Influence of data input in the evaluation of Stress Intensity Factors from Thermoelastic Stress Analysis

Giuseppe Pitarresi1*, Riccardo Cappello1, Giovanni Li Calsi1 and Giuseppe Catalanotti2

1 Dipartimento di Ingegneria (DING), Università degli Studi di Palermo, Viale delle Scienze Ed. 8, 90128 Palermo, Italy
2 Advanced Composite Research Group (ACRG), School of Mechanical and Aerospace Engineering, Queen’s University Belfast, Belfast BT9 5AH, UK
E-mail: giuseppe.pitarresi@unipa.it

Abstract. Thermoelastic Stress Analysis (TSA) is applied to evaluate the Stress Intensity Factor (SIF), T-stress and J-Integral in a Single-Edge-Notched-Tension sample undergoing fatigue cycling. The Williams’ series stress formulation and a least-square fitting (LSF) procedure are used to obtain the SIF and the T-stress. The evaluation is carried out with the aim to investigate the influence of the input data in the system of equations solved with the LSF, and in particular: the number of coefficients used in the Williams’ series and the choice and position of the fitted experimental data points. Three algorithms for the determination of the crack tip position are also evaluated: a coarse grid method with pixel resolution, a refined grid method and a patternsearch method with sub-pixel resolution. In order to establish a criterion for the choice of input parameters for the LSF, the theoretical case of an infinite plate with a central crack has been analysed, for which an exact solution of the isopachics, SIF and T-stress is available. Finally, the stress separation obtained with the fully characterized Williams’ model is also used to evaluate the J-Integral by applying an Energy-Domain-Integral formulation, and the SIF retrieved from the J-Integral is compared with SIF obtained from the first coefficient of the Williams’ series.

Keywords: Infrared Cameras, Thermoelastic Stress Analysis, LockIn Correlation, Fatigue Crack Growing, Stress Intensity Factor, J-Integral, Crack Detection

1. Introduction

Thermoelastic Stress Analysis (TSA) is an ideal technique for the evaluation of fatigue cracks growing in materials and structures. In fact, dynamic loads allow the adiabatic conditions needed for a quantitative interpretation of the thermoelastic effect [1].

The full field experimental TSA stress metric can be used to fully characterise analytical formulations of the crack tip stress field, by means of fitting procedures. In order to fully exploit the rich domain information, an over-deterministic system of equations is usually obtained and solved with a least-square fitting approach [2]. This procedure has been popular with a number of different full-field experimental stress analysis techniques, including Photoelasticity or Digital Image Correlation [3, 4, 5].

A better evaluation of fracture parameters is generally obtained by fitting the experimental data with series expansion formulations of the stress field. In fact, these allow to employ experimental data further from the crack tip, avoiding the inaccuracies of the near crack-tip zone. In the context of TSA, several studies have followed this approach, using three different series expansion stress functions:
Mushkelishvili [6, 7, 8], Lekhnitskii [9, 10] and Williams [2, 11, 12, 13]. In all cases, the system of equations obtained requires the input of the coordinates of the selected data points with respect to the crack tip location. Therefore, if the crack tip location is not available, its coordinates can be treated as further unknowns, together with the unknown coefficients of the stress functions [14].

Results in the literature show that the over-deterministic least-square approach can fit well the experimental isopachics, while requiring low computation efforts [1]. On the contrary, values of SIF and T-stress appear to be significantly affected by the choice of input parameters [2, 11, 13]. Therefore, it would be useful to further investigate this dependency, with the aim to possibly develop a criterion for the optimal choice of input data in order to obtain the most accurate SIF and T-stress evaluations, especially considering how the actual literature does not provide any clear guidelines.

In the present work, the Williams’ series stress formulation is used to evaluate the SIF and T-stress [13] from the thermoelastic signal acquired on a cyclically loaded Single Edge Notched Tension sample, under pure mode I crack loading. A linear LSF implementation is in particular carried out to evaluate the ability to identify the crack tip location, the influence of the extension and position of the input data points (fitting area) and the number of coefficients in the series model. The same evaluation is also carried out in a simulated case. This consists of a central crack tension sample with infinite width, for which an analytical closed-form solution of the stress field is available, providing also exact estimations of the SIF and T-stress. The thermoelastic signal, for this case, is simulated by adding a suitable Gaussian noise, which well simulates the real TSA case. This simulation has also allowed to evaluate the interpolation procedure of Stanley-Chan, where an error arises from neglecting the effects of the T-stress [13, 15].

Finally, the Williams’ formulation characterised from the experimental data is also used to obtain separate stress and strain components and compute the J-Integral by means of an Energy Domain Integral formulation [5]. The SIF obtained from the elastic J-Integral is then compared with that obtained from the coefficient of the singular term (i.e. the first coefficient) of the Williams’ series.

2. Theoretical background

2.1. Thermoelastic effect law and measurement of the thermoelastic signal

The experimental stress metric employed in this work is obtained with TSA. According to the first order thermoelastic effect law [16], the temperature change in a point of an isotropic medium undergoing linear elastic deformation, under adiabatic conditions, is linearly related to the change in the sum of normal stresses by the following relationship:

$$\Delta T = -T_0 \cdot K_{th} \cdot \Delta (\sigma_{xx} + \sigma_{yy})$$

where $T_0$ is the mean body temperature and $K_{th}$ the thermoelastic constant (a physical property of the material). If a sinusoidal load is applied at an angular frequency $\omega_L$, the evolution in time of the temperature can be represented by a Fourier like series expansion:

$$T(t) = T_0 + \Delta T \sin(\omega_L \cdot t + \phi) + D_{SH} \sin(2\omega_L \cdot t + \phi_{SH}) + \cdots + \sum_{i=other\ terms} H_i \sin(\omega_i \cdot t + \phi_i)$$

It is in particular observed that the harmonic at $\omega_L$ carries the thermoelastic signal, and its amplitude can be associated to the expression of Eq. 1. The Harmonic at $2\omega_L$, sometimes referred to as Second Harmonic, is also relevant for thermomechanical analyses, as it can be, to some extents, correlated to the part of mechanical energy dissipated as heat. Furthermore, in a recent study it was shown how the amplitude and phase of the Second Harmonic can reveal the presence of crack closure on the wake of the crack [13].

In the present work, the maps of $\Delta T$, $\phi$ and $D_{SH}$ are obtained with an efficient digital cross correlation algorithm that is fully described in [1].
2.2. Williams’ stress formulation and Least Squares Fitting

The Williams’ series expansion formulation of the stress field, valid for linear elastic isotropic plane media, is one of the most employed stress formulations in problems investigating cracks behavior under Mode I and II [17]. Its characterization has been carried out with many experimental and numerical procedures [4, 17, 18]. In the present work, the Williams’ stress formulation can be used to derive the following expression of the sum of normal stresses (i.e. first stress invariant, or isopachics):

\[
(\sigma_{xx} + \sigma_{yy})_i = \sum_{n=1}^{\infty} C_n \left[ 2n r_i^{n-1} \cos \left( \frac{n}{2} \right) \theta_i \right]
\]

Equation (3)

The subscript \(i\) in Eq. 3 refers to the \(i\)th point, meaning that one can write as many similar equations as the number of available data points where the sum of normal stress is measured. The terms \(r_i\) and \(\theta_i\) represent the polar coordinates of the \(i\)th point relative to the crack tip, and \(C_n\) represents the \(n\)th unknown coefficient in the series expansion. Considering a peak-to-peak range variation of \(\Delta(\sigma_{xx} + \sigma_{yy})\), as arising from a loading cycle with ratio \(R > 0\), and replacing the left hand side term of Eq. 3 with Eq. 1, and writing the equation into matrix form, for a number of \(m\) points and \(N_w\) Williams’ coefficients, one yields:

\[
\begin{bmatrix}
\Delta T_1 \\
\vdots \\
\Delta T_m
\end{bmatrix} = -T_{th} K_{th} \begin{bmatrix}
\sqrt{2} \cos \left( \frac{\theta_1}{2} \right) & 4 \sqrt{r_1} \cos(\theta_1) & 8 r_1 \cos(\theta_1) & 10 r_1 \sqrt{r_1} \cos(\frac{\theta_1}{2}) & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots \\
\end{bmatrix} \begin{bmatrix}
\Delta C_1 \\
\Delta C_2 \\
\Delta C_3 \\
\Delta C_4 \\
\Delta C_5 \\
\vdots \\
\Delta C_{N_w}
\end{bmatrix}
\]

Equation (4)

If \(m > N_w\) then the matrix in Eq. 4 is rectangular and the system of equations is overdetermined. The column vector on the left side of Eq. 4 collects the experimental values of \(\Delta T\). The matrix term on the right side requires the knowledge of the crack tip location, in order to evaluate \((r_i, \theta_i)\). If this is provided, then Eq. 4 is a system of linear equations that can be effectively solved for the vector of unknown coefficients with a least-square procedure. In this work this has been implemented by using the backslash Matlab operator.

Once the unknown coefficients have been determined, a correlation coefficient \(R^2\) can be computed to evaluate the effectiveness of the fitting. \(R^2\) is in particular defined as:

\[
R^2 = 1 - \frac{RSS}{TSS}
\]

Equation (5)

where \(RSS\) is the Residual Sum of Squares and \(TSS\) the Total Sum of Squares. An ideally perfect fitting would yield \(R^2 = 1\). The evaluation of \(R^2\) is performed over an area of the sample domain and in the present work has been used also as an objective function to be optimized in the search for the crack-tip location (see Sec. 4.1). It is finally observed that the solution of Eq. 4 will provide the unknown coefficients \(C_1, \ldots, C_{N_w}\), and the SIF and T-stress are readily obtained from the first and second coefficients, rewritten according to the formalism of the classic Westergaard expressions, i.e.:

\[
\Delta(K_I) = \Delta C_1 \cdot \sqrt{2\pi}; \quad \Delta(\text{T-stress}) = -4 \cdot \Delta C_2
\]

Equation (6)

2.3. Data input analysis

In order to perform the Last-Square Fitting (LSF) procedure for the solution of the linear system of Eq. 4, the following list of data input is in general needed:

(i) coordinates of the crack tip location for the evaluation of \((r_i, \theta_i)\);
(ii) number and location of the \(m\) points;
(iii) number $N_w$ of coefficients $C$ in the Williams’ series model.

In the present work the experimental data points inserted in Eq. 4 are those contained in a Pac-man like shaped area, hereinafter referred to as fitting area, defined as shown in Figure 1. The area is in particular defined as an annulus sector having a minimum and maximum radiuses ($r_{\text{min}}$ and $r_{\text{max}}$) centered on the presumed crack tip, and an angle stretching from $22.5^\circ$ to $337.5^\circ$, in order to avoid collecting points also near the wake of the crack where the signal is either very low and noisy, or corrupted by the opening crack. In this work, every single measured point of the Infrared camera array that falls into the fitting area is inputted in Eq. 4. Studies performed by the authors indicate that a subsampling of points within the fitting area, i.e. using lesser points, can lead to savings in computation time without significantly affecting results, but these outcomes are not reported here [19].

![Image of fitting area](image)

**Figure 1**: three examples of fitting area with increasing $r_{\text{max}}$ superimposed on a map of the thermoelastic signal of the SENT sample.

### 3. Case studies and plan of experiments

Details of the experimental and simulated case studies are given below. It is here reported also that all the signal processing, including the evaluation of the thermoelastic signal, the LSF implementations and the fracture toughness parameters computation, was performed by in-house developed Matlab scripts.

#### 3.1. Experimental case study: Single Edge Notched Tension (SENT) specimen

The experimental data analysed in this work regards a SENT sample with geometric dimensions illustrated in Figure 2. The sample is made of stainless steel AISI 304L, and the analysed surface is painted with a matt black paint.

The sample has been tested in a Servo-hydraulic MTS 810 testing machine with loads as reported in Table 1 (loading ratio $R = 0.1$). The temperature signal was acquired with an infrared camera Flir X6540sc, which uses a cooled photon sensor. Details of the acquisition parameters are summarised in Table 1. It is in particular seen that three acquisitions have been acquired with some differences between them, in terms of mm-to-pixel IFOV resolution and sensor Integration time.

Results reported in Section 4 are referred only to test SENT#1 of Table 1, for the sake of brevity. All conclusions that will be drawn, though, can be extended also to the other two tests. Results in Section 6.3 will instead be reported for all three samples.

In Figure 3 the maps of the Thermoelastic and Second Harmonic signal from SENT#1 are shown. Here it is interesting to point out that the phase map is shifted such to set a zero phase in the ligament zone. This allows to better highlight the zone right ahead of the crack tip, where the phase has a negative shift of about 20 degrees, induced by both non-adiabatic behavior (heat leakage) and plastic strains (ruling out the thermoelastic effect). The map of the Second Harmonic also shows an increase near the crack tip, but there is little evidence of the occurrence of crack closure.
Table 1: Settings of the conducted tests for the different analysed samples.

| Test ID | Load Frequency (LF) | Sampling Frequency (SF) | Integration time (IT) | Frames (N) | Load Range (ratio $R = 0.1$) | Scaling Factor | Irwin Plastic Radius |
|---------|---------------------|-------------------------|-----------------------|------------|-------------------------------|----------------|---------------------|
| SENT#1  | 10                  | 200                     | 659                   | 6020       | $1 \div 10$                  | 0.15           | 9.35                |
| SENT#2  | 10                  | 100                     | 2930                  | 3020       | $1 \div 10$                  | 0.15           | 8.97                |
| SENT#3  | 5                   | 51                      | 659                   | 6120       | $1 \div 10$                  | 0.33           | 3.50                |

Figure 2: dimensions of the AISI 304L SENT sample tested.

Figure 3: maps of the Thermoelastic Signal $\Delta T$ (left), Phase $\phi$ (center) and Second Harmonic amplitude $D_{SH}$ (right).

3.2. Theoretical case study: Central Crack Tension (CCT) infinite plate

A simulated experiment is also proposed in order to investigate the effectiveness of the Matlab scripts with a case study where the exact theoretical solutions of the stress field, SIF and T-stress are available.

The case study consists of a Central Crack Tension infinite plate loaded along the direction orthogonal to the crack, with a remote stress $\sigma_{yy}^\infty = \sigma_0$. For this case a closed form solution of the elastic stress field is provided by applying the Muskhelishvili Complex Potential Method [17]. The sum of normal stresses is given by the following equation:

$$\left(\sigma_{xx} + \sigma_{yy}\right) = \sigma_0 \left[2\Re \left(\frac{z}{\sqrt{z^2 - a^2}}\right) - 1\right]$$

where $2a$ is the crack length and $z$ is a complex variable $x + iy$ providing the coordinates of the field domain with respect to the center crack point. The exact values of SIF and T-stress in this case are
SIF = \sigma_0(\pi a)^{0.5} \text{ and } T\text{-stress} = \sigma_0.

The frame size investigated, the pixel resolution and the remote stress in the simulated case are chosen such to resemble as much as possible the experimental case. In order to do so, the area investigated is a square box of sides $2a \times 2a$, centered on the crack (see Figure 4-left). The crack line is positioned on the right hand side, same as in Figure 3. The resolution is also equal to 0.15 mm/pixel and the remote stress is chosen as $\sigma_0 = 122$ MPa, since such value yields a similar SIF to that of the experimental SENT sample.

Figure 4: simulated isopachics. Theoretical map (left), map after the addiction of noise (centre) and isopachics contour plot after the addiction of noise (right).

A Gaussian noise is finally added to the field data of the theoretical map to simulate the typical noise of the experimental thermoelastic signal. From previous studies of the authors [1], a suitable noise-to-signal ratio, representative of experimental thermoelastic signals, is about 0.06. TSA has also a typical resolution threshold at about $10^{-3}$ °C, which can be determined as the noise bed observed in a power spectrum of the Discrete Fourier Transform of the thermal signal [1]. Such resolution, for steels,
corresponds to about 1 MPa. Therefore, the Gaussian noise added to the theoretical data follows the following logic: if signal < 1 MPa then noise (i.e. standard deviation) is 1 MPa, else noise = 0.065 × signal. The noise is in particular introduced by using the Matlab function normrnd. Figure 4-center and 4-right report an example of the isopachics map and isopachics contours after the introduction of noise.

4. Results #1: SIF and T-Stress from SENT specimen

As outlined in Section 2.3, solving Eq. 4 requires an input of crack tip coordinates. Treating such coordinates as further unknowns would lead to a non-linear optimization problem. Some authors have followed this route, implementing more complex algorithms based on Newton or Simplex methods [7, 14]. The strategy implemented in this work is to explore iteratively several candidate crack-tip locations, so that each iteration requires solving a linear system of equations. An objective function is evaluated at each location. In the present case, the $R^2$ correlation coefficient defined in Eq. 5 is the objective function and the final crack tip coordinates are those which, among all the polled points, yield the value of $R^2$ closer to one.

In this work, three search algorithms have been implanted, which differ for the polling strategy and mesh size of the explored candidate points. The three algorithms are briefly described here:

- **Coarse grid method**: a square grid of $21 \times 21$ pixels is selected on the same mesh of the experimental data array, and positioned over a zone where the crack-tip is likely to fall. Each point is then iteratively polled to identify the one that optimises $R^2$. The successful point will then be located on the data array (i.e. the method has a pixel resolution) [1]

- **Refined grid method**: a new square sub-mesh of $21 \times 21$ points is created between the pixel identified by the coarse grid method, and the surrounding pixels. Each point of the new mesh is polled so that now the successful candidate can be found with a more refined sub-pixel resolution;

- **Pattern-search algorithm**: The polling strategy is more optimised, following now a continuously adjusted path, based on the evaluation of the cost function in the near surroundings. Furthermore, as the optimum point is approached, the resolution of the search area shrinks more and more, so that a better final sub-pixel resolution is obtained. In the present work, the algorithm was applied via the Matlab patternsearch function, and more information about how this works and perform can be found at https://www.mathworks.com/help/gads/how-patternsearch-polling-works.html or in [14, 20].

Figure 5 shows an example of the solved position of the crack tip with the above algorithms, reported over the thermoelastic phase map.

4.1. Identification of the crack tip location

It is observed that all the obtained crack tips falls near the local minimum value of the phase. This result is in good agreement with the findings of Tomlinson et al. [21], which used a non-linear Downhill-Simplex optimization. Authors in [21] have also called this solved crack tip as a “notional” crack tip, since it does not coincide with the real crack tip, but it rather indicates the length of an equivalent crack satisfying the linear elastic fracture mechanics solution.

Since the $R^2$ objective function may depend on the choice of the fitting area and number of Williams’ coefficients, Figure 6 collects more results obtained by varying $r_{\text{max}}$ and $N_w$. The data in particular report the values, in pixels, of $\Delta x$ and $\Delta y$, which represent the distances of the solved crack tip from the position of the minimum phase, measured along the x and y direction. For each calculation, the final value of $R^2$ is also reported. In order to run the three algorithms in a fully automated way, the initial seed point for all procedures was taken as the pixel of local minimum phase.

From the results summarized in Figure 6a number of comments arise. The values of $\Delta y$ are always close to zero, while $\Delta x$ is found to vary a little more. Excluding some outliers, which generally arise with low $N_w$, the range of variation is below 2 pixels (i.e. 0.3 mm). Higher values of $R^2$ are obtained
Figure 5: a) phase map for SENT#1 test; (b) superposition of the crack-tips obtained by the three algorithms upon the phase map.

with increasing $r_{max}$ and $N_w$, and the pattern search method seem to achieve always the higher $R^2$. The location identified by the pattern search method is always very close to that found by the refined grid method, but the pattern search method is an order of magnitude faster (few seconds versus tens of seconds, see also [19]). In general, the variation of the crack-tip has been found to be not significantly influenced by the LSF input parameters. In light of the obtained results, the pattern search algorithm appears to be a robust, reliable and fast tool that can be used in TSA for implementing automated tracking of fatigue growing cracks [1, 19].

4.2. Influence of input data on SIF and T-stress

Figure 7 reports plots of the SIF and T-stress with varying $r_{max}$ and $N_w$. The crack-tip location for these evaluations was the one obtained with the pattern search algorithm when $r_{max} = 60$ px and $N_w = 5$.

In these evaluations, $r_{min}$ was kept fixed and equal to 9 pixels, which is roughly the amount of the Irwin plastic radius obtained with a yield stress of 280 MPa and a SIF of 26 MPa$\sqrt{m}$. Such value is wider than the zone with shifted phase (see Figure 5b), and should be big enough to avoid the input in the LSF of points which may not follow the behavior predicted by the thermoelastic effect law.

It is also significant to report that for values of $N_w > 10$ a rank deficient warning message was given by Matlab, meaning that the rank of the matrix in Eq. 4 becomes too smaller than $N_w$, and no inversion is possible.

Contrary to what observed for the crack-tip identification, results now show a significant influence of input parameters. The range of variation of the SIF goes from 22 to 32 MPa$\sqrt{m}$, and the T-stress shows a range of variation between $-150$ and 50 MPa (see also [13] for a discussion on the T-stress value in SENT samples). The variation in both plots seems to narrow for $N_w > 5$ and $r_{max} > 50$ px, but there is no evidence of a clear convergence behavior.

Figure 8 shows the plot of $R^2$ for the same input data used in Figure 7. It is observed that for $N_w > 6$, $R^2$ has very high and constant values, above 0.99, throughout the whole range of $r_{max}$, while for $r_{max} < 60$ px a $N_w > 3$ is already sufficient to achieve $R^2 > 0.99$.

The high values of $R^2$ indicate that the Williams’ model is able to provide a very good fitting of the experimental data within the fitting area comprised between $r_{min}$ and $r_{max}$ (actual fitting area). Nevertheless, maximum values of $R^2$ embraces varying values of SIF and T-stress, i.e. there is not a clear correlation between $R^2$ and the trend in SIF and T-stress, and therefore a high $R^2$ cannot serve as a criterion for establishing the best input parameters.

Figure 9 shows the superposition of experimental and reconstructed isopachic contours. Figure 9-left
The 49th AIAS Conference (AIAS 2020)  
IOP Conf. Series: Materials Science and Engineering 1038 (2021) 012023  
doi:10.1088/1757-899X/1038/1/012023

| Nw  | 1  | 2  | 3  | 4  | 5  | 8  | 10 | 1  | 2  | 3  | 4  | 5  | 8  | 10 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| rmin| 12 |    |    |    |    |    |    | 12 |    |    |    |    |    |    |
| rmax| 30 |    |    |    |    |    |    | 60 |    |    |    |    |    |    |
| coarse grid method | Δx | 1  | 0  | 1  | 1  | 1  | 1  | 1  | -1 | 1  | 0  | 1  | 2  | 2  |
|                  | Δy | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| R² | 0.982 | 0.99 | 0.99 | 0.992 | 0.992 | 0.992 | 0.992 | 0.993 | 0.943 | 0.979 | 0.989 | 0.99 | 0.991 | 0.992 | 0.992 |
| fine grid method | Δx | 1.4 | 0.3 | 0.7 | 0.7 | 0.7 | 1  | 0.4 | -1 | 1  | 0  | 0.6 | 0.4 | 1  | 1.8 |
|                  | Δy | -0.1 | -0.1 | -0.1 | -0.1 | -0.1 | -0.1 | -0.1 | -0.2 | 0  | -0.2 | -0.2 | -0.2 | -0.2 | -0.2 |
| R² | 0.983 | 0.99 | 0.991 | 0.992 | 0.992 | 0.992 | 0.993 | 0.993 | 0.943 | 0.979 | 0.99 | 0.991 | 0.991 | 0.992 | 0.992 |
| pattern search | Δx | 1.377 | 0.344 | 0.787 | 0.787 | 0.787 | 0.997 | 0.391 | 1  | -1 | 0.617 | 0.504 | 0.988 | 1.736 | 1.95 |
|                  | Δy | -0.02 | -0.05 | -0.13 | -0.13 | -0.13 | -0.07 | -0.11 | -0.13 | -0.02 | -0.18 | -0.23 | -0.18 | -0.22 | -0.22 |
| R² | 0.983 | 0.99 | 0.991 | 0.992 | 0.992 | 0.992 | 0.993 | 0.993 | 0.943 | 0.979 | 0.99 | 0.991 | 0.991 | 0.992 | 0.992 |

Figure 6: values in pixels (1 px = 0.15 mm) of the x and y distances between the location of the minimum thermoelastic phase and the crack-tip obtained varying r_max and Nw.

![Figure 6](image_url)

Figure 7: plots of SIF (a) and T-stress (b) for varying r_max and Nw from SENT#1 test.

![Figure 7](image_url)

is obtained with a high r_max and high Nw, and it shows how the fitting is very good also near the straight edge of the sample. As r_max is reduced, the fitting is still very good within the fitting area comprised between r_min and r_max, but is rather poor on the rest of the domain. Figure 9-right in particular shows how the Williams model has an area of convergence, that in this case is reduced by the particular choice of the couple: r_max, Nw (see also [17]).
5. Results #2: SIF and T-stress from the CCT simulation
This section reports the results of applying the same LSF algorithms used for SENT#1, to the simulated CCT sample of Figure 4, obtained as described in Section 3.2.

5.1. Influence of input data on SIF and T-stress
The calculated values of SIF and T-stress with varying $r_{max}$ and $N_w$ are reported in Figure 10. This time the exact value is reported for comparison (see the horizontal green line). It is noticed that both the SIF and T-stress now show a convergence trend for $N_w > 6$ and growing $r_{max}$. It is also noticed that for small values of $r_{max}$, predictions with $3 < N_w < 6$ still provide low errors, while $N_w > 6$ leads to fluctuations and bigger errors.

Figure 11a also shows the plot of $R^2$ for the simulated case. The behavior is similar to that observed in Figure 8. Figure 11b reports an example of comparison between the simulated isopachs and those reconstructed with the fitted Williams model. As occurred for the experimental case (Figure 9a), also for the simulated case the reconstruction is very good.

Results from the simulated case confirm that the Matlab scripts implementing the LSF procedure work well and are reliable. Moreover an indication is given that a number of $N_w = 5$ or 6 works fairly well over the whole range of $r_{max}$. 

Figure 8: plot of $R^2$ for varying $r_{max}$ and $N_w$ for the tested SENT sample.

Figure 9: superimposed experimental and analytical isopachic contours for varying $r_{max}$ and $N_w$. 
Figure 10: plots of SIF (a) and T-stress (b) for varying $r_{\text{max}}$ and $N_w$ from simulated CCT sample.

Figure 11: plot of $R^2$ for varying $r_{\text{max}}$ and $N_w$ for the simulated CCT sample (a); contour plots of simulated isopachics and their reconstruction by the Williams’ model (b).

5.2. Error evaluation with the Stanley-Chan interpolation method

It is interesting to exploit the present simulated case-study to implement also the Stanley-Chan SIF evaluation procedure [7, 15]. In fact, this has been a very popular method implemented in a significant number of works in the literature. The main reasons for its success are the easy implementation and the unnecessary knowledge of the crack-tip position. On the contrary, the method does neglect the effects of the T-stress and higher order terms.

Figure 12 reports the graphical implementation of the method. A SIF value of 25.69 MPa$\sqrt{m}$ is obtained, against the exact value of 30.49 MPa$\sqrt{m}$. Therefore, the error amounts to about 16%, and is significantly higher than that obtained with the Williams’ Least Square Fitting evaluation of Figure 10.
6. Results #3: Evaluation of the J-Integral

The evaluation of the J-Integral from TSA requires the separation of stress components. This is possible once an analytical stress formulation such as Williams’ is fully characterized. It is also known that the elastic J-Integral is correlated to the SIF. A rationale is then proposed in this work, i.e. since the fitting model seems to be very effective over a wide domain (high $R^2$ values for high $R_{\text{max}}$ and $N_w$, see Section 4.2), it could be the case that the J-Integral evaluated via the fitted model would provide more reliable SIF evaluations, less influenced by input parameters. In the present work some first results are provided to prove the above statement. In particular an Energy Domain Integral formulation has been successfully implemented as described below.

6.1. Definition and numerical implementation of the Energy Domain Integral

The J-Integral for a two-dimensional body can be evaluated along a counter-clockwise arbitrary integration path $\Gamma$ around the crack tip as expressed in [22, 23]:

$$ j = \int_{\Gamma} \left( w \partial_{x_2} - T_i \frac{\partial u_i}{\partial x_1} \right) ds $$

(8)

where $w = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$ is the strain energy density (being $\sigma_{ij}$ and $\varepsilon_{ij}$, respectively, the stress and the strain tensors), $T_i$ are the tractions along the path ($T_i = \sigma_{ij} n_i$ where $n_i$ is the i-th component of $n$, the normal to $\Gamma$). The coordinates $\{x_1, x_2\}$ define the reference system, with $x_1$ aligned with the crack line.

Eq. 8 could be used to evaluate $j$ starting from data obtained by full-field experimental techniques [9, 24, 25]. If techniques such as TSA or Photoelasticity are used, a preliminary separation of stresses is required. The availability of full-field experimental data suggests that the accuracy can be improved by expressing the J-Integral via a domain integral. The Energy Domain Integral (EDI) method is in particular a good candidate choice, since its formulation specifically requires that the domain does not contain the crack tip, where experimental data can be inaccurate. The EDI formulation is derived from Eq. 8 by applying the divergence theorem and defining an auxiliary function ($q$). $j$ can be then be expressed as follows [24, 26]:

$$ j = \int_{\Omega} \left[ \left( \sigma_{ij} \frac{\partial u_j}{\partial x_1} - w \delta_{11} \right) \frac{\partial q}{\partial x_i} \right] dA \quad i, j = 1, 2 $$

(9)
where $\Omega$ is the integration area, $q$ is an arbitrary continuous smooth function that takes the value of 0 on the inner and 1 on the outer integration boundaries and $\delta_{ij}$ is the Kronecker delta.

In this work, the evaluation of the J-Integral has been numerically implemented in the same way as in [5], i.e.:

$$\mathcal{J} = n \cdot \frac{\Delta A}{h} \cdot \sum_{k=1}^{n} \left( \sigma_{ij} \frac{\Delta u_j}{\Delta x_1} - W \cdot \delta_{i1} \right) \frac{\Delta q}{\Delta x_i} \Delta u_i \Delta x_1$$

where in case of TSA $n$ is the number of pixels in $\Omega$, $\Delta A$ is the area of each pixel $h$ is the thickness of the specimen, $\frac{\Delta u_j}{\Delta x_1}$ and $\frac{\Delta q}{\Delta x_i}$ are the numerical derivatives of the displacement and the auxiliary function.

If the material is considered to be linear-elastic, the J-Integral can be used to evaluate the Stress Intensity Factor [23], knowing only the Young Modulus of the material:

$$\mathcal{J} = \frac{K^2}{E}$$

### 6.2. Evaluation of the Energy Domain Integral from TSA data

The evaluation of the EDI from TSA data can be performed according to the following steps:

- stresses and displacements are obtained by using the unknown coefficients “C” determined by means of the LSF procedure of the Williams’ stress series;
- the strains $\varepsilon_{ij}$ can be trivially evaluated starting from the stresses $\sigma_{ij}$ and considering the compliance matrix $[S]$;
- the knowledge of both $\varepsilon_{ij}$ and $\sigma_{ij}$ allows the evaluation of the strain energy density, $W$;
- an appropriate $q$ function is defined, and the required numerical derivatives evaluated, and finally $\mathcal{J}$ is evaluated according to Eq. 10.

A schematic representation of the above steps is also provided in Figure 13.

**Figure 13:** Steps required to evaluate fracture parameters starting from TSA data (symbols: $\sigma_0$ – T-stress, $W$ – strain energy density, $S$ – compliance matrix, $u_i$ – displacements).

The evaluation of the EDI is performed over a rectangular area, excluding from the calculation a certain amount of points around the crack tip (Figure 14).
6.3. Evaluation of the SIF from the J-Integral

The Energy Domain Integral is used to confirm the results obtained by means of the LSF, as long as the fitting procedure is able to correctly retrieve the values of the stresses in an area far from the crack-tip. The parameters used for the LSF and EDI are summarized in Table 2:

**Table 2**: dimensions of the fitting areas for the LSF and EDI procedures

|          | Least Squares Fitting | Energy Domain Integral Parameters |
|----------|------------------------|----------------------------------|
|          | $r_{\text{min}}$ [pixels] | $r_{\text{max}}$ [pixels] | $N_w$ [pixels] | $m$ [pixels] | $n$ [pixels] | $l_0$ [pixels] |
| SENT#1   | 8                      | 65                             | 10             | 80          | 60           | 50               |
| SENT#2   | 8                      | 65                             | 10             | 80          | 50           | 40               |
| SENT#3   | 4                      | 30                             | 10             | 60          | 60           | 40               |

According to [5] a pyramidal $q$ function (see also Figure 15a) has been defined, since it appears to give more reliable results in the evaluation of the EDI, making the calculation less sensitive to local errors. Figure 15b shows a map of the experimental thermoelastic signal where it is possible to notice that the data near the crack-tip are excluded from the calculation.

The Stress Intensity Factor can be obtained from the Energy Domain Integral, according to Eq. 11. The results are summarized in Table 3:

**Table 3**: SIF and EDI values obtained for the analysed specimens.

|          | SIF from LSF [MPa√m] | EDI [N/mm] | SIF from EDI [MPa√m] |
|----------|-----------------------|------------|-----------------------|
| SENT#1   | 27                    | 3.7        | 26.8                  |
| SENT#2   | 26.8                  | 3.7        | 26.8                  |
| SENT#3   | 27.9                  | 4          | 27.8                  |

The results in Table 3 confirm that the EDI approach is able to provide reasonable values of the SIF, very close to that obtained from the singular coefficient $C_1$ of the Williams’ series. Another noteworthy outcome is that a similar result is obtained also from tests SENT#2 and SENT#3 which
Figure 15: Matlab® plot of the pyramidal q function(a); integration area, over the experimental thermoelastic signal map from SENT#1 test(b).

have respectively a lower noise to signal ratio (SENT#2 uses a higher infrared sensor integration time) and a lower resolution (0.33 mm/pixel for SENT#3). It is interesting to observe also that the similar values of the SIF are obtained for a case where the fitting area of the LSF procedure and the integrating area of the EDI have similar dimensions and positions with respect to the crack tip. Future work will be carried out in order to assess if the property of path-independency of the J-Integral is transferred to a domain independency for the EDI, and if this translates in a convergence of the SIF value, i.e. an independency from the input parameters.

7. Conclusions
The present work has investigated the influence of input parameters in the fitting of experimental data, acquired from Thermoelastic Stress Analysis, with the Williams stress formulation, for the evaluation of Stress Intensity Factors. Two input parameters have been explored in particular: the extent of the area where experimental data are collected for the Least-Square Fitting (LSF) procedure, and the number of coefficients in the Williams model. Two cases have been analysed: an experimental case-study of a Single-Edge-Notched-Tension sample made of stainless steel, and a simulated Central-Crack-Tension infinite plate. The following conclusions are drawn:

- The linear LSF of the Williams’ series expansion is able to fit very well the experimental isopachics measured by TSA. The adoption of up to ten coefficients has been explored, resulting in correlation coefficient $R^2$ as high as 0.99;
- The linear LSF, associated to the use of $R^2$ as an objective function and to a pattern-search algorithm have allowed to implement a fast, accurate and robust method for the automatic tracking of the “notional” crack-tip location [21], evidencing also how this is little influenced by the LSF input parameters;
- On the contrary, the values of the SIF and T-stress have exhibited a significant variation the LSF input parameters, thus justifying the motivations of this work of trying to identify a criterion of choice;
- The simulated case study has allowed to verify the effectiveness of the implemented LSF procedures, and has shown a smaller but still present influence of input parameters on the SIF and T-stress. Contrary to the experimental case, a convergence behaviour has been observed when
$N_w > 5$ and $r_{\text{max}}$ increases. The simulated case has also allowed to compute the error made with the Stanley-Chan interpolation procedure. This error largely depends on the value of the T-stress, and for the case of the CCT sample it amounted to about 16%.

- The Williams interpolation has been used to separate stresses in the TSA stress metric and calculate the J-Integral via the Energy Domain Integral formulation. The value of the SIF retrieved from the J-Integral compares well with the SIF obtained from the singular coefficient of the Williams’ series. Future works will explore the possibility to exploit the path-independency of the J-Integral to provide values of the SIF which are less sensitive to the LSF input parameters.

Acknowledgements
The IR thermal camera FLIR X6540sc used in this work has been purchased using funds from the project INTEP – PO FESR 2007/2013 – 4.1.2.A.

References
[1] Pitarresi G, Cappello R and Catalanotti G 2020 Optics and Lasers in Engineering 134 106158
[2] Ju S H, Lesniak J and Sandor B 1997 Experimental mechanics 37 278–284
[3] Simon B N and Ramesh K 2010 Fourth International Conference on Experimental Mechanics vol 7522 (International Society for Optics and Photonics) p 75220D
[4] Reddy M S, Ramesh K and Thiagarajan A 2018 Theoretical and Applied Fracture Mechanics 96 146–159
[5] Cappello R, Pitarresi G, Xavier J and Catalanotti G 2020 Theoretical and Applied Fracture Mechanics 108 102663
[6] Tomlinson R, Nurse A and Patterson E 1997 Fatigue & Fracture of Engineering Materials & Structures 20 217–226
[7] Díaz F, Patterson E, Tomlinson R and Yates J 2004 Fatigue & Fracture of Engineering Materials & Structures 27 571–583
[8] Díaz F, Vasco-Olmo J, López-Alba E, Felipe-Sesé L, Molina-Viedma A and Nowell D 2020 International Journal of Fatigue 105567
[9] Lin S, Feng Z and Rowlands R 1997 Engineering Fracture Mechanics 56 579–592
[10] Haj-Ali R, Wei B S, Johnson S and El-Hajjar R 2008 Engineering Fracture Mechanics 75 58–75
[11] Zanganeh M, Tomlinson R and Yates J 2008 The Journal of Strain Analysis for Engineering Design 43 529–537
[12] Vieira R, González G and Freire J 2017 Residual Stress, Thermomechanics & Infrared Imaging, Hybrid Techniques and Inverse Problems, Volume 9 (Springer) pp 37–45
[13] Pitarresi G, Ricotta M and Meneghetti G 2019 Procedia Structural Integrity 18 330–346
[14] Zanganeh M, Lopez-Crespo P, Tai Y and Yates J 2013 Strain 49 102–115
[15] Pukas S R 1987 Stress Analysis by Thermoelastic Techniques vol 731 (International Society for Optics and Photonics) pp 88–101
[16] Pitarresi G and Patterson E 2003 The Journal of Strain Analysis for Engineering Design 38 405–417
[17] Hello G, Tahar M B and Roelandt J M 2012 International Journal of Solids and Structures 49 556–566
[18] Ramesh K, Gupta S and Kelkar A A 1997 Engineering Fracture Mechanics 56 25–45
[19] Cappello R, Catalanotti G and Pitarresi G 2020 2020 Virtual AIAS Conf. under Eval. publication IOP Conf. Ser. Mater. Sci. Eng
[20] Hooke R and Jeeves T A 1961 Journal of the ACM (JACM) 8 212–229
[21] Tomlinson R A and Patterson E A 2011 Thermomechanics and Infra-Red Imaging, Volume 7 (Springer) pp 123–129
[22] Rice J R 1968
[23] Anderson T L 2017 Fracture mechanics: fundamentals and applications (CRC press)
[24] Sakagami T, Kubo S, Fujinami Y and Kojima Y 2004 JSME International Journal Series A Solid Mechanics and Material Engineering 47 298–304
[25] Catalanotti G, Camanho P, Xavier J, Dávila C and Marques A 2010 Composites Science and Technology 70 1986–1993
[26] Shih C, Moran B and Nakamura T 1986 International Journal of fracture 30 79–102