Chiral transitions in three–dimensional magnets
and higher order $\epsilon$–expansion

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The critical behaviour of helimagnets and stacked triangular antiferromagnets
is analyzed in $(4 - \epsilon)$ dimensions within three–loop approximation. Numerical es-
timates for marginal values of the order parameter dimensionality $N$ obtained by
resummation of corresponding $\epsilon$–expansions rule out the possibility of continuous
chiral transitions in magnets with Heisenberg or planar spins.

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Chiral phase transitions in helical magnets and stacked triangular antiferromagnets with Heisenberg or $XY$–like spins as well as in some other systems attract much attention during the last decade [1–9]. Special interest to these transitions demonstrated both by theorists and experimentalists has been given rise by the conjecture [1–3] that they belong to new universality class which is characterized by critical exponents differing markedly from those of 3D $O(n)$–symmetric model with relevant values of $n$ (2, 4, and 6). This conjecture originates from the renormalization–group (RG) analysis of corresponding $(4 - \epsilon)$–dimensional model performed within the lowest (one- and two-loop, according to quantum field theory language) orders in $\epsilon$. Some results given by the $\frac{1}{n}$–expansion and Monte Carlo simulations were also considered as favoring the abovementioned idea [2,3].

In this Letter, an attempt is made to clear up whether the conclusion about an existence of new universality class for 3D chiral systems with Heisenberg or planar spins survives when higher-order terms in $\epsilon$–expansion are taken into account. Below we calculate $\epsilon$–expansions for quantities of interest up to the three–loop order. To obtain numerical estimates relevant to real 3D magnets we apply resummation procedures to these series before setting $\epsilon = 1$. Such a machinery proved to give good results for plenty of phase transition models. It is believed to be powerful enough to yield reasonable predictions in our case as well.

The Landau–Wilson Hamiltonian describing systems under consideration may be written down in the form (see, e.g. Ref. [9]):

$$H = \frac{1}{2} \int d^D x \left[ m_0^2 \varphi_\alpha \varphi^*_\alpha + \nabla \varphi_\alpha \nabla \varphi^*_\alpha + \frac{u_0}{2} \varphi_\alpha \varphi^*_\alpha \varphi_\beta \varphi^*_\beta + \frac{w_0}{2} \varphi_\alpha \varphi_\alpha \varphi^*_\beta \varphi^*_\beta \right],$$

(1)

where $\varphi_\alpha$ is a complex vector order parameter field, $\alpha, \beta = 1, 2, \ldots, N$, a bare mass squared $m_0^2$ being proportional to the deviation from the mean–field transition point. This model undergoes chiral phase transitions if $w_0 > 0$ [9]. In the opposite case, the transitions into somewhat trivial (linearly polarized or unfrustrated) ordered states take place.

In the critical region, where fluctuations are strong and the system behavior is governed
by the RG equations, the model Eq. (1) can demonstrate four different regimes of RG flow depending on $N$ [3,4]. Correspondingly, three critical (marginal) values of $N$ exist separating these regimes from each other. If $N < N_{c1}$ the RG equations possess three nontrivial fixed points (FP’s) with the $O(2N)$–symmetric point being stable. When $N$ exceeds $N_{c1}$ the Heisenberg FP loses its stability but the other, anisotropic FP with coordinate $w < 0$ acquires it. This point “annihilates” with another, saddle anisotropic FP when $N$ approaches $N_{c2}$, and there is only one nontrivial FP in the domain $N_{c2} < N < N_{c3}$. It is $O(2N)$–symmetric and unstable. At last, when $N$ increases further and crosses over the value $N_{c3}$ the creation of two new anisotropic FP’s with $w > 0$ takes place. One of them is stable and describes the chiral critical behavior. Hence, to answer the question about the relevance of the chiral FP to the critical thermodynamics of real helical magnets and stacked triangular antiferromagnets, one has to estimate $N_{c3}$ and compare the number obtained with physical values $N = 2$ and $N = 3$.

Marginal values of $N$ may be found analyzing RG $\beta$–functions. We calculate these functions for the model Eq. (1) within three–loop approximation in $(4 - \epsilon)$ dimensions using the minimal subtraction scheme (corresponding expansions are too lengthy and not presented here). The $\epsilon$–expansion for $N_{c3}$ resulting from the $\beta$–functions obtained is as follows:

$$N_{c3} = 12 + 4\sqrt{6} - \left[12 + \frac{14\sqrt{6}}{3}\right]\epsilon + \left[\frac{137}{150} + \frac{91\sqrt{6}}{300} + \left(\frac{13}{5} + \frac{47\sqrt{6}}{60}\right)\zeta(3)\right]\epsilon^2$$

$$= 21.80 - 23.43\epsilon + 7.088\epsilon^2$$

(2)

where $\zeta(x)$ is the Riemann $\zeta$–function, $\zeta(3) = 1.20206$. The constant and linear terms in Eq. (2) coincide with those presented by H. Kawamura [3] while the second–order one is essentially new.

Such expansions are known to be asymptotic and physical information may be extracted from them provided some resummation method is applied. The Borel transformation combined with proper procedure of analytical continuation of the Borel transform usually plays
a role of this method leading to precise numerical estimates in cases of long enough original series [10, 11]. To perform the analytical continuation the Pade approximant of \([L/1]\) type may be used which is known to provide rather good results for various Landau–Wilson models (see, e.g. Refs. [9, 12, 13]). The Pade–Borel summation of the expansion (2) gives:

\[
N_{c3} = a - \frac{2b^2}{c} + \frac{4b^3}{c^2} e^{\exp\left(\frac{-2b}{c\epsilon}\right) Ei\left(\frac{2b}{c\epsilon}\right)},
\]

(3)

where \(a, b,\) and \(c\) are coefficients before \(\epsilon^0, \epsilon^1,\) and \(\epsilon^2\) in Eq. (2), respectively, \(Ei(x)\) being the exponential integral. Setting \(\epsilon = 1,\) we obtain from Eq. (3)

\[
N_{c3} = 3.39.
\]

(4)

Making use of the Pade approximant \([1/1]\) itself gives \(N_{c3} = 3.81\) while direct summation of the expansion Eq. (4), being rather crude procedure, results in \(N_{c3} = 5.46.\)

All these numbers, although considerably scattered, are nevertheless greater than 3. Hence, helical magnets and stacked triangular antiferromagnets with Heisenberg and \(XY\)–like spins are not seen to demonstrate new, chiral critical behavior. Instead, they should approach helically ordered state or frustrated antiferromagnetic phase only via first–order phase transitions.

On the other hand, the difference between the number Eq. (4) and \(N = 3\) is not so large. Moreover, numerical estimates for \(N_{c3}\) were obtained from the theory having no small parameter in the limit \(\epsilon = 1.\) How close to the precise value of \(N_{c3}\) they may be?

To clear up this point let us compare Eq. (4) and its Pade counterpart with analogous estimate given by the RG analysis in three dimensions. Three–loop RG expansions for \(\beta–\)functions resummed by means of generalized Pade–Borel technique result in \(N_{c3} = 3.91\) [9]. Since the accuracy provided by this approximation was argued to be rather high [9] (about 1% for FP’s coordinates and critical exponents), resummed \(\epsilon–\)expansion, within three–loop order, seems to yield plausible estimates for the quantities of interest.
This conclusion is of prime importance. It is reasonable, therefore, to look for extra pros and cons it. One can find them calculating the rest of marginal values of $N$ from resummed three–loop $\epsilon$–expansions and making a comparison of numbers obtained with corresponding 3D RG estimates. The inequality $N_{c2} > 2$ proven earlier [3] may be used as a criterion in course of such a study. The $\epsilon$–expansions for $N_{c1}$ and $N_{c2}$ are found to be:

$$N_{c1} = 2 - \epsilon + \frac{5}{24}(6\zeta(3) - 1)\epsilon^2 = 2 - \epsilon + 1.294\epsilon^2 ,$$  

(5)

$$N_{c2} = 12 - 4\sqrt{6} - \left(12 - \frac{14\sqrt{6}}{3}\right)\epsilon + \left[\frac{137}{150} - \frac{91\sqrt{6}}{300} + \left(\frac{13}{5} - \frac{47\sqrt{6}}{60}\right)\zeta(3)\right]\epsilon^2 \nonumber$$

$$= 2.202 - 0.569\epsilon + 0.989\epsilon^2 .$$  

(6)

Their Pade–Borel summation gives for $\epsilon = 1$

$$N_{c1} = 1.50 , \quad N_{c2} = 1.96 .$$  

(7)

These values are close to those obtained in 3D: $N_{c1} = 1.45, N_{c2} = 2.03$ [3], but $N_{c2}$ is seen to be obviously underestimated by the $\epsilon$–expansion since corresponding number is less than 2. The difference $2 - N_{c2}^{(\epsilon)} = 0.04$, however, is small and may be considered as a lower bound for the error produced by this approximation. The most likely estimate for this error is believed to be close to $N_{c2}^{(3D)} - N_{c2}^{(\epsilon)}$, i.e. being of order of 0.1.

For $N_{c1}$ the $\epsilon$–expansion predicts the value which is slightly greater than $N_{c1}^{(3D)}$. At the same time, these numbers differ from each other by only 3% and lie so far from the nearest physical value $N = 2$ that this difference is quite unimportant.

We see that resummed three–loop $\epsilon$–expansion gives good enough numerical estimates for $N_{c1}$ and $N_{c2}$ providing, however, the lower accuracy when used for evaluation of highest critical dimensionality $N_{c3}$. It is not surprising since the structure of series Eq. (2) turns out to be rather unsuitable for yielding reliable quantitative results. Indeed, to obtain precise numerical estimates one has to deal with series which, at least, possess coefficients decreasing
with increasing their number. Instead, the second term in Eq. (2) exceeds, for \( \epsilon = 1 \), the first one. It is clear that such an expansion would not demonstrate good summability. That is why we believe that the true value of \( N_{c3} \) is closer to 3D estimate 3.91 than to 3.39. On the other hand, the latter estimate differs from the former by no more than 15\%. Hence, actually the current status of the \( \epsilon \)-expansion is not so bad provided three-loop contributions are properly taken into account. It seems natural that calculations of higher-order terms will result in further improvement of numerical estimates.

It is worth noting that really three-loop terms added and the summation have changed the situation drastically. The point is that two-loop \( \epsilon \)-expansions for \( N_{c1}, N_{c2}, \) and \( N_{c3} \) directly extrapolated to \( \epsilon = 1 \) violate the inequalities \( N_{c1} < N_{c2} < N_{c3} \) which should hold good according to the definition of \( N_{ci} \); in this approximation \( N_{c3}^{(\epsilon)} < N_{c1}^{(\epsilon)} < N_{c2}^{(\epsilon)} \) when \( \epsilon = 1 \). This is an alarm bell signalizing that with such short series in hand one can not safely penetrate into the three-dimensional world. To the contrary, the numbers given by the Pade and Pade–Borel resummed three-loop \( \epsilon \)-expansions at \( \epsilon = 1 \) meet abovementioned inequalities.

Another point to be specially marked is as follows. Actually, we have now enough information resulting from higher-order RG analysis in three and 4\( - \epsilon \) dimensions to make the firm conclusion about the critical behavior of the \( XY \)-like systems. Indeed, since \( N_{c2} \) has been proven to be greater than 2 \([9]\) and \( N_{c3} \) should exceed \( N_{c2} \), magnets with planar spins certainly can not undergo continuous chiral transitions. Only first-order phase transitions into chiral states are possible, in principle, in these systems.

We conclude with the comment concerning the \( \frac{1}{n} \)-expansion and Monte Carlo simulations. Actually, their predictions do not contradict to those just obtained. Indeed, as was shown in Ref. \([13]\), even for simple, \( O(n) \)-symmetric model the \( \frac{1}{n} \)-expansion begin to yield reasonable numerical estimates only when \( n \) exceeds 20. Since for systems studied
n = 2N = 4, 6, this approach is obviously inapplicable in our case. Monte Carlo simulations, in turn, give the values of critical exponents which are close to tricritical ones, especially for systems with XY–like spins [14]. That is why it is believed [5,9,14] that not chiral critical behavior but tricritical one or tricritical–to–critical crossover are really seen in these computer experiments.

To summarize, we have found three–loop contributions to the $\epsilon$–expansions of critical order–parameter dimensionalities $N_{c1}$, $N_{c2}$, and $N_{c3}$ for the model describing the chiral critical behavior. The Pade–Borel summation of series obtained has yielded, at $\epsilon = 1$, fair numerical estimates for $N_{c1}$ and $N_{c2}$ in 3D. For the lower boundary of the domain where continuous chiral transitions are possible the $\epsilon$–expansion resummed by Pade-Borel and simple Pade methods has given $N_{c3} = 3.39$ and $N_{c3} = 3.81$ respectively. Being close enough to 3D RG estimate $N_{c3} = 3.91$, these numbers are greater than physical values $N = 2$ and $N = 3$. It may be considered as an evidence that in magnets with Heisenberg or planar spins the chiral critical behavior with specific values of critical exponents would not really occur and they can not belong to new, chiral class of universality. Since $N_{c3} > N_{c2}$ and $N_{c2}$ should exceed 2, for the XY–like systems this conclusion sounds as firm.

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