MILNOR MONODROMIES AND MIXED HODGE STRUCTURES FOR NON-ISOLATED HYPERSURFACE SINGULARITIES

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Abstract. We study the Milnor monodromies of non-isolated hypersurface singularities and show that the reduced cohomology groups of the Milnor fibers are concentrated in the middle degree for some eigenvalues of the monodromies. As an application of this result, we give an explicit formula for some parts of their Jordan normal forms.

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1. INTRODUCTION

The Milnor monodromies of complex hypersurface singularities are important subjects in singularity theory. For isolated hypersurface singular points, we have some algorithms or formulas to compute them. However, for non-isolated singular points, there still remain some difficulties in computing them explicitly. In this paper, we will show that even for non-isolated singular points, the generalized eigenspaces of the Milnor monodromies for “good” eigenvalues have some nice properties similar to those for isolated singular points. By this result, we give an explicit formula for some parts of the Jordan normal forms of the Milnor monodromies.

Let $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial of $n(\geq 2)$ variables with coefficients in $\mathbb{C}$ such that $f(0) = 0$ and $V := f^{-1}(0) \subset \mathbb{C}^n$ the hypersurface defined by it. We denote by $F_{f,0}$ the Milnor fiber of $f$ at $0$ and by

$$\Phi_j : H^j(F_{f,0}; \mathbb{C}) \cong H^j(F_{f,0}; \mathbb{C})$$

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the $j$-th Milnor monodromy for $j \in \mathbb{Z}$. If the origin $0 \in V$ is an isolated singular point of $V$, the Milnor fiber $F_{f,0}$ is homotopic to a bouquet of some $(n - 1)$-spheres $S^{n - 1}$ by a celebrated theorem of Milnor [11]. This implies that the reduced cohomology groups $\tilde{H}^j(F_{f,0}; \mathbb{C})$ vanish except for $j = n - 1$. However, if $0 \in V$ is a non-isolated singular point, we cannot expect to have such a concentration in general. On the other hand, for a polynomial $f$ non-degenerate at $0$, Varchenko [19] described explicitly the monodromy zeta function

$$\zeta_{f,0}(t) := \prod_{j \in \mathbb{Z}} \det(Id - t\Phi_j)^{(-1)^j} \in \mathbb{C}(t)$$

in terms of the Newton polyhedron $\Gamma_+(f)$. If $0 \in V$ is an isolated singular point of $V$, the $(n - 1)$-th Milnor monodromy $\Phi_{n-1}$ is the only non-trivial one. Therefore, in this case, we obtain a formula for the characteristic polynomial of $\Phi_{n-1}$. However, if $0 \in V$ is a non-isolated singular point, Varchenko’s formula does not tell us any explicit information about the characteristic polynomial of each Milnor monodromy $\Phi_j$.

For non-isolated singular points, there is a similar difficulty also for the mixed Hodge structures of the cohomologies of the Milnor fibers. Recall that each cohomology group $H^j(F_{f,0}; \mathbb{Q})$ of $F_{f,0}$ is endowed with a mixed Hodge structure $(H^j(F_{f,0}; \mathbb{Q}), F^\bullet, W_\bullet)$ defined by Steenbrink [17] in the case where $0 \in V$ is an isolated singular point and by Navarro [12] and M. Saito [14] in the case where $0 \in V$ is a non-isolated singular point. For $j \in \mathbb{Z}$ and an eigenvalue $\lambda \in \mathbb{C}$ of the Milnor monodromy $\Phi_j$, we denote by

$$H^j(F_{f,0}; \mathbb{C})_\lambda \subset H^j(F_{f,0}; \mathbb{C})$$

the generalized eigenspace of $\Phi_j$ for $\lambda$. For $p, q \in \mathbb{Z}$, we denote by $h^{p,q}_\lambda(H^j(F_{f,0}; \mathbb{C}))$ the $(p, q)$-mixed Hodge number for the eigenvalue $\lambda$ of the $j$-th cohomology group of $F_{f,0}$ i.e. the dimension of $\text{Gr}^p_F \text{Gr}^W_{p+q} H^j(F_{f,0}; \mathbb{C})_\lambda$. For $\lambda \in \mathbb{C}$, we define a polynomial $E_\lambda(F_{f,0}; u, v) \in \mathbb{Z}[u, v]$ with coefficients in $\mathbb{Z}$ by

$$E_\lambda(F_{f,0}; u, v) := \sum_{p, q \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} (-1)^j h^{p,q}_\lambda(H^j(F_{f,0}; \mathbb{C})) u^p v^q.$$

By using a description of the Motivic Milnor fiber of $f$ at 0 (see Theorem 3.3), we can describe $E_\lambda(F_{f,0}; u, v)$ explicitly in terms of certain polynomials defined by the Newton polyhedron $\Gamma_+(f)$ (see Corollary 3.10). Moreover, if $0 \in V$ is an isolated singular point of $V$, as in the previous discussion about Varchenko’s formula, we can describe $h^{p,q}_\lambda(H^{n-1}(F_{f,0}; \mathbb{C}))$ by our formula. In particular, we obtain an explicit formula for the Jordan normal form of $\Phi_{n-1}$ (see Matsui-Takeuchi [10]). On the other hand, if $0 \in V$ is a non-isolated singular point, the formula for $E_\lambda(F_{f,0}; u, v)$ does not tell us any explicit information about each mixed Hodge number $h^{p,q}_\lambda(H^j(F_{f,0}; \mathbb{C}))$.

In this paper, we follow an idea of Takeuchi-Tibar [18] for monodromies at infinity and overcome the above-mentioned difficulties by introducing a finite subset $R_f \subset \mathbb{C}$ of “bad” eigenvalues of the Milnor monodromies (see Definition 3.12) as follows.

**Theorem 1.1** (see Theorem 4.1). Assume that $f$ is non-degenerate at 0. Then, for any $\lambda \notin R_f$ we have a concentration:

$$\tilde{H}^j(F_{f,0}; \mathbb{C})_\lambda \simeq 0 \quad (j \neq n - 1).$$
Note that a more general but less explicit concentration result was given in [3, Corollary 6.1.7]. By this theorem and Varchenko’s formula, we can compute the multiplicities of eigenvalues \( \lambda \notin R_f \) in \( \Phi_{n-1} \) as follows.

**Corollary 1.2.** (see Corollary 4.2) In the situation of Theorem 1.1, for any \( \lambda \notin R_f \) the multiplicity of the eigenvalue \( \lambda \) in the Milnor monodromy \( \Phi_{n-1} \) is equal to that of the factor \((1 - \lambda t)^k\) in a rational function

\[
\prod_{I \neq \emptyset \subset \{1, \ldots, n\}} k_I \prod_{i=1}^{k_I} (1 - t^{d_{I,i}})(-1)^{n-|I|} \text{Vol}_z(\Gamma_{I,i}).
\]

For the definitions of \( \Gamma_{I,i}, d_{I,i} \) and \( \text{Vol}_z(\Gamma_{I,i}) \), see Section 2.

Moreover, for such \( \lambda \) we obtain the mixed Hodge numbers \( h^p_q(\mathcal{H}^{n-1}(F_{f,0}; \mathbb{C})) \) by our formula for \( E_\lambda(F_{f,0}; u, v) \).

If \( 0 \in V \) is an isolated singular point, the filtration on \( \mathcal{H}^{n-1}(F_{f,0}; \mathbb{C})_\lambda \) induced by the weight filtration coincides with the monodromy filtration of \( \Phi_{n-1} \). This implies that the Jordan normal form of \( \Phi_{n-1} \) for an eigenvalue \( \lambda \) can be described by the mixed Hodge numbers \( h^p_q(\mathcal{H}^{n-1}(F_{f,0}; \mathbb{C})) \). On the other hand, to the best of our knowledge, if \( 0 \in V \) is a non-isolated singular point of \( V \), the geometric meaning of the weight filtrations on \( \mathcal{H}^j(F_{f,0}; \mathbb{C})_\lambda \) is not fully understood yet. For this problem, we obtain the following.

**Theorem 1.3** (see Theorem 4.6). In the situation of Theorem 1.1, for \( \lambda \notin R_f \) the filtration on \( \mathcal{H}^{n-1}(F_{f,0}; \mathbb{C})_\lambda \) induced by the weight filtration on \( \mathcal{H}^{n-1}(F_{f,0}; \mathbb{Q}) \) coincides with the monodromy filtration of \( \Phi_{n-1} \) centered at \( n-1 \).

We denote by \( J_{k,\lambda} \) the number of the Jordan blocks with size \( k \) for an eigenvalue \( \lambda \) in the Jordan normal form of the Milnor monodromy \( \Phi_{n-1} \). Combining the above theorem with our description of \( h^p_q(H^{n-1}(F_{f,0}; \mathbb{C})) \), for any eigenvalue \( \lambda \notin R_f \) we can describe it by using the Newton polyhedron \( \Gamma^+(f) \) as follows.

**Corollary 1.4** (see Corollary 5.1). In the situation of Theorems 1.1 and 1.3, for \( \lambda \notin R_f \) we have

\[
\sum_{0 \leq k \leq n-1} J_{n-k,\lambda} u^{k+2} = \sum_{F \in \Gamma^+(f): \text{admissible}} u^{\dim F + 2} l^*_\lambda(\Delta_F, \nu; 1) \cdot \tilde{l}_P(\mathcal{S}_\nu, \Delta_F; u^2),
\]

where in the sum \( \Sigma \) of the right hand side the face \( F \) ranges through the admissible compact ones of \( \Gamma^+(f) \) (see Definition 3.11). For the definition of the polynomials \( l^*_\lambda(\Delta_F, \nu; u) \) and \( \tilde{l}_P(\mathcal{S}_\nu, \Delta_F; u) \), see Section 3.2.

We also apply our results to obtain a formula for the Hodge spectrum of the Milnor fiber \( F_{f,0} \) (see Corollary 5.3).

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2. Milnor fibration

Let \( f(x) \in \mathbb{C}[x_1, \ldots, x_n] \) be a non-constant polynomial of \( n \) variables with coefficients in \( \mathbb{C} \) such that \( f(0) = 0 \). For a natural number \( m \geq 1 \) and a positive real number \( r > 0 \), we denote by \( B(0, r) \) the open ball in \( \mathbb{C}^m \) centered at the origin 0 with radius \( r \). Set \( B(0, r)^* := B(0, r) \setminus \{0\} \).

**Theorem 2.1** (Milnor [11]). Fix a sufficiently small \( \epsilon > 0 \). Then for a sufficiently small \( (\epsilon \gg) \eta > 0 \) the restriction of \( f: \mathbb{C}^n \rightarrow \mathbb{C} \)

\[
f: B(0, \epsilon) \cap f^{-1}(B(0, \eta)^*) \rightarrow B(0, \eta)^*
\]

is a locally trivial fibration. Moreover, if the origin \( 0 \in \mathbb{C}^n \) is an isolated singular point of \( V := f^{-1}(0) \subset \mathbb{C}^n \), its fiber is homotopy equivalent to a bouquet (wedge sum) of some \( (n-1) \)-dimensional spheres.

This fibration is called the Milnor fibration of \( f \) at 0 and its general fiber \( F_{f,0} \) is called the Milnor fiber of \( f \) at 0. By the above theorem, if \( 0 \) is an isolated singular point of \( V \), it follows that the reduced cohomology group \( \tilde{H}^j(F_{f,0}; \mathbb{C}) \) vanishes for \( j \neq n-1 \).

We get an action of the fundamental group \( \pi_1(B(0, \eta)^*) \) on \( H^j(F_{f,0}; \mathbb{C}) \), and thus can define an automorphism

\[
\Phi_j: H^j(F_{f,0}; \mathbb{C}) \xrightarrow{\sim} H^j(F_{f,0}; \mathbb{C})
\]

for each \( j \in \mathbb{Z} \). We call it the \( j \)-th Milnor monodromy of \( f \) at 0. It is well-known that \( \Phi_j \) is a quasi-unipotent linear operator for any \( j \in \mathbb{Z} \). For \( \lambda \in \mathbb{C} \) and \( j \in \mathbb{Z} \) denote by \( H^j(F_{f,0}; \mathbb{C})_\lambda \) the generalized eigenspace of \( \Phi_j \) for the eigenvalue \( \lambda \). Let \( \Phi_{j,\lambda} \) be the restriction of \( \Phi_j \) to \( H^j(F_{f,0}; \mathbb{C})_\lambda \). To study the eigenvalues of the Milnor monodromies \( \Phi_j \) and their multiplicities, we introduce the following rational function.

**Definition 2.2.** In the situation as above, we define the monodromy zeta function \( \zeta_{f,0}(t) \in \mathbb{C}(t) \) of \( f \) at 0 by

\[
\zeta_{f,0}(t) := \prod_{j \in \mathbb{Z}} \det(\text{Id} - t\Phi_j)^{(-1)^j} \in \mathbb{C}(t),
\]

where \( \text{Id} \) is the identity map of \( H^j(F_{f,0}; \mathbb{C}) \) to itself.

Since \( \Phi_j \) are automorphisms, the polynomials \( \det(\text{Id} - t\Phi_j) \) determine the characteristic polynomials of \( \Phi_j \). To introduce a formula for the monodromy zeta functions of Varchenko [19], we prepare some notions.

**Definition 2.3.** Let \( f(x) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n] \) be a Laurent polynomial with coefficients in \( \mathbb{C} \). Then the Newton polytope \( \text{NP}(f) \subset \mathbb{R}^n \) of \( f \) is the convex hull of the set \( \text{supp}(f) := \{ \alpha \in \mathbb{Z}^n \mid a_\alpha 
eq 0 \} \subset \mathbb{R}^n \) in \( \mathbb{R}^n \).

**Definition 2.4.** Let \( f(x) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} a_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n] \) be a polynomial with coefficients in \( \mathbb{C} \) such that \( f(0) = 0 \).

1. The Newton polyhedron \( \Gamma_+(f) \subset \mathbb{R}^n \) of \( f \) at the origin \( 0 \in \mathbb{C}^n \) is the convex hull of \( \bigcup_{\alpha \in \text{supp}(f)} \{ \alpha + \mathbb{R}_{\geq 0}^n \} \subset \mathbb{R}^n \) in \( \mathbb{R}^n \).
(2) The Newton boundary $\Gamma_f \subset \Gamma_+(f)$ of $f$ is the union of the compact faces of $\Gamma_+(f)$.

(3) We say that the polynomial $f$ is convenient if $\Gamma_+(f)$ intersects the positive part of each coordinate axis of $\mathbb{R}^n$.

For a Laurent polynomial $f(x) = \sum_{\alpha \in \mathbb{Z}^n} a_{\alpha} x^\alpha$ and a polytope $F$ in $\mathbb{R}^n$, we set $f_F(x) := \sum_{\alpha \in F} a_{\alpha} x^\alpha$.

**Definition 2.5.** Let $f(x) \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$ be a Laurent polynomial with coefficients in $\mathbb{C}$. We say that $f$ is non-degenerate if for any face $F$ of NP($f$) the hypersurface \( \{ x \in (\mathbb{C}^*)^n \mid f_F(x) = 0 \} \) in $(\mathbb{C}^*)^n$ is smooth and reduced.

**Definition 2.6.** Let $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-constant polynomial with coefficients in $\mathbb{C}$ such that $f(0) = 0$. Then we say that $f$ is non-degenerate at 0 if for any compact face $F$ of $\Gamma_+(f)$ the hypersurface $\{ x \in (\mathbb{C}^*)^n \mid f_F(x) = 0 \}$ in $(\mathbb{C}^*)^n$ is smooth and reduced.

Let $F$ be a lattice polytope in $\mathbb{R}^n$ (i.e. its vertices are in $\mathbb{Z}^n$). We denote by $L_F$ (resp. Aff $F$) the minimal linear (resp. affine) subspace of $\mathbb{R}^n$ which contains $F$ and set $M_F := L_F \cap \mathbb{Z}^n$. Note that $M_F$ is a lattice (i.e. a finite rank free $\mathbb{Z}$-module) and we have $M_F \otimes \mathbb{R} \cong L_F$. Assume that dim $F < n$ and $0 \notin$ Aff $F$. In this case there exists a unique primitive vector $v_F$ in the dual lattice $M_F^*$ of $M_F$ which takes a positive constant value on Aff $F$.

**Definition 2.7.** We define the lattice distance $d_F \in \mathbb{Z}_{\geq 1}$ of $F$ from the origin $0 \in L_F$ to be the value of $v_F$ on Aff $F$.

Let $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-constant polynomial with coefficients in $\mathbb{C}$ such that $f(0) = 0$. For a subset $I \subset \{1, \ldots, n\}$, set $\mathbb{R}^I := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = 0 \ (i \notin I)\} \cong \mathbb{R}^{|I|}$, and let $\Gamma_{I,1}, \ldots, \Gamma_{I,k_I}$ be the ($|I| - 1$)-dimensional compact faces of $\mathbb{R}^I \cap \Gamma_+(f)$. For $1 \leq i \leq k_I$, we define an integer $d_{I,i} \in \mathbb{Z}_{>0}$ to be the lattice distance of $\Gamma_{I,i}$ from the origin $0 \in \mathbb{R}^I$. Let $\operatorname{Vol}_2(\Gamma_{I,i}) \in \mathbb{Z}_{>0}$ be the ($|I| - 1$)-dimensional normalized volume of $\Gamma_{I,i}$. Then we have the following.

**Theorem 2.8 (Varchenko [19]).** Assume that $f$ is non-degenerate at 0. Then we have

\[
\zeta_{f,0}(t) = \prod_{\emptyset \neq I \subset \{1, \ldots, n\}} \prod_{i=1}^{k_I} (1 - t^{d_{I,i}})^{(-1)^{|I|}-1} \operatorname{Vol}_{2}(\Gamma_{I,i}).
\]

(1)

The monodromy zeta function $\zeta_{f,0}(t)$ being an alternating product of the polynomials $\det(\text{Id} - t\Phi_j)$, we can not compute the eigenvalues of each Milnor monodromy $\Phi_j$ and their multiplicities by Varchenko’s formula in general. Recall that if $0 \in V$ is an isolated singular point of $V$ we have $H^j(F_{f,0};\mathbb{C}) = 0$ for $j \neq 0, n - 1$ and $\det(\text{Id} - t\Phi_0) = 1 - t$ (here we assumed $n \geq 2$). In this case we thus obtain

\[
\zeta_{f,0}(t) = (1 - t) \cdot \det(\text{Id} - t\Phi_{n-1})^{(-1)^{n-1}}
\]
and can compute the eigenvalues of \( \Phi_{n-1} \) and their multiplicities by Varchenko’s formula. Let us recall a well-known condition for the origin \( 0 \in V \) to be an isolated singular point.

**Proposition 2.9.** If \( f \) is convenient and non-degenerate at 0, then the origin 0 is a smooth or an isolated singular point of \( V \).

Therefore, if \( f \) is convenient and non-degenerate at 0, we can describe the eigenvalues of the Milnor monodromy \( \Phi_{n-1} \) and their multiplicities by the Newton polyhedron \( \Gamma_+(f) \). In Section 4 we will show that even if \( f \) is not convenient, we can compute the multiplicities of some “good” eigenvalues of \( \Phi_{n-1} \) by Varchenko’s formula (see Corollary 4.2).

Recall that the cohomology group \( H^j(F_{f,0}; \mathbb{Q}) \) of the Milnor fiber \( F_{f,0} \) is endowed with a mixed Hodge structure \( (H^j(F_{f,0}; \mathbb{Q}), F^*, W_\bullet) \) defined by Steenbrink [17] in the case where \( 0 \in V \) is isolated singular point, by Navarro [12] and M.Saito [14] in the case where \( 0 \in V \) is a non-isolated singular point. To introduce a property of the weight filtration \( W_\bullet \), we recall the following notion.

**Definition 2.10.** Let \( r \in \mathbb{Z}_{\geq 1} \) and \( N \) be a nilpotent endomorphism of a finite dimensional \( \mathbb{C} \)-vector space \( H \) such that \( N^{r+1} = 0 \). Then, there exists an increasing filtration \( \{W_k\}_{k \in \mathbb{Z}} \) on \( H \), which is uniquely determined by the following conditions:

1. \( W_{-1} = \{0\} \) and \( W_{2r} = H \),
2. \( N(W_k) \subset W_{k-2} \) and
3. \( N^k: W_{r+k}/W_{r+k-1} \sim W_{r-k}/W_{r-k-1} \) for \( 0 \leq k \leq r \).

We call it the monodromy filtration of \( N \) centered at \( r \).

Note that the number of Jordan blocks (for the eigenvalue 0) with size \( k \in \mathbb{Z}_{\geq 1} \) in the Jordan normal form of \( N \) is equal to \( \dim W_{r+1-k}/W_{r-k} - \dim W_{r-1-k}/W_{r-2-k} \).

We decompose the \((n-1)\)-th Milnor monodromy \( \Phi_{n-1} \) into the semisimple part \( \Phi^s_{n-1} \) and the unipotent part \( \Phi^u_{n-1} \) as \( \Phi_{n-1} = \Phi^s_{n-1} \Phi^u_{n-1} \). If \( 0 \in V \) is an isolated singular point, the filtration on \( H^{n-1}(F_{f,0}; \mathbb{C})_\lambda \) induced by the weight filtration is the monodromy filtration of the logarithm \( \log \Phi^u_{n-1} \) of \( \Phi^u_{n-1} \) centered at \( n - 1 \) (resp. \( n \)) for \( \lambda \neq 1 \) (resp. \( \lambda = 1 \)) (see [17]). Therefore, we can recover the Jordan normal form of \( \Phi_{n-1} \) for the eigenvalue \( \lambda \) from the dimensions of the graded pieces \( \text{Gr}^W_k H^{n-1}(F_{f,0}; \mathbb{C})_\lambda \). On the other hand, if \( 0 \in V \) is a non-isolated singular point, we can not expect such a relationship between the weight filtration \( W_\bullet \) and the Milnor monodromy. In Section 4 we will show that even if \( f \) is not convenient (so \( 0 \in V \) may be a non-isolated singular point) for a “good” eigenvalue \( \lambda \) the filtration on \( H^{n-1}(F_{f,0}; \mathbb{C})_\lambda \) induced by \( W_\bullet \) coincides with the monodromy filtration (see Theorem 4.4).

Finally, we describe the cohomology groups of the Milnor fibers and their mixed Hodge structures in terms of the nearby cycle functors. For details, see [3], [5], [6] and [14]. For a field \( K \), we denote by \( K_{\mathbb{C}^n} \) the constant sheaf on \( \mathbb{C}^n \) with stalk \( K \) and by \( \psi_f(K_{\mathbb{C}^n}) \) the nearby cycle sheaf of \( f \). Recall that \( \psi_f(K_{\mathbb{C}^n}) \) is an object of the derived...
category of constructible sheaves $\mathcal{D}_c^b(V)$ on $V = f^{-1}(0)$. Then, for any $j \in \mathbb{Z}$ there exists an isomorphism
\[
H^j(\psi_f(\mathbb{K}_{\mathbb{C}^n}),\mathbb{Q}) \cong H^j(F_{f,0},\mathbb{C})
\]
(see e.g. Proposition 4.2.2 of [3]). Moreover, there exists an automorphism of the complex $\psi_f(\mathbb{K}_{\mathbb{C}^n})$, called the monodromy automorphism. The automorphism is $\psi_{\lambda,0}(\mathbb{K}_{\mathbb{C}^n})$ induced by it coincides with the Milnor monodromy $\Phi_{\lambda}$. For a complex number $\lambda \in \mathbb{C}$, we denote by $\psi_{f,\lambda}(\mathbb{C}_{\mathbb{C}^n})$ the $\lambda$-part of $\psi_f(\mathbb{C}_{\mathbb{C}^n})$ with respect to the monodromy automorphism. Then the $j$-th cohomology $H^j(\psi_f(\mathbb{K}_{\mathbb{C}^n}),\mathbb{Q})$ is isomorphic to the generalized eigenspace $H^j(F_{f,0},\mathbb{C})_{\lambda}$ for the eigenvalue $\lambda$. Recall that the complexes
\[
\mathbb{K}_{\mathbb{C}^n}[n] \in \mathcal{D}_c^b(\mathbb{C}^n)
\]
are objects of the Abelian categories of perverse sheaves $\text{Perv}(\mathbb{C}^n) \subset \mathcal{D}_c^b(\mathbb{C}^n)$ and $\text{Perv}(V) \subset \mathcal{D}_c^b(V)$ respectively. Moreover, $\mathbb{Q}_{\mathbb{C}^n}[n]$ and $\mathbb{Q}_{\mathbb{C}^n}[n]$ are the underlying perverse sheaves of the mixed Hodge modules $\mathbb{Q}_{\mathbb{C}^n}[n]$ and $\mathbb{Q}_{\mathbb{C}^n}[n]$ respectively. Let $j_0^{-1}\psi_f^H(\mathbb{Q}_{\mathbb{C}^n})$ be the pull-back of the mixed Hodge module $\psi_f^H(\mathbb{Q}_{\mathbb{C}^n}) \in D^b\text{MHM}(V)$ by the inclusion $j_0: \{0\} \hookrightarrow V$. This is an object of the derived category of mixed Hodge modules $D^b\text{MHM}(\{0\})$ (which is equivalent to the derived category of polarizable mixed Hodge structures), and its underlying complex is $p_{\psi_f}(\mathbb{Q}_{\mathbb{C}^n}[n]) = j_0^{-1}(p_{\psi_f}(\mathbb{Q}_{\mathbb{C}^n}[n]))$. This implies that the cohomology groups of $p_{\psi_f}(\mathbb{Q}_{\mathbb{C}^n}[n])$ have mixed Hodge structures. Thus we can endow $H^j(F_{f,0},\mathbb{Q})$ with a mixed Hodge structure for each $j \in \mathbb{Z}$. Note that in the case where $0 \in V$ is an isolated singular point, the mixed Hodge structure of $H^{n-1}(F_{f,0},\mathbb{Q})$ was defined by Steenbrink [17] more elementarily.

3. Motivic Milnor fibers

3.1. Motivic Milnor fibers. Let $\mu_m = \{x \in \mathbb{C} \mid x^m = 1\}$ be the cyclic group of order $m \in \mathbb{Z}_{\geq 1}$. By $\psi_{\mu_m} = \lim_{m \rightarrow \infty} \mu_m$ be the projective limit with respect to the morphisms $\mu_{md} \hookrightarrow \mu_m (x \mapsto x^d)$. Let $K^0_d(\text{Var}_\mathbb{C})$ be the Abelian group generated by the symbols $[X \circ \hat{\mu}]$ for algebraic varieties $X$ over $\mathbb{C}$ with a good $\hat{\mu}$-action (i.e. a $\hat{\mu}$-action induced by a good $\mu_m$-action for some $m \in \mathbb{Z}_{\geq 1}$), and divided by some relations (see [2]). For $[X \circ \hat{\mu}]$ and $[X' \circ \hat{\mu}]$ in $K^0_d(\text{Var}_\mathbb{C})$ we can endow the product $X \times X'$ with a good $\hat{\mu}$-action and define their multiplication $[X \circ \hat{\mu}] \cdot [X' \circ \hat{\mu}]$ by $[X \times X' \circ \hat{\mu}]$. In this way, we can endow $K^0_d(\text{Var}_\mathbb{C})$ with a ring structure and call it the monodromic Grothendieck ring. Set $\mathbb{L} := \{\mathbb{C} \circ \hat{\mu} \in K^0_d(\text{Var}_\mathbb{C})\}$, where $\mathbb{C} \circ \hat{\mu}$ is the affine line with the trivial $\hat{\mu}$-action. We denote by $\mathcal{M}_c^d$ the localization of the ring $K^0_d(\text{Var}_\mathbb{C})$ obtained by inverting $\mathbb{L}$. For a non-constant polynomial $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ such that $f(0) = 0$, Denef-Loeser [2] defined the motivic Milnor fiber $S_{f,0}$ of $f$ as an object in $\mathcal{M}_c^d$ by using the theory of arc spaces. It is an “incarnation of the Milnor fiber of $f$ at 0” in the ring $\mathcal{M}_c^d$ as we shall see in Theorem 3.1 below. Let $X$ be an algebraic variety with a good $\mu_m$-action for some $m \in \mathbb{Z}_{\geq 1}$. The generator of $\mu_m$ defines an automorphism $l: X \xrightarrow{\sim} X$ of $X$ such that $l^m$ is the identity map. Each cohomology group $H^j_c(X; \mathbb{Q})$ with compact
support is endowed with Deligne’s mixed Hodge structure \((H^i_c(X; \mathbb{Q}), F^*, W_*)\) with an automorphism
\[
(I^*)^{-1} : H^i_c(X; \mathbb{Q}) \sim H^i_c(X; \mathbb{Q}) .
\] (2)

For \(\lambda \in \mathbb{C}\), we denote by
\[
H^i_c(X; \mathbb{C})_\lambda \subset H^i_c(X; \mathbb{C})
\]
the generalized eigenspace of the automorphism for the eigenvalue \(\lambda\). Moreover, for \(p, q \in \mathbb{Z}\) and \(\lambda \in \mathbb{C}\) we define \(h^{p,q}_\lambda(H^i_c(X; \mathbb{C})) \in \mathbb{Z}_{\geq 0}\) to be the dimension of \(\text{Gr}^p_F \text{Gr}^W_{p+q} H^i_c(X; \mathbb{C})\).

Then for \(\lambda \in \mathbb{C}\) we can define a ring homomorphism \(E_\lambda(\cdot; u, v)\) of \(\mathcal{M}_C\) to the polynomial ring \(\mathbb{Z}[u,v]\) which sends \([X \circ \mu] \in \mathcal{M}_C\) to the polynomial
\[
E_\lambda([X \circ \mu]; u, v) := \sum_{p,q \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} (-1)^j h^{p,q}_\lambda(H^j_c(X; \mathbb{C})) u^p v^q \in \mathbb{Z}[u,v].
\]

For an element \(\sum_i [X_i \circ \mu_i] \in \mathcal{M}_C\) and \(\lambda \in \mathbb{C}\) we call the polynomial \(\sum_i E_\lambda([X_i \circ \mu_i]; u, v)\) the Hodge realization for \(\lambda\) of \(\sum_i [X_i \circ \mu_i]\). For \(\lambda \in \mathbb{C}\) we denote by \(E_\lambda(F_{f,0}; u, v)\) the equivariant Hodge-Deligne polynomial for the eigenvalue \(\lambda\) of the mixed Hodge structures of the cohomology groups of the Milnor fiber \(F_{f,0}\) with the automorphisms \(\Phi_j\), i.e.
\[
E_\lambda(F_{f,0}; u, v) := \sum_{p,q \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} (-1)^j h^{p,q}_\lambda(H^j(F_{f,0}; \mathbb{C})) u^p v^q \in \mathbb{Z}[u,v],
\]
where \(h^{p,q}_\lambda(H^j(F_{f,0}; \mathbb{C}))\) is the dimension of \(\text{Gr}^p_F \text{Gr}^W_{p+q} H^j(F_{f,0}; \mathbb{C})\). Then we have the following theorem of Denef-Loeser [2].

**Theorem 3.1** (Denef-Loeser [2]). For any \(\lambda \in \mathbb{C}\) we have
\[
E_\lambda(S_{f,0}; u, v) = E_\lambda(F_{f,0}; u, v).
\]

Originally, \(S_{f,0}\) is defined abstractly in [2] by using the theory of arc spaces. However, by using a log resolution of the pair \((\mathbb{C}^n, f^{-1}(0))\), we can describe \(S_{f,0}\) explicitly as follows. Let \(Y\) be a smooth algebraic variety over \(\mathbb{C}\) and \(\pi : Y \rightarrow \mathbb{C}^n\) be a proper morphism such that \(\pi^{-1}(V)\) is a normal crossing divisor of \(Y\) and \(\pi\) induces an isomorphism \(Y \setminus \pi^{-1}(V) \sim \mathbb{C}^n \setminus V\). Let
\[
\pi^{-1}(V) = E_1 \cup \cdots \cup E_m
\]
be the irreducible decomposition of \(\pi^{-1}(V)\). We denote by \(m_i\) the order of zeros along \(E_i\) of \(f \circ \pi\). For a subset \(I \subset \{1, \ldots, m\}\), we define
\[
E_I := \cap_{i \in I} E_i, \quad E_I^0 := E_I \setminus \cup_{i \notin I} E_i
\]
and \(m_I := \gcd_{i \in I}(m_i)\). Moreover, we define a covering \(\overline{E}_I^0\) of \(E_I^0\) in the following way. For a point in \(E_i^0\), we take a Zariski open neighborhood \(U\) of it in \(Y\) on which for any \(i \in I\) there exists a regular function \(h_i\) such that \(E_i \cap U = \{h_i = 0\}\). We have \(f = f_{1}f_{2}^{m_I}\) on \(U\), where we set \(f_{1} = f \prod_{i \in I} h_i^{-m_i}\) and \(f_{2} = \prod_{i \in I} h_i^{m_i/m_I}\). Note that \(f_{1}\) is a unit on \(U\). Then, we have a covering of \(E_I^0 \cap U\) defined by
\[
\{(z, y) \in \mathbb{C} \times (E_I^0 \cap U) \mid z^{m_I} = f_{1}^{-1}(y)\}.
\] (3)
Consider an open covering of $E_i^\circ$ by such affine open sets $E_i^\circ \cap U$. Then by gluing together the varieties (3) in an obvious way, we obtain an $m_I$-fold covering $\widehat{E}_i^\circ$ of $E_i^\circ$. Moreover by the multiplication of $\exp(2\pi\sqrt{-1}/m_I) \in \mathbb{C}$ to the $z$-coordinate of (3), we can endow $\widehat{E}_i^\circ$ with a $\mu_{m_I}$-action (and also a $\mu$-action induced by it). We denote by $\widehat{E}_{i,0}^\circ$ the base change of $\widehat{E}_i^\circ \longrightarrow E_i^\circ$ by $\pi^{-1}(0) \cap E_i^\circ \longrightarrow E_i^\circ$. Then we obtain the following expression of $S_{f,0}$.

**Proposition 3.2** (Denef-Loeser [2]). In the situation as above, we have

$$S_{f,0} = \sum_{\emptyset \neq I \subseteq \{1, \ldots, m\}} (1 - \mathbb{L})^{1/1-1}[\widehat{E}_{I,0}^\circ]$$

(4)

in $\mathcal{M}_E^\circ$.

We denote by $\Sigma_0$ the dual fan of $\Gamma_+(f)$ in $\mathbb{R}^n$. Let $\Sigma$ be a smooth subdivision of $\Sigma_0$, and denote by $X_\Sigma$ the toric variety associated with it. We denote by $\Sigma_1$ the fan which consists of all the faces of $\mathbb{R}_{\geq 0}^n$. Note that the toric variety associated with it is $\mathbb{C}^n$. Then, the morphism of fans $\Sigma \longrightarrow \Sigma_1$ induces a morphism of toric varieties

$$\pi: X_\Sigma \longrightarrow \mathbb{C}^n.$$

If $f$ is non-degenerate at 0, we can take $\pi: X_\Sigma \longrightarrow \mathbb{C}^n$ as a log resolution of $(\mathbb{C}^n, V)$ (In fact, $\pi^{-1}(V)$ is normal crossing only in a neighborhood of $\pi^{-1}(0)$ in $X_\Sigma$. Nevertheless, we can apply Proposition 3.2). Moreover by calculating the right hand side of (4), we can describe the motivic Milnor fiber $S_{f,0}$ explicitly in terms of the Newton boundary $\Gamma_f$ as follows (see Matsui-Takeuchi [10, Section 4]). Assume that $f(x) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} a_\alpha x^\alpha$ ($a_\alpha \in \mathbb{C}$) is non-degenerate at 0. For a compact face $F$ of $\Gamma_+(f)$, we define a lattice polytope $\Delta_F \subset \mathbb{R}^n$ by $\Delta_F := \mathrm{Conv}(F \cup \{0\})$. Moreover, let $f_{\Delta_F}^\circ = \sum_{\alpha \in \Delta_F \cap \mathbb{Z}^n} b_\alpha x^\alpha$ be the polynomial defined by

$$b_\alpha = \begin{cases} 
 a_\alpha & (\alpha \in F) \\
 -1 & (\alpha = 0) \\
 0 & \text{(otherwise)}. 
\end{cases}$$

Then we define a hypersurface $Z_F^\circ$ in $\mathrm{Spec} \mathbb{C}[\mathbb{Z}^n \cap \mathrm{Aff} F] \simeq (\mathbb{C}^*)^{\dim F}$ by

$$Z_F^\circ = \{ x \in \mathrm{Spec} \mathbb{C}[\mathbb{Z}^n \cap \mathrm{Aff} F] \mid f_F(x) = 0 \} \subset (\mathbb{C}^*)^{\dim F},$$

and a hypersurface $Z_{\Delta_F}^\circ$ in $\mathrm{Spec} \mathbb{C}[\mathbb{Z}^n \cap \mathrm{Aff} \Delta_F] \simeq (\mathbb{C}^*)^{\dim F + 1}$ by

$$Z_{\Delta_F}^\circ = \{ x \in \mathrm{Spec} \mathbb{C}[\mathbb{Z}^n \cap \mathrm{Aff} \Delta_F] \mid f_{\Delta_F}(x) = 0 \} \subset (\mathbb{C}^*)^{\dim F + 1}.$$

We endow $Z_F$ with the trivial $\mu$-action, and $Z_{\Delta_F}$ with a good $\mu$-action in the following way. Let $\nu_F$ be the linear function on $\mathrm{Aff} \Delta_F \simeq \mathbb{R}^{\dim F + 1}$ which takes the value 1 on $F$ and $e_F \in (\mathbb{C}^*)^{\dim F + 1} \simeq \mathrm{Spec} \mathbb{C}[\mathbb{Z}^n \cap \mathrm{Aff} \Delta_F] \simeq \mathrm{Hom}_{\text{group}}(\mathbb{Z}^n \cap \mathrm{Aff}(\Delta_F), \mathbb{C}^*)$ be the element which corresponds to the group homomorphism

$$\exp \left(2\sqrt{-1}\nu_F(\cdot)\right) \in \mathrm{Hom}_{\text{group}}(\mathbb{Z}^n \cap \mathrm{Aff}(\Delta_F), \mathbb{C}^*).$$
Then $Z^\Delta_p$ is invariant by the multiplication by $e_F$ and hence we can endow $Z^\Delta_p$ with a $\mu_d$-action. We thus obtain the elements $[Z^\rho_p \circ \hat{\mu}]$ and $[Z^\Delta_p \circ \hat{\mu}]$ in $\mathcal{M}^\mu_C$ for any compact face $F \prec \Gamma_+(f)$. For a compact face $F$ of $\Gamma_+(f)$ we define also a subset $I_F \subset \{1, \ldots, n\}$ to be the minimal one such that $F \subset R^{|I_F|}$, and set $s_F = |I_F|$. Then we have the following theorem.

**Theorem 3.3** (see Matsui-Takeuchi [10]). Assume that $f$ is non-degenerate at 0. Then we have

$$S_{f,0} = \sum_{F \prec \Gamma_+(f): \text{compact}} (1 - L)^{s_F - \dim F - 1} \left\{ (1 - L) \cdot [Z^\rho_p \circ \hat{\mu}] + [Z^\Delta_p \circ \hat{\mu}] \right\} \in \mathcal{M}^\mu_C,$$

where in the sum $\Sigma$ the face $F(\neq \emptyset)$ ranges through the compact ones of $\Gamma_+(f)$.

This theorem was proved only in the case where $f$ is convenient in [10]. However, we can show it even if $f$ is not convenient similarly.

### 3.2. Katz and Stapledon's polynomials

In this subsection, we introduce some polynomials defined by Katz-Stapledon [7], [8] and Stapledon [16]. Let $P \subset \mathbb{R}^n$ be a polytope in $\mathbb{R}^n$. If a subset $F \subset P$ is a face of $P$ (possibly $F = P$ or $F = \emptyset$), we write $F \prec P$. For a pair of faces $F \prec F'$ of $P$, we define $[F,F']$ to be the face poset $\{F'' \prec P \mid F \prec F'' \prec F'\}$, and $[F,F']^*$ to be its opposite poset.

**Definition 3.4.** Let $F \prec F'$ be a pair of faces of $P$. We define one-variable polynomials $g([F,F];t)$ and $g([F,F]^*;t)$ with coefficients in $\mathbb{Z}$ of order less than $(\dim F' - \dim F)/2$ by the following inductive way. If $F = F'$, we set $g([F,F];t) = 1$ and $g([F,F]^*;t) = 1$. If $F \not\prec F'$, we define them by

$$t^{\dim F' - \dim F} g([F,F];t^{-1}) = \sum_{F'' \in [F,F']} (t - 1)^{\dim F' - \dim F''} g([F,F''];t)$$

and

$$t^{\dim F' - \dim F} g([F,F]^*;t^{-1}) = \sum_{F'' \in [F,F]^*} (t - 1)^{\dim F'' - \dim F'} g([F'',F'];t).$$

Let $P$ be a lattice polytope in $\mathbb{R}^n$ (i.e. all the vertices of $P$ are in $\mathbb{Z}^n$) and $\nu$ a convex $\mathbb{R}$-valued function on $P$ which is piecewise $\mathbb{Q}$-affine with respect to a lattice polyhedral subdivision of $P$. Let $S_\nu$ be the coarsest such polyhedral subdivision. A (possibly empty) polytope $F \subset P$ in $S_\nu$ is called a cell. For a cell $F \in S_\nu$, let $\text{lk}_{S_\nu}(F)$ be the set of all cells in $S_\nu$ containing $F$, and we call it the link of $F$. Moreover, we denote by $\sigma(F)$ the smallest face of $P$ containing $F$.

**Definition 3.5.** For a (possibly empty) cell $F \in S_\nu$, the $h$-polynomial $h(\text{lk}_{S_\nu}(F);t)$ of the link $\text{lk}_{S_\nu}(F)$ is defined by

$$t^{\dim P - \dim F} h(\text{lk}_{S_\nu}(F);t^{-1}) = \sum_{F' \in \text{lk}_{S_\nu}(F)} g([F,F'];t)(t - 1)^{\dim P - \dim F'}.$$

The local $h$-polynomial $l_P(S_\nu, F; t)$ of $F$ in $S_\nu$ is defined by

$$l_P(S_\nu, F; t) = \sum_{\sigma(F) < Q < P} (-1)^{\dim P - \dim Q} h(\text{lk}_{S_\nu|Q}(F);t) g([Q,P]^*;t).$$
Note that if $S_\nu$ is the trivial subdivision of $P$ (i.e. $S_\nu$ consists of all the faces of $P$), we have $h(\text{lk}_{S_\nu}(Q); t) = g([Q, P]; t)$ and $l_P(S_\nu, Q; t) = 0$ for a face $Q$ of $P$. For $\lambda \in \mathbb{C}$, $m \in \mathbb{Z}_{\geq 0}$ and $v \in mP \cap \mathbb{Z}^n$, we set
\[
w_\lambda(v) = \begin{cases} 
1 & (\exp(2\pi \sqrt{-1} \cdot m\nu(\frac{Z}{m})) = \lambda) \\
0 & \text{otherwise}.
\end{cases}
\]
For $m \in \mathbb{Z}$, we define an integer $f_\lambda(P, \nu; m) \in \mathbb{Z}$ by
\[f_\lambda(P, \nu; m) := \sum_{v \in mP \cap \mathbb{Z}^n} w_\lambda(v).
\]
Then $f_\lambda(P, \nu; m)$ is a polynomial in $m$ whose degree is less than or equal to $\dim P$.

**Definition 3.6.**
(i) If $P \neq \emptyset$, we define the $\lambda$-weighted $h^*$-polynomial $h^*_\lambda(P, \nu; u) \in \mathbb{Z}[u]$ by
\[h^*_\lambda(P, \nu; u) = (1 - u)^{\dim P + 1} \sum_{m \geq 0} f_\lambda(P, \nu; m)u^m.
\]
We set $h^*_\lambda(\emptyset, \nu; u) = 1$ and $h^*_\lambda(\emptyset, \nu; u) = 0$ for $\lambda \neq 1$.
(ii) If $P \neq \emptyset$, we define the $\lambda$-local weighted $h^*$-polynomial $l^*_\lambda(P, \nu; u) \in \mathbb{Z}[u]$ by
\[l^*_\lambda(P, \nu; u) = \sum_{Q < P} (-1)^{\dim P - \dim Q} h^*_\lambda(Q, \nu|Q; u) \cdot g([Q, P]^*; u).
\]
We set $l^*_\lambda(\emptyset, \nu; u) = 1$ and $l^*_\lambda(\emptyset, \nu; u) = 0$ for $\lambda \neq 1$.
(iii) We define the $\lambda$-weighted limit mixed $h^*$-polynomial $h^*_\lambda(P, \nu; u, v) \in \mathbb{Z}[u, v]$ by
\[h^*_\lambda(P, \nu; u, v) := \sum_{F \in S_\nu} v^{\dim F + 1} l^*_\lambda(F, \nu|F; uv^{-1}) \cdot h(\text{lk}_{S_\nu}(F); uv).
\]
(iv) We define the $\lambda$-local weighted limit mixed $h^*$-polynomial $l^*_\lambda(P, \nu; u, v) \in \mathbb{Z}[u, v]$ by
\[l^*_\lambda(P, \nu; u, v) := \sum_{F \in S_\nu} v^{\dim F + 1} l^*_\lambda(P, \nu|F; uv^{-1}) \cdot l_P(S_\nu, F; uv).
\]
(v) We define the $\lambda$-weighted refined limit mixed $h^*$-polynomial $h^*_\lambda(P, \nu; u, v, w) \in \mathbb{Z}[u, v, w]$ by
\[h^*_\lambda(P, \nu; u, v, w) := \sum_{Q < P} w^{\dim Q + 1} l^*_\lambda(Q, \nu|Q; u, v) \cdot g([Q, P]; uvw^2).
\]
We introduce some properties of these polynomials.

**Proposition 3.7** (see Theorem 4.21 of Stapledon [16]). In the situation as above, the followings hold.
(i) We have
\[l^*_\lambda(P, \nu; u, 1) = l^*_\lambda(P, \nu; u),
\]
\[h^*_\lambda(P, \nu; u, 1) = h^*_\lambda(P, \nu; u) \quad \text{and} \quad h^*_\lambda(P, \nu; u, 1) = h^*_\lambda(P, \nu; u, v).
\]
(ii) We have
\[ l^*_\lambda(P, \nu; u, v) = l^*_\lambda(P, \nu; v; u, v) = (uv)^{\dim P + 1} l^*_\lambda(P, \nu; v^{-1}, u^{-1}) \quad \text{and} \]
\[ h^*_\lambda(P, \nu; u, v, w) = h^*_\lambda(P, \nu; v, u, w) = h^*_\lambda(P, \nu; v^{-1}, u^{-1}, uvw). \]

(iii) We have
\[ h^*_\lambda(P, \nu; u, v) = \sum_{F \in \mathcal{S}_P, \sigma(F) = P} h^*_\lambda(F, \nu|_F; u, v)(uv - 1)^{\dim P - \dim F}. \]

For the practical computation of the \( h^* \)-polynomials, the following example is useful.

**Example 3.8** (Example 4.23 of Stapledon [16]). Assume that \( P \) is an \( n \)-dimensional simplex in \( \mathbb{R}^n \) and \( \nu \) is a \( \mathbb{Q} \)-affine function on it. For a face \((\emptyset \neq) Q \prec P \), we denote by \( \mathcal{C}_Q \subset \mathbb{R}^n \times \mathbb{R} \) the rational polyhedral cone in \( \mathbb{R}^n \times \mathbb{R} \) generated by the vectors \((\{v, 1\})_{v \in Q}\). For the empty face \( Q = \emptyset \), we set \( \mathcal{C}_Q = \{0\} \subset \mathbb{R}^n \times \mathbb{R} \). Let \( v_0, \ldots, v_n \in P \times \{1\} \) be the ray generators of edges of \( C_P \). We define a finite subset \( \text{Box} \) in \( \mathbb{Z}^n \times \mathbb{Z} \) by
\[ \text{Box} := \{v \in C_P \cap (\mathbb{Z}^n \times \mathbb{Z}) \mid v = \sum_{i=0}^n a_i v_i, 0 \leq a_i < 1\}. \]

We denote by \( \text{pr} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) the projection. Then we have
\[ h^*(P, \nu; u, v, w) = \sum_{Q \prec P, v \in \text{Box} \cap \text{rel int}(C_Q)} \sum_{\sigma(F) = P} \omega_\lambda(v) u^{\text{pr}(v)} v^{\dim Q - \dim P + 1} \]
\[ l^*(P, \nu; u, v) = \sum_{v \in \text{Box} \cap \text{rel int}(C_P)} \omega_\lambda(v) u^{\text{pr}(v)} v^{\dim P - \dim P + 1}. \]

### 3.3. The Hodge realizations of the motivic Milnor fibers.

Next, we discuss the Hodge realization of \( S_{f,0} \). We recall the proposition stated below for the Hodge realizations of non-degenerate hypersurfaces in the algebraic torus \((\mathbb{C}^*)^n\). Let \( g(x) = \sum_{\beta \in \mathbb{Z}^n} c_{\beta} x^\beta \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm] \) be a non-degenerate Laurent polynomial (see Definition 2.5) with \( \dim \text{NP}(g) = n \), and \( \nu \) a convex piecewise \( \mathbb{Q} \)-affine function on \( P := \text{NP}(g) \) such that \( \nu(\beta) \in \mathbb{Z} \) if \( c_{\beta} \neq 0 \). The multiplication by the element in \((\mathbb{C}^*)^n\) which correspond to the group homomorphism
\[ \exp(2\pi \sqrt{-1} \nu(\cdot)) \in \text{Hom}_{\text{group}}(\mathbb{Z}^n, \mathbb{C}^*) \]
defines a \( \hat{\mu} \)-action on the hypersurface
\[ Z^0 := \{x \in (\mathbb{C}^*)^n \mid g(x) = 0\} \]

We thus obtain an element \([Z^0 \circ \hat{\mu}]\) in \( \mathcal{M}_{\hat{\mu}}^\ell \). Set
\[ \epsilon(\lambda) = \begin{cases} 1 & (\lambda = 1) \\ 0 & (\lambda \neq 1) \end{cases} \]

**Proposition 3.9** (Stapledon [16], Matsui-Takeuchi [10]). In the situation as above, we have
\[ uv E_\lambda([Z^0 \circ \hat{\mu}]; u, v) = \epsilon(\lambda)(uv - 1)^n + (-1)^{n+1} h^*_\lambda(P, \nu; u, v), \]
for \( \lambda \in \mathbb{C} \).
Let us remark that in [16] and [10] for an algebraic variety \( X \) with a \( \mu_m \)-action for some \( m \in \mathbb{Z}_{\geq 1} \) the authors endowed \( H^j_f(X; \mathbb{Q}) \) with the inverse automorphism of the one which is defined by (2) in Section 3.1. Therefore, Proposition 3.9 is slightly different from the original one in [16] and [10].

Let \( f(x) \in \mathbb{C}[x_1, \ldots, x_n] \) be a non-constant polynomial such that \( f(0) = 0 \) and assume that \( f \) is non-degenerate at 0. We denote by \( P \) the convex hull \( \text{Conv}(\Gamma_f \cup \{0\}) \) of \( \Gamma_f \cup \{0\} \) in \( \mathbb{R}^n \) and define a piecewise \( \mathbb{Q} \)-affine function \( \nu \) on \( P \) which takes the value 0 (resp. 1) at the origin 0 \( \in \mathbb{R}^n \) (resp. on \( \text{Conv}(\Gamma_f) \)) such that for any compact face \( F \) of \( \Gamma_+(f) \) the restriction \( \nu_F \) of \( \nu \) to \( \Delta_F \) is linear. Moreover, for a compact face \( F \) of \( \Gamma_+(f) \) let \( 0_F \) be the zero function on \( F \). Then by Theorem 3.3 and Proposition 3.9 we can calculate the Hodge realization of the motivic Milnor fiber \( S_{f,0} \), and we can describe \( E_\lambda(F_{f,0}; u, v) \) as follows.

**Corollary 3.10.** In the situation as above, for \( \lambda \in \mathbb{C} \) we have

\[
uvE_\lambda(F_{f,0}; u, v) = \sum_{F \in \Gamma_+(f): \text{compact}} (-1)^{\dim F} \left\{ (1 - uv)^{s_F - \dim F} h_\lambda^*(F_{f,0}; u, v) + (1 - uv)^{-\dim F - 1} h_\lambda^*(\Delta_F, u, v) \right\},
\]

where in the sum \( \Sigma \) the face \( F(\neq \emptyset) \) ranges through the compact ones of \( \Gamma_+(f) \).

Note that if \( \lambda \neq 1 \), the polynomial \( h_\lambda^*(F_{f,0}; u, v) \) is zero. The coefficient of \( u^p v^q \) in \( E_\lambda(F_{f,0}; u, v) \) being an alternating sum of \( h_\lambda^{p,q}(H^j(F_{f,0}; \mathbb{C})) \), for each \( j \in \mathbb{Z} \) we can not always compute \( h_\lambda^{p,q}(H^j(F_{f,0}; \mathbb{C})) \) by the formula in Corollary 3.10. Recall that if \( 0 \in V \) is an isolated singular point, we have \( H^j(F_{f,0}; \mathbb{C}) = 0 \) unless \( j = 0 \) or \( n - 1 \), and \( h_\lambda^{0,q}(H^0(F_{f,0}; \mathbb{C})) = 0 \) unless \( \lambda = 1 \) and \( (p, q) = (0, 0) \). Therefore, in this case, we can compute each \( h_\lambda^{p,q}(H^{n-1}(F_{f,0}; \mathbb{C})) \) by our formula. Even if \( f \) is not convenient (in this case, \( 0 \in V \) may be a non-isolated singular point), we will show later that for “good” eigenvalues we have \( H^j(F_{f,0}; \mathbb{C})_\lambda = 0 \) (\( j \neq n - 1 \)) and we can compute \( h_\lambda^{p,q}(H^{n-1}(F_{f,0}; \mathbb{C})) \) (see Theorem 4.1).

Let us explain a symmetry of \( E_\lambda(F_{f,0}; u, v) \).

**Definition 3.11.** We say that a face \( F(\neq \emptyset) \) of \( \Gamma_+(f) \) is extremal if \( F \) is compact and a face of a non-compact face \( G \) of \( \Gamma_+(f) \) not contained in the boundary \( \partial \mathbb{R}^n_{\geq 0} \) of \( \mathbb{R}^n_{\geq 0} \) such that \( F \prec G \). If a compact face \( F \prec \Gamma_+(f) \) is not extremal, we say that \( F \) is admissible.

**Definition 3.12.** For a non-constant polynomial \( f(x) \in \mathbb{C}[x_1, \ldots, x_n] \) such that \( f(0) = 0 \), we define a finite subset \( R_f \) of \( \mathbb{C} \) by

\[
R_f := \bigcup_{F \prec \Gamma_+(f): \text{extremal}} \{ \lambda \in \mathbb{C} \mid \lambda^{d_F} = 1 \},
\]

where \( F \) ranges through the extremal compact faces of \( \Gamma_+(f) \) and \( d_F \in \mathbb{Z}_{\geq 0} \) is the lattice distance of \( F \) from the origin 0 \( \in \mathbb{R}^n \).

**Example 3.13.** Consider the case where \( n = 2 \). Let \( f \) be the polynomial \( f(x_1, x_2) = x_1^2 + x_2^2 + x_1^2 x_2^2 \). Then the Newton polyhedron \( \Gamma_+(f) \subset \mathbb{R}^2 \) of \( f \) is as in Fig. 1. The union of the bold lines in \( \Gamma_+(f) \) in it is the Newton boundary \( \Gamma_f \) of \( f \). In this case, the
only extremal face of $\Gamma_+(f)$ is the 0-dimensional face $\{(2, 4)\}$. Its lattice distance from the origin $(0, 0)$ is 2. Hence, we have
\[ R_f = \{1, -1\}. \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The Newton polyhedra of $f = x_1^7 + x_1^3 x_2 + x_1^2 x_2^4$}
\end{figure}

If $f$ is convenient and $\lambda \neq 1$, then the weight filtration on $H^{n-1}(F_{f,0})$ is the monodromy filtration and we have
\[ h^p,q_{\lambda}(H^{n-1}(F_{f,0}; \mathbb{C})) = h^{n-1-q, n-1-p}_{\lambda}(H^{n-1}(F_{f,0}; \mathbb{C})), \]
for any $p, q \in \mathbb{Z}$. Hence in this case, we have
\[ E_{\lambda}(F_{f,0}; u, v) = (uv)^{n-1} E_{\lambda}(F_{f,0}; v^{-1}, u^{-1}). \]

However, in the case where $f$ is not convenient, this symmetry does not hold in general. Nevertheless, by Corollary 3.10 and some properties in Proposition 3.7 of the $h^*$-polynomials we can easily see the following results.

**Proposition 3.14.** Assume that $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ is non-degenerate at 0 and is not convenient and $\dim P = n$. Then for any $\lambda \notin R_f$ we have
\[ uv E_{\lambda}(F_{f,0}; u, v) = (-1)^{n-1} l^*_{\lambda}(P, v; u, v). \]

In particular, for $\lambda \notin R_f$ we have the symmetry
\[ E_{\lambda}(F_{f,0}; u, v) = (uv)^{n-1} E_{\lambda}(F_{f,0}; v^{-1}, u^{-1}). \]

4. **Main theorem**

Let $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-constant polynomial such that $f(0) = 0$ and set $V := f^{-1}(0) \subset \mathbb{C}^n$ as before. Throughout this section, we assume that $f$ is non-degenerate at 0 (see Definition 2.6). Since our assertion below will become trivial if $\dim P (= \dim \text{Conv}(\Gamma_f \cup \{0\})) < n$, in what follows we assume also that $\dim P = n$. Note that we do not assume $f$ is convenient here, and therefore the origin $0 \in \mathbb{C}^n$ may be a non-isolated singular point of $V$ in general. So we can not expect to have the concentration $\tilde{H}^j(F_{f,0}; \mathbb{C}) \simeq 0$ ($j \neq n-1$). However, we will prove the following result.
Theorem 4.1. In the situation as above, for any \( \lambda \notin R_f \) (see Definition 3.12) we have a concentration
\[
\tilde{H}^j(F_{f,0}; \mathbb{C})_\lambda = 0 \quad (j \neq n - 1).
\]

By this theorem and Theorem 2.8, we obtain the following corollary.

Corollary 4.2. In the situation of Theorem 4.1, for any \( \lambda \notin R_f \) the multiplicity of the eigenvalue \( \lambda \) in the Milnor monodromy \( \Phi_{n-1} \) is equal to that of the factor \((1 - \lambda t)^{k_I} \) in a rational function
\[
\prod_{\emptyset \neq I \subset \{1, \ldots, n\}} (1 - t^{d_{I,i}})^{(-1)^{n-|I|} \text{Vol}_Z(\Gamma_{I,i})}.
\]

For the definitions of \( \Gamma_{I,i} \), \( d_{I,i} \) and \( \text{Vol}_Z(\Gamma_{I,i}) \), see Section 2.

For the proof of Theorem 4.1, we need the following proposition.

Proposition 4.3. In this situation of Theorem 4.1, for any \( \lambda \notin R_f \), \( k \in \mathbb{Z} \) and the inclusion map \( j_0: \{0\} \hookrightarrow V \) the natural morphism in \( \text{D}^b_c(\{0\}) = \text{D}^b(\text{Mod}(\mathbb{C})) \):
\[
\psi_f,\lambda(C^n[n]) \rightarrow j_0^{-1}(\psi_f,\lambda(C^n[n]))
\]
is an isomorphism.

Proof of Proposition 4.3. It suffices to show that for any \( \lambda \notin R_f \) the morphism
\[
j_0^! \psi_f,\lambda(C^n) \rightarrow j_0^{-1} \psi_f,\lambda(C^n)
\]
is an isomorphism. We denote by \( \Sigma_0 \) the normal fan of the Newton polyhedron \( \Gamma_+(f) \). For a subset \( I \subset \{1, 2, \ldots, n\} \) we set \( \mathbb{R}^I_{\geq 0} := \mathbb{R}^I \cap \mathbb{R}^n_{\geq 0} \). We construct a smooth subdivision \( \Sigma \) of \( \Sigma_0 \) without subdividing the cones in \( \Sigma_0 \) of the type \( \mathbb{R}^I_{\geq 0} \). Let \( X_\Sigma \) be the toric variety associated with \( \Sigma \). Recall that \( X_\Sigma \) contains \( T := (\mathbb{C}^*)^n \) as an open dense subset and it acts naturally on \( X_\Sigma \) itself. For a cone \( \sigma \in \Sigma \), we denote by \( T_{\sigma} \) the \( T \)-orbit in \( X_\Sigma \) associated with it. Let \( \Sigma_1 \) be the fan formed by all the faces of \( \mathbb{R}^n_{\geq 0} \). Then the toric variety associated with it is \( \mathbb{C}^n \). Moreover, the morphism of fans \( \Sigma \rightarrow \Sigma_1 \) induces a proper morphism
\[
\pi: X_\Sigma \rightarrow \mathbb{C}^n
\]
of toric varieties. Set \( V' := \pi^{-1}(V) \).

Lemma 4.4. In the situation as above, we have an isomorphism
\[
\psi_f,\lambda(C^n) \simeq \psi_f,\lambda(R\pi_* C_{X_\Sigma})
\]
in \( \text{D}^b_c(V) \) for any \( \lambda \in \mathbb{C} \).

Proof. By the construction of \( \Sigma \), one can easily show that the morphism \( \pi \) induces an isomorphism
\[
X_\Sigma \setminus V' \sim \mathbb{C}^n \setminus V.
\]
Hence we obtain an isomorphism
\[
(R\pi_* C_{X_\Sigma})|_{\mathbb{C}^n \setminus V} \simeq C_{\mathbb{C}^n \setminus V}
\]
in \( \text{D}^b_c(\mathbb{C}^n \setminus V) \). Now the desired assertion immediately follows from the definition of the nearby cycle functor. \( \square \)
In what follows, we fix $\lambda \notin R_f$. Consider the following distinguished triangle in $D^b_c(\{0\}) \simeq D^b(\text{Mod}(\mathbb{C}))$:

$$(R\Gamma_{\{0\}} \psi_{f,\lambda}(\mathbb{C}^n))_0 \to \psi_{f,\lambda}(\mathbb{C}^n)_0 \to (R\Gamma_{V \setminus \{0\}} \psi_{f,\lambda}(\mathbb{C}^n))_0 \to 1.$$ 

The first arrow in it coincides with the natural morphism

$$j'_0 \psi_{f,\lambda}(\mathbb{C}^n) \to j_0^{-1} \psi_{f,\lambda}(\mathbb{C}^n).$$

Therefore, it is enough to show that $(R\Gamma_{V \setminus \{0\}} \psi_{f,\lambda}(\mathbb{C}^n))_0$ is isomorphic to 0. Consider the Cartesian diagram:

$$\begin{array}{ccc}
V \setminus \{0\} & \xrightarrow{i} & V \\
\pi'' & \downarrow & \pi' \\
V' \setminus \pi^{-1}(0) & \xleftarrow{i'} & V',
\end{array}$$

where $i, i'$ are the inclusions and $\pi', \pi''$ are the restrictions of $\pi$. Since $\pi'$ and $\pi''$ are proper, we obtain the following isomorphisms, where in the third isomorphism we used Proposition 4.2.11 of Dimca [3]:

$$(R\Gamma_{V \setminus \{0\}} \psi_{f,\lambda}(\mathbb{C}^n))_0 = (Ri_* i^{-1} \psi_{f,\lambda}(\mathbb{C}^n))_0$$

$$\simeq (Ri_* i^{-1} \psi_{f,\lambda}(R\pi_* \mathcal{C}_{X_S}))_0$$

(by Lemma 4.4)

$$\simeq (Ri_* i^{-1} R\pi'_* \psi_{f_{\sigma,\lambda}}(\mathcal{C}_{X_S}))_0$$

$$\simeq (Ri_* R\pi''_* i^{-1} \psi_{f_{\sigma,\lambda}}(\mathcal{C}_{X_S}))_0$$

$$\simeq (R\pi''_* R_{\sigma} i^{-1} \psi_{f_{\sigma,\lambda}}(\mathcal{C}_{X_S}))_0$$

$$\simeq R\Gamma(\pi^{-1}(0); (Ri''_* i^{-1} \psi_{f_{\sigma,\lambda}}(\mathcal{C}_{X_S}))_{|\pi^{-1}(0)})$$

$$\simeq R\Gamma(\pi^{-1}(0); (R\Gamma_{V' \setminus \pi^{-1}(0)} \psi_{f_{\sigma,\lambda}}(\mathcal{C}_{X_S}))_{|\pi^{-1}(0)})$$ in $D^b(\text{Mod}(\mathbb{C}))$. The following argument is inspired by the proof of Theorem 3.17 in Saito-Takeuchi [15]. Let $\rho_1, \ldots, \rho_N$ be the rays in $\Sigma$. By abuse of notation, we will use the same symbol $\rho_i$ for the primitive vector on the ray $\rho_i$. For $1 \leq i \leq N$, set

$$m_{\rho_i} := \min_{\alpha \in \Gamma_+(f)} \langle \alpha, \rho_i \rangle \in \mathbb{Z}_{\geq 0},$$

where $\langle \alpha, \rho_i \rangle$ is the inner product of $\alpha$ and $\rho_i$ in $\mathbb{R}^n$. We may assume that for some $1 \leq l \leq N$ we have $m_{\rho_i} \neq 0$ and $\lambda^{m_{\rho_i}} = 1$ if and only if $1 \leq i \leq l$, and for some $l \leq l' \leq N$ we have $m_{\rho_i} = 0$ if and only if $l' + 1 \leq i \leq N$. For a cone $\sigma \in \Sigma$ containing some $\rho_i$ with $1 \leq i \leq l'$, we also set

$$m_\sigma := \gcd_{1 \leq i \leq l'} (m_{\rho_i}) \in \mathbb{Z}_{\geq 1}.$$

For $1 \leq i \leq l'$ we denote by $E_i$ the closure $\overline{T_{\rho_i}}$ of $T_{\rho_i}$ in $X_S$. Note that the order of zeros of $f \circ \pi$ along the divisor $E_i$ is equal to $m_{\rho_i}$. Let $Z$ be the strict transform of $V$ in $X_S$. Then $V' = (f \circ \pi)^{-1}(0)$ has the following form

$$V' = E_1 \cup \cdots \cup E_{l'} \cup Z.$$
Since \( f \) is non-degenerate at 0, the divisor \( V' \) is normal crossing in a neighborhood of \( \pi^{-1}(0) \). In particular, \( Z \) is smooth there. Thus, for our discussion below we may assume \( V' \) is normal crossing. For a subset \( I \subset \{1, \ldots, l\} \), we set

\[
E_I := \bigcap_{i \in I} E_i, \quad E^0_I := E_I \setminus \left( \bigcup_{i \leq i' \leq l} E_i \cup Z \right) \quad \text{and} \quad U_I := E_I \setminus \left( \bigcup_{l+1 \leq i' \leq l} E_i \cup Z \right).
\]

Moreover, we write \( i_I \) and \( j_I \) for the inclusion maps \( i_I: E^0_I \hookrightarrow E_I \) and \( j_I: U_I \hookrightarrow E_I \) respectively. We denote by \( \iota \) the inclusion map \( \iota: X_{E} \setminus V' \hookrightarrow X_{E} \) and define a sheaf \( F_{\lambda} \) on \( X_{E} \) by

\[
F_{\lambda} := \iota_* (f \circ \pi|_{X_{E} \setminus V'})^{-1} L_{\lambda^{-1}},
\]

where \( L_{\lambda^{-1}} \) is the \( \mathbb{C} \)-local system on \( \mathbb{C}^* \) of rank 1 whose monodromy is given by the multiplication by \( \lambda^{-1} \). Since for \( 1 \leq i \leq l' \) the monodromy of the local system \( (f \circ \pi)^{-1} L_{\lambda^{-1}} \) around the divisor \( E_i \) is given by \( \lambda^{-m_\rho_i} \), the restriction of \( F_{\lambda} \) to \( U_{I} \) is a local system of rank 1.

For \( I \subset \{1, \ldots, n\} \), we set \( F_{\lambda I} = F_{\lambda}|_{E_I} \) and write \( i_{E_I} \) for the inclusion map \( i_{E_I}: E_I \hookrightarrow V' \). Then, by the primitive decomposition of \( p_{\psi_{f_0, \lambda}}(\mathbb{C}_{X_{E}}[n]) \) each graded piece of \( p_{\psi_{f_0, \lambda}}(\mathbb{C}_{X_{E}}[n]) \) with respect to the filtration \( W_{\bullet} p_{\psi_{f_0, \lambda}}(\mathbb{C}_{X_{E}}[n]) \) is a direct sum of some perverse sheaves

\[
i_{E_i} j_i^* j_i^{-1} F_{\lambda I} |_{\pi^{-1}(0)} \simeq i_{E_i} j_i^* j_i^{-1} F_{\lambda I} |_{\pi^{-1}(0)}
\]

for \( I \subset \{1, \ldots, l\} \) (see Section 1.4 of [4] and Section 4.2 of [1]). Therefore, to show that (8) is isomorphic to 0 in \( D^b(\text{Mod}(\mathbb{C})) \), it suffices to show that for each \( I \subset \{1, \ldots, l\} \), we have

\[
\text{RG}(\pi^{-1}(0); (\text{RG}_{V' \setminus \pi^{-1}(0)}(i_{E_i} j_i^* j_i^{-1} F_{\lambda I})))|_{\pi^{-1}(0)} \simeq 0.
\] (9)

Fix \( I \subset \{1, \ldots, l\} \) such that \( E_I \neq \emptyset \). If for some \( i \in I \) we have \( E_i \subset \pi^{-1}(0) \), the sheaf \( i_{E_i}^{-1} F_{\lambda I} \) is zero. For this reason, in what follows we may assume that \( E_i \not\subset \pi^{-1}(0) \) for any \( i \in I \). Namely, the ray \( \rho_i \) is contained in the boundary \( \partial \mathbb{R}^n_{\geq 0} \) of \( \mathbb{R}^n_{\geq 0} \) for any \( i \in I \). By the property of \( F_{\lambda} \) stated above, we can easily see the isomorphisms on \( E_I \):

\[
\text{RG}(\pi^{-1}(0); (\text{RG}_{V' \setminus \pi^{-1}(0)}(i_{E_i} j_i^* j_i^{-1} F_{\lambda I})))|_{\pi^{-1}(0)} \simeq 0.
\] (9)

We thus obtain

\[
\text{RG}(\pi^{-1}(0); (\text{RG}_{V' \setminus \pi^{-1}(0)}(i_{E_i} j_i^* j_i^{-1} F_{\lambda I})))|_{U_I \cap \pi^{-1}(0)} \simeq 0.
\] (11)

Since \( E_I \neq \emptyset \), the cone \( \tau \) generated by the rays \( \{\rho_i\}_{i \in I} \) is in \( \Sigma \). The subvariety \( U_I \cap \pi^{-1}(0) \) of \( X_{E} \) is the union of \( T_\sigma \setminus Z \) for the cones \( \sigma \in \Sigma \) satisfying the following condition (\(* \)):

\[
(\ast) \left\{ \begin{array}{ll}
(i) \quad \tau \text{ is a face of } \sigma, \\
(ii) \quad \text{rel int } \sigma \subset \text{Int } \mathbb{R}^n_{\geq 0} \\
(iii) \quad \text{any ray } \rho_i \in \Sigma \text{ contained in } \sigma \text{ satisfies } \lambda^{m_\rho_i} = 1,
\end{array} \right.
\]
where $\text{rel.int } \sigma$ is the relative interior of $\sigma$ and $\text{Int } \mathbb{R}^n_{>0}$ is the interior of $\mathbb{R}^n_{>0}$. Note that such $\sigma$ may contain some rays $\rho_i$ such that $i > l'$. Therefore, to show (11), we shall show that for any $\sigma \in \Sigma$ with the condition (\ast) we have

$$
\text{R} \Gamma(\mathcal{T}_s \setminus Z; (R j_{s,t}^{-1} \mathcal{F}_{\lambda,t})|_{T_s \setminus Z}) \simeq 0.
$$

(12)

In what follows, we fix a cone $\sigma \in \Sigma$ with the condition (\ast). Let $\tilde{\sigma} \in \Sigma_\sigma$ be the unique cone in $\Sigma_0$ such that $\text{rel.int } \sigma \subset \text{Int } \tilde{\sigma}$ and $F(\tilde{\sigma}) \prec \Gamma_+(f)$ the face of $\Gamma_+(f)$ which corresponds to it. By the condition $\text{rel.int } \sigma \subset \text{Int } \mathbb{R}^n_{>0}$, the face $F(\tilde{\sigma})$ is compact. We denote by $d_{F(\tilde{\sigma})}$ the lattice distance of $F(\tilde{\sigma})$ from the origin $0 \in \mathbb{R}^n$. Then we can easily show the following assertion.

**Lemma 4.5.** Assume that $\dim \sigma = \dim \tilde{\sigma}$. Then we have $m_\sigma = d_{F(\tilde{\sigma})}$.

Suppose that $\dim \sigma = \dim \tilde{\sigma}$. Since for any $i \in I$ we have $\rho_i \subset \partial \mathbb{R}^n_{\geq 0}$ and $m_{\rho_i} > 0$, there exists a non-compact face $G$ of $\Gamma_+(f)$ containing $F(\tilde{\sigma})$ and $G \not\subset \partial \mathbb{R}^n_{\geq 0}$. Then, by Lemma 4.5 and the assumption that $\lambda \notin R_f$, we have $\lambda^{m_\sigma} = \lambda^{d_{F(\tilde{\sigma})}} \neq 1$. Therefore, there exists a ray $\rho_i$ in $\sigma$ such that $\lambda^{m_{\rho_i}} \neq 1$. This contradicts our condition (\ast). Hence we have

$$
\dim \sigma < \dim \tilde{\sigma}.
$$

Take a cone $\sigma' \in \Sigma$ such that $\sigma \prec \sigma' \subset \tilde{\sigma}$ and $\dim \sigma' = \dim \tilde{\sigma}$. Then by the argument as above, we have $\lambda^{m_{\sigma'}} \neq 1$. Thus, by the condition (\ast) it follows that there exists a ray $\rho_i \prec \sigma'$ such that $\rho_i \notin \sigma$ and

$$
\lambda^{m_{\rho_i}} \neq 1.
$$

(13)

Moreover, take a $n$-dimensional cone $\sigma'' \in \Sigma$ such that $\sigma' \prec \sigma''$. Let $\text{Edge}(\tau)$, $\text{Edge}(\sigma)$, $\text{Edge}(\sigma')$ and $\text{Edge}(\sigma'')$ be the sets of edges (i.e. rays $\rho_i$) of the smooth cones $\tau$, $\sigma$, $\sigma'$ and $\sigma''$ respectively. We assume that for some $1 \leq i_1 \leq i_2 < i_3 \leq i_4 \leq n$ we have

$$
\text{Edge}(\tau) = \{\xi_1, \ldots, \xi_{i_1}\},
$$

$$
\text{Edge}(\sigma) = \{\xi_1, \ldots, \xi_{i_2}\},
$$

$$
\text{Edge}(\sigma') = \{\xi_1, \ldots, \xi_{i_3}\},
$$

$$
\text{Edge}(\sigma'') = \{\xi_1, \ldots, \xi_{i_4}\}
$$

$(\xi_i \in \Sigma)$. Set $s_1 := i_1$, $s_2 := i_2 - i_1$, $s_3 := i_3 - i_2$, $s_4 := n - i_3$. Note that by the condition $\sigma \not\preceq \sigma'$ we have $s_3 > 0$. Since $\sigma''$ is a smooth cone, the affine open subset $\mathbb{C}^n(\sigma'') \simeq \mathbb{C}^n$ of $X_\Sigma$ associated with $\sigma''$ has a natural decomposition:

$$
\mathbb{C}^n(\sigma'') \simeq \mathbb{C}^{s_1} \times \mathbb{C}^{s_2} \times \mathbb{C}^{s_3} \times \mathbb{C}^{s_4}.
$$

Let

$$
(x_1, \ldots, x_{s_1}, y_1, \ldots, y_{s_2}, z_1, \ldots, z_{s_3}, w_1, \ldots, w_{s_4})
$$

be the corresponding coordinates of $\mathbb{C}^n(\sigma'')$. In $\mathbb{C}(\sigma'') \simeq \mathbb{C}^n$, we have

$$
E_i = \{0\} \times \mathbb{C}^{s_2} \times \mathbb{C}^{s_3} \times \mathbb{C}^{s_4}
$$
and

$$
T_\sigma = \{0\} \times \{0\} \times (\mathbb{C}^*)^{s_3} \times (\mathbb{C}^*)^{s_4}.
$$
For any $i_2 + 1 \leq i \leq i_3$ the function $\langle \xi_i, \cdot \rangle$ is constant on $F(\tilde{\sigma})$. Since $f$ is non-degenerate at $0$, $T_\sigma \cap Z$ is smooth and its defining polynomial in $T_\sigma$ can be described by

$$f \circ \pi = \prod_{i=1}^{m_{i_2+1}} z_1^{m_{i_2+1}} \ldots z_{s_3}^{m_{i_3}} g(w_1, \ldots, w_{s_4}),$$

where $g(w_1, \ldots, w_{s_4}) \in \mathbb{C}[w_1, \ldots, w_{s_4}]$. We denote by $W$ the zero set of $g(w_1, \ldots, w_{s_4})$ in $(\mathbb{C}^*)^{s_4}$. Then we have

$$T_\sigma \cap Z = \{0\} \times \{0\} \times (\mathbb{C}^*)^{s_3} \times W,$$

and

$$T_\sigma \setminus Z = \{0\} \times \{0\} \times (\mathbb{C}^*)^{s_3} \times ((\mathbb{C}^*)^{s_4} \setminus W).$$

Let $p_3$ be the projection $p_3: T_\sigma \setminus Z = (\mathbb{C}^*)^{s_3} \times ((\mathbb{C}^*)^{s_4} \setminus W) \rightarrow (\mathbb{C}^*)^{s_3}$, and $p_4$ be the projection $p_4: T_\sigma \setminus Z = (\mathbb{C}^*)^{s_3} \times ((\mathbb{C}^*)^{s_4} \setminus W) \rightarrow (\mathbb{C}^*)^{s_4} \setminus W$. We define $L_3$ by the $\mathbb{C}$-local system on $(\mathbb{C}^*)^{s_3}$ of rank $1$. Let $\lambda$ be the projection $\lambda: (\mathbb{C}^*)^{s_4} \setminus W \rightarrow \{0\}$ given by the multiplication by $\lambda^{-m_{i_2+t}}$ for each $1 \leq t \leq s_3$. Note that by (13) there exists $i_2 + 1 \leq i \leq i_3$ such that

$$\lambda^{-m_{i_2+t}} \neq 1.$$  

By the definition of $F_{\lambda, I}$, one can show that there exist a $\mathbb{C}$-local system $L_4$ on $(\mathbb{C}^*)^{s_4} \setminus W$ and a complex $\mathbb{C}^*$ of $\mathbb{C}$-vector spaces such that

$$(Rj_{s, I} j_0^{-1} F_{I, \lambda})|_{T_\sigma \setminus Z} \simeq C^* \otimes_\mathbb{C} p_3^{-1} L_3 \otimes_\mathbb{C} p_4^{-1} L_4.$$  

Recall that for any non-trivial $\mathbb{C}$-local system $L$ on $\mathbb{C}^*$ of rank $1$, we have $H^j(\mathbb{C}^*; L) \simeq 0$ for all $j \in \mathbb{Z}$. Hence, by the Künneth formula and (14) we deduce the vanishing (12). This completes the proof that the morphism (5) is isomorphism and the proof of Proposition 4.3.

**Proof of Theorem 4.4** Recall that the complex $p^* \psi_{f, \lambda}(\mathbb{C}^*[n]) = \psi_{f, \lambda}(\mathbb{C}^*[n-1]) \in D^b_c(V)$ is a perverse sheaf. By the fact that the functor $j_0^{-1}: D^b_c(V) \rightarrow D^b_c(\{0\})$ (resp. $j_0^!: D^b_c(V) \rightarrow D^b_c(\{0\})$) is right (resp. left) $t$-exact, it follows from Proposition 4.3 that $j_0^{-1}(p^* \psi_{f, \lambda}(\mathbb{C}^*[n]))$ is a perverse sheaf on $\{0\}$. Hence the cohomology group

$$H^j(j_0^{-1}(p^* \psi_{f, \lambda}(\mathbb{C}^*[n]))) \simeq H^{j+n-1}(F_{f, 0}; \mathbb{C})_\lambda$$

vanishes for $j \neq 0$. We thus obtain the desired concentration. 

Recall that the cohomology groups $H^j(F_{f, 0}; \mathbb{Q})$ of the Milnor fiber are endowed with mixed Hodge structures. Since we do not assume here that $f$ is convenient, we cannot expect that their weight filtrations coincide with the monodromy filtrations in general. However, we will obtain the following result.

**Theorem 4.6** In the situation of Theorem 4.4 for any $\lambda \notin R_f$ the filtration on $H^{n-1}(F_{f, 0}; \mathbb{C})_\lambda$ induced by the weight filtration on $H^{n-1}(F_{f, 0}; \mathbb{Q})$ coincides with the monodromy filtration of the logarithm of the unipotent part of $\Phi_{n-1, \lambda}$ centered at $n - 1$.

**Remark 4.7** Theorems 4.1 and 4.6 explain the reason why the coefficients of the polynomial $E_\lambda(F_{f, 0}; u, v)$ for $\lambda \notin R_f$ satisfy the symmetry in Proposition 3.14.
Lemma 4.9. For any $M$ cohomology of $F$ Therefore, we have $\psi_j(C_{C^n}[n])$ for any $i,j,k$. Eventually, to show the vanishing (15), it is enough to show that for any $p \in \psi$ weight filtration of the mixed Hodge module $\psi_j^H(Q_{C^n}[n])$. More generally, by using the exact functors $W_k: MHM(X) \to MHM(X)$ for an object $M^\bullet \in D^b MHM(X)$ we can define new ones $W_k M^\bullet$ in $D^b MHM(X)$.

**Proposition 4.8.** In this situation of Theorems 4.6, for any $\lambda \notin R_f$, $k \in \mathbb{Z}$ and the inclusion map $j_0: \{0\} \hookrightarrow V$ the natural morphism in $D^b(V)$:

$$j_0^! W_k (p \psi_{f,\lambda}(C_{C^n}[n])) \to j_0^{-1} W_k (p \psi_{f,\lambda}(C_{C^n}[n]))$$

is an isomorphism.

**Proof.** It is enough to show that

$$R\Gamma_{V \setminus \{0\}} (W_k (p \psi_{f,\lambda}(C_{C^n}[n])))_0 \simeq 0$$

for any $k \in \mathbb{Z}$. Moreover, since by Lemma 4.4 we have $R\pi'_i (p \psi_{f,\lambda}(C_{C^n}[n])) \simeq p \psi_{f,\lambda}(C_{C^n}[n])$, it suffices to show that

$$R\Gamma_{V \setminus \{0\}} (W_k R\pi'_i Gr^W_i (p \psi_{f,\lambda}(C_{X_{\Sigma}[n])))_0 \simeq 0$$

for any $i,k \in \mathbb{Z}$. Thus we have only to show that

$$R\Gamma_{V \setminus \{0\}} (W_k (p H^j R\pi'_i Gr^W_i (p \psi_{f,\lambda}(C_{X_{\Sigma}[n])))_0 \simeq 0$$

for any $i,j,k \in \mathbb{Z}$, where for $F^\bullet \in D^b(V)$ we denote by $p H^j(F^\bullet)$ the $j$-th perverse cohomology of $F^\bullet$. Note that $p H^j R\pi'_i Gr^W_i (p \psi_{f,\lambda}(C_{X_{\Sigma}[n])}$ has a pure weight $i + j$. Therefore, we have

$$W_k (p H^j R\pi'_i Gr^W_i (p \psi_{f,\lambda}(C_{X_{\Sigma}[n]))) \simeq \begin{cases} 0 & (k < i + j) \\ p H^j R\pi'_i Gr^W_i (p \psi_{f,\lambda}(C_{X_{\Sigma}[n))) & (k \geq i + j). \end{cases}$$

Eventually, to show the vanishing (15), it is enough to show that for any $i,j \in \mathbb{Z}$ we have

$$R\Gamma_{V \setminus \{0\}} (p H^j R\pi'_i Gr^W_i (p \psi_{f,\lambda}(C_{X_{\Sigma}[n])))_0 \simeq 0.$$  

Recall that for an algebraic variety $X$ and an object $M^\bullet \in D^b MHM(X)$ we say that $M^\bullet$ has a pure weight $i$ if $Gr^W_k H^j M^\bullet \simeq 0 (k \neq i + j)$. It is well-known that for such an object $M^\bullet$ there exists a non-canonical isomorphism $M^\bullet \simeq \bigoplus_{j \in \mathbb{Z}} H^j(M^\bullet)[-j]$ in $D^b MHM(X)$. Therefore, to show (16), it remains for us to show that

$$R\Gamma_{V \setminus \{0\}} (R\pi'_i Gr^W_i (p \psi_{f,\lambda}(C_{X_{\Sigma}[n])))_0 \simeq 0$$

for any $i \in \mathbb{Z}$. This was already proved in the proof of Proposition 4.3.

For the proof of Theorem 4.6 we also need the following lemma.

**Lemma 4.9.** For any $\lambda \notin R_f$ and $k \in \mathbb{Z}$ we have an isomorphism in $D^b \left(\{0\}\right) \simeq D^b(\text{Mod}(\mathbb{C})): \nabla k j_0^{-1} (p \psi_{f,\lambda}(C_{C^n}[n])) \simeq j_0^{-1} W_k (p \psi_{f,\lambda}(C_{C^n}[n])).$
Proof. For \( k \in \mathbb{Z} \) we have an exact sequence in \( \text{Perv}(V) \):

\[
0 \to W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])) \to p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n]) \to p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])/W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])) \to 0. 
\]

(17)

By Proposition 4.3 and this sequence, the natural morphism

\[
j_0^i(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])/W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n]))) \longrightarrow j_0^{-1}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])/W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])))
\]

is an isomorphism. Thus, by the proof of Theorem 4.1, the complexes \( j_0^{-1}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])) \), \( j_0^{-1}W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])) \) and \( j_0^{-1}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])/W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n]))) \) are perverse sheaves on \( \{0\} \), i.e. their (perverse) cohomologies are concentrated in the degree 0. Therefore, applying the functor \( j_0^{-1} \) to the sequence (17), we obtain an exact sequence

\[
0 \to j_0^{-1}W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])) \to j_0^{-1}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])) \to j_0^{-1}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])/W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n]))) \to 0
\]

(18)

in \( \text{Perv}(\{0\}) \simeq \text{Mod}(\mathbb{C}) \). Since the functor \( j_0^i \) preserves the property that a complex of mixed Hodge modules has mixed weights \( > k \), \( j_0^{-1}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])/W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n]))) \simeq j_0^{-1}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])/W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n]))) \) has mixed weights \( > k \). Therefore, taking \( W_k \) of the sequence (18), we obtain

\[
W_kj_0^{-1}W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])) \simeq W_kj_0^{-1}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])).
\]

(19)

On the other hand, since the functor \( j_0^{-1} \) preserves the property that a complex of mixed Hodge modules has mixed weights \( \leq k \), \( j_0^{-1}W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])) \) has mixed weights \( \leq k \). Hence we have

\[
W_kj_0^{-1}W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])) \simeq j_0^{-1}W_k(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])).
\]

(20)

Combining the isomorphisms (19) and (20), we get the desired isomorphism.

Proof of Theorem 4.6. Assume that \( \lambda \notin R_f \). We denote by \( N \) the logarithm of the unipotent part of the monodromy automorphism of \( p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n]) \), and \( N_0 \) its restriction to 0, i.e. the logarithm operator of the unipotent part of \( \Phi_{n-1} \) of \( j_0^{-1}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])) \simeq H^{n-1}(F_{f,0};\mathbb{C})_\lambda \). Recall that for any \( k \in \mathbb{Z}_{\geq 1} \) we have

\[
N^k : Gr^{W}_{n-1+k}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])) \xrightarrow{\sim} Gr^{W}_{n-1-k}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])).
\]

(21)

Applying the functor \( j_0^{-1} \) to the both sides of (21), by Proposition 4.8 and Lemma 4.9 we obtain

\[
N_0^k : Gr^{W}_{n-1+k}j_0^{-1}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n])) \xrightarrow{\sim} Gr^{W}_{n-1-k}j_0^{-1}(p\psi_{f,\lambda}(\mathbb{C}_\mathbb{C}^n[n]))
\]

that is

\[
N_0^k : Gr^{W}_{n-1+k}H^{n-1}(F_{f,0};\mathbb{C})_\lambda \xrightarrow{\sim} Gr^{W}_{n-1-k}H^{n-1}(F_{f,0};\mathbb{C})_\lambda
\]

for any \( k \in \mathbb{Z}_{\geq 1} \). This implies that the weight filtration on \( H^{n-1}(F_{f,0};\mathbb{C})_\lambda \) coincides with the monodromy filtration centered at \( n - 1 \).
5. Applications

In this section, we apply Theorems 4.1 and 4.6 to compute the Jordan normal forms of the Milnor monodromies and the Hodge spectra. Let \( f(x) \in \mathbb{C}[x_1, \ldots, x_n] \) be a polynomial such that \( f(0) = 0 \). Assume that it is non-degenerate at 0. Let \( P \) be the convex hull of \( \Gamma_f \cup \{0\} \). Since our formula below will become trivial in the case when the dimension of \( P \) less than \( n \), in what follows we assume that the dimension of \( P \) is equal to \( n \). Moreover, since the case where \( f \) is convenient was already treated by Matsui-Takeuchi [10] and M. Saito [13], we assume that \( f \) is not convenient in this section. Then \( R_f \) is not empty and contains \( 1 \in \mathbb{C} \). For \( \lambda \notin R_f \), by Proposition 3.14, we have

\[
u^2 E_{\lambda}(F_{f0}; u, u) = (-1)^{n-1} l^*_{\lambda}(P, \nu; u, u),
\]

where \( \nu \) is the piecewise linear function on \( P \) defined in Section 3.3. Recall that \( S_{\nu} \) is the polyhedral subdivision of \( P \) defined by \( \nu \). By the definition of the \( h^* \)-polynomial, for \( \lambda \notin R_f \) we have

\[
l^*_{\lambda}(P, \nu; u, u) = \sum_{F < \Gamma_+(f): \text{admissible}} u^{\dim F + 1} l^*_{\lambda}(\Delta_F, \nu; 1) \cdot l_P(S_{\nu}, \Delta_F; u^2),
\]

where in the sum \( \Sigma \) the face \( F \) ranges through the compact admissible ones of \( \Gamma_+(f) \). The polynomial \( l_P(S_{\nu}, \Delta_F; t) \) being symmetric and unimodal, it can be expressed in the form

\[
l_P(S_{\nu}, \Delta_F; t) = \sum_{i=0}^{[(n-1-\dim F)/2]} \tilde{l}_{F,i}(t^i + t^{i+1} + \cdots + t^{n-1-\dim F-i}),
\]

for some non-negative integers \( \tilde{l}_{F,i} \in \mathbb{Z}_{\geq 0} \). We set

\[
\tilde{l}_P(S_{\nu}, \Delta_F, t) := \sum_{i=0}^{[(n-1-\dim F)/2]} \tilde{l}_{F,i} t^i.
\]

For \( k \in \mathbb{Z}_{\geq 0} \) and \( \lambda \in \mathbb{C} \) we denote by \( J_{k,\lambda} \) the number of the Jordan blocks in \( \Phi_{n-1} \) with size \( k \) for the eigenvalue \( \lambda \). Then by Theorems 4.1 and 4.6 we obtain the following formula for them.

**Corollary 5.1.** In the situation as above, for any \( \lambda \notin R_f \) we have

\[
\sum_{0 \leq k \leq n-1} J_{n-k,\lambda} u^{k+2} = \sum_{F < \Gamma_+(f): \text{admissible}} u^{\dim F + 1} l^*_{\lambda}(\Delta_F, \nu; 1) \cdot \tilde{l}_P(S_{\nu}, \Delta_F; u^2),
\]

where in the sum \( \Sigma \) of the right hand side the face \( F \) ranges through the admissible ones of \( \Gamma_+(f) \).

Finally, we introduce our formula for the Hodge spectrum of \( f \) at the origin 0.

**Definition 5.2.** (1) We define a Puiseux polynomial \( sp_{f,0}(t) \) with coefficients in \( \mathbb{Z} \) by

\[
sp_{f,0}(t) = (-1)^{n-1} \sum_{\alpha \in \mathbb{Q} \cap [0,n]} \left\{ \sum_{j \in \mathbb{Z}} (-1)^j \dim Gr^{(n-\alpha)} F_j (F_{f0}; \mathbb{C}) \exp(-2\pi \sqrt{-1} \alpha) \right\} t^\alpha,
\]
where $\text{Gr}^{[n-\alpha]}_F H^j(F_{f,0}; \mathbb{C})_{\exp(-2\pi \sqrt{-1} \alpha)}$ is the graded peace with respect to the Hodge filtration of the mixed Hodge structure of $\tilde{H}^j(F_{f,0}; \mathbb{Q})$. We call it the Hodge spectrum of $f$ at $0$.

(2) For $\beta \in (0, 1) \cap \mathbb{Q}$ we set $\lambda = \exp(2\pi \sqrt{-1} \beta)$ and we define a Puiseux polynomial $\text{sp}^\lambda_{f,0}(t)$ by

$$\text{sp}^\lambda_{f,0}(t) = (-1)^{n-1} \sum_{i=0}^{n-1} \left\{ \sum_{j \in \mathbb{Z}} (-1)^j \dim \text{Gr}^{[n-\beta-i]}_F \tilde{H}^j(F_{f,0}; \mathbb{C})_{\lambda^{-1}} \right\} t^{\beta+i}.$$ 

Since $f$ is non-degenerate at $0$, by setting $v = 1$ in Corollary 3.10 we can express $\text{sp}_{f,0}(t)$ and $\text{sp}^\lambda_{f,0}(t)$ in terms of $\Gamma_f$. Moreover, if $\lambda = \exp(2\pi \sqrt{-1} \beta)$ is not in $R_f$, $\text{sp}^\lambda_{f,0}(t)$ can be rewritten much more simply as follows. For a compact face $F$ of $\Gamma_+(f)$, we define a cone $\text{Cone}(F) \subset \mathbb{R}^n$ by $\text{Cone}(F) := \mathbb{R}_{\geq 0} F$ and the linear function $h_F$ on $\text{Cone}(F)$ which takes the value $0$ at the origin $0 \in \mathbb{R}^n$ and the value $1$ on $F$. Moreover, for $\beta \in (0, 1) \cap \mathbb{Q}$ we define a Puiseux polynomial $P_{F,\beta}(t)$ by

$$P_{F,\beta}(t) := \sum_{i=0}^{\infty} \#\{v \in \text{Cone}(F) \cap \mathbb{Z}_{\geq 0}^n \mid h_F(v) = \beta + i\} t^{\beta+i}.$$ 

Then we obtain the following formula, which generalizes the one for $\text{sp}_{f,0}(t)$ in the case where $0 \in V$ is an isolated singular point obtained by M. Saito [13]. For the corresponding result for the monodromies at infinity, see Theorem 5.16 of Matsui-Takeuchi [9].

**Corollary 5.3.** In the situation as above, assume moreover that $\lambda$ is not in $R_f$. Then we have

$$\text{sp}^\lambda_{f,0}(t) = \sum_{F \prec \Gamma_+(f) : \text{admissible}} (-1)^{n-1-\dim F} (1-t)^{s_F} P_{F,\beta}(t),$$

where in the sum $\sum$ the face $F$ ranges through the admissible ones of $\Gamma_+(f)$ and $s_F \in \mathbb{Z}_{\geq 1}$ is the integer defined in Section 3.7.

**Proof.** By Theorem 4.11 for $\beta \in (0, 1) \cap \mathbb{Q}$ such that $\lambda = \exp(2\pi \sqrt{-1} \beta) \notin R_f$ we have the concentration

$$\tilde{H}^j(F_{f,0}; \mathbb{C})_{\lambda} \simeq 0 \quad (j \neq n - 1).$$

Moreover, for $0 \leq i \leq n - 1$ we have

$$\dim \text{Gr}^{[n-i-\beta]}_F \tilde{H}^{n-1}(F_{f,0}; \mathbb{C})_{\lambda^{-1}} = \sum_{k \in \mathbb{Z}} \dim \text{Gr}^{[n-i-\beta]}_F \text{Gr}_k^W \tilde{H}^{n-1}(F_{f,0}; \mathbb{C})_{\lambda^{-1}}$$

$$= \sum_{k \in \mathbb{Z}} \dim \text{Gr}^{n-1-[n-i-\beta]}_F \text{Gr}_k^W(2(n-1)-k-[n-\alpha]) \tilde{H}^{n-1}(F_{f,0}; \mathbb{C})_{\lambda}$$

$$= \dim \text{Gr}^i_F \tilde{H}^{n-1}(F_{f,0}; \mathbb{C})_{\lambda},$$
Furthermore, we have \( TAKAHIRO SAITO \)

\[
\sum_{i=0}^{n-1} \dim \text{Gr}^{[n-i-\beta]}_F H^{n-1}(F_{i,0}; \mathbb{C})_\lambda t^{\beta+i}
\]

where in the sums \( \lambda \) the faces \( F \) range through the admissible ones of \( \Gamma_+(f) \). Since for an admissible face \( F \prec \Gamma_+(f) \), we have \( \dim \sigma(\Delta_F) = s_F \), this completes the proof. \( \square \)

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