Minimum-length Ricci scalar for null separated events

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We consider spacetime endowed with a zero point length, i.e. with an effective metric structure which allows for a (quantum-mechanically arising) finite distance $L_0$ between events in the limit of their coincidence. Restricting attention to null separated events, we find an expression for the Ricci (bi)scalar in this zero-point-length metric. Taking then the coincidence and further $L_0 \to 0$ limits, we find that this expression does not reduce to the Ricci scalar $R$ of the ordinary metric but to $(D - 1)R_{ab}l^a l^b$ in $D$-dimensional spacetime ($D \geq 4$), where $R_{ab}$ and $l^a$ are the ordinary Ricci tensor and tangent vector to the null geodesics. This adds nicely to the existing results for time and space separations. This finding seems to give further support to the view that the quantity $R_{ab}l^a l^b$, ubiquitous in horizon thermodynamics, embodies something which remains as a relic/remnant/memory of a quantum underlying structure for spacetime in the limit of (actual detectability of) this quantumness fading away, and which as such should enter the scene when aiming to derive/motivate the field equations. Further, it turns out to be the same quantity used in an existing derivation of field equations from a thermodynamic variational principle, thus adding further evidence of an origin as quantum-spacetime relic for the latter.

PACS numbers:

I. MOTIVATION

The existence of a deep connection between gravity and thermodynamics/information theory is by now a well established theoretical fact. It actually came as a genuine surprise when, by combining the purely geometric concept of a general relativistic black hole with the basic tenets of quantum mechanics, it undeniably emerged for the first time [1]: indeed, no trace of any thermodynamic feature can be found in the approach to field equations based on the extremization of the Einstein-Hilbert action with respect to the metric. The link with thermodynamics became even further evident when it was found that Einstein’s equations—which we can think of as what axiomatically defines, together with the equivalence principle, general relativity [2]—are of thermodynamic nature themselves; a result (built on Bekenstein’s notion of horizon entropy [3–5]) which was first derived in [6]. A so much intriguing fact this, which can be seen as raising issues at a fundamental level, for randomness turns out this way to be present at the heart of general relativity, and of all of physics with it (cf. [7]). And which yet, any attempt to gain an understanding of gravity purportedly deeper than that provided by general relativity should face for explanation and/or exploit as useful hint.

Beside attempts to construct full-fledged quantum theories of gravity—and potentially to the benefit of them (think e.g. to the holographic principle [8, 9])—there is then clearly a point in the many approaches to gravity which try to gain insight as much as possible building on thermodynamics. Of these, the attempts which bring to a derivation of field equations in a thermodynamic setting, do clearly imply to go beyond strict recognition of a thermodynamic meaning for these equations, breaking thus new ground with respect to the result [6]. This seems to be definitely the case of [10–12], where the field equations of general relativity (actually, of a large class of theories of gravity) have been fully derived from a thermodynamic variational principle based on a functional $S$ representing total (matter+gravitational) entropy. This approach has come a long way since then, up to the point of providing evidence for geometry to be dismissed as primary language in describing classical gravity, and to be replaced entirely by thermodynamic language [13–15] (16 for review).

These results turn out to be strictly intertwined with a in principle completely independent concept, the so-called qmetric. This, introduced as a mean to endow spacetime with a lower limit zero-point-length $L_0$ (of quantum origin) [17–19], appears as a potentially useful tool when trying to describe spacetime at the small scale [20–23]. Considering in particular the Ricci scalar $R$, it has been found that, in case of time/space separated events, its qmetric expression $R_q(p, P)$ (it depends on the two events $P$ and $p$, i.e. it is a biscalar quantity), when we take the coincidence limit $p \to P$ and further let $L_0 \to 0$, is such that $R_q \neq R$. $R_q$ tends instead to something which, taking the expression
for $R_{(q)}$ in Riemannian spaces and analytically continuing it from the Euclidean sector to the Lorentz sector at the point under consideration, remarkably reproduces (the gravitational part of) the $S$ mentioned above [18 19 24]. In other words, the Lagrangian in Einstein-Hilbert action, a completely geometric object, becomes, in the qmetric and in the limit in which the quantum length $L_0$ vanishes, just heat. Its status would thus be sort of relic of an underlying quantum structure for spacetime. $R_{(q)}$ seems then to play some significant role in the geometry-thermodynamics connection.

On the side of the attempts aiming at constructing a complete quantum description of gravity, a proposal has been recently made, within Causal Dynamical Triangulation, of a new geometric observable capable to play the role of a quantum Ricci curvature [25 26]. This might give the opportunity of cross comparisons among very different approaches to quantum curvature (one proceeding from non-smooth metric spaces, the other (the qmetric) trying to extend a smooth metric down to the quantum limit) and provides further motivation for a as complete as possible account of curvature on the side of the qmetric.

In view of these points, in the present analysis we try to extend the investigation of the qmetric biscalar $R_{(q)}$ seeking to find an expression for it for null separated events, a task complicated a little by the need to handle –when writing down the qmetric transverse to the geodesics connecting the events– both the tangent vector and the auxiliary (null) vector. After obtaining such an expression, we proceed to explore its coincidence and further $L_0 \rightarrow 0$ limits.

II. GAUSS–CODAZZI RELATIONS FOR NULL EQUI-GEODESIC HYPERSURFACES

To evaluate the zero-point-length Ricci scalar for null separated events, we exploit the Gauss–Codazzi framework conveniently generalized to the case of null hypersurfaces. This –taking a cue from the approach of [19 27] for timelike/spacelike separated events– with the aim of having a direct expression for the Ricci scalar, meaning without a need to compute it from the components of Riemann or Ricci tensor, a fact this beneficial in view of the qmetric calculation. We consider a congruence $L$ of affinely parameterized null geodesics $\gamma$ emanating from an assigned point $P$, in $D$-dimensional spacetime $M$ $(D \geq 4)$, with affine parameter $\lambda(p, P)$ $(\lambda(P, P) = 0)$ smoothly varying when moving from one geodesics to another nearby. At any $p \in L$ ($p \neq P$) we introduce a canonical observer $\nu^a$ –parallely transported along each geodesic– such that $l_a \nu^a = -1$, with $l^a = dx^a/d\lambda$ the tangent vector where $\{x^a\}$, $a = 1, \ldots, D$, are coordinates of $p$ (the conventions we use are the metric $g_{ab}$ with signature $(-, +, +, \ldots)$, and the Riemann tensor defined by $[\nabla_a, \nabla_b] \nu^a = R^a_{bca} \nu^b$, $\nu^a$ vector). The $\lambda$s can be thought of, this way, as distances along anyone of these geodesics as measured by observers with (parallely transported) velocity $V^a$; in a local frame at $p$ with velocity $V^a$, $\lambda$ is (spatial) distance or elapsed time from $p$. $L$, which is

$$L = \{p \in M : \text{quadratic distance } \sigma^2(p, P) = 0, \text{ and } p \text{ is in the future of } P\}$$

with $l^a$ normal to it (since, from $0 = \frac{d\sigma^2}{d\lambda} = \frac{dx^a}{d\lambda} \partial_a \sigma^2$, $l_a$ must be parallel to the null normal $\partial_a \sigma^2$), can be locally regarded as a collection of spacelike $(D - 2)$-surfaces $\Sigma(P, \lambda)$, i.e. $L$ is locally $L = \gamma \times \Sigma(P, \lambda)$, with $\Sigma(P, \lambda) = \{p \in L : \lambda(p, P) = \lambda(>0)\}$,

$\lambda$ fixed. $L$ can then be mapped through coordinates $(\lambda, \theta^A)$ where $\theta^A$, $A = 1, \ldots, D - 2$, label the different geodesics $\gamma$. We can be sure this construction is consistent provided we limit ourselves to points $p$ such that $\lambda(p, P)$ is small enough that no focal points develop in the congruence (additional to the starting point $P$) (for on the contrary any such focal point would necessarily have multiple assignments of $\theta^A$).

Introducing $m^a = dx^a/d\nu = 2V^a - l^a$ as an auxiliary null vector at $p$ (then parallely-transported along any $\gamma$, and with $m_a e^a_A = 0$ (with $e^a_A = \partial x^a/\partial \theta^A$) and $m_al^a = -2$ at every $p \in L$), the induced metric on $\Sigma(P, \lambda)$ can be expressed as (cf. e.g. [28])

$$h_{ab} = g_{ab} + \frac{1}{2} l_al_b + \frac{1}{2} m_am_b,$$

(1)

with $h^a_{b}$ acting as a projector into the space $T(\Sigma)$ tangent to $\Sigma(P, \lambda)$, $h^{a}_{\lambda b} l^b = 0 = h^{a}_{\lambda b} b^b$ and $h^{a}_{\lambda b} h^{b}_{\epsilon} = h^{a}_{\epsilon}$.

In [29] a relation connecting the Riemann tensor $R^a_{\epsilon bcd}$ intrinsic to null hypersurfaces with the Riemann tensor $R^a_{b c d}$ of $M$ is given (generalization to null hypersurfaces of so-called first Gauss–Codazzi identity), which we take as our starting point. In our circumstances, it reads
\[ R_{\Sigma\ abcd} = R_{efgh} h^e_{\ a} h^f_{\ b} h^g_{\ c} h^h_{\ d} + K_{bd}^{(r)} K_{ac}^{(r)} - K_{ad}^{(r)} K_{bc}^{(r)} - K_{bd}^{(V)} K_{ac}^{(V)} + K_{ad}^{(V)} K_{bc}^{(V)}, \tag{2} \]

where

\[ K_{ab}^{(r)} \equiv h^c_{\ a} h^d_{\ b} \nabla_c r_d \]
\[ K_{ab}^{(V)} \equiv h^c_{\ a} h^d_{\ b} \nabla_c V_d \]

(cf. equation (50) of \cite{29}), with \( r^a \equiv l^a - V^a \), and \( V^a \), defined as above, extended outside \( L \). Due to integrability of \( V^a \) at any point \( p \) of \( L \) \((V^a \text{ is orthogonal to the hypersurface } \Sigma_{(r)} \times \Sigma(\bar{P}, \lambda), \text{ where } i_{(r)} \text{ is the integral curve of } r^a \text{ through } p)\), \( R_{\Sigma\ a\ bcd} \) in the relation above results defined by

\[ Y^a||cd - Y^a||dc = -R_{\Sigma\ a\ bcd} Y^b \]

for any \( Y^a \in T(\Sigma) \) (cf. equation (55) in \cite{29}), where double stroke means differentiation with respect to the connection relative to the induced metric.

A little algebra permits to express relation \( \text{(2)} \) in terms of the vectors \( l^a, m^a \), defining the induced metric in \( \Sigma \), as

\[ R_{\Sigma\ abcd} = R_{efgh} h^e_{\ a} h^f_{\ b} h^g_{\ c} h^h_{\ d} + \frac{1}{2} \left( -K_{bd} K_{ac} + K_{ad} K_{bc} - K_{bd} K_{ac} + K_{ad} K_{bc} \right), \tag{3} \]

where

\[ K_{ab} \equiv h^c_{\ a} h^d_{\ b} \nabla_c l_d = K_{ab}^{(V)} + K_{ab}^{(r)} \]
\[ \bar{K}_{ab} \equiv h^c_{\ a} h^d_{\ b} \nabla_c m_d = K_{ab}^{(V)} - K_{ab}^{(r)}. \]

Formula \( \text{(3)} \) does coincide with that provided, by other means and in complete generality, in \cite{30} (equation (99) there).

From it, we get in particular the scalar relation

\[ R_{\Sigma} = R_{efgh} h^g_{\ c} h^f_{\ h} - K \bar{K} + K_{ab} \bar{K}_{ab}, \tag{4} \]

\( K = K^a_{\ a}, \bar{K} = \bar{K}^a_{\ a} \), where we used of the symmetry of \( K_{ab} \) (from hypersurface-orthogonality of \( l^a \)). Connection with the Ricci scalar \( R \) is made using the easily algebraically established relation

\[ R_{efgh} h^g_{\ c} h^f_{\ h} = R + 2R_{ab} l^a m^b + \frac{1}{2} R_{efgh} l^c m^f m^g l^h \]

\( (R_{ab} \text{ is the Ricci tensor). Equation } \text{(4)} \text{ becomes} \)

\[ R_{\Sigma} = R + 2R_{ab} l^a m^b + \frac{1}{2} R_{efgh} l^c m^f m^g l^h - K \bar{K} + K_{ab} \bar{K}_{ab}. \tag{5} \]

This relation can be further simplified considering that the extension outside \( L \) of scalars or vectors determined on \( L \) by spacetime geometry alone –like \( l^a, m^a \), puts constraints on the derivative of these objects at \( p \in L \) in directions outside \( L \), in particular along curves \( \gamma' \) with tangent \( m^a \) (which we can take to be geodesics without loss of generality, since we are interested in taking first derivatives at \( p \)).

To see this, we notice preliminarly that any couple of events \( p' \) and \( p'' \) near \( p \), the first along \( \gamma' \) from \( p \) and the second along \( \gamma \), which are simultaneous according to the observer \( V^a \) at \( p \), give

\[ dv(p', p) = d\lambda(p'', p), \tag{6} \]

where \( dv(p', p) \equiv \nu(p') - \nu(p) = \nu(p') \) taking \( \nu(p) = 0 \), and \( d\lambda(p'', p) \equiv \lambda(p'', P) - \lambda(p, P) \).

Let then consider first a static \((M, g_{ab})\). In such a spacetime, \( \gamma' \) brings back from \( p(\lambda) \) to points \( p' = p(\nu) \) which have the same spatial coordinates as points \( p(\lambda) \) with \( \lambda = \lambda - \nu \) along \( \gamma \) when approaching \( p \) from \( P \), but at later times. Since spacetime is static and this time delay has then no effect, this amounts to say that, at \( p \), differentiation along \( m^a \) is the opposite of differentiation along \( l^a \).
If \((M, g_{ab})\) is not static, \(\gamma'\) brings to points \(\tilde{p}'(\nu)\) which do not coincide exactly with the points \(p'(\nu)\) relative to a would-be static metric (see Fig. 1). If we refer to the local Lorentz frame \(V^a\) at \(p\), which provides coordinates around \(p\) approximating an exactly static configuration, we see that the non-staticity amounts to \(O(\nu^3)\) effects on the coordinates with respect to the static configuration, and to \(O(\nu^2)\) effects on the metric tensor and then \(O(\nu)\) effects on the connection. The derivatives at \(p\) of both scalar and vector quantities determined by spacetime geometry alone –i.e. which do not change with time in case the spacetime is approximated to be locally static at \(p\), meaning their first derivative with respect to time in the local frame vanishes at \(p\)– are thus left unaffected by the departure of exact staticity around \(p\), and what we said for the static case extends to the non-static case. We have thus the general result

\[
m^a \partial_a \Phi = -l^a \partial_a \Phi
\]

(7)

and

\[
m^a \nabla_a z^b = -l^a \nabla_a z^b
\]

(8)

at \(p \in L\) for any scalar \(\Phi\) and any vector \(z^b\) completely determined by spacetime geometry. In particular, when \(\Phi = \text{const along } \gamma\) we get

\[
m^a \partial_a \Phi = 0
\]

(9)
and, when $z^a$ is parallely trasported along $\gamma$, 

$$m^a \nabla_a z^b = 0, \quad (10)$$

both at $p \in L$. When $\Phi = \text{const}$ all over $L$, equation (9) gives

$$\partial_a \Phi = 0 \quad (11)$$
at $p \in L$, in particular

$$l^b \nabla_a l_b = 0 \quad (12)$$

cf. [31] and

$$m^b \nabla_a m_b = 0 \quad (13)$$
at $p \in L$, on choosing $\Phi = l^b l_b$ or $\Phi = m^b m_b$ respectively.

In view of this, specifically on using equations (9) and (10), one can verify that in the rhs of equation (5) the 3rd term vanishes i.e.

$$R_{efgh} l^e m^f m^g l^h = 0; \quad (14)$$

moreover, the term involving the Ricci tensor can be expressed entirely in terms of $l^a, m^a, K_{ab}, \bar{K}_{ab}$ as

$$2 R_{ab} l^a m^b = -m^a \nabla_a K - l^a \nabla_a \bar{K} + \nabla_a (m^b \nabla_b l^a) + \nabla_a (l^b \nabla_b m^a) - 2 K^{ab} \bar{K}_{ab}. \quad (15)$$

All this is spelled out in appendix A. Equation (5) can then be given the form

$$R_\Sigma = R - m^a \nabla_a K - l^a \nabla_a \bar{K} + \nabla_a (m^b \nabla_b l^a) + \nabla_a (l^b \nabla_b m^a) - K \bar{K} - K^{ab} \bar{K}_{ab}. \quad (16)$$

We use now of the fact that the vectors $m^b \nabla_b l^a$ and $l^b \nabla_b m^a$ are both identically vanishing on $L$ (the first in view of (10), the second by construction). From this, and from equation (10) as applied alternately to $z^a = m^b \nabla_b l^a$ and $z^a = l^b \nabla_b m^a$, we get

$$\nabla_c (m^b \nabla_b l^a) = 0 \quad (17)$$

and

$$\nabla_c (l^b \nabla_b m^a) = 0 \quad (18)$$
at $p \in L$.

This gives

$$R_\Sigma = R - m^a \nabla_a K - l^a \nabla_a \bar{K} - K \bar{K} - K^{ab} \bar{K}_{ab},$$
or

$$R = R_\Sigma + K \bar{K} + K^{ab} \bar{K}_{ab} + m^a \nabla_a K + l^a \nabla_a \bar{K}. \quad (19)$$

This scalar relation is (kind of) the direct expression of the Ricci scalar (meaning without any resort to components of Ricci or Riemann tensors) we were seeking for.
For later use, we note that if that same induced metric \( h_{ab} \) of equation (1) were given in terms of a vector \( \hat{m}^a \) replacing \( m^a \) just being parallel to it, and assuming it to be parallely transported along \( \gamma \) (in place of \( m^a \)), that is if it were

\[
\begin{align*}
\hat{m}^a &= \frac{dx^a}{d\nu} = \frac{1}{\mu(\lambda)} m^a,
\end{align*}
\]

with \( \hat{m}_a e^a = 0, \hat{m}_a l^a = -2 \) and \( \hat{m}^a \) parallely transported along \( \gamma \), but with

\[
(20)
\]

where \( \mu(\lambda) \) is a scalar (thus with \( \hat{m}^a \partial a \Phi = -\frac{1}{\mu} l^a \partial a \Phi \) (23) and

\[
(21)
\]

would replace equations (7) and (8) at \( p \in L \).

The relation between \( R \) and \( R_\Sigma \) would clearly remain the same as in (19), with \( \bar{K}_{ab} \) given by

\[
\begin{align*}
\bar{K}_{ab} &= h^c_a h^d_b \nabla_c \hat{m}_d \\
&= h^c_a h^d_b \nabla_c (\mu \hat{m}_d) \\
&= \mu h^c_a h^d_b \nabla_c \hat{m}_d \\
&= \mu \bar{K}_{ab}
\end{align*}
\]

(\( \bar{K}_{ab} \equiv h^c_a h^d_b \nabla_c \hat{m}_d \)) when expressed in terms of \( \hat{m}^a \), with the 3rd equality coming from \( h^d_b \hat{m}_d = 0 \). Equation (19) would then read

\[
R = R_\Sigma + \mu K \bar{K} + \mu K^{ab} \bar{K}_{ab} + \mu \hat{m}^a \nabla_a K + l^a \nabla_a (\mu \bar{K})
\]

(\( \bar{K} = K^a_a \)).

III. SCALAR GAUSS–CODAZZI RELATION FOR NULL EQUI-GEODESIC HYPERSURFACES WITH ZERO-POINT-LENGTH

Having found potentially convenient expressions for the Ricci scalar on null equi-geodesic hypersurfaces for the metric \( g_{ab} \), we can proceed now to try to find out how to translate them into the context of quantum minimum-length metric \( q_{ab} \) (qmetric), hoping to extract this way a formula useful for extraction of zero-point-length Ricci scalar for null separated events. To this aim, our first task is to explore how our geometric construction in \((M, g_{ab})\) appears in \((M, q_{ab})\).

Let us first recall how the qmetric \( q_{ab} \) is defined. The qmetric \( q_{ab}(x, x') \) is constructed as a tensorial quantity which is function of two spacetime points \( x, x' \) (i.e. it is a bitensor, see e.g. [32]), where we may think of \( x' \) as a fixed, ‘base’ point, and of \( x \), with coordinates \( x^a \), as a variable point. \( q_{ab}(x, x') \) is regarded as a 2nd rank tensor at \( x \) (for any bi-quantity we shall consider in the paper indices are meant to refer to point \( x \)). Its specific expression stems [17], [27], [19] from requiring its effect is to give a quadratic interval \( S \) between \( x' \) and \( x \) which is function
of $\sigma^2$ alone (with $\sigma^2 = \sigma^2(x, x')$ the quadratic interval for $g_{ab}$), $S = S(x, x') = S(\sigma^2)$ -i.e. $S$ is the same wherever we choose $x'$, and wherever we choose $x$ for given $x'$ at fixed $\sigma^2(x, x')$–, with the properties $i) S \approx \sigma^2$ for large separations and, in case of space- or time-separated events, $ii) S \rightarrow \epsilon \, L_0^2$ in the coincidence limit $x \rightarrow x'$ ($\epsilon = +1$ or $\epsilon = -1$ for spacelike-connected or timelike-connected events, respectively) along the geodesic connecting $x$ and $x'$. Here the parameter $L_0$ is a fundamental length, whose value is assumed is determined by underlying microscopic quantum properties of matter and geometry. In case of null separated events (which clearly give $\sigma^2 = 0 = S$), calling $\lambda = \lambda(x, x')$ the interval between $x'$ and $x$ of any affine parameter of the (future-directed) null geodesic connecting them, $q_{ab}$ must give place \[ \text{to a new quenetic-affine parameterization with interval } \lambda = \lambda(x, x') = \lambda(x) \text{ (meaning } \lambda(x) \text{ is the same irrespective of where we choose } x' \text{ and of whichever other null geodesic we choose from } x' \text{ (or different choice of the affine parameter for the same geodesic) provided } x \text{ is at a same value } \lambda = \lambda(x) \text{ from } x' \text{ of the new affine parameter } \lambda), \text{ with the above properties } i) \text{ and } ii) \text{ specifically becoming } i)' \lambda \approx \lambda \text{ for large separations (meaning when } \lambda \gg L_0) \text{ and } ii)' \lambda \rightarrow L_0 \text{ in the coincidence limit } x \rightarrow x' \text{ along the null geodesic (i.e. when } \lambda \rightarrow 0). \]

Compatibility with $q_{ab}$, uniquely fixes the quenetic-connection $\Gamma^a_{bc}(q)$ in terms of $q_{ab}$ (and $\Gamma^a_{bc}$) as
\begin{equation}
\Gamma^a_{bc}(q) = \Gamma^a_{bc} + \frac{1}{2} \, q^{ad} \left( -\nabla_d q_{bc} + 2 \nabla_{(b} q_{c)d} \right)
\end{equation}
with $\Gamma^a_{bc}$ evaluated at $x$. For time or space separations, properties $i)$ and $ii)$, when joined with the request that in any maximally symmetric spacetime the quenetic Green’s function $G_(q)$ of the quenetic d’Alembertian $\square(q) = \nabla_d(q) \nabla_d(q)$ be given at $x$ by $G(q) = G(S)$ (where $G$ is the Green’s function of $\square$, and $G(S)$ is $G$ evaluated at that $x$ on the geodesic such that $\sigma^2(x, x') = S$), uniquely fix $q_{ab}$ itself as a function of $g_{ab}$ and the tangent $t^a$ to the geodesic at $x$, and of $\sigma^2$ and $S$ \[ \text{[17, 19, 27]. For null separations, } q_{ab} \text{ is uniquely fixed by properties } i)' \text{ and } ii)' \text{, joined with the request that, in the proximity of } x \text{ (on the null geodesic } \gamma \text{ from } x' \text{ through } x, \text{ $G$ diverges), } G(q) = G(\lambda) \text{ (where } G(\lambda) \text{ is } G \text{ evaluated nearby that point } \hat{x} \in \gamma \text{ such that } \lambda(\hat{x}, x') = \hat{\lambda} \text{) \[31].} \]

Focusing in particular on the null case, the case we have at hand, and specializing to the particular geometrical configuration we are considering, namely to geodesics $\gamma$ emerging from an assigned point $P$ sweeping at any $p = p(\lambda)$ the null hypersurface $L = \gamma \times \Sigma(P, \lambda)$, then (on choosing $P$ as $x'$ and $p$ as $x$) $q_{ab}(p, P)$ reads \[31]
\begin{equation}
q_{ab} = A \, g_{ab} - \frac{1}{2} \left( \frac{1}{\alpha} - A \right)(l_a m_b + m_a l_b),
\end{equation}
and, from $q^{ab} q_{bc} = \delta^a_c$,
\begin{equation}
q^{ab} = \frac{1}{A} \, g^{ab} + \frac{1}{2} \left( \frac{1}{\alpha} - A \right)(l^a m^b + m^a l^b),
\end{equation}
where the $g_{ab}$-null vectors $l^a$ and $m^a$ are as defined in previous Section ($l_a = g_{abl} b$ and $m_a = g_{ab} m^b$; as a general rule, any index of a $g_{ab}$-tensor is raised and lowered using $g^{ab}$ and $g_{ab}$ and any index of a quenetic-tensor is raised and lowered using $q^{ab}$ and $q_{ab}$), then both $g_{ab}$-orthogonal to $\Sigma(P, \lambda)$ and with $g_{ab} l^a m^b = -2$. All vectors and tensors on the rhs of \[28] \text{ and } \[29] \text{ are meant as evaluated at } p. \text{ The quantities } \alpha = \alpha(p, P) \text{ and } A = A(p, P) \text{ are instead biscalars, given by the expressions}
\begin{equation}
\alpha = \frac{d\lambda}{d\hat{\lambda}},
\end{equation}
with the derivative meant as taken at $p$, and
\begin{equation}
A = \frac{\hat{\lambda}^2}{\lambda^2} \left( \frac{\Delta}{\hat{\Delta}} \right)^{\frac{2}{2}},
\end{equation}
where
\begin{equation}
\Delta(p, P) = -\frac{1}{\sqrt{g(p)g(P)}} \det \left[ -\nabla_{(p)} \nabla_{(P)} \frac{1}{2} \sigma^2(p, P) \right]
\end{equation}
is the van Vleck determinant (33-36; see 32, 37, 38), and

\[ \tilde{\Delta}(\tilde{p}, P) = \Delta(\tilde{p}, P), \]

(33)

with \( \tilde{p} \in \gamma \) such that \( \lambda(\tilde{p}, P) = \tilde{\lambda} \), both \( \Delta \) and \( \tilde{\Delta} \) being biscalars.

We go now to consider the appearance of our geometric configuration in the qmetric framework. We know, by construction, that any null geodesic \( \gamma \) with affine parameter \( \lambda \) results, according to the qmetric, an affinely-parametrized null geodesic \( \gamma_{(q)} \) with qmetric-affine parameter \( \tilde{\lambda} \). Its qmetric tangent vector \( l^a_{(q)} \) is \( l^a_{(q)} = d\tilde{x}^a = \frac{d\tilde{\lambda}}{d\lambda} dx^a = \alpha l^a \), which can be readily verified to be qmetric-null, i.e. \( q_{ab} l^a_{(q)} l^b_{(q)} = 0 \). As a consequence, the qmetric-null hypersurface \( L_{(q)} \) swept by vectors \( l^a_{(q)} \),

\[ L_{(q)} = \{ p \in M : S(p, P) = 0 \text{ and } p \text{ is in the future of } P \}, \]

does coincide with \( L \), \( L_{(q)} = L \), and indeed, at any \( p \) in the future of \( P \), \( \sigma^2(p, P) = 0 \Leftrightarrow S(p, P) = 0 \). \( l^a_{(q)} \) is clearly qmetric-normal to \( L_{(q)} \) since, from \( 0 = \frac{dS}{d\lambda} = \frac{dx^a}{d\lambda} \partial_a S \), \( l^a_{(q)} \) must be parallel to the (qmetric) null normal \( \partial_a S \). \( L_{(q)} \) is locally \( L_{(q)} = \gamma_{(q)} \times \Sigma_{(q)}(P, \tilde{\lambda}) \), where, for any fixed \( \tilde{\lambda} \),

\[ \Sigma_{(q)}(P, \tilde{\lambda}) = \{ p \in L_{(q)} : \tilde{\lambda}(p, P) = \tilde{\lambda}(> 0) \}, \]

and can be mapped through coordinates \( (\tilde{\lambda}, \theta^A) \) where the same \( \theta^A \) as before label the different geodesics \( \gamma_{(q)} \). We see that \( \Sigma_{(q)}(P, \tilde{\lambda}) = \Sigma(P, \lambda) \), where the affine interval \( \lambda \) from \( P \) is that corresponding to the given \( \tilde{\lambda} \) (at any \( p \in L_{(q)} \), \( \tilde{\lambda}(p, P) = \tilde{\lambda} \Leftrightarrow \lambda(p, P) = \lambda \)).

Introducing \( m^a_{(q)} = dx^a/d\tilde{\nu} \) as an auxiliary qmetric-null vector satisfying \( m^a_{(q)} e^a_A(q) = 0 \) and \( m^a_{(q)} l^a_{(q)} = -2 \) \( (e^A_{(q)} = (\partial x^a/\partial \theta^A)_{\lambda=\text{const}} = (\partial x^a/\partial \theta^A)_{\lambda=\text{const}} = e^A \) \( 31 \)), it results uniquely determined as \( m^a_{(q)} = m^a \) (showing \( d\tilde{\nu} = d\nu \)), \( m^a_{(q)} = q_{ab} m^b_{(q)} = (1/\alpha) m_a \), and parallelly transported along each \( \gamma_{(q)} \) (appendix B). The qmetric \( h^{(q)}_{ab} \)

induced on the surface \( \Sigma_{(q)}(P, \tilde{\lambda}) \) is given by

\[ h^{(q)}_{ab} = q_{ab} + \frac{1}{2} m^r_{(q)} m^r_{(q)} + \frac{1}{2} m^r_{(q)} m^r_{(q)}, \]

(34)

cf. 31.

In exact analogy with the \( g_{ab} \) case, we have the qmetric curvature transverse fields

\[ K^{(q)}_{ab} = h^c_{a} m^d_{b} - \frac{1}{2} \Gamma^{(q)}_{cd} m^r_{(q)} \]

and the qmetric Riemann tensor \( R_{\Sigma_{(q)}}^{(q)}_{\quad bcd} \) intrinsic to \( \Sigma_{(q)}(P, \tilde{\lambda}) \) defined by

\[ Y^{a}_{(q)} \parallel_{cd} - Y^{a}_{(q)} \parallel_{dc} = -R_{\Sigma_{(q)}}^{(q)}_{\quad bcd} Y^{b}_{(q)}, \]

\( Y^{a}_{(q)} \in T(\Sigma_{(q)}) \), where \( \parallel \) means covariant differentiation with respect to the connection relative to \( h^{(q)}_{ab} \).

Given these circumstances, all the derivation we presented in previous Section can in principle be carried on in \((M, q_{ab})\), with \( q_{ab} \)-quantities replacing \( g_{ab} \)-quantities. To do this, there is one last thing however we need to consider, namely what are the constraints imposed on geometric quantities \( p^a_{(q)} \) and \( m^a_{(q)} \), and scalars and vectors constructed from them, when differentiating in directions outside \( L_{(q)} \), in particular along curves \( \gamma_{(q)} \) with tangent \( m^a_{(q)} \). To settle this issue, first we note that, since \( q_{ab} \) is completely determined by \( g_{ab} \) and \( \tilde{\lambda}(\lambda) \), any quantity which is completely determined by qmetric spacetime geometry is also completely determined by \( g_{ab} \) for assigned \( \tilde{\lambda}(\lambda) \). Then, we consider any scalar \( \Phi \) completely determined by qmetric spacetime geometry, we have
\[
m^{(q)\mu}_{\nu} \frac{\partial}{\partial \nu} \Phi = m^\alpha \partial_\alpha \Phi = -l^\alpha \partial_\alpha \Phi = -\frac{d\lambda}{d\lambda} \frac{\partial \Phi}{\partial \lambda} = -\frac{1}{\alpha} \frac{\partial}{\partial \alpha} \Phi
\]

(35)
p \in L_{(q)}, \text{ where we used of } \partial^{(q)} = \partial_\alpha \text{ and, in the 2nd step, of equation } [7]. \text{ We see that, for a geometric scalar } \Phi, \text{ we must have } \frac{d\Phi}{d\nu} = -(1/\alpha) \frac{d\Phi}{d\lambda}, \text{ i.e. that differentiation along } m^{(q)\mu}_{\nu} \text{ is } -1/\alpha \text{ times differentiation along } l^\alpha_{(q)}. \text{ This implies that, in a static } (M, g_{ab}), \gamma_{(q)} \text{ brings back from } p(\lambda) \text{ to points } p'(\tilde{\nu}) \text{ which have the same spatial coordinates as points } p(\lambda') \text{ with } \tilde{\lambda}' = \tilde{\lambda} - (1/\alpha) \tilde{\nu} \text{ along } \gamma_{(q)} \text{ (with } \tilde{\nu} = 0 \text{ at } p) \text{ when approaching } p \text{ from } P, \text{ but at later times. In a static } (M, q_{ab}) \text{ this amounts to say that also covariant differentiation of any (geometric) vector } z^{(q)}_\alpha \text{ along } m^{(q)\mu}_{\nu} \text{ is } -1/\alpha \text{ times covariant differentiation along } l^\alpha_{(q)}, \text{ i.e.}

\[
m^{(q)\mu}_{\nu} \nabla^a_{(q)} z^{(q)}_\alpha = -\frac{1}{\alpha} l^\alpha_{(q)} \nabla_a \Phi \frac{\partial}{\partial \nu} \Phi.
\]

(36)

But, analogously to what we have seen happens for \((M, g_{ab})\), this result extends to a non-static \((M, q_{ab})\) as well, since the effects of non-staticity on the term \(\frac{1}{2} q^{ab} \left( -\nabla_d q_{bc} + 2\nabla(b q_{c})_d \right) \text{ in the qmetric connection in equation } [27], \text{ additional with respect to } \Gamma^a_{bc}, \text{ are } O(\tilde{\nu}) \text{ (the effects on } g_{ab} \text{ being } O(\tilde{\nu}^2)) \text{ and vanish in the } \tilde{\nu} \to 0 \text{ limit.}

We are now in a position to appreciate that the role the vector \(m^{(q)}_\mu\) plays in \((M, q_{ab})\) does not coincide with the role played in \((M, g_{ab})\) by \(m^a\). It is instead exactly that of the vector \(\hat{m}^a\) of equations \([23]\) and \([24]\) with \(\mu = \alpha\). This fact appears to reflect the intrinsic asymmetry of the qmetric in the radial direction, entailed by the request of a finite limiting distance at coincidence. Thus, we can write the zero-point-length Gauss–Codazzi relation for null equigeodesic hypersurfaces from equation \([26]\). We get

\[
R_{(q)} = R_{\Sigma(q)} + \alpha K_{(q)} \tilde{K}'_{(q)} + \alpha K^{ab}_{(q)} K^{(q)}_{ab} + \alpha m^{(q)\mu}_{\nu} \nabla^a_{(q)} K_{(q)} + l^\alpha_{(q)} \nabla_a \left( \alpha K_{(q)} \right),
\]

(37)

with \(K_{(q)} = K^{ab}_{(q)} \equiv q^{ab} K^{(q)}_{ab}, \tilde{K}'_{(q)} = \tilde{K}'_{b(q)}, \text{ and } K^{(q)}_{ab} \text{ playing the role of } \hat{K}_{ab}.

IV. MINIMUM-LENGTH RICCI SCALAR FOR NULL SEPARATED EVENTS

Equation \([37]\) provides the minimum-length Ricci scalar \(R_{(q)}\) for null separated events in terms of other qmetric quantities. We proceed then now to try to gain an expression for \(R_{(q)}\) in terms of quantities which live in in \((M, g_{ab})\), i.e. with reference to the classical metric \(g_{ab}\). This involves to express the rhs of \([37]\) in terms of the quantities which define the qmetric in \((M, g_{ab})\), namely \(\alpha\) and \(A\). Let us start from the term \(R_{\Sigma(q)}\). From equation \([34]\) we get

\[
h_{(q)}^{ab} = A h_{ab}
\]

(38)

(cf. \([31]\)), i.e. \(h_{(q)}^{ab}\) and \(h_{ab}\) are conformally related, in analogy to what happens in the timelike/spacelike case \([27]\). Since the conformal factor \(A\) is constant on \(\Sigma\), this gives (cf. e.g. \([39]\))

\[
R_{\Sigma(q)} = \frac{1}{A} R_{\Sigma}.
\]

(39)

\(K^{(q)}_{ab}\) turns out to be

\[
K^{(q)}_{ab} = h^{(q)\alpha}_{a} h^{(q)\beta}_{b} \nabla^{(q)}_{\epsilon} l^{(q)}_{d} = K_{ab} + \frac{1}{2} \alpha h^{(q)\alpha}_{a} h^{(q)\beta}_{b} l^{(q)}_{d} \nabla q_{dc} + \frac{1}{2} \alpha h^{(q)\alpha}_{a} h^{(q)\beta}_{b} l^{(q)}_{d} \left( \nabla q_{dc} + \nabla d q_{cd} \right)
\]
This gives

\[
K_{(q)} = q^{ab} K_{ab} = \alpha K + \frac{1}{2} \frac{d}{d\lambda} \ln A,
\]

(41)

where in the 2nd step we used of \( l^a K_{ab} = 0, l^a h_{ab} = 0. \)

As for \( \tilde{K}_{ab}^{(q)}, \) we get (the derivation is spelled out in appendix [C])

\[
\tilde{K}_{ab}^{(q)} = h^c_c (q) h^d_d (q) \nabla_c m^{(q)}_d = A \tilde{K}_{ab} - \frac{1}{2} \frac{d}{d\lambda} h_{ab}.
\]

(42)

This gives

\[
\tilde{K}_{(q)} = q^{ab} \tilde{K}_{ab}^{(q)} = \tilde{K} - \frac{1}{2} (D - 2) \frac{d}{d\lambda} \ln A,
\]

(43)

where we used of \( l^a \tilde{K}_{ab} = 0. \)

Using expressions (40), (41), (42) and (43), for various terms in (37) we get

\[
\alpha m^{(q)}_a \nabla_a K_{(q)} = -\alpha \frac{d\alpha}{d\lambda} K - \alpha^2 \frac{dK}{d\lambda} - \frac{1}{2} \frac{d}{d\lambda} \ln A - \frac{1}{2} (D - 2) \alpha^2 \frac{d^2}{d\lambda^2} \ln A,
\]

\[
l^a (q) \nabla_a (\alpha \tilde{K}')_{(q)} = \alpha l^a_a (q) \nabla_a K_{(q)} + l^a_a (q) \nabla_a (\alpha \tilde{K}')_{(q)} = \alpha^2 \frac{d\tilde{K}}{d\lambda} - \frac{1}{2} (D - 2) \alpha^2 \frac{d^2}{d\lambda^2} \ln A + \alpha \frac{d\alpha}{d\lambda} \left( \tilde{K} - \frac{1}{2} (D - 2) \frac{d}{d\lambda} \ln A \right),
\]

\[
\alpha K_{(q)}^{ab} \tilde{K}_{ab}^{(q)} = \alpha K^{ab} \tilde{K}_{ab} + \frac{1}{2} \alpha^2 \left( \frac{d}{d\lambda} \ln A \right) (\tilde{K} - K) - \frac{1}{4} (D - 2) \alpha^2 \left( \frac{d}{d\lambda} \ln A \right)^2
\]

(44)

(the calculation of last equality is detailed in appendix [C]), and equation (37) becomes

\[
R_{(q)} = \frac{1}{A} R_{\Sigma} - \alpha \frac{d\alpha}{d\lambda} K + \alpha^2 \left( \frac{d\tilde{K}}{d\lambda} - \frac{dK}{d\lambda} \right) - \frac{1}{2} (D - 2) \alpha \frac{d\alpha}{d\lambda} \frac{d}{d\lambda} \ln A
\]

\[
- (D - 2) \alpha^2 \frac{d^2}{d\lambda^2} \ln A + \alpha^2 K \tilde{K} + \frac{1}{2} (D - 1) \alpha^2 \left( \frac{d}{d\lambda} \ln A \right) (\tilde{K} - K)
\]

\[
- \frac{1}{4} (D - 2) (D - 1) \alpha^2 \left( \frac{d}{d\lambda} \ln A \right)^2 + \alpha^2 K^{ab} \tilde{K}_{ab} + \alpha \frac{d\alpha}{d\lambda} \tilde{K} - \frac{1}{2} (D - 2) \alpha \frac{d\alpha}{d\lambda} \frac{d}{d\lambda} \ln A.
\]

(45)

We can proceed further by noting that the fact that the fields \( l^a \) and \( m^a \) are determined by spacetime geometry alone, i.e. that same fact which leads, as we saw, to put the constraints (7), (8), (55), (56) on the covariant derivatives and qmetric covariant derivatives along \( m^a \) and \( m^a (q) \) at \( p \in L \), allows to express here the barred quantities in terms of the non-barred ones. In particular, we have

\[
\tilde{K}_{ab} = -K_{ab}
\]

(46)

This is described in appendix [D]. Using this, \( R_{(q)} \) can be given a form manifestly independent of the auxiliary vector \( m^a \). A final additional progress can be made in that the \( d\tilde{K}/d\lambda \) in (45) can conveniently be expressed in terms of \( K_{ab} \) and Ricci tensor as (see again appendix [D]).
\[ \frac{dK}{d\lambda} = -K^{ab}K_{ab} - R_{ab}l^a l^b. \]  

(47)

The expression we eventually find for the qmetric Ricci scalar for null separated events is thus

\[
R_{(q)} = \frac{1}{A} R_{\Sigma} - 2 \alpha \frac{d\alpha}{d\lambda} K + 2 \alpha^2 R_{ab} l^a l^b - (D - 2) \alpha \frac{d\alpha}{d\lambda} \ln A - (D - 2) \alpha^2 \frac{d^2}{d\lambda^2} \ln A
- \frac{1}{4} (D - 2)(D - 1) \alpha^2 \left( \frac{d}{d\lambda} \ln A \right)^2 - \alpha^2 K^2 + \alpha^2 K^{ab} K_{ab} - (D - 1) \alpha^2 \left( \frac{d}{d\lambda} \ln A \right) K.
\]

(48)

V. COINCIDENCE AND \( L_0 \rightarrow 0 \) LIMIT

Having the expression (48) for \( R_{(q)} \) for null separated events, we can ask what this expression becomes in the coincidence limit \( p \rightarrow P \). One might naively expect it turns out to be the Ricci scalar \( R \) at \( P \) plus a quantity \( \eta \) dependent on \( L_0 \), with \( \eta \) vanishing in the \( L_0 \rightarrow 0 \) limit. In case of the qmetric Ricci scalar for time or space separated events there already exists the remarkable result that this is not the case. What has been found is \( \lim_{L_0 \rightarrow 0} \lim_{p \rightarrow P} R_{(q)} = \epsilon D R_{ab} l^a l^b \neq R \), with \( h^a \) the normalized tangent (at \( P \)) to the geodesics connecting \( P \) and \( p \). Our aim here is to see whether something similar happens in the null case, somehow along what might, in view of these results, be by now expected (by analogy with the spacelike/timelike case or from entropy density considerations [18, 24]).

We begin by noting that, from equation (19), \( R_{\Sigma} \) can be expressed entirely in terms of the other quantities already present in (48) and of \( R \) as

\[
R_{\Sigma} = R - K\bar{K} - K^{ab} K_{ab} - m^a \nabla_a K - l^a \nabla_a \bar{K}
= R + K^2 + K^{ab} K_{ab} + 2 \frac{dK}{d\lambda}
= R + K^2 - K^{ab} K_{ab} - 2 R_{ab} l^a l^b,
\]

(49)

where in the 2nd step we used of (46) and of (7) with \( \Phi = K \), and in the 3rd step of equation (47). Since \( A \) is expressed in terms of the van Vleck determinant biscalar, once we have an expansion of this and of \( K_{ab} \) for \( p \) near \( P \) we have all is needed to extract the coincidence limit.

As for the van Vleck determinant, from [37] we get

\[
\Delta^{1/2}(p, P) = 1 + \frac{1}{12} \lambda^2 R_{ab} l^a l^b + O(\lambda^3),
\]

(50)

with \( R_{ab} \) and the vector \( l^a \) evaluated at \( p \). This gives also

\[
\bar{\Delta}^{1/2}(p, P) = 1 + \frac{1}{12} \lambda^2 (R_{ab} l^a l^b)|_{\bar{p}} + O(\lambda^3).
\]

(51)

As for \( K_{ab} \), we have

\[
K_{ab}(p) = h^c_a h^d_b \nabla_c l_d
= h^c_a h^d_b \nabla_c \left( \frac{1}{\lambda} \nabla_d \left( \frac{\sigma^2}{2} \right) \right)
= h^c_a h^d_b \left[ \left( \nabla_c \frac{1}{\lambda} \right) \nabla_d \left( \frac{\sigma^2}{2} \right) + \frac{1}{\lambda} \nabla_c \nabla_d \left( \frac{\sigma^2}{2} \right) \right]
= \frac{1}{\lambda} h^c_a h^d_b \nabla_c \nabla_d \left( \frac{\sigma^2}{2} \right).
\]

(52)

The 2nd equality here, comes about since, for \( p' \) near \( p \), but not exactly on the null geodesic \( \gamma \) through \( P \) and \( p \), \( \nabla_d \left( \sigma^2(p', P)/2 \right) \) can be usefully expressed as \( \nabla_d \left( \sigma^2(p', P)/2 \right) = 2\lambda l_d + 2\nu m_d \), with \( \lambda \) and \( \nu \) meant as curvilinear...
null coordinates of \( p' \), \( \nu = 0 \) on \( \gamma \) [38], from which \( l_d = (1/\lambda) \nabla_d \left( \sigma^2(p', P)/2 \right) - (\nu/\lambda) m_d \). From this equation we get the mentioned equality for \( h' c \nabla_c [-(\nu/\lambda) m_d] = -m_d h' c \nabla_c (\nu/\lambda) - (\nu/\lambda) \nabla_c m_d \) and both terms on the rhs here vanish in the \( \nu \to 0 \) limit (the 1st term vanishes since \( h' c \nabla_c (\nu/\lambda) \to (h' c \nabla_c (\nu/\lambda))_\Sigma \) for \( \nu \to 0 \), \( h' c \nabla_c \) of a scalar is its gradient on \( \Sigma \), and \( \nu, \lambda = \text{const} \) on \( \Sigma \) (\( \nu = 0 \), actually)). The 4th equality in (52) is a consequence of \( \nabla_c 1/\lambda \to (\nabla_c 1/\lambda)_\Sigma \) for \( \nu \to 0 \) and of being \( h' c \nabla_c (1/\lambda) \) the gradient of \( 1/\lambda \) on \( \Sigma \), where \( \lambda = \text{const} \).

Equation (52) shows that what we need is the expansion for \( \nabla_c \nabla_c (\sigma^2/2) \) for \( p \) near \( P \). From [27], we get

\[
\nabla_a \nabla_b \left( \frac{\sigma^2}{2} \right) = g_{ab} - \frac{1}{3} \lambda^2 E_{ab} + \mathcal{O}(\lambda^3),
\]

where \( E_{ab} = R_{abmn} t^m t^n \) and the quantities on rhs are evaluated at \( p \). As for \( K_{ab} \), this gives

\[
K_{ab}(p) = \frac{1}{\lambda} h_{ab} - \frac{1}{3} \lambda h' c h'' b E_{cd} + \mathcal{O}(\lambda^2),
\]

expression quite similar to (the leading terms) of the expansion found for extrinsic curvature of equi-geodesic surfaces for timelike/spacelike case [19, 27].

Using formula (31) for \( A \), the expression (49) for \( R \Sigma \) and the expansions (50), (51) and (54), the 1st term on the rhs of (48) is found to be

\[
\frac{1}{A} R \Sigma = (D - 2)(D - 3) \frac{1}{\lambda^2} \lambda^2 \nabla^2 + \mathcal{O}(\lambda^2)
\]

\[
= (D - 2)(D - 3) \frac{1}{\lambda^2} + \frac{D - 3}{3} E(\bar{p}) + \mathcal{O}(\bar{\lambda}, \lambda^2),
\]

with \( E = E^a a = R_{ab} l^a l^b \). From equation (54) we get also

\[
K = (D - 2) \frac{1}{\lambda} - \frac{1}{3} \lambda E + \mathcal{O}(\lambda^2),
\]

\[
K^2 = (D - 2)^2 \frac{1}{\lambda^2} - \frac{2}{3} (D - 2) E + \mathcal{O}(\lambda),
\]

\[
K_{ab} K_{bc} = (D - 2) \frac{1}{\lambda^2} - \frac{2}{3} E + \mathcal{O}(\lambda).
\]

As for the terms in (48) containing derivatives of \( \ln A \), we can use the following expansions (from (50) and (51))

\[
\frac{d}{d\lambda} \ln A = \frac{2}{\lambda} \frac{1}{\alpha} - \frac{2}{\lambda} \frac{d}{d\lambda} 1/\alpha - \frac{2}{\lambda} \frac{d}{d\lambda} E(\bar{p}) + \frac{2}{3(D - 2)} \lambda E(p) + \mathcal{O}(\bar{\lambda}, \lambda^2)
\]

\[
\frac{d^2}{d\lambda^2} \ln A = -\frac{2}{\lambda^2} \frac{1}{\alpha^2} - \frac{2}{\lambda} \frac{d}{d\lambda} \frac{1}{\alpha^2} + \frac{2}{\lambda^2} \frac{d}{d\lambda} E(\bar{p}) + \frac{2}{3(D - 2)} E(p) + \mathcal{O}(\bar{\lambda}, \lambda).
\]

In particular, we get

\[
-\frac{1}{4} (D - 2)(D - 1) \alpha^2 \left( \frac{d}{d\lambda} \ln A \right)^2 - (D - 1) \alpha^2 \left( \frac{d}{d\lambda} \ln A \right) K
\]

\[
= (D - 2)(D - 1) \frac{1}{\lambda^2} \alpha^2 - (D - 2)(D - 1) \frac{1}{\lambda^2} + \frac{2}{3} (D - 1) E(\bar{p}) - \frac{2}{3} (D - 1) \alpha^2 E(p) + \mathcal{O}(\bar{\lambda}, \lambda).
\]

Substituting expressions (55)-(61) into equation (48), we get

\[
R_{(q)} = \left[ \frac{D - 3}{3} + \frac{2}{3} + \frac{2}{3} (D - 1) \right] E(\bar{p}) + \mathcal{O}(\bar{\lambda}, \lambda)
\]

\[
= (D - 1) \left( R_{ab} l^a l^b \right)_{\bar{p}} + \mathcal{O}(\bar{\lambda}, \lambda),
\]

which gives

\[
\left( \lim_{L_0 \to 0} \lim_{\lambda \to 0} R_{(q)} \right)_{\bar{p}} = (D - 1) \left( R_{ab} l^a l^b \right)_{\bar{p}}.
\]
VI. COMMENTS AND CONCLUSION

The expression on the rhs turns out to be proportional to (the opposite) of what has found interpretation as the gravitational heat density $H$ of a null surface with normal $l^a$ [16] [40].

$$H = -\frac{1}{L_{Pl}^2} R_{ab} l^a l^b,$$

where $L_{Pl}$ is Planck’s length (units, here and below, make the vacuum speed of light $c = 1$ and the reduced Planck’s constant $\hbar = 1$, with Einstein’s equation in the form $G_{ab} = 8\pi L_{Pl}^2 T_{ab}$, where $G_{ab}$ is the Einstein tensor and $T_{ab}$ the matter energy-momentum tensor). Equally, expression (63) turns out to enter the entropy functional $S$ mentioned above in the statistical derivation of Einstein’s equation, in the term which accounts for the gravitational degrees of freedom, which has the form

$$\ln \rho_g = \frac{1}{4} \left( 1 - \frac{L_{Pl}^2}{2\pi} R_{ab} l^a l^b \right),$$

with $\rho_g$ the density of quantum states of spacetime at $P$ [16].

Formula (64) talks about a heat density one has to assign to a horizon, namely about some quantum degrees of freedom for spacetime. If we allow for $L_{Pl} \to 0$ in that equation, the apparent divergence of $H$ is canceled by the scaling of $R_{ab} l^a l^b$ itself as $L_{Pl}^2$ (from Einstein’s equation). This lets the spacetime heat content $H$, yet a notion of quantum origin, to be insensitive to the actual value of $L_{Pl}$. Writing, from (64), $H$ in terms of Newton’s constant as

$$H = -\frac{1}{G} R_{ab} l^a l^b,$$

we see thus that the quantity to the right, even if written entirely in terms of (measurable) quantities with apparently nothing in them talking about quantum mechanics, can be endowed indeed with a quantum mechanical intrinsic significance, which we are led to think of as sort of relic of a quantumness of spacetime (for it survives to $L_{Pl} \to 0$).

This, corresponding to the view that gravity, starting from Newton’s law itself, ought to be considered as intrinsically quantum [41, 42].

This is also what the analysis in the present paper, in analogy of what already pointed out in [18, 24], actually seems to suggest. We have indeed that the Ricci scalar $R(q)$ of the quantum spacetime framed on null geodesics at $P$, in the limit of actual quantumness of spacetime going to be unnoticeable, becomes $R_{ab} l^a l^b$, not $R$. This, corresponding to the view that gravity, starting from Newton’s law itself, ought to be considered as intrinsically quantum [41, 42].

Acknowledgements. I thank Francesco Anselmo and Sumanta Chakraborty for comments, suggestions and improvements on the draft, as well as for providing some references. I thank Dawood Kothawala for having raised the point which this study tries to answer to.

Appendix A: Derivation of equations (14) and (15)

Let start from equation (14). We have

$$R_{efgh} l^e m^f m^g l^h = -R_{fegh} l^e m^f m^g l^h$$

$$= -R_{f e g h} l^e m^f m^g l^h$$

$$= \left( \nabla_h \nabla_g l^f - \nabla_g \nabla_h l^f \right) m_f m^g l^h$$

$$= \left( l^h \nabla_h \nabla_g l^f - l^g \nabla_g \nabla_h l^f \right) m_f m^g - \left( m^g \nabla_g \nabla_h l^f \right) m_f l^h,$$

(A1)

from $\nabla_h \nabla_g l^f - \nabla_g \nabla_h l^f = -R_{f e g h} l^e$. Now,
\[(l^h \nabla_h \nabla_g l^f) m^g = l^h \nabla_h (m^g \nabla_g l^f) - (l^h \nabla_h m^g) \nabla_g l^f = l^h \nabla_h (m^g \nabla_g l^f) = 0, \tag{A2}\]

where the 2nd equality comes from \(m^a\) being parallelly transported along \(\gamma\), and the 3rd from being the vector \(v^f = m^g \nabla_g l^f\) identically vanishing along \(\gamma\) from equation (10) as applied to \(z^f = l^f\).

In the second term of rhs of equation (A1), we have

\[\left(m^g \nabla_g \nabla_h l^f\right) m^h = m^g \nabla_g \left(l^h \nabla_h l^f\right) - (m^g \nabla_g l^h) \nabla_h l^f = m^g \nabla_g \left(l^h \nabla_h l^f\right) = 0, \tag{A3}\]

where the 2nd equality comes from equation (10) as applied to \(z^h = l^h\), and the 3rd from equation (10) as applied to the vector \(z^f = l^h \nabla_h l^f\) which is identically vanishing along \(\gamma\). We get thus equation (14).

As concerns equation (15), we have

\[2R_{ab} l^a m^b = R_{ab} l^a m^b + R_{ab} m^a l^b = -m^b \nabla_b \nabla_a l^a + m^b \nabla_a \nabla_b l^a - l^b \nabla_b \nabla_a m^a + l^b \nabla_a \nabla_b m^a = -m^b \nabla_b \nabla_a l^a + \nabla_a (m^b \nabla_b l^a) - l^b \nabla_b \nabla_a m^a + \nabla_a (l^b \nabla_b m^a) - 2(\nabla b l^a) \nabla_a m^b, \tag{A4}\]

where the 2nd equality is from \(\nabla_b \nabla_a v^a - \nabla_a \nabla_b v^a = -R_{ab} v^a\) for any vector \(v^a\). In (A4),

\[\nabla_a l^a = g^{ab} \delta^c_a \nabla_c l_b = \frac{1}{2} g^{ab} \delta^c_a \nabla_c l_b = \frac{1}{2} g^{ab} \delta^c_a \nabla_c l_b = h^{ab} \delta^c_a \nabla_c l_b = h^{ab} \delta^c_a \nabla_c l_b = h^{ab} \delta^c_a \nabla_c l_b = K^a_a = K, \tag{A5}\]

where in the 2nd and 4th equality we use equation (1), the 3rd equality comes \(l_b\) being parallelly transported along \(\gamma\) and from (10) as applied to \(z_b = l_b\) (or simply from equations (12) and (13)), the 5th from (10) (or from \(h^{ab} m_a = 0 = h^{ab} l_a\)), and the penultimate equality from the definition of \(K_{ab}\). In an analogous manner, we get

\[\nabla_a m^a = \tilde{K}. \tag{A6}\]

As for the last term in the rhs of (A4), we have

\[\nabla_b l^a \nabla_a m^b = g^{af} g^{bg} \nabla_b l_f \nabla_a m_g = (h^{af} - \frac{1}{2} m^a l^f) (h^{bg} - \frac{1}{2} m^g l^b) \nabla_b l_f \nabla_a m_g = h^{af} h^{bg} \nabla_b l_f \nabla_a m_g = h^{af} h^{bg} \nabla_b l_f \nabla_a m_g = K_{ab} \tilde{K}^{ab}. \tag{A7}\]

Here, the 3rd equality stems from parallel transporting along \(\gamma\), and from repeated use of (10) (or of (12) and (13)); the 4th from the relation
\[ \nabla_h l_f = \nabla_f l_b + (\nabla_c l_b - \nabla_b l_c) \frac{1}{2} m^c l_f + (\nabla_f l_c - \nabla_c l_f) \frac{1}{2} m^l l_b \]  
(A8)

(ef. e.g. [28], Section 2.4.3), joined with orthogonality of \( l^a \) to \( T(\Sigma) \), i.e. \( h^{ab} l_a = 0 \); the 5th from

\[ K_{ab} \bar{K}^{ab} = h^{c a} h^{d b} h^{f a} h^{g b} \nabla_c l_d \nabla_f m_g \]
\[ = h^{f c} h^{g d} \nabla_c l_d \nabla_f m_g, \]

(A9)

(from \( h^{c a} h^{f a} = h^{f c} \)).

Substituting expressions (A5), (A6) and (A7) in (A4), we get equation (15).

**Appendix B: Calculation of \( m^a_{(q)} \) and proof of its qmetric parallel transport along any \( \gamma_{(q)} \)**

The expression \( m^a_{(q)} = m^a \) (right above equation (34)) comes, uniquely, from the requests \( m^a_{(q)} \) be null and have \( m^a_{(q)} e_A^{a (q)} = 0 \) and \( m^a_{(q)} l^a_{(q)} = -2 \). For, \( m^a_{(q)} \) qmetric-orthogonal to \( e_A^{a (q)} \) means that \( m^a_{(q)} \) cannot develop components along \( e_A^{a (q)} \), which is \( e_A^{a} \), and then \( m^a_{(q)} = k m^a + c l^a \) with \( k, c \) scalars. \( m^a_{(q)} \) qmetric-null means further that

\[ 0 = q_{ab} m^a_{(q)} m^b_{(q)} \]
\[ = \left[ A g_{ab} - \frac{1}{2} \left( \frac{1}{\alpha} - A \right) (l_a m_b + m_a l_b) \right] (k m^a + c l^a)(k m^b + c l^b) \]
\[ = -4 k c \left( 3 A - \frac{2}{\alpha} \right), \]
(B1)

gives \( k = 0 \) or \( c = 0 \). From

\[ q_{ab} m^a_{(q)} l^b_{(q)} = -2 k, \]
(B2)

the request \( q_{ab} m^a_{(q)} l^b_{(q)} = -2 \) implies \( k = 1 \) and then \( c = 0 \), that gives \( m^a_{(q)} = m^a \).

As for the transport along \( \gamma_{(q)} \), we get

\[ l^b_{(q)} \nabla^c_{(q)} m^c_{(q)} = \alpha l^b \left[ \partial_b m^c - (\Gamma^{c (q)}_{bc} \frac{m_a}{\alpha}) \right] \]
\[ = \frac{1}{2} q^{ad} (- \nabla_d m^c + 2 \nabla_{(b} q_{c) d} m^a) \]
\[ = \alpha m_c \frac{d}{d\lambda} \left( \frac{1}{\alpha} \right) - \frac{1}{2} \alpha m^d l^b (- \nabla_d m^c + 2 \nabla_{(b} q_{c) d}) \]
\[ = \alpha \left( \delta^c + \frac{1}{2} l^b m^c + \frac{1}{2} m^b l_c \right) \nabla_b \frac{1}{\alpha} - \frac{1}{4} (1 - \alpha A) l^b m^d \nabla_d (m_b l_c) \]
\[ = \alpha h^{c b} \nabla_b \frac{1}{\alpha} \]
\[ = 0, \]
(B3)

showing the transport of \( m^a_{(q)} \) along \( \gamma_{(q)} \) is qmetric-parallel. Here the 3rd step comes from \( l^b \nabla_b m_c = 0 \) and \( q^{ad} m^c = \alpha m^d \), in the 4th step we used equation (11) putting \( \Phi = l^a l_a \) and \( \Phi = m^a l_a \), the 5th is from equation (10) with \( z^b = m^b \) and \( z^b = l^b \) alternately, and the last step for we are taking a gradient on \( \Sigma(P, \lambda) \) and \( \alpha = \text{const on it} \).

**Appendix C: Calculation of \( K_{ab}^{(q)}, K_{ab}^{(s)}, \) and of \( K^{(q)}_{ab} K^{(s)}_{ab} \)**

We begin with the expression for \( K^{(q)}_{ab} \) (equation (40) in the main text). Our starting point is

\[ K^{(q)}_{ab} = h^{c a} (q) h^{d b} (q) \nabla^{(q)}_{c d}. \]
Here,

\[ h_{a}^{d}(q) = q^{da} h_{ab}^{(q)} \]

\[ = \left[ \frac{1}{A} q^{da} + \frac{1}{2} \left( \frac{1}{A} - \alpha \right) \left( l^{d} m^{a} + m^{d} l^{a} \right) \right] A h_{ab} \]

\[ = h_{a}^{d}, \quad (C1) \]

where in the last equality we used of \( m^{a} h_{ab} = 0 = l^{a} h_{ab} \). Also,

\[ \nabla_{c}^{(q)l_{d}} = \partial c_{l_{d}} - \left[ \frac{1}{2} q^{al} \left( - \nabla_{l_{q}c_{d}} + \nabla_{c_{d}l_{q}} + \nabla_{d_{q}c_{l}} + \Gamma^{a}_{cd} \right) \right] l_{a} \]

\[ = \nabla_{c_{l_{d}}} + \frac{1}{2} \alpha l_{l} \left( - \nabla_{l_{q}c_{d}} + \nabla_{c_{d}l_{q}} + \nabla_{d_{q}c_{l}} \right), \quad (C2) \]

using, in the 2nd equality, of \( q^{ad} l_{a} = \alpha l^{d} \). From (C1) and (C2) we get

\[ K_{ab}^{(q)} = K_{ab} + \frac{1}{2} \alpha h_{c}^{d} h_{b}^{d} l^{l} \nabla_{l_{q}c_{d}} - \frac{1}{2} \alpha h_{c}^{d} h_{b}^{d} l^{l} \left( \nabla_{c_{q}d} + \nabla_{d_{q}c} \right), \quad (C3) \]

which is the 2nd step in equation (40).

Now,

\[ l^{l} \nabla_{l_{q}c_{d}} = l^{l} \nabla_{l} \left[ A g_{c_{d}} - \frac{1}{2} \left( \frac{1}{\alpha} - A \right) (l_{c} m_{d} + m_{c} l_{d}) \right] \]

\[ = \frac{dA}{d\lambda} g_{c_{d}} - \frac{1}{2} (l_{c} m_{d} + m_{c} l_{d}) \frac{dA}{d\lambda} \left( \frac{1}{\alpha} - A \right), \]

from parallel transport along \( \gamma \). This gives

\[ h_{c}^{d} h_{b}^{d} l^{l} \nabla_{l_{q}c_{d}} = \frac{dA}{d\lambda} h_{ab} \quad (C4) \]

(again using \( m^{a} h_{ab} = 0 = l^{a} h_{ab} \)). Further,

\[ l^{l} \nabla_{c_{q}d} = l_{d} \nabla_{c} A - \frac{1}{2} l^{l} \nabla_{c} \left[ \left( \frac{1}{\alpha} - A \right) (l_{d} m_{l} + m_{d} l_{l}) \right] \]

\[ = l_{d} \nabla_{c} A + l_{d} \nabla_{c} \left( \frac{1}{\alpha} - A \right) - \frac{1}{2} \left( \frac{1}{\alpha} - A \right) l^{l} \nabla_{c} (l_{d} m_{l} + m_{d} l_{l}) \]

\[ = l_{d} \nabla_{c} \frac{1}{\alpha} - \frac{1}{2} \left( \frac{1}{\alpha} - A \right) l^{l} \nabla_{c} (l_{d} m_{l}) \]

\[ = l_{d} \nabla_{c} \frac{1}{\alpha} + \left( \frac{1}{\alpha} - A \right) \nabla_{c l_{d}} - \frac{1}{2} \left( \frac{1}{\alpha} - A \right) l^{l} \nabla_{c} m_{l}, \]

where we used of \( l^{a} m_{a} = -2 \) and, in the 3rd step, of equation (12). From this,

\[ h_{c}^{d} h_{b}^{d} l^{l} \left( \nabla_{c_{q}d} + \nabla_{d_{q}c} \right) = h_{c}^{d} h_{b}^{d} \left( \frac{1}{\alpha} - A \right) \left( \nabla_{c_{l}d} + \nabla_{d_{l}c} \right) \]

\[ = \left( \frac{1}{\alpha} - A \right) (K_{ab} + K_{ba}) \]

\[ = 2 \left( \frac{1}{\alpha} - A \right) K_{ab}, \quad (C5) \]
where last step is from the symmetry of $K_{ab}$ (from hypersurface orthogonality of the congruence $l^a$). Using (C4) and (C5) in (C3), we get the 3rd equality in equation (40) and from it immediately the 4th.

We consider now how the expression (42) for $\bar{K}_{ab}^{(q)}$ comes about. In

$$\bar{K}_{ab}^{(q)} = h^a_c h^d_b \nabla_c m_d^{(q)},$$

the qmetric covariant derivative is given by

$$\nabla_c m_d^{(q)} = \partial_c \left( \frac{1}{\alpha} m_d \right) - \Gamma^{a}_{cd} \frac{1}{\alpha} m_a$$

$$= \frac{1}{\alpha} \nabla_c m_d + m_d \partial_c \frac{1}{\alpha} - \frac{1}{2} m^l \left( - \nabla_l q_{cd} + \nabla_c q_{dl} + \nabla_d q_{lc} \right),$$

where in the 1st equality we used $m_c^{(q)} = (1/\alpha) m_c$, and in the 2nd, besides the connection (27), the relation $q^{ai} m_a = \alpha m^i$. This gives

$$\bar{K}_{ab}^{(q)} = h^a_c h^d_b \left[ \frac{1}{\alpha} \nabla_c m_d + m_d \partial_c \frac{1}{\alpha} - \frac{1}{2} m^l \left( - \nabla_l q_{cd} + \nabla_c q_{dl} + \nabla_d q_{lc} \right) \right]$$

$$= \frac{1}{\alpha} \bar{K}_{ab} + \frac{1}{2} h^c_a h^d_b m^l \nabla_l q_{cd} - \frac{1}{2} h^c_a h^d_b m^l \left( \nabla_c q_{dl} + \nabla_d q_{lc} \right),$$

(C6)

where we used equation (C4) and the fact that $h^d_b m_d = 0$ (or that $h^c_a \partial_c \frac{1}{\alpha} = 0$ since $\alpha = \text{const on } \Sigma(P, \lambda)$).

From

$$m^l \nabla_l q_{cd} = m^l \left[ A g_{cd} - \frac{1}{2} \left( \frac{1}{\alpha} - A \right) (l_c m_d + m_c l_d) \right]$$

$$= \frac{dA}{dv} g_{cd} - \frac{1}{2} (l_c m_d + m_c l_d) \frac{d}{dv} \left( \frac{1}{\alpha} - A \right),$$

where we used equation (10) as applied alternately to $z^a = l^a$ or $z^a = m^a$, we get

$$h^c_a h^d_b m^l \nabla_l q_{cd} = \frac{dA}{dv} h_{ab}$$

(C7)

due to orthogonality of $m^a$ or $l^a$ to $\Sigma(P, \lambda)$ (cf. equation (C4)). Also,

$$m^l \nabla_c q_{dl} = m_d \nabla_c A + m_d \nabla_c \left( \frac{1}{\alpha} - A \right) - \frac{1}{2} \left( \frac{1}{\alpha} - A \right) m^l \nabla_c (l_d m_l + m_d l_l)$$

$$= m_d \nabla_c \frac{1}{\alpha} - \frac{1}{2} \left( \frac{1}{\alpha} - A \right) m^l \nabla_c (m_d l_l)$$

$$= m_d \nabla_c \frac{1}{\alpha} + \left( \frac{1}{\alpha} - A \right) \nabla_c m_d - \frac{1}{2} \left( \frac{1}{\alpha} - A \right) m^l m_d \nabla_c l_l,$$

where in the 2nd step we used equation (13); this gives

$$h^c_a h^d_b m^l \left( \nabla_c q_{dl} + \nabla_d q_{lc} \right) = h^c_a h^d_b \left( \frac{1}{\alpha} - A \right) (\nabla_c m_d + \nabla_d m_c)$$

$$= 2 \left( \frac{1}{\alpha} - A \right) \bar{K}_{ab},$$

(C8)

where the 1st step comes from orthogonality of $m^a$ to $\Sigma(P, \lambda)$, and the 2nd from the symmetry of $\bar{K}_{ab}$ (which is manifest for example thinking to the geodesic congruence of tangent $m^a$ on $\Sigma(P, \lambda)$, which is orthogonal to the
hypersurface \( C \times \Sigma(P, \lambda) \) at any \( p \in L \), with \( C \) any null curve with tangent \( m^a \) at \( p \). Inserting (C7) and (C8) in equation (C6), and using (7) with \( \Phi = A \), we get

\[
\dot{K}^{(q)}_{ab} = A \dot{K}_{ab} + \frac{1}{2} \frac{dA}{d\nu} h_{ab} = A \dot{K}_{ab} - \frac{1}{2} \frac{dA}{d\lambda} h_{ab},
\]

which is equation (12).

Let us consider finally the quantity \( K^{(q)}_{ab} \dot{K}^{(q)}_{ab} \) (of equation (44)). To evaluate it, we need an expression for \( K^{(q)}_{ab} \). This is

\[
K^{(q)}_{ab} = q^{ac} q^{bd} K^{(q)}_{cd} = q^{ac} q^{bd} \left( \alpha A K_{cd} + \frac{1}{2} \alpha \frac{dA}{d\lambda} h_{cd} \right) = \frac{1}{A} g^{ac} \frac{1}{A} g^{bd} \left( \alpha A K_{cd} + \frac{1}{2} \alpha \frac{dA}{d\lambda} h_{cd} \right) = \frac{\alpha}{A} \left[ K^{ab} + \frac{1}{2} \left( \frac{d}{d\lambda} \ln A \right) h^{ab} \right],
\]

where we used equation (40) in the 2nd step, and \( t^a K_{ab} = 0 = m^a K_{ab} \) and \( t^a h_{ab} = 0 = m^a h_{ab} \) in the 3rd. This gives

\[
K^{(q)}_{ab} \dot{K}^{(q)}_{ab} = \frac{\alpha}{A} \left[ K^{ab} + \frac{1}{2} \left( \frac{d}{d\lambda} \ln A \right) h^{ab} \right] \left( A \dot{K}_{ab} - \frac{1}{2} \frac{dA}{d\lambda} h_{ab} \right) = \alpha K^{ab} \dot{K}_{ab} + \frac{1}{2} \alpha \left( \frac{d}{d\lambda} \ln A \right) (\dot{K} - K) - \frac{1}{4} (D - 2) \alpha \left( \frac{d}{d\lambda} \ln A \right)^2,
\]

where we used of \( K^{ab} h_{ab} = K_{ab} h^{ab} = K_{ab} [g^{ab} + (1/2) t^a m^b + (1/2) m^a t^b] = K_{ab} g^{ab} = K \) and, analogously, \( \dot{K}_{ab} h^{ab} = \dot{K} \).

Appendix D: Derivation of equations (46) and (47)

\( l^a \) and \( m^a \) are quantities determined by spacetime geometry alone. This means that they cannot have an own dependence on time if the spacetime is static, and in a non-static spacetime they must have a vanishing first derivative with respect to time in the locally static observer’s frame at \( p \in L \).

Since \( m^a = 2 V^a - l^a \), then \( \nabla_a m_b = 2 \nabla_a V_b - \nabla_a l_b \). But, in the local frame at \( p \), \( V^b = (0, 0, 0, 0) \), \( (D - 1) \) zeros), up to terms which are quadratic in the displacements from \( p \). This gives

\[
\partial_0 m_b = -\partial_0 l_b = 0 \quad \partial_\alpha m_b = -\partial_\alpha l_b
\]

\( (\alpha = 1, ..., D - 1) \) in the local frame, and thus \( \nabla_a m_b = -\nabla_a l_b \) frame-independently. Then, we have \( \dot{K}_{ab} = h^c_a h^d_b \nabla_c m_d = -h^c_a h^d_b \nabla_c l_d = -K_{ab} \), which is equation (46).

Concerning how the expression for \( dK/d\lambda \) (equation (47)) comes about, we have

\[
\frac{dK}{d\lambda} = l^a \nabla_a K = l^a \nabla_a \nabla_b l^b = l^a \nabla_b \nabla_a l^b + l^a R_{nab} l^n = l^a \nabla_b \nabla_a l^b - R_{ab} l^a l^b = \nabla_b (l^a \nabla_a l^b) - \nabla_b l^a \nabla_a l^b - R_{ab} l^a l^b,
\]
where in the 2nd equality we used equation (A5), and in the 3rd the relation $\nabla_d \nabla_c w^m - \nabla_c \nabla_d w^m = -R^m_{ncd} w^n$ for any vector field $w^m$.

Here, $\nabla_b (L^a \nabla^b) = 0$ for the vector $v^b = L^a \nabla_a t^b$ is identically vanishing on $L$ and, from equation (10) as applied to $z^b = v^b$, also its derivative along $m^a$ vanishes at $p$ (cf. what noted right above equations (17), (18)). Moreover,

$$
\nabla_b a^a \nabla_a t^b = g^{af} g^{bg} \nabla_b f \nabla_a g = (h^{af} - \frac{1}{2} l^a m^f - \frac{1}{2} m^a l^f) \left( h^{bg} - \frac{1}{2} l^b m^g - \frac{1}{2} m^b l^g \right) \nabla_b f \nabla_a g = h^{af} h^{bg} \nabla_b f \nabla_a g = h^{af} h^{bg} \nabla_b g \nabla_a f = R^m_{l m},
$$

where the 3rd equality comes from being $l^a$ parallely displaced along $\gamma$ and from equation (12) (or from equation (10) with $z^b = t^b$), the 4th from equation (A8) and $h^{ab} l^a = 0$, and the 5th from $h^{af} h^{bg} = h^{af}$. From this, we get equation 47.

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