A Numerical Approach for Solving of Fractional Emden-Fowler Type Equations

Josef Rebenda *  Zdeněk Šmarda †

© 2018 AIP Publishing. This article may be downloaded for personal use only. Any other use requires prior permission of the author and AIP Publishing. The following article appeared in "Rebenda, J. and Šmarda, Z., A numerical approach for solving of fractional Emden-Fowler type equations, Proceedings of International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2017), AIP Conference Proceedings, Vol. 1978, 2018" and may be found at https://aip.scitation.org/doi/abs/10.1063/1.5043786

Abstract

In the paper, we utilize the fractional differential transformation (FDT) to solving singular initial value problem of fractional Emden-Fowler type differential equations. The solutions of our model equations are calculated in the form of convergent series with fast computable components. The numerical results show that the approach is correct, accurate and easy to implement when applied to fractional differential equations.

1 INTRODUCTION

Differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena [1]-[8]. This is because of the fact that realistic modeling of a physical phenomenon does not depend only on the instant time, but also on the history. This can be successfully achieved by using fractional calculus.

There are many techniques for the solution of fractional differential equations. A good survey of analytical as well as numerical methods is provided in monographs [1], [3], [9], [10].

Recently, Adomian decomposition method (ADM) [11]-[13], Variational Iteration Method (VIM) [11], [12], Homotopy analysis method (HAM) [14] belong

*CEITEC BUT, Brno University of Technology, Purkynova 123, 612 00 Brno, Czech Republic (josef.rebenda@ceitec.vutbr.cz).
†CEITEC BUT, Brno University of Technology, Purkynova 123, 612 00 Brno, Czech Republic (smarda@feec.vutbr.cz).
among the most popular semi-analytical methods. However, these methods re-
quire initial guess or complicated symbolic calculations of integrals and deriva-
tives. We overcome such drawbacks by implementing simple and easy applicable
approach of the fractional differential transformation.

2 PROBLEM STATEMENT

In the paper, we apply the fractional differential transformation (FDT) to solv-
ing fractional Emden-Fowler type differential equations in the form

\[ C_0^D_{\alpha} u + \frac{2}{t^\beta} C_0^D_{\beta} u + f(t)g(u) = 0, \quad t > 0, \quad (1) \]

subject to initial conditions

\[ u(0) = A, \quad u'(0) = 0, \quad (2) \]

where \( \frac{1}{2} < \beta \leq 1 \), \( A \) is a constant, \( f, g \) are continuous functions, \( C_0^D_{\lambda} \), \( \lambda > 0 \)
denotes the fractional derivative of order \( \lambda \) in the Caputo sense as defined in the
following section. The reason for such special choice of \( \beta \) is that the condition
\( u'(0) = 0 \) is used only if \( \frac{1}{2} < \beta \leq 1 \).

The Emden-Fowler type equations have many applications in the fields of
radioactivity cooling and in the mean-field treatment of a phase transition in
critical adsorption, kinetics of combustion or reactants concentration in chemical
reactor and isothermal gas spheres and thermionic currents \[14\]-\[18\].

To find a solution of the singular initial value problem for Emden-Fowler
type differential equations (1), (2) as well as other various singular initial value
problems in quantum mechanics and astrophysics is numerically challenging
because of the singular behavior at the origin.

3 FRACTIONAL DIFFERENTIAL TRANSFOR-
MATION

In this section, we define the fractional differential transformation (FDT). First
we introduce two fractional differential operators.

The fractional derivative in Riemann-Liouville sense is defined by

\[ t_0^{D^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left[ \int_{t_0}^{t} \frac{f(s)}{(t-s)^{1+\alpha-n}} ds \right], \quad (3) \]

where \( n-1 \leq \alpha < n \), \( n \in \mathbb{N}, \ t > t_0 \).

To avoid fractional initial conditions and to be able to use integer order
initial conditions which have a clear physical meaning, we define the fractional
derivative in the Caputo sense:

\[ C_t^D_{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds, \quad (4) \]
The relation between the Riemann-Liouville derivative and the Caputo derivative is given by (see e.g. [1], [3], [10])

\[ C_0^D_t^\alpha f(t) = t_0 D_t^\alpha \left[ f(t) - \sum_{k=0}^{n-1} \frac{1}{k!} (t - t_0)^k f^{(k)}(t_0) \right]. \]  (5)

**Definition 1.** Fractional differential transformation of order \( \alpha \) of a real function \( u(t) \) at a point \( t_0 \in \mathbb{R} \) in Caputo sense is \( C_0^D \{ u(t) \} |_{t_0} = \{ U_\alpha (k) \}_{k=0}^{\infty} \), where \( k \in \mathbb{N}_0 \) and \( U_\alpha (k) \), the fractional differential transformation of order \( \alpha \) of the \( (ak) \)th derivative of function \( u(t) \) at \( t_0 \), is defined as

\[ U_\alpha (k) = \frac{1}{\Gamma(\alpha k + 1)} \left[ C_0^D t_0^{\alpha k} u(t) \right]_{t=t_0}, \]  (6)

provided that the original function \( u(t) \) is analytical in some right neighborhood of \( t_0 \).

**Definition 2.** Inverse fractional differential transformation of \( \{ U_\alpha (k) \}_{k=0}^{\infty} \) is defined using a fractional power series as follows:

\[ u(t) = C_0^{-1} \left( \{ U_\alpha (k) \}_{k=0}^{\infty} \right) |_{t_0} = \sum_{k=0}^{\infty} U_\alpha (k) (t - t_0)^{\alpha k}. \]  (7)

Convergence of the fractional power series (7) in the definition of the inverse FDT was studied in [19]. In applications, we will use some basic FDT formulas also listed in [19]:

**Theorem 1.** Assume that \( \{ F_\alpha (k) \}_{k=0}^{\infty} \), \( \{ G_\alpha (k) \}_{k=0}^{\infty} \) and \( \{ H_\alpha (k) \}_{k=0}^{\infty} \) are differential transformations of order \( \alpha \) of functions \( f(t) \), \( g(t) \) and \( h(t) \), respectively, and \( r, \beta > 0 \).

If \( f(t) = (t - t_0)^r \), then \( F_\alpha (k) = \delta \left( k - \frac{r}{\alpha} \right) \), where \( \delta \) is the Kronecker delta.

If \( f(t) = g(t) h(t) \), then \( F_\alpha (k) = \sum_{l=0}^{k} G_\alpha (l) H_\alpha (k - l) \).

If \( f(t) = \frac{g(t)}{(t - t_0)^r} \), then \( F_\alpha (k) = G_\alpha \left( k + \frac{r}{\alpha} \right) \).

If \( f(t) = C_0 t_0^{\beta} g(t) \), then \( F_\alpha (k) = \frac{\Gamma(\alpha k + \beta + 1)}{\Gamma(\alpha k + 1)} G_\alpha \left( k + \frac{\beta}{\alpha} \right) \).

4 NUMERICAL APPLICATIONS

Consider singular initial value problem (1), (2). Applying the FDT, in particular the formulas of Theorem 1 to equation (1), we obtain the following relation

\[ \frac{\Gamma(\alpha k + 2\beta + 1)}{\Gamma(\alpha k + 1)} U_\alpha \left( k + \frac{2\beta}{\alpha} \right) + 2 \frac{\Gamma(\alpha k + 2\beta + 1)}{\Gamma(\alpha k + 3\beta + 1)} U_\alpha \left( k + \frac{3\beta}{\alpha} \right) + \sum_{l=0}^{k} F_\alpha (l) G_\alpha (k - l) = 0, \]  (8)
where $F_\alpha(k)$, $G_\alpha(k)$ are fractional differential transformations of functions $f(t)$, $g(u)$.

Before we proceed with transformation of initial conditions (2), we need to determine the order of the fractional power series $\alpha$. For this purpose, we suppose that $\beta$ is strictly "fractional", i.e. $\beta \in \mathbb{Q}^+$. Then we choose $\alpha$ which satisfies the following conditions:

1. $0 < \alpha \leq 1$.
2. There is $k_\beta \in \mathbb{N}$ such that $\alpha \cdot k_\beta = \beta$.
3. There is $k_1 \in \mathbb{N}$ such that $\alpha \cdot k_1 = 1$.

The last condition allows us to use integer order derivatives of $u$ at $t_0$ as initial conditions.

There are infinitely many possibilities for the choice of $\alpha$. However, we propose that $\alpha$ should be chosen as reciprocal of the least common denominator of all orders of fractional derivatives which occur in the considered equation. In our case, we have fractional derivatives of orders $2\beta$ and $\beta$ in equation (1). Recall that we assume $\beta = \frac{p}{q}$ for some $p, q \in \mathbb{N}$. The least common denominator of $\left\{\frac{2p}{q}, \frac{p}{q}\right\}$ is $q$, and $\alpha = \frac{1}{q}$.

The transformation of the initial conditions is then defined as

$$U_\alpha(k) = \begin{cases} \frac{1}{\Gamma(\alpha k + 1)} \left[ \frac{d^{\alpha k} u(t)}{dt^{\alpha k}} \right]_{t=t_0}, & \text{if } \alpha k \in \mathbb{N}, \\ 0, & \text{if } \alpha k \notin \mathbb{N}, \end{cases}$$

where $k = 0, 1, 2, \ldots, (\frac{1}{q} - 1)$ and $\lambda$ is the order of a considered fractional differential equation, in our case $\lambda = 2\beta$. In particular, initial conditions (2) give us $U_\alpha(0) = A$ and $U_\alpha(q) = 0$.

**Example 1.** Consider the following singular initial value problem

$$\frac{\text{d}^{2\beta} u}{\text{d}t^{2\beta}} + \frac{2}{\beta} \frac{\text{d}\beta}{\text{d}t^\beta} u + u = 0$$

subject to initial conditions

$$u(0) = 1, \quad u'(0) = 0.$$ 

We already know that $\alpha = \frac{1}{q}$ and $\beta = \frac{2}{q}$. Then recurrence relation (9) has the form

$$\frac{\Gamma(\alpha k + 2\beta + 1)}{\Gamma(\alpha k + 1)} U_\alpha(k + 2p) + 2 \frac{\Gamma(\alpha k + 2\beta + 1)}{\Gamma(\alpha k + \beta + 1)} U_\alpha(k + 2p) + U_\alpha(k) = 0.$$ (11)

From initial conditions we obtain $U_{\alpha}(0) = 1$, $U_{\alpha}(1) = 0$, $U_{\alpha}(q - 1) = 0$, $U_{\alpha}(q) = 0$, $U_{\alpha}(2p - 1) = 0$. Using the recurrence equation (11) we get
nonzero coefficients only for \( k = 0 \) and integer multiples of \( 2p \):

\[
\begin{align*}
\text{k = 0: } & U_\alpha(2p) = -U_\alpha(0) \left( \frac{\Gamma(2\beta + 1)}{\Gamma(\beta + 1)} + 2 \frac{\Gamma(2\beta + 1)}{\Gamma(3\beta + 1)} \right)^{-1}, \\
n\text{k = 1, \ldots, 2p - 1: } & U_\alpha(k + 2p) = 0, \\
n\text{k = 2p: } & U_\alpha(4p) = -U_\alpha(2p) \left( \frac{\Gamma(4\beta + 1)}{\Gamma(2\beta + 1)} + 2 \frac{\Gamma(4\beta + 1)}{\Gamma(3\beta + 1)} \right)^{-1}, \\
n\text{k = 2p + 1, \ldots, 4p - 1: } & U_\alpha(k + 2p) = 0, \\
\vdots
\end{align*}
\]

Choosing \( \beta = 1 \) we get the known Lane-Emden type equation

\[
\frac{d^2 u}{dt^2} + \frac{2}{t} \frac{du}{dt} + u = 0
\]  

(12)

with the exact solution \( u(t) = \sin \frac{t}{t} \). If we substitute \( \alpha = \beta = 1 \) in the coefficients \( U_\alpha(k) \), we have

\[
U_\alpha(0) = 1, \ U_\alpha(1) = 0, \ U_\alpha(2) = -\frac{1}{3!}, \ U_\alpha(3) = 0, \ U_\alpha(4) = \frac{1}{5!}, \ U_\alpha(5) = 0, \ldots,
\]

\[
U_\alpha(2k) = \frac{(-1)^k}{(2k + 1)!} U_\alpha(2k + 1) = 0, \ldots
\]

Thus

\[
u(t) = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots = \frac{1}{t} \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \right) = \frac{1}{t} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} = \sin \frac{t}{t}.
\]

We can observe that the solutions of fractional differential equations (10) converge to the exact solution of differential equation (12) with the integer order derivative \( \beta = 1 \).

5 ACKNOWLEDGMENTS

This research was carried out under the project CEITEC 2020 (LQ1601) with financial support from the Ministry of Education, Youth and Sports of the Czech Republic under the National Sustainability Programme II.

References

[1] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.

[2] G. O. Young, Definition of physical consistent damping laws with fractional derivatives, Z. Angew. Math. Mech. 75, 623–635 (1995).
[3] A. Jannelli, M. Ruggieri, M. P. Speciale, *Exact and Numerical Solutions of Time-Fractional Advection-Diffusion Equation with a nonlinear source term by means of the Lie symmetries*, submitted to Nonlinear Dynamics (2017).

[4] F. Mainardi, "Fractional calculus: Some basic problems in continuum and statistical mechanics," in *Fractals and Fractional calculus in Continuum Mechanics*, edited by A. Carpenter and F. Mainardi, (Springer-Verlag, New York, 1997), pp. 291–348.

[5] A. Jannelli, M. Ruggieri, M. P. Speciale, "Analytical and numerical solutions of fractional type advection-diffusion equation," in *14th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2016)*, AIP Conference Proceedings 1863, edited by T. E. Simos (American Institute of Physics, Melville, NY, 2017), 530005; doi: [http://dx.doi.org/10.1063/1.4992675](http://dx.doi.org/10.1063/1.4992675)

[6] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley and Sons, New York, 1993.

[7] K. B. Oldham, J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[8] S. Das, *Functional Fractional Calculus*, Springer, Berlin, 2011.

[9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[10] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer, Berlin, 2010.

[11] S. Momani, Z. Odibat, *Numerical comparison of methods for solving linear differential equations of fractional order*, Chaos, Solitons, Fractals 31, 1248–1255 (2007).

[12] N. T. Shawagfeh, *Analytical approximate solutions for nonlinear fractional differential equations*, Appl. Math. Comput. 131, 517–529 (2002).

[13] S. Momani, Z. Odibat, *Analytical approach to linear fractional partial differential equations arising in fluid mechanics*, Phys. Lett. A 355, 271–279 (2006).

[14] H. Huan, Wang, Y. Hu, *Solutions of fractional Emden-Fowler equations by homotopy analysis method*, Journal of Advances in Mathematics 13 (1), 1–6 (2017).

[15] H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover, New York, 1962.

[16] S. Chandrasekhar, *Introduction to the Study of Stellar Structure*, Dover, New York, 1967.
[17] J. H. He, *Approximate analytic solution for seepage flow with fractional derivatives in porous media*, Comput. Methods Appl. Mech. Eng. 67, 57–68 (1998).

[18] Z. Šmarda, Y. Khan, *An efficient computational approach to solving singular initial value problems for Lane-Emden type equations*, J. Comput. Appl. Math. 290, 65–73 (2015).

[19] Z. Odibat, S. Kumar, N. Shawagfeh, A. Alsaedi, T. Hayat, *A study on the convergence conditions of generalized differential transform method*, Math. Methods Appl. Sci. 40, 40–48 (2017).