BOUNDARY EFFECTS IN 2+1 DIMENSIONAL MAXWELL-CHERN-SIMONS THEORY

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Abstract

The effects of the sample’s boundaries in the magnetic response of the charged anyon fluid at finite temperature are investigated. For the case of an infinite-strip sample it is shown that the Meissner effect takes place at temperatures lower than the fermion energy gap $\omega_c$. The temperature dependence of the corresponding effective penetration depth is determined. At temperatures much larger than the scale $\omega_c$, a different phase is found, in which the external magnetic field penetrates the fluid.
I. INTRODUCTION

Recently much attention has been given to lower dimensional gauge theories. Such remarkable results as the chiral symmetry breaking \[1\], quantum Hall effect \[2\], spontaneously broken Lorentz invariance by the dynamical generation of a magnetic field \[3\], and the connection between non-perturbative effects in low-energy strong interactions and QCD\[4\], show the broad range of applicability of these theories.

In particular, 2+1 dimensional gauge theories with fractional statistics -anyon systems \[5\] have been extensively studied. One main reason for such an interest has been the belief that a strongly correlated electron system in two dimensions can be described by an effective field theory of anyons \[6\], \[7\]. Besides, it has been claimed that anyons could play a basic role in high-\(T_C\) superconductivity \[6\] - \[9\]. It is known \[10\] that a charged anyon system in two spatial dimensions can be modeled by means of a 2+1 dimensional Maxwell-Chern-Simons (MCS) theory. An important feature of this theory is that it violates parity and time-reversal invariance. However, at present no experimental evidences of P and T violation in high-\(T_C\) superconductivity have been found. It should be pointed out, nevertheless, that it is possible to construct more sophisticated P and T invariant anyonic models \[11\]. In any case, whether linked to high-\(T_C\) superconductivity or not, the anyon system is an interesting theoretical model in its own right.

The superconducting behavior of anyon systems at \(T = 0\) has been investigated by many authors \[8\] - \[15\]. Crucial to the existence of anyon superconductivity at \(T = 0\) is the exact cancellation between the bare and induced Chern-Simons terms in the effective action of the theory.

Although a general consensus exists regarding the superconductivity of anyon systems at zero temperature, a similar consensus at finite temperature is yet to be achieved \[16\] - \[21\]. Some authors (see ref. \[17\]) have concluded that the superconductivity is lost at \(T \neq 0\), based upon the appearance of a temperature-dependent correction to the induced Chern-Simons coefficient that is not cancelled out by the bare term. In ref. \[18\] it is argued,
however, that this finite temperature correction is numerically negligible at $T < 200 \, K$, and that the main reason for the lack of a Meissner effect is the development of a pole $\sim \left( \frac{1}{k^2} \right)$ in the polarization operator component $\Pi_{00}$ at $T \neq 0$. There, it is discussed how the existence of this pole leads to a so called partial Meissner effect with a constant magnetic field penetration throughout the sample that appreciably increases with temperature. On the other hand, in ref. [16], it has been independently claimed that the anyon model cannot superconduct at finite temperature due to the existence of a long-range mode, found inside the infinite bulk at $T \neq 0$. The long range mode found in ref. [16] is also a consequence of the existence of a pole $\sim \left( \frac{1}{k^2} \right)$ in the polarization operator component $\Pi_{00}$ at $T \neq 0$.

The apparent lack of superconductivity at temperatures greater than zero has been considered as a discouraging property of anyon models. Nevertheless, it may be still premature to disregard the anyons as a feasible solution for explaining high-$T_c$ superconductivity, at least if the reason sustaining such a belief is the absence of the Meissner effect at finite temperature. As it was shown in a previous paper [21], the lack of a Meissner effect, reported in ref. [18] for the case of a half-plane sample as a partial Meissner effect, is a direct consequence of the omission of the sample boundary effects in the calculations of the minimal solution for the magnetic field within the sample. To understand this remark we must take into account that the results of ref. [18] were obtained by finding the magnetization in the bulk due to an externally applied magnetic field at the boundary of a half-plane sample. However, in doing so, a uniform magnetization was assumed and therefore the boundary effects were indeed neglected. Besides, in ref. [18] the field equations were solved considering only one short-range mode of propagation for the magnetic field, while as has been emphasized in our previous letter [21], there is a second short-range mode whose qualitative contribution to the solutions of the field equations cannot be ignored.

In the present paper we study the effects of the sample’s boundaries in the magnetic response of the anyon fluid at finite temperature. This is done by considering a sample shaped as an infinite strip. When a constant and homogeneous external magnetic field, which is perpendicular to the sample plane, is applied at the boundaries of the strip, two
different magnetic responses, depending on the temperature values, can be identified. At temperatures smaller than the fermion energy gap inherent to the many-particle MCS model ($T \ll \omega_c$), the system exhibits a Meissner effect. In this case the magnetic field cannot penetrate the bulk farther than a very short distance ($\lambda \sim 10^{-5}\text{cm}$ for electron densities characteristic of the high-$T_c$ superconductors and $T \sim 200\text{ K}$). On the other hand, as it is natural to expect from a physical point of view, when the temperatures are larger than the energy gap ($T \gg \omega_c$) the Meissner effect is lost. In this temperature region a periodic inhomogeneous magnetic field is present within the bulk.

These results, together with those previously reported in ref. [21], indicate that, contrary to some authors’ belief, the superconducting behavior (more precisely, the Meissner effect), found in the charged anyon fluid at $T = 0$, does not disappear as soon as the system is heated.

As it is shown below, the presence of boundaries can affect the dynamics of the system in such a way that the mode that accounts for a homogeneous field penetration [16] cannot propagate in the bulk. Although these results have been proved for two types of samples, the half-plane [21] and the infinite strip reported in this paper, we conjecture that similar effects should also exist in other geometries.

Our main conclusion is that the magnetic behavior of the anyon fluid is not just determined by its bulk properties, but it is essentially affected by the sample boundary conditions. The importance of the boundary conditions in 2+1 dimensional models has been previously stressed in ref. [22].

The plan for the paper is as follows. In Sec. 2, for completeness as well as for the convenience of the reader, we define the many-particle 2+1 dimensional MCS model used to describe the charged anyon fluid, and briefly review its main characteristics. In Sec. 3 we study the magnetic response in the self-consistent field approximation of a charged anyon fluid confined to an infinite-strip, finding the analytical solution of the MCS field equations that satisfies the boundary conditions. The fermion contribution in this approximation is given by the corresponding polarization operators at $T \neq 0$ in the background of a many-
particle induced Chern-Simons magnetic field. Using these polarization operators in the low
temperature approximation \(T \ll \omega_c\), we determine the system’s two London penetration
depths. Taking into account that the boundary conditions are not enough to completely
determine the magnetic field solution within the sample, an extra physical condition, the
minimization of the system free-energy density, is imposed. This is done in Sec. 4. In
this section we prove that even though the electromagnetic field has a long-range mode of
propagation in the charged anyon fluid at \(T \neq 0\) [16], a constant and uniform magnetic
field applied at the sample’s boundaries cannot propagate through this mode. The explicit
temperature dependence at \(T \ll \omega_c\) of all the coefficients appearing in the magnetic field
solution, and of the effective London penetration depth are also found. In Sec. 5, we discuss
how the superconducting behavior of the charged anyon fluid disappears at temperatures
larger than the energy gap \(T \gg \omega_c\). Sec. 6 contains the summary and discussion.

II. MCS MANY-PARTICLE MODEL

The Lagrangian density of the 2+1 dimensional non-relativistic charged MCS system is

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - \frac{N}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + e n_e A_0 + i \bar{\psi} \gamma_0 D_0 \psi - \frac{1}{2m} |D_k \psi|^2 + \bar{\psi} \gamma_\mu \psi
\]  

(2.1)

where \(A_\mu\) and \(a_\mu\) represent the electromagnetic and the Chern-Simons fields respectively.
The role of the Chern-Simons fields is simply to change the quantum statistics of the matter
field, thus, they do not have an independent dynamics. \(\psi\) represents non-relativistic spinless
fermions. \(N\) is a positive integer that determines the magnitude of the Chern-Simons
coupling constant. The charged character of the system is implemented by introducing a
chemical potential \(\mu\); \(n_e\) is a background neutralizing ‘classical’ charge density, and \(m\)
is the fermion mass. We will consider throughout the paper the metric \(g_{\mu\nu} = (1, -\mathbf{T})\). The
covariant derivative \(D_\nu\) is given by

\[
D_\nu = \partial_\nu + i (a_\nu + e A_\nu), \quad \nu = 0, 1, 2
\]  

(2.2)
It is known that to guarantee the system neutrality in the presence of a different from zero fermion density \(n_e \neq 0\), a nontrivial background of Chern-Simons magnetic field \(\mathbf{b} = \mathbf{f}_{21}\) is generated. The Chern-Simons background field is obtained as the solution of the mean field Euler-Lagrange equations derived from (2.1)

\[
-\frac{N}{4\pi} \varepsilon^{\mu\nu\rho} f_{\nu\rho} = \langle j^\mu \rangle \tag{2.3}
\]

\[
\partial_\nu F^{\mu\nu} = e \langle j^\mu \rangle - en_e \delta^{\mu0} \tag{2.4}
\]

considering that the system formed by the electron fluid and the background charge \(n_e\) is neutral

\[
\langle j^0 \rangle - n_e \delta^{\mu0} = 0 \tag{2.5}
\]

In eq. (2.5) \(\langle j^0 \rangle\) is the fermion density of the many-particle fermion system

\[
\langle j^0 \rangle = \frac{\partial \Omega}{\partial \mu}, \tag{2.6}
\]

\(\Omega\) is the fermion thermodynamic potential.

In this approximation it is found from (2.3)-(2.5) that the Chern-Simons magnetic background is given by

\[
\mathbf{b} = \frac{2\pi n_e}{N} \tag{2.7}
\]

Then, the unperturbed one-particle Hamiltonian of the matter field represents a particle in the background of the Chern-Simons magnetic field \(\mathbf{b}\),

\[
H_0 = -\frac{1}{2m} \left[ (\partial_1 + i\mathbf{b}x_2)^2 + \partial_2^2 \right] \tag{2.8}
\]

In (2.8) we considered the background Chern-Simons potential, \(\alpha_k\), \((k = 1, 2)\), in the Landau gauge

\[
\alpha_k = \mathbf{b}x_2 \delta_{k1} \tag{2.9}
\]
The eigenvalue problem defined by the Hamiltonian (2.8) with periodic boundary conditions in the \( x_1 \)-direction: \( \Psi (x_1 + L, x_2) = \Psi (x_1, x_2) \),

\[
H_0 \Psi_{nk} = \epsilon_n \Psi_{nk}, \quad n = 0, 1, 2, \ldots \text{ and } k \in \mathbb{Z}
\] (2.10)

has eigenvalues and eigenfunctions given respectively by

\[
\epsilon_n = \left( n + \frac{1}{2} \right) \omega_c
\] (2.11)

\[
\Psi_{nk} = \frac{\sqrt{b}^{1/4}}{\sqrt{L}} \exp \left( -2\pi ikx_1/L \right) \phi_n \left( x_2 \sqrt{b} - \frac{2\pi k}{L\sqrt{b}} \right)
\] (2.12)

where \( \omega_c = \sqrt{b}/m \) is the cyclotron frequency and \( \phi_n (\xi) \) are the orthonormalized harmonic oscillator wave functions.

Note that the energy levels \( \epsilon_n \) are degenerates (they do not depend on \( k \)). Then, for each Landau level \( n \) there exists a band of degenerate states. The cyclotron frequency \( \omega_c \) plays here the role of the energy gap between occupied Landau levels. It is easy to prove that the filling factor, defined as the ratio between the density of particles \( n_e \) and the number of states per unit area of a full Landau level, is equal to the Chern-Simons coupling constant \( N \).

Thus, because we are considering that \( N \) is a positive integer, we have in this MCS theory \( N \) completely filled Landau levels. Once this ground state is established, it can be argued immediately \[8\], \[9\], \[13\], \[14\], that at \( T = 0 \) the system will be confined to a filled band, which is separated by an energy gap from the free states; therefore, it is natural to expect that at \( T = 0 \) the system should superconduct. This result is already a well established fact on the basis of Hartree-Fock analysis \[8\] and Random Phase Approximation \[9\], \[13\].

The case at \( T \neq 0 \) is more controversial since thermal fluctuations, occurring in the many-particle system, can produce significant changes. As we will show in this paper, the presence in this theory of a natural scale, the cyclotron frequency \( \omega_c \), is crucial for the existence of a phase at \( T \ll \omega_c \), on which the system, when confined to a bounded region, still behaves as a superconductor.

The fermion thermal Green’s function in the presence of the background Chern-Simons field \( \Phi \).
\[ G(x, x') = -\left\langle T_\tau \psi(x) \overline{\psi}(x') \right\rangle \]  

(2.13)

is obtained by solving the equation

\[ \left( \partial_\tau - \frac{1}{2m} D_k^2 - \mu \right) G(x, x') = -\delta_3(x - x') \]  

(2.14)

subject to the requirement of antiperiodicity under the imaginary time translation \( \tau \to \tau + \beta \) (\( \beta \) is the inverse absolute temperature). In (2.14) we have introduced the notation

\[ D_k = \partial_k + i\alpha_k \]  

(2.15)

The Fourier transform of the fermion thermal Green’s function (2.13)

\[ G(p_4, p) = \int_0^\beta d\tau \int d\mathbf{x} G(\tau, \mathbf{x}) e^{i(p_4\tau - p\mathbf{x})} \]  

(2.16)

can be expressed in terms of the orthonormalized harmonic oscillator wave functions \( \varphi_n(\xi) \) as \[ 23 \]

\[ G(p_4, p) = \int_0^\infty d\rho \int_{-\infty}^\infty dx_2 \sqrt{\rho} \exp(-ip_2x_2) \exp \left( ip_4 + \mu - \frac{\beta}{2m} \right) \rho \sum_{n=0}^{\infty} \varphi_n(\xi) \varphi_n(\xi') t^n \]  

(2.17)

where \( t = \exp \frac{\pi}{m} \rho, \xi = \frac{\rho}{\sqrt{\beta}} + \frac{p_2\sqrt{\beta}}{2}, \xi' = \frac{\rho}{\sqrt{\beta}} - \frac{p_2\sqrt{\beta}}{2} \) and \( p_4 = (2n + 1)\pi/\beta \) are the discrete frequencies \( (n = 0, 1, 2, \ldots) \) corresponding to fermion fields.

### III. LINEAR RESPONSE IN THE INFINITE STRIP

#### A. Effective Theory at \( \mu \neq 0 \) and \( T \neq 0 \)

In ref. [16] the effective current-current interaction of the MCS model was calculated to determine the independent components of the magnetic interaction at finite temperature in a sample without boundaries, i.e., in the free space. These authors concluded that the pure Meissner effect observed at zero temperature is certainly compromised by the appearance
of a long-range mode at $T \neq 0$. Our main goal in the present paper is to investigate the magnetic response of the charged anyon fluid at finite temperature for a sample that confines the fluid within some specific boundaries. As we prove henceforth, the confinement of the system to a bounded region (a condition which is closer to the experimental situation than the free-space case) is crucial for the realization of the Meissner effect inside the charged anyon fluid at finite temperature.

Let us investigate the linear response of a charged anyon fluid at finite temperature and density to an externally applied magnetic field in the specific case of an infinite-strip sample. The linear response of the medium can be found under the assumption that the quantum fluctuations of the gauge fields about the ground-state are small. In this case the one-loop fermion contribution to the effective action, obtained after integrating out the fermion fields, can be evaluated up to second order in the gauge fields. The effective action of the theory within this linear approximation \cite{10,18} takes the form

$$\Gamma_{\text{eff}}(A_\nu, a_\nu) = \int dx \left( -\frac{1}{4} F_{\mu\nu}^2 - \frac{N}{4\pi} \varepsilon^{\mu\nu\rho\sigma} a_\mu \partial_\nu a_\rho + e n_e A_0 \right) + \Gamma^{(2)}$$

(3.1)

$$\Gamma^{(2)} = \int dx \Pi^\nu (x) [a_\nu (x) + eA_\nu (x)] + \int dx dy [a_\nu (x) + eA_\nu (x)] \Pi^{\mu\nu} (x, y) [a_\nu (y) + eA_\nu (y)]$$

Here $\Gamma^{(2)}$ is the one-loop fermion contribution to the effective action in the linear approximation. The operators $\Pi^\nu$ and $\Pi^{\mu\nu}$ are calculated using the fermion thermal Green’s function in the presence of the background field $\vec{b}$ (2.17). They represent the fermion tadpole and one-loop polarization operators respectively. Their leading behaviors for static ($k_0 = 0$) and slowly ($k \sim 0$) varying configurations in the frame $k = (k, 0)$ take the form

$$\Pi_k (x) = 0, \quad \Pi_0 (x) = -n_e, \quad \Pi^{\mu\nu} = \begin{pmatrix} \Pi_0 + \Pi_0' k^2 & 0 & \Pi_1 k \\ 0 & 0 & 0 \\ -\Pi_1 k & 0 & \Pi_2 k^2 \end{pmatrix}$$

(3.2)

The independent coefficients: $\Pi_0$, $\Pi_0'$, $\Pi_1$ and $\Pi_2$ are functions of $k^2$, $\mu$ and $\vec{b}$. In order to find them we just need to calculate the $\Pi^{\mu\nu}$ Euclidean components: $\Pi_{44}$, $\Pi_{42}$ and $\Pi_{22}$. In the Landau gauge these Euclidean components are given by \cite{18}. 

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\[
\Pi_{44}\left(k, \mu, \vec{b}\right) = -\frac{1}{\beta} \sum_{p_4} \frac{dp}{(2\pi)^2} G(p) G(p-k),
\]
(3.3)

\[
\Pi_{4j}\left(k, \mu, \vec{b}\right) = \frac{i}{2m\beta} \sum_{p_4} \frac{dp}{(2\pi)^2} \left\{ G(p) \cdot D_j^- G(p-k) + D_j^+ G(p) \cdot G(p-k) \right\},
\]
(3.4)

\[
\Pi_{jk}\left(k, \mu, \vec{b}\right) = \frac{1}{4m^2\beta} \sum_{p_4} \frac{dp}{(2\pi)^2} \left\{ D_k^- G(p) \cdot D_j^- G(p-k) + D_j^+ G(p) \cdot D_k^+ G(p-k) + D_j^+ D_k^- G(p) \cdot G(p-k) + G(p) \cdot D_j^- D_k^+ G(p-k) \right\} \\
- \frac{1}{2m} \Pi_4,
\]
(3.5)

where the notation

\[
D_j^\pm G(p) = \left[ ip_j \mp \frac{\vec{b}}{2} \varepsilon^{jk} \partial_{p_k} \right] G(p),
\]
(3.6)

was used.

Using (3.3)-(3.5) after summing in \(p_4\), we found that, in the \(k/\sqrt{\vec{b}} \ll 1\) limit, the polarization operator coefficients \(\Pi_0, \Pi_0', \Pi_1\) and \(\Pi_2\) are

\[
\Pi_0 = \frac{\beta \vec{b}}{8\pi k^2} \sum_n \Theta_n, \quad \Pi_0' = \frac{2m}{\pi b} \sum_n \Delta_n - \frac{\beta}{8\pi} \sum_n (2n+1) \Theta_n,
\]

\[
\Pi_1 = \frac{1}{\pi} \sum_n \Delta_n - \frac{\beta \vec{b}}{16\pi m} \sum_n (2n+1) \Theta_n, \quad \Pi_2 = \frac{1}{\pi m} \sum_n (2n+1) \Delta_n - \frac{\beta \vec{b}}{32\pi m^2} \sum_n (2n+1)^2 \Theta_n,
\]

\[
\Theta_n = \text{sech} \frac{2(\epsilon_n/2 - \mu)}{2}, \quad \Delta_n = \left( e^{\beta(\epsilon_n/2 - \mu)} + 1 \right)^{-1}
\]
(3.7)

The leading contributions of the one-loop polarization operator coefficients (3.7) at low temperatures \(T \ll \omega_c\) are

\[
\Pi_0 = \frac{2\beta \vec{b}}{\pi} e^{-\beta \vec{b}/2m}, \quad \Pi_0' = \frac{mN}{2\pi b} A, \quad \Pi_1 = \frac{N}{2\pi} A, \quad \Pi_2 = \frac{N^2}{4\pi mA}, \quad A = \left[ 1 - \frac{2\beta \vec{b}}{m} e^{-\beta \vec{b}/2m} \right]
\]
(3.8)

and at high temperatures \(T \gg \omega_c\) are...
\[ \Pi_0 = \frac{m}{2\pi} \left[ \tanh \frac{\beta \mu}{2} + 1 \right], \quad \Pi_0' = -\frac{\beta}{48\pi} \text{sech}^2 \left( \frac{\beta \mu}{2} \right), \quad \Pi_1 = b \Pi_0', \quad \Pi_2 = \frac{1}{12m^2} \Pi_0 \]

(3.9)

In these expressions \( \mu \) is the chemical potential and \( m = 2m_e \) (\( m_e \) is the electron mass).

These results are in agreement with those found in refs. [16], [20].

**B. MCS Linear Equations**

To find the response of the anyon fluid to an externally applied magnetic field, one needs to use the extremum equations derived from the effective action (3.1). This formulation is known in the literature as the self-consistent field approximation [18]. In solving these equations we confine our analysis to gauge field configurations which are static and uniform in the \( y \)-direction. Within this restriction we take a gauge in which \( A_1 = a_1 = 0 \).

The Maxwell and Chern-Simons extremum equations are respectively,

\[ \partial_\nu F^{\nu\mu} = eJ_{\text{ind}}^\mu \]  

(3.10a)

\[ -\frac{N}{4\pi} \varepsilon^{\mu\nu\rho} f_{\nu\rho} = J_{\text{ind}}^\mu \]  

(3.10b)

Here, \( f_{\mu\nu} \) is the Chern-Simons gauge field strength tensor, defined as \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \), and \( J_{\text{ind}}^\mu \) is the current density induced by the anyon system at finite temperature and density. Their different components are given by

\[ J_{\text{ind}}^0 (x) = \Pi_0 \left[ a_0 (x) + eA_0 (x) \right] + \Pi_0' \partial_x (\mathcal{E} + eE) + \Pi_1 (b + eB) \]  

(3.11a)

\[ J_{\text{ind}}^1 (x) = 0, \quad J_{\text{ind}}^2 (x) = \Pi_1 (\mathcal{E} + eE) + \Pi_2 \partial_x (b + eB) \]  

(3.11b)

in the above expressions we used the following notation: \( \mathcal{E} = f_{01}, \ E = F_{01}, \ b = f_{12} \) and \( B = F_{12} \). Eqs. (3.11) play the role in the anyon fluid of the London equations in BCS superconductivity. When the induced currents (3.11) are substituted in eqs. (3.10) we find, after some manipulation, the set of independent differential equations,
\[ \omega \partial_x^2 B + \alpha B = \gamma [\partial_x E - \sigma A_0] + \tau a_0, \quad (3.12) \]

\[ \partial_x B = \kappa \partial_x^2 E + \eta E, \quad (3.13) \]

\[ \partial_x a_0 = \chi \partial_x B \quad (3.14) \]

The coefficients appearing in these differential equations depend on the components of the polarization operators through the relations

\[ \omega = \frac{2\pi}{N} \Pi_0', \quad \alpha = -e^2 \Pi_1, \quad \tau = e \Pi_0, \quad \chi = \frac{2\pi}{eN}, \quad \sigma = -\frac{e^2}{\gamma} \Pi_0, \quad \eta = -\frac{e^2}{\delta} \Pi_1, \]

\[ \gamma = 1 + e^2 \Pi_0' - \frac{2\pi}{N} \Pi_1, \quad \delta = 1 + e^2 \Pi_2 - \frac{2\pi}{N} \Pi_1, \quad \kappa = \frac{2\pi}{N\delta} \Pi_2. \quad (3.15) \]

Distinctive of eq. (3.12) is the presence of the nonzero coefficients \( \sigma \) and \( \tau \). They are related to the Debye screening in the two dimensional anyon thermal ensemble. A characteristic of this 2+1 dimensional model is that the Debye screening disappears at \( T = 0 \), even if the chemical potential is different from zero. Note that \( \sigma \) and \( \tau \) link the magnetic field to the zero components of the gauge potentials, \( A_0 \) and \( a_0 \). As a consequence, these gauge potentials will play a nontrivial role in finding the magnetic field solution of the system.

C. Field Solutions and Boundary Conditions

Using eqs. (3.12)-(3.14) one can obtain a higher order differential equation that involves only the electric field,

\[ a \partial_x^4 E + d \partial_x^2 E + c E = 0, \quad (3.16) \]

Here, \( a = \omega \kappa, \ d = \omega \eta + \alpha \kappa - \gamma - \tau \kappa \chi, \) and \( c = \alpha \eta - \sigma \gamma - \tau \eta \chi. \)

Solving (3.16) we find

\[ E(x) = C_1 e^{-x \xi_1} + C_2 e^{x \xi_1} + C_3 e^{-x \xi_2} + C_4 e^{x \xi_2}, \quad (3.17) \]
where

$$\xi_{1,2} = \left[-d \pm \sqrt{d^2 - 4ac}\right]^{\frac{1}{2}} / \sqrt{2a} \quad (3.18)$$

When the low density approximation $n_e \ll m^2$ is considered (this approximation is in agreement with the typical values $n_e = 2 \times 10^{14} \text{cm}^{-2}$, $m_e = 2.6 \times 10^{10} \text{cm}^{-1}$ found in high-$T_C$ superconductivity), we find that, for $N = 2$ and at temperatures lower than the energy gap ($T \ll \omega_c$), the inverse length scales (3.18) are given by the following real functions

$$\xi_1 \simeq e \sqrt{\frac{m}{\pi}} \left[ 1 + \left( \frac{\pi^2 n_e^2}{m^3} \right)^\beta \exp - \left( \frac{\pi n_e \beta}{2m} \right) \right] \quad (3.19)$$

$$\xi_2 \simeq e \sqrt{\frac{n_e}{m}} \left[ 1 - \left( \frac{\pi n_e}{m} \right) \beta \exp - \left( \frac{\pi n_e \beta}{2m} \right) \right] \quad (3.20)$$

These two inverse length scales correspond to two short-range electromagnetic modes of propagation. These results are consistent with those obtained in ref. [16] using a different approach. If the masses of the two massive modes, obtained in ref. [16] by using the electromagnetic thermal Green’s function for static and slowly varying configurations, are evaluated in the range of parameters considered above, it can be shown that they reduce to eqs. (319), (3.20).

The solutions for $B$, $a_0$ and $A_0$, can be obtained using eqs. (3.13), (3.14), (3.17) and the definition of $E$ in terms of $A_0$,

$$B (x) = \gamma_1 \left( C_2 e^{x\xi_1} - C_1 e^{-x\xi_1} \right) + \gamma_2 \left( C_4 e^{x\xi_2} - C_3 e^{-x\xi_2} \right) + C_5 \quad (3.21)$$

$$a_0 (x) = \chi \gamma_1 \left( C_2 e^{x\xi_1} - C_1 e^{-x\xi_1} \right) + \chi \gamma_2 \left( C_4 e^{x\xi_2} - C_3 e^{-x\xi_2} \right) + C_6 \quad (3.22)$$

$$A_0 (x) = \frac{1}{\xi_1} \left( C_1 e^{-x\xi_1} - C_2 e^{x\xi_1} \right) + \frac{1}{\xi_2} \left( C_3 e^{-x\xi_2} - C_4 e^{x\xi_2} \right) + C_7 \quad (3.23)$$

In the above formulas we introduced the notation $\gamma_1 = (\xi_1^2 \kappa + \eta) / \xi_1$, $\gamma_2 = (\xi_2^2 \kappa + \eta) / \xi_2$.

In obtaining eq. (3.16) we have taken the derivative of eq. (3.12). Therefore, the solution of eq. (3.16) belongs to a wider class than the one corresponding to eqs. (3.12)-(3.14).
exclude redundant solutions we must require that they satisfy eq. (3.12) as a supplementary condition. In this way the number of independent unknown coefficients is reduced to six, which is the number corresponding to the original system (3.12)-(3.14). The extra unknown coefficient is eliminated substituting the solutions (3.17), (3.21), (3.22) and (3.23) into eq. (3.12) to obtain the relation

$$e \Pi_1 C_5 = -\Pi_0 (C_6 + eC_7)$$  \hspace{1cm} (3.24)$$

Eq. (3.24) has an important meaning, it establishes a connection between the coefficients of the long-range modes of the zero components of the gauge potentials \((C_6 + eC_7)\) and the coefficient of the long-range mode of the magnetic field \(C_5\). Note that if the induced Chern-Simons coefficient \(\Pi_1\), or the Debye screening coefficient \(\Pi_0\) were zero, there would be no link between \(C_5\) and \((C_6 + eC_7)\). This relation between the long-range modes of \(B\), \(A_0\) and \(a_0\) can be interpreted as a sort of Aharonov-Bohm effect, which occurs in this system at finite temperature. At \(T = 0\), we have \(\Pi_0 = 0\), and the effect disappears.

Up to this point no boundary has been taken into account. Therefore, it is easy to understand that the magnetic long-range mode associated with the coefficient \(C_5\), must be identified with the one found in ref. [16] for the infinite bulk using a different approach. However, as it is shown below, when a constant and uniform magnetic field is perpendicularly applied at the boundaries of a two-dimensional sample, this mode cannot propagate (i.e. \(C_5 \equiv 0\)) within the sample. This result is crucial for the existence of Meissner effect in this system.

In order to determine the unknown coefficients we need to use the boundary conditions. Henceforth we consider that the anyon fluid is confined to the strip \(-\infty < y < \infty\) with boundaries at \(x = -L\) and \(x = L\). The external magnetic field will be applied from the vacuum at both boundaries \((-\infty < x \leq -L, \ L \leq x < \infty)\).

The boundary conditions for the magnetic field are \(B (x = -L) = B (x = L) = \overline{B}\) (\(\overline{B}\) constant). Because no external electric field is applied, the boundary conditions for this field are, \(E (x = -L) = E (x = L) = 0\). Using them and assuming \(L \gg \lambda_1, \lambda_2\) (\(\lambda_1 = 1/\xi_1\),
\[ \lambda_2 = 1/\xi_2 \), we find the following relations that give \( C_{1,2,3,4} \) in terms of \( C_5 \),

\[
C_1 = Ce^{-L\xi_1}, \quad C_2 = -C_1, \quad C_3 = -Ce^{-L\xi_2}, \quad C_4 = -C_3, \quad C = \frac{C_5 - B}{\gamma_1 - \gamma_2} \tag{3.25}
\]

### IV. STABILITY CONDITION FOR THE INFINITE-STRIP SAMPLE

After using the boundary conditions, we can see from (3.25) that they were not sufficient to find the coefficient \( C_5 \). In order to totally determine the system magnetic response we have to use another physical condition from where \( C_5 \) can be found. Since, obviously, any meaningful solution have to be stable, the natural additional condition to be considered is the stability equation derived from the system free energy. With this goal in mind we start from the free energy of the infinite-strip sample

\[
\mathcal{F} = \frac{1}{2} \int_{-L'}^{L'} dy \int_{-L}^{L} dx \left\{ (E^2 + B^2) + \frac{N}{\pi} a_0 b - \Pi_0 (eA_0 + a_0)^2 - \Pi_0' (eE + \mathcal{E})^2 - 2\Pi_1 (eA_0 + a_0)(eB + b) + \Pi_2 (eB + b)^2 \right\} \tag{4.1}
\]

where \( L \) and \( L' \) determine the two sample’s lengths.

Using the field solutions (3.17), (3.21)-(3.23) with coefficients (3.25), it is found that the leading contribution to the free-energy density \( f = \frac{\mathcal{F}}{A} \), \((A = 4LL' \text{ being the sample area})\) in the infinite-strip limit \((L \gg \lambda_1, \lambda_2, L' \to \infty)\) is given by

\[
f = C_5^2 - \Pi_0 (C_6 + eC_7)^2 + e^2 \Pi_2 C_5^2 - 2e\Pi_1 (C_6 + eC_7) C_5 \tag{4.2}
\]

Taking into account the constraint equation (3.24), the free-energy density (4.2) can be written as a quadratic function in \( C_5 \). Then, the value of \( C_5 \) is found, by minimizing the corresponding free-energy density

\[
\frac{\delta f}{\delta C_5} = \left[ \Pi_0 + e^2 \Pi_1^2 + e^2 \Pi_0 \Pi_2 \right] \frac{C_5}{\Pi_0} = 0, \tag{4.3}
\]

to be \( C_5 = 0 \).
This result implies that the long-range mode cannot propagate within the infinite-strip when a uniform and constant magnetic field is perpendicularly applied at the sample’s boundaries.

We want to point out the following fact. The same property of the finite temperature polarization operator component $\Pi_{00}$ that is producing the appearance of a long-range mode in the infinite bulk, is also responsible, when it is combined with the boundary conditions, for the non-propagation of this mode in the bounded sample. It is known that the nonvanishing of $\Pi_0$ at $T \neq 0$ (or equivalently, the presence of a pole $\sim 1/k^2$ in $\Pi_{00}$ at $T \neq 0$) guarantees the existence of a long-range mode in the infinite bulk [10]. On the other hand, however, once $\Pi_0$ is different from zero, we can use the constraint (3.24) to eliminate $C_6 + eC_7$ in favor of $C_5$ in the free-energy density of the infinite strip. Then, as we have just proved, the only stable solution of this boundary-value problem, which is in agreement with the boundary conditions, is $C_5 = 0$. Consequently, no long-range mode propagates in the bounded sample.

In the zero temperature limit ($\beta \to \infty$), because $\Pi_0 = 0$, it is straightforwardly obtained from (3.24) that $C_5 = 0$ and no long-range mode propagates.

At $T \neq 0$, taking into account that $C_5 = 0$ along with eq. (3.25) in the magnetic field solution (3.21), we can write the magnetic field penetration as

$$B(x) = B_1(T) \left( e^{-(x+L)\xi_1} + e^{(x-L)\xi_1} \right) + B_2(T) \left( e^{-(x+L)\xi_2} + e^{(x-L)\xi_2} \right)$$

where,

$$B_1(T) = \frac{\gamma_1}{\gamma_1 - \gamma_2} B, \quad B_2(T) = \frac{\gamma_2}{\gamma_2 - \gamma_1} B$$

For densities $n_e \ll m^2$, the coefficients $B_1$ and $B_2$ can be expressed, in the low temperature approximation ($T \ll \omega_c$), as

$$B_1(T) \simeq -\left( \frac{\pi n_e}{m^2} \right)^{3/2} \left[ 1/m + \frac{5}{2} \beta \exp\left( \frac{\pi n_e \beta}{2m} \right) \right] B,$$

$$B_2(T) \simeq \left[ 1 + \frac{5\pi n_e}{2m^2} \sqrt{\pi n_e \beta} \exp\left( \frac{\pi n_e \beta}{2m} \right) \right] B$$

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Hence, in the infinite-strip sample the applied magnetic field is totally screened within the anyon fluid on two different scales, \( \lambda_1 = 1/\xi_1 \) and \( \lambda_2 = 1/\xi_2 \). At \( T = 200K \), for the density value considered above, the penetration lengths are given by \( \lambda_1 \approx 0.6 \times 10^{-8} \text{cm} \) and \( \lambda_2 \approx 0.1 \times 10^{-4} \text{cm} \). Moreover, taking into account that \( \xi_1 \) increases with the temperature while \( \xi_2 \) decreases (see eqs. (3.19)-(3.20)), and that \( B_1 (T) < 0 \) while \( B_2 (T) > 0 \), it can be shown that the effective penetration length \( \overline{\lambda} \) (defined as the distance \( x \) where the magnetic field falls down to a value \( B(\overline{\lambda}) / B = e^{-1} \)) increases with the temperature as

\[
\overline{\lambda} \approx \overline{\lambda}_0 \left( 1 + \pi \beta \exp \left( -\frac{1}{2} \frac{\pi \beta}{\xi} \right) \right)
\] (4.8)

where \( \overline{\lambda}_0 = \sqrt{m/n_e e^2} \) and \( \pi = \pi n_e / m \). At \( T = 200K \) the effective penetration length is \( \overline{\lambda} \approx 10^{-5} \text{cm} \).

It is timely to note that the presence of explicit (proportional to \( N \)) and induced (proportional to \( \Pi_1 \)) Chern-Simons terms in the anyon effective action (3.1) is crucial to obtain the Meissner solution (4.4). If the Chern-Simons interaction is disconnected (\( N \to \infty \) and \( \Pi_1 = 0 \)), then \( a = 0 \), \( d = 1 + e^2 \Pi_0' \neq 0 \) and \( c = e^2 \Pi_0 \neq 0 \) in eq. (3.16). In that case the solution of the field equations within the sample are \( E = 0 \), \( B = B \overline{B} \). That is, we regain the QED in (2+1)-dimensions, which does not exhibit any superconducting behavior.

V. HIGH TEMPERATURE NON-SUPERCONDUCTING PHASE

We have just found that the charged anyon fluid confined to an infinite strip exhibits Meissner effect at temperatures lower than the energy gap \( \omega_c \). It is natural to expect that at temperatures larger than the energy gap this superconducting behavior should not exist. At those temperatures the electron thermal fluctuations should make available the free states existing beyond the energy gap. As a consequence, the charged anyon fluid should not be a perfect conductor at \( T \gg \omega_c \). A signal of such a transition can be found studying the magnetic response of the system at those temperatures.

As can be seen from the magnetic field solution (4.4), the real character of the inverse length scales (3.18) is crucial for the realization of the Meissner effect. At temperatures
much lower than the energy gap this is indeed the case, as can be seen from eqs. (3.19) and (3.20).

In the high temperature ($T \gg \omega_c$) region the polarization operator coefficients are given by eq. (3.9). Using this approximation together with the assumption $n_e \ll m^2$, we can calculate the coefficients $a$, $c$ and $d$ that define the behavior of the inverse length scales,

$$a \simeq \pi^2 \Pi_0' \Pi_2$$

$$c \simeq e^2 \Pi_0$$

$$d \simeq -1$$

Substituting with (5.1)-(5-3) in eq. (3.18) we obtain that the inverse length scales in the high-temperature limit are given by

$$\xi_1 \simeq e \sqrt{m/2\pi} \left( \tanh \frac{\beta \mu}{2} + 1 \right)^{\frac{1}{2}}$$

$$\xi_2 \simeq i \left[ 24 \sqrt{\frac{2m}{\beta}} \cosh \frac{\beta \mu}{2} \left( \tanh \frac{\beta \mu}{2} + 1 \right)^{-\frac{1}{2}} \right]$$

The fact that $\xi_2$ becomes imaginary at temperatures larger than the energy gap, $\omega_c$, implies that the term $\gamma_2 \left( C_4 e^{x\xi_2} - C_3 e^{-x\xi_2} \right)$ in the magnetic field solution (3.21) ceases to have a damping behavior, giving rise to a periodic inhomogeneous penetration. Therefore, the fluid does not exhibit a Meissner effect at those temperatures since the magnetic field will not be totally screened. This corroborate our initial hypothesis that at $T \gg \omega_c$ the anyon fluid is in a new phase in which the magnetic field can penetrate the sample.

We expect that a critical temperature of the order of the energy gap ($T \sim \omega_c$) separates the superconducting phase ($T \ll \omega_c$) from the non-superconducting one ($T \gg \omega_c$). Nevertheless, the temperature approximations (3.8) and (3.9) are not suitable to perform the calculation needed to find the phase transition temperature. The field solutions in this new non-superconducting phase is currently under investigation. The results will be published elsewhere.
VI. CONCLUDING REMARKS

In this paper we have investigated the magnetic response at finite temperature of a charged anyon fluid confined to an infinite strip. The charged anyon fluid was modeled by a (2+1)-dimensional MCS theory in a many-particle ($\mu \neq 0$, $\bar{b} \neq 0$) ground state. The particle energy spectrum of the theory exhibits a band structure given by different Landau levels separated by an energy gap $\omega_c$, which is proportional to the background Chern-Simons magnetic field $\bar{b}$. We found that the energy gap $\omega_c$ defines a scale that separates two phases: a superconducting phase at $T \ll \omega_c$, and a non-superconducting one at $T \gg \omega_c$.

The total magnetic screening in the superconducting phase is characterized by two penetration lengths corresponding to two short-range eigenmodes of propagation of the electromagnetic field within the anyon fluid. The existence of a Meissner effect at finite temperature is the consequence of the fact that a third electromagnetic mode, of a long-range nature, which is present at finite temperature in the infinite bulk [16], does not propagate within the infinite strip when a uniform and constant magnetic field is applied at the boundaries. This is a significant property since the samples used to test the Meissner effect in high-$T_c$ superconductors are bounded.

It is noteworthy that the existence at finite temperature of a Debye screening ($\Pi_0 \neq 0$) gives rise to a sort of Aharonov-Bohm effect in this system with Chern-Simons interaction ($N$ finite, $\Pi_I \neq 0$). When $\Pi_0 \neq 0$, the field combination $a_0 + eA_0$ becomes physical because it enters in the field equations in the same foot as the electric and magnetic fields (see eq. (3.12)). A direct consequence of this fact is that the coefficient $C_5$, associated to the long-range mode of the magnetic field, is linked to the coefficients $C_6$ and $C_7$ of the zero components of the potentials (see eq. (3.24)).

When $T = 0$, since $\Pi_0 = 0$ and $\Pi_I \neq 0$, eq. (3.24) implies $C_5 = 0$. That is, at zero temperature the long-range mode is absent. This is the well known Meissner effect of the anyon fluid at $T = 0$. When $T \neq 0$, eq. (3.24) alone is not enough to determine the value of $C_5$, since it is given in terms of $C_6$ and $C_7$ which are unknown. However, when eq.
(3.24) is taken together with the field configurations that satisfy the boundary conditions for the infinite-strip sample (eqs. (3.17), (3.21)-(3.23) and (3.25)), and with the sample stability condition (4.3), we obtain that $C_5 = 0$. Thus, the combined action of the boundary conditions and the Aharonov-Bohm effect expressed by eq. (3.24) accounts for the total screening of the magnetic field in the anyon fluid at finite temperature.

Finally, at temperatures large enough ($T \gg \omega_c$) to excite the electrons beyond the energy gap, we found that the superconducting behavior of the anyon fluid is lost. This result was achieved studying the nature of the characteristic lengths (3.18) in this high temperature approximation. We showed that in this temperature region the characteristic length $\xi_2$ becomes imaginary (eq. (5.5)), which means that a total damping solution for the magnetic field does not exist any more, and hence the magnetic field penetrates the sample.

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