Verdier Stratifications and \((w_f)\)-Stratifications in o-minimal Structures

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Abstract – We prove the existence of Verdier stratifications for sets definable in any o-minimal structure on \((\mathbb{R}, +, \cdot)\). It is also shown that the Verdier condition \((w)\) implies the Whitney condition \((b)\) in o-minimal structures on \((\mathbb{R}, +, \cdot)\). As a consequence the Whitney Stratification Theorem holds. The existence of \((w_f)\)-stratification of functions definable in polynomially bounded o-minimal structures is presented.

0. Introduction.

0.1 Definition. An o-minimal structure on the real field \((\mathbb{R}, +, \cdot)\) is a family \(\mathcal{D} = (\mathcal{D}_n)_{n \in \mathbb{N}}\) such that for each \(n \in \mathbb{N}\):

1. \(\mathcal{D}_n\) is a boolean algebra of subsets of \(\mathbb{R}^n\).
2. If \(A \in \mathcal{D}_n\), then \(A \times \mathbb{R}\) and \(\mathbb{R} \times A \in \mathcal{D}_{n+m}\).
3. If \(A \in \mathcal{D}_{n+1}\), then \(\pi(A) \in \mathcal{D}_n\), where \(\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n\) is the projection on the first \(n\) coordinates.
4. \(\mathcal{D}_n\) contains \(\{x \in \mathbb{R}^n : P(x) = 0\}\) for all polynomials \(P \in \mathbb{R}[X_1, \cdots, X_n]\).
5. Each set belonging to \(\mathcal{D}_1\) is a finite union of intervals and points. (o-minimality)

A set belonging to \(\mathcal{D}\) is called definable (in this structure). Definable maps are maps whose graphs belonging to \(\mathcal{D}\).

Many results in Semialgebraic Geometry and Subanalytic Geometry hold true in the theory of o-minimal structures on the real field. Recently, o-minimality of many interesting structures on \((\mathbb{R}, +, \cdot)\) has been established, for example, structures generated by the exponential function [W1] (see also [LR] and [DM1]), real power functions [M2], Pfaffian functions [W2] or restricted Gevrey functions [DS]. For more details on o-minimal structures we refer the readers to [D] and [DM] (compare with [S]).

0.2 We now outline the main results of this paper. Let \(\mathcal{D}\) be an o-minimal structure on \((\mathbb{R}, +, \cdot)\). In section 1, we prove that \(\mathcal{D}\) admits Verdier Stratification. We also show that the Verdier condition \((w)\) implies the Whitney condition \((b)\) in \(\mathcal{D}\). Thus, Whitney Stratification Theorem holds true in \(\mathcal{D}\). These improve results in [L1] (see also [DM]). Note that the theorems were proved for subanalytic sets in [V] and [LSW] (see also [DW]), the former based on Hironaka’s Desingularization, and the latter on...
Puiseux’s Theorem. But, in general, neither tools can be applied to sets belonging to o-minimal structures (e.g. to the set \( \{(x,y) \in \mathbb{R}^2 : y = \exp(-1/x), x > 0\} \) in the structure generated by the exponential function). Section 2 is devoted to the study of stratifications of definable functions. In general, definable functions cannot be stratified to satisfy the strict Thom condition \((w_f)\). However, if \(D\) is polynomially bounded, then it admits \((w_f)\)-stratification. Our proof of this assertion is based on piecewise uniform asymptotics for definable functions, instead of Paw/Supersucki’s version of Puiseux’s theorem with parameters, that is used in [KP] to prove the assertion for subanalytic functions.

0.3 Notation. Throughout this paper, let \(D\) denote some fixed, but arbitrary, o-minimal structure on \((\mathbb{R},+,\cdot)\). “Definable” means definable in \(D\). If \(\mathbb{R}^k \times \mathbb{R}^l \ni (y,t) \mapsto f(y,t) \in \mathbb{R}^m\) is a differentiable function, then \(D_1f\) denotes the derivative of \(f\) with respect to the first variables \(y\). As usual, \(d(\cdot,\cdot),\|\cdot\|\) denote the Euclidean distance and norm respectively. Besides, Cell Decomposition [DM. Th. 4.2], and Definable Choice [DM. Th. 4.5] will be often referred in our arguments without the citations.

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1. Verdier Stratifications

1.1 Verdier condition. Let \(\Gamma,\Gamma'\) be \(C^1\) submanifolds of \(\mathbb{R}^n\) such that \(\Gamma \subset \mathbb{T}' \setminus \Gamma'\). Let \(y_0\) be a point of \(\Gamma\). We say that the pair \((\Gamma,\Gamma')\) satisfies the Verdier condition at \(y_0\) if the following holds:

\[
(w) \quad \text{There exists a constant } C > 0 \text{ and a neighborhood } U \text{ of } y_0 \text{ in } \mathbb{R}^n \text{ such that } \delta(T_y \Gamma, T_x \Gamma') \leq C \|x - y\| \quad \text{for all } x \in \Gamma' \cap U, y \in \Gamma \cap U,
\]

where \(T_y \Gamma\) denotes the tangent space of \(\Gamma\) at \(y\), and \(\delta(T,T') = \sup_{v \in T, \|v\|=1} d(v,T')\) is the distance of vector subspaces of \(\mathbb{R}^n\).

Note that \((w)\) is invariant under \(C^2\)-diffeomorphisms.

1.2 Definition. Let \(p\) be a positive integer. A \(C^p\) stratification of \(\mathbb{R}^n\) is a partition \(S\) of \(\mathbb{R}^n\) into finitely many subsets, called strata, such that:

(S1) Each stratum is a connected \(C^p\) submanifold of \(\mathbb{R}^n\) and also a definable set.

(S2) For every \(\Gamma \in S\), \(\mathbb{T} \setminus \Gamma\) is a union of some of the strata.

We say that \(S\) is compatible with a class \(A\) of subsets of \(\mathbb{R}^n\) if each \(A \in A\) is a finite union of some strata in \(S\).

A \(C^p\) Verdier stratification is a \(C^p\) stratification \(S\) such that for all \(\Gamma,\Gamma' \in S\), if \(\Gamma \subset \Gamma' \setminus \Gamma'\), then \((\Gamma,\Gamma')\) satisfies the condition \((w)\) at each point of \(\Gamma\).

1.3 Theorem (Verdier Stratification). Let \(p\) be a positive integer. Then given definable sets \(A_1,\cdots,A_k\) contained in \(\mathbb{R}^n\), there exists a \(C^p\) Verdier stratification of \(\mathbb{R}^n\) compatible with \(\{A_1,\cdots,A_k\}\).
We first make an observation similar to that of [LSW. Prop. 2].

Let (P) be a (local) property for the pairs \( (\Gamma, \Gamma') \) at \( y \) in \( \Gamma \), where \( \Gamma, \Gamma' \) being subsets of \( \mathbb{R}^n \). Put \( P(\Gamma, \Gamma') = \{ y \in \Gamma : (\Gamma, \Gamma') \) satisfies (P) at \( y \} \).

1.4 Proposition. Suppose that for every pair \( (\Gamma, \Gamma') \) of definable \( C^p \) submanifolds of \( \mathbb{R}^n \) with \( \Gamma \subset \overline{\mathbb{T}} \setminus \Gamma' \), the set \( P(\Gamma, \Gamma') \) is definable and \( \dim(\Gamma \setminus P(\Gamma, \Gamma')) < \dim \Gamma \). Then given definable sets \( A_1, \ldots, A_k \) contained in \( \mathbb{R}^n \), there exists a \( C^p \) stratification \( \mathcal{S} \) of \( \mathbb{R}^n \) compatible with \( \{ A_1, \ldots, A_k \} \) and has the following

\[ (P) \quad P(\Gamma, \Gamma') = \Gamma \quad \text{for all} \quad \Gamma, \Gamma' \in \mathcal{S} \quad \text{with} \quad \Gamma \subset \overline{\mathbb{T}} \setminus \Gamma'. \]

Proof. We can construct, by decreasing induction on \( d \in \{ 0, \ldots, n \} \), partitions \( \mathcal{S}^d \) of \( \mathbb{R}^n \) into \( C^p \)-cells compatible with \( \{ A_1, \ldots, A_k \} \), such that \( \mathcal{S}^d \) has properties (S1)(S2) and the following property:

\[ (P_d) \quad P(\Gamma, \Gamma') = \Gamma \quad \text{for all} \quad \Gamma, \Gamma' \in \mathcal{S}^d \quad \text{with} \quad \Gamma \subset \overline{\mathbb{T}} \setminus \Gamma' \quad \text{and} \quad \dim \Gamma \geq d. \]

Indeed, by Cell Decomposition and the fact that \( \dim(\overline{\mathbb{T}} \setminus A) < \dim A \), for all definable set \( A \), we can construct a \( C^p \) cell decomposition of \( \mathbb{R}^n \) compatible with \( \{ A_1, \ldots, A_k \} \) and has (S1)(S2). This cell decomposition can be refined to satisfy \( (P_d) \) by the assumption.

Obviously, \( \mathcal{S} = \mathcal{S}^0 \) is a desired stratification. \( \Box \)

By the proposition, Theorem 1.3 follows from the the following,

1.5 Theorem. Let \( \Gamma, \Gamma' \) be definable sets and \( C^p \)-submanifolds of \( \mathbb{R}^n \). Suppose that \( \Gamma \subset \overline{\mathbb{T}} \setminus \Gamma' \). Then \( W = \{ y \in \Gamma : (\Gamma, \Gamma') \) satisfies \( (w) \) at \( y \} \) is definable, and \( \dim(\Gamma \setminus W) < \dim \Gamma \).

To prove Theorem 1.5 we prepare some lemmas.

1.6 Lemma. Under the notation of Theorem 1.5, \( W \) is a definable set.

Proof. Note that the Grassmanian \( G_k(\mathbb{R}^n) \) of \( k \)-dimensional linear subspaces of \( \mathbb{R}^n \) is semialgebraic, and hence definable; \( \delta \) and the tangent map: \( \Gamma \ni x \mapsto T_x \Gamma \in G_{\dim \Gamma}(\mathbb{R}^n) \) are also definable. Therefore,

\[
W = \{ y_0 : \quad y_0 \in \Gamma, \exists C > 0, \exists t > 0, \forall x \in \Gamma', \forall y \in \Gamma \\
\quad (\| x - y_0 \| < t, \| y - y_0 \| < t \Rightarrow \delta(T_y \Gamma, T_x \Gamma') \leq C\| x - y \|) \}
\]

is a definable set. \( \Box \)

1.7 Lemma (Wing Lemma). Let \( V \subset \mathbb{R}^k \) be open and definable, and \( S \subset \mathbb{R}^k \times \mathbb{R}^l \) be definable. Suppose \( V \subset \overline{S} \setminus S \). Then there exists an open subset \( U \) of \( V \), \( \alpha > 0 \), and a definable map \( \bar{\rho} : U \times (0, \alpha) \rightarrow S \), of class \( C^p \), such that \( \bar{\rho}(y, t) = (y, \rho(y, t)) \) and \( \| \rho(y, t) \| = t \), for all \( y \in U, t \in (0, \alpha) \).
1.8 Lemma. To control the tangent spaces we need the following lemma.

Then \( \epsilon : V \to \mathbb{R} \) is definable, and if \( \epsilon(y) > 0 \) then \( (0, \epsilon(y)) \cap \pi_2(A)_y = \emptyset \).

Claim: \( \dim \{ y \in V : \epsilon(y) = 0 \} < \dim V = k \).

Suppose to the contrary that the dimension is \( k \). Then, by Cell Decomposition, there is an open ball \( B \subset V \) and \( c > 0 \) such that \( \epsilon > c \) on \( B \). This implies \( B \not\subset \overline{V} \setminus S \), a contradiction.

Now let \( V_0 = \{ y \in V : \epsilon(y) = 0 \} \). Then \( \dim V_0 = k \), and, by the definition, \( V_0 \times (0, 1) \subset \pi_2(A) \). By Definable Choice and Cell Decomposition, there exists an open set \( V' \subset V_0 \), \( \delta > 0 \), and a continuous definable map: \( V' \times (0, \delta) \to S \), \( (y, t) \mapsto (y, \theta(y, t)) \). Let \( \tau(y) = \sup_{0 < s < \delta} \| \theta(y, s) \| \). Then for \( y \in V' \), \( t < \tau(y) \), there exists \( x \in S \), such that \( \pi(x) = y \) and \( \| x - y \| = t \). Again by Definable Choice and Cell Decomposition it is easy to prove the existence of the \( U, \alpha, \rho \) satisfying the demands of the lemma.

To control the tangent spaces we need the following lemma.

1.8 Lemma. Let \( U \subset \mathbb{R}^k \) be an open definable set, and \( M : U \times (0, \alpha) \to \mathbb{R}^l \) be a \( C^1 \) definable map. Suppose there exists \( K > 0 \) such that \( \| M(y, t) \| \leq K \), for all \( y \in U \) and \( t \in (0, \alpha) \). Then there exists a definable set \( F \), closed in \( U \) with \( \dim F < \dim U \), and continuous definable functions \( C, \tau : U \setminus F \to \mathbb{R}_+ \), such that for all \( y \in U \setminus F \)

\[
\| D_1 M(y, t) \| \leq C(y) \quad \text{for all } t \in (0, \tau(y)).
\]

Proof. It suffices to prove for \( l = 1 \). Suppose the assertion of the lemma is false. Since \( \{ y \in U : \lim_{t \to 0^+} \| D_1 M(y, t) \| = +\infty \} \) is definable, there is an open subset \( B \) of \( U \), such that

\[
\lim_{t \to 0^+} \| D_1 M(y, t) \| = +\infty, \quad \text{for all } y \in B.
\]

By monotonicity [DM. Th. 4.1], for each \( y \in B \), there is \( s > 0 \) such that \( t \mapsto \| D_1 M(y, t) \| \) is strictly decreasing on \( (0, s) \). Let

\[
\tau(y) = \sup \{ s : \| D_1 M(y, \cdot) \| \text{ is strictly decreasing on } (0, s) \}.
\]

Note that \( \tau \) is a definable function, and, by Cell Decomposition, \( \tau \) is continuous on an open subset \( B' \) of \( B \), and \( \tau > \alpha' \) on \( B' \), for some \( \alpha' > 0 \). Let \( \psi(t) = \inf \{ \| D_1 M(y, t) \| : y \in B', 0 < t < \alpha' \} \). Shrinking \( B' \), we can assume that \( \lim_{t \to 0^+} \psi(t) = +\infty \). Then, for each \( y \in B' \), we have

\[
\| D_1 M(y, t) \| > \psi(t), \quad \text{for all } t \in (0, \alpha').
\]
This implies $|M(y,t) - M(y',t)| > \psi(t)\|y - y'\|$, for all $y, y' \in B'$, and $t < \alpha'$.

Therefore, $\psi(t) \leq \frac{2K}{\text{diam} B'}$, for all $t \in (0, \alpha')$, a contradiction.  \hfill \Box$

1.9 Proof of Theorem 1.5. The first part of the theorem was proved in Lemma 1.6. To prove the second part we suppose, contrary to the assertion, that $\dim(\Gamma \setminus W) = \dim \Gamma = k$.

Since (w) is a local property and invariant under $C^2$ local diffeomorphisms, we can suppose $\Gamma$ is an open subset of $\mathbb{R}^k \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$. In this case $T_y \Gamma = \mathbb{R}^k$, for all $y \in \Gamma$. Then by the assumption, applying Lemma 1.7, we get an open sub set $U$ of $\Gamma$, a $C^p$ definable map $\tilde{\rho} : U \times (0, \alpha) \rightarrow \Gamma'$ such that $\tilde{\rho}(y, t) = (y, \rho(y, t))$ and $\|\rho(y, t)\| = t$, and, moreover, for each $y \in U$

$$\frac{\delta(\mathbb{R}^k, T_{(y, \rho(y, t))} \Gamma')}{\|\rho(y, t)\|} \rightarrow +\infty, \text{ when } t \rightarrow 0^+.$$

On the other hand, applying Lemma 1.8 to $M(y, t) := \frac{\rho(y, t)}{t}$, reducing $U$ and $\alpha$, we have

$$\|D_1 \rho(y, t)\| \leq Ct, \text{ for all } y \in U, t \in (0, \alpha),$$

with some $C > 0$.

Note that $T_{(y, \rho(y, t))} \Gamma' \supset \text{graph}D_1 \rho(y, t)$. Therefore,

$$\frac{\delta(\mathbb{R}^k, T_{(y, \rho(y, t))} \Gamma')}{\|\rho(y, t)\|} \leq \frac{\|D_1 \rho(y, t)\|}{\|\rho(y, t)\|} \leq C, \text{ for } y \in U, 0 < t < \alpha.$$

This is a contradiction.  \hfill \Box

Note that Whitney condition (b) (defined in [Wh]) does not imply condition (w), even for algebraic sets (see [BT]). And, in general, we do not have (w) $\Rightarrow$ (b) (e.g. $\Gamma = (0, 0)$, $\Gamma' = \{(x, y) \in \mathbb{R}^2 : x = r \cos r, y = r \sin r, r > 0\}$, or $\Gamma' = \{(x, y) \in \mathbb{R}^2 : y = x \sin(1/x), x > 0\}$). In o-minimal structures, by the finiteness, such spiral phenomena or oscillation cannot occur. The following is a version of Kuo-Verdier’s Theorem (see [K] [V]).

1.10 Proposition. Let $\Gamma, \Gamma' \subset \mathbb{R}^n$ be definable $C^p$-submanifolds ($p \geq 2$), with $\Gamma \subset \overline{\Gamma'} \setminus \Gamma$. If $(\Gamma, \Gamma')$ satisfies the condition (w) at $y \in \Gamma$, then it satisfies the Whitney condition (b) at $y$.

Proof. Our proof is an adaptation of [V. Theorem 1.5] and based on the following observation:

If $f : (0, \alpha) \rightarrow \mathbb{R}$ is definable with $f(t) \neq 0$, for all $t$, and $\lim_{t \rightarrow 0^+} f(t) = 0$, then, by Cell Decomposition and monotonicity [DM. Th.4.1], there is $0 < \alpha' < \alpha$, such that $f$ is of class $C^1$ and strictly monotone on $(0, \alpha')$. By Mean Value Theorem and Definable Choice, there exists a definable function $\theta : (0, \alpha') \rightarrow (0, \alpha')$ with $0 < \theta(t) < t$, such that $f(t) = f'(\theta(t))t$. Since $|f(t)| > |f(\theta(t))|$, by monotonicity, $\lim_{t \rightarrow 0^+} \frac{f(t)}{f'(t)} = 0$. 


Now we prove the theorem. By a $C^2$ change of local coordinates, we can suppose $\Gamma$ is an open subset of $\mathbb{R}^k \subset \mathbb{R}^k \times \mathbb{R}^l \ (l = n - k)$, and $y = 0$. Let $\pi : \mathbb{R}^k \times \mathbb{R}^l \rightarrow \mathbb{R}^l$ be the orthogonal projection. Since $(\Gamma, \Gamma')$ satisfies (w) at 0, there exists $C > 0$ and a neighborhood $U$ of 0 in $\mathbb{R}^n$, such that

\[
(*) \quad \delta(T_y \Gamma, T_x \Gamma') \leq C\|x - y\|, \quad \text{for all } x \in \Gamma' \cap U, y \in \Gamma \cap U.
\]

If the condition (b) is not satisfied at 0 for $(\Gamma, \Gamma')$, then there exists $\epsilon > 0$, such that $0 \in \overline{S} \setminus S$, where

\[
S = \{ x \in \Gamma' : \delta(\pi(x), T_x \Gamma') \geq 2\epsilon \}.
\]

Since $S \cap \{ x : \|x\| \leq t \} \neq \emptyset$, for all $t > 0$, by Curve selection [DM. Th.4.6], there exists a definable curve $\varphi : (0, \alpha) \rightarrow S$, such that $\|\varphi(t)\| \leq t$, for all $t$. By the above observation, we can assume $\varphi$ is of class $C^1$. Write $\varphi(t) = (a(t), \overline{b}(t)) \in \mathbb{R}^k \times \mathbb{R}^l$. Then $\|\overline{b}'(t)\|$ is bounded. Since $\varphi((0, \alpha)) \subset \Gamma'$, $a \neq 0$. Reducing $a$, we can assume $a'(t) \neq 0$, for all $t$. Since $\lim_{t \rightarrow 0^+} a'(t)$ exists, we have $\delta(Ra'(t), Ra(t)) \rightarrow 0$, when $t \rightarrow 0$. Therefore

\[
(**) \quad \delta(Ra'(t), T_{\varphi(t)} \Gamma') \geq \epsilon, \quad \text{for all } t \text{ sufficiently small.}
\]

On the other hand, we have

\[
\delta(Ra'(t), T_{\varphi(t)} \Gamma') = \frac{1}{\|a'(t)\|} \delta(a'(t), T_{\varphi(t)} \Gamma') = \frac{1}{\|a'(t)\|} \delta(b'(t), T_{\varphi(t)} \Gamma') \\
\leq \frac{\|\overline{b}'(t)\|}{\|a'(t)\|} \delta(R\overline{b}'(t), T_{\varphi(t)} \Gamma').
\]

From (*) and (**), we have $\epsilon \leq C\|a(t)\| \frac{\|\overline{b}'(t)\|}{\|a'(t)\|}$.

By the observation, the right-hand side of the inequality tends to 0 (when $t \rightarrow 0$), which is a contradiction. $\diamond$

From Theorem 1.3 and Proposition 1.10 we have

1.11 Corollary. Whitney Stratification Theorem holds true in any o-minimal structure on the real field.

2. ($w_f$)-Stratifications

Thoughout this section, let $X \subset \mathbb{R}^n$ be a definable set and $f : X \rightarrow \mathbb{R}$ be a continuous definable function. Let $p$ be a positive integer.

2.1 Definition. A $C^p$ stratification of $f$ is a $C^p$ stratification $S$ of $X$, such that for every stratum $\Gamma \in S$, the restriction $f|_\Gamma$ is $C^p$ and of constant rank. For each $x \in \Gamma$, $T_{x,f}$ denotes the tangent space of the level of $f|_\Gamma$ at $x$, i.e. $T_{x,f} = \ker d(f|_\Gamma)(x)$.

Let $\Gamma, \Gamma' \in S$ with $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$. We say that the pair $(\Gamma, \Gamma')$ satisfies the Thom condition $(a_f)$ at $y_0 \in \Gamma$ if and only if the following holds:
(a_f) for every sequence \((x_k)\) in \(\Gamma'\), converging to \(y_0\), we have
\[
\delta(T_{y_0,f}, T_{x_k,f}) \longrightarrow 0.
\]
We say that \((\Gamma, \Gamma')\) satisfies the strict Thom condition \((w_f)\) at \(y_0\) if:
\[(w_f)\ there exists a constant \(C > 0\) and a neighborhood \(U\) of \(y_0\) in \(\mathbb{R}^n\), such that
\[
\delta(T_{y,f}, T_{x,f}) \leq C\|x - y\| \quad \text{for all} \ x \in \Gamma' \cap U, y \in \Gamma \cap U.
\]
Note that the conditions are \(C^2\)-invariant.
The existence of stratifications satisfying \((w_f)\) (and hence \((a_f)\)) for subanalytic function was proved in [KP] (see also [B][KR]). For functions definable in o-minimal structures on the real field we have:

2.2 Theorem. There exists a \(C^p\) stratification of \(f\) satisfying the Thom condition \((a_f)\) at every point of the strata.

Proof: see [L2]

2.3 Remark. In general, definable functions cannot be stratified to satisfy the condition \((w_f)\). The following example is given by Kurdyka.
Let \(f : [a, b] \times [0, +\infty) \longrightarrow \mathbb{R}\) be defined by \(f(x, y) = y^x\ (0 < a < b)\). Let \(\Gamma = [a, b] \times 0\), and \(\Gamma' = [a, b] \times (0, +\infty)\). Then the fiber of \(f|\Gamma'\) over \(c \in \mathbb{R}_+\):
\[
\{(x, y(x) = \exp(-\frac{1}{tx})): x \in [a, b]\}, \quad t = -\frac{1}{\ln c}.
\]
Then \(\frac{y'(x)}{y(x)} = \frac{1}{tx} \rightarrow +\infty\), when \(t \rightarrow 0^+\), for all \(x \in [a, b]\),
i.e. \(\frac{\delta(T_{x,f}, T_{(x,y(x))}, f)}{||y(x)||}\) can not be locally bounded along \(\Gamma\).

The remainder of this section is devoted to the proof of the existence of \((w_f)\)-stratification of functions definable in polynomially bounded o-minimal structures.

2.4 Definition. A structure \(D\) on the real field \((\mathbb{R}, +,.)\) is polynomially bounded if for every function \(f : \mathbb{R} \longrightarrow \mathbb{R}\) definable in \(D\), there exists \(N \in \mathbb{N}\), such that
\[
|f(t)| \leq t^N, \quad \text{for all sufficiently large} \ t.
\]
For example, the structure of global subanalytic sets, the structure generated by real power functions [M2], or by restricted Gevrey functions [DS] are polynomially bounded.

2.5 Theorem. Suppose that \(D\) is polynomially bounded. Then there exists a \(C^p\) stratification of \(f\) satisfying the condition \((w_f)\) at each point of the strata.

Note - The converse of the theorem is also true: If \(D\) is not polynomially bounded, then it must contain the exponential function, by [M1]. So the function given in Remark 2.3
is definable in $\mathcal{D}$ which cannot be $(w_f)$-stratified.

2.6 Proposition. There exists a $C^p$ stratification of $f$.

Proof: (c.f. [DM, Th. 4.8]) First note that if $f : \Gamma \rightarrow \mathbb{R}^l$ is a $C^1$ definable map on a $C^1$-submanifold $\Gamma$ of $\mathbb{R}^n$, then the set

$$P = \{ y \in \Gamma : \exists t > 0, \forall x \in \Gamma (\|x - y\| < t \Rightarrow \text{rank} f(x) = \text{rank} f(y)) \}$$

is definable and $\dim(\Gamma \setminus P) < \dim \Gamma$.

Therefore, applying Proposition 1.4, we have a $C^p$ stratification of $f$. $\Diamond$

By the previous proposition and Proposition 1.4, Theorem 2.5 is implied by the following.

2.7 Theorem. Suppose that $\mathcal{D}$ is polynomially bounded. Let $\Gamma, \Gamma'$ be definable $C^p$ submanifolds of $\mathbb{R}^n$. Suppose $\Gamma \subset \overline{\Gamma|\Gamma'}$, and $f : \Gamma \cup \Gamma' \rightarrow \mathbb{R}$ is continuous definable function such that $f$ has constant rank on $\Gamma$ and $\Gamma'$. Then

(i) $W_f = \{ x \in \Gamma : (w_f) \text{ is satisfied at } x \}$ is definable, and

(ii) $\dim(\Gamma \setminus W_f) < \dim \Gamma$.

Proof: The proof is much the same way as that for the condition (a$_f$) in [L2].

(i) Since $x \mapsto d(f|_\Gamma)$ is a definable map (see [DM]), the kernel bundle of $f|_\Gamma$ is definable. Therefore,

$$W_f = \{ y_0 : y_0 \in \Gamma, \exists C > 0, \exists t > 0, \forall x \in \Gamma', \forall y \in \Gamma \}$$

is definable.

(ii) To prove the second assertion there are three cases to consider.

Case 1: $\text{rank} f|_\Gamma = \text{rank} f|_{\Gamma'} = 0$. In this case

$$W_f = \{ y \in \Gamma : (\Gamma, \Gamma') \text{ satisfies Verdier condition (w) at } y \}.$$ 

The assertion follows from Theorem 1.3.

Case 2: $\text{rank} f|_\Gamma = 0$ and $\text{rank} f|_{\Gamma'} = 1$.

Suppose the contrary: $\dim(\Gamma \setminus W_f) < \dim \Gamma$. Since $(w_f)$ is $C^2$ invariant, by Cell Decomposition, we can assume that $\Gamma$ is an open subset of $\mathbb{R}^k \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, and $f|_{\Gamma'} > 0$, $f|_\Gamma \equiv 0$. So $T_{y,f} = \mathbb{R}^k$, for all $y \in \Gamma$. Let

$$A = \{ (y, s, t) : (y, s) \in \Gamma \cup \Gamma', t > 0, f(y, s) = t \}.$$

Then $A$ is a definable set. By Definable Choice and the assumption, there exists an open subset $U$ of $\Gamma$, $\alpha > 0$, and a definable map $\theta : U \times [0, \alpha] \rightarrow \mathbb{R}^{n-k}$, such that $\theta$ is $C^p$ on $U \times (0, \alpha)$, $\theta|_\Gamma \equiv 0$, and $f(y, \theta(y,t)) = t$, and, moreover, for all $y \in U$, we have

$$(*) \quad \frac{\|D_y \theta(y,t)\|}{\|\theta(y,t)\|} \geq \frac{\delta(\mathbb{R}^k, T_{y,\theta(y,t)}, f)}{\|\theta(y,t)\|} \rightarrow +\infty, \text{ when } t \rightarrow 0^+.$$
On the other hand, by [M2. Prop. 5.2], there exists an open subset $B$ of $U$ and $r > 0$, such that

$$\theta(y, t) = c(y)t^r + \varphi(y, t)t^{r_1}, \quad y \in B, t > 0 \text{ sufficiently small},$$

where $c$ is $C^p$ on $B$, $c \not\equiv 0$, $r_1 > r$, and $\varphi$ is $C^p$ with $\lim_{t \to 0^+} \varphi(y, t) = 0$, for all $y \in B$. Moreover, by Lemma 1.8, we can suppose that $D_1\varphi$ is bounded. Substituting (**) to the left-hand side of (*) we get a contradiction.

**Case 3:** $\text{rank } f|_{\Gamma} = \text{rank } f|_{\Gamma'} = 1$.
If $\dim(\Gamma \setminus W_f) = \dim \Gamma$, then the condition $(w_f)$ is false for $(\Gamma, \Gamma')$ over an open subset $B$ of $\Gamma$. It is easy to see that there is $c \in \mathbb{R}$ such that $(w_f)$ is false for the pair $(\Gamma \cap f^{-1}(c), \Gamma')$ over an open subset of $B \cap f^{-1}(c)$, and hence open in $\Gamma \cap f^{-1}(c)$. This contradicts Case 2.

### 2.8 Remark.
If the structure admits analytic cell decomposition, then the theorems hold true with “analytic” in place of “$C^p$”. Our results can be translated to the setting of analytic-geometric categories in the sense of [DM].

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