Fermionic $\theta$ Vacua and Long-Necked Remnants

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ABSTRACT

We study a vacuum polarization effect in the background of a certain dilatonic extremal black hole, known as the cornucopion. Whenever charged fermions are of any nonzero mass, the gravitational backreaction to a generic value of a $CP$ nonconserving vacuum angle $\theta$ is shown to be important owing to a vacuum energy density which does not vanish deep inside the cornucopion. When the vacuum energy density is positive, this effect creates an extremal horizon at finite physical distance, closing off the infinite neck. We study the geometry of this horizon in some detail and find different physical interpretations for small and large fermion mass. Also, we argue that the conclusion is qualitatively correct despite the inevitable strong coupling.

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1. Motivation

It is well-known that a magnetic monopole carries fractional electric charge in the presence of a \( CP \) nonconserving angle \( \theta \). For a spontaneously broken Yang-Mills theory (with monopoles as solitons), this angle parametrizes a continuum of vacua, known as \( \theta \) vacua, the effect of which is naturally incorporated by including the Pontryagin density multiplied by \( \theta \) in the Lagrangian. With such a setup, the origin of the fractional charge becomes quite clear. The Pontryagin density is essentially a product of electric and magnetic fields, through which a classical magnetic field acts as a source to the fluctuating electric field. As a result a monopole carries a long range electric field proportional to its magnetic field, and the corresponding electric charge must be proportional to \( \theta \). Allowing higher excitations, we arrive at the following Witten’s quantization rule for unit magnetic monopoles.

\[
q = N - \frac{\theta}{2\pi}, \quad N \text{ is any integer.} \tag{1}
\]

As usual, the quantization rule tells us nothing about how the electric charge should be realized in such dyons, which must depend on many details of the theory. One example where we can address this question of dyon core structure is a monopole coupled to charged fermions. The lowest partial waves of such fermions, known to experience no potential barrier, can be used to study the static dyonic core structure.

As demonstrated by Callan, the vacuum fluctuation of the charged fermion field tries to shield the core from the radial electric field and, as a result, the fractional electric charge is effectively realized as a vacuum polarization cloud of size \( \sim 1/m_\psi \) around the monopole, where \( m_\psi \) is the mass of the fermion. In particular, for small fermion mass, the electric charge distribution is concentrated on a thick and large shell of radius \( \sim 1/m_\psi \) and the small magnetic core, being shielded from the extra radial electric field, is found to be essentially intact.

Witten’s charge quantization is a topological statement, and the same fractional charge must appear also for magnetic black holes. On the other hand, when the magnetic black hole is much smaller than the fermion lengthscale \( \sim 1/m_\psi \), we may also expect to find similar dilute clouds of vacuum polarization shielding the black hole from the extra radial electric
field so that its geometry near the horizon is that of the pure magnetic black hole.

As far as light fermions are concerned, the leading effect in such a background is from the long-range magnetic field, and once the “core” region is shielded by the resulting dilute cloud, one may argue, there is little to which gravity reacts. While the horizon can affect the dynamics of the fermion fields nearby, this seems to matter only when the fermion fields are massive enough for the charged cloud to approach the event horizon.

However, there is a curious species of extremal magnetic black holes, known as cornucopia, whose “size” is both small and infinite simultaneously. Instead of a compact core, the cornucopion has an infinitely long neck with the transverse radius proportional to the total magnetic flux, as illustrated in Figure 1. It takes literally infinite proper time just to reach the event horizon at the bottom of the neck, let alone to cross it, while the transverse size of the neck can be arbitrarily small.

![Figure 1: Schematic diagrams for (a) a magnetic monopole in a flat space-time, and for (b) a cornucopion in an asymptotically flat space-time. Magnetic flux emanating from the central region is denoted by the arrows. A cornucopion, which is an extremally charged dilatonic black hole, has an infinitely long neck of fixed transverse radius threaded by the magnetic flux.](image-url)
Then we may ask how the vacuum polarization behaves in such an exotic background. Should one expect to find the narrow and infinite neck surrounded by a large and dilute charged cloud of vacuum fluctuation, provided that the fermion mass is small enough? After all, far away from the black hole, there is little indication as to the existence of the infinite neck. In this paper, we want to address this specific question, by studying a bosonized effective action for the S-wave charged fermions coupled to the dilaton gravity.

After the derivation of the effective action for a general spherically symmetric background in section 2, we shall return to the specific case of the cornucopion. The surprising result of section 3 is that the energy cost of the vacuum polarization, which should be balanced against the gain in the electrostatic energy, is actually divergent in such a noncompact background. The inevitable conclusion thereof is that the gravitational backreaction to this vacuum polarization process is never negligible and, for whatever $m_\psi \neq 0$ is, must have the characteristic long neck terminated, by creating an extremal horizon at finite physical distance.

In section 4, we study the effective action of S-wave fermions combined with the dilaton gravity in four dimension to investigate the self-consistent geometries with finite ADM masses. We find a useful and practical way of studying the solutions near the extremal horizon formed by the gravitational backreaction, and, using this method, we clarify the different roles the fractional electric charge plays in large and small fermion mass limit. In particular, we find the expected behaviour in the large fermion mass limit, where most of the fractional charge must be trapped by the black hole’s gravitational pull. We conclude by discussing the generality of these results.

2. Callan-Rubakov Modes in a Magnetic Black Hole Background

To be definite, let us consider a static dilatonic black hole solution with magnetic charge [10], written down in terms of the tortoise coordinate $z$.

$$g = \lambda^2(z)dt^2 - \lambda^2(z)dz^2 - R^2(z)d\Omega^2, \quad e^{-2\phi} = e^{-2\phi(z)}. \quad (2)$$

Asymptotically $R \approx z \to \infty$, while the event horizon is at $z = -\infty$ where the geometry is largely determined by the behaviour of $\lambda$. For a cornucopion which is a purely magnetic
black hole, we can take \( \lambda \equiv 1 \). For black holes with an extremal horizon at finite physical distance, such as those with both electric and magnetic charges inside the event horizon, \( \lambda^2 \sim 1/z^2 \) as we approach the horizon. Finally, \( \phi \) is the dilaton field and \( e^\phi \) plays the role of the coupling. Since we want to couple the matter system to gravity later on, we shall keep \( \lambda, \, R \) and \( \phi \) unspecified for a while. As long as the solution is static and spherically symmetric, the detailed form of it does not enter the derivation in this section.

Now depending on the origin of the magnetic charge, we can introduce different kinds of charged fermions. The simplest case would be a Dirac fermion coupled to the \( U(1) \) gauge field. However, we found it advantageous to work with a spontaneously broken \( SU(2) \) theory so that the fermions are in the fundamental representation of \( SU(2) \) and that the magnetic charge is realized as a topological quantum number of solitons. The background gauge field outside the horizon is still Abelian except that the Abelian \( U(1) \) generator \( Q \) is expressed in terms of the \( SU(2) \) generators \( T_a \).

\[
Q \equiv T_a n^a, \quad \vec{n} \text{ is the unit radial vector field.}
\]

With this choice, the derivation of the effective matter action is quite similar to that of Callan, up to the conventions regarding the spinor and the modifications due to the nontrivial geometry. We shall compare our results to those of Callan whenever appropriate.

Consider a \( SU(2) \) doublet Dirac fermion \( \psi \) with nonzero mass \( m_\psi \). As mentioned in the previous section, it is sufficient to focus on the lowest partial waves of the fermion, called Callan-Rubakov modes, which do not see any potential barrier of the geometry or of the spherically symmetric gauge field. To isolate such modes, we use the following ansatz for \( \psi_\pm \), positive and negative chiral eigenstates of \( \psi \) written in terms of Weyl 2-spinors.

\[
\psi_+ = \frac{1}{\sqrt{4\pi\lambda R}} \chi_+(t, z), \quad \psi_- = \frac{1}{\sqrt{4\pi\lambda R}} \gamma^t \chi_-(t, z).
\]

The upper (lower) component of the two spinor \( \chi_- (\chi_+) \) has charge \( 1/2 \) while the other has \(-1/2\), with respect to the unbroken \( U(1) \) generator \( Q \). We chose \( \gamma^t = \sigma_x \) and \( \gamma^z = i\sigma_y \) as our two-dimensional Dirac matrices. With this ansatz, the fermion action can be reduced to
the following two-dimensional form.

\[ S_\chi = \int dt dz (\overline{\chi}_+ \gamma^i \partial_i \chi_+ + i \overline{\chi}_- \gamma^i \partial_i \chi_- + m_\psi \lambda (\overline{\chi}_+ \chi_+ + \overline{\chi}_- \chi_-)) + \int dt dz 4\pi \lambda^2 R^2 (a_t J_Q^t + a_z J_Q^z), \]

(4)

where \( \tilde{a} \) is the fluctuating part of the radial \( U(1) \) gauge field. We can set \( a_t \) equal to zero using the gauge degree of freedom and then the radial electric field in \((t,z)\) coordinates is simply \( E \equiv \partial_t a_z \). The relevant currents are,

\[ J_Q^t = \frac{1}{8\pi \lambda^2 R^2} (J_+^z - J_-^z), \quad J_Q^z = \frac{1}{8\pi \lambda^2 R^2} (-J_+^t + J_-^t), \]

(5)

where \( J_\pm \) are the two-dimensional vector currents of \( \chi_\pm \). The effective action \( S_\chi \) above is incomplete since we neglected the action for the fluctuating electric field \( E \) so far. By isolating it from the full Yang-Mills action and integrating the angular part, we find

\[ S_E = \int dt dz \left( \frac{\theta}{2\pi} E + \frac{R^2}{2\lambda^2 e^{2\phi}} E^2 \right). \]

(6)

The \( \theta \) term is from the Pontryagin density and can be deduced from the fact that a unit magnetic monopole carries total magnetic flux \( 4\pi \).

This effective action \( S_\chi + S_E \) is different from that of Callan \[4\] in two respects. First, the effective couplings are changed due to the nontrivial geometry and the nonuniform coupling \( e^\phi \). In particular, the fermion mass term acquires a factor of \( \lambda \). Second, the radial coordinate \( z \) extends from \( \infty \) to all the way to \( -\infty \). Because of this, we no longer need to impose a boundary condition at the origin. In fact, it is effectively a theory of fermions in flat 1+1 Minkowski space-time, coupled to a \( U(1) \) field through an axial current, but with position-dependent mass and coupling.

Since the \( U(1) \) field is coupled to a two-dimensional axial current\[3\], we can bosonize \( \chi_\pm \) through the following fundamental relationships between currents \[1\[2\], preserving the \( U(1) \) current automatically,

\[ \tilde{J}_\pm = -\frac{1}{\sqrt{\pi}} \tilde{\phi} j_\pm, \quad \text{with respect to the flat metric } dt^2 - dz^2. \]

(7)

\[ ^3\text{The two spinors } \chi_\pm \text{ are of opposite charges, so that the } U(1) \text{ gauge symmetry is not anomalous.} \]
Furthurmore, to separate the charged and the uncharged sectors, it is convenient to perform a canonical transformation generated from

\[ f \equiv (f_+ - f_-)/\sqrt{2}, \quad \eta \equiv \eta(-\infty) + \int_{-\infty}^z (\dot{f}_+ + \dot{f}_-)/\sqrt{2}. \]  

(8)

Once this is done, we can simply eliminate the electric field strength \( E \) through its equation of motion and express the effective action completely in terms of bosonic fields \( f \) and \( \eta \). If we define \( \mu \) to be the geometric mean of the normal ordering scales \( \mu_f \) and \( \mu_\eta \) for each field, the effective action \( S_\chi + S_E \) is transformed into

\[
\int dt dz \left\{ \frac{1}{2} (\partial f)^2 + \frac{1}{2} (\partial \eta)^2 - \frac{e^{2\phi} \lambda^2}{4\pi R^2} \left( f - \frac{\theta}{\sqrt{2\pi}} \right)^2 + cm_\psi \mu \lambda \cos \sqrt{2\pi} f \cos \sqrt{2\pi} \eta \right\}.
\]  

(9)

The constant \( c \) is a number of order 1 and shall be kept unspecified. The electric charge density of fermions is now simply proportional to the spatial derivative of \( f \) and we find the total charge inside a radial coordinate \( z \) to be

\[
q(z) = \int dz \left( 4\pi \lambda R^2 J_0^2 \right) = \frac{1}{\sqrt{2\pi}} (f - \frac{\theta}{\sqrt{2\pi}})|_z,
\]  

(10)

where we fixed the integration constant by inspecting the electric energy term in the effective action above. Similarly, the fermion number inside \( z \) is, up to an additive constant, given by \( \eta \sqrt{2/\pi} \) evaluated at \( z \).

Coupling to gravity requires further considerations. First, the effective action above is not manifestly covariant. The actual two-dimensional metric \( g^{(2)} \equiv \lambda^2 (dt^2 - dz^2) \) has a conformal factor \( \lambda^2 \) and the only way to reconcile this with the present form of the action is to choose the normal ordering scales to be proportional to \( \lambda \). On the other hand, the only two physical mass parameters of the effective theory are \( e^{\phi} \lambda / R \) and \( m_\psi \lambda \), both of which are proportional to \( \lambda \). Hence it is appropriate to replace \( \mu \) by \( \lambda \mu \), where the specific choice of \( \mu \) should not in principle affect the physics. But in practice, we will study the effective action at tree level only, which requires a judicious choice of the ordering scales. For example, we can take

\[
\mu^2 = \lambda^2 \mu^2 \equiv m_f m_\eta, \quad m_f, m_\eta \text{ are classical masses of } f \text{ and } \eta, \text{ in } (t, z) \text{ coordinate},
\]
automatically ensuring the covariance of the effective action. Secondly, there is the matter of zero-point energy, which is irrelevant before gravity is turned on. To ensure the existence of the cornucopion for trivial values of the vacuum angle $\theta$, it is necessary to have vanishing vacuum energy density, whenever $\cos \theta = 1$, both at the asymptotic infinity and at the other asymptotic region deep inside the cornucopion. This can be achieved by adjusting the fermion mass term to have minimum at zero rather than at $-m_{\psi}\mu \lambda$. The resulting effective action $S_{\text{eff}}$ in an arbitrary coordinate system is

\begin{equation}
\int dx^2 \sqrt{-g(2)} \left\{ \frac{1}{2} \left( \nabla f \right)^2 + \frac{1}{2} \left( \nabla \eta \right)^2 - \frac{e^{2\phi}}{4\pi R^2} \left( f - \frac{\theta}{\sqrt{2\pi}} \right)^2 - cm_{\psi}\mu \left( 1 - \cos \sqrt{2\pi} f \cos \sqrt{2\pi} \eta \right) \right\}.
\end{equation}

(11)

3. Vacuum Energy Distribution and the Gravitational Backreaction

To recover the charge quantization rule, it is sufficient to study the effective potential at spatial infinity. In the asymptotic region, the potential is dominated by the fermionic mass term, whose minima occur at $f = N\sqrt{\pi}/2$ with $N$ even or odd depending on the asymptotic value of $\eta$. Therefore,

\[ q_{\text{total}} = q(z = \infty) = \frac{N}{2} - \frac{\theta}{2\pi}, \quad N \text{ is any integer}. \quad (12) \]

Obviously the dynamical fermion $\psi$ is responsible for the new half-integral part, and odd $N$ must correspond to an odd number of fermions.\footnote{The fermion number $n_{\psi}$ is not conserved in the presence of a black hole. However, the odd and the even fermion numbers require different charge quantizations, and $n_{\psi}$ modulo-two is a good quantum number. It is easy to see that the fermion number $n_{\psi}$ modulo-two is given by $\eta\sqrt{2/\pi}$ evaluated at spatial infinity.} It is not surprising to find the same results as Callan did, since the asymptotic form of the effective theory is insensitive to whether the gravity is turned on or not.

However, the vacuum polarization effect as we approach the black hole can be very different from the case of a nonsingular monopole in a flat space-time. Specifically, we want to concentrate on the fermionic ground state in the background of the cornucopion, a purely magnetic extremal black hole. Suppose we want to find a parameter region where the non-compact core geometry of the cornucopion serves as the zero-th order approximation. Such a configuration would correspond to a narrow and infinite neck of cornucopion surrounded
by harmless and dilute charged cloud of the vacuum polarization, just as the nonsingular monopole core is, according to Callan [4], surrounded by a harmless and dilute charged cloud of the vacuum polarization.

A quick look at the effective potential convinces us that this is not possible unless the fermion mass is actually zero. Deep inside the neck, the electric energy term is dominant so that $\sqrt{2\pi f}$ approaches $\theta$, and as a result the ground state energy density behaves like

$$V_{\text{min}} \simeq cm_\psi \sqrt{1 - |\cos \theta|},$$

Since the total vacuum energy inside the throat region is given by integrating $V_{\text{min}}$ along the infinite neck, the ground state built on this background comes with infinite vacuum energy distributed along the infinite neck.

Actually a similar phenomenon occurs for monopoles in a flat space-time, contributing a vacuum energy which scales like $m_\psi^2 L_c$, where $L_c$ is the distance between the charged shell and the monopole center. On the other hand, the resulting electric charge distribution carries electrostatic energy which scales like $1/R_c$, where $4\pi R_c^2$ is the area of the charged shell. In a flat space-time, $L_c \simeq R_c$ and the balance between the two contributions fixes the order of magnitude of $L_c \simeq R_c$ at $1/m_\psi$, as mentioned earlier.

In a curved black hole geometry, however, $L_c$ is now some measure of the physical distance between the charged shell and the event horizon, which needs not be proportional to $R_c$ (the linear size of the charged shell) any more. With a cornucopion, in particular, $L_c \to \infty$ while $R_c$ remains finite. It is not possible to achieve a balance between the two, and it is necessary to consider the gravitational backreaction to the vacuum polarization, to understand the true nature of the fermionic $\theta$-vacua.

To understand the gravitational backreaction, let us digress a little bit and recall the energetics of the pure cornucopion solution. The infinite neck of a cornucopion is threaded by a constant flux of classical magnetic fields. While one would normally expect a uniform magnetostatic energy density associated with the flux, the energy density actually vanishes.

$^5$While it is conceivable that some other $\theta$-dependent effects may cancel $V_{\text{min}}$, such a cancellation, if possible, could occur only for very special values of parameters in the theory. The $\theta$-independent part is fixed in the previous section by assuming that the purely magnetic configuration is given by the cornucopion.
exponentially deep down the neck. The reason is simply that the electric coupling $e^\phi$, inverse square of which appears in the magnetostatic energy density, is exponentially growing. Note that, in Einstein-Maxwell theory where the coupling is really a constant, an extremal horizon forms, hiding whatever divergent behaviour the energy-momentum may have.

In a sense, the purely magnetic cornucopion of finite ADM mass exists precisely because the divergent coupling prevents the magnetic energy-momentum from accumulating divergently. However, once we turn on the Callan-Rubakov modes with a generic $\theta$, the energy-momentum given by $V_{min}$ eventually dominates and does accumulate divergently deep inside the neck. On the other hand, since the energy-momentum far outside is completely determined by the total magnetic charge and the fractional electric charge, the ADM mass must be finite regardless of the vacuum polarization.

Now it is clear what must happen. The gravitational backreaction to the accumulated effect of $V_{min}$ must eventually create a horizon somewhere down the would-be cornucopion, rendering $L_c$, thus the vacuum energy contribution to the ADM mass, finite. Hence, the infinite neck must be terminated by a zero-temperature horizon at finite physical distance. In such a self-consistent background, one should be able to find the true vacuum state of the fermion sea.

In fact, one can explicitly check this for small $m_\psi$. In this limit, the geometry near the throat region remains unchanged since the energy-momentum there is dominated by the magnetic flux, and the long-neck structure survives until the point where $V_{min}$ is comparable to the magnetostatic energy density. Then as we travel down the would-be cornucopion, the dynamics effectively reduce to that of a 2-D dilatonic gravity. The relevant 2-D action can be easily obtained by dimensionally reducing the complete action (13) to appear in the next section.

$$\int dx^2 \sqrt{-g^{(2)}} \left\{ e^{-2\phi} \left( -R^{(2)} - 4 (\nabla \phi)^2 + \frac{1}{2\kappa^2} \right) - 2V_{min} \right\}$$

For reasonable choices of $\pi$, the static solutions of this action can be easily shown to possess two horizons generically, and the extremal limit thereof corresponds to a zero-temperature horizon at finite physical distance terminating the long neck.

### 4. Self-Consistent Geometries and the Fractional Charge
We concluded above that, for generic $\theta$, the gravitational backreaction to the vacuum energy distribution is always important and that there exists an extremal horizon stopping indefinite growth of the would-be cornucopion.

In the discussion above however, the fractional electric charge itself does not seem to play a role as far as the core geometry is concerned. After all, not only is the charge cloud too large to approach even the throat region, but it is known that any electric charge faces an exponentially divergent potential barrier as it travels down a cornucopion, owing to the electromagnetic backreaction. The electric flux tube attached to the charge costs more and more energy, proportional to the exponentially divergent coupling squared $e^{2\phi}$, and this tends to push away any electric charge.

On the other hand, the gravitational backreaction renders this potential barrier finite, since the coupling cannot be infinite at the regular extremal horizon, and at least part of the fractional charge should be expected to be trapped inside the horizon. This last observation raises a question whether one can explain the newly-formed extremal horizon entirely in terms of this trapped electric charge.

There are known clean dyonic black hole solutions of the dilaton gravity coupled to Maxwell fields, and their extremal limit comes with an extremal horizon. Maybe, the vacuum energy density found above seemed so prominent only because we were using a wrong background. It is a logical possibility that the self-consistent geometry near the extremal horizon is dictated by the charges.

In fact, this is precisely what must happen in the large $m_\psi$ limit. As $m_\psi$ increase, the density of the charge cloud as well as $V_{\text{min}}$ must increase accordingly. The gravitational backreaction to $V_{\text{min}}$ creates the horizon more and more close to the fractional charge cloud which by now is itself dense enough to distort the geometry. With the increasingly weak potential barrier, the increasingly massive lump of the fractional charge will eventually fall into the black hole and the effect of $V_{\text{min}}$ will disappear behind the horizon. Once this happens, an observer outside would be completely oblivious of the vacuum energy distribution and

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In ref [14], electric and magnetic charges belong to different $U(1)$'s, but as far as the static and spherically symmetric geometry is concerned, it does not matter. One should be careful to distinguish these solutions from another known family of solutions with an axionic field [13].
attribute the termination of the would-be cornucopion to the fact that the extremal horizon hides both electric and magnetic charges.

In this section we would like to investigate this possibility in both the small and large fermion mass limits, with the latter serving as a consistency check. The total action dictating the self-consistent geometry is given by the following.\footnote{In this paper, $\kappa^{-2}$ is the gravitational constant, while $c = \hbar = 1$.} 

\[
S = S_{\text{eff}} - \frac{1}{16\pi\kappa^2} \int dx^4 \sqrt{-g^{(4)}} e^{-2\phi} (R + 4 (\nabla\phi)^2 + \kappa^2 F^2). \tag{13}
\]

For the purpose of studying the properties of static and spherically symmetric solutions at the extremal horizon, which will turn out to be very informative, we can reduce the field equations to a set of algebraic ones involving various physical quantities at the horizon. The key to this simplification is the regularity of the horizon.\footnote{In fact, possible mild singularities at the extremal horizon such as observed by Trivedi\cite{12} in semiclassical extremal black holes do not interfere with this derivation.}

If a function of the radial coordinate only is finite and differentiable at an extremal horizon with respect to a local geodesic coordinate, some of its covariant derivatives vanishes there just because $\lambda$ vanish at the horizon. If we denote the evaluation at the extremal event horizon by the subscript $h$,

\[(\nabla^2 f)_h = (\nabla f)_h^2 = 0, \quad \text{the same for } \phi, R, \text{ and } \eta.\]

As a result, only terms without any derivative of $f, \eta, \phi, R$ survive the evaluation of the static field equations at the extremal horizon. For instance, combining the dilaton equation and an angular Einstein equation, we can easily deduce that $R_h^2 = 2\kappa^2$, showing that the transverse size of the neck remains unchanged. On the other hand, the equation for $\eta$ tells us $\cos \sqrt{2\pi} \eta_h = \pm 1$. From some of the remaining equations, we find two algebraic equations for $e^{\phi_h}$ and $f_h$.

\[
e^{2\phi_h} \frac{1}{4\pi\kappa^2} (f_h - \frac{\theta}{\sqrt{2\pi}}) \pm \sqrt{2\pi} c m_{\psi} \eta_h \sin \sqrt{2\pi} f_h = 0 \tag{14}
\]

\[
e^{2\phi_h} \frac{1}{8\pi\kappa^2} (f_h - \frac{\theta}{\sqrt{2\pi}})^2 + c m_{\psi} \eta_h (1 \mp \cos \sqrt{2\pi} f_h) = e^{-2\phi_h} \frac{1}{4\kappa^2} \tag{15}
\]

Now let us consider two limiting cases as promised, to unravel the role of the fractional electric charge in the formation of the extremal horizon.
When $\kappa m_\psi >> 1$, the first equation (14) tells us that the value $(\sin \sqrt{2\pi} f_h)$ is very small and that, for the lowest energy configuration, the left-hand-side of (15) is dominated by the electric energy term $\sim (f_h - \theta/\sqrt{2\pi})^2$. Then, we find the following relation between the coupling and the trapped charge $q_h$ except for integral $\theta/\pi$, when the argument above breaks down.

$$e^{-4\phi_h} = \frac{1}{2\pi}(f_h - \frac{\theta}{\sqrt{2\pi}})^2 + O(\frac{1}{c \kappa^2 m_\psi^2}) = q_h^2 + \cdots$$

(16)

This is a nontrivial and significant piece of information, in that this is exactly what one would expect to be true if the horizon geometry is completely determined by the electromagnetic charges trapped inside. It is a matter of straightforward algebra to show that the clean dyonic black holes of [14] satisfies $(q/p)^2 = e^{-4\phi_h}$ with electric and magnetic charges given by $q$ and $p$. Furthermore, small $(\sin \sqrt{2\pi} f_h)$ implies that most of the fractional charge is inside the black hole, confirming the assertions earlier in this section.

Finally, coming to the small fermion mass limit $\kappa m_\psi << 1$, we can easily see that the first equation (14) now predicts very small $q_h \sim (f_h - \theta/\sqrt{2\pi})$. Because of this, the left-hand-side of (15) is now dominated by the fermion mass term $\sim m_\psi \mu$, and this in turn implies the following characteristic of the self-consistent geometry in the small fermion mass limit.

$$|q_h| \sim e^{-4\phi_h}, \quad \text{rather than } q_h^2 \approx e^{-4\phi_h}.$$  

(17)

The implication is clear when compared to (16). Now the leading energy-momentum contribution closing off the infinite neck of the would-be cornucopion (making $e^{-2\phi}$ nonzero) is generated by $V_{\text{min}}$ rather than by the trapped electric charge. Very small amounts of electric charge $q_h \sim e^{-4\phi_h}$ (not $\sim e^{-2\phi_h}$) are trapped by the newly-formed extremal horizon only as a secondary effect. The presence of the vacuum energy distribution $V_{\text{min}}$ is very real unlike the previous case of large fermion limit.

A couple of remarks are in order. The key formulae (16) and (17) are derived without detailed knowledge of $\mu$. All we needed was rough characteristics of it in each limit, such as $\mu_h/m_\psi << e^{2\phi_h}$ for small $m_\psi$ and $\mu \sim m_\psi$ for large $m_\psi$. This is an important point because physics should not depend on the choice of the normal ordering scale, and indeed
we managed to isolate such a $\mu$-independent characterization of the self-consistent geometry in the form of these key formulae. Another fact we want to mention is the equations (14) and (15) above show the expected behaviour as $m_\psi \to 0$. Though the precise behaviour does depend on $\mu$, the value $e^{-2\phi h}$ can be shown to vanish rapidly as $\kappa m_\psi$ approaches zero, corresponding to a longer and longer neck. Eventually when $m_\psi$ is identically zero, the limiting self-consistent geometry is that of a cornucopion, as it should be.

5. Discussion

To summarize, we found that fermionic $\theta$ vacua tends to terminate the infinite neck of the cornucopion. This fact, by itself, should not be surprising, since a nontrivial $\theta$ implies the existence of the (fractional) electric charge which, if swallowed by the black hole, produces a clean dyonic black hole with an extremal horizon. In fact, this is exactly what happens when $\theta/\pi$ is non-integral and the fermion mass is sufficiently large. On the other hand, somewhat unexpected is the behaviour for a small fermion mass. In this case, the dilute charged clouds hovering far away from the throat region are shown to exist at the cost of a vacuum energy distribution inside. The energy-momentum associated with this energy density induces a strong gravitational backreaction, and the result is again the formation of an extremal horizon at finite distance. Unlike the case of large fermion mass, however, we found that the charge penetration to the black hole is at most of secondary effect.

One advantage of using the bosonic form of the matter is, among others, the exchange of the roles played by the coupling and the mass. The effective matter action (11) is such that the quantum fluctuation of the bosons $f$ and $\eta$ are increasingly costly deep down the would-be cornucopion, and our tree-level estimates of the energy-momentum are reliable in spite of, or we should say, because of the strong coupling which is inevitable for small $m_\psi$. Even though such a strong coupling may induce large gravitational fluctuations near the extremal horizon in small $m_\psi$ limit, this should not disrupt our qualitative results. At most we expect a quantitative modification of the estimate (17).

While we worked with the case of a $SU(2)$ doublet fermion on a unit-charged would-
be cornucopion, similar results should hold for some different models. The vacuum energy
distribution trailing the fractionally charged cloud is a generic property of the screening and
should exist regardless of the model. What makes this vacuum energy distribution dangerous
enough to destabilize the core geometry is the noncompact nature of the unperturbed core
structure, as emphasized in section 3. Therefore one should expect a similar destabilization
to occur whenever he quantizes massive charged fermion field, in the background of $\theta$ and
an infinite-neck geometry threaded by constant magnetic flux.

For instance, if we consider a single fermion coupled to $U(1)$ rather than $SU(2)$, the
resulting bosonized matter action must be similar to our own but involve the charged sector
only (corresponding to the $f$ field above, with $\eta$ frozen out). Since it is the effective potential
associated with $f$ which induces the screening and the vacuum energy distribution thereof,
an analogue of $V_{\text{min}}$ will appear in the background of a unit-charged cornucopion. Again
the resulting gravitational backreaction will close off the would-be cornucopion, just as we
observed above.

Another interesting case to consider is that of cornucopia of larger transverse sizes. Since
the transverse size is proportional to the magnetic flux threading the infinite neck, these
are highly charged magnetic black holes. In such a background, one finds analogues of the
Callan-Rubakov modes in the form of zero-modes on the transverse two-spheres. Since the
number of these zero-modes is proportional to the total magnetic flux threading the two-
sphere, we need to deal with not just a pair of two-spinors $\chi_{\pm}$ but complicated multiplets of
the generalized angular momentum [16]. Nevertheless, the separation of variables and the
dimensional reduction must be possible, and upon a bosonization trick we expect to find only
two kinds of bosonic fields: $\tilde{f}$, an analogue of $f$ which keeps track of the electric charge and
the vacuum energy distributions, and $\tilde{\eta}_a$'s, the rest of them. There are again two effective
potentials: the electric energy term which is minimized for vanishing local electric fields and
the fermionic mass term which is not. The upshot is again that the would-be cornucopion
develops an extremal horizon at finite physical distance.

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