The Convergence of Iterative Delegations in Liquid Democracy

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Abstract

In this paper, we study liquid democracy, a collective decision making paradigm which lies between direct and representative democracy. One main feature of liquid democracy is that voters can delegate their votes in a transitive manner such that: A delegates to B and B delegates to C leads to A delegates to C. We study the stability (w.r.t. voters preferences) of the delegation process in liquid democracy and model it as a game in which the players are the voters and the strategies are possible delegations. This game-theoretic model enables us to answer several questions on the equilibria of this process under general preferences and several types of restricted preferences (e.g., single-peaked preferences).

1 Introduction

Liquid democracy is a collective decision making paradigm which lies between direct and representative democracy. One main feature of liquid democracy is the concept of transitive delegations. Indeed, in this setting each voter may decide to vote directly or to delegate her vote to a representative, also called proxy. In liquid democracy this proxy can in turn delegate her vote and the votes that have been delegated to her to another proxy. As a result, a voter who decides to vote has a weight corresponding to the number of people she represents, i.e., herself and the voters who directly or indirectly delegated to her. Such voter is called the guru of the people she represents.

This approach has been advocated recently by many political parties as the Pirate parties of Germany, Austria, Italy, Switzerland and Brazil, or the Sweden’s Demoex party. One main advantage of this framework is its flexibility, as it enables voters to vote directly for issues on which they feel both concerned and expert and to delegate for others. Another advantage is that, as any voter can be a proxy, voters who do not have the time or the will to invest themselves in the election are more likely to find a representative whom they trust than in a representative democracy setting where the set of possible delegates is much smaller. In this way, liquid democracy provides a middle-ground between direct democracy, which is strongly democratic but which is likely to yield high abstention rates or uninformed votes, and representative democracy which is more practical but which is less democratic [GA15, Wie13]. Indeed, while in direct democracy it is unrealistic to expect voters to have time and interest for every possible issues, in representative democracy the limited number of representatives briddles the representation’s accuracy of the delegation scheme.

Aim of the Paper. In this work, we tackle the problem of the stability of the delegation process in the liquid democracy setting. Indeed, it is likely that the preferences that each voter has over her possible delegates will be motivated by different criteria and possibly contrary opinions. Hence, the iterative process where voters choose their delegate may end up in an unstable situation, i.e., a situation in which some voters would change their delegations. A striking example to illustrate this point is to consider an election where the voters could be positioned on the real line in a way that represents their right-wing left-wing political identity. If voters are close enough, each voter, starting from the left-side, could agree to delegate to her closest neighbor on her right. By transitivity this would lead to all voters, including the extreme-left voters, delegating to an extreme-right voter.
These unstable situations raise the questions: “Under what conditions do the iterative delegations of the voters always reach an equilibrium? Does such an equilibrium even exist? Can we determine equilibria that are more desirable than others?” To answer these questions, we model the delegation process as a game where players are the voters involved in the election and each player seeks to minimize the rank of her guru in her preference order. We answer the questions raised above under general preferences and restricted preferences (e.g., single-peaked preferences).

The stability of the delegation process is one of the several algorithmic issues raised by the liquid democracy setting. These issues have recently raised attention in the AI literature. In Section 2, we present related works that have recently tackled other issues of the liquid democracy setting. Moreover, we also give a short review of other works that are related to game-theoretic models induced by voting problems. In Section 3, we formally define our setting of liquid democracy, introduce our notations and specify the various questions that we address. Then, while Section 4 answers these questions when preferences of the voters over gurus are unrestricted, Sections 5, 6 and 7 answer these questions for several types of restricted preferences. Finally, Section 8 concludes and discuss several possible future works.

2 Related Works

2.1 Related Works on Liquid Democracy

In the work of Green-Armytage [GA15], the author proposes a setting of spacial voting in which voters’ opinions on different issues of a vote are given by their position on the real line (one position per issue). Furthermore, the competence level of each voter is expressed as a random variable which adds noise to the estimates that she has about her position as well as the positions of the other voters. The votes and delegations are then prescribed by the competence levels of the agents, the noisy estimate that each voter has of her position and the noisy estimates that they have about the positions of the other voters. Green-Armytage defines the expressive loss of a voter as the squared distance between her vote and her position and proves that transitive delegations decrease on average the expressive loss of voters in his model. A closely related work is the one of Cohensius et al. [CMM+17]. They considered a (non-transitive) proxy setting and a spacial voting problem where only a small number of people out of a large population finally votes. In this setting, the authors compare voting with and without the use of proxies, and show that using proxies enables to increase the accuracy of the procedure in various cases, where the accuracy is given by a loss function measuring the distance to the outcome that would have resulted from the election if the whole population were to vote. Interestingly, the authors also consider a game-theoretic setting in which the possible voters may strategically decide to become inactive, rejoining the majority of the population that does not vote. The authors showed that their results extend to this strategic setting.

In the work of Christoff and Grossi [CG17], the authors study liquid democracy from the perspective of binary aggregation with abstentions. In a first part, the authors focus on two potential undesirable affects of liquid democracy: a high abstention rate due to cyclic delegations and the loss of rationality constraints when voters should vote on different issues that are logically linked and for which they delegate to different proxies. In a second part, the authors study an interpretation of liquid democracy in which delegations are replaced by influence links. The idea is that instead of delegating to a proxy, each voter may indicate an influencer from whom she will copy her vote. The influence links then define a process in which, at each time step, each voter copies the vote from her influencer. The authors study the conditions under which the induced process converges.

Another acknowledged drawback of liquid democracy is that some agents may amass an enormous voting power. This issue was addressed in a recent working paper by Gölz et al. [GKMP]. In their setting, each voter can decide to vote or to specify multiple delegation options. Then a centralized algorithm should select delegations to minimize the maximum voting power of an agent. The authors give an optimal method to solve this problem based on mathematical programming. They also give a polynomial time \((1 + \log(n))-approximation algorithm (where \(n\) is the number of voters) and show that approximating the problem within a factor \(\frac{1}{2}\log_2(n)\) is NP-hard. Lastly, the authors gave evidence that allowing voters to specify multiple possible delegation options (instead of one) leads to a dramatic decrease of the maximum voting power of a voter.

In the work of Kahng et al. [KMP18], the authors study an election on a binary issue, for
which there is one “correct” answer and one “incorrect” answer, and for which each voter has a competence level (i.e., a probability of choosing the desirable solution). They further assumed that the voters belong to a social network and that each voter only accepts to delegate to her neighbors in the social network who are “sufficiently” more expert than her. The authors investigate the accuracy of delegation procedures. A delegation procedure takes as input the social network and the competence level of every agents and outputs the probabilities with which each agent should vote or delegate to her approved neighbors. Their key result consists in showing that no “local” procedure (i.e., procedures s.t. the probability distribution for each voter only depends on her neighborhood) can guarantee that liquid democracy is, at the same time, never (in large enough graphs) less accurate and sometimes strictly more accurate than direct voting.

Lastly, in the work of Bloembergen et al. [BGL18], the authors consider a liquid democracy setting where voters are connected in a social network and where voters can only delegate to their neighbors in the network. The election is on a binary issue for which some voters should vote for the 0 answer (voters of type $\tau_0$) and the others should vote for the 1 answer (voters of type $\tau_1$). Each voter $i$ does not know exactly her type which is modeled by an accuracy $q_i \in [0.5, 1]$ representing the probability with which voter $i$ makes the correct choice. Similarly, each pair $(i,j)$ of voters do not know if they are of the same type which is also modeled by a probability $p_{ij}$. Hence, a voter $i$ which has $j$ as guru has a probability that $j$ makes the correct vote (according to $i$) which is a formula including the $p_{ij}$ and $q_j$ values. The goal of each voter is to maximize the accuracy of her vote/delegation. Importantly, each voter that votes incurs an $e_i$ loss of satisfaction representing the work required to vote directly. This modeling leads to a class of games, called delegation games. The authors proved the existence of pure Nash equilibria in several types of delegation games and gave upper and lower bounds on the price of anarchy, and the gain (i.e., the difference between the accuracy of the group after the delegation process and the one induced by direct voting) of such games.

The approach considered in this paper is closest to this last work as we will also consider a game on voters’ delegations. However, our point of departure is quite different. We study a liquid democracy setting in which each voter has a preference order over her possible gurus. This preference order may be dictated by competence levels and types of voters as in [BGL18]. However, we do not make such hypothesis as the criteria to choose a delegate are numerous: geographic locality, cultural, political or religious identity, economic situation, etcetera. This more general framework will lead to a different type of delegation games.

2.2 Related Works on Other Voting Games

The problem addressed in this paper is reminiscent of other game-theoretic work undergone in the voting literature. In particular, two game-theoretic models have been studied to capture the tendency of voters and candidates to manipulate elections, namely strategic voting games [MPRJ10] and strategic candidacy games [DJLB01].

Strategic voting games. In strategic voting games, the players are the voters of an election and their possible pure strategies are all possible linear order over the candidates of the election. The linear order chosen by a voter may or may not represent her true preferences over candidates. The outcome of the game is the winner of the election which is chosen according to a scoring rule and a tie-breaking rule. The goal of a voter is to report the “right” linear order to steer the outcome of the election towards the candidates that she likes most. This game-theoretic model encompasses in a natural way the tendency that voters have to misreport their true preferences to manipulate an election. One main question is then to know if such game can reach a Nash equilibrium, i.e., a state in which all voters are satisfied with their votes. However, all Nash equilibria of this voting game are not of interest. For instance, under plurality, a state in which all voters vote for the same candidate—whatever the candidate—is always a Nash equilibrium as changing one vote will not change the outcome of the election.

To filter out these “uninteresting” Nash equilibria, Meir et al. [MPRJ10] proposed to consider only the ones that can be reached by an iterative process of improving moves. This is the subject of iterative voting. In this setting, all voters vote and then, knowing only the result, may change their strategies, one at a time. Each state of this process is a ballot and, at each time step, one

\footnote{Note that a first step to merge these two types of games has been undertaken by Brill and Conitzer [BC15].}
agent can change her vote if this change improves the outcome of the election with respect to her preferences.

Note that, to filter out even more Nash equilibria, other constraints can be imposed to the voters [ROL+15, OMT13]. For instance, the voters can be considered to be truth biased in the sense that they always report their true preferences if misreporting them does not enable them to improve the outcome of the election. Other voters can be considered to be lazy in the sense that they prefer to abstain if their vote cannot improve the outcome of the election.

This iterative voting framework raises several questions: “Under what conditions does such process necessarily converge to a Nash equilibrium? If such process converges, can we have an upper bound on the convergence rate [GG13]? What is the complexity of deciding if a state is an equilibrium reachable from an initial state [ROL+15]?”.

Since the paper of Meir et al. [MPRJ10] which initiated the study of iterative voting, these questions have been answered in numerous settings depending on the scoring rule (e.g., plurality, veto, ...) or non-scoring rule (e.g., Maximin) that is used, the type of tie-breaking rule (i.e., random or deterministic, lexical or arbitrary) [LR12], the dynamics that are allowed [OMP+15] (i.e., the kind of moves that the agents can perform in the iterative process), and the level of information of the different agents [Mei15].

Strategic candidacy games. Differently, in strategic candidacy games, the players are the candidates involved in the election and the strategy set of each player consists of two strategies: run for the election or withdraw. In this setting, both voters and candidates have preferences over the candidates and while the voters are assumed to vote truthfully, any candidate may decide to withdraw from the election if she prefers the winner of the election restricted to the remaining candidates. These games capture possible manipulations induced by the candidates that have often been witnessed in real elections (e.g., in French local elections where the left or right parties may decide to withdraw to block the extreme right party).

The study of strategic candidacy games has been initiated by Dutta et al. [DJLB01, DJLB02], who formulated the game and showed that no voting rule that are non-dictatorial and unanimous can guarantee the stability of the truthful state, i.e., the state where all candidates run for the election. Lang et al. [LMP13] extended these results for various voting rules. In the case of 4 candidates and an odd number of voters, they showed that a Nash equilibrium always exists for Condorcet-consistent rules. Unfortunately, they also showed that this is not the case for most scoring rules and that the voting rules that guarantee the existence of a Nash equilibrium become scarce as the number of candidates increases. One positive result is the case of the Copeland rule which always guarantees the existence of a Nash equilibrium if the number of voters is odd.

The convergence of strategic candidacy games under improvement dynamics has been recently studied by Polukarov et al. [POR+15] under plurality. In their setting, the candidates are free to enter or leave the election. However, if a candidate decides to leave, there is a probability that each voter will not take her under consideration if she decides to run for the election again. This probability is a parameter of the problem capturing the fact that some voters may no longer have confidence in a candidate who has already let them down. In this framework, the authors studied the complexity of 3 problems: deciding if there exists a Nash equilibrium that can be reached under improvement dynamics from an initial state (problem named NE); deciding if there exists such an equilibrium in which a particular candidate is a winner (problem named WINNER); and lastly, deciding if given a Nash equilibrium and an initial state, they can decide if the equilibrium is reachable under improvement dynamics (problem named SET). Considering the extreme cases where the probability of rejection can be worth 0 or 1, they showed that all these problems are NP-complete except for problem NE with a rejection probability of 1.

The research questions we investigate in this paper (e.g., existence of Nash equilibria, analysis of improvement dynamics) are the same as the ones triggered by strategic candidacy and strategic voting games as we also study a game-theoretic problem induced by an election. However, the nature of the game that we will study, and hence its theoretical analysis will be quite different. Indeed, while the goal of the two previous types of games was to capture possible manipulations by the voters or the candidates, our game is motivated by voters wishing to be represented by the best possible guru (w.r.t. their preferences) in a liquid democracy setting.
3 Notations, Settings and Overview of the Results

3.1 Notations and Nash-Stable Delegation Functions

We denote by $N = \{1, \ldots, n\}$ a set of voters that take part in a vote. Each voter $i$ can either vote herself, delegate to another voter $j$, or abstain. We denote by $d: N \to N \cup \{0\}$ a delegation function such that $d(i) = i$ if voter $i$ votes, $d(i) = j$ with $j \in N \setminus \{i\}$ if $i$ delegates to voter $j$, and $d(i) = 0$ if voter $i$ abstains. (Note that by abuse of notation we represent abstention as a delegation to a fictitious voter denoted by 0.) Given a delegation function $d$, we define the set of gurus $\text{Gu}(d)$ as the set of voters that vote given the delegations prescribed by $d$, i.e., $\text{Gu}(d) = \{ i \in N \mid d(i) = i \}$. Delegations are transitive which means that if $d(i) = j$, $d(j) = k$, and $d(k) = k$, then voter $i$ is represented by voter $k$. In the end, the voter who votes for $i$, called the guru of $i$ and denoted by $\text{gu}(i, d)$, is the voter in $\text{Gu}(d) \cup \{0\}$ attained by following the delegations of the voters from $i$. Stated otherwise, $j = \text{gu}(i, d)$ if there exists a sequence of voters $i_1, \ldots, i_\ell$ such that $d(i_k) = i_{k+1}$, $i_1 = i$, $i_\ell = j$ and $j \in \text{Gu}(d) \cup \{0\}$. Note that the successive delegations starting from $i$ may also end up in a circuit (i.e., $i = i_1$ delegates to $i_2$, who delegates to $i_3$, and so on up to $i_\ell$ who delegates to $i_k$ with $k \in \{1, \ldots, \ell - 1\}$). In this situation, we consider that the $\ell$ voters abstain, as no one in the chain of delegations takes the responsibility to vote. More formally, in this situation, $\text{gu}(i_k, d) = 0$ for all $k \in \{1, \ldots, \ell\}$.

We assume each voter $i$ has a preference order $\succ_i$ over who could be their guru in $N \cup \{0\}$. For every voter $i \in N$, and for every $j, k \in N \cup \{0\}$ we have that $j \succ_i k$ if $i$ prefers to delegate to $j$ (or to vote if $j = i$, or to abstain if $j = 0$) rather than to delegate to $k$ (or to vote if $k = i$, or to abstain if $k = 0$). The collection of preference orders of every voters $\{\succ_i \mid i \in N\}$ in turn defines a preference profile $P$.

For illustrative purposes, we now give an example of a preference profile $P$ representing the preferences that voters have over possible gurus.

Example 1. If $n = 3$, a possible preference profile is given by:

$$
\begin{align*}
1 & : 2 \succ_1 1 \succ_1 3 \succ_1 0 \\
2 & : 3 \succ_2 2 \succ_2 1 \succ_2 0 \\
3 & : 1 \succ_3 3 \succ_3 2 \succ_3 0
\end{align*}
$$

In this example, each voter $i$ prefers to delegate to $(i \mod 3) + 1$ rather than to vote directly and each voter prefers to vote rather than to abstain.

As a consequence of successive delegations, a voter might end up in a situation in which she prefers to vote or to abstain or to delegate to another guru than to maintain her current delegation. Such a situation is regarded as unstable as this voter would modify unilaterally her delegation. More formally, a delegation function $d$ is Nash-stable for voter $i$ if

$$
\text{gu}(i, d) \succ_i g \quad \forall g \in (\text{Gu}(d) \cup \{0, i\}) \setminus \{\text{gu}(i, d)\}.
$$

A delegation function $d$ is Nash-stable if it is Nash-stable for every voter in $N$. A Nash-stable delegation function is also called an equilibrium in the sequel.

3.2 Problems Investigated

Stable situations are obviously desirable. Unfortunately, Example 1, gives an example of preference profile for which there is no Nash-stable delegation function.

Observation 1. The preference profile of Example 1 admits no equilibrium.

Proof. Assume by contradiction that there exists a Nash-stable delegation function $d$. Note that for any pair of voters $i, j$ in $N$, there is always one voter that approves the other as possible guru, i.e., she prefers to delegate to this voter rather than to vote or abstain. Hence, $|\text{Gu}(d)|$ cannot be greater than 1, otherwise one of the guru would rather delegate to another guru than vote directly. On the other hand, note that there is no voter that is approved as possible guru by all other agents. Hence, $|\text{Gu}(d)|$ cannot be less than 2, otherwise one of the voter in $N \setminus \text{Gu}(d)$ would rather vote than delegate to one of the gurus (in this example all agents prefer to vote rather than to abstain). We obtain the desired contradiction.

\[\square\]
Hence the first problem, called **EXISTENCE**, that we will investigate in this article is the one of the existence of an equilibrium.

**EXISTENCE (abbreviated by EX)**

**INSTANCE:** A preference profile $P$ describing the preferences of the voters in $N$ over who in $N \cup \{0\}$ will represent them as guru.

**QUESTION:** Does there exist an equilibrium.

We will show in Section 4 that this problem is equivalent to the problem of determining if a digraph admits a kernel (i.e., an independent set of nodes $S$ such that for every other node $u$ not in $S$, there exists an arc $(u,v)$ with $v \in S$) which is NP-complete [Chv73].

Hence, deciding if a preference profile admits an equilibrium is an NP-complete problem. We thus restrict our attention to structured preference profiles (e.g., single-peaked profiles) that ensure that an equilibrium exists. For this structured preference profiles we will investigate if we can compute equilibria verifying particular desirable properties.

Firstly, given a voter $i \in N$ we will try to know if there exists a Nash-stable delegation function $d$ for which $i$ is a guru, i.e., $i \in Gu(d)$. Stated otherwise, we wish to know if voter $i$ can be a guru in an equilibrium. We call this problem **MEMBERSHIP**.

**MEMBERSHIP (abbreviated by MEMB)**

**INSTANCE:** A preference profile $P$ describing the preferences of the voters in $N$ over who in $N \cup \{0\}$ will represent them as guru, and a voter $i \in N$.

**QUESTION:** Does there exist a Nash-stable delegation function $d$ for which $i \in Gu(d)$.

Secondly, we will try to find an equilibrium that satisfies most the voters, where the dissatisfaction of a voter $i \in N$ with respect to a delegation function $d$ is given by $rk(i,d) - 1$ where $rk(i,d)$ is the rank of $gu(i,d)$ in the preference profile of $i$. More formally, we study the problem, named **MIN DISSATISFACTION**, of determining a Nash-stable delegation function $d$ minimizing $\sum_{i \in N}(rk(i,d) - 1)$.

**MIN DISSATISFACTION (abbreviated by MINDIS)**

**INSTANCE:** A preference profile $P$ describing the preferences of the voters in $N$ over who in $N \cup \{0\}$ will represent them as guru.

**SOLUTION:** A Nash-stable delegation function $d$.

**MEASURE:** $\sum_{i \in N}(rk(i,d) - 1)$ (to minimize).

Thirdly, we wish to avoid the situation where a guru would amass too much voting power, where the voting power $vp(i,d)$ of a guru $i \in N$ with respect to a delegation function $d$ is defined as $vp(i,d) = |\{j \in N|gu(j,d) = i\}|$. More formally, we study the problem, named **MIN MAX VOTING POWER**, of determining a Nash-stable delegation function $d$ minimizing $\max_{i \in Gu(d)} vp(i,d)$.

**MIN MAX VOTING POWER (abbreviated by MINMAXVP)**

**INSTANCE:** A preference profile $P$ describing the preferences of the voters in $N$ over who in $N \cup \{0\}$ will represent them as guru.

**SOLUTION:** A Nash-stable delegation function $d$.

**MEASURE:** $\max_{i \in Gu(d)} vp(i,d)$ (to minimize).

Fourthly, in the case where some voters would prefer to abstain rather than to vote, it is desirable to find a Nash-stable delegation function minimizing the number of voters that finally abstain. More formally, we study the problem, named **MIN ABSTENTION**, of determining a Nash-stable delegation function $d$ minimizing $|\{i \in N|gu(i,d) = 0\}|$.

**MIN ABSTENTION**

**INSTANCE:** A preference profile $P$ describing the preferences of the voters in $N$ over who in $N \cup \{0\}$ will represent them as guru.

**SOLUTION:** A Nash-stable delegation function $d$.

**MEASURE:** $|\{i \in N|gu(i,d) = 0\}|$ (to minimize).
This gives a sequence of delegation functions $d_T$ of choosing such a function which specifies that voter $T$ to improve her outcome if possible. In the best response dynamics (BRD) to change her delegation/vote. In the dynamics called better response dynamics $d_j$ for instance $\sigma(T)$ process (necessarily) converges towards such an equilibrium. As classically done in game theory (see for instance [NSVZ11]), we will consider dynamics where iteratively one voter has the possibility to change her delegation/vote. In the dynamics called better response dynamics, the voter will try to improve her outcome if possible. In the best response dynamics (BRD) a voter $i$ chooses $d(i)$ so as to maximize her outcome. Let us define this more formally.

Given a delegation function $d$ and a voter $i$, let:

- $I_d(i)$ be the set of improved “moves” for $i$: $I_d(i) = \{j \in N \cup \{0\} : gu(i, d_{i \rightarrow j}) \succ_i gu(i, d)\}$ where $d_{i \rightarrow j}$ is the same delegation as $d$ up to the fact that $i$ delegates to $j$ (or votes if $j = i$, or abstains if $j = 0$).
- $B_d(i)$ be the set of $j \in I_d(i) \cup \{d(i)\}$ maximizing the outcome for $i$. In particular if there is no possible improvement ($I_d(i) = \emptyset$) then $B_d(i) = \{d(i)\}$ ($i$ will not change her delegation).

In a dynamics, we are given a starting delegation function $d_0$ and a token function $T : \mathbb{N}^* \rightarrow N$ which specifies that voter $T(t)$ has the token at step $t$: she has the right to change her delegation. This gives a sequence of delegation functions $d_t, t \in N$ where for any $t \in N^*$, if $j \neq T(t)$ then $d_t(j) = d_{t-1}(j)$. A dynamics is said to converge if there is a $t^*$ such that for all $t \geq t^*$ $d_t = d_{t^*}$.

We will assume, as usual, that each voter has the token an infinite number of times. A classical way of choosing such a function $T$ is to consider a permutation $\sigma$ over the voters $N$, and to repeat this permutation over the time to give the token (if $t = nq+r$ with $r \in \{1, \ldots, n\}$ then $T(t) = \sigma(r)$). We will call these dynamics permutation dynamics.

Given $d_0$ and $T$, a dynamics is called:

- A better response dynamics if for all $t$, $T(t)$ chooses an improved move if any, otherwise do not change her delegation: if $I_{d_{t-1}}(T(t)) \neq \emptyset$ then $d_t(T(t)) \in I_{d_{t-1}}(T(t))$, otherwise $d_t(T(t)) = d_{t-1}(T(t))$.
- A best response dynamics if for all $t$, $T(t)$ chooses a move in $B_{d_{t-1}}(T(t))$.

Note that a best response dynamics is also a better response dynamics.

The last problems that we investigate, denoted by $\text{IR-CONVERGENCE}$ and $\text{BR-CONVERGENCE}$, can be formalized as:

$\text{MIN ABSTENTION (abbreviated by MINABST)}$

INSTANCE: A preference profile $P$ describing the preferences of the voters in $N$ over who in $N \cup \{0\}$ will represent them as guru.

SOLUTION: A Nash-stable delegation function $d$.

MEASURE: $|\{i \in N | gu(i, d) = 0\}|$ (to minimize).

Note that optimization problems $\text{MINDIS}$, $\text{MINMAXVP}$ and $\text{MINABST}$ first need to determine if their exists a feasible solution (i.e., a Nash-stable delegation functions). Hence they are at least as hard as decision problem $\text{EX}$. The last question that we investigate captures the dynamic nature of delegations. Indeed, it is not because a Nash-stable delegation function exists that an iterative sequence of delegations will necessarily reach an equilibrium.

### 3.3 Convergence of Iterative Delegations

In situations where an equilibrium exists, a natural question is whether a dynamic delegation process (necessarily) converges towards such an equilibrium. As classically done in game theory (see for instance [NSVZ11]), we will consider dynamics where iteratively one voter has the possibility to change her delegation/vote. In the dynamics called better response dynamics, the voter will try to improve her outcome if possible. In the best response dynamics (BRD) a voter $i$ chooses $d(i)$ so as to maximize her outcome. Let us define this more formally.

Given a delegation function $d$ and a voter $i$, let:

- $I_d(i)$ be the set of improved “moves” for $i$: $I_d(i) = \{j \in N \cup \{0\} : gu(i, d_{i \rightarrow j}) \succ_i gu(i, d)\}$ where $d_{i \rightarrow j}$ is the same delegation as $d$ up to the fact that $i$ delegates to $j$ (or votes if $j = i$, or abstains if $j = 0$).
- $B_d(i)$ be the set of $j \in I_d(i) \cup \{d(i)\}$ maximizing the outcome for $i$. In particular if there is no possible improvement ($I_d(i) = \emptyset$) then $B_d(i) = \{d(i)\}$ ($i$ will not change her delegation).

In a dynamics, we are given a starting delegation function $d_0$ and a token function $T : \mathbb{N}^* \rightarrow N$ which specifies that voter $T(t)$ has the token at step $t$: she has the right to change her delegation. This gives a sequence of delegation functions $d_t, t \in N$ where for any $t \in N^*$, if $j \neq T(t)$ then $d_t(j) = d_{t-1}(j)$. A dynamics is said to converge if there is a $t^*$ such that for all $t \geq t^*$ $d_t = d_{t^*}$.

We will assume, as usual, that each voter has the token an infinite number of times. A classical way of choosing such a function $T$ is to consider a permutation $\sigma$ over the voters $N$, and to repeat this permutation over the time to give the token (if $t = nq+r$ with $r \in \{1, \ldots, n\}$ then $T(t) = \sigma(r)$). We will call these dynamics permutation dynamics.

Given $d_0$ and $T$, a dynamics is called:

- A better response dynamics if for all $t$, $T(t)$ chooses an improved move if any, otherwise do not change her delegation: if $I_{d_{t-1}}(T(t)) \neq \emptyset$ then $d_t(T(t)) \in I_{d_{t-1}}(T(t))$, otherwise $d_t(T(t)) = d_{t-1}(T(t))$.
- A best response dynamics if for all $t$, $T(t)$ chooses a move in $B_{d_{t-1}}(T(t))$.

Note that a best response dynamics is also a better response dynamics.

The last problems that we investigate, denoted by $\text{IR-CONVERGENCE}$ and $\text{BR-CONVERGENCE}$, can be formalized as:

$\text{IR-CONVERGENCE (abbreviated by IR-CONV)}$

QUESTION: Does a dynamic delegation process under better-response dynamics necessarily converges towards an equilibrium whatever the preference profile $P$ and token function $T$.

$\text{BR-CONVERGENCE (abbreviated by BR-CONV)}$

QUESTION: Does a dynamic delegation process under best-response dynamics necessarily converges towards an equilibrium whatever the preference profile $P$ and token function $T$. 

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3.4 Summary of the Results

Our results for problems EX, IR-CONV, BR-CONV, MEMB, MINDIS, MINMAXVP and MINABST are presented in Table 1, where the different lines of the table corresponds to different types of structures of preferences. These results are detailed in the following sections (one section per line of the table).

Section 4 investigates the case of general preference profiles. In Section 5, we study single-peaked preference profiles, where agents are ordered on a line and they prefer gurus that are “close” to them on this axis. In Section 6 we investigate symmetrical preference profiles, where all pair of voters always accept each other as guru, or reject each other. In Section 7 we study Distance Based Social Network (DBSN) preference profiles where voters are nodes of a network and they prefer gurus that are close to them in this network.

| Type          | EX | IR-CONV | BR-CONV | MEMB | MINDIS | MINMAXVP | MINABST |
|---------------|----|---------|---------|------|--------|----------|---------|
| General       | NP-Complete | Not Always | Not Always | NP-Complete | NP-Hard | NP-Hard | NP-Hard |
| Single-Peaked | Always Exists | Not Always | Not Always | O(n^2) | O(n^2) | O(n^2) | O(n^2) |
| Symmetrical   | Always Exists | Not Always | Always | Always Exists | NP-Hard | NP-Hard | NP-Hard |
| DBSN          | Always Exists | Not Always | Always | NP-Complete | NP-Hard | NP-Hard | NP-Hard |

Table 1: Synthesis of Results.

4 General Preference Profiles and Kernels

Let $P$ be a preference profile. We define $\mathcal{Acc}(i) = \{ j \in N \mid j \succ_i i \text{ and } j \succ_i 0 \}$ the set of voters to which voter $i$ would rather delegate to than to abstain or vote directly. A necessary condition for a delegation function to be Nash-stable is that $\mathcal{gu}(i, d) \subseteq \mathcal{Acc}(i)$ for every voter $i$ who delegates to another voter. Otherwise, voter $i$ would change her delegation to abstain or vote directly. We refer to $\mathcal{Acc}(i)$ as the set of acceptable gurus for $i$ in a Nash-stable situation.

We say that agent $i$ is an abstainer in $P$ if she prefers to abstain rather than to vote, i.e., if $0 \succ_i i$; she is a non-abstainer otherwise. Note that, in an equilibrium, every abstainer delegates or abstains, but never votes herself. Hence no guru is an abstainer. Similarly, a voter that is a non-abstainer always prefer to vote than to abstain, and never abstains in an equilibrium.

Note also that since we are interested in equilibria, we do not require the whole preference profile as input. Namely, the preferences of agent $i$ below 0 or $i$ in her preference list have no influence on equilibria. In the sequel, we may define a preference profile only by giving, for every agent $i$, if she is an abstainer or not, and her preference profile on $\mathcal{Acc}(i)$.

We define the delegation-acceptability digraph $G_P = (N, A_P)$ by its arc-set $A_P = \{(i, j) \mid j \in \mathcal{Acc}(i)\}$. Stated differently, there exists an arc from voter $i$ to voter $j$ if and only if voter $i$ accepts voter $j$ as a guru. In the sequel, we will also consider the delegation acceptability digraph without abstainers $G^*_P$, that is, the graph obtained from $G_P$ by removing all vertices that correspond to abstainers.

Example 2. Let $n = 5$ and consider the following partial preference profile $P$:

1 : 2 \succ_1 1 \\
2 : 3 \succ_2 2 \\
3 : 4 \succ_3 2 \succ_4 1 \succ_3 0 \\
4 : 4 \\
5 : 2 \succ_5 3 \succ_5 1 \succ_5 5

The delegation-acceptability digraph $G_P$ and the delegation-acceptability digraph without abstainers $G^*_P$ of this preference profile are given in Figure 1 below.

The main result of this section, stated in Proposition 1, is a characterization of all sets of gurus of equilibria, as specific subsets of vertices of the delegation-acceptability digraph. Let us introduce additional graph-theoretic definitions. Given a digraph $G = (V, A)$, a subset of vertices $K \subset V$ is independent if there is no arc between two vertices of $K$. It is absorbing if for every vertex $u \notin K$, there exists $k \in K$ such that $(u, k) \in A$ (then we say that $k$ absorbs $u$). A kernel of $G$ is a subset of vertices that is both independent and absorbing.

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Proposition 1. Given a preference profile $P$ and a subset of agents $K \subset N$, the following propositions are equivalent:

(i) there exists a Nash-stable delegation function $d$ such that $\mathcal{Gu}(d) = K$;
(ii) $K$ contains no abstainer and $K$ is a kernel of $G^*_P$.

Proof. (i) $\implies$ (ii). Let $d$ be a Nash-stable delegation function of $P$. Let us prove that its set of gurus $\mathcal{Gu}(d)$ satisfy condition (ii). It was noted previously that Nash-stability implies the absence of abstainer in $\mathcal{Gu}(d)$. Assume that $\mathcal{Gu}(d)$ is not independent in $G_P$. Then, there exists $i, j \in \mathcal{Gu}(d)$ such that $(i, j)$ is an arc of $G_P$, that is, $j \in \text{acc}(i)$. It implies that $i$ prefers to delegate to $j$ rather than remaining a guru, and $d$ is not Nash-stable. Assume now that $\mathcal{Gu}(d)$ is not absorbing for all non-abstainers. Then there exists a non-abstainer $i \not\in \mathcal{Gu}(d)$ such that for every guru $g$ in $\mathcal{Gu}(d)$, $(i, g)$ is not an arc of $G_P$, that is, $g \not\in \text{acc}(i)$. Such a voter $i$ prefers to vote herself rather than delegate to any guru in $\mathcal{Gu}(d)$. Therefore $d$ is not Nash-stable. This proves that $\mathcal{Gu}(d)$ is a kernel of $G^*_P$.

(ii) $\implies$ (i). Consider a kernel $K$ in the delegation-acceptability digraph without abstainers $G^*_P$. We define a delegation function $d$ by: $d(i) = i$ if $i \in K$; and $d(i) = j$ if $i \not\in K$ where $j$ is the voter that $i$ prefers in $K \cup \{0\}$. It follows that $\mathcal{Gu}(i, d) = d(i)$ for every $i$, and the set of gurus is $\mathcal{Gu}(d) = K$.

Let us check that the delegation function $d$ is Nash-stable. First $d$ is Nash-stable for every abstainer $i$: the guru of $i$ is $d(i)$, which is her preferred guru in $\mathcal{Gu}(d) \cup \{0\}$. Then $d$ is Nash-stable for any $i \in \mathcal{Gu}(d)$. We need to prove that $i$ does not prefer to delegate to any other guru, or to abstain. By assumption the set $\mathcal{Gu}(d) = K$ contains no abstainer. Also because $K$ is independent, for every other guru $g \in K$, the arc $(i, g)$ does not exist in the delegation acceptability digraph, meaning that $g \not\in \text{acc}(i)$ and $i \not>_{i} g$. Finally $d$ is Nash-stable for any non-abstainer $i \not\in \mathcal{Gu}(d)$. Since $i$ has already chosen her preferred guru in $\mathcal{Gu}(d)$, it is only necessary to check that $i$ does not prefer to vote herself. Because the set $K$ is absorbing, there exists $k \in K$ such that $(i, k)$ is an arc of $G_P$. Hence such $k$ is a guru in $\mathcal{Gu}(d)$ such that $k \in \text{acc}(i)$, hence $k \not>_{i} i$. 

For a preference profile $P$ with no abstainer, the sets of gurus of equilibria are exactly the kernels of the delegation-acceptability digraph. Note that any digraph is the delegation-acceptability digraph of a preference profile $P$ (given the digraph, it suffices to build a preference profile $P$ so that every agent prefers to delegate to its out-neighbors, then to vote, then to delegate to other agents, then to abstain). This observation leads to the claimed complexity result.

Theorem 1. Determining if a preference profile admits an equilibrium is equivalent to the problem of determining if a digraph admits a kernel. Thus it is NP-complete.

Consequences for the other decision and optimization problems. As a direct consequence of the results of this section, optimization problems MINDIS, MINMAXVP and MINABST are NP-Hard as it is NP-Hard to decide if their set of admissible solutions are empty or not. We also directly obtain that the decision problem MEMB, is NP-Complete, by a direct reduction from EX. Lastly, as a Nash-stable delegation function may not exist, it is obvious that better response or best response dynamics do not always converge to a Nash-stable delegation function. However, if a Nash-stable delegation function does exist, we do have the following general result.

Proposition 2. For any preference profile, if an equilibrium exists, then we can find a permutation $\sigma$ inducing a permutation dynamics which starts when nobody delegates, and that always converges under better response dynamics to this equilibrium.
Proof. Let $d$ be a Nash-stable delegation function and consider any permutation $\sigma$ such that voters in $\text{Gu}(d)$ are placed in the $|\text{Gu}(d)|$ last positions. Then, under better-response dynamics, after the $|N \setminus \text{Gu}(d)|$ first delegations steps of the delegation process, no voter in $N \setminus \text{Gu}(d)$ has chosen to vote. Indeed, a choice to vote by one of these voters would contradict the fact that $\text{Gu}(d)$ is absorbing in $G^*_P$. Hence, they all have decided to abstain or to delegate and are no more available as possible guru. Furthermore, as $\text{Gu}(d)$ is independent in $G^*_P$ and contains no abstainers, in the $|\text{Gu}(d)|$ next steps, all voters in $\text{Gu}(d)$ decide to vote. The second time each voters in $N \setminus \text{Gu}(d)$ have the token, they choose to delegate to their favorite guru in $\text{Gu}(d) \cup \{0\}$ (if it is not already the case with their current delegation), and each voter $\text{Gu}(d)$ remains a guru. At this point, the delegation process has converged to $d$. \hfill \square

5 Single-peaked Preference Profiles and Interval Catch Di-graphs

5.1 Definition and Existence of an Equilibrium

In this section, we consider that agents are ordered on a line. We assume that the agents are indexed in accordance with this ordering and we identify the agents with their index in $\{1, \ldots, n\}$. This ordering may represent the political positions of the voters on a left-right ladder.

A preference profile is single-peaked for an agent $i \in N$ if for every $j, k \in N$,

$$(i < j < k \text{ or } k < j < i) \implies j \succ_i k.$$ 

A preference profile is single-peaked if it is single-peaked for all agents.

In a single-peaked preference profile, if an agent delegates to a guru on her left (and similarly on her right), she prefers to delegate to the closest possible. Note that in $i$’s preference list, we allow $i$ (vote) and 0 (abstention) to be in any position (differently from the traditional definition of single peakedness.). For an abstainer $i$, it represents the fact that $i$ prefers to delegate to gurus that are close, then, beyond a given threshold on her left (resp. right), she prefers to abstain rather than to delegate to a guru that is too far from her opinions. For a non-abstainer $i$, one may expect $i$ to be her own preferred guru, hence the first element in the preference list. Though, we model the fact that voting herself may have a cost for voter $i$, and hence she would prefer to delegate to a close neighbor rather than to vote. Beyond a given threshold on her left (resp. right), she prefers to vote herself rather than to delegate to a guru that is too far from her opinions.

Example 3. The following partial preference profile $P$ on 4 ordered voters $\{1, 2, 3, 4\}$ is single-peaked.

$\begin{align*}
1 & : 2 \succ_1 1 \\
2 & : 3 \succ_2 4 \succ_2 2 \\
3 & : 2 \succ_3 1 \succ_3 3 \\
4 & : 3 \succ_4 4
\end{align*}$

Its delegation-acceptability digraph $G_P$ (equal to $G^*_P$ since $P$ contains no abstainer) is represented on Figure 2.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (1,0) {2};
\node (3) at (2,0) {3};
\node (4) at (3,0) {4};
\path[->] (1) edge (2);
\path[->] (2) edge (3);
\path[->] (3) edge (4);
\end{tikzpicture}
\caption{Delegation-acceptability digraph $G_P$.}
\end{figure}

Single-peaked preferences represent the most well known restriction of preferences in social choice theory. They where introduced by Black [Bla48] who showed that they enabled to solve
the Condorcet paradox, i.e., they restore transitivity in pairwise elections. Furthermore, single-peaked electorates have many desirable properties: they induce a simple characterization of strategy proof voting schemes [Mou80]; they are easily recognizable [ELÖ08]; and they often lead to more desirable complexity results (e.g., in multiwinner elections, were the goal of the election is to elect a committee representing best a set of voters [BSU13]).

Let us study the properties of the delegation-acceptability digraph $G_P$ for $P$ a single-peaked preference profile. A digraph $D$ is an interval catch digraph [Pri94] with vertex-set $N$ if for every $i \in N$, there exists $l_i, r_i \in N$ such that $l_i \leq i \leq r_i$ and the out-neighborhood of $i$ in $D$ is the subset $\{l_i, \ldots, r_i\} \setminus \{i\}$. These digraphs are related to single-peaked preference profiles by the following proposition.

**Proposition 3.** If $P$ is a single-peaked preference profile, then its delegation-acceptability digraph $G_P$ is an interval catch digraph.

**Proof.** Consider a single-peaked preference profile $P$. Given $i \in N$, let $l_i$ (resp. $r_i$) be the minimal (resp. maximal) agent (or $i$ at worst) that $i$ accepts to delegate to: $l_i = \min \text{acc}(i) \cup \{i\}$ (resp. $r_i = \max \text{acc}(i) \cup \{i\}$). Then all agents $j < l_i$ are unacceptable as gurus for $i$. Single-peakedness also implies that $i$ accepts every agent that is between $l_i$ and $i$. A similar argument holds for $r_i$. Hence, in the delegation-acceptability digraph $G_P$, the out-neighborhood of $i$ is exactly $\{l_i, \ldots, r_i\} \setminus \{i\}$. \hfill $\square$

Considering the preference profile from Example 3, the digraph $G_P^*$ is clearly the interval catch digraph defined by the values $l_1 = 1, r_1 = 2, l_2 = 2, r_2 = 4, l_3 = 1, r_3 = 3, l_4 = 3, r_4 = 4$.

From the equivalence stated in Theorem 1, deciding the existence of an equilibrium is equivalent to deciding the existence of a kernel in the delegation-acceptability digraph without abstainers. An induced subgraph of an interval catch digraph is still an interval catch digraph. Hence the delegation-acceptability digraph without abstainers $G_P$ is still an interval catch digraph. It was proven by Prisner [Pri94] that a kernel in an interval catch digraph always exists and is computable in $O(n^2)$ time. This leads to polynomial algorithms for computing an equilibrium for a single-peaked preference profile.

**Theorem 2.** If preferences are single-peaked, a preference profile always admits an equilibrium. Furthermore, an equilibrium can be computed in $O(n^2)$.

### 5.2 Kernels in Interval Catch Digraphs

Theorem 2 addresses the question of computing one equilibrium. We provide additional structural results on kernels in interval catch digraphs, in order to answer other existence or optimization problems on equilibria.

Consider an interval catch digraph $G = (N, A)$ defined by the values $l_i, r_i, i \in N$. For $U \subset N$, let $G[U]$ denote the subgraph of $G$ induced by the vertices in $U$.

**Lemma 1.** Let $K \subset N$ and $k^1, \ldots, k^p$ be the vertices of $K$ in increasing order. Define $I_0 = \{1, \ldots, k^1\}$, $I_t = \{k^t, \ldots, k^{t+1}\}$ for every $t \in \{1, \ldots, p-1\}$, and $I_p = \{k^p, \ldots, n\}$.

The set $K$ is a kernel of $G$ if and only if $K \cap I_t$ is a kernel of $G[I_t]$ for every $t \in \{0, \ldots, p\}$.

Note that for every $t$, the set $K \cap I_t$ is a singleton or a pair of vertices. The idea of the lemma is that $K$ being a kernel can be expressed by conditions on pairs of successive vertices of $K$.

**Proof.** Assume $K$ is a kernel of $G$. Clearly $K \cap I_t$ is an independent set of $G[I_t]$ for every $t$. Assume there exists $t$ such that $K \cap I_t$ is not absorbing in $G[I_t]$, i.e., there exists $j$ such that $k^t < j < k^{t+1}$ and neither $k^t$ nor $k^{t+1}$ are in $\text{acc}(j)$. It implies $k^t < l_j \leq r_j < k^{t+1}$, and since $k^t$ and $k^{t+1}$ are successive vertices of $K$ it comes $K \cap \text{acc}(j) = \emptyset$. Hence $K$ is not absorbing in $G$.

Conversely, if $K \cap I_t$ is a kernel of $G[I_t]$ for every $t \in \{0, \ldots, p\}$ then $K$ is clearly an absorbing set of $G$. Assume it is not independent: there exist two vertices $k^t$ and $k^{t'}$ in $K$ that are neighbors. Assume w.l.o.g. $t < t'$ and $(k^t, k^{t'}) \in A$. Then $r_{k^t} \geq k^{t'} \geq k^{t+1}$. Hence $(k^t, k^{t+1}) \in A$, and $K \cap I_t$ is not an independent set in $G[I_t]$. \hfill $\square$

Consider the auxiliary digraph $\overrightarrow{G} = (\overrightarrow{V}, \overrightarrow{A})$ associated with $G$ built as follows. The vertex-set $\overrightarrow{V}$ contains the set of voters $\{1, \ldots, n\}$, plus a source $s$ and a sink $t$. The arc-set $\overrightarrow{A}$ contains the arc $(i, j)$ (resp. the arc $(s, j)$, the arc $(i, t)$) if the pair $\{i, j\}$ (resp. the singleton $\{j\}$, the singleton $\{i\}$) is a kernel of $G[\{i, \ldots, j\}]$ (resp. $G[\{1, \ldots, j\}]$, $G[\{i, \ldots, n\}]$).
Let us illustrate the construction of the auxiliary digraph of the preference profile from Example 3. This auxiliary digraph is represented in Figure 3. The two successors of source $s$ are 1 and 2. Indeed the singletons $\{1\}$ and $\{2\}$ are kernels of the subgraph induced by $\{1\}$ and $\{1, 2\}$ respectively. Vertices 3 or 4 do not absorb vertex 1, hence they are not successors of $s$. Between two vertices of $\{1, \ldots, 4\}$ the only arc in $\mathcal{G}$ is $(1, 4)$, because all other pairs of vertices are neighbors, while $\{1, 4\}$ is a kernel of $G_P$.

![Figure 3: Auxiliary digraph of $G_P^*$ for $P$ the preference profile of Example 3.](image)

A consequence of Lemma 1 is the following proposition.

**Proposition 4.** There is a one-to-one correspondence between kernels of the interval catch digraph $G$ and $s-t$ paths in its auxiliary digraph $\mathcal{G}$.

Note that the existence result from Prisner [Pri94] implies that there always exists at least one $s-t$ path in $\mathcal{G}$. We will apply this proposition by taking as input the delegation-acceptability digraph without abstainers $G_P^*$, which is an interval catch digraph, and building its auxiliary digraph $\mathcal{G}$. With Proposition 1 it comes that the $s-t$ paths in the auxiliary digraph $\mathcal{G}$ are exactly the sets of gurus of Nash-stable delegation functions.

Finally we prove that the auxiliary digraph is computable in quadratic time (it is trivially computable in $O(n^3)$ time).

**Proposition 5.** Given a preference profile $P$, the auxiliary digraph of $G_P^*$ is computable in $O(n^2)$ time.

**Proof.** The digraph $G_P^*$ is computable in $O(n^2)$ time. Then we prove that it is possible to compute in linear time the out-neighborhood of every vertex of the auxiliary digraph $\mathcal{G}$ of $G_P^*$. Let $i \in N$. Define for every $j > i$ the value $r_j^* = \min\{r_k \mid k \in \{i + 1, \ldots, j - 1\}, i \notin \text{Acc}(k)\}$. We claim that: the pair $\{i, j\}$ is an absorbing set of $G_P^*[\{i, \ldots, j\}]$ if and only if $j \leq r_j^*$. Indeed, if $j \leq r_j^*$, then for every $k \in \{i+1, \ldots, j-1\}$ either $i \in \text{Acc}(k)$ or $j \leq r_j^* \leq r_k$: then $j \in \text{Acc}(k)$. Hence in both cases the agent $k$ accepts $i$ or $j$ as a guru: $\{i, j\}$ is absorbing. Conversely, if $j > r_j^*$ then by definition of $r_j^*$ there exists $k \in \{i+1, \ldots, j-1\}$ such that $i \notin \text{Acc}(k)$ and $j > r_k$. Then $j \notin \text{Acc}(k)$, and the vertex $k$ is neither absorbed by $i$ nor by $j$.

Using this claim, we prove that the out-neighborhood of $i$ can be computed in linear time by the following procedure. Initialize $j := i + 1$ and $r^* := +\infty$. While $j \leq n$, apply the following:

(i) If $i \notin \text{Acc}(j)$, $j \notin \text{Acc}(i)$, and $j \leq r^*$, then add $(i, j)$ to $\mathcal{A}$. (ii) If $i \notin \text{Acc}(j)$, update $r^*$ by $r^* := \min\{r^*, r_j\}$. (iii) Increment $j$.

**5.3 Decision and Optimization Problems on Equilibria**

In the sequel the digraph $\mathcal{G}$ is the auxiliary digraph associated to the delegation-acceptability digraph without abstainers $G_P^*$.

*Membership problem.*

**Theorem 3.** For single-peaked preference profiles, the problem MEMB is solvable in $O(n^2)$ time.

**Proof.** From Proposition 4 the problem MEMB is equivalent to the problem of deciding the existence of an $s-t$ path in $\mathcal{G}$ that goes through agent $i$, i.e., finding an $s-i$ path and a $i-t$ path. It can be done in $O(n^2)$ time by computing $\mathcal{G}$ then performing two graph traversals.
This result can be extended to the more general problem of deciding if a subset of voters can be part of the set of gurus of a Nash-stable delegation function.

**Optimization problems.** The optimization problems introduced in Section 1 use objective functions that depend not only on the set of gurus \(\mathfrak{g}(d)\) but also on the values of \(\mathfrak{g}(i, d)\) for each voter.

**Observation 2.** Let \(K \subseteq N\) and let \(d\) be an equilibrium with set of gurus \(\mathfrak{g}(d) = K\). Then the guru \(\mathfrak{g}(i, d)\) of every agent \(i \notin K\) is the guru that \(i\) prefers among: the closest agent of \(K\) on her left, the closest agent of \(K\) on her right; and abstention.

**Proof.** Nash-stability implies that for every \(i\) the guru \(\mathfrak{g}(i, d)\) is the agent that \(i\) prefers in \(K \cup \{0, i\}\) (this holds for all preferences and not only single-peaked preferences). Let \(i \notin K\). Since \(K\) is the set of gurus of the equilibrium \(d\), it comes \(\mathfrak{g}(i, d) \neq i\), hence \(\mathfrak{g}(i, d)\) is the element that \(i\) prefers in \(K \cup \{0\}\). By single-peakedness, the preferred guru of \(i\) in the set \(K\) is either the closest guru on her left or the closest guru on her right. The result follows.

A consequence is the following. Assume \(d\) is an equilibrium with set of gurus \(\mathfrak{g}(d)\), associated with an \(s - t\) path of \(\mathfrak{G}\). Let \((k, k')\) be an arc of this path. Then from Observation 2 every \(i \in \{k + 1, \ldots, k'-1\}\) abstains or it has \(k\) or \(k'\) as guru in \(d\), i.e., \(\mathfrak{g}(i, d) \in \{0, k, k'\}\). Let us present into details algorithms for solving optimization problems, based on this remark.

**Theorem 4.** For single-peaked preference profiles, the problem **MINDIS** of finding a Nash-stable delegation function \(d\) minimizing \(\sum_{i \in K} (\mathfrak{r}(i, d) - 1)\) is solvable in \(O(n^3)\) time.

**Proof.** Build the auxiliary digraph \(\mathfrak{G}\). For every arc \((k, k') \in \mathfrak{A}\), compute an arc-weight \(w_{k,k'}\) as follows. Let \(i \in \{k, \ldots, k'-1\}, i \neq s\) (where by abuse of notation \(t = 1\) is \(n\)). Define a value \(r_{k,k'}(i)\) by: if \(i \neq k\), \(r_{k,k'}(i)\) is the rank of the preferred guru of \(i\) in \(\{0, k, k'\} \setminus \{s, t\}\) in \(i\)’s preference list; otherwise, \(r_{k,k'}(i)\) is the rank of \(i\) in \(i\)’s preference list. By Observation 2, if \(d\) is an equilibrium in which \(k\) and \(k'\) are two successive gurus, then for every voter \(i \in \{k, \ldots, k'-1\}\), it holds that \(\mathfrak{r}(i, d) = r_{k,k'}(i)\). Define now \(w_{k,k'} = \sum_{i \in \{k, \ldots, k'-1\}, i \neq s} r_{k,k'}(i)\). All weights \(w\) can be computed in \(O(n^3)\) time.

Then the total dissatisfaction associated with a Nash-stable delegation function \(d\) is then equal to the weight of the \(s - t\) path associated with its set of gurus \(\mathfrak{g}(d)\). An optimal solution to the problem **MINDIS** can then be computed by computing a shortest \(s - t\) path for the weights \(w\).

**Theorem 5.** For single-peaked preference profiles, the problem **MINMAXVP** of finding a Nash-stable delegation function \(d\) minimizing \(\max_{i \in \mathfrak{g}(d)} \mathfrak{v}(i, d)\) is solvable in \(O(n^3)\) time.

**Proof.** For every arc \((i, j)\) of the auxiliary digraph \(\mathfrak{G}\), let \(w_{ij}^j\) (resp. \(w_{ij}^i\)) denote the number of agents in \(\{i + 1, \ldots, j - 1\}\) whose preferred guru in \(\{0, i, j\}\) is \(i\) (resp. \(j\)). Given an \(s - t\) path involving a guru \(j\), the voting power of guru \(j\) is exactly \(w_{ij}^j + w_{jk}^j + 1\), where \((i, j)\) and \((j, k)\) are the arcs of the \(s - t\) path containing \(j\). Hence, the problem **MINMAXVP** is equivalent to the problem of finding an \(s - t\) path in the auxiliary digraph that minimizes the maximum value of \(w_{ij}^j + w_{jk}^j + 1\) over pairs of consecutive arcs \((i, j)\) and \((j, k)\). The proof follows.

For every \(i \in N, w \in \{0, \ldots, n\}\), let \(M(i, w)\) be the minimum value \(W\) such that: there exists an \(s - i\) path where all gurus in \(\{1, \ldots, i\}\) have voting power at most \(W\) and \(i\) has voting power at most \(w\) on her left. We prove that all values \(M(i, w)\) can be computed in \(O(n^3)\) time. Let \(j \in N\). Assume all values have been computed for agents on the left of \(j\). For each predecessor \(i\) of \(j\) in \(\mathfrak{G}\), for each value \(w \in \{0, \ldots, n\}\), form \(M = \max\{M(i, w) + 1\}\), then update \(M(j, w_{ij}^j) := \min\{M(j, w_{ij}^j) + 1\}\). Hence all values \(M(j, \_\_\_)\) can be computed in quadratic time for every \(j\). Finally the value \(M(t, 0)\) is the optimal value of the **MINMAXVP** optimization problem. The auxiliary digraph \(\mathfrak{G}\) can be computed in \(O(n^2)\) time, and the \(w_{ij}^j\) can be computed in \(O(n^3)\). Thus the overall complexity is \(O(n^3)\). Note that the optimal solution can be obtained by standard bookkeeping techniques without increasing the complexity of the method.

**Theorem 6.** For single-peaked preference profiles, the problem **MINABST** of finding a Nash-stable delegation function \(d\) minimizing \(|\{i \in N | \mathfrak{g}(i, d) = 0\}|\) is solvable in \(O(n^3)\) time.
Proof. For every arc \((k, k')\) in the auxiliary digraph \(\overrightarrow{G}\), define \(a_{kk'}\) as the number of voters in \(\{k + 1, \ldots, k' - 1\}\) whose preferred guru in \(\{0, k, k'\}\) is 0. By Observation 2, if \(d\) is an equilibrium in which \(k\) and \(k'\) are two successive gurus, then for every voter \(i \in \{k + 1, \ldots, k' - 1\}\), the guru \(\text{gu}(i, d)\) of \(i\) is the most preferred guru of \(i\) in \(\{0, k, k'\}\), and \(i\) abstains if and only if 0 is her preferred guru in \(\{0, k, k'\}\). Hence, the number of voters of \(\{k + 1, \ldots, k' - 1\}\) who abstain in \(d\) is exactly \(a_{kk'}\). Thus for any equilibrium \(d\), associated with an \(s - t\) path in the auxiliary digraph, the total number of voters who abstain in \(d\) is equal to the sum of arc-weights \(a\) over the path.

Hence an optimal solution to the problem \textsc{MINABST} can then be computed by searching for a shortest \(s - t\) path for the weights \(a\). All arc-weights \(a\) can be computed in \(O(n^3)\) time. Hence an optimal solution to \textsc{MINABST} can be computed in \(O(n^3)\) time. \(\square\)

5.4 Convergence of Dynamics

As Theorem 2 asserts that an equilibrium always exists in the single-peaked case, it is worth considering convergence of dynamics in this setting. Unfortunately, such a convergence is not guaranteed.

**Proposition 6.** There exist a single-peaked profile and a BRD dynamics that do not converge for this profile. This holds also if the BRD dynamics is required to be a permutation dynamic, and start at \(d_0\) where no voter delegates.

Proof. We consider again the single-peaked preference profile on 4 voters from Example 3. We consider the following instance with 4 ordered voters \(\{1, 2, 3, 4\}\) and the following preferences, which clearly fulfill the single-peaked criterion:

\[
\begin{align*}
1 : & 2 \succ 1 \succ 3 \succ 4 \succ 0 \\
2 : & 3 \succ 2 \succ 1 \succ 2 \succ 0 \\
3 : & 2 \succ 3 \succ 4 \succ 3 \succ 0 \\
4 : & 3 \succ 4 \succ 2 \succ 1 \succ 0
\end{align*}
\]

We consider the BRD permutation dynamics with function \(T\) associated to the permutation \(\sigma = \{4, 3, 1, 2\}\), and we start at \(d_0\) where nobody delegates.

- First round: \(d_1(4) = 3, d_2(3) = 2, d_3(1) = 2, d_4(2) = 2\). At the end of this round, 2 is the unique guru.
- Second round: \(d_5(4) = 4, d_6(3) = 2, d_7(1) = 2, d_8(2) = 4\). At the end of this round, 4 is the unique guru.
- Third round: \(d_9(4) = 4, d_{10}(3) = 3, d_{11}(1) = 1, d_{12}(2) = 3\). At the end of this round, 1, 3 and 4 are gurus.
- Fourth round: \(d_{13}(4) = 3, d_{14}(3) = 1, d_{15}(1) = 1, d_{16}(2) = 2\). At the end of this round, 1 and 2 are gurus.
- Fifth round: \(d_{17}(4) = 4, d_{18}(3) = 2, d_{19}(1) = 2, d_{20}(2) = 2\). At the end of this round, 2 and 4 are gurus.
- Sixth round: \(d_{21}(4) = 4, d_{22}(3) = 2, d_{23}(1) = 2, d_{24}(2) = 4\). At the end of this round, the delegation function is exactly the same as the one at the end of round 2. \(\square\)

6 Symmetrical Preference Profiles

6.1 Definition, Existence of Equilibria and Membership Problem

We consider in this section the case where the preferences are symmetrical in the sense that \(i \in \text{Acc}(j)\) if and only if \(j \in \text{Acc}(i)\). As we will see later, this is a particular case of the more general distance-based preference profiles on social networks that we will tackle in Section 7.
In the case of symmetrical preference profiles, the delegation-acceptability digraph has the arc \((i, j)\) iff it has the arc \((j, i)\) (it is symmetrical). Then, any inclusion maximal independent set is a kernel. Hence, constructions of equilibria are easy. Moreover, for any agent \(i\) there exists an equilibrium in which \(i\) is a guru (take a maximal independent set containing \(i\)). In other words, the answer to \textsc{MEMB} is always yes. More generally, given a set of voters, deciding if there exists an equilibrium in which every voter in this set is a guru is easy: we just have to check whether the set is independent or not.

6.2 Equilibria and Optimization

Though the existence of equilibrium is trivial in the case of symmetrical preference profiles, we can first notice that finding a Nash-stable delegation function which minimizes (or maximizes) the number of gurus is NP-hard, as finding an inclusion maximal independent set of maximal or minimal size is NP-hard [GJ90].

Besides, we now show that, in contrast with the results of the single-peaked case, \textsc{MINDIS}, \textsc{MINMAXVP} and \textsc{MINABST} are computationally hard. These results, as well as another hardness result in Section 7.2, are all based on a reduction from the 3-Satisfiability (3-SAT) problem, and use the same gadget digraph that we present now.

Let us first define the problem 3-SAT, known to be NP-complete [GJ90]:

\begin{center}
\begin{tabular}{|c|}
\hline
\textbf{3-SATISFIABILITY} (abbreviated by 3-SAT) \hline
\textbf{INSTANCE:} A set \(U\) of \(n_u\) binary variables, a collection \(C\) of \(n_c\) disjunctive clauses of 3 literals, where a literal is a variable or a negated variable in \(U\). \hline
\textbf{QUESTION:} Does there exist a truth assignment for \(U\) that satisfies all clauses in \(C\). \hline
\end{tabular}
\end{center}

To an instance \((U, C)\) of 3-SAT we associate the symmetric digraph \(G_{U, C}\) defined as follows:

- For each variable \(x_i \in U\), we create two adjacent vertices \(v_i^1\) and \(v_i^2\), called variables vertices, representing respectively the literals \(x_i\) and \(\overline{x_i}\).
- For each clause \(c_j \in C\) we create one vertex \(v_j\), called clause vertex; \(v_j\) is adjacent to the three vertices corresponding to the three literals in \(c_j\).

To illustrate this construction, consider the following 3-SAT instance:

\begin{align*}
U &= \{x_1, x_2, x_3, x_4, x_5\} \\
C &= \{(x_1 \lor x_2 \lor \overline{x_3}), (\overline{x_2} \lor \overline{x_4} \lor x_1), (\overline{x_1} \lor x_3 \lor x_5)\}
\end{align*}

Figure 4 gives the corresponding digraph \(G_{U, C}\).

\textbf{Observation 3.} \(G_{U, C}\) has a kernel containing no clause vertex if and only if \((U, C)\) is satisfiable.

\textbf{Proof.} If \((U, C)\) has a satisfying assignment, then consider in \(G_{U, C}\) the set of variable vertices corresponding to true literals. This set is clearly independent, and absorbing since every clause is satisfied by the assignment. Conversely, a kernel containing no clause vertex must contain exactly one variable vertex among \(v_i^1\) and \(v_i^2\) (for each \(i\)). Since the set is absorbing, the literals corresponding to this kernel satisfy all the clauses. \(\square\)

From this construction we easily derive the following hardness result.

\textbf{Theorem 7.} Given a symmetrical preference profile \(P\), it is NP-hard to decide whether there exists an equilibrium where no voter abstains, or not.

Thus, in particular, \textsc{MINABST} is NP-hard.

\textbf{Proof.} Let us consider a 3-SAT instance with a set \(U\) of variables and a set \(C\) of clauses. We will create a preference profile with \(2n_u\) voters \(v_i^1\) and \(v_i^2\), \(i = 1, \ldots, n_u\), and \(n_c\) voters \(v_j^1\), \(j = 1, \ldots, n_c\). A voter \(v_i^1\) accepts to delegate to the 3 voters corresponding to the three literals in the clause (and they accept her by symmetry), and then \(v_j^1\) prefers to abstain. Moreover, \(v_i^2\) and \(v_j^2\) also accept to delegate to each other. Then they prefer to vote. Then an equilibrium where nobody (no voter \(v_j^2\)) abstains corresponds to a kernel in \(G_{U, C}\) with no clause vertex. The result follows from Observation 3. \(\square\)
Theorem 8. MINDIS is NP-hard in the case of symmetrical preference profiles, even if there are no abstainers.

Proof. Let us consider a 3-SAT instance with a set \( U \) of variables and a set \( C \) of clauses. We will create a preference profile the delegation acceptability digraph of which is made of:

- The digraph \( G_{U,C} \) associated to \((U, C)\);
- A clique \( \{v^*, v_1^*, \ldots, v_{k-1}^*\} \) of \( k \) vertices (the value of \( k \) will be given later). Every vertex of the clique is adjacent to every clause vertex of \( G_{U,C} \).

Thus, we build a profile on \( 2n_u + n_c + k \) voters: \( 2n_u \) ‘variable voters’, \( n_c \) ‘clause voters’, and \( k \) ‘clique voters’. Every voter prefers to vote than to abstain (abstention will be the last preferred option for all voters). Each agent in \( \{v_1^*, \ldots, v_{k-1}^*\} \) have \( v^* \) as their first choice. Then the preferences of voters in \( \{v_1^*, \ldots, v_{k-1}^*\} \) form a Latin-square, i.e., voter \( v_i^* \)’s second choice is \( v_{(i \mod k-1)+1}^* \), third choice is \( v_{(i+1 \mod k-1)+1}^* \) and so on until \( v_{(i+k-3 \mod k-1)+1}^* \) is reached. Then agent \( v_i^* \) prefers to delegate to voters \( v_j^*, i \in \{1, \ldots, n_c\} \), then they prefer to vote. Agent \( v^* \) prefers to delegate to voters in \( \{v_1^*, \ldots, v_{k-1}^*\} \), then she prefers to delegate for voters in \( \{v_i^*| i = 1, \ldots, n_c\} \), then she prefers to vote.

For example, if we assume \( k = 4 \) and the previous 3-SAT instance \((U = \{x_1, x_2, x_3, x_4, x_5\} \text{ and } C = \{(x_1 \lor x_2 \lor \neg x_3), (\neg x_2 \lor \neg x_4 \lor x_1), (\neg x_1 \lor x_3 \lor x_5)\})\), then possible preferences could be:

\[
\begin{align*}
v^*: & \ v_1^* \succ v_2 \succ v_3 \succ v_4 \succ v_5 \succ v_6 \succ v_7 \succ v_8 \\
v_1^*: & \ v^* \succ v_1 \succ v_2 \succ v_3 \succ v_4 \succ v_5 \succ v_6 \succ v_7 \\
v_2^*: & \ v^* \succ v_2 \succ v_3 \succ v_4 \succ v_5 \succ v_6 \succ v_7 \succ v_8 \\
v_3^*: & \ v^* \succ v_3 \succ v_4 \succ v_5 \succ v_6 \succ v_7 \succ v_8 \succ v_1 \\
v_4^*: & \ v^* \succ v_4 \succ v_5 \succ v_6 \succ v_7 \succ v_8 \succ v_1 \succ v_2 \\
v_5^*: & \ v^* \succ v_5 \succ v_6 \succ v_7 \succ v_8 \succ v_1 \succ v_2 \succ v_3 \\
v_6^*: & \ v^* \succ v_6 \succ v_7 \succ v_8 \succ v_1 \succ v_2 \succ v_3 \succ v_4 \\
v_7^*: & \ v^* \succ v_7 \succ v_8 \succ v_1 \succ v_2 \succ v_3 \succ v_4 \succ v_5 \\
v_8^*: & \ v^* \succ v_8 \succ v_1 \succ v_2 \succ v_3 \succ v_4 \succ v_5 \succ v_6 \\
\end{align*}
\]

Every agent in \( \{v_i^*| i = 1, \ldots, n_c\} \) first prefer to delegate to the 3 voters in \( \{v_j^*| n = 1, \ldots, n_u\} \) corresponding to the literals of their clause. Then, they prefer to delegate to the voters in \( \{v_1^*, \ldots, v_{k-1}^*\} \). Then they prefer to delegate to \( v^* \) and then they prefer to vote. For example, if we assume \( k = 4 \) then possible preferences w.r.t. the previous 3-SAT instance could be:

\[
\begin{align*}
v_1^*: & \ v_1^* \succ v_2 \succ v_3 \succ v_4 \succ v_5 \succ v_6 \succ v_7 \succ v_8 \\
v_2^*: & \ v_2^* \succ v_1 \succ v_3 \succ v_4 \succ v_5 \succ v_6 \succ v_7 \succ v_8 \\
v_3^*: & \ v_3^* \succ v_2 \succ v_1 \succ v_4 \succ v_5 \succ v_6 \succ v_7 \succ v_8 \\
v_4^*: & \ v_4^* \succ v_3 \succ v_2 \succ v_1 \succ v_5 \succ v_6 \succ v_7 \succ v_8 \\
v_5^*: & \ v_5^* \succ v_4 \succ v_3 \succ v_2 \succ v_1 \succ v_6 \succ v_7 \succ v_8 \\
v_6^*: & \ v_6^* \succ v_5 \succ v_4 \succ v_3 \succ v_2 \succ v_1 \succ v_7 \succ v_8 \\
v_7^*: & \ v_7^* \succ v_6 \succ v_5 \succ v_4 \succ v_3 \succ v_2 \succ v_1 \succ v_8 \\
v_8^*: & \ v_8^* \succ v_7 \succ v_6 \succ v_5 \succ v_4 \succ v_3 \succ v_2 \succ v_1 \\
\end{align*}
\]
Lastly, each agent $v_i^v$ (resp. $v_i^v$) first prefer to delegate to $v_i^v$ (resp. $v_i^v$), then to delegate to voters in $\{v_i^v| i = 1, \ldots, n_c\}$ corresponding to clauses that include variable $x_i$ (resp. the negation of variable $x_i$), then they prefer to vote directly. For example, possible preferences w.r.t. the previous 3-SAT instance could be:

$$
\begin{align*}
v_{11}^{v} & : v_{1f}^{v} \succ v_{1t}^{v}, v_{1s}^{v} \succ v_{1t}^{v}, v_{1c}^{v} \succ v_{1t}^{v}, v_{1t}^{v} \\
v_{21}^{v} & : v_{2f}^{v} \succ v_{2t}^{v}, v_{1s}^{v} \succ v_{1t}^{v}, v_{1c}^{v} \succ v_{1t}^{v}, v_{1t}^{v} \\
v_{31}^{v} & : v_{3f}^{v} \succ v_{3s}^{v}, v_{1c}^{v} \succ v_{1t}^{v}, v_{1c}^{v} \succ v_{1t}^{v}, v_{1t}^{v} \\
v_{41}^{v} & : v_{4f}^{v} \succ v_{4t}^{v}, v_{1c}^{v} \succ v_{1t}^{v}, v_{1c}^{v} \succ v_{1t}^{v}, v_{1t}^{v} \\
v_{51}^{v} & : v_{5f}^{v} \succ v_{5t}^{v}, v_{1c}^{v} \succ v_{1t}^{v}, v_{1c}^{v} \succ v_{1t}^{v}, v_{1t}^{v} \\
v_{11}^{v} & : v_{1f}^{v} \succ v_{1f}^{v}, v_{1s}^{v} \succ v_{1f}^{v}, v_{1c}^{v} \succ v_{1f}^{v}, v_{1f}^{v} \\
v_{21}^{v} & : v_{2f}^{v} \succ v_{2f}^{v}, v_{1s}^{v} \succ v_{1f}^{v}, v_{1c}^{v} \succ v_{1f}^{v}, v_{1f}^{v} \\
v_{31}^{v} & : v_{3f}^{v} \succ v_{3f}^{v}, v_{1c}^{v} \succ v_{1f}^{v}, v_{1c}^{v} \succ v_{1f}^{v}, v_{1f}^{v} \\
v_{41}^{v} & : v_{4f}^{v} \succ v_{4f}^{v}, v_{1c}^{v} \succ v_{1f}^{v}, v_{1c}^{v} \succ v_{1f}^{v}, v_{1f}^{v} \\
v_{51}^{v} & : v_{5f}^{v} \succ v_{5f}^{v}, v_{1c}^{v} \succ v_{1f}^{v}, v_{1c}^{v} \succ v_{1f}^{v}, v_{1f}^{v}
\end{align*}
$$

We fix $k = 3n_c + n_u + n_u n_c$ and show that the 3-SAT instance is satisfiable if and only if there exists an equilibrium with dissatisfaction at most $2k$.

Assume first that the 3-SAT instance is satisfiable. Then, $v^*$ plus the $n_u$ variable vertices in $G_{U,C}$ corresponding to true literals form a kernel in the delegation acceptability digraph. Let us consider the corresponding delegation function where:

- the $n_u + 1$ voters in the kernel vote. The dissatisfaction of $v^*$ is $(k - 1) + n_c$, the one of each voter corresponding to a true literal is at most $1 + n_c$.
- voters $v_i^v$ delegate to $v^*$; they have dissatisfaction 0;
- clause voters delegate to a variable voter corresponding to a true literal in the clause, thus with a dissatisfaction at most 2.
- (variable) voters corresponding to false literals delegate to the opposite (true) literal and have dissatisfaction 0.

Thus, the dissatisfaction of this equilibrium is at most $k - 1 + n_c + n_u(1 + n_c) + 2n_c = 2k - 1$.

Conversely, assume that there is an equilibrium with dissatisfaction at most $2k$. If a voter $v^*_s$ ($\neq v^*$) in the clique votes, then no other clique voter votes, and this already induces a dissatisfaction at least $\sum_{i=1}^{c-2} 1 = \frac{(k-1)(k-2)}{2} > 2k$ (for $k \geq 7$) for the other $k - 2$ vertices $v^*_j, j \neq s$. Thus this is not possible. Similarly, if a clause voter votes, then no voter in the clique can vote, and this already induces a dissatisfaction at least $(k - 1)$ for each voter in the clique, thus a global dissatisfaction at least $k(k - 1) > 2k$ (for $k > 4$).

Then, in the considered equilibrium, $v^*$ votes, no other voter in the clique votes, and no clause voter votes. Then for any $i$ exactly one voter among $v_{it}^{v}$ and $v_{if}^{v}$ votes. We conclude the proof by showing that the assignment where a literal is true if the corresponding voter votes is a satisfying assignment. Note that $v^*$ has dissatisfaction $(k - 1) + n_c$. If a clause voter delegates to $v^*$, then it has dissatisfaction $k + 2$, so the global dissatisfaction is greater than $2k$, impossible. This means that each clause vertex delegates to a voting variable vertex, and thus all the clauses are satisfied by the assignment.

**Theorem 9.** MINMAXVP is NP-hard in the case of symmetrical preference profiles, even if there are no abstainers.

**Proof.** Let us consider a 3-SAT instance with a set $U$ of variables and a set $C$ of clauses. We will create a preference profile, the delegation acceptability (symmetric) digraph of which is made of:

- The digraph $G_{U,C}$ associated to $(U, C)$;
- A clique $\{v_1, \ldots, v_{n_c+2}\}$ of $n_c + 2$ vertices. Every vertex of the clique is adjacent to every clause vertex of $G_{U,C}$.
• An independent set \( \{v'_1, \ldots, v'_{n_c+2}\} \) of \( n_c + 2 \) vertices. Every vertex \( v'_i \) is adjacent to \( v_i \).

Figure 5 illustrates the digraph corresponding to the following 3-SAT instance:

\[
U = \{x_1, x_2, x_3, x_4, x_5\} \\
C = \{(x_1 \lor x_2 \lor \neg x_3), (\neg x_2 \lor \neg x_4 \lor x_1), (\neg x_1 \lor x_3 \lor x_5)\}
\]

Thus we build a profile on \( 2n_u + 3n_c + 4 \) voters, all of them prefer to vote than to abstain.

We first precise the preferences of voters in \( \{v_1, \ldots, v_{n_c+2}\} \). Each voter in \( \{v_1, \ldots, v_{n_c+2}\} \) prefers first to delegate to voters in \( \{v'_j \mid j = 1, \ldots, n_c\} \) and in the same order, \( v'_1 \) first, \( v'_2 \) second and so on. Then, if they prefer to delegate to the other voters in \( \{v_1, \ldots, v_{n_c+2}\} \), then to the corresponding \( v'_j \) and lastly, they prefer to vote. For example, in the instance described by Equations (3) and (4), possible preferences are given by:

\[
\begin{align*}
v_1 &: v'_1 \succ v_1, v'_2 \succ v_2, v'_3 \succ v_3, v'_4 \succ v_4, v'_5 \succ v_5, v'_1 \succ v_1 \\
v_2 &: v'_1 \succ v_2, v'_2 \succ v_2, v'_3 \succ v_3, v'_4 \succ v_4, v'_5 \succ v_5, v'_2 \succ v_2 \\
v_3 &: v'_1 \succ v_3, v'_2 \succ v_3, v'_3 \succ v_3, v'_4 \succ v_4, v'_5 \succ v_5, v'_3 \succ v_3 \\
v_4 &: v'_1 \succ v_4, v'_2 \succ v_4, v'_3 \succ v_4, v'_4 \succ v_4, v'_5 \succ v_5, v'_4 \succ v_4 \\
v_5 &: v'_1 \succ v_5, v'_2 \succ v_5, v'_3 \succ v_5, v'_4 \succ v_5, v'_5 \succ v_5, v'_5 \succ v_5 \\
\end{align*}
\]

Without defining further the other preferences, we will show the following result: the 3-SAT instance is satisfiable iff there exists a Nash-stable delegation function in which each guru has a voting power which is strictly less than \( n_c + 3 \). Indeed, note that if some voters in \( \{v'_j \mid j = 1, \ldots, n_c\} \) are gurus, then one of them will be endorsed by all voters in \( \{v_1, \ldots, v_{n_c+2}\} \) and will thus have a power of at least \( n_c + 3 \). Additionally, if one voter in \( \{v_1, \ldots, v_{n_c+2}\} \) is a guru, then she will collect the votes of all other voters in \( \{v_1, \ldots, v_{n_c+2}\} \) and the vote of the corresponding \( v'_i \) and will thus have a power of at least \( n_c + 3 \). Hence, a Nash-stable delegation function in which each guru has a power which is strictly less than \( n_c + 3 \) corresponds to a Nash-stable delegation function in which no voters in \( \{v_1, \ldots, v_{n_c+2}\} \) are gurus. This is possible if all voters in \( \{v'_1, \ldots, v'_{n_c+2}\} \) are gurus and if each voter in \( \{v'_j \mid j = 1, \ldots, n_c\} \) delegates to a guru in \( \{v''_i, v''_j \mid i = 1, \ldots, n_u\} \). In this case, the power of a guru is at most \( n_c + 2 \) and the gurus in \( \{v''_i, v''_j \mid i = 1, \ldots, n_u\} \) form a truth assignment that satisfies all clauses.
6.3 Convergence of Dynamics

We now focus on the question of convergence under best response dynamics (BRD) in the case of symmetrical preference profiles. Interestingly, while there might be cycles in the single-peaked case, we show now that under BRD the convergence is guaranteed under symmetrical profiles, and that this convergence occurs within a small number of steps. We will also show that convergence is not guaranteed under better response dynamics, thus providing a notable difference between the two dynamics.

Given a dynamics with token function $T$, let us define rounds as follows:

- The first round is $[1, r_1]$ where $r_1$ is the smallest $r$ such that each voter receives the token at least once in $[1, r]$.
- The $k^{th}$ round is $[r_{k-1} + 1, r_k]$ where $r_k$ is the smallest $r$ such that each voter receives the token at least once in $[r_{k-1} + 1, r]$.

For instance, in the case of permutation dynamics, we have $r_k = kn$.

**Theorem 10.** Given a symmetric preference profile $P$, a BRD dynamics always converges in at most 3 rounds.

**Proof.** Let us consider $j = T(t)$ for some step $t$. Suppose that $j$ decides to vote herself when she has the token at time $t$. Then, for any $t' \geq t$, $j$ remains a guru. Indeed, since she decides to vote at time $t$, it means that no voter in $\text{Acc}(j)$ were gurus. While $j$ is a guru, then a voter $i \in \text{Acc}(j)$ cannot become a guru under BRD since by symmetry $j \in \text{Acc}(i)$.

For a voter $j$, consider a time $t$ in the second round where she receives the token.

- If $j$ decides to vote, from the previous argument she will be a guru forever after time $t$.
- If $j$ delegates (directly or indirectly) to a guru $i$, since we are in the second round $i$ already had the token, already decided to be a guru, and thus by the previous argument will remain a guru forever. Then, $j$ will never be a guru ($i \in \text{Acc}(j)$ will always be available as a guru).
- If $j$ abstains, then she prefers to abstain than to vote so she will never be a guru.

This means that after the second round the set of gurus is fixed. Then, in the third round, gurus remain gurus, and non gurus choose their most preferred guru in the set of gurus, or abstain (if they prefer to abstain than to vote or to delegate to a guru). Thus we reach an equilibrium at the end of the third round.

Note that 3 rounds are necessary. Consider for instance a profile with 3 voters, 1 prefers 2 and then to vote, 3 prefers 2 and then to vote, and 2 prefers 1, then 3, and then to vote. We give the token to 1,2,3 (first round), 2,3,1 (second round), 1, 3, 2 (third round). Under BRD we get $d_1(1) = 2, d_2(2) = 3, d_3(3) = 3$ (end of the first round, 3 is a guru), $d_4(2) = 3, d_5(3) = 3, d_6(1) = 1$ (end of the second round, the set of gurus $\{1,3\}$ is definitive), $d_7(1) = 1, d_8(3) = 3, d_9(2) = 1$ (end of the third round, convergence).

If we use better response instead of best response, then as said earlier the convergence is not guaranteed, even if we start from the delegation $d_0$ where nobody delegates, as shown by the following example.

**Example 4.** Let us consider that the case of 4 voters, where $\text{Acc}(1) = \text{Acc}(3) = \{2, 4\}, \text{Acc}(2) = \text{Acc}(4) = \{1, 3\}$. They all prefer to vote than to abstain.

We give the token to 1,2,1,3,2,4,3,1,4, ... Then the following is compatible with better response: $d_1(1) = 2, d_2(2) = 3, d_3(1) = 1, d_4(3) = 4, d_5(2) = 2, d_6(4) = 1, d_7(3) = 3, d_8(1) = 2, d_9(4) = 4$. At this point $d_9 = d_2$, so this is a cycle. Intuitively, each voter $i$ delegates to its neighbor $i + 1$ (modulo 4); in the following step $i + 1$ delegates to $i + 2$, then we give the token back to $i$ who is no more happy with her delegation and decides to vote herself.
Distance-Based Preference Profiles on Social Networks

7.1 Definition and Existence of Equilibria

In this section, we restrict our attention to another type of structured preference profiles. We assume that voters are connected in a social network. This social network can be represented by an undirected graph $G_{SN} = (N, A)$ such that $(i, j) \in A$ if voters $i$ and $j$ are connected in the social network. We denote by $\text{dist}(i, j)$ the length of a shortest path between nodes $i$ and $j$ in $G_{SN}$. We assume that $G_{SN}$ structures the preference profile such that for each voter $i$ we have a distance bound $db_i$ such that:

$$\forall j \in N\setminus\{i\}, j \in \text{Acc}(i) \iff \text{dist}(i, j) \leq db_i$$

We say that such a preference profile is DBSN (Distance Based on a Social Network).

Example 5. Consider the social network $G_{SN}$ represented in Figure 6 and assume that $db_1 = db_4 = db_5 = 1$ and $db_2 = db_3 = 2$. A partial preference profile which is DBSN w.r.t. this social network and the distance bounds of the voters is:

1 : 2 $\succ$ 3 $\succ$ 5 $\succ$ 1
2 : 1 $\succ$ 2 5 $\succ$ 3 $\succ$ 0
3 : 4 $\succ$ 3 1 $\succ$ 5 $\succ$ 2 $\succ$ 3
4 : 3 $\succ$ 4
5 : 1 $\succ$ 5 3 $\succ$ 5

which in turn yields the following delegation-acceptability digraph (note that the delegation acceptability digraph is unique given the social network and the distance bounds of the voters).

![Figure 6: An example of Social Network.](image)

Figure 7: The delegation-acceptability digraph induced by the social network of Figure 6 and the distance bounds of the voters.

Note that any symmetrical preference profile is DBSN: indeed, consider the social network where $(i, j) \in A$ if and only if $j \in \text{Acc}(i)$ (or, equivalently for a symmetrical profile, $i \in \text{Acc}(j)$), and set $db_i = 1$ for any voter $i$.

From this observation we immediately know that MINDIS, MINMAXVP and MINABST are NP-hard in the case of DBSN preference profiles.

We now show that the existence of equilibrium, which was trivially guaranteed in the symmetrical case, is also guaranteed in this more general case.
Theorem 11. If preferences are DBSN, a preference profile always admits an equilibrium. Furthermore, an equilibrium can be computed in $O(n^2)$.

Proof. We give a $O(n^2)$ procedure that builds a Nash-stable delegation function for any DBSN preference profile.

Build a set $K$ of voters by using the following recursive procedure. Let $S = N$, and remove from $S$ the abstainers. Then, while $S$ is not empty, add to $K$ the voter $i$ of $S$ with smallest $db_i$ value and remove $i$ from $S$ as well as all voters accepting $i$ as guru. At the end of this process, $K = \{i_1, \ldots, i_m\}$ is a Kernel of $G^*_p$. It is absorbing because each voter in $N \setminus K$ has at some point been removed from $S$ because it was absorbed by one element of $K$. It is also independent. Indeed, let us assume by contradiction that $i_k$ accepts to delegate to $i_t$ with $i_k, i_t \in K$. Then, necessarily $i_k$ has been added to $K$ before $i_t$. Otherwise, $i_k$ would have been removed from $S$ at the same time as $i_t$ and would not have been added to $K$. Hence, $db_{i_k} \leq db_{i_t}$ and $i_t$ accepts to delegate to $i_k$ which is not possible by the same argument. Let us conclude by saying that this method builds $K$ in $O(n^2)$ and that the equilibrium induced by $K$ can easily be built in $O(n^2)$.

7.2 Membership Problem

Theorem 12. MEMB is NP-hard in the case of DBSN preference profiles, even if there are no abstainers.

Proof. Let us consider a 3-SAT instance with a set $U$ of variables and a set $C$ of clauses. We consider the social network made of:

- The undirected version of the graph $G_{U,C}$ associated to $(U, C)$ (see Figure 4 in Section 6);
- Two adjacent vertices $v_l$ and $v_{q}$: $v_l$ is moreover adjacent to all clause vertices $v_{j}^c$.

Thus we have $2n_u + n_c + 2$ voters. All voters have a distance bound of $1$ except $v_q$ which has a distance bound of $2$, and they all prefer to vote than to abstain.

Let us show that the 3-SAT instance is satisfiable iff the DBSN preference profile induced by the corresponding social network admits a Nash-stable delegation function in which $v_q$ is a guru.

Assume that there exists a Nash-stable delegation function $d$ in which $v_q$ is a guru. As $v_q$ is a guru, then voters $v_l$ and $v_{q}^c$, $\forall j \in \{1, \ldots, n_c\}$ cannot be gurus as they are at a distance of less than $2$ from $v_q$. Contrarily to voter $v_l$ who accepts to delegate to $v_{q}$, each voter $v^c \in \{v_{q}^c[j = 1, \ldots, n_c]\}$ will necessarily delegate to one of the three voters corresponding to the literals of its clause. Lastly, note that as $v_{q}^c[i]$ and $v_{q}^f[i]$ are connected for all $i \in \{1, \ldots, n_u\}$, then at most one of them can be a guru in a Nash-stable delegation function. Furthermore, as the voters $v^c \in \{v_{q}^c[j = 1, \ldots, n_c]\}$ cannot be gurus, one voter out of $\{v_{q}^c[i], v_{q}^f[i]\}$ will have to be a guru for all $i \in \{1, \ldots, n_u\}$. Now consider the truth assignment that sets to true a variable $x_i$ iff $v_{q}^c[i] \in \text{Gu}(d)$. It is easy to check that this truth assignment satisfies each clause in $C$.

Conversely, if there exists a truth assignment $X$ satisfying each clause in $C$ then consider the delegation function $d$ such that $\text{Gu}(d)$ is composed of $v_q$, all variables $v_{q}^c[i]$ such that $x_i$ is set to true in $X$ and all variables $v_{q}^c[j]$ such that $x_j$ is set to false in $X$. For all voters $j$ not in $\text{Gu}(d)$, $d(j)$ is then given by the voter that $j$ prefers in $\text{Gu}(d)$. It is easy to see that $d$ is Nash-stable.

7.3 Convergence of Dynamics

Since an equilibrium always exists, we consider now the convergence of dynamics. Example 4 shows that, under better response, the convergence is not guaranteed in the case of symmetrical preference profiles. Therefore, it is the same in the case of DBSN preference profiles. We now extend Theorem 10 and show that under BRD the convergence is guaranteed under DBSN preference profiles.

Theorem 13. Given a DBSN preference profile $P$, a BRD dynamics always converges.

Proof. Let us recall that we assume that each voter has the token infinitely many times. Consider a DBSN preference profile $P$, and a BRD dynamics with a starting delegation $d_0$ and a token function $T$. We assume that voters are numbered $1, 2, \ldots, n$ in such a way that $db_i \leq db_{i+1}$, $i = 1, \ldots, n - 1$.

Let us define $G$ as the set of voters which are gurus (vote) infinitely many times in the dynamics: $G = \{i_1, \ldots, i_s\}$ with $i_1 \leq i_2 \leq \cdots \leq i_s$. Note that, obviously, $G$ contains no abstainers. Since
voters in $V \setminus G$ are gurus finitely many times, let us consider a step $t_0$ such that, for any $t \geq t_0$, no voter in $V \setminus G$ are gurus (they always delegate or abstain).

For $k \in \{1, 2, \ldots, s\}$, let us now define $t_k$ as the first time $t > t_{k-1}$ such that $i_k$ has the token and decides to vote (thus it is a guru).

We now show by recurrence on $k$ that for any $k \in \{1, 2, \ldots, s\}$, any $t \geq t_k$, $i_k$ is a guru at time $t$ ($i_k$ remains a guru forever after time $t_k$).

Consider $k = 1$. At $t_1$, $i_1$ decides to vote. This means that no voter in $\text{Acc}(i_1)$ is a guru. Then, while $i_1$ is a guru:

- no voter $j > i_1$ in $\text{Acc}(i_1)$ ever becomes a guru: indeed, since $db_i$ are in non decreasing order, if $j \in \text{Acc}(i_1)$ then $i_1 \in \text{Acc}(j)$. While $i_1$ is a guru $j$ does not decide to vote.
- no voter $j < i_1$ in $\text{Acc}(i_1)$ ever becomes a guru: indeed, these are in $V \setminus G$ and since $t_1 \geq t_0$ we know that they always delegate or abstain.

Then no voter in $\text{Acc}(i_1)$ becomes a guru, so $i_1$ will vote (be a guru) forever.

The inductive step is almost similar. Suppose that the claim is true up to $k - 1$, and consider step $k$. At $t_k$, $i_k$ decides to vote. This means that no voter in $\text{Acc}(i_k)$ is a guru. Then, while $i_k$ is a guru:

- no voter $j > i_k$ in $\text{Acc}(i_k)$ ever becomes a guru, for the same reason as previously.
- no voter $j < i_k$ in $\text{Acc}(i_k)$ ever becomes a guru: if $j \in V \setminus G$ this follows as previously from the fact that $t_k \geq t_0$. If $j \in G$, $j < i_k$ implies that $j = i_f$ for some $f < k$. Then, by induction, $j$ is a guru forever from step $t_f < t_k$, thus she cannot be in $\text{Acc}(i_k)$ (since at time $t_k$, $i_k$ decides to vote).

Thus, at time $t_k$, voters in $G$ are gurus forever, and voters in $V \setminus G$ never become gurus. From $t_s$ we only have to wait for another round that each voter has the token one more time. Gurus will maintain their choice, while non gurus will choose (thanks to BRD) their most preferred guru in $G$, or abstain (if they prefer to abstain than to vote or to delegate to gurus). We reach a Nash-stable delegation function.

\[\square\]

8 Conclusion and Future Works

In this paper, we have proposed a game-theoretic model of the delegation process induced in a liquid democracy setting. Interestingly, this model makes it possible to answer several questions on the equilibria that may be reached by the delegation process of liquid democracy. For this purpose, we have defined and studied several existence and optimization problems defined on the Nash equilibria of the corresponding delegation games. Unfortunately, under unrestricted preferences over gurus, the existence of a Nash-equilibrium is not guaranteed and is even NP-Hard to decide. Hence, we have investigated several types of restricted preferences for which a Nash-equilibrium always exists, namely single-peaked preferences, symmetrical preferences and lastly, preferences induced by a distance on a Social Network.

We have obtained positive and negative results which surprisingly differs for the different types of restricted preferences under study. For instance, while single-peaked preferences are the only ones (under study) that makes it possible to solve efficiently all the optimization problems that we have defined, it is also the only type of restricted preferences (under study) that do not guarantee the convergence of the delegation process under best response dynamics.

We give several possible directions for future works. Firstly, similarly to Brânzei et al. [BCMP13] who studied the price of anarchy of the voting games induced by the iterative voting scheme, it would be interesting to study the price of anarchy of the delegation games developed in this paper.

Another direction would be to impose further constraints on the possible delegations available to a voter. For instance, voters could be part of a network and would only be allowed to delegate directly to their neighbors in the network [BBCV09]. Such restriction could be motivated by the fact that one would want to ensure that delegations rely on a foundation of trust.
A last direction would be to extend our results to the framework of *viscous democracy* that was recently proposed by Boldi et al. \[BBCV11\]. In this setting, the weight of a delegation decreases exponentially with the length of the delegation path. This difference would change our model as the preferences of an agent would not only be defined on her possible gurus but also on the corresponding delegation paths.

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