REGULARITY OF PULLBACK ATTRACTORS AND EQUILIBRIA FOR A STOCHASTIC NON-AUTONOMOUS REACTION-DIFFUSION EQUATIONS PERTURBED BY A MULTIPLICATIVE NOISE

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Abstract. In this paper, a standard about the existence and upper semi-continuity of pullback attractors in the non-initial space is established for some classes of non-autonomous SPDE. This pullback attractor, which is the omega-limit set of the absorbing set constructed in the initial space, is completely determined by the asymptotic compactness of solutions in both the initial and non-initial spaces. As applications, the existences and upper semi-continuity of pullback attractors in $H^1(\mathbb{R}^N)$ are proved for stochastic non-autonomous reaction-diffusion equation driven by a multiplicative noise. Finally we show that under some additional conditions the cocycle admits a unique equilibrium.

1. Introduction

In this paper, we consider the dynamics of solutions to the following reaction-diffusion equation on $\mathbb{R}^N$ driven by a random noise as well as a deterministic non-autonomous forcing:

$$du + (\lambda u - \Delta u)dt = f(x,u)dt + g(t,x)dt + \varepsilon u \circ d\omega(t),$$

(1.1)

with initial condition

$$u(\tau, x) = u_0(x), \quad x \in \mathbb{R}^N,$$

(1.2)

where the initial $u_0 \in L^2(\mathbb{R}^N)$, $\lambda$ is a positive constant, $\varepsilon$ is the intensity of noise, the unknown $u = u(x,t)$ is a real valued function of $x \in \mathbb{R}^N$ and $t > \tau$, $\omega(t)$ is a mutually independent two-sided real-valued Wiener process defined on a canonical Wiener probability space $(\Omega, \mathcal{F}, P)$.

The notion of random attractor, introduced in [7, 17, 9, 8], is an important tool to study the qualitative property of stochastic partial differential equations (SPDE). We can find a large number of literature to investigate the existences of random attractors in an initial space (the initial data located space) for some concrete stochastic partial differential equations, see [4, 12, 15, 21, 23, 24, 31] and the references therein. In particular, [20, 18, 22] discussed the upper semi-continuity of a family of random attractors in the initial space.

As we know, the solutions of SPDE may possess some regularity, for example, higher-order integrability or higher-order differentiability. In these cases, the solutions may escape (or leave) the initial space and enter into another space, which we call a non-initial space. So, the existence and upper semi-continuity of random

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attractors in a non-initial space, usually a higher-regularity space, such as $L^p(p > 2)$ or $H^1$, are necessary for us to understand the dynamics of solutions of SPDE.

In terms of this consideration, some literature attacked this problem recently. In the case of bounded domain, [1, 16, 14, 27, 28, 25] discussed the existence of random attractor in the non-initial spaces $L^p$ and $H^1_0$, respectively. When the state space is unbounded, Zhao and Li [29] proved the existence of random attractors for reaction-diffusion equations in $L^p(\mathbb{R}^N)$, and for the same equation, Li and et al [13] obtained the upper semi-continuity of random attractor in $L^p(\mathbb{R}^N)$. Most recently, Zhao [26, 30] proved the existence of random attractors for semi-linear degenerate parabolic equations in $L^{2p-2}(D)\cap H^1(D)$, where $D$ is an unbounded domain. Bao [3] proved the existence of random attractors for non-autonomous Fitzhugh-Nagumo system in $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. In that paper, the key point is closely related to Lemma 5.1 there, of which the detailed proof is omitted.

It is pointed out that most recently, Li and et al [15] established the theory of bi-spatial random attractors by using the notion of uniform omega-limit compactness, by which SPDE with autonomous forcing can be solved, see also [13]. However, to the best of our knowledge, there are no literature to discuss the existence and upper semi-continuity of pullback attractors in a non-initial space for SPDE with a non-autonomous forcing term, except the literature [3].

In this paper, we study the existence and upper semi-continuity of pullback attractors in the non-initial space $H^1(\mathbb{R}^N)$ for problem (1.1)-(1.2) with a non-autonomous forcing. The nonlinearity $f$ and the deterministic non-autonomous function $g$ satisfy almost the same conditions as [18], in which the author obtained the existence and upper-continuity of pullback attractors in the initial space $L^2(\mathbb{R}^N)$. Here, we strengthen this result and show that the obtained pullback attractors are also compact and attracting in $H^1(\mathbb{R}^N)$ norm. Furthermore, we find that the upper continuity of the obtained pullback attractors happen in $H^1(\mathbb{R}^N)$. The existence of pullback attractor in an initial space for a non-autonomous stochastic partial differential equation is established in [21], where the measurability of pullback attractors is proved. The applications we may see [12, 18, 20, 21]. For the reference on the theory regarding upper semi-continuity of pullback attractors, we may refer to [18, 20, 22] for the stochastic cases and to [6, 11] for the deterministic cases.

In order to solve our problem, we establish a sufficient criteria for the existence and upper semi-continuity of pullback attractors in a non-initial space. It is showed that a family of pullback attractors obtained in an initial space are compact, attracting and upper semi-continuous in a non-initial space if some compactness conditions of the cocycles are satisfied, see Theorem 2.6-2.8 in section 2. This implies that the continuity (or quasi-continuity [14], norm-weak continuity [33]) and absorption in the non-initial space are not necessary ones. This result is a meaningful and convenient tool for us to consider the existence and upper semi-continuity of pullback attractors in some associated non-initial spaces for SPDE with a non-autonomous forcing term.

Consider that the stochastic equation (1.1) is defined on unbounded domains, the asymptotic compactness of solution in $H^1(\mathbb{R}^N)$ can not be derived by the traditional technique. The reasons are as follows. On the one hand, the equation (1.1) is stochastic and the Wiener process $\omega$ is only continuous but not differentiable in $t$. This leads to some difficulties for us to estimate the norm of derivative $u_t$ by the
trick employed in deterministic case, see [33, 32]. Then the asymptotic compactness in $H^{1}(\mathbb{R}^{N})$ can not be proved by estimate of the difference of $\nabla u$ as in [32].

On the other hand, the estimate of $\Delta u$ is not available for our problem (up to now, actually we do not know how to estimate the norm $\Delta u$ of problem (1.1)-(1.2), although this can be achieved in deterministic case by estimate $u_{t}$, see [33]). So we can not use the Sobolev compact embeddings of $H^{2} \hookrightarrow H^{1}$ on bounded domains. Here we surmount these obstacles by checking the uniform smallness of solutions outside a large ball in $H^{1}(\mathbb{R}^{N})$-norms as in [19, 20, 3]. In bounded domains, we prove the asymptotic compactness of solutions by a space splitting technique as in [26, 27, 28, 1], and combination the estimate of the truncation of solutions in $L^{2p-2}$-norm over an integral interval.

Finally, we investigate how the parameters in problem (1.1) affect the pullback attractor. We show that if the parameters satisfy some conditions, then the cocycle admits a unique equilibrium in $L^{2}(\mathbb{R}^{N})$. Furthermore, this equilibrium is also in both $H^{1}(\mathbb{R}^{N})$ and $L^{p}(\mathbb{R}^{N})$, $p > 2$.

In the next section, we recall some notions and prove a sufficient standard for the existence and upper semi-continuity of pullback attractors of non-autonomous system in a non-initial space. In section 3, we give the assumptions on $g$ and $f$, and define a continuous cocycle for problem (1.1). In section 4 and 5, we prove the existence and upper semi-continuity in $H^{1}(\mathbb{R}^{N})$. Finally, in section 6, we prove the existence of equilibria for the cocycle derived from problem (1.1).

2. Preliminaries and abstract results

Let $(X, \|\cdot\|_{X})$ and $(Y, \|\cdot\|_{Y})$ be two complete separable Banach spaces with Borel sigma-algebras $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, respectively. $X \cap Y \neq \emptyset$. For convenience, we call $X$ the initial space (which contains all initial data of a SPDE) and $Y$ the associated non-initial space (usually the regular solutions (of a SPDE) located space).

In this section, we give a sufficient standard for the existence and upper semi-continuity of pullback attractors in the non-initial space $Y$ for the random dynamical system (RDS) over two parametric spaces. The readers may refer to [26, 29, 30, 16, 13, 14, 15, 25] for the existence and semi-continuity of random attractors in the non-initial space $Y$ for a RDS over one parametric space.

We also mention that regarding the existence of random attractors in the initial space $X$ for the RDS over one parametric space, the good references are [2, 4, 7, 17, 9, 8]. However, here we recall from [21] some basic notions regarding RDS over two parametric spaces, one of which is the real numbers space and another of which is the measurable probability space with a measure preserving transformation.

2.1. Preliminaries. The basic notion in RDS is a metric (or measurable) dynamical system (MDS) $\vartheta \equiv (\Omega, \mathcal{F}, P, \{\vartheta_{t}\}_{t \in \mathbb{R}})$, which is a probability space $(\Omega, \mathcal{F}, P)$ with a group $\vartheta_{t}, t \in \mathbb{R}$, of measure preserving transformations of $(\Omega, \mathcal{F}, P)$.

An MDS $\vartheta$ is said to be ergodic under $P$ if for any $\vartheta$-invariant set $F \in \mathcal{F}$, we have either $P(F) = 0$ or $P(F) = 1$, where the $\vartheta$-invariant set is in the sense that $P(\vartheta_{t}F) = (F)$ for $F \in \mathcal{F}$ and all $t \in \mathbb{R}$.

Definition 2.1. Let $(\Omega, \mathcal{F}, P, \{\vartheta_{t}\}_{t \in \mathbb{R}})$ be a measurable dynamical system. A family of measurable mappings $\varphi : \mathbb{R}^{+} \times \mathbb{R} \times \Omega \times X \rightarrow X$ is called a cocycle on $X$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, \{\vartheta_{t}\}_{t \in \mathbb{R}})$ if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $t, s \in \mathbb{R}^{+}$, the following
conditions are satisfied:

\[ \varphi(0, \tau, \omega, \cdot) \text{ is the identity on } X, \]
\[ \varphi(t + s, \tau, \omega, \cdot) = \varphi(t, \tau, \omega, \cdot) \circ \varphi(s, \tau, \omega, \cdot). \]

In addition, if \( \varphi(t, \tau, \omega, \cdot) : X \to X \) is continuous for all \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega \), then \( \varphi \) is called a continuous cocycle on \( X \) over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t \in \mathbb{R}}) \).

**Definition 2.2.** Let \( 2^X \) be the collection of all subsets of \( X \). A set-valued mapping \( K : \mathbb{R} \times \Omega \to 2^X \) is called measurable in \( X \) with respect to \( \mathcal{F} \) in \( \Omega \) if the mapping \( \omega \in \Omega \mapsto \text{dist}_X(x, K(\tau, \omega)) \) is \( (\mathcal{F}, \mathcal{B}(\mathbb{R})) \)-measurable for every fixed \( x \in X \) and \( \tau \in \mathbb{R} \), where \( \text{dist}_X \) is the Hausdorff semi-metric in \( X \). In this case, we also say the family \( \{K(\tau, \omega) ; \tau \in \mathbb{R}, \omega \in \Omega\} \) is measurable in \( X \) with respect to \( \mathcal{F} \) in \( \Omega \). Furthermore if the value \( K(\tau, \omega) \) is a closed nonempty subset of \( X \) for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), then \( \{K(\tau, \omega) ; \tau \in \mathbb{R}, \omega \in \Omega\} \) is called a closed measurable set of \( X \) with respect to \( \mathcal{F} \) in \( \Omega \).

Hereafter, we always assume that \( \varphi \) is a continuous cocycle on \( X \) over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t \in \mathbb{R}}) \) satisfying

(H1) For every fixed \( t \in \mathbb{R}^+, \tau \in \mathbb{R} \) and \( \omega \in \Omega \), \( \varphi(t, \tau, \omega, \cdot) : X \to X \cap Y \);

(H2) If \( \{x_n\}_n \subset X \cap Y \) such that \( x_n \to x \) in \( X \) and \( x_n \to y \) in \( Y \), respectively, then \( x = y \).

Let \( \mathcal{D} \) be a collection of some families of nonempty subsets of \( X \) parametrized by \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \) such that

\[ \mathcal{D} = \{B = \{B(\tau, \omega) \in 2^X ; B(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega\} ; f_B \text{ satisfies some conditions}\}. \]

In particular, for two elements \( B_1, B_2 \in \mathcal{D} \), we say that \( B_1 = B_2 \) if and only if \( B_1(\tau, \omega) = B_2(\tau, \omega) \) for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \).

**Definition 2.3.** Let \( \mathcal{D} \) be a collection of some families of nonempty subsets of \( X \) and \( K = \{K(\tau, \omega) ; \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \). Then \( K \) is called a \( \mathcal{D} \)-pullback absorbing set for a cocycle \( \varphi \) in \( X \) if all \( \tau \in \mathbb{R}, \omega \in \Omega \) and for every \( B \in \mathcal{D} \) there exists a absorbing time \( T = T(\tau, \omega, B) \geq 0 \) such that

\[ \varphi(t, \tau - t, \vartheta_{-t} \omega, B(\tau - t, \vartheta_{-t} \omega)) \subseteq K(\tau, \omega) \quad \text{for all } t \geq T. \]

If in addition, \( K \) is measurable in \( X \) with respect to the \( P \)-completion of \( \mathcal{F} \) in \( \Omega \), then \( K \) is said to be a measurable pullback absorbing set for \( \varphi \).

**Definition 2.4.** Let \( \mathcal{D} \) be a collection of some families of nonempty subsets of \( X \). A cocycle \( \varphi \) is said to be \( \mathcal{D} \)-pullback asymptotically compact in \( X \) (resp. in \( Y \)) if for all \( \tau \in \mathbb{R}, \omega \in \Omega \)

\[ \{\varphi(t_n, \tau - t_n, \vartheta_{-t_n} \omega, x_n)\} \text{ has a convergent subsequence in } X \] (resp. in \( Y \))

whenever \( t_n \to \infty \) and \( x_n \in B(\tau - t_n, \vartheta_{-t_n} \omega) \) with \( B = \{B(\tau, \omega) ; \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \).

**Definition 2.5.** Let \( \mathcal{D} \) be a collection of some families of nonempty subsets of \( X \) and \( \mathcal{A} = \{\mathcal{A}(\tau, \omega) ; \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \). \( \mathcal{A} \) is called a \( \mathcal{D} \)-pullback attractor for a
cocycle \( \varphi \) in \( X \) (resp. in \( Y \)) over \( \mathbb{R} \) and \( (\Omega, F, P, \{ \theta_t \}_{t \in \mathbb{R}}) \) if

(i) \( A \) is measurable in \( X \) with respect to the \( P \)-completion of \( F \), and \( A(\tau, \omega) \) is compact in \( X \) (resp. in \( Y \)) for all \( \tau \in \mathbb{R}, \omega \in \Omega \);

(ii) \( A \) is invariant, that is, for all \( \tau \in \mathbb{R}, \omega \in \Omega \),

\[
\varphi(t, \tau, \omega, A(\tau, \omega)) = A(\tau + t, \theta_t \omega), \forall \ t \geq 0;
\]

(iii) \( A \) attracts every element \( B = \{ B(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega \} \in D \) in \( X \) (resp. in \( Y \)), that is, for all \( \tau \in \mathbb{R}, \omega \in \Omega \),

\[
\lim_{t \to +\infty} \text{dist}_X(\varphi(t, \tau - t, \theta_{-t} \omega, B(\tau - t, \theta_{-t} \omega)), A(\tau, \omega)) = 0
\]

(resp. \( \lim_{t \to +\infty} \text{dist}_Y(\varphi(t, \tau - t, \theta_{-t} \omega, B(\tau - t, \theta_{-t} \omega)), A(\tau, \omega)) = 0 \)).

2.2. Existence of random attractors in a non-initial space \( Y \). This subsection is concerned with the existence of \( D \)-pullback attractor of the cocycle \( \varphi \) in the non-initial space \( Y \). The continuity of \( \varphi \) in \( Y \) is not clear, and the inclusion relation of \( X \) and \( Y \) is also unknown except that \( (H1) \) and \( (H2) \) hold.

**Theorem 2.6.** Let \( D \) be a collection of some families of nonempty subsets of \( X \) which is inclusion closed. Let \( \varphi \) be a continuous cocycle on \( X \) over \( \mathbb{R} \) and \( (\Omega, F, P, \{ \theta_t \}_{t \in \mathbb{R}}) \). Assume that

(i) \( \varphi \) has a closed and measurable (w.r.t. the \( P \)-completion of \( F \)) \( D \)-pullback absorbing set \( K = \{ K(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega \} \in D \) in \( X \);

(ii) \( \varphi \) is \( D \)-pullback asymptotically compact in \( X \). Then the cocycle \( \varphi \) has a unique \( D \)-pullback attractor \( A_X = \{ A_X(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega \} \in D \) in \( X \), given by

\[
A_X(\tau, \omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \varphi(t, \tau - t, \theta_{-t} \omega, K(\tau - t, \theta_{-t} \omega)), \tau \in \mathbb{R}, \omega \in \Omega, \tag{2.1}
\]

where the closure is taken in \( X \).

If further \( (H1)-(H2) \) hold and (iii) \( \varphi \) is \( D \)-pullback asymptotically compact in \( Y \), then the cocycle \( \varphi \) has a unique \( D \)-pullback attractor \( A_Y = \{ A_Y(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega \} \) in \( Y \), given by

\[
A_Y(\tau, \omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \varphi(t, \tau - t, \theta_{-t} \omega, K(\tau - t, \theta_{-t} \omega)), \tau \in \mathbb{R}, \omega \in \Omega. \tag{2.2}
\]

In addition, we have \( A_Y = A_X \cap Y \) in the sense of set inclusion, i.e., for every \( \tau \in \mathbb{R}, \omega \in \Omega \), \( A_Y(\tau, \omega) = A_X(\tau, \omega) \).

**Proof.** The first result is well known and so we are interested in the second result. Indeed, (2.2) makes sense by \( (H1) \) and \( A_Y \neq \emptyset \) by the asymptotic compactness of the cocycle \( \varphi \) in \( Y \). In the following, we show that \( A_Y \) satisfies Definition 2.5.

Step 1. We claim that the set \( A_Y \) is measurable in \( X \) (w.r.t. the \( P \)-completion of \( F \) in \( \Omega \)) and \( A_Y \in D \) is invariant by proving that \( A_Y = A_X \) since \( A_X \) is measurable (with respect to the \( P \)-completion of \( F \) in \( \Omega \)) and \( A_X \in D \) is invariant (the measurability of \( A_X \) is proved by Theorem 2.14 in [20]).
For every fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$, taking $x \in A_X(\tau, \omega)$, by (2.1), there exist two sequences $t_n \to +\infty$ and $x_n \in K(\tau - t_n, \vartheta_{-t_n} \omega)$ such that
\[
\varphi(t_n, \tau - t_n, \vartheta_{-t_n} \omega, x_n) \xrightarrow{n \to \infty} x.
\] (2.3)

Since $\varphi$ is $D$-asymptotically compact in $Y$, then there is a $y \in Y$ such that up to a subsequence,
\[
\varphi(t_n, \tau - t_n, \vartheta_{-t_n} \omega, x_n) \xrightarrow{n \to \infty} y.
\] (2.4)

It implies from (2.2) that $y \in A_Y(\tau, \omega)$. Then by (H2), along with (2.3) and (2.4), we have $x = y \in A_X(\tau, \omega)$ and thus $A_X(\tau, \omega) \subseteq A_Y(\tau, \omega)$ for every fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$. The inverse inclusion can be proved in the same way then we omit it here. Thus $A_X = A_Y$ as required.

**Step 2.** We prove the attraction of $A_Y$ in $Y$ by a contradiction argument. Indeed, if there exist $\delta > 0$, $x_n \in B(\tau - t_n, \vartheta_{-t_n} \omega)$ with $B \in D$ and $t_n \to +\infty$ such that
\[
\text{dist}_Y \left( \varphi(t_n, \tau - t_n, \vartheta_{-t_n} \omega, x_n), A_Y(\tau, \omega) \right) \geq \delta.
\] (2.5)

By the asymptotic compactness of $\varphi$ in $Y$, there exists $y_0 \in Y$ such that up to a subsequence,
\[
\varphi(t_n, \tau - t_n, \vartheta_{-t_n} \omega, x_n) \xrightarrow{n \to \infty} y_0.
\] (2.6)

On the other hand, by condition (i), there exists a large time $T > 0$ such that
\[
y_n = \varphi(T, \tau - t_n, \vartheta_{-t_n} \omega, x_n) = \varphi(T, (\tau - t_n + T) - T, \vartheta_{-T} \vartheta_{-(t_n - T)} \omega, x_n)
\in K(\tau - t_n + T, \vartheta_{-(t_n - T)} \omega).
\] (2.7)

Then by the cocycle property in Definition 2.1, along with (2.6) and (2.7), we infer that as $t_n \to \infty$,
\[
\varphi(t_n, \tau - t_n, \vartheta_{-t_n} \omega, x_n) = \varphi(t_n - T, \tau - (t_n - T), \vartheta_{-(t_n - T)} \omega, y_n) \to y_0 \quad \text{in } Y.
\]

Therefore by (2.2), $y_0 \in A_Y(\tau, \omega)$. This implies that
\[
\text{dist}_Y \left( \varphi(t_n, \tau - t_n, \vartheta_{-t_n} \omega, x_n), A_Y(\tau, \omega) \right) \to 0
\]
as $t_n \to \infty$, which is a contradiction to (2.5).

**Step 3.** It remains to prove the compactness of $A_Y$ in $Y$. Let $\{y_n\}_{n=1}^\infty$ be a sequence in $A_Y(\tau, \omega)$. By the invariance of $A_Y(\tau, \omega)$ which is proved in Step 1, we have
\[
\varphi(t, \tau - t, \vartheta_{-t} \omega, A_Y(\tau - t, \vartheta_{-t} \omega)) = A(\tau, \omega).
\]
Then it follows that there is a sequence $\{z_n\}_{n=1}^\infty$ with $z_n \in A_Y(\tau - t_n, \vartheta_{-t_n} \omega)$ such that for every $n \in \mathbb{Z}^+$,
\[
y_n = \varphi(t_n, \tau - t_n, \vartheta_{-t_n} \omega, z_n).
\]

Note that $A_Y \in D$. Then by the asymptotic compactness of $\varphi$ in $Y$, $\{y_n\}$ has a convergence subsequence in $Y$, i.e., there is a $y_0 \in Y$ such that
\[
\lim_{n \to \infty} y_n = y_0 \quad \text{in } Y.
\]

But $A_Y(\tau, \omega)$ is closed in $Y$, so $y_0 \in A_Y(\tau, \omega)$. 
The uniqueness is easily followed by the attraction property of \( \varphi \) and \( A_Y \in \mathcal{D} \). This completes the total proofs. \( \square \)

Remark. (i) It is pointed out that the assumption \((H1)\) is necessary to guarantee that the closure of the set \( \varphi(t, \tau - t, \vartheta - t, \omega, K(\tau - t, \vartheta - t)) \) in \( Y \) makes sense for all \( t \in \mathbb{R}^+ \), as in (2.2).

(ii) We emphasize that the random attractor \( A_Y \) in the non-initial space is completely determined by the absorbing set constructed in the initial space, without requiring the absorption in the non-initial space. This is different from the construction in [3].

2.3. Upper semi-continuity of random attractors in a non-initial space \( Y \).

Assume that the assumptions \((H1)-(H2)\) hold. Given the indexed set \( I \subset \mathbb{R} \), for every \( \varepsilon \in I \), we use \( D_\varepsilon \) to denote a collection of some families of nonempty subsets of \( X \). Let \( \varphi_\varepsilon (\varepsilon \in I) \) be a continuous cocycle on \( X \) over \( \mathbb{R} \) and \((\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t \in \mathbb{R}})\). We now consider the upper semi-continuous of pullback attractors of a family of cocycle \( \varphi_\varepsilon \) in \( Y \).

Suppose first that for every \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega, \varepsilon_n, \varepsilon_0 \in I \) with \( \varepsilon_n \to \varepsilon_0 \), and \( x_n, x \in X \) with \( x_n \to x \), there holds
\[
\lim_{n \to \infty} \varphi_{\varepsilon_n}(t, \tau - t, \vartheta - t, \omega, x_n) = \varphi_{\varepsilon_0}(t, \tau - t, \vartheta - t, \omega, x) \quad \text{in } X. \tag{2.8}
\]

Suppose second that there exists a map \( R_{\varepsilon_0} : \mathbb{R} \times \Omega \to \mathbb{R}^+ \) such that the family \( B_0 = \{B_\tau(\tau, \omega) = \{x \in X; \|x\|_X \leq R_{\varepsilon_0}(\tau, \omega)\}; \tau \in \mathbb{R}, \omega \in \Omega\} \) belongs to \( D_{\varepsilon_0} \).

And further for every \( \varepsilon \in I \), \( \varphi_\varepsilon \) has \( D_\varepsilon \)-pullback attractor \( A_\varepsilon \in \mathcal{D}_\varepsilon \) in \( X \cap Y \) and a closed and measurable \( D_\varepsilon \)-pullback absorbing set \( K_\varepsilon \in \mathcal{D}_\varepsilon \) in \( X \) such that for every \( \tau \in \mathbb{R}, \omega \in \Omega, \)
\[
\limsup_{\varepsilon \to \varepsilon_0} \|K_\varepsilon(\tau, \omega)\| \leq R_{\varepsilon_0}(\tau, \omega), \tag{2.10}
\]
where \( \|S\|_X = \sup_{x \in S} \|x\|_X \) for a set \( S \). We finally assume that for every \( \tau \in \mathbb{R}, \omega \in \Omega, \)
\[
\bigcup_{\varepsilon \in I} A_\varepsilon(\tau, \omega) \text{ is precompact in } X, \quad \text{and} \tag{2.11}
\]
\[
\bigcup_{\varepsilon \in I} A_\varepsilon(\tau, \omega) \text{ is precompact in } Y. \tag{2.12}
\]

Then we have the upper semi-continuity in \( Y \).

**Theorem 2.7.** If \((2.8)-(2.11)\) hold, then for each \( \tau \in \mathbb{R}, \omega \in \Omega, \)
\[
\lim_{\varepsilon \to \varepsilon_0} \text{dist}_X(A_\varepsilon(\tau, \omega), A_{\varepsilon_0}(\tau, \omega)) = 0.
\]

If further \((H1)-(H2)\) hold and conditions \((2.8)-(2.12)\) are satisfied, then for each \( \tau \in \mathbb{R}, \omega \in \Omega, \)
\[
\lim_{\varepsilon \to \varepsilon_0} \text{dist}_Y(A_\varepsilon(\tau, \omega), A_{\varepsilon_0}(\tau, \omega)) = 0.
\]

**Proof** If \((2.8)-(2.11)\) hold, the upper-continuous in \( X \) is proved in [18]. We only need to prove the upper semi-continuity of \( A_\varepsilon \) at \( \varepsilon = \varepsilon_0 \) in \( Y \).
Suppose that there exist $\delta > 0$, $\varepsilon_n \to \varepsilon_0$ and a sequence $\{y_n\}$ with $y_n \in \mathcal{A}_{\varepsilon_n}(\tau, \omega)$ such that for all $n \in \mathbb{N}$,
\[ \lim_{\varepsilon \to \varepsilon_0} \text{dist}_Y(y_n, \mathcal{A}_{\varepsilon_0}(\tau, \omega)) \geq 2\delta. \] (2.13)

Note that $y_n \in \mathcal{A}_{\varepsilon_n}(\tau, \omega) \subset \mathcal{A}(\tau, \omega) = \bigcup_{\varepsilon \in I} \mathcal{A}_\varepsilon(\tau, \omega)$. Then by (2.11) and (2.12) and using $(H1)$, there exists a $y_0 \in X \cap Y$ such that up to a subsequence,
\[ \lim_{n \to \infty} y_n \equiv y_0 \text{ in } X \text{ (resp. in } Y). \] (2.14)

It suffices to show that $\text{dist}_Y(y_0, \mathcal{A}_{\varepsilon_0}(\tau, \omega)) = 0$. Given a positive sequence $\{t_m\}$ with $t_m \uparrow +\infty$ as $m \to \infty$. For $m = 1$, by the invariance of $\mathcal{A}_{\varepsilon_n}$, there exists a sequence $\{y_{1,n}\}$ with $y_{1,n} \in \mathcal{A}_{\varepsilon_n}(\tau - t_1, \vartheta_{-t_1}(\omega), y_0)$ such that
\[ y_n = \varphi_{\varepsilon_n}(t_1, \tau - t_1, \vartheta_{-t_1}(\omega), y_{1,n}), \] (2.15)

for each $n \in \mathbb{N}$. Since $y_{1,n} \in \mathcal{A}_{\varepsilon_n}(\tau - t_1, \vartheta_{-t_1}(\omega) \subset \mathcal{A}(\tau - t_1, \vartheta_{-t_1}(\omega))$, then by (2.11) and (2.12) and using $(H2)$, there is a $z_1 \in X \cap Y$ and a subsequence of $\{y_{1,n}\}$ such that
\[ \lim_{n \to \infty} y_{1,n} = z_1 \text{ in } X \text{ (resp. in } Y). \] (2.16)

Then (2.8) and (2.16) together imply that
\[ \lim_{n \to \infty} \varphi_{\varepsilon_n}(t_1, \tau - t_1, \vartheta_{-t_1}(\omega), y_{1,n}) = \varphi_{\varepsilon_0}(t_1, \tau - t_1, \vartheta_{-t_1}(\omega), z_1) \text{ in } X. \] (2.17)

Thus combining (2.14), (2.15) and (2.17) we get that
\[ y_0 = \varphi_{\varepsilon_0}(t_1, \tau - t_1, \vartheta_{-t_1}(\omega), z_1). \] (2.18)

Note that $K_{\varepsilon_n}$ as a $\mathcal{D}_{\varepsilon_n}$-pullback absorbing set in $X$ absorbs $\mathcal{A}_{\varepsilon_n} \in \mathcal{D}_{\varepsilon_n}$, i.e., there is a $T = T(\tau, \omega, \mathcal{A}_{\varepsilon_n})$ such that for all $t \geq T$,
\[ \varphi(t, \tau - t, \vartheta_{-t}(\omega), \mathcal{A}_{\varepsilon_n}(\tau - t, \vartheta_{-t}(\omega)) \subset K_{\varepsilon_n}(\tau, \omega). \] (2.19)

Then by the invariance of $\mathcal{A}_{\varepsilon_n}(\tau, \omega)$, it follows from (2.11) that
\[ \mathcal{A}_{\varepsilon_n}(\tau, \omega) \subset K_{\varepsilon_n}(\tau, \omega). \] (2.20)

Since $y_{1,n} \in \mathcal{A}_{\varepsilon_n}(\tau - t_1, \vartheta_{-t_1}(\omega) \subset K_{\varepsilon_n}(\tau - t_1, \vartheta_{-t_1}(\omega)$, then by (2.16) and (2.18), we get that
\[ \|z_1\|_X = \limsup_{n \to \infty} \|y_{1,n}\|_X \leq \limsup_{n \to \infty} \|K_{\varepsilon_n}(\tau - t_1, \vartheta_{-t_1}(\omega)\|_X \leq R_{\varepsilon_0}(\tau - t_1, \vartheta_{-t_1}(\omega). \] (2.21)

By an induction argument, for each $m \geq 1$, there is $z_m \in X \cap Y$ such that for all $m \in \mathbb{N}$,
\[ y_0 = \varphi_{\varepsilon_0}(t_m, \tau - t_m, \vartheta_{-t_m}(\omega), z_m). \] (2.22)
and
\[ \|z_m\|_X \leq R_{\varepsilon_0}(\tau - t_m, \vartheta_{-t_m}(\omega). \] (2.23)
Thus from (2.9) and (2.23), for each $m \in \mathbb{N}$,
\[ z_m \in B_0(\tau - t_m, \vartheta_{-t_m}(\omega). \] (2.24)

Consider that the pullback attractor $\mathcal{A}_{\varepsilon_0}$ attracts every element in $\mathcal{D}_{\varepsilon_0}$ in the topology of $Y$ and connection with $B_0 \in \mathcal{D}_{\varepsilon_0}$. Then $\mathcal{A}_{\varepsilon_0}$ attracts $B_0$ in the topology of $Y$. Therefore by (2.22) and (2.24) we have
\[ \text{dist}_Y(y_0, \mathcal{A}_{\varepsilon_0}(\tau, \omega)) = \text{dist}_Y(\varphi_{\varepsilon_0}(t_m, \tau - t_m, \vartheta_{-t_m}(\omega), z_m), \mathcal{A}_{\varepsilon_0}(\tau, \omega)) \to 0, \]
as \( m \to \infty \). That is to say, \( \text{dist}_Y(y_0,A_{\epsilon_0}(\tau,\omega)) = \inf_{u \in A_{\epsilon_0}(\tau,\omega)} \|y_0 - u\|_Y = 0 \) and thus we can choose a \( u_0 \in A_{\epsilon_0}(\tau,\omega) \) such that
\[
\|y_0 - u_0\|_Y \leq \delta.
\] (2.25)
Therefore, by (2.13) and (2.25), as \( n \to \infty \),
\[
\text{dist}_Y(y_n,A_{\epsilon_0}(\tau,\omega)) \leq \|y_n - u_0\|_Y \leq \|y_n - y_0\|_Y + \delta \to \delta,
\]
which is a contradiction to (2.13). This concludes the proof. \( \Box \)

We next consider a special case of Theorem 2.7 above, in which case the limit cocycle \( \varphi_{\epsilon_0} \) is independent of the parameter \( \omega \in \Omega \). We call such \( \varphi_{\epsilon_0} \) a deterministic non-autonomous cocycle on \( X \) over \( \mathbb{R} \). This is, \( \varphi_{\epsilon_0} \) satisfies the following two statements:

(i) \( \varphi_0(0,\tau,.) \) is the identity on \( X \);
(ii) \( \varphi_0(t+s,\tau,.) = \varphi_0(t,\tau+s,.) \circ \varphi_0(s,\tau,.) \).

If \( \varphi_0(t,\tau,.) : X \to X \) is continuous, then \( \varphi_{\epsilon_0} \) is called a deterministic non-autonomous continuous cocycle on \( X \) over \( \mathbb{R} \).

Let \( \mathcal{D}_{\epsilon_0} \) be a collection of some families of nonempty subsets of \( X \) denoted by
\[
\mathcal{D}_{\epsilon_0} = \{ B = \{ B(\tau) \neq \emptyset ; B(\tau) \in 2^X , \tau \in \mathbb{R} \} \}.
\]
A family \( A_{\epsilon_0} \in \mathcal{D}_{\epsilon_0} \) is called a \( \mathcal{D}_{\epsilon_0} \)-pullback attractor of \( \varphi_{\epsilon_0} \) in \( X \) (resp. in \( Y \)) if
(i) for each \( \tau \in \mathbb{R} \), \( A_{\epsilon_0}(\tau) \) is compact in \( X \) (resp. of \( Y \));
(ii) \( A_{\epsilon_0}(t,\tau,A_{\epsilon_0}(\tau)) = A_{\epsilon_0}(\tau + t) \) for all \( t \in \mathbb{R}^+ \) and \( \tau \in \mathbb{R} \);
(iii) \( A_{\epsilon_0} \) pullback attracts every element of \( \mathcal{D}_{\epsilon_0} \) under the Hausdorff semi-metric of \( X \) (resp. of \( Y \)).

In order to obtain the convergence at \( \epsilon = \epsilon_0 \) in \( Y \), we make some modifications of the conditions used in random case. We assume that for every \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega, \epsilon_n \in I \) with \( \epsilon_n \to \epsilon_0 \), and \( x_n, x \in X \) with \( x_n \to x \), there holds
\[
\lim_{n \to \infty} \varphi_{\epsilon_n}(t,\tau - t, \partial_{-t} \omega, x_n) = \varphi_{\epsilon_0}(t,\tau - t, x) \quad \text{in } X. \tag{2.26}
\]
There exists a map \( R'_{\epsilon_0} : \mathbb{R} \to \mathbb{R} \) such that the family
\[
\mathcal{B}'_0 = \{ B'_0(\tau) = \{ x \in X ; \| x \|_X \leq R'_{\epsilon_0}(\tau) \} ; \tau \in \mathbb{R} \} \text{ belongs to } \mathcal{D}_{\epsilon_0}. \tag{2.27}
\]
For every \( \epsilon \in I \), \( \varphi_{\epsilon} \) has a closed measurable \( \mathcal{D}_{\epsilon} \)-pullback absorbing set \( K_\epsilon = \{ K_\epsilon(\tau,\omega) ; \omega \in \Omega \} \in \mathcal{D}_\epsilon \) in \( X \) such that for every \( \tau \in \mathbb{R}, \omega \in \Omega \),
\[
\lim_{\epsilon \to \epsilon_0} \sup_{\epsilon \in I} \| K_\epsilon(\tau,\omega) \| \leq R'_{\epsilon_0}(\tau). \tag{2.28}
\]
Then we have the following, which can be proved by a similar argument as Theorem 2.7 and so the proof is omitted.

**Theorem 2.8.** If (2.11) and (2.20)-(2.28) hold, then for each \( \tau \in \mathbb{R}, \omega \in \Omega \),
\[
\lim_{\epsilon \to \epsilon_0} \text{dist}_X(A_\epsilon(\tau,\omega),A_{\epsilon_0}(\tau)) = 0.
\]
If further (H1)-(H2) hold and conditions (2.12) and (2.20)-(2.28) are satisfied, then for each \( \tau \in \mathbb{R}, \omega \in \Omega \),
\[
\lim_{\epsilon \to \epsilon_0} \text{dist}_Y(A_\epsilon(\tau,\omega),A_{\epsilon_0}(\tau)) = 0.
\]
3. Non-autonomous reaction-diffusion equation on $\mathbb{R}^N$ with multiplicative noise

For the non-autonomous reaction-diffusion equation (1.1)-(1.2), the nonlinearity $f(x, s)$ satisfies almost the same assumptions as [18], i.e., for $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$,

$$f(x, s)s \leq -\alpha_1 |s|^p + \psi_1(x),$$

$$|f(x, s)| \leq \alpha_2 |s|^{p-1} + \psi_2(x),$$

$$\frac{\partial f}{\partial s}(f(x, s)) \leq \alpha_3,$$

where $\alpha_i > 0 (i = 1, 2, 3)$ are determined constants, $p \geq 2$, $\psi_1 \in L^1(\mathbb{R}^N) \cap L^{p/2}(\mathbb{R}^N)$, $\psi_2 \in L^2(\mathbb{R}^N)$ and $\psi_3 \in L^2(\mathbb{R}^N)$. And the non-autonomous term $g$ satisfies that for every $\tau \in \mathbb{R}$ and some $\delta \in [0, \lambda)$,

$$\int_{-\infty}^\tau e^{\delta s} \|g(s, \cdot)\|_{L^2(\mathbb{R}^N)}^2 ds < +\infty,$$

where $\lambda$ is as in [18], which implies that

$$\int_{-\infty}^0 e^{\delta s} \|g(s + \tau, \cdot)\|_{L^2(\mathbb{R}^N)}^2 ds < +\infty, \quad g \in L^1_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^N)).$$

For the probability space $(\Omega, \mathcal{F}, P)$, we write $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}); \omega(0) = 0\}$. Let $\mathcal{F}$ be the Borel $\sigma$-algebra induced by the compact-open topology of $\Omega$ and $P$ be the corresponding Wiener measure on $(\Omega, \mathcal{F})$. We define a shift operator $\vartheta$ on $\Omega$ by

$$\vartheta_\omega(s) = \omega(s + t) - \omega(t), \quad \text{for every } \omega \in \Omega, t, s \in \mathbb{R}.$$  

Then $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t \in \mathbb{R}})$ is a measurable dynamical system. By the law of the iterated logarithm (see [7]), we know that

$$\frac{\omega(t)}{t} \to 0, \quad \text{as } |t| \to +\infty.$$

For $\omega \in \Omega$, set $z(t, \omega) = z_\tau(t, \omega) = e^{-\tau \omega(t)}$. Then we have $dz + \varepsilon z \circ d\omega(t) = 0$. Put $v(t, \tau, \omega, v_0) = z(t, \omega)u(t, \tau, \omega, u_0)$, where $u$ is a solution of problem (1.1)-(1.2) with initial $u_0$. Then $v$ solves the follow non-autonomous equation

$$\frac{dv}{dt} + \lambda v - \Delta v = z(t, \omega)f(x, z^{-1}(t, \omega)v) + z(t, \omega)g(t, x),$$

with initial condition

$$v(\tau, x) = v_0(x) = z(\tau, \omega)u_0(x).$$

As pointed out in [18], for every $v_0 \in L^2(\mathbb{R}^N)$ we may show that the problem (3.8)-(3.9) possesses a continuous solution $v(\cdot)$ on $L^2(\mathbb{R}^N)$ such that $v(\cdot) \in C([\tau, +\infty), L^2(\mathbb{R}^N)) \cap L^2_{\text{loc}}((\tau, +\infty), H^1(\mathbb{R}^N)) \cap L^p_{\text{loc}}((\tau, +\infty), L^p(\mathbb{R}^N))$. In addition, the solution $v$ is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^N)))$-measurable in $\Omega$. Then formally $u(\cdot) = z^{-1}(\cdot, \omega)v(\cdot)$ is a $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^N)))$-measurable and continuous solution of problem (1.1)-(1.2) on $L^2(\mathbb{R}^N)$ with $u_0 = z^{-1}(\tau, \omega)v_0$.  


Define the mapping \( \varphi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N) \) such that
\[
\varphi(t, \tau, \omega, u_0) = u(t + \tau, \vartheta_{-\tau}\omega, u_0) = z^{-1}(t + \tau, \vartheta_{-\tau}\omega)v(t + \tau, \tau, \vartheta_{-\tau}\omega, z(\tau, \vartheta_{-\tau}\omega)u_0),
\]
for \( u_0 \in L^2(\mathbb{R}^N) \) and \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega \). Then by the measurability and continuity of \( v \) in \( v_0 \in L^2(\mathbb{R}^N) \) and \( t \in \mathbb{R}^+ \), we see that the mappings \( \varphi \) is \( (\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(L^2(\mathbb{R}^N))) \rightarrow \mathcal{B}(L^2(\mathbb{R}^N)) \)-measurable. That is to say, the mappings \( \varphi \) defined by (3.10) is a continuous cocycle on \( L^2(\mathbb{R}^N) \) over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t \in \mathbb{R}}) \). Furthermore, from (3.10) we infer that
\[
\varphi(t, \tau - t, \vartheta_{-\tau}\omega, u_0) = u(\tau, \tau - t, \vartheta_{-\tau}\omega, u_0) = z(-\tau, \omega)v(\tau, \tau - t, \vartheta_{-\tau}\omega, z(\tau - t, \vartheta_{-\tau}\omega)u_0).
\]

We define the collection \( \mathcal{D} \) as
\[
\mathcal{D} = \{ B = \{ B(\tau, \omega) \subseteq L^2(\mathbb{R}^N); \tau \in \mathbb{R}, \omega \in \Omega \}; \lim_{t \to -\infty} e^{-\delta t}||B(\tau - t, \vartheta_{-\tau}\omega)||_2 = 0 \text{ for } \tau \in \mathbb{R}, \omega \in \Omega, \delta < \lambda \} \quad (3.12)
\]
where \( ||B|| = \sup_{\tau \in B} ||v||_{L^2(\mathbb{R}^N)} \) and \( \lambda \) is in (3.8). Note that this collection \( \mathcal{D} \) is much larger than the collection defined by (3.8). That is to say, the collection \( \mathcal{D} \) defined above includes all tempered families of bounded nonempty subsets of \( L^2(\mathbb{R}^N) \).

We can show that all the results in (3.8) hold for this collection \( \mathcal{D} \) defined by (3.12). Thus, the existence and upper semi-continuous of \( \mathcal{D} \)-pullback attractors for the cocycle \( \varphi_\varepsilon \) in the initial space \( L^2(\mathbb{R}^N) \) have been proved by (3.8).

**Theorem 3.1 (3.8).** Assume that (3.7), (3.9) hold. Then the cocycle \( \varphi_\varepsilon \) has a unique \( \mathcal{D} \)-pullback attractor \( \mathcal{A}_\varepsilon = \{ \mathcal{A}_\varepsilon(\tau, \omega), \tau \in \mathbb{R}, \omega \in \Omega \} \) in \( L^2(\mathbb{R}^N) \), given by
\[
\mathcal{A}_\varepsilon(\tau, \omega) = \bigcap_{s > 0} \bigcup_{t > s} \varphi(t, \tau - t, \vartheta_{-\tau}\omega, K_\varepsilon(\tau - t, \vartheta_{-\tau}\omega)) \quad \text{in} \quad L^2(\mathbb{R}^N), \quad \tau \in \mathbb{R}, \omega \in \Omega, \quad (3.13)
\]
where \( K_\varepsilon \) is a closed and measurable \( \mathcal{D} \)-pullback absorbing set of \( \varphi_\varepsilon \) in \( L^2(\mathbb{R}^N) \). Furthermore, \( \mathcal{A}_\varepsilon \) is upper semi-continuous in \( L^2(\mathbb{R}^N) \) at \( \varepsilon = 0 \).

Note that in most cases, we write \( v \) (resp. \( \varphi \) and \( z \)) as the abbreviation of \( v_\varepsilon \) (resp. \( \varphi_\varepsilon \) and \( z_\varepsilon \)).

In the following, we consider the applications of Theorem 2.6-2.8 to the non-autonomous stochastic reaction-diffusion (1.1)-(1.2). We will strengthen the result of Theorem 3.1 holds in the smooth functions space \( H^1(\mathbb{R}^N) \). In particular, we prove the upper semi-continuity of the obtained attractors \( \mathcal{A}_\varepsilon \) in \( H^1(\mathbb{R}^N) \).

**4. Existence of pullback attractor in \( H^1(\mathbb{R}^N) \)**

In this section, we apply Theorem 2.6 to prove the existence of \( \mathcal{D} \)-pullback attractors in \( H^1(\mathbb{R}^N) \) for the cocycle defined in (3.10). To this end, we need to prove the uniform smallness of solutions outside a large ball under \( H^1(\mathbb{R}^N) \) norm (see Lemma 4.4), and in the bounded ball of \( \mathbb{R}^N \), we will prove the asymptotic compactness of solutions by a combined space splitting and function truncation technique (see Lemma 4.5 and Lemma 4.6).

Consider that \( e^{-|\omega(s)|} \leq z(s, \omega) = e^{-\varepsilon \omega(s)} \leq e^{\omega(s)} \) for \( \varepsilon \in (0, 1] \), and \( \omega(s) \) is continuous function in \( s \). Then there exist two positive random constants \( E = E(\omega) \)
and $F = F(\omega)$ depending only on $\omega$ such that for all $s \in [-1,0]$ and $\varepsilon \in (0,1]$.

\begin{equation}
0 < E \leq z(s,\omega) \leq F, \quad \omega \in \Omega.
\end{equation}

Hereafter, we denote by $\|\cdot\|$, $\|\cdot\|_p$, and $\|\cdot\|_{H^1}$ the norms in $L^2(\mathbb{R}^N)$, $L^p(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$, respectively. The number $c$ is a generic positive constant independent of $\tau, \omega, B$ and $\varepsilon$ in any place. We always assume $p > 2$ in the following discussions.

### 4.1. $H^1$-tail estimate of solutions.

This can be achieved by a series of previously proved lemmas. First we stress that Lemma 5.1 in [18] holds on the compact interval $[\tau - 1, \tau]$, which is necessary for us to estimate the tail of solutions in $H^1(\mathbb{R}^N)$.

**Lemma 4.1.** Assume that (3.1)-(3.3) hold. Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $B = \{B(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega\} \subset D$, then there exists a constant $T = T(\tau, \omega, B) \geq 2$ such that for all $t \geq T$, the solution $v$ of problem (3.8)-(3.9) satisfies that for every $\xi \in [\tau - 1, \tau]$,

\begin{equation}
\|v(\xi, t - \tau, \partial_{-\tau} v, \partial_{-\tau} \omega)u_0\|_{H^1(\mathbb{R}^N)}^2 \leq L_1(\tau, \omega, \varepsilon),
\end{equation}

\begin{equation}
\int_{\tau - t}^{\tau} e^{\lambda(s - \tau)} \left( \|v(s, t - \tau, \partial_{-\tau} v, v_0)\|_{H^1}^2 + z^{2-p}(s, \omega)\|v(s, \tau - \tau, \partial_{-\tau} v, v_0)\|_p^p \right) ds \leq L_1(\tau, \omega, \varepsilon),
\end{equation}

where $u_0 \in B(\tau - \tau, \partial_{-\tau} \omega)$ and $L_1(\tau, \omega, \varepsilon) = c z^{-2}(\tau, \omega) \int_0^{1/(\varepsilon^2 + 1)} e^{\lambda z} z^2(s, \omega)(\|g(s + \tau, .)\|^2 + 1) ds$.

**Proof** By (3.3), it is easy to calculate that

\begin{equation}
\frac{d}{dt} \|v\|^2 + \frac{3}{2} \lambda \|v\|^2 + \|\nabla v\|^2 + \alpha_1 z^{2-p}(t, \omega)\|v\|_p^p \leq c z^{2}(t, \omega)(\|g(t, .)\|^2 + \|\psi_1\|_1),
\end{equation}

to which we apply Gronwall’s lemma over the interval $[\tau - t, \xi]$, where $\xi \in [\tau - 1, \tau]$ and $t \geq 2$, we find that, along with $\omega$ replaced by $\partial_{-\tau} \omega$,

\begin{equation}
\|v(\xi, t - \tau, \partial_{-\tau} v, v_0)\|^2 + \frac{\lambda}{2} \int_{\tau - t}^{\xi} e^{\lambda(s - \xi)} \|v(s, t - \tau, \partial_{-\tau} v, v_0)\|^2 ds \\
+ \int_{\tau - t}^{\xi} e^{\lambda(s - \xi)} \left( \|\nabla v(s, t - \tau, \partial_{-\tau} \omega, v_0)\|^2 + \alpha_1 z^{2-p}(s, \partial_{-\tau} \omega)\|v(s, \tau - \tau, \partial_{-\tau} \omega, v_0)\|_p^p \right) ds \\
\leq e^{-\lambda(\xi - \tau + t)} z^{2}(\tau - t, \partial_{-\tau} \omega)\|u_0\|^2 + c \int_{\tau - t}^{\xi} e^{-\lambda(\xi - \sigma)} z^2(\sigma, \partial_{-\tau} \omega)(\|g(\sigma, .)\|^2 + \|\psi_1\|_1) d\sigma.
\end{equation}

If $\xi \in [\tau - 1, \tau]$, then

\begin{equation}
e^{-\lambda \tau} \leq e^{-\lambda \xi} \leq e^{-\lambda(\tau - 1)},
\end{equation}

and therefore (4.5) along with (4.3) implies that

\begin{equation}
\|v(\xi, t - \tau, \partial_{-\tau} v, v_0)\|^2 \\
+ \int_{\tau - t}^{\xi} e^{\lambda(s - \tau)} \left( h\|v(s, t - \tau, \partial_{-\tau} v, v_0)_H^2 + \alpha_1 z^{2-p}(s, \partial_{-\tau} \omega)\|v(s, \tau - \tau, \partial_{-\tau} \omega, v_0)\|_p^p \right) ds \\
\leq e^{\lambda z}(\tau - t, \partial_{-\tau} \omega)\|u_0\|^2 + ce^{\lambda} \int_{\tau - t}^{\tau} e^{-\lambda(\tau - \sigma)} z^2(\sigma, \partial_{-\tau} \omega)(\|g(\sigma, .)\|^2 + \|\psi_1\|_1) d\sigma.
\end{equation}
On the other hand, by (3.8), we deduce that
\[ t \text{ and } T \]
where we have used (4.6). Then by (4.13) and (4.11) it follows that there exists
\[ \xi \]
Then by (4.7)-(4.9), it implies that there exists
\[ T \]
for
\[ \tau \]
and
\[ h \leq 2 \int_{\xi}^{T} v \parallel \nabla z \parallel \delta \lambda t \]
\[ \parallel g(s + \tau, .) \parallel^2 + 1)ds, \]
and
\[ \int_{\tau-t}^{\xi} e^{\lambda(s-\tau)} \left( \left\| v(s, \tau - t, \vartheta_{-r} \omega, v_0) \right\|_{L_1}^2 + 2 - p(s, \omega) \right) \left\| v(s, \tau - t, \vartheta_{-r} \omega, v_0) \right\|_{p}^2 \right)ds \]
\[ \leq c \int_{-\infty}^{0} e^{\lambda\tau_{z}} z^2(s, \omega) (\| g(s + \tau, .) \| + 1)ds. \]

Then by (4.7)-(4.9), it implies that there exists \( T_1 = T_1(\tau, \omega, B) \geq 2 \) such that for all \( t \geq T_1 \) and \( \xi \in [\tau - 1, \tau] \),
\[ \| v(\xi, \tau - t, \vartheta_{-r} \omega, v_0) \|^2 \leq \int_{-\infty}^{0} e^{\lambda\tau_{z}} z^2(s, \omega) (\| g(s + \tau, .) \| + 1)ds, \]
and
\[ \int_{\tau-t}^{\xi} e^{\lambda(s-\tau)} \left( \| v(s, \tau - t, \vartheta_{-r} \omega, v_0) \|_{L_1}^2 + 2 - p(s, \omega) \right) \| v(s, \tau - t, \vartheta_{-r} \omega, v_0) \|_{p}^2 \right)ds \]
\[ \leq c \int_{-\infty}^{0} e^{\lambda\tau_{z}} z^2(s, \omega) (\| g(s + \tau, .) \| + 1)ds. \]

On the other hand, by (3.8), we deduce that
\[ \frac{d}{dt} \| \nabla v \|^2 + \lambda \| \nabla v \|^2 \leq c \| \nabla v \|^2 + z^2(t, \omega) (\| g(t, .) \|^2 + \| \psi_\lambda \|^2). \]

Note that \( \xi - \tau + t \geq t - 1 \geq 1 \) for \( \xi \in [\tau - 1, \tau] \) and \( t \geq 2 \). Then applying Lemma 5.1 in [29] over the interval \([\tau - t, \xi] \) for \( \xi \in [\tau - 1, \tau] \), we get that
\[ \| \nabla v(\xi, \tau - t, \vartheta_{-r} \omega, v_0) \|^2 \leq \int_{\tau-t}^{\tau} e^{\lambda(s-\tau)} \| \nabla v(s, \tau - t, \vartheta_{-r} \omega, v_0) \|^2 ds \]
\[ \leq c \int_{\tau-t}^{\tau} e^{\lambda(s-\tau)} \| \nabla v(s, \tau - t, \vartheta_{-r} \omega, v_0) \|^2 ds \]
\[ + c \int_{\tau-t}^{\tau} e^{\lambda(s-\tau)} \| \nabla v(s, \tau - t, \vartheta_{-r} \omega, v_0) \|^2 ds \]
\[ \leq c \int_{-\infty}^{0} e^{\lambda\tau_{z}} z^2(s, \omega) (\| g(s + \tau, .) \| + 1)ds, \]

where we have used (4.6). Then by (4.13) and (4.11) it follows that there exists \( T_2 = T_2(\tau, \omega, B) \geq 2 \) such that for all \( t \geq T_2 \),
\[ \| \nabla v(\xi, \tau - t, \vartheta_{-r} \omega, v_0) \|^2 \leq \int_{-\infty}^{0} e^{\lambda\tau_{z}} z^2(s, \omega) (\| g(s + \tau, .) \| + 1)ds, \]
which is finite for all $\xi \in [\tau - 1, \tau]$. Taking $T = \max\{T_1, T_2\}$, then for all $t \geq T$, (4.10) and (4.14) together imply the desired. \hfill \Box

Lemma 4.2. Assume that (3.1)-(3.2) hold. Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $B = \{B(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega\} \subset \mathcal{D}$, then for every $\epsilon > 0$, there exist two constants $T = T(\tau, \omega, \epsilon, B) \geq 2$ and $R = R(\tau, \omega, \epsilon) > 1$ such that the weak solution $v$ of problem (3.8)-(3.9) satisfies that for all $t \geq T$ and $k \geq R$,

\[
\int_{|x| \geq k} |v(\tau, t - \tau, \cdot - \tau \omega, z(\tau - t, \cdot - \tau \omega)u_0)|^2 dx \\
+ \int_{t - T}^T e^{\lambda(s - \tau)} \int_{|x| \geq k} |\nabla v(s, \tau - t, \cdot - \tau \omega, z(\tau - t, \cdot - \tau \omega)u_0)|^2 dxds \leq \epsilon,
\]

where $u_0 \in B(\tau - t, \cdot - \tau \omega)$, $R$ and $T$ are independent of $\epsilon$.

Proof. The proof is a simple modification of the proof of Lemma 5.5 in [18]. We first need to define a smooth function $\xi(.)$ on $\mathbb{R}^+$ such that

\[
\xi(s) = \begin{cases} 
0, & \text{if } 0 \leq s \leq 1, \\
0 \leq \xi(s) \leq 1, & \text{if } 1 \leq s \leq 2, \\
1, & \text{if } s \geq 2,
\end{cases}
\]  

(4.15)

which obviously implies that there is a positive constant $C_1$ such that the $|\xi'(s)| + |\xi''(s)| \leq C_1$ for all $s \geq 0$. For convenience, we write $\xi = \xi(\frac{|x|^2}{\lambda})$.

From (3.5), we know that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \xi |v|^2 dx + \lambda \int_{\mathbb{R}^N} \xi |v|^2 dx - \int_{\mathbb{R}^N} \xi \Delta v dx \\
= z(t, \omega) \int_{\mathbb{R}^N} f(x, u)v\xi dx + z(t, \omega) \int_{\mathbb{R}^N} gv\xi dx.
\]

(4.16)

By calculation, we have the following:

\[
\int_{\mathbb{R}^N} \xi \Delta v dx = \int_{\mathbb{R}^N} v(\nabla \xi, \nabla v) dx + \int_{\mathbb{R}^N} \xi |\nabla v|^2 dx
\]

\[
\geq - \int_{|x| \leq \sqrt{2}k} \xi \left| \frac{x}{k^2} \right| |v||\nabla v| dx + \int_{|x| \leq \sqrt{2}k} \xi |\nabla v|^2 dx \geq - \frac{c}{k^2} \|v\|_{H^1}^2 + \int_{\mathbb{R}^N} \xi |\nabla v|^2 dx,
\]

(4.17)

\[
z(t, \omega) \int_{\mathbb{R}^N} f(x, u)v\xi dx \leq -\alpha_1 z^2(t, \omega) \int_{\mathbb{R}^N} \xi |v|^p dx + z^2(t, \omega) \int_{\mathbb{R}^N} \xi \psi_1 dx,
\]

(4.18)

\[
\left| z(t, \omega) \int_{\mathbb{R}^N} gv\xi dx \right| \leq \frac{\lambda}{2} \int_{\mathbb{R}^N} \xi |v|^2 dx + \frac{1}{2\lambda} z^2(t, \omega) \int_{\mathbb{R}^N} \xi y^2 dx.
\]

(4.19)

Then combining (4.16)-(4.19) we get that

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \xi |v|^2 dx + \lambda \int_{\mathbb{R}^N} \xi |v|^2 dx + \int_{\mathbb{R}^N} \xi |\nabla v|^2 dx
\]
Applying the Gronwall lemma to (4.20) over \([\tau - t, \tau]\), we find that, along with \(\omega\) replaced by \(\vartheta - \tau\omega\),

\[
\int_{\mathbb{R}^N} \xi|v(\tau - t, \vartheta - \tau\omega, v_0)|^2 dx + \int_{\tau - t}^{\tau} e^{\lambda(s - \tau)} \int_{\mathbb{R}^N} \xi|\nabla v(\tau - t, \vartheta - \tau\omega, v_0)|^2 dx ds
\leq c z^{-2}(\tau, \omega) \int_{-\infty}^{0} e^{\lambda s} z^2(s, \omega) \int_{|x| \geq k} (|\psi_1| + |g(s + \tau, x)|^2) dx ds
+ \frac{c}{k} \int_{\tau - t}^{\tau} e^{\lambda(\tau - s)} \|v(s, \tau - t, \vartheta - \tau\omega, v_0)\|_{L^2_{\mathbb{H}}}^2 ds + z^2(-\tau, \omega) e^{-\lambda t} z^2(-\tau, \omega) u_0^2.
\]

(4.21)

According to Lemma 4.1, there exist \(T_1 = T_1(\tau, \omega, B) \geq 2\) and \(R_1 = R_1(\tau, \omega, \epsilon) > 2\) such that for all \(t \geq T_1\) and \(k \geq R_1\),

\[
\frac{c}{k} \int_{\tau - t}^{\tau} e^{\lambda(\tau - s)} \|v(s, \tau - t, \vartheta - \tau\omega, v_0)\|_{L^2_{\mathbb{H}}}^2 ds \leq \frac{c L(\tau, \omega, \epsilon)}{k} \leq \frac{\epsilon}{3}.
\]

(4.22)

On the other hand, for each \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(u_0 \in B(\tau - t, \vartheta - \tau\omega)\), by (4.19), there exists \(T_2 = T_2(\tau, \omega, B, \epsilon) > 0\) such that for all \(t \geq T_2\),

\[
z^2(-\tau, \omega) e^{-\lambda t} z^2(-\tau, \omega) u_0^2 \leq \frac{\epsilon}{3}.
\]

(4.23)

By (4.18), there exists a random variable \(a(\omega)\) depending only on \(\omega\) such that

\[
0 < e^{\lambda s} z^2(s, \omega) \leq a(\omega), \quad \text{for } s \in (-\infty, 0].
\]

Then by (4.16), we can deduce that for every \(\tau \in \mathbb{R}\),

\[
\int_{-\infty}^{0} e^{\lambda s} z^2(s, \omega) \int_{\mathbb{R}^N} |g(s + \tau, x)|^2 dx ds = \int_{-\infty}^{0} e^{\lambda s} z^2(s, \omega) e^{\delta s} \|g(s + \tau, .)\|_2^2 ds
\leq a(\omega) \int_{-\infty}^{0} e^{\delta s} \|g(s + \tau, .)\|_2^2 ds < +\infty,
\]

(4.24)

where \(\delta \in [0, \lambda]\). Then by (4.24) and \(\psi_1 \in L^1\), there exists \(R_2 = R_2(\tau, \omega, \epsilon)\) such that for all \(k \geq R_2\),

\[
c z^{-2}(\tau, \omega) \int_{-\infty}^{0} e^{\lambda s} z^2(s, \omega) \int_{|x| \geq k} (|\psi_1| + |g(s + \tau, x)|^2) dx ds \leq \frac{\epsilon}{3}.
\]

(4.25)

Given \(T = \max\{T_1, T_2\}\) and \(R = \max\{R_1, R_2\}\), then combining (4.22)-(4.23) and (4.25) into (4.24), we have for all \(t \geq T\) and \(k \geq R\),

\[
\int_{|x| \geq k} |v(\tau - t, \vartheta - \tau\omega, v_0)|^2 dx + \int_{\tau - t}^{\tau} e^{\lambda(s - \tau)} \int_{|x| \geq k} |\nabla v(\tau - t, \vartheta - \tau\omega, v_0)|^2 dx ds \leq \epsilon.
\]

Then the desired result follows. \(\square\)
Lemma 4.3. Assume that (4.1)-(4.3) hold. Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $B = \{B(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, then there exists $T = T(\tau, \omega, B) \geq 2$ such that the weak solution $v$ of problem (3.8)-(3.9) satisfies that for all $t \geq T$,

$$\int_{\tau-\epsilon}^{\tau} e^{(s-\tau)z^{-2p}(s, \vartheta^{-\tau}(\omega))} \left| \frac{\partial v}{\partial t}(s, \tau, \vartheta^{-\tau}(\omega), \vartheta^{-\tau}(\omega)u_0) \right|^2 \mathrm{d}s \leq L_2(\tau, \omega, \epsilon),$$

$$\int_{\tau-\epsilon}^{\tau} e^{(s-\tau)\|v_s(s, \tau, \vartheta^{-\tau}(\omega), \vartheta^{-\tau}(\omega)u_0)\|^2} \mathrm{d}s \leq L_3(\tau, \omega, \epsilon),$$

where $v_s = \frac{\partial v}{\partial s}$, $u_0 \in B(\tau - t, \vartheta^{-\tau}(\omega))$ and

$$L_2(\tau, \omega, \epsilon) = cz^{-2}(\tau, \omega)F^{2-p}(b(\omega)) \int_{-\infty}^{0} e^{\lambda s z^2(s, \omega)(\|g(s, \tau, .)\|^2 + 1)} \mathrm{d}s + \int_{-\infty}^{0} e^{\lambda s z^2(s, \omega)(\|g(s, \tau, .)\|^2 + 1)} \mathrm{d}s,$$

$$L_3(\tau, \omega, \epsilon) = c(F^{2-p}b(\omega) + 1)L_1(\tau, \omega, \epsilon) + cz^{-2}(\tau, \omega)F^{2-p} \int_{-\infty}^{0} e^{\lambda s z^2(s, \omega)(\|g(s, \tau, .)\|^2 + 1)} \mathrm{d}s,$$

where $F$ is as in (4.1), $L_1(\tau, \omega, \epsilon)$ is as in (4.2) and $b(\omega)$ is as in (4.3).

Proof. We multiply (3.8) by $|v|^{p-2}v$ and then integrate over $\mathbb{R}^N$ to yield that

$$\frac{1}{p} \frac{d}{dt} \|v\|_p^p + \lambda \|v\|_p^p \leq z(t, \omega) \int_{\mathbb{R}^N} f(x, z^{-1}v) |v|^{p-2}v \mathrm{d}x + z(t, \omega) \int_{\mathbb{R}^N} |v|^{p-2}vg \mathrm{d}x. \quad (4.26)$$

By using (5.1), we see that

$$z(t, \omega) \int_{\mathbb{R}^N} f(x, z^{-1}v) |v|^{p-2}v \mathrm{d}x \leq -\alpha_1 z^{-2-p}(t, \omega) \int_{\mathbb{R}^N} |v|^{2p-2}\mathrm{d}x + z^2(t, \omega) \int_{\mathbb{R}^N} \psi_1(x) |v|^{p-2} \mathrm{d}x$$

$$\leq -\alpha_1 z^{-2-p}(t, \omega) \int_{\mathbb{R}^N} |v|^{2p-2}\mathrm{d}x + z^2(t, \omega) \int_{\mathbb{R}^N} \psi_1(x) |v|^{p/2} \mathrm{d}x. \quad (4.27)$$

At the same time, the last term on the right hand side of (4.26) is bounded by

$$z(t, \omega) \int_{\mathbb{R}^N} |v|^{p-2}vg \mathrm{d}x \leq \frac{1}{2} \alpha_1 z^{-2-p}(t, \omega) \int_{\mathbb{R}^N} |v|^{2p-2}\mathrm{d}x + cz^2(t, \omega) \|g(t, .)\|^2. \quad (4.28)$$

Combination (4.26)-(4.28), we obtain that

$$\frac{d}{dt} \|v\|_p^p + 2\lambda \|v\|_p^p + \alpha_1 z^{-2-p}(t, \omega) \|v\|_{2p-2}^2 \leq cz^2(t, \omega) \|g(t, .)\|^2 + 1. \quad (4.29)$$

Applying Lemma 5.1 in [26] over $[\tau - t, \xi]$ for $\xi \in [\tau - 1, \tau]$ and $t \geq 2$, along with $\omega$ replaced by $\vartheta^{-1}(\omega)$, we deduce that

$$\|v(\xi, \tau - t, \vartheta^{-1}(\omega), v_0)\|_p^p \leq c \int_{\tau-\xi}^{\tau} e^{2\lambda(s-\xi)} \|v(s, \tau - t, \vartheta^{-1}(\omega), v_0)\|_p^p \mathrm{d}s$$

$$+ cz^{-2}(\tau, \omega) \int_{-\infty}^{0} e^{\lambda s z^2(s, \omega)(\|g(s + \tau, .)\|^2 + 1)} \mathrm{d}s. \quad (4.30)$$
where we have used (4.6) and \( \xi - \tau + t \geq t - 1 \geq 1 \) for \( \xi \in [\tau - 1, \tau] \) and \( t \geq 2 \). On the other hand, we see that

\[
\int_{\tau-t}^{\tau} e^{2\lambda(s-\tau)}\|v(s, \tau - t, \vartheta_{-\tau}\omega, \nu_0)\|^p ds
\]

\[
= z^{-p}(-\tau, \omega) \int_{\tau-t}^{\tau} e^{2\lambda(s-\tau)} z^{-2p}(s-\tau, \omega) z^{-p}(s, \vartheta_{-\tau}\omega) \|v(s, \tau - t, \vartheta_{-\tau}\omega, \nu_0)\|^p ds.
\]

(4.31)

Consider that when \( s \to -\infty \), \( e^{\lambda s z^p}(s, \omega) \) and \( e^{\lambda s z^2}(s, \omega) \to 0 \). Then there exists a variable \( b(\omega) \) depending only on \( \omega \) such that

\[
0 < e^{\lambda s z^{p-2}}(s, \omega) + e^{\lambda s z^2}(s, \omega) \leq b(\omega), \quad s \in (-\infty, 0].
\]

(4.32)

from which and (4.31), association with Lemma 4.1, it follows that there exists \( T = T(\tau, \omega, B) \geq 2 \) such that for all \( t \geq T \),

\[
\int_{\tau-t}^{\tau} e^{2\lambda(s-\tau)}\|v(s, \tau - t, \vartheta_{-\tau}\omega, \nu_0)\|^p ds
\]

\[
\leq b(\omega) z^{-p}(-\tau, \omega) \int_{\tau-t}^{\tau} e^{\lambda(s-\tau)} z^{-2p}(s, \vartheta_{-\tau}\omega) \|v(s, \tau - t, \vartheta_{-\tau}\omega, \nu_0)\|^p ds
\]

\[
\leq c b(\omega) z^{-p}(-\tau, \omega) \int_{-\infty}^{0} e^{\lambda s z^2}(s, \omega)(\|g(s + \tau, .)\|^2 + 1) ds.
\]

(4.33)

Then by (4.30) and (4.33) we get that for all \( t \geq T \) and \( \xi \in [\tau - 1, \tau] \),

\[
\|v(\xi, \tau - t, \vartheta_{-\tau}\omega, \nu_0)\|^p \leq c z^{-p}(-\tau, \omega) \left( b(\omega) \int_{-\infty}^{0} e^{\lambda s z^2}(s, \omega)(\|g(s + \tau, .)\|^2 + 1) ds + \int_{-\infty}^{0} e^{\lambda s z^p}(s, \omega)(\|g(s + \tau, .)\|^2 + 1) ds \right).
\]

(4.34)

In (4.29), omitting the number 2 of the second term on the left hand side, we multiply (4.29) by \( e^{\lambda(t-\tau)} \) and then integrate (w.r.t \( t \)) from \( [\tau - 1, \tau] \) to yield that, along with \( \omega \) replaced by \( \vartheta_{-\tau}\omega \),

\[
\int_{\tau-1}^{\tau} e^{\lambda(s-\tau)} z^{-2p}(s, \vartheta_{-\tau}\omega) \|v(s, \tau - t, \vartheta_{-\tau}\omega, \nu_0)\|^{2p-2} ds \leq e^{-\lambda} \|v(\tau - 1, \tau - t, \vartheta_{-\tau}\omega, \nu_0)\|^p
\]

\[
+ c \int_{\tau-1}^{\tau} e^{\lambda(s-\tau)} z^p(s, \vartheta_{-\tau}\omega)(\|g(s, .)\|^2 + 1) ds.
\]

(4.35)

Then combination (4.34) and (4.35), we deduce that for all \( t \geq T \),

\[
\int_{\tau-1}^{\tau} e^{\lambda(s-\tau)} z^{-2p}(s, \vartheta_{-\tau}\omega) \|v(s, \tau - t, \vartheta_{-\tau}\omega, \nu_0)\|^{2p-2} ds
\]

\[
\leq c z^{-p}(-\tau, \omega) \left( b(\omega) \int_{-\infty}^{0} e^{\lambda s z^2}(s, \omega)(\|g(s + \tau, .)\|^2 + 1) ds + \int_{-\infty}^{0} e^{\lambda s z^p}(s, \omega)(\|g(s + \tau, .)\|^2 + 1) ds \right)
\]

from which and (4.1) it follows that for all \( t \geq T \),

\[
e^{-\lambda} \int_{\tau-1}^{\tau} e^{\lambda(s-\tau)} z^{2p-2p}(s, \vartheta_{-\tau}\omega) \|v(s, \tau - t, \vartheta_{-\tau}\omega, \nu_0)\|^{2p-2} ds
\]
\[
\leq \int_{\tau-1}^{\tau} e^{2\lambda(s-\tau)} z^{4-2p}(s, \vartheta_{-\tau}\omega) v(s, \tau-t, \vartheta_{-\tau}\omega, v_0) 2^{2p-2} ds \\
= z^{2p-2}(s, \omega) \int_{\tau-1}^{\tau} e^{2\lambda(s-\tau)} z^{2-p}(s-\tau, \omega) z^{2-2p}(s, \vartheta_{-\tau}\omega) v(s) 2^{2p-2} ds \\
\leq z^{2p-2}(s, \omega) F^{2-p} \int_{\tau-1}^{\tau} e^{\lambda(s-\tau)} z^{2-2p}(s, \vartheta_{-\tau}\omega) v(s) 2^{2p-2} ds \\
\leq cz^{2p-2}(s, \omega) F^{2-p} \left(h(\omega) \int_{-\infty}^{0} e^{\lambda s} z^2(s, \omega)(\|g(s, \tau, .)\|^2 + 1) ds \right) \\
+ \int_{-\infty}^{0} e^{\lambda s} z^p(s, \omega)(\|g(s, \tau, .)\|^2 + 1) ds \right). \quad (4.36)
\]

For the estimate of the derivative \(v_t\) in \(L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^N))\), we multiply \(3.8\) by \(v_t\) and integrate over \(\mathbb{R}^N\) to produce that
\[
\|v_t\|^2 + \frac{1}{2} \frac{d}{dt}(\lambda\|v\|^2 + \|\nabla v\|^2) \\
= z(t, \omega) \int_{\mathbb{R}^N} f(x, z^{-1} v) v_t dx + z(t, \omega) \int_{\mathbb{R}^N} g v_t dx \\
\leq \frac{1}{2} \|v_t\|^2 + 2\alpha z^{4-2p}(t, \omega)\|v\|^{2-p} z^2(t, \omega)\|\psi_2\|^2 + z^2(t, \omega)\|g(t, .)\|^2.
\]
i.e., we have
\[
\|v_t\|^2 + \frac{d}{dt}(\lambda\|v\|^2 + \|\nabla v\|^2) + \lambda(\lambda\|v\|^2 + \|\nabla v\|^2) \\
\leq cz^{4-2p}(t, \omega)\|v\|^{2-p} z^2(t, \omega)(\|g(t, .)\|^2 + \|\psi_2\|^2) \\
+ \lambda(\lambda\|v\|^2 + \|\nabla v\|^2). \quad (4.37)
\]
Multiplying \(4.37\) by \(e^{\lambda(t-\tau)}\) then integrating about \(t\) over \([\tau-1, \tau]\), it gives us that, together with \(v_t\) replaced by \(\vartheta_{-\tau}\omega\),
\[
\int_{\tau-1}^{\tau} e^{\lambda(s-\tau)} \|v_s(s, \tau-t, \vartheta_{-\tau}\omega, v_0)\|^2 ds \\
\leq c \int_{\tau-1}^{\tau} e^{\lambda(s-\tau)} z^{4-2p}(s, \vartheta_{-\tau}\omega) v(s, \tau-t, \vartheta_{-\tau}\omega, v_0) 2^{2p-2} ds \\
+ c \int_{\tau-1}^{\tau} e^{\lambda(s-\tau)} \|v(s, \tau-t, \vartheta_{-\tau}\omega, v_0)\|^2 ds \\
+ c \int_{\tau-1}^{\tau} e^{\lambda s} z^2(s, \vartheta_{-\tau})(\|g(s, .)\|^2 + 1) ds \\
+ c \|v(\tau-1, \tau-t, \vartheta_{-\tau}\omega, v_0)\|^2_{H^1}. \quad (4.38)
\]
Then by applying Lemma 4.1 and connection with \(4.36\) and \(4.38\), we deduce that there exists \(T = T(\tau, \omega, B) \geq 2\) such that for all \(t \geq T\),
\[
\int_{\tau-1}^{\tau} e^{\lambda(s-\tau)} \|v_s(s, \tau-t, \vartheta_{-\tau}\omega, v_0)\|^2 ds \\
\leq c(F^{2-p} b(\omega) + 1) L_1(\tau, \omega, \varepsilon) + cz^{2}(-\tau, \omega) F^{2-p} \int_{-\infty}^{0} e^{\lambda s} z^p(s, \omega)(\|g(s, \tau, .)\|^2 + 1) ds\).
This completes the proof.

We now can prove the $H^1$-tail estimate of solutions of problem (3.8)-(3.9), which is one crucial condition for proving the asymptotic compactness in $H^1(\mathbb{R}^N)$.

**Lemma 4.4.** Assume that (3.1)-(3.3) hold. Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $B = \{B(\tau, \omega) ; \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, then for every $\epsilon > 0$, there exist two constants $T = T(\tau, \omega, \epsilon, B) \geq 2$ and $R = R(\tau, \omega, \epsilon) > 1$ such that the weak solution $v$ of problem (3.8)-(3.9) satisfies that for all $t \geq T$,

$$\int_{|x| \geq R} \left( |v(\tau-t, \vartheta-\tau\omega, z(\tau-t, \vartheta-\tau\omega)u_0)|^2 + |\nabla v(\tau-t, \vartheta-\tau\omega, z(\tau-t, \vartheta-\tau\omega)u_0)|^2 \right) dx \leq \epsilon,$$

where $u_0 \in B(\tau-t, \vartheta-\tau\omega)$ and $R, T$ are independent of $\epsilon$.

**Proof.** Given $\xi$ being defined in (4.15), we multiply (3.8) by $-\xi \Delta v$ and integrate over $\mathbb{R}^N$ to find that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} (\nabla \xi, \nabla v)v_t dx + \lambda \int_{\mathbb{R}^N} |\xi \nabla v|^2 dx + \int_{\mathbb{R}^N} (\xi \nabla v) v dx + \int_{\mathbb{R}^N} \xi |\Delta v|^2 dx = -z(t, \omega) \int_{\mathbb{R}^N} f(x, z^{-1}v) \xi \nabla v dx - z(t, \omega) \int_{\mathbb{R}^N} g \xi \Delta v dx. \tag{4.39}$$

Now, we estimate each term in (4.39) as follows. First we have

$$\left| \int_{\mathbb{R}^N} (\nabla \xi, \nabla v)v_t dx + \lambda \int_{\mathbb{R}^N} (\nabla \xi, \nabla v) v dx \right| = \left| \int_{\mathbb{R}^N} (v_t + \lambda v)(\frac{2x}{k^2} \nabla v) \xi dx \right| \leq \frac{C}{k} (\|v_t\|^2 + \|v\|^2_{H^1}), \tag{4.40}$$

where and in the following the constant $c$ is independent of $k$ and $\epsilon$. For the nonlinearity in (4.39), we see that

$$-z \int_{\mathbb{R}^N} f(x, z^{-1}v) \xi \Delta v dx = z \int_{\mathbb{R}^N} f(x, z^{-1}v)(\nabla \xi, \nabla v) dx + z \int_{\mathbb{R}^N} (\frac{\partial}{\partial x} f(x, z^{-1}v, \nabla v) \xi dx$$

$$+ \int_{\mathbb{R}^N} \frac{\partial}{\partial u} f(x, z^{-1}v) |\nabla v|^2 \xi dx. \tag{4.41}$$

On the other hand, by using (6.2), (6.3) and (6.4), respectively, we calculate that

$$\left| z \int_{\mathbb{R}^N} f(x, z^{-1}v)(\nabla \xi, \nabla v) dx \right| \leq \frac{\sqrt{2}C_1}{k} \int_{k \leq |x| \leq \sqrt{2}k} |f(x, z^{-1}v)||\nabla v| dx$$

$$\leq \frac{C}{k} (z^{4-2p} \|v\|_{2p-2}^{2p-2} + z^2 \|\psi_2\|^2 + \|\nabla v\|^2), \tag{4.42}$$
For the last term on the right hand side of (4.39), we have
and

\[
-\omega \tau \mathcal{N} \leq \int R \int \xi |\nabla v|^2 dx,
\]

and

\[
\left| \int \frac{\partial}{\partial x} f(x, z^{-1}v) \nabla v \xi dx \right| \leq \int |\psi_3||\nabla v| \xi dx
\]

\[
\leq \frac{\lambda}{2} \int |\nabla v|^2 dx + cz^2 \int \xi |\psi_3|^2 dx.
\]

Then it follows from (4.41)-(4.44) that

\[
-z \int f(x, z^{-1}v) \xi \Delta v dx \leq \frac{c_k}{z^4-2p} \|v\|^2 + z^2 \|\psi_2\|^2 + \|\nabla v\|^2
\]

\[
+ \frac{\lambda}{2} \int |\nabla v|^2 dx + cz^2 \int \xi |\psi_3|^2 dx + \alpha_3 \int \xi |\nabla v|^2 dx.
\]

(4.45)

For the last term on the right hand side of (4.39), we have

\[
\left| \int \frac{\partial}{\partial x} g \xi \Delta v dx \right| \leq \int |\Delta v|^2 dx + \frac{1}{2\lambda} \int \xi |g|^2 dx.
\]

(4.46)

Then we incorporate (4.40) and (4.45)-(4.46) into (4.39) to find that

\[
\frac{d}{dt} \int |\nabla v|^2 dx + \lambda \int |\nabla v|^2 dx \leq \frac{c_k}{z^4-2p} \|v\|^2 + \|v\|^2 + z^2 \|\psi_2\|^2
\]

\[
+ 2\alpha_3 \int |\nabla v|^2 dx + cz^2 \int \xi (|\psi_3|^2 + |g|^2) dx.
\]

(4.47)

Applying Lemma 5.1 in [26] to (4.47) over \( [\tau - 1, \tau] \), we find that, along with \( \omega \) replaced by \( \vartheta - \omega \),

\[
\int R^N \xi |\nabla v(\tau, \tau - t, \vartheta - \omega, v_0)|^2 dx
\]

\[
\leq \frac{c_k}{z^4} \int_{\tau - 1}^\tau e^{\lambda(s - \tau)} \left( \|v(s)\|^2 + \|v(s)\|^2 + z^4 - 2p \|v(s)\|^2 + z^2 \|\psi_2\|^2 \right)
\]

\[
+ z^2 (s, \vartheta - \omega) \|\psi_2\|^2 dx + c \int_{\tau - 1/2}^\tau e^{\lambda(s - \tau)} \int |\nabla v(s)|^2 dx
\]

\[
+ cz^2 (\tau, \omega) \int_{-\infty}^0 e^{\lambda s} z^2 (s, \omega) \int_{|x| \geq k} (|\psi_3|^2 + |g(s + \tau, x)|^2) dx ds,
\]

(4.48)

where \( v(s) = v(s, \tau - t, \vartheta - \omega, z(\tau - t, \vartheta - \omega)v_0) \). Our task in the following is to show that each term on the right hand side of (4.48) vanishes. First, by Lemma
It is obvious that for all \( t \geq T_1 \) and \( k \geq R_1 \),
\[
c \int_{t-1}^{t} e^{\lambda(s-\tau)} \int_{|x| \geq k} |\nabla v(s, \tau-t, \vartheta_{-\tau} \omega, v_0)|^2 dx ds \leq \frac{\epsilon}{6}.
\] (4.49)

By Lemma 4.1, it follows that there exist \( T_2 = T_2(\tau, \omega, B) \geq 1 \) and \( R_2 = R_2(\tau, \omega, \epsilon) \geq 2 \) such that for all \( t \geq T_2 \) and \( k \geq R_2 \),
\[
c \int_{t-1}^{t} e^{\lambda(s-\tau)} \int_{|x| \geq k} |v(s, \tau-t, \vartheta_{-\tau} \omega, v_0)|^2 dx ds \leq \frac{\epsilon}{6}.
\] (4.50)

By Lemma 4.3, there exist \( T_3 = T_3(\tau, \omega, B) \geq 2 \) and \( R_3 = R_3(\tau, \omega, \epsilon) \geq 2 \) such that for all \( t \geq T_3 \) and \( k \geq R_3 \),
\[
c \int_{t-1}^{t} e^{\lambda(s-\tau)} z^4 \|v(s, \tau-t, \vartheta_{-\tau} \omega, v_0)\|_{L^2}^2 ds \leq \frac{c}{k} L_3(\tau, \omega, \epsilon) \leq \frac{\epsilon}{6}.
\] (4.51)

and
\[
c \int_{t-1}^{t} e^{\lambda(s-\tau)} \|v(s, \tau-t, \vartheta_{-\tau} \omega, v_0)\|_{L^2}^2 ds \leq \frac{c}{k} L_3(\tau, \omega, \epsilon) \leq \frac{\epsilon}{6}.
\] (4.52)

Similar to (4.25), we deduce that there exist \( R_4 = R_4(\tau, \omega, \epsilon) \) such that for all \( k \geq R_4 \),
\[
c \int_{t-1}^{t} e^{\lambda(s-\tau)} z^2(s, \omega) \int_{|x| \geq k} (|\psi_3|^2 + |g(s + \tau, x)|^2) dx ds \leq \frac{\epsilon}{6}.
\] (4.53)

Obviously, there exists \( R_5 = R_5(\tau, \omega, \epsilon) \) such that t for all \( k \geq R_5 \),
\[
c \int_{t-1}^{t} e^{\lambda(s-\tau)} z^2(s, \vartheta_{-\tau} \omega) \psi_2^2 ds \leq \frac{c}{k} \|\psi_2\|_{L^2}^2 \int_{\infty}^{0} e^{\lambda s} z^2(s, \omega) ds \leq \frac{\epsilon}{6},
\] (4.54)

where \( \int_{\infty}^{0} e^{\lambda s} z^2(s, \omega) ds < +\infty \). Finally, take
\[
T = \{T_1, T_2, T_3\}, \quad R = \max\{R_1, R_2, R_3, R_4, R_5\}.
\]

It is obvious that \( R \) and \( T \) are independent of the intension \( \epsilon \). Then we combine (4.49), (4.53) into (4.48) to get that for all \( t \geq T \) and \( k \geq R \),
\[
\int_{|x| \geq \epsilon T k} |\nabla v(\tau, \tau-t, \vartheta_{-\tau} \omega, v_0)|^2 dx \leq \epsilon.
\]

Then connection with Lemma 4.2, the desired result is achieved. \( \square \)

4.2. **Estimate of the truncation of solutions in** \( L^{2p-2} \). Given \( u \) the solution of problem (1.1) - (1.2), for each fixed \( \tau \in \mathbb{R}, \omega \in \Omega \), we write \( M = M(\tau, \omega) > 1 \) and
\[
\mathbb{R}^N (|u(\tau, \tau-t, \vartheta_{-\tau} \omega, u_0)| \geq M) = \{x \in \mathbb{R}^N; |u(\tau, \tau-t, \vartheta_{-\tau} \omega, u_0)| \geq M\}.
\]

We introduce the truncation version of solutions of problem (3.8) - (3.9). Let \((v-M)_+\) be the positive part of \( v-M \), i.e.,
\[
(v-M)_+ = \begin{cases} \quad v-M, & \text{if } v > M; \\ \quad 0, & \text{if } v \leq M. \end{cases}
\]
The next lemma show that the absolute value $|u|$ vanishes in $L^{2p-2}$-norm on the state domain $\mathbb{R}^N(|u(\tau, \tau - t, \vartheta_\tau, \omega), u_0| \geq M)$ for $M$ large enough, which is the second crucial condition for proving the asymptotic compactness of solutions in $H^1(\mathbb{R}^N)$.

**Lemma 4.5.** Assume that (3.1)-(3.3) hold. Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $B = \{B(\tau, \omega)\}; \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, then for any $\eta > 0$, there exist constants $M = M(\tau, \omega, \eta, B) > 1$ and $T = T(\tau, \omega, B) \geq 2$ such that the solution $u_\varepsilon$ of problem (1.1)-(1.2) satisfies that for all $t \geq T$ and all $\varepsilon \in (0, 1]$ and $u_0 \in B(\tau - t, \vartheta_\tau, \omega)$,

$$\int_{\tau - 1}^T \epsilon^\alpha e^{(s-\tau)} \int_{\mathbb{R}^N} |v(s, \tau - t, \vartheta_\tau, \omega, z(\tau - t, \vartheta_\tau, \omega)u_0)|^{2p-2} dx ds \leq \eta,$$

where $p > 2$ and $M, T$ are independent of $\varepsilon$ and

$$q = \rho(\tau, \omega, M) = \alpha_1 E^{2-p} e^{-(p-2)|\omega(\tau)|} M^{p-2}.$$

**Proof** First, we replace $\omega$ by $\vartheta_\tau$ in (3.8) and see that

$$v = v(s) := v(s, \tau - t, \vartheta_\tau, \omega, v_0), \quad s \in [\tau - 1, \tau],$$

is a solution of the following SPDE,

$$\frac{dv}{ds} + \lambda v - \Delta v = \frac{z(s - \tau, \omega)}{z(-\tau, \omega)} f(x, u) + \frac{z(s - \tau, \omega)}{z(-\tau, \omega)} g(s, x), \quad (4.55)$$

with the initial data $v_0 = z(\tau - t, \vartheta_\tau, \omega)u_0$ and $u_0 \in B(\tau - t, \vartheta_\tau, \omega)$.

We multiply (4.55) by $(v - M)^{p-1}_+$ and integrate over $\mathbb{R}^N$ to get that for every $s \in [\tau - 1, \tau],

$$\frac{1}{p} \int_{\mathbb{R}^N} (v - M)^p_+ dx + \lambda \int_{\mathbb{R}^N} v(v - M)^{p-1}_+ dx - \int_{\mathbb{R}^N} \Delta v(v - M)^{p-1}_+ dx = \frac{z(s - \tau, \omega)}{z(-\tau, \omega)} \int_{\mathbb{R}^N} f(x, u)(v - M)^{p-1}_+ dx + \frac{z(s - \tau, \omega)}{z(-\tau, \omega)} \int_{\mathbb{R}^N} g(s, x)(v - M)^{p-1}_+ dx.$$

(4.56)

We now have to estimate every term in (4.56). First, it is obvious that

$$- \int_{\mathbb{R}^N} \Delta v(v - M)^{p-1}_+ dx = (p - 1) \int_{\mathbb{R}^N} (v - M)^{p-2}_+ |\nabla v|^2 dx \geq 0,$$

(4.57)

$$\lambda \int_{\mathbb{R}^N} v(v - M)^{p-1}_+ dx \geq \lambda \int_{\mathbb{R}^N} (v - M)^p_+ dx.$$

(4.58)

If $v > M$, then $u = z^{-1}(s, \vartheta_\tau, \omega) v > 0$, and thus by (3.1) and (3.2), we find that for every $s \in [\tau - 1, \tau],

$$f(x, u) \leq -\alpha_1 \left( \frac{z(s - \tau, \omega)}{z(-\tau, \omega)} \right)^{1-p} |v|^{p-1} + \frac{z(s - \tau, \omega)}{z(-\tau, \omega)} \psi_1(x) v$$

$$\leq -\frac{1}{2} \alpha_1 \left( \frac{E}{z(-\tau, \omega)} \right)^{1-p} M^{p-2}(v - M) - \frac{1}{2} \alpha_1 \left( \frac{E}{z(-\tau, \omega)} \right)^{1-p} (v - M)^{p-1} + \frac{F}{z(-\tau, \omega)} |\psi_1(x)| (v - M)^{-1},$$

by which we find that

$$\frac{z(s - \tau, \omega)}{z(-\tau, \omega)} \int_{\mathbb{R}^N} f(x, u)(v - M)^{p-1}_+ dx$$
The second term on the right hand side of (4.56) is estimated as

\[
\frac{F}{z(-\tau, \omega)} \left| \int_{\mathbb{R}^N} g(s, x)(v(s) - M)^{p-1} \, dx \right| \leq \frac{1}{4} \alpha_1 \left( \frac{E}{z(-\tau, \omega)} \right)^{2-p} \int_{\mathbb{R}^N} (v - M)^{2p-2} \, dx + \frac{1}{4} \alpha_1 \left( \frac{E}{z(-\tau, \omega)} \right)^{2-p} \int_{\mathbb{R}^N} (v - M)^{2p-2} \, dx
\]

\[
\leq \frac{1}{4} \alpha_1 \left( \frac{E}{z(-\tau, \omega)} \right)^{2-p} \int_{\mathbb{R}^N} (v - M)^{2p-2} \, dx + \frac{1}{4} \alpha_1 \left( \frac{F}{z(-\tau, \omega)} \right)^{p} \int_{\mathbb{R}^N(v(s) \geq M)} \left| \psi_1(x) \right|^{p/2} \, dx.
\]

(4.59)

Combination (4.56)-(4.60), we obtain that

\[
\frac{d}{ds} \int_{\mathbb{R}^N} (v(s) - M)^p \, dx + \alpha_1 \left( \frac{E}{z(-\tau, \omega)} \right)^{2-p} M^{p-2} \int_{\mathbb{R}^N} (v(s) - M)^p \, dx
\]

\[
\leq c \left( \frac{F}{z(-\tau, \omega)} \right)^{p} \left( \|g(s, \cdot)\|^2 + \|\psi_1\|_{p/2}^{p/2} \right).
\]

(4.61)

where the positive constant \( c \) is independent of \( \varepsilon, \tau, \omega \) and \( M \). Note that for each \( \tau \in \mathbb{R} \) and \( \varepsilon \in (0, 1] \),

\[
e^{-|\omega(-\tau)|} \leq z^{-1}(-\tau, \omega) = e^{e^{\omega(-\tau)}} \leq e^{\omega(-\tau)}.
\]

(4.62)

For convenience, we put

\[
\varrho = \varrho(\tau, \omega, M) = \alpha_1 E^{2-p} e^{-(p-2)\omega(-\tau) \varepsilon} M^{p-2}, \quad d = d(\tau, \omega) = \alpha_1 E^{2-p} e^{-(p-2)\omega(-\tau) \varepsilon} M^{p-2}
\]

Then (4.61) is rewrote as

\[
\frac{d}{ds} \int_{\mathbb{R}^N} (v(s) - M)^p \, dx + \varrho \int_{\mathbb{R}^N} (v(s) - M)^p \, dx + d \int_{\mathbb{R}^N} (v - M)^{2p-2} \, dx
\]

\[
\leq c \varrho^p e^{p|\omega(-\tau)|} \left( \|g(s, \cdot)\|^2 + 1 \right).
\]

(4.63)

where \( s \in [\tau - 1, \tau] \) and \( \varrho, E, F \) are independent of \( \varepsilon \). By using Lemma 5.1 in [20] to (4.63) over \( [\tau - 1, \tau] \), we find that

\[
d \int_{\tau - 1}^{\tau} \int_{\mathbb{R}^N} e^{\varrho(s-t)} (v(s) - M)^{2p-2} \, dx \, ds \leq \int_{\tau - 1}^{\tau} e^{\varrho(s-t)} \int_{\mathbb{R}^N} (v(s, \tau - t, \vartheta_{-\tau} \omega, \psi_0) - M)^p \, dx \, ds.
\]
\[ + eF^p e^{|\omega(-\tau)|} \int_{\tau-1}^\tau e^{(s-\tau)} \left( \|g(s,.)\|^2 + 1 \right) ds. \]

(4.64)

First by (4.34), there exists \( T = T(\tau, \omega, B) \geq 2 \) such that for all \( t \geq T \),
\[ \int_{\tau-1}^T e^{(s-\tau)} \int_{\mathbb{R}^N} \left( v(s, t\tau - \partial_{-\tau} \omega, v_0) - M \right)^p dx ds \leq N(\tau, \omega, \varepsilon) \frac{1}{\varrho} \to 0, \]
(4.65)
as \( \varrho \to +\infty \), where \( N(\tau, \omega, \varepsilon) \) is defined by the right hand side of (4.34). We then need to show the second term on the right hand side of (4.64) is also small as \( \varrho \to +\infty \). Indeed, choosing \( \varrho > 2 \) and taking \( \xi \in (0, 1) \), we have
\[ \int_{\tau-1}^T e^{(s-\tau)} \left( \|g(s,.)\|^2 + 1 \right) ds = \int_{\tau-1}^{\tau-\xi} e^{(s-\tau)} \left( \|g(s,.)\|^2 + 1 \right) ds + \int_{\tau-\xi}^T e^{(s-\tau)} \left( \|g(s,.)\|^2 + 1 \right) ds \]
\[ = e^{-\varepsilon \tau} \int_{\tau-1}^{\tau-\xi} e^{(s-\xi)} e^{\delta s} \left( \|g(s,.)\|^2 + 1 \right) ds + e^{-\varepsilon \tau} \int_{\tau-\xi}^T e^{\delta s} \left( \|g(s,.)\|^2 + 1 \right) ds \]
\[ \leq e^{-\varepsilon \tau} e^{\delta (\xi - \tau)} \int_{-\infty}^{\tau-\xi} e^{\delta s} \left( \|g(s,.)\|^2 + 1 \right) ds + \int_{\tau-\xi}^T \left( g(s,.) \right)^2 + 1 ds. \]

By (3.5), the first term above vanishes as \( \varrho \to +\infty \), and by \( g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^N)) \) we can choose \( \xi \) small enough such that the second term is small. Then when \( \varrho \to +\infty \), we have
\[ eF^p e^{|\omega(-\tau)|} \int_{\tau-1}^\tau e^{(s-\tau)} \left( \|g(s,.)\|^2 + 1 \right) ds \to 0. \]
(4.66)

Since if \( M \to +\infty \), then \( \varrho \to +\infty \), so by (4.64)-(4.66), we know that for \( M \to +\infty \),
\[ \int_{\tau-1}^T e^{(s-\tau)} \int_{\mathbb{R}^N} (v(s) - M)^{2p-2} dx ds \to 0. \]
(4.67)

Note that \( v - M \geq \frac{2}{\varrho} \) for \( \varrho \geq 2M \). Then by (4.67) it gives that
\[ \int_{\tau-1}^T e^{(s-\tau)} \int_{\mathbb{R}^N(v(s) \geq -2M)} |v(s)|^{2p-2} dx ds \to 0, \]
as \( M \to +\infty \). By a similar argument, we can show that there exists \( T = T(\tau, \omega, B) \geq 2 \) such that for all \( t \geq T \),
\[ \int_{\tau-1}^T e^{(s-\tau)} \int_{\mathbb{R}^N(v(s) \leq 2M)} |v(s)|^{2p-2} dx ds \to 0, \]
as \( M \to +\infty \). Then we finish the total proof. \( \square \)

4.3. Asymptotic compactness on bounded domains. In this subsection, by using Lemma 4.5, we prove the asymptotic compactness of the cocyle \( \varphi \) defined by (3.10) in \( H^1_0(\mathcal{O}_R) \) for any \( R > 0 \), where \( \mathcal{O}_R = \{ x \in \mathbb{R}^N ; |x| \leq R \} \). For this purpose, we define \( \phi(.) = 1 - \xi(.) \), where \( \xi \) is the cut-off function as in (4.15). Then we know that \( 0 \leq \phi(s) \leq 1 \), and \( \phi(s) = 1 \) if \( s \in [0, 1] \) and \( \phi(s) = 0 \) if \( s \geq 2 \). Fix a positive constant \( k \), we define
\[ \tilde{v}(t, \tau, \omega, v_0) = \phi(\frac{x^2}{k}) v(t, \tau, \omega, v_0), \quad \tilde{u}(t, \tau, \omega, u_0) = \phi(\frac{x^2}{k}) u(t, \tau, \omega, u_0), \]
(4.68)
where \( v \) is the solution of problem \((3.8)-(3.9)\) and \( u \) is the solution of problem \((1.1)-(1.2)\) with \( v = z(t,\omega)u \). Then we have

\[
\hat{u}(t,\tau,\omega,u_0) = z^{-1}(t,\omega)\hat{v}(t,\tau,\omega,v_0).
\]

(4.69)

It is obvious that \( \hat{v} \) solves the following equations:

\[
\begin{cases}
\hat{v}_t + \lambda \hat{v} - \Delta \hat{v} = \phi z f(x, z^{-1}v) + \phi z g - v \Delta \phi - 2\nabla \phi \cdot \nabla v, \\
\hat{v}|_{\partial \Omega_{k,\sqrt{T}}} = 0, \\
\hat{v}(\tau,x) = \hat{v}_0(x) = \phi u_0(x),
\end{cases}
\]

(4.70)

where \( \phi = \phi(z^2) \).

It is well-known that the eigenvalue problem on bounded domains \( \Omega_{k,\sqrt{T}} \) with Dirichlet boundary condition:

\[
\begin{cases}
-\Delta v = \lambda v, \\
v|_{\partial \Omega_{k,\sqrt{T}}} = 0
\end{cases}
\]

has a family of orthogonal eigenfunctions \( \{e_j\}_{j=1}^{+\infty} \) in both \( L^2(\Omega_{k,\sqrt{T}}) \) and \( H^1(\Omega_{k,\sqrt{T}}) \) such that the corresponding eigenvalue \( \{\lambda_j\}_{j=1}^{+\infty} \) is non-decreasing in \( j \).

Let \( H_m = \text{Span}\{e_1, e_2, ..., e_m\} \subset H^1_0(\Omega_{k,\sqrt{T}}) \) and \( P_m : H^1_0(\Omega_{k,\sqrt{T}}) \rightarrow H_m \) be the canonical projector and \( I \) be the identity. Then for every \( \hat{u} \in H^1_0(\Omega_{k,\sqrt{T}}) \), \( \hat{u} \) has a unique decomposition: \( \hat{u} = \hat{u}_1 + \hat{u}_2 \), where \( \hat{u}_1 = P_m \hat{u} \in H_m \) and \( \hat{u}_2 = (I - P_m)\hat{u} \in H^1_m \), i.e., \( H^1_0(\Omega_{k,\sqrt{T}}) = H_m \oplus H^1_m \).

**Lemma 4.6.** Assume that \((3.7), (3.8)\) hold. Given \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( B = \{B(\tau,\omega); \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \), then for every \( \epsilon > 0 \), there are \( N_0 = N_0(\tau,\omega,k,\epsilon) \in \mathbb{Z}^+ \) and \( T = T(\tau,\omega,k,\epsilon) \geq 2 \) such that for all \( t \geq T \) and \( m > N_0 \),

\[
||(I - P_m)\hat{u}(\tau,\tau - t, \tau - t, \omega, u_0)||_{H^1_0(\Omega_{k,\sqrt{T}})} \leq \epsilon,
\]

where \( \hat{u}_0 = \phi u_0 \) with \( u_0 \in B(\tau - t, \tau - \tau \omega) \). Here \( \hat{u} \) is as in \((4.69)\) and \( N, T \) are independent of \( \epsilon \).

**Proof** By \((4.69)\), we start at the estimate of \( \hat{v} \). For \( \hat{v} \in H^1_0(\Omega_{k,\sqrt{T}}) \), we write \( \hat{v} = \hat{v}_1 + \hat{v}_2 \) where \( \hat{v}_1 = P_m \hat{v} \) and \( \hat{v}_2 = (I - P_m)\hat{v} \). Then naturally, we have a splitting about \( \hat{u} = \hat{u}_1 + \hat{u}_2 \) where \( \hat{u}_1 = P_m \hat{u} \) and \( \hat{u}_2 = (I - P_m)\hat{u} \). Multiplying \((4.47)\) by \( \Delta \hat{v}_2 \) we get that

\[
\frac{d}{dt}||\nabla \hat{v}_2||^2_{L^2(\Omega_{k,\sqrt{T}})} + \lambda ||\nabla \hat{v}_2||^2_{L^2(\Omega_{k,\sqrt{T}})} + ||\Delta \hat{v}_2||^2_{L^2(\Omega_{k,\sqrt{T}})} = -z \int_{\Omega_{k,\sqrt{T}}} \phi f(x, z^{-1}v) \Delta \hat{v}_2 dx + \int_{\Omega_{k,\sqrt{T}}} (\phi z g - v \Delta \phi - 2\nabla \phi \cdot \nabla v) \Delta \hat{v}_2 dx.
\]

(4.71)

By \((3.2)\), we deduce that

\[
z \int_{\Omega_{k,\sqrt{T}}} \phi f(x, z^{-1}v) \Delta \hat{v}_2 dx \leq \frac{1}{4} ||\Delta \hat{v}_2||^2_{L^2(\Omega_{k,\sqrt{T}})} + cz^{4-2p}||v||^2_{L^{2p-2}(\Omega_{k,\sqrt{T}})} + z^2||\psi_2||^2.
\]

(4.72)
On the other hand,
\[
\int_{\mathcal{O}_{h,\mathcal{F}}} (\phi z g - v \Delta \phi - 2 \nabla \phi \nabla v) \Delta \tilde{v} dx \leq \frac{1}{4} \| \Delta \tilde{v} \|^2_{L^2(\mathcal{O}_{h,\mathcal{F}})} + c(\tau^2 \| g \|^2 + \| v \|^2 + \| \nabla v \|^2).
\]
(4.73)

Then by (4.71)-(4.73) we find that
\[
\frac{d}{dt} \| \nabla \tilde{v}_2 \|^2_{L^2(\mathcal{O}_{h,\mathcal{F}})} + \| \Delta \tilde{v}_2 \|^2_{L^2(\mathcal{O}_{h,\mathcal{F}})} \leq c(z^{4-2p} \| v \|_{L^{2p-2}(\mathcal{O}_{h,\mathcal{F}})}^{2p-2} + \| \psi_2 \|^2 + \| g \|^2 + \| v \|^2_{H^1}).
\]
from which and connection with the Poincaré’s inequality
\[
\| \Delta \tilde{v}_2 \|^2_{L^2(\mathcal{O}_{h,\mathcal{F}})} \geq \lambda_{m+1} \| \nabla \tilde{v}_2 \|^2_{L^2(\mathcal{O}_{h,\mathcal{F}})},
\]
it follows that
\[
\frac{d}{dt} \| \nabla \tilde{v}_2 \|^2_{L^2(\mathcal{O}_{h,\mathcal{F}})} + \lambda_{m+1} \| \nabla \tilde{v}_2 \|^2_{L^2(\mathcal{O}_{h,\mathcal{F}})} \leq c(z^{4-2p} \| v \|_{L^{2p-2}(\mathcal{O}_{h,\mathcal{F}})}^{2p-2} + \| \psi_2 \|^2 + \| g \|^2 + \| v \|^2_{H^1}).
\]
(4.74)

Applying Lemma 5.1 in [26] to $I_{\tau-1}$ over the interval $[\tau-1, \tau]$, we find that, along with $\omega$ replaced by $\vartheta_{\tau, \omega}$,
\[
\| \nabla \tilde{v}_2(\tau, \tau - t, \vartheta_{\tau, \omega}, \tilde{v}_0) \|^2_{L^2(\mathcal{O}_{h,\mathcal{F}})} \\
\leq \int_{\tau-1}^\tau e^{\lambda_m(s-\tau)} \| \nabla \tilde{v}_2(s, \tau - t, \vartheta_{\tau, \omega}, \tilde{v}_0) \|^2_{L^2(\mathcal{O}_{h,\mathcal{F}})} ds \\
+ c \int_{\tau-1}^\tau e^{\lambda_m(s-\tau)} z^{4-2p}(s, \vartheta_{\tau, \omega}) \| v(s, \tau - t, \vartheta_{\tau, \omega}, \tilde{v}_0) \|_{L^{2p-2}(\mathcal{O}_{h,\mathcal{F}})}^{2p-2} ds \\
+ c \int_{\tau-1}^\tau e^{\lambda_m(s-\tau)} (z^2(s, \vartheta_{\tau, \omega}) \| \psi_2 \|^2 + z^2(s, \omega) \| g(s, \cdot) \|^2) ds \\
+ c \int_{\tau-1}^\tau e^{\lambda_m(s-\tau)} \| v(s, \tau - t, \vartheta_{\tau, \omega}, \tilde{v}_0) \|^2_{H^1} ds \\
\leq + c \int_{\tau-1}^\tau e^{\lambda_m(s-\tau)} z^{4-2p}(s, \vartheta_{\tau, \omega}) \| v(s, \tau - t, \vartheta_{\tau, \omega}, \tilde{v}_0) \|_{L^{2p-2}(\mathcal{O}_{h,\mathcal{F}})}^{2p-2} ds \\
+ \int_{\tau-1}^\tau e^{\lambda_m(s-\tau)} \| v(s, \tau - t, \vartheta_{\tau, \omega}, \tilde{v}_0) \|^2_{H^1} ds \\
+ c \int_{\tau-1}^\tau e^{\lambda_m(s-\tau)} z^2(s, \vartheta_{\tau, \omega}) (\| g(s, \cdot) \|^2 + 1) ds \\
= I_1 + I_2 + I_3.
\]
(4.75)

We next show that $I_1$, $I_2$ and $I_3$ converge to zero as $m$ increases to infinite. First we have
\[
I_1 = z^{2p-4}(\tau, \omega) \int_{\tau-1}^\tau e^{\lambda_m(s-\tau)} z^{4-2p}(s - \tau, \omega) \| v(s, \tau - t, \vartheta_{\tau, \omega}, \tilde{v}_0) \|_{L^{2p-2}(\mathcal{O}_{h,\mathcal{F}})}^{2p-2} ds \\
\leq z^{2p-4}(\tau, \omega) F^{4-2p} \int_{\tau-1}^\tau e^{\lambda_m(s-\tau)} \| v(s, \tau - t, \vartheta_{\tau, \omega}, \tilde{v}_0) \|_{L^{2p-2}(\mathcal{O}_{h,\mathcal{F}})}^{2p-2} ds \\
\leq z^{2p-4}(\tau, \omega) F^{4-2p} \int_{\tau-1}^\tau e^{\lambda_m(s-\tau)} \int_{\mathcal{O}_{h,\mathcal{F}} \{ v(s) \geq M \}} \| v(s, \tau - t, \vartheta_{\tau, \omega}, \tilde{v}_0) \|_{L^{2p-2}}^{2p-2} ds dx ds
\]
By Lemma 4.5 there exist $T_1 = T_1(\tau, \omega, B) \geq 2$, $M = M(\tau, \omega, B)$ such that for all $t \geq T_1$,  
\[ \int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)} \int_{\mathcal{O}_{k\sqrt{\tau}}(|v(s)| \leq M)} |v(s, \tau - t, \vartheta_{-\tau} \omega, \tilde{v}_0)|^{2p-2} dx ds \leq \epsilon. \]  
But $\lambda_{m+1} \to +\infty$, then there exists $N' = N'(\tau, \omega) > 0$ such that for all $m > N'$, $\lambda_{m+1} > \varrho$. Hence by (4.77) it gives us that for all $t \geq T_1$ and $m > N'$ there holds  
\[ \int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)} \int_{\mathcal{O}_{k\sqrt{\tau}}(|v(s)| \geq M)} |v(s, \tau - t, \vartheta_{-\tau} \omega, \tilde{v}_0)|^{2p-2} dx ds \leq \epsilon. \]  
For the second term on the right hand side of (4.76), since $\mathcal{O}_{k\sqrt{\tau}}(|v(s)| \leq M)$ is a bounded domain, then there exists $N'' = N''(\tau, \omega) > 0$ such that for all $m > N''$,  
\[ \int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)} \int_{\mathcal{O}_{k\sqrt{\tau}}(|v(s)| \leq M)} |v(s, \tau - t, \vartheta_{-\tau} \omega, \tilde{v}_0)|^{2p-2} dx ds \leq \frac{1}{\lambda_{m+1}} M^{2p-2} \mu(\mathcal{O}_{k\sqrt{\tau}}(|v(s)| \leq M)) \leq \epsilon, \]  
where $\mu(\mathcal{O}_{k\sqrt{\tau}}(|v(s)| \leq M))$ is the finite measure of the bounded domain $\mathcal{O}_{k\sqrt{\tau}}(|v(s)| \leq M)$. Put $N_1 = \max\{N', N''\}$. It follows from (4.76), (4.79) that for all $m > N_1$ and $t \geq T_1$,  
\[ I_1 \leq C_1(\tau, \omega) \epsilon. \]  
By Lemma 4.1, there exists $T_2 = T_2(\tau, \omega)$ and $N_2 = N_2(\tau, \omega) > 0$ such that for all $m > N_2$ and $t \geq T_2$,  
\[ I_2 \leq \frac{L_1(\tau, \omega, \epsilon)}{\lambda_{m+1}} \leq \epsilon. \]  
By a same technique as (4.66), we can show that there exists $N_3 = N_3(\tau, \omega) > 0$ such that for all $m > N_3$,  
\[ I_3 = \int_{\tau-1}^{\tau} e^{\lambda_{m+1}(s-\tau)} z^2(s, \vartheta_{-\tau} \omega) \left( \|g(s, \cdot)\|^2 + 1 \right) ds \leq \epsilon. \]  
Let $N_0 = \max\{N_1, N_2, N_3\}$ and $T = \max\{T_1, T_2\}$. Then combination (4.75) and (4.80), (4.82), we get that there exists a finite constant $\mu = \mu(\tau, \omega) > 0$ such that for all $m > N_0$ and $t \geq T$,  
\[ \|\nabla \tilde{v}_2(\tau, \tau - t, \vartheta_{-\tau} \omega, \tilde{v}_0)\|_{L^2(\mathcal{O}_{k\sqrt{\tau}})} \leq C_1(\tau, \omega) \epsilon. \]  
Then by (4.11) and (4.83), we have  
\[ \|\nabla \tilde{v}_2(\tau, \tau - t, \vartheta_{-\tau} \omega, \tilde{u}_0)\|_{L^2(\mathcal{O}_{k\sqrt{\tau}})} \leq z(\tau, \omega) \|\nabla \tilde{v}_2(\tau, \tau - t, \vartheta_{-\tau} \omega, \tilde{v}_0)\|_{L^2(\mathcal{O}_{k\sqrt{\tau}})} \leq C_2(\tau, \omega) \epsilon, \]  
for all $m > N_0$ and $t \geq T$, which completes the proof. \[ \square \]

Lemma 4.7. Assume that (3.71), (3.72) hold. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, then for every $k > 0$, the sequence $\{\tilde{u}(\tau, \tau - n\omega, \vartheta_{-\tau} \omega, \tilde{v}(\frac{2}{\sqrt{k}})u_{0,n})\}_{n=1}^{\infty}$ has a convergent subsequence in $H_0^1(\mathcal{O}_{k\sqrt{\tau}})$ whenever $t_n \to +\infty$ and $u_{0,n} \in B(\tau - t_n, \vartheta_{-\tau} \omega)$. 


Proof. Given $\epsilon > 0$, by Lemma 4.6, there exist $N_0 \in \mathbb{Z}^+$ such that as $t_n \to +\infty$

$\|(I - P_{N_0})\tilde{u}(\tau, \tau - t_n, \vartheta - \tau\omega, \phi(\frac{x^2}{R^2})u_{0,n})\|_{H^1(\mathcal{O}_{k,v})} \leq \epsilon. \quad (4.84)$

By Lemma 4.1, we deduce that if $t_n$ large enough,

$\|P_{N_0}\tilde{u}(\tau, \tau - t_n, \vartheta - \tau\omega, \phi(\frac{x^2}{R^2})u_{0,n})\|_{H^1(\mathcal{O}_{k,v})} \leq L_1(\tau, \omega, \epsilon). \quad (4.85)$

Note that $H^1(\mathcal{O}_{k,v}) = P_{N_0}H^1(\mathcal{O}_{k,v}) + (I - P_{N_0})H^1(\mathcal{O}_{k,v})$, but $P_{N_0}H^1(\mathcal{O}_{k,v})$ is a finite dimensional space. Then by (4.85), if $n, m$ large enough,

$\|P_{N_0}\tilde{u}(\tau, \tau - t_n, \vartheta - \tau\omega, \phi(\frac{x^2}{R^2})u_{0,n}) - P_{N_0}\tilde{u}(\tau, \tau - t_m, \vartheta - \tau\omega, \phi(\frac{x^2}{R^2})u_{0,m})\|_{H^1(\mathcal{O}_{k,v})} \leq \epsilon. \quad (4.86)$

Then it is easy to finish the proof by means of (4.84) and (4.85) and a standard argument. \qed

4.4. Existence of pullback attractor in $H^1(\mathbb{R}^N)$. In this subsection, we prove the existences of pullback attractors in $H^1(\mathbb{R}^N)$ for problem (1.1)-\( (1.2) \) for every $\epsilon \in (0, 1]$.

Lemma 4.8. Assume that (8.1)-(8.3) hold. Then the cocycle $\varphi$ defined by (3.11) is asymptotically compact in $H^1(\mathbb{R}^N)$, i.e., for every $\tau \in \mathbb{R}, \omega \in \Omega$, the sequence $\{\varphi(t, \tau - t_n, \vartheta - \tau\omega, u_{0,n})\}_{n=1}^{\infty}$ has a convergent subsequence in $H^1(\mathbb{R}^N)$ whenever $t_n \to +\infty$ and $u_{0,n} \in B = B(\tau - t_n, \vartheta - t_n\omega)$ with $B \in \mathcal{D}$.

Proof. Given $R > 0$, denote by $\mathcal{O}_R^c = \mathbb{R}^N - \mathcal{O}_R$, where $\mathcal{O}_R = \{x \in \mathbb{R}^N; |x| \leq R\}$. By Lemma 4.4, for any $\epsilon > 0$, there exist $\epsilon = \epsilon(\tau, \omega, \epsilon) > 0$ and $N_1 = N_1(\tau, \omega, B, \epsilon) \in \mathbb{Z}^+$ such that for all $n \geq N_1$,

$\|v(\tau, \tau - t_n, \vartheta - \tau\omega, z(\tau - t_n, \vartheta - \tau\omega)u_{0,n})\|_{H^1(\mathcal{O}_R^c)} \leq \frac{\epsilon}{8}e^{-|\omega(\tau)|}, \quad (4.87)$

for every $u_{0,n} \in B = B(\tau - t_n, \vartheta - t_n\omega)$. By (3.11) and (4.87), we have

$\|u(\tau, \tau - t_n, \vartheta - \tau\omega, z(\tau - t_n, \vartheta - \tau\omega)u_{0,n})\|_{H^1(\mathcal{O}_R^c)} \leq \frac{\epsilon}{8}. \quad (4.88)$

On the other hand, for this radius $R$, by Lemma 4.7, there exists $N_2 = N_2(\tau, \omega, B, \epsilon) \geq N_1$ such that for all $m, n \geq N_2$,

$\|u(\tau, \tau - t_n, \vartheta - \tau\omega, \phi(\frac{x^2}{R^2})u_{0,n}) - u(\tau, \tau - t_m, \vartheta - \tau\omega, \phi(\frac{x^2}{R^2})u_{0,m})\|_{H^1(\mathcal{O}_R^c)} \leq \frac{\epsilon}{8}, \quad (4.89)$

Then the desired result follows from (4.88) and (4.89) by a standard argument. \qed

Given $\epsilon \in (0, 1]$, by Lemma 4.1, we deduce that the $\mathcal{D}$-pullback absorbing set $K_\epsilon$ of $\varphi_\epsilon$ in $L^2(\mathbb{R}^N)$ is defined by

$K_\epsilon = \{K_\epsilon(\tau, \omega) = \{u \in L^2(\mathbb{R}^N); \|u\| \leq L_\epsilon(\tau, \omega)\}; \tau \in \mathbb{R}, \omega \in \Omega\}, \quad (4.90)$

where

$L_\epsilon(\tau, \omega, \epsilon) = c\left(\int_{-\infty}^{0} e^{\lambda s}e^{-2s\omega(s)}(\|g(s + \tau, \cdot)\|)^2 + 1\right)^{1/2}. \quad (4.91)$
By Lemma 4.8 and Theorem 2.6, we immediately have

**Theorem 4.9.** Assume that \((\text{3.1})-\text{(3.5)}\) hold. Then for every fixed \(\varepsilon \in (0,1]\), the cocycle \(\varphi_\varepsilon\) defined by \((\text{3.10})\) possesses a unique \(\mathcal{D}\)-pullback attractor \(A_{\varepsilon,H}\) in \(H^1(\mathbb{R}^N)\), given by

\[
A_{\varepsilon,H}(\tau,\omega) = \bigcup_{s>0} \bigcup_{t \geq s} \varphi_\varepsilon(t,\tau-t,\vartheta_{-t}\omega,K_\varepsilon(\tau-t,\vartheta_{-t}\omega))_{H^1(\mathbb{R}^N)}, \quad \tau \in \mathbb{R}, \omega \in \Omega.
\]

Furthermore, \(A_{\varepsilon,H}\) is consistent with the \(\mathcal{D}\)-pullback random attractor \(A_\varepsilon\) in \(L^2(\mathbb{R}^N)\), which is defined as in \((\text{3.13})\).

5. **Upper semi-continuity of pullback attractor in \(H^1(\mathbb{R}^N)\)**

From Theorem 4.9, for every \(\varepsilon \in (0,1]\), the cocycle \(\varphi_\varepsilon\) admits a common \(\mathcal{D}\)-pullback attractor \(A_\varepsilon\) in both \(L^2(\mathbb{R}^N)\) and \(H^1(\mathbb{R}^N)\), where \(\mathcal{D}\) is defined by \((\text{3.11})\). From this fact we may investigate the upper semi-continuity of \(A_\varepsilon\) in both \(L^2(\mathbb{R}^N)\) and \(H^1(\mathbb{R}^N)\). Note that \([13]\) only proved the upper semi-continuity in \(L^2(\mathbb{R}^N)\) at \(\varepsilon = 0\). In this section, we strengthen this study and prove that the upper semi-continuity of \(A_\varepsilon\) may happen in \(H^1(\mathbb{R}^N)\) at \(\varepsilon = 0\).

For the upper semi-continuity, we also give a further assumption as in \([13]\), that is, \(f\) satisfies that for all \(x \in \mathbb{R}^N\) and \(s \in \mathbb{R}\),

\[
\left| \frac{\partial}{\partial s} f(x,s) \right| \leq \alpha_4 |s|^{p-2} + \psi_4(x), \quad (5.1)
\]

where \(\alpha_4 > 0, \psi_4 \in L^\infty(\mathbb{R}^N)\) if \(p = 2\) and \(\psi_4 \in L^{\frac{2p}{p-2}}(\mathbb{R}^N)\) if \(p > 2\).

Let \(\varphi_0\) be the continuous cocycle associated with the problem \((\text{1.1})-\text{L.12})\) for \(\varepsilon = 0\). That is to say, \(\varphi_0\) is a deterministic non-autonomous cocycle over \(\mathbb{R}\). Denote by \(\mathcal{D}_0\) the collection of some families of deterministic noneempty subsets of \(L^2(\mathbb{R}^N)\):

\[
\mathcal{D}_0 = \{ B = \{ B(\tau) \subseteq L^2(\mathbb{R}^N); \tau \in \mathbb{R} \}; \lim_{t \to +\infty} e^{-\delta t} \| B(\tau-t) \| = 0, \tau \in \mathbb{R}, \delta < \lambda \},
\]

where \(\lambda\) is as in \((\text{3.8})\). As a special case of Theorem 4.9, under the assumptions \((\text{3.1})-\text{(3.5)}\), \(\varphi_0\) has a common \(\mathcal{D}_0\)-pullback attractor \(A_0 = \{ A_0(\tau); \tau \in \mathbb{R} \}\) in both \(L^\infty(\mathbb{R}^N)\) and \(H^1(\mathbb{R}^N)\).

To prove the upper semi-continuity of \(A_\varepsilon\) at \(\varepsilon = 0\), we have to check that the conditions \((\text{2.8})-\text{(2.12)}\) in Theorem 2.8 hold in \(L^2(\mathbb{R}^N)\) and \(H^1(\mathbb{R}^N)\) point by point. But \((\text{2.8})-\text{(2.11)}\) have been achieved, see Corollary 7.2, Lemma 7.5 and equality \((\text{7.31)}\) in \([13]\). We only need to prove the condition \((\text{2.12)}\) holds in \(H^1(\mathbb{R}^N)\).

**Lemma 5.1.** Assume that \((\text{3.1})-\text{(3.5)}\) hold. Then for every \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\), the union \(\bigcup_{\varepsilon \in (0,1]} A_\varepsilon(\tau,\omega)\) is precompact in \(H^1(\mathbb{R}^N)\).

**Proof** For any \(\varepsilon > 0\), it suffices to show that for every fixed \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\), the set \(\bigcup_{\varepsilon \in (0,1]} A_\varepsilon(\tau,\omega)\) has finite \(\varepsilon\)-nets in \(H^1(\mathbb{R}^N)\). Let \(\chi = \chi(\tau,\omega) \in \bigcup_{\varepsilon \in (0,1]} A_\varepsilon(\tau,\omega)\). Then there exists \(\varepsilon \in (0,1]\) such that \(\chi(\tau,\omega) \in A_\varepsilon(\tau,\omega)\). By the invariance of \(A_\varepsilon(\tau,\omega)\), it follows that there is a \(u_0 \in A_\varepsilon(\tau-t,\vartheta_{-t}\omega)\) such that

\[
\chi(\tau,\omega) = \varphi_\varepsilon(t,\tau-t,\vartheta_{-t}\omega,u_0) = u_\varepsilon(\tau-t,\vartheta_{-t}\omega,u_0) \quad (\text{by } (\text{3.11})),
\]

for all \(t \geq 0\). Give \(R > 0\), denote by \(\mathcal{O}_R^\varepsilon = \mathbb{R}^N - \mathcal{O}_R\), where \(\mathcal{O}_R = \{ x \in \mathbb{R}^N; |x| \leq R \}\). Note that \(A_\varepsilon(\tau,\omega) \subseteq \mathcal{D}\). Then by Lemma 4.4, for every \(\varepsilon > 0\), there exist
$T = T(\tau, \omega, \epsilon) \geq 2$ and $R = R(\tau, \omega, \epsilon) > 1$ such that the solution $u$ of problem (1.1)-(1.2) satisfies for all $t \geq T$,

$$
\|u_\epsilon(t, \tau - t, \vartheta, \omega, u_0)\|_{H^1(\mathcal{O}^\rho_{\epsilon})} \leq \epsilon.
$$

(5.3)

Then by (5.2)-(5.3), we have

$$
\|\chi(\tau, \omega)\|_{H^1(\mathcal{O}^\rho_{\epsilon})} \leq \epsilon, \quad \text{for all } \chi \in \cup_{\epsilon \in (0,1]} \mathcal{A}_\epsilon(\tau, \omega).
$$

(5.4)

On the other hand, by Lemma 4.6, there exist a projector $P_{N_0}$ and a $T = T(\tau, \omega, \epsilon) \geq 2$ such that for all $t \geq T$

$$
\|(I - P_{N_0})\tilde{u}_\epsilon(t, \tau - t, \vartheta, \omega, \tilde{u}_0)\|_{H^1(\mathcal{O}_{\epsilon, \tau})} \leq \epsilon,
$$

(5.5)

where $\tilde{u}_\epsilon$ is the cut-off of $u_\epsilon$ on the domain $\mathcal{O}_{\epsilon, \tau}$, by (4.68). Because $P_{N_0}\tilde{u}_\epsilon \in H_{N_0}$, where $H_{N_0} = \text{span}\{e_1, e_2, ..., e_{N_0}\}$ is a finite dimension space and $P_{N_0}\tilde{u}_\epsilon(\tau - t, \vartheta, \omega, \tilde{u}_0)$ is bounded in $H_{N_0}$ which is compact. Therefore there exist some finite points $v_1, v_2, ..., v_s \in H_{N_0}$ such that

$$
\|P_{N_0}\tilde{u}_\epsilon(t, \tau - t, \vartheta, \omega, \tilde{u}_0) - v_i\|_{H^1(\mathcal{O}_{\epsilon, \tau})} \leq \epsilon.
$$

(5.6)

Thus by (5.2), (5.5) and (5.6) are rewrote as

$$
\| (I - P_{N_0}) \chi(\tau, \omega) \|_{H^1(\mathcal{O}_{\epsilon, \tau})} \leq \epsilon, \quad \text{and} \quad \| P_{N_0} \chi(\tau, \omega) - v_i \|_{H^1(\mathcal{O}_{\epsilon, \tau})} \leq \epsilon,
$$

(5.7)

for all $\chi \in \cup_{\epsilon \in (0,1]} \mathcal{A}_\epsilon(\tau, \omega)$. We now define $\tilde{v}_i = \tilde{v}_i(x) = 0$ if $x \in \mathcal{O}_{\epsilon, \tau}^c$ and $\tilde{v}_i = v_i$ if $x \in \mathcal{O}_{\epsilon, \tau}$. Then for every $i = 1, 2, ..., s$, $\tilde{v}_i \in H^1(\mathbb{R}^N)$. Furthermore, by (5.4) and (5.7), we have

$$
\| \chi(\tau, \omega) - \tilde{v}_i \|_{H^1(\mathbb{R}^N)} \leq \| \chi(\tau, \omega) - \tilde{v}_i \|_{H^1(\mathcal{O}_{\epsilon, \tau})} + \| \chi(\tau, \omega) - \tilde{v}_i \|_{H^1(\mathcal{O}_{\epsilon, \tau}^c)} + \| \chi(\tau, \omega) - \tilde{v}_i \|_{H^1(\mathcal{O}_{\epsilon, \tau})} + \| (I - P_{N_0}) \chi(\tau, \omega) \|_{H^1(\mathcal{O}_{\epsilon, \tau})} \leq 3\epsilon,
$$

for all $\chi \in \cup_{\epsilon \in (0,1]} \mathcal{A}_\epsilon(\tau, \omega)$. Thus $\cup_{\epsilon \in (0,1]} \mathcal{A}_\epsilon(\tau, \omega)$ has finite $\epsilon$-nets in $H^1(\mathbb{R}^N)$, which implies that the union $\cup_{\epsilon \in (0,1]} \mathcal{A}_\epsilon(\tau, \omega)$ is precompact in $H^1(\mathbb{R}^N)$. \hfill \Box

We then obtain that the family of random attractors $\mathcal{A}_\epsilon$ indexed by $\epsilon$ converges to the deterministic $\mathcal{A}_0$ in $H^1(\mathbb{R}^N)$ in the following sense,

**Theorem 5.2.** Assume that (3.1), (3.5) and (5.1) hold. Then for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$
\lim_{\epsilon \downarrow 0} \text{dist}_{H^1}(\mathcal{A}_\epsilon(\tau, \omega), \mathcal{A}_0(\tau)) = 0
$$

where $\text{dist}_{H^1}$ is the Hausdorff semi-metric in $H^1(\mathbb{R}^N)$.

6. Existence of random equilibria for the generated cocycle

It is known that the random equilibrium is a special case of omega-limit sets. The corresponding notion in deterministic case is fixed points or stationary solutions. We can refer to [2] [5] for the definitions and applications. The problem of the construction of equilibria for a general random dynamical system is rather complicated [5]. Recently, [27] [28] obtained the existence of unique random equilibrium for stochastic reaction-diffusion equation with autonomous term on bounded domains.
or a unbounded Poincaré domains. Gu [10] proved that the stochastic FitzHugh-Nagumo lattice equations driven by fractional Brownian motions possesses a unique equilibrium.

However, we here introduce the random equilibrium under the circumstance of non-autonomous stochastic dynamical system. In particular, we have

**Definition 6.1.** Let \((\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t \in \mathbb{R}})\) be a measurable dynamical system. A random variable \(u^* : \mathbb{R} \times \Omega \rightarrow X\) is said to be an equilibrium (or fixed point, or stationary solution) of the cocycle \(\varphi\) if it is invariant under \(\varphi\), i.e., if
\[
\varphi(t, \tau, \omega, u^*(\tau, \omega)) = u^*(\tau + t, \vartheta_t \omega) \quad \text{for all} \ t \geq 0, \ \tau \in \mathbb{R}, \ \omega \in \Omega.
\]

In this paper, we will prove the existence of equilibrium for stochastic non-autonomous reaction-diffusion equation on the whole space \(\mathbb{R}^N\). We assume the coefficient \(\lambda > \alpha_3\), where \(\alpha_3\) is as in (3.3) and \(\lambda\) is as in (3.8). For convenience, here we write \(\varepsilon = 1\). First, we have

**Lemma 6.2.** Suppose that \(g \in L^2(\mathbb{R}^N)\), \(f\) and \(g\) satisfies (3.1)-(3.5) and \(\lambda > \alpha_3\). Let the initial values \(u_{0,i} = u(\tau - t_i, \vartheta_{-\tau} \omega)(i = 1, 2), t_1 < t_2\). Then there exists a constant \(b_0\) such that the solution of problem (1.1) with initial value \(u_{0,i}\) satisfies the following decay property:
\[
\|u(\tau, \tau - t_1, \vartheta_{-\tau} \omega, u_{0,1}) - u(\tau, \tau - t_2, \vartheta_{-\tau} \omega, u_{0,2})\|^2 \leq \\
2\left(e^{-b_0 t_1}e^{-2\omega(t_1)}\|u_{0,1}\|^2 + 2e^{-b_0 t_2}e^{-2\omega(t_2)}\|u_{0,2}\|^2\right) + ce^{-b_0 t_1}\int_{-\infty}^{0} e^{\lambda s}z^2(s, \omega)(\|g(s + \tau, \cdot)\|^2 + 1)ds,
\]
where \(c\) is a deterministic non-random constant.

**Proof** Put \(\bar{v} = v(\tau, \tau - t_1, \vartheta_{-\tau} \omega, v_{0,1}) - v(\tau, \tau - t_2, \vartheta_{-\tau} \omega, v_{0,2})\). Then by (3.8) we have
\[
\frac{d}{dt}\|\bar{v}\|^2 + b\|\bar{v}\|^2 \leq 0, \quad (6.1)
\]
where \(b = \lambda - \alpha_3\). By applying Gronwall lemma to (6.1) over the interval \([\tau - t_1, \tau]\), we immediately get
\[
\|\bar{v}(\tau)\|^2 \leq e^{-bt_1}\|v(\tau - t_1, \tau - t_2, \vartheta_{-\tau} \omega, v_{0,2}) - v_{0,1}\|^2 \\
\leq 2e^{-bt_1}\|v(\tau - t_1, \tau - t_2, \vartheta_{-\tau} \omega, v_{0,2})\|^2 + 2e^{-bt_1}\|v_{0,1}\|^2. \quad (6.2)
\]
Choose
\[
0 < b_0 < b. \quad (6.3)
\]
By (4.4), we have
\[
\frac{d}{dt}\|v\|^2 + b_0\|v\|^2 \leq cz^2(t, \omega)(\|g(t, \cdot)\|^2 + 1). \quad (6.4)
\]
Then by Gronwall lemma again, we find that
\[
\|v(\tau - t_1, \tau - t_2, \vartheta_{-\tau} \omega, v_{0,2})\|^2 \\
\leq e^{b_0(t_1 - t_2)}\|v_{0,2}\|^2 + c \int_{\tau - t_2}^{\tau - t_1} e^{-b_0(t - s)}z^2(s, \vartheta_{-\tau} \omega)(\|g(s, \cdot)\|^2 + 1)ds
\]
\[ e^{b_0(t_1 - t_2)} \|v_{0.2}\|^2 + ce^{b_0 t_1} e^{-2t_3} \int_0^0 e^{b_0 s} z^2(s, \omega)(\|g(s + \tau, .)\|^2 + 1)ds, \]

from which and (6.2) it follows that

\[
\|\bar{v}(\tau)\|^2 \leq 2e^{-bt_1} \|v(\tau - t_1, \tau - t_2, \vartheta - \tau \omega, v_{0.2})\|^2 + 2e^{-bt_1} \|v_{0.1}\|^2
\]

\[
\leq 2e^{-bt_1} \|v_{0.1}\|^2 + 2e^{(b_0 - b)t_1} e^{bt_2} \|v_{0.2}\|^2 + ce^{(b_0 - b)t_1} e^{-2t_3} \int_0^0 e^{b_0 s} z^2(s, \omega)(\|g(s + \tau, .)\|^2 + 1)ds
\]

\[
\leq 2 \left( e^{b_0 t_1} \|v_{0.1}\|^2 + e^{b_0 t_2} \|v_{0.2}\|^2 \right) + ce^{(b_0 - b)t_1} e^{-2t_3} \int_0^0 e^{b_0 s} z^2(s, \omega)(\|g(s + \tau, .)\|^2 + 1)ds, \quad (6.5)
\]

where we have used \( e^{(b_0 - b)t_1} \leq 1 \) for \( b_0 < b \). By the equality \( v(t) = z(t, \omega)u(t) = e^{-\omega(t)}u(t) \), we get

\[
\|\bar{v}(\tau)\|^2 \leq e^{-2\omega(-\tau)} \|\bar{v}(\tau)\|^2
\]

\[
\leq 2e^{-2\omega(-\tau)} \left( e^{b_0 t_1} \|v_{0.1}\|^2 + e^{b_0 t_2} \|v_{0.2}\|^2 \right)
\]

\[
+ ce^{(b_0 - b)t_1} \int_{-\infty}^0 e^{b_0 s} z^2(s, \omega)(\|g(s + \tau, .)\|^2 + 1)ds
\]

\[
= 2e^{-2\omega(-\tau)} \left( e^{b_0 t_1} z^2(\tau - t_1, \vartheta - \tau \omega)\|u_{0.1}\|^2 + 2e^{-b_0 t_2} z^2(\tau - t_2, \vartheta - \tau \omega)\|u_{0.2}\|^2 \right)
\]

\[
+ ce^{(b_0 - b)t_1} \int_{-\infty}^0 e^{b_0 s} z^2(s, \omega)(\|g(s + \tau, .)\|^2 + 1)ds
\]

\[
= 2 \left( e^{b_0 t_1} e^{-2\omega(t_1)}\|u_{0.1}\|^2 + 2e^{-b_0 t_2} e^{-2\omega(t_2)}\|u_{0.2}\|^2 \right)
\]

\[
+ ce^{(b_0 - b)t_1} \int_{-\infty}^0 e^{b_0 s} z^2(s, \omega)(\|g(s + \tau, .)\|^2 + 1)ds, \quad (6.6)
\]

which finishes the proof. \( \square \)

According to Lemma 6.2, we set \( \lambda_0 < b_0 \) and define the collection \( \mathcal{D} \) by

\[
\mathcal{D} = \{ B(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega; \lim_{t \to +\infty} e^{-\lambda_0 t} \|B(\tau - t, \vartheta - \tau \omega)\|^2 = 0, \text{ for } \tau \in \mathbb{R}, \omega \in \Omega \}.
\]

Then we have the convergence result about the solution of problem (1.1)-(1.2) in \( L^2(\mathbb{R}^N) \).

**Lemma 6.3.** Suppose that \( g \in L^2(\mathbb{R}^N) \), \( f \) and \( g \) satisfies (3.1)-(3.5) and \( \lambda > \alpha_3 \). Let \( B = \{ B(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \). Then for \( \tau \in \mathbb{R}, \omega \in \Omega \), there exists a unique element \( u^* = u^*(\tau, \omega) \in L^2(\mathbb{R}^N) \) such that

\[
\lim_{t \to +\infty} u(\tau, \tau - t, \vartheta - \tau \omega, u_0) = u^*(\tau, \omega), \text{ in } L^2(\mathbb{R}^N),
\]

where \( u_0 \in B(\tau - t, \vartheta - \tau \omega) \). Furthermore, the convergence is uniform (w.r.t \( u_0 \in B(\tau - t, \vartheta - \tau \omega) \)).
Proof. If \( u_{0,i} \in B(t-t_i, \vartheta-t_i, \omega) \), then we have
\[
\lim_{t_i \to +\infty} e^{-b_0t_i} e^{-2\omega(t_i)} \| u_{0,i} \|^2 = 0
\]
for \( i = 1, 2 \). Thus the result is derived directly from Lemma 6.2. \( \square \)

**Lemma 6.4.** Suppose that \( g \in L^2(\mathbb{R}^N) \), \( f \) and \( g \) satisfies (6.7)–(6.9) and \( \lambda > \alpha_3 \). Then for \( \tau \in \mathbb{R}, \omega \in \Omega \), the element \( u^* = u^*(\tau, \omega) \) defined in Lemma 6.3 is a unique random equilibrium for the cocycle \( \phi \) defined by (3.10) in \( L^2(\mathbb{R}^N) \), i.e.,
\[
\phi(t, \tau, \omega, u^*(\tau, \omega)) = u^*(\tau + t, \vartheta_t \omega), \quad \text{for every} \quad t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega.
\]
Furthermore, the random equilibrium \( \{ u^*(\tau, \omega), \tau \in \mathbb{R}, \omega \in \Omega \} \) is the unique element of the pullback attractor \( A = \{ A(\tau, \omega); \tau \in \mathbb{R}, \omega \in \Omega \} \) for the cocycle \( \phi \), i.e., for every \( \tau \in \mathbb{R}, \omega \in \Omega \),
\[
A(\tau, \omega) = \{ u^*(\tau, \omega) \}.
\]

Proof. By the definition of the cocycle, \( \phi(t, \tau-t, \vartheta_{-t} \omega, u_0) = u(\tau, \tau-t, \vartheta_{-t} \omega, u_0) \), then for for every \( \tau \in \mathbb{R}, \omega \in \Omega \), we have
\[
u^* (\tau, \omega) = \lim_{t \to +\infty} \phi(t, \tau-t, \vartheta_{-t} \omega, u_0),
\]
where \( u_0 \in B(t-t, \vartheta_{-t} \omega) \). Thus by the continuity and the cocycle property of \( \phi \) and (6.7), we find that for every \( t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega \),
\[
\phi(t, \tau, \omega, u^*(\tau, \omega)) = \phi(t, \tau, \omega, u_0) = \lim_{t \to +\infty} \phi(t, \tau-s, \vartheta_{-s} \omega, u_0)
\]
\[
= \lim_{t \to +\infty} \phi(t+s, \tau-s, \vartheta_{-s} \omega, u_0)
\]
\[
= \lim_{t \to +\infty} \phi(t+s, (\tau+t) - t-s, \vartheta_{-s} \vartheta_{-t} \omega, u_0)
\]
\[
= u^*(\tau+t, \vartheta_{t} \omega),
\]
which also implies the invariance of \( A \), that is, \( \phi(t, \tau, \omega, A(\tau, \omega)) = A(\tau+t, \vartheta_{t} \omega) \).

The compactness of \( A(\tau, \omega) \) is obvious and the attracting property follows from (6.7). \( \square \)

**Remark 6.5.** We notice that by Theorem 4.9, the equilibria \( u^* \in H^1(\mathbb{R}^N) \). In particular, we further have \( u^* \in L^p(\mathbb{R}^N) \).

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