EXPLICIT FORMULAS FOR BIHARMONIC SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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Dedicated to Professor Neculai Papagiuc on his 60-th birthday

Abstract. We classify the biharmonic Legendre curves in a Sasakian space form, and obtain their explicit parametric equations in the $(2n + 1)$-dimensional unit sphere endowed with the canonical and deformed Sasakian structures defined by Tanno. Then, composing with the flow of the Reeb vector field, we transform a biharmonic integral submanifold into a biharmonic anti-invariant submanifold. Using this method we obtain new examples of biharmonic submanifolds in spheres and, in particular, in $S^7$.

1. Introduction

Biharmonic maps between Riemannian manifolds $\phi : (M, g) \to (N, h)$ are the critical points of the bienergy functional $E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \, v_g$. They represent a natural generalization of the well-known harmonic maps \cite{8}, the critical points of the energy functional $E(\phi) = \frac{1}{2} \int_M |d\phi|^2 \, v_g$, and of the biharmonic submanifolds in Euclidean spaces defined by B.-Y. Chen \cite{7}.

The Euler-Lagrange equation for the energy functional is $\tau(\phi) = 0$, where $\tau(\phi) = \text{trace} \, \nabla d\phi$ is the tension field, and the Euler-Lagrange equation for the bienergy functional was derived by G. Y. Jiang in \cite{14}:

$$\tau_2(\phi) = -\Delta \tau(\phi) - \text{trace} \, R^N(d\phi, \tau(\phi)) d\phi$$

$$= 0.$$ 

Since any harmonic map is biharmonic, we are interested in non-harmonic biharmonic maps, which are called proper-biharmonic.

A special case of biharmonic maps is represented by the biharmonic Riemannian immersions, or biharmonic submanifolds. There are several results of classification or construction for such submanifolds in space forms \cite{15, 3}. Then, the next step would be the study of biharmonic submanifolds in Sasakian space forms. In this context J. Inoguchi classified in \cite{13} the proper-biharmonic Legendre curves and Hopf cylinders in a 3-dimensional Sasakian space form $M^3(c)$, and in \cite{11} the explicit parametric equations were obtained. Then, T. Sasahara and his collaborators studied the biharmonic integral surfaces and 3-dimensional biharmonic anti-invariant submanifolds in $S^5$ \cite{1, 17}.

Recent results on biharmonic submanifolds in spaces of nonconstant sectional curvature were obtained by T. Ichiyama, J. Inoguchi and H. Urakawa in \cite{12}, by Y.-L. Ou and Z.-P. Wang in \cite{16}, and by W. Zhang in \cite{20}.

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Biharmonic submanifolds in pseudo-Euclidean spaces were also studied, and many examples and classification results were obtained (for example, see [2], [7]).

The goals of our paper are to obtain new classification results for biharmonic Legendre curves in any dimensional Sasakian space form and to provide a method for constructing biharmonic submanifolds. In order to obtain explicit examples, we use the \((2n + 1)\)-dimensional unit sphere \(S^{2n+1}\) as a model of Sasakian space form.

For a general account of biharmonic maps see [15] and The Bibliography of Biharmonic Maps [18].

**Conventions.** We work in the \(C^\infty\) category, that means manifolds, metrics, connections and maps are smooth. The Lie algebra of the vector fields on \(M\) is denoted by \(C(TM)\).

## 2. Preliminaries

### 2.1. Contact manifolds.** A contact metric structure** on a manifold \(N^{2n+1}\) is given by \((\varphi, \xi, \eta, g)\), where \(\varphi\) is a tensor field of type \((1, 1)\) on \(N\), \(\xi\) is a vector field, \(\eta\) is an 1-form and \(g\) is a Riemannian metric such that

\[
\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,
\]

\[
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = d\eta(X, Y), \quad \forall X, Y \in C(TN).
\]

A contact metric structure \((\varphi, \xi, \eta, g)\) is called normal if

\[
N_\varphi + 2d\eta \otimes \xi = 0,
\]

where

\[
N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y], \quad \forall X, Y \in C(TN),
\]

is the Nijenhuis tensor field of \(\varphi\).

A contact metric manifold \((N, \varphi, \xi, \eta, g)\) is a **Sasakian manifold** if it is normal or, equivalently, if

\[
(\nabla_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X, \quad \forall X, Y \in C(TN).
\]

The contact distribution of a Sasakian manifold \((N, \varphi, \xi, \eta, g)\) is defined by \(\{X \in TN : \eta(X) = 0\}\), and an integral curve of the contact distribution is called Legendre curve. A submanifold \(M\) of \(N\) which is tangent to \(\xi\) is said to be anti-invariant if \(\varphi\) maps any vector tangent to \(M\) and normal to \(\xi\) to a vector normal to \(M\).

Let \((N, \varphi, \xi, \eta, g)\) be a Sasakian manifold. The sectional curvature of a 2-plane generated by \(X\) and \(\varphi X\), where \(X\) is an unit vector orthogonal to \(\xi\), is called \(\varphi\)-sectional curvature determined by \(X\). A Sasakian manifold with constant \(\varphi\)-sectional curvature \(c\) is called a **Sasakian space form** and it is denoted by \(N(c)\).

The curvature tensor field of a Sasakian space form \(N(c)\) is given by

\[
R(X, Y)Z = \frac{c+3}{4} [g(Z, Y)X - g(Z, X)Y] + \frac{c-1}{4} [\eta(Z)\eta(X)Y - \eta(Y)\eta(X)X - g(Z, Y)\xi + g(Z, X)\eta(Y)\xi - g(Z, Y)\eta(X)\xi + g(Z, \varphi X)\varphi Y - g(Z, \varphi Y)\varphi X] + 2g(X, \varphi Y)\varphi Z.
\]

\[\text{(2.1)}\]

Let \(S^{2n+1} = \{z \in \mathbb{C}^{n+1} : |z| = 1\}\) be the unit \((2n + 1)\)-dimensional sphere endowed with its standard metric field \(g_0\). Consider the following structure tensor fields on \(S^{2n+1}\): \(\xi_0 = -\mathcal{I}z\) for each \(z \in S^{2n+1}\), where \(\mathcal{I}\) is the usual almost complex structure on \(\mathbb{C}^{n+1}\) defined by

\[
\mathcal{I}z = (-y^1, ..., -y^{n+1}, x^1, ..., x^{n+1}),
\]
Sasakian 3-structure. The dimension of such a manifold is of the form 4
Definition 3.1.
for an even permutation \((a, b, c)\), note that the maximum dimension of a submanifold of a 3-Sasakian manifold which is an integral submanifold with respect to all three Sasakian structures is

\[ (3.2) \]
\[ R(T, \nabla_T T)T = -\left(\frac{c + 3}{4}\right)\kappa_1 E_2 - \frac{3(c - 1)}{4}\kappa_1 g(E_2, \varphi T)\varphi T, \]

where \(a\) is a positive constant. The structure \((\varphi, \xi, \eta, g)\) is still a Sasakian structure and \((\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)\) is a Sasakian space form with constant \(\varphi\)-sectional curvature

2.2. 3-Sasakian manifolds. If a manifold \(N\) admits three Sasakian structures \((\varphi_a, \xi_a, \eta_a, g)\), \(a = 1, 2, 3\), satisfying

\[ \varphi_c = -\varphi_a \varphi_b + \eta b \otimes \xi_a = \varphi b \varphi_a - \eta_a \otimes \xi_b, \]
\[ \xi_c = -\varphi_a \xi b = \varphi b \xi_a, \quad \eta_c = -\eta_a \circ \varphi b = \eta_b \circ \varphi_a, \]

for an even permutation \((a, b, c)\) of \((1, 2, 3)\), then the manifold is said to have a Sasakian 3-structure. The dimension of such a manifold is of the form \(4n + 3\). We note that the maximum dimension of a submanifold of a 3-Sasakian manifold \(N^{4n+3}\) which is an integral submanifold with respect to all three Sasakian structures is \(n\).

3. Biharmonic Legendre curves in Sasakian space forms

Definition 3.1. Let \((N^m, g)\) be a Riemannian manifold and \(\gamma : I \rightarrow N\) a curve parametrized by arc length, that is \(|\gamma'| = 1\). Then \(\gamma\) is called a Frenet curve of osculating order \(r\), \(1 \leq r \leq m\), if there exists orthonormal vector fields \(E_1, E_2, ..., E_r\) along \(\gamma\) such that

\[
\begin{align*}
E_1 &= \gamma' = T \\
\nabla_T E_1 &= \kappa_1 E_2 \\
\nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3 \\
&\quad \vdots \\
\nabla_T E_r &= -\kappa_{r-1} E_{r-1}
\end{align*}
\]

where \(\kappa_1, ..., \kappa_{r-1}\) are positive functions on \(I\).

Remark 3.2. A geodesic is a Frenet curve of osculating order 1; a circle is a Frenet curve of osculating order 2 with \(\kappa_1 = \text{constant}\); a helix of order \(r\), \(r \geq 3\), is a Frenet curve of osculating order \(r\) with \(\kappa_1, ..., \kappa_{r-1}\) constants; a helix of order 3 is called, simply, helix.

Now let \((N^{2n+1}, \varphi, \xi, \eta, g)\) be a Sasakian space form with constant \(\varphi\)-sectional curvature \(c\) and \(\gamma : I \rightarrow N\) a Legendre Frenet curve of osculating order \(r\). As

\[ \nabla_T^2 T = (-3\kappa_1 \kappa_1') E_1 + (\kappa''_1 - \kappa_1^3 - \kappa_1 \kappa_2^2) E_2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 \]

(3.1)

and

\[ R(T, \nabla_T T)T = -\left(\frac{c + 3}{4}\right)\kappa_1 E_2 - \frac{3(c - 1)}{4}\kappa_1 g(E_2, \varphi T)\varphi T, \]

(3.2)
we get
\[ \tau_2(\gamma) = \nabla^3_T T - R(T, \nabla_T T)T \]
\( (3.3) \)
\[ = (-3\kappa_1\kappa'_1)E_1 + \left( \kappa''_1 - \kappa'_1 - \kappa_1\kappa'_2 + \frac{(c+3)\kappa_1}{4} \right)E_2 \]
\[ + (2\kappa'_1\kappa_2 + \kappa_1\kappa'_2)E_3 + \kappa_1\kappa_2\kappa'_3E_4 + \frac{3(c-1)\kappa_1}{4}g(E_2, \varphi T)\varphi T. \]
In the following we shall solve the biharmonic equation \( \tau_2(\gamma) = 0 \). The problem is to find the relation between \( \varphi T \) and the Frenet frame field. The simplest two cases are provided by \( \frac{3(c-1)\kappa_1}{4}g(E_2, \varphi T) = 0 \). So,

**Case I: \( c = 1 \).**

In this case \( \gamma \) is proper-biharmonic if and only if
\[ \begin{cases} 
\kappa_1 = \text{constant} > 0, & \kappa_2 = \text{constant} \\
\kappa_1^2 + \kappa_2^2 = 1 \\
\kappa_2\kappa_3 = 0
\end{cases} \]

One obtains

**Theorem 3.3.** If \( c = 1 \) and \( n \geq 2 \), then \( \gamma \) is proper-biharmonic if and only if either \( \gamma \) is a circle with \( \kappa_1 = 1 \), or \( \gamma \) is a helix with \( \kappa_1^2 + \kappa_2^2 = 1 \).

**Remark 3.4.** If \( n = 1 \) and \( \gamma \) is a non-geodesic Legendre curve we have \( \nabla_T T = \pm\kappa_1\varphi T \) and then \( E_2 = \pm\varphi T \) and \( \nabla_T E_2 = \pm\nabla_T \varphi T = \pm(\xi \mp \kappa_1 T) = -\kappa_1 T \pm \xi \). Therefore \( \kappa_2 = 1 \) and \( \gamma \) cannot be biharmonic.

**Case II: \( c \neq 1, E_2 \perp \varphi T \).**

In this case \( \gamma \) is proper-biharmonic if and only if
\[ \begin{cases} 
\kappa_1 = \text{constant} > 0, & \kappa_2 = \text{constant} \\
\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} \\
\kappa_2\kappa_3 = 0
\end{cases} \]

Before stating the theorem we need the following

**Lemma 3.5. Let \( \gamma \) be a Legendre Frenet curve of osculating order 3 and \( E_2 \perp \varphi T \). Then \( \{ T = E_1, E_2, E_3, \varphi T, \xi, \nabla_T \varphi T \} \) are linearly independent, in any point, and hence \( n \geq 3 \).**

**Proof.** Since \( \gamma \) is a Frenet curve of osculating order 3, we have
\[ \begin{align*}
E_1 &= \gamma' = T \\
\nabla_T E_1 &= \kappa_1 E_2 \\
\nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3 \\
\nabla_T E_3 &= -\kappa_2 E_2 
\end{align*} \]

It is easy to see that, in an arbitrary point, the system
\[ S_1 = \{ T = E_1, E_2, E_3, \varphi T, \xi, \nabla_T \varphi T \} \]
has only non-zero vectors and
\[ T \perp E_2, \quad T \perp E_3, \quad T \perp \varphi T, \quad T \perp \xi, \quad T \perp \nabla_T \varphi T. \]
Thus \( S_1 \) is linearly independent if and only if \( S_2 = \{ E_2, E_3, \varphi T, \xi, \nabla_T \varphi T \} \) is linearly independent. Further, as
\[
E_2 \perp \xi, \quad E_2 \perp \nabla_T \varphi T, \quad E_3 \perp \xi, \quad E_3 \perp \nabla_T \varphi T, \quad \varphi T \perp \xi, \quad \varphi T \perp \nabla_T \varphi T,
\]
and
\[
E_2 \perp E_3 \perp \varphi T,
\]
it follows that \( S_2 \) is linearly independent if and only if \( S_3 = \{ \xi, \nabla_T \varphi T \} \) is linearly independent. But \( \nabla_T \varphi T = \xi + \kappa_1 \varphi E_2, \kappa_1 \neq 0 \), and therefore \( S_3 \) is linearly independent.

Now we can state

**Theorem 3.6.** Assume that \( c \neq 1 \) and \( \nabla_T T \perp \varphi T \). We have
1) If \( c \leq -3 \) then \( \gamma \) is biharmonic if and only if it is a geodesic.
2) If \( c > -3 \) then \( \gamma \) is proper-biharmonic if and only if either
   a) \( n \geq 2 \) and \( \gamma \) is a circle with \( \kappa_1^2 = \frac{c+3}{2} \). In this case \( \{ E_1, E_2, \varphi T, \xi \} \) are linearly independent,
   or
   b) \( n \geq 3 \) and \( \gamma \) is a helix with \( \kappa_1^2 + \kappa_2^2 = \frac{c+3}{1} \). In this case \( \{ E_1, E_2, E_3, \varphi T, \xi, \nabla_T \varphi T \} \) are linearly independent.

**Case III:** \( c \neq 1, \quad E_2 \parallel \varphi T. \)
In this case \( \gamma \) is proper-biharmonic if and only if
\[
\begin{align*}
\kappa_1 &= \text{constant} > 0, \quad \kappa_2 = \text{constant} \\
\kappa_1^2 + \kappa_2^2 &= c \\
\kappa_2 \kappa_3 &= 0
\end{align*}
\]
We can assume that \( E_2 = \varphi T. \) Then we have \( \nabla_T T = \kappa_1 E_2 = \kappa_1 \varphi T, \nabla_T E_2 = \nabla_T \varphi T = \xi - \kappa_1 T. \) That means \( E_3 = \xi \) and \( \kappa_2 = 1. \) Hence \( \nabla_T E_3 = \nabla_T \xi = -\varphi T = -E_2. \)
Therefore

**Theorem 3.7.** If \( c \neq 1 \) and \( \nabla_T T \parallel \varphi T, \) then \( \{ T, \varphi T, \xi \} \) is the Frenet frame field of \( \gamma \) and we have
1) If \( c \leq 1 \) then \( \gamma \) is biharmonic if and only if it is a geodesic.
2) If \( c > 1 \) then \( \gamma \) is proper-biharmonic if and only if it is a helix with \( \kappa_1^2 = c - 1 \) (and \( \kappa_2 = 1 \)).

**Remark 3.8.** If \( n = 1 \) then \( \nabla_T T \parallel \varphi T \) and we reobtain Inoguchi’s result [13].

**Case IV:** \( c \neq 1 \) and \( g(E_2, \varphi T) \) is not constant 0, 1 or \(-1\).
Assume that \( \gamma \) is a Legendre Frenet curve of osculating order \( r, \) \( 4 \leq r \leq 2n + 1, \)
\( n \geq 2. \) If \( \gamma \) is biharmonic it follows that \( \varphi T \in \text{span}\{ E_2, E_3, E_4 \}. \)
Now, we denote \( f(t) = g(E_2, \varphi T) \) and differentiating along \( \gamma \) we obtain
\[
\begin{align*}
f'(t) &= g(\nabla_T E_2, \varphi T) + g(E_2, \nabla_T \varphi T) = g(\nabla_T E_2, \varphi T) + g(E_2, \xi + \kappa_1 \varphi E_2) \\
&= g(\nabla_T E_2, \varphi T) = g(-\kappa_1 T + \kappa_2 E_3, \varphi T) \\
&= \kappa_2 g(E_3, \varphi T).
\end{align*}
\]
Since $\varphi T = g(\varphi T, E_2)E_2 + g(\varphi T, E_3)E_3 + g(\varphi T, E_4)E_4$, the curve $\gamma$ is proper-biharmonic if and only if

$$
\begin{align*}
\kappa_1 &= \text{constant} > 0 \\
\kappa_1^2 + \kappa_2^2 &= \frac{c+3}{4} + \frac{3(c-1)}{4}f^2 \\
\kappa_2' &= -\frac{3(c-1)}{4}fg(\varphi T, E_3) \\
\kappa_2\kappa_3 &= -\frac{3(c-1)}{4}fg(\varphi T, E_4)
\end{align*}
$$

Using the expression of $f'(t)$ we see that the third equation of the above system is equivalent to

$$\kappa_2^2 = -\frac{3(c-1)}{4}f^2 + \omega_0,$$

where $\omega_0 = \text{constant}$. Replacing in the second equation it follows

$$\kappa_1^2 = \frac{c+3}{4} - \omega_0 + \frac{3(c-1)}{2}f^2,$$

which implies $f = \text{constant}$. Thus $\kappa_2 = \text{constant} > 0$, $g(E_3, \varphi T) = 0$ and then $\varphi T = fE_2 + g(\varphi T, E_4)E_4$. It follows that there exists an unique constant $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ such that $f = \cos \alpha_0$ and $g(\varphi T, E_4) = \sin \alpha_0$.

We can state

**Theorem 3.9.** Let $c \neq 1$, $n \geq 2$ and $\gamma$ a Legendre Frenet curve of osculating order $r \geq 4$ such that $g(E_2, \varphi T)$ is not constant $0, 1$ or $-1$. We have

a) If $c \leq -3$ then $\gamma$ is biharmonic if and only if it is a geodesic.

b) If $c > -3$ then $\gamma$ is proper-biharmonic if and only if $\varphi T = \cos \alpha_0 E_2 + \sin \alpha_0 E_4$ and

$$
\begin{align*}
\kappa_1 &= \text{constant} > 0, \quad \kappa_2 = \text{constant} \\
\kappa_1^2 + \kappa_2^2 &= \frac{c+3}{4} + \frac{3(c-1)}{4}\cos^2 \alpha_0 \\
\kappa_2\kappa_3 &= -\frac{3(c-1)}{8}\sin 2\alpha_0
\end{align*}
$$

where $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ is a constant such that $c + 3 + 3(c-1)\cos^2 \alpha_0 > 0$ and $3(c-1)\sin 2\alpha_0 < 0$.

**Remark 3.10.** In this case we may obtain biharmonic curves which are not helices.

**Remark 3.11.** We note that a preliminary version of the full classification of the proper-biharmonic Legendre curves in Sasakian space forms was obtained in [10].

In the following, we shall choose the unit $(2n + 1)$-dimensional sphere $S^{2n+1}$ with its canonical and modified Sasakian structures as a model for the complete, simply connected Sasakian space form with constant $\varphi$-sectional curvature $c > -3$, and we will find the explicit equations of biharmonic Legendre curves obtained in the first three cases, viewed as curves in $\mathbb{R}^{2n+2}$.

**Theorem 3.12.** Let $\gamma : I \rightarrow (S^{2n+1}, \varphi_0, \xi_0, \eta_0, g_0)$, $n \geq 2$, be a proper-biharmonic Legendre curve parametrized by arc length. Then the equation of $\gamma$ in the Euclidean space $\mathbb{E}^{2n+2} = (\mathbb{R}^{2n+2}, (,))$, is either

$$
\gamma(s) = \frac{1}{\sqrt{2}}\cos \left(\sqrt{2}s\right)e_1 + \frac{1}{\sqrt{2}}\sin \left(\sqrt{2}s\right)e_2 + \frac{1}{\sqrt{2}}e_3
$$
where \( \{e_i, Ie_j\} \) are constant unit vectors orthogonal to each other, or
\[
\gamma(s) = \frac{1}{\sqrt{2}} \cos(As)e_1 + \frac{1}{\sqrt{2}} \sin(As)e_2 + \frac{1}{\sqrt{2}} \cos(Bs)e_3 + \frac{1}{\sqrt{2}} \sin(Bs)e_4,
\]
where
\[
A = \sqrt{1 + \kappa_1}, \quad B = \sqrt{1 - \kappa_1}, \quad \kappa_1 \in (0, 1),
\]
and \( \{e_i\} \) are constant unit vectors orthogonal to each other, with
\[
\langle e_1, Ie_3 \rangle = \langle e_1, Ie_4 \rangle = \langle e_2, Ie_3 \rangle = \langle e_2, Ie_4 \rangle = 0, \quad A\langle e_1, Ie_2 \rangle + B\langle e_3, Ie_4 \rangle = 0.
\]

**Proof.** Let us denote by \( \nabla \) and by \( \bar{\nabla} \) the Levi-Civita connections on \( (S^{2n+1}, g_0) \) and \( (\mathbb{R}^{2n+2}, \langle \cdot, \cdot \rangle) \), respectively.

First, assume that \( \gamma \) is the biharmonic circle, that is \( \kappa_1 = 1 \). From the Gauss and Frenet equations we get
\[
\bar{\nabla}_T T = \nabla_T T - \langle T, T \rangle \gamma = \kappa_1 E_2 - \gamma
\]
and
\[
\bar{\nabla}_T \bar{\nabla}_T T = (-\kappa_1^2 - 1)T = -2T,
\]
which implies
\[
\gamma'' + 2\gamma' = 0.
\]
The general solution of the above equation is
\[
\gamma(s) = \cos \left( \sqrt{2} s \right) c_1 + \sin \left( \sqrt{2} s \right) c_2 + c_3,
\]
where \( \{c_i\} \) are constant vectors in \( \mathbb{E}^{2n+2} \).

Now, as \( \gamma \) satisfies
\[
\langle \gamma, \gamma \rangle = 1, \quad \langle \gamma', \gamma' \rangle = 1, \quad \langle \gamma, \gamma' \rangle = 0, \quad \langle \gamma', \gamma'' \rangle = 0, \quad \langle \gamma'', \gamma'' \rangle = 2, \quad \langle \gamma, \gamma'' \rangle = -1,
\]
and since in \( s = 0 \) we have \( \gamma = c_1 + c_3, \gamma' = \sqrt{2} c_2, \gamma'' = -2 c_1 \), we obtain
\[
c_{11} + 2c_{13} + c_{33} = 1, \quad c_{22} = \frac{1}{2}, \quad c_{12} + c_{23} = 0, \quad c_{12} = 0, \quad c_{11} = \frac{1}{2}, \quad c_{11} + c_{13} = \frac{1}{2},
\]
where \( c_{ij} = \langle c_i, c_j \rangle \). Further, we get that \( \{c_i\} \) are orthogonal vectors in \( \mathbb{E}^{2n+2} \) with \( |c_1| = |c_2| = |c_3| = \frac{1}{\sqrt{2}} \).

Finally, using the fact that \( \gamma \) is a Legendre curve one obtains easily that \( \langle c_1, Ic_j \rangle = 0 \) for any \( i, j = 1, 2, 3 \).

Suppose now \( \gamma \) is the biharmonic helix, that is \( \kappa_1^2 + \kappa_2^2 = 1, \kappa_1 \in (0, 1) \). From the Gauss and Frenet equations we get
\[
\bar{\nabla}_T T = \nabla_T T - \langle T, T \rangle \gamma = \kappa_1 E_2 - \gamma,
\]
\[
\bar{\nabla}_T \bar{\nabla}_T T = \kappa_1 \bar{\nabla}_T E_2 - T = \kappa_1 \left( \kappa_1 T + \kappa_2 E_3 \right) - T = -\left( \kappa_1^2 + 1 \right) T + \kappa_1 \kappa_2 E_3
\]
and
\[
\bar{\nabla}_T \bar{\nabla}_T \bar{\nabla}_T T = -\left( \kappa_1^2 + 1 \right) \bar{\nabla}_T T + \kappa_1 \kappa_2 \bar{\nabla}_T E_3 = -\left( \kappa_1^2 + 1 \right) \bar{\nabla}_T T - \kappa_1 \kappa_2 E_2 = -2\gamma'' - \kappa_2^2 \gamma.
\]

Hence
\[
\gamma^{iv} + 2\gamma'' + \kappa_2^2 \gamma = 0,
\]
and its general solution is
\[
\gamma(s) = \cos(As)c_1 + \sin(As)c_2 + \cos(Bs)c_3 + \sin(Bs)c_4,
\]
where \( A, B \) are given by \( (3.3) \) and \( \{c_i\} \) are constant vectors in \( \mathbb{E}^{2n+2} \).
As $\gamma$ satisfies
\[
\langle \gamma, \gamma \rangle = 1, \langle \gamma', \gamma' \rangle = 1, \langle \gamma', \gamma' \rangle = 0, \langle \gamma'', \gamma'' \rangle = 0, \langle \gamma'', \gamma'' \rangle = 1 + \kappa_1^2,
\]
\[
\langle \gamma, \gamma'' \rangle = -1, \langle \gamma', \gamma''' \rangle = -(1 + \kappa_1^2), \langle \gamma'', \gamma''' \rangle = 0, \langle \gamma', \gamma''' \rangle = 0, \langle \gamma'', \gamma''' \rangle = 3\kappa_1^2 + 1,
\]
and since in $s = 0$ we have $\gamma = c_1 + c_3$, $\gamma' = Ac_2 + Bc_4$, $\gamma'' = -A^2c_1 - B^2c_3$, $\gamma''' = -A^3c_2 - B^3c_4$, we obtain
\[
\begin{align*}
(3.5) & \quad c_{11} + 2c_{13} + c_{33} = 1 \\
(3.6) & \quad A^2c_{22} + 2ABc_{24} + B^2c_{44} = 1 \\
(3.7) & \quad Ac_{12} + Ac_{23} + Bc_{14} + Bc_{34} = 0 \\
(3.8) & \quad A^3c_{12} + AB^2c_{23} + A^2Bc_{14} + B^3c_{34} = 0 \\
(3.9) & \quad A^4c_{11} + 2A^2B^2c_{13} + B^4c_{33} = 1 + \kappa_1^2 \\
(3.10) & \quad A^2c_{11} + (A^2 + B^2)c_{13} + B^2c_{33} = 1 \\
(3.11) & \quad A^4c_{22} + (AB^3 + A^3B)c_{24} + B^4c_{44} = 1 + \kappa_1^2 \\
(3.12) & \quad A^5c_{12} + A^3B^2c_{23} + A^2B^3c_{14} + B^5c_{34} = 0 \\
(3.13) & \quad A^3c_{12} + A^3c_{23} + B^3c_{14} + B^3c_{34} = 0 \\
(3.14) & \quad A^6c_{22} + 2A^3B^3c_{24} + B^6c_{44} = 3\kappa_1^2 + 1
\end{align*}
\]
where $c_{ij} = \langle c_i, c_j \rangle$. Since the determinant of the system given by (3.7), (3.8), (3.12) and (3.13) is $-A^2B^2(A^2 - B^2)^4 \neq 0$ it follows that $c_{12} = c_{23} = c_{14} = c_{34} = 0$.

The equations (3.5), (3.9) and (3.10) give
\[
\begin{align*}
c_{11} & = \frac{1}{2}, \quad c_{13} = 0, \quad c_{33} = \frac{1}{2},
\end{align*}
\]
and, from (3.6), (3.11) and (3.14) follows that
\[
\begin{align*}
c_{22} & = \frac{1}{2}, \quad c_{24} = 0, \quad c_{44} = \frac{1}{2}.
\end{align*}
\]
Therefore, we obtain that $\{c_i\}$ are orthogonal vectors in $\mathbb{E}^8$ with $|c_1| = |c_2| = |c_3| = |c_4| = \frac{1}{\sqrt{2}}$.

Finally, since $\gamma$ is a Legendre curve one obtains the conclusion of the Theorem. 

**Theorem 3.13.** Let $\gamma : I \rightarrow (\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)$, $n \geq 2$, $a > 0$, $a \neq 1$, be a proper-biharmonic Legendre curve parametrized by arc length such that $g(\nabla \gamma', \varphi \gamma') = 0$.

Then the equation of $\gamma$ in the Euclidean space $\mathbb{E}^{2n+2}$, is either
\[
\gamma(s) = \frac{1}{\sqrt{2}} \cos \left( \sqrt{\frac{2}{a}}s \right) e_1 + \frac{1}{\sqrt{2}} \sin \left( \sqrt{\frac{2}{a}}s \right) e_2 + \frac{1}{\sqrt{2}} e_3
\]
for $n \geq 2$ or, for $n \geq 3$,
\[
\gamma(s) = \frac{1}{\sqrt{2}} \cos(As)e_1 + \frac{1}{\sqrt{2}} \sin(As)e_2 + \frac{1}{\sqrt{2}} \cos(Bs)e_3 + \frac{1}{\sqrt{2}} \sin(Bs)e_4,
\]
where
\[
(3.15) \quad A = \sqrt{\frac{1 + \kappa_1 \sqrt{a}}{a}}, \quad B = \sqrt{\frac{1 - \kappa_1 \sqrt{a}}{a}}, \quad \kappa_1 \in \left(0, \frac{1}{a}\right),
\]
and \( \{e_i, Ie_j\} \) are constant unit vectors orthogonal to each other.

**Proof.** Let us denote by \( \nabla, \hat{\nabla} \) and by \( \tilde{\nabla} \) the Levi-Civita connections on \((S^{2n+1}, g), (S^{2n+1}, g_0)\) and \((\mathbb{R}^{2n+2}, \langle , \rangle)\), respectively.

First we consider the case when \( \gamma \) is the biharmonic circle, that is \( \kappa_1^2 = \frac{e+3}{4} \). Let \( T = \gamma' \) be the unit tangent vector field (with respect to the metric \( g \)) along \( \gamma \). Using the two Sasakian structures on \( S^{2n+1} \) we obtain \( \tilde{\nabla}_T T = \hat{\nabla}_T T \) and \( \tilde{\nabla}_T E_2 = \hat{\nabla}_T E_2 \).

From the Gauss and Frenet equations we get

\[
\tilde{\nabla}_T T = \hat{\nabla}_T T - \langle T, T \rangle \gamma = \kappa_1 E_2 - \frac{1}{a} \gamma
\]

and

\[
\tilde{\nabla}_T \tilde{\nabla}_T T = (-\kappa_1^2 - \frac{1}{a}) T = -\frac{2}{a} T.
\]

Hence

\[
a \gamma''' + 2 \gamma' = 0,
\]

with the general solution

\[
\gamma(s) = \cos \left( \sqrt{\frac{2}{a}} s \right) c_1 + \sin \left( \sqrt{\frac{2}{a}} s \right) c_2 + c_3,
\]

where \( \{c_i\} \) are constant vectors in \( \mathbb{E}^{2n+2} \).

As \( \gamma \) verifies the following equations,

\[
\langle \gamma, \gamma \rangle = 1, \quad \langle \gamma', \gamma' \rangle = \frac{1}{a}, \quad \langle \gamma, \gamma' \rangle = 0, \quad \langle \gamma', \gamma'' \rangle = 0, \quad \langle \gamma'', \gamma'' \rangle = \frac{2}{a^2}, \quad \langle \gamma, \gamma''' \rangle = -\frac{1}{a},
\]

and in \( s = 0 \) we have \( \gamma = c_1 + c_3, \quad \gamma' = \sqrt{\frac{2}{a}} c_2, \quad \gamma'' = -\frac{2}{a} c_1, \) one obtains

\[
c_{11} + 2 c_{13} + c_{33} = 1, \quad c_{22} = \frac{1}{2}, \quad c_{12} + c_{23} = 0, \quad c_{12} = 0, \quad c_{11} = \frac{1}{2}, \quad c_{11} + c_{13} = \frac{1}{2},
\]

where \( c_{ij} = \langle c_i, c_j \rangle \). Consequently, we obtain that \( \{c_i\} \) are orthogonal vectors in \( \mathbb{E}^{2n+2} \) with \( |c_1| = |c_2| = |c_3| = \frac{1}{\sqrt{2}} \).

Finally, using the facts that \( \gamma \) is a Legendre curve and \( g(\nabla_\gamma \gamma', \varphi \gamma') = 0 \) one obtains easily that \( \langle c_i, Ic_j \rangle = 0 \) for any \( i, j = 1, 2, 3 \).

Now we assume that \( \gamma \) is a biharmonic helix, that is \( \kappa_1^2 + \kappa_2^2 = \frac{e+3}{4} \), \( \kappa_1^2 \in \left( 0, \frac{e+3}{4} \right) \).

First we obtain \( \tilde{\nabla}_T T = \hat{\nabla}_T T, \tilde{\nabla}_T E_2 = \hat{\nabla}_T E_2 \) and \( \tilde{\nabla}_T E_3 = \hat{\nabla}_T E_3 \).

From the Gauss and Frenet equations we get

\[
\tilde{\nabla}_T T = \hat{\nabla}_T T - \langle T, T \rangle \gamma = \kappa_1 E_2 - \frac{1}{a} \gamma,
\]

\[
\tilde{\nabla}_T \tilde{\nabla}_T T = \kappa_1 \tilde{\nabla}_T E_2 - \frac{1}{a} T = \kappa_1 \left( -\kappa_1 T + \kappa_2 E_3 \right) - \frac{1}{a} T = -\left( \kappa_2^2 + \frac{1}{a} \right) T + \kappa_1 \kappa_2 E_3,
\]

and

\[
\tilde{\nabla}_T \tilde{\nabla}_T \tilde{\nabla}_T T = -\left( \kappa_1^2 + \frac{1}{a} \right) \tilde{\nabla}_T T + \kappa_1 \kappa_2 \tilde{\nabla}_T E_3 = -\left( \kappa_1^2 + \frac{1}{a} \right) \tilde{\nabla}_T T - \kappa_1 \kappa_2 E_2
\]

\[
= -\frac{2}{a} \gamma'' - \frac{1}{a} \kappa_2^2 \gamma.
\]

Therefore

\[
a \gamma^{iv} + 2 \gamma'' + \kappa_2^2 \gamma = 0,
\]

and its general solution is

\[
\gamma(s) = \cos(As) c_1 + \sin(As) c_2 + \cos(Bs) c_3 + \sin(Bs) c_4,
\]

where \( A, B \) are given by (3.15) and \( \{c_i\} \) are constant vectors in \( \mathbb{E}^{2n+2} \).
The curve $\gamma$ satisfies

$$\langle \gamma, \gamma \rangle = 1, \quad \langle \gamma', \gamma' \rangle = 0, \quad \langle \gamma'', \gamma'' \rangle = 0, \quad \langle \gamma'', \gamma''' \rangle = 1 + \frac{1 + a\kappa^2}{a^2},$$

$$\langle \gamma, \gamma'' \rangle = \frac{1}{a}, \quad \langle \gamma', \gamma''' \rangle = -\frac{1 + a\kappa^2}{a^2}, \quad \langle \gamma'', \gamma''' \rangle = 0, \quad \langle \gamma, \gamma''' \rangle = 0,$$

and in $s = 0$ we have

$$\gamma = c_1 + c_3, \quad \gamma' = Ac_2 + Bc_4, \quad \gamma'' = -A^2c_1 - B^2c_3, \quad \gamma''' = -A^3c_2 - B^3c_4.$$Then, it follows

(3.16)  
$$c_{11} + 2c_{13} + c_{33} = 1$$

(3.17)  
$$A^2c_{22} + 2ABc_{24} + B^2c_{44} = \frac{1}{a}$$

(3.18)  
$$Ac_{12} + Ac_{23} + Bc_{14} + Bc_{34} = 0$$

(3.19)  
$$A^3c_{12} + AB^2c_{23} + A^2Bc_{14} + B^3c_{34} = 0$$

(3.20)  
$$A^4c_{11} + 2A^2Bc_{13} + B^4c_{33} = \frac{1 + a\kappa^2}{a^2}$$

(3.21)  
$$A^2c_{11} + (A^2 + B^2)c_{13} + B^2c_{33} = \frac{1}{a}$$

(3.22)  
$$A^4c_{22} + (AB^3 + A^3B)c_{24} + B^4c_{44} = \frac{1 + a\kappa^2}{a^2}$$

(3.23)  
$$A^5c_{12} + A^3B^2c_{23} + A^2B^3c_{14} + B^5c_{34} = 0$$

(3.24)  
$$A^3c_{12} + A^3c_{23} + B^3c_{14} + B^3c_{34} = 0$$

(3.25)  
$$A^6c_{22} + 2A^3B^3c_{24} + B^6c_{44} = \frac{3a\kappa^2 + 1}{a^3}$$

where $c_{ij} = \langle c_i, c_j \rangle$.

The solution of the system given by (3.18), (3.19), (3.23) and (3.24) is

$$c_{12} = c_{23} = c_{14} = c_{34} = 0.$$From equations (3.16), (3.20) and (3.21) we get

$$c_{11} = \frac{1}{2}, \quad c_{13} = 0, \quad c_{33} = \frac{1}{2},$$

and, from (3.17), (3.22) and (3.25),

$$c_{22} = \frac{1}{2}, \quad c_{24} = 0, \quad c_{44} = \frac{1}{2}.$$We obtain that $\{c_i\}$ are orthogonal vectors in $\mathbb{E}^8$ with $|c_1| = |c_2| = |c_3| = |c_4| = \frac{1}{\sqrt{2}}$.

Finally, since $\gamma$ is a Legendre curve and $g(\nabla_{\gamma'}, \varphi_{\gamma'}) = 0$, one obtains the conclusion.

□

Just like for $\mathbb{S}^3$ (see [11]) one obtains for $\mathbb{S}^{2n+1}$ endowed with the modified Sasakian structure, with $0 < a < 1$ (that means $c > 1$), the following result.
Moreover, and we have i.e. ∂ in the Euclidean space $E$.

Since Theorem 4.1. by the following Theorem.

Remark 3.15. The ODE satisfied by proper-biharmonic Legendre curves in the $(2n + 1)$-sphere, in the fourth case, may be also obtained but the computations are rather complicated.

4. Biharmonic submanifolds in Sasakian space forms

A method to obtain biharmonic submanifolds in a Sasakian space form is provided by the following Theorem.

**Theorem 4.1.** Let $(S^{2n+1}, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian space form with constant $\varphi$-sectional curvature $c$ and let $i : M \to N$ be an $r$-dimensional integral submanifold of $N$. Consider

$$F : \tilde{M} = I \times M \to N, \quad F(t, p) = \phi_t(p) = \phi_p(t),$$

where $I = S^1$ or $I = \mathbb{R}$ and $\{\phi_t\}_{t \in \mathbb{R}}$ is the flow of the vector field $\xi$. Then $F : (\tilde{M}, \tilde{g} = dt^2 + \imath^*g) \to N$ is a Riemannian immersion and it is proper-biharmonic if and only if $M$ is a proper-biharmonic submanifold of $N$.

**Proof.** From the definition of the flow of $\xi$ we have

$$dF(t, p)\left(\frac{\partial}{\partial t}\right) = \frac{d}{ds}|_{s=t}\{\phi_p(s)\} = \dot{\phi}_p(t) = \xi(\phi_p(t)) = \xi(F(t, p)),$$

i.e. $\frac{\partial}{\partial t}$ is $F$-correlated to $\xi$ and

$$\left|dF(t, p)\left(\frac{\partial}{\partial t}\right)\right| = |\xi(F(t, p))| = 1 = \left|\frac{\partial}{\partial t}\right|.$$

The vector $X_p \in T_p M$ can be identified to $(0, X_p) \in T_{(t', p)}(I \times M)$, for any $t' \in I$, and we have

$$dF_{(t, p)}(X_p) = (dF)_{(t, p)}(\dot{\gamma}(0)) = \frac{d}{ds}|_{s=0}\{\phi_t(\gamma(s))\} = (d\phi_t)_p(X_p).$$

Since $\phi_t$ is an isometry, we have $|dF_{(t, p)}(X_p)| = |(d\phi_t)_p(X_p)| = |X_p|.$

Moreover,

$$g\left(dF_{(t, p)}\left(\frac{\partial}{\partial t}\right), dF_{(t, p)}(X_p)\right) = g(\xi(\phi_p(t)), (d\phi_t)_p(X_p)) = g((d\phi_t)_p(\xi_p), (d\phi_t)_p(X_p)) = g(\xi_p, X_p) = 0,$$
so $F: (I \times M, dt^2 + i^*g) \to N$ is a Riemannian immersion.

Let $F^{-1}(TN)$ be the pull-back bundle over $M$ and $\nabla^F$ the pull-back connection determined by the Levi-Civita connection on $N$. We shall prove that

$$\tau(F)_{(t,p)} = (d\phi_t)_p(\tau(i)) \quad \text{and} \quad \tau_2(F)_{(t,p)} = (d\phi_t)_p(\tau_2(i)),$$

so, from the point of view of harmonicity and biharmonicity, $\tilde{M}$ and $M$ have the same behaviour.

We start with two remarks. Let $\sigma \in F^{-1}(TN)$ defined by $\sigma_{(t,p)} = (d\phi_t)_p(Z_p)$, where $Z$ is a vector field along $M$, that is $Z_p \in T_pN$, $\forall p \in M$. We have

$$\nabla^F \sigma = (d\phi_t)_p(\nabla^N X) \quad \forall X \in C(TM).$$

Then, if $\sigma \in F^{-1}(TN)$, it follows that $\varphi\sigma \in F^{-1}(TN)$, $(\varphi\sigma)_{(t,p)} = \varphi \phi_t(\sigma_{(t,p)})$, and

$$\nabla^F \varphi\sigma = \varphi \nabla^F \sigma.$$

Now, we consider $\{X_1, \ldots, X_r\}$ a local orthonormal frame field on $U$, where $U$ is an open subset of $M$. The tension field of $F$ is given by

$$\tau(F) = \nabla^F \frac{\partial}{\partial t} dF \left( \frac{\partial}{\partial t} \right) - dF \left( \nabla^M \frac{\partial}{\partial t} \right) + \sum_{a=1}^r \{ \nabla^F_{\nabla^F a} dF(X_a) - dF(\nabla^M_{\nabla^F a} X_a) \}.$$

As

$$\nabla^F \frac{\partial}{\partial t} dF \left( \frac{\partial}{\partial t} \right) = \nabla^N X, \quad \nabla^M \frac{\partial}{\partial t} = \nabla^I \frac{\partial}{\partial t} = 0,$$

$$\nabla^F_{\nabla^F a} dF(X_a) = (d\phi_t)_p(\nabla^N X_a), \quad dF(\nabla^M_{\nabla^F a} X_a) = (d\phi_t)_p(\nabla^M_{\nabla^F a} X_a)$$

we obtain

$$\tau(F)_{(t,p)} = (d\phi_t)_p(\tau(i)).$$

The next step is to prove that $\nabla^F \frac{\partial}{\partial t} \tau(F) = -\varphi(\tau(F))$. Since $[\frac{\partial}{\partial t}, X_a] = 0$, $a = 1, \ldots, r$,

$$\nabla^F \frac{\partial}{\partial t} dF(X_a) = \nabla^F_{\nabla^F a} dF \left( \frac{\partial}{\partial t} \right).$$

But

$$\left( \nabla^F_{\nabla^F a} dF \left( \frac{\partial}{\partial t} \right) \right)_{(t,p)} = \nabla^N_{dF(\nabla^M a)} X_a \xi = \nabla^N_{(d\phi_t)_p(\nabla^M a)} X_a \xi = -\varphi((d\phi_t)_p(X_a)) = -(d\phi_t)_p(\varphi X_a),$$

so

$$\left( \nabla^F \frac{\partial}{\partial t} dF(X_a) \right)_{(t,p)} = -(d\phi_t)_p(\varphi X_a).$$

We note that

$$R^F \left( \frac{\partial}{\partial t}, X_a \right) dF(X_a) = \nabla^F \frac{\partial}{\partial t} \nabla^F_{\nabla^F a} dF(X_a) - \nabla^F_{\nabla^F a} \nabla^F \frac{\partial}{\partial t} dF(X_a)$$

and, on the other hand

$$\left( R^F \left( \frac{\partial}{\partial t}, X_a \right) dF(X_a) \right)_{(t,p)} = R^N_{\phi_t(p)}(\xi, (d\phi_t)_p(X_a))(d\phi_t)_p(X_a) = \xi.$$ 

Therefore

$$\nabla^F \frac{\partial}{\partial t} \nabla^F_{\nabla^F a} dF(X_a) - \nabla^F_{\nabla^F a} \nabla^F \frac{\partial}{\partial t} dF(X_a) = \xi$$

(4.4)
Remark 4.2. The previous result was expected because of the following simple
\[ \phi \] form with constant \( M \) smooth map from a compact Riemannian manifold
\[ (4.11) \text{ trace} \]
and from (4.3)
\[ (4.9) \]
From (4.2) we have
\[ (4.6) \]
Replacing (4.5) and (4.6) in (4.4), we obtain
\[ (4.8) \]
As
\[ (4.7) \]
From (4.1) we get
\[ (4.6) \]
and from (4.3)
\[ (4.9) \]
From (4.2) we have
\[ (4.5) \]
and let
\[ \xi, \eta, g \]
is biharmonic, where
\[ F : M = S^1 \times M \to N, F(t, p) = \phi_t(G(p)). \]
Indeed, an arbitrary variation \( \{G_s\}_s \) of \( G \) induces a variation \( \{F_s\}_s \) of \( F \). We have that \( \tau_{(\alpha, \beta)}(F_s) = (d\phi_t)_{G_s(\tau_t(G_s))} \) and, from the biharmonicity of \( F \) and the Fubini Theorem, we get

\[
0 = \frac{d}{ds}|_{s=0} \{E_2(F_s)\} = \frac{1}{2} \frac{d^2}{ds^2}|_{s=0} \int_M |\tau(F_s)|^2 \, v_g = \frac{\pi}{2} \frac{d}{ds}|_{s=0} \int_M |\tau(G_s)|^2 \, v_g
\]

\[
= \frac{\pi}{2} \frac{d}{ds}|_{s=0} \{E_2(G_s)\}.
\]

Next, consider the unit \((2n+1)\)-dimensional sphere \( S^{2n+1} \) endowed with its canonical or modified Sasakian structure. The flow of \( \xi \) is \( \phi_t(z) = \exp(-it)z \), where \( z = (z^1, ..., z^{n+1}) = (x^1, ..., x^{n+1}, y^1, ..., y^{n+1}) \).

From the above expression of the flow and from Theorems 3.14, 3.13, 3.12 and 4.1, we obtain explicit examples of proper-biharmonic submanifolds in \((S^{2n+1}, \varphi, \xi, \eta, g)\), \( a > 0 \), of constant mean curvature. In particular, we reobtain a result in [1] and, for \( n = 1 \), the result in [11].

**Proposition 4.3** ([1]). Let \( F : \tilde{M} \to S^5 \subset \mathbb{R}^6 \) be a proper-biharmonic anti-invariant immersion. Then the position vector of \( \tilde{M} \) in \( \mathbb{R}^6 \) is

\[
F(t, u, v) = \frac{\exp(-it)}{\sqrt{2}}(\exp(iu), i \exp(-iu) \sin(\sqrt{2}v), i \exp(-iu) \cos(\sqrt{2}v)).
\]

**Proof.** It was proved in [17] that the proper-biharmonic integral surface of \((S^5, \varphi_0, \xi_0, \eta_0, g_0)\) is given by

\[
f(u, v) = \frac{1}{\sqrt{2}}(\exp(iu), i \exp(-iu) \sin(\sqrt{2}v), i \exp(-iu) \cos(\sqrt{2}v)).
\]

Now, composing with the flow of \( \xi_0 \) we reobtain the result in [1]. \qed

**Proposition 4.4** ([11]). Let \( M \) be a surface in \((S^{2n+1}, \varphi, \xi, \eta, g)\) with \( a \in (0, 1) \), with the position vector in the Euclidean space \( \mathbb{E}^{2n+2} \) given by

\[
F(t, s) = \exp \left( -\frac{t}{a} \right) \left( \sqrt{\frac{B}{A+B}} \exp(-iAs)e_1 + \sqrt{\frac{A}{A+B}} \exp(iBs)e_3 \right),
\]

where \( \{e_1, e_3\} \) is an orthonormal system of constant vectors in the Euclidean space \( (\mathbb{R}^{2n+2}, \langle \cdot, \cdot \rangle) \), with \( e_3 \) orthogonal to \( \mathcal{I}e_1 \) and \( A, B \) are given by (3.26).

Then \( M \) is a proper-biharmonic surface in \( S^{2n+1}(c) \).

### 5. Proper-Biharmonic Submanifolds of \((S^7, g_0)\)

We consider the Euclidean space \( \mathbb{E}^8 \) with three complex structures,

\[
\mathcal{I} = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 0 & 0 & I_2 \\ 0 & 0 & -I_2 & 0 \\ 0 & I_2 & 0 & 0 \\ -I_2 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{K} = -\mathcal{I}\mathcal{J},
\]

where \( I_n \) denotes the \( n \times n \) identity matrix. We define three vector fields on \( S^7 \) by

\[
\xi_1 = -\mathcal{I}z, \quad \xi_2 = -\mathcal{J}z, \quad \xi_3 = -\mathcal{K}z, \quad z \in S^7,
\]

and consider their dual 1-forms \( \eta_1 = \eta_0, \eta_2, \eta_3 \). Let \( \varphi_a \) defined by

\[
\varphi_1 = \varphi_0 = s \circ \mathcal{I}, \quad \varphi_2 = s \circ \mathcal{J}, \quad \varphi_3 = s \circ \mathcal{K}.
\]

Then \( (\varphi_a, \xi_a, \eta_a, g_0), a = 1, 2, 3, \) determine a Sasakian 3-structure on \( S^7 \).
Now, we shall indicate a method to construct proper-biharmonic submanifolds in \((S^7, g_0)\). We consider \(\gamma = \gamma(s)\) a proper-biharmonic curve in \((S^7, g_0)\), parametrized by arc-length, which is a Legendre curve for two of the three contact structures (it was proved in [9] that there is no proper-biharmonic curve which is Legendre with respect to all three contact structures on \(S^7\)). For example, assume that \(\gamma\) is a Legendre curve for \(\eta_1\) and \(\eta_2\). Composing with the flow of \(\xi_1\) (or \(\xi_2\)) we obtain a biharmonic surface which is Legendre with respect to \(\eta_2\) (or \(\eta_1\)). Then, composing with the flow of \(\xi_2\) (or \(\xi_1\)) we get a biharmonic 3-dimensional submanifold of \((S^7, g_0)\).

Using this method, from Theorems 3.12 and 4.11 we obtain 4 classes of proper-biharmonic surfaces in \((S^7, g_0)\) and 4 classes of proper-biharmonic 3-dimensional submanifolds of \((S^7, g_0)\), all of constant mean curvature.

For example, from Theorems 3.12 and 4.11 composing first with the flow of \(\xi_1\) and then with that of \(\xi_2\), we get the explicit parametric equations of proper-biharmonic 3-dimensional submanifolds of \((S^7, g_0)\).

**Proposition 5.1.** Let \(M\) be a 3-dimensional submanifold in \(S^7\) such that its position vector field in \(\mathbb{R}^8\) is either

\[
\begin{align*}
x_1 & = x_1(u, t, s) \\
& = \frac{1}{\sqrt{2}} \left( \cos(u) \cos(\sqrt{2} s) \cos(t)e_1 + \cos(u) \sin(\sqrt{2} s) \cos(t)e_2 ight. \\
& \quad + \cos(u) \cos(t)e_3 - \cos(u) \cos(\sqrt{2} s) \sin(t)Ie_1 \\
& \quad - \cos(u) \sin(\sqrt{2} s) \sin(t)Ie_2 - \cos(u) \sin(t)Ie_3 \\
& \quad - \sin(u) \cos(\sqrt{2} s) \cos(t)Je_1 - \sin(u) \sin(\sqrt{2} s) \cos(t)Je_2 \\
& \quad - \sin(u) \cos(t)Je_3 - \sin(u) \cos(\sqrt{2} s) \sin(t)Ke_1 \\
& \quad - \sin(u) \sin(\sqrt{2} s) \sin(t)Ke_2 - \sin(u) \sin(t)Ke_3 \right),
\end{align*}
\]

where \(\{e_i, Ie_j\}, \{e_i, Je_j\}\) are systems of constant orthonormal vectors in \(\mathbb{R}^8\), or

\[
\begin{align*}
x_2 & = x_2(u, t, s) \\
& = \frac{1}{\sqrt{2}} \left( \cos(u) \cos(As) \cos(t)e_1 + \cos(u) \sin(As) \cos(t)e_2 ight. \\
& \quad + \cos(u) \cos(Bs) \cos(t)e_3 + \cos(u) \sin(Bs) \cos(t)e_4 \\
& \quad - \cos(u) \cos(As) \sin(t)Ie_1 - \cos(u) \sin(As) \sin(t)Ie_2 \\
& \quad - \cos(u) \cos(Bs) \sin(t)Ie_3 - \cos(u) \sin(Bs) \sin(t)Ie_4 \\
& \quad - \sin(u) \cos(As) \cos(t)Je_1 - \sin(u) \sin(As) \cos(t)Je_2 \\
& \quad - \sin(u) \cos(Bs) \cos(t)Je_3 - \sin(u) \sin(Bs) \cos(t)Je_4 \\
& \quad - \sin(u) \cos(As) \sin(t)Ke_1 - \sin(u) \sin(As) \sin(t)Ke_2 \\
& \quad - \sin(u) \cos(Bs) \sin(t)Ke_3 - \sin(u) \sin(Bs) \sin(t)Ke_4 \right),
\end{align*}
\]
where

\[ A = \sqrt{1 + \kappa_1}, \quad B = \sqrt{1 - \kappa_1}, \quad \kappa_1 = \text{constant} \in (0, 1), \]

and \( \{e_i\} \) are constant orthonormal vectors in \( \mathbb{E}^8 \) such that

\[ \langle e_1, Ie_3 \rangle = \langle e_1, J e_4 \rangle = \langle e_2, I e_3 \rangle = 0, \]

\[ \langle e_1, J e_3 \rangle = \langle e_1, J e_4 \rangle = \langle e_2, J e_3 \rangle = 0, \]

and

\[ A \langle e_1, I e_2 \rangle + B \langle e_3, J e_4 \rangle = A \langle e_1, J e_2 \rangle + B \langle e_3, J e_4 \rangle = 0. \]

Then \( M \) is a proper-biharmonic submanifold of \( (\mathbb{S}^7, g_0) \).

**Proof.** As the flows of \( \xi_1 \) and \( \xi_2 \) are given by

\[ \phi_1^t(z) = (\cos t)z - (\sin t)Iz, \quad \phi_2^t(z) = (\cos t)z - (\sin t)Jz, \]

the Proposition follows by a straightforward computation. \( \square \)

**Remark 5.2.** Note that there exist vectors \( \{e_i\} \) which satisfy the hypotheses of the previous Proposition. For example the first three or four vectors, respectively, from the canonical basis of \( \mathbb{E}^8 \) satisfy the required properties.

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