Approximation by \((p, q)\)-Lorentz polynomials on a compact disk

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Abstract

In this paper, we introduce a new analogue of Lorentz polynomials based on \((p, q)\)-integers and we call it as \((p, q)\)-Lorentz polynomials. We obtain quantitative estimate in the Voronovskaja’s type theorem and exact orders in simultaneous approximation by the complex \((p, q)\)-Lorentz polynomials of degree \(n \in \mathbb{N}\), where \(q > p > 1\) attached to analytic functions in compact disks of the complex plane.

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1 Introduction and preliminaries

In 1986, G. G. Lorentz \[7\], introduced the following sequence of operators defined for any analytic function \(f\) in a domain containing the origin

\[
L_n(f; z) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{z}{n} \right)^k f^{(k)}(0), \quad n \in \mathbb{N}.
\]

(1.1)

In last two decades, the applications of \(q\)-calculus emerged as a new area in the field of approximation theory.

In \[4\], Gal introduced and studied the \(q\)-analogue of the Lorentz operators for \(q > 1\), for any analytic function \(f\) in a domain containing the origin as follows:

\[
L_{n,q}(f; z) = \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \binom{n}{k} \left( \frac{z}{[n]_q} \right)^k D_q^{(k)}(f)(0), \quad n \in \mathbb{N}, \quad z \in \mathbb{C}
\]

(1.2)

Details on the \(q\)-calculus can be found in \[1, 15\]

Several authors have introduced and studied the approximation properties for different operators in compact disk. For instance, in \[8, 9, 10\] Mahmudov studied \(q\)-Stancu polynomials, \(q\)-Szasz Mirakjan operators and generalised Kantorovich operators; In \[2\] Gal et al studied \(q\)-szasz-Kantorovich operators.

Recently, Mursaleen et al applied \((p, q)\)-calculus in approximation theory and introduced first \((p, q)\)-analogue of Bernstein operators \[11\]. Similarly they introduced and studied approximation properties for \((p, q)\)-Bernstein-Stancu operators \[12\] and \((p, q)\)-Bernstein-Kantorovich operators \[13\].
Let us recall certain notations of \((p, q)\)-calculus.

The \((p, q)\) integer \([n]_{p,q}\) is defined by

\[
[n]_{p,q} := \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots, \ p > q \geq 1.
\]

The \((p, q)\)-binomial expansion is given as

\[
(ax + by)^n_{p,q} := \sum_{k=0}^{n} \binom{n}{k}_{p,q} a^{n-k} b^k x^{n-k} y^k
\]

and the \((p, q)\)-binomial coefficients are defined by

\[
\binom{n}{k}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.
\]

The \((p, q)\)-derivative of the function \(f\) is defined as

\[
D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0,
\]

and \((D_{p,q})f(0) = f'(0)\), provided that \(f\) is differentiable at 0. It can be easily seen that \(D_{p,q}x^n = [n]_{p,q}x^{n-1}\).

Details on \((p, q)\)-calculus can be found in [5, 6, 14, 16, 17].

Our results generalize the results of Gal [4].

### 2 Construction of Operators

Now, with the help of \((p, q)\)-calculus and using above formula, we present \((p, q)\)-analogue of Lorentz operators \((1.1)\) as follows:

\[
L_{n,p,q}(f; z) = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_{p,q} \left(\frac{z}{[n]_{p,q}}\right)^k D_{p,q}^{(k)}(f)(0), \quad n \in \mathbb{N}, \ z \in \mathbb{C}
\]  \(2.1\)

Note that for \(p = 1\), \((p, q)\)-Lorentz operators given by \((2.1)\) turn out to be \(q\)-analogue of Lorentz operators \((1.2)\).

Firstly, we obtain an upper approximation estimate.

**Theorem 2.1** Let \(R > q > p > 1\) and \(D_R = \{z \in \mathbb{C} : |z| < R\}\). Suppose that \(f : D_R \to \mathbb{C}\) is analytic in \(D_R\), i.e., \(f(z) = \sum_{k=0}^{\infty} c_k z^k\), for all \(z \in D_R\).
(i) Let $1 \leq r < \frac{p}{q} < \frac{q}{q}$ be arbitrary fixed. For all $|z| \leq r$ and $n \in \mathbb{N}$. Then, we have the upper estimate as

$$|L_{n,p,q}(f)(z) - f(z)| \leq \frac{p^n}{|n|_{p,q}} M_{r_1,p,q}(f),$$

where $M_{r_1,p,q}(f) = \frac{p(q-p+1)}{(q-p)^2} \sum_{k=0}^{\infty} |c_k|(k+1)r_1^k < \infty$.

(ii) Let $1 \leq r < r^* < \frac{p}{q} < \frac{q}{q}$ be arbitrary fixed. For the simultaneous approximation by complex Lorentz polynomials, for all $|z| \leq r$, $m, n \in \mathbb{N}$, we have

$$|L_{n,p,q}^{(m)}(f)(z) - f^{(m)}(z)| \leq \frac{p^n}{|n|_{p,q}} M_{r_1,p,q}(f)\frac{m! r^*}{(r^*-r)^{m+1}},$$

where $M_{r_1,p,q}(f)$ is given as at the above point (i).

**Proof.** (i) For $e_j(z) = z^j$, it is to see that $L_{n,p,q}(e_0)(z) = 1$, $L_{n,p,q}(e_1)(z) = z$. Then we have

$$L_{n,p,q}(e_j)(z) = q^{j(j-1)/2} n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} [j]_{p,q}! \frac{z^j}{|n|_{p,q}^j}, \quad 2 \leq j \leq n, \text{ for all } j, n \in \mathbb{N}$$

by some simple calculation, we get

$$L_{n,p,q}(e_j)(z) = z^j \left(1 - p^{n-1} \frac{[1]_{p,q}}{|n|_{p,q}} \right) \left(1 - p^{n-2} \frac{[2]_{p,q}}{|n|_{p,q}} \right) \ldots \left(1 - p^{n-(j-1)} \frac{[j-1]_{p,q}}{|n|_{p,q}} \right).$$

It is easy to see that for $j \geq n+1$, we get $L_{n,p,q}(e_j)(z) = 0$. Now it can be easily seen that

$$L_{n,p,q}(f)(z) = \sum_{j=0}^{\infty} c_j L_{n,p,q}(e_j)(z) \text{ for all } |z| \leq r.$$

Hence

$$\left|L_{n,p,q}(f)(z) - f(z)\right| \leq \sum_{j=0}^{n} |c_j||L_{n,p,q}(e_j)(z) - e_j(z)| + \sum_{j=n+1}^{\infty} |c_j||L_{n,p,q}(e_j)(z) - e_j(z)|$$

$$\leq \sum_{j=2}^{n} |c_j| r^j \frac{1}{p^m} \left|\left(1 - p^{n-1} \frac{[1]_{p,q}}{|n|_{p,q}} \right) \left(1 - p^{n-2} \frac{[2]_{p,q}}{|n|_{p,q}} \right) \ldots \left(1 - p^{n-(j-1)} \frac{[j-1]_{p,q}}{|n|_{p,q}} \right) - 1\right|$$

$$+ \sum_{j=n+1}^{\infty} |c_j|r^j, \quad \text{for all } |z| \leq r.$$

On the other hand, the analyticity of $f$ implies $c_j = \frac{f^{(j)}(0)}{j!}$, and by the Cauchy’s estimates of the coefficients $c_j$ in the disk $|z| \leq r_1$, we have $|c_j| \leq \frac{K_{r_1}}{r_1^{j+1}}$, for all $j \geq 0$, where

$$K_{r_1} = \max\{|f(x)| : |x| \leq r_1\} \leq \sum_{j=2}^{n} |c_j|r^j \leq \sum_{j=2}^{n} |c_j|(j+1)r_1^j := K_{r_1}(f) < \infty.$$
Therefore
\[\sum_{j=n+1}^{\infty} |c_j|r^j \leq R_{r_1}(f) \left[ \frac{r}{r_1} \right]^{n+1} \sum_{j=0}^{\infty} \left( \frac{r}{r_1} \right)^j = R_{r_1}(f) \left[ \frac{r}{r_1} \right]^{n+1} \frac{r_1}{r_1 - r},\]
and finally we get
\[|L_{n,p,q}(f)(z) - f(z)| \leq \frac{p^{n+1}(q - p + 1)}{|n|_{p,q} (q - p)^2} R_{r_1}(f)\]
for all \(n \in \mathbb{N}\) and \(|z| \leq r\).

(ii) Let \(\gamma\) be the circle of radius \(r^* > r\) and center 0, since for any \(|z| \leq r\) and \(v \in \gamma\), we have \(|v - z| \geq r^* - r\). By Cauchy’s formula it follows that for all \(n \in \mathbb{N}\)
\[|L_{n,p,q}^{(m)}(f)(z) - f^{(m)}(z)| = \frac{m!}{2\pi} \left| \int_{\gamma} \frac{L_{n,p,q}(f)(v) - f(v)}{(v-z)^{m+1}} dv \right| \leq \frac{p^{n+1}}{|n|_{p,q}} M_{r_1,p,q}(f) \left( \frac{m!}{2\pi \pi} \right) \frac{2\pi r^*}{(r^* - r)^{m+1}} = \frac{p^{n+1}}{|n|_{p,q}} M_{r_1,p,q}(f) \left( \frac{m! \pi r^*}{(r^* - r)^{m+1}} \right).

We have the following quantitative Voronovskaja-type results.

**Theorem 2.2** For \(R > q^4 > p^4 > 1\), let \(f : D_R \to \mathbb{C}\) be analytic in \(D_R\), i.e., \(f(z) = \sum_{k=0}^{\infty} c_k z^k\), for all \(z \in D_R\) and let \(1 < r < \frac{p_1 r^4}{q^4} < \frac{p^2 R}{q^2}\) be arbitrary fixed. Then for all \(n \in \mathbb{N}\), \(|z| \leq r\), we have
\[\left| L_{n,p,q}(f)(z) - f(z) + \frac{S_{p,q}(f)(z)}{|n|_{p,q}} \right| \leq \frac{p^{2n}}{|n|_{p,q}^2} Q_{r_1,p,q}(f),\]
where
\[S_{p,q}(f)(z) = \sum_{k=2}^{\infty} p_n^{n-(k-1)} c_k \frac{[k]_{p,q} - [k]_{q,p}}{q - 1} z^k = \sum_{k=2}^{\infty} p_n^{n-(k-1)} c_k ([1]_{p,q} + \ldots + [k - 1]_{p,q}) \]
and \(Q_{r_1,p,q} = \frac{p_{q+1} q^4}{(p-1)(q-p)} \sum |c_k|(k + 1)(k + 2)^2 \left( \frac{2}{r_1} \right)^k < \infty\).

**Proof.** We have
\[\left| L_{n,p,q}(f)(z) - f(z) + \frac{S_{p,q}(f)(z)}{|n|_{p,q}} \right|\]
\[
\left| \sum_{k=0}^{\infty} c_k \left[ L_{n,p,q}(e_k)(z) - e_k(z) + p^{n-(k-1)} \frac{[k]p - [k]q}{p-1} e_k(z) \right] \right| \\
\leq \sum_{k=0}^{n} c_k \left[ L_{n,p,q}(e_k)(z) - e_k(z) + p^{n-(k-1)} \frac{[k]p - [k]q}{p-1} e_k(z) \right] \\
+ \left| \sum_{k=n+1}^{\infty} p^{n-(k-1)} c_k z^k \left( \frac{[k]p - [k]q}{p-1} - 1 \right) \right| \\
\leq \sum_{k=0}^{n} c_k \left[ L_{n,p,q}(e_k)(z) - e_k(z) + p^{n-(k-1)} \frac{[k]p - [k]q}{p-1} e_k(z) \right] \\
+ \sum_{k=n+1}^{\infty} |c_k| r^k \left( p^{n-(k-1)} \frac{[k]p - [k]q}{p-1} - 1 \right),
\]

for all \(|z| \leq r\) and \(n \in \mathbb{N}\).

In what follows, firstly we will prove by mathematical induction with respect to \(k\) that

\[
0 \leq E_{n,k,p,q}(z) \leq \frac{p^{2n} (k+1)(k-2)^2}{[n]_{p,q}^2} \frac{(qr)}{p}^k,
\]

(2.2)

for all \(2 \leq k \leq n\) (here \(n \in \mathbb{N}\) is arbitrary fixed) and \(|z| \leq r\), where

\[
E_{n,k,p,q}(z) = L_{n,p,q}(e_k)(z) - e_k(z) + p^{n-(k-1)} \frac{[k]p - [k]q}{p-1} e_k(z)
\]

\[
= L_{n,p,q}(e_k)(z) - e_k(z) + p^{n-(k-1)} ([1]_{p,q} + ... + [k-1]_{p,q}) e_k(z).
\]

By mathematical induction, we easily

\[
\frac{[k]p - [k]q}{p-1} = ([1]_{p,q} + ... + [k-1]_{p,q}).
\]

On the other hand, by the formula for \(L_{n,p,q}(e_k)\) in the proof of Theorem 2.1 (i), simple calculation leads to \(E_{n,2,p,q}(z) = 0\), for all \(n \in \mathbb{N}\) and to the recurrence relation

\[
E_{n,k,p,q}(z) = -\frac{z^2}{[n]_{p,q}} p^{n-(k-1)} D_{p,q}[L_{n,p,q}(e_{k-1})(z)-e_{k-1}(z)] + p^{-1} \frac{z}{p} E_{n,k-1,p,q}(z), \quad |z| \leq r.
\]

Now, for \(|z| \leq r\) and \(3 \leq k \leq n\) and applying the mean value theorem in complex analysis, with notation \(|f| = \max\{|f(z)| : |z| \leq r\}\), we get

\[
|E_{n,k,p,q}(z)| = \frac{r^2}{[n]_{p,q}} p^{n-(k-1)} \| (L_{n,p,q}(e_{k-1})(z) - e_{k-1}(z))' \| \frac{p}{r}.
\]
\[ |E_{n,k,p,q}(z)| \leq \left( \frac{p^n}{[n]_{p,q}}(k+1) \right) \left( \frac{p^n}{[n]_{p,q}}(k-2)[k-2]_{p,q} r_1^k + r_1 |E_{n,k-1,p,q}(z)| \right) \]

\[ \leq \frac{p^{2n}}{[n]_{p,q}^2}(k+1)(k-2)[k-2]_{p,q} r_1^k + r_1 |E_{n,k-1,p,q}(z)| \]

Now on taking \( k = 1, 2, 3, \ldots \), step by step, we easily obtain the estimate

\[ |E_{n,k,p,q}(z)| \leq \frac{p^{2n}}{[n]_{p,q}^2} \sum_{j=3}^{k} (j-1)(j-2)[j-2]_{p,q} \]

\[ \leq \frac{p^{2n}}{[n]_{p,q}^2} \left( \frac{k+1}{q-p} \right)^2 \left( \frac{qr_1}{p} \right)^k \]

Now we calculate

\[ \left| \sum_{k=0}^{n} |c_k| \left[ L_{n,p,q}(e_k)(z) - e_k(z) \right] \right| + q^{n-(k-1)} \frac{[k]_{p,q} - [k]_{q}}{p-1} e_k(z) \]

\[ \leq \sum_{k=0}^{n} |c_k| |E_{n,k,p,q}(z)| \]

\[ \leq \frac{p^{2n}}{[n]_{p,q}^2} \frac{1}{q-p} \sum_{k=0}^{n} |c_k| (k+1)(k-2)^2 \left( \frac{qr_1}{p} \right)^k \]

\[ \leq \frac{p^{2n}}{[n]_{p,q}^2} \frac{1}{q-p} \sum_{k=0}^{n} |c_k| (k+1)(k+2)^2 \left( \frac{qr_1}{p} \right)^k \]

On the other hand, since \( \left( \frac{p^{n-(k-1)} [k]_{p,q} - [k]_{q}}{p-1} - 1 \right) \geq 0 \) for all \( k \geq n+1 \), similar to proof of Theorem 2.1 (i), we get

\[ \sum_{k=n+1}^{\infty} |c_k| r^k \left( p^{n-(k-1)} \frac{[k]_{p,q} - [k]_{q}}{p-1} - 1 \right) \leq \sum_{k=n+1}^{\infty} p^{n-(k-1)} |c_k| r^k \frac{[k]_{p,q}}{(p-1)[n]_{p,q}} \]

\[ \leq \sum_{k=n+1}^{\infty} p^{n-(k-1)} |c_k| \frac{1}{(p-1)[n]_{p,q}} \frac{kq^k}{p^k(q-p)} \]

\[ \leq \frac{R_{r_1}(f)p^{n+1}}{(p-1)[n]_{p,q}} \sum_{k=n+1}^{\infty} \frac{r^k}{r_1^k} q^k p^{-k} \]

\[ \leq \frac{R_{r_1}(f)p^{n+1}}{(p-1)[n]_{p,q}} \sum_{k=n+1}^{\infty} \left[ \left( \frac{r}{r_1} \right)^{1/3} \right]^k \left[ \left( \frac{r}{r_1} \right)^{1/3} \right]^2 q^k p^{-k} \]

\[ \leq \frac{R_{r_1}(f)p^{n+1}}{(p-1)[n]_{p,q}} \left( \frac{r}{r_1} \right)^{n+1} \sum_{k=0}^{\infty} \left[ \left( \frac{r}{r_1} \right)^{1/3} \right]^k \]
We have

Using the following inequality

Theorem 2.3

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The following result gives the lower approximation estimate

\[
\begin{align*}
\sum_{k=0}^{n} |c_k| (k+1)(k+2)^2 \left( \frac{q}{p_1} \right)^k,
\end{align*}
\]

(2.3)

where we used the inequalities, \([k]_{p,q} \leq \frac{kq}{p^n}, \frac{p^n}{q^n} \leq \frac{p^n}{(q-p)[n]_{p,q}}\) and \( \frac{r^1}{(1^2)_{p,q}} \leq \frac{p}{(q-p)} \).

Hence, by combining all above estimates, we have

\[
\left| L_{n,p,q}(f)(z) - f(z) + \frac{S_{p,q}(f)(z)}{[n]_{p,q}} \right| \leq \frac{(pq - q + p - 1)}{(p - 1) (q - p)^2} \frac{p^{2n}}{[n]_{p,q}^2} \sum_{k=0}^{n} |c_k| (k+1)(k+2)^2 \left( \frac{q}{p_1} \right)^k.
\]

The following result gives the lower approximation estimate

**Theorem 2.3** Let \( R > p^4/q^4 > 1 \), \( f : D_R \to \mathbb{C} \) be analytic in \( D_R \), i.e., \( f(z) = \sum_{k=0}^{\infty} c_k z^k \), for all \( z \in D_R \) and let \( 1 \leq r < \sqrt[4]{\frac{p^4}{R^4}} \) be arbitrary fixed. If \( f \) is not a polynomial of degree \( \leq 1 \), then for all \( n \in \mathbb{N} \) and \( |z| \leq r \), we have

\[
\| L_{n,p,q}(f) - f \|_r \geq \frac{p^n}{[n]_{p,q}} C_{r,r_1,p,q}(f),
\]

where the constant \( C_{r,r_1,p,q}(f) \) depends only on \( f \), \( r \) and \( r_1 \). Here \( \| f \|_r \) denotes \( \max_{|z| \leq r} \{|f(z)|\} \).

**Proof.** For \( S_{n,p,q}(f)(z) \) as defined in Theorem 2.3, all \( |z| \leq r \) and \( n \in \mathbb{N} \), we have

\[
L_{n,p,q}(f)(z) - f(z)
= \frac{p^n}{[n]_{p,q}} \left\{ -S_{p,q}(f)(z) + \frac{p^n}{[n]_{p,q}} \left[ [n]_{p,q}^2 \frac{p^{2n}}{p^n} \left( L_{n,p,q}(f)(z) - f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z) \right) \right] \right\}.
\]

Using the following inequality

\[
\| F + G \|_r \geq \| F \|_r - \| G \|_r \geq \| F \|_r - \| G \|_r.
\]

We have

\[
\| L_{n,p,q}(f) - f \|_r
\geq \frac{p^n}{[n]_{p,q}} \left\{ \| S_{p,q}(f)(z) \| - \frac{p^n}{[n]_{p,q}} \left[ [n]_{p,q}^2 \frac{p^{2n}}{p^n} \left\| L_{n,p,q}(f)(z) - f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z) \right\| \right] \right\}.
\]

Since by hypothesis \( f \) is not a polynomial of degree \( \leq 1 \) in \( D_R \), we get \( \| S_{p,q}(f) \|_r > 0 \).

Indeed, supposing the contrary it follows that \( S_{p,q}(f)(z) = 0 \) for all \( z \in \overline{D}_R = \{ z \in \mathbb{C} : |z| \leq r \} \).
A simple calculation yields \(S_{p,q}(f)(z) = z \frac{D_{p,q}(f)(z) - f'(z)}{p - q} \), \(S_{p,q}(f)(z) = 0\) implies that \(D_{p,q}(f)(z) = f'(z)\), for all \(z \in D_R \setminus \{0\}\). Taking into account the representation of \(f\) as \(f(z) = \sum_{k=0}^{\infty} c_k z^k\), the last inequality immediately leads to \(c_k = 0\), for all \(k \geq 2\), which means that \(f\) is linear in \(D_r\), a contradiction with hypothesis.

Now, by Theorem \(2.2\) we have
\[
\frac{[n]_p^2}{p^{2n}} \left\| L_{n,p,q}(f)(z) - f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z) \right\|_r \leq Q_{r_1,p,q}(f),
\]
where \(Q_{r_1,p,q}(f)\) is a positive constant depending only on \(f, r_1, p\) and \(q\). Since \(\frac{p^n}{[n]_{p,q}} \to 0\) as \(n \to \infty\), there exists an index \(n_o\) depending only on \(f, r, r_1, p\) and \(q\) such that for all \(n > n_o\), we have
\[
\|S_{p,q}(f)(z)\| - \frac{p^n}{[n]_{p,q}} \left[ \frac{[n]_p^2}{p^{2n}} \left\| L_{n,p,q}(f)(z) - f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z) \right\|_r \right] \geq \frac{1}{2} \|S_{p,q}(f)\|_r,
\]
which immediately implies that
\[
\|L_{n,p,q}(f) - f\|_r \geq \frac{p^n}{[n]_{p,q}} \frac{1}{2} \|S_{p,q}(f)(z)\|_r, \quad \text{for all } n > n_o.
\]

For \(n \in \{1, \ldots, n_o\}\), we have \(\|L_{n,p,q}(f) - f\|_r \geq \frac{p^n}{[n]_{p,q}} M_{r_1,n,p,q} M_{r,r_1,n,p,q} = \frac{[n]_{p,q}}{p^n} \|L_{n,p,q}(f) - f\|_r > 0\) (if \(\|L_{n,p,q}(f) - f\|_r\) would be equal to 0, this would imply that \(f\) is a linear function, a contradiction).

Therefore, finally we get \(\|L_{n,p,q}(f) - f\|_r \geq \frac{p^n}{[n]_{p,q}} C_{r,r_1,p,q}(f)\) for all \(n \in \mathbb{N}\), where
\[
C_{r,r_1,p,q}(f) = \min \left\{ M_{r,r_1,1,p,q}(f), \ldots, M_{r,r_1,n_0,p,q}(f), \frac{1}{2} \|S_{p,q}(f)\| \right\},
\]
which completes the proof.

Combining Theorem \(2.3\) and Theorem \(2.1\) (i), we immediately get the following result.

**Corollary 2.4** Let \(R > p^4/q^4 > 1\), \(f : D_R \to \mathbb{C}\) be analytic in \(D_R\), i.e., \(f(z) = \sum_{k=0}^{\infty} c_k z^k\), for all \(z \in D_R\) and let \(1 \leq r < \frac{q^4 R}{p^4} < \frac{q^4 R}{p^3}\) be arbitrary fixed. If \(f\) is not a polynomial of degree \(\leq 1\), then for all \(n \in \mathbb{N}\) and \(|z| \leq r\), we have
\[
\|L_{n,p,q}(f) - f\|_r \sim \frac{p^n}{[n]_{p,q}},
\]
where the constants in the equivalence depend only on \(f, r, r_1, p\) and \(g\) but are independent of \(n\).
3 Approximation results

Concerning the simultaneous approximation, we prove the following:

**Theorem 3.1** Let $R > p^4/q^4 > 1$, $f : D_R \to \mathbb{C}$ be analytic in $D_R$, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$ and let $1 \leq r < r^* < \frac{p^4 R}{q^4} < \frac{p^4 R}{q^4}$. Also let $m \in \mathbb{N}$. If $f$ is not a polynomial of degree $\leq \max \{1, m - 1\}$, then for all $n \in \mathbb{N}$, we have

$$\|L_{n,p,q}^{(m)}(f) - f^{(m)}\|_r \sim \frac{p^n}{[n]_{p,q}},$$

where the constants in the equivalence depend only on $f$, $r$, $r_1$, $m$, $p$ and $q$ but are independent of $n$.

**Proof.** We already have the upper estimate for $\|L_{n,p,q}^{(m)}(f) - f^{(m)}\|_r$, by Theorem 2.1 (ii), so it remains to find the lower estimate for $\|L_{n,p,q}^{(m)}(f) - f^{(m)}\|_r$,

Let us denote by $\Gamma$ the circle of the radius $r^*$ and center 0. We have that the inequality $|v - z| \geq r^* - r$ holds for all $|z| \leq r$ and $v \in \Gamma$. Cauchy’s formula is expressed by

$$|L_{n,p,q}^{(m)}(f)(z) - f^{(m)}(z)| = m! \left| \frac{1}{2\pi i} \int_{\gamma} \frac{L_{n,p,q}^{(m)}(f)(v) - f(v)}{(v - z)^{m+1}} dv \right|$$

(3.1)

Now, as in the proof of Theorem 2.1 (ii), for all $v \in \Gamma$ and $n \in \mathbb{N}$, we have

$$L_{n,p,q}^{(m)}(f)(z) - f(z) = \frac{p^n}{[n]_{p,q}} \left\{ -S_{p,q}(f)(z) + \frac{p^n}{[n]_{p,q}} \left[ [n]_{p,q}^2 \left( L_{n,p,q}^{(m)}(f)(z) - f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z) \right) \right] \right\}$$

(3.2)

By (3.1) and (3.2), we get

$$L_{n,p,q}^{(m)}(f) - f^{(m)}(f) = \frac{p^n}{[n]_{p,q}} \left\{ \frac{m!}{2\pi i} \int_{\Gamma} - \frac{S_{p,q}(f)(z)}{(v - z)^{m+1}} dv \right\}$$

$$+ \frac{p^n}{[n]_{p,q}} \frac{m!}{2\pi i} \int_{\Gamma} \left[ [n]_{p,q}^2 \left( L_{n,p,q}^{(m)}(f)(z) - f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z) \right) \right] \frac{p^n}{p^{2n}} dv$$

$$= \frac{p^n}{[n]_{p,q}} \left\{ \left[ -S_{p,q}(f)(z) \right]^{(m)} + \frac{p^n}{[n]_{p,q}} \frac{m!}{2\pi i} \int_{\Gamma} \left[ [n]_{p,q}^2 \left( L_{n,p,q}^{(m)}(f)(z) - f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z) \right) \right] \frac{p^n}{p^{2n}} dv \right\}.$$ 

Hence

$$\|L_{n,p,q}^{(m)}(f) - f^{(m)}\|_r \geq \frac{p^n}{[n]_{p,q}} \left\{ \| - \left[ S_{p,q}(f) \right]^{(m)} \|_r \right\} - \frac{p^n}{[n]_{p,q}} \left\{ \frac{m!}{2\pi i} \int_{\Gamma} \left[ [n]_{p,q}^2 \left( L_{n,p,q}^{(m)}(f)(z) - f(z) + \frac{p^n}{[n]_{p,q}} S_{p,q}(f)(z) \right) \right] \frac{p^n}{p^{2n}} dv \right\}.$$
Now by using Theorem 2.2 for all \( n \in \mathbb{N} \), we get
\[
\left\| \frac{[n]_{p,q}^2}{2\pi} \int_{\Gamma} \frac{L_n(p,q,f)(z) - f(z) + p^n S_{p,q}(f)(z)}{p^{2n}(v - z)^{m+1}} \, dv \right\|_r \leq \frac{m!}{2\pi} \frac{2\pi r^*[n]_{p,q}^2}{(r^* - r)^{m+1} p^{2n}} \left\| L_n(p,q,f) - f + p^n S_{p,q}(f) \right\|_{r^*},
\]
leq Q_{r,1,p,q}(f). \frac{m! r^*}{(r^* - r)^{m+1}}.

But by hypothesis on \( f \), we have \( \left\| - [S_{p,q}(f)]^{(m)} \right\|_{r^*} > 0 \). Indeed, supposing the contrary, it would follow that \( [S_{p,q}(f)]^{(m)}(z) = 0 \), for all \( |z| \leq r^* \), where by the statement of Theorem 2.2 we have
\[
S_{p,q}(f)(z) = \sum_{k=2}^{\infty} q^{-(k-1)} \frac{k [k]_{p,q} - [k]_{p,q}}{q - 1} z^k = \sum_{k=2}^{\infty} q^{-(k-1)} c_k ([1]_{p,q} + \ldots + [k - 1]_{p,q}) z^k.
\]
Firstly, supposing that \( m = 1 \), by \( S_{p,q}(f)(z) = \sum_{k=2}^{\infty} q^{-(k-1)} c_k k ([1]_{p,q} + \ldots + [k - 1]_{p,q}) z^{k-1} = 0 \), for all \( |z| \leq r^* \), would follow that \( c_k = 0 \), for all \( k \geq 2 \), that is, \( f \) would be a polynomial of degree \( 1 = \max\{1, m - 1\} \), a contradiction with the hypothesis.

Taking \( m = 2 \), we would get \( S_{p,q}''(f)(z) = \sum_{k=2}^{\infty} q^{-(k-1)} c_k (k - 1) ([1]_{p,q} + \ldots + [k - 1]_{p,q}) z^{k-2} = 0 \), for all \( |z| \leq r^* \), which immediately would imply that \( c_k = 0 \), for all \( k \geq 2 \), that is, \( f \) would be a polynomial of degree \( 1 = \max\{1, m - 1\} \), a contradiction with the hypothesis.

Now, taking \( m > 2 \), for all \( |z| \leq r^* \), we would get
\[
S_{p,q}^{(m)}(f)(z) = \sum_{k=m}^{\infty} q^{-(k-1)} c_k (k - 1) \ldots (k - m + 1) ([1]_{p,q} + \ldots + [k - 1]_{p,q}) z^{k-m} = 0,
\]
which would imply that \( c_k = 0 \), for all \( k \geq m \), that is, \( f \) would be a polynomial of degree \( m - 1 = \max\{1, m - 1\} \), a contradiction with the hypothesis.

Finally, we prove some approximation results for the iterates of \((p, q)\)-Lorentz operators.

For \( f \) analytic in \( D_R \) that is of the form \( f(z) = \sum_{k=0}^{\infty} c_k z^k \), for all \( z \in D_R \), let us define the iterates of complex Lorentz operators \( L_{n,p,q}(f)(z) \), by \( L_{n,p,q}^{(1)}(f)(z) = L_{n,p,q}(f)(z) \) and \( L_{n,p,q}^{(m)}(f)(z) = L_{n,p,q}[L_{n,p,q}^{(m-1)}(f)](z) \), for any \( m \in \mathbb{N} \), \( m \geq 2 \).

Since we have \( L_{n,p,q}^{(m)}(f)(z) = \sum_{k=0}^{\infty} c_k L_{n,p,q}(e_k)(z) \), by recurrence for all \( m \geq 1 \), we get that \( L_{n,p,q}^{(m)}(f)(z) = \sum_{k=0}^{\infty} c_k L_{n,p,q}(e_k)(z) \), where \( L_{n,p,q}^{(m)}(e_k)(z) = e_k \) if \( k = 0 \), \( L_{n,p,q}^{(m)}(e_k)(z) = z \) if \( k = 1 \), \( L_{n,p,q}^{(m)}(e_k)(z) = 0 \), if \( k \geq n + 1 \) and
\[
L_{n,p,q}^{(m)}(e_j)(z) = \left( 1 - p^{-1} \frac{[1]_{p,q}}{[n]_{p,q}} \right)^m \left( 1 - p^{-2} \frac{[2]_{p,q}}{[n]_{p,q}} \right)^m \ldots \left( 1 - p^{-(j-1)} \frac{[j-1]_{p,q}}{[n]_{p,q}} \right)^m z^k,
\]
for $2 \leq k \leq n$.

We present the following:

**Theorem 3.2** Let $R > p > q > 1$ and $1 \leq r < \frac{pq}{q} < \frac{pR}{q}$ be arbitrary fixed. Denoting $D_R = \{z \in \mathbb{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbb{C}$ is analytic in $D_R$, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$, we have the upper estimate

$$\|L_{n,p,q}^{(m)}(f) - f\|_r \leq \frac{mp^n q - p + 1}{(q - p)^2} \sum_{k=0}^{\infty} |c_k|(k + 1)r_1^k.$$ 

Moreover, if $\lim_{n \to \infty} \frac{mp^n}{[n]_{p,q}} = 0$, then

$$\lim_{n \to \infty} \|L_{n,p,q}^{(m)}(f) - f\|_r = 0.$$ 

**Proof.** For all $|z| \leq r$, we easily obtain

$$|f(z) - L_{n,p,q}^{(m)}(f)(z)| \leq \sum_{k=2}^{n} |c_k| r^k \left[ 1 - \left(1 - p^{n-1} \frac{[1]_{p,q}}{[n]_{p,q}} \right) \cdots \left(1 - p^{n-(j-1)} \frac{[j-1]_{p,q}}{[n]_{p,q}} \right) \right] + \sum_{k=n+1}^{\infty} |c_k| r^k.$$ 

Denoting $A_{k,n} = \left(1 - p^{n-1} \frac{[1]_{p,q}}{[n]_{p,q}} \right) \cdots \left(1 - p^{n-(j-1)} \frac{[j-1]_{p,q}}{[n]_{p,q}} \right)$, we get $1 - A_{k,n} = (1 - A_{k,n})(1 + A_{k,n} + A_{k,n}^2 + \cdots + A_{k,n}^{m-1}) \leq m(1 - A_{k,n})$ and therefore since $1 - A_{k,n} \leq p^{n-(k-1)} \frac{[k-1]_{p,q}}{[n]_{p,q}}$, for all $|z| \leq r$, we obtain

$$\sum_{k=2}^{n} |c_k| r^k \left[ 1 - \left(1 - p^{n-1} \frac{[1]_{p,q}}{[n]_{p,q}} \right) \cdots \left(1 - p^{n-(j-1)} \frac{[j-1]_{p,q}}{[n]_{p,q}} \right) \right] + \sum_{k=n+1}^{\infty} |c_k| r^k \leq m \sum_{k=2}^{\infty} |c_k| r^k (k - 1)[k - 1]_{p,q} r^k \leq \frac{mp^n}{[n]_{p,q}} \sum_{k=2}^{\infty} |c_k| r^k \frac{kq}{q - p} \leq \frac{mp^n}{[n]_{p,q}} \sum_{k=2}^{\infty} \frac{1}{q - p} \sum_{k=2}^{\infty} |c_k| (k + 1)(r_1)^k \leq \frac{mp^n}{[n]_{p,q}} \sum_{k=2}^{\infty} \frac{1}{q - p} |c_k| (k + 1)(r_1)^k.$$ 

On the other hand, following exactly the reasonings in the proof of the Theorem 2.1, we get the estimate

$$\sum_{k=n+1}^{\infty} |c_k| r^k \leq \frac{p^n}{[n]_{p,q}} \sum_{k=0}^{\infty} |c_k| (k + 1)(r_1)^k \leq \frac{mp^n}{[n]_{p,q}} \sum_{k=0}^{\infty} |c_k| (k + 1)(r_1)^k.$$ 

Collecting now all the estimates and taking into account that $\frac{1}{q-p} + \frac{1}{(q-p)^2} = \frac{q-p+1}{(q-p)^2}$, we arrive at the desired estimate.

Since $\lim_{n \to \infty} \frac{mp^n}{[n]_{p,q}} = 0$, it follows the conclusion that

$$\lim_{n \to \infty} \|L_{n,p,q}^{(m)}(f) - f\|_r = 0.$$
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