Mathematical modelling of a laminar suspension flow in the flat inclined channel

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Abstract. The laminar flow of a suspension consisting of viscous incompressible fluid with solid spherical particles of the same size in a flat inclined channel is investigated in this work. The mathematical model is formulated in a one-fluid approximation in a three-dimensional statement and solved in the OpenFOAM software package. The results of mathematical modeling are compared with experimental data. The study of the dynamics of the distribution of solid spherical particles in the flow and sedimentation along the length of the channel depending on the size of particles and the angle of inclination of the channel relative to the horizon is carried out.

1. Introduction

Currently, almost all expensive technologies include a stage of preliminary mathematical modeling, which serves to avoid additional costs and more accurately determine the parameters used to achieve a particular goal. For example, when using one of the most popular technologies for enhanced oil recovery - hydraulic fracturing, modeling is often used. Proppant transportation problems in fractures are associated with gravity settling and fracture plugging [1]. In the numerical simulation of suspension flows, it is important to take into account the interaction between particles, their migration, and flow regimes [2].

Among the methods for modeling suspension flows, two approaches may be distinguished: one-fluid and two-continuum [3-5]. The differences between them are significant. The two-continuum approach solves the complete two-velocity model system. However, for most practical purposes, the latter is not only computationally expensive, but unnecessary. To study the laminar flow of a suspension in a flat inclined channel, we choose a one-fluid approach, in which the carrier liquid and the suspended solid particles are considered as a single whole. This approach is applicable when stable flows are characterized by a small time of dynamic relaxation of particles in comparison with the hydrodynamic time or a small Stokes number [5]. Despite its seeming simplicity, the one-fluid model enables quite reliable technical calculations without the use of large computing power. This explains the widespread use of this approach by researchers. In particular, the aforementioned approach was applied in [6] when simulating the flow of a low-concentration suspension. Comparison with experimental data shows good agreement between the simulation results.

At present, the features of the behavior of proppant in a fluid medium remain insufficiently studied. It is particularly unclear how the fluid transports proppant in fractures that are inclined relative to the horizon. At the same time, the use of simulators requires knowledge of certain correlation dependences, which become available only after additional laboratory or computational studies. In
particular, for these purposes, we can apply the results of work [7], in which the process of particle distribution in an inclined channel is studied.

In this work, we carry out a numerical study of the distribution of solid spherical particles in the flow at positive and negative tilt angles. In this case, the fracture is a rectangular inclined channel. The obtained results will serve to study in detail the behavior of the proppant in inclined fractures.

2. Problem statement

The flow of viscous incompressible fluid with suspended solid spherical particles in an inclined channel is investigated. It is considered that the fluid is incompressible, solid spherical particles have the same size and shape, and the flow is laminar. Figure 1 shows a schematic of the computational domain.

![Figure 1. Schematic of the computational domain.](image)

3. Mathematical model

The mathematical model written in the one-fluid approximation, includes the equation of continuity for suspension, equation of motion of the suspension, and the balance equation in the form of a convective-diffusion equation for the transfer of the volume concentration of particles [5, 8]:

\[
\frac{\partial u}{\partial t} + \nabla \cdot (\rho u) = 0, 
\]

\[
\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u u) = -\nabla p + \rho \ddot{g} + \nabla \cdot \Sigma + \nabla \cdot \left( \frac{\rho f}{c} (1 - C) \right) \mathbf{C} \dot{u} + \mathbf{f},
\]

\[
\frac{\partial C}{\partial t} + \nabla \cdot (C u) = -\nabla \cdot \left( \frac{\Sigma}{C} (1 - f_p) \right),
\]

where \( \rho = \rho_p C + \rho_f (1 - C) \) is the suspension density; \( u \) is the suspension velocity; \( p \) is the average suspension pressure; \( C \) is the volume concentration of solid spherical particles; \( f_p = \rho_p / \rho \) is the mass fraction of dispersed phase; \( \ddot{g} = (-g \sin \alpha, 0, -g \cos \alpha) \) is the gravity acceleration; \( \Sigma = \mu \nabla u + \frac{1}{2} \nabla (\nabla u)^T - 2/3 (\nabla \cdot u) I \) is the stress tensor in suspension; \( \dot{u} \) is the relative velocity between phases; \( \mu \) is the effective viscosity ratio of the suspension; and \( I \) is the identity matrix. Hereinafter, index \( p \) denotes parameters related to solid dispersed phase and \( f \) denotes those for the liquid continuous phase.

The coefficient of effective viscosity of the mixture is calculated according to the empirical relationship proposed by Krieger [9]:

\[
\mu = \mu_f \left( 1 - \frac{C}{C_{max}} \right)^{\gamma},
\]

where \( \mu_f \) is the dynamic viscosity coefficient of liquid continuous phase; \( C_{max} \) is the ultimate packing density of particles; and \( \gamma \) is the empirical coefficient.

The relative velocity between the phases is determined by the formula that considers gravitational forces at different particle densities and the carrier phase and the force associated with a change in shear stresses in the system under consideration when moving in a channel [5]:

\[
\dot{u}_r = f^h \left( \dot{u}_{st} - \frac{d^2 (\rho_p - \rho_f)}{18 \mu_f} \frac{d u}{d t} + \frac{d^2}{18 \mu_f C} \nabla \cdot \Sigma \right),
\]
\[ \mathbf{u}_{st} = \frac{d^2(\rho_p - \rho_f)g}{18\mu_f}, \] (6)

\[ \nabla \cdot \Sigma_p = -\gamma \nabla (\mu_f a_n) - K_f a_n \nabla \gamma, \] (7)

\[ K_f = \left( 2 - \frac{K_\eta}{K_c} \right) \left( 1 - \frac{C}{C_{\text{max}}} \right)^p + \frac{K_\eta}{K_c}, \] (8)

where \( \mathbf{u}_{st} \) is the sedimentation velocity according to Stokes; \( d \) is the particle diameter; \( \Sigma_p \) is the stress tensor in the medium of particles; \( \gamma \) is the suspension flow shear rate; \( a_n = 0.75 \left( \frac{C}{C_{\text{max}}} \right)^2 \left( 1 - \frac{C}{C_{\text{max}}} \right)^{-2} \) is the empirical function [10]; \( K_\eta \) and \( K_c \) are the empirical coefficients, the ratio of which according to [11] is defined as \( \frac{K_c}{K_\eta} = 1.042C + 0.1142 \).

The constrained sedimentation function is determined by the formula [5]:

\[ f^h = (1 - C) \frac{\mu_f}{\mu} \left( 1 - \left( \frac{C}{C_{\text{max}}} \right)^2 \right) \] (9)

At the initial moment of time, the channel is filled with liquid, and the system is at rest:

\[ C(x, y, z, t = 0) = 0 \] (10)

\[ \mathbf{u}(x, y, z, t = 0) = 0 \] (11)

Boundary conditions

For the volume concentration on solid walls, the condition of the absence of flow into the wall is set:

\[ \frac{\partial C}{\partial n} = 0 \] (12)

Constant concentration and velocity are set at the entrance to the channel:

\[ C(x = 0, y, z, t) = C_{in} \] (13)

\[ \mathbf{u}(x = 0, y, z, t) = \mathbf{u}_{in} \] (14)

For the tangent component of the mixture velocity to the wall, the condition of partial slip on the wall is specified:

\[ \beta_w d \left( 1 - \frac{C}{C_{\text{max}}} \right) C \frac{\mu}{\mu_f} \frac{\partial \mathbf{u}}{\partial n} = \mathbf{u} \] (15)

where \( \beta_w \) is the slip parameter depending on the sphere radius [12].

For the normal component of the mixture velocity to the wall, the no-flow condition on the solid wall is specified:

\[ u n_w = 0 \] (16)

where \( n_w \) is the normal to the wall surface.

For the concentration and velocity at the outlet from the channel, the flow conditions are set:

\[ \frac{\partial C}{\partial n} = 0 \] (17)

\[ \frac{\partial \mathbf{u}}{\partial n} = 0 \] (18)

4. Results

The system of equations (1) - (3) with closing relations (4) - (9) and boundary conditions (10) - (18) is solved by the control volume method in the OpenFOAM software package. To validate the
mathematical model and test the solution algorithm, the simulation results are compared with experimental data for the velocity and concentration profiles of a fluid flow with spherical solid particles in a flat rectangular channel given in [13]. In particular, Figure 2 shows a comparison of the velocity profile and distribution of the particle concentration across the channel at a distance of 11.2 cm from the entrance, obtained as a result of numerical modeling and experimental studies for $C=0.3$ at the moment of time 30 s. The calculations are carried out for an experiment with the following channel parameters: length of 300 mm, width of 1 mm, and height of 25.4 mm. The densities of the carrier liquid and spherical particles are equal and amount to 1190 kg/m$^3$, the carrier phase viscosity is 0.48 Pa∙s, the particle diameter is 55.4 μm, and the maximum particle packing density is $C_{\text{max}}=0.68$.

The suspension injection rate is $u_{\text{in}}=8.53$ mm/s.

Figure 2. The distribution of the velocity (b) and the concentration of particles (b) in the flat channel for the suspension at $C_{\text{in}}=0.3$.

As a result of comparing the data obtained in mathematical modeling and experimental data, it may be concluded that the model is applicable to predict the behavior of the suspension.

Further, the dynamics of the distribution of the concentration of solid spherical particles (proppant) in a liquid (water) flow in a flat channel is simulated depending on the angle of inclination of the channel relative to the horizon and the diameter of the particles. The results below are obtained with the following system parameters: channel length $L=0.5$ m, width $w=0.002$ m, height $h=0.04$ m, $\rho_f=1000$ kg/m$^3$, $\mu_f=0.001$ Pa∙s, $\rho_p=2650$ kg/m$^3$, $d=50$ and 100 μm, $C_{\text{in}}=0.05$, and $C_{\text{max}}=0.68$.

Figure 3 shows the distribution of the volumetric content of solid spherical particles with a diameter of 100 μm in a laminar flow at a time of 30 s at tilt angles $\alpha=5^\circ$, 0°, and -5° at a suspension injection rate of 0.1 m/s. The tilt angle 0° corresponds to the case when the injection velocity vector is perpendicular to the gravity vector. Figure 3b shows that in the horizontal channel during this time, particles begin to accumulate closer to the exit from the channel. A small shaft of particles with a height equal to 1/4 of the channel height is formed with a concentration varying from 0.2 at the top of the shaft to 0.56 at the bottom of the channel. This is due to the fact that the rate of injection of the suspension into the channel is comparable to the velocity of sedimentation of the particles. In this case, the particles are carried away by the liquid flow in the upper part of the channel. When the angle of inclination of the channel increases by 5°, a shaft of particles begins to form already in the first half of the channel with the maximum height of settled particles at a distance of 20 cm from the entrance. The height of the shaft exceeds 1/3 of the height of the channel with the concentration varying, as in the horizontal channel, from 0.2 to 0.56. This is because the resulting sedimentation velocity of particles in the inclined channel is directed against the flow of the suspension. The settling of particles occurs until the flow rate in the upper part of the channel exceeds the settling rate of the particles. A different picture is observed when the channel is tilted in the opposite direction (figure 3c). In this case, the resulting sedimentation velocity of the particles is co-directed with the flow rate of the suspension. And the bulk of the particles is not retained in the channel and is carried away by the fluid flow.
further. The height of the formed sediment doesn't exceed 1/7 of the channel height and practically doesn't change over time. Similar results were obtained for particles with a diameter of 50 μm.

Figure 3. Distribution of concentration of solid particles along the length of the channel at $u_{in}=0.1$ m/s and $C_{in}=0.05$ at an angle of inclination: a – 5°; b – 0°; c – −5°.

Figure 4 shows the distribution of solid spherical particles along the channel bottom at the time moment of 30 s at an injection rate $u_{in}=0.1$ m/s, with a concentration $C_{in}=0.05$ at different channel tilt angles for two cases: $d=50$ μm and $d=100$ μm. It can be seen from the figures that in the case of smaller particle diameter ($d=50$ μm), the concentration of settled particles in the horizontal channel (curve 2 in figure 4a) doesn't exceed 0.15, while for particles of a larger diameter ($d=100$ μm) this concentration is 0.43 (curve 2 in figure 4b). For an inclined channel with a positive angle, these values are 0.25 and 0.56, respectively. In this case, the concentration of smaller particles size begins to increase closer to the center of the channel (curve 1 in figure 4a). For a negative slope, the maximum concentration values are also different. This picture is explained by the fact that with a decrease in the particle size, the rate of their settling decreases and becomes less than the flow rate of the suspension in the channel. In this case, most of the particles are carried away by the fluid flow and don't have time to settle to the bottom of the channel. An increase in the angle of inclination in the positive direction leads to the growth of the particle shaft in the first half of the channel, up to the termination of the suspension flow.

Figure 4. Distribution of concentration of solid particles diameters 50 μm (a) and 100 μm (b) along the bottom of the channel at $u_{in}=0.1$ m/s and $C_{in}=0.05$ at the time moment of 30 s for different angles: 1 – 5°, 2 – 0°, 3 – −5°.
Conclusions
The study of the dynamics of distribution of solid spherical particles in the flow and sedimentation along the length of the channel depending on the size of particles and the angle of inclination of the channel relative to the horizon has been carried out. It has been found that at a positive angle of inclination of the channel relative to the horizon by 5°, a layer of settled particles is formed in the first half of the channel, its concentration increasing with an increase in the particle diameter. The settling of particles occurs until the flow rate in the upper part of the channel exceeds the sedimentation velocity of the particles.

At a negative angle of channel inclination relative to the horizon, the particles are carried away by the liquid flow into the depth of the channel until they are out of the channel. The height of the formed sediment in this case occupies an insignificant part of the channel.

Acknowledgments
The reported study was funded by RFBR, project number 19-31-90157.

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