Lévy noise induced escape in the Morris-Lecar model

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Abstract

The phenomenon of an excitable system producing a pulse under external or internal stimulation may be interpreted as a stochastic escape problem. This work addresses this issue by examining the Morris-Lecar neural model driven by symmetric $\alpha$-stable Lévy motion. Two deterministic indices: the first escape probability and the mean first exit time, are adopted to analyse the state transition in this stochastic model. We calculate both indices in order to understand the transition from the escape region to the target region, and the area of higher indices in escape region. Additionally, we consider the special case of (Gaussian) Brownian motion to compare with (non-Gaussian) Lévy motion case. Our main results indicate that higher first escape probability promotes the transition, while the mean first exit time reflects the stability of the rest state with the selected escape region. The higher non-Gaussianity index $\alpha$ and relatively small noise intensity are more prone to produce spikes. Moreover, by calculating both deterministic indices as functions of noise intensity ratio and non-Gaussianity index $\alpha$, we find that the effect of ion channel noise is more pronounced on the stochastic Morris-Lecar model than noise in the current. This work provides some mathematical understanding about the impact of non-Gaussian, heavy-tailed, burst-like fluctuations on excitable systems such as the Morris-Lecar system.

Keywords: Lévy motion; Morris-Lecar model; First escape probability; Mean first exit time; Nonlocal partial differential equations.

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1 Introduction

In recent years, there has been an increasing interest in the effects of noise in neuroscience. Because of the noisy environment that neurons live in, there are many sources of noise in the neuronal systems. These noisy sources include, for instance, random attacks caused by spontaneous release of neurotransmitters, synaptic noise from spontaneous postsynaptic potentials, small fluctuations in the electrical potential across the nerve-cell membrane, and the opening and closing of ion channels. Noise may induce various phenomena, such as oscillations [1, 2], chaos-like behaviors [3], state transitions [4], stochastic resonance [5, 6, 7], and spatial coherence resonance [8, 9, 10].

In this paper, we study the Morris-Lecar (ML) neuron model under the disturbance of (non-Gaussian) Lévy noise as well as (Gaussian) Brownian noise. As a simplified version of the Hodgkin-Huxley (HH) system, the ML model was first introduced to account for the electrical activities of the giant barnacle muscle fibers in invertebrates [11]. Since then, it becomes a canonical neuronal model because it can display two different forms of neuronal excitability behaviors under various parameter regimes. One form of excitability is not sensitive to external stimulation intensity, the discharge starting frequency can be very low, and the discharge range is relatively wide. This is called type I excitability. The type II excitability is relatively insensitive to external stimulation, and the discharge frequency is in a certain range. Meanwhile, from the point of view of bifurcation, type I excitability results from a saddle-node (tangent) bifurcation of equilibrium points on an invariant circle in ML model, while type II excitability corresponds to a subcritical Hopf bifurcation. The ML model is widely used in the theoretical research of the excitatory nerve discharge [12, 13, 14, 15, 16, 17]. Moreover, it can be used in cardiac cell modeling [18, 19]. As we known, Na⁺ ions and K⁺ ions cross the ion channels on the cell membrane back and forth, forming a transmembrane current and leading to the generation of an action potential or spike - an abrupt and transient change of membrane voltage. An excitable system may be sensitive to noise [20]. So an excitable membrane can generate action potential when stimulated by a strong enough input or disturbed by noise.

Recent works on the stochastic ML model are mostly concerned with the model under Gaussian noise [21, 22, 23, 24, 25, 26]. In order to understand the information coding in the nervous systems, the influence of additive stochastic perturbation on bifurcation scenarios and the stationary distribution of stochastic ML system were studied in [27], based on random dynamical systems theory. The Gaussian noise induced multiple spatial coherence resonance and spatial patterns in excitable systems were revealed in [23, 24]. The methods based on asymptotic approximations of the stationary density function and most probable path were developed to understand the role of channel noise in spontaneous excitability [28]. However, Gaussian noise can not describe some fluctuations with bursts or intermittence or with heavy-tailed distributions, which are characteristics of α-stable Lévy motions. Indeed, many complex phenomena involve fluctuations of the Lévy type, such as asset prices [29], turbulent motions of rotating annular fluid flows [30], a class of biological evolution [31], and random search [32]. The neuron systems with Lévy noise have attracted some recent attention [33, 34, 35, 36]. In fact, Lévy noise appears to be more reasonable than Gaussian noise, due to jumps by excitatory and inhibitory impulses caused by external disturbances in biological systems.

In this paper, we will consider the escape problem of the ML model with type II excitability under Lévy fluctuations. In this case the undisturbed system has one equilibrium state. More concretely, we will study whether the system trajectory starting from the stable equilibrium point in the ML system reaches other region through a boundary under the influence of α-stable Lévy noise. Two
different deterministic quantities are applied to analyse the problem: first escape probability (FEP) and mean first exit time (MFET). In order to quantify the escape behaviors, we should choose a proper escape region that contains the equilibrium point and a corresponding target region. The FEP is the likelihood for a system trajectory escaping to the target region, while MFET is the expected time for a system trajectory exiting the escape region. It turns out that both deterministic quantities are described by nonlocal partial differential equations (details in Sections 3 and 4). Then we will numerically calculate FEP of the solution stating from the escape region to the target region, and MFET of the solution stating from the escape region to the various outside regions.

The organization of the paper is as follows. In the next section we introduce the undisturbed ML model and the stochastic ML model driven by $\alpha$-stable Lévy noise. We also briefly review the $\alpha$-stable Lévy motion and two deterministic indices FEP and MFET, together with appropriate regions for computing these quantities. In Sections 3 and 4, we report numerical experiments on the effects of Lévy motion as well as Brownian motion in ML system, quantified by FEP and by MFET, respectively. Finally, we end the paper with a summary in Section 5.

2 The Morris-Lecar model

We now recall the undisturbed Morris-Lecar (ML) model and its disturbed version.

2.1 The undisturbed Morris-Lecar model

The deterministic Morris-Lecar (ML) model has been derived to describe giant barnacle (Balanus Nubilus) muscle fibres [27, 37], and it is represented by the following two-dimensional system:

\begin{align*}
C \frac{dv_t}{dt} &= -g_{Ca} m_\infty(v_t)(v_t - V_{Ca}) - g_K w_t(v_t - V_K) - g_L(v_t - V_L) + I, \\
\frac{dw_t}{dt} &= \varphi \frac{w_\infty(v_t) - w_t}{\tau_w(v_t)},
\end{align*}

(2.1)

where

\begin{align*}
m_\infty(v) &= 0.5[1 + \tanh(\frac{v - V_2}{V_2})], \\
w_\infty(v) &= 0.5[1 + \tanh(\frac{v - V_4}{V_4})], \\
\tau_w(v) &= [\cosh(\frac{v - V_3}{2V_4})]^{-1}.
\end{align*}

The variables $v_t$ and $w_t$ represent the membrane potential and the activation variable for the $K^+$ current, respectively. The parameter $C$ stands for the membrane capacitance. The first three terms in the right-hand side of the first equation in the system (2.1) respectively represent the voltage-gated $Ca^{2+}$ current, the voltage-gated delayed rectifier $K^+$ current and the leak current.
The parameters $g_{Ca}$, $g_K$ and $g_L$ are the maximal conductance of the calcium current, potassium current and leak current, respectively. The parameters $V_{Ca}$, $V_K$ and $V_L$ are the reversal potentials of the calcium current, potassium current and leak current, respectively. Input current is represented by $I$. The constant $\varphi$ indicates the change between fast and slow scales of the model. Finally $V_1$, $V_2$, $V_3$, $V_4$ are tuning parameters for steady state and time constant.

![Phase Portrait](image)

Figure 1: The phase portrait of the membrane potential $v_t$ and activation variable $w_t$ in the deterministic Morris-Lecar system (2.1) with type II excitability parameters and $I = 88(\mu A/cm^2)$.

The parameter values for the type II excitability of ML model are \cite{38,39}: $C = 20 \, \mu F/cm^2$, $V_{Ca} = 120 \, mV$, $V_K = -84 \, mV$, $V_L = -60 \, mV$, $g_{Ca} = 4.4 \, \mu S/cm^2$, $g_K = 8 \, \mu S/cm^2$, $g_L = 2 \, \mu S/cm^2$, $V_1 = -1.2 \, mV$, $V_2 = 18 \, mV$, $V_3 = 2 \, mV$, $V_4 = 30 \, mV$ and $\varphi = 0.04$. In this case the system possesses a unique equilibrium state for all values of $I$. This equilibrium is stable for $I < I_H \approx 93.86 \mu A/cm^2$, and unstable beyond $I_H \cite{27,40}. In this study, we choose $I = 88 \, \mu A/cm^2$, so the equilibrium state is stable. Furthermore, the resting cells at $I = 88$ have excitability. Figure 1 shows the phase portrait of system (2.1), in which the $v$-nullcline is divided into three branches, the left branch $f_1$, the middle branch $f_2$ and the right branch $f_3$. Moreover, the resting potential corresponding to the equilibrium $s^*$ is located at the left branch $f_1$.

### 2.2 The stochastic Morris-Lecar model

We consider the Morris-Lecar model driven by symmetric $\alpha$-stable Lévy motion. This stochastic model is described by the following stochastic differential equations:

\begin{align}
    dv_t &= \frac{1}{C} [-g_{Ca} m_\infty(v_t)(v_t - V_{Ca}) - g_K w_t(v_t - V_K) - g_L(v_t - V_L) + I] dt + \sigma_1 dL^\alpha_1, \\
    dw_t &= \varphi \frac{w_\infty(x_t) - w_t}{\tau_w(v_t)} dt + \sigma_2 dL^\alpha_2, \quad (2.2)
\end{align}
where \( L_1^{\alpha_1} \) and \( L_2^{\alpha_2} \) are independent symmetric \( \alpha \)-stable Lévy motions, with nonnegative noise intensities \( \sigma_1, \sigma_2 \), respectively. As a special class of non-Gaussian process with jumps \([41, 42]\), the \( \alpha \)-stable Lévy motion is defined by stable Lévy random variables. The distribution for a stable random variable is denoted as \( S_{\alpha}(\delta, \beta, \gamma) \). Here \( \alpha \) is called the Lévy motion index (non-Gaussianity index or stability index), \( \delta \) is the scale parameter, \( \beta \) is the skewness parameter, and \( \gamma \) is the shift parameter. Let us recall the definition of a symmetric \( \alpha \)-stable Lévy motion.

A symmetric \( \alpha \)-stable Lévy motion \( L_\alpha^t \), with \( 0 < \alpha < 2 \), is a stochastic process with the following properties \([43, 44]\):

(i) \( L_\alpha^0 = 0 \), almost surely (a.s);
(ii) \( L_\alpha^t \) has independent increments;
(iii) \( L_\alpha^t - L_\alpha^s \sim S_{\alpha}((t - s)^{\frac{\alpha}{2}}, 0, 0) \);
(iv) \( L_\alpha^t \) has stochastically continuous sample paths: for every \( s \), \( L_\alpha^t \to L_\alpha^s \) in probability, as \( t \to s \).

When \( \alpha \) is 2, the stable Lévy motion corresponds to the well-known Brownian motion \( B_t \). Moreover, a symmetric \( \alpha \)-stable Lévy motion can be represented as the triplet \((0, 0, \nu_{\alpha})\), where the jump measure \( \nu_{\alpha} \) in the two-dimensional case is defined as \([41, 45]\)

\[
\nu_{\alpha} = \frac{C_{\alpha}dy}{||y||^{2+\alpha}},
\]

with

\[
C_{\alpha} = \frac{\alpha}{2^{1-\alpha}\pi} \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}.
\]

For \( 0 < \alpha < 2 \), the following tail estimate of stable Lévy random variable \( L \) holds \([45]\)

\[
\lim_{y \to \infty} y^{\alpha}P(L > y) = C_{\alpha} \frac{1+\beta}{2} \sigma^{\alpha}.
\]

This estimate indicates that the stable Lévy random variable \( L \) has a “heavy tail”, which decays polynomially, unlike the tail estimate of Gaussian random variable, which decays exponentially.

In this paper, we take \( \alpha_1 = \alpha_2 = \alpha \) and make scale transformation of variables for the convenience of calculation. In the deterministic case (see equation (2.1)), the stable equilibrium point is \( s^* = (-2.7277, 1.2436) \). The noise term \( L_1^{\alpha_1} \) represents current fluctuations in the environment and the noise term \( L_2^{\alpha_2} \) is understood as stochastic ion channel noise. Those processes may be accompanied by discontinuously unpredictable jumps. Thus the Lévy motion \( L_\alpha^t \) is suitable for simulating this type of noise. Moreover, \( \alpha \)-stable Lévy motion has larger jumps with lower jump frequencies when \( \alpha \) closes to 0, while for \( 1 < \alpha < 2 \), it has smaller jumps with higher jump probabilities. Especially, in the following numerical simulations, when \( \alpha = 2 \), we respectively change the Lévy motion \( L_1^{\alpha_1}, L_2^{\alpha_2} \) to independent Brownian motion \( B_1^t, B_2^t \).

If ML system (2.1) is not subject to any disturbance, resting state is a fixed point. When the system is under a perturbation, the solution orbit (or path) starting from the rest state may respond
as a small oscillation near the rest state or produce a spike. We further note that the $v$-nullcline is “cubic”, the middle branch $f_2$ in some sense separates the firing of an action potential from the subthreshold return to equilibrium. If an orbit crosses the separatrix, it will be attracted by the right branch $f_3$. Then we explain whether the orbit from the equilibrium state can be attracted by $f_3$ as an escape problem. Two deterministic indices: FEP and MFET are applied to this problem. In order to calculate FEP of the equilibrium $s^*$, the escape region $D$ (containing $s^*$) and the target region $E$ should be chosen. The two regions are $D : (-5.9277, 1.0723) \times (-1.7564, 5.2436)$ and $E : [1.0723, \infty) \times [-1.7564, 5.2436]$ as shown in Figure 2. The FEP represents the probability of the solution orbit starting at a point in $D$ first escapes to region $E$. The reason we choose the line $\{(v,w)|v = 1.0723, w \in \mathbb{R}\}$ as a boundary line to calculate FEP is that $v > 1.07234$ can be seen as the high electrical potential of a nerve cell. And if the solution orbit crosses this line $\{(v,w)|v = 1.0723, w \in \mathbb{R}\}$, it must be attracted by $f_3$. Meanwhile, the stability of the equilibrium under the stimulation by noise will also be considered. We select region $D$ to compute MFET, which implies the mean first exit time of the solution orbit starting at a point in region $D$ escapes to region $D^c$.

### 3 First escape probability

The general form of the two-dimensional stochastic differential system (2.1) is as follows:

\[
\begin{align*}
    dv_t &= f_1(v_t,w_t) + \sigma_1 dL_t^{\alpha_1}, \\
    dw_t &= f_2(v_t,w_t) + \sigma_2 dL_t^{\alpha_2}.
\end{align*}
\] (3.1)
The infinitesimal generator $A$ of the system (3.1) is

\[
Ap(v, w) = f_1(v, w)p_v(v, w) + f_2(v, w)p_w(v, w) + \sigma_1^{\alpha_1} \int_{\mathbb{R}\setminus\{0\}} [p(v + v', w) - p(v, w)] \nu_{\alpha_1}(dv') \\
+ \sigma_2^{\alpha_2} \int_{\mathbb{R}\setminus\{0\}} [p(v, w + w') - p(v, w)] \nu_{\alpha_2}(dw').
\]

(3.2)

When $L_t^{\alpha_1}, L_t^{\alpha_2}$ are replaced by independent Brownian motions $B_t^1, B_t^2$, the generator becomes

\[
Ap(v, w) = f_1(v, w)p_v(v, w) + f_2(v, w)p_w(v, w) + \frac{\sigma_1^2}{2} p_{vv}(v, w) + \frac{\sigma_2^2}{2} p_{ww}(v, w).
\]

The first escape probability (FEP) here is employed to characterize the escape phenomenon. It is denoted as $p(v, w)$, which is used to characterize the probability of the solution orbit starting at $(v, w)$ in a open region $D$ first escaping to a target region $E$. The escape probability $p(v, w)$ can be solved by the following integral-differential equation with a Balayage-Dirichlet exterior boundary condition [44]:

\[
Ap(v, w) = 0, \quad (v, w) \in D,
\]

\[
p(v, w) = \begin{cases} 1, & (v, w) \in E, \\ 0, & (v, w) \in D^c \setminus E. \end{cases}
\]

Here $D^c$ is the complement set of the bounded region $D$.

Figure 3 shows the numerical simulation of the FEP for different Lévy motion index $\alpha$ and noise intensity $\sigma$ ($\sigma_1 = \sigma_2 = \sigma$). In Figure 3(a)-(d), the noise intensity $\sigma = 0.5$, the area of high FEP (red region) gradually becomes bigger with the increase of $\alpha$. While in Figure 3(e)-(h), when the Lévy motion index $\alpha = 1.25$, the area of high FEP (red region) gradually becomes smaller with the increase of $\sigma$. Figure 4 depicts FEP of the equilibrium $s^*$ about the Lévy motion index $\alpha$ and the noise intensity $\sigma$ ($\sigma_1 = \sigma_2 = \sigma$). In Figure 4(a), noise intensity $\sigma$ is fixed, we can see that the higher the Lévy motion index $\alpha$, the higher FEP and the FEP reaches the maximum at $\alpha = 2$. This means that smaller jumps with higher frequencies are more beneficial to make the solution orbit starting at the equilibrium point get out of region $D$. While in Figure 4(b), for various value of fixed $\alpha$ ($\alpha = 0.5, 1, 1.5, 2$), the lower noise intensity, the higher FEP. When $\alpha = 0.5$, FEP keeps almost unchanged as the noise intensity increases. When $\alpha = 1$ and $\alpha = 1.5$, FEP drops rapidly with small noise intensity and as the noise intensity increases, the trend of FEP tends to be steady. Finally, when $\alpha = 2$, with increasing $\sigma$ from 0.05, FEP remains at 1 until $\sigma_a (=0.185)$, and then it becomes monotonic decreasing. This means when the system is affected by the Brownian motion, the solution path starting at the equilibrium point can definitely escape region $D$ for very small noise intensity. Through the results on Figure 4, we can obtain that when FEP is lager, a solution path starting at the equilibrium point is less ‘stable’ and easier to enter into the region $E$ and to generate a ‘spike’.
Figure 3: FEP $p(v, w)$ from the escape region $D : (-5.9277, 1.0723) \times (-1.7564, 5.2436)$ to target region $E : [1.0723, \infty) \times [-1.7564, 5.2436]$. The color map depends on Lévy motion index $\alpha$ and noise intensity $\sigma$ ($\sigma = \sigma_2 = \sigma$). (a)-(d) Influence of Lévy motion index $\alpha$ on FEP for different values of $\alpha$ with fixed noise intensity $\sigma = 0.5$. (e)-(h) Influence of noise intensity $\sigma$ on FEP for different values of $\sigma$ with fixed Lévy motion index $\alpha = 1.25$. The color bar in all figures is set to the same scale, red making 1 and blue making 0.
Figure 4: FEP $p(v, w)$ at the equilibrium $s^* = (-2.7277, 1.2436)$, $\sigma_1 = \sigma_2 = \sigma$. (a) Effect of Lévy motion index $\alpha$ on FEP at $s^*$ for various values of noise intensity $\sigma$ (red $\sigma = 0.25$, blue $\sigma = 0.5$, green $\sigma = 0.75$, pink $\sigma = 1$). (b) Effect of noise intensity $\sigma$ on FEP at $s^*$ for various values of Lévy motion index $\alpha$ (red $\alpha = 0.5$, blue $\alpha = 1$, green $\alpha = 1.5$, pink $\alpha = 2$).

Figure 5: Effect of noise intensity and Lévy motion index on the $R_{FEP}$. The threshold $p^*$ to compute $R_{FEP}$ is chosen as 0.8. (a) $R_{FEP}$ against Lévy motion index $\alpha$ for various noise intensity $\sigma$ (red $\sigma = 0.25$, blue $\sigma = 0.5$, green $\sigma = 0.75$, pink $\sigma = 1$). (b) $R_{FEP}$ against noise intensity $\sigma$ for various Lévy motion index $\alpha$ (red $\alpha = 0.5$, blue $\alpha = 1$, green $\alpha = 1.5$, pink $\alpha = 2$).

Furthermore, we will investigate the stability of region $D$. Menck et al. [47] presented basin stability, which measured by the volume of the basin of attraction for quantifying the stability of a state in deterministic dynamical systems. Moreover, Serdukova et al. [48] provided a new concept of stochastic basin of attraction (SBA), which is used to quantify the basin stability for stochastic dynamical systems. In our paper, we regard the volume as the area of the basin of attraction in the state plane. We denote the stochastic basin of the attractor $f_1$ in region $D$ to be $\mathcal{B}(f_1) = K(p^*)$. The the collection of all initial points in region $D$ possess $K(p^*) = \{(v, w) \in D | p(v, w) > p^*\}$
Figure 6: FEP $p(v, w)$ at the equilibrium $s^* = (-2.7277, 1.2436)$, the color map depends on noise intensity ratio $r$ and Lévy motion index $\alpha$. The color bar in all figures is set to the same scale, red making 1 and blue making 0. (a) Fixed $\sigma_1 = 0.5$, $\sigma_2 \in [0.05, 1]$. (b) Fixed $\sigma_1 = 1$, $\sigma_2 \in [0.05, 1]$. (c) $\sigma_1 \in [0.05, 1]$, fixed $\sigma_2 = 0.5$. (d) $\sigma_1 \in [0.05, 1]$, fixed $\sigma_2 = 1$.

and $p^*$ indicates a high probability level (‘threshold’). The collection $K(p^*)$ also means an orbit starting from those points reaches to the region $E$ with high probability (we ignore the initial points whose solution have a ‘small’ probability away from by $f_3$). The basin stability in region $D$ can be quantified in terms of its area, that is, $S_{FEP}$ represents the area of $K(p^*)$. The normalized $S_{FEP}$ is

$$R_{FEP} = \frac{S_{FEP}(p^*)}{S_D},$$

where $S_D$ is the area of region $D$.

Now we choose $p^* = 0.8$ to compute $R_{FEP}$ as illustrated in Figure 5. In Figure 5 we let the noise intensity $\sigma_1 = \sigma_2 = \sigma$. From Figure 5(a), we can see that for a fixed noise intensity, the larger Lévy motion index, the larger $R_{FEP}$. When $\alpha = 2$, there is a special case where the curves get maximum and present a jumping growth. This exactly shows a distinct difference between
Brownian motion and Lévy motion. While in Figure 5(b), as can be seen, $R_{FEP}$ decreases with the increase of $\sigma$ for fixed $\alpha$ besides $\alpha = 2$. When $\alpha = 2$, as the noise intensity $\sigma$ increases, $R_{FEP}$ increases monotonously, reaches its maximum at $\sigma_b$, and then decreases as $\sigma$ continues to grow. As a whole, the larger Lévy motion index and the smaller noise intensity are beneficial to more solution orbits with the point in region $D$ as the initial point escape to the target region $E$.

Next, we denote a new parameter named noise intensity ratio

$$r = \sigma_2/\sigma_1.$$  

(3.5)

In order to better understand the influence of noise intensity and Lévy motion index on FEP starting from the equilibrium point, we plot the change graph of FEP shown in Figure 6. In Figure 6(a)-(b), we fix $\sigma_1 = 0.5$ and $\sigma_1 = 1, \sigma_2$ belongs to interval $[0.05, 1]$. It can be seen that the first escape probabilities all have larger value for larger Lévy motion index $\alpha$ and smaller noise intensity ratio $r$. While in Figure 6(c)-(d), we fix $\sigma_2 = 0.5$ and $\sigma_2 = 1, \sigma_1$ belongs to interval $[0.05, 1]$. It can be seen that, for fixed $r$, FEP is larger for the larger Lévy motion index $\alpha$ in Figure 6(c), but FEP is smaller for larger $\alpha$ and smaller $\sigma_1$ (larger noise intensity $r$) with fixed $\sigma_2 = 1$ in Figure 6(d). In general, we can obtain that the noise intensity $\sigma_2$ has a greater influence on escape probability for the same $\alpha$.

4 Mean first exit time

Back to the general two-dimensional stochastic system (3.1) in the previous section. The first exit time for a solution orbit starting at $(v, w)$ in a region $D$ is defined as

$$\tau(\omega, (v, w)) = \inf\{t \geq 0 : (v_0, w_0) = (v, w), (v_t, w_t) \in D^c\},$$

and the mean first exit time (MFET) is

$$u(v, w) := \mathbb{E}\tau(\omega, v, w) \geq 0.$$  

It is the ‘average’ residence time of a solution orbit initially at $(v, w)$ inside region $D$ until it first escapes outside of region $D$. The difference between a Gaussian and a non-Gaussian process is that the orbit typically hits the boundary of region $D$ for the Brownian motion case, while it may jumps outside of region $D$ for Lévy motion case. The MFET $u(v, w)$ satisfies the following nonlocal integral-differential equation with an exterior boundary condition [44]:

$$Au(v, w) = -1, \quad (v, w) \in D,$$

$$u(v, w) = 0, \quad (v, w) \in D^c.$$  

(4.1)

Here $A$ is defined in equation (3.2) and $D^c$ is the complement set of the bounded region $D$.

MFET can be used as a tool to measure the stability of a system. For ML system, we choose the region $D$ that contains the equilibrium point to calculate the MFET of all the solution orbits starting inside region $D$. We plot MFET $u(v, w)$ from escape region $D : (-5.9277, 1.0723) \times (-1.7564, 5.2436)$ to the region $D^c$ as shown in Figure 7. In Figure 7(a)-(d), noise intensity $\sigma$ is fixed as $\sigma = 0.75$, the MFET of points belonging to the region $D$ gradually increases with the increase of Lévy motion.
Figure 7: MFET $u(v,w)$ from the escape region $D : (-5.9277, 1.0723) \times (-1.7564, 5.2436)$ to the region $D^c$. The colour map depends on Lévy motion index $\alpha$ and noise intensity $\sigma$ ($\sigma_1 = \sigma_2 = \sigma$). (a)-(d) correspond to fixed noise intensity $\sigma = 0.75$ and different Lévy motion index $\alpha = 0.5, 1, 1.5, 2$. (e)-(h) correspond to fixed $\alpha = 1.25$ and different $\sigma = 0.25, 0.5, 1.5, 2$. All figures are unified into an identical color map with the same scale, red marking 11.7084 and blue making 0.
Levy motion index $\alpha$

MFET $u(v,w)$

$\sigma = 0.25$

$\sigma = 0.5$

$\sigma = 0.75$

$\sigma = 1$

(a)

Figure 8: MFET $u(x,y)$ of the equilibrium $s^* = (-2.7277, 1.2436)$. Here $\sigma_1 = \sigma_2 = \sigma$. (a) Effect of Lévy motion index $\alpha$ on MFET at $s^*$ with different noise intensity $\sigma$ (red $\sigma = 0.25$, blue $\sigma = 0.5$, green $\sigma = 0.75$, pink $\sigma = 1$). (b) Effect of noise intensity $\sigma$ on MFET at $s^*$ with different Lévy motion index $\alpha$ (red $\alpha = 0.5$, blue $\alpha = 1$, green $\alpha = 1.5$, pink $\alpha = 2$).

index $\alpha$ ($\alpha = 0.5, 1, 1.5, 2$). While in Figure 7(e)-(h), as $\sigma$ increases, the MFET of points in region $D$ is getting less for fixed $\alpha = 1.25$. In fact, the solution orbit starting at the equilibrium point will stay there forever without noise. Now for noisy situations, the solution orbits may stay there in a finite time then run out of the region $D$. So we can use the finite time quantified by MFET to characterize the relative stability of the solution orbits starting at the points in region $D$.

Figure 9: Effect of noise intensity $\sigma$ and Lévy motion index $\alpha$ on the $R_{MFET}$. The threshold $p^*$ to compute $R_{MFET}$ is chosen as 10. Here we choose $\sigma_1 = \sigma_2 = \sigma$. (a) $R_{MFET}$ against Lévy motion index $\alpha$ for various noise intensity $\sigma$ (red $\sigma = 0.25$, blue $\sigma = 0.5$, green $\sigma = 0.75$, pink $\sigma = 1$). (b) $R_{MFET}$ against noise intensity $\sigma$ for various Lévy motion index $\alpha$ (red $\alpha = 0.5$, blue $\alpha = 1$, green $\alpha = 1.5$, pink $\alpha = 2$).

Similarly, Figure 8 depicts the change of MFET of the equilibrium point in some cases. In Figure
we denote the Lévy motion index $\alpha \in (0, 2)$ (the case where $\alpha = 2$ is replaced by the Brownian motion) and the noise intensity $\sigma \in (0, 1] (\sigma_1 = \sigma_2 = \sigma)$. It can be seen from the Figure S(a), the larger $\alpha$, the greater MFET, and this is more noticeable for smaller noise intensity. No matter what the noise intensity is $(\sigma = 0.25, 0.5, 0.75, 1)$, MFET of the equilibrium point reaches the maximum at $\alpha = 2$, which indicates Gaussian noise has more influence on the MFET of equilibrium point in the system than non-Gaussian noise. In addition, Figure S(b) further depicts the effect of noise intensity $\sigma$ on MFET at the equilibrium point with different Lévy motion index $\alpha$. As can be seen, MFET will decrease with the increasing of $\sigma$ for fixed $\alpha$ and the change curve of MFET is not very obvious when $\alpha$ is small, such as $\alpha = 0.5$. On the contrary, when $\alpha$ is lager, the change curve of MFET goes down very fast at first and tends to level off with the increasing of $\sigma$.

We define another index similar to the stochastic basin of attraction in Section 3. Let $M(u^*) = \{(v, w) \in D \mid u(v, w) > u^*\}$, i.e., the solution orbit starting from region $D$ and remaining there for a finite time (remarked by a threshold $u^*$). Then we also quantify the basin stability in region $D$ based on its area $|D|$:

$$R_{MFET} = \frac{S_{MFET}(u^*)}{S_D},$$

(4.2)

where $S_{MFET}$ is the area of $M(u^*)$, and $S_D$ is the area of $D$. $R_{MFET}$ is the normalization of $S_{MFET}$.

In the following, as an example, we make $u^* = 10$ to calculate the $R_{MFET}$, the result is shown in Figure 9. Figure 9(a) depicts $R_{MFET}$ against the Lévy motion index $\alpha$ for various values of $\sigma$ ($\sigma_1 = \sigma_2 = \sigma$). It can be seen that for fixed $\sigma = 0.25$ and $\sigma = 0.5$, the $R_{MFET}$ remains 0 in the beginning, then increases with increasing $\alpha$ besides $\alpha = 2$. While for the other two curves of $R_{MFET}$ with $\sigma = 0.75$ and $\sigma = 1$, the $R_{MFET} = 0$, which means that for high noise intensity the solution orbit starting at a point in region $D$ gets out of region $D$ quickly. Special cases occur when $\alpha = 2$: for $\sigma = 0.5$ and $\sigma = 0.75$, the values of $R_{MFET}$ both have a jumping growth; for $\sigma = 0.25$, $R_{MFET}$ suddenly decreases and the value of $R_{MFET}$ has no change for $\sigma = 1$. Figure 9(b) depicts $R_{MFET}$ against noise intensity $\sigma$ for various values of $\alpha$. It shows that for $\alpha = 1.5$ and $\alpha = 2$, the curve of $R_{MFET}$ transits from increasing to decreasing at $\sigma_5$ and $\sigma_d$, respectively, then goes down to zero as further increasing $\alpha$. While when $\alpha = 1$, $R_{MFET}$ gradually decreases from positive to zero, then keep zero with the further increase of $\sigma$. At the same time, $R_{MFET} = 0$ when $\alpha = 0.5$, which indicates that a solution orbit starting at $(v, w) \in D$ escapes quickly for smaller $\alpha$. Here we choose the initial value $\sigma = 0.05$ and the same partition $\Delta \sigma = 0.005$. Moreover, we can find that the curve with $\alpha = 2$ and $\alpha = 1.5$, $\alpha = 1$ has a intersection point, respectively. For fixed noise intensity, larger $\alpha$ has smaller $R_{MFET}$ before the intersection point and it’s the opposite after the intersection point.

As in the section 3, we also plot MFET at the equilibrium point depending on noise intensity ratio $r$ and Lévy motion index $\alpha$ as shown in the Figure 10. In Figure 10(a)-(b), we respectively fix $\sigma_1 = 0.5$ and $\sigma_1 = 1$ as well $\sigma_2$ belongs to interval $[0.05, 1]$. It can be seen that MFET has larger value for lager Lévy motion index $\alpha$ and smaller noise intensity ratio $r$. Meanwhile, when fix $\alpha < 1$, whether in Figure 10(a) or Figure 10(b), the noise intensity ratio $r$ has a small effect on MFET of the equilibrium point; and when fix $\alpha > 1$, the change of MFET is obvious. This just verifies that the jump property of Lévy motion. While in Figure 10(c)-(d), we respectively fix $\sigma_2 = 0.5$ and $\sigma_2 = 1$ as well $\sigma_1$ belongs to interval $[0.05, 1]$, then we have the lager $\alpha$, the lager MFET. Comparing Figure 10(c) with Figure 10(d), when fix $\alpha > 1$, for the same $\sigma_1$ (the $r$ in (d) is 2 times that in
(a) $r = \sigma_2/\sigma_1$, fixed $\sigma_1 = 0.5$.  

(b) $r = \sigma_2/\sigma_1$, fixed $\sigma_1 = 1$.

(c) $r = \sigma_2/\sigma_1$, fixed $\sigma_2 = 0.5$.  

(d) $r = \sigma_2/\sigma_1$, fixed $\sigma_2 = 1$.

Figure 10: MFET $u(v, w)$ of the equilibrium $s^* = (-2.7277, 1.2436)$, the color graphs depend on noise intensity ratio $r$ and Lévy motion index $\alpha$. (a)-(b) $\sigma_2 \in [0.05, 1]$, fixed $\sigma_1 = 0.5$, 1, respectively. The color bar of (a) and (b) is taken the same scale. (c)-(d) $\sigma_1 \in [0.05, 1]$, fixed $\sigma_1 = 0.5$, 1, respectively. The color bar of (c) and (d) is taken the same scale.

(c)), the MFET with $\sigma_2 = 0.5$ obviously larger than the MFET with $\sigma_2 = 1$, which indicates the solution orbit starting from equilibrium point relatively stable for smaller $\sigma_2$.

5 Conclusion

In summery, we focus on the escape problem driven by a symmetric $\alpha$-stable Lévy noise (non-Gaussian noise) in Morris-Lecar (ML) model. We have provided a method to quantify the dynamics of escape from the rest state of the system by means of two deterministic indices: the first escape probability (FEP) and the mean first exit time (MFET). To be specific, we have used the method to describe the state transition from the rest state to the excitatory state. To simulate the firing behavior of neurons and calculate FEP, we have chosen the appropriate escape region containing
the equilibrium point and the target region. Meanwhile, we have depicted the stability of the escape region in terms of MFET.

Through numerical simulation and analysis, we have found that the noise intensities $\sigma_1$, $\sigma_2$ and the jump size of $\alpha$-stable Lévy motion have significant and delicate influences on the MFET and FEP. We have also discovered that for smaller jumps of the Lévy motion and relatively smaller noise intensity, FEP is larger, which means that they are conducive to the production of spikes. While higher noise intensity and larger jumps of the Lévy motion shortens the MFET. Moreover, the Brownian motion (Gaussian noise) has been also considered and compared with the Lévy case. The FEP and MFET of the equilibrium point in the ML system are significantly different between the cases of Brownian motion and Lévy motion, when the noise intensity is fixed. Meanwhile, the influences of noise on the whole escape region are also different by calculating the area of high FEP and long MFET.

By calculating the impact of the noise intensity ratio $r$ and Lévy motion index $\alpha$ on FEP of the equilibrium point, we have revealed that $\sigma_2$ (intensity for ion channel noise) has more pronounced influence on the system than $\sigma_1$ (intensity for current fluctuations), for fixed non-Gaussianity index $\alpha$. While for the MFET at the equilibrium point, it has larger value for larger $\alpha$ and smaller noise intensity ratio $r$. This means the solution trajectory starting from equilibrium point is relatively stable for larger $\alpha$ and smaller $r$.

This work provides some mathematical understanding about the impact of non-Gaussian, heavy-tailed, burst-like fluctuations on excitable systems such as the Morris-Lecar system.

**Appendix: Numerical simulation**

We use an efficient numerical finite difference scheme [49] to compute the FEP (equation (3.3)) and MFET (equation (4.1)). This method is revised for our model in $(v, w) \in D = (a, b) \times (c, d)$, $E = [a', b'] \times [c, d]$ by a scalar conversion $v = \frac{b-a}{2}s + \frac{a+b}{2}$, and $w = \frac{d-c}{2}k + \frac{c+d}{2}$ for $s \in (-1, 1), k \in (-1, 1)$. 

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16
If we let \( m(s, k) = p\left(\frac{b-a}{2}s + \frac{a+b}{2} \cdot \frac{d-c}{2}k + \frac{c+d}{2}\right) \), the equation (3.2) is discretized as follows:

\[
Ap(v, w) = 2 \left\{ \frac{b-a}{2} f_1\left(\frac{b-a}{2}s + \frac{a+b}{2} \cdot \frac{d-c}{2}k + \frac{c+d}{2}\right)m_s + \frac{\gamma_2^2}{2}(\frac{2}{b-a})^2 m_{ss} \right.
\]
\[
+ \frac{2}{d-c} f_2\left(\frac{b-a}{2}s + \frac{a+b}{2} \cdot \frac{d-c}{2}k + \frac{c+d}{2}\right)m_k + \frac{\gamma_2^2}{2}(\frac{2}{d-c})^2 m_{kk} \right.
\]
\[
- \sigma_1^{\alpha_1} C_{\alpha_1} \left(\frac{b-a}{2}\right)^{\alpha_1} \left[ \frac{1}{(1+s)^{\alpha_1}} + \frac{1}{(1-s)^{\alpha_1}} \right] m(s, k)
\]
\[
+ \sigma_1^{\alpha_1} C_{\alpha_1} \left(\frac{b-a}{2}\right)^{\alpha_1} \int_{-1-s}^{1-s} \frac{m(s+s', k) - m(s, k)}{|s'|^{1+\alpha_1}} ds'
\]
\[
- \sigma_2^{\alpha_2} C_{\alpha_2} \left(\frac{d-c}{2}\right)^{\alpha_2} \left[ \frac{1}{(1+k)^{\alpha_2}} + \frac{1}{(1-k)^{\alpha_2}} \right] m(s, k)
\]
\[
+ \sigma_2^{\alpha_2} C_{\alpha_2} \left(\frac{d-c}{2}\right)^{\alpha_2} \int_{-1-k}^{1-k} \frac{m(s, k+k') - v(s, k)}{|k'|^{1+\alpha_2}} dk'
\]
\[= \psi(s, k), \tag{A.1}\]

where the integral in this equation is taken as the Cauchy principle value integral, and

\[
\psi(s, k) = \frac{\sigma_1^{\alpha_1} C_{\alpha_1}}{\alpha_1} \left[ \frac{1}{(b' - \frac{b-a}{2}s - \frac{a+b}{2})^{\alpha_1}} - \frac{1}{(a' - \frac{b-a}{2}s - \frac{a+b}{2})^{\alpha_1}} \right]
\]

in the case of FEP (equation (3.3)), or \( \psi(s, k) = -1 \) in the case of MFET (equation (4.1)).

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