Research on nonlinear vibration control of laminated cylindrical shells with discontinuous piezoelectric layer

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Abstract A model of laminated cylindrical shells with discontinuous piezoelectric layer is proposed. Based on the first-order shear nonlinear shell theory, the nonlinear vibration control of the piezoelectric laminated cylindrical shell model with point-supported elastic boundary condition is analyzed. In this model, a series of artificial springs are introduced to simulate the arbitrary boundary conditions. And the elastic-electrically coupled differential equations of piezoelectric laminated cylindrical shells are obtained by the Chebyshev polynomials and Lagrange equations and decoupled by using the negative velocity feedback adjustment. Then, the frequency–amplitude responses of the piezoelectric laminated cylindrical shells are obtained by the incremental harmonic balance method. Finally, the influence of the constant gain, size, and position of the piezoelectric layer on the nonlinear amplitude–frequency response is investigated. The results show that the constant gain and the position and size of the piezoelectric layer have a significant influence on the amplitude of the nonlinear amplitude–frequency and time–frequency response.

Keywords Geometrically nonlinear · Piezoelectric · Vibration control · Elastic boundary condition

1 Introduction

In recent years, with the development of piezoelectric materials, the piezoelectric films (PVDF) have been widely used in the fields of machinery, environmental testing, and medicine because of their functions of vibration detection, vibration control, and energy collection. The thin-walled cylindrical shell structure is an essential structure in aerospace and other fields. Because the environment of the cylindrical shell is mostly complicated, therefore, the large amplitude vibration is always inevitable, and the nonlinear vibration of plate and shell structures and their applications have been the focus and hotspot of scholars [1–5].

Since the advent of piezoelectric intelligent mechanisms, the study of the vibration characteristics of piezoelectric laminated cylindrical shells has continued to attract scholar’s attention [6, 7]. The modeling of piezoelectric laminated shells mainly focuses on the finite element method and Hamilton’s principle. Kerur et al. [8] presented a finite element model using a
composite material (AFC) as the piezoelectric actuator and a piezoelectric film (PVDF) as the piezoelectric sensor and used the negative velocity feedback control algorithm to control the dynamic response of the laminated composite plate. Parashar and Kumar [9] used the Rayleigh–Ritz method to study the vibration behavior of the piezoelectric shells, and the results showed that the Rayleigh–Ritz method had a very high convergence rate and can be calculated with relatively less efforts. Sheng and Wang [10] analyzed the dynamics characteristic of functionally graded material (FGM) cylindrical shells with a surface-bonded PZT piezoelectric layer by using the first-order shear deformation theory (FSDT) and Hamilton’s principle, in which the thermal and moving loads are considered in the model. Qin et al. [11] used Sanders shell theory combined with the artificial spring technique to study the free vibration of cylindrical shells under arbitrary boundary conditions and compared the accuracy, convergence rate, and computational efficiency of three different admissible displacement functions, namely the modified Fourier series, the orthogonal polynomials, and the Chebyshev polynomials, and the results showed that Chebyshev polynomials obtained higher computational efficiency. Further, using Chebyshev polynomials as admissible displacement functions and FSDT, Qin et al. [12] proposed a general method to study the free vibration of rotating functionally graded carbon nanotube-reinforced composite (FG-CNTRC) cylindrical shells with arbitrary boundary conditions. The Lagrange equation is simple to use and has a small dimension, which can well handle the dynamic modeling of piezoelectric laminated cylindrical shells. Li et al. [13] presented a model of discontinuous piezoelectric laminated shell with point-supported elastic boundary conditions based on the first-order shear shell theory, the Chebyshev polynomials, and the Lagrange equation. Then, the location of piezoelectric layer was optimized by using the multi-objective particle swarm optimization algorithm. In practical applications, the thin-walled cylindrical shell structures, such as the aero-engine casing, are subjected to complex loads in a relatively complex environment. In this case, the geometric nonlinearity of cylindrical shell structures is inevitable. Therefore, it is of great significance to study the nonlinear amplitude–frequency response of cylindrical shells. With the deepening of the research on piezoelectric laminated cylindrical shells, many researchers have applied piezoelectric materials to the vibration control of thin-walled cylindrical shell structures. Amabili [14] considered the axial symmetry of laminated cylindrical shells and calculated the nonlinear forced vibration response for the first time and compared the classical Novozhilov theory, the usual version of higher-order shear deformation theory, and the higher-order shear deformation theory recently developed by Amabili and Reddy. The results show that the Amabili–Reddy and Novozhilov theories have good results for laminated thin shells. For thick shells, the Amabili–Reddy theory should be used in order to have accurate results. Jansen [15] studied the effects of static loads and imperfections on the nonlinear vibration characteristics of cylindrical shells based on the Donnell-type governing equations. Przekop et al. [16] used structural degrees of freedom as a framework and established a coupled structural–electrical nonlinear modal FE model for laminated composite shallow shells with embedded piezoelectric sensors and actuators, aiming to suppress the large amplitude undamped free vibration. According to Donnell’s shell theory, Galerkin method, and Volmir assumption, Rafiee et al. [17, 18] analyzed the nonlinear vibration of the FGM cylindrical shell with a piezoelectric layer subject to electro-thermal-pneumatic-mechanical coupling load. Then, the effects of piezoelectric layer thickness, geometric parameters, temperature, voltage, and aerodynamic load on its dynamic characteristics were analyzed. Sheng and Wang [19] proposed a new simplifying model of smart FG laminated cylindrical shells with thin piezoelectric layers in which the inner layer is a sensing layer, the functionally graded layer, and the outer layer is the active layer and use the von Kármán nonlinear theory, the Hamilton’s principle, the FSDT, and the multi-term Galerkin method to analyze the active vibration control of smart FG laminated cylindrical shells. Shen and Yang [20] used the higher-order shear deformation shell theory with a von Kármán type of kinematic nonlinearity to study the small and large amplitude flexural vibrations of anisotropic shear deformable laminated cylindrical shells with piezoelectric fiber-reinforced composite (PFRC) actuators in thermal environments. Zhang [21] used various geometrically nonlinear shell theories based on the FSDT, including refined von Kármán nonlinear shell theory (RVK5), moderate rotation shell theory (MRT5), fully geometrically nonlinear shell theory with moderate rotations.
(LRT5), and fully geometrically nonlinear shell theory with large rotations (LRT56). The results showed that the LRT56 can get more accurate results when structures undergo large rotations, and by adopting appropriate feedback actuation voltages, shape and vibration control can be accomplished pretty well in the structures undergoing large deformations and large rotations. Yue et al. [22] established an experimental platform for adaptive modal control of precision paraboloidal shell, conducted an experimental study on the active modal vibration control of a flexible paraboloidal shell with free boundary conditions, and used the positive position feedback algorithm to control the first-order mode as well as the first- and second-order coupled mode of the paraboloidal shell. Ninh et al. [23] established the model of functionally graded carbon nanotube-reinforced composite (FG-CNTRC) cylindrical shells with the piezoelectric actuators surrounded by an elastic medium and used the classical shell theory with geometrical nonlinearity to investigate its electro-thermo-mechanical vibration.

In recent years, many scholars have proposed discontinuous elastic boundaries and nonlinear flexible boundaries to replace traditional classical boundaries. Chen et al. [24] extended the improved Fourier series method to the free vibration analysis of cylindrical shells and studied the free vibration of cylindrical shells with arbitrary and non-uniformly restrained boundary conditions. Xie et al. [25] used the wave propagation method to study the free vibration and forced vibration of a non-uniform constrained cylindrical shell. Li et al. [26, 27] studied the geometrically nonlinear vibration of non-continuous elastic-supported laminated composite cylindrical thin shells by extending a new arcs-supported and points-supported shell model. Tang et al. [28, 29] considered the three contact states of bolted connection, including stick, slip, and separation and established a piecewise linear boundary analytical model of bolted cylindrical shells and studied its dynamic characteristics. Li et al. [30] presented a model of laminated composited cylindrical shells with arc-supported and point-supported elastic boundary condition based on Donnell’s shell theory, the Chebyshev polynomials, and the Lagrange equation.

According to the above literature research, the research on the dynamic characteristic of laminated shells with piezoelectric materials has attracted much attention. In addition, the vibration control of cylindrical shells by piezoelectric materials has been confirmed in some researches, but the influence of piezoelectric materials on the nonlinear vibration of cylindrical shell remains to be studied. Due to the fabrication process and cost of the piezoelectric material, it is very difficult to completely cover the piezoelectric layer on the surface of the cylindrical shell. Therefore, we innovatively proposed the nonlinear dynamic model of laminated cylindrical shells with discontinuous piezoelectric layers to study the influence of piezoelectric layers of different sizes and positions on the frequency–amplitude response. The Chebyshev polynomials, Lagrange equation, first-order shear nonlinear shell theory, and negative speed feedback strategy are used to obtain the differential equation of decoupled motion. Afterward, the IHBM is used to solve the differential equation of motion, and the effects of constant gain, piezoelectric layer size, and piezoelectric layer position on the nonlinear frequency–amplitude response of the piezoelectric laminated cylindrical shell are analyzed.

### Nomenclature

| Symbol | Description |
|--------|-------------|
| $A$    | Stretching stiffness matrix |
| $C_G$, $C_A$ | Damping matrix |
| $D$    | Bending stiffness matrix |
| $D_h$, $D_s$ | Electric displacements |
| $E_{b/a/s}$ | Yong’s modulus of the base layer, actuator layer, and sensor layer |
| $E$    | Electric fields |
| $F$    | Point excitation |
| $G_l$  | Constant gain |
| $K_{qq}$, $K_{qr}$, $K_{q}$, $K_{q}$ | Generalized stiffness matrix, spring stiffness matrix, electrical mechanical coupled stiffness, electrical stiffness matrix |
| $L$    | Length of the shell |
| $M_{qq}$ | Generalized mass matrix |
| $M_r$, $M_p$, $M_{xy}$ | Moments of the in-plane stresses |
| $N$    | Number of terms for circumferential wave |
| $N_r$, $N_p$, $N_{xy}$ | Force of the in-plane stresses |
| $N_A$  | Number of supported point |
| $N_T$  | Number of terms for Chebyshev polynomials |
| $P$    | Applied electrical charge for piezoelectric layers |
| $Q^g$, $Q^e$, $Q^h$ | Plane stresses–strain matrix |
| $Q_{b/a/c}$ | Transverse shear force |
## Nomenclature

| Symbol | Description                                                                 |
|--------|------------------------------------------------------------------------------|
| $Q_N$  | Nonlinear item                                                               |
| $R$    | Radius of the shell                                                          |
| $T$    | Kinetic energy                                                               |
| $T_m(\zeta)$ | Admissible displacement functions                                            |
| $U_e$, $U_{elaspr}$ | Strain energy, potential energy                                               |
| $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, $\mathbf{\Psi}$, $\Psi_0$ | Mode vector satisfying the boundary condition                                 |
| $a_{mn}$, $b_{mn}$, $c_{mn}$, $d_{mn}$, $e_{mn}$, $f_{mn}$ | Unknown corresponding coefficients                                           |
| $\mathbf{u}_{mn}$, $\mathbf{v}_{mn}$ | Dielectric constants                                                        |
| $\mathbf{w}_{mn}$ | Mass normalized mode shape matrix                                            |
| $\mathbf{\kappa}_x$, $\mathbf{\kappa}_y$, $\mathbf{\kappa}_z$, $\mathbf{\kappa}_\theta$ | Thickness of the whole shell                                                 |
| $k_a$, $k_y$, $k_w$, $k_x$, $k_\theta$ | Stiffness of axial, circumferential, radial, torsional spring                |
| $k_c$  | Shear correction factor                                                       |
| $n$    | Circumferential wave number                                                  |
| $u$, $v$, $w$, $\phi_x$, $\phi_\theta$ | Amplitude of excitation, thickness of the shell, actuator layer, and sensor layer |
| $\mathbf{\Psi}_a$, $\mathbf{\Psi}_s$ | Distribution of electric potential along thickness coordinate                 |
| $\mathbf{\Phi}$ | Mass normalized mode shape matrix                                            |
| $\mathbf{\xi}_a$, $\mathbf{\xi}_y$, $\mathbf{\gamma}_x$, $\mathbf{\gamma}_y$, $\mathbf{\gamma}_z$ | Strains of the shell                                                        |
| $\zeta$ | Non-dimensional axial coordinate                                             |
| $(\xi, \theta)$ | Starting and ending coordinates of the 4th piezoelectric layer               |
| $\mathbf{\rho}_b$, $\mathbf{\rho}_a$, $\mathbf{\rho}_s$ | Mass density of the base layer, actuator layer, and sensor layer            |
| $\sigma_x$, $\sigma_\theta$, $\tau_{xy}$, $\tau_{y\theta}$, $\tau_{xz}$ | Stresses of the shell                                                        |
| $\omega$ | Natural frequency of the shell                                               |
| $\omega_{EX}$ | Excitation frequency                                                         |

### 2 Theoretical formulations

#### 2.1 Description of laminated cylindrical shells mode

In Fig. 1, the schematic diagram of laminated cylindrical shells with discontinuous piezoelectric layer is presented, in which radius, length, thickness of the base layer, thickness of the piezoelectric sensor and actuator layers are defined as $R$, $L$, $h$, $h_a$, and $h_s$, respectively. Along the axial, circumferential, and radial directions of the shell, an orthogonal coordinate system $(x, \theta, z)$ is established. In the $x$, $\theta$, $z$ directions, the displacements of a point on the middle surface are denoted as $u$, $v$, and $w$, and in the rotations direction with respect to the $x$ and $\theta$ axes, the transverse normal is denoted as $\phi_x$ and $\phi_\theta$. Parts of the cylindrical shell are covered with the piezoelectric layers, and the number of piezoelectric layers is defined as $NP$. The starting and ending coordinates of the $n$th piezoelectric layer are $(\xi_0, \theta_0)$ and $(\xi_n, \theta_n)$, respectively. $NA$ is the number of restrained points. The arbitrary boundary conditions are represented by using artificial spring technique. $k_{a_{x,2}}, k_{a_{y,2}}, k_{w_{z,2}}, k_{k_{x,2}},$ and $k_{k_{\theta,2}}$ are the stiffness of five groups of boundary springs distributed at the left end $(x = 0)$ of the laminated cylindrical shells at the 4th point. Similarly, $k_{a_{1,2}}, k_{a_{y,2}}, k_{w_{z,2}}, k_{k_{x,2}},$ and $k_{k_{\theta,2}}$ denote the corresponding boundary springs stiffness distributed at the right end $(x = L)$ of the cylindrical shell. In this paper, when the spring stiffness in a certain direction reaches $10^{12}$ N/m, it can be regarded as the constraint of the degree of freedom in that direction.

#### 2.2 Expressions of laminated cylindrical shell’s energy

For the piezoelectric laminated cylindrical shells, the kinetic energy $T$ and the strain energy $U_e$ are written as [13]

$$
T = \frac{LR}{2} \sum_{i=1}^{NP} \int_{\theta_i}^{\theta_{i+1}} \int_{\xi_i}^{\xi_{i+1}} \left[ (\ddot{u}^2 + \ddot{v}^2 + \ddot{w}^2) I_0 + 2(\ddot{u}\ddot{\phi}_x + \ddot{v}\ddot{\phi}_\theta) I_1 + (\ddot{\phi}_x^2 + \ddot{\phi}_\theta^2) I_2 \right] \, d\xi \, d\theta
$$

$$
+ \frac{LR}{2} \sum_{r=1}^{NP} \int_{\theta_r}^{\theta_{r+1}} \int_{\xi_r}^{\xi_{r+1}} \left[ (\ddot{u}^2 + \ddot{v}^2 + \ddot{w}^2) \tilde{I}_0 + 2(\ddot{u}\ddot{\phi}_x + \ddot{v}\ddot{\phi}_\theta) \tilde{I}_1 + (\ddot{\phi}_x^2 + \ddot{\phi}_\theta^2) \tilde{I}_2 \right] \, d\xi \, d\theta
$$

where
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Fig. 1 Schematic diagram of a piezoelectric laminated composite cylindrical shell with elastic boundary conditions: a coordinate system and geometry of the shell; b partial cross-sectional view of the shell with elastic boundary condition

\[ [1_0, 1_1, 1_2] = \int_{h/2}^{h/2} \rho_a [1, z, z^2] dz + \int_{-h/2}^{-h/2} \rho_h [1, z, z^2] dz + \int_{-h/2}^{-h/2} \rho_k [1, z, z^2] dz \]

where \( \rho_i \) is the density of each layer of the shell. The subscript \( i \) is used to represent the sensor layer \( (i = s) \), actuator layer \( (i = a) \), and base layer \( (i = b) \).

\[ U = \frac{LR}{2} \sum_{s=1}^{NP} \int_{0}^{\theta_s} \int_{\xi_s}^{\tilde{\xi}_s} \left( N_x^T \epsilon_x + N_\theta^T \epsilon_\theta + N_x^T \gamma_x \right. \]

\[ + M_k^T \kappa_k + M_a^T \kappa_a + M_s^T \kappa_s + Q_{0x}^T \gamma_{0x} + Q_{0z}^T \gamma_{0z} \]

\[ - \int_{h/2}^{h/2} D_a^{T} E_d d\zeta - \int_{-h/2}^{-h/2} D_a^{T} E_d d\zeta \right) d\xi d\theta \]

\[ = \frac{LR}{2} \sum_{s=1}^{NP} \int_{0}^{\theta_s} \int_{\xi_s}^{\tilde{\xi}_s} \left( N_x^T \epsilon_x + N_\theta^T \epsilon_\theta + N_x^T \gamma_x \right. \]

\[ + M_k^T \kappa_k + M_a^T \kappa_a + M_s^T \kappa_s + Q_{0x}^T \gamma_{0x} + Q_{0z}^T \gamma_{0z} \]

\[ + M_k^T \kappa_k + M_a^T \kappa_a + M_s^T \kappa_s + Q_{0x}^T \gamma_{0x} + Q_{0z}^T \gamma_{0z} \right) d\xi d\theta \]

where \( D_i \) and \( E_i \) are the electric displacements and electric fields of the piezoelectric layer [10], respectively.

where \( e_i \) and \( \xi_i \) are the piezoelectric constants and dielectric constants, respectively.

Considering the poling direction of the piezoelectric layer is coincident with the thickness direction, the electric fields can be written as [10]

\[ E_x^i = - \frac{\partial \phi^i(x, \theta, z, t)}{\partial \theta}, \quad E_\theta^i = - \frac{\partial \phi^i(x, \theta, z, t)}{R \partial \theta}, \quad E_z^i = - \frac{\partial \phi^i(x, \theta, z, t)}{\partial z} \]

where the piezoelectric layer potential due to elastic deformation is given as

\[ \phi^i(x, \theta, z, t) = \left[ z_a^2 - \left( \frac{h_a}{2} \right)^2 \right] \psi_a(x, \theta, t) \]

\[ \phi^i(x, \theta, z, t) = \left[ z_a^2 - \left( \frac{h_a}{2} \right)^2 \right] \psi_a(x, \theta, t) \]
where \( \psi_A(x, \theta, t) \) and \( \psi_s(x, \theta, t) \) are the distribution of the electric potential of the actuator layer along the thickness coordinate, \( z_a \) is the coordinate relative to the middle surface of the actuator layer, \( z_a = z - (h + h_s)/2 \); \( \psi_s(x, \theta, t) \) is the distribution of the sensor layer, \( z_s \) is the coordinate of the sensor layer, \( z_s = z + (h + h_s)/2 \).

In Eq. (3), the strain vector \( \varepsilon^T \) is given by [12]

\[
\varepsilon^T = \{ \varepsilon_x, \varepsilon_\theta, \gamma_{x\theta}, \gamma_{xz}, \kappa_x, \kappa_\theta, \kappa_{x\theta}, \kappa_{xz} \} \tag{7}
\]

The strains of a point of the shell are defined as [12]

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u_0}{L \partial z} + \frac{1}{2L^2} \left( \frac{\partial v_0}{\partial z} \right)^2, \\
\varepsilon_\theta &= \frac{1}{R} \frac{\partial v_0}{\partial \theta} + \frac{w_0}{R} + \frac{1}{2R^2} \left( \frac{\partial w_0}{\partial \theta} \right)^2, \\
\gamma_{x\theta} &= \frac{1}{R} \frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{L \partial z} + \frac{1}{LR} \left( \frac{\partial w_0}{\partial \theta} \frac{\partial w_0}{\partial z} \right), \\
\gamma_{xz} &= \phi_\theta + \frac{1}{R} \frac{\partial v_0}{\partial \theta} - \frac{\partial w_0}{L \partial z}, \\
\kappa_x &= \frac{1}{L} \frac{\partial \phi_\theta}{\partial z}, \\
\kappa_\theta &= \frac{1}{R} \frac{\partial \phi_\theta}{\partial \theta}, \\
\kappa_{x\theta} &= \frac{1}{L} \frac{\partial \phi_\theta}{\partial \theta}, \quad \kappa_{xz} = 0, \quad \kappa_{x\theta} = 0
\end{align*}
\tag{8}
\]

where the subscript \((0)\) is the middle surface. The strain and the curvature can be given as [12]

\[
\begin{align*}
\tilde{A}_{ij} &= \int_{-h/2}^{h/2} Q^b_{ij} dz, \\
\tilde{B}_{ij} &= \int_{-h/2}^{h/2} Q^b_{ij} z dz, \\
\tilde{D}_{ij} &= \int_{-h/2}^{h/2} Q^b_{ij} z^2 dz
\end{align*}
\tag{11}
\]

where the elements of \( Q^b_{ij} \) are given as [12]

\[
Q^b_{ij} = \begin{bmatrix}
Q^b_{11} & Q^b_{12} & 0 & 0 & 0 \\
Q^b_{12} & Q^b_{22} & 0 & 0 & 0 \\
0 & 0 & Q^b_{66} & 0 & 0 \\
0 & 0 & 0 & Q^b_{44} & 0 \\
0 & 0 & 0 & 0 & Q^b_{55}
\end{bmatrix}
\tag{12}
\]

where the plane stresses–strain matrix \( Q_b \) is as [12]

\[
E_b = \frac{E_b}{1 - \mu_b^2}, \quad Q^b_{11} = Q^b_{22} = -Q^b_{66}, \quad Q^b_{44} = Q^b_{55} = Q^b_{66}
\tag{13}
\]

where \( E_b \) is Young’s moduli of the base layer. \( \mu_b \) is the Poisson’s ratios.

In the location of base-piezoelectric layers, the matrix form of the force and moment resultant relations to the strains in the middle surface and curvature changes is defined as [13]
where $N^E_i, N^E_0, M^E_i, M^E_0, Q^E_i, Q^E_0$ are the piezoelectric results of the piezoelectric layer.

$$A_y = \int_{h/2}^{h/2+h_0} Q_y^j dz + \int_{-h/2}^{-h/2-h_0} Q_y^d dz$$
$$B_y = \int_{h/2}^{h/2+h_0} Q_y^e dz + \int_{-h/2}^{-h/2-h_0} Q_y^e dz$$
$$D_y = \int_{h/2}^{h/2+h_0} Q_y^d dz + \int_{-h/2}^{-h/2-h_0} Q_y^d dz$$

where $Q_j$ are the elastic constants.

The force and moment resultant of the piezoelectric layer is given as [10]

$$\left\{ \begin{array}{c} N_i^E \\ N_0^E \\ M_i^E \\ M_0^E \\ Q_i^E \\ Q_0^E \end{array} \right\} = \left\{ \begin{array}{c} A_{11} A_{12} 0 0 B_{11} B_{12} 0 \\ A_{12} A_{22} 0 0 B_{12} B_{22} 0 \\ 0 0 A_{44} 0 0 B_{44} \\ 0 0 B_{44} 0 0 \\ \left[ \begin{array}{c} \gamma_{\varepsilon(0)} \\ \gamma_{\varepsilon(0)} \end{array} \right] \\ \left[ \begin{array}{c} \epsilon_{\varepsilon(0)} \\ \epsilon_{\varepsilon(0)} \end{array} \right] \end{array} \right\} + \left\{ \begin{array}{c} N_{i}^E \\ N_0^E \\ M_i^E \\ M_0^E \\ \left[ \begin{array}{c} \gamma_{\varepsilon(0)} \\ \gamma_{\varepsilon(0)} \end{array} \right] \\ \left[ \begin{array}{c} \epsilon_{\varepsilon(0)} \\ \epsilon_{\varepsilon(0)} \end{array} \right] \end{array} \right\}$$

(14)

2.3 The potential energy of the boundary springs

To simulate arbitrary boundary conditions, we use a series of artificial springs in five directions. The potential energy $U_{spr}$ introduced by the elastic boundary springs can be calculated as follows[30]

$$U_{spr} = \frac{1}{2} \sum_{n=1}^{NA} \left\{ k_{s,2}^0 [u(0, \theta_2(t), t)]^2 + k_{s,2}^0 [v(0, \theta_2(t), t)]^2 \right\}$$

(17)

where $(x_2, \theta_2)$ is used to represent the coordinates of the $n$th point.

2.4 Admissible displacement functions

In Qin’s study [11], the Chebyshev polynomials show the higher computational efficiency, the higher convergence rate, and the higher computational efficiency, compared with the orthogonal polynomials and the modified Fourier series. Therefore, the admissible displacement functions in this paper use the Chebyshev polynomials. Then, the admissible displacement functions are given by

$$u(\xi, \psi, t) = \sum_{m=0}^{NT} \sum_{n=0}^{N} A_{mn} T_m(\xi) \cos n \psi e^{-jmt} = U^T q_u$$
$$v(\xi, \psi, t) = \sum_{m=0}^{NT} \sum_{n=0}^{N} B_{mn} T_m(\xi) \sin n \psi e^{-jmt} = U^T q_v$$
$$w(\xi, \psi, t) = \sum_{m=0}^{NT} \sum_{n=0}^{N} C_{mn} T_m(\xi) \cos n \psi e^{-jmt} = W^T q_w$$
$$\phi_x(\xi, \psi, t) = \sum_{m=0}^{NT} \sum_{n=0}^{N} D_{mn} T_m(\xi) \cos n \psi e^{-jmt} = \Phi^x q_{\phi_x}$$
$$\phi_y(\xi, \psi, t) = \sum_{m=0}^{NT} \sum_{n=0}^{N} E_{mn} T_m(\xi) \sin n \psi e^{-jmt} = \Phi^y q_{\phi_y}$$
$$\psi_x(\xi, \psi, t) = \sum_{m=0}^{NT} \sum_{n=0}^{N} F_{mn} T_m(\xi) \cos n \psi e^{-jmt} = \Psi^x q_{\psi_x}$$
$$\psi_y(\xi, \psi, t) = \sum_{m=0}^{NT} \sum_{n=0}^{N} G_{mn} T_m(\xi) \sin n \psi e^{-jmt} = \Psi^y q_{\psi_y}$$

(18)

where $A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}, F_{mn}, G_{mn}$ are the unknown corresponding coefficients. $T_m(\xi)$ are appropriate admissible displacement functions[26], $NT$ is the number of terms of the admissible displacement functions. $\psi$ is the natural frequency. $q_u, q_v, q_w, q_{\phi_x}, q_{\phi_y}, q_{\psi_x}, q_{\psi_y}$ are the generalized
coordinates, and the expressions are given in Appendix A. \( n \) is the circumferential wave number. \( \mathbf{U}, \mathbf{V}, \mathbf{W}, \mathbf{T}_x, \mathbf{T}_z, \mathbf{\Psi}_a, \mathbf{\Psi}_s, \mathbf{\Psi}_\alpha \) are mode vectors satisfying the boundary conditions.

2.5 Energy expressions and the solution procedure

Substituting Eq. (18) into Eqs. (1), (3) and (17), we obtain the quadratic form of kinetic energy \( T \), the strain energy \( U_c \), and the potential energy \( U_{spr} \) as follows

\[
T = \frac{1}{2} q_i^T M_{ii} \cdot q_i + \frac{1}{2} q_i^T M_{Kii} \cdot q_i + \frac{1}{2} q_i^T M_{\phi_i \phi_i} \cdot q_i + \frac{1}{2} q_i^T M_{\phi_i \phi_i} \cdot q_i + \frac{1}{2} q_i^T M_{\phi_i \phi_i} \cdot q_i
\]

\[
T = \frac{1}{2} q_i^T M_{ii} \cdot q_i + \frac{1}{2} q_i^T M_{Kii} \cdot q_i + \frac{1}{2} q_i^T M_{\phi_i \phi_i} \cdot q_i + \frac{1}{2} q_i^T M_{\phi_i \phi_i} \cdot q_i + \frac{1}{2} q_i^T M_{\phi_i \phi_i} \cdot q_i
\]

\[
U_c = \frac{1}{2} q_i^T K_{ii} \cdot q_i + \frac{1}{2} q_i^T K_{Kii} \cdot q_i + \frac{1}{2} q_i^T K_{\phi_i \phi_i} \cdot q_i + \frac{1}{2} q_i^T K_{\phi_i \phi_i} \cdot q_i + \frac{1}{2} q_i^T K_{\phi_i \phi_i} \cdot q_i
\]

\[
U_c = \frac{1}{2} q_i^T K_{ii} \cdot q_i + \frac{1}{2} q_i^T K_{Kii} \cdot q_i + \frac{1}{2} q_i^T K_{\phi_i \phi_i} \cdot q_i + \frac{1}{2} q_i^T K_{\phi_i \phi_i} \cdot q_i + \frac{1}{2} q_i^T K_{\phi_i \phi_i} \cdot q_i
\]

\[
U_{spr} = \frac{1}{2} q_i^T K_{spr} \cdot q_i + \frac{1}{2} q_i^T K_{spr} \cdot q_i + \frac{1}{2} q_i^T K_{spr} \cdot q_i + \frac{1}{2} q_i^T K_{spr} \cdot q_i + \frac{1}{2} q_i^T K_{spr} \cdot q_i
\]

\[
U_{spr} = \frac{1}{2} q_i^T K_{spr} \cdot q_i + \frac{1}{2} q_i^T K_{spr} \cdot q_i + \frac{1}{2} q_i^T K_{spr} \cdot q_i + \frac{1}{2} q_i^T K_{spr} \cdot q_i + \frac{1}{2} q_i^T K_{spr} \cdot q_i
\]

The Lagrange equation is given by

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial (U_c + U_{spr})}{\partial \dot{\psi}_a} + C_A \dot{q}_i + K_{q q} \dot{q}_i + Q_N + K_{q \psi} \psi = F \\
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial (U_c + U_{spr})}{\partial \dot{\psi}_a} + \frac{\partial D}{\partial \dot{\psi}_a} = f_a \\
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial (U_c + U_{spr})}{\partial \dot{\psi}_a} + \frac{\partial D}{\partial \dot{\psi}_a} = f_a \\
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial (U_c + U_{spr})}{\partial \dot{\psi}_a} + \frac{\partial D}{\partial \dot{\psi}_a} = f_a \\
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial (U_c + U_{spr})}{\partial \dot{\psi}_a} + \frac{\partial D}{\partial \dot{\psi}_a} = f_a
\end{align*}
\]

in which \( D \) is the dissipated energy, expressed as

\[
D = \frac{1}{2} (C_A + C_R) q^2
\]

where \( C_A \) and \( C_R \) are damping matrices, and their specific expressions are described in detail in the following content.

For the piezoelectric laminated cylindrical shell, the differential equations of electro-elastic coupling motion included the elastic displacements \( q \) and the electric potential vector \( \psi \) are given by [31]

\[
M_{qq} \ddot{q} + C_R \dot{q} + (K_{qq} + K_{spr}) q + Q_N + K_{q \psi} \psi = F \\
K_{q \psi} q - K_{q \psi} \psi = P
\]

where \( M_{qq}, K_{qq}, \) and \( K_{spr} \) are the shell’s mass matrix, stiffness matrix, and spring stiffness matrix; \( K_{q \psi} \) is the coupled stiffness; \( K_{q \psi} \) is the electrical stiffness matrix; and \( Q_N \) is the nonlinear item. These matrices are given in Appendixes A, B, C, D, respectively. \( P \) is applied electrical charge for the piezoelectric layers, and \( F \) is the radial harmonic excitation on the position \((x, \theta)\), in which \( \mathbf{W}_0 \) is the admissible displacement functions in the shell’s radial direction, \( f \) is the excitation amplitude, and \( \omega_{EX} \) is the excitation frequency.
\[
\dot{q} = \left[ q_a \quad q_v \quad q_w \quad q_{\phi_v} \quad q_{\phi_w} \right]^T, \quad \dot{\psi} = \left[ \psi_a \quad \psi_v \right]^T
\]
\[
P = \left[ p_a \quad p_s \right], \quad F = \left[ f_u \quad f_v \quad f_w \quad f_{\phi_v} \quad f_{\phi_w} \right]^T
\]
\[
\begin{align*}
f_u(x, \theta, t) &= 0 \\
f_v(x, \theta, t) &= 0 \\
f_w(x, \theta, t) &= \overline{\mathbf{W}}_0 f \cos \omega_{1\text{EXT}} \\
f_{\phi_v}(x, \theta, t) &= 0 \\
f_{\phi_w}(x, \theta, t) &= 0
\end{align*}
\]

(25)

The differential equations of electro-elastic coupling motion can be written as follows to facilitate the decoupling of the equation,
\[
\begin{align*}
M_{qq}\ddot{q} + C_R \dot{q} + (K_{qq} + K_{spr})q + Q_N + K^{S\psi}_q \dot{\psi}_s + K^{\lambda}_q \dot{\psi}_a &= F \\
K^{\lambda\psi}_q \ddot{\psi}_s &= p_a \\
K^{ST}_q \ddot{\psi}_s &= p_s
\end{align*}
\]

(26)

Neglecting the converse piezoelectric effect of the sensor layer, the differential equations of motion are given as [32]
\[
\begin{align*}
M_{qq}\ddot{q} + C_R \dot{q} + \left( K_{qq} + K_{spr} + K^{S\psi}_q K^{S\psi-1}_q K^{ST}_q \right)q + Q_N &= F - K^{\lambda\psi}_q \dot{\psi}_s - K^{S\psi-1}_q K^{ST}_q \dot{q} \\
\end{align*}
\]

(27)

The negative velocity feedback control is used in this study as described in Fig. 2. The input potential of the actuator layer is given as
\[
\psi_a = -G_F \dot{\psi}_s = -G_F \dot{\psi}_s = -G_F K^{S\psi-1}_q K^{ST}_q \dot{q}
\]

(28)

where \( G_F \) is a constant gain. Then, the differential equation of decoupled motion writes the next form
\[
\begin{align*}
M_{qq}\ddot{q} + (C_A + C_R)\dot{q} + \left( K_{qq} + K_{spr} + K^{S\psi}_q K^{S\psi-1}_q K^{ST}_q \right)q + Q_N &= F \\
\end{align*}
\]

(29)

where \( C_R \) is the Rayleigh damping matrix. \( C_A \) is the damping matrix caused by the control potential.
\[
C_R = \alpha M_{qq} + \beta (K_{qq} + K_{spr})
\]

(30)

\[
C_A = -G_F K^{A\psi}_q K^{S\psi-1}_q K^{ST}_q
\]

(31)

where \( \alpha \) and \( \beta \) are the Rayleigh damping parameters, which can be given as
\[
\alpha = 2 \left( \frac{\xi_2}{\omega_2^2} - \frac{\xi_1}{\omega_1^2} \right) \left( \frac{1}{\omega_2^2} - \frac{1}{\omega_1^2} \right)
\]

(32)

where \( \omega_1, \omega_2 \) are the first- and second-order natural frequencies of the laminated cylindrical shells with discontinuous piezoelectric layer, and \( \xi_1 \), and \( \xi_2 \) are the damp coefficients. The differential equation is solved by IHBM in this paper.

The vibration differential equations are solved to obtain the nonlinear response by using the incremental harmonic balance method. The specific process of IHBM is shown in Appendix E.

3 Dynamic model verification

3.1 Linear model verification

In the previous paper [13], we have studied the time-domain response of the discontinuous piezoelectric laminated cylindrical shells subject to pulse excitation and harmonic excitation and optimized the position of the piezoelectric layer based on the system’s retained energy. In the above study, the comparison with the finite element software ANSYS is sufficient to verify the accuracy of the linear part of the model. The linear part refers to the free vibration, and through the verification of the linear part, the stiffness matrix and mass matrix of the model are verified. Therefore, we believed that the linear part of the model present in this paper is right, and the verification is no longer performed here.

3.2 Nonlinear model verification

In order to verify the nonlinear part of the model established in this paper, we compare the results obtained in this paper with those in the literature [14, 15]. By comparing with the nonlinear amplitude-frequency curves in the literature, the correctness of the nonlinear term \( Q_N \) is verified. The geometric parameters of the model in the verification process are:
\[
\begin{align*}
L &= 0.09587 \text{ m}, \\
R &= 0.0678 \text{ m}, \\
H &= 0.678 \text{ mm},
\end{align*}
\]
the damping coefficient $\xi = 0.001$ has been used for $n = 6$. The simply supported–simply supported boundary condition is employed, where $k_{u,i} = k_{\theta,i} = k_{v,i} = 0, k_{w,i} = 10^{12}(i = 0, 1)$. In the response curve in Fig. 3, $\omega$ is the excitation frequency, $\omega_{i(1,6)}$ is the fundamental frequency ($m = 1, n = 6$), $A$ is the amplitude of the mode, and $H$ denotes the thickness of the shell.

In Fig. 3, the amplitude–frequency curve calculated in this paper is compared with the results in the literature, and it can be seen from the figure that the amplitude–frequency curves obtained in this paper have a similar variation trend with those in the references. After the verification of both linear and nonlinear parts, it can be considered that the method we used is correct.

4 Results and discussion

In this section, the IHBM is used to obtain the nonlinear frequency–amplitude response curves of the piezoelectric laminated cylindrical shell. The structural parameters are selected as follows: $h = 0.002$ m, $h_a = 0.001$ m, $h_s = 0.001$ m, $H = h + h_a + h_s = 0.004$, $L = 0.1$ m, $H/R = 1/25$, $D = 0.2$ m, $NT = 5$, $f = 2e3$N, $NA = 16$, $k_{u,i} = k_{\theta,i} = k_{v,i} = k_{w,i} = k_{h,i} = k_{x,i} = 1e12$ N/m(N/rad), $k_{u,0} = k_{\theta,0} = k_{v,0} = k_{w,0} = k_{h,0} = k_{x,0} = 0$ N/m(N/rad), the exciting position $(x, \theta)$ is $(0.9, 3)$, and the response position is $(0.9, 3.2)$. In order to facilitate the comparison of different results, the horizontal and vertical coordinates of the amplitude–frequency response curve are dimensionless; that is, the $x$-axis is $\omega / \omega_{i(1,4)}$ (in this paper, the established model reaches the minimum frequency when $n = 4$; therefore, the first-order frequency $n = 4$ is selected as the research object), and the $y$-axis is $A/h$. The material parameters of the base layer and the piezoelectric layer are described in Table 1.

4.1 Influence of the constant gain for frequency–amplitude response

Fig. 2 The schematic diagram of feedback control

The effect of constant gain $G_F$ on the nonlinear amplitude–frequency response curve of a piezoelectric laminated cylindrical shell is discussed in this section. In Fig. 4, the response curves of the laminated shells with constant gain of 0, 0.1, 0.2, and 0.4 are discussed, respectively. The range of the piezoelectric layer is: (a) the size and position of the randomly selected piezoelectric layer as $\xi \in [0.4, 0.6]$, $\theta \in [0, 2\pi]$; (b) the piezoelectric layer is completely
covered on the surface of the shell, $\xi \in [0.4, 0.6]$, $\theta \in [0, 2\pi]$. According to Fig. 4, when the piezoelectric layer is completely covered on the surface of the shell, the nonlinear amplitude–frequency response curve has no obvious nonlinear characteristics and approaches the linear amplitude–frequency response. As the constant gain $G_F$ increases, the response of amplitude gradually decreases, and the nonlinear characteristics weaken, which proves that the piezoelectric layer has a large influence on the vibration response of the cylindrical shell. When the position and size of the piezoelectric layer are random, as $\xi \in [0.4, 0.6], \theta \in [0, 2\pi]$, the response curve has obvious soft nonlinear characteristics, and as the constant gain increases, the soft nonlinear characteristic weakens, and the amplitude decreases. After that, in Fig. 5, the influence of constant gain $G_F$ on the time-domain response of the laminated shell is calculated, respectively, when the excitation frequency is equal to the corresponding frequency of point A and point B in Fig. 4. Point A and point B are, respectively, the peak of the nonlinear response curve when $G_F = 0.4$ in Fig. 4. It can be seen from Fig. 5 that the response tends to be stable with time, and the nonlinear response curve is no longer a symmetrical curve.

The main reason for the phenomenon shown in Figs. 4 and 5 is that as the gain constant in Eq. (31) increases, the damping matrix $C$ in Eq. (29) increases. However, the mass matrix and stiffness matrix keep unchanged; therefore, the response amplitude decreases.

### 4.2 Influence of the size of the piezoelectric layer for frequency–amplitude response

According to the phenomenon in Sect. 4.1, we believe that it is necessary to study the influence of the size and position of the piezoelectric layer on the frequency–amplitude response. Then, the effects of the size and position of the piezoelectric layer are analyzed. This section discusses the effect of the size of the piezoelectric layer on the nonlinear amplitude–frequency response curve of the piezoelectric laminated...
cylindrical shell. In Fig. 6, the influence of the piezoelectric layer’s size on the nonlinear amplitude–frequency curve of the laminated shell is discussed when the constant gain is \( G_F = 0 \). The circumferential range of the piezoelectric layer is set as \( \xi \in [0, 0.1] \), \( \xi \in [0, 0.3] \), \( \xi \in [0, 0.5] \), \( \xi \in [0, 0.8] \), \( \xi \in [0, 1] \). It can be seen that the nonlinear amplitude–frequency response has obvious soft nonlinear characteristics; as the piezoelectric layer increases, the nonlinear characteristic of the response curve gradually decreases, and the response amplitude is reduced.

In Fig. 7, the influence of the piezoelectric layer’s size on the nonlinear amplitude–frequency curve of the laminated shell is discussed when the constant gain is \( G_F = 0 \). The circumferential range of the piezoelectric layer is set as \( \theta \in [0, 2\pi] \), and the axial range is, respectively, set as \( \xi \in [0, 0.1] \), \( \xi \in [0, 0.3] \), \( \xi \in [0, 0.5] \), \( \xi \in [0, 0.8] \), \( \xi \in [0, 1] \). It can be seen that the nonlinear amplitude–frequency response has obvious soft nonlinear characteristics; as the piezoelectric layer increases, the nonlinear characteristic of the response curve gradually decreases, and the response amplitude is reduced.

In Fig. 7, the influence of the piezoelectric layer’s size on the nonlinear amplitude–frequency curve of the laminated shell is discussed when the constant gain is \( G_F = 0, 0.1, 0.2, \) and \( 0.4 \), respectively, in which \( \xi \in [0, 0.3] \), \( \xi \in [0, 0.5] \), \( \xi \in [0, 0.7] \). In each figure, the size of the piezoelectric layer is kept constant, and only the size of \( G_F \) is changed. In Fig. 7a–c, when \( \xi \in [0, 0.3] \), the response curve changes less with the increase of \( G_F \); when \( \xi \in [0, 0.7] \), the response curve obviously changes with different \( G_F \). Then, in Fig. 8, the influence of constant gain on the time-domain response of the laminated shell is discussed, respectively, when the excitation frequency is equal to the corresponding frequency of points A, B, and C in Fig. 7. The points A, B, and C are the peak of the amplitude–frequency curve when \( G_F = 4 \) in Fig. 7. The response curve tends to be stable with time, and the amplitude of the response curve decreases with the increase in constant gain. According to Figs. 6, 7, and 8, it can be seen that as the size of the piezoelectric layer increases, the amplitude of the nonlinear frequency–amplitude response curve decreases, and the nonlinear characteristic decreases; at the same time, the influence of the constant gain on the vibration response is weakened.

4.3 Influence of the position of the piezoelectric layer for frequency–amplitude response

This section discusses the effect of the position of the piezoelectric layer on the nonlinear frequency–amplitude response curve of the piezoelectric laminated cylindrical shell with the clamped-free boundary condition. The size of the piezoelectric layer remains unchanged, \( \xi \in [0, 0.1], \theta \in [0, 2\pi] \), and the \( G_F = 0 \). In Fig. 9, the influence of the position of the piezoelectric layer on the nonlinear amplitude–frequency curve of the laminated shell is discussed, the position of the
piezoelectric layer is, respectively, calculated as: \( \xi \in [0, 0.1]; \ \xi \in [0.5, 0.6]; \ \xi \in [0.7, 0.8] \), and \( G_F \) keeps unchanged. The nonlinear frequency–amplitude response of the piezoelectric laminated cylindrical shell has obvious soft nonlinear characteristics. When the position of the piezoelectric layer is at the clamped end, the nonlinear characteristic has a weaker amplitude; near the free end, the response’s nonlinear characteristic is enhanced, and the amplitude is increased.

In Fig. 10, the influence of constant gain on the nonlinear amplitude–frequency response curve is discussed when the piezoelectric layer is at different positions, where the constant gain is 0, 0.1, 0.2, 0.4, the size of the piezoelectric layer remains unchanged, and the positions are \( \xi \in [0, 0.1], \ \xi \in [0.5, 0.6], \ \xi \in [0.7, 0.8] \), respectively. In each of the figures, the size and position of the piezoelectric layer are kept unchanged, and only \( G_F \) changes. In Fig. 10a–c, when \( \xi \in [0, 0.1] \), as \( G_F \) increases, the response curve changes less. When \( \xi \in [0.7, 0.8] \), with the rise of \( G_F \), the amplitude and nonlinear characteristics of the response curve are obviously weakened. In Fig. 11,
the influence of the constant gain on the time-domain response of the laminated shell is discussed when the piezoelectric layer is at different positions. The time-domain response at different piezoelectric layer locations is calculated in Fig. 11, and the response curve has significant nonlinearity when the response is stabilized. It can be seen from Figs. 9, 10, and 11 that as the position of the piezoelectric layer approaches the free end from the clamped end, the response amplitude increases, the nonlinear characteristic increases, and the effect of constant gain for the piezoelectric laminated cylindrical shell is enhanced.
5 Conclusions

In this paper, we innovatively present a discontinuous piezoelectric laminated cylindrical shell model with point-supported elastic boundary conditions and study the effect of the discontinuous piezoelectric layer on geometric nonlinear vibration of the shell under radial harmonic excitation. The following conclusions are drawn:

1. With the increase of the constant gain $G_F$, the amplitude of the nonlinear frequency–amplitude response and time-domain response decrease; at the same time, the nonlinear phenomenon weakens.

2. As the size of the piezoelectric layer is more prominent, the nonlinear phenomenon and the amplitude of the nonlinear response decrease.

3. The effect of the piezoelectric layer on response is enhanced by constant gain $G_F$ when the size increases.

4. For the clamped-free piezoelectric laminated shells, the amplitude of amplitude–frequency response increases slightly as the position of the piezoelectric layer is closer to the free end, and the nonlinear characteristics are enhanced.

5. The position of the piezoelectric layer is more obviously affected by the constant gain at the free end.

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Declarations

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this article.

Appendix A: Expressions for the generalized coordinates

\[ q_u = [a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, a_{22}, \cdots, a_{2n}, \cdots, a_{mn}, 0]^T e^{-j\omega t} \]

\[ q_v = [b_{11}, b_{12}, \cdots, b_{1n}, b_{21}, b_{22}, \cdots, b_{2n}, \cdots, b_{mn}, 0]^T e^{-j\omega t} \]

\[ q_w = [c_{11}, c_{12}, \cdots, c_{1n}, c_{21}, c_{22}, \cdots, c_{2n}, \cdots, c_{mn}, 0]^T e^{-j\omega t} \]

\[ q_{\phi_u} = [d_{11}, d_{12}, \cdots, d_{1n}, d_{21}, d_{22}, \cdots, d_{2n}, \cdots, d_{mn}, 0]^T e^{-j\omega t} \]

\[ q_{\phi_v} = [e_{11}, e_{12}, \cdots, e_{1n}, e_{21}, e_{22}, \cdots, e_{2n}, \cdots, e_{mn}, 0]^T e^{-j\omega t} \]

\[ q_{\phi_w} = [f_{11}, f_{12}, \cdots, f_{1n}, f_{21}, f_{22}, \cdots, f_{2n}, \cdots, f_{mn}, 0]^T e^{-j\omega t} \]

\[ q_{\phi_w_a} = [g_{11}, g_{12}, \cdots, g_{1n}, g_{21}, g_{22}, \cdots, g_{2n}, \cdots, g_{mn}, 0]^T e^{-j\omega t} \]

where $\theta$ is a $1 \times m$ zero vector, so that all the vectors $q_u, q_v, q_w, q_{\phi_u}, q_{\phi_v}, q_{\phi_w}, q_{\phi_w_a}$ have the same dimension.

Appendix B: Expressions for the mass matrix $M$

The generalized mass matrix of the piezoelectric laminated shell is expressed as

\[
M_{qq} = \begin{bmatrix}
M^{uu} & 0 & 0 & \frac{1}{2} M^{u\phi_u} & 0 \\
0 & M^{vv} & 0 & 0 & \frac{1}{2} M^{v\phi_v} \\
0 & 0 & M^{ww} & 0 & 0 \\
\frac{1}{2} M^{u\phi_v T} & 0 & 0 & M^{\phi_u \phi_v} & 0 \\
0 & \frac{1}{2} M^{v\phi_v T} & 0 & 0 & M^{\phi_u \phi_v} \\
\end{bmatrix}
\]

where
Appendix C: Expressions of the stiffness matrix \( K \)

The generalized stiffness matrix of the piezoelectric laminated shell is expressed as

\[
K_{qq} = \begin{bmatrix}
K^{uu} & \frac{1}{2} K^{vw} & \frac{1}{2} K^{ww} & \frac{1}{2} K^{wv} \\
\frac{1}{2} K^{vw} & K^{vw} & \frac{1}{2} K^{ww} & 0 \\
\frac{1}{2} K^{ww} & \frac{1}{2} K^{ww} & K^{ww} & \frac{1}{2} K^{ww} \\
\frac{1}{2} K^{wv} & 0 & \frac{1}{2} K^{ww} & K^{wv}
\end{bmatrix}
\]

(35)

where

\[
K^{uu} = LR \sum_{s=1}^{NP} \int_{\xi_0}^{\xi_f} \left( \frac{2A_{12} \partial U \partial U^T}{LR \partial \zeta} + \frac{2A_{66} \partial U \partial U^T}{LR \partial \zeta} \right) d\zeta d\theta
+ LR \sum_{r=1}^{NP} \int_{\xi_0}^{\xi_f} \left( \frac{2A_{12} \partial U \partial U^T}{LR \partial \zeta} + \frac{2A_{66} \partial U \partial U^T}{LR \partial \zeta} \right) d\zeta d\theta
\]

\[
K^{vw} = LR \sum_{s=1}^{NP} \int_{\xi_0}^{\xi_f} \left( \frac{2A_{12} \partial U \partial U^T}{LR \partial \zeta} + \frac{2A_{66} \partial U \partial U^T}{LR \partial \zeta} \right) d\zeta d\theta
+ LR \sum_{r=1}^{NP} \int_{\xi_0}^{\xi_f} \left( \frac{2A_{12} \partial U \partial U^T}{LR \partial \zeta} + \frac{2A_{66} \partial U \partial U^T}{LR \partial \zeta} \right) d\zeta d\theta
\]

\[
K^{ww} = LR \sum_{s=1}^{NP} \int_{\xi_0}^{\xi_f} \left( \frac{2A_{12} \partial U \partial U^T}{LR \partial \zeta} + \frac{2A_{66} \partial U \partial U^T}{LR \partial \zeta} \right) d\zeta d\theta
+ LR \sum_{r=1}^{NP} \int_{\xi_0}^{\xi_f} \left( \frac{2A_{12} \partial U \partial U^T}{LR \partial \zeta} + \frac{2A_{66} \partial U \partial U^T}{LR \partial \zeta} \right) d\zeta d\theta
\]
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The spring stiffness matrix of the piezoelectric laminated shell is shown as

\[ \mathbf{K}_{spr} = \begin{bmatrix} \mathbf{K}_{u u} & \mathbf{K}_{v v} & \mathbf{K}_{w w} & \mathbf{K}_{\phi \phi} \end{bmatrix} \]

where

\[ \mathbf{K}_{u u} = \sum_{p=1}^{N_p} \left( k_{u u}^p U(0, \theta_p) U^T(0, \theta_p) + k_{u u}^p U(1, \theta_p) U^T(1, \theta_p) \right) \]

\[ \mathbf{K}_{v v} = \sum_{p=1}^{N_p} \left( k_{v v}^p V(0, \theta_p) V^T(0, \theta_p) + k_{v v}^p V(1, \theta_p) V^T(1, \theta_p) \right) \]

\[ \mathbf{K}_{w w} = \sum_{p=1}^{N_p} \left( k_{w w}^p W(0, \theta_p) W^T(0, \theta_p) + k_{w w}^p W(1, \theta_p) W^T(1, \theta_p) \right) \]

\[ \mathbf{K}_{\phi \phi} = \sum_{p=1}^{N_p} \left( k_{\phi \phi}^p \Phi_s(0, \theta_p) \Phi_s^T(0, \theta_p) + k_{\phi \phi}^p \Phi_s(1, \theta_p) \Phi_s^T(1, \theta_p) \right) \]

Appendix D: Expressions of the spring stiffness matrix \( \mathbf{K}_{spr} \)

The spring stiffness matrix of the piezoelectric laminated shell is shown as

\[ \mathbf{K}_{spr} = \begin{bmatrix} \mathbf{K}_{u u} & \mathbf{K}_{v v} & \mathbf{K}_{w w} & \mathbf{K}_{\phi \phi} \end{bmatrix} \]

\[ \mathbf{K}_{u u} = \sum_{p=1}^{N_p} \left( k_{u u}^p U(0, \theta_p) U^T(0, \theta_p) + k_{u u}^p U(1, \theta_p) U^T(1, \theta_p) \right) \]

\[ \mathbf{K}_{v v} = \sum_{p=1}^{N_p} \left( k_{v v}^p V(0, \theta_p) V^T(0, \theta_p) + k_{v v}^p V(1, \theta_p) V^T(1, \theta_p) \right) \]

\[ \mathbf{K}_{w w} = \sum_{p=1}^{N_p} \left( k_{w w}^p W(0, \theta_p) W^T(0, \theta_p) + k_{w w}^p W(1, \theta_p) W^T(1, \theta_p) \right) \]

\[ \mathbf{K}_{\phi \phi} = \sum_{p=1}^{N_p} \left( k_{\phi \phi}^p \Phi_s(0, \theta_p) \Phi_s^T(0, \theta_p) + k_{\phi \phi}^p \Phi_s(1, \theta_p) \Phi_s^T(1, \theta_p) \right) \]
Appendix E: Expressions of the nonlinear item $Q_N$

$$Q_N = \frac{LR}{2} \sum_{s=1}^{NP} \int_{0}^{\varepsilon_s} \int_{0}^{\varepsilon_s'} \left( \begin{array}{c}
A_{12} q_u^T \frac{\partial U \partial W^T}{\partial \varepsilon} - q_u q_w \frac{\partial W}{\partial \varepsilon'} + A_{11} q_u^T \frac{\partial U \partial W^T}{\partial \varepsilon} - q_u q_w \frac{\partial W}{\partial \varepsilon'} \\
+ 2A_{66} q_u^T \frac{\partial U \partial W^T}{\partial \varepsilon} - q_u q_w \frac{\partial W}{\partial \varepsilon'} + 2A_{66} q_v^T \frac{\partial V \partial W^T}{\partial \varepsilon} - q_v q_w \frac{\partial W}{\partial \varepsilon'} \\
+ A_{12} q_v^T \frac{\partial V \partial W^T}{\partial \varepsilon} - q_v q_w \frac{\partial W}{\partial \varepsilon'} + A_{22} q_v^T \frac{\partial V \partial W^T}{\partial \varepsilon} - q_v q_w \frac{\partial W}{\partial \varepsilon'} \\
+ A_{12} q_w^T \frac{\partial W^T}{\partial \varepsilon} - q_w q_w \frac{\partial W}{\partial \varepsilon'} + A_{22} q_w^T \frac{\partial W^T}{\partial \varepsilon} - q_w q_w \frac{\partial W}{\partial \varepsilon'} \\
+ B_{12} q_{\phi_s}^T \frac{\partial \Phi_s \partial W^T}{\partial \varepsilon} - q_{\phi_s} q_{\phi_s} \frac{\partial W}{\partial \varepsilon'} + B_{11} q_{\phi_s}^T \frac{\partial \Phi_s \partial W^T}{\partial \varepsilon} - q_{\phi_s} q_{\phi_s} \frac{\partial W}{\partial \varepsilon'} \\
+ 2B_{66} q_{\phi_s}^T \frac{\partial \Phi_s \partial W^T}{\partial \varepsilon} - q_{\phi_s} q_{\phi_s} \frac{\partial W}{\partial \varepsilon'} + 2B_{66} q_{\phi_s}^T \frac{\partial \Phi_s \partial W^T}{\partial \varepsilon} - q_{\phi_s} q_{\phi_s} \frac{\partial W}{\partial \varepsilon'} \\
+ B_{12} q_{\phi_s}^T \frac{\partial \Phi_s \partial W^T}{\partial \varepsilon} - q_{\phi_s} q_{\phi_s} \frac{\partial W}{\partial \varepsilon'} + B_{22} q_{\phi_s}^T \frac{\partial \Phi_s \partial W^T}{\partial \varepsilon} - q_{\phi_s} q_{\phi_s} \frac{\partial W}{\partial \varepsilon'} \\
+ A_{11} q_{\phi_s}^T \frac{\partial \Phi_s \partial W^T}{\partial \varepsilon} - q_{\phi_s} q_{\phi_s} \frac{\partial W}{\partial \varepsilon'} + A_{22} q_{\phi_s}^T \frac{\partial \Phi_s \partial W^T}{\partial \varepsilon} - q_{\phi_s} q_{\phi_s} \frac{\partial W}{\partial \varepsilon'} \\
+ A_{66} q_{\phi_s}^T \frac{\partial \Phi_s \partial W^T}{\partial \varepsilon} - q_{\phi_s} q_{\phi_s} \frac{\partial W}{\partial \varepsilon'} \end{array} \right) d\varepsilon d\varepsilon' \right) \text{d} \varepsilon \text{d} \varepsilon'}$$
Appendix F: Incremental Harmonic Balance Method

By presenting the new non-dimensional time variable \( \tau \), we write the nonlinear vibration differential equation to a modified form

\[
\hat{\omega}^2 M_{qq} \dddot{X} + \omega (C_A + C_R) \ddot{X} + \left( K_{qq} + K_{qr} + K_s^2 K_{q \phi}^{S-1} K_{q \phi}^{ST} + K_N^{(2)} + K_N^{(3)} \right) = F_{\text{cost}}
\]

\[
\tau = \omega_{\text{ext}}, \quad X = [X_1, X_2, X_3, \ldots, X_n]^T, \quad F = [f_u, f_v, f_w, f_{\phi}, f_{\psi}]^T
\]

(39)

Firstly, the differential equations are linearized by using the Newton–Raphson procedure. The corresponding adding increments are used to represent the neighboring state as

\[
X_j = X_{j0} + \Delta X, \quad j = 1, 2, \ldots, n
\]

\[
\omega = \omega_0 + \Delta \omega
\]

(40)

Taking Eqs. (40) to (39), the higher-order terms are neglected, and Eq. (39) is written as

\[
\hat{\omega}^2 M_{qq} \dddot{X} + \omega_0 (C_A + C_R) \ddot{X} + \left( K_{qq} + K_{qr} + K_s^2 K_{q \phi}^{S-1} K_{q \phi}^{ST} + 2K_N^{(2)} + 3K_N^{(3)} \right) = \text{Re} \left( 2\omega_0 M_{qq} \dddot{X} + (C_A + C_R) X' \right) \Delta \omega
\]

(41)
\[ \text{Re} = F \cos \tau - \omega^2_0 M_{yy} X''_0 - \omega_0 (C_A + C_R) X'_0 - \left(K_{yy} + K^S_{yy} K_{yy}^{S-1} K_{yy}^{ST} + 2 K^{(2)}_N + 3 K^{(3)}_N\right) X_0 \]

where \[ X_0 = [X_{10}, X_{20}, X_{30}, \ldots, X_{n0}]^T, \quad \Delta X = [\Delta X_1, \Delta X_2, \Delta X_3, \ldots, \Delta X_n]^T. \] When the solution is the exact value, the residue “\text{Re}” becomes zero.

Next, by using the Galerkin’s technique, the steady-state response with a truncated Fourier series is written as

\[ X_0 = A_0 + \sum_{k=1}^{r} \left( a_{jk} \cos k \tau + b_{jk} \sin k \tau \right) = 2T_c A_j \]

\[ \Delta X_j = \Delta A + \sum_{k=1}^{r} \left( a_{jk} \cos k \tau + b_{jk} \sin k \tau \right) = 2T_c \Delta A_j \]

where

\[ T_c = [1, \cos \tau, \sin \tau, \cos 2\tau, \sin 2\tau, \ldots, \cos r\tau, \sin r\tau]^T \]

\[ A_j = \{a_{j1}, a_{j2}, \ldots, a_{jn}, b_{j1}, b_{j2}, \ldots, b_{jn}\}^T \]

\[ \Delta A_j = \{\Delta a_{j1}, \Delta a_{j2}, \ldots, \Delta a_{jn}, \Delta b_{j1}, \Delta b_{j2}, \ldots, \Delta b_{jn}\}^T \]

(43)

(44)

where \( a_{jk}, b_{jk} \) are the Fourier coefficients, and \( n \) are the numbers of cosine and sine terms. Using these vectors of Fourier coefficients \( A \) and its increments \( \Delta A \), the vectors of the unknown parameters are written as

\[ X_0 = SA, \quad \Delta X = S \Delta A \]

in which

\[ S = \begin{bmatrix} T_c & T_c & \ldots & T_c \end{bmatrix}, \quad A = \{A_1, A_2, \ldots, A_n\}^T, \]

\[ \Delta A = \{\Delta A_1, \Delta A_2, \ldots, \Delta A_n\}^T \]

(45)

(46)

Taking Eqs. (45) to (41), Galerkin’s technique is used.

\[ \int_0^{2\pi} \delta(\Delta X)^T \left[ \omega^2_0 M_{yy} X''_0 + \omega_0 (C_A + C_R) X'_0 \right. \]

\[ \left. + \left(K_{yy} + K^S_{yy} K_{yy}^{S-1} K_{yy}^{ST} + 2 K^{(2)}_N + 3 K^{(3)}_N\right) \Delta X \right] d\tau \]

\[ = \int_0^{2\pi} \delta(\Delta X)^T \left[ \text{Re} - (2\omega_0 M_{yy} X''_0) \Delta \omega - (C_A + C_R) X'_0 \right] d\tau \]

(47)

The linear equation including \( \Delta A \) and \( \Delta \omega \) is written as

\[ K_{mc} \Delta A + R_{mc} \Delta \omega = R_{m1} A + R_{m2} \]

(48)

where

\[ K_{mc} = \int_0^{2\pi} S^T \left[ \omega^2_0 M_{yy} \mathbf{\hat{f}} + \omega (C_A + C_R) \mathbf{\hat{s}} \right. \]

\[ \left. + \left(K_{yy} + K^S_{yy} K_{yy}^{S-1} K_{yy}^{ST} + 2 K^{(2)}_N + 3 K^{(3)}_N\right) \mathbf{S} \right] d\tau \]

\[ = \int_0^{2\pi} S^T \left[ (2\omega_0 M_{yy} \mathbf{\hat{f}} + (C_A + C_R) \mathbf{\hat{s}}) d\tau R_{m1} \right] d\tau R_{m2} \]

(49)

Finally, we use the arc-length method to solve Eq. (48) for obtaining the frequency–amplitude response.

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