UPDATES ON HIRZEBRUCH’S 1954 PROBLEM LIST

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ABSTRACT. We present updates to the problems on Hirzebruch’s 1954 problem list focussing on open problems, and on those where substantial progress has been made in recent years. We discuss some purely topological problems, as well as geometric problems about (almost) complex structures, both algebraic and non-algebraic, about contact structures, and about (complementary pairs of) foliations.

1. INTRODUCTION

In the first week of May, 1953, a conference on fiber bundles and differential geometry was held at Cornell University. A brief report about that conference, written by N. E. Steenrod, appeared in [St53], where the final two sentences read:

The discussions were marked by the presentations of numerous unsolved problems. These were recorded and a report on them is being prepared for publication.

The young postdoc given the task of collecting the problems was F. Hirzebruch, who was then nearing the end of the first year of his two-year stay at the Institute for Advanced Study (IAS) in Princeton. For him, the most important open problem was what he thought of as the high-dimensional Riemann–Roch problem. A few weeks after the Cornell conference, Hirzebruch proved the signature theorem, and, by the end of 1953, he had deduced from it what we now call the Hirzebruch–Riemann–Roch theorem.

By the end of March, 1954, when Hirzebruch submitted the collection of problems for publication, he had replaced the Riemann–Roch problem by several other problems that arose out of his work on the signature and Riemann–Roch theorems. The published collection [Hi54] was reviewed in Mathematical Reviews by S. Chern, who wrote [Ch55]:

The paper lists a set of 34 unsolved problems of current interest concerned with differentiable, almost complex, and complex manifolds. The problems are expertly chosen; their clarification and partial or complete solutions will probably take many years and will certainly mean progress of the field. In stating the problems, the author tries to give their motivation, their relations to known results, and related facts. In this sense the paper is at the same time a resumé and exposition of the subject, or at least of a major part of it.

Chern’s description was certainly prescient. Reading Hirzebruch’s paper [Hi54] now, 60 years after it was written, one cannot help noticing how modern the paper was, and how, even now,
it contains nothing outdated. Hirzebruch’s problems foreshadowed many of the developments in geometry and topology over the intervening decades. A few problems were solved right away, some were solved years or even decades later through major advances in the field, and a few are open to this day.

When Hirzebruch published his collected works in 1987, he compiled solutions and updates to his problems, and these appear in [Hi87, p. 762–784]. Some of the solutions described there draw on unpublished work of M. Puschnigg [Pu88].

This paper is an attempt to collect together further updates to Hirzebruch’s problems, without repeating the information given in [Hi87, Pu88]. Thus we discuss results on Hirzebruch’s problems obtained since 1987, and some earlier results that were not mentioned in [Hi87]. We also want to advertise the problems that have remained open. Some of the problems are quite open-ended, or admit several interpretations that are interesting. Because of this, the question of whether a specific problem on the list is solved or not sometimes does not have a clear yes-or-no answer. Nevertheless, to give an overview of the material discussed below, I would like to give a few simplified pointers in this introduction, which will help readers locate what may be of interest to them.

Open problems. I believe that the following problems are essentially open:

• Problem 3 (elementary proof of the signature theorem),
• Problem 10 (existence of Pfaffians of constant class),
• Problem 13 (complex structures on \( \mathbb{C}P^3 \) and on \( S^6 \)),
• Problem 20 (harmonic theory for almost complex manifolds),
• Problem 28 (complex structures on \( \mathbb{C}P^n \)).

In addition to these, the following problems, though solved to some extent, still offer interesting open questions:

• Problem 9 (integrability of \( G \)-structures),
• Problem 11 (embeddings of \( \mathbb{R}P^n \) in Euclidean spaces),
• Problem 25 (complex structures on the (topological) manifold \( S^2 \times S^2 \)).

Progress and solutions. I believe the only problem from [Hi54] that was wide open in 1987 and has since been solved completely, is Problem 31, about the topological (non-)invariance of characteristic numbers of algebraic varieties, whose solution is described at the end of this report. In addition, there has been important progress on the following problems:

• Problem 3 (geometric interpretation of the functional equation for virtual signatures),
• Problem 10 (existence of Pfaffians of constant class),
• Problem 11 (embeddings of \( \mathbb{R}P^n \) in Euclidean spaces),
• Problem 25 (complex structures on the (smooth) manifold \( S^2 \times S^2 \)).

I also found that in the case of Problems 8 and 12 some results known before 1987 were missed in [Hi87], and so I will record them here.

Thanks and disclaimer. I have known about Hirzebruch’s problem list [Hi54] essentially all of my mathematical life, and I have at different times thought hard about several of the problems. The fact that some of them are fairly open-ended, or require interpretation, is not a drawback, but an advantage that makes perusal of the problem list more stimulating and rewarding.

After I gave a lecture about the solution of Problem 31 at the IAS in Princeton in March 2013, several people asked me about the status of the other problems on this list. The present report was compiled in part as a response to their questions. It also aims to celebrate the 60th anniversary
of the list, and of Hirzebruch’s 1953 work that formed the context for many of the problems. I am grateful to the IAS for its hospitality that made it possible for me to work on Problem 31 and to compile this report. I am also grateful to D. Davis, H. Geiges and T. Vogel for supplying information on the status of certain problems.

This paper does not contain any new theorems, but rather consists of informal discussions of results, some of them my own, but most of them due to other authors. Such informal discussions are always subjective, and are bound to be colored by the biases and prejudices of the author. I just hope that this report does not stray too far from the taste and spirit of Hirzebruch’s mathematics. Finally, I would like to point out that I only discuss 11 of the 34 problems here, not mentioning those that were solved before 1987, unless I have something to add to Hirzebruch’s report in [Hi87].

**Note about references.** Each section of this paper covers between one and three problems from the list from [Hi54], and separate references are given for each section. Although the references given at the end of this introduction are often cited in later sections, they are not included again among the references for those later sections.

**REFERENCES**

[Ch55] S. Chern, Review of [Hi54]. MR0066013 (16,518c).

[Hi54] F. Hirzebruch, *Some problems on differentiable and complex manifolds*, Ann. Math. 60 (1954), 213–236; reprinted in [Hi87].

[Hi87] F. Hirzebruch, *Gesammelte Abhandlungen*, Band I, Springer-Verlag 1987.

[Pu88] M. Puschnigg, *On some problems of “Some problems on differentiable and complex manifolds”*, Diplomarbeit, Bonn 1988.

[St53] N. E. Steenrod, *The conference on fiber bundles and differential geometry in Ithaca*, Bull. Amer. Math. Soc. 59 (1953), 569–570.
2. TWO TOPOLOGICAL PROBLEMS

At the beginning of June, 1953, Hirzebruch proved what he then called the “index theorem”:

$$\tau(M) = \langle L(p_1, \ldots, p_k), [M] \rangle,$$

where $M$ is any closed smooth oriented manifold, $\tau(M)$ is its index or signature, and $L(p_1, \ldots, p_k)$ is the $L$-polynomial in the Pontrygain classes of $M$. Later on the term “index” became monopolized by the Fredholm index appearing in the Atiyah–Singer index theorem, and so $\tau(M)$ is now usually called the signature, and the “index theorem” has become the Hirzebruch signature theorem.

**Problem 3** (Hirzebruch-Thom). *Give an elementary proof of the index theorem and explain the geometrical meaning of the functional equation valid for $\tau$.*

Hirzebruch’s proof was not elementary in the sense that it appealed to R. Thom’s determination of the oriented bordism ring over $\mathbb{Q}$, which is a rather non-elementary result in algebraic topology. Ten years later the signature theorem became a consequence of the Atiyah–Singer index theorem, which is even more sophisticated and “non-elementary”. To this day no proof of the signature theorem is known that would qualify as “elementary”, and so Problem 3 is wide open.

The functional equation referred to in the problem is the following. Let $\dim(M) = n$. Any homology class $x \in H_{n-2}(M; \mathbb{Z})$ can be represented by a smooth closed oriented submanifold, whose signature is independent of the chosen representative. Thus one has a well-defined virtual signature function $\tau: H_{n-2}(M; \mathbb{Z}) \to \mathbb{Z}$. By taking signatures of transverse intersections, this can be extended to

$$\tau: H_{n-2}(M; \mathbb{Z}) \times \ldots \times H_{n-2}(M; \mathbb{Z}) \to \mathbb{Z}.$$  

For elements $x$ and $y \in H_{n-2}(M; \mathbb{Z})$ the functional equation for the virtual signature is

$$(1) \quad \tau(x + y) = \tau(x) + \tau(y) - \tau(x, y, x + y).$$

Hirzebruch proved this identity by using the signature theorem. A geometric interpretation of (1) was given by N. Yu. Netsvetaev [Ne97], who showed that if $x, y$ and $w = x + y$ are represented by transversely intersecting submanifolds $X, Y$ and $W$, then their bordism classes satisfy the equation

$$[W] = [X] + [Y] - [T \times \mathbb{C}P^2],$$

where the last term on the right hand side stands for a linear $\mathbb{C}P^2$-bundle over $T = X \cap Y \cap W$. Taking signatures and using the fact that $T \times \mathbb{C}P^2$ has the same signature as $T$, one deduces (1) from (2).

Netsvetaev’s proof of (2) is direct and geometric, and therefore qualifies as “elementary”. Thus his result does give a nice, elementary and satisfactory interpretation of (1). However, this does not solve Problem 3, as there is still no elementary proof of the signature theorem. All that [Ne97] achieved, was a geometric interpretation of the functional equation, and a proof of this equation that avoids the use of the signature theorem.

In Subsection 1.2 of [Hi54], following Thom, Hirzebruch discusses the representability of integral homology classes in manifolds by submanifolds. He notes that, by Thom’s results, all homology classes in degrees $\leq 6$ are representable by submanifolds, as are those of codimension at most 2 – this was used above to define the virtual signatures. Thus degree 7 is the smallest degree in which there could be a homology class not representable by a submanifold, and the ambient manifold would have to have dimension at least 10. In fact, Thom had already constructed an example where the ambient manifold had dimension 14.
Problem 8 in [Hi54], attributed to Thom, aimed at determining the smallest dimension in which there could be a manifold with an integral homology class not representable by a submanifold. However, the actual formulation of that problem was more technical, focusing on a particular integral Steenrod power operation instead, which is an obstruction for representability by submanifolds. As explained in [Hi87, P. 771/2], this technical formulation of Problem 8 was resolved by M. Kreck, who performed a calculation with the Atiyah–Hirzebruch spectral sequence. His result about integral Steenrod powers in particular implies that there is a manifold of dimension 10 with a degree 7 homology class not representable by a submanifold. However, Kreck’s is purely an existence result, and no manifold is produced which has this property.

In [BHK02], C. Bohr, B. Hanke and I pointed out that there are explicit examples of 10-manifolds, like the group manifold $Sp(2)$, which have degree 7 homology classes not representable by submanifolds. This is detected by the reduced Steenrod powers in cohomology with coefficients mod 3, rather than with integral coefficients. The calculation of Steenrod powers required for $Sp(2)$ had been carried out at the beginning of the 1950s by A. Borel and J-P. Serre, so that the example of [BHK02] was – in some sense – known when Hirzebruch compiled the problem list [Hi54], though clearly neither he nor Thom realized this. Of course the result of Borel and Serre was quite new at the time of [Hi54], but was certainly well known in the mid 1980s when Kreck gave his solution recorded in [Hi87].

REFERENCES

[BHK02] C. Bohr, B. Hanke and D. Kotschick, Cycles, submanifolds, and structures on normal bundles, Manuscripta Math. 108 (2002), 483–494.

[Ne97] N. Yu. Netsvetaev, Cobordism version of the Hirzebruch functional equation for the virtual signature, Topology 36 (1997), 471–480.
3. Existence of $G$-structures and their integrability

Subsection 1.3 of [Hi54] discusses what are nowadays called $G$-structures on manifolds and their integrability. In the terminology of $G$-structures $G$ denotes a subgroup of $GL(n, \mathbb{R})$. However, in [Hi54] the general linear group itself is denoted by $G$, and subgroups are denoted by $H$, so that one has to speak about $H$-structures. We shall follow this terminology here, so as not to confuse readers when referring back to [Hi54, Hi87].

Problem 9, attributed to E. Calabi, asks for a general criterion to determine whether a manifold $M^n$ admits an integrable $H$-structure, for some given closed subgroup $H \subset GL(n, \mathbb{R})$. As explained in [Hi87], many important cases of this problem have been resolved, but the methods and techniques used are very different in the different cases, suggesting that the general formulation attempted by Calabi may in fact not be appropriate.

I want to make a few comments here on some special cases, in part updating and expanding on Hirzebruch’s commentary [Hi87].

### (Almost) complex structures.

For $n = 2k$ and $H = GL(k, \mathbb{C})$ an $H$-structure is an almost complex structure. By the Newlander–Nirenberg theorem, the obstruction to integrability is precisely the Nijenhuis tensor. In real dimension $4$ there are many examples of almost complex manifolds without any complex structure. Such examples can be detected from many different points of view, e.g. from the Enriques–Kodaira classification and Chern number inequalities, or using the fundamental group and the Albanese map, or using gauge theory (Donaldson and Seiberg–Witten invariants).

In high dimensions, meaning real dimensions $\geq 6$, S.-T. Yau has conjectured that every almost complex manifold should admit a complex structure; see, for example, [Ya92, Problem 52]. There is no real evidence in favor of Yau’s speculation, but if it makes any sense at all, then it should perhaps be formulated in an even stronger form, as an $h$-principle: in dimensions $\geq 6$ every almost complex structure should be homotopic to an integrable one.

Note that not only are there almost complex four-manifolds without any complex structure, but there are also many complex surfaces that admit homotopy classes of almost complex structures that contain no integrable structure.

### Symplectic structures.

For $n = 2k$ and $H = Sp(2k, \mathbb{R})$ an $H$-structure is an almost symplectic structure, which is the same as an almost complex structure. One can think of an almost symplectic structure as a non-degenerate two-form, and the integrability condition is then that this two-form be closed.

There are almost symplectic manifolds, like $S^1 \times S^3$, or $S^0$, for which the structure of the cohomology ring obstructs the existence of a symplectic structure. Therefore, in order to have a chance to get a symplectic structure, one needs to assume not only that one has an almost symplectic manifold, but in addition it has to be cohomologically symplectic, or $c$-symplectic for short. For $n = 4$ this is still not enough for integrability. There are many examples of almost symplectic $c$-symplectic four-manifolds that cannot be symplectic, because of obstructions coming from Seiberg–Witten invariants, cf. [Ko97a]. In higher dimensions no further obstructions are known, but it is not known whether every almost symplectic structure on a $c$-symplectic manifold is homotopic to an integrable one. In any case, existence of symplectic structures is known in many cases; see the work of Gompf [Go95] in particular.

### Pairs of complementary foliations.

For $n = p + q$ and $H = GL(p, \mathbb{R}) \times GL(q, \mathbb{R})$ an $H$-structure on $M$ is a splitting of the tangent bundle into a direct sum of subbundles or distributions of ranks
p and q respectively. The question about the existence of such a splitting can often be answered in terms of algebraic topological invariants of M. Such an H-structure is integrable if and only if it is induced from a local product structure given by a pair of complementary foliations.

As mentioned in [Hi87], if one considers the weaker requirement that only one of the two distributions be integrable, then a lot is known. This is the question of the existence of a foliation, of dimension p say, on M, assuming that the tangent bundle of M admits a rank p subbundle. In many cases the integrability of all distributions (up to homotopy) has been proved by W. P. Thurston, for example if \( p = n - 1 \), see [Th76], and also if \( p = 2 \), see [Th74]. (The case \( p = 1 \) is trivial, of course.) In other cases there are additional obstructions coming from the so-called Bott vanishing theorem that forces the vanishing of certain characteristic classes of the normal bundle of a foliation.

Returning to the full integrability of \( GL(p, \mathbb{R}) \times GL(q, \mathbb{R}) \)-structures, it is still a very difficult problem to understand when a splitting of the tangent bundle can be induced by a pair of complementary foliations. Even in situations where both distributions are separately homotopic to integrable ones, for example because of Thurston’s results, it is very unclear whether they can simultaneously be made integrable in such a way that they remain complementary. It is certainly not possible to fix one foliation and then homotope the normal bundle to obtain a second, complementary, foliation. This problem appears already for \( p = 1 \) and \( q = 2 \), since there are circle bundles over surfaces which do not admit any horizontal foliation complementary to the fibers. Although the two-dimensional horizontal foliation is homotopic to a foliation, that foliation will never be complementary to the fibers. Of course in this case one can just switch the roles of the two distributions and argue that one makes the two-dimensional distribution integrable without worrying about the complement since every one-dimensional distribution is integrable. This switching does not work already for \( p = q = 2 \). In this case all distributions are homotopic to integrable ones [Th74], but if we take for \( M^4 \) the non-trivial \( S^2 \)-bundle over \( S^2 \), then again there is no two-dimensional foliation complementary to the fibers of the fibration. If one just homotopes the horizontal foliation to make it integrable (and no longer horizontal), then one does not know whether an integrable complement exists for the homotoped distribution. For general surface bundles over surfaces of positive genus the (non-)existence of a horizontal foliation is an interesting problem that has attracted quite a bit of attention in recent years, but is still open.

Next we come to a problem about G-structures defined by nonvanishing one-forms.

**Problem 10** (Chern). A Pfaffian equation in \( M^{2k+1} \) can be locally reduced to the normal form

\[
dx_{2k+1} + x_1 dx_{k+1} + \ldots + x_r dx_{k+r} = 0, \quad 0 \leq r \leq k,
\]

where \( r \) is the only local invariant, \( 2r + 1 \) being what is usually called the class of the local Pfaffian equation. The number \( r \) being fixed, give a criterion to determine whether \( M^{2k+1} \) admits a Pfaffian equation in the large whose class is \( 2r + 1 \) everywhere.

In more general terminology, the problem asks for criteria to determine whether, for given \( r \), a manifold \( M \) admits a one-form \( \alpha \) for which \( \alpha \wedge (d\alpha)^r \) is nowhere zero, but \( \alpha \wedge (d\alpha)^{r+1} \) vanishes identically. (Having a global defining form for the Pfaffian equation is no significant restriction, and can always be ensured by passage to a two-fold covering, if necessary.) The problem also makes sense for even-dimensional manifolds, not just odd-dimensional ones. A slightly different problem would be to ask for \( \alpha \) to be of constant class – the notion of class for a one-form and for the Pfaffian equation it determines are slightly different. Chern’s problem asks for the Pfaffian equation \( \alpha = 0 \) to have constant class \( 2r + 1 \), which is equivalent to \( \alpha \) having class \( 2r + 1 \) or \( 2r + 2 \),
as a one-form, at every point of $M$. At points where $(d\alpha)^{r+1} = 0$, the class of $\alpha$ is $2r + 1$, and at points where $(d\alpha)^{r+1} \neq 0$, the class of $\alpha$ is $2r + 2$.

For this problem, the cases corresponding to the two extreme values of $r$ have been studied the most. Firstly, if $r = 0$, then one looks for nowhere zero one-forms $\alpha$ such that $\alpha \wedge d\alpha$ vanishes identically. By the Frobenius theorem, this is equivalent to the existence of a codimension one foliation on $M$. It was proved by Thurston [Th76] that the vanishing of the Euler characteristic, which is necessary for the existence of a codimension one distribution, is also sufficient for the existence of a foliation. In fact, in this case every distribution is homotopic to an integrable one.

Now the defining one-form for a codimension one foliation has class 1 or 2 at every point of $M$. A one-form whose class is 1 everywhere is nowhere vanishing and closed. The existence of such a form is equivalent to the existence of a foliation without holonomy, and, by D. Tischler’s theorem [Ti70], is also equivalent to the existence of a smooth fibration of $M$ over the circle. The existence of such a fibration is a much stronger condition than the vanishing of the Euler characteristic. For example, it implies the vanishing of the signature, and it imposes restrictions on the fundamental group.

Instead of the minimal value of $r$, let us now consider the maximal value. The case of even-dimensional manifolds is rather easier than that of odd-dimensional ones. So assume for the moment that $\dim(M) = 2k + 2$. Then the maximal $r$ for which there can be a one-form $\alpha$ with $\alpha \wedge (d\alpha)^r$ nowhere zero, is $r = k$. For this value of $r$ we always have $\alpha \wedge (d\alpha)^{r+1}$ vanishing identically for dimension reasons. Now the kernel of such a one-form $\alpha$ is what is usually called a (cooriented) even contact structure. It is a result of D. McDuff [McD87] that these structures obey the $h$-principle, so that such structure exists for every reduction of the structure group of $TM$ from $GL(2k + 2; \mathbb{R})$ to $GL(k; \mathbb{C}) \times \{1\} \times \{1\}$. In this case of even contact structures, the class of a defining one-form is everywhere $2k + 1$ or $2k + 2$. If $M$ is closed, then the class cannot be $2k + 2$ everywhere, since this would give an exact volume form on $M$. The class of $\alpha$ is $2k + 1$ everywhere if and only if the holonomy of the characteristic foliation $C(\alpha) = \ker(\alpha \wedge (d\alpha)^k)$ is volume-preserving.

Returning to the case $\dim(M) = 2k + 1$ in the statement of the problem, the maximal value of $r$ for which $\alpha \wedge (d\alpha)^r$ can be nowhere zero is again $r = k$. In this case the kernel of $\alpha$ is a (cooriented) contact structure. Contact structures are known to satisfy the $h$-principle only on open manifolds; see M. Gromov [Gr86] and Y. Eliashberg–N. Mishachev [EM02]. Nevertheless, it is a classical result of R. Lutz and J. Martinet that on orientable three-manifolds every hyperplane distribution is in fact homotopic to a contact structure. Furthermore, a relationship between open book decompositions and contact structures was discovered by W. P. Thurston and H. E. Winkelnkemper [TW75]. E. Giroux [Gi02] developed this into a precise correspondence between isotopy classes of contact structures and certain equivalence classes of open book decompositions.

On manifolds of higher dimension, such general results are not known. A necessary condition for the existence of a contact structure is the existence of an almost contact structure, that is, a reduction of the structure group of $TM$ from $GL(2k + 1; \mathbb{R})$ to $GL(k; \mathbb{C}) \times \{1\}$. Already in dimension 5, this is a non-trivial constraint, but one that can be discussed effectively in terms of characteristic classes. More than twenty years ago, H. Geiges [Ge91] proved that on simply connected 5-manifolds every homotopy class of almost contact structures does contain a contact structure. Although it was clear that the fundamental group does not obstruct the existence of contact structures, see [AK94], the generalization of Geiges’s existence result took a very long time. After lots of partial results for special classes of manifolds, many of them obtained by Geiges and
his coauthors, a complete existence result for arbitrary five-manifolds, and all homotopy classes of almost contact structures on them, was only announced very recently by R. Casals, D. Pancholi and F. Presas [CPP12] and by J. B. Etnyre [Et13].

In dimensions \( \geq 7 \), there are few existence results, see however [Bo02, Ge97a, Ge97b]. An extension of the correspondence between open book decompositions and contact structures, which is only known in dimension three [TW75, Gi02], to higher dimensions was at one time announced by E. Giroux and J.-P. Mohsen. The status of that, very technical, work is still unclear at the time of writing, and, therefore, some of the results in the literature that are based on it may have to be taken with a grain of salt. If that correspondence could indeed be put on a sound footing, then it might form the basis for an inductive proof of a general existence theorem, where the induction hypothesis is applied on the spine of an open book decomposition, similarly to the application of the three-dimensional results in the proof of the existence theorem in dimension five in [Et13].

REFERENCES

[AK94] N. A’Campo and D. Kotschick, Contact structures, foliations, and the fundamental group, Bull. London Math. Soc. 26 (1994), 102–106.
[Bo02] F. Bourgeois, Odd dimensional tori are contact manifolds, Int. Math. Research Notices 2002, No. 30, 1571–1574.
[CPP12] R. Casals, D. M. Pancholi and F. Presas, Almost contact 5-folds are contact, Preprint arXiv:1203.2166v3 [math.SG] 20 Oct 2012.
[EM02] Y. Eliashberg and N. Mishachev, Introduction to the h-principle, Graduate Studies in Mathematics, 48. American Mathematical Society, Providence, RI, 2002.
[Et13] J. B. Etnyre, Contact structures on 5-manifolds, Preprint arXiv:1201.5208v2 [math.SG] 4 Feb 2013.
[Ge91] H. Geiges, Contact structures on 1-connected 5-manifolds, Mathematika 38 (1991), 303–311.
[Ge97a] H. Geiges, Constructions of contact manifolds, Math. Proc. Cambridge Philos. Soc. 121 (1997), 455–464.
[Ge97b] H. Geiges, Applications of contact surgery, Topology 36 (1997), 1193–1220.
[Gi02] E. Giroux, Géométrie de contact: de la dimension trois vers les dimensions supérieures, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 405–414, Higher Ed. Press, Beijing, 2002.
[Go95] R. E. Gompf, A new construction of symplectic manifolds, Ann. Math. 142 (1995), 527–595.
[Gr86] M. Gromov, Partial Differential Relations, Springer Verlag 1986.
[Ko97a] D. Kotschick, The Seiberg-Witten invariants of symplectic four–manifolds, Séminaire Bourbaki, 48ème année, 1995–96, no. 812, Astérisque 241 (1997), 195–220.
[McD87] D. McDuff, Applications of convex integration to symplectic and contact geometry, Ann. Inst. Fourier (Grenoble) 37 (1987), 107–133.
[Th74] W. P. Thurston, The theory of foliations of codimension greater than one, Comment. Math. Helv. 49 (1974), 214–231.
[Th76] W. P. Thurston, Existence of codimension-one foliations, Ann. of Math. 104 (1976), 249–268.
[TW75] W. P. Thurston and H. E. Winkelnkemper, On the existence of contact forms, Proc. Amer. Math. Soc. 52 (1975), 345–347.
[Ti70] D. Tischler, On fibering certain foliated manifolds over S^1, Topology 9 (1970), 153–154.
[Ya92] S.-T. Yau, Open problems in geometry, in Chern – A Great Geometer of the Twentieth Century, ed. S.-T. Yau, International Press 1992.
4. Chern’s embedding problems

Subsection 1.4 of \[Hi54\] states two embedding problems posed by S.-S. Chern, the first one about smooth and the second one about isometric embeddings. One can think of both these problems as asking for optimal embedding dimensions for certain types of embeddings of manifolds in Euclidean spaces.

**Problem 11** (Chern). *Let \( n + d(n) \) be the minimum dimension of an Euclidean space in which the \( n \)-dimensional real projective space \( \mathbb{P}^n \) can be differentiably imbedded. H. Hopf proved that (even for topological imbedding) \( d(n) > 1 \); Chern proved that \( d(n) > 2 \) if \( n \neq 2^k - 1 \) \((k \geq 2)\) and \( n \neq 2^k - 2 \) \((k \geq 2)\). Can these bounds be improved? In particular, can \( \mathbb{P}^6 \) be imbedded in an Euclidean space of 8 dimensions?*

As explained by Hirzebruch in his commentary \[Hi87, p. 774\], the explicit questions posed here were answered long ago: yes, the bounds on embedding dimensions proved in the 1940s can often be improved, and no, \( \mathbb{P}^6 \) does not embed in \( \mathbb{R}^8 \). However, the more general question implicit in Chern’s problem, namely to determine for every \( n \) the optimal embedding dimension for \( \mathbb{P}^n \), is still open. For example, it is now known that \( \mathbb{P}^6 \) embeds in \( \mathbb{R}^{11} \), but it is unknown whether it embeds in \( \mathbb{R}^{10} \).

This question has been a test case for many of the developments in algebraic topology and homotopy theory over the past 60 years. Unfortunately, the results that are known to date do not point to there being a clean answer that would be easy to state.

In the early 1960s, M. F. Atiyah attempted a clean answer, by suggesting that the minimal embedding dimension for \( \mathbb{P}^n \) should be \( 2n - \alpha(n) + 1 \), where \( \alpha(n) \) is the number of nonzero terms in the dyadic expansion of \( n \), cf. \[ES62\]. Minimality of this dimension was almost immediately disproved by M. Mahowald \[Ma64\], with the first counterexample being \( n = 12 \), where \( 2n - \alpha(n) + 1 = 23 \) and Mahowald gave an embedding in dimension 21. However, it has turned out that the other half of Atiyah’s suggestion was indeed correct: \( \mathbb{P}^n \) does embed in the Euclidean space of dimension \( 2n - \alpha(n) + 1 \). The first step in this direction was taken by D. B. A. Epstein and R. L. E. Schwarzenberger \[ES62\]. Their work in particular gives the embedding of \( \mathbb{P}^6 \) in \( \mathbb{R}^{11} \) mentioned above. The general case, for \( n \) odd only, was proved by B. Steer \[St70\]. Using Steer’s result for the projective space of one more dimension does not quite resolve the case of even \( n \), but the improvement (by one) needed for this was made by A. J. Berrick \[Be80\].

Although dimension \( 2n - \alpha(n) + 1 \) always works for \( \mathbb{P}^n \), and is minimal for certain values of \( n \), it is sometimes not minimal, as first shown by Mahowald \[Ma64\]. However, there is a dearth of positive results, and most of the progress made beyond the work of Steer and Berrick has been negative, by proving non-embedding results for various smaller dimensions. Some of these results obtained before 1987 were summarized in the commentary \[Hi87, p. 774\]. After 1987 many more incremental results have been obtained, with perhaps the most recent ones due to D. M. Davis in \[Da11\]. We refer the reader to this paper and to the literature quoted there, as well as to the webpage \[Da\] maintained by Davis that keeps track of the progress on this problem.

So far, no substitute for Atiyah’s conjecture has emerged, meaning that a general solution is not in sight. Since people have concentrated on specific, incremental, improvements for a long time, it is amazing that even for \( \mathbb{P}^6 \) one does still not know whether the minimal embedding dimension is 9, or 10 or 11, and no progress has been made on this particular case for about 50 years.
Finally, I cannot resist mentioning a far-reaching generalization of the correct part of Atiyah’s conjecture: S. Gitler [Gi71] conjectured that all $n$-manifolds embed in $\mathbb{R}^{2n-\alpha(n)+1}$. As we saw above, this is now known for real projective spaces, but it is still open in general.

We now come to Chern’s question about isometric embeddings.

**Problem 12** (Chern). A closed surface $M$ in the 4-dimensional Euclidean space $\mathbb{E}^4$ has an induced Riemannian metric and hence a Gaussian curvature $K$. Does there exist such a surface for which $K < 0$ everywhere? In particular, does there exist a surface of genus 2 in $\mathbb{E}^4$ for which $K < 0$ everywhere?

In the 1987 update, Hirzebruch wrote [Hi87, p. 775]:

*Mir ist darüber nichts bekannt.*

The context for this problem was that, on the one hand, a closed surface in $\mathbb{E}^3$ must have a point of positive Gaussian curvature (by the maximum principle), and, on the other hand, in the early 1950’s the isometric embedding problem for Riemannian manifolds (in arbitrary codimension) was very much in the air. Shortly after Hirzebruch’s problem list, J. F. Nash solved the embedding problem, and this implies of course that some $\mathbb{E}^n$, with $n$ large, contains closed surfaces of negative Gaussian curvature, and even of constant negative Gaussian curvature. Eventually the isometric embedding dimension for Riemannian surfaces was reduced to 5, see M. Gromov [Gr86, p. 298–303], showing in particular that $\mathbb{E}^5$ contains surfaces of constant negative curvature for all genera $\geq 2$. In 1962, È. R. Rozendorn [Ro62] constructed a closed surface of higher genus in $\mathbb{E}^4$ with (non-constant) strictly negative Gaussian curvature, thus answering the first part of Problem 12. We refer to [Ro92] for a sketch of Rozendorn’s construction. It is still an open question whether one can reduce the genus to 2, and whether one can make the curvature constant.

**REFERENCES**

[Be80] A. J. Berrick, *Projective space immersions, bilinear maps and stable homotopy groups of spheres*, in *Topology Symposium Siegen 1979*, Springer LNM 788, Springer Verlag 1980.

[Da11] D. M. Davis, *Some new nonimmersion results for real projective spaces*, Bol. Soc. Mat. Mexicana (3) 17 (2011), no. 2, 159–166.

[Da] D. M. Davis, *Table of immersions and embeddings of real projective spaces*, [http://www.lehigh.edu/~dmd1/immtable](http://www.lehigh.edu/~dmd1/immtable).

[ES62] D. B. A. Epstein and R. L. E. Schwarzenberger, *Imbeddings of real projective spaces*, Ann. Math. 76 (1962), 180–184.

[Gi71] S. Gitler, *Immersion and embedding of manifolds*, in *Algebraic Topology*, Proc. Symp. Pure Math. XXII, Amer. Math. Soc. 1971.

[Gr86] M. Gromov, *Partial Differential Relations*, Springer Verlag 1986.

[Ma64] M. Mahowald, *On obstruction theory in orientable fiber bundles*, Trans. Amer. Math. Soc. 110 (1964), 315–349.

[Ro62] È. R. Rozendorn, *On complete surfaces of negative curvature $K \leq -1$ in the Euclidean spaces $\mathbb{E}_3$ and $\mathbb{E}_4$*, (Russian) Mat. Sb. (N.S.) 58 (100) (1962), 453–478.

[Ro92] È. R. Rozendorn, *Surfaces of negative curvature*, in *Geometry III*, eds. Yu. D. Burago and V. A. Zalgaller, p. 87–178, Encyclopaedia Math. Sci. 48, Springer Verlag, Berlin, 1992.

[St70] B. Steer, *On the embedding of projective spaces in Euclidean space*, Proc. London Math. Soc. 21 (1970), 489–501.
5. DOLBEAULT COHOMOLOGY FOR ALMOST COMPLEX MANIFOLDS

Subsection 2.2 of [Hi54] discusses (almost) Hermitian metrics on closed almost complex manifolds. An almost complex structure on \( M \) defines a decomposition \( TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M \), which in turn induces a decomposition of complex differential forms into \((p, q)\)-types, just as in the complex case. One then defines differential operators \( \partial \), respectively \( \bar{\partial} \), as the components of the exterior derivative \( d \) that raise \( p \) respectively \( q \) by one. On functions or zero-forms one always has \( d = \partial + \bar{\partial} \), but this is no longer true on forms of higher degree as there are additional summands arising from the Nijenhuis tensor. Nevertheless, one can consider the \( L^2 \)-adjoints of these operators with respect to an Hermitian metric and define the \( \bar{\partial} \)-Laplacian

\[
\square = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^* .
\]

In [Hi54, Subsection 2.2] the space \( H^{p,q} \) is defined to be the kernel of \( \square \) in the space of \((p, q)\)-forms, and \( h^{p,q} \) is its dimension.

**Problem 20** (Kodaira-Spencer). Let \( M^n \) be an almost-complex manifold. Choose an Hermitian structure and consider the numbers \( h^{p,q} \) defined as above. Is \( h^{p,q} \) independent of the choice of the Hermitian structure? If not, give some other definition of the \( h^{p,q} \) of \( M^n \) which depends only on the almost-complex structure and which generalizes the \( h^{p,q} \) of a complex manifold.

On a complex manifold the \( H^{p,q} \) are the Dolbeault cohomology groups, isomorphic to the sheaf cohomology of the bundle of holomorphic \( p \)-forms. Therefore, in that case the \( h^{p,q} \) are indeed independent of the metric.

As pointed out in the commentary in [Hi87, p. 779], the Atiyah-Singer index theorem implies that for fixed \( p \) the index of \( \partial + \bar{\partial}^* \) is the \( p \)-component of the Hirzebruch–Todd genus, and is therefore metric-independent without assuming integrability. However, the operator \( \square \) is different from \((\partial + \bar{\partial}^*)^2 \) in the non-integrable case.

There seems to have been no progress at all on this problem, which asks for a development of harmonic Dolbeault theory on arbitrary almost complex manifolds. Such a theory could be very useful, particularly for geometrically interesting almost complex structures, like those tamed by a symplectic form. In this context an attempt to develop harmonic theory was made by S. K. Donaldson in [Do90].

M. Verbitsky [Ve11] has developed Hodge theory on strictly nearly Kähler 6-manifolds. These are certain special almost complex manifolds with a Hermitian metric with respect to which the covariant derivative of the almost complex structure is skew-symmetric. For these special Hermitian metrics Verbitsky obtained a Hodge decomposition of the harmonic forms for the usual Riemannian Laplacian \( \Delta = dd^* + d^*d \) into harmonic forms of pure \((p, q)\)-types. However, \( \Delta \) is different from the \( \bar{\partial} \)-Laplacian \( \square \) considered in Problem 20.

**REFERENCES**

[Do90] S. K. Donaldson, *Yang-Mills invariants of four-manifolds*, in Geometry of low-dimensional manifolds, 1 (Durham, 1989), 5–40, London Math. Soc. Lecture Note Ser., 150, Cambridge Univ. Press, Cambridge, 1990.

[Ve11] M. Verbitsky, *Hodge theory on nearly Kähler manifolds*, Geometry & Topology 15 (2011), 2111–2133.
6. EXISTENCE AND CLASSIFICATION OF COMPLEX STRUCTURES

Three problems that are still open concern complex structures on certain simple manifolds. We first state the two problems concerning complex projective spaces.

Problem 13. Does there exist a complex structure on $\mathbb{P}_3$ with vanishing second Chern class? (Such a complex structure cannot carry a Kählerian metric, see [...] Problem 28*.)

Hirzebruch pointed out that an almost complex structure on $\mathbb{P}_3$ with $c_2 = 0$ does exist, and is obtained by blowing up a point in $S^6$ equipped with an almost complex structure. A negative answer to Problem 13 would imply that $S^6$ does not have a complex structure. This problem is still considered to be wide open, although solutions have been claimed repeatedly over the years.

Problem 28. Consider the complex projective space $\mathbb{P}_n$ as a differentiable manifold with the usual differentiable structure. Determine all complex structures of $\mathbb{P}_n$. (See Problem 14.) In particular (28*), determine all complex structures of $\mathbb{P}_n$ which can carry a Kähler metric.

Problem 14 that is alluded to here asked about all the possible Chern classes of almost complex structures on $\mathbb{P}_n$. As described in [Hi87 p. 776], that problem was essentially solved by M. Puschnigg [Pu88].

Hirzebruch also explained in [Hi87 p. 782] how Problem 28* is solved. If one has a Kählerian complex structure on $\mathbb{P}_n$, then by the Kodaira embedding theorem it is projective algebraic. A combination of results of Hirzebruch and Kodaira proved in the mid-1950s with the later result of Yau ruling out structures with ample canonical bundle shows that the standard complex structure is the only algebraic one.

Both Problem 13 and Problem 28 are open for non-Kählerian complex structures as soon as $n \geq 3$. For $n = 2$ one does know that any complex structure on $\mathbb{P}_2$ is automatically Kähler; see [Bu99, La99]. As explained above, it is then standard.

If Yau’s conjecture mentioned in Section 3 above were true, then $S^6$ would be a complex manifold, and $\mathbb{P}_3$ would admit at least one non-Kählerian complex structure. If the strong version of the conjecture were true, in the form of an $h$-principle, then by the classification of almost complex structures due to Puschnigg mentioned above, $\mathbb{P}_n$ would have infinitely many non-homotopic and therefore non-deformation-equivalent complex structures for every $n \neq 1, 2, 4$.

This brings us to a specifically four-dimensional problem:

Problem 25. Determine all complex structures of $S^2 \times S^2$.

After the complex projective spaces, this is the next natural example to look at. In his thesis, Hirzebruch had considered the complex structures on $S^2 \times S^2$ that we now call the (even) Hirzebruch surfaces. They show that $S^2 \times S^2$ has infinitely many non-biholomorphic complex Kähler structures. These structures are all ruled and projective algebraic. Problem 25 asks whether there are any other complex structures on $S^2 \times S^2$ beyond the Hirzebruch surfaces. If we fix the standard smooth structure on $S^2 \times S^2$, then the answer is that there are no other complex structures on it. If we allow all manifolds homeomorphic to $S^2 \times S^2$, then the answer is still not known, and in this sense the problem is still open.

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1. In Section 4, $\mathbb{P}_n$ denotes real projective $n$-space, in this section $\mathbb{P}_n$ denotes complex projective $n$-space.
2. This was the subject of a lecture by Hirzebruch at the Cornell conference, cf. [St53].
To explain all this, let us consider more generally all compact complex surfaces $X$ with the Betti numbers and the intersection form of $S^2 \times S^2$, without assuming for the moment that $X$ is simply connected, and with no assumption on the underlying smooth structure. Then $X$ has to be Kählerian. This can be deduced from classification results of K. Kodaira, but a direct argument is now available, due to N. Buchdahl [Bu99] and A. Lamari [La99]. It then follows from the Kodaira embedding theorem that $X$ is projective algebraic. Its Chern numbers are $c^2_1 = 8$ and $c^2_2 = 4$. Since the intersection form is even, $X$ is minimal. Now the Enriques-Kodaira classification implies that $X$ is either ruled, in which case it is a Hirzebruch surface, or is a minimal surface of general type.

At the time when Hirzebruch wrote the commentary in [Hi87], it was an important problem to decide whether a surface of general type could ever be diffeomorphic to a rational surface. At that time, the so-called Barlow surface was the only known example of a simply connected minimal surface of general type with geometric genus $p_g = 0$, equivalently with $b^+ = 1$. In fact, by M. H. Freedman’s classification up to homeomorphism, the Barlow surface was known to be homeomorphic to the 8-fold blowup of the complex projective plane. In 1988 I proved that the Barlow surface is not diffeomorphic to a rational surface [Ko89]. There were then several attempts to generalize this result to other surfaces of general type, although no other ones were known to exist that were even homeomorphic to rational surfaces. Finally, R. Friedman and Z. Qin [FQ95] proved that no surface of general type can be diffeomorphic to a rational surface, in particular not to $S^2 \times S^2$. (This was an important result in Donaldson theory, but became fairly straightforward after the advent of Seiberg–Witten theory [FM97, Wi94].) Thus, the only complex surfaces diffeomorphic to $S^2 \times S^2$ are the even Hirzebruch surfaces.

Starting in 2006, J. Park, together with several coauthors, constructed new examples of surfaces of general type that are homeomorphic to the complex projective plane blown up in 5, 6 or 7 points. We refer to Park’s survey in [Pa10]. By [FQ95, FM97, Wi94], these surfaces are not diffeomorphic to rational surfaces. No smaller simply connected examples have been found so far, in particular none that have the Betti numbers of $S^2 \times S^2$.

If we do not insist on simple connectivity and consider arbitrary surfaces $X$ of general type with $p_g = 0$, $c^2_1 = 8$ and $c^2_2 = 4$, then examples have been known for a long time, and were mentioned already by Hirzebruch in the commentary in [Hi87]. However, all known examples are uniformized by the polydisk $\mathbb{H}^2 \times \mathbb{H}^2$, and therefore have infinite fundamental group. Some authors have speculated about a general uniformization result, that would force all surfaces with these numerical invariants to be quotients of the polydisk. If that could be proved, then any compact complex surface homeomorphic to $S^2 \times S^2$ would be a Hirzebruch surface.

The speculations about a uniformization result are motivated by the parallel case of ball quotients. Through the work of Y. Miyaoka and S.-T. Yau one knows that any surface with $c^2_1 = 3c_2 > 0$ is either the projective plane or a compact quotient of its non-compact dual $CH^2$. For the case $c^2_1 = 2c_2 > 0$, equivalently zero signature and positive Euler characteristic, some partial uniformization results have been obtained using gauge theory. First, using Yang–Mills theory, C. T. Simpson [Si88] proved that any Kähler surface with $c^2_1 = 2c_2 > 0$ having the additional property that the tangent bundle is the direct sum of two line bundles of negative degrees is uniformized by the polydisk. Second, I proved that a surface $X$ of general type with $c^2_1 = 2c_2 > 0$ for which the Seiberg–Witten invariants of the underlying smooth manifold do not vanish for the orientation opposite to that induced by the complex structure is also uniformized by the polydisk [Ko97b]. Thus, if the underlying manifold has an orientation-reversing self-diffeomorphism, or if it is complex or
symplectic for the opposite orientation in some other way, then \( X \) is uniformized by the polydisk and thus has infinite fundamental group. In particular it cannot be homeomorphic to \( S^2 \times S^2 \).

That the additional assumptions in these uniformization results cannot be dispensed with follows from the existence of several important examples that disprove an unconditional statement for surfaces of general type with zero signature. Among surfaces of general type there are simply connected ones satisfying \( c_1^2 = 2c_2 > 0 \). They can even be chosen to be spin by a result of B. Moishezon and M. Teicher \([MT87]\). These examples are homeomorphic to connected sums of very many copies of \( S^2 \times S^2 \). Very recently some new surfaces of general type with zero signature have been found that are not uniformized by the polydisk \([CMR13]\), although they have infinite fundamental group. In this case \( c_1^2 = 2c_2 = 16 \).

\[ \text{REFERENCES} \]

[Bu99] N. Buchdahl, *On compact Kähler surfaces*, Ann. Inst. Fourier (Grenoble) 49 (1999), 287–302.
[CMR13] C. Ciliberto, M. Mendes Lopes and Z. Roulleau, *On Schoen surfaces*, Preprint arXiv 1303.1750v1 [math.AG] 7 Mar 2013.
[FM97] R. Friedman and J. W. Morgan, *Algebraic surfaces and Seiberg-Witten invariants*, J. Algebraic Geom. 6 (1997), 445–479.
[FQ95] R. Friedman and Z. Qin, *On complex surfaces diffeomorphic to rational surfaces*, Invent. Math. 120 (1995), 81–117.
[Ko89] D. Kotschick, *On manifolds homeomorphic to \( \mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2} \)*, Invent. math. 95 (1989), 591–600.
[Ko97b] D. Kotschick, *Orientations and geometrisations of compact complex surfaces*, Bull. London Math. Soc. 29 (1997), 145–149.
[La99] A. Lamari, *Courants kählériens et surfaces compactes*, Ann. Inst. Fourier (Grenoble) 49 (1999), 263–285.
[MT87] B. Moishezon and M. Teicher, *Simply-connected algebraic surfaces of positive index*, Invent. Math. 89 (1987), 601–643.
[Pa10] J. Park, *A new family of complex surfaces of general type with \( p_g = 0 \)*, Proceedings of the International Congress of Mathematicians. Volume II, 1146–1158, Hindustan Book Agency, New Delhi, 2010.
[Si88] C. T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. 1 (1988), 867–918.
[Wi94] E. Witten, *Monopoles and four-manifolds*, Math. Research Letters 1 (1994), 769–796.
7. TOPOLOGICAL INVARIANCE OF CHARACTERISTIC NUMBERS OF ALGEBRAIC VARIETIES

In the following problem \( h^{p,q} \) denotes the Hodge numbers.

**Problem 31.** Are the \( h^{p,q} \) and the Chern characteristic numbers of an algebraic variety \( V_n \) topological invariants of \( V_n \)? If not, determine all those linear combinations of the \( h^{p,q} \) and the Chern characteristic numbers which are topological invariants.

This problem was motivated by the Hirzebruch–Riemann–Roch theorem, which at the time of [Hi54] was only known for complex projective varieties. Of course the problem is interesting more generally, especially for Kähler manifolds, where the Hodge numbers have the same properties and symmetries as on algebraic varieties.

It is clear from the text in [Hi54] that Hirzebruch was well aware of the fact that Chern numbers of almost complex structures would probably not be topological invariants of the underlying manifold, since there are too many almost complex structures with different Chern classes. The term “topological invariant” can be interpreted in several different ways, for example one can take it to mean homeomorphism- or diffeomorphism-invariance, and one can fix the orientation, or not. It is obvious that the top Chern number is a topological invariant in every sense of the word, since it is the Euler number, and Hirzebruch [Hi54] knew that the Pontryagin numbers are oriented diffeomorphism invariants. In fact, by S. Novikov’s result, they are oriented homeomorphism invariants.

By the Hodge decomposition of the cohomology, there are certain linear combinations of Hodge numbers that reduce to Betti numbers, and are therefore topological invariants.

Up until [Hi87], the only further information in the direction of Problem 31 was the fact that \( c_5^1 \) is not a diffeomorphism invariant of complex projective 5-folds. An example showing this had been found by Borel and Hirzebruch in 1958. Of course one can take products of this example with other varieties to generate examples of diffeomorphic varieties in higher dimensions which have distinct Chern numbers. These examples say nothing about the Hodge numbers, and, as recently as 2010, algebraic geometers were asking each other in internet discussion groups whether the Hodge numbers of Kähler manifolds are diffeomorphism invariants.

After some earlier, partial, results in small dimensions, see [Ko08], I completely solved Problem 31 for Chern numbers in 2009, see [Ko09, Ko12]. I proved that a rational linear combination of Chern numbers is an oriented diffeomorphism invariant of smooth complex projective varieties if and only if it is a linear combination of the Euler and Pontryagin numbers. If one does not fix the orientation, then in complex dimension \( n \geq 3 \) a rational linear combination of Chern numbers is a diffeomorphism invariant of smooth complex projective varieties if and only if it is a multiple of the Euler number \( c_n \). In complex dimension 2 both Chern numbers \( c_2 \) and \( c_1^2 \) are diffeomorphism-invariants of complex-algebraic surfaces, see [Ko97b, Ko08]. Except in complex dimension 2, the results are the same if one considers homeomorphism invariants instead of diffeomorphism invariants.

To prove these results one needs a supply of diffeomorphic projective algebraic varieties with distinct Chern numbers. Using the unitary bordism ring for bookkeeping, one can reduce the problem to the construction of certain special basis sequences for this bordism ring tensored with \( \mathbb{Q} \). Suitable basis sequences involving formal differences of diffeomorphic varieties are constructed in [Ko12] by considering the pairs of orientation-reversingly homeomorphic algebraic surfaces obtained in [Ko92] and algebraic projective space bundles over them, so that the total spaces become diffeomorphic, but have distinct Thom–Milnor numbers (which are special combinations of Chern numbers).
The full solution to Problem 31 was not obtained in [Ko09, Ko12], because I did not discuss Hodge numbers systematically there. I did understand that the only linear relations between Hodge and Chern numbers are the Hirzebruch–Riemann–Roch relations \( \chi(X; \Omega^p) = \text{Td}_p(X) \), where the left-hand side is a combination of Hodge numbers and the right-hand side is a combination of Chern numbers. While the examples in [Ko09, Ko12] do show that certain Hodge numbers are not diffeomorphism-invariant, there was no way to do a good bookkeeping of all possible linear combinations because the Hodge numbers are not bordism invariants.

These issues were resolved in recent joint work with S. Schreieder [KS13], in which we studied the Hodge ring of Kähler manifolds. This is a ring that keeps track of Hodge numbers in the same way that the unitary bordism ring keeps track of Chern numbers. Refining the results of [Ko09, Ko12] on the completeness of the Hirzebruch–Riemann–Roch relations we also studied a mixed Chern–Hodge ring, in which two equidimensional Kähler manifolds have the same image if and only if the have the same Hodge and Chern numbers. Using the Chern–Hodge ring, we completely solved Problem 31 for mixed linear combinations of Hodge and Chern numbers. The answer is, cf. [KS13, Theorem 4]:

A rational linear combination of Hodge and Chern numbers of smooth complex projective varieties is

1. an oriented homeomorphism or diffeomorphism invariant if and only if it reduces to a linear combination of the Betti and Pontryagin numbers after perhaps adding a suitable combination of the \( \chi(\Omega^p) - \text{Td}_p \), and

2. an unoriented homeomorphism invariant in any dimension, or an unoriented diffeomorphism invariant in dimension \( n \neq 2 \), if and only if it reduces to a linear combination of the Betti numbers after perhaps adding a suitable combination of the \( \chi(\Omega^p) - \text{Td}_p \).

REFERENCES

[Ko92] D. Kotschick, Orientation–reversing homeomorphisms in surface geography, Math. Annalen 292 (1992), 375–381.

[Ko97b] D. Kotschick, Orientations and geometrisations of compact complex surfaces, Bull. London Math. Soc. 29 (1997), 145–149.

[Ko08] D. Kotschick, Chern numbers and diffeomorphism types of projective varieties, J. of Topology 1 (2008), 518–526.

[Ko09] D. Kotschick, Characteristic numbers of algebraic varieties, Proc. Natl. Acad. USA 106, no. 25 (2009), 10114–10115.

[Ko12] D. Kotschick, Topologically invariant Chern numbers of projective varieties, Adv. Math. 229 (2012), 1300–1312.

[KS13] D. Kotschick and S. Schreieder, The Hodge ring of Kähler manifolds, Compos. Math. 149 (2013), 637–657.