Quartet excitations in atomic nuclei

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Abstract. The recently invented phenomenologic and semimicroscopic algebraic quartet models, as well as their relations to other approaches are discussed. The semimicroscopic model is applied to the $^{20}$Ne and $^{28}$Si nuclei.

1. Introduction

Quarteting is an important phenomenon in various fields of human culture, among others in the study of nuclear structure. Its remarkable role in nuclei is a consequence of the fact that the effective nucleon-nucleon interaction within an atomic nucleus is attractive and short-ranged. Therefore, the nucleons tend to occupy the same single-particle states. The Pauli-exclusion principle allows two protons and two neutrons to be in the same orbital. They interact with each other very strongly, and form a quartet. These facts are known from the beginning of nuclear research, therefore, several models have been invented for the description of nucleon quartets.

In what follows first we give a very brief review of the history of quartet theories. Then we present the basic concepts of the newly introduced [1] phenomenologic (PAQM) and semimicroscopic (SAQM) algebraic quartet models. In this description two nice characteristic features of the previous frameworks are combined: the exactly defined microscopic (shell model) content of the quartets, and the efficient and easily applicable group-theoretical technology of the spectrum generation. Then applications to the $^{20}$Ne and $^{28}$Si nuclei are considered, as illustrative examples. Finally some conclusions are drawn.

2. Historical background

An experimental signature of quarteting is that the separation energy of a nucleon in an even-even $N = Z$ nucleus is larger than that of an $\alpha$-particle. It was already the motivation for Wigner’s supermultiplet theory [2], which might be considered as the first quartet model. (Much work has been done on the binding energies and quarteting later on, too, see e.g. [3].)

A quartet model of the nucleons was formulated in [4], (based on the stretched scheme), and then it was generalized in several steps. In [5] quartet excitations were considered from one major shell to the other, and the corresponding energies were determined from mass relationships. In this generalized interpretation a quartet is not related to a specific angular momentum coupling scheme: it is made of 2 protons and 2 neutrons, occupying a fourfold degenerate single-particle state ($l, m$ orbit in $L - S$ coupling, or $j, m$ and $j, -m$ orbits in $j - j$ coupling). The internal binding of a quartet is strong, while the quartet-quartet interaction is relatively weak. Arima
and Gillet took into account [6] also pairs of nucleons, as further building blocks, extending the
description to even-even nuclei of different $Z$ and $N$.

In [7] intrashell quartet excitations have been introduced in addition to the intershell
excitations of [5]. This concept leads to a quartet shell model, i.e. one assumes the existence
of a self-consistent quartet potential well, and its states are used to describe the quartet states
in $4n$ nuclei. The $0s, 0p, 1s - 0d, \ldots$ oscillator shells of the nucleon shell model are replaced
by $0s, 0p, 1s - 0d, \ldots$ quartet shells, having 1, 3, 6, \ldots single quartet states, respectively. The
corresponding energies were determined empirically, too.

A further extension was presented in [8] by incorporating any number of particle-hole
excitations (in the language of the nucleon shell model), contrary to the quartet shell model
of [5, 7] which had only 0, 4, 8 \ldots excitation quanta (in terms of nucleon shell model). This
considerable extension of the quartet model space appeared due to the conceptual generalization
of a quartet. Harvey defined [8] it as 2 protons and 2 neutrons having a quartet-symmetry:
permutational symmetry of $\{4\}$, and spin-isospin symmetry of $\{1,1,1,1\}$.

Interacting boson type quartet models were invented [9, 10] for the description of quarteting
in heavy nuclei. In [9] the basic building block quartets are treated as $l = 0$ ($s$) and $l = 2$
($d$) bosons, and the model has a $U(6)$ group structure, like the interacting boson model of the
quadrupole collectivity [11]. This model describes a spectrum of positive parity states. In [10]
the alpha-like correlation is treated in terms of bosons of nucleon pairs, but in addition to the
$s$ and $d$ bosons another set of basic building blocks of $l = 0$ ($s'$) boson and $l = 1$ ($p$) boson
is included, therefore, negative parity states are also involved. These phenomenological models
have the efficiency and elegance of the algebraic methods in generating the spectrum. E.g.
they have dynamical symmetries as limiting cases, which provide us with exact solutions for the
eigenvalue problem.

In [12] a BCS-like study was carried out for bosons of the proton-neutron interacting boson
model [11] and it was concluded that the superfluid condensate is more of a quartet type, rather
than separate superfluid phases of proton and neutron pairs.

The new microscopic quartet model of Sandulescu et al. [13, 14], has also concentrated on
the study of quartet condensates, and it turned out to be a useful description of the nucleon
pairing of isovector as well as isoscalar type (and quarteting). More recently it has been applied
for the calculation of excitation spectrum [15], too. This aspect will be further discussed in the
concluding section, in comparison with the algebraic quartet models.

### 3. The algebraic quartet models

The algebraic quartet models are based on the concepts of shell-model-like quarteting of [5, 7, 8].
Their main purpose is the description of the excitation spectrum. The simpler model, called
phenomenologic algebraic quartet model (PAQM), has the building blocks very similar to that of
the quartet shell model of [7], i.e. the composite nature of the quartet does not appear explicitly.
The more involved approach is the semimicroscopic algebraic quartet model (SAQM), which is
based on the quartet concept of [8], i.e. each of the four nucleons of the quartet is treated. The
novel feature in comparison with the works [5, 7, 8] is that an algebraic framework is formulated
for the description of the detailed spectrum, like in the group theoretical approach of the works
[9, 10, 11]. On the other hand, the new models are different from the interacting boson type
models of [9, 10], because of the nature of their building blocks, and shell-like structure of the
model spaces.

Elliott’s SU(3) scheme is applied [16, 17] for generating the spectrum both in the
phenomenological and in the semimicroscopic descriptions. In the former case structureless
quartets are supposed to occupy the single-particle levels of the harmonic oscillator shells, while
in the latter model nucleons do so. Therefore, the phenomenological model space has only a
spatial part, while the semimicroscopic one contains a space and a spin-isospin components.
In fact, this latter model space is a truncation of that of the $L - S$ coupled no-core shell model [18], based on the spin-isospin formalism.

The physical operators are expressed in terms of the group generators, thus algebraic techniques can be applied in calculating the matrix elements. The building blocks of the description are the nine operators, $\hat{A}_{\alpha\beta} = \frac{1}{2}(\hat{a}_\alpha \hat{a}_\beta + \hat{a}_\beta \hat{a}_\alpha)$, $\alpha, \beta = x, y, z$, $\hat{a}_\alpha = \sum_j \hat{a}_\alpha(j)$, $\hat{a}_\alpha^\dagger = \sum_j \hat{a}_\alpha^\dagger(j)$, $j = 1, \ldots, N$; (here $N$ is the total number of particles), which are number-conserving bilinear products of the creation and annihilation operators of oscillator quanta. They can be rewritten into three spherical tensors: a scalar operator $\hat{n}$, which is the number of oscillator quanta, five components of the quadrupole momentum $\hat{Q}_m$ (acting in a single major shell), and three components of the angular momentum $\hat{L}_m$. The nine operators $\hat{n}, \hat{Q}_m, \hat{L}_m$ generate the U(3) group, the eight operators $\hat{Q}_m, \hat{L}_m$ generate the SU(3) group, and the three $\hat{L}_m$ are generators of the SO(3) group.

The basis states are characterized by the representation labels of the group chain:

$$U(3) \supset SU(3) \supset SO(3) \supset SO(2)$$

$$[\{n_1, n_2, n_3\}, (\lambda, \mu), K, L, M].$$

Here $n = n_1 + n_2 + n_3$ is the eigenvalue of the $\hat{n}$ operator. The angular momentum content of a $(\lambda, \mu)$ representation is as follows [16, 17]:

$L = K, K + 1, \ldots, K + \text{max}(\lambda, \mu)$,

$K = \text{min}(\lambda, \mu), \text{min}(\lambda, \mu) - 2, \ldots, 1 \text{ or } 0$, with the exception of $K_L = 0$, for which $L = \text{max}(\lambda, \mu), \text{max}(\lambda, \mu) - 2, \ldots, 1 \text{ or } 0$. The SU(3) content is given by the $U(k) \supset SU(3)$ decomposition [19], where $k = 3, 6, 10, \ldots$ for the major shell with 1, 2, 3, \ldots quartet excitations. The irreducible representation (irrep) of $U(k)$ is the same as that of the permutational group in the major shell in question. The U(3) symmetry of the whole nucleus is obtained as a direct product of the major shell U(3) irreps. The irreps of the spurious center of mass excitations can be determined exactly due to the fact that the c.m. excitation operator is fully symmetric in particle indices, and has an $\{1, 0, 0\}$ U(3) irreducible tensor character [20, 21, 22].

The operators of physical quantities are obtained in this description in terms of the generators of the U(3) group. In particular the Hamiltonian can be expanded in terms of the generators of the U(3) group, coupled to spherical scalars. The general solution of the eigenvalue problem then involves two steps: i) calculation of matrix elements of the Hamiltonian between the basis states, and ii) numerical diagonalization of the energy matrix. In the special case of the dynamical symmetry, i.e. when the Hamiltonian is expressed in terms of the invariant operators of the group chain (1), an analytical solution is available.

The electromagnetic transition operators are obtained as Hermitian combinations of group generators with appropriate tensorial character. The lowest order transition operators are:

$$\hat{T}^{(E0)} = e^{(0)} \hat{n}, \quad \hat{T}^{(E2)} = e^{(2)} \hat{Q}_m^{(2)}, \quad \hat{T}^{(M1)} = m^{(1)} \hat{L}_m^{(1)}.$$  \hfill (2)

The phenomenologic and the semimicroscopic algebraic quartet models share the same formalism of the physical operators, their difference shows up between the two model spaces.

### 3.1. The phenomenologic algebraic quartet model

In this approach an excitation quantum $(\hbar \omega)_q$ between the major shells is expected to be approximately 4 times that of the nucleon shell model: $(\hbar \omega)_q \approx 4(\hbar \omega)$. All the shells have positive parity, due to their quartet nature. If a single quartet state is occupied, then no other particle can be put there, therefore, the permutational symmetry of the quartets has to be that of a single-columned Young diagram: $\{1,1,\ldots\}$.
3.2. The semimicroscopic algebraic quartet model
On the semimicroscopic level one takes into account the composition of the quartets explicitly. They are considered [8] as 2 protons and 2 neutrons having permutational symmetry of \( \{4\} \), and spin-isospin symmetry of \( \{1,1,1,1\} \). Therefore, in this case the nucleon shell model space is applied, and it is truncated according to these symmetries. Subsequent major shells have opposite parities.

The spectrum is determined by the U(3) spatial and the U\(^{ST}(4)\) spin-isospin irreps. In this case, however, the particles are nucleons, not structureless quartets, as in the phenomenologic model. Therefore, the groups describe the symmetries of the many-nucleon systems. Their relevant irreps are obtained in the following way. In each major shell those U\((k)\) (or permutational) symmetries have to be taken into account which result in the required quartet symmetry \( \{4\}^N \) when calculating their outer product with those of the other major shells. (Equivalently, those U\(^{ST}(4)\) irreps are relevant, which result in the \( \{1,1,1,1\}^N \) quartet symmetry in the direct products.) These U\((k)\) (or U\(^{ST}(4)\)) symmetries determine the relevant U(3) representations, and their direct products define the model space.

The model space is much richer than that of the phenomenological model. Especially remarkable is the angular momentum content of the model space; e.g. already in the lowest-lying major shell of the \(^{20}\)Ne L=8 appears, contrary to the L\(\leq2\) of the PAQM.

4. Applications
The PAQM states correspond to very highly excited states (due to the large excitation quantum of the quartets: \( \langle h\omega \rangle_q \approx 4\langle h\omega \rangle \)). Therefore, it is hard to find a well-established correspondence between the experimental and model states. In case of the semimicroscopic description, on the other hand, it is much more straightforward.

Here we discuss the application of the SAQM for the description of the \(^{20}\)Ne and \(^{28}\)Si nuclei. In both cases the band structure is known experimentally. The U(3) dynamical symmetry approach is used, i.e. the interactions are written in terms of the invariant operators of group-chain (1):

\[
\hat{H} = (h\omega)\hat{n} + a\hat{C}_{SU(3)}^{(2)} + b\hat{C}_{SU(3)}^{(3)} + d\frac{1}{2\theta}\hat{L}^2.
\]

The first term is the harmonic oscillator Hamiltonian (linear invariant of the U(3)), with a strength obtained from the systematics [23] \( h\omega = 45\Lambda^{-\frac{1}{2}} - 25\Lambda^{-\frac{1}{2}} \text{ MeV} \). The second-order invariant of the SU(3) \( \hat{C}_{SU(3)}^{(2)} \) represents the quadrupole-quadrupole interaction, while the third-order Casimir-operator \( \hat{C}_{SU(3)}^{(3)} \) distinguishes between the prolate and oblate shapes. \( \theta \) is the moment of inertia calculated classically for the rigid shape determined by the U(3) quantum numbers (for a rotor with axial symmetry) [1], and the \( a, b \) and \( d \) parameters were fitted to the experimental data. The \( B(E2) \) value is given as [1, 24]:

\[
B(E2, I_i \rightarrow I_f) = \frac{2I_f + 1}{2I_i + 1} \alpha^2 \langle\langle(\lambda,\mu)KI_i, (11)2\rangle\rangle \langle\langle(\lambda,\mu)KI_f\rangle\rangle^2 C(\lambda,\mu),
\]

where \( \langle\langle(\lambda,\mu)KI_i, (11)2\rangle\rangle \langle\langle(\lambda,\mu)KI_f\rangle\rangle \) is the SU(3) \( \supset SO(3) \) Wigner coefficient [25], and \( \alpha^2 \) is a parameter fitted to the experimental value of the \( 2^+_1 \rightarrow 0^+_1 \) transition.

The experimental data of \(^{20}\)Ne are taken from [26], but for the band assignment of the highly-excited alpha-cluster states also the conclusions of [27] are taken into account. The lower part of Figure 1. shows the states with definite band-assignment. All the bands with \( K^\pi \) values of [26] are included, except the one of the very uncertain (and somewhat contradictory) \( 0^+_2 \) band. In case of the \( 7^- \) state of the \( 0^- \) band, and the \( 6^+ \) and \( 8^+ \) states of the \( 0^+_6 \) band, which have more than one experimental candidates, the average energies are indicated. (In [26] there are only three states, which are not included here, for not having corresponding states in the model.
Table 1. The parameters of the SAQM for the description of the experimental spectra. (Note that \( \hbar \omega \) is calculated from the systematics [23].) The coefficients of the Hamiltonian are in MeV, while \( \alpha^2 \) is measured in W.u.

| Nucleus | \( \hbar \omega \) | a    | b    | d    | \( \alpha^2 \) |
|---------|-----------------|------|------|------|---------------|
| \(^{20}\text{Ne}\) | 13.18           | -0.421 | 0.000909 | 0.635 | 2.31          |
| \(^{28}\text{Si}\) | 12.11           | -0.133 | 0.000444 | 1.003 | 0.366         |

The experimentally identified bands are described by the lowest-lying model bands with the appropriate spin-parity content, in both nuclei.

Figure 1. The spectrum of the semimicroscopic algebraic quartet model in comparison with the experimental data of the \(^{20}\text{Ne}\) nucleus. The experimental bands are labeled by the \( K^\pi \), and the model states by the \( n(\lambda, \mu)K^\pi \) quantum numbers. The spin-parity in parenthesis indicates uncertain band assignment. The width of the arrow between the states is proportional to the strength of the \( E2 \) transition.

Figure 2. The spectrum of the semimicroscopic algebraic quartet model in comparison with the experimental data of the \(^{28}\text{Si}\) nucleus. The experimental bands are labeled by the available quantum numbers, and the model states by the \( n(\lambda, \mu)K^\pi \) labels. \( \beta \) means \( \beta \)-instabil, while O, P and SD stand for oblate, prolate and superdeformed respectively.

The reason for choosing the \(^{28}\text{Si}\) nucleus as another illustrative example is manyfold. i) It has a well-established band structure in the low-energy region, and to several bands \( \text{SU}(3) \) quantum numbers could be associated as a joint conclusion of experimental and theoretical investigations [28]. ii) More recently a new candidate was proposed for the superdeformed (SD) band [29]. Theoretical studies predicted the SD band [30, 31] in line with the experimental observation. iii) There are two cluster configurations: \(^{24}\text{Mg} + ^{4}\text{He}\), and \(^{16}\text{O} + ^{12}\text{C}\), belonging to reaction channels in which fine-resolution measurements revealed a rich spectrum of resonances.

The lower part of Figure 2. shows the experimental bands of the \(^{28}\text{Si}\) nucleus, as established in [28] together with the recently found superdeformed (SD) band [29]. Please, note that the
SU(3) quantum numbers are associated to the experimental bands without any reference to the quartet or cluster studies.

5. Summary and conclusions
In this contribution we have presented algebraic quartet models based on the shell-like quarteting concept both on the phenomenologic and on the semimicroscopic levels. The spectrum is generated in both cases by Elliott’s U(3) formalism. The model space of the PAQM is identical with that of [7], while the SAQM model space is similar to that of [8]. The main difference between the approach of [5, 7, 8] and the present one is that in [5, 7, 8] the interaction matrix elements are obtained empirically from the binding energies, while here we construct all the physical operators algebraically. This enables us to calculate the complete spectrum in an easy way.

In [15] a recently developed microscopic quartet model is applied for the description of the excitation spectrum. In that approach the wavefunction of a quartet is determined from a shell model calculation, which serves as an input for the quartet analysis. The present algebraic quartet model does not use such an input, i.e. it is self-contained, and easily applicable, though not fully microscopic: the parameters of its interactions are fitted to the experimental data. The combination of the two methods seems to be promising for finding the exact territory of the relevance of quarteting.

As for the connection to other models, the transparent symmetry properties of the present approach is very helpful. Via its obvious shell-model relation, the connection of the quartet model to the cluster and collective models is also well-defined (see e.g. [32, 33], and references therein for a recent discussion). In this respect the models with algebraic structure are relevant, in particular, the microscopic cluster model applying U(3) basis [34, 35], and the semimicroscopic algebraic cluster model [36], as well as the symplectic shell model [37], and the contracted symplectic model [38] of the quadrupole collectivity.

The semimicroscopic algebraic quartet model can be considered as an effective model [39], which is obtained as a symmetry-governed truncation of the no-core shell model with the $U^{ST}(4) \otimes U(3)$ formalism [18].

Especially promising can be the application of the present semimicroscopic quartet model in combination with the concept of the multichannel dynamical symmetry [40], when the spectra of different cluster configurations are obtained from the quartet spectrum by simple projections. E.g. preliminary studies [41] show that the complete high-lying spectrum of the $^{16}\text{O} + ^{12}\text{C}$ resonances can be obtained as a prediction from the presently determined quartet spectrum of the $^{28}\text{Si}$ nucleus. In particular, the resonance spectrum corresponds to the $^{16}\text{O} + ^{12}\text{C}$ cluster states in the second (superdeformed) minimum of the energy surface.

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