Existence and Stability of the Lamb Dipoles for the Quasi-Geostrophic Shallow-Water Equations

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Abstract

In this paper, we prove the nonlinear orbital stability of vortex dipoles for the quasi-geostrophic shallow-water (QGSW) equations. The vortex dipoles are explicit travelling wave solutions to the QGSW equations, which are analogues of the classical circular vortex of Lamb and Chaplygin for the steady planar Euler equations. We establish a variational characterization of these vortex poles, which provides a basis for the stability result.

1 Introduction and main results

In this paper, we investigate the quasi-geostrophic shallow-water equation which is a nonlinear and nonlocal transport equation generalizing the two-dimensional Euler equations and used to describe large-scale motion for the atmosphere and the ocean circulation.

1.1 The quasi-geostrophic shallow-water equations

The quasi-geostrophic shallow water (QGSW) equations are derived asymptotically from the rotating shallow-water equations, in the limit of rapid rotation and weak variations of the free surface [25], which are given by

\begin{align}
\partial_t q + v \cdot \nabla q &= 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
v &= \nabla^\perp \psi, \quad \nabla^\perp = (\partial_2, -\partial_1), \\
\psi &= (-\Delta + \varepsilon^2)^{-1} q, \\
q|_{t=0} &= q_0,
\end{align}

(1.1)

where \( v \) is the velocity field, \( q \) is the ‘potential’ vorticity, \( \psi \) is the stream function, and \( \varepsilon \geq 0 \) is a parameter.

When the parameter \( \varepsilon = 0 \), we recover the two-dimensional Euler equations. The QGSW equations are a generalisation of the Euler equations and contain an additional parameter \( \varepsilon \).
parameter $\varepsilon$ is known as the inverse ‘Rossby deformation length’, which is a natural length scale arising from a balance between rotation and stratification.

1.2 The Lamb dipole

The Lamb dipole is a special translational vortex pair, which has a steady translating structure with opposite-sign vorticity of compact support in a circular disk. Translating vortex pairs are theoretical models of coherent vortex structures in large-scale geophysical flows; see [12, 26]. Let us assume that a travelling wave solution is of the form

\[
v(x, t) = u(x + u_\infty t) - u_\infty, \\
q(x, t) = \omega(x + u_\infty t),
\]

with a constant velocity $u_\infty \in \mathbb{R}^2$ at space infinity. Vortex pairs are pairs of compactly supported dipoles, symmetrically placed with opposite signs, translating in one direction. Without loss of generality, we may assume that $u_\infty = (-W, 0)$, $W > 0$ by rotation invariance of (1.1). Substituting $(v, q)$ into equation (1.1), we obtain the steady QGSW equations for $(u, \omega)$ in the half plane $\Pi := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$:

\[
\begin{align*}
&u \cdot \nabla \omega = 0, \quad \text{in } \Pi, \\
&u \to u_\infty \quad \text{as } |x| \to \infty.
\end{align*}
\]

In 1906, Lamb [16] noted an explicit vortex pair solution to the two-dimensional Euler equations (i.e., $\varepsilon = 0$ in (1.1)), a solution $\omega_C = \lambda (\Psi_C(x) - Wx_2) + u_C = \nabla \perp \Psi_C - We_1, 0 < \lambda < \infty$, of the form in the polar coordinate $(r, \theta)$

\[
\Psi_C(x) = \begin{cases} 
(A_CJ_1(\lambda^{1/2}r) + Wr)\sin \theta, & r \leq a, \\
\frac{a^2}{r}\sin \theta, & r > a,
\end{cases}
\]

with the constants

\[
A_C = -\frac{2W}{\lambda^{1/2}J_0(c_0)}, \quad a = c_0\lambda^{-1/2},
\]

where $J_m(r)$ is the $m$-th order Bessel function of the first kind and the constant $c_0$ is the first zero point of $J_1$, i.e., $J_1(c_0) = 0, c_0 = 3.8317 \cdots, J_0(c_0) < 0$, $f_+$ denotes the positive part of $f$, and $e_1 = (1, 0)$. This explicit solution is indeed a special case of non-symmetric Chaplygin dipoles, independently founded by S. A. Chaplygin [7, 8, 19]. So it is now generally referred to as the Lamb dipole or Chaplygin–Lamb dipole. The stream function $\Psi_C$ satisfies the following elliptic equation

\[
\begin{cases}
-\Delta \Psi = \lambda (\Psi - Wx_2)_+, & \text{in } \Pi, \\
\Psi \to 0 \text{ as } r \to \infty, & \Psi = 0, \quad \text{on } \partial \Pi, \\
\Psi(x_1, x_2) = -\Psi(x_1, -x_2), & \forall x \in \mathbb{R}^2.
\end{cases}
\]

The Lamb dipole $\omega_C$ has the form

\[
\omega_C(x_1, x_2) = -\omega_C(x_1, -x_2) = \lambda (\Psi_C(x) - Wx_2)_+, \quad \forall x \in \Pi.
\]

In 1996, Burton [4] proved that $\Psi_C$ is the unique solution to (1.4) when viewed in a natural weak formulation by using the method of moving planes. Very recently, Abe and Choi [1] established
nonlinear orbital stability of the Lamb dipole $\omega_C$. For some numerical and experimental studies on stability, see [12, 14].

In the present paper, we are interested in the Lamb dipole for the QGSW equations (1.1) with $\varepsilon > 0$. Without loss of generality, we shall restrict our attention to the case $\varepsilon = 1$. Let $\Psi = (-\Delta + Id)^{-1}\omega$ and $e_1 = (1,0)$, then (1.2) can be rewritten as

$$ \left( \nabla^\perp \Psi - We_1 \right) \cdot \nabla \omega = 0, $$

which is equivalent to

$$ \nabla^\perp (\Psi - Wx_2) \cdot \nabla \omega = 0. $$

(1.5)

As remarked by V. I. Arnol’d [2], a natural way of obtaining solutions to the stationary problem (1.5) is to impose that $\Psi - Wx_2$ and $\omega$ are (locally) functional dependent. Inspired by (1.4), we assume that

$$ \omega = \lambda (\Psi - Wx_2)_+ \text{ in } \Pi $$

for some constant $\lambda$. The problem is thus transformed into finding a solution to the following problem

$$ \begin{cases} 
-\Delta \Psi + \Psi = \lambda (\Psi - Wx_2)_+, & \text{in } \Pi, \\
\Psi \to 0 \text{ as } r \to \infty, & \Psi = 0, \text{ on } \partial \Pi, \\
\Psi (x_1, x_2) = -\Psi (x_1, -x_2), & \forall x \in \mathbb{R}^2.
\end{cases} $$

(1.6)

A solution of (1.6) can be easily found by using the separation of variables method. Indeed, let $1 < \lambda < \infty$ and

$$ \Psi_L(x) = \begin{cases} 
A_L J_1((\lambda - 1)^{1/2}r) + \frac{Wa}{\lambda - 1} \frac{1}{J_1((\lambda - 1)^{1/2}a)} \sin \theta, & r \leq a, \\
\frac{Wa}{K_1(a)} K_1(r) \sin \theta, & r > a,
\end{cases} $$

(1.7)

where $J_1(r)$ is the Bessel function of the first kind of order one, $K_1(r)$ is the modified Bessel function of the second kind of order one,

$$ A_L = -\frac{Wa}{\lambda - 1} \frac{1}{J_1((\lambda - 1)^{1/2}a)}, $$

and $a$ be the smallest positive solution satisfying (1.8)

$$ a \left( \frac{K'_1(a)}{K_1(a)} + \frac{1}{(\lambda - 1)^{1/2}} \cdot \frac{J'_1((\lambda - 1)^{1/2}a)}{J_1((\lambda - 1)^{1/2}a)} \right) = \frac{\lambda}{\lambda - 1}. $$

(1.8)

Then $\Psi_L$ is a desired solution of (1.6). Moreover, $\omega_L = \lambda (\Psi_L - Wx_2)_+, u_L = \nabla^\perp \Psi_L - We_1$ is an explicit solution to (1.2). Its vorticity is positive inside a semicircular region, while outside this region the flow is irrotational. In conjunction with its reflection in the $x_1$-axis, this flow constitutes a circular vortex. We shall call this solution the Lamb dipole to the QGSW equations. It seems that limited work has been done for the Lamb dipole to the QGSW equations. There are some analytical and numerical studies of the vortex patch solution to the QGSW equations. Polvani [20] and Polvani, Zabusky and Flierl [21] computed the generalizations of Kirchhoff ellipses under various values of $\varepsilon$, including doubly-connected patches and multi-layer flows. Later, Plotka and Dritschel [22] numerically studied the equilibrium form and stability of the rotating simply-connected vortex patches for the QGSW equations. Very recently, Dritschel, Hmidi and Renault...
investigated both analytically and numerically the bifurcation diagram of simply-connected rotating vortex patch equilibria for the QGSW equations.

The main purpose of this paper is to study the dynamical stability of the Lamb dipole for the QGSW equations. More precisely, we will establish the nonlinear orbital stability of the Lamb dipole $\omega_L$.

### 1.3 The main result

Similar to Burton [5], we introduce the following $L^p$-regular solution:

**Definition 1.1.** For the function $\zeta \in L^\infty_{loc} ([0, \infty), L^1 (\mathbb{R}^2)) \cap L^\infty_{loc} ([0, \infty), L^p (\mathbb{R}^2))$ is called a $L^p$-regular solution of (1.1), if $\zeta$ satisfies (1.1) in the sense of distributions, such that $E(\zeta(t, \cdot)), I(\zeta(t, \cdot))$ and $\|\zeta(t, \cdot)\|_{L^s}$ for $1 \leq s \leq p$ are constant for $t \in [0, \infty)$. Moreover, if $\zeta_0$ is non-negative and odd symmetric in $x_2$, then we require that $\zeta(t, \cdot)$ is also non-negative and odd symmetric in $x_2$.

Roughly speaking, the $L^p$-regular solution is a weak solution of (1.1), whose kinetic energy, impulse, and $L^s$ norm are conserved when $1 \leq s \leq p$. This is true for sufficiently smooth solutions. In the sequel, we identify a function $\zeta$ in $\Pi$ with an odd extension to $\mathbb{R}^2$ for the $x_2$-variable, i.e., $\zeta(x_1, x_2) = -\zeta(x_1, -x_2)$. We shall denote $\|\zeta\|_{L^1 \cap L^2} := \|\zeta\|_1 + \|\zeta\|_2$.

We have the following stability result:

**Theorem 1.2.** The Lamb dipole $\omega_L$ is orbitally stable in the sense that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any non-negative function $\zeta_0 \in L^1 \cap L^2(\Pi)$ and

$$\inf_{c \in \mathbb{R}} \{ ||\zeta_0 - \omega_L (\cdot + ce_1)||_{L^1 \cap L^2} + \|x_2 (\zeta_0 - \omega_L (\cdot + ce_1))\|_{L^1} \} \leq \delta,$$

if there exists a $L^2$-regular solution $\zeta(t)$ with initial data $\zeta_0$, then

$$\inf_{c \in \mathbb{R}} \{ ||\zeta(t) - \omega_L (\cdot + ce_1)||_{L^1 \cap L^2} + \|x_2 (\zeta(t) - \omega_L (\cdot + ce_1))\|_{L^1} \} \leq \varepsilon, \quad \forall t \in [0, \infty).$$

This paper is organized as follows. In Section 2 we provide a variational formulation for the Lamb dipole $\omega_L$. In Section 3 we establish the existence of maximizers. The uniqueness of maximizers is proved in Section 4. Section 5 is devoted to establishing the orbital stability in Theorem 1.2.

### 2 Variational formulation

We shall use Arnol’d’s idea [2] (see also [1, 6, 10]) to establish the nonlinear stability. The key idea is to give a variational characterization of the Lamb dipole $\omega_L$. Since the desired flows are odd symmetric about the $x_1$-axis, we can restrict our attention henceforth to the upper half-plane $\Pi$. Let $\bar{x} = (x_1, -x_2)$ be the reflection of $x$ in the $x_1$-axis. Denote

$$G_\Pi (x, y) = G(x, y) - G(\bar{x}, y), \quad \forall x, y \in \Pi,$$

where $G(x, y) = G(|x - y|)$ is the fundamental solution of the Bessel operator $-\Delta + Id$. Define

$$\mathcal{G} \omega (x) = \int_\Pi G_\Pi (x, y) \omega (y) \, dy, \quad x \in \Pi.$$
We introduced the kinetic energy of the fluid as follows

\[
E(\omega) = \frac{1}{2} \int_{\Pi} \omega(x) G \omega(x) \, dx,
\]

and its impulse

\[
I(\omega) = \int_{\mathbb{R}^2} x_2 \omega(x) \, dx.
\]

Let \( \lambda > 1, \mu > 0 \) and \( \nu > 0 \). We introduce the following space of admissible functions

\[
A_{\mu,\nu} := \left\{ \omega \in L^2(\Pi) \mid \omega \geq 0, \int_{\Pi} x_2 \omega(x) \, dx = \mu, \int_{\Pi} \omega(x) \, dx \leq \nu \right\},
\]

and the energy functional \( E_{\lambda} \) corresponding to the flows

\[
E_{\lambda}(\omega) = E(\omega) - \frac{1}{2\lambda} \int_{\Pi} \omega^2 \, dx, \quad \omega \in A_{\mu,\nu}.
\]

We will consider the maximization of the energy functional \( E_{\lambda} \) relative to \( A_{\mu,\nu} \). Set

\[
S_{\mu,\nu,\lambda} := \sup_{\omega \in A_{\mu,\nu}} E_{\lambda}(\omega), \quad (2.3)
\]

and

\[
\Sigma_{\mu,\nu,\lambda} := \left\{ \omega \in A_{\mu,\nu} \mid E_{\lambda}(\omega) = S_{\mu,\nu,\lambda} \right\}. \quad (2.4)
\]

Recall that

\[
\omega_{L} = \omega_{L}^{\lambda,W} = \lambda \left( \Psi_{L}^{\lambda,W} - W x_2 \right)_{+},
\]

where \( \Psi_{L}^{\lambda,W} \) is given by (1.7). We will show that the Lamb dipole \( \omega_{L}^{\lambda,W} \) can be re-obtained via the maximization problem (2.3) by appropriately choosing the parameters. More precisely, we have (see also Corollary 4.6 below)

**Proposition 2.1.** Let \( \lambda > 1 \) and \( \mu > 0 \) be given. Then there exists \( \nu_0 > 0 \), such that if \( \nu \geq \nu_0 \), then

\[
\Sigma_{\mu,\nu,\lambda} = \left\{ \omega_{L}^{\lambda,W}(\cdot + c e_1) \mid c \in \mathbb{R} \right\},
\]

where \( W = \mu/I(\omega_{L}^{\lambda,1}) \).

### 3 Existence of Maximizers

In this section, we prove the existence of maximizers for \( E_{\lambda} \) over \( A_{\mu,\nu} \). We first give some basic estimates that will be used frequently later. In what follows, the symbol \( C \) denotes a general positive constant that may change from line to line. We have the following basic estimate:

\[
G(x, y) = \begin{cases} 
C_0 \left( \ln \frac{2}{|x-y|} + O(1) \right), & \text{if } |x-y| \leq 2, \\
O\left( e^{-|x-y|/2} \right), & \text{if } |x-y| > 2,
\end{cases} \quad (3.1)
\]

where \( C_0 \) is a positive number.
Lemma 3.1. There exists a positive constant $C$ such that if $0 \leq \omega \in L^1(\Omega) \cap L^2(\Omega)$, then
\[
\|G\omega\|_{\infty} \leq C\|\omega\|_{1}^{1/2}\|\omega\|_{2}^{1/2},
\] (3.2)
and
\[
E(\omega) \leq C\|\omega\|_{3/2}^{3/2}\|\omega\|_{2}^{1/2}.
\] (3.3)

Proof. Let us first prove (3.2). By Hölder’s inequality, we have
\[
\int_{\Omega} G_{\Omega}(x,y)\omega(y)dy \leq C\|\omega\|_{4/3} \leq C\|\omega\|_{1}^{1/2}\|\omega\|_{2}^{1/2}, \quad \forall x \in \Omega.
\]
By the definition of $E$ and (3.2), we get
\[
E(\omega) \leq C\|G\omega\|_{\infty}\|\omega\|_{1} \leq C\|\omega\|_{3/2}^{3/2}\|\omega\|_{2}^{1/2}.
\]
The proof is thus complete. \qed

Lemma 3.2. Suppose that $0 \leq \omega \in L^1(\Omega) \cap L^2(\Omega)$, we have
\[
G_{\omega}(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\] (3.4)

Proof. For $|x|$ large, by (3.1) and (3.2) we have
\[
0 \leq G_{\omega}(x) \leq \int_{|y| \leq |x|/2} G_{\Omega}(x,y)\omega(y)dy + \int_{|y| \geq |x|/2} G_{\Omega}(x,y)\omega(y)dy
\]
\[
\leq C\left( e^{-|x|/4}\|\omega\|_{1} + \|\omega1_{\Omega\setminus B_{|x|/2}(0)}\|_{1} + \|\omega1_{\Omega\setminus B_{|x|/2}(0)}\|_{2} \right)
\]
\[
= o(1),
\]
which implies (3.4) and completes the proof. \qed

Since the energy $E_\lambda$ is invariant under translations in the $x_1$-direction, to control maximizers, we shall take the Steiner symmetrization in the $x_1$-variable.

We have the following result, whose proof is quite similar to that in [13, 23, 24] and so is omitted.

Lemma 3.3. For $\omega \geq 0$ satisfying $\omega \in L^1(\Omega) \cap L^2(\Omega)$ and $x_2\omega \in L^1(\Omega)$, there exists $\omega^* \geq 0$ such that
\[
\omega^*(x_1, x_2) = \omega^*(-x_1, x_2),
\]
\[
\omega^*(x_1, x_2) \text{ is non-increasing for } x_1 > 0
\] (3.5)
and
\[
\|\omega^*\|_{s} = \|\omega\|_{s}, \quad \forall s \in [1, 2],
\]
\[
\|x_2\omega^*\|_{1} = \|x_2\omega^*\|_{1},
\]
\[
E(\omega^*) \geq E(\omega).
\]

For a Steiner symmetric function, we have the following estimate:
Lemma 3.4. There exists a positive constant $C$ such that if $0 \leq \omega \in L^1(\Pi) \cap L^2(\Pi)$ is Steiner symmetric in the $x_1$-variable, then
\[
G_\omega(x) \leq C \left( |x_1|^{-3/8} \| \omega \|_{L^1}^{1/2} \| \omega \|_{L^2}^{1/2} + e^{-\sqrt{|x_1|}/2} \| \omega \|_1 \right) \tag{3.6}
\]
for any $x = (x_1, x_2) \in \Pi$ with $|x_1| > 4$.

Proof. Let $x = (x_1, x_2) \in \Pi$ satisfy $|x_1| > 4$. Define
\[
\omega_1(y) = \begin{cases} \omega(y), & \text{if } |y_1 - x_1| < \sqrt{|x_1|}, \\ 0, & \text{if } |y_1 - x_1| \geq \sqrt{|x_1|}. \end{cases}
\]
Using Eq. (2.11) in [3], we have
\[
\| \omega_1 \|_p \leq |x_1|^{-\frac{1}{2p}} \| \omega \|_p, \quad 1 \leq p \leq \infty.
\]
Hence, by (3.2), we have
\[
G_{\omega_1}(x) \leq C \| \omega_1 \|_{L^1}^{1/2} \| \omega \|_{L^2}^{1/2} \leq C |x_1|^{-3/8} \| \omega \|_{L^1}^{1/2} \| \omega \|_{L^2}^{1/2}. \tag{3.7}
\]
Letting $\omega_2 = \omega - \omega_1$, we have
\[
G_{\omega_2}(x) \leq C e^{-\sqrt{|x_1|}} \| \omega \|_1. \tag{3.8}
\]
Combining (3.7) and (3.8), we get (3.6).

Set
\[
\varrho(\lambda) = \frac{1}{I(\omega_L^{\lambda_1})} \int_\Pi \omega_L^{\lambda_1}(x) \, dx.
\]
Note that $\omega_L^{\lambda W} = W \omega_L^{\lambda_1}$. We have the following result concerning the supremum value.

Lemma 3.5. If $\mu g(\lambda) \leq \nu$, then
\[
0 < S_{\mu, \nu, \lambda} \leq C. \tag{3.9}
\]
Proof. By (3.3) and Young’s inequality, we have for $\omega \in A_{\mu, \nu}$
\[
E_\lambda(\omega) = E(\omega) - \frac{1}{2\lambda} \int_\Pi \omega^2 \, dx \\
\leq C \| \omega \|_{L^1}^{3/2} \| \omega \|_{L^2}^{1/2} - \frac{1}{2\lambda} \int_\Pi \omega^2 \, dx \\
\leq C \lambda^{1/3} \| \omega \|_1^2 \leq C.
\]
On the other hand, since $\omega_L^{\lambda \tilde{W}}$ with $\tilde{W} = \mu/I(\omega_L^{\lambda_1})$ belongs to $A_{\mu, \nu}$, so
\[
E_\lambda(\omega_L^{\lambda \tilde{W}}) = \frac{1}{2} \int_\Pi \lambda (\Psi_L^{\lambda \tilde{W}} - \tilde{W} x_2) \Psi_L^{\lambda \tilde{W}} \, dx - \frac{1}{2\lambda} \int_\Pi \lambda (\Psi_L^{\lambda \tilde{W}} - \tilde{W} x_2)^2 \, dx \\
= \frac{1}{2} \int_\Pi \lambda \Psi_L^{\lambda \tilde{W}} \, dx - \left( \Psi_L^{\lambda \tilde{W}} - [\Psi_L^{\lambda \tilde{W}} - \tilde{W} x_2]_+ \right) \, dx > 0.
\]
Therefore $S_{\mu, \nu, \lambda} \geq E_\lambda(\omega_L^{\lambda \tilde{W}}) > 0$ and the proof is thus complete.
In the sequel we shall assume that \( \mu \varphi(\lambda) \leq \nu \). Having made all the preparation, we are now able to show the existence of maximizers.

**Lemma 3.6.** It holds \( \Sigma_{\mu,\nu,\lambda} \neq \emptyset \). In addition, each \( \omega \in \Sigma_{\mu,\nu,\lambda} \) satisfies \( \int_{\Pi} x_2 \omega(x)dx = \mu \).

**Proof.** Let \( \{\omega_j\}_{j=1}^{\infty} \subset A_{\mu,\nu} \) be a maximizing sequence. By Lemma 3.5, we may assume that \( \mathcal{E}_{\lambda}(\omega_j) \geq 0 \) for all large \( j \). Using the definition of \( \mathcal{E}_{\lambda} \) and (3.3), we have

\[
\|\omega_j\|^2 \leq 2\lambda (E(\omega_j) - \mathcal{E}_{\lambda}(\omega_j)) \leq 2\lambda E(\omega_j) \leq C\|\omega_j\|^{3/2}\|\omega_j\|^{1/2} \leq C\|\omega_j\|^{1/2}.
\]

Hence \( \|\omega_j\|_2 \) is bounded by a constant independent of \( j \). We are going to show the convergence of energy. According to Lemma 3.3, we may assume that \( \omega_j \) is Steiner symmetric by replacing \( \omega_j \) with its Steiner symmetrisation. We assume \( \omega_j \to \omega \) weekly in \( L^2(\Pi) \) as \( j \to \infty \) by passing to a sub-sequence if necessary (still denoted by \( \{\omega_j\}_{j=1}^{\infty} \)). It is easy to verify that

\[
\int_{\Pi} x_2 \omega dx \leq \mu \quad \text{and} \quad \int_{\Pi} \omega dx \leq \nu.
\]

On the one hand, by Lemmas 3.1 and 3.3, we have

\[
2E(\omega_j) = \int_{\Pi} \int_{\Pi} \omega_j(x)G_{\Pi}(x,y)\omega_j(y)dxdy
\]

\[
\leq \int_{|x_1|<R,0<x_2<R} \int_{|y_1|<R,0<x_2<R} \omega_j(x)G_{\Pi}(x,y)\omega_j(y)dxdy
\]

\[
+ 2 \int_{x_2 \geq R} \omega_j(x)G\omega_j(x)dx + 2 \int_{|x_1| \geq R} \omega_j(x)G\omega_j(x)dx
\]

\[
\leq \int_{|x_1|<R,0<x_2<R} \int_{|y_1|<R,0<x_2<R} \omega_j(x)G_{\Pi}(x,y)\omega_j(y)dxdy
\]

\[
+ C\left( R^{-3/8}\|\omega_j\|_2^2 + e^{-\frac{\sqrt{\mathcal{F}}}{2}}\|\omega_j\|_1^2 \right) + 2R^{-1}\|G\omega_j\|_\infty \int_{\Pi} x_2 \omega_j(x)dx
\]

\[
\leq \int_{|x_1|<R,0<x_2<R} \int_{|y_1|<R,0<x_2<R} \omega_j(x)G_{\Pi}(x,y)\omega_j(y)dxdy + C\left( R^{-3/8} + e^{-\frac{\sqrt{\mathcal{F}}}{2}} + R^{-1} \right).
\]

Thanks to \( G_{\Pi}(x,y) \in L^2_{loc}(\Pi \times \Pi) \), we get

\[
\limsup_{j \to \infty} E(\omega_j) \leq E(\omega)
\]

by first letting \( j \to \infty \) and then \( R \to \infty \).

On the other hand, we have

\[
2E(\omega_j) = \int_{\Pi} \omega_j G\omega_j dx \geq \int_{|x_1|<R,0<x_2<R} \int_{|y_1|<R,0<x_2<R} \omega_j(x)G\omega_j(y)dxdy,
\]

it implies that

\[
\liminf_{j \to \infty} E(\omega_j) \geq E(\omega)
\]

by first letting \( j \to \infty \) and then \( R \to \infty \).
Hence, we conclude that
\[
\lim_{j \to \infty} E(\omega_j) = E(\omega).
\]
and
\[
\mathcal{E}_\lambda(\omega) = E(\omega) - \frac{1}{2\lambda} \int_\Pi \omega^2 dx \geq \lim_{j \to \infty} E(\omega_j) - \frac{1}{2\lambda} \liminf_{j \to \infty} \int_\Pi \omega_j^2 dx = S_{\mu,\nu,\lambda}.
\]

We now check that \( \int_\Pi x_2 \omega dx = \mu \). Indeed, suppose not, then there exists some \( \tau > 0 \) such that
\[
\omega_\tau(x_1, x_2) := \begin{cases} 
\omega(x_1, x_2 - \tau), & \text{if } x_2 > \tau, \\
0, & \text{if } x_2 \leq \tau,
\end{cases}
\]
belongs to \( A_{\mu,\nu} \). By virtue of the facts that \( G_\Pi(x, y) = G(|x - y|) - G(|\bar{x} - y|) \) and \( G(s) \) is strictly decreasing for \( s > 0 \), we check that
\[
S_{\mu,\nu,\lambda} = \mathcal{E}_\lambda(\omega) < \mathcal{E}_\lambda(\omega_\tau) \leq S_{\mu,\nu,\lambda}.
\]
This is a contradiction and the proof is thus complete. \( \square \)

From the proof of Lemma 3.6, we can obtain the monotonicity of \( S_{\mu,\nu,\lambda} \) with respect to \( \mu \).

**Lemma 3.7.** If \( 0 < \mu_1 < \mu_2 \), then \( S_{\mu_1,\nu,\lambda} < S_{\mu_2,\nu,\lambda} \).

### 4 Uniqueness of Maximizers

In the preceding section, we have proved the existence of maximizers for \( \mathcal{E}_\lambda \) over \( A_{\mu,\nu} \). In this section, we will establish the uniqueness of maximizers in the sense that any two maximizers differ by only a translation in the \( x_1 \)-direction.

**Lemma 4.1.** Each \( \omega \in \Sigma_{\mu,\nu,\lambda} \) satisfies
\[
\omega = \lambda(G\omega - Wx_2 - \gamma)_+ \quad \text{(4.1)}
\]
for some constants \( W, \gamma \geq 0 \), uniquely determined by \( \omega \).

**Proof.** By Lemma 3.5, \( S_{\mu,\nu,\lambda} > 0 \). There exists a constant \( \delta_0 > 0 \) such that \( \text{meas}\{\delta_0 < \omega\} > 0 \).
We take functions \( h_1, h_2 \in L^\infty(\Pi) \) with compact support and satisfying
\[
\begin{cases}
\text{supp}(h_1), \text{supp}(h_2) \subset \{\delta_0 \leq \omega\}, \\
\int_\Pi h_1(x)dx = 1, \quad \int_\Pi x_2 h_1(x)dx = 0, \\
\int_\Pi h_2(x)dx = 0, \quad \int_\Pi x_2 h_2(x)dx = 1.
\end{cases}
\]
We take an arbitrary \( \delta \in (0, \delta_0) \) and compactly supported \( h \in L^\infty(\Pi) \), \( h \geq 0 \) on \( \{0 \leq \omega \leq \delta\} \). We consider the test functions
\[
\omega_\varepsilon = \omega + \varepsilon \eta, \quad \varepsilon > 0,
\]
where
\[
\eta = h - \left( \int_\Pi hdx \right) h_1 - \left( \int_\Pi x_2 hdx \right) h_2.
\]
If $\varepsilon$ is small enough, one can verify that $\omega_{\varepsilon} \in A_{\mu, \nu}$. Since $\omega$ is a maximizer,

$$0 \geq \frac{dE_{\lambda}(\omega_{\varepsilon})}{d\varepsilon}\bigg|_{\varepsilon=0} = \int_{\Pi} \left( G\omega - \frac{1}{\lambda} \omega \right) \eta dx.$$ 

We define

$$\gamma := \int_{\Pi} \left( G\omega - \frac{1}{\lambda} \omega \right) h_1 dx, \quad W := \int_{\Pi} \left( G\omega - \frac{1}{\lambda} \omega \right) h_2 dx,$$

and

$$\Psi := G\omega - W x_2 - \gamma.$$ 

Hence we get

$$0 \geq \int_{\Pi} \left( G\omega - \frac{1}{\lambda} \omega \right) \eta dx = \int_{\Pi} \left( \Psi - \frac{1}{\lambda} \right) h dx.$$ 

Since the arbitrariness of $h$, we have

$$\begin{cases} 
\Psi - \frac{1}{\lambda} \omega = 0, & \text{on } \{ \omega > \delta \}, \\
\Psi - \frac{1}{\lambda} \omega \leq 0, & \text{on } \{ 0 \leq \omega \leq \delta \}.
\end{cases}$$

By letting $\delta \to 0$, we obtain $\omega = \lambda \Psi_+$. 

According to $\int_{\Pi} \omega dx \leq \nu$, we can take a sequence $\{x_i\}_{i=1}^{\infty}$ with $x_i = (x_{i1}, x_{i2})$, such that $x_{i1} \to \infty$, $x_{i2} \to 0$ and $\omega(x_i) \to 0$ as $i \to \infty$. By (3.4) in Lemma 3.2 we have

$$\limsup_{n \to \infty} (G\omega(x_i) - W x_{i2} - \gamma) \leq 0.$$ 

Hence $\gamma \geq 0$. Similarly, we can take another sequence $\{x_j\}_{j=1}^{\infty}$ with $x_j = (x_{1j}, x_{2j})$, such that $x_{1j} \to 0$, $x_{2j} \to \infty$ and $\omega(x_j) \to 0$ as $j \to \infty$. By (3.4) in Lemma 3.2 we have

$$0 = \lim_{j \to \infty} (G\omega(x_j) - W x_{2j} - \gamma)_{+} = \lim_{j \to \infty} (-W x_{2j} - \gamma)_{+},$$

which implies $W \geq 0$.

Next, we show the uniqueness of $W$ and $\gamma$. Suppose (4.1) holds with $W_1, \gamma_1 \geq 0$. Then

$$G\omega(x) - W_1 x_2 - \gamma_1 = G\omega(x) - W x_2 - \gamma,$$

for all $x \in \Pi$ satisfying $\omega(x) > 0$. Then,

$$(W_1 - W)x_2 = \gamma - \gamma_1,$$

which implies $W_1 = W$ and $\gamma_1 = \gamma$. \hfill \Box

The following result shows that if $\nu$ is sufficiently large, then $W > 0$ and $\gamma = 0$.

**Lemma 4.2.** Given $\lambda > 1$ and $\mu > 0$, there exists $\nu_0 > \mu \Phi(\lambda)$ such that if $\nu \geq \nu_0$, then the constants $W > 0$, $\gamma = 0$ in Lemma 4.1.
Proof. Let $\lambda > 1$ and $\mu > 0$ be fixed and $\omega \in \Sigma_{\mu, \nu, \lambda}$, we start to prove $\gamma = 0$ for all large $\nu$. Since

$$\mu = \int_{\Pi} x_2 \omega dx \geq \frac{2\mu}{\nu} \int_{x_2 \geq 2\nu} \omega dx,$$

we have

$$\int_{x_2 \geq 2\nu} \omega dx \leq \frac{\nu}{2}. \quad (4.2)$$

By Lemma 4.1, $\omega \leq \lambda G\omega$, so

$$\int_{0 < x_2 < \frac{2\nu}{\nu}} \omega dx \leq \int_{\Pi} \int_{0 < y_2 < \frac{2\nu}{\nu}} \lambda G_{\Pi}(x, y)\omega(x)dy dx.$$

By (3.1), for $\nu$ large we have

$$\int_{0 < y_2 < \frac{2\nu}{\nu}} G_{\Pi}(x, y)dy = o(1),$$

uniformly with respect to $x$. Hence

$$\int_{0 < x_2 < \frac{2\nu}{\nu}} \omega dx = o(1)\nu \quad (4.3)$$

for $\nu$ large. Combining (4.2) and (4.3), we see that for all sufficiently large $\nu$, it holds

$$\int_{\Pi} \omega dx < \nu.$$

Hence, we can take

$$\eta = h - \left( \int_{\Pi} x_2 \omega dx \right) h_2.$$

Consider the test functions $\omega + \varepsilon \eta$. Proceeding as in the proof of Lemma 4.1, we can obtain

$$\omega = \lambda (G\omega - Wx_2)_+,$$

which implies $\gamma = 0$.

Now, we turn to prove $W > 0$ for $\nu$ large. By (4.1), we have

$$0 < \int_{\Pi} \omega G\omega dx - \frac{1}{\lambda} \int_{\Pi} \omega^2 dx$$

$$= \int_{\Pi} \omega G\omega dx - \int_{\Pi} \omega (G\omega - Wx_2)_+ dx$$

$$\leq \int_{\Pi} \omega G\omega dx - \int_{\Pi} \omega (G\omega - Wx_2) dx$$

$$= W\mu,$$

which implies $W > 0$. The proof of Lemma 4.2 is thus finished.

The following result shows that each maximizer has compact support in $\overline{\Pi}$. We denote by $BUC(\overline{\Pi})$ the space of all bounded uniformly continuous functions in $\overline{\Pi}$ and by $C^{\alpha}(\overline{\Pi})$ the space of all Hölder continuous functions of exponent $0 < \alpha < 1$ in $\overline{\Pi}$. For an integer $k \geq 0$, $BUC^{k,\alpha}(\overline{\Pi})$ denotes the space of all $\phi \in BUC(\overline{\Pi})$ such that $\partial^l_x \phi \in BUC(\overline{\Pi}) \cap C^\alpha(\overline{\Pi})$, for $|l| \leq k$. 11
Lemma 4.3. For each \( \omega \in \Sigma_{\mu,\nu,\lambda} \), \( \text{supp}(\omega) \) is a compact set in \( \Pi \).

Proof. Let \( \omega \in \Sigma_{\mu,\nu,\lambda} \). By (4.1), we have \( \text{supp}(\omega) = \{ x \in \Pi \mid G_\omega - W x_2 - \gamma > 0 \} \) for \( W \geq 0 \) and \( \gamma \geq 0 \). If \( \gamma > 0 \), the conclusion follows easily from (3.4). If \( \gamma = 0 \), we must have \( W > 0 \). By (4.1), we have \( \text{supp}(\omega) = \{ x \in \Pi \mid G_\omega - W x_2^2 > 0 \} \). It follows from \( \omega \in L^1 \cap L^2 \) that \( \nabla^2 G_\omega \in L^p \), \( p \in (1, 2) \) and \( \nabla G_\omega \in L^q \), \( 1/q = 1/p - 1/2 \). By (3.4) and (4.1), \( G_\omega \) satisfies the following elliptic equation

\[
\begin{align*}
-\Delta \psi + \psi &= \lambda(\psi - W x_2^2), & \text{in } \Pi, \\
\psi &= 0, & \text{on } \partial \Pi, \\
\psi &\to 0, & \text{as } |x| \to \infty.
\end{align*}
\]

By the Sobolev embedding, we have \( G_\omega \in BUC^{2,\alpha}(\Pi) \). Since \( G_\omega(x_1, 0) = 0 \) and

\[ \frac{G_\omega}{x_2} = \int_0^1 (\partial_2 G_\omega)(x_1, x_2 s) ds, \]

hence \( G_\omega/x_2 \in BUC^{1,\alpha}(\Pi) \). Using Hardy's inequality (15), we get

\[ \| G_\omega/x_2 \|_2 \leq 2 \| \nabla G_\omega \|_2, \]

and hence \( G_\omega/x_2 \in BUC(\Pi) \cap L^2(\Pi) \). It follows that

\[ \frac{G_\omega(x)}{x_2} \to 0 \quad \text{as } |x| \to \infty, \]

which implies that \( \text{supp}(\omega) \) is a compact set of \( \Pi \).

Next, we consider positive solutions to the problem

\[
\begin{align*}
-\Delta \psi + \psi &= \lambda(\psi - W x_2^2), & \text{in } \Pi, \\
\psi &= 0, & \text{on } \partial \Pi, \\
\psi(x) &\to 0, & \text{as } |x| \to \infty.
\end{align*}
\]

(4.4)

Lemma 4.4. Let \( \psi \in BUC^{2,\alpha}(\Pi) \), \( 0 < \alpha < 1 \), be a positive solution of (4.4) for some \( W > 0 \) and \( \lambda > 1 \). Then \( \psi(x) = \psi_L(x + ce_1) \) for some \( c \in \mathbb{R} \), where \( \psi_L = \Psi_L \) and \( \Psi_L \) is defined by (1.7).

Proof. For \( y = (y', y_4) \in \mathbb{R}^4 \), \( y' = (y_1, y_2, y_3) \), we set \( x_1 = y_4, x_2 = |y'| \) and

\[ \phi(y) = \frac{\psi(x_1, x_2)}{x_2}. \]

(4.5)

By a direct calculation, we have

\[
\begin{align*}
-\Delta_y \phi + \phi &= \lambda(\phi - W)_{+}, & \text{in } \mathbb{R}^4, \\
\phi &\to 0, & \text{as } |y| \to \infty.
\end{align*}
\]

Thus \( \phi \) satisfies the integral equation

\[ \phi(x) = \int_{\mathbb{R}^4} G_4(x - y) \lambda(\phi(y) - W)_{+} dy. \]

(4.6)
where $G_4$ is the fundamental solution of the Bessel equation in $\mathbb{R}^4$.

Since $\phi$ is continuous and the support of $(\phi(y) - W)_+$ is compact, one can apply the standard method of moving planes in the integral form to deduce that $\phi$ is radially symmetric with respect to some point $y^0 = (0, c) \in \mathbb{R}^4$, see [9, 15] for more details.

Hence $\varphi(y) = \phi(y', y_4 + c)$ is radially symmetric and $|y| = |x|$, we have

$$\frac{\psi(x_1 + c, x_2)}{x_2} = \varphi(|x|).$$

By translation of $\psi$ for the $x_1$-variable, we may assume that $c = 0$. By the polar coordinate $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, we define

$$\Psi(x) = \psi(x) - Wx_2 = (\varphi(r) - W)r \sin \theta =: \eta(r) \sin \theta.$$

By (4.4), $\Psi$ satisfies

$$\begin{cases}
-\Delta \Psi + \Psi = \lambda \Psi, & \text{in } \Omega, \\
-\Delta \Psi + \Psi = 0, & \text{in } \Pi \setminus \Omega, \\
\Psi = 0, & \text{on } \partial \Pi \cup \partial \Omega, \\
\partial_{x_1} \Psi \to 0, \quad \partial_{x_2} \Psi \to -W, & \text{as } |x| \to \infty.
\end{cases} \tag{4.7}$$

where $\Omega = B_a(0) \cap \Pi$ for some $a > 0$. Using (4.7), we have

$$\begin{cases}
r^2 \eta'' + r \eta' + ((\lambda - 1)r^2 - 1) \eta - Wr^3 = 0, & \eta > 0, \ 0 < r < a, \\
\eta(a) = 0.
\end{cases} \tag{4.8}$$

We take $\eta_0 = \eta - \frac{W}{\lambda - 1}r$, then $\eta_0$ satisfies

$$\begin{cases}
r^2 \eta_0'' + r \eta_0' + ((\lambda - 1)r^2 - 1) \eta_0 = 0, & \eta_0(r) > -\frac{W}{\lambda - 1}r, \ 0 < r < a, \\
\eta_0(a) = -\frac{W}{\lambda - 1}a.
\end{cases} \tag{4.9}$$

Since $\eta_0(0)$ is bounded, we have

$$\eta_0 = \frac{-Wa}{\lambda - 1} \cdot \frac{J_1(\lambda - 1)^{1/2}r)}{J_1(\lambda - 1)^{1/2}a)}.$$

Similarly, in $\Pi \setminus \Omega$, $\eta$ satisfies

$$r^2 \eta'' + r \eta' - (r^2 + 1) \eta - Wr^3 = 0.$$

We take $\eta_1 = \eta + Wr$, then $\eta_1$ satisfies

$$r^2 \eta_1'' + r \eta_1' - (r^2 + 1) \eta_1 = 0.$$

Since $\eta_1$ is decaying at $\infty$ and $\eta(a) = 0$, we obtain

$$\eta_1 = \frac{Wa}{K_1(a)} K_1(r).$$
By $\Psi > 0$ in $B_a(0) \cap \Pi$ and the continuity of $\partial_r \Psi$ at $a$, it follows that $a$ is the smallest positive solution of the following equation
\[ a \left( \frac{K'(a)}{K_1(a)} + \frac{1}{(\lambda - 1)^{1/2}} \cdot \frac{J'_1((\lambda - 1)^{1/2}a)}{J_1((\lambda - 1)^{1/2}a)} \right) = \frac{\lambda}{\lambda - 1}. \tag{4.10} \]

Hence we get
\[ \Psi(x) = \Psi_L(x) - Wx = \begin{cases} A_L J_1((\lambda - 1)^{1/2} r) + \frac{W}{\lambda - 1} r \sin \theta, & r \leq a, \\ \frac{W}{\lambda - 1} \frac{K_1(r - W r)}{K_1(a)} \sin \theta, & r > a, \end{cases} \]
where
\[ A_L = -\frac{W a}{\lambda - 1} \cdot \frac{1}{J_1((\lambda - 1)^{1/2}a)}. \]

**Remark 4.5.** We want to show that equation (4.10) is solvable. Define the set as follows
\[ A = \{ t \in \mathbb{R}_+ \mid J_1((\lambda - 1)^{1/2}t) \neq 0 \} \]
and the function
\[ W(t) = \ln \frac{K_1(t) \cdot |J_1((\lambda - 1)^{1/2}t)|^{1/(\lambda - 1)}}{t^{\lambda/(\lambda - 1)}}, \quad t \in A. \tag{4.11} \]
By the properties of $J_1$, we know that $\mathbb{R}_+ \setminus A$ is at most countable. Suppose
\[ \mathbb{R}_+ \setminus A = \{ x_1, x_2, \ldots, x_n, \ldots \}, \quad \text{for} \ x_{i+1} > x_i > 0, \quad i \in \{ 1, 2, 3, \ldots \}. \]
We find that
\[ \lim_{t \to x_i} W(t) = -\infty, \]
and
\[ W(t) > -\infty, \quad \text{for} \ t \in (x_i, x_{i+1}), \]
where $i \in \{ 1, 2, 3, \ldots \}$. Therefore, on each interval $(x_i, x_{i+1})$, $W$ has at least one extreme point, then (4.10) is solvable. By direct calculation, we obtain
\[ W'(t) = \frac{K'(t)}{K_1(t)} + \frac{1}{(\lambda - 1)^{1/2}} \cdot \frac{J'_1((\lambda - 1)^{1/2}t)}{J_1((\lambda - 1)^{1/2}t)} - \frac{\lambda}{\lambda - 1} \cdot \frac{1}{t}, \]
and
\[ \lim_{t \to 0^+} W'(t) = -\infty. \]
Thus there exists a smallest positive solution $a$ to equation (4.10).

**Corollary 4.6.** For $\lambda > 1$, $\mu > 0$ and $\nu \geq \nu_0$, we have
\[ \Sigma_{\mu, \nu, \lambda} = \{ \omega_{\lambda, W}^\mu (\cdot + ce_1) \mid c \in \mathbb{R} \}, \]
where $W = \mu / I(\omega_{\lambda, L}^1)$. 

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5 Compactness of Maximizing Sequences

In this section, we shall prove the compactness of a maximizing sequence up to translations for the $x_1$-variable by using a concentration compactness principle due to P. L. Lions.

**Theorem 5.1.** Let $\lambda > 1$, $\mu > 0$ and $\nu \geq \nu_0$. Suppose that $\{\omega_n\}_{n=1}^\infty$ is a maximizing sequence in the sense that

\[
\omega_n \geq 0, \quad \omega_n \in L^1 \cap L^2, \quad \int_\Pi \omega_n dx \leq \nu, \quad \|\omega_n\|_2 \leq C, \quad \forall n \geq 1,
\]

and

\[
\mu_n = \int_\Pi x_2 \omega_n dx \rightarrow \mu, \quad \text{as} \quad n \rightarrow \infty, \quad (5.2)
\]

Then there exists $\omega \in \Sigma_{\mu, \nu, \lambda}$, a sub-sequence $\{\omega_{n_k}\}_{k=1}^\infty$ and a sequence of real numbers $\{c_k\}_{k=1}^\infty$ such that as $k \rightarrow \infty$, it holds

\[
\omega_{n_k}(\cdot + c_k e_1) \rightarrow \omega \quad \text{in} \quad L^2(\Pi), \quad (5.4)
\]

and

\[
x_2 \omega_{n_k}(\cdot + c_k e_1) \rightarrow x_2 \omega \quad \text{in} \quad L^1(\Pi). \quad (5.5)
\]

To prove Theorem 5.1, we need the following concentration compactness lemma (see [17]).

**Lemma 5.2.** Let $\{\xi_n\}_{n=1}^\infty$ be a sequence of nonnegative functions in $L^1(\Pi)$ satisfying

\[
\limsup_{n \rightarrow \infty} \int_\Pi \xi_n dx \rightarrow \mu,
\]

for some $0 < \mu < \infty$. Then, after passing to a subsequence, one of the following holds:

(i) (Compactness) There exists a sequence $\{y_n\}_{n=1}^\infty$ in $\overline{\Pi}$ such that for arbitrary $\varepsilon > 0$, there exists $R > 0$ satisfying

\[
\int_{\Pi \cap B_R(y_n)} \xi_n dx \geq \mu - \varepsilon, \quad \forall n \geq 1.
\]

(ii) (Vanishing) For each $R > 0$,

\[
\lim_{n \rightarrow \infty} \sup_{\Pi \cap B_R(y_n)} \int_{B_R(y)} \xi_n dx = 0.
\]

(iii) (Dichotomy) There exists a constant $0 < \alpha < \mu$ such that for any $\varepsilon > 0$, there exist $N = N(\varepsilon) \geq 1$ and $0 \leq \xi_{i,n} \leq \xi_n$, $i = 1, 2$ satisfying

\[
\left\{ \begin{array}{l}
\|\xi_n - \xi_{1,n} - \xi_{2,n}\|_1 + |\alpha - \int_\Pi \xi_{1,n} dx| + |\mu - \alpha - \int_\Pi \xi_{2,n} dx| < \varepsilon, \quad \text{for} \quad n \geq N, \\
\quad d_n := \text{dist} (\text{supp} (\xi_{1,n}), \text{supp} (\xi_{2,n})) \rightarrow \infty, \quad \text{as} \quad n \rightarrow \infty.
\end{array} \right.
\]

**Proof of Theorem 5.1** Let $\xi_n = x_2 \omega_n$. Using Lemma 5.2, we find that for a certain sub-sequence, still denoted by $\{\omega_n\}_{n=1}^\infty$, one of the three cases in Lemma 5.2 should occur. To deal with the three cases, we divide the proof into three steps.
Step 1. (Vanishing excluded) Suppose that for each fixed $R > 0$, 

$$
\limsup_{n \to \infty} \int_{y \in \Pi \setminus B_R(y) \cap \Pi} x_2 \omega_n \, dx = 0. \tag{5.6}
$$

To get a contradiction, it is sufficient to prove that $\lim_{n \to \infty} E(\omega_n) = 0$. We set

$$
2E(\omega_n) = \int_{\Pi} \int_{\Pi} \omega_n(x) G_\Pi(x, y) \omega_n(y) \, dx \, dy
$$

where

$$
G_\Pi(x, y) = \left( \int_{|x-y| \geq R} + \int_{|x-y| \leq R} \right) \omega_n(x) G_\Pi(x, y) \omega_n(y) \, dx \, dy.
$$

By (4.1) we have

$$
\int_{|x-y| \geq R} G_\Pi(x, y) \omega_n(x) \omega_n(y) \, dx \, dy \leq \int_{|x-y| \geq R} G(x, y) \omega_n(x) \omega_n(y) \, dx \, dy
$$

$$
\leq Ce^{-\frac{R^2}{2}} \nu^2 \quad \text{as} \quad R \to \infty.
$$

Set

$$
\int_{|x-y| \leq R} G_\Pi(x, y) \omega_n(x) \omega_n(y) \, dx \, dy = \left( \int_{y_2 \geq \frac{1}{R}} + \int_{y_2 < \frac{1}{R}} \right) \int_{|y-x| \leq R} G_\Pi(x, y) \omega_n(x) \omega_n(y) \, dx \, dy.
$$

By simple calculations, we get that

$$
\int_{|x-y| \leq R} G_\Pi(x, y) \omega_n(x) \omega_n(y) \, dx \, dy \leq \int_{\Pi} \omega_n(x) \, dx \int_{y_2 \geq \frac{1}{R}} \int_{|y-x| \leq R} G(x, y) \omega_n(y) \, dy
$$

$$
\leq C \nu \| \omega_n \|_{L^2} \left( \sup_{y_2 \geq \frac{1}{R}} \int_{|y-x| \leq R} \omega_n(y) \, dy \right)^{1/2}
$$

$$
\leq CR^{1/2} \left( \sup_{y_2 \geq \frac{1}{R}} \int_{|y-x| \leq R} \omega_n(y) \, dy \right)^{1/2} \to 0 \quad \text{as} \quad n \to \infty.
$$

In addition, we have

$$
\int_{|x-y| \leq R} G_\Pi(x, y) \omega_n(x) \omega_n(y) \, dx \, dy \leq \int_{\Pi} \omega_n(x) \, dx \int_{y_2 < \frac{1}{R}} \int_{|y-x| \leq R} G(x, y) \omega_n(y) \, dy
$$

$$
\leq C \nu \| \omega_n \|_{L^2} \left( \sup_{y_2 < \frac{1}{R}} \int_{|y-x| \leq R} G^2(x, y) \, dy \right)^{1/2}
$$

$$
\leq C \nu \left( \sup_{y_2 < \frac{1}{R}} \int_{|y-x| \leq R} G^2(x, y) \, dy \right)^{1/2} \to 0 \quad \text{as} \quad R \to \infty.
$$

Hence

$$
2E(\omega_n) \leq Ce^{-\frac{R^2}{2}} \nu^2 + CR^{1/2} \left( \sup_{y_2 < \frac{1}{R}} \int_{|y-x| \leq R} \omega_n(y) \, dy \right)^{1/2} + C\nu \left( \sup_{y_2 < \frac{1}{R}} \int_{|y-x| \leq R} G^2(x, y) \, dy \right)^{1/2}.
$$

Letting $n \to \infty$ and then $R \to \infty$ implies $\lim_{n \to \infty} E(\omega_n) = 0$. 

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Step 2. (Dichotomy excluded) Suppose that there exists some $\alpha \in (0, \mu)$ such that

$$
\begin{align*}
\omega_n &= \omega_{1,n} + \omega_{2,n} + \omega_{3,n}, \quad 0 \leq \omega_{i,n} \leq \omega_n, \quad i = 1, 2, 3, \\
\|x_2 \omega_{3,n}\|_1 + |\alpha - \alpha_n| + |\mu - \alpha - \beta_n| &\to 0, \quad \text{as } n \to \infty, \\
d_n := \text{dist}(\text{supp}(\omega_{1,n}), \text{supp}(\omega_{2,n})) &\to \infty, \quad \text{as } n \to \infty,
\end{align*}
$$

where $\alpha_n = \|x_2 \omega_{1,n}\|_1$ and $\beta_n = \|x_2 \omega_{2,n}\|_1$. According to the symmetry of $E$, we have

$$
2E(\omega_n) = 2E(\omega_{1,n} + \omega_{2,n} + \omega_{3,n}) = \int_{\Pi} \int_{\Pi} \omega_{1,n}(x)G(x,y)\omega_{1,n}(y)dx \ dy
$$

$$
+ \int_{\Pi} \int_{\Pi} \omega_{2,n}(x)G(x,y)\omega_{2,n}(y)dx \ dy + 2 \int_{\Pi} \int_{\Pi} \omega_{1,n}(x)G(x,y)\omega_{2,n}(y)dx \ dy
$$

$$
+ \int_{\Pi} \int_{\Pi} (2\omega_n - \omega_{3,n}(x))G(x,y)\omega_{3,n}(y)dx \ dy.
$$

For fixed $R > 0$,

$$
\int_{\Pi} \int_{\Pi} (2\omega_n - \omega_{3,n}(x))G(x,y)\omega_{3,n}(y)dx \ dy

\leq C \int_{y_2 \geq 1/R} G(x,y)\omega_{3,n}(y)dy + C \int_{y_2 < 1/R} G(x,y)\omega_{3,n}(y)dy
$$

$$
\leq CR^{\frac{1}{2}} \|x_2 \omega_{3,n}\|_1^\frac{1}{2} + C \left(\sup_{x \in \Pi} \int_{y_2 < 1/R} G^2(x,y)dy\right)^\frac{1}{2}.
$$

By (5.7), we have

$$
\int_{\Pi} \int_{\Pi} \omega_{1,n}(x)G(x,y)\omega_{2,n}(y)dx \ dy \leq Ce^{-d_n/2}.
$$

Hence

$$
\mathcal{E}_\lambda(\omega_n) = E(\omega_n) - \frac{1}{2\lambda} \int_{\Pi} \omega_n^2 dx
$$

$$
\leq \mathcal{E}_\lambda(\omega_{1,n}) + \mathcal{E}_\lambda(\omega_{2,n}) + CR^{\frac{1}{2}} \|x_2 \omega_{3,n}\|_1^\frac{1}{2} + C \left(\sup_{x \in \Pi} \int_{y_2 < 1/R} G^2(x,y)dy\right)^\frac{1}{2} + Ce^{-d_n/2}.
$$

Taking Steiner symmetrization $\omega^*_{i,n}$ of $\omega_{i,n}$ for $i = 1, 2$, we get

$$
\begin{align*}
\mathcal{E}_\lambda(\omega_n) &\leq \mathcal{E}_\lambda(\omega^*_{1,n}) + \mathcal{E}_\lambda(\omega^*_{2,n}) + CR^{\frac{1}{2}} \|x_2 \omega_{3,n}\|_1^\frac{1}{2} + C \left(\sup_{x \in \Pi} \int_{y_2 < 1/R} G^2(x,y)dy\right)^\frac{1}{2} + Ce^{-d_n/2}, \\
\|\omega^*_{i,n}\|_1 + \|\omega^*_{2,n}\|_1 &\leq \nu, \quad \|\omega^*_{1,n}\|_2 + \|\omega^*_{2,n}\|_2 \leq C, \\
\|x_2 \omega^*_{1,n}\|_1 = \alpha_n, \quad \|x_2 \omega^*_{2,n}\|_1 = \beta_n.
\end{align*}
$$

We assume that $\omega^*_{i,n} \to \omega^*_i$ weakly in $L^2(\Pi)$ as $n \to \infty$ for $i = 1, 2$. Proceeding as in the proof of Lemma 3.6, we can obtain the convergence of the kinetic energy

$$
\lim_{n \to \infty} E(\omega^*_{i,n}) = E(\omega^*_i), \quad \text{for } i = 1, 2.
$$

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By first letting \( n \to \infty \), then \( R \to \infty \), we obtain

\[
\begin{aligned}
S_{\mu,\nu,\lambda} & \leq \mathcal{E}_\lambda (\omega_1^*) + \mathcal{E}_\lambda (\omega_2^*) , \\
\|\omega_1^*\|_1 + \|\omega_2^*\|_1 & \leq \nu , \quad \|\omega_1^*\|_2 + \|\omega_2^*\|_2 \leq C , \\
\|x_2\omega_1^*\|_1 & \leq \alpha , \quad \|x_2\omega_2^*\|_1 \leq \mu - \alpha .
\end{aligned}
\]

We set \( \alpha_1 = \|x_2\omega_1^*\|_1 \leq \alpha , \; \nu_1 = \|\omega_1^*\|_1 , \; \beta_1 = \|x_2\omega_2^*\|_1 \leq \mu - \alpha \) and \( \nu_2 = \|\omega_2^*\|_1 \). It holds

\[
\alpha_1 > 0 , \quad \beta_1 > 0 .
\]

In fact, suppose that \( \alpha_1 = 0 \), then we have \( \omega_1^* \equiv 0 \), and hence

\[
S_{\mu,\nu,\lambda} \leq \mathcal{E}_\lambda (\omega_1^*) + \mathcal{E}_\lambda (\omega_2^*) \leq \mathcal{E}_\lambda (\omega_2^*) \leq S_{\beta_1,\nu,\lambda} .
\]

This is a contradiction to Lemma 3.7. Similarly, one can verify \( \beta_1 > 0 \). We choose \( \tilde{\omega}_1 \in \Sigma_{\alpha_1,\nu,\lambda}, \; \tilde{\omega}_2 \in \Sigma_{\beta_1,\nu,\lambda} \). By Lemma 4.3 we have that supports of \( \tilde{\omega}_1, \; i = 1, 2 \) are bounded. Therefore, we may assume that \( \text{supp}(\tilde{\omega}_1) \cap \text{supp}(\tilde{\omega}_2) = \emptyset \) by suitable translations in \( x_1 \)-direction. Letting \( \tilde{\omega} = \tilde{\omega}_1 + \tilde{\omega}_2 \), then we have

\[
\begin{aligned}
\int_{\mathbb{R}} \tilde{\omega} dx = \int_{\mathbb{R}} \tilde{\omega}_1 dx + \int_{\mathbb{R}} \tilde{\omega}_2 dx \leq \nu , \\
\int_{\mathbb{R}} x_2\tilde{\omega} dx = \int_{\mathbb{R}} x_2\tilde{\omega}_1 dx + \int_{\mathbb{R}} x_2\tilde{\omega}_2 dx = \alpha_1 + \beta_1 \leq \mu ,
\end{aligned}
\]

which implies that \( \tilde{\omega} \in A_{\alpha_1+\beta_1,\nu} \). Observing that \( \tilde{\omega}_1 \not\equiv 0 \) and \( \tilde{\omega}_2 \not\equiv 0 \), we have

\[
\begin{aligned}
S_{\mu,\nu,\lambda} & \leq \mathcal{E}_\lambda (\omega_1^*) + \mathcal{E}_\lambda (\omega_2^*) \\
& \leq \mathcal{E}_\lambda (\tilde{\omega}_1) + \mathcal{E}_\lambda (\tilde{\omega}_2) \\
& = \mathcal{E}_\lambda (\tilde{\omega}) - 2 \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{\omega}_1(x)G_{\mathbb{R}}(x,y)\tilde{\omega}_2(y)dydx \\
& < S_{\alpha_1+\beta_1,\nu,\lambda} \leq S_{\mu,\nu,\lambda},
\end{aligned}
\]

which is a contradiction.

Step 3. (Compactness) Assume that there is a sequence \( \{y_n\}_{n=1}^{\infty} \) in \( \mathbb{R} \) such that for arbitrary \( \varepsilon > 0 \), there exists \( R > 0 \) satisfying

\[
\int_{\mathbb{R} \cap B_R(y_n)} x_2\omega_n dx \geq \mu - \varepsilon , \quad \forall \; n \geq 1 . \tag{5.8}
\]

We may assume that \( y_n = (0, y_{n,2}) \) after a suitable \( x_1 \)-translation. We claim that

\[
\sup_{n \geq 1} y_{n,2} < \infty . \tag{5.9}
\]

Indeed, if \( (5.9) \) is false, then there exists a subsequence, still denoted by \( \{y_{n}\} \), such that

\[
\lim_{n \to \infty} y_{n,2} = \infty .
\]

By direct calculation, we have

\[
2E(\omega_n) = \int_{\mathbb{R}} \omega_n(x)G\omega_n(x)dx \\
= \int_{\mathbb{R} \cap B_R(y_n)} \omega_n(x)G\omega_n(x)dx + \int_{\mathbb{R} \setminus B_R(y_n)} \omega_n(x)G\omega_n(x)dx .
\]
Since \( \{ \omega_n \}_{n=1}^{\infty} \) is uniformly bounded in \( L^2(\Pi) \), \( \|x_2 \omega_n\|_1 \leq \mu + o(1) \) and \( \|x_2 \omega_n\| \leq \mu + o(1) \) and (3.6), we have
\[
\int_{\Pi \cap B_R(y_n)} \omega_n(x) G \omega_n(x) dx \leq \frac{C \mu}{(y_n, 2 + 1 - R)^{1/2}} \to 0 \quad \text{as } n \to \infty.
\]
For any fixed \( M > 0 \) large, we have
\[
\int_{\Pi \setminus B_R(y_n)} \omega_n(x) G \omega_n(x) dx \leq \int_{\Pi \setminus B_R(y_n)} x_2 \omega_n(x) G \omega_n(x) dx \leq C M^2 \|x_2 \omega_n 1_{B_R(y_n)}\|_1^{1/2} + C \left( \sup_{x \in \Pi} \int_{y_2 < 1/M} G^2(x, y) dy \right)^{1/2} \leq CM^2 \varepsilon^{1/2} + C \left( \sup_{x \in \Pi} \int_{y_2 < 1/M} G^2(x, y) dy \right)^{1/2}.
\] (5.10)
Hence, by first letting \( n \to \infty \), then \( \varepsilon \to 0 \) and lastly \( M \to \infty \), we obtain
\[
0 < S_{\mu, \nu, \lambda} \leq \lim_{n \to \infty} E(\omega_n) = 0.
\]
The claim (6.9) is thus proved. We may assume that \( y_{n, 2} = 0 \) by taking \( R \) larger. Therefore, we have
\[
\int_{\Pi \setminus B_R(0)} x_2 \omega_n dx \geq \mu - \varepsilon, \quad \forall \ n \geq 1.
\]
Since \( \{ \omega_n \} \) is uniformly bounded in \( L^2 \), by choosing a subsequence, \( \omega_n \to \omega \) weekly in \( L^2 \) for some \( \omega \). By sending \( n \to \infty \),
\[
\int_{\Pi} \omega dx \leq \nu, \quad \int_{\Pi} x_2 \omega dx = \mu.
\]
Hence \( \omega \in A_{\mu, \nu} \). Let us assume that
\[
\lim_{n \to \infty} E(\omega_n) = E(\omega), \quad (5.11)
\]
which implies
\[
S_{\mu, \nu, \lambda} = \lim_{n \to \infty} E_{\lambda}(\omega_n) \\
\leq \lim_{n \to \infty} E(\omega_n) - \frac{1}{2 \lambda} \lim inf_{n \to \infty} \|\omega_n\|_2^2 \\
\leq E_{\lambda}(\omega) \leq S_{\mu, \nu, \lambda}.
\]
Hence \( \lim_{n \to \infty} \|\omega_n\|_2^2 = \|\omega\|_2^2 \) and \( \omega_n \to \omega \) in \( L^2 \) follows. By
\[
\int_{\Pi} x_2 |\omega_n - \omega| dx = \int_{\Pi \setminus B_R(0)} x_2 |\omega_n - \omega| dx + \int_{\Pi \setminus B_R(0)} x_2 |\omega_n - \omega| dx \\
\leq CR^2 \|\omega_n - \omega\|_2 + \int_{\Pi \setminus B_R(0)} x_2 (\omega_n + \omega) dx \\
\leq CR^2 \|\omega_n - \omega\|_2 + \mu_n - \mu + 2 \varepsilon.
\]
Sending \( n \to \infty \) and then \( \varepsilon \to 0 \), the above inequality implies \( x_2 \omega_n \to x_2 \omega \) in \( L^1(\Pi) \). Since \( \mathcal{E}_\lambda(\omega_n) \to \mathcal{E}_\lambda(\omega) \), the limit \( \omega \in A_{\mu,\nu} \) is a maximizer of \( S_{\mu,\nu} \).

It remains to show the assumption (5.11). On the one hand, for any fixed \( M > 0 \) large, we have

\[
2E(\omega_n) = \int_\Pi \int_\Pi \omega_n(x)G_\Pi(x,y)\omega_n(y)dx\,dy \\
\leq \int_{\Pi \cap B_R(0)} \int_{\Pi \cap B_R(0)} \omega_n(x)G_\Pi(x,y)\omega_n(y)dx\,dy \\
+ 2 \int_{\Pi \setminus B_R(0)} \int_{\Pi \setminus B_R(0)} \omega_n(x)G_\Pi(x,y)\omega_n(y)dx\,dy \\
\leq \int_{\Pi \cap B_R(0)} \int_{\Pi \cap B_R(0)} \omega_n(x)G_\Pi(x,y)\omega_n(y)dx\,dy \\
+ CM^{\frac{1}{2}} \left\| x_2 \omega_n \mathbb{1}_{\Pi \setminus B_R(0)} \right\|_1^{\frac{1}{2}} + C \left( \sup_{x \in \Pi} \int_{y < 1/M} G^2(x,y)dy \right)^{\frac{1}{2}}.
\]

Letting \( n \to \infty \), then \( \varepsilon \to 0 \) and lastly \( M \to \infty \), we get

\[
\limsup_{n \to \infty} E(\omega_n) \leq E(\omega).
\]

On the other hand, for any \( L > 0 \), we have

\[
2E(\omega_n) = \int_\Pi \int_\Pi \omega_n(x)G_\Pi(x,y)\omega_n(y)dx\,dy \\
\geq \int_{\Pi \cap B_L(0)} \int_{\Pi \cap B_L(0)} \omega_n(x)G_\Pi(x,y)\omega_n(y)dx\,dy,
\]

which implies

\[
\liminf_{n \to \infty} E(\omega_n) \geq E(\omega).
\]

The proof of (5.11) is thus completed. \( \square \)

# 6 Orbital Stability

In this section, we establish the orbital stability of the Lamb dipoles \( \omega_L \). Recalling Corollary 4.6, Theorem 1.2 follows from the following result.

**Theorem 6.1.** Let \( \lambda > 1 \), \( \mu > 0 \) and \( \nu \geq \nu_0 \). Then for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for any non-negative function \( \zeta_0 \in L^1 \cap L^2(\Pi) \) and

\[
\inf_{\omega \in \Sigma_{\mu,\nu,\lambda}} \left\{ \left\| \zeta_0 - \omega \right\|_{L^1 \cap L^2} + \left\| x_2 (\zeta_0 - \omega) \right\|_{L^1} \right\} \leq \delta,
\]

if there exists a \( L^2 \)-regular solution \( \zeta(t) \) with initial data \( \zeta_0 \), then

\[
\inf_{\omega \in \Sigma_{\mu,\nu,\lambda}} \left\{ \left\| \zeta(t) - \omega \right\|_{L^1 \cap L^2} + \left\| x_2 (\zeta(t) - \omega) \right\|_{L^1} \right\} \leq \delta \tag{6.1}
\]

for all \( t \geq 0 \).
Proof. We argue by contradiction. Suppose that the statement were false. Then there exists $\varepsilon_0 > 0$ such that for $n \geq 1$, there exist $\varepsilon_0 > 0$ such that for $n \geq 1$, there exist $\zeta_{0,n} \in L^1 \cap L^2(\Pi)$ satisfying

$$\inf_{\omega \in \Sigma_{\mu, \nu, \lambda}} \left\{ \| \zeta_{0,n} - \omega \|_{L^1 \cap L^2} + \| x_2 (\zeta_{0,n} - \omega) \|_{L^1} \right\} \leq \frac{1}{n},$$

and

$$\inf_{\omega \in \Sigma_{\mu, \nu, \lambda}} \left\{ \| \zeta(t) - \omega \|_{L^1 \cap L^2} + \| x_2 (\zeta(t) - \omega) \|_{L^1} \right\} \geq \varepsilon_0,$$ \hspace{1cm} (6.2)

where $\zeta_n(t)$ is a $L^2$-regular solution with the initial data $\zeta_{0,n}$. We take $\omega_n \in \Sigma_{\mu, \nu, \lambda}$ such that

$$\| \zeta_{0,n} - \omega_n \|_{L^1 \cap L^2} + \| x_2 (\zeta_{0,n} - \omega_n) \|_{L^1} \to 0 \quad \text{as} \quad n \to \infty.$$

It is not hard to verify that

$$E_{\lambda}(\zeta_{0,n}) \to S_{\mu, \nu, \lambda}.$$ We write $\zeta_n = \zeta_n(t_n)$ by suppressing $t_n$. By the conservation laws, one has

$$\begin{cases}
\zeta_n \geq 0, \quad \zeta_n \in L^1 \cap L^2(\Pi), \quad \int_{\Pi} \zeta_n \, dx \leq \nu, \quad \| \zeta_n \|_2 \leq C, \\
\mu_n = \int_{\Pi} x_2 \zeta_n \, dx \to \mu, \quad \text{as} \quad n \to \infty, \\
E_{\lambda}(\zeta_n) \to S_{\mu, \nu, \lambda}, \quad \text{as} \quad n \to \infty.
\end{cases}$$

By Theorem 5.1, there exist $\omega \in \Sigma_{\mu, \nu, \lambda}$, a subsequence $\left\{ \zeta_{n_k} \right\}_{k=1}^{\infty}$ and a sequence of real number $\{c_k\}_{k=1}^{\infty}$ such that

$$\| \zeta_{n_k} (\cdot + c_k e_1) - \omega \|_2 + \| x_2 (\zeta_{n_k} (\cdot + c_k e_1) - \omega) \|_1 \to 0, \quad \text{as} \quad k \to \infty,$$

which is contrary to (6.2), and the proof of Theorem 6.1 is thus completed. \hfill \Box

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