Perturbation Expansion and $N$-th Order Fermi Golden Rule of the Nonlinear Schrödinger Equations

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Abstract

In this paper we consider generalized nonlinear Schrödinger equations with external potentials. We compute the fourth and the sixth order Fermi Golden Rules (FGR), conjectured in [GS2, GS3], which is used in a study of the asymptotic dynamics of trapped solitons.

1 Introduction

It is well known that the eigenvalues embedded in a continuous spectrum of a self-adjoint operator, $L$, are unstable under generic perturbations. Intuitively, one can think of time periodic solutions (‘bound states’) of the equation, $i \dot{u} = Lu$, corresponding to such eigenvalues, leaking their energy to the continuous spectrum solutions (‘scattering states’) which in turn is radiated to infinity. The coupling between time-periodic and continuous spectrum solutions was computed in the second order of perturbation theory by P. A. M. Dirac in 1927 (see [Di], resulting in an elegant expression. If this expression is nonzero, then the decay mentioned above takes place. E. Fermi called this condition “Golden Rule No. 2”. In physics literature it is known as the Fermi Golden Rule (FGR). The FGR, computed first for atomic and molecular states, appears in many other areas of physics e.g. in non-equilibrium statistical mechanics. Of course, due to energy conservation the isolated eigenvalues are stable under reasonable time independent perturbations.

The Fermi Golden Rule was introduced to nonlinear Hamiltonian PDE in [S] where it was used to prove that time-periodic solutions for linear (and nonlinear) wave equations are unstable under generic nonlinear perturbations. It was used

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extensively in investigating long-time dynamics of solitary waves in the nonlinear Schrödinger [BP2, BuSu, SW1, SW2, SW3, TY1, TY2, TY3] and wave [SW4] equations.

However, in nonlinear problems with a small parameter, \( h \), of the quasi-classical nature, the second order expression obtained by Dirac vanishes and, in principle, one can hope to get non-vanishing expression only in the 2\( N \)th order where \( N = O\left( \frac{1}{h} \right) \). (If one thinks about the FGR as a condition that an 'atom is ionized by a single photon', in this case because the ionization energy is much larger than the photon frequency, this 'ionization' is due to a multi-photon process.) This type of situation occurs in particular in the problem of the relaxation of a solution to the ground state in generalized nonlinear Schrödinger equations with external slowly varying potentials (Gross-Pitaevskii equations) described below.

In this paper we find, for the first time, the 'nonlinear' forth and sixth order FGRs for the relaxation problem mentioned above. As was conjectured in [GS2, GS3], they are of the same form as the second order FGR and as, presumably, the higher order FGRs. To obtain the forth and sixth order FGRs we have to perform additional normal-form-type transformations in the perturbation theory we developed in [GS2, GS3].

Problem. Now we describe the problem in more details. We consider the generalized nonlinear Schrödinger equation (NLS) in dimension \( d \neq 2 \) with an external potential \( V_h : \mathbb{R}^d \to \mathbb{R} \),

\[
\frac{\partial \psi}{\partial t} = -\Delta \psi + V_h \psi - f(\vert \psi \vert^2) \psi. \tag{1}
\]

Here \( h > 0 \) is a small parameter giving the length scale of the external potential in relation to the length scale of the \( V_h = 0 \) solitons (see below), \( \Delta \) is the Laplace operator and \( f(s) \) is a nonlinearity to be specified later. We normalize \( f(0) = 0 \). Such equations arise in the theory of Bose-Einstein condensation \(^1\), nonlinear optics, theory of water waves \(^2\) and in other areas.

To fix ideas we assume the potentials to be of the form \( V_h(x) := V(hx) \) with \( V \) smooth and decaying at \( \infty \). Thus for \( h = 0 \), Equation (1) becomes the standard generalized nonlinear Schrödinger equation (gNLS)

\[
\frac{\partial \psi}{\partial t} = -\Delta \psi + \mu \psi - f(\vert \psi \vert^2) \psi, \tag{2}
\]

where \( \mu := V(0) \). For a certain class of nonlinearities, \( f(\vert \psi \vert^2) \), (see Section 3), there is an interval \( I_0 \subset \mathbb{R} \) such that for any \( \lambda \in I_0 \) Equation (2) has solutions of the form \( e^{i(\lambda - \mu)t} \phi_0^\lambda(x) \) where \( \phi_0^\lambda \in \mathcal{H}_2(\mathbb{R}^n) \) and \( \phi_0^\lambda > 0 \). Such solutions (in general without the restriction \( \phi_0^\lambda > 0 \)) are called the solitary waves or solitons or, to emphasize the property \( \phi_0^\lambda > 0 \), the ground states. For brevity we will use the term soliton applying it also to the function \( \phi_0^\lambda \) without the phase factor \( e^{i(\lambda - \mu)t} \).

\(^1\)In this case Equation (1) is called the Gross-Pitaevskii equation.

\(^2\)In the last two areas \( V_h \) arises if one takes into account impurities and/or variations in geometry of the medium and is, in general, time-dependent.
Equation (2) is translationally and gauge invariant. Hence if \( e^{i(\lambda - \mu)t} \phi_0^\lambda(x) \) is a solution for Equation (2), then so is

\[
e^{i(\lambda - \mu)t} e^{i\alpha} \phi_0^\lambda(x + a), \quad \text{for any } a \in \mathbb{R}^n, \text{ and } \alpha \in [0, 2\pi).
\]

This situation changes dramatically when the potential \( V_h \) is turned on. In general, as was shown in [FW, Oh1, ABC] out of the \((n+2)\)-parameter family

\[
e^{i(\lambda - \mu)t} e^{i\alpha} \phi_0^\lambda(x + a)
\]

only a discrete set of two-parameter families of solutions to Equation (1) bifurcate:

\[
e^{i\lambda t} e^{i\alpha} \phi_0^\lambda(x), \quad \alpha \in [0, 2\pi) \text{ and } \lambda \in I \text{ for some } I \subseteq I_0, \text{ with } \phi_0^\lambda \equiv \phi_h^\lambda \in H_2(\mathbb{R}^n) \text{ and } \phi_0^\lambda > 0.
\]

Each such family centers near a different critical point of the potential \( V_h(x) \). It was shown in [Oh2] that the solutions corresponding to minima of \( V_h(x) \) are orbitally (Lyapunov) stable and to maxima, orbitally unstable. We call the solitary wave solutions described above which correspond to the minima of \( V_h(x) \) trapped solitons or just solitons of Equation (1) omitting the last qualifier if it is clear which equation we are dealing with.

A basic question about soliton solutions is whether they are asymptotically stable, i.e. whether for initial condition of (1) sufficiently close to a trapped soliton \( \{ e^{i\gamma} \phi^\lambda(x) \} \) the solution converges (relaxes) in a local norm, up to a phase factor, to another trapped soliton,

\[
\psi(x, t) - e^{i\gamma(t)} \phi^\lambda_\infty(x) \to 0.
\]

We observe that (1) is a Hamiltonian system with conserved energy (see Section 2) and, though orbital (Lyapunov) stability is expected, the asymptotic stability is a subtle matter. To have asymptotic stability the system should be able to dispose of excess of its energy, in our case, by radiating it to infinity. The infinite dimensionality of a Hamiltonian system in question plays a crucial role here. This phenomenon as well as a general class of classical and quantum relaxation problems was pointed out by J. Fröhlich and T. Spencer [Private Communication].

Another important property is their effective dynamics. Namely, one would like to show that if an initial condition is close (in the weighted norm \( \| u \|_{\nu,1} := \|(1 + |x|^2)^\frac{\nu}{2} u \|_{H^1} \) for sufficiently large \( \nu \)) to the soliton \( e^{i\gamma_0} \phi_0^\lambda_0 \), with \( \gamma_0 \in \mathbb{R} \) and \( \lambda_0 \in I \) (\( I \) as above), then the solution, \( \psi(t) \), of Equation (1) can be written as

\[
\psi(x, t) = e^{i\gamma(t)} \left[ e^{ip(t) \cdot x} \phi^\lambda(t)(x - a(t)) + R(x, t) \right],
\]

where \( \| R(t) \|_{\nu,1} \to 0, \lambda(t) \to \lambda_\infty \) for some \( \lambda_\infty \) as \( t \to \infty \) and the soliton center \( a(t) \) and momentum \( p(t) \) evolve according to certain effective equations of motion.

**Results.** As in [GS2, GS3] we assume that either \( d = 1 \) and the potential is even or \( d > 2 \) and the potential is spherically symmetric and the initial condition symmetric with respect to permutations of the coordinates. In this case the soliton relaxes to the ground state along the radial direction. This limits the number of technical difficulties we have to deal with. We expect that our techniques extend to the general case when the soliton spirals toward its equilibrium.
It is shown in \[GS2, GS3\], under certain conditions, that the ground state is asymptotically stable and the effective equations for the parameters \(a\) and \(p\) are close to Newton’s equations. It was conjectured in \[GS2, GS3\] that one of the conditions (which plays a key role) is equivalent to an explicit \(2N\)th order FGR. This conjecture is true for \(N = 1\) due to the works \[BP2, BuSu, SW1, SW2, SW3, TY1, TY2, TY3\]. In this paper we prove this conjecture for \(N = 2\) and \(3\). In other words we establish the FGR for nonlinear problem in the 4th and 6th orders.

Moreover, we find more precise effective equations for \(a\) and \(p\). We show that there exist a function \(z(t)\) and a constant \(Z_{N+1,N} \in \mathbb{C}\) satisfying
\[
\frac{1}{2} \frac{d}{dt}|z|^2 = \text{Re} Z_{N+1,N} |z|^{2N+2} + O(|z|^{2N+3})
\]
with \(N = O(1/h)\) being an integer depends on \(\lambda\) and \(h\). For \(N = 2, 3\) we find the explicit form of \(\text{Re} Z_{N+1,N}\) which in particular shows that it is always non-positive and negative generically. The explicit form is given by \(2N\)th order FGR mentioned above.

In the present paper by using normal forms we simplify the expressions for \(z\) and the other parameters, thus make the proof more transparent than \[GS2, GS3\].

Previous results. We refer to \[GS1\] for a detailed review of the related literature. Here we only mention results of \[Cu, BP1, BP2, BuSu, SW1, SW2, SW3, TY1, TY2, TY3\] which deal with a similar problem. Like our work, \[SW1, SW2, SW3, TY1, TY2, TY3\] study the ground state of the NLS with a potential. However, these papers deal with the near-linear regime in which the nonlinear ground state is a bifurcation of the ground state for the corresponding Schrödinger operator \(-\Delta + V(x)\). The present paper covers highly nonlinear regime in which the ground state is produced by the nonlinearity (our analysis simplifies considerably in the near-linear case).

Papers \[Cu, BP1, BP2, BuSu\] consider the NLS without a potential so the corresponding solitons, which were described above, are affected only by a perturbation of the initial conditions which disperses with time leaving them free. In our case they, in addition, are under the influence of the potential and they relax to an equilibrium state near a local minimum of the potential.

As was mentioned above, the relaxation of the solution to the ground state (the asymptotic stability of the ground state) was shown in \[GS2, GS3\] under some assumptions on the potential \(V\), nonlinearity \(f\) and the linearized operator \(L(\lambda)\), which in particular include these related to the FGR which was explained above.

The paper is organized in the following way. In Section 1 we introduce the concept of Fermi Golden Rules and outline its applications and importance in proving the asymptotic stability of trapped solitons of nonlinear Schrödinger equations. In Section 2 we show the conservation laws and the local well-posedness of \((1)\). In Section 3 we formulate the conditions on the nonlinearity \(f\) and the potential \(V\) so that there exists a soliton manifold to \((1)\) and it is
stable. In Section 2 we linearize the solution to (1) around the soliton get a linear operator. Moreover we analyze its spectrum and put some assumptions. In Section 4 we state the main theorem. In Section 6 we separate the 'useless' parts from the equations of parameters, to prepare for the computation of forth ($N = 2$) and sixth ($N = 3$) order Fermi Golden Rules which will be proved in Section 7 and 8 respectively.

**Notation.** As customary we often denote derivatives by subindices as in $\phi^\lambda = \frac{\partial}{\partial \lambda} \phi^\lambda$ for $\phi^\lambda = \phi^\lambda(x)$. However, the subindex $h$ signifies always the dependence on the parameter $h$ and not the derivatives in $h$. The Sobolev and $L^2$ spaces are denoted by $H^k$ and $L^2$ respectively.

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## 2 Hamiltonian Structure and GWP

Equation (1) is a Hamiltonian system on Sobolev space $H^1(\mathbb{R}, \mathbb{C})$ viewed as a real space $H^1(\mathbb{R}, \mathbb{R}) \oplus H^1(\mathbb{R}, \mathbb{R})$ with the inner product $(\psi, \phi) = Re \int_{\mathbb{R}} \bar{\psi} \phi$ and with the simplicial form $\omega(\psi, \phi) = Im \int_{\mathbb{R}} \bar{\psi} \phi$. The Hamiltonian functional is:

$$H(\psi) := \int_{\mathbb{R}} \left[ \frac{1}{2} (|\psi_x|^2 + V_h |\psi|^2) - F(|\psi|^2) \right],$$

where $F(u) := \frac{1}{2} \int_0^u f(\xi) d\xi$.

Equation (1) has the time-translational and gauge symmetries which imply the following conservation laws: for any $t \geq 0$, we have

- (CE) conservation of energy: $H(\psi(t)) = H(\psi(0));$
- (CP) conservation of the number of particles: $N(\psi(t)) = N(\psi(0))$, where $N(\psi) := \int |\psi|^2$.

To address the global well-posedness of (1) we need the following condition on the nonlinearity $f$. Below, $s_+ = s$ if $s > 0$ and $= 0$ if $s \leq 0$.

- (fA) The nonlinearity $f$ satisfies the estimate $|f'(\xi)| \leq c(1 + |\xi|^{\alpha-1})$ for some $\alpha \in (0, \frac{2}{d-2})$ and $|f(\xi)| \leq c(1 + |\xi|^\beta)$ for some $\beta \in [0, \frac{d}{2})$.

The following result can be found in [Caz].

**Theorem** Assume that the nonlinearity $f$ satisfies the condition (fA), and that the potential $V$ is bounded. Then Equation (1) is globally well posed in $H^1$, i.e. the Cauchy problem for Equation (1) with a datum $\psi(0) \in H^1$ has a unique solution $\psi(t)$ in the space $H^1$ and this solution depends continuously on $\psi(0)$. Moreover $\psi(t)$ satisfies the conservation laws (CE) and (CP).
3 Existence and Orbital Stability of Solitons

In this section we review the question of existence of the solitons (ground states) for Equation (1). Assume the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies

(fB) There exists an interval $I_0 \in \mathbb{R}^+$ s.t. for any $\lambda \in I_0$, $-\infty \leq \lim_{s \to +\infty} \frac{f(s)}{s^{d-2}} \leq 0$ and $\frac{1}{\xi} \int_0^\xi f(s)ds > \lambda$ for some constant $\xi$, for $d > 2$; and

$$U(\phi, \lambda) := -\lambda \phi^2 + \int_0^{\phi^2} f(\xi)d\xi$$

has a smallest positive root $\phi_0(\lambda)$ such that $U(\phi_{0}, \lambda) \neq 0$, for $d = 1$.

It is shown in [BL, Str] that under Condition (fB) there exists a spherical symmetric positive solution $\phi_{0}(\lambda)$ to the equation

$$-\Delta \phi_{0} + \lambda \phi_{0} - f((\phi_{0})^2)\phi_{0} = 0.$$  \hspace{1cm} (4)

**Remark 1.** Existence of soliton functions $\phi_{0}(\lambda)$ for $d = 2$ is proved in [Str] under different conditions on $f$.

When the potential $V$ is present, then some of the solitons above bifurcate into solitons for Equation (1). Namely, let, in addition, $f$ satisfy the condition

|f'(\xi)| \leq c(1 + |\xi|^p), \text{ for some } p < \infty,

and $V$ satisfy the condition

(VA) $V$ is smooth and 0 is a non-degenerate local minimum of $V$.

Then, similarly as in [FW, Oh1] one can show that if $h$ is sufficiently small, then for any $\lambda \in I_{0V}$, where

$I_{0V} := \{\lambda|\lambda > -\inf_{x \in \mathbb{R}} \{V(x)\}\} \cap \{\lambda|\lambda + V(0) \in I_{0}\}$

there exists a unique soliton $\phi_{\lambda} \equiv \phi_{h}(\lambda)$ (i.e. $\phi_{\lambda} \in \mathcal{H}_2(\mathbb{R})$ and $\phi_{\lambda} > 0$) satisfying the equation

$$-\Delta \phi_{\lambda} + (\lambda + V_{h})\phi_{\lambda} - f((\phi_{\lambda})^2)\phi_{\lambda} = 0$$

and the estimate $\phi_{\lambda} = \phi_{0}^{\lambda+V(0)} + O(h^{3/2})$ where $\phi_{0}^{\lambda}$ is the soliton of Equation (4).

Let $\delta(\lambda) := \frac{1}{2}\|\phi_{\lambda}\|_2^2$. It is shown in [GSS1] that the soliton $\phi_{\lambda}$ is a minimizer of the energy functional $H(\psi)$ for a fixed number of particles $N(\psi) = \text{constant}$ if and only if $\delta'(\lambda) > 0$. Moreover, it is shown in [We2, GSS1] that under the latter condition the solitary wave $\phi_{\lambda} e^{i\lambda t}$ is orbitally stable. Under more restrictive conditions (see [GSS1]) on $f$ one can show that the open set

$$I := \{\lambda \in I_{0V} : \delta'(\lambda) > 0\}$$

is non-empty. Instead of formulating these conditions we assume in what follows that the open set $I$ is non-empty and $\lambda \in I$.

Using the equation for $\phi_{\lambda}$ one can show that if the potential $V$ is radially symmetric then there exist constants $c, \delta > 0$ such that $|\phi_{\lambda}(x)| \leq ce^{-\delta|x|}$ and $\frac{d}{dx} \phi_{\lambda} \leq ce^{-\delta|x|}$, and similarly for the derivatives of $\phi_{\lambda}$ and $\frac{d}{dx} \phi_{\lambda}$.
4 Linearized Equation and Resonances

We rewrite Equation (1) as $d\psi/dt = G(\psi)$ where the nonlinear map $G(\psi)$ is defined by $G(\psi) = -i(-\Delta + \lambda + V_h)\psi + if(|\psi|^2)\psi$. Then the linearization of Equation (1) can be written as $\partial\chi/\partial t = \partial G(\phi^\lambda)\chi$ where $\partial G(\phi^\lambda)$ is the Fréchet derivative of $G(\psi)$ at $\phi$. It is computed to be

$$\partial G(\phi^\lambda)\chi = -i(-\Delta + \lambda + V_h)\chi + if((\phi^\lambda)^2)\chi + 2if'(\phi^\lambda)^2(\phi^\lambda)^2Re\chi. \quad (6)$$

This is a real linear but not complex linear operator. To convert it to a linear operator we pass from complex functions to real vector-functions $\chi \leftrightarrow \vec{\chi} = (\chi_1 \chi_2)$, where $\chi_1 = Re\chi$ and $\chi_2 = Im\chi$. Then $\partial G(\phi^\lambda)\chi \leftrightarrow L(\lambda)\vec{\chi}$ where the operator $L(\lambda)$ is given by

$$L(\lambda) := \begin{pmatrix} 0 & L_-(\lambda) \\ -L_+(\lambda) & 0 \end{pmatrix}, \quad (7)$$

with $L_-(\lambda) := -\Delta + V_h + \lambda - f((\phi^\lambda)^2)$, and $L_+(\lambda) := -\Delta + V_h + \lambda - f((\phi^\lambda)^2) - 2f'(\phi^\lambda)^2(\phi^\lambda)^2$. The operator $L(\lambda)$ is extended to the complex space $H^2(\mathbb{R}, \mathbb{C}) \oplus H^2(\mathbb{R}, \mathbb{C})$. If the potential $V_h$ in Equation (1) decays at $\infty$, then by a general result

$$\sigma_{\text{ess}}(L(\lambda)) = (-i\infty, -i\lambda] \cap [i\lambda, i\infty).$$

The eigenfunctions of $L(\lambda)$ are described in the following theorem (cf [GS1], [GS2]).

**Theorem 4.1.** Let $V$ satisfy Condition (VA) and $|h|$ be sufficiently small. Then the operator $L(\lambda)$ has at least $2d + 2$ eigenvectors and associated eigenvectors with eigenvalues near zero: two-dimensional space with the eigenvalue $0$ and a $2d$-dimensional space with non-zero imaginary eigenvalues $\pm i\epsilon_j(\lambda)$,

$$\epsilon_j(\lambda) := h\sqrt{2\epsilon_j} + o(h),$$

where $\epsilon_j$ are eigenvalues of the Hessian matrix of $V$ at value $x = 0$, $V''(0)$. The corresponding eigenfunctions $\begin{pmatrix} \xi_j \\ \pm i\eta_j \end{pmatrix}$ are related by complex conjugation and satisfy

$$\xi_j = h\sqrt{2\epsilon_j}\phi^\lambda_0 + o(h) \text{ and } \eta_j = -h\sqrt{\epsilon_j}x_j\phi^\lambda_0 + o(h),$$

and $\xi_j$ and $\eta_j$ are real.

**Remark 2.** The zero eigenvector $\begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix}$ and the associated zero eigenvector $\begin{pmatrix} \partial_\lambda \phi^\lambda \\ 0 \end{pmatrix}$ are related to the gauge symmetry $\psi(x,t) \rightarrow e^{i\omega} \psi(x,t)$ of the original equation and the $2d$ eigenvectors $\begin{pmatrix} \xi_j \\ \pm i\eta_j \end{pmatrix}$ with $O(h)$ eigenvalues originate...
from the zero eigenvectors \( \left( \begin{array}{c} \partial_{x_k} \phi_0^\lambda \\ 0 \end{array} \right), \quad k = 1, 2, \ldots, d \), and the associated zero eigenvectors \( \left( \begin{array}{c} 0 \\ x_k \phi_0^\lambda \end{array} \right), \quad k = 1, 2, \ldots, d \), of the \( V = 0 \) equation due to the translational symmetry and to the boost transformation \( \psi(x,t) \rightarrow e^{ib\cdot x} \psi(x,t) \) (coming from the Galilean symmetry), respectively.

For \( d \geq 2 \) we will be interested in permutationally symmetric functions, \( g \in \mathcal{L}^2(\mathbb{R}^d) \), characterized as

\[
g(x) = g(\sigma x) \quad \text{for any } \sigma \in S_d
\]

with \( S_d \) being the group of permutation of \( d \) indices and \( \sigma(x_1, x_2, \ldots, x_d) := (x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(d)}) \).

**Remark 3.** For any function of the form \( e^{ip\cdot x} \phi(|x-a|) \) with \( a \parallel p \), there exists a rotation \( \tau \) such that the function \( e^{ip\cdot \tau x} \phi(|\tau x - a|) = e^{i\tau^{-1} p\cdot x} \phi(|x - \tau^{-1} a|) \) is permutationally symmetric. Such families describe wave packets with the momenta directed toward or away from the origin.

If for \( d \geq 2 \) the potential \( V(x) \) is spherically symmetric, then \( V''(0) = \frac{1}{d} \Delta V(0) \cdot I_{d \times d} \), and therefore the eigenvalues \( e_j \) of \( V''(0) \) are all equal to \( \frac{1}{d} \Delta V(0) \). Thus we have

**Corollary 4.2.** Let \( d \geq 2 \) and \( V \) satisfy Condition (VA) and let \( V \) be spherically symmetric. Then \( L(\lambda) \) restricted to permutational symmetric functions has 4 eigenvectors or associated eigenvectors near zero: two-dimensional space with eigenvalue 0; and two-dimensional space with the non-zero imaginary eigenvalues \( \pm i\epsilon(\lambda) \), where

\[
\epsilon(\lambda) = h \sqrt{\frac{2\Delta V(0)}{d}} + o(h),
\]

and with the eigenfunctions \( \left( \begin{array}{c} \xi(\lambda) \\ \pm i\eta(\lambda) \end{array} \right) \), where \( \xi \) and \( \eta \) are real, and permutational symmetric functions satisfying

\[
\xi(\lambda) = \sqrt{2} \sum_{n=1}^{d} \frac{1}{d} x_n \phi_0^\lambda + O(h) \quad \text{and} \quad \eta(\lambda) = -h \sqrt{\frac{1}{d} \Delta V(0)} \sum_{n=1}^{d} x_n \phi_0^\lambda + O(h^{3/2}).
\]

Besides eigenvalues, the operator \( L(\lambda) \) may have resonances at the tips, \( \pm i\lambda \), of its essential spectrum (those tips are called thresholds). Recall the notation \( \alpha_+ := \alpha \text{ if } \alpha > 0 \) and \( \alpha_0 \text{ if } \alpha \leq 0 \).

**Definition 1.** Let \( d \neq 2 \). A function \( h \) is called a resonance function of \( L(\lambda) \) at \( \mu = \pm i\lambda \) if \( h \notin \mathcal{L}^2 \), \( |h(x)| \leq c|x|^{-(d-2)_+} \) and \( h \) is \( C^2 \) and solves the equation

\[
(L(\lambda) - \mu) h = 0.
\]
Note that this definition implies that for $d > 2$ the resonance function $h$ solves the equation $(1 + K(λ))h = 0$ where $K(λ)$ is a family of compact operators given by $K(λ) := (L_0(λ) - μ + 0)^{-1}V_{big}(λ)$. Here $L_0(λ) := \begin{pmatrix} 0 & -Δ + λ \\ Δ - λ & 0 \end{pmatrix}$ and

$$V_{big}(λ) := \begin{pmatrix} 0 & -V_h + f((φ^λ)^2) + 2f′((φ^λ)^2)(φ^λ)^2 \\ -V_h - f((φ^λ)^2) \end{pmatrix}.$$ (8)

In this paper we make the following assumptions on the point spectrum and resonances of the operator $L(λ)$:

(SA) $L(λ)$ has only 4 standard and associated eigenvectors in the permutation symmetric subspace.

(SB) $L(λ)$ has no resonances at $±iλ$.

The discussion and results concerning these conditions, given in [GS1], suggested strongly that Condition (SA) is satisfied for a large class of nonlinearities and potentials and Condition (SB) is satisfied generically. Elsewhere we show this using earlier results of [CP, CPV]. We also assume the following condition

(FGR) Let $N$ be the smallest positive integer such that $ε(λ)(N + 1) > λ$, $∀λ \in I$. Then $ReZ_{N+1,N} < 0$ where $Z_{m,n}$, $m, n = 1, 2, \ldots$, are the functions of $V$ and $λ$, defined in Equation (14) below (see also (31)).

We expect that Condition (FGR) holds generically. Theorem 5.1 below shows that $ReZ_{n+1,n} = 0$ if $n < N$.

We expect the following is true:

(a) if for some $N_1(≥ N)$, $ReZ_{n+1,n} = 0$ for $n < N_1$, then $ReZ_{N_1+1,N_1} ≤ 0$ and (b) for generic potentials/nonlinearities there exists an $N_1(≥ N)$ such that $ReZ_{N_1+1,N_1} ≠ 0$. Thus Condition (FGR) could have been generalized by assuming that $ReZ_{N_1+1,N_1} < 0$ for some $N_1 ≥ N$ such that $ReZ_{n+1,n} = 0$ for $n < N_1$. We took $N = N_1$ in order not to complicate the exposition.

The following form of $ReZ_{N+1,N}$:

$$ReZ_{N+1,N} = Imσ_1(L(λ) - (N + 1)ie(λ) - 0)^{-1}F, F\rangle ≤ 0$$ (9)

for some function $F$ depending on $λ$ and $V$ and $σ_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, is proved in [BiSu, TY1, TY2, TY3, SW4] for $N = 1$, and in the present paper for $N = 2, 3$. We conjecture that this formula holds for any $N$. [See more explicit forms of $F$ for $N = 1, 2, 3$ in Theorem 5.1 below].

Condition (FGR) is related to the Fermi Golden Rule condition which appears whenever time-(quasi)periodic, spatially localized solutions become coupled to radiation. In the standard case it says that this coupling is effective in the second order ($N = 1$) of the perturbation theory and therefore it leads
to instability of such solutions. In our case these time-periodic solutions are stationary solutions
\[ c_1 \left( \frac{\xi}{i\eta} \right) e^{i\varepsilon(\lambda)t} + c_2 \left( \frac{\xi}{-i\eta} \right) e^{-i\varepsilon(\lambda)t} \]
of the linearized equation \( \frac{\partial^2 \varphi}{\partial t^2} = L(\lambda)\varphi \) and the coupling is realized through the nonlinearity. Since the radiation in our case is "massive" the essential spectrum of \( L(\lambda) \) has the gap \( (-i\lambda, i\lambda) \), \( \lambda > 0 \), the coupling occurs only in the \( N \)-th order of perturbation theory where \( N \) is given in Condition (FGR).

The rigorous form of the Fermi Golden Rule for the linear Schrödinger equations was introduced in \([BS]\). For nonlinear waves and Schrödinger equations the Fermi Golden Rule and the corresponding condition were introduced in \([S]\) and, in the present context, in \([SW4, BuSu, BP2, TY1, TY2, TY3]\).

Recall that a function \( g \in L^2(\mathbb{R}^d) \) is permutational symmetric if
\[ g(x) = g(\sigma x) \]
with \( S_n \) being the group of permutation of \( d \) indices and
\[ \sigma(x_1, x_2, \cdots, x_d) := (x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(d)}). \]

5 Main Theorem

The following are two main theorems of this paper.

**Theorem 5.1.** Let \( h \) be sufficiently small, the potential \( V \) be spherically symmetric and the initial date \( \psi(0) \) be permutational symmetric if \( d \geq 3 \). If \( f \) satisfies the conditions (fA)-(fC) and (GWP); \( V \) satisfies the conditions (VA)-(VC) and (GWP); \( \lambda \in \mathcal{I} \), and the spectral conditions (SA) and (SB) are satisfied, then there exists \( c, c_0 > 0 \) such that, if
\[ \inf_{\gamma \in \mathbb{R}} \{ \| \psi_0 - e^{i\gamma t}(\phi^0 + z_1^{(0)} e^{i\xi} e^{i\xi t}) + i\lambda \| \}_{H^k} + \| (1 + x^2)^{\nu} [\psi_0 - e^{i\gamma t}(\phi^0 + z_1^{(0)} e^{i\xi} e^{i\xi t})] \|_2 \} \leq c \| (z_1^{(0)}, z_2^{(0)}) \|^2 \]

with \( |(z_1^{(0)}, z_2^{(0)})| \leq c_0 \) and \( z_{n}^{(0)} \) being real, some large constant \( \nu > 0 \) and with \( k = \lfloor \frac{d}{2} \rfloor + 3 \) if \( d \geq 3 \), and \( k = 1 \) if \( d = 1 \), then there exist smooth functions \( \gamma, \xi, z_1, z_2 : \mathbb{R}^+ \to \mathbb{R}, \lambda : \mathbb{R}^+ \to \mathcal{I} \) and \( R : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{C} \) such that
\[ \psi(x, t) = e^{i \int_0^t \lambda(s) ds} e^{i\gamma(t)} \left[ \phi^0 + p_1(1 + x^2)^{\nu} e^{i\xi} + ip_2(1 + x^2)^{\nu} e^{i\xi} + p_3(1 + x^2)^{\nu} e^{i\xi} + p_4(1 + x^2)^{\nu} e^{i\xi} + R(t) \right] \]
where \( p_n(z, \bar{z}) : C \to \mathbb{R}, n = 1, 2, 3, 4, \) are polynomials of \( z := z_1 + i z_2 \) and \( \bar{z} \) and of order \( |z|^2 \), the function \( R(t) \) can be decomposed as
\[ R = \sum_{2 \leq m+n \leq k \leq 2N} R_{mn}(\lambda) e^{m z} \bar{z}^n + R_k \]
where for some large constant \( \nu \) \( \langle x \rangle^{-\nu} R_{m,n} \in L^2 \), \( \| \langle x \rangle^{-\nu} R_{2N} \|_2 \lesssim (1 + t)^{-\frac{2N+1}{2N}} \).

The functions \( \lambda, \gamma, z \) have the following properties
(A) There exists a $\lambda_\infty \in \mathcal{I}$ such that $|\lambda(t) - \lambda_\infty| \leq c(1 + t)^{-\frac{1}{2}}$ as $t \to \infty$ and moreover
\[
\dot{\lambda} = \sum_{2 \leq m+n \leq 2N+1} \Lambda_{m,n} z^m \bar{z}^n + \text{Remainder}
\]
where $\Lambda_{m,n} = 0$ if $m, n \leq N + 1$;

(B) \[
\dot{\gamma} = \sum_{2 \leq m+n \leq 2N+1} \Gamma_{m,n} z^m \bar{z}^n + \text{Remainder}
\]
where $\Gamma_{m,n} = 0$ if $m, n \leq N + 1$ and $m \neq n$; $\Gamma_{m,m}$ is real if $m \leq N$;

(C) $|z| \leq c(1 + t)^{-\frac{1}{2}}$ and moreover
\[
\dot{z} = -i\epsilon(\lambda) z + \sum_{2 \leq m+n \leq 2N+1} Z_{m,n} z^m \bar{z}^n + \text{Remainder}
\]
$Z_{n+1,n}(\lambda)$ is purely imaginary if $n < N$; $Z_{m,n}(\lambda) = 0$ if $m, n \leq N + 1$ and $m \neq n + 1$; moreover when $N = 1, 2, 3$ we have
\[
\text{Re}Z_{N+1,N} = \text{Im}\langle \sigma_1(L(\lambda) - (N + 1)i\epsilon(\lambda) - 0)^{-1}F, F \rangle \leq 0
\]
where $F$ is some vector function such that $(L(\lambda) - i(N + 1)i\epsilon(\lambda) - 0)^{-1}F = d_0 R_{N+1}$ for some constant $d_0$ (actually $F = -d_0 N_{N+1,0}$ with $N_{N+1,0}$ defined in Corollary 5.4 below), the matrix $\sigma_1$ is defined as \[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

(D) \[
p_k(z, \bar{z}) = \sum_{2N+1 \geq m+n \geq 2} P_{m,n}^{(k)}(\lambda) z^m \bar{z}^n, \quad k = 1, 2, 3, 4.
\]
If $m, n \leq N$ then $P_{m,n}^{(1)}(\lambda)$ and $P_{m,n}^{(3)}(\lambda)$ are real, $P_{m,n}^{(2)}(\lambda)$ and $P_{m,n}^{(4)}(\lambda)$ are purely imaginary.

The term Remainder is bounded as
\[
|\text{Remainder}(t)| \lesssim |z|^{2N+2}(t) + \|\langle x \rangle^{-\mu} R_N(t)\|_2^2 + \|R_N(t)\|_2^2 + |z(t)||\langle x \rangle^{-\mu} R_{2N}(t)||_2
\]
with $\nu$ being the same in Theorem \ref{thm:remainder_bound}.

In the next we present the properties of the remainder function $R$ in detail. First we define the following functions to measure the asymptotic behavior of the parameters and remainders:

\[
Z(T) := \max_{t \leq T} (T_0 + t) \frac{\lambda^N}{N!} |z(t)|, \quad \mathcal{R}_1(T) := \max_{t \leq T} (T_0 + t) \frac{\lambda^N}{N!} \|\rho^{-\nu} R_N\|_{\mathcal{H}^1},
\]
\[
\mathcal{R}_2(T) := \max_{t \leq T} (T_0 + t) \frac{\lambda^{N+1}}{(N+1)!} \|R_N(t)\|_{\mathcal{H}^\infty}, \quad \mathcal{R}_3(T) := \max_{t \leq T} (T_0 + t) \frac{\lambda^{N+1}}{(N+1)!} \|\rho^{-\nu} R_{2N}(t)\|_2
\]
\[
\mathcal{R}_4(T) := \max_{t \leq T} \|R_N(t)\|_{\mathcal{H}^1}
\]

(18)
\[ l := \left\lfloor \frac{d}{2} \right\rfloor + 3, \quad T_0 := (|z_1^{(0)}| + |z_2^{(0)}|)^{-1}, \quad \rho_\nu = \langle x \rangle^\nu, \]

recall the definitions of \( z_1^{(0)} \) and \( z_2^{(0)} \) and of the large constant \( \nu \) in Theorem 5.1. Before stating the main theorem we introduce the notion of admissible functions.

**Definition 2.** A vector-function \( \vec{u} : \mathbb{R}^d \rightarrow \mathbb{C}^2 \) is admissible if the vector-function \( \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \vec{u} \) has real entries.

**Theorem 5.2.** The function \( R \) in Theorem 5.1 satisfies the equation

\[ \text{Im} \langle R, i\phi^\lambda \rangle = \text{Im} \langle R, i\phi^\lambda \rangle = \text{Im} \langle R, \xi \rangle = 0 \quad (19) \]

and the function \( \vec{R} \) can be decomposed as

\[ \vec{R} = \sum_{2 \leq m+n \leq k \leq 2N} R_{mn}(\lambda)z^m\bar{z}^n + R_k \quad (20) \]

where the functions \( R_{mn} : \mathbb{R}^d \rightarrow \mathbb{C}^2, \quad R_k : \mathbb{R}^d \rightarrow \mathbb{R}^2 \) have the following properties:

(RA) the function \( R_{m,n} \in \mathcal{L}^2 \) is admissible, and decays exponentially fast at \( \infty \) if \( \max\{m, n\} \leq N \); and

(RB) if \( \max\{m, n\} > N \) then the functions \( R_{m,n} \) are of the form

\[ \prod_k (L(\lambda) + ike(\lambda) + 0)^{-n_k} P_c \phi_{mn}, \quad (21) \]

where the function \( \phi_{mn} \) is smooth and decays exponentially fast at \( \infty \), \( 0 \leq \sum_k n_k \leq N \), and \( 2N \geq k \geq N + 1 \); note that the equation (21) makes sense in some weighted \( \mathcal{L}^2 \) space, see [GS2, GS3];

the function \( R_k \) (\( 2 \leq k \leq 2N \)) satisfies the equation

\[ \frac{d}{dt} R_k = L(\lambda)R_k + M_k(z, \bar{z}) R_k + N_{\min(k,N)}(R_{\min(k,N)}, z, \bar{z}) + F_k(z, \bar{z}), \quad (22) \]

where

(1) \( F_k(z, \bar{z}) = O(|z|^{k+1}) \) is a polynomial of \( z \) and \( \bar{z} \) with \( \lambda \)-function-valued coefficients, and each coefficient can be written as the sum of functions of the form (21).

(2) \( M_k(z, \bar{z}) \) is an operator defined by

\[ M_k(z, \bar{z}) := \gamma P_c J + \lambda P_c + A_k(z, \bar{z}), \]

where \( A_k(z, \bar{z}) \) is a \( 2 \times 2 \) matrix bounded in the matrix norm as

\[ |A_k(z, \bar{z})| \leq c|z|^2 e^{-\epsilon_0|x|}. \]
Proof of Theorems 5.1 and 5.2

By plugging Equation (11) into Equation (11) we have

\[
\frac{d}{dt} \tilde{R} = L(\lambda) \tilde{R} + \gamma J \tilde{R} + J \tilde{N}(R, p, z) + \left( (z_2 + p_4)\eta \xi + \hat{\gamma}(z_2 + p_4)\eta \right) + p_1 \phi^\lambda - \hat{\gamma}[\phi^\lambda + \gamma (z_1 + p_3)\xi] \nabla \left[ \partial_i [p_2 \phi^\lambda] + \hat{\gamma} z_2 \xi + \hat{\gamma} [z_1 + p_3]\xi \right] \right),
\]

with \( \tilde{R} = \left( \begin{array}{c} R_1 \\ R_2 \end{array} \right) \), \( R_1 := ReR, R_2 := ImR, J \tilde{N}(R, p, z) := \left( \begin{array}{c} ImN(R, p, z) \\ -ReN(R, p, z) \end{array} \right) \) and

\[
ImN(R, p, z) := f((\phi^\lambda + I_1 + iI_2)[(\phi^\lambda)^2]I_2 - f((\phi^\lambda)^2)I_2
\]

\[
ReN(R, p, z) := [f((\phi^\lambda + I_1 + iI_2)^2) - f((\phi^\lambda)^2)]((\phi^\lambda + I_1) - 2f((\phi^\lambda)^2)(\phi^\lambda)^2)I_1
\]

with

\[
I_1 := p_1 \phi^\lambda + (z_1 + p_3)\xi + R_1, \quad I_2 := p_2 \phi^\lambda + (z_2 + p_4)\eta + R_2.
\]

From Equations (26) and the orthogonality condition (13) we obtain equations for \( z_1, z_2, \lambda, \hat{\gamma} \)

\[
\frac{d}{dt}(z_1 + p_3) - \epsilon(\lambda)(z_2 + p_4)|\xi, \eta) - \langle ImN(R, p, z), \eta \rangle = F_1;
\]

\[
\frac{d}{dt}(z_2 + p_4) + \epsilon(\lambda)(z_1 + p_3)|\xi, \eta) + \langle ReN(R, p, z), \xi \rangle = F_2;
\]

\[
\hat{\gamma} + \partial_i p_2 - p_1 + \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} \langle ReN(R, p, z), \phi^\lambda \rangle = F_3;
\]

\[
\hat{\lambda} + \partial_i p_1 - \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} \langle ImN(R, p, z), \phi^\lambda \rangle = F_4;
\]

where the scalar functions \( F_n, n = 1, 2, 3, 4, \) are defined as

\[
F_1 := \hat{\gamma}(z_2 + p_4)\eta, \hat{\gamma} - \hat{\lambda}(z_1 + p_3)\xi, \eta) - \hat{\gamma}(R_2, \eta) + \hat{\lambda}(R_1, \eta); \]

\[
F_2 := -\hat{\gamma}(z_1 + p_3)\xi, \xi) - \hat{\lambda}(z_2 + p_4)\eta, \xi) + \hat{\gamma}(R_1, \xi) + \hat{\lambda}(R_2, \xi); \]

\[
F_3 := \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} \hat{\lambda}(R_2, \phi^\lambda, \xi, \eta) - \hat{\gamma}(R_2, \phi^\lambda) - p_2 \hat{\lambda}(\phi^\lambda, \phi^\lambda)];
\]
\[
F_4 := \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} [\dot{\lambda}(R_1, \phi_\lambda^\lambda) + \dot{\gamma}(R_1, \phi^\lambda) - p_1 \dot{\lambda}(\phi_\lambda^\lambda, \phi^\lambda)].
\]

Moreover using the fact that \( P_c \frac{d}{dt} \tilde{R} = \frac{d}{dt} \tilde{R} - \dot{\lambda} P_c \tilde{R} \) we rewrite Equation (25) as
\[
\frac{d}{dt} \tilde{R} - L(\lambda) \tilde{R} - P_c J \tilde{N}(R, p, z) = \mathcal{G}
\]
where the function
\[
\mathcal{G} := -\dot{\lambda} P_c \tilde{R} + \dot{\gamma} P_c J \tilde{R} + P_c \left( \dot{\gamma}(z_2 + p_4) \eta - \lambda p_1 \dot{\phi}^\lambda - \dot{\lambda}(z_1 + p_3) \xi \right).
\]

In the next we sketch the proof of the theorem except Equation (15). We start with proving the expansion of \( R \) and the expressions for \( \dot{z}, \lambda, \dot{\gamma} \) and \( p_i, l = 1, 2, 3, 4 \), by induction on \( k = m + n, k = 2, 3, \ldots \), with the main tool Lemma 5.3 below. For the detail of finding \( R_{m,n}, \Lambda_{m,n}, \Gamma_{m,n}, Z_{m,n} \) we refer to [GS2, GS3]. For \( k = 2 \), by Lemma 5.3 we have that \( iN_{m,n}^{(2)} \), defined in (22), below, is admissible (Note \( N_{m,n}^{(2)} \), \( m + n = 2 \), does not depend on any \( R_{m',n'}, m + n \geq 2 \) by Lemma 5.3 and the term \( \mathcal{G} \) has no contribution for terms of order \( |z|^2 \). Thus \( R_{m,n} = -[L(\lambda) + i(m - n)\epsilon(\lambda)]^{-1} N_{m,n}^{(2)} \), \( m + n = 2 \), is admissible if \( N > 1 \). Moreover by the properties of \( Z_{m,n}, \Lambda_{m,n} \) and \( \Gamma_{m,n} \) in the equations for \( \dot{z}, \dot{\gamma}, z = z_1 + i \dot{z}_2 \) in (12)-(14), the equations (20)-(23) and the observation that the term \( F_j, j = 1, 2, 3, 4 \) has no terms of order \( |z|^2 \) we have if \( m + n = 2 \) and \( m \neq n + 1 \) then
\[
\begin{align*}
\left[ -i(m - n)\epsilon(\lambda) P_{m,n}^{(3)} - \epsilon(\lambda) P_{m,n}^{(4)} \right] + i\left[ -i(m - n)\epsilon(\lambda) P_{m,n}^{(4)} + \epsilon(\lambda) P_{m,n}^{(3)} \right] \\
&= \frac{1}{\langle \eta, \eta \rangle} \left[ (Im N_{m,n}^{(2)}, \eta) - i(Re N_{m,n}^{(2)}, \xi) \right]
\end{align*}
\]
and if \( m' + n' = 2 \) and \( m' \neq n' \) then
\[
\begin{align*}
-i(m' - n')\epsilon(\lambda) P_{m',n'}^{(2)} - P_{m',n'}^{(1)} &= \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} (Re N_{m',n'}^{(2)}, \phi^\lambda),
-i(m' - n')\epsilon(\lambda) P_{m',n'}^{(1)} &= \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} (Im N_{m',n'}^{(2)}, \phi^\lambda)
\end{align*}
\]
where \( \begin{pmatrix} Im N_{m,n}^{(2)} \\ -Re N_{m,n}^{(2)} \end{pmatrix} := N_{m,n}^{(2)} \). We observe the solutions satisfying the condition \( P_{m,n}^{(k)} = P_{m,m}^{(k)}, k = 1, 2, 3, 4 \), to the equations above exist, and by the properties of \( N_{m,n}^{(2)} \) we obtain \( P_{m,n}^{(1)}(\lambda) \) and \( P_{m,n}^{(3)}(\lambda) \) are real, \( P_{m,n}^{(2)}(\lambda) \) and \( P_{m,n}^{(4)}(\lambda) \) are purely imaginary. Thus we finish the proof of the step of \( m + n = 2 \). For the step \( m + n = 3 \), \( F_{m,n}, R_{m,n}, m' + n' = 2 \) and Lemma 5.3 enable us to determine \( F_{m,n}, R_{m,n}, m + n = 3 \) and prove all the properties in the theorem. By using the procedures above repeatedly we finish the proof.
By using the same techniques in [GS2, GS3] and Equation (22) we prove the decay of $R_N$ and $R_{2N}$ in appropriate norms; and by (12) there exists a new parameter $\beta = z + O(|z|^2)$ such that
\[
\dot{\beta} = -i\epsilon(\lambda)\beta + \sum_{1 \leq n \leq N} Y_n \beta^{n+1} \bar{\beta}^n + \text{Remainder},
\] (31)
where $Y_n$ is purely imaginary if $n < N$ and $ReY_N = ReZ_{N+1,N}$, consequently we have
\[
\frac{1}{2} \frac{d}{dt} |z|^2 = ReZ_{N+1,N}|z|^{2(N+1)} + |z|\text{Remainder}
\]
which implies the decay of $\beta$, hence $y$, by the assumption $ReZ_{N+1,N} < 0$ and the estimates on $|z|\text{Remainder}$; and similar as in [GS3] by the expressions for $\lambda$ in (12) there exist $c_{m,n}(\lambda)$ such that
\[
\frac{d}{dt} \left[ \lambda - \sum_{2N+1 \geq m+n \geq 2} c_{m,n}(\lambda) z^m \bar{z}^n \right] = \text{Remainder}
\]
with the the term $\text{Remainder}$ satisfies (17), consequently $\lambda$ is convergent due to the decay of $z$ and the fact that $\text{Remainder}(t)$ is integrable at $\infty$.

The proof of Equation (15) is in Sections 7 and 8 below.

\[
\square
\]

In the proof we use the following lemma.

**Lemma 5.3.** (A) If $G_1$ is a vector-function from $\mathbb{R}^d$ to $\mathbb{C}^2$ such that $iG_1$ is admissible and $G_1 \in L^2$, then the vector function $(L(\lambda) + i\epsilon)^{-1} P_i G_1$ is admissible for any $\mu \in \{\lambda, -\lambda\}$.

(B) Let $R_k$, $R_{m,n}$, $p_v$ be the same as those in (20) and (17) respectively and
\[
p_v = \sum_{m+n \geq 2} q_{m,n}^{(v)}(\lambda) z^m \bar{z}^n, \hspace{1em} v = 1, 2, 3, 4
\]
with $q_{m,n}^{(1)}(\lambda)$ and $q_{m,n}^{(3)}(\lambda)$ are real, $q_{m,n}^{(2)}(\lambda)$ and $q_{m,n}^{(4)}(\lambda)$ are purely imaginary if $m, n \leq \min\{k, N\}$. Then $J\tilde{N}(R,p,z)$ can be expanded as
\[
J\tilde{N}(R,p,z) = \sum_{2 \leq m+n \leq 2N} z^m \bar{z}^n N_{m,n}^{(k)}(\lambda) + A_k(z, \bar{z}) R_k
+ N_{\min\{k,N\}}(R_{\min\{k,N\}}, p, z) + \text{Remainder}_1
\] (32)
where $iN_{m,n}^{(k)}(\lambda) : \mathbb{R}^3 \rightarrow \mathbb{C}^2$ is admissible if $m, n \leq N$ and only depends on $R_{m',n'}$, $q_{m',n'}^{(v)}$ with $m' \leq m, n' \leq n$ and $|m' - m| + |n' - n| \neq 0$; the $2 \times 2$-matrix function $A_k(z, \bar{z})$ is bounded in the matrix norm as $|A_k(z, \bar{z})| \leq c|z| e^{-\epsilon_0|x|}$; $N_j(R_j, p, z), j \leq N$, contains all the nonlinear terms in $R_j$, $N_N(R_N, p, z)$ admits the same estimates as in (20) (24); and the term $\text{Remainder}_1$ has the estimate
\[
|\text{Remainder}_1| \leq c|z|^{2N+1} e^{-\epsilon_0|x|}.
\] (33)
Proof. The first part is copied from [GS2, GS3].

For the second part we prove the expansion for $J\vec{N}(\vec{R}, p = 0, z)$ in [GS2, GS3]. The proof is almost identical to that, thus omitted.

The following corollary will be used later, whose proof is obvious by (32), thus omitted.

Corollary 5.4. Let $k = 2N$ in (30), then

$$J\vec{N}(R, p, z) = \sum_{m+n=2}^{2N} N_{m,n}(\lambda)z^m\bar{z}^n + N_N(R, p, z) + \text{Remainder}_1$$

with $iN_{m,n}$ admissible for $m, n \leq N$; $N_N(R, p, z)$ and $\text{Remainder}_1$ have the same estimates as above.

6 The Effective Equations for $\dot{z}$, $\dot{\lambda}$, $\dot{\gamma}$ and $R$

The equations for the parameters $\dot{z}$, $\dot{\lambda}$, $\dot{\gamma}$ and the function $R$ have some terms having no role on determining $\text{Re}Z_{N+1,N}$. In this subsection we separate the effective parts from the "useless" parts. We will use the following lemma to show that the terms on the right hand side of Equations (26)-(29) and (30) are useless.

Lemma 6.1. If we expand the scalar functions $F_1 + iF_2$, $F_3$, $F_4$ and the function $\mathcal{G}$ in $z$ and $\bar{z}$, then we have

$$F_1 + iF_2 = \sum_{2 \leq m+n \leq 2N+1} K_{m,n}^{(1)}(\lambda)z^m\bar{z}^n + \text{Remainder}$$

$$F_3 = \sum_{2 \leq m+n \leq 2N} K_{m,n}^{(2)}(\lambda)z^m\bar{z}^n + \text{Remainder}$$

$$F_4 = \sum_{2 \leq m+n \leq 2N} K_{m,n}^{(3)}(\lambda)z^m\bar{z}^n + \text{Remainder}$$

$$\mathcal{G} = \sum_{2 \leq m+n \leq 2N} \mathcal{G}_{m,n}(\lambda)z^m\bar{z}^n + GN(R, z, \bar{z}) + \text{Remainder}_1$$

where if $m, n \leq N + 1$, then the coefficients $K_{m,n}^{(1)}$, $K_{m,n}^{(3)}$ are purely imaginary, $K_{m,n}^{(2)}$ is real, the function $i\mathcal{G}_{m,n}$ is admissible; $K_{m,n}^{(k)} = 0$ and $\mathcal{G}_{m,n} = 0$ if $(m, n) = (0, l)$ or $(l, 0)$ with $l \leq 2N$; moreover the function $\text{Remainder}_1$ satisfies the estimate (33); the term $GN(R, z, \bar{z})$ admits the same estimates as $N_N(R, z, \bar{z})$ in (28)/(29).

Proof. By the expansions for the scalar functions $\dot{\lambda}$, $\dot{\gamma}$ and the function $\vec{R}$ in Theorems 5.1 and 5.2 we prove the lemma by direct computations. □
7 Proof of Equation (15) when \( N = 2 \)

In this section we prove Equation (15) when \( N = 2 \). For consideration of space we only prove the case of the nonlinearity \( f(x) = x \). One can show that the results hold for more general nonlinearities, but the computations are much messier. The main result is

**Theorem 7.1.** If \( N = 2 \) and the nonlinearity \( f(x) = x \) in (14), then

\[
ReZ_{3,2} = \frac{6}{\langle \xi, \eta \rangle} Im\langle \sigma_1(L(\lambda) - 3i\epsilon(\lambda) - 0)^{-1}P_{c}N_{3,0}, N_{3,0} \rangle, \tag{36}
\]

where, recall the definition of \( N_{3,0} \) from Corollary 5.4.

The proof is in subsection 7.2.

7.1 the coefficient of \( z^3 \bar{z}^2 \) in the expansion of \( \langle ImN(R, p, z), \eta \rangle - i \langle ReN(R, p, z), \xi \rangle \)

Suppose that the expansion of \( \langle ImN(R, p, z), \eta \rangle - i \langle ReN(R, p, z), \xi \rangle \), the part on the left hand side of Equations (26) and (27), in \( z \) and \( \bar{z} \) is

\[
\langle ImN(R, p, z), \eta \rangle - i \langle ReN(R, p, z), \xi \rangle = \sum_{2 \leq m + n \leq 5} X_{m,n} z^m \bar{z}^n + \text{Remainder}.
\]

In the next we compute \( X_{3,2} \). Before starting the proof we define the following functions and constants:

\[
K_1 := \left( \begin{array}{c}
-P_{2,0}^{(2)}(\phi^\lambda)^2 \xi + iP_{2,0}^{(1)}(\phi^\lambda)^2 \bar{\eta} \\
3P_{2,0}^{(2)}(\phi^\lambda)^2 \xi - iP_{2,0}^{(2)}(\phi^\lambda)^2 \bar{\eta}
\end{array} \right), \quad K_2 := \left( \begin{array}{c}
i\phi^\lambda \bar{\eta}, \\
3\phi^\lambda \xi, \\
i\phi^\lambda \bar{\eta}
\end{array} \right) R_{2,0},
\]

\[
K_3 := \frac{1}{8} \left( \begin{array}{c}
-i\eta^3 + i \xi^2 \bar{\eta} \\
-\xi \eta^2 + \xi^3
\end{array} \right), \quad K_4 := \left( \begin{array}{c}
-\phi^\lambda \xi \eta [P_{2,0}^{(4)} - iP_{2,0}^{(3)}] \\
-i\phi^\lambda \eta^2 P_{2,0}^{(4)} + 3\phi^\lambda \xi^2 P_{2,0}^{(3)}
\end{array} \right)
\]

and

\[
D_1(\lambda) := i(P_{3,1}^{(1)}(\phi^\lambda (\eta^2 - 3\xi^2), \phi^\lambda) - iP_{3,1}^{(2)}(2\xi \phi^\lambda, \phi^\lambda)),
\]

\[
D_2(\lambda) := -2i(\sigma_1 R_{3,0}, K_1 + K_2 + 3K_3 + K_4),
\]

\[
D_3(\lambda) := \langle R_{3,1}, \left( -i(\phi^\lambda \eta^2 - 3\phi^\lambda \xi^2) \right) \rangle,
\]

and

\[
D_4 := P_{3,1}^{(5)}[-3i(\phi^\lambda \xi^2, \xi) + i(\phi \xi \eta, \eta)] + \sum_{l=1}^{4} P_{3,1}^{(l)}(2\phi^\lambda \eta^2, \xi)
\]

and the matrix \( \sigma_1 := \left( \begin{array}{cc}
0 & -1 \\
1 & 0
\end{array} \right) \), recall the definitions of \( P_{m,n}^{(l)} \), \( l = 1, 2, 3, 4 \) in Theorems 5.1. The main result is
Remark 4.

\[ \text{Remark 4.} \]

The proof is at the end of this section. In the proof the following observation will be used.

Lemma 7.3. \( P^{(k)}_{m,n} = P^{(k)}_{n,m} \) for any \( m, n \) and \( k \); If \( k = 1, 3 \) and \( m, n \leq 2 \) then \( P^{(k)}_{m,n} \) is real; if \( k = 2, 4 \) and \( m, n \leq 2 \) then \( P^{(k)}_{m,n} \) is purely imaginary. The constants \( P^{(k)}_{m,n} \) have the following expressions:

\[
\begin{align*}
    P_{2,0}^{(1)} &= \frac{1}{4\epsilon(\lambda)} \langle \phi^\lambda \xi \eta, \phi^\lambda \rangle, \\
    P_{2,0}^{(2)} &= -\frac{P_{2,0}^{(1)}}{i2\epsilon(\lambda)} + \frac{3\phi^\lambda \xi^2 - \phi^\lambda \eta^2, \phi^\lambda \rangle}{8i\epsilon(\lambda)} + \langle \phi^\lambda \xi, \phi^\lambda \rangle, \\
    -2iP_{2,0}^{(3)} - P_{2,0}^{(4)} &= \frac{1}{4\epsilon(\lambda)} \langle \phi^\lambda \xi, \phi^\lambda \rangle, \\
    -2iP_{2,0}^{(4)} + P_{2,0}^{(3)} &= \frac{1}{4\epsilon(\lambda)} \langle \phi^\lambda \xi, \phi^\lambda \rangle, \\
    -2iP_{3,1}^{(3)} - P_{3,1}^{(4)} &= \frac{1}{\epsilon(\lambda)} \langle R_{3,0}, \left( -i\phi^\lambda \xi \phi^\lambda \eta \right) \rangle U, \\
    -2iP_{3,1}^{(4)} + P_{3,1}^{(3)} &= \frac{1}{\epsilon(\lambda)} \langle R_{3,0}, \left( -3\phi^\lambda \xi^2 - i\phi^\lambda \xi \phi^\lambda \eta \right) \rangle U, \\
    P_{3,1}^{(1)} &= \frac{1}{2\epsilon(\lambda)} \langle R_{3,0}, \left( -\phi^\lambda \xi^2 \right) \rangle U, \\
    P_{3,1}^{(2)} &= -\frac{P_{3,1}^{(3)}}{2i\epsilon(\lambda)} + \frac{i}{\epsilon(\lambda)} \langle R_{3,0}, \left( 3\phi^\lambda \phi^\lambda \xi \phi^\lambda \eta \right) \rangle U.
\end{align*}
\]

\( U \) stands for real (different) constants throughout the rest of the paper. If \( m, n \leq 3 \) and \( |m - 3| + |n - 3| \neq 0 \) then

\[
R_{m,n} = -(L(\lambda) + i(m - n)e(\lambda) + 0)^{-1}[N_{m,n} - G_{m,n}]. \tag{37}
\]

Proof. The property \( P^{(k)}_{m,n} = P^{(k)}_{n,m} \) following from the fact that \( \lambda, \gamma, z_1, z_2 \) are real functions. The fact \( P^{(k)}_{m,n} \) being real or purely imaginary is based on the properties of \( N_{m,n} \) in Corollary 5.4. By Equations (26)-(29) and the properties of \( \tilde{\lambda}, \tilde{\gamma}, \tilde{z} \) in Theorem 5.1 we compute to get the constants \( P^{(k)}_{m,n} \).

The formula of \( R_{m,n} \) follows from the expansion, we refer to our paper [GS2] [GS3].

We have the important remark for \(-0\) symbol in the expression of \( R_{m,n} \).

Remark 4. If \( |m - n| \leq 2 \) then

\[
(L(\lambda) + i(m - n)e(\lambda) + 0)^{-1} = (L(\lambda) + i(m - n)e(\lambda))^{-1}.
\]
Proof of Proposition 7.2

For the fact
\[ R_{X,3,2} = \sum_{n=1}^{4} \text{Re} D_n(\lambda) \]
we only use Lemma 7.3, the fact that \( R_{m,n} \) is admissible if \( m, n \leq N = 2 \) in Theorem 5.2 and direct computation, thus omit the detail.

For the fact on the right hand side we transform the expression for \( D_n, n = 1, 2, 3, 4 \).

First we prove the \( D_1 \) part. By the expressions of \( P^{(1)}_{2,0}, P^{(2)}_{2,0}, P^{(2)}_{3,1} \) and the properties of \( P^{(k)}_{m,n} \) in Lemma 7.3 we have
\[
\text{Re} D_1 = \text{Re} i\left[ P^{(1)}_{3,1}(\phi^\lambda(\eta^2 - 3\xi^2), \phi^\lambda_\phi) + 2P^{(2)}_{3,1}(-i\xi\eta\phi^\lambda, \phi^\lambda) \right]
= 4\text{Re}\left( R_{3,0}, \begin{pmatrix} 3P^{(1)}_{2,0}i\phi^\lambda \phi^\lambda_\xi + P^{(2)}_{2,0}(\phi^\lambda)^2 \eta \\ P^{(1)}_{2,0} \phi^\lambda \phi^\lambda_\eta + iP^{(2)}_{2,0}(\phi^\lambda)^2 \xi \end{pmatrix} \right)
= -\text{Re}\left( R_{3,0} + 4i\sigma_1 K_1 \right).
\]

For \( D_3 \) we compute \( R_{2,0} \) and \( R_{3,1} \) first. Recall the forms of \( R_{m,n} \) in Lemma 7.3 and the properties of \( G_{m,n} \) in (35). First we have
\[
R_{2,0} = -[L(\lambda) + 2i\epsilon(\lambda)]^{-1} P_c N_{2,0} \tag{38}
\]
with \( N_{2,0} = \frac{1}{4} \begin{pmatrix} -2i\phi^\lambda \xi \eta \\ -3\phi^\lambda \xi^2 + \phi^\lambda \eta^2 \end{pmatrix} \). The function \( R_{3,1} \) has two parts
\[
R_{3,1} = -[L(\lambda) + 2i\epsilon(\lambda)]^{-1} P_c (G_1 + K_5) \tag{39}
\]
where the vector function \( iG_1 \) is admissible, hence \(-[L(\lambda) + 2i\epsilon(\lambda)]^{-1} P_c G_1 \) is admissible (thus “useless”) by Lemma 5.2 and the vector function \( K_5 \) is
\[
K_5 := - \begin{pmatrix} i\phi^\lambda \eta & \phi^\lambda_\xi \\ -3\phi^\lambda \xi & -i\phi^\lambda \eta \end{pmatrix} \tag{40}
\]
which is part of \( N_{3,1} \). By the observations that \( \sigma_1(L(\lambda))^* \sigma_1 = L(\lambda), \sigma_1^2 = -1 \) and the definition of \( K_2 \) we have
\[
\text{Re} D_3 = -\text{Re}\left( R_{3,0} + 4i\sigma_1 K_2 \right).
\]

\( D_4 \) admits the form \( \text{Re} D_4(\lambda) = 4Im(\sigma_1 R_{3,0}, K_4) \), which follows from some manipulation on the expressions of \( P^{(n)}_{2,0} \) and \( P^{(n)}_{3,1} \). The proof is tedious, but not hard, thus omitted.

Collecting the computation above and recalling the form of \( D_2 \) we have
\[
\sum_{n=1}^{4} \text{Re} D_n(\lambda) = 6Im(\sigma_1 R_{3,0}, \sum_{n=1}^{4} K_n).
\]
Thus the proof is complete. \( \square \)
7.2 Proof of Theorem [7.1]

Proof. Recall that $X_{3,2}$ is the coefficient of $z^3\bar{z}^2$ in the expansion of the term $\langle \text{Im}N(R,p,z),\eta \rangle - i\langle \text{Re}N(R,p,z),\xi \rangle$. First by Equations (26), (27) and (34) we have $ReZ_{3,2} = \frac{1}{\langle \xi,\eta \rangle} ReX_{3,2}$, then by Proposition 7.2

$$ReX_{3,2} = \sum_{n=1}^{4} ReD_n(\lambda) = 6Im\langle \sigma_1 R_{3,0}, \sum_{n=1}^{4} K_n \rangle,$$

where, recall the definitions of $K_n$ before (30).

By (37) and the properties of $G$ in (35) for $N=2$

$$R_{3,0} = -(L(\lambda) + 3i\epsilon(\lambda) + 0)^{-1}P_cN_{3,0}$$

and we compute to get

$$N_{3,0} = -\sum_{n=1}^{4} K_n.$$

Consequently we have

$$ReZ_{3,2} = \frac{1}{\langle \xi,\eta \rangle} \sum_{n=1}^{4} ReD_n(\lambda) = \frac{6}{\langle \xi,\eta \rangle} Im\langle \sigma_1(\lambda) - 3i\epsilon(\lambda) - 0 \rangle^{-1}P_cN_{3,0}, N_{3,0} \rangle.$$

Thus the proof is complete. \qed

8 Proof of Equation (15) when $N=3$

The proof of $N=3$ is more involved, but the ideas are the same. Thus in this section we just state the key steps and only prove the case $f(x) = x$ and the potential $V$ is spherical symmetric. The following is the main result. Recall that $\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the definition of $N_{4,0}$ from Corollary 5.4.

Theorem 8.1. If $N=3$ and the nonlinearity $f(x) = x$ and the potential $V$ is spherical symmetric in (4), then the constant $Z_{4,3}$ satisfies the equation

$$ReZ_{4,3} := \frac{8}{\langle \xi,\eta \rangle} Im\langle \sigma_1(\lambda) - 4i\epsilon(\lambda) - 0 \rangle^{-1}N_{4,0}, N_{4,0} \rangle$$

where, recall the definition of $N_{4,0}$ from Corollary 5.4 (for $N=3$).

In the next we state the following observations which will simplify our computation. We defined two spaces

$$S = \{ g : \mathbb{R}^d \to \mathbb{C} | g(-x) = g(x) \}; \quad AS = \{ g : \mathbb{R}^d \to \mathbb{C} | g(-x) = -g(x) \}.$$ 

Obviously $S \perp AS$ and $\phi^\lambda, \phi^\lambda \in S$ and $\xi, \eta \in AS$ (see Corollary 1.2). Recall that $N_{m,n}$ and $G_{m,n}$ are defined in Corollary 5.4 and Equation (39) respectively.
Lemma 8.2. If the potential $V$ is spherical symmetric, then $R_{m,n}$, $N_{m,n} \in \mathbb{S}$ if $m + n = 2, 4, 6$; $R_{m,n}$, $N_{m,n} \in \mathbb{AS}$ if $m + n = 3, 5$. Moreover

$$p_{m,n}^{(3)} = p_{m,n}^{(4)} = p_{m',n'}^{(1)} = p_{m',n'}^{(2)} = 0$$

if $m + n$ is even or $m' + n'$ is odd. If $m, n \leq 3$ then $P_{m,n}^{(1)}$, $P_{m,n}^{(3)}$ are real and $P_{m,n}^{(2)}$, $P_{m,n}^{(4)}$ are purely imaginary, and

$$P_{m,n}^{(k)} = P_{n,m}^{(k)} \text{ if } k = 1, 3, \quad P_{m,n}^{(l)} = -P_{n,m}^{(l)} \text{ if } l = 2, 4. \quad (43)$$

If $m, n \leq 3$ and $|m - 3| + |n - 3| \neq 0$ then

$$R_{m,n} = -(L(\lambda) + i(m - n)\epsilon(\lambda) + 0)^{-1}[N_{m,n} - G_{m,n}].$$

Proof. The facts $P_{m,n}^{(k)}$ being real or purely imaginary follow from similar discussion in Lemma 7.3. The rest follows from direct computation based on the properties of $N_{m,n}$ in Corollary 5.4 and the observations made on $\xi, \eta, \phi^4, \phi^2$ before the lemma, thus we omit the detail. \(\square\)

8.1 The Explicit Forms of $R_{4,0}$ and $N_{4,0}$

We denote the coefficients of $z^m \bar{z}^n$ in the expansion of $\text{Im} N(R, p, z)$, $(\text{Re} N(R, p, z))$ as $\text{Im} N_{m,n}$, $(\text{Re} N_{m,n})$. The following lemma is the main result of this section.

Lemma 8.3.

$$R_{4,0} = -(L(\lambda) + 4i\epsilon(\lambda) + 0)^{-1} N_{4,0}$$

with $N_{4,0}$ given explicitly below.

The proof follows from the definition of $R_{4,0}$ in Lemma 8.2, the property $G_{4,0} = 0$ in (35) for $N = 3$.

In the next we compute $N_{4,0} = \begin{pmatrix} \text{Im} N_{4,0} \\ \text{Re} N_{4,0} \end{pmatrix}$ explicitly. As usual we denote

$$R_{m,n} = \begin{pmatrix} P_{m,n}^{(1)} \\ P_{m,n}^{(2)} \end{pmatrix}$$

and recall that for some $m, n, k$, $P_{m,n}^{(k)} = 0$ from Lemma 8.2.

We compute to get

$$\text{Im} N_{4,0} = \sum_{n=1}^{3} \mathcal{H}_n^{(1)}, \quad \text{Re} N_{4,0} = \sum_{n=1}^{3} \mathcal{H}_n^{(2)}$$

(45)

with the functions $\mathcal{H}_n^{(k)}$, $n = 1, 2, 3$, $k = 1, 2$, defined as

$$\mathcal{H}_1^{(1)} := 2(\phi^4)^2 R_{2,0}^{(1)} P_{2,0}^{(1)} + 2\phi^4 R_{2,0}^{(2)} R_{2,0}^{(2)} - \frac{i}{2} \epsilon \eta R_{2,0}^{(2)}$$

$$+ 2\phi^4 \phi^2 P_{2,0}^{(1)} R_{2,0}^{(2)} + \frac{1}{4} \epsilon^2 R_{2,0}^{(2)} - \frac{3}{2} \eta^2 R_{2,0}^{(2)}$$

$$+ 2(\phi^4)^2 \phi^2 P_{2,0}^{(1)} P_{2,0}^{(2)} + \frac{1}{4} \phi^2 \delta^2 P_{2,0}^{(2)} - \frac{3}{2} \phi^2 \eta^2 P_{2,0}^{(2)} - \frac{3}{2} \phi \delta \eta P_{2,0}^{(2)}$$

$$\mathcal{H}_2^{(1)} := \phi \delta \eta [P_{3,0}^{(4)} - i P_{3,0}^{(3)}];$$

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8.2 The Explicit Form of Lemma 8.4.

The following lemma is the main result of this section.

**Lemma 8.4.**

\[ \text{Re} Z_{4,3} = \frac{1}{\langle \xi, \eta \rangle} \text{Re}[E_1 + E_2 + E_3] \quad (46) \]

where the constants \( E_n, \ n = 1, 2, 3, \) are defined below.

**Proof.** By Lemma 8.1, we have that the right hand sides of (24) and (25) are useless in determining the real part of \( Z_{4,3}. \) Thus \( \text{Re} Z_{4,3} = \frac{1}{\langle \xi, \eta \rangle} \text{Re}[\text{Im} N_{4,3}, \eta] - i(\text{Re} N_{4,3}, \xi) \) by the fact \( z = z_1 + iz_2. \) What is left is just direct computation while deserting useless terms by using the property of \( P_m(k), \ k = 1, 2, 3, 4 \) in Lemma 8.2, and using the fact \( R_{m,n}, m, n \leq 3 \) are admissible. \( \square \)

In the next we give the definitions of \( E_n, \ n = 1, 2, 3, \) and the various related terms.

\[ E_1 := \langle R_{4,0}, \left( F_1 \right) \rangle + \langle R_{4,2}, \left( F_5 \right) \rangle - 4i\langle R_{4,1}, \left( \Omega_1 \Omega_2 \right) \rangle - P^{(3)}_{4,1} G_5 - P^{(4)}_{4,1} G_6; \]

\[ E_2 := P^{(3)}_{4,1} (G_1 + G_5) + P^{(4)}_{4,1} (G_2 + G_6) + P^{(1)}_{4,2} G_3 + P^{(2)}_{4,2} G_4; \]

\[ E_3 := \langle R_{4,1}, \left( F_3 \right) \rangle + 4i\langle R_{4,1}, \left( \Omega_1 \Omega_2 \right) \rangle; \]

\[ F_1 := -2\phi^4 \eta^2 P^{(3)}_{3,0} - i6\phi^4 \xi^2 P^{(3)}_{3,0} - 2\phi^4 \eta R^{(2)}_{3,0} - i6\phi^4 \xi R^{(2)}_{3,0} + i\phi^4 \xi^2 R^{(2)}_{1,0}; \]

\[ F_2 := -2\phi^4 \eta \xi P^{(3)}_{3,0} - i2\phi^4 \xi P^{(3)}_{3,0} - 2\phi^4 \eta \xi R^{(2)}_{3,0} - i2\phi^4 \xi R^{(2)}_{3,0} + i2\phi^4 \xi \eta R^{(2)}_{2,0}; \]

\[ F_3 := -2(\phi^2)^2 \eta P^{(2)}_{2,0} - 2\phi^4 \eta R^{(2)}_{2,0} + 3i\eta^2 \xi^2 - 4\xi^3 - i6\phi^4 \xi R^{(2)}_{1,0} - i6\phi^4 \xi \eta P^{(2)}_{1,0}; \]

\[ F_4 := -2\phi^4 \phi^2 \eta P^{(2)}_{2,0} - 2\phi^4 \eta R^{(2)}_{2,0} - i3\eta^2 \xi^2 - 2iP^{(2)}_{2,0} (\phi^2)^2 \xi + 4\eta^3 - 2i\phi^4 \xi P^{(2)}_{2,0}. \]
\[ \mathcal{F}_5 := i\phi^2\eta^2 - i3\phi^2\xi^2; \]
\[ \mathcal{F}_6 := -2\phi^2\xi\eta; \]
\[ \Omega_1 := 3\phi^2\xi R_{2,0}^{(1)} - i\phi^2\eta R_{2,0}^{(2)}; \]
\[ \Omega_2 := \phi^2\xi R_{2,0}^{(2)} - i\phi^2\eta R_{2,0}^{(1)}; \]
\[ G_1 := -(2(\phi^2\eta^2) P_{0,2}^{(2)} - 2(\phi^2\xi\eta) R_{0,2}^{(2)} - \frac{3i}{4}\xi^2\eta^2\]
\[ + (6i\phi^2\xi^2\xi P_{0,2}^{(1)} + \frac{3i}{4}\xi^4) + 6i(\phi^2\xi^2 R_{0,2}^{(1)}); \]
\[ G_2 := 2i P_{0,2}^{(2)}(\phi^2\eta^2) - (2\phi^2\xi^2 P_{0,2}^{(1)} - 2(\phi^2\eta^2 R_{0,2}^{(1)})
+ \frac{3i}{4}\eta^4 - \frac{3i}{4}\xi^2\eta^2 + i(2\phi^2\xi^2 R_{0,2}^{(1)}); \]
\[ G_3 := -i(\phi^2\xi^2\xi^2) + 3i(\phi^2\xi^2\xi^2); \]
\[ G_4 := -(\phi^2\xi^2\xi^2); \]
\[ G_5 := 12i(\phi^2\xi^2, R_{2,0}^{(1)}) - 4(\phi^2\xi^2, R_{2,0}^{(2)}); \]
\[ G_6 := -4(\phi^2\xi^2, R_{2,0}^{(1)}) + 4(\phi^2\xi^2, R_{2,0}^{(2)}); \]

where we use \( \langle g_1, g_2 \rangle \) to denote \( \int g_1(x)dx, \int g_1g_2dx \) respectively for any function \( g_1 \) and \( g_2 \).

### 8.3 Expansion of \( E_n, \ n = 1, 2, 3 \)

In this section we first relate the terms \( R_{4,1}, R_{4,2}, P_{4,2}^{(k)}, \ k = 1, 2, 3, 4, \) to the function \( R_{4,0} \) so that \( ReZ_{4,3} = Re(R_{4,0}, \Psi) \) for some function \( \Psi \). Then after some manipulation we relate the function \( \Psi \) to \( N_{4,0} \), hence to \( R_{4,0} \). Recall that the relation between \( R_{4,0} \) and \( N_{4,0} \) in Lemma 8.3.

The main result is the following proposition whose proof is in the later part of this section.

**Proposition 8.5.**

\[
Re[E_2] = Re[E_{1,0} + E_{4,0} + P_{4,1}^{(3)} Y_1 + P_{4,1}^{(4)} Y_2], \tag{47}
\]

and

\[
Re[P_{4,1}^{(3)} Y_1 + P_{4,1}^{(4)} Y_2] = Re[(R_{4,0}^{(1)}, -6iH_{2}^{(2)}) + (R_{4,0}^{(2)}, -6iH_{2}^{(1)}), \tag{48}
\]

\[
Re[E_{1} + E_{4,0}] = Re[(R_{4,0}^{(1)}, -8iH_{2}^{(2)} + 2iH_{2}^{(2)} - 2iH_{2}^{(3)})
+ Re[(R_{4,0}^{(2)}, -8iH_{2}^{(1)} - 2iH_{2}^{(1)} - 2iH_{2}^{(3)}), \tag{49}
\]

\[
Re[E_{3} + E_{4,1}] = Re[(R_{4,0}^{(1)}, -6iH_{3}^{(2)} + (R_{4,0}^{(2)}, -6iH_{3}^{(1)})] \tag{50}
\]

where the functions \( Y_1, Y_2, E_{4,0} \) and \( E_{4,1} \) are defined below.
In the decomposition of $E_2$ above we used the following constants

$$E_{4,0} := -4i(R_{4,0}^{(1)}, M_1^{(1)} P_{0,2}^{(2)} + M_1^{(2)} P_{0,2}^{(1)}) - 4i(R_{4,0}^{(2)}, M_1^{(1)} P_{0,2}^{(2)} + M_2^{(2)} P_{0,2}^{(1)})$$

$$E_{4,1} := -4i(R_{4,1}^{(1)}, M_3^{(1)} P_{0,2}^{(2)} + M_3^{(2)} P_{0,2}^{(1)}) - 4i(R_{4,1}^{(2)}, M_4^{(1)} P_{0,2}^{(2)} + M_4^{(2)} P_{0,2}^{(1)})$$

$$Y_1 := -4i W_1^{(1)} P_{2,0}^{(2)} - 4i W_1^{(2)} P_{2,0}^{(1)} + G_1 + G_5,$$

and

$$Y_2 := -4i W_2^{(1)} P_{2,0}^{(2)} - 4i W_2^{(2)} P_{2,0}^{(1)} + G_2 + G_6.$$

Recall the notation $\delta'(\lambda) := \frac{1}{2} \frac{d}{d\lambda} \langle \phi^\lambda, \phi^\lambda \rangle$. Before starting proving the proposition, we compute the explicit form of $P_{4,2}^{(1)}, P_{4,2}^{(2)}$:

$$-2i\epsilon(\lambda) \delta'(\lambda) P_{4,2}^{(1)} = (R_{4,0}^{(1)}, M_1^{(1)}) + (R_{4,0}^{(2)}, M_1^{(1)}) + (R_{4,1}^{(1)}, M_3^{(2)}) + (R_{4,1}^{(2)}, M_4^{(2)}) + P_{4,1}^{(2)} W_1^{(1)} + P_{4,1}^{(1)} W_1^{(2)} + iU. \tag{51}$$

where, $U$ is a real constant and $iU$ includes all the contributions from $F_4$ (see Lemma 6.1),

$$M_1^{(1)} := 2(\phi^\lambda)^3 P_{2,0}^{(2)} + 2(\phi^\lambda)^2 R_{2,0}^{(1)} - \frac{i}{2} \phi^\lambda \xi \eta,$$

$$M_2^{(1)} := 2(\phi^\lambda)^2 \phi^\lambda P_{2,0}^{(2)} + 2(\phi^\lambda)^2 R_{2,0}^{(1)} + \frac{1}{4} \phi^\lambda \xi^2 - \frac{3}{4} \phi^\lambda \eta^2,$$

$$M_3^{(1)} := -i(\phi^\lambda)^2 \eta, \quad M_4^{(1)} := (\phi^\lambda)^2 \xi,$$

$$W_1^{(1)} := i((\phi^\lambda)^2 \xi \eta), \quad W_1^{(2)} := ((\phi^\lambda)^2 \xi \eta).$$

$$-2i\epsilon(\lambda) \delta'(\lambda) P_{4,2}^{(2)} = (R_{4,0}^{(1)}, M_1^{(1)}) + \frac{M_1^{(1)}}{2\epsilon(\lambda)} + (R_{4,0}^{(2)}, M_1^{(1)}) + (R_{4,1}^{(1)}, M_3^{(2)}) + (R_{4,1}^{(2)}, M_4^{(2)}) + \frac{M_3^{(1)}}{2\epsilon(\lambda)} \tag{52}$$

$$+ P_{4,1}^{(2)} W_1^{(1)} + P_{4,1}^{(1)} W_1^{(2)} + U,$$

where $U$ is a real constant including all the contributions from $F_3$ (see Lemma 6.1),

$$W_2^{(2)} := -3(\phi^\lambda)^2 \phi^\lambda \xi \eta^2, \quad W_2^{(2)} := -i(\phi^\lambda)^2 \phi^\lambda \eta^2,$$

$$M_1^{(2)} := -6\phi^\lambda (\phi^\lambda)^2 P_{2,0}^{(2)} - 6\phi^\lambda \phi^\lambda R_{2,0}^{(1)} - \frac{3}{4} \phi^\lambda \xi^2 + \frac{1}{4} \phi^\lambda \eta^2,$$

$$M_2^{(2)} := -2(\phi^\lambda)^2 \phi^\lambda P_{2,0}^{(2)} - 2\phi^\lambda \phi^\lambda R_{2,0}^{(2)} + \frac{i}{2} \phi^\lambda \xi \eta,$$

$$M_3^{(2)} := -3\phi^\lambda \phi^\lambda \xi, \quad M_4^{(2)} := i\phi^\lambda \phi^\lambda \eta,$$

In the following lemma we relate $Y_1$ and $Y_2$ to $P_{3,0}^{(1)}$ and $P_{3,0}^{(2)}$. 24
Lemma 8.6.

\[ Y_1 = 6i\epsilon(\lambda)\langle \xi, \eta \rangle [3iP^{(4)}_{3,0} - P^{(3)}_{3,0}], \]
\[ Y_2 = 6i\epsilon(\lambda)\langle \xi, \eta \rangle [3iP^{(3)}_{3,0} + P^{(4)}_{3,0}], \]

(53)

Proof. Using Equations (20) to (21) to obtain

\[-8\epsilon(\lambda)\langle \xi, \eta \rangle P^{(3)}_{3,0} = -3i\langle ImN_{3,0}, \eta \rangle - \langle ReN_{3,0}, \xi \rangle \]

(54)

and

\[ 8\epsilon(\lambda)\langle \xi, \eta \rangle P^{(4)}_{3,0} = -3i\langle ReN_{3,0}, \xi \rangle + \langle ImN_{3,0}, \eta \rangle. \]

(55)

On the other hand we observe

\[ Y_1 = \frac{3}{4}i\langle \xi^4 \rangle - \frac{3}{4}i\langle \xi^2\eta^2 \rangle + 6\langle (\phi^\lambda)^2\xi\eta \rangle P^{(2)}_{2,0} + 6\langle \phi^\lambda\xi\eta R^{(2)}_{2,0} \rangle + 18i\langle \phi^\lambda\xi^2 R^{(4)}_{2,0} \rangle + \]

\[ + 18i\langle \phi^\lambda\xi R^{(3)}_{1,0} \rangle P^{(1)}_{2,0} \]

\[ = 6i\langle ReN_{3,0}, \xi \rangle \]

and

\[ Y_2 = \frac{3}{4}i\langle \eta^4 \rangle - \frac{3}{4}i\langle \xi^2\eta^2 \rangle - 6iP^{(2)}_{2,0} (\langle (\phi^\lambda)^2\xi\eta \rangle - 6\langle \phi^\lambda\eta^2 R^{(1)}_{1,0} \rangle - 6i\langle \phi^\lambda\xi\eta R^{(2)}_{2,0} \rangle - \]

\[ -6i\langle (\phi^\lambda)^2\xi R^{(4)}_{2,0} \rangle \]

\[ = -6i\langle ImN_{3,0}, \eta \rangle. \]

These observations together with (54) and (55) imply (53). \(\square\)

Proof of Equations (47) and (48) By the definition of \(E_2\) it is sufficient to prove (47) by proving

\[ Re[P^{(1)}_{4,2}G_3 + P^{(2)}_{4,2}G_4] = Re[E_{4,0} + E_{4,1} + P^{(3)}_{4,1}(Y_1 - G_1 - G_3) + P^{(4)}_{4,1}(Y_2 - G_2 - G_6)]. \]

(56)

First we observe that

\[ G_3 - \frac{1}{2i\epsilon(\lambda)} G_4 = -8\epsilon(\lambda)\delta'(\lambda)P^{(2)}_{2,0}, \]

\[ G_4 = -8\epsilon(\lambda)\delta'(\lambda)P^{(1)}_{2,0}. \]

Lemma 8.2 implies that \(P^{(1)}_{2,0}\) and \(P^{(2)}_{2,0}\) are real and purely imaginary respectively, this fact together with Equations (51) (52) implies (56). The computation is straightforward, but tedious, hence we omit the detail.

Now we turn to (48). We start with analyzing \(P^{(3)}_{4,1}\) and \(P^{(4)}_{4,1}\). Equations (20) (21) and the properties of the equation for \(\dot{z}\) in (14) imply

\[-8\epsilon(\lambda)\langle \xi, \eta \rangle P^{(3)}_{4,1} = -3i\langle ImN_{4,1}, \eta \rangle - \langle ReN_{4,1}, \xi \rangle + U \]

(57)

and

\[ 8\epsilon(\lambda)\langle \xi, \eta \rangle P^{(4)}_{4,1} = -3i\langle ReN_{4,1}, \xi \rangle + \langle ImN_{4,1}, \eta \rangle + iU \]

(58)

where \(U\) stands for real (different) constants including all the contribution from the terms \(F_1\) and \(F_2\) (see Lemma 7.4). Moreover we compute to get

\[ N_{4,1} = \begin{pmatrix} ImN_{4,1} \\ -ReN_{4,1} \end{pmatrix} = \begin{pmatrix} i\phi^\lambda\eta R^{(1)}_{4,0} + \phi^\lambda\xi R^{(2)}_{4,0} \\ -3i\phi^\lambda\xi R^{(1)}_{4,0} - i\phi^\lambda\eta R^{(2)}_{4,0} \end{pmatrix} + U \]

(59)

\[ \text{Re} \]

\[ \text{Im} \]


where $U_1$ is a vector function such that $iU_1$ is admissible. Thus we have
\[
\langle \text{Re} N_{4,1}, \xi \rangle = \langle R_{4,0}, \begin{pmatrix} 3i\phi^2 + i\phi^3 \xi^2 \\ -i\phi^3 \xi \eta \end{pmatrix} \rangle + U
\]
and
\[
\langle \text{Im} N_{4,1}, \eta \rangle = \langle R_{4,0}, \begin{pmatrix} -i\phi \eta^2 \\ \phi \lambda \xi \eta \end{pmatrix} \rangle + iU
\]

Collecting the computation above and using the properties of $Y_1$ and $Y_2$ in (53) and the facts that $F_{3,0}^{(3)} = F_{3,0}^{(3)}$, $F_{3,0}^{(4)} = -P_{3,0}^{(4)}$ in Lemma 8.2, we have (48).

**Proof of Equation (49)** We start with rewriting $\langle R_{4,2}, \begin{pmatrix} F_5 \\ F_6 \end{pmatrix} \rangle$ to relate the function $\begin{pmatrix} F_5 \\ F_6 \end{pmatrix}$ to $R_{2,0}$. By the definition of $R_{4,2}$ in Lemma 8.2, we have
\[
R_{4,2} = -(L(\lambda) + 2i\epsilon(\lambda))^{-1}P_c[N_{4,2} + U_1]
\]
where $U_1$ is a vector function such that $iU_1$ is admissible and includes all the contributions of $G$ in (53) (see Lemma 6.1 for $N = 3$).

Recall that $\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The facts $\sigma_1(L(\lambda))^* \sigma_1 = L(\lambda)$ and $\sigma_1^2 = -1$ imply
\[
\langle R_{4,2}, \begin{pmatrix} F_5 \\ F_6 \end{pmatrix} \rangle = \langle N_{4,2}, -\sigma_1(L(\lambda) + 2i\epsilon(\lambda))^{-1}P_c\sigma_1 \begin{pmatrix} F_5 \\ F_6 \end{pmatrix} \rangle + iU
\]
where $U$ is a real constant generated by $U_1$ above (see Lemma 5.3). Moreover, we observe that
\[
R_{2,0} = -(L(\lambda) + 2i\epsilon(\lambda))^{-1}P_cN_{2,0} = \frac{i}{2}(L(\lambda) + 2i\epsilon(\lambda))^{-1}P_c\sigma_1 \begin{pmatrix} F_5 \\ F_6 \end{pmatrix}
\]
consequently
\[
\langle R_{4,2}, \begin{pmatrix} F_5 \\ F_6 \end{pmatrix} \rangle = -4i \langle N_{4,2}, \sigma_1 R_{2,0} \rangle + iU.
\]

In the next we decompose $N_{4,2}$ into various terms. By direct computation we have
\[
N_{4,2} = MR_{4,0} + \begin{pmatrix} i\phi \eta & \phi \lambda \xi \\ -3\phi \lambda \xi & -i\phi \eta \end{pmatrix} R_{4,1}
+ \begin{pmatrix} i\phi \xi \eta & \phi \lambda \xi \eta \\ -3\phi \lambda \xi^2 & -i\phi \lambda \eta^2 \end{pmatrix} \begin{pmatrix} P_{4,3}^{(3)} \\ P_{4,4}^{(4)} \end{pmatrix} + U_1
\]
where $U_1$ is a vector function with $iU_1$ being admissible, $M$ is an $2 \times 2$ matrix defined as
\[
\begin{pmatrix}
2\phi \lambda R_{0,2}^{(2)} + \frac{i}{2} \xi \eta + 2(\phi \lambda)^2 P_{0,2}^{(2)} \\
\frac{i}{4} \xi^2 + 6\phi \lambda \phi \lambda P_{0,2}^{(1)} - \frac{1}{4} \eta^2 + 6\phi \lambda R_{0,2}^{(1)},
\frac{1}{4} \eta^2 - \frac{3}{2} \eta^2 + 2\phi \lambda \phi \lambda P_{0,2}^{(2)} + 2\phi \lambda R_{0,2}^{(1)}
\end{pmatrix}
\]
The definitions of $E_1$ after (46) yields

$$E_1 = \langle R_{4,0}, \left( \frac{F_1}{F_2} \right) \rangle - 4i\langle MR_{4,0}, \sigma_1 R_{2,0} \rangle + iU$$

which together with the definition of $E_{4,0}$, Lemma 8.2 and the fact $R_{m,n}$, $m, n \leq 3$ is admissible implies Equation (49). The computation is straightforward, hence we omit the detail.

\[\square\]

**Proof of Equation (50)** The definitions of $E_3$ and $E_{4,1}$ imply

$$E_3 + E_{4,1} = \langle R_{4,1}, K \rangle + iU$$

with $U$ being a real constant and

$$K := \left( \frac{F_3 - 4i\Omega_1 + 4i[M_4^{(1)} P_{0,2}^{(2)} + M_3^{(2)} P_{0,2}^{(1)}]}{F_4 - 4i\Omega_2 + 4i[M_4^{(1)} P_{0,2}^{(2)} + M_3^{(2)} P_{0,2}^{(1)}]} \right).$$

In the next we relate the function $K$ to $R_{3,0}$.

First we have

$$R_{4,1} = -(L(\lambda) + 3i\epsilon(\lambda))^{-1} P_c [N_{4,1} + U_1]$$

where as usual $U_1$ is a vector function such that $iU_1$ is admissible and includes all the contributions from $G_1$ from (58). On the other hand by Lemma 8.2 and direct computation we have $-6i\sigma_1 N_{3,0} = K$, thus

$$E_3 + E_{4,1} = 6i\langle R_{4,1}, \sigma_1 N_{3,0} \rangle$$

$$= 6i\langle N_{4,1}, -(L(\lambda)^* + 3i\epsilon(\lambda))^{-1} P_c^* \sigma_1 N_{3,0} \rangle + iU$$

with $U$ being a real constant generated by $U_1$ above (see Lemma 8.3). By the facts that $\sigma_1 (L(\lambda))^* \sigma_1 = L(\lambda)$, $\sigma_1^2 = -1$, $-\sigma_1 P_c^* \sigma_1 = P_c$ and the definition $R_{3,0} = -(L(\lambda) + 3i\epsilon(\lambda))^{-1} P_c N_{3,0}$ we have

$$E_3 + E_{4,1} = -6i\langle N_{4,1}, \sigma_1 R_{3,0} \rangle.$$
8.4 Proof of Theorem 8.1

Proof. Recall the explicit form of $Z_{4,3}$ and $R_{4,0}$ in (46). By Equations (47)-(50) and the observation that $N_{4,0} = \left( \sum_{n=1}^{3} H_n^{(1)} - \sum_{n=1}^{3} H_n^{(2)} \right)^T$ in (45) we have

\[
ReZ_{4,3} = \frac{8}{(\xi,\eta)} Im\langle \sigma_1 (L(\lambda) + 4i\epsilon(\lambda) + 0)^{-1} N_{4,0}, N_{4,0} \rangle
\]

\[
= \frac{8}{(\xi,\eta)} Im\langle \sigma_1 (L(\lambda) - 4i\epsilon(\lambda) - 0)^{-1} N_{4,0}, N_{4,0} \rangle
\]

to complete the proof. \qed

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