Entropic N-bound and Maximal Mass Conjecture Violations in Four Dimensional Taub-Bolt(NUT)-dS Spacetimes

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Abstract

We show that the class of four-dimensional Taub-Bolt(NUT) spacetimes with positive cosmological constant for some values of NUT charges are stable and have entropies that are greater than that of de Sitter spacetime, in violation of the entropic N-bound conjecture. We also show that the maximal mass conjecture, which states "any asymptotically dS spacetime with mass greater than dS has a cosmological singularity", can be violated as well. Our calculation of conserved mass and entropy is based on an extension of the path integral formulation to asymptotically de Sitter spacetimes.

1 Introduction

The set of conserved quantities associated with a given physical system is one of its most fundamental features. If the system consists of a spacetime that is either asymptotically flat (aF) or asymptotically anti de Sitter (aAdS) these quantities are generally well understood. In the former case the conserved quantities are the \[ d(d+1)/2 \] conserved charges corresponding to the Poincare generators in \( d \)-dimensions. In the latter case the situation is a bit more problematic since the conserved charges have supertranslation-like ambiguities (due to the coordinate dependence of the formalism). In either case such quantities have been defined relative to an auxiliary spacetime, in which the boundary of the spacetime of interest must be embedded in a reference spacetime. In recent years progress in this area has been made by incorporating expectations from the AdS/CFT correspondence. By introducing additional surface terms that are functionals of geometric invariants on the boundary [1], an alternative approach was developed for computing conserved quantities associated with aAdS spacetimes that was free of the aforementioned difficulties.

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Both aF and aAdS spacetimes have spatial infinity, a property that plays an important role in the construction of conserved charges for both of these cases. However, asymptotically de Sitter (adS) spacetimes do not have a spatial infinity, and so present an interesting puzzle in the calculation of conserved quantities. Such spacetimes also have no global timelike Killing vector; rather the norm of the Killing vector changes sign depending on which side of the horizon one is considering. Calculations for conserved charges and actions/entropies have been carried out for pure and asymptotically de Sitter spacetimes inside the cosmological horizon where the Killing vector is timelike \[2\]. Outside of this horizon the physical meaning of energy and other conserved quantities is not clear; for example to construct the energy one could use the conformal Killing vector \[3\].

Recently a novel prescription was proposed for computing conserved charges (and associated boundary stress tensors) of adS spacetimes from data at early or late time infinity \[4\]. The method is analogous to the Brown-York prescription in asymptotically AdS spacetimes \[1, 5, 6, 7\], suggesting a holographic duality similar to the AdS/CFT correspondence. The specific prescription in ref. \[4\] (which has been employed previously by others but in more restricted contexts \[8, 9\]) presented the counterterms on spatial boundaries at early and late times that yield a finite action for asymptotically dS spacetimes in 3, 4, 5 dimensions. By carrying out a procedure analogous to that in the AdS case \[1, 6\], one could compute the boundary stress tensor on the spacetime boundary, and consequently a conserved charge interpreted as the mass of the asymptotically dS spacetime could be calculated.

The conserved charge associated with the Killing vector \(\partial/\partial t\) – now spacelike outside of the cosmological horizon – was interpreted as the conserved mass \[4\]. Employing this definition, the authors of ref. \[4\] were led to the conjecture that \emph{any asymptotically dS spacetime with mass greater than dS has a cosmological singularity}. We shall refer to this conjecture as the maximal mass conjecture. As stated, the conjecture is in need of clarification before a proof could be considered, but roughly speaking it means that the conserved mass of any physically reasonable adS spacetime must be negative (i.e. less than the zero value of pure dS spacetime). This has been verified for topological dS solutions and their dilatonic variants \[10\] and for the Schwarzschild-de Sitter (SdS) black hole up to dimension nine \[11\]. The maximal mass conjecture was based in part on the Bousso N-bound \[12\], another conjecture stating that \emph{any asymptotically dS spacetime will have an entropy no greater than the entropy \(\pi\ell^2\) of pure dS with cosmological constant \(\Lambda = 3/\ell^2\) in \((3 + 1)\) dimensions}.\[16\]

We have found, however, that locally asymptotically de Sitter spacetimes – with NUT charge \(n\) – provide counterexamples to both of these conjectures under certain circumstances \[13\]. In this paper we explore this situation in detail, illustrating the circumstances under which the conjectures are and are not satisfied. We demonstrate that locally asymptotically de Sitter spacetimes with NUT charge in \((3 + 1)\) dimensions can violate both the N-bound and the maximal mass conjecture. While it has been shown recently that the class of stable spacetimes in the form of \(dS_p \times S^q\) in more than four dimensions \((p + q > 4)\), have entropy greater than that of de Sitter spacetime (in violation of the N-bound) \[14\], to our knowledge this is the first demonstration of spacetimes that violate the maximal mass conjecture.
The paper proceeds as follows. In section 2 we will outline and review the procedure for calculating the conserved mass and entropy, and derive the general expressions for these quantities in \((d + 1)\) dimensions. Since there are several different ways of writing the metric, depending on which set of Wick rotations is chosen, these calculations will in the next section be shown for all such choices. In section 4 we will analyze the quantities in \((3 + 1)\) dimensions, and compare our various approaches - also, we will demonstrate where these solutions violate the Bousso bound and the maximal mass conjectures, as both of these have been formulated for \((3 + 1)\) dimensions.

In section 5, we will analyze the conserved mass and entropy in \((5 + 1)\) dimensions. NUT-charged spacetimes of dimensionality \(4k\) are qualitatively similar to the \((3 + 1)\) dimensional case whereas those of dimensionality \(4k + 2\) are qualitatively similar to the \((5 + 1)\) dimensional case.

2 Path-Integration and Asymptotically de Sitter Space-times

We begin with the path-integral approach, for which

\[
\langle g_2, \Phi_2, S_2 | g_1, \Phi_1, S_1 \rangle = \int D[g, \Phi] \exp (iI[g, \Phi]) \tag{2.1}
\]

represents the amplitude to go from a state with metric and matter fields \([g_1, \Phi_1]\) on a surface \(S_1\) to a state with metric and matter fields \([g_2, \Phi_2]\) on a surface \(S_2\). The quantity \(D[g, \Phi]\) is a measure on the space of all field configurations and \(I[g, \Phi]\) is the action taken over all fields having the given values on the surfaces \(S_1\) and \(S_2\). For asymptotically flat and asymptotically anti de Sitter spacetimes these surfaces are joined by timelike tubes of some finite mean radii, so that the boundary and the region contained within are compact. In the limit that the larger mean radius becomes infinite and the smaller mean radius vanishes one obtains the amplitude for the entire spacetime and matter fields to evolve from \([g_1, \Phi_1, S_1]\) to \([g_2, \Phi_2, S_2]\).

In the case of asymptotically \((d + 1)\)-dimensional de Sitter spacetimes the situation is somewhat different. We replace the surfaces \(S_1, S_2\) with histories \(H_1, H_2\) that have spacelike unit normals and are surfaces that form the timelike boundaries of a given spatial region; they therefore describe particular histories of \(d\)-dimensional subspaces of the full spacetime. The amplitude (2.1) becomes

\[
\langle g_2, \Phi_2, H_2 | g_1, \Phi_1, H_1 \rangle = \int D[g, \Phi] \exp (iI[g, \Phi]) \tag{2.2}
\]

and describes quantum correlations between differing histories \([g_1, \Phi_1]\) and \([g_2, \Phi_2]\) of metrics and matter fields. The correlation between a history \([g_2, \Phi_2, H_2]\) with a history \([g_1, \Phi_1, H_1]\) is obtained from the modulus squared of this amplitude. The surfaces \(H_1, H_2\) are joined by spacelike tubes at some initial and final times, so that the boundary and interior region are compact. In the limit that these times approach past and future infinity one obtains the correlation between the complete histories \([g_1, \Phi_1, H_1]\) and \([g_2, \Phi_2, H_2]\). This correlation is given by summing over all metric and matter field configurations that interpolate between these two histories. The quantity \(\langle g_2, \Phi_2, S_2 | g_1, \Phi_1, S_1 \rangle\) depends only on the metrics \(g_1, g_2\) and the fields \(\Phi_1, \Phi_2\) on the initial
and final surfaces $S_1, S_2$ and not on a special metric $g_i$ and matter fields $\Phi_i$ on an intermediate surface $S_i$ embedded in spacetime; in the evaluation of the amplitude $\langle g_2, \Phi_2, S_2 | g_1, \Phi_1, S_1 \rangle$ all the different possible configurations $g_i, \Phi_i$ in spacetime are summed. Similarly, the quantity $\langle g_2, \Phi_2, H_2 | g_1, \Phi_1, H_1 \rangle$ depends only on the hypersurfaces $H_1$ and $H_2$ and the metrics and matter fields over these hypersurfaces and not on any special hypersurface in the spacetime, between the hypersurfaces $H_1$ and $H_2$.

The action can be decomposed into three distinct parts

$$I = I_B + I_{\partial B} + I_{ct} \tag{2.3}$$

where the bulk ($I_B$) and boundary ($I_{\partial B}$) terms are the usual ones, given by

$$I_B = \frac{1}{16\pi} \int_{\mathcal{M}} d^{d+1}x \, \sqrt{-g} \left( R - 2\Lambda + \mathcal{L}_M(\Phi) \right) \tag{2.4}$$

$$I_{\partial B} = -\frac{1}{8\pi} \int_{\partial M^\pm} d^d x \, \sqrt{\gamma^\pm} \Theta^\pm \tag{2.5}$$

where $\partial \mathcal{M}^\pm$ represents future/past infinity, and $\int_{\partial \mathcal{M}^\pm} = \int_{\partial \mathcal{M}^-}$ represents an integral over a future boundary minus a past boundary, with the respective metrics $\gamma^\pm$ and extrinsic curvatures $\Theta^\pm$ (working in units where $G = 1$). The quantity $\mathcal{L}_M(\Phi)$ in (2.4) is the Lagrangian for the matter fields, which we won’t be considering here. The bulk action is over the $(d + 1)$ dimensional manifold $\mathcal{M}$, and the boundary action is the surface term necessary to ensure well-defined Euler-Lagrange equations.

For an asymptotically dS spacetime, the boundary $\partial \mathcal{M}$ will be a union of Euclidean spatial boundaries at early and late times. The necessity of the boundary term (2.5) can also be understood from the path-integral viewpoint by considering the joint correlation between histories $[g_1, \Phi_1, H_1]$ and $[g_2, \Phi_2, H_2]$ and also between $[g_2, \Phi_2, H_2]$ and $[g_3, \Phi_3, H_3]$. The correlation between $[g_1, \Phi_1, H_1]$ and $[g_3, \Phi_3, H_3]$ should be given by summing over the products of correlations with all possible intermediate histories $[g_2, \Phi_2, H_2]$:

$$\langle g_3, \Phi_3, H_3 | g_1, \Phi_1, H_1 \rangle = \sum_2 \langle g_3, \Phi_3, H_3 | g_2, \Phi_2, H_2 \rangle \langle g_2, \Phi_2, H_2 | g_1, \Phi_1, H_1 \rangle \tag{2.6}$$

which will hold provided

$$I [g_{13}, \Phi] = I [g_{12}, \Phi] + I [g_{23}, \Phi] \tag{2.7}$$

where $g_{12}$ is the metric of the spacetime region between histories $H_1$ and $H_2$ and $g_{23}$ is the metric of that between histories $H_2$ and $H_3$. The metric $g_{13}$ is the metric of the full spacetime between histories $H_1$ and $H_3$ obtained by joining the two regions. In general the metrics $g_{12}$ and $g_{23}$ will have different spacelike normal derivatives, yielding delta-function contributions to the Ricci tensor proportional to the difference between the extrinsic curvatures of the history $H_2$ in the metrics $g_{12}$ and $g_{23}$. The boundary term $I_{\partial B}$ compensates for these discontinuities, and so ensures that the relation (2.7) holds.

We next turn to a consideration of the counter-term action $I_{ct}$ in (2.3). In the context of the dS/CFT correspondence conjecture, it appears due to the counterterm contributions from the
boundary quantum CFT \[1\] [15]. The existence of such terms in de Sitter space is suggested by analogy with the AdS/CFT correspondence, which posits the relationship

\[
Z_{\text{AdS}}[\gamma, \Psi_0] = \int_{[\gamma, \Psi_0]} D[g] D[\Psi] e^{-I[g, \Psi]} = \left\langle \exp \left( \int_{\partial M_4} d^d x \sqrt{g} \mathcal{O}_{[\gamma, \Psi_0]} \right) \right\rangle = Z_{\text{CFT}}[\gamma, \Psi_0] \tag{2.8}
\]

between the partition function of the field theory on AdS\(_{d+1}\) and its quantum conformal field theory on its boundary. The counter-term action \(I_c\) in eq. \(2.3\) appears for similar reasons: the quantum CFT at future/past infinity is expected to have counterterms whose values can only depend on geometric invariants of these spacelike surfaces. The counterterm action can be shown to be universal for both the AdS and dS cases by re-writing the Einstein equations in Gauss-Codacci form, and then solving them in terms of the extrinsic curvature and its derivatives to obtain the divergent terms; these will cancel the divergences in the bulk and boundary actions \[11\] [16]. It can be generated by an algorithmic procedure, without reference to a background metric, and yields finite values for conserved quantities that are intrinsic to the spacetime. The result of employing this procedure in de Sitter spacetime is \[11\]

\[
I_c = - \int d^d x \sqrt{\gamma} \left\{ -\frac{d-1}{\ell} + \frac{\Theta (d-3)}{2(d-2)} R - \frac{\ell^2 \Theta (d-5)}{2(d-2)^2(d-4)} \left( R_{ab} R^{ab} - \frac{d}{4(d-1)} R^2 \right) \right. \\
- \frac{\ell^5 \Theta (d-7)}{(d-2)^3(d-4)(d-6)} \left( \frac{3d+2}{4(d-1)} R R^{ab} R_{ab} - \frac{d(d+2)}{16(d-1)^2} R^3 \right) \\
- 2R^{ab} R^{cd} R_{acbd} - \frac{d}{4(d-1)} \nabla_a R \nabla^a R + \nabla^c R^{ab} \nabla_c R_{ab} \right\} + \ldots \tag{2.9}
\]

with \(R\) the curvature of the induced metric \(\gamma\) and \(\Lambda = \frac{d(d-1)}{2\ell^2}\). The step-function \(\Theta (x)\) is unity provided \(x > 0\) and vanishes otherwise, thereby ensuring that the series only contains the terms necessary to cancel divergences and no more. Hence, for example, in four \((d = 3)\) dimensions, only the first two terms appear, and only these are needed to cancel divergent behavior in \(I_B + I_{\partial B}\) near past and future infinity.

If the boundary geometries have an isometry generated by a Killing vector \(\xi^\pm\), then it is straightforward to show that \(T^{\pm}_{ab} \xi^{\pm b}\) is divergenceless, from which it follows that

\[
Q^\pm = \oint_{\Sigma^\pm} d^{d-1} \varphi^\pm \sqrt{\sigma^\pm} n^{\pm a} T^{\pm}_{ab} \xi^{\pm b} \tag{2.10}
\]

is conserved between histories of constant \(t\), whose unit normal is given by \(n^{\pm a}\). The \(\varphi^a\) are coordinates describing closed surfaces \(\Sigma\), where we write the boundary metric(s) of the spacelike tube(s) as

\[
h^{\pm}_{ab} d\varphi^{\pm a} d\varphi^{\pm b} = d\tilde{s}^{\pm 2} = N^{\pm 2} dT^2 + \sigma^{\pm}_{ab} \left( d\varphi^{\pm a} + N^{\pm a} dT \right) \left( d\varphi^{\pm b} + N^{\pm b} dT \right) \tag{2.11}
\]

where \(\nabla_\mu T\) is a spacelike vector field that is the analytic continuation of a timelike vector field. Physically this means that a collection of observers on the hypersurface all observe the same value of \(Q\) provided this surface has an isometry generated by \(\xi^b\). Alternatively it means that for any two histories \([g_1, \Phi_1, H_1]\) and \([g_2, \Phi_2, H_2]\), the value of \(Q\) is the same for each provided \(\xi\) is a Killing
vector on the surface $\Sigma$. Note that unlike the asymptotically flat and AdS cases the surface $\Sigma$ does not enclose anything; rather it is the boundary of the class of histories that interpolate between $H_1$ and $H_2$. In this sense $Q$ is associated only with this boundary and not with the class of histories that it bounds. This is analogous to the situation in asymptotically flat and AdS spacetimes, in which conserved quantities can be associated with surfaces whose interiors have no isometries [7].

If $\partial/\partial t$ is itself a Killing vector, then we define

$$\mathcal{M}^\pm = \oint_{\Sigma^\pm} d^{d-1} \varphi^\pm \sqrt{\sigma^\pm} N^\pm n^\pm a n^\pm b T^\pm_{ab}$$

as the conserved mass associated with the future/past surface $\Sigma^\pm$ at any given point $t$ on the spacelike future/past boundary. This quantity changes with the cosmological time $\tau$. Since all asymptotically de Sitter spacetimes must have an asymptotic isometry generated by $\partial/\partial t$, there is at least the notion of a conserved total mass $\mathcal{M}^\pm$ for the spacetime in the limit that $\Sigma^\pm$ are future/past infinity. Similarly the quantity

$$J^{\pm a} = \oint_{\Sigma^\pm} d^{d-1} \varphi^\pm \sqrt{\sigma^\pm} \sigma^{\pm ab} n^\pm c T^\pm_{bc}$$

can be regarded as a conserved angular momentum associated with the surface $\Sigma^\pm$ if the surface has an isometry generated by $\partial/\partial \phi^{\pm a}$.

We turn now to an approach for evaluating and interpreting the path integral. The action \[ (2.3) \] will be real for Lorentzian metrics and real matter fields, and so the path integral will be oscillatory and so will not converge. To set the stage for the calculations we perform we briefly review the path-integral approach to quantum gravity and its relationship to gravitational thermodynamics for asymptotically flat or asymptotically AdS spacetimes.

Consider a scalar quantum field $\phi$ – the amplitude for going from a state $|t_1, \phi_1\rangle$ to $|t_2, \phi_2\rangle$ can be expressed as an integral

$$\langle t_2, \phi_2 | t_1, \phi_1 \rangle = \int d[\phi] e^{iH[\phi]}$$

over all possible intermediate field configurations between the initial and final states. However, this amplitude can also be expressed as

$$\langle t_2, \phi_2 | t_1, \phi_1 \rangle = \langle \phi_2 | e^{-iH(t_2-t_1)} | \phi_1 \rangle$$

where $H$ is the Hamiltonian. By imposing the periodicity condition $\phi_1 = \phi_2$ for $t_2 - t_1 = -i\beta$, we sum over $\phi_1$ to obtain

$$\text{Tr}(\exp(-\beta H)) = \int d[\phi] e^{-i\hat{I}[\phi]}$$

The right-hand side is now a Euclidean path integral over all field configurations intermediate between the periodic boundary conditions because of the Wick rotation of the time coordinate, where $\hat{I}$ is the Euclidean action. Inclusion of gravitational effects can be carried out as described above, by considering the initial state to include a metric on a surface $S_1$ at time $t_1$ evolving to another metric on a surface $S_2$ at time $t_2$, yielding the relation \[ (2.1) \].
Note that the left-hand side of (2.16) is simply the partition function $Z$ for the canonical ensemble for a field at temperature $\beta^{-1}$. This connection with standard thermodynamic arguments can be seen as follows. We start with the canonical distribution

$$P_r \equiv \frac{\langle n_r \rangle}{N} = \frac{e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} \quad (2.17)$$

with $\beta$ determined by considering the average total energy $M$

$$M = \frac{\sum_r E_r e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} = -\frac{\partial}{\partial \beta} \ln \left\{ \sum_r e^{-\beta E_r} \right\} = -\frac{\partial}{\partial \beta} \ln Z \quad (2.18)$$

Also, the Helmholtz free energy $W = M - TS$ can be rearranged so that

$$M = W + TS = W - T \left( \frac{\partial W}{\partial T} \right)_{N,V} = -T^2 \left[ \frac{\partial}{\partial T} \left( \frac{W}{T} \right) \right]_{N,V} \quad (2.19a)$$

$$= \frac{\partial}{\partial \beta} (\beta W) \quad (2.19b)$$

Comparing (2.18) and (2.19b), we get

$$-\beta W = \ln \left\{ \sum_r e^{-\beta E_r} \right\} = \ln Z \quad (2.20)$$

which can be interpreted as describing the partition function of a gravitational system at temperature $\beta^{-1}$ contained in a (spherical) box of finite radius.

We therefore compute $Z$ using an analytic continuation of the action in (2.1) so that the axis normal to the surfaces $S_1, S_2$ is rotated clockwise by $\frac{\pi}{2}$ radians into the complex plane [2] (i.e. by rotating the time axis so that $t \to iT$ ) in order to obtain a Euclidean signature. The positivity of the Euclidean action ensures a convergent path integral in which one can carry out any calculations (of action, entropy, etc.). The presumed physical interpretation of the results is then obtained by rotation back to a Lorentzian signature at the end of the calculation.

In the asymptotically de Sitter case these arguments require a greater degree of care because the action is in general negative definite near past and future infinity (outside of a cosmological horizon). The natural strategy would appear to be to analytically continue the coordinate orthogonal to the histories $[g_1, \Phi_1, H_1]$ and $[g_2, \Phi_2, H_2]$ to complex values by rotating the axis normal to the histories $H_1, H_2$ anticlockwise by $\frac{\pi}{2}$ radians into the complex plane. The action becomes pure imaginary and so $\exp (iI [g, \Phi]) \longrightarrow \exp \left( +\hat{I} [g, \Phi] \right)$, yielding a convergent path integral

$$Z' = \int e^{+I} \quad (2.21)$$
since \( \dot{I} < 0 \). Furthermore, since we want a converging partition function, we must change (2.18) to

\[
M = + \frac{\partial}{\partial \beta} \ln \left\{ \sum_r e^{\beta E_r} \right\} = + \frac{\partial}{\partial \beta} \ln Z'
\]  

(2.22)

Now comparing (2.22) with (2.19b) (since (2.19a, 2.19b) won’t change) we will obtain

\[ +\beta W = \ln \left\{ e^{\beta E_r} \right\} = \ln Z' \]  

(2.23)

In the semi-classical approximation this will lead to \( \ln Z' = +I_{cl} \). Substituting this and (2.19a) into (2.23),

\[
\beta (M - TS) = +I_{cl} \\
\beta M - S = I_{cl} \\
S = \beta M - I_{cl}
\]  

(2.24)

As before, the presumed physical interpretation of the results is then obtained by rotation back to a Lorentzian signature at the end of the calculation. However there is an ambiguity here that is not present in the asymptotically flat and AdS cases. This occurs because outside the horizon, near past and future infinity, the signature of any asymptotically dS spacetime becomes \((+, -, +, +)\), and so the spacelike boundary tubes naturally have Euclidean signature. This leads to two possible approaches in evaluating physical quantities.

In the first approach one proceeds in a manner similar to the asymptotically flat and AdS cases, performing all calculations after anticlockwise rotation into the complex plane of the spacelike axis normal to histories. This involves not only a complex rotation of the (spacelike) \( t \) coordinate \((t \rightarrow iT)\), but also an analytic continuation of any rotation and NUT charge parameters, yielding a metric of signature \((-,-,+,+)\). In the calculation of the action, this will give rise to a negative action, and hence a negative definite energy. Our argument for this approach is that it is not the Euclidean signature of the metric that is important, but rather the convergence of the path integral and of the partition function. We also periodically identify \( T \) with period \( \beta \) (given by the surface gravity of the cosmological horizon of the \((+, -)\) section) to ensure the absence of conical singularities. We shall refer to this approach as the C-approach, since it involves a rotation into the complex plane.

In the second approach, we note that at future infinity \( \partial/\partial t = \partial_t \) is asymptotically a spacelike Killing vector. This suggests that rotation into the complex plane is merely formal device whose function is to establish the relationship (2.24); it is not necessary for computational purposes. One can simply evaluate the action at future infinity, imposing periodicity in \( t \), consistent with regularity at the cosmological horizon (given by the surface gravity of the cosmological horizon of the \((+, -)\) section). There is no need to analytically continue either rotation parameters or NUT to complex values. We shall refer to this approach as the R-approach, since no quantities are analytically continued into the complex plane.

In AdS spacetimes with NUT charge there is an additional periodicity constraint in \( t \) that arises from demanding that no Misner-string singularities appear in the spacetime. When incorporated
with the periodicity $\beta$, this yields an additional consistency criterion that relates the mass and NUT parameters, the two solutions of which produce generalizations of asymptotically flat Taub-Bolt space to the asymptotically de Sitter case \cite{13}. These solutions can be classified by the dimensionality of the fixed point sets of the Killing vector $\xi = \partial/\partial t$ that generates a $U(1)$ isometry group. If this fixed point set dimension is $(d - 1)$ the solution is called a Bolt solution; if the dimensionality is less than this then the solution is called a NUT solution. We shall see that in the C-approach this yields a dS-NUT solution analogous to the AdS-NUT case, as well as the Bolt solutions, whereas the R-approach yields Bolt solutions only. Both of these versions of the Taub-NUT-dS spacetime are solutions to the Einstein equations.

Our proposal \cite{22} for extending the path-integral formalism for quantum gravity to describe quantum correlations between differing histories (as opposed to quantum amplitudes between differing spacelike slices) can be physically motivated in the following way. Consider for definiteness an adS spacetime with cylindrical topology. A given history can be interpreted as the collection of worldlines of a set of observers on a compact slice at a given point $t$ performing a variety of experiments in a (cosmologically) evolving universe. The choice of initial point of their history is determined by the time at which they began their experiments and the final point corresponds to the time at which they completed their experiments. Their entire history determines a causal diamond given by the intersection of the causal future of their initial point with the causal past of their final point. If their experiments begin at past infinity and end at future infinity they have obtained the maximal amount of information that they can empirically access.

The path integral \cite{22} then describes the quantum correlation between the information these observers collected within their causal diamond with that obtained by another class of observers at some other point $t'$. A given class of observers could split up, choosing the same initial point but have differing intermediate histories. Similarly, two classes of observers could choose to meet at some final point, or a given class could split up at some initial time and reunite at some final time. In all such cases the interpretation of the path integral \cite{22} would be that it describes the quantum correlations between the information contained in their observations. The modulus squared of the amplitude would represent the probability that the information collected from one history is correlated with that of another.

We note also that although we have derived eq. \cite{224} from the path integral formalism, its thermodynamic interpretation remains to be fully understood. However it would seem reasonable to expect that gravitational entropy $S$ is generated by the presence of past/future cosmological horizons, and that the entropy in eq. \cite{224} counts the degrees of freedom hidden behind such horizons. Of course there is a distinction between the entropy at past infinity and the entropy at future infinity. A future cosmological horizon shields information from observers at or near past infinity somewhat analogously to the manner in which a black hole shields external observers from the information inside. They have the option of actively probing for information behind the horizon, but only at later times. Observers at or near future infinity cannot probe for information from behind the past cosmological horizon; rather they can only passively access it.

As an application of the relation \cite{224}, consider the $(d + 1)$-dimensional SdS spacetimes, with
metric
\[ ds^2 = -\frac{d\tau^2}{\tau^2 + \frac{2m}{\tau^{d-2}}} - 1 + \left(\frac{\tau^2}{\ell^2} + \frac{2m}{\tau^{d-2}} - 1\right) dt^2 + \tau^2 d\Omega^2_{d-1} \] (2.25)

The mass is given by
\[ M_{d+1} = -\frac{V_d}{16\pi} \left\{ (d-1)m - C_d \right\} \]
where \( V_d \) is the volume of the unit \( d \)-sphere, \( C_d \) is the Casimir energy which is non-vanishing for even \( d \) and we obtain \( II \)

\[ S_{d+1} = \frac{\beta \left( \tau_+^d - (d-2)m\ell^2 \right) V_{d-1}}{8\pi\ell^2} = \frac{A_{d-1}}{4} \] (2.26)

for the entropy where \( \tau_+ \) is the largest root of the lapse function and \( m \) is the mass parameter. The gravitational entropy \( S_{d+1} \) is a positive monotonically increasing (decreasing) function of the conserved total mass \( M \) (mass parameter \( m \)) and so the N-bound is satisfied. In the special case of \((2 + 1)\)-dimensions, \( S_3 \) is exactly the same as Cardy formula \( II \). In higher dimensions, the Gibbs-Duhem entropy (2.26) is less than the entropy associated with the cosmological horizon, in agreement with the N-bound.

### 3 General Considerations of NUT-charged Spacetimes

The general form for the NUT-charged dS spacetime in \((d+1)\) dimensions is given by
\[ ds^2 = V(\tau) (dt + nA)^2 - \frac{d\tau^2}{V(\tau)} + (\tau^2 + n^2) d\Gamma^2 \] (3.1)

where \( d = 2k + 1 \) and \( V(\tau) \) is given by the general formula
\[ V(\tau) = \frac{2m\tau}{(\tau^2 + n^2)^k} - \frac{\tau}{(\tau^2 + n^2)^k} \int_{\tau} ds \left[ \frac{(s^2 + n^2)^k}{s^2} - \frac{(2k + 1) (s^2 + n^2)^{k+1}}{s^2} \right] \] (3.2)

with \( n \) the non-vanishing NUT charge and \( \Lambda = \frac{d(d-1)\pi|n|}{2q^2} \).

The one-form \( A \) is a function of the coordinates \((\vartheta_1, \phi_1, \cdots, \vartheta_k, \phi_k)\) of the non-vanishing compact base space of positive curvature (with metric \( d\Omega^2 \)). The coordinate \( t \) parameterizes a circle \( S^1 \) Hopf-fibered over this space; it must have periodicity \( \frac{2(d+1)\pi|n|}{q} \) to avoid conical singularities, where \( q \) is a positive integer. The geometry of a constant-\( \tau \) surface is that of a Hopf fibration of \( S^1 \) over the base space, which is a well defined spacelike hypersurface in spacetime where \( V(\tau) > 0 \) outside of the past/future cosmological horizons. The spacelike Killing vector \( \partial/\partial t \) has a fixed point set where \( V(\tau_c) = 0 \) whose topology is the same as that of the base space.

The general form of the base space is a combination of products of \( S^2 \) and \( CP^2 \), i.e. \( \otimes_{i=1}^s S^2 \otimes_{j=1}^c CP^2 \) such that \( s + 2c = k \). The metric of \( CP^2 \) has the general form
\[ d\Sigma^2 = \frac{1}{(1 + \frac{u^2}{6})^2} \left\{ du^2 + \frac{u^2}{4} (d\psi + \cos \theta d\phi)^2 \right\} + \frac{u^2}{4(1 + \frac{u^2}{6})} (d\theta^2 + \sin^2 \theta d\phi^2) \] (3.3)

for which the one-form \( A \) is
\[ A = \frac{u^2}{2(1 + \frac{u^2}{6})} (d\psi + \cos \theta d\phi) \] (3.4)
Figure 3.1: The Penrose diagram of the TBdS spacetime. We denote the roots of $V$ by the increasing sequence $\tau_1 < 0 < \tau_2 < \tau_3 < \tau_4 = \tau_c$. The vertical and horizontal lines are the $\tau = 0$ and the $\tau = -\infty$ slices of the spacetime, respectively. The double line denotes $\tau = +\infty$ and the solid black dots denote the quasiregular singularities of the spacetime. Our calculation is performed outside the cosmological horizon, located within the triangle denoted by “X”.

whereas

$$A = 2 \cos \theta d\phi$$  \hfill (3.5)

if the base space is a 2-sphere with metric $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. For the general form $\otimes_{i=1}^{\delta} S^2 \otimes_{j=1}^{\delta} CP^2$ the one-form $A$ is a linear combination of metrics of the forms (3.4) and (3.5).

The causal structure of TNdS spacetime can be understood by looking at a typical Penrose diagram (Figure 3). For simplicity, we consider a four-dimensional TNdS with an $S^2$ base space. We denote the roots of $V(\tau)$ by the increasing sequence $\tau_1 < 0 < \tau_2 < \tau_3 < \tau_4 = \tau_c$. The vertical and horizontal lines are the $\tau = 0$ and the past infinity $\tau = -\infty$ slices of the spacetime, respectively and the double line denotes the future infinity $\tau = +\infty$. The solid black dots denote the quasiregular singularities. The region that is outside the cosmological horizon is located inside the triangle denoted by “X”.

Quasiregular singularities are the end points of incomplete and inextendible geodesics which spiral infinitely around a topologically closed spatial dimension. Moreover the world lines of observers approaching these points come to an end in a finite proper time [18]. They are the mildest kinds of singularity in that the Riemann tensor and its derivatives remain finite in all parallelly propagated orthonormal frames. Consequently observers do not see any divergences in
physical quantities when they approach a quasiregular singularity. The flat Kasner spacetimes on the manifolds \( R \times T^3 \) or \( R^3 \times S^1 \), Taub-NUT spacetimes and Moncrief spacetimes are some typical spacetimes with quasiregular singularities.

We consider these quasiregular singularities to be quite different from the cosmological singularities referred to in the maximal mass conjecture. This conjecture states that a timelike singularity will be present for any adS spacetime whose conserved mass \( M > 0 \) is positive (i.e. larger than the zero value of de Sitter spacetime). Using Schwarzschild de Sitter spacetime as a paradigmatic example, it is straightforward to show that scalar Riemann curvature invariants will diverge for \( M > 0 \), yielding a timelike boundary to the manifold upon their excision. Note that such curvature invariants diverge in certain regions even if \( M > 0 \); however observers at future infinity cannot actively probe such regions. We therefore interpret the cosmological singularities in the conjecture of ref. \([4]\) to imply that scalar Riemann curvature invariants will diverge to form timelike regions of geodesic incompleteness whenever the conserved mass of a spacetime becomes positive (i.e. larger than the zero value of pure dS). By this definition quasiregular singularities are clearly not cosmological singularities, and vice-versa.

### 3.1 R-approach

For simplicity we shall consider the form of the metric when the base space is a product \( \bigotimes_{i=1}^k S^2 \) of 2-spheres

\[
ds_R^2 = V(\tau) \left( dt + \sum_{i=1}^k 2n \cos(\theta_i) d\phi_i \right)^2 - \frac{d\tau^2}{V(\tau)} + (\tau^2 + n^2) \sum_{i=1}^k (d\theta_i^2 + \sin^2(\theta_i) d\phi_i^2)
\]

keeping in mind that the results below apply to the more general case \((3.1)\) given above. We shall denote the largest root of \( V(\tau) \) by \( \tau_c \). The subspace for which \( \tau = \tau_c \) is the fixed point set of \( \partial / \partial t \).

Since \( \frac{\partial}{\partial \phi_1}, \ldots, \frac{\partial}{\partial \phi_k} \) are Killing vectors, for any constant \((\phi_1, \ldots, \phi_k)\)-slice near the horizon \( \tau = \tau_c \) additional conical singularities will be introduced in the \((t, \tau)\) Euclidean section unless \( t \) has period

\[
\beta_R = 4\pi / |V'(\tau_c)|
\]

This periodicity must match the one induced by the requirement that the Misner string singularities vanish. This yields

\[
\frac{1}{|V'(\tau_c)|} = \frac{(d + 1)|n|}{2q}
\]

which in general has two solutions for \( \tau_c = \tau^\pm \) as a function of \( n \). For each of these solutions the fixed point set of \( \partial / \partial t \) is \((d - 1)\)-dimensional, and so both are bolt solutions. We shall refer to these solutions as \( R^+ \) and \( R^- \) respectively, denoting their metric \((3.6)\) as \( ds_R^2 \).

The general form for the metric determinant \( g_R \) and the Ricci scalar for arbitrary dimension
The bulk action (2.4) can then be computed for arbitrary \(d\) (recalling that the \(\prod \sin^2(\theta_i)\) will contribute to the volume term) giving

\[
I_{R,B} = \frac{d\beta}{8\pi \ell^2} \int d\tau \left(\tau^2 + n^2\right)^k
\]

where \(|V'(\tau_c)| = \beta > 0\) is the period of \(t\).

We work in the region \(\tau > \tau_c\) near future infinity. Employing the binomial expansion on the integrand, we integrate term by term from \(\tau = \tau_c\) to \(\tau \to \infty\). Since we are only after the finite contributions (the divergent terms being cancelled by the counter-terms), the resultant contribution from the bulk action is

\[
I_{R,B,\text{finite}} = -\frac{d\beta(4\pi)^k}{8\pi \ell^2} \sum_{i=0}^{k} \binom{k}{i} n_i^2 \frac{\tau_c^{2k-2i+1}}{2k-2i+1}
\]

Turning now to the boundary contributions at future infinity, the boundary metric is given by \(\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu\), where \(n_\mu = \left[0, 1, \sqrt{-g_{\tau\tau}}, 0, \ldots\right]\) is the unit norm of a surface of fixed \(\tau\). We obtain for the boundary metric determinant and its associated Ricci scalar

\[
\gamma_R = V(\tau)(\tau^2 + n^2)^{2k} \prod_{i=1}^{k} \sin^2(\theta_i)
\]

\[
R_R(\gamma) = (d-1) \left[ \frac{1}{(\tau^2 + n^2)} - \frac{V(\tau)n^2}{(\tau^2 + n^2)^2} \right]
\]

where the trace of the extrinsic curvature can be obtained from the metric (3.6):

\[
\Theta_R = -\frac{V''(\tau)}{2\sqrt{V(\tau)}} + \frac{(d-1)\tau\sqrt{V'(\tau)}}{(\tau^2 + n^2)}
\]

Expanding (3.13) and (3.15) in (2.5) for large \(\tau\), the finite contribution from the boundary action will be

\[
I_{R,\partial B,\text{finite}} = -\frac{\beta(4\pi)^kd}{8\pi} m
\]

Turning next to the counter-term contributions, which can be found from (3.13) and (3.14), it can be shown, using exactly the same arguments employed in [19], that only the first term in the counter-term action (2.9) contributes a finite term - all the other terms will only cancel the...
divergences in the bulk and boundary actions. Hence, the finite contribution at future infinity from the counter-term action

\[ I_{R,\text{finite}} = \frac{\beta (4\pi)^k (d-1)}{8\pi} m \]  

(3.17)

Adding together (3.12), (3.16) and (3.17), we find

\[ I_{R(\text{finite})} = -\frac{\beta (4\pi)^k}{8\pi} \left[ m + \frac{d}{\ell^2} \sum_{i=0}^{k} \left( \binom{k}{i} n^2 i \frac{\tau_c^{2k-2i+1}}{(2k-2i+1)} \right) \right] \]  

(3.18)

for the general form of the R-approach action.

We now turn to an evaluation of the conserved charges from the formula

\[ \Omega = \int d^{d-1} x \sqrt{\gamma} T_{ab} n^a \xi^b \]  

(3.19)

The only non-vanishing conserved charge will be the conserved mass associated with \( \xi = \partial_t \). Thus we have

\[ \mathcal{M} = \frac{1}{8\pi} \int d^{d-1} x \sqrt{\gamma} \left\{ \Theta_{ab} - \Theta \gamma_{ab} + \frac{(d-1)}{\ell} \gamma_{ab} + \ldots \right\} n^a \xi^b \]  

(3.20)

The extra terms from the variation of the counter-term action can be found in [11]. Using exactly the same arguments as above (from [19]), it can be shown that only the first term \( (d-1)\gamma_{ab} \) contributes to the finite conserved mass. Inserting all of the quantities, we find that the finite conserved mass for general \((d+1) = 2k + 2\) dimensions is given by

\[ \mathcal{M}_R = -\frac{(4\pi)^k 2k}{8\pi} m \]  

(3.21)

\( m \) can be solved for in terms of \( \tau, n \), through the first condition of demanding that \( V(\tau) = 0 \).

Using (3.21), (3.18), and the Gibbs-Duhem relation (2.24), we obtain

\[ S_R = \frac{(4\pi)^k}{8\pi} \left\{ \frac{d}{\ell^2} \sum_{i=0}^{k} \left( \binom{k}{i} n^2 \tau_c^{2k-2i+1} \right) \frac{2k-2i+1}{2k-2i+1} - (2k-1)m \right\} \]  

(3.22)

as the expression for the entropy for the Taub-Bolt-dS spacetime in general dimension \((d+1) = 2k + 2\).

Note that none of the preceding results required imposition of the consistency condition (3.8), which also reads

\[ |V'(\tau_c)| = \frac{2q}{(d+1)|n|} \]  

(3.23)

Eq. (3.23) has in general four solutions for \( \tau_c \), two of which are positive, yielding two possible relationships between the parameters \( m \) and \( n \). This in turn implies two distinct spacetimes, each with its own characteristic entropy and conserved mass for a given \( n \). While eq. (3.23) is easily solvable for specific choice of \( d \), it is cumbersome to solve for arbitrary \( d \), and so we shall postpone analysis of the implementation of this condition.
3.2 C-approach

The form of the metric in this approach is obtained from (3.6) by rotating the time and the NUT parameter \((t \rightarrow iT, \ n \rightarrow iN)\), giving

\[
ds^2_C = -F(\rho) \left( dT + \sum_{i=1}^{k} 2N \cos(\theta_i) d\phi_i \right)^2 - \frac{d\rho^2}{F(\rho)} + (\rho^2 - N^2) \sum_{i=0}^{k} (d\theta_i^2 + \sin^2(\theta_i) d\phi_i^2) \quad (3.24)
\]

where \(F(\rho)\) is now given by

\[
F(\rho) = \frac{2m\rho}{(\rho^2 - N^2)^k} - \frac{\rho}{(\rho^2 - N^2)^k} \int \rho \left[ \frac{(s^2 - N^2)^k}{s^2} - \frac{(2k + 1)(s^2 - N^2)^{k+1}}{s^2} \right] \quad (3.25)
\]

Since these two formulae are the same as in the Bolt case, except for a few signs, the same arguments used above can be used to find the finite action and entropy, as well as the conserved mass. The general metric determinant and Ricci scalar are

\[
g_C = (\rho^2 - N^2)^k \prod_{i=1}^{k} \sin(\theta_i) \quad (3.26)
\]

\[
R_C = \frac{d(d+1)}{\ell^2} \quad (3.27)
\]

The finite contribution to the bulk action can again be found by inserting the above into (2.4) and using the binomial expansion,

\[
I_{C,B\text{finite}} = -\frac{(4\pi)^k \beta d}{8\pi \ell^2} \sum_{i=0}^{k} \binom{k}{i} (-1)^i N^{2i} \frac{\rho_+^{2k-2i+1}}{2k-2i+1} \quad (3.28)
\]

with \(\rho_+\) the largest positive root of \(F(\rho)\), found by the fixed point set of \(\partial_T\).

The quantity \(\beta\) is the period of \(T\), again obtained by setting

\[
\beta_C = \frac{4\pi}{|F'(\rho_+)|} = \frac{2(d+1)\pi |N|}{q}
\]

so as to ensure regularity in the \((T, \rho)\) section. Note that in this approach the functional form of \(F(\rho)\) is altered due to the changes in signs, so that \(\rho_+\) is not equal to \(\tau_c\).

The boundary metric is again given by \(\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu\), with \(n_\mu\) as before, and again we work at future infinity. This gives

\[
\gamma_C = -F(\rho)(\rho^2 - N^2)^{2k} \prod_{i=1}^{k} \sin^2(\theta_i) \quad (3.29)
\]

\[
R_C(\gamma) = (d-2) \left[ \frac{1}{(\rho^2 - N^2)} + \frac{F(\rho)N^2}{(\rho^2 - N^2)^2} \right] \quad (3.30)
\]
for the boundary metric determinant and Ricci scalar. The trace of the extrinsic curvature on the boundary can be found from (3.24)

\[ \Theta_C = -\left[ \frac{F'(\rho)}{2\sqrt{F(\rho)}} + \frac{(d-1)\rho\sqrt{F(\rho)}}{(\rho^2 - N^2)} \right] \] (3.31)

Using the same steps as above, the finite contributions from the boundary and counter-term actions can be found to be the same as (3.16), (3.17), and so, for the Taub-NUT-dS metric, the general action is calculated to be

\[ I_{C \text{ finite}} = -\frac{(4\pi)^k\beta}{8\pi} \left[ m + \frac{d}{\ell^2} \sum_{i=0}^{k} \binom{k}{i} (-1)^i N^{2i} \frac{\rho_+^{2k-2i+1}}{2k - 2i + 1} \right] \] (3.32)

As in the R-approach, the conserved mass can be found from (3.19), using the expansion (3.21). Again, only the three terms given in (3.21) contribute to the finite conserved mass, giving for the C-approach spacetimes

\[ M_C = -\frac{(4\pi)^k2k}{8\pi} m \] (3.33)

Using once more the relation \( S = \beta H_\infty - I \), the expression for the entropy for the general Taub-NUT-dS spacetime is

\[ S_C = \frac{(4\pi)^k\beta}{8\pi} \left[ \frac{d}{\ell^2} \sum_{i=0}^{k} \binom{k}{i} (-1)^i N^{2i} \frac{\rho_+^{2k-2i+1}}{2k - 2i + 1} - (2k - 1)m \right] \] (3.34)

The periodicity conditions for ensuring regularity in the \((T, \tau)\) section and removal of all Misner-string singularities now yields the consistency requirement

\[ |F'(\rho_+)| = \frac{2q}{(d + 1)|N|} \] (3.35)

In this case there are two qualitatively distinct solution classes to (3.35), characterized by the co-dimensionality of the fixed point set of \( \partial_T \). In one class, this co-dimensionality is \( (d - 1) \), yielding a solution \( \rho_+ > N \) – we shall refer to this as the Taub-Bolt-C solution. In the second class \( \rho_+ = N \), and the fixed-point set is of zero-dimensionality. This class shall be referred to as the Taub-NUT-C solution. These different cases will be treated in more detail in specific dimensions in the sequel.

4 Four Dimensional Analysis

4.1 R-approach in 4 dimensions

The \((3 + 1)\)-dimensional metric will in this case (3.6), with \( k = 1 \) have

\[ V(\tau) = \frac{\tau^4 + (6n^2 - \ell^2)\tau^2 + n^2(\ell^2 - 3n^2) + 2m\tau\ell^2}{(\tau^2 + n^2)\ell^2} \] (4.1)
where \( n \) is the non-vanishing NUT charge and \( \Lambda = \frac{3}{\ell^2} \). The coordinate \( t \) parameterizes a circle fibered over the 2-sphere with coordinates \((\theta, \phi)\), and must have a period respecting the condition (3.23), which is

\[
\beta_R = \frac{4\pi}{|V'(\tau_c)|} = \frac{8\pi |n|}{q}
\] (4.2)

where \( q \) is a positive integer, yielding

\[
\beta_R = 2\pi \left| \frac{(\tau_c^2 + n^2)^2 \ell^2}{-2\tau_c \ell^2 n^2 + \tau_c^5 + 2\tau_c^3 n^2 + 9n^4 \tau_c - m\ell^2 \tau_c^2 + m\ell^2 n^2} \right|
\] (4.3)

The geometry of a constant-\( \tau \) surface is that of a Hopf fibration of \( S^1 \) over \( S^2 \), and the metric (4.1) describes the contraction/expansion (for \( q = 1 \)) of this 3-sphere in spacetime regions where \( V(\tau) > 0 \) outside of the past/future cosmological horizons. The condition \( V(\tau_c) = 0 \) yields

\[
m_R = -\frac{\tau_c^4 - \ell^2 \tau_c^2 - n^2 \ell^2 + 6n^2 \tau_c^2 - 3n^4}{2\ell^2 \tau_c}
\] (4.4)

Using the general formula (3.18) with \( k = 1, d = 3 \), the action is

\[
I_{R,Ad} = -\frac{\beta}{2\ell^2}(m_R \ell^2 + \tau_c^3 + 3n^2 \tau_c)
\] (4.5)

and the conserved mass is found to be

\[
\mathfrak{m}_{R,Ad} = -m_R + \frac{\ell^4 - 30n^2 \ell^2 + 105n^4}{8\tau \ell^2} + \mathcal{O} \left( \frac{1}{\tau^2} \right)
\] (4.6)

near future infinity, which for \( n = 0 \) reduces to the total mass of the four dimensional Schwarzschild-dS black hole [11]. From (3.22) or by directly applying the Gibbs-Duhem relation (2.24) \( S = \beta H_\infty - I \), we find

\[
S_{RAd} = -\frac{\beta(m_R \ell^2 - 3n^2 \tau_c - \tau_c^3)}{2\ell^2}
\] (4.7)

for the total entropy.

The preceding results are generic to either of the two solutions \( \tau_c = \tau_c^\pm \) to (3.23), which are

\[
\tau_c^\pm = q\ell^2 \pm \sqrt{q^2 \ell^4 - 144n^4 + 48n^2 \ell^2}
\] (4.8)

Since the discriminant of \( \tau_c^\pm \) must always be positive, we find

\[
|n_{max}| < \frac{\ell}{6} \sqrt{6 + 3\sqrt{4 + q^2}}
\] (4.9)

Note that both the high temperature \((n \to 0)\) and flat space \((\ell \to \infty)\) limits of \( \tau_c^\pm \) are infinite. The high temperature limit of \( \tau_c^- \) is 0, and its flat space limit is \(-\frac{2n}{q}\). From these results we have mass and temperature parameters \( \beta^\pm \) and \( m^\pm \), straightforwardly obtained by insertion of \( \tau_c = \tau_c^\pm \)
Figure 4.1: Plot of the upper ($\tau_b = \tau_{b+}$) and lower ($\tau_b = \tau_{b-}$) TB masses (for $q = 1$).

into eqs. (4.4) and (4.6) respectively. We shall refer to the distinct spacetimes associated with these cases as $R_{+}^\pm$, with action $I_{+}^\pm$ and entropy $S_{+}^\pm$.

Further analysis indicates that the $R_{+}^\pm$ spacetimes provide a counter-example to the maximal mass conjecture of [4] for certain ranges of the parameter $n$. As shown in Figure 4.1 (with $q = 1$), the $M_{+}R$ is always positive, and thus always violates the conjecture. Note that although the $M_{-}R$ is positive for $n < 2360026142\ell$, it doesn’t violate the conjecture for $n$ less than this value, since $R^{-}$ exists only for $|n| > .2658\ell$. Otherwise $V(\tau)$ develops two additional larger real roots, and the periodicity condition cannot be satisfied. Note that for $q > 1$, the lower branch $R^{-}$ always has a negative $M_{-}R$ and so does not violate the conjecture, whereas the $R^{+}$ branch violates the conjecture for all $q$, since $M_{+}R > 0$.

From (4.5), using (4.3) and (4.8), the R action is

$$I_{R,Ad}^\pm(\tau_0 = \tau_0^\pm) = -\frac{\pi \ell^2 (72n^2 + q^2 \ell^2)}{216} \frac{n^2}{n^2} \pm \frac{\pi}{216} \left(-q^2 \ell^4 + 144n^4 - 48n^2 \ell^2 \right) \sqrt{q^2 \ell^4 - 144n^4 + 48n^2 \ell^2}$$

and from (4.7), the entropy is

$$S_{R,Ad}^\pm = \frac{\pi \ell^2 \left(24n^2 + q^2 \ell^2 \right)}{72n^2} \pm \frac{\pi}{72n^2 q \ell^2} \frac{144n^4 + q^2 \ell^4}{\sqrt{q^2 \ell^4 - 144n^4 + 48n^2 \ell^2}}$$

This does satisfy the first law, though each branch must be checked separately. From the entropy and the relation $C_{R,Ad}^\pm = -\beta_{R}^\pm \partial_{\beta_{R}^\pm} S_{R,Ad}^\pm$, we find for the specific heat

$$C_{R}^\pm(\tau_0^\pm) = \frac{\pi \ell^4 q^2}{36n^2} \pm \frac{\pi}{q^2 n^2 \sqrt{q^2 \ell^4 - 144n^4 + 48n^2 \ell^2}} \left(-144q^2 \ell^4 n^4 + 41472n^8 - 10368n^6 \ell^2 + 24n^2 \ell^6 k^2 + k^4 \ell^8 \right)$$

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Figure 4.2: Plot of the upper branch bolt entropy and specific heat (for $q = 1$).

Figure 4.3: Plot of the lower branch bolt entropy and specific heat (for $q = 1$).

The plots for the upper and lower branch entropies/specific heats are in Figures 4.2 and 4.3.

In figure 4.2, the upper branch entropy is always positive (and almost always greater than $\pi \ell^2$, except for NUT charge in a range near the maximal value $n_{\text{max}} = 0.5941\ell$), but the specific heat is positive only outside the range $0.2886751346\ell < n < 0.5\ell$; thus, the upper branch solutions are only stable for $n$ outside this range. Figure 4.3 shows that the lower branch entropy is always negative, and so the lower branch solutions are always unstable.

In both the AdS and dS cases there is a natural correspondence between phenomena occurring near the boundary (or in the deep interior) of either spacetime and UV (IR) physics in the dual CFT. Solutions that are asymptotically (locally) dS lead to an interpretation in terms of renormalization group flows and an associated generalized dS $c$-theorem. This theorem states that in a contracting patch of dS spacetime, the renormalization group flows toward the infrared and in an expanding spacetime, it flows toward the ultraviolet. Since the spacetime (3.6) is asymptotically (locally) dS, we can use the four-dimensional $c$-function [20]

$$c = (G_{\mu\nu}n^\mu n^\nu)^{-1} = \frac{1}{G_{\tau\tau}}$$  \hspace{1cm} (4.13)

where $n^\mu$ is the unit normal vector to a constant $\tau$-slice. In figures (4.4) and (4.5), the diagrams of the R-approach spacetime $c$-functions outside the cosmological horizon with $\ell = 1$ and $n = 0.5$ for two cases $q = 1$ and 3 are plotted.

As one can see from these figures, outside the cosmological horizon, the $c$-function is a monotonically increasing function of $\tau$, indicative of the expansion of a constant $\tau$-surface of the metric (3.6) outside of the cosmological horizon. Since the metric (3.6) at future infinity $\tau \to +\infty$, reduces to

$$ds^2_R \to -du^2 + e^{2u/\ell}d\Sigma^2_3$$  \hspace{1cm} (4.14)

where $u = \ell \ln \tau$ and $d\Sigma^2_3$ is the metric of three-dimensional constant $u$-surface, the scale factor in (4.14) expands exponentially near future infinity. Hence the behavior of $c$-function in figures (4.4) and (4.5) is in good agreement with what one expects from the $c$-theorem. According to
$c$-function for any asymptotically (locally) dS spacetimes, the $c$-function must increase (decrease) for any expanding (contracting) patch of the spacetime.

### 4.2 C-approach in 4 dimensions

The $(3 + 1)$ dimensional C-approach ($t \to iT$, $n \to iN$) implies that the metric has the form (3.24), with $k = 1$, $d = 3$, with $F(\rho)$ given by

\[
F(\rho) = \frac{\rho^4 - (\ell^2 + 6N^2)\rho^2 + 2m\rho\ell^2 - N^2(\ell^2 + 3N^2)}{(\rho^2 - N^2)\ell^2} \quad (4.15)
\]

where $N$ is the nonvanishing NUT charge and $\Lambda = \frac{3}{\ell^2}$. The coordinate $T$ parameterizes a circle fibered over the non-vanishing sphere parameterized by $(\theta, \phi)$ and must have periodicity respecting the following condition

\[
\beta_C = \frac{4\pi}{|F'(\rho)|} = \frac{8\pi|N|}{q} \quad (4.16)
\]

to avoid conical singularities, where $q$ is a positive integer.

For this situation (i.e. with $(- - + +)$ signature) the geometry is, strictly speaking, no longer that of a Hopf fibration of $S^1$ over a 2-sphere since the coordinate $T$ is now timelike. Consequently its physical relevance is less clear. However the metric is independent of the coordinate $T$ and so we are able to proceed to calculate the action, the conserved mass and various other quantities. We shall do so, mindful of the preceding considerations.

Using the method of counter-terms for de Sitter space [11] directly, or using the general formula (3.32) obtained above with $k = 1$ and $d = 3$, we find

\[
I_{C,Ad} = -\frac{\beta_C(\rho_+^3 - 3N^2\rho_+ + m\ell^2)}{2\ell^2} \quad (4.17)
\]
for the action in four dimensions, where $\rho_+$ is the value of $\rho$ that is the largest positive root of $F(\rho)$, determined by the fixed point set of $\partial_T$, and $m$ is the mass parameter, which will have differing values for different $\rho \geq \rho_+$.

Working at future infinity, using either (3.33) or calculating directly from (3.19), (3.20) for the metric (3.21) yields the conserved mass
\[
M_{C,4d} = -m + \frac{105N^4 + 30N^2\ell^2 + \ell^4}{8\ell^2\rho} + O\left(\frac{1}{\rho^2}\right)
\] (4.18)

near future infinity, which for $N = 0$ reduces exactly to the total mass of the four dimensional Schwarzschild-dS black hole [11].

By applying the Gibbs-Duhem relation (2.24) or from (3.34) we obtain
\[
S_{C,4d} = \frac{\beta_H(\rho_+^3 - 3N^2\rho_+ - m\ell^2)}{2\ell^2}
\] (4.19)

for the total entropy. This entropy can be shown to satisfy the first law of gravitational thermodynamics (as required) for both the NUT and bolt cases (see below). The above equations are generic, and can now be analyzed for the specific cases of the “NUT” and “Bolt” solutions (called such in analogue with the Taub-NUT-AdS case [19]).

The metric arising from the C-approach affords two sets of solutions, depending on the fixed point set of $\partial_T$. These arise from the regularity condition (4.16) that ensures the absence of conical singularities. When $\rho_+ = N$, $F(\rho = N) = 0$ and the fixed point set of $\partial_T$ is 0-dimensional, we get the “NUT” solutions; when $\rho_+ = \rho_{b\pm} > N$, the fixed point set is 2-dimensional, we get the “bolt” solutions. Since the thermodynamic and mass analysis yield interesting yet different results for each case we will handle each separately.

### 4.2.1 Taub-NUT-C Solution

For the NUT solution, $\rho_+ = N$, and we can solve for the NUT mass parameter
\[
m_{C,n} = \frac{N(\ell^2 + 4N^2)}{\ell^2}
\] (4.20)

Looking at this equation, it is easily seen that $m_{C,n}$ is always positive, and so (since the conserved mass at future infinity is $-m_{C,n}$ (4.18)) the NUT solution always has a mass less than the de-Sitter mass, always satisfying the Balasubramanian et. al. conjecture [1]. In the flat space limit ($\ell \to \infty$), the NUT mass will go to $N$, and in the high temperature ($N \to 0$) limit, it goes to 0.

The period in four dimensions ($q = 1$) is given by $\beta = 8\pi N$, and so the NUT action and entropy can be found from (4.17) (4.19):
\[
I_{C,NUT4d} = -\frac{4\pi N^2(\ell^2 + 2N^2)}{\ell^2}
\] (4.21)

\[
S_{C,NUT4d} = -\frac{4\pi N^2(\ell^2 + 6N^2)}{\ell^2}
\] (4.22)
It is easy to show that (4.22) and (4.18) with \( m = m_{C,n} \) satisfy the first law \( dS = \beta \, dH \). In the flat space limit, both of these go to \(-4\pi N\), and in the high temperature limit, they both go to 0.

Using (4.22) and the relation \( C = -\beta \partial_\beta S \) yields

\[
C_{C,NUT4d} = \frac{8\pi N^2(\ell^2 + 12N^2)}{\ell^2}
\]

for the NUT specific heat. In the flat space limit, \( C_{C,NUT4d} \rightarrow 8\pi N \), and it approaches 0 in the high temperature limit.

We note that the specific heat is seen to be always positive, and the entropy is always negative for the NUT solution. We interpret this to mean that the NUT solution is not thermodynamically stable (see Figure 4.6). Also, the specific heat always negative means it is always less than the pure dS entropy, thus satisfying the N-bound.

### 4.2.2 Taub-Bolt-C Solution

For the bolt solution, the fixed point set of \( \partial_T \) is 2 dimensional, and we get \( \rho_+ = \rho_{b\pm} > N \). The conditions for a regular bolt solution are (i) \( F(\rho) = 0 \) and (ii) \( F'(\rho) = \pm \frac{q}{2N} \), with (ii) arising from the second equality in (4.16) (and \( N > 0 \)). From (i), we get the bolt mass parameter

\[
m_{C,b} = -\left(\rho_b^4 - (\ell^2 + 6N^2)\rho_b^2 - N^2(\ell^2 + 3N^2)\right)
\]

where from (ii\(^+\)), \( \rho_b \) is

\[
\rho_{b\pm} = \frac{q\ell^2 \pm \sqrt{q^2\ell^4 + 48N^2\ell^2 + 144N^4}}{12N}
\]

The discriminant of \( \rho_{b\pm} \) will always be positive, and so there is no restriction on the range of \( N \) (except \( N > 0 \)). Note that both the flat space and high temperature limits of \( \rho_{b+} \) are infinite; the flat space limit of \( \rho_{b-} \) is \(-\frac{2N}{k}\), and the high temperature limit is 0.
The period for the bolt is found from the first equality in (4.16)

\[ \beta_{C,bolt} = \frac{2\pi}{\rho_b - 2N^2\rho_b^3 + N^2(9N^2 + 2\ell^2)\rho_b - m\ell^2(\rho_b^2 + N^2)} \]  

The temperature for the two solutions is the same, as can be seen by substituting \( m = m_{C,b} \) and either of \( \rho_b = \rho_{b\pm} \) into (4.26).

Substituting in \( \rho_b = \rho_{b\pm} \) into \( m_{C,b} \), we can see that the Taub-Bolt-C solution is a counter-example to the maximal mass conjecture of [4] for certain values of \( N \). As shown in Figure 4.7 (where \( q = 1 \) in the plots), the lower branch \( (\rho_b = \rho_{b\pm}) \) mass is always negative, and since (4.18) is \( -m \), the lower branch bolt conserved mass will always be positive, and thus greater than the de-Sitter mass, violating the conjecture. Also, the upper branch \( (\rho_b = \rho_{b\pm}) \) is negative for \( N < 0.2066200733 \), and thus the upper branch solution also violates the conjecture for \( N \) less than this value. (This trend holds for higher values of \( q \), with the cross-over point for the upper branch solution increasing with increasing \( q \)).

The bolt action is, using (4.17) and (4.26)

\[ I_{C,bolt} = -\frac{\pi(\rho_b^4 + \ell^2\rho_b^2 + N^2(\ell^2 + 3N^2))}{\rho_b} \left[ \frac{\rho_b}{3\rho_b^2 - 3N^2 - \ell^2} \right] \]  

and from (4.19), the bolt entropy is

\[ S_{C,bolt} = -\frac{\pi(3\rho_b^4 - (\ell^2 + 12N^2)\rho_b^2 - N^2(\ell^2 + 3N^2))}{\rho_b} \left[ \frac{\rho_b}{3\rho_b^2 - 3N^2 - \ell^2} \right] \]

Figure 4.7: Plot of the upper \( (\rho_b = \rho_{b+}) \) and lower \( (\rho_b = \rho_{b-}) \) bolt masses \( m_{b\pm} \) (for \( q = 1 \)).
It can again be checked that this satisfies the first law, though each branch must be checked separately. From this entropy, the specific heat can be found for the bolt; explicitly for each branch, it is given by

\[
C_{C,bolt}(\rho_{b\pm}) = \frac{\pi}{36N^2} \left[ q^2\ell^4 \pm \left( 144q^2\ell^4N^4 + 41472N^8 + 10368N^6\ell^2 + 24N^2\ell^6q^2 + q^4\ell^8 \right) \right]^{\frac{1}{2}} \quad (4.29)
\]

Plots of the entropy and specific heat for the upper and lower branch solutions (for \( q = 1 \)) appear in Figures 4.8, 4.9. From Figure 4.8, we can see that the entropy for the upper branch solution is positive for \( N < .3562261982\ell \), and the specific heat is always positive; thus the upper branch solution is thermodynamically stable for \( N < .3562261982\ell \). However, for the lower branch solutions, while the entropy is always positive, the specific heat is always negative, and so the lower branch bolt solution is always thermodynamically unstable. Note that this trend continues for \( q > 1 \). Note also that the lower branch entropy is greater than the pure dS value for \( N > .3716679966\ell \), showing the lower branch violates the N-bound above this value of \( N \). Similarly, note that the upper branch entropy violates the N-bound for \( N < .2180098653\ell \).

5 Six Dimensional Analysis

5.1 R-approach in 6 dimensions

The \((5 + 1)\) dimensional form of the metric (3.6), with \( k = 2 \), will have in the R-approach

\[
V(\tau) = \frac{3\tau^6 + (-\ell^2 + 15n^2)\tau^4 + 3n^2(-2\ell^2 + 15n^2)\tau^2 - 3n^4(-\ell^2 + 5n^2) + 6m\tau^2\ell^2}{3(\tau^2 + n^2)\ell^2} \quad (5.1)
\]

where \( n \) is the non-vanishing NUT charge and \( \Lambda = \frac{10}{\ell^2} \). The coordinate \( t \) parameterizes an \( S^1 \) Hopf fibered over the non-vanishing \( S^2 \times S^2 \) base space, parameterized by \((\theta_1, \phi_1, \theta_2, \phi_2)\) . It must
have periodicity $\frac{12\pi|n|}{q}$ to avoid conical singularities, where $k$ is a positive integer. The geometry of a constant-$\tau$ surface is that of a Hopf fibration of $S^1$ over $S^2 \times S^2$ which is a well defined hypersurface in spacetime where $V(\tau) > 0$ outside of the past/future cosmological horizons. The spacelike Killing vector $\partial/\partial t$ has a fixed point set where $V(\tau_c) = 0$ whose topology is that of a $S^2 \times S^2$ base space. Since $\frac{\partial}{\partial \phi_1}$ and $\frac{\partial}{\partial \phi_2}$ are Killing vectors, for any constant $(\phi_1, \phi_2)$-slice near the horizon $\tau = \tau_c$ additional conical singularities will be introduced in the $(t, \tau)$ Euclidean section unless $t$ has period

$$\beta_{R,6d} = \frac{4\pi}{|V'(\tau_c)|}$$

(5.2)

This period must be equal to $\frac{12\pi|n|}{q}$, which forces $\tau_c = \tau^\pm_c$ where

$$\tau^\pm_c = \frac{q\ell^2 \pm \sqrt{q^2\ell^4 - 900n^4 + 180n^2\ell^2}}{30n}$$

(5.3)

and the spacetime exists only for the following NUT charges:

$$|n| \leq \ell \sqrt{90 + 30 \sqrt{q^2 + 9}}$$

(5.4)

The mass parameters are given by

$$m_R = \frac{-3\tau^6_c - \tau^4_c(\ell^2 - 15n^2) - \tau^2_c n^2(6\ell^2 - 45n^2) + 3n^4(\ell^2 - 5n^2)}{6\ell^2\tau_c}$$

(5.5)

The conserved mass and the action near future infinity are found to be

$$\mathcal{M}_{R,6d} = -8\pi m_R - \frac{\pi}{54\ell^2\tau}(2205n^4\ell^2 - 10773n^6 - \ell^6 - 63n^2\ell^4) + \mathcal{O}\left(\frac{1}{\tau^2}\right)$$

(5.6)

$$I_{R,6d} = -\frac{2\beta_{R,6d}}{3\ell^2}(3\tau^5_c + 10n^2\tau^3_c + 15n^4\tau_c + 3m_R\ell^2) + \mathcal{O}\left(\frac{1}{\tau}\right)$$

(5.7)

Applying the Gibbs-Duhem relation $S_{R,6d} = \beta_{R,6d}\mathcal{M}_{R,6d} - I_{R,6d}$, the total entropy at future infinity can be found

$$S_{R,6d} = \frac{2\pi\beta_{R,6d}(3\tau^5_c + 10n^2\tau^3_c + 15n^4\tau_c - 9m_R\ell^2)}{3\ell^2}$$

(5.8)

where $\beta_{R,6d}$ is given by:

$$\beta_{R,6d} = \frac{6\pi(\tau^2_c + n^2)^3\ell^2}{|3\tau^7_c + 9\tau^5_c n^2 + \tau^3_c n^2(4\ell^2 - 15n^2) - 9m\ell^2\tau^2_c + n^4\tau_c(75n^2 - 12\ell^2) + 3m\ell^2n^2|}$$

(5.9)

Figures (5.1), (5.2), (5.3) and (5.4) show the conserved masses and entropies for two different branches of six dimensional R-approach spacetime with $q = 1$ and $q = 3$. 

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Figure 5.1: Mass of $R^+_6$ with $q = 1$ (solid) and $q = 3$ (dotted).

Figure 5.2: Mass of $R^-_6$ with $q = 1$ (solid) and $q = 3$ (dotted).

Figure 5.3: Entropy of $R^+_6$ with $q = 1$ (solid) and $q = 3$ (dotted).

Figure 5.4: Entropy of $R^-_6$ with $q = 1$ (solid) and $q = 3$ (dotted).
Figure 5.5: \(c\)-function of \(R^+_6\) solution versus \(\tau\) with different values of \(q = 1\) (solid) and \(q = 3\) (dotted). The two plots overlap.

Figure 5.6: \(c\)-function of \(R^-_6\) solution versus \(\tau\) with different values of \(q = 1\) (solid) and \(q = 3\) (dotted). The two plots overlap.

Figure (5.1) shows that for all positive NUT charge, \(R^+_6\) has a positive mass. The mass of \(R^-_6\) (with \(q = 1\)) for \(n < 0\) is also positive.

Using equations (5.6), (5.5) and (5.3), the R mass is

\[
M^+_R = \frac{\pm 2\pi}{759375n^5\ell^2} \left\{ \sqrt{q^2\ell^4 + 180n^2\ell^2 - 900n^4(810000n^8 - 54000n^6\ell^2)} - 1350n^4\ell^4 + 450n^4q^2\ell^4 + 60n^2q^2\ell^6 + q^4\ell^8 \right\} \pm 150n^2q^3\ell^8 \pm q^5\ell^{10}
\]  

and from (5.8), the entropy is

\[
S^+_R = \frac{\pm \pi^2(\pm q^2\ell^2 + 90n^2 + q\sqrt{q^2\ell^4 + 180n^2\ell^2 - 900n^4(810000n^8 - 54000n^6\ell^2)}}{101250n^4q\ell^2} \left\{ \sqrt{q^2\ell^4 + 180n^2\ell^2 - 900n^4 + 150n^4q^2\ell^6 + 4050n^4\ell^6q \pm q^5\ell^{10} \pm 180n^2\ell^8q^3} \right\} / \left\{ \sqrt{q^2\ell^4 + 180n^2\ell^2 - 900n^4(q^5\ell^4 + 675n^4q^4\ell^2 - q^6\ell^6 - 18225n^6)} - \sqrt{q^2\ell^4 + 180n^2\ell^2 - 900n^4(q^5\ell^4 + 6075n^4q + 180n^2\ell^2q^3 - 225n^4q^3)} \right\}
\]  

The entropy for both branches satisfies the first law \(dS^+_R = \beta^\pm dM^\pm_R\).

The six-dimensional \(c\)-function is given by

\[
c = (G_\mu\nu n^\mu n^\nu)^{-2} = \frac{1}{(G_{\tau\tau})^2}
\]  

where \(n^\mu\) is the unit normal vector to a constant \(\tau\)-slice. In figures (5.5) and (5.6), the diagrams of a Taub-Bolt-dS spacetimes \(c\)-functions outside the cosmological horizon with \(\ell = 1\) and \(n = 0.25\) for two cases \(q = 1\) and \(3\) are plotted.

As one can see from these figures, outside the cosmological horizon, the \(c\) -function is a monotonically increasing function of coordinate \(\tau\), showing the expansion of a constant \(\tau\)-surface of the
metric (3.6) outside of the cosmological horizon. We note that the behavior of the $c$-function is rather insensitive to $q$.

### 5.2 C-approach in 6 dimensions

In this approach the metric now has the form (3.24) with $k = 2, d = 5$, and will have

$$F(\rho) = \frac{3\rho^6 - (\ell^2 + 15N^2)\rho^4 + 3N^2(2\ell^2 + 15N^2)\rho^2 + 3N^4(\ell^2 + 5N^2) + 6m\rho\ell^2}{3(\rho^2 - N^2)^2 \ell^2}$$

(5.13)

where $N$ is again the non-vanishing NUT charge and the cosmological constant is now given by $\Lambda = \frac{10}{\ell^2}$. The periodicity condition becomes

$$\beta_{C,6d} = \frac{4\pi}{|F'(\rho)|} = \frac{12\pi |N|}{q}$$

(5.14)

to avoid conical singularities in 6 dimensions.

The geometric interpretation is fraught with the same difficulties as its 4 dimensional counterpart. Notwithstanding these issues, we shall proceed as before.

The action follows from (3.32) with $k = 2$

$$I_{C,6d} = -\frac{2\pi \beta (3\rho_+^5 + 15N^4\rho_+^3 + 3m\ell^2 - 10N^2\rho_+^3)}{3\ell^2}$$

(5.15)

with $\rho_+$ the largest positive root of $F(\rho)$, determined by the fixed point set of $\partial_T$, and $m = m_{C,6d}$ the mass parameter for six dimensions.

Working at future infinity, the conserved mass is

$$m_{C,6d} = -8\pi m_{C,6d} - \frac{\pi (63\ell^4 N^2 + 2205N^4\ell^2 + 10773N^6 - \ell^6)}{54\rho^2} + \mathcal{O}\left(\frac{1}{\rho^2}\right)$$

(5.16)

and the total entropy is

$$S_{C,6d} = \frac{2\pi \beta (3\rho_+^5 - 10N^2\rho_+^3 + 15N^4\rho_+ - 9m\ell^2)}{3\ell^2}$$

(5.17)

These equations are generic, and can be analyzed for the specific 6-dimensional Taub-NUT-C and Taub-Bolt-C cases. When $\rho_+ = N$, $F(\rho = N) = 0$ and the fixed point set of $\partial_T$ is 2-dimensional, giving the NUT solution; when $\rho_+ = \rho_{b\pm} > N$, the fixed point set is 4-dimensional, giving the bolt solutions.

#### 5.2.1 Taub-NUT-C Solution

For the NUT solution, $\rho_+ = N$, and the NUT mass is

$$m_{C,6n} = -\frac{4N^3(\ell^2 + 6N^2)}{\ell^2}$$

(5.18)
It is easily seen from this that \( m_{\text{C,n6}} \) is always negative, which (from (5.16)) will give a positive conserved mass at future infinity. In the flat space limit, the NUT mass will go to \(-\frac{4}{3}N^3\), and in the high temperature limit, \( m_{\text{C,n6}} \) goes to 0.

The period in six dimensions \((q = 1)\) is \( \beta = 12\pi N \), so from (5.15, 5.17),

\[
I_{\text{C,NUT6d}} = \frac{32\pi^2 N^4 (\ell^2 + 4N^2)}{\ell^2} \quad (5.19)
\]

\[
S_{\text{C,NUT6d}} = \frac{32\pi^2 N^4 (3\ell^2 + 20N^2)}{\ell^2} \quad (5.20)
\]

(5.20) and (5.16) with \( m = m_{\text{C,n6}} \) can be shown to satisfy the first law \( dS = \beta dM \). In the flat space limit, \( I_{\text{C,NUT6d}} \rightarrow 32\pi^2 N^4 \), and \( S_{\text{C,NUT6d}} \rightarrow 96\pi^2 N^4 \). Both the action and the entropy go to 0 in the high temperature limit.

The specific heat in six dimensions can also be calculated,

\[
C_{\text{C,NUT6d}} = -\frac{384\pi^2 N^4 (\ell^2 + 10N^2)}{\ell^2} \quad (5.21)
\]

where this will go to \(-384\pi^2 N^4\) in the flat space limit, and will go to 0 in the high temperature limit.

In six dimensions, it can be seen (see Figure 5.7) that the entropy is always positive, and the specific heat is always negative. This is opposite to what occurs in four dimensions, though as in four dimensions, this means that the NUT solution is thermodynamically unstable.

### 5.2.2 Taub-Bolt-C Solution

In this case the fixed point set of \( \partial_T \) is 4 dimensional, giving \( \rho_+ = \rho_{b\pm} > N \). The conditions for a regular bolt solution are now (i) \( F(\rho) = 0 \), and (ii) \( F'(\rho) = \pm \frac{q}{3N} \) (where (ii) comes from the
second equality in (5.14)). From (i), the bolt mass is given by

\[
m_{C,b}^{\pm} = -\frac{3\rho_b^6 - (\ell^2 + 15N^2)\rho_b^4 + N^2(6\ell^2 + 45N^2)\rho_b^2 + 3N^4(\ell^2 + 5N^2)}{6\ell^2\rho_b}
\]  
(5.22)

\(\rho_b^{\pm}\) is given by (ii+)

\[
\rho_b^{\pm} = q\ell^2 \pm \sqrt{q^2\ell^4 + 900N^4 + 180N^2\ell^2}
\]  
(5.23)

Note again that the discriminant of \(\rho_b^{\pm}\) will always be positive, and so there will be no limit on \(N\) (except \(N > 0\)). The flat space and high temperature limits of \(\rho_b^{+}\) are infinite; the flat space limit of \(\rho_b^{-}\) is \(-\frac{3N}{q}\), and the high temperature limit is 0.

The period for the bolt is found from the first equality in (5.14)

\[
\beta_{C,Bolt6d} = 6\pi \left| \frac{(\rho_b^2 - N^2)^3\ell^2}{3\rho_b^6 - 9\rho_b^4N^2 - N^2(4\ell^2 + 15N^2)\rho_b^3 - 9m\ell^2\rho_b^2 - N^4(12\ell^2 + 75N^2)\rho_b - 3m\ell^2N^2} \right|
\]  
(5.24)

The temperature of the NUT and bolt solutions can again be shown to be the same, by substituting in \(m = m_{C,b6}\) and either of \(\rho_b^{\pm}\).

Substituting \(\rho_b = \rho_b^{\pm}\) into (5.22), we can plot the mass vs. \(N\) (see Figure 5.8). The upper branch mass is always negative, and the lower branch mass is always positive. Since the conserved mass is again negative at future infinity (5.16), this will mean that in six dimensions, the bolt upper branch conserved mass will always be positive, and the bolt lower branch conserved mass negative. Note that this is different than the four dimensional case, where the upper branch solution varied from positive to negative.
Now, from (5.15), using (5.22) and (5.23), the action is
\[
I_{C,Bolt} = -\frac{4\pi^2(3\rho^6 + (\ell^2 - 5N^2)\rho^4 - N^2(6\ell^2 + 15N^2)\rho^2 - 3N^4(\ell^2 + 5N^2))}{3|5N^2 - 5\rho^2 + \ell^2|} \tag{5.25}
\]
\[
= \frac{2\pi^2}{253125} \left[ -\ell^4(20250N^4 + 750N^4q^2 + 300q^2\ell^2N^2 + q^4\ell^4) \right. \\
\left. + \frac{N^4}{\ell^2q^4N^4}(-q^2\ell^4 + 600N^4 - 30N^2\ell^2)(q^2\ell^4 + 900N^4 + 180N^2\ell^2)^{3/2} \right]
\]
and from (5.17), the entropy is
\[
S_{C,Bolt} = \frac{4\pi^2(15\rho^6 - (3\ell^2 + 65N^2)\rho^4 + 3N^2(6\ell^2 + 55N^2)\rho^2 + 9N^4(\ell^2 + 5N^2))}{3|5N^2 - 5\rho^2 + \ell^2|} \tag{5.26}
\]
\[
= \frac{2\pi^2}{50625} \left[ \frac{(q^4\ell^4 + 180N^2q^2\ell^2 + 150N^4q^2 + 4050N^4)\ell^4}{N^4} \right. \\
\left. \pm \frac{N^4}{\ell^2q^4N^4}(q^4\ell^8 + 90N^2q^2\ell^6 - 300N^4q^2\ell^4 + 27000N^6\ell^2 + 540000N^8)\sqrt{q^2\ell^4 + 900N^4 + 180N^2\ell^2} \right]
\]
This entropy does satisfy the first law, though note that both branches must be checked separately. The specific heat (explicitly for each branch) in six dimensions is given by
\[
C_{C,Bolt} = \frac{8\pi^2}{50625} \left[ \frac{\ell^6q^2(90N^2 + q^2\ell^2)}{N^4} \right. \\
\left. \pm \frac{(q^2\ell^2 + 30N^2)(q^2 - 30N^2)\ell^4}{N^4q^2\ell^4 + 900N^4 + 180N^2\ell^2} \left( q^4\ell^8 + 180N^2q^2\ell^6 + 1350N^4q^2\ell^4 \\
+ 4050N^4\ell^4 + 162000N^6\ell^2 + 810000N^8 \right) \right] \tag{5.27}
\]
Plots of the entropy and specific heat for the upper and lower branch solutions (for \(q = 1\)) appear in Figures 5.9, 5.10. From Figure 5.9 we can see that the entropy for the upper branch solution is always positive, and the specific heat is positive for \(N < 0.2014312523\ell\); thus the upper branch solution is thermodynamically stable for \(N\) less than this. However, for the lower branch solutions, the entropy is always negative and the specific heat always positive, and so the lower branch bolt solution is always thermodynamically unstable. Note that this trend continues for \(q > 1\).

6 Discussion

We have extended the use of the path-integral formalism to include quantum correlations between timelike histories. By employing this formalism in the semiclassical approximation we have been able to extend our notions of conserved quantities (such as mass and angular momentum), actions and entropies outside of cosmological horizons. Applying this formalism to Schwarzschild de Sitter spacetimes we find that the values of these quantities are in accord with our physical expectations, as previously shown in refs.\[1\] [10] [11].
When we extend this formalism to NUT-charged spacetimes we find that the situation is considerably modified. First, NUT-charged spacetimes present us with two possible ways (the R-approach and the C-approach) in which we can apply our formalism, depending on how the spacetime is analytically continued. Moreover, there exist broad ranges of parameter space for which NUT-charged spacetimes violate both the maximal mass conjecture and the N-bound, in both four dimensions and in higher dimensions. We present in tables 1 and 2 the results for dimensions 4, 6, 8, 10 and general \((d + 1)\) dimensions. We find the thermodynamic behaviour for the \(4k\)-dimensional spacetimes to be qualitatively similar in the R-approach, with the lower branch entropy always negative, and the upper branch solutions always having a range of \(n\) in which both the entropy and the specific heat are positive. Likewise the \((4k + 2)\)-dimensional spacetimes have qualitatively similar thermodynamics, behaving as illustrated in figures (5.1), (5.2), (5.3) and (5.4). In the C-approach we likewise find a similarity in the thermodynamics of the \(4k\)-dimensional spacetimes, distinct from the common behaviour of the \((4k + 2)\)-dimensional ones.

These results suggest that there may be some limitations to the application of the holographic conjecture to spacetimes with \(\Lambda > 0\). For example, one implication of the N-bound (and the maximal mass conjecture) is that a quantum gravity theory with an infinite number of degrees of freedom (such as M-theory) cannot describe spacetimes with \(\Lambda > 0\) \[^{12}\]. Our results suggest that this obstruction is not necessarily an obstruction in principle, but can be overcome in spacetimes with NUT charge that are locally asymptotically dS. One might wish to restrict the appearance of such spacetimes in the spectrum of states of quantum gravity since they contain contain causality-violating regions with closed timelike curves. The mechanism for so doing remains an unsolved problem.

The entropy-area relation \(S = A/4\) is satisfied for any black hole in a \((d+1)\)-dimensional aAdS or aF, where \(A\) is the area of a \((d - 1)\)-dimensional fixed point set of isometry group. However, the entropy can defined for other kinds of spacetimes in which the isometry group has fixed points on surfaces of even co-dimension \[^{21}\]. The best examples of these spacetimes are asymptotically

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**Figure 5.9:** Plot of the upper branch bolt entropy and specific heat (for \(q = 1\)) for six dimensions.

**Figure 5.10:** Plot of the lower branch bolt entropy and specific heat (for \(q = 1\)) for six dimensions.
In locally flat and asymptotically locally AdS spacetimes with NUT charge. In these cases when the isometry group has a two-dimensional fixed set (bolt), the entropy of the spacetime is not given by the area-entropy relation, since there is a contribution to the entropy, coming from the Misner string [22].

In asymptotically dS spacetimes, the Gibbs-Duhem entropy (2.24) is less than the area of the horizon and respects the N-bound (for the case of Schwarzschild-dS spacetime, see [11]). However, for asymptotically locally dS spacetime with NUT charge, we have an additional contribution to the entropy (2.24) from the Misner string. Consequently the entropy need not respect the N-bound, and we find that there are a wide range of situations where it does not.

In fact for positive NUT charge, for $R_+^+$, the fixed-$t$ area of the cosmological horizon exceeds that of pure dS spacetime for certain range of values of the NUT charge. Consequently if one interprets the N-bound in terms of a relationship between horizon areas (as opposed to entropies), we still find that (within this range) the N-bound is violated. For $R_-^-$, the fixed-$t$ area of the cosmological horizon is less than the cosmological horizon area of pure dS spacetime for all values of NUT charge, and so the re-interpreted N-bound is respected. In the $R_+^+$ case, the Gibbs-Duhem entropy is larger than one-quarter of the horizon area, which in turn is larger than the cosmological horizon area $\pi \ell^2$ of pure dS spacetime. In the $R_-^-$ case, these inequalities are reversed, with $\pi \ell^2$ always greater than the Gibbs-Duhem entropy, and both the Gibbs-Duhem entropy and one-quarter the area of cosmological horizon respect N-bound. Figures (6.1) and (6.2) show the behaviour of entropies for the $R_+^+$ and $R_-^-$. Both the robustness of our formalism and its physical relevance remain subjects for future study. The entropy defined by the Gibbs-Duhem relation (2.24) would appear to have the requisite properties: it is positive and monotonically increasing with conserved mass for the Schwarzschild de Sitter case, and obeys the first law of thermodynamics for all cases we have considered so far (indeed, since our definition is built on the path integral formalism, it is hard to see how it could be otherwise). However the applicability of the second law remains an outstanding problem: in what
sense can we say that the entropy always increases in any physical process in this context? Even more intriguing is the relationship between this entropy and the underlying degrees of freedom that it presumably counts.

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Table 1: Summary of General R-approach quantities

| Dim. | Radial ($\tau_{c\pm}$) | Period | $\mathcal{M}$ | Action |
|------|------------------------|--------|---------------|--------|
| 4    | $l^2 \pm \sqrt{l^4 - 144n^4 + 48n^2 l^2}$ | $8\pi N$ | $-m$ | $-\frac{\beta(m^2 + \tau_c^3 + 3\tau_c n^2)}{2l^2}$ |
| 6    | $l^2 \pm \sqrt{l^4 - 900n^4 + 180n^2 l^2}$ | $12\pi N$ | $-8\pi m$ | $-\frac{2\beta\pi}{3l^2} \left( 3m\ell^2 + 12\tau_c^3 + 60n^2\tau_c + 15n^4\tau_c^3 \right)$ |
| 8    | $l^2 \pm \sqrt{l^4 - 3136n^4 + 448n^2 l^2}$ | $16\pi N$ | $-48\pi^2 m$ | $-\frac{2\beta\pi^2}{5l^2} \left( 5m\ell^2 + 21\tau_c^5 + 21n^2\tau_c^5 + 35n^4\tau_c^5 \right)$ |
| 10   | $l^2 \pm \sqrt{l^4 - 8100n^4 + 900n^2 l^2}$ | $20\pi N$ | $-256\pi^3 m$ | $-\frac{32\beta\pi^3}{35l^2} \left( 35m\ell^2 + 35\tau_c^9 + 210n^2\tau_c^9 + 378n^4\tau_c^9 + 420n^6\tau_c^9 + 315n^8\tau_c^9 \right)$ |
| (d+1)| $\frac{2}{n(d+1)} = \frac{4\pi}{|V'(|)}$ | $\frac{2(d+1)\pi n}{q}$ | $\frac{(4\pi)^k k m}{4\pi}$ | $-\frac{\beta(4\pi)^k}{8\pi l^2} \left[ m\ell^2 + d\left( \sum_{i=0}^{k} \binom{k}{i} n^{2i} \tau_c^{2k-2i+1} \right) \right]$ |
Table 2: Summary of General C-approach quantities

| Dim. | $\rho_+$ | Period | $\mathcal{M}$ | Action |
|------|----------|--------|--------------|--------|
| 4    | $\rho_+ = N$ | $8\pi N$ | $-m$ | $\frac{\beta(-m\ell^2-\rho_1^3+3N^2\rho_+)}{2\pi^2}$ |
|      | $\rho_{b\pm} = \frac{\ell^2 \pm \sqrt{\ell^4 + 144N^4 + 48N^2\ell^2}}{12N}$ |          |              |        |
| 6    | $\rho_+ = N$ | $12\pi N$ | $-8\pi m$ | $\frac{-23\pi}{3\ell^2} \left(3m\ell^2 + 3\rho_+^5 - 10N^2\rho_+^3 + 15N^4\rho_+\right)$ |
|      | $\rho_{b\pm} = \frac{\ell^2 \pm \sqrt{\ell^4 + 900N^4 + 180N^2\ell^2}}{30N}$ |          |              |        |
| 8    | $\rho_+ = N$ | $16\pi N$ | $-48\pi^2 m$ | $\frac{8\beta\pi}{5\ell^2} \left(-5m\ell^2 - 5\rho_+^7 + 21N^2\rho_+^5 - 35N^4\rho_+^3 + 35N^6\rho_+\right)$ |
|      | $\rho_{b\pm} = \frac{\ell^2 \pm \sqrt{\ell^4 + 3136N^4 + 448N^2\ell^2}}{56N}$ |          |              |        |
| 10   | $\rho_+ = N$ | $20\pi N$ | $-256\pi^3 m$ | $\frac{-32\beta\pi^3}{35\ell^2} \left(35m\ell^2 + 35\rho_+^9 - 180N^2\rho_+^7 + 378N^4\rho_+^5 - 420N^6\rho_+^3 + 315N^8\rho_+\right)$ |
|      | $\rho_{b\pm} = \frac{\ell^2 \pm \sqrt{\ell^4 + 8100N^4 + 900N^2\ell^2}}{90N}$ |          |              |        |
| (d+1)| $\rho_+ = N$ | $2(d+1)\pi n$ | $-(4\pi)^k km - \frac{\beta(4\pi)^k}{8\pi^2} \left[\frac{m\ell^2}{2(\ell^2)}\right]$ | $+d\left(\sum_{i=0}^{k} \binom{k}{i} (-1)^i N^{2i} \rho_+^{2k-2i+1}\right)$ |
|      | $\frac{\rho_+^2}{N(d+1)} = \frac{4\pi}{|E'(|\rho)|}$ |          |              |        |
| dim. | R-approach | C-approach |
|------|------------|------------|
| 4    | $\frac{\beta(\tau_3^3+3n^2\tau_c-\ell^2)}{2\ell^2}$ | $\frac{\beta(\rho_+^3-3N^2\rho_+-\ell^2)}{2\ell^2}$ |
| 6    | $\frac{2\pi\beta}{3\ell^2} \left( 3\tau_c^5 + 10n^2\tau_c^3 + 15n^4\tau_c - 9\ell^2 \right)$ | $\frac{2\pi\beta}{3\ell^2} \left( 3\rho_+^5 - 10N^2\rho_+^3 + 15N^4\rho_+ - 9\ell^2 \right)$ |
| 8    | $\frac{8\pi^2\beta}{5\ell^2} \left( 5\tau_c^7 + 21n^2\tau_c^5 + 35n^6\tau_c^3 - 25\ell^2 \right)$ | $-\frac{8\pi^2\beta}{5\ell^2} \left( -5\rho_+^7 + 21N^2\rho_+^5 - 35N^4\rho_+^3 + 35N^6\rho_+ + 25\ell^2 \right)$ |
| 10   | $\frac{32\pi^3\beta}{35\ell^2} \left( 35\tau_c^9 + 180n^2\tau_c^7 + 378n^4\tau_c^5 + 420n^6\tau_c^3 + 315n^8\tau_c - 245\ell^2 \right)$ | $\frac{32\pi^3\beta}{35\ell^2} \left( 35\rho_+^9 - 180N^2\rho_+^7 + 378N^4\rho_+^5 - 420N^6\rho_+^3 + 315N^8\rho_+ - 245\ell^2 \right)$ |
| (d+1)| $(\frac{4\pi)^k\beta}{8\pi\ell^2 \left[ d \sum_{i=0}^{k} \binom{k}{i} \frac{n^{2k-2i-1}}{2k-2i+1} - m\ell^2(2k-1) \right]}$ | $(\frac{4\pi)^k\beta}{8\pi\ell^2 \left[ d \sum_{i=0}^{k} \binom{k}{i} (-1)^iN^{2i}\rho_+^{2k-2i+1} - m\ell^2(2k-1) \right]}$ |