A non-iterative method for electrical property tomography based on a simple formula

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Abstract A non-iterative method of reconstruction is proposed from data of MRI system and of a harmonic electro-magnetic field at Larmor frequency. The method is based on the exact analytic formula for the contrast source function. A geometric method for acquisition of the full inductive field is discussed.

Key words: contrast source function, Helmholtz operator, transmit magnetic field, acquisition geometry

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1 Introduction

Electrical properties tomography (EPT) is a noninvasive reconstruction technique for retrieving electrical properties (conductivity and permittivity) of biological tissue from magnetic fields generated by radio frequency coils in a magnetic resonance imaging scanner. The electric properties of tissue are of great interest, since these properties can be used to aid the discrimination of cancerous tissue from benign tissue and characterize various kinds of pathological tissue of human body [4], [8]. The main benefits of EPT over other reconstruction modalities is that it uses the Larmor frequency fields of an MRI system, which can penetrate biological tissues and it does not make use of surface electrode mounting, current injection or additional hardware.

The method called now EPT was proposed by Haacke et al. [1]. The first application of this method was done by Wen [2] who used an approximation for the contrast source function $\gamma/\gamma_0$ (CSF). Song and Seo [6] reduced reconstruction of admittivity $\gamma$ to solution of an elliptic equation under assumption $\partial\gamma/\partial z = 0$ in terms of $B^+$ field. See also [9] for an explicit construction. Ammari et al. [7] studied the problem in terms of the quasilinear system of differential equations. In [10] a survey of local methods of reconstructions is given. C van den Berg et al. [11] developed the iterative method of determination of the CSF of an object making use of a global integral approach. Arduino et al. [12] applied the iterative conjugate gradient method (1000 steps). We describe here a non-iterative algorithm for retrieving CSF which is based on the exact global formula. This is the first algorithm of this kind so far.
2 Basic equations

We assume that the conductivity and permittivity are isotropic at the angular Larmor frequency $\omega$ and ignore spatial permeability variations $\mu = \mu_0$, since they are considered small for biological tissue. Let $\Omega$ be an object domain in the MRI scanner that is disjoint of the antenna (coil) generating the RF wave. The total time harmonic electromagnetic field in $\mathbb{R}^3$ can be written in the form $\{E \exp(i\omega t), H \exp(i\omega t)\}$ where

$$E = E + E^{sc}, \quad H = H + H^{sc},$$

$\{E = E^{inc}, H = H^{inc}\}$ is the incident and $\{E^{sc}, H^{sc}\}$ the scattered fields due to the presence of the object. Following to van der Berg et al \cite{8} we use the integral representations

$$E^{sc} = (k^2 + \nabla\nabla \cdot)A, \quad (1)$$

$$H^{sc} = \gamma_0 \nabla \times A \quad (2)$$

for the scattered fields where $\gamma_0 = i\omega\varepsilon_0$, $\varepsilon_0$ is the permittivity in vacuum and

$$A (p) = \int_\Omega G(p - q) \chi(q) E(q) dV(q), \quad q, p \in \mathbb{R}^3 \quad (3)$$

is the vector potential and

$$G(q) = -\frac{\exp(ik|q|)}{4\pi|q|}, \quad k = \frac{\omega}{c_0} \quad (4)$$

is the scalar Green’s function for the Helmholtz operator $\Delta + k^2$ in 3D background medium. The contrast source function is defined as

$$\chi(q) = \frac{\gamma}{\gamma_0} - 1$$

where admittivity $\gamma = \sigma + i\varepsilon$ is the per meter admittance. Conductivity $\sigma$ and permittivity $\varepsilon$ are profiles of the object, $\Omega$ is a domain that contains the support of $\chi$.

3 Evaluation of the contrast function

We write the electric and magnetic fields as first order differential forms

$$E = E_x dx + ..., \quad B = B_x dx + ...$$

Hodge star operator $*$ acts linearly on differential forms $\alpha, \beta$ on $\mathbb{R}^3$ by the rule (\cite{13}, p.15)

$$\alpha^* \wedge \beta = \alpha \wedge \beta^* = \langle \alpha, \beta \rangle dx \wedge dy \wedge dz,$$

$$(adx)^* = ady \wedge dz, \quad (ady \wedge dz)^* = adx; \quad (x \to y \to z \to x),$$

where $\alpha, \beta$ are arbitrary differential forms in $\mathbb{R}^3$ of equal degree and $\langle, \rangle$ means the natural scalar product of the forms. The dual differential is defined by $d^* = *d*$ and Laplace operator $\Delta = d^*d + dd^*$ commutes with $d = \nabla$ and $d^*$. 

2
Theorem 1. Let $\Omega$ be a domain in $\mathbb{R}^3$ disjoint of the RF antenna. If the fields $E$ and $B$ are known then the contrast source function can be found in $\Omega$ by

$$\chi = -\frac{(E - \ast dB) \wedge (\Delta + k^2) B^*}{(E - \ast dB) \wedge (dE + k^2 B^*)}$$  \hspace{1cm} (5)$$

if the denominator does not vanish.

Corollary 2. We have

$$\frac{\gamma}{\gamma_0} = \chi + 1 = \frac{\langle -\Delta B^* + dE, \ast dB - E \rangle}{\langle k^2 B^* + dE, \ast dB - E \rangle}.$$  \hspace{1cm} (6)$$

Note that $(dE, E) = 0$ since $dE = -i\omega c^{-1}\mu H^*$ and $H^* \wedge E = \langle H, E \rangle = 0$.

Remark. Equation

$$\gamma = \frac{1}{i\omega \mu_0} \frac{\langle \nabla^2 H, \nabla \times H \rangle}{\langle H, \nabla \times H \rangle}$$  \hspace{1cm} (7)$$

is mentioned by Seo [6] and attributed to Nachman et al [3].

Proof of Theorem 1.

Lemma 3. Let $\Omega$ be a compact set in $\mathbb{R}^n$ that can be contracted to a point in itself. For any $k \geq 1$, and arbitrary closed differential $k$-form $a$ on $\Omega$, there exists a $k-1$-form $b$ on $\Omega$ such that $db = a$ on $\Omega$.

For a proof see ”Converse of the Poincaré lemma” [13], p.29.

Note that equation (5) does not depend on $\Omega$. Therefore we may assume that $\Omega$ is a compact set with smooth boundary in $\mathbb{R}^3$ that fulfills the condition of Lemma 3.

Lemma 4. The system of equations for a 1-form $C$

$$dC = B^*, \quad d^* C = 0$$  \hspace{1cm} (8)$$

has a solution defined on $\Omega$. There exists a function $g$ on $\Omega$ such that

$$C + dg = A_0.$$  \hspace{1cm} (9)$$

Proof. Form $B^*$ has bounded coefficients and fulfills the Gauss law $dB^* = 0$ on $\Omega$. By Lemma 3 there exists 1-form $C_0$ on $\Omega$ that satisfies $dC_0 = B^*$. Consider Dirichlet problem

$$\Delta f = -d^* C_0$$

for a function $f$ on $\Omega$. To solve it we define the function $h$ on $\mathbb{R}^3$ that is equal to $-d^* C_0$ on $\Omega$ and $h = 0$ on the complement to $\Omega$. Set $f = G_0 \ast h$ where $G_0$ is the kernel [4] with $k = 0$. Differential form $C = C_0 + df$ fulfills [5] since

$$d^* C = d^* C_0 + d^* df = d^* C_0 + \Delta f = 0.$$  \hspace{1cm} (10)$$

1We may assume that coefficients of $a$ and of $b$ are distributions since $b$ will not appear in the final formula.
Equation (2) implies
\[ B^* = \mu (H^{sc})^* = \frac{i\omega}{c^2} dA = dA_0, \]
where \( A_0 = i\omega c^{-2} A \) and \( A \) is the vector potential (3). By (5) we have \( d (A_0 - C) = 0 \). Again, by Lemma 3 there exists a function \( g \) on \( \Omega \) such that \( A_0 - C = dg \) and (9) follows. \( \triangleright \)

By (3) and (1) we have
\[ A = G_\chi E = G_\chi (E^{sc} + E) = G_\chi \left( k^2 + dd^* \right) A + G_\chi E \]

\[ A_0 = i\omega c^{-2} A \]

Since \( \nabla \nabla \cdot = dd^* \) where \( G_\chi u = G \ast (\chi u) \). This yields
\[ G_\chi E = A - G_\chi E^{sc} = (I - G_\chi (k^2 + dd^*)) A_0. \]

By (9) we have
\[ G_\chi E = \left( I - G_\chi k^2 \right) C + (I - G_\chi (k^2 + \Delta)) dg \]

since \( \nabla \nabla \cdot = dd^* \) where \( G_\chi u = G \ast (\chi u) \). This yields
\[ G_\chi E = A - G_\chi E^{sc} = (I - G_\chi (k^2 + dd^*)) A_0. \]

Differential operators \( d \) and \( k^2 + \Delta \) commute, therefore
\[ d \left( \theta (E - \Delta C) \right) \equiv d\theta \wedge (E - \Delta C) + \theta d (E - \Delta C) = (k^2 + \Delta) dC. \]

Multiplying by form \( E - \Delta C \) we kill the term with \( d\theta : \)
\[ (E - \Delta C) \wedge \theta d (E - \Delta C) = (E - \Delta C) \wedge (k^2 + \Delta) dC. \]

By (10) \( dC = dA_0 = B^* \), hence
\[ \Delta C = d^* dC = d^* B^* = \ast dB, \]
\[ d\Delta C = dd^* dC = dd^* B^* = \Delta B^* \]

which yields
\[ \theta = \frac{(E - \Delta C) \wedge (k^2 + \Delta) dC}{(E - \Delta C) \wedge d (E - \Delta C)} = \frac{(E - \ast dB) \wedge (k^2 + \Delta) B^*}{(E - \ast dB) \wedge (dE - \Delta B^*)}. \]

Finally
\[ \chi = \frac{1}{\theta + 1} - 1 = \frac{(E - \ast dB) \wedge (k^2 + \Delta) B^*}{(E - \ast dB) \wedge (dE + k^2 B^*)} \]

and (5) follows. \( \triangleright \)
4 Acquisition geometries

Any acquisition geometry should be rich enough to guarantee reconstruction of field $B$ on $\Omega$. Let $x, y, z$ be an euclidean positively oriented coordinate system on the physical space such that the static magnetic field has the form $B_0 = (0, 0, b)$, $b > 0$ in this system. The field

$$B^+ = (B_x + iB_y)/2$$

is called positively rotating part of $B$ or transmit field. Several methods (sequences) are known that provide positively rotating part of a magnetic field for ex. [5]. According to [4] the "negatively rotating" part $B^- = (B_x - iB_y)/2$ cannot be determined in this way. The asymmetric role of coordinates $x$ and $y$ in (10) is defined with respect to $z$ coordinate which means that the coordinate system $x, y, z$ is positively oriented. Note that the orientation is indispensable feature of the Maxwell system which is illustrated by Maxwell’s right hand rule. It follows that such measurements of the transmit field can be made for any positively oriented coordinate system that is in the system obtained by rotation of the system $x, y, z$.

1. The simple acquisition geometry for magnetic field $B$ is to fix the body on the bed and rotate both around the central axis $y$ that is parallel to the bed. Let $x, y, z$ be the laboratory euclidean positively oriented (left-handed) system of coordinates, and let $\xi, \eta, \zeta$ be the positively oriented system of coordinates that are constant on the bed and the body when rotating that is

$$\xi = \cos \varphi x + \sin \varphi z, \eta = y, \zeta = -\sin \varphi x + \cos \varphi z$$

where $\varphi$ is the rotation angle see Fig.1.

Write magnetic field $B(\varphi) = \beta$ by means of both coordinate systems.

$$B(\varphi) = B_x(\varphi) \, dx + B_y(\varphi) \, dy + B_z(\varphi) \, dz, \quad \beta = \beta_\xi d\xi + \beta_\eta d\eta + \beta_\zeta d\zeta$$

and note that the body components of $\beta_\xi, \beta_\eta, \beta_\zeta$ of this field do not depend on $\varphi$. Keeping in view that

$$d\xi = \cos \varphi dx + \sin \varphi dz, \quad d\eta = dy, \quad d\zeta = -\sin \varphi dx + \cos \varphi dz$$

we obtain

$$B_x(\varphi) = \cos \varphi \beta_\xi - \sin \varphi \beta_\zeta, \quad B_y(\varphi) = \beta_\eta,$$

$$2B^+(\varphi) = B_x(\varphi) + iB_y(\varphi) = \cos \varphi \beta_\xi - \sin \varphi \beta_\zeta + i\beta_\eta.$$ 

Suppose that measurements of the transmit field are available for $\varphi = -\psi, 0, \psi$ and consider system of linear equations

$$\cos \psi \beta_\xi - \sin \psi \beta_\zeta + i\beta_\eta = 2B^+(\psi),$$

$$\cos \psi \beta_\xi + \sin \psi \beta_\zeta + i\beta_\eta = 2B^+(-\psi),$$

$$\beta_\xi + i\beta_\eta = 2B^+(0).$$


The solution exists and is unique for any \( \psi \neq 0, \pi \):

\[
\beta_\zeta = \frac{1}{\sin \psi} \left( B^+(-\psi) - B^+(\psi) \right), \quad \beta_\xi = \frac{1}{\cos \psi - 1} \left( B^+(\psi) + B^+(-\psi) - B^+(0) \right),
\]

\[
\beta_\eta = \frac{i}{\cos \psi - 1} \left( B^+(\psi) + B^+(-\psi) + (1 - 2 \cos \psi) B^+(0) \right)
\]

hence magnetic form \( \beta \) is uniquely reconstructed by (11). Now we can apply (5) to the fields \( B = B(0) = \beta \) and \( E \).

2. Another geometry for scanning a body is as follows: the bed with the body rotating in a tilted plane around the center \( O \) in \( B_0 \) field. Let \( \alpha, 0 < \alpha < \pi/2 \) be the constant tilting angle and \( e_3 = (0, -\sin \alpha, \cos \alpha) \).

For any \( \varphi, 0 \leq \varphi < 2\pi \), the vectors

\[
e_1 = (\cos \varphi, \sin \varphi \cos \alpha, \sin \varphi \sin \alpha), \quad e_2 = (-\sin \varphi, \cos \varphi \cos \alpha, \cos \varphi \sin \alpha)
\]

and \( e_3 \) form the orthogonal frame since \( e_1 \) and \( e_2 \) belong to the plane \( P \) through the origin orthogonal to \( e_3 \). This frame is a positively oriented. Functions

\[
\xi = \langle (x, y, z), e_1 \rangle, \quad \eta = \langle (x, y, z), e_2 \rangle, \quad \zeta = \langle (x, y, z), e_3 \rangle
\]

are euclidean coordinates that are constant on \( P \). To express the transmit field (10) by means of the body coordinates we write

\[
d\xi = \cos \varphi dx + \sin \varphi \cos \alpha dy + \sin \varphi \sin \alpha dz, \\
d\eta = -\sin \varphi dx + \cos \varphi \cos \alpha dy + \cos \varphi \sin \alpha dz, \\
d\zeta = -\sin \alpha dy + \cos \alpha dz
\]

and substitute in (11). This gives

\[
B_x(\varphi) = \cos \varphi \beta_\xi - \sin \varphi \beta_\eta, \quad B_y(\varphi) = \sin \varphi \cos \alpha \beta_\xi + \cos \varphi \cos \alpha \beta_\eta - \sin \alpha \beta_\zeta
\]

and

\[
2B^+(\varphi) = \cos \varphi \beta_\xi - \sin \varphi \beta_\eta + i \left( \sin \varphi \cos \alpha \beta_\xi + \cos \varphi \cos \alpha \beta_\eta - \sin \alpha \beta_\zeta \right).
\]

Evaluating the transmit field for \( \varphi = \varphi_1, \varphi_2, \varphi_3 \) we obtain the system like (12) for unknown functions \( \beta_\xi, \beta_\eta, \beta_\zeta \) that do not depend on \( \varphi \). The determinant of this system equals

\[
det = 4 \sin^3 \alpha \sin \varphi_{12} \sin \varphi_{23} \sin \varphi_{31},
\]

where \( \varphi_{ij} = (\varphi_i - \varphi_j) / 2 \). It does vanish if \( \alpha \neq 0, \pi \) and \( \varphi_{ij} \neq 0 \). It follows that for arbitrary \( \alpha \neq 0, \pi \), arbitrary different angles \( \varphi_i \), functions \( \beta_\xi, \beta_\eta, \beta_\zeta \) are uniquely determined form data of fields \( B^+(\varphi_i) \).

**Algorithm for computation of Contrast source function**

1. Make three measurements of the transmit field according to geometry 1 or 2.
2. Calculate the field \( \beta \) in the coordinate system of the body. and set \( B = B(0) = \beta \).
3. Apply (5) to the fields \( \beta \) and \( E \) and calculate the quotient as the function \( \chi \) of the body coordinates \( \xi, \eta, \zeta \).
References

[1] Haacke E, Petropoulos L S, Niiges E W, and Wu D H 1991 Extraction of conductivity and permittivity using magnetic resonance imaging, Phys. Med. Biol. 36 723

[2] Wen H 2003 Noninvasive quantitative mapping of conductivity and dielectric distributions using RF wave propagation effects in high-field MRI Proc. SPIE 5030 471–477

[3] Nachman A, Wang D, Ma W, and Joy M 2007 A local formula for inhomogeneous complex conductivity as a function of the RF magnetic field Proc. Intl. Soc. Mag. Reson. Med. 15.

[4] Katscher U et al 2009 Determination of electric conductivity and local SAR via B1 mapping IEEE Trans. Med. Imag. 28 1365–1374

[5] Sacolick L I, Wiesinger F, Hancu I and Vogel M W 2010 B1-Mapping by Bloch-Siegert Shift”, Magnetic Resonance in Medicine 63 1315-1322

[6] Song Y and Seo J K 2013 Conductivity and permittivity image reconstruction at the Larmor frequency using MRI SIAM J. Appl. Math. 73, N6 2262-2280

[7] Ammari H, Kwon H, Lee Y, Kang K, and Seo J K (2015) Magnetic resonance-based reconstruction method of conductivity and permittivity distributions at the Larmor frequency Inverse Problems 31 105001

[8] Kim S Y, Shin J, Kim D H, et al. 2016 Correlation between conductivity and prognostic factors in invasive breast cancer using magnetic resonance electric properties tomography (MREPT)” Eur. Radiol. 26 2317–2326

[9] Palamodov V P 2016 An analytic method for the inverse problem of MREPT Inverse Problems 32, N3 035003

[10] Liu J, Wang J L, Katscher U and He B 2017 Electrical Properties Tomography Based on B1 Maps in MRI: Principles, Applications and Challenges IEEE Trans. biomedical imaging 84 2515-2529

[11] Leijsen R L, Brink W M, van der Berg C, Webb A G and Remis R F 2018 3-D Contrast Source Inversion-Electrical Properties Tomography IEEE Trans. on medical imaging 37 2080-2089

[12] Arduino A, Bottauscio O, Chiampi M, and Zilberti L 2018 Magnetic resonance-based imaging of human electric properties with phaseless contrast source inversion Inverse problems 34 084002

[13] Flanders, H 1989 Differential forms with applications to the physical sciences (New York: Dover Publications)
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