Towards the definition of metric hyperbolicity

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To Ya. G. Sinai for his 70th anniversary.

Abstract

We introduce measure-theoretic definitions of hyperbolic structure for measure-preserving automorphisms. A wide class of $K$-automorphisms possesses a hyperbolic structure; we prove that all $K$-automorphisms have a slightly weaker structure of semi-hyperbolicity. Instead of the notions of stable and unstable foliations and other notions from smooth theory, we use the tools of the theory of polymorphisms. The central role is played by polymorphisms associated with a special invariant equivalence relation, more exactly, with a homoclinic equivalence relation. We call an automorphism with given hyperbolic structure a hyperbolic automorphism and prove that it is canonically quasisimilar to a so-called prime nonmixing polymorphism. We present a short but necessary vocabulary of polymorphisms and Markov operators from [11, 12].

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1 Motivations and statement of the problem

The theory of hyperbolic dynamical systems is one of the main achievements in the theory of dynamical systems of the second half of the last century. Although the basic concept appeared as far ago as in the papers by H. Poincaré and J. Hadamard and was discussed in many subsequent papers, and the main example — the geodesic flow on a surface of constant negative curvature — was known from the very beginning, and some “hyperbolic” effects (such as exponential rate of convergence and divergence, Lyapunov exponents, etc.) were known in terms of concrete differential equations (such as the Van der Pol equation, which was studied by G. Littlewood and M. Cartwright), but in the framework of the modern theory of dynamical systems, ergodic theory, and representation theory, it was considered only in the 40s–50s by E. Hopf and G. Hedlund and I. M. Gelfand and S. V. Fomin. The analysis of concrete examples gave an impulse to the general theory, which was formulated and axiomatized in the 60s by several authors (S. Smale, D. Anosov, and others). The definition of smooth hyperbolic systems involves the notions of Riemannian metric, stable and unstable foliations on a manifold, etc., which use the smooth structure of the phase space.

At the same time, connections of these ideas with the theory of stationary random processes were advocated already in the 40s by A. N. Kolmogorov, who considered this type of dynamical systems in a very wide context; he defined the notion of regular random stationary processes; apparently, he was the first to emphasize that the sigma-fields of the partitions with fixed “past” and “future” of stationary random processes are similar to pairs of horocycle foliations or geodesic flows on the unit tangent bundle of compact
surfaces of constant negative curvature. In 1958, Kolmogorov introduced a Shannon-type entropy as a metric invariant of dynamical systems and solved the long-standing problem on isomorphisms of Bernoulli systems. His pupil Ya. Sinai, together with V. A. Rokhlin and their schools, developed, in the 60s–70s, entropy theory and the theory of $K$-systems in the framework of ergodic theory and dynamical systems. Ya. Sinai’s contributions concerned not only the theory of dynamical systems, but much wider areas, including statistical physics, classical dynamics, ergodic hypothesis, and so on; his activity helped to combine dynamical theory with statistical physics and many others topics.

The link between classical hyperbolic systems and the class of $K$-systems became more clear in the 70s, after the papers by D. Ornstein appeared, who, starting from Sinai’s theorem on weak isomorphisms of Bernoulli automorphisms with the same entropy, proved a fundamental result that entropy is a complete invariant in the class of Bernoulli systems and gave an invariant definition of Bernoulli systems. This result allowed him together with B. Weiss [5] to prove the Bernoulli property of the geodesic flow on a compact surface of constant negative curvature. Bernoulli property of the hyperbolic automorphisms of torus also follows from that theory. Later, Ya. Pesin [6] proved the Bernoulli property for smooth hyperbolic systems in full generality. The existence of non-Bernoulli $K$-systems, which was discovered by Ornstein and Shields, and especially Kalikow’s example opened a new class of problems in dynamical theory.

But we can see a gap between hyperbolicity in the smooth category and $K$-property in the category of measure spaces — we have no purely measure-theoretic analogs of the notions of hyperbolic theory. The vague analogy between stable and unstable foliations on one hand and the “past” and “future” of a stationary $K$-process on the other have not been put into an appropriate general scheme.

To be more concrete, let us formulate our main problem:

The goal of this paper is to suggest a purely measure-theoretic definition of hyperbolic structure of measure-preserving automorphisms and to develop some tools for studying it.

In order to do this, we must overcome difficulties with definitions of objects that use the smooth and metric structures and the corresponding numerical characteristics.

To this end, we use the notion of polymorphism (= Markov, or multival-
ued, map in ergodic theory, see [11]) and the corresponding tools. This allows us to avoid problems with defining foliations and so on: roughly speaking, instead of foliations we consider polymorphisms which are associated with them. This allows us to transfer metrical and topological questions, including estimations, from the manifold (phase space) to the space of transformations or operators. In order to formulate the main definition, we use the weak topology in the space of polymorphisms and Markov operators (Condition H and the definition of hyperbolicity, see Sec. 3).

Roughly speaking, a hyperbolic structure for an automorphism with invariant measure is a polymorphism associated with an invariant equivalence relation which plays the role of the homoclinic equivalence. We called such automorphisms hyperbolic automorphisms; the definition is metrically invariant. An automorphism can have several hyperbolic structures or none at all. Presumably, a hyperbolic automorphism must be a \( K \)-automorphism satisfying an additional property (property \((*)\), see Sec. 3.3 and the problem in Sec. 3.7). We also define a weaker notion of semi-hyperbolic structure, which could be related to the notion of partial hyperbolic systems in the sense of Pesin (see [6]).

An extremely important notion closely related to this topic is the notion of quasi-similarity; one of our main results claims that the hyperbolicity of an automorphism \( T \) implies its quasi-similarity with a prime nonmixing and non-co-mixing polymorphism (see Sec. 3.2). The notion of quasi-similarity came from the theory of contractions in Hilbert spaces (see [4]) and scattering theory ([3]) and was not used earlier in the theory of dynamical systems. One of the motivations of this paper is to study the interrelation between classical dynamical systems and polymorphisms and apply it to hyperbolic theory; in particular, to investigate the notion of quasi-similarity between polymorphisms and automorphisms. The theory of polymorphisms and Markov operators has also a direct contact with the theory of Markov processes, which can be used for refining some of our results. We discussed this question briefly in [11, 12].

We start in Sec. 2 with briefly recalling some of the notions concerning polymorphisms and Markov operators. Section 3 contains our main results. We give definitions and first corollaries of hyperbolicity in Sec. 3.1 and a theorem on quasi-similarity in Sec. 3.2; explain how to include classical examples into our approach in Sec. 3.3; present a geometrical interpretation of quasi-similarity in Sec. 3.4; give the definition of semi-hyperbolic structures in Sec. 3.5; and prove that all \( K \)-automorphisms have a semi-hyperbolic
structure in Sec. 3.6. Two questions from a large list of open problems are presented in Sec. 3.7.

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2 Vocabulary of polymorphisms and Markov operators

2.1 Polymorphisms

In order to make this paper independent, we give a short list of definitions concerning polymorphisms and Markov operators. The reader can find a detailed version in [11]. In other areas of mathematics, notions parallel to that of polymorphism are: correspondence in algebra and algebraic geometry; bifibration in differential geometry, Markov map in probability theory, Young measure in optimal control, etc. Equivalent definitions of the notions under consideration in terms of Markov operators will be presented in the next section.

Definition 1. A polymorphism with invariant measure $\Pi$ of a Lebesgue space $(X, \mu)$ to itself is a diagram consisting of an ordered triple of Lebesgue spaces:

$$(X, \mu) \overset{\pi_1}{\leftarrow} (X \times X, \nu) \overset{\pi_2}{\rightarrow} (X, \mu),$$

where $\pi_1$ and $\pi_2$ stand for the projections to the first and second component of the product space $(X \times X, \nu)$, and the measure $\nu$, which is defined on the $\sigma$-field generated by the product of the $\sigma$-fields of mod0 classes of measurable sets in $X$, is such that $\pi_i \nu = \mu$, $i = 1, 2$. The measure $\nu$ is called the bistochastic measure of the polymorphism $\Pi$.

Let us define the main structures on the set of the polymorphisms and the notions of the theory of the polymorphisms.

1. A polymorphism $\Pi^*$ is called conjugate to the polymorphism $\Pi$ if its diagram is obtained from the diagram of $\Pi$ by reflecting with respect to the central term. If polymorphism is an automorphism then conjugate polymorphism is nothing more than inverse automorphism.

Consider the “vertical” partition $\xi_1$ and the “horizontal” partition $\xi_2$ of the space $(X \times X, \nu)$ into the preimages of points under the projections
\(\pi_1\) and \(\pi_2\), respectively. In terms of bistochastic measures, the value of a polymorphism at a point \(x \in X\) is a conditional measure. More precisely, we have the following definition.

**Definition 2.** In the above notation, the value of the polymorphism \(\Pi : X \to X\) at a point \(x_1 \in X\) is, by definition, the conditional measure \(\nu^{x_1}\) of \(\nu\) on the set \(\{(x_1, \cdot)\}\) with respect to the vertical partition \(\xi_1\) (the transition probability); similarly, the value of the conjugate polymorphism \(\Pi^*\) at a point \(x_2 \in X\) is the conditional measure \(\nu_{x_2}\) of \(\nu\) on the set \(\{\cdot, x_2\}\) with respect to the horizontal partition \(\xi_2\) (the cotransition probability). These systems of conditional measures are well-defined on sets of full \(\nu\)-measure. Thus a polymorphism is a mod 0 class of measurable maps from \((X, \mu)\) to the space of measures on \(X\) equipped with the ordinary Borel structure.

2. The set of “images” of the points \(x \in X\) under a polymorphism is the system of conditional (transition) measures \(\nu^x\) on \(X\). It is important that this system is defined up to measure zero, so there is no sense in the measure-theoretic category of “individual” image, but only the system of images as a whole makes sense. If almost all measures \(\nu^x\) are delta-measures, then we have a deterministic measure-preserving map. Denote the set of polymorphisms of a given space \((X, \mu)\) by \(\mathcal{P}(X, \mu) = \mathcal{P}\).

3. A multiplication in the set of polymorphisms (bistochastic measures) \(\mathcal{P}\) is defined as follows: let \(\Pi_1, \Pi_2\) be two polymorphisms with bistochastic measures \(\nu_1, \nu_2\); then the product \(\Pi_1 \Pi_2\) has the bistochastic measure \(\nu\) defined by

\[
\nu^x(A) = \int \nu_1^y(A) d\nu_2^x(y).
\]

The ordinary weak topology on the set of polymorphisms (a neighborhood of the identity is the set of polymorphisms whose bistochastic measures are in a neighborhood of the diagonal measure) endows it with the structure of a compact topological semigroup. This semigroup has a unity (the identity map), involution (conjugacy), zero element \(\Theta\) (the bistochastic product measure \(\nu = \mu \times \mu\)), and a natural convex structure on the set of bistochastic measures. The subgroup of invertible elements of the semigroup \(\mathcal{P}\) is the group of measure-preserving transformations. The set of all polymorphisms of a finite space is the convex semigroup of bistochastic matrices. The convex semigroup of all polymorphisms of a Lebesgue space with a continuous
measure is the inverse limit of the sequence of convex compact spaces of bistochastic matrices.

Now define the classes of polymorphisms, factor polymorphisms, ergodicity, mixing, primeness and quasisimilarity of polymorphisms.

4. A measurable partition $\xi$ is called invariant under a polymorphism $\Pi$ if for almost all elements $C \in \xi$ there exists another element $D \in \xi$ such that for almost all (with respect to the conditional measure on $C$) points $x \in C$, we have $\mu^x(D) = 1$, where $\mu^x$ is the $\Pi$-image of $x$. In other words, the factor polymorphism $\Pi_\xi$ of $\Pi$ with respect to an invariant partition $\xi$ is an endomorphism of the space $(X_\xi, \mu_\xi)$.

In particular, if for almost all elements $C \in \xi$ of $\xi$ we have $\mu^x(C) = 1$ for almost all $x \in C$, then $\xi$ is called a fixed partition for $\Pi$ and the corresponding factor polymorphism is the identity map on $X_\xi$. A polymorphism is ergodic if it has no nontrivial identity factor automorphism.

5. For a given polymorphism $\Pi$, of a space $(X, \mu)$, with bistochastic measure $\nu$, the factor polymorphism (or quotient) of $\Pi$ by a measurable partition $\xi$ is the polymorphism of the space $(X/\xi, \mu/\xi)$ to itself with the factorized bistochastic measure $\nu/(\xi \times \xi)$. Thus the factor polymorphism of any polymorphism by any measurable partition does exist; in particular, the factor polymorphism of any automorphism by any (not necessarily invariant) partition always does exist.

A polymorphism is called prime if it has no nontrivial invariant partition, or has no nonzero factor endomorphism. A polymorphism is called coprime if its conjugate is prime. (Compare this notions with the notions of a prime automorphism and an exact endomorphism.)

6. A polymorphism $\Pi$ is called mixing if the sequence of its powers tends to the zero polymorphism $\Theta$ in the weak topology: $\lim_{n \to \infty} \Pi^n = \Theta$. Note that it may happen that a polymorphism is mixing while its conjugate is not. We call a polymorphism co-mixing if its conjugate is a mixing polymorphism.

7. A polymorphism $\Pi$ is called semi-dense if a measurable function for which $\int f(y) d\nu^x(y) = 0$ for $\mu$-almost all $x$ is equal to zero; a polymorphism $\Pi$ is dense if both $\Pi$ and $\Pi^*$ are semi-dense. It is more convenient to express density in terms of Markov operators (see the next subsection).

8. A polymorphism $\Pi$ is called nondegenerate if for almost all $x$, the conditional measure $\nu^x$ of $\Pi$ is not a delta-measure.

**Remark.** Sometimes it is more convenient to regard a partition as an equivalence relation (congruence); we will not distinguish an equivalence relation...
on \((X, \mu)\) and the corresponding partition of \((X, \mu)\), and will denote them by the same letter.

9. We say that a polymorphism \(\Pi\) is associated with a partition \(\xi\) (measurable or not) if for almost all \(x\) we have \(\nu_x(\xi(x)) = 1\), where \(\xi(x)\) is the element of \(\xi\) that contains \(x\). In other words, the polymorphism acts along the blocks of the partition. Each automorphism is associated with its orbit partition (see \([11]\)).

10. A polymorphism (in particular, automorphism) \(\Pi\) is a quasi-image (see an analog of this notion in \([4]\)) of a polymorphism or automorphism \(\Psi\) if there exists a dense polymorphism \(\Lambda\) such that \(\Lambda\Pi = \Psi\Lambda\). If \(\Psi\) is also a quasi-image of \(\Pi\), then we say that \(\Pi\) and \(\Psi\) are quasi-similar. Quasi-similarity is a much more rough equivalence than similarity, for example, mixing is not an invariant of quasi-similarity. The classification of automorphisms (e.g., of \(K\)-automorphisms) up to quasi-similarity is a very intriguing problem; one problem of such a type: is entropy of automorphism an invariant under quasisimilarity? But for further discussions it is especially important that an automorphism may be quasi-similar to a polymorphism; our main definition uses this fact in a very crucial way (see the definition of hyperbolicity).

2.2 Markov operators

The functional analog of the notion of polymorphism is the notion of Markov operator in the Hilbert space \(L^2_\mu(X)\) (see the classical theory in \([14, 12]\)).

**Definition 3.** A Markov operator in the Hilbert space \(L^2(X, \mu)\) of complex-valued square integrable functions on a Lebesgue–Rokhlin space \((X, \mu)\) with a continuous normalized measure \(\mu\) is a continuous linear operator \(V\) satisfying the following conditions:

1) \(V\) is a contraction: \(\|V\| \leq 1\) (in the operator norm);
2) \(V1 = V^*1 = 1\), where \(1\) is the function identically equal to one;
3) \(V\) preserves the nonnegativity of functions: \(Vf\) is nonnegative whenever \(f \in L^2(X, \mu)\) is nonnegative.

Note that condition 1) follows from 2) and 3), and the second condition in 2) follows from the other ones. In short: a Markov operator is a unity-preserving positive contraction.

The set \(\mathcal{M}\) of all Markov operators is a convex weakly compact semigroup with involution \(V \to V^*\). Unitary (isometric) Markov operators are precisely the operators generated by measure-preserving auto(endo)morphisms.
Proposition 1. 1. Let $\Pi$ be a polymorphism of a space $(X, \mu)$ with invariant measure; then the formula

$$(V_{\Pi}f)(x) = \int_X f(y)\mu^x(dy)$$

correctly defines a Markov operator in $L^2_{\mu}(X)$.

2. Every Markov operator $V$ in the space $L^2_{\mu}(X)$, where $(X, \mu)$ is a Lebesgue space with continuous finite measure, can be represented in the form $V = V_{\Pi}$, where $\Pi$ is a polymorphism of $(X, \mu)$ with invariant measure.

3. The correspondence $\Pi \mapsto V_{\Pi}$ is a continuous (with respect to the weak topologies) antiisomorphism between the convex compact semigroup with involution of mod 0 classes of polymorphisms and the analogous semigroup of Markov operators.

The Markov operator $1 = W_\Theta$ corresponding to the zero polymorphism $\Theta$ is the orthogonal projection to the one-dimensional subspace of constants. The operator of mathematical expectation is, obviously, also a Markov operator; it corresponds to the polymorphism that sends a point $x$ to the conditional measure of the element of the partition (corresponding to the expectation) that contains $x$.

Now we reformulate the notions introduced for polymorphisms (ergodicity, mixing, primality, density, etc.) in terms of Markov operators.

1. A Markov operator $V$ is called mixing (resp. comixing) if the sequence $V^n$ (resp. $V^*n$) weakly tends, as $n \to \infty$, to the projection onto the subspace of constants:

$$V^n \to 1 = V_\Theta, \text{ (resp. } V^*n \to 1 = V_\Theta).$$

The Markov operator $V_{\Pi}$ is mixing (comixing) if and only if the polymorphism $\Pi$ is mixing (resp. comixing).

2. We will say that a Markov operator $V = V_{\Pi}$ is semi-dense if the $V$-image of the space $L^2_{\mu}(X)$ is dense in $L^2_{\mu}(X)$; this is equivalent to the triviality of the kernel of the conjugate operator and, consequently, to the semi-density of the polymorphism $\Pi$. A Markov operator $V$ is called dense if both ker $V$ and ker $V^*$ are trivial. The density of $V_{\Pi}$ is equivalent to the density of $\Pi$.

3. A Markov operator $V$ is a quasi-image of a Markov operator $W$ if there exists a semi-dense Markov operator $L$ such that $WL = LV$. Two Markov operators are quasi-similar if each of them is a quasi-image of the other.
one. Two Markov operators are quasisimilar if and only if the corresponding polymorphisms are quasisimilar.

4. A Markov operator $V$ is called totally nonisometric if there is no nonzero invariant subalgebra\footnote{More exactly, a subspace that consists of all functions from $L^2$ that are constant a.e. on all elements of some measurable partition, see \cite{7}.} in the orthogonal complement to the subspace of constants in $L^2(X)$ on which $V$ acts isometrically.

**Proposition 2.** A Markov operator $V\Pi$ is totally nonisometric if and only if $\Pi$ is a prime polymorphism.

The dual notions of coisometrical and noncoisometrical Markov operators and connections with coprime and noncoprime polymorphisms are defined in a natural way.

A mixing Markov operator is totally nonisometric, but we are interested in Markov operators that are far from isometries (in other words, in polymorphisms that are far from automorphisms) and far from mixing ones. Examples of prime nonmixing polymorphisms and, equivalently, totally nonisometric nonmixing operators play the key role in our theory; the existence of totally nonisometric nonmixing Markov operators is not a priori obvious.

In the terminology of the book \cite{4}, a totally nonisometric nonmixing Markov operator is a Markov operator of type $C_{1,1}$ (for the one-sided case, $C_{1,1}$, or $C_{1,1}$).

The first example of this type was given in \cite{14}; then in \cite{10} we suggested a general approach related to hyperbolic transformations, which we will use here (see also \cite{11}). We will return to this in Sec. 3.

## 3 Metric hyperbolic structure

### 3.1 Hyperbolic structure of a measure-preserving automorphism

In this section, we formulate the main definitions.

Suppose that $T$ is an ergodic measure-preserving automorphism of a Lebesgue space $(X, \mu)$. The following condition on the automorphism plays the key role in our considerations.
**Condition H.** There exists a $T$-invariant ergodic equivalence relation $\chi$ and a polymorphism $\Pi$ that can be represented as

$$\Pi = \Phi \cdot T,$$

where $\Phi$ is a nondegenerate polymorphism associated with the partition $\chi$; at the same time, the following limits (in the weak topology on the semigroup of polymorphisms) exist:

$$\Lambda = \lim_{n \to \infty} \Pi^n \cdot T^{-n}$$

(1)

and

$$\Gamma = \lim_{n \to \infty} T^{-n} \cdot \Pi^n.$$  

(2)

Besides, both polymorphisms $\Lambda$ and $\Gamma$ are dense.

**Definition 4.** A proper hyperbolic structure for an ergodic automorphism $T$ is an ergodic equivalence relation $\chi$ for which Condition H (the existence of a polymorphism $\Pi$, etc.) holds.

An automorphism $T$ for which there exists at least one hyperbolic structure will be called a hyperbolic automorphism; in this case, we will say that $\chi$ is a homoclinic equivalence relation for the automorphism $T$. The same partition $\chi$ defines a hyperbolic structure for the automorphism $T^{-1}$.

The convergence of the infinite products in formulas (1) and (2) above is the main condition of our construction; in a sense, it is equivalent to the existence of stable and unstable foliations in the classical smooth theory of hyperbolic systems. It is easy to check that the polymorphisms $\Lambda$ and $\Gamma$ are also associated with the partition $\chi$. Note that for a given hyperbolic structure $\chi$, the choice of a polymorphism $\Phi$ and, consequently, of a polymorphism $\Pi$ satisfying Condition H is not unique; of course, both limits $\Lambda$ and $\Gamma$ depend on the choice of $\Phi$.

Nevertheless, technically, the central role is played by the polymorphism $\Phi$ associated with the relation $\chi$; therefore, we will rewrite the above limits in several forms, using the polymorphism $\Phi$ instead of $\Pi$. Let $\Phi_k = T^k \Phi T^{-k}$, where $k \in \mathbb{Z}$, and $\Phi \equiv \Phi_0$; then we can rewrite these limits as

$$\Lambda = \lim_{n \to \infty} \Phi \cdot T \Phi T^{-1} \ldots T^n \Phi T^{-n},$$

(3)

\[\text{footnote:} \text{The notion of homoclinic equivalence relation for automorphisms was introduced and used by M. I. Gordin [1] for other purposes. His definition is different, and we will discuss its connections with our definition below and elsewhere.}\]
or
\[ \Lambda = \lim_{n \to \infty} \prod_{0}^{n} \Phi_k = \prod_{0}^{\infty} \Phi_k; \]  
(3')

analogously,
\[ \Gamma = \lim_{n \to \infty} T^{-n} \Phi T^{m} \cdots T^{-1} \Phi T \cdot \Phi, \]  
(4)

or
\[ \Gamma = \lim_{n \to \infty} \prod_{-n}^{0} \Phi_k = \prod_{-\infty}^{0} \Phi_k. \]  
(4')

Thus we obtain the following proposition.

**Proposition 3.** The convergence of two products in (3') and (4') to dense polymorphisms \( \Lambda \) and \( \Gamma \) (together with the condition that \( \Phi \) is a nondegenerate polymorphism associated with the \( T \)-invariant partition \( \chi \)) is equivalent to Condition \( H \).

In particular, we have the following important corollary.

**Corollary 1.** The sequence of polymorphisms \( \Phi_k = T^k \cdot \Phi \cdot T^{-k} \) weakly tends to the identity automorphism as \( |k| \to \infty \):
\[ \lim_{|k| \to \infty} T^k \cdot \Phi \cdot T^{-k} = \text{Id}. \]  
(\( \diamond \))

The question is what rate of convergence can have the left-hand side of (\( \diamond \)) for various examples.

### 3.2 Quasi-similarity of automorphisms and polymorphisms.

The most essential ingredient of our construction is the polymorphism \( \Pi \).

**Theorem 1.** Under Condition \( H \), the following formulas hold:
\[ \Pi \cdot \Lambda = \Lambda \cdot T \]  
(5)

and
\[ \Gamma \cdot \Pi = T \cdot \Gamma. \]  
(6)

If relations (5), (6) hold, then we can claim that the automorphism \( T \) is quasi-similar to the polymorphism \( \Pi \).
Proof. Because of the importance of equations (5),(6), we present some calculations. Using our notation and the above formulas, we can rewrite these equations as

\[
\Pi \cdot \Lambda = \Phi_0 \cdot T \cdot \lim_{n \to \infty} \prod_{k=0}^{n} \Phi_k = \lim_{n \to \infty} [\Phi T \cdot \Phi \cdot T \Phi^{-1} \ldots \ldots T^n \Phi T^{-n}]
\]

\[
= \lim_{n \to \infty} [\Phi \cdot T \Phi T^{-1} \cdot T^2 \Phi T^{-2} \ldots T^{n+1} \Phi T^{-(n+1)}] \cdot T = \lim_{n \to \infty} \prod_{k=0}^{n+1} \Phi_k \cdot T
\]

\[= \Lambda \cdot T;
\]
in a shorter form,

\[
\Pi \cdot \Lambda = \Pi \cdot \lim_{n \to \infty} \Pi^n T^{-n} = \lim_{n \to \infty} [\Pi^{n+1} T^{-(n+1)}] T = \Lambda \cdot T.
\]

Similarly,

\[
\Gamma \cdot \Pi = \lim_{n \to \infty} \prod_{k=0}^{n} \Phi_k \cdot \Phi_0 \cdot T = T \cdot \lim_{n \to \infty} \prod_{k=0}^{n+1} \Phi_k = T \cdot \Gamma,
\]
or

\[
\Gamma \cdot \Pi = \lim_{n \to \infty} [T^{-n} \cdot \Pi^n] \cdot \Pi = T \lim_{n \to \infty} T^{-(n+1)} \cdot \Pi^{(n+1)} = T \cdot \Gamma.
\]

The theorem follows from these equations and the definition of quasi-similarity, together with the above conditions on the density of the polymorphisms \(\Lambda\) and \(\Gamma\).

Now we can refine the properties of the polymorphism \(\Pi\).

**Theorem 2.** A polymorphism \(\Pi\) that satisfies Condition \(H\) is prime, non-mixing, and non-co-mixing. More exactly, if a polymorphism \(\Pi\) satisfies relations (1) and (2) with some ergodic automorphism \(T\) and dense polymorphisms \(\Lambda\) and \(\Gamma\), then it is prime, coprime, nonmixing, and non-comixing.

Proof. First we will prove that \(\Pi\) (or \(\Pi^*\)) is prime. Suppose that \(\Pi\) is not prime; this means that there exists a nontrivial \(\Pi\)-invariant measurable partition \(\zeta\) of \((X, \mu)\). We have \(T^{-n} \Pi^n \zeta = T^{-n} \zeta\), whence \(\gamma \equiv \Gamma \zeta = \lim_{n \to \infty} T^{-n} \zeta\); the existence of this limit is possible only if the partition \(\zeta\) is \(T\)-invariant and, consequently, \(\Gamma\)-invariant. Therefore, we can consider the actions of the automorphism \(T_\zeta\) and the endomorphism \(\Pi_\zeta\) on the quotient space \(X/\zeta\). Thus we reduce the problem to the following one.
Proposition 4. If the sequence of products $R^n \cdot S^{-n}$, where $R \neq \text{Id}$ is an endomorphism and $S \neq \text{Id}$ is an automorphism, tends to some limit in the weak topology, then $R = S$.

Proof. Consider the endomorphism $Q = S^{-1}R$; then (as above) we have

$$R^n \cdot S^{-n} = (QS)^n S^{-n} = Q \cdot SQS^{-1} \cdot S^2QS^{-2} \ldots S^{n-1}QS^{-n+1} \cdot S^{-1}.$$  

The existence of the limit means that the following weak limit also exists and is equal to the identity:

$$\lim_{n \to \infty} S^n \cdot Q \cdot S^{-n} = \text{Id}.$$  

But if $Q$ is an endomorphism, this can happen only if $S = \text{Id}$, which is not the case, or if $Q = \text{Id}$, which means that $R = S$.

The claim of the proposition is not true if $R$ is a polymorphism.

Now suppose that $\Pi$ is mixing. Recall (see [11, 12]) that each polymorphism naturally defines a Markov chain. We use the following observation (see [12]).

Proposition 5. The shift in the space of realizations of the Markov chain corresponding to a prime mixing polymorphism $\Pi$ is a $K$-automorphism, and the Markov generator is a $K$-generator.

Consequently, if $\Pi$ is mixing, then, in view of the $K$-property, the above limit is again the zero polymorphism:

$$\lim_{n \to \infty} \Pi^n \cdot T^{-n} = \Theta;$$

but this limit is equal to $\Lambda$, which is impossible. Thus $\Pi$ is not mixing; the same is true for $\Pi^*$. 

Note that the assertion converse to that of the proposition is also true, so this gives a criterion of $K$-generators of Markov chains. The question what prime nonmixing and non-co-mixing polymorphism $\Pi$ defines a hyperbolic structure with given polymorphism $\Pi$ requires more information on the properties of the Markov process generated by the polymorphism; we consider the corresponding construction elsewhere (see also [11]).
3.3 Classical examples of hyperbolic structures

**Theorem 3.** A smooth hyperbolic transformation of a compact manifold with finite invariant measure (Anosov system with discrete time) has a natural proper hyperbolic structure in the above sense.

*Proof.* Let $T$ be an Anosov transformation of a smooth compact manifold with an invariant measure; as an ergodic equivalence relation $\chi$ from the definition above, we choose the ordinary homoclinic equivalence relation: two points are equivalent if they belong to the same stable and unstable leaves. In the algebraic case — that of a hyperbolic automorphism of the torus $\mathbb{T}^n$ — the homoclinic partition is the orbit partition of the action of $\mathbb{Z}^{n-1}$, the Dirichlet group. It is an ergodic relation, because the corresponding partition has a trivial measurable hull. As a polymorphism $\Phi$, we can take a polymorphism for which the conditional measure $\nu^x$ at a point $x$ is a nondegenerate measure concentrated on a finite subset of the set of points homoclinic to the point $x$ — such a polymorphism $\Phi$ is associated with $\chi$ in the sense of our definition. Thus the polymorphism $\Pi$ sends a point $x$ to the homoclinic class of the point $Tx$. In order to prove that such a polymorphism $\Phi$ exists, or that one can find a measurable map $x \to \nu^x$, it suffices to choose two different measurable maps on the manifold, each associating with every point $x$ a point $y(x) \neq x$ homoclinic to $x$. The more serious part of the proof, the existence of the limits (1) and (2), or the existence of polymorphisms $\Lambda$ and $\Gamma$ above, was given in [10, 11]; see also Sec. 3.6. □

**Condition (\(*\)).** Let $T$ be a $K$-automorphism, and let $\xi$ be a finite or countable $K$-generator of $T$ satisfying the following additional property:

$$\bigwedge_{n=0}^{\infty} \bigvee_{|k|>n} T^k \xi = \nu; \quad (\ast)$$

here $\nu$ is the trivial measurable partition.

It is well known that not all $K$-generators of a $K$-automorphism, and even of a Bernoulli automorphism, satisfy property (\(*\)).

**Theorem 4.** If a $K$-automorphism satisfies condition (\(*\)), then it is hyperbolic.

*Proof.* Define an equivalence relation $\chi$ (= partition) as the nonmeasurable partition obtained as the set-theoretic intersection of the partitions from the
previous expression:
\[ \chi = \bigcap_{n=0}^{\infty} \bigvee_{|k|>n} T^k \xi. \] (**) 

If we realize the automorphism \( T \) as the right shift in the space of two-sided sequences (states of the process), then two sequences \( x = \{x_i\}, y = \{y_i\}, \) \( i \in \mathbb{Z}, \) belong to the same element of \( \chi \) if there exists \( k \in \mathbb{N} \) such that \( x_i = y_i \) for each \( i > |k|. \) As the limit of a decreasing sequence of measurable partitions with finite or countable blocks, \( \chi \) is a hyperfinite (or tame) partition. Condition (**) means that the equivalence relation \( \chi \) is ergodic. A direct construction of polymorphisms \( \Phi \) and \( \Pi \) with required properties is given in Sec. 3.6. Thus \( \chi \) determines a proper hyperbolic structure for the automorphism \( T. \)

**Remark.** For all \( K \)-automorphisms known at present (2005) there exists a \( K \)-generator satisfying property (\(*\)).\(^3\) The open question is whether such a generator exists for all \( K \)-automorphisms.

### 3.4 Geometrical interpretation

Relations (5) and (6) (together with (1) and (2)) have an important geometrical interpretation. We interpret conditions (5) and (6); below the measure \( \mu^\Psi_z \) is the \( \Psi \)-image of a point \( z \) (see definitions):

\[ \mu_T^{x_\Lambda}(\cdot) = \int \mu^{y_\Lambda}(\cdot) d\mu^x_\Lambda(y), \] (5')

\[ \mu^y_\Gamma(T \cdot) = \int \mu^x_\Lambda(\cdot) d\mu^y_\Gamma(x). \] (6')

But the action of a polymorphism on a given space can be naturally extended to the action on probability measures on the same space (convolution); using this, we can rewrite the formula as follows:

\[ \mu_T^{x_\Lambda}(\cdot) = (\Pi * \mu^x_\Lambda)(\cdot) \]

and, respectively,

\[ \mu^y_\Gamma(T \cdot) = (\mu^y_\Gamma * \Pi)(\cdot). \]

\(^3\)The author is grateful to Professor J.-P. Thouvenot for this information.
These equalities show that the right (resp., left) action of the automorphism $T$ on the set of measures $\mu^x_\Lambda$ (resp., $\mu^x_\Gamma$), $x \in X$, is the same as the left (resp., right) action of the polymorphism $\Pi$ on these sets of measures. This is an explanation of the quasi-similarity between the automorphism $T$ and the polymorphism $\Pi$.

The polymorphisms $\Lambda$ and $\Gamma$ are of special interest from the point of view of the corresponding Markov processes; see [11] and [10].

An ergodic automorphism can have several metrically nonisomorphic hyperbolic structures or have no such structures. The main problem is to characterize automorphisms that have hyperbolic structures and to classify these structures; this problem is new and has no answer up to now. We will present some results in this direction.

The definitions of hyperbolic structures, quasi-similarity, and other notions discussed above could be easily reformulated in terms of unitary and Markov positive operators in the space $L^2$ (see [12]). We restrict ourselves only to an operator reformulation of quasi-similarity.

Let $U_T$ be the unitary operator in $L^2$ corresponding to a measure-preserving automorphism $T$, and let $V_\Pi$ be the Markov operator corresponding to a polymorphism $\Pi$ (see definitions). Then, under conditions (3) and (4), the following limits of Markov operators in the weak operator topology do exist:

\[
\lim_{n \to \infty} V_\Pi^n U_T^{-n} = L, \quad \lim_{n \to \infty} U_T^{-n} V_\Pi^n = G,
\]

and the operators $U_T$ and $V_\Pi$ are quasi-similar:

\[
V_\Pi L = L U_T, \quad GV_\Pi = U_T G.
\]

Recall that the polymorphism $\Pi$ is nonmixing (non-co-mixing) and prime and, consequently, the Markov operator $V_\Pi$ is totally nonisometric and nonmixing (non-co-mixing), so this is an example of quasi-similarity between a unitary and a totally nonisometric operator. This is a positive analog of operators of class $C_{1,1}$ in the sense of [4]. The existence of such examples is not obvious.
3.5 Left and right semi-hyperbolic structures

Now we define structures that are weaker than the hyperbolic one. Let $T$ be an ergodic measure-preserving automorphism. We define left and right semi-hyperbolic structures.

**Condition SH.** There exists a $T$-invariant ergodic equivalence relation $\chi_l$ (resp., $\chi_r$) and a polymorphism $\Pi_l$ (resp., $\Pi_r$) that can be represented as

$$\Pi_l = \Phi_l \cdot T$$

(resp.,

$$\Pi_r = \Phi_r \cdot T$$)

where the polymorphism $\Phi_l$ (resp., $\Phi_r$) is associated with the partition $\chi_l$ (resp., $\chi_r$) and such that the following limits in the weak topology on the semigroup of polymorphisms exist:

$$\Lambda_l = \lim_{n \to \infty} \Pi_l^n \cdot T^{-n}$$

(1$_l$)

(resp.,

$$\Gamma_r = \lim_{n \to \infty} T^{-n} \cdot \Pi_r^n$$)

(2$_r$)

Besides, both polymorphisms $\Lambda_l$ and $\Gamma_r$ are dense.

As in Definition 4, the polymorphism $\Pi_l$ (resp., $\Pi_r$) is prime and non-mixing (resp., non-co-mixing).

**Definition 5.** An ergodic equivalence relation $\chi_l$ (resp., $\chi_r$) defines a left (resp., right) semi-hyperbolic structure of an automorphism $T$ if condition (1$_l$) (resp., 2$_r$) of Condition SH holds. If both conditions hold for some polymorphisms $\Pi_l$ and $\Pi_r$, we will say that they define a semi-hyperbolic structure for the automorphism $T$.

Similarly to Theorem 1, we have the following result.

**Theorem 5.** The following relations hold:

$$\Pi_l \cdot \Lambda_l = \Lambda_l \cdot T,$$

$$\Gamma_r \cdot \Pi_r = T \cdot \Gamma_r.$$
Consequently, the automorphism $T$ is a quasi-image of the polymorphism $\Pi_l$, and the polymorphism $\Pi_r$ is a quasi-image of the automorphism $T$.

If $T$ is the shift in the space of realizations of a stationary random process, one can take as $\chi_l$ or $\chi_r$ the partitions with fixed past or future of this process. In smooth hyperbolic theory, the partitions $\chi_l$ and $\chi_r$ can be chosen to be the stable and unstable foliations, respectively; in that theory, they exist simultaneously. In general, in the above definition there are no connections between the left and right semi-hyperbolic structures. Thus we can consider many variants and examples of semi-hyperbolic structures in the sense of our definition. If the left and right structures agree, in the sense that the supremum of two relations $\chi_l$ and $\chi_r$ (or, in terms of partitions, the product of the partitions $\chi_l$ and $\chi_r$) is an ergodic relation (resp., an ergodic partition), then we have a proper hyperbolic structure in the sense of the main definition of Sec. 3.1, and this product is a homoclinic partition.

### 3.6 Semi-hyperbolic structure of $K$-automorphisms

We will prove that every $K$-automorphism possesses a semi-hyperbolic structure.

**Theorem 6.** For every $K$-automorphism $T$ there exists a semi-hyperbolic structure, i.e., there exists a prime nonmixing polymorphism $\Pi_l$ that defines a left semi-hyperbolic structure of the automorphism $T$ and a prime non-co-mixing polymorphism $\Pi_r$ that defines a right semi-hyperbolic structure.

**Proof.** It is clear from the definition that if $T$ has a left (right) semi-hyperbolic structure, than it is a right (left) semi-hyperbolic structure for the automorphism $T^{-1}$. Now if $T$ is a $K$-automorphism, then $T^{-1}$ is also a $K$-automorphism. Thus it suffices to prove that every $K$-automorphism has a left semi-hyperbolic structure.

We will construct series of polymorphisms that will be random perturbations of special type of the initial $K$-automorphism. Our construction is a detailed version of the previous examples from the papers [8, 10, 11].

Let $T$ be an arbitrary $K$-automorphism. By well-known theorems (see, e.g., [2]), we can realize $T$ as the right shift in the space $\mathcal{X}$ of two-sided sequences in a finite or countable alphabet $A$; the space $\mathcal{X}$ is equipped with a shift-invariant measure $\mu$ and has trivial (in the sense of the measure $\mu$) tail algebras in the past and in the future. This means that if we consider the
one-sided right shift in the space of one-sided sequences \( X = \{\{x_n\}_{n<0}\} \) (we denote it by the same letter), then we have a decreasing sequence of measurable partitions \( \zeta_n \equiv T^{-n}\varepsilon \), and for every \( n \) the partition \( \zeta_n \) has countable or finite fibers; here \( T \) is the right shift in the space \( X \) and \( \varepsilon \) is the partition of the space \( X \) into separate points. We have \( \bigwedge_n \zeta_n = \nu \), where \( \nu \) is the trivial partition. Our first goal, according to the definition of a left hyperbolic structure, is to construct a measure-preserving polymorphism \( \Phi_l = \Phi \) that acts in the space \( X \) and has the following structure: for almost all \( x \), the supports of the measures \( \Phi(x) \equiv \mu_x \) belong to the element \( \zeta_n(x) \) of the partition \( \zeta_n \) for sufficiently large \( n \). We have a space \( X \) with measure \( \mu \) and a decreasing sequence of measurable partitions \( \zeta_n \) that tends to the trivial partition. The desired polymorphism \( \Phi \) (and later \( \Pi \)) will be a kind of random walk over several automorphisms with quasi-invariant measure.

Note that since the intersection \( \bigwedge_n \zeta_n \) is trivial, the number of points in the elements of \( \zeta_n \) tends to infinity (or already equal to infinity). Thus for every small \( \delta > 0 \) there exist a positive integer \( n_\delta \in \mathbb{N} \) and a measurable set \( C_\delta \subset X \) of \( \mu \)-measure greater than \( 1 - \delta \) such that for every point \( x \in C_\delta \), the element of the partition \( \zeta_n \) containing \( x \) contains at least four different points (including \( x \)). Let us take a measurable refinement \( \eta_n \) of the restriction of \( \zeta_n \) to the set \( C_\delta \) with all elements consisting of four points. If the number of points in the elements is not divisible by four, we form the set \( C_\delta' \) from all the remaining points and join it to the set \( X \setminus C_\delta \). Then we restrict our partitions \( \zeta_n \) with \( n > n_\delta \) to the set \( (X \setminus C_\delta) \cup C_\delta' \) and repeat this procedure again. Finally, we obtain a measurable partition \( \eta \) of the whole space \( X \), and each of its elements is a refinement of some element of the partition \( \zeta_n \) for some \( n \).

Each element of the partition \( \eta \) consists of four points, and we label them with the numbers 1, 2, 3, 4 in a measurable way (so that the set of points with label \( i \) \((i = 1, 2, 3, 4) \) is measurable). For every point \( y \), denote by \( n(y) \) the label of \( y \). Define three involutions on four points \( v_1, v_2, v_3 \) as follows: \( v_1 = (1, 2)(3, 4), v_2 = (1, 3)(2, 4), v_3 = (1, 4)(2, 3) \). Denote by \( p_i = p_i(y) \) the conditional measures of points in the element of the partition \( \eta \) that contains \( y \). Using the combinatorial lemma given below, we introduce a polymorphism \( \Phi \) as follows:

\[
\Phi(y) = v_i(y) \quad \text{with probability} \quad q_{i, n(y)}.
\]

Note that the partition \( \eta \) is, by definition, a fixed partition for the polymorphism \( \Phi \). Thus the images of \( y \) under \( \Phi \) are points (not equal to \( y \)) of the
same element of $\eta$ that contains $y$ with some probabilities. The fact that $\Phi$ preserves the measure follows from the construction of $q_{i,n}(y)$; we must only mention that because of the transitivity of conditional measures, the conditional measure of the point $x^i$ with respect to the partition $\eta$ is the same as the conditional measure with respect to the partition $\zeta_n$.

Now we define a polymorphism $\Pi$ on the space $\mathcal{X}$ as follows: $\Pi = \Phi \cdot T$.

Since $\Pi$ is the product of two measure-preserving automorphisms $\Phi$ and $T$, it also preserves the measure $\mu$.

From the definition of $\Pi$ we see that it sends a sequence $x = \{x_n\}$ to the shifted sequence $Tx$ and then changes at random a finite number of digits. The fact that the polymorphism changes a finite number of coordinates follows from the fact that the elements of the partition $\eta$ (in which the involutions $\nu^i$ act) are contained in some element of the partition $\zeta_n$ for some $n$, so at most $n$ coordinates can be changed. Of course, this $n$ depends on $x$, thus it can be arbitrarily large.

In order to finish the proof, we need to prove that
1) the polymorphism $\Pi$ is prime;
2) there exists $\lim_{n \to \infty} \Pi^n \cdot T^{-n}$.

The primality (the absence of nontrivial invariant measurable partitions) follows from the fact that, by definition, an invariant partition for $\Pi$ that does not coincide with $\varepsilon$ must be less (coarser) (for definitions, see, e.g., [13]) than the intersection $\bigwedge_n \zeta_n$, which is the trivial partition $\nu$.

The existence of the limit follows from the structure of $\Pi$. Indeed, the polymorphism $\Pi^n$ shifts every sequence by $n$ and changes at random a finite number of digits so that no digit is changed infinitely many times. Thus $\Pi^n \cdot T^n$ is a polymorphism that for every $x$ changes finitely many digits of $x$, and the coordinates of these digits go to infinity, so each coordinate stabilizes, and the corresponding measures converge. \hfill \Box

Now we formulate a simple combinatorial lemma that we have used in the proof of the theorem.

**Lemma 1.** Let $p_i$, $i = 1, 2, 3, 4$, be an arbitrary probability vector of length four. There exists a matrix $\{q_{i,j}\}_{i,j=1}^4$ with $q_{i,i} = 0$, $q_{i,j} \geq 0$, $i, j = 1, 2, 3, 4$, with given marginal projections: $\sum_i q_{i,j} = p_j$, $\sum_j q_{i,j} = p_i$.

The proof of this lemma is straightforward. We may say that the matrix $\{q_{i,j}\}$ determines a measure-preserving polymorphism of the space. Since the
lemma is valid for any number of points greater than four, the partition \( \eta \) can also be chosen with arbitrarily many points.

For \( K \)-automorphisms satisfying property \((*)\) (see Sec. 3.3), we can choose a polymorphism \( \Pi \) that simultaneously defines a left and right semi-hyperbolic structures and, consequently, a hyperbolic structure. The supports of the conditional measures are the blocks of the homoclinic partition. As we have already mentioned, all known \( K \)-automorphisms have this partition. But for a general \( K \)-automorphism the situation is unclear.

### 3.7 A conjecture and a problem

1. **Conjecture.** Each hyperbolic automorphism is a \( K \)-automorphism satisfying property \((*)\); each automorphism that is quasi-similar to a prime non-mixing and non-co-mixing polymorphism has a hyperbolic structure defined by this polymorphism.

2. **Problem.** Is property \((*)\) equivalent to the \( K \)-property? Or, is every \( K \)-automorphism hyperbolic?

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