Cohomology and deformations of $O$-operators on Hom-associative algebras

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Abstract

In this paper, we introduce the cohomology theory of $O$-operators on Hom-associative algebras. This cohomology can also be viewed as the Hochschild cohomology of a certain Hom-associative algebra with coefficients in a suitable bimodule. Next, we study infinitesimal and formal deformations of an $O$-operator and show that they are governed by the above-defined cohomology. Furthermore, the notion of Nijenhuis elements associated with an $O$-operator is introduced to characterize trivial infinitesimal deformations.

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1. Introduction

The concept of Rota-Baxter operators on associative algebras was introduced in the 1960th by G. Baxter [1] in the study of fluctuation theory in Probability and then the theory was established by G.-C. Rota [2] mainly in connection with Combinatorics. Motivated by the study of associative analogues of Poisson structures, Uchino introduced the notion of generalized Rota-Baxter operator (called also $O$-operator) on a bimodule over an associative algebra in [47]. In [48], Uchino also introduced the notion of a strong Maurer-Cartan equation on a bimodule $(V; \ell, r)$. A solution of the strong Maurer-Cartan equation can be seen as the associative analogue of a closed 2-form complementary to a Poisson structure. Rota-Baxter operators and their generalization $O$-operators are becoming rather popular due to their interest in various domain like nonassociative algebra and applied algebra. Their deformations and their controlling cohomologies have been studied for Lie algebras in [46] and for associative algebras in [20], see also [13] [35].

The first instances of Hom-type algebras appeared in $q$-deformations of algebras of vector fields in physics contexts. It turns out that the Jacobi condition is no longer satisfied in the $q$-deformations of Witt algebra and Virasoro algebra using Jackson derivative. Hartwig, Larsson and Silvestrov [28] introduced a so called Hom-Lie algebras, generalizing Lie algebras and where the usual Jacobi identity is twisted by a homomorphism. The pending Hom-associative algebras were introduced in [12]. Various classical structures were generalized within this framework. A huge research activity was dedicated to Hom-type algebras, due in part to the prospect of having a general framework in which one can produce many types of natural deformations which are of interest to both mathematicians and physicists. Cohomology and deformations of Hom-Lie and Hom-associative algebras were studied in [9, 10, 12]. Notice that Cohomology and deformations of $O$-operators on Hom-Lie algebras were discussed in [3].

The purpose of this paper is to introduce and study a cohomology controlling deformations of $O$-operators on Hom-associative algebras. Therefore, we consider one parameter formal deformations and provide relevant results about infinitesimal deformations and general $n$-order deformations. In addition, infinitesimal deformations are characterized using Nijenhuis elements. In Section 2, relevant definitions and preliminary results are presented. They deal with Hom-associative algebras and Hom-dendriform algebra, bimodule structures, Rota-Baxter structures, $O$-operators, Nijenhuis operators, cohomology and graded Lie algebras. Main results of this paper are given in Section 3 in which we provide Cohomologies of $O$-operators on Hom-associative algebras and their deformations are considered in Section 4.
2. Preliminaries

In this section, we recall the relevant definitions and the notion of $O$-operator on Hom-associative algebra. We also recall Hom-dendriform structures induced from $O$-operators. Finally, we construct a graded Lie algebra with explicit graded Lie bracket whose Maurer-Cartan elements are given by $O$-operators.

2.1. Hom-associative and Hom-dendriform algebras

Definition 2.1. A Hom-associative algebra is a Hom-module $(A,\alpha)$, consisting of a $K$-vector space $A$ and a linear map $\alpha$, together with a bilinear map $\mu : A \times A \to A$, $(a, b) \mapsto a \cdot b$, that satisfies

$$\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c), \quad \text{for all } a, b, c \in A.$$  

A Hom-associative algebra is called multiplicative if $\alpha(a \cdot b) = \alpha(a) \cdot \alpha(b)$. By a Hom-associative algebra, we shall always mean a multiplicative one, unless otherwise specified.

Definition 2.2. Let $(A, \mu, \alpha)$ be a Hom-associative algebra and $M$ a vector space. Let $\phi \in gl(M)$ and $l : A \otimes M \to M$, $r : M \otimes A \to M$ be two linear maps. We say that the tuple $(M, l, r, \phi)$ is a bimodule of $A$ if for any $x, y \in A$ and $v \in M$

$$\phi l(x, v) = l(\alpha(x), \phi(v)), \quad \phi r(v, x) = r(\phi(v), \alpha(x)),$$

$$l(xy, \phi(v)) = l(\alpha(x), l(y, v)), \quad r(\phi(v), xy) = r(r(v, x), \alpha(y)),$$

$$l(\alpha(x), r(v, y)) = r(l(x, v), \alpha(y)).$$

Unless otherwise stated, $l(x, v)$ (resp. $r(x, v)$) may be denoted, depending on the situation, by $xy$ or $l(x)v$ (resp. $vx$ or $r(x)v$).

Recall that if we consider a Hom-associative algebra $(A, \mu, \alpha)$. Then $(M, l, r, \phi)$ is a bimodule of $A$ if and only if the vector space $A \oplus M$ (the split extension of $A$ by $M$) carries a Hom-associative structure with product given by

$$(x, u) : (y, v) \mapsto (x \cdot y, l(x, u) + r(v, y)), \quad \forall x, y \in A, u, v \in M,$$

$$(\alpha, \phi)(x, u) = (\alpha(x), \phi(u)).$$

This is called the semi-direct product of $A$ with $M$.

In [3, 4] the authors define a Hochschild-type cohomology of a Hom-associative algebra and study the one parameter formal deformation theory for these type of algebras. Let $(A, \alpha, \mu)$ be a Hom-associative algebra and $(M, l, r, \phi)$ be a bimodule. For each $n \geq 1$, the group of $n$-cochains is defined as

$$C^n_{\alpha, \phi}(A, M) := \{f : A^\otimes n \to M| \phi \circ f(a_1, \ldots, a_n) = f(\alpha(a_1), \ldots, \alpha(a_n))\}$$

and the differential $\delta_{\alpha, \phi} : C^n_{\alpha, \phi}(A, M) \to C^{n+1}_{\alpha, \phi}(A, M)$ is given by

$$(\delta_{\alpha, \phi}f)(a_1, \ldots, a_{n+1}) = \alpha^{n+1}(a_1) f(a_2, \ldots, a_{n+1}) \quad (2.1)$$
We can easily verify that
\[ \rho \]
where
\[ a \]
\[ \mu \]
This result is still true for Hom-associative algebras. Recall first the notion of an \( \mathcal{O} \)-operator on a Hom-associative algebra.

**Definition 2.3.** Let \((A, \mu, \alpha)\) be a Hom-associative algebra and \((M, l, r, \phi)\) be a bimodule. A linear map \( T : M \to A \) is called an \( \mathcal{O} \)-operator with respect to \( M \) if it satisfies
\[ T\phi = \alpha T, \quad (uT)v = T(l(T(u))v + r(T(v))u), \quad \text{for all } u, v \in M. \]

**Example 2.4.** A Rota-Baxter operator \( \mathcal{R} : A \to A \) of weight 0 is an \( \mathcal{O} \)-operator on \( A \) with respect to the adjoint representation \((A, L, R, \alpha)\).

**Example 2.5.** Let \( \{x_1, x_2, x_3\} \) be a basis of a 3-dimensional vector space \( A \) over \( \mathbb{K} \). Consider the following multiplication \( \mu \) and linear map \( \alpha \) on \( A \) that define Hom-associative algebras over \( \mathbb{K}^3 \):
\[
\begin{align*}
\mu(x_1, x_1) &= a x_1, \\
\mu(x_1, x_2) &= \mu(x_2, x_1) = a x_2, \\
\mu(x_1, x_3) &= \mu(x_3, x_1) = b x_3, \\
\mu(x_2, x_2) &= a x_2, \\
\mu(x_2, x_3) &= b x_3, \\
\mu(x_3, x_2) &= \mu(x_3, x_3) = 0,
\end{align*}
\]
where \( a, b \) are parameters in \( \mathbb{K} \). The algebras are not associative when \( a \neq b \) and \( b \neq 0 \), since \( \mu(\mu(x_1, x_1), x_3) - \mu(x_1, \mu(x_1, x_3)) = (a - b)bx_3 \).

Let \( \mathcal{R} \) be the operator defined with respect to the basis \( \{x_1, x_2, x_3\} \) by
\[ \mathcal{R}(x_1) = \rho_1 x_3, \quad \mathcal{R}(x_2) = \rho_2 x_3, \quad \mathcal{R}(x_3) = 0, \]
where \( \rho_1, \rho_2 \) are parameters in \( \mathbb{K} \). Then \( \mathcal{R} \) is a Rota-Baxter operator of weight 0 on the Hom-associative algebra \((A, \mu, \alpha)\).

**Example 2.6.** Let \((A, \mu, \alpha)\) be a Hom-associative algebra and \((M, l, r, \phi)\) be an \( A \)-bimodule. We can easily verify that \( A \oplus M \) is an \( A \)-bimodule under the following actions:
\[
\begin{align*}
a \cdot (b, m) &= (\mu(a, b), a \cdot m), \quad (b, m) \cdot a = (0, m \cdot a), \quad \forall a, b \in A, \quad m \in M.
\end{align*}
\]
Define the linear map \( T : A \oplus M \to A \), \((a, m) \mapsto a \). Then \( T \) is an \( \mathcal{O} \)-operator on \( A \) with respect to the bimodule \( A \oplus M \).

**Proposition 2.7.** A linear map \( T : M \to A \) is an \( \mathcal{O} \)-operator on a Hom-associative algebra \( A \) with respect to the \( A \)-bimodule \( M \) if and only if the graph of \( T \),
\[ \text{Gr}(T) = \{(T(u), u) \mid u \in M \} \]
is a subalgebra of the semi-direct product algebra \( A \oplus M \).
In the following proposition, we show that an $O$-operator can be lifted up the Rota-Baxter operator.

**Proposition 2.8.** Let $(A, \mu, \alpha)$ be a Hom-associative algebra, $(M, l, r, \phi)$ be an $A$-bimodule and $T : M \to A$ be a linear map. Define $\hat{T} \in \text{End}(A \oplus M)$ by $\hat{T}(a, m) = (T(m), 0)$. Then $T$ is an $O$-operator if and only if $\hat{T}$ is a Rota-Baxter operator on $A \oplus M$.

Another characterization of an $O$-operator can be given in terms of Nijenhuis operator on Hom-associative algebras.

**Definition 2.9.** Let $(A, \mu, \alpha)$ be a Hom-associative algebra. A linear map $N : A \to A$ is said to be a Nijenhuis operator if $N \circ \alpha = \alpha \circ N$ and its Nijenhuis torsion vanishes, i.e.

$$N(x) \cdot N(y) = N(N(x) \cdot y + x \cdot N(y) - N(x \cdot y)), \quad \text{for all } x, y \in A.$$

Note that the deformed multiplication $\mu_N : A \otimes A \to A$ given by

$$\mu_N(x, y) = N(x) \cdot y + x \cdot N(y) - N(x \cdot y),$$

defines a new Hom-associative multiplication on $A$, and $N$ becomes an algebra morphism from $(A, \mu_N, \alpha)$ to $(A, \mu, \alpha)$. In addition $\mu$ and $\mu_N$ are compatible, i.e. $\mu + \lambda \mu_N$ still a Hom-associative structure for each $\lambda \in \mathbb{K}$. Then the pair $(\mu, \mu_N)$ becomes a Hom-quantum bi-Hamiltonian system.

The following result is straightforward, so we omit details.

**Proposition 2.10.** A linear map $T : M \to A$ is an $O$-operator on $(A, \cdot, \alpha)$ with respect to the bimodule $(M, l, r, \phi)$ if and only if $N_T = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} : A \oplus M \to A \oplus M$ is a Nijenhuis operator on the Hom-associative algebra $A \oplus M$.

Dendriform structures were first introduced by Loday in his study of the periodicity phenomenons in algebraic K-theory [36]. They become rather popular in the last 20 years due to their connection with Rota-Baxter algebras, shuffle algebras and combinatorics [3, 48, 27]. In [38], Makhlouf introduced the notion of Hom-dendriform algebra.

**Definition 2.11.** A Hom-dendriform algebra is a vector space $D$ together with three linear maps $\prec, \succ : D \otimes D \to D$ and $\alpha : D \to D$ satisfying the following three identities

$$x \prec y < a(z) = a(x) \prec (y < z + y > z), \quad (2.2)$$

$$x \succ y < a(z) = a(x) > (y < z), \quad (2.3)$$

$$x \prec y + x > y \succ a(z) = a(x) > (y > z), \quad (2.4)$$

for all $x, y, z \in D$.

It follows from (2.2)-(2.4) that the new product $\star : D \otimes D \to D$ defined by $x \star y = x < y + y > x$ turns out to be Hom-associative. Thus, a Hom-dendriform algebra can be seen as a splitting of a Hom-associative algebra.

An $O$-operator has an underlying Hom-dendriform structure on the bimodule.

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Proposition 2.12. Let $T : M \to A$ be an $O$-operator on a Hom-associative algebra $A$ with respect to the bimodule $M$. Then the vector space $M$ carries a Hom-dendriform structure with

$$u < v = r(T(v))u \quad \text{and} \quad u > v = l(T(u))v, \quad \text{for all } u, v \in M.$$ 

Moreover, $(M, \star_T = < + >, \phi)$ is a Hom-associative algebra.

2.2. Graded Lie algebras, Lift and Bidegree

In this section, we discuss a graded Lie algebra related to Hom-associative algebras and its corresponding Maurer-Cartan equation for which a solution characterize an $O$-operator. It is shown in [4] that a Gerstenhaber algebra structure is constructed on the cohomology of Hom-associative algebras using a cup product together with the degree $-1$ graded Lie bracket. We aim to show that an $O$-operator can be characterized by certain solutions of the corresponding Maurer-Cartan equation (see Theorem 3.2).

Definition 2.13. Let $(G = \bigoplus_{k=1}^{\infty} G_k, [\cdot, \cdot], d)$ be a differential graded Lie algebra. A degree 1 element $\theta \in G_2$ is called a Maurer-Cartan element of $G$ if it satisfies the following Maurer-Cartan equation:

$$d \theta + \frac{1}{2} [\theta, \theta] = 0. \quad (2.5)$$

A graded Lie algebra is a differential graded Lie algebra with $d = 0$. Then we have

Proposition 2.14. Let $(G = \bigoplus_{k=1}^{\infty} G_k, [\cdot, \cdot])$ be a graded Lie algebra and let $\mu \in G_2$ be a Maurer-Cartan element. Then the map

$$d_\mu : G \longrightarrow G, \quad d_\mu(v) := [\mu, v], \quad \forall v \in G,$$

is a differential on $G$. For any $v \in G$, the sum $\mu + v$ is a Maurer-Cartan element of the graded Lie algebra $(G, [\cdot, \cdot])$ if and only if $v$ is a Maurer-Cartan element of the differential graded Lie algebra $(G, [\cdot, \cdot], d_\mu)$.

Let $A$ be a vector space and $\alpha : A \to A$ be a linear map. Denote by $V_n^\alpha$ the set of all linear maps $f : A^\otimes n \to A$ satisfying

$$\alpha(f(a_1, \cdots, a_n)) = f(\alpha(a_1), \cdots, \alpha(a_n)), \quad \text{for all } a_i \in A.$$

The graded space $G = \bigoplus_{n \geq 1} V_n^\alpha$ carries a graded Lie algebra structure $[-, -]_\alpha : V_n^\alpha \times V_n^\alpha \to V_{m+n-1}^\alpha$ given by

$$[f, g]_\alpha = f \circ g - (-1)^{(m-1)(n-1)} g \circ f, \quad \text{for all } f \in V_m^\alpha, g \in V_n^\alpha,$$

where

$$(f \circ g)(a_1, \ldots, a_{m+n-1}) =$$

$$\sum_{i=1}^{m} (-1)^{(i-1)(n-1)} f(\alpha^{n-1}(a_1), \ldots, \alpha^{n-1}(a_{i-1}), g(a_i, \ldots, a_{i+n-1}), \ldots, \alpha^{n-1}(a_{m+n-1})).$$
Let $\mu \in V^*_2$. By a direct computation, we have
\[
\frac{1}{2} [\mu, \mu]_a(x, y, z) = \mu \circ \mu(x, y, z) = \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z)),
\]
which implies that $\mu$ defines a Hom-associative structure on $A$ if an only if $[\mu, \mu]_a = 0$, i.e. $\mu$ is a Maurer-Cartan element of the graded Lie algebra $G$.

In the following, we introduce the notion of bidegree and discuss it with respect to Gerstenhaber bracket. Let $A_1$ and $A_2$ be two vector spaces and $c : A_2^{\otimes n} \rightarrow A_1$ be a linear map, or a cochain in $C^n(A_2, A_1)$. We construct a cochain $\hat{c} \in C^n(A_1 \oplus A_2)$ by
\[
\hat{c}((a_1, x_1) \otimes \ldots \otimes (a_n, x_n)) := (c(x_1, \ldots, x_n), 0).
\]
In general, for a given multilinear map $f : A_{i_1} \otimes A_{i_2} \otimes \ldots \otimes A_{i_n} \rightarrow A_{j_1}, \ldots, i_n, j \in \{1, 2\}$, we define a cochain $\hat{f} \in C^n(A_1 \oplus A_2)$ by
\[
\hat{f} := \begin{cases} f & \text{on } A_{i_1} \otimes A_{i_2} \otimes \ldots \otimes A_{i_n}, \\ 0 & \text{all other cases.} \end{cases}
\]
We call the cochain $\hat{f}$ a horizontal lift of $f$, or simply a lift. For instance, the lifts of $\alpha : A_1 \otimes A_1 \rightarrow A_1$, $\beta : A_1 \otimes A_2 \rightarrow A_2$ and $\gamma : A_2 \otimes A_1 \rightarrow A_2$ are defined respectively by
\[
\hat{\alpha}((a, x), (b, y)) = (\alpha(a, b), 0),
\]
\[
\hat{\beta}((a, x), (b, y)) = (0, \beta(a, y)),
\]
\[
\hat{\gamma}((a, x), (b, y)) = (0, \gamma(x, b)).
\]
Let $H : A_2 \rightarrow A_1$ (resp. $H : A_1 \rightarrow A_2$) be a 1 cochain. The lift is defined by
\[
\hat{H}(a, x) = (H(x), 0) \quad \text{(resp. } \hat{H}(a, x) = (0, H(a))).
\]

For any $(a, x) \in A_1 \oplus A_2$,

We denote by $A^{l+k}$ the direct sum of all $(l+k)$-tensor powers of $A_1$ and $A_2$, where $l$ (resp. $k$) is the number of $A_1$ (resp. $A_2$). For instance,
\[
A^{1,2} := (A_1 \otimes A_2) \oplus (A_2 \otimes A_1 \otimes A_2) \oplus (A_2 \otimes A_2 \otimes A_1).
\]

The tensor space $(A_1 \oplus A_2)^{\otimes n}$ is expanded into the direct sum of $A^{l+k}$, $l + k = n$. For instance,
\[
(A_1 \oplus A_2)^{\otimes 2} = A^{2,0} \oplus A^{1,1} \oplus A^{0,2}.
\]
Let $f$ be a $n$-cochain in $C^n(A_1 \oplus A_2)$. We say that the bidegree of $f$ is $kl$ if $f$ is an element in $C^n(A^{k-1, l}, A_1)$ or in $C^n(A^{l-1, k}, A_2)$, where $n = l + k - 1$. We denote the bidegree of $f$ by $\|f\| = kl$. In general, cochains do not have bidegree. We call a cochain $f$ a homogeneous cochain, if $f$ has a bidegree.

We have $k + l \geq 2$, because $n \geq 1$. Thus there are no cochains of bidegree $0|0$ or $1|0$ or $0|1$. We recall $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in C^2(A_1 \oplus A_2)$ in (2.6), (2.7) and (2.8). One can easily see $\|\hat{\alpha}\| = \|\hat{\beta}\| = \|\hat{\gamma}\| = 1|2$.

**Lemma 2.15.** If $\|f\| = k|l$ (resp. $0|k$) and $\|g\| = l|0$ (resp. $0|l$), then $[f, g]_a = 0$.

**Proposition 2.16.** Let $f, g \in C^*(A_1 \oplus A_2)$. If $\|f\| = k|l$, and $\|g\| = k_0|l_0$, then the Gerstenhaber bracket $[f, g]_a$ has the bidegree $k_f + k_g - 1|l_f + l_g - 1$. 

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3. Cohomologies of $O$-operators on Hom-associative algebras

3.1. Cohomology of $O$-operator

Let $A$ and $M$ be two vector spaces equipped with maps $\mu : A^{\otimes 2} \to A$, $l : A \otimes M \to M$, $r : M \otimes A \to M$, $\alpha : A \to A$ and $\phi : M \to M$. Set $V = A \oplus M$. It is, naturally equipped with the linear map $\alpha + \phi : (x, u) \mapsto (\alpha(x), \phi(u))$. Now, consider the graded Lie algebra structure on $g = \oplus_{i \geq 1} V_{n+i}$. Observe that the elements $\mu, l, r \in V_{2+i+\phi}$. Therefore, $\mu + l + r \in V_{2+i+\phi}$.

**Proposition 3.1.** The product $\mu$ defines a Hom-associative structure on $A$ and $l, r$ define an $A$-bimodule structure on $M$ with respect to $\phi$ if and only if $\mu + l + r \in V_{2+i+\phi}$ is a Maurer-Cartan element in $g$.

**Proof.** By a direct computation, we have

$$
\frac{1}{2}[\mu + l + r, \mu + l + r]_a((a_1, m_1), (a_2, m_2), (a_3, m_3)) = (\mu + l + r) \circ (\mu + l + r)((a_1, m_1), (a_2, m_2), (a_3, m_3))
$$

$$
= (\mu + l + r)((\mu + l + r)((a_1, m_1), (a_2, m_2)), (a_3, \phi(m_3))) - (\mu + l + r)((\alpha(a_1), \phi(m_1)), (\mu + l + r)((a_2, m_2), (a_3, m_3)))
$$

$$
= (\mu(\mu(a_1, a_2), a_3)), l((\mu(a_1, a_2), \phi(m_3)) + r(l(a_1, m_2), a_3)) + r(r(m_1, a_2), a_3))
$$

$$
- (\mu(\alpha(a_1), \mu(a_2, a_3)), l(\alpha(a_1), l(a_2, m_3)) + l(\alpha(a_1), r(m_2, a_3)) + r(m_1, a_2), a_3)).
$$

Then $\mu$ defines a Hom-associative structure on $A$ and $l, r$ define an $A$-bimodule structure on $M$ with respect to $\phi$ if and only if $\mu + l + r$ is a Maurer-Cartan element in $g$. \hfill \Box

We aim in the following the Voronov’s derived bracket construction \([15]\) (see also \([13]\)) to get a graded Lie algebra structure on $\bigoplus_{n \geq 1} \text{Hom}(M^{\otimes n}, A)$, where $A$ is a regular Hom-associative algebra and $(l, r)$ defines an $A$-bimodule structure on $M$ with respect to $\phi$ supposed to be invertible.

According to the above proposition, the graded Lie algebra $(g, [\cdot, \cdot])$ together with the differential $d_{\mu+l+r} = [\mu + l + r, \cdot]_a$ becomes a differential graded Lie algebra. Moreover, it is easy to see that the graded subspace $\bigoplus_{n \geq 1} \text{Hom}(M^{\otimes n}, A)$ is an abelian subalgebra.

Define, for $P \in \text{Hom}(M^{\otimes m}, A)$, $Q \in \text{Hom}(M^{\otimes n}, A)$, $m, n \geq 1$, the bracket

$$
\langle P, Q \rangle_a := (-1)^m[[\mu + l + r, P]_a, Q]_a
$$

(3.1)

It is well defined according to Proposition 2.1.3, $\langle -,- \rangle_a$ and may be explicitly given by

$$
\langle P, Q \rangle_a(u_1, \ldots, u_{m+n}) = \sum_{i=1}^{m} (-1)^{i-1} P(\phi^m(u_1), \ldots, \phi^m(u_{i-1}), Q(u_1, \ldots, u_{i+n-1}), \phi^{m-1}(u_{i+n}))
$$

$$
- \sum_{i=1}^{m} (-1)^{i} P(\phi^m(u_1), \ldots, \phi^m(u_{i-1}), r(\phi^{m-1}(u_1), Q(u_{i+1}, \ldots, u_{i+n})), \phi^m(u_{i+n+1}), \ldots, \phi^m(u_{m+n}))
$$

(3.2)
Theorem 3.2.

Consider a differential graded Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$. For any linear map $T : M \rightarrow A$ that satisfies $T(0) = 0$, we define $\{T, T\}_a$ to be the Lie bracket $[a, b]_c = \mu(a, b) - \mu(b, a)$.

Thus, for any $T, T' \in \text{Hom}(A, M)$, we have from (3.2) that

\[ \{T, T'\}_a(u, v) = T(T'(u)v) + T(uT'(v)) + T'(T(u)v) - T(u)T'(v) - T'(u)T(v). \]  

(3.4)

For any $n \geq 0$, we define $C^n(M, A) := \text{Hom}(M^{\otimes n}, A)$ and consider the graded vector space $C^*(M, A) = \bigoplus_{n \geq 0} C^n(M, A) = \bigoplus_{n \geq 0} \text{Hom}(M^{\otimes n}, A)$. Thus, we obtain the following result.

**Theorem 3.2.** The graded vector space $C^*(M, A) = \bigoplus_{n \geq 0} \text{Hom}(M^{\otimes n}, A)$ together with the above defined bracket $\{\cdot, \cdot\}_a$ form a graded Lie algebra. A linear map $T : M \rightarrow A$ is an $\mathcal{O}$-operator on $A$ with respect to the bimodule $M$ if and only if $T \in C^1(M, A)$ is a Maurer-Cartan element in $(C^1(M, A), \{\cdot, \cdot\}_a)$, i.e. $T$ satisfies $\{T, T\}_a = 0$.

Therefore, from the general principle of Maurer-Cartan elements, we obtain.

**Theorem 3.3.** Let $T : M \rightarrow A$ be an $\mathcal{O}$-operator on $A$ with respect to the bimodule $M$. Then $T$ induces a differential $d_T = \{T, \cdot\}_a$, which makes the graded Lie algebra $(C^*(M, A), \{\cdot, \cdot\}_a)$ into a differential graded Lie algebra $(d\mathcal{G}La)$.

Moreover, for any linear map $T' : M \rightarrow A$, the sum $T + T'$ is still an $\mathcal{O}$-operator if and only if $T'$ is a Maurer-Cartan element in the $d\mathcal{G}La$ $(C^*(M, A), \{\cdot, \cdot\}_a, d_T)$, that is

\[ \{T + T', T + T'\}_a = 0 \iff d_T(T') + \frac{1}{2}\{T', T\}_a = 0. \]

It follows from the above theorem that if $T$ is an $\mathcal{O}$-operator, then $(C^*(M, A), d_T = \{T, \cdot\}_a)$ is a cochain complex. Its cohomology is called the **cohomology of the $\mathcal{O}$-operator $T$**.

In the next paragraph, we interpret this cohomology as the Hochschild cohomology of a certain algebra with coefficients in a suitable bimodule. Then we will use the usual notation for Hochschild cohomology to denote the cohomology of an $\mathcal{O}$-operator.
3.2. Cohomology of $O$-operators as Hochschild cohomology

The aim of this section is to show that the cohomology of an $O$-operator can also be described as the Hochschild cohomology of a certain Hom-associative algebra with coefficients in a suitable bimodule.

Let $T : M \to A$ be an $O$-operator on $A$ with respect to the bimodule $(M, l, r, \phi)$. Then by Proposition 2.12 the vector space $M$ carries a Hom-associative algebra structure with the product

$$u \star_T v = r(u, T(v)) + l(T(u), v), \quad \text{for } u, v \in M. \quad (3.5)$$

According to (2.10), there is a Hom-associative algebra structure on $A \oplus M = M \oplus A$ defined as

$$(x, u) \cdot_T (y, v) = (Tu, 0) \cdot_T (y, v) + (x, u) \cdot_T (Tv, 0) - (T(l(x, v) + r(u, y), 0)$$

$$= (Tu \cdot y, l(Tu, v)) + (x \cdot Tv, -r(u, Tv)) - (T(l(x, v) + T(r(u, y), 0),$$

$$= (r_T(u, y) + l_T(v, x), u \star v),$$

where

$$l_T(u, x) = Tu \cdot x - T(r(u, x)), \quad r_T(x, u) = x \cdot Tu - T(l(x, u), \forall x \in A, u \in M. \quad (3.6)$$

Therefore we have the following result.

**Lemma 3.4.** Let $T : M \to A$ be an $O$-operator on $A$ with respect to the bimodule $(M, l, r, \phi)$. Then $(l_T, r_T)$ defines an $M$-bimodule structure on $A$ with respect to $\alpha$.

By Lemma 3.4 we obtain an $M$-bimodule structure on the vector space $A$. Therefore, we may consider the corresponding Hochschild cohomology of $M$ with coefficients in $A$. We define

$$C^n_\alpha(M, A) := \text{Hom}(M^\otimes n, A), \quad \text{for } n \geq 0$$

and the differential by

$$d_H(a)(m) = l_T(m, a) - r_T(a, m) = T(m) \cdot a - T(ma) - a \cdot T(m) + T(am), \quad \text{for } a \in A = C^0_\alpha(M, A) \quad (3.7)$$

and

$$(d_H f)(u_1, \ldots, u_{n+1}) = T(\phi^{n-1}(u_1)) \cdot f(u_2, \ldots, u_{n+1}) - T(r(\phi^{n-1}(u_1), f(u_2, \ldots, u_{n+1}))$$

$$+ \sum_{i=1}^n (-1)^i f(\phi(u_1), \ldots, \phi(u_{i-1}), r(u_i, T(u_{i+1}))) + l(T(u_i), u_{i+1}), \ldots, \phi(u_n))$$

$$+ (-1)^{n+1} f(u_1, \ldots, u_n) \cdot T(\phi^{n-1}(u_{n+1})) - (-1)^{n+1} T(l(f(u_1, \ldots, u_n), \phi^{n-1}(u_{n+1}))).$$

We denote the group of $n$-cocycles by $Z^n_\alpha(M, A)$ and the group of $n$-coboundaries by $B^n_\alpha(M, A)$. The corresponding cohomology groups are defined by

$$H^n_\alpha(M, A) = Z^n_\alpha(M, A)/B^n_\alpha(M, A), \quad n \geq 0.$$
In particular, we have

\[ H^0(M, A) = \{ a \in A \mid d_H(a) = 0 \} = \{ a \in A \mid a \cdot T(m) - T(m) \cdot a = T(l(a, m) - r(m, a)), \forall m \in M \} . \]

From this definition, it is easy to see that if \( a, b \in H^0(M, A) \), then their commutator \([a, b] := a \cdot b - b \cdot a\) is also in \( H^0(M, A) \). This shows that \( H^0(M, A) \) has a Hom-Lie algebra structure induced from that of \( A \).

Note that a linear map \( f : M \to A \) (i.e. \( f \in C^1(M, A) \)) is closed if it satisfies

\[ T(u) \cdot f(v) + f(u) \cdot T(v) - T(uf(v)) + f(uT(v) + T(u)v) = 0, \quad (3.8) \]

for all \( u, v \in M \).

For an \( O \)-operator \( T \) on \( A \) with respect to the bimodule \( M \), we get two coboundary operators \( d_T = \llbracket T, \llbracket_a \) and \( d_H \) on the same graded vector space \( C^\bullet_a(M, A) = \bigoplus_{n \geq 0} C^n_a(M, A) \). The following proposition relates the above two coboundary operators.

**Proposition 3.5.** Let \( T : M \to A \) be an \( O \)-operator on \( A \) with respect to the \( A \)-bimodule \( M \). Then the two coboundary operators are related by

\[ d_T f = (-1)^n d_H f, \quad \text{for } f \in C^n_a(M, A). \]

**Proof.** For any \( f \in C^n_a(M, A) = \text{Hom}(M^{\otimes n}, A) \) and \( u_1, \ldots, u_{n+1} \in M \), we have from (3.2) that

\[
(d_T f)(u_1, \ldots, u_{n+1}) = \llbracket T, f \rrbracket_a(u_1, \ldots, u_{n+1})
= T(l(f(u_1, \ldots, u_n) \phi^{n-1}(u_{n+1}))) - (-1)^n T(r(\phi^{n-1}(u_1), f(u_2, \ldots, u_{n+1})))
\]

\[ - (-1)^n \left\{ \sum_{i=1}^{n} (-1)^{i-1} f(\phi(u_1), \ldots, \phi(u_{i-1}), T(u_i)u_{i+1}, u_{i+2}, \ldots, \phi(u_{n+1})) \right\}
\]

\[ - \sum_{i=1}^{n} (-1)^i f(\phi(u_1), \ldots, \phi(u_{i-1}), u_iT(u_{i+1}), u_{i+2}, \ldots, \phi(u_{n+1})) \]

\[ + (-1)^n T(\phi^{n-1}(u_1)) \cdot f(u_2, \ldots, u_{n+1}) - f(u_1, \ldots, u_n) \cdot T(\phi^{n-1}(u_{n+1})) \]

\[ = (-1)^n (d_H f)(u_1, \ldots, u_{n+1}). \]

The same holds true when \( f = a \in A \). Compare (3.3) with (3.7). Hence the proof. \( \square \)

This shows that the cohomology of the either complex \((C^\bullet_a(M, A), d_T)\) or \((C^\bullet(M, A), d_H)\) are isomorphic. Thus, we may use the same notation \( H^\bullet_a(M, A) \) to denote the cohomology of an \( O \)-operator.

### 4. Deformations of \( O \)-operators

In this section, we study the Infinitesimal and formal deformation theory of \( O \)-operators following the classical approaches initiated by Gerstenhaber [24, 25].
4.1. Infinitesimal deformations

Let \((A, \cdot_A, \alpha_A)\) be a Hom-associative algebra and \((V, l_A, r_A, \phi)\) an \(A\)-bimodule, and \((B, \cdot_B, \alpha_B)\) be a Hom-associative algebra with \((W, l_B, r_B, \psi)\) a \(B\)-bimodule. Suppose \(T : V \to A\) is an \(O\)-operator on \(A\) with respect to the \(A\)-bimodule \((V, l_A, r_A, \phi)\) and \(T' : W \to B\) is an \(O\)-operator on \(B\) with respect to the bimodule \((B, \cdot_B, \alpha_B)\).

**Definition 4.1.** A morphism of \(O\)-operators from \(T\) to \(T'\) consists of a pair \((f, g)\) of an algebra morphism \(f : A \to B\) and a linear map \(g : V \to W\) satisfying

\[
\begin{align*}
\psi g &= g \phi, \\ T' g &= f T, \\ l_B(f(x), g(u)) &= g(l_A(x, u)), \\ r_B(g(u), f(x)) &= g(r_A(u, x)),
\end{align*}
\]

for all \(x \in A\) and \(u \in V\).

It is called an isomorphism if both \(f\) and \(g\) are bijective. According to Lemma 2.12, the vector spaces \(V\) and \(W\) can be endowed with a Hom-dendriform algebra structures and \(g\) becomes then a morphism between Hom-dendriform algebras \(V\) and \(W\) and also a morphism between the sub-adjacent Hom-associative algebras.

The proof of the following result is straightforward so we omit the details.

**Proposition 4.2.** A pair of linear maps \((f : A \to B, g : V \to W)\) is a morphism of \(O\)-operators from \(T\) to \(T'\) if and only if

\[\text{Gr}((f, g)) := \{(a, u), (f(a), g(u))\} | a \in A, u \in V\] \(\subset (A \oplus V) \oplus (B \oplus W)\)

is a Hom-subalgebra, where \(A \oplus V\) and \(B \oplus W\) are equipped with semi-direct product Hom-algebra structures.

In the rest of the paper, we will be most interested in morphisms between \(O\)-operators on the same Hom-algebra with respect to the same bimodule.

Let \(T : M \to A\) be an \(O\)-operator on a Hom-associative algebra \(A\) with respect to the \(A\)-bimodule \(M\).

**Definition 4.3.** An infinitesimal (one-parameter) deformation of \(T\) consists of a sum \(T_t = T + tT\), for some \(T \in \text{Hom}(M, A)\), such that \(T_t\) is an \(O\)-operator on \(A\) with respect to the bimodule \(M\), for all \(t\). In such a case, we say that \(T\) generates an infinitesimal or a linear deformation of \(T\).

It is easy to check that \(T_t = T + tT\) is an infinitesimal deformation of an \(O\)-operator \(T\) if and only if for any \(u, v \in M\),

\[
\begin{align*}
T_{t_1} \phi &= \alpha T_t, \\
T_t(u) \cdot T_t(v) &= T_t(l(T_t(u))v + r(T_t(v))u),
\end{align*}
\]

for all \(t\) and \(u, v \in M\).
By equating coefficients of $t$ and $t^2$ from both side, we obtain
\begin{align}
T \phi &= \alpha T, \quad \text{(4.5)} \\
T(u) \cdot T(v) + T(u) \cdot T(v) &= T(r(T(v))u + l(T(u))v) + T(r(T(v))u + l(T(u))v), \quad \text{(4.6)} \\
T(u) \cdot T(v) &= T(r(T(v))u + l(T(u))v). \quad \text{(4.7)}
\end{align}

The identity (4.6) means that $d_T(T) = 0$, that is $T$ is a 1-cocycle with respect to the cohomology of the $O$-operator $T$. Eqs. (4.5) and (4.7) mean that $T$ is an $O$-operator on $\mathcal{O}$ associated to the bimodule $M$.

Given a Hom-dendriform algebra $(D, <, >, \phi)$. Let $\omega_\prec, \omega_\succ : D \otimes D \to D$ be two linear maps. If for any $t \in \mathbb{K}$, the multiplications $<,>$, defined by
\begin{align}
\omega_\prec v &= u < v = u < v + t \omega_\prec(u, v), \quad \omega_\succ v &= u > v = u > v + t \omega_\succ(u, v), \quad \forall u, v \in D
\end{align}
give a Hom-dendriform algebra structure on $D$, we say that the pair $(\omega_\prec, \omega_\succ)$ generates a (one parameter) infinitesimal deformation of the Hom-dendriform algebra $(D, <, >, \phi)$.

In the following, we show that an infinitesimal deformation of an $O$-operator induces an infinitesimal deformation of the corresponding Hom-dendriform structure on the module.

**Proposition 4.4.** If $T$ generates an infinitesimal deformation of an $O$-operator $T$ on a Hom-associative algebra $A$ with respect to the bimodule $M$, then the multiplications
\begin{align}
<,> = u < v = uT(v) + t uT(v), \quad >, = v > u = T(u)v + t T(u)v
\end{align}
define an infinitesimal deformation of the corresponding Hom-dendriform structure on $M$.

**Corollary 4.5.** If $T$ generates an infinitesimal deformation of an $O$-operator $T$ on a Hom-associative algebra $A$ with respect to the bimodule $M$, then the multiplication
\begin{align}
\ast, = u \ast v = u \ast v + t(uT(v) + T(u)v)
\end{align}
defines an infinitesimal deformation of the corresponding Hom-associative structure on $M$.

**Definition 4.6.** Let $T$ be an $O$-operator on a Hom-associative algebra $A$ with respect to a bimodule $(M, l, r, \phi)$. Two infinitesimal deformations $T^1_t = T + tT$ and $T^2_t = T + tT$ of $T$ are equivalent if there exists an $x \in A$ such that $\alpha(x) = x$ and the pair $(\text{id}_A + l(L_x - R_x), \text{id}_M + l(l_x - r_x))$ is a morphism of $O$-operators from $T^1_t$ to $T^2_t$.

Let us recall from Definition 4.1 that the pair $(\text{id}_A + l(L_x - R_x), \text{id}_M + l(l_x - r_x))$ is a morphism of $O$-operators from $T^1_t$ to $T^2_t$ if the following conditions are satisfied
\begin{enumerate}
\item[(i)] The map $\text{id}_A + l(L_x - R_x)$ is a Hom-associative algebra homomorphism.
\item[(ii)] $\phi \circ (\text{id}_M + l(l_x - r_x)) = (\text{id}_M + l(l_x - r_x)) \circ \phi$.
\item[(iii)] $(T + tT) \circ (\text{id}_M + l(l_x - r_x)) = (\text{id}_A + l(L_x - R_x)) \circ (T + tT)$.
\item[(iv)] $l(y + t[x, y], u + t(l(x, u) - r(u, x))) = l(y, u) + t(l(l(x, l(y, u)) - r(l(y, u), x)))$.
\end{enumerate}
(v) \( r(u + t(l(x, u) - r(u, x)), y + l(x, y)) = r(u, y) + t(l(x, r(u, y)) - tr(r(u, y), x)). \)

From (i), we obtain, for all \( y, z \in A \)

\[
(id_A + t(L_x - R_x))(y \cdot z) = (id_A + t(L_x - R_x))(y) \cdot (id_A + t(L_x - R_x))(z),
\]

or equivalently

\[
(x \cdot y - y \cdot x) \cdot (x \cdot z - z \cdot x) = 0, \quad \forall y, z \in A. \tag{4.8}
\]

Since \( \alpha(x) = x \), then condition (ii) holds. Moreover, from (iii) we get, for any \( u \in M \),

\[
(T + t\mathcal{T}_2) \circ (id_M + t(l_x - r_x))(u) = (id_A + t(L_x - R_x)) \circ (T + t\mathcal{T}_1)(u),
\]

on comparing the coefficients of \( t \) and \( t^2 \), respectively, from both sides of the above identity, we obtain

\[
\mathcal{T}_1(u) - \mathcal{T}_2(u) = T(l(x, u) - r(u, x)) - (x \cdot T(u) - T(u) \cdot x)
\]

\[
= l_2(u, x) - r_T(x, u), \tag{4.9}
\]

\[
x \cdot \mathcal{T}_1(u) - \mathcal{T}_1(u) \cdot x = \mathcal{T}_2(l(x, u) - r(u, x)). \tag{4.10}
\]

From (iv), on comparing just the coefficients of \( t^2 \) (since the coefficients of \( t \) vanish) from both sides of the identity, we obtain

\[
l(x \cdot y - y \cdot x, l(x, u) - r(u, x)) = 0
\]

or equivalently,

\[
l_{x \cdot y - y \cdot x} \circ l_x = l_{x \cdot y - y \cdot x} \circ r_x, \quad \text{for} \quad y \in A. \tag{4.11}
\]

Similarly, (v) gives rise to

\[
r_{x \cdot y - y \cdot x} \circ l_x = r_{x \cdot y - y \cdot x} \circ r_x, \quad \text{for} \quad y \in A. \tag{4.12}
\]

**Theorem 4.7.** Let \( T_1^t = T + t\mathcal{T}_1 \) and \( T_2^t = T + t\mathcal{T}_2 \) be two equivalent infinitesimal deformations of an \( O \)-operator \( T \). Then \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) defines the same cohomology class in \( H^2_o(M, A) \).

**Proof.** The proof follows from equation (4.9). \( \square \)

**Definition 4.8.** Let \( T : M \rightarrow A \) be an \( O \)-operator on a Hom-associative algebra \( A \) with respect to a representation \((M, l, r, \phi)\). An infinitesimal deformation \( T_t = T + t\mathcal{T} \) is said to be trivial if it is equivalent to the deformation \( T_0 = T \).

According to the above computation, \( T_t = T + t\mathcal{T} \) is trivial if there exists an element \( x \in A \) such that \( \alpha(x) = x \) and satisfying

\[
(x \cdot y - y \cdot x) \cdot (x \cdot z - z \cdot x) = 0, \tag{4.13}
\]

\[
x \cdot (l_T(u, x) - r_T(x, u)) - (l_T(u, x) - r_T(x, u)) \cdot x = 0, \tag{4.14}
\]

\[
l_{x \cdot y - y \cdot x} \circ l_x = l_{x \cdot y - y \cdot x} \circ r_x, \tag{4.15}
\]

\[
r_{x \cdot y - y \cdot x} \circ l_x = r_{x \cdot y - y \cdot x} \circ r_x, \tag{4.16}
\]

for any \( y, z \in A \) and \( u \in M \). Then we introduce the notion of a Nijenhuis element associated to an \( O \)-operator on a Hom-associative algebra.
We refer to a Hom-Lie algebra $(A, \alpha)$ with respect to a representation $(M, l, r, \phi)$. An element $x \in A$ is called a Nijenhuis element associated to $T$ if $x$ satisfies $\alpha(x) = x$ and the conditions (4.13)–(4.16) hold.

Let us denote the set of Nijenhuis elements associated to the $O$-operator $T$ by $Nij(T)$.

In [3], the authors introduce Nijenhuis elements associated to an $O$-operator on a Hom-Lie algebra to study their trivial deformations. If $T$ is an $O$-operator on a Hom-Lie algebra $(A, [\cdot, \cdot], \alpha)$ with respect to a representation $(M, \rho, \beta)$. Then an element $x \in A$ is called a Nijenhuis element if $\alpha(x) = x$ and

$$[[x, y], [x, z]] = 0, \quad \rho([x, y])\rho(x) = 0, \quad [x, T \rho(x)v + [Tv, x]] = 0,$$

for all $y, z \in A$ and $v \in M$.

**Proposition 4.10.** Let $x \in A$ be a Nijenhuis element with respect to an $O$-operator $T$ on a Hom-associative $A$. Then $x$ is also a Nijenhuis element for the $O$-operator $T$ on the commutator Hom-Lie algebra $(A, [\cdot, \cdot], \alpha)$.

**Proof.** Straightforward. □

By Eqs. (4.10)–(4.12), it is obvious that a trivial infinitesimal deformation gives rise to a Nijenhuis element. Conversely, a Nijenhuis element can also generate a trivial infinitesimal deformation as the following theorem shows.

**Theorem 4.11.** Let $T$ be an $O$-operator on a Hom-associative algebra $(A, \alpha)$ with respect to a representation $(M, l, r, \phi)$. For any element $x \in Nij(T)$, $T_t := T + t\mathcal{T}$, with $\mathcal{T} = d_H(x)$, is a trivial infinitesimal deformation of $T$.

**Proof.** First, we need to prove that $T_t := T + t\mathcal{T}$ is an infinitesimal deformation generated by $\mathcal{T} = d_H(x)$, where $x \in Nij(T)$. That is $\mathcal{T}$ must satisfy conditions (4.5), (4.6) and (4.7). First,

$$\alpha \mathcal{T}(u) = \alpha d_H(x)(u) = d_H(x)(\phi(u)) = \mathcal{T}(\phi(u)), \forall u \in M.$$

Then $\mathcal{T}$ satisfies (4.3). The identity (4.5) holds immediately since $\mathcal{T} = d_H(x)$. The identity (4.7) is also straightforward. We omit details since it follows from a long computation. We refer to [4] for a similar computation. Finally, since $x$ is a Nijenhuis element, it follows that the pair $(id_A + tL_x - R_x, id_M + t(l_x - r_x))$ is a morphism of $O$-operators from $T_t$ to $T$. Hence the result follows. □

4.2. Formal Deformations

In this section, we study one-parameter formal deformations of O-operators. Let $(A, \mu, \alpha)$ be a Hom-associative algebra with a representation $(M, l, r, \phi)$. Let $\mathbb{K}[[t]]$ be the formal power series ring in one variable $t$ and $A[[t]]$ be the formal power series in $t$ with coefficients in $A$. Then the tuple $(A[[t]], \mu_t, \alpha_t)$ is a Hom-associative algebra, where the product $\mu_t$ and the structure map $\alpha_t$ are obtained by extending $\mu$ and $\alpha$ linearly over the ring $\mathbb{K}[[t]]$. Moreover the maps $l$, $r$ and $\phi$ can be extended linearly over $\mathbb{K}[[t]]$ to obtain $\mathbb{K}[[t]]$-linear maps $l_t : A[[t]] \otimes M[[t]] \rightarrow M[[t]]$, $r_t : M[[t]] \otimes A[[t]] \rightarrow M[[t]]$ and $\phi_t : M[[t]] \rightarrow M[[t]]$. Then $(M[[t]], l_t, r_t, \phi_t)$ is a representation of the Hom-associative algebra $(A[[t]], \mu_t, \alpha_t)$.
We mean by a deformation of a Hom-dendriform algebra, a formal deformation of its which is equivalent to the above identity holds for \(k \in \mathbb{Z}_{\geq 0}\). That is
\[
T_i \phi_i = \alpha_i T_i, \quad \text{(4.17)}
\]
\[
T_i(u) \cdot T_i(v) = T_i(T_i(u)v + uT_i(v)), \quad \forall u, v \in M. \quad \text{(4.18)}
\]
Condition (4.17) is immediate since \(T_i \in C_{\phi,\alpha}^1 (M, A), \quad \forall i\). For any \(k \geq 0\), by equating the coefficients of \(t^k\) from both sides of equation (4.18), we obtain the following system of equations
\[
\sum_{i+j=k} T_i(u) \cdot T_j(v) = \sum_{i+j=k} T_i(T_j(u)v + uT_j(v)). \quad \text{(4.19)}
\]
The above identity holds for \(k = 0\), since \(T\) is an \(O\)-operator. For \(k = 1\), we get
\[
T(u) \cdot T(v) + T_i(u) \cdot T(v) = T(uT_i(v) + T_i(u)v) + T_1(uT(v) + T(u)v), \quad \forall u, v \in M, \quad \text{(4.20)}
\]
which is equivalent to
\[
\{T, T_1\}_a = 0. \quad \text{(4.21)}
\]
That is \(d_T(T_1) = 0\). Therefore, we get the following proposition.

**Proposition 4.13.** Let \(T_i = \sum_{i \geq 0} t^iT_i\) be a formal deformation of an \(O\)-operator \(T\) on \(A\) with respect to the bimodule \(M\). Then the 1-cochain \(T_1 \in C_{\phi,\alpha}^1 (M, A)\) is a 1-cocycle with respect to the cohomology of the \(O\)-operator \(T\).

The 1-cochain \(T_1\) is called the infinitesimal of the deformation \(T_i\). More generally, if \(T_i = 0\) for \(1 \leq i \leq (n-1)\) and \(T_n\) is a non-zero cochain, then \(T_n\) is called the \(n\)-infinitesimal of the deformation \(T_i\).

In [13], the author study formal deformations of Hom-dendriform algebras and their relation with the cohomology. Here we provide a connection between deformations of an \(O\)-operator with deformations of the corresponding Hom-dendriform algebra. We mean by a deformation of a Hom-dendriform algebra, a formal deformation of its defining two multiplications.

**Proposition 4.14.** Let \(T_i = \sum_{i \geq 0} t^iT_i\) be a formal deformation of an \(O\)-operator \(T\). Then the formal sums
\[
u <_1 \nu = \sum_{i \geq 0} t^iuT_i(v) \quad \text{and} \quad \nu >_1 \nu = \sum_{i \geq 0} t^iT_i(u)v
\]
define a formal deformation of the Hom-dendriform structure on \(M\).
Corollary 4.15. It follows that the formal sum \( u \star v = \sum_{i \geq 0} t^i (uT_i(v) + T_i(u)v) \) defines a formal deformation of the Hom-associative structure on \( M \).

Now, we consider the equivalence of two formal deformations of an \( O \)-operator. The definition is motivated by the associative algebra case \([23]\).

Definition 4.16. Two formal deformations \( T_t = \sum_{i \geq 0} t^i T_i \) and \( \overline{T}_t = \sum_{i \geq 0} t^i \overline{T}_i \) of an \( O \)-operator \( T \) are said to be equivalent if there exists an element \( x \in A \) satisfying \( \alpha(x) = x \), linear maps \( f_i : A \to A \) and \( g_i : M \to M \), for \( i \geq 2 \), such that the pair \((f_t, g_t)\) given by

\[
 f_t = id_A + t(L_x - R_x) + \sum_{i \geq 2} t^i f_i \quad \text{and} \quad g_t = id_M + t(l_x - r_x) + \sum_{i \geq 2} t^i g_i
\]
defines a formal morphism of \( O \)-operators from \( T_t \) to \( \overline{T}_t \).

With the above notations, for all \( y, z \in A \) and \( u \in M \), the equivalence of two deformations \( T_t \) and \( \overline{T}_t \) corresponds to the following conditions

\[
 f_t T_1 = T_1 g_t, \quad (4.22)
\]
\[
 f_t (y \cdot z) = f_t(y) \cdot f_t(z), \quad (4.23)
\]
\[
 g_t(yu) = f_t(y) g_t(u), \quad (4.24)
\]
\[
 g_t(uy) = g_t(u) f_t(y), \quad (4.25)
\]
\[
 f_t \alpha = \alpha f_t, \quad g_t \phi = \phi g_t. \quad (4.26)
\]

In particular, a formal deformation \( T_t \) of an \( O \)-operator \( T \) is said to be trivial if it is equivalent to \( T \) (here \( T \) is regarded as a deformation of itself).

Now, on comparing the coefficients of \( t \) from both sides of the condition \((4.22)\), we get

\[
 T_1(u) - \overline{T}_1(u) = T(u) \cdot x - T(ux) - x \cdot T(u) + T(xu) = d_H(x)(u).
\]

Consequently, we obtain the following result.

Proposition 4.17. The infinitesimals of equivalent deformations are cohomologous, i.e. they lie on the same cohomology class.

Definition 4.18. An \( O \)-operator \( T \) is said to be rigid if every deformation \( T_t \) of \( T \) is equivalent to the deformation \( \overline{T}_t = T \).

As a cohomological condition of the rigidity, we have the following result which suggests that the rigidity of an \( O \)-operator is a very strong condition.

Proposition 4.19. Let \( T \) be an \( O \)-operator on a Hom-associative algebra \( A \) with respect to a bimodule \( M \). If \( Z^1(M, A) = d_H(Nij(T)) \) then \( T \) is rigid.
Proof. Let \( T_i = \sum_{i \geq 0} t^i T_i \) be any formal deformation of the \( O \)-operator \( T \). Then by Proposition 4.13, the linear term \( T_1 \) is in \( Z^1(M, A) \). Therefore, by assumption, we have \( T_1 = d_H(x) \), for some \( x \in \text{Nij}(T) \). Set

\[
  f_t := id_A + t(L_x - R_x) \quad \text{and} \quad g_t := id_A + t(l_x - r_x)
\]

and define \( \overline{T}_i = f_t \circ T_i \circ g_t^{-1} \). Then \( T_i \) is equivalent to \( \overline{T}_i \). Moreover, from the definition of \( \overline{T}_i \), we have

\[
  \overline{T}_i(u) = (id_A + t(L_x - R_x)) \circ (\sum_{i \geq 0} t^i T_i) \circ (id_A - t(l_x - r_x) + t^2(l_x - r_x)^2 - \cdots)(u) \quad \text{for } u \in M.
\]

Hence

\[
  \overline{T}_i(u) \quad \text{(mod } t^2) = (id_A + t(L_x - R_x)) \circ (T + tT_1)(u - t(xu - ux)) \quad \text{(mod } t^2) \]
\[
  = (id_A + t(L_x - R_x))(T(u) + tT_1(u) - tT(xu - ux)) \quad \text{(mod } t^2) \]
\[
  = T(u) + t(x \cdot T(u) - T(u) \cdot x - T(xu - ux) + T_1(u)).
\]

The coefficient of \( t \) is zero as \( T_1 = d_H(x) \). See (3.7) for instance. Therefore, \( \overline{T}_i \) is of the form \( \overline{T}_i = T + \sum_{i \geq 2} t^i T_i \). By repeating this argument, one get the equivalence between \( T_i \) and \( T \). Hence the proof. \( \square \)

4.3. Order \( n \) deformation of an \( O \)-operator

In this section, We introduce a cohomology class associated to any order \( n \) deformation of an \( O \)-operator. We prove that an order \( n \) deformation of an \( O \)-operator is extensible if and only if this cohomology class is trivial. This cohomology class is called the obstruction class of an order \( n \) deformation being extensible. Let \((A, \mu, \alpha)\) be a Hom-assiative algebra with a representation \((M, l, r, \phi)\). Let \( T : M \to A \) be an \( O \)-operator on \( A \) with respect to \( M \). An order \( n \) deformation of the \( O \)-operator \( T \) is given by a \( \mathbb{K}[[t]]/(t^{n+1}) \)-linear map

\[
  T_i = T + \sum_{i=1}^n t^i T_i, \quad T_i \in C^i_{\phi, \alpha}(M, A)
\]

such that

\[
  T_i(u) \cdot T_i(v) = T_i(T_i(u)v + uT_i(v)) \quad \text{mod}(t^{n+1}), \ \forall u, v \in M.
\]

Which is equivalent to the following system of equations

\[
  \sum_{i+j=k} T_i(u) \cdot T_j(v) = \sum_{i+j=k} T_i(T_j(u)v + uT_j(v)), \quad \text{for any } k = 0, 1, \cdots n.
\]

or equivalently,

\[
  \left\langle [T, T_k] \right\rangle_n = \frac{-1}{2} \sum_{i+j=k, i \neq j=1} \left\langle T_i, T_j \right\rangle_n, \quad \text{for any } k = 0, 1, \cdots n. \quad (4.27)
\]
Definition 4.20. Let $T_t = T + \sum_{i=1}^n t^i T_i$ be an order $n$ deformation of an $O$-operator $T$ on $A$ with respect to $M$. If there exists a 1-cochain $T_{n+1} \in C^1_{\phi,\alpha}(M, A)$ such that $T_t = T_t + t^{n+1} T_{n+1}$ is an order $(n + 1)$ deformation of $T$, then we say that $T_t$ extends to a deformation of order $(n + 1)$. In this case, we say that $T_t$ is extendable.

Let us observe that the map $\tilde{T}_t := T_t + t^{n+1} T_{n+1}$ is an extension of the order $n$ deformation $T_t$ if and only if

$$\sum_{i+j=n+1; i,j \geq 0} \langle T_i, T_j \rangle _\alpha = 0.$$  \hspace{1cm} (4.28)

Definition 4.21. Let $T_t$ be an order $n$ deformation of the $O$-operator $T$. Let us consider a 2-cochain $\Theta_{T} \in C^2_{\phi,\alpha}(M, A)$ defined as follows

$$\Theta_{T} = -\frac{1}{2} \sum_{i+j=n+1; i,j \geq 0} \langle T_i, T_j \rangle _\alpha.$$  \hspace{1cm} (4.29)

The 2-cochain $\Theta_{T}$ is called the obstruction cochain for extending the deformation $T_t$ of order $n$ to a deformation of order $n + 1$. From equation (4.29) and using graded Jacobi identity of the bracket $\langle \cdot , \cdot \rangle _\alpha$, it follows that $\Theta_{T}$ is a 2-cocycle.

The cohomology class $|\Theta_{T}| \in H^2(M, A)$ is called the obstruction class of $T_t$ being extendable.

Theorem 4.22. An order $n$ deformation $T_t$ extends to a deformation of next order if and only if the obstruction class $|\Theta_{T}|$ is trivial.

Proof. Suppose that a deformation of $O$-operator $T_t$ of order $n$ extends to a deformation of order $n + 1$. Then

$$\sum_{i+j=n+1} (T_i(u) \cdot T_j(v) - T_j(T_i(u)v + uT_j(v))) = 0.$$  \hspace{1cm} (4.30)

As a result, we get $\Theta_{T} = -\frac{1}{2} \sum_{i+j=n+1; i,j \geq 0} \langle T_i, T_j \rangle _\alpha$. So, the obstruction class vanishes. Conversely, if the obstruction class $|\Theta_{T}|$ is trivial, suppose that $\Theta_{T} = -d^* \theta$ for some 1-cochain $\theta$. Put $\tilde{T}_t = T_t + T^{n+1} T_{n+1}$. Then $\tilde{T}$ is a deformation of order $n + 1$, which means that $T_t$ is extendable. $\square$

Corollary 4.23. If $H^2(M, A) = 0$, then every finite order deformation of $T$ extends to a deformation of next order.

Corollary 4.24. If $H^2(M, A) = 0$, then every 1-cocycle in $Z^1(M, A)$ is the infinitesimal of some formal deformation of the $O$-operator $T$. 19
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