PEIFFER ELEMENTS IN THE MOORE COMPLEX OF A BISIMPLICIAL GROUP

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Abstract

In this work, we explain ‘Peiffer pairings’ in the Moore (bi)complex of a bisimplicial group and give their applications for crossed modules and crossed squares.

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Introduction

In [16], Mutlu and Porter generalized Peiffer elements to higher dimensions giving systematic maps of generating them. They used ideas based on the works of Conduché [7] and Carrasco-Cegarra [6] and gave images of the functions $F_{\alpha,\beta}$ in the Moore complex of a simplicial group. For further accounts of these functions in simplicial Lie and commutative algebras see [1] and [3, 4], respectively. The purpose of this paper is to give their results for bisimplicial groups.

Crossed modules were introduced by Whitehead in [17]. Crossed n-cubes (cf. [11]) model connected $(n + 1)$-types and for $n = 2$, they are just considered as crossed squares (cf. [12]). These crossed squares are related to 2-crossed modules introduced by Conduché in [7].

In a letter to Brown and Loday, dated in the mid 1980s, Conduché pointed out that the mapping cone complex of a crossed square, constructed by Loday in [13], has a 2-crossed module structure. As a generalization of this result, Conduché has proven Theorem 0.1 in [9]. In this work, he showed that crossed squares are related to bisimplicial groups in the same way crossed modules are related to simplicial groups. In this paper we aim to shed some light on the 2-crossed modules and crossed squares structures given by Conduché, by use of the functions $F_{\alpha,\beta}$ in the Moore (bi)complex of a bisimplicial group. In particular, we see in Proposition 4.2 that the $h$-map of the crossed square given by Conduché corresponds to the functions $F_{\alpha,\beta}$. There is a crossed module version of this in Proposition 4.1. These results, by a similar way, can be iterated to multisimplicial (or $n$-simplicial) groups and also the relationship between crossed $n$-cubes and $n$-simplicial groups can be constructed in terms of these functions. In this case, a 1-truncated $n$-simplicial group gives a crossed $n$-cube structure.
1 (Bi)simplicial groups

In this section, the definitions and properties related to (bi)simplicial groups, some of which are classical, are recalled. We refer the reader to May’s and Loday’s books (cf. [15] and [14]) and Artin-Mazur’s [2] article for most of the basic properties of (bi)simplicial groups that we will be needing.

Let $\Delta$ be the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$. A simplicial group is a functor from the opposite category $\Delta^{op}$ to the category of groups $\text{Grp}$. That is, a simplicial group $G$ consists of a family of groups $G_n$ together with homomorphisms $d^n_i : G_n \to G_{n-1}$, $0 \leq i \leq n$, $(n \neq 0)$ and $s^n_j : G_n \to G_{n+1}$, $0 \leq j \leq n$, called face and degeneracy maps, satisfying the usual simplicial identities given in [10, 13].

Given a simplicial group $G$, the Moore complex $(NG, \partial)$ of $G$ is the normal chain complex defined by $(NG)_n = \ker d^n_0 \cap \ker d^n_1 \cap \cdots \cap \ker d^n_{n-1}$ with boundaries $\partial_n : NG_n \to NG_{n-1}$ induced from $d^n_n$ by restriction.

A 2-simplicial group or a bisimplicial group $G_{*,*}$ is a functor from the product category $\Delta^{op} \times \Delta^{op}$ to the category of groups $\text{Grp}$, with the face and degeneracy maps given by

$$d^{p,q}_i : G_{p,q} \to G_{p-1,q} \quad s^{p,q}_i : G_{p,q} \to G_{p+1,q} \quad 0 \leq i \leq p$$
$$d^{p,q}_j : G_{p,q} \to G_{p,q-1} \quad s^{p,q}_j : G_{p,q} \to G_{p,q+1} \quad 0 \leq j \leq q$$

such that the maps $d^{p,q}_i, s^{p,q}_i$ commute with $d^{p,q}_j, s^{p,q}_j$ and that $d^{p,q}_i, s^{p,q}_i$ (resp. $d^{p,q}_j, s^{p,q}_j$) satisfy the usual simplicial identities.

We think of $d^{p,q}_j, s^{p,q}_j$ as the vertical operators and $d^{p,q}_i, s^{p,q}_i$ as the horizontal operators. If $G_{*,*}$ is a bisimplicial group, it is convenient to think of an element of $G_{p,q}$ as a product of a $p$-simplex and a $q$-simplex.

The Moore bicomplex of a bisimplicial group $G_{*,*}$ is defined by

$$NG_{n,m} = \bigcap_{(i,j) = (0,0)} \ker d^{n,m}_{i,j} \cap \ker d^{n,m}_{j,i}$$

with the boundary homomorphisms

$$\partial^{n,m}_i : NG_{n,m} \to NG_{n-1,m}$$

and

$$\partial^{n,m}_j : NG_{n,m} \to NG_{n,m-1}$$

induced by the face maps $d^{n,m}_i$ and $d^{n,m}_j$ where $0 \leq i \leq n \neq 0$, $0 \leq j \leq m \neq 0$. 

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This Moore bicomplex is illustrated by the following diagram.

This definition extends easily to multisimplicial groups.

2 Peiffer pairings for bisimplicial groups

This section is devoted to description of the functions $F_{\alpha, \beta}$ in the Moore complex of a bisimplicial group. The following notation and terminology is derived from [6]. For the ordered set $[n] = \{0 < 1 < \cdots < n\}$, let $\alpha^n_i : [n + 1] \to [n]$ be the increasing surjective map given by:

$$\alpha^n_i(j) = \begin{cases} j & \text{if } j \leq i, \\ j - 1 & \text{if } j > i. \end{cases}$$

Let $S(n, n - r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n - r]$. This can be generated from the various $\alpha^n_i$ by composition. The composition of these generating maps is subject to the following rule: $\alpha_j \alpha_i = \alpha_{i-1} \alpha_j$, $j < i$. This implies that every element $\alpha \in S(n, n - r)$ has a unique expression as $\alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_r}$, with $0 \leq i_1 < i_2 < \cdots < i_r \leq n - 1$, where the indices $i_k$ are the elements of $[n]$ such that $\{i_1, \ldots, i_r\} = \{i : \alpha(i) = \alpha(i+1)\}$. We thus can identify $S(n, n - r)$ with the set $\{(i_r, \ldots, i_1) : 0 \leq i_1 < i_2 < \cdots < i_r \leq n - 1\}$. In particular, the single element of $S(n, n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly the only element of $S(n, 0)$ is $(n - 1, n - 2, \ldots, 0)$. For all $n \geq 0$, let

$$S(n) = \bigcup_{0 \leq r \leq n} S(n, n - r).$$

For example

$$S(0) = \{\emptyset\}, S(1) = \{\emptyset, (0)\}, S(2) = \{\emptyset, (1), (0), (1, 0)\}.$$}

The following terminology is derived from [9]. For $n, q \in \mathbb{N}$ with $q \leq n$, let $S(n, q)$ be the set of nondecreasing surjections from $[n]$ to $[q]$ as defined above. For $\sigma \in S(n, q)$ the target of $\sigma$ is called $b(\sigma) : q = b(\sigma)$. The set $S(n)$ is partially ordered set by the following relation: $\sigma \leq \tau$ if, for $i \in [n]$, one has $\sigma(i) \geq \tau(i)$ where $[b(\sigma)]$ and $[b(\tau)]$ are considered as subsets of $\mathbb{N}$. 3
Given \( r \in \mathbb{N}^* \) and \( \underline{a} = (n_1, \ldots, n_r) \in \mathbb{N}^r \), let \( S(\underline{a}) = S(n_1) \times \cdots \times S(n_r) \) with the product partial order. For \( \sigma = (\sigma_1, \ldots, \sigma_r) \in S(\underline{a}) \) let \( b(\sigma) = (b(\sigma_1), \ldots, b(\sigma_r)) \).

The following result was proved by Conduché in \([8]\).

**Theorem 2.1** Let \( G_\ast \) be a \( r \)-simplicial group. Then, for \( \underline{a} \in \mathbb{N}^r \), the group \( G_\ast \) is a \( S(\underline{a}) \)-semi-direct product of the family of subgroups \((s_\sigma(N(G)_{b(\sigma)}))_{\sigma \in S(\underline{a})}\), where \( s_\sigma \) is the composite of the iterated degeneracies \( s_{\sigma_i} \) corresponding to each simplicial structure.

Now we can give the functions \( F_{\alpha,\beta} \) in the Moore complex of a bisimplicial group.

**Proof**

Given \( \underline{a} = (k_1, k_2) \in \mathbb{N} \times \mathbb{N} \), let \( S(\underline{a}) = S(k_1) \times S(k_2) \) with the product (partial) order. Let \( \underline{\alpha}, \underline{\beta} \in S(\underline{a}) \) and \( \underline{\alpha} = (\alpha_1, \alpha_2) ; \underline{\beta} = (\beta_1, \beta_2) \) where \( \alpha_i, \beta_i \in S(k_i) \) for \( 1 \leq i \leq 2 \). The pairings that we will need are given by composing the maps in the following diagram:

\[
\begin{array}{ccc}
NG_{k_1-\#\alpha_1,k_2-\#\alpha_2} \times NG_{k_1-\#\beta_1,k_2-\#\beta_2} & \longrightarrow & NG_{k_1,k_2} \\
\text{\((s_{\underline{\alpha}} s_{\underline{\beta}})\)} & \downarrow & \downarrow \mu \\
G_{k_1,k_2} \times G_{k_1,k_2} & \longrightarrow & G_{k_1,k_2}
\end{array}
\]

where \( s_{\underline{\alpha}} : s_{\alpha_1} s_{\alpha_2} \) and where \( s_{\alpha_1} = s_{i_r} \cdots s_{i_1} \) for \( \alpha_1 = (i_r, \ldots, i_1) \in S(k_1) \) and similarly \( s_{\beta} = s_{j_m} \cdots s_{j_1} \) for \( \beta_1 = (j_m, \ldots, j_1) \in S(k_2) \) and where

\[
p : G_{k_1,k_2} \rightarrow NG_{k_1,k_2}
\]

is defined by the composite projection

\[
p = \left( p_{k_1-1}^{h} \cdots p_{0}^{h} \right) \left( p_{k_2-1}^{v} \cdots p_{0}^{v} \right)
\]

where \( p_j (x) = x^{-1} s_j d_j (x) \) in both horizontal and vertical directions and \( \mu \) is given by the commutator map.

Thus for \( \underline{\alpha} = (\alpha_1, \alpha_2), \underline{\beta} = (\beta_1, \beta_2) \in S(k_1) \times S(k_2) \), we obtain

\[
F_{\underline{\alpha},\underline{\beta}}(x, y) = p \mu (s_{\underline{\alpha}}, s_{\underline{\beta}})(x, y) = p \mu (s_{\alpha_1}^{h} s_{\alpha_2}^{v}(x), s_{\beta_1}^{h} s_{\beta_2}^{v}(y)) = p [s_{\alpha_1}^{h} s_{\alpha_2}^{v}(x), s_{\beta_1}^{h} s_{\beta_2}^{v}(y)]
\]

where \( x \in NG_{k_1-\#\alpha_1,k_2-\#\alpha_2} \) and \( y \in NG_{k_1-\#\beta_1,k_2-\#\beta_2} \).

The construction given above can be easily extended to the \( n \)-simplicial groups as follows.
Given \( n \neq 0, n \in \mathbb{N} \) and \( \underline{n} = (k_1, k_2, \ldots, k_n) \in \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} = \mathbb{N}^n \), let \( S(\underline{n}) = S(k_1) \times S(k_2) \times \cdots \times S(k_n) \) with the product (partial) order as given in \([9]\).

Let \( \underline{\alpha}, \underline{\beta} \in S(\underline{n}) \) and \( \underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n); \underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \) where \( \alpha_i \in S(k_i) \) and \( \beta_j \in S(k_j) \), \( 1 \leq i, j \leq n \).

The pairings that we will need

\[
\left\{ F_{\underline{\alpha}, \underline{\beta}} : NG_{\underline{n}} - \# \alpha \times NG_{\underline{n}} - \# \beta \longrightarrow NG_{\underline{n}} : \underline{\alpha}, \underline{\beta} \in S(\underline{n}) \right\}
\]

are given as composites by the diagram

\[
\begin{array}{c}
NG_{k_1 - \# \alpha_1, k_2 - \# \alpha_2, \ldots, k_n - \# \alpha_n} \times NG_{k_1 - \# \beta_1, k_2 - \# \beta_2, \ldots, k_n - \# \beta_n} \\
\downarrow (t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_n}, t_{\beta_1} t_{\beta_2} \cdots t_{\beta_n}) \\
G_{k_1, k_2, \ldots, k_n} \times G_{k_1, k_2, \ldots, k_n} \\
\downarrow \mu \\\nG_{k_1, k_2, \ldots, k_n}
\end{array}
\]

where \( s_{\alpha_i} : s_{\alpha_i}^1 \cdot s_{\alpha_i}^2 \cdots s_{\alpha_i}^{t_{\alpha_i}} \), for \( 1 \leq i \leq n \); \( s_{\beta_i} : s_{\beta_i}^1 \cdot s_{\beta_i}^2 \cdots s_{\beta_i}^{t_{\beta_i}} \) for \( \alpha_i = (i_r, \ldots, i_1) \in S(k_i) \) and similarly \( s_{\beta_i}^j \), and \( p \) is defined by the composite projection

\[
p = (p_{t_{\alpha_1}^1} \cdots p_{t_{\alpha_1}^{t_{\alpha_1}}}) (p_{t_{\alpha_2}^1} \cdots p_{t_{\alpha_2}^{t_{\alpha_2}}}) \cdots (p_{t_{\alpha_n}^1} \cdots p_{t_{\alpha_n}^{t_{\alpha_n}}})
\]

where \( p_j^k (x) = x^{-1} s_j^k d_j^k (x) \), for any \( j \) and \( k \), and where each \( t_k \) for \( 1 \leq k \leq n \) indicates the directions of \( n \)-simplicial group, and \( \mu \) is given by the commutator map.

3 Calculations of the functions \( F_{\underline{\alpha}, \underline{\beta}} \) in low dimensions

For \( 0 \leq k_1, k_2 \leq 2 \) we consider the sets \( S(k_1) \times S(k_2) \). We shall calculate the images of the functions \( F_{\underline{\alpha}, \underline{\beta}} \) for all \( \underline{\alpha}, \underline{\beta} \in S(k_1) \times S(k_2) \).

First, consider \( (n, m) = (0, 1) \) or \( (n, m) = (1, 0) \). Then, we get the \( F_{\underline{\alpha}, \underline{\beta}} \) functions whose codomain \( NG_{0,1} \) or \( NG_{1,0} \) respectively. First let \( (n, m) = (0, 1) \). Then

\[
S(n, m) = S(0) \times S(1) = \{(\emptyset, \emptyset), (\emptyset, (0))\}.
\]

Let \( \underline{\alpha} = (\emptyset, \emptyset) \) and \( \underline{\beta} = (\emptyset, (0)) \). Then the function

\[
F_{(\emptyset, \emptyset), (\emptyset, (0))} : NG_{0,0} \times NG_{0,0} \longrightarrow NG_{0,1}
\]
can be given as follows:

\[
F(\emptyset, \emptyset)((\emptyset), (\emptyset)) (x, y) = p\mu(s^h_{\emptyset}s^v_{\emptyset}(x), s^h_{\emptyset}s^v_{(0)}(y))
\]

\[
= p^0_\emptyset [id(x), s^0_\emptyset (y)] \quad (\because s^0_\emptyset = s^0_0 = id)
\]

\[
= [x, s^v_{(0)} y]s^0_\emptyset d^{(01)}_0 [s^v_{(0)} y, x]
\]

\[
= [x, s^v_{(0)} y][s^v_{(0)} y, s^v_{(0)} d^{(01)}_0 (x)]
\]

\[
= [x, s^v_{(0)} y][s^v_{(0)} y, 1] \quad (\because x \in \ker d^{(01)}_0 = NG_{0,1})
\]

\[
= [x, s^v_{(0)} y]
\]

for \(x \in NG_{0,1}\) and \(y \in NG_{0,0}\).

Suppose now that \((n, m) = (1, 0)\). Then we take \(S(1) \times S(0) = \{((\emptyset), ((0), \emptyset))\}\). Let \(\alpha = (\emptyset, \emptyset)\) and \(\beta = ((0), \emptyset)\). Then the function

\[
F((\emptyset), ((0), \emptyset)) : NG_{1,0} \times NG_{0,0} \longrightarrow NG_{1,0}
\]

is defined by

\[
F((\emptyset), ((0), \emptyset))(x, y) = p\mu(s^h_{\emptyset}s^v_{\emptyset}(x), s^h_{\emptyset}s^v_{(0)}(y))
\]

\[
= p^h_\emptyset [id(x), s^h_\emptyset (y)]
\]

\[
= [x, s^h_{(0)} y]s^h_{(0)} d^{(10)}_0 [s^h_{(0)} y, x]
\]

\[
= [x, s^h_{(0)} y][s^h_{(0)} y, s^h_{(0)} d^{(10)}_0 (x)]
\]

\[
= [x, s^h_{(0)} y][s^h_{(0)} y, 1] \quad (\because x \in \ker d^{(10)}_0 = NG_{1,0})
\]

\[
= [x, s^h_{(0)} y]
\]

for all \(x \in NG_{1,0}\) and \(y \in NG_{0,0}\).

We give the calculations of other functions listed below in Appendix A.
| $F_{\alpha, \beta}$ : Domain $\rightarrow$ Codomain | $\alpha$ | $\beta$ | $F_{\alpha, \beta}(x, y)$ |
|---|---|---|---|
| $NG_{0,1} \times NG_{0,0} \rightarrow NG_{0,1}$ | $\emptyset, \emptyset$ | $\emptyset, (0)$ | $\{x, s_0^{[1]}(y)\}$ |
| $NG_{1,0} \times NG_{0,0} \rightarrow NG_{1,0}$ | $\emptyset, \emptyset$ | $(0), 0)$ | $\{x, s_0^{[0]}(y)\}$ |
| $NG_{1,1} \times NG_{1,0} \rightarrow NG_{1,1}$ | $\emptyset, \emptyset$ | $(0), (0)$ | $\{x, s_0^{[1]}(y)\}$ |
| $NG_{1,1} \times NG_{0,1} \rightarrow NG_{1,1}$ | $\emptyset, \emptyset$ | $(0), (0)$ | $\{x, s_0^{[0]}(y)\}$ |
| $NG_{0,1} \times NG_{1,0} \rightarrow NG_{1,1}$ | $\emptyset, (0)$ | $(0), (0)$ | $\{x, s_0^{[1]}(y)\}$ |
| $NG_{0,1} \times NG_{0,1} \rightarrow NG_{0,2}$ | $\emptyset, (0)$ | $(0), (1)$ | $\{x, s_0^{[0]}(y)\}$ |
| $NG_{1,0} \times NG_{1,0} \rightarrow NG_{2,0}$ | $\emptyset, (0)$ | $(1), (0)$ | $\{x, s_0^{[0]}(y)\}$ |
| $NG_{1,1} \times NG_{1,1} \rightarrow NG_{2,1}$ | $\emptyset, (0)$ | $(1), (0)$ | $\{x, s_0^{[0]}(y)\}$ |
| $NG_{1,1} \times NG_{0,2} \rightarrow NG_{1,2}$ | $\emptyset, (1)$ | $(0), (0)$ | $\{x, s_0^{[0]}(y)\}$ |
| $NG_{1,1} \times NG_{2,0} \rightarrow NG_{2,1}$ | $\emptyset, (0)$ | $(0), (0)$ | $\{x, s_0^{[0]}(y)\}$ |
| $NG_{1,0} \times NG_{1,1} \rightarrow NG_{2,1}$ | $\emptyset, (0)$ | $(0), (0)$ | $\{x, s_0^{[0]}(y)\}$ |
| $NG_{0,1} \times NG_{1,1} \rightarrow NG_{2,1}$ | $\emptyset, (0)$ | $(0), (0)$ | $\{x, s_0^{[0]}(y)\}$ |
| $NG_{1,2} \times NG_{1,2} \rightarrow NG_{2,2}$ | $\emptyset, (0)$ | $(1), (0)$ | $\{x, s_0^{[0]}(y)\}$ |
| $NG_{2,1} \times NG_{2,1} \rightarrow NG_{2,2}$ | $\emptyset, (0)$ | $(1), (0)$ | $\{x, s_0^{[0]}(y)\}$ |
| $NG_{1,2} \times NG_{2,1} \rightarrow NG_{2,2}$ | $\emptyset, (1)$ | $(0), (0)$ | $\{x, s_0^{[0]}(y)\}$ |
| $NG_{1,1} \times NG_{2,1} \rightarrow NG_{2,2}$ | $\emptyset, (0)$ | $(1), (0)$ | $\{x, s_0^{[0]}(y)\}$ |

**Definition 3.1** Let $G_{*,*}$ be a bisimplicial group and $n, m > 1$, and $D_{n,m}$ the subgroup in $G_{n,m}$ generated by degenerate elements. Let $N_{n,m}$ be the normal subgroup of $G_{n,m}$ generated by elements of the form

$$F_{\alpha, \beta}(x, y)$$

where $x \in NG_{n-\#a_1, m-\#a_2}$ and $y \in NG_{n-\#b_1, m-\#b_2}$.

Muth and Porter [14] have defined a normal subgroup $N_n$ of $G_n$ generated by elements of the forms $F_{\alpha, \beta}(x, y)$. Furthermore they proved that there is an equality

$$\partial_n(NG_n \cap D_n) = \partial_n(N_S^n \cap D_n).$$

By a similar way we can write the following equalities

$$\partial_n^h(NG_{n,m} \cap D_{n,m}) = \partial_n^h(N_{n,m} \cap D_{n,m}),$$
and
\[ \partial_m^v(NG_{n,m} \cap D_{n,m}) = \partial_m^v(N_{n,m} \cap D_{n,m}) \]
in each direction.

Thus we have the following result.

**Proposition 3.2** Let \( G_{*,*} \) be a bisimplicial group. Then for \( n \geq 1, m \geq 2 \) and \( I, J \subseteq [m-1] \) with \( I \cup J = [m-1] \), there is the inclusion
\[ [K_I \cap K_H, K_J \cap K_H] \subseteq \partial_m^v(NG_{n,m} \cap D_{n,m}) \]
where
\[ K_H = \bigcap_{i=0}^{n-1} \ker d_i^{(n-1,m)} \]
and
\[ K_I = \bigcap_{i \in I} \ker d_i^{v(n,m-1)} \quad \text{and} \quad K_J = \bigcap_{j \in J} \ker d_j^{v(n,m-1)} . \]

Similarly, for \( n \geq 2, m \geq 1 \) and \( I', J' \subseteq [n-1] \) with \( I' \cup J' = [n-1] \), there is the inclusion
\[ [K_{I'} \cap K_{V}, K_{J'} \cap K_{V}] \subseteq \partial_n^h(NG_{n,m} \cap D_{n,m}) \]
where
\[ K_V = \bigcap_{i=0}^{m-1} \ker d_i^{h(n,m-1)} \]
and
\[ K_{I'} = \bigcap_{i \in I'} \ker d_i^{h(n-1,m)} \quad \text{and} \quad K_{J'} = \bigcap_{j \in J'} \ker d_j^{h(n-1,m)} . \]

**Proof:** We know that from the results of [16], there are already the following inclusions in both horizontal and vertical directions: if \( m \) is constant in the horizontal direction, then for \( n \geq 2 \), we have
\[ [K_{I'} \cap K_{V}, K_{J'} \cap K_{V}] \subseteq \partial_n^h(NG_{n,m} \cap D_{n,m}) \]
and if \( n \) is constant in the vertical direction, then for \( m \geq 2 \), we have
\[ [K_I \cap K_{H}, K_J \cap K_{H}] \subseteq \partial_m^v(NG_{n,m} \cap D_{n,m}) . \]

Thus the result can be seen easily by using these inclusions. \( \square \)

Now, in low dimensions, we investigate the images of these functions under the boundary homomorphisms \( \partial_n^h \) and \( \partial_m^v. \)

For \( NG_{1,2} \) take \( x, y \in NG_{1,1} = \ker d_0^{(1,1)} \cap \ker \partial_0^v(1,1) \), then we obtain
\[ \partial_2^v(12)(F((\emptyset, (0)), (0, (1))))(x, y) = d_2^{v(12)}([s_0^{v(11)}(x), s_1^{v(11)}(y)] [s_1^{v(11)}(y), s_1^{v(11)}(x)]) = [s_0^{v(01)} d_1^{v(11)} x, y] \]
where \([s_0^{(1)}d_1^{(1)} x, y][y, x] \in [\ker d_0^{(11)}, \ker d_1^{(11)}]\) from [10]. Further we obtain
\[
d_0^{(11)} ([s_0^{(1)}d_1^{(1)} x, y][y, x]) = 1,
\]
hence \([s_0^{(1)}d_1^{(1)} x, y][y, x] \in [\ker d_0^{(11)} \cap \ker d_0^{(11)}, \ker d_1^{(11)} \cap \ker d_1^{(11)}]\).

Similarly for \(y \in NG_{0,2}\) and \(x \in NG_{1,1}\), from \(NG_{1,1} \times NG_{0,2}\) to \(NG_{1,2}\) we obtain
\[
\partial_2^{(12)} (F((\emptyset, (1)), ((0), 0))) (x, y) = \partial_2^{(12)} [s_1^{(11)} (x), s_0^{(02)} (y)] = [x, d_2^{(12)} s_0^{(02)} (y)].
\]

Since \(d_0^{(11)} [x, d_2^{(12)} s_0^{(02)} (y)] = 1\) we have \([x, d_2^{(12)} s_0^{(02)} (y)] \in \ker d_0^{(11)}\). Furthermore, \([x, d_2^{(12)} s_0^{(02)} (y)] \in [\ker d_0^{(11)}, \ker d_1^{(11)}]\).

By a similar way one can show that the images of other generating elements in
\([\ker d_0^{(11)} \cap \ker d_0^{(11)}, \ker d_1^{(11)} \cap \ker d_0^{(11)}]\).

Thus we have the following equality
\[
\partial_2 (NG_{1,2} \cap D_{1,2}) = [\ker d_0^{(11)} \cap \ker d_0^{(11)}, \ker d_1^{(11)} \cap \ker d_0^{(11)}].
\]

For \(NG_{2,1}\) take \(x, y \in NG_{1,1}\). Then we obtain
\[
\partial_2^{(2,1)} (F(((0), 0), ((1), 0))) (x, y) = [s_0^{(01)} d_1^{(11)} (x), y][y, x]
\]
where \([s_0^{(01)} d_1^{(11)} (x), y][y, x] \in [\ker d_0^{(11)}, \ker d_1^{(11)}]\) and
\[
d_0^{(11)} ([s_0^{(01)} d_1^{(11)} (x), y][y, x]) = 1,
\]
hence
\[
[s_0^{(01)} d_1^{(11)} (x), y][y, x] \in \ker d_0^{(11)} \cap [\ker d_0^{(11)}, \ker d_1^{(11)}].
\]

Thus we obtain the following equality
\[
\partial_2^{(1)} (NG_{2,1} \cap D_{2,1}) = [\ker d_0^{(11)} \cap \ker d_0^{(11)}, \ker d_1^{(11)} \cap \ker d_0^{(11)}].
\]

For \((n, m) = (2, 1)\) and \((n, m) = (1, 2)\), we can summarize these situations in the following diagram.

| \(\alpha\) | \(\beta\) | \(I^*\) | \(J^*\) | \(V\) |
|----------|----------|---------|---------|---------|
| \((0, 0)\) | \((1, 0)\) | \{0\}   | \{1\}   | \{0\}   |
| \((\emptyset, (0))\) | \((\emptyset, (1))\) | \{0\}   | \{1\}   | \{0\}   |

For \(NG_{2,2}\) take \(x, y \in NG_{2,1} = \ker d_0^{(2,1)} \cap \ker d_1^{(2,1)} \cap \ker d_0^{(2,1)}\), then we obtain
\[
\partial_2^{(22)} (F(((0), 0), ((0), 1))) (x, y) = d_2^{(22)} \left[ [s_0^{(22)} (x), s_1^{(22)} (y)][s_1^{(22)} (y), s_1^{(22)} (x)] \right] = [s_0^{(20)} d_1^{(21)} x, y][y, x]
\]
where \([s_0^{(20)} d_1^{(21)}(x, y)[y, x] \in [\ker \alpha_0^{(21)}, \ker \alpha_1^{(21)}]\) from (10). Further we obtain
\[
\alpha_0^{(21)} ([s_0^{(20)} d_1^{(21)} x, y][y, x]) = 1,
\]
hence \([s_0^{(20)} d_1^{(21)}(x, y)[y, x] \in [\ker \alpha_0^{(21)} \cap \ker \alpha_1^{(21)}, \ker \alpha_1^{(21)} \cap \ker \alpha_0^{(21)}]\).

Similarly for \(x \in NG_{1,2}\) and \(y \in NG_{2,1}\), from \(NG_{1,2} \times NG_{2,1}\) to \(NG_{2,2}\) we obtain
\[
\partial_2^{(22)} (F((1), \emptyset), (0, 0))(x, y) = d_2^{(22)} [s_1^{(12)}(x), s_0^{(21)}(y)]
\]
\[
= [d_2^{(22)} s_1^{(12)}(x), d_2^{(22)} s_0^{(21)}(y)].
\]
Since \(d_0^{(11)} [x, d_2^{(12)} s_0^{(1)}(y)] = 1\) we have \([x, d_2^{(12)} s_0^{(1)}(y)] \in \ker d_0^{(11)}\). Furthermore, \([x, d_2^{(12)} s_0^{(1)}(y)] \in [\ker d_0^{(11)}, \ker d_1^{(11)}]\).

By a similar way, one can show that the images of other generating elements in
\[
[\ker \alpha_0^{(21)} \cap \ker \alpha_1^{(21)}, \ker \alpha_1^{(21)} \cap \ker \alpha_0^{(21)}].
\]

Thus we have the following equality
\[
\partial_2^{(22)} (NG_{2,2} \cap D_{2,2}) = [\ker \alpha_0^{(21)} \cap \ker \alpha_1^{(21)}, \ker \alpha_1^{(21)} \cap \ker \alpha_0^{(21)}].
\]

We can summarize this in the following diagram for \((n, m) = (2, 2)\).

| α  | β  | I' | J' | V   |
|----|----|----|----|-----|
| (0, 0) | (1, 0) | {0} | {1} | {0, 1} |
| (0, (0)) | (0, 1) | {0} | {1} | {0, 1} |

Using the calculation method given above we obtained the following equalities in low dimensions.
\[
\partial_2^{(22)} (NG_{0,2} \cap D_{0,2}) = [\ker \alpha_0^{(01)}, \ker \alpha_1^{(01)}],
\]
\[
\partial_2^{(22)} (NG_{1,2} \cap D_{1,2}) = [\ker \alpha_0^{(11)} \cap \ker \alpha_1^{(11)}, \ker \alpha_1^{(11)} \cap \ker \alpha_0^{(11)}],
\]
\[
\partial_2^{(22)} (NG_{2,2} \cap D_{2,2}) = [\ker \alpha_0^{(21)} \cap \ker \alpha_1^{(21)}, \ker \alpha_1^{(21)} \cap \ker \alpha_0^{(21)} \cap \ker \alpha_1^{(21)}],
\]
\[
\partial_2^{(22)} (NG_{2,0} \cap D_{2,0}) = [\ker \alpha_0^{(02)}, \ker \alpha_1^{(02)}],
\]
\[
\partial_2^{(22)} (NG_{2,1} \cap D_{2,1}) = [\ker \alpha_0^{(12)} \cap \ker \alpha_1^{(12)}, \ker \alpha_1^{(12)} \cap \ker \alpha_0^{(12)}],
\]
\[
\partial_2^{(22)} (NG_{2,2} \cap D_{2,2}) = [\ker \alpha_0^{(22)} \cap \ker \alpha_1^{(22)} \cap \ker \alpha_0^{(22)} \cap \ker \alpha_1^{(22)}].
\]

4 Applications

4.1 Crossed modules from bisimplicial groups

Recall that a crossed module is a group morphism \(\partial : M \to P\) endowed with a (left) action of \(P\) on \(M\) such that

CM1. the morphism \(\partial\) is \(P\)-equivariant, where \(P\) acts on itself via conjugacy,
CM2. for \(x, y \in M\) we have \(\partial x y = x y x^{-1}\).
Proposition 4.1 Let $G_{*,*}$ be a bisimplicial group with Moore bicomplex $NG_{*,*}$. If $p \geq 1$ and $q \geq 1$, $NG_{p,q} = \{1\}$, then the maps $d_i^{(01)}, d_i^{h(10)}$ and
\[
\partial : NG_{0,1} \times NG_{1,0} \to NG_{0,0} \\
(x,y) \mapsto d_i^{(01)}(x)d_i^{h(10)}(y)
\]
are crossed modules.

Proof: See Appendix B

4.2 Crossed squares from bisimplicial groups

First, we recall detailed definition of a crossed square from [13]. A crossed square of groups is a commutative square of group morphisms

\[
\begin{array}{ccc}
L & \rightarrow & M \\
\lambda \downarrow & & \downarrow \mu \\
N & \rightarrow & P \\
\chi' \downarrow & & \downarrow \nu
\end{array}
\]

with action of $P$ on every other group and a map $h: M \times N \to L$ such that

1. The maps $\lambda$ and $\chi'$ are $P$-equivariant and $\nu, \mu, \lambda$ and $\nu \circ \chi'$ are crossed modules,

2. $\lambda \circ h(x, y) = x^{\nu(y)}x^{-1}$, $\lambda' \circ h(x, y) = (\mu(x)y)y^{-1}$,

3. $h(\lambda(z), y) = z^{\nu(y)}z^{-1}$, $h(x, \lambda'(z)) = (h(x)z)z^{-1}$,

4. $h(xx', y) = \mu(x)h(x', y)h(x, y)$, $h(x, yy') = h(x, y)\nu(y)h(x, y')$,

5. $h(t x, t'y) = t h(x, y)$

for $x, x' \in M$, $y, y' \in N$, $z \in L$ and $t \in P$.

The following proposition was initially given by Conduché in [9]. Here, we see that the $h$-map of the crossed square can be given by the function $F_{(0,0),(0),0} : NG_{01} \times NG_{10} \to NG_{11}$.

Proposition 4.2 Let $G_{*,*}$ be a bisimplicial group and $NG_{*,*}$ its Moore bicomplex. Suppose $NG_{k_1,k_2} = \{1\}$ for any $k_1 \geq 2$ or $k_2 \geq 2$. Then the diagram

\[
\begin{array}{ccc}
NG_{1,1} & \xrightarrow{\partial_i^{h(11)}} & NG_{0,1} \\
\downarrow{\partial_i^{t(11)}} & & \downarrow{\partial_i^{(01)}} \\
NG_{1,0} & \xrightarrow{\partial_i^{h(10)}} & NG_{0,0}
\end{array}
\]
is a crossed square. \( NG_{0,0} \) acts on other groups via the degeneracies \( s^0_h \) and \( s^0_v \). The \( h \)-map is given by the map \( F_{(\emptyset, (0)), ((0), \emptyset)}(x, y) \), namely,

\[
h : NG_{0,1} \times NG_{1,0} \to NG_{1,1}
\]

\[
(x, y) \mapsto h(x, y) = F_{(\emptyset, (0)), ((0), \emptyset)}(x, y)
\]

for \( x \in NG_{0,1}, \ y \in NG_{1,0} \) where \( (\emptyset, (0)), ((0), \emptyset) \in S(1) \times S(1) \) and

\[
F_{(\emptyset, (0)), ((0), \emptyset)}(x, y) = [s^0_h \circ (x), s^0_v \circ (y)].
\]

**Proof:** For the axioms see Appendix B. \( \square \)

**Remark:** This result can be extended to crossed \( n \)-cubes defined by Ellis and Steiner in [11]. In this case, if \( G_{s_1 s_2 \cdots s_n} \) is an \( n \)-simplicial group with Moore \( n \)-complex \( NG_{s_1 s_2 \cdots s_n} \), such that \( NG_{s_1 s_2 \cdots s_n} = \{1\} \) for any \( s_j \geq 2, (1 \leq j \leq n) \), then we can easily say that, as a generalization of the above result, this Moore \( n \)-complex has a crossed \( n \)-cube structure. Then the \( h \)-maps of associated crossed \( n \)-cube are given by the functions \( F_{\alpha, \beta} \) in the Moore \( n \)-complex.

### 4.3 2-Crossed modules from bisimplicial groups

To characterize the simplicial groups having a Moore complex trivial in dimension bigger or equal to three, Conduché has [7]. The definition of a 2-crossed module is also recalled for completeness [7].

A 2-crossed module is a complex of length 2

\[
L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N
\]

of \( N \)-groups with action of \( M \) on \( L \) and a function \( \{,\} : M \times M \to L \), called ‘Peiffer lifting’, such that

1. \( \partial_2 \{y, y'\} = yy'y^{-1}(\partial_1(y)y')^{-1} \),
2. \( \{\partial_2 z, \partial_2 z'\} = zz'z^{-1}z'^{-1} \),
3. \( \{\partial_2 z, y, \partial_2 z\} = z(\partial_1 y z)^{-1} \),
4. \( \{y, y'y''\} = \{y, y'\}\{y, y''\}\{\partial_2 y, y''\}^{-1}, \partial_1 y y'\},
5. \( \{yy', y''\} = \{y, y'y''y'^{-1}\}^1, \partial_1 y \{y', y''\} \),
6. \( x\{y, y'\} = \{x^y, x^y'\} \)

for \( x \in N, y, y', y'' \in M \) and \( z, z' \in L \).

Conduché proved the following proposition in [7].
Proposition 4.3 Let $G_\ast$ be a simplicial group and $\text{NG}_\ast$ its Moore complex. Suppose $\text{NG}_n = \{1\}$ for $n \geq 3$. Then the complex

$$\text{NG}_2 \xrightarrow{\partial_2} \text{NG}_1 \xrightarrow{\partial_1} \text{NG}_0$$

is a 2-crossed module, where $\text{NG}_0$ acts on $\text{NG}_1$ and $\text{NG}_2$ by conjugacy via the degeneracies and the Peiffer lifting is given by

$$\{y, y'\} = s_1(yy'y^{-1})s_0(y)s_1(y')^{-1}s_0(y)^{-1}.$$ 

Conduché also constructed in [9] a 2-crossed module from a crossed square

$$\begin{array}{ccc}
L & \xrightarrow{\lambda} & M \\
\chi & & \mu \\
N & \xrightarrow{\nu} & P
\end{array}$$

as

$$L \xrightarrow{(\lambda^{-1}, \chi')} M \times N \xrightarrow{\mu\nu} P.$$ 

Since

$$\begin{array}{ccc}
\text{NG}_{1,1} & \xrightarrow{\partial_1^h} & \text{NG}_{0,1} \\
\partial_1^v & & \partial_1^v \\
\text{NG}_{1,0} & \xrightarrow{\partial_0^v} & \text{NG}_{0,0}
\end{array}$$

is a crossed square, the complex of morphisms of groups

$$\begin{array}{ccc}
\text{NG}_{1,1} & \xrightarrow{([\partial_1^h])^{-1}, \partial_1^v} & \text{NG}_{0,1} \times \text{NG}_{1,0} \\
\partial_0^v & & \partial_0^v \\
\text{NG}_{1,0} & \xrightarrow{\partial_0^v} & \text{NG}_{0,0}
\end{array}$$

is a 2-crossed module with the Peiffer lifting map

$$\{(x, a), (y, b)\} = F_{\emptyset, (0), (0), \emptyset}(x, ab)$$

for $(x, a), (y, b) \in \text{NG}_{0,1} \times \text{NG}_{1,0}$ as given in [9].

We get the following result.

Proposition 4.4 Let $G_{\ast, \ast}$ be a bisimplicial group with Moore bicomplex $\text{NG}_{\ast, \ast}$. If for any $p \geq 0$ and $q \geq 3$, $\text{NG}_{p,q} = \{1\}$, then the complex of morphisms of groups

$$\begin{array}{ccc}
\text{NG}_{p,2} & \xrightarrow{\partial_2^v} & \text{NG}_{p,1} \\
\partial_1^v & & \partial_1^v \\
\text{NG}_{p,1} & \xrightarrow{\partial_0^v} & \text{NG}_{p,0}
\end{array}$$

is a 2-crossed module. The Peiffer lifting map

$$\{-, -\} : \text{NG}_{p,1} \times \text{NG}_{p,1} \rightarrow \text{NG}_{p,2}$$
is given by
\[
\{x, y\} = (F_{(\emptyset, (0)), (\emptyset, (1))}(x, y))^{-1} = [s^{(p_0)}_1(x), s^{(p_1)}_1(y)] [s^{(p_1)}_1(y), s^{(p_0)}_1(x)]
\]
for \(x, y \in NG_{p,1}\). Similarly, if \(NG_{p,q} = \{1\}\) for any \(q \geq 0\) and \(p \geq 3\), then the complex of morphisms of groups
\[
NG_{2,q} \overset{\partial_2}{\longrightarrow} NG_{1,q} \overset{\partial_1}{\longrightarrow} NG_{0,q}
\]
is a 2-crossed module. The Peiffer lifting map
\[
\{-, -\} : NG_{1,q} \times NG_{1,q} \rightarrow NG_{2,q}
\]
is given by
\[
\{x, y\} = (F_{(0, (0)), ((1), (0))}(x, y))^{-1} = [s^{(1q)}_1(x), s^{(1q)}_1(y)] [s^{(1q)}_1(y), s^{(1q)}_1(x)]
\]
for \(x, y \in NG_{1,q}\).

**Proof:** See Appendix B. \(\Box\)

5 Appendices

5.1 Appendix A

For \((n, m) = (1, 1)\), consider the set
\[
S(1) \times S(1) = \{(\emptyset, 0), (0, (0)), ((0), \emptyset), ((0), (0))\}.
\]

1. Take \(\alpha = (\emptyset, 0)\) and \(\beta = (0, 0)\). In this case, the function \(F_{\alpha, \beta}\) becomes from \(NG_{1,1} \times NG_{1,0}\) to \(NG_{1,1}\). This map can be defined for any \(x \in NG_{1,1}\) and \(y \in NG_{1,0}\) by
\[
F_{(\emptyset, 0), (0, (0))}(x, y) = [x, s^{(10)}_0(y)].
\]

2. Take \(\alpha = (\emptyset, 0)\) and \(\beta = ((0), \emptyset)\). In this case, the function \(F_{\alpha, \beta}\) becomes from \(NG_{1,1} \times NG_{0,1}\) to \(NG_{1,1}\). This map can be defined by
\[
F_{(\emptyset, 0), ((0), \emptyset)}(x, y) = [x, s^{(01)}_0(y)].
\]
for any \(x \in NG_{1,1}\) and \(y \in NG_{0,1}\).

3. For \(\alpha = (\emptyset, 0)\) and \(\beta = ((0), (0))\). The map
\[
F_{(\emptyset, 0), ((0), (0))} : NG_{1,1} \times NG_{0,0} \rightarrow NG_{1,1}
\]
is defined by
\[
F_{(\emptyset, 0), ((0), (0))}(x, y) = [x, (s^{(00)}_0 s^{(01)}_0(y))]
\]
for all \(x \in NG_{1,1}\) and \(y \in NG_{00}\).
4. For $\alpha = ((0), \emptyset)$ and $\beta = (\emptyset, (0))$. Then the map

$$F_{((0),\emptyset),((0),\emptyset))} : NG_{0,1} \times NG_{1,0} \rightarrow NG_{1,1}$$

can be calculated for any $x \in NG_{0,1}$ and $y \in NG_{1,0}$ by

$$F_{((0),\emptyset),((0),\emptyset))}(x, y) = p\mu(s_0^{(h_{01})}(x), s_0^{(e_{10})}(y))$$

$$= p_0^h p_0^v (s_0^{(h_{01})}(x)) s_0^{(e_{10})}(y) s_0^{(h_{01})}(x)^{-1} s_0^{(e_{10})}(y)^{-1}$$

$$= p_0^h ((s_0^{(h_{01})}(y)) s_0^{(e_{10})}(x)^{-1} s_0^{(h_{01})}(x)^{-1})$$

$$= p_0^h (s_0^{(h_{01})}(y)) s_0^{(e_{10})}(x)^{-1} (s_0^{(h_{01})}(x)^{-1})$$

$$= p_0^h (s_0^{(h_{01})}(y)) s_0^{(e_{10})}(x)^{-1}$$

$$= [s_0^{(h_{01})}(x), s_0^{(e_{10})}(y)].$$

5. For $\alpha = ((0), \emptyset)$ and $\beta = ((0), (0))$. Then the map

$$F_{((0),\emptyset),((0),\emptyset))} : NG_{0,1} \times NG_{0,0} \rightarrow NG_{1,1}$$

can be calculated for any $x \in NG_{0,1}$ and $y \in NG_{0,0}$ by

$$F_{((0),\emptyset),((0),\emptyset))}(x, y) = p\mu(s_0^{(2)}, s_2)(x, y)$$

$$= p_0^h p_0^v [s_0^{(0)}(x), s_0^{(0)}(y)]$$

$$= 1$$

6. Similarly for $\alpha = (\emptyset, (0))$ and $\beta = ((0), (0))$, the map

$$F_{((\emptyset),((\emptyset),((0),\emptyset))}) : NG_{1,0} \times NG_{0,0} \rightarrow NG_{1,1}$$

is the identity as given in the previous step.

By taking $(n, m) = (0, 2)$ and $(2, 0)$, we calculate the possible non identity maps with codomain $NG_{0,2}$ and $NG_{2,0}$ respectively.
First \((n, m) = (0, 2)\). Consider the set

\[ S(0) \times S(2) = \{(\emptyset, \emptyset), (\emptyset, (0)), (\emptyset, (1)), (\emptyset, (0, 1))\}. \]

We try to find the functions \(F_{\underline{\alpha}, \underline{\beta}}\) with codomain \(NG_{0, 2}\). In this case the only non identity map \(F_{\underline{\alpha}, \underline{\beta}}\) can be defined by choosing \(\underline{\alpha} = (\emptyset, (0))\) and \(\underline{\beta} = (\emptyset, (1))\). Then this is a map from \(NG_{0, 1} \times NG_{0, 1}\) to \(NG_{0, 2}\). This map is calculated as follows. For \(x, y \in NG_{0, 1}\), we obtain

\[
F_{(\emptyset, (0)), (\emptyset, (1))}(x, y) = p_\mu(s_{\underline{\alpha}}, s_{\underline{\beta}})(x, y)
= p_1p_0[s_0^{(01)}(x), s_1^{(01)}(y)]
= [s_0^{(01)} x, s_1^{(01)} y][s_1^{(01)} y, s_1^{(01)} x] \in NG_{0, 2}.
\]

Now suppose \((n, m) = (2, 0)\). From the set

\[ S(2) \times S(0) = \{(\emptyset, \emptyset), ((0), \emptyset), ((1), \emptyset), ((0, 1), \emptyset)\} \]

we can choose \(\underline{\alpha} = ((0), \emptyset)\) and \(\underline{\beta} = ((1), \emptyset)\). Then \(F_{\underline{\alpha}, \underline{\beta}}\) is a map from \(NG_{1, 0} \times NG_{1, 0}\) to \(NG_{2, 0}\). This map can be given by for \(x, y \in NG_{1, 0}\)

\[
F_{((0), \emptyset), ((1), \emptyset)}(x, y) = p_\mu(s_{\underline{\alpha}}, s_{\underline{\beta}})(x, y)
= p_1p_0[s_0^{(10)}(x), s_1^{(10)}(y)]
= [s_0^{(10)} x, s_1^{(10)} y][s_1^{(10)} y, s_1^{(10)} x] \in NG_{2, 0}.
\]

Now, by taking \((n, m) = (1, 2)\) and \((2, 1)\), we shall define the possible non identity maps \(F_{\underline{\alpha}, \underline{\beta}}\) whose codomain \(NG_{1, 2}\) and \(NG_{2, 1}\) respectively.

First suppose that \((n, m) = (1, 2)\). We set

\[ S(1) \times S(2) = \{(\emptyset, \emptyset), (\emptyset, (1)), (\emptyset, (0)), (\emptyset, (0, 1)), ((0), \emptyset), ((0), (1)), ((0), (0)), ((0), (1, 0))\}. \]

In the following calculations, by taking appropriate \(\underline{\alpha}, \underline{\beta}\) from the set \(S(1) \times S(2)\), we shall give all the non identity maps whose codomain \(NG_{1, 2}\). To obtain these maps, we can choose the possible \(\underline{\alpha}, \underline{\beta}\) from the set \(S(1) \times S(2)\) as follows.

1. \((\underline{\alpha}, \underline{\beta}) = (\emptyset, (0)), (\emptyset, (1)))\)
2. \((\underline{\alpha}, \underline{\beta}) = ((0), \emptyset), ((0), \emptyset))\)
3. \((\underline{\alpha}, \underline{\beta}) = ((0), (0)), (\emptyset, (0)))\)
4. \((\underline{\alpha}, \underline{\beta}) = ((0), (1)), (\emptyset, (0)))\)
5. \((\underline{\alpha}, \underline{\beta}) = ((0), (0)), (\emptyset, (1)))\).

Now we calculate the functions \(F_{\underline{\alpha}, \underline{\beta}}\) for these pairings \((\underline{\alpha}, \underline{\beta})\).

1. For \(\underline{\alpha} = (\emptyset, (0))\) and \(\underline{\beta} = (\emptyset, (1))\), we obtain the map

\[
F_{(\emptyset, (0)), (\emptyset, (1))} : NG_{1, 1} \times NG_{1, 1} \to NG_{1, 2}.
\]
This map can be given by

\[ F(∅, (1)) (x, y) = pµ(s_α, s_β)(x, y) \]
\[ = p_s^v p_h^k [s^{(11)}_0 (x), s^{(11)}_1 (y)] \]
\[ = [s^{(11)}_0 (x), s^{(11)}_1 (y)] [s^{(11)}_1 (y), s^{(11)}_1 (x)] \in NG_{1,2} \]

for \( x, y \in NG_{1,1} \).

2. For \( \alpha = (∅, (1)), \beta = ((0), ∅) \), we get the map

\[ F((0), (1)) : NG_{1,1} \times NG_{0,2} \rightarrow NG_{1,2} \]

given by

\[ F((0), (1), ((0), ∅)) (x, a) = [s^{(11)}_1 (x), s^{(02)}_0 (a)] \in NG_{1,2} \]

for \( x \in NG_{1,1} \) and \( a \in NG_{0,2} \).

3. For \( \alpha = (∅, (0)), \beta = ((0), ∅) \), we have the following map

\[ F((0), (0)) : NG_{1,1} \times NG_{0,2} \rightarrow NG_{1,2} \]

calculated by

\[ F((0), (0), ((0), ∅)) (x, a) = [s^{(11)}_0 (x), s^{(02)}_0 (a)] \in NG_{1,2} \]

for \( x \in NG_{1,1} \) and \( a \in NG_{0,2} \).

4. For \( \alpha = ((0), (1)), \beta = (∅, (0)) \), we get the following map

\[ F((0), (1)) : NG_{0,1} \times NG_{1,1} \rightarrow NG_{1,2} \]

given by

\[ F((0), (1), ((0), ∅)) (x, y) = [s^{(02)}_0 s^{(01)}_1 (x), s^{(11)}_0 (y)] \in NG_{1,2} \]

for \( x \in NG_{0,1} \) and \( y \in NG_{1,1} \).

5. For \( \alpha = ((0), (0)), \beta = (∅, (1)) \), we get the following map

\[ F((0), (0)) : NG_{0,1} \times NG_{1,1} \rightarrow NG_{1,2} \]

given by

\[ F((0), (0), ((0), (1))) (x, y) = [s^{(02)}_0 s^{(01)}_1 (x), s^{(11)}_1 (y)] \in NG_{1,2} \]

for \( x \in NG_{0,1} \) and \( y \in NG_{1,1} \).

Now suppose that \( (n, m) = (2, 1) \). We consider the set \( S(2) \times S(1) \). By choosing appropriate \( \alpha, \beta \) from the set \( S(2) \times S(1) \), we can calculate similarly all the non identity maps with codomain \( NG_{2,1} \). To obtain these maps, we take the possible \( \alpha, \beta \) as follows.
1. \( (\alpha, \beta) = (((0), \emptyset), ((1), \emptyset)) \)
2. \( (\alpha, \beta) = (((1), \emptyset), (\emptyset, (0))) \)
3. \( (\alpha, \beta) = (((0), \emptyset), (\emptyset, (0))) \)
4. \( (\alpha, \beta) = (((1), (0)), ((0), \emptyset)) \)
5. \( (\alpha, \beta) = (((0), (0)), ((1), \emptyset)) \).

For these \( (\alpha, \beta) \), the corresponding \( F_{\alpha, \beta} \) functions can be calculated as follows.

1. For \( \alpha = ((0), \emptyset) \) and \( \beta = ((1), \emptyset) \), we obtain the map

\[
F_{((0), \emptyset), ((1), \emptyset)} : NG_{1,1} \times NG_{1,1} \rightarrow NG_{2,1}.
\]

This map can be given by

\[
F_{((0), \emptyset), ((1), \emptyset)}(x, y) = [s_0^{(11)}(x), s_1^{(11)}(y)] [s_1^{(11)}(y), s_1^{(11)}(x)] \in NG_{2,1}
\]
for \( x, y \in NG_{1,1} \).

2. For \( \alpha = ((1), \emptyset), \beta = (\emptyset, (0)) \), we get the map

\[
F_{((1), \emptyset), (\emptyset, (0))} : NG_{1,1} \times NG_{2,0} \rightarrow NG_{2,1}
\]

given by

\[
F_{((1), \emptyset), (\emptyset, (0))}(x, a) = [s_1^{(11)}(x), s_0^{(02)}(a)] \in NG_{2,1}
\]
for \( x \in NG_{1,1} \) and \( a \in NG_{2,0} \).

3. For \( \alpha = ((0), \emptyset), \beta = (\emptyset, (0)) \), we get the map

\[
F_{((0), \emptyset), (\emptyset, (0))} : NG_{1,1} \times NG_{2,0} \rightarrow NG_{2,1}
\]

given by

\[
F_{((0), \emptyset), (\emptyset, (0))}(x, a) = [s_0^{(11)}(x), s_0^{(02)}(a)] \in NG_{2,1}
\]
for \( x \in NG_{1,1} \) and \( a \in NG_{2,0} \).

4. For \( \alpha = ((1), (0)), \beta = ((0), \emptyset) \), we get the following map

\[
F_{((1), (0)), ((0), \emptyset)} : NG_{1,0} \times NG_{1,1} \rightarrow NG_{2,1}.
\]

It is given by

\[
F_{((1), (0)), ((0), \emptyset)}(x, y) = [s_1^{(11)} s_0^{(10)}(x), s_0^{(11)}(y)] \in NG_{2,1}
\]
for \( x \in NG_{1,0} \) and \( y \in NG_{1,1} \).

5. For \( \alpha = ((0), (0)), \beta = ((1), \emptyset) \), we get the following map

\[
F_{((0), (0)), ((1), \emptyset)} : NG_{1,0} \times NG_{1,1} \rightarrow NG_{2,1}.
\]

This can be defined by

\[
F_{((0), (0)), ((1), \emptyset)}(x, y) = [s_0^{(11)} s_0^{(10)}(x), s_1^{(11)}(y)] \in NG_{2,1}
\]
for \( x \in NG_{1,0} \) and \( y \in NG_{1,1} \).
Let \((n, m) = (2, 2)\). By choosing appropriate \(\alpha, \beta\) from the set \(S(2) \times S(2)\), we can calculate the non identity maps with codomain \(NG_{2,2}\). The possible \(\alpha, \beta\) are given as follows.

1. \(\alpha = (\emptyset, (0))\), \(\beta = ((1), \emptyset)\)
2. \(\alpha = ((1), (0))\), \(\beta = (\emptyset, (0))\)
3. \(\alpha = ((0), (0))\), \(\beta = ((1), (0))\)
4. \(\alpha = ((1), (0))\), \(\beta = (\emptyset, (0))\)
5. \(\alpha = ((0), (0))\), \(\beta = ((1), (0))\).

For these \((\alpha, \beta)\), the corresponding \(F_{\alpha, \beta}\) functions can be calculated similarly. These functions were listed in the table of Section 3.

### 5.2 Appendix B

**The Proof of Proposition 4.1**

Let \(\alpha = NG_{0,0}\) on \((x, y) \in NG_{0,1} \times NG_{1,0}\) can be given by

\[
a(x, y) = (s^{(0)}_0(a)x s^{(0)}_0(a)^{-1}, s^{(0)}_0(a)y s^{(0)}_0(a)^{-1}).
\]

**CM1)** For \(\alpha = NG_{0,0}\) and \(x \in NG_{0,1}, y \in NG_{1,0}\), we obtain

\[
\partial^a(x, y) = \partial(s^{(0)}_0(a)x s^{(0)}_0(a)^{-1}, s^{(0)}_0(a)y s^{(0)}_0(a)^{-1})
\]

\[
= d^{(0)}_1(s^{(0)}_0(a)x s^{(0)}_0(a)^{-1}, s^{(0)}_0(a)y s^{(0)}_0(a)^{-1})
\]

\[
= (a d^{(0)}_1(x)a^{-1}) (a d^{(0)}_1(y)a^{-1})
\]

\[
= a \partial(x, y)a^{-1}.
\]

**CM2)** For \((x_1, y_1), (x_2, y_2) \in NG_{0,1} \times NG_{1,0},

\[
\partial(x_1, y_1)(x_2, y_2) = d^{(1,0)}_1(x_1)d^{(1,0)}_1(y_1)(x_2, y_2)
\]

\[
= \left( s^{(0)}_0 \left( d^{(1,0)}_1(x_1)d^{(1,0)}_1(y_1) \right) x_2 s^{(0)}_0 \left( d^{(0)}_1(x_1)d^{(1,0)}_1(y_1) \right)^{-1} ,
\]

\[
= \left( s^{(0)}_0 \left( d^{(1,0)}_1(x_1)d^{(1,0)}_1(y_1) \right) y_2 s^{(0)}_0 \left( d^{(0)}_1(x_1)d^{(1,0)}_1(y_1) \right)^{-1} \right)
\]

Since \(NG_{1,1} = \{1\}\) for \(x_1, y_2 \in NG_{0,1}\) we have

\[
\partial_2^{(0,2)}(F_{\emptyset, (0)}, (\emptyset, 1))(x_1, x_2) = [s^{(0)}_0 d^{(0)}_1(x_1), x_2][x_2, x_1]
\]

\[
= s^{(0)}_0 d^{(0)}_1(x_1)x_2 s^{(0)}_0 d^{(0)}_1(x_1)^{-1} x_1 x_2^{-1} x_1^{-1} \in \partial_2^{(0,2)}(NG_{0,2} \cap D_{0,2}) = \{1\},
\]

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and we obtain
\[ s_0^{h(00)} d_1^{h(10)} (x_1)x_2 s_0^{h(00)} d_1^{h(01)} (x_1)^{-1} = x_1 x_2 x_1^{-1}. \]

Similarly, for \( y_1, y_2 \in NG_{1,0} \) we have
\[
\partial_2^{h(20)} (F_{(0,0),(1,0)} (y_1, y_2)) = [s_0^{h(00)} d_1^{h(10)} (y_1), y_2] [y_2, y_1] \\
= s_0^{h(00)} d_1^{h(10)} (y_1) y_2 s_0^{h(00)} d_1^{h(10)} (y_1)^{-1} y_1 y_2^{-1} y_1^{-1} \in \partial_2^{h(20)} (NG_{2,0} \cap D_{2,0}) = \{1\},
\]

and we obtain
\[ s_0^{h(00)} d_1^{h(10)} (y_1) y_2 s_0^{h(00)} d_1^{h(10)} (y_1)^{-1} = y_1 y_2 y_1^{-1}. \]

Thus we get
\[
\partial(x_1, y_1) (x_2, y_2) = (x_1 x_2 x_1^{-1}, y_1 y_2 y_1^{-1}) \\
= (x_1, y_1) (x_2, y_2) (x_1, y_1)^{-1}
\]

and this is the second condition of crossed module. □

**The Proof of Proposition 4.2:** The actions of \( NG_{0,1} \) and \( NG_{1,0} \) on \( NG_{1,1} \) are given by

1. \( y x = s_0^{e(00)} d_1^{e(10)} (y) x s_0^{e(00)} d_1^{e(10)} (y)^{-1} = d_1^{e(11)} s_0^{e(10)} (y) d_1^{e(11)} s_0^{e(10)} (y)^{-1} \)
2. \( x y = s_0^{h(00)} d_1^{h(10)} (x) y s_0^{h(00)} d_1^{h(10)} (x)^{-1} = d_1^{h(11)} s_0^{h(10)} (x) d_1^{h(11)} s_0^{h(10)} (x)^{-1} \)
3. \( y z = s_0^{e(10)} (y) z s_0^{e(10)} (y)^{-1} \)
4. \( x z = s_0^{h(01)} (x) z s_0^{h(01)} (x)^{-1} \)

for \( x \in NG_{0,1}, y \in NG_{1,0} \) and \( z \in NG_{1,1} \).

(i) Since \( NG_{1,2} = NG_{2,1} = NG_{0,2} = NG_{2,0} = \{1\} \), by using the method given in the previous proposition, it can be easily shown that the maps \( \partial_1^{h(10)}, \partial_1^{e(01)}, \partial_1^{h(11)} \) and \( \partial_1^{e(11)} \) are crossed modules.

(iv) For \( x \in N_{0,1} \) and \( y \in NE_{1,0} \), we obtain
\[
\partial_1^{h(11)} h(x, y) = \partial_1^{h(11)} (s_0^{h(01)} (x) s_0^{h(01)} (y) s_0^{h(01)} (x)^{-1}) s_0^{h(10)} (y)^{-1} \\
= x \partial_1^{h(11)} s_0^{h(10)} (y) x^{-1} \partial_1^{h(11)} s_0^{h(10)} (y)^{-1} \\
= x s_0^{h(00)} d_1^{h(10)} (y) x^{-1} s_0^{h(00)} d_1^{h(10)} (y)^{-1} \\
= x y x^{-1}.
\]
(v) We obtain for $x \in NE_{0,1}$ and $y \in NE_{1,0}$

\[
\partial_1^{(11)} h(x, y) = \partial_1^{(11)} \left[ s_0^{(01)}(x), s_0^{(10)}(y) \right] \\
= \left[ \partial_1^{(11)} s_0^{(01)}(x), y \right] \\
= \left[ s_0^{(00)} d_1^{(01)}(x), y \right] \\
= s_0^{(00)} d_1^{(01)}(x) y s_0^{(00)} d_1^{(11)}(x) y^{-1} \\
= x y^{-1}.
\]

(vi) For $z \in NG_{1,1}$ and $y \in NG_{1,0}$, we get

\[
b(h_1^{(11)}(z), y) = [s_0^{(01)} d_1^{(11)}(z), s_0^{(10)}(y)].
\]

On the other hand for $z \in NG_{1,1}$ and $s_0^{(10)} y \in NG_{1,1}$, we obtain also

\[
F_{((0,0),(1,0))}(z, s_0^{(10)} y) = [s_0^{(11)}(z), s_1^{(11)}(y)] [s_0^{(11)}(z), s_0^{(10)}(y), s_1^{(11)}(z)] \in NG_{2,1} = \{1\},
\]

and

\[
\partial_2^{(12)} F_{((0,0),(1,0))}(z, s_0^{(10)} y) = [s_0^{(01)} d_1^{(11)}(z), s_0^{(10)}(y)] [s_0^{(10)}(y), z] = 1.
\]

Thus we get

\[
h(h_1^{(11)}(z), y) = [s_0^{(01)} d_1^{(11)}(z), s_0^{(10)}(y)] \\
= [z, s_0^{(10)}(y)] \\
= z s_0^{(10)}(y) z^{-1} s_0^{(10)}(y)^{-1} \\
= z y(z^{-1}).
\]

(vii) For $z \in NG_{1,1}$ and $y \in NG_{0,1}$, we get

\[
h(y, h_1^{(11)}(z)) = [s_0^{(01)}(y), s_0^{(10)} d_1^{(11)}(z)].
\]

On the other hand for $z \in NG_{1,1}$ and $s_0^{(01)} y \in NG_{1,1}$, we obtain also

\[
F_{((0,0),(0,1))}(z, s_0^{(01)} y) = [s_0^{(11)}(z), s_1^{(11)}(y)] [s_0^{(11)}(z), s_0^{(01)}(y), s_1^{(11)}(z)] \in NG_{1,2} = \{1\},
\]

and

\[
\partial_2^{(12)} F_{((0,0),(0,1))}(z, s_0^{(01)} y) = [s_0^{(10)} d_1^{(11)}(z), s_0^{(01)}(y)] [s_0^{(01)}(y), z] = 1.
\]

Thus we get

\[
h(y, h_1^{(11)}(z)) = [s_0^{(01)}(y), s_0^{(10)} d_1^{(11)}(z)] \\
= [s_0^{(01)}(y), z] \\
= s_0^{(01)}(y) z s_0^{(01)}(y) z^{-1} \\
= y(z) z^{-1}.
\]
The Proof of Proposition 4.4 In each direction, by using Proposition 4.3 and the images of the functions $F_{\alpha,\beta}$ similarly to [16], the proof can be easily given. For example, one has:

1. \[
\partial^{(2q)}_2 \{x, y\} = d^{(2q)}_2 [s^{(1q)}_1 (x), s^{(1q)}_1 (y)][s^{(1q)}_0 (y), s^{(1q)}_0 (x)]
\]

\[
= [x, y][y, d^{(2q)}_2 s^{(1q)}_0 (x)]
\]

\[
= xyx^{-1} s^{(0q)}_0 d^{(1q)}_1 (y)(y)^{-1} s^{(0q)}_0 d^{(1q)}_1 (x)^{-1}
\]

\[
= xyx^{-1} (d^{(1q)}_1 (x))^{-1}.
\]

2. Since

\[
\partial^{(3q)}_3 (F((0),\emptyset),(1),\emptyset)(z, z'))^{-1} = [z', z][s^{(1q)}_1 d^{(2q)}_2 (z), s^{(1q)}_1 d^{(2q)}_2 (z')][s^{(1q)}_0 d^{(2q)}_2 (z), s^{(1q)}_0 d^{(2q)}_2 (z')]
\]

\[
\in \partial^{(3q)}_3 (NG_3, q \cup D_3, q) = \{1\}
\]

for $z, z' \in NG_2, q$, we have

\[
\{\partial^{(2q)}_2 z, \partial^{(2q)}_2 z'\} = [s^{(1q)}_1 d^{(2q)}_2 (z), s^{(1q)}_1 d^{(2q)}_2 (z')][s^{(1q)}_0 d^{(2q)}_2 (z'), s^{(1q)}_0 d^{(2q)}_2 (z)]
\]

\[
= [z, z'].
\]
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