Growth and relations in graded rings

D. I. Piontkovsky*

1 Introduction

Let $k$ be a field. We will call a vector $k$–space, a $k$–algebra, or $k$–algebra module *graded*, if it is $\mathbb{Z}_+$–graded and finite–dimensional in every component. For a such space $V$ (in particular, $V$ may be an algebra or a module) we denote by $V(x)$ its Hilbert series $\sum_{i \geq 0} \dim V_i x^i$. A graded algebra $A = A_0 \oplus A_1 \oplus \ldots$ is called *connected*, if its zero component $A_0$ is $k$; a connected algebra is called *standard*, if it is generated by $A_1$ and a unit.

The word "algebra" below denotes an associative graded algebra. All inequalities between Hilbert series are coefficient–wise, i. e., we write $\sum_i a_i t^i \geq \sum_i b_i t^i$ iff $a_i \geq b_i$ for all $i \geq 0$.

We are interested in the following situation. Suppose $A$ is a graded algebra, $\alpha \subset A$ is a subset consisting of homogeneous elements such that $\alpha$ minimally generates an ideal $I \triangleleft A$, and $B = A/I$. What can we say about relations of the Hilbert series $A(t)$, $B(t)$, and the generating function $\alpha(t)$?

If $A$ is a free associative algebra, then a partial answer is given by Golod–Shafarevich theorem (see [GSh] or Corollary 2 below). In particular, it follows that if the number of elements of $\alpha$ of every degree is sufficiently small, then $B$ is infinite-dimensional [GSh] (moreover, if in this case $A$ is standard, then $B$ grows exponentially [P]). On the other hand, V. E. Govorov (see [Gov] or Theorem 5 below) gives some estimates for a number of relations $\#\alpha$, whenever $B(t)$ is known (if all elements of $\alpha$ have the same degree).

Our first goal is to demonstrate that the equality cases in the Golod–Shafarevich theorem and in both of Govorov’s inequalities coincide (Theorem 6). Sets $\alpha$ such that equality holds are called *strongly free* [A1], or

*E-mail: piont@mech.math.msu.su
inert\cite{Hi}; they are non-commutative analogues of regular sequences in commutative ring theory \cite{Al}, \cite{Go}. If \(A\) is free, then the set \(\alpha \subset A\) of homogeneous elements is strongly free iff
\[
\text{gl. dim } A/id(\alpha) \leq 2.
\]
In the general case of an arbitrary connected algebra \(A\), the property of being strongly free is defined by certain conditions on Hilbert series \cite{Al}. E. g., for every set of homogeneous elements \(\alpha \subset A\) the following inequality holds:
\[
(B \ast k(\alpha))(t) \geq A(t);
\]
the equality holds iff \(\alpha\) is strongly free (where star denotes a free product of algebras).

As for our question in the case of an arbitrary algebra \(A\), we need to consider an asymptotical characteristics of the algebra’s growth. General graded algebras (even finitely generated ones) have exponential growth, so, for a such algebra \(A\), \(\text{GK–dim } A = \infty\). Let us introduce the following analogue of Gelfand–Kirillov dimension: if \(a_i = \dim A_i\), then define the exponent of growth of \(A\) by
\[
p(A) = \inf\{q > 0 \mid \exists c > 0 \forall n \geq 0 \ a_n \leq cq^n\}.
\]
At least for a finitely generated algebra \(A\), \(p(A)\) is finite. If \(A(t)\) is known, it is clear how to compute \(p(A)\): if \(r(A)\) is a radius of convergence of the series \(A(t)\), then
\[
p(A) = \lim \sqrt[n]{a_n} = r(A)^{-1}.
\]
Notice that \(p(A)\) depends of the grading on \(A\) (for example, if \(A = k\langle x, y \mid \deg x = \deg y = d \rangle\), then \(p(A) = \sqrt[2]{d}\) depends of \(d\), so it is not a “dimension” in the usual sense.

It is proved in \cite{A2} that over a field of zero characteristic there is no algorithm to decide whether or not a given quadratic subset of standard free algebra is strongly free. Using this fact and our criteria for strongly free sets, we proved in Section 4 the following. Let \(R\) be a finitely presented standard \(s\)-generated algebra with a set of relations \(\alpha\). Then for some particular \(s, \alpha\) for some rational numbers \(q, r\) there is no algorithm to decide, when \(\alpha\) is given, whether or not \(R(q) = r\). Moreover, for some integer \(n\) there is no algorithm to decide, when \(\alpha\) is given, whether or not \(p(R) = n\). It means
that in the simplest case that the algebra $A$ is free and standard, even the asymptotical version of our question is undecidable in general.

In Section 5 below we introduce a concept of extremal algebra: specifically, a graded algebra $A$ is called extremal, if for any proper quotient $B = A/I$

$$p(B) < p(A).$$

If $A$ is an extremal algebra, then it is prime (Theorem 13), and $p(A)$ is finite (Proposition 12). Any free product of two connected algebras is extremal (Theorem 13), as is any algebra that includes a strongly free set (Corollary 17).

Using extremality, we generalize Govorov’s inequality for an arbitrary connected algebra $A$ (Theorem 19): if $A(t)$ and $B(t)$ are known, we obtain an estimate for the generating function $\alpha(t)$. Also, we find a new characterizing of strongly free sets in terms of algebras’ growth (Proposition 18, Theorem 19): if a set $\alpha$ is strongly free, then not only the Hilbert series $A(t)$, but also the exponent of growth $p(A)$ is as small as possible. This characterizing generalizes the one of D. Anick [A1, theorem 2.6].

I am grateful to Professor E. S. Golod for fruitful discussions.

2 Golod–Shafarevich theorem and homology

Let $R = R_0 \oplus R_1 \oplus \ldots$ be a standard associative algebra. Suppose a set $\{x_1, \ldots, x_s\}$ is a basis in the space $R_1$. Then $R = F/I$, where $F = k\langle x_1, \ldots, x_s \rangle$ is a free associative algebra, $I \triangleleft F$ is a two–sided ideal generated by homogeneous elements of degrees $\geq 2$. Let $\alpha = \{f_1, f_2, \ldots\}$ be a minimal system of generators of $I$, and let $u = k\alpha \approx I/PI + IP$ be a vector space generated by $\alpha$ (here $P = F_1 \oplus F_2 \oplus \ldots$ is an augmentation ideal of $F$).

Suppose

$$0 \leftarrow k \leftarrow^{d_0} R \otimes H_0 \leftarrow^{d_1} R \otimes H_1 \leftarrow^{d_2} \ldots$$

(1)

be a minimal free left $R$–resolution of $k$; here $H_i \approx \text{Tor}_i^R(k, k) = H_i(R)$, i. e.

$$H_0 = k, H_1 \approx R_1, H_2 \approx u$$

etc.
Let $\Omega^i = \text{Coker} \ d_{i-1}$ be the $i$–th syzygy module. Since the Hilbert series of a tensor product is a product of the factors’ Hilbert series, using the Euler formula for the exact sequence

$$0 \leftarrow k \xleftarrow{d_0} R \otimes H_0 \xleftarrow{d_1} \ldots \xleftarrow{d_i} R \otimes H_i \leftarrow \Omega^i \leftarrow 0$$

we obtain the following

**Proposition 1** There is an equality of formal power series

$$R(x)(1 - sx + u(x) - H_3(x) + \ldots + (-1)^i H_i(x)) = 1 + (-1)^i \Omega^i(x).$$

In particular, there are inequalities

$$R(x)(1 - H_0(x) + \ldots - H_{2i-1}(x)) \geq 1,$$

$$R(x)(1 - H_0(x) + \ldots + H_{2i}(x)) \leq 1;$$

equalities hold iff gl. dim $R \leq 2i - 1$ (respectively, gl. dim $R \leq 2i$).

**Corollary 2 (Golod–Shaferevich theorem)** There is an inequality

$$R(x)(1 - sx + u(x)) \geq 1. \quad (2)$$

Equality holds iff gl. dim $R \leq 2$.

The inequality is proved in [GSH], the equality condition is proved in [A1]. A non-empty set $\alpha$ such that the equality above holds is called strongly free, or inert in $F$; since in our case $\alpha$ is not empty, these properties are equivalent to the equality gl. dim $R = 2$.

The equality $R(x)(1 - sx + u(x)) = 1 - \Omega^3(x)$ shows that for a finitely presented algebra $R$ we can compute the Hilbert series $R(x)$ whenever $\Omega^3(x)$ is known. So it is interesting to study the module $\Omega^3$.

**Proposition 3** Denote by $L = F\alpha \triangleleft F$ the left ideal generated by $\alpha$. There is an isomorphism of left $R$–modules

$$\Omega^3 \cong \text{Tor}_1^F(R, P/L).$$
**Proof** Suppose \( \{u_1, \ldots, u_s \mid \deg u_i = 1\} \) is a basis of the space \( H_1 \) and \( \{r_1, r_2, \ldots \mid \deg r_i = \deg f_i\} \) is a basis of \( H_2 \). Then we may assume that the map \( d_2 \) in the resolution (\( \Pi \)) is given by the following way: if \( f_i = \sum_{j=1}^{n} a_j^i x_j \), then \( d_2(r_i) = \sum_{j=1}^{n} \overline{a_j^i} u_j \), where the overbar denotes the image of an element of \( F \) in \( R \).

On the other hand, since any left ideal in a free algebra is a free module, taking the long sequence of graded Tor \( \mathbb{F}^* (R, P/L) \) we obtain:

\[
0 \rightarrow \text{Tor}_1^F (R, P/L) \rightarrow R \otimes_F L \xrightarrow{\phi} R \otimes_F P \rightarrow R \otimes_F P/L \rightarrow 0.
\]

Here \( R \otimes_F L \cong R \otimes u, R \otimes_F P \cong R \otimes R_1 \). So the map \( \phi \) induced by the inclusion \( L \hookrightarrow P \) coinsides with the map \( d_2 \) above. Therefore,

\[
\text{Tor}_1^F (R, P/L) \cong \text{Ker} d_2 \cong \Omega^3.
\]

**Corollary 4** The non-empty set \( \alpha \) is strongly free iff \( \text{Tor}_1^F (R, P/L) = 0 \).

**Remark**

All results of this section hold for an arbitrary connected algebra \( R \). The only change is to replace in all formulae the term \( sx \) by a generating function in the algebra’s generators \( \sum_{i \geq 1} t^{\deg x_i} \).

### 3 Estimates for the number of relations

We keep the notation of the previous section.

Suppose that the ideal \( I \) is minimally generated by \( t \) \( (t > 0) \) elements of degree \( l \), i.e., \( \alpha = \{f_1, \ldots, f_t \mid \deg f_i = l\} \). Let \( a_i \) denote the dimension of the space \( R_i \):

\[
R(x) = \sum_{i \geq 0} a_i x^i.
\]

In the situation above, V. E. Govorov proved the following theorem.
Theorem 5 ([Gov]) The series $R(x)$ converges for $x = s^{-1}$. The following inequalities hold:

$$t \geq \frac{s^l}{R(s^{-1})} \quad (3)$$

and

$$t \geq \frac{sa_{n-1} - a_n}{a_{n-1}} \quad (4)$$

for all possible $n$. Equality holds in (4) for all possible $n$ if and only if $Tor^F_1(R, P/L) = 0$.

The following theorem shows that the equality holds simultaneously in (3), (4), and the Golod–Shafarevich theorem.

Theorem 6 The following conditions are equivalent:

(i) The set $\alpha$ is strongly free.

(ii) Equality holds in (3).

(iii) Equality holds in (4) for all possible $n$.

Proof

The implications (i) $\Leftrightarrow$ (iii) follow immediately from Corollary 4 and Theorem 5. Let us prove (i) $\Leftrightarrow$ (ii).

Let us take $x = s^{-1}$ in (3). Since $u(x) = tx^l$, we have the inequality

$$R(s^{-1})ts^{-l} \geq 1,$$

which is equivalent to (3). So, if $\alpha$ is strongly free, then equality holds in (3).

Conversely, if $\alpha$ is not strongly free, then the inequality of formal power series (2) is strict, i.e., the following inequality holds

$$R(x)(1 - sx + tx^l) \geq 1 + ax^n,$$

where $n \geq 0, a > 0$. Put $x = s^{-1}$. We obtain

$$R(s^{-1})ts^{-l} \geq 1 + as^{-n} > 1,$$

or

$$t > \frac{s^l}{R(s^{-1})}.$$

So, the inequality (3) is strict too.
4 Radii of convergence of Hilbert series: non-existence of algorithms

For a graded algebra $A$, let $r(A)$ denotes the radius of convergence of the series $A(t)$.

The following properties of radii of convergence are clear and well known.

**Proposition 7** Let $A$ be a graded algebra.

(i) $r(A) = \infty$ iff $A$ is finite-dimensional.

(ii) If $A$ is finitely generated, then $r(A) > 0$.

(iii) $r(A) = 1$ iff $A$ is not finite-dimensional and $A$ has sub-exponential growth.

(iv) If $B$ is either a subalgebra or a quotient algebra of $A$ with the induced grading, then $r(B) \geq r(A)$.

(v) If $r(A) > 0$, then $\lim_{t \to r(A)} A(t) = \infty$. So, if $A$ is connected, then the function $f(t) = A(t)^{-1}$ is continuous on $[0, r(A)]$ with $f(0) = 1, f(r(A)) = 0$.

Under the notation of previous sections, let $l = 2$. It is shown by D. Anick [A2, Theorem 3.1] that over a field of zero characteristic for some positive integers $s$ and $t$, there is no algorithm which, when given a set $\alpha \subset F$ of $t$ homogeneous quadratic elements, always decides in a finite number of steps whether or not $\alpha$ is strongly free. We will call such a pair of integers $(s, t)$ undecidable.

**Lemma 8** Let $l$ be a positive integer, and let $(s, t)$ be an undecidable pair. Then the pair $(s + l, t)$ is undecidable.

**Proof**

Let $G = F \ast k\langle x_{s+1}, \ldots, x_{s+l} \rangle$ be a free algebra of rank $s + l$. Then a set $\alpha \in F$ is strongly free in $F$ iff it is strongly free in $G$. So there is no algorithm to recognize it.

**Lemma 9** Let $s$ be an even integer. If the pair $(s, t)$ is undecidable, then the pair $(2s, s^2/4)$ is undecidable too.

**Proof**
By [A4], a quadratic strongly free set in F consisting of q elements does exist iff $4q \leq s^2$. So $t \leq s^2/4$. Let $\beta$ be a quadratic strongly free set in the algebra $G = k\langle x_{s+1}, \ldots, x_{2s} \rangle$ consisting of $s^2/4 - t$ elements. Then the set $\alpha \cup \beta$ is strongly free in the algebra $F \ast G = k\langle x_1, \ldots, x_{2s} \rangle$ if and only if the set $\alpha$ is strongly free in $F$.

**Corollary 10** For large enough integer $d$, the pair $(4d, d^2)$ is undecidable.

**Theorem 11** Let $\text{char } k = 0$. Let us denote by $F_s$ the free associative algebra of rank $s$ with standard grading.

(i) Let $s, t$ be an undecidable pair of integers. Then there is no algorithm which, when given a set $\alpha \subset F_s$ of $t$ homogeneous quadratic elements, always decides whether or not the equality

$$R(s^{-1}) = \frac{s^2}{t}$$

holds, where $R = F_s/id(\alpha)$.

(ii) For some positive integers $s$, $t$, and $q$, there is no algorithm which, when given a set $\gamma \subset F_s$ of $t$ homogeneous quadratic elements, always decides whether or not the equality

$$r(R) = q^{-1}$$

holds, where $R = F_s/id(\gamma)$. For large enough integer $d$, we can put $s = 64d$, $t = 241d^2$, and $q = 60d$.

**Proof**

The statement (i) follows from Theorem [I] and Corollary [II]. To prove (ii), let $d$ be an integer such that the pair $(60d, 15^2d^2)$ is undecidable. Let $\alpha \in F_{60d}$ be a quadratic set consisting of $15^2d^2$ elements, and let $B = F_{60d}/id(\alpha)$. Let us denote by $A$ the standard algebra $k\{1, y_1, \ldots, y_{4d}\} = k\langle y_1, \ldots, y_{4d} \mid y_i y_j = 0, 1 \leq i, j \leq 4d \rangle$. Then the algebra $C = k\langle x_1, \ldots, x_{64d} \rangle/id(\gamma)$ is isomorphic to $A \ast B$, where $\gamma = \alpha \cup \{x_i x_j = 0 \mid 60d + 1 \leq i, j \leq 64d\}$. So $C$ is an algebra with $s = 64d$ generators and $t = (4d)^2 + 15^2d^2 = 241d^2$ quadratic relations. It is sufficient to prove that the set $\alpha$ is strongly free in $F_{60d}$ if and only if $r(C) = (60d)^{-1}$.

We have

$$C^{-1}(x) = B^{-1}(x) + A^{-1}(x) - 1 = B^{-1}(x) + \frac{1}{1 + 4dx} - 1 = B^{-1}(x) - \frac{4dx}{1 + 4dx}.$$
If \( \alpha \) is strongly free in \( F_{60d} \), then by Corollary 2 we have \( B^{-1}(x) = 1 - 60dx + 225d^2x^2 \), so

\[
C^{-1}(x) = 1 - 60dx + 225d^2x^2 - \frac{4dx}{1 + 4dx} = \frac{(1 - 60dx)(1 - 15d^2x^2)}{1 + 4dx}.
\]

By Proposition 7, we obtain \( r(C) = (60d)^{-1} \).

Now let \( \alpha \) is not strongly free in \( F_{60d} \). By Theorem 3, we have \( B^{-1}((60d)^{-1}) < 225d^2/(60d)^2 = 1/16 \). So

\[
C^{-1}((60d)^{-1}) = B^{-1}((60d)^{-1}) - \frac{4d(60d)^{-1}}{1 + 4d(60d)^{-1}} < 1/16 - \frac{1}{1 + 1/(4d(60d)^{-1})} = 0.
\]

We obtain \( r(C) \neq (60d)^{-1} \), contradicting Proposition 7. (v).

5 Algebras of extremal growth

Now we introduce the following concept.

**Definition 1** A graded algebra \( A \) is said to be extremal, if for every nonzero homogeneous ideal \( I \triangleleft A \) we have \( r(A/I) > r(A) \).

We will discuss some properties of extremal algebras.

**Proposition 12** If \( A \) is an extremal algebra, then \( 0 < r(A) < \infty \).

**Proof**

Let us proof the other inequality.

Suppose that \( r(A) = 0 \). Let \( a \in A \) be a nonzero homogeneous element of degree \( d > 0 \), let \( I \) be an ideal generated by \( a \), and let \( B = A/I \). Since \( A \) is extremal, \( r(B) > 0 \).

Let \( F = k\langle c | \deg c = d \rangle \). By the obvious inequality of Hilbert series

\[(B \ast F)(t) \geq A(t),\]

\( r(B \ast F) \) must be equal to 0.

On the other hand,

\[(B \ast F)(t) = b(t)^{-1} - dt,\]

where...
where \( b(t) \) is equal to \( B(t) \) (resp., \( B(t) + 1 \)), if \( B \) is unitary (resp., non-unitary, i.e. \( B_0 = 0 \)). Since the right side is an analytical function in a neighborhood of zero, and this function takes 0 into 1, then in a neighborhood of zero its image does not contain 0. So the function \((B \ast F)(t)\) is analitycal in a neighborhood of zero. Therefore \( r(B \ast F) > 0 \).

**Theorem 13** Let \( A \) be an extremal algebra. Then \( A \) is prime.

**Proof**

Obviously, it is sufficient to prove that for any two nonzero homogeneous ideals \( I \) and \( J \) of \( A \), \( I \cdot J \neq 0 \). Without loss of generality, we can assume that \( J \) is a principal ideal generated by an element \( a \) of a degree \( h \).

Suppose \( I \cdot J = 0 \). Let \( B = A/I, C = A/J \), and let \( A = I \oplus V \), there \( V \) is a graded vector space. We have

\[
J = ka + Aa + aA + AaA = ka + Va + aV + VaV.
\]

Therefore, there is an inequality of Hilbert series

\[
J(t) \leq t^h + 2t^hV(t) + t^hV(t)^2,
\]

or

\[
A(t) - C(t) \leq t^h(B(t) + 1)^2.
\]

So, we have

\[
A(t) \leq C(t) + t^h(B(t) + 1)^2.
\]

Thus, for radii of convergence we obtain

\[
r(A) \geq \min\{r(B), r(C)\},
\]

contradicting the extremality of \( A \).

**Remark**

In fact, we have proved that for any two homogeneous ideals \( I, J \) of a graded algebra \( A \), if \( I \cdot J = 0 \), then \( r(A) = \min\{r(A/I), r(A/J)\} \).

**Corollary 14** Let \( A \) be a (non-graded) locally finite filtered algebra such that the associated graded ring \( \text{gr} A \) is extremal. Then \( A \) is prime.
Now, let us consider examples of extremal algebras. In fact, the extremality of nontrivial free algebras is proved by V. E. Govorov in [Gov]. We add the following

**Theorem 15** Let $A, B$ be non-trivial connected algebras such that $r(A) > 0$ and $r(B) > 0$. Then the algebra $A \ast B$ is extremal.

**Proof of Theorem 15**
Let $C = A \ast B$. By the formula

$$C(t)^{-1} = A(t)^{-1} + B(t)^{-1} - 1,$$

the function $C(t)^{-1}$ is analytical and nonzero in a neighborhood of zero, so $r(C) > 0$. For $t \in (0, r(C)]$, we have $0 \leq A(t)^{-1} < 1$ and $0 \leq B(t)^{-1} < 1$. Since $A(r(C))^{-1} + B(r(C))^{-1} - 1 = 0$, we obtain $A(r(C))^{-1} > 0$ and $B(r(C))^{-1} > 0$; hence $r(C) < \min\{r(A), r(B)\}$.

It follows from the standard Gröbner bases arguments that we may assume the algebras $A$ and $B$ to be monomial. (Indeed, if we fix an order on monomials, then, denoting by $\mathcal{R}$ the associated monomial algebra of an algebra $R$, we have $\overline{C} = \overline{A} \ast \overline{B}$; moreover, if $I$ is a homogeneous ideal in $C$, then there exists an ideal $J \triangleleft \overline{C}$ such that $r(C/I) \geq r(\overline{C}/J)$.)

Now, let $A = k\langle x \rangle/I$ and $B = k\langle Y \rangle/J$, there $X, Y$ are homogeneous sets minimally generating algebras $A$ and $B$, and $I, J$ are ideals generated by monomials of elements of $X$ and $Y$. If $A$ and $B$ are two–dimensional, i. e., $A \cong k\langle x | x^2 = 0 \rangle$ and $B \cong k\langle x | x^2 = 0 \rangle$, then there is nothing to prove. So, we can assume that $\dim B \geq 3$.

Let $S \triangleleft C$ be a nonzero principal ideal generated by a non-empty monomial $m$: it is sufficient to prove that $r(C/S) > r(C)$. We will say that a non-empty monomial $a$ is an overlap of two monomials $b, c$ if there are non-empty monomials $f, g$ such that $b = fa, c = ag$. Now we need the following

**Lemma 16** Let $A, B$ be connected monomial algebras such that $\dim_k B \geq 3$, where $B$ is minimally generated by the set $Y = \{y_i\}_{i \in \Gamma}$, and let $S$ be a monomial ideal in the algebra $C = A \ast B$. Then $S$ contains a nonzero monomial $p$ with the following properties: all monomials $p_{ij} = y_i y_j, \ i, j \in \Gamma$ are nonzero, and, moreover, for any two monomials $p_{ij}$ and $p_{kl}$, there are no overlaps in the case $j \neq k$ and there is the unique overlap $y_j$ in the case $j = k$. 
Proof of Lemma 16

It is obvious that $S$ contains a nonzero monomial $n$ such that $n = xn'x$, where $n'$ is a monomial, $x \in X$, where $X$ is the set of generators of $A$. To construct such a monomial $p$, let us consider two cases.

Case 1 Let $\#Y = 1$, i.e., $Y = \{y\}$. Since $\dim B \geq 3$, then $y^2 \neq 0$. Let $l \geq 0$ be the largest integer satisfying $n = (xy^2)^ln_1$, where $n_1$ is a monomial. Put $p = y(xy^2)^qn_1(yx)^q$, where $q > \max\{l, \text{len} n_1 + 3\}$; then $p_{11} = y^2(xy^2)^qn_1(yx)^qy$.

Assume that a monomial $a$ is an overlap for the pair $p_{11}, p_{11}$. Then there exist non-empty monomials $c, d$ such that $p_{11} = ca = ad$. Hence $a$ has the form $(y^2x)^qf(xy)^q$, where $f$ is a monomial; therefore, $\text{len} c = \text{len} d = \text{len} p_{11} - \text{len} a < q$. Since $p_{11} = y^2(xy^2)^qn_1(yx)^qy = c(y^2x)^qf(xy)^q$, $c$ has the form $(y^2x)^r$ for some $r > 0$. By the maximality of $l$, we have $r = 1$, so $\text{len} c = \text{len} d = 3$. On the other hand, it follows from the equality $p_{11} = ad$ that $d$ has the form $(xy)^l$ for some $l$, so $\text{len} d$ must be even.

Case 2 Let $\#Y \geq 2$, i.e., $Y = \{y_1, y_2, \ldots\}$. Put $p = (xy_1)^qn(y_2x)^q$, where $q > \text{len} n$. For some monomials $p_{ij}$ and $p_{kl}$, suppose $a$ is an overlap such that $\text{len} a \geq 2$. Then there exists an overlap of monomials $p, p$. It means that the set $\{p\}$ is not combinatorial free, so, it is not strongly free in a free associative algebra generated by the set $X \cup Y$ [A1]. By [HL, Proposition 3.15], this means that there exist a non-empty monomial $a$ and a monomial $b$ such that $p = aba$

(at least in the monomial case, the proof in [HL] did not really use the assumption char $k = 0$). Then $a$ has the form $a = (xy_1)^qc(y_2x)^q$ for a monomial $c$, so $\text{len} a \geq 4q$ and $\text{len} p \geq 8q$, contradicting the choice of $q$.

Returning to the general proof, let $P \triangleleft C$ be an ideal generated by all of the monomials $p_{ij}$, and let $D = C/P$. Since $P \subset S$, it is sufficient to prove that $r(D) > r(C)$. To prove this, we will compute the homology of the algebra $D$ and obtain its Hilbert series as the Euler characteristic.

Recall how to compute homologies of a monomial algebra (see details in [A3]; we use the terminology of [U]). Suppose $F$ is a free associative algebra generated by a set $X$, $I \triangleleft F$ is an ideal minimally generated by a set of monomials $U$, and $M$ is the quotient algebra $F/I$. Let us define a concept of a chain of a rank $n$ and its tail.
For \( n = 0 \), every generator \( x \in X \) is called a chain of rank 0; it coincides with its tail. For \( n > 0 \), a monomial \( f = gt \) is called a chain of rank \( n \) and \( t \) is called its tail, if the following conditions hold: (i) \( g \) is a chain of rank \( n - 1 \); (ii) if \( r \) is a tail of \( g \), then \( rt = vu \), where \( v, u \) are monomials and \( u \in U \); (iii) excluding the word \( u \) as the end, there are no subwords of \( rt \) lying in \( U \).

Let us denote by \( C_n^M \) the set of chains of rank \( n \); for example, \( C_0^M = X \) and \( C_1^M = U \). Then for all \( n \geq 0 \) there are the following isomorphisms of graded vector spaces:

\[
kC_n^M \simeq \text{Tor}_n^M(k, k).
\]

Letting \( c_j^i \) denote the number of chains of degree \( i \) having a rank \( j \), consider the generating function

\[
C^M(s, t) = \sum_{i \geq 0} \sum_{j \geq 0} c_j^i s^j t^i.
\]

Arguing as in Proposition 1, taking the Euler characteristic of the minimal resolution, we obtain

\[
M(t)^{-1} = 1 - C^M(-1, t);
\]

the formal power series in the right side does exist since every vector space \( \text{Tor}_n^M(k, k) \) is concentrated in degrees \( \geq n \).

By definition, for all \( i \geq 0 \), we have

\[
C_i^C = C_i^A \cup C_i^B
\]

and \( C_0^D = C_0^C \). Therefore,

\[
C^C(s, t) = C^A(s, t) + C^B(s, t),
\]

so

\[
C^D(s, t) = C^A(s, t) + C^B(s, t) + C'(s, t),
\]

there the set \( C' \) consists of chains that have a subword \( p \). Thus the set \( C'_0 \) is empty, and \( C'_1 = \{p_{ij}\} \).

Let us prove that the set \( C' \) consists of all monomials of the form

\[
c_1pc_2\ldots pc_n
\]

for \( n \geq 2 \), where \( c_1, \ldots, c_n \in C^B \). It is clear that all these monomials are chains of \( C' \). Let us prove the converse.
Since \( C^B_0 = Y \), this is obvious for chains of rank 1. Now, let \( f = gt \in C'_n \), where \( g \) is a chain of lesser rank and \( t \) is a tail of the chain \( f \). Let \( r \) be the tail of \( g \). By induction, we may assume that \( g \in C^B \) or \( g \) has the form \((5)\); in the second case, \( r \) is the tail of \( c_n \), or has the form \( py_i \), where \( y_i = c_n \in Y \). If \( t \) is a word of the alphabet \( Y \), then \( gt \in C^B \), or \( c_nt \in C^B \), so \( f \) has the desired form. Otherwise, \( t \) must contain a subword equal to \( p \); hence, \( t = py_j \) for some \( j \). Thus,

\[ f = c_1pc_2p \ldots pc_npy_j. \]

Now, let us compute the generating function. Notice that if a chain \( f \) has the form \((5)\), then the rank of \( f \) is equal to \( k+n-1 \), where \( k \) is the sum of ranks of the chains \( c_1, \ldots, c_n \). Let \( \deg p = b \). By \((3)\), we have

\[ C'(s, t) = \sum_{i \geq 1} (st^b)^i \left( C^B(s, t) \right)^{i+1} = \frac{st^b \left( C^B(s, t) \right)^2}{1 - st^b C^B(s, t)}. \]

Put \( q(t) = 1 - B(t)^{-1} = C^B(t, -1) \). Obviously, for \( 0 < t \leq r(B) \), we have \( q(t) > 0 \).

We obtain

\[ D(t)^{-1} = C(t)^{-1} - C'(-1, t), \]

hence,

\[ D(r(C))^{-1} = -C'(-1, r(C)) = \frac{r(C)^b q(r(C))^2}{1 + r(C)^b q(r(C))} > 0. \]

Since \( r(D) \geq r(C) \) and \( D(r(C)) > 0 \), we obtain \( r(D) > r(C) \). This completes the proof of Theorem 15.

**Corollary 17** Let \( A \) be a connected algebra such that \( r(A) > 0 \). If there exists a strongly free set in \( A \), then \( A \) is extremal.

**Proof**

By [A, Lemma 2.7], any subset of a strongly free set is strongly free; so, there is a strongly free element \( f \in A \). Let \( L \) be the ideal generated by \( f \), and let \( B = A/L \).

If \( A \) is generated by \( f \), then, since every strongly free set generates a free subalgebra, \( A = k\langle f \rangle \); hence, every proper quotient of \( A \) is finitely-dimensional, so \( A \) is extremal. Otherwise, the algebra \( B \) is not trivial, so the algebra \( C = B * k\langle g | \deg g = \deg f \rangle \) is extremal by Theorem 15.
By [11, Section 2], there is an isomorphism of graded vector spaces \( \rho : C \to A \) having the following properties:

(i) the restriction of \( \rho \) to \( B \) is a right inverse to the canonical projection \( A \to B \);

(ii) \( \rho(g) = f \), and \( \rho(a_1 ga_2 \ldots ga_n) = \rho(a_1)f \rho(a_2) \ldots f \rho(a_n) \).

Suppose \( m \in A \) is an arbitrary homogeneous element, \( m = \rho(c) \), and \( I = AmA \) is the ideal generated by \( m \). We need to prove that \( r(A/I) > r(A) \).

Indeed, let \( c' = gcg \), let \( m' = fmf = \rho(c') \), and let \( J \triangleleft A \) (respectively, \( K \triangleleft C \)) be the ideal generated by \( m' \) (resp., \( c' \)). For every \( a, b \in C \) we get

\[
\rho(ac'b) = \rho(aagcb) = \rho(a)f \rho(c)f \rho(b) = \rho(a)m'\rho(b).
\]

Therefore \( \rho(K) \subset J \), so \( (A/J)(t) \leq (C/K)(t) \). We obtain

\[
r(A/I) \geq r(A/J) \geq r(C/K) > r(C) = r(A).
\]

6 How a quotient algebra may grow?

Suppose that \( A \) is a connected algebra such that \( r(A) > 0 \), \( S \subset A \) is a non-empty set of homogeneous elements minimally generating an ideal \( I = ASA \), and \( B = A/I \). Let \( C = B \ast k\langle S \rangle \), and let

\[
D = A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots
\]

with the induced grading; then \( D(t) = A(t) \). By [11, Theorem 2.4], we have a epimorphism

\[
f : C \to D,
\]

which is an isomorphism iff the set \( S \) is strongly free. Since \( C \) is either a free algebra or a free product of non-trivial algebras, it is extremal; so, we obtain the following

**Proposition 18** Using the notation above,

\[
r(A) \geq r(C).
\]

Equality holds if and only if the set \( S \) is strongly free.
Now, by the formula for the Hilbert series of a free product, we have

\[ C(t)^{-1} = B(t)^{-1} - S(t). \]

Since \( C \) is extremal, the series \( B(t) \) and \( S(t) \) converge for \( t \in [0, r(C)] \), so

\[ B(r(C))^{-1} - S(r(C)) = 0, \]

or

\[ B(r(C))S(r(C)) = 1. \]

Since \( r(A) \geq r(C) \), we have

\[ B(r(A))S(r(A)) \geq 1 \]

(where \( \infty > 1 \); the equality holds iff \( r(A) = r(C) \).

Thus we obtain:

**Theorem 19** Using our notation,

\[ B(r(A))S(r(A)) \geq 1, \]

and the following conditions are equivalent:

(i) the equality above holds;

(ii) \( r(A) = r(C) \);

(iii) the set \( S \subset A \) is strongly free.

In particular, if the set \( S \) consists of \( t \) elements of degree \( l \), then we have

\[ t \geq B(r(A))^{-1}r(A)^{-l}, \]

where equality holds iff \( S \) is strongly free. This estimate generalizes Gov-ovrov's inequality (3).

**Remark**

Suppose that \( \text{char} \ k = 0 \), the algebra \( A \) is free of rank \( s \), and \( \alpha \) is a set of \( t \) quadratic elements. If the pair \( (s, t) \) is undecidable, then \( C \) is finitely presented and connected (but non-standard) algebra such that there is no algorithm do decide whether or not \( r(C) = 1/s \).
References

[A1] D. Anick, *Non-commutative graded algebras and their Hilbert series*, J. Algebra, **78** (1982), p. 120–140

[A2] D. Anick, *Diophantine equations, Hilbert series, and undecidable spaces*, Ann. Math., **122** (1985), p. 87–112

[A3] D. Anick, *On the homology of associative algebras*, Trans. Amer. Math. Soc., **296** (1986), 2, p. 641–659

[A4] D. Anick, *Generic algebras and CW–complexes*, Proc. of 1983 Conf. on algebra, topol. and K–theory in honor of John Moore. Princeton Univ., 1988, p. 247–331

[Gol] E.S. Golod, *Non-commutative complete intersections and homologies of Shafarevich complex*, Uspekhi Mat. Nauk, **52** (1997), 4, p. 201–202 [Russian] (will be translated in *Russian Math. Surveys*)

[Gov] V.E. Govorov, *Graded algebras*, Mat. zametki, **12** (1972), 2, p. 197–204 [Russian]; Math. Notes, **12** (1972), p. 552–556 [English]

[GSh] E.S. Golod, I.R. Shafarevich, *On a tower of class fields*, Izv. AN SSSR, Ser. mat., **28** (1964), 2, p. 261–272 [Russian]

[HL] S. Halperin, J.-M. Lemaire, *Suites inertes dans les algèbres de Lie graduées*, Math. Scand., **61** (1987), 1, p. 39–67

[P] D. I. Piontkovsky, *On the growth of graded algebras with a small numbers of defining relations*, Uspekhi Mat. nauk, **48** (1993), 3, p. 199–200 [Russian]; Russian Math. Surveys, **48** (1993), 3, p. 211–212 [English]

[U] V. A. Ufnarovsky, *Combinatorial and asymptotical methods in algebra*, Sovr. probl. mat., Fund. napr., **57** (1990), p. 5–177 [Russian]