Collecting Coded Coupons over Overlapping Generations

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Abstract—Coding over subsets (known as generations) rather than over all content blocks in P2P distribution networks and other applications is necessary for a number of practical reasons such as computational complexity. A penalty for coding only within generations is an overall throughput reduction. It has been previously shown that allowing contiguous generations to overlap in a head-to-toe manner improves the throughput. We here propose and study a scheme, referred to as the random annex code, that creates shared packets between any two generations at random rather than only the neighboring ones. By optimizing very few design parameters, we obtain a simple scheme that outperforms both the non-overlapping and the head-to-toe overlapping schemes of comparable computational complexity, both in the expected throughput and in the rate of convergence of the probability of decoding failure to zero. We provide a practical algorithm for accurate analysis of the expected throughput of the random annex code for finite-length information. This algorithm enables us to quantify the throughput vs. computational complexity tradeoff, which is necessary for optimal selection of the scheme parameters.

I. INTRODUCTION

In randomized network coding applications, such as P2P content distribution [1] or streaming, the source splits its content into blocks. For a number of practical reasons (e.g. computational complexity and delay reduction, easier synchronization, simpler content tracking), these blocks are further grouped into subsets referred to as generations, and, in the coding process at the source and within the network, only packets in the same generation are allowed to be linearly combined. A penalty for coding only within generations is an overall throughput reduction. The goal of this work is to develop a strategy which allows generations to overlap, and hence, to improve the throughput while maintaining the benefits brought up by introducing generations.

Since the idea of coding over generations has been introduced by Chou et al. in [2], a number of issues concerning such coding schemes have been addressed. Maymounkov et al. studied coding over generations with random scheduling of the generation transmission in [3]. They referred to such codes as chunked codes. This random scheduling coding scheme has also been studied in our recent work [4]. The tradeoff between the reduction in computational complexity and the throughput benefits was addressed in [3], [4], and the tradeoff between the reduction in computational complexity and the resilience to peer dynamics in P2P content distribution networks was the topic of [5].

Here, we are particularly interested in recovering some of the throughput that is lost as a consequence of coding over generations by allowing the generations to overlap. It has been observed that with random scheduling, some generations accumulate their packets faster, and can be decoded earlier than others. If generations are allowed to overlap, the decoding of some generations will reduce the number of unknowns in those lagging behind but sharing packets with those already decoded, and thus help them become decodable with fewer received coded packets, consequently improving the throughput. Coding with overlapping generations was first studied in [6] and [7].

In this work, we propose a coding scheme over generations that share randomly chosen packets. We refer to this code as the random annex code. We demonstrate with both a heuristic analysis and simulations that, assuming comparable computational complexity, with its small number of design parameters optimized, the simple random annex coding scheme outperforms the non-overlapping scheme, as well as a “head-to-toe” overlapping scheme. The head-to-toe scheme was the topic of [7], in which only contiguous generations overlap in a fixed number of information packets in an end-around mode, and hence only contiguous generations can benefit from each other in decoding.

The main contribution of this paper is an accurate and practical evaluation of the expected throughput performance of the random annex code for finite information lengths. We first anamorize the overlapping structure of the generations to quantify the benefit of previously decoded generations on those not yet decoded, namely, how many fewer coded packets are needed for each generation to be decoded. With this accomplished, we are then able to place the overlaps in oblivion and analyze the throughput performance of random annex codes under the coupon collection setting. We succeed to optimize the scheme design parameters based on the evaluation of the overhead necessary to decode all the packets.

Our paper is organized as follows: In Section II we describe the coding scheme for the random annex code, and introduce the coupon collector’s model under which our analysis for code throughput is studied. In Section III we analyze the overlapping structure of the random annex code, and give a practical algorithm to evaluate the expected code performance.
for finite information length. In Section IV we present our numerical evaluation and simulation results which illustrate and quantify the throughput vs. computational complexity tradeoff brought up by our scheme. Section V concludes.

II. CODING OVER GENERATIONS

A. Coding over Overlapping Generations in Unicast

We refer to the code we study as the random annex code. This section describes (a) the way the generations are formed, (b) the encoding process, (c) the decoding algorithm, and (d) how the computational complexity for the random annex code is measured.

a) Forming Overlapping Generations: We first divide file $F$ into $N$ information packets $p_1, p_2, \ldots, p_N$. Each packet $p_i$ is represented as a column vector of $d$ information symbols in Galois field $GF(q)$. We then form $n$ overlapping generations in two steps as follows:

1) We partition the $N$ packets into sets $B_1, B_2, \ldots, B_n$ of each containing $h$ consecutive packets. We refer to these $n = N/h$ sets as base generations. Thus, $B_i = \{p(i-1)h+1, p(i-1)h+2, \ldots, p_ih\}$ for $i = 1, 2, \ldots, n$. We assume that $N$ is a multiple of $h$ for convenience. In practice, if $N$ is not a multiple of $h$, we set $n = \lceil N/h \rceil$ and assign the last $\lfloor N - (n-1)h \rfloor$ packets to the last (smaller) base generation.

2) To each base generation $B_i$, we add a random annex $R_i$, consisting of $l$ packets chosen uniformly at random (without replacement) from the $N-h = (n-1)h$ packets in $F \setminus B_i$. The base generation together with its annex constitutes the extended generation $G_i = B_i \cup R_i$. The size of each $G_i$ is $g = h+l$. Throughout this paper, unless otherwise stated, the term “generation” will refer to “extended generation” whenever used alone.

The members of $G_i$ are enumerated as $p_1^{(i)}, p_2^{(i)}, \ldots, p_g^{(i)}$.

b) Encoding: The encoding process is oblivious to overlaps between generations. In each transmission, the source first selects one of the $n$ generations with equal probability. Assume $G_j$ is chosen. Then the source chooses a coding vector $e = [e_1, e_2, \ldots, e_g]^T$ with each entry chosen independently and equally probably from $GF(q)$. A new packet $\tilde{p}$ is then formed by linearly combining packets from $G_j$ by $\tilde{p} = \sum_{i=1}^g e_i p_i^{(j)}$. The coded packet $\tilde{p}$ is then sent over the communication link to the receiver along with the coding vector $e$ and the generation index $j$.

c) Decoding: Decoding starts with a generation for which the receiver has collected $(h+l)$ coded packets with linearly independent coding vectors. The information packets making up this generation are decoded by solving a system of $(h+l)$ linear equations in $GF(q)$ formed by the coded packets and the linear combinations of the information packets by the coding vectors. Each decoded information packet is then removed as an unknown from other generations’ coded packets in which it participates. Consequently, the number of unknowns in all generations overlapping with those that are already decoded is reduced, and some such generations may become decodable even when no new coded packets are received from the source. Again, the newly decoded generations resolve some unknowns of the generations they overlap with, which in turn may become decodable and so on. This is the mechanism through which the generation overlapping potentially improves the throughput. We declare successful decoding when all $N$ information packets have been decoded.

d) Computational Complexity: The computational complexity for encoding is $O((h+l)d)$ (recall that $d$ is the number of information symbols in each packet as defined in Part a)) per coded packet for linearly combining the $(h+l)$ information packets in each generation. For decoding, the largest number of unknowns in the systems of linear equations to be solved is $(h+l)$, and therefore the computational complexity is upper bounded by $O((h+l)^2 + (h+l)d)$ per information packet.

While some may argue that the assumption of random scheduling of generations deviates from reality, we put forward here a few motives behind its adoption: (1) Locality: Uniformly random scheduling assumes knowledge of least information, which to some extent approximates the case with limited-visioned peer nodes in large-scale systems; (2) Ratelessness: in the case of single-hop multicast over erasure channels, the code throughput automatically adapts to all erasure rates. Some previous works on coding over generations, such as [3] and [7], also assumed random scheduling.

We measure code throughput by the number of coded packets necessary for decoding all the information packets.

B. Coupon Collector’s Problem

As in [4], we model the collection of coded packets from $n$ generations as the sampling of a set of $n$ coupons without replacement. In the next section, we will look into the overlapping structure of the extended generations, and use our extension of the coupon collector’s brotherhood problem [8], [9] described in [4] to evaluate the throughput performance of the random annex code. We will also compare the performance of random annex code to the overlapped chunked code proposed in [7], which has the generations overlap in a “head-to-toe” fashion. Note that the random annex code in fact defines a code ensemble that encompasses the overlapped chunked code.

III. OVERLAPPING GENERATIONS-AN ANALYSIS OF EXPECTED PERFORMANCE

A. Overlapping Structure

The decoding of different generations becomes intertwined with each other as generations are no longer disjoint. Our goal here is to unravel the structure of the overlapping generations, in order to identify the condition for successful decoding of random annex codes over a unicast link.

In Claims [1] through [4] we study the overlapping structure of the random annex code. Compared with the head-to-toe overlapping scheme, an extended generation in the random annex code overlaps more evenly with other generations.
Intuitively, this can help with code throughput when random scheduling of generations is used.

Claim 1: For any packet in a base generation $B_k$, the probability that it belongs to annex $R_r$ for some $r \in \{1, 2, \ldots, n\} \setminus \{k\}$ is

$$\pi = \frac{(N-h-1)}{l-1} \cdot \frac{(N-h)}{l} = \frac{l}{N-h} \cdot \frac{(n-1)}{h},$$

while the probability that it does not belong to $R_r$ is $\pi = 1 - \pi$.

Claim 2: Let $X$ be the random variable representing the number of generations an information packet participates in. Then, $X = 1 + Y$ where $Y$ is Binom($n-1, \pi$).

$$E[X] = 1 + (n-1)\pi = 1 + \frac{l}{h},$$

and

$$\text{Var}[X] = (n-1)\pi \bar{\pi}.$$  

Claim 3: In any generation of size $g = h + l$, the expected number of information packets not present in any other generation is $h\bar{\pi}^{(n-1)} = e^{-l/h}$ for $n \gg 1$. The expected number of information packets present at least once in some other generation is

$$l + h[1 - \bar{\pi}^{(n-1)}] \approx l + h[1 - e^{-l/h}] < \min\{g, 2l\}$$

for $n \gg 1$ and $l > 0$.

Claim 4: The probability that two generations overlap is $1 - \left(\frac{N-2h}{l, N-2h-2l} / (N-h)^2\right)$. For any given generation, the number of other generations it overlaps with is then

$$\text{Binom}\left(n-1, 1 - \left(\frac{N-2h}{l, N-2h-2l} / (N-h)^2\right)\right).$$

The following Theorem 5 gives the expected overlap size $\Omega(s)$ between the union of $s$ generations and an $(s+1)$th generation.

Theorem 5: For any $I \subset \{1, 2, \ldots, n\}$ with $|I| = s$, and any $j \in \{1, 2, \ldots, n\} \setminus I$,

$$\Omega(s) = E[| (\bigcup_{i \in I} G_i) \cap G_j |] = g \cdot [1 - \bar{\pi}^s] + sh \cdot \bar{\pi}^s \tag{1}$$

where $|B|$ denotes the cardinality of set $B$, and $\bar{\pi}$ are as defined in Claim 1.

When $n \to \infty$, if $\frac{h}{n} \to \alpha$ and $\frac{h}{n} \to \beta$, and let $\omega(\beta) = \Omega(s)$, then, $\omega(\beta) \approx h \left(1 + \alpha \right) \left(1 - e^{-\alpha \beta}\right) + \alpha \beta e^{-\alpha \beta} \cdot \bar{\pi}^s$.

We provide a proof of Theorem 5 in Appendix A.

B. An Analysis of Overhead Based on Mean Values

We next describe an analysis of the expected number of coded packets a receiver needs to collect in order to decode all $N$ information packets of $\mathcal{F}$ when they are encoded by the random annex code. We base our analysis on Theorem 5 above and Claim 6 and Theorem 7 below, and use the mean value for every quantity involved.

By the time when $s(s=0, 1, \ldots, n-1)$ generations have been decoded, for any one of the remaining $(n-s)$ generations, on the average $\Omega(s)$ of its participating information packets have been decoded, or $(g - \Omega(s))$ of them are not yet resolved.

If the coded packets collected from some generation are enough for decoding its unresolved packets, that generation becomes the $(s+1)$th decoded one; otherwise, if no such generation exists, decoding fails.

The following Claim 6 estimates the number of coded packets needed from a generation for its decoding when $(g-x)$ of its information packets remain to be resolved.

Claim 6: For any generation $G_i$, if $x$ of the $g = h + l$ information packets of $G_i$ have been resolved by decoding other generations, then, the expected number of the number of coded packets $N_i(g,x)$ from $G_i$ needed to decode the remaining $(g-x)$ information packets

$$E[N_i(g,x)] \leq g - x + \frac{q^{-1}}{1-q^{-1}} + \log_q \frac{1}{1-q^{-1}} \eta(x). \tag{2}$$

Proof: In all the coding vectors, remove the entries corresponding to the information packets already resolved. Now we need to solve a system of linear equations of $(g-x)$ unknowns with all coefficients chosen uniformly at random from $GF(q)$. Thus, $N_i(g,x) = N_i(g-x,0)$.

$$E[N_i(g,x)] = \sum_{j=0}^{g-x-1} \left( \frac{q^{g-x} - q^j}{q^{g-x}} \right)^{-1} \cdot \left( \frac{1}{1-q^{-1}} \right) + \log_q \frac{1}{1-q^{-1}} \tag{3}$$

$$\leq g - x + \frac{q^{-1}}{1-q^{-1}} + \log_q \frac{1}{1-q^{-1}} \tag{4}$$

Extending the domain of $\eta(x)$ from integers to real numbers, we can estimate that the number of coded packets needed for the $(s+1)$th decoded generation should exceed $m' \approx \lceil \eta(\Omega(s)) \rceil$. Since in the random annex code, all generations are randomly scheduled with equal probability, for successful decoding, we would like to have at least $m' \approx \lceil \eta(\Omega(s)) \rceil$ coded packets belonging to one of the generations, at least $m' \approx \lceil \eta(\Omega(s)) \rceil$ belonging to another, and so on. We wish to estimate the total number of coded packets needed to achieve the above.

For any $m \in \mathbb{N}$, we define $S_m(x)$ as follows:

$$S_m(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{m-1}}{(m-1)!} \quad (m \geq 1) \tag{6}$$

$$S_\infty(x) = \exp(x) \text{ and } S_0(x) = 0. \tag{7}$$

The following Theorem 7 which is a restatement of Theorem 2 from [4] using the terminology of coding over generations, provides a way to estimate the expected number of coded packets necessary for decoding the whole file $\mathcal{F}$.

Theorem 7: (Theorem 2, [4]) Suppose for some $A \in \mathbb{N}$, integers $k_1, \ldots, k_A$ and $m_1, \ldots, m_A$ satisfy $1 \leq k_1 < \cdots < k_A \leq n$ and $\infty = m_0 > m_1 > \cdots > m_A > m_{A+1} = 0$. Let $\mu_r$ be the number of generations for which at least $r$ coded packets have been collected. Then, the expected number of coded packets necessary to simultaneously have $\mu_m \geq k_j$ for
all \(j = 1, 2, \ldots, A\) is
\[
\int_0^{\infty} e^{-nx} - \sum_{(i_1, i_2, \ldots, i_A):} \prod_{j=0}^n \left( i_j+1 \right) \left( S_{m_j}(x) - S_{m_{j+1}}(x) \right)^{i_j+1-i_j} e^{-nx} dx
\]

A practical method to evaluate \(8\) is provided in Appendix B. The computational complexity for one evaluation of the integrand is \(O(An^2)\) given \(m_1 = O(An^2)\).

The algorithm for our heuristic analysis is listed as follows:

1. Compute \(\Omega(s-1)\) for \(s = 1, 2, \ldots, n\) using Theorem 3.
2. Compute \(m'_s = \eta(\Omega(s-1))\) for \(s = 1, 2, \ldots, n\) using (2) from Claim 6.
3. Map \(m'_s(s = 1, 2, \ldots, n)\) into \(A\) values \(m_j(j = 1, 2, \ldots, A)\) so that \(m_j = m'_k, \text{ for } j = 1, 2, \ldots, A, m_1 > m_2 > \cdots > m_A > m_{A+1} = 0, k_0 = 0 \text{ and } k_A = n\).
4. Evaluate \(8\) in Theorem 7 with the \(A, k_j, s, \text{ and } m_j, s\) obtained in Step 3, as an estimate for the expected number of coded packets needed for successful decoding.

**Remark 1:** The above Step 3 is viable because \(\Omega(s)\) is nondecreasing in \(s\), the righthand side of (2) is non-increasing in \(x\) for fixed \(g\), and thus \(m'_s\) is non-increasing in \(s\).

Although our analysis is heuristic, we will see in the next section that the estimate closely follows the simulated average performance curve of the random annex coding scheme.

IV. **Numerical Evaluation and Simulation Results**

A. **Throughput vs. Complexity in Fixed Number of Generations Schemes**

Our goal here is to find out how the annex size \(l\) affects the throughput performance of the scheme with fixed base generation size \(h\) and the total number of information packets \(N\) (and consequently, the number of generations \(n\)). Note that we may be trading throughput for complexity since the generation size \(g = h+l\) affects the computational complexity of the scheme.

Figure 1 shows the analytical and simulation results when the total number \(N\) of information packets is 1000 and the base generation size \(h\) is 25. Figure 1(a) shows \(h+l-\Omega(s)\) for \(s = 0, 1, \ldots, n\) with different annex sizes. Recall that \(\Omega(s)\) is the expected size of the overlap of the union of \(s\) generations with any one of the rest \(n-s\) generations. After the decoding of \(s\) generations, for any generation not yet decoded, the expected number of information packets that still need to be resolved is then \(h+l-\Omega(s)\). We observe that the \(h+l-\Omega(s)\) curves start from \(h+l\) for \(s = 0\) and gradually descends, ending somewhere above \(h-l\), for \(s = n-1\).

Recall that we measure throughput by the number of coded packets necessary for successful decoding. Figure 1(b) shows the expected performance of the random annex code, along with the head-to-toe overlapping code and the non-overlapping code \((l = 0)\). Figure 1(c) shows the probability of decoding failure of these codes versus the number of coded packets collected.

- Our analysis for the expected number of coded packets required for successful decoding extremely closely matches the simulation results.
- For both the random annex scheme and the head-to-toe scheme, there is an optimal annex size, beyond or
below which throughput is lower than optimal. From the simulation results in Figure 1(b), it is observed that the optimal annex size is 12 for the random annex scheme and 8 for the head-to-toe scheme. Beyond the optimal annex size, throughput cannot be increased by raising computational cost.

- The random annex code outperforms head-to-toe overlapping at their respective optimal points. Both codes outperform the non-overlapping scheme.
- We also plotted the probability of decoding failure versus the number of coded packets received. The probability of decoding failure of the random annex code converges faster than those of the head-to-toe and the non-overlapping scheme.

B. Enhancing Throughput in Fixed Complexity Schemes

Our goal here is to see how we can choose the annex size that optimizes the throughput with negligible sacrifice in complexity. To this end, we fix the extended generation size \( g = h + l \) and vary only the annex size \( l \). Consequently, the computational complexity for coding remains roughly constant (actually decreases with growing \( l \)).

Figure 2 shows the analytical and simulation results for the code performance when the total number \( N \) of information packets is fixed at 1000 and size \( g \) of extended generation fixed at 25.

- Again our analytical results agree with simulation results very well;
- It is interesting to observe that, without raising computational complexity, increasing annex size properly can still give non-negligible improvement to throughput. There is still an optimal annex size that achieves highest throughput. From Figure 2(a) we see that the optimal annex size is 10 for the random annex scheme and 6 for the head-to-toe scheme;
- The random annex code again outperforms head-to-toe overlapping at their optimal points. Both codes outperform the non-overlapping scheme;
- We again observe that the probability of decoding failure of the random annex code converges faster than those of the head-to-toe and the non-overlapping schemes.

V. CONCLUSION AND FUTURE WORK

We proposed a random annex scheme for coding with overlapping generations. We obtained an accurate analytical evaluation of its expected throughput performance for a unicast link using an extension of the coupon collector’s model derived in our recent work [4]. Both the expected throughput and the probability of decoding failure of the random annex code are generally better than those of the non-overlapping and head-to-toe overlapping coding schemes. Under fixed information length and fixed number of generations, there exists an optimal annex size that minimizes the number of coded packets needed for successful decoding. One of our most interesting findings is that when we fix the information length and the generation size, increasing the annex size may still improve throughput without raising computational complexity.

We developed a practical algorithm to numerically evaluate some of our complex analytical expressions. With slight modification of the analytical method used in this work, we can also predict the expected decoding progress, i.e., the number of generations/information packets decodable with the accumulation of coded packets. This will be useful to studies of content distribution with tiered reconstruction at the user. It can also be used to find the best rate of a “precode” [3] applied to coding over generations. It would be interesting to know if the combination of overlapping generations and precode can further improve code throughput.

It is also interesting to study the asymptotic performance of the code, as the information length tends to infinity. We also hope to characterize the optimal annex size in terms of generation size and number of generations.

APPENDIX A

PROOF OF THEOREM 5

Without loss of generality, let \( I = \{1, 2, \ldots, s\} \) and \( j = s + 1 \), and define \( R_s = \cup_{i=1}^s R_i, B_s = \cup_{i=1}^s B_i, \) and \( G_s = \)
And therefore
\[ E[|G_s \cap G_{s+1}|] = E[|B_s \cap R_{s+1}|] + E[|R_s \cap B_{s+1}|] + E[|\{(R_s \setminus B_s) \cap R_{s+1}\}] \]
Using Claim 1 we have
\[ E[|B_s \cap R_{s+1}|] = s \pi, \]
\[ E[|R_s \cap B_{s+1}|] = h(1 - (1 - \pi)^n), \]
\[ E[|\{(R_s \setminus B_s) \cap R_{s+1}\}] = (n - s - 1)h\pi[1 - (1 - \pi)^s], \]
where \( \pi \) is as defined in Claim 1. Bringing (10)-(12) into (9) we obtain (1).

Furthermore, when \( n \to \infty \), if \( l/h \to \alpha \) and \( s/n \to \beta \), then
\[ E[|G_s \cap G_{s+1}|] = g \cdot [1 - \pi^n] + s \pi n \pi^s \]
\[ \to h(1 + \alpha)\left[1 - \left(1 - \frac{\alpha}{n - 1}\right)^n\right] + \]
\[ h\alpha\beta\left(1 - \frac{\alpha}{n - 1}\right)^n \]
\[ \to h\left[(1 + \alpha)(1 - e^{-\alpha}) + \alpha\beta e^{-\alpha\beta}\right] \]
\[ = h\left[1 + \alpha - (1 + \alpha\beta)e^{-\alpha\beta}\right] \]

APPENDIX B
EVALUATION OF EXPRESSION (8)

We give here a method to calculate the integrand in (8) of Theorem 7. The integrand of (8) can be rewritten as
\[ 1 - \sum_{i_A=k_A}^{n} \binom{n}{i_A} [S_{m_A}(x) - S_{m_A+1}(x)]e^{-x} |^{n-i_A} \]
\[ \cdot \sum_{i_{A-1}=k_{A-1}}^{i_A} \binom{i_A}{i_{A-1}} (S_{m_{A-1}}(x) - S_{m_A}(x))e^{-x} |^{A-i_{A-1}} \]
\[ \cdots \]
\[ \cdot \sum_{i_1=k_1}^{i_2} \binom{i_2}{i_1} (S_{m_1}(x) - S_{m_2}(x))e^{-x} |^{2-i_1} \]
\[ \cdot \sum_{i_0=k_0}^{i_1} \binom{i_1}{i_0} (S_{m_0}(x) - S_{m_1}(x))e^{-x} |^{1-i_0} \]
For \( k = k_1, k_1 + 1, \ldots, n \), let
\[ \phi_{0,k}(x) = [(S_{m_0}(x) - S_{m_1}(x))e^{-x}]^k; \]
For each \( j = 1, 2, \ldots, A \), let
\[ \phi_{j,k}(x) = \sum_{w=k_j}^{k} \binom{k}{w} [(S_{m_j}(x) - S_{m_{j+1}}(x))e^{-x}]^{k-w}\phi_{j-1,w}(x), \]
for \( k = k_j+1, k_{j+1} + 1, \ldots, n \).

Then, one can verify that (13) is exactly \( 1 - \phi_{A,n}(x) \).