Nonlinear coherent loss for generating non-classical states

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Abstract
Here, we discuss a generation of non-classical states of bosonic mode with the help of artificially designed loss, namely the nonlinear coherent loss. We show how to generate superpositions of Fock states, and how it is possible to ‘comb’ the initial states leaving only states with certain properties in the resulting superposition (for example, a generation of a superposition of Fock states with odd number of particles). We discuss purity of generated states and estimate maximal achievable generation fidelity.

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1. Introduction

Artificially designed nonlinear loss is a rather novel and highly promising way to generate deterministically non-classical states. Nonlinear loss possesses a number of advantages over more usual nonlinear unitary dynamics. First of all, it can turn a mixed initial state into a final pure one, which is generally impossible with unitary transformations. Then, the artificial loss is actually turning the greatest enemy of the non-classical state generators into their biggest ally. In schemes of generation of non-classical states via unitary dynamics, linear losses are prone to destroy quantum features rather quickly (here one might recall famous ‘Schrödinger-cat states’ turning into mixtures with the rate proportional to the average particle number of the state [1]). However, nonlinear losses can force the system into a certain non-vacuum state (‘pointer state’ [2, 3]), and make it remain there despite possible perturbations (which was epitomized in the concept of ‘protecting’ quantum states [4–6]). Then, the nonlinear loss generation scheme can be extremely robust, and is able to turn a wide variety of initial states into a predefined output state [7]. For example, one can produce ‘Schrödinger-cat states’ [8], and Fock states [9] this way. The nonlinear loss can be a flexible and efficient tool of quantum computation [10].

Recently, significant progress was achieved in the development of practical realizable schemes of artificial nonlinear loss. It was shown that it is possible to produce a wide variety of nonlinear losses for vibrational states of ions in magnetic traps [11, 12], or atoms in optical...
lattices [13]. Using a combination of optical nonlinearities, one can produce nonlinear losses of electromagnetic field modes [14–17]. Implementing correlated linear losses, one can also produce an effective nonlinear loss in Bose–Einstein condensates [18]. Correlated loss gives an opportunity to design nonlinear losses for the generation of single-photon states and strongly sub-Poissonian states in multi-core optical fibers [19].

This work is devoted to the analysis of possible states which can be produced with the help of designed nonlinear coherent loss (NCL) quite commonly encountered in the schemes mentioned above. This kind of loss is able to produce a wide range of non-classical states. We discuss conditions and prerequisites for generation of such states, provided an estimation of fidelity and purity of generated states. In particular, we show how to generate an arbitrary Fock state with fidelity arbitrarily close to unity using NCL and an input coherent state. Also, we analyze the generation of finite or countable superpositions of Fock states. It appears that one cannot reach a unit fidelity for generation of superposition of two Fock states. The fidelity of such superposition generation gets worse when the states of the pair are strongly different in particle numbers. On the other hand, we show that it is possible to cut a countable set of Fock-state components from the initial state with almost perfect fidelity, thus effectively ‘combing’ the input state and producing an almost pure state as the result.

The outline of the paper is as follows. In section 2, we describe nonlinear coherent states and introduce NCL. Here, the master equation for coherent losses is also given.

In section 3, we investigate general properties of NCL and analyze their connection with features of a stationary state which is to be obtained with the help of the kind of NCL considered.

In section 4, we discuss the generation of pure Fock states and consider the problem of coherence preserving for two important classes of states that can be generated by the discussed approach starting from a coherent state: superpositions of two Fock states (either including vacuum state or not) and superpositions of states with photon numbers distributed with equal intervals (e.g. states with even or odd photon numbers). In section 5, the conditions necessary for complete preservation of coherence are derived. It is shown that coherent states taken as starting states for the considered type of evolution do not satisfy the conditions exactly and, therefore, cannot be used for creation of completely pure nonclassical states by nonlinear absorption. However, an optimal classical starting state belongs to the class of coherent states, and the final state, arbitrarily close to pure state, can be generated by choosing sufficiently high amplitude of the starting coherent state.

2. Nonlinear coherent loss

In our work, we consider the dynamics of a single bosonic mode described by the following standard master equation in the Lindblad form:

\[
\frac{d\rho(t)}{dt} = \gamma (2\hat{L}\rho(t)\hat{L}^* - \hat{L}^*\hat{L}\rho(t) - \rho(t)\hat{L}^*\hat{L}),
\]

where \( \gamma > 0 \) is the decay rate and \( \hat{L} \) is the Lindblad operator. We assume that our bosonic mode is described by the creation and annihilation operators \( \hat{a}^+ \) and \( \hat{a} \). We consider the non-unitary dynamics generated by the following general class of Lindblad operators:

\[
\hat{L} = \hat{a} f(\hat{n}),
\]

where \( \hat{n} = \hat{a}^+\hat{a} \) and \( f(n) \) is a non-negative function.

The Lindblad operator (2) can be considered as the annihilation operator of the so-called \( f \)-deformed harmonic oscillator with the commutation relations [22]

\[
[L, L^\dagger] = (\hat{n} + 1)(f(\hat{n} + 1)) - \hat{n}(f(\hat{n}))^2.
\]
Eigenstates of the operator $L$ were termed ‘nonlinear coherent states’ [23] (it is curious to note that a specific subclass termed ‘Mittag–Leffler coherent states’ does actually arise in micromasers in the presence of loss and incoherent pump [24]). For that reason, we refer to the decay described by the Lindblad operator (2) as NCL. It is interesting that any pure state non-orthogonal to an arbitrary Fock state can be exactly represented as a nonlinear coherent state [5]. If it is orthogonal to some Fock states (for example, if this pure state is a finite superposition of Fock states), then one can still build a nonlinear coherent state closely approximating the state in question [5].

Thus, NCL looks highly promising for non-classical state generation. In [5], it was shown how to build a function $f$ leading to an approximate generation of an arbitrary Fock state from the initial coherent one. In more recent work [17], it was shown that the single-particle Fock state, $|1\rangle$, can be generated with arbitrarily high fidelity from the initial coherent state by NCL with the Lindblad operator (2) with $f(\hat{n}) = \hat{n} - 1$.

It should be emphasized that NCL design is completely realistic and could be realized in practice. For vibrational states of ions in magnetic traps NCL is already realized [11, 12]. Recently, it was shown how to produce NCL by simple adjustment of well parameters in the three-well trap configuration for Bose–Einstein condensates [18]. Very recently, the new realistic way of realizing NCL in multi-core optical fiber was suggested [19]. It was shown that for experimentally realistic values of Kerr nonlinearity of chalcogenide fibers with subwavelength core (which is $10^3$ times higher than Kerr nonlinearity of a conventional fused silica optical fiber [20, 21]), it is possible to realize NCL in such a scheme and achieve a deterministic generation of a single-photon state [19].

Note that the losses and imperfections of the scheme, generally, do not spoil the desired form of NCL (however, they can lead to the appearance of other losses, both linear and nonlinear, spoiling the effect of NCL). The form of NCL is defined by the nonlinearity present in the scheme. For illustration in the appendix an example of the nonlinear loss appearance (and NCL appearance, in particular) is given for the system of nonlinear bosonic modes coupled to the strongly dissipative mode).

In this work, we do not intend to discuss practical realization of NCL in more detail. Our aim is more fundamental; we want to discuss the very possibilities offered by NCL. Furthermore, we discuss general limitations on the states that can be generated by NCL starting from a coherent state. We demonstrate when it is possible to generate pure superpositions of Fock states, and when losses lead to the decrease of the state purity.

3. General properties of the function $f(n)$ and corresponding stationary states

In order to analyze dependence between general properties of the function $f(n)$, describing nonlinear absorption, and stationary states that can be obtained as a result of evolution, characterized by the master equation equation (1), it is convenient to decompose the density matrix in terms of Fock states:

$$\rho(t) = \sum_{n,m} \rho_{nm}(t) |n\rangle \langle m|.$$  \hspace{1cm} (3)

Then, the master equation (equation (1)) leads to the following system of equations for density matrix elements:

$$\frac{d\rho_{nm}(t)}{dt} = 2\gamma F(n + 1)F(m + 1)\rho_{n+1,m+1}(t) - \gamma \{F^2(n) + F^2(m)\} \rho_{nm}(t),$$  \hspace{1cm} (4)

where $F(n) = \sqrt{n} f(n) \geq 0$, $F(0) = 0$.  \hspace{1cm} (5)
Note that, according to the system (4), different diagonals of the density matrix evolve independently. This fact can be made more apparent by introducing the notation
\[ \rho_{n,n+k}(t) = c_k \xi_k(n,t), \]
where constants \( c_k \) can be arbitrary and will be fixed later. The quantities \( \xi_k(n,t) \) satisfy the following equation:
\[ \frac{d\xi_k(n,t)}{dt} = 2\gamma F(n+1)F(n+k+1)\xi_k(n+1,t) - \gamma \{ F^2(n) + F^2(n+k) \} \xi_k(n,t), \]
(6)
which does not contain \( \xi_k(n',t) \) for \( k' \neq k \).

The density matrix is the Hermitian one. It is completely defined by elements \( \rho_{nm} \) with \( m \geq n \). Furthermore, we will take into consideration only the main diagonal of the density matrix and the diagonals lying below the main diagonal. We assume \( k \geq 0 \) in equations (5), (6) and similar equations.

**Lemma 1.** If quantities \( \xi_k(n,t) \) are positive (non-negative) for all \( n \) at \( t = 0 \), they remain positive (non-negative) for all \( t > 0 \).

**Proof.** Equation (6) can be rewritten as
\[ \frac{d\tilde{\xi}_k(n,t)}{dt} = 2\gamma F(n+1)F(n+k+1)\tilde{\xi}_k(n+1,t) e^{\gamma t(F^2(n)+F^2(n+k))}, \]
(7)
where \( \tilde{\xi}_k(n,t) = \xi_k(n,t) e^{\gamma t(F^2(n)+F^2(n+k))} \). The right-hand side of equation (7) is non-negative for positive or non-negative values of quantities \( \tilde{\xi}_k(n,t) \). It leads to non-negativity of derivatives on the left-hand side of equation (7) and non-decreasing character of evolution of the quantities \( \xi_k(n,t) \), if they are initially non-negative. The exponential factor, connecting \( \xi_k(n,t) \) and \( \tilde{\xi}_k(n,t) \), is strictly positive, and therefore quantities \( \xi_k(n,t) \) (together with \( \tilde{\xi}_k(n,t) \)) preserve their positivity (non-negativity) during evolution.

If the initial state of the considered field mode is the coherent state \( |\alpha\rangle \), the condition of lemma 1 can be satisfied by setting \( c_k = (\alpha^*)^k \) in equation (5). Then
\[ \xi_k(n,0) = \rho_{n,n+k}(0)/c_k = |\alpha|^2 e^{-|\alpha|^2} \sqrt{n!(n+k)!} > 0 \]
for all \( n, k \).

It should be noted that any density matrix can be represented in diagonal form in terms of coherent states using the Glauber–Sudarshan \( P \)-representation [26]
\[ \rho(0) = \int P(\alpha)|\alpha\rangle\langle\alpha|, \]
(8)
where \( P(x) \) is the Glauber–Sudarshan quasiprobability distribution. Moreover, one can closely approximate the state in question using a discrete set of coherent-state projectors on a square lattice [25]. Therefore, one may conclude that all results derived here for initial coherent state conditions will be valid for an arbitrary initial state.

**Lemma 2.** If \( \xi_k(n,0) > 0 \) for all \( n \), for the stationary value \( \xi_k(n_1) = \lim_{t \to \infty} \xi_k(n_1,t) \) of the quantity \( \xi_k(n_1,t) \) to be non-zero, it is necessary that \( F(n_1) = F(n_1 + k) \).

**Proof.** Summing equation (6) over \( n \) leads to the following relation:
\[ \frac{d}{dt} \sum_{n=0}^{\infty} \xi_k(n,t) = -\gamma \sum_{n=0}^{\infty} (F(n) - F(n+k))^2 \xi_k(n,t), \]
(9)
where we have taken into account that \( F(0) = 0 \). For the stationary state the left-hand side of equation (9) equals zero. According to lemma 1, each term on the right-hand side of equation (9) is non-negative, and the sum can be equal to zero in the stationary state, when for each \( n \) either \( \xi_k(n) = \lim_{t \to \infty} \xi_k(n, t) = 0 \) or \( F(n) = F(n + k) \).

\[ \square \]

Theorem 1. The density matrix element \( \rho_{n,n+k}(t) \) can have non-zero value \( \lim_{t \to \infty} \rho_{n,n+k}(t) \neq 0 \) in the stationary state of evolution, described by the master equation (equation (4)), only if \( F(n) = F(n + k) = 0 \).

Proof. If the initial state of the field mode is a coherent state \( \rho(0) = |\alpha\rangle \langle \alpha| \), then conditions of lemmas 1 and 2 are satisfied by setting \( c_k = (\alpha^*)^k \). Therefore, for \( \lim_{t \to \infty} \rho_{n,n+k}(t) \neq 0 \) it is necessary that \( F(n) = F(n + k) \). Equation (6) implies that the following relation is satisfied:

\[
d t \sum_{m=0}^{n-1} \xi_k(m, t) = 2\gamma F(n)F(n + k)\xi_k(n, t) - \gamma \sum_{m=0}^{n-1} (F(m) - F(m + k))^2 \xi_k(m, t). \tag{11}
\]

In the limit \( t \to \infty \), equation (11) is transformed into

\[
2\gamma F^2(n)\xi_k(n) = 0, \tag{12}
\]

because, according to lemma 2, for all \( m \) either \( F(m) - F(m + k) = 0 \) or \( \xi_k(m, t) \equiv \lim_{t \to \infty} \xi_k(m, t) = 0 \). Therefore, for \( \lim_{t \to \infty} \rho_{n,n+k}(t) = c_k\xi_k(n) \neq 0 \), it is necessary that \( F(n) = F(n + k) \) and \( F(n) = 0 \).

Taking into account that any initial state of the considered mode can be represented using \( P \)-representation, one concludes that the quantity \( \lim_{t \to \infty} \rho_{n,n+k}(t) \) can have non-zero value, only if it is non-zero in the stationary state for at least one coherent state \( |\alpha\rangle \) taken as the initial state. Then, the first paragraph of the proof is applicable, and we obtain \( F(n) = F(n + k) = 0 \) as a necessary condition.

\[ \square \]

In order to derive explicit expressions for stationary values of non-zero elements of the density matrix, one can rearrange the system of equations (6) in the following way. By introducing quantities

\[
T_k(n) = \frac{2F(n)F(n + k)}{F^2(n) + F^2(n + k)},
\]

equation (6) can be transformed as

\[
\frac{d\xi_k(n, t)}{dt} = \gamma T_k(n + 1)\Phi_k(n + 1)\xi_k(n + 1, t) - \gamma \Phi_k(n)\xi_k(n, t), \tag{14}
\]

where \( \Phi_k(n) = F^2(n) + F^2(n + k) \). Then, it is quite easy to show that the following equality holds:

\[
\frac{d}{dt} \sum_{m=n_1}^{n_1-1} \xi_k(m, t)T_k(n_1 + 1) \ldots T_k(m - 1)T_k(m) = 2\gamma F(n_2)F(n_2 + k)\xi_k(n_2, t) - \gamma \Phi_k(n_1)\xi_k(n_1, t). \tag{15}
\]

Let the number \( n_1 \) correspond to the quantity \( \xi_k(n_1, t) \), which has a non-zero stationary value \( \xi_k(n_1) = \lim_{t \to \infty} \xi_k(n_1, t) \) (i.e. the conditions \( F(n_1) = 0 \) and \( F(n_1 + k) = 0 \) are satisfied). We can choose \( n_2 \) to be the minimal number greater than \( n_1 \), for which at least one of the following equations is satisfied: \( F(n_2) = 0 \) or \( F(n_2 + k) = 0 \) (\( n_2 \) can be equal to infinity; then \( \xi_k(n_2, t) \to 0 \)). For this choice of \( n_1 \) and \( n_2 \), the right-hand side of equation (15)
equals zero. The left-hand side of equation (15) remains constant during evolution. In the
limit \( t \to \infty \), we obtain

\[
\xi_k(n_1) = \xi_k(n_1, 0) + \xi_k(n_1 + 1, 0) T_k(n_1 + 1) + \cdots + \xi_k(n_2 - 1, 0) T_k(n_1 + 1) \cdots T_k(n_2 - 1).
\]  

(16)

Using equation (5) and returning to the density matrix elements, one can rewrite equation (16) as

\[
\rho_{n_1, n_1+k}(\infty) = \rho_{n_1, n_1+k}(0) + \rho_{n_1+1, n_1+k+1}(0) T_k(n_1 + 1) + \cdots + \rho_{n_2-1, n_2+k-1}(0) T_k(n_1 + 1) \cdots T_k(n_2 - 1).
\]  

(17)

The obtained expression for non-zero elements of the stationary density matrix can be interpreted in the following simple way. The master equation (1) in the form (4) describes the ‘flow’ of amplitudes of density matrix elements along diagonals in the direction of photon number decreasing. The ‘transmittance’ of the transition between \( \rho_{n_1, n_1+k} \) and \( \rho_{n_2-1, n_2+k-1} \) equals \( T_k(n) \) (see equation (14)). Elements \( \rho_{n,n+k} \) with \( F(n) = F(n+k) = 0 \) ‘accumulate’ the flow (i.e. they do not transmit it to the next elements \( \rho_{n-1,n+k-1} \)). Elements \( \rho_{n,n+k} \) with either \( F(n) = 0 \) or \( F(n+k) = 0 \) (but without the two conditions being satisfied simultaneously) neither transmit the flow, nor accumulate it; for such elements \( T_k(n) = 0 \). This interpretation is illustrated in figure 1.

The left-hand side of equation (15) remains constant during evolution. In the

\[
\xi_k(n_1) = \xi_k(n_1, 0) + \xi_k(n_1 + 1, 0) T_k(n_1 + 1) + \cdots + \xi_k(n_2 - 1, 0) T_k(n_1 + 1) \cdots T_k(n_2 - 1).
\]  

(16)

It should be noted that, according to the definition equation (13) of ‘transmittances’ \( T_k(n) \), amplitudes of diagonal elements of the density matrix are transmitted perfectly: \( T_0(n) \equiv 1 \) (i.e. the trace is expectedly preserved). In order to attain maximal coherence, transmittances
for non-diagonal elements must be also equal to unity (at least those present in expressions of the form of equation (17) for non-zero elements of the density matrix). Therefore, the condition of preserving maximal coherence in the stationary state is

\[ F(n) = F(n + k), \]

for density matrix elements giving non-zero contributions to non-zero elements of the stationary density matrix.

4. Examples

4.1. Generation of Fock states

The most ‘natural’ nonclassical states that can be generated by NCL with arbitrarily high fidelity are pure Fock states. Here, and later in this section, we assume that the initial state is the coherent state, \(|\alpha\rangle\), with the amplitude \(\alpha\).

Suppose that the function \(f(n)\) has only one zero \(n_1\): \(f(n_1) = 0\), \(f(n) \neq 0\) for \(n \neq n_1\). Then, the function \(F(n)\) has two zeros: \(F(0) = 0\) and \(F(n_1) = 0\). According to theorem 1, only elements \(\rho_{00}, \rho_{nn}, \rho_{n1} = \rho^{*}_{1n}\) can have non-zero values in the stationary state for the system with such NCL. Equation (17) implies that stationary values of on-zero diagonal elements of the density matrix are described by the following expressions:

\[ \rho_{00} = \sum_{k=0}^{n_1-1} q_k^2 |\alpha|, \]

where

\[ q_m(|\alpha|) = |\alpha|^m e^{-|\alpha|^2/2} / \sqrt{m!}, \]

and \(\rho_{n1,n1} = 1 - \rho_{00}\).

For large enough amplitudes, \(|\alpha|\), of the initial coherent state the following estimation is valid:

\[ \rho_{00} < n_1 q_{n_1-1}^2 |\alpha| \rightarrow \infty \rightarrow 0. \]

Therefore, the fidelity \(F = \langle n_1 |\rho| n_1 \rangle = \rho_{n_1,n1}\) of generating the Fock state \(|n_1\rangle\) can be made arbitrarily close to unity by choosing a large enough amplitude \(|\alpha|\) of the starting coherent state (see figure 2).
4.2. Superposition of the Fock states $|0\rangle$ and $|n\rangle$

As the first example of coherent superpositions that can be generated by NCL, we consider states maximally close to the state

$$|\Psi_{0n}\rangle = \frac{|0\rangle + e^{i\phi}|n\rangle}{\sqrt{2}}. \tag{22}$$

If we require $F(n) = F(0) = 0$ and $F(m) \neq 0$ for $m \neq 0, n$, only elements $\rho_{00}, \rho_{nn}, \rho_{0n} \neq 0$ (see equation (3)) of the stationary density matrix, $\rho$, will be non-zero (theorem 1). Then, fidelity of generating the state (equation (22)) is

$$F = \langle \Psi_{0n} | \rho | \Psi_{0n} \rangle = \frac{1}{2} + \text{Re}(\rho_{0n} e^{i\phi}) \leq \frac{1}{2} + |\rho_{0n}|. \tag{23}$$

The maximal possible value of fidelity is $F = \frac{1}{2} + |\rho_{0n}|$. Positivity of the density matrix implies that $|\rho_{0n}| \leq \frac{1}{2}$. For convenience, we will characterize coherence of the stationary density matrix by the quantity

$$c_{0n} = 2|\rho_{0n}|, \quad c_{0n} \in [0, 1]. \tag{24}$$

So, maximization of the coherence $c_{0n}$ leads also to maximality of the fidelity $F$.

For the initial coherent state amplitude $\alpha = |\alpha| e^{i\phi} / n$ the following equality holds: $\rho_{0n}(0) = |\rho_{0n}(0)| e^{-i\phi}$. According to lemma 1, this holds also for any moment of time, $t$. Therefore, in order to derive conditions for preserving maximal fidelity by nonlinear absorption, one needs to carry out maximization of the quantity $c_{0n}$ over amplitudes $|\alpha|$. According to equations (17) and (24) (see also figure 1(b)), the coherence $c_{0n}$ equals

$$c_{0n} = 2 \sum_{m=0}^{n-1} q_m(|\alpha|) q_{n+m}(|\alpha|) T_n(m), \tag{25}$$

where $q_m(|\alpha|)$ are given by equation (20). To maximize the coherence $c_{0n}$ one needs to have $T_n(m) = 1$ for $m = 0, \ldots, n - 1$. According to equation (18), this condition implies that

$$F(m) = F(m+n) \quad \text{for} \quad m = 0, \ldots, n - 1. \tag{26}$$

Further maximization of equation (25) can be carried out numerically. Results of numerical calculations are shown in figure 3. One can see that the best performance of this method is achieved for $n = 2$. Then, according to equation (25), the coherence is

$$c_{02} = \sqrt{2}(|\alpha|^2 + |\alpha|^4 / \sqrt{3}) e^{-i|\alpha|^2}. \tag{27}$$

The modulus of optimal amplitude of the initial coherent state is $|\alpha_{opt}| = \sqrt{\frac{1}{2} (2 - \sqrt{3} + \sqrt{7})} \approx 1.2$. Elements of the final density matrix in the optimal case are $\rho_{00} = 0.60, \rho_{22} = 0.40, |\rho_{02}| = 0.44$ (see figure 3(b)), and the coherence is $c_{02} = 0.88$.

4.3. Superpositions of Fock states $|m\rangle$ and $|n\rangle$

The next example is the generation of a state maximally close to the state

$$|\Psi_{nm}\rangle = \frac{|n\rangle + e^{i\phi}|m\rangle}{\sqrt{2}}. \tag{28}$$

Note that there is rather pronounced difference between the current example and the one considered in the previous subsection. According to the definition of the function $F(n)$, $F(0) = 0$ holds for any system. For any amplitude $\alpha$ of the initial coherent state, the element $\rho_{00}$ of the initial density matrix has a non-zero value. Therefore, according to equation (17),
Figure 3. (a) The modulus of the optimal amplitude of the starting coherent state (gray dots) and the maximal achievable coherence (black dots) for generation of the state $|\Psi_{01}\rangle$. (b) Absolute values of the stationary state density matrix generated for the initial coherent state optimal for generation of $|\Psi_{02}\rangle$.

this density matrix element will have a non-zero value in the final state. However, its value for the ideal state $|\Psi_{nm}\rangle$ must be equal to zero.

In order to generate the state sufficiently close to $|\Psi_{nm}\rangle$, we require $F(m) = F(n) = 0$, $F(k) \neq 0$ for $k \neq n, m, 0$. Only by the final density matrix, elements $\rho_{ij}$ with $i, j = 0, n, m$ are non-zero in the stationary state. Similar to the previous example, the fidelity of the desired state is

$$F = \langle \Psi_{nm} | \rho | \Psi_{nm} \rangle$$

$$= \frac{1}{2} + \text{Re}(\rho_{nm} e^{i\phi}) \leq \frac{1}{2} + |\rho_{nm}|.$$  (29)

Again, we define the coherence

$$c_{nm} = 2|\rho_{nm}|, \quad c_{nm} \in [0, 1].$$  (30)

The phase of the optimal initial coherent state in this case is defined by the following equation:

$$\alpha = |\alpha| e^{i\phi/(m-n)}.$$  (31)

For preserving maximal coherence and, therefore, for obtaining maximal fidelity of the generated state, we need

$$F(k) = F(k + m - n) \quad \text{for} \quad k = 0, \ldots, m - n - 1$$  (32)

(see equations (18) and (26)).

If the above conditions are satisfied, the coherence $c_{nm}$ is described by the following expression:

$$c_{nm} = 2 \sum_{k=m}^{m-1} q_k(|\alpha|) q_{k+m-n}(|\alpha|).$$  (33)

Optimal values of $|\alpha|$ and maximal achievable coherence $c_{nm}$ can be found either numerically (dots in figures 4(a) and (b)), or by approximate analytical expressions (lines in figures 4(a) and (b)).

To obtain an analytical expression for the coherence $c_{nm}$, we will take into account that (according to central limit theorem), for large enough values of $|\alpha|$ the following approximation is valid for $q_k(|\alpha|)$ given by equation (20):

$$q_k^2(|\alpha|) \approx \frac{1}{\sqrt{2\pi}|\alpha|} \exp \left\{ -\frac{(k - |\alpha|^2)^2}{2|\alpha|^2} \right\}.$$  (34)
Figure 4. Optimal amplitude of the starting coherent state (a) and maximal achievable coherence (b) for generation of the state $|\Psi_{nm}\rangle$: dots and lines correspond to numerical optimization and approximate analytical expressions (equation (36), equation (37)); the black, dark gray, gray, light gray dots and lines correspond to $m = n + 1, n + 3, n + 5, n + 7$, respectively. Absolute values of elements of stationary state density matrices generated by the schemes optimized for creation of the state $|\Psi_{4,9}\rangle$ (c) and $|\Psi_{10,17}\rangle$ (d).

Then, the coherence is approximately equal to

$$c_{nm} = \frac{2}{\sqrt{2\pi}|\alpha|} \sum_{k=n}^{m-1} \exp \left\{ -\frac{(k - |\alpha|^2)^2 + (k + m - n - |\alpha|^2)^2}{4|\alpha|^2} \right\},$$  

(35)

To maximize the coherence one needs practically the same $|\alpha|$, as required for maximalization of the numerator in the exponent of equation (35). This value is

$$|\alpha_{opt}|^2 \approx m - \frac{1}{2}. \quad (36)$$

Taking into account equation (36) and approximating summation by integration in equation (35), one can derive the following expression for the coherence:

$$c_{nm} \approx 2 \text{erf} \left( \frac{\Delta n}{2\sqrt{2|\alpha_{opt}|}} \right) \exp \left( -\frac{\Delta n^2}{8|\alpha_{opt}|^2} \right),$$  

(37)

where $\Delta n = m - n$. Figure 4 shows that obtained approximate expressions represent results of numerical calculations with sufficiently high accuracy.

Expression (37) for the coherence $c_{nm}$ has a maximal value 0.84 for $\Delta n^2 \approx 6.4|\alpha_{opt}|^2$. Therefore, this method is most suitable for generation of the states with

$$(m - n)^2 \approx 6.4(m - \frac{1}{2}). \quad (38)$$
Using the approximation (equation (34)), one can show that the density matrix elements of the optimal stationary state are described by the following expressions:

\[ \rho_{mm} = \frac{1}{2}, \quad \rho_{nn} = \frac{1}{2} \text{erf}(2\xi), \]

\[ |\rho_{nm}| = \text{erf}(\xi) e^{-\xi^2}, \quad \rho_{00} = \frac{1}{2} \{1 - \text{erf}(2\xi)\}, \]

(39)

where \( \xi = \frac{\Delta n}{2\sqrt{2}|\alpha_{\text{opt}}|}. \)

Examples of density matrices of the states generated by the schemes optimized for \( n = 4, m = 9 \) and \( n = 10, m = 17 \), are shown in figures 4(c) and (d).

4.4. Superpositions of states with equidistant photon numbers

Here, we consider examples of functions \( f(n) \) (and \( F(n) \)) with countable sets of zeros. We focus on the case of functions with equidistantly distributed zeros:

\[ f(jN + n_0) = 0, \quad j = 0, 1, 2, \ldots, \]

(40)

where \( N \) is the distance between neighboring zeros and \( n_0, 0 \leq n_0 < N \), determines the position of the first zero. It should be noted that, according to the definition of \( F(n) \), one also has \( F(0) = 0 \) for any function \( f(n) \).

For such kinds of NCL only elements \( \rho_{nm} \) with \( n, m = 0, n_0, n_0 + N, n_0 + 2N, \ldots \) remain non-zero in the stationary state.

As in previously discussed examples, for preserving the maximal coherence we require that the function \( F(n) \) should satisfy equation (18). This implies

\[ F(n + N) = F(n), \quad \text{for} \quad n = 1, 2, \ldots, \]

(41)

i.e. the function \( F(n) \) must be periodic.

In previous examples, zeros of \( F(n) \) were giving a clue for choosing the amplitude for the initial coherent state. It is not so in the current case. Now the amplitude \( |\alpha| \) will be considered as a free parameter, and the final state will be investigated as a function of the amplitude. For the sake of simplicity, we assume that the amplitude \( \alpha \) is real and positive.

According to equation (17), non-zero elements of the stationary density matrix are described by the following expressions:

\[ \rho_{nm} = \sum_{k=0}^{N-1} q_{n-k}(|\alpha|)q_{m+k}(|\alpha|), \]

(42)

\[ \rho_{0n} = \sum_{k=0}^{n_0} q_k(|\alpha|)q_{n_0+k}(|\alpha|), \quad \text{if} \quad n_0 \neq 0. \]

(43)

For large amplitudes of the initial coherent state \( |\alpha|^2 \gg N \), the approximate expression for \( q_k(|\alpha|) \), provided by equation (34), can be used to simplify equation (42). One can show that in this case

\[ \rho_{nm} = Nq_{n}(|\alpha'|)q_{m}(|\alpha'|) \cdot \left\{1 + O\left(\frac{1}{|\alpha|^2}\right)\right\}, \]

(44)

where \( |\alpha'|^2 = |\alpha|^2 - (N-1)/2 \). Therefore, the stationary density matrix can be approximated with

\[ \rho \approx \left| \Phi_N^{(m)} \right| \left| \Phi_N^{(n)} \right|, \]

(45)
where
\[ |\Phi^{(m_0)}_N\rangle = \text{const} \cdot \sum_{j=0}^{\infty} \frac{\alpha^j N + m_0}{\sqrt{(jN + m_0)!}} e^{-|\alpha|^2/2} |\alpha|^j N + m_0 \rangle, \]
(46)
is the state that can be obtained from a coherent state \(|\alpha\rangle\) by retaining only the states with photon numbers \(m_0, m_0 + N, m_0 + 2N, \ldots\). On the other hand, the state \(|\Phi^{(m_0)}_N\rangle\) can be considered as a superposition of coherent states, distributed on a circle:
\[ |\Phi^{(m_0)}_N\rangle = \text{const} \cdot \sum_{k=0}^{N-1} e^{-2\pi i k m_0/N} |\alpha| e^{2\pi i k/N}. \]
(47)

For example, for \(N = 2\) the states \(|\Phi^{(0)}_2\rangle\) and \(|\Phi^{(1)}_2\rangle\) are superpositions of states with even and odd numbers of photons respectively and correspond to the following superpositions of coherent states with opposite phases: \(|\Phi^{(0)}_2\rangle \sim |\alpha\rangle + |-\alpha\rangle\) and \(|\Phi^{(1)}_2\rangle \sim |\alpha\rangle - |-\alpha\rangle\).

To describe the ‘quality’ of generation of the superpositions, it is convenient to consider the purity of the final state, defined as
\[ P = \text{Tr} \rho^2 = \sum_{n,m} \rho_{nm}\rho_{mn}. \]
(48)

According to equation (42), for large \(|\alpha|\), when the presence of a non-zero element \(\rho_{00}\) can be neglected, this expression can be rewritten as
\[ P \approx 1 - \frac{N^2 - 1}{24|\alpha|^2}. \]
(49)

Therefore, in the limit \(|\alpha| \to \infty\) the obtained stationary state tends to a pure state (figure 5(a)).

Several examples of the density matrix that can be generated by the method are shown in figures 5(b)–(d).

5. Discussion

The discussed examples show that systems with appropriate nonlinear losses can be used for creation of Fock state superpositions with sufficiently high fidelity. However, for all examples obtained stationary states were mixed. It is quite interesting to find the conditions that must be satisfied for the final stationary state to be pure and to determine whether such conditions can be fulfilled.

As stated above, equation (18) is one of the conditions necessary for complete coherence preserving.

Several other conditions are provided by following lemmas.

Lemma 3. If all diagonal elements of the initial state density matrix are positive, the final stationary state can be pure only if the function \(F(n)\) has equidistantly distributed zeros.

Proof. Suppose \(n_1, n_1'\) and \(n_2, n_2'\) are two pairs of successive zeros of the function \(F(n)\), and \(n_1' - n_1 < n_2' - n_2\). Either of \(n_1', n_2'\) can be equal to infinity, if the corresponding zero \(n_1\) or \(n_2\) is the last zero of \(F(n)\).
Then, according to equation (17), the density matrix elements have the following values in the stationary state:

\[
\rho_{n_1,n_1} = \sum_{k=0}^{n_1'-n_1-1} \rho_{n_1+k,n_1+k}(0),
\]

(51)

\[
\rho_{n_1,n_2} = \sum_{k=0}^{n_1'-n_1-1} \rho_{n_1+k,n_2+k}(0),
\]

(52)

\[
\rho_{n_2,n_2} = \sum_{k=0}^{n_2'-n_2-1} \rho_{n_2+k,n_2+k}(0) > \sum_{k=0}^{n_1'-n_1-1} \rho_{n_2+k,n_2+k}(0),
\]

(53)
where we have taken into account positivity of diagonal elements of the initial state density matrix. On the other hand, the following inequality holds:

\[
\left( \sum_k \rho_{n_1+k,n_1+k}(0) \right) \left( \sum_k \rho_{n_2+k,n_2+k}(0) \right) \geq \left( \sum_k \sqrt{\rho_{n_1+k,n_1+k}(0) \rho_{n_2+k,n_2+k}(0)} \right)^2 \geq \left| \sum_k \rho_{n_1+k,n_1+k}(0) \right|^2,
\]

when all the sums are taken over the same range of \( k \).

Therefore, assumption of nonequal distances \( n'_1 - n_1 \) and \( n'_2 - n_2 \) between successive zeros \( n_1, n'_1 \) and \( n_2, n'_2 \) leads to the following inequality:

\[
\rho_{n_1,n_1} \rho_{n_2,n_2} > |\rho_{n_1,n_2}|^2,
\]

which manifests that the state cannot be pure. Finally, the distance between any neighboring zeros must be equal to some constant value, which was denoted by \( N \) in the last example of section 4.

**Lemma 4.** Let \( n_1, n'_1 \) and \( n_2, n'_2 \) be two pairs of successive zeros of the function \( F(n) \), \( n'_1 - n_1 = n'_2 - n_2 \), and let all diagonal elements of the initial state density matrix be positive. The final stationary state can be pure only if the density matrix of the initial state satisfies the following condition:

\[
\frac{\rho_{n_1+k,n_1+k}(0)}{\rho_{n_2+k,n_2+k}(0)} = \frac{\rho_{n_1,n_1}(0)}{\rho_{n_2,n_2}(0)} \quad \text{for} \quad k = 0, \ldots, n'_1 - n_1 - 1.
\]

**Proof.** Elements \( \rho_{n_1,n_1}, \rho_{n_1,n_2}, \rho_{n_2,n_2} \) of the stationary state density matrix are described by equations (51), (52) and the first line of equation (53).

The condition

\[
\rho_{n_1,n_1} \rho_{n_2,n_2} = |\rho_{n_1,n_2}|^2,
\]

which is necessary for the state purity, is satisfied only if all parts of equation (54) are equal to each other. Obviously, the equality can be achieved only if ‘vectors’ \( \{\rho_{n_1+k,n_1+k}(0)\} \) and \( \{\rho_{n_2+k,n_2+k}(0)\} \) are collinear, which implies equation (56).

On the basis of these lemmas the following theorem can be proved.

**Theorem 2.** If the function \( F(n) \) has at least one zero \( n_1, n_1 > 0 \), and the initial state is classical, the final stationary state will be mixed.

**Proof.** Any classical initial state can be represented as a mixture of coherent states with positive weights. Therefore, if we prove that the stationary state will be mixed for an arbitrary initial coherent state, the stationary state will be proven to be mixed for any classical starting state.

For the initial state being a coherent state \( |\alpha\rangle \), the following statement holds:

\[
\frac{\rho_{n_1+k,n_1+k}(0)}{\rho_{n_2+k,n_2+k}(0)} = |\alpha|^{2(n_2-n_1)} \frac{(n_2+k)!}{(n_1+k)!}.
\]

Therefore, equation (56) is not satisfied except for the trivial case, when \( n'_1 - n_1 = 1 \), \( F(n) \equiv 0 \).

However, when the amplitude \( |\alpha| \) of the initial coherent state is large enough, the generated state can be very close to a pure state. Indeed, only the density matrix elements \( \rho_{n_1,n_2}(0) \) with
\(|n_1 - |\alpha|^2| \lesssim |\alpha|\), \(|n_2 - |\alpha|^2| \lesssim |\alpha|\) have significantly nonzero values. For \(n_1, n_2 \gg k\), equation (58) can be transformed into

\[
\frac{\rho_{n_1 + k, n_1 + k}(0)}{\rho_{n_2 + k, n_2 + k}(0)} \approx |\alpha|^{2(n_2 - n_1)} \left( \frac{n_2}{n_1} \right)^k \\
\approx |\alpha|^{2(n_2 - n_1)} \left( 1 + k \frac{n_2 - n_1}{n_1} \right) \\
= |\alpha|^{2(n_2 - n_1)} \left\{ 1 + O \left( \frac{1}{|\alpha|} \right) \right\}.
\]

Therefore, equation (56) can be satisfied with arbitrarily high precision by using high enough amplitudes of the starting coherent state.

6. Conclusions

We have investigated the nonlinear coherent loss (NCL) as a resource for generating non-classical states. We have established conditions for generating a prescribed state (namely, arbitrarily Fock states and specific superpositions of them) from the initial coherent state. We have highlighted a connection between properties of the Lindblad operator of NCL and the generated state. We have demonstrated that the state generated by NCL from the initial classical state will always be mixed. However, for certain classes of states one can generate almost pure states. In some cases, it is possible to reach high fidelity of generation by appropriately choosing an amplitude of the initial coherent state. Fock states belong to such a class; one can generate an almost pure Fock state for an initial coherent state of a sufficiently high amplitude. The situation is more complicated for finite superpositions of Fock states. For example, a superposition of two Fock states cannot be generated by NCL with the arbitrarily high fidelity from the initial coherent state; a strict upper border exists for this case. However, certain infinite superpositions can also be generated with an arbitrarily high fidelity. For example, one can ‘comb’ the initial coherent state cutting off Fock state components with an odd number of particles with an arbitrarily high fidelity.

Finally, it should be noted that though coherent states are ‘nonideal’ for generating states via NCL, they remain optimal initial classical states. Any classical state is a mixture of coherent states with positive weights. Quantum mechanical equations for evolution of the density matrix are always linear. Therefore, any final state obtained from a classical state is necessarily a mixture of final states which are to be obtained from corresponding initial coherent states. Thus, no classical initial state can lead to purity of the final state, greater than the purity, provided by ‘the best choice’ from the possible initial coherent states.

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Appendix. Designing the NCL by correlated loss

To highlight the concept of designing nonlinear loss in systems of coupled bosonic modes (which can be realized, for example, in Bose–Einstein condensates [18] or in optical fibers [19]), let us consider a model of \(N + 1\) bosonic modes, \(a_1 \ldots a_{N+1}\) coupled in the usual linear
way to the same Markovian reservoir. Thus, the model is described by the master equation in
the standard Lindblad form:
\[
\frac{d}{dt} \rho_{N+1} = -\frac{i}{\hbar} [H_{N+1}, \rho_{N+1}] + \gamma L_{N+1} \rho_{N+1},
\]
where \( H_{N+1} \) is the Hamiltonian describing unitary evolution of modes described by the
annihilation, \( a_i \), and creation, \( a_i^\dagger \), operators; \( \gamma > 0 \) is the linear decay rate. Here and in
the derivation below, we define superoperators \( L_j \) on the basis of Lindblad operators \( L_j \) in the
following way:
\[
L_j \rho = 2L_j \rho L_j^\dagger - L_j^\dagger L_j \rho - \rho L_j^\dagger L_j.
\]
In equation (A.1), the Lindblad operator is
\[
L_{N+1} \equiv b_{N+1} = \sum_{j=1}^{N+1} u_{N+1,j} a_j, \quad \sum_{j=1}^{N+1} |u_{N+1,j}|^2 = 1.
\]
It depends linearly on the annihilation operators \( a_j \) and therefore represents a collective mode.
One can introduce a new set of independent bosonic operators, \( b_j \), by the unitary transformation
\( u_{i,j} \):
\[
b_k = \sum_{j=1}^{N+1} u_{k,j} a_j,
\]
where coefficients \( u_{N+1,j} \) are fixed by the interaction between the system and the reservoir.
The Hamiltonian, \( H_{N+1} \), can be decomposed in terms of annihilation, \( b_{N+1} \), and creation, \( b_{N+1}^\dagger \),
operators of the collective mode:
\[
H_{N+1} = \sum_{m,n=0}^{\infty} F_{m,n} (b_{N+1}^\dagger)^m b_{N+1}^n,
\]
where operators \( F_{m,n} \) depend only on operators \( b_j \) and \( b_j^\dagger \) for \( j < N + 1 \).

Now we assume that the state of the collective mode \( b_{N+1} \) decays to the vacuum very
rapidly on the time scale of dynamics prescribed by the Hamiltonian \( H_{N+1} \). Thus, an adiabatic
elimination of the mode \( b_{N+1} \) can be made resulting in the following equation for the reduced
density matrix of modes \( b_j \) for \( j = 1, \ldots, N \):
\[
\frac{d}{dt} \rho_N = -\frac{i}{\hbar} [F_{0,0}, \rho_N] + \sum_{n=1}^{\infty} \frac{n!}{(n+1)\gamma} L_n \rho_N,
\]
where Lindblad operators are \( L_n = F_{0,n} \). The master equation (A.3) describes both possible
interactions between modes \( b_j \) and their nonlinear losses. Note that \( F_{0,0} \) might include
nonlinearities not present in the original Hamiltonian, \( H_{N+1} \).

Even for low-order nonlinearities, the scheme described above is able to lead to the
appearance of NCL. In [19] it was shown that scheme (A.1) for the case of just two bosonic
modes subject to Kerr nonlinearity leads to the appearance of NCL with \( \hat{L} = \hat{a}(\hat{n} - 1) \).

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