Using simulation to estimate parameters and reliability function for extreme value distribution

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Abstract
This study includes estimating scale parameter ($\theta$), location parameter ($\gamma$) and reliability function $R(t)$ for Extreme Value (EXV) distribution by two methods, namely:
- Maximum Likelihood Method (MLE).
- Probability Weighted Moments Method (PWM).

Used simulations to generate the required samples to estimate the parameters and reliability function of different sizes ($n=10, 25, 50, 100$), and give real values for the parameters are $\theta = 0.5, 1, 1.5$ and $\gamma = 3$, replicate the simulation experiments ($R_p=1000$), and estimated the function reliability at two times ($T=2, 5$), adopted mean square error for comparison between the results of estimators, and summarized all the results in tables especially prepared for this purpose. An analysis of the results in the tables it becomes that:

- Estimators of (MLE) better than estimators of (PWM) in most of the cases studied in this research.

Keywords: extreme value distribution, Maximum Likelihood method (MLE), Probability Weighted Moments method (PWM), reliability function, means squared error.
1. Introduction

The extreme value distribution has been used widely in the field of reliability theory, and is considered one of the statistical distributions and the continuing failure models one common task\(^1\).

If the random variable \(T\) follows the extreme value distribution with two parameters \((\theta, \gamma)\), then it's probability density function (p.d.f) and also cumulative distribution function (C.D.F), are as the follows: \(^4\)

\[
f_T(t; \theta, \gamma) = \frac{1}{\theta} \exp\left(\frac{t - \gamma}{\theta}\right) \exp\left(-\exp\left(\frac{t - \gamma}{\theta}\right)\right) \quad \text{... (1)}
\]

\[
F_T(t) = 1 - \exp\left(-\exp\left(\frac{t - \gamma}{\theta}\right)\right) \quad \text{... (2)}
\]

Where
- \(t\) is Value of the random variable, \(-\infty < t < \infty\)
- \(\theta\) Scale parameter, \(\theta > 0\)
- \(\gamma\) Location parameter, \(\gamma \geq 0\)

Reliability \((R)\) is defined as the probability that any system will perform its required function for a given period of time \(t\) when used under stated operating conditions. In a mathematical language that yields:

\[R(t) = \Pr(T > t)\]

The failure law is the probability that a failure occurs before time \(t\), it is the cumulative distribution function (C.D.F) of the failure distribution:

\[F(t) = \Pr(T \leq t) = 1 - R(t)\]

Finally, the reliability function \(R(t)\) is the conditional probability of failure in the time interval \([0, t]\) given that the system has survived to time \(t\): \(^1\)

\[R(t) = \exp\left(-\exp\left(\frac{t - \gamma}{\theta}\right)\right) \quad \text{... (3)}\]

2. Methods of Estimation

In this section, we display two methods for estimating the two parameters and reliability function of extreme value distribution, which are (MLE) and (PWM).
1-2 Probability Weighted Moments method (PWM)

The general formula of the (PWM) method was proposed by (Green Wood) (1979) [8], Although it is an independent method to find estimator for the parameters it is also one of the methods in which can be used its estimator a preliminary estimation for other methods, This has been the use of the estimator of this method as initial estimates of the (MLE) method. [5]

We can depend formula of (PWM) version of the order q, r, s (q, r and s are non-negative integers) of the random variable T which represents the distribution function F(t) as follow.[2]

\[
M_{q,r,s} = E\left[ (t(F))^q [F(t)]^r [1 - F(t)]^s \right] \quad \text{... (4)}
\]

\[
= \int_0^\infty (t(F))^q [F(t)]^r [1 - F(t)]^s \, dF
\]

t(F) is Inverse distribution function F(t), when we offset formula (2) in the formula (4) we get the following formula [2]

\[
M_{q,r,s} = E \left[ \left( \gamma + \theta \ln(-\ln(1-F(t))) \right)^q \left[ 1 - \exp\left(-\exp\left(\frac{t-\gamma}{\theta}\right)\right) \right]^r \right] ^s \quad \text{... (5)}
\]

\[
M_{1,0,s} = E \left[ \left( \gamma + \theta \ln(-\ln(1-F(t))) \right) \left[ \exp\left(-\exp\left(\frac{t-\gamma}{\theta}\right)\right) \right]^s \right] \quad \text{... (6)}
\]
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When we find the expected value of the formula (6) they become as the follow

\[ M_{1,0,s} = \exp(\gamma)(1 + S)^S \Gamma(\delta) \]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \frac{n - i}{S} \right) \exp(t_i) 
\]

... (7)

So that \( \delta = 1 + \theta \)

When the compensation values \( s = 0, 1, 2 \) in the formula (7), respectively, we get the following formulas

\[
M_{1,0,0} = \frac{1}{n} \sum_{i=1}^{n} \exp(t_i) 
\]

\[
M_{1,0,1} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{n - i}{n - 1} \right) \exp(t_i) 
\]

\[
M_{1,0,2} = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{(n - i)(n - i - 1)}{(n - 1)(n - 2)} \right) \exp(t_i) 
\]

In the following two formulas we can estimate scale parameter \( \theta \) and location parameter \( \gamma \) respectively

\[ \hat{\theta}_{pwm} = \hat{\delta} - 1 \]

... (8)

\[ \hat{\gamma}_{pwm} = \log \left( \frac{M_{1,0,0}}{\Gamma(\hat{\delta})} \right) \]

... (9)

So that

\[ \hat{\delta} = \frac{\log \left( \frac{M_{1,0,0}}{M_{1,0,1}} \right)}{\log(2)} \]
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So the (PWM) estimate of the reliability function \( R(t) \) is

\[
\hat{R}_{PWM}(t) = \exp\left(-\exp\left(\frac{t - \hat{\gamma}_{PWM}}{\hat{\theta}_{PWM}}\right)\right)
\]

... (10)

2-2 Maximum Likelihood method (MLE)

The manner (MLE) of the most common method, which can be obtained by which the destinies of the values of efficient when compared with other estimation methods, In order to find estimates of the values of two parameters of the extreme value distribution function, the maximum likelihood \((L_f)\) of the formula (1) are as follows:\[^7\]

\[
L_f(t; \theta, \gamma) = \prod_{i=1}^{n} \left[ \frac{1}{\theta} \exp\left(\frac{t - \gamma}{\theta}\right) \exp\left(-\exp\left(\frac{t - \gamma}{\theta}\right)\right) \right]
\]

\[
= \frac{1}{\theta^n} \exp\left(\sum_{i=1}^{n} \frac{t_i - \gamma}{\theta}\right) \exp\left(-\sum_{i=1}^{n} \exp\left(\frac{t_i - \gamma}{\theta}\right)\right)
\]

... (11)

Taking the natural logarithm of the formula (11) as they become as follow

\[
LL_f(t; \theta, \gamma) = -n \ln \theta + \sum_{i=1}^{n} \frac{t_i - \gamma}{\theta} - \sum_{i=1}^{n} \exp\left(\frac{t_i - \gamma}{\theta}\right)
\]

... (12)

The application of the method (MLE) should be taking first and second partial derivatives of formula (12) with respect to scale and location parameters as follows:

\[
\frac{\partial LL_f}{\partial \gamma} = -n + \frac{1}{\theta} \sum_{i=1}^{n} \exp\left(\frac{t_i - \gamma}{\theta}\right)
\]

\[
\frac{\partial^2 LL_f}{\partial \gamma^2} = -\frac{1}{\theta^2} \sum_{i=1}^{n} \exp\left(\frac{t_i - \gamma}{\theta}\right)
\]

\[
\frac{\partial^2 LL_f}{\partial \gamma \partial \theta} = \frac{n}{\theta^2} - \frac{1}{\theta^2} \sum_{i=1}^{n} \left(\frac{t_i - \gamma}{\theta}\right) \exp\left(\frac{t_i - \gamma}{\theta}\right) - \frac{1}{\theta^2} \sum_{i=1}^{n} \exp\left(\frac{t_i - \gamma}{\theta}\right)
\]

\[
\frac{\partial LL_f}{\partial \theta} = -\frac{n}{\theta} - \frac{1}{\theta} \sum_{i=1}^{n} \left(\frac{t_i - \gamma}{\theta}\right) + \frac{1}{\theta} \sum_{i=1}^{n} \left(\frac{t_i - \gamma}{\theta}\right) \exp\left(\frac{t_i - \gamma}{\theta}\right)
\]
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\[
\frac{\partial^2 LLf}{\partial \theta^2} = \frac{n}{\theta^2} + 2 \sum_{i=1}^{n} \left( \frac{t_i - \gamma}{\theta} \right) - \frac{1}{\theta^2} \sum_{i=1}^{n} \left( \frac{t_i - \gamma}{\theta} \right)^2 \exp\left( \frac{t_i - \gamma}{\theta} \right) - \frac{2}{\theta^2} \sum_{i=1}^{n} \left( \frac{t_i - \gamma}{\theta} \right) \exp\left( \frac{t_i - \gamma}{\theta} \right)
\]

by solving the formulas below, we get the values of \( \delta \gamma_i \) and \( \delta \theta_i \)

\[
\begin{align*}
- \frac{\partial LLf}{\partial \gamma} &= \frac{\partial^2 LLf}{\partial \gamma^2} \delta \gamma_i + \frac{\partial^2 LLf}{\partial \gamma \partial \theta} \delta \theta_i \\
- \frac{\partial LLf}{\partial \theta} &= \frac{\partial^2 LLf}{\partial \gamma \partial \theta} \delta \gamma_i + \frac{\partial^2 LLf}{\partial \theta^2} \delta \theta_i
\end{align*}
\]

And the estimated values for two parameters of the extreme value distribution get it of the two formulas below

\[
\hat{\theta}^{(i+1)}_{MLE} = \hat{\theta}_i + \delta \theta_i \quad \text{... (13)}
\]

\[
\hat{\gamma}^{(i+1)}_{MLE} = \hat{\gamma}_i + \delta \gamma_i \quad \text{... (14)}
\]

So the (MLE) estimate of the reliability function \( R(t) \) is

\[
\tilde{R}_{MLE}(t) = \exp\left( - \exp\left( \frac{t - \hat{\gamma}_{MLE}}{\hat{\theta}_{MLE}} \right) \right) \quad \text{... (15)}
\]

3- Simulation

We obtained, in the above Sections, white and modify white estimates for the scale and location parameters \( \theta \) and \( \gamma \), reliability \( R(t) \) function of the (EXV) distribution.

In order to assess the statistical performances of these estimates, a simulation study is conducted. The mean square errors (MSE’s) using generated random samples of different sizes are computed for each estimator. The random samples are generated as follows:

1. Given values of the scale and threshold parameters \((\theta = 0.5, 1.15)\) and \((\gamma = 3)\).
2. Using $\theta$ and $\gamma$, obtained in step (1), and generate random samples of different sizes: $n=10,25,50$ and $100$, replicate size: $N=1000$ from the (EXV) distribution by using the following:

From the formula (2) we obtained

$$\exp\left(-\exp\left(\frac{t-\gamma}{\theta}\right)\right) = 1 - F(t)$$

so

$$\frac{t-\gamma}{\theta} = \ln(-\ln(1 - F(t)))$$

If $U = F(t)$ where $U$ is continuous random on $[0,1]$ then we obtained a random samples from the following

$$t = \gamma + \theta(\ln(-\ln(1 - U)))$$

3. The white and modify white estimators of the parameters $\theta$ and $\gamma$, $(\hat{\theta}_{PWM}, \hat{\theta}_{MLE})$ and $(\hat{\gamma}_{PWM}, \hat{\gamma}_{MLE})$, are obtained by solving the equations (8), (13), (9) and (14) respectively. The estimators $\hat{R}_{PWM}(t)$ and $\hat{R}_{MLE}(t)$ of the function $R(t)$ are computed at values $T=2,5$ from (10) and (15) respectively.

4. The above steps are repeated 1000 times and the mean square errors (MSE) are computed for different sample sizes ($n$) and run sizes $N$ by using the following:

$$MSE(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\theta}_i - \hat{\theta})^2$$

4- The Results

After applying both the method (MLE) and (PWM) on the data generated to estimate the parameters, the results of extreme value estimate and the values of MSE and the values of Bias to these estimates, as shown in the following tables.
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1-4 Results of $\theta$

Table (1)

Values of $\theta$ estimators

| $\theta$ | $\hat{\theta}_{PWM}$ | $\hat{\theta}_{MLE}$ | n |
|----------|----------------------|----------------------|---|
| 0.5      | 0.50846              | 0.45792              | 10 |
| 0.5      | 0.50109              | 0.48195              | 25 |
| 0.5      | 0.50281              | 0.49331              | 50 |
| 0.5      | 0.50178              | 0.49722              | 100|
| 1        | 1.02004              | 0.91415              | 10 |
| 1        | 1.00393              | 0.96390              | 25 |
| 1        | 1.00731              | 0.98664              | 50 |
| 1        | 1.00423              | 0.99443              | 100|
| 1.5      | 1.48746              | 1.36128              | 10 |
| 1.5      | 1.49988              | 1.44514              | 25 |
| 1.5      | 1.50966              | 1.47996              | 50 |
| 1.5      | 1.50537              | 1.49166              | 100|

Table (2)

Values of (MSE) for $\theta$ estimators

| $\theta$ | $\hat{\theta}_{PWM}$ | $\hat{\theta}_{MLE}$ | n |
|----------|----------------------|----------------------|---|
| 0.5      | 0.01830              | 0.01589              | 10 |
| 0.5      | 0.00713              | 0.00663              | 25 |
| 0.5      | 0.00346              | 0.00321              | 50 |
| 0.5      | 0.00180              | 0.00166              | 100|
| 1        | 0.07820              | 0.06300              | 10 |
| 1        | 0.03078              | 0.02654              | 25 |
| 1        | 0.01497              | 0.01282              | 50 |
| 1        | 0.00767              | 0.00665              | 100|
| 1.5      | 0.17019              | 0.14339              | 10 |
| 1.5      | 0.08159              | 0.05944              | 25 |
| 1.5      | 0.04133              | 0.02885              | 50 |
| 1.5      | 0.02059              | 0.01497              | 100|
## 2-5 Results of $\gamma$

Table (3) 
Values of $\gamma$ estimators

| $\theta$ | $\hat{\gamma}_{PWM}$ | $\hat{\gamma}_{MLE}$ | $n$ |
|----------|----------------------|----------------------|-----|
| 0.5      | 2.98173              | 2.98560              | 10  |
|          | 2.98837              | 2.98966              | 25  |
|          | 2.99098              | 2.99177              | 50  |
|          | 2.99520              | 2.99562              | 100 |
| 1        | 2.92239              | 2.97225              | 10  |
|          | 2.96079              | 2.97932              | 25  |
|          | 2.97391              | 2.98354              | 50  |
|          | 2.98657              | 2.99124              | 100 |
| 1.5      | 2.85473              | 2.96649              | 10  |
|          | 2.92012              | 2.97002              | 25  |
|          | 2.94883              | 2.97532              | 50  |
|          | 2.97407              | 2.98687              | 100 |

Table (4) 
Values of (MSE) for $\gamma$ estimators

| $\theta$ | $\hat{\gamma}_{PWM}$ | $\hat{\gamma}_{MLE}$ | $n$ |
|----------|----------------------|----------------------|-----|
| 0.5      | 0.02836              | 0.02771              | 10  |
|          | 0.01154              | 0.01141              | 25  |
|          | 0.00559              | 0.00557              | 50  |
|          | 0.00328              | 0.00284              | 100 |
| 1        | 0.12445              | 0.11057              | 10  |
|          | 0.04843              | 0.04620              | 25  |
|          | 0.02310              | 0.02230              | 50  |
|          | 0.01172              | 0.01137              | 100 |
| 1.5      | 0.28997              | 0.24178              | 10  |
|          | 0.11755              | 0.10207              | 25  |
|          | 0.05625              | 0.05017              | 50  |
|          | 0.02758              | 0.02559              | 100 |
3-5 Results of Reliability

Table (5)
Values of Reliability estimators

| T | θ | \( \hat{R}_{PWM} \) | \( \hat{R}_{MLE} \) | n |
|---|---|---|---|---|
| 0.5 | 0.5 | 0.86024 | 0.88210 | 10 |
| | | 0.86782 | 0.87693 | 25 |
| | | 0.86891 | 0.87350 | 50 |
| | | 0.87085 | 0.87309 | 100 |
| 2 | 1 | 0.67330 | 0.70897 | 10 |
| | | 0.68384 | 0.69776 | 25 |
| | | 0.68502 | 0.69226 | 50 |
| | | 0.68842 | 0.69190 | 100 |
| 1.5 | 0.57968 | 0.61571 | 10 |
| | | 0.58764 | 0.60278 | 25 |
| | | 0.58950 | 0.59757 | 50 |
| | | 0.59390 | 0.59777 | 100 |
| 0.5 | 1.64022×10^{-6} | 1.78997×10^{-7} | 10 |
| | 1.12533×10^{-9} | 8.74046×10^{-11} | 25 |
| | 1.00593×10^{-11} | 1.72754×10^{-12} | 50 |
| | 5.71287×10^{-14} | 1.49569×10^{-14} | 100 |
| 5 | 1 | 0.00388 | 0.00242 | 10 |
| | | 0.00181 | 0.00133 | 25 |
| | | 0.00127 | 0.00103 | 50 |
| | | 0.00095 | 0.00084 | 100 |
| 1.5 | 0.02556 | 0.02427 | 10 |
| | | 0.02324 | 0.02331 | 25 |
| | | 0.02314 | 0.02244 | 50 |
| | | 0.02280 | 0.02249 | 100 |
### Table (6)

Values of (MSE) for Reliability estimators

| $\theta$ | Est | $\hat{R}_{PWM}$ | $\hat{R}_{MLE}$ | n |
|-----------|-----|-----------------|-----------------|---|
| 0.5       | 0.00881 | 0.00597 | 10 |
|           | 0.00304 | 0.00318 | 25 |
|           | 0.00135 | 0.00128 | 50 |
|           | 0.00074 | 0.00071 | 100 |
| 1         | 0.05292 | 0.06700 | 10 |
|           | 0.00673 | 0.00696 | 25 |
|           | 0.00307 | 0.00289 | 50 |
|           | 0.00170 | 0.00165 | 100 |
| 1.5       | 0.02354 | 0.02890 | 10 |
|           | 0.01546 | 0.01208 | 25 |
|           | 0.00380 | 0.00326 | 50 |
|           | 0.00203 | 0.00318 | 100 |
| 0.5       | 5.36128×10^{-10} | 1.23990×10^{-11} | 10 |
|           | 2.02797×10^{-16} | 1.23523×10^{-18} | 25 |
|           | 3.05654×10^{-20} | 1.20923×10^{-21} | 50 |
|           | 5.97776×10^{-25} | 7.196558×10^{-26} | 100 |
| 1         | 0.00008 | 0.00004 | 10 |
|           | 0.00001 | 0.000006 | 25 |
|           | 0.000003 | 0.000002 | 50 |
|           | 0.000001 | 0 | 100 |
| 1.5       | 0.00089 | 0.00098 | 10 |
|           | 0.00039 | 0.00038 | 25 |
|           | 0.00022 | 0.00020 | 50 |
|           | 0.00012 | 0.00011 | 100 |

### 6- Analysis of the results

a. When we analyze the results of the parameter $\theta$ in table (1) has become quite clear that the probability weighted moments method is better than maximum likelihood method, but the results in table (2) shows the Maximum Likelihood method better than probability weighted moments method.

b. When we analyze the results of the parameter $\gamma$ in tables (3) and (4) has become quite clear that the maximum likelihood method better than probability weighted moments method in all cases.

c. When we analyze the results of the reliability function $R(t)$ in the table (5) shows that (PWM) method is better than (MLE) when T=2, but when T=5 the (MLE) is better than (PWM).
The results in table (6) shows the (MLE) method better than (PWM) method in all cases except in cases (n=25,T=2, $\theta =0.5$), (n=10,T=2, $\theta =1$), (n=25,T=2, $\theta =1$) and (n=10,T=2, $\theta =1.5$).

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