In" on"u - Wigner Contraction of Kac-Moody Algebras

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ABSTRACT

We discuss In" on"u-Wigner contractions of affine Kac-Moody algebras. We show that the Sugawara construction for the contracted affine algebra exists only for a fixed value of the level \( k \), which is determined in terms of the dimension of the uncontracted part of the starting Lie algebra, and the quadratic Casimir in the adjoint representation. Further, we discuss contractions of \( G/H \) coset spaces, and obtain an affine translation algebra, which yields a Virasoro algebra (via a GKO construction) with a central charge given by \( \text{dim}(G/H) \).
1. Introduction

Contractions of finite dimensional Lie algebras were introduced several decades ago by Inönü and Wigner\(^1\), and applied successfully to recover the Galilei group (and its representations) from the Lorentz group. Subsequently, group contractions were used to retrieve the Poincare group from the de-Sitter group in various dimensions. This idea has been used very successfully in obtaining irreducible representations of the super-Poincare algebra from those of the super-de-Sitter algebra which has a compact even (bosonic) subalgebra\(^2\).

With the resurgence of string theory, affine Lie (Kac-Moody) algebras have become the subject of very intense research\(^3,4\). Recently they have come into even more focus (especially in their coset versions) following the formulation of the two dimensional black hole as a gauged \(SL(2, R)/U(1)\) Wess-Zumino-Witten model\(^5\). However, it is somewhat surprising that the issue of contraction of affine Lie algebras has not received any attention in the literature. Since the method of group contractions remains a very powerful tool in obtaining noncompact (and non-semisimple) algebraic structures from compact semisimple Lie algebras, it is conceivable that the generalization of this to the infinite dimensional case of affine Kac-Moody (KM) algebras will have ramifications for string theories. With most attempts at construction of higher (i.e. \(> 2\)) dimensional noncritical strings stymied at present by the so-called \(c = 1\) barrier this may be a worthwhile approach.

Indeed, if this barrier can be attributed to the existence of tachyons in the spectrum, as has been argued most articulately by Seiberg\(^6\), then one might consider introducing spacetime supersymmetry to eliminate them. Thus, noncritical Green-Schwarz type theories, if they exist, might be a viable way out. From a geometrical standpoint, such theories may be formulated as Wess-Zumino-Witten models on (super)group manifolds corresponding perhaps to the super-Poincare group (or some coset). Since, as already mentioned, the super-Poincare algebra can be obtained by contraction of superalgebras which are gradings of compact semisimple Lie algebras, generalization of the contraction procedure to the case of affine (super) Kac-Moody
algebras is likely to become important. This is the motivation of this rather preliminary investigation.

Here we consider the simplest generalization of the Inönü-Wigner contraction applicable to affine Lie algebras (generically denoted by $\hat{g}$) corresponding to finite dimensional semisimple compact Lie groups (generically denoted by $G$). We find rather stringent restrictions on the possible central extensions of the contracted affine algebra, which can be best understood in terms of the Cartan-Killing metric of $G$. A straightforward Sugawara construction of the Virasoro algebra associated with the contracted algebra seems to work only for a fixed value of the level $k$ of the initial KM algebra, which can be determined in terms of the dimension of the uncontracted part of the algebra (assumed semisimple) and its quadratic Casimir operator in the adjoint representation. These restrictions are seen to disappear for affine $G/H$ coset spaces under contraction. In this latter case, the Sugawara construction (via the GKO$^4$ technique) is seen to exist for any value of $k$ and leads to a Virasoro algebra with central charge given by $\text{dim}(G/H)$. As we discuss briefly in the concluding section, this could have applications in non-critical superstring theories.

We should perhaps mention that our approach is not completely rigorous, but we expect that the main results will survive a more careful investigation.

2. Inönü-Wigner Contraction

Let $T^a, a = 1, \ldots \text{dim}G$ be a basis of generators of the Lie algebra $g$ of the compact semisimple Lie group $G$. We make the following decomposition of $g$ in the sense of a vector space:

$$g = V_R \oplus V_C,$$

(2.1)

where, $T^a, \alpha = 1, \ldots \text{dim}V_R$ is a basis for $V_R$ and $T^i, i = 1, \ldots \text{dim}V_C$ is a basis for $V_C$. We make the following transformation on $T^a$

$$\tilde{T}^a = U^{ab}(\epsilon)T^b,$$

(2.2)
where,

\[ U^{\alpha \beta} = \delta^{\alpha \beta} \]
\[ U^{ij} = f(\epsilon)\delta^{ij} \]  
(2.3)
\[ U^{i\alpha} = 0 = U^{\alpha i} . \]

Here \( f(\epsilon) \) has the property that \( f(0) = 0, f(1) = 1 \). Thus the \( T^\alpha \) are left invariant under this transformation. Now, the Lie algebra \( g \) is given by the commutation relations

\[ [T^a, T^b] = if^{abc}T^c , \]  
(2.4)

so that under the decomposition (2.1) and the subsequent transformation (2.2,3) the structure constants \( f^{abc} \) transform into new ones denoted by \( \tilde{f}^{abc} \), where

\[ \tilde{f}^{\alpha \beta \gamma} = f^{\alpha \beta \gamma} , \quad \tilde{f}^{\alpha \beta j} = f^{-1}(\epsilon)f^{\alpha \beta j} \]
\[ \tilde{f}^{\alpha ij} = f(\epsilon)f^{\alpha ij} , \quad \tilde{f}^{ijk} = f^2(\epsilon)f^{ijk} . \]
(2.5)

Thus if the limit \( \epsilon \to 0 \) is to exist, one must assume that the structure constants \( f^{\alpha \beta i} = 0 \). In this limit therefore, one obtains, under this assumption, a new algebra \( \tilde{g} \), given by the commutation relations

\[ [\tilde{T}^\alpha, \tilde{T}^\beta] = if^{\alpha \beta \gamma}\tilde{T}^\gamma \]
\[ [\tilde{T}^\alpha, \tilde{T}^i] = if^{\alpha ij}\tilde{T}^j \]  
(2.6)
\[ [\tilde{T}^i, \tilde{T}^j] = 0 . \]

The Lie algebra \( \tilde{g} \) is the contraction of \( g \) according to Inönü and Wigner. Observe that \( \tilde{g} \) contains a ‘translation’ subalgebra of dimension \( \text{dim}V_C \). However, one should remark that contractions are not unique and one could have certainly contracted \( g \) leaving any other subalgebra \( V'_R \) invariant under the contraction. In the following, we shall restrict this ambiguity to a certain extent by allowing for \( V'_R \)s which are themselves compact and semisimple.

It is of some interest to examine the effect on the Cartan-Killing metric. The Cartan-Killing metric \( K_{ab} \) for \( g \) can be expressed in terms of the structure constants as \( K_{ab} = f^d_{ac}f^e_{bd} \). Thus, under the transformation (2.2,3) and in the limit \( \epsilon \to 0 \), the
only non-vanishing components of the new metric (i.e., that corresponding to \( \tilde{g} \)) are the ones with Greek indices, and for these, \( \tilde{K}_{\alpha\beta} = K_{\alpha\beta} \). This property of the metric of the contracted algebra plays an important role in restricting the possible central extensions of the affinized contracted algebra, as we shall show in the sequel.

3. Contraction of affine KM Algebras

To generalize the foregoing to the case of affine KM algebras, we first consider the classical case, i.e. the affine algebra without the central extension (loop algebra). Following ref. 4, we define maps \( \gamma : S^1 \rightarrow G \) which are given near the identity by

\[
\gamma(z) \approx 1 - iT^a \theta^a(z),
\]

where the parameters \( \theta^a(z) \) are holomorphic functions on the unit circle \(|z| = 1\), admitting the Laurent expansion

\[
\theta^a(z) = \sum_n T^a z^n \theta^a_{-n}.
\]

Defining the affine basis \( T^a_n \equiv T^a z^n \), we immediately get the loop algebra \( \hat{g}_0 \)

\[
[T^a_m, T^b_n] = i f^{abc} T^c_{m+n} . \tag{3.1}
\]

It is now straightforward to proceed exactly as in the last section. One decomposes the basis set of the generators

\[
\hat{g}_0 = \hat{V}_R \oplus \hat{V}_C ,
\]

and applies the transformation (2.2,3) to the corresponding basis elements \( T^\alpha_m \) and \( T^i_m \). In the limit as \( \epsilon \rightarrow 0 \), we get the unique contraction \( \tilde{g}_0 \) of \( \hat{g}_0 \) given by

\[
[\tilde{T}^\alpha_m, \tilde{T}^\beta_n] = i f^{\alpha\beta\gamma} \tilde{T}^\gamma_{m+n},
\]

\[
[\tilde{T}^\alpha_m, \tilde{T}^i_n] = i f^{\alpha ij} \tilde{T}^j_{m+n} , \tag{3.2}
\]

\[
[\tilde{T}^i_m, \tilde{T}^j_n] = 0 .
\]

The parameters \( \tilde{\theta}^\alpha_{-n} \) are related to the original \( \theta^\alpha_{-n} \) through the relations

\[
\theta^\alpha_{-n} = \tilde{\theta}^\alpha_{-n}, \quad \theta^i_{-n} = f(\epsilon) \tilde{\theta}^i_{-n} .
\]
Thus in the limit $\epsilon \to 0$, the $\theta^i_n$ vanish, quite akin to the finite dimensional case.

We next incorporate the central extension for the affine KM algebra which we call $\hat{g}$, given by

$$ [T^\alpha_m, T^\beta_n] = i f^{\alpha \beta \gamma} T^\gamma_{m+n} + \frac{1}{2} km \delta^{\alpha \beta} \delta_{m+n,0} . \quad (3.3) $$

In order to generalize the contraction procedure described above, one is faced with two choices. One could apply the procedure directly to the affine KM algebra (3.3) and thereby construct what one might call $\tilde{\hat{g}}$. Alternatively, one might contract $g$ to $\tilde{g}$ first and then affinize to obtain $\hat{\tilde{g}}$. The key issue therefore is whether $\tilde{\hat{g}} \sim \hat{\tilde{g}}$? Indeed, for a given subalgebra $V_R$ which is left invariant under the contraction, the answer to the above query must be in the affirmative if one is to give any meaning to the contraction procedure in this infinite dimensional case.

The straightforward application of the contraction procedure to (3.3) is seen to lead to the following commutation relations for $\tilde{\hat{g}}$ (we drop the tildes for ease of writing):

$$ [T^\alpha_m, T^\beta_n] = i f^{\alpha \beta \gamma} T^\gamma_{m+n} + \frac{1}{2} km \delta^{\alpha \beta} \delta_{m+n,0} $$

$$ [T^\alpha_m, T^i_n] = i f^{\alpha ij} T^j_{m+n} \quad (3.4) $$

$$ [T^i_m, T^j_n] = 0. $$

Thus, one does not obtain an affine translation algebra, as one might have expected.

The central extension of $\hat{g}$ given in terms of the level $k$ leads to a vanishing central extension for the translation algebra upon contraction. If we were to scale $k$ by a power of $f(\epsilon)$, the first of the equations (3.4) would be adversely affected as $\epsilon \to 0$, which is undesirable.

On the other hand, if we were to contract $g$ first and then affinize in the standard fashion (notwithstanding the non-semisimple nature of $\tilde{g}$), naïvely, the translation subalgebra might be taken to have a structure

$$ [T^i_m, T^j_n] = \frac{1}{2} km \delta^{ij} \delta_{m+n,0} . \quad (3.5) $$

But it is easy to rule out the central extension on the rhs of (3.5) by using the Jacobi
Thus, given a contraction of \( g \) which leaves a subalgebra \( V_R \) invariant, there is a unique contraction given by (3.4) independent of which of the choices in procedure one makes.

This uniqueness can be traced to the Cartan-Killing metric of the contracted algebra. Recall that this is non-vanishing only in directions left unaffected by contraction. It follows that free field realizations of the contracted algebra exist only for the compact, semisimple \textit{uncontracted} subalgebra. E.g., for contraction of \( \widehat{so}(3) \) to \( \widehat{e}(2) \), a free field realization of the latter can only be given for the \( \widehat{so}(2) \) subalgebra. Hence, the fact that one is able to retrieve a translation algebra by this contraction seems not to be very useful from the physical point of view, since all the physics appears to pertain solely to the part of the original algebra that has been left uncontracted.

These restrictions can be better articulated by considering a Sugawara construction of the contracted algebra. The Virasoro generators are defined as usual a la’ Sugawara\(^4\) as

\[
\kappa L_n \equiv \sum_{m=-\infty}^{\infty} : T^\alpha_{n-m} T^\alpha_m : , \tag{3.6}
\]

which, in terms of the generators \( T^\alpha_m \), \( T^i_m \) of the contracted algebra can be rewritten as

\[
\kappa L_n = \sum_m : \{ T^\alpha_{n-m} T^\alpha_m + T^i_{n-m} T^i_m \} . \tag{3.7}
\]

The constant \( \kappa \) is to be determined by requiring that

\[
[L_n, T^\alpha_m] = -m T^\alpha_{m+n} ,
\]

\[
[L_n, T^i_m] = -m T^i_{m+n} . \tag{3.8}
\]

Now, the first of these equations is easily seen to imply that

\[
\kappa = k + c_{V_R} \ , \tag{3.9a}
\]

where \( c_{V_R} \) is the quadratic Casimir operator of the uncontracted subalgebra \( V_R \) in the adjoint representation. The second of the equations (3.8) leads to a strange result

\[
\kappa = \frac{F}{2 \dim V_C} , \tag{3.9b}
\]
where, $F \equiv f^{aij} f^{aij}$. One is thus led to the very important restriction that the Sugawara construction, applied straightforwardly, works only for a specific value of the level $k$ (obtained by solving (3.9) for $k$).

Indeed, for the contraction of $\widehat{so}(3)$ to $\widehat{e}(2)$, this value of $k$ is easily seen to be unity. The Sugawara construction would then imply that the central charge of the Virasoro algebra is $\tilde{c} = 1$, a result that is identical in this case to the Virasoro central charge for $\widehat{so}(3)$. As is well-known, the level 1 $\widehat{so}(3)$ can be realized in terms of a single boson. Thus the contracted algebra in this case also has a free boson realization. This corroborates our earlier remarks regarding possible free field realizations of the contracted affine algebra.

The Virasoro algebra ensues after a straightforward calculation using eqn.s (3.5-6) and yields a central charge $\tilde{c}_k$ given by

$$\tilde{c}_k = \frac{kd\text{im}V_R}{\kappa}.$$  \hspace{1cm} (3.7)

Clearly, for the contraction of $\widehat{so}(3)$ to $\widehat{e}(2)$, we have, with $k = 1$, $\tilde{c}_k = 1$. This therefore implies that our contraction procedure leads uniquely to the realization of the contracted algebra in terms of a single free massless scalar field. In this particular case of $so(3)$, the central charge for the level one KM algebra is the same as that of the contracted algebra $e(2)$. This is not generically true. But the above restriction does have the disconcerting feature that the translation subalgebra (obtained upon contraction) is rendered physically irrelevant. The contraction procedure is then devoid of any physical interest, at least in its simplest version. It is conceivable that a more complicated contraction, e.g., one in which the contraction parameter $\epsilon$ is itself taken to be a function $\epsilon(z)$ on the circle $S^1$, will yield a more physically interesting result. We do not pursue such alternatives in this short note. Instead we analyze contractions of affine coset spaces, generalizing the standard procedure appropriate to finite dimensional $G/H$ cosets\(^7\).
4. Contraction of Coset Spaces

Referring back to our discussion in Section 2, let us identify the subalgebra \( h \) corresponding to the subgroup \( H \) with respect to which \( G \) is to be quotiented, with the subalgebra \( V_R \) which is left undisturbed by the contraction. In that case, the relevant commutation relation is

\[
[T^i, T^j] = i(f^{ij\alpha}T^\alpha + f^{ijk}T^k) .
\] (4.1)

Proceeding as in Section 2, one obtains in the limit \( \epsilon \to 0 \), a translation algebra of dimensionality \( \dim(G/H) \)

\[
[\tilde{T}^i, \tilde{T}^j] = 0 ,
\] (4.2)

where we have put back the tildes to exhibit that these correspond to the contracted generators. One can then affinize (4.2) in the standard fashion and obtain an affine translation algebra with a central extension

\[
[\tilde{T}^i_m, \tilde{T}^j_n] = \frac{1}{2}k\delta^{ij}\delta_{m+n,0} ,
\] (4.3)

where \( \tilde{k} \neq k \) in general.

Before discussing the Sugawara construction for (4.2), let us mention that (4.3) is the unique (in the aforementioned sense) contraction of the affine coset space for \( G/H \). In other words, if we affinize (4.1) first to get

\[
[T^i_m, T^j_n] = if^{ija}T^a + \frac{1}{2}k\delta^{ij}\delta_{m+n,0}
\] (4.4)

and then contract (4.4) as in section 3, we do get back (4.3) provided we rescale the level \( k \) appropriately to \( \tilde{k} \). Since the central extension given by \( k \) is an operator which commutes with all the generators \( T^i \), it is legitimate in the contraction procedure to transform it as well. This was ruled out in the last section because there such a transformation would have affected the central extension corresponding to the uncontracted part of the algebra, a feature that we wanted to avoid.

For the Sugawara construction appropriate to the contracted affine coset algebra (4.3), we follow ref. 4 and define the Virasoro generators as

\[
\tilde{k}L_n \equiv \sum_m :\tilde{T}^i_m \tilde{T}^i_{-n-m} : .
\] (4.5)
Requiring as before that
\[
[\tilde{L}_n, \tilde{T}^i_m] = -m \tilde{T}^i_{m+n} \tag{4.6}
\]
immediately fixes the constant \( \tilde{\kappa} = \tilde{k} \). The central charge of the corresponding Virasoro algebra is then easily calculated by evaluating the \([\tilde{L}_n, \tilde{L}_m]\) commutator explicitly, and one obtains \( \tilde{c} = \text{dim}(G/H) \), a result that one has already anticipated given that (4.3) is an affine translation algebra in a space of dimensionality \( \text{dim}(G/H) \). Notice that the central charge is independent of the level \( \tilde{k} \).

Thus the contracted affine \( G/H \) coset space can be given a free field realization by \( \text{dim}(G/H) \) massless scalar fields in two dimensions, corresponding to a free bosonic string in these dimensions. If \( \text{dim}(G/H) \neq 26 \), such a theory will be consistent only when the Liouville mode is taken into account.

5. Discussion

We have not yet analyzed the consequences of the results of the last section for general gauged WZW models. Presumably, the contraction procedure collapses the interacting WZW theory into a free field theory. What this means for the two-dimensional black hole formulated as a gauged \( \text{SL}(2,R)/\text{U}(1) \) WZW model is an interesting question. A general theory of contraction of irreducible representations and field theoretic realizations of KM algebras is indeed in order, but beyond the scope of this short note.

Another interesting application of the foregoing analysis could be to the contraction of affine \( super \)-Lie algebras and super-cosets, especially the case of the quotient of a supergroup by a bosonic subgroup. Henneaux and Mezincescu\(^8\) have given a formulation of the type II ten dimensional Green Schwarz superstring as a WZW model on a super-coset

\[
N = 2 \quad D = 10 \quad \text{super – Poincare group} \quad \frac{\text{SO}(9,1)}{\text{SO}(9,1)}.
\]

For the \( N = 1 \) case, similar formulations have also been considered, via reduction from super-Chern Simons theories wherein the gauge fields take values in the appropriate
super-coset\textsuperscript{9}. The question now is, whether Green Schwarz superstrings in lower dimensions (3,4, e.g.) can be considered as gauged WZW models on super-cosets. If so, then it might be useful to begin with some supergroup like $SU(2|1)$ which is a grading of a classical group, and then contract the quotient $SU(2|1)/SO(3)$ to get

$$N = 1 \quad D = 3 \quad \text{Euclidean Poincare superalgebra}$$

$SO(3)$.

A WZW model on such a coset might be a candidate for a non-critical ‘Green Schwarz string’ in three dimensional Euclidean superspace. We hope to report on these and other related issues in a future publication.

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