On the existence of paths between points in high level excursion sets of Gaussian random fields

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Let $X = (X(t), t \in \mathbb{R}^d)$ be a real-valued sample continuous Gaussian random field.

Given a level $u$, the excursion set of $X$ above the level $u$ is the random set

$$A_u = \{ t \in \mathbb{R}^d : X(t) > u \}.$$ 

Much is known about the structure of the excursion set when the field is smooth, and $u$ is large (Adler and Taylor (2007), Azaïs and Wschebor (2009)).
Consider a large level $u$.

**Question:** given that two points in $\mathbb{R}^d$ belong to the excursion set, what is the probability that they belong to the same path connected component of the excursion set?

Let $a, b \in \mathbb{R}^d$, $a \neq b$. A path in $\mathbb{R}^d$ connecting $a$ and $b$ is a continuous map $\xi : [0, 1] \rightarrow \mathbb{R}^d$ with $\xi(0) = a$, $\xi(1) = b$.

Denote the collection of all such paths by $\mathcal{P}(a, b)$. Estimate

$$P \left( \exists \xi \in \mathcal{P}(a, b) : X(\xi(v)) > u, \ 0 \leq v \leq 1 \mid X(a) > u, X(b) > u \right).$$
The non-trivial part of the problem: estimate the probability

$$\Psi_{a,b}(u) := P(\exists \xi \in \mathcal{P}(a,b) : X(\xi(v)) > u, \ 0 \leq v \leq 1).$$

If the random field is stationary, we may assume that $b = 0$, and use the notation $\Psi_a$.

If the domain of a random field is restricted to $T \subset \mathbb{R}^d$, and $a, b$ are in $T$, we consider

$$\Psi_{a,b}(u) = P(\exists \xi \in \mathcal{P}(a,b) : \xi(v) \in T \ and \ X(\xi(v)) > u, \ 0 \leq v \leq 1).$$
Large deviations setup

Let $A$ be the open set

$$A \equiv A_{a,b} := \left\{ \omega \in C_0(\mathbb{R}^d) : \exists \xi \in P(a, b), \omega(\xi(v)) > 1, 0 \leq v \leq 1 \right\},$$

$$C_0(\mathbb{R}^d) = \left\{ \omega = (\omega(t), t \in \mathbb{R}^d) \in C(\mathbb{R}^d) : \lim_{\|t\| \to \infty} \omega(t)/\|t\| = 0 \right\}.$$

We can write for $u > 0$

$$\Psi_{a,b}(u) = P\left(u^{-1}X \in A \right),$$

and use the large deviations results for Gaussian measures of Deutschel and Stroock (1989).
The reproducing kernel Hilbert space (RKHS) \( \mathcal{H} \) of the random field \( \mathbf{X} \), is a subspace of \( C(\mathbb{R}^d) \).

Consider the space of finite linear combinations \( \sum_{j=1}^{k} a_j X(t_j) \) \( a_j \in \mathbb{R}, \ t_j \in \mathbb{R}^d \) for \( j = 1, \ldots, k, \ k = 1, 2, \ldots \).

Its closure \( \mathcal{L} \) in the mean square norm is identified with \( \mathcal{H} \) via the injection \( \mathcal{L} \to C(\mathbb{R}^d) \) given by

\[
H \to w_H = \left( E(X(t)H), \ t \in \mathbb{R}^d \right)
\]

and the resulting norm

\[
\|w_H\|_{\mathcal{H}}^2 = E(H^2).
\]
Let $X$ be stationary with spectral measure $F_X$.

The RKHS $\mathcal{H}$ can be identified with the subspace of $L^2(F_X)$ of functions with even real parts and odd imaginary parts, via the injection $L^2(F_X) \to C_0(\mathbb{R}^d)$ given by

$$h \to S(h) = \left( \int_{\mathbb{R}^d} e^{i(t,x)} \bar{h}(x) F_X(dx), \ t \in \mathbb{R}^d \right),$$

with the resulting norm

$$\|S(h)\|_{\mathcal{H}}^2 = \|h\|_{L^2(F_X)}^2 = \int_{\mathbb{R}^d} \|h(x)\|^2 F_X(dx).$$
Theorem 1  Let $X = (X(t), t \in \mathbb{R}^d)$ be a continuous stationary Gaussian random field, with covariance function satisfying

$$\limsup_{\|t\| \to \infty} R_X(t) \leq 0.$$ 

Then

$$\lim_{u \to \infty} \frac{1}{u^2} \log \Psi_a(u) = -\frac{1}{2} C_X(a),$$

where

$$C_X(a) := \inf \left\{ \int_{\mathbb{R}^d} \|h(x)\|^2 F_X(dx) : \text{for some } \xi \in \mathcal{P}(0, a) \right\},$$

$$\int_{\mathbb{R}^d} e^{i(\xi(v), x)} \bar{h}(x) F_X(dx) > 1, 0 \leq v \leq 1.$$
The constraints in the optimization problem in Theorem 1 are not convex. However, for a fixed path, the constraints are convex, and one can use the convex Lagrange duality.

**Theorem 2**  For a continuous stationary Gaussian random field $X$,

$$
\mathcal{C}_X(a) = \left[ \sup_{\xi \in \mathcal{P}(0,a)} \min_{\mu \in M_1^+([0,1])} \int_0^1 \int_0^1 R_X(\xi(u) - \xi(v)) \mu(du) \mu(dv) \right]^{-1}.
$$

Here $M_1^+([0,1])$ is the space of all probability measures on $[0,1]$.

An optimal path is a path of maximal $R_X$ capacity.
Assume a path $\xi \in \mathcal{P}(0, a)$ is fixed. Then the minimization problem

$$\min_{\mu \in \mathcal{M}_1^+([0,1])} \int_0^1 \int_0^1 R_X(\xi(u), \xi(v)) \mu(du) \mu(dv)$$

is the problem of finding a probability measures $\mu$ of minimal energy, or capacitory measures.

The set $\mathcal{W}_\xi \subseteq \mathcal{M}_1^+([0,1])$ over which the minimum is achieved is a weakly compact convex subset of $\mathcal{M}_1^+([0,1])$. 
If the feasible set for the primary problem is non-empty then, for every $\varepsilon > 0$,

$$P \left( \sup_{0 \leq v \leq 1} \left| \frac{1}{u} X(\xi(v)) - x_\xi(v) \right| \geq \varepsilon \bigg| X(\xi(v)) > u, \ 0 \leq v \leq 1 \right) \to 0$$

as $u \to \infty$. Here $H_\xi$ is primary optimal, and

$$x_\xi(v) = E \left[ X(\xi(v)) H_\xi \right], \ 0 \leq v \leq 1.$$

Furthermore, there is a characterization of the optimal $H_\xi \in \mathcal{L}$ and the optimal $\mu \in \mathcal{W}_\xi$. 
**Theorem 3**

(i) For every $\mu \in \mathcal{W}_\xi$ we have

$$H_\xi = C_X(a, b; \xi) \int_0^1 X(\xi(v)) \mu(dv)$$

with probability 1.

(ii) A probability measure $\mu \in M_1^+([0, 1])$ is a measure of minimal energy if and only if

$$\min_{0 \leq v \leq 1} \int_0^1 R_X(\xi(u), \xi(v)) \mu(du) = \int_0^1 \int_0^1 R_X(\xi(u_1), \xi(u_2)) \mu(du_1) \mu(du_2) > 0.$$
Remarks

- By Theorem 3, the function

\[ v \mapsto \int_0^1 R_x(\xi(u), \xi(v)) \mu(du), 0 \leq v \leq 1, \]

is constant on the support of any measure \( \mu \in \mathcal{W}_\xi \). This seems to indicate that the support of any measure of minimal energy may not be ‘large’. However, this intuition holds only in some cases.

- If the random field is stationary, and the spectral measure is of the full support, then the image of any measure \( \mu \in \mathcal{W}_\xi \) on the path \( \xi \) is unique.
**One-dimensional case**

In this case there is, essentially, a single path between two points.

Let $X = (X(t), t \in \mathbb{R})$ be a stationary continuous Gaussian process. We are interested in understanding how the probability

$$
\Psi_a(u) = P\left(X(t) > u, \, 0 \leq t \leq a\right)
$$

changes with $a > 0$, and what happens with the optimal probability measures $\mu_a$ and limiting shapes $x_a$. 
For some processes (in particular, those with a finite second spectral moment), on short intervals we get an easy description.

**Proposition 1** Suppose that for some \( a > 0 \)

\[
R_X(t) + R_X(a - t) \geq R_X(0) + R_X(a) > 0 \quad \text{for all} \quad 0 \leq t \leq a.
\]

Then a measure in \( \mathcal{W}_a \) is given by

\[
\mu = \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1.
\]

Furthermore,

\[
C_X(a) = \frac{2}{R_X(0) + R_X(a)},
\]

\[
x_a(t) = \frac{R_X(t) + R_X(a - t)}{R_X(0) + R_X(a)}, \quad 0 \leq t \leq a.
\]
In any case, the measure \( \mu = (\delta_0 + \delta_1)/2 \) does NOT remain optimal for longer intervals.

**Example 1** Consider the centered stationary Gaussian process with the Gaussian covariance function

\[
R(t) = e^{-t^2/2}, \ t \in \mathbb{R}.
\]

Since the spectral measure has a Gaussian spectral density which is of full support in \( \mathbb{R} \), for every \( a > 0 \) there is a unique (symmetric) measure of minimal energy. Furthermore, the second spectral moment is finite, so that, for \( a > 0 \) sufficiently small this process satisfies the conditions of Proposition 1.
The measure $\mu = (\delta_0 + \delta_1)/2$ remains optimal for $a \leq a_1 \approx 2.2079$. 

![Graphs showing limiting shapes for different values of $a$.](image-url)
In the next regime the optimal measure acquires a point in the middle of the interval. This continues for $a_1 < a \leq a_2 \approx 3.9283$. 

![Graphs showing limiting shapes for different values of $a$.](image)
In the next regime the middle point of the optimal measure splits in two and starts moving away from the middle. This continues for $a_2 < a \leq a_3 \approx 5.4508$. 
**Example 2** Consider an Ornstein-Uhlenbeck process, i.e. a centered stationary Gaussian process with the covariance function

\[ R(t) = e^{-|t|}, \quad t \in \mathbb{R}. \]

For this process the spectral measure has a Cauchy spectral density, which has a full support in \( \mathbb{R} \). Therefore, for every \( a > 0 \) there is a unique (symmetric) measure of minimal energy.

In this case even the first spectral moment is infinite. Proposition 1 does not apply here.
The optimal probability measure is

$$\mu = \frac{1}{a + 2} \delta_0 + \frac{1}{a + 2} \delta_1 + \frac{a}{a + 2} \lambda,$$

where $\lambda$ is Lebesgue measure on $(0, 1)$.

There are no phase transitions. We have

$$C_X(a) = (a + 2)/2 \text{ for all } a > 0$$

and the limiting shape $x_a$ is identically equal to 1 on $[0, a]$. 
Long intervals

As the length $a$ of the interval increases, a difference between short and long memory processes arises. In the short memory case, the uniform measure is asymptotically optimal.

**Theorem 4** Assume that $R_X$ is positive, and satisfies

$$\int_0^\infty R(t) \, dt < \infty.$$ 

Then, with $\lambda$ denoting the uniform probability measure on $[0, 1]$, 

$$\lim_{a \to \infty} \frac{1}{a} C_X(a) = \left( \lim_{a \to \infty} a \int_0^1 \int_0^1 R_X(a(u - v)) \, \lambda(du)\lambda(dv) \right)^{-1}$$ 

$$= \frac{1}{2 \int_0^\infty R(t) \, dt}.$$
In the long memory case the uniform measure is no longer asymptotically optimal. We will assume that the covariance function of the process is regularly varying at infinity:

\[ R_X(t) = \frac{L(t)}{|t|^\beta}, \quad 0 < \beta < 1, \]

where \( L \) is slowly varying at infinity.

Consider the minimization problem with respect to Riesz kernel,

\[
\min_{\mu \in M_1^+([0,1])} \int_0^1 \int_0^1 \frac{\mu(du)\mu(dv)}{|u - v|^\beta} \quad 0 < \beta < 1.
\]

An optimal measure \( \mu_\beta \) exists, but it is different from the uniform measure.
**Theorem 5**  Assume that $R_X$ is positive and regularly varying. Then for any $\mu_\beta \in \mathcal{W}_\beta$, the set of optimal measures for the Riesz kernel,

$$\lim_{a \to \infty} R_X(a)C_X(a) = \left( \int_0^1 \int_0^1 \frac{\mu_\beta(du)\mu_\beta(dv)}{|u - v|^\beta} \right)^{-1}.$$

In particular, $C_X(a)$ is regularly varying with exponent $\beta$. 