GROUP GRADINGS ON FINITE DIMENSIONAL INCIDENCE ALGEBRAS

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Abstract. In this work, we classify the group gradings on finite-dimensional incidence algebras over a field, where the field has characteristic zero, or the characteristic is greater than the dimension of the algebra, or the grading group is abelian.

Moreover, we investigate the structure of $G$-graded $(D_1, D_2)$-bimodules, where $G$ is an abelian group, and $D_1$ and $D_2$ are the group algebra of finite subgroups of $G$. As a consequence, we can provide a more profound structure result concerning the group gradings on the incidence algebras, and we can classify their isomorphism classes of group gradings.

1. INTRODUCTION

When one deals with an algebraic structure, it is usually useful to consider an additional structure on it, when possible. Considering group gradings in algebras has proved to be useful to solve relevant mathematical problems more than once. That is the case of the classification of finite-dimensional semisimple Lie algebras, which are graded by the root system (see e.g. [8], or [15]). That is also the case of the positive solution for the Specht problem given by Kemer in [12]. Furthermore, group gradings naturally appear in several branches of Mathematics and Physics.

We briefly recall the definition of graded algebras. Let $A$ be any algebra (not necessarily associative, not necessarily with unit), and let $G$ be any group. We say that $A$ is $G$-graded if there exists a vector space decomposition $A = \bigoplus_{g \in G} A_g$, where $A_g$ are possibly zero subspaces, such that $A_g A_h \subseteq A_{gh}$, for all $g, h \in G$. The elements of $\bigcup_{g \in G} A_g$ are called homogeneous, and a non-zero $x \in A_g$ is said to have degree $g$, denoted by $\text{deg}_G x = g$, or simply $\text{deg} x = g$ if there is no ambiguity. An extensive theory concerning graded algebras can be found in the monograph [6].

An interesting question is, given an algebra, determine all the possible group gradings on it up to graded isomorphisms. In this direction, the works [1, 3] gave the answer for the matrix algebras (they investigate a more general situation), and then, in [2], the authors show how to obtain group gradings on some simple Lie and Jordan algebras from the knowledge of the group gradings on matrix algebras. These works started an extensive research in the subject. The monograph [2] is a complete state-of-art of the theory.
In this paper, we are interested in the non-simple associative algebras called incidence algebras. These are a generalization for the upper triangular matrices $UT_n$. In short, an incidence algebra is a subalgebra of $UT_n$, generated by matrix units and containing all diagonal matrices (see a precise definition below). Thus, the classification of group gradings on upper triangular matrices is a particular result of the classification of group gradings on the incidence algebras. The classification of group gradings on the upper triangular matrices was obtained in two works [17, 5].

To state the classification, we give some definitions. We call a grading on $UT_n$ good if all matrix units $e_{ij}$ are homogeneous in the grading; and we call a $G$-grading on $UT_n$ elementary if there exists a sequence $(g_1, g_2, \ldots, g_n) \in G^n$ such that every $e_{ij}$ is homogeneous of degree $g_i g_j^{-1}$. Clearly every elementary grading is a good grading, and, in the context of upper triangular matrices, the converse holds. Hence, good and elementary gradings are equivalent notions for the algebra of upper triangular matrices. In [17], it is proved that every group grading on $UT_n$ is elementary, up to an isomorphism. In [5], the isomorphism classes of elementary gradings on $UT_n$ are classified, thus obtaining a complete classification of group gradings on this algebra.

Note that the notions of elementary gradings and good gradings can be defined for subalgebras of matrix algebras generated by matrix units. Hence, in particular, one can define these notions for the incidence algebras. It is worth mentioning that these notions were generalized in [4] for a larger class of algebras. It is natural to conjecture that these two notions are equivalent, and that every grading on an incidence algebra is elementary up to an isomorphism. However, these two notions are not always equivalent: not every group grading on an incidence algebra is elementary or good (see below). Thus, incidence algebra turns out to be a more intricate combinatorial object in the point of view of gradings by a group.

In this paper, we classify group gradings on the incidence algebra $I(X)$ over a field $F$, where $X$ is finite and one of the following conditions hold: the base field has characteristic zero, $\text{char } F > \dim I(X)$, or $G$ is abelian.

Some works were dedicated to study the group gradings on the incidence algebras [13, 10, 14], but none of them provide a complete understanding of its group gradings. Some different phenomenon happens in this algebra. The first surprising fact is that not every good grading on $I(X)$ is necessarily elementary, even if $X$ is finite (see, for instance, [10 Example 12]). Moreover, there exist group gradings on $I(X)$ that are not good (see also [13 Example 1]):

**Example 1.** Let $X = \{1, 2, 3\}$ be partially ordered by $1 \leq 2$ and $1 \leq 3$. Then we obtain the algebra

$$I(X) = \left\{ \begin{pmatrix} * & * & * \\ * & 0 & * \\ * & * & * \end{pmatrix} \right\}.$$

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, and define

$$A_{(0,0)} = \text{Span}\{e_{11}, e_{22} + e_{33}\},$$
$$A_{(0,1)} = \text{Span}\{e_{22} - e_{33}\},$$
$$A_{(1,0)} = \text{Span}\{e_{12} - e_{13}\},$$
$$A_{(1,1)} = \text{Span}\{e_{12} + e_{13}\}.$$

This is a well-defined grading on $I(X)$. Moreover, it is not isomorphic to a good grading, since there are not three orthogonal homogeneous idempotents.
Thus, the study of group gradings on incidence algebras is more complicated than the case of upper triangular matrices. Recall that a grading by a group is equivalent to the action of the group of characters on the algebra, given that the group is abelian and the base field is algebraically closed of characteristic zero (see, for instance, [6, Section 1.4]). The automorphism group of the incidence algebra is known [16, Theorem 7.3.6], and it is more intricate than the automorphism group of the upper triangular matrices (see, for instance, [9]). Hence, the automorphism group of $I(X)$ suggests the existence of new gradings, other than the gradings on upper triangular matrices.

Another surprising fact is the existence of group gradings on incidence algebras without any homogeneous multiplicative basis (see example in Section 7). We recall that a basis $B$ is multiplicative if for all $u, v \in B$, there exists a scalar $\lambda$ such that $\lambda w = uv$, for some $w \in B$. In contrast, it is clear that any good grading admits a multiplicative homogeneous basis. Thus, every group grading on $UT_n$ admits a multiplicative homogeneous basis.

In the next section, we are going to prove our main result:

**Theorem 1.** Let $F$ be a field, $X$ a finite poset, and let $I(X)$ be endowed with a $G$-grading. Assume at least one of the following conditions: $\text{char} F = 0$, $\text{char} F > \text{dim} I(X)$, or $G$ is abelian. Then, up to a graded isomorphism, there exist finite abelian subgroups $H_1, \ldots, H_t \subseteq G$, such that: for each $i = 1, \ldots, t$, $\text{char} F$ does not divide $|H_i|$, and $F$ contains a primitive $\exp H_i$-root of $1$; and $I(X) \cong \begin{pmatrix} FH_1 & M_{1,2} & \cdots & M_{1,t} \\ & FH_2 & \vdots & \\ & \vdots & \ddots & \vdots \\ & \vdots & & FH_t \\ & M_{t-1,2} & \cdots & M_{t-1,t} \\ & & & FH_t \end{pmatrix}$, where each $M_{i,j}$ is a graded $(FH_i, FH_j)$-bimodule.

The set $E = \{1, 2, \ldots, t\}$ has a natural structure of poset, and it plays an essential role in establishing the isomorphism classes of group gradings on the incidence algebras. The partial order of $E$ is given by $i \preceq j$ if $M_{i,j} \neq 0$ (see Remark 19).

Further, if we assume the grading group abelian, then we can prove a stronger statement concerning the bimodules $M_{ij}$:

**Theorem 2.** Assume that $G$ is abelian, and, using the notation of Theorem 1, fix $i < j$ and let $H_{ij} = H_i \cap H_j$. Then, there exist pairwise distinct characters $\chi_1^{(ij)}, \ldots, \chi_{s_{ij}}^{(ij)} \in \hat{H}_{ij}$, and $m_1, \ldots, m_{s_{ij}} \in M_{ij}$ homogeneous such that $M_{ij} = FH_1 m_1 FH_j \oplus \cdots \oplus FH_1 m_{s_{ij}} FH_j$, where $hm_{i} = \chi_{i}^{(ij)}(h)m_{i}h$, for all $h \in H_{ij}$, for $\ell = 1, 2, \ldots, s_{ij}$. Moreover, as a $G$-graded vector spaces, $FH_{\ell} m_{\ell} FH_{\ell} \cong (FH_{\ell} FH_{\ell})^{[\deg m_{\ell}]}$, for each $\ell = 1, 2, \ldots, s_{ij}$. Also, $s_{ij} \leq |H_{ij}|$.

Finally, given that the grading group is abelian, we solve the problem of classifying the isomorphism classes of group gradings on the incidence algebras (Theorem 34).
Lemma 4. Let $char$ follow the conditions:

$\left( \begin{array}{cc} F & M \\ 0 & \mathbb{F}Z_2 \end{array} \right)$,

where $M$ is isomorphic to $\mathbb{F}Z_2$ as a right $\mathbb{F}Z_2$-module.

2. Preliminaries

2.1. Incidence algebras. We provide the definition of an incidence algebra over a field $\mathbb{F}$. Let $(X, \leq)$ be any partially ordered set (poset, for short). Assume that $(X, \leq)$ is locally finite, i.e., for all $x, y \in X$, there exists a finite number of $z \in X$ such that $x \leq z \leq y$. Define $I(X) = \{ f : X \times X \to \mathbb{F} \mid f(x, y) = 0, \forall x \not\leq y \}$. Then $I(X)$ has a natural sum (pointwise sum), and natural scalar multiplication, which gives $I(X)$ a structure of $\mathbb{F}$-vector space. For $f, g \in I(X)$, we define $h = f \cdot g$ as the function $h$, such that $h(x, y) = \sum_{z \in X} f(x, z)g(z, y)$. Note that the unique possibly non-zero elements in the previous sum are the $z \in X$ such that $x \leq z \leq y$; hence, since $X$ is locally finite, the sum is well-defined. So $h \in I(X)$. It is standard to prove that $I(X)$, with the defined operations, is an associative algebra. The algebra $I(X)$ is called an incidence algebra.

Note that the defined multiplication on $I(X)$ is similar to the product of matrices. Moreover, if $X$ is totally ordered and contains $n$ elements, then $I(X) \cong UT_n$. In connection, if $(X, \leq)$ is arbitrary (not necessarily totally ordered) and finite with $n$ elements, then we can rename the elements of $X = \{1, 2, \ldots, n\}$ in such way that $i \leq X j$ implies $i \leq j$ in the usual ordering of the integers. With this identification, we see that $I(X) \subset UT_n$ is a subalgebra. Thus, finite-dimensional incidence algebras are subalgebras of $UT_n$, generated by matrix units, and containing the diagonal matrices. As we are interested in finite-dimensional incidence algebras, we will assume from now on that $I(X) \subset UT_n$.

The incidence algebras are very interesting by its own and, moreover, they are related with other branches of Mathematics. They give rise to interesting and challenging combinatorial problems. For an extensive theory on incidence algebras see, for instance, the book [16].

2.2. Graded Jacobson radical. It is fundamental in our arguments to guarantee that the Jacobson radical of the incidence algebras are graded ideals.

For this, we have the following result, due to Gordienko:

Lemma 3 (Corollary 3.3 of [7]). Let $A$ be a finite-dimensional algebra, graded by any group $G$. Suppose $char \mathbb{F} = 0$ or $char \mathbb{F} > dim A$. Then the Jacobson radical $J(A)$ is a graded ideal.

Now, if $I(X)$ has a $G$-grading, and $G$ is abelian, then the commutator is a homogeneous operation. Moreover, $[I(X), I(X)] = J(I(X))$ is the Jacobson Radical of the incidence algebra $I(X)$. Hence, $J(I(X))$ is a graded ideal. Thus, we proved

Lemma 4. Let $X$ be finite, and let $I(X)$ be $G$-graded. Assume at least one of the following conditions: $char \mathbb{F} = 0$, $char \mathbb{F} > dim I(X)$, or $G$ is abelian. Then the Jacobson radical of $I(X)$ is graded.
2.3. **Group algebras over fields.** We include in this subsection a few facts concerning the group algebras, which will be essential for our purposes. We do not include the proofs, since the results are classical, or an adaptation of classical proofs when the base field is \( \mathbb{C} \), the field of complex numbers.

Let \( F \) be a field, \( G \) a group, and we denote \( \hat{G} = \{ \chi : G \to F \ \text{group homomorphisms} \} \).

**Lemma 5.** Assume that \( G \) is abelian finite. The following assertions are equivalent:

(i) \( G \cong \hat{G} \),
(ii) \( FG \cong F\hat{G} \) (as \( F \)-algebras),
(iii) for every \( g \in G \) having order \( q \), \( \text{char} \ F \) does not divide \( q \), and \( F \) contains a primitive \( q \)-root of unit.

\[ \square \]

For instance, we will have the following situation. A \( G \)-graded algebra \( A \) admits a subalgebra \( D \subseteq A \) which is, simultaneously isomorphic to \( F^m \) as ordinary algebras, and to \( FH \) as graded algebras, for some abelian subgroup \( H \subseteq G \). Then, in this case, the field \( F \) must satisfy the properties (iii) of the lemma above.

**Lemma 6.** Let \( G \) be finite, assume that \( \hat{G} \cong G \), and let \( \hat{G} = \{ \chi_1, \ldots, \chi_m \} \). The map \( \psi : FG \to F^m \), given by

\[
(1) \quad \psi(g) = (\chi_1(g), \ldots, \chi_m(g))
\]

is an isomorphism of algebras. Moreover, every isomorphism of algebras \( \psi' : FG \to F^m \) is given by \( \psi \), up to a permutation of the indexes.

\[ \square \]

3. **Group gradings on incidence algebras**

Let \( (X, \leq_X) \) be a finite poset, and let \( n = |X| \). We fix a field \( F \) and \( G \) any group with multiplicative notation and neutral element 1, and we assume one of the conditions of Lemma\([17]\). As mentioned before, we can rename the elements of \( X \) as \( X = \{1, 2, \ldots, n\} \), and \( i \leq_X j \) implies \( i \leq j \) in the usual ordering of the integers.

Hence, we can identify \( I(X) \subseteq UT_n \). If \( 1 \leq i \leq j \leq n \) and \( x \in I(X) \), \( x(i, j) \) denotes the entry \( (i, j) \) of the element \( x \).

By direct computation, it is easy to see that for any invertible \( M \in I(X) \), we have \( M^{-1} \in I(X) \), and it is a well-known result.

**Lemma 7.** Let \( e \in I(X) \) be an idempotent. Then there exists \( M \in I(X) \) such that \( M^{-1} e M \) is diagonal.

**Proof.** The proof is similar to the proof of Lemma 1 of \([17]\).

We assume that \( I(X) \) acts on the left on the \( n \times 1 \) matrices. For each \( i = 1, 2, \ldots, n \), let

\[
e_i = \begin{pmatrix}
0 \\
0 \\
\vdots \\
1 \\
\vdots \\
0
\end{pmatrix},
\]

having non-zero entry only in the position \( (i, 1) \). One, and only one, between \( e \cdot e_i \) or \( e_i - e \cdot e_i \) will have non-zero entry at \( (i, 1) \), where \( e \cdot e_i \) is the usual matrix.
multiplication. Let $f_i$ be either $e \cdot e_i$, or $e_i - e \cdot e_i$, the unique matrix with the non-zero entry in $(i, 1)$.

Let $M$ be the matrix such that the columns are $f_1, f_2, \ldots, f_n$. By construction, $M \in I(X)$. Moreover, since the diagonal entries of $M$ are non-zero, it is invertible. Also, $f_1, f_2, \ldots, f_n$ are eigenvectors of $e$. Thus, $M^{-1}eM$ is diagonal. \hfill \square

A classical result says that a set of diagonalizable matrices, which pairwise commutes, is simultaneously diagonalizable. We have a similar result for $I(X)$:

**Lemma 8.** Let $e_1, \ldots, e_s \in I(X)$ be orthogonal idempotents. Then there exists $M \in I(X)$ such that $M^{-1}e_iM$ is diagonal, for each $i = 1, 2, \ldots, s$.

**Proof.** The proof will be by induction on $s$. If $s = 1$, then the result follows from the previous lemma. So, assume $s > 1$, and $f_1, \ldots, f_n$ is a basis such that $e_1, \ldots, e_{s-1}$ are diagonal, and the matrix $M$ having $f_1, \ldots, f_n$ as its columns lies in $I(X)$.

Fix $j \in \{1, \ldots, n\}$. If $e_jf_j = 0$, then $f_j$ is an eigenvector of $e_s$ with eigenvalue 0, and there is nothing to do. If there is $i < s$ such that $e_i f_j = f_j$, then $e_s f_j = e_s e_i f_j = 0$, hence we are in the previous case. So, assume that $e_j f_j \neq 0$ and $e_i f_j = 0$, for all $i < s$. As in the previous lemma, we can change $f_j$ to $e_i f_j$ or $f_j - e_i f_j$, in such a way that this new $f_j$ still form a basis $\{f_1, \ldots, f_n\}$ of $\mathbb{F}^n$. By construction, $e_i f_j = 0$, for this new $f_j$, and $e_s f_j$ is either 0 or $f_j$. Continuing the process, we obtain $f_1, \ldots, f_s$ such that the matrix $M$, having these elements as its columns, is in $I(X)$, and every $e_i$ is diagonal in this basis. \hfill \square

**Lemma 9.** Let $x \in UT_n$ be such that $x = e + b$, where $e^2 = e \neq 0$ is a diagonal matrix, and $b$ is a strict upper triangular matrix. Then there exists a non-zero idempotent in $\text{Span}\{x, x^2, \ldots\}$. In other words, there exists a non-zero polynomial $p(\lambda) \in \mathbb{F}[\lambda]$, such that $p(0) = 0$ and $p(x)$ is a non-zero idempotent.

**Proof.** Let $\overline{\mathbb{F}}$ an algebraic closure of $\mathbb{F}$. By Jordan canonical form, there exists an invertible matrix $m$ such that

$$m x m^{-1} = \begin{pmatrix} 1 & * & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & * \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}.$$ 

With an adequate $s > 1$, we obtain

$$m x^s m^{-1} = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix},$$

where $J$ is a triangular matrix, with 1’s in the diagonal. Now, there exists a polynomial $p(\lambda)$ such that $p(J) = 1$, the identity matrix. Indeed, let $q(\lambda) = \sum_{i \geq 0} a_i \lambda^i$ be the characteristic polynomial of $J$. By the Cayley-Hamilton Theorem, $q(J) = 0$. Moreover, since $q(0) = a_0 = \det J = 1$, we obtain that $p(\lambda) = \sum_{i > 0} -a_i \lambda^i$ satisfies $p(J) = 1$.

Let $e' = p(x^s)$, then $m e' m^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, hence $e'^2 = e'$, as desired. \hfill \square
Fix any $G$-grading on $I(X)$. Define
\[ \mathcal{E}(X) = \{ e \in I(X) \text{ non-zero homogeneous idempotent} \}. \]

Of course it is non-empty since $1 \in \mathcal{E}(X)$. For $e_1, e_2 \in \mathcal{E}(X)$, let $e_1 \preceq e_2$ denote (the non-necessarily partial order) $e_1 I(X) \subseteq e_2 I(X)$ (usually, $\preceq$ is called a preordering). Let $e \in \mathcal{E}(X)$ be a minimal element according to $\preceq$ (since $I(X)$ has finite dimension, we can always find such minimal $e$). Note that $\deg e = 1$, since $e^2 = e$. By Lemma 7 we can assume $e$ diagonal, up to a graded isomorphism. The algebra $B = e I(X)e$ has a natural $G$-grading, induced from the grading of $I(X)$. We will prove that $B$ is a (commutative) graded division algebra.

**Lemma 10.** For every homogeneous $x \in B$, $x \notin J(B)$, $\deg x$ has finite order.

**Proof.** Note that, if $x \notin J(B)$, then $x$ has at least one non-zero entry in the diagonal. So, $x^m \notin J(B)$ for all $m \in \mathbb{N}$. In particular, $x, x^2, x^3, \ldots$ are non-zero elements. If $\deg x$ has infinite order, then the elements $x, x^2, \ldots$ have all different degree, thus they are linearly independent. However, $B$ has finite dimension, so we obtain a contradiction. As a conclusion, $\deg x$ has finite order. \( \square \)

Recall, from Lemma 8 that $J(B)$ is graded. Thus $B/J(B)$ has a natural $G$-grading, induced from the grading on $B$.

**Lemma 11.**

(i) The algebra $B/J(B)$ is a commutative graded division algebra.

(ii) Write $B/J(B) = \bigoplus_{g \in G} B_g$. Then $\dim B_g \leq 1$, for all $g \in G$.

**Proof.** If $\dim \bar{B}_1$ is not 1, then we claim that there would exists an idempotent $e' \in \bar{B}_1$ such that $e' \preceq e$. Indeed, let $\bar{x} \in \bar{B}_1$ be a non-zero homogeneous element, not a scalar multiple of $e$, modulo $J(B)$. Then there exists a homogeneous $x \in B$ of degree 1, having non-zero entries in the diagonal. We can write $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_s e_s + b$, where $b \in J(B)$, $e_1, e_2, \ldots, e_s$ are diagonals orthogonal idempotents (sum of diagonal matrix units), and $\alpha_1, \ldots, \alpha_s \in \mathbb{F}$ are non-zero elements such that $\alpha_i \neq \alpha_j$ for all $i \neq j$. Since $x$ has non-zero entry in the diagonal, $s > 0$; and since $x$ and $e$ are linearly independent modulo $J(B)$, we have necessarily $s > 1$. The powers $x, x^2, \ldots, x^s$ are homogeneous elements of degree 1. By Vandermonde argument, we conclude that $y = e_1 + b'$ is a linear combination of $x, x^2, \ldots, x^s$, for some $b' \in J(B)$, hence $y$ is homogeneous of degree 1. Using Lemma 9 we obtain a homogeneous idempotent $e'$, which is a linear combination of $y, y^2, \ldots$. Thus $e' \in B$. Moreover, we have $e'I(X) \subsetneq eI(X)$. But $e$ is minimal, so we obtain a contradiction. As a conclusion, $\dim \bar{B}_1 = 1$.

Now, let $\bar{x} \in B/J(B)$ be non-zero and homogeneous. Thus, $\bar{x}$ has non-zero entry in the diagonal. Hence, $\bar{x}^m \neq 0$, for all $m \in \mathbb{N}$. By Lemma 10, $\bar{x}^m \in \bar{B}_1$, for some $m \in \mathbb{N}$. This means $\bar{x}^m = \lambda \bar{e}$, for some $\lambda \neq 0$. As a consequence, $\bar{x}$ is invertible. Thus, $B/J(B)$ is a graded division algebra. Moreover, since $B/J(B)$ is a sum of copies of $\mathbb{F}$, as an ordinary algebra, then it is necessarily commutative, proving (i).

Since $\dim \bar{B}_1 = 1$ and $B/J(B)$ is a graded division, then $\dim \bar{B}_g \leq 1$, for all $g \in G$, thus proving (ii). \( \square \)

**Corollary 12.** The algebra $B$ is semisimple as an ordinary algebra, that is $J(B) = 0$. In particular, $B$ is a commutative graded division algebra with unit $e$.  

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Proof. Let $J = J(B)$ and $\bar{B} = B/J$. Note that $J/J^2$ is a graded $\bar{B}$-module. The right annihilator $\text{Ann}^\bar{B}_J(J/J^2)$ is a homogeneous subspace of $\bar{B}$. However, no nonzero homogeneous element of $\bar{B}$ annihilates a nonzero homogeneous element of $J/J^2$, since $\bar{B}$ is a graded division algebra, by previous lemma. Hence, $J(B) = 0$.

Since, by previous lemma, $B/J(B)$ is a commutative graded division algebra, then $B$ is a commutative graded division algebra as well. \qed

From now on, $e$ will always denote a minimal idempotent as before. Denote by $\mathcal{D}(e) = \{i \in X \mid e(i, i) \neq 0\}$.

**Corollary 13.** The elements of $\mathcal{D}(e)$ are not pairwise comparable.

**Proof.** If not, suppose $i < j$, with $i, j \in \mathcal{D}(e)$. Recall that $B = eI(X)e \cong I(Y)$, for some poset $Y$ (where $Y = \mathcal{D}(e)$). Then $e_{ij} \in B$. Moreover, one has $e_{ij} \in J(B)$, so $J(B) \neq 0$, contrary to Corollary 12. \qed

Now, if $e \neq 1$, consider the set

$$E_{1}(X) = \{e' \in I(X) \text{ idempotent} \mid e' I(X) \subseteq (1 - e)I(X)\}.$$ 

We remark that $E_{1}(X) \neq \emptyset$, since $1 - e \in E_{1}(X)$. We claim that we can choose the idempotents belonging to $E_{1}(X)$ so that they commute with $e$. Indeed, if $e' \in E_{1}(X)$, then $ee' = ee'^2 \in e(1 - e)I(X) = 0$. Also, let $f_{1} = e' - e'e$. Then $f_{1}^2 = f_{1}$, $f_{1}e = f_{1} = 0$ and $f_{1}I(X) = e'I(X)$. Thus, $f_{1} \in E_{1}(X)$ is the required element. Let $e_{1} = e$, and chose a homogeneous idempotent $e_{2} \in E_{1}(X)$ which is minimal with respect to $\preceq$, and orthogonal to $e_{1}$.

Proceeding by induction, we assume that we have a set $e_{1}, \ldots, e_{r}$ of minimal homogeneous orthogonal idempotents. Then we define

$$E_{r}(X) = \{e' \in I(X) \text{ idempotent} \mid e' I(X) \subseteq (1 - e_{1} \cdots - e_{r})I(X)\}.$$ 

As before, we can find a minimal homogeneous idempotent $e_{r+1} \in E_{r}(X)$ which is orthogonal to $e_{1}, \ldots, e_{r}$. We can continue the process, and, since dim $I(X)$ is finite, it ends within finitely many steps. Let $e_{1}, e_{2}, \ldots, e_{t}$ be the homogeneous idempotents, which are minimal with respect to $\preceq$, obtained by the previous construction. By Lemma 8 we can assume $e_{1}, e_{2}, \ldots, e_{t}$ diagonal.

**Definition 14.** The diagonal homogeneous idempotents $e_{1}, \ldots, e_{t}$ will be called the minimal idempotents. We denote $\mathcal{E} = \{e_{1}, \ldots, e_{t}\}$.

We denote $D_{i} = e_{i}I(X)e_{i}$ which is, according to Corollary 12, a commutative graded division algebra. Moreover, $M_{ij} = e_{i}I(X)e_{j}$ is a graded $(D_{i}, D_{j})$-bimodule. The triple $(D_{i}, D_{j}, M_{ij})$ constuits a triangular algebra.

If $e_{i}$ and $e_{j}$ are two minimal idempotents, we denote $e_{i} \preceq e_{j}$ if there exist $x \in \mathcal{D}(e_{i})$ and $y \in \mathcal{D}(e_{j})$ such that $x \preceq y$.

**Lemma 15.** The relation $\preceq$ is a partial order in $\mathcal{E}$.

Before we prove Lemma 15 we need to investigate the relation between each $M_{ij}$ and the poset.

Recall that the structure of graded modules over graded division algebras is similar to the structure of vector spaces over division rings (see, for instance, [6, page 29]). More precisely, let $D$ be a $G$-graded division algebra and let $M$ be a
graded right $D$-module. Then $M \cong \bigoplus_{h \in G} M_h$, where each $M_h = [h \cdot]D$, for some $h \in G$. Here, if $D = \bigoplus_{g \in G} D_g$, then

$$[h \cdot]D = \bigoplus_{g \in G} [h \cdot]D_g,$$

where $[h \cdot]D_g = D_{h^{-1}g}$.

In particular, every graded right $D$-module is free. A similar discussion is valid for graded left $D$-modules.

Let $e_i$ and $e_j$ be minimal idempotents. For each $x \in \mathcal{D}(e_i)$, we define the link of $x$ to $e_j$ as the non-negative integer $\ell(x, e_j) = |\{y \in \mathcal{D}(e_j) \mid x \leq y\}|$. Similarly, we define the link of $e_i$ to $y$, denoted $\ell(e_i, y)$. The link of $e_i$ to $e_j$ is $\ell(e_i, e_j) = |\{(x, y) \mid x \leq y, x \in \mathcal{D}(e_i), y \in \mathcal{D}(e_j)\}|$.

**Lemma 16.** Let $e_i, e_j$ be minimal idempotents. Then, for any $x \in \mathcal{D}(e_i)$, $y \in \mathcal{D}(e_j)$,

$$\ell(e_i, e_j) = |\mathcal{D}(e_i)|\ell(x, e_j) = \ell(e_i, y)|\mathcal{D}(e_j)|.$$

**Proof.** Let $M_{ij} = e_i \{I(X)\} e_j$, which is a free $G$-graded left $D_i$-module. We shall prove that $\ell(x, e_j) = \dim \mathcal{D} M_{ij}$, for any $x \in \mathcal{D}(e_i)$.

On one hand, as $\mathbb{F}$-linear space, we have

$$M_{ij} = \bigoplus_{(x, y) \in \mathcal{D}(e_i) \times \mathcal{D}(e_j); x \leq y} \mathbb{F} e_{xy}.$$

Thus, $\ell(e_i, e_j) = \dim_{\mathbb{F}} M_{ij} = \dim_{\mathbb{F}} D_i \dim_{D_i} M_{ij}$. On the other hand, let $v_1, \ldots, v_\ell$ be a $D_i$-basis of $M_{ij}$, where $\ell > 0$ (the case $\ell = 0$ is trivial).

**Claim.** For $k \in \{1, \ldots, \ell\}$, let $w \in D_i v_k$, and assume $w(x, y) = 0$, but $v_k(x, y) \neq 0$. Then $w(x, y) = 0$, for all $y' \in \mathcal{D}(e_j)$.

Indeed, in this case, $B = \{e_{xx} v_k \mid x \in \mathcal{D}(e_i)\}$ is a set of $|\mathcal{D}(e_i)|\mathbb{F}$-linearly independent elements. Since $\dim_{\mathbb{F}} D_i v_k = \dim_{\mathbb{F}} D_i = |\mathcal{D}(e_i)|$, $B$ is a (non-homogeneous) $\mathbb{F}$-vector space basis of $D_i v_k$. So, write $w = \sum_{x \in \mathcal{D}(e_i)} \lambda_x e_{xx} v_k$. If $v_k(x, y) \neq 0$, but $w(x, y) = 0$, then necessarily $\lambda_x = 0$. This implies $w(x, y) = 0$, for all $y' \in \mathcal{D}(e_j)$.

As a conclusion, since $v_1, \ldots, v_\ell$ generate $M_{ij}$ as a $D_i$-linear space, every element of $M_{ij}$ is a linear combination of elements in $D_i v_1, \ldots, D_i v_\ell$. By the claim, in order to be possible to fill all the entries of the coordinates of type $(x, \cdot)$ in $M_{ij}$, it is necessary that $\ell \geq \ell(x, e_j)$. Hence, $\ell \geq \max\{\ell(x, e_j) \mid x \in \mathcal{D}(e_i)\}$. And, therefore,

$$\dim_{\mathbb{F}} D_i \dim_{D_i} M_{ij} = \ell(e_i, e_j) = \sum_{x \in \mathcal{D}(e_i)} \ell(x, e_j) \leq |\mathcal{D}(e_i)|\ell = \dim_{\mathbb{F}} D_i \dim_{D_i} M_{ij}$$

which implies $\ell(x, e_j) = \dim_{D_i} M_{ij}$, for any $x \in \mathcal{D}(e_i)$ and we are done. 

Now, we can prove the lemma:

**Proof of Lemma 15.** It is clear that $\preceq$ is reflexive and transitive. So, let $e_i \neq e_j$ (thus, by construction, $\mathcal{D}(e_i) \cap \mathcal{D}(e_j) = \emptyset$), and assume that $e_i \preceq e_j$, and $e_j \preceq e_i$.

Then, by Lemma 16, every $x \in \mathcal{D}(e_i)$ is $\preceq$ to at least one $y \in \mathcal{D}(e_j)$. In the same way, every $y \in \mathcal{D}(e_j)$ is $\preceq$ to at least one $x \in \mathcal{D}(e_i)$. Hence, we can find

$x_1, x_2 \in \mathcal{D}(e_i)$, and $y \in \mathcal{D}(e_j)$ such that $x_1 < y < x_2$, so $x_1 < x_2$. This contradicts Corollary 13. Hence, $\preceq$ is antisymmetric. 

As a consequence of Lemma 15, we can rename the elements of $X$, in such a way that $\mathcal{D}(e_1) = \{1, 2, \ldots, i_1\}$, $\mathcal{D}(e_2) = \{i_1 + 1, i_1 + 2, \ldots, i_2\}$, ..., $\mathcal{D}(e_t) = \{i_{t-1} + 1, \ldots, n\}$. Since $e_i I(X) e_j = \text{Span}\{e_{k\ell} \mid k \in \mathcal{D}(e_i), \ell \in \mathcal{D}(e_j), k < \ell\}$, we
have \( \dim e_i I(X) e_j \leq \dim D_i \dim D_j \). Moreover, as graded vector space, \( I(X) = \sum_{i,j} e_i I(X) e_j \). In this way, we obtain the proof of the Theorem \[ \square \]

**Corollary 17.** Up to a graded isomorphism, there exists commutative graded division algebras \( D_1, D_2, \ldots, D_t \), all of them having all homogeneous subspaces with dimension at most 1, such that

\[
I(X) \cong \begin{pmatrix}
D_1 & M_{1,2} & \cdots & M_{1,t} \\
D_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots \\
D_t & \cdots & M_{t-1,t}
\end{pmatrix},
\]

where each \( M_{i,j} \) is a graded \((D_i, D_j)\)-bimodule of dimension at most \( \dim_F D_i \dim_F D_j \). \[ \square \]

If \( D \) is a commutative graded division algebra, where each homogeneous component has dimension at most 1 and \( D \cong F^{\dim D} \), then \( D \cong F H \), where \( H = \text{Supp} \, D \).

Indeed, a set of nonzero homogeneous elements \( \{ X_u \mid u \in H \} \), where \( \deg X_u = u \) is a vector space basis of \( D \). Also, \( X_u X_v = \sigma(u, v) X_{uv} \), for some (nonzero) \( \sigma(u, v) \in F \).

Since \( D \) is associative, we obtain that \( \sigma \) is a 2-cocycle. Thus, \( \sigma \in Z^2(H, F^\times) \), and \( D \cong F^\sigma H \) (the twisted group algebra). To complete the proof, we need the following result, communicated by M. Kochetov:

**Proposition 18.** Let \( H \) be a finite abelian group and \( \sigma \in Z^2(H, F^\times) \). If \( F^\sigma H \cong F^{\text{Supp} \, H} \), then \( FH \cong F^{\text{Supp} \, H} \).

**Proof.** Given a finite-dimensional unital commutative algebra \( A \) over \( F \), let \( N(A) \) denote the number of algebra homomorphisms from \( A \) to \( F \).

**Claim.** \( A \) is a direct sum of copies of \( F \) if and only if \( N(A) = \dim A \).

Indeed, \((\Rightarrow)\) is obvious, and \((\Leftarrow)\) follows from linear independence of distinct homomorphisms.

Note that both \( N \) and \( \dim \) are multiplicative over tensor products of algebras.

Now, write \( H = (h_1) \times \cdots \times (h_r) \), direct product of cyclic groups, where each \( h_i \) has order \( m_i \). Then, we notice that

\[
F^\sigma H \cong F[x_1]/(x_1^{m_1} - a_1) \otimes \cdots \otimes F[x_r]/(x_r^{m_r} - a_r),
\]

where \( a_i = \prod_{k=1}^{m_i-1} \sigma(h_i, h_i^k) \in F \). In the same way, \( FH \cong \bigotimes_{i=1}^r F[x_i]/(x_i^{m_i} - 1) \).

From the remarks above, \( F^\sigma H \) is a direct sum of copies of \( F \) if and only if each \( F[x_i]/(x_i^{m_i} - a_i) \) is a direct sum of copies of \( F \). The former condition is equivalent to \( F \) contains \( m_i \) distinct \( m_i \)-roots of \( a_i \); and this implies that \( F \) contains a primitive \( m_i \)-th root of 1. Thus, each \( F[x_i]/(x_i^{m_i} - 1) \) is a direct sum of \( F \)'s. Hence, by the remarks above, \( FH \) is a direct sum of copies of \( F \).

As a consequence, \( F^\sigma H \cong FH \) as ungraded algebras. From \[11\] Corollary 1.2], we obtain that \( \alpha \) is a coboundary map. Thus, \( F^\sigma H \cong FH \) as graded algebras.

**Remark 19.** Consider the notation of Corollary \[17\] Let \( e_i \in D_i \) be the unity. Set \( \mathcal{E} = \{ 1, 2, \ldots, t \} \). We define the partial order \( \trianglelefteq \) on \( \mathcal{E} \) by the following (equivalent) conditions (see Lemma \[15\]):

1. \( i \leq j \),
2. \( e_i \trianglelefteq e_j \),
3. there exists \( x \in D(e_i) \) and \( y \in D(e_j) \) such that \( i \leq j \),
(4) $M_{ij} \neq 0$.

Note that we can identify the present definition of $E$ with that given in Definition [14]. We call $E$ the associated poset of the graded algebra $I(X)$.

Note that we have the following equality:

$$|H_i| = \dim D_i = |\mathcal{D}(e_i)|.$$

4. ON THE STRUCTURE OF GRADED BI-VESOR SPACES

In this section, we shall investigate graded bimodules. We are interested in the special case where $G$ is an abelian group, $H_1, H_2 \subseteq G$ are finite subgroups, and $V$ is a $G$-graded $(\mathbb{F}H_1, \mathbb{F}H_2)$-bimodule. We will conclude that, in this situation, $V$ is a $\mathbb{F}H$-vector space, where $H = H_1H_2$. Hence, the bimodules have a nice description in this case.

The notion of tensor product will play a fundamental role in this section. Recall that, if $R$ is any $\mathbb{F}$-algebra, $M, N$ are $\mathbb{F}$-vector spaces, $M$ is a right $R$-module, and $N$ is a left $R$-module, then $M \otimes_R N$ is a uniquely defined $\mathbb{F}$-vector space, satisfying a universal property. It is spanned by $m \otimes n$, where $m \in M$, and $n \in N$, and they satisfy $mr \otimes n = m \otimes rn$, $r \in R$. For the special case where $M$ and $N$ are algebras, then $M \otimes_R N$ is an algebra as well.

We are interested in the following special case: let $R, A_1, A_2$ be $\mathbb{F}$-algebras, and assume $\varphi_i : R \to A_i$ homomorphism of algebras, for $i = 1, 2$. Then $A_1$ becomes a right $R$-module by $a \cdot r = a\varphi_1(r)$, and, similarly, $A_2$ becomes a left $R$-module. So we can construct the tensor product $A_1 \otimes_R A_2$. Since $\varphi_1$ and $\varphi_2$ will play a somewhat significant role, we use the notation $(A_1 \otimes_R A_2, \varphi_1, \varphi_2)$. In this case, $A_1 \otimes_R A_2$ is spanned by $a_1 \otimes a_2$, for $a_i \in \beta_i$, where $\beta_i$ is any basis of $A_i$; also, we have $b_1 \varphi_1(r) \otimes b_2 = b_1 \otimes \varphi_2(r)b_2$, for any $r \in R, b_1 \in A_1, b_2 \in A_2$.

Now, assume further that $R, A_1, A_2$ are $G$-graded algebras, and $\varphi_1, \varphi_2$ are $G$-graded homomorphism of algebras. So $A_1$ and $A_2$ are $G$-graded $R$-modules. The algebra $A_1 \otimes_R A_2$ admits a natural structure of $G$-grading, where the homogeneous component of degree $g \in G$ is spanned by $a_1 \otimes a_2$, $a_i \in A_i$ homogeneous satisfying $\deg a_1 \deg a_2 = g$. Moreover, $A_1 \otimes_R A_2$ becomes a $G$-graded algebra. In what follows, we will see that, in the case of group algebras, the just defined grading has a nicer way to be constructed.

Lemma 20. Let $G$ be an abelian group, and $H_1, H_2 \subseteq G$ be finite subgroups. Let $H = H_1H_2$, and $\mathcal{H} = H_1 \cap H_2$. Let $i_j : \mathbb{F}H \to \mathbb{F}H_j$ be $G$-graded monomorphisms, for $j = 1, 2$. Then, as graded algebras,

$$(\mathbb{F}H_1 \otimes_{\mathbb{F}H} \mathbb{F}H_2, i_1, i_2) \cong \mathbb{F}H.$$

Proof. First we note that $\mathbb{F}H_1 \otimes_{\mathbb{F}H} \mathbb{F}H_2$ has a natural well-defined $G$-grading, given by $\deg h_1 \otimes h_2 = h_1h_2$, where $h_1 \in H_1, h_2 \in H_2$.

Now, since every homogeneous component of $\mathbb{F}H_j$ has dimension at most 1, we see that an inclusion $i_j(h) = \chi_j(h)h$, for each $h \in \mathcal{H}$, for some $\chi_j(h) \in \mathbb{F}$. Moreover, $h \mapsto \chi_j(h) \in \mathbb{F}$ is a character of $H$. Consider the character given by $\chi_j^{-1}$. It is known that every character of $\mathcal{H}$ can be extended to a character of $H_j$. So, let $\bar{\chi}_j$ be an extension of $\chi_j^{-1}$.
Let \( \varphi : F H_1 \otimes F H_2 \to F H \) be defined by \( \varphi(h_1 \otimes h_2) = \bar{\chi}_1 (h_1) \bar{\chi}_2(h_2) h_1 h_2 \). If \( h \in H \), then
\[
\varphi(h_{11}(h) \otimes h_2) = \chi_1(h) \bar{\chi}_1(h_1) \bar{\chi}_2(h_2) h_1 h_2 h_2 = \chi_1(h) \bar{\chi}_1(h_1) \bar{\chi}_2(h_2) h_1 h_2 h_2,
\]
\[
\varphi(h_1 \otimes i_2(h_2)) = \chi_2(h) \bar{\chi}_1(h_1) \bar{\chi}_2(h_2) h_1 h_2 h_2 = \chi_2(h) \bar{\chi}_2(h_1) \bar{\chi}_2(h_2) h_1 h_2 h_2.
\]

Since \( \chi_1(h) \bar{\chi}_1(h) = \chi_2(h) \bar{\chi}_2(h) = 1 \), we obtain that \( \varphi \) is well-defined.

Since \( G \) is abelian, \( \varphi \) is an onto homomorphism of graded algebras.

Let \( \sum_{i=1}^m \lambda_i g_i \otimes h_i \in \text{Ker} \varphi \). As \( \text{Ker} \varphi \) is a graded subspace, we can suppose that \( g_1 h_1 = \cdots = g_m h_m \). Moreover, we can assume that \( h_1, \ldots, h_m \) are \( \mathbb{F} H \)-linearly independent. Thus, \( g_i h_i = g_j h_j \) implies \( h_i h_j^{-1} \in H \), which is a contradiction if \( m > 1 \). Hence, \( m \leq 1 \), and this is sufficient to conclude that \( \text{Ker} \varphi = 0 \). \( \square \)

**Lemma 21.** Let \( G \) be an abelian group, \( H_1, H_2 \subseteq G \) finite subgroups, \( \mathcal{H} = H_1 \cap H_2 \). Let \( V \) be a finite-dimensional \( G \)-graded \( (F H_1, F H_2) \)-bimodule. Then, there exist \( m_1, \ldots, m_s \in V \) homogeneous, and \( \chi_1, \ldots, \chi_s \in \mathcal{H} \), such that
\[
V = F H_1 m_1 F H_2 \oplus \cdots \oplus F H_1 m_s F H_2,
\]
and \( h m_i = \chi_i(h) m_i h \), for all \( i = 1, \ldots, s \), \( h \in \mathcal{H} \).

**Proof.** Let \( W_g \) be the homogeneous component of degree \( g \) of \( V \). Given \( h \in \mathcal{H} \), let \( L_h, R_h : V \to V \) denote the maps \( L_h(m) = h m \), \( R_h(m) = m h \). Then \( L_h \) is a linear isomorphism from \( W_g \to W_{h g} \). Also, \( R_h^{-1} : W_{h g} \to W_g \) is an isomorphism as well. Thus, \( L_h R_h^{-1} \in \text{GL}(W_g) \). It is elementary to see that this defines a linear representation \( h \in \mathcal{H} \to L_h R_h^{-1} \in \text{GL}(W_g) \).

Since \( \mathcal{H} \) is abelian, the representation is completely reducible, and each irreducible representation has degree 1. So, we can find a vector space basis \( v_1, \ldots, v_r \in W_g \), and characters \( \chi_1, \ldots, \chi_r \in \mathcal{H} \) such that \( L_h R_h^{-1}(v_i) = \chi_i(h) v_i \), for all \( i = 1, \ldots, r \). In particular, this gives \( h v_i = \chi_i(h) v_i h \), for all \( h \in \mathcal{H} \).

It is clear that every component \( W_{g'} \), where \( g' \in H_1 g H_2 \) belongs to \( F H_1 v_{i} F H_2 \oplus \cdots \oplus F H_1 v_{r} F H_2 \). Thus, we can take \( g_2 \not\in H_1 g H_2 \) and repeat the process. Since \( \dim V < \infty \), the process ends. \( \square \)

In the notation of the previous lemma, given \( g_1 \in H_1 \) and \( g_2 \in H_2 \), we have
\[
g_1 h m g_2 = \chi_i(h) g_1 m g_2, \forall h \in \mathcal{H}.
\]

**Lemma 22.** Assuming the same hypotheses of the previous lemma, let \( \chi \in \mathcal{H} \), and \( V = F H_1 m F H_2 \), where \( h m = \chi(h) m h \), for all \( h \in \mathcal{H} \). Then \( V \) is a graded \( F H_1 \otimes F H_2 \)-module.

**Proof.** Consider the inclusions
\[
\iota_1 : h \in F H \to h \in F H_1,
\]
\[
\iota_2 : h \in F H \to \chi(h) h \in F H_2.
\]

Define the left action of \( F H_1 \otimes F H_2 \) on \( V \) by \( g_1 \otimes g_2 m = g_1 m g_2 \). It suffices to prove that this action is well-defined. For, assume \( g_1 \in H_1, g_2 \in H_2, h \in \mathcal{H} \). Then \( g_1 h \otimes g_2 = \chi(h) g_1 \otimes h g_2 \), and
\[
(g_1 h \otimes g_2)m = g_1 h m g_2 = \chi(h) g_1 m h g_2 = (g_1 \otimes \chi(h) g_2)m.
\]
\( \square \)
Recall that, if $R$ and $S$ are any rings, $\varphi : R \to S$ is a ring homomorphism, and $M$ is an $S$-module; then $M$ becomes an $R$-module if we set $m \cdot r = m\varphi(r)$, for $m \in M$ and $r \in R$. We summarize the results of this section.

**Corollary 23.** Let $G$ be an abelian group, $H_1, H_2 \subseteq G$ finite subgroups, $\mathcal{H} = H_1 \cap H_2$, $H = H_1 H_2$. Let $V$ be a finite-dimensional $G$-graded $(\mathbb{F}H_1, \mathbb{F}H_2)$-bimodule. Then, there exist homogeneous $m_1, \ldots, m_s \in V$ and characters $\chi_1, \ldots, \chi_s \in \mathcal{H}$ such that

$$V = \mathbb{F}H_1 m_1 \mathbb{F}H_2 \oplus \cdots \oplus \mathbb{F}H_1 m_s \mathbb{F}H_2,$$

where $hm_\ell = \chi_\ell(h)m_\ell h$, for each $h \in \mathcal{H}$. Moreover, for each $i$, one has $\mathbb{F}H_1 m_i \mathbb{F}H_2 \cong (\mathbb{F}H)_{[\deg m_i]}$, as graded vector spaces.

**Proof.** Lemma 21 says that we can find homogeneous $m_1, \ldots, m_s \in V$ such that

$$V = \mathbb{F}H_1 m_1 \mathbb{F}H_2 \oplus \cdots \oplus \mathbb{F}H_1 m_s \mathbb{F}H_2.$$

By Lemma 22 every $\mathbb{F}H_1 m_i \mathbb{F}H_2$ has a structure of graded $(\mathbb{F}H_1 \otimes_{\mathbb{F}H} \mathbb{F}H_2, \iota_{11}, \iota_{12})$-module. By Lemma 20 there exists a graded isomorphism $\varphi_1 : \mathbb{F}H \to (\mathbb{F}H_1 \otimes_{\mathbb{F}H} \mathbb{F}H_2, \iota_{11}, \iota_{12})$. Hence, $\mathbb{F}H_1 m_i \mathbb{F}H_2$ has a structure of graded $\mathbb{F}H$-module, and it is 1-dimensional, since it is $\mathbb{F}H$-spanned by $m_i$. This concludes the proof. $\square$

**Remark 24.** It should be noted that Corollary 23 is no longer valid when the grading group is not abelian. Indeed, let $H$ be a finite abelian group, $C_2 = \{1, w\}$ the cyclic group of order 2, and let $G = H \ast C_2$ be the free product of $H$ and $C_2$. Consider $V = \mathbb{F}H \otimes \mathbb{F}H$, and define the $G$-grading on $V$ by $V = \bigoplus_{g \in G} V_g$, where

$$V_{h_1 \cdot w \cdot h_2} = \text{Span}\{h_1 \otimes h_2\}, \quad h_1, h_2 \in H.$$

Then $V$ is a $G$-graded vector space, and $V$ has a structure of $G$-graded $(\mathbb{F}H, \mathbb{F}H)$-bimodule. For every nonzero homogeneous $v \in V$, we have $\mathbb{F}H \circ \mathbb{F}H = V$; and it is not isomorphic to a shift of $\mathbb{F}H$. Moreover, $V$ is not a $G$-graded left $\mathbb{F}H \otimes_{\mathbb{F}H} \mathbb{F}H$-module.

Now, we are able to prove Theorem 2.

**Proof of Theorem 2.** Fix $i, j$. We consider the triangular algebra

$$A = \left( \begin{array}{cc} \mathbb{F}m_i & V_{ij} \\ \mathbb{F}m_j & 0 \end{array} \right),$$

where $m_\ell = |H_\ell|$, which is graded isomorphic to $\left( \begin{array}{cc} \mathbb{F}H_i & M_{ij} \\ \mathbb{F}H_j & 0 \end{array} \right)$. Let $e_\ell \in H_\ell$ be the unit. Then $A = (e_j + e_j)I(X)(e_i + e_j)$, and it is the incidence algebra of the subposet $Y = \mathcal{D}(e_i) \cup \mathcal{D}(e_j) \subseteq X$.

Let $\mathcal{H} = H_i \cap H_j$, $\hat{\mathcal{H}} = \{\chi_1, \ldots, \chi_m\}$, and assume $\hat{\mathcal{H}} = \hat{H}_i \cap \hat{H}_j$. Write

$$\hat{H}_\ell = \chi_{i}^{(\ell)} \hat{\mathcal{H}} \cup \cdots \cup \chi_{m}^{(\ell)} \hat{\mathcal{H}},$$

where $m_\ell = m n_\ell$, for $\ell \in \{i, j\}$ (we can assume $\chi_1^{(i)} = 1$).

By Lemma 6 up to renaming the entries, the graded isomorphism $\phi : \mathbb{F}H_i \oplus M_{ij} \oplus \mathbb{F}H_j \to A$ is such that, for $h_1 \in H_i$, $h_2 \in H_j$,

$$\phi(h_1 + h_2) = \left( \begin{array}{cc} (\chi_1^{(i)} \chi_{1}^{(j)}(h_1), \ldots, \chi_{m}^{(i)} \chi_{m}^{(j)}(h_1)) \\ 0 \\ (\chi_1^{(j)} \chi_{1}(h_2), \ldots, \chi_{m}^{(j)} \chi_{m}(h_2)) \end{array} \right).$$
Let $M_{ij} = \mathcal{F}H_{m_1} \mathcal{F}H_j \oplus \cdots \oplus \mathcal{F}H_{m_s} \mathcal{F}H_j$, with characters $\mu_1, \ldots, \mu_s$, as in [2]. We only need to prove that $\mu_1, \ldots, \mu_s$ are pairwise distinct characters. By Corollary 23, $\dim \mathcal{F}H_{m_i} \mathcal{F}H_j = |H_i H_j|$, so $\phi(m_i) \in V_{ij}$ has at least $|H_i H_j|$ non-zero entries.

To make notation easier, we will identify the elements of $\mathcal{P}(e_i) \cup \mathcal{P}(e_j)$ with the respective character $\chi_r^{(i)} \chi_s$, as in [3]. So, for instance, to denote the entry $(1, 1)$ of $\phi(h_1 + h_2)$ in [4], we will write $\phi(h_1 + h_2)(\chi_1^{(i)} \chi_1^{(i)})$. Let $\chi_r^{(i)} \chi_s, \chi_r^{(i)} \chi_s \in \hat{H}$ such that $\phi(m_i)(\chi_r^{(i)} \chi_s) \neq 0, r = 1, 2, \ldots, n_j$.

Indeed, let $\mu \in \hat{H}_i$ and let $\mu' \in \hat{H}_j$ be such that $\phi(m_i)(\mu, \mu') \neq 0$. Now, if $\mu'' \in \hat{H}_j$ is such that $\phi(m_i)(\mu, \mu'') \neq 0$, then $\mu'$ and $\mu''$ cannot be in the same left coset, unless they coincide, as we will prove now. For each $h \in \hat{H}$, $hm_i h^{-1} = \mu_\ell(h)m_i$. On the other hand, $h\phi(m_i)h^{-1}(\mu, \mu') = \mu_\ell(h)\phi(m_i)(\mu, \mu')h^{-1}$, and a similar relation holds for $\mu''$. This implies $\mu'|\hat{H}_i = (\mu_\ell^{-1})|\hat{H}_i = \mu''|\hat{H}_i$. Hence, if $\mu'$ and $\mu''$ are in the same coset of $\hat{H}_i$, then $\mu' = \mu''$. So, for each $\mu \in \hat{H}_i$, there exist at most $|H_j|/|\hat{H}_i|$ elements $\mu'$ such that $\phi(m_i)(\mu, \mu') \neq 0$. Thus, $\phi(m_i)$ has at most $|H_j|/|\hat{H}_i|$ non-zero entries.

This implies that $\phi(m_i)$ has exactly $|H_i H_j|$ non-zero entries. Moreover, the non-zero entries of the $\phi(m_1), \ldots, \phi(m_s)$ are disjoint.

Now, let $\ell_i, \ell_j$ with $\mu_{\ell_i} = \mu_{\ell_j}$, and fix $\chi' \in \hat{H} \subseteq \hat{H}_i$, and let $\chi_1', \chi_2' \in \hat{H} \subseteq \hat{H}_j$ be such that $\phi(m_{\ell_i})(\chi', \chi_1') \neq 0, k = 1, 2$. For each $h \in \hat{H}$, again we have $hm_{\ell_i} h^{-1} = \mu_{\ell_i}(h)m_{\ell_i}$. On the other hand, (recall that $\chi_1^{(i)} = 1, \chi_2^{(i)} = 1$)

$$(h\phi(m_{\ell_i})h^{-1})(\chi', \chi_1') = \chi'(h)\chi_1^{-1}(h)\phi(m_{\ell_i})(\chi', \chi_1').$$

Hence, $\chi_1' = \chi_2'$. Thus, this implies $m_{\ell_i} = m_{\ell_j}$.

We also have the following immediate consequence of Corollary 23.

**Corollary 25.** Let $G$ be abelian, and let $i < j$. Then:

(i) if $H_i \cap H_j = \{1\}$, then $M_{ij}$ is either 0 or $\mathcal{F}H_{ij}$, where $H_{ij} \cong H_i \times H_j$.

(ii) if $H_i \subseteq H_j$ (in particular, $H_i = H_j$), then $M_{ij}$ is a graded $\mathcal{F}H_j$-vector space, and $0 \leq \dim_{\mathcal{F}H_j} M_{ij} \leq |H_i|$.

Given a graded $(\mathcal{F}H_1, \mathcal{F}H_2)$-bimodule $M$ with decomposition given by [2], denote $[M] = [(\chi_1, h_1), \ldots, (\chi_s, h_s)]$, where $h_\ell = \deg m_\ell$. We finish this section showing that the knowledge of those characters determines the isomorphism classes of the bimodules, or certain triangular algebras.

**Lemma 26.** Let $V$ and $V'$ be $G$-graded $(\mathcal{F}H_1, \mathcal{F}H_2)$-bimodules, and denote $[V] = [(\chi_1, h_1'), \ldots, (\chi_s, h_s')], [V'] = [(\chi'_1, h'_1), \ldots, (\chi'_{s'}, h'_{s'})]$. Then, as $G$-graded bimodules, $V \cong V'$ if, and only if $s = s'$ and there exists $\sigma \in S_s$ such that $h_{\ell} = h'_{\sigma(\ell)} (\text{mod } H_1 H_2), \chi_{\ell} = \chi'_{\sigma(\ell)}, \forall \ell = 1, 2, \ldots, s$.

**Proof.** Write $V = \mathcal{F}H_1 m_1 \mathcal{F}H_2 \oplus \cdots \oplus \mathcal{F}H_1 m_s \mathcal{F}H_2, V' = \mathcal{F}H_1 m'_1 \mathcal{F}H_2 \oplus \cdots \oplus \mathcal{F}H_1 m'_{s'} \mathcal{F}H_2$, as in [2]. Let $\psi: V \to V'$ be a graded isomorphism of bimodules. Then $\psi$ is an isomorphism of $H_1 \cap H_2$ representations, where the action is given by $L_h \circ R_{h^{-1}}$. Let $V_\chi = \sum_{h \in \chi} \mathcal{F}H_1 h \mathcal{F}H_2$. Note that $V_\chi$ is the sum of all subrepresentations of $V$ isomorphic to $\chi$. So, $\psi(V_\chi) = \sum_{h \in \chi} \mathcal{F}H_1 m'_h \mathcal{F}H_2 =: V'_\chi$. 


Moreover, $\psi : V_\chi \to V'_{\chi'}$ is a graded isomorphism of $\mathbb{F}(H_1H_2)$-modules. This proves one implication.

The converse is immediate. \hfill $\Box$

**Lemma 27.** Let $V$ and $V'$ be $G$-graded $(\mathbb{F}H_1, \mathbb{F}H_2)$-bimodules, and denote $[V] = [(\chi_1, h_1), \ldots, (\chi_s, h_s)]$, $[V'] = [(\chi'_1, h'_1), \ldots, (\chi'_{s'}, h'_{s'})]$. Then, as $G$-graded algebras,

$$A := \left( \begin{array}{c|c} \mathbb{F}H_1 & V \\ \hline \mathbb{F}H_2 & V' \end{array} \right) \cong \left( \begin{array}{c|c} \mathbb{F}H_1 & V' \\ \hline \mathbb{F}H_2 & \mathbb{F}H_2 \end{array} \right) =: A'$$

if and only if $s = s'$ and there exist $\chi \in H_1 \cap H_2$ and $\sigma \in S_s$ such that

$$\chi_{e_{\sigma(\ell)}} \equiv h_{\sigma(\ell)} (\mod \mathcal{H}_1 \mathcal{H}_2), \quad \chi_{e_{\sigma(\ell)}} = \chi'_{e_{\sigma(\ell)}}, \quad \forall \ell = 1, 2, \ldots, s.$$

**Proof.** The converse is an elementary computation.

Let $\psi : A \to A'$ be a graded isomorphism of algebras. Then we have an induced graded isomorphism $\overline{\psi} : A/V \to A'/V'$, moreover, $\overline{\psi}(\mathbb{F}H_1) = \mathbb{F}H_1$. So, there exists $\chi_1 \in \mathcal{H}_1$ such that $\overline{\psi}(h_1) = \chi_1 h_1$, for each $h_i \in \mathcal{H}_1$, for $i = 1, 2$. Write $V' = \mathbb{F}H_1 m'_1 \mathbb{F}H_2 \oplus \cdots \oplus \mathbb{F}H_1 m'_{s'} \mathbb{F}H_2$, and consider the map $\varphi : A' \to A'$ given by

$$\varphi \left( \begin{array}{c} h_1 \\ h'_1 m'_1 \\ h_2 \end{array} \right) = \left( \begin{array}{c} \chi_1^{-1}(h_1) h_1 \\ \chi_1^{-1}(h'_1) \chi_2^{-1}(h'_2) h'_1 m'_1 h'_2 \\ \chi_2^{-1}(h_2) h_2 \end{array} \right).$$

It is elementary to prove that $\varphi$ is a graded isomorphism of algebras. Moreover, given $h_i \in \mathcal{H}_1$ and $m \in V$, we have

$$\varphi \psi(h_1 mh_2) = h_1 \varphi \psi(m) h_2.$$

Thus, $\varphi \psi : V \to \varphi(V')$ is a graded isomorphism of bimodules. Consider the product of the restriction of the characters, $\chi = \chi_1|_{\mathcal{H}_1 \cap \mathcal{H}_2} \cdot \chi_2|_{\mathcal{H}_1 \cap \mathcal{H}_2}$. Note that, if $[V] = [(\chi_1', h'_1), \ldots, (\chi'_{s'}, h'_{s'})]$, then the decomposition of $\varphi(V')$ is $[\varphi(V')] = [(\chi \chi_1', h'_1), \ldots, (\chi \chi_{s'}, h'_{s'})]$. Thus, the result follows from Lemma \ref{26}. \hfill $\Box$

5. The isomorphism problem

Assume that the conditions of Theorem \ref{11} hold. Consider a $G$-grading on $I(X)$ and let $E$ be the associated poset. Denote by $e \triangleleft e'$ if $e, e' \in E$, and $e'$ is a cover for $e$; that is, $e \neq e'$, $e \subseteq e'$, and if $e \subseteq f \subseteq e'$, then either $f = e$ or $f = e'$. Note that the Jacobson Radical $J = J(I(X))$ coincides with $\sum_{e \triangleleft e', e \neq e'} eI(X)e'$. Also, $J^2$ is the sum of all $eI(X)e''$ where there exists a chain of different elements $e \triangleleft e' \triangleleft e''$; $J^3$ is the sum of all $eI(X)e'''$, where $e \triangleleft e' \triangleleft e'' \triangleleft e'''$, and so on. Given $e, e' \in E$, $e \triangleright e'$, we can always construct a chain of elements containing only subsequent covers, starting with $e$, and ending with $e'$; however, we can have two different such chains having different number of elements. Nonetheless, if $e \triangleleft e'$, then $e \triangleleft e'$ is the unique chain of subsequent covers linking $e$ and $e'$. Let $J_1 = \bigoplus_{e \triangleleft e'} eI(X)e'$. The previous discussion is the proof of the following:

**Lemma 28.** Let $I(X)$ be $G$-graded, and let $J_1$ be as above.

(i) The restriction of the natural projection $\pi : J_1 \to J_1^2$ is a graded linear isomorphism.

(ii) $J = J_1 \oplus \sum_{m \geq 1} J_1^m$. \hfill $\Box$
It should be noticed that \( J_1 = \text{Span}\{e_{xy} \mid x \leq_X y\} \), where \( \leq_X \) is the partial order of \( X \).

As a consequence, we obtain

**Corollary 29.** Any \( G \)-grading on \( I(X) \) is completely determined by either one of the following graded spaces:

(i) \( D = \bigoplus_{e \in \mathcal{E}} eI(X)e \), and \( J_1 = \bigoplus_{e \leq e', eI(X)e'} eI(X)e' \), or,

(ii) \( I(X)/J \) and \( J/J^2 \), or,

(iii) \( I(X)/J^2 \).

**Proof.** We prove (i), the others being consequences. Let \( x \in I(X) \) be a homogeneous element. Then \( x = d + r \), where \( d \in D \) and \( r \in J \) are both homogeneous elements. By Lemma 28(ii), \( r \) is a linear combination of products of elements of \( J_1 \). Hence, the element \( x \) is completely determined by the homogeneous elements in \( D \) and in \( J_1 \). \( \square \)

The advantage of (ii) and (iii) above is an assertion independent of \( \mathcal{E} \). We can restate the previous corollary in terms of isomorphisms:

**Corollary 30.** Consider two \( G \)-gradings on \( I(X) \), and denote them by \( A_1 \) and \( A_2 \), and let \( \psi : A_1 \rightarrow A_2 \) be a graded homomorphism of algebras. The following assertions are equivalent:

(i) \( \psi \) is a graded isomorphism,

(ii) the induced maps by \( \psi \) on \( A_1/J(A_1) \) \( \rightarrow \) \( A_2/J(A_2) \) and \( J(A_1)/J^2(A_1) \rightarrow J(A_2)/J^2(A_2) \) are graded isomorphisms,

(iii) the induced map by \( \psi \) on \( A_1/J(A_1)^2 \rightarrow A_2/J(A_2)^2 \) is a graded isomorphism.

\( \square \)

Now, we will prove a necessary condition so that two gradings on \( I(X) \) are isomorphic:

**Proposition 31.** Let \( I(X) \) be endowed with two \( G \)-gradings, say \( A \) and \( A' \), and assume that

\[
A = \left( \begin{array}{cccc}
\mathbb{F}H_1 & M_{12} & \cdots & M_{1t} \\
\vdots & \ddots & \ddots & \vdots \\
\mathbb{F}H_{t-1} & M_{t-1,2} & \cdots & M_{t-1,t} \\
\mathbb{F}H_t & & & \\
\end{array} \right), \quad A' = \left( \begin{array}{cccc}
\mathbb{F}H'_1 & M'_{12} & \cdots & M'_{1t'} \\
\vdots & \ddots & \ddots & \vdots \\
\mathbb{F}H'_{t'-1} & M'_{t'-1,2} & \cdots & M'_{t'-1,t'} \\
\mathbb{F}H'_{t'} & & & \\
\end{array} \right),
\]

Let \( \mathcal{E} \) and \( \mathcal{E}' \) be the respective associated posets (see Remark 18). If \( A \cong A' \) then \( t = t' \), and there exists an isomorphism of posets \( \alpha : \mathcal{E} \rightarrow \mathcal{E}' \) such that \( H_i = H_{\alpha(i)} \), for each \( i = 1, 2, \ldots, t \).

**Proof.** Let \( \psi : A \rightarrow A' \) be a graded isomorphism. Then we have an induced graded isomorphism \( \psi : A/J(A) \rightarrow A'/J(A') \). So, \( t = t' \) and \( \mathbb{F}H_i \cong \mathbb{F}H_{\alpha(i)} \), for all \( i = 1, 2, \ldots, t \), and some permutation \( i \mapsto \alpha(i) \). But \( \mathbb{F}H_i \cong \mathbb{F}H_{\alpha(i)} \) as graded algebras implies \( H_i = H_{\alpha(i)} \).

Since \( \psi(J(A)) = J(A') \) and \( \psi(J(A)^2) = J(A')^2 \), we have an induced graded isomorphism \( \tilde{\psi} : J(A)/J(A)^2 \rightarrow J(A')/J(A')^2 \). Also, given a cover \( e, e' \in \mathcal{E} \), that is, \( e \leq e' \), we have \( \psi(eAe) \neq 0 \). Moreover, \( \psi(eAe + J(A)^2) = \alpha(e)A'\alpha(e') \). Hence, \( e \leq e' \) implies \( \alpha(e) \leq \alpha(e') \). Using a symmetric argument with \( \alpha^{-1} \), we conclude that \( \alpha \) is an isomorphism of posets. \( \square \)
Example 3. Let $UT_n$ be $G$-graded. In this case, $UT_n = I(X)$, where $X$ is a chain of $n$ elements. We saw that $E$ is constructed as a set of subsets of $X$; and every element of $E$ contains pairwise non-comparable elements of $X$. Since $X$ is a chain, it implies that every element of $E$ is a singleton, that is, $\{x\}$, for some $x \in X$. Thus, $E \simeq X$. Hence, $I(X)/J(I(X))$ always has the trivial grading. This fact (with no restrictions on the ground field) was originally proved by Valenti and Zaicev in 2007 [17].

Example 4. Let $\Gamma_1$ and $\Gamma_2$ be two $G$-gradings on $UT_n$, and name $A_1 = (UT_n, \Gamma_1)$ and $A_2 = (UT_n, \Gamma_2)$. By the previous example, $\mathcal{E}_1 \simeq \mathcal{E}_2 \simeq X$ is a chain of $n$ elements. By Proposition 31, $\Gamma_1 \simeq \Gamma_2$ only if $e_{i_i} A_1 e_{i_{i+1},i+1} \simeq e_{i_i} A_2 e_{i_{i+1},i+1}$, for all $i = 1,2,\ldots,n-1$. Equivalently, $\Gamma_1 \simeq \Gamma_2$ only if $\deg_{\Gamma_1} e_{i,i+1} = \deg_{\Gamma_2} e_{i,i+1}$ for all $i = 1,2,\ldots,n-1$. The converse is immediate. This fact (indeed, a stronger statement) was originally proved by Di Vincenzo et al in 2004 [5].

6. Abelian grading group

In this section, we investigate properties of the gradings on $I(X)$, if the grading group is abelian. Let $G$ be an abelian group. Consider any $G$-grading $I(X)$, where $X$ is a finite poset, written as Theorem 1. For any $m \in I(X)$, we denote $s(m) = \{(u,v) \in X^2 \mid m(u,v) \neq 0\}$. We say that $m$ is $G$-primitive if $m$ is homogeneous and there is no homogeneous $y \in I(X)$ with $s(y) \subseteq s(x)$.

Let $\bar{\mathbb{F}}$ be an algebraically closure of $\mathbb{F}$, and consider $I(X) = I(X) \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ with the induced $G$-grading. In this case, if the characteristic of $\bar{\mathbb{F}}$ is adequate, it is well-known that we have a duality between $\bar{G}$-action and $G$-gradings (see, for instance, [6, §1.4]). The automorphism group of $I(X)$ is $\text{Aut}(I(X)) = \text{Int}(I(X)) \cdot \text{mult}(I(X)) \cdot \text{Aut}(X)$ (see, for instance, [16, Theorem 7.3.6]), where $\text{Int}(I(X))$ is the group of inner automorphisms of $I(X)$, $\text{mult}(I(X))$ is the group of the so-called multiplicative automorphisms of $I(X)$ and $\text{Aut}(X)$ are the induced automorphisms from the automorphisms of the poset $X$.

Lemma 32. If $I(X)$ is expressed as in Theorem 1, then $\bar{G} \subset \text{mult}(I(X)) \text{Aut}(X)$.

Proof. Let $f \in \bar{G}$, and let $x \in X$. Then there is unique minimal idempotent $e$ such that $x \in \mathcal{D}(e)$. Thus, denoting by $D = eI(X)e = \text{Span}\{e_{yy} \mid y \in \mathcal{D}(e)\}$, we have $e_{xx} \in D$. Hence, $e_{xx}$ is a linear combination of homogeneous elements in $D$. Since $f(D) = D$, $f(e_{xx})$ is still a linear combination of homogeneous elements in $D$. However, since $f$ is a composition of an inner automorphism, multiplicative automorphism, and an induced automorphism of the poset, actually we have $f(e_{xx}) = e_{yy}$, for some $y \in \mathcal{D}(e)$.

Since this construction works for all $x \in X$ and $f$ is bijective, define $\alpha_f : X \to X$ as the (automorphism) $\alpha_f(x) = y$. Denote the induced automorphism on $I(X)$ by the same name. Then, clearly $f \circ \alpha_f^{-1}(e_{xx}) = e_{xx}$, for all $x \in X$. By [16, Proposition 7.3.2], $f \circ \alpha_f^{-1} \in \text{mult}(I(X))$. □

Let $\mathcal{A} = \{\alpha_f \mid f \in \bar{G}\}$ as above, and let $e_1,\ldots,e_t$ be the minimal idempotents. Let $x \in \mathcal{D}(e_i)$, then
$$\sum_{f \in \bar{G}} f(e_{xx}) = \sum_{f \in \bar{G}} e_{\alpha_f(x),\alpha_f(x)} = \lambda_i e_i,$$

is homogeneous, where $\lambda_i \in \mathbb{Z}_{>0}$.
Now, the above elements are homogeneous in $I(X)$, but they also belong to $I(X)$. Hence, they are homogeneous in $I(X)$ as well. Moreover, the automorphisms induced from $\text{Aut}(X)$ are independent on the base field. Thus, we just proved

**Theorem 33.** Let $\mathbb{F}$ be any field of characteristic zero, $G$ an abelian group and consider any $G$-grading on $I(X)$. Up to a graded isomorphism, let $e_1, e_2, \ldots, e_t$ be the unity of each $\mathbb{F}H_i$, as in Theorem 7. Then there exists a subset $\mathcal{A} \subset \text{Aut}(X)$ such that $\lambda_i e_i = \sum_{a \in \mathcal{A}} \alpha(e_{ax})$, for any $x \in \mathcal{D}(e_i)$, for some $\lambda_i \in \mathbb{Z}_{>0}$, and for all $i = 1, 2, \ldots, t$. □

We conjecture that this theorem is true, even when the grading group is not abelian.

Now, using the results of Sections 4 and 5 we are able to provide a classification of isomorphism classes of group gradings on the incidence algebras, given that the grading group is abelian.

**Theorem 34.** Let $G$ be an abelian group, $X$ a finite poset, and assume two $G$-gradings on $I(X)$, namely $A$ and $A'$. Write

$$A = \left( \begin{array}{cccc} \mathbb{F}H_1 & M_{12} & \cdots & M_{1t} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbb{F}H_{t-1} & M_{t-1,1} & \cdots & M_{t-1,t} \\ \mathbb{F}H_t & \end{array} \right), A' = \left( \begin{array}{cccc} \mathbb{F}H'_1 & M'_{12} & \cdots & M'_{1t'} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbb{F}H'_{t'-1} & M'_{t'-1,1} & \cdots & M'_{t',t'} \\ \mathbb{F}H'_{t'} & \end{array} \right),$$

and denote $\mathcal{E} = \{1, 2, \ldots, t\}$, $\mathcal{E}' = \{1, 2, \ldots, t'\}$. Let $[M_{ij}] = [(\chi_{ij}^{(ij)}, h_{ij}^{(ij)}), \ldots, (\chi_{s_{ij}}^{(ij)}, h_{s_{ij}}^{(ij)})]$, $[M'_{ij}] = [(\chi_{ij}^{(ij)'}, h_{ij}^{(ij)'})], \ldots, (\chi_{s'_{ij}}^{(ij)'}, h_{s'_{ij}}^{(ij)'})]$, where $s_{ij}, s'_{ij} \geq 0$. Then $A \cong A'$, as $G$-graded algebras if and only if $t = t'$ and there exist an isomorphism of posets $\alpha : \mathcal{E} \rightarrow \mathcal{E}'$, and characters $\chi_i \in H_1, \ldots, \chi_t \in H_t$ such that

1. $H_i = H_{\alpha(i)}$, for each $i = 1, 2, \ldots, t$,
2. Given $i \leq j$, let $H_{ij} = H_i \cap H_j$. Then we have $s_{ij} = s'_{\alpha(i), \alpha(j)}$, and there is $\sigma_{ij} \in S_{s_{ij}}$ satisfying (for each $t = 1, 2, \ldots, s_{ij}$)

$$h_{(ij)}^{(t)} \equiv h_{\sigma_{ij}}^{(t)} \pmod{H_i H_j}, \quad \chi_{(ij)}^{(t)} \equiv \chi_i \chi_j \chi_{\sigma_{ij}} (t).$$

**Proof.** The ‘only if’ part follows from Proposition 51 and the proof of Lemma 27. To prove the ‘if’ part, we notice that the isomorphism of posets $\alpha$ induces an algebra isomorphism. Then, we apply Corollary 30 and the proof of Lemma 27. □

7. Examples

**7.1. It is not always possible to reduce to cyclic groups.** Consider a commutative graded division algebra $D$. It is natural to ask if we can reduce such graded algebra to a direct sum of graded subalgebras, where each of them are graded by cyclic groups. The answer is no, as shown by the following example. Let $X = \{1, 2, 3, 4\}$ where $i \leq j$ if and only if $i = j$. We can define a $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ grading on $I(X)$ posing:

$$\deg e_{11} + e_{22} + e_{33} + e_{44} = (0, 0),$$
$$\deg e_{11} - e_{22} - e_{33} + e_{44} = (1, 0),$$
$$\deg e_{11} + e_{22} - e_{33} - e_{44} = (0, 1),$$
$$\deg e_{11} - e_{22} + e_{33} - e_{44} = (1, 1).$$
In particular, there is no homogeneous idempotent \( e' \in I(X) \) with \( s(e') \subset s(e) \) other than \( e \) itself. Hence we can not break \( I(X) \) as a sum of two graded division algebras.

7.2. Non-existence of homogeneous multiplicative basis. Let \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \) and \( X \) be the poset whose Hasse diagram is the following:

![Hasse diagram](image)

Define the following diagonal elements to be homogeneous:

\[
\begin{align*}
\deg e_{11} + e_{22} &= (0, 0, 0), \\
\deg e_{11} - e_{22} &= (1, 0, 0), \\
\deg e_{33} + e_{44} &= (0, 0, 0), \\
\deg e_{33} - e_{44} &= (0, 1, 0), \\
\deg e_{55} + e_{66} &= (0, 0, 0), \\
\deg e_{55} - e_{66} &= (1, 0, 0), \\
\deg e_{77} + e_{88} &= (0, 0, 0), \\
\deg e_{77} - e_{88} &= (1, 0, 0),
\end{align*}
\]

and, the following elements of the Jacobson radical to be homogeneous:

\[
\begin{align*}
\deg e_{13} + e_{14} + e_{23} + e_{24} &= (0, 0, 1), \\
\deg e_{13} + e_{14} - e_{23} - e_{24} &= (1, 0, 1), \\
\deg e_{13} - e_{14} + e_{23} - e_{24} &= (0, 1, 1), \\
\deg e_{13} - e_{14} - e_{23} + e_{24} &= (1, 1, 1), \\
\deg e_{37} + e_{38} + e_{47} + e_{48} &= (0, 0, 1), \\
\deg e_{37} - e_{38} + e_{47} - e_{48} &= (1, 0, 1), \\
\deg e_{37} + e_{38} - e_{47} - e_{48} &= (0, 1, 1), \\
\deg e_{37} - e_{38} - e_{47} + e_{48} &= (1, 1, 1), \\
\deg e_{15} + e_{26} &= (0, 0, 1), \\
\deg e_{15} - e_{26} &= (1, 0, 1), \\
\deg e_{57} + e_{68} &= (0, 0, 1), \\
\deg e_{57} - e_{68} &= (1, 0, 1), \\
\deg e_{17} + e_{28} &= (0, 0, 2), \\
\deg e_{17} - e_{28} &= (1, 0, 2), \\
\deg e_{27} + e_{18} &= (0, 0, 2), \\
\deg e_{27} - e_{18} &= (1, 0, 2).
\end{align*}
\]

Let \( G_1 = \mathbb{Z}_2 \times 0 \times 0 \) and \( G_2 = 0 \times \mathbb{Z}_2 \times 0 \) be subgroups of \( G \). The induced poset, as in Lemma 15 is
where, by abuse of notation, every $FG_i$ represents its homogeneous idempotent generator. Moreover, with the notation of Theorem 1, the above grading is isomorphic to

$$I(X) \cong \begin{pmatrix}
FG_1 & F(G_1 \times G_2) & FG_1 & FG_1 \oplus FG_1 \\
FG_2 & 0 & FG(G_2 \times G_1) & FG_1 \\
FG_1 & FG_1 \oplus FG_1 & FG_1 & FG_1 \\
FG_1 & FG_1 & FG_1 & FG_1
\end{pmatrix}.$$

We claim that $I(X)$ has no multiplicative homogeneous basis. Let us use the notation of Theorem 1, that is, let $M_{12} = F(G_1 \times G_2)$, $M_{13} = FG_1$, $M_{24} = F(G_2 \times G_1)$, $M_{34} = FG_1$. The main idea is that each nonzero element obtained by the product of homogeneous elements $M_{12}M_{24}$ must have 4 nonzero entries; while the nonzero elements of the product $M_{13}M_{34}$ have only 2 nonzero entries. Thus, we cannot construct a homogeneous basis of this algebra.

The proof is tedious, and we list the main steps. Assume that $B$ is a homogeneous basis of $I(X)$.

1. $B$ consists of a homogeneous basis of the diagonal $FG_1 \oplus FG_2 \oplus FG_1 \oplus FG_1$, the space $M_{12} \oplus M_{13} \oplus M_{24} \oplus M_{34}$, and $M_{14}$.
2. $B$ contains a homogeneous basis of $FG_2$.
3. So, $B$ contains a homogeneous basis of $M_{12}$ and $M_{24}$ as well. Hence, the elements of $B$ that belongs to $M_{14}$ must have four non-zero entries.
4. For $i < j$, denote by $\pi_{ij} : I(X) \to M_{ij}$ the projection. Given $v \in B$, then $\pi_{13}(v) \neq 0$ implies $\pi_{12}(v) = 0$. Similarly, $\pi_{34}(v) \neq 0$ implies $\pi_{24}(v) = 0$.
5. So, we can find $v_1, v_2 \in B$ such that $\pi_{13}(v_1) \neq 0$, $\pi_{34}(v_2) \neq 0$ and $v_1v_2 \neq 0$. However, $v_1v_2 \in M_{14}$ contains only two non-zero entries, and it should be a multiple of an element of $B$. This is a contradiction.

### 7.3. Non-abelian grading

Fix a $G$-grading on $I(X)$. If we can find a commutative group $H$, and a $H$-grading on $I(X)$, equivalent to the original $G$-grading (in the sense of [1], Definition 1.14, p. 14]), then we can utilize Theorem 23 and obtain the same statement for non-abelian grading group. But the following example shows that we can not always do it. Let $G = S_3$ be the group of permutations of 3 elements. Let $X = \{1, 2, 3, 4, 5, 6\}$, where $i \leq i + 3$, for all $i$. In Hasse diagram:
Let \( \omega \neq 1 \) be such that \( \omega^3 = 1 \). Define the following elements to be homogeneous:

\[
\begin{align*}
\deg e_{11} + e_{22} + e_{33} &= (1), \\
\deg e_{11} + \omega e_{22} + \omega^2 e_{33} &= (123), \\
\deg e_{11} + \omega^2 e_{22} + \omega e_{33} &= (132), \\
\deg e_{14} + e_{25} + e_{36} &= (12), \\
\deg e_{14} + \omega e_{25} + \omega^2 e_{36} &= (13), \\
\deg e_{14} + \omega^2 e_{25} + \omega e_{36} &= (23), \\
\deg e_{44} + e_{55} + e_{66} &= (1), \\
\deg e_{44} + \omega e_{55} + \omega^2 e_{66} &= (123), \\
\deg e_{44} + \omega^2 e_{55} + \omega e_{66} &= (132),
\end{align*}
\]

This indeed well defines a \( S_3 \)-grading on \( I(X) \). The key point is the product

\[
(e_{11} + \omega e_{22} + \omega^2 e_{33})(e_{14} + e_{25} + e_{36})(e_{44} + \omega e_{55} + \omega e_{66}) = e_{14} + e_{25} + e_{36}
\]

Thus, if \( g = \deg e_{11} + \omega e_{22} + \omega^2 e_{33} \), and \( h = \deg e_{14} + e_{25} + e_{36} \), we obtain the equation \( ghg = h \). So, the grading cannot be equivalent to an abelian grading, unless \( g^2 = 1 \), which is never the case. Hence, the extension of Theorem 33 to the non-abelian case, if possible, is not so direct.

### 7.4. Reconstruction of algebras

A natural question is which algebras can be realized as an incidence algebra endowed with a grading.

In general, this is a hard question and we do not have a complete answer. It seems that we cannot obtain a general answer for all kinds of finite posets.

Every group algebra can be realized as an incidence algebra with a grading, by Lemma [5]. If the associated poset \( \mathcal{E} \) has 2 elements, then we can also realize:

**Proposition 35.** Let \( G \) be an abelian group, and assume that \( \mathbb{F} \) contains enough roots of unit. Let \( H_1, H_2 \) be finite subgroups of \( G \), and \( M \) a graded \( (\mathbb{F} H_1, \mathbb{F} H_2) \)-bimodule, satisfying \( |M| = |\chi_1, h_1), \ldots, (\chi_m, h_m)| \) with \( \chi_i \neq \chi_j \) for \( i \neq j \). Then there exists a finite poset \( X \) and a \( G \)-grading on \( I(X) \) such that

\[
I(X) \cong \left( \begin{array}{cc}
\mathbb{F} H_1 & M \\
\mathbb{F} H_2 & \end{array} \right).
\]

**Proof.** Let \( X = X_1 \cup X_2 \) be a poset, where \( X_1 \) and \( X_2 \) are disjoint sets, and \( X_i \) contains \( |H_i| \) elements. We construct an ordering \( \leq_X \) on \( X \) where \( i \leq_X j \) implies \( i \in X_1 \), and \( j \in X_2 \). Thus

\[
I(X) \cong \left( \begin{array}{cc}
\mathbb{F}^{[H_1]} & V \\
0 & \mathbb{F}^{[H_2]} \end{array} \right).
\]

Since, as ordinary algebras, \( \mathbb{F} H_i \cong \mathbb{F}^{[H_i]} \), we can endow in \( \mathbb{F}^{[H_i]} \) a \( H_i \)-grading, in such a way that \( \mathbb{F}^{[H_i]} \cong \mathbb{F} H_i \) as graded algebras. Recall that, if \( e_i \in \mathbb{F} H_i \) is the homogeneous idempotent, then \( \ell(e_1, e_2) \) is the linking number of \( e_1 \) and \( e_2 \). We shall construct \( \leq_X \) to have \( \ell(e_1, e_2) = \dim_M \).

Write \( M \cong \mathbb{F} H_m \oplus \mathbb{F} H_m \otimes \mathbb{F} H_m \), where \( 0 \leq s \leq \frac{|H_1||H_2|}{|H_1|} \). Now, we construct exactly \( s|H| \) relations \( i \leq_X j \), in such a way that \( \ell(i, e_2) \), \( i \in X_1 \), and \( \ell(e_1, j) \), \( j \in X_2 \), are constants. As a graded bimodule, \( V \) is generated by exactly \( s \) elements, say \( r_1, \ldots, r_s \). Defining \( \deg r_k = \deg m_k \), for \( k = 1, \ldots, s \), we obtain the desired realization. We left to the reader to fill all the details of this construction. \( \square \)
If $|\mathcal{E}| > 2$, then we should deal with compatibility conditions, as the following example suggests.

**Example 5.** Let $G$ be an abelian group containing $H \cong \mathbb{Z}_2$. Assume that $I(X)$ is a $G$-graded incidence algebra such that

$$I(X) \cong \begin{pmatrix} \mathbb{F}H & M_{12} & M_{13} \\ \mathbb{F}H & M_{23} \\ \mathbb{F}H \end{pmatrix},$$

where $\dim_{\mathbb{F}H} M_{ij} = 2$, for all $1 \leq i < j \leq 3$. Write $M_{ij} = \mathbb{F}Hm^{(1)}_{ij} \oplus \mathbb{F}Hm^{(2)}_{ij}$, where each $m^{(k)}_{ij}$ is homogeneous of degree $h^{(k)}_{ij}$. Then $I(X)$ is consistent if, and only if, the following conditions hold:

$$h^{(1)}_{12}h^{(1)}_{23} = h^{(1)}_{13} = h^{(2)}_{12}h^{(2)}_{23},$$

$$h^{(2)}_{12}h^{(1)}_{23} = h^{(2)}_{13} = h^{(1)}_{12}h^{(2)}_{23}.$$

**Question.** Determine which graded algebras can be realized as an incidence algebra endowed with a $G$-grading.

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**References**

[1] Y. A. Bahturin, S. K. Sehgal, M. V. Zaicev, *Group gradings on associative algebras*, Journal of Algebra 241 (2001), 677–698.

[2] Y. A. Bahturin, I. P. Shestakov, M. V. Zaicev, *Gradings on simple Jordan and Lie algebras*, Journal of Algebra 283 (2005), 849–868.

[3] Y. A. Bahturin, M. V. Zaicev, *Group gradings on matrix algebras*, Canadian Mathematical Bulletin 45 (4) (2002), 499–508.

[4] L. Bemm, E. Fornaroli, E. Santulo Jr, *A cohomological point of view on gradings on algebras with multiplicative basis*, Journal of Pure and Applied Algebra 223 (2019), 769–782.

[5] O. M. Di Vincenzo, P. Koshlukov, A. Valenti, *Gradings on the algebra of upper triangular matrices and their graded identities*, Journal of Algebra 275 (2004), 550–566.

[6] A. Elduque, M. Kochetov, *Gradings on simple Lie algebras*, Mathematical Surveys and Monographs, 189. American Mathematical Society (2013).

[7] A. Gordienko, *Co-stability of radicals and its applications to PI-theory*, Algebra Colloquium 23.3 (2016), 481–492.

[8] N. Jacobson, *Lie algebras*. Republication of the 1962 original. Dover Publications, Inc., New York, 1979.

[9] S. Jondrup, *Automorphisms and derivations of upper triangular matrix rings*, Linear algebra and its applications, 221 (1995), 205–218.

[10] M. Jones, *Elementary and good group gradings of incidence algebras over partially-ordered sets with cross-cuts*, Communications in Algebra 34 (2006), 2369–2387.

[11] G. Karpilovsky, *Projective representations of finite groups*. Monographs and Textbooks in Pure and Applied Mathematics, 94. Marcel Dekker, Inc., New York, 1985.

[12] A. R. Kemer, *Ideals of identities of associative algebras*, Translations of Mathematical Monographs, 87. American Mathematical Society (1991).

[13] L Miller, E. Spiegel, *Group gradings in incidence algebras*, Communications in Algebra 38 (2010), 953–963.
[14] K. Price, *Good gradings of generalized incidence rings*, Communications in Algebra **41** (2013), 3668–3678.

[15] L. A. B. San Martin, *Álgebras de Lie*, Unicamp, 2010.

[16] E. Spiegel, C. O’Donnell, *Incidence algebras*, Monographs and Textbooks in Pure and Applied Mathematics, 206. Marcel Dekker, Inc., New York, 1997.

[17] A. Valenti, M. V. Zaicev, *Group gradings on upper triangular matrices*, Archiv der Mathematik **89**(1) (2007), 33–40.

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