Optimum operating regimes for the ideal wind turbine

Valery L. Okulov 1,2, Jens N. Sørensen 1

1 Department of Mechanical Engineering, Technical University of Denmark, DK-2800 Lyngby, Denmark
2 Institute of Thermophysics, SB RAS, Lavrentyev Ave. 1, Novosibirsk, 630090, Russia

E-mail: vlo@mek.dtu.dk

Abstract. We here present new results on the classical work of the optimum rotor. The emphasis is put vortex theory for which we have developed a new analytical method to determine the loading on an optimum wind turbine rotor. The introduction of the work is a repetition of results using momentum theory. This is included in order to validate and compare the new model at simplified situations, such as a rotor operating without swirl and/or a rotor with infinite many blades.

1. Introduction

In the following we consider various classical theories for the optimum rotor. First we show the Betz limit using axial momentum theory. Next we consider a rotor with an infinite number of blades using general momentum theory. Finally, we analyse a realistic rotor with a finite number of blades using a new solution of the Goldstein function.

1.1. Optimum Rotor: Rankine-Froude Theory. We first consider the simple axial momentum theory as it originated by Rankine [1], W. Froude [2] and R.E. Froude [3]. Here we consider axial flow past an actuator disk representing the axial load on a rotor. Denoting by \( V_0 \) the undisturbed wind speed and by \( v_\alpha \) the velocity in the rotor plane, the axial interference factor is defined as

\[
a = 1 - \frac{v_\alpha}{V_0},
\]

From one-dimensional axial momentum theory we get the following expression for the axial load (thrust) and power extraction

\[
T = 2\rho A v_\alpha (V_0 - v_\alpha) = 2\rho AV_0^2 a(1-a),
\]

\[
P = u_\alpha T = 2\rho AV_0^3 a(1-a)^2,
\]

the \( \rho \) is the density of air, \( A \) denotes the rotor area and \( u_\alpha = V_0 - v_\alpha \) is the axial velocity induced by wake in the rotor plane. Introducing dimensionless power and thrust coefficients, \( C_P \) and \( C_T \), defined as, respectively,
the maximum power that can be extracted from a stream of air contained in an area equivalent to that swept out by the rotor corresponds to the maximum value of the power coefficient. We get that

$$C_{p_{\text{max}}} = \frac{16}{27} = 0.59327$$

for

$$a = \frac{1}{3}.$$ (6)

This result is usually referred to as the Betz limit and states the upper maximum for power extraction. However, it does not include the losses due to rotation of the wake and therefore it represents a conservative upper maximum.

1.2. Optimum Rotor: General Momentum Theory. Utilizing general momentum theory Glauert [4] developed a simple model for the optimum rotor that included rotational velocities. In this approach Glauert treated the rotor as a rotating actuator disk, corresponding to a rotor with an infinite number of blades. Denoting by \( \Omega \) the angular velocity of the rotor blade and by \( u_{\theta} \) the azimuthal velocity in the rotor plane, we define the azimuthal interference factor as

$$a' = \frac{u_{\theta}}{\Omega r}.$$ (7)

In figure 1 we show of the velocity vectors for a blade element at radial position \( r \). The local relative velocity, \( U_0 \), is perpendicular to the induced velocity \( \tilde{u} = (u_z, u_{\theta}) \) and the lift \( \tilde{L} \) (ignoring drag). Employing Euler’s turbine equation, we get

$$P = \rho \Omega \int_r u_{\theta} u_{z} dA,$$ (8)

where \( u_{\theta} = u_{\theta}(r) \) is the azimuthal velocity in the wake. For a rotor with an infinite number of blades it can be shown that the induced velocities in the rotor plane approximately take half the values of those in the wake \( (u_{\theta} = 2u_{\theta}) \). Considering the flow through annular elements of area \( dA = 2\pi r dr \), we get

$$P = 4\pi \rho \Omega^2 V_0 \int (1-a') r^3 dr,$$ (9)

\[ \text{Figure 1. Velocity triangle in the rotor plane of a wind turbine} \]
\[ \text{Figure 2 Sketch of undeformed helical screw surface corresponding to the wake of an optimum rotor.} \]
Introducing the tip speed ratio, $\lambda = \Omega R / V_0$, and dimensionless radius, $x = r / R$, where $R$ is the radius of the rotor, the power coefficient reads

$$C_p = \frac{8}{\lambda^2} \int_{x=0}^{1} (1 - a) a' x^3 dx.$$  \hspace{1cm} (10)

Since the integral involves two dependent variables, we need to derive yet a relation between $a$ and $a'$ in order determine the conditions under which the integral attains a maximum. This is accomplished by inspection of the velocity triangle in figure 1. Since the total induced velocity is normal to the relative velocity, we get

$$\tan \Phi = \frac{(1-a)V_0}{(1+a')r\Omega} = \frac{a'r\Omega}{aV_0},$$  \hspace{1cm} (11)

resulting in the following relation

$$\lambda^2 x^3 a'(1+a') - a(1-a) = 0.$$  \hspace{1cm} (12)

Now the task is to optimize equation (10) subject to the constraint (12). Using variational calculus (see Glauert [4] or Wilson and Lissamann [5]), we get

$$a' = \frac{1-3a}{4a-1},$$  \hspace{1cm} (13)

Such that

$$a' x^2 \lambda^2 = (1-a)(4a-1).$$  \hspace{1cm} (14)

From equations (13) and (14) it is readily seen that the operating range for an optimum rotor is $1/4 \leq a \leq 1/3$. The variation in $a$, $a'$, $a' x^2 \lambda^2$ and $x$ are given in table 1 (reproduced from [5]).

| $a$  | $a'$  | $a' x^2 \lambda^2$ | $\lambda x$ |
|------|-------|---------------------|-------------|
| 0.25 | $\infty$ | 0                  | 0           |
| 0.27 | 2.375 | 0.0584              | 0.157       |
| 0.29 | 0.812 | 0.1136              | 0.374       |
| 0.31 | 0.292 | 0.1656              | 0.753       |
| 0.33 | 0.031 | 0.2144              | 2.630       |
| 1/3  | 0     | 0.2222              | $\infty$    |

The variation in $\lambda$ and $C_{P_{\text{max}}}$ for the optimum actuator disk are given in table 2 (reproduced from [5]).

| $\lambda$ | $C_{P_{\text{max}}}$ |
|-----------|----------------------|
| 0.5       | 0.288                |
| 1.0       | 0.416                |
| 1.5       | 0.480                |
| 2.0       | 0.512                |
| 2.5       | 0.532                |
| 5.0       | 0.570                |
| 7.5       | 0.582                |
| 10.0      | 0.593                |
Combining equations (12) and (13) and integrating equation (10), the power coefficient may be obtained as a function of tip speed ratio. The result is shown in table 2, from which it is seen that the power coefficient approaches 0.593 for large tip speed ratios.

It shall be mentioned that these results are valid only for a rotor with an infinite number of blades and the analysis is based on the assumption that the rotor can be optimized by considering each blade element independently of the remaining blade elements.

2. Optimum Rotor: Present Development of Vortex Theory

The flow over a real rotor with a finite number of blades is very different from the properties of the flow models used in the previous to describe the optimum rotor. Indeed, important phenomena such as tip losses and azimuthal dependencies of the induced velocities are neglected in the momentum theory of the optimum rotor. An alternative model is the vortex theory in which each of the rotor blades is represented by a bound vortex line. Using this technique, the bound vorticity serves to produce the local lift on the blades while the trailing vortices induce the velocity field in the rotor plane and in the wake. The induced velocity field is determined using the induction low of Biot-Savart. The fundamental expressions for the forces acting on a rotor blade is most conveniently expressed by the Kutta–Joukowsky theorem, which in vector form reads

\[ dL = \rho U_0 \times \Gamma dr, \tag{15} \]

where \( dL \) is the lift force on a blade element of radial dimension \( dr \), \( U_0 \) is the resultant relative velocity and \( \Gamma \) is the bound circulation. Let \( u_\theta \) and \( u_z \) be the circumferential and axial components of the velocity, induced by the free vortex wake behind the rotor, at a blade element in the rotor plane. Then, in accordance with figure 1, we can write the local torque, \( dQ \), and the local thrust \( dT \) of the rotor as follows

\[ dQ = \rho \Gamma (V_0 - u_\theta) r dr, \tag{16} \]
\[ dT = \rho \Gamma (\Omega r + u_\theta) d\theta. \tag{17} \]

Integrating these quantities along each blade and summing up, we get the following expressions for power and thrust

\[ P = \rho B \Omega \int_0^R \Gamma (V_0 - u_\theta) r dr, \tag{18} \]
\[ T = \rho B \int_0^R \Gamma (\Omega r + u_\theta) d\theta, \tag{19} \]

where \( B \) is the number of blades. In order to derive the relations for an optimum rotor we impose a small but arbitrary continuous circulation perturbation \( \epsilon \Delta \Gamma(r) \) on an existing distribution of circulation. Thus, the additional torque and thrust read

\[ \delta Q = \rho \left( V_0 - u_\theta \right) (r) \epsilon \Delta \Gamma(r) r dr, \tag{20} \]
\[ \delta T = \rho \left( \Omega r + u_\theta \right) (r) \epsilon \Delta \Gamma(r) d\theta, \tag{21} \]

where \( \epsilon \) is small constant. We now seek the conditions under which a change in circulation from the initial distribution, at constant thrust does not change the power yield. Employing variational calculus the change in torque and thrust read
\[
\delta Q = \varepsilon \rho \int_0^r \left( V_0 - u_\phi \right) \Delta \Gamma(r) r dr , \tag{22}
\]

\[
\delta T = \varepsilon \rho \int_0^r \left( \Omega r + u_\theta \right) \Delta \Gamma(r) dr . \tag{23}
\]

Maintaining the torque unchanged, i.e. keeping \( \delta T = 0 \), we seek the conditions for which a maximum is achieved for the total power. This can only be obtained when the perturbed power yield is zero, i.e. for \( \delta P = \Omega \delta Q = 0 \). Instead of going through a formal derivation, Betz [6] showed that this condition is satisfied if

\[
r \tan \Phi_0 = r \left[ V_0 - u_\phi (r) \right] \left[ \Omega r + u_\theta (r) \right] = \Lambda , \tag{24}
\]

where \( \Lambda \) is a constant that does not depend on the radial distance. By inspection it is readily seen that \( \delta T = \delta P = 0 \) while \( \delta Q / \delta T = \Lambda \), indicating that an extremum has been achieved. From equation (24) it is seen that maximum efficiency is obtained when the pitch of the trailing vortices is constant and each trailing vortex sheet translates backward as an undeformed regular helicoidal surface.

Using potential flow theory and infinite series of Bessel functions Goldstein [7] derived the circulation distribution corresponding to an undeformed helical screw surface. In the following we employ Goldstein’s circulation function \( G(r) \) as it was later used in the text book by Theodorsen (see [8] and [9]). This function expresses the optimum circulation corresponding to a helicoidal vortex sheet translated with constant speed, and is in dimensionless form defined as

\[
G(r) = \Gamma(r) / hw = B \Omega / 2 \pi w (V_0 + w) , \tag{25}
\]

where \( h = 2 \pi r \tan \Phi / B = 2 \pi (V_0 + w) / B \Omega \) is the axial distance between adjacent turns of the helicoidal sheets and \( w \) the backward velocity of the vortex system with respect to the surrounding fluid. It should be pointed out that Goldstein, assuming light loading, wrote \( V_0 \) where Theodorsen put \( V_0 + w \). The distribution of \( G(r) \) will be derived in the following, but first we derive an expression for translated axial speed \( w \). In figure 3 we sketch the velocity triangle in a planar cut of the translating helicoidal screw surface.

![Figure 3](image)

**Figure 3** Sketch of the velocity triangle in a planar cut of the helicoidal screw surface.

Since it is translated with constant axial speed \( w \), the induced velocity comprises only the component \( w \cos \Phi \) that is ‘pushed’ normal to the screw surface. The induced velocities are therefore given as

\[
u_\theta = -w \cos \Phi \sin \Phi \quad \text{and} \quad u_z = w \cos^2 \Phi , \tag{26}
\]
where Φ is the angle between the vortex sheet and the rotor plane (see figure 3). Specification of the trigonometric functions in (26) let us to find a correlation between \( u_z, u_\theta \) and \( w \), given by Goldstein [7]

\[
  u_\theta = -\frac{wx}{l}\sqrt{l^2 + x^2} \quad \text{and} \quad u_z = \frac{wx^3}{l^2 + x^2},
\]

where \( x = r/R_0 \) and \( 2\pi l/R_0 \) are the non-dimensional radius and the distance between turns of the fixed helicoidal sheet, respectively. In fact, equations (26) and (27) fix the ratio \( u_\theta/u_z = -l/x = -\tan \Phi \) which, for a given radius, is constant everywhere on the helicoidal sheets, although the induced velocities may change along wake.

Inserting the Goldstein function (25) into equations (18) and (19), and introducing dimensionless variables equations (4) and (5), we get

\[
  C_p = 2\sigma \left( \frac{w}{V_0} - \bar{w} I_1 - \bar{w} I_2 \right) \quad \text{and} \quad C_r = 2\sigma \left( \bar{w} \left( 1 + \frac{w}{V_0} \right) I_1 - \bar{w} I_3 \right)
\]

where \( \bar{w} = \frac{w}{V_0} \); \( \sigma = A_{\text{wake}}/A \) and \( I_1 = 2 \int_0^1 G(x) \, dx \); \( I_2 = 2 \int_0^1 \frac{G(x)}{x^2 + l^2} \, dx \); \( I_3 = 2 \int_0^1 \frac{G(x)}{x^3} \, dx \).

Differentiating \( C_p \) with respect to \( \bar{w} \) yields the maximum value \( C_{p,\text{max}} \), resulting in

\[
  \bar{w} = \frac{1}{3 I_3} \left( \sqrt{I_2^2 + I_3 I_1} - I_1 - I_3 \right).
\]

It now remains to compute the distribution of circulation or the Goldstein circulation function \( G(x) \) for the vortex sheet which corresponds to the helicoidal configuration and to relate this to the distribution at the propeller itself. Goldstein [7] was the first to derive a solution to the general potential flow problem posed by Betz [6]. He considered two-bladed and four-bladed propellers and expressed the function \( G(x) \) via a trigonometrical series of Kaptain type. However, such are the difficulties of computation, even after the way to a solution was found, that Theodorsen resorted to the use of a rheoelectrical analog to evaluate the circulation function, but unfortunately without big success [8]. Accurate tabulated values of a function related to the Goldstein function and covering a wide (but finite) range of parameters became available with an extensive mathematical and computational effort by Tibery & Wrench [10]. Expressions for the torque and thrust of a propeller operating at maximum efficiency are given by (28) and (29). Thus, to determine power and thrust as function of tip speed ratio, and the corresponding circulation, we need to derive an expression for the Goldstein circulation function \( G \). For this purpose we employ Okulov’s representation of the velocity field induced by a single helical vortex filament with strength \( \gamma \), radius \( r_0 \) and pitch \( 2\pi l \) (see Okulov [11]). In cylindrical coordinates \((r, \theta, z)\) the corresponding velocity components read

\[
  u_r \equiv -\frac{\gamma}{2\pi r} \frac{x}{l} \sqrt{l^2 + r^2} \left( l^2 + r_0^2 \right) \frac{e^{ix} + e^{ix}}{2} \left[ \frac{2l^2 + 9r_0^2}{(l^2 + r^2)^{3/2}} - \frac{2l^2 + 9r^2}{(l^2 + r_0^2)^{3/2}} \right] \ln \left( 1 - e^{ix} \right),
\]

\[
  u_\theta \equiv \frac{\gamma}{2\pi l} \left( 1 + \sqrt{l^2 + r^2} \right) \left( l^2 + r_0^2 \right) \frac{e^{ix} + e^{-ix}}{2} \left[ \frac{2l^2 + 9r_0^2}{(l^2 + r^2)^{3/2}} - \frac{2l^2 + 9r^2}{(l^2 + r_0^2)^{3/2}} \right] \ln \left( 1 - e^{ix} \right),
\]

\[
  u_z \equiv (u_0 - u_r) l/r, \quad \text{and} \quad u_0 = \gamma/2\pi l.
\]

\[
  \text{where} \quad x = r/R_0 \quad \text{and} \quad 2\pi l/R_0 \quad \text{are the non-dimensional radius and the distance between turns of the fixed helicoidal sheet, respectively.}
\]

\[
  \text{In fact, equations (26) and (27) fix the ratio} \quad u_\theta/u_z = -l/x = -\tan \Phi \quad \text{which, for a given radius, is constant everywhere on the helicoidal sheets, although the induced velocities may change along wake.}
\]

\[
  \text{Inserting the Goldstein function (25) into equations (18) and (19), and introducing dimensionless variables equations (4) and (5), we get}
\]

\[
  C_p = 2\sigma \left( \frac{w}{V_0} - \bar{w} I_1 - \bar{w} I_2 \right) \quad \text{and} \quad C_r = 2\sigma \left( \bar{w} \left( 1 + \frac{w}{V_0} \right) I_1 - \bar{w} I_3 \right)
\]

\[
  \text{where} \quad \bar{w} = \frac{w}{V_0} \; \sigma = A_{\text{wake}}/A \quad \text{and} \quad I_1 = 2 \int_0^1 G(x) \, dx \; I_2 = 2 \int_0^1 \frac{G(x)}{x^2 + l^2} \, dx \; I_3 = 2 \int_0^1 \frac{G(x)}{x^3} \, dx
\]

\[
  \text{Differentiating} \quad C_p \quad \text{with respect to} \quad \bar{w} \quad \text{yields the maximum value} \quad C_{p,\text{max}} \quad \text{resulting in}
\]

\[
  \bar{w} = \frac{1}{3 I_3} \left( \sqrt{I_2^2 + I_3 I_1} - I_1 - I_3 \right)
\]

\[
  \text{It now remains to compute the distribution of circulation or the Goldstein circulation function} \quad G(x) \quad \text{for the vortex sheet which corresponds to the helicoidal configuration and to relate this to the distribution at the propeller itself. Goldstein [7] was the first to derive a solution to the general potential flow problem posed by Betz [6]. He considered two-bladed and four-bladed propellers and expressed the function} \quad G(x) \quad \text{via a trigonometrical series of Kaptain type. However, such are the difficulties of computation, even after the way to a solution was found, that Theodorsen resorted to the use of a rheoelectrical analog to evaluate the circulation function, but unfortunately without big success [8]. Accurate tabulated values of a function related to the Goldstein function and covering a wide (but finite) range of parameters became available with an extensive mathematical and computational effort by Tibery & Wrench [10]. Expressions for the torque and thrust of a propeller operating at maximum efficiency are given by (28) and (29). Thus, to determine power and thrust as function of tip speed ratio, and the corresponding circulation, we need to derive an expression for the Goldstein circulation function} \quad G \quad \text{. For this purpose we employ Okulov’s representation of the velocity field induced by a single helical vortex filament with strength} \quad \gamma \quad \text{radius} \quad r_0 \quad \text{and pitch} \quad 2\pi l \quad \text{(see Okulov [11]). In cylindrical coordinates} \quad (r, \theta, z) \quad \text{the corresponding velocity components read}
\]

\[
  u_r \equiv -\frac{\gamma}{2\pi r} \frac{x}{l} \sqrt{l^2 + r^2} \left( l^2 + r_0^2 \right) \frac{e^{ix} + e^{ix}}{2} \left[ \frac{2l^2 + 9r_0^2}{(l^2 + r^2)^{3/2}} - \frac{2l^2 + 9r^2}{(l^2 + r_0^2)^{3/2}} \right] \ln \left( 1 - e^{ix} \right),
\]

\[
  u_\theta \equiv \frac{\gamma}{2\pi l} \left( 1 + \sqrt{l^2 + r^2} \right) \left( l^2 + r_0^2 \right) \frac{e^{ix} + e^{-ix}}{2} \left[ \frac{2l^2 + 9r_0^2}{(l^2 + r^2)^{3/2}} - \frac{2l^2 + 9r^2}{(l^2 + r_0^2)^{3/2}} \right] \ln \left( 1 - e^{ix} \right),
\]

\[
  u_z \equiv (u_0 - u_r) l/r, \quad \text{and} \quad u_0 = \gamma/2\pi l.
\]
Where $\chi = \theta - \frac{z}{l}$ and a double sign notation (“±” or “±”) is used in which the upper sign corresponds to the case of $r < r_0$, and the lower sign to $r > r_0$. The exponential term is defined as follows:

$$e^l = \frac{r'}{r_0} = \frac{r}{r_0} \left(1 + \sqrt{l^2 + r_0^2}\right) \exp\left(\sqrt{l^2 + r^2}\right),$$

where $x' = x \exp\left(\sqrt{l^2 + x^2}\right)\left(1 + \sqrt{l^2 + x^2}\right)$ may be interpreted as the radial coordinate in the disturbed space. Note, that if $l \to \infty$, that is the helical filament straightens to become rectilinear, the limit values of equation (30) tend to the velocity field induced by a point vortex.

Although the solution to equation (30) may seem somewhat complicated, it is a simple representation of the original expressions of Goldstein. It should be noted that the present form of the equations only give the induction from a single helical vortex, but by superposing the solution from a series of vortices emanating from the full span of the rotor blades a complete solution can be obtained. In the present work we apply 100 discrete helical vortex filaments. To validate the equations the results were compared to the computations by Tibery & Wrench [10] in figure 4.
The excellent correlations show that the problem can be successfully solved by using (30) and we now have the tools to evaluate optimum rotor at all operating conditions (see, for example, figure 5), and not only for the parameter range computed by Tibery & Wrench. Thus, at any given value of the wake pitch, we can find the Goldstein circulation function $G(x)$ and calculate the integrals introduced in (28) and (29). Finally, to determine the optimum, it is needed to take into account the expansion of the wake. This we include in the expression for $C_P$ by the expansion ratio $\sigma$. If we introduce the average value of the velocity deficit in the wake as $\langle u_z \rangle = 2 \langle u_z \rangle = 2 \langle a \rangle V_0$ the relationship between $A_{wake}$ and $A$ in rotor plane can be estimated from the conservation of mass as

$$\sigma = (1 - \langle u_z \rangle)/(1 - \langle u_z \rangle) = (1 - 2 \langle a \rangle)/(1 - \langle a \rangle). \quad (31)$$

The velocity interference factor $\langle a \rangle$ in (31) can be found as the total contribution from the induction of all helix filaments in the wake (see, e.g. [11])

$$\langle a \rangle = \frac{\langle G \rangle}{hV_0} = \bar{\langle a \rangle} \quad (32)$$

In Figure 6 the dependence of the deficit velocity interference factor and ratio of the area between wake and rotor plane are shown graphically as functions of $1/h$ for different numbers of blades.

Figure 7 shows the power and thrust coefficients for different numbers of blades, given by (29), using of (31) and (32), for different operating regimes of a wind turbine with ideal load distribution.

3. Conclusion

A new analytical method to determine the loading on an optimum wind turbine rotor has been developed. The method enables to determine the optimum circulation distribution from Goldstein’s function at all operating conditions. The proposed procedure for calculation of Goldstein’s function and corresponding torque and thrust coefficients completes the traditional theory as it was formulated by Theodorsen [8]. The optimum characteristics are given as a function the pitch of the wake. In a continuation of the work this will be related to the actual tip speed ratio of the rotor and to practical guide lines for how to design an optimum rotor.
Figure 6. Dependence of the velocity interference factor $\langle a \rangle$ and $\sigma$ as functions of $1/h$ for different numbers of blades: $B = 1$(red); 2(blue); 3 (green); 10(pink); and 20(cyan); dashed black line shows the Betz value $a = 1/3$ and points indicates the data of Table 2.

Figure 7 Comparison the optimum power, $C_P$, and thrust, $C_T$, coefficients by Betz limit (dashed lines) and momentum theory (points) and present theory (color lines) as functions of $1/h$ for different numbers of blades: $B = 1$(red); 2(blue); 3 (green); 10(pink); and 20(cyan).

References
[1] Rankine, W.J. (1865); Transactions, Institute of Naval Architects, vol. 6, p. 13.
[2] Froude, W. (1878); Transactions, Institute of Naval Architects, vol. 19, p. 47.
[3] Froude, R.E. (1889); Transactions, Institute of Naval Architects, vol. 30, p. 390.
[4] Glauert, H. (1935); ‘Aerodynamic Theory’ (W.F. Durand, Editor-in-chief), vol. 6, Division L, p. 324, Julius Springer, Berlin.
[5] Wilson, R.E. and Lissaman, P.B.S. (1974); ‘Applied Aerodynamics of Wind Power Machines’, Oregon State University.
[6] Betz, A. (1919); ‘Schaubenpropeller mit Geringstem Energieverlust’, Dissertation, Gottingen.
[7] Goldstein, S. (1929); Proc R Soc London A; Vol. 123, 440–465.
[8] Theodorsen, T. (1948); Theory of propellers; New York: McGraw-Hill Book Company.
[9] Wald (2006); Progress in Aerospace Sciences; Vol. 42, 85–128
[10] Tibery and Wrench (1964); Report 1534, Applied Mathematics Laboratory, Washington, DC.
[11] Okulov, V.L. (2004); J. Fluid Mech.; Vol. 521, 319 - 342.