Minimal Nondeterministic Finite Automata and Atoms of Regular Languages *

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Abstract. We examine the NFA minimization problem in terms of atomic NFAs, that is, NFAs in which the right language of every state is a union of atoms, where the atoms of a regular language are non-empty intersections of complemented and uncomplemented left quotients of the language. We characterize all reduced atomic NFAs of a given language, that is, those NFAs that have no equivalent states. Using atomic NFAs, we formalize Sengoku’s approach to NFA minimization and prove that his method fails to find all minimal NFAs. We also formulate the Kameda-Weiner NFA minimization in terms of quotients and atoms.

Keywords: regular language, quotient, atom, atomic NFA, minimal NFA

1 Introduction

Nondeterministic finite automata (NFA’s) have played a major role in the theory of finite automata and regular expressions and their applications ever since their introduction in 1959 by Rabin and Scott [10]. In particular, the intriguing problem of finding NFA’s with the minimal number of states has received much attention. The problem was first stated by Ott and Feinstein [8] in 1961. Various approaches have then been used over the years in attempts to answer this question; we mention a few examples here. In 1970, Kameda and Weiner [6] studied this problem using matrices related to the states of the minimal deterministic finite automata (DFA’s) for a given language and its reverse. In 1992, Arnold, Dicky, and Nivat [1] used a “canonical” NFA. In the same year, Sengoku [11] used “normal” NFA’s and “standard formed” NFA’s. In 1995, Matz and Potthoff [7] returned to the “canonical” automaton and introduced the “fundamental” automaton. In 2003, Ilie and Yu [5] applied equivalence relations. In 2005, Polák [9] used the “universal” automaton.

* This work was supported by the Natural Sciences and Engineering Research Council of Canada under grant No. OGP0000871, the ERDF funded Estonian Center of Excellence in Computer Science, EXCS, and the Estonian Ministry of Education and Research target-financed research theme no. 0140007s12.
Our approach is to use the recently introduced atoms and atomic languages [3] for this question; we briefly state some of their basic properties here.

The (left) quotient of a regular language $L$ over an alphabet $\Sigma$ by a word $w \in \Sigma^*$ is the language $w^{-1}L = \{x \in \Sigma^* \mid wx \in L\}$. It is well known that the number of states in the complete minimal deterministic finite automaton recognizing $L$ is precisely the number of distinct quotients of $L$. Also, $L$ is its own quotient by the empty word $\varepsilon$, that is $\varepsilon^{-1}L = L$. A quotient DFA is a DFA uniquely determined by a regular language; its states correspond to left quotients. The quotient DFA is isomorphic to the minimal DFA.

An atom of a regular language $L$ with quotients $K_0, \ldots, K_{n-1}$ is any non-empty language of the form $\overline{K_0} \cap \cdots \cap \overline{K_{n-1}}$, where $\overline{K_i}$ is either $K_i$ or $\overline{K_i}$, and $\overline{K_i}$ is the complement of $K_i$ with respect to $\Sigma^*$. If the intersection with all quotients complemented is non-empty, then it constitutes the negative atom; all the other atoms are positive. Let the number of atoms be $m$, and let the number of positive atoms be $p$. Thus, if the negative atom is present, $p = m - 1$; otherwise, $p = m$.

So atoms of $L$ are regular languages uniquely determined by $L$. They are pairwise disjoint and define a partition of $\Sigma^*$. Every quotient of $L$ (including $L$ itself) is a union of atoms, and every quotient of an atom is a union of atoms. Thus the atoms of a regular language are its basic building blocks. Also, $L$ defines the same atoms as $\overline{L}$. The átomaton is an NFA uniquely determined by a regular language; its states correspond to atoms. An NFA is atomic if the right language of every state is a union of atoms.

Our contributions are as follows:

1. We characterize all trim reduced atomic NFA’s of a given language, where an NFA is reduced if it has no equivalent states.
2. We show that, if $n_0$ is the minimal number of states of any NFA of a language, then the language may have trim reduced atomic NFA’s with as few as $n_0$ states, and as many as $2^p - 1$ states.
3. We demonstrate that the number of atomic minimal NFA’s can be as low as 1, or very high. For example, the language $\Sigma^*ab\Sigma^*$ with 3 quotients has 281 atomic minimal NFA’s, and additional non-atomic ones.
4. We formalize the work of Sengoku [11] in our framework. He had no concept of atoms, but used an NFA equivalent to the átomaton and NFA’s equivalent to atomic NFA’s. Our use of atoms significantly clarifies Sengoku’s method.
5. We prove that Sengoku’s claim that an NFA can be made atomic by adding transitions and without changing the number of states is false. We show that there exist languages for which the minimal NFA’s are all non-atomic. So Sengoku’s claim that his method can always find a minimal NFA is also incorrect.
6. We formulate the Kameda-Weiner NFA minimization method [6] in terms of quotients and atoms.

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The definition in [3] does not consider the intersection of all the complemented quotients to be an atom. Our new definition in [4] adds symmetry to the theory.
In Section 2 we recall some properties of automata and átomata. Atomic
NFA's are then presented in Section 3. Sengoku's method is studied in Section 4,
and the Kameda-Weiner method, in Section 5. Section 6 concludes the paper.

2 Automata and Átomata of Regular Languages

A nondeterministic finite automaton (NFA) is a quintuple \( N = (Q, \Sigma, \eta, I, F) \),
where \( Q \) is a finite, non-empty set of states, \( \Sigma \) is a finite non-empty alphabet,
\( \eta : Q \times \Sigma \rightarrow 2^Q \) is the transition function, \( I \subseteq Q \) is the set of initial states, and
\( F \subseteq Q \) is the set of final states. As usual, we extend the transition function to
functions \( \eta' : Q \times \Sigma^* \rightarrow 2^Q \), and \( \eta'' : 2^Q \times \Sigma^* \rightarrow 2^Q \), but use \( \eta \) for all three.

The language accepted by an NFA \( N \) is \( L(N) = \{ w \in \Sigma^* \mid \eta(I, w) \cap F \neq \emptyset \} \).
Two NFA's are equivalent if they accept the same language. The right language
of a state \( q \) is \( L_{q,F}(N) = \{ w \in \Sigma^* \mid \eta(q, w) \cap F \neq \emptyset \} \). The right language of a
set \( S \) of states of \( N \) is \( L_{S,F}(N) = \bigcup_{q \in S} L_{q,F}(N) \); so \( L(N) = L_{I,F}(N) \). A state
is empty if its right language is empty. Two states are equivalent if their right
languages are equal. An NFA is reduced if it has no equivalent states. The left
language of a state \( q \) is \( L_{I,q} = \{ w \in \Sigma^* \mid q \in \eta(I, w) \} \). A state is unreachable
if its left language is empty. An NFA is trim if it has no empty or unreachable
states. An NFA is minimal if it has the minimal number of states among all the
equivalent NFA's.

A deterministic finite automaton (DFA) is a quintuple \( D = (Q, \Sigma, \delta, q_0, F) \),
where \( Q, \Sigma, \) and \( F \) are as in an NFA, \( \delta : Q \times \Sigma \rightarrow Q \) is the transition function,
and \( q_0 \) is the initial state.

We use the following operations on automata:
1. The determinization operation \( \mathcal{D} \) applied to an NFA \( \mathcal{N} \) yields a DFA \( \mathcal{N}^\mathcal{D} \)
   obtained by the subset construction, where only subsets reachable from the initial
   subset of \( \mathcal{N}^\mathcal{D} \) are used, and the empty subset, if present, is included.
2. The reversal operation \( \mathcal{R} \) applied to NFA \( \mathcal{N} \) yields an NFA \( \mathcal{N}^\mathcal{R} \), where the
   sets of initial and final states are interchanged and all transitions are reversed.
3. The trimming operation \( \mathcal{T} \) applied to an NFA deletes all unreachable and
   empty states.

The following theorem is from [2], and was also discussed in [3]:

**Theorem 1 (Determinization).** If \( \mathcal{D} \) is a DFA accepting a language \( L \), then
\( \mathcal{D}^\mathcal{R} \) is a minimal DFA for \( L^\mathcal{R} \).

Let \( L \) be any non-empty regular language, and let its set of quotients be
\( \mathcal{K} = \{ K_0, \ldots, K_{n-1} \} \). One of the quotients of \( L \) is \( L \) itself; this is called the
initial quotient and is denoted by \( K_{n_0} \). A quotient is final if it contains the
empty word \( \varepsilon \). The set of final quotients is \( \mathcal{F} = \{ K_i \mid \varepsilon \in K_i \} \).

In the following definition we use a 1-1 correspondence \( K_i \leftrightarrow K_i \) between
quotients \( K_i \) of a language \( L \) and the states \( K_i \) of the quotient DFA \( \mathcal{D} \) defined
below. We refer to the \( K_i \) as quotient symbols.
Definition 1. The quotient DFA of $L$ is $\mathfrak{O} = (K, \Sigma, \delta, K_{in}, F)$, where $K = \{K_0, \ldots, K_{n-1}\}$, $K_{in}$ corresponds to $K_{in}$, $F = \{K_i \mid K_i \in F\}$, and $\delta(K_i, a) = K_j$ if and only if $a^{-1}K_i = K_j$, for all $K_i, K_j \in K$ and $a \in \Sigma$.

In a quotient DFA the right language of $K_i$ is $K_i$, and its left language is $\{w \in \Sigma^* \mid w^{-1}L = K_i\}$. The language $L(\mathfrak{O})$ is the right language of $K_{in}$, and hence $L(\mathfrak{O}) = L$. DFA $\mathfrak{O}$ is minimal, since all quotients in $K$ are distinct.

It follows from the definition of an atom, that a regular language $L$ has at most $2^n$ atoms. An atom is initial if it has $L$ (rather than $\overline{L}$) as a term; it is final if it contains $\varepsilon$. Since $L$ is non-empty, it has at least one quotient containing $\varepsilon$. Hence it has exactly one final atom, the atom $\overline{K_0} \cap \cdots \cap \overline{K_{n-1}}$, where $\overline{K_i} = K_i$ if $\varepsilon \in K_i$, and $\overline{K_i} = \overline{F}$ otherwise. Let $A = \{A_0, \ldots, A_{m-1}\}$ be the set of atoms of $L$. By convention, $\mathcal{I}$ is the set of initial atoms, $A_{p-1}$ is the final atom and the negative atom, if present, is $A_{m-1}$. The negative atom is not reachable from $\mathcal{I}$ and can never be final, since there must be at least one final quotient in its intersection.

As above, we use a 1-1 correspondence $A_i \leftrightarrow A_i$ between atoms $A_i$ of a language $L$ and the states $A_i$ of the NFA $\mathfrak{A}$ defined below. We refer to the $A_i$ as atom symbols.

Definition 2. The átomaton of $L$ is the NFA $\mathfrak{A} = (A, \Sigma, \alpha, A_1, \{A_{p-1}\})$, where $A = \{A_i \mid A_i \in A\}$, $A_1 = \{A_i \mid A_i \in \mathcal{I}\}$, $A_{p-1}$ corresponds to $A_{p-1}$, and $A_j \in \alpha(A_i, a)$ if and only if $aA_j \subseteq A_i$, for all $A_i, A_j \in A$ and $a \in \Sigma$.

In the átomaton, the right language of any state $A_i$ is the atom $A_i$.

The results from [3] and our definition of atoms in [4] imply that $\mathfrak{A}^R$ is a minimal DFA that accepts $L^R$. It follows from Theorem 1 that $\mathfrak{A}^R$ is isomorphic to $\mathfrak{O}^{RDR}$. The following result from [4] makes this isomorphism precise:

Theorem 2 (Isomorphism). Let $S$ be the collection of all subsets of the set $K$ of quotient symbols. Let $\varphi : A \rightarrow S$ be the mapping assigning to state $A_j$, corresponding to $A_j = K_i \cap \cdots \cap K_{i-r-1} \cap \overline{K_{i-r}} \cap \cdots \cap \overline{K_{i-r-1}}$ of $\mathfrak{A}^R$, the set $\{K_{i-r}, \ldots, K_{i-r-1}\}$. Then $\varphi$ is a DFA isomorphism between $\mathfrak{A}^R$ and $\mathfrak{O}^{RDR}$.

Corollary 1. The mapping $\varphi$ is an NFA isomorphism between $\mathfrak{A}$ and $\mathfrak{O}^{RDR}$.

3 Atomic NFA’s

A new class of NFA’s was defined in [3] as follows:

Definition 3. An NFA $\mathfrak{N} = (Q, \Sigma, \eta, I, F)$ is atomic if for every $q \in Q$, the right language $L_q, F(\mathfrak{N})$ of $q$ is a union of some positive atoms of $L(\mathfrak{N})$.

The following theorem, slightly restated, was proved in [3]:

4
Since suppose \( B \) but their behaviours are not distinct; hence we consider only reduced NFA’s.

Three of these NFA’s are equivalent, and they accept \( B \) of Table 1 and its reverse are not atomic. NFA \( N \) is atomic. To do this, reverse \( N \) is minimal.

This theorem allows us to test whether an NFA \( N \) accepting a language \( L \) is atomic. To do this, reverse \( N \) and apply the subset construction. Then \( N \) is atomic if and only if \( N^{RD} \) is isomorphic to the minimal DFA of \( L^R \).

All three possibilities for the atomic nature of \( B \) exist: NFA \( N_a \) of Table 1 and its reverse are not atomic. NFA \( N_b \) of Table 2 is atomic, but its reverse is not. NFA \( N_c \) of Table 3 and its reverse are both atomic. Note that all three of these NFA’s are equivalent, and they accept \( \Sigma^*ab\Sigma^* \).

If we allow equivalent states, there is an infinite number of atomic NFA’s, but their behaviours are not distinct; hence we consider only reduced NFA’s. Suppose \( \mathcal{B} = (B, \Sigma, \beta, B_1, B_F) \) is any trim reduced atomic NFA accepting \( L \). Since \( \mathcal{B} \) is atomic, the right language of any state in \( \mathcal{B} \) is a union of positive atoms of \( L \); hence the states of \( \mathcal{B} \) can be represented by sets of positive atom symbols. Because \( \mathcal{B} \) is trim, it does not have a state with the empty set of atom symbols. Since \( \mathcal{B} \) is reduced, no set of atom symbols appears twice. Thus the state set \( B \) is a collection of non-empty sets of positive atom symbols.

**Theorem 4 (Legality).** Suppose \( L \) is a regular language, its atomaton is \( A = (A, \Sigma, \alpha, A_1, \{A_{p-1}\}) \), and \( \mathcal{B} = (B, \Sigma, \beta, B_1, B_F) \) is a trim NFA, where \( B = \{B_1, \ldots, B_r\} \) is a collection of sets of positive atom symbols and \( B_1, B_F \subseteq B \). If \( B_i \subseteq B \), define \( S(B_i) = \bigcup_{B_j \in B} B_i \) to be the set of atom symbols appearing in the sets \( B_j \) of \( B_i \). Then \( \mathcal{B} \) is a reduced atomic NFA of \( L \) if and only if it satisfies the following conditions:

1. \( S(B_1) = A_1 \).
2. For all \( B_i \in B \), \( S(\beta(B_i, a)) = \alpha(B_i, a) \).
3. For all \( B_i \in B \), we have \( B_i \in B_F \) if and only if \( A_{p-1} \in B_i \).

Before proving the theorem, we require the following lemma:

**Lemma 1.** If \( \mathcal{B} \) satisfies Condition 2 of Theorem 4, then \( S(\beta(B_i, w)) = \alpha(B_i, w) \) for every \( B_i \in B \) and \( w \in \Sigma^* \).

**Proof.** For \( w = \varepsilon \), we have \( S(\beta(B_i, \varepsilon)) = S(B_i) = B_i \), and \( \alpha(B_i, \varepsilon) = B_i \); so the claim holds for this case.

Assume that \( S(\beta(B_i, w)) = \alpha(B_i, w) \) for all \( B_i \in B \) and all \( w \in \Sigma^* \) with length less than or equal to \( l \geq 0 \). We prove that \( S(\beta(B_i, wa)) = \alpha(B_i, wa) \) for...
every \( a \in \Sigma \). Let \( \beta(B_i, w) = \{B_{i_1}, \ldots, B_{i_k}\} \) for some \( B_{i_1}, \ldots, B_{i_k} \in \mathcal{B} \). Since \( \beta(B_i, wa) = (\beta(B_i, w), a) = \beta(B_i, a) \cup \cdots \cup \beta(B_{i_k}, a) \), we have \( S(\beta(B_i, wa)) = S(\beta(B_i, a)) \cup \cdots \cup S(\beta(B_{i_k}, a)) \). By Condition 2, the latter is equal to \( \alpha(B_i, a) \cup \cdots \cup \alpha(B_{i_k}, a) = \alpha(B_i, a) = \alpha(S(\beta(B_i, w)), a) \). By the inductive assumption, we get \( \alpha(S(\beta(B_i, w)), a) = \alpha(\alpha(B_i, w), a) = \alpha(B_i, wa) \), which proves our claim. \( \square \)

**Proof of Theorem 4**

Proof. First we prove that any NFA \( \mathcal{B} \) satisfying Conditions 1–3 is an atomic NFA of \( L \). Let \( B_i \in \mathcal{B} \) be a state of \( \mathcal{B} \). If \( w \in L_{B_i,B_F}(\mathcal{B}) \), then by Condition 3, there exists \( B_j \in \beta(B_i, w) \) such that \( A_{p-1} \in B_j \), and we have \( A_{p-1} \in S(\beta(B_i, w)) \). By Lemma 1, we get \( A_{p-1} \in \alpha(B_i, w) \), implying that there is some \( A_k \in \alpha(B_i, w) \) such that \( w \in L_{A_k,A_{p-1}}(\mathcal{B}) \). Conversely, if \( w \in L_{A_k,A_{p-1}}(\mathcal{B}) \) and \( A_k \in B_i \), then \( A_{p-1} \in \alpha(B_i, w) = S(\beta(B_i, w)) \). Hence there exists \( B_j \in \beta(B_i, w) \) such that \( A_{p-1} \in B_j \). Consequently, every word accepted in \( \mathcal{B} \) from state \( B_i \) is in some atom \( A_k \) such that \( A_k \in B_i \), and every word in an atom \( A_k \) such that \( A_k \in B_i \) is also in \( L_{B_i,B_F}(\mathcal{B}) \). Therefore the right language of \( B_i \) in \( \mathcal{B} \) is equal to the union of atoms \( A_k \) such that \( A_k \in B_i \). In particular, \( L_{B_i,B_F}(\mathcal{B}) \) is the union of atoms whose atom symbols appear in the initial collection of \( \mathcal{B} \) which, by Condition 1, is the same as the union of atom symbols in \( \mathcal{B} \). But that last union is precisely \( L_{A_1,A_{p-1}}(\mathcal{B}) = L \). Since any two sets \( B_i \) and \( B_j \) are different, and atoms are disjoint, \( \mathcal{B} \) is reduced. Hence \( \mathcal{B} \) is a reduced atomic NFA of \( L \).

Conversely, we show that if \( \mathcal{B} \) is a reduced atomic NFA of \( L \), then it must satisfy Conditions 1–3. So in the following we assume that \( \mathcal{B} \) is atomic, that is, for every state \( B_i \) of \( \mathcal{B} \), the right language of \( B_i \) is equal to the union of atoms \( A_k \) such that \( A_k \in B_i \).

First, we show that Condition 1 holds. Let \( A_j \in S(\beta(B_i)) \). Then there is a state \( B_j \in B_i \) such that \( A_j \in B_j \). So for any \( a \in A_j \), \( w \in L(\mathcal{B}) \). Since \( L(\mathcal{B}) = L(\mathcal{A}) \), we have \( w \in L(\mathcal{A}) \) for all \( w \in A_j \). Thus \( A_j \in A_1 \). Conversely, if \( A_j \in A_1 \), then for all \( w \in A_j \), \( w \in L(\mathcal{A}) = L(\mathcal{B}) \). Since \( \mathcal{B} \) is atomic, there is an initial state \( B_j \) such that \( A_j \subseteq L_{B_j,B_F}(\mathcal{B}) \). Hence \( A_j \in S(\beta(B_j)) \).

Next, we prove Condition 2. If \( A_j \in S(\beta(B_i, a)) \), then \( L_{B_i,B_F}(\mathcal{B}) \) must contain \( aA_j \). So there must exist some \( A_i \in B_i \) such that \( aA_j \subseteq A_i \). Thus \( A_j \in \alpha(B_i, a) \). Conversely, if \( A_j \in \alpha(B_i, a) \), then there is an atom \( A_i \in B_i \) such that \( A_j \in \alpha(A_i, a) \), implying \( aA_j \subseteq A_i \). Since \( A_i \in B_i \), \( L_{B_i,B_F}(\mathcal{B}) \) must contain \( aA_j \). Hence \( A_j \in S(\beta(B_i, a)) \).

To show that Condition 3 holds, we first suppose that \( B_i \in B_F \). Then \( a \) is in the right language of \( B_i \). Since \( \mathcal{B} \) is atomic, \( a \) must be in one of the atoms of \( B_i \). However, the only atom containing \( a \) is \( A_{p-1} \), so \( A_{p-1} \in B_i \). Conversely, if \( A_{p-1} \in B_i \), then \( a \) is in the right language of \( B_i \), and \( B_i \) is a final state by definition of an NFA. \( \square \)

**Example 1.** Consider the trim automaton \( \mathcal{A}^T \) of Table 4 and the atomic NFA \( \mathcal{B} \) of Table 5. Here \( \mathcal{B} = \{B_0, B_1, B_2\} \), where \( B_0 = \{A_0, A_1\} \), \( B_1 = \{A_2\} \), and \( B_2 = \{A_0, A_2\} \). The initial collection is \( B_I = \{B_0\} = \{\{A_0, A_1\}\} \), and the
Thus, \( w \) follows that some initial state of a set of atom symbols in the \( \mathcal{A} \) is reachable from the initial state that corresponds to the quotient \( K \) of any quotient \( K_j \) of \( L \). Therefore, there exists a trim reduced atomic NFA of \( L \) with state set \( \mathcal{B} \).

Proof. Let \( \mathcal{B} = (\mathcal{B}, \Sigma, \beta, \mathcal{B}_F, \mathcal{B}_F') \) be an NFA in which the state set \( \mathcal{B} \) is the collection of all sets \( \mathcal{B}_i \) such that \( \mathcal{B}_i \) is a non-empty subset of the set of atom symbols \( \bar{A}_h \subseteq K_j \) of any quotient \( K_j \) of \( L \), where \( j \in \{0, \ldots, n-1\} \). Let \( \beta(B_i, a) = \{B_j | B_j \subseteq \alpha(B_i, a)\} \) for every \( B_i \in \mathcal{B} \) and \( a \in \Sigma \). \( \mathcal{B}_i \in \mathcal{B}_F \) if and only if \( \mathcal{B}_i \) is a subset of the set of atom symbols of the initial quotient \( K_{in} \), and \( \mathcal{B}_i \in \mathcal{B}_F' \) if and only if \( \bar{A}_{p-1} \subseteq \mathcal{B}_i \). We claim that \( \mathcal{B} \) is a trim reduced atomic NFA of \( L \).

First, we show that \( \mathcal{B} \) is trim. Let us consider any state \( \mathcal{B}_i \) of \( \mathcal{B} \). Let \( K_j \) be a quotient such that \( \mathcal{B}_i \) is a subset of the set of atom symbols of \( K_j \), and let \( \mathcal{B}_j \) be the set of atom symbols corresponding to \( K_j \). Let \( \mathcal{B}_0 \) be the set of atom symbols corresponding to the initial quotient \( K_{in} \) of \( L \). Note that \( \mathcal{B}_0 = A_I \). Since every set of atom symbols corresponding to some quotient is reachable from the initial set of atom symbols in the \( \mathcal{A} \), there must be a word \( w \in \Sigma^* \), such that \( \mathcal{B}_j \) is reachable from \( \mathcal{B}_0 \) by \( w \) in \( \mathcal{A} \). We show that \( \mathcal{B}_i \) is reachable from some initial state of \( \mathcal{B} \) by \( w \). If \( w = \varepsilon \), then \( K_j = K_{in} \), and since \( \mathcal{B}_j \subseteq \mathcal{B}_i \), it follows that \( \mathcal{B}_i \) is an initial state of \( \mathcal{B} \) reachable from itself by \( \varepsilon \). If \( w = a \) for some \( a \in \Sigma \), then there is a state \( \mathcal{B}_u \) of \( \mathcal{B} \), reachable from \( \mathcal{B}_0 \) by \( a \), such that \( \mathcal{B}_u \) corresponds to the quotient \( u^{-1}L \) of \( L \) and \( \mathcal{B}_j = \alpha(\mathcal{B}_u, a) \). Since \( \mathcal{B}_i \subseteq \mathcal{B}_j \), \( \mathcal{B}_j = \alpha(\mathcal{B}_u, a) \), by the definition of \( \beta \) we have \( \mathcal{B}_i \subseteq \beta(\mathcal{B}_u, a) \). Thus, \( \mathcal{B}_i \) is reachable from \( \mathcal{B}_0 \) in \( \mathcal{B} \) by \( ua \).

We also have to show that there is a word \( w \in \Sigma^* \), such that some final state of \( \mathcal{B} \) is reachable from \( \mathcal{B}_i \) by \( w \). If \( \mathcal{B}_i \) is final, then it is reachable from itself by \( w = \varepsilon \). If \( \mathcal{B}_i \) is not final, then let us consider any \( \bar{A}_h \in \mathcal{B}_i \). Since the right language of the state \( \bar{A}_h \) in the \( \mathcal{A} \) is not empty, and \( \bar{A}_h \) cannot be the final state of \( \mathcal{A} \), there must be some state \( \bar{A}_l \) of \( \mathcal{A} \) and some \( a \in \Sigma \), such

\[
\begin{array}{|c|c|c|}
\hline
& a & b \\
\hline
\mathcal{A}_0 & \{\mathcal{A}_0, \mathcal{A}_1\} & \{\mathcal{A}_0, \mathcal{A}_2\} \\
\hline
\mathcal{A}_1 & \{\mathcal{A}_2\} & \{\} \\
\hline
\mathcal{A}_2 & \{\} & \{\} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
& a & b \\
\hline
\Rightarrow \{\mathcal{A}_0, \mathcal{A}_1\} & \{\{\mathcal{A}_0, \mathcal{A}_1\}, \{\mathcal{A}_2\}\} & \{\{\mathcal{A}_0, \mathcal{A}_2\}\} \\
\hline
\Rightarrow \{\mathcal{A}_2\} & \{\} & \{\} \\
\hline
\Rightarrow \{\{\mathcal{A}_0, \mathcal{A}_2\}\} & \{\{\mathcal{A}_0, \mathcal{A}_1\}\} & \{\{\mathcal{A}_0, \mathcal{A}_2\}\} \\
\hline
\end{array}
\]
that \( A_i \in \alpha(A_k, a) \). Now we know that there is some \( B_j \) such that \( A_i \in B_j \) and \( \alpha(B_i, a) = B_j \). Since \( \beta(B_i, a) \) is the collection of all non-empty subsets of \( B_j \), it follows that \( \{ A_i \} \in \beta(B_i, a) \). Since the final state \( A_{p-1} \) of \( \mathfrak{A} \) is reachable from \( A_i \) by any word \( v \in A_i \), we get \( \{ A_{p-1} \} \in \beta(B_i, av) \) by the definition of \( \beta \). So a final state \( \{ A_{p-1} \} \) of \( \mathfrak{B} \) is reachable from \( B_i \) by \( av \). Thus, \( \mathfrak{B} \) is trim.

To see that \( \mathfrak{B} \) is a reduced atomic NFA, one verifies that Conditions 1–3 of Theorem 4 hold. Thus by Theorem 4, \( \mathfrak{B} \) is a trim reduced atomic NFA of \( L \). □

**Theorem 6 (NFA with \( 2^p - 1 \) states).** A regular language \( L \) has a trim reduced atomic NFA with \( 2^p - 1 \) states if and only if for some quotient \( K_i \) of \( L \), \( K_i = A_0 \cup \cdots \cup A_{p-1} \).

**Proof.** Let \( \mathfrak{B} = (B, \Sigma, \beta, B_0, B_F) \) be a trim reduced atomic NFA of \( L \) with \( 2^p - 1 \) states. Then there must be a state \( B_i \) of \( \mathfrak{B} \) such that \( B_i = \{ A_0, \ldots, A_{p-1} \} \). Since the right language of any state of a trim NFA is a subset of some quotient, we have \( L_{B_i, B_F}(\mathfrak{B}) = A_0 \cup \cdots \cup A_{p-1} \subseteq K_i \) for some quotient \( K_i \) of \( L \). On the other hand, \( K_i \) must be a union of some positive states, so we get \( K_i = A_0 \cup \cdots \cup A_{p-1} \).

Conversely, let \( K_i = A_0 \cup \cdots \cup A_{p-1} \) be a quotient of \( L \) which includes all the positive states of \( L \). Then by Theorem 5, there is a trim reduced atomic NFA of \( L \) in which the state set is the collection of all non-empty subsets of the set of positive atom symbols. This NFA has \( 2^p - 1 \) states. □

The construction of reduced atomic NFA’s is illustrated in the following example. To simplify the notation, we do not use atom symbols in examples.

**Example 2.** Consider the minimal DFA \( \mathfrak{D} \) taken from [6] and shown in Table 6. It accepts the language \( L = \Sigma^*(b\epsilon a a)\cup a \), and its quotients are \( K_0 = \epsilon^{-1}L = L \), \( K_1 = a^{-1}L = \Sigma^*(b \cup a a) \cup a \cup \epsilon \), and \( K_2 = b^{-1}L = \Sigma^*(b \cup a a) \cup \epsilon \). NFA \( \mathfrak{D}_{\text{RDT}} \) and the isomorphic trim âtomaton \( \mathfrak{A}^\uparrow \) with states renamed are shown in Tables 7 and 8. The positive atoms are \( A = \Sigma^*(b \cup a a) \), \( B = a \) and \( C = \epsilon \), and \( K_0 = A \cup B \), \( K_1 = A \cup B \cup C \), and \( K_2 = A \cup C \).

Since the set \{ \( A, B \) \} of initial atoms does not contain all positive atoms, no 1-state NFA exists.

1. For the initial state we could pick one state \{ \( A, B \) \} with two atoms. From there, the âtomaton reaches \{ \( A, B, C \) \} under \( a \), and \{ \( A, C \) \} under \( b \).

(a) If we pick \{ \( A, C \) \} as the second state, we can cover \{ \( A, B, C \) \} by \{ \( A, B \) \} and \{ \( A, C \) \}, as in Table 9. Here the minimal atomic NFA is unique.

| \( a \) | \( b \) | \( a \) | \( b \) | \( a \) | \( b \) |
|---|---|---|---|---|---|
| 01 | 1 | 2 | \{ 012 \} | \{ 012, 01 \} | \{ 012, 12 \} |
| 02 | 0 | 2 | \{ 12 \} | \{ \} | \{ \} |
Table 9. NFA $\mathfrak{B}_1$.

|          | $a$          | $b$          |
|----------|--------------|--------------|
| $\rightarrow$ | $\{A, B\}, \{A, C\}$ | $\{A, C\}$ |
| $\leftarrow$ | $\{A, C\}$ | $\{A, C\}$ |

Table 10. Atomic NFA $\mathfrak{B}_2$.

|          | $a$          | $b$          |
|----------|--------------|--------------|
| $\rightarrow$ | $\{A, B\}, \{C\}$ | $\{A, C\}$ |
| $\leftarrow$ | $\{C\}$ | $\{A, C\}$ |

Table 11. A 5-state NFA.

|          | $a$          | $b$          |
|----------|--------------|--------------|
| $\rightarrow$ | $\{A\}$ | $\{A\}, \{B\}$, $\{A, C\}$ |
| $\rightarrow$ | $\{B\}$ | $\{C\}$ |
| $\leftarrow$ | $\{A, C\}$ | $\{A\}, \{C\}$ |
| $\leftarrow$ | $\{C\}$ | $\{A, B\}$, $\{A, C\}$ |

Table 12. A 7-state NFA.

|          | $a$          | $b$          |
|----------|--------------|--------------|
| $\rightarrow$ | $\{A\}$ | $\{A\}, \{B\}$, $\{A, C\}$ |
| $\rightarrow$ | $\{B\}$ | $\{C\}$ |
| $\leftarrow$ | $\{A, C\}$ | $\{A\}, \{B\}$, $\{A, C\}$ |
| $\leftarrow$ | $\{C\}$ | $\{A, B\}, \{A, C\}, \{B, C\}$ |
| $\leftarrow$ | $\{B, C\}$ | $\{C\}$ |

(b) We can also use $\{A, B, C\}$ as a state. Then we need $\{A, C\}$ for the transition under $b$. This gives an NFA isomorphic to the DFA of Table 6.

(c) We can use state $\{C\}$ as shown in Table 10.

2. We can pick two initial states, $\{A\}$ and $\{B\}$.
   (a) If we add $\{C\}$, this leads to the励志on of Table 8.
   (b) A 5-state solution is shown in Table 11.

3. We can use three initial states, $\{A\}$, $\{B\}$ and $\{A, B\}$. A 7-state NFA is shown in Table 12. This is a largest possible reduced solution.

The number of minimal atomic NFA’s can also be very large.

Example 3. Let $\Sigma = \{a, b\}$ and consider the language $L = \Sigma^* a \Sigma^* b \Sigma^* = \Sigma^* ab \Sigma^*$. The quotients of $L$ are $K_0 = L$, $K_1 = L \cup b \Sigma^*$ and $K_2 = \Sigma^*$. The quotient DFA of $L$ is shown in Table 13, and its automaton, in Tables 14 and 15 (where the atoms have been relabelled). The atoms are $A = L$, $B = b^* ba^*$ and $C = a^*$, and there is no negative atom. Thus the quotients are $K_0 = L = A$, $K_1 = A \cup B$, and $K_2 = A \cup B \cup C$.

We find all the minimal atomic NFA’s of $L$. Obviously, there is no 1-state solution. The states of any atomic NFA are sets of atoms, and there are seven non-empty sets of atoms to choose from. Since there is only one initial atom, there is no choice: we must take $\{A\}$. For the transition $(A, a, \{A, B\})$, we can add $\{B\}$ or $\{A, B\}$. If there are only two states, atom $\{C\}$ cannot be reached. So there is no 2-state atomic NFA. The results for 3-state atomic NFA’s are summarized in Proposition 1.

Proposition 1. The language $\Sigma^* ab \Sigma^*$ has 281 minimal atomic NFA’s.
Table 13. DFA $\mathcal{D}$.

Table 14. Automaton $\mathfrak{A}$.

Table 15. $\mathfrak{A}$ relabelled.

Table 16. NFA $\mathfrak{N}_2$.

Table 17. NFA $\mathfrak{N}_9$.

Proof. We concentrate on 3-state solutions. We drop the curly brackets and commas and represent sets of atoms by words. Thus $\{A, AB, BC\}$ stands for $\{\{A\}, \{A, B\}, \{B, C\}\}$.

State $A$ is the only initial state and so it must be included. To implement the transition $(A, a, \{A, B\})$ from $\mathfrak{A}$, either $B$ or $AB$ must be chosen.

1. If $B$ is chosen, then there must be a set containing $C$ but not $A$; otherwise the transition $(B, b, \{B, C\})$ cannot be realized.
   (a) If $BC$ is taken, then $C$ must be taken, and this would make four states.
   (b) Hence $C$ must be chosen, giving states $A$, $B$, and $C$. This yields the automaton $\mathfrak{A} = \mathfrak{N}_1$.

2. If $AB$ is chosen, then we could choose $C$, $AC$ or $ABC$, since $BC$ would also require $C$. Thus there are three cases:
   (a) $\{A, AB, C\}$ yields $\mathfrak{N}_2$ of Table 16, if the minimal number of transitions is used. The following transitions can also be added: $(A, a, A)$, $(AB, a, A)$, $(AB, b, A)$, since these can be added independently, we have eight more NFA’s. Using the maximal number of transitions, we get $\mathfrak{N}_9$ of Table 17.
   (b) $\{A, AB, AC\}$ results in $\mathfrak{N}_{10}$ with the minimal number of transitions, and $\mathfrak{N}_{25}$ with the maximal one.
   (c) $\{A, AB, ABC\}$ results in $\mathfrak{N}_{26}$ (the quotient DFA) with the minimal number of transitions, and $\mathfrak{N}_{281}$ with the maximal one.

Table 18. NFA $\mathfrak{N}_{10}$.

Table 19. NFA $\mathfrak{N}_{25}$. 

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As well, \( L \) has 3-state non-atomic NFA’s. The determinized version of NFA \( \mathfrak{N}_{10} \) of Table 18 is not minimal. By Theorem 3, \( \mathfrak{N}_{10}^{R} \) is not atomic. But \( L^{R} = \Sigma^{*}ba\Sigma^{*} \); hence we obtain a non-atomic 3-state NFA for \( L \) by reversing \( \mathfrak{N}_{10} \) and interchanging \( a \) and \( b \). That NFA with renamed states is shown in Table 22.

The right languages of the states of \( \mathfrak{N}_{282} \) are: \( L_{0} = L = A \), \( L_{1} = A \cup B \), and \( L_{2} = \varepsilon \cup a \cup aa\Sigma^{*} \cup abb*aa\Sigma^{*} \), which is not a union of atoms. Six more non-atomic NFA’s can be derived from NFA’s between \( \mathfrak{N}_{10} \) and \( \mathfrak{N}_{25} \). \( \square \)

This is a rather large number of NFA’s for a language with 3 quotients. ■

One can verify that there is no NFA with fewer than 3 states which accepts the language \( L = \Sigma^{*}ab\Sigma^{*} \). This implies that every minimal atomic NFA of \( L \) is also a minimal NFA of \( L \). However, this is not the case with all regular languages, as we will see in the next section.

4 Sengoku’s NFA Minimization Method

Sengoku had no concept of atom, but he came very close to discovering it. For a language accepted by a minimal DFA \( \mathcal{D} \), the normal NFA [11](p. 18) is isomorphic to \( \mathcal{D}^{\text{RDRT}} \), and hence to the trim atomaton, by our Corollary 1. Moreover, he defines an NFA \( \mathfrak{N} \) to be in standard form [11](p. 19) if \( \mathfrak{N}^{\text{RD}} \) is minimal. By our Theorem 3, such an \( \mathfrak{N} \) is atomic. Sengoku makes the following claim [11](p. 20):

\[
\text{We can transform the nondeterministic automaton into its standard form by adding some extra transitions to the automaton. Therefore the number of states is unchangeable.}
\]

This claim amounts to stating that any NFA can be transformed to an equivalent atomic NFA by adding some transitions. Unfortunately, the claim is false:
Theorem 7. There exists a language for which no minimal NFA is atomic.

Proof. This example is from [7]. A quotient DFA \( \mathcal{D} \), the NFA \( \mathcal{D}^{RDR} \), and its isomorphic átomaton \( \mathcal{A} \) with relabelled states are in Tables 23–25, respectively (there is no negative atom). We now drop the curly brackets and commas in tables, and represent sets of atoms by words. A minimal NFA \( \mathcal{N}_{\min} \) of this language, having four states, is shown in Table 26; it is not atomic and it is not unique. We try to construct a 4-state atomic NFA \( \mathcal{N}_{\text{atom}} \) equivalent to \( \mathcal{D} \).

| States | \( \mathcal{D} \) |
|--------|----------------|
| 0      | \( \rightarrow 1 \) 2 |
| 1      | \( \rightarrow 3 \) 4 |
| 2      | \( \rightarrow 4 \) |
| 3      | \( \rightarrow 5 \) 4 |
| 4      | \( \rightarrow 6 \) 2 |
| 5      | \( \rightarrow 7 \) 2 |
| 6      | \( \rightarrow 8 \) 7 |
| 7      | \( \rightarrow 8 \) 7 |
| 8      | \( \rightarrow 8 \) 7 |

| States | \( \mathcal{D}^{RDR} \) |
|--------|----------------|
| 0      | \( \rightarrow 257 \) 257, 04578 |
| 1      | \( \rightarrow 04578 \) 12678, 257 |
| 2      | \( \rightarrow 12678 \) 04578, 03 – 8 |
| 3      | \( \rightarrow 03 – 8 \) 12678 |
| 4      | \( \rightarrow 1 – 8 \) 03 – 8 |
| 5      | \( \rightarrow 0 – 8 \) 1 – 8, 0 – 8 |
| 6      | \( \rightarrow 0 – 8 \) 1 – 8, 0 – 8 |
| 7      | \( \rightarrow 0 – 8 \) 1 – 8, 0 – 8 |
| 8      | \( \rightarrow 0 – 8 \) 1 – 8, 0 – 8 |

| States | \( \mathcal{A} \) |
|--------|----------------|
| 0      | \( \rightarrow A \) 257, 04578 |
| 1      | \( \rightarrow B \) 257, 04578 |
| 2      | \( \rightarrow C \) 12678, 257 |
| 3      | \( \rightarrow D \) 04578, 03 – 8 |
| 4      | \( \rightarrow E \) 12678 |
| 5      | \( \rightarrow F \) 03 – 8 |
| 6      | \( \rightarrow E \) 1 – 8, 0 – 8 |
| 7      | \( \rightarrow F \) 0 – 8 |
| 8      | \( \rightarrow F \) 0 – 8 |

First, we note that quotients corresponding to the states of \( \mathcal{D} \) can be expressed as sets of atoms as follows: \( K_0 = \{ B, D, F \} \), \( K_1 = \{ C, E, F \} \), \( K_2 = \{ A, C, E, F \} \), \( K_3 = \{ D, E, F \} \), \( K_4 = \{ B, D, E, F \} \), \( K_5 = \{ A, B, D, E, F \} \), \( K_6 = \{ C, D, E, F \} \), \( K_7 = \{ A, B, C, D, E, F \} \), and \( K_8 = \{ B, C, D, E, F \} \). One can verify that these are the states of the determinized version of the átomaton, which is isomorphic to the original DFA \( \mathcal{D} \). Now, every state of \( \mathcal{N}_{\text{atom}} \) must be a subset of a set of atoms of some quotient, and all these sets of atoms of quotients must be covered by the states of \( \mathcal{N}_{\text{atom}} \). We note that quotients \( \{ B, D, F \} \), \( \{ C, E, F \} \), and \( \{ D, E, F \} \) do not contain any other quotients as subsets, while all the other quotients do. It is easy to see that there is no combination of three or fewer sets of atoms, other than these three sets, that can cover these quotients. So we have to use these sets as states of \( \mathcal{N}_{\text{atom}} \). We also need at least one state containing the atom \( A \). If we use only one set of atoms with \( A \), that set has to be a subset of every quotient having \( A \). So it must be a subset of \( \{ A, E, F \} \). If we use \( \{ A \} \) as a state, then by the transition table of the átomaton, there must be at least one more state to cover \( \{ A, B \} \). Similarly, if we use \( \{ A, E \} \), then we must have another state to cover \( \{ A, B, D \} \). If we use \( \{ A, F \} \), then we must have a state to cover \( \{ A, B, E, F \} \). And if we use \( \{ A, E, F \} \), then we must have a state to cover \( \{ E, F \} \). We conclude that a smallest atomic NFA has at least five states. There is a five-state atomic NFA, as shown in Table 27. It is not unique.

Since there does not exist a four-state atomic NFA equivalent to the DFA \( \mathcal{D} \), it is not possible to convert the non-atomic minimal NFA \( \mathcal{N}_{\min} \) to an atomic NFA by adding transitions. \( \square \)

In summary, Sengoku’s method cannot find the minimal NFA’s in all cases. However, it is able to find all atomic minimal NFA’s. His minimization algorithm
Table 26. \( \mathcal{N}_\text{min} \).

| \( a \) | \( b \) |
|---|---|
| \( \rightarrow 0 \) | \( 1 \), \( 1.2 \) |
| \( 1 \) | \( 3 \), \( 0.3 \) |
| \( \leftarrow 2 \) | \( 0.2, 3 \) |
| \( 3 \) | \( 3 \), \( 1 \) |

Table 27. \( \mathcal{N}_\text{atom} \).

| \( a \) | \( b \) |
|---|---|
| \( \rightarrow BDF \) | \( CEF \), \( AEF \) |
| \( CEF \) | \( DEF \), \( BDF \), \( DEF \) |
| \( \leftarrow AEF \) | \( BDF \), \( AEF \), \( DEF \) |
| \( DEF \) | \( DEF \), \( CEF \) |
| \( EF \) | \( DEF \), \( EF \) |

proceeds by “merging some states of the normal nondeterministic automaton.” This is similar to our search for subsets of atoms that satisfy Theorem 4.

5 The Kameda-Weiner Minimization Method

We present a short and modified outline of the properties of the Kameda-Weiner NFA minimization method [6] using mostly our terminology and notation. They consider a trim minimal DFA \( D = (Q, \Sigma, \delta, q_0, F) \) with \( Q \) of cardinality \( n \), and its reversed determinized and trim version \( D^{RDT} \); the set of states of \( D^{RDT} \) is a subset \( S \) of cardinality \( p \) of \( 2^Q \setminus \emptyset \). They then form an \( n \times p \) matrix \( T \) where the rows correspond to non-empty states \( q_i \in Q \) of the trim minimal DFA of a language \( L \), and columns, to states \( S_j \in S \) of \( D^{RDT} \), which is the trim minimal DFA of the language \( L^R \) by Theorem 1. The entry \( t_{i,j} \) of the matrix \( T \) is 1 if \( q_i \in S_j \), and 0 otherwise.

We use \( D^{RDT} \), the trim automaton, instead of \( D^{RDT} \), since the state sets of these two automata are identical. Interpret the rows of the matrix as non-empty quotients of \( L \) and columns, as positive atoms of \( L \). Then \( t_{i,j} = 1 \) if and only if quotient \( K_i \) contains atom \( A_j \), and it is clear that every regular language defines a unique such matrix, which we will refer to as the quotient-atom matrix.

The ordered pair \((K_i, A_j)\) with \( K_i \in \mathcal{K} \) and \( A_j \in \mathcal{A} \) is a point of \( T \) if \( t_{i,j} = 1 \). A grid \( g \) of \( T \) is the direct product \( g = P \times R \) of a set \( P \) of quotients with a set \( R \) of atoms. If \( g = P \times R \) and \( g' = P' \times R' \) are two grids of \( T \), then \( g \subseteq g' \) if and only if \( P \subseteq P' \) and \( R \subseteq R' \). Thus \( \subseteq \) is a partial order on the set of all grids of \( T \), and a grid is maximal if it is not contained in any other grid. A cover \( C \) of \( T \) is a set \( C = \{g_0, \ldots, g_{k-1}\} \) of grids, such that every point \((K_i, A_j)\) belongs to some grid \( g_i \) in \( C \). A minimal cover has the minimal number of grids.

Let \( f : \mathcal{K} \to 2^C \setminus \emptyset \) be the function that assigns to quotient \( K_i \in \mathcal{K} \) the set of grids \( g = P \times R \) such that \( K_i \in P \). The NFA constructed by the Kameda-Weiner method is \( \mathcal{N}_C = (C, \Sigma, \eta_C, C_I, C_F) \), where \( C \) is a cover consisting of maximal grids, \( C_I = f(K_{in}) \) is the set of grids corresponding to the initial quotient \( K_{in} \), and \( C_F \) is defined by \( g \in C_F \) if and only if \( g \in f(K_i) \) implies that \( K_i \) is a final quotient. For every grid \( g = P \times R \) and \( x \in \Sigma \), we can compute \( \eta_C(g, x) \) by the formula \( \eta_C(g, x) = \bigcap_{K_i \in P} f(x^{-1}K_i) \).

It may be the case that \( \mathcal{N}_C \) is not equivalent to DFA \( \mathcal{D} \). A cover \( C \) is called legal if \( L(\mathcal{N}_C) = L(\mathcal{D}) \). To find a minimal NFA of a language \( L \), the method...
in [6] tests the covers of the quotient-atom matrix of \( L \) in the order of increasing size to see if they are legal. The first legal NFA is a minimal one.

When we apply the Kameda-Weiner method [6] to the example in Theorem 7, we get the NFA of Table 26.

We apply the Kameda-Weiner method [6] to the example in Theorem 7. The quotients in the example are referred to as the integers 0–8, as in Table 23. The atoms are those in Table 24 relabeled as in Table 25. The quotient-atom matrix is shown in Table 28, where the non-blank entries are to be interpreted as 1’s and the blank entries as 0’s. Table 28 also shows a minimal cover \( S = (g_0, g_1, g_2, g_3) \) and \( f(K_i) \) for each quotient \( K_i \) of \( K \).

Table 28. Cover \( C \) for quotient-atom matrix of \( D \).

| \( i \) | \( F \) | \( E \) | \( D \) | \( C \) | \( B \) | \( A \) | \( f(K_i) \) |
|-------|-------|-------|-------|-------|-------|-------|-------|
| \( \rightarrow \) | \( 0 \) | \( g_0 \) | \( g_0 \) | \( \{g_0\} \) |
| \( 1 \) | \( g_1 \) | \( g_1 \) | \( g_1 \) | \( \{g_1\} \) |
| \( \leftarrow \) | \( 2 \) | \( g_1, g_2 \) | \( g_1, g_2 \) | \( g_1 \) | \( g_2 \) | \( \{g_1, g_2\} \) |
| \( 3 \) | \( g_3 \) | \( g_3 \) | \( g_3 \) | \( \{g_3\} \) |
| \( 4 \) | \( g_0, g_3 \) | \( g_0, g_3 \) | \( g_0, g_3 \) | \( g_0 \) | \( \{g_0, g_3\} \) |
| \( \leftarrow \) | \( 5 \) | \( g_0, g_2, g_3 \) | \( g_2, g_3 \) | \( g_0, g_2, g_3 \) |
| \( 6 \) | \( g_1, g_3 \) | \( g_1, g_3 \) | \( g_1 \) | \( g_1 \) | \( \{g_1, g_3\} \) |
| \( \leftarrow \) | \( 7 \) | \( g_0, g_1, g_2, g_3 \) | \( g_1, g_2, g_3 \) | \( g_0, g_1, g_2, g_3 \) |
| \( 8 \) | \( g_0, g_1, g_2, g_3 \) | \( g_1, g_2, g_3 \) | \( g_0, g_1, g_2, g_3 \) | \( g_0, g_1, g_2, g_3 \) |

The construction of the NFA \( \mathcal{N}_{min} \) is shown in Table 29. For each grid \( g = P \times R \), we show its set of quotients \( P \), with \( K_i \in P \) replaced by \( i \). For each input \( x \in \Sigma \), we give \( x^{-1} P \), and then the intersection of the \( f(K_i) \) for \( K_i \in x^{-1}P \). For example, the set \( P \) for \( g_0 \) is expressed as \( \{0, 4, 5, 7, 8\} \), the set of quotients \( a^{-1}P \) of the set \( P \) by \( a \) is \( \{1, 6, 7\} \), and \( \eta_C(g_0, a) = f(1) \cap f(6) \cap f(7) = \{g_1\} \cap \{g_1, g_3\} \cap \{g_0, g_1, g_2, g_3\} = \{g_1\} \). Table 26 shows the constructed NFA \( \mathcal{N}_{min} \), where \( g_i \)'s are replaced by \( i \)'s. Since \( \mathcal{N}_{min} \) is equivalent to \( D \), \( C \) is a legal cover. However, \( \mathcal{N}_{min} \) is not atomic, since the right language of state \( g_2 \) is not a union of atoms, although it includes atoms \( A \) and \( E \) as its subsets. The right languages of the other states of \( \mathcal{N}_{min} \) are sets of atoms: \( L(g_0) = B \cup D \cup F \), \( L(g_1) = C \cup E \cup F \), and \( L(g_3) = D \cup E \cup F \).

We believe that NFA’s defined by grids are a topic for future research.

6 Conclusions

We have studied the properties of atomic NFA’s. We have shown that atoms play an important role in NFA minimization and proved that it is not enough to search for atomic NFA’s only.
Table 29. Construction of NFA $\mathcal{N}_{\text{min}}$.

| $g$   | $P$         | $a^{-1}P$ | $\eta_C(g,a)$ | $b^{-1}P$ | $\eta_C(g,b)$ |
|-------|-------------|-----------|---------------|-----------|---------------|
| $\rightarrow g_0$ | $\{0, 4, 5, 7, 8\}$ | $\{1, 6, 7\}$ | $\{g_1\}$ | $\{2, 7\}$ | $\{g_1, g_2\}$ |
| $g_1$ | $\{1, 2, 6, 7, 8\}$ | $\{3, 5, 6, 7\}$ | $\{g_3\}$ | $\{4, 7, 8\}$ | $\{g_0, g_1\}$ |
| $\leftarrow g_2$ | $\{2, 5, 7\}$ | $\{5, 7\}$ | $\{g_0, g_2, g_3\}$ | $\{2, 4, 7\}$ | $\emptyset$ |
| $g_3$ | $\{3, 4, 5, 6, 7, 8\}$ | $\{3, 6, 7\}$ | $\{g_3\}$ | $\{1, 2, 7, 8\}$ | $\{g_1\}$ |

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