TRILINEAR FORMS WITH DOUBLE KLOOSTERMAN SUMS

IGOR E. SHPARLINSKI

Abstract. We obtain several estimates for trilinear form with double Kloosterman sums. In particular, these bounds show the existence of nontrivial cancellations between such sums.

1. Introduction

1.1. Background and motivation. Let $q$ be a positive integer. We denote the residue ring modulo $q$ by $\mathbb{Z}_q$ and denote the group of units of $\mathbb{Z}_q$ by $\mathbb{Z}_q^*$. For integers $\ell$, $m$ and $n$ we define the double Kloosterman sum

$$K_q(\ell, m, n) = \sum_{x, y \in \mathbb{Z}_q^*} e_q(\ell xy + mx + ny).$$

where $\overline{x}$ is the multiplicative inverse of $x$ modulo $q$ and

$$e_q(z) = \exp(2\pi i z/q).$$

Given three sets

$$\mathcal{L} = \{u + 1, \ldots, u + L\},$$
$$\mathcal{M} = \{v + 1, \ldots, v + M\},$$
$$\mathcal{N} = \{w + 1, \ldots, w + N\},$$

of $L$, $M$, $N$ consecutive integers and a sequence of weights $\alpha = \{\alpha_\ell\}_{\ell \in \mathcal{L}}$, we define the weighted triple sums of double Kloosterman sums

$$S_q(\alpha; \mathcal{L}, \mathcal{M}, \mathcal{N}) = \sum_{\ell \in \mathcal{L}} \alpha_\ell \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} K_q(\ell, m, n).$$

Assuming that $\alpha_\ell = 0$ if $\gcd(\ell, q) > 1$ and using the Weil bound [9, Equation (11.58)], one can easily obtain

$$|S_q(\alpha; \mathcal{L}, \mathcal{M}, \mathcal{N})| \leq MNq^{1+o(1)} \sum_{\ell \in \mathcal{L}} |\alpha_\ell|,$$

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which in the case $|\alpha_\ell| \leq 1$ takes form
\begin{equation}
|S_q(\alpha; \mathcal{L}, \mathcal{M}, \mathcal{N})| \leq L M N q^{1+o(1)}.
\end{equation}

We are interested in studying cancellations amongst Kloosterman sums and thus in improvements of the trivial bound (1.1). This question is partially motivated by a series of recent results concerning various bilinear forms with single Kloosterman sums
\[ K_q(m, n) = \sum_{x,y \in \mathbb{Z}_q^*} e_q(mx + nx), \]
see [1, 2, 5, 11, 12, 13] and references therein for various approaches, and also for generalisation to bilinear forms with more general quantities. The triple sums $S_q(\alpha; \mathcal{L}, \mathcal{M}, \mathcal{N})$ seems to be a new object of study.

1.2. Results. Here we use some ideas from [12, 13] to improve the trivial bound (1.1). Although the approach works in larger generality, to exhibit it in a simplest form we assume that weights $\alpha$ supported only on $\ell \in \mathbb{Z}_q^*$, that is, that $\alpha_\ell = 0$ if $\gcd(\ell, q) = 1$.

**Theorem 1.1.** For any integer $q \geq 1$, and weights $\alpha = \{\alpha_\ell\}_{\ell \in \mathcal{L}}$ with $|\alpha_\ell| \leq 1$ and supported only on $\ell \in \mathbb{Z}_q^*$, we have,
\[ |S_q(\alpha; \mathcal{L}, \mathcal{M}, \mathcal{N})| \leq \min\left\{ \left(L + L^{1/2} M^{1/2}\right) N^{1/2} q^{3/2}, \left(L + L^{3/4} M^{1/4}\right) \left(N^{1/8} q^{7/4} + N^{1/2} q^{3/2}\right) \right\} q^{o(1)}. \]

**Theorem 1.2.** For any fixed real $\varepsilon > 0$ and integer $r \geq 2$, for any sufficiently large $Q \geq 1$, for all but at most $Q^{1-2r\varepsilon+o(1)}$ integers $q \in [Q, 2Q]$ and weights $\alpha_q = \{\alpha_{q, \ell}\}_{\ell \in \mathcal{L}}$, that may depend on $q$, with $|\alpha_{q, \ell}| \leq 1$ and supported only on $\ell \in \mathbb{Z}_q^*$, we have,
\[ |S_q(\alpha_q; \mathcal{L}, \mathcal{M}, \mathcal{N})| \leq \left(L + L^{1-1/2r} M^{1/2r}\right) \left(q^{2-1/2r} + N^{1/2} q^{3/2}\right) q^{o(1)}. \]

Clearly, the roles of $M$ and $N$ can be interchanged in Theorems 1.1 and 1.2.

Now, assuming that $N \leq q^{2/3+o(1)}$ we have $N^{1/8} q^{7/4} \geq N^{1/2} q^{3/2+o(1)}$. Hence by Theorems 1.1 we have
\[ |S_q(\alpha; \mathcal{L}, \mathcal{M}, \mathcal{N})| \leq \left(L + L^{3/4} M^{1/4}\right) N^{1/8} q^{7/4+o(1)}, \]
which improves (1.1) for
\[ N \leq q^{2/3+o(1)} \quad \text{and} \quad \min\{L, M\} M^3 N^{7/2} \geq q^{3+\varepsilon} \]
for some fixed $\varepsilon > 0$. Thus in the symmetric case when $L \sim M \sim N$ this condition becomes $q^{2/3+o(1)} \geq L \geq q^{2/5+\varepsilon}$. 
1.3. **Possible generalisations and open problems.** Analysing the proofs of Theorems 1.1 and 1.2 one can easily see that they can be extended to more general sums of the form

\[ \sum_{x,y \in \mathbb{Z}_q^*} \eta_x \kappa_y \exp(\ell xy + m\overline{x} + n\overline{y}), \]

with complex weights satisfying \(|\eta_x|, |\kappa_y| \leq 1\) (which may depend on \(q\) in the settings of Theorem 1.2).

On the other hand, our approach does not work for the sums

\[ \sum_{\ell \in \mathcal{L}} \alpha_\ell \sum_{m \in \mathcal{M}} \beta_m \sum_{n \in \mathcal{N}} \eta_n \chi_{q}(\ell, m, n) \quad \text{and} \quad \sum_{\ell \in \mathcal{L}} \alpha_\ell \sum_{m \in \mathcal{M}} \beta_m \sum_{n \in \mathcal{N}} \gamma_n \chi_{q}(\ell, m, n) \]

with nontrivial weights attached to the variables \(m\) and \(n\). It is however possible that one can apply to these sums the method of \([1, 5, 11]\).

2. **Preliminaries**

2.1. **General notation.** We always assume that the sequence of weights \(\alpha = \{\alpha_\ell\}_{\ell \in \mathcal{L}}\) is supported only on \(\ell\) with \(\gcd(\ell, q) = 1\), that is, we have \(\alpha_\ell = 0\) if \(\gcd(\ell, q) > 1\) (and the same for the weights \(\alpha_q = \{\alpha_{q,\ell}\}_{\ell \in \mathcal{L}}\) depending on \(q\)).

Throughout the paper, as usual \(A \ll B\) and \(B \gg A\) are equivalent to the inequality \(|A| \leq cB\) with some constant \(c > 0\), which occasionally, where obvious, may depend on the real parameter \(\varepsilon > 0\) and on the integer parameter \(r \geq 1\), and is absolute otherwise.

2.2. **Number of solutions to some multiplicative congruences.** We start with some estimates on power moments of character sums. Let \(\mathcal{X}_q\) be the set of all multiplicative characters \(\chi\) modulo \(q\) and let \(\mathcal{X}_q^* = \mathcal{X}_q \setminus \{\chi_0\}\) be the set of non-principal characters.

The first result is a special case of a bound of Cochrane and Shi \([4, \text{Theorem 1}]\).

**Lemma 2.1.** For any integers \(k\) and \(H\) we have

\[ \sum_{\chi \in \mathcal{X}_q^*} \left| \sum_{z=k+1}^{k+H} \chi(x) \right|^4 \leq H^2 q^{o(1)}. \]

We now derive our main technical tool.

**Lemma 2.2.** For any sets

\[ A = \{s+1, \ldots, s+A\} \quad \text{and} \quad B = \{t+1, \ldots, t+B\} \]
consisting of $A$ and $B$ consecutive integers, respectively, for
\[ E(A, B) = \{ a_1 b_1 \equiv a_2 b_2 \mod q : a_1, a_2 \in A, b_1, b_2 \in B, \gcd(a_1 a_2 b_1 b_2, q) = 1 \} \]
we have
\[ E(A, B) \leq \left( \frac{A^2 B^2}{q} + AB \right) q^{o(1)}. \]

Proof. Using the orthogonality of characters, we write
\[ E(A, B) = \sum_{a_1, a_2 \in A, \gcd(a_1 a_2, q) = 1} \sum_{b_1, b_2 \in A, \gcd(b_1 b_2, q) = 1} \frac{1}{\varphi(q)} \sum_{\chi \in \chi_q} \chi \left( a_1 a_2 b_1 b_2^{-1} \right), \]
where, as usual, $\varphi(q)$ denotes the Euler function. Changing the order summation, and separating the contribution
\[ \frac{1}{\varphi(q)} \sum_{a_1, a_2 \in A, \gcd(a_1 a_2, q) = 1} \sum_{b_1, b_2 \in B, \gcd(b_1 b_2, q) = 1} 1 \leq \frac{A^2 B^2}{\varphi(q)} \]
from the principal character, we obtain
\[ (2.1) \quad E(A, B) \leq \frac{A^2 B^2}{\varphi(q)} + R, \]
where
\[ R = \frac{1}{\varphi(q)} \sum_{\chi \in \chi_q} \sum_{a_1, a_2 \in A, \gcd(a_1 a_2, q) = 1} \sum_{b_1, b_2 \in B, \gcd(b_1 b_2, q) = 1} \chi \left( a_1 a_2 b_1 b_2^{-1} \right). \]
Rearranging, we obtain
\[ R = \frac{1}{\varphi(q)} \sum_{\chi \in \chi_q} \left( \sum_{a \in A} \chi (a) \right)^2 \left( \sum_{b \in B} \overline{\chi} (b) \right)^2, \]
where $\overline{\chi}$ is the complex conjugate character (note the co-primality conditions $\gcd(a, q) = \gcd(b, q) = 1$ are abandoned from the last sum as redundant). Now, using the Cauchy inequality and recalling Lemma 2.1, we conclude that
\[ |R| \leq A B q^{o(1)}. \]
Substituting this in (2.1), and recalling the well-known lower bound
\[ \varphi(q) \gg \frac{q}{\log \log(q + 2)} \]
see [7, Theorem 328], we conclude the proof. \qed
2.3. Number of solutions to some congruences with reciprocals. An important tool in our argument is an upper bound on the number of solutions \( J_r(q; K) \) to the congruence

\[
\frac{1}{x_1} + \ldots + \frac{1}{x_r} \equiv \frac{1}{x_{r+1}} + \ldots + \frac{1}{x_{2r}} \mod q, \quad 1 \leq x_1, \ldots, x_{2r} \leq K,
\]

where \( r = 1, 2, \ldots \).

We start with the trivial bound \( 1 \leq K \leq q \) we have

\[
(2.2) \quad J_1(q; K) \ll K.
\]

For \( r \geq 2 \) and arbitrary \( q \) and \( K \), good upper bounds on \( J_r(q; K) \) are known only for \( r = 2 \) and are due to Heath-Brown [8, Page 368] (see the bound on the sums of quantities \( m(s)^2 \) in the notation of [8]). More precisely, we have:

Lemma 2.3. For \( 1 \leq K \leq q \) we have

\[
J_2(q; K) \leq \left(K^2 q^{-1/2} + K^2\right) q^o(1).
\]

It is also shown by Fouvry and Shparlinski [6, Lemma 2.3] that the bound of Lemma 2.3 can be improved on average over \( q \) in a dyadic interval \([Q, 2Q]\). The same argument also works for \( J_r(q; K) \) without any changes.

Indeed, let \( J_r(K) \) be the number of solutions to the equation

\[
\frac{1}{x_1} + \ldots + \frac{1}{x_r} = \frac{1}{x_{r+1}} + \ldots + \frac{1}{x_{2r}}, \quad 1 \leq x_1, \ldots, x_{2r} \leq K,
\]

where \( r = 1, 2, \ldots \). We recall that by the result of Karatsuba [10] (presented in the proof of [10, Theorem 1]), see also [3, Lemma 4], we have:

Lemma 2.4. For any fixed positive integer \( r \), we have

\[
J_r(K) \leq K^{r+o(1)}.
\]

Now repeating the argument of the proof of [6, Lemma 2.3] and using Lemma 2.4 in the appropriate place, we obtain:

Lemma 2.5. For any fixed positive integer \( r \) and sufficiently large integers \( 1 \leq K \leq Q \), we have

\[
\frac{1}{Q} \sum_{q \leq q \leq 2Q} J_r(q; K) \leq \left(K^{2r} Q^{-1} + K^r\right) Q^o(1).
\]
3. Proofs of Main Results

3.1. Proof of Theorem 1.1. For an integer \( u \) we define
\[
\langle u \rangle_q = \min_{k \in \mathbb{Z}} |u - kq|
\]
as the distance to the closest integer, which is a multiple of \( q \).

Changing the order of summation and then changing the variables
\( x \mapsto \overline{x} \) and \( y \mapsto \overline{y} \),
we obtain
\[
S_q(\alpha; \mathcal{L}, \mathcal{M}, \mathcal{N}) = \sum_{\ell \in \mathcal{L}} \sum_{x \in \mathbb{Z}_q^*} \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} e_q(\ell \overline{xy} + mx + ny)
\]
\[
= \sum_{\ell \in \mathcal{L}} \sum_{x, y \in \mathbb{Z}_q^*} e_q(\ell \overline{xy}) \sum_{m \in \mathcal{M}} e_q(mx) \sum_{n \in \mathcal{N}} e_q(ny).
\]
Hence
\[
S_q(\alpha; \mathcal{L}, \mathcal{M}, \mathcal{N}) = \sum_{\ell \in \mathcal{L}} \sum_{x, y \in \mathbb{Z}_q^*} \mu_x \nu_y e_q(\ell \overline{xy}),
\]
with some complex coefficients \( \mu_x \) and \( \nu_y \), satisfying

\[
|\mu_x| \leq \min \left\{ M, \frac{q}{\langle x \rangle_q} \right\} \quad \text{and} \quad |\nu_y| \leq \min \left\{ N, \frac{q}{\langle y \rangle_q} \right\},
\]
see [9, Bound (8.6)].

We now set \( I = \lceil \log(M/2) \rceil \) and define \( 2(I + 1) \) the sets
\[
Q^+_0 = \{ x \in \mathbb{Z} : 0 < \pm x \leq q/M, \gcd(x, q) = 1 \},
\]
(3.2) \[
Q^+_i = \{ x \in \mathbb{Z} : \min\{q/2, e^i q/M\} \geq \pm x > e^{i-1}q/M, \gcd(x, q) = 1 \},
\]
where \( i = 1, \ldots, I \). Similarly, we set \( J = \lceil \log(N/2) \rceil \) and define \( 2(J + 1) \) the sets
\[
R^+_0 = \{ x \in \mathbb{Z} : 0 < \pm y \leq q/N, \gcd(x, q) = 1 \},
\]
(3.3) \[
R^+_j = \{ y \in \mathbb{Z} : \min\{q/2, e^j q/N\} \geq \pm y > e^{j-1}q/N, \gcd(x, q) = 1 \},
\]
where \( j = 1, \ldots, J \).

Therefore,

\[
S_q(\alpha; \mathcal{L}, \mathcal{M}, \mathcal{N}) \ll \sum_{i=0}^I \sum_{j=0}^J \left( |S_{i,j}^+| + |S_{i,j}^-| \right),
\]
where
\[ S_{i,j}^\pm = \sum_{\ell \in \mathcal{L}} \sum_{x \in \mathcal{Q}_i^\pm} \sum_{y \in \mathcal{R}_j^\pm} \alpha_\ell \mu_x \nu_y e_q(\ell x y) , \quad i = 0, \ldots, I, \ j = 0, \ldots, J. \]

For \( \lambda \in \mathbb{Z}_q \) we denote
\[ T_{i}^\pm(\lambda) = \sum_{\ell \in \mathcal{L}, \ x \in \mathcal{Q}_i^\pm \atop \ell x \equiv \lambda \mod q} \alpha_\ell \mu_x \]
and note that \( T_{i}^\pm(\lambda) = 0 \) unless \( \lambda \in \mathbb{Z}_q^* \). Therefore,
\[ S_{i,j}^\pm = \sum_{\lambda \in \mathbb{Z}_q^*} \sum_{y \in \mathcal{R}_j^\pm} T_{i}^\pm(\lambda) \nu_y e_q(\lambda y) , \quad i = 0, \ldots, I, \ j = 0, \ldots, J. \]

Let us fix some integers \( i \in [0, I] \) and \( j \in [0, J] \).

We now fix some integer \( r \geq 1 \).

Below we present the argument in a general form with an arbitrary integer \( r \geq 1 \). We then apply it with \( r = 1 \) and \( r = 2 \) since we use Lemma 2.3. However in the proof of Theorem 1.2 we use it in full generality.

Writing
\[ |S_{i,j}^\pm| \leq \sum_{\lambda \in \mathbb{Z}_q^*} |T_{i}^\pm(\lambda)|^{1-1/r} |T_{i}^\pm(\lambda)|^{2r} \sum_{y \in \mathcal{R}_j^\pm} \nu_y e_q(my) , \]
by the Hölder inequality, for every choice of the sign ‘+’ or ‘-’, we obtain
\[ |S_{i,j}^\pm| \leq \left( \sum_{\lambda \in \mathbb{Z}_q^*} |T_{i}^\pm(\lambda)| \right)^{1-1/r} \left( \sum_{\lambda \in \mathbb{Z}_q^*} |T_{i}^\pm(\lambda)|^2 \right)^{1/2r} \left( \sum_{\lambda \in \mathbb{Z}_q^*} \sum_{y \in \mathcal{R}_j^\pm} \nu_y e_q(my) \right)^{2r} \]
(3.5)
\[ \sum_{\lambda \in \mathbb{Z}_q^*} \sum_{y \in \mathcal{R}_j^\pm} \nu_y e_q(my) \]

We observe that by (3.1) and (3.2), for \( x \in \mathcal{Q}_i^\pm \) we have
\[ \mu_x \ll e^{-i} M. \]
Hence
\[
\sum_{\lambda \in \mathbb{Z}_q^*} |T_i^\pm(\lambda)| \ll \sum_{\lambda \in \mathbb{Z}_q^*} \sum_{\ell L, x \in \mathbb{Q}_i^*} |\alpha \mu x| \ll e^{-iM} \sum_{\lambda \in \mathbb{Z}_q^*} \sum_{\ell L, x \in \mathbb{Q}_i^*} \ll e^{-iM} \#L \#\mathbb{Q}_i^* \ll e^{-iM} (e^{i q/M}) = qL.
\]

Similarly,
\[
\sum_{\lambda \in \mathbb{Z}_q^*} |T_i^\pm(\lambda)|^2 \ll e^{-2iM^2} \sum_{\lambda \in \mathbb{Z}_q^*} \left( \sum_{\ell L, x \in \mathbb{Q}_i^*} \sum_{\lambda \in \mathbb{Z}_q^*} \right)^2 \ll e^{-2iM^2} E(L, \mathbb{Q}_i^*),
\]

where \( E(A, B) \) is as defined in Lemma 2.2, which implies
\[
\sum_{\lambda \in \mathbb{Z}_q^*} |T_i^\pm(\lambda)|^2 \leq e^{-2iM^2} \left( \frac{L^2 (e^{i q/M})^2}{q} + L (e^{i q/M}) \right) q^{o(1)}
\leq q^{1+o(1)} L^2 + e^{-i q^{1+o(1)} LM}.
\]

Next, opening up the inner exponential sum in (3.5), changing the order of summation and using the orthogonality of exponential functions, we obtain
\[
\sum_{\lambda \in \mathbb{Z}_q^*} \left| \sum_{y \in \mathcal{R}_j^*} \nu_y e_q(\lambda \bar{y}) \right|^4 \leq \sum_{\lambda \in \mathbb{Z}_q^*} \sum_{y_1, \ldots, y_2 \in \mathcal{R}_j^*} \prod_{j=1}^r \nu_{y_j} \nu_{y_{r+j}} e_q \left( \lambda \sum_{j=1}^r (\bar{y}_j - \bar{y}_{r+j}) \right)
\leq \sum_{y_1, \ldots, y_2 \in \mathcal{R}_j^*} \prod_{j=1}^r \nu_{y_j} \nu_{y_{r+j}} \sum_{\lambda \in \mathbb{Z}_q^*} e_q \left( m \sum_{j=1}^r (\bar{y}_j - \bar{y}_{r+j}) \right)
= q \sum_{y_1, \ldots, y_2 \in \mathcal{R}_j^*} \prod_{j=1}^r \nu_{y_j} \nu_{y_{r+j}}.
\]

We observe that by (3.1) and (3.3) for \( y \in \mathcal{R}_j^* \) we have
\[
\nu_y \ll e^{-jN}.
\]
Hence,

\[
\sum_{\lambda \in \mathbb{Z}_q} \left| \sum_{g \in R_j^\pm} \nu_g e_q(\lambda g) \right|^{2r} \leq e^{-2rj} q N^{2r} \sum_{y_1, \ldots, y_{2r} \in \mathbb{Z}_q^\pm} \mathbb{I}_{y_1 + \cdots + y_{2r} \equiv \lambda_{r+1} + \cdots + \lambda_{2r} \mod q} 1 \leq e^{-2rj} q N^{2r} J_r(q; [e^j q/N]).
\]

Substituting (3.6), (3.7) and (3.8) in (3.5), we see that

\[
|S_{i,j}^\pm| \leq (qL)^{1-1/r} (q^{1+o(1)} L^2 + e^{-i} q^{1+o(1)} LM)^{1/2r}
\]

\[
\leq e^{-2rj} q N^{2r} J_r(q; [e^j q/N])^{1/2r}
\]

\[
\leq e^{-j} LN q^{1+o(1)} (1 + e^{-i} M/L)^{1/2r} J_r(q; [e^j q/N])^{1/2r}.
\]

Now using (3.9) with \( r = 1 \) and recalling (2.2), we derive

\[
|S_{i,j}^\pm| \leq e^{-j} LN q^{1+o(1)} (1 + e^{-i/2} (M/L)^{1/2}) e^{i/2} N^{-1/2} q^{1/2} \leq e^{-j/2} (L + e^{-i/2} L^{1/2} M^{1/2}) N^{1/2} q^{3/2+o(1)}.
\]

Summing over all admissible \( i \) and \( j \) yields

\[
S_q(\alpha; \mathcal{L}, \mathcal{M}, \mathcal{N}) \leq (L + L^{1/2} M^{1/2}) N^{1/2} q^{3/2+o(1)}.
\]

Next, using (3.9) with \( r = 2 \) and invoking Lemma 2.3, we obtain

\[
|S_{i,j}^\pm| \leq e^{-j} LN q^{1+o(1)} (1 + e^{-i/4} (M/L)^{1/4}) (e^{7j/8} N^{-7/8} q^{3/4} + e^{j/2} N^{-1/2} q^{1/2}) \leq (L + e^{-i/4} L^{3/4} M^{1/4}) (e^{-j/8} N^{1/8} q^{7/4} + e^{-j/2} N^{1/2} q^{3/2}) q^{o(1)},
\]

and we now derive

\[
S_q(\alpha; \mathcal{L}, \mathcal{M}, \mathcal{N}) \leq (L + L^{3/4} M^{1/4}) (N^{1/8} q^{7/4} + N^{1/2} q^{3/2}) q^{o(1)}.
\]

Combining the bounds (3.11) and (3.12), we obtain the result.

3.2. Proof of Theorem 1.2. We proceed as in the proof of Theorem 1.1, in particular, we set \( J = \lfloor \log(N/2) \rfloor \). We also define \( K_j = [2e^j Q/N] \) and replace \( J_r(q; [e^j q/N]) \) with \( J_r(q; K_j) \) in (3.9), \( j = 0, \ldots, J \).

We now see that by Lemma 2.5 for every \( j = 0, \ldots, J \) for all but at most \( Q^{1-2r \varepsilon + o(1)} \) integers \( q \in [Q, 2Q] \) we have

\[
J_r(q; K_j) \leq (K_j^{2r} q^{-1} + K_j^r) Q^{2r \varepsilon}.
\]

Since \( J = O^{o(1)} \), for all but at most \( Q^{1-2r \varepsilon + o(1)} \) integers \( q \in [Q, 2Q] \), the bound (3.13) holds for all \( j = 0, \ldots, J \) simultaneously.
Now, for every such integer $q$, using (3.13) instead of the bound of Lemma 2.3, we obtain
\[
|S_{i,j}^\pm| \leq e^{-j} LNq^{1+o(1)} \left( 1 + e^{-i} M/L \right)^{1/2r} \\
\left( e^j N^{-1} q^{1-1/2r} + e^j/2 N^{-1/2} q^{1/2} \right) \\
\leq \left( L + e^{-i/2r} L^{1-1/2r} M^{1/2r} \right) \left( q^{2-1/2r} + e^{-j/2} N^{1/2} q^{3/2} \right) q^{o(1)},
\]
instead of (3.10) for every $i = 0, \ldots, I$ and $j = 0, \ldots, J$. Since $I, J = Q^{o(1)}$, the result now follows.

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Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052, Australia

E-mail address: igor.shparlinski@unsw.edu.au