Boundary layer and fundamental problems of hydrodynamics (compatibility of a logarithmic velocity profile in a turbulent boundary layer with the experience values)

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Abstract. The compatibility of the semiempirical turbulence theory of L. Prandtl with the actual flow pattern in a turbulent boundary layer is considered in this article, and the final calculation results of the boundary layer is analyzed based on the mentioned theory. It shows that accepted additional conditions and relationships, which integrate the differential equation of L. Prandtl, associating the turbulent stresses in the boundary layer with the transverse velocity gradient, are fulfilled only in the near-wall region where the mentioned equation loses meaning and are inconsistent with the physical meaning on the main part of integration. It is noted that an introduced concept about the presence of a laminar sublayer between the wall and the turbulent boundary layer is the way of making of a physical meaning to the logarithmic velocity profile, and can be defined as adjustment of the actual flow to the formula that is inconsistent with the actual boundary conditions. It shows that coincidence of the experimental data with the actual logarithmic profile is obtained as a result of the use of not particular physical value, as an argument, but function of this value.

1. Introduction
Due to the fact that all current programs of gas-dynamic calculations are based on the Reynolds equations describing the turbulent flow motions in general, it has enabled the continuous calculation of the velocity and pressure fields in real fluids and gases without their division into the boundary layer regions, where the impact of viscous turbulent stresses and the outer region in relation to this layer is concentrated and where the effect of tangential stresses can be neglected and this region could be regarded on the basis of simple Euler equations of motion.

In this case, the central part of classical theoretic fluid and gas dynamics – a boundary layer theory determining the physics of moving fluids and gases with streamlined surfaces – is excluded.

However, such radical conclusion is justified only where there is guarantee that a theoretical base, which forms the modern programs of the gas-dynamic calculations, is unquestionable.

Unfortunately, in the present case it is far from true as in 1924 A.V. Keller and A.A. Friedman showed the impossibility of correct closing of the Reynolds equations [1].

Some later works [2, 3] contain the similar statements. Considerations given in these works remain topical and give fully scientific assessment to that theoretical base, which is fundamental for modern programs of fluid-dynamics calculations using several phenomenological turbulence theories to close differential equations of the turbulent motion.
In methodological plan, each such theory is tested on the certain type of tasks. Accordingly, when passing to the other type task solution the closing relationships shall be corrected again using just another turbulence theory.

For the first time the similar methodology of the unclosed differential equation solution was used to find the velocity profile in the turbulent boundary layer based on the Prandtl semiempirical turbulence theory.

The results of this classical solution provide the basis for formation of the flow physics in the turbulent near-wall layer that required to introduce into consideration two- and three-layer model of the turbulent boundary layer.

Let’s examine in details to what extent the tasks, assumptions, statements and hypotheses introduced in solving the above mentioned task correspond to the facts, and how correctly the comparison of calculation and experimental data is implemented among themselves.

2. The logarithmic velocity profile in the turbulent boundary layer

Solution to the mentioned task in its classic case is based on known Prandtl relation determining connection of the turbulent shear stress with the averaged transverse velocity gradient in the boundary layer:

\[ \tau = \rho l^2 \left( \frac{du}{dy} \right)^2 \tag{1} \]

where, \( \tau \) – turbulent stress (Pa);
\( \rho \) – density of moving medium (kg/m\(^3\));
\( l \) – certain reference length (m);
\( u \) – longitudinal velocity (m/s);
\( y \) – transverse coordinate (m).

The reduced differential equation is unclosed as apart from the longitudinal velocity \( u \) it contains two more unknown values – stress \( \tau \) and linear dimension \( l \).

Accordingly, to solve it relatively to the velocity \( u \) two more relationships connecting \( \tau \) and \( l \) values with the transverse coordinate \( y \) are required.

L. Prandtl [4] proposed to integrate the equation (1) at the constant transverse value of the shear stress \( \tau \) that equals to this stress on the streamlined surface \( \tau_w \) (\( \tau = \tau_w = \text{const} \)), and take the linear dimension \( l \) proportional to the transverse coordinate \( y \). In other words, \( l = \kappa y, \kappa \) – proportion factor.

Under the conditions specified previously the following elementary differential equation is to be integrated:

\[ \sqrt{\frac{\tau_w}{\rho}} \kappa y \frac{du}{dy} = \text{const} \tag{2} \]

Accepted conditions even before the integration predetermined the law of variation of the transverse velocity gradient. Next equation follows from the relationship (2):

\[ \frac{du}{dy} = \frac{\text{const}}{\kappa y} = \frac{v_*}{\kappa y} \tag{3} \]

where, \( v_* = \sqrt{\frac{\tau_w}{\rho}} \) – dynamic velocity (m/s).

In other words, the hyperbolic law of the transverse velocity gradient variation in the turbulent boundary layer and the resultant logarithmic law of the velocity variation in the transverse section of the mentioned layer were laid on the basis of intuitive considerations.

Equation (3) is not fulfilled near the wall, where its right part is increasing with no limits, and on the outer limit of the boundary layer, where its left part is going to zero.

In this connection it is important to mention that the condition \( \tau(y) = \text{const} \) and linear relationship of displacement length \( l \) from the transverse coordinate \( y \) were taken initially specifically for the near-wall region where the differential equation (3) is not fulfilled. Formally, the integration of this equation results in known logarithmic velocity profile:
\[
\frac{u}{v_s} = \frac{1}{k} \ln y + C \tag{4}
\]

Obtained formula should meet the basic boundary condition resulting from the adhesion theory: at \(y = 0\) and \(u = 0\). However, in this case at \(y = 0\) and \(u \to \infty\). Also this relationship does not fulfill the shear stress finiteness on the wall \(\tau_w\), as at \(y = 0\) \(\frac{\partial u}{\partial y} \to 0\).

Also a boundary condition is not fulfilled on the outer limit of the boundary layer where at \(y = \delta\) (\(\delta\) – thickness of the boundary layer) \(\frac{du}{dy} \neq 0\).

The mentioned circumstance is a natural consequence of the impossibility to provide the equality of left and right equation parts (3) in specified regions.

In other words, the obtained velocity profile cannot be used in the most important flow region determining interaction of moving environment with streamlined surfaces.

There are two options only in situation like this. Either the solution needs to be considered wrong, that is natural as all accepted above assumptions are against the real physics, or the flow pattern itself needs to be corrected to formula (4). Paradoxically, but further development of the turbulent boundary layer theory has opted for the second option – correcting the flow pattern in the near-wall region.

Essence of this correction comes down to adoption of the double-layer and in some cases three-layer structural model of the turbulent boundary layer.

According to this model the turbulent boundary layer is considered as combination of the laminar sublayer contiguous to the streamlined surface where the turbulent tangential stresses are excluded (no velocity and pressure fluctuations), and the value \(\tau = \tau_{\text{lam}}\) is determined only by the molecular viscosity of fluid and the main turbulent part where the velocity profile is described by the equation (4).

The known logarithmic velocity profile was obtained as a result of binding of the velocity in the lower part of the turbulent boundary layer with the velocity on the outer part of the laminar sublayer, which is described by the following equation:

\[
\frac{u}{v_s} = \frac{1}{k} \ln \frac{y v_s}{v} + B = 2.5 \ln \frac{y v_s}{v} + 5.5 \tag{5}
\]

3. **The comparison results of the experimental data with the logarithmic profile data**

Figure 1 illustrates a degree of conformity of formula (5) to the experimental data where the calculation relationship is associated with the experiments of Nikuradze in the pipes at different values of the Reynolds numbers [5].
Figure 1. Experimental data on distribution of the averaged velocities in circular pipes.

The good coincidence of the experimental points with calculations in the main velocity profile part and serious deviation of experimental data from calculations near the pipe walls is clearly seen. However, in coordinates taken in fig. 1 \( \left( \frac{u}{v}, \ln \frac{y}{\delta} \right) \), the velocity profile does not depend on the Reynolds value. In that connection in [6] it is noted: “The fact that logarithmic formula retains its form for all Reynolds numbers of the flow or, in the popular phrase, is universal is most important. From physics the mentioned characteristic of logarithmic formulas is accounted for the laminar sublayer in which the whole viscosity effect is concentrated”. Similar assessments of the logarithmic velocity profile are contained practically in all know publications [7, 8, 9].

Due to this, it seems useful to analyse more precisely the mentioned universality fact of the logarithmic velocity profile and the justification degree of assumptions about the presence of the laminar sublayer in the turbulent boundary layer.

To analyse the comparison results of the logarithmic velocity profile with experimental data let us represent the formula (5) as follow:

\[
\frac{u}{u_t} \cdot \frac{u_t}{v} = \frac{1}{k} \ln \frac{y}{\delta} \cdot \frac{\delta v_v}{v} + B
\]  

where, \( u_t \) – longitudinal velocity on the outer limit of the boundary layer (m/s);
\( \delta \) – its thickness (m).

Due to the fact that the shear stress \( \tau_w \) is inversely proportional to the number \( Re_\delta = \frac{u_t \delta}{v} \) to some extent \( m (\tau_w \sim \frac{1}{Re_\delta^m}) \), then the dynamic velocity \( v_\ast = \sqrt{\frac{\tau_w}{\rho} \sim \frac{1}{Re_\delta^{m/2}}} \), and relationship \( \frac{u_t}{v} \sim Re_\delta^{m/2} \).

Then, taking into account the given considerations, let us write the formula (6) as follow:

\[
\frac{u}{u_t} \cdot \varphi(Re_\delta) = \frac{1}{k} \ln \left( \frac{y}{\delta} \psi(Re_\delta) \right) + B
\]

This indicates that in fig.1 the “floating” coordinates, maximum values of which change depending on the Reynolds value, are used instead of the fixed coordinates.

For circular pipes \( \delta = r_0 (r_0 \text{ – pipe radius}) \) equation (7) takes the following form:
\[
\frac{u}{u_t} \cdot \varphi(Re_{r_0}) = \frac{1}{\kappa} \ln \left( \frac{r_0 - r}{r_0} \psi(Re_{r_0}) \right) + B
\] (8)

From formula (8) it follows that for each value \( Re_{r_0} \) the maximum value of “floating” coordinates, in which the particular velocity profile is drawn, will be changed depending on number \( Re_{r_0} \).

In other words, \( \left( \frac{u}{v} \right)_{\text{max}} = \varphi(Re_{r_0}) \) and \( \ln \left( \frac{r_0}{v} \right)_{\text{max}} = \ln \left( \psi(Re_{r_0}) \right) \) as on the outer boundary layer limit (in this case in the pipe center) \( \frac{r}{r_0} = \frac{r_0 - r}{r_0} = 1 \).

While using these “universal” coordinates an increase in the Reynolds number increases that part of the plane \( \frac{u}{v} - \ln \frac{r_0}{v} \) in which experimental points of profiles are located in fig. 1 used in the mentioned figure to compare with the computational relationship (5).

For the purpose of greater visibility the quadrants corresponding to these profiles are shown in dashed lines in fig. 1.

In concerned “floating” coordinates the first profile is located in the first quadrant and it is determined by the straight-line segment 0-a. The second profile occupies the second quadrant and it is determined by the straight-line segment 0-b and etc. In other words, in the reduced coordinate system with increase in number \( Re \) the velocity profiles given in the mentioned dimensionless coordinates \( \frac{u}{v} - \ln \frac{r_0}{v} \) stretch along the abscise axis in their overlapping.

In this case, to identify the impact of the factors of our interest (in such case numbers \( Re \)) on the examined velocity profiles is basically not possible. Thus, if in fig. 1 the points “m” and “n” related, accordingly, to third (p.”m”) and forth (p.”n”) profiles are in line than it should not be inferred from this that they are “universal” as in the fixed coordinate system \( \frac{u}{v} - \ln \frac{r_0}{\delta} \) they, in general, are referred to different values of the mentioned dimensionless coordinates.

“Universality” of the logarithmic velocity profile mentioned in scientific literature is a result of more general characteristic that in case when not an argument itself is used as a reason, but its function the arrangement of experimental and calculation data along the line defined by two experimental points is initially assured.

In case, if the variation limits of the examined value and an argument determining this value are the same, then all experimental points will be arranged on the bisectrix of right angle.

For example, if the velocity profile series are described by the power function \( \frac{u}{u_t} = (\frac{r_0}{\delta})^n \), where \( n \) – index depending on the profile type and several criteria parameters of the Mach and Reynolds numbers type, then experimental point for all velocity profiles will be arranged at the same line on the plane \( \bar{u} - \bar{y}^n \), and only one point would be sufficient to find the power index \( n \) for each profile.

This showed that an information value of the examined experimental data presentation is extremely low and it is difficult to draw the general conclusions on their basis.

4. Conclusion
According to the above research we can draw the following conclusions:
1. Accepted additional conditions and relationships, which integrate the differential equation of L. Prandtl, associating the turbulent stresses in the boundary layer with the transverse velocity gradient, are fulfilled only in the near-wall region where the mentioned equation loses its meaning and are inconsistent with the physical meaning on the main part of integration.
2. This article shows that “universality of “logarithmic” velocity profile, consisting in its independence from the Reynolds number, is a result of the use of “floating” coordinates the maximum value of which varies depending on the mentioned parameter.

As a result of this, an information value of the comparison of experimental data with calculated values is eliminated as the coincident points on the assumption curve obtained at the different values of the Reynolds numbers refer to the different sections of the boundary layer.
3. The used concept about the presence of laminar sublayer between the streamlined surface and the turbulent part of the boundary layer is the way to give a physical meaning to the logarithmic velocity profile and does not match the interaction physics of moving media with streamlined surfaces as the arising at that additional questions do not have physically reasoned responses.

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