Non-Hermitian Second-Order Skin and Topological Modes

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The skin effect and topological edge states in non-Hermitian system have already been well studied in much previous work, while the second-order non-Hermitian edge states and skin effect have also been proposed recently. We deduce the hybrid skin-topological modes as well as second-order topological edge states in a rigorous manner, for which we construct a nested tight-binding formalism in this paper. We also illustrate that the second-order skin effect originates from the existence of both two direction first-order skin effect which originates from loop topology of the complex energy spectrum under periodic boundary condition. We conclude that the hybrid skin-topological mode is generated by skin effect and localized edge states for each of two directions respectively, while the second-order topological edge states are induced by localized edge states along both two directions.

I. INTRODUCTION

Beyond the conventional hotspot on topological insulators and superconductors \cite{18} and their classification \cite{9,18} in condensed physics last decades, it rapidly ramifications into two patulous fields which involves higher-order topological phases \cite{19,38} and non-Hermitian topological systems \cite{39,59} in recent years. An \textit{n}-th order topological insulator which originates from the topological crystalline insulators has topologically protected gapless states at the \textit{n}-codimension surfaces \cite{20,33}, but is gapped otherwise. For example, a second-order topological insulator in two dimensions has zero energy states at corners but a gapped bulk. Meanwhile the gapless edge states, the significant symbol of the first-order topological phase, are absent. Non-Hermitian Hamiltonian describes the wide applications of open system \cite{60,65} and realizable system of gain and loss \cite{66,77} such as photonic and phonons systems etc. Of all properties in non-Hermitian systems, the exceptional points \cite{44,51,78} at which many complex bands coalesce and the skin effect \cite{46,47,52,56} with localized bulk modes are the most intriguing focus. In addition, the combination of higher-order and non-Hermitian has also been studied \cite{79,83} and two extremely novel states has been proposed that is the second-order skin(\textit{SS}) and skin-topological(\textit{ST}) state \cite{82}.

The abundant localized behavior in first-order non-Hermitian system exploits more possible second-order localized states. The interplay between two direction with topological edge states and skin bulk naturally induces three types second-order corner localized behavior corner states: topological-topological(\textit{TT}), topological-skin(\textit{ST} or \textit{TS}) and skin-skin(\textit{SS}), which has been numerically proposed in Ref \cite{82} and extended to higher-order behavior. We declare here that the nonzero edge states is not topological protected in 1D system but they still contributes to the second-order corner states, hence we identify the protected zero edge states and nonzero edge states isolated from continuous bulk when we search the second-order corner states. In this sense, the defined noun for \textit{ST} and \textit{TT} \cite{82} is suitable. In principle, after understanding clearly the first and second-order topological insulator behavior, the higher-order case can be obtained by induction, merely more and more complicated to be strictly unanalysable but can be left for numerical calculation. Hence we only concentrate on the second-order corner localized behavior in this paper.

In this paper, we rigorously depict the second-order topological(\textit{TT}) and hybrid skin-topological(\textit{ST}) corner modes. We illustrate this based on the nested tight-binding formalism which is a direct dialog to the generic tight-biding model on hand without any additional annexing agent. The paper is organized as follows. In Sec. I\textsuperscript{II} we revive the topological origin of the first skin effect in previous work, and then elicit the second-order skin(\textit{SS}) effect for a simplest 2D model. In Sec. I\textsuperscript{II} we construct the rigorous general formalism of nested tight-binding formalism. Using this method, we analyse the four-band model proposing in Ref. \cite{82} and a novel two-band model \cite{83} to investigate the \textit{TT} and \textit{ST} corner modes. Finally, we conclude this paper in Sec. I\textsuperscript{IV}.

II. WINDING NUMBER AND SECOND-ORDER SKIN EFFECT

The \textit{n}-th order topological insulator in \textit{d}-dimensional system is featured by the topologically protected gapless states at the \textit{n}-codimension surfaces when we take \textit{n}-directions open boundary condition(OBC) and the remaining (\textit{d}−\textit{n})-directions periodic boundary condition(PBC). It indicates that the \textit{n}-th order topological insulators with arbitrary dimension are all ascribed to \textit{n}-dimensional Hamiltonian with full-OBC. The remaining (\textit{d}−\textit{n}) parallel momentums $k'_n$ in \textit{d}-dimensional system are just viewed as the parameters generating the hinge or higher dimensional direction for \textit{n}-th order topological edge states. Given this, we merely need consider a 2D Hamiltonian for second-order skin and topological phases, for which we propose the nested tight-binding formalism to depict this universally in Sec. I\textsuperscript{II}. 

A. Winding number and first-order non-Hermitian skin effect

The first-order skin effect originated from intrinsic non-Hermitian point gap topology \[58\] is determined by winding number of the complex energy contour for a 1D Hamiltonian. For simplicity, we refer the skin effect and edge states to the first-order status and indicate the order for higher-order status hereinafter. We emphasize that the winding number for skin effect is different from that for edge states, which characterize the topological protected edge states at 2n-dimensional surface for a \((2n + 1)\)-dimensional Hamiltonian comes from a homotopy map: \(BZ^{2n+1} \rightarrow U(N)\)

\[
W_{2n+1} = \frac{n!}{(2\pi i)^{n+1}(2n + 1)!} \int_{BZ^{2n+1}} tr(H^{-1}dH)^{2n+1}.
\]

However, the skin effect winding number is always \(W_1\) for a 1D Hamiltonian, which only characterizes the skin effect for fixed \((d - 1)\)-dimensional parallel momentum of a \(d\)-dimensional system. In addition, the skin effect winding number is vanishing for Hermitian Hamiltonian since the energy spectrum is always real in complex plane. Notice that, the topological winding number characterizing the edge states for a 1D chiral symmetric Hermitian Hamiltonian \(H_h\) is actually the winding of the chiral non-Hermitian block Hamiltonian

\[
W^h = \frac{1}{2\pi i} \int_0^{2\pi} dk \frac{d}{dk} \log \det[h(k)], \quad H_h = \begin{bmatrix} 0 & h(k) \\ h^*(k) & 0 \end{bmatrix}.
\]

In conclusion, the non-Hermitian skin effect of a 1D Hamiltonian originates from the point gap topology \[58\], for which the characteristic topological invariant is the winding number of the PBC complex spectrum around the reference skin mode point \(E\), while the topology of edge states inhering from Hermitian counterpart is different from that for skin effect

\[
W(E) = \frac{1}{2\pi i} \int_0^{2\pi} dk \frac{d}{dk} \log \det[H(k) - E].
\]

It reveals that the nontrivial topology is due to the PBC not OBC spectrum for the point gap, since the OBC spectrum is arcs in complex plane inducing vanishing winding number. In addition, the value of winding number \(W(E)\) counts the skin modes degeneracy at reference energy \(E\) \[58\]. In general, we should calculate winding number by adding all the winding number of multiple bands(Riemann energy spectrum sheet) with multiple Brillouin zones

\[
W(E) = \frac{1}{2\pi i} \sum_{\mu=1}^{n} \int_0^{2\pi} dk \frac{d}{dk} \log[E^\mu(k) - E].
\]

The Brillouin zones are degenerated at \(|k| = 1\) for PBC, while the generalized Brillouin zones(GBZs) are not degenerated in general cases \[54\] [see Appendix. A].

As a typical model, the energy spectra of non-Hermitian SSH model \(H_{nSSH}(k) = (t_1 + t_2 \cos k)\sigma_x + (t_2 \sin k + i\gamma/2)\sigma_y\) \[11\] under PBC sketches two Riemann sheet with \pm square of Hamiltonian \(E^\pm_s(k)\) and each sheet encircles half loop(cyan and orange loop in Fig.\ref{fig:1}(a)) of the energy spectra deducing the winding number for each skin mode energy \(E_s\)(points on the black line in Fig.\ref{fig:1}(a))

\[
W(E_s) = W^+(E_s) + W^-(E_s) = 1.
\]

Therefore each point on the black line in Fig.\ref{fig:1}(a) which is divided into two Riemann sheets located at the both side of imaginary axis is the eigenenergy of one skin mode respect to the Hamiltonian under OBC except the origin point which contains two degenerated edge modes.

Another example has two completely separated bands

\[\begin{align*}
\text{Fig. 1. (a)} & \text{The complex energy spectra for non-Hermitian SSH model with } t_1 = 0.8, t_2 = 1, \gamma = 4/3, \text{in which the cyan and orange loop are energy spectra under PBC while black line and point OBC.} \\
\text{The complex energy spectra for two-band model Eq. (5)} & \text{with } t_0 = 1, t_- = 2, t_+ = 1, w_0 = 1, w_- = 1, w_+ = 3, c = 1 \text{ for (b) and } t_0 = 1, t_- = 2, t_+ = 1, w_0 = -1, w_- = 1, w_+ = 3, c = 1 \text{ for (c), in which the orange loops are energy spectra under PBC for } E_\pm(k) \text{ while black line and point OBC.}
\end{align*}\]
whose Hamiltonian is
\[ H_2(k) = \begin{bmatrix} t_0 + t_- e^{-ik} + t_+ e^{ik} & c \\ w_0 + w_- e^{-ik} + w_+ e^{ik} \end{bmatrix}. \] (5)

The two energy bands (Riemann sheets) are \( E_{\pm}(k) = h_{\pm}(k) = \frac{h_1(k) \pm h_2(k)}{2} \) with \( h_{\pm}(k) = (h_1(k) \pm h_2(k))/2 \) and \( h_1(k) = t_0 + t_- e^{-ik} + t_+ e^{ik}, h_2(k) = w_0 + w_- e^{-ik} + w_+ e^{ik}. \)

In Fig. (b) of (c), the complex energy spectra under PBC are plotted as orange loops for \( E_{\pm}(k) \) while OBC black. The skin modes (black lines) only exists in the area with non-vanishing winding number.

### B. The second-order skin effect

Consider the simplest 2D non-Hermitian model possessing second-order skin effect, whose Hamiltonian in momentum space is
\[ H_{2D}(k) = t_{x,y}^a e^{-ik} + t_{x,y}^e e^{ik}, \quad t_{x,y}^a, t_{x,y}^e \rightarrow t_{x,y}^a, t_{x,y}^e \]
where \( t_{x,y}^a \) and \( t_{x,y}^e \) are the real nonreciprocal hopping parameter.

The two energy bands (Riemann sheets) are
\[ E_{\pm}(k) = h_{\pm}(k) = \frac{h_1(k) \pm h_2(k)}{2} \]
with \( h_{\pm}(k) = (h_1(k) \pm h_2(k))/2 \) and \( h_1(k) = t_0 + t_- e^{-ik} + t_+ e^{ik}, h_2(k) = w_0 + w_- e^{-ik} + w_+ e^{ik}. \)

For each fixed \( k \) value, the complex energy spectrum sketches a loop \( C(k) \) which lies in the PBC energy spectrum (blue) along \( x \)-OBC energy spectrum (orange) lying inside double-PBC energy spectrum (cyan) for varying \( k_y \) with 2D and 3D plot in (a) and (b) respectively.

The parameters: \( \gamma^x = 1, \gamma^y = 0.8 \).

\[ E(k) = 2 \sqrt{T_{\pm}^2 \cos k} \] which lies in the PBC energy spectrum loop \( E_{\pm}(k_y) = t_{x,y}^a e^{-ik} + t_{x,y}^e e^{ik} \) indicating the skin effect along \( x \)-direction. Since the internal freedom is 1 in this model, we can directly obtain the effective Hamiltonian for second-order skin effect (see Sec. IIA)
\[ H_{eff}(k_y) = \sum_k (t_{x,y}^a e^{-ik} + E(k) + t_{x,y}^e e^{ik}). \] (7)

For each fixed \( k \) value, the complex energy spectrum sketches a loop \( C(k) \) for which \( E(k) \) assigns the loop center varying in \( \{-2 \sqrt{T_{\pm}^2}, 2 \sqrt{T_{\pm}^2}\} \). According to the topological origin of the first-order skin effect \( \mathcal{M} \), each loop \( C(k) \) surrounds the corresponding second-order skin modes localized on one corner [Fig. 2(d)] under both x-OBC/y-OBC.

### III. NESTED TIGHT-BINDING FORMALISM FOR SECOND-ORDER PHASE

#### A. The nested tight-binding formalism

A simplest perspective to give the second-order corner states is working out the localized states one-by-one along two related direction. It means that we put the localized information of one direction into the other, for which we call the nested process. With the lattice tight-binding model nature, our general formalism for second-order behavior is called nested tight-binding formalism.

A generic tight-binding 2D Hamiltonian with \( L_x, L_y \) lattice sites and \( R_x, R_y \) hopping range along \( x, y \) direction
respectively and \( q \) internal freedom on each site is

\[
\hat{H} = \sum_{x=1}^{L_x} \sum_{y=1}^{L_y} \sum_{\mu=1}^{q} \left( \sum_{\nu=1}^{q} \hat{c}_{x+y,1}^{\mu \dagger} \hat{t}^{x}_{x+y,1} \hat{c}_{x+y,\nu}^{\nu} + \sum_{\nu=1}^{q} \hat{c}_{x+y,\nu}^{\nu \dagger} \hat{t}^{y}_{x+y,\nu} \hat{c}_{x+y,1}^{\nu} \right) + \sum_{j=-R_y}^{R_y} \hat{c}_{x+1,y}^{\mu \dagger} \hat{t}^{y}_{x+1,y} \hat{c}_{x,y}^{\nu}.
\]

We finally arrive at a 1D effective Hamiltonian along \( x \)-direction under the biorthogonal basis along \( x \)-direction

\[
\hat{H}_{\text{eff}} = \sum_{y=1}^{L_y} \sum_{j=-R_y}^{R_y} \sum_{\alpha \beta} \sum_{\mu \nu} \hat{\Phi}^{R,\mu \nu}_{\alpha,\beta} \cdot (T^y_{\alpha,\gamma})_{\alpha \mu, \beta \nu} \cdot \hat{\Phi}^{L,\nu}_{\beta, \gamma},
\]

where \( \hat{\Phi}^{R,\mu \nu}_{\alpha,\beta} = \sum_{L_x} \sum_{\nu=1}^{q} \hat{c}_{x+1,y}^{\mu \dagger} \hat{t}^{y}_{x+1,y} \hat{c}_{x,y}^{\nu} \) and \( \hat{\Phi}^{L,\nu}_{\beta, \gamma} \) is the annihilated operator of the corresponding biorthogonal left eigenstate [see Appendix. B].

Our nested tight-binding formalism is analytically valid to investigate \( T^T \) and hybrid \( S^T \) modes when the edge-state-subspace block of \( T^y \) is independent from the bulk block, in other word, the topological edge states are not coupling with skin bulk states along \( x \)-direction. Fortunately, our nested tight-binding formalism is valid for the typical four-band model (complete block diagonal for typical parameter choice) to analytically obtain \( T^T \) and \( S^T \) corner modes (Sec. III B) which can naturally reduce to Hermitian case for second-order topological corner modes. Moreover, the block diagonal result also appears in the 2D model with extrinsic second-order \( S \) states [83]. For the skin bulk block part, the \( H_{\text{eff}} \) induces the pure second-order skin effect which is the result of combining with skin effect along another \( y \)-direction, namely bulk block of \( H_{\text{eff}} \) also has nontrivial number topology indicating the existence of the skin effect. The simplest 2D model [Eq. (9)] with pure \( S \) modes has already been given in Sec. II whose effective Hamiltonian is easily obtained as Eq. (7). Although it’s hard to analysis the \( S \) modes for more complicated model due to the complexity of bulk skin states, the numerical result also can indicate the \( S \) modes, such as in Fig. [4]. In addition, a deeper sight for \( S \) modes has been just proposed in related work [87]. Hence, we focus on the widely analysable \( S \) and \( T \) modes hereinafter.

**B. The four-band model**

Consider a 2D non-Hermitian four-band model [80] [82]

\[
H(\vec{k}) = \begin{pmatrix}
0 & 0 & H_{z,+} & -H_{z,-} \\
0 & 0 & H_{z,-}^* & H_{z,+}^* \\
H_{z,+} & H_{z,-} & 0 & 0 \\
-H_{z,-} & -H_{z,+} & 0 & 0
\end{pmatrix},
\]

where \( H_{z,\pm} = t_x \pm \delta_j + \lambda e^{\pm i\gamma} \) for \( j = 1,2 \) and \( H_{z,\pm} = t_y \pm \delta_j + \lambda e^{\pm i\gamma} \lambda \) for \( j = 3,4 \) and we set \( t_x = t_y = t \) for simplicity. The Hermitian counterpart of this model \( (\delta_j = 0, j = 1,2,3,4) \) has already been investigated in Ref. [21] [29]. Without any other parameters assignment, this Hamiltonian only preserves sublattice symmetry with \( S^T H(\vec{k}) S = -H(\vec{k}) \), \( S = \tau_z \). We first set \( \delta_1 = -\delta_2 = -\delta_3 = \delta_4 = \gamma \), from which we reminisce the model investigated in Ref. [50] with net nonreciprocity for both \( x \) and \( y \)-direction, i.e.

\[
H(\vec{k}) = (t + \lambda \cos k_x) \tau_x - (\lambda \sin k_x + i\gamma) \tau_y \sigma_z + (t + \lambda \cos k_y) \tau_y \sigma_y + (\lambda \sin k_y + i\gamma) \tau_y \sigma_z.
\]

Besides sublattice symmetry, this Hamiltonian also preserves mirror-rotation symmetry \( M_{xy}^{-1} H(\vec{k}, \vec{k}) M_{xy} = H(\vec{k}, \vec{k}) \) with \( M_{xy} = C_4 M_y \), while the Hermitian counterpart of this model preserves both mirror symmetries
$M_x = \tau_x \sigma_z, M_y = \tau_x \sigma_x$ and four-fold rotational symmetry $C_4 = [(\tau_x - i \tau_y) \sigma_0 - (\tau_x + i \tau_y)(i \sigma_y)]$.

Using our nested tight-binding formalism, we first study a single $x$-layer Hamiltonian

$$\hat{H}_x = \sum_y (\epsilon^+ m_0 \hat{c}_y + \epsilon^+ t^+_y \hat{c}_{y+1} + \hat{c}^+_y t^-_y \hat{c}_y),$$

where

$$m_0 = t(\tau_x + \tau_y \sigma_y) + i \gamma(\tau_y \sigma_x - \tau_y \sigma_z),$$
$$t^+_y = \frac{\lambda}{2} (\tau_y \sigma_y - i \tau_y \sigma_x),$$
$$t^-_y = \frac{\lambda}{2} (\tau_y \sigma_y + i \tau_y \sigma_x).$$

As usual process, we assume the eigenstate of the OBC Hamiltonian is

$$|\psi\rangle = \sum_{y=1}^{L_y} \beta^n |y\rangle |\phi\rangle,$$

where $|\phi\rangle$ is a $4$-component column vector representing the internal freedom. From the eigenequation $\hat{H}_x |\psi\rangle = \epsilon |\psi\rangle$, we can read the characteristic equation of the bulk equation

$$\det(t^-_y \beta^{-1} + m_0 + t^+_y \beta - \epsilon) = 0,$$

which gives

$$\frac{1}{\beta^2} [\lambda(t + \gamma) \beta^2 + (2t^2 - 2\gamma^2 + \lambda^2 - \epsilon^2) \beta + \lambda(t - \gamma)]^2 = 0.$$

The four nonzero finite solutions have the relation

$$\beta_1 \beta_2 = \beta_3 \beta_4 = \frac{t - \gamma}{t + \gamma}.$$

Combining with the continuous condition [46 [50], we obtain

$$|\beta_1| = |\beta_2| = |\beta_3| = |\beta_4| = \sqrt{\frac{t - \gamma}{t + \gamma}}.$$

which indicates the left-localized bulk skin effect along $y$-direction (the same for $x$-direction). In momentum space, the Hamiltonian of this model is

$$H_x(k_y) = t(\tau_x + \tau_y \sigma_y) + i \gamma(\tau_y \sigma_x - \tau_y \sigma_z) + \lambda \cos k_y \tau_y \sigma_y + \lambda \sin k_y \tau_y \sigma_x.$$

Although above Hamiltonian possesses four edge states under OBC, it's topological trivial since the edge states can be continuously absorbed into bulk due to the nonzero energy values but it contributes to the second-order corner-localized behavior. We emphasize that the 1D Hamiltonian in Eq. [15] with $k_y$ being parameter is also topological trivial, therefore we must search the second-order topological behavior for further step.

We now find the edge solutions of Hamiltonian $H_s$ under OBC and consider the left semi-infinite localized states first [40 [86]. The bulk and boundary equations are

$$(t^-_y \beta^{-1} + m_0 + t^+_y \beta) |\phi\rangle = \epsilon |\phi\rangle,$$

$$(m_0 + t^+_y \beta) |\phi\rangle = \epsilon |\phi\rangle,$$

We can easily obtain $|\phi\rangle$ the kernels of $t^-_y$ which are

$$|u_1\rangle = u_1 \cdot |\sigma\rangle,$$
$$|u_2\rangle = u_2 \cdot |\sigma\rangle,$$

where $u_1 = (0, 0, 1, 0), u_2 = (0, 1, 0, 0)$ and we denote $|\sigma\rangle = (|1\rangle, |2\rangle, |3\rangle, |4\rangle)^T$ as the internal freedom particles. Put again the linear combination of $|u_{1,2}\rangle$ into the bulk equation, we can obtain $\beta_1 = -\frac{t - \gamma}{\lambda}$ and $\epsilon_\pm = \pm \sqrt{(t + \gamma)(t - \gamma)}$ which are corresponding to the solutions

$$|\phi^\pm_L\rangle = |u_1\rangle \pm r |u_2\rangle := \phi^\pm_L \cdot |\sigma\rangle,$$

where $r = \sqrt{\frac{t\gamma}{t^2 - \gamma^2}}$. Therefore we obtain two left-localized solution with energies $\epsilon_\pm$ respectively,

$$|\psi^\pm_L\rangle = \sum_{y=1}^{L_y} \beta^n |y\rangle |\phi^\pm_L\rangle.$$

In addition, the left-localized condition $|\beta_1| < 1$ guarantees the above solutions automatically satisfying the right boundary equation for larger enough $L_y$. In the same way, we can solve the right-localized solutions with energies $\epsilon_\pm$ respectively,

$$|\psi^\pm_R\rangle = \sum_{y=1}^{L_y} \beta^{-n} |y\rangle |\phi^\pm_R\rangle,$$

where $\beta_2 = -\frac{t}{t + \gamma}$ and

$$|\phi^\pm_R\rangle = |v_1\rangle \pm r^{-1} |v_2\rangle := \phi^\pm_R \cdot |\sigma\rangle,$$

with

$$|v_1\rangle = v_1 \cdot |\sigma\rangle,$$
$$|v_2\rangle = v_2 \cdot |\sigma\rangle,$$

where $v_1 = (0, 0, 1, 0), v_2 = (1, 0, 0, 0)$.

We find that the numerical result of $U^T L \cdot T_x \cdot U_R$ and $U^T L \cdot T^+_x \cdot U_R$ are both block-diagonal with each block is a $4 \times 4$ matrix in this model, therefore we can independently deal with the edge-state-subspace. However, we have to find the correspondence left eigenstates corresponding to the known right eigenstates $|\psi^+_L\rangle$ due to the biorthogonal normalization of non-Hermitian Hamiltonian. So we solve the edge states for eigenequation $\hat{H}_s^T |\psi\rangle^* = \epsilon |\psi\rangle$. In the same way with the solving for
right eigenstates, we can finally obtain the left eigenstates with energies $\epsilon_\pm$ and left or right-localization

$$\begin{align*}
|\psi_L^{\pm}\rangle^* &= \sum_{y=1}^{L_y} \beta_2^{-y} |y\rangle \left| \phi_L^{\pm} \right>,
|\psi_R^{\pm}\rangle^* &= \sum_{y=1}^{L_y} \beta_1^{L_y-y} |y\rangle \left| \phi_R^{\pm} \right>,
\end{align*}$$

(27)

where

$$\begin{align*}
|\phi_L^{\pm}\rangle &= |u_1\rangle \pm r^{-1} |v_2\rangle := \phi_L^{\pm} \cdot |\sigma\rangle,
|\phi_R^{\pm}\rangle &= |v_1\rangle \pm r |v_2\rangle := \phi_R^{\pm} \cdot |\sigma\rangle.
\end{align*}$$

(28)

Construct the biorthogonal diagonalized matrices of the edge-subspace

$$\begin{align*}
U_R^{edge} &= \left( (\phi_L^+)^T, (\phi_L^-)^T, (\phi_R^+)^T, (\phi_R^-)^T \right),
U_L^{edge\dagger} &= \left( (\phi_r^+)^T, (\phi_r^-)^T, (\phi_r^+)^T, (\phi_r^-)^T \right)^T.
\end{align*}$$

(29)

After biorthogonal normalizing for right and left eigenstates, we can acquire the effective edge-state-subspace Hamiltonian

$$H_j^{edge} = \sum_{x=1}^{L_x} \left( \hat{\phi}_x^{j\dagger} \epsilon_0 \hat{\phi}_x^j + \hat{\phi}_x^{j\dagger} \hat{\phi}_x^{j+1} + \hat{\phi}_x^{j+1} \hat{\phi}_x^{j\dagger} \right),
$$

(30)

where $j = L, R$ corresponding to left or right-localized edge-sub-subspace and

$$\begin{align*}
\hat{\phi}_x^{j\dagger} &= (\hat{\phi}_x^{j+1}, \hat{\phi}_x^{-1}),
\hat{\phi}_x^j &= (\hat{\phi}_x^{j+1}, \hat{\phi}_x^{-1})^T,
\end{align*}$$

(31)

with

$$\begin{align*}
\hat{\phi}_x^{j\dagger} &= \sum_{y=1}^{L_y} \mathcal{N}_j \beta_j^{y-L_y} (\hat{\phi}_x^{j\dagger}, \hat{\phi}_x^{j+1}, \hat{\phi}_x^{j+1}, \hat{\phi}_x^{j\dagger}) \cdot (\phi_j^+)^T,
\hat{\phi}_x^j &= \sum_{y=1}^{L_y} \mathcal{N}_j \beta_j^{-y-L_y} \phi_j^+ \cdot (\hat{\phi}_x^{j+1}, \hat{\phi}_x^{j+1}, \hat{\phi}_x^{j+1}, \hat{\phi}_x^{j\dagger})^T.
\end{align*}$$

(32)

and $j = 1, 2$ for $\beta$ in the first equation ($j = 2, 1$ in the second equation) corresponding to $\delta = 0, 1$ and $j = L, R$, $\mathcal{N}_j$ are the normalized coefficients

$$\begin{align*}
\mathcal{N}_L &= \left[ 2 \sum_{y=1}^{L_y} (\beta_1 \beta_2^{-y}) \right]^{-\frac{1}{2}},
\mathcal{N}_R &= \left[ 2 \sum_{y=1}^{L_y} (\beta_1^{-1} \beta_2^{y}) \right]^{-\frac{1}{2}}.
\end{align*}$$

(33)

The hopping matrices are given by

$$\begin{align*}
\epsilon_0 &= \sqrt{(t + \gamma)(t - \gamma)} \sigma_z
\end{align*}$$

and

$$t^\pm = \frac{1}{2} U_L^{edge\dagger} \cdot t_x \cdot U_R^{edge} = \left[ \begin{array}{cc} t_L^\pm & 0 \\ 0 & t_R^\pm \end{array} \right],$$

where we can directly calculate to obtain

$$t_L^\pm = \frac{\lambda}{2} r^\pm \left[ \begin{array}{cc} 1 & \mp 1 \\ \mp 1 & 1 \end{array} \right],$$

(34)

and

$$t_R^\pm = \frac{\lambda}{2} r^\pm \left[ \begin{array}{cc} 1 & \mp 1 \\ \mp 1 & 1 \end{array} \right].$$

(35)

Transforming the edge effective Hamiltonian Eq. (30) into momentum space

$$H_j^{edge}(k_x) = t_j^e^{-ik_x} + \epsilon_0 + t_j^e^{ik_x},$$

(36)

the energy spectrum under PBC reads as

$$\epsilon_j^e(k_x) = t^2 - \gamma^2 + \lambda^2 + \lambda[(t + \gamma)e^{\pm ik_x} + (t - \gamma)e^{\mp ik_x}],$$

where $j = L, R$. They sketch two orange loops in the complex energy plane locating at both side imaginary axis [Fig. 3(b)], which is exactly projected from the $k_x$ dependence $x$-PBC/ $y$-OBC edge-subspace spectrum(isolated orange line in Fig. 3(a)). It indicates the bulk skin effect for $H_j^{edge}$ under OBC along $x$-direction deducing the ST mode [22] under full-OBC, which is plotted as black line lying in the orange loop in Fig. 3(b).
The skin effect indicator for $H_j^{edge}$ is also $|\rho| = \sqrt{\frac{t - \gamma}{t + \gamma}}$ which manifests all the bulk states located on the left side. Together with the edge-subspace along $y$-direction, we deduce that the four zero energy $TT$ modes are located on the four corners and the $ST$ modes are located on the low-left and up-left corners when $\lambda > t - \gamma, t + \gamma$. The four zero corner modes localized at low-left(LL), low-right(LR), up-left(RL) and up-right(RR) can be written as

$$|\Psi_{ij}\rangle = \mathcal{N}_j \sum_{x=1}^{L_x} \rho_{ij}^x \left[ |\psi_j^x\rangle - (1)^{i+j} |\psi_j^y\rangle \right] |x\rangle$$

$$= \mathcal{N}_j \sum_{x=1}^{L_x} \sum_{y=1}^{L_y} \rho_{ij}^x \beta_y^x \left[ |\phi_j^x\rangle - (1)^{i+j} |\phi_j^y\rangle \right] |x\rangle |y\rangle,$$  \hspace{1cm} (37)

where $i,j = L,R,0,1$ represents the localized behavior along $x$ and $y$-direction respectively and $\rho_1 = \beta_1, \rho_2 = \beta_2$.

However the $TT$ modes are all numerically localized on the low-left corner [Fig.3(c)] while the $ST$ modes on low-left corner with larger amplitude and low-right and up-left corners with smaller amplitude [Fig.3(d)]. In addition, the pure $SS$ modes are also all localized on the low-left corner by numerical result. The analytical and numerical results are seen not consistence but we notice that the linear combinations of energy degenerated states are also the eigenstates of the Hamiltonian with the same energy, for which we just perform a basis transformation. Based on this consideration, we can make our analytical and numerical results consistent and we illustrate this manifestation below.

Let us focus on the 1D Hamiltonian Eq. (30) to explore the tiny difference between analytical and numerical result. Following the process of Eqs. (19)-(23), we can analytically figure out the topological zero edge modes for $H_j^{edge}$ under OBC and write the two zero modes of $H_j^{edge}$ as an example

$$\psi_{0,L} = \sum_{y=1}^{L_y} \left( -\frac{t - \gamma}{\lambda} \right)^y (1,-1)^T,$$

$$\psi_{0,R} = \sum_{y=1}^{L_y} \left( -\frac{\lambda}{t + \gamma} \right)^y (1,-1)^T.$$  \hspace{1cm} (38)

The two solutions are localized on left and right side along $x$-direction for $\lambda > t - \gamma, t + \gamma$. However the four numerical edge modes are dramatically both localized on left side when we set parameters as $t = 0.6, \gamma = 0.4, \lambda = 1.5$. After comparing carefully these solutions, it’s found that the numerical solutions are indeed the precisely linear combination of the analytical two

$$\psi_0 = \pm \alpha_L \psi_{0,L} - \alpha_R \psi_{0,R},$$

but the coefficient $\alpha_R$ is extremely small comparing with $\alpha_L$ so that the two zero modes are both localized on left side. In addition, the two combination solutions are not necessary orthogonal normalization since they are biorthogonal in non-Hermitian system. Although the different between analytical and numerical result, the topological invariant winding number just characterizes the number of zero modes not the localized behavior which depends on the choice of linear combination.

Motivated by the 1D case, the four low-left corner localized second-order zero modes of our model can be obtained by linear combination of the analytical four corner localized zero modes

$$|\Psi_k\rangle = \sum_{i,j=L,R} \alpha_{ij}^k |\Psi_{ij}\rangle,$$  \hspace{1cm} (39)

where $k = 1,2,3,4$ denotes the four zero corner modes. The dominant of the coefficient $\alpha_{LL}$ induces the final four zero modes all localized at low-left corner, which are consistent with the numerical result. For the $ST$ modes, due to the mirror rotation symmetry $M_{xy}$, we also can analytically obtain $ST$ modes by considering single $y$-direction tight-binding model first, which arrive at low-left and low-right localized $ST$ modes with degenerated energies for above $ST$ modes. By properly combining these $ST$ modes with degenerated energy, we can interpret the localized behavior of the numerical $ST$ modes.

In general, it’s analysable when we take $|\delta_1| = |\delta_2|$ and $|\delta_3| = |\delta_4|$ and at least one direction net nonreciprocal. The coupling constant between neighbor lattice can also be different in general, i.e. $\lambda_1, \lambda_2$ for $x$ and $y$-direction respectively. Following our nested tight binding formalism, we first solve the direction with net nonreciprocity for a single layer such as $x$-direction. It’s well known that $\sqrt{|\frac{t_x - \delta_1}{t_x - \delta_2}|} < (>) 1$ indicates the bulk skin modes located on left (right) side along $x$-direction which indeed suggests the net nonreciprocity $\delta_1 \neq \delta_2$ relevant to the skin effect. Moreover, the localized behavior of analytical edge states is determined by $\beta_1 = -\frac{t_x - \delta_1}{\lambda_1}$ and $\beta_2 = -\frac{t_x - \delta_2}{\lambda_2}$ under the nonreciprocity condition. As derived in Ref [80], the merging-into-bulk condition is the topological phase transition point

$$|\beta_1| = |\beta_2| = \sqrt{\frac{|t_x - \delta_1|}{|t_x - \delta_2|}},$$  \hspace{1cm} (40)

this gives $(t_x - \delta_1)(t_x - \delta_2) = \pm \lambda_1^2$. Noticing the nonreciprocity condition $\delta_1 = -\delta_2 = \gamma_1$, the topological phase edge for $x$-direction is $t_x^2 - \gamma_1^2 = \pm \lambda_1^2$.

Fortunately, the edge-state-subspace block effective Hamiltonian is independent from bulk subspace as long as one direction is net nonreciprocal for our four-band model. Following the above derivation for phase edge, we can obtain the similar result for edge-state-subspace block effective Hamiltonian $t_x^2 - \gamma_2^2 = \pm \lambda_2^2$. Therefore we recover the phase diagram $t^2 - \gamma^2 = \pm \lambda^2$ in Ref [80] with $t_x = t_y = t$ and $\gamma_1 = \gamma_2 = \gamma$. Moreover, we introduce another parameters choice for the four-band model in Appendix C.
\[ H_x(\vec{k}) = 2t_x \cos k_x \tau_0 - 2ig_x \sin k_x \tau_z \]
\[ -2t_y \cos k_y \tau_y - 2ig_y \sin k_y \tau_x. \]  

The complex energy spectrum for single \( y \)-layer Hamiltonian \( H_x(k_x) \) forms a loop which indicates skin effect, while \( H_y(k_y) \) forms pure imaginary lines suppressing skin effect. For simplicity, we start from \( H_y(k_y) \) with just two localized zero topological states due to the sublattice symmetry and line gap [55, 59]

\[ H_y(k_y) = -2it_y \cos k_y \tau_y - 2ig_y \sin k_y \tau_x. \]  

We can easily work out the two localized zero modes with odd lattice sites [see Appendix. D for even sites and details]

\[ |\psi_L\rangle = \sum_{y=1}^{L_y/2} \beta^{2y-1} |2y-1\rangle \phi_L, \]
\[ |\psi_R\rangle = \sum_{y=1}^{L_y/2} \beta^{-L_y+2y} |2y-1\rangle \phi_R, \]

where \( |\beta| = \sqrt{t_y/g_y} \) with \( t_y > g_y \) and \( \phi_L = (0, 1)^T, \phi_R = (1, 0)^T \). Hence, it’s easily obtain the edge-state-subspace effective Hamiltonian, which is blocked independent with the bulk and supported by the numerical result. Actually, the effective edge Hamiltonian is exactly the transposition of \( H_x \) under OBC after similarity transformation by the biorthogonal edge-state-matrix [see Sec. III B and Appendix. D], therefore

\[ H_{eff}(k_x) = 2t_x \cos k_x \tau_0 + 2ig_x \sin k_x \tau_z. \]  

The complex energy spectrum of \( H_{eff}(k_x) \) under PBC [orange loop in Fig. 4(a)] surrounds skin bulk complex spectrum under OBC [black part in Fig. 4(a)] which is also the second-order corner localized modes under full-OBC [center part in Fig. 4(b)]. The different between analytical and numerical spectrum is due to the finite lattice site which is change with different sites number and must be consistent in the thermodynamic limit. According to the left(right) localized behavior of the skin modes of \( H_{eff}(k_x) \) under OBC, we can exactly deduce the low-left(up-right) corner modes [83]. We emphasize that the corner modes in this model is categorized into hybrid \( x \)-skin and \( y \)-topological ST modes [82]. Nevertheless, the \( TT \) modes localized at the same corners appear if the lattice site number is larger enough. In addition, the extrinsic feature is due to the extended Hermitian Hamiltonian [83] preserving only chiral symmetry without any crystal symmetry leading the termination dependence for second-order corner modes.

\section*{IV. CONCLUSION AND DISCUSSION}

In this paper, we construct the nested tight-binding formalism to exactly deduce the second-order corner-localized-behavior modes. In the sense of identifying the protected zero edge states and nonzero edge states isolated from continuous bulk, it has been discovered that the corner modes are classified to three types [82]: (i) Pure second-order skin effect (SS) modes which is the result of first skin effect both along two directions. (ii) The pure second-order topological (TT) corner modes which inherits from Hermitian counterpart are interplay between two localized behavior both along two directions. Notice that we should distinguish the topology for edge states from that for skin effect, in which the former inherits from Hermitian counterpart and the latter is pure non-Hermitian product. (iii) The most charming hybrid skin-topological (ST) which are the interplay between edge states and skin effect induced by nonzero winding number, in other word, the Hermitian ramification and pure non-Hermitian product. Utilizing the nested tight-binding formalism, we have strictly illustrated the simplest 2D model [Eq. (15)] with pure SS modes, the ST corner modes for four-band model [Eq. (15)] and the extrinsic ST corner modes for a 2D model given in Ref. [83]. More precisely, we have obtained the complete zero TT corner solutions for four-band model [Eq. (15)].

The typical zero corner modes are numerically unbroken for the relevant crystal symmetry \( M_{xy} \) for four-band model, while the zero corner modes are unbroken at only one symmetry in Hermitian case since \( M_{x} \) and \( M_{y} \) are anticommutation. The numerical unbroken modes are the linear combination of our analytical zero corner modes localized on each corner. The underlying physics for numerically unbroken corner states remains to be explored. More precisely, the nonzero edge states for one direction is not topological which can be absorbed into the bulk by...
continuous transformation \cite{S0}, leading the unstable TT zero corner modes. This instability perhaps is the origin of extrinsic second-order corner modes \cite{S3} which is left for the future work. Moreover, the mechanism of bulk-edge separation after biorthogonal transformation in the nested tight-binding formalism is perhaps related to the crystal symmetry mathematically, which also remains for future work.

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*Note added. — After completion of this work, we became aware of a recent related work \cite{S7} which proposes a deeper sight for SS modes.*

**Appendix A: The exact eigenstates of 1D tight-binding model**

Without loss of generality, any first-order tight binding model can be ascribed to a 1D tight-binding model with the edge parallel momentums $k_{\parallel}$ regard as the parameters.

$$
\hat{H} = \sum_{ij,\mu\nu} \varepsilon_{ij}^\dagger H_{ij,\mu\nu}(k_2, \ldots, k_d, \lambda') \varepsilon_{j\nu},
$$

(A1)

where we choose the first axis with OBC, $k_2, \ldots, k_d, \lambda'$ are all parameters and omit the internal freedom similarly. The Hamiltonian of a 1D tight-binding model with hopping range $-R \rightarrow R$ and internal freedom $q$ on each lattice site is

$$
\hat{H} = \sum_{n=1}^{L} \sum_{i=-R}^{R} \sum_{\mu,\nu=1}^{q} \varepsilon_{n+i,\mu\nu}^\dagger t_{i,\mu\nu} \varepsilon_{n,\nu}.
$$

(A2)

Assuming the solution is

$$
|\Phi\rangle = \sum_{n=1}^{L} |\phi_{n}\rangle |n\rangle = \sum_{n=1}^{L} \sum_{\mu=1}^{q} \beta^{n\mu} |\mu\rangle |n\rangle,
$$

(A3)

with the eigenvalue equation $\hat{H} |\Phi\rangle = E |\Phi\rangle$, we obtain the bulk equation

$$
\sum_{\nu=1}^{q} H(\beta_{\mu\nu}) \phi_{\nu} := \sum_{\nu=1}^{q} \sum_{i=-R}^{R} t_{i,\mu\nu} \beta^{i\nu} \phi_{\nu} = E \phi_{\mu}
$$

(A4)

and the characteristic equation

$$
\det( \sum_{i=-R}^{R} t_{i,\mu\nu} \beta^{i} - E ) = 0.
$$

(A5)

From the above linear equation set of $\phi^0$'s, we can linearly express the $(q-1)$ $\phi^0$'s by the remain one

$$
\phi_{\mu} = J_{\nu\mu}(\beta) \phi_{\nu} \quad \mu = 1, 2, \ldots, \nu, \ldots, q \quad \nu = 1, 2, \ldots, q
$$

(A6)

and $J_{\nu\nu} = 1$ naturally. The characteristic equation of bulk equation can be solved resulting $2qR$ roots of $\beta$ in general in which we briefly ignore the multiple roots case(it has be well studies in Ref \cite{S60}). Now the full solution is

$$
|\Phi\rangle = \sum_{n=1}^{L} \sum_{\mu=1}^{q} |\phi_{n\mu}\rangle |n\rangle = \sum_{n=1}^{L} \sum_{\mu=1}^{q} \sum_{j=1}^{2qR} \beta^{n\mu}_{j} |\mu\rangle |n\rangle.
$$

(A7)

Imposing the boundary condition both on left and right boundaries

$$
\sum_{i=-s}^{R} t_{i} |\phi_{s+i+1}\rangle = E |\phi_{1+s}\rangle, \quad \sum_{i=-R}^{s} t_{i} |\phi_{L-s+i}\rangle = E |\phi_{L-s}\rangle,
$$

(A8)

where $s = 0, 1, \ldots, (R-1)$. It's equivalent to more handy form \cite{S88}

$$
|\phi_{0}\rangle = |\phi_{-1}\rangle = \ldots = |\phi_{-R+1}\rangle = 0
$$

$$
|\phi_{L+1}\rangle = |\phi_{L+2}\rangle = \ldots = |\phi_{L+R}\rangle = 0
$$

(A9)

and then we obtain

$$
\sum_{j=1}^{2qR} \beta^{L-s}_{j} \phi^{\mu}_{j} = 0; s = 0, 1, \ldots, (R-1); \mu = 1, 2, \ldots, q,
$$

$$
\sum_{j=1}^{2qR} \beta^{L+s}_{j} \phi^{\mu}_{j} = 0; s = 1, \ldots, R; \mu = 1, 2, \ldots, q
$$

(A10)

Using Eq. (A6) and fixing a $\nu$, we obtain

$$
\sum_{j=1}^{2qR} f_{s\mu}(\beta_{j}, E) \phi^{\mu}_{j} = 0; s = 0, 1, \ldots, (R-1); \mu = 1, 2, \ldots, q,
$$

$$
\sum_{j=1}^{2qR} g_{s\mu}(\beta_{j}, E) \beta^{L}_{j} \phi^{\mu}_{j} = 0; s = 1, \ldots, R; \mu = 1, 2, \ldots, q
$$

(A11)

where

$$
\begin{align*}
&f_{s\mu}(\beta_{j}, E) = J_{\nu\mu}(\beta) \beta^{L-s}_{j}, \\
g_{s\mu}(\beta_{j}, E) = J_{\nu\mu}(\beta) \beta^{s}_{j}.
\end{align*}
$$

(A12)

We can denote the $2qR$ functions $f_{s\mu}$ and $g_{s\mu}$ as $f_{j}, g_{j}, j = 1, 2, \ldots, qR$ respectively and then the boundary requires \cite{S60}

$$
\begin{vmatrix}
 f_{1}(\beta_{1}, E) & \ldots & f_{1}(\beta_{2qR}, E) \\
 \vdots & \ddots & \vdots \\
 f_{qR}(\beta_{1}, E) & \ldots & f_{qR}(\beta_{2qR}, E) \\
 g_{1}(\beta_{1}, E) & \beta^{L}_{1} & \ldots & g_{1}(\beta_{2qR}, E) \beta^{L}_{2qR} \\
 \vdots & \ddots & \vdots \\
 g_{qR}(\beta_{1}, E) & \beta^{L}_{qR} & \ldots & g_{qR}(\beta_{2qR}, E) \beta^{L}_{2qR}
\end{vmatrix} = 0.
$$

(A13)
When we sort $|\beta_1| \leq \ldots \leq |\beta_{qR}| \leq |\beta_{qR+1}| \leq \ldots \leq |\beta_{2qR}|$ and take limitation $L \to \infty$, the boundary condition restricts two type $\beta$ solutions: discrete and continuous types corresponding to edge if exists and bulk states respectively. If $|\beta_{qR}| < |\beta_{qR+1}|$, only one leading order term can survives when take $L \to \infty$ in Eq. (A13)

$$F(\beta_{i \in P_1}, \beta_{j \in Q_1}, E, \phi_{\beta_{qR}}) := \det \begin{vmatrix} f_1(\beta_1, E) & \ldots & f_1(\beta_{qR}, E) \\ \vdots & \ddots & \vdots \\ f_{qR}(\beta_1, E) & \ldots & f_{qR}(\beta_{qR}, E) \end{vmatrix} \times \det \begin{vmatrix} g_1(\beta_{qR+1}, E) & \ldots & g_1(\beta_{2qR}, E) \\ \vdots & \ddots & \vdots \\ g_{qR}(\beta_{qR+1}, E) & \ldots & g_{qR}(\beta_{2qR}, E) \end{vmatrix} = 0,$$  \hspace{1cm} (A14)

where $P_1 = \{\beta_1, \ldots, \beta_{qR}\}$, $Q_1 = \{\beta_{qR+1}, \ldots, \beta_{2qR}\}$. The above equation gives discrete $\beta$'s deducing the edge states isolated from the continuous bulk states if it exists.

If $|\beta_{qR}| = |\beta_{qR+1}|$, there be two leading order terms surviving. Let $P_0 = \{\beta_1, \ldots, \beta_{qR-1}, \beta_{qR+1}\}$, $Q_0 = \{\beta_{qR}, \beta_{qR+2}, \ldots, \beta_{2qR}\}$, then the continuous $\beta$'s are given (A15)

$$F(\beta_{i \in P_1}, \beta_{j \in Q_1}, E, \phi_{\beta_{qR}}) = \left( \frac{\beta_{qR}}{\beta_{qR+1}} \right)^L.$$  \hspace{1cm} (A15)

Following the above logic, we can obtain the bulk band spectra (or continuous band spectra) and generalized Brillouin zone (GBZ) [51] as

$$E_{bulk} = \{ E \in \mathbb{C} : |\beta_{qR}(E)| = |\beta_{qR+1}(E)| \},$$  \hspace{1cm} (A16)

$$C_{\beta} = \{ \beta \in \mathbb{C} : \forall E \in E_{bulk}, |\beta_{qR}(E)| = |\beta_{qR+1}(E)| \}.$$  \hspace{1cm} (A16)

We emphasize that the GBZs depends on Riemann energy spectrum sheet $\mu = 1, 2, \ldots, q$ in general. In other word, there are $q$ GBZs $C_{\beta}^\mu$ one-to-one correspondence to $q$ Riemann energy spectrum sheet (i.e. complex energy bands) $E^\mu$ deduced from the $q$ internal freedom. However, the multiply GBZs are degenerated in some simple model, such as non-Hermitian SSH model [46]. In this paper, we only consider the degenerated GBZs or single band model leaving the multiply GBZs for numerical calculation in future work.

The above process to solve the eigenstates in non-hermitian system is the non-Bloch band theory without any symmetry constraint proposed in Ref. [50], which has been extended to symplectic class [88, 89] and $Z_2$ skin effect [54] recently.

### Appendix B: Biorthogonal Diagonalization of the single $y$-layer Hamiltonian

The $qL_x$ eigenvalue solutions in main text can also be write as creation fermion operators

$$\Phi_{\alpha \nu}^{R, \mu} = \sum_{x=1}^{L_x} \sum_{\nu=1}^{q} \tilde{\phi}_{\alpha \nu}^{R, \mu} \epsilon_{x, y},$$  \hspace{1cm} (B1)

where $\tilde{\phi}_{\alpha \nu}^{R, \mu}$ is the $\nu$-$\mu$-th row component of $\alpha$-$\mu$-th right eigenstate for general non-Hermitian system. We define the right eigenstate matrix

$$U_R = \begin{bmatrix} \tilde{\phi}_{11}^{R, 11} & \cdots & \tilde{\phi}_{11}^{R, L_x} \\ \vdots & \ddots & \vdots \\ \tilde{\phi}_{11}^{R, L_x, q} & \cdots & \tilde{\phi}_{11}^{R, L_x, q} \end{bmatrix},$$  \hspace{1cm} (B2)

and

$$\tilde{\epsilon}_y = (\epsilon_{1, y}^1, \epsilon_{1, y}^2, \epsilon_{1, y}^3, \ldots, \epsilon_{1, y}^q)^T,$$

$$\tilde{\epsilon}_y = (\epsilon_{1, y}^1, \epsilon_{1, y}^2, \epsilon_{1, y}^3, \ldots, \epsilon_{1, y}^q)^T,$$

$$\Phi_R = (\Phi_{1, y}^{R, 11}, \Phi_{1, y}^{R, 11}, \Phi_{1, y}^{R, 11}, \ldots, \Phi_{1, y}^{R, L_x, q}),$$

$$\Phi_L = (\Phi_{1, y}^{L, 11}, \Phi_{1, y}^{L, 11}, \Phi_{1, y}^{L, 11}, \ldots, \Phi_{1, y}^{L, L_x, y})^T,$$  \hspace{1cm} (B3)

then

$$\tilde{\Phi}_y^R = \tilde{\epsilon}_y \cdot U_R.$$  \hspace{1cm} (B4)

Solving $\tilde{H}^\dagger$ with similar manner for $\tilde{H}$, we can obtain the left eigenstates with the equations

$$\tilde{\Phi}_y^L = \tilde{\epsilon}_y \cdot U_L,$$

$$\tilde{\Phi}_y^L = \tilde{\Phi}_y^R,$$  \hspace{1cm} (B5)

and the biorthogonal relation

$$U_R \cdot U_L^\dagger = U_L \cdot U_R^\dagger = \hat{1}.$$  \hspace{1cm} (B6)

The inverse relation between two fermion operators is then

$$\tilde{\epsilon}_y = \tilde{\Phi}_y^L \cdot U_L^\dagger,$$

$$\tilde{\epsilon}_y = U_R \cdot \tilde{\Phi}_y^R.$$  \hspace{1cm} (B7)

The result transformed to the biorthogonal basis for the single $y$-layer single-particle Hamiltonian $\tilde{H}_y$ is

$$\epsilon = \tilde{U}_L^\dagger \cdot \tilde{H}_y \cdot U_R,$$

$$\tilde{H}_y = \tilde{\Phi}_y^L \cdot \epsilon \cdot \tilde{\Phi}_y^L,$$  \hspace{1cm} (B8)

where

$$\tilde{H}_y = \begin{bmatrix} T_{1}^\dagger & \cdots & T_{1}^\dagger \\ \vdots & \ddots & \vdots \\ T_{L_x}^\dagger & \cdots & T_{L_x}^\dagger \end{bmatrix},$$

$$T_i^\dagger = \begin{bmatrix} T_{1, 11}^\dagger & \cdots & T_{1, 1q}^\dagger \\ \vdots & \ddots & \vdots \\ T_{L_x, 11}^\dagger & \cdots & T_{L_x, 1q}^\dagger \end{bmatrix},$$

$$\epsilon = \begin{bmatrix} \epsilon_1(\beta_{1}) & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \epsilon_1(\beta_{L_x}) & \cdots & 0 \end{bmatrix}.$$  \hspace{1cm} (B9)
Notice that the last \( q \) eigenvalues are edge states energy for nontrivial phase which deducing the \( ST \) modes and \( TT \) corner modes. For Hermitian case, the biorthogonal relation reduces to \( U_L^\dagger = U_R^{-1} \), inducing that the diagonal process become standard in linear algebra \( \epsilon = U^{-1} \cdot \hat{H}_y \cdot U \).

**Appendix C: Other parameters choice for four-band model**

If we set \( \delta_1 = \delta_2 = -\delta_3 = -\delta_4 = \gamma \), the net nonreciprocity only exists along \( y \)-direction. The \( M_{xy} \) is broken in this case. The corner modes in this case contain: four second-order topological \( (TT) \) zero modes and TS modes while the \( SS \) modes are absent. This case is almost the same as the double nonreciprocity case in the edge subspace, in which the tiny different is the form of edge states

\[
|\phi_{L}^{\pm}(s_n)\rangle = |u_1\rangle \pm r|u_2\rangle, \\
|\phi_{R}^{\pm}(s_n)\rangle = |v_1\rangle \pm r|v_2\rangle.
\]

The edge effective Hamiltonian different from the double nonreciprocity one is then deduced

\[
t_L^\pm = \frac{\lambda}{2} r^\pm \begin{bmatrix} 1 & \mp 1 \\ \pm 1 & 1 \end{bmatrix}, \\
t_R^\pm = \frac{\lambda}{2} r^\pm \begin{bmatrix} 1 & \mp 1 \\ \mp 1 & 1 \end{bmatrix}.
\]

The double reciprocity case: \( \delta_1 = \delta_2 = -\delta_3 = -\delta_4 = \gamma \). The \( M_{xy} \) is also broken in this case. Unfortunately, the effective Hamiltonian is not block diagonal in numerical result. Therefore we cannot give the \( TS \) and TS modes analytically at present, but the numerical plots is obvious in Ref \[82\]. Meanwhile the \( SS \) corner modes are also absent.

Asymmetry case: \( \delta_1 = \delta_2 = 0 \) or \( \delta_3 = \delta_4 = 0 \) while the other direction is nonreciprocity. The mirror symmetry \( M_x \) or \( M_y \) is restored. The \( TT \) and \( ST \) corner modes are present while \( SS \) modes absent.

Hermitian case: \( \delta_1 = \delta_2 = \delta_3 = \delta_4 = 0 \). Both \( M_x \) and \( M_y \) are restored as well as the fourfold rotation symmetry \( C_4 \), the only existence corner modes are \( TT \) zero modes.

Non-hermitian onsite gain and loss case: \( \delta_1 = \delta_2 = \delta_3 = \delta_4 = 0 \) with adding term \(-iv\tau_z \), the \( C_4 \) is restored and the only existence corner modes are \( TT \) zero modes.

**Appendix D: Edge states for 2D extrinsic model**

The zero edge modes has very simply form for \( H_y(k_y) \) in main text when the lattice site is odd number. The bulk equation for the Hamiltonian is

\[
det(t_y^+ \beta + t_y^- \beta^{-1}) = 0,
\]

where

\[
t_y^+ = \begin{bmatrix} 0 & -t_y - g_y \\ t_y - g_y & 0 \end{bmatrix}, \quad (D1)
\]

and

\[
t_y^- = \begin{bmatrix} 0 & -t_y + g_y \\ t_y + g_y & 0 \end{bmatrix}. \quad (D2)
\]

Due to the boundary condition

\[
t_y^+ \phi_2 = 0,
\]

\[
t_y^- \phi_{L_y=1} = 0,
\]

the amplitude for exact zero edge states are destroyed on even lattice site which is consistence with the numerical result. The similar amplitude destruction is also found in Ref \[45\]. Utilizing bulk equation to obtain \( \beta = \sqrt{\frac{t_y - g_y}{t_y + g_y}} \), the two edge states are give by Eq. \[43\] in main text. For even lattice site number, the exact edge solutions has same form with odd site case when the site number is large enough. However, the numerical results are tended to lineally combining the two localized edge states, which final numerically results in the diagonal corners localized corner modes. In addition, the edge states form is also valid for the Hamiltonian \( H_x(k_x) \) along \( x \)-direction under OBC.

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