Early-Time and Late-Time Quantum Chaos

Chen-Te Ma$^{a,b}$ 1

$^a$ Institute of Quantum Matter,
School of Physics and Telecommunication Engineering,
South China Normal University, Guangzhou 510006, Guangdong, China.

$^b$ The Laboratory for Quantum Gravity and Strings,
Department of Mathematics and Applied Mathematics,
University of Cape Town, Private Bag, Rondebosch 7700, South Africa.

Abstract

We show the relation between the Heisenberg averaging of regularized two-points out-of-time ordered correlation function and the two-points spectral form factor in bosonic quantum mechanics. The generalization to all even-points is also discussed. We also do the direct extension from the bosonic quantum mechanics to the non-interacting scalar field theory through the oscillator languages. Finally, we find that the coherent state and large-$N$ approaches are useful in the late-time study. We find that the computation of the coherent state can be simplified by the Heisenberg averaging. Therefore, this provides a simplified way to probe the late-time quantum chaos through a coherent state. The large-$N$ result is also comparable to the $N = 3$ numerical result in the large-$N$ quantum mechanics. This can justify that large-$N$ technique in bosonic quantum mechanics can really probe the late time, not the early time. Because the quantitative behavior of large-$N$ can be captured from the $N = 3$ numerical result, the realization in experiments should be possible.

1e-mail address: yefgst@gmail.com
1 Introduction

Quantum chaos is an interesting direction in understanding the classical chaos phenomena in a quantum system [1]. The classical chaos phenomena have irregular dynamics and instability features. The most intuitive and well-known feature is the sensitivity on initial conditions. Since the well-known condition can be included in other features, this is not a conclusive characteristic, but it is easier to realize in a quantum system like the out-of-time ordered correlation function (OTOC) [2]. This provides the early-time chaos. The other expected criteria are random matrix statistics in spectrum [3] and was concretely realized from the Sinai billiard model [4]. Since the spectral statistics in a quantum chaotic model is quite different from the integrable model, this should not be just a coincidence [5]. Now people conjectured that the spectrum of a quantized chaotic system should be exhibited as in the random matrix ensemble [6]. This probes the late-time chaos.

We first introduce the early-time chaos. The criteria of early-time chaos is the exponential growth on the time in OTOC. This is characterized by the exponent of OTOC or Lyapunov exponent. In various chaotic models, the OTOC only has the exponential growth at the early time and saturates (vanishing Lyapunov exponent) at the late time. The Sachdev-Ye-Kitaev (SYK) model [7] is one familiar example recently [8]. The physical reason is possibly due to that a quantum system follows the uncertainty principle [9]. Therefore, we lose the infinitesimal perturbation. Hence the exponential growth cannot persist forever in a chaotic system. This immediately implies that OTOC cannot probe the classical chaos phenomena [10]. Therefore, we should choose one way to connect OTOC to other late-time chaotic quantities for the study of sensitivity on initial conditions at the late time for understanding the classical chaos limit.

The most direct way for observing the random matrix spectrum is to compute the level spacing distribution function. The SYK model exhibits the random matrix statistics [11]. From the transition between the integrable and non-integrable models, ones also observed the Poisson distribution and the random matrix statistics respectively [12]. This showed that the level spacing distribution function is a useful chaos quantity, but this only restricts to quantum mechanics. The recent study in the random matrix spectrum was extensively studied in the spectral form factor (SFF) [13]. Because SFF is defined by the partition function, we can compute the quantity in quantum field theory without only restricting to quantum mechanics. This lets us probe the random matrix
spectrum in two-dimensional conformal field theory (CFT$_2$) [13] or other quantum field theories.

Recently, quantum chaos got much attention because it can be applied to gravity theory or black hole physics. For the analytical computation in quantum field theory, people choose the regularized form to define the OTOC (regularized OTOC) [14]. Ones first showed that the Lyapunov exponent in the regularized OTOC has the universal bound under some assumptions [15]. Because various studies showed that quantum field theory has the deviation of bound if the bulk theory is not Einstein gravity theory, ones conjectured that the boundary theory of Einstein gravity theory should saturate the bound [15]. The SYK model gave the concrete support to the conjecture because the model saturates the bound under the holographic limit from a direct computation [8], and the corresponding bulk theory can be obtained by the compactification from Einstein gravity theory. This motivates various studies for understanding the conjecture through computing the regularized OTOC.

For understanding the validity of the conjecture, we should know whether the Lyapunov exponent is affected by the regularization. The first proof for the universal Lyapunov exponent, defined by some regularize forms, for all theories [16], but the result is not valid for the generic form. The first testable system is the disordered electron system [17]. In the disordered metals [18], ones first found that the Lyapunov exponent is affected by the regularization [19] through the Schwinger-Keldysh formalism [20] and non-linear sigma model [21]. Later a more generic computation also got a similar result and showed that the dependence of regularization is due to the infrared regulator [22]. Nevertheless, the gapless limit possibly does not suffer from the issue [22]. Hence the conjecture does not meet any counterexample.

For the probe of black hole physics or information loss [23], the first study is the two-point correlation function in CFT$_2$ [24]. The correlation function decays exponentially with a fast speed, which exhibits that the corresponding back hole is more thermal than a thermal state in a unitary theory. This interpretation links the information loss problem to correlation functions [25].

The application is not only restricted to high energy physics, and this can also be applied to strongly correlated condensed matter physics [26] and quantum information. For example, the OTOC provides the butterfly velocity, which characterizes the spread
of quantum information [27]. Furthermore, the protocol was devised to measure the OTOC [28] and was already implemented [29]. The measure of regularized OTOC at the finite temperature was also proposed by the similar way [30]. The protocol can also extract the butterfly velocity from the OTOC [31]. This experiment set-up can also be applied to the Jaynes-Cummings (JC) interactions [32] and the Loschmidt echo [33], which can provide the semiclassical calculation of the Lyapunov exponent [34]. These applications provide usefulness to the study of quantum chaos.

Recently, the connection between the early-time chaos and late-time chaos has some new developments. One development is using the Lyapunov spectrum to obtain the statistics of random matrix ensemble. This can be seen as applying the statistics to Lyapunov exponent for connecting the early-time chaos to late-time chaos. Therefore, the averaging should be an important ingredient for the connection [9]. The other approach is to average over all Pauli operators in OTOCs to provide the spectral form factors [35]. This approach has another similar numerical observation to support from the summing over all momentum modes in OTOC, which provides the similar integrable-chaotic transition to the spectral analysis [36]. The first approach only has numerical evidence now. Although the second approach is exact, this is only useful in a finite-dimensional Hilbert space. This motivates the central question that we would like to address in this paper: How do we connect the early-time chaos to late-time chaos in an infinite-dimensional Hilbert space?

In this paper, we apply the Heisenberg averaging to the regularized two-points OTOC for obtaining the two-points SFF [37]. This approach is primarily based on the Haar measure property [38]. We found that applying the Heisenberg averaging to the most generic regularized four-points OTOC does not give the simple relation to the partition function. Nevertheless, we can find the corresponding four-points correlation function with the Heisenberg averaging for getting the four-point SFF. We also discuss the generalization of all even-points. Then we directly extend bosonic quantum mechanics to non-interacting scalar field theory from the Heisenberg group [37]. The extension is not trivial because the dimensions of Hilbert space are infinite. When we consider a Lie algebra with infinite-dimensional generators, we should expect to average over an infinite-dimensional group. For example, $W_\infty$ algebra [39]. Our result explicitly shows that the averaging is only over the phase space [40], not the full Hilbert space.

Since OTOCs and SFFs are hard to compute, the connection seems to be unuseful.
Indeed, introducing the Heisenberg averaging to the OTOC becomes easier to compute at the late-time limit [37]. We first discuss the coherent state [41], which is a quantum state closest to a classical regime. We can find that the coherent state [41] can simplify the computation in the averaged OTOCs because computation can be simplified by using the property of the Heisenberg group. In a generic system, we can also use the saddle-point evaluation to treat the integration in the OTOCs [37]. Our coherent approach is also suitable to the solvable and detectable model, the two-photon non-degenerate JC model with the rotating wave approximation, which ignores the oscillating fast term [42]. The second approach is the large-\(N\) technique [43]. We apply the large-\(N\) approximation to bosonic quantum mechanics [44]. Then the large-\(N\) theory is just harmonic oscillators with a modified frequency. Therefore, two-points SFF has an exact solution. We compare this exact solution to the numerical study. Our \(N = 3\) numerical result is already good enough for capturing the quantitative behavior of the large-\(N\). This precisely justifies that the large-\(N\) technique in bosonic systems is useful for the probe of late time.

This paper is organized as follows. We first introduce the Heisenberg averaging to correlation functions for connecting to SFFs in Sec. 2. Then we discuss the late-time study from the coherent state approach and large-\(N\) technique in Sec. 3. Finally, we discuss and conclude in Sec. 4.

2 Heisenberg Group

We first introduce the Heisenberg averaging to the two-points regularized OTOC at the finite temperature [2], and it leads to the two-points SFF [13]. Then we discuss the generalization to the higher-points correlation functions and also scalar field theory [37].

2.1 Two-Points Spectral Form Factor

The Heisenberg group is a two-dimensional Lie group generated by the canonical position \(X\) and the canonical momentum \(P\). The commutation between \(X\) and \(P\) is

\[
[P, X] = -i. \tag{1}
\]

A element of this group with respect to the variables, \(q_1, q_2\), is

\[
U(q_1, q_2) \equiv e^{iq_1X+iq_2P}. \tag{2}
\]
This element is unitary,

\[ U(q_1, q_2) U^\dagger(q_1, q_2) = 1. \]  

By direct computation:

\[
\int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} dq_2 \langle x_1 | U(q_1, q_2) | x_2 \rangle \langle y_1 | U^\dagger(q_1, q_2) | y_2 \rangle \\
= \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} dq_2 \ e^{iq_1 x_2 - y_1} \langle x_1 | e^{iq_2 P} | x_2 \rangle \langle y_1 | e^{-iq_2 P} | y_2 \rangle \\
= \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} dq_2 \ e^{iq_1 x_2 - y_1} \langle x_1 | x_2 - q_2 \rangle \langle y_1 | y_2 + q_2 \rangle \\
= \int_{-\infty}^{\infty} dq_2 \ \delta(x_2 - y_1) \langle x_1 | x_2 - q_2 \rangle \langle x_2 | y_2 + q_2 \rangle \\
= \delta(x_2 - y_1) \delta(x_1 - x_2 + q_2) \delta(x_2 - y_2 - q_2) \\
= \delta(x_2 - y_1) \delta(x_2 - y_2 + x_1 - x_2) \\
= \delta(x_2 - y_1) \delta(x_1 - y_2),
\]  

in which we used

\[ U(q_1, q_2) \equiv e^{iq_1 X + iq_2 P} = e^{iq_2 P} e^{iq_1 X + \frac{iq_2 q_2}{2}}, \quad e^{iq_1 X} | x \rangle = e^{iq_1 x} | x \rangle \]  

in the first equality, we used

\[ e^{iq_2 P} | x \rangle = | x - q_2 \rangle, \quad \langle p | x \rangle = e^{-ipx} \]  

in the second equality, we used

\[
\int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{iq_1 (x_2 - y_1)} = \delta(x_2 - y_1)
\]  

in the third equality. The computation result shows the same result as in using the Haar measure [38].
Now we compute the Heisenberg average of the regularized two-points OTOC:

\[
C_2 = \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} dq_2 \int_{-\infty}^{\infty} dx \langle x | U(q_1, q_2) e^{-(\beta/2+it)H} U_1^\dagger (q_1, q_2) e^{-(\beta/2-it)H} | x \rangle
\]

\[
= \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} dq_2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3
\]

\[
\times \langle x | U(q_1, q_2) | x_1 \rangle \langle x_1 | e^{-(\beta/2+it)H} | x_2 \rangle \langle x_2 | U_1^\dagger (q_1, q_2) | x_3 \rangle \langle x_3 | e^{-(\beta/2-it)H} | x \rangle
\]

\[
= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3
\]

\[
\times \delta(x - x_3) \delta(x_1 - x_2) \langle x_1 | e^{-(\beta/2+it)H} | x_2 \rangle \langle x_3 | e^{-(\beta/2-it)H} | x \rangle
\]

\[
= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx_1 \langle x_1 | e^{-(\beta/2+it)H} | x \rangle \langle x | e^{-(\beta/2-it)H} | x \rangle.
\]  

Therefore, the two-point SFF at the inverse finite temperature \(\beta/2\) exactly connects to \(C_2\) at the inverse temperature \(\beta\).

### 2.2 Higher-Points Correlation Functions

Now we consider the most generic Heisenberg average of the regularized four-points OTOC [22]

\[
C_{M4} = \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} dq_2 \int_{-\infty}^{\infty} dq_3 \int_{-\infty}^{\infty} dq_4 \int_{-\infty}^{\infty} dx
\]

\[
\times \langle x | U_1(q_1, q_2) e^{-(\beta\sigma+it)H} U_1^\dagger (q_1, q_2) e^{-(\beta(\alpha-\sigma)-it)H} | x \rangle
\]

\[
\times U_2(q_3, q_4) e^{-(\beta\sigma+it)H} U_2^\dagger (q_3, q_4) e^{-(\beta(1-\alpha-\sigma)-it)H} \rangle
\]

\[
+ \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} dq_2 \int_{-\infty}^{\infty} dq_3 \int_{-\infty}^{\infty} dq_4 \int_{-\infty}^{\infty} dx
\]

\[
\times \langle x | e^{-(\beta(1-\alpha-\sigma)+it)H} U_2(q_3, q_4) e^{-(\beta\sigma-it)H} U_2^\dagger (q_3, q_4) e^{-(\beta(\alpha-\sigma)+it)H} | x \rangle
\]

\[
\times U_1(q_1, q_2) e^{-(\beta\sigma-it)H} U_1^\dagger (q_1, q_2) | x \rangle
\],

\[
(9)
\]

where

\[
0 \leq \alpha \leq 1, \quad 0 \leq \sigma \leq \frac{1}{4}.
\]  

\[
(10)
\]
We will use the following relation
\[ \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} dq_2 \langle x_1 | U(q_1, q_2) | x_2 \rangle \langle y_1 | U^\dagger(q_1, q_2) | y_2 \rangle = \delta(x_2 - y_1) \delta(x_1 - y_2) \]  
(11)
to do the above integration:
\[ \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} dq_2 \left\langle x \right| U_1(q_1, q_2)e^{-(\beta_\sigma + it)H} U_1^\dagger(q_1, q_2) \left| y_1 \right\rangle \]
\[ = \delta(x - y_1) \left\langle x \right| e^{-(\beta_\sigma + it)H} \left| y_1 \right\rangle , \]
\[ \int_{-\infty}^{\infty} \frac{dq_3}{2\pi} \int_{-\infty}^{\infty} dq_4 \left\langle y_2 \right| U_2(q_3, q_4)e^{-(\beta_\sigma + it)H} U_2^\dagger(q_3, q_4) \left| y_3 \right\rangle \]
\[ = \delta(y_2 - y_3) \left\langle y_2 \right| e^{-(\beta_\sigma + it)H} \left| y_3 \right\rangle . \]  
(12)
Therefore, we get:
\[ \langle x \left| U_1(q_1, q_2)e^{-(\beta_\sigma + it)H} U_1^\dagger(q_1, q_2)e^{-(\beta(\alpha - \sigma) - it)H} \right. \]
\[ \times U_2(q_3, q_4)e^{-(\beta_\sigma + it)H} U_2^\dagger(q_3, q_4)e^{-(\beta(1 - \alpha - \sigma) - it)H} \left| x \right\rangle \]
\[ = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \int_{-\infty}^{\infty} dy_3 \delta(x - y_1) \delta(y_2 - y_3) \]
\[ \times \left\langle x \right| e^{-(\beta_\sigma + it)H} \left| y_1 \right\rangle \left\langle y_1 \right| e^{-(\beta(\alpha - \sigma) - it)H} \left| y_2 \right\rangle \]
\[ \times \left\langle y_2 \right| e^{-(\beta_\sigma + it)H} \left| y_3 \right\rangle \left\langle y_3 \right| e^{-(\beta(1 - \alpha - \sigma) - it)H} \left| x \right\rangle \]
\[ = \int_{-\infty}^{\infty} dy_2 \left\langle x \right| e^{-(\beta_\sigma + it)H} \left| y_2 \right\rangle \left\langle y_2 \right| e^{-(\beta(\alpha - \sigma) - it)H} \left| y_2 \right\rangle \]
\[ \times \left\langle y_2 \right| e^{-(\beta_\sigma + it)H} \left| y_2 \right\rangle \left\langle y_2 \right| e^{-(\beta(1 - \alpha - \sigma) - it)H} \left| x \right\rangle . \]  
(13)
Finally, we obtain
\[ C_{M4} \]
\[ = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy_2 \left\langle x \right| e^{-(\beta_\sigma + it)H} \left| x \right\rangle \left\langle x \right| e^{-(\beta(\alpha - \sigma) - it)H} \left| y_2 \right\rangle \]
\[ \times \left\langle y_2 \right| e^{-(\beta_\sigma - it)H} \left| y_2 \right\rangle \left\langle y_2 \right| e^{-(\beta(1 - \alpha - \sigma) + it)H} \left| x \right\rangle \]
\[ + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy_2 \left\langle x \right| e^{-(\beta_\sigma - it)H} \left| x \right\rangle \left\langle y_2 \right| e^{-(\beta(\alpha - \sigma) + it)H} \left| x \right\rangle \]
\[ \times \left\langle y_2 \right| e^{-(\beta_\sigma + it)H} \left| y_2 \right\rangle \left\langle y_2 \right| e^{-(\beta(1 - \alpha - \sigma) - it)H} \left| y_2 \right\rangle . \]  
(14)
When we choose
\[
\alpha = \frac{1}{2}, \quad \sigma = \frac{1}{4},
\]
we can go back to the usual regularized-OTOC, and the regularized OTOC becomes

\[
C_{M4} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy_2 \left\langle x | e^{-(\beta/4+it)H} | x \right\rangle \left\langle x | e^{-(\beta/4-it)H} | y_2 \right\rangle 
\times \left\langle y_2 | e^{-(\beta/4-it)H} | y_2 \right\rangle \left\langle y_2 | e^{-(\beta/4+it)H} | x \right\rangle 
+ \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy_2 \left\langle x | e^{-(\beta/4-it)H} | x \right\rangle \left\langle y_2 | e^{-(\beta/4+it)H} | y_2 \right\rangle 
\times \left\langle y_2 | e^{-(\beta/4-it)H} | y_2 \right\rangle \left\langle x | e^{-(\beta/4+it)H} | y_2 \right\rangle.
\]

Therefore, it is hard to find the simple relation to the partition function as in the regularized two-points OTOC. If we want to have a simple relation, we need to include more fluctuation to quantum states by the below replacement:

\[
\left\langle x | e^{-(\beta/4-it)H} | y_2 \right\rangle \rightarrow \left\langle y_3 | e^{-(\beta/4-it)H} | y_3 \right\rangle,
\]
\[
\left\langle y_2 | e^{-(\beta/4-it)H} | x \right\rangle \rightarrow \left\langle y_4 | e^{-(\beta/4-it)H} | y_4 \right\rangle,
\]
\[
\left\langle y_2 | e^{-(\beta/4+it)H} | x \right\rangle \rightarrow \left\langle y_3 | e^{-(\beta/4+it)H} | y_3 \right\rangle,
\]
\[
\left\langle x | e^{-(\beta/4+it)H} | y_2 \right\rangle \rightarrow \left\langle y_4 | e^{-(\beta/4+it)H} | y_4 \right\rangle.
\]

and also introduce integration with respect to the variables, \(y_3\) and \(y_4\). Nevertheless, it is not surprising for this result. For this method, we need the relation in the two-points SFF

\[
A_1 B_1 = I,
\]

where \(A_1\) and \(B_1\) are two operators, and \(I\) is the identity operator. Therefore, we only have one independent operator. We used the Heisenberg group to generate the operator. If we want to generalize the relation to the four-points SFF, we need the below relation

\[
A_1 B_1 A_2 B_2 = I.
\]
Therefore, we need to generate three independent variables, and the operator $B_2$ is
\[ B_2 = A_2^\dagger B_1^\dagger A_1^\dagger. \] (20)

One direct way for obtaining the four-points SFF is to use the Heisenberg group to generate three independent operators, and then we can work the similar generalization [35] by computing the four-points correlation function
\[ U_1(q_1, q_2)U_2(q_3, q_4; t)U_3(q_5, q_6)U_4(q_1, q_2, q_3, q_4, q_5, q_6; t), \] (21)
where
\[ U_2(q_3, q_4; t) \equiv e^{iHt}U_2(q_3, q_4)e^{-iHt}, \]
\[ U_4(q_1, q_2, q_3, q_4, q_5, q_6; t) \equiv e^{iHt}U_4(q_1, q_2, q_3, q_4, q_5, q_6)e^{-iHt}, \] (22)
with the Heisenberg averaging, where
\[ U_4 \equiv U_3^\dagger U_2^\dagger U_1^\dagger. \] (23)

For the generalization to all even-points OTOCs, we need to introduce the condition
\[ A_1B_1A_2B_2 \cdots A_kB_k = I \] (24)
and also use the Heisenberg group to generate all independent operators. After we do the $(2k - 1)$-times Heisenberg averaging, we can get the $2k$-points SFFs. This generalization is more similar to computing $2k$-correlation functions, not our familiar OTOCs, which at most has two independent operators. The physical interpretation of the higher-points correlation functions possibly not be the same as the OTOCs, but it is interesting to study the decay of correlation functions for the information loss issue [24]. Although the generalization was only discussed at $\beta = 0$, it is easy to use the similar way to generalize to $\beta \neq 0$.

### 2.3 Scalar Field Theory

Now we discuss how to extend the above result to quantum field theory. Although bosonic quantum mechanics has an infinite-dimensional Hilbert space, it is countable. In general, quantum field theory does not have a countable Hilbert space. One way for understanding the extension is to consider the scalar field theory in a box, which satisfies the periodic boundary condition, for having the discrete momenta $\vec{k}$. This Hilbert space is countable, but this is truly quantum field theory, not a quantum mechanical
Here we consider the non-interacting scalar field theory. This can be seen as the assembly of harmonic oscillators. Because a direct generalization is easy from the discrete momenta, we use the oscillator languages:

$$a = \frac{(P - i\omega X)}{\sqrt{2\omega}}, \quad a^\dagger = \frac{(P + i\omega X)}{\sqrt{2\omega}}$$  \hspace{1cm} (25)

to rewrite $X$ and $P$ in terms of the creation operator $a$ and annihilation operator $a^\dagger$. The Heisenberg group are the the same, but it is written in terms of $a$ and $a^\dagger$:

$$U[q_1(\cdot), q_2(\cdot)] = e^{i\int a \cdot X - \frac{q_1(\vec{k})}{\sqrt{2\omega}} - \frac{q_2(\vec{k})}{\sqrt{2\omega}} - \frac{q_1(\vec{p})}{\sqrt{2\omega}} + \frac{q_2(\vec{p})}{\sqrt{2\omega}}} e^{\frac{\gamma (q_1)^2}{4\omega} + \frac{\gamma (q_2)^2}{4\omega}}.$$  \hspace{1cm} (26)

The square bracket notation is just to stress that $U$ is the functional of $q_1(\vec{k})$ and $q_2(\vec{k})$. This Heisenberg group in quantum field theory is totally generalized from the below:

$$U(q_1, q_2) = e^{iq_1X + iq_2P} = e^{a \left( i\frac{q_1}{\sqrt{2\omega}} - \frac{q_1(\vec{k})}{\sqrt{2\omega}} \right) \left( i\frac{q_2}{\sqrt{2\omega}} + \frac{q_2(\vec{p})}{\sqrt{2\omega}} \right)} e^{\frac{\gamma (q_1)^2}{4\omega} + \frac{\gamma (q_2)^2}{4\omega}}.$$  \hspace{1cm} (27)

The non-interacting scalar field theory in a box also has the same Hamiltonian form as in the harmonic oscillator

$$H_{NS} = \frac{1}{V} \sum_{\vec{k}} \frac{1}{2} \tilde{a}^\dagger(\vec{k}) \tilde{a}(\vec{k}),$$  \hspace{1cm} (28)

where $V$ is the volume of the box. The $\tilde{a}^\dagger(\vec{k})$ and $\tilde{a}(\vec{k})$ are the standard creation and annihilation operators in the box, and then they satisfy the commutation relation

$$[\tilde{a}(\vec{k}_1), \tilde{a}^\dagger(\vec{k}_2)] = 2V\omega_{\vec{k}_1} \delta_{\vec{k}_1 \vec{k}_2}, \quad [\tilde{a}(\vec{k}_1), \tilde{a}(\vec{k}_2)] = 0, \quad [\tilde{a}^\dagger(\vec{k}_1), \tilde{a}^\dagger(\vec{k}_2)] = 0, \hspace{1cm} (29)$$

where

$$\omega_{\vec{k}_1}^2 \equiv |\vec{k}_1|^2 + m^2 \hspace{1cm} (30)$$

with the mass of the scalar field $m$. Performing the field redefinition

$$\tilde{a}(\vec{k}) \equiv \sqrt{2V\omega(\vec{k})} a(\vec{k})$$  \hspace{1cm} (31)

can exactly see the assembly of harmonic oscillators. Hence the non-interacting scalar field theory in a box must have the same relation between the Heisenberg average of correlation functions and SFFs as in bosonic quantum mechanics.
3 Late-Time Study

Since SFFs [13] and OTOTs [2] are hard to compute, the connection seems to be unuseful practically. Nevertheless, this is not true exactly. What we are interested in is the late-time physics. A late-time limit is also a classical limit, which is useful for the probe of classical chaos. Therefore, the coherent state is one approach. We demonstrate that the two-particles coherent state can simplify the computation of the Heisenberg average of OTOCs [37] and should be easily extended to other particle numbers. The other computable approach is the large-$N$ study. We use the large-$N$ bosonic quantum mechanics to present. The large-$N$ theory is the harmonic oscillators with a modified frequency. Since the large-$N$ quantum chromodynamics (QCD) and various large-$N$ quantum field theory can approach to the non-interacting quantum field theory, our result can be applied to large-$N$ quantum field theory also [37]. Finally, we compare the exact solution of the two-points SFF to the numerical solution. For $N = 3$ numerical solution, we already obtain a qualitative comparison to the large-$N$.

3.1 Coherent State

We first demonstrate why the Heisenberg averaging can simplify the computation of coherent state from the example, two-particles coherent state:

$$a_1|\alpha_1\alpha_2\rangle = |\alpha_1\alpha_2\rangle, \quad a_2|\alpha_1\alpha_2\rangle = |\alpha_2\alpha_1\rangle,$$

$$|\alpha_1\alpha_2\rangle = e^{-\frac{\alpha_1^2 + \alpha_2^2}{2}} e^{\alpha_1 a_1^\dagger + \alpha_2 a_2^\dagger} |0,0\rangle.$$  (32)

The completeness relation of this coherent state is

$$\int \frac{d^2\alpha_1}{\pi} \int \frac{d^2\alpha_2}{\pi} |\alpha_1\alpha_2\rangle \langle \alpha_1\alpha_2| = 1.$$  (33)

Because we have two-particles (two canonical pairs), we have four variables ($q_1$, $q_2$, $r_1$, $r_2$) in the Heisenberg group

$$U(q_1,q_2,r_1,r_2) = e^{iq_1X_1+iq_2P_1+ir_1X_2+ir_2P_2}.$$  (34)

We can find that the computation of the regularized two-points OTOC

$$C_2(t) = \langle \alpha_1\alpha_2|U(q_1,q_2,r_1,r_2)e^{-\frac{\beta}{2} - iHt}U^\dagger(q_1,q_2,r_1,r_2)e^{-\frac{\beta}{2} + iHt}|\alpha_1\alpha_2\rangle.$$  (35)

always meets the matrix element $\langle \alpha_1\alpha_2|U(q_1,q_2,r_1,r_2)|\gamma_1\gamma_2\rangle$. Because the matrix elements related to the coherent state parameters are just Gaussian, and the exponent of
Heisenberg group only has the linear term on \( a \) and \( a^\dagger \), this implies that the integration over the coherent variables are simplified, which entirely relies on the property of the Heisenberg group. If we choose other operators, the Gaussian form can be broken. Therefore, we think that the Heisenberg averaging is a nice simplification for the coherent state. This coherent study can be applied to the observable and exactly solvable model, the two-photon non-degenerate JC model with the rotating wave approximation [42]. Because this model is solvable, we can carry out all integration exactly. In general, we need to do the saddle-point evaluation on the integration at the late time. Here we only discussed the two-particles coherent state, but it is easy to do the similar generalization to other particle numbers.

### 3.2 Large-\( N \) Quantum Mechanics

Now we discuss the second approach for the late-time physics. The large-\( N \) limit gives the factorization to simplify a study [43]. We demonstrate this approach by the large-\( N \) bosonic quantum mechanics

\[
H_{\text{QMN}} = \sum_{j=1}^{N} \frac{P_j P_j}{2} + \mu^2 \frac{X_j X_j}{2} + g \frac{(X_j X_j)^2}{4},
\]

(36)

where \( j = 1, 2, \cdots, N \), and \( g \) is the coupling constant. We first introduce the auxiliary field \( \sigma \), and consider the large-\( N \) approximation (\( \sigma \) is just a constant) to obtain

\[
H_{\text{QNM}} = \sum_{j=1}^{N} \frac{P_j P_j}{2} + \mu^2 \frac{X_j X_j}{2} + \lambda \sigma \frac{X_j X_j}{2},
\]

(37)

in which the ’t Hooft coupling constant \( \lambda \equiv gN \) is fixed when we scale \( N \to \infty \). Because the \( \sigma \) is just a constant under the large-\( N \) limit, this theory is just harmonic oscillators with a modified frequency. We determine the auxiliary field

\[
\sigma = \frac{\sum_{j=1}^{N} (X_j X_j)}{N}
\]

(38)

from the two-point function or the large-\( N \) Schwinger-Dyson equation

\[
\left( \frac{d^2}{dt^2} + \mu^2 + \lambda \sigma \right) \sum_{j=1}^{N} \langle X_j(t) X_j(t') \rangle = -iN \delta(t - t').
\]

(39)

Therefore, we obtain the solution

\[
\sigma = \frac{1}{2\sqrt{\mu^2 + \lambda \sigma}},
\]

(40)
by taking $t - t' = \epsilon$ and choosing $\epsilon \to 0^+$ in the below:

$$
\langle X^i(t)X^j(t') \rangle = \int \frac{d\omega}{2\pi} \frac{iN}{\omega^2 - \mu^2 - \lambda \sigma + i\epsilon} e^{i\omega(t-t')}
$$

$$
= \frac{N\theta(t-t')}{2\sqrt{\mu^2 + \lambda \sigma}} e^{-i\sqrt{\mu^2 + \lambda \sigma}(t-t')} + \frac{N\theta(t'-t)}{2\sqrt{\mu^2 + \lambda \sigma}} e^{i\sqrt{\mu^2 + \lambda \sigma}(t-t')}. \quad (41)
$$

The two-points SFF

$$
g_2(\beta, t) \equiv \left| \sum_{n_1=0}^{\infty} D(n_1) e^{(-\beta+i\omega)E_{n_1}} \right|^2 \quad (42)
$$
can be computed exactly [37]:

$$
\sum_{n=0}^{\infty} D(n) e^{(-\beta+i\omega)E_n} = \sum_{n=0}^{\infty} \frac{(n + N - 1)!}{n!(N-1)!} e^{(-\beta+i\omega)\left(\frac{\omega}{2} + n\omega\right)}
$$

$$
= e^{(-\beta+i\omega)\frac{\omega}{2}} \sum_{n=0}^{\infty} \frac{(n + N - 1)!}{n!(N-1)!} e^{n(-\beta+i\omega)\omega}
$$

$$
= e^{(-\beta+i\omega)\frac{\omega}{2}} (1 - e^{(-\beta+i\omega)\omega})^{-N}, \quad (43)
$$

where

$$
\omega \equiv \sqrt{\mu^2 + \lambda \sigma}, \quad E_n \equiv \frac{\omega N}{2} + n\omega,
$$

$$
\sum_{n=0}^{\infty} \frac{(n + N - 1)!}{n!(N-1)!} x^n = (1 - x)^{-N}, \quad |x| < 1, \quad (44)
$$

$$
g_2(\beta, t) \equiv \left| \sum_{n_1=0}^{\infty} D(n_1) e^{(-\beta+i\omega)E_{n_1}} \right|^2 \left( \frac{1 + e^{-2\omega\beta} - 2e^{-\omega\beta}}{1 + e^{-2\omega\beta} - 2\cos(\omega t)e^{-\omega\beta}} \right)^N. \quad (45)
$$

for $\beta \neq 0$. When we take $\beta = 0$, the two-points SFF is divergent. Therefore, we are careful about the ordering of summation. We choose the same ordering of the summation as in a numerical study that we will work. Therefore, this should give the rigorously consistent result even for the extremely-low $\beta$ for the comparison between the exact solution and the numerical solution. Our numerical result for $N = 1$, $N = 2$, and $N = 3$ with a fixed inverse temperature $\beta = 1$ and ’t Hooft coupling constant $\lambda=2$ are in Fig. 1. We use the naive discretization to treat the derivative term or the momenta

$$
P^2_j \psi_j \equiv -\frac{\psi_{j+1} - 2\psi_j + \psi_{j-1}}{a^2}, \quad (46)
$$
where $P_j$ is the lattice momentum, $\psi_j$ is the lattice eigenfunction, and $a$ is the lattice spacing, in the Hamiltonian. The lattice index is labeled by $j = 1, 2, \cdots, n$, where $n$ is the number of lattice points. Then we do the exact diagonalization to obtain the eigenvalues of the Hamiltonian. If the lattice size

$$2L = n \cdot a$$

(47)

is not large enough, the high-energy modes will not provide a bound state in the numerical study. Therefore, we always choose some numbers in the low-lying eigenenergy modes for obtaining the physically correct result. Here our numerical study always preserves the periodic boundary condition:

$$-L \leq X_j < L, \quad X_1 = -L, \quad X_{j+1} \equiv X_j + a, \quad X_0 \equiv X_n, \quad X_{n+1} \equiv X_1.$$  

(48)

We use the subscript in $X$ to denote the lattice index without confusing to the superscript in $X$, which denotes the number of the canonical positions.

The comparison justifies that increasing $N$ in the numerical solution really helps the probe of late time physics. It is also surprising that the $N = 3$ numerical result is enough to capture the quantitative large-$N$ result. This sheds the light in the hope of the experimental realization on the large-$N$ physics [28, 30].

Figure 1: We fix the inverse temperature $\beta = 1$ while choosing the ’t Hooft coupling constant $\lambda = 2$. The lattice size is 8 in $N=1$, and the lattice size is 4 in $N=2, 3$. The number of lattice points is 128 in $N=1$, and the number of lattice points is 32 in $N=2, 3$. We compute the two-points SFF $g_2(t)$ from 16 low-lying eigenenergy modes for $N=1, 2$, and 3 in the left, middle, and right figures respectively. The numerical solution in $N=3$ matches the large-$N$ perturbation quantitatively.

4 Discussion and Conclusion

We used the Heisenberg averaging to connect the regularized two-points OTOC and the two-points SFF in any bosonic quantum mechanics [37]. The connection used the average of the phase space, not the full Hilbert space. This avoids the naive intuition,
averaging infinite-dimensional variables. This is an exact realization for connecting the early-time chaos to the late-time chaos in an infinite-dimensional Hilbert space. For getting the generalization to the higher-points SFFs, we also found the corresponding higher-points correlation functions with the Heisenberg averaging. The higher-points correlation functions possibly do not have the same physical interpretation as in the OTOCs, but it is still useful to study the decay for the motivation of information loss [24]. Although the bosonic quantum mechanics has the infinite-dimensional Hilbert space, it is still countable. A generic quantum field theory does not have a countable Hilbert space. Therefore, we generalized bosonic quantum mechanics to the non-interacting scalar field for getting closer to quantum field theory [37] from a direct approach.

The OTOCs [2] and SFFs [13] are hard to compute both. Therefore, the connection is not practical for the reason. For the motivation of quantum chaos, we are more interested in the late-time limit. We can use the coherent state and the large-$N$ technique to show the usefulness at the late-time limit. In the coherent state, the Heisenberg averaging simplifies the computation for all coherent states [37]. In general, the coherent state cannot provide an exact solution, but one can use the saddle-point evaluation to do the perturbation. The large-$N$ theory can have the simplification from the factorization [43], and the theory is just the non-interacting theory. Therefore, our result can be obtained from the large-$N$ theory. We demonstrated the large-$N$ study from the large-$N$ bosonic quantum mechanics [37]. This gives the harmonic oscillators with a modified frequency [37]. Therefore, the exact solution in the SFFs can be obtained [37]. We compared the exact solution of the two-points SFF to the numerical solution. This explicitly shows that the large-$N$ technique is really useful for the probe of late-time physics, not the early-time physics. We also found that $N = 3$ result already gives the quantitative result to the large-$N$ study. The experimental realization is impossible for the infinite oscillators, but it is possible for three oscillators. Therefore, the numerical study also confirmed the hope of an experimental realization [28, 30].

The connection between the correlation functions and SFFs is primarily based on the Heisenberg group. The elements of the Heisenberg group can also generate a coherent state. When we computed the coherent states, we also found useful simplification. The use of the coherent state can be seen as a classical limit. We know that the sensitivity on initial conditions is also hidden in the irregular dynamics and instability in one-interval classical chaos. Therefore, the connection through the Heisenberg averaging should not
be just a coincidence, and this possibly has a more fundamental reason behind. This direction should be understood more from the practical computation of quantum chaos using a coherent state. Because the connection purely relies on the Heisenberg group, we should study other groups to know the differences between different groups. This should be helpful to know why using the Heisenberg group is quite special.

We provided a coherent state and the large-$N$ techniques for showing that the connection can be computed practically. The examples in this paper are all solvable and integrable models. Therefore, we did not use practical computation to study quantum chaos. Nevertheless, the demonstrated approaches should be useful for getting the plateau time scale in a non-integrable model from the saddle-point evaluation (coherent state) or large-$N$ perturbation. Because the plateau regime is related to the thermalization, we can study how thermal in a non-integrable model. The speed of the decay of correlation functions is closely related to information loss issue [24]. Now we can use the coherent state and the large-$N$ technique to compute SFFs for understanding the information loss issue from quantum chaos perspective, and we know that SFFs are also related to the correlation functions through the Heisenberg averaging. Therefore, the practical computation of SFFs or the correlation functions should be interesting based on the connection.

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