COMPARING SEVERAL CALCULI FOR FIRST-ORDER INFINITE-VALUED ŁUKASIEWICZ LOGIC

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Abstract. From the viewpoint of provability, we compare some Gentzen-type hypersequent calculi for first-order infinite-valued Łukasiewicz logic and for first-order rational Pavelka logic with each other and with Hájek’s Hilbert-type calculi for these logics. The key aspect of our comparison is a density elimination proof for one of the hypersequent calculi considered.

Keywords: many-valued logic; mathematical fuzzy logic; first-order infinite-valued Łukasiewicz logic; first-order rational Pavelka logic; proof theory; hypersequents; density elimination; conservative extension.

1. Introduction

Mathematical fuzzy logics provide formal foundations for approximate reasoning. Among the most important such logics are first-order infinite-valued Łukasiewicz logic $L\forall$ and its expansion by rational truth constants, first-order rational Pavelka logic $RPL\forall$ (see [16, 11, 12]).

For the logic $L\forall$, as well as for the logic $RPL\forall$, besides equivalent Hilbert-type calculi (see, e.g., [16]), only the Gentzen-type calculi mentioned below are known.

For $L\forall$, the article [2] presents an analytic hypersequent calculus $GL\forall$ with structural inference rules, and establishes that $GL\forall$ extended with the cut rule and a Hilbert-type calculus for $L\forall$ from [16] prove the same $L\forall$-sentences.

With the aim of developing proof search methods for $L\forall$ and $RPL\forall$, we introduced the following calculi. First, excluding all the structural inference rules from $GL\forall$, in [14] we obtained a cumulative$^1$ hypersequent calculus $G^1L\forall$ for $RPL\forall$, and showed that any $GL\forall$-provable

$^1$ We say that a hypersequent calculus is cumulative if so is its every rule; and a hypersequent rule is cumulative if, for its every application, each premise contains the conclusion (cf. [23, Section 3.5.11]).
sentence is $G^1\forall$-provable. (Also, in [14] we introduced a variant $G^2\forall$ of $G^1\forall$, which is suitable for bottom-up proof search for prenex $RPL\forall$-sentences.) Next, in [15] we presented a hypersequent calculus $G^3\forall$ for $RPL\forall$ without structural inference rules; this calculus is repetition-free, in the sense that designations of multisets of formulas are not repeated in any premise of its rules. As shown in [15], $G^3\forall$ is well-suited to bottom-up proof search for arbitrary $RPL\forall$-sentences, and any $G^1\forall$-provable sentence (and so any $GL\forall$-provable sentence) is $G^3\forall$-provable.

In the present article, from the viewpoint of provability, we compare $G^3\forall$ with $GL\forall$ in more detail, and compare $G^3\forall$ with Hilbert-type calculi for $RPL\forall$ and $L\forall$ from [16]. The key part of our comparison is a proof of the admissibility of some variants of the density rule for an auxiliary hypersequent calculus; the features and value of the proof are discussed in the concluding section, in the context of works related to density elimination.

The article is organized as follows. In Section 2 we describe the syntax and semantics of the logics $L\forall$ and $RPL\forall$; then we formulate the calculi $GL\forall$ and $G^3\forall$, as well as the calculus $G^1\forall$ and a new calculus $G^0\forall$, which help us to compare $GL\forall$ and $G^3\forall$. In Section 3 we show that $G^0\forall$ is a conservative extension of $GL\forall$ and that any $G^0\forall$-provable sentence is $G^1\forall$-provable (and hence $G^3\forall$-provable). In Section 4 we establish the admissibility in $G^0\forall$ of two variants of the density rule (they underlie some rules of $G^3\forall$), and using this, show that $G^3\forall$ and $G^0\forall$ are equivalent; hence we conclude that $G^3\forall$ is a conservative extension of $GL\forall$. In Section 5 we formulate Hilbert-type calculi $HRP\forall$ and $HL\forall$ for $RPL\forall$ and $L\forall$, respectively; next we establish that any $HRP\forall$-provable sentence is provable in $G^3\forall$ extended with the cut rule (on $RPL\forall$-formulas), and that any $L\forall$-sentence is provable in $HL\forall$ iff it is provable in $G^3\forall$ extended with the cut rule on $L\forall$-formulas. In Section 6 we summarize our results, discuss our proof of density admissibility, and pose some problems for further research.

2. Preliminaries

Let us define $L\forall$- and $RPL\forall$-formulas of a given signature (it may contain predicate and function symbols of any nonnegative arities). The notion of a term is standard. Atomic $L\forall$-formulas are the truth con-
Comparing calculi for L∀

stant 0 and predicate symbols with terms as their arguments. Atomic RPL∀-formulas are atomic L∀-formulas and truth constants r̄ for all positive rational numbers r ⩽ 1. L∀- and RPL∀-formulas are built up as usual from atomic L∀- and RPL∀-formulas, respectively, using the following logical symbols: the binary connective → and the quantifiers ∀, ∃.

An interpretation ⟨D, μ⟩ of a given signature is defined as in classical logic, except that the map μ takes each n-ary predicate symbol P to a predicate μ(P) : D^n → [0, 1], where [0, 1] is an interval of real numbers. Let M = ⟨D, μ⟩ be an interpretation. Then an M-valuation is a map of the set of all individual variables to D. For an M-valuation ν, an individual variable x, and an element d ∈ D, by ν[x ← d] we denote the M-valuation that may differ from ν only on x and obeys the condition ν[x ← d](x) = d.

The value |t|M,ν of a term t under an interpretation M and an M-valuation ν is defined in the standard manner. The truth value |C|M,ν of an RPL∀-formula C under an interpretation M = ⟨D, μ⟩ and an M-valuation ν is defined as follows:

1. |r̄|M,ν = r;
2. |P(t₁, ..., tₙ)|M,ν = μ(P)(|t₁|M,ν, ..., |tₙ|M,ν) for an n-ary predicate symbol P and terms t₁, ..., tₙ;
3. |A → B|M,ν = min(1 − |A|M,ν + |B|M,ν, 1);
4. |∀x A|M,ν = inf_{d ∈ D} |A|M,ν[x ← d];
5. |∃x A|M,ν = sup_{d ∈ D} |A|M,ν[x ← d].

An RPL∀-formula C is called valid (also written ⊨ C) if |C|M,ν = 1 for every interpretation M and every M-valuation ν.

The result of substituting a term t for all free occurrences of an individual variable x in an RPL∀-formula A is denoted by [A][x ← t]. The provability (resp. unprovability) of an object α in a calculus C is written as ⊢ₓ C (resp. ⊬ₓ C). By a proof in a calculus, we mean a proof tree. In depicting a proof tree D, if we place a designation over a node N of D and do not separate the designation from N by a horizontal line, then we regard the designation as one for the proof tree whose root is N and that is a subtree of D.

The letters k, l, m, n stand for nonnegative integers. An expression k..n denotes the set {k, k + 1, ..., n} if k ⩽ n, and the empty set otherwise.
In what follows, we work with a fixed signature that includes a countably infinite set of nullary function symbols called parameters.

Let us formulate the auxiliary hypersequent calculus \( G^{0\forall} \) and define accompanying notions and notation common to several calculi considered.

We introduce two countably infinite disjoint sets of new words and call such words semipropositional variables of type 0 and of type 1, respectively. An RPL\( ^\forall \)-formula as well as a semipropositional variable (of any type) is called a formula.

An RPL\( ^\forall _0 \)-sequent (or simply a sequent) is written as \( \Gamma \Rightarrow \Delta \) and is an ordered pair of finite multisets \( \Gamma \) and \( \Delta \) consisting of formulas. An RPL\( ^\forall _0 \)-hypersequent (hypersequent for short) is a finite multiset of sequents and is written as \( \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \) or \( [\Gamma_i \Rightarrow \Delta_i]_{i \in 1..n} \).

A sequent not containing logical symbols is called atomic. Suppose that \( \mathcal{H} \) is a hypersequent; then by \( \mathcal{H}_{at} \) we denote the hypersequent obtained from \( \mathcal{H} \) by removing all non-atomic sequents.

We define an hs-interpretation as an interpretation \( \langle D, \mu \rangle \) in which the map \( \mu \) additionally takes each semipropositional variable of type 0 to a real number in \( [0, +\infty) \) and each semipropositional variable of type 1 to a real number in \( (-\infty, 1] \). For a semipropositional variable \( p \), an hs-interpretation \( M = \langle D, \mu \rangle \), and an \( M \)-valuation \( \nu \), the value \( \mu(p) \) will also be written as \( |p|_M \) and as \( |p|_{M, \nu} \).

For a finite multiset \( \Gamma \) of formulas, a sequent \( \Gamma \Rightarrow \Delta \), an hs-interpretation \( M \), and an \( M \)-valuation \( \nu \), we put

\[
\| \Gamma \|_{M, \nu} = \sum_{A \in \Gamma} (|A|_{M, \nu} - 1),
\]

where the summation is performed taking multiplicities of multiset elements into account, and \( \sum_{A \in \emptyset} (\ldots) = 0 \). A sequent \( \Gamma \Rightarrow \Delta \) is called true under an hs-interpretation \( M \) and an \( M \)-valuation \( \nu \) if

\[
\| \Gamma \|_{M, \nu} \leq \| \Delta \|_{M, \nu}.
\]

Following [2, Definition 1], we say that a hypersequent \( \mathcal{H} \) is valid (and write \( \models \mathcal{H} \)) if, for every hs-interpretation \( M \) and every \( M \)-valuation \( \nu \), some sequent in \( \mathcal{H} \) is true under \( M \) and \( \nu \). Note that, for an RPL\( ^\forall \)-formula \( A \), \( \models A \iff \models (\Rightarrow A) \). To denote that a hypersequent \( \mathcal{G} \) is not valid, we write \( \not\models \mathcal{G} \).
Unless otherwise specified, below the letters $A$, $B$, and $C$ denote any RPL∀-formulas, $\Gamma$, $\Delta$, $\Pi$, and $\Sigma$ any finite multisets of formulas, $S$ any sequent, $\mathcal{G}$ and $\mathcal{H}$ any hypersequents, $x$ any individual variable, $t$ any closed term, $a$ any parameter, and $r$ any rational number such that $0 \leq r \leq 1$; all these letters may have subscripts and superscripts. Also $p_i$ ($i = 0, 1$) denotes any semipropositional variable of type $i$.

The language of the calculus $G^0L∀$ consists of all possible hypersequents. A hypersequent $\mathcal{H}$ is called an axiom of $G^0L∀$ if $\models \mathcal{H}$ at $\mathcal{G}$. (Axioms of $G^0L∀$ can be recognized by a polynomial algorithm in much the same way as described in [14, Section 4.2].)

The inference rules of the calculus $G^0L∀$ are:

$$
\begin{align*}
\text{G} & \mid \Gamma, A \rightarrow B \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid \Gamma, B \Rightarrow A, \Delta \\
\text{G} & \mid \Gamma \Rightarrow A \rightarrow B, \Delta \\
\text{G} & \mid \Gamma \Rightarrow A \rightarrow B, \Delta \mid \Gamma \Rightarrow \Delta;
\end{align*}
$$

where $a$ does not occur in the conclusion of $(\Rightarrow ∀)^0$ or $(∃ ⇒)^0$.

For convenience in comparing calculi, we also introduce the calculus $G^1L∀$ that is obtained from $G^0L∀$ by replacing the inference rule $(\rightarrow ⇒)^0$ with

$$
\begin{align*}
\text{G} & \mid \Gamma, A \rightarrow B \Rightarrow \Delta \mid \Gamma, p_1 \Rightarrow \Delta \mid B \Rightarrow p_1, A \\
\text{G} & \mid \Gamma, A \rightarrow B \Rightarrow \Delta
\end{align*}
$$

where $p_1$ does not occur in the conclusion.

The calculus $G^1L∀$ [14] is obtained from $G^1L∀$ by restricting the language of $G^1L∀$ to hypersequents not containing semipropositional variables of type 0; such hypersequents are called RPL∀1-hypersequents.

The rule of $G^iL∀$ ($i = 1, \hat{1}$) that corresponds to a rule of $G^0L∀$ is denoted just as the latter but with the superscript $i$ instead of 0.
Remark 1. It is clear that, for an RPL\(^1\)-hypersequent \(\mathcal{H}\), a \(G^1\text{L}\mathcal{V}\)-proof of \(\mathcal{H}\) is a \(G^1\text{L}\mathcal{V}\)-proof of \(\mathcal{H}\), and conversely.

The calculus \(G^3\text{L}\mathcal{V}\) [15] is obtained from \(G^0\text{L}\mathcal{V}\) by replacing all the inference rules with the following ones:

\[
\frac{\mathcal{G} | \Gamma, \forall \alpha \rightarrow B \Rightarrow \Delta}{\mathcal{G} | \Gamma, \forall \alpha \rightarrow B \Rightarrow \Delta} \quad (\forall \Rightarrow)^3,
\]
\[
\frac{\mathcal{G} | \Gamma \Rightarrow \Delta; \mathcal{G} | \Gamma, \forall \alpha \rightarrow B \Rightarrow \Delta}{\mathcal{G} | \Gamma \Rightarrow \Delta} \quad (\Rightarrow \rightarrow)^3,
\]
\[
\frac{\mathcal{G} | \exists \alpha \rightarrow B \Rightarrow \Delta}{\mathcal{G} | \exists \alpha \rightarrow B \Rightarrow \Delta} \quad (\exists \Rightarrow)^3,
\]
where \(\varphi\) does not occur in the conclusion of \((\Rightarrow \forall)^3\) or \((\forall \Rightarrow)^3\), \(\varphi_0\) does not occur in the conclusion of \((\Rightarrow \exists)^3\), and \(a\) does not occur in the conclusion of \((\Rightarrow \forall)^3\) or \((\exists \Rightarrow)^3\).

For an application of an inference rule of \(G^i\text{L}\mathcal{V}\) \((i = 0, 1, \hat{1}, 3)\), the principal formula occurrence and the principal sequent occurrence are defined in essentially the same manner as in [18, § 49] and [23, Section 3.5.1]. The notion of an ancestor of a sequent occurrence in a \(G^i\text{L}\mathcal{V}\)-proof \((i = 0, 1, \hat{1}, 3)\) is defined much as the notion of an ancestor of a formula occurrence is defined in [18, § 49].

Now we formulate the calculus \(G^i\text{L}\mathcal{V}\) [2], using parameters instead of free individual variables, which are syntactically distinct from bound individual variables in [2]. The language of \(G^i\text{L}\mathcal{V}\) consists of all possible \(\text{L}\mathcal{V}\)-hypersequents, i.e., hypersequents that do not contain semipositional variables and, of truth constants, may contain only \(\tilde{0}\).

The axiom schemes of \(G^i\text{L}\mathcal{V}\) are: \(\Rightarrow A\) \((\text{id})\), \(\Rightarrow (\Lambda)\), \(\tilde{0} \Rightarrow A\) \((\tilde{0} \Rightarrow)\), where \(A\) is an \(\text{L}\mathcal{V}\)-formula.

The inference rules of \(G^i\text{L}\mathcal{V}\) are:

\[
\frac{\mathcal{G} | S}{\mathcal{G} | S} \quad (\text{ew}),
\]
\[
\frac{\mathcal{G} | S \Rightarrow \Delta}{\mathcal{G} | S \Rightarrow \Delta} \quad (\text{ec}),
\]
\[
\frac{\mathcal{G} | \Gamma \Rightarrow \Delta}{\mathcal{G} | \Gamma, C \Rightarrow \Delta} \quad (\text{wl}),
\]
\[
\frac{\mathcal{G} | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} | \Gamma_1 \Rightarrow \Delta_1; \mathcal{G} | \Gamma_2 \Rightarrow \Delta_2} \quad \text{(split)},
\]
\[
\frac{\mathcal{G} | \Gamma_1 \Rightarrow \Delta_1; \mathcal{G} | \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \quad \text{(mix)},
\]
\[
\frac{\mathcal{G} | \Gamma \Rightarrow \Delta; \mathcal{G} | \Gamma, \forall \alpha \rightarrow B \Rightarrow \Delta}{\mathcal{G} | \Gamma \Rightarrow \Delta; \mathcal{G} | \Gamma, \forall \alpha \rightarrow B, \Delta} \quad \text{($\Rightarrow \forall$)},
\]
\[
\frac{\mathcal{G} | \Gamma \Rightarrow \Delta; \mathcal{G} | \Gamma, A \Rightarrow B, \Delta}{\mathcal{G} | \Gamma \Rightarrow \Delta; \mathcal{G} | \Gamma, A \Rightarrow B, \Delta} \quad \text{($\Rightarrow \rightarrow$)},
\]
Comparing calculi for \( \mathcal{L}_\forall \)

\[
\begin{align*}
\frac{\mathcal{G} \mid \Gamma, [A]^x_i \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \forall x A \Rightarrow \Delta} \quad \forall \Rightarrow, \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow [A]^x_i, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \forall x A, \Delta} \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \exists x A, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \forall x A \Rightarrow \Delta} \quad \exists \Rightarrow,
\end{align*}
\]

where all the premises and conclusions are \( \mathcal{L}_\forall \)-hypersequents, and \( a \) does not occur in the conclusion of \( (\Rightarrow \forall) \) or \( (\exists \Rightarrow) \). The first five of these rules are called \textit{structural}; the others, \textit{logical}.

For each calculus formulated above, its every one-premise rule in whose premise \( a, t, \) or \( p_i \) \((i = 0, 1)\) figurates, and for any application of the rule, the \( a, t, \) or \( p_i \) is called, respectively, the \textit{proper} parameter, \textit{proper} term, or \textit{proper} semipropositional variable of the application.

A \textit{proof of (for) an RPL_\forall-formula} \( A \) in any of the hypersequent calculi given above is a proof of the hypersequent \( \Rightarrow A \) in the respective calculus.

3. Initial relationships between the hypersequent calculi considered

**Theorem 1.** \( G^0_{\mathcal{L}\forall} \) is a conservative extension of \( GL_\forall \); i.e., for any \( \mathcal{L}_\forall \)-hypersequent \( \mathcal{H} \), \( \vdash_{G^0_{\mathcal{L}\forall}} \mathcal{H} \) iff \( \vdash_{GL_\forall} \mathcal{H} \).

**Proof.** Let \( \mathcal{H} \) be an \( \mathcal{L}_\forall \)-hypersequent. If \( \vdash_{G^0_{\mathcal{L}\forall}} \mathcal{H} \), then \( \vdash_{GL_\forall} \mathcal{H} \) by Lemma 2 below. Let us prove the converse.

We get the calculus \( G_a_{\mathcal{L}\forall} \) from \( GL_\forall \) by taking \( A \) in the axiom schemes (id) and \( (\bar{0} \Rightarrow) \) to be an atomic \( \mathcal{L}_\forall \)-formula. Lemma 6 in [13] guarantees that \( \vdash_{GL_\forall} \mathcal{H} \) iff \( \vdash_{G_a_{\mathcal{L}\forall}} \mathcal{H} \).

So it suffices to show that \( \vdash_{G_a_{\mathcal{L}\forall}} \mathcal{H} \) implies \( \vdash_{G^0_{\mathcal{L}\forall}} \mathcal{H} \). Any axiom of \( G_a_{\mathcal{L}\forall} \) is obviously an axiom of \( G^0_{\mathcal{L}\forall} \). All the structural rules of \( G_a_{\mathcal{L}\forall} \) are admissible for \( G^0_{\mathcal{L}\forall} \) by Lemma 3 below. Since the rule (ew) is admissible for \( G^0_{\mathcal{L}\forall} \), it follows easily that all the logical rules of \( G_a_{\mathcal{L}\forall} \) are admissible for \( G^0_{\mathcal{L}\forall} \). \( \square \)

**Lemma 2.** For any \( \mathcal{L}_\forall \)-hypersequent \( \mathcal{H} \), if \( \vdash_{G^0_{\mathcal{L}\forall}} \mathcal{H} \), then \( \vdash_{GL_\forall} \mathcal{H} \).

**Proof.** Let \( \mathcal{H} \) be an \( \mathcal{L}_\forall \)-hypersequent.

If \( \mathcal{H} \) is an axiom of \( G^0_{\mathcal{L}\forall} \), then \( \vdash \mathcal{H}_{at} \), and by the completeness of \( GL_\forall \) for quantifier-free \( \mathcal{L}_\forall \)-hypersequents [21, Theorem 6.24], we get \( \vdash_{GL_\forall} \mathcal{H}_{at} \), whence \( \vdash_{GL_\forall} \mathcal{H} \) by the rule (ew).
To conclude the proof, it is sufficient to show that all the rules of $G^0\forall$ are derivable in $GL\forall$ if their premises and conclusions are restricted to $\forall\forall$-hypersequents. For the rule $(\to\Rightarrow)^0$, we have:

$$
\frac{\mathcal{G} | \Gamma, A \to B \Rightarrow \Delta | \Gamma \Rightarrow \Delta | \Gamma, B \Rightarrow A, \Delta}{\mathcal{G} | \Gamma, A \to B \Rightarrow \Delta | \Gamma \Rightarrow \Delta | \Gamma, A \to B \Rightarrow \Delta} \ (\to\Rightarrow)
$$

$$
\frac{\mathcal{G} | \Gamma, A \to B \Rightarrow \Delta | \Gamma \Rightarrow \Delta | \Gamma, A \to B \Rightarrow \Delta}{\mathcal{G} | \Gamma, A \to B \Rightarrow \Delta} \ (wl) \times 2.
$$

For the rule $(\forall\Rightarrow)^0$, we have:

$$
\frac{\mathcal{G} | \Gamma, \forall x A \Rightarrow \Delta | \Gamma, [A]^x_t \Rightarrow \Delta}{\mathcal{G} | \Gamma, \forall x A \Rightarrow \Delta} \ (\forall\Rightarrow)
$$

$$
\frac{\mathcal{G} | \Gamma, \forall x A \Rightarrow \Delta}{\mathcal{G} | \Gamma, \forall x A \Rightarrow \Delta} \ (ec).
$$

The other rules of $G^0\forall$ are treated similarly to $(\forall\Rightarrow)^0$.

**Lemma 3.** The following rules are admissible for $G^0\forall$:

$$
\frac{\mathcal{G} | S}{\mathcal{G} | S} \ (ew)^0, \quad \frac{\mathcal{G} | S}{\mathcal{G} | S} \ (ec)^0, \quad \frac{\mathcal{G} | \Gamma \Rightarrow \Delta}{\mathcal{G} | \Gamma, C \Rightarrow \Delta} \ (wl)^0,
$$

$$
\frac{\mathcal{G} | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} | \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2} \ (split)^0, \quad \frac{\mathcal{G} | \Gamma_1 \Rightarrow \Delta_1; \mathcal{G} | \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \ (mix)^0.
$$

Moreover, the rules $(ew)^0$, $(ec)^0$, and $(split)^0$ are height-preserving admissible, or briefly hp-admissible, for $G^0\forall$.

**Proof.**

1. It is clear that $(ew)^0$ is hp-admissible (for $G^0\forall$).

2. To establish the hp-admissibility of $(ec)^0$, we note that all the rules of $G^0\forall$ are cumulative and proceed just as in the proof of Lemma 5 in [14].

3. To show that $(wl)^0$ is admissible, we use induction on the number of logical symbol occurrences in $C$. Let $\mathcal{H}_1 = (\mathcal{G} | \Gamma \Rightarrow \Delta)$ and $\mathcal{H}_2 = (\mathcal{G} | \Gamma, C \Rightarrow \Delta)$. We can assume that there is a $(G^0\forall\forall)$-proof $D_1$ for $\mathcal{H}_1$ such that no proper parameter from $D_1$ occurs in $C$.

3.1. Suppose that $C$ is atomic or is of the form $(A \to B)$. From $D_1$ we construct a proof search tree $D_2$ for $\mathcal{H}_2$ as follows. For each occurrence $S$ of a sequent of the form $\Pi \Rightarrow \Sigma$, if $S$ is an ancestor of the distinguished occurrence of the sequent $\Gamma \Rightarrow \Delta$ in the root of $D_1$, then we replace $S$ by an occurrence $S'$ of the sequent $\Pi, C \Rightarrow \Sigma$. We also mark $S'$ if $S$ is an atomic sequent occurrence in a leaf of $D_1$. 
Comparing calculi for $\mathsf{L}^\forall$

If $C$ is atomic, then $D_2^0$ is a proof for $\mathcal{H}_2$.

Suppose $C$ is of the form $(A \rightarrow B)$, and $S_0, \ldots, S_{l-1}$ are all distinct marked sequent occurrences in $D_2^0$.

We expand $D_2^0$ by performing the following for each $i = 0, \ldots, l - 1$: on the only branch $B_i$ of $D_i^j$ containing $S_i$, apply the rule $(\rightarrow \Rightarrow)^0$ backward to the ancestor of $S_i$ in the leaf on $B_i$, and denote by $D_i^{j+1}$ the tree obtained as a result of this backward application.

Note that if $S_i$ is an occurrence of a sequent of the form $\Pi_i, C \Rightarrow \Sigma_i$, then the atomic sequent $\Pi_i \Rightarrow \Sigma_i$ is on the continuation of the branch $B_i$ in $D_{i+1}^j$. Therefore, it is easy to see that $D_l^j$ is a proof for $\mathcal{H}_2$.

3.2. Suppose that $C$ is of the form $QxA$, where $Q$ is a quantifier. By the induction hypothesis, there is a proof for $\mathcal{H} = (\mathcal{H}_2 | \Gamma, [A]^a_x \Rightarrow \Delta)$, where $a$ is a parameter not occurring in $\mathcal{H}_2$. By applying the rule $(Q \Rightarrow)^0$ to the distinguished occurrence of $[A]^a_x$ in $\mathcal{H}$, we get a proof for $\mathcal{H}_2$.

4. The proof of the hp-admissibility of $(\text{split})^0$ is very similar to the proof of Lemma 7 in [14].

5. The proof of the admissibility of $(\text{mix})^0$ can be easily obtained from the proof of Lemma 8 in [14] by identifying the notion of a completable ancestor of a sequent occurrence with the notion of an ancestor of a sequent occurrence (the former notion is used in [14]).

\begin{theorem}
If $\vdash_{G^0\mathsf{L}^\forall} \mathcal{H}$, then $\vdash_{G^1\mathsf{L}^\forall} \mathcal{H}$.
\end{theorem}

\begin{proof}
All axioms of $G^0\mathsf{L}^\forall$ are axioms of $G^1\mathsf{L}^\forall$. All the rules of $G^0\mathsf{L}^\forall$, except for the rule $(\rightarrow \Rightarrow)^0$, are rules of $G^1\mathsf{L}^\forall$. Hence, it suffices to establish that $(\rightarrow \Rightarrow)^0$ is admissible for $G^1\mathsf{L}^\forall$.

For this, we use the rules

\[
\frac{\mathcal{G} | \Gamma \Rightarrow \Delta}{\mathcal{G} | \Gamma, p_1 \Rightarrow p_1, \Delta} \quad \text{(sp_1 \Rightarrow sp_1)^0} \quad \text{and} \quad \frac{\mathcal{G} | \Gamma \Rightarrow \Delta}{\mathcal{G} | \Gamma, p_1 \Rightarrow \Delta} \quad \text{(wl)^0_{sp_1}},
\]

whose hp-admissibility for $G^1\mathsf{L}^\forall$ is obvious. We also use the rules $(\text{ec})^0$ and $(\text{split})^0$ from Lemma 3, noting that the proofs of their hp-admissibility for $G^1\mathsf{L}^\forall$ are entirely analogous to the proofs of Lemmas 5 and 7 in [14].

\footnote{See also Section A (of the appendix) on p. 30.}
We obtain the conclusion of the rule \((\to \Rightarrow)^0\) from its premise by rules, which are admissible for \(G^1L\forall\), as shown in Figure 1, where \(p_1\) does not occur in \(\texttt{G} | \Gamma, A \to B \Rightarrow \Delta\). Thus \((\to \Rightarrow)^0\) is admissible for \(G^1L\forall\).

\[\text{Figure 1: Obtaining the conclusion of the rule } (\to \Rightarrow)^0 \text{ from its premise.}\]

**Theorem 5.** If \(\vdash_{G^1L\forall} \mathcal{H}\), then \(\vdash_{G^3L\forall} \mathcal{H}\).

**Proof** is obtained from the proofs of Lemma 6 and Theorem 2 in [15] by substituting the superscript \(\hat{1}\) for the superscript 1 (in \(G^1L\forall\) and the designations of the rules of \(G^1L\forall\)).

**4. The admissibility of variants of the density rule for \(G^0L\forall\) and further relationships between the hypersequent calculi considered**

The primary goal of this section is to show that if a hypersequent is \(G^3L\forall\)-provable, then it is \(G^0L\forall\)-provable. For this, we establish that all the rules of \(G^3L\forall\) are admissible for \(G^0L\forall\).

As shown in the proof of the next lemma, the rules \((\to \Rightarrow)^3\), \((\forall \Rightarrow)^3\), and \((\Rightarrow \exists)^3\) of \(G^3L\forall\) are based on the rules

\[
\frac{\texttt{G} | \Gamma, p_1 \Rightarrow p_1 \Rightarrow C \Rightarrow p_1}{\texttt{G} | \Gamma, C \Rightarrow \Delta} \quad \text{(den}_1) \quad \text{and} \quad \frac{\texttt{G} | \Gamma \Rightarrow p_0 \Rightarrow C \Rightarrow \Gamma \Rightarrow p_0 \Rightarrow \Delta}{\texttt{G} | \Gamma \Rightarrow C, \Delta} \quad \text{(den}_0),
\]

where \(p_i\) does not occur in the conclusion of \((\text{den}_i), i = 0, 1\). The last two rules can be characterized as nonstandard variants of the density rule, cf. [21, Section 4.5].
Comparing calculi for $\mathbf{L}_\forall$

**Remark 2.** The (standard) *density rule* in the hypersequent formulation is:

\[
\frac{\mathcal{G} | \Gamma, p \Rightarrow \Delta | \Pi \Rightarrow p, \Sigma}{\mathcal{G} | \Gamma, \Pi \Rightarrow \Delta, \Sigma} \quad \text{(den)},
\]

where $p$ is a propositional variable not occurring in the conclusion; see [21, Section 4.5]. Given our definition of the validity of a hypersequent, it is not hard to check that (den) is unsound, but becomes sound if we expand the notion of a hypersequent by special variables interpreted by any real numbers, and require $p$ to be such a variable not occurring in the conclusion.\(^3\) Let us refer to this modified rule (den) as the *nonstandard density rule*.

**Lemma 6.** If the rules (den\(_1\)) and (den\(_0\)) are admissible for $G^0\mathbf{L}_\forall$, then $\vdash_{G^3\mathbf{L}_\forall} \mathcal{H}$ implies $\vdash_{G^0\mathbf{L}_\forall} \mathcal{H}$.

**Proof.** Any axiom of $G^3\mathbf{L}_\forall$ is an axiom of $G^0\mathbf{L}_\forall$. Assuming that (den\(_1\)) and (den\(_0\)) are admissible for $G^0\mathbf{L}_\forall$, we establish that all the rules of $G^3\mathbf{L}_\forall$ are admissible for $G^0\mathbf{L}_\forall$. The conclusion of the rule ($\to \to$)\(^3\) is obtained from its premise as follows:

\[
\frac{\mathcal{G} | \Gamma, p_1 \Rightarrow \Delta | B \Rightarrow p_1, A}{\mathcal{G} | \Gamma, p_1 \Rightarrow \Delta | B \Rightarrow p_1, A \to B \Rightarrow p_1} \quad \text{(ew)}^0 \times 2
\]

\[
\frac{\mathcal{G} | \Gamma, p_1 \Rightarrow \Delta | A \to B \Rightarrow p_1}{\mathcal{G} | \Gamma, A \to B \Rightarrow \Delta} \quad \text{(den\(_1\)),}
\]

(ew)\(^0\) being admissible for $G^0\mathbf{L}_\forall$ by Lemma 3. The conclusion of the rule ($\Rightarrow \exists$)\(^3\) is obtained from its premise thus:

\[
\frac{\mathcal{G} | \Gamma \Rightarrow p_0, \Delta | p_0 \Rightarrow \exists x A | p_0 \Rightarrow [A]x}{\mathcal{G} | \Gamma \Rightarrow p_0, \Delta | p_0 \Rightarrow \exists x A} \quad \text{(\(\Rightarrow \exists\))}^0
\]

\[
\frac{\mathcal{G} | \Gamma \Rightarrow \exists x A, \Delta}{\mathcal{G} | \Gamma \Rightarrow \exists x A} \quad \text{(den\(_0\)).}
\]

The rule ($\forall \Rightarrow$)\(^3\) is treated similarly to ($\Rightarrow \exists$)\(^3\), but with an application of (den\(_1\)). Finally, the admissibility for $G^0\mathbf{L}_\forall$ of the rules ($\Rightarrow \to$)\(^3\), ($\Rightarrow \forall$)\(^3\), and ($\exists \Rightarrow$)\(^3\) follows easily from the admissibility of (ew)\(^0\). \(\square\)

Lemmas 7 and 13 below ensure that the rules (den\(_1\)) and (den\(_0\)) are admissible for $G^0\mathbf{L}_\forall$.

\(^3\) See also Section B (of the appendix) on p. 33.
Lemma 7 (admissibility of a generalization of (den) for $G^0L\forall$). Suppose that $m \geq 1$, $n \geq 1$,

$$
\mathcal{H} = \left( \mathcal{S} \mid \left[ \Gamma_i, p_1 \Rightarrow \Delta_i \right]_{i \in 1..m} \left| \left[ \Pi_j \Rightarrow p_1, \Sigma_j \right]_{j \in 1..n} \right) ,
$$

$$
\mathcal{H}' = \left( \mathcal{S} \mid \left[ \Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j \right]_{i \in 1..m} \right) ,
$$

$p_1$ does not occur in $\mathcal{H}'$, the sequent $C \Rightarrow p_1$ occurs in $\mathcal{H}$, and $\vdash_{G^0L\forall} \mathcal{H}$.

Then $\vdash_{G^0L\forall} \mathcal{H}'$.

Proof. By Lemma 8 below, there exists a $(G^0L\forall)$-proof $D$ of $\mathcal{H}$ in which each leaf hypersequent $\mathcal{L}$ contains a sequent of the form $C_\mathcal{L} \Rightarrow p_1$ or $\Rightarrow p_1$, where $C_\mathcal{L}$ is an atomic RPL$\forall$-formula. We transform $D$ into a proof of $\mathcal{H}'$ using induction on the height of $D$.

1. Suppose that $\mathcal{H}$ is an axiom; i.e., $\vdash \mathcal{H}_{at}$. Without loss of generality we assume that

$$
\mathcal{H}_{at} = \left( \mathcal{S}_{at} \mid \left[ \Gamma_i, p_1 \Rightarrow \Delta_i \right]_{i \in 1..k} \left| \left[ \Pi_j \Rightarrow p_1, \Sigma_j \right]_{j \in 1..l} \right) ,
$$

where $0 \leq k \leq m$, $0 < l \leq n$, and the sequent $\Pi_1 \Rightarrow p_1, \Sigma_1$ has the form $C_1 \Rightarrow p_1$ or $\Rightarrow p_1$. We put $\mathcal{H}'_{at} = (\mathcal{H}')_{at}$.

1.1. Consider the case where $k \neq 0$. We have

$$
\mathcal{H}'_{at} = \left( \mathcal{S}_{at} \mid \left[ \Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j \right]_{i \in 1..k} \right) .
$$

We want to show that $\vdash \mathcal{H}'_{at}$. Suppose otherwise; i.e., for some hs-interpretation $M$ and some $M$-valuation $\nu$, there is no true sequent in $\mathcal{S}_{at}$; and for all $i \in 1..k$ and $j \in 1..l$,

$$
\| \Delta_i \|_{M,\nu} - \| \Gamma_i \|_{M,\nu} < \| \Pi_j \|_{M,\nu} - \| \Sigma_j \|_{M,\nu} .
$$

By the density of the set $\mathbb{R}$ of all real numbers, there exists $\xi \in \mathbb{R}$ such that, for all $i \in 1..k$ and $j \in 1..l$,

$$
\| \Delta_i \|_{M,\nu} - \| \Gamma_i \|_{M,\nu} < \xi - 1 < \| \Pi_j \|_{M,\nu} - \| \Sigma_j \|_{M,\nu} .
$$

In particular, $\xi < \| \Pi_1 \|_{M,\nu} - \| \Sigma_1 \|_{M,\nu} + 1 = \| \Pi_1 \|_{M,\nu} + 1 \leq 1$.

Define an hs-interpretation $M_1$ to be like $M$, but set $| p_1 |_{M_1} = \xi$. Since $p_1$ does not occur in $\mathcal{S}_{at}$, $\Gamma_i$, $\Delta_i$ ($i \in 1..k$), $\Pi_j$, $\Sigma_j$ ($j \in 1..l$), we
see that no sequent in $H_{at}$ is true under the hs-interpretation $M_1$ and $M_1$-valuation $\nu$. Hence $\not\models H_{at}$, a contradiction.

Therefore $\models H'_{at}$, and so $H'$ is an axiom.

1.2. Now consider the case where $k = 0$. Then $H_{at} = \left( G_{at} \mid [\Pi_j \Rightarrow p_1, \Sigma_j]_{j \in 1..l} \right)$ and $H'_{at} = G_{at}$. Since $p_1$ does not occur in $G_{at}$, $\Pi_j$, $\Sigma_j$ ($j \in 1..l$), and hs-interpretations can take $p_1$ to negative real numbers whose absolute values are arbitrarily large, we conclude that $\models H_{at}$ implies $\models G_{at}$. Thus $\models H'_{at}$ and $H'$ is an axiom.

2. Suppose that the root hypersequent $H$ in $D$ is the conclusion of an application $R$ of a rule $\mathcal{R}$, and $S$ is the principal sequent occurrence in $R$.

2.1. If $S$ is in the distinguished occurrence of $G$ in $H$, then we apply the induction hypothesis to the proof of each premise of $R$, and next we get a proof of $H'$ by $\mathcal{R}$.

2.2. Now suppose that $S$ is not in the distinguished occurrence of $G$ in $H$, and for definiteness assume that $S$ is the distinguished occurrence of $\Gamma_1, p_1 \Rightarrow \Delta_1$ in $H$.

2.2.1. If $R$ is the rule $(\rightarrow \Rightarrow)^0$, then $\Gamma_1 = (\Gamma'_1, A \rightarrow B)$ for some $\Gamma'_1$, and the proof $D$ has the form:

$$
\begin{array}{c}
D_1 \\
H \mid H' \mid \Gamma'_1, p_1 \Rightarrow \Delta_1 \mid \Gamma'_1, B, p_1 \Rightarrow A, \Delta_1 \\
\hline
(\rightarrow \Rightarrow)^0 \\
\end{array}
$$

By the induction hypothesis, we transform $D_1$ into a proof of

$$
H' \mid [\Gamma'_1, \Pi_j \Rightarrow \Delta_1, \Sigma_j]_{j \in 1..n} \mid [\Gamma'_1, B, \Pi_j \Rightarrow A, \Delta_1, \Sigma_j]_{j \in 1..n},
$$

whence we obtain a proof for $H'$ by $n$ applications of $(\rightarrow \Rightarrow)^0$.

2.2.2. The rules $(\forall \Rightarrow)^0$ and $(\Rightarrow \exists)^0$ are treated as $\mathcal{R}$ similarly to the rule $(\rightarrow \Rightarrow)^0$, see item 2.2.1.

2.2.3. If $R$ is $(\Rightarrow \rightarrow)^0$, then $\Delta_1 = (A \rightarrow B, \Delta'_1)$ for some $\Delta'_1$, and the proof $D$ looks like this:

$$
\begin{array}{c}
D_1 \\
\hline
H \mid H' \mid \Gamma_1, p_1 \Rightarrow \Delta'_1: \quad D_2 \\
\hline
H \mid \Gamma_1, A, p_1 \Rightarrow B, \Delta'_1 \quad (\Rightarrow \rightarrow)^0.
\end{array}
$$
By the induction hypothesis applied to the proofs $D_1$ and $D_2$, we construct proofs of

\[ \mathcal{H}' \mid [\Gamma_1, \Pi_j \Rightarrow \Delta'_1, \Sigma_j]_{j \in 1..n} \quad \text{and} \quad \mathcal{H}' \mid [\Gamma_1, A, \Pi_j \Rightarrow B, \Delta'_1, \Sigma_j]_{j \in 1..n}, \]

respectively; whence we get a proof of $\mathcal{H}'$ by Lemma 9 below.

2.2.4. If $R$ is ($\Rightarrow \forall$)$^0$, then $\Delta_1 = (\forall x A, \Delta'_1)$ for some $\Delta'_1$, and the proof $D$ has the form:

\[
\begin{align*}
D_1 & \quad \mathcal{H} \mid [\Gamma_1, p_1 \Rightarrow [A]_a^\forall, \Delta'_1] \quad (\Rightarrow \forall)^0, \\
\end{align*}
\]

where $a$ does not occur in $\mathcal{H}$ (and hence, $a$ does not occur in $\mathcal{H}'$). Using the induction hypothesis, we transform $D_1$ into a proof of

\[ \mathcal{H}' \mid [\Gamma_1, \Pi_j \Rightarrow [A]_a^\forall, \Delta'_1, \Sigma_j]_{j \in 1..n}, \]

whence we obtain a proof of $\mathcal{H}'$ by Lemma 11.

2.2.5. The rule ($\exists \Rightarrow$)$^0$ is treated similarly to the rule ($\Rightarrow \forall$)$^0$ in item 2.2.4, using Lemma 12.

**Lemma 8.** Suppose that $\mathcal{H} = (\mathcal{G} \mid C \Rightarrow p_1)$ is an axiom of $G^0L\forall$. Then a $G^0L\forall$-proof of $\mathcal{H}$ can be constructed in which each leaf hypersequent $\mathcal{L}$ contains a sequent of the form $C_{\mathcal{L}} \Rightarrow p_1$ or $\Rightarrow p_1$, where $C_{\mathcal{L}}$ is an atomic RPL$\forall$-formula.

**Proof.** The RPL$\forall$-formula $C$ has the form

\[ Q_1x_1 \ldots Q_nx_nC' \quad \text{or} \quad Q_1x_1 \ldots Q_nx_n(A \Rightarrow B), \]

where $Q_1, \ldots, Q_n$ are quantifiers and $C'$ is an atomic RPL$\forall$-formula. The desired proof can be obtained from $\mathcal{H}$ by $n$ backward applications of the rules ($Q_1 \Rightarrow$)$^0$, $\ldots$, ($Q_n \Rightarrow$)$^0$ and if $C = Q_1x_1 \ldots Q_nx_n(A \Rightarrow B)$, by one more backward application of the rule ($\Rightarrow \Rightarrow$)$^0$.

**Lemma 9.** Suppose that $n \geq 1$,

\[ \mathcal{H}'_n = \left( \mathcal{G} \mid [\Gamma_i \Rightarrow \Delta_i]_{i \in 1..n} \right), \quad \mathcal{H}''_n = \left( \mathcal{G} \mid [\Gamma_i, A \Rightarrow B, \Delta_i]_{i \in 1..n} \right), \]

\[ \vdash_{G^0L\forall} \mathcal{H}'_n, \vdash_{G^0L\forall} \mathcal{H}''_n, \text{and} \, [\Gamma_i \Rightarrow A \Rightarrow B, \Delta_i]_{i \in 1..n} \subseteq \mathcal{G}. \, \text{Then} \, \vdash_{G^0L\forall} \mathcal{G}. \]
Proof. We proceed by induction on \( n \). The case \( n = 1 \) is trivial.

Now suppose that \( n \geq 2 \). By Lemma 10 below, from \( \vdash_{G^0L\forall} \cal{H}'_n \) and \( \vdash_{G^0L\forall} \cal{H}''_n \) it follows that the hypersequent

\[
\cal{H}_n = \left( \cal{G} \mid [\Gamma_i \Rightarrow \Delta_i]_{i \in 1..(n-1)} \mid \Gamma_n, A \Rightarrow B, \Delta_n \right)
\]

is \( G^0L\forall \)-provable. Applying the rule \((\Rightarrow \Rightarrow)^0\) to \( \cal{H}'_n \) and \( \cal{H}_n \) gives

\[
\cal{H}'_{n-1} = \left( \cal{G} \mid [\Gamma_i \Rightarrow \Delta_i]_{i \in 1..(n-1)} \right).
\]

Likewise we obtain the \( G^0L\forall \)-provable hypersequent

\[
\cal{H}''_{n-1} = \left( \cal{G} \mid [\Gamma_i, A \Rightarrow \Delta_i]_{i \in 1..(n-1)} \right).
\]

Finally, by applying the induction hypothesis to \( \cal{H}'_{n-1} \) and \( \cal{H}''_{n-1} \), we get \( \vdash_{G^0L\forall} \cal{G} \). \( \square \)

Lemma 10. Suppose that \( n \geq 2 \),

\[
\cal{H}' = \left( \cal{G} \mid [\Gamma_i, \Pi' \Rightarrow \Sigma', \Delta_i]_{i \in 1..n} \right), \quad \cal{H}'' = \left( \cal{G} \mid [\Gamma_i, \Pi'' \Rightarrow \Sigma'', \Delta_i]_{i \in 1..n} \right),
\]

\( \vdash_{G^0L\forall} \cal{H}' \), and \( \vdash_{G^0L\forall} \cal{H}'' \). Then

\[
\vdash_{G^0L\forall} \left( \cal{G} \mid [\Gamma_i, \Pi' \Rightarrow \Sigma', \Delta_i]_{i \in 1..(n-1)} \mid \Gamma_n, \Pi'' \Rightarrow \Sigma'', \Delta_n \right).
\]

Proof. For each \( k \in 1..n \), we put

\[
\cal{H}_k = \left( \cal{G} \mid [\Gamma_i, \Pi' \Rightarrow \Sigma', \Delta_i]_{i \in 1..(n-1)} \mid [\Gamma_i, \Pi'' \Rightarrow \Sigma'', \Delta_i]_{i \in k..n} \right).
\]

We can get \( \cal{H}_1 \) from \( \cal{H}'' \) by the rule \((\text{ew})^0\). For each \( k \in 1..(n-1) \), Figure 2 shows how to obtain \( \cal{H}_{k+1} \) from \( \cal{H}' \) and \( \cal{H}_k \) using the rules \((\text{ew})^0\), \((\text{ec})^0\), \((\text{split})^0\), and \((\text{mix})^0\). Recall that these four rules are admissible for \( G^0L\forall \) by Lemma 3. So \( \vdash_{G^0L\forall} \cal{H}_n \) as required. \( \square \)

Lemma 11. Suppose that \( n \geq 1 \),

\[
\vdash_{G^0L\forall} \left( \cal{G} \mid [\Gamma_i \Rightarrow \forall x A, \Delta_i]_{i \in 1..n} \right),
\]

\([\Gamma_i \Rightarrow \forall x A, \Delta_i]_{i \in 1..n} \subseteq \cal{G} \), and the parameter \( a \) does not occur in \( \cal{G} \). Then \( \vdash_{G^0L\forall} \cal{G} \).
Proof. We can obtain $\mathcal{G}$ from $\mathcal{G} \mid [\Gamma_i \Rightarrow [A]^x_{a_i}, \Delta_i]_{i \in 1..n}$ by $n$ applications of the rule $(\Rightarrow \forall)^0$, provided the parameters $a_1, \ldots, a_n$ are distinct and none of them occurs in $\mathcal{G}$.

Therefore, it suffices to prove the following claim for every $n \geq 1$: suppose that $\mathcal{H}(a) = (\mathcal{G}_0 \mid [\Gamma_i \Rightarrow [A]^x_{a_i}, \Delta_i]_{i \in 1..n})$, $\vdash_{G^0_L\forall} \mathcal{H}(a)$, and the parameters $a, a_1, \ldots, a_n$ are distinct and none of them occurs in $\mathcal{G}_0, A, \Gamma_i, \Delta_i (i \in 1..n)$; then $\vdash_{G^0_L\forall} \left(\mathcal{G}_0 \mid [\Gamma_i \Rightarrow [A]^x_{a_i}, \Delta_i]_{i \in 1..n}\right)$.

We use induction on $n$. In the case $n = 1$, the claim is obvious.

Suppose that $n \geq 2$. Clearly, $\vdash_{G^0_L\forall} \mathcal{H}(a)$ implies $\vdash_{G^0_L\forall} \mathcal{H}(a_n)$. By Lemma 10, from $\vdash_{G^0_L\forall} \mathcal{H}(a)$ and $\vdash_{G^0_L\forall} \mathcal{H}(a_n)$ it follows that

$$\vdash_{G^0_L\forall} \left(\mathcal{G}_0 \mid [\Gamma_i \Rightarrow [A]^x_{a_i}, \Delta_i]_{i \in 1..(n-1)} \mid \Gamma_n \Rightarrow [A]^x_{a_n}, \Delta_n\right),$$

whence by the induction hypothesis, we get what is required. 

Lemma 12. Suppose that $n \geq 1$,

$$\vdash_{G^0_L\forall} \left(\mathcal{G} \mid [\Gamma_i, [A]^x_a \Rightarrow \Delta_i]_{i \in 1..n}\right),$$

$\Gamma_i, \exists x A \Rightarrow \Delta_i]_{i \in 1..n} \subseteq \mathcal{G}$, and the parameter $a$ does not occur in $\mathcal{G}$. Then $\vdash_{G^0_L\forall} \mathcal{G}$.

Proof is similar to the proof of Lemma 11.

For a finite multiset $\Delta$, by $\#(\Delta)$ we denote the number of its elements, taking their multiplicities into account.
Lemma 13 (admissibility of a generalization of (den0) for G0∀). Suppose that m ≥ 1, n ≥ 1,

\[ \mathcal{H} = \left( \mathcal{G} \mid [\Gamma_i, p_0 \Rightarrow \Delta_i]_{i \in 1..m}, [\Pi_j \Rightarrow p_0, \Sigma_j]_{j \in 1..n} \right), \]

\[ \mathcal{H}' = \left( \mathcal{G} \mid [\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j]_{i \in 1..m}^{j \in 1..n} \right), \]

p0 does not occur in \( \mathcal{H}' \), the sequent p0 ⇒ C occurs in \( \mathcal{H} \), and \( \vdash_{G0∀} \mathcal{H} \). Then \( \vdash_{G0∀} \mathcal{H}' \).

Proof. Using Lemma 14 below, we find a (G0∀-)proof D of \( \mathcal{H} \) in which each leaf hypersequent \( \mathcal{L} \) contains an atomic sequent of the form \( \Gamma_{\mathcal{L}}, p_0 \Rightarrow \Delta_{\mathcal{L}} \), where \( \#(\Delta_{\mathcal{L}}) \leq 1 \) and no semipropositional variable occurs in \( \Gamma_{\mathcal{L}} \) or \( \Delta_{\mathcal{L}} \). We show that \( \vdash_{G0∀} \mathcal{H}' \) by induction on the height of D.

1. Suppose that \( \mathcal{H} \) is an axiom; i.e., \( \models \mathcal{H}_{at} \). We can harmlessly assume that

\[ \mathcal{H}_{at} = \left( \mathcal{G}_{at} \mid [\Gamma_i, p_0 \Rightarrow \Delta_i]_{i \in 1..k}, [\Pi_j \Rightarrow p_0, \Sigma_j]_{j \in 1..l} \right), \]

where 0 < k ≤ m, 0 ≤ l ≤ n and \( \#(\Delta_{\mathcal{L}}) \leq 1 \). We put \( \mathcal{H}'_{at} = (\mathcal{H}')_{at} \).

1.1. Consider the case where l ≠ 0. We have

\[ \mathcal{H}'_{at} = \left( \mathcal{G}_{at} \mid [\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j]_{i \in 1..k}^{j \in 1..l} \right). \]

Suppose for a contradiction that \( \not\models \mathcal{H}'_{at} \); i.e., for some hs-interpretation M and some M-valuation \( \nu \), there is no true sequent in \( \mathcal{G}_{at} \), and for all \( i \in 1..k \) and \( j \in 1..l \),

\[ \|\Delta_i\|_{M,\nu} - \|\Gamma_i\|_{M,\nu} < \|\Pi_j\|_{M,\nu} - \|\Sigma_j\|_{M,\nu}. \]

Hence, for some real number \( \xi \) and for all \( i \in 1..k \) and \( j \in 1..l \),

\[ \|\Delta_i\|_{M,\nu} - \|\Gamma_i\|_{M,\nu} < \xi - 1 < \|\Pi_j\|_{M,\nu} - \|\Sigma_j\|_{M,\nu}. \]

In particular, \( \xi > \|\Delta_1\|_{M,\nu} - \|\Gamma_1\|_{M,\nu} + 1 \geq 0 \).

Define an hs-interpretation \( M_0 \) to be the same as \( M \) except that \( \|p_0\|_{M_0} = \xi \). Since \( p_0 \) does not occur in \( \mathcal{G}_{at} \), \( \Gamma_i, \Delta_i \) (\( i \in 1..k \)), \( \Pi_j, \Sigma_j \) (\( j \in 1..l \)), it follows that \( \mathcal{H}_{at} \) has no true sequent under the hs-interpretation \( M_0 \) and \( M_0 \)-valuation \( \nu \). So \( \not\models \mathcal{H}_{at} \), a contradiction.
Thus $\vdash \mathcal{H}'_{at}$ and $\mathcal{H}'$ is an axiom.

1.2. Now consider the case where $l = 0$. Then $\mathcal{H}_{at} = \left( \mathcal{G}_{at} \mid [\Gamma_i, p_0 \Rightarrow \Delta_i]_{i \in 1..k} \right)$ and $\mathcal{H}'_{at} = \mathcal{G}_{at}$. Since $p_0$ does not occur in $\mathcal{G}_{at}$, $\Gamma_i$, $\Delta_i$ ($i \in 1..k$), and $p_0$ can assume arbitrarily large values under hs-interpretations, we see that $\vdash \mathcal{H}_{at}$ implies $\vdash \mathcal{G}_{at}$. So $\vdash \mathcal{H}'_{at}$ and $\mathcal{H}'$ is an axiom.

2. It remains to consider the case where the root hypersequent $\mathcal{H}$ in $D$ is the conclusion of a rule application. But the argument for this case can be obtained from item 2 of the proof of Lemma 7 by replacing $p_1$ with $p_0$.

Lemma 14. Suppose that $\mathcal{H} = (\mathcal{G} \mid \Gamma, p_0 \Rightarrow \Delta)$ is an axiom of $G^0L_{\forall}$, $\#(\Delta) \leq 1$, and no semipropositional variable occurs in $\Gamma$ or $\Delta$. Then a $G^0L_{\forall}$-proof of $\mathcal{H}$ can be constructed in which each leaf hypersequent $\mathcal{L}$ contains an atomic sequent of the form $\Gamma_{\mathcal{L}}, p_0 \Rightarrow \Delta_{\mathcal{L}}$, where $\#(\Delta_{\mathcal{L}}) \leq 1$ and no semipropositional variable occurs in $\Gamma_{\mathcal{L}}$ or $\Delta_{\mathcal{L}}$.

Proof is by induction on the number of logical symbol occurrences in the sequent $S = (\Gamma, p_0 \Rightarrow \Delta)$. If $S$ is atomic, then $\mathcal{H}$ is the desired proof. Otherwise, $S$ has one of the forms given in items 1–4 below.

1. Suppose that $S = (\Gamma', A \rightarrow B, p_0 \Rightarrow \Delta)$. By applying the rule $(\rightarrow \Rightarrow)_0$ backward to the distinguished occurrence of $A \rightarrow B$ in $\mathcal{H}$, we get the axiom $\mathcal{H}_1 = (\mathcal{G} \mid S \mid \Gamma', p_0 \Rightarrow \Delta \mid \Gamma', B, p_0 \Rightarrow A, \Delta)$. By the induction hypothesis applied to $\mathcal{H}_1$ with $S = (\Gamma', p_0 \Rightarrow \Delta)$, we obtain the desired proof of $\mathcal{H}$.

2. Suppose that $S = (\Gamma, p_0 \Rightarrow A \rightarrow B)$. Applying the rule $(\Rightarrow \rightarrow)_0$ backward to the distinguished occurrence of $A \rightarrow B$ in $\mathcal{H}$ yields the axioms $(\mathcal{G} \mid S \mid \Gamma, p_0 \Rightarrow)$ and $(\mathcal{G} \mid S \mid \Gamma, A, p_0 \Rightarrow B)$, to each of which the induction hypothesis applies.

3. Suppose that $S = (\Gamma, p_0 \Rightarrow Qx A)$, where $Q$ is a quantifier. We apply the rule $(\Rightarrow \forall)_0$ or $(\Rightarrow \exists)_0$ backward to the distinguished occurrence of $Qx A$ in $\mathcal{H}$ with a new parameter $a$ as the proper parameter or proper term, respectively. Thus we get the axiom $(\mathcal{G} \mid S \mid \Gamma, p_0 \Rightarrow [A]^x_{a})$ and then use the induction hypothesis.

4. The case where $S = (\Gamma', Qx A, p_0 \Rightarrow \Delta)$, with $Q$ being a quantifier, is treated similarly to case 3.

Remark 3. The proofs of Lemmas 7 and 13 can be easily combined to establish the admissibility of the nonstandard density rule (given
Comparing calculi for $L∀$

in Remark 2 on p. 11) for $G^0L∀$ (with the notion of a hypersequent expanded as mentioned in Remark 2).\(^4\)

**Theorem 15** (equivalence of $G^0L∀$, $G^1L∀$, and $G^3L∀$). $\vdash_{G^0L∀} \mathcal{H}$ iff $\vdash_{G^iL∀} \mathcal{H}$ iff $\vdash_{G^3L∀} \mathcal{H}$.

**Proof.** $\vdash_{G^0L∀} \mathcal{H}$ implies $\vdash_{G^iL∀} \mathcal{H}$ by Theorem 4. If $\vdash_{G^iL∀} \mathcal{H}$, then $\vdash_{G^3L∀} \mathcal{H}$ by Theorem 5. By Lemmas 6, 7, and 13, from $\vdash_{G^3L∀} \mathcal{H}$ it follows that $\vdash_{G^0L∀} \mathcal{H}$. □

**Theorem 16.** For each $i = 0, \hat{1}, 3$, the calculus $G^iL∀$ is a conservative extension of $G^1L∀$; i.e., for any RPL∀-hypersequent $\mathcal{H}$, $\vdash_{G^iL∀} \mathcal{H}$ iff $\vdash_{G^1L∀} \mathcal{H}$.

**Proof.** Follows from Remark 1 on p. 6 and Theorem 15. □

**Theorem 17.** For each $i = 0, 1, \hat{1}, 3$, the calculus $G^iL∀$ is a conservative extension of $GL∀$; i.e., for any $L∀$-hypersequent $\mathcal{H}$, $\vdash_{G^iL∀} \mathcal{H}$ iff $\vdash_{GL∀} \mathcal{H}$.

**Proof.** Immediate from Theorems 1 and 16. □

For RPL∀- and L∀-formulas, the preceding three theorems yield the next result.

**Corollary 18.**
(a) For each $i = 0, 1, \hat{1}$ and any RPL∀-formula $A$, $\vdash_{G^iL∀} A$ iff $\vdash_{G^3L∀} A$.
(b) For each $i = 0, 1, \hat{1}, 3$ and any $L∀$-formula $A$, $\vdash_{G^iL∀} A$ iff $\vdash_{GL∀} A$.

5. Some relationships between $G^0L∀$, $G^3L∀$, and Hilbert-type calculi for RPL∀ and HL∀

In this section we compare the hypersequent calculi $G^0L∀$ and $G^3L∀$ with a Hilbert-type calculus HRP∀ for the logic RPL∀ (cf. [16]), as well as with a Hilbert-type calculus HL∀ for the logic L∀ (cf. [16]). First we formulate HRP∀ and L∀.

The axiom schemes of HRP∀ are:

(L1) $A → (B → A)$;

(L2) $(A → B) → ((B → C) → (A → C))$;

See also Section C (of the appendix) on p. 34.
\((L3) \quad (\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)\), where \(\neg C\) is short for \((C \rightarrow \bar{0})\);
\((L4) \quad ((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)\);
\((tc1) \quad (\bar{r}_1 \rightarrow \bar{r}_2) \rightarrow \bar{r}, \) where \(r = \min(1 - r_1 + r_2, 1)\);
\((tc2) \quad \bar{r} \rightarrow (\bar{r}_1 \rightarrow \bar{r}_2), \) where \(r = \min(1 - r_1 + r_2, 1)\);
\((\forall 1) \quad \forall x A \rightarrow [A]^T_t, \) where \(t\) is a term (not necessarily closed) free for \(x\) in \(A\);
\((\forall 2) \quad \forall x (A \rightarrow B) \rightarrow (A \rightarrow \forall x B), \) where \(x\) does not occur free in \(A\);
\((\exists 1) \quad [A]^T_t \rightarrow \exists x A, \) where \(t\) is a term (not necessarily closed) free for \(x\) in \(A\);
\((\exists 2) \quad \forall x (A \rightarrow B) \rightarrow (\exists x A \rightarrow B), \) where \(x\) does not occur free in \(B\).

The inference rules of \(\text{HRP}^\forall\) are:
\[
\begin{align*}
A; & \quad A \rightarrow B \quad \text{(mp)}; & \quad A \quad \text{(gen)}. \\
\end{align*}
\]

To obtain \(\text{HL}^\forall\) from \(\text{HRP}^\forall\), we require \(A, B,\) and \(C\) to be \(L^\forall\)-formulas in the formulation of \(\text{HRP}^\forall\) and remove the axiom schemes \((tc1)\) and \((tc2)\) from it.

As hypersequent counterparts of the rules \((mp)\) of \(\text{HRP}^\forall\) and \(\text{HL}^\forall\), we consider the following cut rules (cf., e.g., [21, Section 4.2]):
\[
\frac{
\mathcal{G} \mid \Gamma_1 \Rightarrow C, \Delta_1; \quad \mathcal{G} \mid \Gamma_2, C \Rightarrow \Delta_2
}{
\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2
}(\text{cut}), \quad \text{where } C \text{ is an } RPL^\forall\text{-formula},
\]
\[
\frac{
\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2
}{
\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2
}(\text{Lcut}), \quad \text{where } C \text{ is an } L^\forall\text{-formula}.
\]

**Proposition 19.** The rule \((\text{Lcut})\) (and hence \((\text{cut})\)) is admissible neither for \(G^0L^\forall\) nor for \(G^3L^\forall\).

**Proof.** In [19, p. 268], for some \(L^\forall\)-sentence \(A\), it is shown that \(\nu^A_{GL^\forall}\), and a proof is constructed of the form
\[
\frac{D_1 \quad D_2}{\mathcal{H}_1; \quad \mathcal{H}_2 \Rightarrow A}(\text{Lcut}),
\]
where \(D_1\) and \(D_2\) are \(GL^\forall\)-proofs. By Theorem 17, for \(i = 0, 3\), we get \(\vdash_{G^iL^\forall} \mathcal{H}_1\) and \(\vdash_{G^iL^\forall} \mathcal{H}_2\), whence \(\vdash_{G^iL^\forall+(\text{cut})} A\). But by Corollary 18, we have \(\nu^A_{G^iL^\forall}\) for \(i = 0, 3\).

In the rest of this section we establish the following two theorems.

**Theorem 20.** For any \(RPL^\forall\)-sentence \(A\),
\[
\vdash_{\text{HRP}^\forall} A \text{ implies } \vdash_{G^3L^\forall+(\text{cut})} A, \text{ which in turn implies } \vdash_{G^0L^\forall+(\text{cut})} A.
\]
**Theorem 21.** For any $\forall$-sentence $A$, 
\[
\vdash_{\text{HRP}_\forall} A \iff \vdash_{\text{HL}_\forall} A \iff \vdash_{G^3_{\forall+}(L\text{cut})} A \iff \vdash_{G^0_{\forall+}(L\text{cut})} A.
\]

In proving these theorems, we will use the cumulative cancellation rules (cf. [9, Section 4.1] and [21, Section 4.3.5])
\[
\frac{\emptyset \mid \Gamma \Rightarrow \Delta \mid \Gamma, C \Rightarrow C, \Delta}{\emptyset \mid \Gamma \Rightarrow \Delta} \quad \text{(ccan), where $C$ is an RPL$_\forall$-formula,}
\[
\frac{\emptyset \mid \Gamma \Rightarrow \Delta}{\emptyset \mid \Gamma \Rightarrow \Delta} \quad \text{(Lccan), where $C$ is an L$_\forall$-formula;}
\]
and the calculi $\hat{\text{HRP}}_\forall$ and $\hat{\text{HL}}_\forall$ that are obtained from HRP$_\forall$ and HL$_\forall$, respectively, thus: $t$ in the axiom schemes ($\forall 1$) and ($\exists 1$) is taken to be a closed term, and the inference rule (gen) is replaced by the rule
\[
\frac{[A]_a^x}{\forall x A} \quad \text{(gen),}
\]
where $a$ is a parameter not occurring in $A$.

**Lemma 22.** For any RPL$_\forall$-sentence $A$, $\vdash_{\text{HRP}_\forall} A \iff \vdash_{\hat{\text{HRP}}_\forall} A$. For any $\forall$-sentence $B$, $\vdash_{\text{HL}_\forall} B \iff \vdash_{\hat{\text{HL}}_\forall} B$.

We omit the proof of Lemma 22, because the proof is not complicated and does not differ from that of a similar assertion for appropriate variants of classical first-order Hilbert-type calculi.

We will also use the following translations and Lemma 23.

For an RPL$_\forall$-formula $A$ and a hypersequent $\mathcal{H}$, let the translations $A \mapsto \tilde{A}$ and $\mathcal{H} \mapsto \tilde{\mathcal{H}}$ replace each truth constant $\bar{r} \neq \bar{0}$ by a unique propositional variable $p_{\bar{r}}$.

**Lemma 23.** Let $\mathcal{H}$ be a hypersequent not containing semipositional variables, and $\tilde{\mathcal{H}}$ be the $\forall$-hypersequent that results from $\mathcal{H}$ by applying the above translation. If $\vdash_{G_{\forall}} \mathcal{H}$, then $\vdash_{G^i_{\forall}} \mathcal{H}$ for all $i = 0, 1, \hat{1}, 3$.

**Proof.** Fix any $i \in \{0, 1, \hat{1}, 3\}$. By Theorem 17, from $\vdash_{G_{\forall}} \tilde{\mathcal{H}}$ it follows that there exists a $G^i_{\forall}$-proof $D$ of $\tilde{\mathcal{H}}$. For each truth constant $\bar{r} \neq \bar{0}$ occurring in $\mathcal{H}$, we replace the propositional variable $p_{\bar{r}}$ by $\bar{r}$ in $D$; thus we get a $G^i_{\forall}$-proof of $\mathcal{H}$. \[\square\]

---

5 The (noncumulative) cancellation rule was introduced in [9] as a variant of the cut rule to establish cut elimination for the propositional fragment of the calculus GL$_\forall$ via elimination of the cancellation rule.
\[
\begin{align*}
\Rightarrow A; & \quad \Rightarrow A \to B; \quad A, A \to B \Rightarrow B \quad \text{(cut)} \\
\Rightarrow B & \quad \Rightarrow \bar{A}; \quad \bar{B} \Rightarrow \bar{B} \quad \text{(mix)} \\
\Rightarrow \bar{A}, \bar{B} \Rightarrow \bar{A}, \bar{B} & \quad \Rightarrow A, \bar{A} \to B \Rightarrow \bar{B} \quad (\to \Rightarrow)
\end{align*}
\]

Figure 3: Proofs for showing the derivability of (mp) in G\(^3\)L\(\forall\)+(cut).

**Proof of Theorem 20.** For any RPL\(\forall\)-sentence \(A\), the following implications hold:

\[
\begin{align*}
\vdash_{\text{HRP}\forall} A & \quad \vdash_{\text{HRP}\forall} \bar{A}; \quad \vdash_{G\text{L}\forall+(cut)} \bar{A}; \quad \vdash_{G\text{L}\forall+(cut)} A \\
\vdash_{G\text{L}\forall+(ccan)} A & \quad \vdash_{G\text{L}\forall+(cut)} A.
\end{align*}
\]

Over each of these implications, there is a number of the lemma that verifies it.

**Lemma 24.** For any RPL\(\forall\)-formula \(A\), if \(\vdash_{\text{HRP}\forall} A\), then \(\vdash_{G\text{L}\forall+(cut)} A\).

**Proof.** The rule \((\bar{\text{gen}})\) of \(\text{HRP}\forall\) is derivable in G\(^3\)L\(\forall\), because the latter calculus contains the rule \((\Rightarrow \forall)^3\).

On the left in Figure 3, we show how to get the conclusion of the rule (mp) from its premises and the hypersequent \(\not{\mathcal{H}} = (A, A \to B \Rightarrow B)\) using the rule (cut); and on the right, we show a GL\(\forall\)-proof of \(\bar{\not{\mathcal{H}}}\). By Lemma 23, we have \(\vdash_{G\text{L}\forall} \not{\mathcal{H}}\). So (mp) is derivable in G\(^3\)L\(\forall\)+(cut).

To conclude the proof, it is sufficient to establish that all axioms of \(\text{HRP}\forall\) are G\(^3\)L\(\forall\)-provable.

Let \(L\) be an instance of one of the axiom schemes (L1)–(L4), say,

\[
L = (A \to B) \to ((B \to C) \to (A \to C))
\]

for some RPL\(\forall\)-formulas \(A, B, C\). Take the propositional L\(\forall\)-formula

\[
L' = (p_1 \to p_2) \to ((p_2 \to p_3) \to (p_1 \to p_3)),
\]

where \(p_1, p_2, p_3\) are distinct propositional variables. Since \(\vdash L'\) and GL\(\forall\) is complete for quantifier-free L\(\forall\)-hypersequents [21, Theorem 6.24], we have a GL\(\forall\)-proof \(D'\) of \(L'\). In \(D'\) we replace propositional variables \(p_1, p_2, p_3\) with \(\bar{A}, \bar{B}, \bar{C}\), respectively, producing a GL\(\forall\)-proof of \(\bar{L}\). Then by Lemma 23, we get \(\vdash_{G\text{L}\forall} L\).
Comparing calculi for $L_\forall$

Any instance of any of the axiom schemes (tc1) and (tc2) is $G^3L_\forall$-provable, because it is valid and $G^3L_\forall$ is complete for quantifier-free hypersequents [15, Proposition 1].

Finally, let $Q$ be a quantifier axiom of $\hat{HRP}_\forall$. Then we can construct a $GL_\forall$-proof of $Q$. Indeed, in the cases of ($\forall 1)$ and ($\exists 1)$, this is trivial; in the case of ($\exists 2)$, such a $GL_\forall$-proof is given in Figure 4 (where $a$ does not occur in $Q$); and in the case of ($\forall 2)$, a $GL_\forall$-proof of $Q$ is constructed similarly. Hence $\vdash G^3L_\forall Q$ by Lemma 23.

**Lemma 25.** If $\vdash_{G^3L_\forall+(cut)} H_0$, then $\vdash_{G^0L_\forall+(ccan)} H_0$.

**Proof.** It suffices to show that all the rules of $G^3L_\forall+(cut)$ are admissible for $G^0L_\forall+(ccan)$.

1. Let us demonstrate that the rules $(ew)^0$, $(ec)^0$, and $(split)^0$ are hp-admissible for $G^0L_\forall+(ccan)$, and the rule $(mix)^0$ is admissible for $G^0L_\forall+(ccan)$ (these rules are formulated in Lemma 3).

For $(ew)^0$, $(ec)^0$, and $(mix)^0$, these assertions are established just as in items 1, 2, and 5 of the proof of Lemma 3.

The proof of the hp-admissibility of $(split)^0$ for $G^0L_\forall+(ccan)$ is similar to the proof of Lemma 7 in [14], we only need to consider one more case. As in the proof of Lemma 7 in [14], by induction on the height of a proof $D_1$ of $\exists \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ (in $G^0L_\forall+(ccan)$ now), we show that $D_1$ can be transformed into a proof of $\exists \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2$ whose height is not greater than the height of $D_1$. We add the case where the proof $D_1$ has the form:

---

See also Section A (of the appendix) on p. 30.
\[ \frac{D_0}{\frac{G | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 | \Gamma_1, \Gamma_2, A \Rightarrow A, \Delta_1, \Delta_2}{G | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} } \]

In this case, using the induction hypothesis twice, we split the two sequent occurrences distinguished in the lowest hypersequent in the proof \( D_0 \) to obtain a proof of

\[ G | \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2 | \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2, A \Rightarrow A, \Delta_2; \]

whence by the hp-admissible rule \((ec)^0\), we construct a proof of

\[ G | \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2 | \Gamma_2, A \Rightarrow A, \Delta_2; \]

and by \((ccan)\), we get the desired proof of \( G | \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2. \)

2. Let us establish the admissibility of the rules \((den_1)\) and \((den_0)\) for \( G^0L\forall+(ccan) \) (these rules are formulated at the beginning of Section 4). With the results of the preceding item, we do this as in Lemmas 7 and 13, adding to item 2.2 of the proof of Lemma 7 one more case 2.2.6 where \( \mathcal{R} \) is \((ccan)\) and the proof \( D \) (in \( G^0L\forall+(ccan) \) now) looks like:

\[ \frac{D_1}{\frac{H | \Gamma_1, A, p_1 \Rightarrow A, \Delta_1}{H} } \]  

\((ccan)\).

In this case, using the induction hypothesis, we transform \( D_1 \) into a proof of

\[ H' | [\Gamma_1, A, \Pi_j \Rightarrow A, \Delta_1, \Sigma_j]_{j \in 1..n}, \]

whence we get the desired proof of \( H' \) by \( n \) applications of \((ccan)\).

3. Now the admissibility for \( G^0L\forall+(ccan) \) of each rule of \( G^3L\forall \) can be shown just as in the proof of Lemma 6. Finally, \((cut)\) is admissible for \( G^0L\forall+(ccan) \). Indeed, the conclusion of \((cut)\) is obtained from its premises thus:

\[ G | \Gamma_1 \Rightarrow C, \Delta_1; \quad G | \Gamma_2, C \Rightarrow \Delta_2 \quad \frac{G | \Gamma_1, \Gamma_2, C \Rightarrow C, \Delta_1, \Delta_2}{G | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \quad \frac{(mix)^0}{(ew)^0} \]

\( (mix)^0 \) and \( (ew)^0 \) being admissible for \( G^0L\forall+(ccan) \) by item 1 of the present proof. \( \square \)
Lemma 26. \( \vdash_{G^0L\forall+(ccan)} \mathcal{H} \iff \vdash_{G^0L\forall+(cut)} \mathcal{H} \).

Proof. For the left-to-right direction, it is enough to establish that (ccan) is admissible for \( G^0L\forall+(cut) \). The conclusion of (ccan) is obtained from its premise and the hypersequents \( \Rightarrow \) and \( \tilde{\mathcal{H}} = (C \Rightarrow C) \) by rules, which are admissible for \( G^0L\forall+(cut) \), as follows (cf. [9, Section 4.1]):

\[
\begin{align*}
(ew)^0, (\Rightarrow \rightarrow)^0 & \Rightarrow \; C \Rightarrow C; \\
\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma, C \Rightarrow C, \Delta & \Rightarrow \mathcal{G} \mid \Gamma \Rightarrow \Delta \\
\mathcal{G} \mid \Gamma \Rightarrow \Delta & \Rightarrow \mathcal{G} \mid \Gamma \Rightarrow \Delta.
\end{align*}
\]

The hypersequent \( \Rightarrow \) is an axiom of \( G^0L\forall \). The hypersequent \( \tilde{\mathcal{H}} \) is an axiom of \( GL\forall \), and hence by Lemma 23, we get \( \vdash_{G^0L\forall} \mathcal{H} \). Thus (ccan) is admissible for \( G^0L\forall+(cut) \).

For the right-to-left direction, it suffices to show that (cut) is admissible for \( G^0L\forall+(ccan) \). This is done in item 3 of the proof of Lemma 25.

Proof of Theorem 21. Let \( A \) be an \( L\forall \)-sentence. By [17, Theorem 2.4], we have: \( \vdash_{HRP\forall} A \iff \vdash_{HL\forall} A \). Also the following implications hold:

\[
\begin{align*}
\vdash_{HL\forall} A & \xrightarrow{22} \vdash_{HL\forall} A \\
\vdash_{HL\forall} A & \xrightarrow{24} \vdash_{G^3L\forall+(Lcut)} A \\
\vdash_{G^3L\forall+(Lcut)} A & \xrightarrow{25} \vdash_{G^0L\forall+(Lccan)} A \\
\vdash_{G^0L\forall+(Lccan)} A & \xrightarrow{26} \vdash_{GL\forall+(Lcut)} A \\
\vdash_{GL\forall+(Lcut)} A & \xrightarrow{26} \vdash_{HL\forall} A.
\end{align*}
\]

The first implication holds by Lemma 22; the last one, by [2, Theorem 9]. Over each of the other implications, a number (with the symbol \( \approx \)) is given indicating that the implication is proved by analogy with the lemma designated by the number.

6. Conclusion

In the present article, we have established that the calculi \( G^0L\forall \) and \( G^3L\forall \) for the logic RPL\forall are equivalent and are conservative extensions of the calculus \( GL\forall \) for the logic \( L\forall \) (see Theorems 15 and 17).

The crucial part of our argument is the syntactic proofs of the admissibility for \( G^0L\forall \) of the nonstandard variants (den\(_1\)) and (den\(_0\)) of the density rule (see Lemmas 7 and 13). These proofs can be easily adapted to show the admissibility of the nonstandard density rule for
$G^0L\forall$ (see Remark 3 on p. 18). The given proof of the admissibility of $(\text{den}_1)$ for $G^0L\forall$ provides an algorithm for transforming a proof of a hypersequent in $G^0L\forall+(\text{den}_1)$ into a proof of the same hypersequent in $G^0L\forall$, in other words, establishes elimination of $(\text{den}_1)$ for $G^0L\forall+(\text{den}_1)$;\footnote{Suppose that a rule $R$ is not an inference rule of a calculus $\mathcal{C}$. It is said that elimination of $R$ holds for $\mathcal{C}+R$ (as well as for $\mathcal{C}$) if a proof of an object in $\mathcal{C}+R$ can be algorithmically transformed into a proof of the same object in $\mathcal{C}$ (cf., e.g., \cite{9, 20, 21, 22}).} similarly with $(\text{den}_0)$ and the nonstandard density rule.

Density elimination proofs are known for some calculi (and for some classes of calculi), though for logics different from $L\forall$ and $RPL\forall$; see \cite{3, 1, 20, 10, 21, 8, 4, 5, 6, 22, 7}. In all these works except [1], such proofs use the cut rule even if no application of it is in an initial formal proof. In [1] the density elimination proof for a single-conclusion hypersequent calculus for first-order Gödel logic does not introduce cuts if no cuts are in an initial formal proof. Recall that the cut rule is not admissible for $G^0L\forall$ (see Proposition 19). Our technique for proving the admissibility of $(\text{den}_1)$ for $G^0L\forall$ resembles the technique of [1] for proving density elimination, but was rediscovered and elaborated for the multiple-conclusion calculus $G^0L\forall$ for the logic $RPL\forall$.

Further, the book [21] on p. 134 says that it is unclear whether density elimination can be obtained for the propositional fragment of the calculus $GL\forall$. We have given such a density elimination proof for the calculus $G^0L\forall$, which is a conservative extension of $GL\forall$; and let us note that a $GL\forall$-proof can be algorithmically transformed into a $G^0L\forall$-proof of the same $L\forall$-hypersequent, and conversely (see Theorem 1 and its proof). Moreover, to the best of our knowledge, the given proof is the first syntactic proof of density admissibility for a multiple-conclusion hypersequent calculus in which neither the weakening rule nor the contraction rule is admissible.\footnote{The weakening and contraction rules are, respectively:

$$\frac{\mathcal{G} | \Gamma \Rightarrow \Delta}{\mathcal{G} | \Gamma, \Pi \Rightarrow \Sigma, \Delta} \quad \text{and} \quad \frac{\mathcal{G} | \Gamma, \Pi \Rightarrow \Sigma, \Delta}{\mathcal{G} | \Gamma \Rightarrow \Sigma, \Delta},$$

where $\mathcal{G}$ is any hypersequent, and $\Gamma$, $\Delta$, $\Pi$, and $\Sigma$ are any finite multisets of formulas of a language under consideration (see \cite[Section 4.3]{21}).}

It would be nice to generalize the density elimination technique used in [1] and in the present article to as wide a syntactic class of
Comparing calculi for $L\forall$

hypersequent calculi as possible. Besides, notice that how complexity
of formal proofs varies has not been investigated for any density
elimination proof.

Also in the given article, we have established that the calculi $HL\forall$,
$G^0L\forall+(Lcut)$, and $G^3L\forall+(Lcut)$ prove the same $L\forall$-sentences (see Theorem 21). And we have shown that the provability of an $RPL\forall$-
sentence in $HRP\forall$ implies its provability in $G^3L\forall+(cut)$, which in turn
implies its provability in $G^0L\forall+(cut)$ (see Theorem 20).

Thus natural open questions are whether any $RPL\forall$-sentence prov-
able in $G^3L\forall+(cut)$ is provable in $HRP\forall$, and a similar question for
$G^0L\forall+(cut)$. To answer both questions affirmatively, it suffices to es-
tablish that any $RPL\forall$-sentence provable in $G^0L\forall+(cut)$ is provable in
$HRP\forall$, assuming that hypersequents do not contain semipropositional
variables, i.e., are built up only from $RPL\forall$-formulas. With this prepa-
ration, it is worth trying to extend to $RPL\forall$ the algebraic technique
used in [2] for showing that any $L\forall$-sentence provable in $GL\forall+(Lcut)$
is provable in $HL\forall$.

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APPENDIX

A. The admissibility of the rules \((\text{split})^0\) and \((\text{mix})^0\) for \(G^0\mathcal{L}\forall\)

Item 4 of the proof of Lemma 3 says that the proof of the hp-admissibility of \((\text{split})^0\) for \(G^0\mathcal{L}\forall\) is very similar to the proof of Lemma 7 in [14]. Besides, in item 1 of the proof of Lemma 25, we extend the proof of the hp-admissibility of \((\text{split})^0\) for \(G^0\mathcal{L}\forall\) with a new case and obtain the proof of the hp-admissibility of \((\text{split})^0\) for \(G^0\mathcal{L}\forall+(ccan)\).

Next, item 5 of the proof of Lemma 3 says that the proof of the admissibility of \((\text{mix})^0\) for \(G^0\mathcal{L}\forall\) can be easily obtained from the proof of Lemma 8 in [14] by identifying the notion of a completable ancestor of a sequent occurrence with the notion of an ancestor of a sequent occurrence (the former notion being used in [14]).

Below we give the proof of the hp-admissibility of \((\text{split})^0\) for \(G^0\mathcal{L}\forall\) and the proof of the admissibility of \((\text{mix})^0\) for \(G^0\mathcal{L}\forall\), adapting the mentioned proofs in [14] (and correcting some inaccuracies introduced in [14] by a translator of the original Russian article).

**Lemma 27.** The following rule is hp-admissible for \(G^0\mathcal{L}\forall\):

\[
\frac{\mathcal{G} | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} | \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2} \quad (\text{split})^0.
\]

**Proof.** Let

\[
\mathcal{H}_1 = (\mathcal{G} | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2), \quad \mathcal{H}_2 = (\mathcal{G} | \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2).
\]

Using induction on the height of a \((G^0\mathcal{L}\forall-)\)proof \(D_1\) of \(\mathcal{H}_1\), we show that \(D_1\) can be transformed into a proof of \(\mathcal{H}_2\) whose height is not greater than the height of \(D_1\).

1. If \(\mathcal{H}_1\) is an axiom, then it is easy to see that \(\mathcal{H}_2\) is an axiom too.

2. Let the lowest hypersequent \(\mathcal{H}_1\) in \(D_1\) be the conclusion of an application \(R\) of a rule \(\mathcal{R}\). We consider the case where \(\mathcal{R}\) is \((\rightarrow \Rightarrow)^0\); the remaining cases are similar.

2.1. Suppose that the principal sequent occurrence in the application \(R\) is in the distinguished occurrence of \(\mathcal{G}\) in \(\mathcal{H}_1\). Then the premise \(\mathcal{H}_0\) of the application \(R\) has the form \(\mathcal{G}_0 | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2\). By the induction hypothesis for the proof of \(\mathcal{H}_0\) (which is a subtree of the
proof tree $D_1$), we can construct a proof of $\mathcal{G}_0 | \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2$. Applying the rule $\mathcal{R}$, we obtain the required proof of $\mathcal{H}_2$.

2.2. Suppose that the principal sequent occurrence in the application $R$ is the distinguished occurrence of $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ in $\mathcal{H}_1$. For definiteness we assume that the principal occurrence of a formula $A_1 \rightarrow B_1$ in the application $R$ is in $\Gamma_1$. Then $\Gamma_1 = (\Gamma'_1, A_1 \rightarrow B_1)$ for some $\Gamma'_1$. The proof $D_1$ has the form

$$D_0$$

$$\mathcal{G} | \Gamma'_1, A_1 \rightarrow B_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 | \Gamma'_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$$

$$\mathcal{G} | \Gamma'_1, A_1 \rightarrow B_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$$

Applying the rule $\mathcal{R}$, we obtain the required proof of $H_2$.

We use the induction hypothesis twice and split all the three sequent occurrences that are distinguished in the lowest hypersequent in the proof $D_0$. We obtain a proof of the hypersequent

$$\mathcal{G} | \Gamma'_1, A_1 \rightarrow B_1 \Rightarrow \Delta_1 | \Gamma'_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2$$

We use the induction hypothesis twice and split all the three sequent occurrences that are distinguished in the lowest hypersequent in the proof $D_0$. We obtain a proof of the hypersequent

$$\mathcal{G} | \Gamma'_1, A_1 \rightarrow B_1 \Rightarrow \Delta_1 | \Gamma'_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2$$

Finally, we apply the rule $(\rightarrow \Rightarrow)^0$ to the lowest hypersequent in $D'_0$ and obtain the required proof of $\mathcal{G} | \Gamma'_1, A_1 \rightarrow B_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2$.

**Lemma 28.** The following rule is admissible for $G^0L\forall$:

$$\frac{\mathcal{G} | \Gamma_1 \Rightarrow \Delta_1; \mathcal{G} | \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \quad (\text{mix})^0.$$ 

**Proof.** Let

$$\mathcal{H}_1 = (\mathcal{G} | \Gamma_1 \Rightarrow \Delta_1), \quad \mathcal{H}_2 = (\mathcal{G} | \Gamma_2 \Rightarrow \Delta_2),$$

$$\mathcal{H}_3 = (\mathcal{G} | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2).$$

We suppose that $\vdash_{G^0L\forall} \mathcal{H}_1$ and $\vdash_{G^0L\forall} \mathcal{H}_2$, and show that $\vdash_{G^0L\forall} \mathcal{H}_3$. Let $D_1$ be a $(G^0L\forall)$-proof of $\mathcal{H}_1$ such that no proper parameter from $D_1$ occurs in $\Gamma_2 \Rightarrow \Delta_2$. 



We obtain a proof search tree $D_3^0$ for $\mathcal{H}_3$ as follows. In $D_1$, for each occurrence $S$ of a sequent of the form $\Pi_1 \Rightarrow \Sigma_1$, if $S$ is an ancestor of the distinguished occurrence of the sequent $\Gamma_1 \Rightarrow \Delta_1$ in the root of $D_1$, then we replace $S$ by an occurrence $S'$ of the sequent $\Pi_1, \Gamma_2 \Rightarrow \Sigma_1, \Delta_2$. We also mark $S'$ if $S$ is an atomic sequent occurrence in a leaf of $D_1$. Let $S_i, i = 0, \ldots, l - 1$, be all distinct marked sequent occurrences in $D_0^3$.

We expand $D_0^3$, proceeding for each $i = 0, \ldots, l - 1$ as follows.

(0) Let $S_i$ be an occurrence of a sequent of the form $\Pi_1, \Gamma_2 \Rightarrow \Sigma_1, \Delta_2$.

(1) We construct a proof $D_2$ of $H_2$ such that no proper parameter from $D_2$ occurs in $D_i^3$.

(2) We obtain a proof search tree $\hat{D}_2$ for $G \mid \Pi_1, \Gamma_2 \Rightarrow \Sigma_1, \Delta_2$ thus: in $D_2$, for each occurrence of a sequent of the form $\Pi_2 \Rightarrow \Sigma_2$, if this occurrence is an ancestor of the distinguished occurrence of the sequent $\Gamma_2 \Rightarrow \Delta_2$ in the root of $D_2$, then we replace this occurrence by $\Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2$.

(3) We expand each branch of $D_3^i$ containing the occurrence $S_i$ as follows: we identify the top node of this branch, which represents on occurrence of a hypersequent of the form $G \mid \Pi_1, \Gamma_2 \Rightarrow \Sigma_1, \Delta_2 \mid \mathcal{H}$ for some $\mathcal{H}$, with the root of the tree obtained from $\hat{D}_2$ by appending “$\mid \mathcal{H}$” to each node hypersequent. By $D_3^{i+1}$ we denote the tree resulting from this expansion of $D_3^i$.

It is not difficult to see that the tree $D_3^l$ is a proof search tree for $\mathcal{H}_3$. It remains to show that $D_3^l$ is a proof.

We consider an arbitrary leaf $L_3$ of $D_3^l$ and show that $L_3$ is an axiom. Given $L_3$, we find a unique leaf $L_1$ of $D_1$ that transforms into a leaf of $D_3^0$ that, in turn, transforms (in expanding $D_0^3$) into a node of $D_3^l$ belonging to the same branch as $L_3$.

Let $\Pi_{1,i} \Rightarrow \Sigma_{1,i}, i \in I$, be all atomic sequents whose occurrences in $L_1$ are ancestors of the distinguished occurrence of $\Gamma_1 \Rightarrow \Delta_1$ in the root of $D_1$. By the construction of $D_3^l$, for each $i \in I$, there exist a proof $D_2^i$ of $\mathcal{H}_2$ and its leaf $L_2^i$ such that, for each $j \in J_i$, an atomic sequent $\Pi_{1,i}, \Pi_{2,j} \Rightarrow \Sigma_{1,i}, \Sigma_{2,j}$ occurs in $L_3$, where $\Pi_{2,j} \Rightarrow \Sigma_{2,j}, j \in J_i$, are all atomic sequents whose occurrences in $L_2^i$ are ancestors of the distinguished occurrence of $\Gamma_2 \Rightarrow \Delta_2$ in the root of $D_2^i$.

In addition, $L_3$ contains all atomic sequents $S_{1,k}, k \in K$, whose occurrences in $L_1$ are ancestors of sequent occurrences in the distinguished occurrence of $G$ in the root of $D_1$. 

Finally, for each \( i \in I \), the leaf \( L_3 \) contains all atomic sequents \( S^i_{2,m} \), \( m \in M_i \), whose occurrences in \( L^i_2 \) are ancestors of sequent occurrences in the distinguished occurrence of \( \mathcal{G} \) in the root of \( D_2^i \).

The leaf \( L_1 \) of the proof \( D_1 \) is an axiom and contains exactly the following atomic sequents: \( \Pi_{1,i} \Rightarrow \Sigma_{1,i} \) for each \( i \in I \) and \( S_{1,k} \) for each \( k \in K \). For each \( i \in I \), the leaf \( L^i_2 \) of the proof \( D_2^i \) is an axiom and contains exactly the following atomic sequents: \( \Pi_{2,j} \Rightarrow \Sigma_{2,j} \) for each \( j \in J_i \) and \( S^i_{2,m} \) for each \( m \in M_i \). Therefore, the leaf \( L_3 \) of \( D_3^l \), which contains the above-mentioned atomic sequents, is an axiom too.

\[ \square \]

B. The soundness of the nonstandard density rule

Remark 2 on p. 11 says that the rule

\[
\frac{G | \Gamma, p \Rightarrow \Delta | \Pi \Rightarrow p, \Sigma}{G | \Gamma, \Pi \Rightarrow \Delta, \Sigma} \quad (\text{den}),
\]

(1) is unsound if \( p \) is a propositional variable not occurring in the conclusion, but (2) becomes sound if we expand the notion of a hypersequent by special variables interpreted by any real numbers, and require \( p \) to be such a variable not occurring in the conclusion.

Let us prove (1). Recall that the propositional variable \( p \) is interpreted by any real number in \([0, 1]\). Consider the following application of (\text{den}): \[
\frac{p \Rightarrow \bar{0}, \bar{0} | \bar{0} \Rightarrow p}{0 \Rightarrow \bar{0}, \bar{0}}. \]

The premise of this application is valid (because \( \models (\bar{0} \Rightarrow p) \)), but its conclusion is not valid.

\[ \square \]

Now let us prove (2), i.e., that \( \models (G | \Gamma, p \Rightarrow \Delta | \Pi \Rightarrow p, \Sigma) \) implies \( \models (G | \Gamma, \Pi \Rightarrow \Delta, \Sigma) \) under the specified restriction on \( p \). To make this proof shorter, assume harmlessly that the hypersequent \( G \) is empty.

(a) \( \not\models (\Gamma, \Pi \Rightarrow \Delta, \Sigma) \iff \) for some hs-interpretation \( M \) and \( M \)-valuation \( \nu \), \( \|\Delta\|_{M,\nu} - \|\Gamma\|_{M,\nu} < \|\Pi\|_{M,\nu} - \|\Sigma\|_{M,\nu} \iff \) (by the density of the set of all real numbers) for some hs-interpretation \( M \), \( M \)-valuation \( \nu \), and real number \( \xi \), \( \|\Delta\|_{M,\nu} - \|\Gamma\|_{M,\nu} < \xi - 1 < \|\Pi\|_{M,\nu} - \|\Sigma\|_{M,\nu} \).
(b) $\not\in (\Gamma, p \Rightarrow \Delta \mid \Pi \Rightarrow p, \Sigma) \iff$ for some hs-interpretation $M'$ and $M'$-valuation $\nu'$, $\|\Delta\|_{M',\nu'} - \|\Gamma\|_{M',\nu'} < |p|_{M'} - 1 < \|\Pi\|_{M',\nu'} - \|\Sigma\|_{M',\nu'}$.

It is easy to see that (a) implies (b): take $\nu' = \nu$ and define $M'$ to be the same as $M$ but set $|p|_{M'} = \xi$. $\square$

C. The admissibility of the nonstandard density rule for $G^0L\forall$

Remark 3 on p. 18 says that the proofs of Lemmas 7 and 13 can be easily combined to establish the admissibility for $G^0L\forall$ of the rule

\[
\frac{\mathcal{G} \mid \Gamma, p \Rightarrow \Delta \mid \Pi \Rightarrow p, \Sigma}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{(den)},
\]

provided the notion of a hypersequent is expanded by special variables interpreted by any real numbers, and $p$ is such a variable not occurring in the conclusion.

Let us prove the next lemma on the admissibility of a generalization of (den) for $G^0L\forall$, denoting by $p$ a special variable that can assume any real values under hs-interpretations.

**Lemma 29** (admissibility of a generalization of (den) for $G^0L\forall$). Suppose that $m \geq 1$, $n \geq 1$,

\[
\mathcal{H} = \left( \mathcal{G} \mid \left[ \Gamma_i, p \Rightarrow \Delta_i \right]_{i \in 1..m} \mid \left[ \Pi_j \Rightarrow p, \Sigma_j \right]_{j \in 1..n} \right),
\]

\[
\mathcal{H}' = \left( \mathcal{G} \mid \left[ \Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j \right]_{i \in 1..m} \mid \left[ j \in 1..n \right] \right),
\]

$p$ does not occur in $\mathcal{H}'$, and $\vdash_{G^0L\forall} \mathcal{H}$. Then $\vdash_{G^0L\forall} \mathcal{H}'$.

**Proof.** Take a ($G^0L\forall$-)proof $D$ of $\mathcal{H}$ and proceed by induction on the height of $D$.

1. Suppose that $\mathcal{H}$ is an axiom; i.e., $\vdash \mathcal{H}_{at}$. Without loss of generality we assume that

\[
\mathcal{H}_{at} = \left( \mathcal{G}_{at} \mid \left[ \Gamma_i, p \Rightarrow \Delta_i \right]_{i \in 1..k} \mid \left[ \Pi_j \Rightarrow p, \Sigma_j \right]_{j \in 1..l} \right),
\]

where $0 \leq k \leq m$ and $0 \leq l \leq n$. We put $\mathcal{H}'_{at} = (\mathcal{H}')_{at}$. Consider the following cases 1.1–1.4.
Case 1.1: \( k \neq 0 \) and \( l \neq 0 \). We have
\[
\mathcal{H}_{at}' = \left( \mathcal{S}_{at} \mid [\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j]_{i \in 1..k} \right).
\]
We want to show that \( \models \mathcal{H}_{at}' \). Suppose otherwise; i.e., for some hs-interpretation \( M \) and some \( M \)-valuation \( \nu \), there is no true sequent in \( \mathcal{S}_{at} \), and for all \( i \in 1..k \) and \( j \in 1..l \),
\[
\|\Delta_i\|_{M,\nu} - \|\Gamma_i\|_{M,\nu} < \|\Pi_j\|_{M,\nu} - \|\Sigma_j\|_{M,\nu}.
\]
By the density of the set \( \mathbb{R} \) of all real numbers, there exists \( \xi \in \mathbb{R} \) such that, for all \( i \in 1..k \) and \( j \in 1..l \),
\[
\|\Delta_i\|_{M,\nu} - \|\Gamma_i\|_{M,\nu} < \xi - 1 < \|\Pi_j\|_{M,\nu} - \|\Sigma_j\|_{M,\nu}.
\]
Define an hs-interpretation \( M_1 \) to be like \( M \), but set \( |p|_{M_1} = \xi \). Since \( p \) does not occur in \( \mathcal{S}_{at} \), \( \Gamma_i \), \( \Delta_i \) \( (i \in 1..k) \), \( \Pi_j \), \( \Sigma_j \) \( (j \in 1..l) \), we see that no sequent in \( \mathcal{H}_{at} \) is true under the hs-interpretation \( M_1 \) and \( M_1 \)-valuation \( \nu \). Hence \( \notmodels \mathcal{H}_{at} \), a contradiction.
Therefore \( \models \mathcal{H}_{at}' \), and so \( \mathcal{H}' \) is an axiom.

Case 1.2: \( k = 0 \) and \( l \neq 0 \). Then
\[
\mathcal{H}_{at} = \left( \mathcal{S}_{at} \mid [\Pi_j \Rightarrow p, \Sigma_j]_{j \in 1..l} \right)
\]
and \( \mathcal{H}_{at}' = \mathcal{S}_{at} \). Since \( p \) does not occur in \( \mathcal{S}_{at} \), \( \Pi_j \), \( \Sigma_j \) \( (j \in 1..l) \), and hs-interpretations can take \( p \) to negative real numbers whose absolute values are arbitrarily large, we conclude that \( \models \mathcal{H}_{at} \) implies \( \models \mathcal{S}_{at} \). Thus \( \models \mathcal{H}_{at}' \) and \( \mathcal{H}' \) is an axiom.

Case 1.3: \( k \neq 0 \) and \( l = 0 \). Then
\[
\mathcal{H}_{at} = \left( \mathcal{S}_{at} \mid [\Gamma_i, p \Rightarrow \Delta_i]_{i \in 1..k} \right)
\]
and \( \mathcal{H}_{at}' = \mathcal{S}_{at} \). Since \( p \) does not occur in \( \mathcal{S}_{at} \), \( \Gamma_i \), \( \Delta_i \) \( (i \in 1..k) \), and \( p \) can assume arbitrarily large values under hs-interpretations, we see that \( \models \mathcal{H}_{at} \) implies \( \models \mathcal{S}_{at} \). So \( \models \mathcal{H}_{at}' \) and \( \mathcal{H}' \) is an axiom.

Case 1.4: \( k = 0 \) and \( l = 0 \). Then \( \mathcal{H}_{at} = \mathcal{S}_{at} = \mathcal{H}_{at}' \). Thus \( \models \mathcal{H}_{at} \) means that \( \models \mathcal{H}_{at}' \) and \( \mathcal{H}' \) is an axiom.

2. It remains to consider the case where the root hypersequent \( \mathcal{H} \) in \( D \) is the conclusion of a rule application. But the argument for this case can be obtained from item 2 of the proof of Lemma 7 by replacing \( p_1 \) with \( p \). \( \square \)