Heights of mixed motives

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Abstract

We define the height of a mixed motive over a number field extending our previous work for pure motives.

In [8], we defined the height of a pure motive over a number field. In this paper, we extend the definition to mixed motives.

For a mixed motive $M$ over a number field, the height $h(M)$ is defined as the sum of the heights $h_{w,d}(M)$ for $w \in \mathbb{Z}$ and $d \geq 0$, and to define $h_{w,d}$ for $d \geq 1$, we have to fix a polarization of the pure motive $P_{w,k} := (\text{gr}_W^w M)^* \otimes \text{gr}_W^{w-d} M$ where $W$ is the weight filtration of $M$.

In the case $d = 0$, $h_{w,0}(M)$ is the height of the pure motive $h(\text{gr}_W^w M)$ defined in [8].

The case $d = 1$ is the height of the extension $0 \to \text{gr}_W^{w-1} M \to W_w M/W_{w-2} M \to \text{gr}_W^w M \to 0$ defined by Beilinson [1] and Bloch [2]. We review the idea of the definition in 4.3.

For $d \geq 2$, $h_{w,d}(M)$ is determined by $W_w M/W_{w-d-1} M$. It is defined in section 3 as the sum of local heights. We first give the geometric analogue of $h_{w,d}$ ($d \geq 2$) in section 2 for the idea in the number field case can be seen well by comparison.

Ideas in this paper came from a joint work of Spencer Bloch and the author. Also, joint works with Chikara Nakayama and Sampei Usui inspired this work.

Details will be published elsewhere.

1 Review on the splitting of Deligne

This section is a preliminary for sections 2 and 3.

1.1. Let $V$ be an abelian group, let $W = (W_w)_{w \in \mathbb{Z}}$ be a finite increasing filtration on $V$, and let $N : V \to V$ be a nilpotent homomorphism such that $NW_w \subset W_w$ for all $w \in \mathbb{Z}$.

Then a finite increasing filtration $W' = (W'_w)_{w \in \mathbb{Z}}$ on $V$ is called the relative monodromy filtration of $N$ with respect to $W$ if it satisfies the following conditions (i) and (ii).

(i) $NW'_w \subset W'_{w-2}$ for any $w \in \mathbb{Z}$. (ii) For any $w \in \mathbb{Z}$ and $m \geq 0$, we have an isomorphism $N^m : \text{gr}_{w+m} W' \cong \text{gr}_{w-m} W$.

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The relative monodromy filtration of $N$ with respect to $W$ need not exist. If it exists, it is unique (\cite{II} 1.6.13). In the rest of section 1 we assume that the relative monodromy filtration $W'$ of $N$ with respect to $W$ exists.

1.2. Let $m \geq 0$. Then $\text{gr}_{w+m}W' = A \oplus B$, where $A$ is the kernel of $\text{gr}_{w+m}W' \rightarrow \text{gr}_{w-m-2}W$ and $B$ is the image of $N : \text{gr}_{w+m+2}W' \rightarrow \text{gr}_{w+m}W'$. The component $A$ is called the primitive component of $\text{gr}_{w+m}W'$ and will be denoted by $(\text{gr}_{w+m}W')_{\text{prim}}$.

1.3. Denote by $W_*\text{Hom}(V, V)$ (resp. $W'_*\text{Hom}(V, V)$) the filtration on $\text{Hom}(V, V)$ induced by $W$ (resp. $W'$). Then $W_*\text{Hom}(V, V)$ is the relative weight filtration of the nilpotent homomorphism $Ad(N) : \text{Hom}(V, V) \rightarrow \text{Hom}(V, V)$ with respect to $W_*\text{Hom}(V, V)$.

1.4. Assume $V = \bigoplus_w U_w'$ is a splitting of $W'$ (that is, $W' = \bigoplus_{n \leq w} U_w'$ for any $w$) which satisfies $\mathcal{N}(U_w') \subset U_{w-2}'$ for all $w$ and which is compatible with $W$ (that is, $W_w = \sum_n (W_w \cap U_n')$ for any $n$). Then by Deligne (see \cite{II}), there is a unique splitting $V = \bigoplus_w U_w$ of $W$ such that $V = \bigoplus_{m,n} U_{m,n} \cap U_n'$ and such that if $N_w \in \text{gr}_W \text{Hom}(V, V)$ denotes the component of weight $w$ of $N$ with respect to this splitting of $W$ and if $\bar{N}_w$ denotes the class of $N_w$ in $\text{gr}_{-2} \text{gr}_W \text{Hom}(V, V)$, then $\bar{N}_{-1} = 0$ and for any $w \leq -2$, $\bar{N}_w$ belongs to the primitive component of $\text{gr}_{-2} \text{gr}_W \text{Hom}(V, V)$.

Lemma 1.5. Assume that a splitting of $W'$ as in 1.4 exists. Then the element $\bar{N}_w$ of $(\text{gr}_{-2} \text{gr}_W \text{Hom}(V, V))_{\text{prim}}$ is independent of the choice of such splitting of $W'$.

Proof. Assume we are given two such splittings $((U_w')^{(i)})$ ($i = 1, 2$) of $W'$, and let $g$ be the unique automorphism of $(V, W')$ which induces the identity map on $\text{gr}_{w'}$ such that $g(U_w')^{(1)} = (U_w')^{(2)}$. Then $gW = W$ and $g\mathcal{N} = \mathcal{N}g$. For Deligne’s splitting $(U_w^{(i)})$ of $W$ associated to $((U_w')^{(i)})$, we have $U_w^{(2)} = gU_w^{(1)}$. If $N_w^{(i)}$ and $\bar{N}_w^{(i)}$ denote the $N_w$ and $\bar{N}_w$ for $(U_w^{(i)})$, respectively, we have $N_w^{(2)} = gN_w^{(1)}g^{-1}$ and hence $\bar{N}_w^{(2)} = g\bar{N}_w^{(1)}g^{-1} = \bar{N}_w^{(1)}$. □

2 Geometric heights

2.1. We consider the case of variation of MHS (mixed Hodge structure) first.

Let $C$ be a proper smooth curve over $\mathbb{C}$ and let $H = (H_\mathbb{Z}, W, F)$ be a variation of MHS on $C - S$ for some finite subset $S$ of $C$. Assume that the graded quotients $\text{gr}_w H$ are polarized and assume that $H$ is admissible at any point of $S$.

Here the admissibility means that IHMH (infinitesimal mixed Hodge module) in the sense of Kashiwara \cite{II} appears at each point of $C$. In other words, the admissibility means that the following conditions (i) and (ii) are satisfied at each $x \in C$. (i) Let $N_x : H_{\mathbb{Q}, x} \rightarrow H_{\mathbb{Q}, x}$ be the logarithm of the local monodormy at $x$. Then the relative monodromy filtration of $N_x$ with respect to $W_x$ exists. (ii) The limit Hodge filtration appears at $x$. It is known that any variation of MHS with geometric origin satisfies these conditions (i) and (ii).

Let $w, d \in \mathbb{Z}$ and assume $d \geq 2$.

We will define the height $h_{w,d}(H)$ of $H$ as the sum of local heights $h_{w,d,x}(H)$ for all points $x$ of $C$. The local height $h_{w,d,x}(H)$ will be defined as the ”size” of $N_x$. 

2
2.2. Let $W'$ be the relative monodromy filtration of $N_x$ with respect to $W_x$. By Kashiwara [7], there is a splitting of $W'$ as in [1.4]. Hence by [1.5], $\bar{N}_{x,-d} \in (\text{gr}_W^{W'} \text{Hom}(H_x, H_x))_{\text{prim}}$ is defined for $d \geq 2$.

2.3. Now we define $h_{w,d,x}(H)$. Let $P = (\text{gr}_W^W H)^* \otimes \text{gr}_W^{W_d} H$. We have the component $\bar{N}_{x,w,d} \in \text{gr}_W^W \text{Hom}(H_x, H_x)$. Let $\langle \ , \ \rangle : P \times P \to \mathbb{Q}$ be the pairing defined by the given polarizations of $gr_w W$ and $gr_w^{W_d} H$. Then by [3], [7], the induced pairing $\langle \ , \ \rangle_{d-2} : \text{gr}_W^{W_d}(P) \times \text{gr}_W^{W_d}(P) \to \mathbb{Q}$; $(u, v) \mapsto \langle Ad(N_x)^{d-2}(u), v \rangle$
satisfies $\langle \bar{N}_{x,w,d}, \bar{N}_{x,w,d} \rangle_{d-2} \geq 0$.

We define the local geometric height $h_{w,d,x}(H)$ by

$$h_{w,d,x}(H) = (\langle \bar{N}_{x,w,d}, \bar{N}_{x,w,d} \rangle_{d-2})^{1/d} \in \mathbb{R}_{\geq 0}.$$  

(1It is important to take the $d$-th root here, to have formulas like [3.4] [3.5].)

2.4. We define the global geometric height $h_{w,d}(M)$ by

$$h_{w,d}(M) = \sum_{x \in \mathcal{C}} h_{w,d,x}(H).$$

2.5. Let $K$ be a function field in one variable over a field $k$ and let $M$ be a mixed motive over $K$.

If $k$ is of characteristic 0, the definition of the geometric height $h_{w,d}(M)$ ($d \geq 2$) as well as that of the local version $h_{w,d,x}(M)$ goes in the same way as above, by using the $\ell$-adic realization of $M$ in place of the local system $H_Q$ of MHS. The fact it works independently of $\ell$ is reduced to the above story of MHS, by taking a subfield $k'$ of $k$ which is finitely generated over $\mathbb{Q}$ such that $K$ and $M$ are defined over $k'$, and by embedding $k'$ into $\mathbb{C}$ and by using the variation of MHS associated to $M$.

2.6. In the case $k$ is of characteristic $p > 0$, for the $\ell$-adic local system ($\ell \neq p$) associated to $M$, we have the following (WM) by Deligne [4].

(WM) The relative weight filtration $W'$ of $N_x$ with respect to $W_x$ exists, and it coincides with the weight filtration in the sense of mixed sheaf.

Since $W'$ has a splitting as in [1.4] by eigen values of a frobenius, $\bar{N}_{x,d}$ ($d \geq 2$) are defined. The independence of $\ell$ and the positivity in 2.3 may become questions.

3 Heights of mixed motives over number fields

3.1. Now we consider a mixed motive $M$ over a number field $K$ with polarized graded quotients for the weight filtration. The height $h_{w,d}(M)$ ($d \geq 2$) will be defined as the sum of local heights $h_{w,d,v}(M)$ for all places $v$ of $K$. 

3
3.2. For a finite place $v$ of $K$, we assume that the definition in section 2 works by considering the associated $\ell$-adic Galois representation $M_\ell$ for $\ell$ not equal to the characteristic of the residue field of $v$, and (in place of $N_x$ in section 2) the logarithm of the action of a generator of the tame inertia group at $v$. We define $h_{w,d,v}(M)$ to be $\log(N(v))$ times the local height defined by the method of section 2.

(A difficulty comes from the fact the analogue of (WM) in 4.2 (the weight-monodromy conjecture) is not yet proved. The independence of $\ell$ and the positivity in 2.3 are also problems. We can say only that we have a good definition in many cases.)

3.3. Now we consider an Archimedean place $v$. Then $h_{w,d,v}(M)$ is defined by using the Hodge structure associated to $M$ at $v$ as follows.

3.4. Let $H$ be a mixed Hodge structure. We review a homomorphism $\delta : \text{gr}^W_w H_{\mathbb{R}} \to \text{gr}^W_{w-d} H_{\mathbb{R}}$.

As is explained in [3] 2.20, there is a unique pair $(\delta, \tilde{F})$ where $\tilde{F}$ is a decreasing $\mathbb{R}$-linear map $H_{\mathbb{R}} \to H_{\mathbb{R}}$ such that $(H_{\mathbb{R}}, \tilde{F})$ is an $\mathbb{R}$-split mixed Hodge structure, $\delta$ is a nilpotent $\mathbb{R}$-linear map $H_{\mathbb{R}} \to H_{\mathbb{R}}$ such that the original Hodge filtration $F$ of $H$ is expressed as $F = \exp(i\delta)\tilde{F}$ and such that the Hodge $(p,q)$-component $\delta_{p,q}$ of $\delta$ with respect to $\tilde{F}$ is zero unless $p < 0$ and $q < 0$.

Furthermore, a nilpotent linear map $\zeta : H_{\mathbb{R}} \to H_{\mathbb{R}}$ is defined by a universal Lie polynomial in $(\delta_{p,q})_{p,q}$ as in [CKS] 6.60. The $\mathbb{R}$-split mixed Hodge structure $(H_{\mathbb{R}}, \tilde{F})$ with $\tilde{F} = \exp(\zeta)\tilde{F}$ and the $\mathbb{R}$-splitting of $W$ associated to $\tilde{F}$ are important (see [3], [9]).

Let $\delta_{w,d} : \text{gr}^W_w H_{\mathbb{R}} \to \text{gr}^W_{w-d} H_{\mathbb{R}}$ be the linear map induced by the component of $\delta$ of weight $-d$ with respect to the last $\mathbb{R}$-splitting of $W$.

3.5. Let $H$ be a MHS. Let $P = (\text{gr}^W_w H)^* \otimes \text{gr}^W_{w-d} H$ and assume that $P$ is endowed with a polarization $p$. Then we define the height $h_{w,d}(H, p)$ as

$$h_{w,d}(H, p) := ((\delta_{w,d}, \delta_{w,d})_p)^{1/d}$$

where $(\ , \ )_p$ is the Hodge metric (Hermitian form) associated to $p$.

3.6. Let $M$ be a mixed motive over a number field $K$ with polarized graded quotient for the weight filtration.

For a real (resp. complex) place $v$ of $K$, we define the local height $h_{w,d,v}(M)$ as $h_{w,d}(H, p)$ (resp. $2 \cdot h_{w,d}(H, p)$) where $H$ is the mixed Hodge structure associated to $M$ at $v$ and $p$ is the induced polarization on $(\text{gr}^W_w H)^* \otimes \text{gr}^W_{w-d} H$.

We define the global height $h_{w,d}(M)$ by

$$h_{w,d}(M) := \sum_v h_{w,d,v}(M)$$

where $v$ ranges over all places of $K$.

3.7. Examples.

1. Let $a \in K^\times$. Then $a$ corresponds to a mixed $\mathbb{Z}$-motive $M$ over $K$ with an exact sequence $0 \to \mathbb{Z}(1) \to M \to \mathbb{Z} \to 0$. We have $h_{0,2,v}(M) = |\log(|a_v|)|$ for any $v$. 


2. Let $E$ be an elliptic curve over $K$, consider the pure $\mathbb{Z}$-motive $H^1(E)(2)$ over $K$ of weight $−3$, and consider a mixed $\mathbb{Z}$-motive $M$ over $K$ with an exact sequence $0 \to H^1(E)(2) \to M \to \mathbb{Z} \to 0$. Then $M$ gives an element $a$ of $K_2(E) \otimes \mathbb{Q}$, and $h_{0,3,v}(E)$ is expressed by the $K_2$-regulator (resp. the tame symbol) of $a$ at $v$ if $v$ is Archimedean (resp. finite).

4 On the cases $d = 0, 1$

We describe the geometric analogues of $h_{w,0}$, and describe shortly the idea of the definition of $h_{w,1}$ by Beilinson and Bloch.

We define $h_{w,0}$ of a mixed object as the height of the pure object $\text{gr}_w^WH$.

**4.1.** Let $(C, H)$ be as in section 12 and assume $H$ is pure. We define the height $h(H)$ of $H$ by

$$h(H) = \sum_{r \in \mathbb{Z}} r \cdot \text{deg}(\text{gr}^rH_C)$$

where $H_C$ is the vector bundle on $C$ corresponding to $H$ and $\text{gr}^r = F^r/F^{r+1}$ for the Hodge filtration $F$ on $H_C$. Here the degree of a vector bundle means the degree of its highest exterior power.

This invariant is related to the works 6 section 7, 10, 11, 14, etc. The author is thankful to T. Koshikawa for pointing out this relation. Comparing with these works on the geometric analogue, he suggests that any constant $c < 4/(\sum_{p,q} (p-q)^2h^{p,q} )$ may work as $c$ in Conjecture 4.8 of 8.

The case of pure motives over $k(C)$ for $k$ of characteristic 0 is similar.

**4.2.** For a proper smooth curve $C$ over a field $k$ of characteristic $p > 0$, and for a $p$-adically integral $F$-crystal $D$ (with $F : \varphi^*D \to D$) on $C$ with logarithmic poles associated to a pure motive, its height should be defined to be

$$p^{-1} \text{deg}(D/F(\varphi^*D))$$

This will be discussed in our forthcoming paper.

**4.3.** We review the idea of the definition of $h_{w,1}(M)$ by Beilinson and Bloch, shortly. Here to give a short explanation, we assume that usual philosophies on mixed motives are true. Let $P = (\text{gr}_w^W M)^* \otimes \text{gr}_w^{W−1}M$. The extension $0 \to \text{gr}_w^{W−1}M \to W_{w−1}M/W_{w−2} \to \text{gr}_w^{W}M \to 0$ corresponds to an extension $0 \to P \to Q \to \mathbb{Z} \to 0$. By taking the dual, we have an extension $0 \to \mathbb{Z}(1) \to Q^*(1) \to P^*(1) \to 0$. Assume that $P$ is polarized. By pulling back by the polarization $P \to P^*(1)$, the last extension gives an extension $0 \to \mathbb{Z}(1) \to Q' \to P \to 0$. By the exact sequence $\text{Ext}^1(\mathbb{Z}, Q') \to \text{Ext}^1(\mathbb{Z}, P) \to \text{Ext}^2(\mathbb{Z}, \mathbb{Z}(1))$ and by $\text{Ext}^2(\mathbb{Z}, \mathbb{Z}(1)) = 0$, we see that there is an element $a$ of $\text{Ext}^1(\mathbb{Z}, Q')$ whose image in $\text{Ext}^1(\mathbb{Z}, P)$ coincides with the class of $Q$. Then $a$ gives a mixed motive $R$ such that $W_0R = R$, $\text{gr}_0^WR = \mathbb{Z}$, $\text{gr}_1^WR = P$, $\text{gr}_2^WR = \mathbb{Z}(1)$, and $W_{−3}R = 0$. Define

$$h_{w,1}(M) := h_{0,2}(R).$$

This does not depend on the choice of $a \in \text{Ext}^1(\mathbb{Z}, Q')$ as above.

The geometric analogue of $h_{w,1}$ is given similarly.
5 Some topics

5.1. For a mixed $\mathbb{Z}$-motive $M$ over a number field $K$ whose graded quotients for the weight filtration are polarized, we define the (total) height $h(M)$ of $M$ by

$$h(M) := \sum_{w,d \in \mathbb{Z}, d \geq 0} h_{w,d}(M).$$

Like in the pure case, the following finiteness is a basic question. We fix a type $\Psi$ of mixed motive by giving the ranks of $\text{gr}^W_w$ for all $w$.

**Conjecture 5.2.** Fix $c > 0$ and integers $n_w \geq 1$ ($w \in \mathbb{Z}$). Then there are only finitely many isomorphism classes of mixed $\mathbb{Z}$-motives $M$ over $K$ of type $\Psi$ with polarized graded quotients for the weight filtration, with semi-stable reductions, such that $h(M) < c$ and such that the degree of the polarization is $(n_w)_w$.

5.3. If we assume this conjecture, the finite generations of the motivic cohomology groups $\text{Ext}^i_\mathbb{Z}(\mathbb{Z}, M)$ ($i \geq 0$) associated to $\mathbb{Z}$-motives $M$ over $K$ (or their subgroups, for example $(O_K)^{x} \subset K^x = \text{Ext}^1_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}(1))$) are reduced by the standard arguments to the weak Mordell-Weil for them.

5.4. Let $K$ be a number field and let $C$ be a proper smooth curve over $K$. Let $M$ be a mixed $\mathbb{Z}$-motive over the function field $K(C)$ with polarized graded quotients for the weight filtration. Fix $w \in \mathbb{Z}$ and $d \geq 0$. Assume that either the following (i) or (ii) is satisfied. (i) $d \leq 3$. (ii) For any $r \in \mathbb{Z}$ such that $0 \leq r \leq -d$, the local monodromies of $\text{gr}^W_w(M)$ are trivial.

Concerning the relation between geometric heights and arithmetic heights, the following formula was proved in many cases in the joint work of Spencer Bloch and the author:

$$h_{w,d}(M(t))/[L : K] = h_{w,d}(M)h(t) + O(1).$$

Here $L$ is a finite extension of $K$, $t$ is an $L$-rational point of $C$ at which $M$ does not degenerate, $h_{w,d}(M(t))$ is the height of the motive $M(t)$ over $L$ defined in section 3, $h_{w,d}(M)$ is the geometric height of $M$ defined in section 2, and $h(t)$ is the height of the point $t$ defined as $h_{L}(t)/\text{deg}(L)$, where $L$ is an ample line bundle $L$ on $C$ and $h_{L}$ is a height function associated to $L$. The $O(1)$ is a function in $t$ which is bounded below and above independently of $L$ and $t$.

For example, let $E$ be an elliptic curve over $K(C)$, let $a \in E(K(C))$, and let $M$ be the mixed motive over $K(C)$ corresponding to $a$ such that $W_0M = M$, $\text{gr}^W_0 = \mathbb{Z}$, $\text{gr}^W_{-1}$ is the $H_1$ of $E$, and $W_{-2}M = 0$. Then $h_{0,-1}(M(t))$ coincides with the Néron-Tate height of $a(t)$ and $h_{0,-1}(M)$ coincides with the geometric height of $a$ defined by using the intersection theory. The above formula for $h_{0,-1}$ in this case was proved by Tate in [13]. A generalization to abelian varieties was obtained in Green [5].

5.5. Assume furthermore $d \geq 2$. There is a related local version of 5.4 which Spencer Bloch and the author proved in many cases:

$$h_{w,d,v}(M(t)) = h_{w,d,x}(M)h_{x,v}(t) + O(1).$$
Here $x \in C(K_v)$ is fixed, $t \in C(K_v)$, $t \neq x$, $t$ converges to $x$ in $C(K_v)$, $h_{w,d,v}(M(t))$ is the local height of $M(t)$ at $v$, $h_{w,d,x}(M)$ is the local height of $M$ at $x$, and $h_{x,v}(t)$ is the local height function defined as follows. By taking a local coordinate function $q$ on $C$ at $x$, it is defined as

$$h_{x,v}(t) = - \log(|q(t)|_v).$$

5.6. If neither the condition (i) nor (ii) in 5.4 is satisfied, it seems that formulas like 5.4 and 5.5 become more complicated.

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