Density of Roots of the Yamada Polynomial of Spatial Graphs

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Abstract—We recall the construction and survey the properties of the Yamada polynomial of spatial graphs and present formulas for the Yamada polynomial of some classes of graphs. Then we construct an infinite family of spatial graphs for which the roots of the Yamada polynomials are dense in the complex plane.

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1. INTRODUCTION

It is well known that (Laurent) polynomials play an important role in knot theory (see, for example, [12]). We just mention the Alexander and Jones polynomials, which are familiar to every expert in low-dimensional topology. The spatial graph theory arises as a natural generalization of knot theory. Recall that the study of intrinsic knotting and linking of graphs in $S^3$ was initiated in the 1980s by J. H. Conway and C. Gordon with their nice results in [3]. They proved that any embedding of the complete graph $K_7$ in $\mathbb{R}^3$ contains a knotted cycle and that any embedding of the complete graph $K_6$ in $\mathbb{R}^3$ contains a pair of linked cycles. It is natural that the modern theory of spatial graphs combines topological and graph-theoretical methods. The power of polynomial invariants of knots as well as polynomial invariants of graphs was a natural motivation for investigation of polynomial invariants of spatial graphs initiated by L. H. Kauffman [11]. Being motivated by problems of knotting and linking of DNA and chemical compounds, the study of spatial graphs is in the center of interest for last decades.

It is well known due to S. Kinoshita [13, 14] that the Alexander ideal and Alexander polynomial are invariants of spatial graphs which are determined by the fundamental groups of the complements of spatial graphs.

In 1989, S. Yamada [24] introduced a polynomial for spatial graphs in $\mathbb{R}^3$. It is a concise and useful ambient isotopy invariant for graphs with maximal degree less than 4. There are many interesting results on the Yamada polynomial and its generalizations. J. Murakami [18] investigated the two-variable extension $Z_S$ of the Yamada polynomial and constructed an invariant related to the HOMFLY polynomial. In 1994, the crossing number of spatial graphs in terms of the reduced degree of the Yamada polynomial was studied by T. Motohashi, Y. Ohyama, and K. Taniyama [17]. In 1996, A. Dobrynin and A. Vesnin [23] studied the properties of the Yamada polynomial of spatial graphs. For any graph $G$, V. Vershinin and A. Vesnin [22] defined bigraded cohomology groups whose graded Euler characteristic is a multiple of the Yamada polynomial of $G$. Another invariant of spatial graphs associated with $U_q(\mathfrak{sl}(2, \mathbb{C}))$ was introduced by S. Yoshinaga [25]. In [7] a relation

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between the Yamada and Yoshinaga polynomials is described. Nice results on the structure of the Yamada and flow polynomials of cubic graphs were established by I. Agol and V. Krushkal [1].

A polynomial invariant of virtual graphs was constructed by Y. Miyazawa [16] as an extension of the Yamada polynomial in 2006 (see also the paper [8] by T. Fleming and B. Mellor). The generalized Yamada polynomials of virtual spatial graphs were recently introduced by Q. Deng, X. Jin and L. H. Kauffman in [6].

In this paper we will study the zeros of the Yamada polynomials. Recall that the zeros of polynomial invariants of knots and graphs are a question of special interest studied by A. D. Sokal [21] and O. T. Dasbach, T. D. Le, and X.-S. Lin [5] and X. Jin, F. Zhang, F. Dong, and E. G. Tay [10] for the Jones polynomial.

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The following properties of \( G \) hold (see [24] for details):

1. \( H(\bullet) = -1 \).
2. Let \( e \) be a non-loop edge of a graph \( G \). Then \( H(G) = H(G/e) + H(G - e) \), where \( G/e \) is the graph obtained by contracting the edge \( e \) and \( G - e \) is the graph obtained by deleting the edge \( e \).
3. Let \( e \) be a loop edge of a graph \( G \). Then \( H(G) = -\sigma H(G - e) \), where \( \sigma = A + 1 + A^{-1} \).
4. Let \( G_1 \cup G_2 \) be a disjoint union of graphs \( G_1 \) and \( G_2 \). Then \( H(G_1 \cup G_2) = H(G_1)H(G_2) \).
5. Let \( G_1 \cdot G_2 \) be a union of graphs \( G_1 \) and \( G_2 \) that have one common point. Then \( H(G_1 \cdot G_2) = -H(G_1)H(G_2) \).
6. If \( G \) has an isthmus, then \( H(G) = 0 \).
Fig. 1. The graph $\Theta_s$.

It is easy to find directly (see also [23]) the polynomial $H(G)$ for some simple classes of graphs.

**Lemma 2.3.** The following properties hold with $\sigma = A + 1 + A^{-1}$:

(i) Let $T_q$ be a tree with $q$ edges. Then $H(T_q) = 0$ for all $q$.

(ii) Let $C_n$ be the cycle of length $n$. Then $H(C_n) = \sigma$ for all $n$.

(iii) Let $B_q$ be the one-vertex graph with $q$ loops, also known as the $q$-bouquet. Then $H(B_q) = (-1)^{q-1} \sigma^q$.

(iv) Let $\Theta_s$ be the graph consisting of two vertices and $s$ edges between them, also known as the $s$-theta-graph (see Fig. 1). Then

$$H(\Theta_s) = \frac{1}{\sigma^2} [\sigma + (-\sigma)^s].$$

The following property of $H(G)$ was obtained in [15].

**Proposition 2.4** [15]. Let $G_1 : G_2$ be the union of two graphs $G_1$ and $G_2$ that have only two common vertices $u$ and $v$. Let $K_1$ and $K_2$ be the graphs obtained from $G_1$ and $G_2$, respectively, by identifying $u$ and $v$. Then

$$H(G_1 : G_2) = \frac{1}{\sigma^2} [H(K_1)H(K_2) + (\sigma + 1)H(G_1)H(G_2) + H(K_1)H(G_2) + H(K_2)H(G_1)].$$

Before further discussions we recall some properties of the chain polynomial introduced by R. C. Read and E. G. Whitehead [20] (see also [10]). The chain polynomial is defined for edge-labeled graphs with labels being elements of a commutative ring with unity. We will denote edges by the labels associated with them.

**Definition 2.5.** The chain polynomial $\text{Ch}(G)$ of a labeled graph $G$ is defined as

$$\text{Ch}(G) = \sum_{Y \subseteq E} F_{G-Y}(1 - w) \prod_{a \in Y} a,$$

where the sum is taken over all subsets of the edge set $E$ of $G$, $F_{G-Y}(1 - w)$ denotes the flow polynomial of the subgraph $G - Y$ calculated at $1 - w$, and $\prod_{a \in Y}$ denotes the product of edge labels of $Y$.

The chain polynomial can also be defined in the following recursive form.

**Definition 2.6.** The chain polynomial $\text{Ch}(G)(w)$ in a variable $w$ of a labeled graph $G$ is defined by following rules:

1. If $G$ is edgeless, then $\text{Ch}(G) = 1$.
2. Otherwise, suppose $a$ is an edge of $G$ labeled by $a$. Then
   - if $a$ is a loop, $\text{Ch}(G) = (a - w) \text{Ch}(G - a)$;
   - if $a$ is not a loop, $\text{Ch}(G) = (a - 1) \text{Ch}(G - a) + \text{Ch}(G/a)$.
For the reader’s convenience we demonstrate chain polynomials for the simplest classes of graphs.

**Example 2.7.** For the $n$-cycle $C_n$ with edges labeled by $a_1, a_2, \ldots, a_n$, we have
\[
\text{Ch}(C_n) = \prod_{i=1}^{n} (a_i - w).
\]

For the $s$-theta-graph $\Theta_s$ with edges labeled by $a_1, a_2, \ldots, a_s$, we have
\[
\text{Ch}(\Theta_s) = \frac{1}{1 - w} \left[ \prod_{i=1}^{s} (a_i - w) - w \prod_{i=1}^{s} (a_i - 1) \right].
\]

For the $q$-bouquet $B_q$ with loops labeled by $a_1, a_2, \ldots, a_q$, we have
\[
\text{Ch}(B_q) = \prod_{i=1}^{q} (a_i - w).
\]

To explore the relation (inspired by [10]) between the chain polynomial and the Yamada polynomial, let us introduce the following notation. Let $G$ be a connected labeled graph and $K_E$ be a family of connected graphs with two distinguished vertices. Denote by $G(K_E)$ the graph obtained from $G$ by replacing each edge $a = uv$ of $G$ by a connected graph $K_a \in K_E$ with two distinguished vertices $u$ and $v$ that has only the vertices $u$ and $v$ in common with $(G - a)(K_E)$.

Let $K'_a$ be the graph obtained from $K_a$ by identifying $u$ and $v$, the two distinguished vertices of $K_a$. Define
\[
\alpha_a = \alpha(K_a) := \frac{1}{\sigma} \left[ (\sigma + 1)H(K_a) + H(K'_a) \right], \quad \beta_a = \beta(K_a) := \frac{1}{\sigma} \left[ H(K_a) + H(K'_a) \right],
\]
and
\[
\gamma_a = \gamma(K_a) := 1 - \frac{\alpha(K_a)}{\beta(K_a)}.
\]

It is easy to see that
\[
H(K'_a) = (\sigma + 1)\beta_a - \alpha_a \quad \text{and} \quad H(K_a) = \alpha_a - \beta_a.
\]

The following result from [15] gives the relation between the Yamada polynomial and the chain polynomial.

**Theorem 2.8** [15]. Let $G$ be a connected labeled graph, and let $G(K_E)$ be the graph obtained from $G$ by replacing the edge $a$ by a connected graph $K_a \in K_E$ for every edge $a$ of $G$. If we replace $w$ by $-\sigma$ and replace $a$ by $\gamma_a$ for every label $a$ in $\text{Ch}(G)$, then we get
\[
H(G(K_E)) = \frac{\prod_{a \in E(G)} \beta_a}{(-1)^{q(G)} p(G)} \text{Ch}(G),
\]
where $p(G)$ and $q(G)$ are the number of vertices and the number of edges of $G$, respectively.

**Example 2.9.** Consider a cyclic graph $G = C_n$ with all edges labeled by $a$ and replace each of its edges by the graph $K_a = \Theta_s$. These graphs as well as the resulting graph $C_n(\Theta_s)$ are presented in Fig. 2 for $n = 4$ and $s = 3$.

Since in this case $K'_a = B_s$, we get
\[
H(K_a) := H(\Theta_s) = \frac{1}{\sigma + 1} \left[ \sigma + (-\sigma)^s \right], \quad H(K'_a) := H(B_s) = (-1)^{s-1} \sigma^s.
\]
A regular projection of a diagram transversal edges. A projection with overcrossing/undercrossing information for all crossings is a flat vertex graph for each graph. Let $G$ be a graph embedded in $\mathbb{R}^3$; we say that $G$ is a spatial graph. If for each vertex $v$ of $G$ there exists a neighborhood $U_v$ of $v$ and a flat plane $P_v$ such that $G \cap U_v \subset P_v$, then we say that $G$ is a flat vertex graph. For two spatial graphs $G$ and $G'$, if there exists an isotopy $\phi_t: \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,1]$, such that $\phi_0 = \text{id}$ and $\phi_1(G) = G'$, then we say that $G$ and $G'$ are ambient isotopic as pliable vertex graphs (pliably isotopic). For two flat vertex graphs $G$ and $G'$, if there exists an isotopy $h_t: \mathbb{R}^3 \to \mathbb{R}^3$ with $t \in [0,1]$ such that $h_0 = \text{id}$, $h_1(G) = G'$, and $h_t(G)$ is a flat vertex graph for each $t \in [0,1]$, then we say that $G$ and $G'$ are ambient isotopic as flat vertex graphs (flatly isotopic).

Let $G \subset \mathbb{R}^3$ be a spatial graph. According to [24], we say that a projection $p: \mathbb{R}^3 \to \mathbb{R}^2$ is a regular projection corresponding to $G$ if each multiple point of $p(G)$ is a double point with two transversal edges. A projection with overcrossing/undercrossing information for all crossings is a diagram of $G$. The classical Reidemeister moves $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ of knot diagrams and the Reidemeister moves $\mathcal{R}_4$, $\mathcal{R}_5$, and $\mathcal{R}_6$ of neighborhoods of graph vertices are presented in Fig. 3. We say that a deformation generated by $\mathcal{R}_1, \ldots, \mathcal{R}_6$ is pliable deformation, a deformation generated by $\mathcal{R}_1, \ldots, \mathcal{R}_5$ is a flat deformation, and a deformation generated by $\mathcal{R}_0$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$ is a regular deformation.

Let $g$ be a diagram of $G$. For any double point, S. Yamada [24] defined the spins $+1$, $-1$, and 0, which are denoted by $S_+$, $S_-$, and $S_0$, as shown in Fig. 4.

Let $S$ be a plane graph obtained from $g$ by replacing each double point with a spin. The graph $S$ is called a state on $g$. Denote the set of all states by $U(g)$. Put

$$c(g|S) = A^{m_1 - m_2},$$

where $m_1$ and $m_2$ are the numbers of double points with spin $S_+$ and $S_-$, respectively, used to obtain $S$ from $g$. 

**Fig. 2.** Graphs $G = C_n$, $K_\alpha = \Theta_s$, and $C_n(\Theta_s)$ for $n = 4$ and $s = 3$. 

Introduce the notation

$$\alpha := \alpha_n = \frac{1}{n \sigma}[(\sigma + 1)H(\Theta_s) + H(B_s)], \quad \beta := \beta_n = \frac{1}{n \sigma}[H(\Theta) + H(B)], \quad \gamma := \gamma_n.$$

Recall that $\text{Ch}(C_n) = \prod_{i=1}^{n} a_i - w$ and $p(C_n) = q(C_n) = n$. Then

$$H(C_n(\Theta_s)) = \beta^n(\gamma^n - (\sigma))^{n} = (\beta \gamma^n + \sigma \beta^n = (\beta - \sigma)^n + \sigma \beta^n = (-\sigma H(\Theta_s))^n + \sigma \beta^n = 
\left( -\frac{1}{\sigma + 1}[\sigma + (\sigma)^{n}] \right)^n + \frac{1}{\sigma^n} [H(\Theta) + H(B)]^n
\left[ \frac{\sigma + (\sigma)^{n}}{\sigma + 1} + (-1)^{n-1} \right]^n.$$
Definition 3.1 [24]. The Yamada polynomial of a diagram $g$ of a spatial graph $G$ is a Laurent polynomial in a variable $A$ defined as follows:

$$R[g] = R[g](A) := \sum_{S \in U(g)} c(g|S)H(S),$$

where $H(S) = h(S)(-1, -A - 2 - A^{-1})$.

The role of the Yamada polynomial in studying spatial graphs is clear from the following result.

Theorem 3.2 [24]. The Yamada polynomial has the following properties:

1. $R[g]$ is a regular deformation invariant of a diagram $g$;
2. $R[g]$ is a flat deformation invariant of a diagram $g$ up to multiplying by $(-A)^n$ for some integer $n$;
3. if $g$ is a diagram of a graph whose maximum degree is less than 4, then $R(g)$ is a pliable deformation invariant of $g$ up to multiplying by $(-A)^n$ for some integer $n$.

Here we recall only some properties of the Yamada polynomial.

Proposition 3.3 [24]. The following properties hold:

1. Let $g_1 \cup g_2$ be a disjoint union of diagrams $g_1$ and $g_2$. Then
   $$R[g_1 \cup g_2] = R[g_1]R[g_2].$$
(2) Let \( g_1 \cdot g_2 \) be a union of diagrams \( g_1 \) and \( g_2 \) that have one common point. Then
\[
R[g_1 \cdot g_2] = -R[g_1]R[g_2].
\]

(3) If \( g \) has an isthmus, then \( R[g] = 0 \).

**Remark 3.4.** If a diagram \( g \) of \( G \) does not have double points, then \( R[g] = H(G) \).

The properties of \( H(G) \) immediately imply the following properties of \( R(G) \).

**Proposition 3.5** [24]. The following properties hold:

1. \( R[\bigtriangleup] = AR[\bigtriangleup] + A^{-1}R[\bigtriangleup] + R[\bigtriangleup] \)
2. \( R[\bigtriangledown \bigtriangleup] = R[\bigtriangledown \bigtriangleup] + R[\bigtriangledown \bigtriangleup] \)
3. \( R[\bigcirc] = R[\bigcirc] = A + 1 + A^{-1} \)
4. \( R[B_n] = -(-\sigma)^n \), where \( B_n \) is the unlinked one-vertex graph with \( n \) loops and \( \sigma = A + 1 + A^{-1} \)

The formula for \( H(G) \) from Proposition 2.4 implies a similar formula for the polynomial \( R[g] \).

**Proposition 3.6** [15]. Let \( g_1 : g_2 \) be the union of two diagrams \( g_1 \) and \( g_2 \) that have only two common vertices \( u \) and \( v \). Let \( k_1 \) and \( k_2 \) be the diagrams obtained from \( g_1 \) and \( g_2 \), respectively, by identifying \( u \) and \( v \). Then
\[
R[g_1 : g_2] = \frac{1}{\sigma} \left( R[k_1]R[k_2] + (\sigma + 1)R[g_1]R[g_2] + R[k_1]R[g_2] + R[k_2]R[g_1] \right).
\]

Let \( G \) be a connected plane labeled graph in which each edge \( a \) with terminal vertices \( u \) and \( v \) is labeled by a diagram \( g_a \) with two distinguished vertices \( u_{a*} \) and \( v_{a*} \). We define \( G(g_a) \) to be the spatial graph (its diagram) obtained from \( G \) by replacing an edge \( a = u_av_a \) of \( G \) by a connected diagram \( g_a \) and identifying \( u_a \) with \( u_{a*} \) and \( v_a \) with \( v_{a*} \) in such a way that \( g_a \) has only the vertices \( u_a = u_{a*} \) and \( v_a = v_{a*} \) in common with \( G - a \). We define \( G(g_a, g_b) \) to be the spatial graph (its diagram) obtained from \( G(g_a) \) by replacing an edge \( b = u bv_b \) of \( G(g_a) \cap (G - a) \) by a connected diagram \( g_b \) and identifying \( u_b \) with \( u_{b*} \) and \( v_b \) with \( v_{b*} \) in such a way that \( g_b \) has only the vertices \( u_b = u_{b*} \) and \( v_b = v_{b*} \) in common with \( G(g_a) \). Applying the same construction, we can obtain \( G(g_{a_1}, g_{a_2}, g_{a_3}, \ldots) \). In this context we denote by \( g'_a \) the diagram obtained from \( g_a \) by identifying the vertices \( u_{a*} \) and \( v_{a*} \) in such a way that no new intersections appear.

**Theorem 3.7** [15]. The following properties hold:

1. For the \( n \)-cycle graph \( C_n \) with edges labeled by \( a_1, a_2, \ldots, a_n \),
\[
R[C_n(g_{a_1}, g_{a_2}, \ldots, g_{a_n})] = \prod_{i=1}^{n} \left( -R[g_{a_i}] \right) + \sigma \prod_{i=1}^{n} \left( \frac{R[g_{a_i}] + R[g'_{a_i}]}{\sigma} \right).
\]
2. For the \( s \)-theta-graph \( \Theta_s \) with edges labeled by \( a_1, a_2, \ldots, a_s \),
\[
R[\Theta_s(g_{a_1}, g_{a_2}, \ldots, g_{a_s})] = (-1)^s \prod_{i=1}^{s} R[g'_{a_i}] + \sigma \prod_{i=1}^{s} \left( \frac{\sigma + 1}{\sigma} R[g_{a_i}] + \frac{1}{\sigma} R[g'_{a_i}] \right).
\]
3. For the \( q \)-bouquet \( B_q \) with loops labeled by \( a_1, a_2, \ldots, a_q \),
\[
R[B_q(g_{a_1}, g_{a_2}, \ldots, g_{a_q})] = (-1)^{q-1} \prod_{i=1}^{q} R[g'_{a_i}].
\]

If every edge of \( G \) is labeled by the same diagram \( g_a = g \), we denote \( G(g, \ldots, g) \) shortly by \( G(g) \).
Corollary 3.8 [15]. If all edges of a graph $G$ are labeled by the same diagram $g$, then

$$R[C_n(g)] = (-R[g])^n + \sigma \left( \frac{R[g] + R[g']}{\sigma} \right)^n,$$

$$R[\Theta_s(g)] = \left(\frac{(-1)^s}{1 + \sigma} R[g']^s + \frac{\sigma}{1 + \sigma} \left( \frac{\sigma + 1}{\sigma} R[g] + \frac{1}{\sigma} R[g'] \right) \right)^s,$$

$$R[B_q(g)] = (-1)^{q-1} R[g']^q.$$

Remark that $C_n(g_{a_1}, g_{a_2}, \ldots, g_{a_n})$ is the “ring of beads” discussed in [9] and [20]. Formula (3.1) shows that the Yamada polynomial $R[C_n(g_{a_1}, g_{a_2}, \ldots, g_{a_n})]$ of a graph $C_n(g_{a_1}, g_{a_2}, \ldots, g_{a_n})$ is independent of the order in which the subdiagrams occur.

The following example illustrates an application of Theorem 3.7.

Example 3.9. Let $\infty_+$ be the spatial graph diagram with two vertices and one double point signed by “+” (Figs. 5 and 6). We denote the mirror image of the diagram $\infty_+$ by $\infty_-$. Direct calculations give the Yamada polynomials $R[\infty_+] = A^{-2}\sigma$ and $R[\infty'_+] = \sigma$, where $\sigma = A + 1 + A^{-1}$.

By Theorem 3.7, the Yamada polynomial of the diagram $C_n(\infty_+)$ presented in Fig. 5 is as follows:

$$R[C_n(\infty_+)] = (-A^{-2}\sigma)^n + \sigma(A^{-2} + 1)^n.$$

The Yamada polynomial of the diagram $\Theta_s(\infty_+)$ presented in Fig. 6 is as follows:

$$R[\Theta_s(\infty_+)] = \frac{(-\sigma)^s + \sigma((\sigma + 1)A^{-2} + 1)^s}{1 + \sigma}.$$

Similarly, the Yamada polynomial of $B_q(\infty_+)$ is as follows:

$$R[B_q(\infty_+)] = (-1)^{q-1}\sigma^q.$$

4. ZEROS OF THE YAMADA POLYNOMIAL

In this section we will discuss the zeros of the Yamada polynomial. To study the zeros of families of polynomials, we will apply the following results by S. Beraha, J. Kahane, and N. J. Weiss [2] and by A. D. Sokal [21].
The diagrams \( \infty^k_+ \) and \( \infty^k_- \).

**Lemma 4.1** [2]. Let \( \{f_n(x)\} \) be a family of polynomials such that

\[
f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + \ldots + \alpha_l(x)\lambda_l(x)^n,
\]

where \( \alpha_i(x) \) and \( \lambda_i(x) \) are fixed nonzero polynomials such that the equality \( \lambda_i(x) \equiv \omega \lambda_j(x) \) does not hold for any pair \( i \neq j \) and for any complex number \( \omega \) of unit modulus. Then \( z \) is a limit of zeros of \( \{f_n(x)\} \) if and only if one of the following conditions is satisfied:

1. two or more of \( \lambda_i(z) \) are of equal modulus and are strictly greater in modulus than the others;
2. for some \( j \), the modulus of \( \lambda_j(z) \) is strictly greater than those of the others, and \( \alpha_j(z) = 0 \).

**Lemma 4.2** [21]. Let \( F_1 \), \( F_2 \), and \( G \) be analytic functions on a disc \( \{z \in \mathbb{C} : |z| < R\} \) such that \( |G(0)| \leq 1 \) and \( G \) is not constant. Then, for every \( \epsilon > 0 \), there exists \( s_0 < \infty \) such that for all integers \( s \geq s_0 \) the equation

\[
|1 + F_1(z)G(z)^s| = |1 + F_2(z)G(z)^s|
\]

has a solution in the disc \( |z| < \epsilon \).

The study of zeros of the Yamada polynomial was started in [15].

**Theorem 4.3** [15]. The zeros of the Yamada polynomial of spatial graphs are dense in the following region:

\[
\Omega = \{ z \in \mathbb{C} : |z + 1 + z^{-1}| \geq \min\{1, |z^3 + 2z^2 + z + 1|, |1 + z^{-1} + 2z^{-2} + z^{-3}|\} \}.
\]

Thus, a natural question arises: are the zeros of the Yamada polynomial dense in the whole complex plane? The following theorem gives an affirmative answer.

**Theorem 4.4.** The zeros of the Yamada polynomial of spatial graphs are dense in the whole complex plane.

The proof of Theorem 4.4 is constructive. We will explicitly describe the family of spatial graphs with the required property. The construction will consist of a few steps. First, denote by \( \infty^k_+ \) \( (\infty^k_-) \) the diagrams with two vertices and \( k \) positive \( (\text{respectively, negative}) \) double points, as shown in Fig. 7.

For these diagrams we easily get

**Lemma 4.5.** The following properties hold:

\[
R[\infty^k_+] = A^{-2k}\sigma
\]

and

\[
R[(\infty^k_+)^\prime] = (-1)^k A^{-k}\sigma(A + A^{-1}) - A^{-2k}\sigma.
\]

**Proof.** Indeed, denote by \( Q_k \) the diagram with one bivalent vertex and \( k \) double points as presented in Fig. 8 on the left.

To prove (4.1), we observe that by properties (2) and (3) in Proposition 3.3 and property (1) in Proposition 3.5, the following relations hold:

\[
R[Q_k] = AR[Q_{k-1}] + A^{-1}\sigma R[Q_{k-1}] - \sigma R[Q_{k-1}] = (A + A^{-1}\sigma - \sigma)R[Q_{k-1}] = A^{-2}R[Q_{k-1}].
\]
Hence $R[Q_k] = A^{-2k}\sigma$. By property (3) in Proposition 3.5, we have $R[\infty_+^0] = R[Q_0] = \sigma$. The induction hypothesis $R[\infty_+^{k-1}] = R[Q_{k-1}]$ implies (4.1). Indeed,

$$R[\infty_+^k] = AR[\infty_+^{k-1}] + A^{-1}\sigma R[Q_{k-1}] - \sigma R[\infty_+^{k-1}] = A^{-2}R[\infty_+^{k-1}] = R[Q_k] = A^{-2k}\sigma.$$ 

To demonstrate (4.2), we will also apply induction on $k$. Let us introduce the shorter notation $W_k = (\infty_+^k)'$, where $W_k$ is the diagram with one four-valent vertex and $k$ double points as pictured in Fig. 8 on the right. First we verify that (4.2) holds for $k = 1$. Indeed, by Example 3.9, $R[W_1] = R[(\infty_+^1)'] = \sigma$ and the right-hand side of (4.2) is also equal to $\sigma$. It is easy to see that the diagram $W_k$ satisfies the skein relation with the diagrams obtained from $W_k$ by replacing the $k$th double point with spins $+1$, $-1$, and $0$. For the reader’s convenience we present these diagrams for the case $k = 3$ in Fig. 9. It is easy to see that in general we get $S_+(W_k) = W_{k-1}$, $S_-(W_k)$ is a dot-product of $Q_k$ and the loop $B_1$, and $S_0(W_k)$ is a two-vertex diagram with $k - 1$ double points, which we denote by $X_{k-1}$.

By writing this skein relation (see property (1) in Proposition 3.5), we get

$$R[W_k] = AR[W_{k-1}] + A^{-1}R[B_1 \cdot Q_{k-1}] + R[X_{k-1}].$$

Applying property (2) in Proposition 3.3 and twice property (2) in Proposition 3.5, we obtain

$$R[W_k] = AR[W_{k-1}] - A^{-1}\sigma R[Q_{k-1}] + (-\sigma)R[W_{k-1}] + R[W_{k-1}] + R[Q_{k-1}]$$

$$= (A - \sigma + 1)R[W_{k-1}] + (1 - A^{-1}\sigma)R[Q_k]$$

$$= -A^{-1}R[W_{k-1}] + (A^{-1} - A^{-2})A^{-2(k-1)}\sigma.$$ 

Assume that (4.2) holds for $k - 1$. Then we get

$$R[W_k] = -A^{-1}((-1)^{k-2}A^{-k(1-1)}A(A + A^{-1}) - A^{-2(k-1)}\sigma) + (-A^{-1} - A^{-2})A^{-2(k-1)}\sigma$$

$$= (-1)^{k-1}A^{-k}\sigma(A + A^{-1}) + A^{-2k+1}\sigma - A^{-2k+1}\sigma - A^{-2k}\sigma$$

$$= (-1)^{k-1}A^{-k}\sigma(A + A^{-1}) - A^{-2k}\sigma.$$ 

Hence (4.2) holds for $k$ and the lemma is proved. □
**Proof of Theorem 4.4.** To prove Theorem 4.4, we will calculate the Yamada polynomial of the diagram $C_n(\Theta_s(\infty^k_+))$, which is obtained when we replace every edge of the graph $C_n(\Theta_s)$, which has $n$ vertices and $ns$ edges (see Fig. 2), by the diagram $\infty^k_+$ (see Fig. 7).

The Yamada polynomial of such a graph can be calculated by the formula given in Corollary 3.8:

$$R[C_n(\Theta_s(\infty^k_+))] = (-R[\Theta_s(\infty^k_+)])^n + \sigma \left( \frac{R[\Theta_s(\infty^k_+)] + R[(\Theta_s(\infty^k_+))]'}{\sigma} \right)^n.$$ 

By condition (1) of Lemma 4.1, each complex number $A$ satisfying the equation

$$|-R[\Theta_s(\infty^k_+)]| = \left| \frac{R[\Theta_s(\infty^k_+)] + R[(\Theta_s(\infty^k_+))]'}{\sigma} \right|$$

is a limit of zeros of the Yamada polynomials $\{R[C_n(\Theta_s(\infty^k_+))]\}_{n=1}^{\infty}, \infty, \infty$. To shorten the notation in (4.2), we set

$$M_k = (-1)^{k-1}A^{-k}(A + A^{-1}).$$

Then (4.2) can be rewritten as

$$R[(\infty^k_+)'] = M_k\sigma - A^{-2k}\sigma = \sigma(M_k - A^{-2k}).$$

Combining Corollary 3.8 with (4.1) and (4.2), we have

$$R[\Theta_s(\infty^k_+)] = \frac{(-1)^s}{1 + \sigma} R[(\infty^k_+')]^s + \frac{\sigma}{1 + \sigma} \left( \frac{\sigma + 1}{\sigma} R[\infty^k_+] + \frac{1}{\sigma} R[(\infty^k_+)'] \right)^s$$

$$= \frac{(-1)^s}{1 + \sigma} (M_k\sigma - A^{-2k}\sigma)^s + \frac{\sigma}{1 + \sigma} \left( \frac{\sigma + 1}{\sigma} A^{-2k}\sigma + \frac{M_k\sigma + A^{-2k}\sigma}{\sigma} \right)^s$$

$$= \frac{(-1)^s\sigma^s}{1 + \sigma} (M_k - A^{-2k})^s + \frac{\sigma}{1 + \sigma} (M_k + A^{-2k}\sigma)^s$$

(4.4)

and, since $(\Theta_s)' = B_s$, we get

$$R[(\Theta_s(\infty^k_+)')] = (-1)^{s-1}(R[(\infty^k_+)'])^s = (-1)^{s-1}\sigma^s(M_k - A^{-2k})^s.$$ (4.5)

Then (4.3) is equivalent to

$$\left| \frac{(-1)^s}{1 + \sigma} \frac{\sigma^s}{1 + \sigma} (M_k - A^{-2k})^s + \frac{\sigma}{1 + \sigma} (M_k + A^{-2k}\sigma)^s \right|$$

$$= \left| \frac{1}{\sigma} \left( (-1)^s \frac{\sigma^s}{1 + \sigma} (M_k - A^{-2k})^s + \frac{\sigma}{1 + \sigma} (M_k + A^{-2k}\sigma)^s + (-1)^{s-1}\sigma^s(M_k - A^{-2k})^s \right) \right|.$$ 

Hence

$$\left| (-1)^s\sigma^s(M_k - A^{-2k})^s + \sigma(M_k + A^{-2k}\sigma)^s \right| = \left| (-1)^{s-1}\sigma^s(M_k - A^{-2k})^s + (M_k + A^{-2k}\sigma)^s \right|,$$

and then

$$\left| (-1)^s + \frac{\sigma(M_k + A^{-2k}\sigma)^s}{\sigma^s(M_k - A^{-2k})^s} \right| = \left| (-1)^{s-1} + \frac{(M_k + \sigma A^{-2k})^s}{\sigma^s(M_k - A^{-2k})^s} \right|.$$ 

Introducing the notation

$$G(A) = \frac{M_k + \sigma A^{-2k}}{\sigma(M_k - A^{-2k})},$$

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we find that (4.3) is equivalent to
\[ |(-1)^s + \sigma [G(A)]^s| = |(-1)^{s-1} + [G(A)]^s| \]
and each root of this equation is a limit of zeros of the polynomials \( \{ R[C_n(\Theta_s(\infty_k^+))] \}_{n=1,s=1,k=1}^\infty \). Observe that for even and odd \( s \) we have
\[ |1 + \sigma [G(A)]^s| = |1 - [G(A)]^s| \quad \text{and} \quad |1 - \sigma [G(A)]^s| = |1 + [G(A)]^s|, \quad (4.6) \]
respectively, which are both of the form presented in Lemma 4.2. We will discuss one of them, and the other can be considered analogously.

Let us fix a complex number \( z_0 \) such that \( |z_0| \leq 1 \), and set \( a = A - z_0 \). Recall that
\[ \sigma = A + 1 + A^{-1} = (a + z_0) + 1 + (a + z_0)^{-1} \]
and
\[ M_k = (-1)^{k-1} A^{-k}(A + A^{-1}) = (-1)^{k-1}(a + z_0)^{-k}((a + z_0) + (a + z_0)^{-1}). \]
Then the first equation in (4.6) is equivalent to
\[ |1 + ((a + z_0) + 1 + (a + z_0)^{-1})[G(a)]^s| = |1 - [G(a)]^s|, \quad (4.7) \]
where
\[ G(a) = \frac{(-1)^{k-1}(a + z_0)^{-k}((a + z_0) + (a + z_0)^{-1}) + ((a + z_0) + 1 + (a + z_0)^{-1})(a + z_0)^{-2k}}{((a + z_0) + 1 + (a + z_0)^{-1})((-1)^{k-1}(a + z_0)^{-k}((a + z_0) + (a + z_0)^{-1}) - (a + z_0)^{-2k})} \]
and we have
\[ |G(0)| = \left| \frac{(-1)^{k-1}z_0^{-k}(z_0 + z_0^{-1}) + (z_0 + 1 + z_0^{-1})z_0^{-2k}}{(z_0 + 1 + z_0^{-1})((-1)^{k-1}z_0^{-k}(z_0 + z_0^{-1}) - (z_0 + 1 + z_0^{-1})z_0^{-2k})} \right|. \quad (4.8) \]
It follows from Lemma 4.2 that for any \( \varepsilon > 0 \) there exists a \( k_0 \) such that for any \( k \geq k_0 \) equation (4.6) has a root in the disc \( |a| < \varepsilon \) and, therefore, the Yamada polynomial \( R[C_n(\Theta_s(\infty_k^+))] \) has a zero \( A \) in the \( \varepsilon \)-neighborhood of \( z_0 \) if \( |G(0)| \leq 1 \).

Now we consider \( G(0) \) as a function
\[ f(z) = \frac{(-1)^{k-1}z^{-k}(z + z^{-1}) + (z + 1 + z^{-1})z^{-2k}}{(z + 1 + z^{-1})((-1)^{k-1}z^{-k}(z + z^{-1}) - (z + 1 + z^{-1})z^{-2k})} \quad (4.9) \]
calculated at the complex number \( z_0 \). We will show that there exists a complex number \( z_1 \) in the \( \varepsilon \)-neighborhood of \( z_0 \) such that \( |f(z_1)| = 1 \).

It is easy to see that the equality \( |f(z)| = 1 \) is equivalent to the equality
\[ \left| 1 + (-1)^{k-1} \frac{z + z^{-1}}{z + 1 + z^{-1}} z^k \right| = \left| 1 + (-1)^{k}(z + z^{-1})z^k \right|. \]
Since we are looking for roots in a neighborhood of \( z_0 \), set \( \zeta = z - z_0 \) and consider the function
\[ \tilde{f}(\zeta) = \frac{(-1)^{k-1}z^{-k}(z + z^{-1}) + (z + 1 + z^{-1})z^{-2k}}{(z + 1 + z^{-1})((-1)^{k-1}z^{-k}(z + z^{-1}) - (z + 1 + z^{-1})z^{-2k})}, \quad (4.10) \]
where \( z = z_0 + \zeta \). The equality \(|\tilde{f}(\zeta)| = 1\) is equivalent to the equality

\[
1 + (-1)^{k-1} \frac{(z_0 + \zeta) + (z_0 + \zeta)^{-1}}{(z_0 + \zeta) + 1 + (z_0 + \zeta)^{-1}}(z_0 + \zeta)^k = 1 + (-1)^k ((z_0 + \zeta) + (z_0 + \zeta)^{-1})(z_0 + \zeta)^k. \tag{4.11}
\]

Let \( g(\zeta) = z_0 + \zeta \). Since \(|z_0| \leq 1\), we have \(|g(0)| \leq 1\). Therefore, we can apply Lemma 4.2 to (4.11). This yields that for any \( \varepsilon > 0 \) there exists \( k_1 \) such that for any integer \( k \geq k_1 \) equation (4.11) has a root \( \zeta_0 \) with \(|\zeta_0| < \varepsilon\). Thus, we find that there exists \( z_1 = z_0 + \zeta_0 \) such that \(|z_1 - z_0| < \varepsilon\) and \(|f(z_1)| = 1\) in (4.9). Hence, if we replace \( z_0 \) by \( z_1 \) in (4.6) and (4.8), we will find that there exists a number \( A \) which is a limit of zeros of the Yamada polynomials \( \{R[C_\infty(\Theta_s(\infty^k)))\}_{n=1,s=1,k=1}^\infty \) and is such that \(|z_1 - A| < \varepsilon\) and \(|z_0 - A| < 2\varepsilon\). Thus, we have proved that the zeros of the Yamada polynomial are dense in the disc \( \{z \in \mathbb{C}: |z| \leq 1\} \).

The relation between the Yamada polynomials of a spatial graph \( g \) and its mirror image \( \hat{g} \) is presented in [24, Proposition 6]:

\[
R[\hat{g}](A) = R[g](A^{-1}).
\]

Since the diagram \( C_n(\Theta_s(\infty^k)) \) is the mirror image of the diagram \( C_n(\Theta_s(\infty^k)) \), from the previous discussions we see that the zeros of the Yamada polynomials \( \{R[C_\infty(\Theta_s(\infty^k)))\}_{n=1,s=1,k=1}^\infty \) are dense in the region \( \{z \in \mathbb{C}: |z| \geq 1\} \).

Summarizing, we find that the zeros of the Yamada polynomial are dense in the whole complex plane. \( \square \)

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