Central limit theorems for multiple stochastic integrals and Malliavin calculus

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Abstract

We give a new characterization for the convergence in distribution to a standard normal law of a sequence of multiple stochastic integrals of a fixed order with variance one, in terms of the Malliavin derivatives of the sequence. We also give a new proof of the main theorem in [7] using techniques of Malliavin calculus. Finally, we extend our result to the multidimensional case and prove a weak convergence result for a sequence of square integrable random variables.

KEY WORDS: Multiple stochastic integrals. Limit theorems. Gaussian processes. Malliavin calculus. Weak convergence.

RUNNING HEAD: Central limit theorems for multiple stochastic integrals

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1 Introduction

Consider a sequence of random variables $F_k$ belonging to the $n$th Wiener chaos, $n \geq 2$, and with unit variance. In [7], Nualart and Peccati have proved that this sequence converges in distribution to a normal $N(0, 1)$ law if and only if one of the following two equivalent conditions hold:

\begin{enumerate}
\item[i)] $\lim_{k \to \infty} \mathbb{E}(F_k^4) = 3$,
\item[ii)] $\lim_{k \to \infty} f_k \otimes_l f_k = 0$, for all $1 \leq l \leq n - 1$,
\end{enumerate}

where $f_k$ is the square integrable kernel associated with the random variable $F_k$, and $f_k \otimes_l f_k$ denotes the contraction of $l$ indices of both kernels. In a subsequent paper, Peccati and Tudor [10] gave a multidimensional version of this characterization.

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There have been different extensions and applications of these results. In [3] Hu and Nualart have applied this characterization to establish the weak convergence of the renormalized self-intersection local time of a fractional Brownian motion. In two recent papers, Peccati and Taqqu [9, 10] study the stable convergence of multiple stochastic integrals to a mixture of normal distributions.

The aim of this paper is to provide an additional necessary and sufficient condition for the convergence of the sequence $F_k$ to a normal distribution, in terms of the derivative of $F_k$ in the sense of Malliavin calculus. This new condition is

$$\|DF_k\|_{H^L(\Omega)}^2 \rightarrow +\infty \quad k \rightarrow +\infty. \quad (1)$$

On the other hand, we give a simple proof of the fact that condition (1) implies the convergence in distribution to the normal law based on Malliavin calculus. The main ingredient of the proof is the identity $\delta D = -L$, where $\delta$, $D$ and $L$ are the basic operators in Malliavin calculus. In this way, we are able to show the result of Nualart and Peccati with the additional equivalent hypotheses (1), without using the Dambis-Dubins-Schwartz characterization of continuous martingales as a Brownian motion with a time change.

We also discuss the extension of these results to the multidimensional case. In the last section we study the weak convergence of a sequence of centered square integrable random variables. The result assume conditions on the Malliavin derivatives of the chaotic projections of the sequence $\{F_k\}_{k \in \mathbb{N}}$ not on the derivatives of the sequence $\{F_k\}_{k \in \mathbb{N}}$. Therefore, it can be used with non regular random variables in the Malliavin sense.

In [8] Peccati and Taqqu also provide a condition for convergence to the normal law, involving projections of Malliavin derivatives. This type of condition is different from ours, and it is inspired on Clark-Ocone’s formula.

Condition (1) is a useful tool in establishing the central limit theorem for sequences of random variables defined in terms of a fixed function of a Gaussian process. We apply this approach to derive the weak convergence of the normalized sums of odd powers of the increments of a fractional Brownian motion. In this case, condition (1) follows easily from the ergodic theorem.

The paper is organized as follows. In Section 2 we introduce some notation and preliminary results. In section 3 we state and prove the main result of the paper. Section 4 deals with the multidimensional version of the result proved in Section 3, and in Section 5 we apply the previous results to the weak convergence of a sequence of centered and square integrable random variables. Finally, in Section 6 we discuss the application of our approach to an example related to the fractional Brownian motion.

## 2 Preliminaries and notation

Let $H$ be a separable Hilbert space. For every $n \geq 1$ let $H^{\otimes n}$ be the $n$th tensor product of $H$ and denote by $H^{\otimes_{\text{sym}}}n$ the $n$th symmetric tensor product of $H$, endowed with the modified norm $\sqrt{n!} \|\cdot\|_{H^{\otimes n}}$. Suppose that $X = \{X(h) : h \in H\}$
is an isonormal Gaussian process on \( H \). This means that \( X \) is a centered Gaussian family of random variables indexed by the elements of \( H \), defined on some probability space \( (\Omega, \mathcal{F}, P) \), and such that, for every \( h, g \in H \),

\[
\mathbb{E}[X(h)X(g)] = \langle h, g \rangle_H.
\]

We will assume that \( \mathcal{F} \) is generated by \( X \).

For every \( n \geq 1 \), let \( \mathcal{H}_n \) be the \( n \)th Wiener chaos of \( X \), that is, the closed linear subspace of \( L^2(\Omega, \mathcal{F}, P) \) generated by the random variables \( \{H_n(X(h)), h \in H, \|h\|_H = 1\} \), where \( H_n \) is the \( n \)th Hermite polynomial. We denote by \( \mathcal{H}_0 \) the space of constant random variables. For \( n \geq 1 \), the mapping \( I_n(h^{\otimes n}) = n!H_n(X(h)) \) provides a linear isometry between \( H^{\otimes n} \) and \( \mathcal{H}_n \). For \( n = 0 \), \( \mathcal{H}_0 = \mathbb{R} \), and \( I_0 \) is the identity map.

It is well known (Wiener chaos expansion) that \( L^2(\Omega, \mathcal{F}, P) \) can be decomposed into the infinite orthogonal sum of the spaces \( \mathcal{H}_n \). Therefore, any square integrable random variable \( F \in L^2(\Omega, \mathcal{F}, P) \) has the following expansion

\[
F = \sum_{n=0}^{\infty} I_n(f_n),
\]

where \( f_0 = \mathbb{E}[F] \), and the \( f_n \in H^{\otimes n} \) are uniquely determined by \( F \). For every \( n \geq 0 \) we denote by \( J_n \) the orthogonal projection on the \( n \)th Wiener chaos \( \mathcal{H}_n \), so \( I_n(f_n) = J_n(F) \).

Let \( \{e_k, k \geq 1\} \) be a complete orthonormal system in \( H \). Given \( f \in H^{\otimes n} \) and \( g \in H^{\otimes m} \), for \( l = 0, \ldots, n \wedge m \) the contraction of \( f \) and \( g \) of order \( l \) is the element of \( H^{\otimes(n+m-2l)} \) defined by

\[
f \otimes_l g = \sum_{i_1, \ldots, i_l=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_l} \rangle_{H^{\otimes l}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_l} \rangle_{H^{\otimes l}}.
\]

We denote by \( f \tilde{\otimes} g \) its symmetrization. Then, \( f \otimes_0 g = f \otimes g \) equals to the tensor product of \( f \) and \( g \), and for \( n = m \), \( f \otimes_n g = (f, g)_{H^{\otimes n}} \).

Let us introduce some basic facts on the Malliavin calculus with respect the Gaussian process \( X \). We refer the reader to Nualart [6] for a complete presentation of these notions. Consider the set \( \mathcal{S} \) of smooth random variables \( \mathcal{S} \) of the form

\[
F = f(X(h_1), \ldots, X(h_n)),
\]

where \( h_1, \ldots, h_n \in H, f \in C^\infty_b(\mathbb{R}^n) \) (the space of bounded functions which have bounded derivatives of all orders) and \( n \in \mathbb{N} \). The derivative operator \( D \) on a smooth random variable of the form \( \mathcal{S} \) is defined by

\[
DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(h_1), \ldots, X(h_n)) h_i,
\]

which is an element of \( L^2(\Omega; H) \). By iteration one can define \( D^mF \) which is an element of \( L^2(\Omega; H^{\otimes m}) \). For \( m \geq 1 \) we denote by \( D^{m,2} \) the completion of \( \mathcal{S} \).
with respect to the norm $\|F\|_{m,2}$ given by

$$\|F\|_{m,2}^2 = \mathbb{E}[F^2] + \sum_{i=1}^{m} \mathbb{E}[\|D^i F\|_{H^\otimes i}^2].$$

We denote by $\delta$ the adjoint of the operator $D$. That is, $\delta$ is an unbounded operator on $L^2(\Omega; H)$ with values in $L^2(\Omega)$, whose domain, denoted by Dom $\delta$, is the set of $H$-valued square integrable random variables $u \in L^2(\Omega; H)$ such that

$$|\mathbb{E}[\langle DF, u \rangle_H]| \leq c \|F\|_{L^2(\Omega)},$$

for all $F \in \mathbb{D}^{1,2}$. If $u$ belongs to Dom $\delta$, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by

$$\mathbb{E}[F \delta(u)] = \mathbb{E}[\langle DF, u \rangle_H],$$

for any $F \in \mathbb{D}^{1,2}$.

The operator $L$ defined on the Wiener chaos expansion as $L = \sum_{n=0}^{\infty} -nJ_n$ is called the infinitesimal generator of the Ornstein-Uhlenbeck semigroup. The domain of this operator is the set

$$\text{Dom } L = \{F \in L^2(\Omega) : \sum_{n=1}^{\infty} n^2 \|J_n F\|_{L^2(\Omega)}^2 < +\infty\} = \mathbb{D}^{2,2}.$$

The next proposition explains the relationship between the operators $D, \delta$ and $L$.

**Proposition 1** For $F \in L^2(\Omega)$ the statement $F \in \text{Dom } L$ is equivalent to $F \in \text{Dom } \delta D$ (i.e., $F \in \mathbb{D}^{1,2}$ and $DF \in \text{Dom } \delta$), and in this case

$$\delta DF = -LF.$$

In the particular case where $H = L^2(A, A, \mu)$, $(A, A)$ is a measurable space, and $\mu$ is a $\sigma$-finite and non-atomic measure, then $H^\otimes n = L^2(A^n, A^\otimes n, \mu^\otimes n)$ is the space of symmetric and square integrable functions on $A^n$ and for every $f \in H^\otimes n$, $I_n(f)$ is the multiple Wiener-Itô integral (of order $n$) of $f$ with respect to $X$, as defined by Itô in [4]. In this case, $F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathbb{D}^{1,2}$ if and only if

$$\mathbb{E}[\|DF\|_H^2] = \sum_{n=1}^{\infty} n \|f_n\|_{H^\otimes n}^2 < +\infty,$$

and its derivative can be identified as the element of $L^2(A \times \Omega)$ given by

$$D_tF = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)).$$

We need the following technical lemma.
Lemma 2 Consider two random variables $F = I_{n}(f)$, $G = I_{m}(g)$, where $n, m \geq 1$. Then

$$E \left[ \langle DF, DG \rangle^2_{H} \right] = \sum_{r=1}^{n \land m} \frac{(n \land m)!^2}{((n-r)!(m-r)!(r-1)!)^2} \|f \otimes_{r} g\|_{H^{\otimes (n+m-2r)}}^2 \|. (4)$$

Proof. Without loss of generality we can assume that $H = L^2(A, \mathcal{A}, \mu)$, $(A, \mathcal{A})$ is a measurable space, and $\mu$ is a $\sigma$-finite and non-atomic measure. In that case, (3) implies that

$$D_tF = nI_{n-1}(f(\cdot, t)), D_tG = mI_{m-1}(g(\cdot, t))$$

and we have

$$\langle DF, DG \rangle^2_{H} = nm \int_{A} I_{n-1}(f(\cdot, t)) I_{m-1}(g(\cdot, t)) \mu(dt).$$

Thanks to the multiplication formula for multiple stochastic integrals, see, for instance, Proposition 1.1.3. in [6], one obtains

$$\langle DF, DG \rangle^2_{H} = nm \int_{A} \sum_{r=0}^{n \land m-1} r! \begin{pmatrix} n-1 \cr r \end{pmatrix} \begin{pmatrix} m-1 \cr r \end{pmatrix} I_{n+m-2-2r}(f(\cdot, t) \otimes_{r} g(\cdot, t)) \mu(dt).$$

Taking into account the orthogonality between multiple stochastic integrals of different order, we have

$$E[\langle DF, DG \rangle^2_{H}]$$

$$= n^2m^2 \sum_{r=0}^{n \land m-1} (r!)^2 \begin{pmatrix} n-1 \cr r \end{pmatrix}^2 \begin{pmatrix} m-1 \cr r \end{pmatrix}^2$$

$$\times \int_{A^2} \langle f(\cdot, t) \otimes_{r} g(\cdot, t), f(\cdot, s) \otimes_{r} g(\cdot, s) \rangle_{H^{\otimes (n+m-2-2r)}} \mu(dt) \mu(ds).$$

Notice that

$$\int_{A} f(\cdot, t) \otimes_{r} g(\cdot, t) \mu(dt) = f \otimes_{r+1} g,$$

and, as a consequence,

$$\int_{A} f(\cdot, t) \otimes_{r} g(\cdot, t) \mu(dt) = f \otimes_{r+1} g.$$

Therefore, 

$$E[\langle DF, DG \rangle^2_{H}] = n^2m^2 \sum_{r=0}^{n \land m-1} (r!)^2 \begin{pmatrix} n-1 \cr r \end{pmatrix}^2 \begin{pmatrix} m-1 \cr r \end{pmatrix}^2 \|f \otimes_{r+1} g\|_{H^{\otimes (n+m-2-2r)}}^2,$$

which implies the desired result. □
3 Main result

Fix \( n \geq 2, n \in \mathbb{N} \). Consider a sequence \( \{F_k\}_{k \in \mathbb{N}} \) of square integrable random variables belonging to the \( n \)th Wiener chaos. We know that

\[
\mathbb{E}[\|DF_k\|_{H}^2] = n \|f_k\|_{H\otimes n}^2. \tag{5}
\]

The next lemma establishes the equivalence between the convergence in \( L^2(\Omega) \) of \( \|DF_k\|_{H}^2 \) to a constant, and the convergence of \( \mathbb{E}[\|DF_k\|_{H}^4] \) to the square of the same constant.

**Lemma 3** Consider a sequence \( \{F_k = I_n(f_k)\}_{k \in \mathbb{N}} \) of square integrable random variables belonging to the \( n \)th Wiener chaos such that

\[
\mathbb{E}[F_k^2] = \|f_k\|_{H\otimes n}^2 \xrightarrow{k \to +\infty} 1.
\]

The following conditions are equivalent:

1. \( \lim_{k \to +\infty} \mathbb{E}[\|DF_k\|_{H}^4] = n^2 \).
2. \( \lim_{k \to +\infty} \|DF_k\|_{H}^2 = n, \) in \( L^2(\Omega) \).

**Proof.** Notice that, using (5),

\[
\mathbb{E}[(\|DF_k\|_{H}^2 - n)^2] = \mathbb{E}[\|DF_k\|_{H}^4] - 2n\mathbb{E}[\|DF_k\|_{H}^2] + n^2
\]

\[
= \mathbb{E}[\|DF_k\|_{H}^4] - 2n^2 \|f_k\|_{H\otimes n}^2 + n^2
\]

and the result follows easily. \( \blacksquare \)

Now, we establish the main result of this paper.

**Theorem 4** Consider a sequence \( \{F_k = I_n(f_k)\}_{k \in \mathbb{N}} \) of square integrable random variables belonging to the \( n \)th Wiener chaos such that

\[
\mathbb{E}[F_k^2] = \|f_k\|_{H\otimes n}^2 \xrightarrow{k \to +\infty} 1. \tag{6}
\]

The following statements are equivalent.

i) As \( k \) goes to infinity, the sequence \( \{F_k\}_{k \in \mathbb{N}} \) converges in distribution to the normal law \( \mathcal{N}(0, 1) \).

ii) \( \lim_{k \to +\infty} \mathbb{E}[F_k^4] = 3 \).

iii) For all \( 1 \leq l \leq n - 1 \), \( \lim_{k \to +\infty} \|f_k \otimes_l f_k\|_{H\otimes 2(n-1)} = 0 \).

iv) \( \|DF_k\|_{H}^2 \xrightarrow{k \to +\infty} n \) in \( L^2(\Omega) \).
Proof. We will prove the following implications
iv) ⇒ i) ⇒ ii) ⇒ iii) ⇒ iv).

[iv) ⇒ i)] The sequence of random variables \( \{ F_k \}_{k \in \mathbb{N}} \) is tight because it is bounded in \( L^2(\Omega) \) by condition (6). Then, by Prokhorov’s Theorem we have that \( \{ F_k \}_{k \in \mathbb{N}} \) is relatively compact, and it suffices to show that the limit of any subsequence converging in distribution is \( N(0,1) \). Suppose that, for a subsequence \( \{ k_l \}_{l \in \mathbb{N}} \subseteq \{ k \}_{k \in \mathbb{N}} \) we have

\[
F_{k_l} \xrightarrow{l \to +\infty} G.
\] (7)

By condition (6) \( G \in L^2(\Omega) \). Therefore, the characteristic function \( \varphi(t) = \mathbb{E}[e^{itG}] \) is differentiable and \( \varphi'(t) = i\mathbb{E}[Ge^{itG}] \). For every \( k \in \mathbb{N} \), define \( \varphi_k(t) = i\mathbb{E}[F_ke^{itF_k}] \).

By the Continuous Mapping Theorem, (7) implies that

\[
F_{k_l}e^{itF_{k_l}} \xrightarrow{l \to +\infty} Ge^{itG}.
\] (8)

The boundedness in \( L^2(\Omega) \) plus the convergence in law (8) imply convergence of the expectations. Hence, we obtain

\[
\varphi_{k_l}'(t) \xrightarrow{l \to +\infty} \varphi'(t).
\]

On the other hand, using the definition of the operator \( L \), Proposition 1 and the definition of the operator \( \delta \), we have

\[
\mathbb{E}[F_ke^{itF_k}] = -\frac{1}{n}\mathbb{E}[LF_ke^{itF_k}] = -\frac{1}{n}\mathbb{E}[\delta D(F_k)e^{itF_k}]
\]

\[
= \frac{1}{n}\mathbb{E}[\langle DF_k, D(e^{itF_k}) \rangle_H] = \frac{it}{n}\mathbb{E}[e^{itF_k} \| DF_k \|^2_H].
\]

Therefore,

\[
\varphi'_{k_l}(t) = -\frac{t}{n}\mathbb{E}[e^{itF_{k_l}} \| DF_{k_l} \|^2_H].
\]

Furthermore,

\[
\left| \mathbb{E}[e^{itF_{k_l}} \| DF_{k_l} \|^2_H] - n\varphi(t) \right| \leq \mathbb{E}\left[ \| DF_{k_l} \|^2_H - n \right] + n \left| \mathbb{E}[e^{itF_{k_l}}] - \varphi(t) \right|,
\]

which, by the definition of \( \varphi \) and hypothesis iv), gives that

\[
\varphi'_{k_l}(t) \xrightarrow{l \to +\infty} -t\varphi(t).
\]

This implies that \( \varphi(t) \) satisfies the following differential equation

\[
\begin{align*}
\varphi'(t) &= -t\varphi(t) \\
\varphi(0) &= 1,
\end{align*}
\]
which is the differential equation satisfied by the characteristic function of the \( N(0, 1) \).

\( [i) \Rightarrow ii) \) and \( [ii) \Rightarrow iii) \] These implications are proved by Nualart and Pec-cati in Proposition 3, [7]. The proof of the first one is trivial and the proof of the second one involves some combinatorics and it is based on the product formula for multiple stochastic integrals.

\( [iii) \Rightarrow iv) \] By Lemma 3 it is enough to prove that iii) implies \( \lim_{k \to +\infty} E[\|DF_k\|_H^4] = n^2 \). Using (4) we obtain

\[
E[\|DF_k\|_H^4] = \sum_{r=1}^{n-1} \frac{(nl)^4}{((n-r)!)^4((r-1)!)^2} \|f_k \otimes_r f_k\|_{H^{\otimes 2(n-r)}}^2 + n^2(n!)^2 \|f_k\|_{H^{\otimes n}}^4.
\]

It follows that \( E[\|DF_k\|_H^4] \) converges to \( n^2 \) if and only if

\[
\|f_k \otimes_1 f_k\|_{H^{\otimes 2(n-1)}} \to 0, \quad 1 \leq l \leq n-1.
\]

As

\[
\|f_k \otimes_1 f_k\|_{H^{\otimes 2(n-1)}} = (2(n-l)!) \|f_k \otimes_1 f_k\|_{H^{\otimes 2(n-l)}} \leq (2(n-l)!) \|f_k \otimes_1 f_k\|_{H^{\otimes 2(n-1)}},
\]

we conclude the proof. ■

As a consequence, we obtain.

**Corollary 5** Fix \( n \geq 2 \) and \( F \) belonging to the \( n \)th Wiener chaos such that \( E[F^2] = 1 \). Then the distribution of \( F \) cannot be normal and \( E[\|DF\|_H^4] \neq n^2 \).

**Proof.** If \( F \) had a normal distribution or \( E[\|DF\|_H^4] = n^2 \), then, according to Theorem 4 and Lemma 3 we would have \( \text{Var}[\|DF\|_H^4] = 0 \), but this implies \( F = 0 \) or \( F \) belonging to the first chaos. ■

### 4 Multidimensional case

In this section we give a multidimensional version of Theorem 4. For \( d \geq 2 \), fix \( d \) natural numbers \( 1 \leq n_1 \leq \cdots \leq n_d \). Consider a sequence of random vectors of the form

\[
F_k = \left(F_k^1, \ldots, F_k^d\right) = \left(I_{n_1} \left(f_k^1\right), \ldots, I_{n_d} \left(f_k^d\right)\right), \tag{9}
\]

where \( f_k^i \in H^{\otimes n_i} \), and

\[
\gamma_k = \gamma_k^{i,j} 1 \leq i, j \leq d = \left\langle (DF_k^i, DF_k^j) \right\rangle_{H} 1 \leq i, j \leq d.
\]

The following lemma shows that the convergence of the covariance matrix of \( F_k \) to a diagonal matrix plus the convergence in \( L^2(\Omega) \) of the diagonal elements of \( \gamma_k \) to the constant \( n_i \), implies the convergence in \( L^2(\Omega) \) of \( \gamma_k \) to a diagonal matrix.
Lemma 6 Let \( \{F_k\}_{k \in \mathbb{N}} \) be a sequence of random vectors as \(^{[2]}\) such that, for every \( 1 \leq i, j \leq d \),
\[
\lim_{k \to +\infty} \mathbb{E}[F^i_k F^j_k] = \delta_{ij},
\]
(10)
where \( \delta_{ij} \) is the Kronecker symbol. We have that
\[
\|DF_k^i\|_H^2 \overset{L^2(\Omega)}{\to} n_i, 1 \leq i \leq d
\]
implies
\[
\gamma^i_j \overset{L^2(\Omega)}{\to} \sqrt{n_i n_j} \delta_{ij}, 1 \leq i, j \leq d.
\]
Proof. We need to show that, for \( i < j \), one has
\[
\lim_{k \to +\infty} \mathbb{E}[\langle DF^i_k, DF^j_k \rangle_H^2] = 0.
\]
Using (4) we obtain
\[
\mathbb{E}[\langle DF^i_k, DF^j_k \rangle_H^2] = \sum_{r=1}^{n_1} \frac{(n_i! n_j!)^2}{((n_i - r)! (n_j - r)! (r - 1)!)^2} \| f^i_k \otimes_r f^j_k \|^2_{H^\otimes (n_i + n_j - 2r)} \leq \sum_{r=1}^{n_1} \frac{(n_i! n_j!)^2 (n_i + n_j - 2r)!}{((n_i - r)! (n_j - r)! (r - 1)!)^2} \| f^i_k \otimes_r f^j_k \|^2_{H^\otimes (n_i + n_j - 2r)}.
\]
We have reduced the problem to show that
\[
\lim_{k \to +\infty} \| f^i_k \otimes_r f^j_k \|^2_{H^\otimes (n_i + n_j - 2r)} = 0, \quad 1 \leq r \leq n_i.
\]
The next step is to relate the norm of \( f^i_k \otimes_r f^j_k \) with the norms of \( f^i_k \otimes_{n_i - r} f^i_k \) and \( f^j_k \otimes_{n_j - r} f^j_k \). Using the definition of the contractions, we have
\[
\| f^i_k \otimes_r f^j_k \|^2_{H^\otimes (n_i + n_j - 2r)} = \langle f^i_k \otimes_{n_i - r} f^i_k, f^j_k \otimes_{n_j - r} f^j_k \rangle_{H^\otimes (2r)}.
\]
Hence, by Cauchy-Schwarz’s inequality, we obtain
\[
\| f^i_k \otimes_r f^j_k \|^2_{H^\otimes (n_i + n_j - 2r)} \leq \| f^i_k \otimes_{n_i - r} f^i_k \|_{H^\otimes (2r)} \| f^j_k \otimes_{n_j - r} f^j_k \|_{H^\otimes (2r)}.
\]
(13)
In the case \( 1 \leq r \leq n_i - 1 \), by assumption \(^{[11]}\) and Theorem \(^{[3]}\) (implication iv) \( \Rightarrow \) iii)), the right hand side of equation (13) tends to zero as \( k \) tends to infinity. In the case \( r = n_i < n_j \), we have that the right hand side of equation (13) is equal to
\[
\| f^i_k \|^2_{H^\otimes n_i} \| f^j_k \otimes_{n_j - r} f^j_k \|_{H^\otimes (2r)},
\]
which tends to zero as \( k \) tends to infinity, because
\[
\sup_{k \geq 1} \| f^i_k \|^2_{H^\otimes n_i} < +\infty,
\]
(14)
thanks to assumption 10 and, analogously to the previous case,
\[ \left\| f_k^i \otimes_{n_j - r} f_k^j \right\|_{H \otimes (2r)} \to k \to +\infty 0. \]

In the case \( r = n_i = n_j \), the equality 10 gives
\[ \left\| f_k^i \otimes_{r} f_k^j \right\|_{H \otimes (n_i + n_j - 2r)}^2 = \left( \mathbb{E}[F_k^i F_k^j] \right)^2, \]
which tends to zero by assumption 10. 

Let \( V_d \) be the set of all \((i_1, i_2, i_3, i_4) \in (1, ..., d)^4\), such that one of the following conditions is satisfied: (a) \( i_1 \neq i_2 = i_3 = i_4 \), (b) \( i_1 \neq i_2 = i_3 \neq i_4 \) and \( i_4 \neq i_1 \), (c) the elements of \((i_1, i_2, i_3, i_4)\) are all distinct.

The following is a multidimensional version of Theorem 4.

**Theorem 7** Let \( \{F_k\}_{k \in \mathbb{N}} \) be a sequence of random vectors of the form 14 such that, for every \( 1 \leq i, j \leq d \),
\[ \lim_{k \to +\infty} \mathbb{E}[F_k^i F_k^j] = \delta_{ij}, \]
where \( \delta_{ij} \) is the Kronecker symbol. The following statements are equivalent.

i) For every \( j = 1, ..., d \), \( F_k^j \) converges in distribution to a standard Gaussian variable.

ii) For every \( i = 1, ..., d \), \( \lim_{k \to +\infty} \mathbb{E}[(F_k^i)^4] = 3 \).

iii) For all \( 1 \leq i \leq d, 1 \leq l \leq n_i - 1 \), \( \lim_{k \to +\infty} \mathbb{E}[|F_k^i \otimes_l f_k^i|^2]_{H \otimes (n_i - l)} = 0. \)

iv) For all \( 1 \leq i \leq d \), \( \lim_{k \to +\infty} \mathbb{E}[DF_k^i]^2_{L^2(\Omega)} n_i. \)

v) For every \((i_1, i_2, i_3, i_4) \in V_d, \)
\[ \lim_{k \to +\infty} \mathbb{E} \left[ \left( \sum_{i=1}^{d} F_k^i \right)^4 \right] = 3d^2, \]
and
\[ \lim_{k \to +\infty} \mathbb{E} \left[ \prod_{l=1}^{4} F_k^{i_l} \right] = 0. \]

vi) As \( k \) goes to infinity the sequence \( \{F_k\}_{k \in \mathbb{N}} \) converges in distribution to a \( d \)-dimensional standard Gaussian vector \( N_d(0, I_d) \).
Proof. As in the one-dimensional case, we provide a proof to the above theorem using Malliavin calculus and avoiding the Dambis-Dubins-Schwarz theorem. The equivalences i) ⇐⇒ ii) ⇐⇒ iii) ⇐⇒ iv) follow from Theorem 4. The fact that vi) ⇒ v) is easy, and the implication v) ⇒ iii) is proved by Peccati and Tudor in [10] using the product formula for multiple stochastic integrals. Hence, it only remains to show the implication iv) ⇒ vi).

The sequence of random variables \( \{F_k\}_{k \in \mathbb{N}} \) is tight by condition (14). Then, it suffices to show that the limit in distribution of any converging subsequence \( \{F_{k_l}\}_{l \in \mathbb{N}} \) is \( N_d(0, I_d) \). For every \( k \in \mathbb{N} \), define \( \varphi_k(t) = \mathbb{E}[e^{i \langle t, F_k \rangle}] \). Let \( \varphi(t) \) be the limit of \( \varphi_{k_l}(t) \) as \( l \) tends to infinity. As in the proof of Theorem 4 we have for all \( j = 1, \ldots, d \)

\[
\frac{\partial \varphi_{k_l}}{\partial t_j}(t) \xrightarrow{l \to \infty} \frac{\partial \varphi}{\partial t_j}(t).
\]

On the other hand, using the definition of the operator \( L \), Proposition 1 and the definition of the operator \( \delta \), we have

\[
\mathbb{E}[F_k e^{i \langle t, F_k \rangle}] = -\frac{1}{n_j} \mathbb{E}[LF_k e^{i \langle t, F_k \rangle}] = -\frac{1}{n_j} \mathbb{E}[-\delta D(F_k) e^{i \langle t, F_k \rangle}]
\]

\[
= \frac{1}{n_j} \mathbb{E}[(DF^j_k, D(e^{i \langle t, F_k \rangle}))_H] = \frac{i}{n_j} \sum_{h=1}^{d} t_h \mathbb{E}[e^{i \langle t, F_k \rangle} \gamma_k^h].
\]

Therefore,

\[
\frac{\partial \varphi_{k_l}}{\partial t_j}(t) = -\frac{i}{n_j} \sum_{h=1}^{d} t_h \mathbb{E}[e^{i \langle t, F_k \rangle} \gamma_k^h].
\] (15)

Using Lemma 6 and taking the limit of the right-hand side of expression (15) yields

\[
\frac{\partial \varphi}{\partial t_j}(t) = -t_j \varphi(t),
\]

for \( j = 1, \ldots, d \). As a consequence, \( \varphi \) is the characteristic function of the law \( N_d(0, I_d) \). ■

5 Central limit theorem for square integrable random variables

In this section, we will establish a weak convergence result for an arbitrary sequence of centered square integrable random variables.

Theorem 8 Let \( \{F_k\}_{k \in \mathbb{N}} \) be a sequence of centered square integrable random variables with the following Wiener chaos expansions

\[
F_k = \sum_{n=1}^{\infty} J_n(F_k).
\]

Suppose that
\( (i) \lim_{N \to +\infty} \limsup_{k \to +\infty} \sum_{n=N+1}^{\infty} \mathbb{E}[(J_n F_k)^2] = 0, \)

\( (ii) \) for every \( n \geq 1, \lim_{k \to +\infty} \mathbb{E}[(J_n F_k)^2] = \sigma_n^2, \)

\( (iii) \) \( \sum_{n=1}^{\infty} \sigma_n^2 = \sigma^2 < +\infty, \)

\( (iv) \) for all \( n \geq 1, \)

\[
\| D(J_n F_k) \|_{L^2(\Omega)}^2 \xrightarrow{k \to +\infty} n \sigma_n^2.
\]

Then, \( F_k \) converges in distribution to the Normal law \( N(0, \sigma^2) \) as \( k \) tends to infinity.

**Proof.** By Theorem 4, conditions (ii) and (iv) imply that for each fixed \( n \geq 1 \) the sequence \( \{J_n F_k\}_{k \in \mathbb{N}} \) converges in distribution to the normal law \( N(0, \sigma_n^2) \).

Moreover, by Theorem 7 we have, for each \( n \geq 1, \)

\[
(J_1 F_k, ..., J_k F_k) \xrightarrow{k \to +\infty} (\xi_1, ..., \xi_n),
\]

where \( \{\xi_n\}_{n \in \mathbb{N}} \) are independent centered Gaussian random variables with variances \( \{\sigma_n^2\}_{n \in \mathbb{N}} \). For every \( N \geq 1, \)

\[
F_k^N = \sum_{n=1}^{N} J_n (F_k), \quad \xi^N = \sum_{n=1}^{N} \xi_n.
\]

Define also \( \xi = \sum_{n=1}^{\infty} \xi_n \). Let \( f \) be a \( C^1 \) function such that \(|f|\) and \(|f'|\) are bounded by one. Then

\[
|\mathbb{E}[f(F_k)] - \mathbb{E}[f(\xi)]| \\
\leq |\mathbb{E}[f(F_k)] - \mathbb{E}[f(F_k^N)]| + |\mathbb{E}[f(F_k^N)] - \mathbb{E}[f(\xi^N)]| + |\mathbb{E}[f(\xi^N)] - \mathbb{E}[f(\xi)]| \\
\leq \left( \sum_{n=N+1}^{\infty} \mathbb{E}[(J_n F_k)^2] \right)^{1/2} + |\mathbb{E}[f(F_k^N)] - \mathbb{E}[f(\xi^N)]| + |\mathbb{E}[f(\xi^N)] - \mathbb{E}[f(\xi)]|.
\]

Taking first the limit as \( k \) tends to infinity, and then the limit as \( N \) tends to infinity, and applying conditions (i), (iii) and [10] we finish the proof. ■

**Remark 9** If \( F_k \in D^{1,2} \), and

\[
\sup_k \mathbb{E}(\|DF_k\|_{H}^2) < \infty,
\]

then condition (i) holds.
This theorem can be applied to random variables not belonging to $D^{1,2}$, and it requires the convergence of the derivatives of the projections. In this sense, it would be interesting to study the relation between the convergence in distribution of a sequence $F_k$ to a normal law and the convergence in $L^2(\Omega)$ of $\|DF_k\|_H^2$ to a constant. As we have seen, these conditions are equivalent (if $F_k$ are centered and with $\lim_{k \to \infty} E[(F_k^2)] = \sigma^2$) for random variables in a fixed chaos. In the general case, we conjecture that this equivalence does not hold.

6 Example

Suppose that $B^H = \{B^H_t, t \geq 0\}$ is a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$. That is, $B^H$ is a Gaussian stochastic process with zero mean and the covariance function $E(B^H_t B^H_s) = 1/2 \left( t^{2H} + s^{2H} - |t - s|^{2H} \right)$.

Fix $H < \frac{1}{2}$ and an odd integer $\kappa \geq 1$. We are interested in the asymptotic behavior of

$$Z^{(n)}_t = n^{\kappa H - \frac{1}{2}} \sum_{j=1}^{[nt]} \left( B^H_{j/n} - B^H_{(j-1)/n} \right)^\kappa,$$

as $n$ tends to infinity, where $t \in [0, T]$. Set $X_j = B^H_j - B^H_{j-1}$. Then, $\{X_j, j \geq 1\}$ is a stationary Gaussian sequence with zero mean, unit variance and correlation $\rho_H(n) \leq H(2H - 1)n^{2H-2}$ as $n$ tends to infinity. We have the following result.

**Theorem 10** The two-dimensional process $(B^H, Z^{(n)})$ converges in distribution in the Skorohod space $D([0, T])^2$ to $(B^H, cW)$, where $W$ is a Brownian motion independent of $B^H$, and

$$c^2 = \sum_{j=0}^{\infty} E[(X_1 X_{1+j})^\kappa].$$

**Proof.** The proof will be done in two steps.

**Step 1.** We will first show the convergence of finite-dimensional distributions. Let $(a_k, b_k)$, $k = 1, \ldots, N$, be pairwise disjoint intervals contained in $[0, T]$. Define the random vectors $B = (B^H_{b_1} - B^H_{a_1}, \ldots, B^H_{b_N} - B^H_{a_N})$ and $X^{(n)} = (X_1^{(n)}, \ldots, X_N^{(n)})$, where

$$X_i^{(n)} = n^{\kappa H - \frac{1}{2}} \sum_{[na_k] < j \leq [nb_k]} \left( B^H_{j/n} - B^H_{(j-1)/n} \right)^\kappa.$$

We claim that $(B, X^{(n)})$ converges in law to $(B, V)$, where $B$ and $V$ are independent and $V$ is a Gaussian random vector with zero mean and independent
components with variance \( c^2(b_k - a_k) \). By the self-similarity of the fBm, it suffices to show the convergence in distribution of \((B^{(n)}, Y^{(n)})\) to \((B, V)\), where

\[
B_k^{(n)} = n^{-H} \sum_{[na_k] < j \leq [nb_k]} X_j,
\]

\[
Y_k^{(n)} = \frac{1}{\sqrt{n}} \sum_{[na_k] < j \leq [nb_k]} X_j^k,
\]

with \( X_j = B^H - B_{j-1}^H \) and \( 1 \leq k \leq N \).

We denote by \( \mathcal{H} \) the Hilbert space defined as the completion of the step functions on \([0, T]\) under the scalar product

\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = E(B_t^H B_s^H).
\]

Using Theorem \( \ref{thm:main} \) it suffices to show that:

\[
\lim_{n \to \infty} E \left( B_k^{(n)} B_{j}^H \right) = \mathbb{E} \left( (B_{b_k}^H - B_{a_k}^H) (B_{b_h}^H - B_{a_h}^H) \right), \quad (17)
\]

\[
\lim_{n \to \infty} E \left( B_k^{(n)} J_m Y_k^{(n)} \right) = 0, \quad (18)
\]

\[
\lim_{n \to \infty} E \left( J_m Y_k^{(n)} J_m Y_h^{(n)} \right) = \delta_{kh} (b_k - a_k) \sigma_m^2, \quad (19)
\]

and

\[
\lim_{n \to \infty} \left\| DJ_m Y_k^{(n)} \right\|_{\mathcal{H}}^2 = (b_k - a_k) m \sigma_m^2, \quad (20)
\]

in \( L^2 \), for all \( 1 \leq m \leq \kappa \), and \( 1 \leq k, h \leq N \). The variances \( \sigma_m^2 \) must satisfy

\[
\sum_{m=1}^{\kappa} \sigma_m^2 = c^2.
\]

The projection on the \( m \)th Wiener chaos \( J_m Y_k^{(n)} \) has the form

\[
\sum_{[na_k] < j \leq [nb_k]} \frac{c_m^2}{m!} H_m(X_j),
\]

where \( H_m(x) \) denotes the \( m \)th Hermite polynomial. The convergences \( (17) \) and \( (18) \) are immediate. To prove \( (19) \) we write

\[
E \left( J_m Y_k^{(n)} J_m Y_h^{(n)} \right) = \frac{c_m^2}{n} \sum_{[na_k] < j \leq [nb_k]} \mathbb{E} \left( H_m(X_j) H_m(X_{\ell}) \right)
\]

\[
\to \delta_{kh} \sum_{j=0}^{\infty} \frac{1}{m!} \rho_H^m(j),
\]

as \( n \) tends to infinity. On the other hand, we have

\[
\left\| DJ_m Y_k^{(n)} \right\|_{\mathcal{H}}^2 = \frac{c_m^2}{n} \sum_{[na_k] < j \leq [nb_k]} \left\| H_{m-1}(X_j) 1_{(j-1,j]} \right\|_{\mathcal{H}}^2
\]

\[
= \frac{c_m^2}{n} \sum_{[na_k] < i,j \leq [nb_k]} H_{m-1}(X_i) H_{m-1}(X_j) \rho_H(j-i)
\]

\[
= \frac{c_m^2}{n} \sum_{i=[na_k]+1}^{[nb_k]} \sum_{j=0}^{[nb_k]-[na_k]} H_{m-1}(X_i) H_{m-1}(X_{i+j}) \rho_H(j).
\]
We claim that the series \( \xi_i = \sum_{j=0}^{\infty} H_{m-1}(X_i)H_{m-1}(X_{i+j})\rho_H(j) \), converges almost surely and in \( L^2 \), and \( \{\xi_i, i \geq 1\} \) is a stationary ergodic sequence. The convergence in \( L^2 \) follows from the fact that \( \sup_j \mathbb{E}\left[|H_{m-1}(X_i)H_{m-1}(X_{i+j})|^2\right] < \infty \), and \( \sum_{j=0}^{\infty} |\rho_H(j)| < \infty \). On the other hand, the sequence \( \{\xi_i, i \geq 1\} \) is ergodic because \( \{X_i, i \geq 1\} \) is so. Hence, by the ergodic theorem we have in \( L^2 \)

\[
\lim_{n \to \infty} \left\| DJ_m Y_k^{(n)}(t) \right\|_{H^q}^2 = c_m^2(b_k - a_k) \sum_{j=0}^{\infty} \mathbb{E}(H_{m-1}(X_1)H_{m-1}(X_{1+j})) \rho_H(j)
\]

\[
= c_m^2(b_k - a_k) \sum_{j=0}^{\infty} \frac{1}{(m-1)!} \rho_H(j)^m,
\]

which implies (20).

**Step 2.** Taking into account that all \( L^p \) norms, for \( 1 < p < \infty \), are equivalent on a fixed sum of Wiener chaos, in order to show that the sequence \( Z_t^{(n)} \) is tight in \( D([0, T])^2 \) it suffices to show that

\[
\mathbb{E}\left[|Z_t^{(n)} - Z_s^{(n)}|^2\right] = \frac{1}{n} \mathbb{E}\left[\sum_{j=|nt|+1}^{[nt]} X_j^2\right] \leq C|t - s|,
\]

and this follows easily as above. ■

The convergence of the finite dimensional distributions in the above theorem can also be deduced from general central limit theorems for functionals of Gaussian stationary sequences satisfying the Hermite rank condition (see Brauer and Major [1]). A related result for the function \( g(x) = |x|^p - \mathbb{E}(|B_H|^p) \), where \( p > 0 \) and \( H \in (0, \frac{3}{4}) \) was obtained by Corcuera, Nualart and Woerner in [2]. The central limit theorem was proved in this case using the approach of Nualart and Peccati [7] (see [2], Proposition 10).

The above result is motivated by the extension of the Itô formula to the fractional Brownian motion in the critical case \( H = \frac{1}{2} \), using discrete Riemann sums. This problem has been considered by Swanson in [11] in the case of the solution of the one-dimensional stochastic heat equation driven by a space-time white noise \( \{u(t, x), t \geq 0\} \), which behaves as a fractional Brownian motion with Hurst parameter \( H = \frac{1}{2} \), for any fixed \( x \in \mathbb{R} \). In this case, the convergence in law to a Brownian motion is proved for a modified sum of squares of the increments.

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