JOHN-NIRENBERG INEQUALITY AND ATOMIC DECOMPOSITION FOR NONCOMMUTATIVE MARTINGALES

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Abstract. In this paper, we study the John-Nirenberg inequality for \( B_{\infty} \) and the atomic decomposition for \( H_1 \) of noncommutative martingales. We first establish a crude version of the column (resp. row) John-Nirenberg inequality for all \( 0 < p < \infty \). By an extreme point property of \( L_p \)-space for \( 0 < p \leq 1 \), we then obtain a fine version of this inequality. The latter corresponds exactly to the classical John-Nirenberg inequality and enables us to obtain an exponential integrability inequality like in the classical case. These results extend and improve Junge and Musat’s John-Nirenberg inequality. By duality, we obtain the corresponding \( q \)-atomic decomposition for different Hardy spaces \( H_q \) for all \( 1 < q \leq \infty \), which extends the \( 2 \)-atomic decomposition previously obtained by Bekjan et al. Finally, we give a negative answer to a question posed by Junge and Musat about \( B_{\infty} \).

1. Introduction

This paper deals with BMO spaces and atomic decomposition for noncommutative martingales. The modern period of development of noncommutative martingale inequalities began with Pisier and Xu’s seminal paper [18] in which they established the noncommutative Burkholder-Gundy inequalities and Fefferman duality theorem between \( H_1 \) and \( B_{\infty} \). Since then remarkable progress has been made in the field. We refer, for instance, to [6], [9], [11], [20] for other noncommutative martingales inequalities, to [14], [1] for interpolation of noncommutative Hardy spaces and to [16], [17] for the noncommutative Gundy and Davis decompositions. Let us also mention two other works that motivate the present paper. The first one is Junge and Musat’s noncommutative John-Nirenberg theorem [8] and the second the 2-atomic decomposition of the Hardy spaces \( H_1 \) by Bekjan, Chen, Perrin and Yin [1].

Before describing our main results, we recall the classical John-Nirenberg inequalities in the martingale theory. Let \( (\Omega, \mathcal{F}, P) \) be a probability space

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and \((\mathcal{F}_n)_{n \geq 0}\) an increasing sequence of sub-\(\sigma\)-algebras of \(\mathcal{F}\) with the associated conditional expectations \((\mathbb{E}_n)_{n \geq 0}\). The \(BMO(\Omega)\) space is defined as the set of all \(x \in L_1(\Omega)\) with the norm \[\|x\|_{BMO} = \sup_n \|\mathbb{E}_n |x - x_{n-1}|\|_{\infty} < \infty.\] 

The classical John-Nirenberg theorem says that there exist two universal constants \(c_1, c_2 > 0\) such that if \(\|x\|_{BMO} < c_2\), then 

\[\sup_n \|\mathbb{E}_n (e^{c_1 |x - x_{n-1}|})\|_{\infty} < 1.\] 

This statement is equivalent to the following one: There exists an absolute constant \(c\) such that for all \(1 \leq p < \infty\), 

\[\|x\|_{BMO} \leq \sup_n \|\mathbb{E}_n |x - x_{n-1}|^p\|^{\frac{1}{p}}_{\infty} \leq cp\|x\|_{BMO}.\] 

A duality argument yields 

\[\|\mathbb{E}_n |x - x_{n-1}|^p\|^{\frac{1}{p}}_{\infty} = \sup_{b \in L_\infty(\mathcal{F}_n), \|b\|_p \leq 1} \left(\int |x - x_{n-1}|^p b \, d\mathbb{P}\right)^{\frac{1}{p}}\] 

Furthermore, by the extreme point property of \(L_1(\mathcal{F}_n)\) and (1.4), the John-Nirenberg theorem (1.3) can be rewritten as follows 

\[\|x\|_{BMO} \leq \sup_n \|\mathbb{E}_n |x - x_{n-1}|^p\|^{\frac{1}{p}}_{\infty} \leq cp\|x\|_{BMO}.\] 

Accordingly, (1.2) can be reformulated as: For any \(n \geq 1, E \in \mathcal{F}_n\) and \(\lambda > 0\), 

\[\frac{1}{\mathbb{P}(E)} \mathbb{P}\left\{\omega \in E : |x(\omega) - x_{n-1}(\omega)| > \lambda\right\} \leq c_2 \exp(-c_1 \lambda/\|x\|_{BMO}).\] 

Junge and Musat [8] proved a noncommutative version of John-Nirenberg theorem corresponding to (1.5). To state their result we need fix some notation. Let \(\mathcal{M}\) be a von Neumann algebra with a normal faithful tracial state \(\tau\). Let \((\mathcal{M}_n)_{n \geq 1}\) be an increasing sequence of von Neumann subalgebras of \(\mathcal{M}\) such that the union of \(\mathcal{M}_n\)'s is \(w^*\)-dense in \(\mathcal{M}\). Let \(\mathcal{E}_n\) be the conditional expectation of \(\mathcal{M}\) with respect to \(\mathcal{M}_n\). Define 

\[\|x\|_{BMO^c} = \sup_{n \geq 1} \|\mathcal{E}_n |x - x_{n-1}|^2\|^{\frac{1}{2}}_{\infty}\]

and 

\[BMO(\mathcal{M}) = \{x \in L_1(\mathcal{M}) : \|x\|_{BMO} < \infty\}\]

with 

\[\|x\|_{BMO} = \max\{\|x\|_{BMO^c}, \|x^*\|_{BMO^c}\}.\]

Then Junge and Musat’s John-Nirenberg inequality reads as follows: There exists an absolute constant \(c\) such that for all \(2 \leq p < \infty\), 

\[\|x\|_{BMO} \leq B_p(x) \leq cp\|x\|_{BMO^c},\]
where

\[ B_p(x) = \max \left\{ \sup_n \sup_{b \in M_n, \|b\|_p \leq 1} \| (x - x_{n-1})b \|_p, \right. \\
\left. \sup_n \sup_{b \in M_n, \|b\|_p \leq 1} \| b(x - x_{n-1}) \|_p \right\}. \]

However, this theorem does not correspond to the commonly used form of the classical John-Nirenberg inequality. On the other hand, it does not hold (see Remark 3.14 for a counterexample) when considering \( \text{BMO}^c(M) \) or \( \text{BMO}^r(M) \) separately. The first purpose of this paper is to remedy these aspects of Junge and Musat’s theorem. The following is one of our main results. We refer to the next section for all spaces and notations used below. \( \mathcal{P}(M) \) denotes the set of all projections of \( M \).

**Theorem A.** For \( 0 < p < \infty \), we have
\[ \alpha_p^{-1} \|x\|_{\text{BMO}} \leq \mathcal{P}B_p(x) \leq \beta_p \|x\|_{\text{BMO}}, \]
where
\[ \mathcal{P}B_p(x) = \max \left\{ \sup_n \sup_{e \in \mathcal{P}(M_n)} \| (x - x_{n-1}) \frac{e}{(\tau(e))^{1/p}} \|_p, \right. \\
\left. \sup_n \sup_{e \in \mathcal{P}(M_n)} \| \frac{e}{(\tau(e))^{1/p}}(x - x_{n-1}) \|_p \right\}. \]

The two constants \( \alpha_p \) and \( \beta_p \) have the following properties
(i) \( \alpha_p = 1 \) for \( 2 \leq p < \infty \);
(ii) \( \alpha_p \leq C^{1/p-1/2} \) for \( 0 < p < 2 \);
(iii) \( \beta_p \leq cp \) for \( 2 \leq p < \infty \);
(iv) \( \beta_p = 1 \) for \( 0 < p < 2 \).

This result goes beyond Junge/Musat’s result in two aspects. First we extend their result to all \( 0 < p < \infty \). Second, the \( b \)'s in the definition of \( B_p(\cdot) \) are reduced to projections \( e \)'s in \( \mathcal{P}B_p(\cdot) \), which corresponds exactly to the form (1.6) in the classical case. Furthermore, the optimal constants \( \beta_p \) in Theorem A enable us to formulate John-Nirenberg inequality that corresponds to the form (1.7). That is, let \( x \in \text{BMO}(M) \), then for all natural numbers \( n \geq 1 \), all \( e \in \mathcal{P}(M_n) \) and for all \( \lambda > 0 \), we have
\[ \frac{1}{\tau(e)} \tau(\mathbb{1}_{(\lambda, \infty)}(|(x - x_{n-1})e|) + \mathbb{1}_{(\lambda, \infty)}(|e(x - x_{n-1})|)) \leq 4 \exp(-\frac{c\lambda}{\|x\|_{\text{BMO}}}) \]
with \( c \) an absolute constant.

By the essentially same idea, we establish similar results for \( \text{BMO}^c(M) \) and \( \text{BMO}^r(M) \) separately, but only with \( 2 \leq p < \infty \) (see Remark 3.9).

We now turn to the second objective of this paper: the atomic decomposition of different noncommutative Hardy spaces. Let us recall the 2-atomic decomposition obtained in [1]. An element \( a \in L_1(M) \) is said to be a \((1,2)_c\)-atom with respect to \((M_n)_{n\geq 1}\), if there exist \( n \geq 1 \) and \( e \in \mathcal{P}(M_n) \) such that
(i) $\mathcal{E}_n(a) = 0$; (ii) $ae = a$; (iii) $\|a\|_2 \leq (\tau(e))^{-1/2}$.

The atomic Hardy space $h_{1,\text{at}}^c(\mathcal{M})$ is defined as the space of all $x \in L_1(\mathcal{M})$, such that the following norm is finite,

$$\|x\|_{h_{1,\text{at}}^c} = \|\mathcal{E}_1x\|_1 + \inf \sum_j |\lambda_j|.$$ 

Here the infimum is taken for possible decompositions $x - \mathcal{E}_1x = \sum_j \lambda_j a_j$ with $\lambda_j \in \mathbb{C}$, $a_j$ being $(1,2)_c$-atom. It is proved in [1] that $x \in h_{1,\text{at}}^c(\mathcal{M})$ if and only if $x \in h_{1,\text{at}}^c(\mathcal{M})$ and

$$\|x\|_{h_{1,\text{at}}^c} \simeq \|x\|_{h_{1,\text{at}}^c}.$$ 

Together with the equivalence $H_{1,\text{at}}^c(\mathcal{M}) = h_{1,\text{at}}^c(\mathcal{M}) + h_{d,1}(\mathcal{M})$, the authors of [1] also obtained a 2-atomic decomposition for $H_{1,\text{at}}^c(\mathcal{M})$.

Let us briefly recall the argument used in [1]. The dual space of $h_{1,\text{at}}^c(\mathcal{M})$ can be described as

$$\Lambda^c(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : \|x\|_{\Lambda^c} < \infty\}$$

with

$$\|x\|_{\Lambda^c} = \max \{\|\mathcal{E}_1x\|_\infty, \sup_{n \geq 1} \tau(e) \frac{\|x - x_n\|_2}{\sup_{e \in \mathcal{P}_n} \|e\|_1} \}.$$ 

Actually, the supremum in the definition above can be taken for all $b \in L_1(\mathcal{M}_n)$ since the extreme points of the unit ball of $L_1(\mathcal{M}_n)$ are all multiples of projections. Therefore,

$$\|x\|_{\Lambda^c} = \max \{\|\mathcal{E}_1x\|_\infty, \sup_{n \geq 1} \|\mathcal{E}_n|x - x_n|\|_\infty \}.$$ 

Then the duality $h_{1,\text{at}}^c(\mathcal{M}) = bmo^c(\mathcal{M})$ yields $h_{1,\text{at}}^c(\mathcal{M}) = h_{1,\text{at}}^c(\mathcal{M})$.

It is well known in the classical theory that 2-atoms in the previous atomic decomposition can be replaced by $q$-atoms for any $1 < q \leq \infty$. Let us recall these atoms in the commutative case. A function $a \in L_1(\Omega)$ is said to be a $q$-atom if there exist $n \geq 1$ and $E \in \mathcal{F}_n$ such that

(i) $\mathcal{E}_n a = 0$; (ii) $\{a \neq 0\} \subset E$; (iii) $\|a\|_q \leq \mathbb{P}(E)^{-1 + \frac{1}{q'}}$.

We refer to [22] for more information.

The main difficulty to obtain $q$-atomic decompositions in the noncommutative case is that the key equivalence (1.8) no longer holds if one replaces the power indices 2 by $q' \neq 2$, $1 \leq q' < \infty$. We overcome this obstacle by Theorem A.

**Theorem B.** For all $1 < q \leq \infty$,

$$H_1(\mathcal{M}) = h_{1,q}^\text{at}(\mathcal{M})$$
with equivalent norms. Here $h_{1,q}^t(M)$ is the $q$-atomic Hardy spaces with its atoms defined as: $a \in L_1(M)$ is said to be a $(1,q)$-atom with respect to $(M_n)_{n \geq 1}$, if there exist $n \geq 1$ and a projection $e \in P(M_n)$ such that

(i) $E_n(a) = 0$;
(ii) $r(a) \leq e$ or $l(a) \leq e$;
(iii) $\|a\|_q \leq (\tau(e))^{-\frac{1}{q'}}$.

This is exactly the noncommutative analogue of the classical atomic decomposition. Moreover, applying the conditional version of John-Nirenberg inequality for $BMO^c(M)$ (resp. $BMO^r(M)$), we get a $q$-atomic decomposition for $h_{1}^c(M)$ (resp. $h_{r}^c(M)$) with $1 < q \leq \infty$ (see Theorem 4.12), hence recover the 2-atomic decomposition of [1] mentioned above.

As in the classical case (see e.g. [3]), we also find some applications of our results. Indeed, the John-Nirenberg inequality and atomic decomposition built in this paper have been used in [5] to establish $H_1 \to L_1$ boundedness of noncommutative paraproducts or martingale transforms with noncommuting symbols or coefficients.

Our paper is organized as follows. Section 2 is on preliminaries and notations. All the results on John-Nirenberg inequality will be presented in section 3. Section 4 is devoted to the atomic decomposition of Hardy spaces. In section 5, we answer Junge/Musat’s question in [8] which implies that the John-Nirenberg inequality in the classical sense does not hold any more in the noncommutative setting.

In this article, the letter $c$ always denotes an absolute positive constant, while $C$ an absolute constant bigger than 1. They may vary from lines to lines.

2. Preliminaries and notations

Throughout this paper, we will work on a von Neumann algebra $M$ with a normal faithful normalized trace $\tau$. For all $0 < p \leq \infty$, let $L_p(M, \tau)$ or simply $L_p(M)$ be the associated noncommutative $L_p$ spaces. For $x \in L_p(M)$ we denote the right and left supports of $x$ by $r(x)$ and $l(x)$ respectively. $r(x)$ (resp. $l(x)$) is also the least projection $e$ such that $xe = x$ (resp. $ex = x$). If $x$ is selfadjoint, $r(x) = l(x)$, denoted by $s(x)$. We mainly refer the reader to [19] for more information on noncommutative $L_p$ spaces.

Let us recall some basic notions on noncommutative martingales. Let $(M_n)_{n \geq 1}$ be an increasing sequence of von Neumann subalgebras of $M$ such that the union of the $M_n$’s is $w^*$-dense in $M$. Let $E_n$ be the conditional expectation of $M$ with respect to $M_n$. A sequence $x = (x_n)$ in $L_1(M)$ is called a noncommutative martingale with respect to $(M_n)_{n \geq 1}$ if $E_n(x_{n+1}) = x_n$ for every $n \geq 1$. If in addition, all the $x_n$’s are in $L_p(M)$ for some $1 \leq p \leq \infty$, $x$ is called an $L_p$-martingale. In this case we set $\|x\|_p = \sup_{n \geq 1} \|x_n\|_p$. 


If $\|x\|_p < \infty$, $x$ is called a bounded $L_p$-martingale.

Let $x = (x_n)$ be a noncommutative martingale with respect to $(\mathcal{M}_n)_{n \geq 1}$.

Define $dx_n = x_n - x_{n-1}$ for $n \geq 1$ with the convention that $x_0 = 0$ and $\mathcal{E}_0 = \mathcal{E}_1$. The sequence $dx = (dx_n)_n$ is called the martingale difference sequence of $x$. In the sequel, for any operator $x \in L_1(\mathcal{M})$ we denote $x_n = \mathcal{E}_n(x)$ for $n \geq 1$.

The sequence $(\mathcal{M}_n)_{n \geq 1}$ will be fixed throughout the paper. All martingales will be with respect to $(\mathcal{M}_n)_{n \geq 1}$. Let $1 \leq p < \infty$. Define $\mathcal{H}_p^c$ (resp. $\mathcal{H}_p^r$) as the completion of all finite $L_p$-martingales under the norm $\|x\|_{\mathcal{H}_p^c} = \|S_c(x)\|_p$ (resp. $\|x\|_{\mathcal{H}_p^r} = \|S_r(x)\|_p$), where $S_c(x)$ and $S_r(x)$ are defined as

$$S_c(x) = \left( \sum_{k \geq 1} |dx_k|^2 \right)^{1/2}, \quad S_r(x) = S_c(x^*) .$$

The noncommutative martingale Hardy spaces $\mathcal{H}_p(\mathcal{M})$ are defined as follows: if $1 \leq p < 2$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \inf_{x = y + z} \{ \|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r} \} .$$

When $2 \leq p < \infty$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}_p} = \max\{ \|x\|_{\mathcal{H}_p^c}, \|x\|_{\mathcal{H}_p^r} \} .$$

The space $\text{BMO}^c$ is defined as

$$\text{BMO}^c(\mathcal{M}) = \{ x \in L_1(\mathcal{M}) : \|x\|_{\text{BMO}^c} < \infty \}$$

where

$$\|x\|_{\text{BMO}^c} = \sup_{n \geq 1} \|\mathcal{E}_n|x - x_{n-1}|^2\|_\infty^{1/2} ,$$

and

$$\text{BMO}^r(\mathcal{M}) = \{ x : x^* \in \text{BMO}^c(\mathcal{M}) \} .$$

Define

$$\text{BMO}(\mathcal{M}) = \text{BMO}^c(\mathcal{M}) \cap \text{BMO}^r(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\text{BMO}} = \max\{ \|x\|_{\text{BMO}^c}, \|x\|_{\text{BMO}^r} \} .$$

Pisier and Xu [18] proved the two fundamental results: $\mathcal{H}_p(\mathcal{M}) = L_p(\mathcal{M})$ and $(\mathcal{H}_1(\mathcal{M}))^* = \text{BMO}(\mathcal{M})$. Their work triggered a rapid development of the noncommutative martingale theory.

We will also work on the conditional version of Hardy and BMO spaces developed in [9]. Let $x = (x_n)_{n \geq 1}$ be a finite martingale in $L_2(\mathcal{M})$. We set

$$s_c(x) = \left( \sum_{k \geq 1} \mathcal{E}_{k-1}|dx_k|^2 \right)^{1/2} \quad \text{and} \quad s_r(x) = s_c(x^*) .$$
Let $0 < p < \infty$. Define $h^c_p(M)$ (resp. $h^r_p(M)$) as the completion of all finite $L_\infty$-martingales under the (quasi-)norm $\|x\|_{h^c_p} = \|s_c(x)\|_p$ (resp. $\|x\|_{h^r_p} = \|s_r(x)\|_p$). Define $h^d_p(M)$ as the subspace of $\ell_p(L_p(M))$ consisting of all martingale difference sequences, where $\ell_p(L_p(M))$ is the space of all sequences $a = (a_n)_{n \geq 1}$ in $L_p(M)$ such that

$$\|a\|_{\ell_p(L_p(M))} = \left( \sum_{n \geq 1} |a_n|^p \right)^{1/p} < \infty$$

with the usual modification for $p = \infty$. The noncommutative conditional martingale Hardy spaces are defined as follows: if $0 < p < 2$,

$$h_p(M) = h^c_p(M) + h^r_p(M) + h^d_p(M)$$

equipped with the (quasi-)norm

$$\|x\|_{h_p} = \inf_{x = y + z + w} \{\|y\|_{h^c_p} + \|z\|_{h^r_p} + \|w|_{h^d_p}\}.$$  

When $2 \leq p < \infty$,

$$h_p(M) = h^c_p(M) \cap h^r_p(M) \cap h^d_p(M)$$

equipped with the norm

$$\|x\|_{h_p} = \max\{\|x\|_{h^c_p}, \|x\|_{h^r_p}, \|x\|_{h^d_p}\}.$$  

The space $bmo^c$ is defined as

$$bmo^c(M) = \{x \in L_1(M) : \|x\|_{bmo^c} < \infty\}$$

where

$$\|x\|_{bmo^c} = \max \left\{ \|E_1(x)\|_\infty, \sup_{n \geq 1} \|E_n| x - x_n |^2 \|^{1/2} \right\}.$$  

Let

$$bmo^r(M) = \{x : x^* \in bmo^c(M)\}.$$  

Let $bmo^d(M)$ be the subspace of $\ell_\infty(L_\infty(M))$ consisting of all martingale difference sequences. Note that $bmo^d(M) = h^d_\infty(M)$. Define

$$bmo(M) = bmo^c(M) \cap bmo^r(M) \cap bmo^d(M)$$

equipped with the norm

$$\|x\|_{bmo} = \max\{\|x\|_{bmo^c}, \|x\|_{bmo^r}, \|x\|_{bmo^d}\}.$$  

We refer to [9], [12], [20], [21], [7], [17] for more information on these spaces.
3. John-Nirenberg inequality

3.1. A crude version.

**Definition 3.1.** For $0 < p < \infty$, we define

(i) $\text{bmo}_p^c(\mathcal{M}) = \{ x \in L_1(\mathcal{M}) : \| x \|_{\text{bmo}_p^c} < \infty \}$

with

$$\| x \|_{\text{bmo}_p^c} = \max \left\{ \| E_1(x) \|_\infty, \sup_n \sup_{a \in \mathcal{M}_n, \| a \|_p \leq 1} \| (x - x_n)a \|_{b_p^c} \right\};$$

(ii) $\text{bmo}_p^r(\mathcal{M}) = \{ x : x^* \in \text{bmo}_p^c(\mathcal{M}) \}$;

(iii) $\text{bmo}_p(\mathcal{M}) = \text{bmo}_p^c(\mathcal{M}) \cap \text{bmo}_p^r(\mathcal{M}) \cap \text{bmo}_d(\mathcal{M})$

equipped with the (quasi-)norm

$$\| x \|_{\text{bmo}_p} = \max \{ \| x \|_{\text{bmo}_p^c}, \| x \|_{\text{bmo}_p^r}, \| x \|_{\text{bmo}_d} \}.$$

**Remark 3.2.** When $p = 2$, these are exactly the spaces $\text{bmo}_p^c(\mathcal{M})$, $\text{bmo}_p^r(\mathcal{M})$ and $\text{bmo}(\mathcal{M})$.

Below is our first version of the column (resp. row) John-Nirenberg inequality.

**Theorem 3.3.** For all $0 < p < \infty$, there exist two constants $\alpha_p$ and $\beta_p$ such that

$$\alpha_p^{-1} \| x \|_{\text{bmo}_p^c} \leq \| x \|_{\text{bmo}_p^c} \leq \beta_p \| x \|_{\text{bmo}_p^c},$$

with $\alpha_p$ and $\beta_p$ satisfying

(i) $\alpha_p = 1$ for $2 \leq p < \infty$;
(ii) $\alpha_p \leq C^{1/p-1/2}$ for $0 < p < 2$;
(iii) $\beta_p \leq cp$ for $2 \leq p < \infty$;
(iv) $\beta_p = 1$ for $0 < p < 2$.

The similar inequalities hold for $\| \cdot \|_{\text{bmo}_p^r}$ and $\| \cdot \|_{\text{bmo}_d}$.

**Proof.** We only need to prove the column case, since the row case can be done by replacing $x$ with $x^*$. First consider the case $2 < p < \infty$. We will show the following inequalities:

$$\| x \|_{\text{bmo}_p^c} \leq \| x \|_{\text{bmo}_p^c} \leq cp \| x \|_{\text{bmo}_p^c}.$$

The left inequality is obtained directly by Hölder’s inequality. In fact, taking $a \in \mathcal{M}_n$ with $\| a \|_2 \leq 1$, there exists a factorization $a = a_0a_1$ such that

$$\| a_0 \|_p = \| a \|_{2/p}^2 \leq 1 \quad \text{and} \quad \| a_1 \|_{(p-2)/p} = \| a \|_{(p-2)/p} \leq 1,$$

so

$$\| (x - x_n)a \|_{b_p^c}^2 = \tau(a_1^*a_0^*s_n^2(x - x_n)a_0a_1)$$

$$\leq \| a_1^* \|_{p-2} \| a_0^*s_n^2(x - x_n)a_0 \|_{b_p^c} \| a_1 \|_{2/p}$$

$$\leq \| (x - x_n)a_0 \|_{b_p^c}^2.$$
We invoke complex interpolation to prove the right inequality. Fix \( n \), let \( b \in L_p(\mathcal{M}_n) \) with \( \|b\|_p \leq 1 \) and \( S = \{ z \in \mathbb{C} : 0 \leq Rez \leq 1 \} \). Then by interpolation between \( L_p \) spaces \( L_p = (L_2, L_\infty)_\theta \), there exists an operator-valued function \( B \) which is continuous on \( S \) and analytic in the interior of \( S \) such that \( B(\theta) = b \) and

\[
\sup_{t \in \mathbb{R}} \|B(it)\|_2 \leq 1, \quad \sup_{t \in \mathbb{R}} \|B(1 + it)\|_\infty \leq 1.
\]

Define

\[
f(z) = (x - x_n)B(z).
\]

Then on the one hand, by the definition of \( \text{bmo}_2(\mathcal{M}) \), we have

\[
\|f(it)\|_{h^c_2} \leq \|x\|_{\text{bmo}_2^c}.
\]

On the other hand, by a simple calculation, we have

\[
\|f(1 + it)\|_{\text{bmo}_2^c} \leq \|x - x_n\|_{\text{bmo}_2^c} \|B(1 + it)\|_\infty \leq \|x\|_{\text{bmo}_2^c}.
\]

Therefore, by interpolation,

\[
\|f(\theta)\|_{(h^c_2, \text{bmo}_2^c)_\theta} \leq \|x\|_{\text{bmo}_2^c} = \|x\|_{\text{bmo}_2^c}.
\]

However by [1],

\[
(h^c_2, \text{bmo}_2^c)_\theta \subset h^c_\theta
\]

with relevant constant majorized by \( cp \). We then deduce that

\[
(3.1) \quad \|f(\theta)\|_{h^c_\theta} \leq cp \|x\|_{\text{bmo}_\theta^c},
\]

hence the desired inequality holds.

For the case \( 0 < p < 2 \). We show the following inequalities:

\[
\|x\|_{\text{bmo}_\theta^c} \leq \|x\|_{\text{bmo}_2^c} \leq C^{1/p - 1/2} \|x\|_{\text{bmo}_\theta^c}.
\]

Again, the left inequality is obtained by Hölder’s inequality. It remains to prove the right one. We choose \( 2 < p_1 < \infty \) and \( 0 < \theta < 1 \) such that \( 1/2 = (1 - \theta)/p + \theta/p_1 \). Fix \( n \), by the definition of \( \text{bmo}_\theta^c(\mathcal{M}) \), we can view \( x - x_n \) as a bounded operator from \( L_p(\mathcal{M}_n) \) to \( h^c_p(\mathcal{M}) \). Then we have the following two inequalities:

\[
\|x - x_n\|_{L_p(\mathcal{M}_n) \rightarrow h^c_\theta} \leq \|x\|_{\text{bmo}_\theta^c}, \quad \|x - x_n\|_{L_{p_1}(\mathcal{M}_n) \rightarrow h^c_\theta} \leq \|x\|_{\text{bmo}_\theta^c}.
\]

Then by interpolation, we get

\[
\|x - x_n\|_{L_2(\mathcal{M}_n) \rightarrow (h^c_p, h^c_{p_1})_\theta} \leq \|x\|_{\text{bmo}_\theta^c}^{1 - \theta} \|x\|_{\text{bmo}_\theta^c}^\theta.
\]

Now by the trivial contractive inclusion \( (h^c_p, h^c_{p_1})_\theta \subset h^c_\theta \), and the right inequality in the case \( 2 < p_1 < \infty \), we get

\[
\|x - x_n\|_{L_2(\mathcal{M}_n) \rightarrow h^c_\theta} \leq cp_1 \|x\|_{\text{bmo}_\theta^c}^{1 - \theta} \|x\|_{\text{bmo}_\theta^c}^\theta.
\]

Therefore,

\[
\|x\|_{\text{bmo}_\theta^c} \leq (cp_1)^\theta \|x\|_{\text{bmo}_\theta^c}^{1 - \theta},
\]

hence

\[
\|x\|_{\text{bmo}_\theta^c} \leq (cp_1)^{\frac{\theta}{1 - \theta}} \|x\|_{\text{bmo}_\theta^c}.
\]
Noting that $\theta/(1 - \theta) = (1/p - 1/2)/(1/2 - 1/p_1)$, we get the desired estimate by taking $C = (cp_1)^{(1/(1/2 - 1/p_1))}$.

**Remark 3.4.** The constant in (3.1) is optimal. This can be seen as follows. By Lemma 4.3 in [1], $h_p^*(\mathcal{M})$ embeds into $(h_2^*(\mathcal{M}), h_1^*(\mathcal{M}))_\theta$ with constant independent of $p'$. So $((h_2^*(\mathcal{M}))^*, (h_1^*(\mathcal{M}))^*)_\theta$ embeds into $(h_p^*(\mathcal{M}))^*$ with constant independent of $p$ by duality. Finally, by the optimal embedding $(h_p^*(\mathcal{M}))^* \subset h_2^*(\mathcal{M})$ with constant $cp$ in [9] and $bmo^c(\mathcal{M}) \subset (h_2^*(\mathcal{M}))^*$ in [17], $(h_2^*(\mathcal{M}), bmo^c(\mathcal{M}))_\theta$ embeds into $h_2^*(\mathcal{M})$ with optimal constant $cp$.

It is natural to ask whether there is a result similar to Theorem 3.3 for $BMO^c$ by replacing $h_p^c$ and $x - x_n$ in the definition of $bmo_p^c$ by $H_p^c$ and $x - x_{n-1}$ respectively. Using the identity

$$BMO^c(\mathcal{M}) \simeq bmo^c(\mathcal{M}) \cap bmo^d(\mathcal{M})$$

proved in [17], we are reduced to deal with the diagonal space $bmo^d(\mathcal{M})$. Surprisingly, the result is true only for $2 \leq p < \infty$ (see Remark 3.9).

**Definition 3.5.** For $1 \leq p < \infty$, we define

(i) $$BMO^c_p(\mathcal{M}) = \left\{ x \in L_1(\mathcal{M}) : \|x\|_{BMO^c_p} < \infty \right\}$$

with $$\|x\|_{BMO^c_p} = \sup_n \sup_{a \in \mathcal{M}, \|a\|_p \leq 1} \|(x - x_{n-1})a\|_{H_p^c};$$

(ii) $$BMO^c(\mathcal{M}) = \left\{ x^* \in BMO^c_p(\mathcal{M}) \right\};$$

(iii) $$BMO_p(\mathcal{M}) = BMO^c_p(\mathcal{M}) \cap BMO^c(\mathcal{M})$$

equipped with the norm $$\|x\|_{BMO_p} = \max\{\|x\|_{BMO^c_p}, \|x\|_{BMO^c}\}.$$

**Remark 3.6.** For $p = 2$, we recover the spaces $BMO^c(\mathcal{M})$, $BMO^c(\mathcal{M})$ and $BMO(\mathcal{M})$.

The following lemma will allow us to handle with the diagonal space $bmo^d(\mathcal{M})$.

**Lemma 3.7.** For $2 \leq p < \infty$, we have

$$cp^{-1}\|b\|_\infty \leq \sup_{a \in \mathcal{M}, \|a\|_p \leq 1} \|ba\|_{H_p^c} \leq cp^{1/2} \|b\|_\infty.$$

**Proof.** Note that $\| \cdot \|_{H_p^c} \leq cp^{1/2} \| \cdot \|_p$ (see [20], Remark 5.4 as a reference for the constant we use here), we have

$$\sup_{a \in \mathcal{M}, \|a\|_p \leq 1} \|ba\|_{H_p^c} \leq cp^{1/2} \sup_{a \in \mathcal{M}, \|a\|_p \leq 1} \|ba\|_p = cp^{1/2} \|b\|_\infty.$$
For the first inequality, without loss of generality assume $\|b\|_\infty = 1$. Note that for selfadjoint $x \in \mathcal{M}$, $\|x\|_p \leq cp\|x\|_{\mathcal{H}^c_p}$ (see [20], Remark 5.4). Then
\[
\|b^*\|_\infty = \sup_{y \in \mathcal{M}, \|y\|_2 \leq 1} \|gb^*\|_{2p} = \sup_{y \in \mathcal{M}, \|y\|_2 \leq 1} \|b|y|^2b^*\|_p^{\frac{1}{p}} \leq cp\sup_{y \in \mathcal{M}, \|y\|_2 \leq 1} \|b|y|^2b^*\|_{\mathcal{H}^c_p} \leq cp\sup_{a \in \mathcal{M}, \|a\|_p \leq 1} \|ba\|_{\mathcal{H}^c_p}.
\]

And then $cp^{-1}\|b\|_\infty \leq \sup_{a \in \mathcal{M}, \|a\|_p \leq 1} \|ba\|_{\mathcal{H}^c_p}$. □

**Theorem 3.8.** For all $2 \leq p < \infty$, we have
\[
\mathcal{BMO}^c_p(\mathcal{M}) = \mathcal{BMO}^c(\mathcal{M})
\]
with equivalent norms. More precisely,
\[
cp^{-1}\|x\|_{\mathcal{BMO}^c} \leq \|x\|_{\mathcal{BMO}^c_p} \leq cp\|x\|_{\mathcal{BMO}^c}.
\]
Similarly, $\mathcal{BMO}^c_p(\mathcal{M}) = \mathcal{BMO}^c(\mathcal{M})$ with equivalent norms.

Using the previous lemma and the identity $\mathcal{BMO}^c(\mathcal{M}) \simeq \text{bmo}^c(\mathcal{M}) \cap \text{bmo}^d(\mathcal{M})$, we can easily deduce Theorem 3.8 from Theorem 3.3. We will however present a direct proof.

**Proof.** We only prove the inequalities for the column case, the row case can be dealt with similarly. By the previous lemma and Hölder’s inequality, we have
\[
\left\|E_n \sum_{k=n}^{\infty} |dx_k|^2 \right\|_\infty \leq \sup_{b \in \mathcal{M}_n^+, \|b\|_1 \leq 1} \left( \sum_{k=n+1}^{\infty} |dx_k|^2b \right) + \|x_n - x_{n-1}\|_\infty^2 \\
\leq \sup_{b \in \mathcal{M}_n^+, \|b\|_1 \leq 1} \left( \sum_{k=n+1}^{\infty} |(dx_k)b|^{\frac{1}{p}} |2b|^{\frac{p-2}{p}} \right) + cp^2 \sup_{a \in \mathcal{M}_n, \|a\|_p \leq 1} \|x_n - x_{n-1}a\|_{\mathcal{H}^c_p}^2 \\
\leq \sup_{b \in \mathcal{M}_n^+, \|b\|_1 \leq 1} \left\| \sum_{k=n+1}^{\infty} |(dx_k)b|^{\frac{1}{p}} |2b|^{\frac{p-2}{p}} \right\|_{\mathcal{BMO}^c} + cp^2 \sup_{a \in \mathcal{M}_n, \|a\|_p \leq 1} \|x_n - x_{n-1}a\|_{\mathcal{H}^c_p}^2 \\
\leq \sup_{b \in \mathcal{M}_n^+, \|b\|_1 \leq 1} \left\| (x - x_n)b\right\|_{\mathcal{BMO}^c}^2 + cp^2 \sup_{a \in \mathcal{M}_n, \|a\|_p \leq 1} \|x_n - x_{n-1}a\|_{\mathcal{H}^c_p}^2.
\]
Then by $\| \mathcal{E}_n x \|_{\mathcal{H}_p} \leq \| x \|_{\mathcal{H}_p}$,
\[ \| x \|_{\mathcal{BMO}_p^2} \leq cp \sup_{a \in M_n, \| a \|_p \leq 1} \| (x - x_{n-1})a \|_{\mathcal{H}_p} = cp \| x \|_{\mathcal{BMO}_p^c}. \]
Conversely, by the previous lemma,
\[ \| x \|_{\mathcal{BMO}_p^c} \leq \sup_n \sup_{a \in M_n, \| a \|_p \leq 1} \| (x - x_n)a \|_{\mathcal{H}_p} + \sup_n \sup_{a \in M_n, \| a \|_p \leq 1} \| (x_n - x_{n-1})a \|_{\mathcal{H}_p} \]
\[ \leq \sup_n \sup_{a \in M_n, \| a \|_p \leq 1} \| (x - x_n)a \|_{\mathcal{H}_p} + cp^{\frac{1}{2}} \sup_n \| x_n - x_{n-1} \|_{\infty} \]
(3.2) \[ \leq \sup_n \sup_{a \in M_n, \| a \|_p \leq 1} \| (dx_k a)^{\infty}_{k=n+1} \|_{L_p(\ell_p^2)} + cp^{\frac{1}{2}} \| x \|_{\mathcal{BMO}_2^c}. \]

Note that, by the Hahn-Banach theorem and the duality between $\mathcal{H}_p^c(\mathcal{M})$ and $\mathcal{BMO}_p^c(\mathcal{M})$, there exists a sequence $(b_n)_{n=1}^{\infty} \in L_\infty(\mathcal{M}; \ell_p^2)$ such that
\[ \| (b_n)_{n=1}^{\infty} \|_{L_\infty(\ell_p^2)} = \| x \|_{\mathcal{BMO}_2^c}, \]
$dx_k = \mathcal{E}_k b_k - \mathcal{E}_{k-1} b_k$.

Thus by the noncommutative Stein inequality (see [20] for the constant used below) and Hölder’s inequality,
\[ \sup_{a \in M_n, \| a \|_p \leq 1} \| (dx_k a)^{\infty}_{k=n+1} \|_{L_p(\ell_p^2)} \]
\[ \leq \sup_{a \in M_n, \| a \|_p \leq 1} \| (\mathcal{E}_a(b_k))^{\infty}_{k=n+1} \|_{L_p(\ell_p^2)} \]
\[ + \sup_{a \in M_n, \| a \|_p \leq 1} \| (\mathcal{E}_k b_{k+1})^{\infty}_{k=n} \|_{L_p(\ell_p^2)} \]
\[ \leq cp \sup_{a \in M_n, \| a \|_p \leq 1} \| (b_k a)^{\infty}_{k=n+1} \|_{L_p(\ell_p^2)} \]
\[ \leq cp \left( \sum_{k=1}^{\infty} |b_k|^2 \right)^{\frac{1}{2}} \| x \|_{\mathcal{BMO}_2^c}, \]
Combining this with (3.2) we finish the proof.  \hfill \Box

**Remark 3.9.** It is a bit surprising that Theorem 3.8 is actually wrong for any $p < 2$. Indeed, choose a filtration $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, ..., \mathcal{M}_{n-1}$ and $y \in \mathcal{M}_{n-1}$ such that $\| y \|_p = 1$ and $\| y \|_{\mathcal{H}_p} = c_n$. Let $\mathcal{M}_n = L_\infty(\Omega, \mathcal{M}_{n-1})$ with $\Omega = \{0,1\}$ with $\mu\{1\} = \mu\{0\} = 1/2$. We certainly can view $\mathcal{M}_k, k < n$ as the space of constant functions on $\Omega$, so $\mathcal{M}_k \subset \mathcal{M}_n$. Let $x = 1$ on $\{0\}$ and $x = -1$ on $\{1\}$ then $x_{n-1} = 0$. Let $a = y$ on $\{0\}$ and $a = -y$ on $\{1\}$. Then $(x - x_{n-1})a = y$ whose $\mathcal{H}_p$ norm equals $c_n$ and $\| a \|_p = 1$, so $\| x \|_{\mathcal{BMO}_p^c} \geq c_n$. But $\| x \|_{\mathcal{BMO}_2^c} = 1$.

In the rest of this subsection, we turn to Junge/Musat’s type of John-Nirenberg inequality. In [8], Junge and Musat established the inequality for $2 < p < \infty$ in the state case. Later the second author of the present paper gave a simple proof for all $1 \leq p < \infty$ in the tracial setting (see [13]). The
idea of the proof of Theorem 3.3 can be applied to obtain this inequality for all $0 < p < \infty$ (see Corollary 3.13). We start again with $\text{bmo}(M)$.

**Theorem 3.10.** For all $0 < p < \infty$, we have

$$\alpha_p^{-1} \|x\|_{\text{bmo}} \leq b_p(x) \leq \beta_p \|x\|_{\text{bmo}}$$

where

$$b_p(x) = \max \{ \sup_n \|(dx_n)_n\|_\infty, \sup_n \sup_{b \in M_n, \|b\|_p \leq 1} \|b(x - x_n)\|_p \}.$$  

The constant $\alpha_p$ and $\beta_p$ have the same orders as those in Theorem 3.3.

**Proof.** We first treat the case $2 \leq p < \infty$. For $p = 2$, it is trivial. So we can assume $2 < p < \infty$. The inequality

$$\|x\|_{\text{bmo}} \leq b_p(x)$$

follows from Hölder’s inequality. We will prove the reverse inequality by interpolation. By a simple calculation, we have the following estimates

$$\|(x - x_n)b\|_{\text{bmo}} \leq \|x\|_{\text{bmo}} \|b\|_\infty,$$

$$\|(x - x_n)b\|_{\text{bmo}} \leq \|x\|_{\text{bmo}} \|b\|_\infty,$$

$$\|(x - x_n)b\|_{\text{bmo}} \leq \|x\|_{\text{bmo}} \|b\|_\infty.$$  

Then it follows that

$$\|(x - x_n)b\|_{\text{bmo}} \leq \|x\|_{\text{bmo}} \|b\|_\infty.$$  

On the other hand, it is clear that

$$\|(x - x_n)b\|_2 = \|(x - x_n)b\|_{\text{bmo}} \leq \|x\|_{\text{bmo}} \|b\|_2.$$  

Then by the interpolation result of [1], we have

$$\|(x - x_n)b\|_p \leq c \|x\|_{\text{bmo}} \|b\|_{L_2, \text{bmo}} \|b\|_p,$$

In the same way, we obtain

$$\|b(x - x_n)\|_p \leq c \|x\|_{\text{bmo}} \|b\|_p.$$  

Thus we prove the assertion.

Now we turn to the case $0 < p < 2$, by Hölder’s inequality, we obtain the trivial part

$$b_p(x) \leq b_2(x) = \|x\|_{\text{bmo}}.$$  

Let us prove the inverse one, let $2 < p_1 < \infty$ and $\theta$ be such that

$$\frac{1}{2} = \frac{1 - \theta}{p} + \frac{\theta}{p_1}.$$
We view $x - x_n$ and $(x - x_n)^*$ as two operators. By interpolation,
\[
\| (x - x_n) \|_{L^2(M_n) \to L^2(M)} \\
\leq \| (x - x_n) \|_{L^p(M_n) \to L^p(M)}^{1 - \theta} \| (x - x_n) \|_{L^p_1(M_n) \to L^p_1(M)}^\theta,
\]
and similarly for $(x - x_n)^*$. By the estimate for $p_1 > 2$, we have
\[
b_2(x) \leq (c_1)^\theta b_p^{1-\theta}(x) b_2^\theta(x).
\]
Therefore, we obtain
\[
\| x \|_{\text{bmo}} \leq (c_1)^{\theta_p} b_p(x) = C^{1/p - 1/2} b_p(x),
\]
with $C = (c_1)^{1/(2 - 1/p_1)}$. 

**Remark 3.11.** This constant in (3.3) is optimal. This can be seen as follows. By Lemma 4.3 in [1], $h_{\theta}^p(M)$ embeds into $(h_1^\infty(M), h_1(M))_\theta$ with constant independent of $p'$. So $h_{\theta}^p(M)$ embeds into $(h_2(M), h_1(M))_\theta$ with constant independent of $p'$. Now by Theorem 4.1 in [21], $L^p_\theta(M)$ embeds into $h_{\theta}^p(M)$, hence into $(h_2(M), h_1(M))_\theta$ with optimal constant $c/(p' - 1)$. Then by duality, $(h_2(M))^*, (h_1(M))^\theta_\theta$ embeds into $(L^p_\theta(M))^* = L^p_\theta(M)$ with best constant $c_\theta$. At last, by $\text{bmo}(M) \subset (h_1(M))^\theta$ in [17], $(h_2(M), \text{bmo}(M))_\theta$ embeds into $L^p_\theta(M)$ with optimal constant $c_\theta$.

**Remark 3.12.** We can directly compare the norms $\| \cdot \|_{\text{bmo}_p}$ and $b_p(\cdot)$ directly for $1 < p < \infty$ by using Theorem 3.3.

Let us justify this remark. We first deal with the case $2 < p < \infty$. Fix $n$, for any $b \in M_n$ with $\| b \|_p \leq 1$, by the noncommutative Burkholder inequality [9], we have
\[
\| (x - x_n) b \|_{h_{\theta}^p(M)} \leq c_\theta \| (x - x_n) b \|_p, \quad \| b (x - x_n) \|_{h_{\theta}^p(M)} \leq c_\theta \| b (x - x_n) \|_p,
\]
hence
\[
\| (x - x_n) b \|_{h_{\theta}^p(M)} \leq c_\theta \| b (x - x_n) \|_p \leq c \theta b_p(x).
\]
Then by Theorem 3.3,
\[
\| x \|_{\text{bmo}_p} \leq c \theta b_p(x).
\]
Another direction can be done by the way in Theorem 3.10,
\[
b_p(x) \leq c \theta \| x \|_{\text{bmo}} \leq c \theta \| x \|_{\text{bmo}_p},
\]
For the case $1 < p < 2$. The trivial part
\[
b_p(x) \leq c \| x \|_{\text{bmo}_p}
\]
follows from the noncommutative Burkholder inequality in [9]. Now let us prove the inverse one. Take $b \in M_n$ with $\| b \|_2 \leq 1$. By Hölder’s inequality, we have
\[
\| (x - x_n) b \|_2^2 = \tau(b^{2/p}(x - x_n)^* (x - x_n)b^{2/p}) \\
\leq \| b^{2/p}(x - x_n)^* \|_p \| (x - x_n)b^{2/p} \|_p.
\]
and
\[ \|b(x - x_n)\|_2^2 = \tau((x - x_n)^*b^{2/p'}b^{2/p}(x - x_n)) \]
\[ \leq \|(x - x_n)^*b^{2/p'}\|_{p'}\|b^{2/p}(x - x_n)\|_p. \]
So by the result in Theorem 3.3 for \( 2 < p' < \infty \), we have
\[ \|b(x - x_n)\|_2^2, \|(x - x_n)b\|_2^2 \]
\[ \leq \max\{\|b^{2/p'}(x - x_n)^*\|_{p'}, \|(x - x_n)^*b^{2/p'}\|_{p'}\} \cdot \max\{\|(x - x_n)b^{2/p}\|_p, \|b^{2/p}(x - x_n)\|_p\} \]
\[ \leq c\|x\|_{\text{bmo}_p} \cdot \|b_p(x)\| \leq cp'\|x\|_{\text{bmo}_2} \cdot \|b_p(x)\| \]

Then by the definition of \( \text{bmo}_2(\mathcal{M}) \), we finish the proof by Theorem 3.3
\[ \|x\|_{\text{bmo}_p} \leq \|x\|_{\text{bmo}_2} \leq cp'\|b_p(x)\| \]

The following corollary extends Junge/Musat’s theorem to all \( 0 < p < \infty \). It can be proved similarly as Theorem 3.3. However, using the identity \( \mathcal{BMO}(\mathcal{M}) \simeq \text{bmo}(\mathcal{M}) \) proved in [17], we give a simpler proof.

Corollary 3.13. For \( 0 < p < \infty \), we have
\[ \alpha_p^{-1}\|x\|_{\mathcal{BMO}} \leq \mathcal{B}_p(x) \leq \beta_p\|x\|_{\mathcal{BMO}}, \]
where
\[ \mathcal{B}_p(x) = \max\{\sup_n \sup_{b \in M_n, \|b\|_p \leq 1} \|(x - x_n - 1)b\|_p, \sup_n \sup_{b \in M_n, \|b\|_p \leq 1} \|b(x - x_n - 1)\|_p\}. \]
The constant \( \alpha_p \) and \( \beta_p \) have the same orders as those in Theorem 3.3.

Proof. For \( 2 \leq p < \infty \), it is very easy to get
\[ \mathcal{B}_p(x) \leq b_p(x) \leq cp\|x\|_{\text{bmo}} \leq cp\|x\|_{\mathcal{BMO}} \]
from the triangular inequality
\[ \|(x - x_n - 1)b\|_p \leq \|(x - x_n)b\|_p + \|(x_n - x_n - 1)b\|_p, \]
with \( b \in M_n \) and \( \|b\|_p \leq 1 \). And the rest of the proof is the same to Theorem 3.10. \( \square \)

Remark 3.14. The following example shows that Junge/Musat’s John-Nirenberg inequality does not hold for \( \text{bmo}^c \) or \( \mathcal{BMO}^c \). The example is the same as the one given in Remark 3.20 of [8]. Let \( n \) be a positive integer and consider the von Neumann algebra
\[ \mathcal{M} = L_\infty(\mathbb{T}) \otimes M_n, \]
where \( M_n \) is the algebra of \( n \times n \) matrices with normalized trace. For \( k \geq 1 \) let \( \mathcal{F}_k \) be the \( \sigma \)-algebra generated by dyadic intervals in \( \mathbb{T} \) of length \( 2^{-k} \). Denote by \( \mathcal{M}_k \) the subalgebra \( L_\infty(\mathbb{T}, \mathcal{F}_k) \otimes M_n \) of \( \mathcal{M} \) and let \( \mathcal{E}_k = \mathbb{E}_k \otimes \text{id}_{M_n} \).
be the conditional expectation onto $\mathcal{M}_k$. Let $r_k$ be the $k$-th Rademacher function on $\mathbb{T}$ and consider
\[ x = \sum_{k=1}^{n} r_k \otimes e_1. \]
Then $x$ is a martingale relative to the filtration $(\mathcal{M}_k)_{k \geq 1}$ and the martingale differences are given by $dx_k = r_k \otimes e_1$. A simple calculation shows that
\[ \sup_m \|x - x_m\|_p = (n - 1)^{\frac{1}{2}} n^{-\frac{1}{p}}, \]
while
\[ \|x\|_{bmo^c} = \sup_m \left\| \sum_{k=m+1}^{n} \mathcal{E}_m |d_kx| \right\|_\infty^{\frac{1}{2}} = 1. \]
Let $p > 2$. Then for any $c > 0$, there exists $n \geq 1$ such that $(n - 1)^{1/2} n^{-1/p} > c$. Hence
\[ \sup_m \sup_{b \in \mathcal{M}_m, \|b\|_p \leq 1} \|(x - x_m)b\|_p \geq \sup_m \|x - x_m\|_p >> \|x\|_{bmo^c}. \]

3.2. A fine version. Now we can formulate the fine version of the column (resp. row) John-Nirenberg inequality.

**Definition 3.15.** For $0 < p < \infty$, we define
\[ bmo^c_{p,pr}(\mathcal{M}) = \{ x \in L_1(\mathcal{M}) : \|x\|_{bmo^c_{p,pr}} < \infty \} \]
with
\[ \|x\|_{bmo^c_{p,pr}} = \max \left\{ \|\mathcal{E}_1(x)\|_\infty, \sup_n \sup_{e \in \mathcal{P}(\mathcal{M}_n)} \|(x - x_n) \frac{e}{\tau(e)}\|_{h^*_p} \right\}. \]

Similarly,
\[ bmo^c_{p,pr}(\mathcal{M}) = \{ x : x^* \in bmo^c_{p,pr}(\mathcal{M}) \} \quad \text{with} \quad \|x\|_{bmo^c_{p,pr}} = \|x^*\|_{bmo^c_{p,pr}}. \]

Finally,
\[ bmo_{p,pr}(\mathcal{M}) = bmo^c_{p,pr}(\mathcal{M}) \cap bmo^r_{p,pr}(\mathcal{M}) \cap bmo^d(\mathcal{M}) \]
equipped with
\[ \|x\|_{bmo_{p,pr}} = \max\{\|x\|_{bmo^c_{p,pr}}, \|x\|_{bmo^r_{p,pr}}, \|x\|_{bmo^d}\}. \]

The fine version of the column (resp. row) John-Nirenberg inequality is stated as follows.

**Theorem 3.16.** For all $0 < p < \infty$, we have
\[ \alpha_p^{-1} \|x\|_{bmo^c} \leq \|x\|_{bmo^c_{p,pr}} \leq \beta_p \|x\|_{bmo^c}. \]

The constants $\alpha_p$ and $\beta_p$ have the same properties as those in Theorem 3.3. The same inequalities hold for $\|\cdot\|_{bmo^r}$ and $\|\cdot\|_{bmo^r_{p,pr}}$. 
Proof. We first consider the case $0 < p \leq 1$. By Theorem 3.3, the trivial part

$$
\|x\|_{bmo_p^{c^e}} \leq \|x\|_{bmo_p^c} \leq \|x\|_{bmo^c}
$$

follows from the fact that $e/(\tau(e))^{1/p} \in M_n$ and its $L_p$-norm equals 1. Now we turn to the proof of the inverse inequality. Since any $a \in M_n$ with $\|a\|_p \leq 1$ can be approximated by sums $\sum_k \lambda_k e_k/(\tau(e_k))^{1/p}$ with $e_k$'s in $M_n$ and $\sum_k |\lambda_k|^p \leq 1$. Thus we can assume that $a$ itself is such a sum. Then

$$
\| (x - x_n) a \|_{bmo_p^{c^e}}^p = \| \sum_k \lambda_k (x - x_n) \frac{e_k}{(\tau(e_k))^{1/p}} \|_{bmo_p^{c^e}}^p \\
\leq \sum_k |\lambda_k|^p \| (x - x_n) \frac{e_k}{(\tau(e_k))^{1/p}} \|_{bmo_p^{c^e}}^p \\
\leq \sum_k |\lambda_k|^p \| x \|_{bmo_{p,pr}^{c^e}}^p \leq \| x \|_{bmo_{p,pr}^{c^e}}^p.
$$

Therefore by Theorem 3.3,

$$
\|x\|_{bmo^c} \leq C^{1/p-1/2} \|x\|_{bmo_p^c} \leq C^{1/p-1/2} \|x\|_{bmo_{p,pr}^{c^e}}.
$$

Now let $1 < p < \infty$. Again, because of the fact that $e/(\tau(e))^{1/p} \in M_n$ and its $L_p$-norm equals 1, by Theorem 3.3,

$$
(3.4) \quad \|x\|_{bmo_{p,pr}^{c^e}} \leq \|x\|_{bmo_p^c} \leq c_1 p \|x\|_{bmo^c}.
$$

We exploit the result for $p = 1$ to prove the inverse inequality. By Hölder’s inequality, we have

$$
\|x\|_{bmo_p^{c^e}} \leq \|x\|_{bmo_{p,pr}^{c^e}}.
$$

We end the proof by Theorem 3.3 and the result for $p = 1$,

$$
\|x\|_{bmo^c} \leq C^{1/p-1/2} \|x\|_{bmo_1^{c^e}} \leq C^{1/p-1/2} \|x\|_{bmo_1^{c^e}} \leq C^{1/p-1/2} \|x\|_{bmo_{p,pr}^{c^e}}.
$$

Now we give the distributional form of the John-Nirenberg inequality for $bmo^c(M)$ and $bmo^c(M)$.

**Theorem 3.17.** Let $x \in bmo^c(M)$. Then for all natural numbers $n \geq 1$, all $e \in P(M_n)$ and for all $\lambda > 0$, we have

$$
\frac{1}{\tau(e)} \tau(1_{(\lambda, \infty)}(s_1((x - x_n)e))) \leq 2 \exp\left(-\frac{c\lambda}{\|x\|_{bmo^c}}\right),
$$

with $c$ an absolute constant. Here $1_{(\lambda, \infty)}(a)$ denotes the spectral projection of a positive operator $a$ corresponding to the interval $(\lambda, \infty)$.

**Proof.** By homogeneity, we can assume $\|x\|_{bmo^c} = 1$. We first deal with the case $\lambda \geq 2c_1$, where $c_1$ is the constant in inequality (3.4). Let $p = \lambda/(2c_1) \geq 1$.
1, by Chebychev’s inequality and Theorem 3.16,

\[ \tau(\mathbb{1}_{(\lambda, \infty)}(s_c ((x - x_n)e))) \leq \tau(e) \frac{\| (x - x_n)e \|^p_{K_p}}{\lambda^p} \]

\[ \leq \tau(e) (c_1 p \lambda^{-1})^p = \tau(e) \exp(p \ln(c_1 p \lambda^{-1})) = \tau(e) \exp(-\frac{\ln 2}{2c_1 \lambda}). \]

When \( 0 < \lambda < 2c_1 \),

\[ \frac{1}{\tau(e)} \tau(\mathbb{1}_{(\lambda, \infty)}(s_c ((x - x_n)e))) \leq 1 < 2 \exp(-\frac{\ln 2}{2c_1 \lambda}). \]

Therefore, we obtain the desired result by letting \( c = \ln 2/(2c_1) \). \( \square \)

Based on the crude version of Junge/Musat’s John-Nirenberg inequality in Theorem 3.10 (resp. Corollary 3.8) for \( \text{bmo}(M) \) (resp. \( \text{BMO}(M) \)), the argument in the proof of Theorem 3.16 can be adapted to get the fine version of Junge/Musat’s John-Nirenberg inequality.

**Corollary 3.18.** For all \( 0 < p < \infty \), we have

\[ \alpha_p^{-1} \| x \|_{\text{bmo}} \leq \mathcal{P}_b p(x) \leq \beta_p \| x \|_{\text{bmo}}, \]

where

\[ \mathcal{P}_b p(x) = \max \{ \sup_n \| (dx_n)_n \|_{\infty}, \quad \sup_n \sup_{e \in \mathcal{M}_n} \| (x - x_n) e \|_{(\tau(e))^{1/p}} \}
\]

\[ \sup_n \sup_{e \in \mathcal{M}_n} \| \frac{e}{(\tau(e))^{1/p}} (x - x_n) \|_p. \]

The constants \( \alpha_p \) and \( \beta_p \) have the same orders as those in Theorem 3.3.

**Corollary 3.19.** For \( 0 < p < \infty \), we have

\[ \alpha_p^{-1} \| x \|_{\text{BMO}} \leq \mathcal{P}_b p(x) \leq \beta_p \| x \|_{\text{BMO}}, \]

where

\[ \mathcal{P}_b p(x) = \max \{ \sup_n \sup_{e \in \mathcal{M}_n} \| (x - x_{n-1}) e \|_{(\tau(e))^{1/p}} \}
\]

\[ \sup_n \sup_{e \in \mathcal{M}_n} \| \frac{e}{(\tau(e))^{1/p}} (x - x_{n-1}) \|_p. \]

The constant \( \alpha_p \) and \( \beta_p \) have the same orders as those in Theorem 3.3.

Again, based on Corollary 3.19, by arguments similar to the proof of Thoerem 3.17, we obtain the exponential integrability form of the John-Nirenberg inequality for \( \text{BMO}(M) \).

**Theorem 3.20.** Let \( x \in \text{BMO}(M) \). Then for all natural numbers \( n \geq 1 \), all \( e \in \mathcal{P}(\mathcal{M}_n) \) and for all \( \lambda > 0 \), we have

\[ \frac{1}{\tau(e)} \tau(\mathbb{1}_{(\lambda, \infty)}(|(x - x_{n-1})|)) \leq 4 \exp(-\frac{c\lambda}{\| x \|_{\text{BMO}}}) \]

with \( c \) an absolute constant.
4. ATOMIC DECOMPOSITION

4.1. A CRUDE VERSION OF ATOMS. According to the crude version of the noncommutative John-Nirenberg inequality, we introduce the following

**Definition 4.1.** For $1 < q \leq \infty$, $a \in L_1(M)$ is said to be a $(1, q, c)$-atom with respect to $(M_n)_{n \geq 1}$, if there exist $n \geq 1$ and a factorization $a = yb$ such that

(i) $E_n(y) = 0$;
(ii) $b \in L_{q'}(M_n)$ and $\|b\|_{q'} \leq 1$;
(iii) $\|y\|_{\text{bmo}} \leq 1$ for $1 < q \leq \infty$; $\|y\|_{\text{bmo}} \leq 1$ for $q = \infty$.

Similarly, we define the notion of a $(1, q, r)$-atom with $a = yb$ replaced by $a = by$.

**Lemma 4.2.** Let $1 < q \leq \infty$. If $a$ is a $(1, q, c)$-atom, then

$$\|a\|_{h^1_1} \leq 1.$$ 

The analogous inequality holds for $(1, q, r)$-atoms.

**Proof.** We first deal with the case $1 < q < \infty$. By definition, there exists an $n$ such that the $(1, q, c)$-atom $a$ admits a factorization $a = yb$ as in Definition 4.1. Then

$$s^2_c(a) = b^* \sum_{k > n} |E_{k-1} \cdot dy_k|^2 b = b^* s^2_c(y) b.$$ 

Thus by Hölder’s inequality,

$$\|a\|_{h^1_1} = \|s_c(a)\|_1 \leq \|s_c(y)\|_q \|b\|_{q'} \leq 1.$$ 

For the case $q = \infty$, the calculation is a bit different,

$$\|a\|_{h^1_1} = \|b^* s^2_c(y) b\|_{1/2}^{1/2} = \tau(E_n(b^* s^2_c(y) b)^{1/2})$$

$$\leq \tau((E_n(b^* s_c(y) b))^{1/2}) \leq \|E_n(s_c(y))\|_\infty \|b\|_1$$

$$\leq \|y\|_{\text{bmo}} \|b\|_1 \leq 1.$$ 

We have used the trace preserving property of conditional expectations in the fourth equality and the operator Jensen inequality in the first inequality. For the second inequality, we have used the property that $E_n \cdot E_{k-1} = E_n$ for all $k > n$ and Hölder’s inequality.

**Definition 4.3.** We define $h^c_{1, \text{at}_q}(M)$ as the Banach space of all $x \in L_1(M)$ which admit a decomposition $x = \sum_k \lambda_k a_k$, where for each $k$, $a_k$ a $(1, q, c)$-atom or an element in the unit ball of $L_1(M_1)$, and $\lambda_k \in \mathbb{C}$ satisfying $\sum_k |\lambda_k| < \infty$. We equip this space with the norm

$$\|x\|_{h^c_{1, \text{at}_q}} = \inf \sum_k |\lambda_k|,$$

where the infimum is taken over all decompositions of $x$ described above. Similarly, we define $h^r_{1, \text{at}_q}(M)$.
Now, by Lemma 4.2, we have the obvious inclusion $h_{1,atq}^c(\mathcal{M}) \subset h_{1}^c(\mathcal{M})$. In fact, the two spaces coincide thanks to the following theorem.

**Theorem 4.4.** For all $1 < q \leq \infty$, we have

$$h_{1}^c(\mathcal{M}) = h_{1,atq}^c(\mathcal{M})$$

with equivalent norms. Similarly, $h_{1}^c(\mathcal{M}) = h_{1,atq}^c(\mathcal{M})$ with equivalent norms.

We prove this theorem by duality. We require the following lemmas.

**Lemma 4.5.** (i) For all $1 < q \leq 2$, $L_2(\mathcal{M})$ densely and continuously embeds into $h_{1,atq}^c(\mathcal{M})$.

(ii) For all $2 < q \leq \infty$, $L_q(\mathcal{M})$ densely and continuously embeds into $h_{1,atq}^c(\mathcal{M})$.

**Proof.** (i). For any $x \in L_2(\mathcal{M})$, we decompose it as a linear combination of two atoms:

$$x = \|x - E_1(x)\|_2 \frac{x - E_1(x)}{\|x - E_1(x)\|_2} + \|E_1(x)\|_2 \frac{E_1(x)}{\|E_1(x)\|_2}.$$  

Indeed, on the one hand, $E_1(x)/\|E_1(x)\|_2 \in L_2(\mathcal{M}_1) \subset L_1(\mathcal{M}_1)$ and

$$\|E_1(x)/\|E_1(x)\|_2\|_1 = \|E_1(x)/\|E_1(x)\|_2\|_1 \leq 1.$$  

On the other hand,

$$\frac{x - E_1(x)}{\|x - E_1(x)\|_2} = \frac{x - E_1(x)}{\|x - E_1(x)\|_2} \cdot 1 = y \cdot b.$$  

Clearly, $E_1(y) = 0$, $\|b\|_{q'} \leq 1$ and

$$\|y\|_{h_q^c} = \|\frac{x - E_1(x)}{\|x - E_1(x)\|_2}\|_{h_q^c} \leq \|\frac{x - E_1(x)}{\|x - E_1(x)\|_2}\|_{h_q^c} \leq 1.$$  

Thus $x$ is a sum of two atoms and

$$\|x\|_{h_{1,atq}^c} \leq \|x - E_1(x)\|_2 + \|E_1(x)\|_2 \leq \sqrt{2}\|x\|_2.$$  

The density is trivial.

(ii). This case is similar to the previous one. We first deal with the case $2 < q < \infty$. Given $x \in L_q(\mathcal{M})$, we write again:

$$x = c_q \|x - E_1(x)\|_q \frac{x - E_1(x)}{c_q \|x - E_1(x)\|_q} + \|E_1(x)\|_q \frac{E_1(x)}{\|E_1(x)\|_q},$$

where $c_q$ is fixed below. Indeed, $E_1(x)/\|E_1(x)\|_q \in L_q(\mathcal{M}_1) \subset L_1(\mathcal{M}_1)$ and

$$\|\frac{E_1(x)}{\|E_1(x)\|_q}\|_1 = \|\frac{E_1(x)}{\|E_1(x)\|_q}\|_1 \leq 1.$$  

On the other hand,

$$\frac{x - E_1(x)}{c_q \|x - E_1(x)\|_q} = \frac{x - E_1(x)}{c_q \|x - E_1(x)\|_q} \cdot 1 = y \cdot b,$$
\[ \mathcal{E}_1(\frac{x - \mathcal{E}_1(x)}{c_q \|x - \mathcal{E}_1(x)\|_q}) = 0, \quad \|b\|_{q'} \leq 1 \]

and the noncommutative Burkholder inequality in [9] yields
\[ \|y\|_{h^c_q} = \|\frac{x - \mathcal{E}_1(x)}{c_q \|x - \mathcal{E}_1(x)\|_q} \|_{h^c_q} \leq c_q \|\frac{x - \mathcal{E}_1(x)}{c_q \|x - \mathcal{E}_1(x)\|_q}\| \leq 1. \]

Therefore,
\[ \|x\|_{h^c_1,at_q} \leq c_q \|x - \mathcal{E}_1(x)\|_q + \|\mathcal{E}_1(x)\|_q \leq (2c_q + 1)\|x\|_q. \]

The case \( q = \infty \) is proved in the same way just by replacing the noncommutative Burkholder inequality by the trivial fact that \( \| \cdot \|_{bmo^c} \leq \| \cdot \|_\infty \). The density is trivial. \( \square \)

**Lemma 4.6.** Let \( 1 < q < \infty \). Then
\[ (h^c_{1,at_q}(\mathcal{M}))^* = bmo^c_q(\mathcal{M}) \]
with equivalent norms. More precisely,

(i) Every \( x \in bmo^c_q(\mathcal{M}) \) defines a bounded linear functional on \( h^c_{1,at_q}(\mathcal{M}) \) by
\[ \varphi_x(a) = \tau(x^*a), \forall a \in (1, q, c)\text{-atoms}. \]

(ii) Conversely, each \( \varphi \in (h^c_{1,at_q}(\mathcal{M}))^* \) is given as (4.1) by some \( x \in bmo^c_q(\mathcal{M}) \).

Similarly, \( (h^r_{1,at_q}(\mathcal{M}))^* = bmo^r_q(\mathcal{M}) \) with equivalent norms.

**Proof.** (i) Let \( x \in bmo^c_q \), and \( a = yb \) where \( a \) is a \((1, q, c)\)-atom as in Definition 4.1. Then
\[ |\tau(x^*a)| = |\tau(\mathcal{E}_n(x^*y)b)| \]
\[ = |\tau(\mathcal{E}_n((x^* - x^*_n)y)b)| = |\tau(((x - x_n)b^*)^*y)|. \]

Thus, by the duality identity \( h^c_q(\mathcal{M}) = (h^c_q(\mathcal{M}))^* \) (see [9] for the relevant constants),
\[ |\tau(x^*a)| \leq \|(x - x_n)b^*\|_{h^c_q} \|y\|_{h^c_q} \leq \|x\|_{bmo^c_q}. \]

(ii). Let \( \varphi \) be any linear functional on \( h^c_{1,at_q}(\mathcal{M}) \). When \( 1 < q \leq 2 \), by Lemma 4.5 we can find \( x \in L_2(\mathcal{M}) \) such that
\[ \varphi(y) = \tau(x^*y), \quad \forall y \in L_2(\mathcal{M}), \]

and
\[ \|\varphi\| = \sup_{y \in L_2, \|y\|_{h^c_{1,at_q}} \leq 1} |\tau(x^*y)|. \]
When $2 < q < \infty$, by the same Lemma 4.5, we get the same representation of $\varphi$ with an $x \in L^q(M)$. Then fix $n$ and take any $b \in M_n$ with $\|b\|_{q'} \leq 1$. Again, by the duality $h^c_q(M) = (h^c_{q'}(M))^*$, we do the following calculation:

\[
\| (x - x_n)b \|_{h^c_q} = \sup_{\|y\|_{h^c_{q'}} \leq 1} |\tau(b^*(x^* - x^*_n)y)|
\leq \sup_{\|y\|_{h^c_{q'}} \leq cq} |\tau(b^*(x^* - x^*_n)y)|
= \sup_{\|y\|_{h^c_{q'}} \leq cq} |\tau((x^* - x^*_n)(y - y_n)b^*)|
= \sup_{\|y\|_{h^c_{q'}} \leq cq} |\tau((y - y_n)b^*)|
\leq cq \|\varphi\|
\]

Here, we have used the fact that $\tau(x - x_n) = \tau(y - y_n) = 0$ in the second and third equality respectively. The second inequality is due to the fact that $(y - y_n)b^*$ is a $(1, q, c)$-atom.

Now we are at a position to prove Theorem 4.4.

**Proof.** We consider here only the case $1 < q < \infty$ and postpone the case $q = \infty$ to the end of the proof of Theorem 4.12 below. We only need to show the inclusion

$h^c_1(M) \subset h^c_{i, at_q}(M)$.

Take $x \in h^c_{i, at_q}(M)$, by Theorem 3.3 and Lemma 4.6, we can conduct the following calculation,

\[
\|x\|_{h^c_{i, at_q}} = \sup_{\|y\|_{h^c_{i, at_q}} \leq 1} |\tau(x^* y)|
\leq \sup_{\|y\|_{bmo_{q'}} \leq cq} |\tau(x^* y)|
\leq \sup_{\|y\|_{bmo_{q'}} \leq cq} |\tau(x^* y)| \leq cq \|x\|_{h^c_q}.
\]

Then we end the proof with the density of $h^c_{i, at_q}(M)$ in $h^c_1(M)$.

**Definition 4.7.** We define

$h_{1, at_q}(M) = h^c_{1, at_q}(M) + h^r_{1, at_q}(M) + h^d_1(M)$

equipped with the sum norm

\[
\|x\|_{h_{1, at_q}} = \inf_{x = x_c + x_r + x_d} \{\|x_c\|_{h^c_{1, at_q}} + \|x_r\|_{h^r_{1, at_q}} + \|x_d\|_{h^d_1}\}.
\]

Then by Theorem 4.4, we obtain the atomic decomposition of $h_1(M)$.

**Corollary 4.8.** We have

$h_1(M) = h_{1, at_q}(M)$

with equivalent norms.
Combined with Davis’ decomposition presented in [17], the above theorem yields $H_1(M) = h_{1\text{at}, q}(M)$ with equivalent norms. In other words, we obtain an atomic decomposition for $H_1(M)$ too.

4.2. A fine version of atoms.

**Definition 4.9.** For $1 < q \leq \infty$, $a \in L_1(M)$ is said to be a $(1, q, c)_{pr}$-atom with respect to $(M_n)_{n \geq 1}$, if there exist $n \geq 1$ and a projection $e \in P(M_n)$ such that

(i) $E_n(a) = 0$;
(ii) $r(a) \leq e$;
(iii) $\|a\|_{h_1^c} \leq (\tau(e))^{-1/q'}$ for $1 < q < \infty$; $\|a\|_{\text{bmo}^c} \leq (\tau(e))^{-1}$ for $q = \infty$.

Similarly, we define $(1, q, r)_{pr}$-atoms with $r(a)$ replaced by $l(a)$.

**Remark 4.10.** A $(1, q, c)_{pr}$-atom $a$ is necessarily a $(1, q, c)$-atom. Indeed, we can factorize $a$ as $a = yb$ with $y = a(\tau(e))^{1/q'}$ and $b = e(\tau(e))^{-1/q'}$.

**Definition 4.11.** We define $h_{1\text{at}, q, pr}(M)$ to be the Banach space of all $x \in L_1(M)$ which admit a decomposition $x = \sum \lambda_k a_k$, where for each $k$, $a_k$ is a $(1, q, c)_{pr}$-atom or an element in the unit ball of $L_1(M_1)$, and $\lambda_k \in \mathbb{C}$ satisfying $\sum_k |\lambda_k| < \infty$. We equip this space with the norm

$$\|x\|_{h_{1\text{at}, q, pr}} = \inf \sum_k |\lambda_k|,$$

where the infimum is taken over all decompositions of $x$ described above. Similarly, we define $h_{1\text{at}, q, pr}^c(M)$.

Now, by Remark 4.10 and Lemma 4.4, we have the obvious inclusion $h_{1\text{at}, q, pr}^c(M) \subset h_{1\text{at}, q, pr}(M)$. In fact, the two spaces coincide thanks to the following theorem.

**Theorem 4.12.** For all $1 < q \leq \infty$, we have

$$h_{1\text{at}, q, pr}^c(M) = h_{1\text{at}, q, pr}^c(M)$$

with equivalent norms. Similarly, $h_{1\text{at}, q, pr}(M) = h_{1\text{at}, q, pr}(M)$ with equivalent norms.

Again, we prove this theorem for $1 < q < \infty$ by showing $(h_{1\text{at}, q, pr}^c(M))^* = \text{bmo}_{q', pr}(M)$. The latter duality equality is proved in the same way as Theorem 4.6. We leave the details to the reader. However by the argument in Theorem 4.6, we can not prove the theorem in the case $q = \infty$, due to the lack of Riesz representation. Here we provide another way to do it, which seems new, even in the commutative case.

Let $P$ be the set of projections of $M$. Given $e \in P$ let

$$n_e = \min\{k : e \in P(M_k)\}.$$
Note that \( n_e = \infty \) if the set on the right hand side is empty. This case is of no interest in the discussion below. For a family \((g_e)_{e \in P} \subset \text{bmo}^c(M)\) define
\[
\| (g_e)_e \|_{L^p_1(\text{bmo}^c)} = \sum_{e \in P} \tau(e) \| g_e \|_{\text{bmo}^c}.
\]
We will consider the Banach space:
\[
L^p_1(\text{bmo}^c) = \{ (g_e)_e : g_e = g_e, \mathcal{E}_{n_e} g_e = 0, \| (g_e)_e \|_{L^p_1(\text{bmo}^c)} < \infty \}.
\]
We will also need the following space consisting of families in \( h^c(M)\):
\[
L^p_\infty(h^c) = \{ (f_e)_e : f_e e = f_e, \mathcal{E}_{n_e} f_e = 0, \| (f_e)_e \|_{L^p_\infty(h^c)} < \infty \},
\]
where
\[
\| (f_e)_e \|_{L^p_\infty(h^c)} = \sup_{e \in P} \frac{1}{\tau(e)} \| f_e \|_{h^c}.
\]
For convenience, we denote \( L^p_1(\text{bmo}^c) \) by \( X \) and \( L^p_\infty(h^c) \) by \( Z \). We embed \( \text{bmo}^c_{1, pr}(M) \) isomorphically into \( Z \) via the following map
\[
\pi(y) = ((y - y_{n_e})_e)_e.
\]
Set \( Y = \pi(\text{bmo}^c_{1, pr}(M)) \).

**Lemma 4.13.** With the notation above we have
(i) \( Z \) is a subspace of \( X^* \) with equivalent norms, so is \( Y \).
(ii) \( Y \) is \( w^* \)-closed in \( X^* \).

**Proof.** (i). Let \((f_e)_e \in Z\), for any \((g_e)_e \in X\), we have
\[
|\langle (f_e)_e, (g_e)_e \rangle| = |\sum_{e} \tau((f_e)_e) g_e|
\]
\[
\leq \sqrt{2} \sum_{e} \| f_e \|_{h^c} \| g_e \|_{\text{bmo}^c}
\]
\[
\leq \sqrt{2} \sup_{e} \frac{1}{\tau(e)} \| f_e \|_{h^c} \cdot \sum_{e} \tau(e) \| g_e \|_{\text{bmo}^c}
\]
\[
= \sqrt{2} \| (f_e)_e \|_Z \| (g_e)_e \|_X.
\]
Thus we get \( \| (f_e)_e \|_{X^*} \leq \sqrt{2} \| (f_e)_e \|_Z \).

We turn to the proof of the inverse inequality. For any \((f_e)_e \in Z\), fix \( e_0 \in P \), we have
\[
\frac{1}{\tau(e_0)} \| f_{e_0} \|_{h^c} = \sup_{\| g \|_{\text{bmo}^c} \leq 1} \frac{1}{\tau(e_0)} |\tau((f_{e_0})^* g)|
\]
\[
= \sup_{\| g - g_{n_{e_0}} \|_{\text{bmo}^c} \leq 1} \frac{1}{\tau(e_0)} |\tau((f_{e_0})^* (g - g_{n_{e_0}}))_{e_0}|
\]
\[
\leq \sup_{\| g - g_{n_{e_0}} \|_{\text{bmo}^c} \leq 1} \frac{1}{\tau(e_0)} |\tau((f_{e_0})^* (g - g_{n_{e_0}}))_{e_0}|.
\]
Then we define $(g_e)_e$ as $g_e = (g - gn_{e_0})/\tau(e_0)$ if $e = e_0$, otherwise $g_e = 0$. Thus

$$\frac{1}{\tau(e_0)} \|f_{e_0}\|_{\xi} \leq \|(f_e)_e\|_{X^*} \|((g_e)_e)\|_{X} \leq \|(f_e)_e\|_{X^*},$$

which implies $\|(f_e)_e\|_{Z} \leq \|(f_e)_e\|_{X^*}$.

(ii). Since $Y$ is a subspace of $X^*$, by Krein and Smulian’s theorem, we only need to prove that for all $t > 0$, $Y \cap B_t(X^*)$ is $w^*$-closed in $X^*$, where $B_t(X^*)$ is the closed ball of $X^*$ centered at the origin and with radius $t$. Take a net $(y^\alpha)_\alpha \subset bmo^c_{1,p}(\mathcal{M})$ such that $\pi((y^\alpha)_\alpha) \subset Y \cap B_t(X^*)$. Hence $(y^\alpha)_\alpha$ are bounded in $bmo^c_{1,pr}(\mathcal{M})$. Suppose that,

$$\langle \pi(y^\alpha), (g_e)_e \rangle \to \langle \xi, (g_e)_e \rangle, \quad \forall (g_e)_e \in X,$$

for some $\xi \in B_t(X^*)$. We will show that $\xi \in Y$, which will complete the proof. We need two facts. The first one is that $bmo^c_{1,pr}(\mathcal{M})$ is a dual space by Theorem 3.16, so its unit ball is $w^*$-compact. Therefore, the bounded net $(y^\alpha)_\alpha$ in $bmo^c_{1,pr}(\mathcal{M})$ admits a $w^*$-cluster point $y$. Without loss of generality, we assume that $(y^\alpha)_\alpha$ converges to $y$ in the $w^*$-topology:

$$\langle y^\alpha, x \rangle \to \langle y, x \rangle, \quad \forall x \in h^1_t(\mathcal{M}).$$

The second fact is that for any $(g_e)_e \in X$, the sum $\sum_e g_e$ is absolutely summable in $h^1_t(\mathcal{M})$. Indeed, by Lemma 4.2

$$\sum_e \|g_e\|_{h^1_t} \leq \sum_e \tau(e) \|g_e\|_{bmo^c} = \|(g_e)_e\|_{X}.$$ 

Therefore, for any $(g_e)_e \in X$, we have

$$\langle \pi(y^\alpha), (g_e)_e \rangle = \sum_e \tau(((y^\alpha - y^\alpha_{n_e})_e) g_e) $$

$$= \tau((y^\alpha)^* \sum_e g_e)$$

Combining 4.2 and 4.3, we deduce that $\xi = \pi(y) \in Y$, as desired. \qed

We can now prove Theorem 4.12 in the case of $q = \infty$.

**Proof.** Let $Y_\perp$ be the preannihilator of $Y$ in $X^*$:

$$Y_\perp = \{(g_e)_e \in X : \langle \pi(y), (g_e)_e \rangle = 0, \forall y \in bmo^c_{1,pr}(\mathcal{M})\}.$$ 

Then by the bipolar theorem

$$Y \simeq (X/Y_\perp)^*.$$ 

Using the second fact in the proof of the previous lemma, we get

$$Y_\perp = \{(g_e)_e \in X : \tau(y^* \sum_e g_e) = 0, \forall y \in bmo^c_{1,pr}(\mathcal{M})\}$$

$$= \{(g_e)_e \in X : \sum_e g_e = 0 \text{ in } h^1_t(\mathcal{M})\}.$$
Then for \((g_e)_e \in X/Y_\perp\), let

\[ g = \sum_{e \in \mathcal{P}} g_e. \]

Then

\[ \|(g_e)_e\|_{X/Y_\perp} = \inf \{ \sum_{e} \tau(e) \|(g'_e)_e\|_{bmo^c} : g = \sum_{e} g'_e, (g'_e)_e \in X \} \]

\[ = \inf \{ \sum_{e} |\lambda_e| : g = \sum_{e} \lambda_e a_e, (\lambda_e a_e)_e \in X, \|a_e\|_{bmo^c} \leq \frac{1}{\tau(e)} \} \]

\[ = \|g\|_{h_{1,at,pr}^c}. \]

Consequently, for any \(x \in h_{1,at,pr}^c(M)\) and any decomposition \(x = \sum_{e} \lambda_e a_e\),

\[ \|x\|_{h_{1,at,pr}^c} = \|(\lambda_e a_e)_e\|_{X/Y_\perp} \]

\[ = \|(\lambda_e a_e)_e\|_{Y^*} \]

\[ = \sup_{y \in bmo_{pr}^c, \|\pi(y)\|_{Y} \leq 1} |\langle (\lambda_e a_e)_e, \pi(y) \rangle| \]

\[ \leq \sup_{\|y\|_{bmo^c} \leq c} |\tau((\sum_{e} \lambda_e a_e)^* y)| \leq c\|x\|_{h_{1}^c}. \]

Therefore, combined with Lemma 4.2 and Remark 4.10, the density of \(h_{1,at,pr}^c(M)\) in \(h_1^c(M)\) (due to Lemma 4.5) yields the desired duality identity \(h_{1,at,pr}^c(M) = h_1^c(M)\).

Let us return back to the unsettled case \(q = \infty\) in the proof of Theorem 4.4. Since a fine atom is necessarily a crude atom, we get \(h_1^c(M) \subset h_{1,at,pr}^c(M)\), hence \(h_1^c(M) = h_{1,at,pr}^c(M)\) with equivalent norms due to Lemma 4.2. Thus Theorem 4.4 is completely proved.

**Definition 4.14.** We define

\[ h_{1,at,q,pr}(M) = h_{1,at,q,pr}^c(M) + h_{1,at,q,pr}^r(M) + h_1^q(M) \]

equipped with the sum norm

\[ \|x\|_{h_{1,at,q,pr}} = \inf_{x = x_c + x_r + x_d} \{ \|x_c\|_{h_{1,at,q,pr}^c} + \|x_r\|_{h_{1,at,q,pr}^r} + \|x_d\|_{h_1^q} \}. \]

Then by Theorem 4.12 and Perrin’s noncommutative Davis decomposition (see [17]), we get the atomic decomposition of \(h_1(M)\) and \(H_1(M)\).

**Corollary 4.15.** We have

\[ H_1(M) = h_1(M) = h_{1,at,q,pr}(M), \]

for any \(1 < q \leq \infty\), with equivalent norms.

However, using Corollary 3.18, we can obtain another kind of atomic decomposition for \(h_1(M)\) or \(H_1(M)\), which is exactly the noncommutative analogue of the classical case.
Definition 4.16. For $1 < q \leq \infty$, $a \in L_1(M)$ is said to be a $(1, q)$-atom with respect to $(M_n)_{n \geq 1}$, if there exist $n \geq 1$ and a projection $e \in \mathcal{P}(M_n)$ such that

(i) $\mathcal{E}_n(a) = 0$;
(ii) $r(a) \leq e$ or $l(a) \leq e$;
(iii) $\|a\|_q \leq \left(\tau(e)\right)^{\frac{q}{q'}}$.

Definition 4.17. We define $h_{1,q}^1(M)$ as the Banach space of all $x \in L_1(M)$ which admit a decomposition $x = y + \sum_k \lambda_k a_k$, where for each $k$, $a_k$ is a $(1, q)$-atom or an element in the unit ball of $L_1(M_1)$, $\lambda_k \in \mathbb{C}$ satisfying $\sum_k |\lambda_k| < \infty$, and where the martingale differences of $y$ satisfy $\sum_{j \geq 1} \|dy_j\|_1 < \infty$. We equip this space with the norm

$$\|x\|_{h_{1,q}^1} = \inf \left\{ \sum_j \|dy_j\|_1 + \sum_k |\lambda_k| \right\},$$

where the infimum is taken over all decompositions of $x$ as above.

Lemma 4.18. If $a$ is a $(1, q)$-atom, then

$$\|a\|_{h_1} \leq \frac{cq}{q-1}.$$  

Proof. Without loss of generality, suppose $a$ is a $(1, q)$-atom with $r(a) \leq e$. We apply Corollary 3.18 and the duality $(h_1(M))^* = \text{bmo}(M)$.

$$\|a\|_{h_1} \leq c \sup_{\|x\|_{\text{bmo}} \leq 1} \tau(x^*a)$$
$$= c \sup_{\|x\|_{\text{bmo}} \leq 1} \tau((x - x_n)^*a)$$
$$= c \sup_{\|x\|_{\text{bmo}} \leq 1} \tau(((x - x_n)e)^*a)$$
$$\leq c\|a\|_q \|(x - x_n)e\|_{q'} \leq cq'.$$

□

Theorem 4.19. For all $1 < q \leq \infty$, we have

$$\mathcal{H}_1(M) = h_1(M) = h_{1,q}^1(M)$$

with equivalent norms.

By Lemma 4.18, Corollary 3.18 and using arguments similar to those in the proof of Theorem 4.4, we can prove the theorem for the case $1 < q < \infty$. For the case $q = \infty$, we use the argument in Theorem 4.12. Instead of $L_p^p(\text{bmo})$ and $L_p^\infty(h_1^p)$, we consider the following two spaces:

$$L_p^p(L_\infty) = \{(g_e)e : g_e e = g_e, e g_e = g_e, \mathcal{E}_n g_e = 0, \| (g_e)e \|_{L_p^p(L_\infty)} < \infty\},$$

$$L_p^\infty(L_1) = \{(f_e)e : f_e e = f_e, e f_e = f_e, \mathcal{E}_n f_e = 0, \| (f_e)e \|_{L_p^\infty(L_1)} < \infty\},$$
where
\[
\|(g_e)_e\|_{L^p(L^\infty)} = \sum_e \tau(e) \|g_e\|_{\infty},
\]
\[
\|(f_e)_e\|_{L^\infty(L^1)} = \max \left\{ \sup_e \frac{1}{\tau(e)} \|f_e e\|_1, \sup_e \frac{1}{\tau(e)} \|f e\|_1 \right\}.
\]

Then by Lemma 4.18 and Corollary 3.18, we get the announced results. We leave the details to the reader.

**Remark 4.20.** The part of this paper on the crude versions of the John-Nirenberg inequalities and atomic decomposition can be easily extended to the type III case with minor modifications.

5. **An open question of Junge and Musat**

It is an open question asked in [8] (on page 136) that given \(2 < p < \infty\), whether there exists a constant \(c_p\) such that

\[
\sup_k \|\mathcal{E}_k x - \mathcal{E}_{k-1} x\|^p \leq c_p \|x\|_{BMO}\? 
\]

It is easy to see that the answer is negative for matrix-valued functions with irregular filtration. In the following, we show that the answer is negative even for matrix-valued dyadic martingales. Recall that Remark 3.14 already shows that the answer is negative if one considers the column norm \(\|\cdot\|_{BMO_c}\) alone on the right hand side.

Let \(\mathcal{M}\) and \(\mathcal{M}_k\) be as in Remark 3.14. We consider this special case and show that the best constant \(c_p(n)\) such that (5.1) holds is bigger than \(c(\log(n+1))^{1/p}\) for all \(p \geq 3\). Let \(b\) be an \(M_n\)-valued function on \(\mathbb{T}\). We need the so-called “sweep” function of \(b\)

\[
S(b) = \sum_{k=1}^{\infty} |db_k|^2.
\]

Note that it is just the square of the usual square function. Matrix-valued sweep functions have been studied in [2], [4], [13] etc. It is proved in [13] that the best constant \(c_n\) such that

\[
\|S(b)\|_{BMO_c} \leq c_n \|b\|^2_{\infty}
\]

is \(c(\log(n+1))^2\). A similar result had been proved previously by Blasco and Pott (see [2]) by considering \(\|b\|_{BMO_c}^2\) on the right side of (5.2).

**Lemma 5.1.** Assume \(\|f\|_{BMO_c} \leq c(n) \sup_k \|\mathcal{E}_k f - \mathcal{E}_{k-1} f\|_{\infty}\) for any self-adjoint \(f\). Then \(c(n) \geq c(\log(n+1))^2\).
Proof. Under the assumption, we have
\[
\|S(b)\|_{\mathcal{BMO}} \leq c(n) \sup_m \|\mathcal{E}_m S(b) - \mathcal{E}_{m-1} S(b)\|_\infty
\]
\[
= c(n) \sup_m \left\| \mathcal{E}_m \left[ \sum_{k=1}^{\infty} |db_k|^2 - \mathcal{E}_{m-1} \sum_{k=1}^{\infty} |db_k|^2 \right] \right\|_\infty
\]
\[
= c(n) \sup_m \left\| \mathcal{E}_m \left[ \sum_{k=m}^{\infty} |db_k|^2 - \mathcal{E}_{m-1} \sum_{k=m}^{\infty} |db_k|^2 \right] \right\|_\infty.
\]
Let \( x = \sum_{k=m}^{\infty} |db_k|^2 \) and \( y = \mathcal{E}_{m-1} \sum_{k=m}^{\infty} |db_k|^2 \). By the convexity of \(| \cdot |^2\), we get
\[
\left| \frac{x - y}{2} \right|^2 \leq \frac{|x|^2 + |y|^2}{2} \leq \frac{|x|^2 + \|y\|_\infty^2}{2} \leq \frac{(|x| + \|y\|_\infty^2)^2}{2}.
\]
Then by Löwner-Heinz’s inequality,
\[
\left| \frac{x - y}{2} \right| \leq \frac{|x| + \|y\|_\infty}{\sqrt{2}}.
\]
Thus by the triangle inequality, we have
\[
\|S(b)\|_{\mathcal{BMO}} \leq 2c(n) \sup_m \|\mathcal{E}_m x + \|y\|_\infty \mathbb{1}\|_\infty
\]
\[
= 2c(n) \sup_m \|\mathcal{E}_m |b - \mathcal{E}_{m-1} b|^2\|_\infty + 2c(n) \|\mathcal{E}_{m-1} |b - \mathcal{E}_{m-1} b|^2\|_\infty
\]
\[
\leq 2c(n) ||b||_{\mathcal{BMO}}^2 + 2c(n) \|\mathcal{E}_m |b - \mathcal{E}_{m-1} b|^2\|_\infty
\]
\[
\leq 4c(n) ||b||_{\mathcal{BMO}}^2.
\]
We then get \( c(n) \geq c(\log(n+1))^2 \) by \((5.2)\). \(\square\)

Lemma 5.2. Let \( 0 < p < \infty \) and \( \mathcal{E}_m \) be the conditional expectation from \( \mathcal{M} \) onto \( \mathcal{M}_m \), we have
\[
\|\mathcal{E}_m |x|^\frac{p+1}{2}\|_\infty \leq \|\mathcal{E}_m |x|^p\|_\infty^{\frac{1}{2}} \|\mathcal{E}_m |x|\|_\infty^{\frac{1}{2}}.
\]

Proof. By Hölder’s inequality, we get
\[
\|\mathcal{E}_m |x|^\frac{p+1}{2}\|_\infty = \sup_{\|a\|_{L^1_\tau(\mathcal{M}_m)} \leq 1} \tau(\mathcal{E}_m |x|^\frac{p+1}{2} a)
\]
\[
= \sup_{\|a\|_{L^1_\tau(\mathcal{M}_m)} \leq 1} \tau(a|\frac{1}{2}|x|^\frac{p}{2} |x|^\frac{1}{2} a \frac{1}{2})
\]
\[
\leq \sup_{\|a\|_{L^1_\tau(\mathcal{M}_m)} \leq 1} (\tau(a|\frac{1}{2}|x|^p))^{\frac{1}{2}} (\tau(a|x|))^{\frac{1}{2}}
\]
\[
= \|\mathcal{E}_m |x|^p\|_\infty^{\frac{1}{2}} \|\mathcal{E}_m |x|\|_\infty^{\frac{1}{2}}.
\]
\(\square\)
Theorem 5.3. Suppose \( \sup_k \| E_k f - E_{k-1} f \|_p^{1/p} \| f \|_{BMO} \leq c_p(n) \| f \|_{BMO} \) for some \( p \geq 3 \). Then

\[
c_p(n) \geq c (\log(n + 1))^{\frac{2}{p}}.
\]

Proof. Fix a selfadjoint \( M_n \)-valued function \( b \). By the operator Jensen inequality and Lemma 5.2, for \( p \geq 3 \),

\[
\| b \|_{BMO}^2 = \sup_m \| E_m b - E_{m-1} b \|_{\infty}^2 \\
\leq \sup_m \| E_m b - E_{m-1} b \|_{\frac{p+1}{2}}^{p+1} \\
\leq \sup_m \| E_m b - E_{m-1} b \|_{\infty}^{\frac{2}{p+1}} \sup_m \| E_m b - E_{m-1} b \|_{\infty}^{\frac{2}{p+1}} \\
\leq \left( c_p(n) \| b \|_{BMO} \right)^{\frac{2}{p+1}} \sup_m \| E_m b - E_{m-1} b \|_{\infty}^{\frac{2}{p+1}}.
\]

Then

\[
\| b \|_{BMO} \leq (c_p(n))^p \sup_m \| E_m b - E_{m-1} b \|_{\infty}.
\]

By Lemma 5.1, we get

\[
(c_p(n))^p \geq c (\log(n + 1))^{\frac{2}{p}}.
\]

\[\square\]

From Theorem 5.3, we get a negative answer for the open question by letting \( n \to \infty \).

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References

[1] T. Bekjan, Z. Chen, M. Perrin, Z. Yin, Atomic decomposition and interpolation for Hardy spaces of noncommutative martingales. J. Funct. Analysis., 258(7):2483-2505, 2010.
[2] O. Blasco, S. Pott, Embeddings between operator-valued dyadic BMO spaces, Illinois J. of Math., 52(3):799-814, 2008.
[3] R.R. Coifman, R. Rochberg, M.H. Taibleson, G. Weiss, Introduction. Representation theorems for Hardy spaces, pp. 1-9, Astérisque, 77, Soc. Math. France, Paris, 1980.
[4] T. A. Gillespie, S. Pott, S. Treil, A. Volberg, Logarithmic growth for martingale transforms, J. London Math. Soc, 64(3):624-636, 2001.
[5] G. Hong, L.D. López-Sánchez, J.M. Martell, J. Parcet, Calderón-Zygmund operators associated to matrix-valued kernels, arXiv:1201.4351.
[6] M. Junge, Doob’s Inequality for Non-commutative Martingales, J. Reine Angew. Math. 549:149-190, 2002.
[7] M. Junge and T. Mei, Noncommutative Riesz transforms-A probabilistic approach, Amer. J. Math. 132(3):611-680, 2010.
[8] M. Junge, M. Musat, Non-commutative John-Nirenberg theorem, Trans. Amer. Math. Soc. 359(1):115-142, 2007.
[9] M. Junge, Q. Xu, Non-commutative Burkholder/Rosenthal Inequalities, Ann. Prob. 31(2):948-995, 2003.
[10] M. Junge and Q. Xu, On the best constants in some non-commutative martingale inequalities, Bull. London Math. Soc. 37:243C253, 2005.
[11] M. Junge and Q. Xu, Noncommutative maximal ergodic theorems, J. Amer. Math. Soc. 20:385-439, 2006.
[12] M. Junge and Q. Xu, Noncommutative Burkholder/Rosenthal inequalities II: applications, Israel J. Math. 167:227-282, 2008.
[13] T. Mei, Notes on matrix valued paraproducts. Indiana Univ. Math. J. 55(2):747–760, 2006.
[14] M. Musat, Interpolation Between Non-commutative BMO and Non-commutative $L^p$-spaces, J. Funct. Analysis., 202(1):195-225, 2003.
[15] F. Nazarov, G. Pisier, S. Treil, A. Volberg, Sharp Estimates in Vector Carleson Imbedding Theorem and for Vector Paraproducts, J. Reine Angew. Math., 542:147-171, 2002.
[16] J. Parcet, N. Randrianantoanina, Gundy’s Decomposition for Non-Commutative Martingales and Applications, Proc. London Math. Soc., 93(1):227-252, 2006.
[17] M. Perrin, A noncommutative Davis’ decomposition for martingales, J. London Math. Soc., 80(3):627-648, 2009.
[18] G. Pisier, Q. Xu, Non-commutative Martingale Inequalities, Comm. Math. Phys., 189:667-698, 1997.
[19] G. Pisier, and Q. Xu, Non-commutative $L_p$-spaces, pp. 1459-1517 in "Handbook of the Geometry of Banach Spaces", Vol. II, edited by W.B. Johnson and J. Lindenstrauss, Elsevier, 2003.
[20] N. Randrianantoanina, Non-commutative martingale transform, J. Funct. Analysis., 194(1):181-212, 2002.
[21] N. Randrianantoanina. Conditioned square functions for noncommutative martingales. Ann. Probab., 35(3):1039-1070, 2007.
[22] F. Weisz, Atomic Hardy spaces, Anal. Math., 20:65-80, 1994.

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