COMMENTS ON A THEOREM BY ANCONA

ANDREAS WANNEBO

Abstract. This note concerns a theorem by A. Ancona, see [1], which gives two different sufficient conditions on the open set \( \Omega \) in \( \mathbb{R}^N \) in order to make every element in the Sobolev space \( W^{m,p}_0(\Omega) \) a difference of two nonnegative functions in the space. The proof consists of two parts, a main part and an input part. Ancona gives two different inputs as possible – a Hardy inequality by Nečas and a then new Hardy inequality.

The main part is treated here to give two (new) theorems. In this there is no claim of originality. The scope is improved though. Newer results by the present author are put into this scheme and discussed. A (very) far-reaching conjecture on the main theme introduced by Ancona is given.

Introduction

This note has as aim to discuss a theorem in a Compte Rendue note by Alano Ancona, see [1], and most of all its proof, i.e. what is there and what to do with it?

Two new theorems are given based on his proof. Since Ancona is brief we are detailed. There is no claim of originality on our part here. The matter is put into the context of later development.

The note by Ancona had – already as a manuscript – a big impact on the present author. The note seems to have been rather neglected by most authors in the field of Sobolev space theory and Hardy inequalities.

His note was prompted by that Brezis and Browder needed some positive result in this direction in order to conclude their paper, [2], a sequel to a paper on Schrödinger operators. Complications in this second paper depends on the setting in higher order Sobolev space. One of the main complications then was the absence of truncation. The Ancona note was made to provide a remedy.

The proof by Ancona involves a general part together with Hardy inequalities for domains as input. He gives two cases for the theorem. One uses a Hardy inequality for Lipschitz domains by Nečas, see [9], as input and the second one uses a new Hardy inequality that he proved in the note.

The latter inequality was a major conceptual break through since no regularity of the boundary is used at all. At this time it was commonly believed that smoothness of the boundary of some kind was needed for such Hardy inequalities.

These input results are far from optimal though but the main part can be reformulated into a statement, which is optimal in some settings.

The purpose of this note is mainly to show the importance of the Ancona C.R. note and to relate to later progress.
Main part

First we fix some notation and definitions. Let \( \Omega \) denote an open set in \( \mathbb{R}^N \). Let \( W^{m,p}(\Omega) \) denote the usual Sobolev space. The definition here is that the Sobolev space is the set of all distributions with each derivative up to order \( m \) equivalent (as distribution) to a real function in \( L^p(\Omega) \). This is a Banach space with a norm, the Sobolev space norm

\[
||u||_{W^{m,p}(\Omega)} = \sum_{k=0}^{m} ||\nabla^k u||_{L^p(\Omega)}.
\]

The Sobolev space \( W^{m,p}_0(\Omega) \) is the closure of \( C_0^\infty(\Omega) \), the infinitely times differentiable functions with compact support in \( \Omega \), and the closure is taken in the Sobolev norm, and thus is a closed subspace of \( W^{m,p}(\Omega) \).

Furthermore let + in the expression

\[
W^{m,p}_0(\Omega)_+
\]

denote that the nonnegative cone is taken.

The following is what Ancona gives as theorem in [1].

1. **Theorem. Ancona.**

   If it holds that \( \Omega \) is open and bounded in \( \mathbb{R}^N \) and if it holds that \( \Omega \) either is a Lipschitz domain or if \( p > N \), then every \( u \in W^{m,p}_0(\Omega) \) can be written as \( u = u_1 - u_2 \) with \( u_i \in W^{m,p}_0(\Omega)_+ \).

   Furthermore there is some constant \( c = c(m, p, \Omega) \) such that

\[
||u_i||_{W^{m,p}(\Omega)} \leq c||u||_{W^{m,p}(\Omega)}.
\]

The first of these properties can be written

\[
W^{m,p}_0(\Omega) = W^{m,p}_0(\Omega)_+ - W^{m,p}_0(\Omega)_+.
\]

We will give two theorems by a closer look at the proof by Ancona.

For more generality we introduce a weighted Sobolev spaces based on a standard kind of weight functions.
Let $d_{\partial \Omega}(x) = dist(x, \partial \Omega)$. Then the weighted Sobolev space norm

$$||u||_{W^{m,p}(\Omega, d_{\partial \Omega}(x)^s)}$$

is defined by introducing the weight $d_{\partial \Omega}(x)^s$ into all the seminorms that occur as $L^p(\Omega)$-norms in the definition the Sobolev space norm.

However the norm by itself does not determine the elements in a Banach space. In the unweighted case there is a calibrated theorem by Meyers and Serrin to the effect that you get the same Sobolev space $W^{m,p}(\Omega)$ if you use the definition given above or use the set of elements defined as the closure of $C^\infty(\Omega)$ in the norm.

The corresponding question for weighted Sobolev space with the norm above appears not to be settled, i.e. the question if and/or when equality holds.

This question is avoided here since a subspace is used.

Define the weighted Sobolev space $W^{m,p}_0(\Omega, d_{\partial \Omega}(x)^s)$ as the closure of $C^\infty_0(\Omega)$ in the corresponding weighted Sobolev norm.

**Definition.** If the supremum of $d_{\partial \Omega}(x)$ is finite when $x \in \Omega$ then $\Omega$ is said to have finite (inner) width. This is the same as to say that balls inside $\Omega$ have bounded radii.

Now we turn to the new formulations based on the proof by Ancona.

2. **Theorem.**

Let $p > 1$ and $-\infty < s < \infty$. Let $\Omega$ be of finite width,

For $u \in W^{m,p}_0(\Omega, d_{\partial \Omega}(x)^s)$ it then holds that

$$\int_\Omega |u|^p d_{\partial \Omega}(x)^{-mp+s} dx < \infty$$

implies $u = u_1 - u_2$ with $u_i \in W^{m,p}_0(\Omega, d_{\partial \Omega}(x)^s)_+.$

This follows together with a norm estimate as in Theorem 1.

Denote by $W^{m,p}(\Omega)_{loc}$ the set of functions $\{u\}$ on $\Omega$ with

$$||u||_{W^{m,p}(\omega)} < \infty$$

for every open $\omega$ with $\overline{\omega}$ a compact subset of $\Omega$.

Then the second theorem reads as follows.
3. Theorem. Let $p \geq 1$ and $-\infty < s < \infty$. If $u \in W^{m,p}(\Omega)_{\text{loc}}$, then
\[ \|u\|_{L^p(\Omega,d\partial\Omega(x)^{-mp+s})} < \infty \]
and
\[ \|\nabla^m u\|_{L^p(\Omega,d\partial\Omega(x)^s)} < \infty \]
together implies
\[ u \in W^{m,p}_0(\Omega, d\partial\Omega(x)^s). \]

Proofs are given at the end of the paper.

There is some relation here to early papers by Kadlec and Kufner, see [4] and [5], also referred to by Ancona.

A remark on the history and development of Hardy inequalities is given in Wannebo [13]. It contains a general outline as well as the role of the present author. See also Wannebo [10], [11], [12] and [13].

The main question raised by Ancona is:
When holds
\[ W^{m,p}_0(\Omega) = W^{m,p}_0(\Omega)_{+} - W^{m,p}_0(\Omega)_{+}. ? \]
Again more generally:
When holds
\[ W^{m,p}_0(\Omega, d\partial\Omega(x)^s) = W^{m,p}_0(\Omega, d\partial\Omega(x)^s)_{+} - W^{m,p}_0(\Omega, d\partial\Omega(x)^s)_{+}. ? \]

One way to get positive answers is to give appropriate Hardy inequalities to use as inputs to Theorem 2.
The presently known answers can be found in Wannebo [12]. The results involve conditions given as uniform capacity/uniform polynomial capacity conditions. However a discussion of these concepts is beyond the scope of this note.

The following far-reaching conjecture would require more ideas and technique.

4. Conjecture. Wannebo. Let $\Omega \subset \mathbb{R}^N$ be open.
(i) For any $m$ odd, any $N$, any $p > 1$ and $\Omega$, it holds
\[ W^{m,p}_0(\Omega) = W^{m,p}_0(\Omega)_{+} - W^{m,p}_0(\Omega)_{+}. \]
(ii) For any $m$ even and positive there exists an $N$, an $\Omega$ and a $p > 1$ such that
\[ W^{m,p}_0(\Omega) \neq W^{m,p}_0(\Omega)_{+} - W^{m,p}_0(\Omega)_{+.} \]
This conjecture is trivial for \( m = 1 \) by truncation. The case \( m = 2 \) is well-known. But this information is too thin. The conjecture rests instead on theoretical ideas.

The following result is a combination of Theorem 2 and a result in Wannebo [12].

5. **Theorem.** Given a “certain uniform capacity condition” on \( \partial \Omega \), then if \( u \in W_0^{2,p}(\Omega) \), it holds

\[
u \in W_0^{2,p}(\Omega) - W_0^{2,p}(\Omega) +
\]

if and only if

\[
\int_{\Omega} |u|^p d_{\partial \Omega}(x)^{-2p} dx < \infty.
\]

**Proofs**

The proofs of Theorem 2 and 3 given below follows from a careful reading the proof by Ancona.

We will need a definition and some notation.

A Whitney decomposition of \( \Omega \), open, proper subset of \( \mathbb{R}^N \), is a covering of \( \Omega \) with closed cubes. These have pair-wise disjoint interior and they shrink as they tend to the boundary of \( \Omega \) according to the formula (\( Q \) any such cube)

\[
diam Q \leq \text{dist}(Q, \partial \Omega) \leq 4diam Q.
\]

We will also use \( A \) as a generic constant which is nonnegative but may vary at each occurrence. (Standard.)

**Proof of Theorem 2.**

The goal is to take any function \( u \in W_0^{m,p}(\Omega, d_{\partial \Omega}(x)^s) \) and then to construct a nonnegative majorization with finite norm. Then the nonpositive part arises as the difference between these two functions which also has finite norm of course.

The way to accomplish this is by cutting the original function into pieces with cut-off functions, to majorize each of these pieces with a nonnegative function and then to sum this new pieces into a function with finite norm.

Now we set out to do this procedure.

Let \( Q_0 \) be a cube with unit diameter and with centre at the origin.

Choose \( \eta \leq 0 \) with \( \eta \in C_0^\infty(\frac{4}{3}Q) \) and \( \eta|_{Q_0} = 1 \). In order to translate to any Whitney cube \( Q \), let \( \eta_Q \) be defined as follows \( \eta_Q = \eta(diam(Q)^{-1}(x-x_Q)) \) with \( x_Q \) centre of \( Q \). Then for \( u \in W_0^{m,p}(\Omega, d_{\partial \Omega}(x)^s) \) denote \( u_Q \) for \( \eta_Q u \).

This construction ensures that \( u_Q \in W_0^{m,p}(\frac{5}{3}Q) \), i.e. \( u_Q \in W_0^{m,p}(\Omega, d_{\partial \Omega}(x)^s) \).
For $p > 1$ there is a representation of Sobolev functions as Bessel potentials. Hence $u_Q = G_m * f_Q$ almost everywhere, where $f_Q \in L^p$ and $G_m$ is a certain Bessel kernel. Here it is important that $G_m$ is a nonnegative kernel.

This two representations of the Sobolev function and they have similar sized norms, i.e.

$$||u_Q||_{W^{m,p}} \sim ||f_Q||_{L^p}.$$

Denote $f_{Q,+} = \max[f_Q,0]$ and $v_Q = \eta_Q(G_m * f_{Q,+})$.

This way $v_Q \geq \max\{0,u_Q\}$, $v_Q \in W_0^{m,p}(\frac{5}{3} Q)_+$, $v_Q \in W_0^{m,p}(\Omega, d\partial\Omega(x)^s)_+$.

In order to avoid messy formulas we denote by $Q'$ the dilation made that maps $Q$ to a unit cube. When it is clear which cube is refered to, we write $u'$ etc. for the furter effects of this dilation.

Now we give a row of inequalities together with the reason that each hold. The order is the same in both lists. We do the argument first for a unit cube $Q'$.

- Equivalent norms for Sobolev space;
- Equivalent norms for Sobolev space and Bessel potentials;
- Leibnitz’ rule expansion and the triangle inequality;
- A Poincaré inequality using Bessel potentials;
- A triviality for $f_{Q,+}$ and $f_Q$;
- The triangle inequality and Poincaré inequalities.

The inequality row is as follows,

$$||\nabla^m u'_Q||_{L^p}$$

$$\sim ||v'_Q||_{W^{m,p}}$$

$$\sim ||\nabla^m \eta'_Q(G_m * f_{Q,+})||_{L^p}$$

$$\leq A \sum_{k=0}^m (||\nabla^k (G_m * f_{Q,+})||_{L^p}$$

$$\leq A ||f_{Q,+}||_{L^p}$$

$$\leq A ||f'_Q||_{L^p}$$

$$\leq A ||u'_Q||_{W^{m,p}}$$

$$\leq A ||\nabla^m u'_Q||_{L^p}$$
Summing up

\[ \| \nabla^m v'_Q \|_{L^p} \leq A \| \nabla^m u'_Q \|_{L^p}. \]

The expressions here are now dilation homogeneous. Hence we conclude

\[ \| \nabla^m v_Q \|_{L^p} \leq A \| \nabla^m u_Q \|_{L^p}. \]

Next denote

\[ v = \sum v_Q, \]

which is the candidate as the majorizing function.

It remains to check the norm. Since the Sobolev norm is built of some seminorms it is enough to check them.

This is done by estimations as a sequence of inequalities. As before the list of inequalities is preceded by the list of arguments used.

Observe that by the very construction and the properties of Whitney cubes it follows that there is at most a fixed number of overlaps from the \( \{ v_Q \} \).

List of arguments.
- The definition of \( v \);
- Whitney cube properties;
- A Poincaré inequality;
- Interpolation between Sobolev seminorms;
- The bounded inner width of \( \Omega \);
- The previous result;
- Leibnitz’ rule, the triangle inequality finite overlap;
- Interpolation cube-wise between seminorms.

The \( p \)-power of a seminorm is estimated
\[ \| \nabla^k v \|_{L^p(\Omega, d_{\partial \Omega}^s(x)^s)}^p \leq \| \sum_Q \nabla^k v_Q \|_{L^p(\Omega, d_{\partial \Omega}^s(x)^s)}^p \]
\[ \leq A \sum_Q \| \nabla^m v_Q \|_{L^p}^p \cdot l(Q)^s \]
\[ \leq A \sum_Q \| \nabla^m v_Q \|_{L^p}^p \cdot l(Q)^{m-k}p+s \]
\[ \leq A \sum_Q \| \nabla^m v_Q \|_{L^p}^p \cdot l(Q)^s \]
\[ \leq A \sum_Q \| \nabla^m u_Q \|_{L^p}^p \cdot l(Q)^s \]
\[ \leq A \sum_{r=0}^{m} \sum_Q \| \nabla^r u \|_{L^p}^p \cdot l(Q)^{-r}p+s \]
\[ \leq A(\| u \|_{L^p(\Omega, d_{\partial \Omega}^s(x)^s)^s}^p + \| \nabla^m u \|_{L^p(\Omega, d_{\partial \Omega}^s(x)^s)}^p). \]

Now the first expression is finite according to the assumption on \( u \) and the second is finite since \( u \in W^{m,p}(\Omega, d_{\partial \Omega}^s(x)^s). \)

End of proof.

**Proof of Theorem 3.**

The proof is quite the same. Just take \( u \in W^{m,p}(\Omega)_{loca} \) instead and then proceed as before. The summation of the \( v_Q \) then gives a result in the closure of \( C_0^{m,p}(\Omega) \) by the convergence and with the right norm because of the estimates.

Furthermore the procedure can be simplified since nonnegativity and the Bessel kernel is not needed. The argument can be made solely by usual seminorms in Sobolev space.

End of proof.

For the record we give a list of misprints in Wannebo [10]. The list was given in Wannebo 1991 thesis: “Some topics in Sobolev space theory”. (All items have been publicized.)

p.90 line -10 ball has as centre the centre of \( Q' \);

p.90 line -17 reads \( x \) in the interior;

p.90 line -7 reads \( k + a \);

p.91 line -7 reads \( \int_Q |\nabla^k (u - P)|^pdx \) with \( P \) polynomial of degree \( \leq m - 1 \);

p.91 line -1 reads \( 2^{(m-1)p} \);

p.92 line -12 reads \( \sum_{n=a}^{s+a} \).
p.92 line -9 reads with extra factor $e^{a\alpha}$ (no consequence);
p.92 line 10 reads $(q(x) - n + a + 1)$ in integrand;
p.93 line 16 reads $|\gamma|$;
p.93 line 93 line -6 reads $\sum_{r=0}^{k-1}$;
p.94 line -5 reads $A \cdot \gamma_{m,m-1,p}(K, 2Q)$.

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