Quasi-continuous atom laser in the presence of gravity

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Abstract

We analyse the extraction and the subsequent fall of a coherent atomic beam from a trapped Bose-Einstein condensate using a rf transition to a non-trapping state, at T=0 K. Our treatment fully takes gravity into account but neglects the details of the mean-field potential exerted on the free falling beam by the trapped atoms. We focus on the weak coupling regime, where analytical expressions for the output rate and the output mode of the “atom laser”, can be obtained. Comparison with experimental data of Bloch et al. [Phys. Rev. Lett. 82, 3008 (1999)] without any adjustable parameter is satisfactory.
Bose-Einstein condensates (BEC) of dilute alkali vapors constitute a potential source of matter waves for atom interferometry, since it has been proven that they are inherently first-order coherent. Various schemes for “atom lasers” have been used to extract a coherent matter wave out of a trapped BEC. Pulsed devices were demonstrated by using an intense spin-flip radio frequency (rf) pulse, Raman transitions or gravity-induced tunneling from an optically trapped BEC. Later on, a quasi-continuous atom laser has been demonstrated by using a weak rf field that continuously couples atoms into a free falling state. This “quasi-continuous” atom laser promises spectacular improvements in application of atom optics, for example in the performances of atom-interferometer-based inertial sensors.

Gravity plays a crucial role in outcouplers with spin-flip rf transitions: it determines the direction of propagation of the extracted matter wave, its amplitude and its phase. However, most of the theoretical studies do not take it into account (for an up-to-date review, see Ref.). Although gravity has been included in numerical treatments relevant for the pulsed atom laser of Mewes et al., to our knowledge, only the 1D simulations of Refs. treat the quasi-continuous case in presence of gravity, and the results of compare only qualitatively to the experimental data of.

In this Letter, we present a 3D analytical treatment fully taking gravity into account. The extraction of the atoms from the trapped BEC and their subsequent propagation under gravity are analytically treated in the weak coupling regime relevant to the quasi-continuous atom laser. We derive an expression for the atom laser wave function and a generalized rate equation for the trapped atoms that agrees quantitatively with the experimental results of, without any adjustable parameter.

We consider a $^{87}$Rb BEC in the $F = 1$ hyperfine level at $T = 0$ K. The $m = -1$ state is confined in a harmonic magnetic potential $V_{\text{trap}} = \frac{1}{2} M (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$. A rf magnetic field $B_{\text{rf}} = B_{\text{rf}} \cos (\omega_{\text{rf}} t) e_x$ can induce transitions to $m = 0$ (non-trapping state) and $m = +1$ (expelling state). The condensate three-component spinor wavefunction $\Psi' = [\psi'_m]_{m=-1,0,+1}$ obeys a set of coupled non-linear Schrödinger equations. Following Steck et al., we consider the “weak coupling limit”. In this regime to be defined more precisely later, the coupling strength is low enough that the populations $N_m$ of the three Zeeman sublevels obey the following inequality: $N_{+1} \ll N_0 \ll N_{-1}$. In the rest of this paper, we therefore restrict ourselves to $m = -1$ and $m = 0$, and set the total atomic density $n(\mathbf{r}) \approx |\psi_{-1}(\mathbf{r}, t)|^2$. 

At this stage, the components \( \psi_m = \psi_{m}e^{-im\omega_{rt}t} \) obey, under the rotating wave approximation, the following two coupled equations:

\[
\begin{align*}
 i\hbar \frac{\partial \psi_{-1}}{\partial t} &= [\hbar \delta_{\text{rf}} + \mathcal{H}_{-1}]\psi_{-1} + \frac{\hbar \Omega_{\text{rf}}}{2}\psi_0 \\
 i\hbar \frac{\partial \psi_0}{\partial t} &= \mathcal{H}_0\psi_0 + \frac{\hbar \Omega_{\text{rf}}}{2}\psi_{-1}
\end{align*}
\]

with \( \mathcal{H}_{-1} = p^2/2M + V_{\text{trap}} + U | \psi_{-1} |^2 \) and \( \mathcal{H}_0 = p^2/2M - Mgz + U | \psi_{-1} |^2 \). The strength of interactions is fixed by \( U = 4\pi \hbar^2 aN/M, \) \( a \approx 5 \) nm being the diffusion length that we suppose equal for all collision processes, and \( N \) the initial number of trapped atoms. The rf outcoupler is described by the detuning from the bottom of the trap \( \hbar \delta_{\text{rf}} = V_{\text{off}} - \hbar \omega_{\text{rf}} \) and the Rabi frequency \( \hbar \Omega_{\text{rf}} = \mu_B B_{\text{rf}}/2\sqrt{2} \) (taking the Landé factor \( g_{\text{r}}=-1/2 \)). The origin of the \( z \) axis is at the center of the condensate, displaced by gravity from the trap center by \( z_{\text{sag}} = g/\omega_{\perp}^2 \). We have taken the zero of energy at \( z = 0 \) in \( m = 0 \), so that the level splitting at the bottom of the trap is \( V_{\text{off}} = \mu_B B_0/2 + Mg^2/2\omega_z^2 \) (\( B_0 \) is the bias field).

The evolution of the \( m = -1 \) sublevel at \( T = 0 \) K is described \([14]\) by a decomposition of the wave function into a condensed part and an orthogonal one that describes elementary excitations (quasiparticles). As \( \mathcal{H}_{-1} \) depends on time through \( | \psi_{-1} | \), both the coefficients and the eigenmodes depend on time. However, we assume an adiabatic evolution of the trapped BEC \([13]\), so that the uncondensed component remains negligible and:

\[
\psi_{-1}(r, t) \approx \alpha(t)\phi_{-1}(r, t)e^{-i\int_0^t(\mu(t')/\hbar + \delta_{\text{rf}})dt'}
\]

where \( \alpha(t)=(N_{-1}(t)/N)^{1/2} \). The time-dependent ground state \( \phi_{-1} \), of energy \( \mu \) is, in the Thomas-Fermi (TF) approximation \([14]\), \( \phi_{-1}(r, t)=(\mu/U)^{1/2}[ 1 - \tilde{r}_1^2 - \tilde{z}^2 ]^{1/2} \), where \( \tilde{r}_1^2=(x/x_0)^2 + (y/y_0)^2 \) and \( \tilde{z}=z/z_0 \) are such that \( \tilde{r}_1^2 + \tilde{z}^2 \leq 1 \). The BEC dimensions are respectively \( x_0=(2\mu/M\omega_z^2)^{1/2} \) and \( y_0=z_0=(2\mu/M\omega_z^2)^{1/2} \). The condensate energy is given by \( \mu=(\hbar \omega/2)(15aN_{-1}/\sigma)^{2/5} \) (we have set \( \omega=(\omega_x\omega_z^2)^{1/3} \) and \( \sigma=(\hbar/M\omega)^{1/2} \)). The time-dependence of \( \phi_{-1} \) is contained in \( \mu \), \( x_0 \), \( y_0 \) and \( z_0 \), which decrease with \( N_{-1}(t) \). In the following, we will take typical values corresponding to the situation of \([3]\) (reported in \([10]\)): \( N = 7 \times 10^5 \) atoms initially, \( \omega_x=2\pi \times 13 \) Hz and \( \omega_z=2\pi \times 140 \) Hz, which gives \( x_0 \approx 55.7 \mu \)m, \( z_0 \approx 5.3 \mu \)m and \( \mu/\hbar \approx 2.2 \) kHz.

For the \( m=0 \) state, the hamiltonian of equation \((3)\) with \( \Omega_{\text{rf}} =0 \) becomes in the TF approximation \([13]\): \( \mathcal{H}_0 = p^2/2M - Mgz + \mu \text{Max}[0,1 - \tilde{r}_1^2 - \tilde{z}^2] \). There is a clear hierarchy
in the energy scales associated with each term in \( H \). First, in the TF regime, the kinetic energy in the trap is negligible as compared to the mean field energy. Second, for the atom numbers involved in current experimental setups, \( \mu/Mgz_0 \) is small, and the mean field term is only a perturbation as compared with gravity. The leading term is therefore the gravitational potential, that subsequently converts into kinetic energy along \( z \). On the contrary, the transverse kinetic energy (along \( x \) and \( y \)) remains small (we will make use of this to compute the output rate). In addition, we neglect for simplicity the spatial variations of the mean field potential: we replace it by the mean-field contribution to the energy \( 2 \langle \mu \rangle \)

The approximated hamiltonian \( H_0 \approx p^2/2M - Mgz + 4\mu/7 \).

The eigenstates of \( H_0 \) are factorized products of one dimensional wavefunctions along the three axis. In the horizontal plane \((x,y)\), the eigenstates are plane waves with wavevectors \( k_x, k_y \) that we quantize with periodic boundaries conditions in a 2D box of size \( L \). Consequently, the wavefunction is \( \phi_0^{(+)}(x,y) = L^{-1}e^{i(k_xx + k_yy)} \) and the density of states \( \rho_{xy} = L^2/4\pi^2 \).

Along the vertical direction \( z \), the normalizable solution of the 1D Schrödinger equation in a gravitational field is \( \phi_0^{(z_{E_z})} = A\text{Ai}(-\zeta_{E_z}) \), where \( \text{Ai} \) is the Airy function of the first kind and \( \zeta_{E_z} = (z - z_{E_z})/l \). The classical turning point \( z_{E_z} = -E_z/Mg \) associated with the vertical energy \( E_z \) labels the vertical solution; \( l = (h^2/2M^2g)^{1/3} \) is a length scale, such that \( l \ll x_0, y_0, z_0 \) (for \( ^{87}\text{Rb} \), \( l \approx 0.28\mu m \)). In order to normalize \( \text{Ai} \) and to work out the density of states, we restrict the wavefunction to the domain \([-\infty, H] \), where the boundary at \( z=H \) can be arbitrarily far from the origin. In \([ -\infty, z_{E_z} ] \), \( \text{Ai} \) falls off exponentially over a distance \( l \), while it can be identified with its asymptotic form \( \text{Ai}(-s) \approx \pi^{1/2}s^{-1/4}\cos(2s^{3/2}/3 - \pi/4) \) for \( s \gtrsim 0 \). To leading order in \( H \), by averaging the fast oscillating \( \cos^2 \) term to 1/2, we obtain the normalization factor \( A = (\pi H^{-1/2}/l)^{1/2} \). The longitudinal density of states follows from \( \rho_z(E_z) = h^{-1}d\mathcal{V}/dE_z \), where \( \mathcal{V} = \int_{z \leq E_z} dz dp_z \theta(p_z^2/2M - Mgz - E_z) \) (with \( \theta \) the step function) is the volume in phase space associated with energies lower than \( E_z \). We restrict the \( z \) space to the domain defined above, and the \( p_z \) space to \( p_z \geq 0 \). In this way, we do not count the component of \( \text{Ai} \) that propagates opposite to gravity \([17]\). We obtain \( \rho_z(z_{E_z}) = (1/2\pi l)H^{1/2} \).

Finally, the output modes are given by \( \phi_0^{(n)}(r) = \phi_0^{(+)}(x,y)\phi_0^{(z_{E_z})}(z) \), where \( n \) stands for the quantum numbers \((k_x, k_y, z_{E_z})\), and the density of modes is \( \rho_{3D}^{(n)} = (1/8\pi^3)L^2H^{1/2}/l \).

Thus, the problem is reduced to the coupling of an initially populated bound state \( m=-1 \),
of energy \( E_{-1} = \hbar \delta_{rf} + \mu \) to a quasi-continuum of final states \( m=0 \), with a total energy \( E_0^{(n)} = \hbar^2 (k_x^2 + k_y^2)/2M - Mg zE_z + 4\mu/7 \). A crucial feature in this problem is the resonant bell-shape of the coupling matrix element \( W_k=(\hbar \Omega_{rf}/2)\langle \phi_0^{(n)} | \phi_{-1} \rangle \). We work out the overlap integral \( I^{(n)}=\langle \phi_0^{(n)} | \phi_{-1} \rangle \) in reduced cylindrical coordinates \( (r_\perp, \theta, z) \) and set \( \tilde{k}^2 = (k_x x_0)^2 + (k_y y_0)^2 \). We integrate over \( \theta \) and \( r_\perp \), and transform the sum over \( z \) with the Parseval relation \[8\] to obtain: \( I^{(n)} = 2\pi A x_0 y_0 / L (\mu / U)^{(1/2)} \int_{-\infty}^{\infty} \tilde{g}(\tilde{k}, v) e^{i(z_0 l)/(v^3/3-vz/E_z)} dv \). In this integral, \( \tilde{g} \) is the Fourier transform with respect to \( z \) of \( g(\tilde{k}, z) = (p \cos p - \sin p)/\tilde{k}^3 \), with \( p(\tilde{k}, \tilde{z}) = \tilde{k}(1 - \tilde{z}^2)^{1/2} \), and \( 2\pi (-1/2) e^{i(v^3/3-vz/E_z)} \) the Fourier transform of \( A_i \). Since \( z_0/l \gg 1 \), the integrand averages to zero out of a small neighbourhood of the origin, where the linear term in \( v \) is dominant. We obtain in this way:

\[
I^{(n)} = 2\pi A l L \sqrt{\mu / U} x_0 y_0 g(\tilde{k}, z \tilde{E}_z) \tag{4}
\]

This overlap integral is non-vanishing only if the accessible final energies \( E_0^{(n)} \approx E_{-1} \) are restricted to an interval \( [-\Delta/2, \Delta/2] \), where \( \Delta \approx 2M g z_0 \approx 22.7 \) kHz. This gives a resonance condition for the frequency \( \omega_{rf} \) \[3\] \[3\]:

\[
| \hbar \delta_{rf} | \lesssim Mg z_0 \tag{5}
\]

Two different behaviour can be expected in such a situation \[13\] \[13\]. In the strong coupling regime \( (\hbar \Gamma \gg \Delta) \), where \( \Gamma \) is the decay rate worked out with the Fermi Golden Rule), Rabi oscillations occur between the BEC level and the narrow-band continuum. This describes the pulsed atom laser experimentally realized by Mewes et al. \[3\]. On the contrary, in the weak coupling limit \( (\hbar \Gamma \ll \Delta) \), oscillations persist only for \( t \leq t_c = \hbar / \Delta \), while for \( t \geq t_c \), the evolution of the BEC level is a monotonous decay. We have numerically verified this behaviour on a 1D simulation analogous to \[14\] (see Fig.\((1a))

Eq.\((2)\) can be formally integrated \[13\] with the help of the propagator \( G_0 \) of \( H_0 \): \( \Psi_0(\mathbf{r}, t) = \frac{\Omega_{rf}}{2i} \int_0^t dt' \int d^3 \mathbf{r}' G_0(\mathbf{r}, t; \mathbf{r}', t') \Psi_{-1}(\mathbf{r}', t') \). Together with Eq.\((1)\) and Eq.\((3)\), we can derive an exact integro-differential equation on \( \alpha^2 \), the fraction of atoms remaining in the BEC. In the weak coupling regime, the condition \( \hbar \Gamma \ll \Delta \) expresses that the memory time of the continuum \( t_C \) is much shorter than the damping time of the condensate level. This allows one to make a Born-markov approximation \[13\] \[22\], which yields the rate equation:

\[
\frac{dN_{-1}}{dt} = -\Gamma(N_{-1}) N_{-1} \tag{6}
\]
The output rate $\Gamma$ is given by the Fermi golden rule:

$$\frac{\Gamma}{\Omega_{rf}^2} \approx \frac{15\pi}{32} \frac{\hbar}{\Delta} \text{Max}[0, 1 - \eta^2]^2$$  \hspace{1cm} (7)

with $\eta = 2(h\delta_{rf} + 4\mu/7)/\Delta$. The rate equation (6) is non linear, since $\Gamma$ depends on $N_{-1}$ through $\Delta = 2Mg\bar{z}_0$. To obtain Eq.(7), we have neglected the transverse kinetic energy as compared to $E_{-1}$, and used the relations $\mu/Ux_0y_0z_0 = 15/8\pi$ (valid in the TF approximation) and $\int_{0}^{+\infty} dw \left| w \cos(w) - \sin(w) \right|^2 w^{-5} = 1/4$. A quasi-continuous output corresponds to the weak coupling regime ($h\Gamma \ll \Delta$). From Eq.(7), we deduce a critical Rabi frequency $\Omega_{rf}^C \sim 0.8\Delta/\hbar$ for which $h\Gamma \sim \Delta$. However, we have assumed from the beginning that the BEC decay was adiabatic. This requires $|\partial h_{-1}/\partial t| \sim \Gamma \mu \ll \epsilon_{(i)}^2/\hbar$, where $\epsilon_{(i)}$ is the energy of the $i$th quasiparticle level in the trap. Taking $\epsilon_{(i)} \gtrsim \hbar\omega_{\perp}$, we deduce from Eq.(7) a condition on the Rabi frequency: $\Omega_{rf} \ll 1.6(g/\bar{z}_0)^{1/2}$. This condition turns out to be much more stringent than the condition for weak coupling $\Omega_{rf} \ll \Omega_{rf}^C$.

To compare our model to the data of Bloch et al [6], we have numerically integrated Eq.(6) with the output rate (7) and their experimental parameters. We show in Fig.(2) the number of atoms remaining in the condensate after a fixed time as a function of the detuning. The agreement with the experimental data for the $|F = 1; m_F = -1\rangle$ state is good (Fig.(2a)). To treat the $|F = 2; m_F = 2\rangle$ case, the second trapping state $|F = 2; m_F = 1\rangle$ must be included. We use the result of [19] for the situation where two discrete levels are mutually coupled, and only one of them is also coupled to a continuum (with a decay rate $\Gamma_{2,1}$)[19].

For near resonant coupling, the upper level acquires a decay rate $\Gamma_{2,2} \sim \Gamma_{2,1}/2$ if $\Omega_{rf} \gg \Gamma_{2,1}$. We have taken this value as a first approximation. This leads to an already good agreement with the experimental data (Fig.2b), although better refinements is needed to describe the full dynamics.

Finally, we want to point out that, since $h\Gamma \ll \Delta$, the spatial region where outcoupling takes place, of vertical extension $\delta z \sim h\Gamma/mg$, is very thin compared to the BEC size. It allows us to think of the outcoupling process in a semi classical way, in analogy with a Franck-Condon principle [20]: the coupling happens at the turning point of the classical trajectory of the free falling atoms. Neglecting the transverse kinetic energy in the calculation of $\Gamma$ amounts to approximate the Franck-Condon surfaces by planes. Since over the size of the BEC, the surface curvature is small, the deviation from this plane is negligible compared to $\bar{z}_0$.  

6
We have analyzed so far the extraction of the atoms from the trapped BEC. We address now the question of the propagation of the outcoupled atoms under gravity. For times \( t \gg t_c \), the explicit expression of the propagator \( G_0 \) of \( H_0 \) is [21]:

\[
G_0(\mathbf{r}, \mathbf{r}', \tau = t - t') = \left( \frac{M}{2\pi \hbar \tau} \right)^{3/2} e^{iS_{\text{class}}(\mathbf{r}, \mathbf{r}', \tau = t - t') / \hbar} \theta(\tau)
\]

(8)

Here, the classical action is given by

\[
S_{\text{class}}(\mathbf{r}, \mathbf{r}', \tau = t - t') = \left( \frac{M}{2\bar{\hbar} \tau} \right) \left( \mathbf{r} - \mathbf{r}' \right)^2 + g\tau^2 (z + z') - g^2 \tau^4 / 12 + 4\mu_t / 7\bar{\hbar}
\]

and \( \theta(\tau) \) is the step function. We compute the output wavefunction \( \Psi_0 \) with stationary phase approximations. This amounts to propagate the wavefunction along the classical trajectory: in the limit where \( S_{\text{class}} \gg \bar{\hbar} \) (this is true as soon as the atom has fallen from a height \( l \)), this is the Feynman path that essentially contributes. We obtain the following expression for the outcoupled atomic wave function, i.e. the atom laser mode [22] (see Fig. (1b)):

\[
\psi_0(\mathbf{r}, t) \approx -\sqrt{\frac{\hbar}{\pi M g l}} \phi_{-1}(x, y, z_{\text{res}}) e^{\frac{i}{\hbar} \left[ \frac{3}{2} \zeta_{\text{res}}^2 - i \frac{E - 1}{\hbar} \right]} F(t, t_{\text{fall}})
\]

(9)

In this expression, \( z_{\text{res}} = \eta z_0 / 2 \) is the point of extraction, \( t_{\text{fall}} = (2/g)^{1/2} (z - z_{\text{res}})^{1/2} \) is the time of fall from this point, \( \zeta_{\text{res}} = (z - z_{\text{res}}) / l \), and \( F(\beta) = \alpha(\beta) M(\beta) \): \( M \) is the top hat function, that describes the finite extent of the atom laser due to the finite coupling time. We can deduce from Eq.(9) the size of the laser beam in the \( x \) direction \( x_{\text{out}} = x_0 (1 - (2E_{-1}/\Delta)^2)^{1/2} \), and a similar formula for \( y_{\text{out}} \). Because the semi-classical approximation neglects the quantum velocity spread due to the spatial confinement of the trapped BEC, this wavefunction has plane wavefronts. This property will not persist beyond a distance \( z_R \approx y_{\text{out}}^4 / 4l^3 \sim \) a few mm, analog to the Rayleigh length in photonic laser beams, where the transverse diffraction \((\hbar / M y_{\text{out}}) (2z_R / g)^{1/2}\) becomes comparable to the size \( y_{\text{out}} \). Beyond \( z_R \), diffraction has to be taken into account. However, this discussion neglects the (weak) transverse acceleration provided by the repulsive mean field potential that is likely to affect the transverse profile.

In conclusion, we have obtained analytical expressions for the output rate and the output mode of a quasi-continuous atom laser based on rf outcoupling from a trapped BEC. Our treatment, which fully takes the gravity and the 3D geometry into account, leads to a good agreement with the experimental results of Ref. [6]. This points out the crucial role played by gravity in the atom laser behaviour. More elaborated treatments should take the interaction energy between the trapped BEC and the atom laser, as well as finite temperature effects [13] into account. Nevertheless, we believe that the present zero-temperature theory is a
valuable starting point to describe experiments with the atom laser.

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FIG. 1: Atom laser in the weak coupling regime ($\Omega_{rf} = 300$ Hz). 1a : time evolution of the laser intensity starting with $\sim 2000$ atoms. For $t \gg t_c$, the numerical integration of Eq.(1) agrees with the output rate of Eq.(7). 1b : Spatial intensity profile of the atom laser at $t \gg t_c$ according to Eq.(9)
FIG. 2: Number of trapped atoms after 20 ms of rf outcoupling, starting with $\approx 7.2 \times 10^5$ condensed atoms, $\Omega_{\text{rf}}=312$ Hz for the $|1; -1\rangle$ sublevel (left), and $\approx 7.0 \times 10^5$ atoms, $\Omega_{\text{rf}}=700$ Hz for the $|2; 2\rangle$ sublevel (right). Diamonds are the experimental points from Bloch et al., solid line is the prediction based upon our model using the experimental parameters. Theoretical and experimental curves have been shifted in frequency to match each other, since $V_{\text{off}}$ is not experimentally known precisely enough (a precision of $10^{-3}$ Gauss for the bias field $B_0$ is required to know $V_{\text{off}}$ within a kHz uncertainty).
