Bogolyubov Measure in Quantum Equilibrium Statistical Mechanics

D.P. Sankovich

Abstract

Application of the functional integration methods in equilibrium statistical mechanics of quantum Bose-systems is considered. We show that Gibbs equilibrium averages of Bose-operators can be represented as path integrals over a special Gauss measure defined in the corresponding space of continuous functions. This measure arises in the Bogolyubov $T$-product approach and is non-Wiener. We consider problems related to integration with respect to the Bogolyubov measure in the space of continuous functions and calculate some functional integrals with respect to this measure. Approximate formulas that are exact for functional polynomials of a given degree and also some formulas that are exact for integrable functionals belonging to a broader class are constructed. We establish the nondifferentiability of the Bogolyubov trajectories in the corresponding function space and prove a theorem on the quadratic variation of trajectories and study the properties implied by this theorem for the scale transformations. We construct some examples of semigroups related to the Bogolyubov measure. Independent increments are found for this measure. We consider the relation between the Bogolyubov measure and parabolic partial differential equations. An inequality for some traces is proved, and an upper estimate is derived for the Gibbs equilibrium mean square of the coordinate operator in the case of a one-dimensional nonlinear oscillator with a positive symmetric

1 Introduction

The purpose of this article is to provide a mathematical treatment of the Bogolyubov functional integral and to introduce some possible applications of this integral to the equilibrium quantum statistical mechanics.

Studying integration problems for functions on an abstract set was initiated by Fréchet [1], who appropriately generalized the Lebesgue method. Somewhat later, these problems were studied by Daniell [2, 3], who used the idea of extending linear functionals. The Daniell theory is based on the family $H(X)$ of elementary functions $h(x)$ on a set $X$ with an elementary integral $I(h)$ defined for them. Under some conditions, this family can be extended to a broader family $L$ to which the integral $I$ is extended such that $L$ becomes a Banach space with the norm $\|\varphi\| = I(\|\varphi\|)$. This is the essence of the construction of the Lebesgue integral in the Daniell scheme [4, 5].

The early results by Wiener [6] have much in common with the theory of the Daniell integral. He defined the integration process for functionals and showed that the integral he considered is the Daniell integral. We note that from 1921 on, the problem of functional integration in all works by Wiener is related to studying the Brownian motion of particles. The set $C = C[0, 1]$ of continuous real functions $x(t)$ satisfying the condition $x(0) = 0$ is defined on the interval $[0, 1]$, where $x(t)$ is the coordinate of a particle issuing from the origin at $t = 0$ and undergoing Brownian motion along the $x$ axis under the action of random impulses. The Wiener measure has a zero mean and a correlation function $\min(t, s)$. This measure belongs to a more general class of measures called Gaussian measures.

Feynman [7] was the first to use functional integration in quantum physics. The construction of the Feynman functional (continual) integral has some properties in common with the Wiener integral. However, these integrals are essentially different [8].

The idea of writing physical observables as continual integrals was developed in quantum field theory for representing the Green’s function. In due course, two such representation methods appeared almost simultaneously. One of them was based on formal integration of equations in variational derivatives for Green’s functions [9–12]. Bogolyubov developed a different approach [13] proceeding from the representation of Green’s functions in terms of vacuum expectations of chronological products, and the averaging operation over the boson vacuum was interpreted as a functional integral. In [14], the Bogolyubov functional integration method was used.
to study problems of gradient transformations for electrodynamic Green’s functions and to investigate the Bloch–Nordsiek model. Bogolyubov returned to this construction in the framework of statistical mechanics to investigate the polaron model [15]. It was shown in [16] that the measure appearing in the Bogolyubov approach is the Gaussian measure in the related space of continuous functions. The Gibbs equilibrium means of chronological products of operators are expressed in the form of functional integrals with respect to this measure.

In Section 2, conception of the $T$-product is considered and the Bogolyubov measure is introduced. In Section 3, the main results of the integration theory in abstract spaces as applied to the specific case of the Bogolyubov measure are presented. In Section 4, some simplest functional integrals with respect to the Bogolyubov measure are calculated. In Section 5, formulas of approximate integration are considered. In Section 6, we give a brief discussion of a probabilistic approach to the Bogolyubov process. In Section 7, some properties of the Bogolyubov trajectories are studied and scale transformations in the Bogolyubov space are considered. In Section 8, examples of semigroups related to the Bogolyubov measure are constructed, independent increments for this measure are found and relation between the Bogolyubov measure and parabolic partial differential equations is considered. In Section 9, an inequality for traces that is used in phase transition theory is proved.

2 Gaussian functional integrals and Gibbs equilibrium averages

2.1 $T$-product

The notion of the chronological product ($T$-product) of operators appeared in quantum mechanics in the analysis of the Schrödinger equation with a time-dependent Hamiltonian [17]. This equation emerges in the so-called interaction representation and is

$$i\frac{d\Phi(t)}{dt} = \tilde{H}(t)\Phi(t),$$

where

$$\tilde{H}(t) = e^{iH_0(t-t_0)}Ve^{-iH_0(t-t_0)}$$

and $H = H_0 + V$ is the time-independent Hamiltonian of the dynamic system under consideration. If $\Phi$ is a time-independent state vector in the Heisenberg representation, then $\Phi(t) = S(t,t_0)\Phi$, where

$$S(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}, \quad S(t_0,t_0) = I.$$  

The evolution operator $S(t,t_0)$ satisfies the equation and the initial condition

$$i\frac{\partial}{\partial t}S(t,t_0) = \tilde{H}(t)S(t,t_0), \quad S(t_0,t_0) = I. \quad (1)$$

In quantum mechanics, the operator $S(+\infty, -\infty)$ is called the scattering matrix [18]. The evolution operator $S(t,t_0)$ is a unitary propagator [19], i.e., it satisfies the conditions that

a) $S(t,t_1)S(t_1,t_0) = S(t,t_0),$

b) $S(t,t) = I,$ and

c) $S(t,t_0)$ is strongly continuous in all the variables $t$ and $t_0.$

The equation with initial condition (1) is formally equivalent to the integral equation

$$S(t,t_0) = I - i \int_{t_0}^{t} \tilde{H}(\tau)S(\tau,t_0) d\tau.$$  

Using consecutive substitutions, we can establish the Dyson expansion

$$S(t,t_0) = \sum_{n=0}^{\infty} S_n(t,t_0), \quad (2)$$
where
\[ S_n(t, t_0) = (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_{n} \tilde{H}(t_1)\tilde{H}(t_2)\ldots\tilde{H}(t_n). \] (3)

It is convenient to write this as
\[ S_n(t, t_0) = \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_{n} T[\tilde{H}(t_1)\tilde{H}(t_2)\ldots\tilde{H}(t_n)], \]

where we introduce the T-product
\[ T[V(t_1)V(t_2)\ldots V(t_n)] = \sum \pm \theta(t_1 > t_2 > \ldots > t_n) V(t_1)V(t_2)\ldots V(t_n), \] (4)

with
\[ \theta(t_1 > t_2 > \ldots > t_n) = \begin{cases} 1 & \text{if } t_1 \geq t_2 \geq \ldots \geq t_n, \\ 0 & \text{otherwise}. \end{cases} \]

The sum in Eqs. (4) is taken over all possible permutations of the indices 1, 2, \ldots, n. The minus sign corresponds to the Fermi case and is determined by the number of Fermi transpositions that are necessary for the derivation of the corresponding term.

Using Eqs. (4), we can write expansion (2) in the symbolic form
\[ S(t, t_0) = T \exp \left[ -i \int_{t_0}^{t} \tilde{H}(\tau) d\tau \right]. \]

It follows from the definition of the T-product that operators commute under the T-product sign.

The general conditions for the existence of the solution of the evolution equation
\[ \frac{d\varphi(t)}{dt} = A(t)\varphi(t) \]

with an unbounded operator \( A(t) \) were first found in [19].

Using the T-product, we can obtain an important formula of equilibrium statistical mechanics [15]. We consider the operator equation
\[ \frac{dU(s)}{ds} = -[H_0 + H_1(s)]U(s), \quad U(0) = I, \] (5)

which is solved by
\[ U(\beta) = T \exp \left\{ -\int_{0}^{\beta} \left[ H_0 + H_1(\sigma) \right] d\sigma \right\}. \] (6)

We assume that \( U(s) = e^{-sH_0}C(s) \) in (5). Then the equation and the initial condition satisfied by \( C(s) \) are
\[ \frac{dC(s)}{ds} = -e^{sH_0}H_1(s)e^{-sH_0}C(s), \quad C(0) = I \]

and are solved by
\[ C(s) = T \exp \left[ -\int_{0}^{s} d\sigma e^{\sigma H_0}H_1(\sigma)e^{-\sigma H_0} \right]. \]

Therefore,
\[ U(\beta) = e^{-\beta H_0}T \exp \left[ -\int_{0}^{\beta} ds e^{sH_0}H_1(s)e^{-sH_0} \right]. \] (7)

For the special case where the operator \( H_1(s) = H_1 \) is independent of \( s \), we compare solutions (6) and (7) and thus obtain the Bogolyubov formula
\[ e^{-\beta(H_0+H_1)} = e^{-\beta H_0}T \exp \left[ -\int_{0}^{\beta} ds e^{sH_0}H_1e^{-sH_0} \right]. \]

This formula is necessary for representing the partition function as a path integral.
2.2 Gibbs equilibrium averages

If $\hat{A}$ is a linear span of Bose operators and $\hat{\Gamma}$ is a positive-definite quadratic Hamiltonian, we have the formula \[2 \ln \langle e^{\hat{A}} \rangle = \langle \hat{A}^2 \rangle, \tag{8}\]

where

$$\langle \cdot \rangle = \frac{\text{Tr} \cdot e^{-\beta \hat{\Gamma}}}{\text{Tr} e^{-\beta \hat{\Gamma}}}$$

denotes the Gibbs average with the Hamiltonian $\hat{\Gamma}$.

We consider the average

$$\left\langle T \exp \left[ i \sum_{k=1}^{N+1} \nu_k \hat{Q}(s_k) \right] \right\rangle, \tag{9}$$

where $\nu_k$ are real numbers and

$$0 = s_1 < s_2 < \ldots < s_k < \ldots < s_N < s_{N+1} = \beta. \tag{10}$$

The operators $\hat{Q}(s)$ and $\hat{\Gamma}$ are given by

$$\hat{Q}(s) = e^{i \hat{q} s} \hat{q} e^{-i \hat{q} s}, \quad \hat{\Gamma} = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2,$$

which means that we consider the one-dimensional harmonic oscillator. Taking Eq. (8) into account, we can write

$$\left\langle T \exp \left[ i \sum_{k=1}^{N+1} \nu_k \hat{Q}(s_k) \right] \right\rangle = \exp \left\{-\frac{1}{2} \sum_{n=1}^{N+1} \sum_{m=1}^{N+1} \nu_n \nu_m \left\langle T[\hat{Q}(s_n)\hat{Q}(s_m)] \right\rangle \right\}. \tag{11}$$

We evaluate the average in the right-hand side of the last relation using $T$-product definition (4), which leads us to

$$\left\langle T[\hat{Q}(s_n)\hat{Q}(s_m)] \right\rangle = (2m\omega(1 - e^{-\beta \omega}))^{-1} (e^{-\omega |s_n - s_m|} + e^{-\beta \omega + \omega |s_n - s_m|}).$$

Thus, average (9) can be represented as

$$\left\langle T \exp \left[ i \sum_{k=1}^{N+1} \nu_k \hat{Q}(s_k) \right] \right\rangle = \exp \left\{-\frac{1}{2} \sum_{n,m=1}^{N+1} \nu_n \nu_m \left(2m\omega(1 - e^{-\beta \omega}))^{-1} (e^{-\omega |s_n - s_m|} + e^{-\beta \omega + \omega |s_n - s_m|}) \right\}. \tag{11}$$

We now write the last formula in a more convenient form for the future analysis. We consider the expression

$$K(s_n, s_m) = e^{-\omega |s_n - s_m|} + e^{-\beta \omega + \omega |s_n - s_m|}, \quad 0 < s_n, s_m < \beta,$$

as a function of $s_n$. This function, which we represent by $y(s_n)$, satisfies the differential equation

$$\frac{d^2 y(s_n)}{ds_n^2} - \omega^2 y(s_n) = -2\omega (1 - e^{-\beta \omega}) \delta(s_n - s_m) \tag{11}$$

and the boundary conditions

$$y(0) = y(\beta), \quad y'(0) = y'(\beta).$$

We seek the solution of Eq. (11) in the form

$$y(s) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i ns/\beta}.$$
It follows that
\[ K(s_j, s_k) = 2\omega \frac{1 - e^{-\beta\omega}}{\beta} \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n(s_j - s_k)/\beta}}{\omega^2 + (2\pi n/\beta)^2}. \]

For average (9), we thus have the representation
\[ \left\langle T \exp \left[ i \sum_{k=1}^{N+1} \nu_k \bar{Q}(s_k) \right] \right\rangle = e^{-\Omega(\nu_k)} \]
with the quadratic form in \( \nu_k \) given by
\[ \Omega(\nu_k) = \frac{1}{2m\beta} \sum_{n=-\infty}^{\infty} \frac{\left| \sum_{k=1}^{N+1} \nu_k e^{2\pi i n s_k/\beta} \right|^2}{\omega^2 + (2\pi n/\beta)^2}. \]

Obviously, \( \Omega \geq 0 \). In addition, \( \Omega = 0 \) if and only if \( \nu_1 + \nu_{N+1} = 0 \) and \( \nu_2 = 0, \ldots, \nu_N = 0 \).

Introducing new variables \( \eta_1 = \nu_1 + \nu_{N+1}, \eta_2 = \nu_2, \ldots, \eta_N = \nu_N \), we can rewrite Eq.(12) as
\[ \left\langle T \exp \left[ i \sum_{k=1}^{N+1} \nu_k \bar{Q}(s_k) \right] \right\rangle = \exp \left( -\frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} A_{jk} \eta_j \eta_k \right), \]
where
\[ \sum_{j,k=1}^{N} A_{jk} \eta_j \eta_k = \frac{1}{m\beta} \sum_{n=-\infty}^{\infty} \frac{\left| \sum_{k=1}^{N+1} \eta_k e^{2\pi i n s_k/\beta} \right|^2}{\omega^2 + (2\pi n/\beta)^2} \]
and the covariance matrix entries are
\[ A_{jk} = \frac{1}{2m\omega \sinh(\beta\omega/2)} \cosh \left( \frac{\beta\omega}{2N} |j - k| \right). \]

In deriving the last formula, a partition of form (10) was defined by the simple relation \( s_j = 2\beta N^{-1}(j - 1) \).

We now apply Eqs. (13) and (14) to find the relation between the Gibbs equilibrium averages of Bose operators and the path integral.

### 2.3 Gaussian path integrals

We consider the expression
\[ \int \left\langle T \exp \left[ i \sum_{k=1}^{N+1} \nu_k \bar{Q}(s_k) \right] \right\rangle \exp \left\{-i \sum_{k=1}^{N+1} \nu_k q_k \right\} d\nu_1 \ldots d\nu_N d\nu_{N+1}, \]
where \( q_k \) are real numbers and the integration with respect to each variable \( \nu_k \) goes over the entire real axis.

Taking Eq. (13) and the known values of Gaussian integrals into account, we obtain
\[ \frac{1}{(2\pi)^{N+1}} \int \left\langle T \exp \left[ i \sum_{k=1}^{N+1} \nu_k \bar{Q}(s_k) \right] \right\rangle \exp \left\{-i \sum_{k=1}^{N+1} \nu_k q_k \right\} d\nu_1 \ldots d\nu_N d\nu_{N+1} = \rho(q_1, q_2, \ldots, q_{N+1}), \]
where
\[ \rho(q_1, q_2, \ldots, q_{N+1}) = \frac{1}{\sqrt{(2\pi)^N \det A}} \exp \left( -\frac{1}{2} \sum_{j,k=1}^{N} (A^{-1})_{jk} q_j q_k \right). \]
\( \delta(q) \) is the Dirac delta function, and \( A^{-1} \) is the inverse covariance matrix with the entries
\[ (A^{-1})_{ij} = \frac{m\omega}{\sinh(\beta\omega/N)} \left( 2\cosh \frac{\beta\omega}{N} \delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1} \right). \]
The determinant of the inverse covariance matrix is

$$\det(A^{-1}) = \frac{1}{2}(m\omega)^N \left[ \left( \cosh \frac{\beta \omega}{N} + 1 \right)^{N+1} - \left( \cosh \frac{\beta \omega}{N} - 1 \right)^{N+1} \right].$$

It follows from (16) that

$$\rho \geq 0, \quad \int \rho \, dq_1 \ldots dq_{N+1} = 1. \quad (17)$$

Using relation (15), we can evaluate the averages of the form

$$\langle T[f(\hat{Q}(s_1), \ldots, \hat{Q}(s_{N+1}))] \rangle.$$ 

Indeed, we recall the complex Fourier formula

$$f(Q_1, \ldots, Q_{N+1}) = \frac{1}{(2\pi)^{N+1}} \int f(q_1, \ldots, q_{N+1}) \times$$

$$\times \exp\left\{ i \sum_{j=1}^{N} \nu_j (Q_j - q_j) \right\} \, dq_1 \ldots dq_{N+1} \, d\nu_1 \ldots d\nu_{N+1}.$$ 

Because the operators $\hat{Q}(s_j)$ commute under the $T$-product sign, we have

$$\langle T[f(\hat{Q}(s_1), \ldots, \hat{Q}(s_{N+1}))] \rangle = \int f(q_1, \ldots, q_{N+1}) \rho(q_1, \ldots, q_{N+1}) \, dq_1 \ldots dq_{N+1}. \quad (18)$$ 

Now using properties (17), we see that

$$0 \leq \langle T[f(\hat{Q}(s_1), \ldots, \hat{Q}(s_{N+1}))] \rangle \leq M, \quad \text{if} \quad 0 \leq f(\hat{Q}(s_1), \ldots, \hat{Q}(s_{N+1})) \leq M. \quad (19)$$ 

We now consider the functionals $F(q)$ of real functions (“trajectories”) $q(s)$ defined on the segment $0 \leq s \leq \beta$. We construct the integral

$$I \equiv \int F(q) \, d\mu \quad (20)$$

over the corresponding measure.

We first consider the subset of “special functionals” $[\mathbb{E}]$ that are continuous functions of a finite number $N$ of variables,

$$F^{(N)}(q) \equiv \Phi(q_1, q_2, \ldots, q_N),$$

where $q_j = q(s_j)$. By definition, we then have

$$I^{(N)} = \int \Phi(q_1, q_2, \ldots, q_N) \rho(q_1, q_2, \ldots, q_N) \, dq_1 \, dq_2 \ldots dq_N. \quad (21)$$

It follows then from (18) and (19) that

$$\langle T[F^{(N)}(\hat{Q})] \rangle = \int F^{(N)}(q) \, d\mu$$

and $\langle T[F^{(N)}(\hat{Q})] \rangle \geq 0$ if $F^{(N)}(q) \geq 0$ for arbitrary real numbers $q_1, q_2, \ldots, q_N$. We now consider the sequence of functions $\{q_N(s)\}$, $N = 1, 2, \ldots$, defined as

$$q_N(s) = q(s_j) \quad \text{for} \quad s_j \leq s < s_{j+1}, \quad j = 1, 2, \ldots, N, \quad q_N(\beta) = q(\beta). \quad (22)$$

The set of points $\{s_j\}$ is the partition (10) of the segment $[0, \beta]$. We assume that $|s_{j+1} - s_j| \leq \Delta s$ for $j = 1, 2, \ldots, N$ and also that $\Delta s \to 0$ as $N \to \infty$. Then the sequence of step-functions (22) uniformly tends to the function $q(s)$. Path integral (20) can be defined as the $N \to \infty$ limit of integrals (21), which are defined on the subset of “special functionals,” because the functionals $F(q_N(s))$ belong to this subset; therefore,

$$I = \lim_{N \to \infty} I^{(N)}.$$
We consider the space $C^q[0, \beta]$ of continuous functions $q(s)$ defined on the segment $[0, \beta]$ that satisfy the condition $q(0) = q(\beta)$. This is a metric space with respect to the uniform metric

$$
\rho(q, p) = \sup_{s \in [0, \beta]} |q(s) - p(s)|.
$$

The square order-$N$ matrix $A = (A_{jk})$ is positive and symmetrical, i.e., the mapping $(j, k) \rightarrow A_{jk}$ is a positive-type kernel on the set $\{1, 2, \ldots, N\}$. Therefore, we can speak of the Gaussian measure $\gamma_A$ on the space $R^N$ with the covariance $A$. By the Stone–Weierstrass theorem, the corresponding set of “special functionals” is dense in the set of all continuous functions defined on the space $C^q[0, \beta]$. In $C^q[0, \beta]$, we can introduce a $\sigma$-algebra generated by quasi intervals (cylindrical sets). This $\sigma$-algebra is the same as the $\sigma$-algebra generated by the sets that are open in the metric $\rho$. Extending the Gaussian measure from the quasi intervals to their Borel closure, we obtain a Gaussian measure in the space $C^q[0, \beta]$.

### 3 The Bogolyubov measure in the space of continuous functions

So we see that the Gaussian measure $\mu_B$ with zero average and the correlation function

$$
B(t, s) = \frac{1}{2m \omega \sinh(\beta \omega/2)} \cosh \left( \omega |t - s| - \frac{\beta \omega}{2} \right)
$$

is defined in the space $X = C^q[0, \beta]$ of continuous functions on the interval $[0, \beta]$ with the uniform metric $\rho = \max_{t \in [0, \beta]} |x(t) - y(t)|$ that satisfy the condition $x(0) = x(\beta)$. Measurable functionals $F(x)$ are considered on the space with measure $\{X, G, \mu_B\}$, where $G$ is an isolated $\sigma$-algebra of subsets in this space. In this case, the formula

$$
\langle T[F(\hat{\xi}(t))] \rangle_\Gamma = \int_X F(x(t)) \, d\mu_B(x)
$$

(24)

holds for the Gibbs equilibrium mean of the $T$-product taken with respect to the Hamiltonian $\hat{\Gamma}$ of the harmonic oscillator; the integral is understood as the Daniell integral over the space $X$,

$$
\hat{\Gamma} = \frac{\bar{\gamma}^2}{2m} + \frac{m \omega^2}{2} q^2, \quad \hat{\xi}(t) = e^{it\hat{p}} x e^{-it\hat{p}}, \quad \langle \cdot \rangle_\Gamma = \frac{\text{Tr}(\cdot e^{-\beta \hat{T}})}{\text{Tr} e^{-\beta \hat{T}}},
$$

where $\hat{p}$ and $\hat{q}$ are the respective coordinate and momentum operators of a particle with mass $m$ that satisfy the commutation relation $[\hat{q}, \hat{p}] = i$ ($\hbar = 1$ is assumed), $\beta$ is the reciprocal of the temperature, and $\omega$ is the eigenfrequency of the oscillator ($\beta > 0, \omega > 0$). The mean in formula (24) exists and is finite for an integrable functional $F(x)$. The measure $\mu_B$ thus defined is called the **Bogolyubov measure**.

The kernel $B(t, s)$ of the correlation operator $B$ is symmetric and Hermitian. It belongs to the space $L^2$ of square summable functions of two variables with respect to the Lebesgue measure in the domain $0 \leq t \leq \beta, 0 \leq s \leq \beta$. By the Schmidt theorem, every square summable function $A(t, s)$ that is symmetric with respect to its arguments can be expanded as a series

$$
A(t, s) = \sum_n \lambda_n \Phi_n(t) \Phi_n(s)
$$

(25)

in the sense of the convergence in the mean, where $\{\Phi_n(t)\}$ is an orthonormalized sequence of eigenfunctions and $\{\lambda_n\}$ is the sequence of the corresponding eigenvalues of the operator $\hat{A}$ generated by the kernel $A(t, s)$. According to [10], correlation function (23) has an expansion of form (25), where

$$
\Phi_n(t) = \frac{1}{\sqrt{2\pi n}} \frac{\sin(n \pi t/\beta)}{\beta}, \quad \lambda_n = \frac{1}{m \omega^2 + (2\pi n \beta^{-1})^2},
$$

and $n$ ranges the set of all integers from $-\infty$ to $\infty$. By the Mercer theorem, series (25) for the kernel $B(t, s)$ is uniformly convergent because the operator $\hat{B}$ generated by $B(t, s)$ is positive. We also note that this operator is completely continuous. Series (25) for the correlation function $B(t, s)$ can be written in the space of real functions in the form

$$
B(t, s) = \sum_{n=\pm \infty} \lambda_n \varphi_n(t) \varphi_n(s),
$$

7
where

\[ \varphi_n(t) = \begin{cases} \sqrt{\frac{2}{\beta}} \cos \frac{2\pi nt}{\beta}, & n > 0; \\ \sqrt{\frac{2}{\beta}} \sin \frac{2\pi nt}{\beta}, & n < 0; \\ \frac{1}{\sqrt{\beta}}, & n = 0. \end{cases} \]

The conjugate space of \( X \), \( X' = V_0[0, \beta] \), is the space of functions of bounded variation on \([0, \beta]\) that satisfy the conditions

\[ g(0) = 0, \quad g(t) = \frac{1}{2} [g(t + 0) + g(t - 0)] \quad \text{for} \ t \in (0, \beta). \]

By the Riesz representation theorem \([3]\), the linear functionals in \( X \) have the form

\[ \langle \varphi, x \rangle = \int_0^\beta x(t) d\varphi(t), \]

where the integral is understood as the Stieltjes integral, \( x(t) \in X \), and \( \varphi(t) \in V_0[0, \beta] \). The correlation functional in the space \( X' \) can be written as

\[ K(\varphi, \psi) = \int_0^\beta \int_0^\beta B(t, s) d\varphi(t) d\varphi(s), \]

where the correlation function for the measure has the form

\[ B(t, s) = \int_X x(t)x(s) d\mu(x). \]

By the Kuelbs theorem \([23]\), the Hilbert space \( H \) generated by the measure \( \mu \) is the linear span of the eigenfunctions \( \{\varphi_n(t)\} \) of the kernel \( B(t, s) \). This linear span is closed with respect to the norm corresponding to the inner product

\[ (x, y)_H = \sum_{n=0}^{+\infty} \frac{1}{\lambda_n} \left( \int_0^\beta x(t)\varphi_n(t) dt \right) \left( \int_0^\beta y(t)\varphi_n(t) dt \right). \]

The functions \( \{e_n(t) = \sqrt{\lambda_n} \varphi_n(t)\}_{n=0}^{+\infty} \) form a basis in the space \( H \), and the expansion

\[ x(t) = \sum_{n=0}^{+\infty} \frac{1}{\sqrt{\lambda_n}} \left( \int_0^\beta x(t)\varphi_n(t) dt \right) \varphi_n(t) \]

holds for almost all \( x \in X \). The general form of a linear measurable functional on \( X \) is given by the expression

\[ (a, x) = \sum_{n=0}^{+\infty} \frac{1}{\sqrt{\lambda_n}} \left( \int_0^\beta a(t)\varphi_n(t) dt \right) \left( \int_0^\beta x(t)\varphi_n(t) dt \right), \]

where \( a \in H \) and \( x \in X \). The functions

\[ e_n(t) = \int_0^\beta B_{1/2}(t, u)\alpha_n(u) du \]

also form a complete orthonormalized system in \( H \), where \( B_{1/2}(t, u) \) is the kernel of the operator \( B^{1/2} \) and \( \alpha_n(t) \) is an arbitrary complete orthonormal system in the space \( L_2[0, \beta] \).

We note that the closure \( \overline{H} \) of the Hilbert space \( H \) is the support of the measure \( \mu \) and is dense almost everywhere in \( X \) \([2]\). The triple \( (X, H, \mu) \) is called an abstract Wiener space, and the measure \( \mu \) is called an abstract Wiener measure \([23]\).

We also note that in the case of the Bogolyubov measure, \( G(t, s) = -mB(t, s) \) is the Green’s function of the boundary value problem

\[ \begin{cases} y'' - \omega^2 y = 0, \\ y(0) = y(\beta), \\ y'(0) = y'(\beta) \end{cases} \]

on the interval \([0, \beta]\).
4 Functional integral with respect to the Bogolyubov measure

Let \( a_1, a_2, \ldots, a_n \) be linearly independent elements in a separable Hilbert space \( H \) whose closure is the support of a measure \( \mu \) and which is dense almost everywhere in \( X \). Then

\[
\int_X F[(a_1, x), (a_2, x), \ldots, (a_n, x)] d\mu(x) =
\]

\[
= (2\pi)^{-n/2} \frac{1}{\sqrt{\det A}} \int_{R^n} e^{-(A^{-1}u, u)/2} F(u) du
\]

if one of the integrals in (26) exists, where \( A \) is the matrix of the elements \( a_{ij} = (a_i, a_j)_H \), \( i, j = 1, 2, \ldots, n \), \( u = (u_1, u_2, \ldots, u_n) \), and \( du = du_1 du_2 \cdots du_n \). If orthonormalized vectors in \( H \) are taken as the elements \( a_j \), then (26) becomes

\[
\int_X F[(a_1, x), (a_2, x), \ldots, (a_n, x)] d\mu(x) = (2\pi)^{-n/2} \int_{R^n} e^{-(u, u)/2} F(u) du.
\]

The form of formula (26) is particularly simple if the functional \( F(x) \) depends only on the values of the function \( x(t) \) at finitely many points. For example, if

\[
F(x(t)) = x(t_1)x(t_2) \cdots x(t_n),
\]

then the Wick theorem holds, by which

\[
\int_X x(t_1)x(t_2) \cdots x(t_n) d\mu(x) = \sum B(t_{i_1}, t_{i_2})B(t_{i_3}, t_{i_4}) \cdots B(t_{i_{2k-1}}, t_{i_{2k}}),
\]

where \( n = 2k \) and the summation extends over all \((2k)!/(2^k k!)\) decompositions of the numbers \( 1, 2, \ldots, 2k \) into \( k \) different unordered pairs,

\[
(i_1, i_2), (i_3, i_4), \ldots, (i_{2k-1}, i_{2k}).
\]

This integral vanishes for \( n = 2k + 1 \). In particular, for the case of the Bogolyubov measure, we have

\[
\langle q^2 \rangle_T = \int_X x^2(t) d\mu_B(x) = B(t, t) = \frac{1}{2m\omega} \coth \frac{\beta\omega}{2},
\]

\[
\langle e^{aq^2} \rangle_T = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n (n!)^2} (a^2 \langle q^2 \rangle_T)^n = \frac{1}{\sqrt{1 - a \coth(\beta\omega/2)/(m\omega)}},
\]

where it is necessary to assume that \(-m\omega \tanh(\beta\omega/2) \leq a < m\omega \tanh(\beta\omega/2)\) in the second formula.

We consider the quadratic functional

\[
A(x, x) = \sum_{k,j=1}^{\infty} a_{kj}(e_k, x)(e_j, x)
\]

on \( X \), where \( a_{kj} = (A e_k, e_j)_H \), \( A \) is a self-adjoint kernel operator from \( H \) into \( H \), and \( \{e_k\}_{k=1}^{\infty} \) is a basis in \( H \). Using formula (26), we can then calculate the integrals

\[
\int_X A(x, x) d\mu(x) = \text{Tr} A, \quad \int_X A^2(x, x) d\mu(x) = (\text{Tr} A)^2 + 2 \sum_{k=1}^{\infty} \lambda_k^2,
\]

where \( \lambda_k \) are the eigenvalues of the operator \( A \). The relation

\[
\int_X e^{\lambda A(x, x)/2} d\mu(x) = \frac{1}{\sqrt{D_A(\lambda)}}
\]

also holds, where \( D_A(\lambda) \) is the characteristic determinant of \( A \) at the point \( \lambda \),

\[
\text{Re} \lambda < \frac{1}{\lambda}, \quad \lambda_1 > \lambda_2 > \ldots,
\]
and
\[
\sqrt{D_A(\lambda)} = \sqrt{|D_A(\lambda)|} \exp \left[ -\frac{i}{2} \arg D_A(\lambda) \right].
\]

We take
\[
A(x, x) = \int_0^\beta x^2(t) \, dt = \sum_{k=\pm\infty} \lambda_k(e_k, x)^2
\]
as the quadratic functional in (27), where \(e_k = \sqrt{\lambda_k} \varphi_k\), \(\lambda_k\) are the eigenvalues of the kernel \(B(t, s)\), and \(\varphi_k\) are the corresponding eigenfunctions. We use the formula [5]
\[
-\frac{d}{d\lambda} \ln D_B(\lambda) = \int_0^\beta B(t, t) \, dt + \lambda \int_0^\beta B^{(2)}(t, t) \, dt + \cdots + \lambda^k \int_0^\beta B^{(k)}(t, t) \, dt + \ldots,
\]
where \(B^{(k)}\) are the corresponding iterated kernels, which have the form
\[
B^{(k)}(t, t) = \frac{1}{\beta m^k} \sum_{n=\pm\infty} \sum_{k=1}^{\infty} \left( \frac{\lambda}{m} \right)^k \frac{1}{\left( \omega^2 + (2\pi n \beta^{-1})^2 \right)^k}
\]
in the case of the Bogolyubov measure. This results in
\[
-\frac{d}{d\lambda} \ln D_B(\lambda) = \frac{1}{\lambda^2} \sum_{n=\pm\infty} \sum_{k=1}^{\infty} \left( \frac{\lambda}{m} \right)^k \frac{1}{\left( \omega^2 + (2\pi n \beta^{-1})^2 \right)^k}
\]
where \(\lambda < m\omega^2\). Integrating this equation, we obtain
\[
D_B(\lambda) = \frac{\sinh^2 \left( \beta \sqrt{\omega^2 - \lambda/m} / 2 \right)}{\sinh^2(\beta\omega/2)},
\]
whence follows the formula
\[
\int_X \exp \left( \frac{\lambda}{2} \int_0^\beta x^2(t) \, dt \right) \, d\mu_B(x) = \frac{\sinh(\beta\omega/2)}{\sinh \left( \beta \sqrt{\omega^2 - \lambda/m} / 2 \right)}, \quad \lambda < m\omega^2. \quad (28)
\]

We note that the moments
\[
m_k = \int_X A(x, x)^k \, d\mu_B(x) = \int_X \left( \int_0^\beta x^2(t) \, dt \right)^k \, d\mu_B(x)
\]
can be determined using formula (28) and the relation
\[
m_{k+1} = 2^{k+1} \frac{d^{k+1}}{d\lambda^{k+1}} \frac{1}{\sqrt{D_B(\lambda)}} \bigg|_{\lambda=0}.
\]

Taking the relation
\[
D_B(\lambda) = \prod_{n=\pm\infty} (1 - \lambda\lambda_n)
\]
into account, we derive the following formula for the infinite product from the above value of the Fredholm determinant of the kernel \(B(t, s)\):
\[
\prod_{n=1}^{\infty} \left(1 + \frac{a}{n^2 + b^2} \right) = \frac{1}{\sqrt{1 + a/b^2}} \frac{\sinh \left( \pi b \sqrt{1 + a/b^2} \right)}{\sinh(\pi b)}, \quad a > -b^2.
\]
5 Approximate calculation of functional integrals

We consider approximate formulas that are exact for functional polynomials of a given degree. Let $X$ be the space $C[a,b]$ of continuous functions $x(t)$ on $[a,b]$. We assume that a Gaussian measure $\mu$ with a zero average and a correlation function $B(t,s)$ is defined in $X$. An arbitrary continuous functional polynomial $P_n(x)$ of degree $n$ on $C$ has the form

$$P_n(x) = p_0 + \sum_{j=1}^{n} \int_a^b \cdots \int_a^b x(t_1) \cdots x(t_j) d_{t_1} \cdots d_{t_j} g(t_1, \ldots, t_j),$$

where $p_0 = \text{const}$ and the other terms are multiple Stieltjes integrals.

**Theorem 1** [26]. Let $\nu$ be a symmetric probability measure on the Borel sets in $R$, and let $\rho(u,t)$ be a function on $R \times [a,b]$ such that

1) $\rho(u,t) = -\rho(-u,t)$,

2) $\prod_{j=1}^{m} \rho(u,t_j) \in L(R,\nu)$ for $1 \leq m \leq 2n+1$,

3) $\int_{-\infty}^{\infty} \rho(u,t)\rho(u,s) d\nu(u) = B(t,s).$ \hspace{1cm} (29)

Then the formula

$$\int_C F(x) d\mu(x) \approx \int_{R^n} F(\theta_n(u,\cdot)) d\nu_n(u),$$ \hspace{1cm} (30)

where

$$\theta_n(u,t) = \sum_{j=1}^{n} c_j \rho(u_j,t),$$

$c_j^2$ are the roots of the polynomial

$$Q_n(z) = \sum_{k=0}^{n} \frac{z^{n-k}}{k!},$$

and $\nu_n$ is a measure in $R^n$ that is an $n$-fold Cartesian product of the measures $\nu$, is exact for functional polynomials of degree $2n+1$.

**Theorem 2** [26]. Let the assumptions in Theorem 1 hold. Then the formula

$$\int_C F(x) d\mu(x) \approx (-1)^n(A-n)^n \frac{n!}{n!} F(0) +$$

$$+ \sum_{k=1}^{n} (-1)^{n-k} \frac{(A-n+k)^n}{k!(n-k)!} \int_{R^k} F(\theta_k^{(n)}(u,\cdot)) d\nu_k(u),$$ \hspace{1cm} (31)

where

$$\theta_k^{(n)}(u,t) = \frac{1}{\sqrt{A-n+k}} \sum_{j=1}^{k} \rho(u_j,t), \hspace{1cm} R^k = R \times R \times \ldots \times R,$$

$$d\nu_k(u) = d\nu(u_1) \cdots d\nu(u_k), \hspace{1cm} k = 1,2,\ldots,n,$$

and $A$ is an arbitrary constant, is exact for all functional polynomials of degree $2n+1$.

If $A = n$, then formula (31) becomes [27]

$$\int_C F(x) d\mu(x) \approx I_n(F),$$

where

$$I_n(F) = \sum_{k=1}^{n} (-1)^{n-k} \frac{k^n}{k!(n-k)!} \int_{R^k} F(\theta_k(u,\cdot)) d\nu_k(u)$$
\[ \theta_k(u, t) = \frac{1}{\sqrt{k!}} \sum_{j=1}^{k} \rho(u_j, t). \]

It is easy to verify that the recursive relation
\[ I_n(F) = \frac{n^n}{n!} \int_{\mathbb{R}^n} F(\theta_n(u, \cdot)) \, d\nu_n(u) - \sum_{k=1}^{n-1} \frac{kn^{n-k}}{(n-k)!} I_k(F) \]
holds for \( I_n(F) \).

Deriving formulas (30) and (31) for approximately calculating integrals with respect to the Gaussian measure relates to finding a function \( \rho(u, t) \) possessing properties 1–3 in Theorem 1. The most difficult task here is solving Eq. (29).

We first seek the solution of this equation for the case of a purely discrete measure \( \nu \) on the line. We recall that a measure entirely concentrated on a finite or countable set of points on the line is said to be discrete.

Let a finite or countable set of points \( \{x_n\}_{n=-\infty}^{\infty} \) be given on an interval \([a, b]\), and let a positive number \( h_n \) satisfying the condition
\[ \sum_n h_n < \infty \]
be associated with each \( x_n \). We define a function \( f \) on \([a, b]\) by setting
\[ f(x) = \sum_{x_n < x} h_n. \]

The function \( f(x) \) does not decrease and is left-continuous. If \( x \) coincides with one of the points \( x_n \), with \( x = x_{n_0} \) for example, then
\[ f(x_{n_0} + 0) - f(x_{n_0} - 0) = h_{n_0}. \]

If \( x \) does not coincide with any of the points \( x_n \), then \( f(x) \) is continuous at \( x \). The function \( f(x) \) is called a jump function.

We define a measure \( \nu \) on \( \mathbb{R} \) in the form
\[ \nu((-\infty, x]) = f(x) \]
and assume that
\[ \sum_n h_n = 1, \quad h_n = h_{-n}, \quad n = 0, \pm 1, \pm 2, \ldots. \]

If we set \( x_n = n \), then
\[ \int_R \rho(u, s)\rho(u, t) \, d\nu(u) = \sum_{n=-\infty}^{\infty} h_n \rho(n, s)\rho(n, t). \]

Expanding the correlation function in a series with respect to the complete system of orthonormalized eigenfunctions,
\[ B(t, s) = \sum_{n=-\infty}^{+\infty} \lambda_n \varphi_n(t) \varphi_n(s), \]
we see that all assumptions of Theorem 1 hold if we set
\[ \rho(u, t) = \begin{cases} 0 & \text{for } -1 < u < 1, \\ \sqrt{\lambda_n} \varphi_n(t) & \text{for } u \in [n, n+1), \\ -\sqrt{\lambda_n} \varphi_n(t) & \text{for } u \in (-n-1, -n], \quad n = 1, 2, \ldots. \end{cases} \]

The solution of Eq. (29) in the case of an absolutely continuous measure \( \nu \) was found for the Wiener measure, the conditional Wiener measure, and some other measures. The following solution of (29) can be constructed for the Bogolyubov measure. We take the normalized Lebesgue measure on the closed interval \([-\beta, \beta]\) as \( \nu \), i.e.,
\[ d\nu(u) = \frac{1}{2\beta} du. \]
Then
\[
\rho(u, t) = \sqrt{\frac{\beta}{m}} \frac{1}{e^{\beta \omega} - 1} e^{\omega(t - |u|)} \{ \theta(t - |u|) + e^{\beta \omega} \theta(|u| - t) \} \text{sgn } u.
\] (32)

It can be verified that the measure \( \nu \) and the function \( \rho(u, t) \) thus chosen satisfy all assumptions in Theorem 1. Hence, in the case of the Bogolyubov measure in question, we have
\[
\int_X F(x) d\mu(x) \approx \frac{1}{(2\beta)^n} \int_{-\beta}^\beta \cdots \int_{-\beta}^\beta du_1 \cdots du_n F \left( \sum_{j=1}^n c_j \rho(u_j, t) \right)
\]
by Theorem 1, where \( \rho(u, t) \) is given by (32).

We consider approximate formulas that are exact for functional polynomials of the third degree and for some functionals of a special form. As before, let \( X \) be the space \( C[a, b] \) of continuous functions on \([a, b]\), let \( \nu \) be a measure on the Borel sets in the real line \( R \), let \( A(u) \) be a positive function on \( R \), and let \( p(x) \) be a weight functional. We assume that the conditions
\[
\int_R A(u) d\nu(u) = A < \infty,
\]
\[
\int_X p(x)x(t) d\mu(x) = \int_X p(x)x(t)x(s) d\mu(x) \equiv 0,
\]
\[
r(t, s) = \frac{1}{p_0} \int_X p(x)x(t)x(s) d\mu(x) < \infty
\]
hold, where
\[
p_0 = \int_X p(x) d\mu(x).
\]

**Theorem 3** \( 28 \). Let a symmetric function \( r(t, s) \) be representable in the form
\[
r(t, s) = \int_R x(u, t)x(u, s) d\nu(u),
\]
where the function \( x(u, t) \) belongs to the space \( L_2[R, \nu] \) relative to the argument \( u \). Then the formula
\[
\int_X p(x)F(x) d\mu(x) \approx p_0(1 - A)F(0) +
\]
\[
+\frac{1}{2} p_0 \int_R A(u) \left[ F \left( \frac{x(u, \cdot)}{\sqrt{A(u)}} \right) + F \left( -\frac{x(u, \cdot)}{\sqrt{A(u)}} \right) \right] d\nu(u)
\]
(33)
is exact if \( F(x) \) is an arbitrary functional polynomial of the third degree.

If the measure \( \nu \) is discrete, then
\[
A = \sum_k A_k, \quad r(t, s) = \sum_k x_k(t)x_k(s),
\]
and formula (33) becomes
\[
\int_X p(x)F(x) d\mu(x) \approx p_0(1 - A)F(0) + \frac{p_0}{2} \sum_k A_k \left[ F \left( \frac{x_k(\cdot)}{\sqrt{A_k}} \right) + F \left( -\frac{x_k(\cdot)}{\sqrt{A_k}} \right) \right].
\] (34)

We consider an example. Let the weight be
\[
p(x) = \int_a^b x^2(t) dt.
\]
Then
\[
r(t, s) = B(t, s) + \frac{2}{\text{Tr } B} \int_a^b B(t, \tau)B(\tau, s) d\tau, \quad \text{Tr } B = \int_a^b B(t, t) dt.
\]
If the expansion of the correlation function in a series with respect to its eigenfunctions is used, then we obtain
\[ r(t, s) = \sum_k \left( \lambda_k + \frac{2}{\text{Tr} B} \lambda_k^2 \right) \varphi_k(t)\varphi_k(s), \]
where \( \lambda_k \) are the eigenvalues of the kernel \( B(t, s) \). Formula (34) becomes
\[ \int_X \left( \int_a^b x^2(t) \, dt \right) F(x) \, d\mu(x) \approx \text{Tr} B \left\{ (1 - A) F(0) + \frac{1}{2} \sum_k A_k \left[ F(b_k \varphi_k(\cdot)) + F(-b_k \varphi_k(\cdot)) \right] \right\}, \]
where
\[ b_k = \left[ \frac{1}{A_k} \left( \lambda_k + \frac{2}{\text{Tr} B} \lambda_k^2 \right) \right]^{1/2}, \quad \sum_k A_k = A < \infty. \]

**Theorem 4** [28]. Let the functions \( r(t, s) \) and \( \rho(t, s) \) be representable in the form
\[ r(t, s) = \sum_k x_k(t)x_k(s), \quad \rho(t, s) = \sum_k B_k x_k(t)x_k(s), \]
where \( B_k \) are such that the equation
\[ V \left( \frac{x_k(\cdot)}{\sqrt{A_k}} \right) = B_k \]
for each value of \( k \) has a positive solution \( A_k \) satisfying the condition
\[ A = \sum_k A_k < \infty. \]

Then formula (34) is exact for all functional polynomials of the third degree and also for the functionals \( F(x) \) of the form
\[ F(x) = V(x)p_2(x), \]
where \( p_2(x) \) is an arbitrary homogeneous functional of the second degree.

As an example, we consider the case of the Bogolyubov measure. Let
\[ p(x) \equiv 1, \quad V(x) = \|x\|^2 = \int_0^\beta x^2(t) \, dt. \]

Then
\[ \rho(t, s) = \sum_k B_k x_k(t)x_k(s), \]
where
\[ B_k = \text{Tr} B + 2\lambda_k, \quad x_k(t) = \sqrt{\lambda_k} \varphi_k(t), \]
and \( \lambda_k \) and \( \varphi_k(t) \) are the eigenvalues and eigenfunctions of the kernel \( B(t, s) \). In the case under consideration, the other quantities in formula (34) are given by the relations
\[ \text{Tr} B = \frac{\beta}{2m\omega} \coth \frac{\beta \omega}{2}, \quad A_k = \left( 2 + \frac{\beta}{2\omega} \coth \frac{\beta \omega}{2} \left[ \omega^2 + (2\pi k\beta^{-1})^2 \right] \right)^{-1}, \]
\[ A = \frac{1}{\sqrt{1 + 4 \tanh(\beta \omega/2)/(\beta \omega)}} \cdot \coth \left( \frac{\beta \omega \sqrt{1 + 4 \tanh(\beta \omega/2)/(\beta \omega)}}{2} \right). \]
6 Stochastic processes and the Bogolyubov measure

The notions and methods of the theory of stochastic processes are widely used to study probability measures in function spaces. We assume that a probability space \( \{ \Omega, G, P \} \) is fixed, where \( \Omega \) is a space of elementary events \( \omega \) with a selected \( \sigma \)-algebra of subsets of events \( G \) and a measure, namely, the probability \( P \) of events on \( G \). The numerical functions \( f(\omega) \) on \( \Omega \) measurable with respect to \( P \) are called random variables. For integrable functions with respect to the measure \( P \), the integral (mathematical expectation)

\[
Mf(\omega) = \int_{\Omega} f(\omega) dP(\omega)
\]

is defined. By a random element with a range in \( X \), we mean a weakly measurable mapping \( x(\omega) \) of \( \Omega \) into \( X \), i.e., a mapping under which a functional \( \langle \xi, x(\omega) \rangle \) is measurable with respect to the measure \( P \) for any \( \xi \in X' \), where \( X' \) is the adjoint space of \( X \). If a random element \( x(\omega) \) with a range in \( X \) is given, then the probability measure

\[
\mu\{x \in X : [(\xi_1, x), \ldots, (\xi_n, x)] \in A_n \} = P\{\omega \in \Omega : [(\xi_1, x(\omega)), \ldots, (\xi_n, x(\omega))] \in A_n \}
\]

can be defined on the \( \sigma \)-algebra generated by the cylindrical sets in \( X \). Here, \( A_n \) is an arbitrary Borel set in \( R^n \), and the vectors \( \xi_1, \ldots, \xi_n \) \( (n = 1, 2, \ldots) \) belong to the adjoint space \( X' \). In this case,

\[
\int_{\Omega} F[x(\omega)] dP(\omega) = \int_{X} F(x) d\mu(x)
\]

for any functional \( F \) such that one of the above integrals exists for it.

Let \( X \) be a space of real functions of the argument \( t \in T \), where \( T \) is a subset in \( R \). Then a random element \( x(\omega) \) is called a random function and is denoted by \( x(\omega, t) \). The argument \( \omega \) in \( x(\omega, t) \) is often omitted. If \( t \) is interpreted as time, then the related random functions are called random or stochastic processes. A random function is regarded as being defined if its finite-dimensional distributions are known. A random function \( x(t) = x(\omega, t) \) \( (t \in T) \) with a range in \( X \) in a probability space \( \{X, G, P\} \) is called a Gaussian process if all its finite-dimensional distributions are Gaussian. This means that the joint distribution of the values \( x(t_1), x(t_2), \ldots, x(t_n) \) of this random process are defined by the density function

\[
p(u_1, \ldots, u_n) = (2\pi)^{-n/2}(\det B)^{-1/2} \exp\left[ -\frac{1}{2} \sum_{i,j=1}^{n} b_{ij}^{(-1)}(u_i - m(t_i))(u_j - m(t_j)) \right]
\]

with the mathematical expectation \( m(t) = M[x(\omega, t)] \) and the correlation function

\[
B(t, s) = M[(x(\omega, t) - m(t))(x(\omega, s) - m(s))],
\]

where \( B \) is a matrix with the elements \( B(t_i, t_j) \) \( (i, j = 1, 2, \ldots, n) \) and \( b_{ij}^{(-1)} \) are the elements of the matrix \( B^{-1} \) inverse to \( B \). Therefore, if \( X \) is a function space, then the relation

\[
\int_{\Omega} F[x(\omega, t)] dP(\omega) = \int_{X} F[x(t)] d\mu(x)
\]

holds, and the problem of integrating with respect to the Gaussian measure in the function space is equivalent to the problem of integrating with respect to the measure generated by the corresponding Gaussian random process. In what follows, we constantly use this relation between the theory of Gaussian random processes and the theory of functional integration with respect to Gaussian measures.

Let \( \bar{t} = (t_1, t_2, \ldots, t_n) \), \( 0 < t_1 < t_2 < \ldots < t_n \leq \beta \), be a set of real numbers. For an arbitrary given subset \( E \subset R^n \), we define the cylindrical set \( Q_{\bar{t}}(E) = \{ x \in X : (x(t_1), \ldots, x(t_n)) \in E \} \). The sets \( E \) and \( Q_{\bar{t}}(E) \) uniquely each other for a fixed \( \bar{t} \). By definition, the centered Gaussian measure of the given cylindrical set \( Q_{\bar{t}}(E) \) is

\[
\mu\{Q_{\bar{t}}(E)\} = (2\pi)^{-n/2}(\det K)^{-1/2} \int_{E} \exp\left( -\frac{1}{2} \sum_{i,j=1}^{n} k_{ij}^{(-1)} u_i u_j \right) du_1 \cdots du_n.
\]

(35)
In the case of the Bogolyubov measure \( X = C_0^v[0, \beta] \), we have \( X = C_0^v[0, \beta] \) is the space of continuous functions on the closed interval \([0, \beta]\) with the uniform metric

\[
\rho = \max_{t \in [0, \beta]} |x(t) - y(t)|
\]

that satisfy the condition \( x(0) = x(\beta) \). The bilinear functional \( K(\varphi, \psi) \) on the adjoint space \( X' \) has the form

\[
K(\varphi, \psi) = \int_0^\beta \int_0^\beta B(t, s) \, d\varphi(t) \, d\psi(s),
\]

where \( \varphi(t) \in X' = V_0[0, \beta] \) and \( X' = V_0[0, \beta] \) is the space of functions of bounded variation on \([0, \beta]\) satisfying the condition

\[
\varphi(0) = 0, \quad \varphi(t) = \frac{1}{2} [\varphi(t + 0) + \varphi(t - 0)] \quad \text{for} \ t \in (0, \beta).
\]

The elements of the variance matrix have the form \([16]\)

\[
k_{ij} = B(t_i, t_j) = \frac{1}{2m\omega \sinh(\beta \omega/2)} \cosh \left( \omega |t_i - t_j| - \frac{\beta \omega}{2} \right). \quad (36)
\]

The Bogolyubov measure has a zero mean.

We consider some special cases of formula (35) for the Bogolyubov measure.

Let \( 0 < t \leq \beta \). We calculate the function

\[
F_{x(t)} = \mu_B \{ x \in X : x(t) \leq \gamma \},
\]

where \( \gamma \) is an arbitrary real number. Using (35) and (36), we write

\[
F_{x(t)} = \frac{1}{\sqrt{2\pi K(\varphi, \varphi)}} \int_0^\gamma \exp \left( -\frac{1}{2} \frac{u^2}{K(\varphi, \varphi)} \right) \, du =
\]

\[
= \frac{1}{\sqrt{\left( \frac{\pi}{m\omega} \right) \coth(\beta \omega/2)}} \int_0^\gamma \exp \left( -\frac{u^2}{2} \frac{2m\omega}{\coth(\beta \omega/2)} \right) \, du.
\]

This formula shows that the random variable \( G(x) = x(t) \) is normally distributed with a zero mean and the variance \((2m\omega)^{-1} \coth(\beta \omega/2)\).

Let \( 0 < t_1 < t_2 \leq \beta \), and let \( \gamma \) be an arbitrary real number. We find the function

\[
F_{x(t_2) - x(t_1)} = \mu_B \{ x \in X : x(t_2) - x(t_1) \leq \gamma \}.
\]

We can write

\[
F_{x(t_2) - x(t_1)} = \mu_B \{ x \in X : (x(t_1), x(t_2)) \in E \},
\]

where \( E = \{ (u_1, u_2) \in R^2 : u_2 - u_1 \leq \gamma \} \). Using (35), we obtain

\[
F_{x(t_2) - x(t_1)} = \frac{1}{\sqrt{2\pi \det K}} \int_B d\nu_1 \, d\nu_2 \exp \left[ -\frac{1}{2} \left( k_{11}^{-1} u_1^2 + k_{12}^{-1} u_1 u_2 + k_{21}^{-1} u_2 u_1 + k_{22}^{-1} u_2^2 \right) \right]. \quad (37)
\]

The elements of the inverse matrix \( K^{-1} \) are calculated quite simply in this case. They have the forms

\[
k_{11}^{-1} = k_{22}^{-1} = \frac{k_{11}}{\det K}, \quad k_{12}^{-1} = k_{21}^{-1} = -\frac{k_{12}}{\det K},
\]

where

\[
k_{11} = \frac{1}{2m\omega \sinh(\beta \omega/2)} \cosh \frac{\beta \omega}{2}, \quad k_{12} = \frac{1}{2m\omega \sinh(\beta \omega/2)} \cosh \left( \omega |t_1 - t_2| - \frac{\beta \omega}{2} \right),
\]

16
\[
\text{det } K = \frac{1}{4m^2 \omega^2 \sinh^2(\beta \omega/2)} \left[ \cosh^2 \frac{\beta \omega}{2} - \cosh^2 \left( \omega |t_1 - t_2| - \frac{\beta \omega}{2} \right) \right].
\]

After substituting these expressions in (37) and performing some elementary transformations, we obtain
\[
F_{x(t_2) - x(t_1)} = \sqrt{\frac{m \omega \sinh(\beta \omega/2)}{2\pi \cosh(\beta \omega/2) - \cosh(\beta \omega/2 - \omega(t_2 - t_1))}} \times \int_{-\gamma}^{\gamma} du \exp \left[ -\frac{u^2}{2} \frac{m \omega \sinh(\beta \omega/2)}{\cosh(\beta \omega/2) - \cosh(\beta \omega/2 - \omega(t_2 - t_1))} \right].
\]

Consequently, the random variable \( G(x) = x(t_2) - x(t_1) \) is normally distributed with a zero average and the variance
\[
\frac{\cosh(\beta \omega/2) - \cosh(\beta \omega/2 - \omega(t_2 - t_1))}{m \omega \sinh(\beta \omega/2)}.
\]

7 Metric properties of Bogolyubov trajectories

7.1 Nondifferentiability of Bogolyubov trajectories

We consider the properties of the support of the Bogolyubov measure in the space \( C^0[0, \beta] \). As in the case of the Wiener measure, the measure \( \mu_B \) is concentrated on continuous paths rather than on continuously differentiable ones. Hence, along with the Wiener measure, the Bogolyubov measure gives another important example of continuous functions that are almost everywhere nondifferentiable.

We introduce the set
\[
C_h^\gamma(t, t') = \{ x \in X : |x(t) - x(t')| \leq h|t - t'|^\gamma \},
\]
where \( h > 0, 0 < \gamma \leq 1 \), and \( t, t' \in [0, \beta] \). We seek the Bogolyubov measure of this set. Using (35), we can write
\[
\mu_B \{ C_h^\gamma(t, t') \} = \frac{1}{2\pi \sqrt{\text{det } K}} \int_B du_1 du_2 e^{-(u, K^{-1}u)/2},
\]
where \( B = \{(u_1, u_2) \in \mathbb{R}^2 : |u_1 - u_2| \leq h|t - t'|^\gamma \} \). The matrix \( K \) in (39) coincides with the matrix \( K \) used in the preceding section in formula (37). Performing a suitable linear change of integration variables in (39), we obtain
\[
\mu_B \{ C_h^\gamma(t, t') \} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-v^2/2} dv,
\]
where
\[
a = \sqrt{\frac{m \omega \sinh(\beta \omega/2)}{\cosh(\beta \omega/2) - \cosh(\beta \omega/2 - \omega|t - t'|)}} h|t - t'|^\gamma.
\]

Formula (40) implies an upper estimate for the desired measure,
\[
\mu_B \{ C_h^\gamma(t, t') \} \leq \frac{\sqrt{2/a}}{\pi}.
\]

We now consider the sets
\[
C_h^\gamma(t) = \bigcap_{t' \in [0, \beta]} C_h^\gamma(t, t') = \{ x \in X : |x(t) - x(t')| \leq h|t - t'|^\gamma \text{ for all } t' \in [0, \beta] \}
\]
and
\[
C_h^\gamma(t) = \bigcap_{t' \in [0, \beta]} C_h^\gamma(t, t') = \{ x \in X : |x(t) - x(t')| \leq h|t - t'|^\gamma \text{ for all } t, t' \in [0, \beta] \}.
\]
It can be proved \( \mathbb{R} \) that \( C_h^\gamma(t, t'), C_h^\gamma(t), \) and \( C_h^\gamma \) are closed subsets in \( X = C^0[0, \beta] \).
We consider a sequence of points \( \{t_k\} \) in the closed interval \([0, \beta]\) such that they do not coincide with \( t \) and \( t_k \to t \) as \( k \to \infty \). Then definition (42), inequality (41), and the downward convexity of \( \cosh x \) imply that

\[
\mu_B \{ C_h^\gamma(t) \} \leq \mu_B \{ C_h^\gamma(t, t_k) \} \leq \sqrt{\frac{2m \sinh(\beta \omega/2)}{\pi \sinh(\beta \omega/2 - \omega|t - t_k|)}} h |t - t_k|^{-1/2}.
\]

The resulting inequality shows that \( \mu_B \{ C_h^\gamma(t) \} = 0 \) for \( \gamma > 1/2 \), and consequently

\[
\mu_B \{ C_h^\gamma \} = 0.
\]

We recall that a function \( x : [0, \beta] \to R \) is said to be Hölder continuous of order \( \gamma \) if there is a positive constant \( h \) such that \( |x(t) - x(t')| \leq h|t - t'|^\gamma \) for all \( t, t' \in [0, \beta] \) and \( \gamma \in (0, 1] \). Because

\[
\Gamma^\gamma = \{ x \in X : x \text{ is a Hölder continuous function of order } \gamma \} = \bigcup_{h=1}^{\infty} C_h^\gamma,
\]

condition (43) implies that \( \Gamma^\gamma, 1/2 < \gamma \leq 1 \), is a Borel subset in \( X \) with the Bogolyubov measure (or probability) zero. In other words, the Bogolyubov trajectories are not Hölder continuous of order \( \gamma > 1/2 \) almost everywhere with respect to the measure. (Consequently, they cannot be continuously differentiable.)

Let \( 0 \leq t \leq \beta \). We consider the set \( D_t = \{ x \in X : x'(t) \text{ exists} \} \), where \( x'(t) \) denotes the ordinary derivative of \( x \) with respect to \( t \) for \( t \in (0, \beta) \) and the one-sided derivative for \( t = 0 \) or \( t = \beta \). It can then be shown \( 29 \) that \( D_t \subset \bigcup_{h=1}^{\infty} C_h^1(t) \), whence \( \mu_B(D_t) = 0 \).

We define a function \( F = X \times [0, \beta] \to R \) by the relation

\[
F(x, t) = \begin{cases} 
1, & \text{if } x'(t) \text{ exists (as a finite function),} \\
0, & \text{otherwise.}
\end{cases}
\]

It can be proved \( 30 \) that \( F \) is measurable as a function of \( x \) and \( t \). Therefore, by the Fubini theorem,

\[
\int_X \left( \int_0^\beta F(x, t) \, dt \right) \, d\mu_B(x) = \int_0^\beta \left( \int_X F(x, t) \, d\mu_B(x) \right) \, dt = \int_0^\beta \mu_B(D_t) \, dt = 0.
\]

This formula shows that the relation

\[
\int_0^\beta F(x, t) \, dt = 0
\]

holds for almost all functions \( x \) with respect to the measure \( \mu_B \). Consequently, the relation \( F(x, t) = 0 \) holds for almost all functions \( x \) with respect to the Bogolyubov measure \( \mu_B \) and for almost all values of \( t \) with respect to the Lebesgue measure. We have thus proved that the trajectories \( x \in X \) are differentiable with probability 1 on at most a subset in \([0, \beta]\) of Lebesgue measure zero. Because every function \( x \) of bounded variation on any interval is always everywhere differentiable with respect to the Lebesgue measure on this interval \( X \), the Bogolyubov trajectories have unbounded variation with probability 1 on any subinterval of \([0, \beta]\).

### 7.2 Scale transformations in the Bogolyubov space

In the theory of Feynman continual integrals, the scale transformation \( x \to \sigma x \) with the parameter \( \sigma \in C \) in the related function space is important. An essential role is played here by the well-known Lévy theorem on the quadratic variation of Wiener trajectories \( 31 \) and by the special case that was investigated somewhat later in \( 32 \). In view of the possible analytic continuation with respect to temperature or mass in the Bogolyubov continual integral, it is interesting to apply the Lévy scheme to the case of the Bogolyubov measure.

We introduce Lévy quadratic variations of trajectories. We consider a partition \( \Pi \) of the closed interval \([0, \beta]\), \( 0 = t_0 < t_1 < \ldots < t_k = \beta \), and a function \( x \in X \). We define the function

\[
S_{\Pi}(x) = \sum_{j=1}^{k} \left[ x(t_j) - x(t_{j-1}) \right]^2.
\]
If the interval \([0, \beta]\) is partitioned into \(k\) equal subintervals, then we simply write \(S_k(x)\) instead of \(S_B(x)\). We note that

\[
\lim_{n \to \infty} S_{2^n}(x) = 0 \tag{45}
\]

for sufficiently smooth trajectories, for example, for those satisfying the Lipschitz condition with constant \(k\). However, as shown in the preceding section, the Bogolyubov measure is concentrated on nondifferentiable trajectories. Therefore, as in the case of the Wiener measure, it can be expected that condition (45) does not hold for Bogolyubov trajectories.

**Theorem 5.** The Bogolyubov trajectories \(x \in X\) satisfy the relation

\[
\lim_{n \to \infty} S_{2^n}(x) = \frac{\beta}{m}
\]

almost everywhere.

**Proof.** We first show that

\[
I_N \equiv \left\| S_N - \frac{\beta}{m} \right\|_2^2 = \frac{2\beta^2}{m^2} \frac{1}{N} + O\left(\frac{1}{N^2}\right) \tag{46}
\]

for any sufficiently large positive integer \(N\), where \(\| \cdot \|_2 \equiv \| \cdot \|_{L^2(X, \mu_B)}\) is the \(L^2\)-norm in the space \(X\) with the measure \(\mu_B\). It follows from definition (44) that

\[
S_N(x) = \sum_{j=1}^{N} [x(t_j) - x(t_{j-1})]^2, \quad t_j = \frac{j\beta}{N}, \quad j = 0, 1, \ldots, N.
\]

We have

\[
I_N = \int_X (S_N(x) - \frac{\beta}{m})^2 \, d\mu_B(x) = \int_X S_N^2(x) \, d\mu_B(x) - \frac{2\beta}{m} \int_X S_N(x) \, d\mu_B(x) + \frac{\beta^2}{m^2} \tag{47}
\]

for the desired expression \(I_N\). The random variable \(x(t_j) - x(t_{j-1})\) is distributed with a zero mean and variance (38), and therefore

\[
\int_X S_N(x) \, d\mu_B(x) = \sum_{j=1}^{N} \frac{\cosh(\beta \omega/2) - \cosh(\beta \omega/2 - \omega|t_j - t_{j-1}|)}{m \omega \sinh(\beta \omega/2)} = \frac{N \cosh(\beta \omega/2) - \cosh(\beta \omega/2 - \beta \omega/N)}{m \omega \sinh(\beta \omega/2)}.
\]

In particular,

\[
\int_X S_N(x) \, d\mu_B(x) = \frac{\beta}{m} - \frac{\beta^2 \omega}{2m} \coth \frac{\beta \omega}{2} \frac{1}{N} + O\left(\frac{1}{N^2}\right)
\]

as \(N \to \infty\). We now calculate the integral of \(S_N^2(x)\) in formula (47),

\[
\int_X S_N^2(x) \, d\mu_B(x) = \sum_{m,n=1}^{N} \int_X d\mu_B(x) \left[ x^2(t_n)x^2(t_m) + 2x^2(t_n)x^2(t_{m-1}) + x^2(t_{n-1})x^2(t_{m-1}) - 4x^2(t_n)x(t_{m-1})x(t_{m-1}) - 4x^2(t_n)x(t_m)x(t_{m-1})x(t_{m-1}) \right]. \tag{48}
\]

In calculating the integrals of individual terms in the right-hand side of (48), it is necessary to use the Wick theorem and the corresponding tabular values of finite sums in [3]. For example,

\[
\int_X d\mu_B(x) x^2(t_n)x^2(t_m) = B^2(t_n, t_m) + 2B^2(t_n, t_m).
\]
Using (36), we find
\[ \sum_{n,m=1}^{N} \int_{X} d\mu_{B}(x) x^{2}(t_{n})x^{2}(t_{m}) = \left( 2m\omega \sinh \frac{\beta\omega}{2} \right)^{-2} \times \]
\[ \times \left( N^{2} \cosh^{2} \frac{\beta\omega}{2} + N^{2} + N \sinh(\beta\omega) \coth \frac{\beta\omega}{N} \right). \]

Accordingly, calculating the other terms in the right-hand side of (48), we obtain
\[ \int_{X} S_{N}^{2}(x) d\mu_{B}(x) = \left( 2m\omega \sinh \frac{\beta\omega}{2} \right)^{-2} \left[ 4N^{2} \cosh^{2} \frac{\beta\omega}{2} + 4N^{2} \cosh^{2} \left( \frac{\beta\omega}{2} - \frac{\beta\omega}{N} \right) - \right. \]
\[ - 8N^{2} \cosh \frac{\beta\omega}{2} \cosh \left( \frac{\beta\omega}{2} - \frac{\beta\omega}{N} \right) + 6N^{2} - 8N^{2} \cosh \frac{\beta\omega}{N} + \]
\[ + 2N(N - 1) \cosh \frac{2\beta\omega}{N} + 2N + 2N \cosh \left( \frac{\beta\omega}{N} - \frac{2\beta\omega}{N} \right) + \]
\[ + 6N \sinh(\beta\omega) \cosh(\beta\omega/N) \sinh(\beta\omega/N) - 8N \frac{\sinh(\beta\omega)}{\sinh(\beta\omega/N)} \frac{\sinh(\beta\omega - (\beta\omega/N))}{\sinh(\beta\omega/N)}. \] (49)

With regard to passage to the limit as \( N \to \infty \), we obtain
\[ \int_{X} S_{N}^{2}(x) d\mu_{B}(x) = \frac{\beta^{2}}{m^{2}} + \frac{\beta^{2}}{2m^{2}} \frac{2\cosh(\beta\omega) - \beta\omega \sinh(\beta\omega) - 2}{\sinh^{2}(\beta\omega/2)} \frac{1}{N} + \varepsilon_{N} \]
from formula (49), where \( \varepsilon_{N} \sim O(1/N^{2}) \), is a positive number. As a result, we obtain relation (46) for the desired expression \( I_{N} \). In particular, it follows from (46) that
\[ \left\| S_{2n} - \frac{\beta}{m} \right\|_{2}^{2} = \frac{\beta^{2}}{m^{2}} \frac{1}{2^{n-1}} + \varepsilon_{2^{n}}. \] (50)

We consider the set
\[ E_{n} = \left\{ x \in X : \left| S_{2n}(x) - \frac{\beta}{m} \right| \geq \frac{\beta}{m} \frac{1}{2^{n/3}} + \mu_{n} \right\}, \] (51)
where
\[ \mu_{n} = \frac{\beta}{m} \frac{1}{2^{n/3}} \left( \sqrt{1 + \frac{m^{2}}{\beta^{2}} 2^{n-1} \varepsilon_{n} - 1} \right) \sim O \left( \frac{1}{24^{n/3+1}} \right). \]

We prove that
\[ \mu_{B}(E_{n}) \leq \frac{2}{2^{n/3}}. \] (52)

We suppose the contrary. Then
\[ \int_{X} \left| S_{2n}(x) - \frac{\beta}{m} \right|^{2} d\mu_{B}(x) \geq \int_{E_{n}} \left| S_{2n}(x) - \frac{\beta}{m} \right|^{2} d\mu_{B}(x) > \]
\[ > \left( \frac{\beta}{m} \frac{1}{2^{n/3}} + \mu_{n} \right)^{2} \frac{2}{2^{n/3}} = \frac{\beta^{2}}{m^{2}} \frac{1}{2^{n-1}} + \varepsilon_{2^{n}}, \]
which contradicts (50). We set
\[ F_{n} = \bigcup_{k=n}^{\infty} E_{k}. \]

Then it follows from (52) that
\[ \mu_{B}(F_{n}) \leq \sum_{k=n}^{\infty} \mu_{B}(E_{k}) \leq \frac{c}{2^{n/3}}, \] (53)
where \( c = 2^{4/3}(2^{1/3} - 1)^{-1} \). Formula (51) now implies that the inequality
\[
\left| S_{2k}(x) - \frac{\beta}{m} \right| < \frac{\beta}{m}2^{-k/3}
\]
holds for \( x \in X \setminus F_n = \bigcap_{k=n}^{\infty} E_k \), where \( E_k \) is the complement of the set \( E_k \) in \( X \), and for any \( k = n, n+1, \ldots \). Consequently, if there is an \( n \) such that \( x \notin F_n \), then
\[
\lim_{k \to \infty} S_{2k}(x) = \frac{\beta}{m}.
\] (54)
Therefore, condition (54) holds for all \( x \) possibly except for \( x \in F = \bigcap_{n=1}^{\infty} F_n \). However, inequality (53) implies that
\[
\mu_B(F) \leq \mu_B(F_n) \leq \frac{c}{2^{m/3}}
\]
for any \( n \), i.e., \( \mu_B(F) = 0 \). Theorem 5 is proved.

We consider the set
\[
\Omega_\sigma = \left\{ x \in X : \lim_{n \to \infty} S_{2n}(x) = \sigma^2 \frac{\beta}{m} \right\},
\]
where \( \sigma \) is a given positive number. Let \( \mu_B^{\sigma} (\sigma > 0) \) denote the image of the measure \( \mu_B \equiv \mu_B^1 \) under the mapping \( \varphi_\sigma : X \to X, \varphi_\sigma = \sigma x \). The measure \( \mu_B^{\sigma} = \mu_B^1 \circ \sigma^{-1} \) is defined on the Borel \( \sigma \)-algebra \( B(X) \), and the relation \( \mu_B^{\sigma}(B) = \mu_B^1(\sigma^{-1}B) \) holds for any \( B \in B \).

**Proposition 1 [29].**
1. The set \( \Omega_\sigma \) is Borel measurable for any \( \sigma > 0 \).
2. The relation \( \sigma_1 \Omega_{\sigma_2} = \Omega_{\sigma_1, \sigma_2} \) holds for any \( \sigma_1, \sigma_2 > 0 \); in particular, \( \sigma \Omega_1 = \Omega_\sigma \) for any \( \sigma > 0 \).
3. For any \( \sigma > 0 \), \( \mu_B^{\sigma}(\Omega_\sigma) = 1 \).
4. If \( \sigma_1 \neq \sigma_2 (\sigma_1, \sigma_2 > 0) \), then \( \Omega_{\sigma_1} \cap \Omega_{\sigma_2} = 0 \), i.e., the measures \( \mu_B^{\sigma_1} \) and \( \mu_B^{\sigma_2} \) are mutually orthogonal.

Assertion 3 in the proposition implies that \( \Omega_\sigma \) is a set of full \( \mu_B^{\sigma} \)-measure. Following [29], we say that the measure \( \mu_B^{\sigma} \) is concentrated on the set \( \Omega_\sigma \). We note that \( \text{supp} \mu_B^{\sigma} = X \) for any \( \sigma > 0 \) and \( \Omega_\sigma \subset X \).

A subset \( A \) in \( X \) is called a scale-invariant measurable set if \( \sigma A \in S_\infty \) for all \( \sigma > 0 \), where \( S_\infty \) is the \( \sigma \)-algebra of sets in the space \( X \) that are measurable with respect to the Bogolyubov measure \( \mu_B^1 \). A scale-invariant measurable set \( N \) is called a zero-scale-invariant measurable set if \( \mu_B^1(\sigma N) = 0 \) for all \( \sigma > 0 \). The classes of scale-invariant and zero-scale-invariant sets are denoted by \( S \) and \( N \) respectively. We let \( S_\sigma \) denote the \( \sigma \)-algebra obtained by completing the measurable space \( (X, B(X), \mu_B^{\sigma}) \) and \( N_\sigma \) the class of \( \mu_B^{\sigma} \)-zero sets. It can be shown that \( N_\sigma = \sigma N_\infty, S_\sigma = \sigma S_\infty \), and \( \mu_B^{\sigma}(E) = \mu_B^1(\sigma^{-1}E) \) for any \( E \in S_\sigma \). Moreover, the algebra \( S \) is a \( \sigma \)-algebra, \( S = \bigcap_{\sigma > 0} S_\sigma \), and \( N = \bigcap_{\sigma > 0} N_\sigma \). It can be easily seen that \( B(X) \subset S \subset S_\sigma \) for each \( \sigma > 0 \). We have \( E \in S \) if and only if \( E \cap \Omega_\sigma \in S_\sigma \) for any \( \sigma > 0 \), and \( N \in N \) if and only if \( N \cap \Omega_\sigma \in N_\sigma \) for any \( \sigma > 0 \).

The structure of scale-invariant and zero-scale-invariant sets is described in the following proposition.

**Proposition 2 [29].**
1. The inclusion \( E \in S \) holds if and only if the set \( E \) can be represented in the form
\[
E = \left( \bigcup_{\sigma > 0} E_\sigma \right) \cup L,
\] (55)
where each \( E_\sigma \) is an \( \mu_B^{\sigma} \)-measurable subset in \( \Omega_\sigma \) and \( L \) is an arbitrary subset of the set
\[
X \setminus \bigcup_{\sigma > 0} \Omega_\sigma.
\] (56)
Relation \( \mu_B^{\sigma}(E) = \mu_B^{\sigma}(E_\sigma) \) holds for all \( \sigma > 0 \) and any set \( E \) of form (55).
2. The inclusion \( N \in N \) holds if and only if the set \( N \) can be represented in the form
\[
N = \left( \bigcup_{\sigma > 0} N_\sigma \right) \cup L,
\]
where each set \( N_\sigma \) is an \( \mu_B^{\sigma} \)-measurable subset in \( \Omega_\sigma \) and \( L \) is an arbitrary subset of the set in (56).
8 Dynamic properties of Bogolyubov measure

8.1 Semigroups with respect to the Bogolyubov measure

Let \( \mathcal{L}(\mathcal{X}) \) denote the space of bounded linear operators in a Banach space \( X \). We recall that a family of operators \( \{T(t) : 0 \leq t < \infty\} \) in \( \mathcal{L}(\mathcal{X}) \) is called a strongly continuous semigroup of operators on \( X \) if \( T(0) = I \), \( T(t,s) = T(t)T(s) \) for all \( t, s \in [0, \infty) \), and the mapping \( t \mapsto T(t)x \) from \( [0, \infty) \) into \( X \) is continuous for each \( x \in X \). As is known \cite{34}, in the case of a strongly continuous semigroup, the generating operator (generator)

\[
L = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (T(\epsilon) - I)
\]

of the semigroup has a dense domain \( D \) in \( X \), is a closed linear operator, and

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} (T(t + \epsilon) - T(t))f = LT(t)f = T(t)Lf
\]

for \( f \in D \). For a strongly continuous semigroup \( \{T(t) : 0 \leq t < \infty\} \) with the generator \( L \) and an arbitrary vector \( f \in D \), it can be shown \cite{35} that the function \( u(t) = T(t)f \in X \) is continuously differentiable on the half-infinite interval \([0, \infty)\) and satisfies the initial condition \( u(0) = f \) and that the differential equation \( du/dt = Lu \) is satisfied for all \( t > 0 \).

In the case of Gaussian measures, there is a universal example of a strongly continuous semigroup known as the Ornstein–Uhlenbeck semigroup. Let \( \mu \) be a centered Gaussian measure on a locally convex space \( X \). The Ornstein–Uhlenbeck semigroup on the space \( L^p(\mu) \) is given by the formula

\[
T(t)f(x) = \int_X f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \, d\mu(y).
\] (57)

It was proved \cite{38} that for every \( p \geq 1 \), the family of operators \( \{T(t) : 0 \leq t < \infty\} \) defined by formula (57) forms a strongly continuous semigroup on the Banach space \( L^p(\mu) \) with the operator norm

\[
\|T(t)\|_{\mathcal{L}(L^p(\mu))} = 1.
\]

Moreover, the operators \( T(t) \) are nonnegative for \( p = 2 \).

In the case of the Bogolyubov measure, the form of the generator of the Ornstein–Uhlenbeck semigroup can be found under the assumption that \( f \in C^\infty_0(R) \), where \( C^\infty_0(R) \) is the space of infinitely differentiable functions compactly supported in \( R \). We have

\[
T(t)f(x) = \frac{1}{\sqrt{\pi \coth(\beta\omega/2)/(m\omega)}} \int_{-\infty}^{\infty} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \exp \left( -\frac{m\omega y^2}{\coth(\beta\omega/2)} \right) \, dy.
\]

The change of the integration variable \( e^{-t}x + \sqrt{1 - e^{-2t}}y = z \) results in

\[
\frac{T(t)f(x) - f(x)}{t} = \frac{1}{t} \left\{ \frac{m\omega \tanh(\beta\omega/2)}{\pi(1 - e^{-2t})} \int_{-\infty}^{\infty} (f(z) - f(x)) \times \right.
\]

\[
\left. \times \exp \left[ \frac{m\omega \tanh(\beta\omega/2)}{1 - e^{-2t}}(z - e^{-t}x)^2 \right] \, dz \right\}.
\]

We now use the Taylor theorem to expand \( f(z) \) under the integral sign,

\[
f(z) = f(x) + f'(x)(z - x) + \frac{1}{2} f''(x)(z - x)^2 + \frac{1}{6} f^{(3)}(x)(z - x)^3 + \frac{f^{(4)}(x)}{24}(z - x)^4.
\]

Furthermore, calculating the elementary Gaussian integrals, we obtain

\[
\frac{T(t)f(x) - f(x)}{t} = -xf'(x) + \frac{1}{2} f''(x) \frac{1}{2t} \frac{1 - e^{-2t}}{m\omega \tanh(\beta\omega/2)} + o(t),
\]

22
i.e., the generator of semigroup (57) in the case of the Bogolyubov measure is given by

\[ L = -x \frac{d}{dx} + \coth(\beta\omega/2) \frac{d^2}{dx^2}. \]

We next consider the family of operators \( \{ T(\beta) : 0 \leq \beta < \infty \} \) acting in the space \( L^2(R) \) according to the formula

\[ (T(\beta)f)(x) = \int_x^\infty d\mu_B(y) f\left( \int_0^\beta y(t) \, dt + x \right). \]  

(58)

It is clear that \( T(0) = I \). Moreover, using the formulas for integration with respect to Gaussian measures [37], we obtain

\[ (T(\beta)f)(x) = \sqrt{\frac{m\omega}{2\pi \beta}} \int_{-\infty}^\infty f(y) \exp \left[ -\frac{(y-x)^2m\omega^2}{2\beta} \right] \, dy. \]  

(59)

Formula (59) gives the well-known expression for the free semigroup in the case of the heat conduction equation. Hence, family of operators (58) is in fact a strongly continuous semigroup in \( L^2(R) \). In this case, the generator of the semigroup has the form

\[ L = \frac{1}{2m\omega^2} \frac{d^2}{dx^2}, \]

and for any \( f \in L^2(R) \), the function \( u(\beta, x) = (T(\beta)f)(x) \) is the solution of the Bloch equation

\[ \frac{\partial u}{\partial \beta} = \frac{1}{2m\omega^2} \frac{\partial^2 u}{\partial x^2} \]

with the initial condition \( u(0, x) = f(x) \). Formula (58) implies the relation between the Bogolyubov and Wiener measures

\[ \int_{C_\beta} f\left( x + \int_0^{m\omega^2t} y(\tau) \, d\tau \right) \, d\mu_B(y) = \int_{C_\beta} f(y(t) + x) \, d\mu_W(y), \]

where \( C_\beta \) is the space of continuous functions on \([0, t]\) vanishing at zero.

### 8.2 Independent increments

The classical Wiener process on the interval \([a, b]\) has independent increments, i.e., for any \( a < t_1 < t_2 < \ldots < t_n \leq b \), the random variables \( \xi_{t_2} - \xi_{t_1}, \ldots, \xi_{t_n} - \xi_{t_{n-1}} \) are independent. To prove this assertion, because the Wiener process is Gaussian, it suffices to show that these increments are pairwise independent. In the case of the Bogolyubov measure, the increments \( x(t_i) - x(t_{i-1}) \), \( i = 1, 2, \ldots, n \), are not independent, which substantially hampers an analysis of the corresponding random process. However, the relation between the Wiener and Bogolyubov measures established in Subsec. 8.1 permits constructing a system of independent increments for the Bogolyubov random process as well.

We consider the random variable

\[ y(t) = \lambda x(t) + \int_0^t x(\tau) \, d\tau, \quad 0 \leq t \leq \beta, \]

where \( \lambda \) is a constant to be defined below. The mathematical expectation \( M(\{y(t) \mid y(s)\}) \) is given by

\[ M(\{y(t) \mid y(s)\}) = \left[ \frac{\lambda^2}{2m\omega \sinh(\beta\omega/2)} \right] - \frac{1}{2m\omega^3 \sinh(\beta\omega/2)} \cosh \left( \omega |t - s| - \frac{\beta\omega}{2} \right) + \]

\[ + \frac{1}{2m\omega^2 \sinh(\beta\omega/2)} \left[ 2 \sinh\left( \frac{\beta\omega}{2} \right) \sin(\omega t) - \frac{1}{\omega} \cosh\left( \frac{\beta\omega}{2} \right) \right] + \]

\[ + \frac{1}{\omega} \cosh\left( \omega s - \frac{\beta\omega}{2} \right) + \frac{1}{\omega} \cosh\left( \omega t - \frac{\beta\omega}{2} \right) \]  

\[ + \frac{\lambda}{2m\omega^2 \sinh(\beta\omega/2)} \left[ 2 \sinh\left( \frac{\beta\omega}{2} \right) + \sinh\left( \omega s - \frac{\beta\omega}{2} \right) + \sinh\left( \omega t - \frac{\beta\omega}{2} \right) \right]. \]
Setting $\lambda = \omega^{-1}$, we can now easily show that

$$M[(y(t) - y(s))(y(\tau) - y(\sigma))] = 0 \quad \text{for} \quad s < t < \sigma < \tau.$$ 

Because the Bogolyubov process is Gaussian, we can state the above result in the form of the following theorem.

**Theorem 6.** A Gaussian random process with a Bogolyubov measure has independent increments, i.e., the random variables $y(t_2) - y(t_1), \ldots, y(t_n) - y(t_{n-1})$, where

$$y(t) = \omega^{-1}x(t) + \int_0^t x(\tau) \, d\tau, \quad 0 \leq t \leq \beta,$$

are independent for any $0 < t_1 < t_2 < \ldots < t_n \leq \beta$.

The random process $\{y(t), 0 \leq t \leq \beta\}$ is a Gaussian process with a zero average and the correlation function

$$M(y(t)y(s)) = \frac{1}{2m\omega^2\sinh(\beta\omega/2)} \left\{ 2 \left[ \omega^{-1} + \min(s,t) \right] \sinh \left( \frac{\beta\omega}{2} - \frac{1}{\omega} \cosh \left( \frac{\beta\omega}{2} \right) \right) \right.$$

$$+ \frac{1}{\omega} \left[ \cosh \left( \omega s - \frac{\beta\omega}{2} \right) + \sinh \left( \omega s - \frac{\beta\omega}{2} \right) \right] +$$

$$+ \frac{1}{\omega} \left[ \cosh \left( \omega t - \frac{\beta\omega}{2} \right) + \sinh \left( \omega t - \frac{\beta\omega}{2} \right) \right] \right\}. \quad \text{(61)}$$

Formula (61) permits proving that the Gaussian random variable $G \equiv y(t) - y(s)$ is normally distributed with a zero mean and the variance $(t-s)/(m\omega^2)$, where $t > s$, i.e.,

$$G \sim N \left( 0, \frac{t-s}{m\omega^2} \right).$$

We note that if $x(t)$ is regarded as a random function, then the integral

$$\int_0^t x(\tau) \, d\tau \quad \text{(62)}$$

introduced in previous sections is a stochastic integral defined as the limit in the mean with respect to the given measure for the corresponding integral sums. Integral (62) exists if and only if the mean value $M(y^2)$ exists. This condition is fulfilled for the Bogolyubov measure, which, in particular, follows from formula (61).

To conclude this subsection, we note that because

$$\int_X d\mu_B(x) \left( \frac{1}{\beta} \int_0^\beta x(t) \, dt \right)^2 = \frac{1}{\beta^2} \int_0^\beta \int_0^\beta d\tau B(t-\tau) = \frac{1}{\beta m\omega^2},$$

we have

$$\lim_{\beta \to \infty} M \left( \frac{1}{\beta} \int_0^\beta x(t) \, dt \right)^2 = 0.$$ 

Because the Bogolyubov random process has a zero mathematical expectation, $m \equiv Mx(t) = 0$, we can say that this is an ergodic process in the sense that the “temporal” means (with respect to $\beta$) converge in the squared mean to the “phase” means.

### 8.3 Bogolyubov measure and differential equations

We define the function

$$\delta_{\beta,\xi}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \, e^{iz[y(\beta) - y(0)] - iz\xi}, \quad \text{(63)}$$

where $x(t) \in X$ is an arbitrary function, $\xi$ is an arbitrary real number, $\beta$ is a positive number, and $y(t)$, $0 \leq t \leq \beta$, is defined in (60). Function (63) is an analogue of the Donsker–Lions function [38], which was introduced some time ago to investigate the Wiener measure.
Lemma. The mathematical expectation of function (63) is given by

\[ E_{\mu} \{ \delta_{\beta,\xi}(x) \} = \sqrt{\frac{m\omega^2}{2\pi\beta}} \exp \left( -\frac{m\omega^2}{2\beta} \xi^2 \right). \] (64)

Proof. We consider the mathematical expectation

\[ E_{\mu} \{ \delta_{\beta,\xi}(x) \} = \int_X \delta_{\beta,\xi}(x) \, d\mu_B(x) = \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz  e^{-iz\xi} \int_X d\mu_B(x) \exp \left[ iz \int_0^\beta x(t) \, dt \right] = \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz  e^{-iz\xi} \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} e^{izu} e^{-u^2/(2a)} \, du, \] (65)

where

\[ a = \int_0^\beta \int_0^\beta B(t,s) \, dt \, ds = \frac{\beta}{m\omega^2}. \]

Calculating the integrals in (65), we derive (64). The lemma is proved.

We introduce the function

\[ u(\beta,\xi) = E_{\mu_B} \left\{ \delta_{\beta,\xi}(x) \exp \left( -\int_0^\beta V(y(s) - y(0)) \, ds \right) \right\}, \] (66)

where \( V \) is a real function bounded from below.

Theorem 7. Function (66) is a solution of the partial differential equation

\[ \frac{\partial u}{\partial \beta} = \frac{1}{2m\omega^2} \frac{\partial^2 u}{\partial \xi^2} - V(\xi)u \] (67)

with the initial condition \( u(0,\xi) = \delta(\xi) \) and the boundary conditions \( u(\beta,\pm\infty) = 0 \).

Proof. We use the obvious formula

\[ \exp \left( -\int_0^\tau V(z(s)) \, ds \right) = 1 - \int_0^\tau V(z(\tau)) \exp \left( -\int_0^\tau V(z(s)) \, ds \right) \, d\tau. \]

This gives

\[ u(\beta,\xi) = E_{\mu_B} \left\{ \delta_{\beta,\xi}(x) \right\} - \int_0^\beta E_{\mu_B} \left\{ \delta_{\beta,\xi}(x) V(y(\tau) - y(0)) \exp \left( -\int_0^\tau V(y(s) - y(0)) \, ds \right) \right\} \, d\tau = \]

\[ = E_{\mu_B} \left\{ \delta_{\beta,\xi}(x) \right\} - \frac{1}{2\pi} \int_0^\beta d\tau \int_{-\infty}^{\infty} dz  e^{-iz\xi} \times \]

\[ \times E_{\mu_B} \left\{ V(y(\tau) - y(0)) \exp \left( -\int_0^\tau V(y(s) - y(0)) \, ds + iz(y(\beta) - y(0)) \right) \right\}. \]

At the same time, we have

\[ E_{\mu_B} \left\{ V(y(\tau) - y(0)) \exp \left( -\int_0^\tau V(y(s) - y(0)) \, ds + iz(y(\beta) - y(0)) \right) \right\} = \]

\[ = E_{\mu_B} \left\{ V(y(\tau) - y(0)) \exp \left( -\int_0^\tau V(y(s) - y(0)) \, ds + iz(y(\tau) - y(0)) \right) \right\} \times \]

\[ \times \exp \left( iz(y(\beta) - y(0)) - iz(y(\tau) - y(0)) \right) \]
\[
E_{\mu B} \left\{ V(y(\tau) - y(0)) \exp \left( - \int_0^\tau V(y(s) - y(0)) \, ds + iz(y(\tau) - y(0)) \right) \right\} \times \\
E_{\mu B} \left\{ \exp \left( iz(y(\beta) - y(\tau)) \right) \right\} = \\
= \exp \left( - \frac{\beta - \tau}{2m\omega^2} z^2 \right) E_{\mu B} \left\{ V(y(\tau) - y(0)) \times \\
\times \exp \left( - \int_0^\tau V(y(s) - y(0)) \, ds + iz(y(\tau) - y(0)) \right) \right\} = \\
= \exp \left( - \frac{\beta - \tau}{2m\omega^2} z^2 \right) \int_{-\infty}^{\infty} d\eta V(\eta) e^{iz\eta} \delta_{\tau,\eta}(x),
\]
which follows from Theorem 6 and the properties of function (63). Therefore,

\[
u(\beta, \xi) = E_{\mu B} \left\{ \delta_{\beta, \xi}(x) \right\} - \frac{1}{2\pi} \int_0^\beta d\tau \int_{-\infty}^{\infty} dz e^{-iz\xi} \exp \left( - \frac{\beta - \tau}{2m\omega^2} z^2 \right) \times \\
\times \int_{-\infty}^{\infty} d\eta V(\eta) e^{iz\eta} u(\tau, \eta).
\]

In view of

\[
\int_{-\infty}^{\infty} dz \exp \left( -iz\xi - \frac{\beta - \tau}{2m\omega^2} z^2 + iz\eta \right) = \sqrt{\frac{2\pi m\omega^2}{\beta - \tau}} \exp \left[ -\frac{2m\omega^2}{4} (\xi - \eta)^2 \right],
\]

we use the lemma to obtain

\[
u(\beta, \xi) = \sqrt{m\omega^2 \pi \beta} \exp \left( -\frac{m\omega^2}{2\beta} \xi^2 \right) - \\
- \int_0^\beta \int_{-\infty}^{\infty} V(\eta) u(\tau, \eta) \sqrt{\frac{m\omega^2}{2\pi(\beta - \tau)}} \exp \left[ -\frac{m\omega^2}{2(\beta - \tau)} (\xi - \eta)^2 \right] d\eta d\tau.
\]

Direct verification now readily shows that function (68) satisfies Eq. (67). The corresponding initial and boundary conditions are obviously satisfied. Theorem 7 is proved.

9 Inequalities for equilibrium averages

We consider a system with a Hamiltonian

\[
\hat{H} = \hat{\Gamma} + \hat{V},
\]
where \(\hat{V} = V(\hat{q})\) is an interaction term, and also a one-dimensional family of Hamiltonians,

\[
\hat{H}(h) = \hat{\Gamma}(h) + \hat{V}, \quad h \in R,
\]

\[
\hat{\Gamma}(h) = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2} (\hat{q} - h)^2.
\]
The statistical sum

\[
Z(h) = \text{Tr} e^{-\beta \hat{H}(h)}
\]
of the system under consideration becomes

\[
Z(h) = \text{Tr} e^{-\beta [\hat{\Gamma} + V(\hat{q} + h)]}
\]
after the canonical transformation \(\hat{q} - h \rightarrow \hat{q}\). We assume that the interaction potential is nonnegative and symmetric, i.e.,

\[
V(x) \geq 0, \quad V(x) = V(-x).
\]
Using the chronological-ordering operator, we can write

\[
e^{-\beta \hat{\Gamma} + \hat{\mathcal{V}}} = e^{-\beta \hat{\Gamma}} \exp \left( - \int_0^\beta ds \ e^{\hat{s} \hat{\mathcal{V}} e^{-s \hat{\Gamma}}} \right).
\]

Therefore,

\[
R(h) \equiv \frac{\Tr e^{-\beta \hat{H}(h)}}{\Tr e^{-\beta \hat{\Gamma}}} = \left\langle T \exp \left[ - \int_0^\beta ds \ V(\hat{Q}(s) + h) \right] \right\rangle_{\hat{\Gamma}}.
\]

Expressing relation (69) via the Bogolyubov functional integral, we obtain

\[
\int_X d\mu_B(x) = e^{-\|a\|^2/\hbar} \int_X F(x + a) e^{-\langle a, x \rangle} d\mu(x)
\]

for an integrable functional \(F(x)\) and a function \(a \in H\). In this situation, we use formula (70) for the case of the Bogolyubov measure and the constant functions \(a\) that belong to \(H\). This results in

\[
\int_X F(x + a) d\mu_B(x) = e^{-\beta m \omega^2 a^2 / 2} \int_X F(x) \exp \left\{ m \omega^2 \int_0^\beta x(t) dt \right\} d\mu_B(x).
\]

We now consider the Fourier–Gauss transform

\[
\tilde{f}(y) \equiv F(f; y) = \int_X f(x + iy) d\mu_B(x)
\]

of a functional \(f(x)\) and the Parseval relation

\[
\int_X f \left( \frac{x}{\sqrt{2}} \right) g^* \left( \frac{x}{\sqrt{2}} \right) d\mu_B(x) = \int_X F \left( f; \frac{y}{\sqrt{2}} \right) F^* \left( g; \frac{y}{\sqrt{2}} \right) d\mu_B(y)
\]

for the the case of functionals

\[
f(x) = F(x) \equiv \exp \left\{ - \int_0^\beta dt V(x(t)) \right\}
\]

and

\[
g(x) = \exp \left\{ m \omega^2 \int_0^\beta x(t) dt \right\}.
\]

Relation (71) becomes

\[
e^{-\beta m \omega^2 / 2} \int_X F \left( \frac{x}{\sqrt{2}} \right) \exp \left\{ \frac{i}{\sqrt{2}} m \omega^2 \int_0^\beta x(t) dt \right\} d\mu_B(x) = \int_X \tilde{F} \left( \frac{y}{\sqrt{2}} \right) \exp \left\{ \frac{i}{\sqrt{2}} m \omega^2 \int_0^\beta y(t) dt \right\} d\mu_B(y),
\]

whence we see that if the inequality

\[
\tilde{F}(y) \geq 0
\]

holds for all \(y\), then

\[
R(h) = \tilde{F}(-ih) \leq R(0) = \tilde{F}(0).
\]
Condition (72) is proved as follows. We have

\[ \bar{F}(y) = e^{(y, y)/2} \int_X \exp \left\{ - \int_0^\beta V(x) \, dt + i(x, y) \right\} \, d\mu_B(x) = \]

\[ = e^{(y, y)/2} \int_X \exp \left\{ - \int_0^\beta V(x) \, dt - i(x, y) \right\} \, d\mu_B(x) = \bar{F}^*(y) \]

for symmetric functionals, i.e., the Fourier–Gauss transform is real in this case. We prove that it is nonnegative. In view of

\[ e^{\|y\|^2/2} \int_X e^{i(x, y)} \, d\mu_B(x) = 1, \]

applying the Jensen inequality yields

\[ \bar{F}(y) \geq \exp \left\{ - e^{\|y\|^2/2} \int_X \exp \left\{ - \int_0^\beta \beta_0 V(x) \, dt - i(x, y) \right\} \, d\mu_B(x) \right\} \geq 0, \]

which precisely completes the proof of (72).

In particular, condition (73) implies that

\[ (\hat{q}, \hat{q})_{\hat{H}} \leq \frac{1}{\beta m \omega^2}, \]

where the Bogolyubov inner product of arbitrary operators \( \hat{A} \) and \( \hat{B} \) is defined as

\[ (\hat{A}, \hat{B})_{\hat{H}} = \frac{1}{\beta \text{Tr} e^{-\beta \hat{H}}} \int_0^\beta ds \, \text{Tr} [ e^{-s\hat{H}} \hat{A} e^{-(\beta-s)\hat{H}} \hat{B} ] = (\hat{B}, \hat{A})_{\hat{H}}. \]

Using the relations

\[ \hat{q} = \frac{1}{\sqrt{2m \omega}} (\hat{b} + \hat{b}^\dagger), \quad \hat{p} = i \sqrt{\frac{m \omega}{2}} (\hat{b}^\dagger - \hat{b}) \]

to pass from the operators \( \hat{q} \) and \( \hat{p} \) to \( \hat{b} \) and \( \hat{b}^\dagger \) and taking the selection rules for equilibrium averages with respect to the quadratic Hamiltonian into account, we bring inequality (74) to the form

\[ (\hat{b}^\dagger, \hat{b})_{\hat{H}} \leq (\beta \omega)^{-1}. \]

Relation (74) can be used to derive an inequality for the Gibbs equilibrium average \( \langle \hat{q}^2 \rangle_{\hat{H}} \). For this, the Falk–Bruch inequality [39] should be used. Let

\[ g = \langle \hat{q}^2 \rangle_{\hat{H}}, \quad b = (\hat{q}, \hat{q})_{\hat{H}}, \quad c = \langle [\hat{q}, [\beta \hat{H}, \hat{q}]] \rangle_{\hat{H}}. \]

We also assume that the upper estimates \( b \leq b_0 \) and \( c \leq c_0 \) hold. Then

\[ g \leq g_0 \equiv \frac{1}{2} \sqrt{c_0 b_0} \coth \frac{\beta \omega}{4b_0}. \]

In the case under study, we have \( b_0 = (\beta m \omega^2)^{-1} \) and \( c_0 = \beta / m \), and the above inequality gives

\[ \langle \hat{q}^2 \rangle_{\hat{H}} \leq \frac{1}{2m \omega} \coth \frac{\beta \omega}{2} = \langle \hat{q}^2 \rangle_{\hat{\Gamma}}. \]  

(75)

Condition (73) is an example of the so-called Gaussian domination condition [40], and condition (74) implied by (73) is an example of the so-called local Gaussian domination condition [41], which plays an important role in phase transition theory. An estimate of type (75) was previously found for a less general case of a one-dimensional nonlinear oscillator [42].

Partial financial support by RFBR, grant 99–01–00887 (Russia).
References

[1] M. Fréchet, Bull. Soc. Math. France, 43, 249 (1915).
[2] P. J. Daniell, Ann. Math., 19, 279 (1917–1918).
[3] P. J. Daniell, Ann. Math., 20, 281 (1918–1919).
[4] L. H. Loomis, An Introduction to Abstract Harmonic Analysis, Van Nostrand, Toronto (1953).
[5] F. Riesz and B. Sz.-Nagy, Leçons d’analyse fonctionnelle, (6th ed.), Akadémiai Kiadó, Budapest (1972).
[6] N. Wiener, Ann. Math., 22, 66 (1920).
[7] R. Feynman, Rev. Mod. Phys., 20, 367 (1948).
[8] R. Cameron, J. Anal. Math., 10, 287 (1962–1963).
[9] S. F. Edwards and R. E. Pierls, Proc. Roy. Soc., A224, 24 (1954).
[10] K. Symanzik, Z. Naturforsch, 9a, 809 (1954).
[11] I. M. Gelfand and R. A. Minlos, Dokl. Akad. Nauk SSSR, 97, 209 (1954).
[12] E. S. Fradkin, Dokl. Akad. Nauk SSSR, 98, 47 (1954).
[13] N. N. Bogolyubov, Dokl. Akad. Nauk SSSR, 99, 225 (1954).
[14] A. V. Svidzinskii, Zh. Eksp. Teor. Fiz., 31, 324 (1956).
[15] N. N. Bogolyubov and N. N. Bogolyubov, Jr., Aspects of the polaron theory [in Russian] (Commun. No. R17-81-65), Joint Inst. Nucl. Res., Dubna (1981)
[16] D. P. Sankovich, Theor. Math. Phys., 119, 345 (1999).
[17] F. J. Dyson, Phys. Rev., 75, 486 (1949).
[18] N. N. Bogolyubov and D. V. Shirkov, Introduction to the Theory of Quantum Fields (2nd ed.), Wiley, New York (1959).
[19] M. Reed and B. Simon, Methods of Modern Mathematical Physics, V.2, Acad. Press, New York (1975).
[20] T. Kato, J. Math. Soc. Japan, 5, 208 (1953).
[21] N. Bourbaki, Éléments de mathématique. Première partie. Les structures fondamentales de l’Analyse. Livre VI. Integration, Hermann, Paris.
[22] Yu. L. Dalecky and S. V. Fomin, Measures and Differential Equations in Infinite-Dimensional Space, Kluwer, Dordrecht (1991).
[23] J. Kuelbs, J. Funct. Anal., 5, 354 (1970).
[24] B. S. Rajput, J. Multivariate Anal., 2, 282 (1972).
[25] L. Gross, ”Abstract Wiener spaces”, in: Proc. 5th Berkeley Symp. Math. Stat. and Prob. (L. M. LeCam and J. Neyman, eds.), Vol.2, Univ. of California Press, Berkeley (1967), p.31.
[26] A. D. Egorov and L. A. Yanovich, Dokl. Akad. Nauk Belorus. SSR, 14, 873 (1970).
[27] M. Kac, ”On some connections between probability theory and differential and integral equations”, in: Proc. 2nd Berkeley Symp. Math. Stat. and Prob. (J. Neyman, ed), Univ. of California Press, Berkeley (1951), p.189.
[28] P. I. Sobolevskii and L. A. Yanovich, Dokl. Akad. Nauk Belorus. SSR, 18, 965 (1974).
[29] G. W. Johnson and M. L. Lapidus, *The Feynman Integral and Feynman’s Operational Calculus*, Clarendon, Oxford (2000).

[30] H. L. Royden, *Real Analysis*, (3rd ed.), Macmillan, New York (1988).

[31] P. Lévy, *Am. J. Math.*, 62, 487 (1940).

[32] R. H. Cameron and W. T. Martin, *Bull. Am. Math. Soc.*, 53, 130 (1947).

[33] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series, Vol. 1. Elementary Functions*, Gordon and Breach, New York (1988).

[34] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Am. Math. Soc., Providence, RI (1957).

[35] E. B. Davies, *One-Parameter Semigroups*, Acad. Press, London (1980).

[36] V. I. Bogachev, *Gaussian Measures*, Am. Math. Soc., Providence, RI (1998).

[37] L. A. Yanovich, *Approximate Calculation of Continual Integrals with Respect to Gaussian Measures* [in Russian], Nauka i Tekhnika, Minsk (1976).

[38] Hui-Hsiung Kuo, *Gaussian Measures in the Banach Spaces*, Springer, Berlin (1975).

[39] H. Falk and L. W. Bruch, *Phys. Rev.*, 180, 442 (1969).

[40] J. Fröhlich, *Bull. Am. Math. Soc.*, 84, 165 (1978).

[41] D. P. Sankovich, *Theor. Math. Phys.*, 79, 656 (1989).

[42] N. N. Bogolyubov Jr. and D. P. Sankovich, *Phys. Lett. A*, 137, 179 (1989).