Instability in near-horizon geometries of even-dimensional Myers–Perry black holes

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Abstract
We study the gravitational, electromagnetic and scalar field perturbations on the near-horizon geometries of the even-dimensional extremal Myers–Perry black holes. By dimensional reduction, the perturbation equations are reduced to effective equations of motion in AdS$_2$. We find that some modes in the gravitational perturbations violate the Breitenl"ohner–Freedman bound in AdS$_2$. This result suggests that the even-dimensional (near-)extremal Myers–Perry black holes are unstable against gravitational perturbations. We also discuss implications of our results to the Kerr–CFT correspondence.

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1. Introduction

Black hole solutions have been fundamental objects in the study of general relativity theory. Higher dimensional spacetimes introduced in some particle physics models motivated to consider their generalizations to higher dimensions, and as a result diverse solutions of black objects were found [1, 2]. A feature specific to those higher dimensional black hole solutions is that they may be unstable, even though the Schwarzschild and Kerr black holes in four dimensions are stable. Those instabilities signal the appearance of new branches of black hole solutions and also tell us that some other solutions will be realized as a result of time evolution. As such, the stability analysis of the higher dimensional black objects is one of the most important steps to understand the phase structure of the solutions in higher dimensions and also the final states of higher dimensional spacetimes.

Generalizations of (four-dimensional) Kerr black holes in $d$-dimensional spacetime are the Myers–Perry (MP) black holes [3], which may have $\lfloor (D - 1)/2 \rfloor$ different angular momenta. By an intuitive argument, it was predicted that ‘ultraspinning’ MP black holes should suffer from an instability [4]. Here, by ultraspinning, we mean that the Hessian matrix, $H_{ab} = -\partial^2 S / \partial x^a \partial x^b$ ($x^a = M, J_i$), for a black hole with mass $M$, angular momenta $J_i$ and horizon entropy...
$S$ has at least two negative eigenvalues [5]4. Authors of [6–9] have clarified the conditions for the ultraspinning instability of the singly spinning MP black holes, and also the authors of [5, 10–13] studied that the odd-dimensional MP black holes with equal angular momenta or all but one angular momenta are equal. Since also the even-dimensional MP black holes with equal angular momenta have an ultraspinning regime [5], we can expect the instability to occur even for these black holes. The instability of these even-dimensional MP black holes is, however, yet to be shown so far. A part of the reason is that the geometries of the even-dimensional ones are cohomogeneity-2 and less symmetric compared to those of the odd-dimensional ones, which are cohomogeneity-1, and then the stability analysis becomes more involved.

One of our purposes is to fill this gap in our knowledge about the MP black holes by assessing the stability of the even-dimensional MP black holes with equal angular momenta. Since the stability analysis of the full spacetime requires to solve complicated partial differential equations, it is desirable that we have more simpler ways to obtain implications of instability. Authors of [12, 14] initiated a study on a new analysis method for such a demand. Their method is based on the generalization of the Geroch–Held–Penrose formalism to higher dimensions [15]. They chose some combinations of the Weyl tensor components as perturbation variables and found that the variables satisfy decoupled equations of motion when the near-horizon geometries of extreme black holes are taken as the background spacetimes. Their method can be regarded as the higher dimensional generalization of the Teukolsky formalism [16], although it is applicable only to the Kundt spacetimes5. In [12], it was conjectured that the instability of the near-horizon geometries implies the instability of the original full spacetime. A proof for this statement is provided for the scalar field perturbations, and some pieces of evidence were found for the gravitational perturbations, especially in the background of the odd-dimensional MP black holes. In this paper, we assume this conjecture to be correct and study the stability of the even-dimensional MP black holes with equal angular momenta by examining the perturbations on their near-horizon geometries.

We also have a motivation from the Kerr–CFT correspondence [17], which is a conjecture that near-extreme black holes are described by 2D CFTs. From the fall-off behavior of the perturbations on near-horizon geometries, we can read off the conformal weights of right sectors in the dual CFTs [12, 18–20]. In four and five dimensions, some interesting properties of the conformal weights have been found for the operators corresponding to the axisymmetric perturbations. One of them is the integrerness of the conformal weights: all the conformal weights for vacuum near-horizon geometries take integral values. Another one is the universality of the conformal weights: all the $U(1) \times U(1)$ symmetric five-dimensional vacuum near-horizon geometries with vanishing cosmological constant share the same sequences of conformal weights6. This result is surprising because these five-dimensional near-horizon geometries contain dimensionless parameters on which the conformal weights may depend in principle. In addition to that, five-dimensional extreme black holes can have $S^3$ or $S^2 \times S^1$ horizon topology [21, 23]. Nevertheless, those various near-horizon geometries share the common sequences of the conformal weights. This result suggests that there is a ‘universal sector’ present in all the CFTs dual to extreme rotating black holes in five dimensions. These features have been observed only in four and five dimensions. It was also observed that the

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4 In [5], it was shown that any vacuum stationary black holes have at least one negative mode in the Hessian matrix. It is a manifestation of the property that black holes have negative specific heat. Ultraspinning MP black holes have at least one more negative mode in addition to that.

5 The Kundt spacetime is the spacetime admitting a null geodesic congruence with vanishing expansion, rotation and shear. All known near-horizon geometries are the Kundt spacetimes.

6 In four dimensions, the near-horizon extreme Kerr geometry is the unique $U(1)$ symmetric vacuum near-horizon geometry [21, 22] and does not have any dimensionless parameters. Thus, the universality is trivial in four dimensions.
effective mass of the perturbative fields becomes non-integer but rational numbers for any parameters when the background near-horizon geometry is unstable [12]. We will examine if these properties persist even for the even-dimensional MP black holes we are focusing on.

In our analysis method, which was used also in [12], the perturbation equations of the near-horizon geometries are Kaluza–Klein reduced to an equation of motion of a massive charged scalar field on AdS$_2$ spacetime whose effective mass is determined by the eigenvalues of the angular part of the perturbation equations. For the background of the even-dimensional MP black holes, those angular part equations are reduced to one-dimensional ordinary differential equations (ODEs) which are coupled to each other if we classify the perturbations into scalar/vector/tensor modes according to their transformation properties on $\mathbb{C}P^{n-1}$ contained in the induced geometry on the horizon. We solve those equations using the numerical technique employed in [18], which analyzed the conformal weights for the general near-horizon geometries in five dimensions.

This paper is organized as follows. After introducing the near-horizon geometries and the perturbation equations in section 2, we show the results of the stability analysis for the gravitational perturbations in section 3. As a result, we find that certain modes in these perturbations become unstable. In section 4, we show the results for the scalar and electromagnetic field perturbations. These fields are stable in the full geometries, and we observe that the analysis of the near-horizon geometries gives results consistent with this fact. As a byproduct of our analysis, we find that the spectrum of the mass of the effective AdS$_2$ field, which is given by the eigenvalues of the angular part of the perturbations, shows a well-organized structure. We give comments on this property in section 5. We also discuss the implications of our results to the Kerr–CFT correspondence in section 6, especially about the integrerness and universality of the conformal weights mentioned above. Section 7 is devoted to the summary and discussions, and appendices show the technical details of our analysis.

2. Perturbations of near-horizon geometries

In this section, we summarize the near-horizon geometries of the even-dimensional MP black holes with equal angular momenta and also the perturbation equations in this background. We also introduce the instability condition for the near-horizon geometries which is argued in [12].

2.1. Near-horizon geometry

In our study, we focus on the extremal MP black holes in $d = 2n + 2$ ($n \geq 2$) dimensions with all the angular momenta set equal. Their near-horizon geometries can be written as a fibration over AdS$_2$ given by [24]

$$\begin{align*}
\text{d}s^2 &= L(\theta)^2 \left( -r^2 \, \text{d}t^2 + \frac{\text{d}r^2}{r^2} \right) + \Theta^2(\theta) \, \text{d}\theta^2 \\
&\quad + \Psi^2(\theta) \hat{g}_{\alpha\beta} \, \text{d}x^\alpha \, \text{d}x^\beta + \Phi^2(\theta) (\text{d}\phi + \Omega r \, \text{d}t)^2,
\end{align*}$$

(1)

where $0 \leq \theta \leq \pi$, $L^2(\theta) = \frac{a^2 f_\alpha(\theta)}{(2n-1)^2}$, $\Theta^2(\theta) = \frac{a^2 f_\alpha(\theta)}{2n-1}$, $\Psi^2(\theta) = \frac{2n a^2 \sin^2 \theta}{2n-1}$, $\Phi^2(\theta) = \frac{4n^2 a^2 \sin^2 \theta}{(2n-1)f_\alpha(\theta)}$, $f_\alpha(\theta) = 1 + (2n-1) \cos^2 \theta$, $k^\phi = -\frac{1}{n \sqrt{2n-1}} \equiv \Omega$, $\phi$ is $2\pi$-periodic and $\hat{g}_{\alpha\beta} \, \text{d}x^\alpha \, \text{d}x^\beta (\alpha, \beta = 1, \ldots, 2n-2)$ is the Fubini–Study metric on $\mathbb{C}P^{n-1}$ with the Kähler form $\mathcal{J} = \frac{1}{2} \, \text{d}\mathcal{A}$ normalized to have $R_{\alpha\beta} = 2n \hat{g}_{\alpha\beta}$. From the near-horizon
metric, we can read off the induced metric on the horizon (the surface of constant $t$ and $r$) as
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \Theta^2(\theta) d\theta^2 + \Psi^2(\theta) g_{\alpha\beta} dx^\alpha dx^\beta + \Phi^2(\theta) (d\phi + A)^2. \] (3)

2.2. Perturbation equations

Reference [12] gave a prescription to obtain separated perturbation equations for general near-horizon geometries. We explain it briefly below.

To describe the perturbations of the near-horizon geometry, we introduce a null basis \( \{\ell, n, m_i\} \), where \( \ell = L(\theta)(-dr + dr/r)/\sqrt{2} \), \( n = L(\theta)(r dr + dr/r)/\sqrt{2} \) and \( m_i \) are the orthonormal space-like vectors orthogonal to both \( \ell \) and \( n \). We define \( \Omega_{ij} \equiv C_{abcd} e^a\mu e^b\nu e^c\tau e^d\rho \) and \( \phi_i \equiv F_{ab} e^a\mu e^b\nu \), where \( C_{abcd} \) is the Weyl tensor and \( F_{ab} \) is the electromagnetic field strength, and use them as the perturbation variables. As for the scalar field satisfying \( \Box - M^2 \psi = 0 \), we use the scalar field itself as the perturbation variable. In this paper, we focus only on axisymmetric modes along the \( \phi \) direction: \( \partial_\phi \Omega_{ij} = \partial_\phi \phi_i = \partial_\phi \psi = 0 \). For these perturbations, we can take the separation ansatz as
\[ \psi = \chi_0(t, r) Y(t), \quad \phi_i = \text{Re} \{X_1(t, r) Y(t)\}, \quad \Omega_{ij} = \text{Re} \{X_2(t, r) Y(t)\}. \] (4)

Then, the perturbation equation is separated as follows. The radial equations are given by massive charged Klein–Gordon equations in AdS$_2$ with homogeneous electric fields
\[ \left[ (\nabla_2 - iq_2 A_2)^2 - q_2^2 - \lambda_2 \right] \chi_s = 0, \quad (s = 0, 1, 2), \] (5)
where \( \nabla_2 \) is the covariant derivative on AdS$_2$ \( (dr^2 = -r^2 dr^2 + \frac{dr^2}{r^2}) \), \( A_2 = -r dr \) is the effective gauge field which appeared as a result of the dimensional reduction and \( q_2 \equiv \) is the effective \( U(1) \) charges. The angular equations are given by
\[ O^{(0)} Y = \lambda_0 Y, \quad (O^{(1)})_{\mu} = \lambda_1 Y_{\mu}, \quad (O^{(2)})_{\mu\nu} = \lambda_2 Y_{\mu\nu}, \] (6)
where the operators \( O^{(s)} \) are defined as
\[ O^{(s)} Y = -\nabla_\mu (L^2 \nabla^\mu Y) + L^2 M^2 Y, \] (7)
\[ (O^{(1)})_{\mu} = -\frac{1}{L^2} \nabla_\mu (L^2 \nabla^\mu Y_{\mu}) + \left( 2 - \frac{5}{4L^2} k_0 k^0 \right) Y_{\mu} + L^2 \left( R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} \right) Y^\nu + \left( -\frac{1}{2} (dk)_{\mu\nu} + 2(k - d(L^2)(\mu) \nabla_{\nu} - \frac{1}{L^2} (dL^2)(\mu) k_{\nu} \right) Y^\nu, \] (8)
\[ (O^{(2)})_{\mu\nu} = -\frac{1}{L^4} \nabla^\rho (L^6 \nabla_\mu Y_{\rho}) + \left( 6 - \frac{4}{L^2} k_0 k^0 \right) Y_{\mu\nu} + 2L^2 (R_{\mu\nu} + R g_{\mu\nu}) Y^\rho_{\rho} - 2L^2 R_{\mu\nu}^\rho g Y_{\rho\sigma} - \left[ (-dk)_{\mu\rho} + \frac{2}{L^2} (dL^2) \wedge k)_{\mu\rho} + 2(k - d(L^2)_{\mu}) \nabla_{\rho} - 2(k - d(L^2))_{\rho} \nabla_{\mu} Y^\rho_{\nu} \right] Y^\rho_{\nu}, \] (9)
where the covariant derivative \( \nabla_\mu \) and the curvature tensors such as \( R_{\mu\nu\rho\sigma} \) are defined with respect to the induced metric on the horizon (3).

The solution of equation (5) at large \( r \) behaves as \( \chi_s \sim r^{-\Delta_+} \), where
\[ \Delta_+ = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda_s}. \] (10)
Following the arguments of [12], we shall call the near-horizon geometry to be unstable when $\lambda_s$ violates the effective Breitenlohner–Freedman (BF) bound, i.e. $\lambda_s < -1/4$, and $\Delta_\perp$ becomes complex. Reference [12] conjectured that the sufficient condition for the full black hole geometry to be unstable is that the near-horizon geometry is unstable against axisymmetric perturbations. This conjecture is proved in the case of the scalar field, and it was suggested that a similar argument applies also to the gravitational perturbations.

In this paper, we argue the stability of the even-dimensional MP black holes with equal angular momenta based on this conjecture. We solve equations (6) analytically if possible and numerically otherwise to obtain $\lambda_s$. Applying the conjecture to the resultant $\lambda_s$, we will argue the stability of the full black hole geometries.

3. Instability against gravitational perturbations

We start our discussions from the gravitational perturbations on the near-horizon geometries. The gravitational perturbations can be decomposed by tensor, vector and scalar harmonics on the base space $\mathbb{C}P^{n-1}$, which are labeled by a principal quantum number $\kappa = 0, 1, 2, \ldots$. By the mode decompositions, the eigenvalue equations (9) reduce to coupled ODEs which depend only on the $\theta$ coordinate. Solving the ODEs, we obtain the eigenvalues for the operator $\mathcal{O}^{(2)}$. The computation is straightforward but too technical to show the details here. Hence, we describe only some important results in this section and defer the details of the calculations to the appendices.

3.1. Gravitational tensor modes

We start from the simplest components, which are the gravitational tensor modes. The tensor modes are decomposed by tensor harmonics in $\mathbb{C}P^{n-1}$, and as a result the eigenvalue equations (9) reduce to an ODE for a single coordinate $\theta$ given by equation (B.19). For $n = 2$, the tensor harmonics do not exist on $\mathbb{C}P^1 = S^2$. For $n \geq 3$, we obtain an analytical expression for the eigenvalues of $\mathcal{O}^{(2)}$ as

$$
\lambda_2 = \frac{2 (\kappa + n (\ell + 1) - 1) (\kappa + n (\ell + 1))}{n (2n - 1)} - \ell (\ell + 1) - \frac{2 (n^2 - 3n + 1)}{n (2n - 1)} - \frac{2 \sigma}{n}, \quad (\ell = 0, 1, 2, \ldots),
$$

where $\kappa = 0, 1, 2, \ldots$ is the principal quantum number which labels the tensor harmonics. The parameter $\sigma = \mp 1$ separates the tensor harmonics into Hermitian and anti-Hermitian ones. See appendix B.3.1 for the further details of the tensor harmonics. Another integer $\ell$ is the quantum number along the $\theta$ direction, which parametrizes the $\theta$ dependence of $Y_{\mu \nu}$. This $\lambda_2$ is always non-negative and does not violate the BF bound. Thus, there is no indication of instability in the tensor modes.

3.2. Gravitational vector modes

Next, we study the gravitational vector modes. The vector modes are decomposed by vector harmonics in $\mathbb{C}P^{n-1}$ and their derivatives. For $n = 2$, the vector harmonics do not exist on $\mathbb{C}P^1 = S^2$. For $n \geq 3$, we find the eigenvalues are given by

$$
\lambda_2 = \frac{2 (\kappa + n (\ell + 1) + 1) (\kappa + n (\ell + 1) + 2)}{n (2n - 1)} - \ell (\ell + 1) + \frac{C}{2n - 1}.
$$

5
where $\kappa = 0, 1, 2, \ldots, \ell = \ell_0, \ell_0 + 1, \ldots$ and $(C, \ell_0)$ are the set of integers given by

$$(C, \ell_0) = (4 - 2n, 0), \ (4, -1), \ (4, 0), \ (4, +1). \ (13)$$

The gravitational vector modes are described by four free variables, as we see in appendix B.3.2. This is why we have four kinds of the eigenvalues described by different integer sets $(C, \ell_0)$ as listed in equation (13). The eigenvalues are all positive, and hence no instability is implied.

### 3.3. Gravitational scalar modes

Finally, we consider the gravitational scalar modes. We can expand the scalar modes by scalar harmonics in $\mathbb{C}^{p_a-1}$ and their derivatives. As a result of mode decomposition, we obtain the coupled ODEs of ten variables in general cases ($\kappa > 1$ and $n > 2$) as the eigenvalue equations to solve. In the cases of $\kappa = 0, 1$ and/or $n = 2$, some of the unknown variables drop out and we have to treat those cases separately. In any case, we find that the eigenvalues are written in a unified expression as

$$\lambda_2 = \frac{2[\kappa + n (\ell + 1) - 1][\kappa + n (\ell + 1)]}{n(2\ell - 1)} - \ell(\ell + 1) + \frac{C}{2n - 1}, \ (14)$$

where $\kappa = 0, 1, 2, \ldots$ and $\ell = \ell_0, \ell_0 + 1, \ldots$. For a general case with $\kappa > 1$ and $n > 2$, we have ten eigenvalues described by the integer sets $(C, \ell_0)$ given by

$$(C, \ell_0) = (-2(n - 1), 0), \ (0, -1), \ (0, 0), \ (0, +1), \ (2n, -2), \ (2n, -1), \ (2n, 0), \ (2n, +1), \ (2n, +2). \ (15)$$

Note that the constant $C$ takes only three different values, that is, $C = -2(n - 1)$ and $2n$, and each of them is associated with one, three and six eigenvalues labeled by different $\ell_0$, respectively. For the special values of $\kappa$ and $n$ mentioned above, the eigenvalues are described by

$\kappa = 0:$ \hspace{1cm} $(C, \ell_0) = (-2(n - 1), +2), \ (0, +2), \ (2n, +2)$ \hspace{1cm} (16)

$\kappa = 1, n = 2:$ \hspace{1cm} $(C, \ell_0) = (-2(n - 1), 0), \ (0, 0), \ (0, 0), \ (0, +1), \ (2n, 0), \ (2n, +1)$ \hspace{1cm} (17)

$\kappa = 1, n > 2:$ \hspace{1cm} $(C, \ell_0) = (-2(n - 1), 0), \ (0, 0), \ (0, 0), \ (0, +1), \ (2n, 0), \ (2n, +1), \ (2n, +2)$ \hspace{1cm} (18)

$\kappa > 1, n = 2:$ \hspace{1cm} $(C, \ell_0) = (-2(n - 1), 0), \ (0, -1), \ (0, 0), \ (0, +1), \ (2n, -2), \ (2n, -1), \ (2n, 0), \ (2n, +1), \ (2n, +2).$ \hspace{1cm} (19)

Comparing equations (16)–(19) to equation (15), we note that some of the eigenvalue sequences described by the integers listed in equation (15) drop out in the special cases with low $\kappa$ and $n$. We also note that $\ell_0$ increases in some of the remaining sequences, which implies that low $\ell$ modes existing in general cases are truncated in the special cases.

From the expression of the eigenvalues, we can find the parameter region for the instability, that is, $\lambda_2 < -1/4$. The eigenvalues other than $(C, \ell_0) = (2n, -2)$ in $\kappa > 1$ and $n \geq 2$ are shown to be non-negative, and thus they do not imply instability. Now, let us examine the eigenvalues for $(C, \ell_0) = (2n, -2)$. For $\ell \geq -1$, we can show that the eigenvalues are non-negative again. However, for the lowest mode $\ell = -2$, we have

$$\lambda_2 = \frac{2(k - 1)(k - 2n)}{n(2n - 1)}. \ (20)$$
Table 1. $\kappa_{\pm}$ against the number of dimensions $d$. The near-horizon geometries are unstable against gravitational scalar perturbation satisfying $\kappa_- < \kappa < \kappa_+$. 

| $d$ | 6   | 8   | 10  | 12  | 14  | 16  | 18  | 20  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\kappa_-$ | 1.28 | 1.41 | 1.54 | 1.68 | 1.81 | 1.94 | 2.08 | 2.21 |
| $\kappa_+$  | 3.72 | 5.59 | 7.46 | 9.32 | 11.19| 13.06| 14.92| 16.79|

This eigenvalue can be smaller than $-1/4$ depending on $\kappa$ and $n$. The instability condition can be written as

$$\kappa_- < \kappa < \kappa_+, \quad \kappa_{\pm} \equiv \frac{4n + 2 \pm \sqrt{2(2n-1)(3n-2)}}{4}. \quad (21)$$

We show the values of $\kappa_{\pm}$ for some $n \geq 2$ ($d = 2n + 2 \geq 6$) in table 1. We find that a several numbers of $\kappa$ satisfy the instability condition for any $n \geq 2$. We also find that the parameter region for the instability widens as we increase the number of dimensions $n$. It would be interesting to search for instability in the full spacetime geometry corresponding to these unstable modes on the near-horizon geometries.

4. Scalar and electromagnetic field perturbations

Having finished the study on the gravitational perturbations, we move on to the scalar and electromagnetic field perturbations. These perturbations are guaranteed to be stable on the full geometry by the following argument. These fields satisfy dominant energy conditions and, in addition to that, there is a global time-like Killing vector in the full geometry. Then, we can construct an energy integral whose integrand is non-negative everywhere [25, 26]. Hence, any instability cannot occur. Therefore, if we show the stability of the near-horizon geometry against these perturbations, we can give a non-trivial check of the conjecture in [12].

As for the scalar field perturbations, we may take the scalar field as the perturbation variable, and it is decomposed by the scalar harmonics on $\mathbb{C}P^{n-1}$. We show the result for this case in section 4.1. As for the electromagnetic field perturbations, the perturbation variables are decomposed into the vector and scalar modes on $\mathbb{C}P^{n-1}$. We show the results for each mode in section 4.2 and section 4.3, respectively. We will see that any perturbations are stable, which is consistent with the conjecture mentioned above.

4.1. Scalar field perturbations

The scalar field can be decomposed by scalar harmonics in $\mathbb{C}P^{n-1}$. As a result, we obtain a single ODE as the perturbation equation, which is given by equation (B.3). When the scalar field is massless ($M = 0$), we analytically find the eigenvalues to be given by

$$\lambda_0 = \frac{2(k + n(\ell + 1) - 1)\{k + n(\ell + 1)\}}{n(2n - 1)} - \ell(\ell + 1) - \frac{2(n - 1)}{2n - 1}, \quad (22)$$

where $k = 0, 1, 2, \ldots$ and $\ell = 0, 1, 2 \ldots$. This $\lambda_0$ is non-negative for any $k$, $\ell$ and $n \geq 2$, and thus the AdS$_2$ BF bound ($\lambda_0 \geq -1/4$) is not violated. This is consistent with the stability of the scalar field perturbation on the full geometry. We found evidence that this property persists even for a massive scalar field ($M > 0$). See appendix B.1 for details.
4.2. Electromagnetic vector modes

Electromagnetic fields can be decomposed by vector and scalar harmonics in $\mathbb{CP}^{n-1}$. Here, we consider the vector modes. For $n = 2$, there is no vector harmonics on $\mathbb{CP}^1$. For $n > 2$, we analytically find the eigenvalues for the vector modes are given by

$$\lambda_1 = \frac{2 \left\{ \kappa + n(\ell + 1) \right\} \left\{ \kappa + n(\ell + 1) + 2 \right\}}{n(2n - 1)} - \ell(\ell + 1) + \frac{4 - 2n}{2n - 1},$$

where $\ell = 0, 1, 2, \ldots$. These eigenvalues are always positive, and thus these modes are expected to be stable.

4.3. Electromagnetic scalar modes

Finally, we consider the electromagnetic scalar modes, which can be decomposed by scalar harmonics on $\mathbb{CP}^{n-1}$ and their derivatives. As a result of the mode decomposition, we obtain coupled ODEs for four variables in a general case ($\kappa > 1$) as the eigenvalue equations to solve. When $\kappa = 0$, the scalar harmonics becomes a constant on $\mathbb{CP}^{n-1}$ and the scalar-derived vectors vanish. Then, we have only two variables to solve. For both cases of $\kappa \geq 1$ and $\kappa = 0$, the eigenvalues are written by a unified expression as

$$\lambda_1 = \frac{2 \left\{ \kappa + n(\ell + 1) - 1 \right\} \left\{ \kappa + n(\ell + 1) \right\}}{n(2n - 1)} - \ell(\ell + 1) + \frac{C}{2n - 1},$$

where $\ell = \ell_0, \ell_0 + 1, \ldots$. For $\kappa > 0$, the integer sets $(C, \ell_0)$ given by

$$(C, \ell_0) = (-2(n - 1), 0), (0, -1), (0, 0), (0, +1).$$

(25)

For $\kappa = 0$, they reduce to

$$(C, \ell_0) = (-2(n - 1), +1), (0, +1).$$

(26)

The eigenvalues for these parameter sets are non-negative, and instability is not implied. This result along with that in the previous section is consistent with the stability of the full geometry against the electromagnetic perturbations.

5. Comments on the effective mass spectrum

In this section, we give brief comments on the spectrum of the effective mass of the radial field on AdS$_2$, namely the relationships among the eigenvalues we have clarified in the previous sections.

Comparing equation (13) to equation (23), we note that the eigenvalues of the gravitational vector modes corresponding to the first one in equation (13), $(C, \ell_0) = (4 - 2n, 0)$, coincide exactly with those of the electromagnetic vector modes given by equation (23). Since the gravitational vector modes have also the eigenvalues with $C = 4$, the eigenvalue sets of the electromagnetic vector modes are completely covered by those for the gravitational vector modes.

We also find similar relationships among the eigenvalues of the scalar perturbations of the various fields as follows.

- **Scalar field and electromagnetic scalar modes.** Comparing equation (24) with equation (22), we find that the eigenvalues of the electromagnetic scalar modes coincide with the those of the scalar field perturbations if we set $C = -2(n - 1)$. The first ones in equations (25) and (26) for the electromagnetic scalar modes share such a property, while the eigenvalue for $(C, \ell) = (-2(n - 1), 0)$ does not appear in the electromagnetic scalar modes with $\kappa = 0$. It means that the eigenvalues of the scalar field and electromagnetic...
scalar mode perturbations for \( C = -2(n - 1) \) are almost identical to each other for any \( \kappa \) and \( n \), and the only exception is the eigenvalues for \( \ell = 0 \) of the electromagnetic scalar modes with \( \kappa = 0 
\).

- **Electromagnetic and gravitational scalar modes.** The eigenvalues corresponding to the first four integer sets in equation (15) for the gravitational scalar modes, which have \( C = 0 \) and \( C = -2(n - 1) \), coincide exactly with those of the electromagnetic scalar modes given by equations (24) and (25). When \( \kappa = 0 \) and/or \( n = 2 \), we find some of the eigenvalues of the electromagnetic scalar modes, especially those for low \( \ell \) modes, are missing in the eigenvalues with \( C = 0 \) and \( C = -2(n - 1) \) of the gravitational scalar modes, as we can see from equations (16)–(19).

To summarize, we find the inclusion relations among the eigenvalues for the vector perturbations expressed as

\[
[\lambda_1 \text{ of EM vector with } C = 4 - 2n] \subseteq [\lambda_2 \text{ of grav. vector with } C = 4 - 2n],
\]

(27)

\[
[\lambda_1 \text{ of EM vector}] \subseteq [\lambda_2 \text{ of grav. vector}].
\]

(28)

Similarly, if we do not care about a small number of the exceptions for small \( \kappa \) and \( n \) mentioned above, we have the relations among the eigenvalues for the scalar perturbations given by

\[
[\lambda_0 \text{ with } C = -2(n - 1)] = [\lambda_1 \text{ of EM scalar with } C = -2(n - 1)]
\]

\[
= [\lambda_2 \text{ of grav. scalar with } C = -2(n - 1)].
\]

(29)

\[
[\lambda_1 \text{ of EM scalar with } C = 0] = [\lambda_2 \text{ of grav. scalar with } C = 0],
\]

(30)

\[
[\lambda_0] \subseteq [\lambda_1 \text{ of EM scalar}] \subseteq [\lambda_2 \text{ of grav. scalar}].
\]

(31)

If we care about the missing eigenvalues for \( \kappa = 0, 1 \) and/or \( n = 2 \), the equality in equations (29) and (30) should be changed into ‘\( \supset \)’, and equation (31) needs to be modified accordingly.

We also note that the set of eigenvalues for fixed \( C \) have simple organizations which are described by \( \ell_0 \) being integers around zero. This property is most notable in the eigenvalue list (15) for the gravitational scalar modes. There is only one eigenvalue with \( C = -2(n - 1) \), which is associated with \( \ell_0 = 0 \). As for the eigenvalues which share \( C = 0 \), there are three of them, and they have \( \ell_0 = -1, 0 \) and \(+1\). Similarly, there are six eigenvalues which share 
\( C = 2n \), and they have \( \ell_0 = -2, -1, 0, 0, +1 \) and \(+2\). The eigenvalue sets for the scalar field and electromagnetic scalar modes inherit this structure of the eigenvalues, and we can see a similar relationship to hold also between gravitational and electromagnetic vector modes.

Similar relations for the eigenvalues can also be seen in the results of [12, 18]. The eigenvalues for various near-horizon geometries in five dimensions are studied in [18], and it was observed that the eigenvalues of the scalar field perturbations are included in those for the electromagnetic perturbations, and those for the gravitational perturbations cover both of them. Reference [12] studied the eigenvalues for the odd-dimensional MP black holes with equal angular momenta. By rewriting their results in the form similar to equation (14), we find the eigenvalue of the vector perturbations in that case is expressed by a unified formula

\[
\lambda = \frac{2\kappa + n(C' + \frac{1}{2})}{n(n - 1)} \left[\kappa + n(C' + \frac{1}{2}) + 2\right] + \frac{C}{2(n - 1)},
\]

(32)

where \( n \geq 2 \) corresponds to \( N + 1 \) in [12]. The eigenvalues of the electromagnetic vector modes are expressed by \((C, C') = (-(n - 4), 0)\), while the eigenvalues for the gravitational vector modes are expressed by

\[
(C, C') = (-(n - 4), 0), \quad (4, -\frac{1}{2}), \quad (4, +\frac{1}{2}).
\]

(33)
The inclusion relation between those two modes and also the simple composition of the eigenvalues for \( C = 4 \) resemble to those in our case to some extent, that is, the eigenvalues for the electromagnetic vector modes appear as a part of the eigenvalues for the gravitational vector modes, and the eigenvalues with \( C = 4 \) in the latter are described by some numbers \( C' \) around zero which defer by 1 from each other. Similarly, the eigenvalues for the scalar field, electromagnetic and gravitational scalar modes are expressed by a single formula

\[
\lambda = \frac{2 \left\{ \kappa + n \left( C' + \frac{1}{2} \right) - 1 \right\} \left\{ \kappa + n \left( C' + \frac{1}{2} \right) \right\}}{n(n-1)} + \frac{C}{2(n-1)}, \quad (34)
\]

where the constants \( (C, C') \) are given by

- **Scalar field:** \( (C, C') = (-(n-2), 0) \) \hspace{1cm} (35)
- **EM scalar:** \( (C, C') = (-(n-2), 0), \ (0, -\frac{1}{2}), \ (0, +\frac{1}{2}) \) \hspace{1cm} (36)
- **Grav. scalar:** \( (C, C') = (-(n-2), 0), \ (0, -\frac{1}{2}), \ (0, +\frac{1}{2}), \ (-(n-2), -1), \ (-(n-2), +1), \ (3n-2, 0) \) \hspace{1cm} (37)

It is fair to say that the inclusion relations for the eigenvalues with \( C = -(n-2) \) and 0, and also the composition of the eigenvalues with \( C = 0 \) are similar to those in our cases. The only exceptions are the eigenvalues which appear only for the gravitational scalar modes, the last three sets in equation (37). They do not share the same constant \( C \), while the counterparts in our cases (the last six eigenvalues in equation (15) with \( C = 2 \)) did. Instead of that, the first, fourth and fifth eigenvalues in equation (37) share the same \( C = -(n-2) \) in the case of the odd-dimensional MP black holes.

The origin for the composition of the eigenvalues described above is unclear so far, though their simplicity tempts us to, naively thinking, suspect the existence of a hidden background mechanism to generate it. It would be interesting to study the mathematical and physical origins of these properties studying, e.g., the perturbations of other near-horizon geometries.

### 6. Conformal weight in the Kerr/CFT correspondence

In any four- and five-dimensional vacuum near-horizon geometry, operators dual to the gravitational, electromagnetic and massless scalar field perturbations preserving rotational symmetry have integer conformal weights [12, 18–20]. Let us study if such a property persists in the case of the even-dimensional MP black holes we have studied.

As we have seen in section 3, the near-horizon geometries have unstable modes in the gravitational scalar perturbations. Conformal weights become complex for the unstable modes, as we can see from equation (10). From the formulae for the eigenvalues we have shown in sections 3 and 4, we find that the conformal weights for the most of stable modes take irrational numbers. For example, a stable gravitational scalar modes for \( \kappa = 0, \ n = 2 \) and \( (C, \ell_0) = (2n, 2) \) have a sequence of conformal weights given by \( h_R - 1/2 = \sqrt{201}/6, \ \sqrt{33}/2, \ \sqrt{417}/6, \ldots \) This result suggests the following scenarios about the integerness of the conformal weights: the conformal weights become integers only in four and five dimensions, or they may take integer values even in higher dimensions, while they do so only when the background black holes are stable.

Below, we discuss another property of the conformal weights, the universality, mentioned in section 1. In any five-dimensional vacuum near-horizon geometry, operators dual to axisymmetric perturbations have universal conformal weights. That is, they do not depend on parameters nor horizon topologies of extreme black holes. We consider six dimensions for simplicity and check if the universality holds in this case.
To compare with the six-dimensional MP black holes, we consider the MP black string solution whose metric is given by \( ds^2 = ds^2(\text{MP}_5) + dz^2 \) where \( ds^2(\text{MP}_5) \) is the five-dimensional MP metric. By the dimensional reduction along the \( z \)-direction, the gravitational perturbation on the six-dimensional MP black string solution is decomposed into five-dimensional massless scalar, electromagnetic and gravitational perturbations on the background of \( ds^2(\text{MP}_5) \). The conformal weights for those fields have been studied in [12, 18], and it was found that all the eigenvalues for these perturbations are integers. On the other hand, the gravitational perturbations on the near-horizon geometries of six-dimensional MP black holes have rational but non-integer eigenvalues as we have seen in section 3. This result again suggests that the universality is a property that exists only in four and five dimensions or only for stable black holes.

7. Summary and discussion

We studied the gravitational, electromagnetic and scalar field perturbations on the near-horizon geometries of the even-dimensional extreme MP black holes with equal angular momenta. As a result of our study, we find that some modes in the gravitational scalar modes become unstable, while all others do not. This result implies that the instability for the corresponding modes in the full background geometries, assuming the claim of [12] to be correct. Reference [13] raised a question if there is any (near-)extremal Myers–Perry black hole which is stable in \( d \geq 6 \), and our results suggest that there is no such in even dimensions if the angular momenta are set equal. If the black holes are unstable in the extreme limit, it is reasonable to consider that they are also unstable sufficiently near extremality. It is important to confirm this expectation analyzing the stability of the full geometry.

Recently, a new method for demonstrating dynamical instability was established [27, 28]. They found an inequality whose violation implies black hole instability. The method based on this inequality makes the stability analysis dramatically easier, because the inequality can be evaluated only from initial data that describe a small perturbation of the black hole, and we do not need to solve time evolution from those data. This method is a hopeful approach for demonstrating the instability of the full geometries of the even-dimensional MP black holes with equal angular momenta.

For small angular momenta, the MP black holes are stable since higher dimensional Schwarzschild black holes are stable [29, 30]. Thus, at critical values of angular momenta, the stability changes and there should be a static perturbation. The static perturbation indicates the existence of a new family of solutions which bifurcates from that of the even-dimensional MP black holes. We found instability in the gravitational scalar modes with several \( \kappa \) satisfying \( \kappa \geq 2 \). These perturbations break all of the symmetries in \( \mathbb{C}P^{n-1} \) [5] and, thus, the branched solutions will have only \( U(1) \times R \) isometry. To confirm this picture by constructing nonlinear solutions of the full geometry would be another direction of the future research.

We also found that the mass of the effective field on AdS_2, namely the eigenvalues of the angular part of the perturbations are given by simple rational expressions. This property can also be found in the four- and five-dimensional extreme black holes and odd-dimensional MP black holes [12, 18–20]. These results prompt us to conjecture that the eigenvalues for the angular part of the perturbations are given by rational numbers for any vacuum near-horizon geometries with vanishing cosmological constant, as long as we assume the perturbations to be axisymmetric. Further studies on more general near-horizon geometries will be useful to falsify such a conjecture. One possible recipient for such an analysis is the near-horizon geometries of the odd-dimensional MP black holes with all but one angular momenta are.
equal, which have a cohomogeneity-1 near-horizon geometries. The method used in this paper is directly applicable to these geometries, while it involves 15 unknown variables in general.

As a byproduct of our analysis, we found that the expressions of the eigenvalues are essentially governed by the transformation property of the fields: other than a few exceptions, the eigenvalues of the scalar field coincide with a part of those of the electromagnetic scalar modes, and the latter coincide with a part of those of the gravitational scalar mode. We also found that the eigenvalues have simple organization described by two integers \((C, \ell_0)\). We find similar properties also for the vector modes, whose eigenvalues are described by equation (12). It would be interesting to clarify the mathematical and physical backgrounds for these properties. It would also be nice if we can find microscopic interpretation for them based on the dual CFTs.

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**Appendix A. Background variables**

In the appendices, we show the details of the method to solve the eigenvalue equations (6). In this appendix, we introduce the basic objects which appear in the perturbation equations (7)–(9). Using the formulae in this appendix, we show the explicit forms of the perturbation equations and the analytic solutions for them when they exist in appendix B. In the cases where we cannot find analytic solutions, we solve equations numerically to obtain the eigenvalues. In appendix C, we summarize the numerical technique we employed.

Below, we work in the frame adapted to the induced metric on the horizon (3) given by

\[
(e_\theta)_\mu = \Theta (d\theta)_\mu, \quad (e_\phi)_\mu = \Phi (d\phi + A)_\mu, \quad (e_\alpha)_\mu = \Psi (\hat{e}_\alpha)_\mu, \quad (A.1)
\]

where \(\hat{e}_\alpha\) is the real orthonormal frame for \(\mathbb{CP}^{n-1}\). We show the tetrad components of each tensor quantity below unless otherwise noted. The spin connections \((\omega_{ab})_\mu \equiv (e_a)^{\nu} \nabla_\mu (e_b)_\nu\) for this frame are given by

\[
\begin{align*}
\omega_{\theta\phi} &= -\frac{(\log \Phi)'^2}{\Theta^2} e_\theta \wedge e_\phi, \\
\omega_{\theta\alpha} &= -\frac{\log \Theta'}{\Theta} e_\theta, \\
\omega_{\phi\beta} &= \frac{\Phi}{\Psi^2} J_{\alpha\beta} e_\alpha \wedge e_\beta, \\
\omega_{\alpha\beta} &= -\frac{\Phi}{\Psi^2} J_{\alpha\beta} e_\phi + \frac{1}{\Psi} \hat{\omega}_{\alpha\beta}.
\end{align*}
\]

(A.2)

Assuming the axisymmetry in our case, we have

\[
k = \Omega \Phi e_\phi, \quad k_I m^I = 0, \quad k_s k^s = \Omega^2 \Phi^2,
\]

\[
dk = \Omega \left( \frac{2\Phi}{\Theta} (\log \Phi)' e_\phi \wedge e_\phi + \frac{\Phi^2}{\Psi^2} J_{\alpha\beta} e_\alpha \wedge e_\beta \right), \quad dL^2 = \frac{2\Theta}{2\pi - 1} (\log \Theta)' e_\theta.
\]

(A.3)

Using \(R_{ab} = do_{ab} + \omega_{ac} \wedge \omega_{cb}\) and \(de_a + \omega_{ab} \wedge e_b = 0\), we find the curvature 2-forms are given by

\[
R_{\phi\phi} = -\frac{1}{2\Theta^2} \left( 2(\log \Phi)' + (\log \Phi)' \left( \frac{\log \Phi}{\Theta} \right)' \right) e_\theta \wedge e_\phi - \frac{\Phi}{2\Psi^2} \left( \frac{\log \Phi}{\Psi} \right)' J_{\alpha\beta} e_\alpha \wedge e_\beta.
\]

(A.4)
\begin{equation}
R_{\phi\alpha} = -\frac{1}{2\Theta^2} \left(2(\log \Psi)' - (\log \Psi)\left(\log \frac{\Theta}{\alpha}\right)\right) e_\phi \wedge e_\alpha - \frac{\Phi}{2\Psi^2\Theta} \left(\log \frac{\Phi}{\Psi}\right)' J_{\alpha\beta} e_\phi \wedge e_\beta,
\end{equation}

(A.5)

\begin{equation}
R_{\alpha\beta} = \frac{\Phi}{2\Psi^2\Theta} \left(\log \frac{\Phi}{\Psi}\right)' J_{\alpha\beta} e_\phi \wedge e_\phi + \left(\frac{\Phi^2}{\Psi^2} - \frac{(\log \Phi)'(\log \Psi)'}{\Theta^2}\right) e_\phi \wedge e_\alpha.
\end{equation}

(A.6)

\begin{equation}
R_{\alpha\beta} = -\frac{\Phi}{\Psi^2} \left(\log \frac{\Phi}{\Psi}\right)' J_{\alpha\beta} e_\phi \wedge e_\phi - \frac{(\log \Psi)^2}{\Theta^2} e_\phi \wedge e_\alpha + \frac{1}{2\Psi^2} \hat{R}_{\alpha\beta}

- \frac{\Phi^2}{\Psi^2} \left(J_{\alpha\beta} J_{\gamma\delta} + J_{\alpha\gamma} J_{\beta\delta} - J_{\alpha\delta} J_{\beta\gamma} + 2J_{\alpha\beta} J_{\gamma\delta}\right) e_\gamma \wedge e_\delta,
\end{equation}

(A.7)

where the Riemann and Ricci tensors of \(\mathbb{CP}^{n-1}\) are given by

\begin{equation}
\hat{R}_{\alpha\beta\gamma\delta} = \delta_{\alpha\gamma} \delta_{\beta\delta} - \delta_{\alpha\delta} \delta_{\beta\gamma} + J_{\alpha\gamma} J_{\beta\delta} - J_{\alpha\delta} J_{\beta\gamma} + 2J_{\alpha\beta} J_{\gamma\delta}, \quad \hat{R}_{\alpha\beta} = 2n\delta_{\alpha\beta}.
\end{equation}

(A.8)

**Appendix B. Perturbation equations and eigenvalues**

We sketch the procedure to construct the eigenvalue equations for \(\lambda_s\) in this appendix. The procedure is parallel to that for the odd-dimensional MP black holes with equal angular momenta [12]. We show the analytic solutions and eigenvalues for them when they exist, and numerical ones otherwise. We defer a part of derivations of the eigenvalue equations and most of the technical details to solve them to appendix C. The calculations to obtain the tetrad components of the eigenvalue equations mentioned below are performed partially with the aid of the computer algebra system Cadabra [31, 32].

**B.1. Scalar field perturbations**

Since our interest is in the axisymmetric perturbations, we take the separation ansatz as

\begin{equation}
\psi = \chi_0 (t, r) f(\theta) \tilde{Y}(s^a),
\end{equation}

(B.1)

where \(\tilde{Y}\) is the scalar harmonics on \(\mathbb{CP}^{n-1}\) defined with respect to the covariant derivative on \(\mathbb{CP}^{n-1}\), \(D_a\), as

\begin{equation}
(D^2 + \lambda_s^2) \tilde{Y} = 0, \quad \lambda_s^2 = 4\kappa (\kappa + n - 1), \quad (\kappa = 0, 1, \ldots).
\end{equation}

(B.2)

Decomposing equation (7) into \(\theta\) and \(\mathbb{CP}^{n-1}\) parts and using equation (B.2), we find the eigenvalue equation to be

\begin{align}
\lambda_0 f' \tilde{Y} &= \mathcal{O}^{(0)} f' \tilde{Y} = \frac{1}{2n - 1} \left\{-\partial_{\theta} f' \tilde{Y} - (\log(\Theta \Phi \alpha^{2(\kappa - 1)})')\partial_{\theta} f' \tilde{Y} - \frac{\Theta^2}{\Psi^2} D^2 f' \tilde{Y} + \Theta^2 M^2 f' \tilde{Y}\right\}

= \frac{1}{2n - 1} \left\{-f'' - (2n - 1) \cot \theta f' + f_\alpha(\theta) \left(\frac{4\kappa (\kappa + n - 1)}{2n \sin^2 \theta} + \frac{(aM)^2}{2n - 1}\right) f\right\} \tilde{Y},
\end{align}

(B.3)

where \(f' = df/d\theta\). Note that the parameter \(a\) in the metric (1) only appears as a multiplicative factor of the mass term in equation (B.3). This property originates from the fact that \(\lambda_0\) is dimensionless and thus only the dimensionless combination of the quantities, \(aM\), may appear on the right-hand side.
Table B1. Numerical and exact values of $\lambda_0$ for $M = 0$ and $\kappa = 0$.

| Mode | $n - 2$ | 0       | 1       | 2       | 3       | 4       |
|------|---------|---------|---------|---------|---------|---------|
|      | [Numerical] |         |         |         |         |         |
| 1    | 0.000 01 | 0.000 01 | 0.000 01 | 0.000 01 | 0.000 01 | 0.000 01 |
| 2    | 1.333 30 | 1.200 01 | 1.142 87 | 1.111 12 | 1.090 91 |         |
| 3    | 3.333 34 | 2.800 00 | 2.571 43 | 2.444 45 | 2.363 64 |         |
| 4    | 5.999 97 | 4.800 01 | 4.285 73 | 4.000 01 | 3.818 18 |         |
| 5    | 9.333 33 | 7.200 01 | 6.285 71 | 5.777 78 | 5.454 55 |         |
| 6    | 13.333 30 | 10.000 01 | 8.571 44 | 7.777 79 | 7.272 73 |         |
|      | [Exact (equation (22))] |         |         |         |         |         |
| 1    | 0.000 00 | 0.000 00 | 0.000 00 | 0.000 00 | 0.000 00 |         |
| 2    | 1.333 33 | 1.200 00 | 1.142 86 | 1.111 11 | 1.090 91 |         |
| 3    | 3.333 33 | 2.800 00 | 2.571 43 | 2.444 44 | 2.363 64 |         |
| 4    | 6.000 00 | 4.800 00 | 4.285 71 | 4.000 00 | 3.818 18 |         |
| 5    | 9.333 33 | 7.200 00 | 6.285 71 | 5.777 78 | 5.454 55 |         |
| 6    | 13.333 33 | 10.000 00 | 8.571 43 | 7.777 79 | 7.272 73 |         |

Table B2. Numerical value of $\lambda_0$ for $aM = 1$ and $\kappa = 0$.

| Mode | $n - 2$ | 0       | 1       | 2       | 3       | 4       |
|------|---------|---------|---------|---------|---------|---------|
| 1    | 0.176 31 | 0.068 20 | 0.036 15 | 0.022 39 | 0.015 23 |         |
| 2    | 1.586 02 | 1.306 22 | 1.202 04 | 1.149 00 | 1.117 30 |         |
| 3    | 3.600 44 | 2.920 48 | 2.641 83 | 2.491 09 | 2.396 96 |         |
| 4    | 6.271 70 | 4.927 23 | 4.362 40 | 4.052 04 | 3.856 09 |         |
| 5    | 9.607 15 | 7.330 99 | 6.366 26 | 5.833 39 | 5.495 66 |         |
| 6    | 13.608 30 | 10.133 27 | 8.654 52 | 7.835 88 | 7.316 14 |         |

When the scalar field is massless ($M = 0$), equation (B.3) has an exact solution given by

$$f(\theta) = C_1 P_\mu(\cos \theta) + C_2 Q_\mu(\cos \theta),$$  \hspace{1cm} (B.4)

where $P_\mu(x)$ and $Q_\mu(x)$ are the associated Legendre functions and

$$\mu = 2\kappa + n - 1, \quad \nu = \frac{1}{2} \left[ -1 + \sqrt{1 + \frac{1}{n} (2n + 1) \left( 2(2\kappa + n - 1)^2 - 2 + n(4\lambda_0 + 3) \right) } \right].$$  \hspace{1cm} (B.5)

The regularity of the solution at $\theta = 0$ requires $C_2$ to be zero and $\nu$ to be an integer equal to or larger than $\mu$. This requirement gives the eigenvalue $\lambda_0$ as shown in equation (22).

When $M \neq 0$, we need to calculate $\lambda_0$ numerically. From equation (B.3), we find that a regular solution near $\theta = 0$ behaves as $f \sim \theta^{2\kappa}$. Behavior near $\theta = \pi$ is the same after $\theta \mapsto \pi - \theta$ due to the symmetry about $\theta = \pi/2$ of the near-horizon geometry (1). Then, defining a new variable $\tilde{f}$ by $f = \tilde{f} \sin^2 \theta$ and imposing the Neumann boundary condition at $\theta = 0$ and $\pi$ to $\tilde{f}$, we may obtain regular numerical solutions by the relaxation method. We defer the further details of the numerical method to appendix C.

We show the numerical value of $\lambda_0$ for $M = 0$ and $\kappa = 0$ along with the exact value given by equation (22) in table B1, which shows that the relative error of our numerical results from the exact results is maintained to be $\lesssim \mathcal{O}(10^{-5})$. We also show the result for $aM = 1$ and $\kappa = 0$ in table B2. As we can see from these tables, $\lambda_0$ tends to increase as $M$ increases. As far
as we have checked numerically, this property holds for any $M$, $\kappa$ and $n$. If it holds in general, then there is no instability for any $M^2 \geq 0$.

### B.2. Electromagnetic perturbations

Next, we discuss the perturbations of the Maxwell field on the near-horizon geometries. For the analysis, it is useful to decompose the perturbation variables $Y_\mu$ into the scalar and vector modes according to the transformation properties with respect to $\mathbb{C}P^{n-1}$. We show the derivations of the eigenvalue equations and results mode by mode below.

#### B.2.1. Electromagnetic vector modes

The electromagnetic vector modes are characterized by

\[ Y_i = 0, \quad D^\pm Y_\alpha = 0, \quad (i = \theta, \phi), \quad (B.6) \]

where $D^\pm$ is the covariant derivative on $\mathbb{C}P^{n-1}$ projected to the $\mp i$ eigenspaces of the complex structure for $\mathbb{C}P^{n-1}$, $J = \frac{1}{2} J^\alpha_\beta \hat{e}_\alpha \hat{e}_\beta$, defined as

\[ D^\pm_\alpha \equiv P^\pm_\alpha_\beta D_\beta, \quad P^\pm_\alpha_\beta = \frac{1}{2} (\hat{g}^\alpha_\beta \pm i J^\alpha_\beta). \quad (B.7) \]

We may parametrize this component of the perturbations as $Y_\alpha = f(\theta) Y^\alpha$, where $Y^\alpha$ is the divergence-free vector harmonic satisfying

\[ (D^2 + \lambda^V) Y_\alpha = 0, \quad \lambda^V = 4\kappa (\kappa + 2) + 2n(2\kappa + 3). \quad (B.8) \]

For these modes, we find that $i$ components of equation (8) become trivial, and $\mathbb{C}P^{n-1}$ components reduce to an equation of $f(\theta)$ given by

\[
-n(2n-1) \lambda_1 \tan \theta \frac{f''(\theta)}{f'(\theta)} + \left\{ \frac{(2n-1)}{(2n-1) \cos^2 \theta + 1} \right\} f'(\theta) \\
+ \left\{ \frac{(2n-1)^2 \cos^2 \theta + 4n-1}{(2n-1) \cos^2 \theta + 1} \right\} \frac{2(2n-1)((n+2)\kappa + n + \kappa^2)}{n} f(\theta). \]

This equation has an analytic solution given by

\[ f(\theta) = C_1 P_\nu(\cos \theta) + C_2 (\sin \theta)^{-\mu} Q_\nu(\cos \theta), \quad (B.10) \]

where

\[ \mu = 2\kappa + n + 2, \]

\[ \nu = \frac{1}{2} \left[ -1 + \sqrt{1 - \frac{n}{4} \left\{ 4n^3 + 4n^2(4\kappa + 2\lambda_1 + 5) + n(8\kappa(2\kappa + 3) - 4\lambda_1 - 7) - 8\kappa(\kappa + 2) \right\}} \right]. \quad (B.11) \]

For the solution to be regular at $\theta = 0$, $C_2$ should be set to zero and $\nu$ should be an integer equal to or larger than $\mu$. This requirement fixes $\lambda_1$ as given in equation (23).
B.2.2. Electromagnetic scalar modes. Next, we discuss the electromagnetic scalar modes for which \( i \) components of the perturbations are turned on. We may expand the scalar mode perturbations as

\[
Y_i = f_i(\theta) Y^+, \quad Y_a = g^+ (\theta) Y^a_+ + g^- (\theta) Y^a_-, \tag{B.12}
\]

where \( Y^\pm_a \) are the scalar-derived vector eigenfunctions defined by

\[
Y^\pm_a = -\frac{D^{\pm} Y}{\sqrt{\lambda}} \quad Y^\pm\mp_{\alpha\beta} = \frac{D^{\pm}}{D^{\pm}} (\alpha Y^{\pm}_{\beta}) = D^{\pm}(\alpha Y^{\pm}_{\beta}) - \frac{\sqrt{\lambda}}{2(n - 1)} \hat{g}_{\alpha\beta} Y. \tag{B.13}
\]

We further assume that \( Y_{\alpha} \) are real-valued, and introduce the real quantities by

\[
g^{\pm} = g^R \pm ig^I, \quad Y^{\pm}_{\alpha} = \frac{1}{2} Y^R_{\alpha} \pm i Y^I_{\alpha}. \tag{B.14}
\]

As a result, we find coupled ODEs for \((f_{\theta}, f_{\phi}, g^R, g^I)\) as the eigenvalue equations when \( \kappa > 0 \). For \( \kappa = 0 \), \( Y \) becomes a constant and the variables reduce to \((f_{\theta}, f_{\phi})\). We solve those equations using the numerical technique demonstrated in appendix C.

In table B3, we show an example of the numerically obtained eigenvalues \( \lambda_1 \). As far as we have examined, our numerical result is consistent with a hypothesis that the eigenvalues are expressed in general by equations (24)–(26).

B.3. Gravitational perturbations

Finally, we study the gravitational perturbations. The procedure is parallel to that for the electromagnetic perturbations, while the gravitational perturbations involve ten unknown variables in general. We show the procedures and results for each of tensor, vector and scalar modes below.

B.3.1. Gravitational tensor modes. We start from the simplest components, which are the gravitational tensor modes defined by

\[
Y_{ij} = Y_{a} = 0, \quad \hat{g}^{\beta\gamma} Y_{\alpha\beta} = 0, \quad D^{\pm\alpha} Y_{\alpha(\beta)} = 0. \tag{B.15}
\]

Under these assumptions, only the \( \mathbb{CP}^{m-1} \) components of equation (9) remain nontrivial. We introduce a decomposition given by

\[
Y_{\alpha\beta} = f(\theta) Y_{\alpha\beta}. \tag{B.16}
\]
and we also decompose $\mathcal{Y}_{a\beta}$ into Hermitian and anti-Hermitian parts by $(\mathcal{J}\mathcal{Y})_{a\beta} = \sigma \mathcal{Y}_{a\beta}$, where $\sigma = -1 (+1)$ for the Hermitian (anti-Hermitian) modes.

To deal with the $D^2 Y_{a\beta}$ terms in the equations, we need to take $Y_{a\beta}$ to be an eigenstate of the generalized Lichnerowicz operator on $\mathbb{CP}^{n-1}$, that is,

$$\lambda^X_{\epsilon} Y_{a\beta} = (\Delta^X_{\epsilon} Y)_{a\beta} = -D^2 Y_{a\beta} - 2\hat{R}_{a\gamma\beta\delta} Y^{\gamma\delta} + 4n Y_{a\beta}$$

$$= -D^2 Y_{a\beta} + 2(n+1)\mathcal{Y}_{a\beta} - 3 (\mathcal{J}\mathcal{Y})_{a\beta}.$$  \hspace{1cm} (B.17)

For $n = 2$, the tensor harmonics do not exist on $\mathbb{CP}^1 = S^2$. For $n \geq 3$, the harmonics exist and the eigenvalues of $\Delta^X_{\epsilon}$ are given by $[10]$

$$\lambda_{\epsilon} = 4\kappa (\kappa + n - 1) + 4 (n - 1 + \sigma).$$  \hspace{1cm} (B.18)

This allows us to rewrite $D^2 Y_{a\beta}$ in terms of $Y_{a\beta}$. Using the above equations, the eigenvalue equation (9) is rewritten as

$$-(2n-1)\lambda_2 f(\theta) = f''(\theta) + (2n-1) \left[ \cot \theta - \frac{2 \sin 2\theta}{(2n-1) \cos^2 \theta + 1} \right] f'(\theta)$$

$$+ 2 \left[ 2n \left( \frac{2}{(2n-1) \cos^2 \theta + 1} + \kappa - 1 \right) - \frac{2\kappa (n + \kappa - 1)}{\sin^2 \theta} \right] + \left( -\kappa^2 + \kappa + 1 \right) + \kappa (2\kappa - 3) \right] f(\theta).$$  \hspace{1cm} (B.19)

This equation has an exact solution given by

$$f(\theta) = C_1 P_{\mu}^\nu (\cos \theta) + C_2 (\sin \theta)^{1-\nu} Q_{\mu}^\nu (\cos \theta),$$  \hspace{1cm} (B.20)

where

$\mu = 2\kappa + n - 1$,

$$\nu = \frac{1}{2} \left[ -1 + \sqrt{\frac{2n-1}{n} (2(2n + n - 1)^2 - 10 + n(4\lambda_2 + 3) + 8\sigma)} \right].$$  \hspace{1cm} (B.21)

Setting $C_2 = 0$ and $\nu \geq \mu$ to be an integer for the sake of the regularity at $\theta = 0$, we find $\lambda_2$ to be given by equation (11).

### B.3.2. Gravitational vector modes.

The gravitational vector modes are composed of divergence free vectors $Y_{\alpha}^a$ and traceless $Y_{a\beta}$ satisfying

$$Y_{ij} = 0, \quad D^a Y_{\alpha a} = 0, \quad Y_{a}^a = 0.$$  \hspace{1cm} (B.22)

We expand the perturbations as

$$Y_{\alpha} = g_\alpha^j (\theta) \mathcal{Y}_j, \quad Y_{a\beta} = h^+ (\theta) \mathcal{Y}_{a\beta}^+ + h^- (\theta) \mathcal{Y}_{a\beta}^- = Y_{a\beta}^+ + Y_{a\beta}^-,$$  \hspace{1cm} (B.23)

where $\mathcal{Y}_a$ is a divergence-free vector harmonics defined by equation (B.8). We further decompose $\mathcal{Y}_a$ into the eigenvectors of $\mathcal{J}_{a\beta}$ by $\mathcal{J}_{a\beta} \mathcal{Y}_a = -i \varepsilon \mathcal{Y}_a$ with $\varepsilon = \pm 1$.

Since $\mathcal{Y}_a$ are complex, the variables $(g_\alpha, (h^+ + h^-)/2, (h^+ - h^-)/2i)$ are not real-valued. To rewrite the variables in terms of real quantities, we further decompose the variables as

$$Y_{\alpha} = g^+_{\alpha} (\theta) \mathcal{Y}_a^+ + g^-_{\alpha} (\theta) \mathcal{Y}_a^-, \quad Y_{a\beta} = h^{++} (\theta) \mathcal{Y}_{a\beta}^{++} + h^{+-} (\theta) \mathcal{Y}_{a\beta}^{+-} + h^{-+} (\theta) \mathcal{Y}_{a\beta}^{-+} + h^{--} (\theta) \mathcal{Y}_{a\beta}^{--},$$  \hspace{1cm} (B.24)
where $\tilde{Y}_a^\pm$ are the vector harmonics corresponding to $\epsilon = \pm 1$ (that is, $\mathcal{J}_a^{\pm} \tilde{Y}_a^\pm = \mp i \tilde{Y}_a^\pm$) and

$$\tilde{Y}_{a\beta}^{\pm1\pm2} = -\frac{1}{\sqrt{\lambda_{\kappa}}} D_{(a\beta)}^{\pm1\pm2}(\tilde{Y}_a^\pm) \quad (B.25)$$

Then, replacing the quantities by

$$g_{ij}^\pm = g_{ij}^R \pm i g_{ij}^I, \quad h_{ij}^{\pm1\pm2} = (h_{ij}^{RR} \pm 2 i h_{ij}^{RI}) \pm 1 i (h_{ij}^{IR} \pm 2 i h_{ij}^{II}),$$

$$\tilde{Y}_{i}^\pm = \tilde{Y}_{i}^R \pm i \tilde{Y}_{i}^I, \quad \tilde{Y}_{a\beta}^{\pm1\pm2} = (\tilde{Y}_{a\beta}^{RR} \pm 2 i \tilde{Y}_{a\beta}^{RI}) \pm 1 i (\tilde{Y}_{a\beta}^{IR} \pm 2 i \tilde{Y}_{a\beta}^{II}), \quad (B.26)$$

we obtain real-valued eigenvalue equations in terms of $(g_{ij}^{R}, g_{ij}^{I}, h_{ij}^{RR}, h_{ij}^{RI}, h_{ij}^{IR}, h_{ij}^{II})$.

Due to the self-adjointness of the operators $O^{(\nu)}$ with respect to suitable inner products, $\lambda_{2}$ is guaranteed to be real numbers [12]. This fact implies that the $\epsilon = \pm 1$ modes share the same eigenvalue $\lambda_{2}$.

We show the numerically obtained eigenvalues in table B4. Note that this result contains the eigenvalues for both $\epsilon = 1$ and $\epsilon = -1$ modes, which are equal to each other. This numerical result suggests that there are four species of the eigenvalues, other than the multiplicity coming from $\epsilon = \pm 1$ modes, and the expression of the eigenvalues for general $n$ and $\kappa$ is given by equations (12) and (13). These eigenvalues are all positive, and hence no instability is implied.

### B.3.3. Gravitational scalar modes

Finally, we comment on the gravitational scalar modes, for which the analysis procedure is similar to the previous examples while it is more involved.

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**Table B4.** $\lambda_{2}$ of gravitational vector modes for $n = 5$ and $\kappa = 0, \ldots, 3$.

| Mode | $\kappa$ | $\epsilon = 0$ | $\epsilon = 1$ | $\epsilon = -1$ |
|------|----------|----------------|----------------|----------------|
| 1    | 0.4444   | 0.5778         | 0.8000         | 1.1111         |
| 2    | 0.4444   | 0.5778         | 0.8000         | 1.1111         |
| 3    | 0.8889   | 1.4667         | 2.1333         | 2.8889         |
| 4    | 0.8889   | 1.4667         | 2.1333         | 2.8889         |
| 5    | 2.0000   | 2.5778         | 3.2444         | 4.0000         |
| 6    | 2.0000   | 2.5778         | 3.2444         | 4.0000         |
| 7    | 2.0000   | 2.5778         | 3.2444         | 4.0000         |
| 8    | 2.0000   | 2.5778         | 3.2444         | 4.0000         |
| 9    | 2.6667   | 3.6889         | 4.8000         | 6.0000         |
| 10   | 2.6667   | 3.6889         | 4.8000         | 6.0000         |
| 11   | 3.7778   | 4.8000         | 5.9111         | 7.1111         |
| 12   | 3.7778   | 4.8000         | 5.9111         | 7.1111         |
| 13   | 3.7778   | 4.8000         | 5.9111         | 7.1111         |
| 14   | 3.7778   | 4.8000         | 5.9111         | 7.1111         |
| 15   | 3.7778   | 4.8000         | 5.9111         | 7.1111         |
| 16   | 3.7778   | 4.8000         | 5.9111         | 7.1111         |
| 17   | 4.6667   | 6.1333         | 7.6889         | 9.3333         |
| 18   | 4.6667   | 6.1333         | 7.6889         | 9.3333         |
| 19   | 5.7778   | 7.2444         | 8.8000         | 10.4444        |
| 20   | 5.7778   | 7.2444         | 8.8000         | 10.4444        |
| 21   | 5.7778   | 7.2444         | 8.8000         | 10.4444        |
| 22   | 5.7778   | 7.2444         | 8.8000         | 10.4444        |
| 23   | 5.7778   | 7.2444         | 8.8000         | 10.4444        |
| 24   | 5.7778   | 7.2444         | 8.8000         | 10.4444        |
We expand the metric perturbations using the scalar harmonics $Y$ as

$$ Y_{ij} = f_{ij}(\theta)Y^\alpha, $$

$$ Y_{\alpha} = g^{+}_{i}(\theta)Y_{\alpha}^{+} + g^{-}_{i}(\theta)Y_{\alpha}^{-}, $$

$$ Y_{\alpha\beta} = -\frac{1}{\sqrt{\lambda_S}}(h^{++}(\theta)Y_{\alpha\beta}^{++} + h^{--}(\theta)Y_{\alpha\beta}^{--} + h^{+-}(\theta)Y_{\alpha\beta}^{+-}) $$

$$ -\frac{1}{2(n-1)}(f_{\alpha\phi}(\theta) + f_{\phi\phi}(\theta))\delta_{\alpha\beta}Y. $$

(B.27)
where $Y^\pm$ and $Y^{\pm\pm}$ are scalar-derived vector/tensor eigenfunctions defined by equation (B.13). To construct the real-valued equations, we redefine the variables and mode functions as

$$g^\iota_i \equiv g^\iota_i \pm ig^\jmath_j,$$

$$\kappa^\pm \equiv \kappa^R \pm i\kappa^I,$$

$$\kappa^{\pm\pm} \equiv \kappa^{R\iota}_a \pm i\kappa^{I\iota}_a.$$  

(B.28)

As a result, we obtain coupled ODEs of $(f_{00}, f_{\phi\phi}, f_{\phi0}, g^R_{\iota\iota}, g^R_{\iota\iota}, h^R, h^I, h^{++})$ in a general case ($\kappa > 1$ and $n > 2$) as the eigenvalue equations to solve. Depending on the values of $\kappa$ and $n$, some of the unknown variables drop out as follows (see also [12]).

- $\kappa = 0$: $\kappa$ becomes a constant and there is neither scalar-derived vector nor tensor in this case. We have only $(f_{ij})$ as the unknown variables.
- $\kappa = 1$ and $n = 2$: all of $Y^{\pm\pm}$ and $Y^{+\pm}$ vanish in this case, and we have $(f_{ij}, g^R_{\iota\iota})$ as the unknown variables.
- $\kappa = 1$ and $n > 2$: $Y^{\pm\pm}$ vanish in this case, and we have $(f_{ij}, g^R_{\iota\iota}, h^{++})$ as the unknown variables.
- $\kappa > 1$ and $n = 2$: $Y^{+\pm}$ vanishes in this case. As a result, we have $(f_{ij}, g^R_{\iota\iota}, h^{++})$ as the unknown variables.

The numerically obtained eigenvalues for $\kappa > 1$ and $n > 2$ suggest that they are described by equations (14) and (15). We show some examples of the numerically obtained eigenvalues $\lambda_2$ in table B.5. For the special cases mentioned above, we find that the eigenvalues are given by equation (14) in any cases, and only some of the integer sets $(C, \ell_0)$ are modified. We list the integer sets for each case in equations (16)–(19). In short, some of the eigenvalues listed in equation (15) vanishes, and also $\ell_0$ is shifted in some of the remaining ones.

Appendix C. Technical details

We summarize the numerical technique to solve the eigenvalue equations in this appendix. This technique is essentially the same as that used in [18], which studied the conformal weights for the general near-horizon geometries in five dimensions.

C.1. Perturbation equations and boundary conditions for electromagnetic scalar modes

We explain the numerical technique to find the eigenvalues taking the electromagnetic scalar modes for example. For the electromagnetic scalar modes defined by equation (B.12), we obtain a system of coupled ODEs for the unknown variables $(f_{00}, f_{\phi\phi}, g^R, g^I)$ if $\kappa > 0$, where these variables are defined by equations (B.12) and (B.14). If $\kappa = 0$, then the scalar harmonics $\kappa$ becomes a constant, and the unknown variables reduce to $(f_{00}, f_{0\phi})$.

Below, we take the $\kappa > 0$ case as an example and explain the boundary conditions to be imposed on the unknown variables. The resultant eigenvalue equations may be expressed as

$$(\partial^2_{\phi\phi} + A(\theta)\partial_{\phi} + B(\theta)) \vec{v} = -(2n - 1)\lambda_1 \vec{v},$$

(C.1)

where $\vec{v} \equiv (f_{0\phi}, f_{\phi\phi}, g^R, g^I)^T$, and $A(\theta)$ and $B(\theta)$ are $4 \times 4$ coefficient matrices which behave near $\theta = 0$ as

$$A = \frac{2n - 1}{\theta} E + O(\theta^0), \quad B = \frac{1}{\theta^2} B_{-2} + O(\theta^{-1}),$$

(C.2)

where $E$ is the unit matrix. To find out the fall-off behavior of the solution at $\theta = 0$, we need to diagonalize the coefficient matrix $B_{-2}$ by introducing new variables $\tilde{\vec{v}}$ by $\vec{v} = \Xi \tilde{\vec{v}}$, where $\Xi$ is the matrix composed of the eigenvectors of $B_{-2}$. The equations in terms of these new variables are

$$(\partial^2_{\phi\phi} + \tilde{A}(\theta)\partial_{\phi} + \tilde{B}(\theta)) \tilde{\vec{v}} = -(2n - 1)\lambda_1 \tilde{\vec{v}},$$

(C.3)
where $\hat{A} = \frac{2n-1}{\theta} E + O(\theta^0)$ and $\hat{B}$ behaves for $\theta \to 0$ as
\[
\hat{B} = \frac{1}{\theta^2} \hat{B}_{-2} + O(\theta^{-1}), \quad \hat{B}_{-2} = \text{Diag}[\epsilon_-, \epsilon_-, \epsilon_+, \epsilon_+],
\]
\[
\epsilon_{\pm} \equiv -(2\kappa \pm 1)(2(n-1) + 2\kappa \pm 1).
\] (C.4)

Solving equation (C.3) near $\theta = 0$, we find the asymptotic behavior of the solution near $\theta = 0$ to be given by
\[
\tilde{v}_i \propto \theta^{p_i}, \quad p_i = \begin{cases} 2\kappa - 1 & (i = 1, 2) \\ 2\kappa + 1 & (i = 3, 4) \end{cases},
\] (C.5)
where we chose the decaying solutions from the two linearly independent solutions so that the solutions are regular at $\theta = 0$. Based on this observation, we introduce new variables $\tilde{v}$ by $\tilde{v} = \Pi \hat{v}$, where $\Pi$ is the diagonal matrix whose components are given by $\Pi_i = \sin^{p_i} \theta$. For these new variables $\tilde{v}$, we find that the eigenvalue equations become
\[
\left( \partial_\theta^2 + \hat{A}(\theta) \partial_\theta + \hat{B}(\theta) \right) \tilde{v} = -(2n - 1)\lambda_i \tilde{v},
\] (C.6)
where $\hat{A} = O(\theta^{-1})$ and $\hat{B} = O(\theta^0)$. This equation has a regular singularity at $\theta = 0$, and we need to impose the Neumann boundary condition there, namely $\tilde{v} = \tilde{v}_0 + O(\theta^2)$, to maintain the regularity of the solutions.

Since the perturbation equations (8) become symmetric with respect to $\theta = \pi/2$ for the near-horizon metric (1), we may solve equation (C.6) by imposing the Neumann boundary condition at both $\theta = 0$ and $\theta = \pi$. Alternatively, we separate the variables into the even/odd modes with respect to $\theta = \pi/2$ as follows. In equation (C.1), we find that the parities with respect to $\theta = \pi/2$ of the coefficient matrix components are given as
\[
A = \begin{cases} (\text{odd}) & i = 1, 2 \\ (\text{even}) & i = 3, 4 \end{cases}, \quad B = \begin{pmatrix} (\text{even}) & (\text{odd}) \\ (\text{odd}) & (\text{even}) \end{pmatrix}.
\] (C.7)
This property originates from the fact that the original equations are symmetric with respect to $\theta = \pi/2$, and also that the variable $v_1 = f_0$ is an odd function with respect to $\theta = \pi/2$, since it is proportional to the $e_0$ component of the perturbation variables, while $v_{2,3,4}$ are not. This property implies that an eigenvector takes a form either of
\[
\tilde{v} = \begin{cases} v_i^{(o)} & i = 1, 2 \\ v_i^{(e)} & i = 3, 4 \end{cases} \quad \tilde{v} = \begin{pmatrix} v_i^{(o)} \\ v_i^{(e)} \end{pmatrix},
\] (C.8)
where $v_i^{(o)}$ and $v_i^{(e)}$ are odd and even functions with respect to $\theta = \pi/2$, respectively. Since both $A$ and $B$ are regular at $\theta = \pi/2$, we may impose the Dirichlet boundary condition on $v_1$ to set it zero and the Neumann boundary condition on $v_{2,3,4}$ to obtain solutions corresponding to the left one in equation (C.8). Solutions corresponding to the right one in equation (C.8) can be obtained by switching the boundary condition types. By decomposing into the even/odd modes, we may reduce the calculation region $0 < \theta < \pi$ to the half, and also the number of modes needs to be reduced at each time step in the relaxation method, which will be introduced in the following section, to the half. It helps increasing the precision of the numerical results in a shorter calculation time.

In practice, the even/odd decomposition is more simply implemented by introducing new variables $\tilde{v}$ by $\tilde{v} = \Upsilon \tilde{v}$, where $\Upsilon$ is a diagonal matrix whose components are equal to

\footnotetext[7]{The function $w = \cos(\frac{\pi}{2}(1 - \cos \theta))$ is chosen so that $w$ and $w/(\theta - \pi/2)$ are sufficiently close to constants near $\theta = 0$ and $\theta = \pi/2$, respectively. For example, simpler choices such as $w = \cos \theta$ or $w = \cos(2\theta/\pi)$ give rise to $O(\theta^{-1})$ or $O(\theta^{-1})$ terms in the components of the matrix $\hat{B}$ in some cases. Such terms in $\hat{B}$ become obstacles for our scheme.}
\[
\cos\left(\frac{x}{r}(1 - \cos \theta)\right) \text{ for } i = 1 (i = 2, 3, 4) \text{ and } 1 \text{ for } i = 2, 3, 4 (i = 1)\text{ if we are to find the solutions of the left (right) type in equation (C.8). In terms of } \vec{v}, \text{ the eigenvalue equations become}
\]
\[
\left(\frac{\partial^2}{\partial \theta^2} + \hat{A}(\theta) \partial_\theta + \hat{B}(\theta)\right)\vec{v} = -(2n - 1)\lambda_i \vec{v},
\]
where \(\hat{A} = \mathcal{O}\left(\left(\frac{x}{r} - \theta\right)^{-1}\right)\) and \(\hat{B}(\theta) = \mathcal{O}\left(\left(\frac{x}{r} - \theta\right)^0\right)\), and we may impose the Neumann boundary condition on all the components of \(\vec{v}\) to find the solutions of the left (right) type in equation (C.8).

Following the procedures in the previous sections after the introduction of \(\Upsilon\), we define and use the variables given by \(\vec{v} = (\Upsilon \Xi \Pi)^{-1} \vec{v}\), and solve equation (C.6) imposing the Neumann boundary condition at \(\theta = 0\) and \(\pi/2\).

For \(\kappa = 0\), the unknown variables reduce to \(\vec{v} = (f_\theta, f_\phi)\) as we mentioned previously. We find \(\vec{B}_{-2} = \text{Diag}[-(2n - 1), -(2n - 1)]\) without the diagonalization, and we find that the regular solutions behave as \(v_i \propto \theta^0\). Thus, we may solve the eigenvalue equations using the original variables \(v_i\), or alternatively introducing only the even/odd decomposition defined by \(\Upsilon = \text{Diag}(w, 1)\) or \(\Upsilon = \text{Diag}(1, w)\) with \(w \equiv \cos\left(\frac{x}{r}(1 - \cos \theta)\right)\).

### C.2. Numerical implementations

We solve the eigenvalue equation (C.6) numerically as follows. We employ the relaxation method, which was used also in [18] to treat similar problems.

Firstly, we introduce a diffusion equation given by
\[
\partial_\tau \vec{v}(\tau, \theta) = M(\theta) \vec{v}, \quad M(\theta) \equiv \frac{\partial^2}{\partial \theta^2} + \hat{A}(\theta) \partial_\theta + \hat{B}(\theta), \quad (C.10)
\]
whose solution behaves as
\[
\vec{v}(\tau, \theta) = e^{\tau M} \vec{v}_{\text{init}} \rightarrow e^{\tau \lambda_i} \vec{v}_i,
\]
where \(\lambda_i\) is the largest eigenvalue of \(M\) and \(\vec{v}_i\) is the corresponding eigenvector. It implies that we may obtain the largest eigenvalue by following the time evolution described by equation (C.10) for a sufficiently long time. The smaller eigenvalues can be obtained successively by the same calculations if we project out the eigenvectors for larger eigenvalues at each step of the time evolution.

To make the equation amenable to numerics, we need to discretize the diffusion equation (C.10) on the grids given by \(\tau = \tau_n \equiv n\Delta \tau\) \((n = 0, 1, \ldots)\) and \(\theta = \theta_i \equiv i\Delta \theta\) \((i = 0, 1, \ldots, N, \Delta \theta \equiv \pi/2N)\). We do it implicitly as, for \(1 \leq i \leq N - 1\),
\[
\begin{align*}
\frac{\vec{v}_i^{n+1} - \vec{v}_i^n}{\Delta \tau} &= \frac{1}{\Delta \theta^2} \left[ \frac{\Delta \tau}{\Delta \theta^2} E - \Delta \tau \hat{B} \right] \vec{v}_i^n + \left( -\frac{\Delta \tau}{\Delta \theta^2} E + \frac{\Delta \tau}{\Delta \theta} \hat{A}_i \right) \vec{v}_i^{n-1} \\
&\equiv a_i \vec{v}_{i-1}^n + b_i \vec{v}_i^n + c_i \vec{v}_{i+1}^n.
\end{align*}
\]
An advantage of the implicit scheme is that the time evolution is stable for any \(\Delta \tau\), and we may take it large to speed up the numerical calculations. Equation (C.12) may be transformed into
\[
\begin{align*}
\vec{v}_{i+1}^{n-1} &= \left( \frac{-\Delta \tau}{\Delta \theta^2} E + \frac{\Delta \tau}{2\Delta \theta} \hat{A}_i \right) \vec{v}_{i-1}^n \\
&\quad + \left( \left( 1 + \frac{2\Delta \tau}{\Delta \theta^2} E - \Delta \tau \hat{B} \right) \vec{v}_i^n + \left( -\frac{\Delta \tau}{\Delta \theta^2} E - \frac{\Delta \tau}{2\Delta \theta} \hat{A}_i \right) \vec{v}_i^{n-1} \right) \\
&\equiv a_i \vec{v}_{i-1}^n + b_i \vec{v}_i^n + c_i \vec{v}_{i+1}^n.
\end{align*}
\]
We need to impose the Neumann boundary condition at \(\theta = 0\) and \(\theta = \pi/2\). At \(\theta = 0\), it implies \(\vec{v} \simeq \vec{v}_0^+ + \frac{1}{2} \vec{v}_0^0 \theta^2\), and equation (C.10) at \(\theta = 0\) may be expressed as
\[
\partial_\tau \vec{v} = (E + \hat{A}_0) \vec{v} + \hat{B}_0 \vec{v},
\]
(14.1)
Similarly, we may obtain the results for the results summarized in table B3 and also as equation (24) supplemented with equation (25).

\[
\vec{v}_0' = \left( 1 + \frac{2\Delta\tau}{\Delta\theta^2} \right) E + \frac{2\Delta\tau}{\Delta\theta^2} A_0 - \Delta\tau \hat{B}_0 \vec{v}_0' \equiv b_0 \vec{v}_0' + c_0 v_1'.
\]

(C.15)

Following the same procedure for \( \theta = \pi/2 \), we have the equation at \( i = N \) as

\[
v_N'^{-1} = -\frac{2\Delta\tau}{\Delta\theta^2} \left( E + A_N \right) \vec{v}_{N-1}' + \left( \left( 1 + \frac{2\Delta\tau}{\Delta\theta^2} \right) E + \frac{2\Delta\tau}{\Delta\theta^2} A_N - \Delta\tau \hat{B}_N \right) \vec{v}_N' = a_N \vec{v}_{N-1}' + b_N \vec{v}_N'.
\]

(C.16)

where \( \hat{A}_N \equiv \lim_{\theta \to \pi} \left( \theta - \frac{\pi}{2} \right) \hat{A}(\theta) \).

The above equations are summarized as follows:

\[
\begin{pmatrix}
\hat{v}_0'^{-1} \\
\hat{v}_1' \\
\vdots \\
\hat{v}_N'^{-1} \\
\end{pmatrix} = \begin{pmatrix}
b_0 & c_0 \\
a_1 & b_1 & c_1 \\
\vdots & \ddots & \ddots \\
& a_{N-1} & b_{N-1} & c_{N-1} \\
\end{pmatrix} \begin{pmatrix}
\hat{v}_0' \\
\hat{v}_1' \\
\vdots \\
\hat{v}_{N-1}' \\
\end{pmatrix}.
\]

(C.17)

This equation is readily solved by the forward and backward substitution to obtain \( \vec{v}_0' \) from \( \vec{v}_N'^{-1} \). To obtain the full spectrum, we need to conduct two calculations for the two \( \Gamma \) matrices, each of which corresponds to each mode in equation (C.8).

Applying this method to equation (C.6) for the electromagnetic scalar modes, we obtain the results summarized in table B3 and also as equation (24) supplemented with equation (25). Similarly, we may obtain the results for the \( \kappa = 0 \) case, and the results are summarized as equation (26). In this calculation, we took \( N = 4000 \) and \( \Delta\tau = 10^{-2} \), and truncated the time evolution once the condition \( \left| \lambda_{\pi}^{-1} \frac{d\theta}{d\pi} \right| \lesssim 10^{-8} \) is achieved. After the truncation, we progressed the time evolution one more step using smaller \( \Delta\tau = 10^{-8} \) to obtain the correct value of \( \lambda_1 \).

C.3. Gravitational vector modes

The gravitational vector modes have the following properties. The unknown variables are

\[
\vec{v} = (\vec{gR}, \vec{g'}_0, \vec{gR'}, \vec{g'R}, \vec{g'R'}, \vec{hR}, \vec{h'R}, \vec{h'R'})^T,
\]

(C.18)

whose definitions are found in equation (B.26). Following the procedure of section C.1 in this case, we find the diagonalized coefficient matrix \( \hat{B}_{-2} \) is given by

\[
\hat{B}_{-2} = \text{Diag}[\epsilon_-, \epsilon_-, \epsilon_-, \epsilon_+, \epsilon_+, \epsilon_+],
\]

(C.19)

Solving the equation corresponding to equation (C.3), we find the asymptotic behaviors for \( \theta \to 0 \) of the decaying solutions to be

\[
\vec{v} \sim (\theta^{p_-}, \theta^{p_-}, \theta^{p_-}, \theta^{p_-}, \theta^{p_-}, \theta^{p_+}, \theta^{p_+})^T, \quad p_- = 2(\kappa + 1), \quad p_+ = 2(\kappa + 2).
\]

(C.20)

Then, we define and use new variables \( \vec{v} \equiv \Pi^{-1} \vec{v} \), where \( \Pi \) is a diagonal matrix whose components are given by \( \Pi_{ii} = \sin \pi^{p_-} \) for \( i = 1, \ldots, 4 \) and \( \Pi_{ii} = \sin \pi^{p_+} \) for \( i = 5, \ldots, 8 \). Following the procedure in sections C.1 and C.2, we obtain the eigenvalues summarized in table B4 and also as equations (12) and (13).
C.4. Gravitational scalar modes

The gravitational scalar modes have the following properties. For a general case with $\kappa > 1$ and $n > 2$, the unknown variables are given by

$$\vec{v} = \begin{pmatrix} f_\theta \phi, f_\phi \theta, f_\theta \phi, g_\theta, g_\phi, g_\theta, g_\phi, h^0, h^1, h^{+-} \end{pmatrix}^T,$$

whose definitions are found in equation (B.28). $B_{-2}$ for this case becomes

$$\tilde{B}_{-2} = \text{Diag}[\epsilon_-, \epsilon_-, \epsilon_-, \epsilon_0, \epsilon_0, \epsilon_0, \epsilon_+, \epsilon_+, \epsilon_+] , \quad \epsilon_3 = -4(\kappa + \delta)(\kappa + n - 1 + \delta),$$

which give rise to the asymptotic behavior described as

$$\vec{v} \sim (\theta^{0-}, \theta^{p-}, \theta^{p0}, \theta^{p0}, \theta^{p+}, \theta^{h+}, \theta^{h-})^T , \quad p_3 \equiv 2(\kappa + \delta).$$

We define and use the matrix $\Pi$ using $p_3$ in this equation. The numerical results of the eigenvalue $\lambda_2$ are summarized as equation (24) with equation (25).

Below, we comment on the special cases for the gravitational scalar modes.

- $\kappa = 0$: $\gamma$ constant in this case, and the unknown variables are given by $\vec{v} = (f_\theta \phi, f_\phi \theta, f_\theta \phi, g_\theta, g_\phi, g_\theta, g_\phi, h^0, h^1, h^{+-})^T$. Without introducing the diagonalization, we find $B_{-2} = \text{Diag}[-4n, -4n, -4n]$, from which we find that the regular solutions behave as $v_i \propto \theta^2$.

- $\kappa = 1$ and $n = 2$: all of $\gamma_{\alpha \beta}^{\pm \pm}$ and $\gamma_{\alpha \beta}^{\pm -}$ vanish, and as a result we have only $(f_\theta \phi, f_\phi \theta, g_\theta, g_\phi)$ as the free variables in this case. It follows $\tilde{B}_{-2} = \text{Diag}[0, -8, -8, -24, -24, -24]$, from which we find $\tilde{v}_i \propto \theta^2$ with $p_1 = 0$, $p_{2,3,4} = 2$ and $p_{5,6,7} = 4$.

- $\kappa = 1$ and $n > 2$: $\gamma_{\alpha \beta}^{\pm \pm}$ vanishes in this case, and we have $\vec{v} = (f_\theta \phi, f_\phi \theta, g_\theta, g_\phi, g_\theta, g_\phi, h^0, h^1, h^{+-})^T$ as the free variables. We find $\tilde{B}_{-2} = \text{Diag}[0, -4n, -4n, -4n, -8(n + 1), -8(n + 1), -8(n + 1)]$, which results in $\tilde{v}_i \propto \theta^2$ with $p_1 = 0$, $p_{2,3,4} = 2$ and $p_{5,6,7} = 4$.

- $\kappa > 1$ and $n = 2$: $\gamma_{\alpha \beta}^{\pm -}$ vanishes in this case, and as a result we have $(f_\theta \phi, f_\phi \theta, g_\theta, g_\phi, g_\theta, g_\phi, h^0, h^1, h^{+-})$ as the free variables. We find $\tilde{B}_{-2} = \text{Diag}[\epsilon_-, \epsilon_-, \epsilon_-, \epsilon_0, \epsilon_0, \epsilon_0, \epsilon_+, \epsilon_+, \epsilon_+]$, where $\epsilon_4$ are those in equation (C.22) with $n = 2$ plugged in. The asymptotic behavior of the regular solutions in this case is given by $\vec{v}_i \propto \theta^2$ with $p_{1,2,3} = 2(\kappa - 1), p_{4,5,6} = 2\kappa$ and $p_{7,8,9} = 2(\kappa + 1)$.

The numerical results of the eigenvalues $\lambda_2$ for these cases are summarized by equations (16)–(19).

References

[1] Emparan R 2008 Black holes galore in $D > 4$ Fortsch. Phys. 56 723
[2] Emparan R and Reall H S 2008 Black holes in higher dimensions Living Rev. Rel. 11 6 (arXiv:0801.3471 [hep-th])
[3] Myers R C and Perry M J 1986 Black holes in higher dimensional space-times Ann. Phys. 172 304
[4] Emparan R and Myers R C 2003 Instability of ultra-spinning black holes J. High Energy Phys. JHEP09(2003)025 (arXiv:hep-th/0308056)
[5] Dias O J C, Figueras P, Monteiro R, Reall H S and Santos J E 2010 An instability of higher-dimensional rotating black holes J. High Energy Phys. JHEP05(2010)076 (arXiv:1001.4527 [hep-th])
[6] Kodama H, Konoplya R A and Zhidenko A 2010 Gravitational stability of simply rotating Myers–Perry black holes: tensorial perturbations Phys. Rev. D 81 044007 (arXiv:0904.2154 [gr-qc])
[7] Dias O J C, Figueras P, Monteiro R, Santos J E and Emparan R 2009 Instability and new phases of higher-dimensional rotating black holes Phys. Rev. D 80 111701 (arXiv:0907.2248 [hep-th])
[8] Shibata M and Yoshino H 2010 Bar-mode instability of rapidly spinning black hole in higher dimensions: numerical simulation in general relativity Phys. Rev. D 81 104035 (arXiv:1004.4970 [gr-qc])

[9] Dias O J C, Figueras P, Monteiro R and Santos J E 2010 Ultraspinning instability of rotating black holes Phys. Rev. D 82 104025 (arXiv:1006.1904 [hep-th])

[10] Kunduri H K, Lucietti J and Reall H S 2006 Gravitational perturbations of higher dimensional rotating black holes: tensor perturbations Phys. Rev. D 74 084021 (arXiv:hep-th/0606076)

[11] Murata K and Soda J 2008 Stability of five-dimensional Myers–Perry black holes with equal angular momenta Prog. Theor. Phys. 120 561 (arXiv:0803.1371 [hep-th])

[12] Durkee M and Reall H S 2011 Perturbations of near-horizon geometries and instabilities of Myers–Perry black holes Phys. Rev. D 83 104044 (arXiv:1012.4805 [hep-th])

[13] Dias O J C, Monteiro R and Santos J E 2011 Ultraspinning instability: the missing link J. High Energy Phys. JHEP08(2011)139 (arXiv:1106.4554 [hep-th])

[14] Durkee M and Reall H S 2011 Perturbations of higher-dimensional spacetimes Class. Quantum Grav. 28 035011 (arXiv:1009.0015 [gr-qc])

[15] Durkee M, Pravda V, Pravdova A and Reall H S 2010 Generalization of the Geroch–Held–Penrose formalism to higher dimensions Class. Quantum Grav. 27 215010 (arXiv:1002.4826 [gr-qc])

[16] Teukolsky S A 1972 Rotating black holes–separable wave equations for gravitational and electromagnetic perturbations Phys. Rev. Lett. 29 1114

[17] Guica M, Hartman T, Song W and Strominger A 2009 The Kerr/CFT correspondence Phys. Rev. D 80 124008 (arXiv:0908.4266 [hep-th])

[18] Murata K 2011 Conformal weights in the Kerr/CFT correspondence J. High Energy Phys. JHEP05(2011)117 (arXiv:1103.5635 [hep-th])

[19] Dias O J C, Reall H S and Santos J E 2009 Kerr–CFT and gravitational perturbations J. High Energy Phys. JHEP08(2009)101 (arXiv:0906.2380 [hep-th])

[20] Amzel A J, Herowitz G T, Marolf D and Roberts M M 2009 No dynamics in the extremal Kerr–Throat J. High Energy Phys. JHEP09(2009)044 (arXiv:0906.2376 [hep-th])

[21] Kunduri H K and Lucietti J 2009 A classification of near-horizon geometries of extremal vacuum black holes J. Math. Phys. 50 082502 (arXiv:0806.2051 [hep-th])

[22] Kunduri H K, Lucietti J and Reall H S 2007 Near-horizon symmetries of extremal black holes Class. Quantum Grav. 24 4169 (arXiv:0705.4214 [hep-th])

[23] Hollands S and Ishibashi A 2010 All vacuum near horizon geometries in arbitrary dimensions Ann. Henri Poincare 10 1537 (arXiv:0909.3462 [gr-qc])

[24] Figueras P, Kunduri H K, Lucietti J and Rangamani M 2008 Extremal vacuum black holes in higher dimensions Phys. Rev. D 78 044042 (arXiv:0803.2998 [hep-th])

[25] Hawking S W and Reall H S 2000 Charged and rotating AdS black holes and their CFT duals Phys. Rev. D 61 024014 (arXiv:hep-th/9908109)

[26] Kodama H 2008 Superradiance and instability of black holes Prog. Theor. Phys. Suppl. 172 11 (arXiv:0711.4184 [hep-th])

[27] Figueras P, Murata K and Reall H S 2011 Black hole instabilities and local Penrose inequalities Class. Quantum Grav. 28 225030 (arXiv:1107.5785 [gr-qc])

[28] Hollands S and Wald R M 2012 Stability of black holes and black branes arXiv:1201.0463 [gr-qc]

[29] Ishibashi A and Kodama H 2003 Stability of higher dimensional Schwarzschild black holes Prog. Theor. Phys. 110 901 (arXiv:hep-th/0305185)

[30] Konoplya R A and Zhidenko A 2007 Stability of multidimensional black holes: complete numerical analysis Nucl. Phys. B 777 182 (arXiv:hep-th/0703231)

[31] Peeters K 2007 Introducing Cadabra: a symbolic computer algebra system for field theory problems arXiv:hep-th/0701238

[32] Peeters K 2007 A field-theory motivated approach to symbolic computer algebra Comput. Phys. Commun. 176 550 (arXiv:cs.SC/0608005)