Estimation and Comparison of Correlation-based Measures of Concordance

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Abstract

We address the problem of estimating and comparing transformed rank correlation coefficients defined as Pearson’s linear correlation between two random variables transformed by a so-called concordance-inducing function. The class of transformed rank correlations includes Spearman’s rho, Blomqvist’s beta and van der Waerden’s coefficient as special cases by taking uniform, Bernoulli and normal distributions as concordance-inducing functions, respectively. We propose a novel framework for comparing transformed rank correlations in terms of the asymptotic variance of their canonical estimators. A general criterion derived from this framework is that concordance-inducing functions with smaller variances of squared random variables are more preferable. In particular, we show that Blomqvist’s beta attains the optimal asymptotic variance and Spearman’s rho outperforms van der Waerden’s coefficient. We also find that the optimal bounds of the asymptotic variance are attained by Kendall’s tau.

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1 Introduction

Since the work of Embrechts et al. (2002), copulas have been widely adopted in insurance and risk management to model dependence between random variables. To quantify dependence in terms of a single number, measures of concordance, such as Spearman’s rho and Kendall’s tau, have been widely used as alternatives to Pearson’s linear correlation coefficient since measures of concordance do not depend on the marginal distributions of the underlying random variables whereas Pearson’s correlation does not possess this desirable property of a measure of dependence. Interestingly, popular measures of concordance such as Spearman’s rho and Blomqvist’s beta can be represented as Pearson’s linear correlation $\rho$ between random variables transformed by some real-valued functions $g_1, g_2$. Hofert and Koike (2019) showed that, for such a transformed correlation coefficient to be a measure of concordance, the two functions $g_1$ and $g_2$ necessarily have to be generalized inverses of nondegenerate radially symmetric (cumulative) distribution functions $G_1, G_2 : \mathbb{R} \rightarrow [0, 1]$ of the same type with finite second moments, and that one can take $G_1 = G_2 =: G$ without loss of generality. For such a distribution function $G$ with $G^{-1}$ being the generalized inverse of $G$, the resulting measure

$$\kappa_G(C) = \kappa_G(U, V) = \rho(G^{-1}(U), G^{-1}(V))$$

for a copula $C$ and $(U, V) \sim C$, is called the $G$-transformed rank correlation coefficient with concordance-inducing function $G$. Thanks to this representation via Pearson’s correlation, this class of measures of concordance has various appealing properties, such as interpretability and ease of studying compatibility and attainability problems; see Hofert and Koike (2019). Another advantage of this class is its ease of estimation since one can estimate $\kappa_G$ by the sample correlation of pseudo-observations from $C$ transformed by $G^{-1}$ for a given $G$.

For a given class of transformed rank correlation coefficients, natural questions are which concordance-inducing function is best to use and how to compare different measures of concordance. De Winter et al. (2016) compared Pearson’s linear correlation and Spearman’s rho by numerical experiments in terms of bias, variance and robustness to outliers. Various measures of concordance were compared in terms of their power in tests of independence; see, for example, Bhuchongkul (1964), Behnen (1971), Behnen (1972), Luigi Conti and Nikitin (1999), Rödel and Kössler (2004) and Genest and Verret (2005).

This paper tackles the problem of comparing measures of concordance from the theoretical viewpoint of statistical estimation of $\kappa_G$. In our proposed framework, a concordance-inducing
function $G$ is more preferable than another one $G'$ if the largest (worst) or smallest (best) asymptotic variance of a canonical estimator $\hat{\kappa}_G$ of $\kappa_G$ is smaller than that of $G'$ for a certain set of copulas $\mathcal{D}$. Simply put, $G$ is more preferable than $G'$ if $\hat{\kappa}_G$ tends to estimate $\kappa_G$ more accurately than $\hat{\kappa}_{G'}$ estimates $\kappa_{G'}$ if the underlying copula belongs to $\mathcal{D}$. A general criterion derived from this framework is that concordance-inducing functions with smaller variance $\text{Var}_G(X^2)$ where $X \sim G$ is more preferable. Therefore, heavy-tailed concordance-inducing functions, such as a Student $t$ distribution function, are not recommended in comparison to normal ones. We also find that Spearman’s rho, for which $G$ is the uniform distribution, can be outperformed by rank correlations transformed by Beta distributions. Moreover, under certain conditions on $\mathcal{D}$, we prove that Blomqvist’s beta attains the optimal worst and best asymptotic variances among all transformed rank correlation coefficients, and Spearman’s rho is more preferable than van der Waerden’s coefficient. Considering the drawback of Blomqvist’s beta that it only depends on the local value $C(1/2, 1/2)$ of a copula $C$, we also compare transformed rank correlations with Kendall’s tau. Based on the representation of Kendall’s tau in terms of Pearson’s linear correlation coefficient, we find that Kendall’s tau also attains the optimal worst and best asymptotic variances if estimators of these measures are compared without being standardized by sample size. Since the correlation-representation of Kendall’s tau depends on two independent copies of random vectors following $C$, Kendall’s tau is not optimal any more if the asymptotic variances of these estimators are standardized by sample size. Finally, in a simulation study, we find that the choice of concordance-inducing function $G$ and the strength of dependence of the underlying copula $C$ affect the asymptotic variance of $\hat{\kappa}_G$ more than the kinds of copulas.

This paper is organized as follows. In Section 2, we briefly review measures of concordance, the class of $G$-transformed rank correlations and their basic properties. In Section 3 we introduce a framework for comparing $G$-transformed rank correlations in terms of their asymptotic variances. A canonical estimator of a transformed rank correlation is presented in Section 3.1. Section 3.2 addresses effects of location-scale transforms of $G$ on the asymptotic variance. The worst and best asymptotic variances among fundamental and Fréchet copulas are provided in Section 3.3, and the optimality of Blomqvist’s beta is given in Section 3.4. Transformed rank correlations and Kendall’s tau are compared in Section 4. In Section 5, a simulation study is conducted to compare asymptotic variances for various parametric copulas and concordance-inducing fuctions. Section 6 concludes this work with discussions about directions for future research.
2 Transformed rank correlations

Let $C_2$ be the set of all bivariate copulas, that is, all bivariate distributions functions with standard uniform univariate marginal distributions; see Nelsen (2006). We call $C' \in C_2$ more concordant than $C \in C_2$, denoted by $C \preceq C'$, if $C(u, v) \leq C'(u, v)$ for all $(u, v) \in [0, 1]^2$. The survival function of $C$ is given by $\bar{C}(u, v) = \mathbb{P}(U > u, V > v)$, $(u, v) \in [0, 1]$ where $(U, V) \sim C$. For any map $\kappa : C_2 \to \mathbb{R}$, we identify $\kappa(C)$ with $\kappa(U, V)$ for a random vector $(U, V) \sim C$ defined on a fixed atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A map $\kappa$ on $C_2$ is called a measure of concordance if it satisfies the following seven axioms proposed in Scarsini (1984).

Definition 2.1 (Axioms for measures of concordance). A map $\kappa : C_2 \to \mathbb{R}$ is called a measure of concordance if it satisfies the following seven axioms.

1. Domain: $\kappa(C)$ is defined for any $C \in C_2$.
2. Symmetry: $\kappa(V, U) = \kappa(U, V)$ for any $(U, V) \sim C \in C_2$.
3. Monotonicity: If $C \preceq C'$ for $C, C' \in C_2$, then $\kappa(C) \leq \kappa(C')$.
4. Range: $-1 \leq \kappa(C) \leq 1$ for any $C \in C_2$ and the bounds $\kappa(C) \pm 1$ are attainable for some $C \in C_2$.
5. Independence: $\kappa(\Pi) = 0$ for the independence copula $\Pi \in C_2$, where $\Pi(u, v) = uv$, $(u, v) \in [0, 1]^2$.
6. Change of sign: $\kappa(U, 1 - V) = -\kappa(U, V)$ for any $(U, V) \sim C \in C_2$.
7. Continuity: Let $C_n \in C_2$, $n \in \mathbb{N}$, and $C \in C_2$ with $C_n$ converging pointwise to $C$ as $n \to \infty$. Then $\lim_{n \to \infty} \kappa(C_n) = \kappa(C)$.

The comonotonic and countermonotonic copulas are denoted by $M(u, v) = \min(u, v)$ and $W(u, v) = \max(u + v - 1, 0)$, $(u, v) \in [0, 1]^2$, respectively. As is known from the Fréchet-Hoeffding bounds, $M$ and $W$ satisfy $W \preceq C \preceq M$, and thus $\kappa(M) = 1$ and $\kappa(W) = -1$ for any measure of concordance $\kappa$ by the monotonicity and range axioms.

Consider a class of maps on $C_2$ written as $\kappa_{g_1, g_2}(U, V) = \rho(g_1(U), g_2(V))$ for $g_1, g_2 : [0, 1] \to \mathbb{R}$. By Hofert and Koike (2019, Proposition 1 and Theorem 1), for the map $\kappa_{g_1, g_2}$ to be a measure of concordance, it suffices to consider the case when $g_1 = g_2 = G^{-1}$ for some
nondegenerate radially symmetric distribution function $G : \mathbb{R} \to [0, 1]^2$ with finite second moment, where $G^{-1}$ is the \textit{generalized inverse} of $G$ defined by

$$G^{-1}(p) = \inf \{ x \in \mathbb{R} : G(x) \geq p \}, \quad p \in (0, 1);$$

see Hofert and Koike (2019, Section 2) for more detailed discussion on the characterization of $\kappa_{g_1,g_2}$. We thus obtain the following definition of $G$-transformed rank correlation coefficients.

\textbf{Definition 2.2} ($G$-transformed rank correlation coefficient). For a nondegenerate radially symmetric distribution function $G : \mathbb{R} \to [0, 1]$ with finite second moment, define a measure $\kappa : \mathcal{C}_2 \to [0, 1]$ by

$$\kappa_G(C) = \kappa_G(U,V) = \rho(G^{-1}(U), G^{-1}(V)).$$  \hfill (1)

We call (1) the $G$-transformed rank correlation coefficient of $(U,V) \sim C \in \mathcal{C}_2$ and $G$ the concordance-inducing function. The set of all concordance-inducing functions is denoted by $\mathcal{G}$.

The class of $G$-transformed rank correlation coefficients includes popular measures of concordance as special cases. For example, $\kappa_G$ reduces to Spearman’s rho $\rho_S(U,V) = 12\mathbb{E}[UV] - 3$ (Spearman, 1904) if $G$ is the distribution function of the standard uniform distribution $\text{Unif}(0,1)$, and $\kappa_G$ is Blomqvist’s beta (also called the median correlation coefficient) $\beta(C) = 4C(1/2,1/2) - 1$ (Blomqvist, 1950) if $G$ is a symmetric Bernoulli distribution $\text{Bern}(1/2)$ on $\{0,1\}$. When $G$ is the standard normal distribution $N(0,1)$, then $\kappa_G$ is known as van der Waerden’s coefficient (also known as normal score correlation) $\zeta(U,V) = \rho(\Phi^{-1}(U), \Phi^{-1}(V))$ (Sidak et al., 1999) where $\Phi$ is the distribution function of $N(0,1)$.

The following proposition summarizes basic properties of $\kappa_G$ needed later on in this work.

\textbf{Proposition 2.3} (Basic properties of $\kappa_G$). For any $G \in \mathcal{G}$, the $G$-transformed rank correlation coefficient $\kappa_G$ satisfies the following properties.

1. $\kappa_G$ is a measure of concordance.

2. $\kappa_G$ is invariant under location-scale transforms of $G$, that is, $\kappa_{G_{\mu,\sigma}}(C) = \kappa_G(C)$ for all $C \in \mathcal{C}_2$, where $\mu \in \mathbb{R}$, $\sigma > 0$ and $G_{\mu,\sigma}(x) = G\left(\frac{x-\mu}{\sigma}\right)$.

3. For $n \in \mathbb{N}$, let $C_1, \ldots, C_n \in \mathcal{C}_2$ and $\alpha_1, \ldots, \alpha_n$ be non-negative numbers such that $\alpha_1 + \cdots + \alpha_n = 1$. Then

$$\kappa_G\left(\sum_{i=1}^{n} \alpha_i C_i\right) = \sum_{i=1}^{n} \alpha_i \kappa_G(C_i).$$
Proof. Part 1 is Theorem 1 and Part 3 is Proposition 3 of Hofert and Koike (2019). Part 2 comes from the invariance of Pearson’s linear correlation coefficient under location-scale transform.

3 Estimation of $\kappa_G$ and their comparison

In this section, we propose a novel framework for comparing $G$-transformed rank correlations to answer the question which concordance-inducing function is best to be used. In the proposed framework, transformed correlations are compared in terms of the asymptotic variances of their canonical estimators, and one concordance-inducing function $G \in \mathcal{G}$ is considered better than another $G' \in \mathcal{G}$ if the largest (worst) or smallest (best) asymptotic variance of an estimator $\hat{\kappa}_G$ of $\kappa_G$ among a set of copulas $\mathcal{D} \subseteq \mathcal{C}_2$ is smaller than that of $\kappa_{G'}$.

3.1 Canonical estimator of $\kappa_G$

Based on Proposition 2.3 Part 2, we first consider standardized concordance-inducing functions $G$ with mean zero and variance one. Effects of location-scale transforms of $G$ to estimators of $\kappa_G$ will be discussed in Section 3.2. Assuming that an i.i.d. sample $(U_i, V_i), i = 1, \ldots, n, n \in \mathbb{N}$, from $C$ is available, we consider the following canonical estimator of $\kappa_G$:

$$\hat{\kappa}_G = \frac{1}{n} \sum_{i=1}^{n} G^{-1}(U_i) G^{-1}(V_i).$$

By the central limit theorem (CLT), $\hat{\kappa}_G$ satisfies the following asymptotic normality:

$$\sqrt{n} \{\hat{\kappa}_G - \kappa_G(C)\} \overset{d}{\rightarrow} N(0, \sigma^2_G(C)), \quad \sigma^2_G(C) = \text{Var}(G^{-1}(U)G^{-1}(V)),$$

provided that the fourth moment of $G$ is finite. We consider the sets of optimal concordance-inducing functions and the corresponding optimal bounds in terms of the worst and best asymptotic variances of $\hat{\kappa}_G$, defined by

$$G_\ast(\mathcal{H}, \mathcal{D}) = \arg \inf_{G \in \mathcal{H}} \sigma^2_G(\mathcal{D}), \quad \sigma^2_\ast(\mathcal{H}, \mathcal{D}) = \inf_{G \in \mathcal{H}} \sigma^2_G(\mathcal{D}),$$

$$\overline{G}_\ast(\mathcal{H}, \mathcal{D}) = \arg \inf_{G \in \mathcal{H}} \overline{\sigma}^2_G(\mathcal{D}), \quad \overline{\sigma}^2_\ast(\mathcal{H}, \mathcal{D}) = \inf_{G \in \mathcal{H}} \overline{\sigma}^2_G(\mathcal{D}),$$
respectively, for $\mathcal{H} \subseteq \mathcal{G}_4$ and $\mathcal{D} \subseteq \mathcal{C}_2$, where

$$
\sigma_G^2(\mathcal{D}) = \inf_{C \in \mathcal{D}} \sigma_G^2(C), \quad \sigma_G^2(\mathcal{H}) = \sup_{C \in \mathcal{H}} \sigma_G^2(C)
$$

and

$$\mathcal{G}_4 = \{ G \in \mathcal{G} : \mathbb{E}_G[X] = 0, \ \text{Var}_G(X) = 1 \ \text{and} \ \mathbb{E}_G[X^4] < \infty \}$$

with $\mathbb{E}_G[X]$ and $\text{Var}_G(X)$ being the mean and variance of $X \sim G$, respectively. Calculating the optimal worst and best asymptotic variances $\sigma^2_4(\mathcal{H}, \mathcal{D})$ and $\sigma^2_4(\mathcal{H}, \mathcal{D})$ is not straightforward since neither $C \mapsto \sigma^2_G(C)$ nor $G \mapsto \sigma^2_G(C)$ have simple (sub-/super-)linearity; see Appendix B for details of $C \mapsto \sigma^2_G(C)$. Although an ideal choice of $\mathcal{H}$ is $\mathcal{H} = \mathcal{G}_4$, other choices can also be of interest; for example, $\mathcal{H} = \mathcal{G}_4^c$ where $\mathcal{G}_4^c$ is the set of continuous concordance-inducing functions in $\mathcal{G}_4$, and $\mathcal{H} = \mathcal{G}_4^0$ where $\mathcal{G}_4^0$ is the set of concordance-inducing functions in $\mathcal{G}_4$ with bounded supports. Note that one-sided distributions such that $\text{esssup}(G) = \infty$ and $\text{essinf}(G) < \infty$, or $\text{esssup}(G) < \infty$ and $\text{essinf}(G) = -\infty$, cannot be concordance-inducing since they cannot be radially symmetric. Therefore, the set $\mathcal{G}_4^b$ excludes concordance-inducing functions whose supports are $\mathbb{R}$, and $\mathcal{G}_4 \setminus \mathcal{G}_4^b$ is a set of concordance-inducing functions in $\mathcal{G}_4$ with supports $\mathbb{R}$. The sets of optimal concordance-inducing functions $\overline{\mathcal{G}}_*(\mathcal{H}, \mathcal{D})$ and $\overline{\mathcal{G}}_*(\mathcal{H}, \mathcal{D})$ are considered as the best choices among the set of concordance-inducing functions $\mathcal{H} \subseteq \mathcal{G}_4$ to accurately estimate $\kappa_G$ if one believes that $\mathcal{D}$ is the set of underlying copulas which one wants to quantify and compare in terms of their concordance.

**Remark 3.1** (Reflection invariance of $\sigma^2_G(C)$). Let $\nu_1, \nu_2 : \mathcal{C}_2 \rightarrow \mathcal{C}_2$ be partial reflections of copulas defined by

$$\nu_1(C)(u, v) = v - C(1 - u, v) \quad \text{and} \quad \nu_2(C)(u, v) = u - C(u, 1 - v), \quad C \in \mathcal{C}_2,$$

respectively, with their composition given by $\nu_1 \circ \nu_2(C)(u, v) = u + v - 1 + C(1 - u, 1 - v)$. For an operator $\varphi : \mathcal{C}_2 \rightarrow \mathcal{C}_2$, let $C_\varphi = \varphi(C)$. Then $(1 - U, V) \sim C_{\nu_1}$, $(U, 1 - V) \sim C_{\nu_2}$ and $(1 - U, 1 - V) \sim C_{\nu_1 \circ \nu_2}$ for $(U, V) \sim C$. By radial symmetry of $G \in \mathcal{G}$, we have that $G^{-1}(1 - U) = -G^{-1}(U)$ and $G^{-1}(1 - V) = -G^{-1}(V)$. Therefore, $\sigma_G^2(C)$ is invariant under the reflections $\nu_1, \nu_2, \nu_1 \circ \nu_2$ in the sense that $\sigma_G^2(C) = \sigma_G^2(C_{\nu_1}) = \sigma_G^2(C_{\nu_2}) = \sigma_G^2(C_{\nu_1 \circ \nu_2})$. This property follows intuitively since $|\kappa_G(C)|$ is also invariant under reflections, and thus one can estimate each of the quantities $\kappa_G(C)$, $\kappa_G(C_{\nu_1})$, $\kappa_G(C_{\nu_2})$ and $\kappa_G(C_{\nu_1 \circ \nu_2})$ from any other.

**Remark 3.2** (Asymptotic variance of Blomqvist’s beta). Schmid and Schmidt (2007) derived an asymptotic variance of Blomqvist’s beta. Their asymptotic variance is in general different.
from ours since we standardize the Bernoulli concordance-inducing function so that it has mean zero and variance one. As they stated, one of the advantages of Blomqvist’s beta over other measures of concordance is that Blomqvist’s beta admits an explicit form if the copula can be written explicitly. In fact, this advantage can be passed on to a wider class of discrete concordance-inducing functions; see Appendix A for details.

3.2 Optimal location shift of $G$

Although, by Proposition 2.3 Part 2, $\kappa_G$ is invariant under location-scale transforms of $G$, the asymptotic variance $\sigma^2_G(C)$ of its canonical estimator $\hat{\kappa}_G$ is not location invariant. To see this, let $G_0 \in \mathcal{G}_4$ be a concordance-inducing function with mean zero and variance one, and let $G_{\mu, \sigma}(x) = G_0(x - \mu) / \sigma$ be the corresponding concordance-inducing function of the same type as $G_0$ but with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. A canonical estimator of $\kappa_{G_{\mu, \sigma}}$ for known $\mu$ and $\sigma$ is then given by

$$\hat{\kappa}_{G_{\mu, \sigma}} = \frac{1}{n} \sum_{i=1}^{n} \frac{G_{\mu, \sigma}^{-1}(U_i)G_{\mu, \sigma}^{-1}(V_i)}{\sigma^2} - \left(\frac{\mu}{\sigma}\right)^2.$$  

By the CLT, $\hat{\kappa}_{G_{\mu, \sigma}}$ is asymptotically normal with asymptotic variance given by

$$\sigma^2_{G_{\mu, \sigma}}(C) = \text{Var} \left( \frac{G_{\mu, \sigma}^{-1}(U)G_{\mu, \sigma}^{-1}(V)}{\sigma^2} \right).$$

Since $G_{\mu, \sigma}^{-1}(U)/\sigma = G_{\mu/\sigma, 1}^{-1}(U)$ and $G_{\mu, \sigma}^{-1}(V)/\sigma = G_{\mu/\sigma, 1}^{-1}(V)$, one can assume that $\sigma = 1$ without changing the asymptotic variance $\sigma^2_{G_{\mu, \sigma}}(C)$. Therefore, $\sigma^2_{G}(C)$ for $G \in \mathcal{G}_4$ is invariant under scale transforms of $G$. On the other hand, $\sigma^2_{G}(C)$ changes under location transforms of $G$ since shifting $G^{-1}$ by $\mu \in \mathbb{R}$ leads to the asymptotic variance $\text{Var}((X + \mu)(Y + \mu)) = \text{Var}(XY + \mu(X + Y))$ for $X = G^{-1}(U)$ and $Y = G^{-1}(V)$, and it is in general not equal to $\text{Var}(XY)$.

Since the canonical estimator $\hat{\kappa}_{G_{\mu, \sigma}}$ estimates the same quantity $\kappa_{G_0}$ regardless of the mean $\mu$ and variance $\sigma$ of $G$, a natural choice of $\mu$ under $\sigma = 1$ is such that it minimizes the asymptotic variance $\sigma^2_{G_{\mu, 1}}(C)$. For a fixed concordance-inducing function $G_0 \in \mathcal{G}_4$ with mean zero and variance one, denote by $G_{\mu}(x) = G_0(x - \mu)$ the concordance-inducing function of the same type as $G_0$ but with mean $\mu \in \mathbb{R}$. For $X = X_0 + \mu \sim G_{\mu}$ and $Y = Y_0 + \mu \sim G_{\mu}$
with $X_0 = G_0^{-1}(U)$ and $Y_0 = G_0^{-1}(V)$, the asymptotic variance
\[
\sigma^2_{G_0}(C) = \text{Var}(XY) = \text{Var}(X_0 + \mu)(Y_0 + \mu)) = \text{Var}(X_0Y_0 + \mu(X_0 + Y_0))
\]
\[
= \text{Var}(X_0Y_0) + 2\mu \text{Cov}(X_0Y_0, X_0 + Y_0) + \mu^2 \text{Var}(X_0 + Y_0)
\]
is a quadratic function of $\mu \in \mathbb{R}$ provided that $\text{Var}(X_0 + Y_0) > 0$, and thus is minimized when
\[
\mu = \mu^* = \mu^*(G_0, C) = -\frac{\text{Cov}(X_0Y_0, X_0 + Y_0)}{\text{Var}(X_0 + Y_0)}.
\]
We call $\mu^*(G_0, C)$ an optimal shift of $G_0 \in G_4$ under $C \in C_2$. The degenerate case $\text{Var}(X_0 + Y_0) = 0$ occurs if and only if $\rho(X_0, Y_0) = -1$, and it is also equivalent to $C = W$; see Embrechts et al. (2002). In this case, $X_0 + Y_0 \overset{a.s.}{=} 0$ ("a.s." stands for almost surely) and thus $\text{Var}(XY) = \text{Var}(X_0Y_0)$, that is, location transforms of $G_0$ do not change $\sigma^2_{G_0}(C)$. Provided $\text{Var}(X_0 + Y_0) > 0$, that is, $C \neq W$, the optimal asymptotic variance is given by
\[
\sigma^2_{G_0^*}(C) = \text{Var}(X_0Y_0) - \frac{\text{Cov}(X_0Y_0, X_0 + Y_0)^2}{\text{Var}(X_0 + Y_0)}.
\]
\[(2)\]
The following proposition states that $\mu^*_2 = 0$ for a certain class of copulas.

**Proposition 3.3** (Sufficient condition for $\mu^*_2 = 0$). For a copula $C \in C_2$ and a concordance-inducing function $G_0 \in G_4$ with mean zero and variance one, $\mu^*_2(G_0, C) = 0$ holds if $C$ is radially symmetric $C = C_{\nu_1 \circ \nu_2}$, that is, $(U, V) \overset{d}{=} (1 - U, 1 - V)$ for $(U, V) \sim C$.

**Proof.** For $X_0 = G_0^{-1}(U)$ and $Y_0 = G_0^{-1}(V)$ with $(U, V) \sim C$, we have that $\mathbb{E}[X_0 + Y_0] = \mathbb{E}[X_0] + \mathbb{E}[Y_0] = 0$, and thus $\text{Cov}(X_0Y_0, X_0 + Y_0) = \mathbb{E}[X_0Y_0(X_0 + Y_0)] - \mathbb{E}[X_0Y_0] \mathbb{E}[X_0 + Y_0] = \mathbb{E}[X_0Y_0(X_0 + Y_0)]$. Therefore, it suffices to show that $\mathbb{E}[X_0Y_0(X_0 + Y_0)] = 0$ when $C$ is radially symmetric.

For $\varphi \in \{\iota, \nu_1, \nu_2, \nu_1 \circ \nu_2\}$ where $\iota : C_2 \to C_2$ is the identity $\iota(C) = C$, denote $(U_{\varphi}, V_{\varphi}) \sim C_{\varphi}$. Since $G_0$ is radially symmetric, we have that
\[
(G_0^{-1}(U), G_0^{-1}(V)) \overset{d}{=} (-G_0^{-1}(U_{\nu_1}), G_0^{-1}(V_{\nu_1})) \overset{d}{=} (G_0^{-1}(U_{\nu_2}), -G_0^{-1}(V_{\nu_2}))
\]
\[
\overset{d}{=} (-G_0^{-1}(U_{\nu_1 \circ \nu_2}), -G_0^{-1}(V_{\nu_1 \circ \nu_2})).
\]
Moreover, when $C$ is radially symmetric, we have that $C_{\nu_1} = C_{\nu_2}$. Therefore, $(U, V) \overset{d}{=} (U_{\nu_1 \circ \nu_2}, V_{\nu_1 \circ \nu_2})$ and $(U_{\nu_1}, V_{\nu_1}) \overset{d}{=} (U_{\nu_2}, V_{\nu_2})$ for $(U, V) \sim C$. Together with the identity
\[
1 = 1\{U > 1/2, V > 1/2\} + 1\{U \leq 1/2, V \leq 1/2\} + 1\{U > 1/2, V \leq 1/2\} + 1\{U \leq 1/2, V > 1/2\}
\]
\[
= 1\{U > 1/2, V > 1/2\} + 1\{U_{\nu_1} > 1/2, V_{\nu_1} > 1/2\} + 1\{U_{\nu_2} > 1/2, V_{\nu_2} > 1/2\} + 1\{U_{\nu_1 \circ \nu_2} > 1/2, V_{\nu_1 \circ \nu_2} > 1/2\},
\]
we have that

\[
\mathbb{E}[X_0Y_0(X_0 + Y_0)] = \sum_{\varphi \in \{U_{\nu_1,0,\nu_2,0}, V_{\nu_1,0,\nu_2,0}\}} \mathbb{E}[1_{\{U_{\varphi}>1/2, V_{\varphi}>1/2\}} G_0^{-1}(U)G_0^{-1}(V)(G_0^{-1}(U) + G_0^{-1}(V))]
\]

\[
= \mathbb{E}[1_{\{U>1/2, V>1/2\}} G_0^{-1}(U)G_0^{-1}(V)(G_0^{-1}(U) + G_0^{-1}(V))]
\]

\[
- \mathbb{E}[1_{\{U_{\nu_1,0,\nu_2,0}>1/2, V_{\nu_1,0,\nu_2,0}>1/2\}} G_0^{-1}(U_{\nu_1,0,\nu_2,0})G_0^{-1}(V_{\nu_1,0,\nu_2,0})(G_0^{-1}(U_{\nu_1,0,\nu_2,0}) + G_0^{-1}(V_{\nu_1,0,\nu_2,0}))]
\]

\[
+ \mathbb{E}[1_{\{U_{\nu_1}>1/2, V_{\nu_1}>1/2\}} G_0^{-1}(U_{\nu_1})G_0^{-1}(V_{\nu_1})(G_0^{-1}(U_{\nu_1}) - G_0^{-1}(V_{\nu_1}))]
\]

\[
- \mathbb{E}[1_{\{U_{\nu_2}>1/2, V_{\nu_2}>1/2\}} G_0^{-1}(U_{\nu_2})G_0^{-1}(V_{\nu_2})(G_0^{-1}(U_{\nu_2}) - G_0^{-1}(V_{\nu_2}))]
\]

\[
= 0,
\]

where the last equality comes from \((U, V) \overset{d}{=} (U_{\nu_1,0,\nu_2,0}, V_{\nu_1,0,\nu_2,0})\) and \((U_{\nu_1}, V_{\nu_1}) \overset{d}{=} (U_{\nu_2}, V_{\nu_2})\). \(\Box\)

The conditions \(C = C_{\nu_1,0,\nu_2} \) and \(C_{\nu_1} = C_{\nu_2} \) in Proposition 3.3 hold, for example, if \(C\) is \(M, W, \Pi\), a Gaussian copula, \(t\) copula or one of their mixtures. For these copulas, location shifts of \(G_0\) do not change the asymptotic variance \(\sigma^2_{C_{\nu_0}}(C)\), and thus \(\sigma^2_{C_{\nu_0}}(C)\) is invariant under location-scale transforms of \(G_0\). On the other hand, shifting \(G_0\) may improve \(\sigma^2_{C_{\nu_0}}(C)\) if \(C\) is, for example, a Clayton or Gumbel copula. Nevertheless, we will empirically observe in Section 5 that the reduction of the asymptotic variance \(\sigma^2_{C_0}(C)\) by the optimal shift \(\mu_*\) is typically ignorable compared with the first term \(\text{Var}(X_0Y_0)\) in (2). Based on this observation, in this paper we focus on the case \(\mu = 0\) and compare asymptotic variances of \(\hat{k}_C\) only for standardized concordance-inducing functions \(G\) with mean zero and variance one, and comparing the asymptotic variance \(\sigma^2_{C_{\mu_*}}(C)\) under the optimal shift is left for future research.

### 3.3 Asymptotic variance for fundamental and Fréchet copulas

In this section we investigate optimal concordance-inducing functions and the corresponding bounds of the best and worst asymptotic variances when \(\mathcal{D} \subseteq \mathcal{C}_2\) is a set of fundamental copulas or their mixtures.

**Proposition 3.4** (Optimal asymptotic variances for sets of fundamental copulas).

1. If \(\mathcal{D} = \{\Pi\}\), then \(\sigma^2_x(\mathcal{H}, \{\Pi\}) = \sigma^2_y(\mathcal{H}, \{\Pi\}) = 1\) and \(\overline{C}_x(\mathcal{H}, \{\Pi\}) = \overline{C}_y(\mathcal{H}, \{\Pi\}) = \mathcal{H}\) for any \(\mathcal{H} \subseteq \mathcal{G}_4\).
2. Suppose \( \mathcal{D} = \{M\}, \{W\} \) or \( \{M, W\} \). Then, for \( \mathcal{H} \subseteq \mathcal{G}_4 \),

\[
\sigma_s^2(\mathcal{H}, \mathcal{D}) = \sigma_s^2(\mathcal{H}, \mathcal{D}) = \inf_{G \in \mathcal{H}} \text{Var}_G(X^2), \\
\sigma_s^2(\mathcal{H}, \mathcal{D}) = \sigma_s^2(\mathcal{H}, \mathcal{D}) = \arg \inf_{G \in \mathcal{H}} \text{Var}_G(X^2).
\]

3. If \( \mathcal{D} = \{\Pi, M, W\} \), then, for \( \mathcal{H} \subseteq \mathcal{G}_4 \),

\[
\sigma_s^2(\mathcal{H}, \{\Pi, M, W\}) = 1 \wedge \inf_{G \in \mathcal{H}} \text{Var}_G(X^2), \\
\sigma_s^2(\mathcal{H}, \{\Pi, M, W\}) = 1 \vee \inf_{G \in \mathcal{H}} \text{Var}_G(X^2).
\]

4. Let \( \mathcal{H}_N, \mathcal{H}_{\text{Unif}} \) and \( \mathcal{H}_{\text{Bern}} \) be singletons of normal, uniform and Bernoulli distributions with mean zero and variance one, respectively. Then

\[
\sigma_s^2(\mathcal{H}_{\text{Bern}}, \{M, W\}) < \sigma_s^2(\mathcal{H}_{\text{Unif}}, \{M, W\}) < \sigma_s^2(\mathcal{H}_N, \{M, W\}), \\
\sigma_s^2(\mathcal{H}_{\text{Bern}}, \{M, W\}) < \sigma_s^2(\mathcal{H}_{\text{Unif}}, \{M, W\}) < \sigma_s^2(\mathcal{H}_N, \{M, W\}), \\
\sigma_s^2(\mathcal{H}_{\text{Bern}}, \{\Pi, M, W\}) < \sigma_s^2(\mathcal{H}_{\text{Unif}}, \{\Pi, M, W\}) < \sigma_s^2(\mathcal{H}_N, \{\Pi, M, W\}), \\
\sigma_s^2(\mathcal{H}_{\text{Bern}}, \{\Pi, M, W\}) = \sigma_s^2(\mathcal{H}_{\text{Unif}}, \{\Pi, M, W\}) = \sigma_s^2(\mathcal{H}_N, \{\Pi, M, W\}).
\]

Proof. Writing \( X = G^{-1}(U) \) and \( Y = G^{-1}(V) \) for \( (U, V) \sim C \) and using that \( \text{Var}(XY) = \mathbb{E}((XY)^2) - \mathbb{E}[XY]^2 \), \( \text{Cov}(X^2, Y^2) = \mathbb{E}((XY)^2) - \mathbb{E}[X^2]\mathbb{E}[Y^2] = \mathbb{E}((XY)^2) - 1 \) and \( \text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[XY] \), we have that

\[
\sigma_s^2(C) = \text{Var}(XY) = \text{Cov}(X^2, Y^2) + 1 - \text{Cov}(X, Y)^2. \quad (3)
\]

Part 1). \( (X^2, Y^2) \) and \( (X, Y) \) are both independent random vectors when \( (U, V) \sim \Pi \). Therefore, \( \text{Cov}(X^2, Y^2) = \text{Cov}(X, Y) = 0 \) and thus \( \sigma_s^2(\Pi) = 1 \) for all \( G \in \mathcal{H} \) by (3), which gives \( \mathcal{G}_s(\mathcal{H}, \{\Pi\}) = \mathcal{G}_s(\mathcal{H}, \{\Pi\}) = \mathcal{H} \) and \( \sigma_s^2(\mathcal{H}, \{\Pi\}) = \sigma_s^2(\mathcal{H}, \{\Pi\}) = 1 \).

Part 2). When the copula of \( (X, Y) \) is \( M \) or \( W \), we have that \( \text{Cov}(X, Y) = \pm 1 \), respectively. Moreover, the copula of \( (X^2, Y^2) \) is \( M \) since \( (X^2, Y^2) \overset{d}{=} (G^{-1}(U^2), G^{-1}(U^2)) \) for \( U \sim \text{Unif}(0, 1) \) when \( C = M \), and \( (X^2, Y^2) \overset{d}{=} (G^{-1}(U^2), G^{-1}(1-U^2)) = (G^{-1}(U^2), (-G^{-1}(U))^2) = (G^{-1}(U^2), G^{-1}(1-U)^2) \) for \( U \sim \text{Unif}(0, 1) \) when \( C = W \). Therefore, by (3), we have that

\[
\sigma_s^2(M) = \bar{p}(X^2, Y^2) \text{Var}_G(X^2) + 1 = \bar{p}(X^2, Y^2) \text{Var}_G(X^2), \\
\sigma_s^2(W) = \bar{p}(X^2, Y^2) \text{Var}_G(X^2) + 1 - (1)^2 = \text{Var}_G(X^2),
\]
where $\rho(X^2, Y^2)$ is the maximum correlation coefficient attained by the copula $M$ with the marginal distributions $X^2$ and $Y^2$, and $\rho(X^2, Y^2) = 1$ since $X^2$ and $Y^2$ are of the same type; see Embrechts et al. (2002). Since $\sigma^2_G(M) = \sigma^2_G(W) = \text{Var}_G(X^2)$, we obtain the desired results.

Part 3. The results immediately follow from Part 1 and Part 2.

Part 4. By Part 2, we have that $\sigma^2_*(H_N, \{M,W\}) = \text{Var}(X^2) = 2$ for $X \sim N(0,1)$, $\sigma^2_*(H_{\text{Unif}}, \{M,W\}) = \text{Var}(12(U - 0.5)^2) = 0.8$ for $U \sim \text{Unif}(0,1)$ and $\sigma^2_*(H_{\text{Bern}}, \{M,W\}) = \text{Var}((2B - 1)^2) = 0$ for $B \sim \text{Bern}(1/2)$. Together with Part 2 and Part 3 we have the desired inequalities.

Proposition 3.4 Part 1 implies that the choice of the function $G$ does not affect the accuracy of the estimation of $\kappa_G$ when the underlying copula is the independence copula.

Proposition 3.4 Part 2 shows that the optimal worst and best asymptotic variances are obtained as the variance of $X^2$ where $X \sim G \in G_4$ when the underlying copula is $M$ or $W$.

Proposition 3.4 Part 3 gives the optimal worst and best asymptotic variances when $\mathcal{D}$ is a set of fundamental copulas. Since a small variance of $X^2$ for $X \sim G$ is preferable in terms of best and worst asymptotic variances when the set of underlying copulas is $\{M, W\}$ or $\{\Pi, M, W\}$, heavy-tailed concordance-inducing functions, such as a Student $t$ distribution with degrees of freedom $4 < \nu < \infty$, are not recommendable choices. Finally, Proposition 3.4 Part 4 means that Blomquist’s beta outperforms Spearman’s rho and van der Waerden’s coefficient, and van der Waerden’s coefficient performs worst in terms of the optimal best and worst asymptotic variances when the set of underlying copulas is $\{M, W\}$ or $\{\Pi, M, W\}$.

We now consider a more general class of copulas defined as combinations of the fundamental copulas $M$, $\Pi$ and $W$. A bivariate Fréchet copula is defined by

\[ C^F_p = p_M M + p_\Pi \Pi + p_W W, \quad p = (p_M, p_\Pi, p_W) \in \Delta_3, \]

where $\Delta_3 = \{(p_1, p_2, p_3) \in \mathbb{R}^3 : p_1, p_2, p_3 \geq 0, p_1 + p_2 + p_3 = 1\}$ is the standard unit simplex on $\mathbb{R}^3$. Denote by $C^F = \{C^F_p : p \in \Delta_3\}$ the set of all Fréchet copulas. In addition to applications in insurance and finance, Fréchet copulas can be used to approximate bivariate copulas; see Yang et al. (2006). Moreover, for any $G \in \mathcal{G}$, the transformed rank correlation $\kappa_G$ can take any value in $[-1, 1]$ since, by Proposition 2.3 Part 3,

\[ \kappa_G(C^F_p) = p_M \kappa_G(M) + p_\Pi \kappa_G(\Pi) + p_W \kappa_G(W) = p_M - p_W \in [-1, 1]. \]
The following proposition provides the worst and best asymptotic variances and their attainers when $D = C^F$.

**Proposition 3.5** (Worst and best asymptotic variances for Fréchet copulas). For a concordance-inducing function $G \in G_4$, the worst and best asymptotic variances on $C^F$ are given by

$$
\sigma^2_G(C^F) = 1 + \text{Var}_G(X^2) \quad \text{and} \quad \underline{\sigma}^2_G(C^F) = 1 \wedge \text{Var}_G(X^2)
$$

with the maximum and minimum attained by

$$
C = \begin{cases}
\frac{M+W}{2} & \text{if } \text{Var}_G(X^2) > 0, \\
p\frac{M+W}{2} + (1-p)\Pi & \text{for any } p \in [0,1] \text{ if } \text{Var}_G(X^2) = 0,
\end{cases}
$$

and

$$
C = \begin{cases}
M,W & \text{if } 0 \leq \text{Var}_G(X^2) < 1, \\
M,W,\Pi & \text{if } \text{Var}_G(X^2) = 1, \\
\Pi & \text{if } 1 < \text{Var}_G(X^2),
\end{cases}
$$

respectively.

**Proof.** Fix $G \in G_4$ and $C^F_p \in C^F$ with $p = (p_M, p_W, p) \in \Delta_3$. For $X = G^{-1}(U)$ and $Y = G^{-1}(V)$ with $(U,V) \sim C^F_p$, we have that $\text{Cov}(X^2,Y^2) = (p_M + p_W)\text{Var}_G(X^2)$ and $\text{Cov}(X,Y) = p_M - p_W$. Therefore, by (3),

$$
\sigma^2_G(C^F_p) = (p_M + p_W)v + 1 - (p_M - p_W)^2 =: f(p_M,p_W),
$$

where $v = \text{Var}_G(X^2)$. Since the Hessian of $f$

$$
H(p_M,p_W) = \begin{pmatrix}
\frac{\partial}{\partial p_M} f(p_M,p_W) & \frac{\partial}{\partial p_M} f(p_M,p_W) \\
\frac{\partial}{\partial p_W} f(p_M,p_W) & \frac{\partial}{\partial p_W} f(p_M,p_W)
\end{pmatrix} = \begin{pmatrix}
-2 & 2 \\
2 & -2
\end{pmatrix},
$$

is nonpositive definite, $f$ is a concave function. For $(p_M,p_W) \in \mathbb{R}^2$ such that $0 \leq p_M, p_W$ and $p_M + p_W \leq 1$, consider the reparametrization $(p,0) + r(-1,1) = (p-r,r)$ where $0 \leq r \leq p \leq 1$. Then

$$
f(p-r,r) = pv + 1 - (p-r)^2 - r^2 + 2(p-r)r = -4 \left(r - \frac{p}{2}\right)^2 + pv + 1,$$
and thus $f$ represents a parabolic cylinder. For a fixed $p \in [0, 1]$, the function $r \mapsto f(p - r, r)$ has a maximum $\overline{T}(p) = pv + 1$ when $r = p/2$, and a minimum $\underline{T}(p) = -p^2 + pv + 1$ when $r = 0$ or $r = p$. Since $v \geq 0$, the maximum of $f$ is given by $v + 1$ with the maximum attained by $p = 1$ when $v > 0$, and by any $p \in [0, 1]$ when $v = 0$. Therefore, we have that $\sigma_G^2(C^F) = v + 1 = \sigma_H^2(C)$ with $C = M + W$ when $v > 0$, and with $C = p \left( \frac{M + W}{2} \right) + (1 - p)\Pi$ for any $p \in [0, 1]$ when $v = 0$. For the minimum of $f$, notice that the function $\underline{T}(p) = -p^2 + pv + 1$, $0 \leq p \leq 1$, is a concave parabola, and thus the minimum of $\underline{T}(p)$ is attained at $p = 0$ or $p = 1$. With $\underline{T}(0) = 1$ and $\underline{T}(1) = v$, the minimum of $f$ and its attainers are given by $\sigma_G^2(C^F) = 1 \land v = \sigma_H^2(C)$ with $C = M$ or $W$ when $0 \leq v < 1$, with $C = M$, $W$ or $\Pi$ when $v = 1$ and with $C = \Pi$ when $v > 1$. 

Note that although $(p_M, p_W) = (1/2, 1/2)$ is the unique point attaining the maximum $v + 1$ of $f$ when $v > 0$, $f$ takes the value $v$ at the points $(p_M, p_W) = (1, 0)$ and $(0, 1)$, and is greater than $v$ on $\{(p_M, p_W) \in [0, 1]^2 : p_M + p_W = 1\}$. Therefore, if $\text{Var}_G(X^2)$ is sufficiently large, the asymptotic variance $\sigma_H^2(C)$ takes large values in $[\text{Var}_G(X^2), \text{Var}_G(X^2) + 1]$ if $C = pM + (1 - p)W$ for $p \in [0, 1]$.

Proposition 3.5 immediately leads to the optimal worst and best asymptotic variances on $D = C^F$ by the following corollary.

**Corollary 3.6** (Optimal worst and best asymptotic variances for Fréchet copulas). For $\mathcal{H} \subseteq \mathcal{G}_4$, the optimal worst and best asymptotic variances are given by

$$\sigma_*^2(\mathcal{H}, C^F) = 1 + \inf_{G \in \mathcal{H}} \text{Var}_G(X^2) \quad \text{and} \quad \underline{\sigma}^2(\mathcal{H}, C^F) = 1 \land \inf_{G \in \mathcal{H}} \text{Var}_G(X^2),$$

with the sets of attaining concorance inducing functions

$$\overline{G}_*(\mathcal{H}, C^F) = \text{arginf}_{G \in \mathcal{H}} \text{Var}_G(X^2),$$

$$\underline{G}_*(\mathcal{H}, C^F) = \begin{cases} \text{arginf}_{G \in \mathcal{H}} \text{Var}_G(X^2), & \text{when } \inf_{G \in \mathcal{H}} \text{Var}_G(X^2) < 1, \\ \mathcal{H}, & \text{when } \inf_{G \in \mathcal{H}} \text{Var}_G(X^2) \geq 1, \end{cases}$$

respectively.

Compared with the optimal worst and best asymptotic variances from Proposition 3.4 Part 3, the lower bound $\underline{\sigma}^2(\mathcal{H}, D)$ obtained in Proposition 3.6 remains unchanged but the upper bound $\overline{\sigma}^2(\mathcal{H}, D)$ increases since the attaining copulas $p \left( \frac{M + W}{2} \right) + (1 - p)\Pi$, $p \in [0, 1]$, are
not included in the set $\mathcal{D}$ in Proposition 3.4. Similar to the results obtained in Proposition 3.4, a small variance of $X^2$ for $X \sim G$ is preferable in terms of optimal worst and best asymptotic variances.

**Remark 3.7** (Restrictions of $\mathcal{C}^F$). For a concordance-inducing function $G \in \mathcal{G}_4$, consider the set of Fréchet copulas such that its transformed rank correlation $\kappa_G$ takes values in $[k, \overline{k}]$ for $-1 \leq k \leq \overline{k} \leq 1$, that is,

$$\mathcal{C}^F_{k, \overline{k}}(G) = \{ C \in \mathcal{C}^F : k \leq \kappa_G(C) \leq \overline{k} \}.$$  

By (4), the restriction $k \leq \kappa_G(C) \leq \overline{k}$ reduces to $k \leq p_M - p_W \leq \overline{k}$ and thus $\mathcal{C}^F_{k, \overline{k}}(G)$ does not depend on the choice of $G$. Consequently, the maximum and minimum of the asymptotic variance $\sigma^2_C(C)$ on $\mathcal{C}^F_{k, \overline{k}}(G)$ can be found by calculating $\max f(p_M, p_W)$ and $\min f(p_M, p_W)$ subject to $0 \leq p_M, p_W, p_M + p_W \leq 1$ and $k \leq p_M - p_W \leq \overline{k}$. This maximum and minimum always exist since $(p_M, p_W) \mapsto f(p_M, p_W)$ is bounded, concave and the feasible set is compact in $\mathbb{R}^2$.

### 3.4 Optimality of Blomqvist’s beta

In this section, we show that Blomqvist’s beta attains the optimal best and worst asymptotic variances under mild conditions on $\mathcal{D} \subseteq \mathcal{C}_2$. The conditions are related to the following properties of copulas.

**Definition 3.8** (Balancedness of copulas). A copula $C \in \mathcal{C}_2$ is called **balanced** if

$$p(C) = C(1/2, 1/2) + \bar{C}(1/2, 1/2) = 1/2,$$

**imbalanced** if $p(C) \neq 1/2$, **totally positively imbalanced** (TPI) if $p(C) = 1$ and **totally negatively imbalanced** (TNI) if $p(C) = 0$.

It is straightforward to check that $\Pi$ is balanced, $M$ is TPI and $W$ is TNI. The following proposition provides the optimal bounds of the worst and best asymptotic variances of Blomqvist’s beta.

**Proposition 3.9** (Asymptotic variance for Blomqvist’s beta). For any $\mathcal{D} \subseteq \mathcal{C}_2$, we have that

$$0 \leq \sigma^2_{\ast}(\mathcal{H}_{\text{Bern}}, \mathcal{D}) \leq \sigma^2_{\ast}(\mathcal{H}_{\text{Bern}}, \mathcal{D}) = 1.$$
The upper bound \( \sigma^2_*(\mathcal{H}_{\text{Bern}}, D) = 1 \) is attained if and only if \( D \) contains a balanced copula, and the lower bound \( \sigma^2_*(\mathcal{H}_{\text{Bern}}, D) = 0 \) is attained if and only if \( D \) contains a TPI or TNI copula.

**Proof.** For \( \mathcal{H}_{\text{Bern}} = \{ G_{\text{Bern}} \} \), we have that \( G^{-1}_{\text{Bern}}(u) = 21_{\{u>1/2\}} - 1, u \in [0,1] \). Therefore, for \((X,Y) = (G^{-1}(U), G^{-1}(V))\) with \((U,V) \sim C\), we have that

\[
XY = \begin{cases} 
1, & \text{if } \{U \leq 1/2, V \leq 1/2\} \cup \{U > 1/2, V > 1/2\}, \\
-1, & \text{if } \{U > 1/2, V \leq 1/2\} \cup \{U \leq 1/2, V > 1/2\}.
\end{cases}
\]

Denoting \( p(C) = \mathbb{P}(\{U \leq 1/2, V \leq 1/2\} \cup \{U > 1/2, V > 1/2\}) = C(1/2, 1/2)+\bar{C}(1/2, 1/2) \), the asymptotic variance of \( \kappa_{G_{\text{Bern}}}(C) \) is given by \( \sigma^2_{G_{\text{Bern}}}(C) = \text{Var}(XY) = 4p(C)(1-p(C)) \), which attains its maximum 1 if and only if \( p(C) = 1/2 \) and attains its minimum 0 if and only if \( p(C) = 0 \) or 1. Therefore, the desired results follow. \( \square \)

With the bounds obtained in Proposition 3.9, we can prove the following optimality of Blomqvist’s beta.

**Corollary 3.10** (Optimality of Blomqvist’s beta). Consider \( D \subseteq C_2 \) and \( \mathcal{H}_{\text{Bern}} \subseteq \mathcal{H} \) for \( \mathcal{H} \subseteq G_4 \). If \( \Pi \in D \), then

\[
\sigma^2_*(\mathcal{H}, D) = 1 \quad \text{and} \quad \mathcal{H}_{\text{Bern}} \subseteq \overline{G_4}(\mathcal{H}, D).
\]

(5)

If \( D \) includes at least one TPI or TNI copula, then

\[
\sigma^2_*(\mathcal{H}, D) = 0 \quad \text{and} \quad \mathcal{H}_{\text{Bern}} \subseteq \underline{G_4}(\mathcal{H}, D).
\]

(6)

**Proof.** Since \( \Pi \in D \) is balanced, Proposition 3.4 Part 1 and Proposition 3.9 imply that

\[
\sup_{C \in D} \sigma^2_{G_{\text{Bern}}}(C) = \sigma^2_{G_{\text{Bern}}}(\Pi) = 1 = \sigma^2_G(\Pi) \leq \sup_{C \in D} \sigma^2_G(C)
\]

for any \( G \in G_4 \). Therefore, the desired results in (5) follow. If \( D \) includes at least one TPI or TNI copula denoted by \( C^* \), then Proposition 3.9 means that

\[
\inf_{C \in D} \sigma^2_{G_{\text{Bern}}}(C) = \sigma^2_{G_{\text{Bern}}}(C^*) = 0 \leq \inf_{C \in D} \sigma^2_G(C)
\]

for any \( G \in G_4 \), and thus the results in (6) follow. \( \square \)
Corollary 3.10 states that Blomqvist’s beta attains the optimal worst asymptotic variance when $\Pi \in \mathcal{D}$, and it attains the optimal best asymptotic variance when some TPI or TNI copula is contained in $\mathcal{D}$. These conditions on $\mathcal{D}$ are mild and satisfied for typical choices of $\mathcal{D}$, such as $\mathcal{C}_2$, $\mathcal{C}_2^\geq = \{ C \in \mathcal{C}_2 : C \succeq \Pi \}$ or $\mathcal{C}_2^\leq = \{ C \in \mathcal{C}_2 : C \preceq \Pi \}$. Therefore, Blomqvist’s beta is typically an optimal choice among transformed rank correlations in terms of both the worst and best asymptotic variances.

Uniqueness of the optimality of Blomqvist’s beta $H_{\text{Bern}} = G^\ast(H, \mathcal{D}) = G^\ast(H, \mathcal{D})$ is typically not fulfilled. In the remainder of this section, we discuss non-uniqueness of the optimality of Blomqvist’s beta.

We first consider the optimal best asymptotic variance. For given $\mathcal{H} \subseteq \mathcal{G}_4$ and $\mathcal{D} \subseteq \mathcal{C}_2$, assume that $H_{\text{Bern}} \subseteq \mathcal{H}$ and that $\mathcal{D}$ contains at least one TPI or TNI copula. Then a given $G \in \mathcal{H}$ satisfies $G \in G^\ast(\mathcal{H}, \mathcal{D})$ if and only if $\sigma^2_G(\mathcal{D}) = \sigma^2_{G_{\text{Bern}}}(\mathcal{D}) = 0$. Therefore, the following equivalence holds:

$$G \in G^\ast(\mathcal{H}, \mathcal{D}) \iff \text{there exists } C \in \mathcal{D} \text{ s.t. } \sigma^2_G(C) = 0 \iff G^{-1}(U)G^{-1}(V) \overset{a.s.}{=} a \text{ for some } a \in \mathbb{R} \text{ and } (U, V) \sim C \in \mathcal{D}. \quad (7)$$

The following proposition provides necessary conditions on $a \in \mathbb{R}$, $G \in \mathcal{H}$ and $C \in \mathcal{D}$ under (7).

**Proposition 3.11** (Necessary conditions on $G \in G^\ast(\mathcal{H}, \mathcal{D})$). Let $G \in \mathcal{H}$ be a given concordance-inducing function such that $G \in G^\ast(\mathcal{H}, \mathcal{D})$. Then $C \in \mathcal{D}$ and $a \in \mathbb{R}$ in (7) satisfy the following conditions.

(C1) If $\mathbb{P}(X = 0) > 0$ for $X \sim G$, then $a = 0$ and $\mathbb{P}(X = 0) \geq 1/2$.

(C2) If $\mathbb{P}(X = 0) = 0$, then $a \neq 0$ and the copula $C$ is either TPI or TNI with $0 < a \leq 1$ if $C$ is TPI and $-1 \leq a < 0$ if $C$ is TNI. Moreover, the distribution function $G_+(x) = 2G(x) - 1$, $x > 0$ satisfies $\mathbb{E}_{G_+}[Z] \geq |a|^{1/2}$, $Z \sim G_+$, and

$$G_+(x) = 1 - G_+\left(\frac{|a|}{x}\right), \quad x > 0. \quad (8)$$

In particular it holds that $\mathbb{P}(Z > |a|^{1/2}) = \mathbb{P}(Z < |a|^{1/2})$ for $Z \sim G_+$.

**Proof.** For $G \in \mathcal{H}$ and $C \in \mathcal{D}$ in (7), write $(X, Y) = (G^{-1}(U), G^{-1}(V))$. Under (C1), we have that $\mathbb{P}(XY = 0) > 0$ and thus $a \in \mathbb{R}$ in (7) necessarily has to be $a = 0$. If $XY \overset{a.s.}{=} 0$ holds,
then $X \neq 0$ implies that $Y = 0$. Together with $X \overset{d}{=} Y$, we have that

$$\mathbb{P}(X \neq 0) \leq \mathbb{P}(Y = 0) = \mathbb{P}(X = 0),$$

which leads to the condition $\mathbb{P}(X = 0) \leq 1/2$.

Next we consider (C2). Since

$$XY \begin{cases} > 0, & \text{if } \{U \leq 1/2, V \leq 1/2\} \cup \{U > 1/2, V > 1/2\} \\ < 0, & \text{if } \{U \leq 1/2, V > 1/2\} \cup \{U > 1/2, V \leq 1/2\} \end{cases},$$

we have that $\mathbb{P}(XY = 0) = 0$ and thus $a \in \mathbb{R}$ in (7) necessarily has to be $a \neq 0$. Since $\mathbb{P}(XY > 0) = p(\{U \leq 1/2, V \leq 1/2\} \cup \{U > 1/2, V > 1/2\}) = p(C)$ and $\mathbb{P}(XY < 0) = p(\{U \leq 1/2, V > 1/2\} \cup \{U > 1/2, V \leq 1/2\}) = 1 - p(C)$, the product $XY$ can never be a constant a.s. if $0 < p(C) < 1$. Therefore, $p(C) = 0$ or 1, and thus $C$ is either TPI or TNI.

Assume that $C$ is TPI. Then $a > 0$ since $\mathbb{P}(XY > 0) = 1$. By the TPI assumption of $C$, we have that

$$X_+ = X \mid \{U > 1/2, V > 1/2\} = X \mid \{U > 1/2\} \sim G_+,$$
$$Y_+ = Y \mid \{U > 1/2, V > 1/2\} = Y \mid \{V > 1/2\} \sim G_+,$$
$$X_- = X \mid \{U \leq 1/2, V \leq 1/2\} = X \mid \{U \leq 1/2\} \sim G_-,$$
$$Y_- = X \mid \{U \leq 1/2, V \leq 1/2\} = Y \mid \{V \leq 1/2\} \sim G_-,$$

where

$$G_+(x) = \begin{cases} 2G(x) - 1, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0, \end{cases} \quad \text{and} \quad G_-(x) = \begin{cases} 1, & \text{if } x > 0, \\ 2G(x), & \text{if } x \leq 0. \end{cases}$$

In addition to the equalities $X_+ \overset{d}{=} Y_+$ and $X_- \overset{d}{=} Y_-$, we have that $X_+ \overset{d}{=} -X_-$ and $Y_+ \overset{d}{=} -Y_-$ since

$$\mathbb{P}(-X_- \leq x) = \mathbb{P}(X_+ \geq -x) = 1 - G_-((-x)-)$$

$$= \begin{cases} 1 - 1 = 0 & \text{if } x < 0 \\ 1 - 2G((-x)-) = 1 - 2(1 - G(x)) = 2G(x) - 1 & \text{if } x \geq 0 \end{cases}$$

$$= G_+(x)$$
by radial symmetry of $G$ and the assumption $P(X = 0) = 0$. Moreover, since $XY \overset{a.s.}{=} a$, it holds that

$$X_+Y_+ = XY \mid \{U > 1/2, V > 1/2\} \overset{a.s.}{=} a,$$

$$X_-Y_- = XY \mid \{U \leq 1/2, V \leq 1/2\} \overset{a.s.}{=} a.$$ 

Since $X_+Y_+ \overset{a.s.}{=} a$ and $Y_+ > 0$ a.s., we have that $X_+ \overset{a.s.}{=} a/Y_+$. Therefore, Jensen’s inequality implies that

$$E[X_+] = E\left[\frac{a}{Y_+}\right] = aE\left[\frac{1}{Y_+}\right] \geq \frac{a}{E[Y_+]} = \frac{a}{E[X_+]},$$

which yields the mean condition $E[X_+] \geq \sqrt{a}$.

Since $X_+ \overset{d}{=} -X_- \text{ and } \text{Var}(X) = E[X^2] = 1$, we have that

$$1 = E[X^2] = P\left(U > \frac{1}{2}\right)E\left[X^2 \mid U > \frac{1}{2}\right] + P\left(U \leq \frac{1}{2}\right)E\left[X^2 \mid U \leq \frac{1}{2}\right]$$

$$= \frac{1}{2}E[X_+^2] + \frac{1}{2}E[X_-^2] = E[X_+^2],$$

and thus $E[X_+^2] = 1$. Using $X_+^2 \overset{a.s.}{=} (a/Y_+)^2 > 0$ a.s. and Jensen’s inequality, we have that

$$1 = E[X_+^2] = E\left[\left(\frac{a}{Y_+}\right)^2\right] \geq \frac{a^2}{E[Y_+^2]} = \frac{a^2}{E[X_+^2]},$$

which yields $-1 \leq a \leq 1$. Together with $a > 0$, we have the inequalities $0 < a \leq 1$. Moreover, $X_+ \overset{a.s.}{=} a/Y_+$ implies that, for $x > 0$,

$$G_+(x) = P(X_+ \leq x) = P\left(\frac{a}{Y_+} \leq x\right) = 1 - P\left(Y_+ < \frac{a}{x}\right) = 1 - G_+\left(\frac{a}{x}\right),$$

which leads to identity (8). The symmetry $P(Z > a^{1/2}) = P(Z < a^{1/2})$ for $Z \sim G_+$ is obtained as a special case by taking $x = \sqrt{a} > 0$ in (8).

Next assume that $C$ is TNI. Then $a < 0$ since $P(XY < 0) = 1$. By the TNI assumption, we have that

$$X_+ = X \mid \{U > 1/2, V \leq 1/2\} = X \mid \{U > 1/2\} \sim G_+,$$

$$Y_+ = Y \mid \{U \leq 1/2, V > 1/2\} = Y \mid \{V > 1/2\} \sim G_+,$$

$$X_- = X \mid \{U \leq 1/2, V > 1/2\} = X \mid \{U \leq 1/2\} \sim G_-,$$

$$Y_- = X \mid \{U > 1/2, V \leq 1/2\} = Y \mid \{V \leq 1/2\} \sim G_-.$$
As in the TPI case, it holds that \( X_+ \overset{d}{=} Y_+, \) \( X_- \overset{d}{=} Y_-, \) \( X_+ \overset{d}{=} -X_+ \) and \( Y_+ \overset{d}{=} -Y_+. \) Moreover, \( XY \overset{a.s.}{=} a \) implies that

\[
X_+Y_- = XY \mid \{ U > 1/2, V \leq 1/2 \} \overset{a.s.}{=} a, \\
X_-Y_+ = XY \mid \{ U \leq 1/2, V > 1/2 \} \overset{a.s.}{=} a.
\]

From these equalities, all the necessary conditions derived in the TPI case hold since \( X_+(-Y_-) \overset{a.s.}{=} (-X_-)Y_+ \overset{a.s.}{=} -a \) with \(-Y_-, -X_- \sim G_+ \) and \(-a > 0.\) □

By Proposition 3.11, not any concordance-inducing function and copula can attain the optimal best asymptotic variance \( \sigma_G^2(C) = 0. \) The following examples show non-Bernoulli concordance-inducing functions attaining this lower bound.

**Example 3.12** (Non-Bernoulli concordance-inducing functions in \( G_4(G_4, D) \)).

1. **The case when \( \mathbb{P}(X = 0) > 0: \)** Let \( X \sim G \) be an equally weighted mixture of 0 and \( \text{Unif}(-\sqrt{6}, \sqrt{6}). \) Then \( \mathbb{E}[X] = 0, \) \( \text{Var}(X) = 1 \) and \( \mathbb{E}[X^4] < \infty, \) and thus \( G \in G_4. \) The case falls under (C1) since \( \mathbb{P}(X = 0) = 1/2. \) Let \( M(n, \{J_i\}, \pi, w) \) denote a shuffle-of-\( M \) with \( n \) being the number of connected components in its support, \( \{J_i\} = \{J_1, \ldots, J_n\} \) being a finite partition of \([0,1]\) into \( n \) closed subintervals, \( \pi \) being a permutation of \( \{1, \ldots, n\} \) and \( w : \{1, \ldots, n\} \rightarrow \{-1, 1\} \) being a function indicating whether the strip \( J_i \times J_{\pi(i)} \) is flipped \( (w(i) = 1) \) or not \( (w(i) = -1); \) see Nelsen (2006, Section 3.2.3). Consider \( C_1 = M(4, \bigcup_{i=1}^4 ([i-1)/4, i/4], \{2, 1, 4, 3\}, 14), \) \( C_2 = M(4, \bigcup_{i=1}^4 ([i-1)/4, i/4], \{3, 4, 1, 2\}, 14) \) and \( C_3 = M(4, \bigcup_{i=1}^4 ([i-1)/4, i/4], \{2, 4, 1, 3\}, 14). \) Then \( C_1 \) is TPI, \( C_2 \) is TNI and \( C_3 \) is neither TPI nor TNI. Moreover, all these shuffle-of-\( M \)s satisfy \( \sigma^2_{C_k}(C_k) = 0 \) for \( k = 1, 2, 3 \) since \( G^{-1}(U)G^{-1}(V) \overset{a.s.}{=} 0 \) with \((U, V) \sim C_k \) for \( k = 1, 2, 3. \)

2. **The case when \( \mathbb{P}(X = 0) = 0: \)** Let \( X \sim G \) be a discrete uniform distribution on the four points \( \{-a/b, -b, b, a/b\} \) where \( a = 1/\sqrt{2} \) and \( b = \sqrt{1 - \sqrt{2}/2} \) with \( b \approx 0.541 \) and \( a/b \approx 1.307. \) Then it is straightforward to check that \( G \in G_4. \) Define \((X, Y) = (G^{-1}(U), G^{-1}(V)) \) with \((U, V) \sim C_4 = M(4, \bigcup_{i=1}^4 ([i-1)/4, i/4], \{2, 1, 4, 3\}, -14). \) Then \((X, Y) = (-a/b, -b), (b, -a/b), (b, a/b) \) and \((a/b, b) \) are equiprobable, and thus \( \sigma^2_{C_4}(C_4) = 0 \) since \( XY \overset{a.s.}{=} a. \) This case belongs to (C2) since \( C_4 \) is TPI, \( 0 < a \leq 1 \) and \( \mathbb{E}_{G_+}[Z] \approx 0.924 > 0.841 \approx \sqrt{a}. \)
Compared to the optimal best asymptotic variance, much less is known for the optimal worst asymptotic variance. Assuming that $\Pi \in D$, a given $G \in H$ attains the optimal worst asymptotic variance if and only if $\sup_{C \in D} \sigma^2_G(C) = 1$. Since $\sup_{C \in D} \sigma^2_G(C) \geq \sigma^2_G(\Pi) = 1$, we have that $G \in G^*(H, D) \iff \sigma^2_G(C) \leq 1$ for all $C \in D$.

If $M$ or $W$ is in $D$, then $\text{Var}_G(X^2) \leq 1$ is a necessary condition for $G \in G^*(H, D)$ since $\sigma^2_G(C) = \text{Var}_G(X^2)$ for $C = M$ or $W$ as seen in the proof of Proposition 3.4 Part 2. Although the condition $\sigma^2_G(C) \leq 1$ can be rewritten by a more intuitive condition $\text{Cov}(X^2, Y^2) \leq \text{Cov}(X, Y)^2 = \kappa_G(C)^2$, it does not seem straightforward to derive tractable conditions for such $G \in G_4$ and $C \in D$.

4 Comparison of $\kappa_G$ and Kendall’s tau

In Section 3.4, we showed that Blomqvist’s beta provides optimal best and worst asymptotic variances under mild conditions on $D \subseteq C_2$. However, one of the drawbacks of Blomqvist’s beta is that it depends only on the local value $C(1/2, 1/2)$ of the underlying copula $C$, which attributes to the fact that the corresponding concordance-inducing function $G$ is supported only on two points. In this section, we show that Kendall’s tau, a popular measure of concordance, attains the same optimal worst and best asymptotic variances as Blomqvist’s beta although Kendall’s tau is not a transformed rank correlation coefficient.

Kendall’s tau $\tau : C_2 \rightarrow \mathbb{R}$ is defined by

$$\tau(C) = 4 \int_{[0,1]^2} C(u, v) \ dC(u, v) - 1,$$

and is a measure of concordance; see Scarsini (1984). Moreover, it is not a $G$-transformed rank correlation since $\tau$ is not linear with respect to a mixture of copulas; see Hofert and Koike (2019, Remark 2). Since $\tau(C) = \rho(1_{U \leq \tilde{U}}, 1_{V \leq \tilde{V}})$ where $(U, V) \sim C$ and $(\tilde{U}, \tilde{V}) \sim C$ are independent, Kendall’s tau admits the alternative representation

$$\tau(C) = \rho(g(U, \tilde{U}), g(V, \tilde{V})), \quad \text{where} \quad g(l, m) = \begin{cases} 1 & \text{if } l \leq m, \\ -1 & \text{if } l > m, \end{cases}$$
by invariance of $\rho$ under location-scale transforms. According to (10), we consider the estimator of $\tau(C)$

$$\hat{\tau} = \frac{1}{n} \sum_{i=1}^{n} g(U_i, \tilde{U}_i) g(V_i, \tilde{V}_i)$$

where $(U_i, V_i)$ and $(\tilde{U}_i, \tilde{V}_i)$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, are two i.i.d. samples from $C$. Note that we adopt the estimator $\hat{\tau}$ which is different from the standard estimator defined based on all pairs of samples so that $\hat{\tau}$ is a sum of i.i.d. samples. By the CLT, $\hat{\tau}$ satisfies the following asymptotic normality:

$$\sqrt{n} \{\hat{\tau} - \tau(C)\} \overset{d}{\rightarrow} \text{N}(0, \sigma^2_\tau(C)), \quad \sigma^2_\tau(C) = \text{Var}(g(U, \tilde{U}) g(V, \tilde{V})).$$

Similar to the case of $G$-transformed rank correlations, we consider the following best and worst asymptotic variances among the set of copulas $\mathcal{D} \subseteq \mathcal{C}_2$ defined by

$$\underline{\sigma}^2_\tau(\mathcal{D}) = \inf_{C \in \mathcal{D}} \sigma^2_\tau(C) \quad \text{and} \quad \overline{\sigma}^2_\tau(\mathcal{D}) = \sup_{C \in \mathcal{D}} \sigma^2_\tau(C),$$

respectively. The following proposition provides the best and worst asymptotic variances of Kendall’s tau.

**Proposition 4.1** (Best and worst asymptotic variances of Kendall’s tau).

1. The asymptotic variance of Kendall’s tau satisfies $0 \leq \sigma^2_\tau(C) \leq 1$ for all $C \in \mathcal{C}_2$.
2. For a given $C \in \mathcal{C}_2$, the upper bound $\sigma^2_\tau(C) = 1$ is attained if and only if $\tau(C) = 0$, which holds, for example, when $C = \Pi$ or $C = (M + W)/2$. More generally, $\sigma^2_\tau(C) = 1$ if $C$ satisfies $(U, 1 - V) \overset{d}{=} (U, V)$ or $(1 - U, V) \overset{d}{=} (U, V)$ for $(U, V) \sim C$.
3. For a given $C \in \mathcal{C}_2$, the lower bound $\sigma^2_\tau(C) = 0$ is attained if and only if $C = M$ or $W$, that is, $\tau(C) = 1$ or $-1$, respectively.
4. Suppose $\mathcal{H} \subseteq \mathcal{G}_4$ and $\mathcal{D} \subseteq \mathcal{C}_2$ satisfy $\mathcal{H}_{\text{Bern}} \subseteq \mathcal{H}$ and $\Pi \in \mathcal{D}$. Then $\overline{\sigma}^2_\tau(\mathcal{D}) = \sigma^2_\tau(\mathcal{H}, \mathcal{D}) = 1$.
5. Suppose $\mathcal{H} \subseteq \mathcal{G}_4$ and $\mathcal{D} \subseteq \mathcal{C}_2$ satisfy $\mathcal{H}_{\text{Bern}} \subseteq \mathcal{H}$, and $M \in \mathcal{D}$ or $W \in \mathcal{D}$. Then $\underline{\sigma}^2_\tau(\mathcal{D}) = \sigma^2_\tau(\mathcal{H}, \mathcal{D}) = 0$. 
Proof. Part 1). Writing $X = g(U, \tilde{U})$ and $Y = g(V, \tilde{V})$, we have that $XY = 1$ when $\{U \leq \tilde{U}, V \leq \tilde{V}\} \cup \{U > \tilde{U}, V > \tilde{V}\}$, and $XY = -1$ when $\{U \leq \tilde{U}, V > \tilde{V}\} \cup \{U > \tilde{U}, V \leq \tilde{V}\}$. Therefore, $\sigma^2_\tau(C) = \text{Var}(XY) = 4p_\tau(C)(1 - p_\tau(C))$ where

$$p_\tau(C) = \mathbb{P}(\{U \leq \tilde{U}, V \leq \tilde{V}\} \cup \{U > \tilde{U}, V > \tilde{V}\})$$

$$= \mathbb{P}(\{U \leq \tilde{U}, V \leq \tilde{V}\}) + \mathbb{P}(\{U > \tilde{U}, V > \tilde{V}\})$$

$$= 2 \int_{[0,1]^2} C(u,v) \, dC(u,v) = \frac{\tau(C) + 1}{2},$$

with the last equality implied by (9). Since $0 \leq p_\tau(C) \leq 1$, we have that $0 \leq \sigma^2_\tau(C) \leq 1$.

Part 2). The upper bound $\sigma^2_\tau(C) = 1$ is attained if and only if $p_\tau(C) = 1/2$, that is, $\tau(C) = 0$. When $C$ satisfies $(U, 1 - V) \overset{d}{=} (U, V)$ or $(1 - U, V) \overset{d}{=} (U, V)$ for $(U, V) \sim C$, then the change of sign axiom of measures of concordance in Definition 2.1 implies that $\tau(U, V) = \tau(U, 1 - V) = -\tau(U, V)$ or $\tau(U, V) = \tau(1 - U, V) = -\tau(U, V)$, either of which yields $\tau(U, V) = 0$. The copulas $\Pi$ and $(M + W)/2$ are examples of copulas satisfying $(U, 1 - V) \overset{d}{=} (U, V)$ and $(1 - U, V) \overset{d}{=} (U, V)$.

Part 3). The lower bound $\sigma^2_\tau(C) = 0$ is attained if and only if $p_\tau(C) = 1$ or 0, that is, $\tau(C) = 1$ or $-1$, respectively. By Embrechts et al. (2002, Theorem 3), $\tau(C) = 1$ or $-1$ if and only of $C = M$ or $W$, respectively.

Part 4). and Part 5). They are immediate consequences of Part 2, Part 3 and Corollary 3.10. 

Proposition 4.1 Parts 1, 2, 3 imply that the asymptotic variance of Kendall’s tau has the same upper and lower bounds as those of Blomqvist’s beta although different copulas may attain their bounds. By Proposition 4.1 Parts 4 and 5, Kendall’s tau attains the optimal worst and best asymptotic variances of transformed rank correlations which are also attained by Blomqvist’s beta as seen in Proposition 3.10. Taking into account the drawback of Blomqvist’s beta that it depends only on the local value $C(1/2, 1/2)$ of a copula $C$, Kendall’s tau can be a good alternative of Blomqvist’s beta in terms of worst and best asymptotic variances.

Copulas attaining the lower bound $\sigma^2_\tau(D) = 0$ are completely characterized by Proposition 4.1 Part 3. Although a given copula $C \in D$ attains the upper bound $\sigma^2_\tau(D) = 1$ if and only if $\tau(C) = 0$ as seen in the proof of Proposition 4.1 Part 2, no characterization of such copulas is known to the best of our knowledge. The following proposition provides a characterization of copulas attaining the upper bound $\sigma^2_\tau(D) = 1$ when $D$ is the set of Fréchet copulas.
Proposition 4.2 (Characterization of copulas attaining $\sigma^2_t(C^F)$). A Fréchet copula $C = C^F_{(p_M,p_W)} \in C^F$ attains the worst asymptotic variance $\sigma^2_t(C^F) = 1$ of Kendall’s tau if and only if $p_M = p_W \in [0,1/2]$. Equivalently, $C$ is of the form

$$C = p \frac{M + W}{2} + (1 - p)\Pi, \quad p \in [0,1].$$

Proof. By Proposition 4.1 Part 2, a given copula $C \in C^F$ attains the upper bound $\sigma^2_t(C^F) = 1$ if and only if $\tau(C) = 0$. For a Fréchet copula, we have that $\tau(C^F_{(p_M,p_W)}) = (p_M - p_W)(p_M + p_W + 2)/3$; see Nelsen (2006, Example 5.3). Therefore, $\tau(C^F) = 0$ holds if and only if $p_M = p_W$. \hfill \Box

We now compare Kendall’s tau and $G$-transformed rank correlations when taking the sample size into account. Since Representation (10) of Kendall’s tau in terms of Pearson’s correlation coefficient depends on two independent copies of $(U,V) \sim C$, the estimator $\tilde{\tau}$ of $\tau(C)$ requires $2n$ samples from $C$ to construct the estimator with an $n$-sum. Therefore, if the estimators $\tilde{\tau}$ and $\tilde{\kappa}_G$ are compared based on their actual variances (instead of their asymptotic variances) $\text{Var}(\tilde{\tau}) = \sigma^2_\tilde{\tau}(C)/n$ should be multiplied by 2 to be compared with $\text{Var}(\tilde{\kappa}_G) = \sigma^2_{\tilde{\kappa}}(C)/n$. Based on this discussion, suppose that $\sigma^2_\tilde{\tau}(C)$ in Proposition 4.1 is replaced by $\sigma^2_{\tilde{\kappa}}(C) = 2\sigma^2_\tau(C)$. With this modification, optimality of Kendall’s tau in terms of the best asymptotic variance (Proposition 4.1 Part 5) remains valid since $\sigma^2_{\tilde{\kappa}}(D) = 2\sigma^2_\tau(D) = 0 = \sigma^2_\tau(G_4, D)$. On the other hand, optimality of Kendall’s tau in terms of the worst asymptotic variance (Proposition 4.1 Part 4) becomes invalid since $\sigma^2_{\tilde{\kappa}}(D) = 2\sigma^2_\tau(D) = 2 > 1 = \sigma^2_\tau(G_4, D)$.

Remark 4.3 (Alternative estimators of Kendall’s tau). One could compare an estimator of the $G$-transformed rank correlation $\tilde{\kappa}_G$ with other estimators of Kendall’s tau, such as

$$\tilde{\tau}^\star = \frac{1}{n} \sum_{i=1}^n g(U_i, U_{i+1})g(V_i, V_{i+1})$$

where $(U_i, V_i), \ i = 1, \ldots, n + 1$, is an i.i.d. sample from $C$. Since $Z_n = X_n Y_n \in \{-1,1\}$ with $X_n = g(U_i, U_{i+1})$ and $Y_n = g(V_i, V_{i+1})$ is a Markov chain with $Z_l$ and $Z_m$ being independent whenever $|l - m| \geq 2$, the Markov chain CLT yields $\sqrt{n} \left\{ \tilde{\tau}^\star - \tau(C) \right\} \xrightarrow{d} \text{N}(0, \sigma^2_{\tilde{\tau}^\star}(C))$ with

$$\sigma^2_{\tilde{\tau}^\star}(C) = \text{Var}(g(U_1, U_2)g(V_1, V_2)) + \text{Cov}(g(U_1, U_2)g(V_1, V_2), g(U_2, U_3)g(V_2, V_3)),$$

where $(U_1, V_1), (U_2, V_2), (U_3, V_3) \overset{\text{iid}}{\sim} C$. Since $n + 1$ samples are required to construct the estimator $\tilde{\tau}^\star$, the modification factor $(n + 1)/n$ is asymptotically 1 as $n \to \infty$, and thus we
can directly compare \( \sigma^2_{\tau^*}(C) \) with the asymptotic variance \( \sigma^2_G(C) \) of \( \hat{\kappa}_G \). One can show that the covariance term in (11) equals zero when \( C = M \) or \( W \), and thus \( \sigma^2_{\tau^*}(D) = 0 = \sigma^2(G_4, D) \) if \( M \in D \) or \( W \in D \). Therefore, the conclusion that Kendall’s tau attains the best asymptotic variance remains valid for the estimator \( \hat{\tau}^* \). Although the upper bound \( \sigma^2_{\tau^*}(D) \) is not known, \( \hat{\tau} \) cannot be more preferable than \( \hat{\tau}^* \) since

\[
\sigma_{\tau^*}(C) \leq \text{Var}(g(U_1, U_2)g(V_1, V_2)) + \text{Var}(g(U_1, U_2)g(V_1, V_2)) \leq 1 + 1 = 2 = \sigma^2_{\tau^*}(D).
\]

5 Simulation study

In this section, we conduct a simulation study to compare the asymptotic variances \( \sigma^2_G(C) \) for various copulas \( C \in \mathcal{C}_2 \) and concordance-inducing functions \( G \in \mathcal{G}_4 \). For concordance-inducing functions, we consider Bernoulli, uniform and normal distribution functions which correspond to Blomqvist’s beta, Spearman’s rho and van der Waerden’s coefficient, respectively. For comparison, we also consider a Student \( t \) distribution function \( t(\nu) \) with \( \nu = 10 \) degrees of freedom and a Beta distribution with shape parameters \((0.5, 0.5)\) note that both are radially symmetric, have finite fourth moments, and thus belong to \( \mathcal{G}_4 \). The Beta\((0.5, 0.5)\) concordance-inducing function has a different shape from the others since it puts an increasing probability mass as locations farther away from the center. Kendall’s tau is also considered for comparison. Besides standardized concordance-inducing functions (mean zero and variance one), we also consider optimally shifted ones as introduced in Section 3.2. As underlying copulas, we consider Gaussian \( C_{\rho}^{Ga} \), Student \( t \) \( C_{\rho, \nu}^{t} \) and Clayton copulas \( C_{\theta}^{Cl} \) where \( \rho \in [-1, 1] \) is a correlation parameter, \( \nu > 0 \) is the degrees of freedom and \( \theta \geq -1 \) is a shape parameter. The experiment consists of the following three steps.

1. Set \( \rho = -0.99 + 1.98k/49 \) for \( k = 0, 1, \ldots, 49 \), \( \nu = 5 \) and \( \theta = 2\rho/(1 - \rho) \) (which yields \( \tau(C_{\theta}^{Cl}) = \rho \)) in \( C = C_{\rho}^{Ga}, C_{\rho, \nu}^{t} \) and \( C_{\theta}^{Cl} \).

2. For each copula \( C \) in step 1, simulate \((U_1, V_1), \ldots, (U_n, V_n) \) \( \overset{\text{iid}}{\sim} C \) with \( n = 10^5 \).

3. Based on the samples generated in Step 2, estimate \( \sigma^2_G(C) \) and \( \sigma^2_{\tau^*}(C) \) by the sample variances of \( G^{-1}(U_i)G^{-1}(V_i) \), \( i = 1, \ldots, n \), and of \( g(U_i, U_{i+n/2})g(V_i, V_{i+n/2}) \), \( i = 1, \ldots, n/2 \), where \( G \) is a standardized, and optimally shifted Bernoulli, uniform, normal, \( t(10) \) and Beta\((0.5, 0.5)\) distribution function.
Figure 1: Estimates of asymptotic variances $\sigma^2_G(C)$ and $\sigma^2_C(C)$ against correlation parameters $\rho \in [-0.99, 0.99]$ of $C = C^{Ga}_\rho$ (red), $C^t_{\rho, \nu}$ (blue) with $\nu = 5$ and $C^G_{\theta}$ (green) with $\theta = 2\rho/(1-\rho)$ for $G$-transformed rank correlation coefficients $\kappa_G$ (all except bottom-right) and Kendall’s tau $\tau$ (bottom-right). The concordance-inducing function $G$ is set to be standardized (solid lines) and optimally shifted (dotted lines) Bernoulli, uniform, normal, $t(10)$ and Beta(0.5, 0.5). The black dotted lines represent $y = 1$, $\text{Var}_G(X^2)$ and $\text{Var}_G(X^2) + 1$ with $\text{Var}_{G_{\text{Bern}}}(X^2) = 0$, $\text{Var}_{G_{U}}(X^2) = 0.8$, $\text{Var}_{G_{N}}(X^2) = 2$, $\text{Var}_{G_{t(10)}}(X^2) = 3$, $\text{Var}_{G_{\text{Beta}(0.5,0.5)}}(X^2) = 0.5$ and $\text{Var}_\tau(X^2) = \text{Var}_{G_{\text{Bern}}}(X^2) = 0$. 
The estimates of \( \sigma_{G}^2(C) \) and \( \sigma_{T}^2(C) \) computed in Step 3 are plotted in Figure 1. In the remainder of this section, we discuss the observations from these plots.

**Symmetry, convexity and concavity.** For all copulas \( C \), the curves of \( \sigma_{G}^2(C) \) and \( \sigma_{T}^2(C) \) against the correlation parameter \( \rho \) were almost symmetric around \( \rho = 0 \), convex when \( \text{Var}_G(X^2) > 1 \) (which holds if \( G \) is normal or \( t(10) \)), and concave when \( \text{Var}_G(X^2) < 1 \) (which holds if \( G \) is Bernoulli, uniform and Beta(0.5, 0.5), and if Kendall’s tau is considered). For \( C = C_{\rho}^{Ga} \) and \( C_{\rho,\nu}^{t} \), the symmetry of the curves is a consequence of the invariance of \( \sigma_{G}^2(C) \) under partial reflections since

\[
\sigma_{G}^2(C_{\rho}^{Ga}) = \sigma_{G}^2(\nu_1(C_{\rho}^{Ga})) = \sigma_{G}^2(C_{-\rho}^{Ga}) \quad \text{and} \quad \sigma_{G}^2(C_{\rho,\nu}^{t}) = \sigma_{G}^2(\nu_1(C_{\rho,\nu}^{t})) = \sigma_{G}^2(C_{-\rho,\nu}^{t})
\]

for \( \rho \in [-1, 1] \) by Remark 3.1. This argument does not apply to Clayton copulas, and thus the curves \( \rho \mapsto \sigma_{G}^2(C_{2\rho/(1-\rho)}^{Cl}) \) and \( \rho \mapsto \sigma_{G}^2(C_{2\rho/(1-\rho)}^{Cl}) \) are nearly but not precisely symmetric.

**Best and worst asymptotic variances.** To see the best and worst asymptotic variances, the bounds \( 1 \land \text{Var}_G(X^2) \), \( 1 \lor \text{Var}_G(X^2) \) and \( 1 + \text{Var}_G(X^2) \) derived in Proposition 3.4, Corollary 3.6 and Proposition 4.1 are plotted for each case of \( \kappa_G \) and Kendall’s tau with \( \text{Var}_r(X^2) = \text{Var}_{G_{\text{bern}}}(X^2) \). For all cases of \( C = C_{\rho}^{Ga} \), \( C_{\rho,\nu}^{t} \) and \( C_{\theta}^{Cl} \), the best (smallest) \( \sigma_{G}^2(C) \) and \( \sigma_{T}^2(C) \) were roughly \( 1 \land \text{Var}_G(X^2) \) and \( 1 \lor \text{Var}_r(X^2) \) = 0 with the lower bound \( \sigma_{G}^2(C) = 1 \) attained at \( \rho = 0 \) when \( \text{Var}_G(X^2) > 1 \) (normal or \( t(10) \)), and \( \sigma_{G}^2(C) = \text{Var}_G(X^2) \) attained at \( \rho = \pm 1 \) when \( \text{Var}_G(X^2) < 1 \) (Bernoulli, uniform, Beta(0.5, 0.5) and Kendall). For all cases of copulas, the worst (largest) \( \sigma_{G}^2(C) \) and \( \sigma_{T}^2(C) \) were approximately \( 1 \lor \text{Var}_G(X^2) \) and \( 1 \lor \text{Var}_r(X^2) = 1 \) although the curve was slightly above this value for Spearman’s rho with \( C = C_{\rho,\nu}^{t} \). The upper bound \( \sigma_{G}^2(C) = \text{Var}_G(X^2) \) was attained at \( \rho = \pm 1 \) when \( \text{Var}_G(X^2) > 1 \), and \( \sigma_{G}^2(C) = 1 \) was attained at \( \rho = 0 \) when \( \text{Var}_G(X^2) < 1 \) and in the case of Kendall’s tau. Since we only consider specific classes of copulas, the upper bound \( 1 + \text{Var}_G(X^2) \) derived in Corollary 3.6 was not attained except in the cases of Blomqvist’s beta and Kendall’s tau where \( \text{Var}_G(X^2) = 0 \) and thus \( 1 + \text{Var}_G(X^2) = 1 \lor \text{Var}_G(X^2) \).

**Choice of \( G \), normal or Student \( t \) and uniform or Beta distributions.** As seen for the best and worst asymptotic variances, the variance \( \text{Var}_G(X^2) \) is an important quantity determining the maximum and minimum of the asymptotic variance \( \sigma_{G}^2(C) \). As theoretically indicated, concordance-inducing functions with smaller \( \text{Var}_G(X^2) \) are more preferable in terms of the asymptotic variance of \( \kappa_G \). Therefore, the normal concordance-inducing function is more preferable than the \( t(10) \) since \( \text{Var}_{G_{N}}(X^2) = 2 < 3 = \text{Var}_{G_{t(10)}}(X^2) \). In fact, for all
copulas considered, $G_N$ had a smaller asymptotic variance than $G_{t(10)}$ even though $t(10)$ is already rather close to $N(0,1)$. Interestingly, the Beta(0.5, 0.5) concordance-inducing function typically had smaller asymptotic variance than the uniform distribution since $\text{Var}_{G_{\text{Beta}(0.5,0.5)}}(X^2) = 0.5 < 0.8 = \text{Var}_{G_{\text{Unif}}}(X^2)$. Therefore, Beta concordance-inducing functions, possibly with different parameters, can be good alternatives to Spearman’s rho.

**Similarity of Blomqvist’s beta and Kendall’s tau.** The curves of asymptotic variances for Blomqvist’s beta and Kendall’s tau seem to coincide for all choices of $C$. Moreover, the curves for $C = C_{\rho}^G$ and $C_{\rho,\nu}^t$ overlap in the the cases of Blomqvist’s beta and Kendall’s tau. These observations are partially verified since it holds that

$$\kappa_{G_{\text{Bern}}}(C_{\rho}^G) = \tau(C_{\rho}^G) = \kappa_{G_{\text{Bern}}}(C_{\rho,\nu}^t) = \tau(C_{\rho,\nu}^t) = \frac{2}{\pi} \arcsin(\rho)$$

and that

$$\sigma^2_{G_{\text{Bern}}}(C) = 1 - \kappa_{G_{\text{Bern}}}(C)^2 = 1 - \frac{4}{\pi^2} \arcsin^2(\rho)$$

by Schmid and Schmidt (2007, Proposition 9), which is applicable to the asymptotic variance $\sigma^2_{G_{\text{Bern}}}(C)$ since $\sigma^2_G(C)$ is invariant under location-scale transforms of $G \in \mathcal{G}_4$ when $C = C_{\rho}^G$ or $C = C_{\rho,\nu}^t$; see Section 3.2.

**Strength of dependence and kind of copula.** Compared with the choice of concordance-inducing function, the strength of dependence $\rho$ and the kind of $C$ seem to be less influential on the asymptotic variance $\sigma^2_G(C)$. Furthermore, for any concordance-inducing function, the difference of $\sigma^2_G(C)$ among different copulas $C = C_{\rho}^G$, $C_{\rho,\nu}^t$ and $C_{\theta}^{\text{Cl}}$ was typically smaller than difference of $\sigma^2_G(C)$ among different levels of dependence $\rho$.

**Effect of optimal shifts.** When $C = C_{\rho}^G$ or $C_{\rho,\nu}^t$, the solid and dotted curves of asymptotic variances overlap (and thus the dotted curves are not visible). When $C = C_{\theta}^{\text{Cl}}$, the dotted curves do not coincide with, but were close to the solid ones except in the Bernoulli (top left) and Kendall (bottom right) case when two curves seem to overlap. These observations are consistent with Proposition 3.3 stating that the asymptotic variance is not reduced by the optimal shift of $G$ when $C = C_{\rho}^G$ or $C_{\rho,\nu}^t$. Even when the copula is $C_{\theta}^{\text{Cl}}$, only a small reduction of the asymptotic variance was observed when optimally shifting $G$.

In summary, when $\text{Var}_G(X^2) < 1$, the curve of $\sigma^2_G(C)$ is typically symmetric and concave, with the maximum 1 being attained when $\rho = 0$ and the minimum $\text{Var}_G(X^2)$ being attained when $\rho = \pm 1$. When $\text{Var}_G(X^2) > 1$, the curve of $\sigma^2_G(C)$ is typically symmetric and convex.
with the maximum $\text{Var}_G(X^2)$ attained when $\rho = \pm 1$ and the minimum 1 attained when $\rho = 0$. The curves are not significantly different among different choices of $C$ when the strength of dependence remains the same. Compared with the kind of $C$, the strength of dependence and the choice of $G$ are more influencial on $\sigma^2_G(C)$. Normal and Beta(0.5, 0.5) concordance-inducing functions are more preferable than $t(10)$ and uniform distributions, respectively. Moreover, Blomqvist’s beta and Kendall’s tau perform almost the same in terms of their asymptotic variance. Finally, even when $C$ does not satisfy the sufficient conditions of Proposition 3.3, the optimal shift of $G$ may not significantly reduce the asymptotic variance $\sigma^2_G(C)$.

6 Concluding remark

We addressed the question which measures of concordance to use in terms of best and worst asymptotic variances of their canonical estimators. We proved that Blomqvist’s beta attains the optimal best and worst asymptotic variances among all transformed rank correlation coefficients including Spearman’s rho and van der Waerden’s coefficient. Considering the drawback of Blomqvist’s beta that it depends only on the local value $C(1/2, 1/2)$ of a copula $C$, we also compared transformed rank correlations with the popular measure of concordance Kendall’s tau. Based on the representation of Kendall’s tau in terms of Pearson’s linear correlation coefficient, we found that Kendall’s tau also attains the optimal best and worst asymptotic variances if estimators of these measures are compared without being standardized by sample size. Since the correlation-representation of Kendall’s tau depends on two independent copies from the underlying copula, Kendall’s tau is not optimal any more if the asymptotic variances of these estimators are standardized by sample size. Through a simulation study, we observed that the curve of the asymptotic variance of a $G$-transformed rank correlation against the strength of dependence of the underlying copula was typically symmetric and parabolic. Moreover, convexity, maximum and minimum of the asymptotic variance seemed to be determined by $\text{Var}_G(X^2)$ where $X \sim G$. The results of the simulation study supported that concordance-inducing functions $G$ with smaller $\text{Var}_G(X^2)$ are more preferable. Consequently, heavy-tailed concordance-inducing functions, such as Student $t$ distributions with small degrees of freedom, are not recommended in comparison to the normal distribution, and Beta distributions can be good alternatives for uniform distributions (corresponding to Spearman’s rho).
Other than Kendall’s tau, there are still important measures of concordance which are not included in the class of transformed rank correlations, such as Gini’s gamma. Studying a broader framework of comparing these measures of concordance is a part of future research. Further investigation is also required for parametric classes of concordance-inducing functions such as Beta\((\alpha, \beta)\) with \(\alpha = \beta > 0\). Given the limitations of fundamental copulas in practice, another direction of future work is to investigate optimal concordance-inducing functions under more practical choices of sets of underlying copulas, such as the set of parametric copulas or balls of copulas around a given reference copula. A comparison of multivariate measures of concordance, and of matrices of pairwise bivariate measures of concordance are also interesting directions for future research. Finally, it is of interest whether and how the results in this paper change if all measures of concordance are compared in terms of their asymptotic variance without assuming that marginal distributions are known (and thus only pseudo-samples from copulas are available), and/or if the optimal location shift is applied to the concordance-inducing functions.

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### A A class of discrete concordance-inducing functions

As stated in Schmid and Schmidt (2007), one of the advantages of Blomqvist’s beta is that it admits an explicit form whenever the copula is given analytically. This advantage can be extended to a wider class of discrete concordance-inducing functions. For $m \in \mathbb{N}$,
\( z = (z_1, \ldots, z_m) \in \mathbb{R}^m_+ \) and \( p = (p_0, p_1, \ldots, p_m) \in \mathbb{R}^m_+ \) such that \( 0 < z_1 < \cdots < z_m, \) \( p_0 + 2 \sum_{i=1}^m p_i = 1 \) and \( \sum_{i=1}^m p_i z_i^2 = 1/2, \) consider a discrete distribution \( G_{m, z, p} \) supported on \( -z_m, \ldots, -z_1, 0, z_1, \ldots, z_m \) with corresponding probabilities \( p_m, \ldots, p_1, p_0, p_1, \ldots, p_m. \) Then \( G_{m, z, p} \) is a concordance-inducing function with mean zero and variance one. As a special case, Blomqvist’s beta arises when \( m = 1, z_1 = 1 \) and \( (p_0, p_1) = (0, 1/2). \) Let \( p_+ = p_1 + \cdots + p_m, \) \( I_i = [p_+ - \sum_{j=1}^i p_j, p_+ - \sum_{j=1}^{i-1} p_j], \) \( I_0 = [p_+, p_+ + p_0] \) and \( I_i = [p_+ + p_0 + \sum_{j=1}^{i-1} p_j, p_+ + p_0 + \sum_{j=1}^i p_j] \) for \( i = 1, \ldots, m. \) Then

\[
\kappa_{G_{m, z, p}}(C) = \mathbb{E}[G_{m, z, p}^{-1}(U)G_{m, z, p}^{-1}(V)] = \sum_{(i,j) \in \{-m, \ldots, m\}} z_iz_j V_C(I_i \times I_j),
\]

and

\[
\sigma^2_{G_{m, z, p}}(C) = \text{Var}(XY) = \mathbb{E}[(XY)^2] - (\mathbb{E}[XY])^2
\]

\[
= \sum_{(i,j) \in \{-m, \ldots, m\}} z_i^2 z_j^2 V_C(I_i \times I_j) - \left( \sum_{(i,j) \in \{-m, \ldots, m\}} z_i z_j V_C(I_i \times I_j) \right)^2,
\]

where \( z_i = -z_i \) for \( i = 1, \ldots, m \) and \( V_C(A), A \subseteq [0,1]^2 \) is a volume of \( A \) measured by \( C. \) Therefore, \( \kappa_{G_{m, z, p}}(C) \) and \( \sigma^2_{G_{m, z, p}}(C) \) admit explicit forms if \( V_C(I_i \times I_j) \) can be written explicitly for all \( (i, j) \in \{-m, \ldots, m\}^2. \)

### B Properties of \( C \mapsto \sigma^2_G(C) \)

In this section, we investigate the map \( C \mapsto \sigma^2_G(C) \) for a given \( G \in \mathcal{G}_4. \) To this end, we first study the distribution of \((X^2, Y^2)\) for \((X, Y) = (G^{-1}(U), G^{-1}(V))\) and \((U, V) \sim C\) since the asymptotic variance can be written as \( \sigma^2_G(C) = \text{Var}(XY) = \text{Cov}(X^2, Y^2) + 1 - \text{Cov}(X, Y)^2 \) by (3). First, by radial symmetry of \( G, \) the marginal distribution of \( X^2 \) (and that of \( Y^2 \)) is given by

\[
G^{[2]}(x) = \mathbb{P}(X^2 \leq x) = \mathbb{P}(-\sqrt{x} \leq X \leq \sqrt{x}) = G(\sqrt{x}) - G(-\sqrt{x}) = 2G(\sqrt{x}) - 1, \quad x \geq 0.
\]

The following proposition describes the copula of \((X^2, Y^2)\) when \( G \) is continuous.

**Proposition B.1** (Copula of \((X^2, Y^2))\). Let \( G \in \mathcal{G}_4^\circ \) be a continuous concordance-inducing function. For a copula \( C \in \mathcal{C}_2 \) and \((X, Y) = (G^{-1}(U), G^{-1}(V))\) with \((U, V) \sim C, \) the copula
of \((X^2, Y^2)\) is given by

\[
C^{[2]}(u, v) = \sum_{\varphi \in \{u, v, \bar{u}, \bar{v}, u \cup v\}} \tilde{C}_\varphi \left( \frac{1}{2}, \frac{1}{2} \right) C_{\varphi,(1/2,1/2)} \left( \frac{u + 1}{2}, \frac{v + 1}{2} \right),
\]

where \(C_{\varphi,(1/2,1/2)}(u, v) = \mathbb{P}(U_\varphi \leq u, V_\varphi \leq v \mid U_\varphi > 1/2, V_\varphi > 1/2)\) for \((U_\varphi, V_\varphi) \sim C_\varphi\).

**Proof.** By continuity of \(G\), we have that \(X > 0\) when \(U > 1/2\) and \(X \leq 0\) when \(U \leq 1/2\). Therefore,

\[
G^{[2]}(X^2) = 2G(\sqrt{X^2}) - 1 = 2G(|X|) - 1
\]

\[
= \begin{cases} 
2G(X) - 1 = 2U - 1, & \text{when } U > 1/2, \\
2G(-X) - 1 = 2(1 - G(X)) - 1 = 1 - 2U, & \text{when } U \leq 1/2.
\end{cases}
\]

Using this relationship, we have that

\[
C^{[2]}(u, v) = \mathbb{P}(G^{[2]}(X^2) \leq u, G^{[2]}(Y^2) \leq v)
\]

\[
= \mathbb{P} \left( U > \frac{1}{2}, V > \frac{1}{2} \right) \mathbb{P} \left( 2U - 1 \leq u, 2V - 1 \leq v \mid U > \frac{1}{2}, V > \frac{1}{2} \right)
\]

\[
+ \mathbb{P} \left( U \leq \frac{1}{2}, V > \frac{1}{2} \right) \mathbb{P} \left( 1 - 2U \leq u, 2V - 1 \leq v \mid U \leq \frac{1}{2}, V > \frac{1}{2} \right)
\]

\[
+ \mathbb{P} \left( U > \frac{1}{2}, V \leq \frac{1}{2} \right) \mathbb{P} \left( 2U - 1 \leq u, 1 - 2V \leq v \mid U > \frac{1}{2}, V \leq \frac{1}{2} \right)
\]

\[
+ \mathbb{P} \left( U \leq \frac{1}{2}, V \leq \frac{1}{2} \right) \mathbb{P} \left( 1 - 2U \leq u, 1 - 2V \leq v \mid U \leq \frac{1}{2}, V \leq \frac{1}{2} \right)
\]

\[
= \mathbb{P} \left( U > \frac{1}{2}, V > \frac{1}{2} \right) \mathbb{P} \left( U \leq \frac{u + 1}{2}, V \leq \frac{v + 1}{2} \mid U > \frac{1}{2}, V > \frac{1}{2} \right)
\]

\[
+ \mathbb{P} \left( 1 - U > \frac{1}{2}, V > \frac{1}{2} \right) \mathbb{P} \left( 1 - U \leq \frac{u + 1}{2}, V \leq \frac{v + 1}{2} \mid 1 - U > \frac{1}{2}, V > \frac{1}{2} \right)
\]

\[
+ \mathbb{P} \left( U > \frac{1}{2}, 1 - V > \frac{1}{2} \right) \mathbb{P} \left( U \leq \frac{u + 1}{2}, 1 - V \leq \frac{v + 1}{2} \mid U > \frac{1}{2}, 1 - V > \frac{1}{2} \right)
\]

\[
+ \mathbb{P} \left( 1 - U > \frac{1}{2}, 1 - V > \frac{1}{2} \right) \mathbb{P} \left( 1 - U \leq \frac{u + 1}{2}, 1 - V \leq \frac{v + 1}{2} \mid 1 - U > \frac{1}{2}, 1 - V > \frac{1}{2} \right)
\]

\[
= \sum_{\varphi \in \{u, v, \bar{u}, \bar{v}, u \cup v\}} \tilde{C}_\varphi \left( \frac{1}{2}, \frac{1}{2} \right) C_{\varphi,(1/2,1/2)} \left( \frac{u + 1}{2}, \frac{v + 1}{2} \right).
\]

\[\square\]

As an application of Proposition B.1, let \((X, Y)\) have a fundamental copula. Then the
The copula of \((X^2, Y^2)\) is given by
\[
C^{[2]} = \begin{cases} 
\frac{1}{4} \Pi + \frac{1}{4} \Pi + \frac{1}{4} \Pi = \Pi, & \text{when } C = \Pi, \\
\frac{1}{2} M + 0 + 0 + \frac{1}{2} M = M, & \text{when } C = M, \\
0 + \frac{1}{2} M + \frac{1}{2} M + 0 = M, & \text{when } C = W.
\end{cases}
\]

Next we study the copula \(C^{[2]}\) when \(C\) is a convex combination of copulas.

**Lemma B.2** (Convex combination of \(C^{[2]}\)). For a convex combination \(\tilde{C}_p = pC + (1-p)C'\) of \(C\) and \(C'\) where \(p \in [0, 1]\) and \(C, C' \in \mathcal{C}_2\), we have that
\[
\tilde{C}_p^{[2]} = pC^{[2]} + (1-p)C'^{[2]},
\]
provided that the concordance-inducing function \(G \in \mathcal{G}_4^*\) is continuous.

**Proof.** Consider a random vector \((\tilde{U}, \tilde{V}) = B(U, V) + (1-B)(U', V') \sim \tilde{C}_p\) where \((U, V) \sim C\), \((U', V') \sim C'\) and \(B \sim \text{Bern}(p)\) are independent of each other. For \((X, Y) = (G^{-1}(U), G^{-1}(V))\), \((X', Y') = (G^{-1}(U'), G^{-1}(V'))\) and \((\tilde{X}, \tilde{Y}) = (G^{-1}(\tilde{U}), G^{-1}(\tilde{V}))\) we have that
\[
\begin{align*}
(\tilde{X}, \tilde{Y}) &= (G^{-1}(BU + (1-B)U'), G^{-1}(BV + (1-B)V')) \\
&= B(G^{-1}(U), G^{-1}(V)) + (1-B)(G^{-1}(U'), G^{-1}(V')) \\
&= B(X, Y) + (1-B)(X', Y'),
\end{align*}
\]
and thus \((\tilde{X}^2, \tilde{Y}^2) = B(X^2, Y^2) + (1-B)(X'^2, Y'^2)\). Since \(\tilde{X}^2 \overset{d}{=} \tilde{Y}^2 \sim C^{[2]}\), the copula of \((\tilde{X}^2, \tilde{Y}^2)\) is given by
\[
\tilde{C}_p^{[2]}(u, v) = \mathbb{P}(G^{[2]}(BX^2 + (1-B)X'^2) \leq u, G^{[2]}(BY^2 + (1-B)Y'^2) \leq v) \\
= \mathbb{P}(B = 1)\mathbb{P}(G^{[2]}(X^2) \leq u, G^{[2]}(Y^2) \leq u) + \mathbb{P}(B = 0)\mathbb{P}(G^{[2]}(X'^2) \leq u, G^{[2]}(Y'^2) \leq u) \\
= pC^{[2]}(u, v) + (1-p)C'^{[2]}(u, v).
\]

Together with Hoeffding’s lemma (McNeil et al., 2015, Lemma 7.27), Lemma B.2 implies that the map \(C \mapsto \text{Cov}(X^2, Y^2)\) is linear with respect to convex combinations of copulas. This is not the case for the map \(C \mapsto \sigma_\mathcal{C}^2(C)\) since \(C \mapsto \text{Cov}(X, Y)^2\) is non-linear. The following proposition shows that the map \(C \mapsto \sigma_\mathcal{C}^2(C)\) is linear on a restricted space of \(\mathcal{C}_2\).
**Proposition B.3** (Linearity of $\sigma^2_G(C)$). For a continuous concordance-inducing function $G \in \mathcal{G}_4$ and a constant $k \in [-1, 1]$, the map $C \mapsto \sigma^2_G(C)$ is linear with respect to convex combinations of copulas on $\mathcal{C}_G(k) = \{ C \in \mathcal{C}_2 : \kappa_G(C) = k \}$ for any $G \in \mathcal{G}_4$ and $k \in [-1, 1]$.

**Proof.** For $\tilde{C}_p = pC + (1 - p)C' \in \mathcal{C}_2$ and $(\tilde{U}, \tilde{V}) \sim \tilde{C}_p$, we have that $(\tilde{X}, \tilde{Y}) = (G^{-1}(\tilde{U}), G^{-1}(\tilde{V})) = (G^{-1}(BU + (1 - B)U'), G^{-1}(BV + (1 - B)V')) = B(X, Y) + (1 - B)(X', Y')$ where $(X, Y) = (G^{-1}(U), G^{-1}(V))$ and $(X', Y') = (G^{-1}(U'), G^{-1}(V'))$ with $(U, V) \sim C, (U', V') \sim C'$ and $B \sim \text{Bern}(p)$ being independent of each other. From this representation, we have that $\text{Cov}(\tilde{X}^2, \tilde{Y}^2) = p \text{Cov}(X^2, Y^2) + (1 - p) \text{Cov}(X'^2, Y'^2)$ and $\text{Cov}(\tilde{X}, \tilde{Y}) = p \text{Cov}(X, Y) + (1 - p) \text{Cov}(X', Y')$. If $C \in \mathcal{C}_G(k)$, then $\text{Cov}(X, Y) = \kappa_G(C) = k$ and $\text{Cov}(X', Y') = \kappa_G(C') = k$. Therefore, we have that $\text{Cov}(\tilde{X}, \tilde{Y}) = k$ and thus

$$\sigma^2_G(\tilde{C}_p) = \text{Var}(\tilde{X}\tilde{Y}) = \text{Cov}(\tilde{X}^2, \tilde{Y}^2) + 1 - \text{Cov}(\tilde{X}, \tilde{Y})^2$$

$$= p \text{Cov}(X^2, Y^2) + (1 - p) \text{Cov}(X'^2, Y'^2) + 1 - (p \text{Cov}(X, Y) + (1 - p) \text{Cov}(X', Y'))^2$$

$$= p(\text{Cov}(X^2, Y^2) + 1 - k^2) + (1 - p)(\text{Cov}(X'^2, Y'^2) + 1 - k^2)$$

$$= p \text{Var}(XY) + (1 - p) \text{Var}(X'Y') = p\sigma^2_G(C) + (1 - p)\sigma^2_G(C'),$$

which shows the desired property. \qed

By Proposition B.3, maximum and minimum of $\sigma^2_G(C)$ on $\mathcal{C}_G(k)$ are attained at extremal points of $\mathcal{C}_G(k)$. 

