EULER-POINCARÉ FORMULAE FOR POSITIVE DEPTH
BERNSTEIN PROJECTORS

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Abstract. Work of Bezrukavnikov–Kazhdan–Varshavsky uses an equivariant system of trivial idempotents of Moy–Prasad groups to obtain an Euler–Poincaré formula for the r-depth Bernstein projector. Barbasch–Ciubotaru–Moy use depth-zero cuspidal representations of parahoric subgroups to decompose the Euler–Poincaré presentation of the depth-zero projector. For positive depth r, we establish a decomposition of the Euler–Poincaré presentation of the r-depth Bernstein projector based on a notion of associate classes of cuspidal pairs for Moy–Prasad quotients. We apply these new Euler–Poincaré presentations to the obtain decompositions of the resolutions of Schneider–Stuhler and Bestvina–Savin.

1. Introduction

Suppose k is a non-archimedean local field and G is a reductive group defined over k. Let G = G(k) denote the group of k-rational points. Let Ω(G) denote the category of smooth (complex) representations of G. Bernstein (see [B]) used parabolic induction to define natural full subcategories Ω(\{(M, σ)\}) of Ω(G) indexed by equivalence classes \{(M, σ)\} of cuspidal data consisting of a Levi subgroup M (of G) and an irreducible cuspidal representation of M. The cuspidal data equivalence is (M, σ) ∼ (M', σ'), if (i) there exists g ∈ G, and (ii) there exists an unramified character χ of M so that M' = Ad(g)(M) and σ' and (σ ⊗ χ) ◦ Ad(g) are equivalent. These subcategories are called Bernstein components. Furthermore (see [BD]), to a Bernstein component Ω(\{(M, σ)\}), there is an unique essentially compact G-invariant distribution \(P_{Ω(\{(M, σ)\})}\), i.e., an element of the Bernstein center, with the property that:

(i) \(P_{Ω(\{(M, σ)\})}\) is idempotent,
(ii) for any smooth representation (π, V_π), the subrepresentation π(\(P_{Ω(\{(M, σ)\})}\))(V_π) lies in Ω(\{(M, σ)\}), and
(iii) if (π, v_π) is a smooth representation in Ω(\{(M, σ)\}), then π(\(P_{Ω(\{(M, σ)\})}\)) = Id_{V_π}.

The nomenclature Bernstein projector is used for such a distribution.

As a consequence of the work [MPa,MPb] of Moy-Prasad on unrefined minimal K-types, to any irreducible representation (π, V_π) of G, there is a nonnegative rational number \(ρ(π)\) (called the depth of π). It can be characterized as the smallest rational number r so that exists \(x ∈ B(\mathcal{G})\) so that \(V_π^{\mathcal{G}_{x,r}^+} ≠ \{0\}\) (and the action of \(\mathcal{G}_{x,r}/\mathcal{G}_{x,r}^+\) on \(V_π^{\mathcal{G}_{x,r}^+} \neq \{0\}\) can be shown to be nondegenerate). The depth is the same for all irreducible representations in a

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Bernstein component $\Omega = \Omega(\{(M, \sigma)\})$. We use the notation $\rho(\Omega)$ to denote this rational number. The $r$-th depth projector $P_{\leq r}$ is defined to be (the finite sum):

$$P_{\leq r} := \sum_{\rho(\Omega(\{(M, \sigma)\})) \leq r} P_{\Omega(\{(M, \sigma)\})}.$$

(1.1)

If $N \in \mathbb{N}$, and $r \in \mathbb{Q}N$, Bezrukavnikov–Kazhdan–Varshavsky [BKV] replace the usual simplicial structure on the Bruhat–Tits building $\mathcal{B} = \mathcal{B}(\mathfrak{g})$, with a refinement. For ease of exposition, we assume $\mathfrak{g}$ is split and therefore $\mathcal{B}(\mathfrak{g})$ has a hyperspecial point. The new structure, which we denote as $\mathcal{B}_N$, is obtained by taking each affine apartment $\mathcal{A} \subset \mathcal{B}$ and refining the simplicial structure on $\mathcal{A}$. One selects a hyperspecial point as an origin and then scales the locus of affine root hyperplanes which define the original simplicial structure on $\mathcal{A}$ by a factor of $\frac{1}{N}$. The resulting refined simplicial structure on $\mathcal{A}$ is denoted $\mathcal{A}_N$ and is independent of the hyperspecial point selected. An important feature of this refined simplicial structure is that the Moy-Prasad groups $\mathcal{G}_{x,r}$ are constant on the interior of any facet of $F \subset \mathcal{B}_N$. Denote this group as $\mathcal{G}_{F,r,+}$. Fix a Haar measure on $\mathfrak{g}$, and define the $C_c^\infty(\mathfrak{g})$ idempotent:

$$e^r_F := \frac{1}{\text{meas}(\mathcal{G}_{F,r,+})} 1_{\mathcal{G}_{F,r,+}}.$$

(1.2)

Inspired by work of Meyer–Solleveld in [MS], Bezrukavnikov–Kazhdan–Varshavsky use $\mathcal{B}_N$ to give a presentation of the distribution $P_{\leq r}$ as the Euler–Poincaré sum:

$$P_{\leq r} = \sum_{F \subset \mathcal{B}_N} (-1)^{\dim(F)} e^r_F.$$

(1.3)

A natural question to ask is whether there are similar Euler-Poincaré presentations for other linear combinations of Bernstein projectors. An extreme case would be to ask if there is an Euler-Poincaré presentation for an individual projector $P_{\Omega(\{(M, \sigma)\})}$. Evidence of the latter is the work in [BCM]. We recall the building $\mathcal{B}(\mathfrak{m})$ of a Levi subgroup $\mathfrak{m}$ is the union of the apartments $\mathcal{A}(\mathfrak{s}) \subset \mathcal{B}(\mathfrak{g})$ as $\mathfrak{s}$ runs over the maximal split tori in $\mathfrak{m}$, and that for $x \in \mathcal{B}(\mathfrak{m})$ we have $\mathfrak{m}_{x,r} = (\mathcal{G}_{x,r} \cap \mathfrak{m})$. When the class $\{(M, \sigma)\}$ has depth zero, it is a consequence of work in [MPb] that there is a facet $F \subset \mathfrak{g}$ so that

(i) $F \subset \mathcal{B}(\mathfrak{m}) \subset \mathcal{B}(\mathfrak{g})$ (hence that $\mathcal{G}_{F,0}/\mathcal{G}_{F,0,+} = \mathfrak{m}_{F,0}/\mathfrak{m}_{F,0,+}$).

(ii) a cuspidal representation $\tau$ of the finite field group $\mathfrak{m}_{F,0}/\mathfrak{m}_{F,0,+}$ so that the cuspidal representation $\sigma$ contains $\tau$ (inflated to $\mathfrak{m}_{F,0}$).

Such a pair $(F, \tau)$ is called a cuspidal pair. As indicated in [MPb], uniqueness of the pair $(F, \tau)$ is up to the equivalence of associate pairs, which we now recall. Two facets $F$ and $F'$ are associate if $\dim(F) = \dim(F')$ and there exists $g \in \mathfrak{g}$ so that the convex facet closure $C(F, g.F')$ of $F$ and $g.F'$ also has dimension this common dimension. We say the facets $F$ and $g.F'$ are aligned. We define two cuspidal pairs $(F, \tau)$ and $(F', \tau')$ to be associate if they satisfy the following:

(i) The facets $F$ and $F'$ are associate, e.g., assume $F$ and $g.F'$ are aligned. The alignment of $F$ and $g.F$ yields a canonical identification of $\mathcal{G}_{F,0}/\mathcal{G}_{F,0,+}$ and $\mathcal{G}_{g.F',0}/\mathcal{G}_{g.F',0,+}$.

(ii) The element $g$ in part (i) can be taken so that $\tau \circ \text{Ad}(g)$ is to be equivalent to $\tau'$.
A cuspidal pair \((F, \tau)\) defines an associate class \(\mathcal{C} = \{(F, \tau)\}\), and to the associate class \(\mathcal{C}\) there is an \(\mathcal{G}\)-equivariant system of idempotents \(e_K^{\mathcal{C}}\) (\(K\) a facet in \(\mathcal{B}(\mathcal{G})\)) so that
\[
P_0^\mathcal{C} := \sum_{K \in \mathcal{B}} (-1)^{\dim(K)} e_K^{\mathcal{C}}
\] (1.4)
is an Euler–Poincaré presentation of the sum \(\left( \sum' P_0(\{(\mathcal{L}, \kappa)\}) \right)\) over the classes \(\{(\mathcal{L}, \kappa)\}\) which contain a pair \((\mathcal{M}, \theta)\) so that the restriction \(\theta|_{\mathcal{G}(F,0)}\) contains \(\tau\). We caution the double use of notation (which should be clear from context): the class \(\{(\mathcal{L}, \kappa)\}\) is that of a Levi subgroup \(\mathcal{L}\) and a cuspidal representation \(\kappa\) of \(\mathcal{L}\), while class \(\{(F, \tau)\}\) is a facet \(F\), and a cuspidal representation \(\tau\) of \(\mathcal{G}_{F,0}/\mathcal{G}_{F,0} + \mathcal{G}_{F,0}\) inflated to \(\mathcal{G}_{F,0}\).

A key result of [BCM] is the (orthogonal) decomposition:
\[
P_{\leq 0} = \sum_{\mathcal{C}} P_0^\mathcal{C} .
\] (1.5)
We view this decomposition of \(P_{\leq 0}\) as an embodiment of Harish-Chandra’s philosophy of cusp forms [HC] over finite field groups.

Our goal here is to show a partial analogue of (1.5) for positive depth. We assume \(N \in \mathbb{N}_+\) and \(r \in \frac{1}{N} \mathbb{N}_+\). Let \(\mathcal{B}_N\) denote the Bruhat–Tits building with the refined simplicial structure mentioned above. In addition to constancy of the groups \(\mathcal{G}_{x,r}\) on the interior of a facet \(F\), the groups \(\mathcal{G}_{x,r}\) are constant too (we use the notations \(\mathcal{G}_{F,r}\) and \(\mathcal{G}_{F,r}\) for these groups), and if \(E\) is a subfacet of \(F\), then \(\mathcal{G}_{E,r} \supset \mathcal{G}_{F,r} \supset \mathcal{G}_{F,r+} \supset \mathcal{G}_{E,r+}\). The condition \(r > 0\) has the consequence that the quotient \(\mathcal{G}_{F,r}/\mathcal{G}_{F,r+}\) is commutative.

Let \(F \supset E\) be a pair of facets in \(\mathcal{B}_N\). A facet \(\bar{F} \supset E\) is called opposite to \(F\) if there exists an apartment \(\mathcal{A}(\mathcal{S})\) that contains \(F\) and \(\bar{F}\) is the reflection of \(F\) about the affine subspace of \(\mathcal{A}(\mathcal{S})\) generated by \(E\). Two opposite facets \(F, F'\) yield (see [2.5.5] in Proposition 2.5.4) an Iwahori decomposition of \(\mathcal{G}_{E,r}/\mathcal{G}_{E,r+}\) as:
\[
\mathcal{G}_{E,r}/\mathcal{G}_{E,r+} = \mathcal{G}_{F,r+}/\mathcal{G}_{F,r+} \oplus \mathcal{G}_{F,r}/\mathcal{G}_{F,r+} \oplus \mathcal{G}_{F,r+}/\mathcal{G}_{E,r+} .
\] (1.6)
Since \(F\) and \(\bar{F}\) are aligned, note that we have a canonical identification
\[
\mathcal{G}_{F,r}/\mathcal{G}_{F,r+} = \mathcal{G}_{\bar{F},r}/\mathcal{G}_{F,r+} .
\] (1.7)

Let \(E\) be a facet of \(\mathcal{B}_N\). In homage to the usual notions of parabolic induction and restriction, we define a character \(\chi\) of \(\mathcal{G}_{E,r}/\mathcal{G}_{E,r+}\) to be cuspidal if, for any facet \(F \supseteq E\), the restriction of \(\chi\) to the summand \(\mathcal{G}_{F,r+}/\mathcal{G}_{E,r+}\) in (1.6) is non-trivial. Propositions 3.1.7 and 3.1.10 say to any character \(\chi\) of \(\mathcal{G}_{E,r}/\mathcal{G}_{E,r+}\), there exists a facet \(E\) containing \(E\) and a cuspidal character \(\phi\) of \(\mathcal{G}_{F,r}/\mathcal{G}_{F,r+}\) so that the inflation of \(\phi\) to \(\mathcal{G}_{F,r}\) equals \(\chi|_{\mathcal{G}_{F,r}}\); or, equivalently, that \(\chi\) is contained in the induced representation \(\text{Ind}_{\mathcal{G}_{F,r}}^{\mathcal{G}_{E,r}} \phi\).

We define a cuspidal pair \((E, \chi)\) as consisting of a facet \(E \subset \mathcal{B}_N\) and a cuspidal character of \(\mathcal{G}_{E,r}/\mathcal{G}_{E,r+}\). For this definition, there is a precise analogue of the depth zero equivalence relation of associate cuspidal pairs. As in the depth zero situation, we call the equivalence class of a cuspidal pair \((F, \chi)\), its associate class \(\mathcal{C} = \{(E', \chi') \mid (E', \chi') \sim (E, \chi)\}\). To the associate class \(\mathcal{C}\), in a fashion analogous to the depth zero situation, there is an \(\mathcal{G}\)-equivariant system of idempotents \(e_J^{\mathcal{C}}\) (\(J\) a facet in \(\mathcal{B}_N\)). The support of the equivariant system is on the
facets which are contained in a facet of the associate class. Our main Theorem (Theorem 6.3.10) is that the Euler-Poincaré sum

\[ P^C := \sum_{J \subset B_N} (-1)^{\dim(J)} e_J^C \]  

is a distribution in the Bernstein center and idempotent (hence a finite sum of Bernstein projectors), and the depth \( r \) project \( P_{\leq r} \) has the decomposition

\[ P_{\leq r} = \sum_C P^C. \]  

We note that there is an associate class \( N \) associated to cuspidal pairs of the type \((C, \chi_{\text{triv}})\), where \( C \) is a chamber in \( B_N \), and that (in the obvious notation) \( P_{<r} = P_{\leq r_0} \). The Euler–Poincaré presentation \( P_N \) of \( P_{<r} \), and the Euler–Poincaré presentation (1.3) of \( P_{\leq r_0} \) are two presentations of the same distribution. It would be interesting to have a direct proof of this connection.

For respectively the Bruhat-Tits building \( B \) and the refined simplicial structure \( B_N \), the idempotents \( e_J^r \) appear in the work of Schneider–Stuhler and Bestvina–Savin on resolutions of a smooth representation \((\pi, V_\pi)\). Indeed, for \( r \in \mathbb{N} \), Schneider–Stuhler independently discovered the groups \( G_{F,r} \) and used the notation \( U_{F}^{r} \) for these groups. Both Schneider–Stuhler and Bestvina–Savin assemble the vector spaces \( \pi(e_J^r)(V_\pi) \) into a smooth representation of \( G \)

\[ W_\pi^{r,k} = \bigoplus_{J \subset B_N, \dim(J) = k} (J, \pi(e_J^r)(V_\pi)) \]  

which is projective. They define natural boundary maps \( W^{r,0} \rightarrow W^{r,(k-1)} \) and an augmentation map \( W^{r,0} \rightarrow V_\pi \) so that if \( V_\pi \) is generated by the image of the augmentation map, then the resulting complex is a resolution of \( V_\pi \).

For an associate class \( C = \{ (F, \chi) \} \), a key property of the idempotent distribution \( P^C \) (see Theorem 6.3.10 part (iii)) is that

\[ P^C \star e_H^r = e_J^C \quad \text{for any facet } J \subset B_N. \]

A consequence of this is that when \( P^C \) is applied to a resolution of Schneider–Stuhler or of Bestvina–Savin, we obtain

\[ \pi(P^C) (\pi(e_J^r)(V_\pi)) = \pi(e_J^C)(V_\pi). \]

Thus, if we define \( W^{C,k} \) as in (1.10) by replacing \( \pi(e_J^r)(V_\pi) \) with \( \pi(e_J^C)(V_\pi) \), we obtain a resolution of \( \pi(P^C)(V_\pi) \).

We outline our presentation of results. In section 2 we introduce notation and the simplicial refinement \( B_N \) \((N \in \mathbb{N})\) of the Bruhat–Tits building \( B \). If \( E \) and \( E' \) are facets of \( B_N \), we introduce the set \( C(E, F) \) which is the convex facet closure of \( E \) and \( F \). The set \( C(E, F) \) is used to defined the notion of aligned facets and associate facets. It is essential in formulating the important convolution result Proposition 2.6.2.

In section 3 we define cuspidal characters of the quotient groups \( G_{F,r}/G_{F,r+} \) (and their inflations to \( G_{F,r} \)), and establish that if \( \chi \) is a character of \( G_{E,r}/G_{E,r+} \) then there is a facet \( F \supseteq \)
and a cuspidal character \( \phi \) of \( \mathcal{G}_{F,r} / \mathfrak{g}_{F,r} \), so that \( \chi|_{\mathcal{G}_{F,r}} \) is the inflation of \( \phi \). Furthermore if the same is true for another cuspidal pair \( (F', \phi') \), then it is associate to \( (F, \phi) \). We also establish useful convolution properties of cuspidal characters. We end the section defining the \( \mathcal{G} \)-equivariant system of idempotents \( e_J^C \) (\( J \) a facet of \( \mathcal{B}_N \)) associated to an associate class \( C \) of a cuspidal pair.

In section 4, we fix a chamber \( C \subset \mathcal{B}_N \) and establish formulae for the Euler–Poincaré sums over certain subfacets of \( C \). In section 5, we fix a base chamber \( C_0 \subset \mathcal{B}_N \) and establish Proposition 5.3 which is key to obtaining what we believe to be an elegant proof that the Euler–Poincaré sum (1.8) is in the Bernstein center. In section 6, we prove our main Theorem 6.3.10 about the distributions \( P^C \) of (1.8). In section 7, we apply the distributions \( P^C \) to obtain decompositions of the resolutions (see [SS,BS]) of Schneider–Stuhler and Bestvina–Savin. In section 8, we exposit the change necessary in the definition of virtual affine roots to extend the split arguments to nonsplit groups.

2. Refined simplicial structure on a Bruhat–Tits building

2.1. Notation and Preliminaries.

As in the introduction, we assume \( k \) is a non-archimedean local field. We denote by \( \mathcal{O}_k \), \( \mathfrak{p}_k \), and \( \mathbb{F}_q = \mathcal{O}_k / \mathfrak{p}_k \) respectively, the ring of integers, prime ideal, and residue field of \( k \). We use similar notation for a maximal unramified extension \( k^{un} / k \). The residue field \( \mathcal{O}_k^{un} / \mathfrak{p}_k^{un} \) is an algebraic closure of \( \mathbb{F}_q \) and we denote it as \( \mathbb{F}_q^{un} \). Let \( G \) be a connected reductive linear algebraic group defined over \( k \). For convenience, we assume \( G \) is \( k \)-split and quasisimple. Set \( \ell = \text{rank}(G) \). If \( H \) is a \( k \)-subgroup of \( G \), we write \( \mathcal{H} \) for the group \( H(k) \) of \( k \)-rational points of \( H \), e.g., \( \mathcal{G} = G(k) \). We set \( \mathcal{G}^{un} := G(k^{un}) \), and \( \mathcal{G} := G(k) \).

Let \( S \) be a maximal \( k^{un} \)-split torus of \( G \) (so, by definition, \( \mathcal{S}^{un} = S(k^{un}) \)), and let (see [T,BTa,BTb]) \( \mathcal{A}(\mathcal{S}^{un}) \) be the apartment associated to \( \mathcal{S}^{un} \subset \mathcal{G}^{un} \). Let

\[
\Phi(S) := \text{the set roots of } G \text{ with respect to } S, \\
\Psi(S) := \{ \alpha + k | \alpha \in (\Phi \cup \{0\}), \ k \in \mathbb{Z} \}
\tag{2.1.1}
\]

the set of affine roots (with respect to \( S \)).

When clear, we abbreviate these two sets to \( \Phi \) and \( \Psi \) (as well as \( \Phi(S) \) and \( \Psi(S) \)) respectively. To a root \( \alpha \in \Phi \) (resp. affine root \( \psi \in \Psi \)), let \( U_\alpha \) (resp. \( X_\psi \)) denote the attached root group (resp. affine root group).

When \( \xi \) is a nonconstant affine function on \( \mathcal{A}(\mathcal{S}^{un}) \), we define

\[
H_\xi := \text{the zero hyperplane of } \xi.
\tag{2.1.2}
\]

We recall that an affine root hyperplane is a hyperplane \( H_\psi \) attached to a nonconstant affine root \( \psi \in \Psi \). Our hypotheses on \( G \) (\( k \)-split, quasisimple) and \( S \) (maximal \( k^{un} \)-split torus) means the apartment \( \mathcal{A}(k^{un}) \) is a simplicial complexes under the decomposition by the zero hyperplanes of the nonconstant affine roots. The Bruhat-Tits building \( \mathcal{B}(\mathcal{S}^{un}) \) of \( \mathcal{S}^{un} \) is
obtained by gluing apartments $A(S)$ (S running over all $k^\text{un}$-split tori) along (convex) sets of subsimplices. Thus, $\mathcal{B}(S)$ is a simplicial complex and metric space, with a simplicial and isometrical action of $S$. The Galois group $\text{Gal}(k^\text{un}/k)$ acts on $S$ with fixed points $(S)^{\text{Gal}(k^\text{un}/k)}$ equal to $S$. So, $S$ acts on the Bruhat-Tits building:

$$\mathcal{B}(S) := \mathcal{B}(S)^{\text{Gal}(k^\text{un}/k)}.$$  

(2.1.3)

In fact, $\mathcal{B}(S)$ can be obtained by gluing apartments $A(S)$ of maximal $k$-split tori $S$. We conveniently denote such an apartment also as $A(S)$.

The group $G_{\text{un}}$ (resp. $G$) acts transitively on the chambers, i.e., $\ell$-simplices, of $\mathcal{B}(S)$ (resp. $\mathcal{B}(S)$). The choice of a hyperspecial point $x_0 \in A = A(S)$ corresponds to the choice of a Chevalley basis for the Lie algebra $g$ of $\mathcal{G}$. Such a choice gives an identification of $A(S)$ with $\text{Hom}(G_m, S) \otimes \mathbb{Z} \otimes \mathbb{R}$ whereby the point $x_0$ becomes 0.

2.2. Simplicial refinement by $N \in \mathbb{N}_+$.

For $N \in \mathbb{N}_+$, the identification of $A(S)$ with $\text{Hom}(G_m, S) \otimes \mathbb{Z} \otimes \mathbb{R}$ allows us to scale the simplicial structure on $A(S)$ by a factor of $\frac{1}{N}$. Each facet of $A(S)$ is a union of facets of the new simplicial structure. So, the new simplicial structure is a refinement of the original structure. We use the notation $A(S)_N$ to denote $A(S)$ with this refined simplicial structure.

Equivalently, $A(S)_N$ can be described as the refined simplicial structure on $A(S)$ arising from the zero hyperplanes of the set of virtual affine roots defined as:

$$\Psi(S)_N := \left\{ \frac{\alpha + \ell}{N} \in \Phi, \alpha \in \Phi, \ell \in \frac{1}{N} \mathbb{Z} \right\} \supset \Psi(S).$$

(2.2.1)

We shall sometimes refer to these hyperplanes as refined affine hyperplanes, and sometimes lapse into calling virtual affine roots refined affine roots. The refined simplicial structure $A(S)_N$ of each apartment $A(S)$ yields a refined simplicial structure $\mathcal{B}(S)_N$ of $\mathcal{B}(S)$.

2.3. Simplicial closure of two facets.

If $E, F \subset \mathcal{B}$ are facets, we define their convex simplicial closure is defined as:

$$C(E, F) := \text{smallest convex union of facets of } \mathcal{B}$$

containing $E$ and $F$

$$= \bigcap_{\text{apartment } A, \text{so that } E, F \subset A} A.$$  

(2.3.1)

Suppose $N \in \mathbb{N}_+$. The above can be extrapolated to refined facets $E, F \subset \mathcal{B}_N$ as follows.

$$C(E, F) := \text{smallest convex union of facets of } \mathcal{B}_N$$

containing $E$ and $F$.  

(2.3.2)

**Definition 2.3.3.** Suppose $E$ and $F$ are facets of $\mathcal{B}_N$.

- Define facets $E$ and $F$ to be aligned if

$$\dim(C(E, F)) = \dim(E) = \dim(F).$$

(2.3.4)
• Define a facet $E$ of $\mathcal{B}_N$ is subaligned to a facet $F$ if:

(i) $\dim(E) \leq \dim(F)$

(ii) $\dim(C(E,F)) = \dim(F)$. \hfill (2.3.5)

We use the term strictly subaligned to mean $\dim(E) < \dim(F)$ and subaligned.

A consequence of facets $E$ and $F$ being aligned is that there is a canonical equality of $\mathcal{G}_{E,0}/\mathcal{G}_{E,0}^+$ and $\mathcal{G}_{F,0}/\mathcal{G}_{F,0}^+$ and also of $\mathcal{G}_{E,r}/\mathcal{G}_{E,r}^+$ and $\mathcal{G}_{F,r}/\mathcal{G}_{F,r}^+$. If $E$ is subaligned to $F$, then we can view $\mathcal{G}_{F,0}/\mathcal{G}_{F,0}^+$ and $\mathcal{G}_{F,r}/\mathcal{G}_{F,r}^+$ as canonically inside $\mathcal{G}_{E,0}/\mathcal{G}_{E,0}^+$ and $\mathcal{G}_{E,r}/\mathcal{G}_{E,r}^+$ respectively.

Lemma 2.3.6. Given facets $E, F \subset \mathcal{B}_N$, there are unique facets $D_E, D_F \subset C(E,F)$ so that:

(i) $\dim(D_E) = \dim(D_F) = \dim(C(E,F))$; thus, $D_E$ and $D_F$ are aligned (so both $\mathcal{G}_{D_E,0}/\mathcal{G}_{D_E,0}^+ = \mathcal{G}_{D_F,0}/\mathcal{G}_{D_F,0}^+$ and $\mathcal{G}_{D_E,r}/\mathcal{G}_{D_E,r}^+ = \mathcal{G}_{D_F,r}/\mathcal{G}_{D_F,r}^+$).

(ii) $E \subset D_E$ and $F \subset D_F$; thus, the parahoric subgroup $\mathcal{G}_{D_E,0}$ (resp. $\mathcal{G}_{D_F,0}$) is contained in $\mathcal{G}_{E,0}$ (resp. $\mathcal{G}_{F,0}$).

Proof. We recall that any apartment $\mathcal{A}$ which contains the (refined) facets $E$ and $F$ also contains their convex closure $C(E,F)$. Let $D_E \subset C(E,F)$ be a refined facet of maximal dimension $\dim(C(E,F))$ containing $E$. In $\mathcal{A}$, the affine subspace Aff$(E)$ contains $C(E,F)$. Suppose $D'_E \subset C(E,F)$ is another refined facet containing $E$ of dimension $\dim(C(E,F))$. If $D_E \neq D'_E$, there must be an virtual affine root $\psi$ so that (i) the intersection $H_{\psi} \cap$ Aff$(C(E,F))$ is codimension one in Aff$(C(E,F))$, and so divides it into two halfspaces, and (ii) $D_E$ and $D'_E$ are in opposite halfspaces. Obviously $C(E,F)$ must be in the closure of exactly one of the halfspaces which contradicts $D_E$ and $D'_E$ being subsets of $C(E,F)$. Thus, $D_E$ is unique. Similarly for $F$. \hfill $\Box$

2.4. Iwahori factorizations.

We fix an apartment $\mathcal{A} = \mathcal{A}(\mathcal{S})$ of $\mathcal{B} = \mathcal{B}(\mathcal{G})$. If $\alpha \in \Phi = \Phi(\mathcal{S})$, let $\mathcal{U}_\alpha$ denote the $\alpha$-root group. Similarly, if $\psi \in \Psi$, let $\mathcal{X}_\psi$ denote the affine root group indexed by $\psi$. Fix $x \in \mathcal{A}$ and $r > 0$. For any $\alpha \in \Phi \cup \{0\}$, set

$$\psi_{\alpha,x,r} := \text{the smallest affine root } \psi \text{ with grad}(\psi) = \alpha, \text{ and } \psi(x) \geq r.$$ 

We recall for any choice $\Phi^+$ and $\Phi^- = -\Phi^+$ of positive and negative roots, that the group $\mathcal{G}_{x,r}$ has the Iwahori factorization:

Proposition 2.4.1. (Iwahori factorization) $\mathcal{G}_{x,r} = \mathcal{U}_{x,r} \mathcal{S}_r \mathcal{U}_{x,r}^+$, where

$$\mathcal{U}_{x,r}^- = \prod_{\alpha \in \Phi^-} \mathcal{X}_{\psi_{\alpha,x,r}} \text{ and } \mathcal{U}_{x,r}^+ = \prod_{\alpha \in \Phi^+} \mathcal{X}_{\psi_{\alpha,x,r}}.$$
If $\psi = \alpha + \ell$ is a virtual affine root, define the (virtual) affine root group $X_\psi$ as:

$$X_\psi := X_{(\alpha + [\ell])}.$$  

The (virtual) affine root groups satisfy analogous commutation relations as the affine root groups. In the definition of the groups $G_{x,r}$, we can replace affine root groups with virtual affine root groups, but we obviously do not get any different groups.

**Lemma 2.4.2.** Given $x \in A = A(S)$, and $r > 0$:

(i) $G_{x,r} = \prod_{\psi \text{ virtual affine } \psi(x) \geq r} X_\psi.$

(ii) For a choice of positive roots $\Phi^+$:

$$U_{x,r}^- = \prod_{\psi \text{ virtual affine } \text{ grad}(\psi) \in \Phi^- \psi(x) \geq r} X_\psi \quad \text{and} \quad U_{x,r}^+ = \prod_{\psi \text{ virtual affine } \text{ grad}(\psi) \in \Phi^+ \psi(x) \geq r} X_\psi.$$

**Proof.** Elementary verification. \(\square\)

**Lemma 2.4.3.** Fix $N \in \mathbb{N}_+$. For $r \in \frac{1}{N} \mathbb{N}$, and a (refined) facet $F \subset B(\mathfrak{g})_N$:

(i) The groups $G_{x,r}$ and $G_{x,r}$ are constant on the interior of any facet of $F$. Let $G_{F,r}$ and $G_{F,r}$ denote these groups.

(ii) Suppose $E \subset F$ is a (refined) subfacet of $F$, then

$$G_{E,r} \supset G_{F,r} \supset G_{F,r} \supset G_{E,r}.$$  

**Proof.** To prove (i), suppose $\psi$ is a virtual affine root. If suffices to show

$$\forall x, y \in \text{int}(F) : X_\psi \subset G_{x,r} \iff X_\psi \subset G_{y,r}.$$  

This is obvious if $\psi$ is constant on int$(F)$. If $\psi$ is not constant on int$(F)$, then there exists $j \in \mathbb{Z}$ so that $\forall x \in \text{int}(F) : \frac{j}{N} < \psi(x) < \frac{j+1}{N}$. This is because otherwise the values of $\psi$, on int$(F)$, would contain an open neighborhood of some $\frac{j}{N}$, and hence the hyperplane $H_{(\psi - \frac{j}{N})}$ would cut $F$ into two strictly smaller pieces, contradicting the assumption $F$ is a refined facet. So, (2.4.4) is true, and the two assertions of the statement (i) hold.

Statement (ii) follows from (i). \(\square\)

Statement (ii) of the Lemma for the value $r = 0$ is the assertion the parahoric subgroup $G_{E,0}$ contains the parahoric subgroup $G_{F,0}$.

For a facet $E \subset B(\mathfrak{g})_N$, define the virtual sphere VSph($E$) of $E$ as the set:

$$\text{VSph}(E) := \{ \text{facet } F \subset B(\mathfrak{g})_N \mid F \supset E \}.$$  

(2.4.5)

For $N = 1$, and $B(\mathfrak{g})_N = B(\mathfrak{g})$ the building with its original simplicial structure, we shall also refer to the above set as the sphere of $E$ and use the notation Sph($E$). Let

$$E \to E'$$  

(2.4.6)
be the map which takes a facet $E \subset \mathcal{B}(\mathcal{G})_N$ to its facet closure in $\mathbb{B}(\mathcal{G})$. Obviously
\[ \mathcal{G}_{E,0} = \mathcal{G}_{E',0}, \quad \mathcal{G}_{E,0} = \mathcal{G}_{E',0}, \quad \text{so} \quad \mathcal{G}_{E,0}/\mathcal{G}_{E,0}^+ = \mathcal{G}_{E',0}/\mathcal{G}_{E',0}^+. \]
Each facet $F \subset \text{VSph}(E)$ defines a parabolic subgroup $\mathcal{G}_{F,0}/\mathcal{G}_{E,0}^+$ of $\mathcal{G}_{E,0}/\mathcal{G}_{E,0}^+$, and this map is a surjection but for $N \geq 2$ not a bijection because the facet closure map from $\mathbb{B}(\mathcal{G})_N$ to $\mathbb{B}(\mathcal{G})$ is not a bijection. For $N = 1$, we have

**Proposition 2.4.7.** The facets $F \subset \mathbb{B}(\mathcal{G})$ which properly contain a $E \subset \mathcal{B}(\mathcal{G})$ are in bijection with the spherical Tits building of $\mathcal{G}_{E,0}/\mathcal{G}_{E,0}^+$.

**Proof.** The proper parahoric subgroups contained in $\mathcal{G}_{E,0}$ are in one-to-one correspondence with the proper parabolic subgroups of the finite field group $\mathcal{G}_{E,0}/\mathcal{G}_{E,0}^+$, and the later are parametrized by the facets of the spherical Tits building of $\mathcal{G}_{E,0}/\mathcal{G}_{E,0}^+$. □

### 2.5. Iwahori decompositions.

Fix $N \in \mathbb{N}_+$, and a facet $E \subset \mathcal{B}(\mathcal{G})_N$. Suppose $A = A(S)$ is an apartment containing $E$. Let $r \in \frac{1}{N}\mathbb{Z}$. The set
\[ \Phi_E := \{ \text{grad}(\psi) \mid \psi \text{ a virtual affine root, } \psi = r \text{ on } E \} \quad (2.5.1) \]
is a root sub-system of a $\Phi$, independent of $r$. For example, the extremes are:

(i) If the facet $E$ is a special point then $\Phi_E = \Phi$.
(ii) If $E$ is a chamber, then $\Phi_E = \emptyset$.

For $\alpha \in \Phi \cup \{0\}$, define:

\[
\psi_{\alpha,E,r} := \begin{cases} 
\text{the smallest virtual affine root (with gradient } \alpha \text{) so that } \psi_{\alpha,E,r}(x) \geq r & \forall x \in E \\
\end{cases}
\]

\[
\psi_{\alpha,\text{int}(E),r^+} := \begin{cases} 
\text{the smallest virtual affine root (with gradient } \alpha \text{) so that } \psi_{\alpha,\text{int}(E),r^+}(x) > r & \forall x \in \text{int}(E) \\
\end{cases}
\]

Then,
\[
\mathcal{G}_{E,r} = \left( \prod_{\alpha \in (\Phi(S) \cup \{0\})} X_{\psi_{\alpha,E,r}} \right) \left( \prod_{\alpha \in \Phi_E \cup \{0\}} X_{\psi_{\alpha,E,r}} \right) \left( \prod_{\alpha \notin \Phi_E} X_{\psi_{\alpha,E,r}} \right)
\]
\[
\mathcal{G}_{E,r^+} = \left( \prod_{\alpha \in (\Phi(S) \cup \{0\})} X_{\psi_{\alpha,\text{int}(E),r^+}} \right) \left( \prod_{\alpha \in \Phi_E \cup \{0\}} X_{\psi_{\alpha,\text{int}(E),r^+}} \right) \left( \prod_{\alpha \notin \Phi_E} X_{\psi_{\alpha,\text{int}(E),r^+}} \right)
\]

We note that:

(i) If $\alpha \in \Phi_E \cup \{0\}$, then the value of $\psi_{\alpha,E,r}$ (resp. $\psi_{\alpha,\text{int}(E),r^+}$) on $E$ is $r$ (resp. $> r$).
(ii) Otherwise $\psi_{\alpha,E,r} = \psi_{\alpha,\text{int}(E),r^+}$.
Therefore,
\[ \mathcal{G}_{E,r}/\mathcal{G}_{E,r+} = \prod_{\alpha \in (\Phi_{E}\cup\{0\})} \mathcal{X}_{\psi_{\alpha,E,r}} / \mathcal{X}_{\psi_{\alpha,\text{int}(E),r+}}. \quad (2.5.2) \]

We recall that if \( \mathcal{A} \) is a (real orthogonal) affine space, and \( \mathcal{E} \) is an affine subspace of \( \mathcal{A} \), then there is a unique affine orthogonal transformation \( R_{\mathcal{E}} \) which reflects a point of \( \mathcal{A} \) across \( \mathcal{E} \).

Fix \( N \in \mathbb{N} \), and let \( E \subset \mathcal{B}(\mathcal{S})_{N} \) be a (refined) facet. Define two (refined) facets \( F \) and \( \bar{F} \) containing \( E \) to be opposite with respect to \( E \) if there exists an apartment \( \mathcal{A} = \mathcal{A}(\mathcal{S}) \) containing both \( F \) and \( \bar{F} \), so that in \( \mathcal{A} \):

(i) The affine subspaces \( \text{Aff}(F) \) and \( \text{Aff}(\bar{F}) \) are equal.

(ii) \( R_{\text{Aff}(E)}(F) = \bar{F} \).

Since the parahoric subgroup \( \mathcal{G}_{F,0} \) acts transitively on the apartments containing \( F \), if \( F \) and \( F' \) are opposite with respect to \( E \) in one apartment apartment, they will be opposite with respect to \( E \) in any apartment containing them.

Suppose \( F \) and \( \bar{F} \) are \( E \)-opposite, and \( \mathcal{A} = \mathcal{A}(\mathcal{S}) \) contains \( F \) and \( \bar{F} \). If \( \psi \) is a virtual affine root such that \( \psi = r \) on \( F \) then \( \psi = r \) on \( \bar{F} \) and vice versa. Hence
\[ \Phi_{\bar{F}} = \Phi_{F}. \quad (2.5.3) \]

Let \( \psi \) be a virtual affine root such that \( \psi = r \) on \( E \), hence its gradient is an element in \( \Phi_{E} \). We consider two possibilities for \( \psi \):

(i) \( \psi \) is constant on \( F \) (and on \( \bar{F} \)). Then \( \psi = r \) on \( F \) (and on \( \bar{F} \)), and the gradient of \( \phi \) is in \( \Phi_{F} = \Phi_{\bar{F}} \).

(ii) \( \psi \) is nonconstant on \( F \) (which is true if and only if it is nonconstant on \( \bar{F} \)). Here we have the dichotomy that either \( \psi > r \) on \( \text{int}(F) \) (and \( \psi < r \) on \( \text{int}(\bar{F}) \)), or the other way around. Set \( \Phi_{E,F} \) to be the set of gradients of \( \psi \) such that \( \psi = r \) on \( E \) and \( \psi > r \) on \( \text{int}(F) \) roots. It is a choice of positive roots (determined by the facet \( F \)) in the root system \( \Phi_{E} \). We have \( \Phi_{E,F} = -\Phi_{E,F} \), and \( \Phi_{E} \) is the disjoint union of \( \Phi_{F} = \Phi_{\bar{F}} \), \( \Phi_{E,F} \) and \( \Phi_{E,F} \).

As a consequence, we have:

**Proposition 2.5.4.** Under the hypothesis on \( N \) and \( r \) of Lemma 2.4.3, suppose \( E \) is a (refined) facet of \( \mathcal{B}(\mathcal{S})_{N}, F, \) and \( \bar{F} \) are (refined) facets which are opposite with respect to \( E \). Let \( \mathcal{A}(\mathcal{S}) \) be an apartment containing \( F \) and \( \bar{F} \). Then (Iwahori decomposition)
\[ \mathcal{G}_{E,r}/\mathcal{G}_{E,r+} = \mathcal{G}_{F,r+}/\mathcal{G}_{E,r+} \oplus \mathcal{G}_{F,r}/\mathcal{G}_{F,r+} \oplus \mathcal{G}_{F,r+}/\mathcal{G}_{E,r+}. \quad (2.5.5) \]

We recall each of piece of the Iwahori decomposition (2.5.5) is a \( \mathbb{F}_{q} = \Omega_{k}/\varphi_{k} \) vector space. Set
\[ V_{E} := \mathcal{G}_{E,r}/\mathcal{G}_{E,r+} \quad (\text{the notation suppresses the dependence on } r) \]
\[ V_{F} := \mathcal{G}_{F,r}/\mathcal{G}_{F,r+} = \mathcal{G}_{F,r}/\mathcal{G}_{F,r+} \]
\[ V_{E,F} := \mathcal{G}_{F,r+}/\mathcal{G}_{E,r+}, \quad V_{E,F} := \mathcal{G}_{F,r+}/\mathcal{G}_{E,r+}. \quad (2.5.6) \]
2.6. Convolution of \( e_{g_{E, r^+}} \) and \( e_{g_{F, r^+}} \).

Henceforth we fix the natural number \( N \). To simplify notation and terminology, affine roots, facets etc will mean their refined versions. Furthermore, if \( \psi \) is an affine root and \( F \) a facet \( \psi > r \) on \( E \) shall typically mean on the interior of \( E \).

We fix a Haar measure on \( G \), and therefore a convolution structure on \( C^\infty_c(G) \). For any open compact subgroup \( J \subset G \) we define the idempotent:

\[
e_J := \frac{1}{\text{meas}(J)} 1_J.
\]

**Proposition 2.6.2.** Suppose \( E, F \subset B_N \). Let \( C(E, F) \) be the combinatorial convex hull of \( E \) and \( F \). Let \( D_E, D_F \subset C(E, F) \) be the unique facets so that (i) \( E \subset D_E, F \subset D_F \), and (ii) \( \dim(D_E) = \dim(D_F) = \dim(C(E, F)) \). Then,

\[
e_{g_{E, r^+}} \ast e_{g_{F, r^+}} = e_{g_{D_{E, r^+}}} \ast e_{g_{D_{F, r^+}}} \ast e_{g_{F, r^+}}.
\]

**Proof.** If \( D_E = D_F \) (denote by \( D \)), then \( E \) and \( F \) are subfacets of \( D \). If \( \psi \) is an affine root so that it values are greater than \( r \) on either \( E \), or \( F \), then obviously, since \( E \) and \( F \) are subfacets of \( D \), the values of \( \psi \) on \( D \) must also be greater than \( r \). The assertion \((2.6.2)\) follows. So, we can and do assume \( D_E \neq D_F \). Set

\[
S := \{ \psi \text{ affine root } \mid \psi \text{ has constant value on } D_E \text{ (and therefore on } D_F \text{ too) } \}
\]

\[
P := \{ \psi \text{ affine root } \mid \psi(D_F) > \psi(D_E) \}
\]

\[
N := \{ \psi \text{ affine root } \mid \psi(D_F) < \psi(D_E) \}
\]

Then, the convolution \((2.6.3)\) follows from the decompositions (in any order)

\[
\mathcal{S}_{D_{E, r^+}} = \left( \prod_{\psi \in N} \mathcal{X}_\psi \right) \left( \prod_{\psi \in S} \mathcal{X}_\psi \right) \left( \prod_{\psi \in P} \mathcal{X}_\psi \right)
\]

\[
\mathcal{S}_{D_{F, r^+}} = \left( \prod_{\psi \in N} \mathcal{X}_\psi \right) \left( \prod_{\psi \in S} \mathcal{X}_\psi \right) \left( \prod_{\psi \in P} \mathcal{X}_\psi \right),
\]

and the inclusions

\[
(i) \quad \left( \prod_{\psi \in P} \mathcal{X}_\psi \right) \subset \left( \prod_{\psi \in P} \mathcal{X}_\psi \right) \quad \text{and} \quad \left( \prod_{\psi \in N} \mathcal{X}_\psi \right) \subset \left( \prod_{\psi \in N} \mathcal{X}_\psi \right).
\]

(ii) \( \left( \prod_{\psi \in N} \mathcal{X}_\psi \right) \subset \mathcal{S}_{E, r^+} \) and \( \left( \prod_{\psi \in P} \mathcal{X}_\psi \right) \subset \mathcal{S}_{F, r^+} \)

The inclusions \((2.6.5)\) are obvious. To check the first inclusion of \((2.6.6)\), we need to show that \( \psi(E) > r \). If not, then \( \psi(E) = r \). Since \( \psi \in N \), if follows that \( r > \psi(D_F) \); hence both \( E \) and \( F \) are contained in the half-space \( \psi \leq r \), while \( D_F \) is not. This contradicts \( D_F \) being in the convex hull of \( E \) and \( F \). Similarly for the second inclusion. \( \square \)
Remarks: (i) Since $E \subset D_E$, we have $\mathcal{S}_{D_E,r^+} \supset \mathcal{S}_{E,r^+}$, and therefore $(e_{G,E,r^+} \ast e_{G,E,r^+}) = e_{G,E,r^+}$. Similarly $(e_{D_F,r^+} \ast e_{D_F,r^+}) = e_{D_F,r^+}$.

(ii) Although not needed here, we note the proof of Proposition (2.6.2) in fact shows that if $K$ is a subfacet of $C(E,F)$, then

$$e_{G,E,r^+} \ast e_{G,E,r^+} = e_{G,E,r^+} \ast e_{K,r^+} \ast e_{D_F,r^+}.$$

3. Representations of $\mathcal{S}_{E,r}/\mathcal{S}_{E,r^+}$

3.1. Parabolic inflation and cuspidal characters.

Let $V$ be a finite dimensional $\mathbb{F}_q$ vector space, and $V^* = \text{Hom}_{\mathbb{F}_q}(V, \mathbb{F}_q)$ denote the dual space. The choice of a non-trivial additive character $\chi$ of $\mathbb{F}_q$ determines an identification of the Pontryagin dual $\hat{V}$ and the dual space $V^*$

$$V^* \longrightarrow \hat{V} \quad A \longrightarrow \chi_A = \chi \circ A.$$ (3.1.1)

Suppose $N \in \mathbb{N}_+$, $r \in \frac{1}{N}\mathbb{N}_+$, $E \subset \mathcal{B}(\mathfrak{g})_N$ and $F, \bar{F} \in \text{VSp}(E)$ are opposite with respect to $E$. The projection maps of the Iwahori decomposition

$$\mathcal{S}_{E,r}/\mathcal{S}_{E,r^+} \supset \mathcal{S}_{F,r}/\mathcal{S}_{F,r^+} \supset \mathcal{S}_{F,r^+}/\mathcal{S}_{E,r^+}$$

$$V^{r}_{E} \quad V^{r}_{E,F} \quad V^{r}_{F} \quad V^{r}_{E,F}$$ (3.1.2)

to the subspaces $V^{r}_{E,F}$, $V^{r}_{F}$ and $V^{r}_{E,F}$ give a natural identification of the dual spaces $(V^{r}_{E,F})^*$, $(V^{r}_{F})^*$, and $(V^{r}_{E,F})^*$ as subspaces of $(V^{r}_{E})^*$ so that

$$(V^{r}_{E})^* = (V^{r}_{E,F})^* \oplus (V^{r}_{F})^* \oplus (V^{r}_{E,F})^*.$$ (3.1.3)

Given $A \in (V^{r}_{E})^*$, we write its (3.1.3) decomposition as:

$$A = A_{E,F} + A_{F} + A_{E,F}.$$ (3.1.4)

Definition 3.1.5. Suppose $N \in \mathbb{N}_+$, $r \in \frac{1}{N}\mathbb{N}_+$, and $E \subset \mathcal{B}(\mathfrak{g})_N$. Define $A \in (V^{r}_{E})^*$ to be parabolically inflated from a pair $(F,B)$ consisting of a facet $F \in \text{VSp}(E) \cup \{E\}$ and $B \in V^{r}_{F}$, if:

(i) The restriction of the character $\chi_A \in \mathcal{S}_{E,r}/\mathcal{S}_{E,r^+}$ to $\mathcal{S}_{F,r}/\mathcal{S}_{F,r^+}$ is trivial; therefore $\chi_A$ is the inflation of a character of $\mathcal{S}_{E,r}/\mathcal{S}_{F,r^+}$.

(ii) The restriction of $\chi_A$ to $\mathcal{S}_{F,r}/\mathcal{S}_{F,r^+}(= V_F)$ is $\chi_B$.

If the facet $F$ belongs to VSp(E), the above is equivalent to the existence of a facet $\bar{F} \in \text{VSp}(E)$, which is $E$-opposite to $F$, so that the decomposition (3.1.4) of $\chi_A$, has $A_{E,F} = 0$, and $A_{F} = B$, i.e.,

$$A = B + A_{E,F} \quad \text{with} \quad A_{E,F} \in V^{r}_{E,F}.$$
We observe the property of being parabolically inflated is transitive, i.e., if \( E \subset F \subset H \) are facets of \( \mathcal{B}(G) \), and (i) \( A \in V^*_E \) is parabolically inflated from \( B \in V^*_F \) and (ii) \( B \) is parabolically inflated from \( C \in V^*_H \), then \( A \) is parabolically inflated from \( C \in V^*_H \).

**Definition 3.1.6.** Define \( A \in (V^*_E)^* \) to be non-cuspidal if there exists facet \( F \in \text{VSph}(E) \), i.e., \( F \supseteq E \), so that \( A \) is parabolically induced from \( V_F \), and cuspidal otherwise.

Equivalently, as a character on \( G_{E,r} \), \( A \) is cuspidal if for any facet \( F \) properly containing \( E \), the integral \( \int_{G_{F,r}^+} \chi_A(u) \, du \) vanishes.

**Remarks and Examples.**

(i) When \( N = 1 \), \( E \subset \mathcal{B}(G) \), and depth \( r \in \mathbb{N}_+ \), the quotient group \( G_{E,r}/G_{E,r}^+ \) is isomorphic with the finite field Lie algebra \( \text{Lie}(G_{E,0}/G_{E,0}^+) \) of the group \( G_{E,0}/G_{E,0}^+ \). The existence of a cuspidal character \( A \) is equivalent to existence on an element in \( \text{Lie}(G_{E,0}/G_{E,0}^+) \) which does not lie in any proper parabolic subalgebra.

(ii) As an example, take the group \( \text{SL}(3) \)

![Figure 1. A2 chamber and refined A2 chamber (N = 3)](image)

(ii.1) Take the point \( x_0 \) (\( \alpha(x_0) = \frac{1}{3} \), \( \beta(x_0) = \frac{1}{3} \)), and depth \( r = \frac{1}{3} \). Then:

\[
G_{x_0,0}/G_{x_0,0}^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G_{x_0, \frac{1}{3}}/G_{x_0, \frac{1}{3}}^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

with the boxed matrix entries indicating where the quotient is nontrivial. For the \( x_0 \)-opposite chambers \( C, C' \), the Iwahori decomposition of \( G_{x_0, \frac{1}{3}}/G_{x_0, \frac{1}{3}}^+ \) is:

\[
G_{x_0, \frac{1}{3}}/G_{x_0, \frac{1}{3}}^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \{0\} \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
For $x_0$-opposite edges $E, E'$, the Iwahori decomposition is:

$$G_{x_0,\frac{1}{2}}/G_{x_0,\frac{1}{2}^+} = \left[ \begin{array}{c} \Box \\ \Box \\ \Box \end{array} \right] \oplus \left[ \begin{array}{c} \Box \\ \Box \\ \Box \end{array} \right] \oplus \left[ \begin{array}{c} \Box \\ \Box \end{array} \right]$$

(ii.2) Take the facet $E$ ( $\alpha(x_0) = \frac{1}{3} < \frac{1}{3} < \frac{2}{3}$ ). Then:

$$G_{E,\frac{1}{2}}/G_{E,\frac{1}{2}^+} = \left[ \begin{array}{c} \Box \\ \Box \end{array} \right].$$

For $E$-opposite chambers $C, C''$, we have

$$G_{E,\frac{1}{2}}/G_{E,\frac{1}{2}^+} = \left[ \begin{array}{c} \Box \\ \Box \end{array} \right] \oplus \{0\} \oplus \left[ \begin{array}{c} \Box \\ \Box \end{array} \right].$$

Here, every $A \in V_E^+$ is non-cuspidal.

**Proposition 3.1.7.** Suppose $(E, B)$ is a cuspidal pair.

(i) Take any apartment $A$ containing $E$.

Suppose $\xi$ is a nonconstant affine function so that the hyperplane $H_{\xi}$ contains $E$;

i.e., $\text{grad}(\xi) \perp E$.

Then, there exists an affine root $\chi$ which have constant value $r$ on $E$, and

(i.1) $\langle \text{grad}(\xi), \text{grad}(\psi) \rangle > 0$,

(i.2) $\mathcal{X}_{(\psi-r)}/\mathcal{X}_{(\psi-r)^+}$ is non-zero and $\chi_B$ restricted to $\mathcal{X}_{(\psi-r)}$ is non-trivial.

Conversely, if $B \in (V_E^r)^*$ satisfies the above, then $(E, B)$ is a cuspidal pair.

(ii) Under the inclusion $V_E^r \hookrightarrow V_E^r \otimes_{\mathbb{F}_q} (\mathbb{F}_q) = \text{Hom}_{\mathbb{F}_q}(G_{E,r}^{\text{un}}/G_{E,0}^{\text{un}}, \mathbb{F}_q)$, the $G_{E,0}^{\text{un}}/G_{E,0}^{\text{un}+}$-orbit of $B$ in $\text{Hom}_{\mathbb{F}_q}(G_{E,r}^{\text{un}}/G_{E,0}^{\text{un}}, \mathbb{F}_q)$ is closed.

**Proof.** To prove assertion (i), in the given apartment $A$, set:

$$F := \left\{ \begin{array}{l} \text{the minimal facet containing } E \text{ so that for any interior point } y \\ \text{of } E, \text{ the point } y + t\text{grad}(\xi) \in F \text{ for } t > 0 \text{ sufficiently small}, \end{array} \right.$$  

$$\bar{F} := \left\{ \begin{array}{l} \text{the minimal facet containing } E \text{ so that for any interior point } y \\ \text{of } E, \text{ the point } y + t\text{grad}(\xi) \in F \text{ for } t < 0 \text{ sufficiently small}. \end{array} \right.$$  

The facets $F$ and $\bar{F}$ are $E$-opposite facets and provide an Iwahori decomposition of $V_{E}^{r}$ as in [3.1.3]. The cuspidality hypothesis on $(E, B)$ means $\chi_B$ is not trivial on $V_{E,F}$ (and $V_{E,\bar{F}}$ too). The existence of the affine root $\psi$ satisfying (i.1) and (1.2) follows.

To prove the converse, suppose $(E, B)$ satisfies the assumption of (i) (for any apartment $A$ that contains $E$). If $F$ is a facet properly containing $E$, let $\xi$ be an affine functional which is constant on $E$, and grad($\xi$) moves generically along $F$, i.e., for $x$ an interior point of $E$, then $x + t\text{grad}(\xi)$ is in the interior of $F$ for small positive $t$. Let $\psi$ be an affine root satisfying (i.1) and (i.2). The hypothesis (i.1) that $\langle \text{grad}(\xi), \text{grad}(\psi) \rangle > 0$ means $(\psi - r) > 0$ on $F$, so $\mathcal{X}_{(\psi-r)} \subset G_{F,r^+}$. That hypothesis (i.2) that $\chi_B$ restricted to $\mathcal{X}_{(\psi-r)}$ is nontrivial, means $\chi_B$ is...
not inflated from $G_{E,r}/G_{F,r}$. As this is true for all $F \supseteq E$, we deduce $(E, B)$ is a cuspidal pair.

Assertion (ii) is a consequence of the 2nd fundamental result proved in [K]. □

The following noncuspidality Lemma also follows from the 2nd fundamental result proved in [K].

**Lemma 3.1.8.** Suppose $E \subset B_N$, and $B \in V^*_E \mapsto V^*_E \otimes_{F_q} (\overline{F_q}) = \text{Hom}_{F_q}(G_{E,r}^{un}/G_{E,r}^{un}, \overline{F_q})$. If there exists a one-parameter multiplicative subgroup $\lambda$ of $G_{E,0}^{un}/G_{E,0}^{un}$ which centralizes $B$, and acts nontrivially on $V^*_E$, then the pair $(F, B)$ is not cuspidal.

**Proof.** The non-triviality of action of $\lambda$ on $V^*_E$ gives a nontrivial triangular decomposition $V^*_E = (V^*_E)^0 \oplus (V^*_E)^-$ where $(V^*_E)^0$ is the fixed points of $\lambda$ and $\lambda(t)$ acts as by positive powers of $t$ on $(V^*_E)^+$ and negative on $(V^*_E)^-$. But, this decomposition arises from a pair of opposite chambers containing $E$, namely, take a point $x$ in the interior of $E$ and move along the line through $x$ slightly in the direction $\lambda$ (in both positive and negative direction) and this will give the interior points of $F$ and its $E$-opposite. □

Fix $N \in \mathbb{N}_+$, and $r \in \frac{1}{N}\mathbb{N}_+$. For any pair $(E, A)$ of depth $r$, let $e_{E,A}$ denote the idempotent

$$e_{E,A} = \begin{cases} \frac{1}{\text{meas}(G_{E,r})} \chi_A & \text{on } G_{E,r} \\ 0 & \text{off } G_{E,r} \end{cases} \quad (3.1.9)$$

When the character $\chi$ is given, without the corresponding $A$, we shall write $e_{E,\chi}$.

**Proposition 3.1.10.** Given $A \in V^*_E$, there exists a cuspidal pair $(F, B)$ so that $A$ is parabolically inflated from the pair $(F, B)$. In such a situation,

$$e_{E,A} \ast e_{F,B} = e_{E,A}.$$

**Proof.** Use the transitivity of parabolic inflation, to deduce that there is a facet $F$ of maximal dimension so that $A$ is parabolically inflated from some $B \in V^*_F$. That $B$ is cuspidal is because of the maximality assumption on $\dim(F)$. The convolution formula is the orthogonality or characters $\hat{V}_E$.

□

3.2. Some convolutions. We now come to the main result of this section.

**Proposition 3.2.1.** Suppose $(E, A)$ and $(F, B)$ are two cuspidal pairs (of depth $r > 0$). If $e_{E,A} \ast e_{F,B} \neq 0$ then $E$ and $F$ are aligned and $A = B$.

**Proof.** Let $D_E$ and $D_F$ be the unique facets (see Lemma 2.3.6) of $C(E, F)$ of dimension $\dim(C(E, F))$, and which contain $E$ and $F$ respectively. Since $e_{E,A} \ast e_{F,B} \neq 0$, it follows that

$$e_{E,A} \ast e_{G_{F,r+}} \neq 0.$$
Assume that \( \dim(C(E, F)) > \dim(E) \). We have
\[
e_{E,A} \ast e_{g_{F,r^+}} = (e_{E,A} \ast e_{g_{E,r^+}}) \ast e_{g_{F,r^+}} = e_{E,A} \ast (e_{g_{E,r^+}} \ast e_{g_{F,r^+}})
= e_{E,A} \ast (e_{g_{DE,r^+}} \ast e_{g_{F,r^+}}) \quad \text{(by Proposition 2.6.2)}
= (e_{E,A} \ast e_{g_{DE,r^+}}) \ast e_{g_{F,r^+}}.
\]
Since the pair \((E, A)\) is cuspidal and \(E \subsetneq D_E\), we deduce the convolution \((e_{E,A} \ast e_{g_{DE,r^+}})\) is zero; so, \(e_{E,A} \ast e_{g_{F,r^+}}\) is zero, a contradiction. Replacing the roles of \(E\) and \(F\), we conclude that
\[
\dim(F) = \dim(C(E, F)) = \dim(E),
\]
that is, \(E\) and \(F\) are aligned. Thus the cosets of \(g_{E,r^+} \in G_E\) and of \(g_{F,r^+} \in G_F\) are represented by the same group elements, and it is easy to verify that
\[
e_{E,A} \ast e_{g_{F,r^+}} = e_{E,A} \ast e_{F,A}.
\]
Now
\[
e_{E,A} \ast e_{F,B} = e_{E,A} \ast (e_{g_{F,r^+}} \ast e_{F,B})
= (e_{E,A} \ast e_{g_{F,r^+}}) \ast e_{F,B}
= (e_{E,A} \ast e_{F,A}) \ast e_{F,B}
= e_{E,A} \ast (e_{F,A} \ast e_{F,B}).
\]
The convolution \(e_{F,A} \ast e_{F,B}\) is nonzero precisely when \(A = B\).

\[\square\]

**Corollary 3.2.2.** Assume that \((E_1, A_1)\) and \((F_1, B_1)\) are parabolically inflated from cuspidal pairs \((E, A)\) and \((F, B)\), respectively. If \(e_{E_1,A_1} \ast e_{F_1,B_1} \neq 0\) then \(E\) and \(F\) are aligned and \(A = B\). In particular, this is true if \((E_1, A_1) = (F_1, B_1)\).

**Proof.** By Proposition 3.1.10 and associativity of convolution,
\[
e_{E_1,A_1} \ast e_{F_1,B_1} = (e_{E_1,A_1} \ast e_{E,A}) \ast (e_{F,B} \ast e_{F_1,B_1}) = e_{E_1,A_1} \ast (e_{E,A} \ast e_{F,B}) \ast e_{F_1,B_1}.
\]
Thus \(e_{E,A} \ast e_{F,B} \neq 0\). The corollary follows from Proposition 3.2.1.
Proposition 3.3.1.
(i) Suppose $F_1$, $F_2$ and $F_3$ are three facets with $F_1$ and $F_2$ associate to $F_3$. Then, there exists $r, s \in \mathcal{S}$ so that $F_1$, $r.F_2$ and $s.F_3$ lie in an apartment $\mathcal{A}$, and each of these facets generates the same affine subspace of $\mathcal{A}$.

(ii) Associativity is an equivalence relation on the facets of $\mathcal{B}_N$.

Proof. To prove assertion (i), take $g, h \in \mathcal{S}$ so that $F_1$ and $g.F_2$ are aligned, and $F_2$ and $h.F_3$ are aligned (and thus $g.F_2$ and $(gh).F_3$ are aligned). Let $\mathcal{A}$ be an apartment containing $F_1$ and $g.F_2$, and let $\mathcal{A}'$ be an apartment containing $g.F_2$ and $(gh).F_3$. Take $k \in \mathcal{S}_{g.F_2,0}$ so that $k.\mathcal{A}' = \mathcal{A}$. Then $F_1$, $g.F_2$ and $k.(gh).F_3$ are aligned in $\mathcal{A}$.

Assertion (ii) obviously follows from assertion (i).

□

Fix $N \in \mathbb{N}_+$ and $r \in \frac{1}{N} \mathbb{N}_+$. Define two cuspidal pairs $(E_1, A_1)$ and $(E_2, A_2)$ of depth $r$ to be associate if:

(i) The facets $E_1$ and $E_2$ are associate, i.e., there exists $g \in \mathcal{S}$ so that $g.E_1$, $E_2$ are aligned – so, $\mathcal{S}_{g.E_1,r}/\mathcal{S}_{g.E_1,r+} = \mathcal{S}_{E_2,r}/\mathcal{S}_{E_2,r+}$, and $Ad(g)$ provides an isomorphism between $\mathcal{S}_{g.E_1,r}/\mathcal{S}_{g.E_1,r}$ and $\mathcal{S}_{E_1,r}/\mathcal{S}_{g.E_1,r+}$.

(ii) For an element $g$ as in part (i), there exists $j \in \mathcal{S}_{E_2,0}$ so that, as characters, $Ad(j)E_2 = Ad(g)(E_1)$.

Example. Take $\mathcal{S} = SL(2, k)$, and let $\mathcal{S}$ be the diagonal torus. For $r \in \mathbb{N}_+$, the groups $\mathcal{S}_{x,r}$ and $\mathcal{S}_{x,r+}$ are constant in the interior of the chambers of $\mathcal{A}(\mathcal{S})$. If $x$ an interior point of such a chamber, then the quotient group $\mathcal{S}_{x,r}/\mathcal{S}_{x,r+}$ is supported on the diagonal elements and is isomorphic to $\mathbb{F}_q^*$. Fix a chamber $F \subset \mathcal{A}(\mathcal{S})$. Then, with $A \in \mathbb{F}_q^*$ the cuspidal pairs $(F, A)$ and $(F, -A)$ are associate. Note that for $p \neq 2$, and $A \neq 0$, the characters $A$ and $-A$ are not in the same $Ad(\mathcal{S}_{x,0})$-orbit in $\mathcal{S}_{x,r}/\mathcal{S}_{x,r+}$.

Proposition 3.3.2. Associativity of cuspidal pairs is an equivalence relation.

Proof. It suffices to prove transitivity. Suppose $(F_2, A_2)$ is associate to $(F_1, A_1)$ and $(F_3, A_3)$. We can assume:

- $F_1$ and $g.F_2$ are aligned, and $k \in (\mathcal{S}_{F_1,0} \cap \mathcal{S}_{g.F_2,0})$ satisfies $A_1 = Ad(k)(Ad(g)A_2)$.
- $F_2$ and $h.F_3$ are aligned, and $Ad(g)A_3 = Ad(\ell)(Ad(gh)A_3)$ with $\ell \in (\mathcal{S}_{g.F_2,0} \cap \mathcal{S}_{gh,F_3})$.

As in the proof of Proposition 3.3.1, take an apartment containing $g.F_2$ and $g.(h.F_3)$, and a $t$ translation of the apartment so that the facets $F_1$, $g.F_2$ and $t(gh).F_3$ are simultaneously aligned. Then, the above elements $k$ and $\ell$ can be assume to be in the intersection $(\mathcal{S}_{F_1,0} \cap \mathcal{S}_{g.F_2,0} \cap \mathcal{S}_{t(gh),F_3})$, and we then deduce $(F_1, A_1)$ and $(F_3, A_3)$ are associate.

☐

Set $\mathcal{C} = \mathcal{C}(E, A)$ to be the equivalence class of cuspidal pairs associate to a given cuspidal pair $(E, A)$. The set $\mathcal{C}$ is trivially $\mathcal{S}$-equivariant, i.e., for any $(F, B) \in \mathcal{C}$, and $g \in \mathcal{S}$, then $(gF, Ad(g)B)$ belongs to $\mathcal{C}$ too, and obviously $\mathcal{C}$ has an attached equivalence class of associate facets $\mathcal{F}(\mathcal{C})$, namely the associativity class of $E$. We call $\mathcal{F}(\mathcal{C})$ the support of $\mathcal{C}$.
For $E' \in \mathcal{F}(C)$, we define the idempotent:

$$e_{E'}^C := \sum_{(E', B) \in \mathcal{C}} e_{E', B}.$$  \hfill (3.3.3)

This is a system of $\mathcal{G}$-equivariant idempotents supported on the facets $\mathcal{F}(C)$.

Define the facet closure $\overline{\mathcal{F}(C)}$ of $\mathcal{F}(C)$ to be

$$\overline{\mathcal{F}(C)} := \{ D \text{ a (refined) facet} \mid D \text{ is a subfacet of a facet in } \mathcal{F}(C) \}.$$  \hfill (3.3.4)

If $D \in \overline{\mathcal{F}(C)}$, set

$$\text{Star}_C(D) := \{ \text{ facets } F \subset \mathcal{F}(C) \mid D \subset F \} ,$$  \hfill (3.3.5)

and

$$e_D^C := \sum_{\chi \in \chi(D)} e_{D, \chi} \quad \text{(an idempotent).}$$  \hfill (3.3.6)

Note that if the facet $D$ is a facet of $\mathcal{F}(C)$, then the above definition agrees with $e_D^C$ as defined in (3.3.3).

For any $D \in \overline{\mathcal{F}(C)}$ and $F \in \text{Star}_C(D)$, then trivially,

$$e_F^C * e_D^C = e_D^C * e_F^C .$$

More generally, we have:

**Proposition 3.3.7.** Suppose $D, E \in \overline{\mathcal{F}(C)}$, with $D \subset E$. Then

$$e_E^C * e_D^C = e_E^C = e_D^C * e_E^C .$$

**Proof.** The assertion follows from the property that $D \subset E$ means

$$\text{Star}_C(D) \supset \text{Star}_C(E) .$$

As a complement, we have:

**Proposition 3.3.8.** Suppose $E, F \in \overline{\mathcal{F}(C)}$, with $E \cap F \neq \emptyset$. Then

$$e_E^C * e_F^C = e_F^C * e_E^C .$$

**Proof.** The assertion follows from the fact that the idempotents $e_E^C$ and $e_F^C$ factor to functions on the abelian quotient group $\mathcal{G}_{(E \cap F), r} / \mathcal{G}_{(E \cap F), r^+}$.
We fix $N \in \mathbb{N}_+$, and $r \in \frac{1}{N}\mathbb{N}_+$. Suppose $C$ is a chamber in $\mathcal{B}_N$, and $K$ is a facet of $C$. Define
\begin{equation}
\text{VSp}_C(K) := \text{set of subfacets of } C \text{ which contain } K. \tag{4.1}
\end{equation}

Let
\begin{equation}
\Delta_C(K) := \{ F \in \text{VSp}_C(K) \mid \dim(F) = (\dim(C) - 1) \} \subset \text{VSp}_C(K) \tag{4.2}
\end{equation}
of facets of codimension one. We have a bijection between subsets of $\Delta_C(K)$ and $\text{VSp}_C(K)$ given by
\begin{equation*}
Q = \{ F_{j_1}, \ldots, F_{j_s} \} \subset \Delta_C(K) \longleftrightarrow \bigcap_{F_{j_i} \in Q} F_{j_i} \in \text{VSp}_C(K),
\end{equation*}
with the convention that when $Q = \emptyset$, the empty intersection is the chamber $C$.

**Proposition 4.3.** Suppose $N \in \mathbb{N}_+$, $r \in \frac{1}{N}\mathbb{N}_+$. Suppose $C$ is a chamber of $\mathcal{B}_N$, and $J \not\subset K$ are facets of $C$. Then
\begin{enumerate}[(i)]
\item \[ \left( \sum_{F \in \text{VSp}_C(K)} (-1)^{\dim(F)} e_{g_{F,r^+}} \right) \ast e_{g_{J,r^+}} = 0. \]
\item Suppose $C = \{(E, A)\}$ is an equivalence class of cuspidal pairs of depth $r$.
\[ \left( \sum_{F \in \text{VSp}_C(K)} (-1)^{\dim(F)} e_F^C \right) \ast e_{g_{J,r^+}} = 0. \]
\end{enumerate}

**Proof.** We observe that any two facets $E_1, E_2$ of $C$, the convolution $e_{g_{E_1,r^+}} \ast e_{g_{E_2,r^+}}$ equals $e_{g_{C(E_1,E_2),r^+}}$, where $C(E_1, E_2)$ is the convex closure of $E_1$ and $E_2$. We deduce from this observation, that we can replace $J$ with the facet $C(J,K) \supseteq K$ (equality is not possible since $J \not\subset K$). So we assume $J \supseteq K$. This assumption means we have a map
\begin{equation}
\text{VSp}_C(K) \longrightarrow \text{VSp}_C(J)
E \longrightarrow C(E,J). \tag{4.4}
\end{equation}
As above, let $\Delta_C(K) := \{ F_1, \ldots, F_k \}$ be the facets of dimension $(\dim(C) - 1)$. We assume the numbering of the facet is so that the intersection $(F_1 \cap \cdots \cap F_j)$ equals $J$, i.e., $\{ F_1, \ldots, F_j \}$ equals $\Delta_C(J)$. Then, the map of (4.4) is
\begin{equation*}
Q \subset \Delta_C(K) \longrightarrow (Q \cap \Delta_C(J)) \subset \Delta_C(J),
\end{equation*}
from which see immediately see that the fiber of a subset $Q' \subset \Delta'_j$ consists of the $2^{(k-j)}$ subsets
\begin{equation*}
Q' \cup X, \text{ where } X \text{ is a subset of } \Delta_C(K) \setminus \Delta_C(J).
\end{equation*}
Since $k > j$ (the assumption $J \supseteq K$), the convolution sum over each fiber of (4.4) is obviously 0, and assertion (i) follows.

The proof of assertion (ii) is similar to that of assertion (i). We use the fact that
\begin{equation*}
e_F^C \ast e_{g_{J,r^+}} = e_F^C e_{C(F,J)}.
\end{equation*}
5. A Key Proposition

We fix a maximal $k$-split torus $S = S(k) \subset G(k)$. Let $\Phi$ denote the set of roots of $S$. Given a simple set of roots $\Delta \subset \Phi$, set:

- $\Phi^+_\Delta := \text{set of positive for } \Delta$
- $\mathcal{A}_\Delta := \text{positive Weyl chamber for } \Delta$

Let $\mathcal{B}(G)_N$ be the refined Bruhat–Tits building of $G$ and $\mathcal{A} = \mathcal{A}(S)$ the apartment of the maximal split torus $S$. Let $r \in \mathbb{N} \cap \mathbb{Z}$ and $K$ a (refined) facet in $\mathcal{A}$. Recall that $\Phi_K$ is the set of gradients of affine roots $\psi$ such that $\psi = r$ on $K$. We record the following well known lemma.

**Lemma 5.1.** Set $\ell = \dim(\mathcal{A}) (= \text{rank}(G))$. Let $C \subset \mathcal{A}$ be a (refined) chamber and let $\mathcal{F}(C) = \{ F_0, F_1, \ldots, F_\ell \}$ be the faces of $C$. For any nonempty proper subset $\mathcal{F}_K \subsetneq \mathcal{F}(C)$, set:

- $K := \bigcap_{F \in \mathcal{F}_K} F$.
- For $F \in \mathcal{F}_K$, let $\alpha_F$ be the roots of $S$ so that $\alpha_F \perp F$, and points outward from $C$.

Then the set $\Delta_K := \{ \alpha_F \mid F \in \mathcal{F}_K \}$ is a set of simple root for the root system $\Phi_K$.

**Proof.** Elementary. The chamber $C$ maps to a Weyl chamber of $G_{K,0}/G_{K,0}^+$. 

Fix a chamber $C_0 \subset \mathcal{A}$, and let $\text{ht}_{C_0}$ be the Bruhat height/length function (with respect to $C_0$) on the chambers of $\mathcal{B}$. For a chamber $D \neq C_0$, we recall the faces $\mathcal{F}(D)$ of $D$ can be partitioned into two nonempty sets:

- $c_{C_0}(D)$ (children of $D$) := chambers adjacent to $D$ and of length $(\text{ht}_{C_0}(D) + 1)$
- $p_{C_0}(D)$ (parents of $D$) := chambers adjacent to $D$ and of length $(\text{ht}_{C_0}(D) - 1)$.

For a chamber $D \subset \mathcal{A}$ and $F$ a face of $D$, denote by $s_F(D)$, the chamber obtained by reflecting (in $\mathcal{A}$) the chamber $D$ across the face $F$, and define:

$$\alpha_F := \begin{cases} 
\text{to be the root which is perpendicular to } F \text{ and which points } & \\
\text{inwards to (resp. outwards from) } D \text{ if } \text{ht}_{C_0}(s_F(D)) = (\text{ht}_{C_0}(D) - 1) & \text{(5.3)}
\end{cases}$$

Note that the notation $\alpha_F$ suppressed the dependence on the base affine chamber $C_0$ and apartment $\mathcal{A}$.

Let $\mathcal{A} \subset \mathcal{B}$ be as Lemma 5.1. To a set of simple roots $\Delta$ of $S$, and a chamber $C_0 \subset \mathcal{A}$, there is an associated sector $S(C_0, \Delta)$ in $\mathcal{A}$ whose definition we recall as:
\begin{equation}
S(C_0, \Delta) := \left\{ \begin{array}{ll}
\text{smallest union of affine chambers so that if } x \in S(C_0, \Delta), \text{ and } \\
v \text{ is in the positive Weyl chamber of the simple roots } \Delta, \text{ then } (x + v) \in S(C_0, \Delta) \text{ too. }
\end{array} \right.
\end{equation}

The apartment \( \mathcal{A} \) is the union of the sectors \( S(C_0, \Delta) \) as \( \Delta \) runs over the possible sets of simple roots. See Figure 2 for the example of C2 (where the eight sectors are color coded).

A chamber \( D \) belongs to \( S(C_0, \Delta) \) if there is a sequence of chambers \( C_0, C_1, \ldots C_r = D \) with \( i = \text{ht}_{C_0}(C_i) \) so that \( C_{i+1} \) and \( C_i \) share a face \( F \) so that \( \alpha_F \) is positive for \( \Delta \).

**Proposition 5.5.** Let \( \mathcal{A} \subset \mathcal{B} = \mathcal{B}(\mathcal{S}) \) be as Lemma 5.1. For any chamber \( D \neq C_0 \) in \( \mathcal{A} \), there exists a set (possibly more than one) of simple roots \( \Delta \) of \( \mathcal{S} \) so that:

\begin{enumerate}[(i)]
\item For every face \( F \subset D \), the root \( \alpha_F \) belongs to \( \Phi^+_\Delta \).
\item \( D \) is contained in \( (C_0 + A_\Delta) \subset S(C_0, \Delta) \) (\( A_\Delta \) is the positive Weyl chamber of \( \Delta \)).
\item For every \( \alpha \in \Delta \), there exists a face \( F \subset D \) so that:
   \begin{itemize}
   \item \( \alpha_F \) is outwards,
   \item the root \( \alpha \) appears in the expression of \( \alpha_F \) in terms of \( \Delta \).
   \end{itemize}
\end{enumerate}

**Proof.** Pick a special vertex \( x_0 \) of \( C_0 \) to be the origin. There exists a unique set of simple roots \( \Pi \subset \Phi \) such that \( D \subset A_\Pi \). Suppose \( F \) is a face of \( D \):

- If \( F \) is not contained in the boundary of \( A_\Pi \), let \( x \) be an interior point of \( F \). It easily follows, from the definition of \( \alpha_F \), that \( \alpha_F(x) > 0 \) hence \( \alpha_F \in \Phi^+_\Pi \).
- If \( F \) is contained in the boundary of \( A_\Pi \), then (since the boundary walls of \( A_\pi \) are contained in simple root hyperplanes), either \( \alpha_F \) or \( -\alpha_F \) is in \( \Pi \), and the boundary hyperplane containing \( F \) is \( \alpha_F = 0 \). If \( D \) and \( C_0 \) are on the different sides of the hyperplane \( \alpha_F = 0 \), then \( \alpha_F \in \Pi \), in particular, it is positive. Otherwise \( -\alpha_F \in \Pi \).

Let
\[ \Theta := \left\{ \begin{array}{ll}
\text{the set of faces } F \text{ of } D \text{ contained in the boundary of } A_\Pi \text{ such that } D \text{ and } C_0 \text{ are on the same side of the hyperplane } \alpha_F = 0 .
\end{array} \right. \]

We identify \( \Theta \) with a set of simple roots \( \Theta \subset \Pi \), by the map \( F \mapsto -\alpha_F \). Summarizing, for every face \( F \) of \( D \), either \( \alpha_F \in \Phi^+_\Pi \) or \( -\alpha_F \in \Theta \), and every element of \( -\Theta \) occurs as \( \alpha_F \).

**Claim:** if \( \alpha_F \in \Phi^+_\Pi \) then it is not a linear combination of roots in \( \Theta \).

**Proof of claim:** Since \( D \neq C_0 \), \( \Theta \) is a proper subset of \( \Pi \). Since \( D \) is a simplex no proper subset of \( \alpha_F \) is linearly dependent. The claim now follows since \( |\Theta| < |\Pi| \) and the number of faces of \( D \) is \( |\Pi| + 1 \).

Summarizing again, the set of all \( \alpha_F \) is a union of \( -\Theta \) and a subset of \( \Phi^+_\Pi \setminus \Phi^+_\Theta \), where \( \Phi_\Theta \) denotes the root subsystem spanned by \( \Theta \). Let \( W_\Theta \) be the Weyl group of this root system, and let \( w_\Theta \in W_\Theta \) be the longest element. We observe that \( w_\Theta \) has order 2 so we can write \( w_\Theta \) at places where naturally one should have \( w_\Theta^{-1} \).

We are now ready to prove the three statements of the Proposition. Define
\[ \Delta := w_\Theta(\Pi) . \]
Since $w_\Theta(\Theta) = -\Theta$ and $w_\Theta$ permutes $\Phi^+_\Pi \setminus \Phi^+_\Theta$, it follows that
\[
\Phi^+_\Delta = (\Phi^+_\Pi \setminus \Phi^+_\Theta) \cup \Phi^+_{-\Theta}
\]
hence all $\alpha_F \in \Phi^+_\Delta$, so statement (i) holds.

We now show statement (ii), $D$ is contained in $S(C_0, \Delta)$. To this end, we consider the facet $E$ of $D$ defined by
\[
E = \bigcap_{F \in \Theta} F.
\]
and observe that $w_\Theta$ fixes $E$. Let $y$ be an interior point in $E$. Since $w_\Theta(y) = y$, it follows that $y \in A_\Delta$. Since $C_0$ is on the same side as $D$ with respect to hyperplanes spanned by faces $F \in \Theta$, it follows that $x + y$ is in the interior of $D$ for all $x$ in the interior of $C_0$, sufficiently close to $x_0$, so (ii) holds.

It remains to show statement (iii), for every $\alpha \in \Delta$ there exists an outward $\alpha_F$ such that $\alpha$ appears in the expression of $\alpha_F$ in terms of $\Delta$. If $\alpha \in -\Theta$, then $\alpha = \alpha_F$, for some face $F$, and $\alpha_F$ is outward by the definition of $\Theta$. Now assume that $\alpha \notin -\Theta$. Let $v \in D$ be the (unique) point maximizing the distance from $x_0$. Let $E$ be the facet of $D$ containing $v$ as the interior point. We identify $v$ with (half of) the gradient of the distance function at $v$. Then $v$ is perpendicular to $E$, and $v$ can be uniquely written down as a linear combination of vectors perpendicular to faces $F$ containing $v$ and pointing out of $D$. Thus,
\[
v = \sum_{F \supseteq E} x_F \alpha_F
\]
where $x_F > 0$ and $x_F < 0$ if $\alpha_F$ is outward and inward, respectively. Let $\lambda$ be the fundamental co-weight corresponding to $\alpha$, that is, $\lambda(\alpha) = 1$ and $\lambda(\beta) = 0$ for all other $\beta \in \Delta$. If $\lambda(v) > 0$ then
\[
\lambda(v) = \sum_{F \supseteq E} x_F \lambda(\alpha_F) > 0
\]
so there must be an outward $\alpha_F$ containing $\alpha$ in its linear expansion in terms of $\Delta$. It remains to prove that $\lambda(v) > 0$. Note that $v \in A_{\Pi}$, since $D \subset A_{\Pi}$. Observe that $A_{\Pi} \setminus 0$ is contained in the interior of the cone dual to $A_{\Pi}$. Since $A_{\Pi}$ is spanned by fundamental co-weights, one of which is $w_\Theta(\lambda)$, it follows that $w_\Theta(\lambda)(v) > 0$. Hence
\[
0 < w_\Theta(\lambda)(v) = \lambda(w_\Theta(v)) = \lambda(v)
\]
where the second equality follows from the fact that $v - w_\Theta(v)$ is a linear combination of roots in $\Theta$, so perpendicular to $\lambda$, since $\alpha \notin -\Theta$.

\[\square\]

**Example.** Figure 2 illustrates Proposition 5.5 for C2. For the shown chamber $C_0$ the eight sectors $S(C, \Delta)$ are color coded with an indication of the positive roots in the circles. Chambers of a single color have a unique set of simple roots $\Delta$ which satisfy the conditions of Proposition 5.5. Chambers with a blend of two colors have two possible adjacent set of simple roots satisfying Proposition 5.5.
6. Essential Compactness of Euler-Poincaré Sums

6.1. Assumptions.

In the following, we assume:

- $k$ is a non-archimedean local field.
- $G = G(k)$ is the group of $k$-rational points of an absolutely quasisimple split linear algebraic group defined over $k$.
- $B = B(G)$ is the Bruhat–Tits building of $G$. The assumption $G$ is quasisimple means the chambers of $B$ are simplicies.
- For $N \in \mathbb{N}$, let $B_N$ denote the refined building, and suppose $r \in \frac{1}{N} \mathbb{N}_+$. 
6.2. Bruhat length/height on a refined building.

The fixture of a chamber \( C_0 \subset \mathcal{B}_N \), leads to an elementary and obvious generalization of the Bruhat length/height function on the chambers of \( \mathcal{B} \) to a (generalized) Bruhat length/height function \( \text{ht}_{C_0} \) on the chambers of \( \mathcal{B}_N \). As in [BCM:§2], if \( \mathcal{A} = \mathcal{A}(\mathfrak{g}) \) is an apartment containing \( C_0 \), and \( \pm \alpha \in \Phi(\mathfrak{g}) \) is a pair of opposite roots, we define

\[
\text{ht}_{C_0}^{\pm \alpha}(D) := \left\{ \text{the number of refined affine hyperplanes } H_\psi \text{ perpendicular to } \pm \alpha \text{ and separating } C_0 \text{ and } D \right\}.
\]

Then, \( \text{ht}_{C_0}(D) \) is the sum of the \( \text{ht}_{C_0}^{\pm \alpha}(D) \)'s over all pairs of opposite roots of \( \Phi(\mathfrak{g}) \). For \( m \in \mathbb{N} \), define, as in [BCM:§2], the ball of radius \( m \):

\[
\text{Ball}(C_0, m) := \left\{ D \text{ chamber of } \mathcal{B}_N \mid \text{ht}_{C_0}(D) \leq m \right\}.
\]

Suppose \( D \) is a chamber with \( \text{ht}_{C_0}(D) = m \). We recall (see [BCM:§2]):

- A face \( F \) of \( D \) is called inward with respect to the base chamber \( C_0 \) if \( F \) is a subfacet of \( \text{Ball}(C_0, (m - 1)) \). A face \( F \) of \( D \) which is not inward is called outward.

- Set

\[
\begin{align*}
\mathcal{C}_0(D) &:= \text{set of outward (child) faces of } D, \\
\mathcal{P}_0(D) &:= \text{set of inward (parent) faces of } D.
\end{align*}
\]

and define the facet \( \mathcal{E}_{C_0}(D) \) as:

\[
\mathcal{E}_{C_0}(D) := \bigcap_{F \in \mathcal{C}_0(D)} F.
\]

Then (see [BCM:§2.2]), the subsets of \( \mathcal{C}_0(D) \) parametrize the facets of \( D \) not in \( \text{Ball}(C_0, (m - 1)) \) as follows:

\[
T \subset \mathcal{C}_0(D) \quad \longleftrightarrow \quad \text{Facet}(T) := \bigcap_{F \in T} F.
\]

The empty intersection is the chamber \( D \) itself. These facets of \( D \) are also precisely the facets of \( D \) containing \( \mathcal{E}_{C_0}(D) \).

6.3. Convolution.

**Lemma 6.3.1.** Under the assumptions of subsection [6.1], fix a (base) chamber \( C_0 \subset \mathcal{B}_N \), and an open compact subgroup \( J \subset \mathcal{G} \). Suppose \( \mathcal{C} = \{(E, A)\} \) is an associate class of cuspidal pairs, and \( e_\mathcal{C}^J \) is the corresponding \( \mathcal{G} \)-equivariant system of idempotents defined in (3.3.6). Then, there exists \( L_0 \in \mathbb{N}_+ \) (dependent on \( C_0 \) and \( J \)) so that for any chamber \( D \), with \( \text{ht}_{C_0}(D) \geq L_0 \), the Euler-Poincaré convolution

\[
e_J * \left( \sum_{\mathcal{E}_{C_0}(D) \subset K \subset D} (-1)^{\dim(K)} e_\mathcal{C}^D_K \right)
\]

vanishes.

**Proof.** Let \( r \in \frac{1}{N} \mathbb{N}_+ \) be the depth of the associate class \( \mathcal{C} \). As preliminary reduction, we choose \( s \in \frac{1}{N} \mathbb{N}_+ \) so that both \( s \geq r \) and \( \mathcal{G}_{C_0,s} \subset J \). If suffices to show the assertion with \( J \) replaced by \( \mathcal{G}_{C_0,s} \).
We remark that because the Iwahori subgroup $G_{C_0}$ acts transitively on the apartments containing $C_0$ it is sufficient to fix an $A = \mathcal{A}(S)$ containing $C_0$, and establish the assertion of the Lemma for chambers in $A$ sufficiently far from $C_0$. So, we fix $A = \mathcal{A}(S) \supset C_0$.

Suppose $D \neq C_0$ is a chamber of $A$. By Proposition 5.5 there is a simple set of roots $\Delta \subset \Phi(S)$ so that

$$D \subset C_0 + A_\Delta \ (A_\Delta \text{ the positive Weyl chamber } A_\Delta \text{ of } \Delta),$$

and

(i) For each face $F \subset D$, the root $\alpha_F$ (notation of Proposition 5.5) belongs to the positive roots $\Phi_+(S)$.

(ii) For any simple root $\alpha \in \Delta$, there exists an outwards face $F \subset D$ (i.e., $\alpha_F$ is outwards from $D$), so that the root $\alpha$ appears in the expression of $\alpha_F$ in terms of $\Delta$.

For any choice of a set of simple roots $\Delta \subset \Phi(S)$, we show the vanishing assertion of the Lemma is true for a $D \subset C_0 + A_\Delta$ satisfying (i), (ii) and $ht_{C_0}(D)$ sufficiently large.

For any chamber $D(\neq C_0) \subset A$, define, as in (6.2.4):

$$E := E_{C_0}(D) = F_0 \cap F_1 \cap \ldots \cap F_l,$$

the intersection of all outward faces $F_i$ of $D$. Let $\Phi_E$ be the set of gradients of affine roots $\psi$ such that $\psi = r$ on $E$. The set $\Phi_E$ is a root system, is independent of $r \in \frac{1}{L} \mathbb{Z}$, and

$$\{\alpha_{F_0}, \alpha_{F_1}, \ldots, \alpha_{F_l}\}$$

is a set of simple roots for $\Phi_E$. Thus, if $\psi$ is an affine root constant on $E$, then the gradient of $\psi$ is a sum $\sum_i x_i \alpha_{F_i}$ where all $x_i$ are either non-negative or non-positive integers. Let $F \supset E$ be the intersection of all outward faces with $F_0$ removed. (If $F_0$ is the only outward face then we set $F = D$.)

We make two remarks:

- If $\psi$ is an affine root such that $\psi = r$ on $E$ and $\psi > r$ on $F$ then the gradient of $\psi$ is $\sum_i x_i \alpha_{F_i}$ where $x_0 < 0$ and $x_i \leq 0$ for all $i$. To see this, let $\delta$ be a translation of $A$ such that $\alpha_{F_i}(\delta) = 1$ and $\alpha_{F_i}(\delta) = 0$ for $i \neq 0$. Let $x$ be an interior point of $E$. Since $\alpha_{F_0}$ is outward, the ray starting at $x$ in the direction of $-\delta$ passes through the interior of $F$. Since $\psi > r$ on $F$, it follows that $x_0 < 0$, and $x_i \leq 0$ for all $i$, since all coefficients have the same sign.

- Given $a \geq 0$, there exists $L_a \geq 0$ such that: if $D$ is a chamber in $C_0 + A_\Delta$ and $ht_{C_0}(D) \geq L_a$ then there exists a simple root $\beta \in \Delta$ such that $\beta(x - y) > a$ for all $y \in C_0$ and $x \in D$. To see this, consider the distance on $A$ defined by

$$d(x, y) = \max_{\alpha \in \Delta} |\alpha(x - y)|,$$

where $x, y \in A$. Now observe that there exists $\epsilon > 0$ such that $\alpha(x - y) > -\epsilon$ for all $x \in C_0$, $y \in C_0 + A_\Delta$ and all $\alpha \in \Delta$. Let $b = \max(a, \epsilon)$. Observe that a bounded set in $A$ contains only finitely many chambers. Thus if $L_a$ is large enough then for all $D$ such that $ht_{C_0}(D) \geq L_a$ we have $d(x, y) > b$ for all $x \in D$ and all $y \in C_0$. If $D$ is contained in $C_0 + A_\Delta$ this simply means that there exists $\beta \in \Delta$ such that $\beta(x - y) > b \geq a$. 
We continue the proof of the Lemma under our stated assumptions on the chamber $D$ (that $D \subset C_0 + A_\Delta$ satisfies hypotheses (i), (ii)). Let $a = (s - r) + \lambda h$, where $h$ is the height of the highest root, and $\epsilon$ as in the second bullet above. Then, by the same bullet, there exists $L_a$ be such that if $ht_{C_0}(D) \geq L_a$, then there exists $\beta \in \Delta$ so that $\beta(x) - \beta(y) \geq a$ for all $y \in C_0$ and $x \in D$.

By hypothesis (ii), there exists an outward face $F_0$ of $D$ so that when $\alpha_{F_0}$ is written as an (integer) sum of simple roots (in $\Delta$), the coefficient of $\beta$ is $\geq 1$. Let $F$ be the intersection of the outward faces of $D$ different from $F_0$. (So $F \supset E := E_{C_0}(D)$.) We claim that

$$e_{g_{C_0,s}} * e_{g_{E,r^+}} = e_{g_{C_0,s}} * e_{g_{F,r^+}} .$$

Let $\psi$ be a virtual affine root $\psi > r$ on $F$ but $\psi = r$ on $E$. To prove the identity it suffices to show that $X_\psi \subset G_{C_0,s}$ for every such $\psi$. By the first bullet above, the gradient of $\psi$ is a sum $\sum_i x_i \alpha_{F_i}$ where $x_i \leq 0$ and $x_0 < 0$. Since all $\alpha_{F_i}$ are positive roots, the gradient of $\psi$ is a negative root containing a non-zero (integer) multiple of $\beta$ in its expression as a sum of roots in $\Delta$. Recall that if $x \in F$ and $y \in C_0$, then $\beta(x - y) > a$ and $\alpha(x - y) > -\epsilon$ for all other $\alpha \in \Delta$. It follows that $\psi(y - x) > s - r$. Thus

$$\psi(y) = \psi(x) + \psi(y - x) > s,$$

hence $X_\psi \subset G_{C_0,s}$ which proves the identity. We apply it (and associativity of convolution) to obtain

$$e_{g_{C_0,s}} * \left( \sum_{E \subset K \subset D} (-1)^{\dim(K)} e_{K}^c \right) = e_{g_{C_0,s}} * \left( e_{g_{E,r^+}} * \left( \sum_{E \subset K \subset D} (-1)^{\dim(K)} e_{K}^c \right) \right)$$

$$= e_{g_{C_0,s}} * \left( e_{g_{F,r^+}} * \left( \sum_{E \subset K \subset D} (-1)^{\dim(K)} e_{K}^c \right) \right) .$$

By Proposition 4.3 the inner convolution of the last line vanishes.

**Corollary 6.3.2.** Suppose $f \in C^\infty_c(\mathcal{G})$. Under the assumptions of Lemma 6.3.1, the convolution sums

$$f * \left( \sum_{K \subset \text{Ball}(C_0,m)} (-1)^{\dim(F)} e_{K} \right)$$

stabilize, i.e., are constant as a function of $m$, for $m$ sufficiently large.

**Proof.** The difference in the Euler-Poincaré convolution over the ball Ball($C_0, (m + 1)$) and the ball Ball($C_0, m$) is the sum of over the chambers $D \subset (\text{Ball}(C_0, (m + 1)) \setminus \text{Ball}(C_0, m))$ of the Euler-Poincaré convolutions:

$$e_{g_{C_0,s}} * \left( \sum_{E_{C_0}(D) \subset K \subset D} (-1)^{\dim(K)} e_{K}^c \right) ,$$

which vanish by the Lemma 6.3.1.
Proposition 6.3.3. Under the assumptions of Corollary (6.3.2), suppose \( C_1 \) is a chamber of \( \mathcal{B}_N \) which is adjacent to \( C_1 \), i.e., \( C_0 \) and \( C_1 \) share a common face. Then, for \( m \) sufficiently large:

\[
f * \left( \sum_{K \subset \text{Ball}(C_0,m)} (-1)^{\dim(K)} e_K \right) = f * \left( \sum_{K \subset \text{Ball}(C_1,m)} (-1)^{\dim(K)} e_K \right).
\]

Proof. We first observe the fixture of a base chamber \( C_0 \) results in the following: Suppose \( F \subset \mathcal{B}_N \) is a face. The convex closure \( C(C_0,F) \) contains a unique chamber \( D \) which contains \( F \). Furthermore, if \( D' \neq D \) is another chamber containing \( F \), then \( \text{ht}_{C_0}(D') = \text{ht}_{C_0}(D) + 1 \). So, for any face \( F \subset \mathcal{B}_N \), we define:

\[
\mathcal{C}_{C_0}(F) := \text{the chamber } D \text{ containing } F, \text{ so that } F \text{ is an outward face of } D. \quad (6.3.4)
\]

The chamber \( \mathcal{C}_{C_0}(F) \) is the unique chamber of \( C(C_0,F) \) containing \( F \) (denoted as \( D_F \) in (2.3.6)).

As a second observation, we note that the adjacency of \( C_0 \) and \( C_1 \) means \( |\text{ht}_{C_0} - \text{ht}_{C_0}| \leq 1 \). In fact, an explicit relationship is the following. Take a chamber \( D \subset \mathcal{B}_N \) (possibly equal to \( C_0 \) or \( C_1 \)), and consider two possibilities depending if there is or there is no apartment containing the three chambers \( C_0, C_1, \) and \( D \).

Case \( \nexists \) apartment \( \mathcal{A} \). Let \( F \) denote the common face of \( C_0 \) and \( C_1 \). Here, the relationships between the convex closures \( C(D,F), C(D,C_0), \) and \( C(D,C_1) \) is

\[
C(D,C_0) = C(D,F) \cup C_0 \quad \text{and} \quad C(D,C_1) = C(D,F) \cup C_1.
\]

In particular, there exists \( g \in \mathcal{A} \) which fixes \( C(D,F) \) and takes \( C_0 \) to \( C_1 \), and therefore \( \text{ht}_{C_0}(D) = \text{ht}_{C_1}(D) \).

Case \( \exists \) apartment \( \mathcal{A} \). In the apartment \( \mathcal{A} \), let \( H \) denote the (refined) affine root hyperplane which separates the (adjacent) chambers \( C_0 \) and \( C_1 \). If \( D \) and \( C_0 \) (resp. \( C_1 \)) are on the same side of \( H \), then \( \text{ht}_{C_0}(D) = \text{ht}_{C_0}(D) + 1 \) (resp. \( \text{ht}_{C_0}(D) = \text{ht}_{C_1}(D) + 1 \)).

We deduce, from this explicit height relationship, the relationship between the outward faces of \( D \) with respect to \( C_0 \) and \( C_1 \) is the following:

- If \( \nexists \) apartment \( \mathcal{A} \), then the outward facing faces of \( D \) with respect to \( C_0 \) is the same as with respect to \( C_1 \).
- If \( \exists \) apartment \( \mathcal{A} \), and \( D \) has a face \( E \) in the (refined) affine root hyperplane \( H \), then \( E \) is outward for \( C_0 \) (resp. \( C_0 \)) when \( D \) and \( C_0 \) (resp. \( C_1 \)) are on the same side of \( H \). If \( D \) does not have a face on \( H \), then the outward facing faces of \( D \) with respect to \( C_0 \) is the same as with respect to \( C_1 \).

Suppose \( C_0 \) and \( C_1 \) are adjacent chambers with common face \( F \), and \( F' \neq F \) is any other face.

- If \( F \) and \( F' \) are not aligned, then \( \mathcal{C}_{C_0}(F') = \mathcal{C}_{C_1}(F') \). To see this, we consider \( C(F,F') \). That \( F \) and \( F' \) are not aligned means the maximal dimension facet \( Y \) (denoted as \( D_{F'} \) in Lemma (2.3.3) in \( C(F,F') \) containing \( F' \) is a chamber, and \( F \) is an outward face \( Y \). Since \( C(C_0,F') \supset C(F,F') \) as well as \( C(C_1,F') \supset C(F,F') \), it follows \( Y \) is \( \mathcal{C}_{C_0}(F') \) and \( \mathcal{C}_{C_1}(F') \).
• If $F$ and $F'$ are aligned there exists an apartment $A'$ containing $F$, $F'$, $\mathcal{C}_0(F')$, and $\mathcal{C}_{C_1}(F')$, and in $A'$ the reflection across the affine hyperplane generated by $F$ swaps $\mathcal{C}_0(F')$, and $\mathcal{C}_{C_1}(F')$. To see this, we note that $\dim(C(F, F'))$ is that of a face. Pick an apartment $A''$ containing $C_0$ and $F$. We use $F \subset C_0$, to say $A''$ contains $C(F, F')$, and the later generates a hyperplane in $A''$. The Iwahori subgroup $S_{C_0}$ acts transitively on the apartments containing $C(C_0, F')$. There exists $h \in S_{C_0}$ fixing $C(C_0, F')$ and moving $C_1$ to $A''$. The apartment $A' := h^{-1} A''$ satisfies the assertion.

We are now ready to prove the Proposition. Consider the intersection

$$\text{Int}(m) := \text{Ball}(C_0, m) \cap \text{Ball}(C_1, m). \quad (6.3.5)$$

For $a \in \{0, 1\}$, the Euler-Poincaré sum of the idempotents $e_K^C$ over the facets of the ball $\text{Ball}(C_a, m)$ equals:

$$\sum_{K \subset \text{Int}(m)} (-1)^{\dim(K)} e_K^C + \sum_{D \subset (\text{Ball}(C_a, m) \setminus \text{Int}(m))} \sum_{\mathcal{C}_a(D) \subset K \subset D} (-1)^{\dim(K)} e_K^C.$$ 

So the difference of the Euler-Poincaré sums over $\text{Ball}(C_0, m)$ and $\text{Ball}(C_1, m)$ is

$$\sum_{D \subset (\text{Ball}(C_0, m) \setminus \text{Int}(m))} \sum_{\mathcal{C}_0(D) \subset K \subset D} (-1)^{\dim(K)} e_K^C \quad (6.3.6)$$

- Suppose $m \geq 1$ and $D \subset (\text{Ball}(C_0, m) \cup \text{Ball}(C_1, m))$.
  - **CASE** 2 apartment $A$ containing $C_0, C_1$ and $D$. Here $D \subset \text{Int}(m)$ and therefore it does not occur as a summation index value in (6.3.6).
  - **CASE** $\exists$ apartment $A$ containing $C_0, C_1$ and $D$. Here, $D$ occurs as a summation index value in the first (resp. second) term of (6.3.3) when $\text{ht}_{C_0}(D) = m$ and $\text{ht}_{C_1}(D) = (m - 1)$ (resp. $\text{ht}_{C_1}(D) = m$ and $\text{ht}_{C_0}(D) = (m - 1)$, and the Euler-Poincaré sum

$$f \star \left( \sum_{\mathcal{C}_0(D) \subset K \subset D} (-1)^{\dim(K)} e_K^C \right) \quad \text{resp. } f \star \left( \sum_{\mathcal{C}_1(D) \subset K \subset D} (-1)^{\dim(K)} e_K^C \right)$$

vanishes for $m$ sufficiently large.

Given the above, we deduce that for $m$ sufficiently large that:

$$f \star \left( \sum_{K \subset \text{Ball}(C_0, m)} (-1)^{\dim(K)} e_K^C \right) = f \star \left( \sum_{K \subset \text{Int}(m)} (-1)^{\dim(K)} e_K^C \right) = f \star \left( \sum_{K \subset \text{Ball}(C_1, m)} (-1)^{\dim(K)} e_K^C \right).$$

$\square$
Corollary 6.3.7. Under the assumptions of subsection 6.1, suppose $C_0$ and $C_1$ are two chambers of $\mathcal{B}_N$, and $f \in C^\infty_c(\mathcal{G})$. Then, for $m$ sufficiently large:

$$f \star \left( \sum_{K \subset \text{Ball}(C_0,m)} (-1)^{\dim(K)} e^K_C \right) = f \star \left( \sum_{K \subset \text{Ball}(C_1,m)} (-1)^{\dim(K)} e^K_C \right).$$

Proof. We apply Proposition 6.3.3 to a sequence of chambers $C_0 = D_0, D_1, \ldots, D_n = C_1$ in which has the property that $D_i$ and $D_{i+1}$ are adjacent. 

We combine the stabilization Corollary (6.3.2) and the independence Corollary (6.3.7) to deduce the existence of a distribution $D_C$, which we write descriptively as:

$$D_C = \left( \sum_{F \subset C_N} (-1)^{\dim(F)} e_F^C \right),$$

so that for any $f \in C^\infty_c(\mathcal{G})$ and any chamber $C_0$, there is an integer $N(f, C_0)$ so that

$$f \star D_C = f \star \left( \sum_{K \subset \text{Ball}(C_0,m)} (-1)^{\dim(K)} e^K_C \right) \quad \text{for} \quad m \geq N(f, C_0).$$

Theorem 6.3.10. Suppose $k$ is a non-archimedean local field and $G$ is an absolutely quasisimple linear algebraic group defined over $k$. Let $\mathcal{G} = G(k)$ be the group of $k$-rational points, and $\mathcal{B} = \mathcal{B}(\mathcal{G})$ the Bruhat–Tits building of $\mathcal{G}$. For $N \in \mathbb{N}$, let $\mathcal{B}_N$ be the refined building, and suppose $\mathcal{C} = \{(F, \chi)\}$ is the equivalence class of a cuspidal associate pair of depth $r \in \frac{1}{N}\mathbb{N}_+$. Let $\{e_F^C\}$ be the $\mathcal{G}$-equivariant system of idempotents defined in (3.3.6). Then, the distribution

$$D_C = \sum_{E \subset \mathcal{B}_N} (-1)^{\dim(E)} e_E^C$$

of (6.3.3) satisfies the following:

(i) $D_C$ is $\mathcal{G}$-invariant.

(ii) $D_C$ is essentially compact.

(iii) For a facet $K \subset \mathcal{B}_N$, define

$$e^r_K := \frac{1}{\text{meas}(\mathcal{G}_{K,+})} 1_{\mathcal{G}_{K,+}}.$$

Then,

$$D_C \star e^r_K = e^C_K.$$

(iii.1) If $(K, \xi) \notin \mathcal{C}$, then $D_C \star e_{K,\xi} = 0$.

(iii.2) If $(K, \xi) \in \mathcal{C}$, then $D_C \star e_{K,\xi} = e_{K,\xi}$.

(iv) If $\mathcal{D} = \{(F', \chi')\} \neq \mathcal{C}$ is another cuspidal associate pair of depth $r \in \frac{1}{N}\mathbb{N}_+$, then

$$D_C \star D^\mathcal{D} = 0.$$
(v.1) The depth \( \leq r \) projector \( P_{\leq r} := \left( \sum_{F \subset B_N} (-1)^{\dim(F)} e_{g_{F,r^+}} \right) \) is decomposed as

\[
P_{\leq r} = \sum_{C \text{ of depth } r} D_C.
\]

(v.2) \( D_C \) is idempotent.

As a preparation to the proof of Theorem 6.3.10, we state and prove:

**Lemma 6.3.11.** Fix a base chamber \( C_0 \subset B(\mathfrak{g})_N \), and a facet \( K \subset C_0 \). For any chamber \( D \neq C_0 \):

\[
\dim(C(\mathcal{E}_{C_0}(D), K)) > \dim(\mathcal{E}_{C_0}(D)) .
\]

Proof. Since \( \dim(C(\mathcal{E}_{C_0}(D), K)) \geq \dim(\mathcal{E}_{C_0}(D)) \), if suffices to show \( \dim(C(\mathcal{E}_{C_0}(D), K)) = \dim(\mathcal{E}_{C_0}(D)) \) leads to a contradiction. Recall,

\[
\text{Aff}(\mathcal{E}_{C_0}(D)) = \bigcap_{F \in C_0(D)} \text{Aff}(F)
\]

is the affine subspace generated by \( \mathcal{E}_{C_0}(D) \). If \( \dim(C(\mathcal{E}_{C_0}(D), K)) = \dim(\mathcal{E}_{C_0}(D)) \), then the facet \( K \) must lie in \( \text{Aff}(\mathcal{E}_{C_0}(D)) \). In opposition to the facet \( \mathcal{E}_{C_0}(D) \), set

\[
\mathcal{F}_{C_0} := \bigcap_{F \in pC_0(D)} F \quad \text{(the facet of } D \text{ opposite to } \mathcal{E}_{C_0}(D)\text{)}, \quad \text{and}
\]

\[
\text{Aff}(\mathcal{F}_{C_0}(D)) = \bigcap_{F \in pC_0(D)} \text{Aff}(F) \quad \text{(the affine subspace generated by } \mathcal{F}_{C_0}(D)\text{)} .
\]

That \( \mathcal{E}_{C_0}(D) \) and \( \mathcal{F}_{C_0}(D) \) are opposite facets of \( D \) means the two affine subspaces \( \text{Aff}(\mathcal{E}_{C_0}(D)) \) and \( \text{Aff}(\mathcal{F}_{C_0}(D)) \) have empty intersection. By definition, each face \( F \subset D \), is contained in the zero hyperplane \( H_{\pm \psi_F} \) of a pair of virtual affine root. We choose the sign so that \( \text{grad}(\psi_F) = \alpha_F \), i.e., the outward/inward direction of the face. Then,

\[
K \subset \left( \bigcap_{F \in pC_0(D)} H_{\psi_F \leq 0} \right) \quad \text{and} \quad \text{Aff}(\mathcal{F}_{C_0}(D)) \subset \left( \bigcap_{F \in cC_0(D)} H_{\psi_F \geq 0} \right) .
\]

But, the above two intersections are cones whose intersection is empty– a contradiction to the assumption \( K \subset \text{Aff}(\mathcal{F}_{C_0}(D)) \). Thus, the lemma is proved.

\( \square \)

**Proof of Theorem 6.3.10.** Assertion (i) on \( \mathfrak{g} \)-invariance of \( D^C \) is a consequence of the independence of base chamber Corollary (6.3.7).

Assertion (ii) that \( D^D \) is an essentially compact distribution is the stabilization Corollary (6.3.2).

To prove assertion (iii), we use the independence of base chamber Corollary (6.3.7) to assume that the facet \( K \) is contained in \( C_0 \). Then,
\[ \varepsilon^r_K \star \left( \sum_{J \subset C_0} (-1)^{\dim(J)} e^c_J \right) = \sum_{J \subset C_0} (-1)^{\dim(J)} e^r_K \star e^c_J \]
\[ = \sum_{J \subset C_0} (-1)^{\dim(J)} e^r_K \star (e^r_J \star e^c_J) = \sum_{J \subset C_0} (-1)^{\dim(J)} (e^r_K \star e^r_J) \star e^c_J \]
\[ = \sum_{J \subset C_0} (-1)^{\dim(J)} e^r_{C(K,J)} \star e^c_J = \sum_{J \subset C_0} (-1)^{\dim(J)} e^c_{C(K,J)} = e^c_K. \]

Thus, a sufficient condition to prove statement (iii) is that if \( D \) is a chamber not equal to \( C_0 \), it is the case that the convolution
\[ \varepsilon^r_K \star \left( \sum_{\varepsilon C_0(D) \subset J \subset D} (-1)^{\dim(J)} e^c_J \right) \text{ vanishes}. \]

To see this, let \( L \) be the unique facet of \( C(K, \varepsilon C_0(D)) \) which contains \( \varepsilon C_0(D) \) and is of maximal dimension. we calculate:
\[ \varepsilon^r_K \star \left( \sum_{\varepsilon C_0(D) \subset J \subset D} (-1)^{\dim(J)} e^c_J \right) = \sum_{\varepsilon C_0(D) \subset J \subset D} (-1)^{\dim(J)} e^r_K \star e^c_J \]
\[ = \sum_{\varepsilon C_0(D) \subset J \subset D} (-1)^{\dim(J)} e^r_K \star e^r_J \star e^c_J = \sum_{\varepsilon C_0(D) \subset J \subset D} (-1)^{\dim(J)} e^r_K \star e^r_L \star e^r_J \star e^c_J \]
\[ \text{The summation of the last line vanishes since } \varepsilon C_0(D) \subset L \text{ creates repetition in the summation.} \]

Assertions (iii.1) and (iii.2) are immediate consequences of (iii).

To complete the proof of the Theorem, we note assertion (iv) is a consequence of (iii), assertion (v.1) is obvious, and (v.2) follows from the orthogonality relation (iv) and (v.1).

\[ \square \]

7. Resolutions

7.1. Review of work of Schneider–Stuhler and Bestvina–Savin.

We assume the hypotheses of section (6.1). For a facet \( K \subset B_N \), set
\[ e^r_K = \frac{1}{\text{meas}(\mathcal{G}_{K,r}^+)} 1_{\mathcal{G}_{K,r}^+}. \tag{7.1.1} \]

If \( (\pi, V_\pi) \) is a smooth representation of \( \mathcal{G}_j \), note that
\[ \pi(e^r_K)(V_\pi) = V_\pi^{e^r_K}. \]
Let $\mathcal{K} = \text{Orb}_G(K)$ be the $G$-orbit of the facet $K$. Define the dimension of the orbit $\mathcal{K}$ to be the dimension of the facet $K$. The vector space

$$W^{r,\mathcal{K}}_\pi := \{ (J, v) \mid J \in \mathcal{K}, v_J \in \pi(e_J^r)(V_\pi), \text{ and } v_J = 0 \text{ for almost all } J \}$$

(7.1.2)

is a smooth representation of $G$ with the action

$$g \cdot (J, v) := (g \cdot J, \pi(g)(v_J)) .$$

It is equivalent to the smooth representation

$$(\text{c-Ind}^G_{\mathcal{K},r+} (\text{triv})) \otimes \mathcal{V}_\pi,$$

(7.1.3)

and therefore it is a projective $G$-module.

For $k \in \{1, 2, \ldots, (\ell = \text{rank}(\mathcal{G}))\}$, set

$$W^{r,k}_\pi := \bigoplus_{\dim(\mathcal{K}) = k} W^{r,\mathcal{K}}_\pi .$$

(7.1.4)

A key property of the idempotents $e^K_r$'s is

$$\forall \text{ facet } E, \text{ and subfacet } F \subset \partial(E), \text{ it is the that case } \pi(e^K_r)(V_\pi) \subset \pi(e_E^r)(V_\pi) .$$

This means, the boundary map

$$\partial : W^{r,\mathcal{K}}_\pi \longrightarrow W^{r,(\dim(\mathcal{K})-1)}_\pi \oplus (\partial(E), v_E) \quad (\text{same } v_E's \text{ on both sides});$$

(7.1.5)

is defined, and a $G$-map. Consequently, there is a boundary $G$-map

$$W^{r,k}_\pi \longrightarrow W^{r,(k-1)}_\pi \quad k \in \{1, 2, \ldots, \ell\} .$$

Let

$$\begin{array}{ccc}
W^{r,0}_\pi & \longrightarrow & V_\pi \\
\bigoplus_{x_0 \text{ vertex}}(x_0, v_{x_0}) & \longrightarrow & \sum v_{x_0}
\end{array}$$

be the augmentation map (a $G$-map).

**Theorem 7.1.6.** (Schneider–Stuhler, Bestvina–Savin)

Under the assumptions of section (6.1), let $\ell = \dim(\mathcal{B}(\mathcal{G}))$, and assume a (smooth) $G$-module $V_\pi$ is generated by the subspaces $\pi(e^K_r)(V_\pi)$ as $x_0$ runs over the vertices of $\mathcal{B}(\mathcal{G})_N$, i.e., the augmentation map is surjective. Then

$$0 \longrightarrow W^{r,\ell}_\pi \longrightarrow W^{r,(\ell-1)}_\pi \longrightarrow \cdots \longrightarrow W^{r,1}_\pi \longrightarrow W^{r,0}_\pi \longrightarrow V_\pi$$

is a (projective) resolution of $V_\pi$. 
We remark that Schneider–Stuhler formulated and proved Theorem 7.1.6 for \( N = 1 \) in [SS]. Bestvina–Savin proved Theorem 7.1.6 for general \( N \) in [BS].

Suppose \((F, \chi)\) is a cuspidal pair of depth \( r \in \frac{1}{N}\mathbb{N}_+\), and let \( C = \{(F', \chi')\}\) the equivalence class of pairs associate to \((F, \chi)\). Let \( \mathbb{D}^C \) be the Bernstein center idempotent of Theorem 6.3.10. Application of \( \mathbb{D}^C \) to the resolution of Theorem 7.1.6 gives a resolution:

\[
0 \rightarrow \mathbb{D}^C \cdot W^{r,t}_\pi \overset{\partial}{\rightarrow} \mathbb{D}^C \cdot W^{r,(t-1)}_\pi \overset{\partial}{\rightarrow} \ldots \overset{\partial}{\rightarrow} \mathbb{D}^C \cdot W^{r,1}_\pi \overset{\partial}{\rightarrow} \mathbb{D}^C \cdot W^{r,0}_\pi \rightarrow \mathbb{D}^C \cdot V \pi .
\]

Via the definitions (7.1.2) and (7.1.4), and statement (iii) of Theorem 6.3.10, the \( G \)-module \( \mathbb{D}^C \cdot W^{r,k}_\pi \) has the description

\[
\mathbb{D}^C \cdot W^{r,k}_\pi = \bigoplus_{\dim(K) = k} \mathbb{D}^C \cdot W^{r,k}_\pi = \bigoplus_{\dim(K) = k} \{ (J, v) | J \in K, v_J \in \pi(e^{iJ}(V)) \} \quad \text{and} \quad v_J = 0 \text{ for almost all } J \}
\]

We define

\[
W^{C,k}_\pi := \bigoplus_{\dim(K) = k} \{ (J, v) | J \in K, v_J \in \pi(e^{iJ}(V)) \} \quad \text{and} \quad v_J = 0 \text{ for almost all } J
\]

and note that \( \dim(W^{C,k}_\pi) = 0 \) unless \( k \leq \dim(F) \). Then, in summary:

**Proposition 7.1.8.** Under the assumptions of section (6.1), and \( C = \{(F, \chi)\} \) the equivalence class of a cuspidal pair \((F, \chi)\) of depth \( r \), let \( L = \dim(F) \). Assume a \( \mathcal{S} \)-module \( V \pi \) is generated by the subspaces \( \pi(e^{iJ}(V)) \) as \( x_0 \) runs over the vertices of \( \mathcal{B} (\mathcal{S})_N \), i.e., the augmentation map is surjective. Then,

\[
0 \rightarrow W^{C,L}_\pi \overset{\partial}{\rightarrow} W^{C,(L-1)}_\pi \overset{\partial}{\rightarrow} \ldots \overset{\partial}{\rightarrow} W^{C,1}_\pi \overset{\partial}{\rightarrow} W^{C,0}_\pi \rightarrow V \pi
\]

is a (projective) resolution of \( V \pi \).

**Corollary 7.1.9.** Under the assumptions of section (6.1), suppose \( (\pi, V \pi) \) is an irreducible smooth representation of \( \mathcal{S} \) and \((F, \chi)\) is a cuspidal pair with \( F \subset \mathcal{B} (\mathcal{S})_N \), so that \( V \pi \chi \neq \{0\} \). Then, \( V \pi \) has cohomological dimension \( \leq \dim(F) \).

**Proof.** Let \( C \) denote the equivalence class of the cuspidal pair \((F, \chi)\). The hypothesis \( V \pi \chi \neq \{0\} \) means \( \pi(e^{iJ}(V)) \neq \{0\} \) for any vertex \( x_0 \) of \( F \), and so by the irreducibility hypothesis on \( V \pi \), it is generated by \( \pi(e^{iJ}(V)) \). The resolution of \( V \pi \) provided by \( C \) has length \( L \); thus \( V \pi \) has cohomological dimension \( \leq \dim(F) \).
8. Nonsplit groups

In this section, we explain the minor modifications needed in the proofs under our earlier assumption that $k$-defined group $G$ is split quasisimple to when it is connected, absolutely quasisimple (possibly non-split). We assume $G$ is connected, absolutely quasisimple. Set $\Sigma = G(k)$. We follow [BCM:§6]. (We correct here a misstatement in the displayed line following (6.1.2) of [BCM], namely that the constant parts of affine roots do not run over $\mathbb{Z}$ but rather $\frac{1}{R} \mathbb{Z}$ for some positive integer $R$).

Let $k^{un}$ be the maximal unramified extension of $k$, and let $\text{Gal}(k^{un}/k)$ denote the Galois group. Since $G$ is assumed to be absolutely quasisimple, the Bruhat–Tits building $B(G(k^{un}))$ is a simplicial complex, with simplicial actions by $G(k^{un})$ and $\text{Gal}(k^{un}/k)$. The building $B(\Sigma)$ is the $\text{Gal}(k^{un}/k)$-fixed points of $B(G(k^{un}))$.

The refined building $B_N$ is also defined in terms of a set of virtual affine roots. Take a torus $S$ defined over $k$ satisfying: (i) $S$ is a maximal split $k^{un}$-torus, and (ii) $S := S^{\text{Gal}(k^{un}/k)}$ is a maximal split $k$-torus. Set $A = A(S(k^{un}))^{\text{Gal}(k^{un}/k)}$, an apartment in $B$. The affine root system $\Psi$ on $A$ is the nonconstant restrictions to $A$ of affine roots in $\Psi(S(k^{un}))$. The root system $\Phi$ of $\Sigma$ with respect to $S$ is the set of gradients of affine roots. Let $x_0$ in $A$ be a special point such that $\Phi_{x_0}$, the set of gradients of affine roots vanishing at $x_0$, contains all short roots in $\Phi$. Note that the group of automorphisms of the affine Dynkin diagram acts transitively on such special vertices (see tables in [T:§4]). Consider $A$ a vector space with $x_0$ its origin. For every $\psi \in \Psi$ define an affine functional

$$\psi_N(x) := \frac{1}{N} \psi(Nx).$$

Let $\Psi_N$ be the set of these affine functionals. This set is independent of the choice of $x_0$. In terms of the graphs of affine functionals in $A \times \mathbb{R}$, the graphs of the functionals in $\Psi_N$ are obtained by taking two parallel graphs of affine roots in $\Psi$, and adding $(N-1)$ equally spaced parallel graphs. Denote by $A_N$ the refined simplicial decomposition of $A$ by the zero loci of the virtual affine roots $\Psi_N$, and by $B_N$ the resulting refined simplicial decomposition of $B$. Observe that if $\psi \in \Psi$ then $\psi + 1 \in \Psi$. Indeed, this is true for affine roots of the quasi-split groups $G(k^{un})$, by observation (see the example of odd unitary group below), and therefore for $\Sigma$ by restriction. In particular, it is clear that $\psi_N + 1/N \in \Psi_N$. If $F$ is a facet of $B_N$, and $r \in \frac{1}{N} \mathbb{Z}_{\geq 0}$, then

$$\forall x, y \in \text{interior of } F : \quad g_{x,r} = g_{y,r} \quad \text{and} \quad g_{x,r+} = g_{y,r+}.$$

**Example.** Assume $p \neq 2$. Take $G = SU_{2n+1}$ to be the special unitary group over a ramified quadratic extension. Here $A = \mathbb{R}^n$ and $\Psi$ has simple roots $e_1 - e_2, \ldots, e_{n-1} - e_n, e_n$ and $1/2 - 2e_1$. The affine roots are $2e_i + c + 1/2, \pm e_i \pm e_j + c/2, i \neq j$, and $e_i + c/2$ where $c$ is an integer. The root system $\Phi$ is of the type $BC_n$. The affine chamber corresponding to the given set of simple roots has two special vertices, $x_0 = (0, \ldots, 0)$ and $y_0 = (\frac{1}{2}, \ldots, \frac{1}{2})$. Observe that $\Phi_{x_0}$ has type $B_n$, while $\Phi_{y_0}$ has type $C_n$, thus we use $x_0$ to define virtual affine roots. If $N = 2$ these are $2e_i + c + 1/4, \pm e_i \pm e_j + c/2, i \neq j$, and $e_i + c/2$ where $c$ is half-integer. In particular, we note the refined simplicial structure of $B_2$ is that of $B$. This is caused by the root system $\Phi$ being nonreduced.
An important difference between the split and non-split case is that the set of affine roots may be non-reduced. An affine root $\psi \in \Psi$ is called divisible if $\psi/2$ is also an affine root (in which case the two affine roots obviously have the same zero locus). In the context of Lemma 5.1, we take the root $\alpha_F$, perpendicular to the face $F$ of a chamber, to be the gradient of a non-divisible root vanishing on $F$. With this convention, the formulations and proofs for the split case then hold for the non-split case too.

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