Holographic renormalization of cascading gauge theories

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Abstract

We perform a holographic renormalization of cascading gauge theories. Specifically, we find the counter-terms that need to be added to the gravitational action of the backgrounds dual to the cascading theory of Klebanov and Tseytlin, compactified on an arbitrary four-manifold, in order to obtain finite correlation functions (with a limited set of sources). We show that it is possible to truncate the action for deformations of this background to a five dimensional system coupling together the metric and four scalar fields. Somewhat surprisingly, despite the fact that these theories involve an infinite number of high-energy degrees of freedom, we find finite answers for all one-point functions (including the conformal anomaly). We compute explicitly the renormalized stress tensor for the cascading gauge theories at high temperature and show how our finite answers are consistent with the infinite number of degrees of freedom. Finally, we discuss ambiguities appearing in the holographic renormalization we propose for the cascading gauge theories; our finite results for the one-point functions have some ambiguities in curved space (including the conformal anomaly) but not in flat space.

June 2005
1 Introduction

The traditional definition of quantum field theories starts from a fixed point of the renormalization group (free or interacting) at high energies, which is some local conformal field theory, and defines the theory by a renormalization group flow starting from that fixed point. One of the interesting “side-effects” of the progress in our understanding of string theory in the last few years is the realization that there exist consistent quantum field theories which cannot be defined in this way. In some cases these theories may be described by a decoupling limit of some sector of string theory, and more generally they can always be defined by a background of type IIB string theory which is holographically dual to these theories, in the same sense that the $AdS_5 \times S^5$ background of string theory is dual to the $d = 4 \, \mathcal{N} = 4$ supersymmetric Yang-Mills theory [1].

The new types of theories which were discovered seem to fall into two classes. One class of theories is “little string theories” – non-local theories which share some features with string theories (though they do not include gravity). In this paper we focus on the second class, of “cascading theories” 1. The prototypical example of such theories (which we will focus on here, though we expect to be able to generalize our methods also to other cascading theories, including theories in other dimensions) is the theory related to fractional 3-branes at a conifold singularity, first studied in [3–5] (see [6,7] for reviews). These theories do not have a direct definition in field theory terms (since they do not seem to have a UV fixed point), so their only known direct definition is via the holographic duality; in this paper we will attempt to understand this definition better and verify that it can be used for well-defined computations in these theories. When one introduces a finite high-energy cutoff, these theories at the cutoff scale resemble $\mathcal{N} = 1$ supersymmetric $SU(K) \times SU(K + P)$ gauge theories with two bifundamental and two anti-bifundamental chiral superfields and some superpotential; when one flows down in energy from this cutoff one of the gauge theories becomes strongly coupled and the theory seems to undergo a series of Seiberg duality [8] “cascades”, ending with a confining theory in the IR (which is in the same universality class as the $\mathcal{N} = 1$ supersymmetric pure $SU(P)$ Yang-Mills theory when $K$ is a multiple of $P$). However, these gauge theories are never asymptotically free, so they cannot be used to define the theory – rather it seems that more and more degrees of freedom are needed to define

1An interesting relation between these two types of theories was recently discovered in [2].
the theory at higher energies, and that the ultimate definition of the cascading theories requires a theory with an infinite number of fields. It may be possible to define the cascading gauge theories by a limiting procedure, starting from a theory with a finite number of degrees of freedom which can flow to the cascade (as in [9]) and taking the limit in which the number of degrees of freedom goes to infinity – it would be interesting to make such a definition precise. In this paper we will not discuss the interpretation of the cascade as a gauge theory, but rather we will take the holographic dual to define the cascading theory, and attempt to see if such a definition makes sense.

In order to use a holographic dual as a definition of a field theory we need to have a prescription for the computation of all correlation functions in the field theory. For the AdS/CFT correspondence such a prescription was given in [10, 11], and it can be generalized to other holographic dualities as well. In principle one should be able to perform the computations of correlation functions directly in string theory on the holographic dual background, but in practice, since this string theory is very complicated, computations can only be done in a low-energy gravitational approximation, and we will use this approximation in this paper. In this approximation the correlation functions may be defined as derivatives of the (super)gravity action on the holographic dual background with respect to sources at the boundary of space-time. When one tries to perform such computations one encounters divergences, which from the gravity point of view are IR divergences related to the infinite distance to the boundary. These divergences may be dealt with by the process of “holographic renormalization” [12–25]. First, one regularizes the theory by imposing a cutoff on the radial direction (the precise meaning of this cutoff in the field theory language is not clear, but it certainly provides a good regularization). Next, one adds counter-terms to the gravitational action, which are local functions of the fields at the cutoff, in such a way that the action remains finite when the cutoff is taken to infinity. This process is very analogous to renormalization in field theory, and it seems that it should be mapped to this by the holographic duality; in both cases in a well-defined theory there is just a finite number of divergences which need to be canceled, after which correlation functions may be computed, depending on a finite number of parameters (some of which are coupling constants of the theory, while others are related to vacuum expectation values of fields or to ambiguities in the definitions of operators).

The process of holographic renormalization is by now conceptually well-understood in asymptotically anti-de Sitter spaces, where several examples have been analyzed in
detail [21, 22, 24], though the general renormalization for theories with many fields in the bulk is quite complicated and has not yet been performed. In principle one would expect a similar renormalization process to apply in other holographic dualities such as those of cascading gauge theories\(^2\), and our goal in this paper is to understand how this works. One interesting question which immediately arises is the following. In standard field theories there are some correlation functions, such as the one-point function of the trace of the stress-energy tensor (related to the conformal anomaly), which are proportional to the number of degrees of freedom in the theory. As discussed above, cascading theories do not seem to have a finite number of degrees of freedom, but rather more and more degrees of freedom as one goes to shorter and shorter distance scales. So, should the correlation functions of cascading theories be finite as one takes the cutoff to infinity or should some of them diverge?

We have attempted to perform a holographic renormalization of the cascading theories both under the assumption that all correlation functions must be finite, and under the assumption that correlation functions are allowed to diverge as the cutoff is taken to infinity, in a way which depends on the effective number of degrees of freedom. Somewhat surprisingly, we found that it is possible to renormalize the theory with finite correlation functions, but we were not able to renormalize the theory using the other assumption. Thus, we claim that cascading gauge theories should be renormalized just like standard theories, with all correlation functions finite. At first sight this seems to contradict the fact that these theories have an infinite number of high-energy degrees of freedom. We claim that this is not the case, and that these theories have an infinite number of high-energy degrees of freedom even though all correlation functions (including the conformal anomaly) are finite. We illustrate this by analyzing the thermodynamics of the cascading theories, showing that the effective number of degrees of freedom diverges at high temperatures even though all correlation functions are finite (at any fixed temperature).

At this point we should describe the precise assumptions under which we perform the holographic renormalization of the cascading gauge theories. Usually holographic renormalization is performed using a consistent truncation of the theory to a small number of fields\(^3\), for which any sources are allowed. In our case we also truncate the

\(^2\)Some general properties of the stress tensor in holographic renormalization, which apply also to cascading backgrounds, were derived in [26]. It is not obvious whether holographic renormalization should be possible also in the duals to non-local theories such as “little string theories”.

\(^3\)As far as we know no more than two fields (coupled to gravity) were analyzed until now.
full spectrum of fields in the holographic dual background to a finite number of fields. One important truncation we make is that we only include fields which preserve the $SU(2) \times SU(2) \times \mathbb{Z}_{2P} \times \mathbb{Z}_2$ isometry which is present at high energies in the cascading background of [4], related to the global symmetry in the field theory (the metric actually has a $U(1)$ isometry but this is broken to a $\mathbb{Z}_{2P}$ subgroup by the fluxes [27]). In infinite space the cascading theories spontaneously break the $\mathbb{Z}_{2P}$ symmetry, as found in [5], so our analysis cannot be directly used to analyze the infinite volume cascading theories. However, at finite (high enough) temperature or at finite (small enough) volume (for instance on $S^3 \times \mathbb{R}$, $S^4$ or $dS_4$) this symmetry is expected to be preserved [28–31], so our analysis may be directly used for such backgrounds. We expect that it should be straightforward (though technically difficult) to extend our analysis to include also fields which break the $SU(2) \times SU(2) \times \mathbb{Z}_{2P} \times \mathbb{Z}_2$ symmetry.

Even with the truncation to the $(SU(2) \times SU(2) \times \mathbb{Z}_{2P} \times \mathbb{Z}_2)$-invariant sector, we are left with a large number of five dimensional fields in the bulk – the metric and four scalar fields. These fields all mix together so we were not able to truncate the theory further. Moreover, some of these fields are dual to irrelevant operators $^4$, so it is not known how to introduce arbitrary sources for these fields, as is usually done in the holographic renormalization process in order to systematically compute the counter-terms. Thus, we do not consider the most general sources; we allow a generic source for the five dimensional metric (this source is identified with the four dimensional metric of the space-time on which the cascading theory lives), but only constant sources for the other scalar fields (and, in particular, vanishing sources for the two scalar fields corresponding to irrelevant operators). This simplifies the analysis considerably, but there are three disadvantages. First, usually in holographic renormalization the finiteness of the action (for arbitrary sources) guarantees the finiteness of all correlation functions, but we do not allow arbitrary sources so we have to separately check that the correlation functions are finite in addition to the finiteness of the action (we check this only for one-point functions; additional counter-terms may be needed to ensure the finiteness of all correlation functions). Second, we can no longer translate the divergent terms in the action directly to counter-terms, as usually done in holographic renormalization. Therefore, we are forced to use a different procedure, of guessing the

$^4$Since the cascading theories are close to conformal field theories at high energies, with the characteristic power law behavior of conformal field theories replaced by powers multiplying logs, we will use the standard terminology of conformal field theories.
counter-terms and verifying that they lead to finite correlation functions (with a finite number of ambiguities). Again, we expect that it should be possible to generalize our analysis to include arbitrary sources (at least for all the marginal and relevant operators), though this will be technically complicated. We hope to return to this problem in the future. The counter-terms that we find in this method are far from being unique, and are certainly not the precise counter-terms that lead to finiteness of all correlation functions. However, we expect that the difference between the counter-terms we find and the correct counter-terms will not affect the unambiguous results which we obtain. Third, since we do not have arbitrary sources we cannot compute arbitrary correlation functions, but only the derivatives of the action with respect to the sources we include. Thus, our procedure allows us to compute any correlation functions of the stress-energy tensor (dual to the bulk metric), but in the scalar sector we can only compute one-point functions.

In this paper we show that, in the truncation described above, it is possible to holographically renormalize the cascading gauge theory background and to obtain finite one-point functions. We begin in section 2 by describing the background, the ansatz we use for the solutions with the sources described above, and the solutions we find. In section 3 we describe in detail the holographic renormalization process and the form of the counter-terms we find. In section 4 we discuss the example of cascading theories at finite (high) temperature, following [28, 30, 31], and compute their thermodynamic properties. We end in section 5 with our conclusions and a discussion of future directions. Various technical results are relegated to the appendix.

2 The action and asymptotic behavior of cascading backgrounds

In this section we construct the asymptotic (near the boundary) solutions corresponding to cascading gauge theories compactified on arbitrary manifolds, generalizing the flat space asymptotic solution found in [4].
2.1 The gravitational action and its KK reduction and truncation

We will work in the gravitational approximation to type IIB string theory, using the type IIB supergravity action. This action takes the form (in the Einstein frame)

\[
S_{10} = \frac{1}{16\pi G_{10}} \int_{\mathcal{M}_{10}} \left( R_{10} \wedge \ast 1 - \frac{1}{2} d\Phi \wedge \ast d\Phi - \frac{1}{2} e^{-\phi} H_3 \wedge \ast H_3 - \frac{1}{2} e^\phi F_3 \wedge \ast F_3 \\
- \frac{1}{4} F_5 \wedge \ast F_5 - \frac{1}{2} C_4 \wedge H_3 \wedge F_3 \right),
\]

(2.1)

where \( \mathcal{M}_{10} \) is the ten dimensional bulk space-time, \( G_{10} \) is the ten dimensional gravitational constant, and we have consistently set the axion \( C_0 \) to zero (it vanishes in all the solutions we are interested in). In this action

\[
F_3 = dC_2, \quad F_5 = dC_4 - C_2 \wedge H_3,
\]

(2.2)

where \( C_2 \) and \( C_4 \) are the Ramond-Ramond (RR) potentials. The equations of motion following from the action (2.1) have to be supplemented by the self-duality condition

\[
\ast F_5 = F_5.
\]

(2.3)

It is important to remember that the self-duality condition (2.3) can not be imposed at the level of the action, as this would lead to wrong equations of motion.

Next, we perform a Kaluza-Klein (KK) reduction of this action to five dimensions, using a specific ansatz for the metric and for the various forms. This ansatz includes in particular the solution of [4], and it is the most general ansatz describing a deformation of this solution which preserves the \( SU(2) \times SU(2) \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry of this solution\(^5\) (the discrete \( \mathbb{Z}_2 \) symmetry acts by exchanging the two global \( SU(2) \) factors). We take \( \mathcal{M}_{10} \) to be a direct warped product of \( \mathcal{M}_5 \) with metric \( g_{\mu\nu}(y) \) and the ‘squashed’ \( T^{1,1} \) coset appearing in the solution of [4]. So, the Einstein-frame metric ansatz is

\[
d s_{10}^2 = g_{\mu\nu}(y) dy^\mu dy^\nu + \Omega_1^2(y) e_\psi^2 + \Omega_2^2(y) \sum_{a=1}^{2} (e_{\theta_a}^2 + e_{\phi_a}^2),
\]

(2.4)

where \( y \) denotes the coordinates of \( \mathcal{M}_5 \) (greek indices \( \mu, \nu \) will run from 0 to 4) and the one-forms \( e_\psi, e_{\theta_a}, e_{\phi_a} \) \((a = 1, 2)\) are given by (see also [29]):

\[
e_\psi = \frac{1}{3} \left( d\psi + \sum_{a=1}^{2} \cos \theta_a \ d\phi_a \right), \quad e_{\theta_a} = \frac{1}{\sqrt{6}} d\theta_a, \quad e_{\phi_a} = \frac{1}{\sqrt{6}} \sin \theta_a \ d\phi_a.
\]

(2.5)

\(^5\)Except for two modes of the RR fields which we consistently set to zero as mentioned in the text.
Additionally, we assume the following ansatz for the fluxes $H_3 \equiv dB_2$, $F_3$ and the dilaton $\Phi$:

\[ F_3 = P \, e_\psi \wedge (e_{\theta_1} \wedge e_{\phi_1} - e_{\theta_2} \wedge e_{\phi_2}), \quad B_2 = \tilde{k}(y) \, (e_{\theta_1} \wedge e_{\phi_1} - e_{\theta_2} \wedge e_{\phi_2}), \quad \Phi = \Phi(y), \]

(2.6)

where $P$ is an integer corresponding to the RR 3-form flux on the compact 3-cycle (and to the number of fractional branes on the conifold). Special care should be taken with the RR 5-form. From (2.2) we get the Bianchi identity

\[ dF_5 = -F_3 \wedge H_3, \]

(2.7)

which for the background fluxes (2.6) is solved by

\[ F_5 = dC_4 - \left( \tilde{K}_0 + 2P\tilde{k}(y) \right) e_\psi \wedge e_{\theta_1} \wedge e_{\phi_1} \wedge e_{\theta_2} \wedge e_{\phi_2} \]

(2.8)

with some constant $\tilde{K}_0$. In our ansatz the RR four-form does not depend on the compact coordinates, that is $C_4 \equiv C_4(y)$ (note that $C_4 \wedge F_3 \wedge H_3 \neq 0$), and the RR five-form is proportional to the volume form of $\mathcal{M}_5$ (plus its dual). We define $K(y)$ by

\[ dC_4 = \frac{K(y)}{\Omega_1 \Omega_2} \, \text{vol}_{\mathcal{M}_5} \equiv \frac{K(y)}{\Omega_1 \Omega_2} \sqrt{-\det(g_{\mu\nu})} \, dy^1 \wedge \cdots \wedge dy^5, \]

(2.9)

and then the self-duality condition (2.3) implies

\[ K(y) = \tilde{K}_0 + 2P\tilde{k}(y) \]

(2.10)

(again, in deriving the effective action we should keep $C_4$ unconstrained and impose this equation later). Altogether, from the five-dimensional perspective we allow fluctuations in the metric $g_{\mu\nu}(y)$, in the scalar fields $\Omega_1(y), \Omega_2(y), \tilde{k}(y), \Phi(y)$ and in the four-form $C_4(y)$ (which is determined in terms of the others by the self-duality condition). We have set to zero various fluctuations of the form fields which are $p$-forms on $\mathcal{M}_5$, and also fluctuations of $C_2$ of the same form as the fluctuation of $B_2$ in (2.6), even though they are allowed by the symmetries. This is a consistent truncation of the full ten dimensional supergravity action.

We now perform the KK reduction of (2.1) by plugging into it the ansatz described above. Recall that

\[ \text{vol}_{T^{1,1}} \equiv \int e_\psi \wedge e_{\theta_1} \wedge e_{\phi_1} \wedge e_{\theta_2} \wedge e_{\phi_2} = \frac{16\pi^3}{27}. \]

(2.11)
First, we have
\[ \int_{\mathcal{M}_{10}} 1 \wedge \star 1 = \text{vol}_{\mathcal{T}^{1,1}} \int_{\mathcal{M}_5} \Omega_1 \Omega_2^4 \text{vol}_{\mathcal{M}_5}. \]  
(2.12)

With a straightforward but somewhat tedious computation we find that in the background (2.4)
\[ R_{10} = R_5 - 2 \Omega_1^{-1} g^{\lambda \nu} \left( \nabla_\lambda \nabla_\nu \Omega_1 \right) - 8 \Omega_2^{-1} g^{\lambda \nu} \left( \nabla_\lambda \nabla_\nu \Omega_2 \right) \]
\[ - 4 g^{\lambda \nu} \left( 2 \Omega_1^{-1} \Omega_2^{-1} \nabla_\lambda \Omega_1 \nabla_\nu \Omega_2 + 3 \Omega_2^{-2} \nabla_\lambda \Omega_2 \nabla_\nu \Omega_2 \right) \]
\[ + 24 \Omega_2^{-2} - 4 \Omega_1^2 \Omega_2^{-4}, \]
(2.13)
where \( R_5 \) is the five dimensional Ricci scalar of the metric
\[ ds_5^2 = g_{\mu \nu}(y) dy^\mu dy^\nu. \]  
(2.14)

In (2.13), \( \nabla_\lambda \) denotes the covariant derivative with respect to the metric (2.14), explicitly given by
\[ \nabla_\lambda \Omega_i = \partial_\lambda \Omega_i, \]
\[ \nabla_\lambda \nabla_\nu \Omega_i = \partial_\lambda \partial_\nu \Omega_i - \Gamma^\rho_{\lambda \nu} \partial_\rho \Omega_i. \]  
(2.15)

Now, by plugging our ansatz into (2.1) we find that it reduces to the following effective action:
\[ S_5 = \frac{1}{16 \pi G_5} \int_{\mathcal{M}_5} \text{vol}_{\mathcal{M}_5} \left\{ \Omega_1 \Omega_2^4 \left( R_{10} - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi \right) - \Omega_1 e^{-\Phi} \left( \nabla_\mu \tilde{k} \nabla^\mu \tilde{k} + \frac{P^2 e^{2\Phi}}{\Omega_1^2} \right) \right. \]
\[ - \left. \frac{1}{4} \left( \ell \kappa_0 + 2 P \tilde{k} \right)^2 + \frac{5}{24} \Omega_1 \Omega_2^4 \mathcal{F}_{\mu_1 \cdots \mu_5} \mathcal{F}^{\mu_1 \cdots \mu_5} \right\} \]
\[ + \frac{1}{16 \pi G_5} P \int_{\mathcal{M}_5} \tilde{d} \kappa \wedge C_4. \]  
(2.16)

where
\[ \mathcal{F}_{\mu_1 \cdots \mu_5} \equiv \partial_{[\mu_1} C_{\mu_2 \cdots \mu_5]} = \frac{1}{5} \frac{K}{\Omega_1 \Omega_2^4} \sqrt{- \det(g_{\mu \nu})} \epsilon_{\mu_1 \cdots \mu_5} \]  
(2.17)
\([\cdots]\) denotes anti-symmetrization with weight one) and \( G_5 \) is the five dimensional effective gravitational constant
\[ G_5 \equiv \frac{G_{10}}{\text{vol}_{\mathcal{T}^{1,1}}}. \]  
(2.18)
Note that our gravitational action is not the standard five dimensional action because of the factor of $\Omega_1 \Omega_2^4$ in front of the five dimensional Einstein-Hilbert term.

In the five dimensional action it turns out to be possible to “integrate out” the field $C_4$ using the self-duality equation (2.10) and to obtain an action involving only the other fields. This leads to the action we will be using in this paper

$$S_5 = \frac{1}{16\pi G_5} \int_{\mathcal{M}_5} \text{vol}_{\mathcal{M}_5} \left\{ \Omega_1 \Omega_2^4 \left( R_{10} - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi \right) - P^2 \Omega_1 e^{-\Phi} \left( \frac{\nabla_\mu K \nabla^\mu K}{4P^4} + \frac{e^{2\Phi}}{\Omega_1^2} \right) \right. \\
- \frac{1}{2} \frac{K^2}{\Omega_1 \Omega_2^4} \right\},$$

(2.19)

where $R_{10}$ is given by (2.13) and $K(y)$ is related to $\tilde{k}(y)$ by (2.10).

### 2.2 The equations of motion and the ansatz for the solution

From the effective action (2.19) we obtain the following equations of motion:

$$0 = \frac{1}{\sqrt{-g}} \partial_\mu \left[ e^{-\Phi} \Omega_1 \frac{1}{2P^2} \sqrt{-g} g^{\mu\nu} \partial_\nu K \right] - \frac{K}{\Omega_1 \Omega_2^4},$$

(2.20)

$$0 = \frac{1}{\sqrt{-g}} \partial_\mu \left[ \Omega_1 \Omega_2^4 \sqrt{-g} g^{\mu\nu} \partial_\nu \Phi \right] + \frac{\Omega_1 e^{-\Phi} (\partial K)^2}{4P^2} - \frac{P^2 e^\Phi}{\Omega_1},$$

(2.21)

$$0 = \Omega_2^4 R_5 - 12 \Omega_2^2 (\partial \Omega_2)^2 + 24 \Omega_2^2 - 12 \Omega_1^2 - 8 \Omega_2^3 \Box_5 \Omega_2$$

$$- \frac{1}{2} \frac{\Omega_2^4 (\partial \Phi)^2}{\Omega_1} + \frac{P^2 e^\Phi}{\Omega_1^2} - \frac{e^{-\Phi} (\partial K)^2}{4P^2} + \frac{K^2}{2 \Omega_1 \Omega_2^2},$$

(2.22)

$$0 = 4 \Omega_1 \Omega_2^3 R_5 - 8 \Omega_2^3 \Box_5 \Omega_1 - 24 \Omega_1 \Omega_2^2 \Box_5 \Omega_2 - 24 \Omega_2^2 \partial \Omega_1 \partial \Omega_2 - 24 \Omega_1 \partial \Omega_2 (\partial \Omega_2)^2 + 48 \Omega_1 \Omega_2$$

$$- 2 \Omega_1 \Omega_2^3 (\partial \Phi)^2 + \frac{2K^2}{\Omega_1 \Omega_2^5},$$

(2.23)

$$\Omega_1 \Omega_2^4 R_{5\mu\nu} = \frac{g_{\mu\nu}}{3} \left\{ \frac{P^2 e^\Phi}{\Omega_1} + \frac{K^2}{2 \Omega_1 \Omega_2^2} + \Box_5 (\Omega_1 \Omega_2^4) - 24 \Omega_1 \Omega_2^2 + 4 \Omega_2^3 \right\}$$

$$+ \nabla_\mu \nabla_\nu (\Omega_1 \Omega_2^4) - 4 \Omega_2^3 (\partial_\mu \Omega_1 \partial_\nu \Omega_2 + \partial_\nu \Omega_1 \partial_\mu \Omega_2) - 12 \Omega_1 \Omega_2^2 \partial_\mu \Omega_2 \partial_\nu \Omega_2$$

$$+ \frac{\Omega_1 e^{-\Phi}}{4P^2} \partial_\mu K \partial_\nu K + \frac{1}{2} \Omega_1 \Omega_2^4 \partial_\mu \Phi \partial_\nu \Phi,$$

(2.24)

where $(\partial F)^2$ denotes $g^{\mu\nu} \partial_\mu F \partial_\nu F$ and $\Box$ is the Laplacian in the metric (2.14).
In order to proceed further we make a convenient gauge choice for the five dimensional metric which separates the radial direction, which we will call $\rho$, from the four space-time dimensions of the cascading theory which we will denote by $x^i$:

$$ds_5^2 = h^{-1/2}(x, \rho)\rho^{-2}\left(G_{ij}(x, \rho)dx^i dx^j\right) + h^{1/2}(x, \rho)\rho^{-2}(d\rho)^2.$$  \hspace{1cm} (2.25)

The boundary of the space will be taken to be at $\rho \to 0$. In this gauge choice some of the off-diagonal components of the metric vanish, partly fixing the diffeomorphism symmetry. We also define new scalar fields $f_2$ and $f_3$ related to the $\Omega_i$ fields by

$$\Omega_1^2 = h^{1/2}f_2, \quad \Omega_2^2 = h^{1/2}f_3.$$  \hspace{1cm} (2.26)

The motivation for this parameterization is that in the solution of [4] the function $h$ diverges logarithmically near the boundary $\rho \to 0$, but $G_{ij}$, $f_2$ and $f_3$ approach constant values. It is not difficult to rewrite the equations of motion (2.20)-(2.24) using the new variables $G_{ij}$, $h$, $f_2$, $f_3$, $K$ and $\Phi$, and using the parameterization (2.25) of the metric, and we present the results in appendix A.1.

We now wish to find the solutions for the cascading theories on arbitrary space-time manifolds, in an expansion near the boundary at $\rho \to 0$. In the case of asymptotically anti-de Sitter spaces, fields in the bulk are dual to operators in the field theory of some dimension $\Delta$, and they may be expanded in a power series in the radial $\rho$ coordinate, with a leading term of order $\rho^{4-\Delta}$ corresponding to the source of the operator, multiplied by a power series in $\rho^2$, and then a subleading term of order $\rho^\Delta$ corresponding to the one-point function of the operator (again multiplied by a power series in $\rho^2$). In the cascading gauge theory we expect a similar picture to arise, but with logarithmic corrections to all the terms corresponding to the logarithmic deviation from conformal invariance, and we will see below that this is indeed the case.

Let us first analyze the dimensions of the fields in the action (2.19) in the conformal case of $P = 0$ [32]; note that naively the action (2.19) we wrote is singular as $P \to 0$, but if we change variables from $K(y)$ to $\tilde{k}(y)$ (using (2.10)) the action becomes non-singular, so our analysis everywhere in this paper should be valid also in the $P = 0$ case. Obviously, the metric $G_{ij}$ is dual to a dimension four operator which is just the stress-energy tensor. The dilaton $\Phi$ and $\tilde{k}$ both correspond to dimension four scalar operators (which are the real parts of the two exactly marginal single-trace deformations of the SCFT of [32] which preserve the $SU(2) \times SU(2)$ global symmetry), and one can show

\footnote{For integer values of $\Delta$ there are also some logarithmic terms.}
that combinations of the scalars $\Omega_1$ and $\Omega_2$ (or $f_2$ and $f_3$) correspond to operators of dimensions six and eight\footnote{Note that $h$ does not correspond to an operator in the dual field theory, since it can be gauged away by reparametrizations of the radial coordinate.}. For $P = 0$ all these fields are decoupled (at leading order in the deformation from the solution of \cite{32}), but for non-zero $P$ the equations of motion couple them all together, and we need to analyze all of them at the same time.

The usual procedure of holographic renormalization starts by finding the solution for arbitrary sources, continues by computing the divergences of the action as a function of the sources, and then introduces counter-terms to cancel these divergences by expressing them as a function of the local fields (the transformation from the sources to the fields is invertible). In our case we have a problem with implementing this procedure because some of the operators involved in our action are irrelevant, meaning that we cannot find the solution with arbitrary sources for these operators as a power series with bounded powers of the radial coordinate as usual. In order to find the sources we need to expand the equations of motion around some solution to linear order and look at the solution to the linearized equations of motion which is larger near the boundary. When expanding the equations of motion of appendix A.1 around the solution of \cite{4}, which in our parametrization is given (to leading order in $\rho$) by

\begin{align}
G_{ij}(x, \rho) &= \eta_{ij}, \\
\Phi(x, \rho) &= \ln(p_0), \\
h(x, \rho) &= \frac{1}{8} P^2 p_0 + \frac{1}{4} K_0 - \frac{1}{2} P^2 p_0 \ln \rho, \\
K(x, \rho) &= K_0 - 2 P^2 p_0 \ln \rho, \\
f_2(x, \rho) &= 1, \\
f_3(x, \rho) &= 1, 
\end{align}

(2.27)

(with some constants $p_0$ and $K_0$ which are the parameters of the solution), we find the following independent solutions to the linearized equations of motion which we identify
with the sources for the various operators:

\begin{enumerate}
\item \( \delta G_{ij} = \tilde{G}_{ij}(x) \);
\item \( \delta K = \tilde{K}(x), \delta h = \frac{1}{4} \tilde{K}(x) \);
\item \( \delta \Phi = \tilde{\rho}(x)/p_0, \delta h = \frac{1}{8} P^2 \tilde{\rho}(x) - \frac{1}{4} P^2 \tilde{\rho}(x) \ln \rho, \delta K = -2 P^2 \tilde{\rho}(x) \ln \rho; \)
\item \( \delta f_3 = \alpha_6(x) \rho^{-2}, \delta f_2 = -4 \alpha_6(x) \rho^{-2}, \delta h = \alpha_6(x) P^2 p_0 \rho^{-2}, \delta K = 2 \alpha_6(x) P^2 p_0 \rho^{-2}; \)
\item \( \delta h = \alpha_8(x) \rho^{-4}. \)
\end{enumerate}

The first three sources are those of the operators which are marginal for \( P = 0 \) – the stress-energy tensor and the two scalar operators of dimension four – and the last two are the sources of the two irrelevant scalar operators\(^8\). In all cases we wrote down only the leading \( \rho \)-dependence of the solutions – in general there are corrections to the expressions above which involve powers of \( \rho^2 \ln^n(\rho) \) (for some integer \( n \)) multiplying the \( \rho \)-dependence of the terms we wrote, and which generally involve all the fields (not just the ones which are turned on at the leading order).

Finding a solution to the full non-linear equations involving all the sources above is an ill-defined question since some of these sources are irrelevant. In order to have a well-defined solution we need to set the sources \( \alpha_6(x) = \alpha_8(x) = 0 \) (later we will take these sources to be infinitesimal in order to compute the correlation functions of the corresponding operators, but we cannot take them to be more than infinitesimal). Once we do this there are no negative powers of \( \rho \) in any of the fields, so all fields have a well-defined expansion in powers of \( \rho \) (and \( \ln \rho \)). We will also make another simplifying assumption and we will not introduce any sources for the other two scalar operators, leaving the corresponding fields to take (as \( \rho \to 0 \)) the \((x\text{-independent})\) values they take in (2.27). It would be interesting to analyze the solutions with arbitrary sources for these fields, but we postpone this to future work. Thus, the only field that we allow an arbitrary source for is the metric, which we take to be of the form \( G_{ij}(x, \rho) = G_{ij}^{(0)}(x) + \mathcal{O}(\rho^2 \ln^n(\rho)). \) This means that the solutions we construct will describe the cascading gauge theory compactified on an arbitrary manifold (with metric \( G_{ij}^{(0)} \), since the source for the stress-energy tensor is just the metric of the cas-

\(^8\)The fifth source \( \alpha_8(x) \) naively couples only to \( h \) which, as we mentioned, is not a physical field, but in fact by a reparametrization of the \( \rho \) coordinate one can rewrite the corresponding solution in a way which involves \( f_2 \) and \( f_3 \). We chose to write the solution in the form above for convenience.
cading field theory), but without any deformations to its Lagrangian. Of course, the
fact that we do not allow arbitrary sources means that even though we will be able to
perform the second step of holographic renormalization – expressing the divergences
of the action in terms of the sources in our equations – we will not be able to uniquely
translate these divergences into functions of the local fields (since we have many fields
but just one arbitrary source), and we will be forced to use other methods to determine
the counter-terms. We will discuss this further in the next section.

Next, we would like to find the solution with the source described above. We
do this in a perturbative expansion in \( \rho^2 \), as usual in holographic renormalization.
The difference from the usual case is that already our leading order solution contains
logarithms of \( \rho \), and when we solve the equations we find that we need even higher
powers of logarithms at the higher orders in \( \rho \). We use the following parametrization
for the solution:

\[
G_{ij}(x, \rho) = G_{ij}^{(0)}(x) + \rho^2 \left[ G_{ij}^{(2,0)}(x) + \ln \rho \ G_{ij}^{(2,1)}(x) \right] \\
+ \rho^4 \left[ G_{ij}^{(4,0)}(x) + \ln \rho \ G_{ij}^{(4,1)}(x) + \ln^2 \rho \ G_{ij}^{(4,2)}(x) + \ln^3 \rho \ G_{ij}^{(4,3)}(x) \right] \\
+ \mathcal{O}(\rho^6 \ln^5 \rho)
\]

\[
h(x, \rho) = \frac{1}{8} P^2 p_0 + \frac{1}{4} K_0 - \frac{1}{2} P^2 p_0 \ln \rho + \rho^2 \left[ h^{(2,0)}(x) + \ln \rho \ h^{(2,1)}(x) + \ln^2 \rho \ h^{(2,2)}(x) \right] \\
+ \rho^4 \left[ h^{(4,0)}(x) + \ln \rho \ h^{(4,1)}(x) + \ln^2 \rho \ h^{(4,2)}(x) + \ln^3 \rho \ h^{(4,3)}(x) + \ln^4 \rho \ h^{(4,4)}(x) \right] \\
+ \mathcal{O}(\rho^6 \ln^6 \rho)
\]

\[
K(x, \rho) = K_0 - 2 P^2 p_0 \ln \rho + \rho^2 \left[ K^{(2,0)}(x) + \ln \rho \ K^{(2,1)}(x) \right] \\
+ \rho^4 \left[ K^{(4,0)}(x) + \ln \rho \ K^{(4,1)}(x) + \ln^2 \rho \ K^{(4,2)}(x) + \ln^3 \rho \ K^{(4,3)}(x) \right] \\
+ \mathcal{O}(\rho^6 \ln^5 \rho)
\]

\[
\Phi(x, \rho) = \ln p_0 + \rho^2 \left[ p^{(2,0)}(x) + \ln \rho \ p^{(2,1)}(x) \right] \\
+ \rho^4 \left[ p^{(4,0)}(x) + \ln \rho \ p^{(4,1)}(x) + \ln^2 \rho \ p^{(4,2)}(x) + \ln^3 \rho \ p^{(4,3)}(x) \right] \\
+ \mathcal{O}(\rho^6 \ln^5 \rho)
\]
\[
\begin{align*}
f_2(x, \rho) &= 1 + \rho^2 \left[ a^{(2,0)}(x) + \ln \rho \ a^{(2,1)}(x) \right] \\
&+ \rho^4 \left[ a^{(4,0)}(x) + \ln \rho \ a^{(4,1)}(x) + \ln^2 \rho \ a^{(4,2)}(x) + \ln^3 \rho \ a^{(4,3)}(x) \right] \tag{2.33} \\
&+ \mathcal{O}(\rho^6 \ln^5 \rho) \\
f_3(x, \rho) &= 1 + \rho^2 \left[ b^{(2,0)}(x) + \ln \rho \ b^{(2,1)}(x) \right] \\
&+ \rho^4 \left[ b^{(4,0)}(x) + \ln \rho \ b^{(4,1)}(x) + \ln^2 \rho \ b^{(4,2)}(x) + \ln^3 \rho \ b^{(4,3)}(x) \right] \tag{2.34} \\
&+ \mathcal{O}(\rho^6 \ln^5 \rho)
\end{align*}
\]

where the leading order terms are specified by the parameters \( G^{(0)}_{ij}(x) \), \( p_0 \) and \( K_0 \) as above. Note that there is a residual reparametrization ambiguity associated with the choice of \( h \) in (2.25). We (partially) fix \( h \) order by order in the perturbative expansion in such a way that at each order in \( \rho \) all fields are given by a finite order polynomial in \( \ln \rho \), as indicated in (2.29)-(2.34). This still leaves some diffeomorphisms, of the form

\[
\rho \to \hat{\rho} = \rho \left[ 1 + \rho^2 \left( \delta_{20} + \delta_{21} \ln \rho \right) + \rho^4 \left( \delta_{40} + \delta_{41} \ln \rho + \delta_{42} \ln^2 \rho + \delta_{43} \ln^3 \rho \right) + \cdots \right], \tag{2.35}
\]

unfixed, and this will result in some freedom in the solutions which we will find.

We can now find the solution by solving the equations of motion of appendix A.1 order by order in \( \rho \). We have found the general solution with the boundary conditions described above up to fourth order in \( \rho \), and it is explicitly given in appendix A.2. At the second order in \( \rho \) we find that the solution depends on two arbitrary functions of \( x \). This is related to the reparametrization freedom (2.35), which involves two arbitrary constant parameters at second order. The fact that we find two arbitrary functions rather than two arbitrary parameters is related to the fact that \( x \)-derivatives of fields show up in the equations at higher orders in \( \rho \) than the fields themselves, so we expect the non-constant part of these functions to be determined at the next order, and indeed it is (as described in appendix A.2). At the fourth order we similarly find four arbitrary functions associated to the reparametrization freedom, and we also find additional arbitrary functions associated (as usual in holographic renormalization) to the one-point functions of the dimension four operators (the two scalar operators and the traceless part of the stress-energy tensor) which are not determined by the UV expansion near the boundary (but which must be determined by the behavior of the solution at large values of \( \rho \), which we do not discuss here).
The solution we found is rather complicated, and its precise form is not very illuminating. It is useful to check this solution using some of the symmetries of the problem. First, the reduced type IIB action (2.19) is invariant under shifting the dilaton together with rescaling $P : P \to \alpha P, \ e^\Phi \to \alpha^{-2} e^\Phi$. This means that our solution must also be invariant under the same transformation, and all fields except the dilaton cannot depend on $P$ and $p_0$ separately but only on the combination $P^2 p_0$; it is easy to verify that this is indeed the case. Two additional symmetries involve reparametrizations. The reparametrization (2.35) is a symmetry of our ansatz and boundary condition, so it must take one solution to another, and we verify in appendix A.3 that this is indeed the case. We can also consider a scaling symmetry $\rho \to \lambda \rho$; this does not leave our asymptotic solution invariant, but we can make it into a symmetry if we also give an appropriate transformation to the metric $G^{(0)}_{ij}$, $p_0$ and $K_0$. In appendix A.3 we verify that this symmetry is also satisfied by our solution of appendix A.2. The truncated action (2.19) also has another interesting scaling property: it scales by a factor of $\beta^2$ when we take

$$K \to \beta K, \ e^\Phi \to \beta e^\Phi, \ \Omega_1^4 \to \beta \Omega_1^4, \ \Omega_2^4 \to \beta \Omega_2^4, \ g_{\mu\nu} \to \beta^{1/2} g_{\mu\nu}. \ (2.36)$$

Since this rescales the action by a constant (without acting on the coordinates, just on the fields), it is a symmetry of the equations of motion so it should also be a symmetry of our solution. In appendix A.3 we verify that the symmetry (2.36) is satisfied by our solution of appendix A.2 as well.

It is easy to verify that the solution of appendix A.2 has a good $P \to 0$ limit, where it describes the asymptotic behavior of the conformal field theory of [32] compactified on an arbitrary four-manifold. In this limit the solution simplifies considerably; no logarithms appear at second order, while at fourth order some logarithmic terms can appear in the expansions of fields which are dual to operators of dimension four. The reparametrization freedom (2.35) is reduced in this limit just to the terms with no logs, and correspondingly we have less arbitrariness in the solutions (we should set the functions $a^{(2,1)}$, $a^{(4,1)}$, $a^{(4,2)}$ and $a^{(4,3)}$ which appear in the solution to zero, and we should also set $(a^{(4,0)} - b^{(4,0)}) = O(P)$).

3 Holographic renormalization

In this section we describe the holographic renormalization of the cascading gauge theories. We begin by regularizing the action and computing the divergences of the
regularized action. We find that in addition to the familiar power-law divergences of asymptotically anti-de Sitter (AdS) geometries one encounters various logarithmic divergences. The logarithmic divergences are represented by finite order polynomials in \( \ln \rho \), at least in the specific reparametrization choice\(^9\) of \( h \) that we made in (2.25).

In the previous section we obtained the asymptotic solution of the holographic dual to the cascading gauge theory for arbitrary finite (not infinitesimal) boundary metric \( G^{(0)}_{ij}(x) \) and for constant parameters \( \{p_0, K_0\} \), up to order \( \rho^4 \). As discussed above, since the sources we introduced are not arbitrary, even if we find counter-terms which give rise to a finite regularized action we are not guaranteed that all correlation functions (given by derivatives of the action with respect to arbitrary sources) will be finite. Instead, we have to directly check that we can find counter-terms that will make all the correlation functions finite. Of course, using just the asymptotic solutions that we computed above we cannot compute arbitrary correlation functions, since generic \( n \)-point functions depend on knowing the full solution and not just its asymptotic form. However, in order to compute one-point functions the asymptotic solutions are actually enough\(^10\), since they are just given by the derivatives of the action with respect to infinitesimal sources which can be translated into derivatives of the action with respect to fields near the boundary. In particular, the one-point functions corresponding to the first three sources in (2.28) are simply given by the variation of the action with respect to the parameters \( G^{(0)}_{ij}(x), K_0 \) and \( p_0 \), respectively.

So, our procedure to determine the counter-terms will be to compute the one-point functions of the operators coupling to the sources (2.28) and to require that they are all finite. This will not determine the counter-terms uniquely, and we will see that some ambiguities will remain even in the one-point functions which we compute, but some general properties will be independent of these ambiguities and we expect them to be true for any consistent counter-terms (including the correct ones which renormalize the theory for arbitrary sources). Note that, as discussed above, some of our sources correspond to dimension four operators (\( T^{ij} \) coupling to \( \tilde{G}_{ij} \), \( O_{p_0} \) coupling to \( \tilde{p} \) and \( O_{K_0} \) coupling to \( \tilde{K} \)) and we will be able to compute their one-point functions using our solution directly. On the other hand, the standard holographic operator-state

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\(^9\)Recall that \( h \) is fixed order-by-order in the perturbative solution in such a way that only a finite number of powers of \( \ln \rho \) appears at each order in \( \rho \), see (2.29)-(2.34).

\(^10\)The one-point functions will often depend on arbitrary functions appearing in the asymptotic solutions related to expectation values as mentioned above, but no additional information about the large \( \rho \) behavior of the solution is needed to compute the one-point functions.
correspondence (generalized to our background) implies that in order to compute the one-point correlation functions of the operators $O_6$ and $O_8$ coupling to the sources $\alpha_6(x)$ and $\alpha_8(x)$, respectively, one needs to know the asymptotic holographic background to order $\rho^6$ and $\rho^8$, respectively. Since we know the supergravity geometry only to order $\rho^4$, we will not be able to compute these one-point functions, but we can still require that the contributions to them from the terms we computed should not lead to divergences. So, we can require that the renormalized one-point correlation functions of the subtracted operators$^{11}$ $O^s_6$ and $O^s_8$ satisfy (up to possible logarithmic corrections)

$$\langle O^s_6 \rangle = O(1), \quad \langle O^s_8 \rangle = O(\rho^{-2}), \quad \text{as } \rho \to 0,$$

which is equivalent to saying that these operators do not have the leading and the first two subleading power-law divergences. We find that requiring that all dimension four one-point functions are finite and that there are no power-law divergences in $O^s_6$ and $O^s_8$ significantly reduces the ambiguities in the renormalized one-point functions of the stress-energy tensor.

We begin by regularizing the action in subsection 3.1, and the computation of the regularized one-point correlation functions is explained in subsection 3.2. In subsection 3.3 we discuss the local counter-terms that are needed for the renormalization of $\langle T_{ij} \rangle$, $\langle O_{p_0} \rangle$, $\langle O_{K_0} \rangle$ and for the cancellation of power-law divergences in $\langle O_6 \rangle$ and $\langle O_8 \rangle$. In this subsection we use a particular ansatz for the counter-terms which we call the ‘minimal subtraction scheme’.

In subsection 3.4 we present results for the one-point correlation functions of the operators $T_{ij}$, $O_{p_0}$, $O_{K_0}$ in the minimal subtraction renormalization scheme. We also discuss the $P \to 0$ limit of the minimal subtraction regularization scheme. Already in the minimal subtraction scheme there are some ambiguities in the results, and in more general renormalization schemes additional ambiguities appear. In subsection 3.5 we comment on the ambiguities which appear in general schemes for the renormalization.

Finally, in subsection 3.6 we discuss other possible renormalization prescriptions. Our main result in this section is that the cascading gauge theory can be renormalized with finite one-point functions, and in particular with a finite stress-energy tensor. In section 4 we show that this does not contradict the expectation$^{12}$ that at high temperature the number of effective degrees of freedom of the cascading gauge theory

$^{11}$To be defined in subsection 3.2.

$^{12}$This was originally proposed in [28], and further evidence was presented in [31].
grows as $K_{\text{eff}}^2 \propto \ln^2(T/\Lambda)$, where $\Lambda$ is the strong coupling scale of the cascading gauge theory. One might naively think (given the known thermodynamic properties of the cascading gauge theories [28, 30, 31]) that a different renormalization scheme would be more natural, in which the renormalized one-point functions are not finite but rather depend on the combination $K(\rho) = K_0 - 2P^2 \ln \rho$ (evaluated at the cutoff) rather than on $K_0$, but this does not appear to be possible. It would be interesting to explore this second renormalization scheme in more detail.

3.1 The regularized action

We write the effective action (2.19) as

$$S_5 = \frac{1}{16\pi G_5} \int_{\mathcal{M}_5} \text{vol}_{\mathcal{M}_5} \mathcal{L}(5)$$

(3.2)

where, when evaluated on a solution to the equations of motion,

$$\mathcal{L}(5) = 2\Omega_1 \Omega_2 \left( \frac{1}{8} [\ln \Omega_2] + 4(\partial[\ln \Omega_2])^2 + \partial[\ln \Omega_1] \partial[\ln \Omega_2] \right)$$

$$+ 4\Omega_1 (\Omega_1^2 - 3\Omega_2^2).$$

(3.3)

We wish to regularize the theory by imposing a cutoff on the space $\mathcal{M}_5$, putting in a boundary $\partial \mathcal{M}_5$ at some $\rho = \rho_0$. In some cases the effective action (2.19) evaluated on the equations of motion is a total derivative – for instance, this is true for the cascading gauge theory on $\mathbb{R} \times S^3$ or $\mathbb{R} \times S^1 \times S^2$ – and in such cases we can rewrite (2.19) as an integral just over the boundary, but in general (for instance on $dS_2 \times S^2$ or $dS_4$) this is not the case. With the ansatz (2.25) we find that we can write the action evaluated on a solution to the equations of motion as

$$\sqrt{-g} \mathcal{L}(5) = \frac{1}{2} \left[ \rho^{-3} \sqrt{-G} f_2^{1/2} f_3^2 [\ln h] \right]'$$

$$+ \rho^{-5} \sqrt{-G} f_2^{1/2} f_3^2 \left\{ \delta_0 + \delta_2 + \delta_4 + \delta_6 \right\},$$

(3.4)

where the prime denotes a derivative with respect to $\rho$ and the subscript in $\delta_i$ indicates the power-law scaling of the terms as $\rho \to 0$, i.e., $\delta_i \propto \rho^i$. From here on derivative

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13 Notice that in this prescription certain $\ln \rho$ divergences are allowed, as long as they come from $K(\rho)$.  

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operators and Laplacians will be with respect to the four dimensional metric $G_{ij}$ rather than the five dimensional metric. We find

$$\delta_0 = 4 f_3^{-2} (f_2 - 3 f_3),$$  \hspace{1cm} (3.5)

$$\delta_2 = \rho^2 \left( f_3^{-1} f'_3 - 3 \rho^{-1} f_3^{-1} f'_3 + \frac{1}{2} \Box h \right),$$  \hspace{1cm} (3.6)

$$\delta_4 = \rho^2 \left( f_3^{-1} f'_3 \ln \sqrt{-G} \right)' + f_3^{-2} (f'_3)^2 + 2 f_3^{-1} \partial f_3 \partial h + h f_3^{-1} \Box f_3$$

$$+ \frac{1}{2} f_2^{-1} f_3^{-1} f'_2 f'_3 + \frac{1}{4} f_2^{-1} \partial f_2 \partial h \right),$$  \hspace{1cm} (3.7)

$$\delta_6 = \rho^2 h \left( f_3^{-2} (\partial f_3)^2 + \frac{1}{2} f_2^{-1} f_3^{-1} \partial f_2 \partial f_3 \right).$$  \hspace{1cm} (3.8)

Once we introduce the cutoff as a boundary, in order to get consistent equations of motion we must introduce also a generalized Gibbons-Hawking (GH) term

$$S_{GH} = \frac{1}{8 \pi G_5} \int_{\partial \mathcal{M}_5} d^4 x \sqrt{-\det(\gamma_{\mu\nu})} \left( \nabla_\mu n^\mu + n^\mu \nabla_\mu \ln (\Omega_1 \Omega_2) \right),$$  \hspace{1cm} (3.9)

where $n^\mu$ is a unit space-like vector orthogonal to the four-dimensional boundary $\partial \mathcal{M}_5$, and $\gamma_{\mu\nu}$ is the induced metric on $\partial \mathcal{M}_5$

$$\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu.$$  \hspace{1cm} (3.10)

Note that $\nabla_\mu n^\mu$ is nothing but the extrinsic curvature of the boundary $\partial \mathcal{M}_5$, calculated with the metric (2.14), and that the whole boundary term (3.9) coincides with the Kaluza-Klein reduction of the standard Gibbons-Hawking term for the action (2.1) with a nine dimensional boundary $\partial \mathcal{M}_{10}$. In the ansatz (2.25) we have

$$n^\mu = - \delta_\rho \rho h^{-1/4},$$

$$\gamma_{ij} = \rho^{-2} h^{-1/2} G_{ij},$$

$$\sqrt{-\det(\gamma_{\mu\nu})} = \rho^{-4} h^{-1} \sqrt{-G},$$

(evaluated at $\rho = \rho_0$) and we find

$$S_{GH} = \frac{1}{16 \pi G_5} \int_{\partial \mathcal{M}_5} d^4 x \sqrt{-G} \mathcal{L}_{GH},$$  \hspace{1cm} (3.12)

where

$$\mathcal{L}_{GH} = -2 \rho^{-3} f_2^{1/2} f_3 \left\{ \frac{1}{4} \ln h' - 4 \rho^{-1} + [\ln \sqrt{-G}]' + \frac{1}{2} f_2^{-1} f'_2 + 2 f_3^{-1} f'_3 \right\}.$$  \hspace{1cm} (3.13)
The total regularized effective action is
\[ S^\rho_4 = \frac{1}{16\pi G_5} \int_{\partial M_5} d^4x \sqrt{-G} \mathcal{L}^\rho_4, \]
where
\[ \sqrt{-G} \mathcal{L}^\rho_4 = \sqrt{-G} \mathcal{L}_{GH} + \int_{\rho_0} d\rho \sqrt{-g} \mathcal{L}_5. \]
Generally this will diverge, and we will need to add to it some counter-term Lagrangian. We define the subtracted action to be
\[ S_{\text{tot}} = \frac{1}{16\pi G_5} \int_{\partial M_5} d^4x \sqrt{-G} \left( \mathcal{L}^\rho_4 + h^{-1} \rho^{-4} \mathcal{L}_{\text{counter}} \right), \]
where the (local) counter-term Lagrangian must be chosen in such a way that correlation functions computed from \( S_{\text{tot}} \) remain finite in the limit \( \rho_0 \rightarrow 0 \). The renormalized action is then simply
\[ S_{\text{eff}} = \lim_{\rho_0 \rightarrow 0} S_{\text{tot}}. \]
As explained in [22], one should distinguish between \( S_{\text{eff}} \) and \( S_{\text{tot}} \), as the variations required to obtain correlation functions should be performed before the limit \( \rho_0 \rightarrow 0 \) is taken. This is necessary in order to implement the subtraction covariantly.

### 3.2 Regularized one-point correlation functions

As explained in the previous subsection, we can write the subtracted effective action as
\[ S_{\text{tot}} = S_5 + S_{GH} + S_{ct} \]
where \( S_5 \) is the bulk term (3.2), \( S_{GH} \) is the generalized Gibbons-Hawking term (3.9), and we still need to determine the counter-term action
\[ S_{ct} = \frac{1}{16\pi G_5} \int_{\partial M_5} d^4x \sqrt{-\gamma} \mathcal{L}_{\text{counter}}. \]
The holographic renormalization is implemented by assuming that \( \mathcal{L}_{\text{counter}} \) is a local functional of the fields \( \{ \gamma_{ij}, K, \Phi, \Omega_1, \Omega_2 \} \) on the regularization boundary \( \partial M_5 \). Under a generic variation of the fields in the action we have
\[ \delta S_{\text{tot}} = \int_{M_5} \sqrt{-g} \left\{ [\cdots]_{\mu\nu} \delta g^{\mu\nu} + [\cdots] \delta \Phi + [\cdots] \delta K + [\cdots] \delta \Omega_1 + [\cdots] \delta \Omega_2 \right\} \]
\[ + \int_{\partial M_5} \sqrt{-\gamma} \left\{ [\cdots]_{ij} \delta \gamma_{ij} + [\cdots] \delta \Phi + [\cdots] \delta K + [\cdots] \delta \Omega_1 + [\cdots] \delta \Omega_2 \right\}, \]
where $\cdots$ in the bulk $\mathcal{M}_5$ integral stand for the corresponding five dimensional equations of motion (2.20)-(2.24), while the $\cdots$ in the boundary $\partial \mathcal{M}_5$ integral in (3.19) involve only the boundary metric $\gamma_{ij}$ and the boundary values of the fields $K, \Phi, \Omega_1, \Omega_2$. Clearly, evaluated on a solution to the bulk equations of motion, $\delta S_{\text{tot}}$ does not depend on $\delta g_{55}$, and thus we have the general expression

$$\delta S_{\text{tot}} = \delta S_{\text{tot}} \left[ \delta \gamma_{ij}, \delta K, \delta \Phi, \delta \Omega_1, \delta \Omega_2 \right]$$  
(3.20)

depending only on the values of the fields on $\partial \mathcal{M}_5$.

In order to compute one-point functions we need to take derivatives of this action with respect to our sources. As mentioned above, the one-point function of the stress-energy tensor is the variation with respect to $G^{(0)}_{ij}$, and we can write the one-point functions of $\mathcal{O}_{p_0}$ and $\mathcal{O}_{K_0}$ as variations with respect to $p_0$ and $K_0$, respectively. Notice that $\{\delta \gamma_{ij}, \delta K, \delta \Phi, \delta \Omega_1, \delta \Omega_2\}$ depend implicitly on the variation of the source boundary metric $\delta G^{(0)}_{ij}$ and on $\delta p_0, \delta K_0$. Given (3.20), the subtracted one-point correlation functions of the operators dual to $\{G^{(0)}_{ij}, K_0, p_0\}$ can be evaluated as

$$\langle \mathcal{O}^{s}_{p_0} \rangle \equiv \frac{16\pi G_5}{\sqrt{-G}} \frac{\delta S_{\text{tot}}}{\delta p_0} = \frac{16\pi G_5}{\sqrt{-\gamma}} \frac{1}{\rho^4 h} \left[ \frac{\delta S_{\text{tot}}}{\delta \Phi} \frac{\delta \Phi}{\delta p_0} + \frac{\delta S_{\text{tot}}}{\delta K} \frac{\delta K}{\delta p_0} + \frac{\delta S_{\text{tot}}}{\delta \gamma_{ij}} \frac{\delta \gamma_{ij}}{\delta p_0} \right],$$  
(3.21)

$$\langle \mathcal{O}^{s}_{K_0} \rangle \equiv \frac{16\pi G_5}{\sqrt{-G}} \frac{\delta S_{\text{tot}}}{\delta K_0} = \frac{16\pi G_5}{\sqrt{-\gamma}} \frac{1}{\rho^4 h} \left[ \frac{\delta S_{\text{tot}}}{\delta \Phi} \frac{\delta \Phi}{\delta K_0} + \frac{\delta S_{\text{tot}}}{\delta K} \frac{\delta K}{\delta K_0} + \frac{\delta S_{\text{tot}}}{\delta \gamma_{ij}} \frac{\delta \gamma_{ij}}{\delta K_0} \right],$$  
(3.22)

$$\langle \mathcal{O}^{ij \, s}_{G^{(0)}_{ij}} \rangle \equiv \frac{16\pi G_5}{\sqrt{-G}} \frac{\delta S_{\text{tot}}}{\delta G^{(0)}_{ij}} = \frac{16\pi G_5}{\sqrt{-\gamma}} \frac{1}{\rho^4 h} \left[ \frac{\delta S_{\text{tot}}}{\delta \Phi} \frac{\delta \Phi}{\delta G^{(0)}_{ij}} + \frac{\delta S_{\text{tot}}}{\delta K} \frac{\delta K}{\delta G^{(0)}_{ij}} + \frac{\delta S_{\text{tot}}}{\delta \gamma_{kl}} \frac{\delta \gamma_{kl}}{\delta G^{(0)}_{ij}} \right],$$  
(3.23)

The renormalized one-point correlation functions of the corresponding operators are then simply evaluated as

$$\langle \mathcal{O}_{p_0} \rangle = \lim_{\rho_0 \to 0} \langle \mathcal{O}^{s}_{p_0} \rangle,$$

$$\langle \mathcal{O}_{K_0} \rangle = \lim_{\rho_0 \to 0} \langle \mathcal{O}^{s}_{K_0} \rangle,$$

$$8\pi G_5 \langle T^{ij} \rangle = \lim_{\rho_0 \to 0} \langle \mathcal{O}^{ij \, s}_{G^{(0)}_{ij}} \rangle,$$

(3.24)
where we have defined a standard normalization for \( T^{ij} \). For each subtracted correlator we find it convenient to separate the regularized contribution from \( S_5 + S_{GH} \), and the counter-term contribution from \( S_{ct} \). We can do this separation separately for every term in the equations – for example, we can write

\[
\mathcal{O}_\Phi^e \equiv \frac{16\pi G_5}{\sqrt{-\gamma}} \frac{\delta S_{tot}}{\delta \Phi} \equiv \frac{1}{\rho^4 h} \left( \mathcal{O}_\Phi^e + \mathcal{O}_\Phi^c \right) = \frac{1}{\rho^4 h} \left( \frac{16\pi G_5}{\sqrt{-\gamma}} \frac{\delta(S_5 + S_{GH})}{\delta \Phi} + \frac{16\pi G_5}{\sqrt{-\gamma}} \frac{\delta S_{ct}}{\delta \Phi} \right).
\]

Using (2.19), (3.9) we then find that the contributions from the original action are given by

\[
\mathcal{O}_\rho^e \equiv \frac{16\pi G_5}{\sqrt{-\gamma}} \frac{\delta(S_5 + S_{GH})}{\delta \Phi} = \rho h f_2^{1/2} f_3^2 \Phi',
\]

\[
\mathcal{O}_K^e = \frac{16\pi G_5}{\sqrt{-\gamma}} \frac{\delta(S_5 + S_{GH})}{\delta K} = \frac{f_2^{1/2} \rho K'}{2P^2 e^\Phi},
\]

\[
\mathcal{O}_{\Omega_1}^e = \frac{16\pi G_5}{\sqrt{-\gamma}} \frac{\delta(S_5 + S_{GH})}{\delta \Omega_1} = -2\rho h^{3/4} f_3^2 \left( [\ln \sqrt{-\gamma}]' + 2 f_3^{-1} f_3' - 4 \rho^{-1} \right),
\]

\[
\mathcal{O}_{\Omega_2}^e = \frac{16\pi G_5}{\sqrt{-\gamma}} \frac{\delta(S_5 + S_{GH})}{\delta \Omega_2} = -8\rho h^{3/4} f_2^{1/2} f_3^{-1/2} \left( [\ln \sqrt{-\gamma}]' + \frac{3}{2} f_3^{-1} f_3' + \frac{3}{2} f_3^{-1} f_3' - 4 \rho^{-1} \right),
\]

and

\[
\mathcal{O}_{\gamma ij}^e = (-\Theta_{ij} + \Theta_{\gamma ij}) \Omega_1 \Omega_2^i + n^\lambda \nabla_\lambda \left( \Omega_1 \Omega_2^i \right) \gamma_{ij}
\]

\[
= \rho^{-2} h^{1/2} \left\{ \left[ f_2^{1/2} f_3^2 \left( \frac{1}{2} \rho G_{ij} + G_{ij} \left( 3 - \rho \left[ \ln \sqrt{-\gamma} \right]' + \frac{3}{4} \rho [\ln h]' \right) \right) \right] \right\}
\]

where all expressions should be evaluated at \( \rho = \rho_0 \) and

\[
\Theta^{ij} = \frac{1}{2} \left( \nabla^i n^j + \nabla^j n^i \right), \quad \Theta = \Theta^{ij} \gamma_{ij}.
\]

Note that all the regularized correlation functions contain power-law and logarithmic divergences as the cutoff is removed, \( \rho_0 \to 0 \). Thus, counter-terms must be determined to remove these divergences. Once the counter-term Lagrangian \( \mathcal{L}^{counter} \) is specified, one can compute counter-term contributions to the subtracted operators such as \( \mathcal{O}_\Phi^e \). Then, using the asymptotic solution explicitly given in appendix A.2, the remaining
variational derivatives can be evaluated to obtain the subtracted one-point functions. For example,

\[
\langle O_{p_0} \rangle = \frac{1}{\rho^4 h} \left[ \left( O^\rho_{\Phi} + O^c_{\Phi} \right) \frac{\delta \Phi}{\delta \rho_0} + \left( O^\rho_{K} + O^c_{K} \right) \frac{\delta K}{\delta \rho_0} + \left( O^\rho_{\Omega_1} + O^c_{\Omega_1} \right) \frac{\delta \Omega_1}{\delta \rho_0} \right.
\]

\[
+ \left( O^\rho_{\Omega_2} + O^c_{\Omega_2} \right) \frac{\delta \Omega_2}{\delta \rho_0} + \left( O^{ij \rho} + O^{ij c} \right) \frac{\delta \gamma}{\delta \rho_0} \right],
\]

with similar expressions for \( O^s_{K_0}, O^{ij s}_{G^{(0)}} \).

Similarly we can analyze the subtracted one-point correlation functions of the operators \( O_6 \) and \( O_8 \). Given the constant infinitesimal sources \( \alpha_6 \) and \( \alpha_8 \) of (2.28) for the corresponding dual supergravity fields, we have

\[
\langle O^s_8 \rangle = \frac{1}{\rho^4 h} \left[ \left( O^\rho_{\Phi} + O^c_{\Phi} \right) \frac{\delta \Phi}{\delta \alpha_8} + \left( O^\rho_{K} + O^c_{K} \right) \frac{\delta K}{\delta \alpha_8} + \left( O^\rho_{\Omega_1} + O^c_{\Omega_1} \right) \frac{\delta \Omega_1}{\delta \alpha_8} \right.
\]

\[
+ \left( O^\rho_{\Omega_2} + O^c_{\Omega_2} \right) \frac{\delta \Omega_2}{\delta \alpha_8} + \left( O^{ij \rho} + O^{ij c} \right) \frac{\delta \gamma}{\delta \alpha_8} \right],
\]

\[
\langle O^s_6 \rangle = \frac{1}{\rho^4 h} \left[ \left( O^\rho_{\Phi} + O^c_{\Phi} \right) \frac{\delta \Phi}{\delta \alpha_6} + \left( O^\rho_{K} + O^c_{K} \right) \frac{\delta K}{\delta \alpha_6} + \left( O^\rho_{\Omega_1} + O^c_{\Omega_1} \right) \frac{\delta \Omega_1}{\delta \alpha_6} \right.
\]

\[
+ \left( O^\rho_{\Omega_2} + O^c_{\Omega_2} \right) \frac{\delta \Omega_2}{\delta \alpha_6} + \left( O^{ij \rho} + O^{ij c} \right) \frac{\delta \gamma}{\delta \alpha_6} \right],
\]

where the boundary field variations represent the response of the fields to turning on infinitesimal sources \( \delta \alpha_8 \) and \( \delta \alpha_6 \). Note that in equation (2.28) we only wrote down the source at leading order in \( \rho^2 \), while naively higher orders in the source will also contribute to the one-point functions using the equations above. However, since the divergences have to cancel order by order in \( \rho^2 \), it is easy to see that after we have canceled the divergences (in all the operators) at some order in \( \rho \), the higher order terms in the sources (which naively could contribute to a divergence at the next order) multiply a vanishing expression, so they do not contribute. The first contribution of the higher order terms in the sources comes with one higher power of \( \rho^2 \) than the contribution of the leading terms, but since the latter is required to be finite as \( \rho_0 \to 0 \), the higher order terms never contribute in this limit.

In the next subsection we describe the construction of the counter-terms that lead to finite one-point correlation functions (3.24). As we explained earlier in the section, since we know the asymptotic solution of the dual supergravity background only to order \( \rho^4 \), we cannot compute precisely the subtracted operators \( O^s_6 \) and \( O^s_8 \): at best, we expect to be able to remove only the leading \( (O(\rho^{-6} \ln \# \rho) \text{ and } O(\rho^{-8} \ln \# \rho)) \), next-to-leading \( (O(\rho^{-4} \ln \# \rho) \text{ and } O(\rho^{-6} \ln \# \rho)) \), and next-to-next-to-leading \( (O(\rho^{-2} \ln \# \rho) \text{ and } O(\rho^{-4} \ln \# \rho)) \).
and $\mathcal{O}(\rho^{-4}\ln^#\rho)$ power-law divergences in their one-point correlation functions, see (3.1).

### 3.3 Local counter-terms

The counter-term Lagrangian in (3.18) must be a local functional of the fields on $\partial\mathcal{M}_5$. It is useful to separate the dependence of the counter-terms on the metric $\gamma_{ij}$ from the dependence on the other fields. Given the structure of the divergences of the regularized correlation functions (3.26)-(3.30), it is clear that the most general form of $\mathcal{L}^{\text{counter}}$ must be

$$
\mathcal{L}^{\text{counter}} = \mathcal{L}_0 + \mathcal{L}_R \mathcal{R}_\gamma + \mathcal{L}_R^2 \mathcal{R}^2_{\gamma} + \mathcal{L}_{\mathcal{R}ic} \mathcal{R}_{\gamma}^{ab} \mathcal{R}^{ab}_{\gamma} + \mathcal{L}_R \Box \mathcal{R}_\gamma + \mathcal{L}_{\text{kinetic}},
$$

where $\{\mathcal{L}_0, \mathcal{L}_R, \mathcal{L}_R^2, \mathcal{L}_{\mathcal{R}ic}, \mathcal{L}_R \Box \mathcal{R}\}$ are functions of any local fields at the boundary except for the metric $\gamma_{ij}$. Notice that (3.35) contains counter-terms proportional to $\mathcal{R}_\gamma$. Even though for constant $K_0$, $p_0$ these are total derivatives (up to order $\mathcal{O}(\rho^4\ln^#\rho)$), so they do not contribute to the stress-energy one-point correlation function, they do contribute to the renormalization of the $\mathcal{O}_{p_0}$ and $\mathcal{O}_{K_0}$ operators. Finally, $\mathcal{L}_{\text{kinetic}}$, which scales as $\mathcal{O}(\rho^4\ln^#\rho)$, contains ‘kinetic’ invariants of the boundary scalars $\mathcal{F}_i = \{\Phi, K, \Omega_1, \Omega_2\}$ of the type

$$
\mathcal{F}_1 \Box \mathcal{F}_2 = \rho^2 \mathcal{F}_3 \Box R + \cdots = \Box \left[\rho^2 \mathcal{F}_3 R\right] + \cdots
$$

(3.36)

(where we used the form of the solution in which the leading non-constant terms in the scalar fields are proportional to the curvature $R$) and, thus, it is also a total derivative to order $\mathcal{O}(\rho^4\ln^#\rho)$. Again, even though counter-terms of the type (3.36) do not contribute to the stress-energy one-point function, they are necessary for removing $\ln \rho$ divergences in the one-point functions of $\mathcal{O}_{p_0}$ and $\mathcal{O}_{K_0}$.

Let us discuss in detail the evaluation of $\mathcal{O}_{ij}^{\gamma\gamma}$ ( $\mathcal{O}_{\Phi}$, $\mathcal{O}_{K}$, $\mathcal{O}_{\Omega_1}$ and $\mathcal{O}_{\Omega_2}$ can be evaluated analogously). We will need the following asymptotic expansions:

$$
\mathcal{R}_\gamma = h^{1/2} \rho^2 \left( R_p + \frac{3}{2} \Box \ln h \right) + \mathcal{O}(\rho^6\ln^#\rho),
$$

$$
\mathcal{R}_{ij} = R_{ij} + \frac{1}{2} h^{-1} \nabla_i \nabla_j h + \frac{1}{4} h^{-1} G_{ij} \Box h + \mathcal{O}(\rho^4\ln^#\rho),
$$

$$
\mathcal{R}^2 = h \rho^4 R^2 + \mathcal{O}(\rho^6\ln^#\rho),
$$

$$
\mathcal{R}_{ab} \gamma \mathcal{R}_{\gamma}^{ab} = h \rho^4 R_{ab} R^{ab} + \mathcal{O}(\rho^6\ln^#\rho),
$$

Counter-terms of the form $\mathcal{F}_1 \partial \mathcal{F}_2 \partial \mathcal{F}_3$ may also be added.

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where

$$R_{ij} = R_{ij} + \rho^2 \left( \frac{h}{4} \left( \square R_{ij} - 2 \left( R_{iabj} R^{ab} + R_{ik} R^{kj}_i \right) \right) \right. $$

$$+ \left( \frac{\nabla_i \nabla_j R}{96} + \frac{G^{(0)}_{ij} \square R}{192} \right) \left( -2 K_0 - P^2 p_0 + 4 P^2 p_0 \ln \rho \right) $$

$$+ \frac{1}{4} \left( 2 \nabla_i \nabla_j a^{(2,0)} + \nabla_i \nabla_j a^{(2,1)} \right) \left( 1 + 2 \ln \rho \right) $$

$$+ \frac{1}{8} G^{(0)}_{ij} \left( 2 \square a^{(2,0)} + \square a^{(2,1)} \right) \left( 1 + 2 \ln \rho \right) \right) + \mathcal{O}(\rho^4) \right) \right) + \mathcal{O}(\rho^4)$$

(3.38)

and

$$R_{ij} = G^{ij} R_{ij} \rho \left( 3.39 \right)$$

In (3.37) and (3.38) the differential operators on the right hand side are evaluated with the metric $G_{ij}$.

Given (3.35) we can compute the contribution of $S_{cd}$ to the subtracted operator $\mathcal{O}_{ij}^s$,

$$\mathcal{O}_{ij}^s = \frac{16 \pi G_5}{\sqrt{-\gamma}} \frac{\delta S_{cd}}{\delta \gamma_{ij}}$$

$$= \frac{1}{2} L_0 \gamma_{ij} + \left( -R_{ij} + \frac{1}{2} R_{ij} \gamma_{ij} - \gamma_{ij} \square \gamma + \nabla_i \nabla_j \gamma \right) L_{\gamma}$$

$$+ L_{Ric} \left( \frac{1}{2} R_{ab} \gamma \gamma^{ab} R_{ij} + \nabla_i \nabla_j R_{ij} - \square R_{ij} - \frac{1}{2} \gamma_{ij} \square \gamma + 2 R_{\gamma ij} \gamma_{ij} \right)$$

$$+ L_{R^2} \left( \frac{1}{2} R_{ij} \gamma^2 R_{ij} - 2 R_{ij} R_{ij} + 2 \nabla_i \nabla_j R_{ij} \right) \gamma_{ij} \square \gamma + \mathcal{O}(\rho^8 \ln \rho^2). \right) \right) \right)$$

(3.40)

Notice that the terms which are relevant for removing divergences in the one-point functions of $\mathcal{O}_{p0}$, $\mathcal{O}_{K0}$ and $T_{ij}$ are terms up to order $\rho^4$ in $L_0$, terms up to order $\rho^2$ in $L_{\gamma}$, and terms up to order $\rho^0$ in $L_{R^2}$, $L_{Ric}$, $L_{\square \gamma}$ and $L_{kinetic}$.

Ideally we would like all the counter-terms to be functions just of the local fields at the boundary. However, already in the asymptotically AdS case it turns out that it is not possible to do this, and terms which explicitly involve $\ln \rho$ are necessary; these terms are related to the conformal anomaly. Since our theory reduces to an asymptotically AdS theory as $P \rightarrow 0$, we expect that such explicit $\rho$-dependent terms will be required in our case as well, so we will allow them in our ansatz. We will
find that terms involving up to three powers of \( \ln \rho \) are required for renormalizing the cascading gauge theories.

In this subsection we will discuss a specific ansatz for the counter-terms which turns out to suffice for obtaining finite one-point functions; more general possibilities will be discussed in subsection 3.5. We begin by noting that the simple counter-term Lagrangian

\[
\mathcal{L}_{\text{counter}} = -K - 2 \Omega_1^4 - 8 \Omega_2^4 + \mathcal{R}_\gamma \Omega_1^2 \left( -\frac{1}{12} K + \frac{1}{12} P^2 e^\Phi - \frac{1}{6} \Omega_1^4 \right) \tag{3.41}
\]

removes all power law divergences in \( \mathcal{O}_{p_0}^s, \mathcal{O}_{K_0}^s \) and \( \mathcal{O}_{G_{ij}^{(0)}}^s \), leaving only logarithmic divergences which still need to be canceled. Thus, it is convenient to parameterize the complete counter-term Lagrangian as follows (making a specific choice for the form of \( \mathcal{L}_{\text{kinetic}} \) that will turn out to be sufficient):

\[
\mathcal{L}_{\text{counter}} = K - 2 \Omega_1^4 - 8 \Omega_2^4 + A_4 + A_6 + \mathcal{R}_\gamma \Omega_1^2 \left( -\frac{1}{12} K + \frac{1}{12} P^2 e^\Phi - \frac{1}{6} \Omega_1^4 + B_2 + B_4 \right) + \mathcal{R}_\gamma^2 (\mathcal{L}_{\mathcal{R}^2}^0 + \mathcal{L}_{\mathcal{R}^2}^2) + \mathcal{R}_{ab} \mathcal{R}_{\gamma}^{ab} (\mathcal{L}_{\mathcal{R}ic^2}^0 + \mathcal{L}_{\mathcal{R}ic^2}^2) + \mathfrak{R}_\gamma \mathcal{L}_{\mathcal{R}^0}^0 + \mathfrak{R}_\gamma \mathcal{L}_{\mathcal{R}^2}^2 \\
+ \delta_1 \ln \rho \Omega_1^6 \mathfrak{R}_\gamma^6 \Phi + \delta_2 \Omega_1^6 \mathfrak{R}_\gamma^6 \Phi, \tag{3.42}
\]

where the subscript in \( A, B \) and the superscript in \( \mathcal{L} \) indicates the scaling near the boundary of the local field configuration represented by that coefficient, for instance \( A_n \propto \mathcal{O}(\rho^n \ln^6 \rho) \). Note that the counter-terms containing \{\( A_6, B_4, \mathcal{L}_{\mathcal{R}^2}^2, \mathcal{L}_{\mathcal{R}ic^2}^2, \mathcal{L}_{\mathcal{R}^2}^0 \)\} do not affect the one-point functions of \( \mathcal{O}_{p_0}, \mathcal{O}_{K_0} \) and \( \mathcal{O}_{G_{ij}^{(0)}} \), but they can contribute to the renormalization of power-law divergences in the one-point functions of the irrelevant operators \( \mathcal{O}_6 \) and \( \mathcal{O}_8 \). Counter-terms scaling as higher powers of \( \rho \) do not contribute to any of the one-point functions we compute so we ignore them.

Since, as discussed above, we do not have a systematic way to determine the counter-terms, we will write down an ansatz for the counter-terms and check if it can lead to finite one-point functions (including (3.1)). There are two scaling symmetries which we can use to constrain the form of the counter-terms. As mentioned in the previous section, the type IIB action has a scaling symmetry of \( P \to \alpha P, \ e^\Phi \to \alpha^{-2} e^\Phi \), and this will also be a property of the divergent terms in the action, so we can choose our counter-terms to depend on the dilaton and on \( P \) only through the combination \( P^2 e^\Phi \). Furthermore, both the truncated action (2.19) and the generalized Gibbons-Hawking term (3.9) have weight two under the scaling symmetry (2.36), \( S_5 \to \beta^2 S_5 \).
and $S_{GH} \rightarrow \beta^2 S_{GH}$. Thus, the divergences will have the same scaling, so in order to cancel them we need to have the same scaling also for the counter-terms that we add – this means that $\mathcal{L}^{\text{counter}}$ in (3.18) should scale with a factor of $\beta$ under this transformation, and we will use this to constrain our counter-terms.

In addition we will assume that the counter-terms contain only non-negative integer powers of $K$ and of $P^2$; again this is consistent with the structure of the divergences of the action, so it is reasonable to assume that the “correct” counter-terms (which render all correlation functions finite) should have the same property.

It is convenient to introduce the following short-hand notations

$$X_a \equiv \left(1 - \frac{\Omega_2^2}{\Omega_1^2}\right),$$
$$X_b \equiv K - 4\Omega_1^4 + \frac{1}{2}P^2e^\Phi,$$  \tag{3.43}

for field configurations that scale as $\rho^2$. Using the asymptotic solution of section A.2 we can verify that as $\rho \rightarrow 0$

$$X_a = \mathcal{O}(\rho^2), \quad X_b = \mathcal{O}(\rho^2 \ln \rho).$$ \tag{3.44}

We can replace any dependence of the counter-terms on $\Omega_1$ and $\Omega_2$ by a dependence on $X_a$ and $X_b$, and it will be convenient to do this in many places because of the scaling (3.44).

Our “minimal subtraction ansatz” for the counter-terms involves choosing the kinetic terms to take the specific form they take in (3.42). In addition, when we use the counter-terms (3.41) we find that the one-point functions of $\mathcal{O}_6^s$ and $\mathcal{O}_8^s$ do not contain divergent terms with negative powers of $h$ that are proportional to $R^2$ or $R_{ab}R^{ab}$, and we require that all counter-terms that we add should preserve this property. This turns out to restrict $A_4$ to be proportional to $X_a^2\Omega_1^4$ and $B_2$ to be proportional to $X_a$, and also to restrict $A_6 = B_4 = 0$. Finally, we restrict all the counter-terms to grow no faster than $\ln^4 \rho$ (multiplying the appropriate power of $\rho$) near the boundary, consistent with the fact that this is the scaling of the divergences in the action. Together with the scaling symmetries described above, this leads to an ansatz containing 82 independent coefficients (including $\delta_1, \delta_2$ in (3.42)). By computing the one-point functions using the counter-terms in this ansatz we find that it is possible to get finite one-point functions for the operators $O_{K\sigma}, O_{p\sigma}, T_{ij}$ and to satisfy (3.1); these requirements give 75 constraints on the 82 coefficients of the ansatz, leading to a 7-parameter ambiguity in
the counter-terms. In the parameterization (3.42) the resulting counter-terms take the form:

\[
\delta_1 = -\frac{50}{21},
\]

\[
A_4 = \frac{18}{5} X^2_3 \Omega^4_1,
\]

\[
B_2 = X_a \left( \frac{1}{6} K - \frac{1}{30} P^2 e^\Phi \right),
\]

\[
\mathcal{L}^0_{R^2} = -\frac{1}{144} P^4 e^{2\Phi} \ln^3 \rho - \frac{1}{96} P^2 e^\Phi \ln^2 \rho K - \frac{1}{192} \ln \rho K^2
\]

\[
+ \left( \frac{1}{96} + 4\kappa_1 \right) P^4 e^{2\Phi} \ln^2 \rho + \left( \frac{1}{96} + 4\kappa_1 \right) P^2 e^\Phi \ln \rho K
\]

\[
+ \left( \kappa_1 + \frac{1}{1152} \right) K^2 + \left( 2\kappa_2 - \frac{43}{2304} \right) P^4 e^{2\Phi} \ln \rho
\]

\[
+ \left( \kappa_2 - \frac{13}{1152} \right) P^2 e^\Phi K + \kappa_3 P^4 e^{2\Phi},
\]

\[
\mathcal{L}^0_{Ric^2} = \frac{1}{48} P^4 e^{2\Phi} \ln^3 \rho + \frac{1}{32} P^2 e^\Phi \ln^2 \rho K + \frac{1}{64} \ln \rho K^2 + \left( -\frac{1}{32} - 12\kappa_1 \right) P^4 e^{2\Phi} \ln^2 \rho
\]

\[
+ \left( \frac{1}{32} - 12\kappa_1 \right) P^2 e^\Phi \ln \rho K + \left( -\frac{1}{256} - 3\kappa_1 \right) K^2
\]

\[
+ \left( \frac{43}{768} - 6\kappa_2 \right) P^4 e^{2\Phi} \ln \rho + \left( \frac{5}{192} - 3\kappa_2 \right) P^2 e^\Phi K
\]

\[
+ \left( \frac{541}{138240} - 3\kappa_3 \right) P^4 e^{2\Phi},
\]

\[
\mathcal{L}^0_{\square R} = \frac{1}{144} P^4 e^{2\Phi} \ln^3 \rho + \frac{1}{96} P^2 e^\Phi \ln^2 \rho K + \frac{1}{192} \ln \rho K^2
\]

\[
+ \left( \frac{383}{5760} + 4\kappa_4 \right) P^4 e^{2\Phi} \ln^2 \rho + \left( \frac{2231}{40320} + 4\kappa_4 \right) P^2 e^\Phi \ln \rho K
\]

\[
+ \left( \kappa_4 + \frac{1}{64} \right) K^2 + \left( -\frac{17}{26880} + 2\kappa_5 + \frac{7}{320} \delta_2 \right) P^4 e^{2\Phi} \ln \rho
\]

\[
+ \left( \kappa_5 + \frac{1}{64} \delta_2 + \frac{29}{2160} \right) P^2 e^\Phi K + \kappa_6 P^4 e^{2\Phi},
\]
\[ L_{R}^{2} = X_{a} \left( -\frac{1}{240} P^{4} e^{2\Phi} \ln^{2} \rho - \frac{1}{240} P^{2} e^{\Phi} \ln \rho - \frac{1}{720} K^{2} + P^{4} e^{2\Phi} \left( \frac{1}{240} + \frac{8}{5} \kappa_{1} \right) \ln \rho \right. \\
+ P^{2} e^{\Phi} \left( \frac{4}{5} \kappa_{1} + \frac{1}{1152} \right) K + P^{4} e^{2\Phi} \left( \frac{2}{5} \kappa_{2} + \frac{43}{34560} \right) \right) \\
+ X_{b} \left( -\frac{1}{1152} K + \frac{1}{2304} P^{2} e^{\Phi} \right), \]

(3.49)

\[ L_{Rc}^{2} = X_{a} \left( \frac{1}{80} P^{4} e^{2\Phi} \ln^{2} \rho + \frac{1}{80} P^{2} e^{\Phi} \ln \rho - \frac{1}{240} K^{2} + P^{4} e^{2\Phi} \left( -\frac{1}{80} - \frac{24}{5} \kappa_{1} \right) \ln \rho \right. \\
- \frac{12}{5} P^{2} e^{\Phi} K \kappa_{1} - \frac{6}{5} P^{4} e^{2\Phi} \kappa_{2} \right) + X_{b} \left( \frac{1}{384} K - \frac{1}{384} P^{2} e^{\Phi} \right), \]

(3.50)

\[ L_{Rd}^{2} = X_{a} \left( \frac{1}{240} P^{4} e^{2\Phi} \ln^{2} \rho + \frac{1}{240} P^{2} e^{\Phi} \ln \rho - \frac{1}{480} K^{2} \right. \\
+ P^{4} e^{2\Phi} \left( \frac{8}{5} \kappa_{4} + \frac{533}{14400} \right) \ln \rho + \frac{4}{5} P^{2} e^{\Phi} \kappa_{4} K + \frac{2}{5} P^{4} e^{2\Phi} \kappa_{5} \right) \]

(3.51)

\[ + X_{b} \left( \frac{25}{672} P^{2} e^{\Phi} \ln \rho - \frac{1}{768} K - \left( \frac{167}{23040} + \frac{1}{64} \delta_{2} \right) P^{2} e^{\Phi} \right). \]

This result depends on seven parameters: \( \kappa_{i} (i = 1, \cdots, 6) \) and \( \delta_{2} \). In appendix A.4 we give a simple argument explaining why the coefficients \( \kappa_{i} \) turned out to be ambiguous. In addition, we expect to find an ambiguity in the counter-terms corresponding to reparametrizing \( \rho \to \lambda \rho \), because of the explicit \( \rho \)-dependence in the counter-terms. This is present already in the asymptotic AdS case, and it explains why the parameter \( \delta_{2} \) turned out to be ambiguous (since this reparametrization shifts \( \delta_{2} \), in addition to modifying the \( \kappa_{i} \) parameters).

### 3.4 Renormalized one-point correlation functions

In this subsection we describe in detail our results in the minimal subtraction renormalization scheme; more general schemes are described in the next subsection.
The one-point function of the stress energy tensor is given by

\[
8\pi G_5 \langle T_{ij} \rangle = G^{(0)}_{ij} \left( R_{ab} R^{ab} \left( \frac{1921}{276480} p_0^2 P^4 - \frac{1}{512} K_0^2 + \frac{1}{96} K_0 P^2 p_0 \right) - R^2 \left( \frac{1}{4608} K_0^2 + \frac{337}{51840} p_0^2 P^4 + \frac{175}{27648} K_0 P^2 p_0 \right) + R \left( \frac{1}{16} K_0 a^{(2,0)} + \frac{1}{128} p_0^2 p_0 a^{(2,0)} + \frac{5}{256} P^2 p_0 a^{(2,1)} \right) + \square R \left( \frac{391}{82944} p_0^2 P^4 - \frac{53}{23040} K_0^2 + \frac{323}{46080} K_0 P^2 p_0 \right) + R_{aijb} R^{ab} \left( \frac{17}{8640} p_0^2 P^4 - \frac{1}{32} K_0^2 + \frac{7}{192} K_0 P^2 p_0 \right) - R_i^a R_{aj} \left( \frac{1}{64} K_0^2 + \frac{1}{256} p_0^2 P^4 + \frac{1}{64} K_0 P^2 p_0 \right) + R R_{ij} \left( \frac{1691}{103680} p_0^2 P^4 - \frac{1}{576} K_0^2 + \frac{13}{432} K_0 P^2 p_0 \right) - R_{ij} \left( \frac{1}{16} P^2 p_0 a^{(2,1)} + \frac{1}{4} K_0 a^{(2,0)} \right) - \nabla_i \nabla_j R \left( \frac{2773}{207360} p_0^2 P^4 + \frac{5}{3456} K_0 P^2 p_0 + \frac{7}{1152} K_0^2 \right) + \square R_{ij} \left( - \frac{17}{17280} p_0^2 P^4 - \frac{7}{384} K_0 P^2 p_0 + \frac{1}{64} K_0^2 \right) + \nabla_i \nabla_j a^{(2,0)} \left( \frac{1}{16} P^2 p_0 + \frac{1}{16} K_0 \right) + \nabla_i \nabla_j a^{(2,1)} \left( \frac{7}{128} p_0^2 p_0 + \frac{3}{64} K_0 \right) + 2 G^{(4,0)}_{ij} - \frac{1}{2} G^{(0)}_{ij} G^{(4,0)} - \frac{3}{2} G^{(0)}_{ij} (b^{(4,0)} - a^{(4,0)}) + T^{\text{ambiguity}}_{ij}, \right)
\]

where

\[
T^{\text{ambiguity}}_{ij} = \left( \frac{1}{2} p_0^2 P^4 k_3 + \frac{1}{2} p_0 P^2 k_2 K_0 + \frac{1}{2} k_1 K_0^2 \right) \times \left( -2 \nabla_i \nabla_j R + 6 \square R_{ij} - 12 R_{aijb} R^{ab} - 3 G^{(0)}_{ij} R_{ab} R^{ab} + R^2 G^{(0)}_{ij} - 4 R R_{ij} - \square R G^{(0)}_{ij} \right). \]

The one-point function of the trace of the stress-energy tensor in the “minimal
subtraction” ansatz is unambiguously given by

\[
8\pi G_5 \langle T_i^i \rangle = R_{ab} R^{ab} \left( -\frac{1}{96} K_0 P^2 p_0 + \frac{101}{4608} p_0^2 p^4 + \frac{1}{128} K_0^2 \right) \\
+ R^2 \left( \frac{11}{2304} K_0 P^2 p_0 - \frac{67}{6912} p_0^2 P^4 - \frac{1}{384} K_0^2 \right) \\
+ R P^2 p_0 \left( \frac{1}{32} a^{(2,0)} + \frac{1}{64} a^{(2,1)} \right) \\
+ \square R \left( \frac{43}{2304} K_0 P^2 p_0 + \frac{151}{11520} p_0^2 P^4 + \frac{1}{384} K_0^2 \right) \\
+ 6(b^{(4,0)} - a^{(4,0)}). 
\] (3.54)

- The one-point function of \( O_{p_0} \) is given by

\[
\langle O_{p_0} \rangle = R_{ab} R^{ab} \left( \frac{263}{34560} p_0 P^4 + \frac{3}{64} K_0 P^2 \right) + R^2 \left( -\frac{79}{4608} K_0 P^2 - \frac{83}{138240} p_0 P^4 \right) \\
+ R P^2 \left( -\frac{19}{192} a^{(2,0)} + \frac{13}{384} a^{(2,1)} \right) + \square R \left( \frac{77}{3456} K_0 P^2 + \frac{33}{1280} p_0 P^4 \right) \\
+ \frac{1}{p_0} \left( 4p^{(4,0)} - 3(b^{(4,0)} - a^{(4,0)}) \right) + O_{p_0}^{\text{ambiguity}}, 
\] (3.55)

where

\[
O_{p_0}^{\text{ambiguity}} = (3 R_{ab} R^{ab} - R^2) \left( -2 P^4 \kappa_3 p_0 - P^2 \kappa_2 K_0 \right) \\
+ \square R \left( 2 P^4 p_0 \kappa_6 + P^2 \kappa_5 K_0 + \delta_2 P^2 \left( \frac{7}{320} K_0 - \frac{3}{640} P^2 p_0 \right) \right). 
\] (3.56)

- The one-point function of \( O_{K_0} \) is given by

\[
\langle O_{K_0} \rangle = R_{ab} R^{ab} \left( -\frac{1}{192} K_0 + \frac{7}{1152} P^2 p_0 \right) + R^2 \left( \frac{1}{2304} K_0 + \frac{5}{13824} P^2 p_0 \right) \\
- R \left( \frac{1}{32} a^{(2,0)} + \frac{1}{64} a^{(2,1)} \right) + \square R \left( \frac{53}{1920} K_0 + \frac{11}{34560} P^2 p_0 \right) \\
+ \frac{6}{P^2 p_0} (a^{(4,0)} - b^{(4,0)}) + O_{K_0}^{\text{ambiguity}}, 
\] (3.57)

where

\[
O_{K_0}^{\text{ambiguity}} = (3 R_{ab} R^{ab} - R^2) \left( -2 K_0 \kappa_1 - \kappa_2 p_0 P^2 \right) \\
+ \square R \left( 2 K_0 \kappa_4 + p_0 P^2 \kappa_5 \right). 
\] (3.58)
The conformal anomaly is the transformation of the action under scaling transformations (see [33] for a review). It is generally defined as (for non-conformal theories that have some beta functions $\beta_j$ for the couplings to operators $O_j$)

$$\text{conformal anomaly} = \langle T_i^i \rangle - \frac{1}{2} \sum_j \beta_j \langle O_j \rangle. \quad (3.59)$$

In our case the form of the asymptotic solution (2.27) (and the analysis in the appendix of the scaling transformation (A.40)) indicates that the coupling $K_0$ depends on the scale, and that its derivative with respect to the logarithm of the scale is given by $(-2P^2p_0)$. Thus, we have

$$\text{conformal anomaly} = \langle T_i^i \rangle + P^2p_0 \langle O_{K_0} \rangle. \quad (3.60)$$

Using the previous results we find that this is given by

$$\text{conformal anomaly} = (3R_{ab}R^{ab} - R^2) \left( \frac{1}{384} K_0^2 - \frac{1}{192} K_0 P^2 p_0 + \frac{43}{4608} p_0^2 p^4 \right)$$

$$+ \Box R \left( \frac{1}{384} K_0^2 + \frac{533}{11520} K_0 P^2 p_0 + \frac{9}{2160} p_0^2 p^4 \right)$$

$$+ P^2 p_0 \mathcal{O}_{K_0}^{\text{ambiguity}}. \quad (3.61)$$

Note that, as expected of the conformal anomaly, this is independent of any parameters (like $(a^{(4,0)} - b^{(4,0)})$) associated to the IR behavior of the theory, and that in our ansatz all the ambiguities in the conformal anomaly are related to the ambiguities in the one-point function of $O_{K_0}$. Somewhat surprisingly, we find a finite result for the conformal anomaly, though its precise value is ambiguous because of the ambiguities in our counter-terms.

In any local field theory (see, for instance, [33]) the conformal anomaly is a linear combination of the Euler density, the Weyl tensor squared and $\Box R$. As in other theories dual to gravitational backgrounds the conformal anomaly that we computed does not contain terms proportional to $R_{abcd}R^{abcd}$, so this requires the conformal anomaly to be a linear combination of a term proportional to $(3R_{ab}R^{ab} - R^2)$ and a term proportional to $\Box R$. We find that in our minimal subtraction ansatz this is indeed the case, even though it did not necessarily have to be the case because of the explicit $\rho$-dependence of our counter-terms.

It is straightforward to analyze the $P \to 0$ limit of the minimal subtraction renormalization scheme; this provides a holographic renormalization for a truncation of the
conformal field theory of [32]. The $P \to 0$ limit of the asymptotic solution was discussed in section 2.2. It is easy to verify that the one-point functions of the stress tensor and of $O_{p0}$ have a good $P \to 0$ limit, but we should be more careful with $O_{K_0}$. Note that in this limit equations (2.10) and (2.6) imply that we should change variables from $K(y)$ to the variable $\tilde{k}(y)$ related to the two-form, given by $K(y) = \tilde{K}_0 + P\tilde{k}(y)$, which remains finite in the $P \to 0$ limit. Let us denote by $\langle O_\tilde{k} \rangle$ the operator dual to $\tilde{k}(y)$. Then, it is easy to see from (3.57) that $\langle O_\tilde{k} \rangle$ is finite, provided that as $P \to 0$

$$a^{(4,0)} = b^{(4,0)} + \frac{1}{6} Pp_0 \tilde{k}^{(4,0)} + O(P^2), \quad (3.62)$$

where the parameter $\tilde{k}^{(4,0)}$ is precisely the expectation value

$$\langle O_\tilde{k} \rangle = \tilde{k}^{(4,0)}. \quad (3.63)$$

Notice that in the $P \to 0$ limit of the minimal subtraction scheme, with the scaling (3.62), the scalar one-point functions $\langle O_\tilde{k} \rangle$, $\langle O_{p0} \rangle$ and the conformal anomaly $\langle T^i_i \rangle$ are unambiguous. The conformal anomaly we find reproduces the known results in the AdS/CFT correspondence [12], up to the well-known [13] ambiguity in the term in the conformal anomaly proportional to $\Box R$. This ambiguity arises from finite (in the $P \to 0$ limit) counter-terms $L^{0}_{Ric^2}, L^{0}_{R^2}$, and it is proportional to

$$\langle T^i_i \rangle \propto (L^{0}_{Ric^2} + 3L^{0}_{R^2}) \Box R. \quad (3.64)$$

In the minimal subtraction scheme, even though both $L^{0}_{Ric^2}$ and $L^{0}_{R^2}$ are ambiguous, there is no ambiguity in the combination (3.64). If we work directly in the conformal $P = 0$ theory, the ambiguity can be reintroduced by simply shifting

$$L^{0}_{R^2} \to L^{0}_{R^2} + \delta_{R^2}, \quad L^{0}_{Ric^2} \to L^{0}_{Ric^2} + \delta_{Ric^2} \quad (3.65)$$

where $\delta_{Ric^2}$ and $\delta_{R^2}$ are arbitrary constants with $\delta_{Ric^2} + 3\delta_{R^2} \neq 0$. However, such a simple modification is not possible in the $P \to 0$ limit of the holographic renormalization of the cascading gauge theories. The problem is that the one-point functions of the irrelevant operators (see (3.33), (3.34)) are sensitive to the renormalized $\langle T^i_i \rangle$, and thus to its ambiguity, which is $\propto \Box R$. So, if a given set of counter-terms renormalizes the irrelevant operators, one would expect that a generic shift as in (3.65) would reintroduce divergences $\propto \Box R$ in these one-point functions. This is indeed what we find.
We would like to emphasize that the fact that we find an unambiguous $\langle T_i^i \rangle$ is only a feature of the minimal subtraction scheme – an ambiguity proportional to $\Box R$ does appear in more general renormalization schemes discussed in the next subsection.

Finally, we would like to mention that the $P \to 0$ limit of the counter-terms we found in our holographic renormalization correctly reproduces the unambiguous counter-terms of the conformal holographic renormalization [12, 13], including also the counter-terms with explicit $\ln \rho$ dependence [21]. To see this it is useful to note that in this limit $\Omega_1^4 = \Omega_2^4 = h = K/4$, so the first line of (3.42) is simply $-\frac{5}{2} K - \frac{1}{3} K R$, while the second line includes the terms $\ln(\rho^2)K \frac{1}{32} (R_{ab}R^{ab} - \frac{1}{3} R^2)$, in agreement with equation (5.42) of [21] up to overall normalization factors in the metric and in $G_5$ (recall that $K$ is a constant in the $P \to 0$ limit).

### 3.5 Ambiguities in the choice of counter-terms

In the previous subsection we presented the results of the “minimal subtraction ansatz” which leads to specific finite one-point functions; this ansatz and the resulting one-point functions have a 7-parameter ambiguity. The ansatz we used is the simplest one we could find, but we do not have a good argument that it is correct (in the sense that it leads to all correlation functions being finite); in particular it is not hard to find more general choices for $L_{\text{kinetic}}$, and to add terms with more negative powers of the $\Omega$’s, in a way which preserves the finiteness of all one-point functions.$^{15}$

Since with our limited choice of the sources we do not have a way to uniquely determine the counter-terms, we have studied various possibilities for the counter-terms in order to see which of our results for the one-point functions are modified by more general counter-terms and which remain true.$^{16}$ We find that in a flat background (with $R_{ij} = 0$) the one-point functions are completely independent of the choice of counter-terms; they are finite and given by the unambiguous results that we found in the previous subsection when $R_{ij} = 0$. In curved space there are ambiguities in

---

$^{15}$One rather ugly feature of our choice of $L_{\text{kinetic}}$ is that when we include non-constant sources for the scalars, there is a term with an explicit $\ln \rho$ dependence appearing already at order $\rho^2$. It is possible to choose other forms of $L_{\text{kinetic}}$ which do not have an explicit dependence on $\rho$, but they do not have a good $P \to 0$ limit. The correct choice should presumably be determined by introducing more general sources for the scalar operators, which we hope to do in future work.

$^{16}$We have studied the most general possible counter-terms which do not contain very large explicit powers of $\ln \rho$ and do not contain large negative powers of the $\Omega$ fields; we believe that these should be properties of the “correct” counter-terms but we do not have a rigorous argument for this.
some one-point functions. Recall that already in asymptotically anti-de Sitter spaces the stress-energy tensor has a 2-parameter ambiguity \cite{13} related to a freedom in the definition of the stress-energy tensor in curved space. In the minimal subtraction ansatz we found one of these two ambiguities in the stress tensor (3.53) (multiplied by an arbitrary linear combination of $P^4p_0^2$, $P^2p_0K_0$ and $K_0^2$), and in more general renormalization schemes we find also the other ambiguity (again multiplied by an arbitrary linear combination of $P^4p_0^2$, $P^2p_0K_0$ and $K_0^2$), which contributes also a term proportional to $\Box R$ to $\langle T^i_i \rangle$. It turns out that once these ambiguities are determined (by choosing a specific definition for the stress-energy tensor in curved space) the other one-point functions are also determined, up to a freedom in shifting the one-point functions of $\mathcal{O}_{p_0}$ and $\mathcal{O}_{K_0}$ by terms proportional to $\Box R$; presumably this freedom can also be interpreted as some sort of ambiguity in the definition of these operators in curved space.

Overall, with the most general ansatz for the counter-terms that we checked we find an 11-parameter ambiguity in the results for the one-point functions. We expect that some of the ambiguities would remain also once all the correlation functions are renormalized, but some may disappear. One ambiguity which will always remain in any theory of this type (involving an explicit $\ln \rho$-dependence in the counter-terms) corresponds to the freedom of redefining the radial coordinate by $\rho \rightarrow \lambda \rho$. Using the most general counter-terms we find that the conformal anomaly is not necessarily given by a linear combination of $3R_{ab}R^{ab} - R^2$ and $\Box R$, as it must be in any local field theory with a gravitational dual; as mentioned above this is consistent because of the explicit $\rho$-dependence of our counter-terms. If we impose this as an additional constraint on the counter-terms we find only a 9-parameter ambiguity in the one-point functions. Generic counter-terms do not necessarily have a good $P \rightarrow 0$ limit, and this could also be imposed as an additional constraint.

In summary, in flat space we find unambiguous results for all one-point functions, while in curved space there is a finite number of ambiguity parameters (as in asymptotically AdS space). The conformal anomaly turns out to be finite but ambiguous. This is not surprising given that it is related to the number of degrees of freedom in the theory at high energies (it is independent of the IR behavior), which seems to be ill-defined in the cascading theories – the surprise is that in any specific renormalization scheme we actually find a finite answer for the conformal anomaly, as well as for all other one-point functions.
3.6 Other possible renormalization schemes

The most intriguing feature of our results is that we find finite results (without any ln $\rho_0$ divergences) for all one-point correlation functions. This is somewhat surprising since some one-point functions, such as the conformal anomaly, are supposed to count the number of degrees of freedom in the theory, which is believed to diverge at high energies. In particular, in the conformal theory with $P = 0$ one has (up to $\Box R$ ambiguities)

$$8\pi G_5 \langle T^i_{\text{conformal}} \rangle = -\frac{K_0^2}{16} \left[ -\frac{1}{8} R_{ab} R^{ab} + \frac{1}{24} R^2 \right], \quad (3.66)$$

and the coefficient is proportional to the central charge (in general there are two independent coefficients in the conformal anomaly, but in theories with gravitational duals they are always equal to each other). On the other hand, the behavior of the 5-form flux in the solution of [4] suggests that the number of degrees of freedom in the cascading theories diverges at high energies, and also thermodynamical studies of cascading gauge theories [28, 31] suggest that the number of effective degrees of freedom of the theory accessed at temperature $T$ is proportional to $K_{\text{eff}}^2 \propto (P^2 p_0 \ln \frac{T}{\Lambda})^2$. Thus, it may be natural to guess that the conformal anomaly of the cascading gauge theory would be as in (3.66), but with a replacement

$$K_0 \rightarrow K_{\text{eff}}(\rho_0) \equiv K_0 - 2P^2 p_0 \ln \rho_0 \quad (3.67)$$

depending explicitly on the cutoff scale. This is quite different from the finite result that we found above (3.61). This leads to a natural question: in the context of the holographic renormalization, is it possible to ‘renormalize’ the cascading gauge theory in such a way that all one-point correlation functions contain $K_{\text{eff}}$ instead of $K_0$? Here, by ‘renormalize’ we mean that there are no explicit ln $\rho_0$ divergences in one-point correlation functions, apart from the ones appearing implicitly in $K_{\text{eff}}$.

With the tool-set of counter-terms as in section 3.3, requiring that all the counter-terms have a good limit as $P \to 0$, and that the leading and the first two subleading power-law divergences of $O_6$ and $O_8$ are removed (3.1), it is possible to show that such a renormalization prescription is not possible. Specifically, one finds that the stress energy one-point function contains certain ln $\rho_0$ terms (even on manifolds with $\Box R = \Box R_{ij} = \nabla_i \nabla_j R = 0$) which cannot be subtracted by any counter-terms. Needless to say, it would be very interesting to explore these issues further.
4 Application: cascading gauge theories at finite temperature

In this section we study the high-temperature thermodynamics of cascading gauge theories, and we verify that the finite results which we found in the previous section (which are unambiguous for the thermodynamics of the theory in flat space) are consistent with the expectation that the cascading theories will have an effective “running rank” $K_{\text{eff}} \propto P^2 p_0 \ln \frac{T}{\Lambda}$.

The thermodynamics of cascading gauge theories was studied in [28, 30, 31]. It was noted there that the $\mathbb{Z}_2P$ chiral symmetry is restored in the black brane solutions which are dual to the cascading theories at high temperatures (compared to the strong coupling scale $\Lambda$), and thus the high temperature solutions can be described using the ansatz we use in this paper. In the previous studies the stress-energy tensor was not renormalized, so the only thermodynamic property which could be extracted was the entropy density (which depends only on the horizon area). The high-temperature solutions involve a parameter $K_*$ which is the value of the five-form flux at the horizon (the minimal value of the radial coordinate); the solution is constructed in a perturbation expansion in $P^2/K_*$ (valid at high temperatures), and the leading term in this expansion is explicitly known [31]. The parameter $K_*$ should in principle be determined in terms of the temperature, and it appears in the computation of [31] of the entropy density. The authors of [31] used physically reasonable (albeit somewhat ad-hoc) arguments to argue that

$$K_* = 2P^2 p_0 \ln \frac{T}{\Lambda} + \cdots$$  \hspace{1cm} (4.1)

where $\cdots$ indicate sub-dominant terms in the high temperature limit $T \gg \Lambda$. In this section we will use the renormalized one-point functions of the stress-energy tensor to determine rigorously the relation between $K_*$ and $T$, and thus the high-temperature thermodynamics of the cascading gauge theories.

Given the results of the previous section for the renormalized one-point functions of the stress energy tensor, one can compute the ADM mass-density (the energy density) and the pressure of the black brane solution which is holographically dual to the cascading gauge theory at finite temperature, in addition to the entropy density, to leading order in $P^2/K_*$ (as in [31]). This allows us to explicitly verify (to leading order
in $P^2/K_*$) the relation \(^{17}\)
\[ f = -P = \epsilon - Ts, \tag{4.2} \]
where $f$ and $\epsilon$ are the free energy and the energy densities, $s$ is the entropy density, and $P$ is the pressure. Additionally, requiring the first law of thermodynamics
\[ d\epsilon = Tds \tag{4.3} \]
gives an equation which leads to (4.1).

The rest of this section is organized as follows. In subsection 4.1 we discuss the thermodynamics of cascading gauge theories to leading order in $P^2/K_*$. In subsection 4.2 we briefly comment on the hydrodynamical properties of the cascading gauge theory plasma.

### 4.1 Thermodynamics of cascading gauge theories

Throughout this section we will use the notations and results of [31] \(^{18}\). In particular, we use $K_*$ to denote the 5-form flux $K$ evaluated at the horizon, and $a$ for the non-extremality parameter (which will be related to the temperature below). The ten-dimensional Einstein frame metric of the non-extremal cascading solution is
\[ ds_{10}^2 = \left( \frac{8a}{K_* v} \right)^{1/2} e^{2P^2\eta} \left[ -(1-v)(dx_0)^2 + (dx_\alpha)^2 \right] + \frac{\sqrt{K_*}}{32} e^{-2P^2(\eta-\xi)} \frac{d\eta}{v^2(1-v)} \tag{4.4} \]
\[ + \frac{\sqrt{K_*}}{2} e^{-2P^2(\eta-\xi)} \left[ e^{-8P^2\omega} e_\psi^2 + e^{2P^2 \omega} \left( e_{\theta_1}^2 + e_{\phi_1}^2 + e_{\theta_2}^2 + e_{\phi_2}^2 \right) \right], \]
where $\alpha = 1, 2, 3$, $v$ is a radial coordinate such that $v \to 1_-$ at the horizon and $v \to 0_+$ at the boundary, and $\eta, \xi, \omega$ are functions of $v$. To leading order in $P^2/K_*$ the 5-form and dilaton are given by
\[ K = K_* - \frac{P^2}{2} \ln v, \]
\[ \Phi = \Phi_* + \frac{1}{4K_*} \text{Li}_2(1-v), \tag{4.5} \]
where $\Phi_*$ is the dilaton value at the horizon, and $\text{Li}_n(z)$ is the polylogarithm function. As in [31], we choose the dilaton to vanish at the boundary, corresponding to choosing

\(^{17}\)Recall that in the absence of a chemical potential, the free energy density equals minus the pressure.

\(^{18}\)We have independently verified all the results which we actually use.
\( p_0 = 1 \) in the notations of the previous sections; we will write our results in this section using this choice of \( p_0 \), and the \( p_0 \)-dependence can always be reinstated by recalling that factors of \( p_0 \) come together with factors of \( P^2 \). The functions \( \{ \eta, \xi, \omega \} \) satisfy the following ordinary differential equations:

\[
\begin{align*}
&v(1-v)\omega'' - v\omega' - \frac{3}{4v}\omega = \frac{1}{40K_*}, \\
v(1-v)\xi'' - v\xi' - \frac{2}{v}\xi = -\frac{1}{40K_*}, \\
v(1-v)\eta'' - v\eta' - \frac{2}{v}\eta = \frac{1}{16K_*v} (2 - v - 4 \ln v).
\end{align*}
\] (4.6)

The singularity-free solution with the correct asymptotics in the UV is uniquely determined in terms of \( \{a, K_*\} \). For our purposes it will be necessary to know the asymptotics of the solution near the boundary. As computed in [31], near the boundary \( v \to 0 \)

\[
\begin{align*}
\xi &\sim \frac{v}{80K_*} + \cdots, \\
\eta &\sim \frac{\ln v - 1}{8K_*} + \cdots, \\
\omega &\sim -\frac{1}{30K_*}v + \cdots, \\
\Phi &\sim \frac{v}{4K_*} (\ln v - 1) + \cdots,
\end{align*}
\] (4.7)

where \( \cdots \) denote sub-dominant terms. It is straightforward to compute the Hawking temperature and the entropy density of the black brane solution (to leading order in \( P^2/K_* \)). We find

\[
\begin{align*}
T &= \frac{(2a)^{1/4}}{2\pi K_*^{3/2}} \left( 4K_* - P^2 \right), \\
n &= \frac{\pi^3 K_*}{64G_5} (K_* + P^2).
\end{align*}
\] (4.8)

In order to compute the energy and the pressure of the black brane we need to relate the \( v \) coordinate to the \( \rho \) coordinate that we used in our solution to evaluate the stress
tensor. This is done by comparing the product of warp factors in front of $\text{dx}_0^2$ and $e_{ij}^2$

$$h^{-1/2} \rho^{-2} \times h^{1/2} f_2 = \left( \frac{2a}{v} \right)^{1/2} (1 - v) e^{2P^2(\xi - 4\omega)},$$

(4.9)

$$\rho^{-2} (1 + \cdots) = \left( \frac{2a}{v} \right)^{1/2} (1 + \cdots).$$

Thus, to compare we need to define the radial coordinate $\rho$ by

$$\rho^4 \equiv \frac{v}{2a}.$$  

(4.10)

Given (4.10), by translating the asymptotic form of the solution (4.7) to our ansatz (2.29)-(2.34) we find

$$b^{(4,0)} - a^{(4,0)} = -\frac{2aP^2}{3K_*},$$

$$p^{(4,0)} = \frac{a(\ln(2a) - 1)}{2K_*},$$

$$G^{(4,0)}_{00} = -\frac{P^2a}{4K_*},$$

$$G^{(4,0)}_{\alpha\alpha} = \frac{a}{4K_*} (8K_* + P^2).$$

(4.11)

For the black brane geometry the renormalized one-point function of the stress tensor (3.52) is given by

$$8\pi G_5 \langle T_{ij} \rangle = -\frac{1}{2} G^{(0)}_{ij} \alpha^{(4,0)} + 2G^{(4,0)}_{ij} + \frac{3}{2} G^{(0)}_{ij} (b^{(4,0)} - a^{(4,0)}),$$

(4.12)

where in this background $G^{(0)}_{ij} = \eta_{ij} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. We are now in a position to compute the energy density $\epsilon$ and the pressure $P$. We find

$$\epsilon = \frac{a}{8\pi G_5 K_*} (P^2 + 3K_*),$$

$$P = \frac{a}{8\pi G_5 K_*} (K_* - P^2).$$

(4.13)

With (4.8), (4.13) it is straightforward to verify (4.2) (to leading order in $P^2/K_*$). Given (4.8) we can evaluate $a$ in terms of $T$ and $K_*$. We expect that $K_* = K_*(T)$. Enforcing the first law of thermodynamics (4.3) leads to a differential equation on $K_*$

$$0 = 2P^2 - T \frac{dK_*}{dT},$$

(4.14)

Notice that the product of warp factors in (4.9) does not contain $\ln \rho$ factors, and thus can be consistently evaluated using the (uniformly small in $\rho$) $\mathcal{O}(P^2)$ solution (4.7) for $\xi$ and $\omega$. 

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which leads to (at leading order in \( P^2/K_{\text{star}} \))

\[
K_*(T) = 2P^2 \ln \frac{T}{\Lambda} \tag{4.15}
\]

for some constant \( \Lambda \), as found from different considerations in [31]. This allows us to write the energy density and the pressure (4.13) purely in terms of the temperature (to leading order in \( P^2/K_*(T) \)), and they exhibit the expected behavior of an almost-conformal theory with a number of degrees of freedom proportional to \( K_*(T)^2 \).

The fact that we obtain a finite result for the free energy density should be useful in analyzing the deconfinement phase transition in this theory (of course, this requires going beyond the limit of \( P^2 \ll K_*(T) \), but our renormalization works independently of this limit).

### 4.2 Hydrodynamics of cascading gauge theories

Small low-energy deviations from the thermodynamic equilibrium in a strongly coupled gauge theory plasma are expected to be well described by hydrodynamics. In this paper we advocated the definition of cascading gauge theories in terms of the dual string theory. Thus, the appropriate description of relaxation processes in cascading gauge theory plasma is in terms of “holographic hydrodynamics”, introduced for conformal gauge theory plasma in [34,35]. The effective hydrodynamic description of relaxation of density perturbations in plasma is completely specified by two viscosity coefficients\(^{20}\), the shear viscosity \( \eta \) and the bulk viscosity \( \xi \), and the speed of sound waves \( c_s \). In a conformal gauge theory plasma \( \epsilon = 3P \), and thus using the well-known relation

\[
c_s^2 = \frac{\partial P}{\partial \epsilon} \tag{4.16}
\]

we find

\[
c_s^{\text{conformal}} = \frac{1}{\sqrt{3}} \tag{4.17}
\]

Furthermore, conformal invariance guarantees that the bulk viscosity vanishes

\[
\xi^{\text{conformal}} = 0, \tag{4.18}
\]

while holographic hydrodynamics predicts [36] the ratio of the shear viscosity to entropy density in the planar limit and at strong 't Hooft coupling (namely, in the gravity

\(^{20}\)Not to be confused with \( \{\eta, \xi\} \) of the previous section.
approximation)\textsuperscript{21} to be
\[ \frac{\eta}{s} \bigg|_{\text{conformal}} = \frac{1}{4\pi}. \quad (4.19) \]

It is interesting to generalize these results to non-conformal theories such as the cascading gauge theories. In particular, recall that QCD has a mass scale, and in some regimes the quark-gluon plasma of QCD could be described by strongly coupled hydrodynamics. It is thus of practical importance\textsuperscript{22} to obtain hydrodynamic predictions for strongly coupled non-conformal gauge theory plasma (even though the theory we discuss here is of course very different from QCD).

Somewhat surprisingly, the ratio of shear viscosity to entropy density in all gauge theory plasmas which are dual to gravitational theories (including the cascading gauge theories), was found \textsuperscript{[41–43]} to be universal in the supergravity approximation,\textsuperscript{23} and given by (4.19). On the contrary, the speed of sound and the sound wave attenuation (which is determined in part by the bulk viscosity) are not expected to be universal. Indeed, in \textsuperscript{[44]} it was found that explicit breaking of conformal invariance in strongly coupled gauge theory plasma by fermionic (bosonic) mass terms \(m_f\) (\(m_b\)) leads to a modified dispersion relation, with the speed of sound given in the high temperature \((T \gg m_f, T \gg m_b)\) regime by
\[ c_s = \frac{1}{\sqrt{3}} \left(1 - \delta_f \frac{m_f^2}{T^2} - \delta_b \frac{m_b^2}{T^4}\right). \quad (4.20) \]

In (4.20), \(\delta_f\) and \(\delta_b\) are positive coefficients. The cascading gauge theory plasma differs from the non-conformal plasma discussed in \textsuperscript{[44]} in that its scale invariance is broken by a dynamically generated scale \(\Lambda\). At high temperatures the cascading gauge theory plasma is expected to resemble a conformal plasma. Indeed, using the results of the previous subsection we find that the speed of sound in this plasma is given by
\[ c_s^2 = \frac{\partial P}{\partial \epsilon} = \frac{\partial P}{\partial T} = \frac{1}{3} - \frac{4P^2}{9K_*} + O(P^4) = \frac{1}{3} - \frac{2}{9\ln \frac{T}{\Lambda}} + O(P^4). \quad (4.21) \]

It is amusing to note that the appearance of \(\ln T\) in this correction, suggesting that the cascading gauge theory plasma has (at least some) hydrodynamic properties similar to

\textsuperscript{21}Finite 't Hooft coupling corrections to (4.19) were discussed in \textsuperscript{[37]}.

\textsuperscript{22}A possible application is in hydrodynamics models describing elliptic flows in heavy ion collision experiments at RHIC \textsuperscript{[38–40]}.

\textsuperscript{23}Of course, neither the entropy nor the shear viscosity by itself is universal.
those of weakly interacting relativistic systems (since such a correction is expected to arise in asymptotically free gauge theories at high temperature, i.e., at weak coupling).

In would be very interesting to further study the hydrodynamics of cascading gauge theory plasma, and in particular to evaluate its bulk viscosity.

5 Conclusions

In this paper we have performed a holographic renormalization of the cascading gauge theory of Klebanov and Tseytlin [4] compactified on an arbitrary four-manifold, assuming that the $\mathbb{Z}_2^P$ global symmetry is unbroken; we have found counter-terms that can be added to the action of this theory so that the one-point functions of all operators (in the truncated action) are finite (on an arbitrary four-manifold). As discussed in detail above, the holographic renormalization in these theories is complicated by the fact that we cannot introduce arbitrary sources for the fields in the truncated action, since some of them correspond to irrelevant operators. This is an interesting problem already for asymptotically anti-de Sitter spaces, where it is also not clear how to perform a holographic renormalization for correlation functions of irrelevant operators; the difference in our case is that in the cascading theory these operators mix with the metric so we cannot consistently ignore them. Because of this complication we have performed the analysis with sources for only some of the operators, and we were not able to uniquely determine the counter-terms. However, we have proved that there exist counter-terms that make all the one-point functions finite. The choice of these counter-terms is ambiguous, and we believe that this ambiguity can be resolved (up to the usual freedom related to redefinitions of operators) by requiring that arbitrary correlation functions are finite – it would be interesting to renormalize more general correlation functions and verify that this is correct. Within our limited ansatz for the counter-terms we found that some one-point functions were uniquely determined, but others were ambiguous – we conjecture that the unambiguous one-point functions would remain the same for any consistent choice of counter-terms (including the correct one which renders all correlation functions finite), again it would be interesting to verify this.

Note that even though we discussed the ambiguity in the language of one-point functions, the ambiguity in the one-point functions that we found is independent of the state, so it may be thought of as an ambiguity in the definition of the operators themselves.
As we discussed in the introduction, our main result is that the renormalization of these theories leads to finite one-point functions despite the infinite number of high-energy degrees of freedom in these theories; we discussed in section 4 how this is consistent with the finite temperature behavior. Another possible renormalization of the cascading gauge theories involves flowing to them (at some finite scale $\mu$) from finite rank $N$ gauge theories as discussed in [9], assuming that the construction of [9] is valid also at strong coupling where the gravity approximation that we have been working in is valid. It may be possible to define the confining gauge theories as a limit of the construction of [9] in which $N$ and $\mu$ are both taken to infinity in a correlated way. Then, if one would compute correlation functions keeping the cutoff scale always above the scale $\mu$ one would get infinite results (say, for the conformal anomaly) because of the diverging rank of the high-energy group. However, keeping the cutoff scale above $\mu$ is problematic in the limit in which $\mu$ goes to infinity; we believe that our prescription in which the cutoff scale goes to infinity only at the end is more natural (and more analogous to the usual holographic renormalization performed in asymptotically AdS backgrounds). It would be interesting to study further various alternative renormalization schemes and to understand where they agree and where they disagree, in order to understand better how to define the cascading gauge theories. In the absence of an alternative definition for the cascading gauge theories we suggest that they can be defined in terms of their correlation functions, which one can compute using the procedure we described in this paper (at least in the gravity approximation; it should be possible to generalize this to the full string theory, but this involves understanding string theory in Ramond-Ramond backgrounds).

There are many interesting generalizations of our results. It would be interesting to analyze more general correlation functions; of course, the precise computation of higher $n$-point functions involves additional information beyond just the asymptotic solution that we found, but the counter-terms needed for the finiteness of these correlation functions can be found purely by using our asymptotic solutions. More precisely, $n$-point functions of the stress-energy tensor can be analyzed using our solution, while $n$-point functions of other operators require the generalization of our solution to more general sources. We expect the resulting 2-point functions to agree with the results of [45, 46] which were computed without carefully regulating the theory. Another direction is to add additional fields to our truncated action; in particular, in order to study backgrounds like that of [5] where the $\mathbb{Z}_{2p}$ symmetry is spontaneously broken,
we would need to add to our action fields that are charged under this symmetry. We believe that it should be possible in such backgrounds to add a finite number of fields to our effective action (2.19) and to perform the holographic renormalization as we did in this paper; it would be interesting to verify this. Finally, the asymptotic form of many other cascading backgrounds (generalizing the work of [4]) has recently been found [47], and it should be possible to generalize our results to these backgrounds.

Even without these generalizations there are several interesting applications of our results, which allow us to compute the (finite) stress-energy tensor in any solution corresponding to the compactified cascading theory which preserves the $\mathbb{Z}_2 \mathbb{P}$ symmetry. One example of such a solution is the cascading theory at finite temperature, in the high temperature phase in which the $\mathbb{Z}_2 \mathbb{P}$ chiral symmetry is unbroken. We have computed the thermodynamical properties of this theory at very high temperatures (compared to the strong coupling scale) in section 4 above. The solution for arbitrary temperatures is not known, but given such a solution (which can in principle be found numerically) our results allow us to precisely compute its free energy. In particular, our results would allow us to determine the temperature at which the free energy vanishes, which should be interpreted as the deconfinement temperature of the cascading gauge theory (as in [48]; note that in the case analyzed in [48] a simple subtraction of the action in the two competing backgrounds was sufficient to renormalize the action, but we do not expect this to be true in more complicated backgrounds such as those of the cascading gauge theories\footnote{It is known that background subtraction as a method for computing the free energy does not work for charged black holes in $AdS_5$, and for the supergravity dual to mass deformed $\mathcal{N} = 4$ SYM theory [49–51].}). Other interesting backgrounds, analyzed in [29], describe the confining gauge theory on $S^3 \times \mathbb{R}$, $S^4$ and $dS_4$. Again, given any solution of this type our results allow us to precisely compute the stress-energy tensor of that solution (for example, the Casimir energy of the cascading gauge theory on $S^3$, which our results guarantee will be finite despite the infinite number of high-energy degrees of freedom).

Acknowledgments

It is a pleasure to thank Vijay Balasubramanian, Marcus Berg, Nick Dorey, Eduard Gorbar, Michael Haack, Gary Horowitz, Igor Klebanov, Volodya Miransky, Rob Myers, Kostas Skenderis, Dam Son, Andrei Starinets, Matt Strassler, Marika Taylor and
Arkady Tseytlin for valuable discussions. This work was supported in part by the Albert Einstein Minerva Center for Theoretical Physics. OA would like to thank the Aspen Center for Physics, the Kavli Institute for Theoretical Physics, and the Perimeter Institute for hospitality during various stages of this project. The work of OA was supported in part by the Israel-U.S. Binational Science Foundation, by the Israel Science Foundation (grant number 1399/04), by the Braun-Roger-Siegl foundation, by the European network HPRN-CT-2000-00122, by a grant from the G.I.F., the German-Israeli Foundation for Scientific Research and Development, and by Minerva. AB would like to thank the Weizmann Institute of Science, the University of Pennsylvania, the Aspen Center for Physics and the Kavli Institute for Theoretical Physics for hospitality during various stages of this project. Research at Perimeter Institute is supported in part by funds from NSERC of Canada. AB acknowledges support by an NSERC Discovery grant. AY would like to thank the Kavli Institute for Theoretical Physics for hospitality during this project. The work of AY is supported in part by a Kreitman foundation fellowship.

A Details of the solution

A.1 Equations of motion

In what follows we denote by a prime the derivative with respect to $\rho$ and by $\partial_i$, $i = 0, \cdots, 3$, the partial derivative with respect to $x^i$. Also, we denote for arbitrary functions $g_1, g_2$ on $M_5$

\begin{align}
\partial g_1 \partial g_2 &\equiv G^{ij} \partial_i g_1 \partial_j g_2, \\
\Box g_1 &\equiv \frac{1}{\sqrt{-G}} \partial_\tau \left[ \sqrt{-G} G^{ij} \partial_j g_1 \right].
\end{align}

(A.1)

Note that this is a different notation than the one we used in the beginning of section 2.2.

The equations of motion of the five dimensional supergravity (2.19) dual to the cascading gauge theory on an arbitrary curved manifold $\partial M_5$, in the metric parameter-
terization (2.25), are:

\[
0 = \left[ e^{-2\Phi}(-G) f_2 h^{-2} \rho^{-6} (K')^2 \right]' + e^{-\Phi} \sqrt{-G} K' h^{-1} \rho^{-6} \left\{ 2 f_2 \partial_i \left( e^{-\Phi} \sqrt{-G} G^{ij} \partial_j K \right) \right.
\]
\[
+ e^{-\Phi} \sqrt{-G} \partial f_2 \partial K \left\} - 4 P^2 e^{-\Phi} (-G) K K' h^{-3} f_3^{-2} \rho^{-8}
\]

(A.2)

\[
0 = \left[ (-G) f_2 f_3^4 \rho^{-6} (\Phi')^2 \right]' + \sqrt{-G} \Phi' f_3^2 \rho^{-6} \left\{ 2 f_2 \partial_i \left( h f_3^2 \sqrt{-G} G^{ij} \partial_j \Phi \right) \right.
\]
\[
+ h f_3^2 \sqrt{-G} \partial f_2 \partial \Phi \left\} + \frac{1}{2} (-G) \Phi' P^{-2} h^{-1} f_3^2 \rho^{-8} \left\{ e^{-\Phi} f_2 \rho^2 (K')^2 + h(\partial K)^2 \right\} (A.3)
\]
\[
- 4 e^\Phi P^4 \right\}
\]
\[
\frac{1}{\sqrt{-G}} \left[ \sqrt{-G} h^{-2} \rho^{-3} (hf_2^2)' \right]' - \frac{3}{2} f_2^{-1} h^{-2} \rho^{-3} f_2' (hf_2^2)' - \rho^{-3} (\partial f_2)^2
\]
\[
+ 2 \rho^{-3} f_2 \Box f_2 + \frac{1}{2} \rho^{-3} f_2 h^{-1} \partial f_2 \partial h + f_2^2 \rho^{-3} \Box \ln h
\]
\[
= -3 P^2 e^\Phi \rho^{-5} h^{-2} f_2 f_3^{-2} - \rho^{-5} K^2 h^{-3} f_2 f_3^{-4}
\]
\[
+ \frac{1}{4} h^{-2} \rho^{-3} f_2^2 f_3^{-2} P^{-2} e^{-\Phi} \left\{ (K')^2 + h(\partial K)^2 \right\}
\]
\[
- \rho^{-3} h^{-3} f_3^{-2} \left\{ (hf_3^2)'(hf_3^2)' + h \partial(hf_3^2) \partial(hf_3^2) \right\} + 16 \rho^{-5} h^{-1} f_3^{-2} f_2^3
\]

(A.4)

\[
\frac{1}{\sqrt{-G}} \left[ \sqrt{-G} h^{-2} \rho^{-3} (hf_3^2)' \right]' - \frac{3}{2} f_3^{-1} h^{-2} \rho^{-3} f_3' (hf_3^2)' - \rho^{-3} (\partial f_3)^2
\]
\[
+ 2 \rho^{-3} f_3 \Box f_3 + \frac{1}{2} \rho^{-3} f_3 h^{-1} \partial f_3 \partial h + f_3^2 \rho^{-3} \Box \ln h
\]
\[
= -P^2 e^\Phi \rho^{-5} h^{-2} f_2^{-1} - \rho^{-5} K^2 h^{-3} f_2^{-1} f_3^{-2} - \frac{1}{4} h^{-2} \rho^{-3} P^{-2} e^{-\Phi} \left\{ (K')^2 + h(\partial K)^2 \right\}
\]
\[
- \frac{1}{4} \rho^{-3} h^{-3} f_2^{-2} \left\{ (hf_2^2)'(hf_2^2)' + h \partial(hf_2^2) \partial(hf_2^2) \right\}
\]
\[
- \frac{3}{4} \rho^{-3} h^{-3} f_3^{-2} \left\{ (hf_3^2)'(hf_3^2)' + h \partial(hf_3^2) \partial(hf_3^2) \right\} + 4 \rho^{-5} h^{-1} (6 f_3 - 2 f_2)
\]

(A.5)
\[ R_{5ij} = G_{ij} \left\{ \frac{1}{3} \rho^{-2} h^{-2} f_{2}^{-1} f_{3}^{-2} P^{2} e^{\phi} + \frac{1}{6} \rho^{-2} h^{-3} f_{2}^{-1} f_{3}^{-2} P^{2} e^{\phi} + \frac{1}{12} \rho^{3} h^{-1} f_{2}^{-1} f_{3}^{-2} \right\} \]

\[ - \frac{1}{\sqrt{-G}} \left[ \sqrt{-G} \rho^{-3} f_{3}^{-6} h^{-5} [h^{5} f_{2}^{f_{3}^{8}}] \right] - \frac{1}{\sqrt{-G}} \partial_i \left[ \sqrt{-G} \rho^{-3} f_{3}^{-6} h^{-4} G^{ij} \partial_j [h^{5} f_{2}^{f_{3}^{8}}] \right] \]

\[ - \frac{3}{2} \rho^{-3} f_{3}^{-6} h^{-5} f_{2}^{-1} f_{2}' [h^{5} f_{2}^{f_{3}^{8}}] - \frac{3}{2} \rho^{-3} f_{3}^{-6} h^{-4} f_{2}^{-1} f_{2}' [h^{5} f_{2}^{f_{3}^{8}}] \]

\[ - 8 \rho^{-2} h^{-1} f_{3}^{-1} + \frac{4}{3} \rho^{-2} h^{-1} f_{2} f_{3}^{-2} \right\} \]

\[ + \left\{ \nabla_i \nabla_j \left[ \frac{5}{4} \ln h + \frac{1}{2} \ln f_{2} + 2 \ln f_{3} \right] + \frac{5}{8} h^{-2} \partial_{i} h \partial_{j} h \right\} \]

\[ + \frac{1}{h} f_{2}^{-1} \left( \partial_{i} h \partial_{j} f_{2} + \partial_{j} f_{2} \partial_{i} h \right) + \frac{1}{2} h^{-1} f_{3}^{-1} \left( \partial_{i} h \partial_{j} f_{3} + \partial_{j} f_{3} \partial_{i} h \right) \]

\[ - G_{ij} \left( \frac{5}{16} h^{-2} (\partial h)^{2} + \frac{1}{2} h^{-1} f_{2}^{-1} \partial h \partial f_{2} + \frac{1}{2} h^{-1} f_{3}^{-1} \partial h \partial f_{3} \right) \]

\[ + \left( \frac{1}{2} h^{-1} G_{ij}^{' \prime} - \frac{1}{4} h^{-2} G_{ij} h' - \rho^{-1} h^{-1} G_{ij} \right) \left( \frac{5}{4} h^{-1} h' + \frac{1}{2} f_{2}^{-1} f_{2}' + 2 f_{3}^{-1} f_{3}' \right) \}

\[ + \left\{ \frac{5}{16} h^{-2} \partial_{i} h \partial_{j} h + \frac{1}{8} h^{-1} f_{2}^{-1} \left( \partial_{i} h \partial_{j} f_{2} + \partial_{j} f_{2} \partial_{i} h \right) \right\} \]

\[ + \frac{1}{h} f_{3}^{-1} \left( \partial_{i} h \partial_{j} f_{3} + \partial_{j} f_{3} \partial_{i} h \right) + \frac{1}{4} f_{2}^{-2} \partial_{i} f_{2} \partial_{j} f_{2} + f_{3}^{-2} \partial_{i} f_{3} \partial_{j} f_{3} \right\} \]

\[ + \left\{ \frac{1}{4} P^{-2} h^{-1} f_{3}^{-2} e^{-\phi} \partial_{i} K \partial_{j} K + \frac{1}{2} \partial_{i} \Phi \partial_{j} \Phi \right\} \]

\[ R_{5\nu\rho} = \left\{ \partial_{i} \left[ \frac{5}{4} h^{-1} h' + \frac{1}{2} f_{2}^{-1} f_{2}' + 2 f_{3}^{-1} f_{3}' \right] \right\} \]

\[ - \frac{1}{2} G_{\mu \nu}^{kn} G_{\nu}^{\mu} \left( \frac{5}{4} h^{-1} \partial_{k} h + \frac{1}{2} f_{2}^{-1} \partial_{k} f_{2} + 2 f_{3}^{-1} \partial_{k} f_{3} \right) \]

\[ + \left( \frac{1}{4} h^{-1} h' + \rho^{-1} \right) \left( \frac{5}{4} h^{-1} \partial_{i} h + \frac{1}{2} f_{2}^{-1} \partial_{i} f_{2} + 2 f_{3}^{-1} \partial_{i} f_{3} \right) \]

\[ - \frac{1}{4} h^{-1} \partial_{i} h \left( \frac{5}{4} h^{-1} h' + \frac{1}{2} f_{2}^{-1} f_{2}' + 2 f_{3}^{-1} f_{3}' \right) \}

\[ + \left\{ \frac{5}{16} h^{-2} \partial_{i} h h' + \frac{1}{8} h^{-1} f_{2}^{-1} \left( \partial_{i} h f_{2} + \partial_{i} f_{2} h' \right) + \frac{1}{2} h^{-1} f_{3}^{-1} \left( \partial_{i} h f_{3}' + \partial_{i} f_{3} h' \right) \right\} \]

\[ + \frac{1}{4} f_{2}^{-2} \partial_{i} f_{2} f_{2} + f_{3}^{-2} \partial_{i} f_{3} f_{3}' \}

\[ + \left\{ \frac{1}{4} h^{-1} f_{3}^{-2} P^{-2} e^{-\phi} \partial_{i} K K' + \frac{1}{2} \partial_{i} \Phi \Phi' \right\} \]

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\[ R_{5\rho\rho} = \left\{ \frac{1}{3} \rho^{-2} h^{-1} f_2^{-1} f_3^{-2} P^2 e^\rho + \frac{1}{6} \rho^{-2} h^{-2} f_2^{-1} f_3^{-2} K^2 + \frac{1}{12} \rho^3 f_2^{-2} f_3^{-2} \left( \frac{1}{\sqrt{-G}} \left[ \sqrt{-G} \rho^{-3} f_3^{-6} h^{-5} [h^5 f_2^8 f_3^8] \right] \right) + \frac{1}{\sqrt{-G}} \partial_i \left[ \sqrt{-G} \rho^{-3} f_3^{-6} h^{-4} G_{ij} \partial_j [h^5 f_2^8 f_3^8] \right] \right. \\
\left. - \frac{3}{2} \rho^{-3} f_3^{-6} h^{-5} f_2^{-1} f_3' [h^5 f_2^8 f_3^8]' - \frac{3}{2} \rho^{-3} f_3^{-6} h^{-4} f_2^{-1} \partial f_2 \partial [h^5 f_2^8 f_3^8] \right) \\
- 8 \rho^{-2} f_3^{-1} + \frac{4}{3} \rho^2 f_2 f_3^{-2} \right\} \\
+ \left\{ \left[ \frac{5}{4} h^{-1} h' + \frac{1}{2} f_2^{-1} f_2' + 2 f_3^{-1} f_3 \right]' \right. \\
+ \left( \frac{5}{16} h^{-1} (\partial h)^2 + \frac{1}{8} f_2^{-1} \partial h \partial f_2 + \frac{1}{2} f_3^{-1} \partial h \partial f_3 \right) \\
- \left( \frac{5}{16} h^{-2} (h')^2 + \frac{1}{8} f_2^{-1} h^{-1} f_2' + \frac{1}{2} f_3^{-1} h^{-1} f_3' \right) \\
+ \left( \frac{5}{4} \rho^{-1} h^{-1} f_2' + \frac{1}{2} f_2^{-1} \rho^{-1} f_2' + 2 f_3^{-1} \rho^{-1} f_3' \right) \right\} \\
+ \left\{ \left( \frac{1}{4} h^{-1} h' + \frac{1}{2} f_2^{-1} f_2' \right)^2 + 4 \left( \frac{1}{4} h^{-1} h' + \frac{1}{2} f_3^{-1} f_3' \right)^2 \right\} \\
+ \left\{ \frac{1}{4} h^{-1} f_3^{-2} P^{-2} e^{-\Phi} (K')^2 + \frac{1}{2} (\Phi')^2 \right\} \tag{A.8}
\]

A.2 Coefficients of the asymptotic solution

A.2.1 Next-to-leading order solution, \( O(\rho^2) \)

To next-to-leading order, the solution with the ansatz \((2.29)-(2.34)\) depends on two undetermined functions, which we choose to be \( a^{(2,0)}(x) \) and \( a^{(2,1)}(x) \). We find that the other coefficients in the solution are then given by

\[ G^{(2,1)}_{ij} = -\frac{1}{24} \left( 12 a^{(2,1)} G^{(0)}_{ij} - P^2 p_0 \left( 6 R_{ij} - G^{(0)}_{ij} R \right) \right) \tag{A.9} \]

\[ G^{(2,0)}_{ij} = -\frac{1}{96} \left( 48 a^{(2,0)} G^{(0)}_{ij} + 24 a^{(2,1)} G^{(0)}_{ij} + (2 K_0 + P^2 p_0) \left( 6 R_{ij} - G^{(0)}_{ij} R \right) \right) \tag{A.10} \]

\[ p^{(2,1)} = 0 \tag{A.11} \]

\[ p^{(2,0)} = -\frac{1}{24} P^2 p_0 R \tag{A.12} \]

\[ h^{(2,2)} = P^2 p_0 a^{(2,1)} \tag{A.13} \]
where the Ricci tensor $R_{ij}$ and scalar $R$ are computed using the boundary metric $G_{ij}^{(0)}$.  

A.2.2 Next-to-next-to-leading order solution, $O(\rho^4)$

At this order, we have several arbitrary functions, which we choose to be $p^{(4,0)}$, $a^{(4,3)}$, $a^{(4,2)}$, $a^{(4,1)}$, $a^{(4,0)}$, $b^{(4,0)}$, and $G_{ij}^{(4,0)} - \frac{1}{4} G_{ij}^{(0)} G_{kij}^{(4,0)k}$ (with $G_{kij}^{(4,0)k} \equiv G_{ij}^{(4,0)} G_{0ij}$; note that $G_{kij}^{(4,0)k}$ is not arbitrary but will be determined below). On the other hand, in order to find a solution we find that the Laplacians of the second order coefficients $\{a^{(2,0)}, a^{(2,1)}\}$ are constrained to satisfy

\begin{align}
\square a^{(2,0)} &= -\frac{1}{2160}(161 P^2 p_0 + 78 K_0) \square R, \\
\square a^{(2,1)} &= \frac{13}{180} P^2 p_0 \square R,
\end{align}

(A.20)

where $\square$ is the Laplacian operator with the metric$^{26}$ $G_{ij}^{(0)}$. The constant parts of $a^{(2,0)}$ and $a^{(2,1)}$ remain unfixed, as expected. The remaining functions in the solution are then given by

\begin{align}
G_{ij}^{(4,3)} &= \frac{1}{96} P^4 p_0^2 \square R_{ij} - \frac{1}{288} P^4 p_0^2 \nabla_i \nabla_j R - \frac{1}{48} P^4 p_0^2 R_{ij} R_{ab} R^{ab} - \frac{1}{144} P^4 p_0^2 R_{ij} R \\
&+ G_{ij}^{(0)} \left( -\frac{1}{576} P^4 p_0^2 \square R - \frac{1}{192} P^4 p_0^2 R_{ab} R^{ab} - \frac{3}{4} a^{(4,3)} + \frac{1}{576} P^4 p_0^2 R^2 \right).
\end{align}

(A.21)

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$^{26}$We make a convention that $\square$ is evaluated with respect to $G_{ij}^{(0)}(x)$ whenever it acts on the coefficients of the perturbative solution (2.29)-(2.34), rather than on the supergravity fields. In the latter case the operator $\square$ is evaluated with respect to the full four dimensional metric $G_{ij}(x, \rho)$, as in the equations of motion (A.2)-(A.8).
\[ G_{ij}^{(4,2)} = -\frac{1}{64} P^2 p_0 \left( K_0 + P^2 p_0 \right) \Box R_{ij} + \frac{1}{576} P^2 p_0 \left( 3 K_0 + 4 P^2 p_0 \right) \nabla_i \nabla_j R \\
+ \frac{1}{32} P^2 p_0 \left( K_0 + P^2 p_0 \right) R_{a i j b} R^{a b} + \frac{1}{32} P^4 p_0^2 R_{i a} R^a_j - \frac{1}{16} P^2 p_0 \nabla_i \nabla_j a^{(2,1)} \\
+ R_{ij} \left( -\frac{1}{4} P^2 p_0 a^{(2,1)} + \frac{1}{288} P^2 p_0 \left( 3 K_0 - 2 P^2 p_0 \right) R \right) \\
+ G_{ij}^{(0)} \left( \frac{1}{1152} P^2 p_0 \left( 3 K_0 + 4 P^2 p_0 \right) \Box R + \frac{1}{128} P^2 p_0 \left( K_0 + P^2 p_0 \right) R_{a b} R^{a b} \right) \\
+ \frac{1}{16} \left( -12 a^{(4,2)} + 7 (a^{(2,1)})^2 - 3 a^{(4,3)} \right) + \frac{1}{24} P^2 p_0 a^{(2,1)} R \\
- \frac{1}{2304} P^2 p_0 \left( 6 K_0 + P^2 p_0 \right) R^2 \right) \]
\[ G_{ij}^{(4,1)} = \frac{1}{13824} \left( -36 K_0^2 - 96 K_0 P^2 p_0 - 149 P^4 p_0^2 \right) \nabla_i \nabla_j R \\
+ \frac{1}{512} \left( 4 K_0^2 + 8 K_0 P^2 p_0 + 5 P^4 p_0^2 \right) \Box R_{ij} \\
+ \frac{1}{256} \left( -4 K_0^2 - 8 K_0 P^2 p_0 - 5 P^4 p_0^2 \right) R_{a i j b} R^{a b} \\
- \frac{1}{64} P^2 p_0 \left( 2 K_0 + P^2 p_0 \right) R_{i a} R^a_j \\
+ \frac{1}{64} \left( -4 P^2 p_0 \nabla_i \nabla_j a^{(2,0)} + \left( 2 K_0 + 5 P^2 p_0 \right) \nabla_i \nabla_j a^{(2,1)} \right) \\
+ R_{ij} \left( \frac{1}{8} \left( -2 P^2 p_0 a^{(2,0)} + K_0 a^{(2,1)} \right) + \frac{1}{6912} \left( -36 K_0^2 + 48 K_0 P^2 p_0 + 79 P^4 p_0^2 \right) R \right) \\
+ G_{ij}^{(0)} \left( \frac{1}{27648} \left( -36 K_0^2 - 96 K_0 P^2 p_0 - 119 P^4 p_0^2 \right) \Box R \\
- \frac{1}{3072} \left( 12 K_0^2 + 24 K_0 P^2 p_0 + P^4 p_0^2 \right) R_{a b} R^{a b} \right) \\
+ \frac{1}{32} \left( -24 a^{(4,1)} - 4 a^{(4,2)} + 28 a^{(2,0)} a^{(2,1)} + 7 (a^{(2,1)})^2 + 3 a^{(4,3)} \right) \\
+ \frac{1}{48} \left( 2 P^2 p_0 a^{(2,0)} - K_0 a^{(2,1)} \right) R + \frac{1}{27648} \left( 36 K_0^2 + 12 K_0 P^2 p_0 - 85 P^4 p_0^2 \right) R^2 \right) \]
\[ C^{(4,0)}_k = \frac{1}{27648} P^2 p_0 \left( -6 K_0 + 31 P^2 p_0 \right) R \]
\[ + \frac{1}{3072} \left( 12 K_0^2 + 12 K_0 P^2 p_0 - 5 P^4 p_0^2 \right) R_{ab} R^{ab} + \left( \frac{1}{256} (448 a^{(2,0)})^2 - 64 a^{(4,1)} \right) \]
\[ + 32 a^{(4,2)} + 224 a^{(2,0)} a^{(2,1)} + 8 (a^{(2,1)})^2 - 24 a^{(4,3)} - 768 b^{(4,0)} + 8 K_0 a^{(2,0)} \]
\[ + 8 P^2 p_0 \left[ a^{(2,0)} - 6 K_0 a^{(2,1)} - 7 P^2 p_0 a^{(2,1)} \right) \]
\[ + \frac{1}{384} \left( 2 (8 K_0 - 3 P^2 p_0) a^{(2,0)} + P^2 p_0 a^{(2,1)} \right) R \]
\[ + \frac{1}{27648} \left( -24 K_0^2 - 54 K_0 P^2 p_0 - 37 P^4 p_0^2 \right) R^2 \] (A.24)

\[ p^{(4,3)} = \frac{1}{576} P^4 p_0^2 \left( \Box R + 3 R_{ab} R^{ab} - R^2 \right) \] (A.25)

\[ p^{(4,2)} = -\frac{1}{768} P^2 p_0 \left( 2 K_0 + 11 P^2 p_0 \right) \Box R - \frac{1}{256} P^2 p_0 \left( 2 K_0 + P^2 p_0 \right) R_{ab} R^{ab} \]
\[ + \frac{1}{768} P^2 p_0 \left( 2 K_0 + P^2 p_0 \right) R^2 \] (A.26)

\[ p^{(4,1)} = \frac{1}{27648} P^2 p_0 \left( 138 K_0 + 373 P^2 p_0 \right) \Box R \]
\[ + \frac{1}{4608} P^2 p_0 \left( 42 K_0 - 19 P^2 p_0 \right) R_{ab} R^{ab} + 3 \left( a^{(4,0)} - b^{(4,0)} \right) \] (A.27)
\[ + \frac{1}{384} P^2 p_0 \left( -6 a^{(2,0)} + 5 a^{(2,1)} \right) R + \frac{1}{27648} P^2 p_0 \left( -102 K_0 + 59 P^2 p_0 \right) R^2 \]
\[ h^{(4,4)} = P^2 p_0 a^{(4,3)} \] (A.28)

\[ h^{(4,3)} = -\frac{1}{512} P^6 p_0^3 \Box R - \frac{3}{512} P^6 p_0^3 R_{ab} R^{ab} + P^2 p_0 a^{(4,2)} + \frac{1}{512} P^6 p_0^3 R^2 \]
\[ + \frac{1}{16} \left( -24 P^2 p_0 (a^{(2,1)})^2 - (8 K_0 + 3 P^2 p_0) a^{(4,3)} \right) \] (A.29)

\[ h^{(4,2)} = \frac{1}{110592} P^4 p_0^2 \left( 324 K_0 + 965 P^2 p_0 \right) \Box R + \frac{25}{576} P^4 p_0^2 a^{(2,1)} R \]
\[ + \frac{1}{36864} P^4 p_0^2 \left( 324 K_0 + 23 P^2 p_0 \right) R_{ab} R^{ab} + \frac{1}{12288} P^4 p_0^2 \left( -36 K_0 + 5 P^2 p_0 \right) R^2 \]
\[ + \frac{1}{64} \left( 64 P^2 p_0 a^{(4,1)} - 4 (8 K_0 + 3 P^2 p_0) a^{(4,2)} - 192 P^2 p_0 a^{(2,0)} a^{(2,1)} \right. \]
\[ \left. + 48 K_0 (a^{(2,1)})^2 + 5 P^2 p_0 (a^{(2,1)})^2 - 3 P^2 p_0 a^{(4,3)} \right) \] (A.30)
\[ h^{(4,1)} = -\frac{1}{442368} P^2 p_0 \left( 408 K_0^2 + 2140 K_0 P^2 p_0 + 2259 P^4 p_0^2 \right) R \\
+ \frac{1}{147456} P^2 p_0 \left( -408 K_0^2 - 412 K_0 P^2 p_0 + 11 P^4 p_0^2 \right) R_{ab} R^{ab} \\
+ \frac{1}{128} \left( -192 P^2 p_0 (a^{(2,0)})^2 - 32 P^2 p_0 a^{(4,0)} - 64 K_0 a^{(4,1)} - 24 P^2 p_0 a^{(4,1)} \\
- 4 P^2 p_0 a^{(4,2)} + 4 \left( 48 K_0 + 5 P^2 p_0 \right) a^{(2,0)} a^{(2,1)} + 19 P^2 p_0 (a^{(2,1)})^2 + 3 P^2 p_0 a^{(4,3)} \\
+ 160 P^2 p_0 b^{(4,0)} \right) - \frac{1}{13824} 5 P^2 p_0 \left( -138 P^2 p_0 a^{(2,0)} + \left( 60 K_0 + 101 P^2 p_0 \right) a^{(2,1)} \right) R \\
+ \frac{1}{1327104} P^2 p_0 \left( 1224 K_0^2 + 780 K_0 P^2 p_0 - 1057 P^4 p_0^2 \right) R^2 \] (A.31)

\[ h^{(4,0)} = \frac{1}{3538944} P^2 p_0 \left( 480 K_0^2 + 3244 K_0 P^2 p_0 + 4657 P^4 p_0^2 \right) R \\
+ \frac{1}{1179648} P^2 p_0 \left( 600 K_0^2 + 908 K_0 P^2 p_0 - 59 P^4 p_0^2 \right) R_{ab} R^{ab} \\
+ \frac{1}{512} \left( 8 \left( 48 K_0 + 5 P^2 p_0 \right) (a^{(2,0)})^2 + 8 \left( 2 K_0 + 13 P^2 p_0 \right) a^{(4,0)} - 8 P^2 p_0 a^{(4,1)} \\
+ 4 P^2 p_0 a^{(4,2)} + 76 P^2 p_0 a^{(2,0)} a^{(2,1)} - 3 P^2 p_0 (a^{(2,1)})^2 - 3 P^2 p_0 a^{(4,3)} \\
- 272 K_0 b^{(4,0)} - 200 P^2 p_0 b^{(4,0)} - 32 P^2 p_0 p^{(4,0)} \right) \\
+ \frac{1}{221184} P^2 p_0 \left( -2 \left( 2706 K_0 + 4001 P^2 p_0 \right) a^{(2,0)} + \left( 174 K_0 + 31 P^2 p_0 \right) a^{(2,1)} \right) R \\
+ \frac{1}{10616832} P^2 p_0 \left( -1392 K_0^2 + 220 K_0 P^2 p_0 + 5231 P^4 p_0^2 \right) R^2 \] (A.32)

\[ b^{(4,3)} = a^{(4,3)} \] (A.33)

\[ b^{(4,2)} = -\frac{1}{576} P^4 p_0^2 \Box R - \frac{1}{192} P^4 p_0^2 R_{ab} R^{ab} + a^{(4,2)} + \frac{1}{576} P^4 p_0^2 R^2 \] (A.34)

\[ b^{(4,1)} = \frac{1}{6912} P^2 p_0 \left( 12 K_0 + 43 P^2 p_0 \right) \Box R + \frac{1}{576} P^2 p_0 \left( 3 K_0 - 2 P^2 p_0 \right) R_{ab} R^{ab} \\
- a^{(4,1)} - \frac{1}{192} P^2 p_0 a^{(2,1)} R + \frac{1}{6912} P^2 p_0 \left( -12 K_0 + 11 P^2 p_0 \right) R^2 \] (A.35)

\[ K^{(4,3)} = -\frac{1}{288} P^6 p_0^3 \Box R - \frac{1}{96} P^6 p_0^3 R_{ab} R^{ab} + \frac{1}{4} P^2 p_0 a^{(4,3)} + \frac{1}{288} P^6 p_0^3 R^2 \] (A.36)
\[ K^{(4,2)} = \frac{1}{192} P^4 p_0^2 \left( K_0 + 4 P^2 p_0 \right) \Box R + \frac{1}{256} P^4 p_0^2 \left( 4 K_0 + 3 P^2 p_0 \right) R_{ab} R^{ab} \]
\[ + \frac{1}{16} P^2 p_0 \left( 4 a^{(4,2)} - 3 (a^{(2,1)})^2 + a^{(4,3)} \right) + \frac{1}{48} P^4 p_0^2 a^{(2,1)} R \]
\[ - \frac{1}{384} P^4 p_0^2 \left( 2 K_0 + P^2 p_0 \right) R^2 \]
\[ K^{(4,1)} = -\frac{1}{13824} P^2 p_0 \left( 18 K_0^2 + 159 K_0 P^2 p_0 + 308 P^4 p_0^2 \right) \Box R \]
\[ - \frac{1}{9072} P^2 p_0 \left( 36 K_0^2 + 156 K_0 P^2 p_0 + 7 P^4 p_0^2 \right) R_{ab} R^{ab} \]
\[ + \frac{1}{32} P^2 p_0 \left( -96 a^{(4,0)} + 8 a^{(4,1)} - 4 a^{(2,1)} a^{(2,0)} + 3 a^{(2,1)} a^{(4,0)} \right) + \frac{1}{384} P^2 p_0 \left( 14 P^2 p_0 a^{(2,0)} - (4 K_0 + 11 P^2 p_0) a^{(2,1)} \right) R \]
\[ + \frac{1}{13824} P^2 p_0 \left( 18 K_0^2 + 69 K_0 P^2 p_0 - 29 P^4 p_0^2 \right) R^2 \]
\[ K^{(4,0)} = \frac{1}{221184} P^2 p_0 \left( 132 K_0^2 + 944 K_0 P^2 p_0 + 1371 P^4 p_0^2 \right) \Box R \]
\[ + \frac{1}{36864} P^2 p_0 \left( 84 K_0^2 + 124 K_0 P^2 p_0 - 21 P^4 p_0^2 \right) R_{ab} R^{ab} \]
\[ + \frac{1}{128} \left( -24 P^2 p_0 (a^{(2,0)})^2 + 16 \left( 6 K_0 + 11 P^2 p_0 \right) a^{(4,0)} - 8 P^2 p_0 a^{(4,1)} + 4 P^2 p_0 a^{(4,2)} \right) \]
\[ + 12 P^2 p_0 a^{(2,1)} - 3 P^2 p_0 (a^{(2,1)})^2 - 3 P^2 p_0 a^{(4,3)} - 96 K_0 b^{(4,0)} - 144 P^2 p_0 b^{(4,0)} \]
\[ - 64 P^2 p_0 b^{(4,0)} \right) - \frac{1}{3072} P^2 p_0 \left( 2 K_0 + 3 P^2 p_0 \right) \left( 22 a^{(2,0)} - 5 a^{(2,1)} \right) R \]
\[ + \frac{1}{663552} P^2 p_0 \left( -396 K_0^2 - 252 K_0 P^2 p_0 + 713 P^4 p_0^2 \right) R^2 \]
\[ (A.37) \]

where all the differential operators are evaluated with respect to \( G_{ij}^{(0)} \).

### A.3 Symmetries of the asymptotic solution

There are three symmetries we can use to check our general solution to (A.2)-(A.8).

- First, there is a scaling symmetry

\[ \rho \to \lambda \rho, \quad G_{ij} \to \lambda^2 G_{ij}, \quad h \to h \]
\[ K \to K, \quad \Phi \to \Phi, \quad f_2 \to f_2, \quad f_3 \to f_3 \]
\[ (A.40) \]

which translates into the following scaling symmetry of the parameters of the asymptotic solution (2.29)-(2.34)

\[ p_0 \to p_0 \]
\[ (A.41) \]
\( K_0 \rightarrow K_0 + 2P^2p_0 \ln \lambda \)  
\( \text{(A.42)} \)

\[ G_{ij}^{(0)} \rightarrow \lambda^2 G_{ij}^{(0)} \]  
\( \text{(A.43)} \)

\[ a^{(2,1)} \rightarrow \frac{1}{\lambda^2} a^{(2,1)} \]  
\( \text{(A.44)} \)

\[ a^{(2,0)} \rightarrow \frac{1}{\lambda^2} (a^{(2,0)} - a^{(2,1)} \ln \lambda) \]  
\( \text{(A.45)} \)

\[ a^{(4,3)} \rightarrow \frac{1}{\lambda^4} a^{(4,3)} \]  
\( \text{(A.46)} \)

\[ a^{(4,2)} \rightarrow \frac{1}{\lambda^4} (a^{(4,2)} - 3a^{(4,3)} \ln \lambda) \]  
\( \text{(A.47)} \)

\[ a^{(4,1)} \rightarrow \frac{1}{\lambda^4} (a^{(4,1)} - 2a^{(4,2)} \ln \lambda + 3a^{(4,3)} \ln^2 \lambda) \]  
\( \text{(A.48)} \)

\[ a^{(4,0)} \rightarrow \frac{1}{\lambda^4} (a^{(4,0)} - \ln \lambda (a^{(4,1)} + \ln \lambda (-a^{(4,2)} + a^{(4,3)} \ln \lambda))) \]  
\( \text{(A.49)} \)

\[ b^{(4,0)} \rightarrow \frac{1}{\lambda^4} b^{(4,0)} - \frac{1}{\lambda^4} a^{(4,1)} \ln \lambda - \frac{\ln \lambda}{576 \lambda^4} P^2 p_0 (3K_0 - 2P^2p_0 + 3P^2p_0 \ln \lambda) R_{ab} R^{ab} \] 
\[- \frac{\ln \lambda}{6912 \lambda^4} P^2 p_0 (12K_0 + 43P^2p_0 + 12P^2p_0 \ln \lambda) \Box R \] 
\[ + \frac{\ln \lambda^2}{\lambda^4} (a^{(4,2)} - a^{(4,3)} \ln \lambda) + \frac{\ln \lambda}{192 \lambda^4} P^2 p_0 a^{(2,1)} R \] 
\[ + \frac{\ln \lambda}{6912 \lambda^4} P^2 p_0 (12K_0 - 11P^2p_0 + 12P^2p_0 \ln \lambda) R^2 \]  
\( \text{(A.50)} \)
\[ G^{(4,0)}_{ij} \rightarrow \frac{1}{\lambda^2} G^{(4,0)}_{ij} + \frac{\ln \lambda}{\lambda^2} \left( \frac{1}{16} P^2 p_0 \nabla_i \nabla_j a^{(2,0)} + \frac{1}{64} (-2 K_0 - 5 P^2 p_0) \nabla_i \nabla_j a^{(2,1)} \right) \]
\[ + \frac{1}{512} (-4 K_0^2 - 8 K_0 P^2 p_0 - 5 P^4 p_0^2) \nabla R_{ij} + \frac{1}{8} \left( 2 P^2 p_0 a^{(2,0)} - K_0 a^{(2,1)} \right) R_{ij} \]
\[ + \frac{1}{27648} \left( 36 K_0^2 + 96 K_0 P^2 p_0 + 119 P^4 p_0^2 \right) G^{(0)}_{ij} \nabla R \]
\[ + \frac{1}{13824} \left( 36 K_0^2 + 96 K_0 P^2 p_0 + 149 P^4 p_0^2 \right) \nabla_i \nabla_j R \]
\[ + \frac{1}{3072} \left( 12 K_0^2 + 24 K_0 P^2 p_0 + P^4 p_0^2 \right) G^{(0)}_{ij} R_{ab} R^{ab} \]
\[ + \frac{1}{256} \left( 4 K_0^2 + 8 K_0 P^2 p_0 + 5 P^4 p_0^2 \right) R_{iab} R^{ab} + \frac{1}{64} P^2 p_0 \left( 2 K_0 + P^2 p_0 \right) R_{ia} R_j^a \]
\[ + \frac{1}{27648} \left( -36 K_0^2 - 12 K_0 P^2 p_0 + 85 P^4 p_0^2 \right) G^{(0)}_{ij} R^2 \]
\[ + R \left( \frac{1}{6912} \left( 36 K_0^2 - 48 K_0 P^2 p_0 - 79 P^4 p_0^2 \right) R_{ij} \right) \]
\[ + \frac{1}{48} G^{(0)}_{ij} \left( -2 P^2 p_0 a^{(2,0)} + K_0 a^{(2,1)} \right) \right) + \frac{1}{32} \left( 24 a^{(4,1)} + 4 a^{(4,2)} - 28 a^{(2,0)} a^{(2,1)} \right) \]
\[ - 7 a^{(2,1)^2} - 3 a^{(4,3)} G^{(0)}_{ij} + \frac{\ln^2 \lambda}{\lambda^2} \left( - \frac{1}{16} P^2 p_0 \nabla_i \nabla_j a^{(2,1)} \right) \]
\[ - \frac{1}{64} P^2 p_0 \left( K_0 + P^2 p_0 \right) \nabla R_{ij} + \frac{1}{576} P^2 p_0 \left( 3 K_0 + 4 P^2 p_0 \right) \]
\[ + \frac{1}{1152} P^2 p_0 \left( 3 K_0 + 4 P^2 p_0 \right) G^{(0)}_{ij} \nabla R + \frac{1}{128} P^2 p_0 \left( K_0 + P^2 p_0 \right) G^{(0)}_{ij} R_{ab} R^{ab} \]
\[ + \frac{1}{32} P^2 p_0 \left( K_0 + P^2 p_0 \right) R_{iab} R^{ab} + \frac{1}{64} P^4 p_0^2 R_{ia} R_j^a \]
\[ - \frac{1}{2304} P^2 p_0 \left( 6 K_0 + P^2 p_0 \right) G^{(0)}_{ij} R^2 - \frac{1}{4} P^2 p_0 a^{(2,1)} R_{ij} \]
\[ + R \left( \frac{1}{288} P^2 p_0 \left( 3 K_0 - 2 P^2 p_0 \right) R_{ij} + \frac{1}{24} P^2 p_0 a^{(2,1)} G^{(0)}_{ij} \right) \]
\[ + \frac{1}{16} \left( -12 a^{(4,2)} + 7 a^{(2,1)^2} - 3 a^{(4,3)} \right) G^{(0)}_{ij} \right) + \frac{\ln^3 \lambda}{\lambda^2} \left( - \frac{1}{96} P^4 p_0^2 \nabla R_{ij} \right) \]
\[ + \frac{1}{288} P^4 p_0^2 \nabla_i \nabla_j R \right) + \frac{1}{576} G^{(0)}_{ij} P^4 p_0^2 \nabla R + \frac{1}{192} P^4 p_0^2 G^{(0)}_{ij} R_{ab} R^{ab} \]
\[ + \frac{1}{48} P^4 p_0^2 R_{iab} R^{ab} + \frac{1}{144} P^4 p_0^2 R R_{ij} - \frac{1}{576} P^4 p_0^2 G^{(0)}_{ij} R^2 + \frac{3}{4} a^{(4,3)} G^{(0)}_{ij} \right) \]
\[ (A.51) \]
\[ p^{(4,0)} \rightarrow - \frac{\ln \lambda}{4608 \lambda^4} \left( 42 K_0 - 19 P^2 p_0 + 18 (2 K_0 + P^2 p_0) \ln \lambda + 24 P^2 p_0 \ln^2 \lambda \right) + 24 P^2 p_0 \ln^2 \lambda \bigg] \]

\[ + \frac{\ln \lambda}{27648 \lambda^4} \left( 138 K_0 + 373 P^2 p_0 \right) R + \frac{p^{(4,0)}}{\lambda^4} \]

\[ + 36 (2 K_0 + 11 P^2 p_0) \ln \lambda + 48 P^2 p_0 \ln \lambda^2 \bigg] P^2 p_0 R + \frac{p^{(4,0)}}{\lambda^4} \]

\[ + \frac{3 \ln \lambda}{\lambda^4} \left( b^{(4,0)} - a^{(4,0)} \right) + \frac{\ln \lambda}{\lambda^4} \left( \frac{1}{64} P^2 p_0 a^{(2,0)} - \frac{5}{384} P^2 p_0 a^{(2,1)} \right) R \]

\[ + \frac{\ln \lambda}{27648 \lambda^4} \left( 102 K_0 - 59 P^2 p_0 + 36 (2 K_0 + P^2 p_0) \ln \lambda \right) + 48 P^2 p_0 \ln^2 \lambda \bigg] P^2 p_0 R^2 \]

The solution transforms covariantly under this transformation, and in addition, the constraints (A.20) and (A.24) are also invariant under the symmetry transformations (A.40).

Another symmetry is the residual gauge freedom associated with unfixed diffeomorphisms

\[ \rho \rightarrow \hat{\rho} \equiv \rho \left[ 1 + \rho^2 \left( \delta_{20} + \delta_{21} \ln \rho \right) + \rho^4 \left( \delta_{40} + \delta_{41} \ln \rho + \delta_{42} \ln^2 \rho + \delta_{43} \ln^3 \rho \right) \right] \]

\[ K \rightarrow K, \quad \Phi \rightarrow \Phi, \quad h \rightarrow h \left( \frac{\hat{\rho}}{\rho} \right)^4 \left( \frac{\partial \hat{\rho}}{\partial \rho} \right)^{-4}, \quad f_2 \rightarrow f_2 \left( \frac{\hat{\rho}}{\rho} \right)^{-2} \left( \frac{\partial \hat{\rho}}{\partial \rho} \right)^{2} \]

\[ f_3 \rightarrow f_3 \left( \frac{\hat{\rho}}{\rho} \right)^{-2} \left( \frac{\partial \hat{\rho}}{\partial \rho} \right)^{2}, \quad G_{ij} \rightarrow G_{ij} \left( \frac{\hat{\rho}}{\rho} \right)^4 \left( \frac{\partial \hat{\rho}}{\partial \rho} \right)^{-2} \]
where \( \{ \delta_{20}, \cdots, \delta_{43} \} \) are arbitrary constants. This symmetry is realized on the parameters of the asymptotic solution as follows

\[
\begin{align*}
p_0 &\rightarrow p_0 \quad \text{(A.54)} \\
K_0 &\rightarrow K_0 \quad \text{(A.55)} \\
G_{ij}^{(0)} &\rightarrow G_{ij}^{(0)} \quad \text{(A.56)} \\
a^{(2,0)} &\rightarrow 4\delta_{20} + 2\delta_{21} + a^{(2,0)} \quad \text{(A.57)} \\
a^{(2,1)} &\rightarrow 4\delta_{21} + a^{(2,1)} \quad \text{(A.58)} \\
a^{(4,3)} &\rightarrow 8\delta_{43} + a^{(4,3)} \quad \text{(A.59)} \\
a^{(4,2)} &\rightarrow -8\delta_{21}^2 + 8\delta_{42} + 6\delta_{43} + a^{(4,2)} + 2\delta_{21} a^{(2,1)} \quad \text{(A.60)} \\
a^{(4,1)} &\rightarrow -16\delta_{20}\delta_{21} - 6\delta_{21}^2 + 8\delta_{41} + 4\delta_{42} + 2\delta_{21} a^{(2,0)} + a^{(4,1)} + 2\delta_{20} a^{(2,1)} + \delta_{21} a^{(2,1)} \quad \text{(A.61)} \\
a^{(4,0)} &\rightarrow -8\delta_{20}^2 - 6\delta_{20}\delta_{21} + \delta_{21}^2 + 8\delta_{40} + 2\delta_{41} + 2(\delta_{20} + \delta_{21}) a^{(2,0)} + a^{(4,0)} - \delta_{20} a^{(2,1)} \quad \text{(A.62)} \\
b^{(4,0)} &\rightarrow -8\delta_{20}^2 - 6\delta_{20}\delta_{21} + \delta_{21}^2 + 8\delta_{40} + 2\delta_{41} + 2(\delta_{20} + \delta_{21}) a^{(2,0)} + b^{(4,0)} - \delta_{20} a^{(2,1)} - \frac{1}{48} P^2 p_0 (\delta_{20} + \delta_{21}) R \quad \text{(A.63)} \\
p^{(4,0)} &\rightarrow p^{(4,0)} + \frac{1}{12} P^2 p_0 \delta_{20} R \quad \text{(A.64)} \\
G_{ij}^{(4,0)} &\rightarrow G_{ij}^{(4,0)} + \frac{1}{8} \left( P^2 p_0 \delta_{21} + 2 K_0 (2\delta_{20} + \delta_{21}) \right) R_{ij} + G_{ij}^{(0)} \left( -\frac{1}{48} (P^2 p_0 \delta_{21} + 2 K_0 (2\delta_{20} + \delta_{21})) R + (2\delta_{20} + \delta_{21}) a^{(2,0)} + 13\delta_{20}^2 + 16\delta_{20}\delta_{21} + 3\delta_{21}^2 - 6\delta_{40} - 2\delta_{41} + \frac{1}{2} (3\delta_{20} + \delta_{21}) a^{(2,1)} \right) \quad \text{(A.65)}
\end{align*}
\]

Again, in addition to the solution transforming covariantly, the constraints (A.20) and (A.24) are also invariant under the symmetry transformations (A.53).

- Finally, there is a symmetry which rescales the action (and thus is a symmetry of any solution to the equations of motion) given by (2.36), which in terms of our variables is given by

\[
\begin{align*}
G_{ij} &\rightarrow \beta G_{ij}, \quad h \rightarrow \beta h, \quad e^\Phi \rightarrow \beta e^\Phi, \\
K &\rightarrow \beta K, \quad f_2 \rightarrow f_2, \quad f_3 \rightarrow f_3. \quad \text{(A.66)}
\end{align*}
\]

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This symmetry is realized on the parameters of the asymptotic solution as follows

\[ p_0 \to \beta p_0, \quad K_0 \to \beta K_0, \quad G_{ij}^{(0)} \to \beta G_{ij}^{(0)}, \quad G_{ij}^{(4,0)} \to \beta G_{ij}^{(4,0)}, \] (A.67)

with all other parameters remaining unchanged. Again, in addition to the solution transforming covariantly, the constraints (A.20) and (A.24) are also invariant under the symmetry transformations (A.66). The symmetry transformation (2.36) is very useful in constraining the possible counter-terms in the holographic renormalization of the cascading gauge theories, as discussed in section 3.3.

### A.4 Ambiguities of the minimal subtraction

In this appendix we discuss a specific simple ambiguity of the counter-term action, which is present in our minimal subtraction ansatz.

Consider a counter-term ansatz of the form

\[ \delta L_{\text{counter}} = \delta L_1(K, P^2 e^\Phi) \left( R_\gamma^2 - 3 R_{ab} \gamma^{\gamma ab} \right) + \delta L_2(K, P^2 e^\Phi) \square \mathcal{R}. \] (A.68)

This ansatz was chosen so that it does not contribute to \( \langle T_i^i \rangle \), and it is easy to verify that (since \( \alpha_8 \) is a source only for \( h \)) it also does not contribute to \( \langle \mathcal{O}_8 \rangle \).

It is straightforward to verify that (A.68) leads to

\[ \langle \delta \mathcal{O}_{p_0} \rangle = \frac{\partial \delta L_1}{\partial p_0} \left( R^2 - 3 R_{ab} R^{ab} \right) + \frac{\partial \delta L_2}{\partial p_0} \square R, \] (A.69)

\[ \langle \delta \mathcal{O}_{K_0} \rangle = \frac{\partial \delta L_1}{\partial K_0} \left( R^2 - 3 R_{ab} R^{ab} \right) + \frac{\partial \delta L_2}{\partial K_0} \square R. \] (A.70)

From (A.69), (A.70) it follows that \( \langle \delta \mathcal{O}_{p_0} \rangle \) and \( \langle \delta \mathcal{O}_{K_0} \rangle \) will be finite if the \( \delta L_i \) are finite in the limit \( \rho \to 0 \). Under the symmetry (2.36) we must have the scaling

\[ \delta L_i \to \beta^2 \delta L_i. \] (A.71)

The requirement of finiteness in the \( \rho \to 0 \) limit, along with (A.71), lead to the following choice of counter-terms :

\[ \delta L_1 = \kappa_1 \left( K + 2 P^2 e^\Phi \ln \rho \right)^2 + \kappa_2 \left( K + 2 P^2 e^\Phi \ln \rho \right) P^2 e^\Phi + \kappa_3 P^4 e^{2\Phi}, \] \[ \delta L_2 = \kappa_4 \left( K + 2 P^2 e^\Phi \ln \rho \right)^2 + \kappa_5 \left( K + 2 P^2 e^\Phi \ln \rho \right) P^2 e^\Phi + \kappa_6 P^4 e^{2\Phi}. \] (A.72)

The arguments above show that the only possible divergences arising from the counter-terms (A.72) are in \( \langle \mathcal{O}_6^e \rangle \). We find that these counter-terms indeed lead to
extra divergences in $\langle O_6^6 \rangle$, but they can be removed (preserving what has been achieved thus far) by

$$\delta L_{\text{counter}} \to \delta L_{\text{counter}} + \delta L_{\text{extra}} \left( R^2_{\gamma} - 3 R_{ab} \gamma R^{ab}_{\gamma} \right) + \delta L_{\text{extra}} \Box R, \quad (A.73)$$

with

$$\delta L_{1\text{extra}} = X_a \left( 4 \kappa_1 \left( K + 2 P^2 e^\Phi \ln \rho \right) P^2 e^\Phi + \frac{2}{5} \kappa_2 P^4 e^{2\Phi} \right),$$

$$\delta L_{2\text{extra}} = X_a \left( 4 \kappa_4 \left( K + 2 P^2 e^\Phi \ln \rho \right) P^2 e^\Phi + \frac{2}{5} \kappa_5 P^4 e^{2\Phi} \right). \quad (A.74)$$

Thus, we can always add to our action the counter-terms (A.72) and (A.73) without generating any divergences.

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