Stability and existence analysis of static black holes in pure Lovelock theories

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Abstract

In this paper, we study the existence and stability of static black holes in Lovelock theories with a particular focus on pure Lovelock black holes. We derive the equation of stability from action without using the S-deformation approach. Although a pure-Lovelock black hole in an even dimension is always unstable, introduction of $\Lambda$ stabilizes it by prescribing a lower threshold mass while there also exists an upper bound on mass that is given by the existence of a horizon. We also study the stability of dimensionally continued black holes as well as of the pure Lovelock analogue of BTZ black holes in all odd dimensions.

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(Some figures may appear in colour only in the online journal)

1. Introduction

It is well known that string theory, which is proclaimed as a theory of everything, can only apply in higher dimensions and one of the key issues, besides unification of all forces, it is supposed to address is quantum gravity. Hence higher dimensional probing of gravity is quite in line with the current work in fundamental physics. The most natural higher dimensional generalization of Einstein gravity is Lovelock polynomial gravity \cite{1}, which includes linear order Einstein gravity for $N = 1$ and quadratic Gauss–Bonnet (GB) for $N = 2$, where $N$ is the degree of homogeneous polynomial in Riemann curvature. This is the unique generalization
that retains the second order character of the equation of motion, and the \( N \)th order term makes a non-zero contribution to the equation only in dimension \( d \geq 2N + 1 \). It is therefore truly a higher dimensional generalization.

It may be noted that among others, the quadratic GB term has been shown to be relevant to the low energy limit of string theory [2–4]. This may perhaps indicate that the high energy limit of classical gravity is Lovelock gravity. One of us has made this argument [5, 6] for quite some time on a purely classical basis for higher dimensions. To include high energy effects within the classical framework, we should involve higher orders of Riemann curvature; yet if we demand the equation remain second order, we are uniquely led to the Lovelock polynomial and higher dimensions [6]. This is how Lovelock gravity may perhaps arise as the high energy limit of classical gravity and hence may serve as an intermediary state between classical and quantum gravity [7]. Thus, notwithstanding the strong string theory basis for higher dimensional study of gravity, the point we would like to make is that there is also an equally pertinent classical basis.

Furthermore it has been argued that pure Lovelock gravity exhibits one of the universal features of gravity in that it is kinematical in all odd dimensions as Einstein gravity is in three dimensions [8]. What it means is that \( R_{ab}^{(N)} = 0 \) implies \( R_{abcd}^{(N)} = 0 \) in all \( d = 2N + 1 \) dimensions. Here \( R_{abcd}^{(N)} \) is defined such that the vanishing of the trace of its Bianchi derivative gives the same divergence-free, second-rank symmetric tensor as that obtained from variation of the \( N \)th order Lovelock polynomial action [9]. It also turns out that a pure Lovelock \( \Lambda \)-vacuum solution asymptotically transforms into the corresponding Einstein Schwarzschild-dS/AdS solution even though the action was free of the Einstein term [10]. In addition, there is thermodynamical universality for pure Lovelock static black holes; for instance, entropy in even the \( d = 2N + 2 \) dimension would always go as \( A^{1/N} \) where \( A \) is the area of the horizon [11]. Therefore, a very strong case has been made [12] for the pure Lovelock equation,

\[
G_{ab}^{(N)} = - \kappa T_{ab} - \Lambda g_{ab},
\]

which includes only one \( N \)th order term. There exists a general solution for the vacuum of this equation describing static black holes [10]. Among other general features of Lovelock gravity, very recently it has also been shown that bound orbits around a static black hole exist in all even \( d = 2N + 2 \) dimensions [13].

It should be noted that the pure Lovelock equation is a classical gravitational equation relevant for \( d = 2N + 1, 2N + 2 \) dimensions for a given \( N \). It is not a correction arising out of some other theory such as string theory. This is a basic difference in perspective between pure Lovelock and Einstein–Lovelock theories. From this perspective, Einstein gravity is relevant only for \( d = 3, 4 \) dimensions and similarly GB is relevant for \( d = 5, 6 \), is Gauss–Bonnet, and hence there is no question of inclusion of Einstein or other terms < \( N \). We thus have only one coupling constant for one force that would be determined experimentally as \( G \) is determined for Einstein gravity. When there is more than one coupling constant, higher order couplings arise as a measure of corrections to the first order Einstein gravity. Pure Lovelock is therefore on an entirely different plane.

Black holes are by far the most interesting solutions of gravitational theories. Hence their existence, uniqueness and stability are of utmost importance and have been intensely discussed in the literature. In higher dimensions the vacuum solution with the usual spherical topology \( S^{d-2} \) is static and unique [14]. This is very important for black hole stability analysis.

In this paper, we focus on the study of stability of pure Lovelock black holes with horizons having spherical topology. We studied the linear stability for four-dimensional Einstein black holes. The metric perturbations are decomposed according to their transformation properties under two-dimensional rotations. They are classified by transformation
under parity, namely odd (axial) [15] and even (polar) [16]. The two modes give rise to the same Schrödinger-type differential equation for perturbations. Finally, stability of four-dimensional black holes has been thoroughly investigated by several authors in [17–19].

In higher dimensions, there exists an additional tensor mode [20]. Following the discovery of this mode a gauge-invariant formalism was developed in [21, 22] where perturbations are decomposed into three types of gravitational variables, depending on how they transform with respect to the horizon. Hence we have three types of perturbations; tensor, vector and scalar. The last two types correspond, respectively, to the odd and even modes in the four-dimensional case, whereas tensor perturbations are new and emerge only in higher dimensions.

Following the scheme proposed in [21, 22], stability of higher dimensional black holes has been an active topic of research in recent years. In the case of Einstein–Gauss–Bonnet (EGB) gravity, stability analysis under scalar and vector perturbations was carried out in [23] and later generalized to any Lovelock [24]. Also, tensor perturbations were studied in [25, 26] for the EGB case, then generalized to third order Lovelock in [27] and to any Lovelock order in [28]. It was shown in [29] that vector perturbations are stable as long as tensor perturbations are stable. Also, there exists an instability of Lovelock black holes with small mass under tensor perturbations in even dimensions and under scalar perturbations in odd dimensions.

It may be noted that in all odd $d = 2N + 1$ dimensions, there exist analogues of three-dimensional BTZ black hole [30]; let us call them pure Lovelock odd $d = 2N + 1$ dimensional black holes. Then $N = 1$ is the BTZ black hole. It is, however, possible to have BTZ-like black holes in even dimensions with a nontrivial geometry of the horizon [31, 32]. For stability analysis we shall employ a new approach in which the stability equation directly follows from the second order of the action without the use of the S-deformation method. One of the important results of our analysis is recognition of the key role played by $\Lambda$ in imparting stability to the otherwise unstable pure Lovelock black hole. But this result comes at a price: The mass of the black hole is bounded on either side, and the upper bound is dictated by the existence of a horizon while the lower in terms of stability.

Notice that we focus on the particular case of tensor perturbations because we work in the critical even $2N + 2$ dimensions for pure Lovelock gravity. In fact, as alluded to in the beginning of this discussion, Lovelock gravity in odd and even dimensions is radically different [8–13]. Also, scalar perturbations do not provide an additional constraint for even dimensions [29]; hence we focus on tensor perturbations in critical even $2N + 2$ dimensions.

This paper is organized as follows. In section 2 we give the basic Lovelock formulation followed by recall of static black hole solutions in section 3. In section 4, we present the Ishibashi–Kodama formalism, derive the master equation for tensor perturbations, and discuss stability by using S-deformation technique, and in section 5 we derive the same equation by expanding action in the second order, which also gives us a no-ghost condition without use of the S-deformation. In section 6, we obtain the upper bound on black hole mass in terms of $\Lambda$ as a condition for existence of a horizon. Next we consider stability of pure Lovelock even and odd dimensions in section 7 and of dimensionally continued black holes in section 8. In section 9 we compare the stability of Einstein, EGB, dimensionally continued and pure Lovelock black holes and the paper is rounded off by discussion.
2. Lanczos–Lovelock gravity

We consider the Lagrangian in d dimensions

\[ \mathcal{L} = \sqrt{-g} \sum_{m=0}^{N} \alpha_m \mathcal{L}_m, \]

where we define the maximum integer \( N \equiv [(d-1)/2] \), \( \alpha_m \) are arbitrary constants which represent the couplings and \( \mathcal{L}_m \) is given by

\[ \mathcal{L}_m = \frac{1}{2^n} \delta_{a_1 \cdots a_{2m}}^{b_1 \cdots b_{2m}} R_{a_1 b_1}^{a_2 b_2} \cdots R_{a_{2m-1} b_{2m-1}}^{a_{2m} b_{2m}}, \]

where \( R^{ab} \) is the Riemann tensor and \( \delta_{a_1 \cdots a_{2m}}^{b_1 \cdots b_{2m}} \) is the generalized Kronecker delta of order \( 2m \).

In the following, we use \((a, b)\) as generic indices, whereas \((i, j, k, l)\) = \((2, 3, \cdots, n + 1)\) and \(n = d - 2\).

We consider spacetime of the following form\(^4\)

\[ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \]

with \( f(r) = \kappa - r^2 \psi(r) \), and \( \bar{g}_{ij} \) is the metric\(^5\) of the \( n = d - 2 \)-dimensional constant curvature space with a curvature \( \kappa = 1, 0 \) or \(-1\).

The vacuum equation then reduces to solving the master algebraic equation \([33–35]\)

\[ \sum_{m=0}^{N} \hat{a}_m \psi^m = \frac{\mu}{r^{d-1}}, \]

where the coefficients \( \hat{a}_m \) are given by

\[ \hat{a}_m = \frac{\alpha_m (d-2)!}{(d-1-2m)!}. \]

\( \mu \) is a constant of integration related to the mass by \( \mu = 16\pi G M_{\text{ADM}} / \Omega_n \), \( \Omega_n = 2\pi^{(n+1)/2}/\Gamma((n+1)/2) \) is the area of a unit \( n \)-sphere \((n = d - 2)\), \( G \) denotes the \( d \)-dimensional Newton constant and \( M_{\text{ADM}} \) is the Arnowitt–Deser–Misner mass.

3. Perturbations and stability

In higher dimensions, perturbations can be decomposed into scalar, vector and tensor modes according to how they transform under \( SO(d - 1) \) \([21, 22]\). In \( d = 4 \), a vector perturbation corresponds to the axial (odd) mode while the scalar perturbation corresponds to the polar (even) mode. Finally, an additional tensor mode is present in dimensions \( > 4 \) because there are no suitable tensor harmonics in \( d = 4 \) \([36]\).

Here we study tensor perturbations around solution \((4)\) of the form

\[ g_{ab} \rightarrow g_{ab} + f_{ab}, \]

where \( f_{ab} = 0 \) unless \((a, b) = (i, j)\), and

\[ f_{ij} (t, r, x) = r^2 \psi(t, r) \bar{h}_{ij}(x). \]

\( ^4\) Note that for vacuum solution, it is \( R^t_t = R^r_r \) that implies \( g_{tt} g_{rr} = -1 \).

\( ^5\) A bar will be added to all the tensors associated with the \( n \)-manifold.
Here \( \tilde{h}_{ij} \) is a TT tensor (traceless-transverse) with respect to the metric \( \tilde{\gamma}_{ij} \), solving the eigenvalue problem

\[
p^j \tilde{h}_{ij} = 0, \quad \tilde{\nabla}^j \tilde{h}_{ij} = 0, \quad \Box \tilde{h}_{ij} = p \tilde{h}_{ij},
\]

where \( \tilde{\nabla}^j \) is a covariant derivative with respect to \( \tilde{\gamma}_{ij} \) and \( \Box = \tilde{\nabla}^i \tilde{\nabla}_i \).

Now the Riemann tensor at the first order gives

\[
R_{ab} = R_{ab} + \delta R_{ab},
\]

where

\[
\delta R_{ab} = \frac{1}{2} \left( R_{ab}^e [c f d] - \nabla_v [c f d] \right),
\]

from which we have

\[
\delta R_{ij} = - \frac{1}{2} \left( \phi \partial_j \tilde{\gamma}^{[i} \partial_{]j} \tilde{f}_{ikl} \right) + \left( \frac{\phi \partial^j \tilde{f}_{ikl}}{r^2} \right) \delta_{ij} \tilde{f}_{ikl},
\]

From the last equation, we have the useful relation

\[
\delta_i \delta_{ij} = - \frac{1}{2} \left( (n-2) \frac{\partial \tilde{f}_{ij}}{r} + \frac{\gamma - 2\alpha}{r^2} \phi \right) \tilde{f}_{ij}.
\]

By using the previous formulae, it can be shown that \( \delta G^a_0 = \delta G^a_1 = 0 \) and

\[
\delta G^1_i = - \sum_{m=0}^N \frac{ma_n}{2m+1} \delta R_{a2m+1} \delta R_{b2m+1} \delta R_{c2m+1} \delta R_{d2m+1} + \delta R_{ijk} \delta_{ijk} \delta_i
\]

which after some calculations along with \( \gamma \tilde{h}_{ij} = 0 \), yields

\[
\delta G^1_i = A \left( \delta R_{0i} + \delta R_{1i} \right) + B \delta R_{ik} \delta_{ik}^k
\]

where

\[
A = \sum_{m=0}^N \frac{ma_n}{2m+1} \frac{(n-2)!}{(n+1-2m)!} \psi^{n-2} \left[ (n-1) \psi + (m-1) \psi_0 \right]
\]

\[
B = \sum_{m=0}^N \frac{ma_n}{2m+1} \frac{(n-3)!}{(n+1-2m)!} \psi^{n-3} \left[ (n-1)(n-2) \psi^2 + (m-1)(m-2) \psi^2 + (m-1)(n-2) \psi \psi_0 \right]
\]

As noted in [28], the equations are simpler if we define the function \( h = r^{n-2}A \). Hence we write
\[ \delta G^i_j = \frac{h}{r^{n+2}} \left( \delta R^{(0)}_{ij} + \delta R^{(1)}_{ij} \right) + \frac{h'}{(n-2)r^{n+3}} \delta R^{(2)}_{ik} \delta R^{(2)}_{jl}. \]  

After setting \( \phi(t, r) = e^{\omega \tau} \chi(r) \), which is possible because the background is static (we use the separation of variables and integrate over time), \( \delta G^i_j = 0 \) gives

\[ -f^2 \chi'' - \left( \frac{f^2 h'}{h} + \frac{2f^2}{r} + ff' \right) \chi' + \frac{(2\gamma - \gamma)fh'}{\left(n - 2 \right) r^h} \chi = -\omega^2 \chi. \]  

By further introducing \( \chi = \Phi(\sqrt{\frac{1}{h}}) \) for \( h > 0 \) (or \( \chi = \Phi(\sqrt{\frac{1}{-h}}) \) for \( h < 0 \)) and switching to ‘tortoise’ coordinate \( r^* \), defined by \( dr^* / dr = 1 / f \), we can rewrite the previous equation in the Schrödinger form with ‘energy’ eigenvalue \( E = -\omega^2 \),

\[ -\frac{d^2 \Phi}{dr^*^2} + V \left( r \left( r^* \right) \right) \Phi = -\omega^2 \Phi \equiv E \Phi \]  

where potential is given by,

\[ V(r) = \frac{(2\gamma - \gamma)fh'}{\left(n - 2 \right) r^h} + f \frac{d}{r \sqrt{h}} \frac{d}{dr} \left( f \frac{d}{dr} \left( r \sqrt{h} \right) \right). \]  

and \( \mathcal{H} \) is a differential operator in the Hilbert space of square integrable functions. The solution will be perturbatively stable if and only if the differential operator \( \mathcal{H} \equiv -\frac{d^2}{dr^*^2} + V \) acting on functions defined in the region \( f > 0 \) is a positive self-adjoint operator so that it has no negative eigenvalues \( (E < 0) \). In fact, let us suppose that we have an unstable mode with \( \omega \in \mathbb{R}^* \), which means \( E < 0 \); hence we have

\[ E \int |\Phi|^2 dr^* = \int \Phi^* \mathcal{H} \Phi dr^* \]

\[ = \int \left( \left( \frac{d\Phi}{dr^*} \right)^2 + V|\Phi|^2 \right) dr^* \]

where integration is defined in the region \( f > 0 \) and we have performed a partial integration.

If the potential is positive, this leads to a contradiction because the left-hand side is negative whereas the right-hand side is positive. Thus, the mode is stable if the potential is positive. In contrast, if the potential is negative, we cannot conclude anything about stability. Before we can do that, we must further analyze the equation by using the ‘S-deformation’ technique [21]. This deformation is useful for transforming a partly negative potential to a positive-definite one. We have

\[ \int \left( \left( \frac{d\Phi}{dr^*} \right)^2 + V|\Phi|^2 \right) dr^* = \int \left( |D\Phi|^2 + W|\Phi|^2 \right) dr^*, \]

where

\[ D = \frac{d}{dr^*} + S, \]

\[ W = V + \frac{dS}{dr^*} - S^2, \]

and \( S \) is ‘S-deformation’ function to be defined. Hence, as shown in [28] and by choosing

\[ S = -\frac{d}{dr^*} \ln \left(r \sqrt{h} \right), \]
we get
\[
W = \frac{(2\kappa - \gamma)h'}{(n - 2)rh}.
\] (29)

The ‘new’ potential depends linearly on the factor \(2\kappa - \gamma\), which is positive and can be very large. For example, when \(\kappa = 1\), we have \(2\kappa - \gamma = l(l + n - 1)\), which can be sufficiently large for high harmonics \(l\).

In the case where \(h'/h > 0\), the deformed potential \(W\) is positive, which implies stability of the solution, whereas if \(h'/h < 0\), the potential can be sufficiently negative for high harmonics and therefore equation (25) will be negative, hence unstable modes with negative energy states \((\omega > 0)\). Because the potential can be as negative as desired, we can always find a negative mode solution that is normalizable. A sufficient condition of the instability is \(\int Wdr < 0\) [37], which is trivially satisfied for high harmonics. Thus stability can be read from the sign of the deformed potential \(W\), which implies that the spacetime is stable iff \(h'/h > 0\) on the domain of definition \(f > 0\).

4. No ghost condition

In this section, we adopt another approach, by expanding the action at the second order of perturbations. It is more convenient from a computational point of view and easily gives the conditions for the avoidance of ghosts and Laplacian instabilities. The approach has been widely used for the stability analysis of various spacetimes, starting with the seminal work for the Schwarzschild spacetime [38]; see also the same approach in the context of modified gravity [39] or cosmology [40]. The relevant formulae are given in the appendix. After some lengthy calculations, we have for action on the shell
\[
S = \int d^4x\sqrt{-g} \frac{r^2}{4f} \left[ \dot{\phi}^2 - f^2 \dot{\phi}^2 - W\phi^2 \right] h_i^j \bar{h}^i_j,
\] (30)
from which we can derive no ghost condition as \(h > 0\). Hence, a ghost mode is always present around a spherically symmetric static background in vacuum if \(h < 0\). However, the existence of a ghost does not necessarily imply instability if the mode is massive enough. In fact, the theory might be considered as valid below some cutoff scale and should be completed at higher energies. Therefore, if the of the ghost mode is larger than a cutoff scale \(M_{\text{cutoff}}\) (let us say \(M_{\text{Pl}}\)), instability can be disregarded. But in this particular case, we will have instability at least for the monopole perturbation \(l = 0\), which is massless. Hence we will also consider an additional condition \(h > 0\).

Notice that \(W\) appears as effective mass squared. Hence we can deduce positivity of \(W\) without using the ‘\(S\)-deformation’. Defining new variable \(\phi = \Phi/(r\sqrt{h})\), and switching to the tortoise coordinate, we have
\[
S = \frac{1}{4} \int d^4ydrdr^* \sqrt{-g} \left[ \left( \frac{d\Phi}{dt} \right)^2 - \left( \frac{d\Phi}{d\rho^*} \right)^2 - V\phi^2 \right] h_i^j \bar{h}^i_j,
\] (31)
from which equation (22) can be easily derived.

5. Pure Lovelock

In the following discussion, we will consider the pure Lovelock vacuum solution [41], for which we have
\[ \alpha N \psi^N + \alpha_0 = \frac{\mu}{r^{d-1}}, \]  
\[ \text{at fixed } N. \] So we have the solution,
\[ f(r) = \kappa \pm r^2 \left( \frac{M}{r^{d-1}} + \Lambda \right)^{1/N}, \]
where we have defined \( M = \mu \alpha N \) and \( \Lambda = -\alpha_0 \alpha N \). There are two families of solutions corresponding to the sign in the metric function. The positive branch exists when \( N \) is even, while the negative branch exists for all dimensions.

5.1. Pure Lovelock in even dimensions

In this case, \( d = 2N + 2 \), we can rewrite the solution (33) as
\[ f(r) = \kappa \pm r^2 \left( \frac{M}{r^{2N+2}} + \Lambda \right)^{1/N}. \]
The positive mass solution has a central spacelike (timelike) singularity for the negative (positive) branch.

5.1.1. \( \Lambda = 0 \) case: in this case, we need to assume positivity of mass \( M \geq 0 \). The spacetime with \( M = 0 \) corresponds to Minkowski (hyperbolic) spacetime for \( \kappa = 1 \) (\( \kappa = -1 \)). There is no horizon when \( \kappa = 0, 1 \) for the positive branch or when \( \kappa = -1 \) for the negative branch. The solution represents a spacetime with a globally naked singularity. When \( \kappa = -1 \) for the positive branch, there is a cosmological horizon at \( r = M \). Therefore, this solution represents a spacetime with a globally naked singularity, whereas there is an event horizon at \( r = M \) for the negative branch with \( \kappa = 1 \).

5.1.2. \( \Lambda < 0 \) case: for a well-defined theory, the condition \( M > 0 \) should be satisfied except for \( d = 4 \) or \( N = 1 \). Besides the central singularity at \( r = 0 \), there also occurs a branch singularity at \( r = r_b = (-M/\Lambda)^{1/(2N+1)} \) as in the Einstein GB gravity [42]. The metric is finite at \( r = r_b \), but its derivative blows up and it becomes complex when \( r > r_b \). Around the branch singularity, the Kretschmann scalar behaves as \( (r - r_b)^{-2N/3} \) and the singularity is timelike, null or spacelike for \( 1, 0, -1 \), respectively.

For \( \kappa = 0, 1 \), for the positive branch and \( \kappa = -1 \) for the negative branch, the solution has no horizon and hence represents the spacetime with a globally naked singularity.

For the negative branch and \( \kappa = 1 \), the solution has one event horizon at \( r < M \) (see figure 1), whereas for the positive branch and \( \kappa = -1 \), there is one cosmological horizon at \( r < M \). The latter case represents the spacetime with a naked singularity. Notice that the branch singularity is always outside the horizon.

5.1.3. \( \Lambda > 0 \) case: if \( M = 0 \), it is the de Sitter spacetime. If mass is negative, it is defined only for \( r > r_b \), where \( r_b \) is the location of the branch singularity. If, further, \( \kappa = 0 \), the horizon coincides with the singularity and for \( \kappa = 1, -1 \) there is a naked singularity for the positive (negative) branch. The negative branch with \( \kappa = 1 \) has one horizon (see figure 1), but \( f(r_b) = 1 \); hence the branch singularity lies inside the cosmological horizon \( (f'(r_{\text{horizon}}) < 0) \). For the positive branch and \( \kappa = -1 \), we have an event horizon containing the branch singularity.

When \( M > 0 \) only the central singularity exists. It is a naked singularity for \( \kappa = 0 \) and for the positive (negative) branch with \( \kappa = 1, -1 \). The situation is more complicated for the
positive (negative) branch and $\kappa = -1, 1$. In fact, horizons are located at $f(r) = 0$, which gives

$$r = M + \Lambda r^{d-1} = M + \Lambda r^{2d+1}. \quad (35)$$

Hence we have zero or two real positive solutions of the previous equation as can be seen from figure 1.

In the case where there is no solution, we have a naked singularity. The second situation has two positive real roots of the equation (35); the smallest root is an event horizon (inner horizon) for the negative (positive) branch and $\kappa = 1 (\kappa = -1)$ and the second root is a cosmological (event) horizon.

The existence of the horizons depends on the parameters of the model. ($\Lambda, M$) should be small enough to have a black hole. We notice that (35) is a polynomial of order $d - 1$ for which the discriminant ($\Delta$) can be calculated. Contrary to the quadratic case, the sign of the discriminant will not be relevant. But when $\Delta = 0$, at least two roots will be equal whether real or not. In our particular case, the two real roots will be equal when $\Delta = 0$. Hence the discriminant will give us the conditions of existence of the horizons. We can directly calculate the discriminant, which is a determinant of an $(2d - 3) \times (2d - 3)$ Sylvester matrix, or we can use the standard theorems of the resultant (res) of the polynomials ($P, P'$) where $P \equiv \Lambda r^{d-1} - r + M$ and we have

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{Solutions of the equation $r = M + \Lambda r^{2d+1}$ in the case $\Lambda > 0$ (dashed line) and $\Lambda < 0$ (dotted line). When $M > 0$, we have 0 or 2 horizons depending on the values of the parameters, whereas when $M < 0$ we have always one horizon. We see that for $M > 0$, the horizons are at $r > M$. For $\Lambda < 0$, we always have one positive solution at $r < M$.}
\end{figure}
\[ \Lambda (-1)^{(d-1)(d-2)/2} \Delta = \text{res}_{d-1,d-2}(P, P') = \text{res}_{d-1,d-2}(P + QP', P'), \quad Q \equiv -\frac{r}{d-1} \]

\[ = (-1)^{(d-2)/2} \Lambda (d-1)^{(d-2)/2} \text{res}_{d-2}(\frac{2-d}{d-1} + M, P') \]

\[ = \Lambda (1 - d)^{d-2}\left(\frac{2-d}{d-1}\right)^{d-2} P'\left(\frac{d-1}{d-2} M\right) \]

\[ = \Lambda (d-2)^{d-2}\left[\frac{\Lambda M^{d-2} (d-1)^{d-1}}{(d-2)^{d-2}} - 1\right]. \quad (36) \]

This gives the condition of the existence of two horizons

\[ \Lambda M^{d-2} < \frac{(d-2)^{d-2}}{(d-1)^{d-1}}. \quad (37) \]

Hence we found an extreme mass \( M_{\text{ex}} = (d-2)/(d-1)^{(d-1)/(d-2)} \). We have two or zero horizons for \( M < M_{\text{ex}} \) and \( M > M_{\text{ex}} \), respectively.

Notice that we also have a maximum mass for the Einstein case in all dimensions if \( \Lambda > 0 \) and \( \kappa = 1 \); then

\[ \mu < 2^{2-d} d^{-2} (d-2) (d-3)^{d-1/d-2} A^{1-d/d}. \quad (38) \]

Finally, if \( M = M_{\text{ex}} \), the two solutions coincide and the horizon is degenerate; it represents the extreme black hole spacetime. For this solution, we have \( f'(r_H) = 0 \) at the horizon \( (r_H) \) and

\[ f''(r_H) = \pm \frac{2}{r^3} \left(r - M^d - \frac{1}{d-2}\right). \quad (39) \]

which easily gives the position of the horizon \( r_H = M(d-1)/(d-2) \). Hence we have an additional condition on the position of the horizons in the case where \( M < M_{\text{ex}} \); the first horizon is at \( M < r < M(d-1)/(d-2) \) and the outer horizon is at \( r > M(d-1)/(d-2) \).

### 5.2. Pure Lovelock in odd dimensions

In the following discussion, we consider pure Lovelock vacuum solution odd dimensions \( d = 2N + 1 \) for which we have

\[ f(r) = \kappa \pm \frac{r^2}{r^2} \left(\frac{M}{r^{2N}} + \Lambda\right)^{1/N}, \quad (40) \]

As previously, the positive branch exists when \( N \) is even, whereas the negative branch exists for all \( N \). Notice that the spacetime is regular; we do not have a central singularity. But we can have existence of horizon and branch singularity. The discussion about branch singularity follows the same idea as for even dimensions; therefore, we will not reproduce it here. The existence of the horizon demands \( k = 1 \) for the negative branch and \( k = -1 \) for the positive branch. Considering the equation \( f(r) = 0 \), we have a unique solution \( r_H = (1 - M)\Lambda^{1/2N} \). Hence the horizon exists for \( \Lambda > 0 \) if \( M < 1 \) and for \( \Lambda < 0 \) if \( M > 1 \). Also, the horizon is an event horizon if \( f'(r_H) > 0 \), which gives

- **Positive branch**: \( k = -1, \ \Lambda > 0 \) and \( M < 1 \),
- **Negative branch**: \( k = 1, \ \Lambda < 0 \) and \( M > 1 \),

and we have a cosmological horizon for the opposite \( \Lambda \).
Notice that the negative branch will always have a branch singularity outside the event horizon. In all other cases we have a smooth spacetime, if no branch singularity. Obviously, we assumed \( M > 0 \) in our discussion.

### 6. Stability of pure Lovelock black holes

We turn now to the stability of the black hole derived previously (33). We have

\[
\frac{h'(r)}{h(r)} = \frac{d-4}{r} + \frac{M(d-1)}{N(Mr + \Lambda r^d)} - \frac{M(d-1)(d-2N-1)}{M(d-2N-1)r + (d-3)\Lambda N r^d} \tag{41}
\]

while \( h'/h = (d-3N-1)/(Nr) \) for \( \Lambda = 0 \) and \( d \neq 2N + 1 \). Considering the condition \( h'/h > 0 \), we see that we always have instability for the case \( \Lambda = 0 \) in critical even dimensions \( d = 2N + 2 \). In the case \( d = 2N + 1 \), the last term of (41) is zero and we have \( h'/h = nr > 0 \). Notice that the difference of stability is because the limits when \( d \to 2N + 1 \) and \( \Lambda \to 0 \) do not commute for (41).

We turn now to the case \( \Lambda \neq 0 \).

#### 6.1. Stability of pure Lovelock in even dimensions

As we have seen previously, only three nontrivial solutions are interesting. These three are the solutions for \( \Lambda > 0 \) and

\[
f(r) = 1 - r^2 \left( \frac{M}{r^{2N+1}} + \Lambda \right)^{1/N}, \quad 0 < M < M_{\text{ex}} \tag{42}
\]

\[
f(r) = -1 + r^2 \left( \frac{M}{r^{2N+1}} + \Lambda \right)^{1/N}, \quad 0 < M < M_{\text{ex}} \tag{43}
\]

\[
f(r) = -1 + r^2 \left( \frac{M}{r^{2N+1}} + \Lambda \right)^{1/N}, \quad M < 0 \tag{44}
\]

We have stability \( (h'/h > 0) \) when \( r > r_c \), where \( r_c = c_d(M/\Lambda)^{1/(d-1)} \) is a critical radius and \( c_d \) is a coefficient that depends only on the dimension \( d \). The spacetime will be stable if \( r_c > r_e \), where \( r_e \) is the position of the event horizon.

For the first case (42) and using \( h'/h > 0 \) along with (41), we want \( f(r_c) < 0 \) and \( f'(r_c) > 0 \), which translates to

\[
\frac{\Lambda M^{d-2}}{(d-1)^{d-1} (d^2 - 2d + 6)^{d-2}} \left( 6 - d + \frac{\sqrt{d(d - 4)} + 12}{d} \right)^d \tag{45}
\]

In the other cases (43), (44) we can have \( h'/h > 0 \), but we will always have \( h < 0 \) for the relevant dimensions \( d = \{6, 10, 14, \ldots \} \). Hence these cases are excluded according to the no-ghost condition. Therefore we conclude that the unique stable and ghost-free spacetime is the solution (42) for which a critical mass (minimum value) can be defined. The stability is guaranteed if \( M > M_{\text{ex}} \), where \( M_{\text{ex}} \) is easily derived from (45).
6.2. Stability of pure Lovelock in odd dimensions

Considering only the case $\Lambda > 0$ for which we do not have a branch singularity, we have

$$\frac{h'}{h} = \frac{M (2N - 1) + (2N - 3) \Lambda r^{2N}}{r \left( M + \Lambda r^{2N} \right)}$$

(46)

which is positive iff $-\Lambda r^{2N}/M < (2N - 1)/(2N - 3)$. This is always true ($N > 1$). Also we have $h \propto \Lambda/\psi > 0$ if $\psi > 0$ (negative branch). Therefore, the only ghost-free stable solution under tensor perturbations is

$$f = 1 - r^2 \left( \frac{M}{r^{2N}} + \Lambda \right)^{1/N},$$

(47)

which has a cosmological horizon. This means there exists no stable pure Lovelock black hole in odd dimensions, which is however understandable because potential due to mass is constant.

7. Dimensionally continued BTZ black holes

In this section, we consider a constraint on the parameters of the model. In this case, we have

$$\tilde{a}_m = \hat{a}_N \left( \frac{N}{m} \right) \Lambda^{dN-m},$$

which gives from (5)

$$\sum_{m=0}^{N} \tilde{a}_m \psi^m = \hat{a}_N \sum_{m=0}^{N} \left( \frac{N}{m} \right) \Lambda^{dN-m} \psi^m = \hat{a}_N \left( \Lambda^2 + \psi \right)^N = \frac{\mu}{r^{d-1}}.$$  

(48)

Hence we have

$$f = k - \frac{M^{1/N}}{r^{(d-1-2N)/N}} + \Lambda^2 r^2,$$

(49)

where $M = \mu/\hat{a}_N$. Notice that in the special case of odd dimensions $d = 2N + 1$, we have

$$f = k - M^{2/(d-1)} + \Lambda^2 r^2.$$  

(50)

The existence of a horizon demands $M^{2/(d-1)} > k$.

In the general case, we have after some algebra

$$h = \hat{a}_N \frac{N}{d-2} r^{d-4} \left[ (\Lambda^2 + \psi)^{N-1} + \frac{r}{n-1} \frac{d}{dr} (\Lambda^2 + \psi)^{N-1} \right].$$

$$= \hat{a}_N \frac{(d-1-2N)M^{(N-1)/N}}{(d-2)(d-3)} r^{(2N-d+3)/N}.$$  

(51)

Hence we have instability for even dimensions ($h' < 0$). But for critical odd dimensions, we have $h = 0$. This is simply because potential due to mass is constant, and that is why there is no central singularity. It is therefore neutral to perturbations.

More generically, we will have $h = 0$ (from (18)) iff

$$\sum_{m=1}^{N} m \tilde{a}_m \psi^{m-1} = \frac{\alpha}{r^{d-3}}.$$  

(52)
where $\alpha$ is a constant, along with the equation for $\psi$

$$\sum_{\mu=0}^{N} \hat{a}_{\mu} \psi^{\mu} = \frac{\mu}{r^{d-1}}.$$  

(53)

Differentiating the last equation w.r.t $\psi$ and using (52), we have

$$\frac{\alpha}{r^{d-3}} = -\frac{\mu (d-1)}{\psi' r^{d}}$$

(54)

which gives $\psi = \beta + \frac{\mu (d-1)}{2\alpha r^{d-3}}$, hence the unique black hole for which $h = 0$ is of the form

$$f = a_1 + a_2 r^2$$

(55)

where ($a_1$, $a_2$) are two constants.

8. Einstein vs EGB vs pure GB

Considering the theory in 6D, we have for $L = \alpha_1 R + \alpha_2 R_{\text{EGB}} - 2\lambda$

$$h = r^2 (\alpha_1 + 4\alpha_2 (3\psi + r\psi'))$$

(56)

In the case of Einstein, we always have stability if $\alpha_1 > 0$; in fact $h = \alpha_1 r^2 > 0$ and $h' > 0$. The case of pure GB has been discussed previously. In fact $\psi = (\frac{M}{\sqrt{\Lambda}} + \Lambda)^{1/2}$, where $\Lambda = \lambda/60\alpha_2$

$$h = \frac{2\alpha_2}{r^9} \left( M + 6\Lambda r^2 \right),$$

(57)

$$h' = \frac{\alpha_2}{r^2 \psi} \left( 24 - \frac{25 M^2}{(M + \Lambda r^2)^2} \right).$$

(58)

We have stability if $r > r_c = ((5/2\sqrt{6} - 1)M/\Lambda)^{1/5}$. In order to have stability of the spacetime we need to impose $r_c < r$, which implies $M^2 \Lambda > 0.019$.

For the most general case of EGB in 6D, we have

$$\psi = -\frac{M}{\sqrt{\Lambda}} + \alpha_1 r^2 + \frac{2\alpha_2}{\sqrt{\Lambda}} \frac{\alpha_2}{r^2} r^2 \left( \frac{M^2}{(M + \Lambda r^2)^2} \right)^{1/5}.$$  

(59)

The constraint can be generalized easily; e.g., we have for $\Lambda > 0$,

$$M < \frac{4 \sqrt{3} (1 + \sqrt{10} + \sqrt{15})}{(-1 + \sqrt{10} + \sqrt{15})^{1/2}} \frac{\alpha_2}{\alpha_1}.$$  

(60)

where $X = 144\alpha_2^2 \Lambda/\alpha_1^2$.

Hence we see that except for the Einstein case, we will always have an instability for small masses. Also, the constraint is stronger in EGB compared with pure GB if $\Lambda$ is large ($\Lambda > 2.6 \times 10^{-3}\alpha_1^2/\alpha_2^2$). On the contrary, when $\Lambda$ is small, EGB allows a larger spectrum of stability for the mass.
9. Conclusion

In this paper we have derived the master equation for tensor perturbations of black holes in any order Lovelock theory, that is, in any dimensions, by expanding action to the second order of perturbations. Hence we have derived no-ghost and tachyonic stability conditions for a spherically symmetric solution in Lovelock gravity. The relevant perturbation-defining function $h$ must be positive to avoid a ghost and $h' > 0$ to have the stability of a black hole (tachyonic instability). We apply this stability analysis to pure Lovelock theory in even $d = 2N + 2$ dimensions. It turns out that pure Lovelock black hole spacetime is always unstable unless $\Lambda$ is brought in. $\Lambda$ therefore plays a stabilizing role for even dimensional pure Lovelock black holes; and stability is achieved by prescribing a lower bound on its mass while an upper bound on mass comes from the existence of a horizon. The latter is derived by using the standard theorems of resultant for the polynomial $P = Ar^{d-1} - r + M$. Thus, black hole mass is bounded on either end.

Intuitively, lower and upper mass bounds could perhaps be understood as follows: Without $\Lambda$, a pure Lovelock black hole for $N > 1$ is always unstable. Note that the gravitational potential for a pure Lovelock black hole goes as $1/r^{d-2}/2$, which is weaker than the Einstein potential, going as $1/r^{d-3}$. Now when $\Lambda$ is introduced, which implies repulsion going as $r$, it effectively defines the instability threshold radius, which could be pushed inside the black hole horizon if it is sufficiently massive. This is how the lower mass threshold comes about. The presence of $\Lambda$ would, however, always define the upper threshold for mass in all cases, even for Einstein gravity (see also [44]).

For Einstein gravity, there is always stability and so is also the case for an odd dimensional (BTZ-like) pure Lovelock black hole. The latter, however, has no central singularity; it essentially means that dS-like space is stable under tensor perturbations but it may not be stable for scalar perturbations. In particular, a six-dimensional pure GB black hole is stable for $M^4\Lambda > 0.019$. There also exists a lower mass bound for the stability of an EGB black hole. In the case of dimensionally continued black holes; even dimensional ones are unstable, whereas for odd dimensional ones, $h = 0$, and hence they are neutral for tensor perturbations.

In Einstein–Lovelock theory, there are as many coupling constants as the degree of polynomial $N$ that cannot be determined, since there is only one force that can determine only one coupling constant. Hence, besides $\Lambda$, which is a constant of a spacetime structure [6], there should be only one free coupling. This is what is done in pure Lovelock gravity, whereas for the dimensionally continued case all the couplings are related to the unique vacuum defined by $\Lambda$ [43]. Then dimensionally continued black holes are unstable, whereas pure Lovelock ones could be made stable by $\Lambda$. If stability is the determining criterion, pure Lovelock black holes score higher than dimensionally continued ones. We thus have a remarkably interesting result for pure Lovelock black holes indicating that the existence of a horizon and stability binds its mass between two upper and lower thresholds; i.e., $M_c < M < M_s$.

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Appendix A. First order

The first order variation of the Riemann tensor is given by

\[ \delta R_{ij} = \left( \frac{\dot{\phi}}{2f} - \frac{f'}{4\phi'} \right) \bar{\delta}_{ij}, \]  
\[ \delta R_{ij} = -\frac{f}{2} \frac{\phi}{f'} \left( \frac{f'}{4} + \frac{\phi}{r} \right) \bar{\delta}_{ij}, \]

\[ \delta R_{ij}^{kl} = -\frac{1}{2} \left[ \frac{\phi}{r^2} \delta_{ij}^{[k} \bar{\delta}_{lj]} + \left( \frac{\kappa \phi}{r^2} + \frac{f \phi'}{r} \right) \delta_{ij}^{[k} \bar{\delta}_{lj]} \right], \]

\[ \delta R_{ij} = -\frac{f}{2} \frac{\phi}{f'} \left( \frac{f'}{4} - \frac{\phi}{2r} \right) \bar{\delta}_{ij}, \]

\[ \delta R_{ij}^{0j} = \frac{\phi}{2r^2} \left( \bar{\delta}_{ij}^{+} \bar{\delta}_{ij}^{+} \right). \]

Appendix B. Second order

The second order variation of the Riemann tensor is given by

\[ \delta^2 R_{abcd} = \frac{1}{2} \left[ \phi \left( \phi - \frac{1}{2} \frac{f'}{f} \right) \bar{\delta}_{ij}^{+} \bar{\delta}_{ij}^{+} \right] h_{ij}^{+} h_{ij}^{+}, \]

which gives for the relevant formulas

\[ \delta^2 R_{ab}^{01} = 0, \]

\[ \delta^2 R_{0i}^{0j} = -\frac{1}{2} \left[ \phi \left( \phi - \frac{1}{2} \frac{f'}{f} \right) + \frac{\phi^2}{2} \right] h_{ij}^{+} h_{ij}^{+}, \]

\[ \delta^2 R_{ij}^{0j} = \left[ \frac{f}{2} \phi \phi' + \frac{f'}{4} \phi' + \frac{f'}{4} \phi \phi' \right] \bar{\delta}_{ij}^{+} \bar{\delta}_{ij}^{+}. \]
\[ \delta^i R_{ij}^{\text{BI}} = \left( \frac{\phi_2^2}{4f} - \frac{f}{4} \phi^2 \right) h^{[i}_{(\ell} h_{j)l]} + \left( \frac{k}{2r^2} \phi^2 + \frac{f}{2r} \phi \phi_f \right) h^{[i}_{e} \delta^{j}_{l] e} + \frac{\phi_2^2}{4r^2} \left( \tilde{V}_{k} h^{[i}_{j]} \tilde{V}^{k}_{j} h^{i}_{1} \right) \\
+ \tilde{V}_{k} h^{[i}_{j]} \tilde{V}^{k}_{j} h^{i}_{1]} + \tilde{V}_{k} h^{[i}_{j]} \tilde{V}^{k}_{j} h^{i}_{1] e} + \tilde{V}_{k} h^{[i}_{j]} \tilde{V}^{k}_{j} h^{i}_{1]} + \tilde{V}_{k} h^{[i}_{j]} \tilde{V}^{k}_{j} h^{i}_{1] e} \right) \] (B.4)

From the last equation, we can derive a useful formula:

\[ \delta^i R_{ij}^{\text{BI}} = \frac{\phi_2^2}{2r^2} \left( 3 \tilde{V}_{k} h^{[i}_{j]} \tilde{V}^{k}_{j} h^{i}_{1} - 2 \tilde{V}_{k} h^{[i}_{j]} \tilde{V}^{k}_{j} h^{i}_{1] e} \right) + \left[ \frac{f}{2} \phi_f^2 - \frac{\phi_2^2}{2f} \right] h^{i}_{1} h^{i}_{1} \] (B.5)

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