Corrigendum: A limit formula for the quantum fidelity (2013 J. Phys. A: Math. Theor. 46 025304)

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On page 5, there is an error in the proof of theorem 1. This theorem is still valid but the actual proof is as follows.

Proof. Specify the definition (14) to the case where \(\sigma = |\varphi\rangle\langle\varphi|\) i.e.,

\[
C_s = \text{Tr} \left( \rho' |\varphi\rangle\langle\varphi| \right). \tag{22}
\]

For every \(s \in (0, 1)\) we can use the property of the projector

\[
|\varphi\rangle\langle\varphi|^{s} = |\varphi\rangle\langle\varphi|. \tag{23}
\]

and write

\[
C_s = \langle\varphi | \rho' |\varphi\rangle. \tag{24}
\]

Now, we can always decompose \(\rho\) as

\[
\rho = \sum_k p_k^{1/2} |k\rangle\langle k|. \tag{25}
\]

where \(p_k \in [0, 1]\) for any \(k\), and \(\{|k\rangle\}\) is an orthonormal set

\[
\langle k | k' \rangle = \delta_{kk'}. \tag{26}
\]

Taking the \(s\)-power of (25), we get

\[
\rho' = \sum_k p_k^{s/2} |k\rangle\langle k|. \tag{27}
\]
for any \( s \in (0,1) \). Using the latter expression in (24), we achieve

\[
C_s = \sum_k \rho_k^{1/2} \langle k | \varphi \rangle^2.
\]  

(28)

Finally, taking the limit of \( s \to 1^- \), we derive

\[
\lim_{s \to 1^-} C_s = \sum_k \rho_k^{1/2} \langle k | \varphi \rangle^2
\]

(29)

\[
= \langle \varphi | \rho | \varphi \rangle = F(\rho, | \varphi \rangle),
\]  

(30)

which corresponds to the result of equation (21).

Correspondingly, the proof of the corollary on page 6 must also be modified. It must be replaced by the following:

**Proof.** This is a trivial consequence of the previous theorem. In fact, we have that \( C_s \) in equation (28) is manifestly non-increasing in \( s \). As a consequence, we have

\[
C = \inf_{s \in (0,1)} C_s = \lim_{s \to 1^-} C_s = F(\rho, | \varphi \rangle),
\]  

(33)

which completes the proof.
A limit formula for the quantum fidelity

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Abstract

Quantum fidelity is a central tool in quantum information, quantifying how much two quantum states are similar. By using the notion of generalized overlap, which occurs in the definition of the quantum Chernoff bound, we propose here a limit formula for the quantum fidelity between a mixed state and a pure state. As an example of an application, we apply this formula to the case of multimode Gaussian states, achieving a simple expression in terms of their first- and second-order statistical moments.

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1. Introduction

Very often the performance of a quantum information protocol is measured in terms of similarity between two states. This happens both in discrete (qubit) quantum information [1] and continuous-variable quantum information [2, 3], for instance, with Gaussian states [4, 5].

One of the most well-known examples is that of quantum teleportation between two stations, where the perfect execution of the protocol corresponds to having an output state at the receiver’s station which is equal to the input state originally processed at the sender’s station [6–15]. Another important scenario is that of quantum cloning [16–20], where an unknown input quantum state is transformed into two or more clones. Because of the no-cloning theorem, the output clones cannot be identical to the input state [21, 22]. As a result, a measure of similarity between the input state and output clones is fundamental in order to quantify the performance of a quantum cloning machine.

In these kinds of protocols, the typical measure of similarity between two quantum states is their fidelity. Quantum fidelity was introduced and characterized in two seminal papers by Uhlmann [23] and Jozsa [24]. Its general definition can be regarded as an extension of the wavefunction overlap to generally mixed quantum states. Despite its usefulness, simple closed formulas are not always easy to derive. For instance, we know analytical formulas for the
Quantum fidelity also plays a central role in quantum hypothesis testing, where the basic problem is the discrimination between two equiprobable quantum states of a system by means of an optimal measurement. In this framework, quantum fidelity has been related to various other quantities such as the Helstrom bound [30–32] and the quantum Chernoff bound [33–35], which provide a direct quantification of the minimum error probability affecting the state discrimination.

In this paper, we start from these connections to derive a new formula for the quantum fidelity between a generally mixed state and a pure state. This formula is expressed in the form of a limit and involves a generalized overlap between the two quantum states. This is the same kind of overlap which intervenes in the definition of the quantum Chernoff bound.

As an example of an application of this formula, we consider the bosonic setting and the case of multimode Gaussian states. Here we first introduce the notion of symplectic action, which enables us to simplify the symplectic manipulations of the second order moments. Then, by elaborating a result from [35], we derive a simple analytical expression for the fidelity between a mixed and a pure Gaussian state in terms of their first- and second-order moments.

The paper is organized as follows. In section 2 we provide a brief review of the basic facts regarding quantum fidelity and its connections with the various bounds used in quantum hypothesis testing. In section 3, we derive the limit formula for the quantum fidelity. Then, in section 4, we consider bosonic continuous-variable systems. After a brief review of the basic notions on Gaussian states, we introduce the symplectic action and we derive the formula of the fidelity for Gaussian states. Finally, section 5 is for conclusion and discussion.

2. General notions on quantum fidelity

Consider a quantum system with separable Hilbert space $\mathcal{H}$. In general, the states of this system are described by density operators $\rho : \mathcal{H} \to \mathcal{H}$ forming a corresponding state space $\mathcal{D}(\mathcal{H})$. Given two arbitrary states, $\rho$ and $\sigma$, their similarity can be quantified by the Uhlmann–Jozsa fidelity

$$F(\rho, \sigma) := (\text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}})^2.$$  

(1)

This is a positive number $0 \leq F \leq 1$, where $F = 1$ corresponds to identical states, and $F = 0$ corresponds to orthogonal states, i.e. density operators with orthogonal supports in $\mathcal{D}(\mathcal{H})$.

In the case where one of the states is pure $\sigma = |\psi\rangle \langle \psi|$, the fidelity assumes the sandwich expression

$$F(\rho, |\psi\rangle) = \langle \psi | \rho | \psi \rangle,$$  

(2)

which becomes the overlap

$$F(|\psi\rangle, |\psi\rangle) = |\langle \psi | \psi \rangle|^2,$$  

(3)

if also the other state is pure $\rho = |\psi\rangle \langle \psi|$.  

$^4$ An alternative definition which is commonly adopted in quantum information is the square-root or Bures’ fidelity. This corresponds to the square root of the Uhlmann–Jozsa fidelity

$$F_B(\rho, \sigma) := \sqrt{F(\rho, \sigma)} = \text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}.$$  

Bures’ fidelity is the direct generalization of the classical fidelity. In fact, if the two density operators commute, we can write $\rho = \sum_k p_k |k\rangle \langle k|$ and $\sigma = \sum_k q_k |k\rangle \langle k|$ for an orthonormal basis $\{|k\rangle\}$. Then, we have $F_B(\rho, \sigma) = \sum_k \sqrt{p_k q_k}$ which is the classical fidelity $F(p_k, q_k)$ between the two probability distributions $p_k$ and $q_k$. 

$^2$
Most of the properties of the quantum fidelity can be derived from Uhlmann’s theorem which states that

\[ F(\rho, \sigma) = \max_{|\varphi_\rho\rangle} |\langle \varphi_\rho | \varphi_\sigma \rangle|^2, \]

where \(|\varphi_\rho\rangle\) and \(|\varphi_\sigma\rangle\) are purifications of \(\rho\) and \(\sigma\), respectively. For instance, immediate consequences of this theorem are the positive range \(0 \leq F \leq 1\), the symmetry property \(F(\rho, \sigma) = F(\sigma, \rho)\), and the invariance \(F(\rho, \sigma) = F(\rho \rho^U, \sigma \sigma^U)\) under a generic unitary \(U\).

Despite being a measure of similarity between two quantum states, the quantum fidelity is not properly a metric in the state space \(\mathcal{D}(\mathcal{H})\). In fact, by definition, a metric in \(\mathcal{D}(\mathcal{H})\) is a map \(D: (\rho, \sigma) \rightarrow \mathbb{R}\) with the following properties:

(i) Positive definiteness, i.e., \(D(\rho, \sigma) \geq 0(= 0 \iff \sigma = \rho)\).
(ii) Symmetry, i.e., \(D(\rho, \sigma) = D(\sigma, \rho)\).
(iii) Subadditivity or triangle inequality, i.e., \(D(\rho, \gamma) \leq D(\rho, \sigma) + D(\sigma, \gamma)\), for any triplet \(\rho, \sigma\) and \(\gamma\).

In this list, the fidelity fails both the first property (since \(F(\rho, \rho) = 1\)) and the triangle inequality\(^5\).

Even if it is not a metric by itself, we can easily connect the quantum fidelity to a metric in \(\mathcal{D}(\mathcal{H})\). For instance, we can consider Bures’ distance \(^[32]\)

\[ D_B(\rho, \sigma) = \sqrt{2 - 2F(\rho, \sigma)}, \]

or the angular distance \(^[1]\)

\[ D_A(\rho, \sigma) = \arccos \sqrt{F(\rho, \sigma)}. \]

Most importantly, the quantum fidelity can be connected with the trace distance, which is the standard metric adopted in quantum information for its direct interpretation in quantum hypothesis testing.

Given two quantum states, \(\rho\) and \(\sigma\), their trace distance is defined as \(^6\) \([1]\)

\[ D(\rho, \sigma) = \frac{1}{2} \|\rho - \sigma\|_1, \]

where

\[ \|O\|_1 := \text{Tr}|O| = \text{Tr}\sqrt{O^*O} \]

is the trace norm of an arbitrary trace-class operator \(O\).\(^7\) The trace distance ranges in the positive interval \([0, 1]\), with \(D = 0\) for identical states and \(D = 1\) for orthogonal states. \(D(\rho, \sigma)\) determines the error probability which affects the discrimination of the two states, \(\rho\) and \(\sigma\), by means of an optimal quantum measurement. Suppose that a system is prepared in a state \(|\psi\rangle\) and we want to distinguish it from a possible hypothetical state \(|\varphi\rangle\). The best way to perform this task is to use a quantum measurement that maximizes the error probability. For this reason, it is useful to introduce the error probability for each state, which is given by the trace distance \(D(\rho, \sigma)\).

\(^5\) For instance, consider the qubit states \(\gamma = |1\rangle\langle 1|\), \(\sigma = |0\rangle\langle 0|\) and \(\rho = |\psi\rangle\langle \psi|\) with \(|\psi\rangle = a|0\rangle + \beta|1\rangle\). Then, we have \(F(\rho, \gamma) = |\langle \psi | 1\rangle|^2 = |\beta|^2\), \(F(\rho, \sigma) = |\langle \psi | 0\rangle|^2 = |a|^2\) and \(F(\sigma, \gamma) = |\langle 0 | 1\rangle|^2 = 0\). Now the triangle inequality becomes \(|\beta|^2 \leq |a|^2\) which can be easily violated by infinite choices of \(|\psi\rangle\).

\(^6\) This is the quantum version of the Kolmogorov distance \(D(p_k, q_k) := \frac{1}{2} \sum |p_k - q_k|\) which is defined for two classical probability distributions \(\{p_k\}\) and \(\{q_k\}\).

\(^7\) Roughly speaking, a trace-class operator is a linear operator for which we can define the trace. More rigorously, a bounded linear operator \(O\) over a separable Hilbert space \(\mathcal{H}\) is called ‘trace-class’ if for some (and hence all) orthonormal bases \(|\psi_k\rangle\) we have \(\sum |\langle \psi_k | O | \psi_k \rangle| < \infty\). In such a case, both the trace \(\text{Tr}O\) and the trace norm \(\|O\|_1 := \text{Tr}|O|\) are correctly defined and finite. Furthermore, we can define a whole class of Schatten \(p\)-norms \(\|O\|_p := (\text{Tr}|O|^p)^{1/p}\) for every real \(p \geq 1\). This class includes the trace norm for \(p = 1\) and the Hilbert–Schmidt norm for \(p = 2\).
one of two equiprobable states, $\rho$ and $\sigma$, then the optimal positive operator valued measure provides the correct answer with an error probability given by the Helstrom bound [30]

$$P_{\text{err}} = \frac{1 - D}{2}. \quad (9)$$

According to [31], we can use the trace distance to write the following upper bound for the fidelity:

$$F \leq 1 - D^2. \quad (10)$$

Then, according to [1] we can also write the lower bound

$$(1 - D)^2 \leq F. \quad (11)$$

In particular, if one of the states is pure $\sigma = |\varphi\rangle\langle\varphi|$ we have the tighter lower bound [1]

$$1 - D \leq F. \quad (12)$$

Finally, if both the states are pure, i.e. $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\varphi\rangle\langle\varphi|$, then we have the equality [30]

$$F = 1 - D^2. \quad (13)$$

Besides the trace distance and the Helstrom bound, the quantum fidelity possesses important relations with other crucial quantities in quantum hypothesis testing: the quantum Chernoff bound [33, 34] and the quantum Battacharyya bound [35]. Let us consider the quantity

$$C_s(\rho, \sigma) := \text{Tr}(\rho^s \sigma^{1-s}) \leq 1, \quad (14)$$

which represents a generalized $s$-overlap between the two states $\rho$ and $\sigma$. Using (14), we can define the Chernoff term

$$C(\rho, \sigma) := \inf_{s \in (0,1)} C_s(\rho, \sigma), \quad (15)$$

and the Battacharyya term

$$B(\rho, \sigma) := C_{1/2}(\rho, \sigma) = \text{Tr}\sqrt{\rho}\sqrt{\sigma}. \quad (16)$$

Up to a factor 2, these terms provide the single-shot formulae for the quantum Chernoff and Battacharyya bounds, which are used to estimate the minimum error probability in the discrimination of $\rho$ and $\sigma$ via a single quantum measurement, i.e. we have

$$P_{\text{err}} \leq C, \quad P_{\text{err}} \leq B. \quad (17)$$

It is straightforward to prove the following chain of inequalities involving the fidelity:

$$C \leq B \leq \sqrt{F}. \quad (18)$$

In fact $C \leq B$ is trivial, while $B \leq \sqrt{F}$ comes from the fact that [24, 31]

$$\text{Tr}\sqrt{\rho}\sqrt{\sigma} = |\text{Tr}\sqrt{\rho}\sqrt{\sigma}| \leq \text{Tr}|\sqrt{\rho}\sqrt{\sigma}| = \text{Tr}\sqrt{\rho\sigma}\sqrt{\rho}. \quad (19)$$

where we exploit the inequality $|\text{Tr}O| \leq \text{Tr}|O|$ valid for any trace-class operator $O$. 


3. Quantum fidelity between a pure state and a mixed state

In this section we focus our attention to the case of a mixed state $\rho$ and a pure state $\sigma = |\psi\rangle\langle\psi|$. In this specific case, we prove that the fidelity can be simply expressed as a limit formula involving the $s$-overlap.

Before stating this result, it is important to note that the Chernoff term (15) is defined in terms of an infimum over the open interval $(0, 1)$. In fact, despite the $s$-overlap $C_s(\rho, \sigma) \leq 1$ is correctly defined for any $s$ in the closed interval $[0, 1]$, the two border points $s = 0$ and $s = 1$ can be excluded from its minimization, since we always have

$$C_0 = \text{Tr}\langle\psi|\psi\rangle = 1, \quad C_1 = \text{Tr}\rho = 1. \quad (20)$$

Besides the restriction of the interval $[0, 1] \rightarrow (0, 1)$, it is also essential to consider an infimum instead of a minimum in (15). In fact, there are nontrivial situations where a minimum does not exist and an infimum is defined in the limit of $s \rightarrow 0^+$ or $s \rightarrow 1^-$. This is exactly what happens when one of the two states is pure. In this case the $s$-overlap $C_s$ tends to the quantum fidelity, which becomes equal to the Chernoff term. These are the main contents of the following results.

**Theorem 1.** Given a mixed state $\rho$ and a pure state $|\psi\rangle\langle\psi|$, their quantum fidelity can be expressed as

$$F(\rho, |\psi\rangle) = \lim_{s \to 1^-} C_s(\rho, |\psi\rangle) = \lim_{s \to 1^-} \text{Tr}(\rho^s|\psi\rangle\langle\psi|^{1-s}). \quad (21)$$

**Proof.** Specify the definition (14) to the case where $\sigma = |\psi\rangle\langle\psi|$, i.e.

$$C_s = \text{Tr}(\rho^s|\psi\rangle\langle\psi|^{1-s}). \quad (22)$$

For every $s \in (0, 1)$ we can use the property of the projector

$$|\psi\rangle\langle\psi|^{1-s} = |\psi\rangle\langle\psi|,$$

and write

$$C_s = |\psi\rangle\langle\psi|^s. \quad (24)$$

Now, we can always decompose $\rho$ as

$$\rho = z|\psi\rangle\langle\psi| + z_\perp \rho_\perp, \quad (25)$$

where $0 \leq z, z_\perp \leq 1$, and $\rho_\perp$ is a state whose support is orthogonal to $|\psi\rangle\langle\psi|$. For instance, we can consider

$$\rho_\perp = \sum_k p_k |\psi_k\rangle\langle\psi_k|,$$

with $\langle\psi_k|\psi\rangle = 0$ and $\langle\psi_k|\psi_k\rangle = \delta_{kk}$. In particular, from equation (25), we see that

$$z = \langle\psi|\rho|\psi\rangle = F(\rho, |\psi\rangle). \quad (27)$$

Taking the $s$-power of (25), we get

$$\rho^s = z^s|\psi\rangle\langle\psi| + z_\perp^s \rho_\perp^s,$$

for any $s \in (0, 1)$. Using the latter expression in (24), we achieve

$$C_s = z^s. \quad (29)$$

Finally, taking the limit of $s \to 1^-$ we get

$$\lim_{s \to 1^-} C_s = z = F(\rho, |\psi\rangle),$$

which corresponds to the result of equation (21). \qed
Note that we can equivalently write
\[ F(|\psi\rangle, \rho) = \lim_{s \to 0^+} C_s(|\psi\rangle, \rho) = \lim_{s \to 0^+} \text{Tr}(|\psi\rangle\langle\rho|^{1/s}). \] (31)
As an application of the previous theorem, we have the following corollary, which is a result already known in the literature (e.g., see [33, 36]).

**Corollary.** Given a mixed state \( \rho \) and a pure state \( |\psi\rangle\langle\psi| \), their quantum fidelity can be expressed as
\[ F(\rho, |\psi\rangle) = C(\rho, |\psi\rangle). \] (32)

**Proof.** This is a trivial consequence of the previous theorem. In fact, we can write
\[ C_s = z^s \]
where 
\[ z = F(\rho, |\psi\rangle). \] Since \( 0 \leq z \leq 1 \), we have that \( C_s \) is non-increasing in \( s \). As a consequence, we have
\[ C = \inf_{s \in (0,1)} C_s = \lim_{s \to 1^-} C_s = z = F(\rho, |\psi\rangle), \] (33)
which completes the proof. \( \square \)

The limit formula of theorem 1 is useful in all those scenarios where the \( s \)-overlap \( C_s \) is easy to compute. For instance, one of these scenarios is that of Gaussian states. As we show in the next section, we can derive a very simple formula for the fidelity between a pure and a mixed Gaussian state in terms of their statistical moments.

## 4. Formula for Gaussian states

In this section we apply the limit formula to the case of multimode Gaussian states. We first review some basic facts about bosonic systems, symplectic algebra and Gaussian states. Then, we introduce the notion of symplectic action, that we use to reformulate the expression of the \( s \)-overlap between two arbitrary Gaussian states. From this expression, we finally derive the formula for the fidelity between two multimode Gaussian states, in the case where one of the two states is pure.

### 4.1. Basic notions about Gaussian states

Let us consider a bosonic system of \( n \) modes. This quantum system is described by a tensor product Hilbert space \( \mathcal{H}^{\otimes n} \) and a vector of quadrature operators
\[ \hat{x}^T := (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n), \] (34)
satisfying the commutation relations
\[ [\hat{x}, \hat{x}^T] = 2i\Omega, \] (35)
where
\[ \Omega := \bigoplus_{i=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \] (36)
The matrix (36) defines a symplectic form in \( \mathbb{R}^{2n} \). Correspondingly, a real matrix \( S \) is called 'symplectic' when it preserves \( \Omega \) by congruence, i.e.
\[ S\Omega S^T = \Omega. \] (37)

8 With the notation \([\hat{x}, \hat{x}^T]\) we mean a matrix with entries \([\hat{x}_i, \hat{x}_j]\), where \( i, j = 1, \ldots, 2n \).
By definition a quantum state $\rho$ of a bosonic system is called ‘Gaussian’ when its phase-space representation is Gaussian [4]. In such a case, the quantum state is completely described by the first two statistical moments. Thus, a Gaussian state $\rho$ of $n$ bosonic modes is characterized by a displacement vector

$$\bar{x} := \text{Tr}(\hat{x}\rho),$$

and a covariance matrix (CM)

$$V := \frac{1}{2}\text{Tr}((\hat{x}, \hat{x}^T)\rho) - \bar{x}\bar{x}^T,$$

where $\{,\}$ denotes the anticommutator. According to the definition, a CM is a $2n \times 2n$ real and symmetric matrix. Furthermore, it must satisfy the uncertainty principle [37]

$$V + i\Omega \succeq 0.$$

A Gaussian state is pure if and only if its CM has unit determinant. In fact, one can easily prove that

$$\text{Tr} \rho^2 = \frac{1}{\sqrt{\det V}},$$

for a Gaussian state.

4.2. Symplectic action

An important tool in the study of Gaussian states is Williamson’s theorem [38], which assures the symplectic decomposition of a generic CM. In fact, for every CM $V$, there exists a symplectic matrix $S$ such that

$$V = SWS^T,$$

where

$$W = \bigoplus_{i=1}^{n} v_i I, \quad I := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The matrix $W$ is called the ‘Williamson form’ of $V$, and the set $\{v_i\} = \{v_1, \ldots, v_n\}$ is called the ‘symplectic spectrum’ of $V$. As a consequence of the uncertainty principle, each symplectic eigenvalue $v_i$ must be greater than or equal to the quantum shot-noise (here corresponding to 1). More precisely, the uncertainty principle (40) is equivalent to the conditions [4, 39]

$$V > 0, \quad v_i \geq 1 \quad \text{for any } i.$$

In particular, a Gaussian state is pure if and only if its symplectic spectrum is all equal to one ($v_i = 1$ for any $i$). In other words, for a pure Gaussian state, the Williamson form is equal to the identity. This is a direct consequence of equations (41) and (44) plus the fact that the determinant is a global symplectic invariant (i.e. $\det V = \det W$).

Now, consider a real function $f : \mathbb{R} \to \mathbb{R}$ and generic CM $V$ with symplectic decomposition

$$V = S \left[ \bigoplus_{i=1}^{n} v_i I \right] S^T.$$

Then, we define the ‘symplectic action’ $f(V)_s$ of $f$ over $V$ the following matrix:

$$f(V)_s = S \left[ \bigoplus_{i=1}^{n} f(v_i) I \right] S^T.$$

9 Similarly to before, with the notation $\{\hat{x}, \hat{x}^T\}$ we mean a matrix with entries $\{x_i, x_j\}$.
Since the symplectic decomposition is unique (up to uninfluential local rotations), the output matrix \( f(V) \), is unambiguously defined. In particular, this matrix is a CM if and only if \( f(\nu_i) \geq 1 \) for every \( i \). It is also clear that \( f(SVS^T)_s = SF(V)_sS^T \) for every CM \( V \) and symplectic matrix \( S \).

It is important to note that this operation is different from the standard notion of function of a matrix \( f(V) \), where \( f \) is applied to the standard eigenvalues of the spectral decomposition of \( V \). We have \( f(V)_s = f(V) \) only if spectral and symplectic decompositions coincide, which happens when the symplectic matrix \( S \) is a proper rotation (so that \( S^T = S^{-1} \)). In general, the symplectic action is a useful tool which enables us to simplify the formalism in the manipulation of the CMs.

4.3. From the s-overlap to the quantum fidelity

According to \([35]\), we can write a closed formula for the \( s \)-overlap between two arbitrary multimode Gaussian states. Here we briefly review this formula by adopting the formalism of the symplectic action.

First of all, let us define the two real functions
\[
G_p(x) := \frac{2^p}{(x + 1)^p - (x - 1)^p},
\]
and
\[
\Lambda_p(x) := \frac{(x + 1)^p + (x - 1)^p}{(x + 1)^p - (x - 1)^p},
\]
which are finite and non-negative for every \( x \geq 1 \) and \( p > 0 \). Using these functions, we can easily express the \( s \)-overlap between two arbitrary \( n \)-mode Gaussian states, \( \rho_0 \) and \( \rho_1 \), with statistical moments \( \{\bar{x}_0, V_0\} \) and \( \{\bar{x}_1, V_1\} \), and associated symplectic spectra \( \{\nu_0^j\} \) and \( \{\nu_1^j\} \).

In fact, for any \( 0 < s < 1 \), their \( s \)-overlap is given by \([35]\)
\[
C_s(\rho_0, \rho_1) = \Pi_s(\det \Sigma_s)^{-1/2} \exp \left( -\frac{d^T \Sigma_s^{-1} d}{2} \right),
\]
where \( d := \bar{x}_0 - \bar{x}_1, \) \( \Sigma_s := \Lambda_s(V_0)_s + \Lambda_{1-s}(V_1)_s, \)
and
\[
\Pi_s := 2^p \prod_{i=1}^n G_s(\nu_0^i)G_{1-s}(\nu_1^i).
\]

Note that the symplectic action intervenes in equation (50). Explicitly, we have
\[
V_0 = S_0 \left[ \bigoplus_{i=1}^n \nu_0^i I \right] S_0^T \to \Lambda_s(V_0)_s = S_0 \left[ \bigoplus_{i=1}^n \Lambda_s(\nu_0^i) I \right] S_0^T,
\]
and
\[
V_1 = S_1 \left[ \bigoplus_{i=1}^n \nu_1^i I \right] S_1^T \to \Lambda_{1-s}(V_1)_s = S_1 \left[ \bigoplus_{i=1}^n \Lambda_{1-s}(\nu_1^i) I \right] S_1^T.
\]

The formula of the \( s \)-overlap can be greatly simplified in the presence of pure Gaussian states, on which the two functions \( \Lambda_s \) and \( G_p \) have a trivial action. In fact, suppose that a Gaussian state \( \rho \) is pure. This means that its symplectic spectrum is all equal to one, i.e. \( \nu_i = 1 \) for any \( i \). In other words, its CM has symplectic decomposition
\[
V = S \left[ \bigoplus_{i=1}^n I \right] S^T,
\]
where the Williamson form corresponds to the $n$-mode identity matrix. Then, for every $p > 0$, we have

$$\Lambda_p(V)_s = V,$$

i.e. the symplectic action of $\Lambda_p$ does not change pure CMs. In fact, explicitly we have

$$\Lambda_p(V)_s = S \left[ \bigoplus_{i=1}^{n} \Lambda_p(1)I \right] S^T = S \left[ \bigoplus_{i=1}^{n} I \right] S^T = V,$$

where we use the fact that $\Lambda_p(1) = 1$ for any $p > 0$. Also the computation of $G_p$ becomes trivial. In fact, for any $p > 0$ we have

$$G_p(V_0) = G_p(1) = 1.$$

Coming back to the formula (49), if one of the two Gaussian states is pure, e.g., $\rho_1 = \ket{\varphi_1}\bra{\varphi_1}$, then we have the simplifications

$$\Pi_s = 2^n \prod_{i=1}^{n} G_x(v_0^i),$$

and

$$\Sigma_s = \Lambda_s(V_0) + V_1,$$

for every $s \in (0, 1)$. Now, by taking the limit of $s \to 1^-$, we can derive the formula for Gaussian states.

**Theorem 2.** Let us consider two $n$-mode Gaussian states, $\rho_0$ and $\rho_1$, where $\rho_0$ is generally mixed (with moments $\tilde{x}_0$ and $V_0$) and $\rho_1 = \ket{\varphi_1}\bra{\varphi_1}$ is pure (with moments $\bar{x}_0$ and $V_1$). Their fidelity $F = F(\rho_0, |\varphi_1\rangle)$ can be computed via the formula

$$F = \frac{2^n}{\sqrt{\det(V_0 + V_1)}} \exp \left[ -\frac{d^T(V_0 + V_1)^{-1}d}{2} \right],$$

where $d := \bar{x}_0 - \tilde{x}_1$.

**Proof.** This proof simply combines the limit formula of the fidelity (21) with the analytical formula of $C_s$ for Gaussian states. Given two Gaussian states $\rho_0$ and $\rho_1 = \ket{\varphi_1}\bra{\varphi_1}$, their $s$-overlap $C_s(\rho_0, |\varphi_1\rangle)$ is expressed by (49) with the simplified terms (58) and (59), where $\{v_0^i\}$ is the symplectic spectrum of $\rho_0$. Now taking the limit of $s \to 1^-$ in $C_s(\rho_0, |\varphi_1\rangle)$ is equivalent to taking the limit of $s \to 1^-$ in the terms $\Pi_s$ and $\Sigma_s$ of equations (58) and (59).

Since $G_x$ and $\Lambda_s$ are continuous at $s = 1$, we have

$$\lim_{s \to 1^{-}} G_x(s) = G_1(x) = 1,$$

$$\lim_{s \to 1^{-}} \Lambda_s(s) = \Lambda_1(x) = x,$$

for every $x \geq 1$. In particular, for every CM $V$, we can write

$$\lim_{s \to 1^{-}} \Lambda_s(V)_s = \Lambda_1(V)_s = V.$$

By applying these properties to the state $\rho_0$, we get

$$\lim_{s \to 1^{-}} G_x(v_0^0) = 1, \lim_{s \to 1^{-}} G_x(V_0) = V_0.$$

As a consequence, we can write

$$\lim_{s \to 1^{-}} \Pi_s = 2^n, \lim_{s \to 1^{-}} \Sigma_s = V_0 + V_1.$$

Now, using equations (65) and (49), we get

$$\lim_{s \to 1^{-}} C_s(\rho_0, |\varphi_1\rangle) = \frac{2^n}{\sqrt{\det(V_0 + V_1)}} e^{-\frac{d^T(V_0 + V_1)^{-1}d}{2}},$$

which provides the result of the theorem. \qed
5. Conclusion and discussion

In conclusion, we have provided a limit formula for computing the quantum fidelity between a mixed and a pure state. This formula involves a generalized $s$-overlap between the two quantum states, a quantity used in the definition of the quantum Chernoff bound. As an application of the formula, we have considered the case of Gaussian states, for which we have derived a simple expression in terms of their first- and second-order statistical moments.

An alternative formula for the computation of the quantum fidelity can be useful in many scenarios, including protocols of quantum teleportation [6–15], entanglement swapping [40–42], and quantum cloning [16–20]. Clearly, other important areas of application are quantum state discrimination (i.e. quantum hypothesis testing) and quantum channel discrimination, where the latter includes practical problems such as the quantum illumination of targets [43, 44] and the quantum reading of classical digital memories [45–51].

Specifically, our formula for Gaussian states can be applied to any scenario where a multimode Gaussian state must be distinguished from a pure multimode Gaussian state. This situation naturally arises in the contexts of bosonic state discrimination and bosonic channel discrimination [4, 52]. For instance, consider a multi-copy pure Gaussian state at the input of a black box which contains either the identity channel $I$ or a Gaussian channel $G$, with the same probability. The aim of a decoder is to discriminate between $I$ and $G$ measuring the two possible output states from the box. Applying our formula of the fidelity to the output Gaussian states, we can easily estimate the error probability affecting this kind of channel discrimination. An explicit example of this application can be found in the recent model of locally constrained quantum reading [53], where the black box to be decoded corresponds to the cell of an optical memory.

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