Self-consistent Description of Bose-Bose Droplets: Harmonically Trapped Quasi-2D Droplets

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We describe a quantum droplet of a Bose-Bose mixture squeezed by an external harmonic forces in one spatial direction. Our approach is based on the self-consistent method formulated in [1]. The true spatial droplet profile in the direction of confinement is accounted for, however local density approximation is assumed in the free directions. We define a numerical approach to find the beyond-mean-field contribution to the chemical potential (Lee-Huang-Yang chemical potential) - the quantity that determines the droplet’s profile. In addition to the numerical approach, we find the Lee-Huang-Yang potential in the analytic form in two limiting cases: a perturbative result for a strong confinement and a semiclassical expression when confinement is very weak.

I. INTRODUCTION

Problem of finding the ground state energy of weakly interacting, homogeneous Bose gas belongs to classic issues of quantum many body theory. It attracts attention of researchers since discovery of superfluidity of liquid Helium. Huang and Yang [5] studying energy of N-particle quantum system of hard-spheres and exploring a concept of Fermi pseudo-potential [6] have shown that the ground state energy depends on the s-wave phase shift, proportional to the s-wave scattering length $a$. This observation allows to use as theoretical tools some other potentials, for instance smooth and positive defined, instead of realistic van der Waals ones, provided that the scattering length is the same for both scattering potentials. Next to leading term in the expansion of energy in powers of density is known as the Lee-Huang-Yang (LHY) energy, [7]. The energy of homogeneous weakly interacting Bose gas can be approximated by:

$$E_0/N + E_{LHY}/N = \frac{2\pi a^2}{m_n} \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{a^3}\right) \times 10^{-15} \text{ergs}.$$  

(1)

In the above formula $E_0/N$ and $E_{LHY}/N$ are mean-field and LHY energy per atom, $a$ is the s-wave scattering length, $m_n$ is the mass of the atom and $n$ is atomic density. Further investigations of exited states proved phonon-like momentum-energy spectrum at low energies [8][11], and lack of energy gap separating ground and exited states. This feature was proved in [12], and is referred to as the Hugenholztz-Pines theorem. Ground state energy of one and two dimensional systems are also considered, [13][14], including exact N-particle solution of the Lieb-Liniger model [15].

Advent of experiments with ultracold atomic gases leading to achievement of Bose-Einstein condensation renewed interest in studies of ground state energy of these dilute, weakly interacting systems [16][17]. The formula giving the mean-field energy per atom $E_0/N$ present in Eq. (1) was proved with mathematical rigorousness [18] only in 1998 by E.H. Lieb and J. Yngvason who commented on difficulties which are pertinent to this issue: "Owing to the delicate and peculiar nature of bosonic correlations, four decades of research have failed to establish this plausible formula rigorously." Nowadays not only 3D but also lower dimensional systems are available to experiments with ultracold atoms. Theory follows this experimental progress. First correction beyond Bogoliubov theory in the case of the ground state of two-dimensional weakly interacting Bose gas was derived analytically in a systematic way [19] in excellent agreement with numerical Monte-Carlo calculations [20][21].

In most of experimental arrangements, the leading term in expression giving the ground state energy is sufficient to describe observations. The reason is that Bose condensates are very dilute and observation of subtle effects of the LHY contribution is beyond experimental precision. Scientists used to share a folk wisdom that higher order effects are to be observed at strong interactions. This way of reasoning was challenged by D. Petrov [22], who noticed that one should rather look for such situations where leading term is small or vanishes. In such a case the higher order corrections will be dominating at low densities, thereby in the case of weakly interacting systems.

To make a men-field energy negligible one needs a system where attractive and repulsive interaction coexist and nearly cancel each other. Such conditions can be met in a two-component mixture where intraspecies scattering lengths are positive, $a_{11}, a_{22} > 0$ but interspecies scattering length is negative, $a_{12} < 0$, [22]. Here $i = 1,2$ enumerates the components.

Alternatively one might consider one component systems where dipole-dipole interactions equalize zero-range repulsion. Such situation can be arranged in a prolate geometry when head-to-tail orientation of magnetic moments of atoms ensures effective attraction of dipoles [23].

Indeed, while decreasing strengths of repulsive contact interactions using a Feshbach resonance, instead of
expected collapse of atomic cloud, the Rosensweig instability leading to formation of an array of self-bound droplets was observed [24]. This experiment triggered intensive experimental and theoretical studies of quantum droplets. They can be formed not only in dipolar systems [25–28] but also in a mixture of two bosonic species [29–32].

Mechanism of formation of droplets is described in [22]. When effective attraction overcomes repulsion, a collapse of the system (increasing of its density) is expected. The mean-field energy is very small and negative, therefore the LHY term comes into a play. Collapse is arrested because of stabilizing, repulsive character of the higher order contribution to the energy.

Formation of quantum droplets is a very spectacular manifestation of higher order corrections to the system energy. Precise knowledge of these contributions is crucial for understanding of droplet’s properties. The LHY contribution originates in quantum fluctuations of Bogoliubov vacuum. It strongly depends on a phase space density, therefore dimensionality of the system matters. Contribution of quantum fluctuations to the mean field-energy of two-component Bose-Bose mixtures in reduced dimensions [33] as well as at dimensional crossover [34, 35] are one of the central issues of theoretical investigations. Similarly, the LHY energy of a one-component Bose gas with dipole-dipole interactions in 1D and 2D geometries [36, 37] and at 2D-3D crossover [38] was found.

Most of theoretical results concerning systems in lower dimensions assume uniform density. However, low dimensional configurations are reached by squeezing atomic clouds in one or two directions by harmonic potentials. Density profiles are not uniform thus. Effect of external optical lattice potential is discussed in [39] but only situations where homogeneous approximation is justified are considered.

The only to date result which accounts for a true density profile is given in [55] where the LHY energy term of a droplet squeezed by 1D or 2D harmonic potential is given. In quasi-2D exclusively strong confinement case is studied. The authors assume an universal regime, where a ratio of the mean-field energy to the confining potential excitation is the single parameter of the system. While calculating Bogoliubov modes, responsible for stabilization of droplet, the local density approximations is used in unconfined directions. Uniform lower-dimensional density, \( n_{2d} \) or \( n_{1d} \), depending on a number of free directions is assumed then. The approach allows to find the LHY energy (or alternatively LHY chemical potential) being the function of the lower-dimensional density. This LHY energy becomes an important contribution to the effective lower-dimensional energy functional, which enables to find droplets density profile in the unconfined direction.

This paper is, to some extend, continuation of studies presented in [55]. We find contribution to the chemical potential originating in quantum fluctuations for a Bose-Bose droplet squeezed by a harmonic potential in one spatial dimension, while not confined in two remaining dimensions. We consider geometrical settings in the entire range of values of aspect ratio, from two to three dimensions – at the whole crossover regime, the most interesting from experimental point of view.

Modified Gapless Hartree-Fock Bogoliubov method (MGHFB), introduced by us in [1] allows to find the LHY energy at the crossover, not assuming uniform density profile in confined direction. The method relies on coupled Generalized Gross-Pitaevski equation (GGP) and Bogoliubov-de Gennes equations. GGP equation accounts for energy related to quantum depletion and anomalous (regularized) density.

We pay a special efforts to assure gapless phonon-like spectrum of excitation. This is highly nontrivial issue, as the most natural attempt to improve over the Bogoliubov approach fails with this respect and contradicts Hugenholtz-Pines theorem. Accurate description of low energy physics is crucial to get right values of beyond mean-field energy. The issue is extensively discussed in [40] in a context of single component Bose system at temperatures close to a critical one and also in [17].

In our studies we focus on a symmetric mixture i.e. \( g = g_{11} = g_{22}, \delta g = g_{12} + g, \ N = N_1 = N_2 \) where \( g = \frac{4\pi R^2 a}{m} \) and so on. We present a method which enables to calculate droplet’s density profile. It depends on four parameters \( g, \delta g, N, \) and harmonic confinement of frequency \( \omega_z \). This is too many for a general treatment. Therefore we focus on a special case.

We assume that mean field interaction energy \( \delta g n \) (here \( n \) is a peak atomic density) as well as contribution to the total energy originating in quantum fluctuations, \( g_{in}, g_{inR} \), are much smaller than harmonic excitation energy \( \hbar \omega_z \) (\( \delta n \) and \( m^R \) are quantum depletion and regularized anomalous density, respectively). In such case deviations of droplet’s density profile from a density of ground state of harmonic confinement is negligible. Similarly as in [55] we assume uniform \( n_{2d} \) density and use the local density approximation in solutions of Bogoliubov equations. Effectively we solve a one-dimensional problem, though.

In the regime described above (which we call universal regime), the three (dimensionless) parameters which control the system, reduce to a single one, \( y = \frac{g_{2d} n_{2d}}{a_{ho} \hbar \omega_z} \approx \frac{g_{2d} \omega_z}{a_{ho}}, \) where 2D atomic density, \( n_{2d} \), is of the order of \( n_{2d} \approx n_{aho}, a_{ho} \) being harmonic oscillator length. We shall mention that mean-field energy, \( g_n \), can be much larger than \( \hbar \omega_z \). As a consequence \( y \) can be much larger than unity.

Our paper is organized as follows. In Sec. II we briefly describe MGHFB method. In Sec. III based on MGHFB approach, we find quantum-fluctuation-contribution (LHY chemical potential) to the chemical potential of the system \( \mu_{LHY}(y) \) as a function of \( y \). In the quasi-2D limit, i.e. if \( y \ll 1 \), a perturbative approach can be used and analytical formula for \( \mu_{LHY} \) can be obtained. On the other hand if \( y \gg 1 \) we expect to recover
where interaction strengths approximated by a contact potentials \( U \sum H \) of the kinetic and potential energy terms, two droplet’s components. Total Hamiltonian includes where chemical potentials of the species and mean fields are equal to the s-wave scattering lengths, \( a \) when inter-scatterings lengths are positive (effective interaction is repulsive) and \( a_{12} \) is negative (effective interaction is attractive). In addition, to simplify calculations, we assume that the two components have equal masses \( m \). Mean values of the field operators are assumed to be different than zero because both species are Bose-condensed. Accordingly, we split these operators to explicitly distinguish the mean field, \( \psi_i = \langle \hat{\psi}_i \rangle \), and small quantum perturbations, \( \delta_i \), \( \hat{\psi}_i = \psi_i + \delta_i \), (3)

The Hamilton equations lead to the following set of the two coupled stationary GP equations:

\[ \mu_1 \psi_1(r) = H_0 \psi_1 + g_{11} |\psi_1|^2 \psi_1 + 2g_{12} |\psi_2|^2 \psi_1, \]
\[ \mu_2 \psi_2(r) = H_0 \psi_2 + g_{22} |\psi_2|^2 \psi_2 + 2g_{12} |\psi_1|^2 \psi_2, \]

if quantum fluctuations are neglected. In the above \( \mu_i \) are chemical potentials of the species and mean fields are normalized to the total number of atoms of each kind \( \int dr |\psi_i|^2 = N_i \). It is quite intuitive that if attractive interaction is weak the gas would fill the whole space allowed by a confining potential. If attractive interaction grows the instability appears at a certain critical value. After crossing this point the gas tends to increase the density. The above mean field approach predicts a transition from a stationary solution to a state which eventually collapses (tends to infinite density). In a simplest case of an uniform mixture of species with equal masses the instability occurs when \( \sqrt{g_{11}g_{12}} + g_{12} \leq 0 \).

D. Petrov \[22\] noticed that at transition point, \( \sqrt{g_{11}g_{12}} + g_{12} = 0 \), where mean field energy vanishes, the higher order contributions to the system energy must be accounted for in Eqs. (4) and (5). These terms originate in quantum fluctuations and are responsible for stopping a collapse and formation of droplets.

In the following, we assume a symmetric situation, i.e. the same interparticle interaction potential for both species, \( U_{11}(r) = U_{22}(r) = U(r) \), and the same number of atoms. Obviously both mean fields are equal then, \( \psi_1(r) = \psi_2(r) \). Excitations can be divided into soft and hard modes which in a symmetric case are:

\[ \hat{\psi}_\pm = \frac{1}{\sqrt{2}} (\hat{\psi}_1 \pm \hat{\psi}_2), \]
\[ \hat{\delta}_\pm = \frac{1}{\sqrt{2}} (\hat{\delta}_1 \pm \hat{\delta}_2). \]

Introducing similar combinations of the mean-fields, \( \psi_+ = \psi_1 + \psi_2 \) and \( \psi_- = \psi_1 - \psi_2 = 0 \) and using Eq. (3) we find that:

\[ \hat{\psi}_+ = \psi_+ + \hat{\delta}_+, \]
\[ \hat{\psi}_- = \hat{\delta}_-. \]

The \( \psi_+(r) \)-component corresponds to the soft-mode mean-field, and \( \psi_-(r) = 0 \) is the mean-field of the hard mode, equal to zero in the symmetric case. It follows from Bogoliubov equations \[22\] \[24\] that fluctuations breaking the symmetry between the species, i.e. described by \( \hat{\delta}_-(r) \), are energetically very costly. These hard mode excitations are characterized by a large sound velocity giving a large contribution to the energy of quantum fluctuations. On the contrary, excitations of the soft mode, \( \hat{\delta}_+(r) \), are characterized by a small sound velocity. Their impact on the LHY energy, close to the critical point, is small. As we shall work close to the critical point we neglect the contribution of the soft modes to the LHY energy.

To simplify notations we define \( \psi(r) \equiv \psi_+(r) \) and \( \hat{\delta}(r) \equiv \hat{\delta}_-(r) \). We additionally assume \( \psi \circ \) be a real function. This way instead of the two mean fields \( \psi_1(r) \) and two quantum fields \( \hat{\delta}_i(r) \) we consider only one mean field - the soft mode mean-field, and one quantum-fluctuation operator - hard mode fluctuations. The problem is simplified thus to a single component Bose field \( \hat{\psi}(r) \) having \( 2N \) atoms in total.

Generalized Gross Pitaevskii equation accounting for quantum fluctuations of the hard mode takes the form:
0 = (H_0 - \mu)\psi(r) + \int d\mathbf{r}' U_s(r - r') \left( \psi^2(r') + \delta n(r', r') \right) \psi(r) \\
+ \int d\mathbf{r}' U_d(r - r') \left( \delta n(r', r) \psi(r') + m(r', r)\psi(r') \right) 

\text{where } \delta n(r', r) = \langle \hat{\delta}^+(r', t)\hat{\delta}(r, t) \rangle, \ m(r', r) = \langle \hat{\delta}(r', t)\hat{\delta}(r, t) \rangle, \ U_s = (U + U_{12})/2 \text{ and } U_d = (U - U_{12})/2. \ 

Quantum contributions \delta n \text{ and } m \text{ are real functions do not depending on time.}

Accounting for quantum fluctuations forces us to treat with a special care both low and high energy components of the interaction potentials \( U_{ij} \). In the following we assume that all potentials \( U_{ij} \) have a bell-like shape of a characteristic widths \( \sigma_{ij} \) respectively, and all widths are of the same order, \( \sigma_{ij} \sim \sigma \), being much larger than the all s-wave scattering lengths, \( |a_{ij}| \sim a \) and much smaller than two other length scales, i.e.:

1) the healing length \( \xi(r) = \frac{\hbar}{\sqrt{m_a n[\sigma]}}, \text{ a quantity determining a radial size of a droplet’s surface, where } n(r) \text{ is atomic density}, \delta g = \sqrt{g_{11}g_{22}} + g_{12} < 0 \text{ and } g_{ij} = \frac{4\pi \hbar^2 a_{ij}}{m_n}, \ 

\text{ii) Characteristic distance } \nu \text{ of density variations. The nonuniform density profile results from squeezing of a droplet in a direction of external potential.}

If \( \alpha \ll \sigma \ll \xi, \nu \), all properties of a droplet depend exclusively on low energy scattering properties of the interaction potentials, namely the s-waves scattering lengths \( a_{ij} \). Therefore potentials \( U_{ij} \) can be approximated by their lowest order Fourier components \( \tilde{u}_{ij}(0) \) which, in turn, can be related to the T-matrix expansion of the scattering potentials, \( g_{ij} = \tilde{u}_{ij}(0) - \frac{1}{i \nu} \int d\mathbf{k} \tilde{\alpha}_u^2(k) \),

where \( E_k = \frac{\hbar^2 k^2}{2m_n} \). Whenever \( \tilde{u}_{ij}(0) \) multiplies a small quantity, like \( \delta n \), i.e. fluctuations of density of atoms, it is sufficient to approximate \( \tilde{u}_{ij}(0) \approx g_{ij} \). This is equivalent to substitution \( U_{ij}(r) = g_{ij}\delta(r) \). However if \( \tilde{u}_{ij}(0) \) multiplies condensate density in a given component, \( n_i \), we shall keep also the second order term in the Born expansion of the T-matrix.

According to the above discussion, the GGPE depends only on the low energy scattering properties of the interaction potentials:

\[
0 = \left( H_0 - \mu + \frac{\delta g}{2} \psi^2(r) + g\delta n(r) + gm^R(r) \right) \psi(r),
\]

where the normalization condition is:

\[
\int d\mathbf{r} \left( \psi^2(r) + \delta n(r) \right) = 2N.
\]

In Eq.\[11\] we introduced \( g = -g_{12} + \delta g \) and assumed \( \psi(r, t) = e^{-i\nu t}\Phi(r) \).

The second order terms depending on high energy modes conspire together with the anomalous density to give a regularized anomalous density (assuming \( U_{12}(r) \simeq -U(r) \)):

\[
gm^R(r) = \int \frac{dk}{(2\pi)^3} \tilde{\alpha}_u^2(k) \psi^2(r) + \int d\mathbf{r}' U(r - r') m(r', r),
\]

which depends only on low momenta \( k \simeq 0 \) part of the interaction potentials i.e. on their scattering length only.

To get stationary Bogoliubov equations we factorize a time dependence of fluctuation operator and use standard expansion into eigenmodes \( u_\nu(r), u_\nu(r), \) i.e. \( \delta(r, t) = e^{-i\nu t} \left( \sum_\nu u_\nu(r) e^{-i\nu t} \hat{\alpha}_\nu + v_\nu(r) e^{i\nu t} \hat{\alpha}_\nu^\dagger \right) \). Combining this expansion with linear Heisenberg equations for \( \delta(r, t) \) gives:

\[
\begin{align*}
(H_0 - \mu_0 + g\psi^2(r)) u_\nu(r) + g\psi^2(r)v_\nu(r) &= \varepsilon_\nu u_\nu(r) \\
(H_0 - \mu_0 + g\psi^2(r)) v_\nu(r) + g\psi^2(r)u_\nu(r) &= -\varepsilon_\nu v_\nu(r).
\end{align*}
\]

Note that in Eq.\[14\] the chemical potential \( \mu \) is substituted by \( \mu_0 \). This is a crucial element of MGHFB method. The chemical potential \( \mu_0 \) has to be found from Bogoliubov equation determining the zero-mode wavefunction, \( u_0(r) = -v_0(r) \):

\[
(H_0 - \mu_0)u_0(r) = 0.
\]

Note, that excitation energy is set to zero, \( \varepsilon_0 = 0 \) in Eq.\[15\]. As discussed in details in \[1\] the replacement \( \mu \rightarrow \mu_0 \) is necessary to get a consistent gapless approach and phononic branch in the excitation spectrum. It ensures that amplitudes of Bogoliubov modes have a correct limit at low energies. Substitution of \( \mu \) by \( \mu_0 \) is justified because \( |\mu - \mu_0| \) is much smaller than characteristic interaction energy term, \( gn \), which enters Bogoliubov equations \[14\]. The replacement is consistent with other approximations, though.

Solutions of the Bogoliubov equations allow to find quantum depletion:

\[
\delta n(r) = \sum_{\nu \neq 0} |u_\nu(r)|^2,
\]

and renormalized anomalous density:

\[
m^R(r) = \frac{\partial}{\partial |\Delta r|} \langle |\Delta r|m(r, \Delta r) \rangle_{\Delta r \rightarrow 0},
\]

where

\[
m(r, r') = \sum_{\nu \neq 0} u_\nu(r)v_\nu^*(r'),
\]
and $R = \frac{x + y}{2}$, and $\Delta r = r - r'$.

We now consider a system where numerical solution of the Bogoliubov equations is necessary. In such case quantum fluctuation terms are to be split into low and high energy parts: $\delta n = \delta n_L + \delta n_H$ and $m_R = m_L + m_H^R$. Low energy components are to be found numerically directly from definitions while the high energy components can be obtained using semiclassical approximation:

$$m_H^R(r) = g\psi^2(r) \frac{m}{2\pi \hbar^2} k_c(R) + g\psi^2(r) \int d\Omega k \int_{k_c(R)}^\infty \frac{k^2 dk}{(2\pi)^3} \left( \frac{1 - \frac{k^2}{m_e}}{2} - \frac{1}{2\varepsilon(k, R)} \right)$$

where

$$\varepsilon(k, r) = \sqrt{(A(k, r))^2 - (g\psi^2(r))^2},$$

where $k_c(r)$ is given by equation $\varepsilon_c(k_c(R), r) = E_c$ and $A(k, r) = \frac{k^2}{m_e} + V(r) - \mu_0 + g\psi^2(r)$. Using the same method we find

$$\delta n_H(r) \simeq \int d\Omega k \int_{k_c(r)}^\infty \frac{k^2 dk}{(2\pi)^3} \left( \frac{A(k, r)}{\varepsilon(k, r)} - 1 \right).$$

Note, that the problem has to be solved self-consistently because $\delta n$ and $m_R$ depend on $\psi$ which in turn is a solution of GGPE, Eq. (11), which involves $\delta n$ and $m_R$ as essential ingredients. The above equations define the self consistent method that enables to determine droplet’s wave function $\psi(r)$.

A short comment on a validity of the ansatz $\psi(r) \simeq \psi_\perp(r_\perp)\phi_0(z)$ is now in order. In fact, assumption that $z$-dependence of the mean-field wavefunction $\psi(r)$ is the same as those of harmonic oscillator $\phi_0(z)$ is approximate. The terms $\frac{\hbar}{2m_\perp} \psi^2(r), g\delta n(r), gm^R(r)$ in Eq. (11) introduce some deviations of $\phi_0(z)$ from the ground state of harmonic oscillator. The ‘back-action’, i.e. an effect of modification the quantities of interest, $\delta n(r)$ and $m^R(r)$, by a ‘disturbed’ $\phi_0(z)$ is negligible because they do not appear explicitly in the Bogoliubov equations, Eq. (14). Neither $\mu_0$ depends on them, Eq. (15). Here we do not take into account this small modification.

### III. Harmonically Confined Quasi-2D Bose-Bose System

#### A. Universal regime

We now move to physically important case of a Bose-Bose mixture confined in one spatial direction (we choose it to be the $z$-direction) by a harmonic potential $V(z) = \frac{1}{2}m_\omega z^2$. The system has a ‘pancake’ geometry and $r_\perp = xe_x + ye_y$ is a vector in a plane perpendicular to $z$-axis. Such a system was analyzed in [33] in a strong confinement limit. Here we want to describe the system in the entire range of possible arrangements, from quasi-2D to 3D geometry what can be achieved by changing strength of the confinement.

To find the LHY contribution to a chemical potential we shall use numerical solutions of Bogoliubov equations to obtain $\delta n$ and $m_R$ at the transition point. We additionally restrict our considerations to such arrangements for which the excitation energy in a tight direction is much larger than ‘low energies of the problem’: $\frac{\hbar}{2m_\perp} \psi^2(r), g\delta n(r), gm^R(r) \ll \hbar_\omega$. Note, however that we do not assume that $\hbar_\omega$ must be larger than $g\psi^2(r)$, therefore our considerations include also 3D case.

Under the above conditions a solution of Eq. (11) is well approximated by

$$\psi(r) = \psi_\perp(r_\perp)\phi_0(z)$$

where $\phi_0(z)$ is normalized to unity ground state of the harmonic oscillator, $\int dz \phi_0^2(z) = 1$. GGPE following from Eq. (11) is

$$\left( \frac{\hbar^2}{2m} \Delta_\perp - \mu + \frac{1}{2} \hbar_\omega z + \frac{\delta q}{2} \psi^2_\perp(r_\perp) \right) \int dz \phi_0^2(z) + g \int dz \left( \delta n(r_\perp, z) + m^R(r_\perp, z) \right) \phi_0^2(z) \psi_\perp(r_\perp) = 0.$$  

(23)

to harmonic oscillator unit of distance $a_{ho} = \sqrt{\frac{\hbar}{m_\omega \omega_\perp}}$ and energy $\hbar_\omega$. Therefore, from now on, wavefunctions $\phi_0(z)$ and $\psi_\perp(r_\perp)$ as well as quantum depletion and anomalous (renormalized) density are dimensionless. We do not introduce new notation for dimensionless quantities, however.

We now notice that characteristic length scale associated with changes of droplet’s density in free directions, $r_\perp$, roughly equals to $\xi = \hbar / \sqrt{m_\omega |\delta q| n}$, and is much larger than $a_{ho}$ (this condition follows from assumption that $|\delta q| n \ll \hbar_\omega$). Therefore we can use local density approximation in $r_\perp$ directions only. We solve Bogoliubov equations using this approximation.

### B. Local density approximation in $r_\perp$-direction

Eq. (23) is written in natural units in order to give a clear physical picture of individual terms. Here we switch
where
\[ u_{k_z,\nu}(r) = e^{ik_zr}u_{k_z,\nu}(z), \]
\[ v_{k_z,\nu}(r) = e^{ik_zr}v_{k_z,\nu}(z) \]
and normalization condition is, \( \int dz \left| u_{k_z,\nu}(z) \right|^2 - \left| v_{k_z,\nu}(z) \right|^2 = 1 \). The above are obtained from Eq. (14) where we substituted \(-\Delta_z\) by \(k_z^2\) and used the ansatz Eq. (22). \( r_\perp \) dependence is hidden in dimensionless parameter, \( y \), given by the ratio of mean-field energy per atom to oscillator excitation energy:

\[ y = \frac{g\phi_0^2(r_\perp)}{a_\hbar\omega_z} = \frac{4\pi a}{a_\hbar\omega_z} \phi_\perp^2(r_\perp). \]

Similarly, Eq. (23) written in the oscillatory units is

\[ \left( -\frac{1}{2}\Delta_z + 2 + \frac{\delta g}{2g} \sqrt{2\pi} + \mu \right) \phi_\perp (r_\perp) = 0. \]

In the above we used \( \phi_0^2(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2) \) and \( \int dz \phi_0^2(z) = \frac{1}{\sqrt{2\pi}} \). From the ground state solution of Eq. (15) we found \( \mu_0 = \frac{1}{2} \). A contribution to the chemical potential originating in quantum fluctuations (related to the Lee-Huang-Yang energy) is denoted by

\[ \Delta \mu(y) = \frac{4\pi a}{a_\hbar \omega_z} \mu_{LHY}(y), \]

where \( \mu_{LHY} \) is:

\[ \mu_{LHY}(y) = \int dz \left( \delta n(y,z) + m_R(y,z) \right) \phi_0^2(z), \]

The chemical potential Eq. (30) is averaged with density profile \( |\phi_0(z)|^2 \) because we reduced 3D GGP equation to 2D form, Eq. (25), by integrating over \( z \)-direction, assuming fixed harmonic oscillator \( z \)-component wavefunction of a droplet.

Formalism presented in this section defines a method of finding droplet’s wave function \( \psi_\perp(r_\perp) \). First, Bogoliubov equations are to be solved (24). The solutions allow to obtain \( \delta n(y,z) \) and \( m_R(y,z) \) (in what follows we describe this calculation in more detail). Using Eq. (30) we calculate \( \mu_{LHY}(y) \) for all values of \( y \). Now, for given values of \( a/a_\hbar \) and \( \delta g/g \), all ingredients of the left hand side of Eq. (28) are uniquely defined, and droplet’s profile \( \psi_\perp(r_\perp) \) and chemical potential \( \mu \) can be found as the eigenstate and the eigenenergy of the GGP equation. The wavefunction gives mean-filed of both species, thus normalization condition reads:

\[ \int dr \left( \psi_\perp^2(r_\perp) \phi_\perp^2(z) + \delta n(y,z) \right) = 2N, \]

where \( \psi_\perp \) enters definition of \( y \). This completes the method of determination of \( \psi_\perp(r_\perp) \).

Finally, we stress that our result are restricted to systems being tightly confined in the \( z \)-direction: \( \frac{7\pi y\delta g}{2m\hbar \omega_z} \ll 1 \) and similarly \( m \approx m_L + m_H \). Low energy regime is defined by conditions \( |k_\perp| \gg k_c \) and \( \varepsilon_{k_\perp,\nu} \leq \frac{k_c^2}{2} + \frac{k_2^2}{2} \).

In the case of nonuniform system when LDA cannot be used the analytic calculations are much more complicated. At high energy sector the semiclassical method can be applied. It gives (for detailed calculations see Appendix A):

\[ m_H(y,z) \simeq -y\phi_0^2(z) \frac{1}{4\pi |\Delta r|} \]

\[ + y\phi_0^2(z) k_c \pi + \frac{\log 4}{8\pi^2} h(z,y,k_c) + m_{R,H}(y,z), \]

where \( h(z,y,k_c) \) and \( m_{R,H}(y,z) \) is given by Eq. (A8) and Eq. (A9) respectively. The main message of Eq. (34) is to singled-out a singular term, \( \propto \frac{1}{4\pi |\Delta r|} \), and obtain a regular but cut-off depend contribution:

\[ m_L(y,z) = m_{R,H}(y,z) + y\phi_0^2(z) k_c \frac{\pi + \log 4}{8\pi^2} h(z,y,k_c), \]

where \( k_c \) is the Fermi wave number. \( m_{L}(y,z) \) is:

\[ m_L(y,z) = \frac{1}{2\pi} \int_0^{k_c} \left| k_c \right| \sum_{\nu \in L} u_{k_z,\nu}(y,z) v_{k_z,\nu}(y,z). \]

"L" denotes a set of these \( \varepsilon_{k_\perp,\nu} \) which satisfy \( \varepsilon_{k_\perp,\nu} \leq \frac{k_c^2}{2} + \frac{k_2^2}{2} \). Evidently the low energy contribution depends
on the cut-off. Similarly, the high energy part of quantum depletion \( \delta n_H(z, y) \) is obtained using semiclassical method in Appendix A and given by Eq. (A14):

\[
\delta n_H(y, z) = \frac{\left(y \phi_0^2(z)\right)^{3/2}}{3\pi^2} f(z, y, k_c).
\]

Low energy component can be found directly from the definition:

\[
\delta n_L(y, z) = \frac{1}{2\pi} \int_0^{k_c} \int_{k-z} \left| v_{k-L}(y, z) \right|^2 dk z.
\]

Finally, contribution of the Bogoliubov-vacuum fluctuations to the chemical potential of the system, after averaging over the \( z \)-direction density profile, Eq. (30) is:

\[
\mu_{LHY}(y) = \mu_{LHY}^H(y) + \mu_{LHY}^L(y),
\]

where (see Eq. (A22)):

\[
\mu_{LHY}^H(y) = \frac{y^{3/2}}{3\pi^{11/4}} \sqrt{\frac{2}{5}} \left(3G(y, k_c) + F(y, k_c)\right) + y k_c \frac{\pi}{8} \frac{4}{\sqrt{2\pi^{3/2}}} H(y, k_c),
\]

and:

\[
\mu_{LHY}^L(y, k_c) = \int dz \phi_0^2(z) (\delta n_L(y, z) + \delta m_L(y, z)).
\]

Although, all individual terms in Eq. [39] depend on the cut-off momentum, \( k_c \), the whole expression does not. In general case \( \mu_{LHY}(y) \) has to be evaluated numerically, however the high energy contribution is regularized and expressed in a form of several integrals leading to smooth functions, \( G(y, k_c), F(y, k_c), H(y, k_c) \) specified in Appendix A, see Eqs. (A19), (A20) and (A21).

This general formalism can be simplified in two regimes. First, when the harmonic confinement is weak, \( y > 1 \) and the system has ‘almost continuous’ spectrum, thus the semiclassical approximation, valid in principle for high momenta components, can be extended over the entire range of energies - from zero up to a cut-off energy, which after renormalization of the anomalous density, can be sent to infinity. This procedure leads to the semiclassical expression for \( \mu_{LHY}^{semi}(y) \), see Appendix A, Eqs. (A35):

\[
\mu_{LHY}^{semi}(y) = \frac{y^{3/2}}{3\pi^{11/4}} \sqrt{\frac{2}{5}} \left(3G_{semi}(y) + F_{semi}(y)\right).
\]

On the other hand if \( y \ll 1 \), perturbative calculations are possible. This is a situation when the harmonic confinement is very tight and Bogoliubov amplitudes can be found in the lowest order perturbation of eigenstates of harmonic oscillator. Details of these tedious calculations are presented in Appendix B. The final result can be summarized as follows:

\[
\mu_{LHY}^{pert}(y) = \frac{y}{8\pi^2} \log \left(\frac{y}{4\pi} C_{2d}^h \sqrt{\epsilon}\right),
\]

where \( C_{2d}^h \approx 28.69 \). Eq. (43) is in agreement with the result given in [33]. This is a very important test of our approach.

Contribution to the chemical potential originating in quantum fluctuations is plotted in Fig. (1). Here we plot \( \mu_{LHY}(y) \) as a function of \( y \). Black line indicates numerical result given by Eq. (39). Semiclassical result obtained with the help of Eq. (42) is depicted by the red line, \( \mu_{LHY}^{semi}(y) \), Eq. (43).

In inset of Fig. (1) we show results for \( y \ll 1 \), where perturbative calculations are in order \( \mu_{LHY}(y) \approx \mu_{LHY}^{pert}(y) \), Eq. (43). This result is plotted by the blue line. If \( y \ll 1 \) the LHY contribution to the chemical potential \( \mu_{LHY} \) is negative. We notice that both black and blue curves are practically identical if \( y < 0.02 \). The strong confinement limit \( y \ll 1 \) is clearly visible in the numerical result, Eq. (43).

Now we shortly discuss the semiclassical result Eq. (42). Naively, it should agree with the LDA expression if \( y \gg 1 \), which can be obtained by replacement of atomic density, \( n \), in the 3D expression describing a uniform system, \( \Delta \mu_{3D} = \frac{3}{2} \sqrt{n} \frac{2\pi}{\hbar} \omega_n \), by a local nonuniform density accounting in the case of 1D harmonic confinement, \( n = n_\perp \phi_0^2(z) \). However, to use this expression in the quasi-2D formalism (2D GGP equation), it has to be integrated over \( z \)-coordinate with 1D density profile, \( \phi_0^2(z) = \frac{1}{a_h \sqrt{\pi}} \exp(-z^2/a_h^2) \), compare Eq. (30):

\[
\Delta \mu_{SI}^{LDA} = \int dz \phi_0^2(z) \Delta \mu_{3D}(n_\perp \psi_0^2(z)).
\]
Please note, that we depart here from dimensionless quantities and $\Delta \mu_{LD}$ and similarly the LDA expression, $\Delta \mu_{LD}^{PA}$ are dimensional quantities (SI units for instance). In order to compare it with $\mu_{LHY}^{semi}$, given by Eq. (A25), one has to divide it by $\hbar \omega z a_0^2$, i.e. we define $\mu_{LHY}^{semi} = \frac{a_0}{\hbar \omega z 4 \pi a_0} \Delta \mu_{LHY}^{semi}$, which after integration, Eq. (44), reads:

$$\mu_{LHY}^{semi} = y^{3/2} \frac{4}{3 \pi^{11/4}} \sqrt{\frac{2}{5}}. \quad (45)$$

By inspection of Eq. (A23) and (A24) one can find that $F_{semi}^{y} = G_{semi}^{y} \approx 1$ if $y \gg 1$. Due to this fact, comparing Eqs. (42) and (45), we find:

$$\mu_{LHY}^{semi} \approx \mu_{LHY}^{semi}$$

for $y \gg 1$.

In the above we found analytic formulas that correctly reproduce the numerical result in the limit $y \ll 1$ and $y \gg 1$. To reconstruct the numerical results in the entire range of variations of the parameter $y$ we introduce the following empirical formula allowing for interpolation between the two above mentioned regions:

$$\mu_{LHY}^{semi} = \left[ \mu_{LHY}^{semi} + Ay^2 \log \left( B y \right) \right] e^{-py^2} + \left( 1 - e^{-py^2} \right) \mu_{LHY}^{semi}(y), \quad (46)$$

where $A, B, p$ are fitted parameters $A = -0.0109$, $B = 1.56$ and $p = 1.93$. The formula Eq. (46) gently switches between perturbative and semiclassical expression. The perturbative result, $y \ll 1$, is ‘enriched’ by the term $\propto y^2 \log \left( B y \right)$. This modification is a smart guess for the next order of the perturbation term. The above fit is compared to the full numerical result in Fig. (2). We believe that this universal smooth expression can be very useful in analysis of experimental data.

**IV. SUMMARY AND FINAL REMARKS**

In this paper we find the beyond mean-field contribution to the chemical potential, the so-called Lee-Huang-Yang term of a two-component quantum droplet which is squeezed by a 2D harmonic potential of frequency $\omega_z$. No local density approximation is assumed in the confined direction. The Lee-Huang-Yang contribution originates in quantum fluctuations of Bogoliubov vacuum and is proportional to sum the of quantum depletion, and the renormalized anomalous density.

The approach is based on the Modified Gapless Hartree-Fock-Bogoliubov method which utilizes generalized Gross-Pitaevskii equation, self-consistently coupled to Bogoliubov equations. To account for a phonon-like excitations and avoid a nonphysical gap in the spectrum we modify Bogoliubov equations by introducing a chemical potential $\mu_0$ being a solution of the zero-energy mode eigenproblem. The quantum LHY term has to be found numerically. The high energy contribution is hardly accessible because of finite spatial grid used in numerical calculations and a singular contribution, to be treated with care. We follow the renormalization procedure introduced in [1]. The semiclassical method is utilized to account for regularized high energy terms.

We focus here on the universal regime, when all low energy terms $g n$, $g m^l$, $g m n$ are much smaller than $\hbar \omega_z$. In such a situation there are only two relevant energy scales - the single component mean-field energy, $g n$, and one-particle energy, $\hbar \omega_z$, separating ground and the first excited state of external potential. Physics of the problem depends then on one universal parameter, $y = g n / \hbar \omega_z$ only. We study the Lee-Huang-Yang chemical potential in the whole extend of possible values of $y$, i.e. our results are valid in the entire range of geometric configurations, from $3D$ oblate shape $\hbar \omega_z < g n$, to $2D$ systems where $\hbar \omega_z \gg g n$.

Results presented here contribute to the longstanding issue of the ground state energy of a weakly interacting Bose system. The Lee-Huang-Yang result is generalized to the case of nonuniform systems. This allow for a theoretical description of quantum droplets not assuming the local density approximation. We focus on the universal regime. The obtained expressions cover the entire range of geometries form quasi-2D to $3D$ settings. Of particular importance are low dimensional systems. In such geometric arrangements the three-body losses should be suppressed. Long-lived droplets will make possible many experiments were not only static droplets, but also dynamical situations are subject to observation.
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Appendix A: Harmonic confinement - semiclassical approach

Here we calculate a high energy contribution to \(\mu_{LHY}\) using semiclassical approximation. From Eq. (24) we find that the quasiparticle energies are:

\[ \varepsilon = \sqrt{\left(\frac{k^2}{2} + \frac{1}{2} z^2 - \frac{1}{2}\right) \left(\frac{k^2}{2} + \frac{1}{2} z^2 - \frac{1}{2} + 2y\phi^2_0(z)\right)}, \]  
(A1)

In the above we notice that if \(z = 0\) and \(k^2 < 1\) the quasiparticle energy is imaginary. That is due to the presence of the zero-point energy of the harmonic oscillator, i.e. the \(\frac{1}{2}\) term. This is 'purely' quantum contribution which is not consistent with the semiclassical approximation and should be neglected:

\[ \varepsilon(k, z) = \frac{1}{2} \sqrt{\left(k^2 + z^2\right) \left(k^2 + z^2 + 4y\phi^2_0(z)\right)}. \]  
(A2)

The low energy region is defined as \(0 \leq k_\perp \leq k_c\) and \(\varepsilon_{k_\perp, z} < \frac{k^2}{2} + \frac{1}{2}\), where obviously, \(k^2 = k_{0z}^2 + k_{0z}^2\). In the semiclassical approximation a value of the \(z\)-component of momentum, \(k_{0z, c}\), which separates low and high energy regions can be found from:

\[ 2\varepsilon(k_{0z, c}, z) = k_{0z}^2 + k_c^2, \]  
(A3)

which gives:

\[ k_{0z, c}^2(z, k_{z, c}) = \sqrt{(k_{\perp}^2 + k_c^2)^2 + (2y\phi^2_0(z))^2} - (k_{\perp}^2 + z^2 + 2y\phi^2_0(z)) \]  
\[ - \left(\frac{1}{2}\right)^{2}\left(k_{\perp}^2 + k_c^2\right) \]  
(A4)
or \(k_{0z, c} = 0\), if the above value is negative. The high energy part of anomalous density is:

\[ m_H(z) = -\frac{y\phi^2_0(z)}{(2\pi)^{\frac{1}{3}}} \int d\mathbf{k} e^{ik_{\perp}x + i[k_{0z} + \Delta z] + i[k_{0z} - \frac{1}{2}]} \frac{1}{\varepsilon(k, z)}. \]  
(A5)

Now we observe that

\[ \frac{1}{\varepsilon(k, z)} = \frac{1}{k_{\perp}^2 + k_c^2} + \frac{1}{\varepsilon(k, z) - \frac{1}{k_{\perp}^2 + k_c^2}} \]  
\[ = \frac{1}{k_{\perp}^2 + k_c^2}. \]  
(A6)

Using this equality we get:

\[ m_H(z) \approx -\frac{y\phi^2_0(z)}{4\pi |\Delta r|} \]  

\[ + m_0^H(z) + \frac{y\phi^2_0(z)}{8\pi^2} k_c (\pi + \log 4) h(z, y, k_c), \]  

This way the singular (in the limit \(|\Delta r| \to 0\)) contribution to the anomalous density was found, \(\sim 1/\Delta r\). The remaining two terms are regular. In particular we introduced:

\[ m_0^H(z) = -\frac{y\phi^2_0(z)}{(2\pi)^{\frac{1}{3}}} \int d\mathbf{k} \left( \frac{1}{2\varepsilon(k, z)} - \frac{1}{k_{\perp}^2 + k_c^2} \right). \]  
(A7)

and function \(h(z, y, k_c)\):

\[ h(z, y, k_c) = \frac{1}{k_c \pi (\pi + \log 4)} \int d\mathbf{k} \frac{1}{k_{\perp}^2 + k_c^2}. \]  
(A8)

Here we split the integration region \(\int d\mathbf{k} = \int_L d\mathbf{k} + \int_H d\mathbf{k}\), where \(\int_L d\mathbf{k} = \int_{k_{\perp} < k_c} d\mathbf{k} + \int_{k_{\perp} + k_c} d\mathbf{k} \) \(\int_H d\mathbf{k}\).

Utilizing explicit expressions we can bring \(m_0^H(z)\) to the form:

\[ m_0^H(z) = \frac{(y\phi^2_0(z))^{3/2}}{\pi^2} g(z, y, k_c), \]  
(A9)

Where function \(g(z, y, k_c)\) is defined as follows:

\[ g(z, y, k_c) = \frac{1}{4} \int_0^{\xi_c} \int_0^{\infty} d\tilde{k}_z t(\tilde{x}_c, \tilde{k}_c) + \frac{1}{4} \int_0^{\infty} d\tilde{x}_c \int_0^{\infty} d\tilde{k}_z t(\tilde{x}_0, \tilde{k}_c) \]  
(A10)

We introduced scaled variables: \(\tilde{x} = \frac{k_{\perp}^2}{2y\phi^2_0}, \tilde{x}_c = \frac{z^2}{2y\phi^2_0}, \tilde{x}_c = \frac{k_{\perp}^2}{2y\phi^2_0}, \tilde{k}_c = \frac{z}{2y\phi^2_0}\), and \(k_{\perp} = \frac{k_{\perp}}{\sqrt{2y\phi^2_0}}\).

In the above the function \(t(\tilde{x}_0, \tilde{x}, \tilde{k}_c)\) is defined:

\[ t(\tilde{x}_0, \tilde{x}, \tilde{k}_c) = \frac{1}{\tilde{x} + \tilde{k}_c^2/2} - \frac{1}{\sqrt{\left(\tilde{x} + \tilde{k}_c^2/2 + \tilde{x}_0 + 1\right)^2 - 1}} \]  

Summarizing the above discussion, we found that regular part of the high energy contribution to the anomalous regularized density, \(m_R^H(z)\), is:

\[ m_R^H(z) = -\frac{y\phi^2_0(z)}{8\pi^2} k_c (\pi + \log 4) h(z, y, k_c) \]  
\[ + \frac{(y\phi^2_0(z))^{3/2}}{\pi^2} g(z, y, k_c). \]  
(A11)

This high-energy component to the regularized anomalous density has to be supplemented by the low energy contribution, \(m_L(z)\). This should be calculated directly from the definition provided that numerical solutions of the Bogoliubov equations are found.

Instead, we can extend the semiclassical calculations to the entire range of excitations energies. These procedure is not legitimate in the entire range of variation of \(y\). On the the hand if \(y \geq 1\) the harmonic confinement is weak and \(\hbar \omega_z \ll gn\), therefore excitation spectrum is dominated by a spectrum of free particle. In such a situation semiclassical approach can be extended over entire energy range. Semiclassical estimation of quantum depletion and anomalous density can be obtained without solving numerically Bogoliubov equations. It might be illuminating though to find semiclassical anomalous density and quantum depletion and compare to rigorous results.
Semiclassical expression for a renormalized anomalous density is analogue to Eq. (A6). The singular term there should be omitted (renormalization) and the cut-off dependent term vanishes. Moreover, the integration in expression giving $m^0_H$, Eq. (A9) has to be extended over whole momenta range. It is convenient therefore, to introduce the function $g_{semi}(\tilde{x}_0) = g_{semi}(z, y)$:

$$g_{semi}(\tilde{x}_0) = \frac{1}{4} \int_0^\infty d\tilde{x} \int_0^\infty d\tilde{k}_z t(\tilde{x}_0, \tilde{x}, \tilde{k}_z). \quad (A12)$$

Renormalized anomalous density calculated semiclassically in the entire range of energies is given by

$$m^R_{semi}(z, y) = -\frac{y \phi_0^2(z)}{(2\pi)^3} \int dk \left( \frac{1}{2\varepsilon(k, z)} - \frac{1}{k^2} \right)$$

$$= \frac{1}{\pi^2} (y \phi_0^2(z))^{3/2} g_{semi} \left( \frac{z^2}{2y \phi_0^2(z)} \right). \quad (A13)$$

Now we turn to calculation of the quantum depletion. The high-energy part of quantum depletion obtained within the semiclassical approximation takes the form

$$\delta n_H(z, y, k_c) = \frac{(y \phi_0^2(z))^{3/2}}{3\pi^2} f(z, y, k_c)$$

$$= \frac{1}{2} \int H \left( \frac{k_0^2 + k_c^2}{2\varepsilon(k, z)} \right) + (z^2 - 1) + z \phi_0^2(z) - 1 \right), \quad (A14)$$

where we introduced function $f(\tilde{x}_0, \tilde{x}_c) = f(z, y, k_c)$:

$$f(\tilde{x}_0, \tilde{x}_c) = \frac{3}{4} \int_0^{\tilde{x}_c} d\tilde{x} \int_{k_{c,}\tilde{k}_z}^{\infty} d\tilde{k}_z s(\tilde{x}_0, \tilde{k}_z)$$

$$+ \frac{3}{4} \int_{\tilde{x}_c}^{\infty} d\tilde{x} \int_{k_{c,}\tilde{k}_z}^{\infty} d\tilde{k}_z s(\tilde{x}_0, \tilde{k}_z), \quad (A15)$$

While calculating $\mu_{LHY}$ we arrive at a number of integrals for which we introduce a shorthand notation:

$$\int dz \ m_H^R(z) |\phi_0(z)|^2 = \frac{y^{3/2}}{\pi^2} \int dz \ \phi_0^5(z) g \left( \frac{z^2}{2y \phi_0^2(z)}, \frac{k_c^2}{2y \phi_0^2(z)} \right) \equiv \frac{y^{3/2}}{\pi^{11/4}} \sqrt{\frac{2}{5}} G(y, k_c). \quad (A19)$$

$$\int dz \ \delta n_H(z) |\phi_0(z)|^2 = \frac{y^{3/2}}{3\pi^2} \int dz \ \phi_0^5(z) f \left( \frac{z^2}{2y \phi_0^2(z)}, \frac{k_c^2}{2y \phi_0^2(z)} \right) \equiv \frac{y^{3/2}}{3\pi^{11/4}} \sqrt{\frac{2}{5}} F(y, k_c). \quad (A20)$$

$$\int dz \ y k_c \frac{\pi + \log 4}{8\pi^2} \phi_0^4(z) h(z, y, k_c) \equiv y k_c \frac{\pi + \log 4}{8\sqrt{2}\pi^{5/2}} H(y). \quad (A21)$$

These integrals, while brought together, give a high-energy contribution, $\mu_{LHY}^H$, to the chemical potential of a symmetric two-component quantum droplet squeezed in $x$-direction by an external harmonic potential:

$$\mu_{LHY}^H = \frac{y^{3/2}}{3\pi^{11/4}} \sqrt{\frac{2}{5}} (3G(y, k_c) + F(y, k_c))$$

$$+ y k_c \frac{\pi + \log 4}{8\sqrt{2}\pi^{5/2}} H(y). \quad (A22)$$

and $s(\tilde{x}_0, \tilde{k}_z) = s(z, y, k_c)$:

$$s(\tilde{x}_0, \tilde{k}_z) = \sqrt{\left( \frac{\tilde{x} + \tilde{k}_z^2 + \tilde{x}_0 + 1}{\tilde{x} + \tilde{k}_z^2 + \tilde{x}_0 + 1} \right)^2 - 1}. \quad (A16)$$

Again, the high-energy contribution to the quantum depletion has to be supplemented by the low-energy term $\delta n_L$ which depends on solutions of the Bogoliubov equations and should be found directly from the definition by numerical computations.

Similarly as in the case of anomalous regularized density, we can extend integration over the entire range of energies and obtain the semiclassical expression:

$$\delta n_{semi}(z, y) = \frac{1}{3\pi^2} \left( y \phi_0^2(z) \right)^{3/2} f_{semi} \left( \frac{z^2}{2y \phi_0^2(z)} \right), \quad (A17)$$

where we introduced $f_{semi}(\tilde{x}_0) = f_{semi}(z, y)$:

$$f_{semi}(\tilde{x}_0) = \frac{3}{2\sqrt{2}} \int_{\tilde{x}_0}^{\infty} d\tilde{x} \sqrt{\tilde{x} - \tilde{x}_0} \left( \frac{\tilde{x} + 1}{\sqrt{\tilde{x} + 2}} - 1 \right). \quad (A18)$$
If the semiclassical approach is extended over the whole range of the energy spectrum, the following integrals are essential:

\[
\int dz \, m_{\text{semi}}^R(z) \, |\phi_0(z)|^2 = \frac{y^{3/2}}{\pi^{1/4}} \frac{2}{5} G_{\text{semi}}(y), \tag{A23}
\]

\[
\int dz \, \delta n_{\text{semi}}(z) |\phi_0(z)|^2 = \frac{y^{3/2}}{3\pi^{1/4}} \frac{2}{5} F_{\text{semi}}(y). \tag{A24}
\]

A semiclassical expression for the chemical potential resulting from quantum fluctuations is:

\[
\mu_{\text{LHY}}(y) = \frac{y^{3/2}}{3\pi^{1/4}} \frac{2}{5} (3G_{\text{semi}}(y) + F_{\text{semi}}(y)). \tag{A25}
\]

We want to stress that all integrals involved in the final results are regular and free of any singularities. Their numerical evaluation does not present any technical problems. Their evaluation directly from definitions is practically impossible. Finite spatial greed introduces a cut-off in high-momentum space and, because of a singularity at high momenta, gives uncontrolled results.

Appendix B: Harmonic confinement - perturbative approach

In this section we consider a special case of a very strong harmonic confinement in \(z\)-direction. Because oscillator excitation energy, \(\hbar \omega_z\), is large as compared to all other energy scales, the chemical potential in particular, we will use perturbative solutions of Bogoliubov equations. We assume that Bogoliubov modes are very similar to the oscillator eigenstates and are only slightly perturbed by the mean-field interaction. A ratio of the mean-field interaction energy to the excitation energy \(y = g_{\perp}^2/(\hbar \omega_z) \ll 1\) is the small parameter. We introduced here the 2D atomic density \(n_\perp\) in the \(x-y\) plane. Perturbative calculations allow for a comparison of our results to those of [33], obtained using another method.

We start from rewriting the Bogoliubov equations Eq. (24) and introducing \(f^{\pm}_{x,\nu} = u_{\nu} \mp v_{\nu}\):

\[
(x + H_z) f^+_{x,\nu} = \varepsilon_{x,\nu} f^+_{x,\nu}, \tag{B1}
\]

\[
(x + H_z) f^-_{x,\nu} = \varepsilon_{x,\nu} f^-_{x,\nu}, \tag{B2}
\]

where \(H_z = -1/2 \partial_x^2 + 1/2 z^2 - 1/2\) and \(x = \frac{z^2}{2}\).

Next we find that:

\[
(x + H_z) f^+_{x,\nu} f^-_{x,\nu} = \varepsilon_{x,\nu}^2 f^+_{x,\nu} f^-_{x,\nu} = \varepsilon_{x,\nu}^2. \tag{B3}
\]

\[
(x + H_z) (x + H_z) f^+_{x,\nu} f^-_{x,\nu} = \varepsilon_{x,\nu}^2 f^+_{x,\nu} f^-_{x,\nu}. \tag{B4}
\]

By bringing Bogoliubov equations Eqs. (B1), (B2) to the form above, Eqs. (B3), (B4), we effectively ‘squared’ them. We observe now that these squared equations are Bogoliubov equations for the ‘squared’ free Hamiltonian:

\[
H_0 = (x + H_z)^2, \tag{B5}
\]

with effective interactions

\[
H_+ = 2y \phi_0^2(z) (x + H_z), \tag{B6}
\]

\[
H_- = (x + H_z) 2y \phi_0^2(z), \tag{B7}
\]

so that Eqs. (B3) and (B4) take the form

\[
(H_0 + H_+) f^+_{x,\nu} = E_{x,\nu} f^+_{x,\nu}, \tag{B8}
\]

\[
(H_0 + H_-) f^-_{x,\nu} = E_{x,\nu} f^-_{x,\nu}, \tag{B9}
\]

where \(E_{x,\nu} = \varepsilon_{x,\nu}^2\). The zero order equation is:

\[
f^+\nu_0(z) = f^{\nu_0}_x(0) f^+\nu_0(z), \tag{B10}
\]

and has solutions:

\[
f_{x,\nu}^{\pm}(0) = f_{x,\nu}^{\pm}(0) \phi_\nu(z), \tag{B11}
\]

\[
E_{x,\nu} = (\nu + 1)^2, \tag{B12}
\]

where \(\phi_\nu(z)\) is the eigenstate of the harmonic oscillator normalized to unity i.e. \(\int dz \, |\phi_\nu(z)|^2 = 1\).

Now we turn to the the first order perturbation theory. For \(\nu = 0\) the Bogoliubov energy is:

\[
\varepsilon_{x,0}^2 = E_{x,0}^{(0)} + E_{x,0}^{(1)} = x^2 + 2xy c_{0,0}, \tag{B13}
\]

and corresponding eigenmodes are:

\[
f_0^+ x,0(\nu) \approx f^+_{x,0} \left( \phi_0(z) + \sum_{\nu > 0} \frac{2y \nu c_{\nu,0}}{x^2 - (\nu + \nu)^2} \phi_\nu(z) \right), \tag{B14}
\]

\[
f_0^- x,0(\nu) \approx f^-_{x,0} \left( \phi_0(z) + \sum_{\nu > 0} \frac{2y (\nu + \nu) c_{\nu,0}}{x^2 - (\nu + \nu)^2} \phi_\nu(z) \right), \tag{B15}
\]

where \(c_{\nu,0} = \int dz \, \phi_\nu^* (z) \phi_0^2(z) \phi_\nu(z)\). In particular \(c_{0,0} = 1/\sqrt{2\pi}\).

Inserting the above into Eq. (B2) we obtain

\[
x f_{x,0}^+ \phi_0(z) + \sum_{\nu > 0} \frac{2y (\nu + \nu) c_{\nu,0}}{x^2 - (\nu + \nu)^2} \phi_\nu(z) \tag{B14}
\]

\[
= \varepsilon_{x,0} \phi_{x,0}^+ \phi_0(z) + \sum_{\nu > 0} \frac{2y (\nu + \nu) c_{\nu,0}}{x^2 - (\nu + \nu)^2} \phi_\nu(z), \tag{B15}
\]

where \(\varepsilon_{x,0}\) is given by Eq. (B13). From the above we find that

\[
x f_{x,0}^+ = \varepsilon_{x,0} f_{x,0}^- \tag{B15}
\]

We additionally have the normalization condition which in the first order of perturbation reads

\[
E_{x,0}^{(0)} f_{x,0}^{(0)} = 1. \tag{B16}
\]
This completes the first order calculation for \( \nu = 0 \) component. We found:
\[
\varepsilon_{x,0} = \sqrt{x^2 + 2xye_{0,0}},
\]  
(\text{B17})
\[
f_{x,0}^{\pm(0)} = \left( \frac{\varepsilon_{x,0}}{x} \right)^{\pm 1/2},
\]  
(\text{B18})
which together with Eq. (\text{B14}) give explicit expressions for \( f_{x,0}^{\pm}(z) \). Thus \( \nu = 0 \) mode contribution to \( \mu_{LHY} \) is given by the term \( \sim \left( m(z) + \delta n(z) \right) \) \( \nu = 0 \)
\[
- \frac{1}{4\pi} \int dx F_{x,0}(z).
\]  
In Eq. (\text{B26}) we account only for low energy excitation of the system. We set the upper limit of integration as well as summation finite. We remind that we introduced notation \( \frac{k_{\perp}^2}{2} = x \) and \( \frac{k_{z}^2}{2} = x_c \). In sec. III the low energy sector is defined as \( \varepsilon_{k_{\perp},\nu} \leq \frac{k_{\perp}^2}{2} + \frac{k_{z}^2}{2} \) and \( k_{\perp} \leq k_c \). Now, we must set the upper limit of summation in Eq. (\text{B26}).
From Eq. (\text{B22}) we have
\[
\varepsilon_{x,\nu} - (x + \nu) \simeq y \int dz \phi_{\nu}^2(z) \phi_{\nu}^0(z) \propto \frac{y}{\sqrt{\nu}},
\]  
(\text{B27})
i.e. for large \( \nu \) the energy \( \varepsilon_{x,\nu} \) can be approximated by \( \varepsilon_{x,\nu} \approx (x + \nu) \). In the view of this equality one can clearly see that the condition \( \varepsilon_{k_{\perp},\nu} \leq \frac{k_{\perp}^2}{2} + \frac{k_{z}^2}{2} = x + x_c \) will be met for \( \nu \leq x_c \). This justifies why the upper limit of summation is \( \nu = [x_c] \), where \([x_c]\) is integer part of \( x_c \). On the other hand summation over \( \nu' \) – the intermediate states, is extended up to infinity.
Writing explicitly contribution form \( \nu = 0 \) mode, Eq. (\text{B26}) becomes
\[
m_L(z) + \delta n_L(z) = - \frac{1}{4\pi} \int_0^{x_c} dx \left( \phi_0^2(z)(1 - (f_{x,0}^{-}(z))^2)ight) + y \sum_{\nu' > 0} \phi_0(z)\phi_{\nu'}(z) \frac{c_{\nu,\nu'}}{x + \frac{\nu}{2}} + y \sum_{0 < \nu \leq x_c} \sum_{\nu' > 0} \phi_{\nu}(z)\phi_{\nu'}(z) \frac{c_{\nu,\nu'}}{x + \frac{\nu + \nu'}{2}}.
\]  
(\text{B28})
Contribution to the chemical potential from the LHY energy is \( \mu_{LHY} = \int dz \phi_0(z)^2(m_L(z) + \delta n_L(z)) \), therefore:
\[
-4\pi \mu_{LHY}^{\text{stag}} = \int_0^{x_c} dx \left( (c_{0,0}(1 - (f_{x,0}^{-}(z))^2) + y \sum_{\nu' > 0} \frac{c_{0,\nu'}^2}{x + \frac{\nu}{2}} + y \sum_{x_c \geq \nu' > 0} \sum_{0 < \nu} \frac{c_{\nu,\nu'}^2}{x + \frac{\nu + \nu'}{2}} \right).
\]  
(\text{B29})
\( \mu_{LHY}^{\text{stag}} \) contains a singular contribution. LHY chemical potential \( \mu_{LHY} \) involves not anomalous, \( m_L(z) \), but regularized anomalous density, \( m^R(z) \). As follows from Eq. (\text{35}), the regularization in limit of \( k_c \to \infty \) amounts to addition of a cut-off depending term (the final result does not depend on cut-off, however): \( m^R(z) = m_L(z) + \phi_0^2(z) \frac{k_c^2}{8\pi} (\pi + \log 4) \). Therefore chemical potential \( \mu_{LHY} \) is:
\[
\mu_{LHY} = \left( \mu_{LHY}^{\text{stag}} + yc_{0,0} \frac{k_c^2}{8\pi} (\pi + \log 4) \right).
\]  
(\text{B30})
The second term in Eq. (B30) can be written in the integral form:

\[
y c_0,0 \frac{k_e}{8\pi^2} (\pi + \log 4) = \\
y c_0,0 \frac{1}{4\pi} \int_0^{x_c} dx \frac{1}{\pi \sqrt{2}} \int_0^{x_c} d\nu_z \frac{1}{\sqrt{\nu_z x + \nu_z}}.
\]

As a result we have:

\[
\mu_{\text{LHY}} = \\
- \frac{1}{4\pi} \int_0^{x_c} dx \left( c_{0,0} (1 - (f_{x,0,0})^2) + y \sum_{\nu' > 0} \frac{c_{\nu',0}^2}{x + \frac{1}{2}} \right) \\
y \sum_{x_c > \nu > 0} \frac{c_{\nu',0}^2}{x + \frac{1}{2} x + \nu_z} - y \frac{1}{\pi} \int_0^{x_c} d\nu_z \frac{1}{\sqrt{2\nu_z x + \nu_z}} c_{0,0}.
\]

Using \((f_{x,0,0})^2 = \left( \frac{x}{x_{x,0}} \right) \), Eq. (B17), and \( \varepsilon_{x,0} = \sqrt{x^2 + 2xy_{c,0,0}} \), Eq. (B18), we get:

\[
\int_0^{x_c} dx c_{0,0} (1 - (f_{x,0,0})^2) = \\
\int_0^{x_c} dx \frac{2y_{c,0,0} x}{(x + \varepsilon_{x,0}) \varepsilon_{x,0}} = y c_{0,0}^2 \left( -1 + \log \left( \frac{2x_c}{y_{c,0,0}} \right) \right).
\]

This is the perturbative expression for the LHY contribution to the chemical potential, valid in the limit of a very strong confinement in z-direction, \( y \ll 1 \). Summations and integrations in Eq. (B33) can be completed in quite tedious calculations in the limit of \( x_c \to \infty \) giving the value \( c_{2d} = 28.69 \) which is in agreement with [33].

In the above one can identify a term proportional to \( y \log y \) and terms proportional to to \( y \). Therefore the final formula can be simplified and written in the form:

\[
\mu_{\text{LHY}}^{\text{pert}} = \mu_{\text{LHY}}(y) = \frac{y}{8\pi^2} \log \left( \frac{C_{2d} \sqrt{y}}{4\pi} \right).
\]
[22] D. S. Petrov, Phys. Rev. Lett. 115, 155302 (2015)
[23] K. Góral, L. Santos, and M. Lewenstein, Phys. Rev. Lett. 88, 170406 (2002)
[24] H. Kadau, M. Schmitt, M. Wenzel, C. Wink, T. Maier, I. Ferrier-Barbut, and T. Pfau, Nature 530, 194 (2016)
[25] I. Ferrier-Barbut, H. Kadau, M. Schmitt, M. Wenzel, and T. Pfau, Phys. Rev. Lett. 116, 215301 (2016)
[26] M. Schmitt, M. Wenzel, F. Böttcher, I. Ferrier-Barbut, and T. Pfau, Nature 539, 259 (2016)
[27] I. Ferrier-Barbut, M. Schmitt, M. Wenzel, H. Kadau, and T. Pfau, Journal of Physics B: Atomic, Molecular and Optical Physics 49, 214004 (2016)
[28] L. Chomaz, S. Baier, D. Petter, M. J. Mark, F. Wächtler, L. Santos, and F. Ferlaino, Phys. Rev. X 6, 041039 (2016)
[29] C. R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, P. Cheiney, and L. Tarruell, Science 359, 301 (2018)
[30] P. Cheiney, C. R. Cabrera, J. Sanz, B. Naylor, L. Tanzi, and L. Tarruell, Phys Rev Lett 120, 135301 (2018)
[31] G. Semeghini, G. Ferioli, L. Masi, C. Mazzinghi, L. Wolswijk, F. Minardi, M. Modugno, G. Modugno, M. Inguscio, and M. Fattori, Phys Rev Lett 120, 235301 (2018)
[32] C. D’Errico, A. Burchianti, M. Prevedelli, L. Salasnich, F. Ancilotto, M. Modugno, F. Minardi, and C. Fort, Phys. Rev. Research 1, 033155 (2019)
[33] D. S. Petrov and G. E. Astrakharchik, Phys. Rev. Lett. 117, 100401 (2016)
[34] P. Zin, M. Pylak, T. Wasak, M. Gajda, and Z. Idziaszek, Phys. Rev. A 98, 051603 (2018)
[35] T. Ilg, J. Kumlin, L. Santos, D. S. Petrov, and H. P. Büchler, Phys. Rev. A 98, 051604 (2018)
[36] D. Edler, C. Mishra, F. Wächter, R. Nath, S. Sinha, and L. Santos, Phys Rev. Lett. 119, 050403 (2017)
[37] K. Jachymski and R. Oldziejewski, Phys. Rev. A 98, 043601 (2018)
[38] P. Zin, M. Pylak, T. Wasak, K. Jachynski, and Z. Idziaszek, Journal of Physics B: Atomic, Molecular and Optical Physics 54, 165302 (2021)
[39] J. Kumlin, K. Jachymski, and H. P. Büchler, Phys. Rev. A 99, 033622 (2019)
[40] S. A. Morgan, Journal of Physics B: Atomic, Molecular and Optical Physics 33, 3847 (2000)
[41] L. Pitaevskii and S. Stringari, Bose-Einstein Condensation and Superfluidity, International Series of Monographs on Physics (Oxford University Press, Oxford, 2016) p. 576.
[42] B. Oleś and K. Sacha, Journal of Physics A: Mathematical and Theoretical 41, 145005 (2008)