Test of an Equivalence Theorem at One-Loop

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ABSTRACT

We show that the equivalence theorem approximating one-loop gauge sector diagrams by including only Goldstone bosons in the loop gives a remarkably poor approximation to the amplitude for the decay $H \rightarrow \gamma\gamma$ and for the process $\gamma\gamma \rightarrow HH$. At one loop, large logarithms can arise that evade power counting arguments.
The standard equivalence theorem [1] has become a popular method for approximating difficult calculations. Amplitudes involving Goldstone bosons substituted for external longitudinal gauge bosons are much easier to calculate. It has been proven to all orders in perturbation theory using power counting arguments at least for external gauge bosons. Another possible application of this general concept is to truncate one-loop calculations involving internal gauge bosons to only those diagrams with just Goldstone bosons (no internal gauge bosons or ghosts) [2,3]. This results in a separately finite and gauge invariant sum and is clearly a simpler task than performing the full calculation. In this short note we present examples where this equivalence theorem (ET) performs poorly. We find that the large logarithms that can appear at one-loop destroy the approximation. We choose processes that are absent at tree level and first occur at one-loop. In this way we are able to avoid any subtleties related to renormalization and concentrate on the aspects that arise beyond tree level but are not specifically related to any renormalization scheme. We do not argue that this ET is invalid; rather the asymptotic approach to the limit can be gradual and the predicted rates in physically interesting processes can be unreliable.

We discuss two processes in the Standard Model that are phenomenologically interesting. One is well-known [4] and the other has been considered relatively recently [3]. First consider Higgs decay to two photons, $H \rightarrow \gamma\gamma$. The full one-loop amplitude has been known for some time [4], and this process may serve as an interesting theoretical laboratory for the ET. The ET can be employed to calculate the Feynman diagrams involving the gauge boson sector in the loop. Generic diagrams are shown in Figure 1. In the full calculation there are 26 diagrams, while only three diagrams contribute to the ET approximation. We have calculated these diagrams in the Feynman gauge using the symbolic manipulation programs FORM and MATHEMATICA. We have used the algorithms for reducing one-loop integrals developed by van Oldenborgh and Vermaseren [5]. This technique gives entirely analytic expressions for the matrix elements. Gauge invariance is checked analytically for the resulting expressions.

We find that the one-loop decay rate for the $W$ boson loops (not counting fermion loops)
determined by using the ET is
\[ \Gamma^{ET} = \frac{\alpha^2 G_F M_H^3}{16 \sqrt{2} \pi^3} \left| 1 + 2 M_W^2 C(p_1, p_2) \right|^2, \] (1)

The full calculation including all 26 diagrams yields
\[ \Gamma^{FULL} = \frac{\alpha^2 G_F M_H^3}{16 \sqrt{2} \pi^3} \xi_1 \left( 1 + 2 M_W^2 C(p_1, p_2) \right) - 8 M_W^2 C(p_1, p_2)^2, \] (2)

where
\[ \xi_1 = \left[ 1 + 6 \frac{M_W^2}{M_H^2} \right], \] (3)

and \( C(p_1, p_2) \) is the usual scalar three-point integral with two massless external lines (see below) and can be expressed in logarithms alone,
\[ C(p_1, p_2) = \frac{1}{2 M_H^2} \ln^2 \left( \frac{-z}{1 - z} \right), \] (4)

with
\[ z = \frac{1}{2} \left[ 1 + \sqrt{1 - \frac{4 M_W^2 - i \epsilon}{M_H^2}} \right]. \] (5)

In the very small \( M_W^2/M_H^2 \) limit, \( C(p_1, p_2) \) behaves as \( 1/(2 M_H^2) \ln^2(M_W^2/M_H^2) \), and the sub-leading term in \( \Gamma^{FULL} \) cannot be neglected even for a heavy Higgs boson. It is perfectly natural to expect logarithms and dilogarithms to arise in one-loop graphs where integration over the loop momentum is performed.

In Figure 2 we plot the ratio \( \Gamma^{ET}/\Gamma^{FULL} \) against the Higgs mass \( M_H \). Even for a Higgs boson as heavy as 1 TeV, the decay rate has not begun to display the asymptotic behavior prescribed by the ET. Eventually the ratio approaches one but only for unrealistically large Higgs masses.

The argument presented so far might be considered only academic since the full calculation for \( H \rightarrow \gamma \gamma \) is well-known and easily obtained, so we have also explored the effectiveness of the ET in the more complicated process \( \gamma \gamma \rightarrow HH \). This process has been suggested as a possible method of measuring the triple-Higgs vertex in the Standard Model. The ET calculation of the \( W \) boson loop contribution has been recently computed \[7\]. We have performed
the full one-loop calculation and find the ET calculation to be inaccurate in the region of interest. There are 188 one-loop Feynman diagrams in the full calculation compared to 22 in the ET approximation. See Figure 3.

We have confidence in our result for the following reasons: (1) We have checked analytically that $p^\mu T_{\mu\nu} = 0$ and $p^\nu T_{\mu\nu} = 0$ where $T_{\mu\nu}$ is the polarization tensor for $\gamma\gamma \to HH$. (2) We have used our program to reproduce published helicity amplitudes for $gg \to HH$ (quark loop) [6], $gg \to ZZ$ (quark loop) [7], and the equivalence theorem part of $\gamma\gamma \to HH$ [3]. (3) The simple diagrams in Figure 3 (but not the boxes) were checked versus the program FeynCalc/FeynArts [8].

The helicity amplitudes can be expressed in a compact form using the results obtained in the ET approximation in Ref. [3]. We find

$$M_{+++}^{\text{FULL}} = \xi_2 M_{+++}^{\text{ET,3a}} + \xi_1 M_{+++}^{\text{ET,3b}} + 2\left( -Y D(p_1, p_3, p_2) + 2t_1 C(p_1, p_3) + 2u_1 C(p_2, p_3) + 6s \frac{M_H^2}{s_1} C(p_1, p_2) \right),$$

where

$$\xi_2 = \left[ 1 - 4 \frac{M_W^2}{M_H^2} \left( M_H^2 - 3M_W^2 + s \right) \right],$$

$$\xi_3 = \left[ 1 - 4 \frac{M_W^2}{M_H^2} \left( M_H^2 - 3M_W^2 - s \right) \right],$$

and $s_1 = s - M_H^2$, $t_1 = t - M_H^2$, $u_1 = u - M_H^2$, $Y = tu - M_H^2$. The matrix elements $M_{+++}^{\text{ET,3a}}$, $M_{+++}^{\text{ET,3b}}$ and $M_{+++}^{\text{ET,3a}}$ are those obtained in the ET approximation and given in Ref. [3] as $M_0(\text{box})$, $M_0(\text{triangle})$ and $-M_2(\text{box})$ respectively (The minus sign is simply a matter of our convention for the polarization vectors. The $M_W^2$ in the last line of $M_2(\text{box})$ should be $M_H^2$). The indices $3a$ and $3b$ refer to the diagrams in Figure 3. The scalar triangle and box diagrams are defined as
The momenta of the incoming photons are $p_1$ and $p_2$ while the outgoing Higgs bosons have momenta $p_3$ and $p_4$. In the limit $M_W^2/M_H^2 \to 0, M_W^2/s \to 0$, the factors $\xi_1$, $\xi_2$ and $\xi_3$ approach one with only power law corrections. On the other hand the additional terms do not become negligible immediately.

We do not list separately the contributions from the diagrams in Figure 3a and Figure 3b because in the full calculation they are no longer separately gauge invariant. The graphs for $\gamma\gamma \to H$ shown in Figure 1 are certainly a gauge invariant set, but once the Higgs is allowed off-shell as in Figure 3b, a contribution from the graphs in Figure 3a must be included to obtain gauge invariance. This is not the case for either the subset of diagrams in the ET or for fermion loop contributions. Details of this calculation and issues of phenomenological interest will be presented in a longer paper.

In Figure 4 we compare the cross sections given by the full calculation and given by the ET. The two converge in the appropriate limit, but this convergence is quite mild. The disagreement is most severe for unequal photon helicities, $\lambda_1 = -\lambda_2$. As the center of mass energy increases the approximation gets better, but even at $s = 2$ TeV there is a substantial discrepancy.

We have found that large logarithms that arise at one-loop limit the effectiveness of this ET. We believe this behavior is a general feature of such calculations, and one must be careful not to place too much confidence in such ET calculations beyond the tree level. At tree level with internal gauge boson lines, this type of logarithm is absent, and the ET should converge quite rapidly to the full result in the appropriate limit. We have not specifically addressed the issue of one-loop diagrams with the external longitudinal gauge bosons replaced with Goldstone bosons. We believe large logarithms potentially plague these approximations as well.
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Figures

**Figure 1**: Generic diagrams in the $W$ boson loop contribution to $H \rightarrow \gamma\gamma$. The loops consist of all possible combinations of $W$ bosons, Goldstone bosons and ghosts. The number of nonzero diagrams is shown, and the number of diagrams in the equivalence theorem calculation is given in parentheses.

**Figure 2**: Comparison of the full calculation to the equivalence theorem approximation for $H \rightarrow \gamma\gamma$. The approximation only becomes good for unrealistic Higgs masses.

**Figure 3**: Generic diagrams in the $W$ boson loop contribution to $\gamma\gamma \rightarrow HH$. The loops consist of all possible combinations of $W$ bosons, Goldstone bosons and ghosts. The number of nonzero diagrams is shown, and the number of diagrams in the equivalence theorem calculation is given in parentheses.

**Figure 4**: Comparison of the full calculation to the equivalence theorem approximation for $\gamma\gamma \rightarrow HH$. The full calculation is given by solid lines and the ET result by dashed lines. (a) The cross sections for constant center of mass energy $s = 1$ TeV. (b) The cross sections for constant center of mass energy $s = 2$ TeV. (c) The cross section for $s = 8M_H$ for the case where the two photons have unequal helicity.