INFINITE GRAPHS THAT DO NOT CONTAIN CYCLES OF LENGTH FOUR

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Abstract. We construct an infinite graph $G$ that does not contain cycles of length four having the property that the sequence of graphs $G_n$ induced by the first $n$ vertices has minimum degree $\delta(G_n) \geq n^{\sqrt{2} - 1 + o(1)}$.

1. Introduction

The maximum number of edges that a graph $G(V, E)$ of order $n$ can have without containing cycles of length four is denoted by $\text{ex}(n, C_4)$ and it is known \cite{5} that

$$\text{(1.1)} \quad \text{ex}(n, C_4) = \frac{1}{2} n^{3/2}(1 + o(1)).$$

Let $G(V, E)$ be an infinite $C_4$-free graph and consider the sequence of graphs $G_n(V_n, E_n)$ induced by the first $n$ vertices. Peng and Timmons \cite{6} proved that

$$\liminf_{n \to \infty} |E_n|n^{-3/2} \leq 0.41$$

and constructed an example with $|E_n| \geq 0.23n^{3/2}(1 + o(1))$.

A more interesting problem appears when instead of $|E_n|$ we consider the minimum degree of $G$. We recall that the minimum degree of finite graph $G(V, E)$ is defined by $\delta(G) = \min\{\deg v : v \in V\}$. It is not difficult to prove that for any $n$ there exists a $C_4$-free graph $G$ of order $n$ with $\delta(G) = \sqrt{n}(1 + o(1))$. The analogous problem for infinite graphs seems to be more difficult. The upper bounds in (1.1) and (1.2) imply $\delta(G_n) \leq \sqrt{n}(1 + o(1))$ and $\liminf_{n \to \infty} \delta(G_n)/\sqrt{n} \leq 0.82$ respectively. However it is not clear if an infinite $C_4$-free graph $G$ with $\delta(G_n) \gg \sqrt{n}$ may exists.

The main obstruction is that the good constructions of finite $C_4$-free graphs come from algebraic constructions and they are too rigid to be extended to an infinite graph. A similar problem appears in the analogous problem for Sidon sequences. While any interval $[1, n]$ contains a Sidon set $A$ with $|A| = \sqrt{n}(1 + o(1))$, Erdős \cite{10} proved that the counting function of an infinite Sidon sequence $A$ cannot satisfy $A(n) \gg \sqrt{n}$. We are not able to prove the analogous result for infinite $C_4$-free graphs, so we state it as a conjecture.

Conjecture 1. If $G$ is an infinite $C_4$-free graph, then

$$\liminf_{n \to \infty} \delta(G_n)/\sqrt{n} = 0$$

for the sequence of the graphs $G_n$ induced by the first $n$ vertices.
We propose a second conjecture in the opposite direction.

**Conjecture 2.** For any $\epsilon > 0$ there exists an infinite $C_4$-free graph $G$ with 
\[ \delta(G_n) \gg n^{1/2-\epsilon}. \]

The probabilistic method provides an infinite $C_4$-free graph $G$ with $\delta(G_n) \gg (n/\log n)^{1/3}$ and we show in the next section how to construct one with $\delta(G_n) \gg n^{1/3}$. The exponent $1/3$ seems to be a natural barrier for this problem with the standard methods. We use some of the ideas behind the construction of the Sidon sequence in [3] to prove our main result.

**Theorem 1.1.** There exists an infinite $C_4$-free graph $G$ with 
\[ \delta(G_n) \geq n^{\sqrt{2}-1+o(1)}, \]
where $G_n$ is the graph induced by the first $n$ vertices.

### 2. Some remarks

An easy way to construct finite $C_4$-free graphs is using Sidon sets in finite abelian groups.

**Lemma 2.1.** Let $G$ be a finite abelian group. If $A \subset G$ is a Sidon set, the graph $G(V,E)$ with $V = G$ and 
\[ E = \{\{x,y\} : x \neq y, x+y \in A\} \]
is a $C_4$-free graph with minimum degree $\delta(G) \geq |A| - 1$.

**Proof.** We observe that if $(x,y,u,v)$ is a $C_4$ then $x+y = a_1$, $y+u = a_2$, $u+v = a_3$, $v+x = a_4$ for some $a_1,a_2,a_3,a_4 \in A$. Since $(x+y)+(u+v) = (y+u)+(v+x)$ then $a_1 + a_3 = a_2 + a_4$. Thus $a_1 = a_2$ or $a_1 = a_4$ and then $x = u$ or $y = v$. On the other hand it is clear that $\deg(x) = |A| - 1$ if $2x \in A$ and $\deg(x) = |A|$ otherwise. \square

There are several families of abelian groups $G$ containing Sidon sets $A$ with $|A| \sim \sqrt{|G|}$. These families and Lemma 2.1 provide constructions of $C_2$-free graphs $G$ of order $n$ with $\delta(G) \sim \sqrt{n}$ for special sequences of values of $n$. While this is well known we have not found in the literature a proof that works for all $n$. The following probabilistic construction was communicated to us by Alon [2]. Just choose a prime $p$ so that $p^2 + p + 1$ is at least $n$ and at most $n + o(n)$ and take the induced subgraph on a random set of $n$ vertices in the usual example on $p^2 + p + 1$ vertices (polarity graph of projective plane): an easy probabilistic argument shows that all degrees will stay $(1 + o(1))\sqrt{n}$.

The explicit construction that we present here uses the Sidon set $A = \{(x,x^2) : x \in \mathbb{F}_p\} \subset \mathbb{F}_p \times \mathbb{F}_p$.

**Proposition 1.** Let $\theta$ be a real number having the property that for any $x$ large enough, the interval $[x,x+2^\theta]$ contains a prime number. Then 
\[ n^{1/2} + O(n^{\theta/2}) \leq \delta(n;C_4) \leq n^{1/2} + 1/2, \]
where \( \delta(n, C_4) = \max\{\delta(G) : G \text{ is } C_4 \text{-free of order } n\} \).

It is known that we can take \( \theta = 0.525 \).

**Proof.** It is well known that if \( G(V, E) \) is \( C_4 \)-free of order \( n \) then
\[
|E| \leq \frac{1}{2}n^{3/2} + \frac{4}{3}.
\]
So we have
\[
\delta(G) \leq \frac{1}{n} \sum_v \deg(v) = \frac{2|E|}{n} \leq \frac{2\text{ex}(n, C_4)}{n} \leq n^{1/2} + 1/2.
\]

For the lower bound, assume that \( n \) is large enough and let \( p \) be a prime
\( p \in [\sqrt{n}, \sqrt{n} + n^{\theta/2}] \). We consider the Sidon set \( A = \{(x, x^2) : x \in \mathbb{F}_p\} \subset \mathbb{F}_p^2 \)
and the \( C_4 \)-free graph \( G(V, E) \) induced by \( A \) with \( V = \mathbb{F}_p^2 \) and
\[
E = \{(x_1, y_1), (x_2, y_2) \} : (x_1, x_2) + (y_1, y_2) \in A, \ (x_1, y_1) \neq (x_2, y_2) \}.
\]
We define \( r, s \) by \( p^2 - n = rp + s \), \( 0 \leq s \leq p - 1 \), \( 0 \leq r \) and remove the set of vertices
\( V^* = \{(a, b) : 0 \leq a \leq r - 1, \ 0 \leq b \leq p - 1\} \cup \{(r, b) : 0 \leq b \leq s - 1\} \).
Let \( G_0(V_0, E_0) \) be the graph induced by the vertices \( V_0 = V \setminus V^* \). First we observe that
\[
|V_0| = |V| - |V^*| = p^2 - rp - s = n.
\]
Consider a vertex \((x, y) \in V_0\). It is easy to check that
\[
\deg_G(x, y) = \begin{cases} |A| - 1 & \text{if } y = 2x^2 \\ |A| & \text{otherwise}. \end{cases}
\]
Thus
\[
\deg_{G_0}(x, y) = \deg_G(x, y) - |\{(u, v) \in V^* : (x, y) + (u, v) \in A\}|
\geq |A| - 1 - |\{(u, v) : 0 \leq u \leq r, \ y + v = (x + u)^2\}|
\geq p - 1 - (r + 1) \geq p - 1 - (p - \frac{n}{p} + 1) = \frac{n}{p} - 2
\geq \frac{n}{\sqrt{n} + n^{\theta/2}} - 2 = \sqrt{n} + O\left(n^{\theta/2}\right).
\]

An easy way to construct an infinite \( C_4 \)-free graph with \( |E_n| \asymp n^{3/2} \) was communicated to us by Simonovits \cite{9}. Consider the graph \( G(V, E) \) that is the infinite union of independent graphs \( G(k) \), where \( G(k) \) is an extremal \( C_4 \)-free graph with \( 2^k \) vertices. Obviously the graph \( G \) is \( C_4 \)-free. If \( 2^{k+1} \leq n < 2^{k+2} \) then the vertex labeled with \( n \) is in \( G(k + 1) \), so
\[
|E_n| \geq |E(G(k))| \gg 2^{\frac{k}{2}} \gg n^{3/2}.
\]

The graph above and similar constructions, as the graph in \cite{6}, do not work
if we are interested in infinite graphs with large \( \delta(G_n) \) because the degree of each vertex is bounded in the whole graph \( G \). An alternative attempt is to use infinite Sidon sequences.
Proposition 2. Let $A$ be an infinite Sidon sequence of positive integers. The graph $G(V,E)$ with $V = \mathbb{N}$ and $E = \{(x,y) : x \neq y, \ x+y \in A\}$ is $C_4$-free and

$$\delta(G_n) \geq \min_{1 \leq x \leq n-1} A(n+x) - A(x) - 1.$$ 

Proof. Lemma 2.1 implies that $G$ is a $C_4$-free graph. For the degree condition we observe that

$$\deg_{G_n}(x) = |\{y : x < y \leq n, \ x+y \in A\}| + |\{y : y < x, \ x+y \in A\}| = A(n+x) - A(2x) + A(2x-1) - A(x) \geq A(n+x) - A(x) - 1.$$ 

Proposition 2 is useful when $A(x)$ has an asymptotic behavior. For example, Proposition 4.1 applied to the Sidon sequence $B$ in Theorem 1.1 provides an infinite $C_4$-free graph with $\delta(G_n) \sim Cn^{1/3}$ for some constant $C > 0$.

It is a difficult problem to find an infinite Sidon sequence $A$ with large counting function. Erdős observed that the greedy algorithm provides a sequence with $A(x) \gg x^{1/3}$. Almost 50 years later, Ajtai, Komlós and Szemerédi [1] proved the existence of one with $A(x) \gg (x \log x)^{1/3}$. Using a clever method, Ruzsa [8] proved the existence of an infinite Sidon sequence with $A(x) \gg x^{1/3-o(1)}$, improving the exponent $1/3$ for the first time. Recently the author [3] has constructed an explicit Sidon sequence with similar counting function. The constructions in [8] and [3] are distinct but both use the fact that the set of the primes is a multiplicative Sidon sequence.

Unfortunately we cannot apply Proposition 1.1 to these last sequences. The reason is that their counting function is quite irregular. Nevertheless we will use some of the ideas behind the construction of the Sidon sequence in [3] to construct the infinite graph $G$ in Theorem 1.1.

We insist that the main obstruction to extend the algebraic constructions for finite $C_4$-free graphs to infinite $C_4$-free graphs is that they are too rigid. We finish this section with a semialgebraic construction of a finite $C_4$-free graph that can be considered the finite version of the construction of the infinite graph in Theorem 1.1

Theorem 2.1. Given $q$ a prime number, let $\mathbb{F}_q$ be the finite field of $q$ elements. The graph $G(V,E)$ where $V = \mathbb{F}_q^*$ and $E = \{(x,y) : xy = p \text{ for some prime } p \leq \sqrt{q}\}$ is a $C_4$-free graph with $\delta(G) \sim \sqrt{q} / \log \sqrt{q}$.

Proof. First we prove that $G$ is $C_4$-free. If $(x,y,u,v)$ is a $C_4$ in $G$ then there exist primes $p_1, p_2, p_3, p_4 \leq \sqrt{q}$ such that $xy = p_1, \ yu = p_2, \ uv = p_3, \ vx = p_4$. It implies that $p_1p_3 = p_2p_4$ in $\mathbb{F}_q$ and then $p_1p_3 \equiv p_2p_4 \pmod{q}$. Since $1 < p_1 p_3, \ p_2 p_4 \leq q$ we have the equality $p_1 p_3 = p_2 p_4$ in the integers and then $p_1 = p_2$ or $p_1 = p_4$, so $x = u$ or $y = v$.

For the degree condition it is clear that $\delta(G) \geq \pi(\sqrt{q}) - 1 \sim \sqrt{q} / \log \sqrt{q}$. □
We call this construction semialgebraic because it uses an algebraic part (the finite field $\mathbb{F}_q$) and a non algebraic part, the sequence of the prime numbers, which is common for any $q$. This construction is not so good as other algebraic constructions with $\delta(G) \ge q^{1/2}(1 + o(1))$. We lose a logarithm factor but we gain the possibility of combining these constructions for distinct $\mathbb{F}_q$. Roughly, the strategy to construct the graph $G$ in Theorem 1.1 is to paste the graphs described in Theorem 2.1 for infinite $\mathbb{F}_q$ using the Chinese remainder theorem. The construction of the graph in Theorem 2.1 for distinct $\mathbb{F}_q$ and how to paste them are some of the key ideas in the construction of the graph $G$ in Theorem 1.1.

The asymptotic estimate $\pi(x) \sim \frac{x}{\log x}$ for the number of primes $p \le x$, is known as the prime number theorem. We will need also other well known equivalent forms.

Lemma 2.2. Let $q_k$ the $k$-th prime number and $Q_k = \prod_{j=1}^k q_j$. The asymptotic estimate $\pi(x) \sim \frac{x}{\log x}$ is equivalent to the followings:

i) $q_k \sim k \log k$

ii) $\log Q_k \sim k \log k$.

It is usual to denote by $\pi(x; q, a)$ to the number of primes $p \le x$ such that $p \equiv a \pmod{q}$. The prime number theorem for arithmetic progressions in his strongest version says that for any $a, q$, $(a, q) = 1$ and any $A > 0$ we have

$$\pi(x; q, a) \sim \frac{x}{\phi(q) \log x}$$

uniformly for any $q \ll (\log x)^A$.

A last remark about the statement of Theorem 1.1. What we mean for $\delta(G_n) \ge n^{\frac{2}{e-1} + o(1)}$ is that there exists a function $\epsilon(n) \to 0$ when $n \to \infty$ and a positive integer $n_0$ such that $\delta(G_n) \ge n^{\frac{2}{e-1} - \epsilon(n)}$ for $n \ge n_0$. It could be that $\deg(G_n) = 0$ for a finite number of positive integers $n$. If we are not happy with this we can modify slightly the graph $G$ to get a new graph $G'$ with $\delta(G'_n) \ge n^{\frac{2}{e-1} - \epsilon(n)}$ for maybe a distinct function $\epsilon'(n) \to 0$ and such that $\delta(G'_n) \ge 1$ for all $n \ge 2$. To see this, let $G(V, E)$ be the graph in Theorem 1.1 and suppose that there exist some vertices $v_1, \ldots, v_m$ such that $\deg_{G_n}(t_m) = 0$ for some $n \ge t_m$. Consider the graph $G'(V', E')$ where

$$V' = V \cup \{w_0, w_1, \ldots, w_m\} \quad \text{and} \quad E' = E \cup \bigcup_{j=1}^m \{\{w_0, w_j\}, \{w_j, v_i\}\}.$$  

Now we relabel the vertices of the new graph starting with $w_0, w_1, \ldots, w_m$ and then we continue with the vertices of the original graph. It is easy to check that $G'$ is also $C_4$-free and has the property that $\deg_{G'_n}(i) \ge 1$ for all $i \le n$ and $n \ge 2$. 

3. Sketch of the construction

We start with a sketch of the construction of the graph we will construct to prove Theorem 1.1. The graph $\mathcal{G}(V, E)$ is the following:

The set of vertices $V$ is the union of a sequence of finite sets of vertices $V_k$ that we will describe later:

$$V = \bigcup_{k=2}^{\infty} V_k.$$  

We label the vertices starting with the vertices in $V_2$, then the vertices of $V_3$ and so on. The order of the vertices in the same $V_k$ is arbitrary.

We consider a special Sidon sequence $A$ (the sequence $A$ described in Proposition 3) and then we construct certain edges between between $V_j$ and $V_k$ but only when $j \neq k$ and they satisfy some of these two conditions:

1) $j + k \in A$, $j \equiv k \pmod{3}$, $|j - k| \geq 4$
2) $j$ and $k$ are consecutive numbers.

We can describe conditions 1) and 2) using an auxiliary graph $\mathcal{H}$ (see the picture below) which we call the graph of the indices.

![Graph](image)

Thanks to the conditions 1) and 2) and because $A$ is a Sidon sequence, the auxiliary graph $\mathcal{H}$ above is $C_4$-free (Lemma 4.1). This is nice because then the four vertices of a possible cycle $C_4$ in $\mathcal{G}(V, E)$ cannot belong to four distinct $V_k$.

Vertices in $\mathcal{G}(V, E)$: Let $(q_k)$ be the sequence of the prime numbers. We consider for each $k \geq 2$ the set

$$V_k = \mathbb{F}_{q_1}^\times \times \cdots \times \mathbb{F}_{q_k}^\times.$$  

Actually we will remove the element 1 in the last $\mathbb{F}_{q_k}^\times$ for technical reasons.

Edges between $V_j$ and $V_k$. Assume that $\{j, k\} \in \mathcal{E}(\mathcal{H})$ (otherwise there are not edges between $V_j$ and $V_k$).

Given $x = (x_1, \ldots, x_j) \in V_j$ and $y = (y_1, \ldots, y_k) \in V_k$, $j < k$, we construct the edge $\{x, y\}$ if and only if there exits a prime $p$ in a certain set of primes $P_k$ such that

$$x_i y_i = p \quad (\text{in } \mathbb{F}_{q_i}), \quad 1 \leq i \leq j$$

$$y_i = p \quad (\text{in } \mathbb{F}_{q_i}), \quad j < i \leq k.$$  

(3.1)

Sometimes we write $x \overset{p}{\sim} y$ or we draw $x \bullet \overset{p}{\sim} \bullet y$ to emphasize the prime $p$ involved in the edge.
Since the graph of the indices $H$ is $C_4$-free, a typical shape of a cycle $(x, y, v, u)$ in $G(V, E)$ is the following (assume for example that $y, u \in V_{k_2}$ and $k_1 < k_2 < k_3$):

$$\begin{array}{c}
\text{p_1} \\
\text{V_{k_1}} \\
\text{p_2} \\
\text{y} \\
\text{p_3} \\
\text{V_{k_2}} \\
\text{u} \\
\text{p_4} \\
\text{V_{k_3}} \nonumber
\end{array}$$

The conditions (3.1) for the four edges imply certain congruences between the primes $p_1, p_2, p_3, p_4$, $p_1, p_2 \in P_{k_2}$, $p_3, p_4 \in P_{k_3}$ (Proposition 5) and there are not too many sets $(p_1, p_2, p_3, p_4)$ satisfying these congruences. In fact, in the definition of $P_{k_3}$ we remove previously those $p_3$ involved to destroy all the possible $C_4$ in $G$, so the graph $G$ is $C_4$-free. To be more precise we define for each $k \geq 2$ the set $P_k^* = \hat{P}_k \setminus P_k^*$ where

$$\hat{P}_k = \left\{ p : \frac{(q_1 \cdot \ldots \cdot q_{k-1})^c}{k-1} < p \leq \frac{(q_1 \cdot \ldots \cdot q_k)^c}{k} \right\},$$

c = \sqrt{2} - 1 and roughly we can say that $P_k^*$ is the set of the bad primes in $\hat{P}_k$ that we have to remove to destroy all the cycles $C_4$. We will see that the set $P_k^*$ is smaller than $\hat{P}_k$ when $c = \sqrt{2} - 1$.

Finally we prove that $\delta(G_n) = n^{c + o(1)}$ with $c = \sqrt{2} - 1$. If we try to use a constant $c$ greater than $\sqrt{2} - 1$ (and then a better lower bound for $\delta(G_n)$) the upper bound we get for $|P_k^*|$ is greater than $|\hat{P}_k|$ and our argument does not work. This is the limit of our method.

4. Proof of Theorem 1.1

4.1. An auxiliar Sidon sequence. Pollington and Eyden [7] used a variant of the usual greedy algorithm to get a Sidon sequence whose counting function has an asymptotic behavior.

**Theorem 4.1 ([7]).** There exists an absolute constant $C_0$ and a Sidon sequence $B$ such that for any real $x \geq 1$, the interval $(x, x + C_0 x^{2/3})$ contains an unique element of $B$.

It is easy to check that the counting function of the sequence $B$ in Theorem 4.1 has an asymptotic behavior $B(x) \sim C x^{1/3}$ with $C = C_0^3 / 27$. This sequence can be used in Proposition 2 to construct an infinite $C_4$-free graph $G$ with $\delta(G_n) \sim C n^{1/3}$ as we have mentioned in the introduction.

**Proposition 3.** There exists an absolute constant $C$ and a Sidon sequence $A$ of even numbers with $|a - a'| \geq 4$ when $a \neq a'$ and such that for any $y \geq 1$, the interval $(y, y + C y^{2/3})$ contains three elements $a_0, a_1, a_2 \in A$ which are congruent with 0, 1, 2 (mod 3) respectively.
Proof. Let $B = (b_k)$ be the Sidon sequence in Theorem 4.1 and define $A = (a_k)$ where

\[
\begin{align*}
a_{3k} &= 12b_{3k} \\
a_{3k+1} &= 12b_{3k+1} + 4 \\
a_{3k+2} &= 12b_{3k+2} + 2.
\end{align*}
\]

It is clear that all of them are even numbers, that any three consecutive elements of $A$ satisfy the congruent conditions in some order and that $|a - a'| \geq 4$ for any $a \neq a'$, $a, a' \in A$.

The sequence $A$ is a Sidon sequence. If $a_i - a_j = a_r - a_s$ then

\[
|12(b_i - b_j + b_s - b_r)| \leq 8.
\]

Thus $b_i - b_j = b_r - b_s$ and then $\{b_i, b_s\} = \{b_j, b_r\}$, so $\{a_i, a_s\} = \{a_j, a_r\}$.

Take $C = \frac{36^3}{12^2}C_0^3$. We will prove that any interval $(y, y + Cy^2/3]$ contains three elements of $A$. If it is not the case then one the intervals $I_j = (y + (j - 1)(C/3)y^{2/3}, y + j(C/3)y^{2/3}]$, $j = 1, 2, 3$ does not contain any element of $A$. Since any element $a \in A$ is of the form $a = 12b + \delta$ for some $b \in B$ and some $\delta \in \{0, 2, 4\}$ then for some $j = 1, 2, 3$ the interval

\[
\left(\frac{y - \delta + (j - 1)(C/3)y^{2/3}}{12}, \frac{y - \delta + j(C/3)y^{2/3}}{12}\right)
\]

does not contain any element $b \in B$ for some $\delta \in \{0, 2, 4\}$. Now we apply Theorem 4.1 and then we have that

\[
(C/36)y^{2/3} < C_0 \left(\frac{y - \delta + (j - 1)(C/3)y^{2/3}}{12}\right)^{2/3}.
\]

This implies that

\[
(C/36) < C_0 \left(\frac{1 - \delta/y + 2(C/3)y^{-1/3}}{12}\right)^{2/3}.
\]

We can assume that $1 < C/3$ and then we have

\[
(C/36) < C_0 \left(\frac{C}{12}\right)^{2/3} \leq C_0 \left(\frac{C}{12}\right)^{2/3} \Rightarrow C < \frac{36^3}{12^2}C_0^3.
\]

4.2. The graph of the indices, $\mathcal{H}(K, \mathcal{E})$. We use the sequence $A$ in Proposition 3 to construct the graph of indices $\mathcal{H}(K, \mathcal{E})$.

For the graph of indices we mean that $K = \{2, 3, \ldots\}$ is the set on indices in the union $V = \bigcup_{k \in K} V_k$ and that there are edges between $V_j$ and $V_k$ only if $\{j, k\} \in \mathcal{E}(\mathcal{H})$.

We define the graph $\mathcal{H}(K, \mathcal{E})$ as follows:
We write $c$.

The construction of $G(V,E)$ is the set of the indices of the sequence $V_k$.

Edges: $E = E_1 \cup E_2$ where

$$E_1 = \{ (j,k) : j \equiv k \pmod{3}, |k-j| \geq 4 \text{ and } j+k \in A \}$$

$$E_2 = \{ (k,k+1) : k \geq 2 \}$$

Lemma 4.1. The graph $G(V,E)$ defined above is $C_4$-free.

Proof. We distinguish several cases:

- The four edges in the cycle belong to $E_2$. It is clear that it is not possible.
- The cycle contains exactly three edges in $E_2$. In this case the only possibility is a $C_4$ of the form $(k,k+1,k+2,k+3)$, but the edge $(k,k+3) \notin E_1$ because $|(k+3) - k| < 4$.
- The cycle contains exactly two edges in $E_2$. There exist three possible configurations: $(k,k+1,k+2,j)$, $(k,k+1,j,j+1)$ and $(k,k+1,j+1,j)$.
  - Case $(k,k+1,k+2,j)$. It implies that $(k+2,j), (j,k) \in E_1$ which is not possible because if $j \equiv k \pmod{3}$ then $k+2 \not\equiv j \pmod{3}$.
  - Case $(k,k+1,j,j+1)$. It implies that $(k+1,j), (j+1,k) \in E_1$ which is not possible because if $k+1 \equiv j \pmod{3}$ then $j+1 \not\equiv k \pmod{3}$.
  - Case $(k,k+1,j+1,j)$. It implies that $(k+1,j+1), (j,k) \in E_1$ and then $2(k+2j+2), 2k+2j \in A$, which is not possible because $|a-a'| \geq 4$ when $a,a' \in A$, $a \neq a'$.
- The cycle contains exactly one edge in $E_2$. In this case the cycle is of the form $(k,k+1,j,l)$ and $(k+1,j), (j,l), (l,k) \in E_1$. Then $k+1+j = a_1$, $j+l = a_2$, $l+k = a_3$ for some $a_1,a_2,a_3 \in A$. Thus $2(k+j+l) + 1 = a_1 + a_2 + a_3$, which is not possible because all the elements of $A$ are even numbers.
- The four edges in the cycle $(j,k,l,m)$ belong to $E_1$. Then $j+k = a_1$, $k+l = a_2$, $l+m = a_3$, $m+j = a_4$ for some $a_1,a_2,a_3,a_4 \in A$. It implies that $a_1+a_3 = a_2+a_4$ and then $a_1 = a_2$ or $a_1 = a_4$ that implies that $j = l$ or $k = m$.

\[ \square \]

4.3. The construction of $G(V,E)$. Let $(q_i)$ be the sequence of the prime numbers and let $Q_k = \prod_{i=1}^k q_i$ denotes the product of the first $k$ prime numbers. We write $c = \sqrt{2} - 1$.

We consider, for any $k \geq 2$, the set $P_k = \hat{P}_k \setminus P_k^*$ where

$$P_k = \left\{ p \text{ prime : } \frac{Q_{k-1}}{k-1} < p \leq \frac{Q_k}{k} \right\},$$

and $P_k^*$ is the set formed by the primes $p \in \hat{P}_k$ such that there exists some $j$ with $Q_j < Q_k^{2r}$ and primes $p' \in \hat{P}_k$, $r, r' \in \hat{P}_j$, $pr \neq p'r'$ satisfying the
congruences

\[
\begin{align*}
pr & \equiv p'r' \pmod{Q_j} \\
p & \equiv p' \pmod{Q_k/Q_j}.
\end{align*}
\]

The strange definition of $P_k^*$ will be clear later, but roughly we can say that it is the subset of primes we have to remove from $\hat{P}_k$ to destroy all the $C_4$ that may appear in the graph.

Now we define the graph $G = (V, E)$ as follows:

- **Vertices:**
  \[ V = \bigcup_{k \geq 2} V_k \quad \text{where} \quad V_k = F_{q_1}^* \times \cdots \times F_{q_k-1}^* \times (F_{q_k}^* \setminus \{1\}). \]

- **Edges:** If \( x = (x_1, \ldots, x_j) \in V_j, \quad y = (y_1, \ldots, y_k) \in V_k \) with \( j \leq k \), then \( \{x, y\} \in E(G) \) if and only if \( \{j, k\} \in E(H) \) and there exists a prime \( p \in P_k \) such that

  \[
  \begin{align*}
  x_i y_i &= p \pmod{F_{q_i}} \quad \text{for all} \quad i \leq j \\
  y_i &= p \pmod{F_{q_i}} \quad \text{for all} \quad j < i \leq k.
  \end{align*}
\]

We write \( x \overset{p}{\sim} y \) or we draw \( x \bullet p \bullet y \) to show the prime \( p \) involved in the edge.

4.4. **Properties of** $G(V, E)$.

**Proposition 4.** If $G$ contains a $C_4$, say \( (x, y, u, v) \), then there is a $V_k$ containing the vertices $x, u$ (but not $y, v$) or the vertices $y, v$ (but not $x, u$).

**Proof.** Since $H$ is $C_4$-free (Lemma 4.1), it is not possible that the four vertices belong to four distinct $V_k$. On the other hand the two vertices of an edge cannot belong to the same $V_k$. \[ \square \]

Proposition 4 implies that at most three distinct $V_k$ are involved in a $C_4$, say $V_{k_1}, V_{k_2}, V_{k_3}, \ k_1 \leq k_2 \leq k_3$. We say that the cycle is of type $[k_1, k_2, k_3]$.

**Proposition 5.** Let \( (x, y, u, v) \) be a $C_4$ in $G$ of type $[k_1, k_2, k_3]$, \( k_1 \leq k_2 \leq k_3 \) and suppose that \( x \overset{k_2}{\sim} y \overset{k_2}{\sim} u \overset{k_3}{\sim} v \overset{k_3}{\sim} x \). Then \( p_2, p_2' \in P_{k_2}, \ p_3, p_3' \in P_{k_3}, \ p_2p_3' \neq p_2'p_3 \) and

\[
\begin{align*}
p_2p_3' &\equiv p_2'p_3 \pmod{Q_{k_2}} \\
p_3' &\equiv p_3 \pmod{Q_{k_3}/Q_{k_3}}.
\end{align*}
\]

**Proof.** We have to consider three cases according Proposition 4

**First case:** \( x, u \in V_{k_1}, \ y \in V_{k_2}, \ v \in V_{k_3}, \ k_1 < k_2 \leq k_3 \).
The Chinese remainder theorem implies

\[ x_iy_i = p_2 \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_1 \]
\[ y_i = p_2 \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_2 \]
\[ x_iu_i = p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_1 \]
\[ u_iy_i = p'_2 \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_1 \]
\[ y_i = p'_2 \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_2 \]
\[ x_iu_i = p'_3 \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_1 \]
\[ u_iy_i = p'_3 \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_1 \]
\[ y_i = p'_3 \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_2 \]
\[ v_i = p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_3 \]
\[ v_i = p'_3 \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_3. \]

Combining the equalities above we have

\[ x_iy_iu_iy_i = p_2p'_3 = p'_2p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_1 \]
\[ y_iu_i = p_2p'_3 = p'_2p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_2 \]
\[ v_i = p_3 = p'_3 \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_3. \]

The Chinese remainder theorem implies

\[ p_2p'_3 \equiv p'_2p_3 \pmod{q_1 \cdots q_{k_2}} \]
\[ p'_3 \equiv p_3 \pmod{q_{k_2+1} \cdots q_{k_3}}. \]

It remains to prove that \( p_2p'_3 \neq p'_2p_3 \). If \( p_2p'_3 = p'_2p_3 \) then \( p_2 = p'_2 \) or \( p_2 = p_3 \).

- If \( p_2 = p'_2 \) then \( x_iy_i = u_iy_i \ (\text{in } \mathbb{F}_{q_i}), \ i \leq k_1 \) which implies that \( x_i = u_i \ (\text{in } \mathbb{F}_{q_i}), \ i \leq k_1 \) and then that \( x = u \).
- If \( p_2 = p_3 \) then \( k_2 = k_3 \) (since \( P_{k_2} \cap P_{k_3} = \emptyset \) if \( k_2 < k_3 \)) and the equations \( x_iy_i = x_iu_i \ (\text{in } \mathbb{F}_{q_i}), \ i \leq k_1 \) and \( y_i = v_i \ (\text{in } \mathbb{F}_{q_i}), \ k_1 < i \leq k_2 \) imply that \( y_i = v_i \ (\text{in } \mathbb{F}_{q_i}), \ i \leq k_2 \) and then that \( u = v \).

**Second case:** \( y \in V_{k_1}, \ x, u \in V_{k_2}, \ v \in V_{k_3}, \ k_1 < k_2 < k_3. \)
In this case there exist $p_2, p_2' \in \mathcal{P}_{k_2}$ and $p_3, p_3' \in \mathcal{P}_{k_3}$ such that

\[
y_i x_i = p_2 \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_1 \quad y_i u_i = p_2' \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_1
\]
\[
x_i = p_2 \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_2 \quad u_i = p_2' \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_2
\]
\[
x_i v_i = p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_2 \quad u_i v_i = p_3' \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_2
\]
\[
v_i = p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad k_2 < i \leq k_3 \quad v_i = p_3' \ (\text{in } \mathbb{F}_{q_i}), \quad k_2 < i \leq k_3.
\]

Combining the equalities above we have

\[
x_i y_i u_i v_i = p_2 p_3' = p_2' p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_1
\]
\[
x_i u_i v_i = p_2 p_3' = p_2' p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_2
\]
\[
v_i = p_3 = p_3' \ (\text{in } \mathbb{F}_{q_i}), \quad k_2 < i \leq k_3.
\]

The Chinese remainder theorem implies

\[
p_2 p_3' \equiv p_2' p_3 \ (\text{mod } q_1 \cdots q_{k_2})
\]
\[
p_3' \equiv p_3 \ (\text{mod } q_{k_2+1} \cdots q_{k_3}).
\]

It remains to prove that $p_2 p_3' \neq p_2' p_3$. If $p_2 p_3' = p_2' p_3$ then $p_2 = p_2'$ or $p_2 = p_3$. The case $p_2 = p_3$ is not possible because $\mathcal{P}_{k_2} \cap \mathcal{P}_{k_3} = \emptyset$. If $p_2 = p_2'$ then $y_i x_i = y_i u_i \ (\text{in } \mathbb{F}_{q_i}), \ i \leq k_1$ and $x_i = u_i \ (\text{in } \mathbb{F}_{q_i}), \ k_1 < i \leq k_2$ which implies that $x_i = u_i \ (\text{in } \mathbb{F}_{q_i}), \ i \leq k_2$ and then that $x = u$.

**Third case:** $y \in V_{k_1}$, $v \in V_{k_2}$, $x, u \in V_{k_3}$, $k_1 \leq k_2 < k_3$.

Actually we will prove that this third case is not possible. Otherwise there exist $p_2, p_2', p_3, p_3' \in \mathcal{P}_{k_3}$ such that

\[
y_i x_i = p_2 \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_1 \quad y_i u_i = p_2' \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_1
\]
\[
x_i = p_2 \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_3 \quad u_i = p_2' \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_3
\]
\[
v_i x_i = p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_2 \quad v_i u_i = p_3' \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_2
\]
\[
x_i = p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad k_2 < i \leq k_3 \quad v_i = p_3' \ (\text{in } \mathbb{F}_{q_i}), \quad k_2 < i \leq k_3.
\]

Combining the equalities above we have

\[
y_i x_i v_i u_i = p_2 p_3' = p_2' p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad i \leq k_1
\]
\[
x_i v_i u_i = p_2 p_3' = p_2' p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad k_1 < i \leq k_2
\]
\[
x_i u_i = p_2 p_3' = p_2' p_3 \ (\text{in } \mathbb{F}_{q_i}), \quad k_2 < i \leq k_3
\]
The Chinese remainder theorem implies
\[ p_2p'_3 \equiv p'_2p_3 \pmod{q_1 \cdots q_k}. \]

If \( p_2p'_3 \neq p'_2p_3 \) we use that \( p_2, p_3, p'_2, p'_3 \leq Q^c_{k_3}/k_3 \) to get
\[ \frac{Q^c_{k_3}}{k_3^3} > |p_2p'_3 - p'_2p_3| \geq Q_{k_3}, \]
which is not possible because \( 2c < 1 \).

If \( p_2p'_3 = p'_2p_3 \) then \( p_2 = p'_2 \) or \( p_2 = p_3 \).

- If \( p_2 = p'_2 \) we have that \( x_i = u_i \) (in \( F_{q_i} \)), \( k_1 < i \leq k_3 \) and \( y_ix_i = y_iu_i \), (in \( F_{q_i} \)), \( i \leq k_1 \) so \( x_i = u_i \), (in \( F_{q_i} \)), \( i \leq k_3 \) and then \( x = u \).
- If \( p_2 = p_3 \) and \( k_1 = k_2 \) then \( y_ix_i = v_ix_i \), (in \( F_{q_i} \)), \( i \leq k_2 \) and then \( y = v \).
- If \( p_2 = p_3 \) and \( k_1 < k_2 \) then \( x_i = v_ix_i \) (in \( F_{q_i} \)), \( k_1 < i \leq k_2 \). In particular \( v_{k_2} = 1 \), which is not possible because \( (v_1, \ldots, v_{k_2-1}, 1) \notin V_{k_2} = F_{q_1} \times \cdots \times F_{q_{k_2-1}} \times (F_{q_{k_2}} \setminus \{1\}) \).

\( \square \)

**Proposition 6.** The graph \( G(V, \mathcal{E}) \) is \( C_4 \)-free.

**Proof.** If \( G \) contains a \( C_4 \) of type \([k_1, k_2, k_3], k_1 \leq k_2 \leq k_3 \), Proposition 5 implies that the cycle is of the form
\[ x \overset{p_2}{\sim} y \overset{p'_2}{\sim} u \overset{p'_3}{\sim} v \overset{p_3}{\sim} x \]
for some \( p_2, p'_2 \in \mathcal{P}_{k_2} \) and \( p_3, p'_3 \in \mathcal{P}_{k_3} \) satisfying \( p_2p'_3 \neq p'_2p_3 \) and
\[ p_2p'_3 \equiv p'_2p_3 \pmod{Q_{k_2}} \quad p'_3 \equiv p_3 \pmod{Q_{k_3}/Q_{k_2}}. \]
Using that \( p_2, p'_2 \leq Q^c_{k_2}/k_2 \) and \( p_3, p'_3 \leq Q^c_{k_3}/k_3 \) we have
\[ Q^c_{k_2}Q^c_{k_3}/(k_2k_3) \geq |p_2p'_3 - p'_2p_3| \geq Q_{k_2}, \]
which implies that \( Q_{k_2} < Q^c_{k_3}. \)

By construction of \( \mathcal{P}_k^c \) it implies that \( p_3 \in \mathcal{P}_k^c \) and then that \( p_3 \notin \mathcal{P}_{k_2} \). \( \square \)

**Proposition 7.** We have
\[ |\widehat{\mathcal{P}}_k| \sim \frac{Q^c_k}{ck^2 \log k} \quad \text{and} \quad |\mathcal{P}_k^c| \ll \frac{Q^c_k}{k^4}. \]

**Proof.** We use \( \frac{Q^c_{k-1}}{k-1} = o \left( \frac{Q^c_k}{k} \right) \) and Lemma 2.2 to obtain
\[ |\widehat{\mathcal{P}}_k| = \pi \left( \frac{Q^c_k}{k} \right) - \pi \left( \frac{Q^c_{k-1}}{k-1} \right) \sim \pi \left( \frac{Q^c_k}{k} \right) \sim \frac{Q^c_k}{k \log(Q^c_k/k)} \sim \frac{Q^c_k}{ck^2 \log k}. \]
The upper bound for $|\mathcal{P}_k^*|$ is more involved. If $p_1 \in \mathcal{P}_k^*$ then, by construction, there exists $j$ with $Q_j < Q_k^{\frac{c}{1-c}}$ and primes $p'_1 \in \mathcal{P}_k$, $p_2, p'_2 \in \mathcal{P}_j$ satisfying \(4.2\). We can write
\[
p_1(p_2 - p'_2) = p_1p_2 - p'_1p_2 + (p'_1 - p_1)p'_2 = \frac{p_1p_2 - p'_1p_2}{Q_j}Q_j + \frac{(p'_1 - p_1)p'_2}{Q_k/Q_j}Q_k/Q_j.
\]
The conditions \(4.2\) imply that $s_1 = \frac{p_1p_2 - p'_1p_2}{Q_j}$ and $s_2 = \frac{(p'_1 - p_1)p'_2}{Q_k/Q_j}$ are nonzero integers satisfying
\[
|s_1| = \frac{|p_1p_2 - p'_1p_2|}{Q_j} \leq \frac{Q_jQ_k^c}{jkQ_j}, \quad |s_2| = \frac{|(p'_1 - p_1)p'_2|}{Q_k/Q_j} \leq \frac{Q_jQ_k^c}{jkQ_k/Q_j}.
\]
Thus, if $p_1 \in \mathcal{P}_k^*$ then $p_1$ must be a divisor prime of some integer $s \neq 0$ in some set
\[
S_{j,k} = \left\{ s = s_1Q_j + s_2Q_k/Q_j : 1 \leq |s_1| \leq \frac{Q_jQ_k^c}{jkQ_j}, \quad 1 \leq |s_2| \leq \frac{Q_jQ_k^c}{jkQ_k/Q_j} \right\}
\]
for some $j$ with $Q_j < Q_k^{\frac{c}{1-c}}$. If $\omega_k(s)$ denotes the number of primes $p_1 \in \mathcal{P}_k$ dividing $s$ we have that
\[
|\mathcal{P}_k^*| \leq \sum_{Q_j < Q_k^{\frac{c}{1-c}}} \sum_{s \in S_{j,k}} \omega_k(s).
\]

We claim that $\omega_k(s) \leq 1$ for $k$ large enough. We observe that if some $s \in S_{j,k}$ has two distinct primes divisors $p, p' \in \mathcal{P}_k$ then we would have that
\[
Q_k^{c-1} < pp' \leq |s| \leq 2kQ_k^c
\]
and then that $Q_k^{2c-1} < 2Q_k^{\frac{c}{1-c}}Q_k^c = 2Q_k^{\frac{c}{1-c}}Q_k^{\frac{c}{1-c}}$, which implies that $Q_k^{s(1-2q)} \leq 2Q_k^{\frac{c}{1-c}}$. Taking logarithms we would have that $\log Q_k - 1 \leq \log q_k$ and using Lemma \(2.2\) we can see that it is not possible.

Then we have that
\[
|\mathcal{P}_k^*| \leq \sum_{Q_j < Q_k^{\frac{c}{1-c}}} |S_{j,k}| \ll \sum_{Q_j < Q_k^{\frac{c}{1-c}}} \frac{Q_j^{2c}Q_k^{2c}}{j^2k^{2c}Q_k} \ll \frac{Q_k^{\frac{3c-1}{1-c}}}{k^4}.
\]

Finally notice that $\frac{3c-1}{1-c} = c$ for $c = \sqrt{2} - 1$. \(\Box\)

We label the vertices of $G$ starting with the vertices in $V_2$, then the vertices in $V_3$ and so on.

**Proposition 8.** If a vertex in $V_k$ is labeled with $n$ then
\[
\phi(Q_{k-1}) - 1 < n \leq \phi(Q_k) - 1
\]
where $\phi$ is the Euler function. In particular we have that $Q_k = n^{1+o(1)}$ and $k \sim \log n / \log \log n$. 
Proposition 9. \( \phi(m) = m^{1+o(1)} \) and Proposition 2.2 imply the last part of the Proposition. \( \square \)

**Proof.** We have to prove that for any vertex labeled with \( m \leq n \) in \( V_k \) then

\[
|V_2| + \cdots + |V_{k-1}| < n \leq |V_2| + \cdots + |V_k|
\]

and we have

\[
|V_2| + \cdots + |V_k| = (q_1 - 1)(q_2 - 2) + \cdots + (q_1 - 1)\cdots (q_{k-1} - 1)(q_k - 2)
\]

\[
= -1 + (q_1 - 1)\cdots (q_{k-1} - 1) = \phi(Q_k) - 1.
\]

The well known estimate \( \phi(m) = m^{1+o(1)} \) and Proposition 2.2 imply the last part of the Proposition. \( \square \)

**Proposition 9.** \( \delta(G_n) \geq n^{\sqrt{2}-1+o(1)} \).

**Proof.** We have to prove that for \( n \) large enough then \( \deg_{G_n}(y) = n^{\sqrt{2}-1+o(1)} \) for any vertex \( y \) labeled with \( m \leq n \). Suppose that the vertex labeled with \( n \) is in \( V_k \). Then

\[
y = (y_1, \ldots, y_j) \in V_j = \mathbb{F}_{q_1}^* \times \cdots \times (\mathbb{F}_{q_j}^* \setminus \{1\})
\]

for some \( j \leq k \). Let \( C \) be the constant in Proposition 3 and distinguish two cases:

**Case** \( j \geq k - 2C(2k)^{2/3} \). In this case we use that

\[
\deg_{G_n}(y) \geq |\{x \in V_{j-1} : (x, y) \in E\}|.
\]

**Claim 1.** For any prime \( p \in \mathcal{P}_j \) such that \( p \equiv y_j \mod q_j \) and \( p \not\equiv y_{j-1} \mod q_{j-1} \)

\[
x = (x_1, \ldots, x_{j-1}) = (y_1^{-1}p, \ldots, y_{j-1}^{-1}p) \in V_{j-1} = \mathbb{F}_{q_1}^* \times \cdots \times (\mathbb{F}_{q_{j-1}}^* \setminus \{1\})
\]

is a neighbor of \( y \).

**Proof.** The condition \( p \not\equiv y_{j-1} \mod q_{j-1} \) guaranites that \( x_{j-1} \not\equiv 1 \) and notice that

\[
x_iy_i = p \mod \mathbb{F}_{q_i} \quad \text{for any } i \leq j - 1
\]

\[
y_j = p \mod \mathbb{F}_{q_j}.
\]

Since in addition \( \{j, j-1\} \in E(H) \) we conclude that \( x \) is a neighbor of \( y \). \( \square \)

Observe also that each prime \( p \) in Claim 1 provides a distinct neighbor \( x \) of \( y \).

Otherwise, if \( x \not\in E \) and \( x \not\in \mathcal{E} \) then \( p \equiv p' \mod Q_j \) and \( Q_j/j \geq |p - p'| \geq Q_j \),

which is not possible. Thus we have

\[
\deg_{G_n}(y) \geq |\{p \in \mathcal{P}_j : p \equiv y_j \mod q_j, p \not\equiv y_{j-1} \mod q_{j-1}\}|.
\]

Denoting by \( z_j \) the solution \( \mod q_{j-1}q_j \) of the two congruences \( x \equiv y_j \mod q_j \) and \( x \equiv y_{j-1} \mod q_{j-1} \) we get

\[
\deg_{G_n}(y) \geq |\hat{\mathcal{P}}_j(q_j, y_j)| - |\hat{\mathcal{P}}_j(q_j, q_{j-1}, z_j)| - |\mathcal{P}_j^*|
\]

where

\[
\hat{\mathcal{P}}_j(q_j, y_j) = \{p \in \hat{\mathcal{P}}_j : p \equiv y_j \mod q_j\};
\]

\[
\hat{\mathcal{P}}_j(q_j, q_{j-1}, z_j) = \{p \in \hat{\mathcal{P}}_j : p \equiv z_j \mod q_{j-1}q_j\}.
\]
Thus we have that
\[ |\tilde{P}_j(q_j, y_j)| = \pi(Q_j^c/j; q_j, y_j) - \pi(Q_{j-1}^c/(j-1); q_j, y_j) \]
\[ |\tilde{P}_j(q_j q_{j-1}, z_j)| = \pi(Q_j^c/j; q_j q_{j-1}, z_j) - \pi(Q_{j-1}^c/(j-1); q_j q_{j-1}, z_j) \]

Lemma 2.2 implies that \( q_j \sim \log Q_j \), so \( q_j \ll \log(Q_j^c/j) \) and we can apply \( 2.4 \) with \( A = 1 \) to get
\[ \pi(Q_j^c/j; q_j, y_j) \sim \frac{Q_j^c}{q_j \log(Q_j^c/j)} \sim \frac{Q_j^c}{cj^3 \log^2 j} \]
\[ \pi(Q_{j-1}^c/j; q_j, y_j) \sim \frac{Q_{j-1}^c}{q_j \log(Q_{j-1}^c/j)} \sim \frac{Q_{j-1}^c}{cj^3 \log^2 j} = o(\pi(Q_j^c/j; q_j, y_j)). \]

Then we have
\[ (4.4) \quad |\tilde{P}_j(q_j, y_j)| \sim \frac{Q_j^c}{cj^3 \log^2 j}. \]

Analogously, since \( q_j q_{j-1} \ll \log^2(Q_j^c/j) \) we can apply \( 2.4 \) with \( A = 2 \) to deduce that
\[ (4.5) \quad |\tilde{P}_j(q_j q_{j-1}, z_j)| \sim \frac{Q_j^c}{cj^4 \log^3 j}. \]

Proposition 7 implies that
\[ (4.6) \quad |\tilde{P}_j^*| \ll Q_j^c/j^4 = o\left(|\tilde{P}_j(q_j, y_j)|\right). \]

Putting \((4.4), (4.5)\) and \((4.6)\) in \((4.3)\) we have
\[ \deg_{G_n}(y) \geq (1 + o(1)) \frac{Q_j^c}{cj^3 \log^2 j}. \]

Since \( n < Q_k \) (see Proposition 8) we have
\[ \deg_{G_n}(y) \geq (1 + o(1)) \frac{n^c}{cj^3 \log^2 j (Q_k/Q_j)^c}. \]

Notice that \((Q_k/Q_j) \leq q_k^{k-j} \). We use that \( k - 2C(2k)^{2/3} \leq j < k \) and the asymptotic \( k \sim \log n/\log \log n \) in Proposition 8 to get
\[ \log(cj^3 \log^2 j (Q_k/Q_j)^c) \ll \log k + (k - j) \log q_k \ll k^{2/3} \log k = o(\log n), \]
so \( cj^3 \log^2 j (Q_k/Q_j)^c \leq n^{o(1)} \) and then \( \deg_{G_n}(y) \gg n^{c+o(1)} \).

**Case** \( j < k - 2C(2k)^{2/3} \).

**Claim 2.** There exists \( l, k - C(2k)^{2/3} < l < k \) such that \( \{j, l\} \in \mathcal{E}(H) \).

**Proof.** The interval \((k + j - C(2k)^{2/3}, k + j)\) contains an element \( a \in A \) with \( a \equiv 2j \pmod{3} \). Otherwise, applying Proposition 8 with \( y = k + j \) we would have that \( C(2k)^{2/3} < C(k + j - C(2k)^{2/3})^{2/3} < C(2k - 3C(2k)^{2/3})^{2/3} < C(2k)^{2/3} \).

Take \( l = a - j \). It is clear that
\[ l = a - j \in \left(k - C(2k)^{2/3}, k\right). \]
and that \( l \equiv a - j \equiv j \pmod{3} \). Since \( l - j \geq C(2k)^{2/3} > 4 \) we conclude that \( \{j, l\} \in \mathcal{E}(H) \).

Since \( l < k \) and the vertex labeled with \( n \) is in \( V_k \) then
\[
\deg_{G_n}(y) \geq |\{(x, y) \in \mathcal{E} : x \in V_l\}|.
\]

**Claim 3.** For any \( p \in \mathcal{P}_l \) with \( p \not\equiv 1 \pmod{q_l} \), the vertex
\[
x = (x_1, \ldots, x_l) = (py_1^{-1}, \ldots, py_j^{-1}, p, \ldots, p) \in V_l = \mathbb{F}_{q_1}^* \times \cdots \times (\mathbb{F}_{q_l}^* \setminus \{1\})
\]
is a neighbor of \( y \).

**Proof.** We observe that the condition \( p \not\equiv 1 \pmod{q_l} \) guarantees that \( x_l \neq 1 \). On the other hand it is clear that
\[
x_i y_i = p \quad (\text{in } \mathbb{F}_{q_i}), \quad 1 \leq i \leq j
\]
\[
x_i = p \quad (\text{in } \mathbb{F}_{q_i}), \quad j < i \leq l.
\]
Since \( j \equiv l \pmod{3} \), \(|j - l| \geq 4\) and \( j + l \in A\) then \( \{j, l\} \in \mathcal{E}(H) \), so \( x \) is a neighbor of \( y \). \( \square \)

Since distinct primes in Claim 3 provide distinct neighbors of \( y \) we have that
\[
\deg_{G_n}(y) \geq |\{p \in \mathcal{P}_l : p \not\equiv 1 \pmod{q_l}\}|,
\]
so we have
\[
\deg_{G_n}(y) \geq |\mathcal{P}_l| - |\hat{\mathcal{P}}_l(q_l, 1)| = |\hat{\mathcal{P}}_l| - |\mathcal{P}_l^*| - |\hat{\mathcal{P}}_l(q_l, 1)|
\]
The prime number theorem implies that
\[
|\hat{\mathcal{P}}_l| \sim \frac{Q_l^e}{cl^2 \log l}
\]
and the prime number theorem for arithmetic progressions (see (2.41)) that
\[
|\hat{\mathcal{P}}_l(q_l, 1)| \sim \frac{Q_l^e}{lq_l \log(Q_l^e/l)} \sim \frac{Q_l^e}{cl^3 \log^2 l}.
\]
Since Proposition 7 says that \(|\mathcal{P}_l^*| \ll Q_l^e/1^4\) we conclude that
\[
\deg_{G_n}(y) \geq \frac{Q_l^e}{cl^2 \log l} (1 + o(1)).
\]
Now we use that \( n < Q_k \) (see Proposition 3) to write the inequality
\[
\deg_{G_n}(y) \geq \frac{n^c}{cl^2 \log l (Q_k/Q_l)^c} (1 + o(1)).
\]
We observe that \((Q_k/Q_l) \leq q_k^{k-1}\). Using also that \( k - C(2k)^{2/3} < l < k \) and that \( k \sim \log n/\log \log n \) we have that
\[
\log(cl^2 \log l (Q_k/Q_l)^c) \ll \log k + (k - l) \log q_k \ll k^{2/3} \log k = o(\log n),
\]
which implies that
\[
\deg_{G_n}(y) \geq n^{c + o(1)}.
\]
\( \square \)

Theorem 1.1 is consequence of Propositions 6 and 9.
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