Possible quantum obstructions to the process of Abelian conversion

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Abstract

The procedure for Abelian conversion of second class constraints due to Batalin, Fradkin, Fradkina and Tyutin is considered at quantum level, by using the field-antifield formalism. It is argued that quantum effects can obstruct the process. In this case, Wess-Zumino fields may be introduced in order to restore the lost symmetries.

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1 Introduction

First order systems with first and second class constraints\cite{1,2} can be quantized along several lines. The replacement of Dirac brackets (DB’s) by (anti)commutators, which is the cornerstone of canonical quantization, can only be done for simple systems, due to the usual complexity of the DB structure. Other methods, as the BRST operatorial quantization\cite{3} or its functional counterpart, the Batalin, Fradkin and Vilkovisky (BFV) procedure\cite{3,4}, essentially keep the difficulties associated to the existence of second class constraints\cite{2}. Some years ago, Batalin, Fradkin, Fradkina and Tyutin (BFFT)\cite{6} have introduced an algorithm which implements the Abelian conversion of the second class constraints, by extending in a proper way the phase space and also redefining the dynamic variables of the theory to be converted. This results in a system with a gauge structure with only first class constraints and a trivial symplectic structure. Its quantization can then be implemented avoiding DB’s and related difficulties. This route has been extensively followed in recent works\cite{7}, where in general first order Hamiltonian systems have their second class constraints Abelianized by the BFFT method and quantized along functional procedures. After integrating over the generalized momenta, effective Lagrangians are obtained, generating in the configuration space the terms responsible for the Abelian conversion of the original second class sector. In a work by Fujiwara, Igarashi and Kubo\cite{8}, the BFFT procedure is applied in order to convert second class constraints which satisfy anomalous gauge algebras as a starting point. In a more systematic way, Banerjee, Rothe and Rothe\cite{9} have considered a related problem, by implementing the BFFT procedure directly at quantum level. These last authors use anomalous commutators\cite{10} as the fundamental building blocks for the implementation of the BFFT procedure. In references\cite{8,9}, however, it is not considered the situation where the process of conversion of second class constraints introduces in the considered theory a gauge invariance which is itself obstructed at quantum level. Can the BFFT procedure of conversion present any obstruction due to quantum effects? Another point that seems to be relevant to be understood is the influence in the BFFT conversion procedure in situations with more general gauge structures, such as those with open algebras.

To investigate these points we utilize here the tools of the field-antifield formalism\cite{2,11,12,13,14,15}, which naturally takes into account systems with general gauge algebras. Once one regularization procedure is chosen, possible gauge obstructions due to the presence of anomalies are also naturally considered in that formalism. Although the field-antifield formalism is essentially a Lagrangian procedure, it is powerful enough to treat also first order (Hamiltonian) systems\cite{16}. In this work we consider the field-antifield quantization of gauge invariant first order systems that had their second class constraints converted by the BFFT procedure. In section 2 we analyze the situation where only second class constraints are present in the theory. In section 3, this is extended to the mixed case, where also first class constraints can be originally present. Section 4 is devoted to apply the ideas introduced in the first sections to gauge-fixed chiral electrodynamics. We show in this example that it is necessary to introduce not only BFFT fields, but also Wess-Zumino fields.
and antifields in order to find a local quantum action without obstruction to its gauge symmetries. In section 5 we make some additional comments and conclusions.

2 Pure second class constraints

In order to introduce the ideas in a simpler way, let us first consider the case of a first order system with only second class constraints in a phase-space $P$. For simplicity, we are assuming only discrete bosonic phase-space coordinates $y^\mu$, $\mu = 1, 2, ..., 2N$, and bosonic second class constraints $\chi_\alpha$, $\alpha = 1, 2, ..., 2n$. The extension to more general situations can be trivially done. Defining the fundamental Poisson brackets (PB’s) by

$$\{y^\mu, y^\nu\} = f^{\mu\nu},$$

where $f^{\mu\nu}$ is an antisymmetric and invertible matrix, the PB between any two functions $A(y)$ and $B(y)$ on $P$ is given by

$$\{A, B\} = \frac{\partial A}{\partial y^\mu} f^{\mu\nu} \frac{\partial B}{\partial y^\nu}.$$ 

In this way, the first class Hamiltonian $H = H(y)$ and the constraints $\chi_\alpha = \chi_\alpha(y)$ satisfy the PB structure

$$\{\chi_\alpha, \chi_\beta\} = \Delta^{\alpha\beta},$$

$$\{H, \chi_\alpha\} = V_\alpha^\beta \chi_\beta.$$ 

Since $\chi_\alpha$ are second class, the constraint matrix $\Delta^{\alpha\beta}$ is regular.

The functional quantization of a system like the one appearing in (3) can be done along the lines introduced by Senjanovic. The vacuum functional, for instance, is defined by

$$Z = \int [dy^\mu] |\det f|^{-\frac{1}{4}} |\delta[\chi_\alpha]| |\det \Delta|^{\frac{i}{2}} \exp\{i \int dt [B_\mu \dot{y}^\mu - H] \}.$$ 

We note that in the measure appears the determinant of the second class constraint matrix as well as the determinant of the symplectic matrix given by (4). As a consequence the measure becomes invariant under canonical transformations on $P$. In the argument of the exponential, $B_\mu$ is related to $f^{\mu\nu}$ through...
$$f_{\mu\nu} = \frac{\partial B_\mu}{\partial y^\nu} - \frac{\partial B_\nu}{\partial y^\mu}$$

(5)

and $f^{\mu\nu}$ is the inverse of $f_{\mu\nu}$.

To implement the BFFT procedure, we extend the original phase space $P$ (coordinates $y^\mu$) with BFFT variables $\psi^\alpha$, ($\alpha = 1, 2, ..., 2n$) with PB structure given by

$$\{\psi^\alpha, \psi^\beta\} = \omega^{\alpha\beta}.$$  

(6)

In (6), $\omega^{\alpha\beta}$ is a constant, antissymmetric and invertible matrix. It follows that in the BFFT extended phase space, the PB between two quantities $A(y, \psi)$ and $B(y, \psi)$ is given by

$$\{A, B\} = \frac{\partial A}{\partial y^\mu} f^{\mu\nu} \frac{\partial B}{\partial y^\nu} + \frac{\partial A}{\partial \psi^\alpha} \omega^{\alpha\beta} \frac{\partial B}{\partial \psi^\beta},$$

(7)

as the two sectors of the extended phase space are assumed to be independent.

The general idea of BFFT is to define new constraints $\tilde{\chi}_\alpha = \tilde{\chi}_\alpha(y, \psi)$ and Hamiltonian $\tilde{H} = \tilde{H}(y, \psi)$ in such a way that

$$\{\tilde{\chi}_\alpha, \tilde{\chi}_\beta\} = 0,$$

$$\{\tilde{H}, \tilde{\chi}_\alpha\} = 0.$$  

(8)

By requiring that $\tilde{A}(y, 0) = A(y)$ for any quantity $A$ defined on $P$ (the unitary gauge implemented by the choice $\psi^\alpha = 0$), the original theory is recovered. In references [3] it is proved that eqs. (8), submitted to the above condition, always have a power series solution in the BFFT variables, with coefficients with only $y^\mu$ dependence. The second class constraints can be extended to

$$\tilde{\chi}_\alpha(y, \psi) = \chi_\alpha(y) + X_{\alpha\beta}(y) \psi^\beta + X_{\alpha\beta\gamma}(y) \psi^\beta \psi^\gamma + \ldots .$$

(9)

The condition that $\tilde{\chi}_\alpha$ satisfy (8) imposes restrictions in their expansion coefficients. As an example which will be useful later, the regular matrices $X_{\alpha\beta}$ must satisfy the identity

$$X_{\alpha\beta} \omega^{\beta\gamma} X_{\delta\gamma} = -\Delta_{\alpha\delta}.$$  

(10)

If some quantity $A(y)$ is not a second class constraint, it can also be extended to $\tilde{A}(y, \psi)$ in order to be involutive with the converted constraints $\tilde{\chi}_\alpha$. BFFT show that in this situation
\[ \tilde{A}(y, \psi) = A(y) - \psi^\alpha \omega_{\alpha\beta} X^{\beta\gamma} \{ \chi_\gamma, A \} + ... \] (11)

where the dots represent all least second order corrections to \( A(y) \). Now it is possible to prove that the first order action

\[ \tilde{S}_0 = \int dt \left[ B_\mu \dot{y}^\mu + B_\alpha \dot{\psi}^\alpha + \lambda^\alpha \bar{\chi}_\alpha - \tilde{H} \right] \] (12)

is invariant under the gauge transformations

\[
\begin{align*}
\delta y^\mu &= \{ y^\mu, \bar{\chi}_\alpha \} \epsilon^\alpha, \\
\delta \psi^\alpha &= \{ \psi^\alpha, \bar{\chi}_\beta \} \epsilon^\beta, \\
\delta \lambda^\alpha &= -\dot{\epsilon}^\alpha.
\end{align*}
\]

(13)

Close to what occurs in (5), in (13) \( B_\alpha \) is related with the inverse of \( \omega^{\alpha\beta} \) through

\[ \omega_{\alpha\beta} = \frac{\partial B_\alpha}{\partial \psi^\beta} - \frac{\partial B_\beta}{\partial \psi^\alpha}. \] (14)

By using some of the above equations, it is not difficult to show that actually

\[ \delta [B_\mu \dot{y}^\mu + B_\alpha \dot{\psi}^\alpha + \lambda^\alpha \bar{\chi}_\alpha - \tilde{H}] = \frac{d}{dt} \left\{ [B_\mu f^{\mu\nu} \frac{\partial \bar{\chi}_\alpha}{\partial y^\nu} + B_\beta \omega^{\beta\rho} \frac{\partial \bar{\chi}_\alpha}{\partial \psi^\rho} + \bar{\chi}_\alpha] \epsilon^\alpha \right\}, \]

(15)

and consequently we prove that (12) is indeed invariant under (13), provided boundary terms can be discarded.

As we have already observed, the quantization of the system described by action (12) can be done along several different but equivalent lines. Under the field-antifield formalism [11], it is necessary to introduce the antifields \( \phi_\alpha^* = (y^\mu_\alpha, \psi^\alpha_\beta, \lambda^\alpha_\beta, c^\alpha_\alpha) \) corresponding respectively to the fields \( \phi^A = (y^\mu, \psi^\alpha, \lambda^\alpha, c^\alpha) \), the ghosts \( c^\alpha \) considered here in equal foot to the previous fields. It is then easy to see that the field-antifield action

\[ S = S_0 + \int dt \left[ y^\mu_\beta \{ y^\beta, \bar{\chi}_\alpha \} c^\alpha + \psi^\beta_\beta \{ \psi^\beta, \bar{\chi}_\alpha \} c^\alpha + \lambda^\alpha_\beta c^\alpha \right] \]

(16)

satisfy the classical master equation

\[ \frac{1}{2} (S, S) = 0. \] (17)

In the above equation we have introduced the antibracket \( (X, Y) = \frac{\delta X}{\delta \phi_A} \frac{\delta Y}{\delta \phi^*_A} - \frac{\delta X}{\delta \phi^*_A} \frac{\delta Y}{\delta \phi_A} \) for any two quantities \( X \) and \( Y \). As it is well known, (17) contains all the
gauge structure associated to the action $S$. To fix the gauge we need to introduce the trivial pairs $\bar{c}_\alpha, \bar{\pi}_\alpha$ as new fields, and the corresponding antifields $c^{*\alpha}, \pi^{*\alpha}$, as well as a gauge-fixing fermion. It is always possible to choose

$$\Psi = \bar{c}_\alpha \psi^\alpha,$$  \hspace{1cm} (18)

which implements the unitary gauge, but different choices are available. It is also necessary to extend the field-antifield action to

$$S \to S_\Psi = S + \int dt \bar{\pi}_\alpha c^{*\alpha}.$$  \hspace{1cm} (19)

in order to implement the gauge fixing introduced by $\Psi$. The gauge-fixed vacuum functional is now defined as

$$Z_\Psi = \int [d\phi^A][d\phi^*_A][\det \omega]^{-\frac{1}{2}} \delta[\phi^*_A - \frac{\partial \Psi}{\partial \phi^*_A}] \exp\{i S_\Psi\}.$$  \hspace{1cm} (20)

In the unitary gauge, we observe that besides the identifications $c^{*\alpha} = \psi^\alpha, \psi^{*\alpha} = \bar{c}_\alpha$, all the other antifields vanish. With this and the use of eqs. (9-10), it is not difficult to see that (20) reduces exactly to (4), as expected.

A fundamental point to be considered at the quantum level of any gauge theory is if quantum effects can obstruct the gauge symmetry. Under the field-antifield formalism the non-obstruction is related with the independence of the (vacuum) functional with respect to redefinitions of the gauge-fixing fermion $\Psi$. This independence occurs if the classical field-antifield action $S$ can be replaced by some quantum action $W$ satisfying the so-called quantum master equation

$$< \frac{1}{2} (W, W) - i\hbar \Delta W >_\Psi = 0,$$  \hspace{1cm} (21)

where $<\mathcal{O}>_\Psi$ means the expected value of $\mathcal{O}$ calculated with the use of a specific $\Psi$. In expression (21) we have introduced the potentially singular operator

$$\Delta \equiv \frac{\delta_r}{\delta \phi^A} \frac{\delta_l}{\delta \phi^*_A}.$$  \hspace{1cm} (22)

If now we expand $W$ in powers of $\hbar$,

$$W[\phi^A, \phi^*_A] = S[\phi^A, \phi^*_A] + \sum_{p=1}^{\infty} \hbar^p M_p[\phi^A, \phi^*_A],$$  \hspace{1cm} (23)

we can write the quantum master equation (21) in loop order. For the two first terms we have
\[(S, S) = 0, \quad (M_1, S) = i \Delta S. \tag{24}\]

As expected, the tree approximation gives (17). Eq. (25) is only formal, since the operator \(\Delta\) must be regularized. If it vanishes when applied on \(S\), the quantum action \(W\) can be identified with \(S\). If its action on \(S\) gives a non-trivial result but there exists some \(M_1\) expressed in terms of local fields such that (25) is satisfied, gauge symmetries are not obstructed at one loop order. Otherwise, the theory presents anomalies which can be defined by

\[\mathcal{A}[\phi, \phi^*] = \Delta S + \frac{i}{\hbar} (S, M_1) = a_\alpha e^\alpha + \ldots. \tag{26}\]

It can be shown \[13\] that \(a_\alpha\) is the usual gauge anomaly for closed algebra gauge theories. So if \(\mathcal{A}\) cannot be set to zero, the process of conversion is obstructed at quantum level. If one can introduce WZ fields in order to restore the lost symmetry, then the process is successful, but using more fields than those originally prescribed by BFFT, which should be equal to the number of second class constraints originally present in the theory. In the field-antifield formalism, the introduction of WZ fields \[17\] are necessary to construct some \(M_1\) which is a local functional of the extended set of fields in order to have the quantum master equation satisfied if true gauge anomalies are found. This kind of procedure depends on the regularization prescription adopted as well as on the specific model considered. Further discussions, at this stage, would be only formal and we reserve section 4 to discuss some of these points in the context of a specific example.

### 3 The mixed case

In order to generalize the situation treated in section 2 we consider first-order systems that can present from the beginning first class constraints, say, \(\gamma_a(y), a = 1, 2, \ldots, m\). Keeping the PB structure already introduced in (32), such a system in general presents a constraint algebra given by (2)

\[
\begin{align*}
\{\chi_\alpha, \chi_\beta\} &= \Delta_{\alpha\beta}, \\
\{\chi_\alpha, \gamma_b\} &= \bar{C}_{cb} \gamma_c + \bar{C}_{ab} \chi_\beta, \\
\{\gamma_a, \gamma_b\} &= \bar{C}_{ab} \gamma_c + \bar{T}_{ab} \chi_\alpha \chi_\beta, \\
\{H, \gamma_\alpha\} &= \bar{V}_a^{\beta} \gamma_b + \bar{V}_a^{\alpha\beta} \chi_\alpha \chi_\beta, \\
\{H, \chi_\alpha\} &= \bar{V}_a^{\alpha} \gamma_b + \bar{V}_a^{\beta} \chi_\alpha \chi_\beta, \tag{27}
\end{align*}
\]

where \(H\) and \(\chi_\alpha, \alpha = 1, 2, \ldots, 2n\), are respectively the first class Hamiltonian and the second class constraints. Defining the Dirac Brackets between any two quantities \(A\) and \(B\) in \(P\) by
\[ \{ A, B \} = \{ A, B \} - \{ A, \chi_\alpha \} \Delta^{\alpha \beta} \{ \chi_\beta, B \}, \quad (28) \]

and choosing gauge-fixing conditions \( \Theta_a = 0 \) such that the matrix \( \{ \gamma_a, \Theta_b \} \) is non-singular, the Faddeev-Senjanovic path-integral \[ 18 \] \[ 19 \]

\[ Z = \int [dy^\mu] \det f \left[ \frac{1}{2} \delta \{ \chi_\alpha \} \delta \{ \gamma_\alpha \} \delta \{ \Theta_a \} \right] \det \Delta \left[ \frac{1}{2} \det \{ \chi_\beta, B \} \right] \exp \left\{ i \int dt [B_\mu \dot{y}^\mu - H] \right\}. \quad (29) \]

defines the quantization of such a system, provided the algebra is irreducible. Let us keep the possibility of having open algebras, this is to say, the consistence of the gauge structure given by (27) demands the introduction of higher rank structure functions. This is also associated to the existence of gauge algebras that close only on shell. Open algebras will be considered later in this section. As in section 2, we continue assuming that the Abelian conversion is implemented with the introduction of the 2n variables \( \psi^\alpha \) that have the same symplectic structure defined in (6) and (14). Also any phase space function \( A(y) \) can be properly extended to a corresponding function \( \tilde{A}(y, \psi) \) submitted to the condition \( \tilde{A}(y, 0) = A(y) \), and having null PB’s (defined as in (7)) with any converted constraint \( \tilde{\chi}_\alpha \), also given by (9,10). Once this process is implemented, the algebraic structure defined by (27) is modified to

\[ \{ \tilde{\chi}_\alpha, \tilde{\chi}_\beta \} = 0, \]

\[ \{ \tilde{\chi}_\alpha, \tilde{\gamma}_a \} = 0, \]

\[ \{ \tilde{\gamma}_a, \tilde{\gamma}_b \} = \tilde{C}_{ab} \tilde{\gamma}_c + \tilde{C}^a_{ab} \tilde{\chi}_a, \]

\[ \{ \tilde{H}, \tilde{\gamma}_a \} = \tilde{V}_a^b \gamma_b + \tilde{V}_a^\alpha \tilde{\chi}_a, \]

\[ \{ \tilde{H}, \tilde{\chi}_\alpha \} = 0, \quad (30) \]

Introducing the compact notation \( \phi^i = (y^\mu, \psi^\alpha) \), \( \lambda^A = (\lambda^a, \lambda^\alpha) \), \( \tilde{\gamma}^A = (\tilde{\gamma}^a, \tilde{\gamma}^\alpha) \), \( \tilde{C}_{ab} = (\tilde{C}^a_{bc}, \tilde{C}^\alpha_{bc}) \) and \( \tilde{V}_a^B = (\tilde{V}_a^b, \tilde{V}_a^\alpha) \), \( B_A = (B_\mu, B_\alpha) \) and \( \epsilon^A = (\epsilon^a, \epsilon^\alpha) \), we see that close to what happens to action (12), in the mixed case the first order action

\[ \tilde{S}_0 = \int dt [B_A \phi^A + \lambda^A \tilde{\gamma}_A - \tilde{H}] \quad (31) \]

is also invariant under some set of gauge transformations, now given by

\[ \delta \phi^i = R_A^i \epsilon^A, \]

\[ \delta \lambda^A = R_A^B \epsilon^B, \quad (32) \]

where
\[ R_A^i = \{ \phi^i, \tilde{\gamma}_A \}, \quad R_B^A = -\delta_B^A \frac{d}{dt} + \delta_B^a \lambda^b C_{ab} + \delta_B^b \tilde{V}_a. \] (33)

Now, it is not difficult to see that
\[
[\delta_1, \delta_2] \phi^i = \left( R_A^i \tilde{C}_{ab} - \tilde{S}_{0,i} \{ \tilde{C}_{ab}, \phi^i \} \right) \epsilon^{(1)}_{(1)} \epsilon^{(2)}_{(2)},
\]
\[
[\delta_1, \delta_2] \lambda^A = R_B^A \tilde{C}_{ab} \epsilon^{(1)}_{(1)} \epsilon^{(2)}_{(2)} + \left( \{ \tilde{C}_{ab}, \phi^i \} \tilde{S}_{0,i} + (\lambda^C \tilde{U}_{AB} + \tilde{V}_{AB}) \tilde{S}_{0,B} \right) \epsilon^{(1)}_{(1)} \epsilon^{(2)}_{(2)},
\] (34)

where we have defined the second order structure functions through the relations
\[
\tilde{U}_{ABC} \tilde{\gamma}_B = \tilde{C}_{ab} \tilde{C}_{cd} + \tilde{C}_{bc} \tilde{C}_{ad} + \tilde{C}_{ca} \tilde{C}_{bd},
\]
\[
\tilde{V}_{ABC} \tilde{\gamma}_B = \tilde{C}_{ab} \tilde{V}_{c} + \tilde{C}_{ac} \tilde{V}_{b} + \tilde{C}_{cb} \tilde{V}_{a}
\]
\[
+ \{ \tilde{H}, \tilde{C}_{ab} \} + \{ \tilde{\gamma}_a, \tilde{\gamma}_b \} - \{ \tilde{\gamma}_b, \tilde{\gamma}_a \}. \quad \text{(35)}
\]

As usual, \( \tilde{S}_{0,i} \) and \( \tilde{S}_{0,A} \) mean the functional variations of action \( \tilde{S}_0 \) with respect to \( \phi^i \) and \( \lambda^A \). The terms in (35) depending on them represent trivial gauge transformations [14]. Higher order structure functions are calculated in a similar way, by imposing consistence of gauge variations with Jacobi identity.

To quantize such a theory along the lines of the field-antifield formalism, we first introduce the classical field-antifield action
\[
S = \tilde{S}_0 + \int dt [\phi^i R_A^i c^A + \lambda^A R_B^A c^B + \frac{1}{2} c^A \tilde{C}_{ab} c^a c^b + \frac{1}{2} \lambda^A \phi^i \{ \phi^i, \tilde{C}_{ab} \} c^a c^b + \frac{1}{4} \lambda^A \lambda_B^B \left( \lambda^C \tilde{U}_{cab} + \tilde{V}_{ab} \right) c^b + \ldots], \quad \text{(36)}
\]

where the dots represent contributions to possible higher rank structure functions. A proper gauge fixing can be implemented by \( \Psi = \tilde{c}_a \Theta^a + \tilde{c}_a \tilde{\Theta}^a \), where \( \tilde{\Theta}^a \) are related to \( \Theta^a \) appearing in (29) through the process of extension defined for instance in (11). The unitary gauge is naturally implemented if we choose \( \Theta^a = \psi^a \). Defining a non-minimal action through \( \tilde{S}_\Psi = S + \int dt \pi_A c^A, \) we write the vacuum functional as in (20), but with the set of fields and antifields consistent with the present case. By using (11) and the form assumed for \( \Psi \) for the implementation of the unitary gauge, we can show that the functional analogous to (20) reduces to the form (24). Now, gauge obstructions can occur not only in the primitive first class
sector, but they also can appear in the process of Abelian conversion of the second class constraints. The discussion of this situation is parallel to that one done in the end of section 2 and will not be repeated here. At this point it is useful to observe that the field-antifield quantization of first order systems that present open gauge algebras can be implemented by using the BFFT procedure without any special restrictions, since expression (36) gives a well-defined functional.

4 An example

A model where the ideas discussed above can be applied in a simple way is given by the first order action

\[ S_0 = \int d^Dx \left[ \frac{1}{2} \pi^i A_i - \frac{1}{2} \pi^2 - \frac{1}{4} F_{ij}^2 + i \bar{\psi} \gamma^\mu D_\mu \psi + \lambda^1 (J^0 + \partial_i \pi^i) + \lambda^2 (\partial_i A^i) \right], \tag{37} \]

where \( \mu, \nu,.. = 0,1,..,D-1 \) and \( i,j,.. = 1,2,..,D-1 \). \( \bar{\psi}, \psi \) and \( \gamma^\mu \) are usual Dirac spinors and matrices in \( D \) dimensions. We are here assuming that \( D \) is even. The Faraday tensor is given by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), the fermionic (chiral) current is defined through \( J^\mu = g_2 \bar{\psi} \gamma^\mu (1 - \gamma_5) \psi \) and the covariant derivative \( D_\mu = \partial_\mu - \frac{i}{2} (1 - \gamma_5) A_\mu \). Action (37) of course represents chiral electrodynamics in \( D \) dimensions in the Coulomb gauge \([2][20]\), where the pair \( A_0, \pi^0 \) has been integrated out. Instead of looking on it from this point of view, we can just consider action (37) as a consistent second class system which is a good candidate for the process of Abelian conversion. First, we observe from the symplectic structure of \( S \) that (see for instance \([3]\))

\[ \left. \frac{\partial}{\partial \psi(x)} \right|_{\psi(y)} \left( i \int d^{D-1}z \, \bar{\psi}(z) \gamma^0 \dot{\psi}(z) \right) = i \gamma^0 \delta^{D-1}(x-y), \]

where the superscripts \( l \) and \( r \) mean actions from left and from right. As we are using the metric \( \eta = \text{diag}((-1,+,+,...+,)) \), \( i \gamma^0 \) is itself its inverse and we get directly the bracket structure \( \{ \psi(x), \psi(y) \} = i \gamma^0 \delta^{D-1}(x-y) \). For the bosonic sector, similar arguments show that \( \{ A_i(x), \pi^j(y) \} = \delta_i^j \delta^{D-1}(x-y) \). It is interesting to observe that if we had chosen the chiral covariant derivative to be defined as \( D_\mu = \frac{i}{2} (1 - \gamma_5) (\partial_\mu - ig A_\mu) \), the symplectic matrix would have no inverse and the bracket structure could not be defined for all the components of \( \psi \). With our choice, it is easy to see that the constraints

\[ \chi_1 = J^0 + \partial_i \pi^i \]
\[ \chi_2 = \partial_i A^i \] \tag{38}

and the Hamiltonian

\[ H = \int d^{D-1}x \left[ \frac{1}{2} \pi^2 + \frac{1}{4} F_{ij}^2 - i \bar{\psi} \gamma^i D_i \psi \right] \tag{39} \]
form a consistent set of second class constraints and first class Hamiltonian since
\[
\begin{align*}
\{\chi_\alpha(x), H\} & = 0, \\
\{\chi_\alpha(x), \chi_\beta(y)\} & = \epsilon^{\alpha\beta} \nabla^2 \delta^{D-1}(x-y). \tag{40}
\end{align*}
\]

We are using equal time brackets, \(\alpha, \beta = 1, 2\) and \(\epsilon^{12} = -\epsilon_{12} = 1\).

To implement the BFFT procedure, we introduce a pair of variables \(\phi^\alpha\) such that
\[
\{\phi^\alpha(x), \phi^\beta(y)\} = \delta^{12} \delta(x-y). \tag{41}
\]

where we have defined
\[
\tilde{\pi} = \pi^i + \partial^i \phi^1 \tag{42}
\]

and the functional form of \(H\) is given in (39), but in (41) replacing \(\pi^i\) by \(\tilde{\pi}^i\). It is trivial now to verify that the constraints \(\tilde{\chi}_\alpha\) satisfy an Abelian algebra and are involutive with respect to \(\tilde{H}\). As a consequence, the first order action
\[
\tilde{S}_0 = \int d^D x \left[ \tilde{\pi}^i \dot{A}_i + i \bar{\psi} \gamma^0 \psi + \phi^2 \phi^1 - \tilde{H} + \tilde{\lambda}^\alpha \tilde{\chi}_\alpha \right] \tag{43}
\]

is gauge invariant. Actually, if \(\delta y^\mu(x) = \{y^\mu(x), \int d^{D-1} y \chi_\alpha(y) \epsilon^\alpha(y)\}\) for any field \(y^\mu\) and \(\delta \lambda^\alpha = -\tilde{\epsilon}^\alpha\) for the multipliers, \(\delta \tilde{S}\) vanishes identically. For convenience, we observe that
\[
\begin{align*}
\delta A_i & = -\partial_i \epsilon^1, \\
\delta \pi^i & = \partial^i \epsilon^2, \\
\delta \psi & = -\frac{ig}{2} (1 - \gamma_5) \psi \epsilon^1, \\
\delta \bar{\psi} & = \frac{ig}{2} \bar{\psi} (1 + \gamma_5) \epsilon^1, \\
\delta \phi^1 & = -\epsilon^2, \\
\delta \phi^2 & = -\nabla^2 \epsilon^2, \\
\lambda^\alpha & = -\tilde{\epsilon}^\alpha. \tag{44}
\end{align*}
\]
It is interesting to note that the quantities $\pi^i$ have non-trivial transformations, contrarily to what is expected for electrodynamics. Now, if we introduce the quantities $\tilde{\pi}^0 = \phi^2 - \partial_i A^i$, $A_0 = \phi^1$ and $\tilde{\lambda}^1 = \lambda^1 - A_0$, we can rewrite action (43) as

$$\tilde{S} = \int d^D x [\tilde{\pi}^\mu \dot{A}_\mu + i \tilde{\psi}^0 \bar{\psi} - \frac{1}{2} \tilde{\pi}^{i2} - \frac{1}{4} F^2_{ij} + i \tilde{\psi}^i D_i \psi + (\tilde{\lambda}^1 - A_0)(\partial_i \tilde{\pi}^i + J^0) - \lambda^2 \tilde{\pi}^0].$$

(45)

Not only action (45) can be written in terms of $\tilde{\pi}^\mu$, but also the path integral, since the Jacobian of the transformation is trivially well defined. Also, from (42) and the above definitions, we note that

$$\delta A_0 = -\epsilon^2, \quad \delta \tilde{\pi}^\mu = 0, \quad \delta \tilde{\lambda}^1 = \epsilon^2 - \dot{\epsilon}^1,$$

(46)

the other variations given by (44). These are just the gauge variations and action of chiral electrodynamics when written in first order. By looking at (45), we observe that $A_\mu$ and $\tilde{\pi}^\mu$ can be taken as canonical pairs. It is also useful to note that due to definitions of $A_0$ and $\tilde{\pi}^0$, the unitary gauge implemented by $\phi^{\alpha} = 0$ now is expressed by $A_0 = \partial_i A^i = 0$.

From what has been discussed above we see that at classical level, the BFFT formalism was able to reverse the gauge fixation and phase space reduction present in (37). At quantum level, however, the conversion of the constraints (38) is obstructed, since we know that chiral electrodynamics is an anomalous theory. To investigate this point a bit closer, let us follow the lines discussed in sections 2 and 3, starting by defining a classical field-antifield action corresponding to (45):

$$S_\Psi = \tilde{S} - \int d^D x \left[ A^{\ast i} \partial_i + i g^2 \bar{\psi}^* (1 - \gamma^5) \psi + i g^2 \bar{\tilde{\psi}} (1 + \gamma^5) \tilde{\psi}^* + \tilde{\lambda}^1 \partial_0 \right] c^1 + \left( A^{0} - \tilde{\lambda}^1 + \lambda^2 \partial_0 \right) c^2 - \bar{\pi}_\alpha \bar{c}^\alpha,$$

(47)

where some proper gauge fixing fermion is assumed. As discussed above, the unitary gauge is here implemented by $\Psi_{unitary} = \int d^D x (\bar{c}^1 A_0 + \bar{c}^2 \partial_i A^i)$, but other choices are available. An interesting choice is given by $\Psi_{covariant} = \int d^D x (\bar{c}^1 \lambda^1 + \bar{c}^2 \Theta(A_\mu))$, where $\Theta$ is some unspecified gauge fixing functional. The choice given above makes the identifications $\lambda^1_1 \equiv \bar{c}^1$ and $\bar{c}^1_1 \equiv \lambda^1$. So in (47) it will appear the terms $\int d^D x (\bar{\pi}_1 \lambda^1 + \bar{c}^1 (\bar{c}^1 - c^2))$. The integrations over $\bar{\pi}_1$ and $\bar{c}^1$ implies not only that $\lambda^1$ vanishes but that the ghost $c^2$ must be identified with $\bar{c}^1$. Integrating over $\tilde{\pi}^\mu$ and over $\lambda^1$ makes action (47) be written in its usual covariant Lagrangian form.
All of this can be done without problems since there is no anomaly in the bosonic sector of (chiral) electrodynamics.

At this stage it is necessary to fix some space-time dimension in order to extract concrete results from the field-antifield machinery. Due to its simplicity, let us consider the case where \( D = 2 \). Action (47) or its partially integrated form that describes the chiral Schwinger model. By using a consistent regularization \[14\] \[17\], it is not difficult to see that

\[
\Delta S_\Psi = \frac{ig^2}{4\pi} \int d^2 x c^1 [(1 - a)e^{\mu\nu} - \eta^{\mu\nu}] \partial_\mu A_\nu
\]  

(48)

where \( a \) is some parameter depending on the regularization and is here assumed to be greater than 1 \[9\]. We observe that there is no local \( M_1 \) satisfying (25). So the BFFT process of conversion of second class constraints, in this example, is obstructed. Following however the procedure introduced by Braga and Montani as well as by Gomis and París in references \[17\], we enlarge furthermore the space of fields and antifields introducing a WZ field \( \theta \) as well as its corresponding antifield \( \theta^\ast \). As the classical action (43) or equivalently (45) does not depend on \( \theta \), it is trivially invariant under shifts on it \[21\]. So we can extend the field-antifield action \[17\] to \( S = S_\Psi + \int d^2 x \theta^\ast c^1 \) and introduce the WZ term

\[
M_1 = -\frac{1}{4\pi} \int d^2 x \left\{ \frac{a - 1}{2} \partial_\mu \theta \partial^\mu \theta + \theta [(a - 1)\partial_\mu A^\mu + e^{\mu\nu} \partial_\mu A_\nu] \right\}
\]  

(49)

such that (25) is satisfied. Since further terms are identically satisfied if we define \( M_p = 0 \) for \( p \) greater than 1, we obtain a closed form for \( W \) at one loop order.

Resuming, to convert the system described by the first order action (37) into a gauge invariant system, the quantum action \( W = S + \hbar M_1 \) had to be extended not only with the aid of the two expected BFFT fields \( \phi^\alpha \), but also with a pair of WZ field and antifield which had origin in quantum obstructions of the gauge symmetry classically introduced with the aid of the BFFT fields \( \phi^\alpha \).

5 Conclusions

We have considered the implementation of the BFFT procedure for converting first order systems with first and second class constraints at quantum level in a general way, by using the field-antifield formalism. We argue that this process can be obstructed due to the occurrence of gauge anomalies. When this is the case, the introduction of further auxiliary (WZ) fields may be considered. We also have shown  

\[1\] We observe that since the field \( \theta \) is absent at classical level, its corresponding antifield could be introduced in the action multiplied by some indefinite ghost \( d \). We choose the quantum action where it appears multiplied by \( c^1 \) because in this situation the theory can be made anomaly free.
that open gauge algebras play no special role in the process of Abelian conversion, being considered in the usual way under the field-antifield formalism. An example based on quantum chiral electrodynamics has been included. Specific results have been presented for the case where D=2. Presently we are studying other models where second class constraints may appear in a somehow more fundamental way. Also the cohomological version of these procedures is under study. Results will be reported elsewhere.

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