The Chow Ring of the Hilbert Scheme of Rational Normal Curves
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0. Introduction

0.1. Summary. Let $\mathbb{C}$ be the ground field of complex numbers. A rational normal curve in $\mathbb{P}^d$ is an irreducible, nonsingular, non-degenerate, degree $d$ rational curve. For $d \geq 1$, let $H(d)$ be the open Hilbert scheme of rational normal curves of degree $d$ in $\mathbb{P}^d$. $H(d)$ is a nonsingular, irreducible, quasi-projective, algebraic variety. Let $A^\ast(d)$ be the integral Chow ring of $H(d)$. In case $d = 1$, there is a unique rational normal curve in $\mathbb{P}^1$. Hence, $H(1)$ is a point. $H(2)$ is the space of nonsingular plane conics. The dimension of $H(d)$ is $d^2 + 2d - 3$.

In this paper, a presentation of $A^\ast(d)$ is computed via the theory of equivariant Chow groups. The idea is to exhibit $H(d)$ as a quotient of an appropriate variety $X$ by a free $G$-action. For free actions, the equivariant Chow ring $A^\ast_G(X)$ is isomorphic to the ordinary Chow ring $A^\ast(X/\mathbb{G}) \cong H(d)$. The equivariant Chow ring $A^\ast_G(X)$ is then computed in the required cases via Chow rings of projective bundles and Chow ideals of degeneracy loci.

The geometry of $H(d)$ depends significantly on the parity of $d$. The quotient approaches and the presentations of $A^\ast(d)$ differ for $d$ even and odd. In the even case, a $\text{PGL}(2)$-quotient approach is taken. The geometry of algebraic $\text{BPG}(2)$ is studied as a necessary first step. There is an isomorphism of linear algebraic groups: $\text{PGL}(2) \cong \text{SO}(3)$. The space $\text{BSO}(3)$ is analyzed via conic geometry in projective space. It is no more difficult to study algebraic $\text{B}(n)$ and $\text{BSO}(n = 2k + 1)$ via higher dimensional quadrics. The equivariant Chow rings of these two series are computed. The $d$ odd case is simpler. In this case, the quotient group is taken to be a central extension of $\text{SL}(2)$ in $\text{GL}(2)$. Presentations of $A^\ast(d)$ in case $d$ is even and odd are determined in Theorems 1 and 2 respectively. The equivariant Chow rings of the groups $\text{O}(n)$ and $\text{SO}(n = 2k + 1)$ are computed in Theorem 3.

The equivariant Chow ring of $\text{O}(n)$ was first determined by B. Totaro using complex cobordism theory and topology. After the initial algebraic calculation of the ring of $\text{SO}(3)$ presented here, it was realized both Totaro’s methods and the $\text{SO}(3)$ computation generalize

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to $\text{SO}(2k+1)$. An algebraic approach to $B\text{SO}(3)$ is required for the application to Theorem 1.

In [P], Chow rings (with $\mathbb{Q}$-coefficients) of certain moduli spaces of maps are computed via equivariant Chow groups. The integral computations presented here were motivated by the calculations in [P]. These arguments show the equivariant constructions in [T] and [EG] can be used effectively to compute ordinary Chow rings of quotients.

0.2. Presentations of $A^*(d)$. Equivariant Chow theory is reviewed in section 2. Let $G$ be a reductive algebraic group. Let $G \times X \rightarrow X$ be a linearized algebraic group action on a nonsingular quasi-projective variety $X$. An equivariant Chow ring $A^*_G(X)$ is defined via algebraic approximations to $E^*G$ and $B^*G$.

Let $V$ be a fixed 2-dimensional $\mathbb{C}$-vector space. Let $P^1 \cong P(V)$. There is a canonical isomorphism $H^0(P^1, \mathcal{O}_{P^1}(d)) \cong \text{Sym}^d(V^*)$. Let $U \subset \bigoplus_0^d \text{Sym}^d(V^*)$ denote the non-degenerate locus (this is the open set consisting of linearly independent $(d+1)$-tuples of vectors of $\text{Sym}^d(V^*)$). $U$ parameterizes bases of the linear series of $\mathcal{O}_{P^1}(d)$ on $P^1$. There is a canonical $\text{GL}(V)$-action on $U$ with geometric quotient $H(d)$. The required existence results for the algebraic quotient problems encountered in this paper are developed in the Appendix (section 8). $\text{GL}(V)$ acts with finite stabilizers on $U$ (the stabilizer of a point $u \in U$ is the subgroup of scalar $d^{th}$ roots of unity). By a theorem of D. Edidin and W. Graham ([EG]), there is a canonical isomorphism of graded rings

$$A^*(d) \otimes_{\mathbb{Z}} \mathbb{Q} \cong A^*_{\text{GL}(V)}(U) \otimes_{\mathbb{Z}} \mathbb{Q}.$$ 

The equivariant Chow ring $A^*_{\text{GL}(V)}(U)$ is determined in section 4.3. $A^*_{\text{GL}(V)}(U)$ is generated (as a ring) in codimensions 1, 2 by elements $c_1$, $c_2$ respectively. There are $d+1$ relations given as follows. Let $S$ be a rank 2 bundle with Chern classes $c_1$ and $c_2$. The $d+1$ Chern classes of $\text{Sym}^d(S)$ are the relations. It is not difficult to see $A^*_{\text{GL}(V)}(U) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for $i > 0$.

**Proposition 1.** $A^*(d)$ is torsion in codimension $i > 0$.

Note that $\text{GL}(\text{Sym}^d(V^*))$ acts transitively on $U$ and $H(d)$ is a homogeneous space for $\text{GL}(\text{Sym}^d(V^*)) = \text{GL}(d+1)$.

Let $P(U) \subset P(\bigoplus_0^d \text{Sym}^d(V^*))$ be the projective non-degenerate locus. $P(U)$ is exactly the space of parameterized rational normal curves. There is a canonical $\text{PGL}(V)$-action on $P(U)$ with geometric quotient
This is a free action. Hence, there is a canonical isomorphism of graded rings (see [EG]):

\[ A^*(d) \cong A_{PGL(V)}^*(P(U)). \]

Assume \( d \geq 2 \) is even. Let \( d = 2n \) (where \( n \geq 1 \)). \( P(U) \to H(d) \) is a principal \( PGL(V) \)-bundle. Let \( S \) be the rank 3 algebraic vector bundle on \( H(d) \) obtained from the principal bundle \( P(U) \to H(d) \) and the representation \( Sym^2(V) \) of \( PGL(V) \). A discussion of algebraic principal bundles can be found in the Appendix. For \( 1 \leq i \leq 3 \), let \( c_i \in A^*(d) \) be the Chern classes of \( S \). Let \( H \in A^1(d) \) be the divisor class of curves meeting a fixed codimension 2 linear space in \( P^d \). Let \( L = nH \). In section 5, the equivariant Chow ring \( A_{PGL(V)}^*(P(U)) \) is evaluated in the even case.

**Theorem 1.** \( A^*(d = 2n) \) is generated by \( c_1, c_2, c_3 \), and \( L \). The first relations are:

\[
\begin{align*}
  c_1 &= 0 \\
  2c_3 &= 0.
\end{align*}
\]

There are \( d+1 \) additional relations given by the first \( d+1 \) Chern classes of the formal expansion:

\[
\frac{(1 + L)^{d+1} \cdot c(Sym^{n-2}(S))}{c(Sym^n(S))}.
\]

(If \( n = 1 \) or 2, then \( c(Sym^{n-2}(S)) = 1 \).)

It is easily seen from Theorem [1] that \( A^1(d = 2n) \cong \mathbb{Z}/(d + 1)\mathbb{Z} \) with generator \( L \). The equation \( L = nH \) can then be uniquely solved to obtain \( H = 2dL = -2L \).

Now assume \( d \geq 1 \) is odd. Let \( d = 2n - 1 \) (where \( n \geq 1 \)). Let

\[ det: GL(V) \to \mathbb{C}^* \]

be the determinant homomorphism. Let \( \mathbb{Z}/n\mathbb{Z} \subset \mathbb{C}^* \) be the subgroup of the \( n^{th} \) roots of unity. Let \( SL(V, n) = det^{-1}(\mathbb{Z}/n\mathbb{Z}) \). Consider again \( \bigoplus_{0}^{2n-1} Sym^{2n-1}(V^*) \). There is a canonical, \( GL(V) \)-equivariant, multilinear map

\[
\mu : \bigoplus_{0}^{2n-1} Sym^{2n-1}(V^*) \to \bigwedge^{2n} Sym^{2n-1}(V^*)
\]

given by the exterior product:

\[
(\omega_0, \omega_1, \ldots, \omega_{2n-1}) \mapsto \omega_0 \wedge \omega_1 \wedge \ldots \wedge \omega_{2n-1}.
\]

\( SL(V, n) \) acts trivially on the 1 dimensional space \( \bigwedge^{2n} Sym^{2n-1}(V^*) \). Let \( Y = \mu^{-1}(p) \) where \( 0 \neq p \in \bigwedge^{2n} Sym^{2n-1}(V^*) \). There is an
$\text{SL}(V,n)$-action on $Y$. In Lemma [2], it is shown this is a free action with geometric quotient $H(d)$. Hence, there is a canonical isomorphism of graded rings

$$A^*(d = 2n - 1) \cong A_{\text{SL}(V,n)}^*(Y).$$

Let $S$ now denote the rank 2 algebraic vector bundle obtained from the principal $\text{SL}(V,n)$-bundle $Y \to H(d)$ and the standard representation $V$. For $1 \leq i \leq 2$, let $c_i \in A^*(d)$ be the Chern classes of $S$. The equivariant Chow ring $A_{\text{SL}(V,n)}^*(Y)$ is evaluated in section 6.

**Theorem 2.** $A^*(d = 2n - 1)$ is generated by $c_1$ and $c_2$. The first relation is

$$nc_1 = 0.$$ 

There are $d+1$ additional relations given by the first $d+1$ Chern classes of $\text{Sym}^d(S)$.

It is easily seen that $A^1(d = 2n - 1) \cong \mathbb{Z}/n\mathbb{Z}$.

### 0.3. Chow rings of the Orthogonal Groups.

The Chow ring of a reductive algebraic group $G$ is, by definition, the equivariant Chow ring $A_G^*(\text{point})$. Let $O(n)$ and $\text{SO}(n)$ denote the orthogonal and special orthogonal algebraic groups. The equivariant calculations of Theorem 1 require knowledge of $B\text{PGL}(2)$. $\text{PGL}(2)$ is isomorphic to $\text{SO}(3)$.

The following Theorem will be established:

**Theorem 3.** The integral Chow ring of $O(n)$ is generated by the Chern classes $c_1, \ldots, c_n$ of the standard representation. The odd classes are 2-torsion:

$$A_{O(n)}^*(\text{point}) = \mathbb{Z}[c_1, \ldots, c_n]/(2c_1, 2c_3, 2c_5, \ldots).$$

The integral Chow ring of $\text{SO}(n = 2k + 1)$ is generated by the Chern classes $c_1, \ldots, c_n$ of the standard representation. The odd classes are 2-torsion and $c_1 = 0$:

$$A_{\text{SO}(n)}^*(\text{point}) = \mathbb{Z}[c_1, \ldots, c_n]/(c_1, 2c_3, 2c_5, \ldots).$$

The Chow ring of $\text{SO}(2k)$ is not generated by the Chern classes of the standard representation. The main difference in the odd and even cases is that $\text{SO}(2k + 1) \cong \mathbb{Z}/2\mathbb{Z} \times O(2k + 1)$ while such a product decomposition does not hold for $\text{SO}(2k)$. The methods of this paper do not yield a computation of $A_{\text{SO}(2k)}^*(\text{point})$. The Chow ring of $\text{SO}(n)$ has been computed with $\mathbb{Q}$-coefficients in [EG2].
0.4. Acknowledgments. Equivariant Chow groups were first defined in [T]. Thanks are due to D. Edidin, W. Graham, and B. Totaro for conversations in which the theory of equivariant Chow groups was explained. The author particularly wishes to thank B. Totaro for his insights on $O(n)$ and $SO(n)$. Discussions with W. Fulton on many related issues have also been helpful.

1. Chow Ideals of Degeneracy Loci

1.1. Presentations. For the Chow computations in this paper, presentations of four ideals associated to tautological degeneracy loci are needed.

Let $E$ be a rank $e$ vector bundle on a nonsingular algebraic variety $M$. We will consider two affine and two projective fibrations over $M$:

(i) $\oplus^1 E \to M$,
(ii) $P(\oplus^1 E) \to M$,
(iii) $\text{Sym}^2 E^* \to M$,
(iv) $P(\text{Sym}^2 E^*) \to M$.

The subspace projectivization is taken in (ii) and (iv). Let $r = e^2$ denote the rank of $\oplus^1 E$. Let $L$ in $A^1(P(\oplus^1 E))$ be the class of $O_{P(1)}$ obtained from the projectivization. The Chow ring of $P(\oplus^1 E)$ has a standard presentation:

$$A^*(M)[L] / (L^r + c_1(\oplus^1 E) \cdot L^{r-1} + \ldots + c_r(\oplus^1 E)).$$

Similarly, the Chow ring of $P(\text{Sym}^2 E^*)$ has a presentation:

$$A^*(M)[L] / (L^s + c_1(\text{Sym}^2 E^*) \cdot L^{s-1} + \ldots + c_s(\text{Sym}^2 E^*))$$

where $s = \frac{1}{2}(e^2 + e)$ is the rank of $\text{Sym}^2 E^*$ and $L$ is again the class of $O_{P(1)}$ obtained from the projectivization. The Chow rings of the affine fibrations (i) and (iii) are canonically isomorphic to $A^*(M)$.

There are intrinsic, fiberwise degeneracy loci in these fibrations. Let $D_1 \subset \oplus^1 E$ and $P(D_1) \subset P(\oplus^1 E)$ be the closed subvariety of linearly dependent $e$-tuples of vectors in the fibers of $E$. Let $D_2 \subset \text{Sym}^2 E^*$ and $P(D_2) \subset P(\text{Sym}^2 E^*)$ be the closed subvariety of degenerate quadratic forms on the fibers of $E$. Let

$$I_1 \subset A^*(\oplus^1 E) \cong A^*(M), \quad J_1 \subset A^*(P(\oplus^1 E)),$$

$$I_2 \subset A^*(\text{Sym}^2 E^*) \cong A^*(M), \quad J_2 \subset A^*(P(\text{Sym}^2 E^*))$$

be the ideals generated by classes supported on the degeneracy loci $D_1$, $P(D_1)$, $D_2$, and $P(D_2)$ respectively. In this section, simple sets of generators of the ideals $I_1$, $J_1$, $I_2$, and $J_2$ are determined.

The results of this section are essentially special cases of Pragacz’s presentations of the ideals of Chow classes supported on degeneracy
loci of bundle maps ([Pr]). Pragacz considers more general degenerate loci and obtains presentations of their universal Chow ideals via Schur $S$-polynomials. Actual (not universal) Chow ideal presentations are needed here. Since the geometry of the cases (i)-(iv) is particularly simple, the actual and the universal presentations coincide. A full proof will be given here.

For a rank $f$ bundle $F$, let $c(F) = 1 + c_1(F) + \ldots + c_f(F)$.

Lemma 1. $I_1 \subset A^*(M)$ is generated by $(\alpha_1, \ldots, \alpha_e)$ where

$$\frac{1}{c(E^*)} = 1 + \alpha_1 + \ldots + \alpha_e + \ldots.$$ 

Lemma 2. $J_1 \subset A^*(P(\bigoplus^e_1 E))$ is generated by $(\alpha'_1, \ldots, \alpha'_e)$ where

$$\frac{c(\bigoplus^e_1 O_P(1))}{c(E^*)} = 1 + \alpha'_1 + \ldots + \alpha'_e + \ldots.$$ 

Lemma 3. $I_2 \subset A^*(M)$ is generated by $(\beta_1, \ldots, \beta_e)$ where

$$\frac{c(E^*)}{c(E)} = 1 + \beta_1 + \ldots + \beta_e + \ldots.$$ 

Lemma 4. $J_2 \subset A^*(P(Sym^2 E^*))$ is generated by $(\beta'_1, \ldots, \beta'_e)$ where

$$\frac{c(E^* \otimes O_P(1))}{c(E)} = 1 + \beta'_1 + \ldots + \beta'_e + \ldots.$$ 

The proofs of Lemmas 1 – 4 are essentially the same. The first step is to find a tower of bundles dominating the degeneracy loci $D_1$, $P(D_1)$, $D_2$, and $P(D_2)$.

First consider $D_1$ and $P(D_1)$. Let $\eta : P(E^*) \to M$ be the projective bundle. A point $\xi \in P(E^*)$ is a pair $(m, h)$ where $m \in M$ and $h \in P(E^*_m)$. Let $B$ be the vector bundle on $P(E^*)$ determined as follows. The fiber of $B$ at the point $(m, h)$ is the linear subspace of $\bigoplus^e_1 E_m$ consisting of $e$-tuples of vectors annihilated by $h$. $B$ is a sub-bundle of $\eta^*(\bigoplus^e_1 E)$. There are canonical, proper, surjective projections:

$$\rho : B \to D_1 \subset \bigoplus^e_1 E,$$

$$P(\rho) : P(B) \to P(D_1) \subset P(\bigoplus^e_1 E).$$

There are stratifications of $D_1$ and $P(D_1)$ by the rank of the span of the $e$-tuple of vectors. Over these strata, $\rho$ and $P(\rho)$ are projective bundles. Hence $\rho$ and $P(\rho)$ induce surjections on the integral Chow rings via push-forward:

$$\rho_* : A^*(B) \to A^*(D_1),$$
Let this class be denoted by $J$.

Similarly, there is a presentation of $B$.

To prove Lemmas 1 and 2, it is sufficient to establish the equalities:

$$\rho_*(\zeta^i) = \alpha_i, \quad \rho_*(\zeta^{i-1}) = \alpha'_i$$

for $1 \leq i \leq e$.

First the equalities (3) for Lemma 1 are proven. By definition, $B \subset \eta^*(\oplus_1^e E)$. In fact, there is a natural exact sequence on $P(E^*)$:

$$0 \to B \to \eta^*(\oplus_1^e E) \to \oplus_1^e \mathcal{O}_{P(E^*)}(1) \to 0.$$  

As a first step, the class of $[B] \in A^*(\eta^*(\oplus_1^e E))$ is computed. Since $\eta^*(\oplus_1^e E)$ is a projective bundle over $\oplus_1^e E$, $A^*(\eta^*(\oplus_1^e E))$ is generated over $A^*(M)$ by $\zeta$ (which satisfies the Chern relation). By sequence (4) and Lemma 1 below, it follows that $[B] = \zeta_e \in A^*(\eta^*(\oplus_1^e E))$. Denote the natural projection $\eta^*(\oplus_1^e E)) \to \oplus_1^e E$ by $\phi$. There is a fundamental equality:

$$\rho_*(\zeta^i) = \phi_*(\zeta^{i-1} \cap [B]) \in A^*(M).$$

The right side is easy to calculate.

$$\phi_*(\zeta^i \cap [B]) = \phi_*(\zeta^{i-1+i}).$$

For $1 \leq i \leq e$, the latter is simply the $i$th Segre class of $E^*$. Lemma 1 is proved.

Lemma 2 is only slightly more complicated. The class of $[P(B)] \in A^*(P(\eta^*(\oplus_1^e E)))$ is computed. Again $A^*(P(\eta^*(\oplus_1^e E)))$ is generated...
over $A^*(P(\oplus_1 E))$ by $\zeta$. By sequence (4) and Lemma 3, it follows that $[P(B)] = (L + \zeta)^e \in A^*(P(\eta^*(\oplus_1 E)))$ where $L$ is the class of $O_P(\oplus_1 E)(1)$. Denote the natural projection $P(\eta^*(\oplus_1 E)) \to P(\oplus_1 E)$ by $P(\phi)$. There are equalities:

$$P(\rho)_*(\zeta^i - 1) = P(\phi)_*(\zeta^i - 1 \cap [B]) = P(\phi)_*(\zeta^i - 1 \cap (L + \zeta)^e)$$

in $A^*(P(\oplus_1 E))$. Lemma 3 now yields Lemma 2.

The degeneracy loci $D_2$ and $P(D_2)$ are considered next. The notation will parallel the notation used in the proofs of Lemmas 1 and 2. Let $\eta : P(E) \to M$ be the projective bundle. A point $\xi \in P(E)$ is a pair $(m, p)$ where $m \in M$ and $p \in P(E_m)$. Let $B$ be the vector bundle on $P(E)$ determined as follows. The fiber of $B$ at the point $(m, p)$ is the linear subspace of quadratic forms on $E_m$ singular at $p$. $B$ is a sub-bundle of $\eta^*(Sym^2 E^*)$. There are canonical, proper, surjective projections:

$$\rho : B \to D_2 \subset Sym^2 E^*,$$

$$P(\rho) : P(B) \to P(D_2) \subset Sym^2 E^*.$$ 

There are stratifications of $D_2$ and $P(D_2)$ by the rank of the quadratic form. Over these strata, $\rho$ and $P(\rho)$ are projective bundles. Hence $\rho$ and $P(\rho)$ induce surjections on the integral Chow rings via push-forward:

$$\rho_* : A^*(B) \to A^*(D_2),$$

$$P(\rho)_* : A^*(P(B)) \to A^*(P(D_2)).$$

Lemmas 3 and 4 are proven by computing the images of the generators of $A^*(B)$ and $A^*(P(B))$ respectively. As before,

$$I_2 = (\rho_*(1), \rho_*(\zeta^1), \rho_*(\zeta^2), \ldots, \rho_*(\zeta^{e-1})), $$

$$J_2 = (P(\rho)_*(1), P(\rho)_*(\zeta^1), P(\rho)_*(\zeta^2), \ldots, P(\rho)_*(\zeta^{e-1})).$$

To prove Lemma 3 and 4 it is sufficient to establish the equalities

$$(5) \quad \rho_*(\zeta^i - 1) = \beta_i, \quad P(\rho)_*(\zeta^i - 1) = \beta'_i$$

for $1 \leq i \leq e$. 

First the equalities (3) for Lemma 3 are proven. There is an exact sequence on $P(E)$:

$$(6) \quad 0 \to B \to \eta^*(Sym^2 E^*) \to E^* \otimes O_P(E)(1) \to 0.$$ 

The class of $[B] \in A^*(\eta^*(\oplus_1 E))$ is computed. $A^*(\eta^*(Sym^2 E^*))$ is generated over $A^*(M)$ by $\zeta$. By sequence (3) and Lemma 4 below, it follows that

$$[B] = c_1(E^* \otimes O_P(E)(1)) \in A^*(\eta^*(Sym^2 E)).$$
Denote the natural projection \( \eta^*(\text{Sym}^2 E^*) \to \text{Sym}^2 E^* \) by \( \phi \). There is an equality:
\[
\rho_*(\zeta^{i-1}) = \phi_*(\zeta^{i-1} \cap [B]) \in A^*(M).
\]

Lemma 3 now yields Lemma 4.

Lemma 4 is established next. By sequence (6) and Lemma 5 below, it follows that
\[
[P(B)] = c_q \left( \frac{c(E^* \otimes \mathcal{O}_{P(F)}(1))}{c(\mathcal{O}_{P(Sym^2 E^*)}(-1))} \right) \in A^*(P(\eta^*(\text{Sym}^2 E^*))).
\]

There is an equality (since \( E^* \otimes \mathcal{O}_{P(F)}(1) \) is a rank \( e \) bundle):
\[
c_e \left( \frac{c(E^* \otimes \mathcal{O}_{P(F)}(1))}{c(\mathcal{O}_{P(Sym^2 E^*)}(-1))} \right) = c_e(\text{c_g}(E^* \otimes \mathcal{O}_{P(F)}(1)) \otimes \mathcal{O}_{P(Sym^2 E^*)}(1)).
\]

Denote the natural projection \( P(\eta^*(\text{Sym}^2 E^*)) \to P(\text{Sym}^2 E^*) \) by \( P(\phi) \). There is an equality:
\[
P(\rho_*) (\zeta^{i-1}) = P(\phi_*) (\zeta^{i-1} \cap [B]) \in A^*(P(\text{Sym}^2 E^*)).
\]

Lemma 3 now yields Lemma 4.

1.2. **Lemmas.** The following Lemmas were used in the proofs of Lemmas 1 – 4. Let \( F \to N \) be a vector bundle on a nonsingular algebraic variety \( N \).

**Lemma 5.** Let \( 0 \to B \to F \to Q \to 0 \) be an exact sequence of bundles on \( N \). Let \( q \) be the rank of \( Q \). The class \( [B] \in A^*(F) \cong A^*(N) \) is determined by
\[
[B] = c_q(Q).
\]
The class \([P(B)] \in A^*(P(F))\) is determined by
\[
[P(B)] = c_q \left( \frac{c(Q)}{c(\mathcal{O}_{P(F)}(-1))} \right).
\]

**Proof.** This is an application of the Thom-Porteous formulas for degeneracy loci of bundle maps (see [F]). \( \square \)

Let \( f \) be the rank \( F \). Let \( \phi : P(F) \to N \) be the projection.

**Lemma 6.** Let \( G \) be a bundle of rank \( g = f \) on \( N \). Let \( \zeta = c_1(\mathcal{O}_{P(F)}(1)) \). Let \( \gamma_i \) be determined by
\[
\frac{c(G)}{c(F)} = 1 + \gamma_1 + \ldots + \gamma_f + \ldots.
\]
Then, for \( 1 \leq i \leq f \), \( \gamma_i = \phi_*(\zeta^{i-1} \cap c_f(G \otimes \mathcal{O}_{P(F)}(1))) \).

**Proof.** A simple Segre class argument yields the result. \( \square \)
2. Equivariant Chow Groups

Let $G$ be a group. Let $G \times X \to X$ be a left group action. In topology, the $G$-equivariant cohomology of $X$ is defined as follows. Let $EG$ be a contractible topological space equipped with a free left $G$-action and quotient $EG/G = BG$. Consider the left action of $G$ on $X \times EG$ defined by:

$$g(x, b) = (g(x), g(b)).$$

$G$ acts freely on $X \times EG$. Let $X^GEG$ be the (topological) quotient. The $G$-equivariant cohomology of of $X$, $H^*_G(X)$, is defined by:

$$H^*_G(X) = H^*_{sing}(X^GEG).$$

If $X$ is a locally trivial principal $G$-bundle, then $X^GEG$ is a locally trivial fibration of $EG$ over the quotient $X/G$. In this case, $X^GEG$ is homotopy equivalent to $X/G$ and

$$H^*_G(X) = H^*_{sing}(X^GEG) \cong H^*_{sing}(X/G).$$

For principal bundles, computing the equivariant cohomology ring is equivalent to computing the cohomology of the quotient.

There is an analogous equivariant theory of Chow groups developed by B. Totaro in case $X$ is a point and generalized by D. Edidin and W. Graham to arbitrary $X \ (\cite{T}, \cite{EG})$. Let $G$ be a reductive algebraic group. Let $G \times X \to X$ be a linearized algebraic $G$-action. The algebraic analogue of $EG$ is attained by approximation. Let $V$ be a $\mathbb{C}$-vector space. Let $G \times V \to V$ be an algebraic representation of $G$. Let $W \subset V$ be a $G$-invariant open set satisfying:

(i) The complement of $W$ in $V$ is of codimension greater than $q$.
(ii) $G$ acts freely on $W$ (see the Appendix for the definition).
(iii) There exists a geometric quotient $W \to W/G$.

$W$ is an approximation of $EG$ up to codimension $q$. By (iii) and the assumption of linearization, a geometric quotient $X^GW$ exists as an algebraic variety. Let $d = dim(X)$, $e = dim(X^GW)$. The equivariant Chow groups are defined by:

$$A^G_{d-j}(X) = A_{e-j}(X^G W)$$

for $0 \leq j \leq q$. An argument is required to check these equivariant Chow groups are well-defined (see [EG]). The basic functorial properties of equivariant Chow groups are established in [EG]. In particular, if $X$ is nonsingular, there is a natural intersection ring structure on $A^G_1(X)$. 

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Let $Z$ be a variety of dimension $z$. For notational convenience, a superscript will denote the Chow group codimension:

$$A^G_{z-j}(Z) = A^j_G(Z), \ A_{z-j}(Z) = A^j(Z).$$

In particular, equation (7) becomes:

$$\forall 0 \leq j \leq q, \ A^j_G(X) = A^j(X \times^G W).$$

The following result of [EG] will be used.

**Proposition 2.** Let $C$ be the ground field of complex numbers. Let $X$ be a quasi-projective variety. Let $G$ be a reductive group. Let $G \times X \to X$ be a linearized proper $G$-action. Let $X \to X/G$ be a quasi-projective geometric quotient.

(i) If the action is free, then there is a canonical isomorphism of graded rings:

$$A^*_G(X) \sim = A^*(X/G).$$

(ii) If $G$ acts with finite stabilizers on $X$, then there is a canonical isomorphism of graded rings:

$$A^*_G(X) \otimes Q \sim = A^*(X/G) \otimes Q.$$

Proposition 2 is a characteristic 0 specialization of Theorem 2 of [EG].

3. The Chow Rings of $O(k)$ and $SO(2k + 1)$

3.1. $BO(V)$ and $BSO(V)$. Let $V$ be a complex vector space equipped with a non-degenerate quadratic form. Let $O(V)$, $SO(V)$ be the orthogonal and special orthogonal groups respectively. Approximations to $EO(V)$ and $ESO(V)$ are obtained via direct sums of the representation $V^*$. Let $m \gg 0$ and let

$$W_m \subset \bigoplus^m V^*$$

denote the spanning locus. $W_m$ is the locus of $m$-tuples of vectors of $V^*$ which span $V^*$. The natural actions of $O(V)$ and $SO(V)$ on $W_m$ are free and have a geometric quotients (see section 8). The codimension of the complement of $W_m$ in $\bigoplus^m V^*$ is $m - \dim(V^*) + 1$. $W_m$ is an approximation of $EO(V)$ and $ESO(V)$ up to codimension $m - \dim(V^*)$.

By the general theory of equivariant Chow groups (section 8), we have approximations:

$$BO(V) = m \xrightarrow{\lim} \infty \ W_m/O(V),$$

$$BSO(V) = m \xrightarrow{\lim} \infty \ W_m/SO(V).$$
In this section, equivariant Chow rings of $\text{O}(k)$ and $\text{SO}(2k + 1)$ are computed via the approximations

$$A^*_\text{O}(V)(\text{point}) = \lim_{m \to \infty} A^*(W_m/\text{O}(V)),$$

$$A^*_\text{SO}(V)(\text{point}) = \lim_{m \to \infty} A^*(W_m/\text{SO}(V)),$$

and the degeneracy loci results of section 1.1.

3.2. The Chow Ring of $\text{O}(k)$. Let $k \geq 1$. Let $V \cong \mathbb{C}^k$ be equipped with a non-degenerate quadratic form $Q$ preserved by $\text{O}(k)$. The quotient $W_m/\text{O}(k)$ can be explicitly realized as follows. Let $G(k, m)$ be the Grassmannian of linear $k$-spaces in $\mathbb{C}^m$. Let $S \to G(k, m)$ be the tautological sub-bundle. Let $Y_m \subset \text{Sym}^2 S^*$ be the open locus of non-degenerate quadratic forms on the fibers of $S$.

Lemma 7. There is canonical $\text{O}(k)$-invariant map $\tau : W_m \to Y_m$ which induces an isomorphism $W_m/\text{O}(k) \cong Y_m$.

Proof. Let $w \in W_m$. By the definitions, $w$ naturally induces an injection $\iota_w : V \to \mathbb{C}^m$. The quadratic form $Q$ then induces a non-degenerate quadratic form $\iota_w(Q)$ on $\iota_w(V)$. Let

$$\tau(w) = \iota_w(Q) \in Y_m.$$ 

It is easily checked that $\tau$ is an algebraic morphism. Let $g \in \text{O}(k)$. Then,

$$\iota_{g(w)} = \iota_w \circ g : V \to \mathbb{C}^m.$$

Hence, $\tau$ is $\text{O}(k)$-invariant. Since the fibers of $\tau$ are exactly the $\text{O}(k)$ orbits, the induced map

$$W_m/\text{O}(k) \to Y_m$$

is a bijective morphism of nonsingular complex algebraic varieties and thus an algebraic isomorphism.

$Y_m$ is an approximation to $B\text{O}(k)$ up to codimension $m - k$. $W_m \to Y_m$ is a principal $\text{O}(k)$-bundle (see the Appendix). The pull-back of the tautological sub-bundle $S \to G(k, m)$ to $Y_m$ is the vector bundle on $Y_m$ induced by the principal $\text{O}(k)$-bundle $W_m \to Y_m$ and the representation $V$. The Chow ring of the Grassmannian $G(k, m)$ is freely generated by the Chern classes $c_1, \ldots, c_k$ of $S$ up to codimension $m - k$ (the relations start in codimension $m - k + 1$). By Lemma 3, the Chow ring of $Y_m$ is isomorphic to

$$\mathbb{Z}[c_1, \ldots, c_k] / (\beta_1, \ldots, \beta_k)$$
up to codimension $m - k$ where
\[
\frac{c(S^*)}{c(S)} = 1 + \beta_1 + \ldots + \beta_k + \ldots.
\]
Induction and simple algebra establishes:
\[
(\beta_1, \ldots, \beta_k) = (2c_1, 2c_3, 2c_5, \ldots).
\]
The Chow ring limit $m \to \infty$ of $A^*(Y_m)$ is now easily seen to yield:
\[
A^*_{O(k)}(\text{point}) = \mathbb{Z}[c_1, \ldots, c_k] / (2c_1, 2c_3, 2c_5, \ldots).
\]
Theorem 3 is proven for $O(k)$.

3.3. The Chow ring of $SO(2k + 1)$. Let $k \geq 0$. Let $V \cong \mathbb{C}^{2k+1}$ be equipped with a non-degenerate quadratic form preserved by $SO(2k + 1) \subset GL(V)$.

Let $\mathbb{C}^* \subset GL(V)$ be the scalars. Since $V$ is odd dimensional
\[
\mathbb{C}^* \cap SO(2k + 1) = \{1\}, \quad \mathbb{C}^* \times SO(2k + 1) \subset GL(V).
\]
The approximations $W_m$ to $ESO(2k + 1)$ are used. There is a natural free $GL(V)$-action on $W_m$ which induces a free $SO(2k+1)$-action and a free scalar $\mathbb{C}^*$-action on $W_m$. The $SO(2k+1)$-action and the $\mathbb{C}^*$-action commute. There is a commutative diagram:
\[
\begin{array}{ccc}
W_m & \longrightarrow & W_m/\text{SO}(2k + 1) \\
\downarrow & & \downarrow \\
W_m/\mathbb{C}^* & \longrightarrow & W_m/\mathbb{C}^* \times \text{SO}(2k + 1)
\end{array}
\]
All morphisms are group quotients: the horizontal maps are free $SO(2k+1)$-quotients, the vertical maps are free $\mathbb{C}^*$-quotients. See the Appendix for a discussion of these algebraic quotient problems.

The quotients in diagram (9) are analyzed. Let $S \to G(2k+1, m)$ be the tautological sub-bundle over the Grassmannian. By an argument identical to Lemma 7, it is seen that
\[
W_m/\mathbb{C}^* \times \text{SO}(2k + 1) \cong Z_m
\]
where $Z_m \subset P(Sym^2 S^*)$ is the locus of non-degenerate quadratic forms on the fibers of $S$. Hence, $W_m/\text{SO}(2k + 1) \to Z_m$ is a $\mathbb{C}^*$-bundle. Let $N \to Z_m$ be the line bundle associated to this $\mathbb{C}^*$-bundle. On the left side of the diagram,
\[
W_m/\mathbb{C}^* \subset P(\oplus_1^m V^*)
\]
is the projective spanning locus. $A^1(W_m/\mathbb{C}^*) = \mathbb{Z}$ and $W_m \to W_m/\mathbb{C}^*$ is the $\mathbb{C}^*$-bundle associated to the generator $O_P(-1)$ of $A^1(W_m/\mathbb{C}^*)$.

Lemma 8. $A^1(Z_m) \cong \mathbb{Z}$ and $c_1(N)$ is a generator.
Proof. Consider the inclusion $Z_m \subset \mathbf{P}(\text{Sym}^2 S^*)$. Let
\[
\tau : W_m / \mathbb{C}^* \rightarrow Z_m \subset \mathbf{P}(\text{Sym}^2 S^*)
\]
be the natural map. Let \( \overline{N} \) denote an extension of \( N \) to \( A^1(\mathbf{P}(\text{Sym}^2 S^*)) \). Since \( \tau^*(\overline{N}) = \mathcal{O}_\mathbf{P}(-1) \) generates \( A^1(W_m / \mathbb{C}^*) \sim \mathbb{Z} \), the kernel \( K \) of \( \tau^* : A^1(\mathbf{P}(\text{Sym}^2 S^*)) \rightarrow A^1(W_m / \mathbb{C}^*) \) is isomorphic to \( \mathbb{Z} \). The class \([D]\) of the locus of degenerate quadratic forms is in \( K \). \( A^1(\mathbf{P}(\text{Sym}^2 S^*)) \) is generated by \( c_1(S) \) and \( L \) is the canonical class \( \mathcal{O}_\mathbf{P}(1) \). The class of \([D]\) is \(-2c_1 + (2k + 1)L\) which is not divisible in \( A^1(\mathbf{P}(\text{Sym}^2 S^*)) \). Hence \( K \) is generated by \([D]\). Therefore \( \tau^* : A^1(Z_m) \rightarrow A^1(W_m / \mathbb{C}^*) \) is an isomorphism and \( c_1(N) \) is a generator of \( A^1(Z_m) \). 

There is now enough information to compute the Chow ring of the approximation \( W_m / \text{SO}(2k + 1) \) to \( B\text{SO}(2k + 1) \). As before, \( W_m / \text{SO}(2k + 1) \) is an approximation up to codimension \( m - (2k + 1) \). The Chow ring of \( G(2k + 1, m) \) is freely generated by the Chern classes \( c_1, \ldots, c_{2k + 1} \) of the tautological sub-bundle \( S \) up to codimension \( m - (2k + 1) \). By Lemma 9, the Chow ring of \( Z_m \) (up to codimension \( m - (2k + 1) \)) has a presentation:
\[
\mathbb{Z}[c_1, \ldots, c_{2k + 1}, L]/(p(L), \beta'_1, \ldots, \beta'_{2k+1})
\]
where \( L \) is the class of \( \mathcal{O}_\mathbf{P}(1) \), \( p(L) \) is the Chern polynomial satisfied by \( L \), and
\[
\frac{c(S^* \otimes \mathcal{O}_\mathbf{P}(1))}{c(S)} = 1 + \beta'_1 + \ldots + \beta'_{2k+1} + \ldots.
\]
Finally, since \( W_m / \text{SO}(2k + 1) \) is the total space of the \( \mathbb{C}^* \)-bundle associated to the line bundle \( N \rightarrow Z_m \),
\[
(10) \quad \mathbb{Z}[c_1, \ldots, c_{2k + 1}, L]/(c_1(N), p(L), \beta'_1, \ldots, \beta'_{2k+1})
\]
is a presentation of the Chow ring of \( W_m / \text{SO}(2k + 1) \) (up to codimension \( m - (2k + 1) \)). Since \( c_1(N) \) generates \( A^1(Z_m) \) and the pair \{\( c_1, L \)\} also generate \( A^1(Z_m) \), \([\mathcal{U}]\) is equivalent to:
\[
\mathbb{Z}[c_1, \ldots, c_{2k + 1}, L]/(c_1, L, p(L), \beta'_1, \ldots, \beta'_{2k+1}).
\]
By the definitions of \( p(L) \) and the elements \( \beta'_i \), there is an equality of ideals
\[
(c_1, L, p(L), \beta'_1, \ldots, \beta'_{2k+1}) = (c_1, L, c_s(\text{Sym}^2 S^*), \beta_1, \ldots, \beta_{2k+1})
\]
where \( s = \text{rank}(\text{Sym}^2 S^*) \) and the \( \beta_i \) are determined by \([\mathcal{S}]\).

Lemma 9. \( c_s(\text{Sym}^2 S^*) \in (\beta_1, \ldots, \beta_{2k+1}) \).
Proof. Consider the total space $\text{Sym}^2 S^*$. There is an isomorphism $A^*(\text{Sym}^2 S^*) \cong A^*(G(2k + 1, m))$. The pull-back of the bundle 
$$\text{Sym}^2 S^* \to G(2k + 1, m)$$
to the total space $\text{Sym}^2 S^*$ has a canonical section $\tau$. The zero scheme of $\tau$ is contained in the locus of degenerate quadratic forms $D \subset \text{Sym}^2 S^*$. Also, the zero scheme of $\tau$ represents the class $c_\sigma(\text{Sym}^2 S^*) \in A^*(\text{Sym}^2 S^*)$. Therefore, $c_\sigma(\text{Sym}^2 S^*) \in I_2$. The proof is complete by Lemma 3. □

As before, $(\beta_1, \ldots, \beta_{2k+1}) = (2c_1, 2c_3, 2c_5, \ldots, 2c_{2k+1})$. Hence, the Chow ring of $W_m/\text{SO}(2k + 1)$ up to codimension $m - (2k + 1)$ has a presentation:

$$\mathbb{Z}[c_1, \ldots, c_{2k+1}]/(c_1, 2c_3, 2c_5, \ldots, 2c_{2k+1}).$$
The limit process yields Theorem 3 for $\text{SO}(2k + 1)$.

4. The Proof of Proposition 1

4.1. We follow the notation of section 0.2. Let $V$ be a fixed 2-dimensional $\mathbb{C}$-vector space. Let $P^1 \cong \mathbb{P}(V)$. Let

$$U \subset \bigoplus_{0}^{d} \text{Sym}^d(V^*)$$
denote the non-degenerate locus parameterizing bases of the linear series of $\mathcal{O}_{P^1}(d)$ on $P^1$. $\text{GL}(V)$ acts on $U$ properly with finite stabilizers and geometric quotient (see the Appendix) isomorphic to $H(d)$. By Proposition 4,

$$A^*(d) \otimes \mathbb{Q} \cong A^*_{\text{GL}(V)}(U) \otimes \mathbb{Q}. \tag{11}$$

By the definition of equivariant Chow groups,

$$A^*_{\text{GL}(V)}(U) = A^*(U \times^{\text{GL}(V)} \text{EGL}(V)).$$

4.2. The Chow Rings of $\text{GL}(V)$ and $\text{SL}(V, n)$. Algebraic approximations to $\text{EGL}(V)$ are easily found. Since related results about the groups $\text{SL}(V, n)$ are need in section 6 a unified development is presented here. Recall $SL(V, n) \subset \text{GL}(V)$ is defined to be $\text{det}^{-1}(\mathbb{Z}/n\mathbb{Z})$ where $\text{det} : \text{GL} \to \mathbb{C}^*$ is the determinant homomorphism and $\mathbb{Z}/n\mathbb{Z}$ is the group of $n^{th}$ roots of unity.

As in the orthogonal cases, the easiest approach to $\text{EGL}(V)$ and $\text{ESL}(V, n)$ is via sums of the representation $V^*$. As before, let $m \gg 0$ and let

$$W_m \subset \bigoplus_{1}^{m} V^*.$$
be the spanning locus. The induced $\text{GL}(V)$ and $\text{SL}(V, n)$-actions on $W_m$ are free and have geometric quotients (see the Appendix) which approximate $\text{BGL}(V)$ and $\text{BSL}(V, n)$ up to codimension $m − 2$.

It is easily seen that $W_m/\text{GL}(V) \cong G(2, m)$. Since the Chow ring of this Grassmannian (up to codimension $m − 2$) is freely generated by the Chern classes $c_1$ and $c_2$ of the tautological sub-bundle,

$$A^*(W_m/\text{GL}(V)) \cong \mathbb{Z}[c_1, c_2]$$

up to codimension $m − 2$. Taking the $m \to \infty$ limit,

$$A^*_\text{GL}(V)\text{(point)} \cong \mathbb{Z}[c_1, c_2].$$

Similarly, $W_m/\text{SL}(V, n)$ is the total space of the $n^{th}$ tensor power of the line bundle $\bigwedge^2 S$ over $G(2, m)$. Hence up to codimension $m − 2$,

$$A^*(W_m/\text{SL}(V, n)) \cong \mathbb{Z}[c_1, c_2]/(nc_1).$$

Taking the $m \to \infty$ limit,

$$A^*_\text{SL}(V, n)\text{(point)} \cong \mathbb{Z}[c_1, c_2]/(nc_1).$$

4.3. Proposition. The quotient $U \times^{\text{GL}(V)} E \text{GL}(V)$ is analyzed via approximation. $V \times^{\text{GL}(V)} W_m$ is the tautological sub-bundle $S$ over $G(2, m)$.

$$U \times^{\text{GL}(V)} W_m \subset \oplus_0^d \text{Sym}^d(V^*) \times^{\text{GL}(V)} E$$

is the non-degenerate open locus in the total space of the bundle $\oplus_0^d \text{Sym}^d(S^*)$ over $G(2, m)$. By Lemma 1, there is an isomorphism

$$A^*(U \times^{\text{GL}(V)} W_m) \cong \mathbb{Z}[c_1, c_2]/(\alpha_1, \ldots, \alpha_{d+1})$$

up to codimension $m − 2$ where

$$\frac{1}{c(\text{Sym}^d(S))} = 1 + \alpha_1 + \ldots + \alpha_{d+1} + \ldots.$$

The ideal generated by $(\alpha_1, \ldots, \alpha_{d+1})$ is equal to the ideal generated by the first $d + 1$ Chern classes of $\text{Sym}^d(S)$.

Taking the $m \to \infty$ limit, a presentation of $A^*_{\text{GL}(V)}(U)$ is obtained. $A^*_{\text{GL}(V)}(U)$ is generated (as a ring) in codimensions 1, 2 by elements $c_1$, $c_2$ respectively. There are $d + 1$ relations given as follows. Let $S$ be a rank 2 bundle with Chern classes $c_1$ and $c_2$. The $d + 1$ Chern classes of $\text{Sym}^d(S)$ are the relations.

**Lemma 10.** $A^*_{\text{GL}(V)}(U) \otimes \mathbb{Q}$ is zero is positive codimension.
Proof. A standard calculation yields:

\[
\begin{align*}
  c_1(\text{Sym}^d(S)) &= \frac{d(d+1)}{2}c_1, \\
  c_2(\text{Sym}^d(S)) &= \frac{d(d-1)(d+1)(3d+2)}{24}c_1^2 + \frac{d(d+1)(d+2)}{6}c_2.
\end{align*}
\]

Since the coefficients of \(c_1\) and \(c_2\) never vanish for positive \(d\) in equations (12) and (13) respectively, the first two Chern classes of \(\text{Sym}^d(S)\) generate the ideal \((c_1, c_2)\) in \(\mathbb{Q}[c_1, c_2]\).

Lemma 10 and the isomorphism (11) establish Proposition 1.

5. \(A^*(d)\), \(d\) Even

The notation of section 0.2 is used. Let \(d = 2n\) (where \(n \geq 1\)). Let 
\[V \cong \mathbb{C}^2.\] There is a free \(\text{PGL}(V)\)-action on \(\mathbb{P}(U) \subset \mathbb{P}(\oplus_0^d \text{Sym}^d V^*)\) with geometric quotient (see the Appendix) isomorphic to \(H(d)\). By Proposition 4,

\[A^*(d) \cong A^*_{\text{PGL}(V)}(\mathbb{P}(U)).\]

By the definition of equivariant Chow groups,

\[A^*_{\text{PGL}(V)}(\mathbb{P}(U)) \cong A^*(\mathbb{P}(U) \times_{\text{PGL}(V)} \text{EPGL}(V)).\]

The Chow ring \(A^*(\mathbb{P}(U) \times_{\text{PGL}(V)} \text{EPGL}(V))\) is computed in this section for \(d = 2n\).

Consider the 3-dimensional representation \(\text{Sym}^2(V)\) of \(\text{PGL}(V)\). This representation leaves invariant a unique (up to \(\mathbb{C}^*\)) quadratic form \(Q\) on \(\text{Sym}^2(V)\). A group isomorphism \(\text{PGL}(V) \cong \text{SO}(3)\) is induced by this quadratic form. The dual of the standard 3-dimensional representation of \(\text{SO}(3)\) corresponds to the representation \(\text{Sym}^2(V^*)\) of \(\text{PGL}(V)\). Let 

\[A_m \subset \oplus_1^m \text{Sym}^2(V^*)\]

be the spanning locus. The approximations \(A_m/\text{PGL}(V)\) to \(B\text{PGL}(V)\) correspond exactly to the approximations \(W_m/\text{SO}(3)\) to \(B\text{SO}(3)\) defined in section 3.4. \(A_m/\text{PGL}(V)\) is therefore the total space of a \(\mathbb{C}^*\)-bundle \(N \to Z_m\). \(Z_m\) is the open set of non-degenerate quadratic forms in \(\mathbb{P}(\text{Sym}^2(S^*))\) over the Grassmannian \(G(3, m)\). Let \(B_m\) denote this approximation to \(B\text{PGL}(V)\).

\(\text{Sym}^d(V^*)\) is a \(\text{PGL}(V)\) representation for \(d\) even \((\text{not for} \ d \text{ odd})\). Hence,

\[\text{Sym}^d(V^*) \times_{\text{PGL}(V)} A_m \]

is a rank \(d + 1\) vector bundle \(F_d \to B_m\). The quotient

\[\mathbb{P}(U) \times_{\text{PGL}(V)} A_m \subset \mathbb{P}(\oplus_0^d \text{Sym}^d(V^*)) \times_{\text{PGL}(V)} A_m\]
is simply the projective non-degenerate locus in $\mathbf{P}(\oplus_0^d F_d)$.

The first step is to identify the bundle $F_d \to B_m$. There is a tautological sub-bundle $S \to B_m$ obtained from the Grassmannian. There is a tautological equivalence $S^* \cong F_2$. More generally, there is a tautological sequence on $\mathbf{P}(Sym^2(S^*))$:

$$0 \to \mathcal{O}_\mathbf{P}(-1) \otimes Sym^{n-2}(S^*) \to Sym^n(S^*) \to Q_n \to 0$$

for all $n \geq 2$. Let $([q], P) \in \mathbf{P}(Sym^2(S^*)))$ where $P \subset \mathbb{C}^m$ is a linear 3-space and $0 \neq q \in Sym^2(P^*)$. The fiber of $\mathcal{O}_\mathbf{P}(-1)$ over $([q], P)$ is simply $\mathbb{C} \cdot q$. The left inclusion in sequence (14) is determined by the canonical multiplication map:

$$0 \to \mathbb{C} \cdot q \otimes Sym^{n-2}(P^*) \to Sym^n(P^*)$$

Again, there is a tautological equivalence $F_{2n} \cong Q_n$ on $B_m$.

Note $A^1(B_m) = 0$ by Lemma [3]. The Chern polynomial of $F_d$ on $B_m$ is therefore:

$$c(F_d) = \frac{c(Sym^n(S^*))}{c(Sym^{n-2}(S^*))}.$$ 

Now, by Lemma [2] a presentation of $A^*(\mathbf{P}(U) \times_{\text{PGL}(V)} B_m)$ up to codimension $m - 3$ is obtained by

$$A^*(B_m)[\mathcal{L}]/(p(\mathcal{L}), \alpha_1', \ldots, \alpha_{d+1}')$$

where $\mathcal{L}$ is the class of $\mathcal{O}_\mathbf{P}(\oplus_0^d F_d)(1)$ and

$$\frac{(1 + \mathcal{L})^{d+1}}{c(F_d^*)} = \frac{(1 + \mathcal{L})^{d+1} \cdot c(Sym^{n-2}S)}{c(Sym^nS)} = 1 + \alpha_1' + \ldots + \alpha_{d+1}'. \ldots$$

By the presentation of $A^*(B_m)$ in section 3.3, it follows

$$A^*(\mathbf{P}(U) \times_{\text{PGL}(V)} B_m) \cong \mathbb{Z}[c_1, c_2, c_3, \mathcal{L}]/(p(\mathcal{L}), c_1, 2c_3, \alpha_1', \ldots, \alpha_{d+1}')$$

up to codimension $m - 3$. Taking the $m \to \infty$ limit,

$$A^*(d) \cong A^*_{\text{PGL}(V)}(\mathbf{P}(U)) \cong \mathbb{Z}[c_1, c_2, c_3, \mathcal{L}]/(p(\mathcal{L}), c_1, 2c_3, \alpha_1', \ldots, \alpha_{d+1}').$$

The relation $p(\mathcal{L})$ is of codimension $(d + 1)^2$. Since the dimension of $H(d)$ is $d^2 + 2d - 3 = (d + 1)^2 - 4$ and the generators $c_1, c_2, c_3,$ and $\mathcal{L}$ have dimension at most 3, $p(\mathcal{L})$ is a relation among classes that are already zero. Hence,

$$A^*(d) \cong A^*_{\text{PGL}(V)}(\mathbf{P}(U)) \cong \mathbb{Z}[c_1, c_2, c_3, \mathcal{L}]/(c_1, 2c_3, \alpha_1', \ldots, \alpha_{d+1}').$$

Following [EG], there is natural map

$$\mathbf{P}(U) \times_{\text{PGL}(V)} A_m \to \mathbf{P}(U)/\text{PGL}(V) \cong H(d)$$

which expresses $\mathbf{P}(U) \times_{\text{PGL}(V)} A_m$ as an open set of a vector bundle over $H(d)$. This fibration induces an isomorphism on Chow rings (up
to codimension \( m - 3 \). The classes \( c_i \in A^*(d) \) are easily identified via this isomorphism (up to codimension \( m - 3 \)):

\[
A^*(\mathbf{P}(U) \times^{\text{PGL}(V)} A_m) \cong A^*(d).
\]

They are the Chern classes of the vector bundle obtained from the principal \( \text{PGL}(V) \)-bundle \( \mathbf{P}(U) \to H(d) \) and the representation \( \text{Sym}^2(V) \). Let \( \mathcal{H} \in A^1(d) \) be the class of curves meeting a fixed codimension 2 linear space \( P \) of \( \mathbf{P}^d \). \( \mathcal{H} \) corresponds via the isomorphism (10) to a resultant class in \( A^1(\mathbf{P}(U) \times^{\text{PGL}(V)} A_m) \). Routine calculations show \( \mathcal{H} = 2dL \) where \( 2d \) is the degree of the resultant of degree \( d \) polynomials. Since \( 2nH = nH = 2dL = (d^2 - 1 + 1)L = (d - 1)(d + 1)L + L = L \).

The proof of Theorem 5 is complete.

6. \( A^*(d) \), \( d \) Odd

Let \( d = 2n - 1 \) (where \( n \geq 1 \)). Let \( V \cong \mathbb{C}^2 \). There is a canonical, \( \text{GL}(V) \)-equivariant, multilinear map

\[
\mu : \bigoplus_{0}^{2n-1} \text{Sym}^{2n-1}(V^*) \to \bigwedge^{2n} \text{Sym}^{2n-1}(V^*)
\]
given by the exterior product (see section 0.2).

**Lemma 11.** The \( \text{SL}(V, n) \)-action on \( \bigwedge^{2n} \text{Sym}^{2n-1}(V^*) \) is trivial.

**Proof.** Since the 1-dimensional representations of \( \text{SL}(V) \) are trivial, the action of \( \text{SL}(V) \) on \( \bigwedge^{2n} \text{Sym}^{2n-1}(V^*) \) is certainly trivial. Let \( H \subset \text{SL}(V, n) \) be the subgroup of scalars. \( H \) is the multiplicative group of scalar \( 2n^{th} \) roots of unity. Let \( \xi \in H \) be a scalar. \( \xi \) acts on \( \bigwedge^{2n} \text{Sym}^{2n-1}(V^*) \) by the scalar \( \xi^{(2n)(2n-1)} = 1 \). It is easily checked that \( \text{SL}(V, n) \) is generated (as a group) by \( H \) and \( \text{SL}(V) \). Hence, the \( \text{SL}(V, n) \)-action is trivial. \( \square \)

Let \( Y = \mu^{-1}(p) \) where \( 0 \neq p \in \bigwedge^{2n} \text{Sym}^{2n-1}(V^*) \). There is an \( \text{SL}(V, n) \)-action on \( Y \).

**Lemma 12.** The \( \text{SL}(V, n) \)-action on \( Y \) is free with geometric quotient \( H(d) \).

**Proof.** Certainly \( \text{SL}(V, n) \) acts on \( Y \) since the \( \text{SL}(V, n) \)-action on \( \bigwedge^{2n} \text{Sym}^{2n-1}(V^*) \) is trivial. Let

\[
U \subset \bigoplus_{0}^{2n-1} \text{Sym}^{2n-1}(V^*)
\]


be the non-degenerate locus. First, it is shown that the $\text{SL}(V,n)$-action on $U$ is free. Since $Y \subset U$, $\text{SL}(V,n)$ acts freely on $Y$.

Let $u \in U$. Suppose $g \in \text{SL}(V,n)$ satisfies $g \cdot u = u$. $\text{PGL}(V)$ acts freely on $\text{P}(U)$. Let $\pi : \text{SL}(V,n) \to \text{PGL}(V)$. Then,

$$\pi(g) \cdot \text{P}(u) = \text{P}(u).$$

Hence, $\pi(g) = 1 \in \text{PGL}(V)$. The element $g$ is therefore a scalar in $\text{SL}(V,n)$ equal to a $2n$th root of unity $\xi$. Then, $g$ acts on $u$ by the scalar $\xi^{2n-1}$. Since $g \cdot u = u$, $\xi^{2n-1} = 1$. Since $(2n, 2n - 1) = 1$, $\xi^{2n-1} = 1$ implies $\xi = 1$. Therefore, $g = 1 \in \text{SL}(V,n)$. The $\text{SL}(V,n)$-action on $U$ is free.

It is now shown the quotient $Y/\text{SL}(V,n)$ is isomorphic to $H(d)$. There are natural, equivariant, algebraic projection maps:

$$Y \to \text{P}(U),$$

$$\pi : \text{SL}(V,n) \to \text{PGL}(V).$$

These maps induce a natural surjective map on quotients:

$$\phi : Y/\text{SL}(V,n) \to \text{P}(U)/\text{PGL}(V) \cong H(d).$$

It suffice to prove $\phi$ is injective. (A bijective map of nonsingular complex algebraic varieties is an algebraic isomorphism.)

Let $y_1, y_2 \in Y$ be points. Let $[y_1], [y_2] \in \text{P}(U)$ denote the corresponding points. Suppose there exists an element $\gamma \in \text{PGL}(V)$ satisfying $\gamma \cdot [y_1] = [y_2]$. To prove $\phi$ is injective, it must be shown that $y_1$ and $y_2$ are in the same $\text{SL}(V,n)$ orbit.

Let $g \in \text{SL}(V,n)$ satisfy $\pi(g) = \gamma$. Then, $[g \cdot y_1] = [y_2]$. Hence $g \cdot y_1 = (\lambda y_2)$ where $\lambda \in \mathbb{C}^*$ is a scalar. By the conditions $g \cdot y_1, y_2 \in Y$, it follows $\lambda^{2n} = 1$. Since $(2n, 2n - 1) = 1$, a $2n$th root of unity $\xi \in \text{SL}(V,n)$ can be found satisfying $\xi^{2n-1} = \lambda^{-1}$. Let $h \in \text{SL}(V,n)$ be determined by $h = \xi \cdot g$.

$$h \cdot y_1 = \xi \cdot g \cdot y_1 = \xi \cdot (\lambda y_2) = \xi^{2n-1}(\lambda y_2) = y_2.$$ 

Therefore $y_1$ and $y_2$ are in the same $\text{SL}(V,n)$ orbit. \hfill \Box

There is a canonical isomorphism of graded rings

$$A^*(d = 2n - 1) \cong A^*_{\text{SL}(V,n)}(Y).$$

The equivariant Chow ring $A^*_{\text{SL}(V,n)}(Y)$ is computed in this section.

The approximations $W_m$ and $W_m/\text{SL}(V,n)$ to $E\text{SL}(V,n)$ and $B\text{SL}(V,n)$ determined in section 1.2 are used here. Recall $W_m/\text{SL}(V,n) \to G(2,m)$ is the $\mathbb{C}^*$-bundle associated to the $n$th tensor power of $\lambda^2 S$ (where $S$ is the tautological sub-bundle over $G(2,m)$). Since

$$Y \subset U \subset \oplus_0^{2n-1} \text{Sym}^{2n-1}(V^*),$$

...
there are inclusions:
\[ Y \times_{\text{SL}(V,n)} W_m \subset U \times_{\text{SL}(V,n)} W_m \subset \bigoplus_{0}^{2n-1} \text{Sym}^{2n-1}(V^*) \times_{\text{SL}(V,n)} W_m. \]

Let \( F_n = V^* \times_{\text{SL}(V,n)} W_m \). \( F_n \) is an algebraic vector bundle over \( W_m/\text{SL}(V,n) \). \( F_n \) is easily identified as the pull-back of \( S^* \) to \( W_m/\text{SL}(V,n) \). \( U \times_{\text{SL}(V,n)} W_m \) is the affine non-degenerate locus (i) of section 1.1 associated to the bundle \( \text{Sym}^{2n-1} F_n \cong \text{Sym}^{2n-1} S^* \).

**Lemma 13.** There is an isomorphism
\[ \epsilon : \mathbb{C}^* \times (Y \times_{\text{SL}(V,n)} W_m) \cong U \times_{\text{SL}(V,n)} W_m. \]

**Proof.** Let \( \text{SL}(V,n) \) act trivially on \( \mathbb{C}^* \). Define a \( \text{SL}(V,n) \) equivariant isomorphism
\[ \delta : \mathbb{C}^* \times Y \rightarrow U \]
by the following:
\[ \delta(\lambda, (\omega_0, \omega_1 \ldots, \omega_{2n-1})) = (\lambda \omega_0, \omega_1, \ldots, \omega_{2n-1}). \]

The isomorphism \( \delta \) induces isomorphisms:
\[ \mathbb{C}^* \times (Y \times_{\text{SL}(V,n)} W_m) \cong (\mathbb{C}^* \times Y) \times_{\text{SL}(V,n)} W_m \cong U \times_{\text{SL}(V,n)} W_m. \]

Let \( \epsilon \) be the composition. \( W_m/\text{SL}(V,n) \) approximates \( B\text{SL}(V,n) \) up to codimension \( m - 2 \). The Chow ring of \( Y \times_{\text{SL}(V,n)} W_m \) is now computed (up to codimension \( m - 2 \)). By Lemma 13, there is an isomorphism:
\[ A^*(Y \times_{\text{SL}(V,n)} W_m) \cong A^*(U \times_{\text{SL}(V,n)} W_m). \]

Since \( U \) is the affine non-degenerate locus associated to the bundle
\[ \text{Sym}^{2n-1} S^* \rightarrow (W_m/\text{SL}(V,n)), \]

Lemma 2 can be applied. Recall the Chow ring of \( W_m/\text{SL}(V,n) \) (up to codimension \( m - 2 \)) has a presentation \( \mathbb{Z}[c_1, c_2]/(nc_1) \). Hence, there is an isomorphism (up to codimension \( m - 2 \)):
\[ A^*(Y \times_{\text{SL}(V,n)} W_m) \cong \mathbb{Z}[c_1, c_2]/(nc_1, \alpha_1, \ldots, \alpha_{d+1}) \]

where
\[ \frac{1}{c(Sym^dS)} = 1 + \alpha_1 + \ldots + \alpha_{d+1} + \ldots. \]

The proof of Theorem 2 is complete.
7. Examples

Since $H(1)$ is a point, $A^*(1)$ is the trivial $\mathbb{Z}$-algebra (which agrees
with the presentation of Theorem 2). $H(2)$ is the space of nonsingular
plane conics. By Theorem 1, $A^*(2)$ is generated by $c_2$, $c_3$, and $\mathcal{L} = \mathcal{H}$
subject to 4 relations:

$$2c_3 = 0,$$
$$3\mathcal{H} = 0, -c_2 + 3\mathcal{H}^2 = 0, -c_3 + \mathcal{H}^3 = 0.$$  

Since $\mathcal{H}$ is 3-torsion, $c_2 = 0$. Since $c_3$ is two torsion, the last equation
can be reduced to $c_3 = \mathcal{H}^3 = 0$. Therefore $A^*(2)$ is given by

$$\mathbb{Z}[\mathcal{H}] / (3\mathcal{H}, \mathcal{H}^3).$$

Since $H(2)$ is an open set of the projective space of plane conics, another
approach to $A^*(2)$ is possible. The class $\mathcal{H}$ is simply the restriction of
the hyperplane class which necessarily generates $H(2)$. The relation
$3\mathcal{H}$ can be obtained from the degree 3 degeneracy locus of singular
plane conics. The relation $\mathcal{H}^3$ is a consequence of the fact that the
locus of conics singular at a fixed point in $\mathbb{P}^2$ is a linear $\mathbb{P}^2$ in the $\mathbb{P}^5$
of conics.

Let $d = 3$, $n = 2$, $d = 2n - 1$. By Theorem 2, $A^*(3)$ is generated by $c_1$ and $c_2$ with relations:

$$2c_1 = 0,$$
$$6c_1 = 0, 11c_1^2 + 10c_2 = 0, 6c_1^3 + 30c_1c_2 = 0, 18c_1^2c_2 + 9c_2^2 = 0.$$  

These relations simplify to yield the presentation:

$$A^*(3) = \mathbb{Z}[c_1, c_2] / (2c_1, c_1^2 + 10c_2, c_1^3, c_1^2c_2, c_2^2).$$

In particular, $A^i(3) = 0$ for $i \geq 4$.

8. Appendix On Algebraic Quotients

Let $\mathbb{C}$ be the ground field of complex numbers. The geometric invariant
theory terminology of [MFK] is used here. Let $G$ be a reductive
linear algebraic group. A group action $G \times X \to X$ is proper if the
natural map

$$\Psi : G \times X \to X \times X$$

(given by the action and projection onto the second factor) is a proper
morphism. The main result needed is the following:

**Proposition 3.** Let $X$ be a quasi-projective variety with a linearized $G$-action satisfying $X^{\text{stable}}_{(0)} = X$. Then, the $G$-action on $X$ is proper and there is a quasi-projective geometric quotient $X \to X/G$. 

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Proof. Properness of the action is exactly Corollary 2.5 of [MFK].

The geometric quotient is the main construction in geometric invariant theory (Theorem 1.10 of [MFK]).

The stable locus $X_{(0)}^{stable}$ is detected by the Numerical Criterion.

Let $V \cong \mathbb{C}^k$ be a vector space equipped with a quadratic form. All of the linear algebraic groups considered in this paper are reductive: $\text{GL}(V), \text{SL}(V), \text{PGL}(V), \text{SL}(V,n), \text{O}(V), \text{SO}(V), \mathbb{C}^*, \mathbb{C}^* \times \text{SO}(V)$.

Let

$$Span_m(V,d) \subset \oplus_1^m \text{Sym}^d(V^*)$$

be the spanning locus (the locus of $m$-tuples of vector of $\text{Sym}^d(V^*)$ which span $\text{Sym}^d(V^*)$). The spanning loci $U \subset \oplus_0^d \text{Sym}^d(\mathbb{C}^2)$ and $W_m \subset \oplus_1^m V^*$ are special cases of $Span_m(V,d)$. The group actions considered in the paper are of three forms:

(i) The natural $G$-action on $X = Span_m(V,d)$ where $G \subset \text{GL}(V)$ is a reductive subgroup.

(ii) The $G$-action on a $G$-invariant subvariety $Y \subset Span_m(V,d)$ where $G \subset \text{GL}(V)$ is a reductive subgroup.

(iii) The natural $\text{PGL}(V)$-action on $X = \mathbb{P}(Span_m(V,d))$.

For example, the $\text{SL}(V,n)$-action on $Y$ considered in section 3 is of form (ii). Consider first (i) and (ii). A linearization of the $\text{GL}(V)$-action can be found on $X$ satisfying $X_{(0)}^{stable} = X$. Such a linearization is found in section 1 of [P]. Since the stable locus is detected by the Numerical Criterion, the result for $\text{GL}(V)$ implies $X_{(0)}^{stable} = X$ for the induced action of any reductive subgroup $G \subset \text{GL}(V)$. It is similarly simple to find a linearization in case (iii) satisfying $X_{(0)}^{stable} = X$. Therefore, Proposition 3 applies to all the quotient problems in the paper.

In [MFK], the $G$-action $G \times X \to X$ is defined to be free if the natural map $\Psi$ is a closed embedding. An action is set-theoretically free if the stabilizers are trivial. For the set-theoretically free actions considered in this paper, the following Lemma is utilized.

**Lemma 14.** Let $X$ be nonsingular. Let $G \times X \to X$ be a proper action. In this case, set-theoretically free implies free.

Proof. Let $I \subset X \times X$ be the image of $\Psi$. $I$ is a closed subvariety since $\Psi$ is proper. It must be shown that $G \times X \to I$ is an isomorphism. First it is shown that $I$ is nonsingular. For this, it suffices to prove the differential of $\Psi$ is injective at each point of $G \times X$. It is well known (over $\mathbb{C}$) that set-theoretically trivial stabilizers are also scheme-theoretically trivial. From this, the injectivity of the differential $d\Psi$ is
easily deduced. Now $\Psi : G \times X \to I$ is a bijective map of nonsingular complex algebraic varieties and thus an algebraic isomorphism.

A result of [MFK] (Proposition 0.9) relates free quotients to principal $G$-bundles.

**Proposition 4.** Let $G \times X \to X$ be an algebraic group action with geometric quotient $X \to Y$. If the action is free, then $X \to Y$ is a (étale locally trivial) principal $G$-bundle.

A principal $G$-bundle $X \to Y$ and a representation $G \to GL$ together yield a principal $GL$-bundle $X \times^G GL \to Y$. Since every principal $GL$-bundle is Zariski locally trivial ($GL$ is a special group in the sense of Grothendieck (see *Anneau de Chow et Applications*, Seminaire Chevalley 1958)), an algebraic vector bundle over $Y$ is obtained.

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