THE EFFECT OF THE HALL TERM ON THE NONLINEAR EVOLUTION OF THE MAGNETOROTATIONAL INSTABILITY. II. SATURATION LEVEL AND CRITICAL MAGNETIC REYNOLDS NUMBER

Takayoshi Sano and James M. Stone

Department of Astronomy, University of Maryland, College Park, MD 20742-2421; sano@astro.umd.edu

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ABSTRACT

The nonlinear evolution of the magnetorotational instability (MRI) in weakly ionized accretion disks, including the effect of the Hall term and ohmic dissipation, is investigated using local three-dimensional MHD simulations and various initial magnetic field geometries. When the magnetic Reynolds number, Re\(_M\) = \(v_A^2/\eta\Omega\) (where \(v_A\) is the Alfvén speed, \(\eta\) is the magnetic diffusivity, and \(\Omega\) is the angular frequency), is initially larger than a critical value Re\(_{M,\text{crit}}\), the MRI evolves into MHD turbulence in which angular momentum is transported efficiently by the Maxwell stress. If Re\(_M\) < Re\(_{M,\text{crit}}\), however, ohmic dissipation suppresses the MRI, and the stress is reduced by several orders of magnitude. The critical value is in the range of 1–30 depending on the initial field configuration. The Hall effect does not modify the critical magnetic Reynolds number by much but enhances the saturation level of the Maxwell stress by a factor of a few. We show that the saturation level of the MRI is characterized by \(v_A^2/\eta\Omega\), where \(v_A\) is the Alfvén speed in the nonlinear regime along the vertical component of the field. The condition for turbulence and significant transport is given by \(v_A^2/\eta\Omega\gtrsim1\), and this critical value is independent of the strength and geometry of the magnetic field or the size of the Hall term. If the magnetic field strength in an accretion disk can be estimated observationally and the magnetic Reynolds number \(v_A^2/\eta\Omega\) is larger than about 30, this would imply that the MRI is operating in the disk.

Subject headings: accretion, accretion disks — diffusion — instabilities — MHD — turbulence

On-line material: color figures

1. INTRODUCTION

The nonlinear regime of the magnetorotational instability (MRI; Balbus & Hawley 1991) can strongly affect the structure and evolution of accretion disks. In ideal MHD, the MRI initiates and sustains MHD turbulence in which angular momentum is transported outward by Maxwell (magnetic) stress. Thus, the MRI is thought to be the most promising source of anomalous viscosity in disks. In weakly ionized disks, however, the coupling between the gas and magnetic field may be so poor that nonideal MHD effects must be considered.

When nonthermal processes (such as irradiation by cosmic rays or high-energy photons) dominate the ionization rate in the disk, the abundance of charged particles decreases as the number density of the neutral gas \(n_n\) increases. At high densities \((n_n \gtrsim 10^{19} \text{ cm}^{-3})\), ohmic dissipation dominates the evolution of the MRI (Jin 1996; Fleming, Stone, & Hawley 2000; Sano & Inutsuka 2001). At low densities \((n_n \lesssim 10^{18} \text{ cm}^{-3})\), ambipolar diffusion dominates (Blaes & Balbus 1994; Hawley & Stone 1998). However, at intermediate densities, the ions are decoupled from the magnetic field and can drift relative to the electrons (which remain frozen-in to the field). Thus, in this regime Hall currents can significantly alter the MHD of the plasma, and the Hall term dominates the other nonideal MHD effects. Detailed calculations reveal that the Hall term could be important in dwarf nova disks in quiescence and in protoplanetary disks around young stellar objects (Sano & Stone 2002, hereafter Paper I).

The properties of the MRI are strongly affected by the Hall term (Wardle 1999; Balbus & Terquem 2001). In Hall MHD the critical wavenumber and maximum growth rate of the MRI both depend on the direction of the magnetic field with respect to the angular frequency vector \(\Omega\). The Hall term can increase the maximum growth rate when the disk is threaded by a uniform vertical field in the same direction as \(\Omega\), whereas the MRI can be completely suppressed if the field is oppositely directed to \(\Omega\).

In Paper I, the effect of the Hall term on the nonlinear evolution of the MRI was investigated using axisymmetric numerical MHD simulations. These calculations included ohmic dissipation as well as the Hall effect because at some densities both processes may be important. In two dimensions, depending on the relative amplitude of the Hall and ohmic dissipation terms in the induction equation, the MRI evolves into either a two-channel flow without saturation or MHD turbulence that eventually dies away.

In this paper we continue our study of the Hall effect on the MRI using fully three-dimensional numerical MHD simulations. Previous studies have shown that only in three dimensions is sustained MHD turbulence generated by the MRI (e.g., Hawley, Gammie, & Balbus 1995). Moreover, the effect of nonaxisymmetric modes on the saturation amplitude and resulting stress can only be explored in three dimensions. Previous three-dimensional simulations including only ohmic dissipation have shown that there exists a critical value for the magnetic Reynolds number Re\(_{M,\text{crit}}\) for significant turbulence and stress to be generated in the saturated state (Fleming et al. 2000; Sano & Inutsuka 2001). Moreover, this critical value depends on the field geometry in the disk (Fleming et al. 2000). The value of Re\(_{M,\text{crit}}\) has important implications for the structure and evolution of accretion disks (Gammie 1996; Glassgold, Najita, & Igea
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1997; Sano et al. 2000). For example, the possibility of layered accretion and the size of the putative dead zone in protoplanetary disks depend on the critical value \( \text{Re}_{M, \text{crit}} \) (Fleming & Stone 2002). In addition to studying the properties of the nonlinear regime of the MRI in three-dimensional Hall MHD, an important goal of this paper is to investigate whether the inclusion of the Hall term significantly modifies the value of \( \text{Re}_{M, \text{crit}} \).

In a weakly ionized plasma, the induction equation should include terms that represent three nonideal MHD effects: the Hall effect, ohmic dissipation, and ambipolar diffusion. The importance of these terms relative to the inductive term is determined mainly by the magnitude of the neutral density, the ionization fraction, and the field strength. In Paper I, we calculated the ratios of these terms to the inductive term by solving the ionization equilibrium equations in both dwarf nova and protoplanetary disks. At the typical density of a dwarf nova disk (\( n_n \sim 10^{18} \text{ cm}^{-3} \)), the ambipolar diffusion term is much smaller than the inductive term, while the Hall and ohmic dissipation terms are equally important and of order the inductive term. These two terms are also important in the inner, dense regions of protoplanetary disks. Thus, to study the MRI in these systems, we solve the induction equation including the Hall and ohmic dissipation terms, but neglecting the ambipolar diffusion term.

Several definitions of the magnetic Reynolds number are possible; in this paper we define \( \text{Re}_M \equiv \nu \Omega / \eta \), where \( \nu \) is the Alfvén speed, \( \eta \) is the magnetic diffusivity, and \( \Omega \) is the angular frequency. This definition uses the wavelength of the fastest growing mode of the MRI as the typical length scale, i.e., \( L = \nu \Omega / \eta \), and therefore directly captures the effect of resistivity on the linear dispersion relation (Jin 1996; Sano & Miyama 1999), as well as the local properties of the saturated state (Paper I). Note, however, that with this definition \( \text{Re}_M \) depends on the magnetic field strength. Moreover, it is different from the magnetic Reynolds number \( \text{Re}_M \equiv c_s^2 / \eta \Omega \) used in Fleming et al. (2000) by a factor \( \nu \Omega / c_s^2 \), where \( c_s \) is the sound speed.

The plan of this paper is as follows. In § 2 the numerical method and initial conditions used in the calculations are described. The results from simulations using an initially uniform vertical field are discussed in § 3, from simulations using an initially vertical field with zero net flux in § 4, and from simulations using an initially uniform toroidal field in § 5. The criteria for significant angular momentum transport, and the application of the results to protoplanetary disks, are discussed in § 6, and our results are summarized in § 7.

2. NUMERICAL METHOD

We adopt the local shearing box approximation (Hawley et al. 1995) for our calculations. The MHD equations are written in a local Cartesian frame of reference \((x, y, z)\) corotating with the disk at angular frequency \( \Omega \), where \( x \) is oriented in the radial direction, \( y \) is in the azimuthal direction, and \( z \) is in the vertical direction. Vertical gravity is ignored in this analysis, since our computational domain represents a region much smaller than the thickness of the disk. The gas is assumed to be partially ionized and composed of ions, electrons, and neutrals. Charge neutrality is assumed, so that \( n_i = n_e \), where \( n_i \) and \( n_e \) are the number density of ions and electrons, respectively. The equations we solve are then

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{v},
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = - \frac{\mathbf{P}}{\rho} + \frac{\mathbf{J} \times \mathbf{B}}{\rho \eta} - 2 \Omega \times \mathbf{v} + 2 \eta \Omega^2 \mathbf{x} \hat{x},
\]

\[
\frac{\partial \epsilon}{\partial t} + \mathbf{v} \cdot \nabla \epsilon = - \frac{\mathbf{P} \mathbf{v}}{\rho} + \frac{4 \pi \eta J^2}{c^2} \rho,
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{v} \times \mathbf{B} - \frac{4 \pi \eta J}{c} \mathbf{E} - \frac{\mathbf{J} \times \mathbf{B}}{en_e} \right),
\]

where \( \mathbf{v} \) is the neutral velocity, \( \epsilon \) is the specific internal energy,

\[
\mathbf{J} = \frac{e}{4\pi} \left( \nabla \times \mathbf{B} \right)
\]

is the current density, and \( e \) is the speed of light. The equation of motion (eq. [2]) includes the Coriolis force, \(-2 \Omega \times \mathbf{v}\), and the tidal expansion of the effective potential, \( 2 \eta \Omega^2 x \), with \( \eta = 1/2 \) for a Keplerian disk. The pressure is given by \( P = (\gamma - 1) \rho \epsilon \) with \( \gamma = 5/3 \). The last two terms in the induction equation (4) represent ohmic dissipation and the Hall effect, respectively, where \( \eta \) is the magnetic diffusivity and \( e \) the elementary electric charge.

These equations are solved using a finite-difference code (Sano, Inutsuka, & Miyama 1999). The hydrodynamics module of our scheme is based on the second-order Godunov scheme (van Leer 1979) using a nonlinear Riemann solver modified to account for the effect of tangential magnetic fields. The evolution of magnetic fields is calculated with the constrained transport (CT) method (Evans & Hawley 1988), which guarantees that the divergence-free condition, \( \nabla \cdot \mathbf{B} = 0 \), is satisfied throughout the calculation. Each term of the electromotive force in the induction equation (4) is solved by an operator split solution procedure (see Paper I).

The initial physical quantities are assumed to be spatially uniform (\( \rho = \rho_0 \) and \( P = P_0 \)) except for the Keplerian shear flow \( \mathbf{v}_0 = -\eta \Omega x \). The initial magnetic field is very weak for all models, so that radial force balance in the initial state is between Coriolis and tidal forces. Three initial magnetic field configurations are considered in this paper: a uniform vertical field \( \mathbf{B}_z = B_0 \), a zero net flux vertical field \( \mathbf{B}_z(x) = B_0 \sin(2\pi x/L_x) \), where \( L_x \) is the size of the computational domain in the radial direction, and a uniform toroidal field \( \mathbf{B}_\phi = B_0 \).

The calculations are performed in a local volume bounded by \( x = \pm H/2 \), \( y = \pm 2H \), and \( z = \pm H/2 \), where \( H \equiv (2/\gamma - 1)\epsilon_{\text{io}} / \Omega \) is the scale height of the disk and \( \epsilon_{\text{io}} \) is the initial sound speed. Most of the runs use a standard grid resolution of \( 32 \times 128 \times 32 \) uniform zones. In the azimuthal and vertical directions, periodic boundary conditions are used. For the radial boundary, a sheared periodic boundary condition (Hawley & Balbus 1992) is adopted. The magnetic flux within the shearing box must be conserved unless a net flux of radial field exists (Hawley et al. 1995). However, we have found that numerical errors in the net flux of the vertical and azimuthal fields are nonnegligible (more than 10%) when the electromotive force in the ghost zones at the radial boundary is constructed from applying the sheared periodic boundary condition to the magnetic field and velocity. Instead, if we apply the sheared periodic...
boundary condition to the azimuthal component of the electromotive force, the error in the vertical flux is reduced to \(\sim 0.1\%\).

We choose the normalizations \(\rho_0 = 1\), \(H = 1\), and \(\Omega = 10^{-3}\). Then the sound speed and gas pressure are initially \(2c_0^2/\gamma = 10^{-6}\) and \(P_0 = 5 \times 10^{-7}\), respectively. Initial perturbations are introduced as spatially uncorrelated pressure and velocity fluctuations. These fluctuations have a zero mean value with a maximum amplitude of \(\epsilon\) and \(\delta \Phi\).

Ohmic dissipation. Finally, the third parameter is \(\epsilon\). This parameter measures the importance of the Hall term; note that the sign of \(\epsilon\) depends on the direction of the magnetic field.

\[ X_0 = \frac{eB_0\Omega}{2\pi\rho_0v_{A0}}, \]

\[ \beta_0 = \frac{8\pi P_0}{B_0^2} = \frac{2c_0^2}{\gamma v_A}, \]

We assume that the magnetic diffusivity \(\eta\) is constant in our calculations. This parameter measures the importance of ohmic dissipation. Finally, the third parameter is \(X_0\), where \(n_\text{e0}\) is the initial number density of electrons. We assume that the electron abundance is constant throughout our calculations, thus \(n_\text{e} = n_\text{e0}/\rho_0\). The value of \(X_0\) measures the importance of the Hall term; note that the sign of \(X_0\) depends on the direction of the magnetic field.

### 3. Simulations with a Uniform Vertical Field

Simulations that begin with a uniform vertical field are listed in Tables 1–3. The critical wavelength \(\lambda_{\text{crit}}\) and the maximum growth rate \(\sigma_{\text{max}}\) are obtained from the dispersion relation derived by Balbus & Terquem (2001) for axisymmetric perturbations. Disturbances with a wavelength longer than \(\lambda_{\text{crit}}\) are unstable to the MRI. The critical wavelength is of the order of \(2\pi v_A/\Omega\) in most cases when ohmic dissipation is inefficient (\(R_{\text{EMO}} > 1\)). However, when the Hall parameter is just below zero \((-4 < X_0 < 0)\), \(\lambda_{\text{crit}} \rightarrow 0\) and small-scale perturbations become unstable to the MRI. Ohmic dissipation, on the other hand, can stabilize small-scale perturbations and thus make \(\lambda_{\text{crit}}\) longer and \(\sigma_{\text{max}}\) smaller when \(R_{\text{EMO}} < 1\). In two dimensions, when \(X_0 < 0\) or \(R_{\text{EMO}} < 1\), saturation of the instability occurs, whereas exponential growth of a two-channel flow without saturation persists for other models (Paper I). In three dimensions without the Hall term, the nonlinear evolution of the MRI is characterized by the recurrent appearance of the two-channel flow (Fleming et al. 2000; Sano & Inutsuka 2001). We now examine how this evolution is changed by the inclusion of the Hall effect.

In addition to model parameters, Tables 1–3 contain several fundamental quantities that are referred to in the following discussion. Hereafter, the single brackets \(\langle f \rangle\) imply a volume average of quantity \(f\), whereas double brackets \(\langle\langle f \rangle\rangle\) denote a time and volume average. If not otherwise
stated, the time average is taken over the last 10 orbits of the calculation for the models in this section.

3.1. The Fiducial Models

We describe three models in detail: models Z2 \((X_0 = 2)\), Z3 \((X_0 = 0)\), and Z4 \((X_0 = -2)\). These fiducial models have the same initial field strength and magnetic Reynolds number, \(\beta_0 = 3200\) and \(R_{m0} = 100\). The only difference between the nonzero \(X_0\) models (Z2 and Z4) is the direction of the vertical field. Ohmic dissipation is so weak in these models that the evolution is dominated by the Hall effect.

Figure 1 shows the time evolution of the volume-averaged magnetic energy \(B^2/8\pi = \langle b \rangle = P_0\) for these models, where time is measured in orbits \(t_{\text{rot}} = 2\pi/\Omega\). During the linear phase of the MRI, magnetic energy is amplified exponentially. As expected from the linear analysis, the \(X_0 = 2\) model has a larger growth rate than the \(X_0 = -2\) run initially. In the nonlinear regime, MHD turbulence is sustained in all three models. The saturated magnetic energy shows large fluctuations whose amplitude depends on the size of the Hall parameter: as \(X_0\) decreases, the amplitude of the variability becomes smaller. The saturation level of the magnetic energy is higher in the positive \(X_0\) model than in the negative \(X_0\) model.

The efficiency of angular momentum transport is given by the turbulent stress,

\[
\omega_{xy} = -\frac{B_x B_y}{4\pi} + \rho v_x \delta v_y,
\]

where the first and second terms are the Maxwell stress \(\omega_M \equiv -B_x B_y/4\pi\) and Reynolds stress \(\omega_R \equiv \rho v_x \delta v_y\), respectively. The total stress \(\omega_{xy}\) is related to the \(\alpha\) parameter of Shakura & Sunyaev (1973) by \(\alpha = \omega_{xy}/P_0\). Figure 2 shows the time evolution of the Maxwell stress normalized by the initial pressure \(P_0\) for the fiducial models. The time-

| Table 2: Uniform B_z Simulations (High Resolution) |
|-----------------------------------------------|
| Model | \(\beta_0\) | \(Re_{m0}\) | \(X_0\) | \(\lambda_{\text{crit}}/H\) | \(\sigma_{\text{max}}/\Omega\) | Orbits | \(\langle \omega_M \rangle/P_0\) | \(\langle \omega_R \rangle/P_0\) | \(\langle \sigma_{A_{\text{Al}}/\rho I}^2 \rangle\) |
|-------|--------|---------|------|----------------|--------------------|-------|----------------|----------------|----------------|
| Z2H   | 3200   | 100     | 2    | 0.11           | 0.75               | 10    | 0.128          | 0.0262         | 5.36 \times 10^3 |
| Z4H   | 3200   | 100     | -2   | 0              | 0.70               | 10    | 0.0301         | 0.0102         | 1.35 \times 10^3 |

Note.—Box size is \(H \times 4H \times H\), and grid resolution is \(64 \times 256 \times 64\).

| Table 3: Uniform B_z Simulations (Large Box Size) |
|-----------------------------------------------|
| Model | \(\beta_0\) | \(Re_{m0}\) | \(X_0\) | \(\lambda_{\text{crit}}/H\) | \(\sigma_{\text{max}}/\Omega\) | Orbits | \(\langle \omega_M \rangle/P_0\) | \(\langle \omega_R \rangle/P_0\) | \(\langle \sigma_{A_{\text{Al}}/\rho I}^2 \rangle\) |
|-------|--------|---------|------|----------------|--------------------|-------|----------------|----------------|----------------|
| Z2L   | 3200   | 100     | 2    | 0.11           | 0.75               | 25    | 0.159          | 0.0428         | 6.91 \times 10^3 |
| Z4L   | 3200   | 100     | -2   | 0              | 0.70               | 25    | 0.0676         | 0.0215         | 2.47 \times 10^3 |

Note.—Box size is \(2H \times 8H \times 2H\), and grid resolution is \(64 \times 256 \times 64\).
averaged Maxwell and Reynolds stresses are listed in Table 1. In MHD turbulence generated by the MRI, the Maxwell stress is a few times larger than the Reynolds stress. Large time variability can be seen in the evolution of the stress, with the amplitude of fluctuations in the $x_0 = 2$ run about $\Delta \langle w_{M0} \rangle / P_0 \approx 0.4$, which is much larger than the time-averaged value of the stress $\langle \langle w_{M0} \rangle \rangle / P_0 = 0.16$. In the model with negative $x_0$, on the other hand, the variation is much smaller throughout the evolution.

The spike-shaped variations in the magnetic energy and stress correspond to the recurrent appearance and breakup of the two-channel flow. These same variations could be seen in the three-dimensional simulations of the MRI without the Hall effect studied by Sano & Inutsuka (2001). Figure 3 shows images of the magnetic energy distribution in the $x_0 = 0$ run (Z2) at 18, 21, and 25 orbits. The velocity field is also shown by arrows. The top and bottom panels are chosen at times near the peak of spikes in the magnetic energy, whereas the middle panel is near a minimum. The scale structure of the flow in the top and bottom panels clearly shows the axisymmetric two-channel flow. The spatial dispersion in the magnetic pressure is very high during these phases: at 18 orbits $\langle \delta P_{mag}^2 \rangle^{1/2} / \langle P_{mag} \rangle = 1.5$. The magnetic pressure is comparable to the gas pressure $\langle P_{mag} \rangle / \langle P \rangle = 0.65$, and the gas pressure and density have large spatial dispersions $\langle \delta P^2 \rangle^{1/2} / \langle P \rangle = 0.52$ and $\langle \delta \rho^2 \rangle^{1/2} / \langle \rho \rangle = 0.38$.

The axisymmetric channel flow is unstable to nonaxisymmetric modes of the parasitic instability (Goodman & Xu 1994). After the breakup of the channel flow, organized large-scale structure disappears and the disk is occupied by disorganized MHD turbulence (Fig. 3, middle panel). The magnetic energy at this phase is an order of magnitude smaller than at the peak of the spikes, so that $\langle P \rangle / \langle P_{mag} \rangle = 15$ at 21 orbits. The magnetic pressure is still spatially highly fluctuating, $\langle \delta P_{mag}^2 \rangle^{1/2} / \langle P_{mag} \rangle = 1.1$, but this fluctuation is small relative to the gas pressure, $\langle \delta P_{mag}^2 \rangle^{1/2} / \langle P \rangle = 0.075$. Since the density fluctuation decreases as the fluctuation in the magnetic pressure relative to the gas pressure $\langle \delta P_{mag}^2 \rangle^{1/2} / \langle P \rangle$ decreases (Turner, Stone, & Sano 2002), the spatial dispersion in the gas pressure and density is small at this phase: $\langle \delta P^2 \rangle^{1/2} / \langle P \rangle = 0.14$ and $\langle \delta \rho^2 \rangle^{1/2} / \langle \rho \rangle = 0.085$.

The snapshots of the magnetic energy in the negative $x_0$ model (Z4) are shown in Figure 4. In contrast to the $x_0 = 2$ run (Z2) shown in Figure 3, this model has little time variability. To allow direct comparison, the contour levels are the same in both figures. The magnetic energy is typically comparable to or smaller than that during the low-energy phase of the $x_0 = 2$ run. The emergence of the channel flow cannot be seen, and disorganized MHD turbulence is sustained throughout the evolution. Because the critical wavenumber $k_{crit} \rightarrow \infty$ in this case, small-scale disturbances are noticeable in all directions. The gas pressure is always much larger than the magnetic pressure, and the ratio of their time average is $\langle P \rangle / \langle P_{mag} \rangle = 32$. The spatial dispersion in the magnetic energy is very large, $\langle \delta P_{mag}^2 \rangle^{1/2} / \langle P_{mag} \rangle = 1.5$, but the gas pressure and density are almost uniformly distrib-

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**Fig. 3.**—Slices in the $x-z$ plane at $y = -2H$ and in the $y-z$ plane at $x = -0.5H$ of the magnetic energy, $\log(\delta B^2 / 8 \pi P_0)$ (gray scale), and perturbed velocity field, $\delta v$ (arrows), in model Z2 ($\beta_0 = 3200$, $Re_{M0} = 100$, and $x_0 = 2$) at 18, 21, and 25 orbits. [See the electronic edition of the Journal for a color version of this figure.]
uted, \(\langle \delta P^2 \rangle^{1/2} / \langle P \rangle = 0.093\) and \(\langle \delta P^2 \rangle^{1/2} / \langle \rho \rangle = 0.054\). These quantities are quite similar to those during the low magnetic energy phases of the \(X_0 = 2\) run.

To compare the characteristics of MHD turbulence in the fiducial models, Table 4 lists a number of time- and volume-averaged quantities. Although the magnetic and kinetic energies in the positive \(X_0\) run are about 4 times larger than those in the negative \(X_0\) run, interestingly the turbulence of three fiducial runs has many similarities. For example, the magnetic and perturbed kinetic energies are almost equal for all the models, with the ratio \(\langle \rho \delta v^2 / 2 \rangle / \langle B^2 / 8\pi \rangle \approx 0.4\). Because of the shear motion, the azimuthal component of the magnetic field is amplified most efficiently. Each component of the magnetic field energy has a similar ratio for all cases, \(\langle B_i^2 \rangle / \langle B_z^2 \rangle : \langle B_y^2 \rangle : \langle B_x^2 \rangle \approx 3 : 20 : 1\). The perturbed kinetic energy is slightly anisotropic, \(\langle \rho \delta v^2 \rangle : \langle \rho \delta v^2 \rangle : \langle \rho \delta v^2 \rangle \approx 3 : 2 : 1\).

The model parameters, \(\beta_0\), \(Re_{M0}\), and \(X_0\), are defined by the initial state, but they are variable in time. The time-averaged values of these parameters in the fiducial models are listed in Table 4. The magnetic pressure is smaller than the gas pressure even in the nonlinear stage, so that the time- and volume-averaged plasma beta \(\langle \beta \rangle = \langle P / B^2 / 8\pi \rangle\) is of the order of 100. However, the gas pressure is increasing linearly throughout the evolution as a result of the dissipation of magnetic field. Then, the plasma beta \(\langle \beta \rangle\) must be increasing in time unless some cooling processes are included. When the magnetic field is amplified, the magnetic Reynolds number becomes larger than the initial value. The time-averaged value is \(\langle \text{Re}_M \rangle = \langle v^2 / \eta \Omega \rangle > 10^4\) for the fiducial models, so that the ohmic dissipation at the nonlinear stage is less efficient. We define the effective Hall parameter in the nonlinear regime as

\[
X_{\text{eff}} = \frac{cB_z\Omega}{2\pi en_e c_B^2} = \frac{2c\rho B_z\Omega}{en_B^2}.
\]

Because the Hall parameter is inversely proportional to the field strength, the volume-averaged value \(\langle X_{\text{eff}} \rangle\) decreases as the field is amplified. The time-averaged value is \(\langle X_{\text{eff}} \rangle = 0.003\) and \(-0.03\) for the \(X_0 = 2\) and \(-2\) runs, respectively. Therefore, the effect of the Hall term may be reduced in the nonlinear regime for these cases.

3.2. Saturation Level

In this subsection we explore in more detail the influence of the model parameters on the time- and volume-averaged stress in the saturated state. Time-averaged Maxwell and Reynolds stresses are listed in Tables 1–3 for all models. As shown by the fiducial models, the stress exhibits large time variability in the turbulent state. For all the models, the time averaging is taken over the last 10 orbits of the calculation, which is longer than the typical timescale of variations in the stress. The time-averaged quantities taken over the last 40 orbits of the fiducial models (Z2 and Z4) are similar to the 10 orbit averages, and the difference in the Maxwell and Reynolds stress is less than 10%. The time-averaged magnetic Reynolds number defined by the vertical magnetic field
\( \langle \frac{\nu A^2}{\eta \Omega} \rangle \) is also listed in the tables. The meaning of \( \langle \frac{\nu A^2}{\eta \Omega} \rangle \) is discussed in § 6.1.

### 3.2.1. Effect of Initial Field Strength

First we study the effect of the initial field strength. Figure 5 depicts the saturated Maxwell stress as a function of the Hall parameter \( X_0 \) for various models with different \( \beta_0 \). All the models shown in this figure have the same magnetic Reynolds number \( \text{Re}_M = 1 \). For any \( \beta_0 \), the saturation levels in the positive \( X_0 \) runs are higher than those in the negative \( X_0 \) runs. The ratio of the stress between the \( X_0 = \pm 2 \) runs is 28, 4.1, and 2.6 for \( \beta_0 = 800, 3200, \) and 12,800, respectively. All models with \( X_0 \geq 0 \) show large time variability as a result of the nonlinear growth of the channel flow, while no growth of the two-channel flow can be seen in all models with \( X_0 < 0 \). If we compare models with the same \( X_0 \), larger \( \beta_0 \) models have a lower saturation level, which means that the magnetic energy and stress increase as the initial field strength increases, as in the ideal MHD cases (Hawley et al. 1995).

Since the linear growth rate of the MRI is higher for \( X_0 > 0 \) (Balbus & Terquem 2000; Paper I), this may account for the higher saturation level in this case. In addition, the evolution of the MRI shows the recurrent growth of the channel flow when the Hall parameter is \( X_0 \geq 1 \). Since the channel flow can amplify the magnetic field more efficiently than disorganized MHD turbulence, this could also be a reason for the larger saturation level in the positive \( X_0 \) runs. This result may also be understood in terms of the linear properties of the MRI: if \( X_0 \geq 0 \), the critical wavelength is proportional to the field strength \( \lambda_{\text{crit}} \sim v_A / \Omega \), so that \( \lambda_{\text{crit}} \) increases as the magnetic energy is amplified, leading to the emergence of large-scale channel flows. When the Hall parameter is negative, on the other hand, the critical wave-number for the MRI is infinity, so that small-scale perturbations are unstable. The MRI continuously excites small-scale disturbances in this case, and these small fluctuations impede the nonlinear growth of the two-channel flow.

### 3.2.2. Effect of Magnetic Reynolds Number

When the magnetic Reynolds number is very small, ohmic dissipation can dramatically reduce the linear growth rate (Jin 1996) and the nonlinear saturation level of the MRI (Sano & Inutsuka 2001). The dependence of the saturated stress on the magnetic Reynolds number is illustrated in Figure 6, which shows the time-averaged stress for the models with \( \text{Re}_M = 100, 1, \) and 0.1. The same field strength

![Figure 5. Saturation level of the Maxwell stress as a function of the Hall parameter \( X_0 \) for the models with \( \beta_0 = 800, 3200, \) and 12,800. The magnetic Reynolds number is \( \text{Re}_M = 1 \) for all the models.](image-url)
is used for all the models in this figure ($\beta_0 = 3200$). We find that the positive $X_0$ runs always have a larger stress than the negative $X_0$ runs. The differences are by a factor of 4–6 in the $Re_{M0} = 100$ and 1 runs. For very resistive models with $Re_{M0} = 0.1$, the ratio of the stress between the $X_0 = 4$ and $-2$ runs is about 100. However, the stress is of the order of $10^{-5}$ to $10^{-4}$, and this is more than 2 orders of magnitude smaller than the less resistive cases ($Re_{M0} \geq 1$). This suggests that strong dissipation can weaken the turbulence even when the Hall term is included.

3.2.3. Effect of Resolution and Box Size

Table 2 lists the model parameters and saturation levels for high-resolution runs designed to study the effect of numerical resolution. Except for the grid resolution, the model parameters of Z2H and Z4H are the same as those of the nonzero $X_0$ fiducial models Z2 and Z4. The high-resolution models are followed only to 10 orbits, so that the time average is taken over the last five orbits. In the standard resolution models (Z2 and Z4), the time-averaged stress over every five orbits after 10 orbits has up to 70% difference compared to the average through the last 40 orbits. It may be that the time-averaged stress in the high-resolution models has uncertainty of the similar size. We find that most of the nonlinear features shown by the fiducial models are independent of the grid resolution. The magnetic energy and stress in the $X_0 = 2$ run (Z2H) are larger than those in the $X_0 = -2$ run (Z4H) by a factor of about 4. The saturation levels in models Z2H and Z4H are close to those in the standard resolution cases. The positive $X_0$ run shows the nonlinear growth of the channel flow, and the negative $X_0$ run evolves into disorganized MHD turbulence with many small-scale fluctuations.

Table 3 lists the models computed with a larger computational box than the fiducial models, namely, $2H \times 8H \times 2H$. The grid resolution is $64 \times 256 \times 64$, so that the grid spacing is the same as the standard models listed in Table 1. The other parameters of models Z2L and Z4L are the same as the fiducial models Z2 and Z4. As in the standard box size cases, the magnetic energy and stress in the positive $X_0$ run (Z2L) are larger than those in the negative $X_0$ run (Z4L) throughout the evolution. The frequency of spike-shaped variations in the stress is reduced with a larger box size, and thus disorganized MHD turbulence lasts most of the time. The stress in the large box model with $X_0 = 2$ (Z2L) is comparable to that in the standard model Z2, but the $X_0 = -2$ model (Z4L) is twice as large as the standard model Z4. Although the saturation level for models with a uniform vertical field may be proportional to the box size for ideal MHD simulations (Hawley et al. 1995), our sample of simulations is too small to confirm this dependence in this study.

3.3. Characteristics of Saturated MHD Turbulence

It is of interest to study the properties of the MHD turbulence driven by the MRI in Hall MHD. Figures 7a and 7b show the Fourier power spectra of the magnetic energy along the $k_x$, $k_y$, and $k_z$ axes at 25 orbits for the fiducial models Z2 ($X_0 = 2$) and Z4 ($X_0 = -2$), respectively. The spectra are averaged over 10 snapshots within 0.1 orbits. In the $X_0 = 2$ run, the large-scale channel flow is growing at that time. Thus, smaller wavenumbers have larger power, especially in $k_z$, and the power at the inertial range declines as $k^{-4}$, similar to the ideal MHD case (Hawley et al. 1995). For the $X_0 = -2$ run (Fig. 7b), the amplitude of the power is smaller than the $X_0 = 2$ run, although the shape of the spectra is quite similar to the positive $X_0$ run. The slope of the power is slightly gentler in the $X_0 = -2$ run, probably because modes with larger $k$ are unstable to the MRI in this case. The power spectra at the low magnetic energy phase of the $X_0 = 2$ run (which is dominated by disorganized MHD turbulence) are similar to those of the negative $X_0$ model.

When ohmic dissipation is inefficient ($Re_{M0} > 1$), the MHD turbulence generated by the MRI appears to have the
the diffusion scale $k_{\text{diff}} \equiv v_A/\eta$ and at the critical wavenumber for the MRI $k_{\text{crit}}$, is shown on the plot of the power spectra of the magnetic energy (Fig. 8a). The diffusion length is close to the critical wavelength in this case. For scales smaller than the diffusion length, fluctuations in the magnetic field dissipate faster than the Alfvén timescale. Therefore, the spectrum shows a very steep decline, and the slope is proportional to $k^{-8}$ in the dissipation regime $k \gg k_{\text{diff}}$. Figure 8b shows the power spectra of the perturbed kinetic energy, which also is a steeply decreasing function of $k$.

We find that the energy distribution of the turbulence in the $Re_{M0} = 0.1$ runs is also quite different from that in the less resistive models. In Table 4, time- and volume-averaged quantities for the $Re_{M0} = 0.1$ runs (Z12, Z13, and Z14) are listed. The saturated level of the magnetic energy is much lower than the $Re_{M0} = 100$ runs. The magnetic field is amplified by the MRI during the linear phase but dies away as a result of the ohmic dissipation, so that in the nonlinear regime the magnetic energy returns to its initial value. Because the net flux through the computational volume is conserved, the magnetic energy can never completely die away. The energy in the perturbed velocity is larger than in the magnetic field, $\langle (\rho \delta v^2/2) \rangle \gg \langle (B^2/8\pi) \rangle$, and the Reynolds stress is a few times larger than the Maxwell stress for the $Re_{M0} = 0.1$ models. The large kinetic energy is a remnant of the linear growth of the MRI. Although the perturbed magnetic field generated by the growth of the MRI is efficiently dissipated in these models by the large resistivity, the perturbed kinetic energy remains large as a result of the small viscosity. Therefore, if the disk is linearly unstable and the magnetic Reynolds number is small ($Re_{M0} < 1$), the Reynolds stress could dominate the Maxwell stress, but the efficiency of angular momentum transport is very small ($\alpha \sim 10^{-5}$ to $10^{-3}$).

3.4. Critical Magnetic Reynolds Number

In this subsection we consider the critical value of the magnetic Reynolds number for significant angular momentum transport in accretion disks with uniform vertical fields (as shown by previous studies, this critical value depends on the initial field geometry; Fleming et al. 2000). The dependence of the saturation level on the magnetic Reynolds number $Re_{M0}$ is shown by Figure 9. Open circles and triangles denote the results without the Hall term (models Z1, Z12, and Z14) show little difference in the energy spectra compared to the $X_0 = 0$ run. The typical diffusion scale defined as $k_{\text{diff}} \equiv v_A/\eta$, as well as the critical wavenumber for the MRI, $k_{\text{crit}}$, is shown on the plot of the power spectra of the MRI.
The models including the Hall term are shown by filled circles (\(\triangle\)) or triangles (\(\square\)). The models without Hall term (Paper I). Thus, the dependence of the saturation level on the ohmic dissipation and the Hall effect is consistent with the two-dimensional results. When \(\text{Re}_{x0} \geq 1\), the magnetic Reynolds number \(<(v^2_A/\eta\Omega)>\) at the nonlinear stage is larger than the initial value because the magnetic field is amplified by the MRI. If \(\text{Re}_{x0} < 1\), the magnetic energy is unchanged by the instability, and thus \(<(v^2_A/\eta\Omega)\) remains less than unity. Therefore, the critical value \(<(v^2_A/\eta\Omega)\) ~ 1 is valid even in the nonlinear regime for uniform \(B_z\) models.

4. SIMULATIONS WITH A ZERO NET FLUX

VERTICAL FIELD

We next consider simulations that begin with a zero net flux vertical field, that is, \(B_z(x) = B_0 \sin(2\pi x/L_x)\), where \(B_0\) is a positive constant. For this case, the Hall parameter is given by \(X(x) = x_0/\sin(2\pi x/L_x)\), where \(x_0 = 2\zeta_{\Omega 0}/en_{\Omega 0}B_0\). Thus, the region \(x < 0\) has positive \(X\) while the region \(x > 0\) has negative \(X\), and the minimum of the absolute value \(|X(x)|\) is \(x_0\). Table 5 lists the models computed with a zero net flux \(B_z\). The initial plasma beta \(\beta_0\), the critical wavelength \(\lambda_{\text{crit}}\), and the maximum growth rate \(\sigma_{\text{max}}\) in this table are given for \(B_z = B_0\) and \(X = x_0\). The saturation level of the Maxwell and Reynolds stresses and the magnetic Reynolds number are also listed in the table. No data in the columns for the saturation level mean that it is less than \(10^{-8}\). The time average is taken over the last 20 orbits for all models in this section.

Figure 10 shows the time evolution of the magnetic stress for models S1 \((x_0 = 0\), S2 \((x_0 = 2\), and S3 \((x_0 = 4\). All the other parameters for these models are identical \((\beta_0 = 3200\) and \(\text{Re}_{x0} = 100\)). Because the magnetic Reynolds number for these models is very large, the Hall effect

![Graph](image)

**TABLE 5**

**ZERO NET FLUX \(B_z\) SIMULATIONS (STANDARD BOX SIZE AND RESOLUTION)**

| Model | \(\beta_0\) | \(\text{Re}_{x0}\) | \(x_0\) | \(\lambda_{\text{crit}}/H\) | \(\sigma_{\text{max}}/\Omega\) | Orbits | \(<(w_B)/\rho_0\)>/\(\rho_0\) | \(<(w_\Omega)/\rho_0\)>/\(\rho_0\) | \(<(v^2_A/\eta\Omega)\>)/\(\rho_0\) |
|-------|--------------|-----------------|--------|-----------------|-----------------|--------|-----------------|-----------------|-----------------|
| S1    | 3200         | 100             | 0      | 0.064           | 0.74            | 50     | 0.0168          | 0.0130           | 630             |
| S2    | 3200         | 100             | 2      | 0.11            | 0.75            | 50     | 0.0215          | 0.00673          | 647             |
| S3    | 3200         | 100             | 4      | 0.14            | 0.75            | 50     | 0.0520          | 0.0133           | 1.92 \times 10^3|
| S4    | 3200         | 100             | 0      | 0.064           | 0.70            | 50     | 0.0177          | 0.00837          | 77.4            |
| S5    | 3200         | 100             | 2      | 0.11            | 0.72            | 50     | 0.0244          | 0.00677          | 68.3            |
| S6    | 3200         | 100             | 4      | 0.14            | 0.73            | 50     | 0.0427          | 0.0118           | 166             |
| S7    | 3200         | 3               | 0      | 0.068           | 0.60            | 50     | 2.01 \times 10^{-5}| 2.28 \times 10^{-6}| 0.00401          |
| S8    | 3200         | 1               | 0      | 0.091           | 0.43            | 50     | 3.95 \times 10^{-6}| 3.81 \times 10^{-4}| 2.63 \times 10^{-2}| 6.66 \times 10^{-6}|
| S9    | 3200         | 1               | 2      | 0.12            | 0.51            | 50     | 8.83 \times 10^{-7}| 1.03 \times 10^{-6}| 6.66 \times 10^{-6}|
| S10   | 3200         | 1               | 4      | 0.15            | 0.57            | 50     | 0.0326          | 0.00881          | 13.4            |
| S11   | 3200         | 0.3             | 0      | 0.22            | 0.20            | 50     | ...             | ...              | ...             |
| S12   | 3200         | 0.3             | 2      | 0.21            | 0.28            | 50     | ...             | ...              | ...             |
| S13   | 3200         | 0.3             | 4      | 0.21            | 0.34            | 50     | ...             | ...              | ...             |
| S14   | 3200         | 0.3             | 100    | 0.65            | 0.70            | 50     | 7.02 \times 10^{-5}| 1.86 \times 10^{-7}| 0.00629          |
| S15   | 3200         | 0.1             | 100    | 0.66            | 0.62            | 50     | ...             | ...              | ...             |
| S16   | 3200         | 0.1             | 1000   | 2.0             | 0.74            | 50     | ...             | ...              | ...             |
| S17   | 800          | 100             | 0      | 0.13            | 0.74            | 50     | 0.0173          | 0.0106           | 194             |
| S18   | 800          | 30              | 0      | 0.13            | 0.73            | 50     | 0.0159          | 0.0130           | 59.8            |
| S19   | 800          | 10              | 0      | 0.13            | 0.70            | 50     | 1.41 \times 10^{-5}| 2.44 \times 10^{-6}| 7.37 \times 10^{-4}|
| S20   | 800          | 3               | 0      | 0.14            | 0.60            | 50     | 1.74 \times 10^{-7}| 3.29 \times 10^{-6}| 1.75 \times 10^{-5}|
| S21   | 12800        | 100             | 0      | 0.032           | 0.74            | 100    | 0.0265          | 0.0134           | 4.31 \times 10^3|
| S22   | 12800        | 10              | 0      | 0.032           | 0.70            | 100    | 0.0239          | 0.0152           | 358             |
| S23   | 12800        | 3               | 0      | 0.034           | 0.60            | 100    | 0.0349          | 0.0153           | 160             |
| S24   | 12800        | 1               | 0      | 0.045           | 0.43            | 100    | 9.90 \times 10^{-6}| 4.15 \times 10^{-6}| 0.00179          |
| S25   | 12800        | 0.3             | 0      | 0.11            | 0.20            | 100    | ...             | 1.27 \times 10^{-6}| 1.66 \times 10^{-6}|

Note.—Box size is \(H \times 4H \times H\), and grid resolution is \(32 \times 128 \times 32\).
plasma beta and the magnetic Reynolds number of these models are very small and negative for this case, dominantly distributed. The effective Hall parameter at the non-

dominates the evolution of the MRI. The stress in the $X_0 = 4$ run has large-amplitude time variations and a higher saturation level than that in the $X_0 = 0$ run. Thus, the Hall effect enhances the saturation level of the stress, as in the uniform $B_z$ cases. The difference in the time average $\langle \langle \omega_{M} \rangle \rangle$ is a factor of 3 between the $X_0 = 0$ and 4 runs. This saturation level is an order of magnitude lower than the uniform $B_z$ models with the same $\beta_0$ and Re$_{MB}$. During the linear phase, the MRI grows most rapidly in the half-region with positive $X$ because most of the region with $X < 0$ is linearly stable for the MRI or has a smaller growth rate (see Paper I). The amplified magnetic field gradually affects the structure of the other ($X < 0$) half-region, and after several orbits the entire region becomes turbulent. Figure 11 shows images of the magnetic energy during the turbulent phase for model S3 ($X_0 = 4$) at 35, 40, 45, and 50 orbits. The magnetic field vectors in the $x$-$z$ plane are also shown by arrows. A large time variation in the stress occurs near 35 orbits (see Fig. 10); however, from Figure 11 no channel flow is evident at this time. Instead disorganized MHD turbulence is sustained throughout the evolution. The amplitude of time variation in the stress is slightly smaller compared with uniform $B_z$ cases. At the turbulent phase, the initial distribution of the vertical field disappears completely, and the positive and negative $X$ regions are randomly distributed. The effective Hall parameter at the nonlinear stage is very small and negative for this case, $\langle \langle \langle X_{eff} \rangle \rangle \rangle = -0.018$. At 50 orbits, the spatial dispersion is large in the magnetic pressure $\langle \delta P_{mag} \rangle^{1/2} / \langle P_{mag} \rangle = 1.0$ but small in the density and gas pressure $\langle \delta \rho \rangle^{1/2} / \langle \rho \rangle = 0.12$ and $\langle \delta P_{gas} \rangle^{1/2} / \langle P \rangle = 0.069$. These numbers are similar to the uniform $B_z$ cases during phases of low magnetic energy.

Since there is no net flux for this field configuration, the magnetic field within the shearing box can completely die away. In fact, the magnetic energy during the nonlinear regime is decreasing in time for models with small Re$_{MB}$. Figure 12 shows the time evolution of the magnetic energy for models with Re$_{MB} = 1$. Ohmic dissipation affects the linear characters of the MRI in these cases. When the Hall parameter is small, models S8 ($X_0 = 0$) and S9 ($X_0 = 2$), the magnetic energy is amplified by the MRI during the linear phase, but this amplified field is not sustained. After a few tens of orbits, the magnetic energy and stress decrease as a result of ohmic dissipation until the end of the calculation. In the $X_0 = 4$ run (S10), however, the amplified magnetic energy is sustained for at least 50 orbits even with Re$_{MB} = 1$. The time evolution of run S10 is quite similar to the less resistive models shown in Figure 10. The saturation level of the Maxwell stress in model S10 is $\langle \langle \omega_M \rangle \rangle / \langle \omega \rangle = 0.033$, and this is comparable to those in models S3 (Re$_{MB} = 100$) and S6 (Re$_{MB} = 10$). This enhancement is caused by a faster linear growth rate for the MRI at larger $X_0$ (Paper I). Once the field is amplified, the efficiency of the Hall term and ohmic dissipation is reduced, and thus turbulence can be sustained.

Table 6 lists the time- and volume-averaged quantities for the models with Re$_{MB} = 100$ (S1 and S3) and Re$_{MB} = 1$ (S8 and S10). Models S1 and S8 do not include the Hall effect ($X_0 = 0$), whereas models S3 and S10 have $X_0 = 4$. The properties of the turbulence in the $X_0 = 4$ runs (S3 and S10) are nearly identical. The saturation level of the magnetic energy is $\langle \langle \omega_M \rangle \rangle / \langle \omega \rangle = 0.1$, which is 3 times larger than the perturbed kinetic energy. The ratio of the Maxwell and Reynolds stresses is $\langle \langle \omega_M \rangle \rangle / \langle \omega \rangle \approx 4$. The ratio of each component of the magnetic energy in the turbulence is $\langle \langle \omega_1 \rangle \rangle : \langle \langle \omega_2 \rangle \rangle : \langle \langle \omega_3 \rangle \rangle = 2 : 18 : 1$, and this is quite similar to that in the uniform $B_z$ models. Since the magnetic Reynolds number in the nonlinear regime $\langle \langle \omega_M \rangle \rangle$ is large, ohmic dissipation is ineffective. Moreover, the Hall effect may also be unimportant in the nonlinear regime because the effective Hall parameter is small, $\langle \langle \omega_\theta \rangle \rangle \approx -0.02$. The saturated quantities in model S1 are also close to the $X_0 = 4$ runs. For these three models, the saturated level of $\alpha$ is of the order of 0.01.

The turbulence in model S8, on the other hand, is strongly affected by ohmic dissipation. The stress $\alpha = 7.8 \times 10^{-6}$ at 50 orbits is much smaller than the other runs and is decreasing in time. The properties of the turbulence, such as $\langle \langle B_1 \rangle \rangle / \langle \langle B_2 \rangle \rangle$, are also very different from the other runs. The magnetic Reynolds number in the nonlinear regime is still small $\langle \langle \omega_M \rangle \rangle = 1.2$, which means that the ohmic dissipation is important throughout the evolution of this model.

### 4.1. Critical Magnetic Reynolds Number

Here we estimate the critical value of the magnetic Reynolds number $\text{Re}_{ MB, crit}$ for zero net flux $B_z$ models. Figure 13 shows the saturation level of the Maxwell stress for the models without the Hall term ($X_0 = 0$). Three different cases with $\beta_0 = 800$, 3200, and 12,800 are shown in this figure. When $\text{Re}_{ MB} = 100$, the stress for all the models is more than 0.01, so that angular momentum transport is efficient. If the magnetic Reynolds number is below a critical value, the stress drops dramatically. For the $\beta_0 = 3200$ runs, the stress is $\langle \langle \omega_M \rangle \rangle / \langle \omega \rangle = 0.018$ and $\langle \langle \omega_1 \rangle \rangle / \langle \omega \rangle = 2.0 \times 10^{-5}$ at $\text{Re}_{ MB} = 10$ and 3, respectively. Thus, the stress decreases about 3 orders of magnitude with a small difference in $\text{Re}_{ MB}$.

The critical value is $\text{Re}_{ MB, crit} \approx 10$ for $\beta_0 = 3200$, but this value is found to depend on the initial field strength.
The condition for \( \Re \sim 0 > 0.01 \), for example, is \( \Re > 30, 10, \) and \( 3 \) for \( \beta_0 = 800, 3200, \) and \( 12,800, \) respectively. When \( \Re \sim 0 > \Re_{\text{crit}} \), the saturation level of the Maxwell stress is nearly independent of \( \beta_0 \) \((\text{Hawley, Gammie, \& Balbus 1996})\), and this is different from the uniform \( B_0 \) case. In order to initiate the turbulence, the MRI must grow faster than the dissipation rate of the initial field. Thus, the critical value for the instability should be given by

\[
\frac{v_{A0}L}{\eta} = \Re_{\text{M0}} \sqrt{\beta_0} \left( \frac{L}{L_\alpha} \right) \sim \text{const} \quad (13)
\]

(where \( L \) is the length over which the initial field varies), so that the critical value of \( \Re_{\text{M0}} \) is proportional to \( \beta_0^{-1/2} L^{-1} \). This idea is roughly consistent with the results shown in Figure 13. If \( L \) decreases, the critical value should increase. If fact, when we use a field distribution \( B_0(x) = B_0 \sin(6\pi x/L_\alpha) \) \((\text{i.e., } L = L_\alpha/3)\), the criterion becomes \( \Re_{\text{M0}} > 30 \) for \( \beta_0 = 3200 \). This suggests that if the zero net flux magnetic field has small-scale structure initially, then the ohmic dissipation can suppress the initiation of the MRI with a small magnetic diffusivity \( \eta \).

Next we examine the effect of the Hall term on the critical magnetic Reynolds number. Figure 14 shows the saturated magnetic energy as a function of the initial magnetic Reynolds number \( \Re_{\text{M0}} \). For comparison, the magnetic Reynolds number \( \Re_{\text{M0}} \) used in Fleming et al. (2000) is also shown in the figure. The initial field strength for all the models in this figure is \( \beta_0 = 3200 \). Open circles denote the models without the Hall term. For \( \chi_0 = 0 \), as discussed above, the saturation level is almost constant for \( \Re_{\text{M0}} \sim 10 \) but decreases as the magnetic Reynolds number decreases if \( \Re_{\text{M0}} < 10 \). When \( \Re_{\text{M0}} = 0.3 \), the magnetic energy does not show any increase during the evolution and almost dies out: \( \langle B^2/(8\pi) \rangle/P_0 \sim 10^{-12} \) at the end of the calculation. The critical value for significant turbulence, \( \langle B^2/(8\pi) \rangle/P_0 > 0.01 \), is \( \Re_{\text{M0, crit}} \sim 10 \). This value corresponds to \( \Re_{\text{M0, crit}} \sim 3 \times 10^4 \), which is consistent with the results of Fleming et al. (2000).

For the \( \chi_0 = 2 \) runs depicted by filled triangles, the effect of the Hall term is not large; the critical magnetic Reynolds number is still \( \Re_{\text{M0, crit}} \sim 10 \). However, if the size of the Hall term increases, we find that \( \Re_{\text{M0, crit}} \) shifts to smaller values. The \( \chi_0 = 4 \) runs are shown by filled circles in this figure. The saturation level in the \( \Re_{\text{M0}} = 1 \) run is increased to

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**Fig. 11.**—Slices in the \( x-z \) plane at \( y = 0 \) of the azimuthal component of the magnetic energy, \( \log(B_0^2/(8\pi P_0)) \) (gray scale), and magnetic field vector, \( B \) (arrows), in model S3 (\( \beta_0 = 3200, \Re_{\text{M0}} = 100, \) and \( \chi_0 = 4 \)) at 35, 40, 45, and 50 orbits. [See the electronic edition of the Journal for a color version of this figure.]
The plasma beta and the magnetic Reynolds number of these models are $\beta_0 = 3200$ and $Re_{M0} = 1$.

The Hall parameter $X_0$ can take a large range of values when $Re_{M0}$ is small because the maximum value is $X_0 \sim 100/Re_{M0}$ in the Hall regime (Paper I). The negative $X$ region ($x > 0$) is linearly stable when $X_0 > 4$; however, this is potentially offset by the larger linear growth rate in the regions with a large positive Hall parameter. We have calculated models with $X_0 = 100$ (Fig. 14, filled squares) and 1000 (cross). For $Re_{M0} = 0.3$, the saturation level is enhanced by many orders of magnitude when $X_0 = 100$, but the stress is still very small ($\langle\langle B'z/8\pi\rangle\rangle/P_0 = 7.0 \times 10^{-5}$) compared with the less resistive models ($Re_{M0} \geq 10$). For more resistive models ($Re_{M0} = 0.1$), the MRI does not operate.

### Table 6

| Quantity | S1       | S3       | S8       | S10      | Average$^a$ |
|----------|----------|----------|----------|----------|-------------|
| $\beta_0$ | 3200     | 3200     | 3200     | 3200     | ...         |
| $Re_{M0}$ | 100      | 100      | 1        | 1        | ...         |
| $X_0$ | 0        | 4        | 0        | 4        | ...         |
| $\langle\langle B'/8\pi\rangle\rangle/P_0$ | 0.00584  | 0.0138   | 1.11 $\times 10^{-2}$ | 0.00938 | 0.00821 $\pm$ 0.00274 |
| $\langle\langle B'/8\pi\rangle\rangle/P_0$ | 0.0404   | 0.115    | 3.71 $\times 10^{-4}$ | 0.0743  | 0.0604 $\pm$ 0.0237 |
| $\langle\langle B'/8\pi\rangle\rangle/P_0$ | 0.0207   | 0.0605   | 8.24 $\times 10^{-5}$ | 0.00416 | 0.00339 $\pm$ 0.00129 |
| $\langle\langle \rho v^2/2\rangle\rangle/P_0$ | 0.0281   | 0.0206   | 1.28 $\times 10^{-4}$ | 0.0146  | 0.0266 $\pm$ 0.0119 |
| $\langle\langle \rho v^2/2\rangle\rangle/P_0$ | 0.0199   | 0.0143   | 9.69 $\times 10^{-7}$ | 0.0102  | 0.0176 $\pm$ 0.0075 |
| $\langle\langle \rho v^2/2\rangle\rangle/P_0$ | 0.00686  | 0.00766  | 7.95 $\times 10^{-7}$ | 0.00565 | 0.00824 $\pm$ 0.00304 |
| $\langle\langle P'\rangle\rangle/P_0$ | 2.58     | 1.94     | 1.04     | 3.32     | ...         |
| $\langle\langle \rho w\rangle\rangle/\langle\langle w\rangle\rangle$ | 1.29     | 3.91     | 1.04     | 3.70     | 2.51 $\pm$ 0.98 |
| $\langle\langle \rho w^2/2\pi\rangle\rangle$ | 0.362    | 0.403    | 0.0106   | 0.390    | 0.394 $\pm$ 0.018 |
| $\langle\langle (P'/8\pi)^2\rangle\rangle/P_0$ | 55.5     | 15.0     | 2.81 $\times 10^{3}$ | 39.8    | ...         |
| $\langle\langle (\rho v^2/2)/(\rho w^2/2\pi)\rangle\rangle$ | 1.18     | 0.330    | 0.349    | 0.364    | 0.869 $\pm$ 0.447 |
| $\langle\langle (P'/8\pi)/(\rho w^2/2\pi)\rangle\rangle$ | 2.83     | 2.28     | 0.134    | 2.26     | 2.46 $\pm$ 0.22 |
| $\langle\langle (\rho v^2/2)/(\rho w^2/2\pi)\rangle\rangle$ | 19.6     | 19.0     | 4.51 $\times 10^{3}$ | 17.9    | 18.0 $\pm$ 3.1 |
| $\langle\langle \rho w\rangle\rangle/\langle\langle \rho w\rangle\rangle$ | 4.10     | 2.69     | 161      | 2.58     | 3.15 $\pm$ 0.46 |
| $\langle\langle \rho w^2/2\pi\rangle\rangle/\langle\langle \rho w^2/2\pi\rangle\rangle$ | 4.10     | 2.69     | 161      | 2.58     | 3.15 $\pm$ 0.46 |
| $\langle\langle \rho v^2/2\rangle\rangle/\langle\langle \rho v^2/2\rangle\rangle$ | 2.90     | 1.86     | 1.22     | 1.80     | 2.10 $\pm$ 0.32 |
| $\langle\langle \rho \rangle\rangle$ | 1.23 $\times 10^{3}$ | 204     | 2.81 $\times 10^{3}$ | 562     | ...         |
| $\langle\langle \rho \rangle\rangle$ | 1.52 $\times 10^{2}$ | 4.51 $\times 10^{4}$ | 1.19 | 288     | ...         |
| $\langle\langle \rho \rangle\rangle$ | 0        | -0.0181  | 0        | -0.0335  | ...         |
| $\langle\langle \rho \rangle\rangle$ | 0.0298   | 0.0653   | 7.76 $\times 10^{-6}$ | 0.0414  | 0.0385 $\pm$ 0.0120 |

$^a$ Models with $\langle\langle \rho \rangle\rangle > 1$ are considered for the average (S1–S6, S10, S17, S18, and S21–S23).
Fig. 14.—Saturation level of the magnetic energy as a function of the magnetic Reynolds number $R_{eM0}$ for zero net flux $B_z$ models ($\beta_0 = 3200$). Open circles denote the models with only the ohmic dissipation ($X_0 = 0$), and the other symbols are including also the Hall effect ($X_0 = 2, 4, 100, \text{and} 1000$).

The simulation results show that the saturation level of the magnetic energy is very sensitive to the Hall parameter. Without the Hall effect because ohmic dissipation is inefficient even with Hall parameters as large as $X_0 = 100$ and 1000. Note that in the model with $R_{eM0} = 0.1$ and $X_0 = 1000$ ($S16$), the critical wavelength is longer than the scale height of the disk, meaning that such large values of $X_0$ cannot result in enhanced transport in real disks. Therefore, the change in the critical magnetic Reynolds number due to the Hall term is at most an order of magnitude for zero net flux vertical fields.

5. SIMULATIONS WITH A UNIFORM TOROIDAL FIELD

Finally, we investigate the evolution of the MRI starting with a uniform toroidal field, $B_z = B_0$. Table 7 lists the models calculated with this field geometry. Axisymmetric perturbations are stable if the field is purely toroidal. Since no linear dispersion relation has been obtained for the MRI for nonaxisymmetric perturbations including nonideal MHD effects, the characteristic length of the MRI defined as $\lambda_{MRI} \equiv 2\pi v_{A0}/\Omega$ is listed in the table instead of $\lambda_{\text{crit}}$ and $\sigma^\text{max}$. The length $\lambda_{MRI}$ corresponds to the wavelength in the azimuthal direction of the most unstable mode. The table gives the saturation level of the Maxwell stress $\langle \omega_M \rangle /P_0$, the Reynolds stress $\langle \omega_R \rangle /P_0$, and the magnetic Reynolds number $\langle v_{A0}^2/\eta \Omega \rangle$, if they are more than $10^{-8}$. In some cases the uniform $B_z$ models must be evolved for very long times to reach a saturated nonlinear regime. Time averages are taken over the last 40 orbits for all the models in this section.

We still use the Hall parameter $X_0$ defined by equation (8) as a model parameter. Balbus & Terquem (2001) examined the Hall effect on the nonaxisymmetric behavior of linear perturbations. The critical wavenumber for the MRI becomes longer or shorter depending on the sign and size of the Hall parameter. The effective Hall parameter, $H_a$, for nonaxisymmetric disturbances with a wavenumber $\mathbf{k}$ is given by

$$H_a \equiv \frac{c(\mathbf{k} \cdot \mathbf{B})(\mathbf{k} \cdot \Omega)}{2\pi \Omega v_{A0}^2} \equiv X_0 \left( \frac{k_x v_{A0}}{\Omega} \right) \left( \frac{k_z v_{A0}}{\Omega} \right).$$

Thus, the parameter $H_a$ depends on the wavenumber of each mode. We begin the simulations with small random perturbations, so that modes with every allowed $k$ are included initially. We expect the most unstable mode to dominate the linear phase of the MRI. In the ideal MHD limit for a uniform $B_z$, the most unstable mode has a characteristic wavenumber $k_z \sim v_{A0}/\Omega$ in the azimuthal direction but no scale in the vertical direction $k_z \to \infty$ (Balbus & Hawley 1991). In the numerical calculations, the grid resolution constrains the minimum length in the vertical direction, which is typically $k_z v_{A0}/\Omega \sim 10$. Because the parameter $H_a$ is larger than $X_0$ by the factor $k_z v_{A0}/\Omega$, we expect that the Hall term will have an effect even with small $X_0$.

Figure 15 shows the time evolution of the Maxwell stress for the models with $X_0 = 0, 0.2$, and 0.4, which are models Y2, Y3, and Y4, respectively. All the models have the same field strength and the same magnetic Reynolds number, $\beta_0 = 100$ and $R_{eM0} = 100$. The evolution is dominated by the Hall effect because ohmic dissipation is inefficient ($R_{eM0} = 100 >> 1$). We find that the timescale to reach saturation is very sensitive to the Hall parameter. Without the Hall term ($X_0 = 0$) it requires 15 orbits, but in the $X_0 = 0.4$ run the turbulent state begins at about eight orbits. Thus, as discussed above, the linear phase of the nonaxisymmetric MRI is affected by even small $X_0$.

Despite the sensitivity of the linear growth rates to $X_0$, the properties of the saturated turbulence are almost independent of $X_0$. The saturation level with $X_0 = 0.4$ is slightly higher than $X_0 = 0$. The amplitude of time variability is
much smaller than those in uniform vertical field cases, and
the frequency is higher (see Fig. 2). Figure 16 illustrates the
evolution of the magnetic energy in model Y4 (β₀ = 100, 
Reₘ₀ = 100, and X₀ = 0.4) taken along a slice of constant
y. The contours show the azimuthal component of the mag-
netic energy with logarithmic spacing, and the velocity field
is shown by arrows. The growth of disturbances starts from
modes with large k, typically kₓ ≈ kᵧ ≈ 5 for this case (Fig.
16, top left panel). At about 10 orbits the MRI saturates,
and MHD turbulence persists until the end of the simula-
tion at 100 orbits. No growth of the two-channel flow is evi-
dent. Small-scale growth of the MRI occurs everywhere,
and this makes small-amplitude and frequent time varia-
tions. At 20 orbits, the spatial dispersion in the magnetic
pressure (∂P mag 1/2 /P mag = 1.3) is larger than those in
the density and gas pressure (∂P 1/2 /P = 0.15 and
∂ρ 1/2 /ρ = 0.094). These numbers are almost constant
throughout the evolution and similar to the zero net flux B₀
cases.

Ohmic dissipation is found to make a large difference in
the saturation level of the magnetic energy and stress in
these pure azimuthal field runs, as it did for models with
other field configurations. Because the most unstable mode
for nonaxisymmetric perturbation has a larger kₓ than the
axisymmetric mode on a vertical field, the effect of ohmic
dissipation is expected to be important for smaller η or a
larger Reₘ₀. Figure 17 shows the time evolution of the po-
loidal component of the magnetic energy (∂B₀² + ∂B₀²)/8π)
for models with different magnetic Reynolds number Reₘ₀.

The magnetic field strength and the Hall parameter are
β₀ = 100 and X₀ = 0.4 for all the models in this figure. The
initial magnetic field is purely toroidal, ∂B₀²/8π)/P₀ = 0.01.

Initially, the poloidal field is generated by small perturba-
tions from the toroidal field and thus much smaller than the
azimuthal component (∂B₀² + ∂B₀²)/8π)/P₀ ∼ 10⁻⁷.

When Reₘ₀ ≥ 30, the magnetic energy is amplified by the
MRI in the linear phase, with the linear growth starting ear-
erlier with larger Reₘ₀. The saturation level of the poloidal
field energy is comparable to the initial toroidal field energy.
The toroidal field is also amplified by an order of magni-
tude, and it dominates the other components. The
Reₘ₀ = 100 and 30 runs evolve to the same saturation level,
and the Maxwell stress (∂w₁²)/P₀ is of the order of 0.01.

When Reₘ₀ = 10, the amplified magnetic energy is sus-
tained, but the saturation level of the poloidal field energy
(∂B₀² + ∂B₀²)/8π)/P₀ is 3 orders of magnitude smaller. The
toroidal field is not amplified at all, so that the total mag-
etic energy is unchanged from its initial value. If the mag-
netic Reynolds number is less than 3, the initial perturbations
are decaying until the end of the calculation.

The time- and volume-averaged properties of the turbu-
ience driven by the MRI in the azimuthal field runs are listed
in Table 8. Models Y2 (X₀ = 0) and Y4 (X₀ = 0.4) show lit-
tle effect from ohmic dissipation (Reₘ₀ = 100). The more
resistive (Reₘ₀ = 10) models Y8 (X₀ = 0) and Y10
(X₀ = 0.4) are also listed in the table. Quantities in models
with the same Reₘ₀ are similar, which suggests that the
nonlinear effect of the Hall term is small for uniform B₀
models. In fact, the effective Hall parameter is small,
(∂X₁₀ = −0.01 and −0.0012 for models Y4 and Y10. For
the Reₘ₀ = 100 runs, the turbulence is significant and the
stress α is larger than 0.01. The magnetic energy distri-

| Model  | β₀  | Reₘ₀ | X₀   | λₘ₁₁/H₀ | Orbits | (∂w₁²)/P₀ | (∂w₁²)/P₀ |
|--------|-----|------|------|---------|--------|------------|------------|
| Y1     | 100 | 100  | 0    | 0.63    | 100    | 0.0356     | 0.00852    |
| Y2     | 100 | 100  | 0    | 0.63    | 100    | 0.0353     | 0.00833    |
| Y3     | 100 | 100  | 0.2  | 0.63    | 100    | 0.0364     | 0.00870    |
| Y4     | 100 | 100  | 0.4  | 0.63    | 100    | 0.0541     | 0.0125     |
| Y5     | 100 | 30   | 0    | 0.63    | 100    | 0.0256     | 0.00616    |
| Y6     | 100 | 30   | 0.2  | 0.63    | 100    | 0.0264     | 0.00626    |
| Y7     | 100 | 30   | 0.4  | 0.63    | 100    | 0.0325     | 0.00739    |
| Y8     | 100 | 10   | 0    | 0.63    | 100    | 2.23 × 10⁻⁵| 5.95 × 10⁻⁵|
| Y9     | 100 | 10   | 0.2  | 0.63    | 100    | 2.79 × 10⁻⁵| 8.88 × 10⁻⁵|
| Y10    | 100 | 10   | 0.4  | 0.63    | 100    | 4.39 × 10⁻⁵| 1.48 × 10⁻⁵|
| Y11    | 100 | 10   | 1.0  | 0.63    | 200    | 3.41 × 10⁻⁷| ...         |
| Y12    | 100 | 3    | 0    | 0.63    | 100    | ...        | ...        |
| Y13    | 100 | 3    | 0.2  | 0.63    | 100    | ...        | ...        |
| Y14    | 100 | 3    | 0.4  | 0.63    | 100    | ...        | ...        |
| Y15    | 100 | 3    | 1.0  | 0.63    | 100    | ...        | ...        |
| Y16    | 100 | 3    | 10   | 0.63    | 100    | ...        | ...        |
| Y17    | 400 | 100  | 0    | 0.31    | 200    | 0.0279     | 0.00606    |
| Y18    | 400 | 100  | 0    | 0.31    | 200    | 0.0286     | 0.00616    |
| Y19    | 400 | 30   | 0    | 0.31    | 200    | 0.0261     | 0.00564    |
| Y20    | 400 | 10   | 0    | 0.31    | 200    | 5.34 × 10⁻⁶| 2.98 × 10⁻⁶|
| Y21    | 400 | 3    | 0    | 0.31    | 200    | 5.59 × 10⁻⁷| 1.28 × 10⁻⁷|
| Y22    | 1600| 1000 | 0    | 0.16    | 200    | 0.0223     | 0.00541    |
| Y23    | 1600| 1000 | 0.16 | 0.18    | 200    | 0.0189     | 0.00467    |
| Y24    | 1600| 30   | 0    | 0.16    | 400    | 0.0234     | 0.00512    |
| Y25    | 1600| 10   | 0    | 0.16    | 200    | 3.46 × 10⁻⁶| 4.51 × 10⁻⁶|
| Y26    | 1600| 3    | 0    | 0.16    | 200    | 2.20 × 10⁻⁶| 4.13 × 10⁻⁶|

\[ \text{Table 7} \]

**Uniform \( B_0 \) Simulations (Standard Box Size and Resolution)**

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**Note:** Box size is \( H \times 4H \times H \), and grid resolution is 32 × 128 × 32.

\[ \lambda_{MRI} \equiv 2\pi \varphi \Omega \]
bution \( \langle B_x^2 \rangle : \langle B_y^2 \rangle : \langle B_z^2 \rangle \approx 3 : 22 : 1 \) and the perturbed kinetic energy distribution \( \langle \rho u_x^2 \rangle : \langle \rho u_y^2 \rangle : \langle \rho u_z^2 \rangle \approx 2 : 2 : 1 \) are similar to those in active turbulence of the other initial field geometries. For the \( \text{Re}_{M0} = 10 \) runs, the poloidal field energy is much smaller than the toroidal field energy, and the ratio is \( \langle B_x^2 \rangle / \langle B_z^2 \rangle \approx 3 \times 10^3 \). The kinetic energy is comparable to the poloidal field energy. The stress \( \alpha \) is about \( 10^{-5} \) so that the angular momentum transport is inefficient for these models.

5.1. Critical Magnetic Reynolds Number

Here we consider the critical magnetic Reynolds number for a disk with a toroidal field. Figure 18 shows the saturation level of the Maxwell stress as a function of the initial magnetic Reynolds number \( \text{Re}_{M0} \). All the models in this figure are without the Hall term; only the effect of ohmic dissipation is included. The results with different field strength \( \beta_0 = 100, 400, \) and \( 1600 \) are shown. The critical magnetic Reynolds number for significant stress (e.g., \( \langle w_M \rangle / P_0 > 0.01 \)) is \( \text{Re}_{M0,\text{crit}} \approx 30 \), and this is independent of the initial field strength \( \beta_0 \). The saturation level is also independent of \( \beta_0 \), and typically \( \langle w_M \rangle / P_0 \approx 0.04 \). For ideal MHD, the saturation level is close to this value (Hawley et al. 1995).

If \( \text{Re}_{M0} \) is less than 30, the evolution of the disk is quite different. Ohmic dissipation suppresses the MRI, and the stress in the nonlinear regime is very small and of the order of \( 10^{-6} \) to \( 10^{-5} \). The timescale to reach the saturation depends on both the initial field strength and the magnetic Reynolds number. When \( \text{Re}_{M0} = 30 \), for example, the saturation occurs at 25 orbits for the \( \beta_0 = 100 \) run (Y5), but it takes more than 200 orbits for the \( \beta_0 = 1600 \) run (Y24).

Next, we consider the effect of the Hall term on the critical magnetic Reynolds number. Figure 19 shows the saturation level of the poloidal magnetic energy \( \langle (B_x^2 + B_z^2) / 8\pi \rangle / P_0 \) for models including the Hall effect. The initial field strength is the same \( (\beta_0 = 100) \) for all the models. Open circles depict the saturation level in models without the Hall term. When \( \text{Re}_{M0} \geq 30 \), the saturation level is independent of \( \text{Re}_{M0} \), and the poloidal field energy is of the order of 0.01. If \( \text{Re}_{M0} < 30 \), the turbulence in the nonlinear regime is reduced by ohmic dissipation, and the saturation level drops dramatically. Filled triangles and circles denote the \( X_0 = 0.2 \) and 0.4 runs, respectively. We find that the critical magnetic Reynolds number is not affected by the size of the...
Hall parameter. Even for very large $X_0$ cases ($X_0 = 10$ and 100), the behavior at $Re_{M0} \leq 10$ is unchanged. Therefore, the critical value for uniform $B_y$ models is $Re_{M0,\text{crit}} \sim 30$ for any $\beta_0$ and $X_0$. This is an order of magnitude larger than for uniform $B_x$ models. For comparison, the magnetic Reynolds number $Re_{M0}$ used in Fleming et al. (2000) is also shown in this figure. The critical value for $Re_{M0}$ is about $3 \times 10^3$; however, this value should depend on the initial field strength.

6. DISCUSSION

6.1. Critical Magnetic Reynolds Number at the Nonlinear Stage

The critical magnetic Reynolds number discussed in §§ 3.4, 4.1, and 5.1 is defined using the magnetic field strength in the initial state. However, the initial magnetic field in accretion disks is highly uncertain, and only the field strength in the nonlinear, saturated state of the instability

**TABLE 8**

| Quantity | Y2 | Y4 | Y8 | Y10 | Average* |
|----------|----|----|----|-----|---------|
| $\beta_0$ | 0.00890 | 0.0141 | 3.41 \times 10^{-6} | 5.63 \times 10^{-6} | 0.00743 \pm 0.00240 |
| $Re_{M0}$ | 0.0702 | 0.109 | 0.00907 | 0.00943 | 0.0610 \pm 0.0177 |
| $X_0$ | 0.00288 | 0.00538 | 2.44 \times 10^{-6} | 6.32 \times 10^{-6} | 0.00247 \pm 0.00102 |
| $(B^2/8\pi)/P_0$ | 0.0139 | 0.0193 | 8.98 \times 10^{-6} | 3.10 \times 10^{-5} | 0.0117 \pm 0.0029 |
| $(\mu^2/2)/P_0$ | 0.0105 | 0.0157 | 2.42 \times 10^{-5} | 3.66 \times 10^{-5} | 0.00872 \pm 0.00259 |
| $(\mu^2/2)/P_0$ | 0.00583 | 0.00833 | 1.20 \times 10^{-5} | 3.84 \times 10^{-5} | 0.00490 \pm 0.00134 |
| $(P)/P_0$ | 6.44 | 5.83 | 1.01 | 1.02 |
| $(|\omega|)/(|v_0|)$ | 4.23 | 4.32 | 3.74 | 2.95 | 4.33 \pm 0.20 |
| $(\omega^2/8\pi)/(B^2)$ | 0.447 | 0.433 | 0.00245 | 0.00465 | 0.442 \pm 0.022 |
| $(P)/(|\omega|)$ | 81.7 | 46.6 | 111 | 108 |
| $\langle (B^2/8\pi)/P_0 \rangle$ | 0.383 | 0.347 | 0.00498 | 0.0112 | 0.372 \pm 0.018 |
| $\langle (B^2/8\pi)/P_0 \rangle$ | 3.09 | 2.61 | 1.40 | 0.891 | 3.10 \pm 0.28 |
| $\langle (B^2/8\pi)/P_0 \rangle$ | 24.4 | 20.3 | 3.73 \times 10^3 | 1.49 \times 10^3 | 25.6 \pm 2.5 |
| $\langle \langle \nu \rangle \rangle / \langle \langle \mu \rangle \rangle$ | 2.38 | 2.31 | 0.747 | 0.806 | 2.42 \pm 0.13 |
| $\langle \langle \nu \rangle \rangle / \langle \langle \mu \rangle \rangle$ | 1.80 | 1.89 | 2.02 | 0.953 | 1.78 \pm 0.10 |
| $\langle \beta \rangle$ | $1.39 \times 10^3$ | 688 | 114 | 115 |
| $\langle Re_{M0} \rangle$ | 8.25 | 1.29 \times 10^3 | 9.08 | 9.45 |
| $\langle X_0 \rangle$ | 0 | -0.0112 | 0 | -0.00119 |
| $\alpha$ | 0.0436 | 0.0666 | 2.82 \times 10^{-4} | 5.88 \times 10^{-5} | 0.0372 \pm 0.00106 |

* Models with $\langle (v_0^2/\rho \Omega) \rangle > 1$ are considered for the average (Y1–Y7, Y17–Y19, and Y22–Y24).
Fig. 19.—Saturation level of the poloidal component of the magnetic energy as a function of the magnetic Reynolds number $R_{c,0}$ for uniform $B_t$ models ($X_0 = 100$). Open circles denote the models with only the ohmic dissipation ($X_0 = 0$), and the other symbols are including also the Hall effect ($X_0 = 0.2, 0.4, 10,$ and $100$).

may be observable. We find that the magnetic Reynolds number defined using the time- and volume-averaged vertical magnetic field strength in the nonlinear regime, i.e., $\langle \langle v_{A_2}^2/\eta\Omega \rangle \rangle$, characterizes the saturation amplitude of the MRI very well. Tables 1–3, 5, and 7 list $\langle \langle v_{A_2}^2/\eta\Omega \rangle \rangle$ for all models calculated in this paper. Figure 20 shows the saturation level of the stress $\alpha = \langle \langle w_{xy} \rangle \rangle/P_0$ as a function of $\langle \langle v_{A_2}^2/\eta\Omega \rangle \rangle$ for all the models. Circles, triangles, and squares denote models started with a uniform vertical, zero net flux vertical, and a uniform toroidal field, respectively. Filled symbols denote models that include the Hall term.

The dependence of $\alpha$ on the magnetic Reynolds number is clearly evident. A stress larger than 0.01 requires $\langle \langle v_{A_2}^2/\eta\Omega \rangle \rangle > 1$. The solid arrows denote the saturation levels obtained for ideal MHD ($v_{A_2}^2/\eta\Omega \rightarrow \infty$), where the upper and lower arrows ($\alpha = 0.29$ and 0.044) are the average of uniform $B_z$ runs and uniform $B_t$ runs, respectively, taken from Hawley et al. (1995). The saturation level when $\langle \langle v_{A_2}^2/\eta\Omega \rangle \rangle > 1$ is almost the same as the ideal MHD cases, and thus the stress is nearly independent of the magnetic Reynolds number in this regime. However, when $\langle \langle v_{A_2}^2/\eta\Omega \rangle \rangle < 1$, the stress decreases as the magnetic Reynolds number decreases. If turbulence cannot be sustained by the MRI, the magnetic energy and stress both decrease in time. In some models with $\langle \langle v_{A_2}^2/\eta\Omega \rangle \rangle < 1$, the magnetic energy is decaying at the end of the calculation, and the system is evolving toward the direction $\langle \langle w_{xy} \rangle \rangle \sim \langle \langle v_{A_2}^2/\eta\Omega \rangle \rangle$ shown by the dashed arrow in the figure.

From Figure 20, the saturation level of the stress is approximately given by

$$\alpha \approx \alpha_{MRI} \min \left( 1, \frac{v_{A_2}^2}{\eta\Omega} \right),$$

where $\alpha_{MRI}$ is the stress in the ideal MHD limit ($v_{A_2}^2/\eta\Omega \gg 1$). For a vertical field with zero net flux or a purely toroidal field, the dispersion in the saturation level is quite small: from Tables 6 and 8 the averages of the stress are nearly the same, $\alpha_{MRI} = 0.0395 \pm 0.0120$ and 0.0372 \pm 0.0106, for these runs. On the other hand, when the disk has net flux of $B_z$, the saturation levels are less uniform because they depend on the vertical field strength, the vertical box size (Hawley et al. 1995), and the Hall parameter. The stress is $\alpha_{MRI} \sim 0.011$ and can be larger than zero net vertical flux models. The stress obtained in Fleming et al. (2000) is shown by crosses in this figure; our results are consistent with this earlier work.

Figure 20 shows that the vertical field is a key quantity for predicting the saturation amplitude of the MRI. Nonaxisymmetric modes of the MRI are unstable when azimuthal field is present, but because these modes have a large wave-number in the vertical direction, they are suppressed by a smaller resistivity than axisymmetric modes. Therefore, the suppression of the axisymmetric instability, which depends on the vertical field strength, determines the critical magnetic Reynolds number. This is true so long as the azimuthal field gives an Alfvén speed that is subthermal, $v_{A_t} < c_s$ (Blaes & Balbus 1994), which is always true in our calculations. Thus, the effect of nonideal MHD on the saturation amplitude of the MRI is well characterized by a magnetic Reynolds number defined as $v_{A_2}^2/\eta\Omega$, with a value larger than unity required for turbulence and significant transport.

The critical value $\langle \langle v_{A_2}^2/\eta\Omega \rangle \rangle \sim 1$ is independent of the initial field strength and geometry, as well as the Hall parameter. The toroidal component of the magnetic energy
is dominant and an order of magnitude larger than the vertical component in the turbulence driven by the MRI. The ratio $\langle \mathbf{B}_z^2 \rangle / \langle \mathbf{B}_z^2 \rangle \sim 30$ when $\langle v^2_A \rangle / n \Omega > 1.0$. Thus, the critical value can be written as $\langle v^2_A \rangle / n \Omega \sim 30$. If the magnetic Reynolds number $v^2_A \Omega$ is larger than about 30, this means that the MRI would be operating in the disk.

6.2. Limit on the Parameter Range of Numerical Simulations

The constraint on the time step in the numerical algorithm used here (see Paper I) is inversely proportional to the initial Hall parameter and the field strength. Thus, when the saturation level of the magnetic field is high, and when the Hall parameter is large, this constraint is very severe. Thus, we have computed only a few models with large $X_0$ and large $R_{\text{MHD}}$.

In Paper I we estimated the ratio of the Hall parameter to the magnetic Reynolds number in a weakly ionized gas composed by ions, electrons, and neutrals. When the Hall effect is dominant, the Hall parameter takes on values $R_{\text{MHD}} \leq X_0 \leq 100 R_{\text{MHD}}$. We have computed models with small $R_{\text{MHD}}$ ($\sim 0.1$) and with the maximum value of $X_0$ ($\sim 1000$) that show inefficient angular momentum transport in this case. Thus, we conclude that increasing the Hall parameter $X_0$ within the allowed range of values will not change the lower limit of the critical $R_{\text{MHD}}$. In fact, with large $X_0$ (model S16) the critical wavelength of the MRI is larger than the scale height of the disk, so that growth of the MRI cannot be expected for such a situation.

For the $R_{\text{MHD}} > 1$ runs, we examined the effect of a Hall parameter that is of the order of unity, the maximum value allowed when $R_{\text{MHD}} \geq 100$. However, if $1 \leq R_{\text{MHD}} \leq 10$, the Hall parameter $X_0$ can take any value between 10 and 100. We have found in this work that a larger Hall parameter enhances the saturation level of the stress, especially in the case of a uniform $B_z$. But note that even without the Hall effect, significant turbulence is sustained and the stress $\alpha$ is more than 0.01 in this regime. Thus, other effects (such as stronger vertical fields) could be as important as the Hall effect for enhancing the stress.

6.3. Application to Protoplanetary Disks

At the low temperatures expected in protoplanetary disks, thermal ionization is inefficient, except for the innermost regions within about $r \sim 0.1$ AU of the central star. The dominant ionization sources in this case are nonthermal processes, such as X-rays, cosmic rays, and radioactive elements. By definition, protoplanetary disks contain dust grains that eventually will agglomerate into planetesimals. Because recombination process on the surface of grains can be important, the number density and size distribution of dust grains have a large effect on the ionization fraction of the gas. Unfortunately, there are many uncertainties regarding dust grains in protoplanetary disks, and the characteristics of the grains vary in time as a result of evolutionary effects. For example, grains may grow in size through mutual collisions, while the abundance of grains may decrease as a result of sedimentation toward the midplane. Thus, the effect of dust grains on the ionization state of the gas could be reduced in disks in the late stages of evolution. Therefore, it is important to investigate the ionization state of the disk in situations both with and without dust grains.

Assuming that there are no dust grains in the disk, the distribution of the Hall parameter and the magnetic Reynolds number were calculated at the midplane of some disk models in Paper I. Inside $r_H \sim 80$ AU, the Hall parameter is $|X| > 1$ and the Hall effect is the dominant nonideal MHD effect when the field strength is $c_s / v_A = 10$ in the disk. Ohmic dissipation is more efficient in higher density (inner) regions. The magnetic Reynolds number is less than unity at $r < r_O \sim 6$ AU. The critical radius $r_O$ depends on the field strength in the disk, e.g., $r_O \sim 2$ AU and $r_H \sim 12$ AU for $c_s / v_A = 1$. In the outer disk $r > r_H$, both the Hall effect and ohmic dissipation are unimportant, and thus the disk is unstable to the MRI.

According to the results of the nonlinear simulations presented in this paper, angular momentum transport is inefficient ($\alpha < 0.01$) when $R_{\text{MHD}} \leq 1$ for any initial field strength $\beta_0$ and for any size of the Hall parameter $X_0$. Therefore, at the region $0.1$ AU $< r \leq 6$ AU, the MRI is suppressed by ohmic dissipation and this region probably forms a “dead zone” (Gammie 1996). The critical magnetic Reynolds number $R_{\text{MHD, crit}}$ for the onset of active turbulence is $10^3$ depending on the strength and geometry of the magnetic field. However, because $R_{\text{MHD}}$ is a steeply increasing function of $r$, the corresponding uncertainty in the critical radius $r_O$ is at most a factor of 3 (see Fig. 2 in Paper I).

In the region $r_O < r < r_H$, the Hall parameter is larger than unity, and thus the Hall term can affect the evolution of the MRI. Moreover, ohmic dissipation is too small to suppress the MRI. Although the nonlinear behavior of the MRI depends on the field geometry, the MRI will operate for most of the cases in this region. If there is no net flux in the vertical field (zero net flux $B_z$ or uniform $B_z$ models), MHD turbulence is initiated by the MRI and sustained for the values of $R_{\text{MHD}}$ and $X_0$ in this region. Although this region is unstable even without the Hall term, the stress could be enhanced as a result of the Hall effect. If the disk is threaded by a uniform vertical field oriented in the opposite direction to the angular velocity vector $\Omega$, i.e., $X_0 < -1$, the linear growth of the MRI is suppressed by the Hall effect, forming a “dead zone” with no angular momentum transport via magnetic stress. Since suppression of the MRI requires that the Hall parameter is $X_0 < -4$ everywhere, this may be difficult in actual accretion disks. If the vertical field is oriented in the same sense as $\Omega$, even in only a small part of the disk, the MRI can grow from this region. The unstable region could spread wider and eventually fill the entire region, as demonstrated in the zero net flux $B_z$ simulations.

In summary, we expect the outer regions of protoplanetary disks $r > r_O$ to be unstable to the MRI, with angular momentum transported effectively by Maxwell stress. Inside of $r_O$ is a dead zone unless the temperature is $T > 10^3$ K. The critical radius $r_O$ is a few AU, and this is determined mainly by the ohmic dissipation. The typical size of protoplanetary disks is about 100 AU. Most of the disk is unstable if dust grains are not present as a result of settling or are too large to affect the ionization fraction. The critical radius $r_O$, which determines where the disk is unstable to the MRI and therefore turbulent, depends strongly on the size distribution and abundance of dust grains (Sano et al. 2000). The extent to which grains are mixed vertically through the disk in turn depends strongly on whether the gas is turbulent. Hence, the evolution of gas and dust grains must be solved simultaneously in order to make a consistent scenario of grain settling, growth, and ultimately planet formation.
7. SUMMARY

The Hall effect on the nonlinear evolution of the MRI has been investigated using local three-dimensional nonideal MHD simulations. Various models with different field strengths $\beta_0$, magnetic Reynolds number $Re_M = v^2_A / \eta \Omega$, and Hall parameter $X_0$ have been computed. Our findings are summarized as follows:

1. For uniform $B_z$ models, a positive (negative) $X_0$ enhances (suppresses) the nonlinear turbulence in the disk generated by the MRI. When $X_0 \geq 0$, the nonlinear evolution shows recurrent appearances of the channel flow, and the saturated Maxwell stress is larger than that in the negative $X_0$ case. The saturation level of the magnetic energy and stress decreases dramatically when $Re_M < Re_{M,\text{crit}}$. This critical value is independent of both the field strength and the size of the Hall term.

2. For zero net flux $B_z$ models, disorganized MHD turbulence is sustained in the nonlinear regime without the growth of the channel flow. The critical magnetic Reynolds number $Re_{M,\text{crit}}$ for the initial state depends on the initial field strength $\beta_0$ and the Hall parameter $X_0$, and is in the range of 1–30 for our models. The Hall effect can reduce the critical value $Re_{M,\text{crit}}$; however, the difference is at most an order of magnitude.

3. For uniform $B_y$ models, the effect of the Hall term can be seen only during the linear growth phase of the instability. Disorganized turbulence lasts more than 100 orbits when the initial magnetic Reynolds number is $Re_{M,\text{crit}} \geq 30$, and this critical value is independent of both the field strength and the size of the Hall term. The characteristics of the saturated turbulence are quite similar to those in the zero net flux $B_z$ models.

4. The condition for turbulence and significant transport in the nonlinear regime is found to be given by $\langle (v_A^2 / \eta \Omega) \rangle > 1$ (or $\langle v_A^2 / \eta \Omega \rangle > 30$), where $v_A$ is the Alfvén speed computed from only the vertical field component in the nonlinear regime. This is independent of the strength and geometry of the initial magnetic field and the Hall parameter. If the magnetic field strength in a disk is estimated observationally and the magnetic Reynolds number $v_A^2 / \eta \Omega$ is larger than about 30, this would imply that the MRI is operating in the disk.

We have applied the results of our simulations to the dynamics of protoplanetary disks. We conclude that the stability of such disks is determined mainly by the distribution of the magnetic Reynolds number. The Hall effect makes little change in the critical value of $Re_M$. The size of the dead zone where the MRI may be suppressed by ohmic dissipation is sensitive to the characteristics of dust grains in the disk. When small dust grains are well mixed vertically in the disk, the dead zone extends to a few tens of AU from the central star. However, the dead zone is considerably smaller if small grains are not present.

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