Research Article

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Copula-based dependence measures

Abstract: The aim of the present paper is to examine two wide classes of dependence coefficients including several well-known coefficients, for example Spearman’s $\rho$, Spearman’s footrule, and the Gini coefficient. There is a close relationship between the two classes: The second class is obtained by a symmetrisation of the coefficients in the former class. The coefficients of the first class describe the deviation from monotonically increasing dependence. The construction of the coefficients can be explained by geometric arguments. We introduce estimators of the dependence coefficients and prove their asymptotic normality.

Keywords: dependence measures; Spearman’s $\rho$; Spearman’s footrule; estimators for dependence measures

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1 Introduction

From the literature, we know various coefficients based on copulas which describe the dependence of two random variables: Spearman’s $\rho$, Spearman’s footrule, Gini coefficient, Kendall’s $\tau$ etc. It has been proved to be an advantage that these dependence coefficients do not depend on the marginal distributions. Dependence measures based on copulas do not alter the value in case of strictly increasing transformations of the variables. For all the mentioned coefficients, there exist sample versions which are rank statistics. We refer to chapter 5 of the book by Nelsen [9], where a thorough discussion about dependence can be found. More recently, Schmid et al. [13] published a survey paper about the topic with an emphasis on multivariate (i.e. dimension greater than 2) association measures.

The aim of the present paper is to examine two wide classes of dependence coefficients including most of the mentioned coefficients. There is a close relationship between the two classes: The second class (cf. Cifarelli et al. [2]) is obtained by a symmetrisation of the coefficients in the former class. In comparison to other publications on association coefficients, a special feature of our approach is that the construction of the coefficients can be explained by geometric arguments. The coefficients of the first class describe the deviation from monotonically increasing dependence in contrast to other association coefficients. Furthermore, we introduce estimators of a rather simple structure for the dependence coefficients and study their asymptotic behaviour. At the end of the paper, we examine multivariate extensions of the dependence coefficients.

Let $X, Y$ be random variables with joint distribution function $H$. $F$ and $G$ are the distributions of $X, Y$, respectively. It is assumed that $F$ and $G$ are continuous. In view of Sklar’s Theorem (see Sklar [16]), we have

$$H(x, y) = C(F(x), G(y)) \text{ for } x, y \in \mathbb{R}.$$  

Hereby $C$ is the uniquely determined copula of $X, Y$. We are interested in quantities which describe the dependence of $X, Y$ regardless of the marginal distributions. Since the copula is invariant under strictly increasing transformations (see Nelsen [9], Theorem 2.4.3), the dependence coefficients we want to focus on should be

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constructed by using the copula. These coefficients will measure the deviation from a monotonically increasing function; i.e. from the situation, where \( Y = \lambda(X) \) a.s. and \( \lambda \) is strictly increasing.

Next we give some well-studied dependence coefficients:

1. **Spearman’s \( \rho \)**
   \[
   \rho = 12 \int_0^1 \int_0^1 uv \, dC(u,v) - 3, \tag{1}
   \]

2. **Gini coefficient \( \gamma \)**
   \[
   \gamma = 2 \int_0^1 \int_0^1 (|u + v - 1| - |u - v|) \, dC(u,v), \tag{2}
   \]

3. **Kendall’s \( \tau \)**
   \[
   \tau = 4 \int_0^1 \int_0^1 C(u,v) \, dC(u,v) - 1.
   \]

The idea behind the construction of Kendall’s \( \tau \) is that of concordance and differs so from the construction ideas of the present paper. Other coefficients describe the deviation from the independence, for example Schweizer/Wolff’s \( \sigma \) and Hoeffding’s \( \Phi^2 \). These coefficients are analysed in Schweizer and Wolff [14].

The paper is organised as follows: In Section 2 we introduce a general class of bivariate association coefficients and study their properties. This class includes Spearman’s \( \rho \) and Spearman’s footrule. In Section 3 we consider symmetric association coefficients, which were introduced in a similar form by Cifarelli et al. [2]. The definition of the coefficients in Section 3 covers Spearman’s \( \rho \) and the Gini coefficient. In Section 3 we study the properties of the coefficients utilising statements of Section 2. The statistical estimation of the coefficients is discussed in Section 4. We show asymptotic normality of the estimators.

Section 5 is devoted to multivariate measures of dependence. The general concept of measures of multivariate concordance was studied in a series of papers, see Joe [7, 8], Dolati and Úbeda-Flores [3] and Taylor [17]. Recently, a number of publications on multivariate measures appeared generalising ideas from dimension 2. Unfortunately, several different generalisations exist with few connections to each other. Multivariate extensions of Spearman’s \( \rho \) are discussed in Schmid and Schmidt [11], multivariate extensions of Hoeffding’s \( \Phi^2 \) in Gaißer et al. [4]. The papers by Schmid and Schmidt [12] and Úbeda-Flores [18] deal with multivariate versions of Blomqvist’s beta and Spearman’s footrule. A multivariate extension of Gini coefficient was investigated in Behboodian et al. [1]. In the paper by Genest et al. [5], extensions of Spearman’s \( \rho \) to discontinuous distributions are considered. The measurement of the association between random vectors is examined in Grothe et al. [6]. However, the emphasis of our paper is on measures describing the monotonically increasing dependence and having the potential of a geometric interpretation. This is the main difference to other approaches by Joe, Dolati, Úbeda-Flores and Taylor mentioned before. The association coefficients of Section 5 are introduced as multivariate extensions of the coefficients in Section 2. All proofs are deferred to Section 6.

### 2 A class of bivariate coefficients for monotonically increasing dependence

Let \( \psi : [-1, 1] \to [0, \infty) \) be a function with \( \psi(-x) = \psi(x) \), and \( \psi(0) = 0 \). Assume that \( \psi \) is strictly increasing on \([0, 1]\). The generalised coefficient of monotonically increasing dependence of \( X \) and \( Y \) is now defined by:

\[
\zeta_{X,Y} = \zeta(C) = 1 - \int_0^1 \int_0^1 \psi(u-v) \, dC(u,v) \cdot \bar{\psi}^{-1}
= 1 - E[\psi(F(X) - G(Y)) \cdot \bar{\psi}^{-1}], \tag{3}
\]
where
\[
\tilde{\psi} = \int_0^1 \int_0^1 \psi(u - v) \, du \, dv = 2 \int_0^1 (1 - u) \psi(u) \, du.
\] (4)

This identity is proved in Lemma 6.1 in Section 6. Define \( U = F(X) \) and \( V = G(Y) \). Obviously, the random variables \( U \) and \( V \) have continuous uniform distributions on \([0, 1]\), and the random vector \((U, V)\) has distribution \( C \). Recall that \( M(u, v) = \min(u, v) \) is the upper Fréchet-bound of copulas and the corresponding measure has its support on the diagonal \( D = \{(u, v) \in [0, 1]^2 : u = v\} \). Copulas \( C \) and \( M \) are equal iff \( U = V \) almost surely (comonotonicity) which in turn is equivalent to the property that \( Y = \lambda(X) \) a.s. with a strictly increasing function \( \lambda \).

Now some comments on the geometric interpretation. Observe that \(|u-v|/\sqrt{2}\) is the distance of point \((u, v)\) as a realisation of \((U, V)\) to the diagonal \( D \). Point \((U, V)\) occurs with distribution measure \( C \). Therefore the double integral in (3) is just the expectation of the transformed distances of \((U, V)\) \((\tilde{\psi} \) is the transformation function) to the diagonal \( D \) representing monotonically increasing dependence. Hence \( \zeta_{X,Y} \) measures the distance between \( C \) and \( M \) which represents monotonically increasing dependence of \( X \) and \( Y \). On the other hand, we can rewrite the formula of \( \zeta \) as
\[
\zeta_{X,Y} = \int_0^1 \int_0^1 \psi(u - v) \, du \, dv \frac{D(u, v)}{C(u, v) - uv} \cdot \tilde{\psi}^{-1}.
\]

This representation shows that \( \zeta_{X,Y} \) measures also the distance to the independence. The construction of \( \zeta \) is similar to that of correlation coefficients in regression analysis: one minus a ratio of a discrepancy measure and a normalising value. This normalising value (here \( \tilde{\psi} \)) is chosen such that in a special situation, the resulting value is zero. Here the independence represents this special situation.

Let \( W \) be the lower Fréchet-bound: \( W(u, v) = \max\{u + v - 1, 0\} \). We use the following notation for one-sided limits: \( h(x+0) := \lim_{y \to x+0} h(y) \) and \( h(x-0) := \lim_{y \to x-0} h(y) \). Theorem 2.1 provides important properties of the dependence coefficient \( \zeta_{X,Y} \):

**Theorem 2.1.** Assume that \( \psi' \) exists, is absolutely continuous and nondecreasing on \([0, 1]\). Then we have

a) \[
\zeta_{X,Y} = 1 - \left( 2 \int_0^1 \left( \psi(u) - \psi'(0 + 0)C(u, u) \right) \, du - \int_0^1 \int_0^1 \psi''(u - v)C(u, v) \, du \, dv \right) \cdot \tilde{\psi}^{-1}.
\]

b) **Concordance:** \( C_1 \prec C_2 \) implies \( \zeta(C_1) \leq \zeta(C_2) \). Hereby \( C_1 \prec C_2 \) means \( C_1(u, v) \leq C_2(u, v) \) for all \( u, v \in [0, 1] \).

c) **Normalisation I:** \( \zeta_{\min} \leq \zeta_{X,Y} \leq 1 \) where
\[
\zeta_{\min} = \zeta(W) = 1 - \int_0^1 \psi(u) \, du \cdot \tilde{\psi}^{-1}.
\]

d) **Normalisation II:** \( \zeta_{X,Y} = 1, \zeta_{-X,X} = \zeta_{\min} \)

e) **Normalisation III:** The identity \( \zeta_{X,Y} = 1 \) holds iff \( U = V \) a.s. which in turn is equivalent to \( Y = G^{-1}(F(X)) \) a.s. \( (G^{-1}(t) := \inf\{x : G(x) \geq t\} \) is the generalised inverse of \( G \); i.e. \( Y \) is a strictly increasing function of \( X \) in this case.

f) **Normalisation IV:** If \( X \) and \( Y \) are independent, then \( \zeta_{X,Y} = 0 \).

g) **Increasing Transformations:** \( \zeta_{g(X), h(Y)} = \zeta_{X,Y} \) holds for every strictly increasing functions \( g, h \).

h) **Permutations:** \( \zeta_{Y,X} = \zeta_{X,Y} \).
\[ \zeta_{X,Y} = 1 - \mathbb{E}(\psi(F(X) + G(Y) - 1) \cdot \tilde{\psi}^{-1} \]

j) **Continuity:** Let \( \{C_n\} \) be a sequence of copulas tending pointwise to a copula \( C \). Then \( \zeta(C_n) \to \zeta(C) \).

For parts d) to i) of Theorem 2.1, the assumptions on function \( \psi \) stated in this theorem are not needed. Part j) of this theorem holds true under the weaker assumption that \( \psi \) is continuous. The dependence measure fulfills the conditions 1,3,4,6,7, condition 2 concerning \( \zeta_{X,X} \), and the upper bound 1 in the definition of a concordance measure introduced by Scarsini [10], see Nelsen [9], pp.168ff. The coefficient \( \zeta \) represents a weak concordance measure which means that the lower bound \( \zeta_{\text{min}} \) is not necessarily \(-1\). In Theorem 2.1, the assumption on \( \psi \) can be slightly weakened but in the present version of Theorem 2.1, formula a) can be written in a more convenient form. Moreover, all interesting cases are covered by the actual assumptions of Theorem 2.1.

**Example 2.2.** \( \psi(u) = u^2 \): Then we have \( \tilde{\psi} = \frac{1}{6} \) and

\[
\zeta_{X,Y} = 1 - 6 \int_0^1 \int_0^1 (u - v)^2 \, dC(u,v) \\
= 12 \int_0^1 \int_0^1 uv \, dC(u,v) - 3 = \rho.
\]

In this case, \( \zeta_{X,Y} \) coincides with Spearman’s rho in (1). Hence Spearman’s \( \rho \) uses a quadratic distance of points \((U, V)\) to the diagonal like least squares approaches in regression.

The next examples show that our approach applies to distributions of data with strong deviations from the diagonal \( U = V \) (outlier-like data points). Then it is favourable to use a function \( \psi \) different from the square function which gives less weight to large distances to the diagonal.

**Example 2.3.** \( \psi(u) = |u| \): We deduce

\[
\tilde{\psi} = 2 \int_0^1 (1 - u) \, du = \frac{1}{3},
\]

\[
\zeta_{X,Y} = 1 - 3 \int_0^1 \int_0^1 |u - v| \, dC(u,v) = -2 + 6 \int_0^1 C(v,v) \, dv.
\]

This dependence measure is called Spearman’s footrule. For comparisons, we provide the relationship to measure \( Q \) of Nelsen [9]:

\[
Q(C, M) = 4 \int_0^1 C(v,v) \, dv - 1 = \frac{2}{3} \zeta_{X,Y} + \frac{1}{3},
\]

which is a linear function of \( \zeta_{X,Y} \). \( Q(C, M) \) is one part of Gini’s concordance measure. Moreover, we have \( \zeta_{\text{min}} = -\frac{1}{2} \).

**Example 2.4.** Assume that \( \psi(u) = |u|^p \) for some \( p > 1 \). We obtain

\[
\tilde{\psi} = 2 \int_0^1 (1 - u) \, u^p \, du = \frac{2}{p^2 + 3p + 2}.
\]
and
\[ \zeta_{X,Y} = 1 - \frac{p^2}{2} + 3p + 2 \int_0^1 \int_0^1 |u - v|^p \, dC(u, v). \]

Further
\[ \zeta_{\min} = 1 - \int_0^1 u^p \, du \cdot \tilde{\psi}^{-1} = -\frac{1}{2}p. \]

**Example 2.5** (Huber function). Let us consider
\[ \psi(u) = \begin{cases} \frac{1}{2}u^2 & \text{for } |u| \leq \kappa, \\ \kappa (|u| - \frac{1}{2} \kappa) & \text{otherwise} \end{cases} \]
for some \( \kappa \in (0, 1) \). Then \( \tilde{\psi} = -\frac{1}{172} \kappa (\kappa^3 - 4\kappa^2 + 6\kappa - 4) \) and
\[ \zeta_{\min} = 1 - \left( \frac{1}{2} \int_0^\kappa u^2 \, du + \kappa \int_0^{\kappa} \left( u - \frac{1}{2} \kappa \right) \, du \right) \cdot \tilde{\psi}^{-1} = \frac{\kappa^3 - 2\kappa^2 + 2}{\kappa^3 - 4\kappa^2 + 6\kappa - 4}. \]

It can be shown that \(-1 < \zeta_{\min} < -\frac{1}{2}\). By means of Huber function, the idea of robustness is incorporated in our approach. This function \( \psi \) weights data points of the copula having a distance from the diagonal greater than \( \kappa \) weaker than the usual quadratic function.

### 3 A class of bivariate symmetric dependence coefficients

On the basis of the previous section, we can consider a measure of dependence which gives a distance to a monotonically increasing dependence and a monotonically decreasing one simultaneously:
\[ \xi_{X,Y} = \xi(C) = \int_0^1 \int_0^1 (\psi(u + v - 1) - \psi(u - v)) \, dC(u, v) \cdot \tilde{\psi}^{-1} \]
\[ = \mathbb{E} \left( \psi(F(X) + G(Y) - 1) - \psi(F(X) - G(Y)) \right) \cdot \tilde{\psi}^{-1}, \]

where \( \tilde{\psi} = \int_0^1 \psi(u) \, du \). Let us discuss the geometric interpretation of \( \xi \). One can see that \( |v - (1 - u)|/\sqrt{2} \) is the distance of point \((u, v)\) as a realisation of \((U, V)\) to the second diagonal \((u, v) \in [0, 1]^2 : v = 1 - u\). This diagonal is just the support of \( W \) representing the monotonically decreasing relationship: \( V = 1 - U \ a.s. \iff Y = \lambda(X) \ a.s. \) with a monotonically decreasing function \( \lambda \). Thus, taking the geometric considerations of the previous section into account, \( \xi_{X,Y} \) measures the distance from \( C \) to \( M \) and \( W \) simultaneously, and
\[ \xi_{X,Y} = \frac{\tilde{\psi}}{\psi} (\zeta_{X,Y} - \zeta_{X,-Y}) \]
holds true. This identity relates the symmetric coefficient \( \xi \) to the previously analysed coefficient \( \zeta \) and we can utilise its properties proved before. The measure \( \xi_{X,Y} \) was already introduced by Cifarelli et al. [2] in a similar form. The next Theorem 3.1 states important properties of \( \xi_{X,Y} \):

**Theorem 3.1.** Under the assumption of Theorem 2.1, we have
a) **Concordance:** \( C_1 \prec C_2 \) implies \( \xi(C_1) \leq \xi(C_2) \).
b) **Normalisation I:** \(-1 \leq \xi_{X,Y} \leq 1, \) and \( \xi_{X,X} = 1, \xi_{X,-X} = -1.\)
c) **Normalisation II**: The identity $\xi_{X,Y} = 1$ holds iff $U = V$ a.s. which in turn is equivalent to $Y = G^{-1}(F(X))$ a.s.; i.e. $Y$ is a strictly increasing function of $X$ in this case.

d) **Normalisation III**: If $X$ and $Y$ are independent, then $\xi_{X,Y} = 0$.

e) **Transformations**: $\xi_{\Phi(X), \Psi(Y)} = \xi_{X,Y}$ holds for strictly increasing functions $\Phi, \Psi$. Moreover, $\xi_{X,-Y} = -\xi_{X,Y}$ holds.

f) **Permutations**: $\xi_{Y,X} = \xi_{X,Y}$.

g) **Continuity**: Let $\{C_n\}$ be a sequence of copulas which tends pointwise to a copula $C$. Then $\xi(C_n) \to \xi(C)$.

This proposition shows that $\xi$ satisfies all conditions of a measure of concordance as introduced by Scarsini [10]. Next we consider some examples.

**Example 3.2.** $\psi(u) = u^2$: In this case, $\xi$ coincides with Spearman’s $\rho$ according to (1):

$$\xi(C) = 3 \int \int \frac{1}{2} \bigg((u + v - 1)^2 - (u - v)^2\bigg) \ dC(u,v)$$

$$= 12 \int \int u v \ dC(u,v) - 3 = \rho_3(C)$$

with $\tilde{\psi} = \frac{1}{2}$.

**Example 3.3.** $\psi(u) = |u|$: In view of (2) and $\tilde{\psi} = \frac{1}{2}$, $\xi$ is just Gini’s association measure $\gamma$.

**Example 3.4.** $\psi(u) = |u|^p$ with $p > 1$: Since $\tilde{\psi} = \frac{1}{p+1}$ in this case, we obtain

$$\xi(C) = \int \int \frac{1}{2} \bigg(|u + v - 1|^p - |u - v|^p\bigg) \ dC(u,v) \cdot (p+1).$$

**Example 3.5** (Huber function). We have

$$\tilde{\psi} = \frac{1}{2} \int_0^\kappa u^2 \ du + \kappa \int_0^\frac{1}{2} \bigg(u - \frac{1}{2}\bigg) \ du = \frac{1}{6}\kappa \left(\kappa^2 - 3\kappa + 3\right) \in (0, \frac{1}{6}). \quad \square$$

### 4 Estimation of the dependence coefficients

In this section we deal with properties of estimators for the association measures $\zeta$ and $\xi$. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a sample of independent random vectors with distribution $C$. First we introduce estimators for the distribution functions $F$ and $G$:

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_i \leq x), \quad G_n(x) = \frac{1}{n} \sum_{i=1}^n I(Y_i \leq x).$$

Natural estimators for the dependence coefficients $\zeta$ and $\xi$ are given by

$$\hat{\xi}_{X,Y} = 1 - \frac{1}{n} \sum_{i=1}^n \psi(F_n(X_i) - G_n(Y_i)) \cdot \tilde{\psi}^{-1},$$

$$\hat{\zeta}_{X,Y} = \frac{1}{n} \sum_{i=1}^n \left(\psi(F_n(X_i) + G_n(Y_i) - 1) + \psi(F_n(X_i) - G_n(Y_i))\right) \cdot \tilde{\psi}^{-1}.$$
Define

\[
A(x, y) = \psi(F(x) - G(y)) - \xi_{x, y} + \int_{[0,1]^2} \psi'(u - v) \left(1 \{ F(x) \leq u \} - u - 1 \{ G(y) \leq v \} + v \right) \, dC(u, v),
\]

where \( \xi_{x, y} = E\psi(F(X) - G(Y)) \). To avoid problems of interpretation, we set \( \psi'(t) = 0 \) if the derivative of \( \psi \) does not exist at \( t \) in the usual sense. We prove the following theorem about asymptotic normality of \( \xi_{x, y} \):

**Theorem 4.1.** Suppose that \( \psi \) is continuous on \([-1, 1] \), and \( \psi' \) exists and is piecewise Hölder-continuous on \([-1, 1] \); i.e. there are real numbers \(-1 = \tau_1 < \tau_2 < \ldots < \tau_m = 1 \) (\( m \geq 2 \)) such that for each \( j : 2 \leq j \leq m \), \( \psi' \) is Hölder-continuous on \((\tau_{j-1}, \tau_j) \). Then for \( \xi_{x, y} < 1 \), we have

\[
\sqrt{n} \left( \hat{\xi}_{x, y} - \xi_{x, y} \right) \xrightarrow{d} N(0, \sigma^2),
\]

where \( \sigma^2 = 4E\Lambda^2(X, Y) \hat{\psi}^{-2} \).

In Theorem 4.1, the assumptions on \( \psi \) can be weakened slightly. The actual assumptions are used to simplify the presentation of the proof. Theorem 4.1 can be applied to develop confidence intervals for \( \xi \). For this purpose, we need an estimator for \( \sigma^2 \). The variance \( \sigma^2 \) can be estimated by

\[
\hat{\sigma}^2 = \frac{1}{n\hat{\psi}^2} \sum_{i=1}^{n} \left( \psi(F_n(X_i) - G_n(Y_i)) \right) + \frac{1}{n} \sum_{j=1}^{n} \psi(F_n(X_j) - G_n(Y_j)) \left(1 \{ F_n(X_i) \leq F_n(X_j) \} - F_n(X_i) - 1 \{ G_n(Y_j) \leq G_n(Y_i) \} + G_n(Y_j) \right) \right)^2. \tag{6}
\]

In the second part of Section 4 we deal with the estimator \( \hat{\xi}_{x, y} \) for the symmetric dependence coefficient \( \xi \). In an analogous way one proves a statement about asymptotic normality of the coefficient \( \xi \):

**Theorem 4.2.** Let the assumption on \( \psi \) as in Theorem 4.1 be satisfied. Then for \(|\xi_{x, y}| < 1 \), we have

\[
\sqrt{n} \left( \hat{\xi}_{x, y} - \xi_{x, y} \right) \xrightarrow{d} N(0, \sigma^2),
\]

where \( \sigma^2 = 4E\Lambda^2(X, Y) \hat{\psi}^{-2} \), \( \hat{\xi}_{x, y} = E(\psi(F(X) + G(Y) - 1) - \psi(F(X) - G(Y))) \), and

\[
\Lambda(x, y) = \psi(F(x) + G(y) - 1) - \psi(F(x) - G(y)) - \xi_{x, y} + \int_{[0,1]^2} \left( \psi'(u + v - 1) \cdot \left(1 \{ F(x) \leq u \} - u - 1 \{ G(y) \leq v \} + v \right) \right) \, dC(u, v).
\]

Asymptotic normality of \( \xi \) was already proved in Cifarelli et al. [2] under assumptions which are more restrictive in a certain sense. In comparison to the paper by Cifarelli et al. [2], the advantage of Theorem 4.2 is that it gives an explicit formula for the variance \( \sigma^2 \) from which a formula for an estimator can be derived easily (analogously to (6)). Moreover, we provide an alternative proof of asymptotic normality which is simpler and shorter.
5 Multivariate extensions

Let $X = (X_1, \ldots, X_d)^T$ be the random vector with continuous marginals. We denote the distribution functions of $X$ and $X_i$ by $H$ and $F^{(i)}$, respectively. By Sklar’s theorem, we obtain

$$H(x) = C(F^{(1)}(x_1), \ldots, F^{(d)}(x_d))$$

where $C$ is the multivariate copula. We assume that the marginal distribution functions are continuous so that $C$ is uniquely determined. Let $M(u) = \min\{u_i : i = 1 \ldots d\}$ be the multivariate upper Fréchet bound. We introduce $U_i = F^{(i)}(X_i)$, $F(X) = (F^{(1)}(X_1), \ldots, F^{(d)}(X_d))^T$. The copula $C$ is equal to $M$ iff

$$U_1 = U_2 = \ldots = U_d \text{ a.s.} \iff X_i = \lambda_i(X_1) \text{ a.s. for } i \geq 2$$

(comonotonicity) where $\lambda_2, \ldots, \lambda_d$ are strictly increasing functions. The corresponding distribution measure of $M$ is concentrated on the diagonal $D = \{u \in [0,1]^d : u_1 = u_2 = \ldots = u_d\}$. Copula $M$ describes the perfect strictly increasing relationship. A multivariate copula as counterpart to copula $W$ does not exist. Here the aim is to define an association coefficient which describes the discrepancy between $C$ and copula $M$, and to examine its properties. From the considerations above, it is clear that a direct multivariate extension of $\xi$ does not exist. Therefore, we focus on the extension of $\zeta$.

Let $\psi : [0,1] \to [0,\infty)$ be a strictly increasing function with $\psi(0) = 0$. We introduce the multivariate dependence coefficient:

$$\zeta_X = \zeta(C) = 1 - \int_{[0,1]^d} \psi \left( \min_{t \in [0,1]} \| u - t 1 \| \kappa_d \right) \, dC(u) \cdot \tilde{\psi}^{-1}$$

$$= 1 - \int_{[0,1]^d} \psi \left( \| u - \bar{u} \cdot 1 \| \kappa_d \right) \, dC(u) \cdot \tilde{\psi}^{-1}$$

$$= 1 - \mathbb{E} \psi \left( \left\| F(X) - F(X) \cdot 1 \right\| \kappa_d \right) \cdot \tilde{\psi}^{-1}, \quad (7)$$

where $\| . \|$ is the Euclidean norm, $1 = (1, \ldots, 1)^T$, $\bar{u} = \frac{1}{d} \sum_{i=1}^d u_i$, $\kappa_d = 2/\sqrt{d}$ if $d$ is even, $\kappa_d = \sqrt{\frac{d}{d-1}}$ if $d$ is odd, and

$$\psi = \int_{[0,1]^d} \psi \left( \| u - \bar{u} \cdot 1 \| \kappa_d \right) \, du.$$

The term $\| u - \bar{u} \cdot 1 \|$ in the argument of function $\psi$ in (7) gives the minimum distance of $u$ to the diagonal $D$. The dependence coefficient is actually equal to one minus the normalised expectation of the transformed distances of copula data points $U = (U_1, \ldots, U_d)^T$ to the diagonal $D$. The normalising factor $\tilde{\psi}$ is just the integral in (7) in the case where $C$ is the independent copula $II$ with $II(u) = \prod_{i=1}^n u_i$. Moreover we can write

$$\zeta_X = -\int_{[0,1]^d} \psi \left( \min_{t \in [0,1]} \| u - t 1 \| \kappa_d \right) \, du \cdot (C(u) - II(u)) \cdot \tilde{\psi}^{-1}.$$

This formula shows that $\zeta_X$ measures also the distance between $C$ and $II$.

In general, it is hard to find an explicit formula for $\tilde{\psi}$. In such a situation it is recommended to perform simulations to get this value $\tilde{\psi}$. In Lemma 6.3 in Section 6, it is shown that $\max_{u \in [0,1]^d} \| u - \bar{u} \cdot 1 \| = \kappa_d^{-1}$. Thus $\kappa_d$ is defined in such a way that the largest argument of $\psi$ is equal to $1$.

Now we summarise important properties of the measure $\zeta_X$ in the following theorem:

**Theorem 5.1.**

a) **Normalisation I:** $1 - \psi(1) \cdot \tilde{\psi}^{-1} \leq \zeta_X \leq 1$ holds.

b) **Normalisation II:** The identity $\zeta_X = 1$ holds iff $U_1 = U_2 = \ldots = U_d$ a.s. which in turn is equivalent to $X_j = F^{(j)}(F^{(1)}(X_1))$ a.s. for $j > 1$; i.e. $X_j$ is a strictly increasing function of $X_1$ in this case.
c) Normalisation III: If \( X_1, \ldots, X_d \) are independent, then \( \zeta_X = 0 \).

d) Transformations: \( \zeta_{h_1(X_1), \ldots, h_d(X_d)} = \zeta_X \) holds for every strictly increasing functions \( h_1, \ldots, h_d \).

e) Permutations: \( \zeta_{\pi(X)} = \zeta_X \) for any permutation \( \pi \) of the components of \( X \).

f) Duality: \( \zeta_{X_1, \ldots, X_d} = \zeta_X \).

g) Continuity: Let \( \{C_n\} \) be a sequence of copulas which tends pointwise to a copula \( C \). Assume that \( \psi \) is continuous. Then \( \zeta(C_n) \to \zeta(C) \).

The theorem shows that important properties of the measure can be conferred to the multivariate case. In view of (7), the lower bound \( 1 - \psi(1) \cdot \hat{\psi}^{-1} \) for \( \zeta_X \) in Theorem 5.1a) could only be achieved in the case \( \|U - \bar{U} \cdot 1\| = k^d \) a.s. which requires a discontinuous \( C \). Remember that we assumed the copula to be continuous. Unfortunately, there is no connection between the multivariate coefficient \( \zeta \) and the idea of concordance in general. Next we consider the special case of a quadratic function \( \psi \) and mention that another generalisation of Spearman’s footrule is examined in Úbeda-Flores [18].

**Example 5.2.** \( \psi(u) = u^2 \); generalisation of Spearman’s \( \rho_S \).

Note that \( \int_{[0,1]^d} u_i^2 \ dC(u) = \frac{1}{d} \). We denote the pairwise Spearman’s \( \rho \) of \( X_i \) and \( X_j \) by \( \rho_{ij} \). Then

\[
\int_{[0,1]^d} \psi(\|u - \bar{u} \cdot 1\| ) \ dC(u) = \kappa_d^2 \int_{[0,1]^d} \left( \sum_{i=1}^d u_i^2 - \frac{1}{d} \left( \sum_{i=1}^d u_i \right)^2 \right) \ dC(u) \nonumber \\
= \kappa_d^2 \int_{[0,1]^d} \left( \frac{d-1}{d} \sum_{i=1}^d u_i^2 - \frac{1}{d} \sum_{i,j \in \{1, \ldots, d\}, i \neq j} u_i u_j \right) \ dC(u) \nonumber \\
= \kappa_d^2 \left( \frac{d-1}{12} \sum_{i,j \in \{1, \ldots, d\}, i \neq j} \rho_{ij} \right),
\]

and

\[
\bar{\psi} = \kappa_d^2 \frac{d-1}{12}
\]

holds true. Hence

\[
\zeta_X = \frac{1}{d(d-1)} \sum_{i \neq j} \rho_{ij},
\]

which says that \( \zeta \) is the mean of the pairwise Spearman’s \( \rho \). This coefficient \( \zeta \) is a special case of the average pairwise measure of concordance defined in Example 3.1 of Dolati and Úbeda-Flores [3]. From the investigations of that paper, it follows that \( \zeta \) is a concordance measure of association.

**Example 5.3.** \( \psi(u) = |u| \); generalised Spearman’s footrule.

For dimension 3, we obtain \( \hat{\psi} = \frac{\ln 27}{16} \) which in turn implies

\[
\hat{\zeta}_X = 1 - 2.6859 \mathbb{E} \left\| F(X) - \bar{F}(X) \cdot 1 \right\|
\]

In the last part of this section, we deal with the estimation of the multivariate coefficient \( \zeta_X \). Let \( Z_1, \ldots, Z_n \) be a sample of independent random vectors having the distribution of \( X \) and copula \( C \). Let \( F_n^{(k)} \) be the empirical distribution function of the \( k \)-th component of the random vector \( X \) under consideration. \( F_n \) denotes the vector of empirical distribution functions of the components: \( F_n = (F_n^{(1)}, \ldots, F_n^{(d)})^T \). The following estimator can be used to estimate the multivariate dependence coefficient \( \zeta_X \):

\[
\hat{\zeta}_X = 1 - \frac{1}{n} \sum_{i=1}^n \psi \left( \left\| F_n(Z_i) - \bar{F}_n(Z_i) \cdot 1 \right\| \right) \kappa_d \cdot \hat{\psi}^{-1}.
\]

Similarly to the proofs of above statements, on can show asymptotic normality of this estimator.
Theorem 5.4. Suppose that $\psi$ is continuous on $[0, 1]$, and $\psi'$ exists and is piecewise Hölder-continuous on $[0, 1]$. Then for $\xi_x < 1$, we have

$$\sqrt{n} \left( \xi_x - \xi_x \right) \overset{D}{\rightarrow} N(0, \bar{\sigma}^2),$$

where

$$\bar{\sigma}^2 = 4E\bar{\lambda}^2(X) \bar{\psi}^{-2},$$

$$\bar{\lambda}(x) = \psi \left( \left\| F(x) - \bar{F}(x) \cdot 1 \right\| \kappa_d \right) - \mathbb{E} \left[ \left\| F(X) - \bar{F}(X) \cdot 1 \right\| \kappa_d \right] + \int \kappa_d \psi' \left( \left\| u - \bar{u} \cdot 1 \right\| \kappa_d \right) \cdot \left\| u - \bar{u} \cdot 1 \right\|^{-1} [0,1]^d \cdot \sum_{k=1}^d (u_k - \bar{u}) \left( 1 \left( F(x_k) \leq u_k \right) - \frac{1}{d} \sum_{l=1}^d 1 \left( F(x_l) \leq u_l \right) - u_k + \bar{u} \right) dC(u),$$

$$\bar{F}(x) = \frac{1}{d} \sum_{k=1}^d F^{(k)}(x_k).$$

The asymptotic variance $\bar{\sigma}^2$ can be estimated similarly to (6):

$$\tilde{\sigma}^2 = 4\bar{\psi}^{-2} \frac{1}{n} \sum_{i=1}^n \bar{\lambda}_n^2(Z_i),$$

where

$$\bar{\lambda}_n(x) = \psi \left( \left\| F_n(x) - F_n(x) \cdot 1 \right\| \kappa_d \right) - \frac{1}{n} \sum_{i=1}^n \psi \left( \left\| F_n(Z_i) - F_n(Z_i) \cdot 1 \right\| \kappa_d \right)$$

$$+ \frac{K_d}{n} \sum_{i=1}^n \psi' \left( \left\| F_n(Z_i) - F_n(Z_i) \cdot 1 \right\| \kappa_d \right) \cdot \left\| F_n(Z_i) - F_n(Z_i) \cdot 1 \right\|^{-1}$$

$$+ \frac{d}{n} \sum_{k=1}^d \left( F_n^{(k)}(Z_i) - F_n(Z_i) \right) \left( 1 \left( F_n^{(k)}(x_k) \leq F_n^{(k)}(Z_i) \right) - \frac{1}{d} \sum_{l=1}^d 1 \left( F_n^{(k)}(x_l) \leq F_n^{(k)}(Z_i) \right) - u_k + \bar{u} \right) dC(u).$$

By means of this formula, confidence intervals can be established.

6 Proofs

In this section we start with the proofs of properties of the coefficients $\zeta_{x,y}$ and $\xi_{x,y}$ (Theorems 2.1 and 3.1).

Lemma 6.1. Equation (4) holds true.

Proof. Observe that

$$\bar{\psi} = \int_0^1 \int_{-v}^{1-v} \psi(u) \, du \, dv = \int_{-1}^{1} \int_{\max\{1,1-u\}}^{\min\{1,1-u\}} \psi(u) \, du \, dv$$

$$= \int_{-1}^{1} (1+u) \psi(u) \, du + \int_{0}^{1} (1-u) \psi(u) \, du = 2 \int_{0}^{1} (1-u) \psi(u) \, du,$$

which proves the lemma. \qed
Proof. of Theorem 2.1:
a) Define
\[ I(C) := \int \int \psi(u - v) \, dC(u, v). \]
Then \( \zeta(C) = 1 - I(C) \cdot \bar{\psi}^{-1} \). Let \( C_u(u, v) = \frac{\partial}{\partial u} C(u, v) \). By partial integration, we obtain
\[
I(C) = \int \left( \int_0^1 \psi(u - v) \, dC_u(u, v) \right) du
= \int \psi(u - 1) \, du + \int \left( \int_0^1 \psi(u - v) \, dC_u(u, v) \right) dv
= \int \psi(u) \, du + \int \left( \psi'(1 - v) + \left( \psi'(0 - 0) - \psi'(0 + 0) \right) C(v, v) \right) \, dv
- \int \int \psi''(u - v) C(u, v) \, dv \, du
= \int \psi(u) \, du - \psi(1 - v) v^{1}_{u=0} + \int (\psi(1 - v) - 2 \psi'(0 + 0) C(v, v)) \, dv
- \int \int \psi''(u - v) C(u, v) \, dv \, du,
\]
which implies assertion a).
b) Let \( C_1 \prec C_2 \). Observe that \( \psi \) is convex by assumption. Then \( I(C_1) \geq I(C_2) \) which proves part b).
c) Obviously, \( \zeta_{X,Y} \leq 1 \). Since \( W \prec C \), we have
\[
\zeta_{X,Y} = \zeta(W) = 1 - \int \psi(2u - 1) \, du \cdot \bar{\psi}^{-1} = 1 - \int \psi(u) \, du \cdot \bar{\psi}^{-1}.
\]
d) It is easy to see that
\[
\zeta_{X,X} = 1 - \psi(0) \cdot \bar{\psi}^{-1} = 1.
\]
On the other hand, we have
\[
\zeta_{X,X} = 1 - \int \int \psi(u - v) \, dW(u, v) \, \bar{\psi}^{-1} = 1 - \int \psi(2u - 1) \, du \, \bar{\psi}^{-1}
\]
since \( W \) is the copula of \((-X, X)\).
e) If \( \zeta_{X,Y} = 1 \) holds then
\[
\int \int \psi(u - v) \, dC(u, v) = 0
\]
and the support of the \( C \)-measure is on the diagonal \( u = v \); i.e. \( C = M \) and \( U = V \) a.s. The inverse direction of the assertion follows from \( C = M \iff U = V \) a.s.
f) Let \( X \) and \( Y \) be independent. Then
\[
\zeta_{X,Y} = 1 - \int \int \psi(u - v) \, dv \, \bar{\psi}^{-1} = 0.
\]
g) follows from the fact that \((g(X), h(Y))\) and \((X, Y)\) have the same copula \( C \). The symmetry property of \( \psi \) implies the validity of h). We obtain part j) immediately from (3). Part j) is a consequence of the Portmanteau theorem (see e.g. Van der Vaart [19] p. 6).
Proof. of Theorem 3.1:

a) Note that \( u - C(u, 1 - v) \) is the copula of \((X, -Y)\). An application of (5) and Theorem 2.1a) leads to

\[
\xi(C) \cdot \dot{\psi} = 2\psi'(0 + 0) \int_0^1 (C(u, u) - u + C(u, 1 - u)) \, du + \int_0^1 \int_0^1 \psi''(u-v) \cdot (C(u, v) - u + C(u, 1 - v)) \, dudv \\
= 2\psi'(0 + 0) \left( \int_0^1 (C(u, u) + C(u, 1 - u)) \, du - \frac{1}{2} \right) - \psi(1) - \psi'(0 + 0) \\
+ \int_0^1 \int_0^1 (\psi''(u-v) + \psi''(u+v-1)) \, C(u, v) \, dudv.
\]

Assertion a) follows immediately from this identity.

b) In view of part a), the maximum value of \( \xi(C) \) is given by

\[
\xi(M) = \int_0^1 \psi(2u - 1) \, du \cdot \dot{\psi}^{-1} = 1.
\]

The minimum value of \( \xi(C) \) can be deduced as follows

\[
\xi(W) = -\int_0^1 \psi(2u - 1) \, du \cdot \dot{\psi}^{-1} = -1.
\]

c) If \( \xi_{X,Y} = 1 \) then \( \xi(C) = 1 \). We have

\[
0 = \left( \xi(C) - \xi(M) \right) \cdot \dot{\psi} \\
= 2\psi'(0 + 0) \int_0^1 (C(u, u) - M(u, u) + C(u, 1 - u) - M(u, 1 - u)) \, du \\
+ \int_0^1 \int_0^1 (\psi''(u+v-1) + \psi''(u-v)) \, (C(u, v) - M(u, v)) \, dudv.
\]

Hence using \( C(u, v) = M(u, v) \), we obtain \( C(u, v) = M(u, v) \) for all \( u, v \in [0, 1] \) and \( U = V \) a.s.

Parts d), e) and f) follow from (5) and Proposition 2.1. Part g) is a consequence of the Portmanteau theorem (see e.g. Van der Vaart [19], p. 6).

We proceed with proving the Theorems 4.1 and 4.2 on asymptotic normality of the estimated coefficients in the two-dimensional case. In the proofs we use notations \( O_p \) and \( O_2: Y_n = o_p(Z_n) \iff Y_n/Z_n \xrightarrow{P} 0 \); \( Y_n = O_p(Z_n) \iff Y_n/Z_n \) is bounded in probability \( \iff \) for every \( \varepsilon > 0 \) there exists a constant \( M \) such that \( \limsup_{n \to \infty} P(\{|Y_n/Z_n| > M\} < \varepsilon. \)

Lemma 6.2. We have

\[
\sum_{i=1}^n \left( |F_n(X_i) - F(X_i)|^2 + |G_n(Y_i) - G(Y_i)|^2 \right) = O_p(1).
\]

Proof. Observe that

\[
Z_n = \sum_{i=1}^n \left( |F_n(X_i) - F(X_i)|^2 \right) = n^{-2} \sum_{i,j,k=1}^n (I(X_j < X_i) - F(X_i)) (I(X_k < X_i) - F(X_i)) .
\]
Further,

\[ P \{ Z_n > \varepsilon \} = \varepsilon^{-1} n^{-2} \sum_{i,j,k=1}^{n} \mathbb{E} \left( (I(X_j \leq X_i) - F(X_i)) (I(X_k \leq X_i) - F(X_i)) \right) \varepsilon^{-1} n^{-2} \sum_{i,j=1}^{n} \mathbb{E} \left( (I(X_j \leq X_i) - F(X_i))^2 \right) \leq \varepsilon^{-1} \]

by Markov's inequality. \qed

Proof. of Theorem 4.1:
Let \{ Y_n \} and \{ Z_n \} be two sequences of random variables. Obviously,

\[ \sqrt{n} \left( \tilde{\zeta}_{X,Y} - \zeta_{X,Y} \right) = -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\psi(F_n(X_i) - G_n(Y_i)) - \tilde{\zeta}_{X,Y}) . \]

In the following we represent this term as a sum of a \( U \)-statistic and a remainder. Then we show the asymptotic normality of the \( U \)-statistic and prove that the remainder tends to zero. Note that by assumption,

\[ \sup_{t,u \in (t_1, t_2]} |\psi'(t) - \psi'(u)| |t - u|^{-q} < +\infty \]

for \( j = 2, \ldots, m \) with an appropriate \( q \in (0, 1) \), \( q \) is the Hölder exponent. Therefore

\[ |\psi(t) - \psi(u) - \psi'(u)(t - u)| \leq C_1 \cdot |t - u|^{1+q} \]

holds for \( t, u \in [-1, 1] \setminus \{ \tau_1, \ldots, \tau_m \} \) whenever \( \tau_1, \ldots, \tau_m \) do not lie between \( t \) and \( u \),

\[ |\psi(t) - \psi(u) - \psi'(u)(t - u)| \leq C_1 \cdot |t - u| \quad \text{otherwise} \]

with a constant \( C_1 > 0 \). Moreover, laws of iterated logarithms hold true for the empirical processes \( F_n \) and \( G_n \) (see e.g. van der Vaart [19], p. 268):

\[ \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \leq C_2 \sqrt{\frac{\ln \ln n}{n}}, \quad \sup_{x \in \mathbb{R}} |G_n(x) - G(x)| \leq C_2 \sqrt{\frac{\ln \ln n}{n}} \]

for \( n \geq n_0(\omega) \) with an appropriate constant \( C_2 > 0 \). We deduce

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\psi(F(X_i) - G(X_i)) - \tilde{\zeta}_{X,Y}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\psi(F(X_i) - G(Y_i)) - \tilde{\zeta}_{X,Y}) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi'(F(X_i) - G(Y_i)) F_n(X_i) - F(X_i) - G_n(Y_i) + G(Y_i) + B_n \]

\[ = A_n + B_n, \]

where

\[ A_n = \frac{1}{n^{1/2}} \sum_{i=1}^{n} \sum_{j=1}^{n} (\psi(F(X_i) - G(Y_j)) - \tilde{\zeta}_{X,Y} + \psi'(F(X_i) - G(Y_j)) \cdot (I(X_j \leq X_i) - F(X_i) - I(Y_j \leq Y_i) + G(Y_i)), \]

\[ |B_n| \leq O_p(n^{-1/2}) \sum_{i=1}^{n} \left( |F_n(X_i) - F(X_i)|^{1+q} + |G_n(Y_i) - G(Y_i)|^{1+q} \right) \]

\[ + O_p(n^{-1/2}) \sum_{i=1}^{n} \left( |F_n(X_i) - F(X_i)| + |G_n(Y_i) - G(Y_i)| \right) \cdot \sum_{k=1}^{m} \left( |F(X_k) - G(Y_k) - \tau_k| \leq 3C_2 \sqrt{\frac{\ln \ln n}{n}} \right) \]

\[ \leq O_p(1) + O_p(1) \cdot \left( \sum_{i=1}^{n} \left( |F_n(X_i) - F(X_i)|^2 + |G_n(Y_i) - G(Y_i)|^2 \right) \right)^{1/2} \]

\[ \cdot \left( n^{-1} \sum_{i=1}^{n} \sum_{k=1}^{m} \left( |F(X_k) - G(Y_k) - \tau_k| \leq 3C_2 \sqrt{\frac{\ln \ln n}{n}} \right) \right)^{1/2} \]
for \( n \geq n_0(\omega) \). An application of Lemma 6.2 and the Glivenko-Cantelli theorem (see e.g. van der Vaart [19], p. 266) leads to

\[
|B_n| \leq \alpha_p(1) + \alpha_p(1) \left( \sum_{k=1}^{m} \mathbb{P} \left( |F(X_k) - G(Y_k) - \tau_k| \leq 3C_2 \sqrt{\frac{\ln \ln n}{n}} \right) \right)^{1/2} = \alpha_p(1).
\]  

(8)

We introduce \( H(\eta_1, \eta_2) = H_0(\eta_1, \eta_2) + H_0(\eta_2, \eta_1) \), where \( \eta_i = (X_i, Y_i)^T \),

\[
H_0((x, y)^T, (z, w)^T) = \psi(F(x) - G(y)) - \xi_{X, Y} + \psi'(F(x) - G(y)) \left( 1 (z \leq x) - F(x) - 1 (w \leq y) + G(y) \right).
\]

Further in view of

\[
A_n = \frac{1}{n^{1/2} \sqrt{n}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} H((X_i, Y_i), (X_j, Y_j)),
\]

\( A_n \) is a U-statistic with symmetric kernel \( H \). Observe that

\[
\mathbb{E}H_0((x, y), (X_i, Y_i)) = \psi(F(x) - G(y)) - \xi_{X, Y} \quad \text{and}
\]

\[
\mathbb{E}H_0((X_i, Y_i), (z, w)) = \int_{[0, 1]^d} \psi'(u - v) \left( 1 (F(z) \leq u) - u - 1 (G(w) \geq v) + v \right) \, dC(u, v).
\]

An application of Theorem 5.5.1A in Serfling [15] leads to

\[
A_n \overset{D}{\to} \mathcal{N}(0, \sigma^2),
\]

which proves the theorem in connection with (8).

\( \square \)

**Proof.** of Theorem 4.2:

Analogously to the previous proof, we obtain

\[
\sqrt{n} \left( \xi_{X, Y} - \xi_{X, Y} \right) = \frac{1}{\sqrt{n} \psi} \sum_{i=1}^{n} \left( \psi(F(X_i) + G(Y_i) - 1) - \psi(F(X_i) - G(Y_i)) - \xi_{X, Y} \right)
\]

\[
= \frac{1}{\sqrt{n} \psi} \sum_{i=1}^{n} \left( \psi(F(X_i) + G(Y_i) - 1) - \psi(F(X_i) - G(Y_i)) - \xi_{X, Y} \right)
\]

\[
+ \frac{1}{\sqrt{n} \psi} \sum_{i=1}^{n} \left( \psi'(F(X_i) + G(Y_i) - 1) \left( F_n(X_i) - F(X_i) + G(Y_i) \right) - F(X_i) - G(Y_i) \right) + o_p(1)
\]

\[
= \bar{A}_n \cdot \psi^{-1} + o_p(1),
\]

where \( \bar{H}(\eta_1, \eta_2) = \bar{H}_0(\eta_1, \eta_2) + \bar{H}_0(\eta_2, \eta_1) \) for \( \eta_i \in \mathbb{R}^2 \),

\[
\bar{A}_n = \frac{1}{n^{1/2} \sqrt{n}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \bar{H}((X_i, Y_i), (X_j, Y_j)),
\]

\[
\bar{H}_0((x, y)^T, (z, w)^T) = \psi(F(x) + G(y) - 1) - \psi(F(x) - G(y)) - \xi_{X, Y}
\]

\[
+ \psi'(F(x) + G(y) - 1) \cdot \left( 1 (z \leq x) - F(x) + 1 (w \leq y) - G(y) \right)
\]

\[
- \psi'(F(x) - G(y)) \cdot \left( 1 (z \leq x) - F(x) - 1 (w \leq y) + G(y) \right).
\]

The asymptotic normality of \( A_n \) follows by applying Theorem 5.5.1A in Serfling [15].

\( \square \)

In the remainder of this section, we consider the multivariate coefficient \( \xi_X \) introduced in Section 5 and its estimator. First we prove an auxiliary statement. Then properties of \( \xi_X \) (Theorem 5.1) and the asymptotic normality of \( \hat{\xi}_X \) (Theorem 5.4) are shown.
Lemma 6.3. We have
\[
\max_{u_1, \ldots, u_d \in [0,1]} \Phi(u) = \kappa_d^2 \text{ where } \Phi(u) := \sum_{i=1}^{d} \left( u_i - \frac{1}{d} \sum_{j=1}^{d} u_i \right)^2.
\]

Proof. Let \( \bar{u}_i = u_i - \frac{1}{d} \sum_{j=1}^{d} u_j \). First we show that \( \Phi \) is convex:
\[
\Phi(\lambda u + (1 - \lambda)v) = \sum_{i=1}^{d} (\lambda \bar{u}_i + (1 - \lambda)\bar{v}_i)^2 \leq \sum_{i=1}^{d} (\lambda \bar{u}_i^2 + (1 - \lambda)\bar{v}_i^2) = \lambda \Phi(u) + (1 - \lambda)\Phi(v)
\]
for \( u, v \in [0,1]^d, \lambda \in [0,1] \). Hence maximum is achieved in the edges \( u \) of the domain; i.e. in points of the set \( \{u : u_i \in \{0,1\}\} \). Let \( N \) be the number of ones in the coordinates of edge \( u \). Then
\[
\Phi(u) = h(N) = (d - N) \left( \frac{N}{d} \right)^2 + N (1 - \frac{1}{d})^2 = \frac{N}{d} (d - N).
\]

(1) case \( d \) even: \( \arg \max_N h(N) = d/2 \Rightarrow \max_{u \in [0,1]^d} \Phi(u) = \frac{d}{2} \)

(2) case \( d \) odd: \( \max_N h(N) \in \{(d - 1)/2, (d + 1)/2\} \Rightarrow \max_{u \in [0,1]^d} \Phi(u) = \frac{1}{md} \left( d^2 - 1 \right). \)

Proof. of Theorem 5.1:

a) follows from (7) since \( 0 \leq \psi(t) \leq \psi(1) \) for \( t \in [0,1] \).

b) Condition \( \zeta_X = 1 \) implies that the integral in (7) is zero which in turn is only the case if \( C \) is concentrated on the diagonal. The inverse direction follows immediately.

c) and f) are obvious.

Part d) is a consequence of the fact that monotonically increasing transformations of the marginals do not change the copula.

e) A permutation of the components of \( X \) do not change the integral in (7). Part g) is a consequence of the Portmanteau theorem (see e.g. Van der Vaart [19], p. 6).

Proof. of Theorem 5.4:

Let \( Z_i = (Z_{i1}, \ldots, Z_{in})^T \) and
\[
\tilde{\zeta}_X = \mathbb{E} \psi \left( \left\| F(X) - \bar{F}(X) \cdot 1 \right\| \kappa_d \right).
\]

We have
\[
\sqrt{n} \left( \tilde{\zeta}_X - \zeta_X \right) = -\frac{1}{\sqrt{n} \lambda_n} \sum_{i=1}^{n} \left( \psi \left( \left\| F_n(Z_i) - \bar{F}(Z_i) \cdot 1 \right\| \kappa_d \right) - \tilde{\zeta}_X \right) = -\lambda_n \cdot \bar{\psi}^{-1},
\]

where
\[
\lambda_n = \frac{1}{\sqrt{n} \lambda_n} \sum_{i=1}^{n} \left( \psi \left( \left\| F(Z_i) - \bar{F}(Z_i) \cdot 1 \right\| \kappa_d \right) - \tilde{\zeta}_X + \kappa_d \psi \left( \left\| F(Z_i) - \bar{F}(Z_i) \cdot 1 \right\| \kappa_d \right) - \left( \left\| F(Z_i) - \bar{F}(Z_i) \cdot 1 \right\| \kappa_d \right)^{-1} \right)
\]
\[
\cdot \sum_{k=1}^{d} \left( F^{(k)}(Z_i^{(k)}) - \bar{F}(Z_i) \right) \left( \left\| F^{(k)}(Z_i^{(k)}) - \bar{F}(Z_i) \right\| \right) + o_p(1)
\]
\[
= \frac{1}{n \sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \psi \left( \left\| F(Z_i) - \bar{F}(Z_i) \cdot 1 \right\| \kappa_d \right) - \tilde{\zeta}_{X,Y} + \kappa_d \psi \left( \left\| F(Z_i) - \bar{F}(Z_i) \cdot 1 \right\| \kappa_d \right) - \left( \left\| F(Z_i) - \bar{F}(Z_i) \cdot 1 \right\| \kappa_d \right)^{-1} \right)
\]
\[
\cdot \sum_{k=1}^{d} \left( F^{(k)}(Z_i^{(k)}) - \bar{F}(Z_i) \right) \left( I \left( Z_i^{(k)} \leq Z_i^{(k)} \right) - \frac{1}{d} \sum_{i=1}^{d} I \left( Z_i^{(k)} \leq Z_i^{(k)} \right) - \left( \left\| F^{(k)}(Z_i^{(k)}) - \bar{F}(Z_i) \right\| \right) + o_p(1)
\]
\[
= \frac{1}{n \sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{n} H(Z_i, Z_j) + o_p(1),
\]
\[ \bar{H}(\eta_1, \eta_2) = \bar{H}_0(\eta_1, \eta_2) + \bar{H}_0(\eta_2, \eta_1) \] for \( \eta_i \in \mathbb{R}^d \), and
\[
\bar{H}_0(x, y) = \psi\left( \left\| F(x) - F(x) \cdot 1 \right\| \kappa_d - \bar{\zeta}_x + \kappa_d \psi'\left( \left\| F(x) - F(x) \cdot 1 \right\| \kappa_d \right) \cdot \left\| F(x) - F(x) \cdot 1 \right\|^{-1} \right) \\
\cdot \sum_{k=1}^{d} \left( F^{(k)}(x_k) - F(x) \right) \left( 1 \cdot y_k \leq x_k \right) - \frac{1}{d} \sum_{i=1}^{d} 1 \cdot y_i \leq x_i - F^{(k)}(x_k) + F(x) \right).
\]

An application of Theorem 5.5.1A in Serfling [15] yields the assertion of the theorem. □

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