The weak Cartan property for the $p$-fine topology on metric spaces

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Abstract. We study the $p$-fine topology on complete metric spaces equipped with a doubling measure supporting a $p$-Poincaré inequality, $1 < p < \infty$. We establish a weak Cartan property, which yields characterizations of the $p$-thinness and the $p$-fine continuity, and allows us to show that the $p$-fine topology is the coarsest topology making all $p$-superharmonic functions continuous. Our $p$-harmonic and superharmonic functions are defined by means of scalar-valued upper gradients and do not rely on a vector-valued differentiable structure.

Key words and phrases: capacity, coarsest topology, doubling, fine topology, finely continuous, metric space, $p$-harmonic, Poincaré inequality, quasicontinuous, superharmonic, thick, thin, weak Cartan property, Wiener criterion.

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1. Introduction

The aim of this paper is to study the $p$-fine topology and the fine potential theory associated with $p$-harmonic functions on a complete metric space $X$ equipped with a doubling measure $\mu$ supporting a $p$-Poincaré inequality, $1 < p < \infty$.

Nonlinear potential theory associated with $p$-harmonic functions has been studied since the 1960s. For extensive treatises and notes on the history, see the monographs Adams–Hedberg [1] and Heinonen–Kilpeläinen–Martio [32], the latter developing the theory on weighted $\mathbb{R}^n$ (with respect to $p$-admissible weights). Starting in the 1990s a lot of attention has been given to analysis on metric spaces, see e.g. Hajłasz [24], [25], Hajłasz–Koskela [28], Heinonen [29], [30], and Heinonen–Koskela [33]. Around 2000 this initiated studies of $p$-harmonic and $p$-superharmonic functions on metric spaces without a differentiable structure, by e.g. Shanmugalingam [56], Kinnunen–Martio [39], Kinnunen–Shanmugalingam [41], Björn–MacManus–Shanmugalingam [20] and Björn–Björn–Shanmugalingam [12],
[13]. The theory has later been further developed by these and other authors, see the monograph Björn–Björn [7] and the references therein.

While p-harmonic functions are known to be locally Hölder continuous (even on metric spaces, see [41]), p-superharmonic functions are in general only lower semicontinuous. However, at points of discontinuity they still exhibit more regularity than just lower semicontinuity, namely, the limit \( \lim u(x) \), as \( x \to x_0 \), exists along a substantial (in a capacitary sense) part of \( x_0 \)'s neighbourhood and equals \( u(x_0) \). The topology giving rise to such neighbourhoods and limits is called the p-fine topology. Together with the associated fine potential theory it goes back to Cartan in the 1940s in the linear case \( p = 2 \), which has been later systematically studied, see e.g. Fuglede [22], [23] andLukeš–Malý–Zajíček [48].

The nonlinear fine potential theory started in the 1970s, with papers by e.g. Maz'ya [50], Maz'ya–Havin [51], [52], Hedberg [26], Adams–Meyers [3], Meyers [53], Hedberg–Wolff [27], Adams–Lewis [2] and Lindqvist–Martio [47]. See also the notes to Chapter 12 in Heinonen–Kilpeläinen–Martio [32] and Section 2.6 in Malý–Ziemer [49]. In the 1990s the fine potential theory associated with p-harmonic functions was developed further in Heinonen–Kilpeläinen–Martio [31], Kilpeläinen–Malý [36], [37], Latvala [44], [45], [46], and the monograph Malý–Ziemer [49] for unweighted \( \mathbb{R}^n \). The monograph [32] is the main source for fine potential theory on weighted \( \mathbb{R}^n \) (note that Chapter 21, which is only in the second addition, contains some more recent results). See also Mikkenen [54] for related results (in weighted \( \mathbb{R}^n \)) on the Wolf potential. In fact, the Wolf potential appeared already in Maz'ya–Havin [52].

The fine potential theory in metric spaces is more recent, starting with Kinnunen–Latvala [38], J. Björn [18] and Korte [42], where it was shown that p-superharmonic functions on open subsets of metric spaces are p-finely continuous. There are also some related more recent results in Björn–Björn [8] and [9]. As in the classical situation, the p-fine topology on metric spaces is defined by means of p-capacity and p-thin sets, see Section 4.

From now on we drop the \( p \) from the notation and just write e.g. fine and superharmonic even though the notions depend on \( p \). Our first main result complements the results in [18], [31], [38] and [42] as follows.

**Theorem 1.1.** The fine topology is the coarsest topology making all superharmonic functions on open subsets of \( X \) continuous.

The superharmonic functions considered in this and most of the earlier papers on metric spaces are defined through upper gradients (see later sections for precise definitions), which in particular means that we have no equation, only variational inequalities, to work with. In this way the results do not depend on any differentiable structure of the metric space.

The proofs of our main results are based on pointwise estimates of capacitary potentials. These estimates lead in a natural way to a central property which we call the weak Cartan property, see Theorem 5.1. The following consequence is a slight reformulation and extension of the weak Cartan property.

**Theorem 1.2.** Let \( E \subset X \) be an arbitrary set, and let \( x_0 \in \overline{E} \setminus E \). Then the following are equivalent:

(a) \( E \) is thin at \( x_0 \);
(b) \( x_0 \notin \overline{E}^p \), where \( \overline{E}^p \) is the fine closure of \( E \);
(c) \( X \setminus E \) is a fine neighbourhood of \( x_0 \);
(d) there are \( k \geq 2 \) superharmonic functions \( u_1, \ldots, u_k \) in an open neighbourhood of \( x_0 \) such that the function \( v = \max\{u_1, \ldots, u_k\} \) satisfies

\[
v(x_0) < \liminf_{E \ni x \to x_0} v(x);
\]  
(1.1)
(e) condition (d) holds with \( k = 2 \) nonnegative bounded superharmonic functions.

Here and elsewhere, a set \( U \) is a fine neighbourhood of a point \( x_0 \) if it contains a finely open set \( V \ni x_0 \); it is not required that \( U \) itself is finely open. Note also that if \( x_0 \in E \), then \( E \) is thin at \( x_0 \) if and only if \( C_p(\{x_0\}) = 0 \) and \( E \setminus \{x_0\} \) is thin at \( x_0 \). This is a consequence of the following generalization of Theorem 6.33 in Heinonen–Kilpeläinen–Martio \[32\].

**Proposition 1.3.** If \( C_p(\{x_0\}) > 0 \), then \( \{x_0\} \) is thick at \( x_0 \).

Note that the converse statement is trivially true. At points with positive capacity we further improve Theorem 1.2 and obtain the usual Cartan property (with \( k = 1 \)), see Proposition 6.3. (Note that in weighted \( \mathbb{R}^n \) and in metric spaces it can happen that some points have positive capacity while others do not. A sharp condition for when \( C_p(\{x_0\}) > 0 \) is given in Proposition 8.3 in Björn–Björn–Lehrbäck \[10\].) Proposition 6.3 also shows that \( E \) is thin at \( x_0 \in \mathbb{R}^n \setminus E \) with \( C_p(\{x_0\}) > 0 \) if and only if the seemingly weaker condition

\[
\lim_{\rho \to 0} C_p(E \cap B(x_0, \rho)) = 0
\]

holds. This characterization fails for points with zero capacity.

The classical Cartan property says that if \( E \subset \mathbb{R}^n \) is thin at \( x_0 \in \mathbb{R}^n \setminus E \), then for every \( r > 0 \) there is a nonnegative bounded superharmonic function \( u \) on \( B(x_0, r) \) such that

\[
u(x_0) < \liminf_{E \ni x \to x_0} u(x),
\]

see Theorem 1.3 in Kilpeläinen–Malý \[37\] or Theorem 2.130 in Malý–Ziemer \[49\] for the nonlinear case on unweighted \( \mathbb{R}^n \), and Theorem 21.26 in Heinonen–Kilpeläinen–Martio \[32\] (only in the second edition) for weighted \( \mathbb{R}^n \). In the generality of this paper, for superharmonic functions defined through upper gradients on metric spaces, it is not known whether the classical Cartan property (with \( k = 1 \)) holds, since its proof is based on the equation rather than on the minimization problem. Using variational methods, we have only been able to prove it for points with positive capacity in Proposition 6.3. However, the weak Cartan property provides us with two superharmonic functions whose maximum in many situations can be used instead of the usual Cartan property (but not always, since the maximum need not be superharmonic). In particular Theorem 1.1 follows quite easily.

The (strong) Cartan property is closely related to the necessity part of the Wiener criterion, as it provides a superharmonic function which is not continuous at \( x_0 \), and can thus be used to obtain a p-harmonic function which does not attain its continuous boundary values at \( x_0 \). The weak Cartan property only leads to the necessity part of the Wiener criterion for certain domains, see Remark 5.6. Due to the lack of equation, the necessity part of the Wiener criterion for general domains in metric spaces is not known for p-harmonic functions defined by means of upper gradients, while for Cheeger p-harmonic functions based on a vector-valued differentiable structure it was proved in J. Björn \[17\]. The sufficiency part of the Wiener criterion in metric spaces was proved in Björn–MacManus–Shanmugalingam \[20\] and J. Björn \[18\]. In Euclidean spaces, the Wiener criterion was obtained in Maz'ya \[50\], Lindqvist–Martio \[47\], Heinonen–Kilpeläinen–Martio \[32\], Kilpeläinen–Malý \[36\] and Mikkoenen \[54\].

The outline of the paper is as follows: In Sections 2 and 3 we introduce the necessary background on metric spaces, upper gradients, Newtonian spaces, capacity and superharmonic functions. In Section 4 we introduce the fine topology, cite the necessary background results, and establish a number of auxiliary results not requiring the weak Cartan property nor the capacitary estimates used to establish
Theorem 1.4. (a) Any quasiopen set $U \subset X$ can be written as $U = V \cup E$, where $V$ is finely open and $C_p(E) = 0$.

(b) Let $u$ be a quasicontinuous function on a quasiopen or finely open set $U$. Then $u$ is finely continuous q.e. in $U$.

A fundamental step in the proof is the fact that the capacity of a set coincides with the capacity of its fine closure, see Lemma 4.8 which generalizes Corollary 4.5 in J. Björn [18].

Section 5 is devoted to the proof of the weak Cartan property (Theorem 5.1). Also Theorem 1.2 is established. In the last section, Section 6, we draw a number of consequences of the weak Cartan property, including Theorem 1.1 and Proposition 1.3, and end the paper by proving the following characterization of fine continuity, which as pointed out in Malý–Ziemer [49] is by no means trivial.

Theorem 1.5. Let $u$ be a function on a fine neighbourhood $U$ of $x_0$. Then the following conditions are equivalent:

(a) $u$ is finely continuous at $x_0$;

(b) the set $\{x \in U : |u(x) - u(x_0)| \geq \varepsilon\}$ is thin at $x$ for each $\varepsilon > 0$;

(c) there exists a set $E$ which is thin at $x_0$ such that

$$u(x_0) = \lim_{U \setminus E \ni x \to x_0} u(x),$$

where the limit is taken with respect to the metric topology.

Many of the results in this paper are known on weighted $\mathbb{R}^n$, but as far as we know, Theorem 1.4 and Proposition 6.3 are new on weighted $\mathbb{R}^n$ and Lemma 4.8 is new even on unweighted $\mathbb{R}^n$. Note also that many of our proofs in Sections 5 and 6 differ from the proofs on weighted $\mathbb{R}^n$, since our approach is purely based on variational inequalities, not on an equation. The proofs of the auxiliary results in Section 4 are analogous to the Euclidean ones, but we have given proofs whenever some technical modifications are required.

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2. Notation and preliminaries

We assume throughout the paper that $1 < p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric $d$ and a positive complete Borel measure $\mu$ such that $0 < \mu(B) < \infty$ for all (open) balls $B \subset X$. The $\sigma$-algebra on which $\mu$ is defined is obtained by the completion of the Borel $\sigma$-algebra. It follows that $X$ is separable. Towards the end of the section we further assume that $X$ is complete and supports a $p$-Poincaré inequality, and that $\mu$ is doubling, which are then assumed throughout the rest of the paper. We also always assume that $\Omega \subset X$ is a nonempty open set.
We say that $\mu$ is doubling if there exists a doubling constant $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in $X$,
\[ 0 < \mu(2B) \leq C\mu(B) < \infty. \]

Here and elsewhere we let $\delta B = B(x_0, \delta r)$. A metric space with a doubling measure is proper (i.e. closed and bounded subsets are compact) if and only if it is complete. See Heinonen [29] for more on doubling measures.

A curve is a continuous mapping from an interval, and a rectifiable curve is a curve with finite length. We will only consider curves which are nonconstant, compact and rectifiable. A curve can thus be parameterized by its arc length.

A curve $\gamma$ is an upper gradient of an extended real-valued function $f$ on $X$ if for all nonconstant, compact and rectifiable curves $\gamma : [0, l] \to X$,
\[ |f(\gamma(0)) - f(\gamma(l))| \leq \int_{\gamma} g\, ds, \tag{2.1} \]
where we follow the convention that the left-hand side is $\infty$ whenever at least one of the terms therein is infinite. If $g$ is a nonnegative measurable function on $X$ and if (2.1) holds for $p$-almost every curve (see below), then $g$ is a $p$-weak upper gradient of $f$.

Here we say that a property holds for $p$-almost every curve if it fails only for

\[ \int_{\Gamma} \rho\, ds = \infty \]
for every curve $\gamma \in \Gamma$. Note that a $p$-weak upper gradient need not be a Borel function, it is only required to be measurable. On the other hand, every measurable function $g$ can be modified on a set of measure zero to obtain a Borel function, from which it follows that $\int_{\gamma} g\, ds$ is defined (with a value in $[0, \infty]$) for $p$-almost every curve $\gamma$. For proofs of these and all other facts in this section we refer to Björn–Björn [7] and Heinonen–Koskela–Shanmugalingam–Tyson [34].

The $p$-weak upper gradients were introduced in Koskela–MacManus [43]. It was also shown there that if $g \in L^p_{\text{loc}}(X)$ is a $p$-weak upper gradient of $f$, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of $f$ such that $g_j - g \to 0$ in $L^p(X)$. If $f$ has an upper gradient in $L^p_{\text{loc}}(X)$, then it has a minimal $p$-weak upper gradient $g_f \in L^p_{\text{loc}}(X)$ in the sense that for every $p$-weak upper gradient $g \in L^p_{\text{loc}}(X)$ of $f$ we have $g_f \leq g$ a.e., see Shanmugalingam [56] and Hajłasz [25]. The minimal $p$-weak upper gradient is well defined up to a set of measure zero in the cone of nonnegative functions in $L^p_{\text{loc}}(X)$. Following Shanmugalingam [55], we define a version of Sobolev spaces on the metric measure space $X$.

**Definition 2.1.** A nonnegative Borel function $g$ on $X$ is an upper gradient of an extended real-valued function $f$ on $X$ if for all nonconstant, compact and rectifiable curves $\gamma : [0, l] \to X$,
\[ |f(\gamma(0)) - f(\gamma(l))| \leq \int_{\gamma} g\, ds, \tag{2.1} \]
where we follow the convention that the left-hand side is $\infty$ whenever at least one of the terms therein is infinite. If $g$ is a nonnegative measurable function on $X$ and if (2.1) holds for $p$-almost every curve (see below), then $g$ is a $p$-weak upper gradient of $f$.

Here we say that a property holds for $p$-almost every curve if it fails only for a curve family $\Gamma$ with zero $p$-modulus, i.e. there exists $0 \leq \rho \in L^p(X)$ such that $\int_{\Gamma} \rho\, ds = \infty$ for every curve $\gamma \in \Gamma$. Note that a $p$-weak upper gradient need not be a Borel function, it is only required to be measurable. On the other hand, every measurable function $g$ can be modified on a set of measure zero to obtain a Borel function, from which it follows that $\int_{\gamma} g\, ds$ is defined (with a value in $[0, \infty]$) for $p$-almost every curve $\gamma$. For proofs of these and all other facts in this section we refer to Björn–Björn [7] and Heinonen–Koskela–Shanmugalingam–Tyson [34].

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**Definition 2.2.** Let
\[ \|f\|_{N^{1,p}(X)} = \left( \int_X |f|^p\, d\mu + \inf_g \int_X g^p\, d\mu \right)^{1/p}, \]
where the infimum is taken over all upper gradients of $f$. The Newtonian space on $X$ is
\[ N^{1,p}(X) = \{f : \|f\|_{N^{1,p}(X)} < \infty\}. \]

The space $N^{1,p}(X)/\sim$, where $f \sim h$ if and only if $\|f - h\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see Shanmugalingam [55]. In this paper we assume that functions in $N^{1,p}(X)$ are defined everywhere, not just up to an equivalence class in the corresponding function space. For a measurable set $E \subset X$, the Newtonian
space $N^{1,p}(E)$ is defined by considering $(E, d|_{E}, \mu|_{E})$ as a metric space on its own. We say that $f \in N^{1,p}_{\text{loc}}(\Omega)$ if for every $x \in \Omega$ there exists a ball $B_x \ni x$ such that $B_x \subset \Omega$ and $f \in N^{1,p}(B_x)$. If $f, h \in N^{1,p}_{\text{loc}}(X)$, then $g_f = g_h$ a.e. in \{x $\in X : f(x) = h(x)\}$, in particular $g_{\min\{f,c\}} = g_f \chi_{\{f < c\}}$ for $c \in \mathbb{R}$.

**Definition 2.3.** The **Sobolev capacity** of an arbitrary set $E \subset X$ is

$$C_p(E) = \inf_u \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on $E$.

The capacity is countably subadditive. We say that a property holds **quasieverywhere (q.e.)** if the set of points for which the property does not hold has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, Corollary 3.3 in Shanmugalingam [55] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

A set $U \subset X$ is **quasiopen** if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) \leq \varepsilon$ and $G \cup U$ is open. A function $u$ on a quasiopen set $U \subset X$ is **quasicontinuous** if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) < \varepsilon$ and $u|_{\Omega \setminus G}$ is finite and continuous.

**Definition 2.4.** We say that $X$ supports a **p-Poincaré inequality** if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all integrable functions $f$ on $X$ and all upper gradients $g$ of $f$,

$$\frac{1}{|B|} \int_B |f - f_B| \, d\mu \leq C \lambda^{1/p} B \left( \int_{\partial B} g^p \, d\mu \right)^{1/p}, \quad (2.2)$$

where $f_B := \frac{1}{|B|} \int_B f \, d\mu := \frac{1}{|B|} \int_B f \, d\mu/\mu(B)$.

In the definition of Poincaré inequality we can equivalently assume that $g$ is a $p$-weak upper gradient—see the comments above. If $X$ is complete and supports a $p$-Poincaré inequality and $\mu$ is doubling, then Lipschitz functions are dense in $N^{1,p}(X)$, see Shanmugalingam [55]. Moreover, all functions in $N^{1,p}(X)$ and those in $N^{1,p}_{\text{loc}}(\Omega)$ are quasicontinuous, see Björn–Björn–Shanmugalingam [14]. This means that in the Euclidean setting, $N^{1,p}(\mathbb{R}^n)$ is the refined Sobolev space as defined in Heinonen–Kilpeläinen–Martio [32, p. 96], see Björn–Björn [7] for a proof of this fact valid in weighted $\mathbb{R}^n$. This is the main reason why, unlike in the classical Euclidean setting, we do not need to require the functions admissible in the definition of capacity to be 1 in a neighbourhood of $E$.

In Section 4 the fine topology is defined by means of thin sets, which in turn use the variational capacity $\text{cap}_p$. To be able to define the variational capacity we first need a Newtonian space with zero boundary values. We let, for an arbitrary set $A \subset X$,

$$N^{1,p}_0(A) = \{f|_A : f \in N^{1,p}(X) \text{ and } f = 0 \text{ on } X \setminus A\}.$$ 

One can replace the assumption “$f = 0$ on $X \setminus A$” with “$f = 0$ q.e. on $X \setminus A$” without changing the obtained space $N^{1,p}_0(A)$. Functions from $N^{1,p}_0(A)$ can be extended by zero in $X \setminus A$ and we will regard them in that sense if needed.

**Definition 2.5.** Let $A \subset X$ be arbitrary. The **variational capacity** of $E \subset A$ with respect to $A$ is

$$\text{cap}_p(E, A) = \inf_u \int_X g^p_u \, d\mu,$$

where the infimum is taken over all $u \in N^{1,p}_0(A)$ such that $u \geq 1$ on $E$. 

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Remark 2.6. The infimum above can equivalently be taken over $u \in N^{1,p}(X)$ such that $u \geq 1$ q.e. on $E$ and $u = 0$ q.e. outside $A$. We will call such functions admissible for the capacity $\text{cap}_p(E, A)$.

Similarly, one can test the capacity $C_p(E)$ by any function $u \in N^{1,p}(X)$ such that $u \geq 1$ q.e. on $E$, and we will call such a function admissible for $C_p(E)$.

We will mainly be interested in the variational capacity with respect to open sets $A$, but in Lemma 4.8 we will generalize an earlier result for the variational capacity to arbitrary sets. The variational capacity with respect to nonopen sets was recently studied and used in Björn–Björn [8] and [9]. (Note that the roles of $A$ and $E$ are reversed in [8] and [9] compared with this paper.)

Throughout the rest of the paper, we assume that $X$ is complete and supports a $p$-Poincaré inequality, and that $\mu$ is doubling.

The following lemma from J. Björn [16] compares the capacities $C_p$ and $\text{cap}_p$, and the measure $\mu$. Here and elsewhere, the letter $C$ denotes various positive constants whose values may vary even within a line.

Lemma 2.7. Let $E \subset B = B(x_0, r)$ with $0 < r < \frac{1}{6} \text{diam}(X)$. Then

$$\frac{\mu(E)}{Cr^p} \leq \text{cap}_p(E, 2B) \leq \frac{C\mu(B)}{r^p}$$

and

$$\frac{C_p(E)}{C(1 + r^p)} \leq \text{cap}_p(E, 2B) \leq 2^p \left(1 + \frac{1}{r^p}\right)C_p(E).$$

In particular,

$$\frac{\mu(B)}{C r^p} \leq \text{cap}_p(B, 2B) \leq \frac{C\mu(B)}{r^p}.$$

We will also need the following result from Björn–Björn–Shanmugalingam [14]. (It was recently extended to arbitrary bounded sets $\Omega$ in Björn–Björn [9], but we will not need that generality here.) Recall that $E \subset \Omega$ if $E$ is a compact subset of $\Omega$.

Theorem 2.8. Let $\Omega \subset X$ be a bounded open set. The variational capacity $\text{cap}_p$ is an outer capacity for sets $E \subset \Omega$, i.e.

$$\text{cap}_p(E, \Omega) = \inf_{G \subset G \subset \Omega} \text{cap}_p(G, \Omega).$$

3. Superminimizers and superharmonic functions

In this section we introduce superminimizers and superharmonic functions, as well as obstacle problems, which all will be needed in later sections. For further discussion and references on these topics see Kinnunen–Martio [39] and [40], and also Björn–Björn [7] (which also contains proofs of the facts mentioned in this section, but for Lemma 3.7).

Definition 3.1. A function $u \in N^{1,p}_{\text{loc}}(\Omega)$ is a (super)minimizer in $\Omega$ if

$$\int_{\{\varphi \neq 0\}} g^p \, d\mu \leq \int_{\{\varphi \neq 0\}} g^p_{\mu + \varphi} \, d\mu \quad \text{for all (nonnegative) } \varphi \in N^{1,p}_{0}(\Omega).$$

A function $u$ is a subminimizer if $-u$ is a superminimizer. A $p$-harmonic function is a continuous minimizer.
For characterizations of minimizers and superminimizers see A. Björn [5]. Minimizers were first studied for functions in $N^{1,p}(X)$ in Shanmugalingam [36]. For a superminimizer $u$, it was shown by Kinnunen–Martio [39] that its lower semicontinuous regularization

$$u^*(x) := \text{ess lim inf}_{y \to x} u(y) = \lim_{r \to 0} \text{ess inf}_{B(x,r)} u$$

(3.1)

is also a superminimizer and $u^* = u$ q.e. For an alternative proof of this fact see Björn–Björn–Parvainen [11]. If $u$ is a minimizer, then $u^*$ is continuous, and thus $p$-harmonic, see Kinnunen–Shanmugalingam [41].

We will need the following weak Harnack inequalities.

**Theorem 3.2.** (Weak Harnack inequality for subminimizers) Let $q > 0$. Then there is $C > 0$ such that for all subminimizers $u$ in $\Omega$ and all balls $B \subset 2B \subset \Omega$,

$$\text{ess sup}_B u \leq C \left( \frac{1}{2B} \int_{2B} u^q \, d\mu \right)^{1/q}.$$

Here $u_+ := \max\{u, 0\}$.

**Theorem 3.3.** (Weak Harnack inequality for superminimizers) There are $q > 0$ and $C > 0$, such that for all nonnegative superminimizers $u$ in $\Omega$,

$$\left( \frac{1}{2B} \int_{2B} u^q \, d\mu \right)^{1/q} \leq C \text{ess inf}_B u$$

(3.2)

for every ball $B \subset 50\lambda B \subset \Omega$.

These Harnack inequalities were in metric spaces first obtained for minimizers by Kinnunen–Shanmugalingam [41], using De Giorgi’s method, whereas Kinnunen–Martio [39] soon afterwards modified the arguments for sub- and superminimizers. See Björn–Marola [15], p. 363, for some necessary modifications of the statements in [41] and [39], and for alternative proofs using Moser iteration.

For a nonempty bounded open set $G \subset X$ with $C_p(X \setminus G) > 0$ we consider the following obstacle problem. (If $X$ is unbounded then the condition $C_p(X \setminus G) > 0$ is of course immediately fulfilled.)

**Definition 3.4.** For $f \in N^{1,p}(G)$ and $\psi : G \to \mathbb{R}$ let

$$\mathcal{K}_{\psi,f}(G) = \{ v \in N^{1,p}(G) : v - f \in N^{1,p}_0(G) \text{ and } v \geq \psi \text{ q.e. in } G \}.$$

A function $u \in \mathcal{K}_{\psi,f}(G)$ is a solution of the $\mathcal{K}_{\psi,f}(G)$-obstacle problem if

$$\int_G g_w^p \, d\mu \leq \int_G g_{w^*}^p \, d\mu \text{ for all } v \in \mathcal{K}_{\psi,f}(G).$$

A solution to the $\mathcal{K}_{\psi,f}(G)$-obstacle problem is easily seen to be a superminimizer in $G$. Conversely, a superminimizer $u$ in $\Omega$ is a solution of the $\mathcal{K}_{u,u}(G)$-obstacle problem for all open $G \Subset \Omega$ with $C_p(X \setminus G) > 0$.

If $\mathcal{K}_{\psi,f}(G) \neq \emptyset$, then there is a solution of the $\mathcal{K}_{\psi,f}(G)$-obstacle problem, and this solution is unique up to equivalence in $N^{1,p}(G)$. Moreover, $u^*$ is the unique lower semicontinuously regularized solution. If the obstacle $\psi$ is continuous, then $u^*$ is also continuous. The obstacle $\psi$, as a continuous function, is even allowed to take the value $-\infty$. For $f \in N^{1,p}(G)$, we let $H_Gf$ denote the continuous solution of the $\mathcal{K}_{-\infty,f}(G)$-obstacle problem; this function is $p$-harmonic in $G$ and has the same boundary values (in the Sobolev sense) as $f$ on $\partial G$, and hence is also called the solution of the Dirichlet problem with Sobolev boundary values.
Definition 3.5. A function $u : \Omega \to (-\infty, \infty]$ is superharmonic in $\Omega$ if

(i) $u$ is lower semicontinuous;
(ii) $u$ is not identically $\infty$ in any component of $\Omega$;
(iii) for every nonempty open set $G \subseteq \Omega$ with $C_p(X \setminus G) > 0$ and all functions $v \in \text{Lip}(X)$, we have $H_G v \leq u$ in $G$ whenever $v \leq u$ on $\partial G$.

This definition of superharmonicity is equivalent to the ones in Heinonen–Kilpeläinen–Martio [32] and Kinnunen–Martio [39], see A. Björn [4]. A locally bounded superharmonic function is a superminimizer, and all superharmonic functions are lower semicontinuously regularized. Conversely, any lower semicontinuously regularized superminimizer is superharmonic.

We will need the following comparison lemma for solutions to obstacle problems from Björn–Björn [6].

Lemma 3.6. (Comparison principle) Assume that $\Omega$ is bounded and such that $C_p(X \setminus \Omega) > 0$. Let $\psi_j : \Omega \to \overline{\mathbb{R}}$ and $f_j \in N^{1,p}(\Omega)$ be such that $K_{\psi_j,f_j} \neq \emptyset$, and let $u_j$ be the lower semicontinuously regularized solution of the $K_{\psi_j,f_j}$-obstacle problem, $j = 1, 2$. If $\psi_1 \leq \psi_2$ q.e. in $\Omega$ and $(f_1 - f_2)_+ \in N^{1,p}_c(\Omega)$, then $u_1 \leq u_2$ in $\Omega$.

The following simple localization lemma will be useful in the coming proofs. For a proof in the metric space setting see Farnana [21], Lemma 3.6.

Lemma 3.7. Let $u$ be the lower semicontinuously regularized solution of the $K_{\psi,f}(\Omega)$-obstacle problem and let $\Omega^\prime \subset \Omega$ be open. Then $u$ is the lower semicontinuously regularized solution of the $K_{\psi,u}(\Omega^\prime)$-obstacle problem.

4. Fine topology

In this section we introduce the main concepts of this paper and present the necessary auxiliary results. At the end of the section, we prove Theorem 1.4.

A set $E \subset X$ is thin at $x \in X$ if

$$\int_0^1 \left( \frac{\text{cap}_p(E \cap B(x,r), B(x,2r))}{\text{cap}_p(B(x,r), B(x,2r))} \right)^{1/(p-1)} \frac{dr}{r} < \infty. \quad (4.1)$$

A set $U \subset X$ is finely open if $X \setminus U$ is thin at each point $x \in U$. It is easy to see that the finely open sets give rise to a topology, which is called the fine topology, see Proposition 11.36 in Björn–Björn [7]. Every open set is finely open, but the converse is not true in general.

For any $E \subset X$, the base $b_p(E)$ is the set of all points $x \in X$ so that $E$ is thick, i.e. not thin, at $x$. We also let $E^{\text{fin}}$ be the fine closure of $E$ and fine-int $E$ be the fine interior of $E$, both taken with respect to the fine topology.

In the definition of thinness, and in the sum (4.2) below, we make the convention that the integrand is 1 whenever $\text{cap}_p(B(x,r), B(x,2r)) = 0$. This happens e.g. if $X = B(x,2r)$ is bounded, but never e.g. if $r < \frac{1}{2}\text{diam} X$. Note that thinness is a local property, i.e. $E$ is thin at $x$ if and only if $E \cap B(x,\delta)$ is thin at $x$, where $\delta > 0$ is arbitrary.

Definition 4.1. A function $u : U \to \overline{\mathbb{R}}$, defined on a finely open set $U$, is finely continuous if it is continuous when $U$ is equipped with the fine topology and $\overline{\mathbb{R}}$ with the usual topology.

Note that $u$ is finely continuous in $U$ if and only if it is finely continuous at every $x \in U$ in the sense that for all $\varepsilon > 0$ there exists a finely open set $V \ni x$ such...
that \(|u(y) - u(x)| < \varepsilon\) for all \(y \in V\), if \(u(x) \in \mathbb{R}\), and such that \(\pm u(y) > 1/\varepsilon\) for all \(y \in V\), if \(u(x) = \pm \infty\), or equivalently if and only if the sets \(\{x \in U : u(x) > k\}\) and \(\{x \in U : u(x) < k\}\) are finely open for all \(k \in \mathbb{R}\).

Since every open set is finely open, the fine topology generated by the finely open sets is finer than the metric topology. In fact, it is so fine that all superharmonic functions become finely continuous. This is the content of the following theorem.

**Theorem 4.2.** Let \(u\) be a superharmonic function in an open set \(\Omega\). Then \(u\) is finely continuous in \(\Omega\).

By Theorem 1.1, which we prove in Section 6, the fine topology is the coarsest topology with this property. Together with its consequence Theorem 4.3 below, Theorem 4.2 was obtained by J. Björn \[18\], Theorems 4.4 and 4.6, and independently by Korte \[42\], Theorem 4.3 and Corollary 4.4, (they can also be found in Björn–Björn \[7\] as Theorems 11.38 and 11.40).

**Theorem 4.3.** Let \(\Omega\) be open. Then every quasicontinuous function \(u : \Omega \to \mathbb{R}\) is finely continuous at q.e. \(x \in \Omega\). In particular, this is true for all \(u \in N^{1,p}_{\text{loc}}(\Omega)\).

At the end of this section (when proving Theorem 1.4), we extend the first part of the above result to finely open and quasipanopen sets.

We next give some auxiliary lemmas. The following characterization was essentially obtained in Björn–Björn \[8\].

**Lemma 4.4.** Let \(E \subset X\) and \(x \in X\). Then \(x \in \text{fine-int } E\) if and only if \(x \in E\) and \(X \setminus E\) is thin at \(x\). Moreover, we have \(E^p = E \cup b_p(E)\).

**Proof.** For the characterization of fine interior points, see Proposition 7.8 in \[8\]. Accordingly, \(x \notin E^p\) if and only if \(x\) is a fine interior point of \(X \setminus E\), i.e. \(x \notin E\) and \(E\) is thin at \(x\). Thus \(x \in E^p\) if and only if \(x \in E\) or \(E\) is thick at \(x\).

In Sections 5 and 6 we will use the fact that the integral (4.1) can be replaced by a sum and the factor 2 in (4.1) can be replaced by an arbitrary factor greater than 1. To prove this (see Lemma 4.6), we need the following simple lemma, whose proof can be found e.g. in Björn–Björn \[7\], Lemma 11.22.

**Lemma 4.5.** Let \(B = B(x_0, r)\) and \(E \subset B\). Then for every \(1 < \tau < t < \frac{1}{4}\) \(\text{diam } X\),

\[
\text{cap}_p(E, t B) \leq \text{cap}_p(E, \tau B) \leq C \left(1 + \frac{t^p}{(\tau - 1)^p}\right) \text{cap}_p(E, t B).
\]

**Lemma 4.6.** Let \(E \subset X\), \(x \in X\), \(r_0 > 0\) and \(\sigma > 1\). Then \(E\) is thin at \(x\) if and only if

\[
\sum_{j=1}^{\infty} \left(\frac{\text{cap}_p(E \cap B(x, \sigma^{-j} r_0), B(x, \sigma^{-j} r_0))}{\text{cap}_p(B(x, \sigma^{-j} r_0), B(x, \sigma^{-j} r_0))}\right)^{1/(p-1)} < \infty. \tag{4.2}
\]

**Proof.** Let \(B_s = B(x, s)\) for \(s > 0\). Let \(\rho < \frac{1}{4}\) \(\text{diam } X\) and \(\rho/\sigma \leq r \leq \rho\). Then, by Lemma 4.5 and the monotonicity of the capacity,

\[
\frac{1}{C} \text{cap}_p(E \cap B_{\rho/\sigma}, B_{\rho}) \leq \text{cap}_p(E \cap B_{\rho}, B_{2\rho}) \leq C \text{cap}_p(E \cap B_{\rho}, B_{\rho}),
\]

which together with the doubling property of \(\mu\) and Lemmas 2.7 and 4.5 shows that

\[
\frac{1}{C} \left(\frac{\text{cap}_p(E \cap B_{\rho/\sigma}, B_{\rho})}{\text{cap}_p(B_{\rho/\sigma}, B_{\rho})}\right)^{1/(p-1)} \leq \int_{\rho/\sigma}^{\rho} \left(\frac{\text{cap}_p(E \cap B_{r}, B_{2r})}{\text{cap}_p(B_{r}, B_{2r})}\right)^{1/(p-1)} dr \leq C \left(\frac{\text{cap}_p(E \cap B_{\rho}, B_{\rho})}{\text{cap}_p(B_{\rho}, B_{\rho})}\right)^{1/(p-1)}.
\]

Hence (4.1) converges if and only if (4.2) converges. \(\square\)
Lemma 4.7. Let $E \subset X$ be thin at $x \in \overline{E} \setminus E$. Then there is an open neighbourhood $G$ of $E$ such that $G$ is thin at $x$ and $x \notin G$.

Proof. Let $B_j = B(x, 2^{-j})$, $j = 1, 2, \ldots$. By Lemma 4.5,

$$\text{cap}_p(E \cap \overline{B}_j, 2B_j) \leq C \text{cap}_p(E \cap \overline{B}_j, 4B_j) \leq C \text{cap}_p(E \cap 2B_j, 4B_j).$$

Since the variational capacity is an outer capacity, by Theorem 2.8, we can find open sets $G_j \supset E \cap \overline{B}_j$ such that

$$\left(\frac{\text{cap}_p(G_j, 2B_j)}{\text{cap}_p(B_j, 2B_j)}\right)^{1/(p-1)} \leq \left(\frac{\text{cap}_p(E \cap \overline{B}_j, 2B_j)}{\text{cap}_p(B_j, 2B_j)}\right)^{1/(p-1)} + 2^{-j}.$$

Let

$$G = (X \setminus \overline{E}_1) \cup (G_1 \setminus \overline{B}_2) \cup ((G_1 \cap G_2) \setminus \overline{B}_3) \cup ((G_1 \cap G_2 \cap G_3) \setminus \overline{B}_4) \cup \ldots.$$

Then $G$ is open and contains $E$, and $x \notin G$. Moreover $G \cap B_j \subset G_j$ and thus, by combining the estimates and using Lemmas 2.7 and 4.6,

$$\sum_{j=1}^{\infty} \left(\frac{\text{cap}_p(G \cap B_j, 2B_j)}{\text{cap}_p(B_j, 2B_j)}\right)^{1/(p-1)} \leq C \sum_{j=1}^{\infty} \left(\frac{\text{cap}_p(E \cap 2B_j, 4B_j)}{\text{cap}_p(2B_j, 4B_j)}\right)^{1/(p-1)} + 1 < \infty.$$

Hence the claim follows from Lemma 4.6. \qed

Theorem 4.3 can be used to prove the following generalization of Corollary 4.5 in J. Björn [18] (which can also be found as Corollary 11.39 in [7]), where (4.3) was obtained for bounded open $A$ with $C_p(X \setminus A) > 0$ and $E \subset A$. There is also an intermediate version in Björn–Björn [9], Corollary 4.7. In [18], Corollary 4.5 was used to obtain Theorem 4.3. Here we instead use Theorem 4.3 to obtain Lemma 4.8, i.e. to improve Corollary 4.5 from [18].

Lemma 4.8. If $E \subset X$, then $C_p(\overline{E}) = C_p(E)$. Moreover, if $E \subset A$, then

$$\text{cap}_p(E, A) = \text{cap}_p(\overline{E} \cap A, A).$$

(4.3)

If furthermore $\text{cap}_p(E, A) < \infty$, then $C_p(\overline{E} \setminus \text{fine-int } A) = 0$ and

$$\text{cap}_p(E, A) = \text{cap}_p(\overline{E} \cap A, A) = \text{cap}_p(\overline{E} \setminus \text{fine-int } A, \text{fine-int } A).$$

(4.4)

Proof. The inequality $C_p(\overline{E}) \leq C_p(E)$ follows since any $v \in \mathcal{N}^{1,p}(X)$ admissible for the capacity $C_p(E)$ is also admissible for the capacity $C_p(\overline{E})$. Indeed, if $x \in \overline{E}$ is a finite continuity point of $v$, then

$$v(x) = \lim_{y \to x} v(y) \geq 1.$$

Since q.e. point in $X$ is a fine continuity point for $v \in \mathcal{N}^{1,p}(X)$, by Theorem 4.3, we conclude that $v \geq 1$ q.e. in $\overline{E}$. The converse inequality is trivial.

Similarly, if $u \in \mathcal{N}^{1,p}_0(A)$ is admissible for $\text{cap}_p(E, A)$ then $u \geq 1$ q.e. in $\overline{E}$ and $u = 0$ q.e. in $X \setminus \text{fine-int } A$. This proves the nontrivial inequality in (4.4), and also that $C_p(\overline{E} \setminus \text{fine-int } A) = 0$ if there exists such a $u$. Finally, (4.3) is trivial if $\text{cap}_p(E, A) = \infty$. \qed

As a main consequence of Lemma 4.8, we end this section by proving Theorem 1.4.
Lemma 5.3. There exist a ball $x$ is of capacity zero. If $F$ is finely open, and thus $V := U \cup G_j$ is open. By Lemma 4.8, we have $C_p(G_j) < 2^{-j}$ so that $u_{|V \setminus G_j}$ is continuous. By Lemma 4.8, we have $C_p(G_j) = C_p(G_j) < 2^{-j}$. Hence the set

$$A := E \cap \left( V \cap \bigcap_{j=1}^{\infty} C_j \right)$$

is of capacity zero. If $x \in U \setminus A$, then $x$ belongs to the finely open set $V \setminus C_k$ for some $k$, and the fine continuity of $u$ at $x$ follows from the continuity of $u_{|V \setminus G_j}$ since the fine topology is finer than the metric topology.

\[\Box\]

5. The weak Cartan property

Our aim in this section is to obtain the following weak Cartan property.

Theorem 5.1. (Weak Cartan property) Assume that $E$ is thin at $x_0 \notin E$. Then there exist a ball $B$ centred at $x_0$ and superharmonic functions $u, u' \in N_1^p(B)$ such that

$$0 \leq u \leq 1, \quad 0 \leq u' \leq 1, \quad u(x_0) < 1, \quad u'(x_0) < 1 \quad \text{and} \quad E \cap B \subset F \cup F',$$

where $F = \{x \in B : u(x) = 1\}$ and $F' = \{x \in B : u'(x) = 1\}$.

In particular, with $v = \max\{u, u'\}$ we have $v(x_0) < 1$ and $v = 1$ in $E \cap B$.

Note that $u, u'$ and $v$ above are lower semicontinuous, quasicontinuous and finely continuous in $B$. In the proof we will use two lemmas which are also of independent interest (see e.g. the proof of Proposition 1.3). We shall frequently use the following notion.

Definition 5.2. We say that a function $u$ is the capacitary potential of a set $E$ in $B \supset E$ if it is the lower semicontinuously regularized solution of the $K_{\lambda, u, 0}(B)$-obstacle problem.

Lemma 5.3. Let $B = B(x_0, r)$ and $B_0$ be balls such that $50\lambda B \subset B_0$ and $C_p(X \setminus B_0) > 0$. Also let $E \subset \frac{1}{2}B_0$ be such that $E \cap (2B \setminus \frac{1}{2}B) = \emptyset$ and let $u$ be the capacitary potential of $E$ in $B_0$. Then

$$\sup_{\partial B} u \leq C' \left( \frac{\text{cap}_p(E, B_0)}{\text{cap}_p(B, B_0)} \right)^{1/(p-1)}. \quad (5.1)$$

Proof. Let $m = \inf_{B} u$. If $m = 0$, then the left-hand side in (5.1) is 0, by Theorem 3.3, and (5.1) follows. If $m = 1$, then $\text{cap}_p(B, B_0) \leq \int_{B_0} g^p \, d\mu = \text{cap}_p(E, B_0)$ and (5.1) holds for any $C' \geq 1$. Assume therefore that $0 < m < 1$. Thus the functions

$$u_1 = \min\left\{ \frac{u}{m}, 1 \right\} \quad \text{and} \quad u_2 = \frac{u - mu_1}{1 - m}$$

are superharmonic in $B$ and satisfy $u_1 \leq u_2 \leq u$. Hence $u_1 \leq v = \max\{u, u_2\}$ in $E \cap B$. Arguing as in the second part of the proof of Lemma 4.8, we find that $u_1$ is the capacitary potential of $E \cap B$ in $B_0$. Applying Theorem 3.3 gives $u_1 \leq C' \left( \frac{\text{cap}_p(E, B_0)}{\text{cap}_p(B, B_0)} \right)^{1/(p-1)}$, as desired.
are admissible in the definition of \( \text{cap}_p(B, B_0) \) and \( \text{cap}_p(E, B_0) \), respectively, (in view of Remark 2.6). Note that for a.e. \( x \in B_0 \), at least one of \( g_{u_1}(x) \) and \( g_{u_2}(x) \) vanishes. As \( u \) is the capacitary potential of \( E \) in \( B_0 \), we therefore obtain that

\[
\text{cap}_p(E, B_0) = \int_{\{u \leq m\}} g^p_u \, d\mu + \int_{\{u > m\}} g^p_u \, d\mu
\]

\[
= m^p \int_{B_0} g^p_{u_1} \, d\mu + (1 - m)^p \int_{B_0} g^p_{u_2} \, d\mu
\]

\[
\geq m^p \text{cap}_p(B, B_0) + (1 - m)^p \text{cap}_p(E, B_0).
\]

It follows that

\[
\text{cap}_p(B, B_0) \leq \frac{1 - (1 - m)^p}{m^p} \text{cap}_p(E, B_0) \leq pm^{1 - p} \text{cap}_p(E, B_0)
\]

and equivalently,

\[
m \leq \left( \frac{p \text{cap}_p(E, B_0)}{\text{cap}_p(E, B_0)} \right)^{1/(p - 1)}.
\]

(5.2)

Now, let \( B' = B(x', r') \) be such that \( x' \in \partial B, \ r' = \frac{1}{\sigma} r \), and \( \sup_{\partial B} u \leq \sup_{B'} u \). Then \( 2B' \in 2B' \setminus \frac{1}{2}B \) and as \( u \) is a nonnegative lower semicontinuously regularized minimizer in \( 2B' \setminus \frac{1}{2}B \), the weak Harnack inequality for subminimizers (Theorem 3.2) implies that for every \( q > 0 \), there exists a constant \( C_q \), independent of \( u \) and \( B' \), such that

\[
\sup_{B'} u \leq C_q \left( \int_{2B'} u \, d\mu \right)^{1/q}.
\]

(5.3)

Finally, as \( u \) is a superminimizer in \( B_0 \), the weak Harnack inequality for superminimizers (Theorem 3.3) and the doubling property of \( \mu \) imply that for some \( q > 0 \) and \( C > 0 \), independent of \( u \), \( B \), and \( B' \),

\[
m = \inf_B u \geq C \left( \int_{2B} u \, d\mu \right)^{1/q} \geq C \left( \int_{2B'} u \, d\mu \right)^{1/q}.
\]

(5.4)

Combining (5.2)–(5.4) gives (5.1). \( \square \)

**Remark 5.4.** Lemma 3.9 in J. Björn [18] (Lemma 11.20 in [7] or Lemma 5.6 in Björn–MacManus–Shanmugalingam [20] in linearly locally connected spaces) provides us with the converse inequality to (5.1), viz.

\[
\inf_{\partial B} u \geq C'' \left( \frac{\text{cap}_p(E, B_0)}{\text{cap}_p(B, B_0)} \right)^{1/(p - 1)}.
\]

(5.5)

**Proposition 5.5.** For a ball \( B = B(x_0, r) \) with \( C_p(X \setminus B) > 0 \) let \( B_j = \sigma^{-1} B_j \), \( j = 0, 1, \ldots \), where \( \sigma \geq 50 \lambda \) is fixed. Assume that \( E \subset \frac{1}{2}B \) is such that \( E \cap \left( 2B_j \setminus \frac{1}{2}B_j \right) = \emptyset \) for all \( j = 0, 1, \ldots \), and let \( u \) be the capacitary potential of \( E \) in \( B \). Then

\[
1 - \prod_{j=0}^{\infty} (1 - ca_j) \leq u(x_0) \leq 1 - \prod_{j=0}^{\infty} (1 - a_j),
\]

where

\[
a_j = \min \left\{ 1, \left( \frac{\text{cap}_p(E \cap \frac{1}{2}B_j, B_j)}{\text{cap}_p(B_j, B_j)} \right)^{1/(p - 1)} \right\},
\]

c = C''/C' > 0 and \( C' \) and \( C'' \) are as in (5.1) and (5.5).
Remark 5.6. (a) The case $c \geq 1$ is not excluded in Proposition 5.5. However, by (5.1) and (5.5), the case $c > 1$ holds true only if $a_j = 0$ for all $j = 0, 1, \ldots$. By (5.7), the case $c = 1$ holds true only if

$$\inf_{\partial B_{j+1}} u_j = \sup_{\partial B_{j+1}} u_j$$

for all $j = 0, 1, \ldots$. See the proof below for the notation here.

(b) The first inequality in Proposition 5.5 can be obtained from Lemma 5.7 in Björn–MacManus–Shanmugalingam [20] (in linearly locally connected spaces) or from Proposition 3.10 in J. Björn [18] (alternatively Theorem 11.21 in [7]). In this paper we will not need it, but we have chosen to include it here as the proof below shows that both inequalities can be obtained simultaneously.

In fact, by taking logarithms, the left estimate in Proposition 5.5 implies

$$1 - u(x_0) \leq \exp\left(-c \sum_{j=0}^{\infty} a_j\right),$$

which in particular shows that if $E$ is thick at $x_0$ then $u(x_0) = 1$. As for the right estimate in Proposition 5.5, it is easily shown by induction that $1 - \prod_{j=0}^{n} (1 - a_j) \leq \sum_{j=0}^{n} a_j$ and hence we obtain the qualitative estimate

$$u(x_0) \leq C \sum_{j=0}^{\infty} \left(\frac{\text{cap}_p(E \cap \frac{1}{2} B_j)}{\text{cap}_p(B_{j+1} \setminus B_j)}\right)^{1/(p-1)},$$

which in $\mathbb{R}^n$, with $p < n$, reduces to a special case of the estimate (6.1) in Maz’ya–Havin [52]. It corresponds to the Wolff potential estimates for superharmonic functions in e.g. Kilpeläinen–Malý [37], Mikkonen [54] and Björn–MacManus–Shanmugalingam [20] and partly generalizes Theorem 3.6 in J. Björn [19]. More precisely, the Wolff potential for the capacitary measure of $E$ is easily seen to be comparable to the sum in (5.6). The estimates for general superharmonic functions in [37], [54] and [20] contain an additional term, such as $(\int_{B_0} u^{p'} d\mu)^{1/p}$, but since the potential $u$ has boundary values 0 on $\partial B_0$, this term can be avoided in this case, cf. [19, Theorem 3.6].

In particular, (5.6) implies the necessity part of the Wiener criterion in certain domains (such that $2B_j \setminus \frac{1}{2} B_j \setminus \Omega = \emptyset$ for all sufficiently large $j$ and some $c > 0$), since for a sufficiently small ball $B = B(x_0, r)$, the capacitary potential of $\frac{1}{2} B \setminus \Omega$ in $B$ will not attain its boundary value 1 at $x_0$. Note that the necessity part of the Wiener criterion is still open for $p$-harmonic functions (based on upper gradients) in metric spaces.

Proof. For $j = 0, 1, \ldots$, let $u_j$ be the capacitary potential of $E_j = E \cap \frac{1}{2} B_j$ in $B_j$. Then $u = u_0$. Lemma 5.3 and Remark 5.4 imply that for all $j = 0, 1, \ldots$,

$$ca_j \leq \inf_{\partial B_{j+1}} u_j \leq \sup_{\partial B_{j+1}} u_j \leq a_j.$$  \hfill (5.7)

We shall show by induction that for all $k = 1, 2, \ldots$,

$$1 - \sup_{\partial B_k} u \geq \prod_{j=0}^{k-1} (1 - a_j) := b_k \quad \text{and} \quad 1 - \inf_{\partial B_k} u \leq \prod_{j=0}^{k-1} (1 - ca_j) := b'_k.$$ \hfill (5.8)

By (5.7), this clearly holds for $k = 1$. Assume that (5.8) holds for some $k \geq 1$ and let $G_k = \{x \in B_k : u(x) > 1 - b_k\}$. Then $G_k$ is open by the lower semicontinuity of $u$, and since $\sup_{\partial B_k} u \leq 1 - b_k$, we have $u_k := (u - (1 - b_k)), \in N^{1,p}_0(G_k)$. 

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Lemma 3.7 shows that $v_k$ is the lower semicontinuously regularized solution of the $K_{\psi_k,0}(G_k)$-obstacle problem, where $\psi_k = (\chi_{E_k} - (1 - b_k))_+$ = $b_k \chi_{E_k}$ in $B_k$.

On the other hand, by the minimum principle for superharmonic functions, we have $u \geq 1 - b'_k$ in $B_k$ and Lemma 3.7 again shows that $v'_k := u - (1 - b'_k) \geq 0$ is the lower semicontinuously regularized solution of the $K_{\psi'_k,0}(B_k)$-obstacle problem, where $\psi'_k = (\chi_{E_k} - (1 - b'_k))_+ = b'_k \chi_{E_k}$ in $B_k$.

Since $0 \leq u_k \in N_{A, p}(B_k)$ is the lower semicontinuously regularized solution of the $K_{\chi_{E_k}, 0}(B_k)$-obstacle problem, the comparison principle (Lemma 3.6) yields that $v'_k \geq b'_k u_k$ in $B_k$ and that $v_k \leq b_k u_k$ in $G_k$, and hence in $B_k$. In particular, by (5.7),

$$\sup_{\partial B_{k+1}} v_k \leq \sup_{\partial B_{k+1}} b_k u_k \leq a_k b_k \quad \text{and} \quad \inf_{\partial B_{k+1}} v'_k \geq \inf_{\partial B_{k+1}} b'_k u_k \geq c a_k b'_k.$$  

Hence

$$\sup u \leq \sup_{\partial B_{k+1}} v_k + 1 - b_k \leq a_k b_k + 1 - b_k = 1 - b_k (1 - a_k) = 1 - b_{k+1}$$

and

$$\inf u = \inf_{\partial B_{k+1}} v'_k + 1 - b'_k \geq c a_k b'_k + 1 - b'_k = 1 - b'_k (1 - ca_k) = 1 - b'_{k+1},$$

which proves (5.8) for $k + 1$. By induction, (5.8) holds for all $k = 1, 2, \ldots$. Since $u$ is lower semicontinuously regularized, letting $k \to \infty$ gives

$$u(x_0) = \liminf_{x \to x_0} u(x) \leq 1 - \lim_{k \to \infty} b_k = 1 - \prod_{j=0}^{\infty} (1 - a_j)$$

and, by the minimum principle,

$$u(x_0) \geq 1 - \lim_{k \to \infty} b'_k = 1 - \prod_{j=0}^{\infty} (1 - c a_j).$$  

We are now ready to prove the weak Cartan property. The proof uses a separation argument which has been inspired by Theorem 3.2 in Heinonen–Kilpeläinen–Martio [31], and whose idea goes back to Lindqvist–Martio [47].

**Proof of Theorem 5.1.** By Lemma 4.7, we can assume that $E$ is open. For $r > 0$ let $B_r = \sigma^{-1} B(x_0, r)$ with $\sigma = 50\lambda$ be as in Proposition 5.5. Also let $D_j = \left\{ \frac{1}{2} B_j \setminus \frac{1}{2} B_{j+1} \right\} \cap E \text{ and } E_j = \bigcup_{i=j}^{\infty} D_i$, $j = 0, 1, \ldots$. Note that $E_0 \cap (2B_j \setminus \frac{1}{2} B_j) = \emptyset$ for all $j = 0, 1, \ldots$. Proposition 5.5 then implies that the capacitary potential $u$ of $E_0$ in $B_0 = B(x_0, r)$ satisfies

$$u(x_0) \leq 1 - \prod_{j=0}^{\infty} (1 - a_j),$$

where

$$a_j = \min \left\{ 1, C' \left( \frac{\text{cap}_p(E_j, B_j)}{\text{cap}_p(B_{j+1}, B_j)} \right)^{1/(p-1)} \right\}$$

and $C'$ is as in Lemma 5.3. Since $E$ is thin at $x_0$, we can find $r > 0$ so that all $a_j \leq \frac{1}{2}$ and $\sum_{j=0}^{\infty} a_j < \infty$ (by Lemma 4.6). Hence the series $\sum_{j=0}^{\infty} \log(1 - a_j)$ converges as well, which implies that $\prod_{j=0}^{\infty} (1 - a_j) > 0$, i.e. that $u(x_0) < 1$. On the other hand, we have $u = 1$ in $E_0$, as $E_0$ is open.

Similarly, since $2B_{j+1} \setminus \frac{1}{2} B_{j+1} \subset \frac{1}{2} (B_{j+1} \setminus \frac{1}{2} B_{j+1})$, replacing $r$ by $r' = \frac{1}{2} r$ in the above argument provides us with the capacitary potential $u'$ in $B(x_0, r')$ which satisfies $u'(x_0) < 1$ and $u' = 1$ in $\left( E \cap B(x_0, \frac{1}{2} r') \right) \setminus E_0$. Letting $B = B(x_0, \frac{1}{2} r')$ concludes the proof. 

\[\square\]
We end this section by proving Theorem 1.2.

Proof of Theorem 1.2. (a) ⇔ (b) ⇔ (c) This follows directly from Lemma 4.4.
(a) ⇒ (e) This follows from the weak Cartan property (Theorem 5.1).
(e) ⇒ (d) This is trivial.
(d) ⇒ (b) We can find δ and a ball B ⊃ x₀ such that v(x₀) < δ < v(x) for all x ∈ B ∩ E. As v is finely continuous, by Theorem 4.2, V := {x ∈ B : v(x) < δ} is a finely open fine neighbourhood of x₀. Since E ∩ V = ∅, we see that x₀ ∈ E'.

6. Consequences of the weak Cartan property

In this section we establish several consequences of the weak Cartan property. First, we prove Theorem 1.1, i.e. that the fine topology is the coarsest topology making all superharmonic functions continuous, and that the base of its neighbourhoods is given by finite intersections of level sets of superharmonic functions.

The coarsest topology related to Theorem 1.1 is traditionally formulated using global superharmonic functions on \( \mathbb{R}^n \). This definition relies on the following extension result: If \( u \) is superharmonic in \( \Omega \subset \mathbb{R}^n \) and \( G \Subset \Omega \), then there is a superharmonic function \( v \) on \( \mathbb{R}^n \) such that \( v = u \) in \( G \), see Theorem 3.1 in Kilpeläinen [35] (for unweighted \( \mathbb{R}^n \)) and Theorem 7.30 in Heinonen–Kilpeläinen–Martio [32] (for weighted \( \mathbb{R}^n \)). Such an extension result is not known for unbounded metric spaces, while it is false for bounded metric spaces as there are only constant superharmonic functions on \( X \) if \( X \) is bounded. Therefore we directly prove the following local formulation.

Theorem 6.1. A set \( U \subset X \) is a fine neighbourhood of \( x₀ \) if and only if there exist constants \( c_j \) and bounded superharmonic functions \( u_j \) in some ball \( B \ni x₀, j = 1, 2, \ldots, k \), such that

\[
x₀ \in \bigcap_{j=1}^{k} \{ x \in B : u_j(x) < c_j \} \subset U. \tag{6.1}
\]

The proof shows that the neighbourhood base condition always holds with \( k = 2 \). Recall that a set \( U \) is a fine neighbourhood of a point \( x₀ \) if it contains a finely open set \( V \ni x₀ \); it is not required that \( U \) itself is finely open.

Proof. Let \( U \subset X \). First, we assume that there exist constants \( c_j \) and bounded superharmonic functions \( u_j \) in a ball \( B \ni x₀, j = 1, 2, \ldots, k \), such that (6.1) holds. By Theorem 4.2, each \( u_j \) is finely continuous and hence

\[
V_j := \{ x \in B : u_j(x) < c_j \}
\]

is finely open. It follows that \( \bigcap_{j=1}^{k} V_j \) is finely open and hence \( U \) is a fine neighbourhood of \( x₀ \).

To prove the converse, let \( E = X \setminus U \). Then \( x \notin E \) and \( E \) is thin at \( x \). Let \( B, F, F', u, u' \) be as given by the weak Cartan property (Theorem 5.1). Then

\[
B \cap U = B \setminus E \supset B \setminus (F \cup F') = \{ x \in B : u(x) < 1 \} \cap \{ x \in B : u'(x) < 1 \},
\]

i.e. the fine neighbourhood base condition holds with \( k = 2 \).

Proof of Theorem 1.1. By Theorem 4.2, the fine topology makes all superharmonic functions on all open subsets of \( X \) continuous. To show that it is the coarsest topology with this property, let \( \mathcal{T} \) be such a topology on \( X \), and let \( U \subset X \) be finely open. We shall show that for every \( x₀ \in U \) there exists \( V \in \mathcal{T} \) such that
$x_0 \in V \subset U$. Indeed, let $u_1$ and $u_2$ be the superharmonic functions provided by Theorem 6.1 and so that (6.1) holds. Since $T$ makes all superharmonic functions continuous, we get that the level sets $\{x \in B : u_j(x) < c_j\}$ belong to $T$, and so does their intersection. In view of (6.1) this concludes the proof.

Note that here it is not enough to only consider all superharmonic functions on $X$, as these may be just the constants (if $X$ is bounded). Therefore, superharmonic functions on all open sets (or balls) in $X$ have to be considered in Theorem 1.1.

As a consequence of Proposition 5.5 we can also deduce Proposition 1.3.

**Proof of Proposition 1.3.** Let $\sigma = 50\lambda$, $E = \{x_0\}$, $B = B(x_0, r)$, $B_j$ and $u$ be as in Proposition 5.5. Since $C_p(\{x_0\}) > 0$, we have $u(x_0) = 1$. Proposition 5.5 yields

$$u(x_0) \leq 1 - \prod_{j=0}^{\infty} (1 - a_j),$$

where

$$a_j = \min\left\{1, C' \left( \frac{\text{cap}_p(x_0, B_j)}{\text{cap}_p(B_j+1, B_j)} \right)^{1/(p-1)} \right\}$$

and $C'$ is as in Lemma 5.3. If $E$ were thin at $x_0$, we could find $r > 0$ so that all $a_j \leq \frac{1}{2}$ and $\sum_{j=0}^{\infty} a_j < \infty$ (by Lemma 4.6). Hence the series $\sum_{j=0}^{\infty} \log(1 - a_j)$ would converge as well, implying that $\prod_{j=0}^{\infty} (1 - a_j) > 0$, i.e. that $u(x_0) < 1$, which is a contradiction. Thus $\{x_0\}$ is thick at $x_0$.

The proof of the following lemma has been inspired by the proof of Lemma 12.24 in Heinonen–Kilpeläinen–Martio [32], but here we make use of the weak Cartan property to simplify the argument.

**Lemma 6.2.** If a set $E$ is thin at $x_0$ then for every ball $B \ni x_0$

$$\lim_{\rho \to 0} \text{cap}_p(E \cap B(x_0, \rho), B) = 0.$$  

**Proof.** Without loss of generality we may assume that $\text{diam} B < \frac{1}{\epsilon} \text{diam} X$. Since the variational capacity is an outer capacity, by Theorem 2.8, we see that

$$\text{cap}_p(E \cap B(x_0, \rho), B) \leq \text{cap}_p(B(x_0, \rho), B) \to \text{cap}_p(\{x_0\}, B), \quad \text{as } \rho \to 0,$$

and thus the result is trivial if $\text{cap}_p(\{x_0\}, B) = 0$. If $x_0 \in E$ and $\text{cap}_p(\{x_0\}, B) > 0$, then $C_p(\{x_0\}) > 0$, by Lemma 2.7. Proposition 1.3 then implies that $E$ is thick at $x_0$, a contradiction. We can therefore assume that $x_0 \notin E$ and $\text{cap}_p(\{x_0\}, B) > 0$.

Let $0 < \epsilon < \text{cap}_p(\{x_0\}, B)$ be arbitrary. By the weak Cartan property (Theorem 5.1), there exist a ball $B' \subset 2B' \subset B$, containing $x_0$, and $v \in N^{1,p}(B')$ such that $v(x_0) < 1$ and $v = 1$ in $E \cap B'$. Since $v \in N^{1,p}(B')$ it is quasicontinuous in $B'$, see the discussion after Definition 2.4. Thus Lemma 2.7 shows that there is an open set $G \subset B'$ such that $\text{cap}_p(G, B) < \epsilon$ and $v|_{B' \setminus G}$ is continuous. As $\epsilon < \text{cap}_p(\{x_0\}, B)$, we see that $x_0 \notin G$ and $v|_{B' \setminus G}$ is continuous at $x_0$. Thus, there exists $\rho > 0$ such that $B(x_0, \rho) \subset G$ and $v < 1$ in $B(x_0, \rho) \setminus G$. Since $v = 1$ in $E \cap B'$, we must have $E \cap B(x_0, \rho) \subset G$, and hence

$$\text{cap}_p(E \cap B(x_0, \rho), B) \leq \text{cap}_p(G, B) < \epsilon.$$  

As a corollary of Lemma 6.2 we obtain the following strong Cartan property at points of positive capacity, which also gives a new characterization of thin sets at such points.
Proposition 6.3. Assume that $C_p(\{x_0\}) > 0$ and that $x_0 \in \overline{E} \setminus E$. Then the following are equivalent.

(a) $E$ is thin at $x_0$;
(b) for every (some) ball $B \ni x_0$ with $C_p(X \setminus B) > 0$,
\[ \lim_{\rho \to 0} \text{cap}_p(E \cap B(x_0, \rho), B) = 0; \]
(c) for every (some) ball $B \ni x_0$ with $C_p(X \setminus B) > 0$ there exists a nonnegative superharmonic function $u$ in $B$ such that
\[ \lim_{E \ni x \to x_0} u(x) = \infty > u(x_0). \]

Remark 6.4. By letting $v := \min\{u, u(x_0) + 1\}$, we obtain a bounded superharmonic function satisfying (1.1).

Proof. (a) $\Rightarrow$ (b) This is a special case of Lemma 6.2.
(b) $\Rightarrow$ (c) For $j = 1, 2, \ldots$, find $r_j > 0$ such that
\[ \text{cap}_p(E \cap B(x_0, r_j), B) < 2^{-jp}. \]
Since $\text{cap}_p$ is an outer capacity, by Theorem 2.8, there exist open sets $G_j \ni x_0$ such that $G_j \ni E \cap B(x_0, r_j)$ and $\text{cap}_p(G_j, B) < 2^{-jp}$. Let $v_j$ be the capacitary potential of $G_j$ in $B$. The Poincaré inequality for $N_0^{1,p}$ (also known as Friedrichs’ inequality), see Corollary 5.54 in Björn–Björn [7], shows that
\[ \int_B v_j^p \, d\mu \leq C_B \int_B p v_j^p \, d\mu < C_B 2^{-jp}, \]
and hence $\|v_j\|_{N_0^{1,p}(\Omega)} \leq C_B 2^{-j}$. It follows that $v := \sum_{j=1}^{\infty} v_j \in N_0^{1,p}(B)$.

Let $u$ be the lower semicontinuously regularized solution of the $K_u,0(B)$-obstacle problem. Then $u \in N_0^{1,p}(B)$ is a nonnegative superharmonic function in $B$ and (as $G_j$ are open) $u \geq k$ in $G_1 \cap \ldots \cap G_k$, $k = 1, 2, \ldots$. It follows that $\lim_{E \ni x \to x_0} u(x) = \infty$. On the other hand, as $u \in N_0^{1,p}(B)$ and $C_p(\{x_0\}) > 0$, we have $u(x_0) < \infty$ by Definition 2.3.
(c) $\Rightarrow$ (a) Since superharmonic functions are finely continuous, by Theorem 4.2, the set $U = \{x \in B : u(x) < u(x_0) + 1\}$ is finely open. As $x_0 \in U$, we get that $B \setminus U$ is thin at $x_0$, and hence $E$ is also thin at $x_0$.

Another consequence of Lemma 6.2 is the following result, which is proved in the same way as the first part of Lemma 2.138 in Malý–Ziemer [49], although we use the variational capacity instead of the Sobolev capacity. We include a short proof for the reader’s convenience.

Lemma 6.5. If $E$ is thin at $x_0$ and $\varepsilon > 0$, then there exists $\rho > 0$ such that
\[ \int_0^1 \left( \frac{\text{cap}_p(E \cap B(x_0, \rho) \cap B(x_0, r), B(x_0, 2r))}{\text{cap}_p(B(x_0, r), B(x_0, 2r))} \right)^{1/(p-1)} \frac{dr}{r} < \varepsilon. \]

Proof. Lemma 6.2 implies that the functions
\[ f_j(r) := \left( \frac{\text{cap}_p(E \cap B(x_0, 1/j) \cap B(x_0, r), B(x_0, 2r))}{\text{cap}_p(B(x_0, r), B(x_0, 2r))} \right)^{1/(p-1)} \frac{1}{r} \]
decrease pointwise to zero on $(0, 1)$. As $E$ is thin at $x_0$, we see that $f_j$ is integrable on $(0, 1)$, and hence by dominated convergence, $\int_0^1 f_j(r) \, dr \to 0$, as $j \to \infty$. Choosing $\rho = 1/j$ for some sufficiently large $j$ concludes the proof.
Now we can deduce the following result which we will need when proving Theorem 1.5.

**Lemma 6.6.** Assume that the sets $E_j$, $j = 1, 2, \ldots$, are thin at $x_0$. Then there exist radii $r_j > 0$ such that the set

$$E = \bigcup_{j=1}^{\infty}(E_j \cap B(x_0, r_j))$$

is thin at $x_0$.

Note that in general the union $\bigcup_{j=1}^{\infty} E_j$ need not be thin at $x_0$. This happens e.g. if $E_j = \partial B(x_0, 1/j)$. To obtain a similar example where $x_0 \in E_j$, let $E_j = \partial B(x_0, 1/j) \cup E_0$, where $E_0$ is an arbitrary set thin at $x_0$ and such that $x_0 \in E_0$.

**Proof.** The proof of the corresponding result for weighted $\mathbb{R}^n$ in Heinonen–Kilpeläinen–Martio [32], Lemma 12.25, carries over verbatim to metric spaces. However, instead of appealing to their Lemma 12.24 (i.e. our Lemma 6.2), it is more straightforward to appeal to our Lemma 6.5. □

We end this paper with the proof of Theorem 1.5.

**Proof of Theorem 1.5.** (a) ⇒ (c) For each $j = 1, 2, \ldots$ there is a finely open set $U_j \ni x_0$ such that $|u(x) - u(x_0)| < 1/j$ for every $x \in U_j$. Since the sets $E_j := X \setminus U_j$ are thin at $x_0$, Lemma 6.6 implies that there are radii $r_j > 0$ such that the set

$$E = \bigcup_{j=1}^{\infty}(E_j \cap B(x_0, r_j))$$

is thin at $x_0$. It follows that $|u(x) - u(x_0)| < 1/j$ for every $x \in U \cap B(x_0, r_j) \setminus E$, and we conclude that (c) holds.

The implication (c) ⇒ (b) is immediate and (b) ⇒ (a) follows from Lemma 4.4. □

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