M-theory on a Time-dependent Plane-wave

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Abstract: We propose a matrix model on a homogeneous plane-wave background with 20 supersymmetries. This background is anti-Mach type and is equivalent to the time-dependent background. We study supersymmetries in this theory and calculate the superalgebra. The vacuum energy of the abelian part is also calculated. In addition we find classical solutions such as graviton solution, fuzzy sphere and hyperboloid.

Keywords: Penrose limit and pp-wave background, M(atrix) Theories, M-Theory.
1. Introduction

Superstring theories and M-theory can be consistently realized on various classical solutions of supergravity theories. However, it is difficult to concretely study theories on these backgrounds because of complicated interaction terms. In the recent progress of string theory, plane-wave backgrounds are focused upon. The type IIB pp-wave solution was found and it was shown that this background can be described as the approximation of the \( AdS \) space geometry around a certain null geodesics via the Penrose limit. The superstring theory on this background is exactly solvable in the Green-Schwarz formulation, in spite of the presence of the Ramond-Ramond fluxes. Moreover, this string theory was utilized for the study of the \( AdS/CFT \) correspondence at the stringy level beyond the supergravity analysis.

On the other hand, the matrix model on the eleven-dimensional pp-wave background was proposed by Berenstein, Maldacena and Nastase, which is closely related to a supermembrane theory on this background via the matrix regularization. This pp-wave matrix model includes mass terms and the Myers term, and hence one can expect interesting physics intrinsic to the pp-wave case. Because of mass terms, all of the flat directions are completely lifted up. So it might be expected that a single supermembrane should be stabilized. In addition, the presence of the Myers term leads to many classical
solutions living only on the pp-wave. These solutions are fuzzy sphere type ones. In particular, the fuzzy sphere solution with zero energy exists, and it can appear in the classical vacuum of the system. As the result, the vacuum of this matrix model is very enriched.

Thus, superstring theories and M-theory on plane-wave backgrounds are quite interesting and so have been intensively studied. However, most of the studies have considered the time-independent backgrounds. A recent interest in supergravity, superstring theories and M-theory is to include the time-dependent background in our consideration. In general, it is much difficult to treat the time-dependent backgrounds and so one can hardly study string theories and M-theory on such backgrounds. Under such a circumstance, the time-dependent plane-wave background was considered by Papadopoulos, Russo and Tseytlin \[12\], and they showed that the string theory on this background is exactly solvable. The solvability of this model comes from supersymmetries which are preserved as a characteristic of plane-wave type backgrounds. Thus, we can find that the plane-wave backgrounds are very available to study the physics on time-dependent backgrounds.

The main focus of this paper is to study a matrix model on a homogeneous plane-wave background and study various features such as supersymmetries, vacuum structure and classical solutions in this model. The homogeneous plane-wave background \[13\] we will consider here leads to the time-dependent background through a coordinate transformation. Hence the study of the matrix model on this homogeneous background is equivalent to that on the time-dependent background. On the other hand, this background can be obtained from an M-theory Gödel universe* via the Penrose limit and this is also an anti-Mach type solution† \[15\]. Therefore it can be expected that the study here should be closely connected to the M-theory on the Gödel universe.

Our paper is organized as follows: In Section 2, we will briefly review a homogeneous plane-wave background which leads to a time-dependent background. In Section 3, the action of the matrix model on this background will be proposed. Section 4 is devoted to the study of the vacuum structure of the abelian part of the matrix model. In Section 5, examples of classical solutions in this model will be presented. We will discuss graviton solutions in our model and find fuzzy sphere and hyperbolic type solutions. Section 6 is devoted to a conclusion and discussions. In Appendix A, we will discuss a matrix model on the general homogeneous plane-wave. Appendix B is devoted to the detail study of the energy spectrum.

2. Time-dependent Homogeneous Plane-wave

In this section we will briefly review homogeneous plane-wave backgrounds, which are Cahen-Wallach space \[16\] with rotation terms. This type of backgrounds are related to time-dependent homogeneous plane-waves via time-dependent coordinate transformations.

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*The Gödel universe was originally proposed in the paper \[14\].

†M-theory Gödel universe can be constructed by lifting up a maximally supersymmetric Gödel solution in the minimal supergravity in five dimensions \[17\]. Also, string theories on Gödel universe and homogeneous plane-wave background are discussed in \[18\] and \[19\], respectively.
These are solutions of the eleven-dimensional supergravity and the family of them is given (in the Brinkmann coordinates) by

\[
\begin{align*}
    ds^2 &= -2dx^+ dx^- + \mu^2 A_{ij} x^i (dx^j)^2 + 2\mu f_{ij} x^i dx^j dx^+ + (dx^i)^2, \\
    F_4 &= \frac{\mu}{3!} dx^+ \wedge \xi_{ijk} dx^{ijk},
\end{align*}
\]  

where \(A_{ij}\) and \(f_{ij}\) are constant symmetric and anti-symmetric matrices respectively, and \(\xi_{ijk}\) is a completely anti-symmetric third-rank tensor. The nonzero-components of spin connection for this background are

\[
\omega^{+i} = A_{ij} x^j dx^+ + f_{ij} dx^j, \quad \omega^{ij} = -f_{ij} dx^+.
\]

The metric (2.1) explicitly describes the time-dependent background if \(A_{ij}\) and \(f_{ij}\) do not commute, as mentioned in [13]. In particular, the above type of plane-wave backgrounds is related to the explicitly time-dependent metric

\[
ds^2 = -2dx^+ dx^- + \left(e^{x^+ f} A_0 e^{-x^+ f}\right)_{ij} z^i z^j (dx^+)^2 + (dz^i)^2.
\]

By the use of the coordinate transformation of transverse directions

\[
    z^i \rightarrow w^i = \left(e^{-x^+ f}\right)_{ik} z^k,
\]

the metric (2.4) takes the stationary form

\[
ds^2 = -2dx^+ dx^- + ((A_0)_{ij} - f_{ik} f_{kj}) w^i w^j (dx^+)^2 + (dw^i)^2 - 2w^i f_{ik} dw^k dx^+,
\]

and we see that the time-dependent plane-wave (2.4) can be mapped to the stationary plane-wave as in (2.1).

In this paper, we concentrate on the special case which admits 20 supersymmetries [20], where the matrices \(A_{ij}\) and \(f_{ij}\) are given by

\[
    A_{22} = 4(P^2 - 1), \quad A_{33} = A_{44} = -1, \quad f_{12} = -P,
\]

and otherwise zero. The \(\xi_{ijk}\) is written as

\[
    \xi_{129} = -2, \quad \xi_{349} = 2P, \quad \xi_{256} = -2(1 - P^2)^{1/2}, \quad \xi_{278} = -2(1 - P^2)^{1/2}.
\]

Notably, the above \(A_{ij}\) and \(f_{ij}\) do not commute, and so this background is equivalent to the time-dependent background. We can obtain this background from the M-theory Gödel universe by the use of the Penrose limit [4]. The M-theory Gödel universe is constructed by lifting up a maximally supersymmetric Gödel solution of minimal five-dimensional supergravity [17] (\(\beta\) is an arbitrary parameter):

\[
ds_{5G}^2 = -(dt + \beta(r_1^2 d\phi_1^2 + r_2^2 d\phi_2^2))^2 + dr_1^2 + r_1^2 d\phi_1^2 + dr_2^2 + r_2^2 d\phi_2^2,
\]

as the product space with the six-dimensional flat space \(\mathbb{R}^6\):

\[
ds_M^2 = ds_{5G}^2 + dz^2 + \sum_{i=5}^{9} (dx^i)^2,
\]
where this eleven-dimensional background is supported by the four-form field-strength:

\[
F_{r_1 \phi_1 56} = F_{r_1 \phi_1 78} = F_{r_1 \phi_1 9z} = F_{r_2 \phi_2 56} = F_{r_2 \phi_2 78} = F_{r_2 \phi_2 9z} = -2\beta. \quad (2.11)
\]

The plane-wave solution (2.1) and (2.2) admits 20 supersymmetries (not maximal 32), which are inherited from that of the M-theory Gödel background (2.7), and its Killing spinor was explicitly presented in [20]. That is, this background admits extra Killing spinors satisfying

\[
\left[ \gamma_i \theta^2 + 6\theta \gamma_i \theta + 9\theta^2 \gamma_i + 6\gamma_i [\theta, \phi] + 18[\theta, \phi] \gamma_i - 144(A_{ij} \gamma^j + f_{ij} f_{jk} \gamma^k) \right] \epsilon = 0, \\
\epsilon(x^+) = e^{2/3} (\phi + \theta) \epsilon_0 
\quad (2.12)
\]

where we have defined as

\[
\theta \equiv \frac{1}{3!} \xi_{ijk} \gamma^{ijk}, \quad \phi \equiv \frac{1}{2} f_{ij} \gamma^{ij},
\quad (2.13)
\]

and \( \epsilon_0 \) is a constant spinor. It is worth noting that the parameter \( P \) characterizes the null geodesic in taking the Penrose limit. The null geodesic is chosen to have a momentum \( P \) along the six directions transverse to the five-dimensional Gödel metric. If we consider the \( P = 0 \) case, then the null geodesic is in the Gödel space in five dimensions.

### 3. Action of Matrix Model on a Time-dependent Plane-wave

Here we shall discuss the matrix model on the general homogeneous plane-wave background (anti-Mach type space) whose metric is (2.1) with (2.7) and the four-form flux (2.2) with (2.8) is equipped. By generalizing the result of work [21], we can propose the action of matrix model in the light-cone gauge as follows:

\[
S = \int d\tau \frac{1}{2} \text{Tr} \left[ (D_\tau X^i)^2 + \frac{1}{2} [X^i, X^j]^2 + i\psi^T D_\tau \psi - \psi^T \gamma_i [X^i, \psi] \\
-4\mu^2 (1 - P^2) (X^2)^2 - \mu^2 (X^3)^2 - \mu^2 (X^4)^2 \\
+2\mu PX^1 D_\tau X^2 - 2\mu PX^2 D_\tau X^1 + \frac{\mu}{2} i\psi^T (W - P\gamma^{12}) \psi \\
-\frac{4}{3} i\mu \cdot 3! \{X[1]X^2 X^9] - PX[3]X^4 X^9] \\
+(1 - P^2)^{1/2} X[2]X^5 X^6] + (1 - P^2)^{1/2} X[2]X^7 X^8] \right],
\quad (3.1)
\]

\[
W = \gamma^{129} - P\gamma^{349} + (1 - P^2)^{1/2} \gamma^{256} + (1 - P^2)^{1/2} \gamma^{278},
\quad (3.2)
\]

where \( \psi \) is the 16-components \( SO(9) \) spinor and \( \gamma^i \)'s are the \( SO(9) \) gamma matrices. The \( \tau \) is the world-volume time and the covariant derivative \( D_\tau \) is defined as \( D_\tau \star = \partial_\tau \star - i[\omega, \star] \). The gauge connection \( \omega \) describes the area preserving diffeomorphism in terms of membrane theory\(^4\). It is an easy task to generalize the above action to the background (2.1) with

\(^4\)We restrict ourselves to the closed membrane case. It is an interesting issue to study open supermembrane theories on a time-dependent background.
and present the matrix model on the general homogeneous plane-wave (anti-Mach) background. This generalization will be discussed in the appendix A.

The above matrix model includes the Myers terms due to the presence of the constant four-form flux. Hence the fuzzy sphere solutions should appear as classical solutions as in the pp-wave matrix model [7]. These solutions will be discussed in detail later.

Here we will discuss supersymmetries of the matrix model on the time-dependent background. The 4 dynamical and 16 kinematical supersymmetries are preserved in this background, and so the same number of supersymmetries should be preserved in the matrix model on this background.

The transformation law of 4 dynamical supersymmetries is given by

\[
\delta \epsilon X^i = i \psi^T \gamma^i \epsilon(\tau), \quad \delta \epsilon \omega = i \psi^T \epsilon(\tau), \quad \epsilon(\tau) \equiv e^{i \frac{\mu}{2} (W + 3P \gamma^{12})^\tau} P_1 P_2 \epsilon_0, \quad (3.3)
\]

\[
\delta \epsilon \psi = \left[ D_\tau X^i \gamma_i - \frac{i}{2} [X^i, X^j] \gamma_{ij} + 2 \mu P (X^1 \gamma^2 - X^2 \gamma^1) - \frac{\mu}{6} X^i \gamma_i (W + 3P \gamma^{12}) - \frac{\mu}{2} P (W - P \gamma^{12}) \gamma_i \right] \epsilon(\tau),
\]

where \( \epsilon_0 \) is a constant spinor and we have introduced the projection operators:

\[
P_1 = \frac{1}{2} (\mathbb{1} - \gamma_{5678}), \quad (3.4)
\]

\[
P_2 = \frac{1}{2} (\mathbb{1} + c \gamma_{1956} - s \gamma_{1234}), \quad (3.5)
\]

where \( c \equiv (1 - P^2)^{1/2} \) and \( s \equiv P \). The following properties:

\[
P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_1 P_2 = P_2 P_1.
\]

are satisfied, and \( P_1 \) and \( P_2 \) are independent each other. Hence a quarter of dynamical supersymmetries (i.e., 4 dynamical supersymmetries) can survive.

The transformation law of 16 kinematical supersymmetries is represented by

\[
\delta \eta X^i = \delta \eta \omega = 0, \quad \delta \eta \psi = \eta(\tau), \quad \eta(\tau) = e^{-\frac{\mu}{2} (W - P \gamma^{12})^\tau} \eta_0,
\]

where \( \eta_0 \) is a constant spinor.

We can calculate the superalgebra in our model by using the standard Dirac bracket. In fact, our matrix model contains complicated fluxes and the characteristic parameter \( P \) is turned on, and hence its superalgebra is expected to contain the generators of rotations (angular momenta) and nontrivial terms intrinsic to our system. This interesting topic will be studied in detail in another place [22].

4. Vacuum Energy of the Abelian Part

In this section we will consider the abelian part of the matrix model proposed in Section 3. This part can be exactly solved and so one can calculate the vacuum energy of this part. We will see below that this energy is negative.
First, let us consider the $P = 0$ case. The abelian part of the action is given by
\begin{equation}
S = \frac{1}{2} \int d\tau \left[ (\dot{X}^i)^2 - 4\mu^2 (X^2)^2 - \mu^2 (X^3)^2 - \mu^2 (X^4)^2 + i\psi^T \dot{\psi} + \frac{\mu}{2} i\psi^T W_0 \psi \right].
\end{equation}

(4.1)

where we introduced the notations $W_0 \equiv \gamma_{129} + \gamma_{256} + \gamma_{278}$ and $\dot{X}^i \equiv \partial_\tau X^i$.

The dynamical supersymmetry transformation is given by
\begin{equation}
\delta_\epsilon X^i = i\psi^T \gamma^i \epsilon(\tau), \quad \epsilon(\tau) \equiv e^{\frac{\mu}{2} W_0 \tau} P_1 P_2 \epsilon_0,
\end{equation}

\begin{equation}
\delta_\epsilon \psi = \left[ \dot{X}^i \gamma_i - \frac{\mu}{6} X^i \gamma_i W_0 - \frac{\mu}{2} X^i W_0 \gamma_i \right] \epsilon(\tau),
\end{equation}

(4.2)

On the other hand, the kinematical supersymmetry transformation is described by
\begin{equation}
\delta_\eta X^i = 0, \quad \delta_\eta \psi = \eta(\tau), \quad \eta(\tau) \equiv e^{-\frac{\mu}{2} W_0 \tau}.
\end{equation}

(4.3)

Here we shall decompose the 16-components spinor $\psi$ as
\begin{equation}
\psi = \psi^{(++)} + \psi^{(+-)} + \psi^{(-+)} + \psi^{(--)},
\end{equation}

(4.4)

where each of the spinors $\psi^{(++)}, \psi^{(+-)}, \psi^{(-+)}$ and $\psi^{(--)}$ have only 4 non-trivial components and satisfy the following two chirality conditions:
\begin{equation}
\gamma_{5678}\psi^{(\pm \bullet)} = \pm \psi^{(\bullet \pm)}, \quad \gamma_{1956}\psi^{(\bullet \pm)} = \pm \psi^{(\pm \bullet)}.
\end{equation}

(4.5)

Notably, the matrix $W_0$ commute with $\gamma_{5678}$ and $\gamma_{1956}$, and hence the chirality is preserved under the action of $W_0$ (for example, the chirality of $W_0 \psi^{(++)}$ is the same with $\psi^{(++)}$). By using this decomposition, we can rewrite the fermionic action as
\begin{equation}
S_F = \int d\tau \left[ i\psi^{(++)} \dot{\psi}^{(++)} + i\psi^{(+-)} \dot{\psi}^{(+-)} \\
+ i\psi^{(-+)} \dot{\psi}^{(-+)} + i\psi^{(--)} \dot{\psi}^{(--)} \\
+ \frac{\mu}{2} i\psi^{(++)} \Pi \psi^{(++)} - \frac{\mu}{2} i\psi^{(+-)} \Pi \psi^{(+-)} \\
+ \frac{3}{2} \mu i\psi^{(--)} \Pi \psi^{(--)} + \frac{\mu}{2} i\psi^{(-+)} \Pi \psi^{(-+)} \right],
\end{equation}

(4.6)

where the matrix $\Pi$ is defined as $\Pi \equiv \gamma_{256}$.

The vacuum energy of bosons comes from the massive directions: $X^2, X^3$ and $X^4$, and the net contribution is expressed as
\begin{equation}
E_B = \frac{\mu}{2} (2 + 1 + 1) = +2\mu.
\end{equation}

(4.7)
The vacuum energy of fermions are evaluated as (for example, by following the procedure in the work [24])

\[ E_F = -\frac{\mu}{2} (1 + 1 + 1 + 3) = -3\mu. \] (4.8)

Thus the total vacuum energy is given by

\[ E_{\text{tot}} = -\mu. \] (4.9)

In conclusion, the vacuum energy of the abelian part is negative\(^\S\).

Next, we will consider the \( P \neq 0 \) case. The action of the abelian part is written as

\[
S = \frac{1}{2} \int d\tau \left[ (\dot{X}^i)^2 - 4\mu^2 (1 - P^2)(X^2)^2 - \mu^2 (X^3)^2 - \mu^2 (X^4)^2 \\
+ 2\mu P(X^1 \dot{X}^2 - X^2 \dot{X}^1) + i\psi^T \dot{\psi} + \frac{\mu}{2} i\psi^T (W - P\gamma^{12}) \psi \right],
\] (4.10)

where the expression of \( W \) in the \( P \neq 0 \) case is

\[ W = \gamma^{129} - s\gamma^{349} + c\gamma^{256} + c\gamma^{278}. \] (4.11)

The transformation law of 4 dynamical supersymmetries is given by

\[
\delta_\epsilon X^i = i\psi^T \gamma^i \epsilon(\tau), \quad \epsilon(\tau) \equiv \epsilon_0^{W+(3P\gamma^{12})} P_1 P_0, \] (4.12)

\[
\delta_\epsilon \psi = \left[ \dot{X}^i \gamma_i + 2\mu P(X^1 \gamma^2 - X^2 \gamma^1) \\
- \frac{\mu}{6} X^i \gamma_i (W + 3P\gamma^{12}) - \frac{\mu}{2} X^i (W - P\gamma^{12}) \gamma_i \right] \epsilon(\tau),
\]

where \( \epsilon_0 \) is a constant spinor and the projection operators are written as

\[ P_1 = \frac{1}{2} (\mathbb{I} - \gamma_{5678}) , \] (4.13)

\[ P_2 = \frac{1}{2} (\mathbb{I} + \gamma_{1956} - s\gamma_{1234}) . \] (4.14)

The transformation law of 16 Kinematical supersymmetries is given in (3.7).

To begin with, we will consider the bosonic part. In the \( P \neq 0 \) case the analysis of \( X^1 \) and \( X^2 \) directions are nontrivial. We can find that the effect of \( P \) is to shift the origin of oscillator \( X^2 \) and to rescale the continuous spectrum of \( X^1 \), as we will discuss in Appendix. Thus, the resulting zero-point energy for bosons is identical with that in the \( P = 0 \) case:

\[ E_B = 2\mu. \] (4.15)

\(^\S\)The negative vacuum energy of the abelian part of the matrix model on less supersymmetric pp-waves is also shown in [23].
Next let us consider the fermionic part and focus upon the fermion mass term:

$$\frac{\mu}{2} i \bar{\psi} (W - P \gamma^{12}) \psi. \quad (4.16)$$

We decompose the spinor $\psi$ according to the chirality in terms of the matrices $\gamma^{5678}$ and $c\gamma^{1956} + s\gamma^{1234}$* as in the $P = 0$ case. Remarkably, these matrices commute with $W - P \gamma^{12}$. We can easily see this property if we rewrite the $W - P \gamma^{12}$ as

$$W - P \gamma^{12} = \gamma^{129} (I + (I - \gamma^{5678})(c\gamma^{1956} + s\gamma^{1234})),$$

where we have used the gamma matrices $\gamma^{ij}$'s satisfy $\gamma^{123456789} = I$. Thus, the chiralities of spinor are preserved under the action of $W - P \gamma^{12}$ even in the $P \neq 0$ case. Now we can rewrite the fermion mass term as

$$\frac{\mu}{2} i \bar{\psi}^{(++)} \Pi' \psi^{(++)} + \frac{\mu}{2} i \bar{\psi}^{(+-)} \Pi' \psi^{(+-)} + \frac{3}{2} \mu i \bar{\psi}^{(-+)} \Pi' \psi^{(-+)} - \frac{\mu}{2} i \bar{\psi}^{(--) \Pi' \psi^{(--)}}, \quad (4.17)$$

where $\Pi' \equiv \gamma^{129}$ and the spinors $\psi^{++}$, $\psi^{+-}$, $\psi^{-+}$, and $\psi^{--}$ satisfy the following chirality conditions with respect to the matrices $\gamma^{5678}$ and $c\gamma^{1956} + s\gamma^{1234}$:

$$\gamma^{5678} \psi^{\pm} = \pm \psi^{\pm}, \quad (c\gamma^{1956} + s\gamma^{1234}) \psi^{\pm} = \pm \psi^{\mp}. \quad (4.18)$$

Thus the vacuum energy of fermion part is represented by

$$E_F = -\frac{\mu}{2} (1 + 1 + 1 + 1) = -3\mu. \quad (4.19)$$

In conclusion, the net vacuum energy is

$$E_{tot} = -\mu. \quad (4.20)$$

Thus the zero-point energy is completely identical with that in the $P = 0$ case and it is independent of the parameter $P$.

We can derive the spectrum of the abelian part, and we might expect that the resulting spectrum would agree with that in the linearized supergravity in eleven dimensions around the time-dependent background consider here (as is shown in case of the eleven-dimensional pp-wave background [25]).

5. Classical Solutions

In this section we will study the classical solutions of our matrix model. There are the extra terms with $f_{ij}$, and so we can expect that graviton solutions would be modified. Also, fuzzy sphere solutions should exist due to the presence of the Myers term.

In order to study classical solutions we need equations of motion. We will set $\omega = \psi = 0$ for simplicity in the following consideration. By taking a variation of action, the

*The choice of chirality matrices is not relevant to the projection operator $P_1$ and $P_2$. 
We can evaluate the energy of the classical solution by using the classical potential $V$ in our model:

$$2V = \text{Tr} \left[ -\frac{1}{2} [X^i, X^j]^2 + 4\mu^2 (1 - P^2)(X^i)^2 + \mu^2 (X^3)^2 + \mu^2 (X^4)^2 \\
+ \frac{4}{3} i\mu \cdot 3![X^i X^j X^k X^l] - PX^i [X^j X^k X^l] \\
+ (1 - P^2)^{1/2} X^i [X^j X^k X^l] + (1 - P^2)^{1/2} X^i [X^j X^k X^l] \right].$$  (5.10)

We will discuss below some types of classical solutions and evaluate the classical energies by using the above classical equations of motion and potential.

### 5.1 Graviton Solutions

To begin with, we will consider the graviton solutions. In the case of a matrix model in flat space, the graviton solution is a free particle moving with a constant velocity, while in the pp-wave case this solution is modified to be described by an oscillator because of the presence of mass terms in the action. In our case the graviton solution is slightly different from that in the usual pp-wave cases as we will explain.

Let us investigate the graviton solution by considering the diagonal matrix. The graviton solutions for $X^5, \cdots, X^9$ are free particles as in flat space. For $X^3$ and $X^4$ the solution is a couple of harmonic oscillators as in the usual pp-wave case because of mass terms. The remaining parts are $X^1$ and $X^2$. These directions couple each other as follows:

$$\ddot{X}^1 = 2\mu P \dot{X}^2, \quad \ddot{X}^2 = -2\mu P \dot{X}^1 - 4\mu^2 (1 - P^2) X^2. \quad (5.11)$$

We can easily solve this system of equations and the solution is

$$X^1 = P x_2 \sin(2\mu \tau) - \frac{v_2}{2\mu} \cos(2\mu \tau) + (1 - P^2) c_0 \tau + c'_0, \quad (5.12)$$

$$X^2 = x_2 \cos(2\mu \tau) + \frac{v_2}{2\mu} \sin(2\mu \tau) - \frac{P c_0}{2\mu}. \quad (5.13)$$
The $x_2$ and $v_2$ are the initial position and velocity of $X^2$, respectively. The $c_0$ and $c'_0$ are arbitrary constants related to the initial velocity and position of $X^1$, respectively. It should be remarked that the $X^1$ direction behaves as a harmonic oscillator in spite of the absence of its mass term, because of the contribution of the term $f_{ij}$. Notably, the $X^1$ behaves as a freely moving particle in the $P = 0$ case and as a harmonic oscillator in the $P = 1$ case. This result shows a specific feature of our model.

In addition, we can easily see that there exist a rotating solution:

$$X^3(\tau) + iX^4(\tau) = (X^3(0) + iX^4(0))e^{i\mu \tau}, \quad [X^3(0), X^4(0)] = 0, \quad (5.14)$$

as in the matrix model on the pp-wave background \cite{[7]}. The energy of this solution is given by

$$2V = \mu^2 \text{Tr} \left[ (X^3(0))^2 + (X^4(0))^2 \right]. \quad (5.15)$$

The above solutions describe the motion of a D-particle in our model. For non-diagonal cases, the D-particle can expand to form a higher dimensional object with two dimensions (i.e., membrane) because of the Myers terms. The expanded fuzzy membrane solutions will be discussed in the next subsection.

### 5.2 Fuzzy Sphere and Hyperboloid Solutions

Here we would like to find fuzzy sphere type solutions by imposing the ansatz:

$$X^1 = \alpha J^1, \quad X^2 = \beta J^2, \quad X^9 = \alpha J^3, \quad (5.16)$$

where the $J^a$ ($a = 1, 2, 3$)'s are the $SU(2)$ generators and satisfy the $SU(2)$ Lie algebra $[J^a, J^b] = i\epsilon_{abc}J^c$. By putting the ansatz into the equations of motion, we obtain two conditions

$$\alpha \beta^2 + \alpha^3 - 2\mu \alpha \beta = 0, \quad \alpha^2 \beta - \mu \alpha^2 + 2\mu^2 (1 - P^2) \beta = 0. \quad (5.17)$$

These algebraic equations can be easily solved, and we obtain two nontrivial solutions:

$$\alpha^2 = -2\mu^2 (1 - P^2) + \frac{\mu^2}{2} (1 + \sqrt{1 + 8(1 - P^2)}),$$

$$\beta = \frac{3}{2} \mu - \frac{\mu}{2} \sqrt{1 + 8(1 - P^2)}, \quad (5.18)$$

and

$$\alpha^2 = -2\mu^2 (1 - P^2) + \frac{\mu^2}{2} (1 - \sqrt{1 + 8(1 - P^2)}),$$

$$\beta = \frac{3}{2} \mu + \frac{\mu}{2} \sqrt{1 + 8(1 - P^2)}, \quad (5.19)$$

and a trivial solution $\alpha = \beta = 0$. The first nontrivial solution describes a fuzzy ellipsoidal sphere and the second one represents a fuzzy hyperboloid\footnote{The ellipsoidal sphere and hyperbolic type solutions were constructed in the pp-wave matrix model by D. Bak \cite{[26]}.} in the range $0 \leq P \leq 1$\footnote{If we consider the region $P < 0$ for the first case, then the shape of the solution becomes a fuzzy hyperboloid. On the other hand, if we consider the region $P > 1$ for the second case, then the solution becomes fuzzy ellipsoidal sphere.}. In

\cite{[7]}.
the second case the value of $\alpha$ becomes purely imaginary and the compact $SU(2)$ group is replaced with the noncompact group $SO(2,1)$. The energies of the fuzzy ellipsoidal sphere (5.18) and hyperboloid (5.19) are represented by

$$
2V = \left( \alpha^2 \beta^2 - \frac{4}{3} \mu \alpha^2 \beta \right) \text{Tr}[(J^1)^2 + (J^3)^2] \\
+ \left( \alpha^4 + 4 \mu^2 (1 - P^2) \beta^2 - \frac{4}{3} \mu \alpha^2 \beta \right) \text{Tr}(J^2)^2 \\
= -\frac{\alpha^2 \beta^2}{N},
$$

(5.20)

where we have normalized the trace for the $N$-dimensional representation of $SU(2)$ generators as

$$
\text{Tr}(J^a J^b) = \frac{1}{N} \delta^{ab}.
$$

(5.21)

The energy for the fuzzy ellipsoidal sphere (5.18) is negative while that for the fuzzy hyperboloid (5.19) is positive.

Other fuzzy solutions are given by

$$
X^2 = c \cdot \beta J^1, \quad X^5 = c \cdot \alpha J^2, \\
X^6 = c \cdot \alpha J^3, \quad \text{otherwise zero},
$$

(5.22)

and

$$
X^2 = c \cdot \beta J^1, \quad X^7 = c \cdot \alpha J^2, \\
X^8 = c \cdot \alpha J^3, \quad \text{otherwise zero}.
$$

(5.23)

In the same way, these two cases lead to the same conditions:

$$
2c^3 \left( \alpha^2 \beta + 2 \mu^2 \beta - \mu \alpha^2 \right) = 0, \quad c^3 \alpha \left( 2 \mu \beta - \beta^2 - \alpha^2 \right) = 0.
$$

(5.24)

The solutions of the above algebraic equations are

$$
\alpha^2 = -3\mu^2, \quad \beta = 3\mu,
$$

(5.25)

and the trivial solution $\alpha = \beta = 0$. Thus, both of the nontrivial solutions (5.22) and (5.23) with (5.25) are fuzzy hyperboloids and have the same energy given by

$$
2V = c^4 \cdot \left( 4 \mu^2 \beta^2 + \alpha^4 - \frac{4}{3} \mu \alpha^2 \beta \right) \text{Tr}(J^1)^2 \\
+ \left( \alpha^2 \beta^2 - \frac{4}{3} \mu \alpha^2 \beta \right) \text{Tr}[(J^2)^2 + (J^3)^2] \\
= c^4 \cdot \left( 4 \mu^2 \beta^2 + \alpha^4 - 4 \mu \alpha^2 \beta + 2 \alpha^2 \beta^2 \right) / N, \\
= 27 \mu^4 c^4 / N.
$$

(5.26)

We also can apply the above consideration to the $X^3$, $X^4$ and $X^9$ directions by supposing the following ansatz:

$$
X^3 = \alpha J^1, \quad X^4 = \alpha J^2, \quad X^9 = \beta J^3, \quad \text{otherwise zero}.
$$

(5.27)
By inserting this ansatz into the equations of motion, we can obtain the two constraint conditions:

\[
\mu \alpha^2 P + \alpha^2 \beta = 0, \quad \alpha^3 + \alpha \beta^2 + \mu^2 \alpha + 2 \mu \alpha \beta = 0.
\] (5.28)

These equations can be solved and the solution is

\[
\alpha^2 = -\mu^2 (1 - P^2), \quad \beta = -\mu P.
\] (5.29)

This is also fuzzy hyperboloid solution, whose energy is represented by

\[
2V = \left( \alpha^2 \beta^2 + \mu^2 \alpha^2 + \frac{4}{3} \mu P \alpha^2 \beta \right) \text{Tr}([J^1]^2 + [J^2]^2) + \left( \alpha^4 + \frac{4}{3} \mu P \alpha^2 \beta \right) \text{Tr}(J^3)^2
\]
\[
= -\mu^4 (1 - P^2)^2 / N \leq 0.
\] (5.30)

As the result, the energy of the fuzzy hyperboloid solution has negative energy.

Now we will discuss the range of the parameter \( P \). For example, if we consider the \( P^2 > 1 \) case in the solution (5.29), then the values of \( \alpha \) and \( \beta \) give the ellipsoidal fuzzy sphere solution. If \( P^2 < 1 \), then the shape of the solution is hyperbolic. This result implies that the shape of the solution should be modified as the parameter \( P \) changes. It should be remarked that the parameter \( P \) might run from 0 to 1. If \( P \) is out of this range, then the bosonic mass term would be tachyonic and it seems that some trouble would appear. However, such an issue cannot be caused in the Rosen coordinates since it would be intrinsic to the system described in terms of Brinkmann coordinates.

We will not discuss further classical solutions but it would be expected that other interesting solution can be constructed in this model, and so the construction of such solutions are an interesting future problem. The stability of the fuzzy hyperboloids, which would be non-supersymmetric, would be also interesting††.

### 5.3 BPS Equations

In the previous subsection, we have presented examples of classical solutions. Now we will consider the conditions which lead to supersymmetric classical solutions in terms of BPS equations.

We can derive the BPS equations from the supersymmetric condition: \( \delta \psi = 0 \). The conditions preserving all of dynamical supersymmetries are given by

\[
[X^2, X^9] = i \frac{4}{3} \mu X^1, \quad [X^9, X^1] = i \frac{4}{3} \mu X^2, \quad [X^1, X^2] = i \frac{4}{3} \mu X^9,
\]
\[
[X^4, X^9] = -is \cdot i \frac{4}{3} \mu X^3, \quad [X^9, X^3] = -is \cdot i \frac{4}{3} \mu X^4, \quad [X^3, X^4] = -is \cdot i \frac{4}{3} \mu X^9,
\]
\[
[X^5, X^6] = ic \cdot i \frac{2}{3} \alpha \mu X^2, \quad [X^6, X^2] = ic \cdot i \frac{4}{3} \mu X^5, \quad [X^2, X^5] = ic \cdot i \frac{4}{3} \mu X^6,
\]
\[
[X^7, X^8] = ic \cdot i \frac{2}{3} \mu (1 - \alpha) X^2, \quad [X^8, X^2] = ic \cdot i \frac{4}{3} \mu X^7, \quad [X^2, X^7] = ic \cdot i \frac{4}{3} \mu X^8,
\]
\[
D_\tau X^2 = -2 \mu s X^1, \quad D_\tau X^1 = 2 \mu s X^2, \quad D_\tau X^4 = \mu X^3, \quad D_\tau X^3 = -\mu X^4,
\]
\[
D_\tau X^5 = D_\tau X^6 = D_\tau X^7 = D_\tau X^8 = D_\tau X^9 = 0.
\] (5.31)

††The quantum stability of fuzzy sphere is discussed in [27].
where a constant parameter $\alpha$ takes the value in the range $0 < \alpha < 1$. The commutators which do not appear in the conditions (5.31) should vanish (for example, $[X^1, X^5] = 0$). We note that the equations (5.31) are not necessary but sufficient conditions for the BPS conditions. Because we are dealing with a non-maximally supersymmetric case, some equations are satisfied trivially.

Now we can find a solution of the above BPS conditions (5.31) represented by

\begin{align*}
X^1 &= \left( \frac{4}{3} \mu J^2 \right) \otimes 1 \otimes 1 \otimes 1, \quad X^9 = \left( \frac{4}{3} \mu J^1 \right) \otimes \left( -s \cdot \frac{4}{3} \mu J^1 \right) \otimes 1 \otimes 1, \\
X^2 &= \left( \frac{4}{3} \mu J^3 \right) \otimes 1 \otimes \left( c \cdot \frac{4}{3} \mu J^3 \right) \otimes \left( c \cdot \frac{4}{3} \mu J^3 \right), \\
X^3 &= 1 \otimes \left( -s \cdot \frac{4}{3} \mu J^2 \right) \otimes 1 \otimes 1, \quad X^4 = 1 \otimes \left( -s \cdot \frac{4}{3} \mu J^3 \right) \otimes 1 \otimes 1, \\
X^5 &= 1 \otimes 1 \otimes \left( \frac{4}{3} \mu \cdot c \sqrt{\frac{\alpha}{2}} J^1 \right) \otimes 1, \quad X^6 = 1 \otimes 1 \otimes \left( \frac{4}{3} \mu \cdot c \sqrt{\frac{\alpha}{2}} J^2 \right) \otimes 1, \\
X^7 &= 1 \otimes 1 \otimes 1 \otimes \left( \frac{4}{3} \mu \cdot c \sqrt{\frac{1 - \alpha}{2}} J^1 \right), \quad X^8 = 1 \otimes 1 \otimes 1 \otimes \left( \frac{4}{3} \mu \cdot c \sqrt{\frac{1 - \alpha}{2}} J^2 \right), \\
\omega &= (2\mu \cdot s J^1) \otimes (-\mu J^1) \otimes 1 \otimes 1, \quad D_\tau X^5 = \cdots D_\tau X^9 = \psi = 0.
\end{align*}

The symbol $\otimes$ means the embedding of the $SU(2)$ generators into the matrix, and the $i$th generator belongs to the $N_i$-dimensional representation where $N = N_1 + N_2 + N_3 + N_4$. For example, $\left( \frac{4}{3} \mu J^3 \right) \otimes 1 \otimes \left( c \cdot \frac{4}{3} \mu J^3 \right) \otimes \left( c \cdot \frac{4}{3} \mu J^3 \right)$ is represented by the following matrix

\begin{align*}
\begin{pmatrix}
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\end{pmatrix}
\otimes 
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix}
\otimes 
\begin{pmatrix}
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\end{pmatrix}
\otimes 
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix}
\otimes 
\begin{pmatrix}
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\frac{4}{3} \mu J^3 \\
\end{pmatrix}.
\end{align*}

The above solution (5.32) preserves 4 dynamical supersymmetries while the 16 kinematical supersymmetries are broken. Hence this is a $1/5(=4/20)$ BPS object\textsuperscript{14}. In our case this configuration has zero energy as in the pp-wave matrix model (i.e., $V = 0$). Therefore, this configuration can appear in the classical vacuum without the cost of energy, although this is not a classical solution of equation of motion.

6. Conclusion and Discussion

We have discussed a matrix model on a homogeneous plane-wave background. The action of the matrix model has been proposed. This theory has 20 supersymmetries, and we have explicitly constructed the transformation laws of 4 dynamical and 16 kinematical supersymmetries in this model. Then, the vacuum energy of the abelian part has been

\textsuperscript{14}In the pp-wave matrix model, such fuzzy sphere solutions preserves 16 dynamical supersymmetries and are the $1/2(=16/32)$ BPS objects.
calculated, and we have shown that the vacuum energy in the $P \neq 0$ is identical with that in the $P = 0$ case. In particular, the effect of a parameter $P$ is to shift the origin of the harmonic oscillator $X^2$ and to rescale the continuous spectrum of $X^1$. We have also found classical solutions. The graviton solution is slightly different from that in the usual pp-wave cases because of the presence of the additional parameter $P$. It has been shown that the fuzzy ellipsoidal sphere solution (5.16) with (5.18) and the fuzzy hyperboloid solution (5.27) with (5.29) have negative energies.

It would also be interesting to study a supermembrane theory by taking a large $N$ limit (for the pp-wave case, see [9]). By assuming the matrix regularization [10], the action of supermembrane theory can formally be written down as follows:

$$S = \int d\tau d^2\sigma w(\sigma) \left[ \frac{1}{2} (D_\tau X^i)^2 - \frac{1}{2} \{X^i, X^j\}^2 + i\psi^T D_\tau \psi - i\psi^T \gamma_i \{X^i, \psi\} ight.$$\n$$-4\mu^2(1 - P^2)(X^2)^2 - \mu^2(X^3)^2 - \mu^2(X^4)^2 + 2\mu PX^1 D_\tau X^2 - 2\mu PX^2 D_\tau X^1$$\n$$+\frac{\mu}{2} i\psi^T (W - P\gamma^{12}) \psi + \frac{2}{3} \mu \cdot \left\{ \sum_{I,J,K=1,2,9} -P \sum_{I,J,K=3,4,9} \right. \$$\n$$\left. + (1 - P^2)^{1/2} \sum_{I,J,K=2,5,6} + (1 - P^2)^{1/2} \sum_{I,J,K=2,7,8} \right\} \epsilon_{IJK} X^K \{X^I, X^J\} \right], \quad (6.1)$$

where the covariant derivative is $D_\tau \ast = \partial_\tau \ast + \{\omega, \ast\}$ and $\{A, B\} \equiv (\epsilon_{ab}/w) \partial_a A \partial_b B$ is the Lie bracket. The indices $a, b$ represent the spacial directions of membrane world-volume.

An interesting subject to study is the brane charge in our model. The brane charges are modified in the nontrivial background [9, 28]. In addition, it would be interesting to study an open supermembrane theory on the time-dependent plane-wave background as in [9, 30] because this background has quite special properties and so the classification of Dirichlet branes are also nontrivial. All of the above topics will be reported in our next paper [22].

One of the most interesting subjects is the stability of a single supermembrane on the time-dependent background. In the flat case a single supermembrane is unstable as is well known [29]. This fact is deeply based on the cancellation of zero-point energies between bosons and fermions. However, it is not clear in general whether the bosonic zero-point energy cancels out the fermionic one. Remarkably, our result on the vacuum of the abelian part of the matrix model here should support that the zero-point energies cannot cancel out. Moreover, the flat directions are left in the time-dependent homogeneous plane-wave in contrast with the maximally supersymmetric pp-wave case. This issue is an interesting future work.

Finally, we comment on the relation of our matrix model to the time-dependent background as a closing remark. The homogeneous plane-wave background we have considered can be mapped to the explicitly time-dependent plane-wave background via the time-dependent coordinate transformation. However, once we move to the stationary frame in which the time dependence is not manifest, the effect of the time dependence can be
expressed by a kind of the vector potential, which leads to the constant magnetic field. Thus, we might expect the relationship of our model to the noncommutative geometry or Seiberg-Witten map \[31\]. In addition, the plane-wave geometry considered here is closely related to the Lewis-Riesenfeld invariant theory \[32\] as noted in \[13\]. This theory leads us to the quantization of the system of time-dependent harmonic oscillators. Study of such a system would be a useful laboratory to promote the understanding of the time-dependent backgrounds and to develop the techniques to treat theories on them. We believe that our model should be a clue to shed light on the physics on the time-dependent backgrounds.

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A. Matrix model on the general homogeneous plane-wave

Here we shall discuss the matrix model on the general homogeneous plane-wave background (anti-Mach space) whose metric is \(\mathcal{L}_1\) and the four-form flux \(\mathcal{L}_2\) is equipped. As discussed in \[20\], this background admits extra Killing spinors satisfying

\[
\left[\gamma_i, \theta^2 + 6\theta \gamma_i, \theta + 9\theta^2 \gamma_i + 6\gamma_i [\theta, \phi] + 18[\theta, \phi] \gamma_i - 144(A_{ij} \gamma^j + f_{ij} f_{jk} \gamma^k)\right] \epsilon = 0, \tag{A.1}
\]

\[
\epsilon(x^+) = e^{\pm \frac{1}{2}(\phi + \frac{1}{6} W)} \epsilon_0 \tag{A.2}
\]

where we have defined as

\[
\theta \equiv \frac{1}{3!} \xi_{ijk} \gamma^{ijk}, \quad \phi \equiv \frac{1}{2} f_{ij} \gamma^{ij}, \tag{A.3}
\]

and \(\epsilon_0\) is a constant spinor.

The matrix theory action on the background is given by

\[
S = \frac{1}{2} \int d\tau \text{Tr} \left[ (D_\tau X^i)^2 + \frac{1}{2} [X^i, X^j]^2 + \mu^2 A_{ij} X^i X^j - 2\mu f_{ij} X^i D_\tau X^j + i\psi^T D_\tau \psi - \psi^T \gamma_i [X^i, \psi] + \frac{\mu}{2} i\psi^T (W + \phi) \psi + 2 \frac{i\mu}{3} \xi_{ijk} X^i X^j X^k \right], \tag{A.4}
\]

where the matrix \(W\) and the covariant derivative are defined as, respectively,

\[
W \equiv -\frac{1}{2} \theta, \quad D_\tau * = \partial_\tau * - i[\omega, *]. \tag{A.5}
\]

The action (A.4) is invariant under the dynamical supersymmetry transformations

\[
\delta_\epsilon X^i = i\psi^T \gamma^i \epsilon, \quad \delta_\epsilon w = i\psi^T \epsilon, \tag{A.6}
\]

\[
\delta_\epsilon \psi = \left[ D_\tau X^i \gamma_i - 2\mu f_{ij} X^i \gamma^j - \frac{i}{2} [X^i, X^j] \gamma_{ij} - \frac{\mu}{6} X^i \gamma_i (W - 3\phi) - \frac{\mu}{2} X^i (W + \phi) \gamma_i \right] \epsilon, \tag{A.7}
\]
provided that
\[
[\gamma_i W^2 + 6W\gamma_i W + 9W^2\gamma_i - 3\gamma_i [W, \phi] - 9[W, \phi] \gamma_i 
- 36(A_{ij} \gamma^j + f_{ij} f_{jk} \gamma^k)]\epsilon = 0,
\]
\[\epsilon = e^{\frac{i}{2}(W-3\phi)\tau} \epsilon_0.\]  
(A.8)

The Eqs. (A.8) and (A.9) are nothing but (A.1) and (A.2) respectively. The action (A.4) is also invariant under the kinematical supersymmetries
\[
\delta \eta X_i = \delta \eta \omega = 0, \quad \delta \eta \psi = \eta(\tau), \quad \eta(\tau) = e^{-\frac{\eta_0}{2}(W+\phi)\tau} \eta_0 \]
(A.10)
where \(\eta_0\) is a constant spinor.

B. Analysis of \(X^1\) and \(X^2\) directions in the \(P \neq 0\) case

Here we will examine the \(X^1\) and \(X^2\) directions in the \(P \neq 0\) case where the analysis of these directions are quite nontrivial and complicated due to the presence of the nonzero \(f_{ij}\). The Lagrangian of this part is given by
\[
L = \frac{1}{2} \left((\dot{X}^1)^2 + (\dot{X}^2)^2 - 4\mu^2(1 - P^2)(X^2)^2 + 2\mu P(X^1 \dot{X}^2 - X^2 \dot{X}^1)\right). \]
(B.1)

By using the standard procedure, the Hamiltonian can be derived as follows:
\[
H = \frac{1}{2}(\pi^1 + \mu PX^2)^2 + \frac{1}{2}(\pi^2 - \mu PX^1)^2 + 2\mu^2(1 - P^2)(X^2)^2. \]
(B.2)

Here we quantize the system by imposing the commutation relation:
\[
[X^i, \pi^j] = i\delta^{ij},
\]
and will solve the energy eigenvalue problem:
\[
H\Psi = E\Psi. \]

When we decompose the \(\Psi\) as \(\Psi = e^{i\mu PX^1 X^2} \chi\), we can obtain the equation for the \(\chi\) described as
\[
\tilde{H}\chi = E\chi, \quad \tilde{H} \equiv \frac{1}{2}(\pi^1 + 2\mu PX^2)^2 + \frac{1}{2}(\pi^2)^2 + 2\mu^2(1 - P^2)(X^2)^2. \]
(B.3)

The expression \(\chi = e^{ip_1 X^1} \phi\), where \(p_1\) is the momentum for the \(X^1\)-direction, allows us to rewrite the Hamiltonian \(\tilde{H}\) as
\[
\tilde{H} = \frac{1}{2}(\pi^2)^2 + \frac{1}{2}(2\mu)^2 \left(X^2 + \frac{1}{2\mu}p_1 P\right)^2 + \frac{1}{2}(p_1)^2(1 - P^2). \]
(B.4)

Thus, we can see that the effect of a parameter \(P\) is to shift the origin of harmonic oscillation of \(X^2\)-direction and to rescale the kinetic energy (continuous spectrum) of \(X^1\).

Finally, we would like to note that the final expression of Hamiltonian is closely related to the invariant in the context of the Lewis and Riesenfeld theory [32], which can quantize the system of time-dependent harmonic oscillator.
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