Quantum Fields

à la Sylvester and Witt

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Abstract

A structural explanation of the coupling constants in the standard model, i.e the fine structure constant and the Weinberg angle, and of the gauge fixing contributions is given in terms of symmetries and representation theory. The coupling constants are normalizations of Lorentz invariantly embedded little groups (spin and polarization) arising in a harmonic analysis of quantum vector fields. It is shown that the harmonic analysis of massless fields requires an extension of the familiar Fourier decomposition, containing also indefinite unitary nondecomposable time representations. This is illustrated by the nonprobabilistic contributions in the electromagnetic field.

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1 Introduction

The standard model dynamics for the electroweak interactions is quantified by masses and coupling constants - especially the Fermi mass $M$ in the symmetry breakdown mechanism, the electromagnetic coupling constant (Sommerfeld’s fine structure constant $\alpha_e$) and its relation to the weak coupling constants (Weinberg angle $\theta_w$). This paper is an attempt to understand in the standard model - at least qualitatively in terms of symmetries and representation theory - the origin of those structurally and numerically important scales. Although this structural explanation does not allow to calculate these parameters within the standard model, we believe that it is a necessary step towards a more fundamental quantum field theory in which they may prove to be derivable and therefore calculable.

All relevant constants appear as normalizations for the Casimir invariants of the little groups in the Lorentz group defined by a harmonic analysis. Due to Wigner [Wig39], one classifies relativistic quantum fields with respect to particles by mass-momenta $(m, \vec{q})$ on causal energy-momenta orbits $q^2 = m^2 \geq 0$. Two different particle types are possible because of the indefinite signature sign $g = (1, 3)$ of the Lorentz bilinear form $g$. Particles with strictly positive mass $m^2 > 0$ obey an $SO(3)$ spin classification in the Lorentz group $SO(1, 3)$, particles with vanishing mass $m^2 = 0$ are classified with respect to a polarization $SO(2)$ only. These little groups prove to be the stability groups of decompositions for Minkowski translations, in the massive case a Sylvester decomposition into time and space translations, in the massless case a Witt decomposition [Bou59] into two lightlike translations and 2-dimensional space translations. Therefore massive and massless fields will be called fields à la Sylvester and Witt resp.

The time axis is defined in the translations decompositions. Then the stability groups are compatible with the representation of the time translations, classified with respect to their unitarity properties as follows.

Massive fields with spin have an irreducible positive unitary $U(1)$ classification for their time development as familiar from quantum mechanical bound states, e.g. the harmonic oscillator. The invariant energies (frequencies) of the quantum mechanical case arise as invariant masses in the case of massive relativistic fields.

The situation is completely different for massless fields with circularity (polarization). Here, especially in massless vector fields, e.g. the electromagnetic field, harmonic components without a probabilistic particle interpretation occur (interactions without particle parametrization). New structures arise, e.g. dipole contributions in the propagator or 'gauge fixing constants' which have to be nontrivial on the one hand, but which are experimentally irrelevant on the other hand. This shows that the time representation structure of relativistic fields is more general than their classification into particle interpretable parts ('every particle is a field, but not every field is a particle'). Therefore, in extension of Wigners definition of particles as irreducible positive unitary representations also nondecomposable indefinite unitary ones have to be considered in the time development classification of relativistic fields [Sal94b]. We shall give in some detail the
connection between those indefinite $U(1,1)$ structures and the corresponding harmonic components in the massless fields - e.g. the Coulomb components. It will become clear in this analysis that the 'gauge fixing constant' has its structural origin in the basic dependent and, therefore, physically irrelevant normalization of the nilpotent contribution in the indefinite unitary Hamiltonian. It is also shown that even in the massless case the harmonic analysis requires a specification of a rest system and therewith the introduction of a mass scale.

2 Quantum Fields à la Sylvester

Relativistic quantum fields à la Sylvester are completely parametrizable with massive particles. As a prominent example we discuss the properties of a massive vector field, e.g. the $Z$-boson field in the standard model.

A free massive vector field $(Z^i(x))_{j=0,1,2,3}$ can be introduced with a Lagrangian leading to its dynamical behaviour

$$\mathcal{L}(Z^i, G^{jk}) = \frac{1}{2} G^{jk} \partial_j Z_k - \partial_k Z^i + m (\lambda \frac{G^{jk} G_{jk}}{4} + \frac{Z^i Z^j}{2\lambda})$$

$$\epsilon^{jk}_{\ell r} \partial^r Z^\ell = \partial^j Z^k - \partial^k Z^j = m \lambda G^{kj}, \quad \partial_k G^{jk} = -\frac{m}{\lambda} Z^j, \quad m \lambda > 0$$

In this 1st order derivative formalism $(Z, G)$ is a canonical pair. $\epsilon^{jk}_{\ell r}$ are the Clebsch-Gordan coefficients for the projection of the $D(\frac{1}{2}|\frac{1}{2})$ Lorentz vector representations $\partial$ and $Z$ to the $D^{(1|0)} \oplus D^{(0|1)}$ Lorentz tensor representation $G$.

By using three 'natural' scales, $\hbar$ (Plancks action unit), $c$ (highest action velocity) and $M$ - an unspecified mass scale, e.g. the symmetry breakdown mass of the standard model, all parameters and fields are dimensionless.

The dynamics involves two characteristic numbers: The parameter $m$ is the particle mass. The parameter $\lambda$ can be normalized away in a free theory with a dilatation transformation (the omitted Lorentz indices have to be inserted as above)

$$\sqrt{m \lambda} Z = Z', \quad \sqrt{m \lambda} G = G'$$

$$\mathcal{L}(Z, G) = \frac{G'}{2} \partial Z' + \frac{G' G'}{4} + m^2 Z' Z'$$

However in an interaction, e.g. of gauge type with a Dirac field $\Psi$, the dilatation factor arises as a coupling constant, e.g.

$$Z^i \overline{\gamma} \gamma^j \Psi = g_Z Z^i \overline{\Psi} \gamma^j \Psi, \quad g_Z^2 = m \lambda$$

The dynamical structure of the massive field should be seen in comparison with a harmonic oscillator with the canonical position-momentum pair $x(t), p(t)$

$$L(x, p) = px' - \frac{p^2}{2M} + k x^2$$
\[ \frac{dx}{dt} = \frac{p}{M}, \quad \frac{dp}{dt} = -kx, \quad kM > 0 \]  

(5)

In the quantum mechanical model \( \hbar \) is used as natural unit with equal time commutator \([ip(t), x(t)] = 1\). The mass \( M \) and the spring constant \( k \) give the scales for time \( \omega \) and length \( \ell \)

\[
\omega^2 = \frac{k}{M}, \quad \ell^4 = \frac{1}{kM} = \frac{\omega^2}{k^2} \\
L(x, p) = p \frac{dx}{dt} - \omega \left( \frac{1}{\sqrt{kM}} \frac{p^2}{2} + \sqrt{kM} \frac{x^2}{2} \right) 
\]  

(6)

The frequency decomposition of the harmonic oscillator displays its unitary time development \( e^{i\omega t} \in U(1) = e^{iR} \) (phase property) and its dilatation property \( \omega e^{i\omega t} = e^{i\omega t} \in \mathbb{R} \)

\[
x(t) = \sqrt{\frac{\omega}{2}} \frac{e^{i\omega t} u + e^{-i\omega t} u^*}{\sqrt{2}}, \quad -i\frac{p(t)}{\sqrt{2}} = \sqrt{\frac{\omega}{2}} \frac{e^{i\omega t} u - e^{-i\omega t} u^*}{\sqrt{2}} 
\]  

(7)

The analogue structure is reflected in the harmonic analysis of the relativistic field with \( q_0 = \sqrt{m^2 + q^2} \)

\[
Z^j(x) = \int \frac{d^3q}{\sqrt{(2\pi)^3q_0}} \Lambda(q,m)^j_a \frac{e^{i\omega q} U^a(q) + e^{-i\omega q} U^a(q)}{\sqrt{2}} \\
iG^{kj}(x) = \int \frac{d^3q}{\sqrt{(2\pi)^3q_0}} \epsilon^{kj}_{lp} \Lambda(q,m)^p_a \frac{e^{i\omega q} U^a(q) - e^{-i\omega q} U^a(q)}{\sqrt{2}} 
\]  

(8)

\( \Lambda(q,m) \) transforms from the general momenta \( q \) with \( q^2 = m^2 \) to the rest frame defined by the massive field. It mediates between the regime of the orthochronous Lorentz group with variables \( Z^j(x) \) and the spin regime with variables \( U^a(q), U^a(q) \). We call \( \Lambda(q,m) \) a transmutator. It is a representative \( \Lambda(q,m) \) for a class of the related real 3-dimensional coset space, completely parametrizable by three noncompact momenta \( (\frac{q_0^a}{m})_{a=1,2,3} \)

\[
\Lambda(q,m)^k_{0,a} \cong \frac{1}{\sqrt{2}} \left( \begin{array}{c} q^0 \\ \bar{q} \\ \delta^{bc} \frac{q^c}{m} + \frac{q^a q^c}{q^2 m} \end{array} \right) \in SO^+(1,3)/SO(3) 
\]  

(9)

The Lorentz-orbits for the \( SO(3) \)-invariant bilinear form \( \delta^{ab} \) give the spin-1-projectors

\[ \begin{align*}
\Lambda(q,m)^b_{kj} m^2 \delta^{ab} \Lambda(q,m)^j_b &= -m^2 \eta^{kj} + q^k q^j, & -\eta^{kj} \cong \left( \begin{array}{cccc}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 
\end{array} \right) \\
\Lambda(q,m)^b_{kj} &= q^k, & \Lambda(q,m)^b_{kj} \eta_{kj} \Lambda(q,m)^j_b &= 0
\end{align*} 
\]  

(10)

The mass \( m^2 \) of the vector particle turns out to be the intrinsic scale for the spin group \( SO(3) \), i.e. the normalization of its Casimir invariant \( \sigma^{ab} = m^2 \delta^{ab} \) in the vector field representation.

\footnote{For a group \( G \) with subgroup \( H \) a coset \( gH \in G/H \) has representatives \( x \in gH \), denoted by \( x \in G/H \).}
The time dependent quantization of the harmonic oscillator with the shorthand notation \([a, b](t - s) = [a(s), b(t)]\)

\[
\Rightarrow \begin{pmatrix}
[u^*, u] = 1 \\
[u, p] = \frac{\cos \omega t}{i} \sin \omega t \\
[p, u] = \left(\begin{array}{cc}
\cos \omega t & i \sin \omega t \\
-\frac{i}{m} \omega t & \frac{\cos \omega t}{i} \sin \omega t
\end{array}\right) = \left(\begin{array}{cc}
\frac{\cos \omega t}{i} & \frac{\coth \omega t}{i} \\
-i \frac{\coth \omega t}{i} & \frac{\cos \omega t}{i}
\end{array}\right) i \sin \omega t
\end{pmatrix}
\]

arises for the vector field as follows

\[
\Rightarrow \begin{pmatrix}
[U^a(q), U^b(p)] = \delta^{ab} \delta(p - q) \\
[U^a(q), Z^b(x)] = \left(\begin{array}{cc}
-\frac{i}{m} \omega t & \frac{\cos \omega t}{i} \sin \omega t \\
-\frac{i}{m} \omega t & \frac{\cos \omega t}{i} \sin \omega t
\end{array}\right)\frac{[Z^a(x), Z^b(x)]}{m \lambda}
\end{pmatrix}
\]

with the vector field quantization as the analogue to \([x, x] = \frac{\omega}{i} \sin \omega t\) given by

\[
[Z^a, Z^b](x) = \int \frac{d^4 q}{(2\pi)^3} e^{ixq} m \lambda \langle -q \rangle \delta(q^2 - m^2)
\]

\[
= \int \frac{d^4 q}{(2\pi)^3} m \lambda \delta(q^2 - m^2)
\]

\[
[Z^{kl}, Z^{ij}](x) = m \lambda \delta^{ab} \sin q^0 x_0
\]

The Fock expectation values, abbreviated by \(\langle \{a, b\} \rangle_F(t - s) = \langle 0 | \{a(s), b(t)\} | 0 \rangle\), for the creation operator \(u\) and its annihilation partner \(u^*\)

\[
\Rightarrow \begin{pmatrix}
\langle 0 | u^* u | 0 \rangle = 1 \\
\langle \{p, p\} \rangle_F(t) = \left(\begin{array}{cc}
\frac{\cos \omega t}{i} & \frac{\sin \omega t}{i} \\
-\frac{i}{m} \omega t & \frac{\cos \omega t}{i} \sin \omega t
\end{array}\right) \frac{\langle Z^a(x), Z^b(x) \rangle_F(x)}{m \lambda}
\end{pmatrix}
\]

read for the particle field

\[
\Rightarrow \begin{pmatrix}
\langle 0 | U^{*a} q U^b p | 0 \rangle = \delta^{ab} \delta(p - q) \\
\langle \{Z^a, Z^b\} \rangle_F(x) = \left(\begin{array}{cc}
-\frac{i}{m} \omega t & \frac{\sin \omega t}{i} \\
\frac{\omega t}{m} & -\frac{i}{m} \omega t \sin \omega t
\end{array}\right)\frac{\langle [Z^a, Z^b] \rangle_F(x)}{m \lambda}
\end{pmatrix}
\]

The relativistic analogue of \(\langle \{x, x\} \rangle_F(t) = \frac{\omega}{t} \cos \omega t\) is given by

\[
\langle [Z^a, Z^b] \rangle_F(x) = \int \frac{d^4 q}{(2\pi)^3} e^{ixq} m \lambda (q^2 - m^2)
\]

\[
= \int \frac{d^4 q}{(2\pi)^3} m \lambda \delta(q^2 - m^2)
\]

\[
\langle [Z^{kl}, Z^{ij}] \rangle_F(x) = m \lambda \delta^{ab} \cos q^0 x_0
\]

The intrinsic length scale of the oscillator

\[
\langle \{x, x\} \rangle_F(0) = 2 \langle 0 | x^2 | 0 \rangle = 2 \|x\|^2 = \frac{\omega}{k} = \ell^2
\]

has its analogue in the coupling constant \(g_Z^2 = m \lambda\). It is suggested to identify the square of the massive vector field is identified with the \(SO(3)-\)Casimir invariant. Then the coupling constant coincides with the mass

\[
\langle [Z^{kl}, Z^{ij}] \rangle_F(0) = \sigma^{ab} = m^2 \delta^{ab} \Rightarrow \lambda = m, \ g_Z^2 = m^2
\]
3 Decompositions à la Sylvester and Witt

Relativistic fields map space-time translations $\mathbb{M} \cong \mathbb{R}^4$ into vectors of complex spaces $V \cong \mathbb{C}^n$. Being complex valued is essential for the quantum structure of fields (phases, probability 'amplitudes', scalar products etc.). Relativistic fields are compatible with the action of the orthochronous Lorentz group $SL(\mathbb{C}^2)$ on the Minkowski space $\mathbb{M}$ - as phaseless real group $SO^+(1, 3) \cong SL(\mathbb{C}^2)/\{\pm 1_2\}$ and on the value space $V$ for the field, e.g. for a vector field $\Lambda^i Z^j(x) = Z^i(\Lambda(x))$ with $\Lambda = D(\frac{i}{2}|\frac{1}{2})$. 

The harmonic analysis classifies a relativistic field with respect to unitary representations of time translations, valued in the general linear group $GL(V)$. In extension of Wigners definition of particles with positive unitary representations also indefinite unitary ones will be considered. A harmonic analysis relies on finite dimensional representations of a direct product subgroup $G_{\text{stab}} \times \mathbb{R}$ of the whole semidirect Poincaré group $SO^+(1, 3) \times_\mathbb{R} \mathbb{M}$ with only the time translations $\mathbb{R} \cong \mathbb{R}$. The subgroup $G_{\text{stab}}$ of the homogeneous Lorentz group proves to be the stability group $^{\mathbb{R}}G_{\text{stab}}$ of a decomposition of the Minkowski translations, determined by the 'masses' $m^2 > 0, = 0, < 0$ of the relativistic fields.

For the causal case with masses $m^2 \geq 0$ - then only real energies $q_0 = \sqrt{m^2 + \vec{q}^2}$ occur - the space-time translations can be decomposed either à la Sylvester for $m^2 > 0$ or à la Witt for $m^2 = 0$.

From the momentum orbit $q^2 = m^2$ of a massive field one can distinguish one timelike vector $e_0$

$$q^2 = m^2 > 0 \Rightarrow q = e_0 \cong (m, 0, 0, 0) \quad (19)$$

This leads to the definition of a rest system and therewith a Sylvester decomposition \cite{Bou59} $\mathbb{M} \cong \mathbb{T} \oplus \mathbb{S}$ into time and space translations. The fixgroup of the distinguished time translation $e_0$ is also the stability group of the decomposition with $\mathbb{T} = \mathbb{R}e_0$, it is the compact rotation group (spin group)

$$\text{FIX}_T SO^+(1, 3) = \text{STAB}_{T \oplus S} SO^+(1, 3) = SO(3) \cong SU(2)/\{\pm 1_2\}$$

$$G_{\text{stab}} \times \mathbb{T} \cong SO(3) \times \mathbb{R} \quad (20)$$

From the momentum orbit $q^2 = 0$ of a massless field one can distinguish one lightlike vector $e_+$

$$q^2 = 0, q \neq 0 \Rightarrow q = e_+ \cong \frac{\mu}{\sqrt{2}} (1, 0, 0, 1) \quad (21)$$

The mass parameter $\mu^2 > 0$ is not given with the massless orbit. The fixgroup of one lightlike translation, e.g. $e_+$, and therewith of the 1-dimensional subspace $\mathbb{L}_+ = \mathbb{R}e_+$ is

\footnote{The fixgroup in a group $G$ acting on a set $S$ keeps every element of the set invariant $\text{FIX}_S G = \{g \in G \mid g \cdot x = x \text{ for all } x \in S\}$, the stability group only the whole set $\text{STAB}_S G = \{g \in G \mid g \cdot S = S\} \supseteq \text{FIX}_S G.$}

\footnote{James Joseph Sylvester (1814-1897), Ernst Witt (1911-...)}
the noncompact semidirect Euclidean group in two real dimensions

$$\text{FIX}_{L^+}SO^+(1, 3) = SO(2) \times_x \mathbb{R}^2$$  \hspace{1cm} (22)$$

This fixgroup is no stability group for a vector space decomposition, since the direct complement $L_- \oplus S^2$ is not invariant.

Since a time axis cannot be distinguished by using only $e_+^\pm$, one has to consider for massless fields the fixgroup of two independent lightlike translations, e.g. $e_+^\pm$. It is the fixgroup of all lightlike translations $L = L_+ \oplus L_- \cong \mathbb{R}^2$ and therewith the stability group of a Witt decomposition\[Bou59\] into three direct summands $M \cong L_+ \oplus L_- \oplus S^2$ with 2-dimensional space translations $S^2$ and is given by the compact axial group (polarization or circularity group)

$$\text{FIX}_{L^+}SO^+(1, 3) = \text{STAB}_{L_+ \oplus L_- \oplus S^2}SO^+(1, 3) = SO(2) \cong U(1)$$

$$G_{\text{stab}} \times L \cong SO(2) \times \mathbb{R}$$  \hspace{1cm} (23)$$

The definition of $L$ needs two orbits, e.g. a timelike and a lightlike one

$$p^2 = \frac{M^2}{m^2} > 0 \Rightarrow p = e_0 \cong (\frac{M}{\sqrt{m}}, 0, 0, 0)$$

$$q^2 = 0, \ q \neq 0 \Rightarrow q = e_+ \cong \frac{\mu}{\sqrt{2}}(1, 0, 0, 1)$$

$$\Rightarrow \frac{\mu}{\sqrt{2}}\sqrt{2} - q = e_- \cong \frac{\mu}{\sqrt{2}}(1, 0, 0, -1)$$  \hspace{1cm} (24)$$

Neither the mass parameter $\frac{M}{m}$ nor the mass parameter $\mu$ arise in the lightlike orbit $q^2 = 0$.

A Witt decomposition is a subdecomposition of a Sylvester space-time decomposition as seen in the Lorentz-Sylvester-Witt chain $SO^+(1, 3) \supset SO(3) \supset SO(2)$.

4 Quantum Spinor Fields à la Witt

In the case of a Sylvester decomposition of the space-time translations with stability group $SO(3)$, the real 3-dimensional Sylvester manifold $SO^+(1, 3)/SO(3) \cong SL(\mathbb{C}^2)/SU(2)$ describes the transition to any Sylvester decomposition. For a harmonic analysis of massive fields one starts from a complex 2-dimensional Weyl representation $s(q, m)$ of the coset representatives (boosts), parametrized by the spacelike momenta $\frac{\vec{q}}{m}$

$$s(q, m) \circ 1_2 \circ s^*(q, m) \cong \rho_k q_k = \frac{\rho_k q_k^k}{m} = \frac{1}{m} \left( \begin{array}{ccc} q^0 + \frac{q^3}{q^1 + iq^2} & q^1 - iq^2 \\ q^1 + iq^2 & q^0 - q^3 \end{array} \right), \ \ \ \rho_k = (1_2, \vec{\sigma}), \ \ \ \hat{\rho}_k = (1_2, -\vec{\sigma})$$

$$\Rightarrow \left\{ \begin{array}{l}
\frac{1}{\sqrt{2m(q^2 + m)}} \left( \begin{array}{ccc}
q^0 + m + q^3 & q^1 + iq^2 \\
q^1 + iq^2 & q^0 + m - q^3
\end{array} \right) \in SL(\mathbb{C}^2)/SU(2), \ q^2 = m^2
\end{array} \right.$$  \hspace{1cm} (25)
The Lorentz-Sylvester transmutators in the harmonic analysis of massive fields are representations of $s(q,m)$ as used in chapter 1. for the massive vector field in the vector representation $\Lambda(q,m) = D^{\frac{3}{2}i\frac{1}{2}}(s(q,m))$.

Massless spinor fields with a Witt decomposition of the space-time translations involve two helicity projectors which can be obtained by the limit $m \to 0$ from the Weyl transmutators of the Sylvester decompositions

$$
p_+(q) = p_+(q)^* = \lim_{m \to 0} \sqrt{\frac{2\pi}{x_0}} s(q,m) = \frac{1}{2}(1 + \frac{q^2}{m^2})$$

$$
p_-(q) = p_-(q)^* = \lim_{m \to 0} \sqrt{\frac{2\pi}{x_0}} s(q,m)^{-1} = \frac{1}{2}(1 - \frac{q^2}{m^2}), \quad q^2 = 0 \quad (26)
$$

with two 'compact' momentum parameters $\frac{q}{q_0}$.

A prominent spinor Witt field is a massless neutrino field $(1^A(x))_{A=1,2}$ with a left handed Weyl representation $D^\frac{1}{2}I^0$ and Lagrangian

$$
\mathcal{L}(l) = -\frac{i}{2} \tilde{\rho}_j \tilde{\sigma}_k \tilde{l} \quad \text{i} \tilde{\rho}_j \tilde{l} = 0 \quad (27)
$$

The harmonic analysis with particle and antiparticle creation operators $U(q), A(q)$ and particle and antiparticle annihilation operators $U^*(q), A^*(q)$ looks like

$$
1^A(x) = \int \frac{d^3q}{(2\pi)^3} p_+(q)^A_C \left[ e^{ixq} U(q) + e^{-ixq} A^*(q) \right] \nonumber
$$

$$
\mathbf{l}_A^A(x) = \int \frac{d^3q}{(2\pi)^3} p_+(q)^C_A \left[ e^{-ixq} U(q) + e^{ixq} A_C(q) \right] \quad (28)
$$

We shall show in chapter 4. that this Fourier decomposition is not quite complete.

One has the anticommutators for the Weyl fields

$$
\{U^*_A(q), U^B_B(p)\} = \delta^B_A \delta(p - q) = \{A^*_A(q), A_A(p)\} \nonumber
$$

$$
\Rightarrow \quad \{\mathbf{l}_B^A, \mathbf{l}_A^A\}(x) = \rho^3_B \int \frac{d^3q}{(2\pi)^3} e^{ixq} \delta(q^2) \delta(q^2) = -i \rho^3_B \tilde{\psi} \int \frac{d^3q}{(2\pi)^3} e^{ixq} \sin(q_0) \quad (29)
$$

and commutator expectation values in the Fock form

$$
\langle [U^*_A(q), U^B_B(p)] \rangle_v = \delta^B_A \delta(p - q) = \langle [A^*_A(q), A_A(p)] \rangle_v \nonumber
$$

$$
\Rightarrow \quad \langle [\mathbf{l}_B^A, \mathbf{l}_A^A] \rangle_v (x) = -i \rho^3_B \int \frac{d^3q}{(2\pi)^3} e^{ixq} \delta(q^2) = -i \rho^3_B \tilde{\psi} \int \frac{d^3q}{(2\pi)^3} e^{ixq} \cos(q_0) \quad (30)
$$

5 Quantum Vector Fields à la Witt

Relativistic quantum vector fields à la Witt may contain contributions without particle interpretation [Sal94b]. The most familiar Witt vector field is the massless electromagnetic field with both the nonparticle like Coulomb interaction and the polarized photons, reflecting the lightlike and spacelike subspaces $\mathbb{L} = \mathbb{L}_+ \oplus \mathbb{L}_-$ and $\mathbb{S}^2$ resp. in the Witt decomposition of the Minkowski translations $\mathbb{M} \cong \mathbb{L}_+ \oplus \mathbb{L}_- \oplus \mathbb{S}^2$. 

8
5.1 The Field Equations

For a free massless quantum vector field $A_j(x)$ one has as Lagrangian and field equations

$$L(A_j, F_{jk}, L) = \mathcal{L}_j A_j + F_{jk} \frac{\partial_j A_k - \partial_k A_j}{2} + \mu^2 \frac{F_{jk} F_{jk}}{4} + \epsilon \sigma^2 \frac{L^2}{2}$$

$$\epsilon_{ir} \partial_i A^r = \partial_j A^k - \partial_k A^j = \mu^2 F_{kj}, \quad \partial_j A^j = -\epsilon \sigma^2 L$$

(31)

The canonical partners of the Lorentz vector $A$ are a Lorentz tensor $F$ and a Lorentz scalar field $L$, often called 'gauge fixing' field.

Again, all fields and parameters are dimensionless with three universal scales $\hbar, c$ and $M$. The nontrivial, but otherwise arbitrary constant $\epsilon \sigma^2$ is usually called the 'gauge fixing' parameter, its value is experimentally irrelevant in a 'gauge invariant' dynamics. The parameter $\mu^2$ can be normalized away in a free theory

$$\frac{1}{\mu} A = A', \quad \mu F = F', \quad \mu L = L'$$

(32)

Again however in an interaction, e.g. of gauge type, the parameter $\mu^2$ arises as experimentally relevant gauge coupling constant

$$A_j \overline{\Psi} \gamma^j \Psi = g_A A^j \overline{\Psi} \gamma^j \Psi, \quad g^2_A = \mu^2$$

(33)

5.2 The quantum mechanical analogue

There is a quantum mechanical analogue\cite{Sal89, Sal92a} to the non particle contributions ('gauge' and Coulomb degree of freedom) in the time development of the massless vector field using two positions $(x, x')$ and as canonical partners two momenta $(p, p')$. The appropriate Lagrangian

$$L(x, x', p, p') = \frac{p}{dt} dx + \frac{p'}{dt} dx' - \left[ \frac{p^2}{2M} + \frac{p'^2}{2M'} + k \left( \frac{xp'}{M'} - \frac{x'p}{M} \right) \right]$$

$$\frac{dx}{dt} = -k \frac{x'}{M'}, \quad \frac{dx'}{dt} = k \frac{x}{M}, \quad \frac{dp}{dt} = -k \frac{p'}{M'}, \quad \frac{dp'}{dt} = k \frac{p}{M'}, \quad MM' > 0, \quad k \in \mathbb{R}$$

(34)

contains two masses $M, M'$ and a frequency $\omega = \frac{k}{\sqrt{MM'}}$, e.g. derivable from the equation of motion for the momenta $(\frac{dp}{dt} + \omega^2 p = 0$. $\hbar$ is assumed as universal action scale.
For vanishing frequency $\omega = 0$, the dynamics is decomposable into the motion of two free mass points $t = p_{\delta \mu}^{M_F} - \frac{p_{\delta \nu}}{2M_T}$.

The time dependent quantization reads for the positions and momenta

$$
\begin{pmatrix}
[i,p,x] & [p',x] & [-ix',x] & [x,x] \\
[-p,x'] & [p',ix'] & [x',x'] & [x',ix'] \\
[i,p,p'] & [p',p'] & [-ix',p'] & [x',p'] \\
[p,p] & [p',-ip] & [x',-ip] & [x,-ip]
\end{pmatrix}(t) =
\begin{pmatrix}
\cos \omega t & i\sqrt{\frac{MM'}{M}} \sin \omega t & -i\frac{1}{\sqrt{MM'}} \sin \omega t & \frac{i}{\sqrt{MM'}} \cos \omega t \\
i\sqrt{\frac{MM'}{M}} \sin \omega t & \cos \omega t & i\frac{1}{\sqrt{MM'}} \sin \omega t & -\frac{i}{\sqrt{MM'}} \cos \omega t \\
0 & 0 & \cos \omega t & i\sqrt{\frac{MM'}{M}} \sin \omega t \\
0 & 0 & i\sqrt{\frac{MM'}{M}} \sin \omega t & \cos \omega t
\end{pmatrix}
\begin{pmatrix}
\cos \omega t & i\sqrt{\frac{MM'}{M}} \sin \omega t & -i\frac{1}{\sqrt{MM'}} \sin \omega t & \frac{i}{\sqrt{MM'}} \cos \omega t \\
i\sqrt{\frac{MM'}{M}} \sin \omega t & \cos \omega t & i\frac{1}{\sqrt{MM'}} \sin \omega t & -\frac{i}{\sqrt{MM'}} \cos \omega t \\
0 & 0 & \cos \omega t & i\sqrt{\frac{MM'}{M}} \sin \omega t \\
0 & 0 & i\sqrt{\frac{MM'}{M}} \sin \omega t & \cos \omega t
\end{pmatrix}
$$

(35)

The harmonic analysis is more complicated as for the compact $U(1)$-dynamics since one has a complex 2-dimensional, reducible, but nondecomposable time representation in an indefinite unitary group[Bre95] $\begin{pmatrix} 1 & \frac{d}{M_0} \\ 0 & 1 \end{pmatrix} e^{it\omega} \in U(1,1)$. The representations involves both time eigenvectors $g, g^\times$ ('good') and time nilvectors $b, b^\times$ ('bad')

$$
\begin{pmatrix}
b \\
g
\end{pmatrix}(t) = e^{it\omega} \begin{pmatrix} 1 & \frac{d}{M_0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b \\
g \end{pmatrix}, \quad
(g^\times, b^\times)(t) = (g^\times, b^\times)e^{-it\omega} \begin{pmatrix} 1 & -\frac{d}{M_0} \\ 0 & 1 \end{pmatrix}
$$

(36)

The Hamiltonian consists of a semisimple part with invariant and therewith physically relevant frequency $\omega$ and a nilpotent part where the mass $M_0$ is a basis dependent constant, physically irrelevant (unobservable)

$$
H = \omega(\sqrt{\frac{M}{M_F}} p' - \sqrt{\frac{MM'}{M}} p) + \frac{d}{M_0}(\sqrt{\frac{MM'}{M}} p^2 + \sqrt{\frac{M}{M_F}} p^2)
$$

$$
\omega = \frac{1}{\sqrt{MM'}}, \quad M_0 = \epsilon(M)\sqrt{MM'} \neq 0
$$

$$
H = \omega(b g^\times + g b^\times) + \frac{g g^\times}{M_0}
$$

(37)

The two representations of the Hamiltonian can be transformed into each other by using the indefinite unitary harmonic expansion

$$
x(t) = \sqrt{\frac{M}{M_F}} b(t) + b^\times(t), \quad -ip(t) = \sqrt{\frac{M}{M_0}} g(t) - g^\times(t)
$$

$$
ix'(t) = \sqrt{\frac{M}{M_F}} b(t) - b^\times(t), \quad p'(t) = \sqrt{\frac{M}{M_0}} g(t) + g^\times(t)
$$

(38)

The corresponding nontrivial massless field commutators are

$$
\left( \begin{array}{cc}
[iF^{kl}, A^j] & [A^k, A^j] \\
[F^{1k}, F^{jn}] & [A^k, -iL]
\end{array} \right)(x) =
\begin{pmatrix}
-\epsilon_{\mu\nu} \delta^i_j \frac{\partial_\mu}{\partial_\nu} & \delta^k_j \delta^i_s \\
-\epsilon_{\mu\nu} \epsilon^{ij} \frac{\partial \mu}{\partial_\nu} & i\delta^k_j \delta^i_s \frac{\partial_\nu}{\partial_\tau}
\end{pmatrix}[A^t, A^s](x)
$$

(39)
The basic field commutator contains in addition to the pole structure $\delta(q^2)$, relevant for the $U(1)$-time development (photons), the characteristic indefinite dipole structure $\delta'(q^2)$, related to the $U(1,1)$-time development as will be shown later in more detail

$$[A^k, A^l](x) = \int \frac{dq}{(2\pi)^2} e^{ixq} \epsilon(q_0)|-\mu^2 \eta^k| \delta(q^2) - (\mu^2 + \epsilon \sigma^2) q^k q^l \delta'(q^2)$$

$$= \int \frac{dq}{(2\pi)^2} \left[-\mu^2 \eta^k j \sin x_0 q_0 + (\mu^2 + \epsilon \sigma^2) \partial^k \partial^l \frac{2\sin x_0 q_0 - \sin x_0 q_0 e^{-ixq}}{2q_0^2} \right]$$

$$\left[\mathbf{F}^{kl}, A_j(x) = \int \frac{dq}{(2\pi)^2} e^{ixq} q_0 \epsilon(q_0) \delta(q^2), \quad [A^k, iL](x) = \int \frac{dq}{(2\pi)^2} e^{ixq} q_0 \epsilon(q_0) \delta'(q^2) \right]$$

(40)

To perform the energy $q_0$-integration, i.e. to decompose $q^2 = q_0^2 - \vec{q} \cdot \vec{a}$, there has to exist a time-space decomposition using a rest system. The terms proportional to $x_0 q_0 e^{ixq} q_0$ are characteristic for the noncompact time representations involved. The relation between compact and noncompact time representations on the one hand and distributions, dipoles etc. on the other hand is obvious in the mechanical model

$$e^{itE} = \int dq_0 e^{iq_0} \delta(q_0 - E), \quad \delta(q_0 - E) = \operatorname{Re} \frac{i}{\pi E + i\epsilon - q_0}$$

$$-ite^{itE} = \int dq_0 e^{iq_0} \delta'(q_0 - E), \quad \delta'(q_0 - E) = \operatorname{Re} \frac{i}{\pi (E + i\epsilon - q_0)^2}$$

(41)

The 'gauge fixing' parameter in the combination $\mu^2 + \epsilon \sigma^2$ will turn out to be the analogue to the inverse mass scale $\frac{1}{M^2}$ in the mechanical model.

### 5.3 Lorentz-Witt-Transmutation

The Lorentz-Witt transmutation is performed in two steps. First, a Lorentz-Sylvester transmutation $s_{(p;M)} \in SL(\mathbb{Q}^2)/SU(2)$ or $\Lambda_{(p;M)} \in SO^+(1,3)/SO(3)$ with $p^2 = M^2 > 0$ fixes a time axis (laboratory system) and a reference mass $\Lambda$. For clarity we denote

vector indices for the

\[
\begin{align*}
&SO^+(1,3)-\text{Lorentz regime: } j = (0,1,2,3) \\
&SO(3)-\text{Sylvester regime: } j = (0; a) = (0; 1,2,3)
\end{align*}
\]

(42)

In addition, one has to use a Sylvester-Witt transmutator $u(q)$ as element of the manifold $SU(2)/U(1)$ which - in the vector representation - rotates the two distinguished lightlike vectors $\mathbf{e}_\pm \cong \frac{1}{\sqrt{2}}(1,0,0,\pm 1)$ into two general lightlike vectors with a general space direction $\frac{q_0}{q_0}$

$$u(q) \circ \frac{1}{2} \pm \frac{\sigma^3}{2} \circ u^*(q) = \frac{\rho_j q_\pm^j}{q_0} = \frac{1}{\sqrt{2p^2(q^0 + q_0)}} \begin{pmatrix} q^0 \pm q^2 & \pm (q^1 - iq^2) \\ q^1 + iq^2 & q^0 - q^2 \end{pmatrix} \in SU(2)/U(1), \quad q^2 = 0$$

(43)
u(q) with \( q^2 = 0 \) is constructed with the two parameters \( \frac{\sigma}{q} \) for the compact manifold \( SO(3)/SO(2) \) (2-sphere). The three additional 'noncompact' parameters of the real 5-dimensional Witt manifold \( SO^+(1, 3)/SO(2) \) are contained in \( s(p, M) \) used above.

The fundamental Sylvester-Witt transmutators are unitary \( u(q)^* = u(q)^{-1} \), not hermitian. For half-integer Lorentz representations, e.g. Weyl spinor fields (chapter 2.), the projections to the lightspaces \( L_{\pm} \) or \( L_{\mp} \) define the hermitian Sylvester-Witt projectors \( p_{\pm}(q) \) as used in chapter 3.

\[
\begin{align*}
    u(q) \circ \frac{1 + \sigma^3}{2} \circ u^*(q) &= p_+(q) \circ 1_2 \circ p_+(q) = \frac{\rho_2 q^2}{2q_0} \\
    u(q) \circ \frac{1 - \sigma^3}{2} \circ u^*(q) &= p_-(q) \circ 1_2 \circ p_-(q) = \frac{\rho_2 q^2}{2q_0} 
\end{align*}
\]

(44)

The integer spin Sylvester-Witt-transmutators are hermitian. They start from the vector representation of the coset \( SU(2)/U(1) \)

\[
O(q) = D^{(\frac{1}{2}, \frac{1}{2})}(u(q)) = \begin{pmatrix} 1 & 0 & -q_0(q_0^2 + q_0^3 + q_0^4) & 0 \\
0 & 1 & q_0^2(q_0^2 + q_0^3) & q_0^3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix} \in SO(3)/SO(2)
\]

(45)

Instead of using a Sylvester basis with diagonal Lorentz matrix \( \eta \), it is more convenient for a Witt decomposition to consider a Witt basis with Lorentz matrix \( \iota \)

\[
\begin{pmatrix} -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \end{pmatrix} \quad \text{Sylvester: } -\eta \cong \begin{pmatrix} 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \end{pmatrix} \quad \text{Witt: } -\iota \cong
\]

(46)

The two bases are transformed into each other by \( w \)

\[
\begin{array}{c}
\begin{pmatrix} 1 \\
0 \\
0 \\
1 \end{pmatrix} \end{array} = w \begin{pmatrix} \frac{1}{\sqrt{2}} \\
0 \\
0 \\
-\frac{1}{\sqrt{2}} \end{pmatrix} \quad \begin{array}{c}
\begin{pmatrix} 1 \\
0 \\
0 \\
-1 \end{pmatrix} \end{array} = w \begin{pmatrix} 0 \\
0 \\
0 \\
1 \end{pmatrix} \quad w = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}
\end{array}
\]

(47)

To distinguish the \( SO(2) \)-regime from the \( SO(3) \)- and the \( SO^+(1, 3) \)-regime doubled underlined and Greek indices are used

\[
\text{vector indices for the } SO(2) \text{-Witt regime in a } \begin{cases}
\text{Sylvester basis: } j = (0; 1, 2, 3) \\
\text{Witt basis: } \alpha = (0; 1, 2, 3)
\end{cases}
\]

(48)
The 2-parametric Sylvester-Witt transmutator \( H(q) \)

\[
H(q)^k_{\alpha} = O(q)^k_{\alpha} \circ w^j_{\alpha} \approx \frac{1}{q^0} \begin{pmatrix}
\sqrt{\frac{q^0}{q^2}} & 0 & 0 & -\sqrt{\frac{q^0}{q^2}} \\
\frac{q^1}{q^2} & \sqrt{\frac{q^0}{q^2}} & 0 & -\frac{q^1}{q^2} \\
\frac{q^2}{q^0} & \frac{q^0}{q^2} & \sqrt{\frac{q^0}{q^2}} & -\frac{q^2}{q^0} \\
\frac{q^3}{q^0} & \frac{q^0}{q^3} & -\frac{q^3}{q^0} & \sqrt{\frac{q^0}{q^3}}
\end{pmatrix} \in SO(3)/SO(2)
\] (49)

relates the bilinear forms for Sylvester and Witt bases

\[
H(q)^k_{\alpha} \epsilon^{\alpha\beta} H(q)^j_{\beta} = \eta^{k,j}, \quad H(q)^k_{\alpha} \eta_{k\alpha} H(q)^j_{\beta} = \epsilon_{\alpha\beta}
\] (50)

### 5.4 Harmonic Analysis of Massless Vector Fields

The time representations in the commutator of the massless vector field are transformed from the Lorentz to the Sylvester regime by a transmutator \( \Lambda(p, M) \)

\[
[A^k, A^j](x) = \Lambda(p, M)^k_{\mu} \int \frac{d^3 \kappa}{(2\pi)^3} \mathcal{A} \Lambda^j_{\nu}(\kappa) \Lambda(p, M)^j_{\nu}
\] (51)

The transformation \( \Lambda(p, M)(x) = \mathcal{x} = (\mathcal{x}_0, \vec{\mathcal{x}}) \) and \( \Lambda(p, M)(q) = \mathcal{q} = (\mathcal{q}_0, \vec{\mathcal{q}}) \) allows the use of separate time-energy and space-momentum coordinates. This rest system, especially the mass scale \( M \), has to be introduced by an additional structure \( p^2 = M^2 > 0 \). In contrast to the massive case, it is not determined by the momentum orbit of the massless field.

The commutator in the rest frame is given as follows

\[
[\mathcal{A}^k, \mathcal{A}^j](\mathcal{x}_0, \mathcal{q}) = -\mu^2 \gamma^{k0} \sin \mathcal{x}_0 \mathcal{q}_0 - \frac{\mu^2 + i \mathcal{q}_0}{2} \begin{pmatrix}
\mathcal{x}_0 \mathcal{q}_0 \cos \mathcal{x}_0 \mathcal{q}_0 + \sin \mathcal{x}_0 \mathcal{q}_0
\
\mathcal{x}_0 \mathcal{q}_0 \sin \mathcal{x}_0 \mathcal{q}_0 - \cos \mathcal{x}_0 \mathcal{q}_0
\
\frac{\mu^2 + i \mathcal{q}_0}{2} i (\mathcal{x}_0 \mathcal{q}_0 \cos \mathcal{x}_0 \mathcal{q}_0 + \sin \mathcal{x}_0 \mathcal{q}_0)
\end{pmatrix}
\] (52)

The rotation \( O(q) \) transforms from the \( SO(3) \) to the \( SO(2) \) regime, resulting in \( \mathcal{q}^a \mapsto \delta^a_3 \mathcal{q}^0 \). Then one introduces by \( w \) a Witt basis, \( H(q) = O(q) \circ w \)

\[
[\mathcal{A}^k, \mathcal{A}^j](\mathcal{x}_0, \mathcal{q}) = H(q)^k_{\alpha} [\mathcal{A}^\alpha \mathcal{A}^\beta](\mathcal{x}_0) H(q)^j_{\beta}
\] (53)

In the Witt-regime the 1st and 2nd component have the commutators

\[
\alpha, \beta \in \{1, 2\} : [\mathcal{A}^k, \mathcal{A}^j](\mathcal{x}_0) \cong \mu^2 i \sin \mathcal{x}_0 \mathcal{q}_0 \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\] (54)

They have nontrivial circularity (polarization) \( SO(2) \cong U(1) \). They carry two compact \( U(1) \) time representations of energy \( \mathcal{q}_0 \) with 'positions' \( a^{1,2} \) analogous to the harmonic
oscillator

\[
\begin{align*}
\alpha, \beta \in \{1, 2\} : \\
\begin{bmatrix} a^{\alpha}(q, x) \end{bmatrix}^{\beta}(x_0 - y_0) \delta(q - p) &= \begin{bmatrix} a^{\alpha}(q, y_0), a^{\beta}(q, x_0) \end{bmatrix} \\
U^{\alpha}(q, x_0) &= U^{\alpha}(q) e^{i x_0 q} \\
[U^{\alpha}(p), U^{\beta}(q)] &= \delta^{\alpha \beta} \delta(q - p)
\end{align*}
\]

(55)

and, therefore, have a particle interpretation (polarized photons) with a Fock state

\[
\alpha, \beta \in \{1, 2\} : \langle\{A A\}^{\alpha \beta}\rangle_v(0) = \mu^2 \delta^{\alpha \beta}
\]

(56)

The normalization of the SO(2)-invariant is given by the square of the massless photons

\[
\alpha, \beta \in \{1, 2\} : \langle\{A A\}^{\alpha \beta}\rangle_v(0) = \mu^2 \delta^{\alpha \beta}
\]

(57)

The SO(2) trivial unpolarized contributions (0th and 3rd component)

\[
\begin{align*}
\alpha, \beta \in \{0, 3\} : \\
\begin{bmatrix} a^{\alpha}(x_0) \end{bmatrix}^{\beta}(x_0) &= \mu^2 \left(\begin{array}{c}
\frac{ix_0 q}{M_0} e^{-ix_0 q} N_0 i \sin x_0 q \\
N_0 i \sin x_0 q \\
\frac{ix_0 q}{M_0} e^{ix_0 q}
\end{array}\right) \\
\frac{1}{M_0} &= -\frac{\mu^2 \sigma q}{\mu^2}, \quad N_0 = \frac{3 \mu^2 \sigma q}{\mu^2}
\end{align*}
\]

(58)

carry one indefinite \(U(1, 1)\) time representation as discussed in the quantum mechanical model (section 4.2.)

\[
\begin{align*}
\alpha, \beta \in \{0, 3\} : \\
\begin{bmatrix} a^{\alpha}(q, x_0) \end{bmatrix}^{\beta}(q, x_0) &= \left(\begin{array}{cc}
a^{0, a^0} & [a^0, a^0] \\
[a^0, a^0] & a^{0, a^0}
\end{array}\right)(q, x_0) \\
a^{0}(q, x_0) &= \mu^{B(q, x_0) + N_0 G^x(q, x_0)} \\
\left(\begin{array}{c}
B(q, x_0) \\
G(q, x_0)
\end{array}\right) &= e^{ix_0 q} \left(\begin{array}{c}
1 \\
0
\end{array}\right) \left(\begin{array}{c}
B(q) \\
G(q)
\end{array}\right)
\end{align*}
\]

(59)

The therewith harmonic analysis\(^5\) of the massless vector field exhibits both definite unitary \(U(1)\) and indefinite unitary \(U(1, 1)\) time representations in the (1, 2) components (photons) and (0, 3) components (Coulomb and ‘gauge’ degree of freedom) resp.

\[
A^{k}(x) = \Lambda(p, M)_{\frac{k}{k}} \int \frac{d^3 q}{(2\pi)^{3/2}} H(q)_{\Lambda} a^{\alpha}(q, x)
\]

\(^5\) The complete harmonic analysis of a massless Weyl field (chapter 3.) has to include the Lorentz-Sylvester transmutator

\[
\begin{align*}
1^{A}(x) &= \int \frac{d^3 q}{(2\pi)^{3/2}} p_+ (p, M, q)_C e^{-i q A^C(q)} + e^{-i q A^C(q)} \\
p_+(p, M, q) &= s(p, M) \circ p_+(q)
\end{align*}
\]

(60)
As we have seen in the former chapters, the masses and coupling constants, connected with vector fields à la Sylvester and Witt and their harmonic analysis, can quantify characteristic invariants of the symmetries involved.

To have an experimentally relevant illustration, the masses and coupling constants of the vector fields in the standard model \[\text{Wei67}\] are considered. The \(Z\)-boson coupling constant \(g_Z\) and the electromagnetic coupling constant \(g_e\) with \(g_e^2 \simeq \frac{4\pi}{137}\) measure the hypotenues and height resp. of the rectangular electroweak triangle with the Weinberg angle \(\theta_w\). This triangle \[\text{Sal92b}\,\text{Sal93}\,\text{Sal94a}\] gives the vector boson masses in units of the symmetry breakdown Fermi mass \(M \simeq 123\,\text{GeV}\).

\[
(m_Y, m_W, m_Z|m_e) = (g_Y, g_W, g_Z|g_e)\frac{M}{\omega^\nu}, \quad \begin{align*}
g_Y^2 + g_W^2 &= g_Z^2 \\
g_Y g_W &= g_Z g_e \\
\tan \theta_w &= \frac{g_Y}{g_W}
\end{align*}
\] (62)

Only \(m_Z\) and \(m_W\) are masses of particles.

A time representation as compact unitary group \(U(1)\)

\[
u(t) = e^{i\omega t} u, \quad e^{i\omega t} \in U(1)
\] (63)

comes with a frequency \(\omega\) as \(U(1)\) invariant scale. The \(U(1)\) scalar product defines the probability inducing Fock form in a quantum theory

\[
\langle u|u \rangle = \langle u^* u \rangle_v = \langle 0|u^* u|0 \rangle = 1
\] (64)

For the hermitian position and momentum combinations in a Bose quantum theory, the \(U(1) \cong SO(2)\) time development symmetry leads to the invariance of the \(2 \times 2\)-matrix

\[
\begin{pmatrix}
0 & \{x, x\}

\{0| \{x, x\}|0\}

\{0| \{x, x\}|0\}

\{0| \{p, x\}|0\}

\{0| \{p, x\}|0\}

\{0| \{p, p\}|0\}

\{0| \{p, p\}|0\}

\{0| \{p, p\}|0\}
\end{pmatrix} = \begin{pmatrix}
\omega & 0

0 & \kappa
\end{pmatrix}
\] (65)

For the massive vector fields (Sylvester), e.g. the \(Z\)-boson, the mass \(m\) enters the time representations via \(\omega = \sqrt{m^2 + \kappa^2}\). The \(SO(2) \cong U(1)\) time development normalization
is given by the analogue matrix replacing \((x, p)\) by \((Z^a, G_{0a})\)

\[
a, b \in \{1, 2, 3\} : \begin{pmatrix} m \lambda \delta^{ab} & 0 \\ 0 & \frac{1}{m \lambda} \delta_{ab} \end{pmatrix}
\]

(66)

Here \(\delta^{ab}, \delta_{ab}\) take into account the three components of the spin \(SO(3)\) representation. \(\sqrt{m \lambda}\) is the analogue to \(\frac{\omega}{k}\) in the mechanical model. The Lorentz-Sylvester transmutator \(\Lambda(q, m) \in SO^+(1, 3)/SO(3)\) is constructed only with \(m\) - not with \(\lambda\). It embeds the Casimir invariant \(m^2 \delta^{ab}\) of \(SO(3)\) in its Lorentz orbit

\[
\Lambda(q, m)^{k, m} m^2 \delta^{ab} \Lambda(q, m) = -m^2 \eta^{kj} + q^k q^j
\]

(67)

Since the symmetry \(SO(3)\) has only one independent invariant, the component \(m \lambda \delta^{ab}\) in the invariant time development matrix above is identified with \(m^2 \delta^{ab}\).

For the standard model, the \(Z\)-coupling constant \(m^2 = \frac{m^2_{Z}}{M^2} = g_2^2\) turns out to be the invariant spin \(SO(3) \cong SU(2)/\{\pm 1_2\}\) normalization in the Lorentz group \(SO^+(1, 3)\).

The indefinite unitary \(U(1, 1)\) time development in the massless electromagnetic field does not introduce additional masses or coupling constants: Such a time representation

\[
\begin{pmatrix} b(t) \\ g(t) \end{pmatrix} = e^{i \omega t} \begin{pmatrix} 1 & \frac{it}{M_0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ g \end{pmatrix}, \quad e^{i \omega t} \begin{pmatrix} 1 & \frac{it}{M_0} \\ 0 & 1 \end{pmatrix} \in U(1, 1)
\]

(68)

contains two parameters, a frequency \(\omega\) as the invariant trace of the time translation generator (up to \(i\) the Hamiltonian matrix)

\[
\frac{1}{2} \text{tr} \left( \begin{array}{cc} \omega & \frac{1}{M_0} \\ 0 & \omega \end{array} \right) = \omega
\]

(69)

and a mass \(M_0\) in the nilpotent traceless contribution. The mass \(M_0\), however, reflects only the choice of a basis in the complex 2-dimensional time representation space and - as a basis dependent quantity - is physically irrelevant.

For massless gauge vector fields the frequencies \(\omega = \sqrt{q^2}\) in the harmonic analysis are not Lorentz invariant. The gauge fixing sector reflects the choice of a basis with the physically irrelevant nontrivial gauge fixing parameter \(\epsilon \sigma^2\) in the combination

\[
\frac{1}{M_0} = \frac{\epsilon^2 + \sigma^2}{\omega^2}
\]

(70)

Unlike in the particle case with its definite \(U(1)\) Fock state, the indefinite \(U(1, 1)\) sesquilinear form of the noncompact time representation

\[
\begin{pmatrix} \langle b|b \rangle & \langle b|g \rangle \\ \langle g|b \rangle & \langle g|g \rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

(71)

16
does not contribute to the probabilities in a quantum theory. Especially for nonabelian quantum gauge vector fields, e.g. the Fadeev-Popov fields\cite{Kug78} \cite{Nak90} take care of the metrical structure of $U(1,1)$.

For the massless electromagnetic field, the $U(1)$ time representations for the photons arise with frequencies $q_0 = \sqrt{\vec{q}^2}$. The $U(1)$ time development normalization is given by

$$\alpha, \beta \in \{1, 2\} : \begin{pmatrix} \mu^2 \delta^{\alpha\beta} & 0 \\ 0 & \mu \delta^{\alpha\beta} \end{pmatrix}$$

Here $\delta^{\alpha\beta}, \delta_{\alpha\beta}$ takes into account the two components of the polarization $SO(2)$ representation. The Sylvester-Witt transmutor $O(q) \in SO(3)/SO(2)$ embeds the $SO(2)$ invariant $\mu^2 \delta^{\alpha\beta}$ in its $SO(3)$-orbit, followed by an embedding with the Lorentz-Sylvester transmutor $\Lambda(p, M)$ in its Lorentz $SO^+(1,3)$-orbit.

For the standard model, the electromagnetic coupling constant $\mu^2 = \frac{m_Z^2}{M^2} = g_e^2$ turns out to be the circularity $SO(2) \cong U(1)$ normalization in the Lorentz group $SO^+(1,3)$-orbit.

The three mass parameters involved - or one mass parameter and two mass ratios - can be related to the Lorentz-Sylvester-Witt chain of the three associated groups

$$SO^+(1,3) \supset SO(3) \supset SO(2) : \begin{cases} M, & m_Z, & m_e \\ M, & \sin 2\theta_w = 2m_e/m_Z, & \alpha_e = \frac{m_e^2}{4\pi M^2} \end{cases}$$

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\footnote{Also the fermionic Fadeev-Popov fields have a harmonic analysis with $U(1,1)$ time representations.}
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