RECOVERING NONLOCAL DIFFERENTIAL PENCILS

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Abstract. Inverse problems for differential pencils with nonlocal conditions are investigated. Several uniqueness theorems of inverse problems from the Weyl-type function and spectra are proved, which are generalizations of the well-known Weyl function and Borg’s inverse problem for the classical Sturm-Liouville operator.

1. Introduction

Problems with nonlocal conditions arise in various fields of mathematical physics, biology and biotechnology, and in other fields. Nonlocal conditions come up when value of the function on the boundary is connected to values inside the domain. Recently problems with nonlocal conditions are paid much attention for them in the literature.

In this paper we study inverse spectral problems for differential pencils
\[ y''(x) + [\lambda^2 - 2\lambda p(x) - q(x)]y(x) = 0, \quad x \in (0, T), \]
and with nonlocal linear conditions
\[ U_j(y) := \int_0^T y(t)d\sigma_j(t) = 0, \quad j = 1, 2. \]

Here \( p \in AC[0, T] \) (absolutely continuous function) and \( q \in L(0, T) \) are complex-valued functions, \( \sigma_j(t) \) are complex-valued functions of bounded variations and are continuous from the right for \( t > 0 \). There exist finite limits \( H_j := \sigma_j(+0) - \sigma_j(0) \).

Linear forms \( U_j(y) \) in (2) can be written as forms
\[ U_j(y) := H_jy(0) + \int_0^T y(t)d\sigma_{j0}(t), \quad j = 1, 2, \]
where \( \sigma_{j0}(t) \) in (3) are complex-valued functions of bounded variations and are continuous from the right for \( t \geq 0 \), and \( H_1 \neq 0 \).

A complex number \( \lambda_0 \) is called an eigenvalue of the problem (1) and (2) if equation (1) with \( \lambda = \lambda_0 \) has a nontrivial solution \( y_0(x) \) satisfying conditions (2); then \( y_0(x) \) is called the eigenfunction of the problem (1) and (2) corresponding to the eigenvalue \( \lambda_0 \). The number of linearly independent solutions of the problem (1) and (2) for a given eigenvalue \( \lambda_0 \) is called the multiplicity of \( \lambda_0 \).

Classical inverse problems for Eq. (1) with two-point boundary conditions have been studied fairly completely in many works (see the monographs and the references therein). The theory of nonlocal inverse spectral problems now is only at the beginning because of its complexity. Results of the inverse
problem for various nonlocal operators can be found in [14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

In this work by using Yurko’s ideas of the method of spectral mappings [11] we prove uniqueness theorems for the solution of the inverse spectral problems for Eq. (1) with nonlocal conditions [2]. In Section 2 we formulate our main results (Theorems 1 and 2). Section 3 introduces some properties of spectral characteristics. The proofs of Theorems 1 and 2 are given in Section 4. In Section 5 we provide two counterexamples related to the statements of the inverse problems (see also [16, 24]). In Section 6, as a consequence of Theorem 1, we consider the inverse problem of recovering the double functions \( p \) and \( q \) from the given three spectra.

2. MAIN RESULTS

Let \( X_k(x, \lambda) \) and \( Z_k(x, \lambda), \ k = 1, 2, \) be the solutions of Eq. (1) with the initial conditions

\[
X_1(0, \lambda) = X_2'(0, \lambda) = Z_1(T, \lambda) = Z_2'(T, \lambda) = 1, \\
X_1'(0, \lambda) = X_2(0, \lambda) = Z_1'(T, \lambda) = Z_2(T, \lambda) = 0.
\]

Denote by \( L_0 \) the boundary value problem (BVP) for Eq. (1) with the conditions

\[
U_1(y) = U_2(y) = 0,
\]

and \( \omega(\lambda) := \det[U_j(X_k)]_{j,k=1,2}, \) and assume that \( \omega(\lambda) \neq 0. \) The function \( \omega(\lambda) \) is an entire function of exponential type with order 1, and its zeros \( \Xi := \{\xi_n\}_{n \in \mathbb{Z}} \) (counting multiplicities) coincide with the eigenvalues of \( L_0. \) The function \( \omega(\lambda) \) is called the characteristic function for \( L_0. \)

Denote \( V_j(y) := \int_0^y (j-1)(T), \ \ j = 1, 2. \) Consider the BVP \( L_j, \ j = 1, 2, \) for Eq. (1) with the conditions \( U_j(y) = V_j(y) = 0. \) The eigenvalue sets \( \Lambda_j := \{\lambda_n\}_{n \in \mathbb{Z}} \) (counting multiplicities) of the BVP \( L_j \) coincide with the zeros of the characteristic function \( \Delta_j(\lambda) := \det[U_j(X_k), V_j(X_k)]_{k=1,2}. \)

For \( \lambda \neq \lambda_n, \) let \( \Phi(x, \lambda) \) be the solution of Eq. (1) under the conditions \( U_1(\Phi) = 1 \) and \( V_1(\Phi) = 0. \) Denote Weyl-type function \( M(\lambda) := U_2(\Phi). \) It is known [26] that for differential pencils with classical two-point separated boundary conditions, the specification of the Weyl function uniquely determines the double functions \( p(x) \) and \( q(x). \) In particular, in [25] it is proved that differential pencils with classical two-point separated boundary conditions is uniquely determined by specifying its Weyl function, which is equivalent to specification of the spectra of two boundary value problems with one common boundary condition, and a constructive procedure for solving the inverse problem is given. However, in the case with nonlocal conditions, it is not true; the specification of the Weyl-type function \( M(\lambda) \) does not uniquely determine the functions \( p(x) \) and \( q(x) \) (see counterexamples in Section 5). For the nonlocal conditions the inverse problem is formulated as follows.

Throughout this paper the functions \( \sigma_j(t) \) and the value \( \int_0^T p(x)dx \) are known a priori. And condition \( S: \Lambda_1 \cap \Xi = \emptyset. \)

Inverse problem 1. Given \( M(\lambda) \) and \( \omega(\lambda), \) construct the functions \( p(x) \) and \( q(x). \)

Let us formulate a uniqueness theorem for Inverse problem 1. For this purpose, together with \( (p, q) \) we consider another \( (\tilde{p}, \tilde{q}), \) and we agree that if a certain symbol \( \alpha \) denotes an object related to \( (p, q), \) then \( \tilde{\alpha} \) will denote an analogous object related to \( (\tilde{p}, \tilde{q}). \)
Moreover, by calculation, Eqs. (6)-(7) yield that
conditions.
lem [26] for Sturm-Liouville operators with classical two-point separated boundary
on
\[ \frac{\lambda}{n} \]
and
\[ \pm \]
Denote \( \Lambda \)
Thus, under condition
Lemma 1. (See [24])
Inverse problem 2. Given \( \{\lambda_n, \lambda_n^1\}_{n \in \mathbb{Z}}, \) construct \( p(x) \) and \( q(x) \).
This inverse problem is a generalization of the well-known Borg’s inverse problem [26] for Sturm-Liouville operators with classical two-point separated boundary conditions.

**Theorem 2.** If \( \lambda_n = \lambda_n^1, \lambda_n^1 = \lambda_n, \) \( n \in \mathbb{Z}, \) then \( p(x) = \tilde{p}(x) \) and \( q(x) \overset{a.e.}{=} \tilde{q}(x) \) on \((0, T)\).

### 3. Auxiliary Lemmas

Denote \( \Lambda^\pm := \{ \lambda : \pm \text{Im} \lambda \geq 0 \} \). It is known (see, for example, [10, 27]) that there exists a fundamental system of solutions \( \{Y_k(x, \lambda)\}_{k=1,2} \) of Eq. (1) such that for \( \lambda \in \Lambda^\pm, |\lambda| \to \infty, \nu = 0, 1:\)

\[
Y_1^{(\nu)}(x, \lambda) = (i \lambda)^\nu \exp \left( i \left( \lambda x - \int_0^x p(t) dt \right) \right) (1 + O(\lambda^{-1})),
\]

\[
Y_2^{(\nu)}(x, \lambda) = (-i \lambda)^\nu \exp \left( -i \left( \lambda x - \int_0^x p(t) dt \right) \right) (1 + O(\lambda^{-1})),
\]

and

\[
\det[Y^{(\nu-1)}(x, \lambda)]_{k,\nu=1,2} = -2i\lambda(1 + O(\lambda^{-1})).
\]

**Lemma 1.** (See [24]) Let \( \{W_k(x, \lambda)\}_{k=1,2} \) be a fundamental system of solutions of Eq. (1), and let \( Q_j(y) \), \( j = 1, 2, \) be linear forms. Then

\[
\det[Q_j(W_k)]_{k,j=1,2} = \det[Q_j(X_k)]_{k,j=1,2} \det[W^{(\nu-1)}_k(x, \lambda)]_{k,\nu=1,2}.
\]

It follows from (5)-(6) that

\[
\det[Q_j(Z_k)]_{k,j=1,2} = \det[Q_j(X_k)]_{k,j=1,2}.
\]

and

\[
\det[Q_j(Y_k)]_{k,j=1,2} = -2i\lambda(1 + O(\lambda^{-1})) \det[Q_j(X_k)]_{k,j=1,2}.
\]

Introduce the functions

\[
\varphi(x, \lambda) = -\det[X_k(x, \lambda), U_1(X_k)]_{k=1,2}, \quad \theta(x, \lambda) = \det[X_k(x, \lambda), U_2(X_k)]_{k=1,2},
\]

\[
\psi(x, \lambda) = \det[X_k(x, \lambda), V_1(X_k)]_{k=1,2}.
\]

Then

\[
U_1(\varphi) = 0, \quad U_2(\varphi) = \omega(\lambda), \quad V_1(\varphi) = \Delta_1(\lambda), \quad V_2(\varphi) = \Delta_{11}(\lambda),
\]

\[
U_1(\theta) = \omega(\lambda), \quad U_2(\theta) = 0, \quad V_1(\theta) = -\Delta_2(\lambda), \quad V_2(\theta) = -1.
\]

Moreover, by calculation, Eqs. (5)-(7) yield that

\[
\det[\theta^{(\nu-1)}(x, \lambda), \varphi^{(\nu-1)}(x, \lambda)]_{\nu=1,2} = \omega(\lambda),
\]

\[
\det[\psi^{(\nu-1)}(x, \lambda), \varphi^{(\nu-1)}(x, \lambda)]_{\nu=1,2} = \Delta_1(\lambda),
\]
\[ \Delta_1(\lambda) = -U_1(Z_2), \Delta_2(\lambda) = -U_2(Z_2), \Delta_{11}(\lambda) = U_1(Z_1). \] (10)

Note that the functions \( \Phi, \psi, \varphi \) and \( \theta \) are all the solutions of Eq. (11) with some conditions, comparing boundary conditions on \( \Phi, \psi, \varphi \) and \( \theta \), we arrive at

\[
\Phi(x, \lambda) = \frac{\psi(x, \lambda)}{\Delta_1(\lambda)},
\] (11)

\[
\Phi(x, \lambda) = \frac{1}{\omega(\lambda)} \left( \theta(x, \lambda) + \frac{\Delta_2(\lambda)}{\Delta_1(\lambda)} \varphi(x, \lambda) \right).
\] (12)

Hence,

\[
M(\lambda) := U_2(\Phi) = \frac{\Delta_2(\lambda)}{\Delta_1(\lambda)},
\] (13)

\[
\det[\Phi^{(\nu-1)}(x, \lambda), \varphi^{(\nu-1)}(x, \lambda)]_{\nu=1,2} = 1.
\] (14)

Let \( v_1(x, \lambda) \) and \( v_2(x, \lambda) \) be the solutions of Eq. (11) with the conditions

\[
v_1(T, \lambda) = v'_2(T, \lambda) = 1, \quad v'_1(T, \lambda) = 0, \quad U_1(v_2) = 0.
\]

Obviously,

\[
v_1(x, \lambda) = Z_1(x, \lambda), \quad v_2(x, \lambda) = Z_2(x, \lambda) + N(\lambda)Z_1(x, \lambda),
\] (15)

\[
det[v^{(\nu-1)}_k(x, \lambda)]_{k,\nu=1,2} = 1,
\]

where

\[
N(\lambda) = \frac{\Delta_1(\lambda)}{\Delta_{11}(\lambda)} = \frac{U_1(Z_2)}{U_1(Z_1)}.
\] (16)

Denote

\[
U_1^a(y) := \int_0^a y(t) d\sigma_1(t), \quad a \in (0, T].
\]

Clearly, \( U_1 = U_1^T \), and if \( \sigma_1(t) \equiv C \) (constant) for \( t \geq a \), then \( U_1 = U_1^a \).

For sufficiently small \( \delta > 0 \), we denote

\[
\Pi_\delta := \{ \lambda : \arg \lambda \in [\delta, \pi - \delta]\}, \quad G_\delta := \{ \lambda : |\lambda - \lambda_{n1}| \geq \delta, \quad \forall n \in \mathbb{Z}\},
\]

and

\[
G_\delta' := \{ \lambda : |\lambda - \lambda_{n1}^1| \geq \delta, \quad \forall n \in \mathbb{Z}\},
\]

where \( \lambda_{n1} \in \Lambda_1 \) and \( \lambda_{n1}^1 \in \Lambda_{11} \).

**Lemma 2.** For \( \lambda \in \Pi_\delta, |\lambda| \to \infty \), we have

\[
\Phi^{(\nu)}(x, \lambda) = \frac{(i\lambda)^\nu}{H_1} \exp \left( i \left( \lambda x - \int_0^T p(t) dt \right) \right) (1 + o(1)), \quad x \in [0, T),
\] (17)

\[
v^{(\nu)}_1(x, \lambda) = \frac{(i\lambda)^\nu}{2} \exp \left( -i \left( \lambda(T - x) - \int_0^{T-x} p(t) dt \right) \right) (1 + O(\lambda^{-1})), \quad x \in [0, T),
\] (18)

\[
\Delta_1(\lambda) = -\frac{H_2}{2\lambda} \exp \left( -i \left( \lambda T - \int_0^T p(t) dt \right) \right) (1 + o(1)),
\]

\[
\Delta_{11}(\lambda) = \frac{H_2}{\lambda} \exp \left( -i \left( \lambda T - \int_0^T p(t) dt \right) \right) (1 + o(1)).
\] (19)
Let $\sigma_1(t) \equiv C$ (constant) for $t \geq a$ (i.e. $U_1 = U_1^a$). Then for $\lambda \in \Pi_\delta$, $|\lambda| \to \infty$,

$$\varphi^{(\nu)}(x, \lambda) = \frac{H_1}{2} (-i\lambda)^{\nu-1} \exp \left( -i \left( \lambda x - \int_0^x p(t) dt \right) \right) \times \left[ 1 + o(1) + O(\exp(i\lambda(2x - a))) \right], \ x \in (0, T], \quad (20)$$

$$v_2^{(\nu)}(x, \lambda) = (-i\lambda)^{\nu-1} \exp \left( i \left( \lambda(T - x) - \int_0^{T-x} p(t) dt \right) \right) \times \left[ 1 + o(1) + O(\exp(i\lambda(2x - a))) \right], \ x \in [0, T]. \quad (21)$$

**Proof.** The function $\Phi(x, \lambda)$ can be expressed as

$$\Phi(x, \lambda) = A_1(\lambda) Y_1(x, \lambda) + A_2(\lambda) Y_2(x, \lambda), \quad (22)$$

together with $U_1(\Phi) = 1$ and $V_1(\Phi) = 0$, which yields that

$$A_1(\lambda) U_1(Y_1) + A_2(\lambda) U_1(Y_2) = 1, \ A_1(\lambda) V_1(Y_1) + A_2(\lambda) V_1(Y_2) = 0. \quad (23)$$

Using (4), one gets that for $\lambda \in \Pi_\delta$, $|\lambda| \to \infty$:

$$U_1(Y_1) = H_1 (1 + o(1)), \ U_1(Y_2) = O(\exp(-i\lambda T)), \quad (24)$$

$$V_1(Y_1) = \exp \left( i \left( X - \int_0^X p(t) dt \right) \right) (1 + O(\lambda^{-1})), \ V_1(Y_2) = \exp \left( -i \left( X - \int_0^X p(t) dt \right) \right) (1 + O(\lambda^{-1})). \quad (25)$$

Solving linear algebraic system (23) by using (24)-(25), we obtain

$$A_1(\lambda) = H_1^{-1} (1 + o(1)), \ A_2(\lambda) = O \left( \exp \left( 2i \left( X - \int_0^X p(t) dt \right) \right) \right). \quad (26)$$

Substituting these relations into (22), we have proved (17). Formulas (18)-(21) can be proved similarly, and are omitted. \hfill \Box

By the well-known method (see, for example, [10]) the following estimates hold for $x \in (0, T)$, $\lambda \in \Lambda^+$:

$$v_1^{(\nu)}(x, \lambda) = O \left( \lambda^{\nu} \exp \left( -i \left( \lambda x - \int_0^x p(t) dt \right) \right) \right), \quad (26)$$

$$\Phi^{(\nu)}(x, \lambda) = O \left( \lambda^{\nu} \exp \left( i \left( \lambda x - \int_0^x p(t) dt \right) \right) \right), \quad \rho \in G_\delta. \quad (27)$$

Moreover, if $\sigma_1(t) \equiv C$ (constant) for $t \geq a$ (i.e. $U_1 = U_1^a$), then for $x \geq a/2$, $\lambda \in \Lambda^+$:

$$\varphi^{(\nu)}(x, \lambda) = O \left( \lambda^{\nu-1} \exp \left( -i \left( \lambda x - \int_0^x p(t) dt \right) \right) \right), \quad (28)$$

$$v_2^{(\nu)}(x, \lambda) = O \left( \lambda^{\nu-1} \exp \left( i \left( \lambda(T - x) - \int_0^{T-x} p(t) dt \right) \right) \right), \quad \rho \in G_\delta. \quad (29)$$
4. Proofs of Theorems

Proof of Theorem 2 We know that the characteristic function $\Delta_1(\lambda)$ of the BVP $L_1$ is an entire function of order one with respect to $\lambda$. Following the theory of Hadamard’s factorization (see [28]), $\Delta_1(\lambda)$ can be expressed as an infinite product as

$$\Delta_1(\lambda) = c_1 e^{a_1 \lambda} \prod_{n \in \mathbb{Z}} \left(1 - \frac{\lambda}{\lambda_{n1}}\right) e^{\frac{\lambda}{\lambda_{n1}}} + \frac{\lambda}{\lambda_{n1}}^{2} + \ldots + \frac{\lambda}{\lambda_{n1}}^{p},$$

where $\lambda_{n1}$ are the eigenvalues of the problem $L_1$, $c_1$ and $a_1$ are constants. Since for the order $\rho$ of $\Delta_1(\lambda)$, $p \leq \rho \leq p + 1$ (see [28]), and $\Delta_1(\lambda)$ is an entire function of exponential type with order 1, we find that the genus of $\Delta_1(\lambda)$ is 0 or 1 (that is, $p = 0 \lor 1$). Thus $\Delta_1(\lambda)$ can be rewritten by

$$\Delta_1(\lambda) = c_1 e^{a_1 \lambda} \prod_{n \in \mathbb{Z}} \left(1 - \frac{\lambda}{\lambda_{n1}}\right) e^{\frac{\lambda}{\lambda_{n1}}}.$$

Since $\Delta_1(\lambda)$ and $\tilde{\Delta}_1(\lambda)$ are both entire functions of order one with respect to $\lambda$, and $\lambda_{n1} = \tilde{\lambda}_{n1}$ for all $n \in \mathbb{Z}$, by the Hadamard’s factorization theorem, we may suppose (the case when $\Delta_1(0) = 0$ requires minor modifications)

$$\Delta_1(\lambda) = c_1 e^{a_1 \lambda} \prod_{n \in \mathbb{Z}} \left(1 - \frac{\lambda}{\lambda_{n1}}\right) e^{\frac{\lambda}{\lambda_{n1}}}$$

and

$$\tilde{\Delta}_1(\lambda) = \tilde{c}_1 e^{\tilde{a}_1 \lambda} \prod_{n \in \mathbb{Z}} \left(1 - \frac{\lambda}{\tilde{\lambda}_{n1}}\right) e^{\frac{\lambda}{\tilde{\lambda}_{n1}}}$$

for some constants $c_1, \tilde{c}_1, a_1, \tilde{a}_1$ and $p, \tilde{p}$, which can be determined from the asymptotics. From this we get for all $\lambda \in \mathbb{C}$

$$\frac{\tilde{\Delta}_1(\lambda)}{\Delta_1(\lambda)} = \frac{\tilde{c}_1}{c_1} e^{\left[(\tilde{a}_1 - a_1) + (\tilde{p} - p) \sum_{n \in \mathbb{Z}} \frac{1}{\lambda_{n1}}\right] \lambda}.$$

The expression (19) implies that

$$\Delta_1(\lambda) = -\frac{H}{2i\lambda} \exp \left(-i \left(\lambda T - \int_0^T p(t)dt\right)\right) (1 + o(1))$$

and

$$\tilde{\Delta}_1(\lambda) = -\frac{H}{2i\lambda} \exp \left(-i \left(\lambda T - \int_0^T \tilde{p}(t)dt\right)\right) (1 + o(1)).$$

We get that, using the assumption that $\int_0^T p(t)dt = \int_0^T \tilde{p}(t)dt$,

$$\frac{\tilde{\Delta}_1(\lambda)}{\Delta_1(\lambda)} = 1 + o(1) \equiv \frac{\tilde{c}_1}{c_1} e^{\left[(\tilde{a}_1 - a_1) + (\tilde{p} - p) \sum_{n \in \mathbb{Z}} \frac{1}{\lambda_{n1}}\right] \lambda},$$

which yields that

$$(\tilde{a}_1 - a_1) + (\tilde{p} - p) \sum_{n \in \mathbb{Z}} \frac{1}{\lambda_{n1}} = 0, \quad \tilde{c}_1 = c_1.$$

Consequently, $\Delta_1(\lambda) \equiv \tilde{\Delta}_1(\lambda)$. Analogously, from $\lambda_{n1}^1 = \tilde{\lambda}_{n1}^1$ for all $n \in \mathbb{Z}$ we get $\Delta_{11}(\lambda) \equiv \tilde{\Delta}_{11}(\lambda)$. By virtue of (16), this yields

$$N(\lambda) \equiv \tilde{N}(\lambda).$$

(30)
Thus, for each fixed \( \delta \) where
\[
x \quad \frac{1}{\sqrt{T}} \leq \frac{\xi}{\sqrt{T}}.
\]
On the other hand, taking (18) and (21) into account we calculate for each fixed \( x \):
\[
\frac{\xi}{\sqrt{T}} = \int_{0}^{T} (p(t) - \bar{p}(t)) dt.
\]
Again, using the asymptotic expression (18) for \( v \), from (31) it yields
\[
Z_1(x, \lambda) = (Z_1(x, \lambda) \bar{Z}'_1(x, \lambda) - \bar{Z}'_1(x, \lambda) Z_2(x, \lambda)) + (\bar{N}(\lambda) - N(\lambda)) Z_1(x, \lambda) \bar{Z}'_1(x, \lambda).
\]
Using the maximum modulus principle and Liouville’s theorem for entire functions, we conclude that
\[
P_k(x, \lambda) - \delta_{1k} \Omega_1(x) = o(1), \quad |\rho| \to \infty, \quad \rho \in \Pi_\delta,
\]
where \( \delta_{1k} \) is the Kronecker symbol and
\[
\Omega_1(x) = \frac{\exp[i \int_{0}^{T-x} (p(t) - \bar{p}(t)) dt] + \exp[-i \int_{0}^{T-x} (p(t) - \bar{p}(t)) dt]}{2}.
\]
Also, applying (26) and (29), we get for \( k = 1, 2 \):
\[
P_k(x, \lambda) = O(1), \quad |\rho| \to \infty, \quad \rho \in G'_{\delta}'.
\]
Using the maximum modulus principle and Liouville’s theorem for entire functions, we conclude that
\[
P_1(x, \lambda) \equiv \Omega_1(x), \quad P_2(x, \lambda) \equiv 0, \quad x \geq T/2.
\]
From (31) it yields
\[
v_1(x, \lambda) = \Omega_1(x) \bar{v}_1(x, \lambda), \quad v_2(x, \lambda) = \Omega_1(x) \bar{v}_2(x, \lambda).
\]
Again, using the asymptotic expression (18) for \( v_1(x, \lambda) \) and \( \bar{v}_1(x, \lambda) \), we have
\[
\exp \left( -i(\lambda(T - x) - \int_{0}^{T-x} p(t) dt) \right) [1 + o(1)] = \Omega_1(x) \exp \left( -i(\lambda(T - x) - \int_{0}^{T-x} \bar{p}(t) dt) \right) [1 + o(1)],
\]
which leads to
\[
\exp \left( i \int_{0}^{T-x} (p(t) - \bar{p}(t)) dt \right) = \Omega_1(x).
\]
This deduces for \( x \geq T/2, \)
\[
\int_{0}^{T-x} (p(t) - \bar{p}(t)) dt = 0,
\]
which yields
\[
p(x) = \bar{p}(x) \text{ for } x \in \left[ 0, \frac{T}{2} \right].
\]
At this case we have \( \int_{0}^{T-x} (p(t) - \bar{p}(t)) dt = 0 \) for \( x \geq T/2 \). Thus \( \Omega_1(x) = 1 \) for \( x \geq T/2 \).
Together with (31) this yields that for $x \geq T/2$,

$$v_k(x, \lambda) = \tilde{v}_k(x, \lambda), \quad Z_k(x, \lambda) = \tilde{Z}_k(x, \lambda), \quad p(x) = \tilde{p}(x), \quad q(x) \overset{a.e.}{=} \tilde{q}(x). \quad (32)$$

Next let us now consider the BVPs $L_k^q$ and $L_k^q$ for Eq. (11) on the interval $(0, T)$ with the conditions $U_k^a(y) = V_1(y) = 0$ and $U_k^a(y) = V_2(y) = 0$, respectively. Then, according to Eq. (19), the functions $\Delta_k^{T_1}(\lambda) := -U_k^a(Z_2)$ and $\Delta_k^{T_1}(\lambda) := U_k^a(Z_1)$ are the characteristic functions of $L_k^q$ and $L_k^q$, respectively. And

$$U_k^{a/2}(Z_k) = U_k^a(Z_k) - \int_{a/2}^a Z_k(t, \lambda)d\sigma_1(t), \quad k = 1, 2,$$

hence

$$\Delta_k^{T/2}(\lambda) = \Delta_k^a(\lambda) + \int_{a/2}^a Z_2(t, \lambda)d\sigma_1(t), \quad \Delta_k^{T/2}(\lambda) = \Delta_k^a(\lambda) - \int_{a/2}^a Z_1(t, \lambda)d\sigma_1(t). \quad (33)$$

Let us use (33) for $a = T$. Since $\Delta_k^T(\lambda) = \Delta_k^1(\lambda)$, $\Delta_k^{T_1}(\lambda) = \Delta_k^{11}(\lambda)$, it follows from (32) - (33) that

$$\Delta_k^{T/2}(\lambda) = \tilde{\Delta}_k^{T/2}(\lambda), \quad \Delta_k^{T/2}(\lambda) = \tilde{\Delta}_k^{T/2}(\lambda).$$

Repeating preceding arguments subsequently for $a = T/2, T/4, T/8, \ldots$, we conclude that $p(x) = \tilde{p}(x)$ and $q(x) \overset{a.e.}{=} \tilde{q}(x)$ on $(0, T)$. Theorem 2 is proved. \qed

**Proof of Theorem 1**

Define the functions

$$R_1(x, \lambda) := \Phi(x, \lambda)\tilde{\varphi}'(x, \lambda) - \tilde{\Phi}'(x, \lambda)\varphi(x, \lambda),$$

$$R_2(x, \lambda) := \varphi(x, \lambda)\tilde{\Phi}'(x, \lambda) - \tilde{\varphi}(x, \lambda)\Phi(x, \lambda). \quad (34)$$

Since $\Lambda_1 \cap \Xi = \emptyset$ we can infer that $\Lambda_1 \cap \Lambda_2 = \emptyset$. Otherwise, if a certain $\lambda \in \Lambda_1 \cap \Lambda_2$ then $\lambda \in \Xi$. Thus $\lambda \in \Lambda_1 \cap \Xi$; this leads to a contradiction to the assumption that $\Lambda_1 \cap \Xi = \emptyset$. Moreover, Eqs. $M(\lambda) = \tilde{M}(\lambda), \quad M(\lambda) = \frac{\Delta_2(\lambda)}{\Delta_1(\lambda)}$, and $\tilde{M}(\lambda) = \frac{\tilde{\Delta}_2(\lambda)}{\tilde{\Delta}_1(\lambda)}$ imply that

$$\Delta_1(\lambda) = \tilde{\Delta}_1(\lambda), \quad \Delta_2(\lambda) = \tilde{\Delta}_2(\lambda).$$

It follows from (11) and (34) that

$$R_1(x, \lambda) = \frac{1}{\Delta_1(\lambda)} \left( \psi(x, \lambda)\tilde{\varphi}'(x, \lambda) - \tilde{\psi}'(x, \lambda)\varphi(x, \lambda) \right),$$

$$R_2(x, \lambda) = \frac{1}{\Delta_1(\lambda)} \left( \varphi(x, \lambda)\tilde{\psi}(x, \lambda) - \tilde{\varphi}(x, \lambda)\psi(x, \lambda) \right).$$

The above equations imply that for each fixed $x$, the functions $R_k(x, \lambda)$ are meromorphic in $\lambda$ with possible poles only at $\lambda = \lambda_n$. On the other hand, taking (12) into account, we also get

$$R_1(x, \lambda) = \frac{1}{\omega(\lambda)} \left( \theta(x, \lambda)\tilde{\varphi}'(x, \lambda) - \tilde{\theta}'(x, \lambda)\varphi(x, \lambda) \right), \quad (35)$$

$$R_2(x, \lambda) = \frac{1}{\omega(\lambda)} \left( \varphi(x, \lambda)\tilde{\theta}(x, \lambda) - \tilde{\varphi}(x, \lambda)\theta(x, \lambda) \right). \quad (36)$$

The assumption that $\Lambda_1 \cap \Xi = \emptyset$ tells us that the functions $R_k(x, \lambda)$ are regular at $\lambda = \lambda_n$. Thus, for each fixed $x$, the functions $R_k(x, \lambda)$ are entire in $\lambda$. Using (17) and (20), we can obtain for $x \geq T/2$:

$$R_k(x, \lambda) - \delta_k \Omega_2(x) = o(1), \quad |\rho| \to \infty, \quad \rho \in \Pi_\delta,$$
where
\[ \Omega_2(x) = \frac{\exp[i \int_0^x (p(t) - \hat{p}(t))dt] + \exp[-i \int_0^x (p(t) - \hat{p}(t))dt]}{2}. \]

Also, using (27)-(28), we obtain for \( x \geq T/2 \):
\[ R_k(x, \lambda) = O(1), \quad |\rho| \to \infty, \quad \rho \in G_\delta. \]

Therefore, \( R_1(x, \lambda) \equiv \Omega_2(x) \), \( R_2(x, \lambda) \equiv 0 \) for \( x \geq T/2 \). Using the asymptotic expression [20] for \( \varphi(x, \lambda) \) and \( \tilde{\varphi}(x, \lambda) \), we have
\[ \exp \left(-i(\lambda x - \int_0^x p(t)dt)\right)[1 + o(1)] = \Omega_2(x) \exp \left(-i(\lambda x - \int_0^x \hat{p}(t)dt)\right)[1 + o(1)]. \]

Thus
\[ \exp \left(i \int_0^x (p(t) - \hat{p}(t))dt\right) = \Omega_2(x), \]
which deduces for \( x \geq T/2 \),
\[ \int_0^x (p(t) - \hat{p}(t))dt = 0. \]

this yields
\[ p(x) = \tilde{p}(x) \text{ for } x \in \left[\frac{T}{2}, T\right]. \]

At this case we have \( \int_0^x (p(t) - \hat{p}(t))dt = 0 \) for \( x \geq T/2 \). Thus \( \Omega_2(x) = 1 \) for \( x \geq T/2 \). Together with [13] and [31], it yields
\[ \varphi(x, \lambda) = \tilde{\varphi}(x, \lambda), \quad \psi(x, \lambda) = \tilde{\psi}(x, \lambda), \quad p(x) = \tilde{p}(x), \quad q(x) \equiv \tilde{q}(x), \quad x \geq T/2. \]

Also, we obtain
\[ Z_k(x, \lambda) = \tilde{Z}_k(x, \lambda), \quad k = 1, 2, \quad x \geq T/2. \]

Since
\[ \varphi(x, \lambda) = U_1(Z_1)Z_2(x, \lambda) - U_1(Z_2)Z_1(x, \lambda) \]
and
\[ \tilde{\varphi}(x, \lambda) = U_1(\tilde{Z}_1)\tilde{Z}_2(x, \lambda) - U_1(\tilde{Z}_2)\tilde{Z}_1(x, \lambda) \]
we have
\[ \varphi(x, \lambda) = \Delta_{11}(\lambda)Z_2(x, \lambda) + \Delta_1(\lambda)Z_1(x, \lambda) \]
and
\[ \tilde{\varphi}(x, \lambda) = \tilde{\Delta}_{11}(\lambda)\tilde{Z}_2(x, \lambda) + \tilde{\Delta}_1(\lambda)\tilde{Z}_1(x, \lambda). \]

Taking \( x = T \) we get
\[ \varphi(T, \lambda) = \Delta_1(\lambda), \quad \tilde{\varphi}(T, \lambda) = \tilde{\Delta}_1(\lambda), \quad \varphi'(T, \lambda) = \Delta_{11}(\lambda), \quad \tilde{\varphi}'(T, \lambda) = \tilde{\Delta}_{11}(\lambda). \]

It follows from \( \varphi(x, \lambda) = \tilde{\varphi}(x, \lambda) \) for \( x \geq T/2 \) that
\[ \Delta_1(\lambda) = \tilde{\Delta}_1(\lambda), \quad \Delta_{11}(\lambda) = \tilde{\Delta}_{11}(\lambda). \]

Using Theorem 2, we conclude that \( p(x) = \tilde{p}(x) \) and \( q(x) \equiv \tilde{q}(x) \) on \((0, T)\). Theorem 1 is proved. \( \square \)
5. Counterexamples

Example 1  (To illustrate that if condition $S$ does not hold then Theorem 1 is false)

Suppose that $T = \pi$, $U_1(y) = y(0)$, $U_2(y) = y(\pi/2)$, $p(x) = p(x + \pi/2)$ and $q(x) = q(x + \pi/2)$ for $x \in (0, \pi/2)$, $p(x) \neq p(\pi - x)$ and $q(x) \neq q(\pi - x)$ for $x \in (0, \pi)$.

Take $\hat{p}(x) := p(\pi - x)$ and $\hat{q}(x) := q(\pi - x)$ for $x \in (0, \pi)$. Then BVP $L_1$: Eq. (11) with $\hat{p}(x) = p(\pi - x)$ and $\hat{q}(x) = q(\pi - x)$, $\hat{y}(x) = y(\pi - x)$, and the conditions $U_1(\hat{y}) = V_1(\hat{y}) = 0$;

BVP $L_2$: Eq. (11) with $\hat{p}(x) = p(\pi - x)$ and $\hat{q}(x) = q(\pi - x)$, $\hat{y}(x) = y(\pi - x)$, and the conditions $U_1(\hat{y}) = V_1(\hat{y}) = 0$;

BVP $L_0$: Eq. (11) with $\hat{p}(x) = p(\pi/2 - x)$ (also equal to $p(\pi - x)$) and $\hat{q}(x) = q(\pi/2 - x)$ (also equal to $q(\pi - x)$), $\hat{y}(x) = y(\pi/2 - x)$, and the conditions $U_1(\hat{y}) = U_2(\hat{y}) = 0$.

Here $\lambda_1(\lambda)$ is the characteristic function for Eq. (11) with $y(0) = 0 = y(\pi)$; $\Delta_2(\lambda)$ is the characteristic function for Eq. (11) with $y(\pi/2) = 0 = y(\pi)$; $\omega(\lambda)$ is the characteristic function for Eq. (11) with $y(0) = 0 = y(\pi/2)$. From the above fact the following relations are true:

$$\lambda_1(\lambda) = \hat{\lambda}_1(\lambda), \quad \Delta_2(\lambda) = \hat{\Delta}_2(\lambda), \quad \omega(\lambda) = \hat{\omega}(\lambda),$$

and, in view of (13), $M(\lambda) = \hat{M}(\lambda)$.

Note that $\Lambda_1$, $\Lambda_2$, and $\Xi$ are sets of zeros for characteristic functions $\lambda_1(\lambda)$, $\Delta_2(\lambda)$ and $\omega(\lambda)$, respectively. Since $p(x) = p(x + \pi/2)$ and $q(x) = q(x + \pi/2)$ for $x \in (0, \pi/2)$, there holds $\omega(\lambda) = \Delta_2(\lambda)$, i.e. $\Xi = \Lambda_2$. Thus for all $\lambda \in \Xi(= \Lambda_2)$ then it yields $\lambda \in \Lambda_1$, which implies that $\Xi \cap \Lambda_1 \neq \emptyset$.

Now $M(\lambda) = \hat{M}(\lambda)$ and $\omega(\lambda) = \hat{\omega}(\lambda)$, but $\Xi \cap \Lambda_1 \neq \emptyset$. In Theorem 1 condition $S$ does not hold. In fact, at this case $p(x) \neq \hat{p}(x) := p(\pi - x)$ and $q(x) \neq \hat{q}(x) := q(\pi - x)$. This means, that the specification of $M(\lambda)$ and $\omega(\lambda)$ does not uniquely determine the functions $p(x)$ and $q(x)$.

Example 2  (To illustrate that even if condition $S$ and $M(\lambda) = \hat{M}(\lambda)$ hold without the assumption that $\omega(\lambda) = \hat{\omega}(\lambda)$ then Theorem 1 is false)

Suppose that $T = \pi$, $U_1(y) = y(0)$, $U_2(y) = y(\pi - \alpha)$, where $\alpha \in (0, \pi/2)$.

Let $p(x) \neq p(\pi - x)$ and $q(x) \neq q(\pi - x)$, and $(p(x), q(x)) \equiv (0, 0)$ for $x \in [0, \alpha_0] \cup [\pi - \alpha_0, \pi]$, where $\alpha_0 \in (0, \pi/2)$. If $\alpha < \alpha_0$, then $\lambda_{n2} = n\pi/\alpha$, $n \in \mathbb{Z} \setminus \{0\}$.

Choose a sufficiently small $\alpha < \alpha_0$ such that $\Lambda_1 \cap \Lambda_2 = \emptyset$. Clearly, such choice is possible. Then $\Lambda_1 \cap \Xi = \emptyset$, i.e. condition $S$ holds. Otherwise, if a certain $\lambda^* \in \Lambda_1 \cap \Xi$, then $\lambda^* \in \Lambda_1 \cap \Lambda_2$; this contradicts to the fact that $\Lambda_1 \cap \Lambda_2 = \emptyset$.

Take $\hat{p}(x) := p(\pi - x)$ and $\hat{q}(x) := q(\pi - x)$. Note that $\Delta_2(\lambda)$ is the characteristic function for the problem

$$-y''(x) = \lambda^2 y(x), \quad y(\pi - \alpha) = 0 = y(\pi);$$

and $\hat{\Delta}_2(\lambda)$ is the characteristic function for the problem

$$-y''(x) = \lambda^2 y(x), \quad y(0) = 0 = y(\alpha).$$

A simple calculation shows $\Lambda_2 = \hat{\Lambda}_2 = \{ n\pi/\alpha, n \in \mathbb{Z} \setminus \{0\} \}$.

At this case $\lambda_1(\lambda) = \hat{\lambda}_1(\lambda), \Delta_2(\lambda) = \hat{\Delta}_2(\lambda)$, and consequently, $M(\lambda) = \hat{M}(\lambda)$.

Now in Theorem 1 condition $S$ holds, and $M(\lambda) = \hat{M}(\lambda)$. In fact, in this example,
p(x) ≠ ̃p(x) := p(π − x) and q(x) ≠ ̃q(x) := q(π − x). So the true condition S and the specification of M(λ) does not uniquely determine the functions p(x) and q(x).

6. INVERSE PROBLEM FROM THREE SPECTRA

Fix a ∈ (0, T). Consider Inverse problem 1 in the case when \( U_1(y) := y(0), U_2(y) := y(a) \). Then the boundary value problems \( L_0, L_1, L_2 \) take the forms

\[
L'_0 : \text{Eq. (1) with } y(0) = y(a) = 0,
\]

\[
L'_1 : \text{Eq. (1) with } y(0) = y(T) = 0,
\]

\[
L'_2 : \text{Eq. (1) with } y(a) = y(T) = 0.
\]

Denote by \( \Lambda'_j = \{\lambda'_{nj}\} \) the spectrum of \( L'_j \) (\( j = 0, 1, 2 \)), and assume that \( \Lambda'_0 \cap \Lambda'_1 = \emptyset \) (condition \( S' \)).

**Inverse problem 4.** Given three spectra \( \Lambda'_0, \Lambda'_1 \text{ and } \Lambda'_2 \), construct \( p(x) \) and \( q(x) \).

The following theorem is a consequence of Theorem 1.

**Theorem 4.** Let condition \( S' \) hold. If \( \Lambda'_j = \tilde{\Lambda}'_j, j = 0, 1, 2 \), then \( p(x) = \tilde{p}(x) \) and \( q(x) = \tilde{q}(x) \) on \( (0, T) \).

Comparing Theorem 4 with Theorem 1 we note that \( \Lambda'_0, \Lambda'_1 \text{ and } \Lambda'_2 \) correspond to \( \Xi, \Lambda_1 \text{ and } \Lambda_2 \) in Theorem 1, respectively. In particular, Inverse problem 4 with \( p(x) \equiv 0 \text{ on } [0, T] \) was studied by many authors (see, for example, [29, 30]).

**Acknowledgments.** The research work of the second author was supported by the Russian Ministry of Education and Science (Grant 1.1436.2014K), and by Grant 13-01-00134 of Russian Foundation for Basic Research. The first author was supported in part by the National Natural Science Foundation of China (11171152) and Natural Science Foundation of Jiangsu Province of China (BK 20141392).

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