A new construction of
the Drinfeld–Sokolov Hierarchies

Paolo Casati,

Dipartimento di Matematica e applicazioni
Università di Milano-Bicocca
Via Cozzi 55, I-20125 Milano, Italy

E–mail: paolo.casati@unimib.it

Abstract
The Drinfeld–Sokolov hierarchies are integrable hierarchies associated with every affine Lie algebra. We present a new construction of such hierarchies, which only requires the computations of a formal Laurent series.

Key Words: Integrable Hierarchies of PDE, Lie algebras, Irreducible Representations, Drinfeld–Sokolov construction.

Mathematics Subjects Classification: Primary 37K10 Secondary 53D5
1 Introduction

The discovery made by Drinfeld and Sokolov in their celebrated paper [18], that to any affine Kac–Moody Lie algebra and to any vertex of its (extended) Dynkin diagram corresponds a hierarchy of completely integrable nonlinear partial differential equations, is a hallmark in the theory of integrable systems. These hierarchies (usually called Drinfeld–Sokolov hierarchies) have been further and intensively studied in the last twenty years by many mathematicians with the help of various techniques; [11] [2] [3] [5] [6] [8] [9] [20] [13] [21] [19] [14] [23] [4] [27] [17] [25] (in particular see [17] for a very beautiful and recent paper on this subject).

The aim of this paper is to provide a new simple and algorithmic way to construct the equations and the conserved quantities of the Drinfeld and Sokolov hierarchies corresponding to the untwisted affine Kac–Moody Lie algebras $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and the vertex $c_0$ of the Dynkin diagram (i.e., the one added to the Dynkin diagram of the associated simple Lie algebra).

Such construction may be briefly described as follows. Let $h(z) = z + \sum_{i>0} h_i z_i$ be a formal Laurent series in $z$ whose coefficients $h_i$ are smooth functions from the unit circle $S^1$ in $\mathbb{C}$: $h_i \in C^\infty(S^1, \mathbb{C})$ and define its Faà di Bruno iterates as $h^{(k)}(1)$ where $\partial_x$ is the derivative along the coordinate $x$ on $S^1$ (thus $h^{(0)} = 1$, $h^{(1)} = h(x)$, $h^{(2)} = h_x + h^2$...). Then consider the following constrains in terms of ODE’s of Riccati type on the Laurent series $h(x)$:

$$
z^{n+1} = h^{(n+1)} - \sum_{i=0}^{n-1} u_k h^{(i)}$$

for the Lie algebra $A_n^{(1)}$.

$$
z^{2n} = h^{(2n+1)} - \sum_{k=0}^{n-1} u_k h^{(2k+1)} = \sum_{k=0}^{n-1} (\partial_x + h)^{(2k+1)}(u_k)$$

for the Lie algebra $B_n^{(1)}$.

$$
z^{2n+1} = h^{(2n)} - \sum_{k=0}^{n-1} u_k h^{(2k)} - \sum_{k=0}^{n-1} (\partial_x + h)^{(2k)}(u_k)$$

for the Lie algebra $C_n^{(1)}$.

$$
z^{2n-2} = h^{(2n-1)} - \sum_{k=1}^{n-1} u_k h^{(2k-1)} - \sum_{k=0}^{n-1} (\partial_x + h)^{(2k-1)}(u_k)$$

$- u_0 h_{u_0}^{(-1)}$

for the Lie algebra $D_n^{(1)}$

where the Laurent series $h_{u_0}^{(-1)}$ is uniquely defined by the relation $(\partial_x + h)(h_{u_0}^{(-1)}) = u_0$.

The coefficients of the formal Laurent series, which solve such equations, can be computed recursively in a pure algebraic way (i.e., without performing any integrations) as differential polynomials in the free fields $u_i$.

The above constructed Faà di Bruno polynomials define on the space $\mathcal{L}$ of all Laurent series which are truncated from above i. e., the space

$$
\mathcal{L} := \{ \sum_{-\infty}^{N} l_i z_i | l_i \in \mathbb{C}, N \in \mathbb{Z} \}
$$

the splitting

$$
\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_- = \text{span}[h^{(k)} | k \geq 0] \oplus \text{span}[z^k | k < 0].
$$
Let $\pi_+$ be the corresponding projection of $\mathcal{L}$ onto $\mathcal{L}_+$ and let $\mathcal{N}_{\hat{g}}$ be the subset of $\mathbb{N}$ equal to $\{i \in \mathbb{N} | i \neq 0 \mod (n + 1)\}$ if the considered Kac–Moody algebra $\hat{g}$ is of type $A_1^{(n)}$ and $\{2i + 1 | i \in \mathbb{N}\}$ if it is of type $B, C$, or $D$. Define $H^{(k)} \in \mathcal{L}$ as

$$H^{(k)} = \pi_+ (z^k) \quad k \in \mathcal{N}_{\hat{g}}.$$

Then the equations of the corresponding hierarchies of integrable PDE will have the form

$$\frac{\partial h}{\partial t_k} = \partial_x H^{(k)} \quad k \in \mathcal{N}_{\hat{g}}.$$

To achieve such results we shall mainly use an approach to the Drinfeld–Sokolov reduction based on a bihamiltonian reduction theorem \cite{10} \cite{11}, \cite{4} and the theory of simple and affine Lie algebras \cite{22}.

The paper is organized as follows. In the second section, divided in two subsections, we briefly review the bihamiltonian theory of the integrable system, recall the bihamiltonian reduction theorem, and define a bihamiltonian structure on the affine Kac–Moody Lie algebras describing its property. In the third and final section, divided in three subsections, we finally described into details the construction of the Drinfeld–Sokolov hierarchies outlined above.

The author wishes to thank Andriy Panasyuk for many useful discussions about the geometry of the bihamiltonian manifolds, and professor Laszlo Feher, for appreciating this work and for drawing my attention to the papers \cite{20} \cite{13} \cite{21} \cite{19} \cite{14}.

## 2 Preliminary results

In this section we recall some background material mainly concerning the bihamiltonian theory of the integrable systems and the affine Kac–Moody Lie algebras and their bihamiltonian structures. Detailed descriptions of these topics may be found in the papers \cite{10} \cite{11} \cite{9} \cite{5} \cite{17} \cite{26} or in the books \cite{16} \cite{1}.

### 2.1 The bihamiltonian theory of the integrable systems

A Poisson manifold is a smooth manifold $\mathcal{M}$ endowed with a Poisson bracket $\{\cdot, \cdot\}$ i.e., a bilinear skew symmetric composition law of $C^\infty(\mathcal{M}, \mathbb{C})$ fulfilling the Leibniz rule and the Jacobi identity. The corresponding Poisson tensor $P$ is the bivector tensor field on $\mathcal{M}$, viewed as a linear skew symmetric map between the cotangent and the tangent space bundle: $P : T^* \mathcal{M} \to T \mathcal{M}$, defined at any point $m \in \mathcal{M}$ by

$$\langle F, G \rangle (m) = \langle dF, P_m dG \rangle_m \quad \forall F, G \in C^\infty(\mathcal{M}, \mathbb{C})$$

where $\langle \cdot, \cdot \rangle_m$ is the pairing between $T^*_m \mathcal{M}$ and $T_m \mathcal{M}$.

A bihamiltonian manifold $\mathcal{M}$ is a smooth manifold equipped with two compatible Poisson brackets, i.e., two Poisson brackets $\{\cdot, \cdot\}_0$ and $\{\cdot, \cdot\}_1$ such that the pencil

$$\{\cdot, \cdot\}_1 = \{\cdot, \cdot\}_1 - \lambda \{\cdot, \cdot\}_0$$

for all $\lambda \in \mathbb{R}$.
is a Poisson bracket for any $\lambda \in \mathbb{C}$. We denote by $P_{\lambda}$ the corresponding pencil of Poisson tensors. A bihamiltonian vector field $X$ on $M$ is a vector field which is Hamiltonian with respect to both Poisson brackets (and therefore with respect to any Poisson bracket $\{\cdot,\cdot\}_{\lambda}$).

The central idea of the bihamiltonian theory of the hierarchies of PDE’s is to view them as collections of bihamiltonian vector fields on a (usually infinite dimensional) bihamiltonian manifold $M$.

There are two classical strategy to construct such integrable systems. The first one assumes the nondegeneracy of one of the Poisson tensor associated to the two compatible Poisson brackets and uses the so–called recursion operator. The second one, which will be used in this paper, uses the Casimir functions of the (Poisson) pencil. More precisely we shall look for formal Laurent series $H(\lambda) = \sum_{k \geq -N} H_k \lambda^{-k}, H_k \in C^\infty(M, \mathbb{C})$ which are Casimirs of the Poisson pencil:

$$\{F, H(\lambda)\}_{\lambda} = 0 \quad \forall F \in C^\infty(M, \mathbb{C}).$$

In fact developing this equation in powers of $\lambda$ we obtain a hierarchy $\{X_k\}_{k \geq N}$ of bihamiltonian vector fields:

$$\{F, H_N\}_0 = 0$$

$$X_k(F) = \{F, H_{k-1}\}_0 = \{F, H_k\}_1 \quad \forall F \in C^\infty(M, \mathbb{C}) \quad k \leq N.$$

In this paper we shall tackle this very hard problem in the contest of bihamiltonian brackets defined on affine Kac–Moody Lie algebras, where it can be solved by using the generalization of the dressing method of Zakharov and Shabat proposed by Drinfeld and Sokolov [18].

Actually the bihamiltonian hierarchies, which are interesting in, are not directly defined on such algebras but rather on reduced bihamiltonian manifolds, constructing by means of the

**Theorem 2.1** ([10] Prop 1.1). Let $(M, \{\cdot,\cdot\}_0, \{\cdot,\cdot\}_1)$ be a bihamiltonian manifold, $S$ a symplectic submanifold of $P_0$, $D$ and $E$ the distributions $D = P_1 \text{Ker}(P_0)$, $E = TS \cap D$. Then the distribution $E$ is integrable and, if the quotient space $N = S/E$ is a manifold, it is a bihamiltonian manifold endowed with the reduced Poisson pencil $\{\cdot,\cdot\}_1^N$ defined uniquely by the relation

$$\{f, g\}_1^N \circ \pi = \{F, G\}_1^M \circ i \quad \forall f, g \in C^\infty(N, \mathbb{C})$$

where $i$ and $\pi$ are the canonical injection of $S$ in $M$ and the canonical projection of $M$ onto $N$ respectively, and $F$ and $G$ are any pair of smooth functions, which extend the functions $f$ and $g$ of $N$ into $M$, and are constant on $D$ (i.e., $F \circ i = f \circ \pi$ and $\{F, K\}_1 = 0$ for any Casimir $K$ of $P_0$).
2.2 Affine Lie algebras as Bihamiltonian Manifolds

Let \( \mathfrak{g} \) be a simple Lie algebra over \( \mathbb{C} \), \( G \) the corresponding connected and simply connected Lie group. Fix a nondegenerate symmetric invariant bilinear form \( \langle \cdot, \cdot \rangle_\mathfrak{g} \) on \( \mathfrak{g} \). The associated (non twisted) affine Lie algebra \( \hat{\mathfrak{g}} \) is a semidirect product of the central extensions of the loop algebra \( L(\mathfrak{g}) = C^\infty(S^1, \mathfrak{g}) \) and a derivation \( d \), more precisely

**Definition 2.2** The affine (non twisted) Lie algebra \( \hat{\mathfrak{g}} \) associated with the finite dimensional simple Lie algebra \( \mathfrak{g} \) is the complex vector space

\[
\hat{\mathfrak{g}} = C^\infty(S^1, \mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}d
\]

dowed with the following Lie bracket:

\[
\left[ P(x) + \mu_1 K + \nu_1 d, Q(x) + \mu_2 K + \nu_2 d \right]_{\hat{\mathfrak{g}}} = \left[ P(x), Q(x) \right] + \nu_1 x \frac{dP(x)}{dx} - \nu_2 x \frac{dQ(x)}{dx} + \left( \int_{S^1} \langle \frac{dP(x)}{dx}, Q(x) \rangle_\mathfrak{g} \right) K
\]

where \( [\cdot, \cdot] \) is the bracket in the loop Lie algebra \( L(\mathfrak{g}) = C^\infty(S^1, \mathfrak{g}) \).

In what follows the derivation \( d \) will not play any role.

Let \( \hat{\mathfrak{g}}^* \) be the space of linear functionals on \( \hat{\mathfrak{g}} \) of the following form

\[
\mathcal{L}_{(P(x), aK)}(Q(x) + bK) = \langle (P(x) + aK), Q(x) + bK \rangle = \int_{S^1} \langle P(x), Q(x) \rangle_\mathfrak{g} dx + ab.
\]

We identify \( \hat{\mathfrak{g}}^* \) with \( \hat{\mathfrak{g}} \). Being \( \hat{\mathcal{M}} = \hat{\mathfrak{g}} \) a flat manifold we may further identify the tangent space at any point with \( \mathcal{M} \) itself. Using these identifications it is easy to compute the canonical Lie Poisson tensor of \( \hat{\mathfrak{g}} \), as

\[
P_{(S,K)}(V) = K \partial_x V + [S, V].
\]

It can be easily shown \([28]\) that this Poisson tensor is compatible with the constant Poisson tensors obtained by freezing it in any point of \( \hat{\mathfrak{g}} \).

In order to specify the choice which will lead to the Drinfeld Sokolov hierarchies we must briefly describe the structure of the simple (finite dimensional) Lie algebra \( \mathfrak{g} \).

Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \), and let \( \Delta \) be the corresponding root system. Let \( \Pi = \{ \alpha_1, \ldots, \alpha_n \} \) be a set of simple roots in \( \Delta \) (where \( n \) is the rank of \( \mathfrak{g} \)), and \( \Delta = \Delta^+ \cup \Delta^- \) the associated decomposition of \( \Delta \) in positive (resp. negative) roots, then we have the Cartan decomposition

\[
\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}
\]

where \( \mathfrak{g}_\alpha = \{ X \in \mathfrak{g} | [H, X] = \alpha(H)X \forall H \in \mathfrak{h} \} \) is the root space of \( \alpha \). Let, for any root \( \alpha \in \Delta \), \( H_\alpha \) be the corresponding coroot with respect to the bilinear form \( \langle \cdot, \cdot \rangle_\mathfrak{g} \), and for any
α ∈ Δ⁺ let Xα be a basis of gα and let Yα in g−α be defined by the requirement [Xα, Yα] = Hα. Let θ be the maximal root of Δ with respect the above defined decomposition (i.e, \( θ + α_i \notin Δ \) for all \( α_i \in Π \)); and denote by A the constant function of \( C^∞(S^1, g) \) whose value is \( Y_θ \).

The hierarchies of Drinfeld and Sokolov are bihamiltonian with respect to the reduction of the bihamiltonian pair \( P_1, P_0 \) where \( P_1 \) is the canonical Poisson tensor \( (2.3) \) and \( P_0 \) is the constant Poisson tensor

\[
(P_0)_{S,K}(V) = [A, V].
\]  

(2.4)

To perform the bihamiltonian reduction we have only to choose the appropriate symplectic leaf of \( P_0 \). Following Drinfeld and Sokolov let us choose that passing through the point

\[
B = \sum_{β \in Π} X_β + K.
\]

(In what follows we normalize the value of the central extension to −1). In this setting the integrable distribution \( E \) can be characterized by the

\[ \textbf{Theorem 2.3} \quad \text{[11]} \quad \text{The subspace} \ g_{AB} := \{ V ∈ g_0 | [V, X] + [V, B] ∈ g_α \} \text{ is a subalgebra of } g \text{ contained in the nilpotent subalgebra of loops with values in the maximal nilpotent subalgebra } n^- = \oplus_{α ∈ Δ} g_α. \text{ Therefore the corresponding group } G_{AB} = \exp(g_{AB}) \text{ is well defined. The distribution } E \text{ is spanned by the vector fields } (P_1)_B(V) \text{ with } V \text{ belonging to } g_{AB}, \text{ and its integral leaves are the orbits of the gauge action of } G_{AB} \text{ on } S \text{ defined by:}
\]

\[
S' = JSJ^{-1} + J_0J^{-1}.
\]  

(2.5)

The characterization of the chosen symplectic leaf \( S \) given in the previous Theorem plays a pivotal role in the present work. In particular it allows us to compute a submanifold of \( S \) transversal to the distribution \( E \).

\[ \textbf{Definition 2.4} \quad \text{A transversal submanifold to the distribution } E \text{ is a submanifold } Q \text{ of } S, \text{ which intersects every integral leaves of the distribution } E \text{ and therefore the orbit of the group action in one and only one point. This condition implies the following relations on the tangent space:}
\]

\[
T_qS = T_qQ ⊕ E_q \quad \forall q ∈ Q
\]  

(2.6)

3 The Drinfeld Sokolov Hierarchies

This section is devoted to present the main result of this paper: the new algorithmic construction of the Drinfeld–Sokolov hierarchies outlined in the introduction.
3.1 The Bihamiltonian setting of the Drinfeld Sokolov hierarchies

As we have already seen, we need to find Casimirs of the Poisson pencil \( P_\lambda = P_1 - \lambda P_0 \). More precisely we shall look for solutions of the Casimir’s equation

\[
P_\lambda(V(\lambda)) = V_x + [V, S + \lambda A] = 0 \quad S \in \mathcal{S}
\]

which are formal Laurent series \( V(\lambda) = \sum_{k=-\infty}^{\infty} V_k \lambda^{-k}, \ m \in \mathbb{Z} \) whose coefficients are one forms defined at least on the points of \( \mathcal{S} \) and which are exact when restricted on \( \mathcal{S} \).

It is well known that the simple Lie algebras \( A_n, B_n, C_n, D_n \) can be realized as the matrix Lie algebras \( \mathfrak{sl}(n+1, \mathbb{C}), \mathfrak{so}(2n+1, \mathbb{C}), \mathfrak{sp}(2n, \mathbb{C}), \mathfrak{so}(2n, \mathbb{C}) \). Correspondingly there exist a loop representation of \( \hat{\mathfrak{g}} \) on the infinite dimensional linear space \( \mathcal{C}^{\infty}(S^1, \mathbb{C}^N) \) (where \( N = n + 1, 2n + 1, 2n \) if \( \mathfrak{g} \) is \( A_n, B_n, C_n \) or \( D_n \) resp.) \cite{12}. In this setting it is immediate to observe that the one form \( V(\lambda) \) is a solution of equation (3.1) if and only if as operator \( \partial_x + S + \lambda A \) at any point \( S \in \mathcal{S} \) \cite{6}. To find the elements \( V(\lambda) \) commuting with \( \partial_x + S + \lambda A \) let us observe that \( \Lambda = (B + \lambda A) \) can be viewed as an element of the tensor Lie algebra \( \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}] \) where \( \mathbb{C}[\lambda, \lambda^{-1}] \) is the commutative ring of Laurent polynomials in the indeterminate \( \lambda \) over \( \mathbb{C} \). In this algebra \( \Lambda \) is a regular semisimple element and therefore its isotropic subalgebra \( \mathfrak{g}_{\mathfrak{h} \in \mathfrak{h}} \) is spanned in the case of the Kac–Moody Lie algebra of type \( A_n^{(1)}, B_n^{(1)}, C_n^{(1)} \), by the matrices \cite{18}:

\[
\begin{align*}
\Lambda^m & \quad m \in \mathbb{Z}, \ m \neq 0 \text{mod}(n+1) & \text{if } \mathfrak{g} \text{ is of type } A_n \\
\Lambda^{2m+1} & \quad m \in \mathbb{Z}, \ \text{where for } m < 0, \ \Lambda^{2m+1} \overset{\text{def}}{=} \lambda^{-k} \Lambda^{2m+1+2nk} & \text{if } \mathfrak{g} \text{ is of type } B_n \\
\Lambda^{2m+1} & \quad m \in \mathbb{Z} & \text{if } \mathfrak{g} \text{ is of type } C_n.
\end{align*}
\]

While in the more complicate case of the Lie algebra \( D_n^{(1)} \) the “Heisenberg” Lie algebra \( \mathfrak{h} \) is spanned by the matrices

\[
\Lambda^{2m+1} \quad m \in \mathbb{Z}, \ \text{where for } m < 0, \ \Lambda^{2m+1} \overset{\text{def}}{=} \lambda^{-k} \Lambda^{2m+1+(2n-2)k}
\]

together with the matrices

\[
\lambda^m F \quad m \in \mathbb{Z}, \quad F = \Phi + (-1)^n \Phi^T \quad \Phi = e_{n,1} - 2e_{n+1,1} - 2e_{n,2n} + 4e_{n+1,2n},
\]

here \( e_{ij} \) is the matrix in \( M(N, \mathbb{C}) \) with 1 in the \( ij \) position and zero otherwise and \( X^T \) denotes the transpose of \( X \). From these facts Drinfeld and Sokolov proves indeed the

**Proposition 3.1** Let \( \tilde{G} \) be the Kac–Moody group \( \tilde{G} = \mathcal{C}^{\infty}(S^1, G \otimes \mathbb{C}[\lambda, \lambda^{-1}]) \). Then for any operator of the form \( -\partial_x + S + \lambda A \) with \( S \in \mathcal{S} \) there exists a element \( T \) in \( \tilde{G} \) such that:

\[
T(-\partial_x + S + \lambda A)T^{-1} = \partial_x + (B + \lambda A) + H, \quad H \in \mathfrak{h}.
\]

Therefore the set of the elements in \( \hat{\mathfrak{g}} \otimes \mathbb{C}[\lambda, \lambda^{-1}] \) commuting with \( -\partial_x + S + \lambda A \) is given (up the central charge) by \( T^{-1} \hat{\mathfrak{h}} T \).
From Proposition 3.1 follows

**Proposition 3.2** Let $Z \in \mathcal{X}$ then:

1. The element $V_Z = T^{-1}CT$ solves equation (3.1).

2. Its hamiltonian on $S$ is the function $H_Z = \langle K, Z \rangle$ where $K$ is defined by the relation

   $$K = T(S + \lambda A)T^{-1} + T_xT^{-1}. \quad (3.3)$$

3. In particular: if $Z$ is $\Lambda^j$, $j = Nq + r$ with $q \in \mathbb{Z}$ and $0 < r < N$, then $V_{\Lambda^j}$ has the Laurent expansion

   $$V_{\Lambda^j} = \lambda^q \sum_{p \geq -2} \frac{1}{\lambda^{p+1}} V_1^{Np+r}; \quad \text{if } \tilde{g} \text{ is of the type } A_n^{(1)}, \text{ or } C_n^{(1)} \quad (3.4)$$

   $$V_{\Lambda^j} = \lambda^q \sum_{p \geq -2} \frac{1}{\lambda^{p+1}} V_1^{(N-1)p+q+r}; \quad \text{if } \tilde{g} \text{ is of type } B_n^{(1)} \quad (3.5)$$

   $$V_{\Lambda^j} = \lambda^q \sum_{p \geq -2} \frac{1}{\lambda^{p+1}} V_1^{(N-2)p+2q+r}; \quad \text{if } \tilde{g} \text{ is of type } D_n^{(1)} \quad (3.6)$$

   where $V_{1}^{k}$ denote the coefficient of $\lambda^{-1}$ of $V_{\Lambda^j}$.

4. Finally if $Z$ is $\lambda^jF$, and therefore $g$ is of the type $D_n$ then $V_{\Lambda^j}F$ has the Laurent expansion

   $$V_{\Lambda^j}F = \lambda^j \sum_{p \geq -2} \frac{1}{\lambda^{p+1}} V_1^{j+p-1}; \quad \text{where } V_{1}^{k} \text{ denote the coefficient of } \lambda^{-1} \text{ of } V_{\Lambda^j}F.$$

**Proof**

1. It follows immediately from the previous proposition.

2. Using equation (3.3) we can rewrite equation (3.2) in the form $T(-\partial_x + S + \lambda A)T^{-1} = -\partial_x + J$ showing the $K$ commutes with $C$ then:

   $$\frac{d}{dt}H_Z = \langle K, Z \rangle = \langle T \dot{S}T^{-1} + \left[TT^{-1}, K \right], Z \rangle$$

   but since $Z$ commutes with $K$ we have

   $$\frac{d}{dt}H_Z = \langle T \dot{S}T^{-1}, Z \rangle = \langle \dot{S}, T^{-1}ZT \rangle = \langle \dot{S}, V_Z \rangle.$$

3. First we observe that $T$ may be chosen of the form [18] [22]

   $$T = \exp(t) \quad \text{with } t = \sum_{k \geq 0} T_k \lambda^{-k}$$
and therefore that $V_{\lambda j}$ has the expansion

$$V_{\lambda j} = \lambda^{m(j)} V_{\lambda j}^{m(j)} + \sum_{k \geq -m(j)} V_{\lambda j}^k \lambda^{-k},$$

where $m(j) = \left[ \frac{j+n}{N} \right] + 1$ if $\sim g$ is $A_n^{(1)}$ or $C_n^{(1)}$, $m(j) = \left[ \frac{j}{N} \right] + 1$ if $\sim g$ is $B_n^{(1)}$, and $\left[ x \right]$ denotes the integer part of $x$. Then for example equation (3.6) follows from

$$V_{\lambda j}^{p-q+1} = \text{res}(\lambda^{p-q} V_{\lambda j} | Nq+r) = \text{res}(\lambda^{p-q} T^N \Lambda^{Nq+r} T^{-1})
= \text{res}(\lambda^{p-q} T^q \Lambda^{2q+r} T^{-1}) = \text{res}(T^q \Lambda^{2q+r} T^{-1}) = \text{res}(T^{(N-2)p+2q+r} T^{-1})
= \text{res}(V_{\lambda j}^{(N-2)p+2q+r}).$$

While similar computations prove formulas (3.4) and (3.6).

4. It is almost trivial.

Using this proposition it is easy to give a bihamiltonian formulations of the hierarchies.

**Lemma 3.3** The Drinfeld Sokolov hierarchies corresponding to the element $\Lambda^k$ ($k \in \mathcal{N}_{\tilde{g}}$) can be written in the bihamiltonian form:

$$\dot{S}_k = [A, V^k_1] = \left( (V_{\Lambda^k})_+ S + \lambda A \right) \quad k \in \mathcal{N}_{\tilde{g}}, \quad (3.7)$$

where $V^k_1$ is the residuum of $V_{\Lambda^k}$ and $(V_{\Lambda^k})_+$ and $(V_{\Lambda^k})_-$ are respectively the projection on the regular and singular part of the Laurent series $V_{\Lambda^k}$. 

**Proof** Using the expansion of $V_{\lambda j}$ in Proposition 3.2 it easy to see that any flow of the hierarchy may be written as

$$\dot{S}_k = [A, V^k_1] \quad k \in \mathcal{N}_{\tilde{g}}, \quad (3.8)$$

Then since $V_{\Lambda^k}$ is a solution of (3.1) we have that

$$((V_{\Lambda^k})_+ S + \lambda A) + ((V_{\Lambda^k})_+ S + \lambda A) = -((V_{\Lambda^k})_+ S + \lambda A).$$

This latter equation implies that

$$[A, V^k_1] = \left( (V_{\Lambda^k})_+ S + \lambda A \right) \quad k \in \mathcal{N}_{\tilde{g}}.$$

Observe that, in the present picture, in the case of $D_n^{(1)}$ is missing the integrable hierarchy corresponding to the element $F$. Such hierarchy has been recently described and constructed by Liu, Wu, and Zhang in their beautiful paper [25].

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3.2 The Riccati equations

We could now go ahead along this path of thoughts, by computing explicitly the matrices \( V_\Lambda \) and projecting them on the quotient space \( \mathcal{N} \) to find the Drinfeld Sokolov hierarchies. However we shall see that there exists a different and maybe easier way to achieve the same result.

To begin with, let us find an eigenvector and an eigenvalue of the the linear differential operators \( -\partial_x + S + \lambda A \).

Let \( \{ e_j \}_{j=0, \ldots, N-1} \) with
\[
\begin{align*}
e_0 &= (1, 0, \ldots, 0)^T \\
e_j &= (0, \ldots, 0, 1, 0, \ldots, 0)^T \\
\end{align*}
\]
the canonical basis of \( \mathbb{C}^N \), \( (\cdot, \cdot) \) be the pairing of \( \mathbb{C}^N \) defined by the relation \( (e_i, e_j) = \delta_{ij} \) where \( \delta_{ij} \) is the usual Kronecker delta. Let further set \( v^{(0)} = (1, 0, \ldots, 0)^T \in C^\infty(S^1, \mathbb{C}^N) \) and define recursively
\[
\begin{align*}
v^{(j+1)}(S) &= \partial_x v^{(j)}(S) + (S + \lambda A)^T v^{(j)}(S) \quad (v^{(0)}(S) = v^{(0)}),
\end{align*}
\]
then it holds

**Theorem 3.4**  
1. Let \( g \) be the simple Lie algebra \( A_n, B_n \) or \( C_n \) then
   a) the subset \( \{ v^{(j)} \}_{j=0, \ldots, N-1} \) is for any \( S \) in \( S \) a basis for \( \mathbb{C}^N \). We may therefore develop the (for any fixed \( S \) ) first dependent vector, namely \( v^{(N)}(S) \), obtaining the relation
   \[
   v^{(N)}(S) = \sum_{k=0}^{N-1} c_k(S) v^{(k)}(S) \tag{3.10}
   \]
   called the “characteristic equation” of the operator \( -\partial_x + S + \lambda A \).
   b) Let \( \psi \) be the element of \( C^\infty(S^1, \mathbb{C}^N) \) defined by the relations \( \langle v^{(0)}, \psi \rangle = 1, \langle v^{(1)}, \psi \rangle = h \) and \( \langle v^{(k)}, \psi \rangle = h^{(k)} \) \( k = 2, \ldots, N-1 \), where the function \( h^{(k)} \) are defined by the recurrence: \( h^{(1)} = h, h^{(k+1)} = h^{(k)} + h^{(k)} h \). Then if \( h \) satisfies the “Riccati–type” equation
   \[
   h^{(N)} = \sum_{k=0}^{N-1} c_k(x, \lambda) h^{(k)} \tag{3.11}
   \]
   whose coefficients \( c_k(x, \lambda) \) are those of equation (3.10), \( \psi \) is an eigenvector of \( -\partial_x + S + \lambda A \) with eigenvalue \( h(z) \).
2. Let \( g \) be the simple Lie algebra \( D_n \) then
   a) the subset \( \{ v^{(j)} \}_{j=0, \ldots, 2n-2} \) together with the vector
   \[
   w^{(n)} = (0, \ldots, 0, 1, 0, \ldots, 0)^T
   \]
is for any $S$ in $S$ a basis for $C^{2n}$, developing the first dependent vector $v^{(2n-1)}$ with respect to this basis we obtain the relation

$$v^{(2n-1)}(S) = \sum_{k=0}^{2n-2} c_k(S)v^{(k)}(S) + c_w(S)w^{(n)}. \quad (3.12)$$

b) There exist functions $d_i \in C^\infty(S, \mathbb{C})$, $0 \leq i \leq n-1$ such that if $\psi \in C^\infty(S^1, \mathbb{C}^{2n})$ is defined by the relations $(v^{(0)}, \psi) = 1, (v^{(1)}, \psi) = h, (v^{(k)}, \psi) = h^{(k)}$ for $k = 2, \ldots, N-1$, and $(w^{(n)}, \psi) = \sum_{k=0}^{n-1} d_i h^{(k)} - c_v h^{(-1)}(d)$ where the functions $h^{(k)}$ are defined as above, and $h^{(-1)}(d)$ is defined by the relation $\partial_x + h(h^{(-1)}(d) = d_0$, and $h$ satisfies the “Riccati–type” equation

$$h^{(2n-1)}(S) = \sum_{k=0}^{n-1} c_k(x, A) - c_w d_k h^{(k)} + \sum_{k=n}^{2n-2} c_k(x, A)h^{(k)} - c_w h^{(-1)}(d) \quad (3.13)$$

where the coefficients $c_k(x, A)$ are those of equation (3.10), then $\psi$ is an eigenvector of $-\partial_x + S + \lambda A$ with eigenvalue $h(z)$.

Proof

1.a) Using the “canonical” basis of $C^N$: $\{e_j\}_{j=0,\ldots,N-1}$ and the form of the matrices $S + \lambda A$ with respect to this basis we have easily

$$v^{(j)} = e_j + \sum_{k<j} c_k(x)e_k \quad j = 0, \ldots, N-1,$$

which proves the linear independence of the first $N$ elements $v^{(j)}$, $j = 0, \ldots, N-1$.

1.b) Pairing the relation $-\partial_x + (S + \lambda A)\psi = h\psi$ with the vectors $v^{(j)}(S)$ $j = 0, \ldots, N-1$ we obtain:

$$-\left(\partial_x v^{(j)}, \psi\right) + \left((v^{(j)}, (S + \lambda A)\psi\right) = h \left(v^{(j)}, \psi\right) \quad j = 0, \ldots, N-1$$

which can be written as

$$-\partial_x \left(v^{(j)}, \psi\right) + \left(\partial_x v^{(j)}, \psi\right) + \left((S + \lambda A)^T v^{(j)}, \psi\right)$$

$$= -\partial_x \left(v^{(j)}, \psi\right) + \left((v^{(j+1)}, \psi\right) = h \left(v^{(j)}, \psi\right) \quad j = 0, \ldots, N-1$$

These latter equations are for $j = 0, \ldots, N-2$ satisfied if

$$h^{j+1} = \left((v^{(j+1)}, \psi\right) = \partial_x \left(v^{(j)}, \psi\right) + h \left(v^{(j)}, \psi\right) = h^{(j)}_x + hh^{(j)}$$

as required in the Hypothesis, while for $j = N-1$ using (3.10) we have

$$h^{(N)} = \left((v^{(N-1)}, \psi\right)_x + h \left((v^{(N-1)}, \psi\right) = \left((v^{(N)}, \psi\right)$$

$$= \left(\sum_{k<N} c_k(S) v^{(k)}(S), \psi\right) = \sum_{k=0}^{N-1} c_k(x, A)h^{(k)}.$$
2. a) Using again the canonical basis \( \{e_j\}_{j=0, \ldots, 2n-1} \) and the form of the matrices \( S + AA \) with respect to this basis we have

\[
\begin{align*}
\nu^{(j)} &= e_j + \sum_{k<j} c_k(x)e_k & j = 0, \ldots n - 2, \\
\nu^{(n-1)} &= \frac{1}{2} e_n + e_{n-1} + \sum_{k<n-2} c_k(x)e_k \\
\nu^{(j)} &= e_j + \sum_{k<j} c_k(x)e_k & j = n, \ldots 2n - 2
\end{align*}
\]

and therefore together with the element \( w^{(n)} \) form a basis for \( C^{2n} \) at any point \( s \) of \( S \).

2. b) Reasoning like in the point 1. b) we have that \( \psi \) is an eigenvector of \(-\partial_x + S + AA\) with eigenvalue \( H \) if and only if \( h \) satisfies the relation

\[
h^{(2n-1)} = \sum_{k=0}^{2n-2} c_k h^{(k)} + c_w (w^{(n)}, \psi).
\]

Now since \( S - B \subset p(x)H + C^0(S^1, \mathbb{R}) \) and it easily checked that \( H_0 w^{(n)} = 0 \) and \( B w^{(n)} = e_{n+1} \) it yields

\[
(\partial_x + h) (w^{(n)}, \psi) = (w^{(n)}, \psi)_x + (w^{(n)}, h \psi) = (w^{(n)}, \psi)_x + (w^{(n)}, (-\partial_x + S + \lambda A) \psi)
\]

\[
= ((-\partial_x + (S + \lambda A)^T w^{(n)}, \psi) = \nu^{(n)} - \sum_{i=0}^{n-2} f_i(x) \nu^{(i)}(x, \psi)
\]

\[
= h^{(n)} - \sum_{i=1}^{n-2} f_i(x) h^{(i)}(x) - f_0(x).
\]

Therefore \((w^{(n)}, \psi)\) must have the form

\[
(w^{(n)}, \psi) = h^{(n-1)} - \sum_{i=1}^{n-2} d_i(x) h^{(i)}(x) - h^{(i-1)}.
\]

\[\square\]

To explicitly compute the coefficients of the Riccati equations we shall use

**Proposition 3.5** The vectors \( \nu^{(j)}(S) \) \( j \geq 0 \) are covariant i.e., \( \nu^{(j)}(S') = (J^T)^{-1} \nu^{(j)}(S) \) whenever \( S' = JSJ^{-1} - J_xJ^{-1} \) with \( J \in G_{AB} \).

**Proof** The first statement follows by induction over \( j \). For \( j = 0 \) we have indeed \( (J^T)^{-1} \nu^{(0)} = \nu^{(0)} \) from the definition of the group \( G_{AB} \). While if the statement is true for \( i \) for \( i + 1 \) we have

\[
\nu^{(i+1)}(S') = \partial_x \nu^{(i)}(S') + (S' + \lambda A)^T \nu^{(i)}(S') = \partial_x ((J^T)^{-1} \nu^{(i)}(S))
\]

\[
+(J^T)^{-1} (S^T + \lambda A^T) J^T (J^T)^{-1} \nu^{(i)}(S) - (J^T)^{-1} J_x^T (J^T)^{-1} \nu^{(i)}(S) = (J^T)^{-1} J_x^T (J^T)^{-1} \nu^{(i)}(S)
\]

\[
+(J^T)^{-1} (S^T + \lambda A^T) \nu^{(i)}(S) - (J^T)^{-1} J_x (J^T)^{-1} T \nu^{(i)}(S) = (J^T)^{-1} (S + \lambda A)^T \nu^{(i)}(S).
\]

\[\square\]
Thus the coefficients of the Riccati equations are invariant under the gauge action of the group \( G_{AB} \), and can be computed using the transversal manifold \( Q \) which in our setting can be characterized as follows \[18\] \[11\] \[17\].

Let \( \tilde{\mathfrak{g}} \) be a affine Lie algebra of the type specified above and let \( \mathfrak{g} \) be the corresponding simple Lie algebras then the map \[24\]

\[ \text{ad}_B : \mathfrak{n}^{-} \rightarrow \mathfrak{b} = \mathfrak{n}^{-} \oplus \mathfrak{h} \]

is injective. We fix a subspace \( \mathfrak{q} \) of \( \mathfrak{b} \) such that

\[ \mathfrak{b} = \mathfrak{q} \oplus [B, \mathfrak{n}], \quad (3.14) \]

so \( \dim(\mathfrak{q}) = \dim(\mathfrak{b}) - \dim(\mathfrak{n}) = n = \text{rank}\mathfrak{g} \). Then the manifold:

\[ Q = C^\infty(S^1, \mathfrak{q}) \]

is the transversal manifold defined in section 2.2. To determine it explicitly let consider the decomposition of \( \mathfrak{g} \) w.r.t. the principal gradation \[17\] \[11\]

\[ \mathfrak{g} = \bigoplus_{1-h \leq j \leq h-1} \mathfrak{g}^j \quad \mathfrak{g}^j = \begin{cases} \mathfrak{h} & \text{if } j = 0 \\ \bigoplus_{\text{ht}(\alpha) = j} \mathfrak{g}_\alpha & \text{if } j \neq 0 \end{cases} \]

where \( \text{ht} \) is the the height function of roots: \( \text{ht}(\alpha) = \sum_{\alpha_i \in \Pi} n_i \alpha_i \) and \( h \) is the Coxeter number of \( \mathfrak{g} \). Then we specify the choice of the complement \( \mathfrak{q} \) of the subspace \( [I, \mathfrak{n}] \) of \( \mathfrak{b} \) so that \[11\] \[17\]:

\[ \mathfrak{q} = \bigoplus_{j=0}^{h-1} \mathfrak{q}_j \]

where the subspaces \( \mathfrak{q}_j \) satisfy

\[ \mathfrak{q}_j \subset \mathfrak{b}_j = \mathfrak{b} \cap \mathfrak{g}^j, \quad \mathfrak{b}_j = \mathfrak{q}_j \oplus [B, \mathfrak{b}_{j+1}] \].

Note that \( \mathfrak{q}_j \) is not a null space if and only if \( j \) is one of the exponents

\[ 1 = m_1 \leq m_2 \leq \cdots \leq m_n = h - 1 \]

of the simple Lie algebra \( \mathfrak{g} \). For all simple Lie algebras except the ones of \( D_n \) type with even \( n \) the exponents have multiplicity one, i.e. \( \dim(\mathfrak{a}_{m_i}) = 1 \) and the exponents are distinct. For the \( D_n \) (with even \( n \)) case, the exponents \( m_i \) for \( i \neq \frac{n}{2}, \frac{n}{2} + 1 \) have multiplicity one, \( m_2 = m_2 + 1 = n - 1 \) and \( \dim(\mathfrak{a}_{n-1}) = 2 \). Using these facts it is not difficult to determine a transversal manifold. Denoting with \( e_{ij} \), \( i, j = 0, \ldots, N-1 \) the \( N \times N \) matrix with 1 in the entry \((i, j)\) and zero elsewhere we have indeed:
For example we have that the transversal manifold for the Lie algebra \( C^{(1)}_2 \) (resp. \( B^{(1)}_3 \)) is

\[
q_{c^{(1)}_2} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
u_0 & u_1 & 0 & 1 \\
u_0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
q_{b^{(1)}_3} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
u_0 & u_2 & 0 & 1 & 0 & 0 \\
u_0 & u_1 & u_2 & 0 & 1 & 0 \\
u_0 & 0 & u_1 & 0 & 0 & 1 \\
u_0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
while the transversal manifold for the Lie algebra $D_3^{(1)}$ (resp. $D_4^{(1)}$) is

\[
q_{D_3^{(1)}} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
u_1 & u_2 & 0 & 0 & 1 \\
u_0 & 0 & u_2 & 0 & 0 \\
0 & u_0 & -u_1 & 0 & 0
\end{pmatrix}
\quad q_{D_4^{(1)}} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\
u_2 & 0 & u_3 & 0 & 0 & 1 & 0 \\
u_0 & 0 & u_1 & 0 & 0 & 0 & 0 \\
u_0 & 0 & u_2 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Using the explicit form of the transversal manifold it is easy to check that $c_0 = \lambda + g_0(Q)$ and all the other coefficients $c_k$ are independent if $\hat{g}$ is of the type $A_n^{(1)}$ or $C_n^{(1)}$ and that $c_1 = \lambda + g_1(Q)$ and all the other coefficients are independent of $\lambda$ if $\hat{g}$ is of the type $B_n^{(1)}$ or $D_n^{(1)}$. Moreover if $m$ is the index of the highest Faa di Bruno polynomial appearing in the Riccati equation then $c_{m-1} = 0$. For example the first Riccati equations are

\[
h^{(2)} = \lambda + u_0 \quad \text{if } \hat{g} \text{ is } A_1^{(1)}
\]
\[
h^{(4)} = u_1 h^{(2)} + u_1 h + \lambda + u_0 \quad \text{if } \hat{g} \text{ is } C_1^{(2)}
\]
\[
h^{(7)} = 2u_2 h^{(5)} + 5u_2 h^{(4)} + (2u_1 + 4u_2xx) h^{(3)} + (3u_1x + u_2xxx) h^{(2)} + (\lambda + 2u_0 + u_{1xx}) h = u_0 x
\quad \text{if } \hat{g} \text{ is } B_1^{(3)}
\]
\[
h^{(5)} = u_2 h^{(3)} + \frac{3}{4} u_2 h^{(2)} + (\lambda + 2u_0 - u_{1x} + \frac{1}{2} u_{2xx} - \frac{1}{4} u_2^2) h^{(1)}
\quad \text{if } \hat{g} \text{ is } D_1^{(3)}
\]
\[
h^{(7)} = u_3 h^{(5)} + \frac{3}{4} u_3 h^{(4)} + (2u_3xx + 2u_1 + u_2 - \frac{1}{4} u_3^2) h^{(3)}
\quad \text{if } \hat{g} \text{ is } D_1^{(5)}
\]
Proposition 3.6  Any Riccati equation (3.11) or (3.13) admits a solution of the form \( h(z) = z + \sum_{i=0}^n h_i z^{-i} \) where \( z^{n+1} = \lambda \) if \( \tilde{g} = A_n^{(1)} \), \( z^{2n} = \lambda \) if \( \tilde{g} = B_n^{(1)} \), or \( C_n^{(1)} \) and finally \( z^{2n-2} = \lambda \) if \( \tilde{g} = D_n^{(1)} \) whose coefficients \( h_k \) are obtained iteratively in a pure algebraic way.

Proof  It is immediate to show by induction that if \( h(x) \) is a formal Laurent series of the form \( h(z) = z + \sum_{i>0} h_i z^{-i} \) then \( h^{(k)} \) has the form

\[
h^{(k)} = z^k + \sum_{j=k+1} h_j z^{-j}
\]

with \( h_j = kh_{j-k+1} + q_j \)

where \( q_j \) is a differential polynomial in the coefficients \( h_i \), \( i = 1, \ldots, j-1 \). Therefore by substituting these expressions in equation (3.11) and developing it in powers of \( z \) we obtain that the equation corresponding to the powers \( N \) and \( N-1 \) are automatically satisfied while that corresponding to the power \( i, i \leq N-2 \) is of the type

\[
Nh_{N-i-1} = \text{differential polynomial in } h_1, \ldots, h_{N-i-2} \text{ and } u_0, \ldots, u_{n-1}.
\]

The same argument works in the case of the equation (3.13) once we have shown that condition \( (\partial_x + h)h^{(-1)}(u) = g(u) \) determines completely \( h^{(-1)}(u) \) as Laurent series \( h^{(-1)}(u) = k_1 z^{-1} + \sum_{j \geq 1} k_j z^{-j} \), whose coefficients \( k_j \) are differential polynomials in the function \( g(u) \) and in the coefficients \( h_j \) with \( j < i \), which, however, still follows immediately by induction.

We can write the Riccati equation in a more compact form using the following

Proposition 3.7  The Riccati equations (3.11) and (3.13) can be written as

\[
\begin{align*}
h^{(n+1)} &= \lambda + \sum_{k=0}^{n-1} u_k h^{(k)} & \text{if } \tilde{g} = A_n^{(1)} \\
h^{(2n+1)} &= \lambda h^{(1)} + \sum_{k=0}^{n-1} u_k h^{(2k+1)} + \sum_{k=0}^{n-1} (\partial_x + h)^{(2k+1)}(u_k) & \text{if } \tilde{g} = B_n^{(1)} \\
h^{(2n)} &= \lambda + \sum_{k=0}^{n-1} u_k h^{(2k)} + \sum_{k=0}^{n-1} (\partial_x + h)^{(2k)}(u_k) & \text{if } \tilde{g} = C_n^{(1)} \\
h^{(2n-1)} &= \lambda h^{(1)} + \sum_{k=1}^{n-1} u_k h^{(2k-1)} + \sum_{k=0}^{n-1} (\partial_x + h)^{(2k-1)}(u_k) + u_0 h^{(-1)} & \text{if } \tilde{g} = D_n^{(1)}
\end{align*}
\]

Proof  The form of the Riccati equation corresponding to the Lie algebra \( A_n^{(1)} \) follows immediately from the very form of the transversal manifold \( Q \) in this case \[6\].

In the other cases the operator \(-\partial_x + (S + \lambda A)\) is skew symmetric with respect the bilinear form \[18\]

\[
\langle \cdot, \cdot \rangle : \mathcal{C}^m(S^1, \mathbb{C}^N) \to \mathbb{C}
\]

\[
(v, w) \mapsto \langle v, w \rangle = \int_{S^1}(v, \Omega w)dx
\]

where \( \Omega = \text{diag}(1, -1, \ldots, -1, 1) \) if \( \tilde{g} = B_n^{(1)} \), \( \Omega = \text{diag}(1, -1, \ldots, 1, -1) \) if \( \tilde{g} = C_n^{(1)} \), and finally \( \Omega = \text{diag}((1, -1 \ldots, (-1)^{n-1}, (-1)^{n-1}, (-1)^n, \ldots, 1) \) if \( \tilde{g} = D_n^{(1)} \). This in turn
implies using the proof of Theorem 3.4 that if \( h(x) \) satisfies the relation (3.11) (resp. (3.13)) it must also satisfies the relation \((-1)^{N-1} h^{N-1} = \sum_{k=0}^{N-1} (-1)^k (\partial_x + h)^k (c_k(u))\) (resp. \((-1)^{N-1} h^{N-1} = \sum_{k=0}^{N-1} (-1)^k (\partial_x + h)^k (c_k(u)) - d_0 h_{\infty}^{-1}\)). Therefore with respect to the adjoint operation \((\partial_x + h)^* = - (\partial_x + h)\), \( f(u)^* = f(u) \) the relation (3.13) is skew symmetric while (3.11) is skew symmetric if \( g \) is of type \( B \) and symmetric if \( g \) is of type \( C \).

\[ \square \]

### 3.3 The construction of the hierarchies

Actually the proof of Theorem 3.4 and Proposition 3.6 shows that \( h_p = h(\omega^p z) \) and \( \psi_p = \psi(\omega^p z) \) are respectively eigenvalues and eigenvectors of \(-\partial_x + S + \lambda A\), with \( \omega = \exp \left( \frac{2\pi i}{N'} \right) \) and \( p = 0, 1, \ldots, N' \), where \( N' = n + 1 \) if \( \frac{\pi}{N} = A_n^{(1)} \), \( N' = 2n \) if \( \frac{\pi}{N} = C_n^{(1)} \), or \( \frac{\pi}{N} = C_n^{(1)} \) and finally \( N' = 2n + 2 \) if \( \frac{\pi}{N} = D_n^{(1)} \). Moreover since the eigenvalues \( h(\omega^p z) \) are distinct we have that the corresponding eigenvectors are linear independent. Further equation (3.2) shows that the operator \(-\partial_x + (S + \lambda A)\) has together with the non trivial eigenvectors \( \psi(\omega^p z) \) an eigenvector \( \chi_0 \) (resp. two eigenvectors \( \chi_1, \chi_2 \)) with eigenvalue zero when \( g \) is of type \( B \) (resp. of type \( D \)). Therefore at any point of \( S \in \mathcal{S} \) the eigenvectors \( \psi(\omega^p z) \) \( p = 0, \ldots, N - 1 \) if \( \frac{\pi}{N} = A_n^{(1)} \) or \( \frac{\pi}{N} = C_n^{(1)} \), \( \psi(\omega^p z) \), \( p = 0, \ldots, N - 1, \chi_0 \) if \( \frac{\pi}{N} = B_n^{(1)} \), and finally \( \psi(\omega^p z) \), \( p = 0, \ldots, N - 1, \chi_1, \chi_2 \) if \( \frac{\pi}{N} = D_n^{(1)} \), form a basis of \( \mathbb{C}^N \). This fact allows us to prove

**Theorem 3.8** The Drinfeld–Sokolov hierarchies corresponding to the Casimirs of type \( V_{\Lambda_j} \) can be written in the form

\[ \partial_t h = \partial_x H^{(j)} \quad j \in \mathcal{N}_g \]  

(3.15)

where the Laurent series \( H^{(j)} \) are given by \( H^{(j)} = \langle \psi^{(0)}, (V_{\Lambda_j})_+ \psi_0 \rangle \).

**Proof** We shall present here the proof in the case of the affine Lie algebra \( D_n^{(1)} \) being the other cases with the obvious changes similar.

The Drinfeld–Sokolov hierarchies in this case can be written as 3.3

\[ [-\partial_x + (S + \lambda A), -\partial_t + (V_{\Lambda_2^{(1)}})_+] = 0 \quad j \in \mathbb{N} \]

which together with \(-\psi_0 + (S + \lambda A)\psi_0 = h_0 \psi_0 \) gives

\[ (-\partial_x + S + \lambda A) \phi = h \phi - h_{\lambda x + \psi_0} \]  

(3.16)

where

\[ \phi = (-\partial_t \psi_0 + V_{\Lambda_2^{(1)}}) \psi_0. \]  

(3.17)

Decomposing now \( \phi \) with respect to the basis \( \chi_1, \chi_2, \psi_0, \ldots, \psi_{2n-2} \): \( \phi = a_1 \chi_1 + a_2 \chi_2 + \sum_{l=0}^{2n-1} c_l \psi_l \), equation (3.16) becomes

\[ -a_1 \chi_1 - a_2 \chi_2 + \sum_{l=0}^{2n-1} (-c_{1l} + c_l h_t) \psi_l = h a_1 \chi_1 + h a_2 \chi_2 + \sum_{l=0}^{2n-1} h c_l \psi_l - h_{\lambda x + \psi_0} \]

16
which implies therefore \(-a_{ilx} = h a_i \) \(i = 1, 2\) and \(-c_{ilx} - c_{ihl} = c_l h (h_0 = h)\) for \(l = 1, \ldots, 2n - 2\). Then the Laurent series \(c_l\) \(l = 0, \ldots 2n - 1\) must fulfill the conditions \(c_{ilx} = c_l(h_l - h)\), and since the Laurent series \(h_l - h\) have degree 1 in \(z\) these imply \(c_l = 0\) if \(l \neq 0\). Hence \(\phi = c_0 \psi\) which taking into account (3.17), the definition of \(H^{(j)}\) and the normalization of \(\psi_0\) implies \(c_0 = H^{(j)}\). Therefore \(\phi = H^{(j)} \psi_0\) which plugged in (3.16) yields \(h_l = \partial_s H^{(j)}\). □

We can now show the Laurent series \(H^{(j)}\) can be computed directly from their eigenvalue \(h(z)\) avoiding the construction of the matrix \((V_\Lambda)_+\) or the use of the Poisson pencil.

**Theorem 3.9** The Laurent series of \(H^{(j)}\) is given by

\[
H^{(j)} = z^j + \sum_{l \geq 1} H^{(j)}_l z^{-l} \tag{3.18}
\]

and has a Faà di Bruno expansion

\[
H^{(j)} = h^{(j)} + \sum_{i=0}^{j-1} s^{(j)}_i(x) h^{(i)} \tag{3.19}
\]

where the coefficients \(s^{(j)}_i(x)\) are independent of \(z\). Therefore if we define

\(H_+ = \text{span}(h^{(j)}| j \geq 0)\) the functions \(H^{(j)}\) are the projections

\[
H^{(j)} = \pi_+(z^j)
\]

of \(z^j\) onto \(H_+\) along splitting of the space of Laurent series in the direct sum of \(H_+\) and of the subspace of strictly negative Laurent series.

**Proof** From formula (3.2) and Theorem 3.4 follows that \(T \psi_0\) is an eigenvector of \(\Lambda\) with eigenvalue \(z\) and hence \(V_\Lambda \psi_0 = z \psi_0\). Then from the definition of \(H^{(j)}\) we have \(H^{(j)} = (v^{(0)}, (V_\Lambda)_+ \psi_0) = (v^{(0)}, V_\Lambda / \psi_0) - (v^{(0)}, (V_\Lambda)_- \psi_0) = z^j - (v^{(0)}, (V_\Lambda)_- \psi_0)\). Now from the definition of the \(h^{(k)}\) and the Riccati equations follows that \(\psi_0 = (1, z + O(1), z^2 + O(z), \ldots, z^{N-1} + O(z^{N-2})^2)\) and therefore we have that

\[-(v^{(0)}, (V_\Lambda)_- \psi_0) = \frac{H^{(j)}_i}{z} + \ldots\]

because in the case of the Lie algebras of type \(A\) or \(C\), \(\Lambda = z^N\); while in the case of the Lie algebras of type \(B\) or \(D\) we have \((v^{(0)}, (V_\Lambda)_- (\psi_0)_{N-1}) = 0\), because \(((V_\Lambda)_-)_{0,N-1} = 0\).

Now if \(g\) is of either \(A_n\) or \(C_n\):

\[
(v^{(0)}, (V_\Lambda)_+ \psi_0) = ((V_\Lambda)^T v^{(0)}, \psi_0) = \left( \sum_{i=0}^{N-1} r_i v^{(i)}, \psi_0 \right) = \sum_{i=0}^{N-1} r_i H^{(i)} \tag{3.20}
\]
where the coefficients \( r_i^j \) are polynomials in \( \lambda \). In this case using the equation (3.11) we get

\[
\lambda = h^{(N)} - \sum_{k=0}^{N-2} c_k(x)h^{(k)}
\]  
(3.21)

where the coefficients \( c_k \) are independent of \( \lambda \) and therefore that \( \lambda \in H_+ \). Moreover acting iteratively on (3.21) with \((\partial_x + h)\) we get

\[
\lambda h^{(j)} = h^{(N+j)} - \sum_{k=0}^{N-j-1} c_k(x)h^{(k)}
\]

which shows \( \lambda h^{(j)} \in H_+ \) and by induction that \( \lambda h^{(i)} \in H_+ \), \( i, j \in \mathbb{N} \). Therefore form equation (3.20) it easily seen \( h^{(j)} \in H_+, j \in \mathbb{N} \).

Unfortunately in the case of Lie algebras of type \( B \) and \( D \) we can not use the above argument, but we need to introduce the map \( \phi \) from the space of Laurent series to the space of pseudo differential operator defined in [7], which maps the Faà di Bruno Polynomial \( h^{(k)} \) to the operator \( \partial_x^k \) together with the results obtained in such paper. In particular we have that \( P_{2k+1} = \phi(H^{(2k+1)}) \) is the unique pseudo differential operator such that \( P_{2k+1} = (L^{\frac{2k}{2k+1}})_+ \) if \( \tilde{g} \) is \( B_n^{(1)} \) (resp. \( P_{2k+1} = (L^{\frac{2k}{2k+1}})_+ \) if \( \tilde{g} \) is \( D_n^{(1)} \)) where \( L \) is the Lax operator associated to \( B_n^{(1)} \) (resp. \( D_n^{(1)} \)) [13] [17] and \( (P)_+ \) denotes the differential part of a pseudo differential operator \( P \). But since [15] \( P_{2k+1} = \sum_{j=1}^{2k+1} p_j \partial_x^j \) it follows that in both cases of affine Kac–Moody Lie algebras of type \( B \) or \( D \), \( H^{(2j+1)} \) has the expansion in term of Faà di Bruno Polynomials given by equation (3.19).

\[ \square \]

It is here worth to note that also in the case \( \tilde{g} = B_n^{(1)} \) it is possible to show that \( H^{(2j+1)} \) is completely determined by the Faà di Bruno Polynomial \( h \) without referring to the pseudo differential approach to the Drinfeld–Sokolov hierarchies, it holds indeed.

**Proposition 3.10** If \( \tilde{g} \) is \( B_n^{(1)} \) then \( H^{(2j+1)} \) can be written as

\[
H^{(2j+1)} = \sum_{i=0}^{2n-1} \left( \sum_{l=0}^{\left[ \frac{2i+1}{2n} \right]} r_i^l(x) \lambda^l \right) h^{(i)},
\]  
(3.22)

the coefficients \( r_i \) are independent from \( \lambda \) and are completely determined by the fact that \( H^{(2j+1)} \) has the Laurent expansion \( H^{(2j+1)} = z^{2j+1} + (\text{ negative terms in } z) \).

**Proof** From \( H^{(2j+1)} = (V_{\Lambda^{2j+1}})^T + \psi^{(0)} \psi \), \( V_{\Lambda^{2j+1}} = T \Lambda^{2j+1} T^{-1} \) the Proposition 3.2 and the fact that \( ((V_{\Lambda^i})_+)_{0,N-1} = 0 \) follows that

\[
H^{(2j+1)} = \sum_{i=0}^{2n-1} \left( \sum_{l=0}^{\left[ \frac{2i+1}{2n} \right]} r_i^l(x) \lambda^l \right) h^{(i)}.
\]
Now since the maximal power of $H^{(2j+1)}$ is $z^{2j+1}$, it is easy to show by induction that $r^j_i = 0$ if $l > \left\lfloor \frac{2j+1-i}{2n} \right\rfloor$ i.e., that equation (3.22) holds.

Now since $i$ runs from 0 to $2n - 1$ there exists exact an odd number say $2k + 1$, $0 \leq k \leq n - 1$ such that $2j - 2k$ is divisible by $2n$ we have the number of the coefficients $r^j_i$ in (3.22) is

$$\left( \frac{j-k}{n} + 1 \right) (2k+2) + \frac{j-k}{n} (2n-1-2k-1) = 2(j-k)+2k = 2j+2.$$ 

But this is exactly the number of independent condition imposed on those coefficients by the very form of $H^{(2j+1)}$ as Laurent series: $H^{(2j+1)} = z^{2j+1} + (\text{negative terms in } z)$.

\[ \square \]

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