\nu K) - \text{sn}^2(\alpha K)} = 0,
\]

and, as \( k \to 1 \), by Lemma 3, \( a := k'/4 \),

\[
\log \left( \frac{\Theta(\lambda K)}{\Theta(\mu K)} \right) \sim \log(1 - 2a^2\mu) - \log(1 - 2a^2\lambda)
\]

\[
- 2\log(a)(1 - 2a^2\alpha) \int_{0}^{\beta} \frac{d\nu}{4(a^2\alpha - a^2\nu)} + 2\beta(1 - \alpha - 2a^2\alpha) \log(a)
\]

\[
\sim 2\log(a) \int_{0}^{\beta} \frac{d\nu}{a^{2(\nu-\alpha)} - 1} + 2\beta(1 - \alpha) \log(a)
\]

\[
= -2\beta \log(a) + \log(a^{2(\beta-\alpha)} - 1) - \log(a^{-2\alpha} - 1) + 2\beta(1 - \alpha) \log(a)
\]

\[
\sim -2\beta \log(a) + 2(\beta - \alpha) \log(a) + 2\alpha \log(a) + 2\beta(1 - \alpha) \log(a)
\]

\[
= 2\beta(1 - \alpha) \log(a).
\]

If \( \lambda > \mu \), note that by the above proved result

\[
\log \frac{\Theta(\lambda K)}{\Theta(\mu K)} = - \log \frac{\Theta(\mu K)}{\Theta(\lambda K)} \sim - (\lambda - \mu)(1 - (\lambda + \mu)/2) \log a
\]

\[
= (\mu - \lambda)(1 - (\lambda + \mu)/2) \log a.
\]

(iii) Obviously, \( f(0) = 1 \) and, by (1051.02) and (1051.03) of [2],

\[
f(1) = \frac{\Theta(\mu K - K)}{\Theta(\mu K + K)} = \frac{\Theta_1(-\mu K)}{\Theta_1(\mu K)} = 1.
\]

Further, by Lemma 4,

\[
f''(\lambda) = K^2 f(\lambda) \left( \text{zn}(\mu K) + \text{zn}(\mu K) \right)^2
\]

\[
+ \text{dn}^2(\mu K) - \text{dn}^2(\mu K) + \frac{2E}{K^2} > 0
\]

for every \( \lambda \in (0, 1) \), which gives the assertion.
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