Well-posedness and asymptotic behavior of stochastic convective Brinkman–Forchheimer equations perturbed by pure jump noise

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Abstract
This paper is concerned about stochastic convective Brinkman–Forchheimer (SCBF) equations subjected to multiplicative pure jump noise in bounded or periodic domains. Our first goal is to establish the existence of a pathwise unique strong solution satisfying the energy equality (Itô’s formula) to SCBF equations. We resolve the issue of the global solvability of SCBF equations, by using a monotonicity property of the linear and nonlinear operators and a stochastic generalization of the Minty–Browder technique. The major difficulty is that an Itô’s formula in infinite dimensions is not available for such systems. This difficulty is overcome by the technique of approximating functions using the elements of eigenspaces of the Stokes operator in such a way that the approximations are bounded and converge in both Sobolev and Lebesgue spaces simultaneously. Due to some technical difficulties, we discuss the global in time regularity results of such strong solutions in periodic domains only. Once the system is well-posed, we look for the asymptotic behavior of strong solutions. For large effective viscosity, the exponential stability results (in the mean square and pathwise sense) for stationary solutions is established. Moreover, a stabilization result of SCBF equations by using a multiplicative pure jump noise is also obtained. Finally, we prove the existence of a unique ergodic and strongly mixing invariant measure for SCBF equations subject to multiplicative pure jump noise, by using the exponential stability of strong solutions.

Keywords Convective Brinkman–Forchheimer equations · Jump noise · Strong solution · Exponential stability · Invariant measure

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1 Introduction

The global solvability of the classical 3D Navier–Stokes equations (NSE) is one of the biggest mysteries in Mathematical Physics (see [24,26,37,60,70,71], etc). Several mathematicians came forward with some modifications of this classical model and they established the well-posedness of such modified systems. The authors in [10] showed that the Cauchy problem for NSE with damping $r|u|^{r-1}u$ in the whole space has global weak solutions, for any $r \geq 1$. The existence and uniqueness of a smooth solution to a tamed 3D NSE in the whole space is established in [63]. Moreover, the authors showed that if there exists a bounded smooth solution to the classical 3D NSE, then this solution satisfies the tamed equation. The authors in [2] considered the Navier–Stokes problem in bounded domains with compact boundary, modified by the absorption term $|u|^{r-2}u$, for $r > 2$. For this modified problem, they proved the existence of weak solutions in the Leray-Hopf sense, for any dimension $d \geq 2$ and its uniqueness for $d = 2$. But in three dimensions, they were not able to establish the energy equality satisfied by the weak solutions, later this issue was resolved in [23,34], etc.

The convective Brinkman–Forchheimer (CBF) equations in bounded or periodic domains is considered in this work. The model we are going to describe is characterized on bounded domains, but one can rewrite the same model in periodic domains also (cf. [31]). Let $O \subset \mathbb{R}^d$ ($2 \leq d \leq 4$) be a bounded domain with a smooth boundary $\partial O$. The CBF equations are given by (see [34] for Brinkman–Forchheimer equations with fast growing nonlinearities)

$$
\begin{align*}
\frac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla)u + \alpha u + \beta |u|^{r-1}u + \nabla p &= f, \quad \text{in } O \times (0, T), \\
\nabla \cdot u &= 0, \quad \text{in } O \times (0, T), \\
u &= 0 \quad \text{on } \partial O \times [0, T], \\
u(0) &= u_0 \quad \text{in } O, \\
\int_O p(x, t)dx &= 0, \quad \text{in } (0, T).
\end{align*}
$$

(1.1)

The CBF equations (1.1) describe the motion of incompressible fluid flows in a saturated porous medium (see [68] for more details on physical background). The model given in (1.1) is recognized to be more accurate when the flow velocity is too large for the Darcy’s law to be valid alone, and in addition, the porosity is not too small, so that we use the term non-Darcy models for these types of flows (see [44]). In (1.1), $u(t, x) \in \mathbb{R}^d$ represents the velocity field at time $t$ and position $x$, $p(t, x) \in \mathbb{R}$ denotes the pressure field, $f(t, x) \in \mathbb{R}^d$ is an external forcing. The final condition in (1.1) is imposed for the uniqueness of the pressure $p$. The constant $\mu$ represents the positive Brinkman coefficient (effective viscosity), the positive constants $\alpha$ and $\beta$ represent the
Darcy (permeability of porous medium) and Forchheimer (proportional to the porosity of the material) coefficients, respectively. The absorption exponent \( r \in [1, \infty) \) and the case \( r = 3 \) is known as the critical exponent. It has been shown in [31, Proposition 1.1] that the critical homogeneous CBF equations have the same scaling as the NSE only when \( \alpha = 0 \) and no scale invariance property for other values of \( \alpha \) and \( r \). Note that for \( \alpha = \beta = 0 \), we obtain the classical \( d \)-dimensional NSE.

Let us now discuss some of the solvability results available in the literature for 3D CBF equations and related models in the whole space as well as periodic domains. The authors in [10] considered the Cauchy problem corresponding to (1.1) in the whole space (with \( \alpha = 0 \) and \( \beta = r \)) and showed that the system has global weak solutions, for any \( r \geq 1 \), global strong solutions, for any \( r \geq 7/2 \) and that the strong solution is unique, for any \( 7/2 \leq r \leq 5 \). The authors in [74] improved this result and they showed that the above mentioned problem possesses global strong solutions, for any \( r > 3 \) and the strong solution is unique, when \( 3 < r \leq 5 \). Later, the authors in [75] proved that the strong solution exists globally for \( r \geq 3 \), and they established two regularity criteria for \( 1 \leq r < 3 \). Moreover, for any \( r \geq 1 \), they proved that the strong solution is unique even among weak solutions. A simple proof for the existence of global-in-time smooth solutions for the CBF equations (1.1) with \( r > 3 \) on a 3D periodic domain is obtained in [31]. The authors also proved that unique global, regular solutions exist also for the critical value \( r = 3 \), provided that the coefficients satisfy the relation \( 4\beta\mu \geq 1 \). Furthermore, they showed that in the critical case every weak solution verifies the energy equality and hence is continuous into the phase space \( L^2(O) \). Recently, the authors in [32] showed that the strong solutions of the 3D CBF equations in periodic domains with the absorption exponent \( r \in [1, 3] \) remain strong under small enough changes of initial condition and forcing function.

Whereas in the case of bounded domains, as we discuss earlier, the existence of a global weak solution to the damped NSE was established in [2]. Unlike whole space or periodic domains, in bounded domains, there is a technical difficulty in obtaining strong solutions to (1.1) with the regularity given in (1.3) for the velocity field \( u(\cdot) \). It is discussed in the paper [34] that the major difficulty in working with bounded domains is that \( \mathcal{P}(|u|^{r-1}u) \) (here \( \mathcal{P} \) is the Helmholtz-Hodge projection) need not be zero on the boundary, and \( \mathcal{P} \) and \(-\Delta\) are not necessarily commuting (for a counter example, see [61, Example 2.19]). Moreover, \( \Delta u \cdot n|_{\partial O} \neq 0 \) in general and the term with pressure may not disappear while taking inner product with \( \Delta u \) to the first equation in (1.1) (see [34]). Therefore, the equality

\[
\int_O (-\Delta u(x)) \cdot |u(x)|^{r-1}u(x)dx = \int_O |\nabla u(x)|^2|u(x)|^{r-1}dx + 4 \left[ \frac{r-1}{(r+1)^2} \right] \int_O |\nabla u(x)|^{\frac{r+1}{r}}^2dx
= \int_O |\nabla u(x)|^2|u(x)|^{r-1}dx + \frac{r-1}{4} \int_O |u(x)|^{r-3}|\nabla u(x)|^2dx, \tag{1.2}
\]

may not be useful in the context of bounded domains. The authors in [34] considered the Brinkman–Forchheimer equations with fast growing nonlinearities and they
showed the existence of regular dissipative solutions and global attractors for the system \( (1.1) \) with \( r > 3 \) and \( f \in H^1(0, T; \mathbb{L}^2(\mathcal{O})) \). Note that the existence of global smooth solution assures that the energy equality is satisfied by the weak solutions in bounded domains. But the critical case of \( r = 3 \) was open until it was resolved by the authors in [23]. They were able to construct functions that can approximate functions defined on smooth bounded domains by elements of eigenspaces of linear operators (for example, the Laplacian or the Stokes operator) in such a way that the approximations are bounded and converge in both Sobolev and Lebesgue spaces simultaneously. As a simple application of this result, they proved that all weak solutions of the critical CBF equations \((r = 3)\) posed on a bounded domain in \( \mathbb{R}^3 \) satisfy the energy equality.

Let us now discuss the stochastic counterpart of the problem \( (1.1) \) and related models in the whole space or on a torus. The existence of a unique strong solution

\[
\mathbf{u} \in L^2 \left( \Omega; L^\infty \left( 0, T; \mathbb{H}^1 (\mathcal{O}) \right) \right) \cap L^2 \left( 0, T; \mathbb{H}^2 (\mathcal{O}) \right),
\]

with \( \mathbb{P} \)-a.s. continuous paths in \( C \left( [0, T]; \mathbb{H}^1 (\mathcal{O}) \right) \), for \( \mathbf{u}_0 \in L^2 \left( \Omega; \mathbb{H}^1 (\mathcal{O}) \right) \), to the stochastic tamed 3D NSE perturbed by multiplicative Gaussian noise in the whole space as well as in the periodic boundary case is obtained in [64]. They proved the existence of a unique invariant measure for the corresponding transition semigroup also. Recently, [8] improved their results for a slightly simplified system. The authors in [42] established the global existence and uniqueness of strong solutions for general stochastic nonlinear evolution equations with coefficients satisfying some local monotonicity and generalized coercivity conditions subjected to multiplicative Gaussian noise. In [43], the author showed the existence and uniqueness of strong solutions for a large class of SPDEs perturbed by multiplicative Gaussian noise, where the coefficients satisfy the local monotonicity and Lyapunov condition, and he provided the stochastic tamed 3D NSE as an example. A large deviation principle of Freidlin-Wentzell type for the stochastic tamed 3D NSE driven by multiplicative Gaussian noise in the whole space or on a torus is established in [65]. The works described above established the existence and uniqueness of strong solutions in the regularity class given in (1.3). The authors in [6] described the global solvability of 3D NSE in the whole space with a Brinkman–Forchheimer type term subject to an anisotropic viscosity and a random perturbation of multiplicative Gaussian type. In the paper [20], authors showed the existence and uniqueness of a strong solution to stochastic 3D tamed NSE driven by multiplicative Lévy noise with periodic boundary conditions, based on Galerkin’s approximation and a kind of local monotonicity of the coefficients. They also established a large deviation principle of the strong solution using a weak convergence approach.

Let us now discuss the results available in the literature for stochastic convective Brinkman–Forchheimer (SCBF) and related models in bounded domains. The authors in [66] showed the existence and uniqueness of strong solutions to stochastic 3D tamed NSE perturbed by multiplicative Gaussian noise on bounded domains with Dirichlet boundary conditions. They also proved a small time large deviation principle for the solution. The existence of a random attractor for 3D damped NSE in bounded domains with additive noise by verifying the pullback flattening property is obtained in [73].
The existence of a random attractor \((r > 3, \text{ for any } \beta > 0)\) as well as the existence of a unique invariant measure \((3 < r \leq 5, \text{ for any } \beta > 0 \text{ and } \beta \geq \frac{1}{2} \text{ for } r = 3)\) for stochastic 3D NSE with damping driven by a multiplicative Gaussian noise is established in [39]. By using classical Faedo-Galerkin approximation and compactness method, the existence of martingale solutions for stochastic 3D NSE with nonlinear damping subjected to multiplicative Gaussian noise is obtained in [40]. The exponential behavior and stabilizability of the strong solutions of stochastic 3D NSE with damping is discussed in the work [41]. A class of stochastic 3D NSE with damping driven by pure jump noise is considered in the paper [27]. They have proved the existence and uniqueness of strong solutions for the SPDEs like stochastic 3D NSE with damping, stochastic tamed 3D NSE, stochastic 3D Brinkman–Forchheimer-extended Darcy model, etc. By using the exponential stability of solutions, the existence of a unique invariant measures is proved for such models perturbed by additive jump noise.

As far as strong solutions in bounded domains are concerned, most of these works used the regularity results in the space given in (1.3), and also the estimate given in (1.2), which may not hold true always (see the discussions before (1.2)).

Recently, the author in [57] established the global existence and uniqueness of pathwise strong solutions satisfying the energy equality (Itô’s formula) for SCBF equations subjected to multiplicative Gaussian noise by exploiting a monotonicity property of the linear and nonlinear operators as well as a stochastic generalization of the Minty–Browder technique. The exponential stability results (in the mean square and pathwise sense) for the stationary solutions is also established in [57] for large effective viscosity. A stabilization result for SCBF equations by using a multiplicative Gaussian noise is also obtained. Finally, the author proved the existence of a unique ergodic and strongly mixing invariant measure for SCBF equations subjected to multiplicative Gaussian noise, by making use of the exponential stability of strong solutions. Our aim in this work is to extend these results to the case of multiplicative pure jump noise. In comparison with Gaussian noise, the major difficulty in the case of pure jump noise arises, when we establish energy equality (Itô’s formula). As the jumps occur at a countable number of times, we need to consider the times where the jump occurs separately, while establishing Itô’s formula (see Step (4) of Theorem 3.6). The next difference comes in the choice of multiplicative noise used in the stabilization result for SCBF equations.

In this work, we consider the following SCBF equations perturbed by multiplicative jump noise. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space equipped with an increasing family of sub-sigma fields \(\{\mathcal{F}_t\}_{t \geq 0}\) of \(\mathcal{F}\) satisfying the usual conditions. The SCBF equations perturbed by pure jump noise are given by

\[
\begin{aligned}
\frac{du(t)}{dt} - \mu \Delta u(t) + (u(t) \cdot \nabla) u(t) + \beta |u(t)|^{r-1} u(t) + \nabla p(t) &= \int_Z \sigma(t-, u(t-), z) \tilde{\pi}(dt, dz), \quad \text{in } \mathcal{O} \times (0, T], \\
\nabla \cdot u(t) &= 0, \quad \text{in } \mathcal{O} \times (0, T], \\
u(t) &= 0 \quad \text{on } \partial \mathcal{O} \times [0, T], \\
u(0) &= u_0 \quad \text{in } \mathcal{O},
\end{aligned}
\]  

(1.4)
Table 1  Bounded and periodic domains

| Dimension | Absorption exponent | Result |
|-----------|---------------------|--------|
| \(d = 2\) | \(r \geq 1\)       | for any \(\mu, \beta > 0\) | Theorem 3.6 |
| \(d = 3, 4\) | \(r = 3\)       | \(2\beta \mu \geq 1\) | Theorem 3.8 |
| \(d = 3, 4\) | \(r > 3\)       | for any \(\mu, \beta > 0\) | Theorem 3.6 |

where \(\tilde{\pi}(\cdot, \cdot)\) is the compensated Poisson random measure, \((Z, \mathcal{B}(Z))\) is a measurable space and \(\sigma : [0, T] \times \mathbb{L}^2(\Omega) \times Z \rightarrow \mathbb{L}^2(\Omega)\) is the noise coefficient. We show the existence and uniqueness of strong solutions to the system (1.4) in a larger space than the one given in (1.3) and discuss its asymptotic behavior. We obtained the main motivation from the works [23] and [31], for establishing an Itô’s formula in infinite dimensions for the solution process. The work [31] helped us to construct functions that can approximate functions defined on smooth bounded domains by elements of eigenspaces of the Stokes operator in such a way that the approximations are bounded and converge in both Sobolev and Lebesgue spaces simultaneously. On the \(n\)-dimensional torus, one can approximate functions in \(L^p\)-spaces using truncated Fourier expansions (see [23] for more details). Due to the difficulty explained before (see (1.2)), we may not be able to obtain the regularity of \(u(\cdot)\) given in (1.3) in bounded domains. Now, we list some of the major contributions of this work. The existence and uniqueness of strong solutions to SCBF equations perturbed by multiplicative jump noise in bounded domains with \(u_0 \in L^2(\Omega; \mathbb{L}^2(\mathcal{O}))\) is obtained in the space

\[
L^2(\Omega; L^\infty(0, T; \mathbb{L}^2(\mathcal{O}))) \cap L^2(0, T; H^0_0(\mathcal{O})) \cap L^{r+1}(\Omega; L^{r+1}(0, T; L^{r+1}(\mathcal{O}))),
\]

with \(\mathbb{P}\)-a.s. paths in \(D([0, T]; \mathbb{L}^2(\mathcal{O}))\), where \(D([0, T]; \mathbb{L}^2(\mathcal{O}))\) is the space of all càdlàg functions (right continuous functions with left limits) from \([0, T]\) to \(\mathbb{L}^2(\mathcal{O})\).

The results of this paper are valid for the cases given in Table 1.

Moreover, in the case of periodic domains, for the above mentioned cases, we also obtain further regularity results for the strong solution (Theorem 3.11). The energy equality (Itô’s formula) satisfied by the SCBF equations (1.4) driven by multiplicative jump noise is established by approximating the strong solution using the finite-dimensional space spanned by the first \(n\) eigenfunctions of the Stokes operator. The exponential stability (in the mean square and almost sure sense) of the stationary solutions is obtained for large effective viscosity \(\mu\) and the lower bound on \(\mu\) does not depend on the bounds of the stationary solutions. A stabilization result for SCBF equations by using a multiplicative jump noise is also obtained. The existence of a unique ergodic and strongly mixing invariant measure for SCBF equations perturbed by multiplicative jump noise is established by using the exponential stability of strong solutions.

The rest of the paper is organized as follows. In Sect. 2, we define the linear and nonlinear operators, and provide the necessary function spaces needed to obtain the global solvability results of the system (1.1). For \(r > 3\), we show that the sum of linear and nonlinear operators is monotone (Theorem 2.3), and for the critical case \(r = 3\) and \(2\beta \mu \geq 1\), we show that the sum is globally monotone (Theorem 2.4).
The important properties like demicontinuity and hence hemicontinuity properties of these operators is also obtained in the same section (Lemma 2.6). In Sect. 3, we first provide an abstract formulation of the SCBF equations (1.4) in bounded or periodic domains. Then, we establish the existence and uniqueness of a global pathwise strong solution by using the monotonicity property of the linear and nonlinear operators as well as a stochastic generalization of the Minty–Browder technique (see Proposition 3.4 for a-priori energy estimates satisfied by a Faedo-Galerkin approximated system and Theorem 3.6 for global solvability results). We overcome the major difficulty of establishing the energy equality (Itô’s formula) for the solution of SCBF equations by approximating the solution using the finite-dimensional space spanned by the first \(n\) eigenfunctions of the Stokes operator (cf. [23,31]). Due to the technical difficulties explained earlier, we prove the regularity results of the global strong solutions under smoothness assumptions on the initial data and further assumptions on noise coefficient in periodic domains (on torus) only (Theorem 3.11). The Sect. 4 is devoted for establishing the exponential stability (in the mean square and almost sure sense) of the stationary solutions (Theorems 4.7 and 4.8) for large effective viscosity \(\mu\). In both Theorems, the lower bound of \(\mu\) does not depend on the bounds of the stationary solutions. A stabilization result for SCBF equations by using a multiplicative jump noise is also obtained in the same section (Theorem 4.10). In the final section, we prove the existence of a unique ergodic and strongly mixing invariant measure for SCBF equations driven by multiplicative jump noise by making use of the exponential stability of strong solutions (Theorem 5.5).

2 Mathematical formulation

The necessary function spaces needed to obtain the global solvability results of the system (1.4) is provided in this section. We also discuss the monotonicity as well as hemicontinuity properties of the linear and nonlinear operators in this section. In our analysis, the Darcy parameter \(\alpha\) appearing in (1.1) does not play a major role and we set \(\alpha\) to be zero in the rest of the paper.

2.1 Function spaces

Let \(C_0^\infty(\mathcal{O}; \mathbb{R}^d)\) be the space of all infinitely differentiable functions (\(\mathbb{R}^d\)-valued) with compact support in \(\mathcal{O} \subset \mathbb{R}^d\). Let us define

\[
\mathcal{V} := \{u \in C_0^\infty(\mathcal{O}, \mathbb{R}^d) : \nabla \cdot u = 0\},
\]

\(\mathbb{H} := \) the closure of \(\mathcal{V}\) in the Lebesgue space \(L^2(\mathcal{O}) = L^2(\mathcal{O}; \mathbb{R}^d)\),

\(\mathbb{V} := \) the closure of \(\mathcal{V}\) in the Sobolev space \(H^1(\mathcal{O}) = H^1(\mathcal{O}; \mathbb{R}^d)\),

\(\widetilde{L}^p := \) the closure of \(\mathcal{V}\) in the Lebesgue space \(L^p(\mathcal{O}) = L^p(\mathcal{O}; \mathbb{R}^d)\),

for \(p \in (2, \infty)\). Then under some smoothness assumptions on the boundary (for instance, one can take \(C^2\)-boundary), we characterize the spaces \(\mathbb{H}, \mathbb{V}\) and \(\widetilde{L}^p\) as \(\mathbb{H} = \mathbb{V} = \widetilde{L}^p\).
Let us define a bilinear form \( \mathbf{u} \in L^2(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n}|_{\partial \mathcal{O}} = 0 \), with norm \( \| \mathbf{u} \|^2_H := \int_{\mathcal{O}} |\mathbf{u}(x)|^2 \, dx \), where \( \mathbf{n} \) is the unit outward normal to \( \partial \mathcal{O} \), and \( \mathbf{u} \cdot \mathbf{n}|_{\partial \mathcal{O}} \) should be understood in the sense of trace in \( H^{-1/2}(\partial \mathcal{O}) \) (cf. [70, Theorem 1.2, Chapter 1]), \( \mathcal{V} = \{ \mathbf{u} \in H^1_0(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0 \} \), with norm \( \| \mathbf{u} \|^2_V := \int_{\mathcal{O}} |\nabla \mathbf{u}(x)|^2 \, dx \), and \( \tilde{\mathcal{L}}^p = \{ \mathbf{u} \in L^p(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n}|_{\partial \mathcal{O}} = 0 \} \), with norm \( \| \mathbf{u} \|^p_{\tilde{\mathcal{L}}^p} = \int_{\mathcal{O}} |\mathbf{u}(x)|^p \, dx \), respectively. Let \( (\cdot, \cdot) \) denote the inner product in the Hilbert space \( H \) and \( \langle \cdot , \cdot \rangle \) represent the induced duality between the spaces \( \mathcal{V} \) and its dual \( \mathcal{V}' \) as well as \( \tilde{\mathcal{L}}^p \) and its dual \( \tilde{\mathcal{L}}^p' \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Remember that \( H \) can be identified with its dual \( H' \).

Note that the sum space \( L^p' + \mathcal{V}' \) is well defined (see [22, subsection 2.1]), as \( L^p' \) and \( \mathcal{V}' \) are subspaces of the topological vector space \( \mathcal{D}'(\mathcal{O}) \), where \( \mathcal{D}'(\mathcal{O}) \) is the space of distributions on \( \mathcal{O} \) with the usual topology, and the embedding \( L^p' \subset \mathcal{D}'(\mathcal{O}) \) and \( \mathcal{V}' \subset \mathcal{D}'(\mathcal{O}) \) are continuous. Furthermore, we have

\[
(L^p + \mathcal{V})' = L^p \cap \mathcal{V} \quad \text{and} \quad (L^p \cap \mathcal{V})' = L^p' + \mathcal{V}',
\]

where \( \| \mathbf{u} \|_{L^p \cap \mathcal{V}} = \max \{ \| \mathbf{u} \|_{L^p}, \| \mathbf{u} \|_{\mathcal{V}} \} \), which is equivalent to the norm \( \| \mathbf{u} \|_{L^p} + \| \mathbf{u} \|_{\mathcal{V}} \), and

\[
\| \mathbf{u} \|_{L^p' + \mathcal{V}'} = \inf \{ \| \mathbf{u}_1 \|_{L^p} + \| \mathbf{u}_2 \|_{\mathcal{V}'} : \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 \in L^p', \mathbf{u}_2 \in \mathcal{V}' \}
= \sup \left\{ \frac{|\langle \mathbf{u}_1 + \mathbf{u}_2, \mathbf{f} \rangle|}{\| \mathbf{f} \|_{L^p \cap \mathcal{V}}} : 0 \neq \mathbf{f} \in L^p \cap \mathcal{V} \right\}.
\]

Note that \( L^p \cap \mathcal{V} \) and \( L^p' + \mathcal{V}' \) are Banach spaces. Moreover, we have the continuous embedding \( \mathcal{V} \cap L^p \hookrightarrow H \hookrightarrow \mathcal{V}' + L^p' \). For the functional set up in periodic domains, interested readers are referred to see [31,53,71], etc.

### 2.2 Linear operator

Let us define a bilinear form \( a : \mathcal{V} \times \mathcal{V} \to \mathbb{R} \) by \( a(\mathbf{u}, \mathbf{v}) := \langle \nabla \mathbf{u}, \nabla \mathbf{v} \rangle \), for \( \mathbf{u}, \mathbf{v} \in \mathcal{V} \).

From the definition of \( a(\cdot, \cdot) \), it is clear that \( a(\cdot, \cdot) \) is \( \mathcal{V} \)-continuous. That is, \( |a(\mathbf{u}, \mathbf{v})| \leq C \| \mathbf{u} \|_{\mathcal{V}} \| \mathbf{v} \|_{\mathcal{V}} \), for all \( \mathbf{u}, \mathbf{v} \in \mathcal{V} \) and some \( C > 0 \). Hence, by the Riesz representation theorem, there exists a unique linear operator \( \mathcal{A} : \mathcal{V} \to \mathcal{V}' \), where \( \mathcal{V}' \) is the dual of \( \mathcal{V} \), such that

\[
a(\mathbf{u}, \mathbf{v}) = \langle \mathcal{A} \mathbf{u}, \mathbf{v} \rangle, \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathcal{V}. \tag{2.1}
\]

Moreover, the form \( a(\cdot, \cdot) \) is \( \mathcal{V} \)-coercive, that is, it satisfies \( a(\mathbf{u}, \mathbf{u}) \geq \alpha \| \mathbf{u} \|^2_{\mathcal{V}}, \) for all \( \mathbf{u} \in \mathcal{V} \) and some \( \alpha > 0 \) (here precisely \( \alpha = 1 \)). Therefore, by means of the Lax-Milgram theorem, the operator \( \mathcal{A} : \mathcal{V} \to \mathcal{V}' \) is an isomorphism. Now we define an unbounded linear operator \( \mathcal{A} \) in \( H \) as follows:

\[
\begin{aligned}
\text{D}(\mathcal{A}) : & = \{ \mathbf{u} \in \mathcal{V} : \mathcal{A} \mathbf{u} \in H \}, \\
\mathcal{A} \mathbf{u} : & = \mathcal{A} \mathbf{u}, \quad \mathbf{u} \in \text{D}(\mathcal{A}).
\end{aligned} \tag{2.2}
\]
For a given domain $\mathcal{O}$, which is open, bounded, and has $C^2$-boundary, the operator $A$ is a non-negative self-adjoint operator in $\mathbb{H}$ (cf. [72, page 56]). Also, from page 57, [72], we infer that

It is well-known from [25,35] that every vector field $u \in L^p(\mathcal{O})$, for $1 < p < \infty$ can be uniquely represented as $u = v + \nabla q$, where $v \in L^p(\mathcal{O})$ with $\text{div} \ v = 0$ in the sense of distributions in $\mathcal{O}$ with $v \cdot n = 0$ on $\partial \mathcal{O}$ and $q \in W^{1,p}(\mathcal{O})$ (Helmholtz-Weyl or Helmholtz-Hodge decomposition). For smooth vector fields in $\mathcal{O}$, such a decomposition is an orthogonal sum in $L^2(\mathcal{O})$. Note that $u = v + \nabla q$ holds for all $u \in L^p(\mathcal{O})$, so that we can define the projection operator $P_p$ by $P_p u = v$. Let us consider the set $E_p(\mathcal{O}) := \{ \nabla q : q \in W^{1,p}(\mathcal{O}) \}$ equipped with the norm $\| \nabla q \|_{L^p}$. Then, from the above discussion, we obtain $L^p(\mathcal{O}) = E_p(\mathcal{O}) \oplus E_p(\mathcal{O})$. From [67, Theorem 1.4], we further have

$$
\| \nabla q \|_{L^p} \leq C \| u \|_{L^p}, \quad \| v \|_{L^p} \leq (C + 1) \| u \|_{L^p}
$$

and

$$
\| \nabla q \|_{L^p} + \| v \|_{L^p} \leq (2C + 1) \| u \|_{L^p},
$$

where $C = C(\mathcal{O}, p) > 0$ is a constant such that

$$
\| \nabla q \|_{L^p} \leq C \sup_{0 \neq \nabla \varphi \in E_p(\mathcal{O})} \frac{|\langle \nabla q, \nabla \varphi \rangle|}{\| \nabla \varphi \|_{L^p}}, \quad \text{for all } \nabla q \in E_p(\mathcal{O}), \quad (2.3)
$$

with $\frac{1}{p} + \frac{1}{p'} = 1$. Setting $P_p u := v$, we obtain a bounded linear operator $P_p : L^p(\mathcal{O}) \to L^p(\mathcal{O})$ such that $P_p^2 = P_p$ (projection). For $p = 2$, $P_p := P_2 : L^2(\mathcal{O}) \to \mathbb{H}$ is an orthogonal projection. Since $\mathcal{O}$ is of class $C^2$, from [70, Remark 1.6], we also infer that $P$ maps $H^1(\mathcal{O})$ into itself and is continuous for the norm of $H^1(\mathcal{O})$.

Note that the space $D(A)$ given in (2.2) can be characterized using the regularity theory of linear elliptic systems (cf. [72, page 107]). Thus, we have (cf. [72, page 107])

$$
\begin{align*}
Au &:= -\Delta u, \quad u \in D(A), \\
D(A) &:= \mathbb{V} \cap H^2(\mathcal{O}),
\end{align*}
$$

and it should be noted that $\langle Au, v \rangle = (\nabla u, \nabla v)$, for all $u, v \in \mathbb{V}$. We can also show that $Au = f, \quad u \in D(A), \quad f \in \mathbb{H}$ is equivalent to say that there exists $p \in H^1(\mathcal{O})$ such that

$$
\begin{align*}
-\Delta u + \nabla p &= f \quad \text{in } \mathcal{O}, \\
\nabla \cdot u &= 0 \quad \text{in } \mathcal{O}, \\
u &= 0 \quad \text{in } \partial \mathcal{O}.
\end{align*}
$$

Furthermore, the operator $A$ is invertible and its inverse $A^{-1}$ is bounded, self-adjoint and compact in $\mathbb{H}$. Thus, using the spectral theorem, the spectrum of $A$ consists of an infinite sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$ with $\lambda_k \to \infty$ as $k \to \infty$ of eigenvalues (see [72, page 56] and [59, Chapter VII] for spectral properties of

\[ Springer \]
An integration by parts gives
\[ \| \nabla u \|_2^2 = \langle Au, u \rangle = \sum_{k=1}^{\infty} \lambda_k |(u, w_k)|^2 \geq \lambda_1 \sum_{k=1}^{\infty} |(u, w_k)|^2 = \lambda_1 \| u \|_2^2, \tag{2.4} \]
which is the Poincaré inequality.

**Remark 2.1**
1. In general, one can define \( A_p u := -\mathcal{P}_p \Delta u, \ u \in D(A_p) = \mathbb{W}^{1,p}(\Omega) \cap \mathbb{W}^{2,p}(\Omega) \) (cf. [28] for more details).
2. The following interpolation inequality is frequently in the paper. Assume \( 1 \leq s \leq r \leq t \leq \infty, \theta \in (0, 1) \) such that \( \frac{1}{t} = \frac{\theta}{s} + \frac{1-\theta}{r} \) and \( u \in \mathbb{L}^s(\Omega) \cap \mathbb{L}^t(\Omega) \), then we have
\[ \| u \|_{L^r} \leq \| u \|_{L^s}^{\theta} \| u \|_{L^t}^{1-\theta}. \tag{2.5} \]

### 2.3 Bilinear operator

Let us define the **trilinear form** \( b(\cdot, \cdot, \cdot) : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R} \) by
\[ b(u, v, w) = \int_\Omega (u(x) \cdot \nabla) v(x) \cdot w(x) dx = \sum_{i,j=1}^n \int_\Omega u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x) dx. \]

If \( u, v \) are such that the linear map \( b(u, v, \cdot) \) is continuous on \( \mathbb{V} \), the corresponding element of \( \mathbb{V}' \) is denoted by \( B(u, v) \). We also denote \( B(u) = B(u, u) = \mathcal{P}[(u \cdot \nabla)u] \).

An integration by parts gives
\[ \begin{cases} b(u, v, v) = 0, & \text{for all } u, v \in \mathbb{V}, \\ b(u, v, w) = -b(u, w, v), & \text{for all } u, v, w \in \mathbb{V}. \end{cases} \tag{2.6} \]

In the trilinear form, an application of Hölder’s inequality yields
\[ |b(u, v, w)| = |b(u, v, w)| \leq \| u \|_{L^{2r+1}} \| v \|_{L^{2(2r+1)}} \| w \|_{\mathbb{V}}, \]
for all \( u \in \mathbb{V} \cap \mathbb{L}^{r+1}, v \in \mathbb{V} \cap \mathbb{L}^{2(r+1)} \) and \( w \in \mathbb{V} \), so that we obtain
\[ \| B(u, v) \|_{\mathbb{V}'} \leq \| u \|_{L^{2r+1}} \| v \|_{L^{2(2r+1)}}. \tag{2.7} \]

Hence, the trilinear map \( b : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R} \) has a unique extension to a bounded trilinear map from \( (\mathbb{V} \cap \mathbb{L}^{r+1}) \times (\mathbb{V} \cap \mathbb{L}^{2(r+1)}) \times \mathbb{V} \) to \( \mathbb{R} \). It can also be seen that B...
maps $V \cap \widetilde{L}^{r+1}$ into $V' + \widetilde{L}^{r+1}$ and using interpolation inequality (see (2.5)), we get
\begin{align}
|\langle B(u, u), v \rangle| &= |b(u, v, u) - b(u + v, u)| \\
&\leq \|b(u, v, u)\|_{L^{2(r+1)}} \|v\|_V \\
&\leq \|u\|^{r+1}_{L^{r+1}} \|u\|^{r-3}_{L^{r+1}} \|v\|_V.
\end{align}
(2.8)
for $r > 3$ and all $v \in V \cap \widetilde{L}^{r+1}$. Thus, we deduce that
\begin{align}
\|B(u)\|_{V' + \widetilde{L}^{r+1}} \leq \|u\|^{r+1}_{L^{r+1}} \|u\|^{r-3}_{L^{r+1}}.
\end{align}
(2.9)
For $r \in [1, 3]$, using Hölder’s inequality, we have $|\langle B(u, u), v \rangle| = |\langle b(u, v, u) \rangle| \leq \|u\|^{2}_{L^{4}} \|v\|_V$, for all $v \in V$ so that
\begin{align}
\|B(u)\|_{V'} \leq \|u\|^{2}_{L^{4}}, \text{ for all } u \in \widetilde{L}^{4},
\end{align}
and hence $B(\cdot) : V \cap \widetilde{L}^{4} \to V' + \widetilde{L}^{4}$. Using (2.7), for $u, v \in V \cap \widetilde{L}^{r+1}$, we also have
\begin{align}
\|B(u) - B(v)\|_{V' + \widetilde{L}^{r+1}} &\leq \|B(u - v, u)\|_{V'} + \|B(v, u - v)\|_{V'} \\
&\leq \left( \|u\|^{2}_{L^{2(r+1)}} + \|v\|^{2}_{L^{2(r+1)}} \right) \|u - v\|_{L^{r+1}} \\
&\leq \left( \|u\|^{2}_{L^{4}} + \|v\|^{2}_{L^{4}} \right) \|u - v\|_{L^{r+1}},
\end{align}
(2.10)
for $r > 3$, by using the interpolation inequality. Thus, the map $B(\cdot) : V \cap \widetilde{L}^{r+1} \to V' + \widetilde{L}^{r+1}$ is locally Lipschitz. For $r \in [1, 3]$, a calculation similar to (2.10) yields
\begin{align}
\|B(u) - B(v)\|_{V' + \widetilde{L}^{4}} \leq \left( \|u\|_{L^{4}} + \|v\|_{L^{4}} \right) \|u - v\|_{L^{4}},
\end{align}
and hence $B(\cdot) : V \cap \widetilde{L}^{4} \to V' + \widetilde{L}^{4}$ is a locally Lipschitz operator. For more details on trilinear operator, see [70,71].

### 2.4 Nonlinear operator

Let us now consider the operator $C_r(u) := P_{r+1}(|u|^{r-1}u)$. Note that for all $u, v \in \widetilde{L}^{r+1}$, we have
\begin{align}
|\langle C_r(u), v \rangle| = |\langle P_{r+1}(|u|^{r-1}u), v \rangle| = |\langle |u|^{r-1}u, v \rangle| \leq \|u\|^{r}_{L^{r+1}} \|v\|_{L^{r+1}},
\end{align}
so that \( \|C_r(u)\|_{\tilde{L}^{r+1}} \leq \|u\|_{\tilde{L}^{r+1}} \) and \( C_r : \tilde{L}^{r+1} \to \tilde{L}^{r+1} \). Since \( \mathcal{P} = \mathcal{P}_2 \) maps \( \mathbb{H}^1(\mathcal{O}) \) into itself and is bounded ([70, Remark 1.6]), we define the map

\[
\mathcal{C}(\cdot) : \mathbb{V} \cap \tilde{L}^{r+1} \to \mathbb{V}' + \tilde{L}^{r+1} \quad \text{by} \quad \mathcal{C}(u) = \mathcal{P}(\|u|^{-1} u).
\]

It is immediate that \( \langle \mathcal{C}(u), u \rangle = \| u \|_{\tilde{L}^{r+1}}^{r+1} \), for all \( u \in \mathbb{V} \cap \tilde{L}^{r+1} \). Furthermore, for all \( u \in \mathbb{V} \cap \tilde{L}^{r+1} \), the map is Gateaux differentiable with the Gateaux derivative

\[
\mathcal{C}'(u)v = \begin{cases}
\mathcal{P}(v), & \text{for } r = 1, \\
\mathcal{P}(\|u|^{-1} v) + (r - 1) \mathcal{P} \left( \frac{u}{\|u\|^{1-r}} (u \cdot v) \right), & \text{if } u \neq 0, \\
0, & \text{if } u = 0,
\end{cases}
\]

for \( v \in \mathbb{V} \cap \tilde{L}^{r+1} \). For \( u, v \in \mathbb{V} \cap \tilde{L}^{r+1} \), it can be easily seen that

\[
\langle \mathcal{C}'(u)v, v \rangle = \int_{\mathcal{O}} |u(x)|^{-1} |v(x)|^2 dx + (r - 1) \int_{\mathcal{O}} |u(x)|^{-3} |u(x) \cdot v(x)|^2 dx \geq 0,
\]

for \( 3 \leq r < \infty \). Note that for \( 1 < r < 3 \) also the same result holds, since in that case, the second integral becomes \( \int_{\{x \in \mathcal{O} : u(x) \neq 0\}} \frac{1}{|u(x)|^{3-r}} |u(x) \cdot v(x)|^2 dx \geq 0 \). For \( u, v \in \mathbb{V} \cap \tilde{L}^{r+1} \), using Taylor’s formula ([15, Theorem 7.9.1]), we have

\[
\begin{align*}
|\mathcal{P}(\|u|^{-1} u) - \mathcal{P}(\|v|^{-1} v)| &= \mathcal{P}(\|u\|^{-1} u) - \mathcal{P}(\|v\|^{-1} v) \\
&\leq \|\mathcal{P}(\|u\|^{-1} u) - \mathcal{P}(\|v\|^{-1} v)\|_{\tilde{L}^{r+1}} \|w\|_{\tilde{L}^{r+1}} \\
&\leq C \sup_{0 \leq \theta < 1} \left[ \|\theta u + (1 - \theta) v\|_{\tilde{L}^{r+1}}^{r-1} \|u - v\|_{\tilde{L}^{r+1}}^{r-1} \\
&\quad + (r - 1) \|\theta u + (1 - \theta)v\| \|\theta u + (1 - \theta)v\|^{r-3} \times \|((\theta u + (1 - \theta)v) \cdot (u - v))\|_{\tilde{L}^{r+1}} \|w\|_{\tilde{L}^{r+1}} \\
&\quad \leq Cr \sup_{0 \leq \theta < 1} \|\theta u + (1 - \theta)v\|_{\tilde{L}^{r+1}}^{r-1} \|u - v\|_{\tilde{L}^{r+1}}^{r-1} \|w\|_{\tilde{L}^{r+1}} \\
&\quad \leq Cr \left( \|u\|_{\tilde{L}^{r+1}}^{r-1} + \|v\|_{\tilde{L}^{r+1}}^{r-1} \right)^{r-1} \|u - v\|_{\tilde{L}^{r+1}}^{r-1} \|w\|_{\tilde{L}^{r+1}}^{r-1},
\end{align*}
\]

for \( r \geq 3 \) and all \( u, v, w \in \mathbb{V} \cap \tilde{L}^{r+1} \). The case of \( 1 \leq r < 3 \) can be established in a similar way. Thus the operator \( \mathcal{C}(\cdot) : \mathbb{V} \cap \tilde{L}^{r+1} \to \mathbb{V}' + \tilde{L}^{r+1} \) is locally Lipschitz. Moreover, for any \( r \in [1, \infty) \), we have (cf. [57])

\[
\langle \mathcal{P}(\|u|^{-1} u) - \mathcal{P}(\|v|^{-1} v), u - v \rangle \geq \frac{1}{2} \|u\|_{\tilde{L}^{r+1}}^{r-1} \|u - v\|_{\tilde{L}^{r+1}}^2 + \frac{1}{2} \|v\|_{\tilde{L}^{r+1}}^{r-1} \|u - v\|_{\tilde{L}^{r+1}}^2 \geq 0,
\]

\([2.13]\)
for \( r \geq 1 \). It is important to note that
\[
\| u - v \|_{H^{r+1}}^{r+1} = \int_{\Omega} |u(x) - v(x)|^{r-1} |u(x) - v(x)|^2 \, dx \\
\leq 2^{r-2} \int_{\Omega} (|u(x)|^{r-1} + |v(x)|^{r-1}) |u(x) - v(x)|^2 \, dx \\
\leq 2^{r-2} \| u \|_{H^r}^{r-1} (u - v) \|_{H^r}^2 + 2^{r-2} \| v \|_{H^r}^{r-1} (u - v) \|_{H^r}^2,
\]
(2.14)
for \( r \geq 1 \) (replace \( 2^{r-2} \) with 1, for \( 1 \leq r \leq 2 \), so that
\[
\langle C(u) - C(v), u - v \rangle \geq \frac{1}{2^{r-1}} \| u - v \|_{\tilde{H}^{r+1}}^{r+1},
\]
(2.15)
for all \( u, v \in V \cap \tilde{L}^{r+1} \).

### 2.5 Monotonicity

Let us now discuss the monotonicity as well as the hemicontinuity properties of the linear and nonlinear operators, which plays a crucial role in this paper.

**Definition 2.2** ([5]) Let \( X \) be a Banach space and let \( X' \) be its topological dual. An operator \( G : D \to X' \), \( D = D(G) \subset X \) is said to be **monotone** if
\[
\langle G(x) - G(y), x - y \rangle \geq 0, \quad \text{for all } x, y \in D.
\]
The operator \( G(\cdot) \) is said to be **hemicontinuous**, if for all \( x, y \in D \) and \( w \in X \),
\[
\lim_{\lambda \to 0} \langle G(x + \lambda y), w \rangle = \langle G(x), w \rangle.
\]
The operator \( G(\cdot) \) is called **demicontinuous**, if for all \( x \in D \) and \( y \in X \), the functional \( x \mapsto \langle G(x), y \rangle \) is continuous, or in other words, \( x_k \to x \) in \( X \) implies \( G(x_k) \rightharpoonup G(x) \) in \( X' \). Clearly demicontinuity implies hemicontinuity.

**Theorem 2.3** ([57, Theorem 2.2]) Let \( u, v \in V \cap \tilde{L}^{r+1} \), for \( r > 3 \). Then, for the operator \( G(u) = \mu A u + B(u) + \beta C(u) \), we have
\[
\langle (G(u) - G(v), u - v) + \eta \| u - v \|_{H^r}^2 \geq 0,
\]
(2.16)
where
\[
\eta = \frac{r - 3}{2\mu(r - 1)} \left( \frac{2}{\beta \mu(r - 1)} \right)^{\frac{2}{r-3}}.
\]
(2.17)
That is, the operator \( G + \eta I \) is a monotone operator from \( V \cap \tilde{L}^{r+1} \) to \( V' + \tilde{H}^{r+1} \).
Theorem 2.4 ([57, Theorem 2.3]) For the critical case \( r = 3 \) with \( 2\beta\mu \geq 1 \), the operator \( G(\cdot) : \mathbb{V} \cap \tilde{L}^{r+1} \rightarrow \mathbb{V}' + \tilde{L}^{\frac{r+1}{r}} \) is globally monotone, that is, for all \( u, v \in \mathbb{V} \), we have

\[
\langle G(u) - G(v), u - v \rangle \geq 0. \tag{2.18}
\]

Remark 2.5 For \( d = 2 \) and \( r \in [1, 3] \), we obtain (cf. [57, Remark 2.4])

\[
\langle (G(u) - G(v), u - v) + \frac{27}{32\mu^3} N^4 \| u - v \|_{\tilde{H}}^2 \rangle \geq 0, \tag{2.19}
\]

for all \( v \in \hat{B}_N \), where \( \hat{B}_N \) is an \( \tilde{L}^4 \)-ball of radius \( N \), that is, \( \hat{B}_N := \{ z \in \tilde{L}^4 : \| z \|_{\tilde{L}^4} \leq N \} \). Thus, the operator \( G(\cdot) \) is locally monotone in this case (see [52,54], etc).

Next result proves that the operator \( G : \mathbb{V} \cap \tilde{L}^{r+1} \rightarrow \mathbb{V}' + \tilde{L}^{\frac{r+1}{r}} \) is hemicontinuous, which is useful in establishing the global solvability of the system (1.1).

Lemma 2.6 ([57, Lemma 2.5]) The operator \( G : \mathbb{V} \cap \tilde{L}^{r+1} \rightarrow \mathbb{V}' + \tilde{L}^{\frac{r+1}{r}} \) is demicontinuous.

### 3 Stochastic Navier–Stokes–Brinkman–Forchheimer equations

In this section, we establish the existence and pathwise uniqueness of strong solutions of the system (1.4). We first provide an abstract formulation of the system (1.4). On taking orthogonal projection \( P \) onto the first equation in (1.4), we get

\[
\begin{cases}
\quad du(t) + [\muAu(t) + B(u(t))]dt = \int_{Z} \gamma(t- \cdot, u(t- \cdot), z)\pi(dt, dz), \quad t \in (0, T], \\
\quad u(0) = u_0,
\end{cases}
\]

where \( u_0 \in L^2(\Omega; \mathbb{H}) \) and \( \gamma = P\sigma \).

#### 3.1 Stochastic setting

Let \((Z, \mathcal{B}(Z))\) be a measurable space and let \( \lambda \) be a \( \sigma \)-finite positive measure on it. Let \( \pi : \Omega \times \mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(Z) \rightarrow \mathbb{N} \cup \{0\} \) be a time homogeneous Poisson random measure with intensity measure \( \lambda \) defined over the probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). The intensity measure \( \lambda(\cdot) \) on \( Z \) satisfies the conditions \( \lambda(\{0\}) = 0 \) and

\[
\int_{Z} \left( 1 \wedge |z|^2 \right) \lambda(dz) < +\infty.
\]
We denote $\widetilde{\pi} = \pi - \nu$ as the compensated Poisson random measure associated to $\lambda$, where the compensator $\nu$ is given by $\mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(\mathbb{Z}) \ni (I, A) \mapsto \nu(I, A) = \text{d}(I)\lambda(A) \in \mathbb{R}^+$, where $\text{d}(\cdot)$ is the Lebesgue measure.

Let us denote by $D([0, T]; \mathbb{H})$, the space of all $\mathbb{H}$-valued functions defined on $[0, T]$, which are right continuous and have left limits (càdlàg functions) for every $t \in [0, T]$. Also, let

$$\begin{align*}
L^2 \left( \Omega; L^2 \left( 0, T; L^2_\lambda (Z; \mathbb{H}) \right) \right)
:= L^2 \left( \Omega \times (0, T] \times Z, \mathcal{F} \times \mathcal{B}((0, T]) \times \mathcal{B}(Z), \mathbb{P} \otimes dt \otimes \lambda; \mathbb{H} \right),
\end{align*}$$

be the space of all $\mathcal{F} \times \mathcal{B}((0, T]) \times \mathcal{B}(Z)$ measurable functions $\gamma : [0, T] \times \Omega \times Z \to \mathbb{H}$ such that

$$\mathbb{E} \left[ \int_0^T \int_Z \| \gamma(t, \cdot, z) \|^2_H \lambda(dz)dt \right] < +\infty.$$

Let $\gamma : [0, T] \times \mathbb{H} \times Z \to \mathbb{H}$ be a measurable and $\mathcal{F}_t$-adapted process satisfying

$$\mathbb{E} \left[ \left\| \int_0^T \int_Z \gamma(t-, u(t-), z) \widetilde{\pi}(dt, dz) \right\|^2_H \right] < +\infty,$$

for all $u \in \mathbb{H}$. The integral $M(t) := \int_0^t \int_Z \gamma(s-, u(s-), z) \widetilde{\pi}(ds, dz)$ is an $\mathbb{H}$-valued martingale and there exists an increasing càdlàg process so-called quadratic variation process $[M]_t$ and Meyer process $\langle M \rangle_t$ such that $[M]_t - \langle M \rangle_t$ is a local martingale (see [36, Section 1.6] and [51, Section 2.2]). For the process $M(\cdot)$, it can be shown that $[M]_t = \int_0^t \| \gamma(s, u(s), z) \|^2_H \lambda(dz)ds$ and $\langle M \rangle_t = \int_0^t \| \gamma(s, u(s), z) \|^2_H \lambda(dz)ds$ ([49, Example 2.8]). Indeed, $\mathbb{E}\|M(t)\|^2_H = \mathbb{E}[M]_t = \mathbb{E} \langle M \rangle_t$, so that we get

$$\mathbb{E} \left[ \int_0^t \int_Z \| \gamma(s, u(s), z) \|^2_H \lambda(dz)ds \right] = \mathbb{E} \left[ \int_0^t \int_Z \| \gamma(s, u(s), z) \|^2_H \lambda(dz)ds \right],
\quad (3.3)$$

for all $t \in [0, T]$. Moreover, the following Itô isometry holds:

$$\mathbb{E} \left[ \left\| \int_0^T \int_Z \gamma(t-, u(t-), z) \widetilde{\pi}(dt, dz) \right\|^2_H \right] = \mathbb{E} \left[ \int_0^T \int_Z \| \gamma(t, u(t), z) \|^2_H \lambda(dz)dt \right],$$

for all $u \in \mathbb{H}$. For more details on jump processes, one may refer to [4,46,58], etc.

**Hypothesis 3.1** The noise coefficient $\gamma(\cdot, \cdot)$ satisfies:

(H.1) The function $\gamma \in L^2(\Omega; L^2(0, T; L^2_\lambda (Z; \mathbb{H})))$;
(H.2) (Growth condition) There exists a positive constant $K$ such that for all $t \in [0, T]$ and $u \in H$,

$$\int_{Z} \|\gamma(t, u, z)\|_{H}^{2} \lambda(dz) \leq K \left(1 + \|u\|_{H}^{2}\right);$$

\[ (H.3) \] (Lipschitz condition) There exists a positive constant $L$ such that for any $t \in [0, T]$ and all $u_1, u_2 \in H$,

$$\int_{Z} \|\gamma(t, u_1, z) - \gamma(t, u_2, z)\|_{H}^{2} \lambda(dz) \leq L \|u_1 - u_2\|_{H}^{2};$$

\[ (H.4) \] We fix the measurable subsets $Z_m$ of $Z$ with $Z_m \uparrow Z$ and $\lambda(Z_m) < +\infty$ such that for any $t \in [0, T]$,

$$\sup_{\|u\|_{H} \leq M} \int_{Z_m} \|\gamma(t, u, z)\|_{H}^{2} \lambda(dz) \to 0, \text{ as } m \to \infty, \ u \in H, \ M > 0.$$ 

### 3.2 Global strong solution

Let us now provide the definition of unique global strong solutions to the system (3.1).

**Definition 3.2** (Global strong solution) Let $u_0 \in L^2(\Omega; H)$ be given. An $H$-valued $(\mathcal{F}_t)_{t \geq 0}$-adapted stochastic process $u(\cdot)$ is called a strong solution to the system (3.1) if the following conditions are satisfied:

(i) the process

$$u \in L^2 \left(\Omega; L^\infty(0, T; H) \cap L^2(0, T; V)\right) \cap L^{r+1}(\Omega; L^{r+1}(0, T; \tilde{L}^{r+1}))$$

and $u(\cdot)$ has a $V \cap \tilde{L}^{r+1}$-valued modification, which is progressively measurable with càdlàg paths in $H$ and $u \in D([0, T]; H) \cap L^2(0, T; V) \cap L^{r+1}(0, T; \tilde{L}^{r+1})$, $\mathbb{P}$-a.s.,

(ii) the following equality holds for every $t \in [0, T]$, as an element of $V' + \tilde{L}^{r+1}$, $\mathbb{P}$-a.s.

$$u(t) = u_0 - \int_{0}^{t} \left[\mu A u(s) + B(u(s)) + \beta C(u(s))\right] ds$$

$$+ \int_{0}^{t} \int_{Z} \gamma(s -, u(s -), z) \pi(dz, dz), \quad (3.4)$$

(iii) the following Itô formula holds true:

$$\|u(t)\|_{H}^{2} + 2\mu \int_{0}^{t} \|u(s)\|_{V}^{2} ds + 2\beta \int_{0}^{t} \|u(s)\|_{\tilde{L}^{r+1}}^{r+1} ds.$$
\[
\|u_0\|_{\mathcal{H}}^2 + \int_0^t \|\gamma(s, u(s), z)\|_{\mathcal{H}}^2 \pi(ds, dz) \\
+ 2 \int_0^t \int_\Omega (\gamma(s, u(s), z), u(s)) \tilde{\pi}(ds, dz),
\]
for all \(t \in (0, T), \mathbb{P}\)-a.s.

An alternative version of (3.4) is to require that for any \(v \in \mathcal{V} \cap \tilde{\mathcal{H}}^{r+1}\):

\[
(u(t), v) = (u_0, v) - \int_0^t \langle \mu A(u(t)) + B(u(s)) + \beta C(u(s)), v \rangle ds \\
+ \int_0^t \int_\Omega (\gamma(s, u(s), z), v) \tilde{\pi}(ds, dz), \quad \mathbb{P}\text{-a.s.} (3.6)
\]

**Definition 3.3** A strong solution \(u(\cdot)\) to the system (3.1) is called a **pathwise unique strong solution** if \(\tilde{u}(\cdot)\) is an another strong solution, then

\[
\mathbb{P}\left\{ \omega \in \Omega : u(t) = \tilde{u}(t), \text{ for all } t \in [0, T] \right\} = 1.
\]

### 3.3 Energy estimates

In this subsection, we formulate a finite dimensional system and establish some a-priori energy estimates. Let \(\{w_1, \ldots, w_n, \ldots\}\) be a complete orthonormal system in \(\mathcal{H}\) belonging to \(\mathcal{V}\) and let \(\mathcal{H}_n\) be the span \(\{w_1, \ldots, w_n\}\). Let \(P_n\) denote the orthogonal projection of \(\mathcal{V}'\) to \(\mathcal{H}_n\), that is, \(P_n x = \sum_{i=1}^n \langle x, w_i \rangle w_i\). Since every element \(x \in \mathcal{H}\) induces a functional \(x^* \in \mathcal{V}'\) by the formula \((x^*, y) = (x, y), y \in \mathcal{V}\), then \(P_n|_{\mathcal{H}_n}\), the orthogonal projection of \(\mathcal{H}\) onto \(\mathcal{H}_n\) is given by \(P_n x = \sum_{i=1}^n \langle x, w_i \rangle w_i\). Hence in particular, \(P_n\) is the orthogonal projection from \(\mathcal{H}\) onto \(\text{span} \{w_1, \ldots, w_n\}\). We define \(B^n(u^n) = P_n B(u^n), C^n(u^n) = P_n C(u^n)\) and \(\gamma^n(t, u^n, \cdot) = P_n \gamma(t, u^n, \cdot)\). We consider the following system of ODEs:

\[
\begin{align*}
&d \left( u^n(t), v \right) = -\langle \mu A u^n(t) + B^n(u^n(t)) + \beta C^n(u^n(t)), v \rangle dt \\
&\quad + \int_{\mathcal{Z}_n} (\gamma^n(t, u^n(t), z), v) \tilde{\pi}(dt, dz),
\end{align*}
\]

with \(u^n_0 = P_n u_0\), for all \(v \in \mathcal{H}_n\). Since \(B^n(\cdot)\) and \(C^n(\cdot)\) are locally Lipschitz (see (2.10) and (2.12)), and \(\gamma^n(\cdot, u^n, \cdot)\) is globally Lipschitz, the system (3.7) has a unique \(\mathcal{H}_n\)-valued local strong solution \(u^n(\cdot)\) and \(u^n \in L^2(\Omega; L^\infty(0, T^*; \mathcal{H}_n))\) with \(\tilde{\mathcal{P}}\)-adapted càdlàg sample paths ([4, Theorem 6.2.3]). Now we discuss the a-priori energy estimates satisfied by the solution to the system (3.7) and show that \(T^*\) can be extended to \(T\). Note that the energy estimates established in the next proposition are true for \(r \geq 1\).
Proposition 3.4 (Energy estimates) Under Hypothesis 3.1, let $u^n(\cdot)$ be the unique solution of the system of stochastic ODEs (3.7) with $u_0 \in L^2(\Omega; \mathbb{H})$. Then, we have

$$
\mathbb{E} \left[ \sup_{t \in [0,T]} \| u^n(t) \|_{\mathbb{H}}^2 + 4\mu \int_0^T \| u^n(t) \|_{\mathbb{H}}^2 \, dt + 4\beta \int_0^T \| u^n(t) \|_{\mathbb{H}}^{r+1} \, dt \right]
\leq \left( 2\mathbb{E} \left[ \| u_0 \|_{\mathbb{H}}^2 \right] + 14KT \right) e^{28KT}.
$$

(3.8)

Proof Step (1): Let us first define a sequence of stopping times $\tau^n_N$ by

$$
\tau^n_N := \inf_{t \geq 0} \left\{ t : \| u^n(t) \|_{\mathbb{H}} > N \right\},
$$

for $N \in \mathbb{N}$. Applying the finite dimensional Itô formula to the process $\| u^n(\cdot) \|_{\mathbb{H}}^2$, we obtain

$$
\| u^n(t \wedge \tau^n_N) \|_{\mathbb{H}}^2 = \| u^n(0) \|_{\mathbb{H}}^2 - 2 \int_0^{t \wedge \tau^n_N} (\mu A u^n(s) + B^n(u^n(s))
\hspace{1cm} + \beta C^n(u^n(s)), u^n(s)) \, ds
\hspace{1cm} + \int_0^{t \wedge \tau^n_N} \int_{\mathbb{H}} \left\| \gamma^n(s, u^n(s), z) \right\|_{\mathbb{H}}^2 \pi(ds, dz)
\hspace{1cm} + 2 \int_0^{t \wedge \tau^n_N} \int_{\mathbb{H}} \left( \gamma^n(s-, u^n(s-), z), u^n(s-) \right) \widetilde{\pi}(ds, dz),
$$

(3.10)

where we have used $\langle B^n(u^n), u^n \rangle = \langle B(u^n), u^n \rangle = 0$. Note that $\| u^n(0) \|_{\mathbb{H}} \leq \| u_0 \|_{\mathbb{H}}$ and the term

$$
\int_0^{t \wedge \tau^n_N} \int_{\mathbb{H}} 2 \left( \gamma^n(s-, u^n(s-), z), u^n(s-) \right) \widetilde{\pi}(ds, dz)
$$

is a martingale with zero expectation. Moreover, we know that ([49], see (3.3) also)

$$
\mathbb{E} \left[ \int_0^{t \wedge \tau^n_N} \int_{\mathbb{H}} \left\| \gamma^n(s, u^n(s), z) \right\|_{\mathbb{H}}^2 \pi(ds, dz) \right] = \mathbb{E} \left[ \int_0^{t \wedge \tau^n_N} \int_{\mathbb{H}} \left\| \gamma^n(s, u^n(s), z) \right\|_{\mathbb{H}}^2 \lambda(dz)ds \right],
$$

(3.11)

where $[M]_t$ and $\langle M \rangle_t$, respectively are the quadratic variation process and Meyer process of $M_t = \int_0^t \int_{\mathbb{H}} \gamma^n(s-, u^n(s-), z) \widetilde{\pi}(dz, ds)$. Thus, taking expectation in (3.10),
we get
\[
\mathbb{E}
\left[
\|u^n(t \wedge \tau^n_N)\|_{H^1}^2 + 2\mu \int_0^{t \wedge \tau^n_N} \|u^n(s)\|_{V^*}^2 \, ds + 2\beta \int_0^{t \wedge \tau^n_N} \|u^n(s)\|_{F_{t+1}}^2 \, ds
\right]
\leq \mathbb{E}
\left[
\|u^n(0)\|_{H^1}^2
\right] + \mathbb{E}
\left[
\mu \int_0^{t \wedge \tau^n_N} \|\gamma^n(s, u^n(s), z)\|_{H^1}^2 \, \lambda(dz) \, ds
\right]
\leq \mathbb{E}
\left[
\|u_0\|_{H^1}^2
\right] + K \mathbb{E}
\left[
\int_0^{t \wedge \tau^n_N} \left(1 + \|u^n(s)\|_{H^1}^2\right) \, ds
\right],
\tag{3.12}
\]
where we have used Hypothesis 3.1 (H.2) and (H.4). Applying Gronwall's inequality in (3.12), we find
\[
\mathbb{E}
\left[
\|u^n(t \wedge \tau^n_N)\|_{H^1}^2
\right] \leq \left(\mathbb{E}
\left[
\|u_0\|_{H^1}^2
\right] + K T\right) e^{K T},
\tag{3.13}
\]
for all \(t \in [0, T]\). Note that for the indicator function \(\chi\),
\[
\mathbb{E}
\left[
\chi_{\{\tau^n_N < t\}}
\right] = \mathbb{P}\{\omega \in \Omega : \tau^n_N(\omega) < t\},
\]
and using (3.9), we obtain
\[
\mathbb{E}
\left[
\|u^n(t \wedge \tau^n_N)\|_{H^1}^2
\right] = \mathbb{E}
\left[
\|u^n(\tau^n_N)\|_{H^1}^2 \chi_{\{\tau^n_N < t\}}
\right] + \mathbb{E}
\left[
\|u^n(t)\|_{H^1}^2 \chi_{\{\tau^n_N \geq t\}}
\right]
\geq \mathbb{E}
\left[
\|u^n(\tau^n_N)\|_{H^1}^2 \chi_{\{\tau^n_N < t\}}
\right] \geq N^2 \mathbb{P}\{\omega \in \Omega : \tau^n_N < t\}. \tag{3.14}
\]
Using the energy estimate (3.13), we deduce that
\[
\mathbb{P}\{\omega \in \Omega : \tau^n_N < t\} \leq \frac{1}{N^2} \mathbb{E}
\left[
\|u^n(t \wedge \tau^n_N)\|_{H^1}^2
\right] \leq \frac{1}{N^2} \left(\mathbb{E}
\left[
\|u_0\|_{H^1}^2
\right] + K T\right) e^{K T}.
\tag{3.15}
\]
Hence, we have
\[
\lim_{N \to \infty} \mathbb{P}\{\omega \in \Omega : \tau^n_N < t\} = 0, \quad \text{for all } t \in [0, T],
\tag{3.16}
\]
and \(t \wedge \tau^n_N \to t\) as \(N \to \infty\). Taking limit as \(N \to \infty\) in (3.13) and using the monotone convergence theorem, we get
\[
\mathbb{E}
\left[
\|u^n(t)\|_{H^1}^2
\right] \leq \left(\mathbb{E}
\left[
\|u_0\|_{H^1}^2
\right] + K T\right) e^{K T},
\tag{3.17}
\]
for \(0 \leq t \leq T\). Substituting (3.17) in (3.12), we arrive at
\[
\mathbb{E}
\left[
\|u^n(t)\|_{H^1}^2 + 2\mu \int_0^t \|u^n(s)\|_{V^*}^2 \, ds + 2\beta \int_0^t \|u^n(s)\|_{F_{t+1}}^2 \, ds
\right]
\]
Young’s inequalities to deduce that

\[ \leq \left( \mathbb{E} \left[ \| u_0 \|_{H^1}^2 \right] + KT \right) e^{2KT}, \quad (3.18) \]

for all \( t \in [0, T] \).

**Step (2):** Let us now prove (3.8). Taking the supremum from 0 to \( T \wedge \tau_N \) before taking expectation in (3.10), we obtain

\[
\mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_N^n]} \| u^n(t) \|_{H^1}^2 + 2\mu \int_0^{T \wedge \tau_N^n} \| u^n(t) \|_{V'}^2 dt + 2\beta \int_0^{T \wedge \tau_N^n} \| u''(t) \|_{L^{r+1}}^2 dt \right] \\
\leq \mathbb{E} \left[ \| u_0 \|_{H^1}^2 \right] + \mathbb{E} \left[ \int_0^{T \wedge \tau_N^n} \int_{Z_n} \| \gamma^n(t, u^n(t), z) \|_{H^1}^2 \lambda(dz) dt \right] \\
+ 2\mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_N^n]} \int_0^t \int_{Z_n} \left( \| \gamma^n(s, u^n(s), z) \|_{H^1}^2 + \| u^n(s) \|_{H^1}^2 \right) \tilde{\pi}(ds, dz) \right]. \quad (3.19)
\]

Now we take the final term from the right hand side of the inequality (3.19) and use Burkholder–Davis–Gundy’s ((33, Theorem 1), (45, Theorem 1.1)), Hölder’s and Young’s inequalities to deduce that

\[
2\mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_N^n]} \int_0^t \int_{Z_n} \left( \| \gamma^n(s, u^n(s), z) \|_{H^1}^2 + \| u^n(s) \|_{H^1}^2 \right) \tilde{\pi}(ds, dz) \right] \\
\leq 2\sqrt{3} \mathbb{E} \left[ \int_0^{T \wedge \tau_N^n} \int_{Z_n} \| \gamma^n(t, u^n(t), z) \|_{H^1}^2 \| u^n(t) \|_{H^1}^2 \pi(dt, dz) \right]^{1/2} \\
\leq 2\sqrt{3} \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_N^n]} \| u^n(t) \|_{H^1} \left( \int_0^{T \wedge \tau_N^n} \int_{Z_n} \| \gamma^n(t, u^n(t), z) \|_{H^1}^2 \pi(dt, dz) \right)^{1/2} \right] \\
\leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_N^n]} \| u^n(t) \|_{H^1}^2 \right] + 6\mathbb{E} \left[ \int_0^{T \wedge \tau_N^n} \int_{Z_n} \| \gamma^n(u^n(t), z) \|_{H^1}^2 \lambda(dz) dt \right]. \quad (3.20)
\]

Substituting (3.20) in (3.19), we find

\[
\mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_N^n]} \| u^n(t) \|_{H^1}^2 + 4\mu \int_0^{T \wedge \tau_N^n} \| u^n(t) \|_{V'}^2 dt + 4\beta \int_0^{T \wedge \tau_N^n} \| u''(t) \|_{L^{r+1}}^2 dt \right] \\
\leq 2 \mathbb{E} \left[ \| u_0 \|_{H^1}^2 \right] + 14 \mathbb{E} \left[ \int_0^{T \wedge \tau_N^n} \int_{Z_n} \| \gamma^n(t, u^n(t), z) \|_{H^1}^2 \lambda(dz) dt \right] \\
\leq 2 \mathbb{E} \left[ \| u_0 \|_{H^1}^2 \right] + 14K \int_0^{T \wedge \tau_N^n} \mathbb{E} \left[ 1 + \| u^n(t) \|_{H^1}^2 \right] dt, \quad (3.21)
\]
where we have used Hypothesis 3.1 (H.2). Applying Gronwall’s inequality in (3.21), we obtain
\[
E \left[ \sup_{t \in [0, T \wedge \tau_n^N]} \| u^n(t) \|_{H}^2 \right] \leq \left( 2 \| u_0 \|_{H}^2 + 14K T \right) e^{14K T}.
\] (3.22)
Passing \( N \to \infty \), using the monotone convergence theorem and then substituting (3.22) in (3.21), we finally obtain (3.8).

**Lemma 3.5** For \( r > 3 \) and any functions \( u, v \in L^2(\Omega; L^\infty(0, T; H) \cap L^2(0, T; V)) \cap L^{r+1}(\Omega; L^{r+1}(0, T; \tilde{L}^{r+1})) \), we have
\[
\int_0^T e^{-\eta t} \left[ \mu (A(u(t) - v(t)), u(t) - v(t)) + \langle B(u(t)) - B(v(t)), u(t) - v(t) \rangle 
+ \beta (C(u(t)) - C(v(t)), u(t) - v(t)) \right] dt + \left( \eta + \frac{L}{2} \right) \int_0^T e^{-\eta t} \| u(t) - v(t) \|_{H}^2 dt 
\geq \frac{1}{2} \int_0^T e^{-\eta t} \int_Z \| \gamma(t, u(t), z) - \gamma(t, v(t), z) \|_{H}^2 \lambda(dz) dt,
\] (3.23)
where \( \eta \) is defined in (2.17).

**Proof** Multiplying (2.16) with \( e^{-\eta t} \), integrating over time \( t \in (0, T) \), and then using Hypothesis 3.1 (H.3), we obtain (3.23).

**3.4 Existence and uniqueness of strong solutions**

Let us now show that the system (3.1) has a unique global strong solution in the sense of Definition 3.2. We exploit the monotonicity property (see (3.23)) and a stochastic generalization of the Minty–Browder technique to obtain such a result. This method is widely used to prove the existence of global strong solutions to the stochastic partial differential equations. The local monotonicity property of the linear and nonlinear operators and a stochastic generalization of the Minty–Browder technique has been used to obtain global solvability results of various mathematical physics models perturbed by Gaussian or Lévy noise, see for instance [7,14,47,52,54,55,69], etc and references therein.

**Theorem 3.6** For \( 2 \leq d \leq 4 \), let \( \mathcal{O} \subset \mathbb{R}^d \) be a bounded domain with \( C^2 \)-boundary. Let \( u_0 \in L^2(\Omega; H) \) and \( r > 3 \) be given. Then under Hypothesis 3.1, there exists a pathwise unique strong solution \( u(\cdot) \) to the system (3.1) such that
\[
u \in L^2 \left( \Omega; L^\infty(0, T; H) \cap L^2(0, T; V) \right) \cap L^{r+1}(\Omega; L^{r+1}(0, T; \tilde{L}^{r+1})),
\]
with \( \mathbb{P}\text{-a.s., càdlàg trajectories in } \mathbb{H} \) and \( u \in D([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \cap L^{r+1}(0, T; \mathbb{V}) \), \( \mathbb{P}\text{-a.s.} \).

**Proof:** We prove the global solvability of the system (3.1) in the following steps.

**Step (1):** Finite-dimensional (Galerkin) approximation of the system (3.1): Let us first consider the following Galerkin approximated Itô stochastic differential equation satisfied by \( \{u^n(\cdot)\} \):

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\mathrm{d}u^n(t) = -G(u^n(t))\mathrm{d}t + \int_{Z_n} (\gamma^n(t-, u^n(t-), z), v) \tilde{\pi}(\mathrm{d}t, \mathrm{d}z), \\
u^n(0) = u^n_0,
\end{array} \right.
\end{aligned}
\tag{3.24}
\]

where \( G(u^n(\cdot)) = \mu A u^n(\cdot) + B^n(u^n(\cdot)) + \beta C^n(u^n(\cdot)) \). Let us first apply finite-dimensional Itô’s formula to the process \( e^{-2nt} \|u^n(\cdot)\|_{\mathbb{H}}^2 \) to get

\[
e^{-2nt} \|u^n(t)\|_{\mathbb{H}}^2 = \|u^n(0)\|_{\mathbb{H}}^2 - \int_0^t e^{-2ns} \left\{ 2G(u^n(s)) + 2\eta u^n(s), u^n(s) \right\} \mathrm{d}s \\
+ \int_0^t \int_{Z_n} e^{-2ns} \|\gamma^n(s, u^n(s), z)\|_{\mathbb{H}}^2 \pi(\mathrm{d}s, \mathrm{d}z) \\
+ 2 \int_0^t \int_{Z_n} e^{-2ns} \left( \gamma^n(s-, u^n(s-), z), u^n(s-) \right) \tilde{\pi}(\mathrm{d}s, \mathrm{d}z),
\tag{3.25}
\]

\( \mathbb{P}\text{-a.s.}, \text{for all } t \in [0, T] \). Note that the final term from the right hand side of the equality (3.25) is a martingale and the fifth term satisfies (3.11). Taking expectation, we get

\[
\mathbb{E}\left[ e^{-2nt} \|u^n(t)\|_{\mathbb{H}}^2 \right] = \mathbb{E}\left[ \|u^n(0)\|_{\mathbb{H}}^2 \right] \\
- \mathbb{E}\left[ \int_0^t e^{-2ns} \left\{ 2G(u^n(s)) + 2\eta u^n(s), u^n(s) \right\} \mathrm{d}s \right] \\
+ \mathbb{E}\left[ \int_0^t e^{-2ns} \int_{Z_n} \|\gamma^n(s, u^n(s), z)\|_{\mathbb{H}}^2 \pi(\mathrm{d}s, \mathrm{d}z) \right],
\tag{3.26}
\]

for all \( t \in [0, T] \).

**Step (2):** Weak convergence of the sequences \( u^n(\cdot), G(u^n(\cdot)), \) and \( \gamma^n(\cdot, \cdot, \cdot) \). Our aim is to extract subsequences from the uniformly bounded (independent of \( n \)) energy estimate (3.8) in Proposition 3.4. We know that \( L^2(\Omega; L^\infty(0, T; \mathbb{H})) \) is the dual of \( L^2(\Omega; L^1(0, T; \mathbb{H})) \) and the space \( L^2(\Omega; L^1(0, T; \mathbb{V})) \) is separable. Moreover, the spaces \( L^2(\Omega; L^2(0, T; \mathbb{V})) \) and \( L^{r+1}(\Omega; L^{r+1}(0, T; \mathbb{V})) \) are reflexive (that is, \( X'' = X \)). Thus, we are in a position to apply the Banach-Alaoglu theorem. From the energy estimate (3.8) given in Proposition 3.4, we know that the sequence \( \{u^n(\cdot)\} \) is bounded independent of \( n \) in the spaces \( L^2(\Omega; L^\infty(0, T; \mathbb{H})) \), \( L^2(\Omega; L^2(0, T; \mathbb{V})) \) and \( L^{r+1}(\Omega; L^{r+1}(0, T; \mathbb{V})) \). Then applying the Banach-Alaoglu theorem, we can extract a subsequence \( \{u^{n_k}\} \) of \( \{u^n\} \) such that (for simplicity, we denote the index \( n_k \) by \( n \)):
\[
\begin{align*}
\{ u^n \to u \} & \quad \text{in } L^2(\Omega; L^\infty(0, T; H)), \\
\{ u^n \to u \} & \quad \text{in } L^2(\Omega; L^2(0, T; V)), \\
\{ u^n \to u \} & \quad \text{in } L^{r+1}(\Omega; L^{r+1}(0, T; \tilde{H}^{r+1})), \\
\{ u^n(T) \to \xi \} & \quad \text{in } L^2(\Omega; H), \quad \text{(3.27)} \\
G(u^n) & \to G_0 \quad \text{in } L^2(\Omega; L^2(0, T; V')) + L^{r+1}(\Omega; L^{r+1}(0, T; \tilde{H}^{r+1})).
\end{align*}
\]

Using Hölder’s inequality, interpolation inequality (see (2.5)) and Proposition 3.4, one can justify the final convergence in (3.27) (see [57, Theorem 3.7] for more details). Using Hypothesis 3.1 (H.2) and the energy estimates in Proposition 3.4, we also have

\[
\begin{align*}
\mathbb{E} \left[ \int_{Z^n} \| \gamma^n(s, u^n(s), z) \|_{H^3}^2 \lambda(dz) dt \right] & \leq K \mathbb{E} \left[ \int_0^T \left( 1 + \| u^n(t) \|_{H^3}^2 \right) dt \right] \\
& \leq KT \left( 1 + \left( 2 \mathbb{E} \left[ \| u_0 \|_{H^3}^2 \right] + 24KT \right) e^{48KT} \right) < +\infty. \quad \text{(3.28)}
\end{align*}
\]

Note that the right hand side of the estimate (3.28) is independent of \( n \) and thus we can extract a subsequence \( \{ \gamma^{n_k}(\cdot, u^{n_k}(\cdot), \cdot) \} \) (re-labeled as \( \{ \gamma^n(\cdot, u^n, \cdot) \} \) of \( \{ \gamma^n(\cdot, u^n, \cdot) \} \) such that

\[
\gamma^n(\cdot, u^n, \cdot) \wto \gamma(\cdot, \cdot) \quad \text{in } L^2(\Omega; L^2(0, T; L^2(\Omega; H))), \quad \text{as } n \to \infty. \quad \text{(3.29)}
\]

**Step (3): Itô stochastic differential satisfied by \( u(\cdot) \).** Due to technical reasons, we extend the time interval from \([0, T]\) to an open interval \((-\varepsilon, T + \varepsilon)\) with \( \varepsilon > 0 \), and set the terms in the Eq. (3.24) equal to zero outside the interval \([0, T]\). Let \( \phi \in H^1(-\mu, T + \mu) \) be such that \( \phi(0) = 1 \). For \( v_m \in V \cap \tilde{H}^{r+1} \), we define \( v_m(t) = \phi(t) v_m \).

Let us apply finite dimensional Itô’s formula to the process \( (u^n(t), v_m(t)) \) to get

\[
(u^n(T), v_m(T)) = (u^n(0), v_m) + \int_0^T (u^n(t), \dot{v}_m(t)) dt - \int_0^T (G(u^n(t)), v_m(t)) dt + \int_0^T \int_{Z^n} \left( \gamma^n(t- , u^n(t-), z), v_m(t) \right) \tilde{\pi}(dt, dz),
\]

\( \Pr \)-a.s., where \( \dot{v}_m(t) = \frac{d\phi(t)}{dt} v_m \). One can take the term by term limit as \( n \to \infty \) in (3.30) by making use of the weak convergences given in (3.27) and (3.29).

We consider the stochastic integral present in the final term from the right hand side of the equality (3.30) with \( m \) fixed. Let \( \mathcal{P}_T \) denote the class of predictable processes with values in \( L^2(\Omega; L^2(0, T; L^2(\Omega; \mathbb{H}))) \) (see (3.2) for definition and [46, Chapter 3]) associated with the inner product

\[
(\gamma, \xi)_{\mathcal{P}_T} = \mathbb{E} \left[ \int_0^T \int_{Z^n} (\gamma(s, z), \xi(s, z)) \lambda(dz) ds \right], \quad \text{for all } \gamma, \xi \in \mathcal{P}_T \text{ and } t \in [0, T].
\]
Moreover, we have
\[
\mathbb{E} \left[ \int_0^t \left( \gamma^n(s, u^n(s), z) - \gamma(s, z) \right) \lambda(dz) ds \right] - \mathbb{E} \left[ \int_0^t \left( \gamma^n(s, u^n(s), z) - \gamma(s, z) \right) \lambda(dz) ds \right]
\]
\[
\leq \mathbb{E} \left[ \int_0^t \left( \gamma^n(s, u^n(s), z) - \gamma(s, z) \right) \lambda(dz) ds \right] + \mathbb{E} \left[ \int_0^t \left( \gamma(s, z) \right) \lambda(dz) ds \right].
\] (3.31)

The weak convergence of \( \gamma^n(\cdot, u^n, \cdot) \rightharpoonup \gamma(\cdot, \cdot) \) in \( L^2(\Omega; L^2(0, T; L^2(Z; \mathbb{H})) \) (see (3.29)) implies that \( \gamma^n(t, u^n(t), z, \xi) \rightharpoonup (\gamma(t, z), \xi) \) for all \( t \in [0, T] \) and \( \xi \in \mathcal{P}_T \) as \( n \to \infty \). In particular, for \( \xi = \nu_m(\cdot) \), we find that the first term in the right hand side of the inequality (3.31) converges to zero as \( n \to \infty \). Using Hölder’s inequality and Hypothesis 3.1 (H.4), we estimate
\[
\mathbb{E} \left[ \int_0^t \int_{Z \setminus Z_n} (\gamma(s, z) \nu_m(s) \lambda(dz) ds \right] 
\]
\[
\leq \sup_{t \in [0, T]} |\phi(t)| \mathbb{E} \left[ \int_0^t \left( \lambda(Z \setminus Z_n) \right)^{1/2} \left( \int_{Z \setminus Z_n} |\gamma(s, z) \lambda(dz) |^2 \right)^{1/2} ds \right] 
\]
\[
\leq \sup_{t \in [0, T]} |\phi(t)| \mathbb{E} \left[ \int_0^t \left( \lambda(Z \setminus Z_n) \right)^{1/2} \int_Z |\gamma(t, z) \lambda(dz) |^2 \right]^{1/2} \to 0,
\]
as \( n \to \infty \). Moreover, we have
\[
\mathbb{E} \left[ \int_0^t \int_{Z_n} (\gamma^n(s, u^n(s), z) - \gamma(s, z) \nu_m(s) \lambda(dz) ds \right] 
\]
\[
\leq \mathbb{E} \left[ \int_0^t \left( \gamma^n(s, u^n(s), z) - \gamma(s, z) \right) \lambda(dz) ds \right] + \mathbb{E} \left[ \int_0^t \left( \gamma(s, z) \right) \lambda(dz) ds \right].
\] (3.32)
as \( n \to \infty \), where we have used the convergence of (3.31). Let us now define the map
\( \Gamma : \mathcal{P}_T \to L^2(\Omega; L^2(0, T)) \) by
\[
\Gamma(\gamma) = \int_0^t \int_Z (\gamma(s, \omega, z) \nu_m(s) \lambda(dz), d\omega).
for all \( t \in [0, T] \). It can be easily seen that the map \( \Gamma \) is linear and continuous. Thus, as \( n \to \infty \), we have

\[
\Gamma(y^n(\cdot, u^n(\cdot), \cdot)) = \int_0^t \int_Z (y^n(s-, u^n(s-), z), v_m(s-)) \tilde{\eta}(ds, dz)
\]

for all \( t \in [0, T] \) and for each fixed \( m \). Using this convergence and calculations similar to (3.31) and (3.32) yield

\[
\int_0^t \int_Z (y^n(s-, u^n(s-), z), v_m(s-)) \tilde{\eta}(ds, dz) -> \int_0^t \int_Z (y(s-, z), v_m(s-)) \tilde{\eta}(ds, dz)
\]

as \( n \to \infty \), for all \( t \in [0, T] \) and for each fixed \( m \).

Passing to limit as \( n \to \infty \) term wise in the Eq. (3.30), we get

\[
(\xi, v_m) \phi(T) = (u_0, v_m) + \int_0^T (u(t), \dot{v}_m) dt - \int_0^T \phi(t) (G_0(t), v_m) dt + \int_0^T \phi(t) \int_Z (y(t-, z), v_m) \tilde{\eta}(dt, dz).
\]

Let us now choose a subsequence \( \{\phi_k\} \in H^1(-\mu, T + \mu) \) with \( \phi_k(0) = 1 \), for \( k \in \mathbb{N} \), such that \( \phi_k \to \chi_t \) and the time derivative of \( \phi_k \) converges to \( \delta_t \), where \( \chi_t(s) = 1 \), for \( s \leq t \) and 0 otherwise, and \( \delta_t(s) = \delta(t-s) \) is the Dirac \( \delta \)-distribution. Using \( \phi_k \) in place of \( \phi \) in (3.33) and then letting \( k \to \infty \), we obtain

\[
(u(t), v_m) = (u_0, v_m) - \int_0^t (G_0(s), v_m) ds + \int_0^t \int_Z (y(s-, z), v_m) \tilde{\eta}(ds, dz).
\]

for all \( 0 < t < T \) with \( (u(T), v_m) = (\xi, v_m) \) and for any \( v_m \in \mathbb{V} \cap \tilde{\mathbb{V}}^{r+1} \). It should be noted that \( \mathbb{V} \subset \mathbb{V} \cap \tilde{\mathbb{V}}^{r+1} \subset \mathbb{H} \) and \( \mathbb{V} \) is dense in \( \mathbb{H} \). Therefore, \( \mathbb{V} \cap \tilde{\mathbb{V}}^{r+1} \) is dense in \( \mathbb{H} \) and the above equation holds for any \( v \in \mathbb{V} \cap \tilde{\mathbb{V}}^{r+1} \). Thus, we have

\[
(u(t), v) = (u_0, v) - \int_0^t (G_0(s), v) ds + \int_0^t \int_Z (y(s-, z), v) \tilde{\eta}(ds, dz), \quad \mathbb{P} \text{-a.s.,}
\]

for all \( 0 < t < T \) with \( (u(T), v) = (\xi, v) \), for all \( v \in \mathbb{V} \cap \tilde{\mathbb{V}}^{r+1} \). Hence, \( u(\cdot) \) satisfies the following stochastic differential:

\[
\begin{cases}
du(t) = -G_0(t) dt + \int_Z \gamma(t-, z) \tilde{\eta}(dt, dz), \\
u(0) = u_0,
\end{cases}
\]

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for $u_0 \in L^2(\Omega; \mathbb{H})$.

**Step (4): Energy equality satisfied by $u(\cdot)$**. Let us now establish the energy equality (Itô’s formula) satisfied by $u(\cdot)$, which is the crucial step in proving the main result. It should be noted that such an energy equality is not immediate due to the final convergence in (3.27) and we cannot apply the infinite dimensional Itô formula available in the literature for semimartingales (see [29, Theorem 1], [51, Theorem 6.1]). We follow the approximations given in [23] to obtain such an energy equality. In [23], the authors established an approximation of $u(\cdot)$ in bounded domains such that the approximations are bounded and converge in both Sobolev and Lebesgue spaces simultaneously (one can see [31] for such an approximation of $L^p$-space valued functions using truncated Fourier expansions in periodic domains). We approximate $u(t)$, for each $t \in (0, T)$ and $\mathbb{P}$-a.s., by using the finite-dimensional space spanned by the first $n$ eigenfunctions of the Stokes operator as ([23, Theorem 4.3])

$$u_n(t) := \frac{1}{P_{1/n}}u(t) = \sum_{\lambda_j < n^2} e^{-\lambda_j/n} \langle u(t), w_j \rangle w_j. \quad (3.37)$$

For notational convenience, we use the approximations given in (3.37) as $u_n(\cdot)$ and the Galerkin approximations in steps (1)-(3) as $u^n(\cdot)$. Note first that

$$\|u_n\|^2_{\mathbb{H}} = \|P_{1/n}u\|^2_{\mathbb{H}} = \sum_{\lambda_j < n^2} e^{-2\lambda_j/n} |\langle u, w_j \rangle|^2 \leq \sum_{j=1}^{\infty} |\langle u, w_j \rangle|^2 = \|u\|^2_{\mathbb{H}} < +\infty, \quad (3.38)$$

for all $u \in \mathbb{H}$. Moreover, we have

$$\|(I - P_{1/n})u\|^2_{\mathbb{H}} = \|u\|^2_{\mathbb{H}} - 2 \langle u, P_{1/n}u \rangle + \|P_{1/n}u\|^2_{\mathbb{H}}$$

$$\begin{align*}
\quad &= \sum_{j=1}^{\infty} |\langle u, w_j \rangle|^2 - 2 \sum_{\lambda_j < n^2} e^{-\lambda_j/n} |\langle u, w_j \rangle|^2 + \sum_{\lambda_j < n^2} e^{-2\lambda_j/n} |\langle u, w_j \rangle|^2 \\
\quad &= \sum_{\lambda_j < n^2} (1 - e^{-\lambda_j/n})^2 |\langle u, w_j \rangle|^2 + \sum_{\lambda_j \geq n^2} |\langle u, w_j \rangle|^2. \quad (3.39)
\end{align*}$$

for all $u \in \mathbb{H}$. It should be noted that the final term in the right hand side of the equality (3.39) tends zero as $n \to \infty$, since the series $\sum_{j=1}^{\infty} |\langle u, w_j \rangle|^2$ is convergent. The first term on the right hand side of the equality can be made bounded above by

$$\sum_{j=1}^{\infty} (1 - e^{-\lambda_j/n})^2 |\langle u, w_j \rangle|^2 \leq 4 \sum_{j=1}^{\infty} |\langle u, w_j \rangle|^2 = 4\|u\|^2_{\mathbb{H}} < +\infty.$$
Using the dominated convergence theorem, one can interchange the limit and sum, and hence we obtain

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} (1 - e^{-\lambda_{j}/n})^2 |\langle u, w_j \rangle|^2 = \sum_{j=1}^{\infty} \lim_{n \to \infty} (1 - e^{-\lambda_{j}/n})^2 |\langle u, w_j \rangle|^2 = 0.$$  

Hence $$\| (I - P_{1/n}) u \|_{\mathcal{H}} \to 0$$ as $$n \to \infty$$, which implies $$\| I - P_{1/n} \|_{\mathcal{L}(\mathcal{H})} \to 0$$ as $$n \to \infty$$. Moreover, for $$u \in \mathcal{V}'$$, we have

$$\| (I - P_{1/n}) u \|_{\mathcal{V}'}^2 = \| A^{-1/2} (I - P_{1/n}) A^{-1/2} u \|_{\mathcal{H}}^2 = \| (I - P_{1/n}) A^{-1/2} u \|_{\mathcal{H}}^2 \leq \| I - P_{1/n} \|_{\mathcal{L}(\mathcal{H})} \| u \|_{\mathcal{V}'} \to 0 \text{ as } n \to \infty. \quad (3.40)$$

Let us now discuss the properties of the approximation given in (3.37). The authors in [23] showed that such an approximation satisfies:

1. $$u_n(t) \to u(t)$$ in $$\mathcal{H}$$ with $$\| u_n(t) \|_{\mathcal{H}} \leq C \| u(t) \|_{\mathcal{H}}$$, $$\mathbb{P}$$-a.s. and a.e. $$t \in [0, T]$$,
2. $$u_n(t) \to u(t)$$ in $$\mathbb{L}^p(\mathcal{O})$$ with $$\| u_n(t) \|_{\mathbb{L}^p} \leq C \| u(t) \|_{\mathbb{L}^p}$$, for any $$p \in (1, \infty)$$, $$\mathbb{P}$$-a.s. and a.e. $$t \in [0, T]$$,
3. $$u_n(t)$$ is divergence free and zero on $$\partial\mathcal{O}$$, $$\mathbb{P}$$-a.s. and a.e. $$t \in [0, T]$$.

In (1) and (2), C is an absolute constant. It should be noted that for $$2 \leq d \leq 4$$, $$D(A) \subset \mathbb{H}^2(\mathcal{O}) \subset \mathbb{L}^p(\mathcal{O})$$, for all $$p \in (1, \infty)$$ (cf. [23]). Since $$w_j$$’s are the eigenfunctions of Stokes’ operator A, we get $$w_j \in D(A) \subset \mathcal{V}$$ and $$w_j \in D(A) \subset \mathbb{L}^{d+1}$$ Taking $$v = w_j$$ in (3.35), multiplying by $$e^{-\lambda_{j}/n} w_j$$ and then summing over all $$j$$ such that $$\lambda_{j} < n^2$$, we see that $$u_n(\cdot)$$ satisfies the following Itô stochastic differential:

$$u_n(t) = u_{0n} - \int_0^t G_0 n(s) ds + \int_0^t \int_{\mathcal{Z}} \gamma n(s-, z) \tilde{\pi}(ds, dz), \quad (3.41)$$

$$\mathbb{P}$$-a.s., for all $$t \in (0, T)$$, where $$u_{0n} = P_{1/n} u_0$$, $$G_0 n = P_{1/n} G_0$$, and $$\gamma_n = P_{1/n} \gamma$$. It is clear that the equation (3.41) has a unique solution $$u_n(\cdot)$$ (cf. [4,58], etc). We apply finite dimensional Itô’s formula to the process $$\| u_n(\cdot) \|_{\mathcal{H}}^2$$ to find

$$\| u_n(t) \|_{\mathcal{H}}^2 = \| u_{0n} \|_{\mathcal{H}}^2 - 2 \int_0^t (G_0 n(s), u_n(s)) ds + \int_0^t \int_{\mathcal{Z}} \gamma n(s, z) \| u_n(s-, z) \|_{\mathcal{H}}^2 \tilde{\pi}(ds, dz)$$

$$+ 2 \int_0^t \int_{\mathcal{Z}} (\gamma n(s-, z), u_n^2(s-)) \tilde{\pi}(ds, dz), \quad (3.42)$$

for all $$t \in (0, T)$$.

Using the convergence given in (3.39), we get

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \| u(t) - u_n(t) \|_{\mathcal{H}}^2 \right] \leq \| I - P_{1/n} \|_{\mathcal{L}(\mathcal{H})}^2 \mathbb{E} \left[ \sup_{t \in [0, T]} \| u(t) \|_{\mathcal{H}}^2 \right] \to 0, \quad (3.43)$$
as \( n \to \infty \), since \( u \in L^2(\Omega; L^\infty(0, T; \mathbb{H})) \). Thus, it is immediate that

\[
\mathbb{E}\left[ \|u_n(t)\|_{\mathbb{H}}^2 \right] \to \mathbb{E}\left[ \|u(t)\|_{\mathbb{H}}^2 \right], \quad \text{as } n \to \infty, \tag{3.44}
\]

for a.e. \( t \in [0, T] \). A calculation similar to (3.43) yields

\[
\mathbb{E}\left[ \|u_{0n}\|_{\mathbb{H}}^2 \right] \to \mathbb{E}\left[ \|u_0\|_{\mathbb{H}}^2 \right] \quad \text{as } n \to \infty. \tag{3.45}
\]

We also need the fact

\[
\|u_n - u\|_{L^{r+1}(\Omega; L^{r+1}(0, T; \mathbb{H}))}, \quad \text{as } n \to \infty, \tag{3.46}
\]

which follows from (2). Since \( u \in L^{r+1}(\Omega; L^{r+1}(0, T; \mathbb{H})) \) and the fact that \( \|u^n(t, \omega) - u(t, \omega)\|_{L^{r+1}} \to 0 \), for a.e. \( t \in [0, T] \) and a.a. \( \omega \in \Omega \), one can obtain the above convergence by an application of the dominated convergence theorem (with the dominating function \((1 + C)\|u(t, \omega)\|_{L^{r+1}}\)).

Since \( G_0 \in L^2(\Omega; L^2(0, T; \mathbb{V}')) + L^{\frac{r+1}{r}}(\Omega; L^{\frac{r+1}{r}}(0, T; \mathbb{H}^{r+1})) \), we can write down \( G_0 \) as \( G_0 = G_0^1 + G_0^2 \), where \( G_0^1 \in L^2(\Omega; L^2(0, T; \mathbb{V}')) \) and \( G_0^2 \in L^{\frac{r+1}{r}}(\Omega; L^{\frac{r+1}{r}}(0, T; \mathbb{H}^{r+1})) \). Note that \( 1 < \frac{r+1}{r} < \frac{4}{3} \). We use the approximation

\[
G_0^{1n}(t) := P_{1/n} G_0^1(t) = \sum_{\lambda_j < n^2} e^{-\lambda_j/n} \langle G_0^1(t), w_j \rangle w_j,
\]

and by using (3.40), we get

\[
\mathbb{E}\left[ \int_0^T \|G_0^{1n}(t) - G_0^1(t)\|_{\mathbb{V}'}^2 dt \right] \to 0, \quad \text{as } n \to \infty. \tag{3.47}
\]

Similarly, for \( G_0^2 \), we use the approximation

\[
G_0^{2n}(t) := P_{1/n} G_0^2(t) = \sum_{\lambda_j < n^2} e^{-\lambda_j/n} \langle G_0^2(t), w_j \rangle w_j,
\]

and by using (2), we get

\[
\mathbb{E}\left[ \int_0^T \|G_0^{2n}(t) - G_0^2(t)\|_{L^{\frac{r+1}{r}}}^\frac{r+1}{r} dt \right] \to 0, \quad \text{as } n \to \infty. \tag{3.48}
\]

By defining \( G_{0n} = G_0^1 + G_0^2 \), one can easily see that

\[
\|G_{0n} - G_0\|_{L^2(\Omega; L^2(0, T; \mathbb{V}')) + L^{\frac{r+1}{r}}(\Omega; L^{\frac{r+1}{r}}(0, T; \mathbb{H}^{r+1}))} \to 0, \quad \text{as } n \to \infty. \tag{3.49}
\]
Furthermore using (3.46) and (3.49), it is immediate that (cf. [57, Theorem 3.7])

\[
\mathbb{E} \left[ \int_0^t (G_0(s), u_n(s)) ds - \int_0^t (G_0(s), u(s)) ds \right] \to 0, \quad (3.50)
\]

as \( n \to \infty \), for all \( t \in (0, T) \). Next, we establish the convergence of the stochastic integral. Using (3.38) and (3.39), we get

\[
\| \gamma_n(t, z) \|_{\mathbb{H}} \leq \| \gamma(t, z) \|_{\mathbb{H}} \quad \text{and} \quad \| \gamma_n(t, z) - \gamma(t, z) \|_{\mathbb{H}} \to 0, \quad \text{as} \quad n \to \infty, \quad (3.51)
\]

\( \mathbb{P} \)-a.s., for a.e. \( t \in [0, T] \) and a.a. \( z \in Z \). Let us now consider

\[
\mathbb{E} \left[ \int_0^t \int_Z \| \gamma_n(s, z) \|^2_{\mathbb{H}} \pi(ds, dz) - \int_0^t \int_Z \| \gamma(s, z) \|^2_{\mathbb{H}} \pi(ds, dz) \right]
\]

\[
\leq \mathbb{E} \left[ \int_0^t \int_Z \| \gamma_n(s, z) \|^2_{\mathbb{H}} - \| \gamma(s, z) \|^2_{\mathbb{H}} \| \pi(ds, dz) \right]
\]

\[
\leq \left\{ \mathbb{E} \left[ \int_0^t \int_Z \| \gamma_n(s, z) - \gamma(s, z) \|^2_{\mathbb{H}} \pi(ds, dz) \right] \right\}^{1/2}
\]

\[
\times \left\{ \mathbb{E} \left[ \int_0^t \int_Z (\| \gamma_n(s, z) \|_{\mathbb{H}} + \| \gamma(s, z) \|_{\mathbb{H}}) \| \pi(ds, dz) \right] \right\}^{1/2}
\]

\[
= \left\{ \mathbb{E} \left[ \int_0^t \int_Z \| \gamma_n(s, z) - \gamma(s, z) \|^2_{\mathbb{H}} \lambda(dz)ds \right] \right\}^{1/2}
\]

\[
\times \left\{ \mathbb{E} \left[ \int_0^t \int_Z (\| \gamma_n(s, z) \|_{\mathbb{H}} + \| \gamma(s, z) \|_{\mathbb{H}})^2 \lambda(dz)ds \right] \right\}^{1/2}
\]

\[
\leq 2 \left\{ \mathbb{E} \left[ \int_0^t \int_Z \| \gamma_n(s, z) - \gamma(s, z) \|^2_{\mathbb{H}} \lambda(dz)ds \right] \right\}^{1/2}
\]

\[
\times \left\{ \mathbb{E} \left[ \int_0^t \int_Z \| \gamma(s, z) \|^2_{\mathbb{H}} \lambda(dz)ds \right] \right\}^{1/2}
\]

\[
\to 0, \quad \text{as} \quad n \to \infty, \quad (3.52)
\]

by using the Cauchy-Schwarz inequality, (3.3) and an application of Lebesgue’s dominated convergence theorem. Finally, we consider

\[
\mathbb{E} \left[ \int_0^t \int_Z (\gamma_n(s, z), u_n(s)) \tilde{\pi}(ds, dz) - \int_0^t \int_Z (\gamma(s, z), u(s)) \tilde{\pi}(ds, dz) \right]
\]

\[
\leq \mathbb{E} \left[ \int_0^t \int_Z (\gamma_n(s, z) - \gamma(s, z), u_n(s)) \tilde{\pi}(ds, dz) \right]
\]

\[
+ \mathbb{E} \left[ \int_0^t \int_Z (\gamma(s, z), u_n(s) - u(s)) \tilde{\pi}(ds, dz) \right]. \quad (3.53)
\]
Applying Burkholder–Davis–Gundy’s and Hölder’s inequalities, we find

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \int_Z (\gamma_n(s-, z) - \gamma(s-, z), \mathbf{u}^n(s-)) \tilde{\pi}(ds, dz) \right| \right] \\
\leq \sqrt{3} \mathbb{E} \left[ \int_0^T \int_Z \|\gamma_n(t, z) - \gamma(t, z)\|_H^2 \|\mathbf{u}^n(t)\|_H^2 \tau(dr, dz) \right]^{1/2} \\
\leq \sqrt{3} \mathbb{E} \left[ \sup_{t \in [0,T]} \|\mathbf{u}_n(t)\|_H \left( \int_0^T \int_Z \|\gamma_n(t, z) - \gamma(t, z)\|_H^2 \tau(dr, dz) \right)^{1/2} \right] \\
\leq \sqrt{3} \left\{ \mathbb{E} \left[ \sup_{t \in [0,T]} \|\mathbf{u}(t)\|_H^2 \right] \right\}^{1/2} \left\{ \mathbb{E} \left[ \int_0^T \int_Z \|\gamma_n(t, z) - \gamma(t, z)\|_H^2 \lambda(dz)dr \right] \right\}^{1/2} \\
\to 0 \text{ as } n \to \infty, \quad (3.54)
\end{align*}
\]

using (3.38), (3.51) and Lebesgue’s dominated convergence theorem. Once again an application of the Burkholder–Davis–Gundy inequality and (3.43) yield

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \int_Z (\gamma(s-, z), \mathbf{u}_n(s-) - \mathbf{u}(s-)) \tilde{\pi}(ds, dz) \right| \right] \\
\leq \sqrt{3} \mathbb{E} \left[ \int_0^T \int_Z \|\gamma(s, z)\|_H^2 \|\mathbf{u}_n(s) - \mathbf{u}(s)\|_H \tau(ds, dz) \right]^{1/2} \\
\leq \sqrt{3} \mathbb{E} \left[ \sup_{t \in [0,T]} \|\mathbf{u}_n(s) - \mathbf{u}(s)\|_H \left( \int_0^T \int_Z \|\gamma(s, z)\|_H^2 \tau(ds, dz) \right)^{1/2} \right] \\
\leq \sqrt{3} \left\{ \mathbb{E} \left[ \sup_{t \in [0,T]} \|\mathbf{u}_n(s) - \mathbf{u}(s)\|_H^2 \right] \right\}^{1/2} \left\{ \mathbb{E} \left[ \int_0^T \int_Z \|\gamma(s, z)\|_H^2 \lambda(dz)ds \right] \right\}^{1/2} \\
\to 0 \text{ as } n \to \infty. \quad (3.55)
\end{align*}
\]

Combining (3.54) and (3.55), we obtain that the right hand side of (3.53) tends to zero as \( n \to \infty \). Using the convergences given in (3.44), (3.45), (3.50), (3.52) and (3.53), along a subsequence one can pass to limit in (3.42) to get the energy equality:

\[
\begin{align*}
\|\mathbf{u}(t)\|_H^2 &= \|\mathbf{u}_0\|_H^2 - 2 \int_0^t \langle \mathbf{G}_0(s), \mathbf{u}(s) \rangle ds + \int_0^t \int_Z \|\gamma(s, z)\|_H^2 \tau(ds, dz) \\
&\quad + 2 \int_0^t \int_Z (\gamma(s-, z), \mathbf{u}(s-)) \tilde{\pi}(ds, dz), \quad (3.56)
\end{align*}
\]

for a. e. \( t \in (0, T) \), \( \mathbb{P} \)-a.s.

We need to prove that the energy equality (3.56) is true for all \( t \in [0, T] \). For this purpose, we use an approximation available in [26]. Let \( \eta(t) \) be an even, positive, smooth function with compact support contained in the interval \((-1, 1)\), such that \( \int_{-\infty}^{\infty} \eta(s)ds = 1 \). Let us denote by \( \eta^h \), a family of mollifiers related to the function \( \eta \).
as

$$\eta^h(s) := h^{-1} \eta(s/h), \text{ for } h > 0.$$ 

In particular, we get $\int_0^h \eta^h(s) \, ds = \frac{1}{2}$. For any function $v \in L^p(0, T; \mathbb{X}), \mathbb{P}$-a.s., where $\mathbb{X}$ is a Banach space, for $p \in [1, \infty)$, we define its mollification in time $v^h(\cdot)$ as

$$v^h(s) := (v * \eta^h)(s) = \int_0^T v(\tau) \eta^h(s - \tau) \, d\tau, \text{ for } h \in (0, T).$$

From [26, Lemma 2.5], we know that this mollification has the following properties. For any $v \in L^p(0, T; \mathbb{X}), v^h \in C^k([0, T]; \mathbb{X}), \mathbb{P}$-a.s. for all $k \geq 0$ and

$$\lim_{h \to 0} \|v^h - v\|_{L^p(0, T; \mathbb{X})} = 0, \mathbb{P}$-a.s. (3.57)

Moreover, $\|v^h(t) - v(t)\|_{\mathbb{X}} \to 0$, as $h \to 0$, for a.e. $t \in [0, T], \mathbb{P}$-a.s. For some time $t_1 > 0$, we set

$$u^h(t) = \int_0^{t_1} \eta^h(t - s) u(s) \, ds := (\eta^h * u)(t),$$

with the parameter $h$ satisfying $0 < h < T - t_1$ and $h < t_1$, where $\eta_h$ is the even mollifier given above. Note that $u^h(\cdot)$ satisfies the following Itô stochastic differential:

$$u^h(t) = u^h(0) + \int_0^t (\eta^h * u)(s) \, ds. \quad (3.58)$$

Applying Itô’s product formula to the process $(u^h(\cdot), u(\cdot))$, we obtain

$$(u^h(t), u(t)) = (u(0), u^h(0)) - \int_0^t \langle u^h(s), G_0(s) \rangle \, ds$$

$$+ \int_0^t \int_{\mathbb{Z}} (u^h(s -), \gamma(s -, z)) \pi(d\tau, dz)$$

$$+ \int_0^t (u(s), (\eta^h * u)(s)) \, ds + [u^h, u]_t, \quad (3.59)$$

where $[u^h, u]_t$ is the quadratic variation between the processes $u^h(\cdot)$ and $u(\cdot)$. Using stochastic Fubini’s theorem ([46, Lemma A.1.1]), we find

$$[u^h, u]_t = \left[ \int_0^{t_1} \eta^h(t - s) \left( \int_0^s \int_{\mathbb{Z}} \gamma(\tau -, z) \pi(d\tau, dz) \right) ds, \int_0^{t_1} \left( \int_{\mathbb{Z}} \gamma(\tau -, z) \pi(d\tau, dz) \right) ds \right]_t$$

$$= \left[ \int_0^{t_1} \int_{\mathbb{Z}} \left( \int_{\tau}^{t_1} \eta^h(t - s) \, ds \right) \gamma(\tau -, z) \pi(d\tau, dz), \int_0^{t_1} \int_{\mathbb{Z}} \gamma(\tau -, z) \pi(d\tau, dz) \right]_t,$$
and hence
\[ [u^h, u]_{t_1} = \int_0^{t_1} \int_Z \left( \int_0^{t_1} \eta^h(t_1 - s)ds \right) \|\gamma(\tau, z)\|_{H^2 \varpi}^2 (d\tau, dz). \] (3.60)

Since the function \( \eta^h \) is even in \((-h, h)\), we obtain \( \hat{\eta}^h(r) = -\hat{\eta}^h(-r) \). Changing the order of integration, we get (see [31])
\[
\int_0^{t_1} (u(s), (\eta^h * u)(s))ds = \int_0^{t_1} \int_0^{t_1} \hat{\eta}^h(s - \tau)(u(s), u(\tau))d\tau ds
\]
\[
- \int_0^{t_1} \int_0^{t_1} \hat{\eta}^h(\tau - s)(u(s), u(\tau))d\tau ds
\]
\[
= - \left( \int_0^{t_1} \int_0^{t_1} \hat{\eta}^h(s - \tau)(u(s), u(\tau))d\tau ds \right) = 0. \] (3.61)

Thus, from (3.59), it is immediate that
\[
(u(t_1), u^h(t_1)) = (u(0), u^h(0)) - \int_0^{t_1} \langle u^h(s), G_0(s) \rangle ds
\]
\[
+ \int_0^{t_1} \int_Z (u^h(s), \gamma(s, z))\bar{\pi}(ds, dz)
\]
\[
+ \int_0^{t_1} \int_Z \left( \int_0^{t_1} \eta^h(t_1 - s)ds \right) \|\gamma(\tau, z)\|_{H^2 \varpi}^2 (d\tau, dz). \] (3.62)

Next, we let \( h \to 0 \) in (3.50), by considering the points in \((0, T)\), where the jump occurs. Note that countable number of points only jump occurs. Let \( t_1 \) be a point in \((0, T)\), where the jump does not occur. For \( G_0 = G_0^1 + G_0^2 \), where
\( G_0^1 \in L^2(\Omega; L^2(0, T; V')) \) and \( G_0^2 \in L^{\frac{\gamma+1}{\gamma}}(\Omega; L^{\frac{\gamma+1}{\gamma}}(0, T; H^{\frac{\gamma+1}{\gamma}})) \), we consider
\[
\mathbb{E} \left[ \int_0^{t_1} (G_0(s), u^h(s))ds - \int_0^{t_1} \langle G_0(s), u(s) \rangle ds \right]
\]
\[
\leq \mathbb{E} \left[ \int_0^{t_1} |(G_0^1(s), u^h(s) - u(s))|ds \right] + \mathbb{E} \left[ \int_0^{t_1} |(G_0^2(s), u^h(s) - u(s))|ds \right]
\]
\[
\leq \mathbb{E} \left[ \int_0^{t_1} \|G_0^1(s)\|_{V'} \|u^h(s) - u(s)\|_{V'} ds \right]
\]
\[
+ \mathbb{E} \left[ \int_0^{t_1} \|G_0^2(s)\|_{L^{\frac{\gamma+1}{\gamma}}} \|u^h(s) - u(s)\|_{L^{\frac{\gamma+1}{\gamma}}} ds \right]
\]
\[
\leq \left\{ \mathbb{E} \left[ \int_0^{t_1} \|G_0^1(s)\|_{V'}^2 ds \right] \right\}^{1/2} \left\{ \mathbb{E} \left[ \int_0^{t_1} \|u^h(s) - u(s)\|_{V'}^2 ds \right] \right\}^{1/2}.
\]
+ \left\{ \mathbb{E} \left[ \int_0^{t_1} \| G_0^2(s) \|_{L^{p+1}} \right] ds \right\}^{\frac{1}{p+1}} \left\{ \mathbb{E} \left[ \int_0^{t_1} \| u^h(s) - u(s) \|_{L^{p+1}} \right] ds \right\}^{\frac{1}{p+1}} \rightarrow 0 \text{ as } h \rightarrow 0.

Thus along a subsequence, we obtain

\[
\lim_{h \rightarrow 0} \int_0^{t_1} \langle G_0(s), u^h(s) \rangle ds = \int_0^{t_1} \langle G_0(s), u(s) \rangle ds, \quad \mathbb{P}\text{-a.s.} \quad (3.63)
\]

Using Burkholder–Davis–Gundy’s and Hölder’s inequalities, we have

\[
\mathbb{E} \left[ \int_0^{t_1} \int_Z (u^h(s), \gamma(s, z)) \tilde{\pi}(ds, dz) - \int_0^{t_1} \int_Z (u(s), \gamma(s, z)) \tilde{\pi}(ds, dz) \right] \\
\leq \mathbb{E} \left[ \sup_{t_1 \in [0, T]} \left| \int_0^{t_1} \int_Z (u^h(s) - u(s), \gamma(s, z)) \tilde{\pi}(ds, dz) \right| \right] \\
\leq \sqrt{3} \mathbb{E} \left[ \int_0^T \int_Z \| u^h(s) - u(s) \|_{L^2}^2 \| \gamma(s, z) \|_{L^2}^2 \tilde{\pi}(ds, dz) \right]^{1/2} \\
\leq \sqrt{3} \mathbb{E} \left[ \sup_{s \in [0, T]} \| u^h(s) - u(s) \|_{L^2} \left( \int_0^T \int_Z \| \gamma(s, z) \|_{L^2}^2 \tilde{\pi}(ds, dz) \right)^{1/2} \right] \\
\leq \sqrt{3} \mathbb{E} \left[ \left\{ \sup_{s \in [0, T]} \| u^h(s) - u(s) \|_{L^2} \right\}^{1/2} \left\{ \mathbb{E} \left[ \int_0^T \int_Z \| \gamma(s, z) \|_{L^2}^2 \tilde{\pi}(ds, dz) \right] \right\}^{1/2} \right] \\
\rightarrow 0 \text{ as } h \rightarrow 0. \quad (3.64)
\]

Once again along subsequence, we get

\[
\lim_{h \rightarrow 0} \int_0^{t_1} \int_Z (u^h(s), \gamma(s, z)) \tilde{\pi}(ds, dz) = \int_0^{t_1} \int_Z (u(s), \gamma(s, z)) \tilde{\pi}(ds, dz), \quad \mathbb{P}\text{-a.s.} \quad (3.65)
\]

Finally, using the fact that \( \int_0^h \eta^h(s) ds = \frac{1}{2} \), we estimate

\[
\mathbb{E} \left[ \int_0^{t_1} \int_Z \left( \int_0^{t_1} \eta^h(t_1 - s) ds \right) \| \gamma(\tau, z) \|_{L^2}^2 \tilde{\pi}(d\tau, dz) \right] \\
= \mathbb{E} \left[ \int_0^{t_1} \eta^h(t_1 - s) \int_0^s \int_Z \| \gamma(\tau, z) \|_{L^2}^2 \tilde{\pi}(d\tau, dz) ds \right] \\
= \mathbb{E} \left[ \int_0^{t_1} \eta^h(s) \int_0^{t_1 - s} \int_Z \| \gamma(\tau, z) \|_{L^2}^2 \tilde{\pi}(d\tau, dz) ds \right] \\
= \mathbb{E} \left[ \int_0^{t_1} \eta^h(s) \left( \int_0^{t_1} \int_Z \| \gamma(\tau, z) \|_{L^2}^2 \tilde{\pi}(d\tau, dz) \right) \right]
\]
\[
- \int_{t_1-s}^{t_1} \int_Z \|\gamma(\tau, z)\|^2_{H^2} \pi(d\tau, dq) \, ds
\]
\[
= \frac{1}{2} \mathbb{E} \left[ \int_0^{t_1} \int_Z \|\gamma(\tau, z)\|^2_{H^2} \pi(d\tau, dq) \right]
- \mathbb{E} \left[ \int_0^h \eta^h(s) \int_{t_1-s}^{t_1} \int_Z \|\gamma(\tau, z)\|^2_{H^2} \pi(d\tau, dq) ds \right]
\rightarrow \frac{1}{2} \mathbb{E} \left[ \int_0^{t_1} \int_Z \|\gamma(\tau, z)\|^2_{H^2} \pi(d\tau, dq) \right] \quad \text{as } h \to 0, \tag{3.66}
\]

since jumps are not occurring at \(t_1\). Thus, along a subsequence, we further have

\[
\lim_{h \to 0} \int_0^{t_1} \int_Z \left( \int_0^{t_1} \eta^h(t_1 - s) ds \right) \|\gamma(\tau, z)\|^2_{H^2} \pi(d\tau, dq) = \frac{1}{2} \int_0^{t_1} \int_Z \|\gamma(\tau, z)\|^2_{H^2} \pi(d\tau, dq) \tag{3.67}
\]

Using the convergences (3.63)–(3.67) in (3.62), along a subsequence, we get

\[
\int_0^{t_1} \langle u(s), G_0(s) \rangle ds - \int_0^{t_1} \int_Z (u(s) - \gamma(s, z)) \pi(d\tau, dq)
- \frac{1}{2} \int_0^{t_1} \int_Z \|\gamma(s, z)\|^2_{H^2} \pi(d\tau, dq)
= - \lim_{h \to 0} \langle u(t_1), u^h(t_1) \rangle + \lim_{h \to 0} \langle u(0), u^h(0) \rangle \tag{3.68}
\]

Using the \(L^2\)-weak continuity (form right) of \(u(\cdot)\) around zero and the fact that \(\int_0^h \eta^h(s) ds = \frac{1}{2}\), we find

\[
(u(0), u^h(0)) = \int_0^{t_1} \eta^h(-s)(u(0), u(s)) ds
= \int_0^{t_1} \eta^h(s)(u(0), u(0) + u(s) - u(0)) ds
= \frac{1}{2} \|u(0)\|^2_{H^2} + \int_0^h \eta^h(s)(u(0), u(s) - u(0)) ds
\rightarrow \frac{1}{2} \|u(0)\|^2_{H^2} \quad \text{as } h \to 0, \tag{3.69}
\]

\(\mathbb{P}\)-a.s. Since jump does not occur at \(t_1\), using the fact that \(u(\cdot)\) is \(L^2\)-weakly continuous in time, we get

\[
(u(t_1), u^h(t_1)) = \int_0^{t_1} \eta^h(s)(u(t_1), u(t_1 - s)) ds
\]
\[ \begin{align*}
&= \frac{1}{2} \| u(t_1) \|_{H^2}^2 + \int_0^h \eta^h(s)(u(t_1), u(t_1 - s) - u(t_1)) ds \\
&\to \frac{1}{2} \| u(t_1) \|_{H^2}^2, \text{ as } h \to 0,
\end{align*} \tag{3.70} \]

\( \mathbb{P} \)-a.s. Combining the above convergences, we finally obtain the energy equality

\[ \| u(t_1) \|_{H^2}^2 = \| u(0) \|_{H^2}^2 - 2 \int_0^{t_1} \langle G_0(s), u(s) \rangle ds + 2 \int_0^{t_1} \int_Z (u(s -), \gamma(s, z) \tilde{\tau}(ds, dz) \\
+ \int_0^{t_1} \int_Z \gamma(s, z)\| \|_{H^2}^2 \tau(ds, dz), \tag{3.71} \]

for all \( t_1 \in (0, T) \), where the jump does not occur.

Now, let \( t_1 \in (0, T) \) be a point where jump occurs. Let \( \tilde{t}_1 \) be the point in \((0, T)\) where the jump occurs before \( t_1 \) (take \( \tilde{t}_1 = 0 \), if the first jump occurs at \( t_1 \)). Let \( u(t_1-) \) denote the left limit of \( u(\cdot) \) at the point \( t_1 \). From (3.62), we have

\[ (u(t_1), u^h(t_1)) = (u(0), u^h(0)) - \int_0^{t_1} (u^h(s), G_0(s)) ds + \int_0^{\tilde{t}_1} \int_Z (u^h(s), \gamma(s, z) \tilde{\tau}(ds, dz) \\
+ (u^h(t_1-), u(t_1) - u(t_1-)) - \int_{\tilde{t}_1}^{t_1} \int_Z (u^h(s), \gamma(s, z) \lambda(dz) ds \\
+ \int_0^{t_1} \eta^h(t_1 - s) \int_0^{t_1} \int_Z \gamma(t, z) \| \|_{H^2}^2 \tau(d\tau, dz) ds, \tag{3.72} \]

where we have used the fact that \( u(t_1) - u(t_1-) = \gamma(t_1, u(t_1) - u(t_1-)) \chi_{u(t_1) - u(t_1-)} \in Z \) (see [4, Chapter 4]). Note that the convergences (3.63) and (3.69) hold true in this case also. Once again using he fact that \( \int_0^h \eta^h(s) ds = \frac{1}{2} \) and the \( L^2 \)-weak continuity at \( t_1- \), we find

\[ \begin{align*}
&\left| \int_0^h \eta^h(s)(u(t_1), u(t_1) - u(t_1 - s)) ds - \frac{1}{2} (u(t_1), u(t_1) - u(t_1-)) \right| \\
&= \left| \int_0^h \eta^h(s) [(u(t_1), u(t_1-) - u(t_1 - s))] ds \right| \\
&\to 0 \text{ as } h \to 0, \mathbb{P} \text{-a.s.} \tag{3.73} \]

Thus the convergence given in (3.70) becomes

\[ (u(t_1), u^h(t_1)) \to \frac{1}{2} \| u(t_1) \|_{H^2}^2 - \frac{1}{2} (u(t_1), u(t_1) - u(t_1-)), \text{ as } h \to 0, \mathbb{P} \text{-a.s.} \tag{3.74} \]
The convergence given in (3.64) implies
\[
\lim_{h \to 0} \int_{\tilde{t}_1}^{t_1} \int_{Z} (u^h(s-), \gamma(s-, z)) \tilde{\pi}(ds, dz) = \int_{\tilde{t}_1}^{t_1} \int_{Z} (u(s-), \gamma(s-, z)) \tilde{\pi}(ds, dz), \ \mathbb{P}\text{-a.s.}
\]
(3.75)

Let us now discuss the convergence of \((u^h(t_1-), u(t_1) - u(t_1-))\). A calculation similar to (3.73) gives
\[
(u^h(t_1-), u(t_1) - u(t_1-)) = \int_{0}^{t_1} \eta^h(s)(u((t_1-)-s), u(t_1) - u(t_1-))ds
\]
\[
= \frac{1}{2}(u(t_1), u(t_1) - u(t_1-)) + \int_{0}^{h} \eta^h(s)(u((t_1-)-s) - u(t_1), u(t_1) - u(t_1-))ds
\]
\[
\to \frac{1}{2}(u(t_1), u(t_1) - u(t_1-)) - \frac{1}{2}\|u(t_1) - u(t_1-)\|^2_{\mathbb{H}}, \ \text{as} \ h \to 0, \ \mathbb{P}\text{-a.s.}
\]
(3.76)

As there are no jumps in \((\tilde{t}_1, t_1)\), we consider
\[
\mathbb{E} \left[ \int_{\tilde{t}_1}^{t_1} \int_{Z} (u^h(s), \gamma(s, z)) \lambda(ds, dz) - \int_{\tilde{t}_1}^{t_1} \int_{Z} (u(s), \gamma(s, z)) \lambda(ds, dz) \right]
\]
\[
= \mathbb{E} \left[ \int_{\tilde{t}_1}^{t_1} \int_{Z} (u^h(s) - u(s), \gamma(s, z)) \lambda(ds, dz) \right]
\]
\[
= \mathbb{E} \left[ \int_{\tilde{t}_1}^{t_1} \int_{Z} (u^h(s) - u(s), \gamma(s, z)) \tilde{\pi}(ds, dz) \right]
\]
\[
\leq \sqrt{3} \mathbb{E} \left[ \int_{0}^{T} \int_{Z} \|u^h(s) - u(s)\|^2_{\mathbb{H}} \|\gamma(s, z)\|^2_{\mathbb{H}} \tilde{\pi}(ds, dz) \right]^{1/2}
\]
\[
\leq \sqrt{3} \left\{ \mathbb{E} \left[ \sup_{s \in [0, T]} \|u^h(s) - u(s)\|^2_{\mathbb{H}} \right] \right\}^{1/2} \left\{ \mathbb{E} \left[ \int_{0}^{T} \int_{Z} \|\gamma(s, z)\|^2_{\mathbb{H}} \lambda(ds, dz) \right] \right\}^{1/2}
\]
\[
\to 0, \ \text{as} \ h \to 0,
\]
(3.77)

where we have used the Burkholder–Davis–Gundy inequality. Thus, along a subsequence, we have the following convergence:
\[
\lim_{h \to 0} \int_{\tilde{t}_1}^{t_1} \int_{Z} (u^h(s), \gamma(s, z)) \lambda(ds, dz) = \int_{\tilde{t}_1}^{t_1} \int_{Z} (u(s), \gamma(s, z)) \lambda(ds, dz), \ \mathbb{P}\text{-a.s.}
\]
(3.78)

Since a jump occurs at the point \(t_1\) and the jumps are isolated, a calculation similar to (3.66) yields
\[
\mathbb{E} \left[ \int_{0}^{t_1} \eta^h(t_1-s) \int_{0}^{s} \int_{Z} \|\gamma(t, z)\|^2_{\mathbb{H}} \pi(d\tau, dz)ds \right]
\]
Thus an application of Itô's formula to the process

\[
= \frac{1}{2} \mathbb{E} \left[ \int_0^{t_1} \int_Z \| \gamma(\tau, z) \|_{H}^2 \pi(d\tau, dz) \right]
\]

and the Itô formula (3.71) holds true for all

\[
\text{Combining the convergences (3.74)–(3.80), substituting it in (3.72) and then taking}
\]

Thus, along a subsequence, we have

\[
\int_0^{t_1} \eta^h(t_1 - s) \int_0^s \int_Z \| \gamma(\tau, z) \|_{H}^2 \pi(d\tau, dz) ds
\]

\[
\int_0^{t_1} \int_Z \| \gamma(\tau, z) \|_{H}^2 \pi(d\tau, dz) \]

\[
\int_0^{t_1} \int_Z \| \gamma(\tau, z) \|_{H}^2 \pi(d\tau, dz) - \frac{1}{2} \| u(t_1) - u(t_1 -) \|_{H}^2, \text{ as } h \to 0, \mathbb{P} \text{-a.s.}
\]

Combining the convergences (3.74)–(3.80), substituting it in (3.72) and then taking

limit along a subsequence as \( h \to 0 \), we find

\[
\frac{1}{2} \| u(t_1) \|_{H}^2 = \frac{1}{2} \| u(0) \|_{H}^2 - \int_0^{t_1} \langle G_0(s), u(s) \rangle ds + \frac{1}{2} \int_0^{t_1} \int_Z \| \gamma(\tau, z) \|_{H}^2 \pi(d\tau, dz)
\]

\[
+ \int_0^{t_1} \int_Z (u(s -), \gamma(s - , z) \lambda)(ds, dz) + \langle u(t_1 -), u(t_1) - u(t_1 -) \rangle
\]

\[
- \int_0^{t_1} \int_Z (u(s), \gamma(s, z)) \lambda(dz) ds
\]

\[
= \frac{1}{2} \| u(0) \|_{H}^2 - \int_0^{t_1} \langle G_0(s), u(s) \rangle ds + \int_0^{t_1} \int_Z (u(s -), \gamma(s - , z) \lambda)(ds, dz)
\]

\[
+ \frac{1}{2} \int_0^{t_1} \int_Z \| \gamma(\tau, z) \|_{H}^2 \pi(d\tau, dz), \mathbb{P} \text{-a.s.},
\]

and the Itô formula (3.71) holds true for all \( t_1 \in (0, T) \).

Taking expectation and noting the fact that the final term in the right hand side of

the equality (3.56) is a martingale, we find

\[
\mathbb{E} \left[ \| u(t) \|_{H}^2 \right] = \mathbb{E} \left[ \| u_0 \|_{H}^2 \right] - 2 \mathbb{E} \left[ \int_0^t \langle G_0(s), u(s) \rangle ds \right]
\]

\[
+ \mathbb{E} \left[ \int_0^t \int_Z \| \gamma(s, z) \|_{H}^2 \lambda(dz) ds \right].
\]

Thus an application of Itô’s formula to the process \( e^{-2\eta t} \| u(\cdot) \|_{H}^2 \) yields
\[
\mathbb{E}\left[ e^{-2\eta t} \| \mathbf{u}(t) \|_{\mathbb{H}}^2 \right] = \mathbb{E}\left[ \| \mathbf{u}_0 \|_{\mathbb{H}}^2 \right] - \mathbb{E}\left[ \int_0^t e^{-2\eta s} (2G_0(s) + 2\eta \mathbf{u}(s), \mathbf{u}(s))ds \right] \\
+ \mathbb{E}\left[ \int_0^t e^{-2\eta s} \int_{Z_m} \| \gamma(s, z) \|_{\mathbb{H}}^2 \lambda(\mathrm{d}z) \mathrm{d}s \right],
\]

for all \( t \in [0, T] \). Finally, we note that the initial value \( \mathbf{u}^n(0) \) converges to \( \mathbf{u}_0 \) strongly in \( L^2(\Omega; \mathbb{H}) \), that is,

\[
\lim_{n \to \infty} \mathbb{E}\left[ \| \mathbf{u}^n(0) - \mathbf{u}_0 \|_{\mathbb{H}}^2 \right] = 0.
\]

**Step (5): Minty–Browder technique and global strong solution.** Now, we are ready to prove the existence of strong solution to the system (3.1). It is now left to show that

\[
G(\mathbf{u}(-)) = G_0(\cdot) \quad \text{and} \quad \gamma(\cdot, \mathbf{u}(\cdot), \cdot) = \gamma(\cdot, \cdot).
\]

In order to achieve this aim, we make use of Lemma 3.5. For \( \mathbf{v} \in L^2(\Omega; L^\infty(0, T; \mathbb{H}_m)) \), with \( m < n \), using the local monotonicity result (see (3.23)), we get

\[
\mathbb{E}\left[ \int_0^T e^{-2\eta t} (2\langle G(\mathbf{v}(t)) \rangle - G(\mathbf{u}^n(t)), \mathbf{v}(t) - \mathbf{u}^n(t)) \\
+ 2\eta \left( \mathbf{v}(t) - \mathbf{u}^n(t), \mathbf{v}(t) - \mathbf{u}^n(t) \right) \mathrm{d}t \right] \\
\geq \mathbb{E}\left[ \int_0^T e^{-2\eta t} \int_{Z_m} \| \gamma^n(t, \mathbf{v}(t), z) - \gamma^n(t, \mathbf{u}^n(t), z) \|_{\mathbb{H}}^2 \lambda(\mathrm{d}z) \mathrm{d}t \right].
\]

Rearranging the terms in (3.85) and then using the energy equality (3.26), we obtain

\[
\mathbb{E}\left[ \int_0^T e^{-2\eta t} (2G(\mathbf{v}(t)) + 2\eta \mathbf{v}(t), \mathbf{v}(t) - \mathbf{u}^n(t)) \mathrm{d}t \right] \\
- \mathbb{E}\left[ \int_0^T e^{-2\eta t} \int_{Z_m} \| \gamma^n(t, \mathbf{v}(t), z) \|_{\mathbb{H}}^2 \lambda(\mathrm{d}z) \mathrm{d}t \right] \\
+ 2\mathbb{E}\left[ \int_0^T e^{-2\eta t} \int_{Z_m} \left( \gamma^n(t, \mathbf{v}(t), z), \gamma^n(t, \mathbf{u}^n(t), z) \right) \lambda(\mathrm{d}z) \mathrm{d}t \right] \\
\geq \mathbb{E}\left[ \int_0^T e^{-2\eta t} (2G(\mathbf{u}^n(t)) + 2\eta \mathbf{u}^n(t), \mathbf{v}(t)) \mathrm{d}t \right] \\
- \mathbb{E}\left[ \int_0^T e^{-2\eta t} (2G(\mathbf{u}^n(t)) + 2\eta \mathbf{u}^n(t), \mathbf{u}^n(t)) \mathrm{d}t \right] \\
+ \mathbb{E}\left[ \int_0^T e^{-2\eta t} \int_{Z_m} \| \gamma^n(t, \mathbf{u}^n(t), z) \|_{\mathbb{H}}^2 \lambda(\mathrm{d}z) \mathrm{d}t \right].
\]
\[
\begin{align*}
&= \mathbb{E} \left[ \int_0^T e^{-2\eta t} (2G(u^n(t)) + 2\eta u^n(t), v(t)) \,dt \right] \\
&\quad + \mathbb{E} \left[ e^{-2\eta T} \|u^n(T)\|_{H^2}^2 - \|u^n(0)\|_{H^2}^2 \right].
\end{align*}
\]

(3.86)

Let us now discuss the convergence of the terms involving noise coefficient. Note that

\[
\mathbb{E} \left[ \int_0^T e^{-2\eta t} \int_{Z^n} \left( 2\left( \gamma^n(t, v(t), z), \gamma^n(t, u^n(t), z) \right) - \|\gamma^n(t, v(t), z)\|_{H^2}^2 \right) \lambda(\mathrm{d}z) \,dt \right]
\]

\[
= \mathbb{E} \left[ \int_0^T e^{-2\eta t} \int_{Z^n} 2 \left( \gamma(t, v(t), z), \gamma^n(t, u^n(t), z) \right) \lambda(\mathrm{d}z) \,dt \right]
\]

\[
+ \mathbb{E} \left[ \int_0^T e^{-2\eta t} \int_{Z^n} \left( 2\left( \gamma^n(t, v(t), z) - \gamma(t, v(t), z), \gamma^n(t, u^n(t), z) \right) \lambda(\mathrm{d}z) \,dt \right]
\]

\[
- \mathbb{E} \left[ \int_0^T e^{-2\eta t} \int_{Z^n} \|\gamma^n(t, v(t), z)\|_{H^2}^2 \lambda(\mathrm{d}z) \,dt \right]
\]

\[
\leq \mathbb{E} \left[ \int_0^T e^{-2\eta t} \int_{Z^n} 2 \left( \gamma(t, v(t), z), \gamma^n(t, u^n(t), z) \right) \lambda(\mathrm{d}z) \,dt \right] + 2C \left( \mathbb{E} \left[ \int_0^T e^{-4\eta t} \int_Z \|\gamma^n(t, v(t), z) - \gamma(t, v(t), z)\|_{H^2}^2 \lambda(\mathrm{d}z) \,dt \right] \right)^{1/2}
\]

\[
- \mathbb{E} \left[ \int_0^T e^{-2\eta t} \int_{Z^n} \|\gamma^n(t, v(t), z)\|_{H^2}^2 \lambda(\mathrm{d}z) \,dt \right],
\]

(3.87)

where \( C = \left( \mathbb{E} \left[ \int_0^T e^{-4\eta t} \int_Z \|\gamma^n(t, u^n(t), z)\|_{H^2}^2 \lambda(\mathrm{d}z) \,dt \right] \right)^{1/2}. \) Then, applying the weak convergence of \( \{\gamma^n(\cdot, u^n(\cdot), \cdot) : n \in \mathbb{N} \} \) given in (3.29) to the first term and using the Lebesgue dominated convergence theorem (an argument similar to (3.31)) to the second and final terms on the right hand side of the inequality (3.87), we deduce that (cf. [7] for more details)

\[
\mathbb{E} \left[ \int_0^T e^{-2\eta t} \int_{Z^n} (2\left( \gamma^n(t, v(t), z), \gamma^n(t, u^n(t), z) \right) - \|\gamma^n(t, v(t), z)\|_{H^2}^2) \lambda(\mathrm{d}z) \,dt \right]
\]

\[
\rightarrow \mathbb{E} \left[ \int_0^T e^{-2\eta t} \int_Z (2(\gamma(t, v(t), z), \gamma(t, z)) - \|\gamma(t, v(t), z)\|_{H^2}^2) \,dt \right],
\]

(3.88)

as \( n \to \infty. \) Taking liminf on both sides of (3.86), and using (3.88), we obtain

\[
\mathbb{E} \left[ \int_0^T e^{-2\eta t} (2G(v(t)) + 2\eta v(t), v(t) - u(t)) \,dt \right]
\]

\[
- \mathbb{E} \left[ \int_0^T e^{-2\eta t} \int_Z \|\gamma(t, v(t), z)\|_{H^2}^2 \lambda(\mathrm{d}z) \,dt \right]
\]
We use of the energy equality (3.83) and (3.90) in (3.89) to get

\[
\mathbb{E}\left[\int_0^T e^{-2\eta t} (2G(v(t)) + 2\eta v(t), v(t) - u(t))dt\right] \\
\geq \mathbb{E}\left[\int_0^T e^{-2\eta t} \int_Z \|\gamma(t, v(t), z)\|_{\mathbb{H}}^2 \lambda(dz)dt\right] \\
- 2\mathbb{E}\left[\int_0^T e^{-2\eta t} \int_Z (\gamma(t, v(t), z) - \gamma(t, z)) \lambda(dz)dt\right] \\
+ \mathbb{E}\left[\int_0^T e^{-2\eta t} \int_Z \|\gamma(t, z)\|_{\mathbb{H}}^2 \lambda(dz)dt\right] \\
+ \mathbb{E}\left[\int_0^T e^{-2\eta t} (2G_0(t) + \eta u(t), v(t) - u(t))dt\right].
\]

(3.91)

Rearranging the terms in (3.91), we obtain

\[
\mathbb{E}\left[\int_0^T e^{-2\eta t} (2G(v(t)) - 2G_0(t) + 2\eta(v(t) - u(t)), v(t) - u(t))dt\right] \\
\geq \mathbb{E}\left[\int_0^T e^{-2\eta t} \int_Z \|\gamma(t, v(t), z) - \gamma(t, z)\|_{\mathbb{H}}^2 \lambda(dz)dt\right] \geq 0.
\]

(3.92)

Note that the estimate (3.92) holds true for any \( v \in L^2(\Omega; L^\infty(0, T; \mathbb{H})) \) and for any \( m \in \mathbb{N} \), since the estimate is independent of \( m \) and \( n \). Using a density argument, the inequality (3.92) remains true for any

\[
v \in L^2\left(\Omega; L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V})\right) \cap L^{r+1}(\Omega; L^{r+1}(0, T; \tilde{L}^{r+1})) =: \mathcal{G}.
\]

Indeed, for any \( v \in \mathcal{G} \), there exists a strongly convergent subsequence \( v_m \in \mathcal{G} \), which satisfies the inequality (3.92). Taking \( v(\cdot) = u(\cdot) \) in (3.92) immediately gives
\( \gamma(\cdot, v(\cdot), \cdot) = \gamma(\cdot, \cdot) \). Next, we take \( v(\cdot) = u(\cdot) + \lambda w(\cdot), \lambda > 0 \), where \( w \in \mathcal{G} \), and substitute for \( v \) in (3.92) to find
\[
\mathbb{E} \left[ \int_0^T e^{-2nt} \langle \mathcal{G}(u(t) + \lambda w(t)) - \mathcal{G}_0(t) + \eta \lambda w(t), \lambda w(t) \rangle dt \right] \geq 0. \tag{3.93}
\]
Dividing the above inequality by \( \lambda \), using the hemicontinuity property of \( \mathcal{G}(\cdot) \) (see Lemma 2.6), and passing \( \lambda \to 0 \), we obtain
\[
\mathbb{E} \left[ \int_0^T e^{-2nt} \langle \mathcal{G}(u(t)) - \mathcal{G}_0(t), w(t) \rangle dt \right] \geq 0, \tag{3.94}
\]
since the final term in (3.93) tends to 0 as \( \lambda \to 0 \). Thus from (3.94), we finally obtain \( \mathcal{G}(u(\cdot)) = \mathcal{G}_0(\cdot) \) and hence \( u(\cdot) \) is a strong solution of the system (3.1) and \( u \in \mathcal{G} \).

From (3.56), it is immediate that \( u(\cdot) \) satisfies the following energy equality (Itô’s formula):
\[
\|u(t)\|_{\mathbb{H}}^2 + 2\mu \int_0^t \|u(s)\|_{\mathbb{V}}^2 ds + 2\beta \int_0^t \|u(s)\|_{\mathbb{V}_{\gamma+1}}^2 ds
\]
\[
= \|u_0\|_{\mathbb{H}}^2 + \int_0^t \int_\mathcal{Z} \|\gamma(s, u(s), z)\|_{\mathbb{H}}^2 \pi(ds, dz)
\]
\[
+ 2 \int_0^t \int_\mathcal{Z} (\gamma(s-, u(s-), z), u(s-)) \tilde{\pi}(ds, dz), \tag{3.95}
\]
for all \( t \in (0, T), \mathbb{P}\)-a.s. Moreover, the following energy estimate holds true:
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|u(t)\|_{\mathbb{H}}^2 + 4\mu \int_0^T \|u(t)\|_{\mathbb{V}}^2 dt + 4\beta \int_0^T \|u(t)\|_{\mathbb{V}_{\gamma+1}}^2 dt \right]
\]
\[
\leq 2\mathbb{E} \left[ \|u_0\|_{\mathbb{H}}^2 \right] + 14KT e^{28KT}. \tag{3.96}
\]
Furthermore, since \( u(\cdot) \) satisfies the energy equality (3.95), one can show that the \( \mathcal{F}_t \)-adapted paths of \( u(\cdot) \) are càdlàg with trajectories in \( D([0, T]; \mathbb{H}) \), \( \mathbb{P}\)-a.s. (see [29,30,51], etc).

**Step (6): Uniqueness.** Finally, we show that the strong solution established in step (5) is pathwise unique. Let \( u_1(\cdot) \) and \( u_2(\cdot) \) be two strong solutions of the system (3.1). For \( N > 0 \), let us define
\[
\tau_1^N = \inf_{t \geq 0} \left\{ t : \|u_1(t)\|_{\mathbb{H}} > N \right\}, \quad \tau_2^N = \inf_{t \geq 0} \left\{ t : \|u_2(t)\|_{\mathbb{H}} > N \right\} \quad \text{and} \quad \tau_N := \tau_1^N \wedge \tau_2^N.
\]
Using the energy estimate (3.96), it can be shown in a similar way as in step (1), Proposition 3.4 that \( \tau_N \to T \) as \( N \to \infty, \mathbb{P}\)-a.s. Let us define \( w(\cdot) := u_1(\cdot) - u_2(\cdot) \) and \( \gamma(\cdot, w(\cdot), \cdot) := \gamma(\cdot, u_1(\cdot), \cdot) - \gamma(\cdot, u_2(\cdot), \cdot) \). Then, \( w(\cdot) \) satisfies the following...
Itô stochastic differential:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
dw(t) = -[\mu Aw(t) + B(u_1(t)) - B(u_2(t)) + \beta(C(u_1(t)) - C(u_2(t)))]dt \\
&\quad + \int_Z \gamma(t, w(t), z) \tilde{\pi}(dt, dz), \\
w(0) = w_0.
\end{array} \right.
\end{aligned}
\tag{3.97}
\]

Then, in a similar way as in (3.95), one can show that \(w(\cdot)\) satisfies the following energy equality:

\[
\|w(t \wedge \tau_N)\|_{\mathbb{H}_N}^2 + 2\mu \int_0^{t \wedge \tau_N} \|w(s)\|_{\mathbb{V}}^2 ds \\
= \|w(0)\|_{\mathbb{H}_N}^2 - 2 \int_0^{t \wedge \tau_N} \langle B(u_1(s)) - B(u_2(s)), w(s) \rangle ds \\
- 2 \int_0^{t \wedge \tau_N} \langle C(u_1(s)) - C(u_2(s)), u_1(s) - u_2(s) \rangle ds \\
+ \int_0^{t \wedge \tau_N} \int_Z \|\gamma(s, w(s), z)\|_{\mathbb{H}_N}^2 \tilde{\pi}(ds, dz) \\
+ 2 \int_0^{t \wedge \tau_N} \int_Z (\gamma(s-, w(s-), z), w(s-)) \tilde{\pi}(ds, dz).
\tag{3.98}
\]

Using Hölder’s and Young’s inequalities, we estimate \(\|B(u_1) - B(u_2), w\| = B(u_1 - u_2, u_1 - u_2, u_2)\) as

\[
|\langle B(u_1 - u_2, u_1 - u_2), u_2 \rangle| \leq \|u_1 - u_2\|_{\mathbb{V}} \|u_2(u_1 - u_2)\|_{\mathbb{H}_N} \\
\leq \frac{\mu}{2} \|u_1 - u_2\|_{\mathbb{V}}^2 + \frac{1}{2\mu} \|u_2(u_1 - u_2)\|_{\mathbb{H}_N}^2.
\tag{3.99}
\]

We take the term \(\|u_2(u_1 - u_2)\|_{\mathbb{H}_N}^2\) from (3.99) and use Hölder’s and Young’s inequalities to estimate it as (see [31] also)

\[
\begin{aligned}
&\int_{\mathcal{O}} |u_2(x)|^2 |u_1(x) - u_2(x)|^2 dx \\
&= \int_{\mathcal{O}} |u_2(x)|^2 |u_1(x) - u_2(x)|^{\frac{4}{r-1}} |u_1(x) - u_2(x)|^{\frac{2(r-3)}{r-1}} dx \\
&\leq \left( \int_{\mathcal{O}} |u_2(x)|^{r-1} |u_1(x) - u_2(x)|^2 dx \right)^{\frac{2}{r-1}} \left( \int_{\mathcal{O}} |u_1(x) - u_2(x)|^2 dx \right)^{\frac{r-3}{r-1}} \\
&\leq \beta \mu \left( \int_{\mathcal{O}} |u_2(x)|^{r-1} |u_1(x) - u_2(x)|^2 dx \right) \\
&\quad + \frac{r-3}{r-1} \left( \frac{2}{\beta \mu (r-1)} \right)^{\frac{2}{r-3}} \left( \int_{\mathcal{O}} |u_1(x) - u_2(x)|^2 dx \right),
\end{aligned}
\tag{3.100}
\]
for $r > 3$. Using (3.100) in (3.99), we find

$$|\langle B(u_1 - u_2, u_1 - u_2), u_1 - u_2 \rangle| \leq \frac{\mu}{2} \|u_1 - u_2\|_V^2 + \frac{\beta}{2} \|u_2\|_V^{r-1} \|u_1 - u_2\|_H^2$$

$$+ \frac{r - 3}{2\mu(r - 1)} \left( \frac{2}{\beta\mu(r - 1)} \right)^{\frac{2}{r - 3}} \|u_1 - u_2\|_H^2. \tag{3.101}$$

From (3.101), we obtain

$$|\langle B(u_1) - B(u_2), w \rangle| \leq \frac{\mu}{2} \|w\|_V^2 + \frac{\beta}{2} \|u_2\|_V^{r-1} \|w\|_H^2 + \eta \|w\|_H^2,$$

where $\eta = \frac{r - 3}{2\mu(r - 1)} \left( \frac{2}{\beta\mu(r - 1)} \right)^{\frac{2}{r - 3}}$ and from (2.13), we get

$$\beta \langle C(u_1) - C(u_2), w \rangle \geq \frac{\beta}{2} \|u_2\|_V^{r-1} \|w\|_H^2.$$

Thus, using the above two estimates in (3.98), we infer that

$$\|w(t \wedge \tau_N)\|_H^2 + \mu \int_0^{t \wedge \tau_N} \|w(s)\|_V^2 \, ds$$

$$\leq \|w(0)\|_H^2 + 2\eta \int_0^{t \wedge \tau_N} \|w(s)\|_H^2 \, ds + \int_0^{t \wedge \tau_N} \int_Z \|\tilde{\gamma}(s, w(s), z)\|_H^2 \lambda(\, d\!s, d\!z)$$

$$+ 2 \int_0^{t \wedge \tau_N} \int_Z (\tilde{\gamma}(s -, w(s -), z), w(s -)) \tilde{\pi}(\, ds, d\!z). \tag{3.102}$$

It should be noted that the final term in the right hand side of the inequality (3.102) is a martingale. Taking expectation in (3.102), and then using Hypothesis 3.1 (H.2), we obtain

$$\mathbb{E} \left[ \|w(t \wedge \tau_N)\|_H^2 + \mu \int_0^{t \wedge \tau_N} \|w(s)\|_V^2 \, ds \right]$$

$$\leq \mathbb{E} \left[ \|w(0)\|_H^2 \right] + 2\eta \mathbb{E} \left[ \int_0^{t \wedge \tau_N} \|w(s)\|_H^2 \, ds \right]$$

$$+ \mathbb{E} \left[ \int_0^{t \wedge \tau_N} \int_Z \|\tilde{\gamma}(s, w(s), z)\|_H^2 \lambda(d\!z) \, ds \right]$$

$$\leq \mathbb{E} \left[ \|w(0)\|_H^2 \right] + (L + 2\eta) \mathbb{E} \left[ \int_0^{t \wedge \tau_N} \|w(s)\|_H^2 \, ds \right]. \tag{3.103}$$

Applying Gronwall’s inequality in (3.103), we arrive at

$$\mathbb{E} \left[ \|w(t \wedge \tau_N)\|_H^2 \right] \leq \mathbb{E} \left[ \|w(0)\|_H^2 \right] e^{(L + 2\eta)T}. \tag{3.104}$$
Thus the initial data \( u_1(0) = u_2(0) = u_0 \) leads to \( w(t \wedge \tau_N) = 0 \), \( \mathbb{P} \)-a.s. But using the fact that \( \tau_N \to T \), \( \mathbb{P} \)-a.s., implies \( w(t) = 0 \) and hence \( u_1(t) = u_2(t) \), \( \mathbb{P} \)-a.s., for all \( t \in [0, T] \), and hence the uniqueness follows.

**Remark 3.7** Recently authors in [30] (Theorem 1) obtained Itô’s formula (semimartingales) for processes taking values in intersection of finitely many Banach spaces. But it appears to the author that this result may not be applicable in our context for establishing the energy equality (3.95), as our operators \( B(\cdot), C(\cdot) : \mathbb{V} \cap \tilde{L}^{r+1} \to \mathbb{V'} + \tilde{L}^{r+1} \) and one can show the local integrability in the sum of Banach spaces only.

**Theorem 3.8** For \( d = 3, 4 \), let \( \mathcal{O} \subset \mathbb{R}^d \) be a bounded domain with \( C^2 \)-boundary. For \( r = 3 \) and \( 2 \beta \mu \geq 1 \), let \( u_0 \in L^2(\Omega; \mathbb{H}) \) be given. Then under Hypothesis 3.1, there exists a pathwise unique strong solution \( u(\cdot) \) to the system (3.1) such that

\[
 u \in L^2\left(\Omega; L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V})\right) \cap L^4(\Omega; L^4(0, T; \tilde{\mathbb{L}}^4)),
\]

with \( \mathbb{P} \)-a.s., càdlàg paths in \( \mathbb{H} \).

**Proof** Proof of Theorem 3.8 follows similarly as in Theorem 3.6, by using the global monotonicity result (2.18) and the fact that

\[
 \int_0^T \langle G(u(t)) - G(v(t)), u(t) - v(t) \rangle + \frac{L}{2} \int_0^T \|u(t) - v(t)\|^2_{\mathbb{H}} \, dt \\
 \geq \frac{1}{2} \int_0^T \int_Z \|\gamma(t, u(t), z) - \gamma(t, v(t), z)\|^2_{\mathbb{H}} \lambda(dz) \, dt,
\]

for \( 2 \beta \mu \geq 1 \). The estimate

\[
 |\langle B(u_1 - u_2, u_1 - u_2), u_2 \rangle| \leq \|u_2(u_1 - u_2)\|_{\mathbb{H}} \|u_1 - u_2\|_{\mathbb{V}} \\
 \leq \mu \|u_1 - u_2\|^2_{\mathbb{V}} + \frac{1}{4 \mu} \|u_2(u_1 - u_2)\|^2_{\mathbb{H}},
\]

helps us to obtain the uniqueness.

**Remark 3.9** 1. For \( d = 2 \), \( r \in [1, 3] \) and \( u_0 \in L^4(\Omega; \mathbb{H}) \), one can obtain the existence and uniqueness of pathwise strong solution

\[
 u \in L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))
\]

with \( \mathbb{P} \)-a.s. càdlàg paths in \( \mathbb{H} \) to the system (3.1) can be obtained by using local monotonicity result given in (2.19) (see [7, 47, 54], etc for similar techniques).

2. If the domain is a \( d \)-dimensional torus, then one can approximate functions in \( L^p \)-spaces using the truncated Fourier expansions in the following way (see [61, Theorem...
Let $\mathcal{O} = [0, 2\pi]^d$ and $Q_k := [-k, k]^d \cap \mathbb{Z}^d$. For every $w \in \mathbb{L}^1(\mathcal{O})$ and every $k \in \mathbb{N}$, we define

$$R_k(w) := \sum_{m \in Q_k} \hat{w}_j e^{im \cdot x},$$

where the Fourier coefficients $\hat{w}_j$ are given by $\hat{w}_j := \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} w(x) e^{-im \cdot x} dx$. Then, for every $1 < p < \infty$, there exists a constant $C_p$, independent of $k$, such that

$$\|R_k w\|_{\mathbb{L}^p(\mathcal{O})} \leq C_p \|w\|_{\mathbb{L}^p(\mathcal{O})}, \text{ for all } w \in \mathbb{L}^p(\mathcal{O}),$$

and

$$\|R_k w - w\|_{\mathbb{L}^p(\mathcal{O})} \to 0 \text{ as } k \to \infty.$$

### 3.5 Regularity of strong solution

In order to obtain the regularity results of the strong solution to (3.1), we restrict ourselves to periodic domains with $\int_{\mathcal{O}} u(x) dx = 0$. As discussed in the introduction, the main difficulty in working with bounded domains is that $\mathcal{P}(|u|^{-1} u)$ need not be zero on the boundary, and $\mathcal{P}$ and $-\Delta$ are not necessarily commuting (see [61]). Moreover, $\Delta u \cdot n |_{\partial \mathcal{O}} \neq 0$ in general and the term with pressure will not disappear. Thus, the estimate (1.2) may not be useful in bounded domains. In periodic domains or on the whole space $\mathbb{R}^d$, the operators $\mathcal{P}$ and $-\Delta$ commute, so that we can use (1.2) and we have the following result also (see [31, Lemma 2.1]):

$$0 \leq \int_{\mathcal{O}} |\nabla u(x)|^2 |u(x)|^{r-1} dx \leq \int_{\mathcal{O}} |u(x)|^{r-1} u(x) \cdot Au(x) dx$$

$$\leq r \int_{\mathcal{O}} |\nabla u(x)|^2 |u(x)|^{r-1} dx. \quad (3.107)$$

Note that the estimate (3.107) is true even in bounded domains (with Dirichlet boundary conditions) if one replaces $Au$ with $-\Delta u$ and (3.109)–(3.111) (see below) holds true in bounded domains as well as on the whole space $\mathbb{R}^d$. From [62], we have

$$\|u\|_{\mathbb{L}^\frac{p+q}{d-p}(\mathcal{O})}^{p+q} \leq C \int_{\mathcal{O}} |\nabla u(x)|^p |u(x)|^q dx, \quad (3.108)$$

for all $u \in W^{1,m}_0(\mathcal{O})$ with $m = d(p + q)/(d + q)$, $p < d$. Thus, for $d = 3$, we find

$$\|u\|_{\mathbb{L}^2_{r+1}(\mathcal{O})} \leq C \int_{\mathcal{O}} |\nabla u(x)|^2 |u(x)|^{r-1} dx, \quad \text{for } r \geq 1, \quad (3.109)$$
and for $d = 4$, we obtain
\[
\|u\|_{L^{p(r+1)}(\Omega)}^{r+1} \leq C \int_{\Omega} |\nabla u(x)|^2 |u(x)|^{r-1} \, dx, \quad \text{for } r \geq 1, \tag{3.110}
\]
for all $u \in D(A)$. One can handle the case $d = 2$ in the following way: Let us take $u \in C_0^\infty(\Omega)$. Then $|u|^{p/2} \in H_0^1(\Omega)$, for all $p \in [2, \infty)$, and from the Sobolev embedding, $H_0^1(\Omega) \subset L^p(\Omega)$, for all $p \in [2, \infty)$, we find
\[
\|u\|_{L^{p(r+1)}(\Omega)}^{r+1} = \|u\|_{L^{2p,2}(\Omega)}^{2} \leq C \int_{\Omega} |\nabla u(x)|^{r+1} \, dx.
\]
But from [62, Lemma 2.2], we infer that $|\nabla u|^{r+1} \leq C_r |u|^{r-1} |\nabla u|^2$. Thus, we further have
\[
\|u\|_{L^{p(r+1)}(\Omega)}^{r+1} \leq C \int_{\Omega} |\nabla u(x)|^2 |u(x)|^{r-1} \, dx, \tag{3.111}
\]
for any $p \in [2, \infty)$.

In order to obtain the regularity results, we assume that the noise coefficient satisfies the following:

**Hypothesis 3.10** There exist a positive constant $\tilde{K}$ such that for all $t \in [0, T]$ and $u \in \mathbb{V}$,
\[
\int_Z \|A^{1/2} \gamma(t, u, z)\|^2_{H^1} \lambda(\,dz\,) \leq \tilde{K} \left(1 + \|u\|_V^2\right).
\]

**Theorem 3.11** Let $\Omega \subset \mathbb{R}^d$, $2 \leq d \leq 4$ be a periodic domain. Let $u_0 \in L^2(\Omega; \mathbb{V})$ be given. Then under Hypotheses 3.1 and 3.10, for $r \geq 3$, the pathwise unique strong solution $u(\cdot)$ to the system (3.1) satisfies the following regularity:
\[
\|u\|_{L^2(\Omega; L^\infty(0, T; \mathbb{V}))} \cap L^{r+1}(\Omega; L^{r+1}(0, T; \overline{L^p(r+1)}))
\]

\[
\tag{3.112}
\]
where $p \in [2, \infty)$ for $d = 2$, $p = 3$ for $d = 3$, and $p = 2$ for $d = 4$, with $\mathcal{F}_t$-adapted paths of $u \in D([0, T]; \mathbb{V}) \cap L^2(0, T; D(A)) \cap L^{r+1}(0, T; \overline{L^p(r+1)})$, $\mathbb{P}$-a.s., and (3.6) is satisfied $\mathbb{P}$-a.s., for all $v \in \mathbb{H}$ and $t \in [0, T]$.

Moreover, the following Itô formula is satisfied:
\[
\|u(t)\|^2_{\mathbb{V}} + 2\mu \int_0^t \|Au(s)\|_{H^1}^2 \, ds
\]
\[
= \|u_0\|^2_{\mathbb{V}} - \int_0^t (B(u(s)), Au(s)) \, ds - 2\beta \int_0^t (C(u(s)), Au(s)) \, ds
\]
Proof Let us take \( \mathbf{u}_0 \in L^2(\Omega; \mathcal{V}) \) to obtain further regularity results of the strong solution to the system \( (3.1) \) for \( r \geq 3. \)

**Step I:** Finite dimensional approximation and limit for \( r > 3. \) Firstly, we consider the finite dimensional approximation given in \( (3.7) \) or equivalently \( (3.24). \) Let us define a sequence of stopping times
\[
\tilde{\tau}_N^n := \inf_{t \geq 0} \{ t : \| \mathbf{u}^n(t) \|_{\mathcal{V}} > N \},
\]
for \( N \in \mathbb{N}. \) Applying the finite dimensional Itô formula to the process \( \| A^{1/2} \mathbf{u}^n(\cdot) \|_{\mathcal{H}}^2, \)
we get
\[
\| \mathbf{u}^n(t \wedge \tilde{\tau}_N^n) \|_{\mathcal{V}}^2 + 2\mu \int_0^{t \wedge \tilde{\tau}_N^n} \| A \mathbf{u}^n(s) \|_{\mathcal{H}}^2 ds
\]
\[
= \| \mathbf{u}^n_0 \|_{\mathcal{V}}^2 - 2 \int_0^{t \wedge \tilde{\tau}_N^n} (B^n(\mathbf{u}^n(s)), A \mathbf{u}^n(s)) ds - 2\beta \int_0^{t \wedge \tilde{\tau}_N^n} (C^n(\mathbf{u}^n(s)), A \mathbf{u}^n(s)) ds
\]
\[
+ \int_0^{t \wedge \tilde{\tau}_N^n} \int_{\mathcal{V}} \| \gamma^n(s, \mathbf{u}^n(s), z) \|_{\mathcal{H}}^2 \pi(ds, dz)
\]
\[
+ 2 \int_0^{t \wedge \tilde{\tau}_N^n} \int_{\mathcal{V}} (A^{1/2} \gamma^n(s-, \mathbf{u}^n(s-), z), A^{1/2} \mathbf{u}^n(s)) \tilde{\pi}(ds, dz).
\]

We estimate \(|(B^n(\mathbf{u}^n), A \mathbf{u}^n)|\) using Hölder’s, and Young’s inequalities as
\[
|(B^n(\mathbf{u}^n), A \mathbf{u}^n)| \leq \| \mathbf{u}^n \|_{\mathcal{V}}\| \nabla \mathbf{u}^n \|_{\mathcal{H}}\| A \mathbf{u}^n \|_{\mathcal{H}} \leq \frac{\mu}{2} \| A \mathbf{u}^n \|_{\mathcal{H}}^2 + \frac{1}{2\mu} \| \mathbf{u}^n \|_{\mathcal{V}}\| \nabla \mathbf{u}^n \|_{\mathcal{H}}^2.
\]

For \( r > 3. \), we estimate the final term from \( (3.116) \) using Hölder’s and Young’s inequalities as
\[
\int_\Omega |\mathbf{u}^n(x)|^2 |\nabla \mathbf{u}^n(x)|^2 dx
\]
\[
= \int_\Omega |\mathbf{u}^n(x)|^2 |\nabla \mathbf{u}^n(x)|^{\frac{4}{r-1}} |\nabla \mathbf{u}^n(x)|^{\frac{2(r-3)}{r-1}} dx
\]
\[
\leq \left( \int_\Omega |\mathbf{u}^n(x)|^{r-1} |\nabla \mathbf{u}^n(x)|^2 dx \right)^{\frac{2}{r-1}} \left( \int_\Omega |\nabla \mathbf{u}^n(x)|^2 dx \right)^{\frac{r-3}{r-1}}
\]
\[ \leq \beta \mu \left( \int_{\Omega} |u^n(x)|^{r-1} |\nabla u^n(x)|^2 \, dx \right) \]
\[ + \frac{r - 3}{r - 1} \left( \frac{2}{\beta \mu (r - 1)} \right)^{\frac{2}{r - 3}} \left( \int_{\Omega} |\nabla u^n(x)|^2 \, dx \right). \]

Making use of the estimates (3.107) and (3.116) in (3.115), taking supremum over time from 0 to \( T \) and then taking expectation, we find

\[
\mathbb{E} \left[ \sup_{t \in [0, T \wedge T_N]} \|u^n(t)\|_{V}^2 + \mu \int_{0}^{T \wedge T_N} \|Au^n(t)\|_{H}^2 \, dt \right] + \beta \int_{0}^{T \wedge T_N} \|u^n(t)\|_{\frac{r-1}{2}} \|\nabla u^n(t)\|_{H}^2 \, dt \]
\[
\leq \mathbb{E} \left[ \|u^n_0\|_{V}^2 \right] + 2\eta \mathbb{E} \left[ \int_{0}^{T \wedge T_N} \|u^n(t)\|_{V}^2 \, dt \right] + \mathbb{E} \left[ \int_{0}^{T \wedge T_N} \int_{Z} \|A^{1/2} \gamma^n(t, u^n(t), z)\|_{H}^2 \lambda(dz) \, dt \right]
\[
+ 2\mathbb{E} \left[ \sup_{t \in [0, T \wedge T_N]} \left| \int_{0}^{t} \int_{Z_n} \left( A^{1/2} \gamma^n(s-, u^n(s-), z), A^{1/2} u^n(s-) \right) \tilde{\pi}(ds, dz) \right| \right],
\](3.117)

where \( \eta \) is defined in (2.17). Applying the Burkholder–Davis–Gundy inequality to the final term appearing in the right hand side of the inequality (3.117), we obtain

\[
2\mathbb{E} \left[ \sup_{t \in [0, T \wedge T_N]} \left| \int_{0}^{t} \int_{Z_n} \left( A^{1/2} \gamma^n(s-, u^n(s-), z), A^{1/2} u^n(s-) \right) \tilde{\pi}(ds, dz) \right| \right]
\[
\leq 2\sqrt{3} \mathbb{E} \left[ \int_{0}^{T \wedge T_N} \int_{Z} \|A^{1/2} \gamma^n(t, u^n(t), z)\|_{H}^2 \|A^{1/2} u^n(t)\|_{H}^2 \pi(dt, dz) \right]^{1/2}
\[
\leq 2\sqrt{3} \mathbb{E} \left[ \sup_{t \in [0, T \wedge T_N]} \|A^{1/2} u^n(t)\|_{H} \left( \int_{0}^{T \wedge T_N} \int_{Z} \|A^{1/2} \gamma^n(t, u^n(t), z)\|_{H}^2 \pi(dt, dz) \right)^{1/2} \right]
\[
\leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0, T \wedge T_N]} \|A^{1/2} u^n(t)\|_{H}^2 \right]
\[
+ 6\mathbb{E} \left[ \int_{0}^{T \wedge T_N} \int_{Z} \|A^{1/2} \gamma^n(t, u^n(t), z)\|_{H}^2 \lambda(dz) \, dt \right].
\](3.118)
Substituting (3.118) in (3.117), we deduce that

\[
\mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau^n_N]} \|u^n(t)\|_V^2 + 2 \mu \int_0^{T \wedge \tau^n_N} \|Au^n(t)\|_{H^1}^2 \, dt \right. \\
+ 2 \beta \int_0^{T \wedge \tau^n_N} \|u^n(t)\|_V^{r+1} \|\nabla u^n(t)\|_{H^1}^2 \, dt \left. \right] \\
\leq 2 \mathbb{E} \left[ \|u_0^n\|_V^2 \right] + 4 \eta \mathbb{E} \left[ \int_0^{T \wedge \tau^n_N} \|u^n(t)\|_V^2 \, dt \right] \\
+ 14 \mathbb{E} \left[ \int_0^{T \wedge \tau^n_N} \int_Z \|A^{1/2} \gamma^n(t, u^n(t), z)\|_{H^{1/2}}^2 \lambda(dz) \, dt \right] \\
\leq 2 \mathbb{E} \left[ \|u_0^n\|_V^2 \right] + 4 \eta \mathbb{E} \left[ \int_0^{T \wedge \tau^n_N} \|u^n(t)\|_V^2 \, dt \right] \\
+ 14 \tilde{K} \mathbb{E} \left[ \int_0^{T \wedge \tau^n_N} (1 + \|u^n(t)\|_V^2) \, dt \right], \tag{3.119}
\]

where we have used Hypothesis 3.1 (H.1). Applying Gronwall’s inequality in (3.119), we find

\[
\mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau^n_N]} \|u^n(t)\|_V^2 \right] \leq \left\{ 2 \mathbb{E} \left[ \|u_0^n\|_V^2 \right] + 14 \tilde{K} T \right\} e^{(4 \eta + 14 \tilde{K}) T}. \tag{3.120}
\]

Passing \( N \to \infty \), we know that \( T \wedge \tau^n_N \to T \), and hence taking \( N \to \infty \) in (3.120), we deduce that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|u^n(t)\|_V^2 \right] \leq \left\{ 2 \mathbb{E} \left[ \|u_0^n\|_V^2 \right] + 14 \tilde{K} T \right\} e^{(4 \eta + 14 \tilde{K}) T}. \tag{3.121}
\]

Using (3.111) (respectively (3.109), (3.110) also) and (3.120) in (3.119), we finally get

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|u^n(t)\|_V^2 + 2 \mu \int_0^T \|Au^n(t)\|_{H^1}^2 \, dt + 2 \beta \int_0^T \|u^n(t)\|_{L^{p(r+1)}}^2 \, dt \right] \\
\leq \left\{ 2 \mathbb{E} \left[ \|u_0^n\|_V^2 \right] + 14 \tilde{K} T \right\} e^{(4 \eta + 7 \tilde{K}) T}, \tag{3.121}
\]

where \( p \in [2, \infty) \) for \( d = 2 \), \( p = 3 \) for \( d = 3 \), and \( p = 2 \) for \( d = 4 \). Note that the right hand side of the inequality (3.121) is independent of \( n \). Thus, making use of the
Banach-Alaoglu theorem, we infer from (3.121) that

\[
\begin{align*}
  u^n & \xrightarrow{w^*} \tilde{u} \quad \text{in } L^2(\Omega; L^\infty(0, T; \mathbb{V})), \\
  u^n & \xrightarrow{w} \bar{u} \quad \text{in } L^2(\Omega; L^2(0, T; D(A))), \\
  u^n & \xrightarrow{w} \bar{u} \quad \text{in } L^{r+1}(\Omega; L^{r+1}(0, T; \mathbb{H}^{p(r+1)})).
\end{align*}
\]  

(3.122)

Because the sequence \( \{u^n\} \) also satisfies the convergence (3.27) and the weak limit is unique, we obtain \( u = \bar{u} \), where \( u(\cdot) \) satisfies (3.6). Hence, we get the regularity given in (3.112).

**Step II**: \( u(\cdot) \) satisfies (3.6), \( \mathbb{P} \text{-a.s.}, \) for all \( \mathbb{v} \in \mathbb{H} \) and \( t \in [0, T] \). Our next aim is to show that \( u(\cdot) \) satisfies (3.4) as an equality in \( \mathbb{H}, \mathbb{P} \text{-a.s.} \), that is, the equation (3.6) is satisfied for all \( \mathbb{v} \in \mathbb{H} \) and \( t \in [0, T] \). From the second convergence in (3.122), we easily have

\[
\mathbb{E} \left[ \left| \int_0^t (Au^n(s), \mathbb{v}) \, ds \right| \right] \rightarrow \mathbb{E} \left[ \left| \int_0^t (Au(s), \mathbb{v}) \, ds \right| \right] \quad \text{as } n \to \infty, \quad \text{for all } \mathbb{v} \in \mathbb{H}.
\]

Thus, along a subsequence \( \{u^{nk}\} \) of \( \{u^n\} \), we obtain

\[
\int_0^t (Au^n(s), \mathbb{v}) \, ds \xrightarrow{a.s.} \int_0^t (Au(s), \mathbb{v}) \, ds \quad \text{as } n \to \infty,
\]

(3.123)

for all \( \mathbb{v} \in \mathbb{H} \) and \( t \in [0, T] \). Now, we need to verify the convergences of the nonlinear terms only. Before establishing that, we first obtain some strong convergences. Let us define an another sequence of stopping times by

\[
\hat{\tau}_N^n := \inf_{t \geq 0} \left\{ t : \|u^n(t)\|_{\mathbb{H}} + \|u(t)\|_{\mathbb{H}} > N \right\},
\]

(3.124)

for \( N \in \mathbb{N} \). Since \( u(\cdot) \) is the unique strong solution of (3.4) satisfying the energy equality (3.95), we find that the process \( \|u^n(\cdot) - u(\cdot)\|_{\mathbb{H}}^{\infty} \) satisfies

\[
\begin{align*}
\|u^n(t \vee \hat{\tau}_N^n) - u(t \vee \hat{\tau}_N^n)\|_{\mathbb{H}}^2 & + 2\mu \int_0^{t \vee \hat{\tau}_N^n} \|u^n(s) - u(s)\|_{\mathbb{H}}^2 \, ds \\
& = \|u^n_0 - u_0\|_{\mathbb{H}}^2 - 2\beta \int_0^{t \vee \hat{\tau}_N^n} \langle C(u^n(s)) - C(u(s)), u^n(s) - u(s) \rangle \, ds \\
& \quad - \int_0^{t \vee \hat{\tau}_N^n} \langle B(u^n(s)) - B(u(s)), u^n(s) - u(s) \rangle \, ds \\
& \quad + \int_0^{t \vee \hat{\tau}_N^n} \int_{\mathbb{Z}_n} \|\gamma^n(s, u^n(s), z) - \gamma(s, u(s), z)\|_{\mathbb{H}}^2 \, d\pi(ds, dz) \\
& \quad + 2 \int_0^{t \vee \hat{\tau}_N^n} \int_{\mathbb{Z}_n} \|\gamma^n(s-, u^n(s-), z) - \gamma(s-, u(s-), z), u^n(s-) - u(s-)\|_{\mathbb{H}}^2 \, d\pi(ds, dz)
\end{align*}
\]
\[ + \int_{0}^{t \wedge \hat{\tau}^n_N} \int_{Z \setminus Z_n} \| \gamma(s, u(s), z) \|_{H^2}^2 \pi(ds, dz) \]
\[ + 2 \int_{0}^{t \wedge \hat{\tau}^n_N} \int_{Z \setminus Z_n} (\gamma(s-, u(s-), z), u^n(s-) - u(s-)) \hat{\pi}(ds, dz), \]

\[ \mathbb{P}\text{-a.s., for all } t \in [0, T]. \] A calculation similar to (3.102) gives (using (2.14))

\[ \| u^n(t) - u(t) \|_{H^2}^2 + \mu \int_{0}^{t \wedge \hat{\tau}^n_N} \| u^n(s) - u(s) \|_{H^2}^2 ds + \frac{\beta}{2^{r-1}} \int_{0}^{t \wedge \hat{\tau}^n_N} \| u^n(s) - u(s) \|_{T_{r+1}}^{r+1} ds \]
\[ \leq \| u^n_0 - u_0 \|_{H^2}^2 + 2 \hat{\eta} \int_{0}^{t \wedge \hat{\tau}^n_N} \| u^n(s) - u(s) \|_{H^2}^2 ds \]
\[ + \int_{0}^{t \wedge \hat{\tau}^n_N} \int_{Z_n} \| \gamma^n(s, u^n(s), z) - \gamma(s, u(s), z) \|_{H^2}^2 \pi(ds, dz) \]
\[ + 2 \int_{0}^{t \wedge \hat{\tau}^n_N} \int_{Z_n} (\gamma^n(s-, u^n(s-), z) - \gamma(s-, u(s-), z), u^n(s-) - u(s-)) \hat{\pi}(ds, dz) \]
\[ + \int_{0}^{t \wedge \hat{\tau}^n_N} \int_{Z \setminus Z_n} \| \gamma(s, u(s), z) \|_{H^2}^2 \pi(ds, dz) \]
\[ + 2 \int_{0}^{t \wedge \hat{\tau}^n_N} \int_{Z \setminus Z_n} (\gamma(s-, u(s-), z), u^n(s-) - u(s-)) \hat{\pi}(ds, dz), \]

where \( \hat{\eta} = \frac{r-3}{2\mu (r-1)} \left( \frac{4}{\beta \mu (r-1)} \right)^{\frac{2}{r-3}} \). Taking supremum over \( t \) in \([0, T]\) and then taking expectation, we find

\[ \mathbb{E} \left[ \sup_{t \in [0, T \wedge \hat{\tau}^n_N]} \| u^n(t) - u(t) \|_{H^2}^2 \right] + \mu \mathbb{E} \left[ \int_{0}^{T \wedge \hat{\tau}^n_N} \| u^n(t) - u(t) \|_{H^2}^2 dt \right] \]
\[ + \frac{\beta}{2^{r-1}} \mathbb{E} \left[ \int_{0}^{T \wedge \hat{\tau}^n_N} \| u^n(t) - u(t) \|_{T_{r+1}}^{r+1} dt \right] \]
\[ \leq \mathbb{E} \left[ \| u^n_0 - u_0 \|_{H^2}^2 \right] + 2 \hat{\eta} \mathbb{E} \left[ \int_{0}^{T \wedge \hat{\tau}^n_N} \| u^n(t) - u(t) \|_{H^2}^2 dt \right] \]
\[ + \mathbb{E} \left[ \int_{0}^{T \wedge \hat{\tau}^n_N} \int_{Z} \| \gamma^n(s, u^n(s), z) - \gamma(t, u(t), z) \|_{H^2}^2 \lambda(dz) dt \right] \]
\[ + 2 \mathbb{E} \left[ \sup_{t \in [0, T \wedge \hat{\tau}^n_N]} |I| \right] \]
\[ + \mathbb{E} \left[ \int_{0}^{T \wedge \hat{\tau}^n_N} \int_{Z \setminus Z_n} \| \gamma(t, u(t), z) \|_{H^2}^2 \lambda(dz) dt \right] \]
where

\[ I = \int_0^t \int_{Z_n} (\gamma''(s-, u^n(s-, z)) - \gamma(s-, u(s-, z)), u^n(s-) - u(s-)) \widehat{\Pi} (ds, dz). \]

Applying the Burkholder–Davis–Gundy inequality, we get

\[
2 \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_n^N]} \left| I \right| \right] \leq \frac{1}{4} \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_n^N]} \left\| u^n(t) - u(t) \right\|_{H}^2 \right] + 12 \mathbb{E} \left[ \int_0^{T \wedge \tau_n^N} \int_{Z \setminus Z_m} \left\| \gamma''(s, u^n(s), z) - \gamma(t, u(t), z) \right\|_{H}^2 \lambda(dz) dt \right].
\]

and

\[
2 \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_n^N]} \left| \int_0^{T \wedge \tau_n^N} \int_{Z \setminus Z_m} (\gamma(s-, u(s-, z)), u^n(s-) - u(s-)) \widehat{\Pi} (ds, dz) \right| \right] \leq \frac{1}{4} \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_n^N]} \left\| u^n(t) - u(t) \right\|_{H}^2 \right] + 12 \mathbb{E} \left[ \int_0^{T \wedge \tau_n^N} \int_{Z \setminus Z_m} \left\| \gamma(t, u(t), z) \right\|_{H}^2 \lambda(dz) dt \right].
\]

Thus, from (3.125), it can be easily deduced that

\[
\mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_n^N]} \left\| u^n(t) - u(t) \right\|_{H}^2 \right] + \mu \mathbb{E} \left[ \int_0^{T \wedge \tau_n^N} \left\| u^n(t) - u(t) \right\|_{H}^2 dt \right] + \frac{\beta}{2^{r-1}} \mathbb{E} \left[ \int_0^{T \wedge \tau_n^N} \left\| u^n(t) - u(t) \right\|_{L^{r+1}}^2 dt \right] \leq \mathbb{E} \left[ \left\| u^n_0 - u_0 \right\|_{H}^2 \right] + 2(\eta + 13L) \mathbb{E} \left[ \int_0^{T \wedge \tau_n^N} \left\| u^n(t) - u(t) \right\|_{H}^2 dt \right] + 26 \mathbb{E} \left[ \int_0^{T \wedge \tau_n^N} \int_{Z \setminus Z_m} \left\| \gamma(t, u(t), z) \right\|_{H}^2 \lambda(dz) dt \right],
\]

where we have used Hypothesis 3.1 (H.3). An application of Gronwall’s inequality in (3.126) yields (cf. [21] for similar arguments in the case of 2D Navier–Stokes
equations perturbed by Lévy noise) 

\[
\mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_{N}^n]} \| u^n(t) - u(t) \|_{\mathbb{H}}^2 \right] 
\leq \left\{ \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_{N}^n]} \| u^n(t) - u(t) \|_{\mathbb{H}}^2 \right] + 26 \mathbb{E} \left[ \int_0^{T \wedge \tau_{N}^n} \int_{\mathbb{Z} \setminus \mathbb{Z}_n} \| \gamma(t, u(t), z) \|_{\mathbb{H}}^2 \lambda(dz) dt \right] \right\} e^{2(\tilde{\gamma} + 13L)T} 
\leq \left\{ \mathbb{E} \left[ \sup_{\|u\|_{\mathbb{H}} \leq N} \int_{\mathbb{Z}_n} \| \gamma(t, u, z) \|_{\mathbb{H}}^2 \lambda(dz) \right] \right\} e^{2(\tilde{\gamma} + 13L)T}. \tag{3.127}
\]

From Hypothesis 3.1 (H.4), (3.84), (3.125) and (3.126), we infer that

\[
\lim_{n \to \infty} \left\{ \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_{N}^n]} \| u^n(t) - u(t) \|_{\mathbb{H}}^2 \right] + \mu \mathbb{E} \left[ \int_0^{T \wedge \tau_{N}^n} \| u^n(t) - u(t) \|_{\mathbb{Y}}^2 dt \right] \right\} + \frac{\beta}{2^{r-1}} \mathbb{E} \left[ \int_0^{T \wedge \tau_{N}^n} \| u^n(t) - u(t) \|_{L^1_{\mathbb{Y}}} dt \right] = 0, \tag{3.128}
\]

for each fixed \( N \). It can be easily seen that

\[
\mathbb{P} \{ t > \tau_{N}^n \} \leq \mathbb{P} \{ \| u^n(t) \|_{\mathbb{H}} \vee \| u(t) \|_{\mathbb{H}} \geq N \} \leq \frac{1}{N^2} \left\{ \mathbb{E} \left[ \| u^n(t) \|_{\mathbb{H}}^2 \right] + \mathbb{E} \left[ \| u(t) \|_{\mathbb{H}}^2 \right] \right\} 
\leq \frac{2}{N^2} \left( 2 \mathbb{E} \left[ \| u_0 \|_{\mathbb{H}}^2 \right] + 14 KT \right) e^{28KT} \leq \frac{C(T)}{N^2}.
\]

Then, for \( 1 \leq p < 2 \), we immediately have

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \| u^n(t) - u(t) \|_{\mathbb{H}}^p \right] 
= \mathbb{E} \left[ \sup_{t \in [0, T]} \| u^n(t) - u(t) \|_{\mathbb{H}}^p \chi_{\{T \leq \tau_{N}^n\}} \right] + \mathbb{E} \left[ \sup_{t \in [0, T]} \| u^n(t) - u(t) \|_{\mathbb{H}}^p \chi_{\{T > \tau_{N}^n\}} \right] 
\leq \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_{N}^n]} \| u^n(t) - u(t) \|_{\mathbb{H}}^p \right] 
+ \left\{ \mathbb{E} \left[ \sup_{t \in [0, T]} \| u^n(t) - u(t) \|_{\mathbb{H}}^2 \right] \right\}^{\frac{p}{2}} \mathbb{P} \{ T > \tau_{N}^n \} \left\{ \mathbb{P} \{ T > \tau_{N}^n \} \right\}^{\frac{2-p}{2}} 
\leq \left\{ \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_{N}^n]} \| u^n(t) - u(t) \|_{\mathbb{H}}^2 \right] \right\}^{\frac{p}{2}} + \frac{C(T)}{N^{2-p}}.
\]
Similarly, for $1 \leq p < 2$ and $1 \leq q < r + 1$, we obtain

$$
\mathbb{E}\left[ \left( \int_0^T \| u^n(t) - u(t) \|^2_{V} dt \right)^{\frac{p}{2}} \right] \leq \left\{ \begin{array}{c} \mathbb{E}\left[ \int_0^T \| u^n(t) - u(t) \|^2_{V} dt \right] \\
\frac{C(T)}{N^{2-p}} \\
+ \frac{\beta}{2^{r-1}} \mathbb{E}\left[ \left( \int_0^T \| u^n(t) - u(t) \|_{L^{r+1}}^{r+1} dt \right)^{\frac{q}{r+1}} \right] \\
\end{array} \right\}.
$$

Note that the first term in the right hand side of last three inequalities tends to zero as $n \to \infty$, for each fixed $N$, and the second term goes to 0 as $N \to \infty$. Thus, for each fixed $T$, we get

$$
\lim_{n \to \infty} \left\{ \mathbb{E}\left[ \sup_{t \in [0,T]} \| u^n(t) - u(t) \|_{\mathbb{H}} \right] + \mu \mathbb{E}\left[ \left( \int_0^T \| u^n(t) - u(t) \|^2_{V} dt \right)^{1/2} \right] \\
+ \frac{\beta}{2^{r-1}} \mathbb{E}\left[ \left( \int_0^T \| u^n(t) - u(t) \|_{L^{r+1}}^{r+1} dt \right)^{\frac{q}{r+1}} \right] \right\} = 0. \tag{3.129}
$$

From the above convergence, we infer that

$$
\left\{ \begin{array}{l}
u^n \to u \text{ in } L^p(\Omega; L^\infty(0, T; \mathbb{H})), \\
u^n \to u \text{ in } L^p(\Omega; L^2(0, T; \mathbb{V})), \\
u^n \to u \text{ in } L^q(\Omega; L^{r+1}(0, T; \tilde{\mathbb{L}}^{r+1})),
\end{array} \right.
$$

as $n \to \infty$, for any $1 \leq p < 2$ and $1 \leq q < r + 1$. For any $\psi \in C^1(\overline{\mathcal{O}})$, we have

$$
\mathbb{E}\left[ \left| \int_0^t \left[ b(u^n(s), u^n(s), \psi) ds - \int_0^t b(u(s), u(s), \psi) ds \right] \right| \right]
= \mathbb{E}\left[ \int_0^t |b(u^n(s) - u(s), \psi, u^n(s))| ds + \int_0^t |b(u(s), \psi, u^n(s) - u(s))| ds \right]
\leq \| \nabla \psi \|_{L^\infty} \mathbb{E}\left[ \int_0^t \| u^n(s) - u(s) \|_{\tilde{\mathbb{L}}^{r+1}} \| u^n(s) \|_{\tilde{\mathbb{L}}^{r+1}} ds \right]
\leq \| \nabla \psi \|_{L^\infty} \mathbb{E}\left[ \int_0^t \| u^n(s) \|_{\tilde{\mathbb{L}}^{r+1}} \| u^n(s) - u(s) \|_{\tilde{\mathbb{L}}^{r+1}} ds \right]
\leq \| \nabla \psi \|_{L^\infty} \mathbb{E}\left[ \int_0^t \sup_{s \in [0,t]} \| u^n(s) \|_{\mathbb{H}} \left( \int_0^t \| u^n(s) - u(s) \|_{\tilde{\mathbb{L}}^{r+1}} \right)^{\frac{1}{r+1}} ds \right].
$$

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\[ + \| \nabla \psi \|_{L^\infty} T^{r-1} \| \mathcal{O} \|_{\mathcal{L}^{r+1}} \mathbb{E} \left[ \sup_{s \in [0, t]} \| u(s) \|_H \left( \int_0^t \| u^n(s) - u(s) \|_{L^{r+1}} \right)^{\frac{1}{r+1}} \right] \]

\[ \leq \| \nabla \psi \|_{L^\infty} T^{r-1} \| \mathcal{O} \|_{\mathcal{L}^{r+1}} \left\{ \mathbb{E} \left[ \left( \int_0^T \| u^n(t) - u(t) \|_{L^{r+1}} \right)^{\frac{q}{q+1}} \right] \right\} \]

\[ \times \left\{ \mathbb{E} \left( \sup_{t \in [0, T]} \| u^n(t) \|_H^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} + \mathbb{E} \left( \sup_{t \in [0, T]} \| u(t) \|_H^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \right\} \]

\[ \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (3.130) \]

for all \( t \in [0, T] \) and any \( 2 \leq q < r + 1 \), where \( |\mathcal{O}| \) is the measure of \( \mathcal{O} \). Since \( C^1(\overline{\mathcal{O}}) \) is dense in \( H \), for any given \( v \in H \), there exists a sequence of functions \( \psi_m \in C^1(\overline{\mathcal{O}}) \) such that \( \| \psi_m - v \|_H \rightarrow 0 \) as \( m \rightarrow \infty \). Let us now consider

\[ \mathbb{E} \left[ \int_0^t b(u^n(s), u^n(s), v)ds - \int_0^t b(u(s), u(s), v)ds \right] \]

\[ \leq \mathbb{E} \left[ \int_0^t b(u^n(s), u^n(s), v - \psi_m)ds \right] + \mathbb{E} \left[ \int_0^t b(u(s), u(s), v - \psi_m)ds \right] \]

\[ + \mathbb{E} \left[ \int_0^t b(u^n(s), u^n(s), \psi_m)ds - \int_0^t b(u(s), u(s), \psi_m)ds \right]. \quad (3.131) \]

Note that the final term in the right hand side of the inequality (3.131) tends to zero as \( k \rightarrow \infty \) for each fixed \( m \), by using (3.130). We estimate the first term using Hölder’s and Agmon’s inequalities as

\[ \mathbb{E} \left[ \int_0^t b(u^n(s), u^n(s), v - \psi_m)ds \right] \]

\[ \leq \mathbb{E} \left[ \int_0^t \| u^n(s) \|_{L^\infty} \| u^n(s) \|_V \| v - \psi_m \|_H \right] \]

\[ \leq C \| u - \psi_m \|_H \mathbb{E} \left[ \int_0^t \| \Lambda u^n(s) \|_H^d \| u^n(s) \|_{L^4} \right] \]

\[ \leq C T^{\frac{9-d}{8}} \| v - \psi_m \|_H \mathbb{E} \left[ \sup_{t \in [0, T]} \| u^n(t) \|_H^{\frac{4-d}{8}} \sup_{t \in [0, T]} \| u^n(t) \|_V \left( \int_0^T \| \Lambda u^n(t) \|_V^2 \right)^{\frac{2}{d}} \right] \]

\[ \leq C T^{\frac{9-d}{8}} \| v - \psi_m \|_H \left\{ \mathbb{E} \left[ \int_0^T \| \Lambda u^n(t) \|_V^2 dt \right] \right\}^{\frac{d}{8}} \left\{ \mathbb{E} \left[ \sup_{t \in [0, T]} \| u^n(t) \|_V^2 \right] \right\}^{\frac{1}{2}} \]

\[ \times \left\{ \mathbb{E} \left[ \sup_{t \in [0, T]} \| u^n(t) \|_2^2 \right] \right\} \]

\[ \rightarrow 0 \text{ as } m \rightarrow \infty, \]
for $2 \leq d \leq 4$. Similarly, one can show that the second term from the right hand side of the inequality (3.131) tends to 0 as $m \to \infty$. Thus, we deduce from (3.131) that

$$
\mathbb{E} \left[ \int_0^t b(u^n(s), u^n(s), v) ds \right] \to \mathbb{E} \left[ \int_0^t b(u(s), u(s), v) ds \right] \quad \text{as} \quad k \to \infty,
$$

for all $v \in \mathbb{H}$. Thus, along a subsequence $\{u^{nk}\}$, we obtain

$$
\int_0^t b(u^{nk}(s), u^{nk}(s), v) ds \xrightarrow{a.s.} \int_0^t b(u(s), u(s), v) ds, \quad \text{as} \quad k \to \infty, \quad (3.132)
$$

for all $v \in \mathbb{H}$ and $t \in [0, T]$. Note that for $2 \leq d \leq 4$, $D(A) \subset \mathbb{H}^2(\mathcal{O}) \subset \mathbb{L}^p(\mathcal{O})$, for all $p \in (1, \infty)$. Thus for all $\psi \in D(A)$, we get

$$
\mathbb{E} \left[ \int_0^t (C(u^n(s)), \psi) ds - \int_0^t (C(u(s)), \psi) ds \right] \\
\leq \mathbb{E} \left[ \int_0^t \int_0^1 \left( u^n(s) + (1 - \theta)u(s) \right) (u^n(s) - u(s)) d\theta \right] ds \\
\leq r \mathbb{E} \left[ \int_0^t \left( \|u^n(s)\|_L^{r+1} + \|u(s)\|_L^{r+1} \right)^{r-1} \|u^n(s) - u(s)\|_L^{r+1} ds \right] \\
\leq C \|A\psi\|_{\mathbb{H}^2} \mathbb{E} \left[ \left( \int_0^t \|u^n(s) - u(s)\|_L^{r+1} ds \right)^{\frac{r}{r+1}} \left( \int_0^t \|u^n(s)\|_L^{\frac{(r-1)(r+1)}{r+1}} ds \right)^{\frac{r}{r+1}} \right] \\
+ C \|A\psi\|_{\mathbb{H}^2} \mathbb{E} \left[ \left( \int_0^t \|u^n(s) - u(s)\|_L^{r+1} ds \right)^{\frac{r}{r+1}} \left( \int_0^t \|u(s)\|_L^{\frac{(r-1)(r+1)}{r+1}} ds \right)^{\frac{r}{r+1}} \right] \\
\leq C \|A\psi\|_{\mathbb{H}^2} T \frac{1}{q} \left\{ \mathbb{E} \left[ \left( \int_0^T \|u^n(t) - u(t)\|_L^{r+1} dt \right)^q \right]^{\frac{1}{q}} \right\} \frac{q-1}{q} \\
\times \left\{ \mathbb{E} \left[ \left( \int_0^T \|u^n(t)\|_L^{r+1} dt \right)^{\frac{(r-1)q}{(r+1)(q-1)}} \right] \right\}^{\frac{q-1}{q}} \\
+ \left\{ \mathbb{E} \left[ \left( \int_0^T \|u(t)\|_L^{r+1} dt \right)^{\frac{(r-1)q}{(r+1)(q-1)}} \right] \right\}^{\frac{q-1}{q}} \\
\to 0 \quad \text{as} \quad n \to \infty, \quad (3.133)
$$

for all $t \in [0, T]$ and any $\frac{r+1}{q} \leq q < r + 1$. Since $D(A)$ is dense in $\mathbb{H}$, for any given $v \in \mathbb{H}$, there exists a sequence of functions $\psi_m \in D(A)$ such that $\|\psi_m - v\|_{\mathbb{H}} \to$
0 as \( m \to \infty \). Let us now consider
\[
\mathbb{E} \left[ \left| \int_0^t (C(u^n(s)), v) ds - \int_0^t (C(u(s)), v) ds \right| \right] \\
\leq \mathbb{E} \left[ \left| \int_0^t (C(u^n(s)), v - \psi_m) ds \right| \right] + \mathbb{E} \left[ \left| \int_0^t (C(u(s)), v - \psi_m) ds \right| \right] \\
+ \mathbb{E} \left[ \left| \int_0^t (C(u^n(s)), \psi_m) ds - \int_0^t (C(u(s)), \psi_m) ds \right| \right].
\]
(3.134)

It should be noted that the final term in the right hand side of the inequality (3.134) tends to zero as \( k \to \infty \) for each fixed \( m \), by using (3.133). We estimate the first term using Hölder’s inequality as
\[
\mathbb{E} \left[ \left| \int_0^t (C(u^n(s)), v - \psi_m) ds \right| \right] \leq \|v - \psi_m\|_\mathcal{H} \mathbb{E} \left[ \left| \int_0^t \|u(s)\|^{p+1}_{\mathcal{L}^2} ds \right|^{\frac{1}{p+1}} \right]
\]
\[
\leq T^{\frac{1}{p+1}} \|v - \psi_m\|_\mathcal{H} \left\{ \mathbb{E} \left[ \left| \int_0^T \|u^n(t)\|^{p+1}_{\mathcal{L}^2} dt \right| \right] \right\}^{\frac{1}{p+1}}
\]
\[
\to 0 \text{ as } m \to \infty.
\]
(3.135)

Note that the final term in the right hand side of (3.135) is bounded by using the final estimate in (3.122). Hence, along a subsequence \( \{u^{n_k}\} \), we find
\[
\int_0^t (C(u^{n_k}(s)), v) ds \xrightarrow{a.s.} \int_0^t (C(u(s)), v) ds, \text{ as } k \to \infty, \text{ for all } v \in \mathcal{H}. \]
(3.136)

Finally, for any \( t \in [0, T \wedge \tau^N_N] \), we consider
\[
\mathbb{E} \left[ \left| \int_0^t \int_{Z_n} (\gamma(s - , u^n(s), z), v)\tilde{\pi}(ds, dz) - \int_0^t \int_{Z} (\gamma(s - , u(s), z), v)\tilde{\pi}(ds, dz) \right| \right] \\
\leq \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau^N_N]} \left| \int_0^t \int_{Z_n} (\gamma(s - , u^n(s), z) - \gamma(s - , u(s), z), v)\tilde{\pi}(ds, dz) \right| \right] \\
+ \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau^N_N]} \left| \int_0^t \int_{Z \setminus Z_n} (\gamma(s - , u(s), z), v)\tilde{\pi}(ds, dz) \right| \right] \\
\leq \|v\|_{\mathcal{H}} \left\{ \mathbb{E} \left[ \int_0^{T \wedge \tau^N_N} \int_{Z_n} \|\gamma(t, u^n(t), z) - \gamma(t, u(t), z)\|^2_{\mathcal{H}} \pi(dt, dz) \right] \right\}^{1/2} \\
+ \|v\|_{\mathcal{H}} \left\{ \mathbb{E} \left[ \int_0^{T \wedge \tau^N_N} \int_{Z \setminus Z_n} \|\gamma(t, u(t), z)\|^2_{\mathcal{H}} \pi(dt, dz) \right] \right\}^{1/2} \\
\leq \|v\|_{\mathcal{H}} \left\{ \mathbb{E} \left[ \int_0^{T \wedge \tau^N_N} \int_{Z} \|\gamma(t, u^n(t), z) - \gamma(t, u(t), z)\|^2_{\mathcal{H}} \lambda(dz) dt \right] \right\}^{1/2}.
\]
\[
\begin{align*}
&+ \|v\|_{\mathbb{H}} \left\{ \mathbb{E} \left[ \int_0^{T \wedge \hat{\tau}_N} \int_{Z_n} \|\gamma(t, u(t), z)\|_{\mathbb{H}} \lambda(dz) dt \right] \right\}^{1/2} \\
\leq & \|v\|_{\mathbb{H}} \left[ \sqrt{L} \left\{ \mathbb{E} \left[ \int_0^{T \wedge \hat{\tau}_N} \|u^n(t) - u(t)\|_{\mathbb{H}}^2 dt \right] \right\}^{1/2} \\
&+ \left\{ \sup_{\|u\|_{\mathbb{H}} \leq N} \int_{Z_n} \|\gamma(t, u, z)\|_{\mathbb{H}} \lambda(dz) \right\}^{1/2} \\
\leq & \|v\|_{\mathbb{H}} \left[ \sqrt{2LT} \left\{ \mathbb{E} \left[ \sup_{t \in [0, T \wedge \hat{\tau}_N]} \|u^n(t) - u(t)\|_{\mathbb{H}}^2 \right] \right\}^{1/2} \\
&+ \left\{ \sup_{\|u\|_{\mathbb{H}} \leq N} \int_{Z_n} \|\gamma(t, u, z)\|_{\mathbb{H}} \lambda(dz) \right\}^{1/2} \\
\rightarrow & 0 \text{ as } n \rightarrow \infty,
\end{align*}
\]

for each fixed \(N\), where we have used (3.128), Hypothesis 3.1 (H.3) and (H.4), Burkholder–Davis–Gundy’s and Hölder’s inequalities. For any \(t \in [0, T]\), let us now consider

\[
\mathbb{E} \left[ \int_0^t \int_{Z_n} (\gamma(s-, u^n(s-), z), v) \tilde{\pi}(ds, dz) \right. \\
- \int_0^t \int_{Z_n} \left. (\gamma(s-, u(s-), z), v) \tilde{\pi}(ds, dz) \right] \\
= \mathbb{E} \left[ \int_0^t \int_{Z_n} \left( (\gamma(s-, u^n(s-), z), v) \tilde{\pi}(ds, dz) \\
- \int_0^t \int_{Z_n} \left( (\gamma(s-, u(s-), z), v) \tilde{\pi}(ds, dz) \right) \chi_{[t \leq \hat{\tau}_N]} \right] \\
+ \mathbb{E} \left[ \int_0^t \int_{Z_n} \left( (\gamma(s-, u^n(s-), z), v) \tilde{\pi}(ds, dz) \\
- \int_0^t \int_{Z_n} \left( (\gamma(s-, u(s-), z), v) \tilde{\pi}(ds, dz) \right) \chi_{[t > \hat{\tau}_N]} \right] \\
\leq \mathbb{E} \left[ \sup_{t \in [0, T \wedge \hat{\tau}_N]} \left| \int_0^t \int_{Z_n} (\gamma(s-, u^n(s-), z), v) \tilde{\pi}(ds, dz) \\
- \int_0^t \int_{Z_n} (\gamma(s-, u(s-), z), v) \tilde{\pi}(ds, dz) \right| \\
+ \sqrt{2} \left\{ \mathbb{E} \left[ \int_0^T \int_{Z_n} (\gamma(t-, u^n(t-), z), v) \tilde{\pi}(dt, dz) \right]^2 \right\}^{1/2} \right.
\]
where we have used Itô’s isometry. The first term on the right hand side of the above inequality tends to zero as \( n \to \infty \) and the second term tends to zero as \( N \to \infty \), since

\[
\mathbb{E} \left[ \int_0^T \int_Z |(\gamma(t, u^n(t), z), v)|^2 \lambda(dz) dt \right] + \mathbb{E} \left[ \int_0^T \int_Z |(\gamma(t, u(t), z), v)|^2 \lambda(dz) dt \right] \\
\leq \|v\|_{H^2}^2 \left\{ \mathbb{E} \left[ \int_0^T \int_Z \|\gamma(t, u^n(t), z)\|^2 \lambda(dz) dt \right] \\
+ \mathbb{E} \left[ \int_0^T \int_Z \|\gamma(t, u(t), z)\|^2 \lambda(dz) dt \right] \right\} \\
\leq K \|v\|_{H^2}^2 \left\{ \mathbb{E} \left[ \int_0^T \left( 1 + \|u^n(t)\|_{H^2}^2 \right) dt \right] + \mathbb{E} \left[ \int_0^T \left( 1 + \|u(t)\|_{H^2}^2 \right) dt \right] \right\} \\
\leq K \|v\|_{H^2}^2 C(T) < \infty,
\]

where we have used Hypothesis 3.1 (H.2). Thus, along a subsequence, we obtain

\[
\int_0^t \int_{Z_{n_k}} (\gamma(s-, u^{n_k}(s-), z), v) \tilde{\pi}(ds, dz) \xrightarrow{a.s.} \int_0^t \int_Z (\gamma(s-, u(s-), z), v) \tilde{\pi}(ds, dz),
\]

as \( k \to \infty \), for all \( v \in \mathbb{H} \) and \( t \in [0, T] \). Combining (3.123), (3.132), (3.136) and (3.137), we finally obtain that (3.6) is satisfied \( \mathbb{P} \)-a.s. for all \( v \in \mathbb{H} \) and \( t \in [0, T] \). Since \( u(\cdot) \) is the unique strong solution to the system (3.1), the whole sequence \( \{u_n\} \) converges. Moreover, the \( \mathcal{F}_t \)-adapted paths of \( u(\cdot) \) are càdlàg with \( \mathbb{P} \)-a.s. trajectories in \( \mathbb{V} \), satisfying the Itô formula (3.113) (cf. [29], since the embedding of \( D(A) \subset \mathbb{V} \subset \mathbb{H} \) is continuous and \( u(\cdot) \) is an \( \mathbb{H} \)-valued semi-martingale satisfying (3.4)).
Step III: The case $r = 3$ and $2\beta\mu \geq 1$. For $r = 3$, we estimate $|(B(u^n), Au^n)|$ as
\[
|(B(u^n), Au^n)| \leq \|u^n \cdot \nabla u^n\|_{H^1} \|Au^n\|_{H^1} \leq \frac{1}{4\theta} \|Au^n\|_{H^1}^2 + \theta \|u^n\|_{H^1} \|\nabla u^n\|_{H^1}^2.
\]
(3.138)

A calculation similar to (3.119) gives
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|u^n(t)\|_{V}^2 + 2 \left( \mu - \frac{1}{2\theta} \right) \int_0^T \|Au^n(t)\|_{H^1}^2 \, dt + 4(\beta - \theta) \int_0^T \|u^n(t)\|_{H^1} \|\nabla u^n(t)\|_{H^1}^2 \, dt \right] \\
\leq 2\mathbb{E} \left[ \|u_0^n\|_{V}^2 \right] + 14\tilde{K} \mathbb{E} \left[ \int_0^T (1 + \|u^n(t)\|_{V}^2) \, dt \right].
\]
(3.139)

For $2\beta\mu \geq 1$, it is immediate that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|u^n(t)\|_{V}^2 + 2 \left( \mu - \frac{1}{2\theta} \right) \int_0^T \|Au^n(t)\|_{H^1}^2 \, dt + 4(\beta - \theta) \int_0^T \|u^n(t)\|_{H^1} \|\nabla u^n(t)\|_{H^1}^2 \, dt \right] \\
\leq \left\{ 2\mathbb{E} \left[ \|u_0^n\|_{V}^2 \right] + 14\tilde{K} T \right\} e^{28\tilde{K} T}.
\]
(3.140)

Thus, $u^n \in L^2(\Omega; L^\infty(0, T; V)) \cap L^2(0, T; D(A))$ and using the estimate (3.109), we also get $u^n \in L^4(\Omega; L^4(0, T; \tilde{H}^4))$, where $p \in [2, \infty)$ for $d = 2$, $p = 3$ for $d = 3$, and $p = 2$ for $d = 4$. Proceeding similarly as in the case of $r > 3$, we obtain that
\[
u \in L^2(\Omega; L^\infty(0, T; V)) \cap L^2(0, T; D(A)) \cap L^4(\Omega; L^4(0, T; \tilde{H}^3))
\]
with $\mathbb{P}$-a.s. càdlàg paths in $V$ and (3.6) is satisfied $\mathbb{P}$-a.s. for all $v \in H$ and $t \in [0, T]$. □

4 Stationary solutions and stability

In this section, we consider the deterministic stationary system corresponding to CBF equations. We discuss the existence and uniqueness of weak solutions to the steady state equations and examine the exponential stability as well as stabilization by pure jump noise results.
4.1 Existence and uniqueness of weak solutions to the stationary system

Let us consider the following stationary system:

\[
\begin{align*}
-\mu \Delta u_\infty + (u_\infty \cdot \nabla) u_\infty + \beta |u_\infty|^{r-1} u_\infty + \nabla p_\infty &= f, \quad \text{in} \ \mathcal{O}, \\
\nabla \cdot u_\infty &= 0, \quad \text{in} \ \mathcal{O}, \\
u_\infty &= 0 \quad \text{on} \ \partial \mathcal{O},
\end{align*}
\]

(4.1)

where \( \mathcal{O} \subset \mathbb{R}^d, \ 2 \leq d \leq 4. \) Taking the Helmholtz-Hodge projection onto the system (4.1), one can write down the abstract formulation of the system (4.1) as

\[
\mu A u_\infty + B(u_\infty) + \beta C(u_\infty) = f \quad \text{in} \ V'.
\]

(4.2)

Given any \( f \in V' \), our problem is to find \( u_\infty \in V \cap \tilde{L}_r^{r+1} \) such that

\[
\mu (\nabla u_\infty, \nabla v) + \langle B(u_\infty), v \rangle + \beta \langle C(u_\infty), v \rangle = \langle f, v \rangle, \quad \text{for all} \ v \in V \cap \tilde{L}_r^{r+1},
\]

is satisfied. The following theorem shows that there exists a unique weak solution to the system (4.2) in \( V \cap \tilde{L}_r^{r+1} \), for \( r \geq 3 \).

**Theorem 4.1** ([57, Theorem 4.1]) For every \( f \in V' \) and \( r \geq 1 \), there exists at least one weak solution of the system (4.1) satisfying

\[
\mu \|u_\infty\|_V^2 + \beta \|u_\infty\|_{\tilde{L}_r^{r+1}}^{r+1} = \langle f, u_\infty \rangle.
\]

(4.4)

For \( r > 3 \), if

\[
\mu > \frac{2\eta}{\lambda_1},
\]

(4.5)

where \( \eta \) is defined in (2.17), then the solution of (4.2) is unique. For \( r = 3 \), the condition given in (4.5) becomes \( \mu \geq \frac{1}{2\beta} \).

4.2 Exponential stability in the deterministic case

Let us now discuss the exponential stability of the stationary solution obtained in Theorem 4.1. We first discuss the deterministic case.

**Definition 4.2** For \( r \geq 3 \), a function

\[ u \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; V) \cap L^{r+1}(0, T; \tilde{L}_r^{r+1}), \]

\( \mathbb{H} \) Springer
with \( \partial_t u \in L^2(0, T; V') + L^{r+1}_t(0, T; L^{r+1}_w) \), is called a weak solution to the deterministic system:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{du(t)}{dt} + \mu Au(t) + B(u(t)) + \beta C(u(t)) = f, \\
u(0) = u_0,
\end{array} \right.
\end{aligned}
\] (4.6)

if for \( f \in V' \) and \( u_0 \in H \), \( u(\cdot) \) satisfies:

\[
(u(t), v) = (u(t), v) - \int_0^t (\mu Au(s) + B(u(s)) + \beta C(u(s)), v)ds + (f, v)t, \quad (4.7)
\]

for all \( v \in V \cap \hat{L}^{r+1} \) and \( t \in [0, T) \). A function \( u \) is called a global weak solution if it is a weak solution for all \( T > 0 \).

It has been established in [23,56] that for \( u_0 \in H \) and \( f \in V' \), there exists a unique weak solution of the system (4.6) with \( u \in C([0, T]; H) \) satisfying the following energy equality:

\[
\|u(t)\|_H^2 + 2\mu \int_0^t \|u(s)\|_V^2 ds + 2\beta \int_0^t \|u(s)\|_{L^{r+1}_w}^2 ds = \|u_0\|_H^2 + 2\int_0^t (f, u(s))ds,
\]

(4.8)

for all \( t \in [0, T) \) and \( r \geq 3 \) (\( 2\beta \mu \geq 1 \) for \( r = 3 \)). Therefore, we have

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 + \mu \|u(t)\|_V^2 + \beta \|u(t)\|_{L^{r+1}_w}^2 = (f, u(t)),
\]

(4.9)

for a.e. \( t \in [0, T) \).

**Definition 4.3** A weak solution \( u(t) \) of the deterministic system (4.6) converges to \( u_\infty \) is exponentially stable in \( H \) if there exist a positive number \( \kappa > 0 \), such that

\[
\|u(t) - u_\infty\|_H \leq \|u_0 - u_\infty\|_H e^{-\kappa t}, \quad t \geq 0.
\]

In particular, if \( u_\infty \) is a stationary solution of system (4.2), then \( u_\infty \) is called exponentially stable in \( H \) provided that any weak solution to (4.6) converges to \( u_\infty \) at the same exponential rate \( \kappa > 0 \).

**Theorem 4.4** Let \( u_\infty \) be the unique solution of the system (4.2). If \( u(\cdot) \) is any weak solution to the system (4.6) with \( u_0 \in H \) and \( f \in V' \) arbitrary, then we have \( u_\infty \) is exponentially stable in \( H \) and \( u(t) \to u_\infty \) in \( H \) as \( t \to \infty \), for \( \mu > \frac{2\eta}{\lambda_1} \), for \( r > 3 \) and \( \mu \geq \frac{1}{2\beta} \), for \( r = 3 \).

**Proof** Let us define \( w = u - u_\infty \), so that \( w \) satisfies the following system:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{dw(t)}{dt} + \mu Aw(t) + \beta(B(u(t)) - B(u_\infty)) + \beta(C(u(t)) - C(u_\infty)) = 0, \\
w(0) = u_0 - u_\infty.
\end{array} \right.
\end{aligned}
\] (4.10)
in $\mathcal{V}' + \mathcal{H}^{\frac{r+1}{r}}$ for all $t \in [0, T]$. Since $u(\cdot)$ is the unique weak solution of the system (4.6) satisfying the energy equality (4.8) and $u_\infty$ is the unique weak solution of (4.3) satisfying (4.4), it is immediate that the unique weak solution $w(\cdot)$ to the system (4.10) satisfies the following energy equality:

$$
\frac{1}{2} \frac{d}{dt} \|w(t)\|^2_{\mathcal{H}} + \mu \|w(t)\|^2_{\mathcal{V}} = -\beta \langle (B(u(t)) - B(u_\infty)), w(t) \rangle - \beta \langle (C(u(t)) - C(u_\infty)), w(t) \rangle \\
\leq \frac{\mu}{2} \|w(t)\|^2_{\mathcal{V}} + \eta \|w(t)\|^2_{\mathcal{H}},
$$

(4.11)

for $r > 3$ and for a.e. $t \in (0, T)$, where we have used (2.13) and (3.101). Thus, it is immediate that

$$
\frac{d}{dt} \|w(t)\|^2_{\mathcal{H}} + (\lambda_1 \mu - 2\eta) \|w(t)\|^2_{\mathcal{H}} \leq 0.
$$

(4.12)

Thus, an application of variation of constants formula yields

$$
\|u(t) - u_\infty\|^2_{\mathcal{H}} \leq e^{-\kappa t} \|u_0 - u_\infty\|^2_{\mathcal{H}},
$$

(4.13)

where $\kappa = (\lambda_1 \mu - 2\eta) > 0$. Thus for $\mu > \frac{2\eta}{\lambda_1}$, the exponential stability of $u_\infty$ follows. For $r = 3$ and $\mu \geq \frac{1}{2\beta}$, one can use the estimates (2.13) (for $r = 3$) and (3.106) to get the required result.

**4.3 Exponential stability in the stochastic case**

Now we discuss the exponential stability results in the stochastic case.

**Definition 4.5** A strong solution $u(t)$ of the system (3.1) converges to $u_\infty \in \mathbb{H}$ is **exponentially stable in mean square** if there exists a positive number $a > 0$, such that

$$
\mathbb{E} \left[ \|u(t) - u_\infty\|^2_{\mathbb{H}} \right] \leq \mathbb{E} \left[ \|u_0 - u_\infty\|^2_{\mathbb{H}} \right] e^{-at}, \quad t \geq 0.
$$

In particular, if $u_\infty$ is a stationary solution of system (4.1), then $u_\infty$ is called **exponentially stable in the mean square** provided that any strong solution to (3.1) converges in $L^2$ to $u_\infty$ at the same exponential rate $a > 0$.

**Definition 4.6** A strong solution $u(t)$ of the system (3.1) converges to $u_\infty \in \mathbb{H}$ is called **almost surely exponentially stable** if there exists $\alpha > 0$ such that

$$
\lim_{t \to +\infty} \frac{1}{t} \log \|u(t) - u_\infty\|_{\mathbb{H}} \leq -\alpha, \quad \mathbb{P}\text{-a.s.}
$$

In particular, if $u_\infty$ is a stationary solution of system (4.1), then $u_\infty$ is called **almost surely exponentially stable** provided that any strong solution to (3.1) converges in $\mathbb{H}$ to $u_\infty$ with the same constant $\alpha > 0$.

---

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Let us now show the exponential stability of the stationary solutions to the system (4.1) in the mean square as well as almost sure sense. The authors in [41] obtained similar results for the multiplicative Gaussian noise case. Apart from the technical difficulties discussed in the introduction regarding the model defined on bounded domains, the authors in [41] established the exponential stability results with the help of more regular stationary solutions as well as under the assumption that the lower bound of \( \mu \) depends on the stationary solutions. We need less regular stationary solutions and relax the extra assumptions. The following results are true for all \( r \geq 3 \), and one has to take \( 2\beta \mu \geq 1 \), for \( r = 3 \). In this section, we consider \( u(\cdot) \) as the solution to the system

\[
\begin{aligned}
du(t) + [\mu A u(t) + B(u(t)) + \beta C(u(t))] dt &= f dt + \int_{Z} \gamma(t-,u(t-),z) \widetilde{\pi}(dt, dz), \\
u(0) &= u_0,
\end{aligned}
\]

for \( f \in V', t \in (0, T) \) and \( u_0 \in L^2(\Omega; H) \).

**Theorem 4.7** Let \( u_\infty \) be the unique stationary solution of (4.1) and \( \gamma(t, u_\infty, z) = 0 \), for all \( t \geq 0 \) and \( z \in Z \). Suppose that the conditions in Hypothesis 3.1 are satisfied, then for \( \theta = \mu \lambda_1 - (2\eta + L) > 0 \), we have

\[
E\left[ \|u(t) - u_\infty\|_{H}^2 \right] \leq e^{-\theta t} E\left[ \|u_0 - u_\infty\|_{H}^2 \right],
\]

provided

\[
\mu > \frac{2\eta + L}{\lambda_1},
\]

where \( L \) is the constant appearing in Hypothesis 3.1 (H.3), \( \eta \) is defined in (2.17) and \( \lambda_1 \) is the Poincaré constant.

**Proof** Let us define \( w := u - u_\infty \) and \( \theta = \mu \lambda_1 - (2\eta + L) > 0 \). Then \( w(\cdot) \) satisfies the following Itô stochastic differential:

\[
\begin{aligned}
dw(t) + \mu A w(t) + (B(u(t)) - B(u_\infty)) + \beta (C(u(t)) - C(u_\infty)) \\
&= \int_{Z} (\gamma(t, u(t), z) - \gamma(t, u_\infty, z)) \widetilde{\pi}(ds, dz), \quad t \in (0, T), \\
w(0) &= u_0 - u_\infty,
\end{aligned}
\]

since \( \gamma(t, u_\infty, z) = 0 \), for all \( t \in (0, T) \) and \( z \in Z \). Then \( w(\cdot) \) satisfies the following energy equality:

\[
e^{\theta t} \|w(t)\|_{H}^2 = \|w_0\|_{H}^2 - 2 \int_{0}^{t} e^{\theta s} \langle B(u(s)) - B(u_\infty(s)), w(s) \rangle ds
\]
\[ + \theta \int_0^t e^{\theta s} \| w(s) \|_{H}^2 \, ds - 2 \int_0^t e^{\theta s} \langle C(u(s)) - C(u_\infty(s)), w(s) \rangle \, ds \]
\[ + \int_0^t e^{\theta s} \int_Z \| \tilde{\gamma}(s, w(s), z) \|_{H}^2 \pi(ds, dz) \]
\[ + 2 \int_0^t e^{\theta s} \int_Z (\tilde{\gamma}(s-, w(s-), z), w(s-)) \tilde{\pi}(ds, dz), \tag{4.18} \]

where \( \tilde{\gamma}(\cdot, w, \cdot) = \gamma(\cdot, u(\cdot), \cdot) - \gamma(\cdot, u_\infty, \cdot) \). A calculation similar to (3.102) yields

\[ e^{\theta t} \| w(t) \|_{H}^2 \leq \| w_0 \|_{H}^2 + (\theta + 2\eta - \mu \lambda_1) \int_0^t e^{\theta s} \| w(s) \|_{H}^2 \, ds \]
\[ + \int_0^t e^{\theta s} \int_Z \| \tilde{\gamma}(s, w(s), z) \|_{H}^2 \pi(ds, dz) \]
\[ + 2 \int_0^t e^{\theta s} \int_Z (\tilde{\gamma}(s-, w(s-), z), w(s-)) \tilde{\pi}(ds, dz). \tag{4.19} \]

Taking expectation, and using Hypothesis 3.1 (H.3) and the fact that the final term is a martingale, we find

\[ e^{\theta t} \mathbb{E} \left[ \| w(t) \|_{H}^2 \right] \leq \mathbb{E} \left[ \| w_0 \|_{H}^2 \right] + (\theta + \eta + L - \mu \lambda_1) \int_0^t e^{\theta s} \mathbb{E} \left[ \| w(s) \|_{H}^2 \right] \, ds \]. \tag{4.20} \]

Since \( \mu \) satisfies (4.16) implies \( \theta = \mu \lambda_1 - (2\eta + L) > 0 \) and an application of Gronwall’s inequality yields that (4.15) is satisfied and hence \( u(t) \) converges to \( u_\infty \) exponentially in the mean square sense. \( \square \)

**Theorem 4.8** Let all conditions given in Theorem 4.7 are satisfied and

\[ \mu > \frac{2\eta + 3L}{\lambda_1}. \tag{4.21} \]

Then the strong solution \( u(\cdot) \) of the system (3.1) converges to the stationary solution \( u_\infty \) of the system (4.2) almost surely exponentially stable.

**Proof** Let us take \( n = 1, 2, \ldots, h > 0 \). Then the process \( \| u(\cdot) - u_\infty \|_{H}^2 \), for \( t \geq nh \) satisfies:

\[ \| u(t) - u_\infty \|_{H}^2 + 2\mu \int_{nh}^t \| u(s) - u_\infty \|_{H}^2 \, ds \]
\[ = \| u(nh) - u_\infty \|_{H}^2 - 2 \int_{nh}^t \langle B(u(s)) - B(u_\infty), u(s) - u_\infty \rangle ds \]
\[ - 2 \int_{nh}^t \langle C(u(s)) - C(u_\infty), u(s) - u_\infty \rangle ds \]

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+ 2 \int_{nh}^{t} \int_{Z} (\gamma(s-, u(s-), z) - \gamma(s-, u_{\infty}, z), u(s-) - u_{\infty}) \tilde{\pi}(ds, dz)
+ \int_{nh}^{t} \int_{Z} \|\gamma(s, u(s), z) - \gamma(s, u_{\infty}, z)\|_{\mathbb{H}}^{2} \pi(ds, dz) . \tag{4.22} 

Taking the supremum from \( nh \) to \( (n+1)h \) and then taking expectation in (4.22), we find

\[
\begin{align*}
\mathbb{E} \left[ \sup_{nh \leq t \leq (n+1)h} \| u(t) - u_{\infty} \|_{\mathbb{H}}^{2} + \mu \int_{nh}^{(n+1)h} \| u(s) - u_{\infty} \|_{\mathbb{H}}^{2} ds \right] \\
\leq \mathbb{E} \left[ \| u(nh) - u_{\infty} \|_{\mathbb{H}}^{2} \right] + 2\eta \mathbb{E} \left[ \int_{nh}^{(n+1)h} \| u(s) - u_{\infty} \|_{\mathbb{H}}^{2} ds \right] \\
+ \mathbb{E} \left[ \int_{nh}^{(n+1)h} \int_{Z} \|\gamma(s, u(s), z) - \gamma(s, u_{\infty}, z)\|_{\mathbb{H}}^{2} \lambda(dz)ds \right] \\
+ 2\mathbb{E} \left[ \sup_{nh \leq t \leq (n+1)h} \int_{nh}^{t} \int_{Z} ((\gamma(s-, u(s-), z) - \gamma(s-, u_{\infty}, z)), u(s-) - u_{\infty}) \tilde{\pi}(ds, dz) \right], \tag{4.23} 
\end{align*}
\]

where we have used (2.13) and (3.101). We estimate the final term in the right hand side of the inequality (4.23) using Burkholder–Davis–Gundy’s, Hölder’s and Young’s inequalities as

\[
\begin{align*}
2\mathbb{E} \left[ \sup_{nh \leq t \leq (n+1)h} \int_{nh}^{t} \int_{Z} ((\gamma(s-, u(s-), z) - \gamma(s-, u_{\infty}, z)), u(s-) - u_{\infty}) \tilde{\pi}(ds, dz) \right] \\
\leq 2\sqrt{3} \mathbb{E} \left[ \int_{nh}^{(n+1)h} \int_{Z} \|\gamma(s, u(s), z) - \gamma(s, u_{\infty}, z)\|_{\mathbb{H}}^{2} \|u(s) - u_{\infty}\|_{\mathbb{H}}^{2} \pi(ds, dz) \right]^{1/2} \\
\leq 2\sqrt{3} \mathbb{E} \left[ \sup_{nh \leq s \leq (n+1)h} \|u(s) - u_{\infty}\|_{\mathbb{H}} \right. \\
\times \left. \left( \int_{nh}^{(n+1)h} \int_{Z} \|\gamma(s, u(s), z) - \gamma(s, u_{\infty}, z)\|_{\mathbb{H}}^{2} \pi(ds, dz) \right)^{1/2} \right] \\
\leq \frac{1}{2} \mathbb{E} \left[ \sup_{nh \leq s \leq (n+1)h} \|u(s) - u_{\infty}\|_{\mathbb{H}}^{2} \right] \\
+ 6\mathbb{E} \left[ \int_{nh}^{(n+1)h} \int_{Z} \|\gamma(s, u(s), z) - \gamma(s, u_{\infty}, z)\|_{\mathbb{H}}^{2} \lambda(dz)ds \right] . \tag{4.24} 
\end{align*}
\]
Substituting (4.24) in (4.23), and then using Hypothesis 3.1 (H.2), we get

\[
\mathbb{E} \left[ \sup_{nh \leq t \leq (n+1)h} \| u(t) - u_\infty \|^2_H \right] + \vartheta \mathbb{E} \left[ \int_{nh}^{(n+1)h} \| u(s) - u_\infty \|^2_H ds \right] \\
\leq 2 \mathbb{E} \left[ \| u(nh) - u_\infty \|^2_H \right],
\]

where

\[
\vartheta = 2 (\mu \lambda_1 - (2\eta + 6L)) > 0.
\]

Let us now use (4.15) in (4.25) to obtain

\[
\mathbb{E} \left[ \sup_{nh \leq t \leq (n+1)h} \| u(t) - u_\infty \|^2_H \right] \leq 2 \mathbb{E} \left[ \| u_0 - u_\infty \|^2_H \right] e^{-\vartheta nh}.
\]

Using Chebychev’s inequality, for \(\epsilon \in (0, \theta)\), we also have

\[
P \left\{ \omega \in \Omega : \sup_{nh \leq t \leq (n+1)h} \| u(t) - u_\infty \|_H > e^{-\frac{1}{2}(\theta - \epsilon)nh} \right\} \\
\leq e^{(\theta - \epsilon)nh} \mathbb{E} \left[ \sup_{nh \leq t \leq (n+1)h} \| u(t) - u_\infty \|^2_H \right] \leq 2 \mathbb{E} \left[ \| u_0 - u_\infty \|^2_H \right] e^{-\epsilon nh},
\]

and

\[
\sum_{n=1}^{\infty} P \left\{ \omega \in \Omega : \sup_{nh \leq t \leq (n+1)h} \| u(t) - u_\infty \|_H > e^{-\frac{1}{2}(\theta - \epsilon)nh} \right\} \\
\leq 2 \mathbb{E} \left[ \| u_0 - u_\infty \|^2_H \right] \frac{1}{e^{\epsilon h} - 1} < +\infty.
\]

Thus by using the Borel–Cantelli lemma, there is a finite integer \(n_0(\omega)\) such that

\[
\sup_{nh \leq t \leq (n+1)h} \| u(t) - u_\infty \|_H \leq e^{-\frac{1}{2}(\theta - \epsilon)nh}, \ \mathbb{P}\text{-a.s.},
\]

for all \(n \geq n_0\), which completes the proof. \(\square\)

### 4.4 Stabilization by a multiplicative jump noise

It is an interesting question to ask about the exponential stability of the stationary solution for small values of \(\mu\). For similar results regarding 2D Oldroyd models perturbed by Lévy noise, interested readers are referred to see [54]. We assume that

\[
\gamma(\cdot, u, z) = \gamma(z)(u - u_\infty)
\]
such that
\[ \int_Z |\gamma(z)|^2 \lambda(dz) < +\infty \quad \text{and} \quad \int_Z (\gamma(z) - \log(1 + \gamma(z))) \lambda(dz) = \rho > 0. \quad (4.27) \]

Remember that \( \log(1 + x) \leq x \), for \( x > -1 \). The following form of strong law of large numbers is needed to establish the stabilization result.

**Lemma 4.9** (Strong law of large numbers, [38, Theorem 1]) Let \( M = \{M(t)\}_{t \geq t_0} \) be a local càdlàg martingale. If
\[ P \{ \omega \in \Omega : \lim_{t \to \infty} \int_{t_0}^t \frac{d\langle M \rangle_s}{(1 + s)^2} = 1 \} = 1, \]
then
\[ P \{ \omega \in \Omega : \lim_{t \to \infty} \frac{M(t)}{t} = 1 \} = 1. \quad (4.28) \]

We use the noise given in (4.26) to obtain the stabilization of the SCBF equations (4.14).

**Theorem 4.10** Let the equation (4.14) be perturbed by the jump noise given in (4.27) and noise coefficient (4.26). Then, there exists \( \Omega_0 \subset \Omega \) with \( P(\Omega_0) = 0 \) such that for all \( \omega \notin \Omega_0 \), there exists \( T(\omega) > 0 \) such that any strong solution \( u(t) \) to the system (4.14) satisfies
\[ \|u(t) - u_\infty\|_{\mathcal{H}}^2 \leq \|u_0 - u_\infty\|_{\mathcal{H}}^2 e^{-\zeta t}, \quad \text{for any} \quad t \geq T(\omega), \quad (4.29) \]
where \( \zeta = (\mu \lambda_1 - \eta + \rho) > 0 \). In particular, the exponential stability of sample paths with probability one holds if \( \zeta > 0 \).

**Proof** We know that the process \( u(\cdot) - u_\infty \) satisfies the following energy equality:
\[
\|u(t) - u_\infty\|_{\mathcal{H}}^2 \leq 2 \mu \int_0^t \|u(s) - u_\infty\|_{\mathcal{V}}^2 ds \\
= \|u_0 - u_\infty\|_{\mathcal{H}}^2 - 2 \int_0^t (C(u(s)) - C(u_\infty), u(s) - u_\infty)ds \\
- 2 \int_0^t (C(u(s)) - C(u_\infty), u(s) - u_\infty)ds + \int_0^t \int_Z \|\gamma(z)(u(s) - u_\infty)\|_{\mathcal{H}}^2 \lambda(dz)ds \\
+ \int_0^t \int_Z \left[ 2 (\gamma(z)(u(s) - u_\infty), u(s) - u_\infty) + \|\gamma(z)(u(s) - u_\infty)\|_{\mathcal{H}}^2 \right] \Pi(ds, dz). \quad (4.30)
\]

Applying Itô’s formula to the process \( \log \|u(\cdot) - u_\infty\|_{\mathcal{H}}^2 \), we find
\[
\log \|u(t) - u_\infty\|_{\mathcal{H}}^2 \\
= \log \|u_0 - u_\infty\|_{\mathcal{H}}^2 - 2 \mu \int_0^t \|u(s) - u_\infty\|_{\mathcal{V}}^2 ds.
\]
\[
- 2 \int_0^t \frac{(B(u(s)) - B(u_\infty), u(s) - u_\infty)\|u(s) - u_\infty\|_H^2}{ds}
- 2 \int_0^t \frac{(C(u(s)) - C(u_\infty), u(s) - u_\infty)\|u(s) - u_\infty\|_H^2}{ds}
+ \int_0^t \int_Z \frac{\|\gamma(z)(u(s) - u_\infty)\|_H^2}{\lambda(dz)ds}
+ \int_0^t \int_Z \left(\log \|u(s) - u_\infty\|_H^2 + \gamma(z)(u(s) - u_\infty)\|u(s) - u_\infty\|_H^2 - \log \|u(s) - u_\infty\|_H^2\right)\lambda(dz)ds
\leq \log \|u_0 - u_\infty\|_H^2 + (-2\mu\lambda_1 + 2\eta)t + 2 \int_0^t \int_Z (\log(1 + \gamma(z)) - \gamma(z))\lambda(dz)ds
+ 2 \int_0^t \int_Z \log(1 + \gamma(z))\tilde{\pi}(dz, dt)
\leq \log \|u_0 - u_\infty\|_H^2 + 2(-\mu\lambda_1 + \eta - \rho)t + 2 \int_0^t \int_Z (\log(1 + \gamma(z))\tilde{\pi}(dz, dt),
\]

where we have used (2.13), (3.101) and (4.27). Note that the term

\[
\mathbb{M}(t) = \int_0^t \int_Z \log(1 + \gamma(z))\tilde{\pi}(dz, dt),
\]

is a real martingale. We know that

\[
\langle \mathbb{M} \rangle_t = \int_0^t \int_Z [\log(1 + \gamma(z))]^2 \lambda(dz)ds \leq \int_Z |\gamma(z)|^2 \lambda(dz)t,
\]

and

\[
\lim_{t \to \infty} \int_0^t \frac{d\langle \mathbb{M} \rangle_s}{(1 + s)^2} \leq \int_Z |\gamma(z)|^2 \lambda(dz) < +\infty.
\]

Thus by means of the strong law of large numbers (see Lemma 4.9), we have

\[
\lim_{t \to +\infty} \frac{\mathbb{M}(t)}{t} = 0, \text{ } \mathbb{P}\text{-a.s.}
\]
One can assure the existence of a set $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 0$ such that for every $\omega \notin \Omega_0$, there exists $T(\omega) > 0$ such that for all $t \geq T(\omega)$, we have

$$\frac{2}{t} \int_0^t \int_Z \log(1 + \gamma(z))\pi(ds, dz) \leq (\mu\lambda_1 - \eta + \rho).$$

Hence from (4.31), we finally have

$$\log \|u(t) - u_\infty\|_H^2 \leq \log \|u_0 - u_\infty\|_H^2 - (\mu\lambda_1 - \eta + \rho)t, \quad \text{(4.34)}$$

for any $t \geq T(\omega)$, which completes the proof. \(\square\)

5 Invariant measures and ergodicity

In this section, we discuss the existence and uniqueness of invariant measures and ergodicity results for the SCBF equations (3.1).

5.1 Preliminaries

Let us first provide the definitions of invariant measures, ergodic, strongly mixing and exponentially mixing invariant measures. Let $X$ be a Polish space (complete separable metric space).

**Definition 5.1** A probability measure $\eta$ on $(X, \mathcal{B}(X))$ is called an **invariant measure** or a **stationary measure** for a given transition probability function $P(t, x, dy)$, if it satisfies

$$\eta(A) = \int_X P(t, x, A)d\eta(x),$$

for all $A \in \mathcal{B}(X)$ and $t > 0$. Equivalently, if for all $\phi \in C_b(X)$ (the space of bounded continuous functions on $X$), and all $t \geq 0$,

$$\int_X \phi(x)d\eta(x) = \int_X (P_t\phi)(x)d\eta(x),$$

where the Markov semigroup $(P_t)_{t \geq 0}$ is defined by

$$P_t\phi(x) = \int_X \phi(y)P(t, x, dy).$$

**Definition 5.2** ([17, Theorem 3.2.4], [53, Theorem 3.4.2]) Let $\eta$ be an invariant measure for $(P_t)_{t \geq 0}$. We say that the measure $\eta$ is an **ergodic measure**, if for all $\phi \in \tilde{L}^2(X; \eta)$, we have

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T (P_t\phi)(x)dt = \int_X \phi(x)d\eta(x) \text{ in } L^2(X; \eta).$$
The invariant measure $\eta$ for $(P_t)_{t \geq 0}$ is called \textit{strongly mixing} if for all $\varphi \in L^2(X; \eta)$, we have

$$\lim_{t \to +\infty} P_t \varphi(x) = \int_X \varphi(x) d\eta(x) \text{ in } L^2(X; \eta).$$

The invariant measure $\eta$ for $(P_t)_{t \geq 0}$ is called \textit{exponentially mixing}, if there exists a constant $k > 0$ and a positive function $\Psi(\cdot)$ such that for any bounded Lipschitz function $\varphi$, all $t > 0$ and all $x \in X$,

$$\left| P_t \varphi(x) - \int_X \varphi(x) d\eta(x) \right| \leq \Psi(x) e^{-kt} \| \varphi \|_{Lip},$$

where $\| \cdot \|_{Lip}$ is the Lipschitz constant.

Clearly exponentially mixing implies strongly mixing. [17, Theorem 3.2.6] states that if $\eta$ is the unique invariant measure for $(P_t)_{t \geq 0}$, then it is ergodic. The interested readers are referred to see [17] for more details on the ergodicity for infinite dimensional systems and [19] for ergodicity results for stochastic Navier–Stokes equations.

### 5.2 Existence and uniqueness of invariant measures

Let us now show that there exists a unique invariant measure for the Markovian transition probability associated with the system (3.1). Moreover, we establish that the invariant measure is ergodic and strongly mixing (in fact exponentially mixing). The authors in [27] also obtained similar results for the system (3.1) perturbed by additive Lévy noise in bounded domains, but the technical difficulty explained in the introduction still persists. Let $u(t, u_0)$ denote the unique strong solution to the system (3.1) with the initial condition $u_0 \in H$. Let $(P_t)_{t \geq 0}$ be the Markovian transition semigroup in the space $C_b(H)$ associated with the system (3.1) defined by

$$P_t \varphi(u_0) = \mathbb{E} \left[ \varphi(u(t, u_0)) \right] = \int_H \varphi(y) P(t, u_0, dy) = \int_H \varphi(y) \eta_{t, u_0}(dy), \quad (5.1)$$

for $\varphi \in C_b(H)$, where $P(t, u_0, d y)$ is the transition probability of $u(t, u_0)$ and $\eta_{t, u_0}$ is the law of $u(t, u_0)$. The semigroup $(P_t)_{t \geq 0}$ is Feller, since the solution to (3.1) depends continuously on the initial data (see (3.104)). From (5.1), we also have

$$P_t \varphi(u_0) = \langle \varphi, \eta_{t, u_0} \rangle = \langle P_t \varphi, \eta \rangle, \quad (5.2)$$

where $\eta$ is the law of the initial data $u_0 \in H$. Thus from (5.2), we have $\eta_{t, u_0} = P_t^* \eta$. We say that a probability measure $\eta$ on $H$ is an \textit{invariant measure} if

$$P_t^* \eta = \eta, \text{ for all } t \geq 0. \quad (5.3)$$

That is, if a solution has law $\eta$ at some time, then it has the same law for all later times. For such a solution, it can be shown by Markov property that for all $(t_1, \ldots, t_n)$
and \( \tau > 0 \), \( (u(t_1 + \tau, u_0), \ldots, u(t_n + \tau, u_0)) \) and \( (u(t_1, u_0), \ldots, u(t_n, u_0)) \) have the same law. Then, we say that the process \( u \) is stationary. For more details, the interested readers are referred to see [17,19], etc.

**Theorem 5.3** Let \( u_0 \in \mathbb{H} \) be given. Then under Hypothesis 3.1, for \( \mu > \frac{K}{2\lambda_1} \), there exists an invariant measure for the system \( (3.1) \) with support in \( \mathbb{V} \).

**Proof** Let us use the energy equality obtained in (3.95) to find

\[
\begin{align*}
\|u(t)\|_{\mathbb{H}}^2 &+ 2\mu \int_0^t \|u(s)\|_{\mathbb{V}}^2 \, ds + 2\beta \int_0^t \|u(s)\|_{E_{r+1}}^2 \, ds \\
&= \|u_0\|_{\mathbb{H}}^2 + \int_0^t \int_{\mathbb{Z}} \|\gamma(s, u(s), z)\|_{\mathbb{H}}^2 \pi(ds, dz) \\
&\quad + 2 \int_0^t \int_{\mathbb{Z}} (\gamma(s-, u(s-), z), u(s-)) \tilde{\pi}(ds, dz).
\end{align*}
\tag{5.4}
\]

Taking expectation in (5.4), using Hypothesis 3.1 (H.2), Poincaré inequality and the fact that the final term is a martingale having zero expectation, we obtain

\[
\mathbb{E} \left\{ \|u(t)\|_{\mathbb{H}}^2 + \left(2\mu - \frac{K}{\lambda_1}\right) \int_0^t \|u(s)\|_{\mathbb{V}}^2 \, ds + 2\beta \int_0^t \|u(s)\|_{E_{r+1}}^2 \, ds \right\} \leq \mathbb{E} \left[ \|u_0\|_{\mathbb{H}}^2 \right].
\tag{5.5}
\]

Thus, for \( \mu > \frac{K}{2\lambda_1} \), we have

\[
\frac{2\mu - \frac{K}{\lambda_1}}{t} \mathbb{E} \left[ \int_0^t \|u(s)\|_{\mathbb{V}}^2 \, ds \right] \leq \frac{1}{T_0} \|u_0\|_{\mathbb{H}}^2, \text{ for all } t > T_0.
\tag{5.6}
\]

Using Markov’s inequality, we get

\[
\lim_{R \to \infty} \sup_{T > T_0} \left[ \frac{1}{T} \int_0^T \mathbb{P} \left\{ \|u(t)\|_{\mathbb{V}} > R \right\} \, dt \right] \leq \lim_{R \to \infty} \sup_{T > T_0} \frac{1}{R^2} \int_0^T \|u(t)\|_{\mathbb{V}}^2 \, dt = 0.
\tag{5.7}
\]

Hence along with the estimate in (5.7), using the compactness of \( \mathbb{V} \) in \( \mathbb{H} \), it is clear by a standard argument that the sequence of probability measures

\[
\eta_{t,u_0}(-) = \frac{1}{t} \int_0^t \Pi_{s,u_0}(-) \, ds, \text{ where } \Pi_{t,u_0}(\Lambda) = \mathbb{P} \{ u(t, u_0) \in \Lambda \}, \Lambda \in \mathcal{B}(\mathbb{H}),
\]

is tight, that is, for each \( \delta > 0 \), there is a compact subset \( K \subset \mathbb{H} \) such that \( \eta_{t}(K^c) \leq \delta \), for all \( t > 0 \), and so by the Krylov-Bogoliubov theorem (or by a result of Chow and Khasminskii see [12]) \( \eta_{t,n,u_0} \to \eta \), weakly for \( n \to \infty \), and \( \eta \) results to be an invariant measure for the transition semigroup \( (P_t)_{t \geq 0} \), defined by

\[
P_t \varphi(u_0) = \mathbb{E} \left[ \varphi(u(t, u_0)) \right],
\]
for all $\varphi \in C_b(\mathbb{H})$, where $u(\cdot)$ is the unique strong solution of the system (3.1) with the initial condition $u_0 \in \mathbb{H}$. 

Now we establish the uniqueness of invariant measures for the system (3.1). Similar results for 2D stochastic Navier–Stokes equations is established in [19], for stochastic 2D magneto-hydrodynamic system is proved in [48] and for stochastic 2D Oldroyd models is obtained in [54]. The following result provides the exponential stability results for the system (3.1).

**Theorem 5.4** Let $u(\cdot)$ and $v(\cdot)$ be two solutions of the system (3.1) with $r > 3$ and the initial data $u_0, v_0 \in \mathbb{H}$, respectively. Then under Hypothesis 3.1, for the condition given in (4.16), we have

$$E\left[\|u(t) - v(t)\|_{\mathbb{H}}^2\right] \leq \|u_0 - v_0\|_{\mathbb{H}}^2 e^{-(\mu\lambda_1 - (2\eta + L))t},$$

where $\eta$ is defined in (2.17).

**Proof** Let us define $w(t) = u(t) - v(t)$. Then, $w(\cdot)$ satisfies the following energy equality:

$$\|w(t)\|_{\mathbb{H}}^2 = \|w_0\|_{\mathbb{H}}^2 - 2\mu \int_0^t \|w(s)\|_{\mathbb{H}}^2 ds - 2\beta \int_0^t \langle C(u(s)) - C(v(s)), w(s) \rangle ds$$

$$- 2 \int_0^t \langle B(u(s)) - B(v(s)), w(s) \rangle ds + \int_0^t \int_\mathbb{Z} \|\tilde{\gamma}(s, w(s), z)\|_{\mathbb{H}}^2 \lambda(dz) ds$$

$$+ 2 \int_0^t \int_\mathbb{Z} \langle \tilde{\gamma}(s-, w(s-), z), w(s-) \rangle \tilde{\pi}(ds, dz),$$

where $\tilde{\gamma}(\cdot, w(\cdot), \cdot) = \gamma'(\cdot, u(\cdot), \cdot) - \gamma'(\cdot, v(\cdot), \cdot)$. Taking expectation in (5.9) and then using the Poincaré inequality, Hypothesis (H.3), (2.13) and (3.101), one can easily see that

$$E\left[\|w(t)\|_{\mathbb{H}}^2\right] \leq \|w_0\|_{\mathbb{H}}^2 - \mu\lambda_1 \int_0^t E\left[\|w(s)\|_{\mathbb{H}}^2\right] ds + (2\eta + L) \int_0^t E\left[\|w(s)\|_{\mathbb{H}}^2\right] ds,$$

where $\eta$ is defined in (2.17). Thus, an application of Gronwall’s inequality yields

$$E\left[\|w(t)\|_{\mathbb{H}}^2\right] \leq \|w_0\|_{\mathbb{H}}^2 e^{-(\mu\lambda_1 - (2\eta + L))t},$$

and for $\mu > \frac{2\eta + L}{\lambda_1}$, we obtain the required result (5.8). 

For $2\beta\mu \geq 1$, the results obtained in Theorem 5.4 can be established for $\mu > \frac{L}{\lambda_1}$, using the estimate (3.106). Let us now establish the uniqueness of invariant measures for the system (3.1) obtained in Theorem 5.3. We prove the case of $r > 3$ only and the case of $r = 3$ follows similarly.
Theorem 5.5 Let the conditions given in Theorem 5.4 hold true and $u_0 \in \mathbb{H}$ be given. Then, for the condition given in (4.16), there is a unique invariant measure $\eta$ to system (3.1). The measure $\eta$ is ergodic and strongly mixing, that is,

$$
\lim_{t \to \infty} P_t \varphi(u_0) = \int_{\mathbb{H}} \varphi(v_0) d\eta(v_0), \ \eta\text{-a.s., for all } u_0 \in \mathbb{H} \text{ and } \varphi \in C_b(\mathbb{H}).
$$

(5.12)

Proof For $\varphi \in \text{Lip}(\mathbb{H})$ (Lipschitz $\varphi$), since $\eta$ is an invariant measure, we have

$$
\left| P_t \varphi(u_0) - \int_{\mathbb{H}} \varphi(v_0) \eta(dv_0) \right|
= \left| \mathbb{E}[\varphi(u(t, u_0))] - \int_{\mathbb{H}} P_t \varphi(v_0) \eta(dv_0) \right|
= \left| \int_{\mathbb{H}} \mathbb{E}[\varphi(u(t, u_0)) - \varphi(u(t, v_0))] \eta(dv_0) \right|
\leq L\varphi \int_{\mathbb{H}} \mathbb{E} \left[ \|u(t, u_0) - u(t, v_0)\|_{\mathbb{H}} \right] \eta(dv_0)
\leq L\varphi e^{-\frac{(\mu_1 - (2\eta + L))t}{2}} \int_{\mathbb{H}} \|u_0 - v_0\|_{\mathbb{H}} \eta(dv_0)
\leq L\varphi e^{-\frac{(\mu_1 - (2\eta + L))t}{2}} \left( \|u_0\|_{\mathbb{H}} + \int_{\mathbb{H}} \|v_0\|_{\mathbb{H}} \eta(dv_0) \right)
\rightarrow 0 \text{ as } t \to \infty,
$$

(5.13)

since $\int_{\mathbb{H}} \|v_0\|_{\mathbb{H}} \eta(dv_0) < +\infty$. Hence, we deduce (5.12), for every $\varphi \in C_b(\mathbb{H})$, by the density of Lip(\mathbb{H}) in C_b(\mathbb{H}). Note that we have a stronger result that $P_t \varphi(u_0)$ converges exponentially fast to equilibrium, which is the exponential mixing property. This easily gives uniqueness of the invariant measure also. Indeed, if $\tilde{\eta}$ is another invariant measure, then

$$
\left| \int_{\mathbb{H}} \varphi(u_0) \eta(du_0) - \int_{\mathbb{H}} \varphi(v_0) \tilde{\eta}(dv_0) \right|
= \left| \int_{\mathbb{H}} P_t \varphi(u_0) \eta(du_0) - \int_{\mathbb{H}} P_t \varphi(v_0) \tilde{\eta}(dv_0) \right|
= \left| \int_{\mathbb{H}} \int_{\mathbb{H}} [P_t \varphi(u_0) - P_t \varphi(v_0)] \eta(du_0) \tilde{\eta}(dv_0) \right|
\leq L\varphi e^{-\frac{(\mu_1 - (2\eta + L))t}{2}} \int_{\mathbb{H}} \int_{\mathbb{H}} \|u_0 - v_0\|_{\mathbb{H}} \eta(du_0) \tilde{\eta}(dv_0)
\rightarrow 0 \text{ as } t \to \infty.
$$

(5.14)

By [17, Theorem 3.2.6], since $\eta$ is the unique invariant measure for $(P_t)_{t \geq 0}$, we know that it is ergodic. \qed

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