Optimal input signal distribution for nonlinear optical fiber channel with small Kerr nonlinearity.

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We consider the information channel described by Schrödinger equation with additive Gaussian noise. We introduce the model of the input signal and the model of the output signal receiver. For this channel, using perturbation theory for the small nonlinearity parameter, we calculate the first three terms of the expansion of the conditional probability density function in the nonlinearity parameter. At large signal-to-noise power ratio we calculate the conditional entropy, the output signal entropy, and the mutual information in the leading and next-to-leading order in the nonlinearity parameter and in the leading order in the parameter $1/\text{SNR}$. Using the mutual information we find the optimal input signal distribution and channel capacity in the leading and next-to-leading order in the nonlinearity parameter. Finally, we present the method of the construction of the input signal with the optimal statistics for the given shape of the signal.

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I. INTRODUCTION

Nowadays the fiber-optic communication channels are actively developed. That is why it is important to know their maximum information transmission rate, i.e., the channel capacity. For small powers of the outgoing signal, these channels are well described by the linear models. For the powers in question, the capacity of the noisy channel was found analytically in Shannon’s famous papers [1, 2]:

$$C \propto \log (1 + \text{SNR}) ,$$

where $\text{SNR} = P/N$ is the signal-to-noise power ratio, $P$ is the average input signal power, and $N$ is the noise power. As one can see from this relation, to increase the channel capacity with the noise power being fixed it is necessary to increase the signal power. However, as the signal power increases, the nonlinear effects become more important. In this case, a simple expression for the capacity of this noisy nonlinear channel is unknown. The reason is that the expression should take into account all the details of the mathematical model of the nonlinear channel. This model implies the following components: the input signal model, i.e., the method for encoding incoming information, the physical signal propagation model across the fiber wire, the receiver model (i.e., the signal detection features, frequency filtering), and the procedure of the signal post-processing. Thus, for different channel models the expressions for the capacity will also be different even for some matching components. It is very difficult to find an explicit expression for the capacity even for a specific (and often highly simplified) model of a nonlinear communication channel. A more realistic problem for such channels is to find the upper or lower bounds for the capacity. For example, in the case of commonly used models involving the signal propagation governed by the nonlinear Schrödinger equation (NLSE) with additive white Gaussian noise, see Refs. [3–6] and references therein, an expression for the capacity has not yet been obtained. However, in the case when a small parameter is present in the model, often it is possible to invoke the perturbation theory for this parameter if the zero-parameter model turns out to be solvable. For instance, in the case of the low noise power in the channel (i.e., for the large $\text{SNR}$ parameter), one can develop an analog of the
semiclassical approach in quantum mechanics [7]. Further, in the case when the coefficient of the second dispersion in the channel is small, it is also possible to construct the perturbation theory based on this parameter [3, 10], since the zero-parameter model is well developed: [11, 19]. Finally, in the case of the moderate input power, it is also possible to develop the perturbation theory for the Kerr nonlinearity parameter of the fiber: see [20]. In different approaches, the capacity for an optical fiber channel with nonzero dispersion and Kerr nonlinearity has been studied both analytically and numerically, see Refs. [21–28] and references therein.

In the present paper, we focus on the study of the nonlinear channel described by NLS with additive Gaussian noise using the perturbation theory in the small parameter of Kerr nonlinearity and large SNR. To this end, we consistently build the model of the input signal $X(t)$, we study the impact of the spectral noise width on the output signal $Y(t)$ (i.e., the raw signal in the receiving point), and we investigate the influence of the signal detection procedure in the receiver and the post-processing, i.e., the procedure of the input data extraction from the received signal $Y(t)$. We carry out all our calculations in the leading and next-to-leading orders in the Kerr nonlinearity parameter.

To study the mutual information we use the representation for the conditional probability density function $P[Y(t)|X(t)]$, i.e., the probability density to get the output signal $Y(t)$, if the input signal is $X(t)$, through the path-integral [7]. This representation for $P[Y(t)|X(t)]$ is especially convenient to use the perturbation theory. Generally speaking, the function spaces of the input signal $X(t)$ and the output signal $Y(t)$ are infinite-dimensional. However, information is transmitted using some finite set pulses of a certain shape, spread either in time or in frequency space. For example, the input signal $X(t)$ can be constructed as follows

$$X(t) = \sum_{k=-M}^{M} C_k s(t - kT_0).$$

Here $s(t)$ is a fixed envelope function of time, $C_k$ are complex variables that carry information about the input signal, $T_0$ is a time interval between two successive pulses. The problem of information transfer is reduced to recover the coefficients $\{C_{-M}, \ldots, C_M\}$ from the signal $Y(t)$ received at the output. To find the informational characteristics of the communication channel, we need to reduce the density functional $P[Y(t)|X(t)]$ to the functional $P[\{\tilde{C}\}||C\}]$, i.e., the conditional probability density to get a set of coefficients $\{C_k\}$, if the input signal was encrypted by the coefficients $\{C_k\}$. The functional $P[\{\tilde{C}\}||C\}]$ depends both on the physical laws of the signal propagation along the communication channel, and on the detection procedure with the post-processing of the signal. Generally, the functional $P[\{\tilde{C}\}||C\}]$ can not be reduced to a factorized form

$$P[\{\tilde{C}\}||C\}] = \prod_{k=-M}^{M} P^{(k)}[\tilde{C}_k|C_k]$$

due to the dispersion effects in the first place. This means that we deal with a communication channel with memory (commonly, with infinite one). Fiber optical channels with memory were previously considered in a bulk of papers, see for example [29].

In our work we calculate the density $P[\{\tilde{C}\}||C\}]$ for nonlinear fiber optic communication channel, in which the signal propagation is governed by the nonlinear Schrödinger equation with additive Gaussian noise of finite spectral width. Our model also includes a receiver model and post-processing procedure of the extraction of the coefficients $\{\tilde{C}_k\}$ from the detected signal $Y(t)$. The density functional $P[Y(t)|X(t)]$, as well as the density $P[\{\tilde{C}\}||C\}]$ were found with the use of two different methods. The first method is based on the direct calculation of the path-integral representing $P[Y(t)|X(t)]$ via the effective two-dimensional action [7] in the leading and next-to-leading orders in the parameter 1/SNR and in the parameter of Kerr nonlinearity, correspondingly. The second method is based on the independent calculation of the correlators of the solution of the nonlinear Schrödinger equation with additive Gaussian noise for a fixed input signal $X(t)$.

Using the found density $P[\{\tilde{C}\}||C\}]$ we calculated the entropy of the output signal and the conditional entropy. It allowed us to find mutual information in a leading and next-to-leading orders in the parameter 1/SNR and in the parameter of the Kerr nonlinearity. Then we found the extremum of the mutual information and calculated the probability density of the input signal $P_{opt}[\{C\}]$ that delivers this extremum. We demonstrated that in the first non-vanishing order in the Kerr nonlinearity, the probability density $P_{opt}[\{C\}]$ is not factorized, i.e., already in the leading order in the nonlinearity parameter, the fiber optic channel is the channel with memory. The optimal distribution $P_{opt}[\{C\}]$ allowed us to construct the conditional probabilities $P_{opt}[C_k|C_{-M}, \ldots, C_{k-1}, C_{k+1}, \ldots, C_M]$, which, in turn, are needed to construct the input signal with the given statistics $P_{opt}[\{C\}]$. Using the explicit form of the distribution $P_{opt}[\{C\}]$ we demonstrated that the difference between the mutual information found using the optimal statistics and the mutual information calculated using the Gaussian distribution occurs only in the fourth order in the small parameter of the Kerr nonlinearity. To demonstrate our analytical results, we performed the numerical calculations.
of mutual information, optimal distribution function and correlators of the output signal for various parameters of the second dispersion, as well as for pulse sequences of different lengths.

The article is organized as follows. The next Section is dedicated to the channel model description: we describe the structure of the input signal, then we introduce the procedures of the receiving and post-processing. We introduce the conditional probability density function in the case of small Kerr nonlinearity in the third Section. In the third Section we propose two approaches to the perturbative calculation of the conditional PDF. The details of this calculation are presented in the Appendix A. The fourth Section is devoted to the derivation of the mutual information. The resulting expression for the mutual information uses the tensor notations for the coefficients calculated in detail in the Appendix B. These universal coefficients allow us to present the optimal input signal distribution in the fifth Section. In the Section we present the theoretical and numerical results. We present the statistical method of the construction of the optimal input signal in the sixth Section: we describe the correlations of the input symbols resulting in the optimal distribution. The Conclusion finalizes our consideration of the optimal input signal distribution for the nonlinear channel with small Kerr nonlinearity.

II. MODEL OF THE CHANNEL

Let us start the consideration from the input signal representation.

A. The input signal model

In our model the input signal $X(t)$ has the following form

$$X(t) = \sum_{k=-M}^{M} C_k s(t - kT_0).$$

(4)

Thus, the signal is the sequence of $2M + 1$ pulses of the shape $s(t)$ spaced by time $T_0$. The complex coefficients $C_k$ carry the transmitted information. We chose the pulse envelope $s(t)$ possessing two properties. The first property is the normalization condition:

$$\int_{-\infty}^{\infty} \frac{dt}{T_0} s^2(t) = 1.$$  

(5)

The second property is orthogonality condition:

$$\int_{-\infty}^{\infty} \frac{dt}{T_0} s(t - kT_0)s(t - mT_0) = \delta_{km},$$

(6)

where $\delta_{km}$ is Kronecker $\delta$-symbol. Below we will consider two different types of the function $s(t)$. The first one is the sinc-type function

$$s(t) = \text{sinc} \left[ \frac{Wt}{2} \right] = \frac{2\sin(Wt/2)}{Wt},$$

(7)

where $W = 2\pi/T_0$ is the input signal bandwidth. Note that these envelopes are overlapping, however the properties (6) and (7) are fulfilled. We focus our attention in the following calculations primarily on the sinc type of the envelope.

The second type is the Gaussian function

$$s(t) = \sqrt{\frac{T_0}{\tau\sqrt{\pi}}} \exp \left( -\frac{t^2}{2\tau^2} \right),$$

(8)

where $\tau$ is characteristic signal duration. Below we imply that $\tau \ll T_0$. It is the parameter $\tau$ that determines the frequency bandwidth of the input signal. So the orthogonality condition (6) can be satisfied only approximately with any specified precision by choosing the value of the time $\tau$. 
The complex coefficients $C_k$ are distributed with probability density function (PDF) $P_X[\{C\}]$. Below we refer the function $P_X[\{C\}]$ as the input signal PDF, where $\{C\} = \{C_{-M}, C_{-M+1}, \ldots, C_M\}$ is the ordered set of the coefficients $C_k$. In our model we will consider the continuous PDF $P_X[\{C\}]$ normalized by the condition:

$$
\int \left( \prod_{k=-M}^{M} d^2C_k \right) P_X[\{C\}] = 1, \tag{9}
$$

where $d^2C_k = d\text{Re}C_k d\text{Im}C_k$. We also restrict our consideration by the input signal $X(t)$ with the fixed averaged power $P$:

$$
P = \int \left( \prod_{k=-M}^{M} d^2C_k \right) P_X[\{C\}] \frac{1}{2M+1} \int_{-\infty}^{\infty} \frac{dt}{T_0} |X(t)|^2. \tag{10}
$$

### B. The signal propagation model

In our model the propagation of the signal $\psi(z, t)$ is described by the NLSE with additive Gaussian noise, see [3–6]:

$$
\partial_z \psi(z, t) + i\beta \partial_t^2 \psi(z, t) - i\gamma |\psi(z, t)|^2 \psi(z, t) = \eta(z, t), \tag{11}
$$

with the input condition $\psi(0, t) = X(t)$. In Eq. (11) $\beta$ is the second dispersion coefficient, $\gamma$ is the Kerr nonlinearity coefficient, $\eta(z, t)$ is an additive complex noise with zero mean

$$
\langle \eta(z, t) \rangle_\eta = 0. \tag{12}
$$

Here $\langle \ldots \rangle_\eta$ is the averaging over the realization of the noise $\eta(z, t)$. We also imply that the correlation function $\langle \eta(z, t) \bar{\eta}(z', t') \rangle_\eta$ has the following form:

$$
\langle \eta(z, t) \bar{\eta}(z', t') \rangle_\eta = Q \frac{\hat{W}}{2\pi} \text{sinc} \left( \frac{\hat{W}(t-t')}{2} \right) \delta(z-z'). \tag{13}
$$

Here and below the bar means complex conjugation. The parameter $Q$ in Eq. (13) is a power of the noise $\eta(z, t)$ per unit length and per unit frequency. The parameter $\hat{W}$ is the bandwidth of the noise. Note that, $\lim_{\hat{W} \to \infty} \frac{\hat{W}}{2\pi} \text{sinc} \left( \frac{\hat{W}(t-t')}{2} \right) = \delta(t-t')$.

Below we imply that the noise bandwidth $\hat{W}$ is much greater than the bandwidth $W$ of the input signal $X(t)$ and much greater than the bandwidth $W'$ of the solution $\Phi(z = L, t)$ of the Eq. (11) with zero noise. Here $L$ is the signal propagation distance. So in our consideration we set that

$$
\hat{W} \gg W > W'. \tag{14}
$$

The solution $\Phi(z, t)$ of the Eq. (11) with zero noise and with the input boundary condition $\Phi(z = 0, t) = X(t)$ will play an important role in our further consideration. The details of the perturbative calculation of the solution $\Phi(z, t)$ and its properties are presented in the Subsection 2 of the Appendix A.

### C. The receiver model and the post-processing

To recover the transmitted information we perform the procedure of the output signal detection at $z = L$. Our receiver detects the noisy signal $\psi(L, t)$ (the solution of the Eq. (11) with noise), then it filters the detected signal in the frequency domain. After that we removes the phase incursion $e^{i\beta \omega^2 L}$ related with the second dispersion coefficient and obtain the signal $Y_d(t)$. So in the frequency domain we finally obtain the detected signal $Y_d(\omega)$:

$$
Y_d(\omega) = e^{-i\beta \omega^2 L} \theta(W_d/2 - |\omega|) \int_{-\infty}^{\infty} dte^{i\omega t} \psi(L, t). \tag{15}
$$
where $W_d$ is the frequency bandwidth of our receiver. In our model the bandwidth $W_d$ is much less than $\tilde{W}$ as well as in Eq. [14]. Besides, it is reasonable to consider the receiver with the bandwidth $W_d \geq W$, so it is our case.

To obtain the information we should recover the coefficients $\{C\}$ from the signal $Y_d(\omega)$. To this aim we project the signal $Y_d(t)$ on the shape functions $s(t - kT_0)$:

$$
\tilde{C}_k = \frac{1}{T_0} \int_{-\infty}^{\infty} dt s(t - kT_0) Y_d(t) = \frac{1}{2\pi T_0} \int_{W} d\omega \overline{s^{(k)}(\omega)} Y_d(\omega),
$$

(16)

where $s^{(k)}(\omega)$ is the Fourier transform of the function $s(t - kT_0)$:

$$
s^{(k)}(\omega) = \int_{-\infty}^{\infty} dt s(t - kT_0) e^{i\omega t} = e^{i\omega kT_0} s(\omega).
$$

(17)

Due to the noise and nonlinearity of the Eq. [11] the recovered coefficient $\tilde{C}_k$ does not coincide with the coefficient $C_k$. However, in the case of zero nonlinearity and the zero noise our detection procedure allows us to recover all coefficients $\{C\}$.

The informational characteristics of the channel are described by the conditional probability density function $P[\{\tilde{C}\}|\{C\}]$ to receive the sequence $\{\tilde{C}\}$ for the input sequence $\{C\}$. So we have to find the conditional PDF $P[\{\tilde{C}\}|\{C\}]$.

### III. CONDITIONAL PDF $P[\{\tilde{C}\}|\{C\}]$.

In this section we find the conditional PDF $P[\{\tilde{C}\}|\{C\}]$ using two approaches. The first one is based on the result of Ref. [7] where the conditional PDF $P[Y(\omega)|X(\omega)]$ to receive the output signal $Y(\omega)$ for the input signal $X(\omega)$ was represented in the form of path-integral. The second approach is based on the calculation of the output signal correlators in the leading and the next-to-leading orders in the parameter $Q$. Let us briefly discuss the first and the second approaches.

The base of the path-integral approach is the representation for the conditional PDF $P[Y(\omega)|X(\omega)]$ in the frequency domain, see Ref. [7]:

$$
P[Y(\omega)|X(\omega)] = \int_{\psi(0,\omega)=X(\omega)} \psi(L,\omega)=Y(\omega) D\psi(z,\omega) \exp \left[ -\frac{S[\psi]}{Q} \right],
$$

(18)

where the effective action $S[\psi]$ reads

$$
S[\psi] = \int_{0}^{L} dz \int_{W} \frac{d\omega}{2\pi} \left[ \partial_z \psi(z,\omega) - i\beta\omega^2 \psi(z,\omega) - i\gamma \int_{W} \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^3} \delta(\omega_1 + \omega_2 - \omega_3 - \omega) \psi(z,\omega_1) \psi(z,\omega_2) \overline{\psi}(z,\omega_3) \right]^2,
$$

(19)

and the integration measure $D\psi(z,\omega)$ is defined in such a way to obey the normalization condition $\int D\psi(\omega)P[Y(\omega)|X(\omega)]=1$, for details see [7]. As it was mentioned above, the function $P[Y(\omega)|X(\omega)]$ contains a redundant degrees of freedom, since the receiver does not detect all frequencies of the output signal $Y(\omega)$. That is why we have to introduce the conditional PDF $P_d[Y_d(\omega)|X(\omega)]$ which is the result of the integration of the function $P[Y(\omega)|X(\omega)]$ over redundant degrees of freedom $Y(\omega)$, $|\omega|>W_d/2$:

$$
P_d[Y_d(\omega)|X(\omega)] = \int [DY(\omega)]_{|\omega|>W_d/2} P[Y(\omega)|X(\omega)].
$$

(20)

So, the function $P_d[Y_d(\omega)|X(\omega)]$ contains only detectable degrees of freedom $Y_d(\omega)$, $|\omega|<W_d/2$, see Eq. (15). If one knows the function $P_d[Y_d(\omega)|X(\omega)]$, it is easy to calculate an arbitrary correlator $\langle \tilde{C}_{k_1} \ldots \tilde{C}_{k_N} \rangle$, where

$$
\langle \tilde{C}_{k_1} \ldots \tilde{C}_{k_N} \rangle = \int DY_d(\omega) P_d[Y_d(\omega)|X(\omega)] \tilde{C}_{k_1} \ldots \tilde{C}_{k_N},
$$

(21)
where \( \tilde{C}_k \) is defined in Eq. (16). For our purposes we should know correlators in the leading order in the noise parameter \( Q \), and up to the second order in the nonlinearity parameter \( \gamma \). Knowledge of these correlators allows us to construct the conditional PDF \( P[\{ \tilde{C} \}|\{ C \}] \) which reproduces all correlators with necessary accuracy. The details of this calculation are presented in the Appendix A.

The second approach allows us to calculate the same correlators \( \{ \tilde{C}_k \} \) by solving the equation (11) up to the second order in parameter \( \gamma \) and up to the first order in the noise parameter \( Q \) (i.e., the second order in function \( \eta(z,t) \)). We substitute the solution \( \psi(L,t) \) of the equation (11) to the Eq. (15), then the result of Eq. (15) is substituted to Eq. (16) and we arrive at the expression for the measured coefficient \( \tilde{C}_k \). Note that, since the solution \( \psi(L,t) \) depends on the noise, the coefficient \( \tilde{C}_k \) depends on the noise function \( \eta(z,t) \) as well. To calculate any correlator \( \langle \tilde{C}_{k_1} \ldots \tilde{C}_{k_N} \rangle \) we should average the product \( \tilde{C}_{k_1} \ldots \tilde{C}_{k_N} \) over the noise realizations using the Eqs. (12). As it should be, the results for the correlators \( \langle \tilde{C}_{k_1} \ldots \tilde{C}_{k_N} \rangle \) are the same for both approaches. The results for the correlators are presented in the Subsection 4 of the Appendix A.

Using the obtained correlators we build the conditional PDF \( P[\{ \tilde{C} \}|\{ C \}] \):

\[
P[\{ \tilde{C} \}|\{ C \}] = \Lambda_c \exp \left\{ -\frac{T_0}{QL} \sum_{k,k'=\pm M} \left[ \delta \tilde{C}_{k'} F^{k',k} - \delta \tilde{C}_{k} F^{k,k'} + \delta \tilde{C}_{k'} G^{k',k'} \delta \tilde{C}_k + \delta \tilde{C}_{k} F^{k,k'} \right] \right\}, \tag{22}
\]

here \( F^{k',k} = \tilde{F}^{k',k} + F_2^{k',k} \), \( H^{k,k'} = H_{\tilde{C},k}^{k,k'} + H_2^{k,k'} \), \( G^{k',k'} = G_{\tilde{C},k}^{k',k'} + G_2^{k',k'} \) are dimensionless coefficients with \( k,k' = -M, \ldots, M \). The indexes 1 and 2 indicate terms proportional to \( \gamma \) and \( \gamma^2 \), respectively. The quantity \( \delta \tilde{C}_k \) is defined as follows

\[
\delta \tilde{C}_k = \tilde{C}_k - \langle \tilde{C}_k \rangle. \tag{23}
\]

Here the correlator \( \langle \tilde{C}_k \rangle \) is known function of \( \{ C \} \), see Eq. (A33). Note that the quantity \( \langle \tilde{C}_k \rangle \) contains bandwidth of the noise \( \tilde{W} \).

The dimensionless coefficients can be presented through pair correlators as

\[
F^{k',k} = \tilde{F}^{k',k} + F_2^{k',k}, \quad F_2^{k',k} = \sum_{m=-M}^{M} \tilde{H}_{1,m}^{k',m} H_{1,m}^{k,k} - \gamma^2 \frac{T_0}{2QL} \frac{\partial^2}{\partial \gamma^2} \langle \tilde{C}_{k'} \tilde{C}_k \rangle |_{\gamma=0}, \tag{24}
\]

where

\[
H_{1,m}^{k,k} = -\gamma \frac{T_0}{2QL} \frac{\partial}{\partial \gamma} \langle \tilde{C}_{k} \tilde{C}_{m} \rangle |_{\gamma=0}, \quad H_2^{m,k} = -\gamma^2 \frac{T_0}{4QL} \frac{\partial^2}{\partial \gamma^2} \langle \tilde{C}_k \tilde{C}_m \rangle |_{\gamma=0}. \tag{25}
\]

All needed correlators \( \langle \tilde{C}_k \rangle, \langle \tilde{C}_k \tilde{C}_{m} \rangle, \langle \tilde{C}_k \tilde{C}_m \rangle \) are presented explicitly in the Appendix A see Eqs. (A33), (A34), (A35). The normalization factor \( \Lambda_c \) reads up to \( \gamma^2 \) order

\[
\Lambda_c = \left( \frac{T_0}{\pi QL} \right)^{2M+1} \left[ 1 + \sum_{k=-M}^{M} \sum_{k'=-M}^{M} G_{1,k,k'}^{1,k'} H_{1,k,k'}^{1,k} \right]. \tag{26}
\]

At first sight, the found PDF (22) has the Gaussian form, and it might be suggested that we have reduced the channel to the linear one. But it is not the case, since the dimensionless coefficients depends nonlinearly on the input signal coefficients \( \{ C \} \). The Gaussian structure is the consequence of the consideration of the problem in the leading order in the parameter \( Q \).

Now we turn to the consideration of the channel entropies \( H[\tilde{C}] \) and \( H[\{ \tilde{C} \}|\{ C \}] \) which are necessary for the mutual information calculation.

### IV. MUTUAL INFORMATION

The conditional entropy reads

\[
H[\{ \tilde{C} \}|\{ C \}] = - \int d\tilde{C} d\tilde{C} P_X[\{ C \}] P[\{ \tilde{C} \}|\{ C \}] \log P[\{ \tilde{C} \}|\{ C \}], \tag{27}
\]
where

\[ dC = \prod_{k=-M}^{M} d\text{Re}C_k d\text{Im}C_k, \quad d\tilde{C} = \prod_{k=-M}^{M} d\text{Re}\tilde{C}_k d\text{Im}\tilde{C}_k. \]  

(28)

To calculate the conditional entropy \( H[\{\tilde{C}\}|\{C\}] \) we substitute the conditional PDF \( \{22\} \) to the expression \( \{27\} \), then perform the integration over \( \{\tilde{C}\} \), and we obtain

\[ H[\{\tilde{C}\}|\{C\}] = - \int dC P_x[\{C\}] (\log \Lambda_c - (2M + 1)) \).  

(29)

To perform the integration in Eq. \( \{29\} \) we expand \( \log \Lambda_c \) up to \( \gamma^2 \) terms then integrate over \( \{C\} \) and arrive at

\[ H[\{\tilde{C}\}|\{C\}] = (2M + 1) \log (\pi e Q L / T_0) - \gamma^2 L^2 J^{s_1,s_2,s_3,s_4}_\Lambda \int dC P_x[\{C\}] C_{s_1} C_{s_2} \tilde{C}_{s_3} \tilde{C}_{s_4}. \]  

(30)

To obtain Eq. \( \{30\} \) we have used the normalization condition

\[ \int dC P_x[\{C\}] = 1. \]  

(31)

In Eq. \( \{30\} \) and below, unless otherwise stated, we imply that there is the summation over the repeated indices. The explicit expression for coefficients \( J^{s_1,s_2,s_3,s_4}_\Lambda \) is cumbersome, therefore we present it in the Appendix A see Eq. \( \{A3\} \). Now we proceed to the calculation the output signal entropy

\[ H[\{\tilde{C}\}] = - \int d\tilde{C} P_{out}[\{\tilde{C}\}] \log P_{out}[\{\tilde{C}\}], \]  

(32)

where the output signal distribution reads

\[ P_{out}[\{\tilde{C}\}] = \int dC P[\{\tilde{C}\}|\{C\}] P_x[\{C\}]. \]  

(33)

To calculate the PDF of the output signal we change the integration variables in Eq. \( \{33\} \) from \( C_k \) to \( \delta \tilde{C}_k = \tilde{C}_k - \langle \tilde{C}_k \rangle \).

Since in our model the average noise power is much less than the average input signal power \( (QL / T_0 \ll P) \), we calculate the integral \( \{33\} \) using the Laplace method \( \{30\} \) and obtain the following result in the leading order in the parameter \( 1/\text{SNR} = Q L / (T_0 P) \):

\[ P_{out}[\{\tilde{C}\}] \approx \left. \frac{\partial(C, \tilde{C})}{\partial(\tilde{C}^{(0)}, \tilde{C}^{(0)})} \right|_{\tilde{C}^{(0)}} P_x[\{F\}], \]  

(34)

where \( \tilde{C}_k^{(0)} \) is the known function of \( C_k \), see Eq. \( \{A8\} \) in Appendix A.

\[ \tilde{C}_k^{(0)}[\{C\}] = C_k + i \gamma LC_k^* C_k^* \tilde{C}_k \delta_1 \delta_2 \delta_3 \delta_4 - \gamma^2 L^2 C_m C_{m_2} C_{m_3} C_{m_4} \tilde{C}_m \tilde{a}_2^{m_1, m_2, m_3, m_4, m_5, k}, \]  

(35)

and \( F[\{\tilde{C}\}] \) is the solution of the equation

\[ \tilde{C}_k = \tilde{C}_k^{(0)}[\{F\}]. \]  

(36)

The solution \( F \) of this equation can be found using perturbation theory in the parameter \( \gamma \). One can see, that the distribution of the output signal coincides with the input signal distribution \( P_x[\{F\}] \) up to the Jacobian determinant \( \left| \frac{\partial(C, \tilde{C})}{\partial(\tilde{C}^{(0)}, \tilde{C}^{(0)})} \right|_{\tilde{C}^{(0)}} \). It allows us to calculate the output signal entropy in the leading order in parameter \( Q \), or \( 1/\text{SNR} = Q L / (T_0 P) \) in the dimensionless quantities.

Substituting the result \( \{34\} \) into the expression for the output signal entropy \( \{32\} \), we perform the integration over \( \tilde{C} \) and arrive at

\[ H[\{\tilde{C}\}] = H[\{C\}] + \int dC P_x[\{C\}] \log \left| \frac{\partial(C^{(0)}, \tilde{C}^{(0)})}{\partial(C, C)} \right|, \]  

(37)
where \( H[\{C\}] \) is the input signal entropy:

\[
H[\{C\}] = -\int dCP_X[\{C\}] \log P_X[\{C\}].
\]

(38)

Therefore to find the output entropy we should calculate the logarithm of the Jacobian determinant in Eq. (37). The straightforward calculation in the first non-vanishing order in the parameter \( \gamma \) leads to the following result:

\[
\log \left| \frac{\partial(\tilde{C}^{(0)}, \tilde{C}^{(0)})}{\partial(C, C)} \right| = \gamma^2 L^2 C_{s_1} C_{s_2} C_{s_3} \bar{C}_{s_4} J_{s_1, s_2, s_3, s_4},
\]

(39)

where dimensionless coefficients \( J_{s_1, s_2, s_3, s_4} \) are given by the Eq. (A16).

To calculate the mutual information we subtract the conditional entropy (30) from the output signal entropy (37):

\[
I_{P_X} = H[\{\tilde{C}\}] - H[\{\tilde{C}\}|\{C\}] = -(2M + 1) \log [\pi eQL/T_0] - \int dCP_X[\{C\}] \log P_X[\{C\}] + \gamma^2 L^2 J_{s_1, s_2, s_3, s_4} \int dCP_X[\{C\}] C_{s_1} C_{s_2} C_{s_3} \bar{C}_{s_4},
\]

(40)

where coefficients

\[
J_{s_1, s_2, s_3, s_4} = J_{s_1, s_2, s_3, s_4} + J_{s_1, s_2, s_3, s_4}.\]

(41)

These two contributions to the coefficients \( J_{s_1, s_2, s_3, s_4} \) are presented explicitly in Eqs. (A16) and (A42). The method of the numerical calculation of these coefficients is presented in Appendix B. The first two terms in the mutual information (40) coincide with that for linear channel (\( \gamma = 0 \)). The third term describes the contribution of the Kerr nonlinearity effects. One can see that the first nonlinear correction to the mutual information is of the order of \( \gamma^2 \). Since the coefficients \( J_{s_1, s_2, s_3, s_4} \) depend on the envelope function \( s(t) \), therefore the mutual information also depends on this envelope function. It is worth noting, that the mutual information depends on the bandwidth of the input signal \( W \) via coefficients \( J_{s_1, s_2, s_3, s_4} \) and does not depend on the detector bandwidth \( W_d \). The reason for that is the bandwidth \( W_d \) of the receiver is greater than or equal to the bandwidth of the input signal \( W \): \( W_d \geq W \). Therefore, all integrals over frequency in the interval \([-W_d/2, W_d/2]\) with the envelope \( s(k)(\omega) \) are reduced to the integrals over the frequency interval \([-W/2, W/2]\) determined by the function \( s(\omega) \).

V. OPTIMAL INPUT SIGNAL DISTRIBUTION

Now we can calculate the optimal input signal distribution \( P_{opt}[\{C\}] \) which maximizes the mutual information (40). The optimal distribution \( P_{opt}[\{C\}] \) obeys the normalization condition:

\[
\int dCP_{opt}[\{C\}] = 1,
\]

(42)

and the condition of the fixed average power:

\[
\int dCP_{opt}[\{C\}] \frac{1}{2M + 1} \sum_{k=-M}^{M} |C_k|^2 = P.
\]

(43)

To find \( P_{opt}[\{C\}] \) we solve the variational problem \( \delta \mathcal{K}[P_X] = 0 \) for the functional

\[
\mathcal{K}[P_X] = I_{P_X} - \lambda_1 \left( \int DCp_X[\{C\}] - 1 \right) - \lambda_2 \left( \int DCp_X[\{C\}] \frac{1}{\sum_{k=-M}^{M} |C_k|^2 - P} \right),
\]

(44)

where \( \lambda_{1,2} \) are the Lagrange multipliers, related with the restrictions (42), (43). The solution of the variational problem in the first and in the second orders in the parameter \( \gamma \) and in the leading order in the parameter \( 1/\text{SNR} \) has the form:

\[
P_{opt}[\{C\}] = \begin{cases} P^{(0)}[\{C\}] \left\{ 1 + \gamma^2 L^2 J_{s_1, s_2, s_3, s_4} \bar{C}_{s_3} \bar{C}_{s_4} + \right. \\
\left. (\gamma LP)^2 (J^{s_1, s_2, s_3, s_4} + J^{s_1, s_2, s_3, s_4}) \left( 1 - \frac{2}{P(2M + 1)} \sum_{k=-M}^{M} |C_k|^2 \right) \right\}
\end{cases}
\]

(45)
where \( P^{(0)}[\{C\}] \) is the optimal input signal distribution for the channel with zero nonlinearity parameter \( \gamma \):

\[
P^{(0)}[\{C\}] = \left( \frac{1}{\pi P} \right)^{2M+1} \exp \left[ -\frac{1}{P} \sum_{k=-M}^{M} |C_k|^2 \right].
\] (46)

One can see, that \( P^{(0)}[\{C\}] \) is the Gaussian distribution, whereas the distribution (45) is not Gaussian due to the nonlinear corrections. Thus, \( P_{\text{opt}}[\{C\}] \) leads to the nonzero correlations between coefficients \( C_k \) with different \( k \).

To find the maximal value of the mutual information (40) we substitute \( P_{\text{opt}}[\{C\}] \) into the expression (40), perform the integration over \( C \) and obtain

\[
I_{P_{\text{opt}}} = (2M + 1) \left( \log \left[ \frac{PT_0}{QL} \right] + (\gamma LP)^2 J_{\Sigma} \right),
\] (47)

where

\[
J_{\Sigma} = \frac{J_{I}^{r,s,r,s} + J_{I}^{s,s,r,r}}{2M + 1}.
\] (48)

One can see that the mutual information is proportional to the number of the coefficients \( C_k \), i.e. \( 2M + 1 \). The first term in the second brackets in Eq. (47) coincides with the Shannon’s result for the linear channel at large SNR. The second term is the first nonzero nonlinear correction. Below we will demonstrate numerically that the quantity \( J_{\Sigma} \) depends weakly on \( 2M + 1 \).

We emphasize that the calculation of the mutual information (40) using the Gaussian distribution (46) leads to the result \( I_{P^{(0)}} \) which coincides with the result (47). It means that in this order in the parameter \( \gamma \) both distributions give the same result for the mutual information. Thus, one might think that it doesn’t matter what distribution, the optimal (45) or the Gaussian (46), is used for the calculation of the mutual information. However, the optimal input signal distribution (45) leads to the mutual information \( I_{P_{\text{opt}}} \) that is greater than \( I_{P^{(0)}} \) in the higher orders in the nonlinearity parameter \( \gamma \). To demonstrate that we have calculated the difference between \( I_{P_{\text{opt}}} \) and \( I_{P^{(0)}} \) in the leading nonzero order in the nonlinearity parameter \( \gamma \) and obtain:

\[
I_{P_{\text{opt}}} - I_{P^{(0)}} = \frac{(\gamma LP)^4}{2} \left( \langle A^2 \rangle_{P^{(0)}} - \langle A \rangle_{P^{(0)}}^2 - \frac{4}{2M + 1} \langle A \rangle_{P^{(0)}}^2 \right),
\] (49)

where

\[
A = \frac{J_{I}^{s_1,s_2,s_3,s_4}}{P^2} C_{s_1} C_{s_2} C_{s_3} \tilde{C}_{s_4},
\] (50)

and here we introduce the averaging over the zero-order distribution (46):

\[
\langle (\ldots) \rangle_{P^{(0)}} = \int dCP^{(0)}[\{C\}] (\ldots).
\] (51)

Performing the averaging in Eq. (49) we arrive at the following result

\[
I_{P_{\text{opt}}} - I_{P^{(0)}} = 2(\gamma LP)^4 \left( 4J_{I}^{a,b,c,d} \tilde{J}_{I}^{d,k,a,k} + J_{I}^{a,b,c,d} \tilde{J}_{I}^{d,a,b} - \frac{4}{2M + 1} \tilde{J}_{I}^{a,b,c,a} \tilde{J}_{I}^{c,d;a,d} \right),
\] (52)

where

\[
\tilde{J}_{I}^{a,b,c,d} = \left( J_{I}^{a,b,c,d} + J_{I}^{b,a,c,d} + J_{I}^{a,b,c,d} + J_{I}^{b,a,d,c} \right) / 4.
\] (53)

We have checked numerically that the right-hand side of (52) is positive for all considered in the present paper dispersions \( \beta \).

Below we present results for the mutual information for different envelopes \( s(t) \) and for different values of dispersion.
A. Zero dispersion case

The direct calculation of the mutual information (47) in the case of zero dispersion and non-overlapping envelopes $s(t)$, obeying the condition (6), see for instance the envelope (8), gives the result

$$I_{P_{\text{opt}}} = \log \left[ \frac{PT_0}{QL} \right] - \left( \frac{\gamma LP}{3} \right)^2 \frac{22N_6 - 21N_4^2}{3},$$

(54)

where $N_\lambda$ is the integral

$$N_\lambda = \frac{1}{T_0} \int_{-\infty}^{\infty} ds^\lambda(t).$$

(55)

We note that $22N_6 - 21N_4^2 > 0$ due to Cauchy-Schwarz-Bunyakovsky inequality. For the case of the rectangular pulse $s(t) = \theta(T_0/2 - |t|)$ (which corresponds to the case of the per-sample model, see [16]) the Eq. (54) passes to

$$I_{P_{\text{opt}}} \left|_{\beta=0} = \log \left[ \frac{PT_0}{QL} \right] - \left( \frac{\gamma LP}{3} \right)^2 \frac{22N_6 - 21N_4^2}{18},$$

(56)

This result coincides with Eq. (53) in Ref. [16].

The difference (49) for the case $\beta = 0$ and non-overlapping envelopes $s(t)$, see Eq. (8), has the form

$$I_{P_{\text{opt}}} - I_{P(0)} \left|_{\beta=0} = (2M + 1)\left( \frac{\gamma LP}{3} \right)^2 \frac{22N_6 - 21N_4^2}{18},$$

(57)

One can see that the difference is positive and in agreement with the general results of Ref. [16] that is valid for the arbitrary nonlinearity.

For the case of the envelope of the sinc form,

$$s(t) = \text{sinc}(Wt/2),$$

(58)

we obtain the following result for the maximum value of the mutual information (47), see details in the Appendix B, subsection 3:

$$I_{P_{\text{opt}}} = \log \left[ \frac{PT_0}{QL} \right] + \left( \frac{\gamma LP}{3} \right)^2 \frac{22}{3} \int_{-\infty}^{\infty} d\tau S^6(\tau, \tau) + \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \left( 3S^4(\tau_1, \tau_2) + 4S^2(\tau_1, \tau_2)S^2(\tau_1, \tau_1)S^2(\tau_2, \tau_2) \right),$$

(59)

where

$$S^2(\tau_1, \tau_2) = \sum_{r=-M}^{M} \text{sinc}(\pi(\tau_1 + r))\text{sinc}(\pi(\tau_2 + r)), \quad S^2(\tau, \tau) = \sum_{r=-M}^{M} \text{sinc}^2(\pi(\tau + r)).$$

(60)

The numerical result for the quantity (59) has the form

$$I_{P_{\text{opt}}} = \log \left[ \frac{PT_0}{QL} \right] - 1.26 \left( \frac{\gamma LP}{3} \right)^2,$$

(61)

where the coefficient at the nonlinearity factor $(\gamma LP)^2$ weakly depends on the parameter $M$.

B. Nonzero dispersion case

Here we present the results for the nonzero dispersion parameter. For this consideration we choose the following parameters of the channel: $\beta = 2 \times 10^{-23} \sec^2/km$, the propagation length is equal to $L = 800$ km, and different values of the input signal bandwidth $W$. The dispersion effects can be described by the dimensionless parameter $\tilde{\beta}$:

$$\tilde{\beta} = \beta LW^2/2.$$
Below we present the numerical results for the mutual information for various values of \( \bar{\beta} \). Fig. 1 presents the dependence of the quantity \( J_\Sigma \), see Eq. (47), on different values of the parameter \( \bar{\beta} \) for \( M = 5 \), see Eq. (47). We checked that the quantity \( J_\Sigma \) weakly depends on \( M \) for \( M > 5 \). The points in the Fig. 1 were obtained by two different numerical approaches, see Appendix B. Both approaches leads to the same results and it is the guarantee of correctness of the numerical calculations. One can see that \( J_\Sigma \) has the minimum at \( \bar{\beta} = 0 \). It means that nonlinear correction to the mutual information, see (47), has the maximum absolute value at \( \bar{\beta} = 0 \). At small \( \bar{\beta} \) the quantity \( J_\Sigma \) demonstrates the quadratic dependence on the dispersion parameter, see the inset in Fig. 1. At large \( \bar{\beta} \) the quantity \( J_\Sigma \) goes to zero. Our estimations results in the dependence \( J_\Sigma \sim \frac{\log \bar{\beta}}{\bar{\beta}} \) for large \( \bar{\beta} \), see [31]. Therefore, the nonlinear correction decreases with increasing \( \bar{\beta} \), as \( (\gamma LP)^2/\bar{\beta} \). It means that effective nonlinear parameter at large \( \bar{\beta} \) is not \( \gamma LP \), but \( \gamma LP/\sqrt{\bar{\beta}} \).

VI. CONSTRUCTION OF THE INPUT SIGNALS

To increase the mutual information we should be able to create the signals which obey the optimal input signal distribution [49]. To create the input sequence which has the statistics determined by the PDF \( P_{opt}([C]) \), we represent this PDF in the form [32]:

\[
P_{opt}([C]) = P_{opt}[C_1] \times P_{opt}[C_{i_2}|C_{i_1}] \times P_{opt}[C_{i_3}|C_{i_2}, C_{i_1}] \times \ldots \times P_{opt}[C_{i_{2M+1}}|C_{i_{2M}}, \ldots C_{i_2}, C_{i_1}],
\]

where

\[
P_{opt}[C_1] = \int dC_{i_2} \ldots dC_{i_{2M+1}} P_{opt}([C]),
\]

\[
P_{opt}[C_{i_2}|C_{i_1}] = \frac{\int dC_{i_3} \ldots dC_{i_{2M+1}} P_{opt}([C])}{P_{opt}[C_{i_1}]},
\]

\[
P_{opt}[C_{i_3}|C_{i_2}, C_{i_1}] = \frac{\int dC_{i_4} \ldots dC_{i_{2M+1}} P_{opt}([C])}{P_{opt}[C_{i_1}] \times P_{opt}[C_{i_2}|C_{i_1}]},
\]

\[
P_{opt}[C_{i_{2M}}|C_{i_{2M-1}}, \ldots C_{i_2}, C_{i_1}] = \frac{\int dC_{i_{2M+1}} P_{opt}([C])}{P_{opt}[C_{i_1}] \times \ldots \times P_{opt}[C_{i_{2M-1}}|C_{i_{2M-2}}, \ldots C_{i_1}]},
\]

\[
P_{opt}[C_{i_{2M+1}}|C_{i_{2M}}, \ldots C_{i_2}, C_{i_1}] = \frac{P_{opt}([C])}{P_{opt}[C_{i_1}] \times \ldots \times P_{opt}[C_{i_{2M}}|C_{i_{2M-1}}, \ldots C_{i_1}]}.
\]

Using Eqs. (64)-(68) we can build the sequences which have the necessary statistics by the following way. At first, we choose the first element \( C_1 \) of the sequence distributed with PDF (63). The statistics of the second element \( C_2 \) depends on the value of \( C_1 \), and should be distributed with PDF (65), etc. In our approximation (\( \gamma^2 \) order of the calculation) the optimal PDF \( P_{opt}([C]) \) contains the fourth order polynomial in the coefficients \( C_k \). So, we have two nontrivial correlators \( (C_k \bar{C}_m)_{P_{opt}} \), and \( (C_k C_q \bar{C}_m \bar{C}_p)_{P_{opt}} \), which determine all higher order correlators.

For a very long sequence the correlation between the first and the last coefficients should be neglectable. To find the characteristic length \( |k - m| \) of the correlation between elements \( C_k \) and \( C_m \) of the input sequence, we calculate
the correlator \( \langle C_k \tilde{C}_m \rangle_{P_{opt}} \). After the straightforward calculation we obtain:

\[
\langle C_k \tilde{C}_m \rangle_{P_{opt}} = P_{\delta_{km}} \left( 1 - (\gamma LP)^2 \frac{2}{2M+1} \sum_{r,s=0}^{M} (J_{r}^{s;r,s} + J_{r}^{s;s,r}) \right) + P(\gamma LP)^2 \sum_{r=-M}^{M} \left[ J_{r}^{m;r,k} + J_{r}^{m;k,r} + J_{r}^{m;r,k} + J_{r}^{m;r,k} \right].
\]

(69)

The first term in the right-hand side of this equation contains the Kronecker delta-symbol, i.e. it is zero for \( k = m \). The second term describes the correlation between different elements of the input sequence. To find the correlation length we should investigate the dependence of this term on the parameter \( m - k \). The correlation length depends on the parameter \( \beta \). For the small \( \beta \) only the nearest neighbors are correlated since the spreading of the input signal due to the dispersion is small. When increasing the dispersion parameter \( \beta \), the correlation length is increasing. The numerical values of the coefficients \( J_{r}^{i;j,k,l} \) are presented in Supplementary materials. So, one can calculate any necessary correlators.

As an example, we consider the sequence where only the nearest elements are correlated. To build the sequence we should know only two distributions: \( P_{opt}[C_i] \) and \( P_{opt}[C_i|C_j] = P_{opt}[C_i,C_j]/P_{opt}[C_j] \). We have performed the calculation of these distributions and obtain:

\[
P_{opt}[C_q] = P^{(0)}[C_q] \left( 1 + (\gamma LP)^2 D_{x}^{(q)} \left( |C_q|/\sqrt{P} \right) \right),
\]

(70)

where \( P^{(0)}[C_q] = \frac{1}{\pi P} \exp \left\{ -\frac{|C_q|^2}{P} \right\} \) is the Gaussian distribution, and \( D_{x}^{(q)}(x) \) is the following polynomial function:

\[
D_{x}^{(q)}(x) = \left[ (1 - x^2)^2 \sum_{r,s=-M}^{M} \frac{J_{r}^{s;r,s} + J_{r}^{s;s,r}}{2M+1} - \sum_{r=-M}^{M} \left[ J_{r}^{q;r,q} + J_{r}^{q;q,r} + J_{r}^{r;r,q} + J_{r}^{r;q,r} \right] \right] + J_{0}^{q;q,q} \left( x^4 - 4x^2 + 2 \right),
\]

(71)

\[
P_{opt}[C_i,C_j] = P_{opt}[C_j,C_i] = \int dC_{i_3} \ldots dC_{i_{2M+1}} P_{opt}([C]) = P_{opt}[C_i] P_{opt}[C_j] \left( 1 + (\gamma PL)^2 |D_{x}^{(q)}(\sqrt{P})| \right),
\]

(72)

\[
D_{x}^{(q)}(x) = J_{i}^{j;i,j} x^2 \ddot{y} + \sum_{r,s=-M}^{M} \left( \frac{J_{r}^{j;i,j} + J_{r}^{j;j,i}}{2M+1} \right) \left( |x|^2 - 1 \right) \left( |y|^2 - 1 \right) \left( |y|^2 - 2 \right) \left( J_{r}^{q;r,q} + J_{r}^{q;q,r} + J_{r}^{r;r,q} + J_{r}^{r;q,r} \right) + \sum_{m=-M}^{M} \left( \ddot{x} \dddot{y} \left[ J_{i}^{m;i,m} + \dddot{y} J_{i}^{m;i,m} \right] \right) + \dddot{x} \dddot{y} \left[ J_{i}^{m;i,m} + \dddot{y} J_{i}^{m;i,m} \right]
\]

(73)

One can see that the corrections to these PDFs are the fourth order polynomials in parameter \( C_q/\sqrt{P} \). Let us consider these polynomials. In Fig. 2 we plot the function \( D_{x}^{(q)}(x) \), see Eq. (71), for different values of \( \hat{\beta} \). The function \( D_{x}^{(q)}(x) \) for different \( \hat{\beta} \) has the maximum in the vicinity of the value \( x \approx 1.5 \). For \( x > 1.5 \) this function decreases for all values of \( \hat{\beta} \). For smaller \( \hat{\beta} \) the absolute value of the function \( D_{x}^{(q)}(x) \) is larger for \( x > 2 \). It means the applicability region determined by the relation \( (\gamma LP)^2 D_{x}^{(q)}(\sqrt{P}) \ll 1 \) is wider for larger \( \hat{\beta} \). The reason is the decreasing character of the coefficients \( J_{r}^{i;j,k,l} \) for increasing \( \hat{\beta} \), see e.g. Fig. 2.

To demonstrate the behavior of the function \( P_{opt}[C_q] \) we plot it for the different dispersion parameter \( \hat{\beta} \), see Fig. 3. In Fig. 3 we chose the nonlinear parameter \( (\gamma LP)^2 = 0.2 \). One can see that the function \( P_{opt}[C_0] \) decreases slowly for smaller \( \hat{\beta} \). It means that nonlinear correction decreases with increasing \( \hat{\beta} \). The difference \( |P^{(0)}[C_0] - P_{opt}[C_0]| \)
is getting smaller when increasing \( \tilde{\beta} \). The reason is that for the larger dispersion parameter the signal spreading is larger. It results in the decreasing of the effective nonlinearity parameter, i.e., decreasing the coefficients \( J_{i,j}^{k,l} \).

The expression in the big curly brackets in Eq. (72) is symmetric in the coefficients \( C_i \) and \( C_j \). Since we know the function \( P_{opt}[C_j,C_i] \) the probability \( P_{opt}[C_j|C_i] \) can be easily obtained using Eq. (65):

\[
P_{opt}[C_i|C_j] = P_{opt}[C_i] \left\{ 1 + \left( \gamma LP \right)^2 D^{i,j} \left( \frac{C_i \beta}{\sqrt{P}}, \frac{C_j \beta}{\sqrt{P}} \right) \right\}.
\]

In Fig. 3 we plot the function \( P_{opt}[C_0|C_{-1}] \) for different values of coefficient \( C_{-1}/\sqrt{P} \) (real and imaginary), nonlinearity parameter \( (\gamma LP)^2 = 0.2 \), and dispersion parameter \( \tilde{\beta} \) equals to 1 and 5. We plot the dependence of \( P_{opt}[C_0|C_{-1}] \) on the dimensionless variable \( C_0/\sqrt{P} \), where \( P \) is chosen to be equal to the unity. One can see that the function \( P_{opt}[C_0|C_{-1}] \) differs from \( P_{opt}[C_0] \) and depends on the value \( C_{-1} \) essentially. Also, the PDFs deviate from the Gaussian distribution greater for larger absolute values of \( C_{-1} \). The negative value of the PDF has no sense, but it demonstrates the region of applicability of our approximation. For smaller parameters \( \gamma LP \) or for larger parameter \( \tilde{\beta} \) our perturbative result \( (72) \) is valid wherever the function \( P_{opt}[C_j,C_i] \) is not small. For the large values of \( \tilde{\beta} \) it is necessary to consider not only PDF \( P_{opt}[C_j,C_i] \), but others...
PDFs presented in Eqs. (64)–(68), since the spreading effects become significant. If necessary, PDFs \( P_{\text{opt}}[C_i|C_{j1}, C_{j2}, \ldots, C_{j2M}] \) can be derived from (45) analogously to Eq. (74). We do not present these PDFs here because of its cumbersomeness. We attach the files with coefficients \( J_{i,j}^{k,l} \) for different \( M \) and \( \tilde{\beta} \) to the possibility to calculate these PDFs.

VII. CONCLUSION

In the present paper we develop the method of the calculation of the conditional probability density function \( P([\tilde{C}]|[C]) \) for the channel describing by the nonlinear Schrödinger equation with additive noise and with nonzero second dispersion coefficient \( \beta \). To illustrate our method we calculate the PDF \( P([\tilde{C}]|[C]) \) in the leading and next-to-leading order in the Kerr nonlinearity parameter \( \gamma \) and in the leading order in the parameter \( 1/\text{SNR} \). To obtain \( P([\tilde{C}]|[C]) \) we calculated \( P[Y(\omega)|X(\omega)] \) using two different approaches. The first approach is based on the direct calculation of the path-integral, see Eq. (18). In the second approach we calculate the output signal correlators for the fixed input signal \( X(t) \) and then construct the conditional PDF. Both approaches give the same result (A22). To take into account the envelope of the input signal and the detection procedure of the receiver we integrate the PDF \( P[Y(\omega)|X(\omega)] \) over the redundant degrees of freedom and obtain the conditional PDF \( P([\tilde{C}]|[C]) \). Using the PDF \( P([\tilde{C}]|[C]) \) we calculate the mutual information, solve the variational problem, and find the optimal input signal.
distribution $P_{\text{opt}}[\{C\}]$ in the leading order in the parameter $1/\text{SNR}$ and in the second order in the parameter of the Kerr nonlinearity $\gamma$. We demonstrate that $P_{\text{opt}}[\{C\}]$ differs from the Gaussian distribution. Using the distribution $P_{\text{opt}}[\{C\}]$, we calculated the maximal value of the mutual information in the leading order in the parameter $1/\text{SNR}$ and in the second order in the parameter $\gamma$ for the given pulse envelope, average power, and detection procedure. We demonstrate that the $\gamma^2$-correction to the mutual information is negative. Its absolute value is maximal for the zero dispersion, and it decreases for increasing dispersion parameter $\beta$, see Fig. 1. We also prove that the mutual information calculate using the Gaussian distribution and that calculated with the optimal one coincide in the $\gamma^2$ order. The difference appears only in the $\gamma^4$ order. It means that the Gaussian distribution of the input signal is a good approximation of the optimal distribution for the small nonlinearity parameter. However, for not extremely small correlation length it is necessary to take into account the exact PDF $P_{\text{opt}}[\{C\}]$. So, we are able to construct the sequences $\{C\}$ obeying the statistics $P_{\text{opt}}[\{C\}]$. In the Section VII we propose the method of this construction using the conditional PDFs (64)–(68). For the channels with the small correlation length we calculated explicitly $P_{\text{opt}}[C_i]$ and $P_{\text{opt}}[C_i|C_j]$, and demonstrate the dependence of the probability of the subsequent coefficient $C_i$ on the previous one $C_j$, see Fig. 4.

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Appendix A: Details of the conditional PDF $P[\{\tilde{C}\}|\{C\}]$ calculation

In the following we present the calculation of the conditional PDF $P[Y(\omega)|X(\omega)]$ resulting in the correlators (21). In turn, these correlators enable us to construct the conditional PDF $P[\{\tilde{C}\}|\{C\}]$ in the form (22). In this Appendix we present the explicit expression for the coefficients $F^{'k',k}$, $H^{k',k}$, and $G^{k',k} = \tilde{H}^{k',k}$ appearing in the formula (22). Finally, we present the explicit expression for the normalization factor $\Lambda_\omega$ in Eq. (22).

We start our calculation from the representation for the conditional PDF $P[Y(\omega)|X(\omega)]$ in the form of the path-integral (18), where the effective action $S[\psi]$ is given by the formula (19), see 7. The measure $D\psi(z,\omega)$ can be presented in a specific discretization scheme in such a way that

$$\int DY(\omega)P[Y(\omega)|X(\omega)] = 1,$$

where $DY(\omega) = \prod_{k=-M}^{N-1} d\text{Re}Y(\omega_k) d\text{Im}Y(\omega_k)$, here $2M + 1$ is the total number of the discrete points in frequency domain $\tilde{W}$ with the spacing $2\pi\delta_\omega = \tilde{W}/(2M)$. Let us stress, that the bandwidth $\tilde{W}$ is associated with the noise bandwidth, see Eq. (13). For the discretization in $z$ coordinate with $N$ points with spacing $\delta_z = L/(N - 1)$ one has

$$D\psi(z,\omega) = \left(\frac{\delta_\omega}{\delta_z \pi Q}\right)^{2M+1} \prod_{k=1}^{M} \prod_{j'=1}^{N-1} \left\{ \frac{\delta_\omega}{\delta_z \pi Q} d\text{Re}\psi(z_k,\omega_{j'}) d\text{Im}\psi(z_k,\omega_{j'}) \right\}.$$

1. Short notations

First of all, for brevity sake let us introduce the following notations:

$$e^{i\beta z(\omega_1^2 + \omega_2^2 - \omega_3^2 - \omega_4^2)} d\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4),$$

(A3)
and for any compactly supported functions of frequencies \( F^{(1)}(\omega_1), F^{(2)}(\omega_2), \) and complex conjugated functions \( F^{(3)}(\omega_3), F^{(4)}(\omega_4) \) we use notations:

\[
\begin{align*}
(F^{(1)}, F^{(2)}; F^{(3)}; F^{(4)})_z &= \int_{\mathbb{R}} d\omega_1 d\omega_2 d\omega_3 d\omega_4 (\omega_1, \omega_2; \omega_3, \omega_4) z F^{(1)}(\omega_1) F^{(2)}(\omega_2) F^{(3)}(\omega_3) F^{(4)}(\omega_4), \\
(F^{(1)}, F^{(2)}; F^{(3)}, \omega_4)_z &= \int_{\mathbb{R}} d\omega_1 d\omega_2 d\omega_3 (\omega_1, \omega_2; \omega_3, \omega_4) z F^{(1)}(\omega_1) F^{(2)}(\omega_2) F^{(3)}(\omega_3), \\
(F^{(1)}, F^{(2)}; \omega_3, \omega_4)_z &= \int_{\mathbb{R}} d\omega_1 d\omega_2 (\omega_1, \omega_2; \omega_3, \omega_4) z F^{(1)}(\omega_1) F^{(2)}(\omega_2), \\
(F^{(1)}, \omega_2; \omega_3, \omega_4)_z &= \int_{\mathbb{R}} d\omega_1 (\omega_1, \omega_2; \omega_3, \omega_4) z F^{(1)}(\omega_1). 
\end{align*}
\]

(A4)

It is obvious that these notations have the properties:

\[
\begin{align*}
(\omega_1, \omega_2; \omega_3, \omega_4)_z &= (\omega_2, \omega_1; \omega_3, \omega_4)_z = (\omega_1, \omega_2; \omega_4, \omega_3)_z, \\
(F^{(1)}, F^{(2)}; F^{(3)}, \omega)_z &= (\omega, F^{(3)}; F^{(1)}, F^{(2)})_z.
\end{align*}
\]

(A5)

2. The solution \( \Phi(z, \omega) \) of the NLSE with zero noise. The Jacobian \( \left| \frac{\partial (C^{(0)}, \bar{C}^{(0)})}{\partial (C, \bar{C})} \right| \).

The solution \( \Phi(z, \omega) \) of the NLSE (11) in the frequency domain with zero noise and with the input condition \( \Phi(0, \omega) = X(\omega) \) plays an important role in our consideration. Here we present the perturbative result for this function for the small Kerr nonlinearity \( \gamma \) (\( \gamma L P \ll 1 \)). The solution \( \Phi(z, \omega) \) reads in the first and in the second order in the Kerr nonlinearity parameter \( \gamma \):

\[
\Phi(z, \omega) = e^{i\beta z \omega^2} \bigg\{ X(\omega) + i\gamma \int_0^z d\zeta' (X(X; \bar{X}; \omega)_{z'}) - \gamma^2 \int_0^z dz_1 \int_0^{z_1} dz_2 \int_{\mathbb{R}} d\omega_0 \left[ 2(\omega_0, X; \bar{X}; \omega)_{z_1} (X; \bar{X}; \omega)_{z_2} - (X, X; \bar{X}, \bar{X})_{z_1} (\omega_0, X; \bar{X}, \bar{X})_{z_2} \right] \bigg\},
\]

(A6)

where we have used the notations (A4). The function \( X(\omega) \) appearing in the representation (A6) is the Fourier integral of the input signal, see Eq. (4):

\[
X(\omega) \equiv \int_{-\infty}^{\infty} dt X(t) e^{i\omega t} = \sum_{k = -M}^{M} C_k s^{(k)}(\omega), \quad s^{(k)}(\omega) = s(\omega) e^{ik\omega T_0}. 
\]

(A7)

Note that \( s(\omega) \) is not zero only within the frequency domain \( W: |\omega| < W/2 \).

Let us introduce the coefficients \( \tilde{C}^{(0)}_k \) obtained from the function \( \Phi(z, \omega) \) according to the procedure Eq. (16):

\[
\tilde{C}^{(0)}_k = \frac{1}{T_0} \int_{W_0} \frac{d\omega}{2\pi} e^{-i\beta L \omega^2} \Phi(L, \omega) \tilde{s}^{(k)}(\omega) = C_k + i\gamma L C_k \bar{C} \tilde{C}^{(0)}_{k_1, k_2, k_3, k} - \gamma^2 L^2 C_{m_1} C_{m_2} C_{m_3} \tilde{C}_{m_4} \tilde{C}_{m_5} a_2^{m_1, m_2, m_3; m_4, m_5, k}, 
\]

(A8)

where hereinafter the summation is assumed over all repeated indexes from \( -M \) to \( M \), and in Eq. (A8) we introduce the dimensionless coefficients:

\[
a_1^{k_1, k_2, k_3, k_4} = \frac{1}{2\pi T_0} \int_0^L \frac{dz}{L} \tilde{s}^{(k_1)}(\omega), \bar{s}^{(k)}(\omega), \bar{s}(k_3), \bar{s}(k_4) \bigg|_{z}, 
\]

(A9)
\[ a_{2}^{m_1,m_2,m_3,m_4,m_5,m_6} = \frac{1}{2\pi T_{0}} \left[ \frac{d\zeta_1}{L} \left\{ \frac{d\zeta_2}{L} \right\} \int_{W} d\omega \left[ 2\theta(z_1 - z_2)(\omega_a, s^{(m_3)}, \bar{s}^{(m_5)}, s^{(m_6)}) z_1 s^{(m_1)}, s^{(m_2)} : \omega_a, s^{(m_4)} \right] z_2 - \theta(z_2 - z_1)(\omega_a, s^{(m_3)}, \bar{s}^{(m_4)}, s^{(m_5)}) z_1 s^{(m_1)}, s^{(m_2)} : \omega_a, s^{(m_4)} \right] . \] (A10)

The meaning of the coefficients \( \tilde{C}_{k}^{(0)} \) is as follows: for the nonlinear channel with zero noise the coefficients \( \tilde{C}_{k}^{(0)} \) are the recovered coefficients \( \{ C \} \) on the base of the output signal \( \Phi(L, t) \) according to our procedure [16]. We can restore all \( \{ C \} \) unequivocally in the perturbation theory, if we know all \( \tilde{C}_{k}^{(0)} \).

The Jacobian determinant \( \left| \frac{\partial (\tilde{C}_{k}^{(0)}, \tilde{C}_{k}^{(0)})}{\partial (C, C)} \right| \) plays an important role in the output entropy calculation, see Eq. (37).

Therefore to find the output entropy we should calculate the logarithm of the determinant in Eq. (37). To calculate it we use Eq. (A8), and with \( \gamma^2 \) accuracy we obtain

\[ \log \left| \frac{\partial (\tilde{C}_{k}^{(0)}, \tilde{C}_{k}^{(0)})}{\partial (C, C)} \right| = \gamma^2 L^2 C_{s_1} C_{s_2} C_{s_3} C_{s_4} J^{s_1,s_2,s_3,s_4}, \] (A11)

where dimensionless coefficients \( J^{s_1,s_2,s_3,s_4} \) have the form:

\[ J^{s_1,s_2,s_3,s_4} = \hat{S}_{12,34} \left[ 2a_1^{r',s_1,s_3,r} a_{1}^{s_2,s_4,r} - \frac{1}{2} a_1^{s_1,s_2,r,r'} a_{1}^{r',s_3,s_4} - 2a_2^{r',s_1,s_2,s_3,s_4,r} - a_{2}^{s_1,s_2,r,r';s_3,s_4,r} \right] \]

(A12)

where \( \hat{S}_{12,34} \) is the symmetrization operator that makes the left hand side to be symmetric under substitutions \( s_1 \leftrightarrow s_2 \) and \( s_3 \leftrightarrow s_4 \). We present the convenient combination of coefficients (A10):

\[ a_{2}^{m_1,m_2,m_3,m_4,m_5,m_6} = 2A_{2}^{m_1,m_2,m_3,m_4,m_5,m_6} - (A_{2}^{m_4,m_5,m_6,m_3,m_1,m_2})^*, \] (A13)

where \( A_{2}^{m_1,m_2,m_3,m_4,m_5,m_6} \) is the ninefold integral

\[ A_{2}^{m_1,m_2,m_3,m_4,m_5,m_6} = \frac{1}{2\pi T_{0}} \left[ \frac{d\zeta_1}{L} \left\{ \frac{d\zeta_2}{L} \right\} \int_{W} d\omega a_{1}(s^{(m_3)}, s^{(m_5)}, s^{(m_6)}) z_1 s^{(m_1)}, s^{(m_2)} : \omega_a, s^{(m_4)} \right] . \] (A14)

After simplifying (A12), we present coefficients \( J^{s_1,s_2,s_3,s_4} \) in the form

\[ J^{s_1,s_2,s_3,s_4} = \hat{S}_{12,34} \left[ 4a_{1}^{r',s_1,s_3,r'a_{1}^{s_2,s_4,r'} - a_{1}^{s_1,s_2,r,r'} a_{1}^{r',r';s_3,s_4} - 4A_{2}^{r/s_1,s_2,s_3,s_4,r} - A_{2}^{s_1,s_2,r,r';s_3,s_4} \right] \]

(A15)

where \( \hat{S}_{12,34} \) is the symmetrization operator under the changes \( s_1 \leftrightarrow s_2 \) and \( s_3 \leftrightarrow s_4 \). The explicit result of the symmetrization reads:

\[ J^{s_1,s_2,s_3,s_4} = 2a_1^{r',s_1,s_3,r} a_{1}^{s_2,s_4,r'} + 2a_1^{r',s_2,s_3,r} a_{1}^{s_1,s_4,r'} - a_{1}^{s_1,s_2,r,r'} a_{1}^{r',r';s_3,s_4} + \left[ A_{2}^{r,s_1,s_2,s_3,s_4} - A_{2}^{r,s_1,s_2,s_3,s_4,r} - A_{2}^{r,s_1,s_2,s_3,r,s_4} - A_{2}^{r,s_1,s_2,s_4,r,s_3} \right] + \left[ s_1 \leftrightarrow s_3, s_2 \leftrightarrow s_4 \right] . \] (A16)

The methods of the numerical calculation of these coefficients \( a_{1}^{k_1,k_2,k_3,k_4} \) and \( A_{2}^{s_1,s_2,s_3,s_4,s_5,s_6} \) for the sinc envelope [58] are presented in the Appendix B.

Finally, let us note here that the perturbation theory in the nonlinearity parameter \( \gamma \) allows us to estimate the spectral broadening of the signal which propagation is governed by the NSLE with zero noise. We can define the effective spectral bandwidth of the input signal \( X(t) \) as follows

\[ W_{i}^{2} = \int d\omega \frac{2\pi}{2\pi} |X(\omega)|^{2} \]

(A17)
For the sinc envelope we have $W_t^2 = \frac{W^2}{2\pi}$. In the same manner we can define the effective bandwidth of the output noiseless signal $\Phi(L, t)$:

$$W_f^2 = \int \frac{d\omega}{2\pi} \omega^2 |\Phi(L, \omega)|^2 / \int \frac{d\omega}{2\pi} |\Phi(L, \omega)|^2,$$

where $\Phi(z, \omega)$ is given in Eq. (A10) in perturbation expansion. From this representation it is easy to find that

$$W_f = W_t \left(1 + \frac{\gamma L}{4(\beta L W_t^2)} \int dt |X(t)|^4 - \int dt |\Phi(\gamma = 0, L, t)|^4 \right) + \mathcal{O} (\gamma^2),$$

where $\Phi(\gamma = 0, z, t)$ is the solution of the linear noiseless Schrödinger equation (i.e. equation (11) with $\gamma = 0$ and $\eta = 0$). It is especially simple in the frequency domain: $\Phi(\gamma = 0, z, \omega) = e^{i \beta \bar{z} \omega} X(\omega)$, and in the time domain it reads

$$\Phi(\gamma = 0, z, t) = \int \frac{d\omega}{2\pi} X(\omega) e^{i \beta \bar{z} \omega - i \omega t} = \frac{\theta (z)}{\sqrt{4 \pi \beta}} \int dt' X(t') e^{i \frac{z}{2} - \frac{1}{4\pi \beta} \frac{1}{t'}},$$

(20)

3. Integration over fields $\psi(z, \omega)$ and over $Y(\omega)$ with $|\omega| > W_d$ in path-integral

To calculate the path-integral we perform the change of variables from $\psi(z, \omega)$ to $\phi(z, \omega)$, where

$$\psi(z, \omega) = e^{i \beta \bar{z} \omega} \phi(z, \omega) + \Phi(z, \omega) + \frac{z}{L} B(\omega),$$

(21)

where $\Phi(z, \omega)$ is the solution of the Eq. (11) with zero noise and with the input condition $\Phi(0, \omega) = X(\omega)$, and $B(\omega) = \overline{Y(\omega)} - \Phi(L, \omega)$. The integration over new variables $\phi(z, \omega)$ is performed with the boundary conditions $\phi(0, \omega) = \phi(L, \omega) = 0$. Since we calculate $P_d[Y_d(\omega)|X(\omega)]$ in the leading order in the parameter $1/\text{SNR}$, after substitution of the function $\psi(z, \omega)$ in the form (A21) to Eq. (19) we should retain only terms of the orders of $\phi^0(z, \omega)$, $\phi^1(z, \omega)$, and $\phi^2(z, \omega)$. Then we expand the exponent in Eq. (18) in the parameter $\gamma$ up to $\gamma^2$ terms, and perform the Gaussian integration over $\phi(z, \omega)$.

The result of Gaussian integration over fields $\phi(z, \omega)$ depends on the function $X(\omega)$ and $B(\omega)$, where the frequency $\omega \in [-W/2, W/2]$. According to Eq. (20), to obtain the conditional PDF $P_d[Y_d(\omega)|X(\omega)]$ we should integrate over $B(\omega) = \overline{Y(\omega)} - \Phi(L, \omega)$ for $\omega \in [-W/2, W/2]$ and we arrive at the result:

$$P_d[Y_d|X] = \Lambda_d \exp \left\{ - \frac{1}{Q L} \int_{W_d} d\omega \int_{W_d} d\omega' \left( \delta \tilde{Y}(\omega_1) F(\omega_1, \omega_2) \delta \tilde{Y}(\omega_2) + \delta \tilde{Y}(\omega_1) G(\omega_1, \omega_2) \delta \tilde{Y}(\omega_2) \right) \right\},$$

(22)

One can see that frequency variables in Eq. (A22) are from the interval $\omega \in [-W_d/2, W_d/2]$, rather than $[-W/2, W/2]$ as in Eq. (18). In Eq. (A22) we have introduced the following notations:

$$\delta \tilde{Y}(\omega) = e^{-i \beta \bar{z} \omega^2} \left( Y_d(\omega) - \mathcal{Y}_d(\omega) \right),$$

(23)

where the quantity $\mathcal{Y}_d(\omega)$, see Eq. (A32) below, reads

$$\mathcal{Y}_d(\omega) = \Phi(L, \omega) + i \frac{QLW}{2\pi} L \gamma \Phi(L, \omega) -$$

$$- 4\pi Q \gamma^2 L e^{i \beta \bar{z} \omega^2} \int_0^L dz_1 \int_0^L dz_2 \int \frac{d\omega_1}{W_d} \int \frac{d\omega_2}{W_d} \left( \omega_1 \omega_2 \theta(z_1 - z_2) \frac{\gamma}{L} \theta(\omega_1, \omega_2; \bar{X}, \omega) z_1 (X, \omega_1, \omega_2) z_2 \right),$$

(24)

where $F(\omega_1, \omega_2) = F_0(\omega_1, \omega_2) + F_1(\omega_1, \omega_2) + F_2(\omega_1, \omega_2)$, $H(\omega_1, \omega_2) = H_0(\omega_1, \omega_2) + H_1(\omega_1, \omega_2) + H_2(\omega_1, \omega_2)$, and $G(\omega_1, \omega_2) = H(\omega_1, \omega_2)$, here

$$F_0(\omega_1, \omega_2) = 2\pi \delta(\omega_1 - \omega_2), \quad F_1(\omega_1, \omega_2) = 0;$$

$$F_2(\omega_1, \omega_2) = 4\pi \gamma^2 \int_0^L dz_1 \int_0^L dz_2 \left( \frac{\gamma}{L} z_1 z_2 \right) \int \frac{d\omega}{W_d} \left( \omega_1, \omega_1; \bar{X}, \tilde{X} \right) z_1 (X, \omega_1, \omega_2) z_2,$$

(25)
\[ H_0(\omega_1, \omega_2) = 0, \quad H_1(\omega_1, \omega_2) = -2\pi i \gamma \int_0^L \frac{dz}{L} (X, X; \omega_1, \omega_2)_z; \]

\[ H_2(\omega_1, \omega_2) = 4\pi^2 \int_0^L \int_0^L \frac{d\omega_1}{W} \frac{d\omega_2}{W} \left[ \frac{z_1}{L} (\omega_a, X; \omega_1, \omega_2)_z (X, X; \omega_a, \bar{X})_{z_2} + \frac{z_2}{L} (\omega_a, X; \bar{X}, \omega_2)_z (X, X; \omega_a, \omega_1)_{z_2} \right]. \] (A26)

The normalization factor \( \Lambda_d \) in Eq. (A22) reads

\[ \Lambda_d = \left( \frac{\delta_\omega}{\pi Q L} \right)^{2M_d+1} (1 + \lambda_2), \] (A27)

where \( \delta_\omega = W_d/(2M_d) \) is the frequency grid spacing in the discretization of the path-integral in the frequency domain, \( 2M_d + 1 \) is the frequency discretization number. It means that the integrals over frequencies in Eq. (A22) are understood as the sums: \( \int_{W_d} \frac{d\omega}{2\pi} g(\omega) = \delta_\omega \sum_{k=-M_d}^{M_d} g(\omega_k) \), where \( \omega_k = k\delta_\omega \) and \( g(\omega) \) is any function. The explicit result for \( \lambda_2 \) reads

\[ \lambda_2 = 2\gamma^2 \int_0^L \int_0^L \frac{d\omega_1}{W_d} \frac{d\omega_2}{W_d} \left[ \int \frac{d\omega_a}{W_d} \frac{d\omega_b}{W_d} (\omega_a, \omega_b; \bar{X}, \bar{X})_{z_1} (X, X; \omega_a, \omega_b)_{z_2} - \right. \]

\[ \left. 2\gamma^2 \int_0^L \int_0^L \frac{d\omega_1}{W_d} \frac{d\omega_2}{W_d} \frac{\min(z_1, z_2)}{L} \int \frac{d\omega_a}{W_d} \int \frac{d\omega_b}{W_d} (\omega_a, \omega_b; \bar{X}, \bar{X})_{z_1} (X, X; \omega_a, \omega_b)_{z_2} \right]. \] (A28)

Note that in Eq. (A22) the exponent is understood as a series with the retained terms up to \( \gamma^2 \) only. If we set \( \gamma = 0 \) in Eq. (A22) we result in the Gaussian distribution:

\[ \lim_{\gamma \to 0} P_d[Y_d|X] = \left( \frac{\delta_\omega}{\pi Q L} \right)^{2M_d+1} \exp \left[ -\frac{1}{QL} \int_{W_d} N \left( \frac{d\omega}{2\pi} \right) Y_d(\omega) - e^{i\beta L \omega^2} X(\omega) \right]^2. \] (A29)

One can check that in the limit \( Q \to 0 \) the conditional PDF (A22) reduces to the Dirac delta-function:

\[ \lim_{Q \to 0} P_d[Y_d|X] = \prod_{k=-M_d}^{M_d} \delta(Y_d(\omega_k) - \Phi(L, \omega_k)), \] (A30)

where \( \omega_k = k\delta_\omega \). Also the conditional PDF (A22) obeys the normalization condition:

\[ \int DY_d P_d[Y_d|X] = 1, \] (A31)

where the functional measure reads \( DY_d = \prod_{k=-M_d}^{M_d} dReY_d(\omega_k) dImY_d(\omega_k) \). Note that the quantity \( Y_d(\omega) \) depends only on the input signal \( X(t) \) and has the meaning of the average of the output signal over the distribution (A22):

\[ Y_d(\omega) = \langle Y_d(\omega) \rangle = \int DY_d P_d[Y_d|X] Y_d(\omega). \] (A32)

Let us emphasize that the terms proportional to the noise power \( Q \) in the expression (A24) for \( Y_d(\omega) \) contain the whole noise bandwidth \( W \) rather than the detector bandwidth \( W_d \). This is the consequence of the effects of nonlinearity: due to the Kerr nonlinearity the signal mixes with the noise in the whole frequency interval.
4. The correlators of $\delta \tilde{C}_k$ and the construction of the PDF $P[\{\tilde{C}\}|\{C\}]$. 

Now we can proceed to the calculation of the correlators of the coefficients $\tilde{C}_k$, see Eq. (16). Substituting Eq. (A22), Eq. (16) to the correlator expression (21) and performing integration over $Y_0$ we obtain the following correlator in Kerr nonlinearity expansion:

$$
\langle \tilde{C}_k \rangle = \tilde{C}_k^{(0)} + i \frac{Q L \gamma}{2 \pi} + \gamma L \tilde{C}_k^{(0)} - \frac{2 Q L \gamma^2}{T_0} \int_0^L dz_1 \int_0^L dz_2 \int \frac{d\omega_a}{W} \int d\omega_b \theta(z_1 - z_2) \frac{z_2}{L} \left( \omega_a, \omega_b; \tilde{s}^{(k)}, \tilde{s}(k) \right) z_1 \left( s^{(k)}, s(k); \omega_a, \omega_b \right) z_2 C_k, C_{k_2} C_{k_3},
$$

(A33)

where coefficients $\tilde{C}_k^{(0)}$ are defined in Eq. (A8) through the coefficients $\{C\}$ of the input signal.

For further calculations it is convenient to introduce the quantity $\delta \tilde{C}_k$ in the form $\delta \tilde{C}_k = \tilde{C}_k - \langle \tilde{C}_k \rangle$. This difference is of order of $\sqrt{Q}$. Therefore to construct the conditional PDF $P[\{\tilde{C}\}|\{C\}]$ in the leading order in parameter $1/SNR \propto Q$ it is sufficient to calculate two correlators $\langle \delta \tilde{C}_m \delta \tilde{C}_k \rangle$ and $\langle \delta \tilde{C}_m \delta \tilde{C}_k \rangle$. After straightforward but cumbersome calculations we arrive at following result:

$$
\langle \delta \tilde{C}_m \delta \tilde{C}_k \rangle = \frac{Q L}{T_0} \delta_{mk} + \frac{Q \gamma^2}{T_0} \int_0^L dz_1 \int_0^L dz_2 \int \frac{d\omega_a}{W} \int d\omega_b \theta(z_1 - z_2) \int d\omega_c \left[ \frac{z_2}{L} \left( \omega_a, X; \tilde{s}^{(m)}, \tilde{s}^{(k)} \right) z_1 \left( X, X; \omega_a, \tilde{s}^{(k)} \right) z_2 + \frac{z_2}{L} \left( \omega_a, X; \tilde{s}^{(m)}, \tilde{s}^{(k)} \right) z_1 \left( X, \omega_a, \tilde{s}^{(m)} \right) z_2 \right].
$$

(A34)

$$
\langle \delta \tilde{C}_m \delta \tilde{C}_k \rangle = \frac{Q L}{T_0} \delta_{mk} + \frac{2 Q \gamma^2}{T_0} \int_0^L dz_1 \int_0^L dz_2 \int \frac{d\omega_a}{W} \int d\omega_b \theta(z_1 - z_2) \int d\omega_c \left[ \frac{z_2}{L} \left( \omega_a, \tilde{s}^{(m)}, \tilde{s}^{(k)} \right) z_1 \left( X, X; \omega_a, \tilde{s}^{(m)} \right) z_2 \right].
$$

(A35)

Using the correlators (A33), (A34), and (A35) we can calculate any correlator $\langle \tilde{C}_k, \ldots \tilde{C}_{k_M} \rangle$ in the leading ($Q^0$) and next-to-leading ($Q^1$) order in parameter $Q$. We construct the distribution $P[\{\tilde{C}\}|\{C\}]$ which reproduces all these correlators in the leading and next-to-leading order in parameter $Q$:

$$
P[\{\tilde{C}\}|\{C\}] = \Lambda, \exp \left\{ - \frac{T_0}{QL} \sum_{k,k'=\pm M}^M \delta \tilde{C}_k, F^{k',k} \delta \tilde{C}_k + \delta \tilde{C}_k, G^{k',k} \delta \tilde{C}_k + \delta \tilde{C}_k, H^{k',k} \delta \tilde{C}_k \right\},
$$

(A36)

here $F^{k',k} = F^{k',k} + F^{k',k} + F^{k',k}$, $H^{k',k} = H^{k',k} + H^{k',k}$, $G^{k',k} = G^{k',k} + G^{k',k}$ are dimensionless coefficients with $k, k' = \pm M, \ldots, M$. The subindexes 1 and 2 indicate terms proportional to $\gamma^1$ and $\gamma^2$, respectively. These quantities depend on the input signal via coefficients $C_k$. We have

$$
H^{k,m} = H^{m,k} = \frac{T_0}{2QL} \langle \delta \tilde{C}_k \delta \tilde{C}_m \rangle,
$$

(A37)

therefore using Eq. (A34) we obtain explicit expressions for $H^{k',k}_1$ and $H^{k',k}_2$:

$$
H^{1,k}_m = -\gamma L C_k, C_{k_2} \frac{1}{2 \pi T_0} \int_0^L \frac{dz}{L} \left( s^{(k)}, s^{(m)}, \tilde{s}(k) \right) z_1,
$$

(A38)

$$
H^{2,k}_m = \gamma^2 L^2 C_k, C_{k_2} C_{k_3} \frac{1}{\pi T_0} \int_0^L \frac{dz_1}{L} \int_0^L \frac{dz_2}{L} \int \frac{d\omega_a}{W} \int d\omega_b \left[ \frac{z_2}{L} \left( \omega_a, s^{(k)}, \tilde{s}(k) \right) z_1 \left( s^{(m)}, s^{(k)} \right) z_2 \right] + \frac{z_2}{L} \left( \omega_a, s^{(k)}, \tilde{s}(k) \right) z_1 \left( s^{(m)}, s^{(k)} \right) z_2 + \frac{z_2}{L} \left( \omega_a, s^{(k)}, \tilde{s}(k) \right) z_1 \left( s^{(m)}, s^{(k)} \right) z_2,
$$

(A39)
\[ F_2^{k', k} = 4G_1^{k', m} H_1^{m, k} - \gamma^2 \frac{T_0}{2QL} \frac{\partial^2}{\partial \gamma^2} (\delta \hat{C}_{k'} \delta \hat{C}_k) \bigg|_{\gamma=0} = \]
\[ \gamma^2 L^2 C_k C_{k'} \hat{C}_{k_1} \hat{C}_{k_2} \frac{1}{(2\pi T_0)^2} \int_0^L \frac{dz_1}{L} \int_0^L \frac{dz_2}{L} \left[ \frac{z_1 z_2}{L^2} (\bar{s}(k_1), \bar{s}(k_2), \bar{s}(k), \bar{s}(r')) z_1 (\bar{s}(r'), \bar{s}(k'), \bar{s}(k_3), \bar{s}(k_4))_{z_2} - \right. \]
\[ 4\pi T_0 \frac{\min(z_1, z_2)}{L} \int \frac{d\omega_a}{W} (\bar{s}(k_1), \bar{s}(k_2), \bar{s}(k), \bar{s}(r')) z_1 (\omega_a, \bar{s}(k'), \bar{s}(k_3), \bar{s}(k_4))_{z_2} \right]. \]  

(A40)

The normalization factor \( \Lambda_c \) determines the conditional entropy, see Eq. \([29]\). It has the following form:

\[ \Lambda_c = \left( \frac{T_0}{\pi QL} \right)^{2M+1} \left[ 1 + \left( F_2^{k', k} - 2G_1^{k', k'} H_1^{k, k'} \right) \right] = \left( \frac{T_0}{\pi QL} \right)^{2M+1} [1 + \gamma^2 L^2 C_k C_{k'} \hat{C}_{k_1} \hat{C}_{k_2} \hat{C}_{k_3} \hat{C}_{k_4} J^{s_1, s_2, s_3, s_4}_k], \]  

(A41)

where the real dimensionless coefficients \( J^{s_1, s_2, s_3, s_4}_\Lambda \) are expressed in the following way:

\[ J^{s_1, s_2, s_3, s_4}_\Lambda = \frac{1}{2\pi T_0} \int \frac{dz_1}{L} \int_0^L \frac{dz_2}{L} \left[ \frac{z_1 z_2}{L^2} (s'(s_1), s'(s_2), \bar{s}(r'), \bar{s}(r')) z_1 (s(r'), s'(r'), \bar{s}(s_3), \bar{s}(s_4))_{z_2} - \right. \]
\[ 2\pi T_0 \frac{\min(z_1, z_2)}{L} \int \frac{d\omega_a}{W} (s'(s_1), s'(s_2), \bar{s}(r'), \bar{s}(r')) z_1 (\omega_a, s'(r'), \bar{s}(s_3), \bar{s}(s_4))_{z_2} \right]. \]  

(A42)

Note, as noted previously, the sum over indexes \( r \) and \( r' \) is assumed here.

For the further numerical calculations can present this quantity as follows

\[ J^{s_1, s_2; s_3, s_4}_\Lambda = 2b_1^{s_1, s_2; s_3, s_4} b_1^{r', r'; s_3, s_4} - 2b_2^{s_1, s_2; s_3, s_4}, \]  

(A43)

where

\[ b_1^{s_1, s_2; s_3, s_4} = \frac{1}{2\pi T_0} \int \frac{dz}{L} (s(s_1), s(s_2), \bar{s}(s_3), \bar{s}(s_4)) z \frac{L}{L}, \]  

(A44)

\[ b_2^{s_1, s_2; s_3, s_4} = \frac{1}{2\pi T_0} \int \frac{dz_1}{L} \int \frac{dz_2}{L} \frac{\min(z_1, z_2)}{L} \int \frac{d\omega_a}{W} (\bar{s}(s_1), \bar{s}(s_2), \bar{s}(s_3), \bar{s}(s_4)) z_1 (s(r'), \bar{s}(s_3), \bar{s}(s_4))_{z_2}. \]  

(A45)

The method of the numerical calculation of these coefficients \( b_1^{s_1, s_2; s_3, s_4} \) and \( b_2^{s_1, s_2; s_3, s_4} \) is presented in the Appendix B.

**Appendix B: Calculation of the coefficients** \( J^{s_1, s_2; s_3, s_4}_\Lambda = J^{s_1, s_2; s_3, s_4} + J^{s_1, s_2; s_3, s_4}_\Lambda \)

In the following we will use the dimensionless parameter \([62]\):

\[ \tilde{\beta} = \frac{\beta L W^2}{2}. \]  

(B1)

We choose the envelope of the signal in the sinc form \([82]\). Thus, the input signal \( X(t) \), see Eq. \((4)\), has the form:

\[ X(t) = \sum_{k=-M}^{M} C_k \text{sinc} [W(t - kT_0)/2], \]  

(B2)

where for the first model we use \( T_0 = 2\pi/W \), here \( W \) is given frequency bandwidth of the input signal. The Fourier transform of the signal \([B2]\) has the form

\[ X(\omega) = \int_{-\infty}^{\infty} dt X(t) e^{i\omega t} = \frac{2\pi}{W} \theta(W/2 - |\omega|) \sum_{k=-M}^{M} C_k e^{i\omega T_0}. \]  

(B3)
To obtain the coefficient $C_k$ we can use the following expressions

\[ C_k = \int \frac{dt}{T_0} X(t) \text{sinc} \left[ \frac{W}{2} (t - kT_0) \right], \quad (B4) \]

\[ C_k = \int \frac{d\omega}{2\pi} X(\omega) e^{-i\omega kT_0}. \quad (B5) \]

Note, that the property (6) is exact as resulting from the orthogonality for the complete set of functions $s(t - kT_0)$:

\[ \int_{-\infty}^{\infty} \frac{dt}{T_0} \text{sinc} [W(t - k_1T_0)/2] \text{sinc} [W(t - k_2T_0)/2] = \delta_{k_1,k_2}. \quad (B6) \]

1. Calculation of the Jacobian determinant via coefficients $J^{s_1,s_2,s_3,s_4}$

To find coefficients $J^{s_1,s_2,s_3,s_4}$ from representation (A16):

\[ J^{s_1,s_2,s_3,s_4} = 2a_1^{1',s_1,s_3}a_1^{s_2,s_4,r'} + 2a_1^{r',s_2,s_3}a_1^{s_1,s_4,r'} - a_1^{s_1,s_2;r',s_3,s_4} + \]

\[ A_2^{s_1,s_2;r',s_3,s_4} - A_2^{s_1,s_2;2s_3,s_4} - A_2^{s_1,s_2,s_3,s_4} - A_2^{s_1,s_2,s_4,s_3} - A_2^{s_1,s_2;2s_4,s_3} \]

we should calculate two sums (here we write them explicitly): $\sum_{r=0}^{M} A_2^{r,s_1,s_3,s_4}$ and $\sum_{r=-M}^{M} A_2^{r,s_1,s_2,r,s_3,s_4}$. This summation can be performed at first under the integral, see Eq. (B37) below. The second sum is automatically symmetric under substitutions $s_1 \leftrightarrow s_2$, $s_3 \leftrightarrow s_4$.

To find the mutual information (47) one should find the sum $J^{r,s;r,s} + J^{s,r;s,r}$:

\[ J^{r,s;r,s} + J^{s,r;s,r} = 2J^{r,s;r,s} = \]

\[ 2 \text{Re} \left[ 2a_1^{1',r;r'}a_1^{r',s_1,r'}a_1^{r,r';s_3,s_4} + a_1^{r,r';s_2,s_4}a_1^{r',s_1,r'} - 2A_2^{r',s_3,r,s,r} - 2A_2^{r,s;r',s,r,r} \right]. \quad (B8) \]

a. The calculation of the coefficients $a_1^{n,m,p,k}$

For the sinc envelope (68) the coefficient $a_1^{n,m,p,k}$, see the definition in Eq. (A9), can be rewritten in the form:

\[ a_1^{n,m,p,k} = i \int_{-1}^{1} dx dx_1 dx_2 \theta(1 - |x_1 + x_2 - x|) e^{i\pi x_1(n+p)+x_2(m-p)-x(k-p)} G(\tilde{\beta}(x_1 - x)(x_2 - x)), \quad (B9) \]

where we have introduced the function $G(x)$:

\[ G(x) = -i \int_{0}^{1} dz e^{-izx} = \frac{\cos(x) - 1}{x} - i \frac{\sin(x) - x}{x} - i = \sum_{k=0}^{\infty} \frac{(-ix)^k}{k!(k+1)}. \quad (B10) \]

It is obvious that for the real argument

\[ G(x) = -\overline{G}(-x). \quad (B11) \]

Integration by part in (B9) and the relation (B11) allows us to reduce the representation (B9) to the following one

\[ a_1^{n,m,p,k} = \frac{i(-1)^{n-m-p-k}}{2\pi(n+m-p-k)} \int_{0}^{2-y} dy \int_{0}^{2-y} dt G(\tilde{\beta}yt) \cos \left[ \frac{\pi}{2} (k-p)(t+y) \right] \sin \left[ \frac{\pi t}{2} (k+p-2m) + \frac{\pi y}{2} (k+p-2n) \right] + \]

\[ \overline{G}(\tilde{\beta}yt) \cos \left[ \frac{\pi}{2} (m-n)(t+y) \right] \sin \left[ \frac{\pi t}{2} (m+n-2p) + \frac{\pi y}{2} (m+n-2k) \right] \quad \text{for} \quad n + m - p - k = 0, \quad (B12) \]
and if \( n + m - k - p = 0 \) one has
\[
a_{1,n,m,p,k} = \frac{i}{2} \int_0^y \int_0^{2-y} dt (1-y) \left\{ G(\tilde{\beta}yt) \cos \left[ \frac{\pi}{2} (k - p)(t + y) \right] \cos \left[ \frac{\pi t}{2} (k + p - 2m) + \frac{\pi y}{2} (k + p - 2n) \right] - \right.
\]
\[
\left. G(\tilde{\beta}yt) \cos \left[ \frac{\pi}{2} (m - n)(t + y) \right] \cos \left[ \frac{\pi t}{2} (m + n - 2p) + \frac{\pi y}{2} (m + n - 2k) \right] \right\}, \quad n + m - p - k = 0. \quad (B13)
\]

b. The calculation of the coefficients \( A_{2,m_1,m_2,m_3,m_4,m_5,m_6} \)

Quantities \( A_{2,m_1,m_2,m_3,m_4,m_5,m_6} \) are presented through the nine-fold integrals \( [A14] \). For the sinc envelope \( (B8) \) they take the following form:
\[
A_{2,m_1,m_2,m_3,m_4,m_5,m_6} = \frac{1}{2\pi T_0} \int_0^L \frac{dz_1}{L} \int_0^{z_1} \frac{dz_2}{L} \int_0^{z_2} \frac{dz_3}{L} \int_0^{z_3} \frac{dz_4}{L} \int_0^{z_4} \frac{dz_5}{L} \int_0^{z_5} \frac{dz_6}{L} \int_0^{z_6} \frac{dz_7}{L} \int_0^{z_7} \frac{dz_8}{L} \int_0^{z_8} \frac{dz_9}{L} \left\{ \right.
\]
\[
\left. \exp \left[ 2i\tilde{\beta}x_1 (x_1^2 + x_2^2 - x_5^2 - x_6^2) + 2i\tilde{\beta}x_2 (x_1^2 + x_2^2 - x_2^2) \right] \right\} e^{2\pi i(x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 - x_5 x_6 - x_6 x_7) \frac{3}{2}}. \quad (B14)
\]

Here we have used that the noise bandwidth \( W \gg W \) and introduced the dimensionless variables \( \zeta_{1,2} = z_{1,2}/L \), \( x_i = \omega_i/W \), \( x = \omega_n/W \).

For the case of large \( \tilde{\beta} \) it is convenient to present \( [B14] \) in the form where both delta functions in Eq. \( [B14] \) are integrated out and the quadratic form is reduced to a diagonal form. To this end, we set \( x_6 = 0 \) equal to \( x_1 + x_2 + x_3 - x_5 \) and perform the change of variables:
\[
x_1 = y_5, \quad x_2 = 2y_4 + y_5, \quad x_3 = 2y_2 + y_3 + y_4 + y_5, \quad x_4 = -y_4 + y_4 + y_5, \quad x_5 = y_1 + y_2 + y_3 + y_4 + y_5. \quad (B15)
\]

These transformations of Eq. \( [B14] \) lead to the first presentation of the quantity \( [B14] \)
\[
A_{2,m_1,m_2,m_3,m_4,m_5,m_6} = 4 \int_0^1 \int_0 \int_0 \int_0 \int_0 \int_0 \int_0 \int_0 \int_0 \int_0 \left\{ \right.
\]
\[
\left. \exp \left[ 4i\tilde{\beta}x_1 (y_1^2 + y_2^2) + 4i\tilde{\beta}x_2 (y_3^2 + y_4^2) \right] e^{2\pi i(x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 - x_5 x_6 - x_6 x_7) \frac{3}{2}} \right\} e^{-2\pi i(y_1 y_3 - y_4 y_5) + 2\pi i(y_1 y_4 + y_3 y_5 + y_1 y_5 + y_2 y_4 + y_3 y_5) + 2\pi i(y_1 y_4 + y_2 y_3 + y_1 y_5) + 2\pi i(y_2 y_3 + y_1 y_5) + 2\pi i(y_2 y_4 + y_3 y_5) + 2\pi i(y_3 y_5 + y_1 y_4)} \cdot \quad (B16)
\]

Note that for inner integral (over \( dy_1 \)) the point \( y_1 = 0, y_2 = 0, y_3 = 0, y_4 = 0, y_5 \) is always inside the integration domain. We can perform integration in Eq. \( [B16] \) over \( \zeta_1 \) and \( \zeta_2 \) and then integrate by part over \( y_5 \). Thus, this procedure results in nine four-fold integrals:
\[
A_{2,m_1,m_2,m_3,m_4,m_5,m_6} = \sum_{i=1}^9 I_i^{(m)} \cdot \quad (B17)
\]

We introduce \( N = m_1 + m_2 + m_3 - m_4 - m_5 - m_6 \). For the case \( N = 0 \) the terms \( I_i^{(m)} \) have the following form:
\[
I_1^{(m)} = \frac{2e^{i\pi N}}{i\pi N} \int_0^{1+y_4} \int_0^{2+y_4} \int_0^{1+y_4} \int_0^{1+y_4} \int_0^{1+y_4} \int_0^{1+y_4} \int_0^{1+y_4} \int_0^{1+y_4} \int_0^{1+y_4} \int_0^{1+y_4} \left\{ \right.
\]
\[
\left. \exp \left[ -2\pi i(y_1 y_3 - y_4 y_5) + 2\pi i(y_1 y_4 + y_3 y_5 + y_1 y_5 + y_2 y_4 + y_3 y_5) + 2\pi i(y_2 y_3 + y_1 y_5) + 2\pi i(y_2 y_4 + y_3 y_5) + 2\pi i(y_3 y_5 + y_1 y_4) \right] e^{2\pi i(x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 - x_5 x_6 - x_6 x_7) \frac{3}{2}} \right\} e^{-2\pi i(y_1 y_3 - y_4 y_5) + 2\pi i(y_1 y_4 + y_3 y_5 + y_1 y_5 + y_2 y_4 + y_3 y_5) + 2\pi i(y_2 y_3 + y_1 y_5) + 2\pi i(y_2 y_4 + y_3 y_5) + 2\pi i(y_3 y_5 + y_1 y_4)} \cdot \quad (B18)
\]
\[ I_2^{(m)} = -\frac{2e^{-i\pi N}}{i\pi N} \int_0^{1/2} dy_4 \int_{-1+iy_4}^y \min\{1/2-(y_3+y_4/2),1-(y_3+y_4)\} dy_2 \int_{-1+iy_4}^y dy_1 \times \\
G_2(y_2^2-y_1^2, y_2^2-y_3^2)E_x^{(m)}(y_4, y_3, y_2, y_1), \] (B19)

\[ I_3^{(m)} = \frac{1}{i\pi N} \int e^{2i\pi Ny_5} dy_5 \int_{-1/2}^{1/2} dy_3 \int_{-1/2}^{1/2} dy_2 \int_{-1/2}^{1/2} dy_1 \times \\
G_2(y_2^2-y_1^2, (1/4-y_5/2)^2-y_3^2)E_x^{(m)} \left( \frac{1}{4} - \frac{y_5}{2}, y_3, y_2, y_1 \right), \] (B20)

\[ I_4^{(m)} = -\frac{1}{i\pi N} \int e^{2i\pi Ny_5} dy_5 \int_{-1/2}^{1/2} dy_3 \int_{-1/2}^{1/2} dy_2 \int_{-1/2}^{1/2} dy_1 \times \\
G_2(y_2^2-y_1^2, (1/4+y_5+1/2)^2-y_3^2)E_x^{(m)} \left( y_4, \frac{1}{2} + y_4 + y_5, y_2, y_1 \right), \] (B21)

\[ I_5^{(m)} = -\frac{2}{i\pi N} \int e^{2i\pi Ny_5} dy_5 \int_{-1/2}^{1/2} dy_4 \int_{-1/2}^{1/2} dy_2 \int_{-1/2}^{1/2} dy_1 \times \\
G_2(y_2^2-y_1^2, y_2^2-(y_4+y_5+1/2)^2)E_x^{(m)} \left( y_4, \frac{1}{2} + y_4 + y_5, y_2, y_1 \right), \] (B22)

\[ I_6^{(m)} = \frac{2}{i\pi N} \int e^{2i\pi Ny_5} dy_5 \int_{-1/2}^{1/2} dy_4 \int_{-1/2}^{1/2} dy_2 \int_{-1/2}^{1/2} dy_1 \times \\
G_2(y_2^2-y_1^2, y_2^2-(y_4+y_5-1/2)^2)E_x^{(m)} \left( y_4, \frac{1}{2} + y_4 + y_5, y_2, y_1 \right), \] (B23)

\[ I_7 = \frac{1}{i\pi N} \int e^{2i\pi Ny_5} dy_5 \int_{-1/2}^{1/2} dy_4 \int_{-1/2}^{1/2} dy_3 \theta \left( \frac{1}{2} - (y_3+y_4+y_5) \right) dy_1 \times \\
G_2 \left( \frac{1}{4} - \frac{y_3+y_4+y_5}{2} \right)^2 \left( y_1, y_2^2 - y_3^2 \right)E_x^{(m)} \left( y_4, y_3, \frac{1}{4} + \frac{y_3+y_4+y_5}{2}, y_1 \right), \] (B24)

\[ I_8^{(m)} = -\frac{1}{i\pi N} \int e^{2i\pi Ny_5} dy_5 \int_{-1/2}^{1/2} dy_4 \int_{-1/2}^{1/2} dy_3 \theta \left( \frac{1}{2} + y_3+y_4+y_5 \right) dy_1 \times \\
G_2 \left( \frac{1}{4} + \frac{y_3+y_4+y_5}{2} \right)^2 \left( y_1, y_2^2 - y_3^2 \right)E_x^{(m)} \left( y_4, y_3, -\frac{1}{4} - \frac{y_3+y_4+y_5}{2}, y_1 \right), \] (B25)
\[ I_0^{(m)} = \frac{2}{i\pi N} \int_{-1/2}^{1/2} e^{2i\pi N y_5} dy_5 \int_{-1/2}^{1/2} dy_4 \int_{-1/2}^{1/2} dy_3 \min \left\{ \frac{1}{2} - \frac{y_4 + y_5}{2}, \frac{1}{2} - \frac{y_3 + y_4 + y_5}{2}, \frac{1}{2} - \frac{(y_3 + y_4 + y_5)}{2} \right\} dy_2 \text{sign}(y_2 + y_3 + y_4 + y_5) \times \]

\[ G_2 \left( y_2^2 - \left( \frac{1}{2} - |y_2 + y_3 + y_4 + y_5| \right)^2, y_2^2 - y_2^2 \right) \left[ E_x^{(m)} \left( y_4, y_3, y_2, \frac{1}{2} - |y_2 + y_3 + y_4 + y_5| \right) + \right. \]

\[ E_x^{(m)} \left( y_4, y_3, y_2, -\frac{1}{2} + |y_2 + y_3 + y_4 + y_5| \right) \right], \quad (B26) \]

where we have introduced the functions

\[ E_x^{(m)}(y_4, y_3, y_2, y_1) = e^{\frac{2\pi i y_1}{2\pi N} + \frac{2\pi i y_2}{2\pi N - y_1}} + \frac{2\pi i y_3}{2\pi N - y_2} + \frac{2\pi i y_4}{2\pi N - y_3}, \quad (B27) \]

\[ G_2(a, b) = \int_0^1 d\zeta_1 \int_0^1 d\zeta_2 e^{4i\beta(\zeta_1 a + \zeta_2 b)} = \frac{1}{16\beta^2 b} \left( e^{4i\beta(a+b)} - 1 - \frac{e^{4i\beta a} - 1}{a} \right). \quad (B28) \]

For the case \( N = 0 \) we obtain:

\[ I_1^{(m)} = 2 \int_{-1/2}^{1/2} dy_4 \int_{-1/2}^{1/2} dy_3 \min \left\{ \frac{1}{2} - \frac{y_4 + y_3}{2}, \frac{1}{2} - \frac{y_3 + y_4}{2} \right\} \int_{-1/2}^{1/2} dy_2 \int_{-1/2}^{1/2} dy_1 \times \]

\[ G_2(y_2^2 - y_1^2, y_4^2 - y_3^2) E_x^{(m)}(y_4, y_3, y_2, y_1), \quad (B29) \]

\[ I_2^{(m)} = 2 \int_{-1/2}^{1/2} dy_4 \int_{-1/2}^{1/2} dy_3 \min \left\{ \frac{1}{2} - \frac{y_4 + y_3}{2}, \frac{1}{2} - \frac{y_3 + y_4}{2} \right\} \int_{-1/2}^{1/2} dy_2 \int_{-1/2}^{1/2} dy_1 \times \]

\[ G_2(y_2^2 - y_1^2, y_4^2 - y_3^2) E_x^{(m)}(y_4, y_3, y_2, y_1), \quad (B30) \]

and other integrals \( I_1 - I_11 \) for the case \( N = 0 \) can be obtained from ones in the case \( N \to 0 \) by the change \( e^{2i\pi N y_5} \to y_5 \) under the integral over \( y_5 \).

The second representation of the quantity \( (B14) \) reads as the four-fold integral resulting from the eliminating \( \delta \)-functions in Eq. \( (B14) \) by virtue of the integral representation (specifically, \( \delta(x + x_3 - x_5 - x_6) = \int_{-\infty}^{\infty} da_1 \exp[2\pi i a_1(x + x_3 - x_5 - x_6)] \)) followed by the Gaussian integration over \( x \), and the integration (see, Eq. \( (B33) \) below) over \( x_i \):

\[ A_2^{m_1, m_2, m_3, m_4, m_5, m_6} = \int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \int_{-\infty}^{\infty} da_1 \int_{-\infty}^{\infty} da_2 \sqrt{2\beta(\zeta_1 - \zeta_2)} \exp \left[ -\frac{\pi^2(\alpha_1 - \alpha_2)^2}{2\beta(\zeta_1 - \zeta_2)} + \frac{i\pi}{4} \right] \times \]

\[ E(2\pi(\alpha_1 + m_3), 2\beta\zeta_1) \tilde{E}(2\pi(\alpha_1 + m_3), 2\beta\zeta_1) \tilde{E}(2\pi(\alpha_1 + m_6), 2\beta\zeta_1) \times \]

\[ E(2\pi(\alpha_2 + m_1), 2\beta\zeta_2) \tilde{E}(2\pi(\alpha_2 + m_2), 2\beta\zeta_2) \tilde{E}(2\pi(\alpha_2 + m_4), 2\beta\zeta_2), \quad (B31) \]

where we have introduced the functions

\[ E(a, b) = \int_{-1/2}^{1/2} dy e^{iby^2 + iay}, \quad \tilde{E}(a, b) = \int_{-1/2}^{1/2} dy e^{-iby^2 - iay}, \quad (B32) \]

where one has the following representations for \( E(a, b) \) in the case of positive \( b > 0 \):

\[ E(a, b) = \frac{\sqrt{b}}{\sqrt{b}} \int_{-\sqrt{b}/2}^{\sqrt{b}/2} du e^{i\frac{\pi^2}{2} - \frac{a^2}{2b} \text{ sinc} \left( \frac{a}{2} \right)} = -\sqrt{\frac{\pi}{b}} \exp \left[ -\frac{a^2}{4b} + \frac{i\pi}{4} \right] \frac{\text{erf} \left( e^{i\frac{\pi}{2} \frac{a - b}{2\sqrt{b}}} \right) + \text{erf} \left( e^{i\frac{\pi}{2} \frac{a + b}{2\sqrt{b}}} \right)}{2}. \quad (B33) \]
we perform our calculation for the sin envelope (58). where coefficients \( b \) rotation in the complex plane in Eq. (B31). In this wise we arrive at the following representation of the integral (B31):

\[
E(a, b) \approx e^{ib/4} \text{sinc}(a/2) + O(1/a^2),
\]

(B34)

which directly follows from the second representation in Eq. (B33). Note that \( \hat{E}(a, b) = \hat{E}(a, b) \) only for the real arguments.

There are some difficulties in the numerical calculations of the error function of a complex variable. To avoid these difficulties we use the method of the approximate calculation described in Ref. 33. From the representation (3) in Ref. 33 we have the approximate identity

\[
e^{iu^2} \approx \frac{1}{\sqrt{\pi}} \left( \frac{1}{2} + \sum_{n=1}^{\infty} e^{-n^2/4} \cosh(nue^{i\pi/4}) \right),
\]

(B35)

where the accuracy of the approximation is on the level \( 10^{-7} \) for all complex \( u \): \( |u| < 2.9 \). Using the first representation in Eq. (B33) one arrives at approximation

\[
E(a, b) \approx \frac{1}{\sqrt{\pi}} \left( \frac{\text{sinc}(\frac{a}{2})}{2} + 2 \sum_{n=1}^{\infty} \frac{a \sin \left( \frac{a}{2} \right) \cosh \left( \frac{\sqrt{\pi}}{2} e^{i\pi/4} n \right) + \sqrt{b} \cos \left( \frac{a}{2} \right) e^{i\pi/4} n \sinh \left( \frac{\sqrt{\pi}}{2} e^{i\pi/4} n \right) e^{-\frac{n^2}{4}}}{a^2 + ib n^2} \right),
\]

(B36)

where for \( |a| < 100 \) and \( |b| < 15 \) the accuracy of the approximation (B36) is on the level \( 10^{-7} \).

To avoid the oscillating function when numerically integrating we perform the obvious change of variables and the rotation in the complex plane in Eq. (B31). In this wise we arrive at the following representation of the integral (B31):

\[
A_{n_{1},m_{2};m_{3};m_{4},m_{5},m_{6}}^{m_1} = \int_{0}^{1} dt_{1} \int_{0}^{1} dt_{2} \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{\pi}} E \left( 2\pi m_{1} + \alpha - \zeta \sqrt{2t_{1}t_{2}b}e^{-i\pi/2}, 2\tilde{\beta}t_{1}(1-t_{2}) \right) \times
E \left( 2\pi m_{2} + \alpha - \zeta \sqrt{2t_{1}t_{2}b}e^{-i\pi/2}, 2\tilde{\beta}t_{1}(1-t_{2}) \right) \times
\hat{E} \left( 2\pi m_{4} + \alpha - \zeta \sqrt{2t_{1}t_{2}b}e^{-i\pi/2}, 2\tilde{\beta}t_{1}(1-t_{2}) \right) \times
\hat{E} \left( 2\pi m_{6} + \alpha + \zeta \sqrt{2t_{1}t_{2}b}e^{-i\pi/2}, 2\tilde{\beta}t_{1} \right),
\]

(B37)

where one can use the approximation (B36) for the dispersion parameter \( \tilde{\beta} < 7.5 \) (with the guaranteed approximation level \( 10^{-7} \)).

2. Calculation of the normalization factor via coefficients \( J_{\alpha_{1}}^{\alpha_{2};\alpha_{3};\alpha_{4}} \)

Here we present the calculation strategy for the contribution of the normalization factor (A41) to the mutual information, i.e. the quantities \( J_{\alpha_{1}}^{\alpha_{2};\alpha_{3};\alpha_{4}} \), see Eq. (A43):

\[
J_{\alpha_{1}}^{\alpha_{2};\alpha_{3};\alpha_{4}} = J_{\alpha_{1}}^{\alpha_{2};\alpha_{3};\alpha_{4}} = 2b_{1}^{\alpha_{1};\alpha_{2};\alpha_{3};\alpha_{4}} b_{2}^{\alpha_{2};\alpha_{3};\alpha_{4}} - 2b_{2}^{\alpha_{1};\alpha_{2};\alpha_{3};\alpha_{4}},
\]

where coefficients \( b_{1}^{\alpha_{1};m;p;k} \) and \( b_{2}^{k_{1};k_{2};k_{3};k_{4}} \) are defined in the Eqs. (A44), (A45) respectively.

To find the mutual information (47) one should find the following sum

\[
J_{\alpha_{1}}^{\alpha_{2};\alpha_{3};\alpha_{4}} + J_{\alpha_{1}}^{\alpha_{2};\alpha_{3};\alpha_{4}} = 2J_{\alpha_{1}}^{\alpha_{2};\alpha_{3};\alpha_{4}} = 4b_{1}^{m;p;r,s} b_{2}^{r,s,m,p} - 4b_{2}^{m;p;m,p}.
\]

We perform our calculation for the sin envelope (58).
a. The calculation of the coefficients \( b_{1,n,m:p,k} \)

The method of the calculation of the coefficients \( b_{1,n,m:p,k} \) as defined in Eq. (A44) is identical with the method of calculation of the coefficients \( a_{1,n,m:p,k} \). Thus, we introduce the auxiliary function

\[
G_1(x) = -i \int_0^1 dz e^{-ixz} = \frac{\cos x - 1}{x} + \frac{x - \sin x}{x^2} + i \frac{1 - \cos x - x \sin x}{x^2} = -i \sum_{k=0}^{\infty} \frac{(-ix)^k}{k!(k + 2)},
\]

(B39)

It is obvious that for the real argument

\[
G_1(x) = -G_1(-x).
\]

(B40)

We have from the definition (A44):

\[
b_{1,n,m:p,k} = i \int_{-1}^1 dx dx_1 dx_2 \theta(1 - |x_1 + x_2 - x|) e^{i \pi (x_1(n-p)+x_2(m-p)-(k-p))} G_1(\tilde{\beta}(x_1-x)(x_2-x)). \tag{B41}
\]

Integration by part in (B41) and the relation (B40) allows us to reduce the representation (B41) to the following one

\[
b_{1,n,m:p,k} = \frac{i(-1)^{n+m-p-k}}{2\pi(n+m-p-k)} \int dy \int dy' \left\{ G_1(\tilde{\beta}yt) \cos \left[ \frac{\pi}{2} (k-p)(t+y) \right] \sin \left[ \frac{\pi t}{2} (k+p-2m) + \frac{\pi y}{2} (k+p-2n) \right] + \right. \]

\[
G_1(\tilde{\beta}yt) \cos \left[ \frac{\pi}{2} (m-n)(t+y) \right] \sin \left[ \frac{\pi t}{2} (m+n-2p) + \frac{\pi y}{2} (m+n-2k) \right] \left\}, \quad n+m-p-k = 0, \tag{B42}
\]

and if \( n+m-p-k = 0 \) one has

\[
b_{1,n,m:p,k} = \frac{i}{2} \int dy \int d(1-y) \left\{ G_1(\tilde{\beta}yt) \cos \left[ \frac{\pi}{2} (k-p)(t+y) \right] \cos \left[ \frac{\pi t}{2} (k+p-2m) + \frac{\pi y}{2} (k+p-2n) \right] - \right. \]

\[
G_1(\tilde{\beta}yt) \cos \left[ \frac{\pi}{2} (m-n)(t+y) \right] \cos \left[ \frac{\pi t}{2} (m+n-2p) + \frac{\pi y}{2} (m+n-2k) \right] \right\}, \quad n+m-p-k = 0. \tag{B43}
\]

These two-fold integrals (B42) (B43) and are easy to numerically calculate by the standard methods.

b. The calculation of the coefficients \( b_{2,k_1,k_2,k_3,k_4} \)

Now we proceed to the numerical calculation of the coefficients \( b_{2,k_1,k_2,k_3,k_4} \) as defined in Eq. (A45):

\[
b_{2,k_1,k_2,k_3,k_4} = \frac{1}{2\pi T_0} \int_0^L dz_1 \int_0^L dz_2 \min(z_1,z_2) \int d\omega_\alpha(\omega_\alpha,s(r);\bar{s}(k_3),\bar{s}(k_4)) z_1(s(k_1),s(k_2);\omega_\alpha,\bar{s}(r)) z_2,
\]

where \( \min(z_1,z_2) = z_1 \theta(z_2-z_1) + z_2 \theta(z_1-z_2) \). Note, this representation for the coefficients \( b_{2,k_1,k_2,k_3,k_4} \) differs from the representation (B14) for the coefficients \( A_{2,m_1,m_2,m_3,m_4,m_5,m_6} \) by the change of the integrations over \( z_1 \) and \( z_2 \) \((\int_0^L dz_1 \int_0^L dz_2 \min(z_1,z_2) \to \int_0^L dz_1 \int_0^L dz_2 \min(z_1,z_2))\), and by the changes of indexes \( m_1 \to k_1, m_2 \to k_2, m_3 \to r, m_4 \to r, m_5 \to k_3, m_6 \to k_4 \) followed by the summation over \( r \) from \( -M \) to \( M \). Therefore, we can obtain the following representation from Eq. (B14):

\[
b_{2,k_1,k_2,k_3,k_4} = \frac{1}{2\pi T_0} \int_0^L dz_1 \int_0^L dz_2 \min(z_1,z_2) \int d\omega_\alpha(\omega_\alpha,s(r);\bar{s}(k_3),\bar{s}(k_4)) z_1(s(k_1),s(k_2);\omega_\alpha,\bar{s}(r)) z_2 =
\]

\[
\int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \frac{\min(\zeta_1,\zeta_2)}{L} \int dx \int dx_1 \ldots \int dx_6 \delta(x+x_2-x-x_4) \delta(x+x_3-x_5-x_6) \times \exp \left[ 2i \tilde{\beta}_1(x_1^2+x_3^2-x_5^2-x_6^2) + 2i \tilde{\beta}_2(x_2^2+x_4^2-x_2^2-x_3^2) \right] \sum_{r=-M}^M e^{2\pi i (k_1 x_1 + k_2 x_2 + r (x_3-x_4) - k_3 x_5 - k_4 x_6)}. \tag{B44}
\]
Here we have used that the noise bandwidth $\tilde{W} \gg W$ and employed the dimensionless variables $\zeta_{1,2} = z_{1,2}/L$, $x_i = \omega_i/W$, $x = \omega_a/W$.

The first representation of the coefficients $b_{k_1,k_2;k_3,k_4}^{\alpha_1}$ reads similar to Eqs. (B16) and (B17)

$$b_{k_1,k_2;k_3,k_4}^{\alpha_1} = \sum_{i=1}^{9} \tilde{f}_i^{(k)},$$

(B45)

where four-fold integrals $\tilde{f}_i^{(k)}$ can be obtained from Eqs. (B18–B26) and (B29), (B30) by the changes

$$N \to \tilde{N} = k_1 + k_2 - k_3 - k_4, \quad G_2(a,b) \to G_3(a,b), \quad E^{(m)}(y_4,y_3,y_2,y_1) \to E_{x_{(k)}}(y_4,y_3,y_2,y_1),$$

(B46)

where

$$E_{x_{(k)}}(y_4,y_3,y_2,y_1) = \sum_{r=-M}^{M} e^{-2\pi i y_1(k_3-k_4)+2\pi i(y_2+y_1)(2r-k_3-k_4)+2\pi i y_4(2k_2-k_3-k_4)},$$

(B47)

and

$$G_3(a,b) = \int_{0}^{1} d\zeta_1 \int_{0}^{1} d\zeta_2 \min(\zeta_1,\zeta_2)e^{4i\beta(\zeta_1 a+\zeta_2 b)} =$$

$$\frac{i}{64\beta^3 a^2 b^2 (a+b)} \left[ e^{4i\beta(a+b)} \left( 4i\beta ab (a+b) - a^2 - ab - b^2 \right) + (a+b)(ae^{4i\beta a} + be^{4i\beta b}) - ab \right].$$

(B48)

For the case $\tilde{N} = k_1 + k_2 - k_3 - k_4 = 0$ we use the (B46) for Eqs. (B18–B26) to obtain $\tilde{f}_i^{(k)}$. For the case $\tilde{N} = 0$ we use the change (B46) for Eqs. (B29) and (B30) to obtain $\tilde{f}_1^{(k)}$ and $\tilde{f}_2^{(k)}$, respectively, and the change $\frac{e^{2\pi i N y_5}}{2\pi i N} \to y_5$ under the integral over $y_5$ together with the change (B46) in Eqs. (B20),(B26) to obtain others $\tilde{f}_i^{(k)}$ ($3 \leq i \leq 9$).

The second representation of the quantity (B44) reads as the four-fold integral:

$$b_{k_1,k_2;k_3,k_4}^{\alpha_1} = \int_{0}^{1} d\zeta_1 \int_{0}^{1} d\zeta_2 \min(\zeta_1,\zeta_2) \int_{-\infty}^{\infty} d\alpha_1 \int_{-\infty}^{\infty} d\alpha_2 \sqrt{\frac{\pi}{2\beta(\zeta_1-\zeta_2)}} \exp \left[ -i \frac{\pi^2 (\alpha_1-\alpha_2)^2}{2\beta(\zeta_1-\zeta_2)} + i\frac{\pi}{4} \text{sign}(\zeta_1-\zeta_2) \right] \times$$

$$\sum_{r=-M}^{M} E(2\pi(\alpha_1 + r),2\beta\zeta_1)\tilde{E}(2\pi(\alpha_2 + r),2\beta\zeta_2) \times$$

$$E(2\pi(\alpha_2 + k_1),2\beta\zeta_2)\tilde{E}(2\pi(\alpha_2 + k_2),2\beta\zeta_2)\tilde{E}(2\pi(\alpha_1 + k_3),2\beta\zeta_1)\tilde{E}(2\pi(\alpha_1 + k_4),2\beta\zeta_1),$$

(B49)

where functions $E(a,b) = \int_{-1/2}^{1/2} dy e^{iay^2+iyb}$ and $\tilde{E}(a,b) = \int_{-1/2}^{1/2} dy e^{-iay^2-iyb}$ are defined through Eq. (B33). Let us stress once

$$\tilde{E}(a,b) = \overline{E(a,b)},$$

(B50)

where the overline means the complex conjugation, i.e., $\tilde{E}(a,b) = \overline{E(a,b)}$ only for reals $a$ and $b$. Now we perform the change of variables in the inner integrals over $\alpha_1$ and $\alpha_2$:

$$\alpha_1 = \frac{\alpha + \zeta \sqrt{2\beta i(\zeta_2 - \zeta_1)}}{2\pi}, \quad \alpha_2 = \frac{\alpha - \zeta \sqrt{2\beta i(\zeta_2 - \zeta_1)}}{2\pi},$$

(B51)

where here and below we assume the following branches of the square root analytical function

$$\sqrt{2\beta i(\zeta_2 - \zeta_1)} = e^{\frac{i\pi}{2} \text{sign}(\zeta_2-\zeta_1)} \sqrt{2\beta(\zeta_2 - \zeta_1)}.$$
In such a way, we arrive at the following representation that is more convenient for the numerical calculations by the standard methods

\[
b_{k_{1}, k_{2}; k_{3}, k_{4}} = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \int_{-\infty}^{\infty} d\zeta \frac{e^{-\zeta^2}}{\sqrt{\pi}} \times 
\sum_{r=-M}^{M} E \left(2\pi r + \alpha - \zeta \sqrt{2\beta i (\zeta - \zeta_1), 2\beta c_2} \right) \tilde{E} \left(2\pi r + \alpha + \zeta \sqrt{2\beta i (\zeta - \zeta_1), 2\beta c_2} \right) \times 
\sum_{r=-M}^{M} E \left(2\pi k_1 + \alpha - \zeta \sqrt{2\beta i (\zeta - \zeta_1), 2\beta c_2} \right) \tilde{E} \left(2\pi k_2 + \alpha - \zeta \sqrt{2\beta i (\zeta - \zeta_1), 2\beta c_2} \right) \times 
\sum_{r=-M}^{M} E \left(2\pi k_3 + \alpha + \zeta \sqrt{2\beta i (\zeta - \zeta_1), 2\beta c_2} \right) \tilde{E} \left(2\pi k_4 + \alpha + \zeta \sqrt{2\beta i (\zeta - \zeta_1), 2\beta c_2} \right). 
\]

(B53)

The further numerical calculation of (B53) is based on the approximation (B36) for the functions \(E\) and \(\tilde{E}\).

3. Zero dispersion limit

For the case of zero dispersion \(\tilde{\beta} = 0\) one has the following results:

\[
a_{1}^{n,m,p,k} \bigg|_{\tilde{\beta}=0} = \int_{-\infty}^{\infty} d\tau \text{sinc}(\pi(\tau + n))\text{sinc}(\pi(\tau + m))\text{sinc}(\pi(\tau + k))\text{sinc}(\pi(\tau + p)) = 
\text{sinc}(\pi(n - p))\text{sinc}(\pi(m - p))\text{sinc}(\pi(k - p)) + \frac{1}{2\pi(k - p)} \left( \text{sinc}(\pi(n-p)) \cos(\pi(k + m - 2p)) \right) - 
\text{sinc}(\pi(n - m)) \frac{\cos(\pi(k - p))}{\pi(m - k)} + \text{sinc}(\pi(n - m)) \frac{\cos(\pi(k - p))}{\pi(m - k)} \bigg|_{\tilde{\beta}=0}, 
\]

(B54)

Using the second representation in Eq (B33) it is easy to obtain the simple (for numerical calculation) result for the nondispersive channel from Eq. (B37):

\[
A^{m_1,m_2,m_3,m_4,m_5,m_6}_{2} \bigg|_{\tilde{\beta}=0} = \frac{1}{2} \int_{-\infty}^{\infty} d\alpha \prod_{i=1}^{6} \text{sinc} (\pi(\alpha + m_i)), 
\]

(B55)

For the zero dispersion case it is easy to obtain the following representations

\[
b_{1}^{n,m,p,k} \bigg|_{\tilde{\beta}=0} = \frac{1}{2} a_{1}^{n,m,p,k} \bigg|_{\tilde{\beta}=0}, 
\]

(B56)

where the analytical result for \(a_{1}^{n,m,p,k}\) is given by Eq. (B54).

\[
b_{1}^{n,m,p,k} \bigg|_{\tilde{\beta}=0} = \frac{1}{2} a_{1}^{n,m,p,k} \bigg|_{\tilde{\beta}=0}, 
\]

(B57)

For zero dispersion \(\tilde{\beta}\) it is easy to obtain the one-fold integral representation

\[
b_{2}^{k_{1},k_{2};k_{3},k_{4}} \bigg|_{\tilde{\beta}=0} = \frac{2}{3} \sum_{r=-M}^{M} A^{k_{1},k_{2},r;k_{3},k_{4},r}_{2} \bigg|_{\tilde{\beta}=0} = \frac{1}{3} \int_{-\infty}^{\infty} \frac{dt}{T_0} s^{(k_1)}(t)s^{(k_2)}(t)s^{(k_3)}(t)s^{(k_4)}(t) \sum_{r=-M}^{M} s^{(r)}(t)^2 = 
\]

(B58)

where \(A^{k_{1},k_{2},r;k_{3},k_{4},r}_{2} \bigg|_{\tilde{\beta}=0}\) is taken from the Eq. (B55).
These formulae lead to the following representations:

\[
2 J_{\lambda}^{r,s,r,s} = 4 b_1^{m,p,r,r'} b_1^{r,r';m,p} - 4 b_2^{m,p,m,p} = \int_{-\infty}^{+\infty} d \tau_1 \int_{-\infty}^{+\infty} d \tau_2 S^4(\tau_1, \tau_2) - \frac{4}{3} \int d \tau S^8(\tau, \tau), \tag{B59}
\]

\[
2 J_{\lambda}^{r,s,r,s} = 16 b_1^{m,p,r,r'} b_1^{r,r';m,s,s} + 8 b_1^{m,p,r,r'} b_1^{r,r';m,m} - 18 b_2^{m,p,m,p} = \int_{-\infty}^{+\infty} d \tau_1 \int_{-\infty}^{+\infty} d \tau_2 (2 S^8(\tau_1, \tau_2) + 4 S^4(\tau_1, \tau_2) S^2(\tau_1, \tau_1) S^2(\tau_2, \tau_2)) - 6 \int d \tau S^6(\tau, \tau), \tag{B60}
\]

where

\[
S^2(\tau_1, \tau_2) = \sum_{r=-M}^{M} \sin(\pi(\tau_1 + r)) \sin(\pi(\tau_2 + r)), \quad S^2(\tau, \tau) = \sum_{r=-M}^{M} \sin^2(\pi(\tau + r)). \tag{B61}
\]

And in the sum one has

\[
2 J_{\lambda}^{r,s,r,s} + 2 J_{\lambda}^{r,s,r,s} = 12 b_1^{m,p,r,r'} b_1^{r,r';m,p} + 16 b_1^{m,p,p,m} b_1^{n,s,s,m} - 22 b_2^{m,p,m,p} = \int_{-\infty}^{+\infty} d \tau_1 \int_{-\infty}^{+\infty} d \tau_2 (3 S^8(\tau_1, \tau_2) + 4 S^4(\tau_1, \tau_2) S^2(\tau_1, \tau_1) S^2(\tau_2, \tau_2)) - \frac{22}{3} \int d \tau S^6(\tau, \tau). \tag{B62}
\]

There is no necessity to calculate the two-fold integral, since it represents the sum \(12 b_1^{m,p,r,r'} b_1^{r,r';m,p} + 16 b_1^{m,p,p,m} b_1^{n,s,s,m}\), and for all \(b_1 = \frac{1}{2} a_1\) we have the explicit representation (B54). However the formula (B62) is useful to understand how the expression in the r.h.s. of (B62) turns into \(-(2M + 1)N_0 N_0 - \frac{21}{3} N_0^2\) for the case of the non-overlapping signals, see Eq. (54).

[1] C. Shannon, "A mathematical theory of communication", Bell System Techn. J., vol. 27, no. 3, pp. 379–423, 1948; vol. 27, no. 4, pp. 623–656, 1948.

[2] C. E. Shannon, Communication in the presence of noise, Proc. Institute of Radio Engineers, vol. 37, 1, (1949).

[3] H. A. Haus, J. Opt. Soc. Am. B, V. 8, No 5, 1122 (1991).

[4] A. Mecozi, J.Lightw. Technol., V. 12, No. 11, 1993 (1994).

[5] E. Iannoe, F. Matura, A. Mecozi, and M. Settembre, Nonlinear Optical Communication Networks, John Wiley & Sons, New York, (1998).

[6] A. V. Reznichenko, I.S. Terekhov, IEEE Xplore, IEEE Information Theory Workshop 2017, 186-190 (2017).

[7] A.V. Reznichenko, I.S. Terekhov, Journal of Physics: Conference Series Volume 1206, Issue 1, 17 April 2019, 012013 [arXiv: 1811.10315].

[8] A. V. Reznichenko, I.S. Terekhov, Journal of Physics: Conference Series Volume 1206, Issue 1, 17 April 2019, 012013 [arXiv: 1811.10315].

[9] A. V. Reznichenko, I.S. Terekhov, Journal of Physics: Conference Series Volume 1206, Issue 1, 17 April 2019, 012013 [arXiv: 1811.10315].

[10] A. V. Reznichenko, I.S. Terekhov, Journal of Physics: Conference Series Volume 1206, Issue 1, 17 April 2019, 012013 [arXiv: 1811.10315].

[11] A. Mecozi, J. Lightwave Technol. 12, 1993 (1994).

[12] A. Mecozi and M. Shtai, IEEE Photonics Technol. Lett. 13, 1029 (2001).

[13] J. Tang, J. Lightwave Technol. 19, 1104 (2001).

[14] K. S. Turitsyn, S. A. Derevyanko, I. V. Yurkevich, and S. K. Turitsyn, Phys. Rev. Lett. 91, 203901 (2003).

[15] M. I. Youssef and F. R. Kschischang, IEEE Trans. Inf. Theory 57, 7522 (2011).

[16] I. S. Terekhov, A. V. Reznichenko, Ya. A. Kharkov, and S. K. Turitsyn, Phys. Rev. E 95, 062133 (2017).

[17] A. A. Panarin, A. V. Reznichenko, I. S. Terekhov, Phys. Rev. E 95, 012127 (2016).

[18] G. Kramer, Submitted to the IEEE Transactions on Information Theory, (2018) [arXiv:1705.00454v2].

[19] A. V. Reznichenko, A. I. Chernyukh, S. V. Smirnov, and I. S. Terekhov, "Log-log growth of channel capacity for nondispersive nonlinear optical fiber channel in intermediate power range: Extension of the model", Phys. Rev. E 99, 012133 (2019), ArXiv: 1810.00513.

[20] F. J. García-Gómez, G. Kramer, Journal of Lightwave Technology, Volume: 38, Issue: 24, Dec.15, 15, (2020) [http://arxiv.org/abs/2004.04709v3].

[21] P. Mitra and J. B. Stark, Nature 411, 1027 (2001).
[22] E. E. Narimanov and P. Mitra, J. Lightwave Technol. 20, 530 (2002).
[23] J. M. Kahn and K.-P. Ho, IEEE. J. Sel. Topics Quant. Electron. 10, 259 (2004).
[24] R.-J. Essiambre, G. J. Foschini, G. Kramer, and P. J. Winzer, Phys. Rev. Lett. 101, 163901 (2008).
[25] R.-J. Essiambre, G. Kramer, P. J. Winzer, G. J. Foschini, and B. Goebel, J. Lightwave Technol. 28, 662 (2010).
[26] R. Killey and C. Behrens, J. Mod. Opt. 58, 1 (2011).
[27] E. Agrell, A. Alvarado, G. Durisi, and M. Karlsson, IEEE/OSA J. Lightwave Technol. 32, 2862 (2014).
[28] M. A. Sorokina and S. K. Turitsyn, Nat. Commun. 5, 3861 (2014).
[29] E. Agrell, A. Alvarado, G. Durisi, M. Karlsson, Journal of Lightwave Technology, vol. 32(16), pp. 2862-2876 (2014).
[30] M. A. Lavrentiev and B.V. Shabat, Method of Complex Function Theory (Nauka, Moscow, 1987); M. Lavrentiev and B. Chabot, Methodes de la Theorie des fonctions d‘une variable complexe (Mir, Moscow, 1977).
[31] I. S. Terekhov, A. V. Reznichenko, and S. K. Turitsyn, "Calculation of mutual information for nonlinear communication channel at large signal-to-noise ratio", Phys. Rev. E, vol. 94, no. 4, p. 042203, October 2016.
[32] A.V. Voytishek, Foundations of the Mote Carlo Methods in Algorithms and Problems. Parts 1-5. - Novosibirsk State University, 1997-99 (in Russian).
[33] H. E. Salzer, Mathematical Tables and Other Aids to Computation, Vol. 5, No. 34 (Apr., 1951), pp.67-70.