Research article

On the super edge-magic deficiency of some graphs

Vira Hari Krisnawati a,*, Anak Agung Gede Ngrah b,*, Noor Hidayat a, Abdul Rosuf Alghofari a

a Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Brawijaya, Jl. Veteran Malang, Jawa Timur, Indonesia
b Department of Civil Engineering, Faculty of Engineering, Universitas Mercu Mading, Jl. Taman Agung No.1 Malang, Jawa Timur, Indonesia

1. Introduction

Let G be a finite and simple graph having vertex set V(G) and edge set E(G), where p = |V(G)| and q = |E(G)|. A labeling of G is a bijection f : V(G) → {1, 2, ..., p} such that f(u) + f(uv) + f(v) is a constant for every edge uv ∈ E(G). Such a case, f is called a super edge magic labeling of G. A bipartite graph G with partite sets A and B is called consecutively super edge magic if there exists a super edge-magic labeling f with the property that f(A) = {1, 2, ..., |A|} and f(B) = {1 + |A|, 2 + |A|, ..., |V(G)|}. The super edge-magic deficiency of a graph G, denoted by μ(G), is either the minimum nonnegative integer n such that G ∪ nK₂ is super edge-magic or +∞ if there exists no such n. The consecutively super edge-magic deficiency of a bipartite graph G, denoted by μ(G), is either the minimum nonnegative integer n such that G ∪ nK₂ is consecutively super edge-magic or +∞ if there exists no such n. In this paper, we study the super edge-magic deficiency of some graphs. We investigate the (consecutively) super edge-magic deficiency of forests with two components. We also investigate the super edge-magic deficiency of a 2-regular graph 2C₃ ∪ C₂ and join product of K₁,₄ ∪ P₅ with an isolated vertex.

Lemma 1.1. [5] A graph G is SEM if and only if there exists a bijection f : V(G) → {1, 2, ..., p} such that the set of all edge-sums S = {f(x) + f(y) : xy ∈ E(G)} consists of q consecutive integers. In this case, f extends to be a SEM labeling of G with magic constant k = p + q + min(S).

The next lemma proved by Enomoto et al. [2] gives sufficient condition for non-existence of a SEM labeling of a graph.

Lemma 1.2. [2] If G is a SEM graph then q ≤ 2p − 3.

Moreover, Kotzig and Rosa [1] also proved that for every graph G there exists a nonnegative integer n such that G ∪ nK₂ is an edge-magic graph. This fact leads them to introduce the concept of edge-magic deficiency of a graph. The edge-magic deficiency of a graph G, μ(G), is defined as the minimum nonnegative integer n such that G ∪ nK₂ is an edge-magic graph. This concept motivated Figueueroa-Centeno et al. [6] to introduce the concept of super edge-magic deficiency of a graph. The super edge-magic deficiency (SEMD) of a graph G, μ(G), is defined as either the minimum nonnegative integer n such that G ∪ nK₂ is a SEM graph or +∞ if there exists no such n. In 2016, Ichishima et al. [7] give the notion of consecutively super edge-magic deficiency of a bipartite graph.

© 2020 Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (https://creativecommons.org/licenses/by-nc-nd/4.0/).
The consecutively super edge-magic deficiency (consecutively SEMD) of a bipartite graph $G$, $\mu_{c}(G)$, is defined to be either the smallest nonnegative integer $n$ with the property that $G \cup nK_1$ is consecutively SEM or $+\infty$ if there exists no such $n$.

Many researchers have investigated the SEMD of some classes of graphs. The complete results on this subject can be seen in a dynamic survey of graph labeling by Gallian [8]. In this paper, we study the (consecutively) SEMD of forests with two components, where its component are non isomorphic combs, isomorphic a subdivision of $K_{1,3}$ and $K_{1,4}$, and union of a comb and a subdivision of $K_{1,3}$ or $K_{1,4}$. Moreover, we find SEMD of 2-regular graph $2C_1 + C_2$. By applying Ichishima's et al. [9] and Gichacz’s et al. [10] methods to this result, we prove that some 2-regular graphs with a large order have zero SEMD, and then we also obtain the SEMD of union of cycles and paths. In addition, we study the SEMD of graph $(K_{1,3} \cup P_n) + K_1$ for any integer $n \geq 1$ and $m \geq 3$.

2. The SEMD of forests with two components

Figueroa-Centeno et al. [6] proved that the SEMD of all forests is finite. Next, the same authors [11] investigated the SEMD of some classes of forests with two components, such as $P_n \cup K_{1,n}, K_{1,n} \cup K_{1,m}$, and $P_n \cup P_n$. Based on these results, they gave the following conjecture.

Conjecture 2.1. [11] If $F$ is a forest with two components then $\mu_{c}(F) \leq 1$.

Inspired by Conjecture 2.1, many researchers have investigated the SEMD of some forests with two components. Baig et al. [12] found the SEMD of union of combs and stars. Javed et al. [13] studied the SEMD of forests with two components consisting of combs, generalized combs, and stars. In particular, they found the SEMD of two isomorphic combs. Imran and Mukhtar [14] showed that forests with two components consisting of stars and subdivision of stars have zero SEMD. Krishanawati et al. [15] investigated SEMD of two non isomorphic of a subdivision of $K_{1,3}$ and $K_{1,4}$.

In this section, we study SEMD of disjoint union of two non isomorphic combs. We also continue Krishanawati et al. [15] work to find SEMD of disjoint union of two isomorphic of a subdivision of $K_{1,3}$ and $K_{1,4}$. We prove that these graphs have zero SEMD. Some of these graphs also have zero consecutively SEMD. Additionally, we investigate the (consecutively) SEMD of union of a comb and a subdivision of $K_{1,3}$ or $K_{1,4}$.

A comb, $C_b$, with $n \geq 1$, is a tree consisting of path $P_{n+1}$, whose vertices are $u_0, u_1, u_2, \ldots, u_n$, together with edges $\{u_i, v_i ; 1 \leq i \leq n\}$. We investigate zero (consecutively) SEMD of two non isomorphic combs in the following theorem.

Theorem 2.2. Let $G_{n,m} = C_b \cup C_{b'}$. For any $n, m \geq 1$ and $n \neq m$, $\mu_c(G_{n,m}) = \mu_c(G_{n,m}) = 0$.

Proof. Let $G_{n,m}$ be a graph with vertex set and edge set as follows.

$$V(G_{n,m}) = \{u_i : 0 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup \{w_j : 0 \leq j \leq m\} \cup \{x_j : 1 \leq j \leq m\}$$

and

$$E(G_{n,m}) = \{u_i u_{i+1} : 0 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n\} \cup \{w_j w_{j+1} : 0 \leq j \leq m-1\} \cup \{w_j x_j : 1 \leq j \leq m\}$$

Thus, $G_{n,m}$ is a graph of order $2(n+m+1)$ and size $2(n+m)$.

For any $n, m \geq 1$ and $n \neq m$, define a labeling $\gamma : V(G_{n,m}) \rightarrow \{1, 2, \ldots, 2(n+m+1)\}$. Without loss of generality, we assume $n < m$. We consider the proof based on the value of $n$.

Case 1. $n$ is odd.

$$f(u_i) = \begin{cases} 
1 + 2, & \text{if } a = u_i \text{ and } i \text{ is even}, \\
0 & \text{if } a = u_i \text{ and } i \text{ is odd}, \\
i + 2, & \text{if } a = v_i \text{ and } i \text{ is odd}, \\
n + m + i + 1, & \text{if } a = v_i \text{ and } i \text{ is even}, \\
2n + m + j + 2, & \text{if } a = w_j \text{ and } j \text{ is even}, \\
n + j + 2, & \text{if } a = w_j \text{ and } j \text{ is odd, } 1 \leq j \leq n, \\
n + j + 1, & \text{if } a = u_j \text{ and } j \text{ is odd, } n+2 \leq j \leq m, \\
2n + m + j + 2, & \text{if } a = x_j \text{ and } j \text{ is odd, } 2 \leq j \leq n-1, \\
1, & \text{if } a = x_{n+1}, \\
n + j + 1, & \text{if } a = x_j \text{ and } j \text{ is even, } n+3 \leq j \leq m.
\end{cases}$$

Case 2. $n$ is even.

$$f(u_i) = \begin{cases} 
n + m + i + 2, & \text{if } a = u_i \text{ and } i \text{ is even,} \\
i + 1, & \text{if } a = u_i \text{ and } i \text{ is odd,} \\
n + m + i + 2, & \text{if } a = v_i \text{ and } i \text{ is odd,} \\
i + 1, & \text{if } a = v_i \text{ and } i \text{ is even}, \\
n + j + 2, & \text{if } a = u_j \text{ and } j \text{ is even, } 0 \leq j \leq n, \\
n + j + 1, & \text{if } a = u_j \text{ and } j \text{ is odd, } n+2 \leq j \leq m, \\
2n + m + j + 2, & \text{if } a = w_j \text{ and } j \text{ is odd, } 2 \leq j \leq n-1, \\
n + j + 2, & \text{if } a = w_j \text{ and } j \text{ is odd, } n+3 \leq j \leq m, \\
2n + m + j + 2, & \text{if } a = x_j \text{ and } j \text{ is even.}
\end{cases}$$

It is not hard to verify that the set of all edge-sums generated by $f$ is $S = \{(n + m + 4, n + m + 5, \ldots, 3(n + m + 1))\}$. By Lemma 1.1, $f$ extends to a SEM labeling of $G_{n,m}$ with magic constant $k = 5(n+m) + 6$. Hence, for any $n, m \geq 1$ and $n \neq m$, $\mu_c(G_{n,m}) = 0$.

Next, let $A$ and $B$ be partide sets of $G_{n,m}$.

For odd $n$,

$$A = \{u_i : i \equiv 0 \mod(2)\} \cup \{v_i : i \equiv 1 \mod(2)\} \cup \{w_j : j \equiv 1 \mod(2)\} \cup \{x_j : j \equiv 0 \mod(2)\}$$

and

$$B = \{u_i : i \equiv 1 \mod(2)\} \cup \{v_i : i \equiv 0 \mod(2)\} \cup \{w_j : j \equiv 0 \mod(2)\} \cup \{x_j : j \equiv 1 \mod(2)\}.$$ 

For even $n$,

$$A = \{u_i : i \equiv 1 \mod(2)\} \cup \{v_i : i \equiv 0 \mod(2)\} \cup \{w_j : j \equiv 0 \mod(2)\} \cup \{x_j : j \equiv 1 \mod(2)\}$$

and

$$B = \{u_i : i \equiv 0 \mod(2)\} \cup \{v_i : i \equiv 1 \mod(2)\} \cup \{w_j : j \equiv 1 \mod(2)\} \cup \{x_j : j \equiv 0 \mod(2)\}.$$ 

Since $f(A) = \{1, 2, \ldots, n+m+1\}$ and $f(B) = \{n+m+2, n+m+3, \ldots, 2(n+m+1)\}$, so $G_{n,m}$ is a consecutively SEM graph. Thus, for any $n, m \geq 1$ and $n \neq m$, $\mu_c(G_{n,m}) = 0$.

As an illustration of the proof of Theorem 2.2, see the Fig. 1.

A subdivision of a star $K_{1, r}$, denoted by $T(n_1, n_2, \ldots, n_r)$, where $n_i \geq 1$, $1 \leq i \leq r$, is a graph obtained by inserting $n_i - 1$ vertices to each edge of the star $K_{1, r}$. Now, we study (consecutively) SEMD of disjoint union of two isomorphic of subdivision of $K_{1,3}$ and $K_{1,4}$ in the following theorems.

Theorem 2.3. Let $G_n = 2T(n, n - 1, n - 1)$. For any even $n \geq 2$, $\mu_c(G_n) = \mu_c(G_n) = 0$ and for any odd $n \geq 3$, $\mu_c(G_n) \leq 1$. 

Proof. Let $G_n$ be a graph with

$$V(G_n) = \{ y, y_1, y_2, y_3, z_1, z_2, z_3 : 1 \leq j \leq n, 1 \leq l \leq n-1 \} \cup$$

$$\{ z, z_1, z_2, z_3 : 1 \leq j \leq n, 1 \leq l \leq n-1 \}$$

and

$$E(G_n) = \{ y y_1, z z_1 : 1 \leq l \leq 3 \} \cup \{ y_1 y_{j+1}, z_1 z_{j+1} : 1 \leq j \leq n-1 \} \cup$$

$$\{ y_{j+1}, z_{j+1} z_{j+2}, z_1 z_{j+4} : 1 \leq l \leq n-2 \}.$$ 

Thus, $G_n$ is a graph of order $6n-2$ and size $6n-4$. Let $n \geq 2$ be an even integer. Define a labeling $f : V(G_n) \rightarrow \{1, 2, \ldots, 6n-2\}$ as follows.

$$f(y) = \frac{1}{2}(7n), \quad f(z) = \frac{1}{2}(11n-4).$$

The set of all edge-sums generated by $f$ is $S = \{3n+2, 3n+3, \ldots, 9n-3\}$. By Lemma 1.1, $f$ extends to a SEM labeling of $G_n$ with magic constant $k = 15n - 3$. Hence, for any even $n \geq 2$, $\mu_n(G_n) = 0$.

Moreover, let $A$ and $B$ be partite sets of $G_n$, where

$$A = \{ y_1, y_2, y_3, z_1, z_2, z_3 : l, l \equiv 0 (\mod 2) \} \cup \{ z_1, z_2, z_3 : l, l \equiv 0 (\mod 2) \} \cup \{ z, z_1, z_2, z_3 : l, l \equiv 0 (\mod 2) \},$$

$$B = \{ y, y_1, y_2, y_3, z_1, z_2, z_3 : l, l \equiv 1 (\mod 2) \} \cup \{ z, z_1, z_2, z_3 : l, l \equiv 1 (\mod 2) \}. $$

It is easy to see that $f(A) = \{1, 2, \ldots, 3n\}$ and $f(B) = \{3n+1, 3n+2, \ldots, 6n-2\}$. So, $G_n$ is a consecutively SEM graph. Thus, $\mu_n(G_n) = 0$ for any even $n \geq 2$.

Now, to show $\mu_n(G_n) \leq 1$ for any odd $n \geq 3$, let us consider the graph $H_n = G_n \cup K_1$, where $V(H_n) = V(G_n) \cup \{w\}$ and $E(H_n) = E(G_n)$. Define a labeling $g : V(H_n) \rightarrow \{1, 2, \ldots, 6n-1\}$ as follows.

$$g(y) = \frac{1}{2}(7n+1), \quad g(z) = 6n-2, \quad g(w) = \frac{1}{2}(11n-3).$$

To label the rest of the vertices, we consider the following cases.

(i) For $n \equiv 3 \pmod{4}$ and $n > 5$, set

$$g(z_j) = \begin{cases} \frac{1}{2}(6n-l-2) & \text{for } l = 4r, \ 1 \leq r \leq \frac{n-1}{4}, \\ \frac{1}{2}(6n-l+2) & \text{for } l = 4r + 2, \ 1 \leq r \leq \frac{n-3}{4}. \end{cases}$$

(ii) For $n \equiv 1 \pmod{4}$ and $n > 5$, set

$$g(z_j) = \begin{cases} \frac{3n-l+2}{2} & \text{for } l = 2r + 2, \ 1 \leq r \leq \frac{n-1}{4}, \\ \frac{1}{2}(7n-2l+1) & \text{for } l = \frac{1}{2}(n+4r+3), \ 1 \leq r \leq \frac{n-3}{4}. \end{cases}$$

The set of all edge-sums generated by $g$ is $S' = \{3n+2, 3n+3, \ldots, 9n-3\}$. By Lemma 1.1, $g$ extends to a SEM labeling of $H_n$ with magic constant $k = 15n - 3$. Hence, for any odd $n \geq 3$, $\mu_n(G_n) \leq 1$. 

Theorem 2.4. Let $G_n = 2T(n, n-1, n-1, n)$. For any $n \geq 2$, $\mu_n(G_n) = 0$.

Proof. The graph $G_n$ is a graph of order $8n-2$ and size $8n-4$. Let

$$V(G_n) = \{ y, y_1, y_2, y_3, z_1, z_2, z_3, j : 1 \leq j \leq n, 1 \leq l \leq n-1 \} \cup$$

$$\{ z, z_1, z_2, z_3, j : 1 \leq j \leq n, 1 \leq l \leq n-1 \}$$

and

$$E(G_n) = \{ y y_1, z z_1 : 1 \leq l \leq 4 \} \cup$$

$$\{ y_{j+1}, z_{j+1} z_{j+2}, z_1 z_{j+4} : 1 \leq l \leq n-2 \}.$$ 

Define a labeling $f : V(G_n) \rightarrow \{1, 2, \ldots, 8n-2\}$ as follows.

$$f(y) = 5n + 1, \quad f(z) = 6n - 2, \quad f(w) = \frac{1}{2}(11n-3).$$

$$f(a) = \begin{cases} \frac{1}{2}(n-j+2), & \text{if } a = y_{j+1}, \ j \text{ is odd}, \\ \frac{1}{2}(7n-j+1), & \text{if } a = y_{j+1}, \ j \text{ is even}, \\ \frac{1}{2}(n+l+2), & \text{if } a = y_{j+1}, \ l \text{ is odd}, \\ \frac{1}{2}(7n+l+1), & \text{if } a = y_{j+1}, \ l \neq n - 1 \text{ is even}, \\ 6n - 1, & \text{if } a = y_{2n-1}. \end{cases}$$

$$f(a) = \begin{cases} \frac{1}{2}(3n-l), & \text{if } a = y_{j+1}, \ l \text{ is odd}, \\ \frac{1}{2}(9n-l-1), & \text{if } a = y_{j+1}, \ l \text{ is even}, \\ \frac{1}{2}(4n + l + 1), & \text{if } a = z_{j+1}, \ l \text{ is odd}, \\ \frac{1}{2}(10n - l - 2), & \text{if } a = z_{j+1}, \ l \text{ is even}, \\ \frac{1}{2}(4n + l + 1), & \text{if } a = z_{j+1}, \ l \text{ is even}, \\ \frac{1}{2}(10n + l - 4), & \text{if } a = z_{j+1}, \ l \text{ is odd}, \\ 3n + l - 3, & \text{if } a = z_{j+1}, \ l \text{ is odd}, \end{cases}$$

$$\begin{cases} \mu_n(G_n) \leq 1. \end{cases}$$

\[ \mu_n(G_n) \leq 1. \]
To label the vertices of the second component of \( G_n \), we consider two following cases.

Case 1. \( n \) is even.

For \( n = 4 \), set \( f(z) = 27 \) and
\[
\begin{align*}
\text{For } (z_{1,1}) = 10, \quad f(z_{1,2}) = 26, \quad f(z_{1,3}) = 9, \quad f(z_{1,4}) = 25,
\text{For } (z_{2,1}) = 11, \quad f(z_{2,2}) = 28, \quad f(z_{2,3}) = 12,
\text{For } (z_{3,1}) = 14, \quad f(z_{3,2}) = 29, \quad f(z_{3,3}) = 13,
\text{For } (z_{4,1}) = 17, \quad f(z_{4,2}) = 16, \quad f(z_{4,3}) = 30, \quad f(z_{4,4}) = 15.
\end{align*}
\]
For \( n \neq 4 \), set \( f(z) = 7n \) and
\[
f(a) = \begin{cases} 
\frac{1}{2}(4n + j + 1), & \text{if } a = z_{1,j} \text{ and } j \text{ is odd}, \\
\frac{1}{2}(12n + j), & \text{if } a = z_{1,j} \text{ and } j \text{ is even}, \\
\frac{1}{2}(6n - l + 1), & \text{if } a = z_{2,j} \text{ and } l \text{ is odd}, \\
\frac{1}{2}(14n - l), & \text{if } a = z_{2,j} \text{ and } l \text{ is even}, \\
\frac{1}{2}(8n - j), & \text{if } a = z_{3,j} \text{ and } j = 3, 5, \\
\frac{1}{2}(16n + j - 9), & \text{if } a = z_{4,j} \text{ and } j \geq 7 \text{ is odd}, \\
4n + 1, & \text{if } a = z_{4,1}.
\end{cases}
\]

For even \( j \geq 4 \), label \( z_{4,j} \) as follows.

(i) For \( n \equiv 2 \pmod{4} \),
\[
f(z_{4,j}) = \begin{cases} 
\frac{1}{2}(8n - j) & \text{for } j = 4r, 1 \leq r \leq \frac{n-2}{2}, \\
\frac{1}{2}(8n - j + 4) & \text{for } j = 4r + 2, 1 \leq r \leq \frac{n-4}{2}.
\end{cases}
\]

(ii) For \( n \equiv 0 \pmod{4} \),
\[
f(z_{4,j}) = \begin{cases} 
4n - j + 3 & \text{for } j = 2r + 2, 1 \leq r \leq \frac{n}{4}, \\
\frac{1}{2}(9n - 2j + 4) & \text{for } j = \frac{1}{2}(n + 4r + 4), 1 \leq r \leq \frac{n-4}{4}.
\end{cases}
\]

Case 2. \( n \) is odd.

Set \( f(z) = 7n \) and
\[
f(a) = \begin{cases} 
\frac{1}{2}(4n + j + 1), & \text{if } a = z_{1,j} \text{ and } j \text{ is odd}, \\
\frac{1}{2}(12n + j), & \text{if } a = z_{1,j} \text{ and } j \text{ is even}, \\
\frac{1}{2}(6n - l + 1), & \text{if } a = z_{2,j} \text{ and } l \text{ is odd}, \\
\frac{1}{2}(14n - l), & \text{if } a = z_{2,j} \text{ and } l \text{ is even}, \\
\frac{1}{2}(16n - l - 1), & \text{if } a = z_{3,j} \text{ and } l \geq 3 \text{ is odd}, \\
\frac{1}{2}(8n - l + 2), & \text{if } a = z_{3,j} \text{ and } l \geq 3 \text{ is even}, \\
\frac{1}{2}(6n + j + 1), & \text{if } a = z_{4,j} \text{ and } j \text{ is odd}, \\
\frac{1}{2}(14n + j), & \text{if } a = z_{4,j} \text{ and } j \text{ is even}.
\end{cases}
\]

For all cases, the set of all edge-sums generated by the labeling \( f \) is \( S = (4n + 3, 4n + 4, \ldots, 12n - 2) \). By Lemma 1.1, \( f \) extends to a SEM labeling of \( G_n \) with magic constant \( k = 20n - 3 \). Therefore, for any \( n \geq 2 \), \( \mu\_1(G_n) = 0 \). □

To clarify the proof of Theorems 2.3 and 2.4, see Fig. 2.

Next, we give the upper bound of (consecutively) SEMD of \( C_{b_n} \cup T(n, n - 1, n - 1) \) and \( C_{b_{n-1}} \cup T(n, n - 1, n - 1, n) \).

Theorem 2.5. Let \( G_n = C_{b_n} \cup T(n, n - 1, n - 1) \). For any \( n \geq 2 \), \( \mu\_1(G_n) \leq \mu\_1(G_n) \leq \frac{n - 2}{2} \).

Proof. Let \( G_n \) be a graph having vertex and edge sets as follows.
\[
\begin{align*}
V(G_n) &= \{u_i : 0 \leq i \leq n - 1\} \cup \{v_i : 1 \leq i \leq n - 1\} \cup (y_1, y_{1,1} - 1, y_{1,2} - 1, y_{3,1} - 1, y_{3,2} - 1, y_{3,3} - 1, y_{3,4} - 1) \cup (y_{1,1}, y_{1,2} - 1, y_{1,3} - 1, y_{1,4} - 1), \\
E(G_n) &= \{u_i u_{i+1} : 0 \leq i \leq n - 2\} \cup (u_i v_i : 1 \leq i \leq n - 1\} \cup (v_i y_{i,1} - 1, y_{i,2} - 1, y_{i,3} - 1, y_{i,4} - 1, 1 \leq j \leq 3, 1 \leq r \leq n - 1, 1 \leq s \leq n - 2).
\end{align*}
\]

For any \( n \geq 2 \), let \( H_n = G_n \cup \{\frac{n - 2}{2}\}K_1 \) be a graph with \( V(H_n) = V(G_n) \cup \{z_i : 1 \leq i \leq \frac{n - 2}{2}\} \) and \( E(H_n) = E(G_n) \).

Thus, \( H_n \) has \( \frac{11n - 6}{2} \) vertices and \( 5n - 4 \) edges.

For \( n = 2 \), let \( (u_0, u_1, v_1) \) and \( (y_3, y_{3,1} - 1, y_{3,2} - 1, y_{3,3} - 1, y_{3,4} - 1) \) with \((6,1,7)\) and \((8,4,5,2,3)\), respectively. These vertex labelings can be extended to a SEM labeling of \( H_2 \) with magic constant \( k = 21 \).

For \( n \geq 3 \), define a labeling \( f : V(H_n) \rightarrow \{1, 2, \ldots, \frac{11n - 6}{2}\} \) as follows.
\[
f(u_i) = \begin{cases} 
3n + i - 1 & \text{for even } i, \\
3n + i & \text{for odd } i.
\end{cases}
\]

\[
f(v_i) = \begin{cases} 
\frac{1}{2}(2n + r - 1) & \text{if } a = y_{1,j} \text{ and } r \text{ is odd}, \\
\frac{1}{2}(4n - s) & \text{if } a = y_{2,s} \text{ and } s \text{ is odd}, \\
\frac{1}{2}(4n - s + 1) & \text{if } a = y_{3,s} \text{ and } s \text{ is odd}.
\end{cases}
\]

Case 1. \( n \) is even.

\[
f(y) = 5n - 1, \quad f(z_1) = 4n - 1, \quad f(z_i) = \frac{1}{2}(5n + 2r - 2) \text{ for } t = 2, 3, \ldots, \frac{n - 2}{2} \text{, and}
\]

\[
f(b) = \begin{cases} 
\frac{1}{2}(8n + 2r - 2) & \text{if } b = y_{1,r} \text{ and } r \text{ is even}, \\
\frac{1}{2}(10n - s - 2) & \text{if } b = y_{2,s} \text{ and } s \text{ is even}, \\
\frac{1}{2}(10n + s - 2) & \text{if } b = y_{3,s} \text{ and } s \neq n - 2 \text{ is even}, \\
\frac{1}{2}(5n) & \text{if } b = y_{3,n-2}.
\end{cases}
\]
Case 2. $n$ is odd.

\[ f(y) = 5n - 2, \quad f(z_i) = \begin{cases} \frac{1}{2}(5n + 2r - 1) & \text{for } i = 1, 2, \ldots, \left\lfloor \frac{5n - 2}{2} \right\rfloor, \\
\frac{1}{2}(8n - r - 4), & \text{if } b = y_{1,1}, \text{ and } r \text{ is even} \\
\frac{1}{2}(10n - s - 4), & \text{if } b = y_{2,1}, \text{ and } s \text{ is even} \\
\frac{1}{2}(10n + s - 4), & \text{if } b = y_{3,1}, \text{ and } s \neq n - 1 \text{ is even,} \\
\frac{1}{2}(5n - 1), & \text{if } b = y_{3,1,1}. 
\end{cases} \]

For all cases, it can be verified that the set of all edge-sums generated by the labeling $f$ is $S = \{3n, 3n + 1, \ldots, 8n - 5\}$. By Lemma 1.1, $f$ extends to a SEM labeling of $H_n$ with magic constant $k = \left\lceil \frac{6n - 2}{2} \right\rceil$. Hence, $\mu_i(G_n) \leq \left\lceil \frac{6n - 2}{2} \right\rceil$ for any $n \geq 2$.

Furthermore, let $A$ and $B$ be partite sets of $H_n$ where

\[ A = \{u_i : i \equiv 1 \pmod{2}\} \cup \{v_i : i \equiv 0 \pmod{2}\} \]
\[ B = \{u_i : i \equiv 0 \pmod{2}\} \cup \{v_i : i \equiv 1 \pmod{2}\} \]

and $\mu_i(G_n) \leq \left\lceil \frac{6n - 2}{2} \right\rceil$.

**Theorem 2.6.** Let $G_n = C_{hn} \cup T(n, n - 1, n, n - 1)$. For any $n \geq 2$,

\[ \mu_i(G_n) \leq \left\lceil \frac{6n - 2}{2} \right\rceil. \]

**Proof.** Define the vertex and edge sets of $G_n$ as follows.

\[ V(G_n) = \{u_i : 0 \leq i \leq n - 1\} \cup \{v_i : 1 \leq i \leq n - 1\} \]
\[ \{y_{1,1}, y_{2,1}, y_{3,1}, y_{3,1,1} : 1 \leq r \leq n, 1 \leq s \leq n - 1\} \]
\[ E(G_n) = \{u_{n+1} : 0 \leq i \leq n - 2\} \cup \{u_i : 1 \leq i \leq n - 1\} \]
\[ \{y_{i,1}, y_{i,1+1}, y_{i,2}, y_{i,2+1}, y_{i,3}, y_{i,3+1}, y_{i,3,1}, y_{i,3,1,1} : 1 \leq j \leq 4, 1 \leq r \leq n - 1, 1 \leq s \leq n - 2\}. \]

For any even $n \geq 2$, let $H_n = G_n \cup (n - 1)K_1$ be a graph with

\[ V(H_n) = V(G_n) \cup \{z_i : 1 \leq i \leq n - 1\} \]
\[ E(H_n) = E(G_n) \]

Next, define a labeling $f : V(H_n) \rightarrow \{1, 2, \ldots, 7n - 3\}$ as follows.

\[ f(u_i) = \begin{cases} 4n + i \quad &\text{for even } i, \\ i &\text{for odd } i. \end{cases} \]
\[ f(v_i) = \begin{cases} 4n + i &\text{for odd } i, \\ i &\text{for even } i. \end{cases} \]
\[ f(z_i) = 3n + t \quad &\text{for } i = 1, 2, \ldots, n - 1. \]

For $n = 4$, set $f(y) = 22$ and

\[ f(y_{1,1}) = 5, \quad f(y_{1,2}) = 21, \quad f(y_{1,3}) = 4, \quad f(y_{1,4}) = 20, \]
\[ f(y_{2,1}) = 6, \quad f(y_{2,2}) = 23, \quad f(y_{2,3}) = 7, \]
\[ f(y_{3,1}) = 9, \quad f(y_{3,2}) = 24, \quad f(y_{3,3}) = 8, \]
\[ f(y_{4,1}) = 12, \quad f(y_{4,2}) = 11, \quad f(y_{4,3}) = 28, \quad f(y_{4,4}) = 10. \]

For $n \neq 4$, set $f(y) = 6n - 1$ and

\[ f(a) = \begin{cases} \frac{1}{2}(2n + r - 1), &\text{if } a = y_{1,1} \text{ and } r \text{ is odd,} \\
\frac{1}{2}(10n + r - 2), &\text{if } a = y_{1,1} \text{ and } r \text{ is even,} \\
\frac{1}{2}(4n - s - 1), &\text{if } a = y_{1,1} \text{ and } s \text{ is odd,} \\
\frac{1}{2}(12n - s - 2), &\text{if } a = y_{1,1} \text{ and } s \text{ is even,} \\
\frac{1}{2}(4n + s + 1), &\text{if } a = y_{1,1} \text{ and } s \text{ is odd,} \\
\frac{1}{2}(12n + s - 2), &\text{if } a = y_{1,1} \text{ and } s \text{ is even,} \\
3n - 1, &\text{if } a = y_{1,1}, \\
\frac{1}{2}(14n + r - 11), &\text{if } a = y_{1,1} \text{ and } r = 3, 5, \\
\frac{1}{2}(13n + r - 9), &\text{if } a = y_{1,1} \text{ and } r \geq 7 \text{ is odd,} \\
3n, &\text{if } a = y_{1,2}. \]

For even $r \geq 4$, label $y_{1,2}$ as follows.

(i) For $n \equiv 2 \pmod{4}$,

\[ f(y_{1,2}) = \begin{cases} \frac{1}{2}(6n - r - 2), &\text{for } r = 4l, 1 \leq l \leq \frac{n - 2}{4}, \\
\frac{1}{2}(10n - r + 2), &\text{for } r = 4l + 2, 1 \leq l \leq \frac{n + 2}{4}. \end{cases} \]

(ii) For $n \equiv 0 \pmod{4}$,

\[ f(y_{1,2}) = \begin{cases} 3n - r + 2 &\text{for } r = 2l + 2, 1 \leq l \leq \frac{n}{2}, \\
\frac{1}{2}(7n - 2r + 2) &\text{for } r = \frac{1}{2}(n + 4l + 4), 1 \leq l \leq \frac{n - 2}{4}. \end{cases} \]

It can be checked that the set of all edge-sums is $S = \{4n + 1, 4n + 2, \ldots, 10n - 4\}$. Hence by Lemma 1.1, $f$ extends to a SEM labeling of $H_n$ with magic constant $k = 17n - 6$. Therefore, $\mu_i(G_n) \leq n - 1$ for any even $n \geq 2$.

Now, for any odd $n \geq 3$, let $H_n^* = G_n \cup (n - 2)K_1$ be a graph with $V(H_n^*) = V(G_n) \cup \{z_i : 1 \leq i \leq n - 2\}$ and $E(H_n^*) = E(G_n)$.

Next, define a labeling $g : V(H_n^*) \rightarrow \{1, 2, \ldots, 7n - 4\}$ as follows.

\[ g(u_i) = \begin{cases} 4n + i - 2 &\text{for even } i, \\
4n + i &\text{for odd } i. \end{cases} \]
\[ g(v_i) = \begin{cases} 4n + i &\text{for odd } i, \\
4n + i - 2 &\text{for even } i. \end{cases} \]
\[ g(y) = 6n - 3 \quad &\text{and } g(z_i) = 3n + t - 1 \quad \text{for } i = 1, 2, \ldots, n - 2. \]

Under the labeling $g$, the set of all edge-sums is $S = \{4n - 1, 4n, \ldots, 10n - 6\}$. Hence by Lemma 1.1, $g$ extends to a SEM labeling of $H_n^*$ with magic constant $k = 17n - 9$. Therefore, $\mu_i(G_n) \leq n - 2$ for any odd $n$.

Fig. 3 shows an illustration of the proof of Theorems 2.5 and 2.6.

3. The SEMD of 2C3 ∪ Cn and related graphs

Enomoto et al. [2] stated that a cycle $C_n$ is SEM if and only if $n$ is odd. Figueroa-Centeno et al. [6] investigated the SEMD of the cycle $C_n$. They also proved the following result.

**Theorem 3.1.** [6] If $G$ is a graph of size $q \equiv 2 \pmod{4}$ such that every vertex in $V(G)$ has even degree then $\mu_i(G) = +\infty$.  

5
In 2005, Figueroa-Centeno et al. [11] investigated the SEMD of $2C_n$, $3C_n$, and $4C_n$, and conjectured that

$$\mu_+(mC_n) = \begin{cases} 
0, & \text{if } mn \equiv 1 \pmod{2}, \\
1, & \text{if } mn \equiv 0 \pmod{4}, \\
\infty, & \text{if } mn \equiv 2 \pmod{4}.
\end{cases}$$

In 2009, Holden et al. [16] showed that $C_4 \cup (2i-1)C_3$ for any integer $i \geq 3$, $C_3 \cup (2i)C_3$ for any integer $i \geq 3$, and $C_2 \cup (2i)C_3$ for any integer $i \geq 1$ have a strong vertex-magic labeling, which is equivalent to a SEM labeling. Based on their results, they proposed the following conjecture.

**Conjecture 3.2.** [16] A 2-regular graph of odd order is SEM if and only if it is not one of $C_3 \cup C_n$, $3C_3 \cup C_n$, or $2C_3 \cup C_n$.

Motivated by Conjecture 3.2, Figueroa-Centeno et al. [17] showed that some 2-regular graphs with two components are SEM. They proved that $C_3 \cup C_n$ is SEM if and only if $n \geq 4$ is even, $C_4 \cup C_n$ is SEM if and only if $n \geq 5$ is odd, $C_3 \cup C_n$ is SEM if and only if $n \geq 4$ is even, $C_2 \cup C_n$ is SEM for any even $m \geq 4$ and odd $n \geq \frac{m+4}{2}$. Ichishima and Oshima [18] determined the SEMD of $C_m \cup C_n$ for even $m$ and $n$, and for arbitrary $n$ when $m = 3, 4, 5$ and 7. In this section, we investigate the SEMD of $2C_3 \cup C_n$. 

**Theorem 3.3.** The SEMD of a 2-regular graph $2C_3 \cup C_n$ is given by

$$\mu_+(2C_3 \cup C_n) = \begin{cases} 
0, & \text{if } n \equiv 1,3 \pmod{4} \text{ and } n \neq 5, \\
1, & \text{if } n \equiv 2 \pmod{4}, \\
\infty, & \text{if } n \equiv 0 \pmod{4}.
\end{cases}$$

and $\mu_+(2C_3 \cup C_n) \leq 2$.

**Proof.** First, let $2C_3 \cup C_n$ be a graph with vertex and edge sets

$$V(2C_3 \cup C_n) = \left\{ u_1, u_2, u_3 \right\} \cup \left\{ v_1, v_2, v_3 \right\} \cup \left\{ w_j : 1 \leq j \leq n \right\}$$

and $E(2C_3 \cup C_n) = \left\{ u_1u_2, u_1u_3, u_2u_3, v_1v_2, v_1v_3, v_2v_3 \right\} \cup \left\{ w_jw_{j+1}, w_jw_{j+3} : 1 \leq j \leq n-1 \right\}$.

Next, we show that $\mu_+(2C_3 \cup C_n) = 0$ for $n \equiv 1 \pmod{4}$. Let $n = 4r + 1$ where $r \geq 2$. Define a labeling $f : V(2C_3 \cup C_n) \rightarrow \{1, 2, 3, 4, 5, 6\}$ as follows.

$$f(u_1) = 2r, \quad f(u_2) = 2r + 2, \quad f(u_3) = 2r + 3, \quad f(v_1) = 3r + 4, \quad f(v_2) = 3r + 5, \quad f(v_3) = 3r + 6,$$

$$f(w_j) = \begin{cases} 
2r + j + 3 & \text{for } j = 2i-1 \text{ and } 1 \leq i \leq r-1, \\
3r + 3 & \text{for } j = 2r-1.
\end{cases}$$

To label $w_j$, for $2r \leq j \leq 4r + 1$, we consider the following cases. 

**Case 1.** $r \geq 2$ is even.

$$f(u_j) = r + 2i - 1 \quad \text{for } j = 2r + 4i - 4 \text{ and } 1 \leq i \leq \frac{r}{2},$$

$$f(u_j) = 3r + 2i + 6 \quad \text{for } j = 2r + 4i - 3 \text{ and } 1 \leq i \leq \frac{r}{2} - 1,$$

$$f(u_j) = r + 2i - 2 \quad \text{for } j = 2r + 4i - 2 \text{ and } 1 \leq i \leq \frac{r}{2} - 1,$$

$$f(u_j) = 3r + 2i + 7 \quad \text{for } j = 2r + 4i + 1 \text{ and } 1 \leq i \leq \frac{r}{2} - 1,$$

$$f(u_j) = 4r - i + 8 \quad \text{for } j = 4r + 2i - 5 \text{ and } 1 \leq i \leq 3,$$

$$f(u_j) = 2r - 3i + 4 \quad \text{for } j = 4r + 2i - 4 \text{ and } 1 \leq i \leq 2.$$

**Case 2.** $r \geq 3$ is odd.

$$f(u_j) = r + 2i - 1 \quad \text{for } j = 2r + 4i - 4 \text{ and } 1 \leq i \leq \frac{r}{2} - 1,$$

$$f(u_j) = r + 2i - 2 \quad \text{for } j = 2r + 4i - 2 \text{ and } 1 \leq i \leq \frac{r}{2} - 1,$$

$$f(u_j) = 3r + 2i + 6 \quad \text{for } j = 2r + 4i + 1 \text{ and } 1 \leq i \leq \frac{r-3}{2},$$

$$f(u_j) = 4r - i + 9 \quad \text{for } j = 4r + 2i - 3 \text{ and } 1 \leq i \leq 2,$$

$$f(u_j) = 2r + 1 \quad \text{for } j = 4r - 2.$$

It can be checked that for all cases, the set of all edge-sums is $S = \{2r + 3, 2r + 6, \ldots, 6r + 11\}$. By Lemma 1.1, $f$ extends to a SEM labeling of $2C_3 \cup C_n$ with magic constant $k = 10r + 19$. Hence, $\mu_+(2C_3 \cup C_n) = 0$ for $n \equiv 1 \pmod{4}$.

Next, we prove that $\mu_+(2C_3 \cup C_n) = 0$ for $n \equiv 3 \pmod{4}$. Let $n = 4r + 3$, $r \geq 0$. For $r = 0$, label $(u_1, u_2, u_3, v_1, v_2, v_3)$, and $(w_1, w_2, w_3)$ with $(1, 5), (2, 6, 7), (3, 4, 8)$, respectively. For $r \geq 1$, define a labeling $f : V(2C_3 \cup C_n) \rightarrow \{1, 2, 3, 4, 5, 6\}$ as follows.

$$f(u_1) = 2r + 2, \quad f(u_2) = 2r + 3, \quad f(u_3) = 2r + 4, \quad f(v_1) = 3r + 5, \quad f(v_2) = 3r + 6, \quad f(v_3) = 3r + 7,$$

$$f(w_i) = \begin{cases} 
i & \text{for } j = 2i - 1 \text{ and } 1 \leq i \leq 2r-1, \\
2r + i + 5 & \text{for } j = 2i \text{ and } 1 \leq i \leq r - 1, \\
2r + i + 8 & \text{for } j = 2i \text{ and } r \leq i \leq 2r + 1, \\
2r + 5 & \text{for } j = 4r + 3.
\end{cases}$$

It is not hard to verify that the set of all edge-sums is $S = \{2r + 5, 2r + 6, \ldots, 6r + 14\}$. Hence by Lemma 1.1, $f$ extends to a SEM labeling of $2C_3 \cup C_n$ with magic constant $k = 10r + 24$. Therefore, $\mu_+(2C_3 \cup C_n) = 0$ for $n \equiv 3 \pmod{4}$.

Now, let $G = 2C_3 \cup C_n \cup K_1$ for $n = 4r + 2$, $r \geq 1$. Define a labeling $f : V(G) \rightarrow \{1, 2, 3, 4, 5, 6\}$ as follows.

$$f(u_1) = 2r + 4, \quad f(u_2) = 2r + 5, \quad f(u_3) = 4r + 8, \quad f(v) = 3r + 7,$$

where $z$ is the vertex of $K_1$. For the other vertices, we consider the following cases.

**Case 1.** For any odd $r \geq 1$, set

$$f(v) = r + 2, \quad f(v_1) = r + 4, \quad f(v_2) = 3r + 8, \quad \text{and}$$

$$f(v_3) = 3r + 8.$$
Corollary 3.5. [9] Let m be an odd integer such that \( (m, n) \neq n \), \( 1 \leq i \leq k \). If \( G = (\bigcup_{i=1}^{k} C_{m_i}) \cup k_{2} K_{1} \) is any pseudo super edge-magic graph then \( H = (\bigcup_{i=1}^{k} C_{m_i}) \cup k_{2} C_{n} \) is a SEM graph.

By applying Theorem 3.4 and Corollary 3.5 to a part of Theorem 3.3, we have the following Lemma.

Lemma 3.6. For any odd m and \( n \equiv 2 \) (mod 4), the following SEMD of some graphs are hold.

(i) \( \mu_{s}(2C_{3m} \cup (m,n)C_{m} \cup C_{n}) = 0 \). (iii) \( \mu_{s}(2C_{3m} \cup 2C_{m} \cup 2C_{n}) = 0 \).

(ii) \( \mu_{s}(n+6)P_{m} \cup C_{n} = 0 \). (iv) \( \mu_{s}(12P_{m} \cup C_{n}) = 0 \).

Proof. It is easy to verify that the labeling \( f \) in the proof of Theorem 3.3 is a pseudo SEM labeling of \( 2C_{3} \cup C_{n} \cup K_{1} \) for \( n \equiv 2 \) (mod 4) and \( 2C_{3} \cup C_{n} \cup 2K_{1} \). By applying Theorem 3.4 and Corollary 3.5 to these 2-regular pseudo SEM graphs, we obtain (i) and (iii). By removing \( n \) or 6 edges of the SEMD 2-regular graph \( 2C_{3m} \cup (m,n)C_{m} \cup C_{n} \) with the smallest edge-sums \( a; a+1; \ldots; a+n+5 \), where \( a = \lceil n(n+10)+(-(n+1)) \rfloor \), we get (ii). In a similar way, we obtain the result (iv) by removing 12 edges of the SEMD 2-regular graph \( 2C_{3m} \cup 2C_{n} \cup 2C_{m} \) with the smallest edge-sums \( 0; b, b+1, \ldots, b+11 \), where \( b = \lceil 7m \rceil \).

As an illustration of the proof of Lemma 3.6 (i) and (ii), see Fig. 4.

Gichacz et al. [10] in 2017 introduced a new method to expand the classes of known (strong) vertex-magic labeling of 2-regular graphs as the following theorem.

Theorem 3.7. [10] For \( 1 \leq i \leq \frac{n}{2} \), let \( n \geq 3 \) be an integer and \( G = \bigcup_{i=1}^{k} C_{n} \). If \( G \) is (strong) vertex-magic then \( H = \bigcup_{i=1}^{k} C_{n}(2i+1) \) is (strong) vertex-magic for every integer \( r \).

Ngurah [19] discovered that the method introduced in the proof of Theorem 3.7 is also valid for pseudo SEM graphs. By applying this fact to our results, we have the following lemma.

Lemma 3.8. For any odd m and \( n \equiv 2 \) (mod 4), the following SEMD of some graphs are satisfied.

(i) \( \mu_{s}(2C_{3m} \cup C_{n} \cup C_{m}) = 0 \). (v) \( \mu_{s}(2C_{3m} \cup C_{n} \cup C_{m}) \leq m \).

(ii) \( \mu_{s}((m+6)P_{m} \cup C_{n}) = 0 \). (vi) \( \mu_{s}(2C_{3m} \cup mP_{m} \cup C_{n}) \leq m \).

(iii) \( \mu_{s}(mP_{m} \cup C_{n} \cup mP_{m} \cup C_{n}) = 0 \). (vii) \( \mu_{s}(mP_{m} \cup C_{n} \cup mP_{m} \cup C_{n}) \leq m \).

(iv) \( \mu_{s}(2C_{3m} \cup 2C_{n} \cup 2C_{m}) = 0 \). (viii) \( \mu_{s}(2mP_{m} \cup mP_{m} \cup C_{n}) \leq m \).

Proof. The result (i) and (iv) are obtained by applying Theorem 3.7 to the fact that \( 2C_{3} \cup C_{n} \cup K_{1} \) for \( n \equiv 2 \) (mod 4) and \( 2C_{3} \cup C_{n} \cup 2K_{1} \) are pseudo SEM, respectively. We gain (ii) and (iii) from (i) by removing edges with the edge-sums \( a; a+1; \ldots; a+m-1 \) and \( (a+1; \ldots; a+2m-1) \), where \( a = \lceil \frac{(m+7)m+3} \rceil \). Moreover, we obtain the result (v), (vi), (vii), and (viii) from (iv) by deleting edges with the edge-sums \( b, b+1, \ldots, b+m-1 \), \( (b, b+1, \ldots, b+2m-1) \), \( (b, b+1, \ldots, b+3m-1) \), and \( (b, b+1, \ldots, b+4m-1) \), where \( b = \lceil \frac{7m}{2} \rceil +3 \).

Fig. 5 shows the construction given in the proof of Lemma 3.8 (i), (ii), and (iv).

4. The SEMD of join product of union of a star and a path with an isolated vertex

To present our results, we need the concept of dual labeling. A dual labeling \( f' \) of a SEM labeling \( f \) is defined as \( f'(x) = p + 1 - f(x) \) for all
Fig. 4. (a) The pseudo SEM labeling of $2C_3 \cup C_{10} \cup K_1$; (b) The SEM labeling of $2C_9 \cup C_{30} \cup C_3$, which is obtained by applying Corollary 3.5 to Fig. 4(a) for $m = 3$; (c) The SEM labeling of $16P_3 \cup C_3$, which is obtained by removing 16 edges with the smallest edge-sums $\{27, 28, \ldots, 42\}$ of the graph in Fig. 4(b).

Fig. 5. (a) The SEM labeling of $2C_9 \cup C_{30} \cup C_3$, which is obtained by applying Theorem 3.7 to Fig. 4(a) for $r = 1$; (b) The SEM labeling of $3P_3 \cup C_9 \cup C_{30} \cup C_3$, which is obtained by removing 3 edges with the smallest edge-sums $\{27, 28, 29\}$ in Fig. 5(a); (c) The SEM labeling of $3P_3 \cup C_9 \cup 3P_3 \cup C_3$, which is obtained by removing 3 edges with the smallest edge-sums $\{30, 31, 32\}$ in Fig. 5(b).

$x \in V(G)$ and $f^*(xy) = 2p + q + 1 - f(xy)$ for all $xy \in E(G)$. Baskoro et al. [20] proved that the dual of a SEM labeling is also a SEM labeling.

Ngurah and Simanjuntak [21] studied the SEMD of join products of a path, a star, and a cycle, respectively, with isolated vertices. Generally, they showed that the join product of a SEM graph with isolated vertices has finite SEMD. In [22], the same authors investigated the SEMD of join product of a graph $G$ which has certain properties with an isolated vertex. They gave a necessary condition for $G + K_1$ to have zero SEMD as the following lemma.

**Lemma 4.1.** [22] Let $G$ be a graph with no cycle and minimum degree one. If $\mu_s(G + K_1) = 0$ then $G$ is a tree or a forest.

They showed that Lemma 4.1 is attainable. In particular, they proved that the join product of some forests with an isolated vertex has zero SEMD, such as $\mu_s((P_n \cup P_2) + K_1) = 0$ if and only if $3 \leq n \leq 5$ and $\mu_s((K_{1,n} \cup P_3) + K_1) = 0$ if and only if $n = 2$. In this section, we study the SEMD of join product graph $(K_{1,n} \cup P_n) + K_1$ for any integer $n \geq 1$ and $m \geq 3$. 

$\mu_s(G + K_1)$
For any integer $n \geq 1$ and $m \geq 3$, let $G_{n,m} = (K_{2,m} \cup P_n) \cup K_1$ be a graph having vertex and edge sets:

$$V(G_{n,m}) = \{c,z\} \cup \{x_i : 1 \leq i \leq m\} \cup \{y_j : 1 \leq j \leq n\}$$

and

$$E(G_{n,m}) = \{c,z\} \cup \{c x_i : 1 \leq i \leq m\} \cup \{c y_j : 1 \leq j \leq n\} \cup \{y_j y_{j+1} : 1 \leq j \leq m-1\}.$$ 

Thus, the graph $G_{n,m}$ has $n+m+2$ vertices and $2n+2m$ edges.

Now, we give necessary and sufficient conditions of $G_{n,m}$ to have zero SEMD for $m = 3, 4, 5$.

**Theorem 4.2.** $\mu_i(G_{3,3}) = 0$ if and only if $n = 1, 2, 3$; and for $m = 4, 5$, $\mu_i(G_{n,m}) = 0$ if and only if $n = 1, 2$.

**Proof.** Firstly, we show that for $n = 1, 2, 3$, $\mu_i(G_{n,3}) = 0$. For $n = 1, 2, 3$, label $(c, z), (x_1, \ldots, x_m, y_1, y_2, y_3)$ with $(3, 2), (1, 4, 6, 5, 7)$ and $(2, 3), (1, 4, 5, 7, 6, 8)$, respectively. Moreover, for $n = 1, 2, 3$ and $m = 5$, label $(c, z), (x_1, \ldots, x_m, y_1, y_2, y_3, y_4, y_5)$ with $(2, 1), (3), (4, 5, 7, 6, 8)$ and $(2, 1), (4, 5, 8, 6, 7)$, respectively. It can be checked that these labelings extend to a SEM labeling of $G_{n,4}$ and $G_{n,5}$ for $n = 1, 2$. Conversely, suppose that $\mu_i(G_{n,4}) = \mu_i(G_{n,5}) = 0$ for $n \geq 3$. Then for $m = 4$, we have

$$(n+3)f(z) + (n-1)f(c) + f(y_1) + f(y_2) = n^2 + 8n + 10$$

and for $m = 5$,

$$(n+4)f(z) + (n-1)f(c) + f(y_1) + f(y_2) = n^2 + 10n + 19.$$ 

By a similar argument as in the proof of $G_{n,3}$, we can show that $\mu_i(G_{n,4}) > 0$ and $\mu_i(G_{n,5}) > 0$ for every $n \geq 3$. Hence, for $m = 4, 5$, $\mu_i(G_{n,m}) = 0$ if and only if $n = 1, 2$.

Since $G_{n,3}, G_{n,4}$, and $G_{n,5}$ do not admit zero SEMD, we try to find its SEMD. Our result is as follows.

**Theorem 4.3.** For any integer $n, m \geq 3$, the SEMD of $G_{n,m}$ is given by:

(i) $\mu_i(G_{n,3}) = 1$ if $m = 3$ and $n \geq 1, 4, 5; m = 4$ and $n = 3, 4, 5, 6; m = 5$ and $n = 4, 5, 6$. 

(ii) $\mu_i(G_{n,m}) \leq n - m$ if $m \geq 3$ and $m = r \lceil \frac{m-1}{2} \rceil$, where $r$ is any positive integer.

**Proof.** (i) As a consequence of Theorem 4.2, we obtain that $\mu_i(G_{n,3}) \geq 1$ for $n \geq 4$, $\mu_i(G_{n,4}) \geq 1$ for $n \geq 3$, and $\mu_i(G_{n,5}) \geq 1$ for $n \geq 3$. Hence, the remaining case is to show that $\mu_i(G_{n,3}) \leq 1$ for $n = 4, 5, 6$; and $\mu_i(G_{n,4}) \leq 1$ for $n = 4, 5, 6$. For this, let us consider the graph $G_{n,m} \cup K_1$ and suppose $w$ is the vertex of $K_1$.

- For $n = 4, 5$ and $m = 3$, label $(c, z), (x_1, x_2, \ldots, x_m, y_1, y_2, y_3, y_4, y_5)$ with $(1, 5), (2, 3, 4, 9), (7, 6, 10, 8)$ and $(1, 6), (2, 3, 4, 5, 11), (7, 8, 10, 9)$, respectively.

- For $n = 3, 4, 5$ and $m = 4$, label $(c, z), (x_1, x_2, \ldots, x_n, y_1, y_2, y_3, y_4, y_5)$ with $(1, 4), (2, 3, 7), (6, 10, 5, 8, 9)$ and $(1, 5), (2, 3, 4, 9), (6, 11, 7, 8, 10)$, respectively.

- For $n = 3, 4, 5$ and $m = 5$, label $(c, z), (x_1, x_2, \ldots, x_n, y_1, y_2, y_3, y_4, y_5, y_6)$ with $(1, 4), (2, 3, 7), (5, 8, 10, 6, 11, 9), (7, 6, 10, 8, 12, 7)$, and $(1, 6), (2, 3, 4, 5, 11), (7, 9, 13, 8, 12, 10)$, respectively.

It is not hard to check that these labelings extend to a SEM labeling of $G_{n,m} \cup K_1$ for $n = 4, 5, 6$; for $n = 3, 4, 5$, and $G_{n,5} \cup K_1$ for $n = 3, 4, 5$.

(ii) Let $H = G_{n,m} \cup (n-1)K_1$, where $m \geq 3$ and $m = r \lceil \frac{m-1}{2} \rceil$. Define a labeling $f : V(H) \rightarrow \{1, 2, 3, \ldots, 2m + n + 1\}$ as follows:

$$f(c) = m + 1, \quad f(z) = \frac{3m + 1}{2}, \quad f(x_i) = \frac{(2l+1)m - 2i + 5}{2} + i - \left(\begin{array}{l}l \\
\end{array}\right)\frac{(m+1)}{2} + 1.$$ 

it is odd

$$f(x_i) = \frac{(2l+1)m - 2i + 6}{2} + i - \left(\begin{array}{l}l \\
\end{array}\right)\frac{(m+2)}{2} + 1.$$ 

it is even

$$f(y_j) = \left\{ \begin{array}{ll}
\left\lceil \frac{m+1}{2} \right\rceil + 1 \quad & \text{for } j = 2l - 1 \text{ and } 1 \leq l < \left\lfloor \frac{m+1}{2} \right\rfloor \\
\left\lceil \frac{m+2}{2} \right\rceil + 1 \quad & \text{for } j = 2l \text{ and } 1 \leq l < \left\lfloor \frac{m+2}{2} \right\rfloor 
\end{array} \right.$$ 

The remaining labels are used to label isolated vertices of $H$. It can be verified that the set of all edge-sums is a consecutive integer $S = [a, a+1, \ldots, a+2m+2n]$, where $a = \frac{m+1}{2}$. By Lemma 1.1, $f$ extends to a SEM labeling of $H$ with magic constant $k = 4n + 3m + \frac{a}{4} + 1$. Therefore, for any $m \geq 3$ and $n = r \lceil \frac{m-1}{2} \rceil$, $r \in \mathbb{Z}$, $\mu_i(G_{n,m}) \leq n - 1$. \qed
5. Conclusion

In this paper, we study the (consecutively) SEMD of some graphs. We find the exact value or upper bound of the (consecutively) SEMD of forests with two components. We also find the exact value of the SEMD of a 2-regular graph $2C_n$ for almost all $n$. By using this and previous known results, we obtain the exact value or upper bound of the SEMD of some 2-regular graphs and union of cycles and paths. Moreover, we provide the necessary and sufficient conditions of $(K_{1, n} \cup P_m) + K_1$ to gain zero SEMD and the upper bound of SEMD of this graph for the remaining cases.

Declarations

Author contribution statement

V.H. Krisnawati: Conceived and designed the analysis; Analyzed and interpreted the data; Wrote the paper.
A.A.G. Ngurah: Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.
N. Hidayat, A.R. Alghofari: Conceived and designed the analysis; Contributed analysis tools or data.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

Acknowledgements

The authors would like to thank to the reviewers for their valuable comments and suggestions.

References

[1] A. Kotzig, A. Rosa, Magic valuation of finite graphs, Can. Math. Bull. 13 (1970) 451–461.

[2] H. Enomoto, A. Llado, T. Nakamigawa, G. Ringel, Super edge-magic graphs, SUT J. Math. 34 (1998) 105–109.

[3] F.A. Muntaner-Batle, Special super edge-magic labelings of bipartite graphs, J. Comb. Math. Comb. Comput. 39 (2001) 107–120.

[4] A. Oshima, Consecutively super edge-magic tree with diameter 4, Rev. Bull. Calcutta Math. Soc. 15 (2007) 87–90.

[5] R. Figueroa-Centeno, R. Ichishima, F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, Discrete Math. 231 (2001) 153–168.

[6] R. Figueroa-Centeno, R. Ichishima, F.A. Muntaner-Batle, On the super edge-magic deficiency of graphs, Electron. Notes Discrete Math. 11 (2002) 299–314.

[7] R. Ichishima, F.A. Muntaner-Batle, A. Oshima, The consecutively super edge-magic deficiency of graphs and related concepts, Electron. J. Graph Theory Appl. 8 (2020) 71–92.

[8] J.A. Gallian, A dynamic survey of graph labeling, Electron. J. Comb. DS6 (2019).

[9] R. Ichishima, F.A. Muntaner-Batle, A. Oshima, Enlarging the classes of super edge-magic 2-regular graphs, AKCE Int. J. Graphs Comb. 340 (2013) 129–146.

[10] S. Gichaz, D. Froncek, I. Singgih, Vertex magic total labelings of 2-regular graphs, Discrete Math. 340 (2017) 3117–3124.

[11] R. Figueroa-Centeno, R. Ichishima, F.A. Muntaner-Batle, Some new results on the super edge-magic deficiency of graphs, J. Comb. Math. Comb. Comput. 55 (2005) 17–31.

[12] A.Q. Baig, A. Ahmad, E.T. Baskoro, R. Simanjuntak, On the super edge-magic deficiency of forests, Util. Math. 86 (2011) 147–159.

[13] S. Javed, A. Riasat, S. Kanwal, On super edge-magicness and deficiencies of forests, Util. Math. 98 (2015) 149–169.

[14] M. Imran, A. Mukhtar, On super edge-magic total labeling of forests consisting of stars and subdivided stars, Int. J. Math. Soft Comput. 7 (2017) 1–14.

[15] V.H. Krisnawati, A.A.G. Ngurah, N. Hidayat, A.R. Alghofari, On the super edge-magic deficiency of forests, AIP Conf. Proc. 2192 (2019) 040007.

[16] J. Holden, D. McQuillan, J.M. McQuillan, A conjecture on strong magic labelings of 2-regular graphs, Discrete Math. 309 (2009) 4130–4136.

[17] R. Figueroa-Centeno, R. Ichishima, F.A. Muntaner-Batle, A. Oshima, A magical approach to some labeling conjectures, Discuss. Math., Graph Theory 31 (2011) 79–113.

[18] R. Ichishima, A. Oshima, On the super edge-magic deficiency of 2-regular graphs with two components, Ars Combin. 129 (2016) 437–447.

[19] A.A.G. Ngurah, On (super) edge-magic deficiency of some classes of graphs, Electron. J. Graph Theory Appl. (2019), submitted for publication.

[20] E.T. Baskoro, I.W. Sedarsana, Y.M. Cholily, How to construct new super edge-magic graphs from some old ones, J. Indones. Math. Soc. 11 (2005) 155–162.

[21] A.A.G. Ngurah, R. Simanjuntak, Super edge-magic deficiency of join product graphs, Util. Math. 105 (2017) 279–289.

[22] A.A.G. Ngurah, R. Simanjuntak, On the super edge-magic deficiency of join product and chain graphs, Electron. J. Graph Theory Appl. 7 (2019) 157–167.