A Mathematical Model of the Dynamic Interaction of a Locomotive’s Wheelset and Track

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Abstract. The interaction of the wheel and the rail is the physical basis for the movement of rolling stock on railways. In order to reduce the wear rate of wheel sets and rails, it is necessary to create a mathematical model for the interaction of the “wheel-rail” pair, on which one could look at various methods to reduce catastrophic wear. Currently widely used mathematical algorithms with many degrees of freedom of dynamic interaction of a pair of “wheelset rolling stock – railway track” have a number of disadvantages that affect the accuracy of the calculation results. It is necessary to first build the dynamic equations of dynamics in a basic fixed coordinate system, and then introduce a transformation into local coordinate systems. The algorithm proposed by the authors for constructing dynamic equations, taking into account the inclination and elevation of the outer rail over the inner rail, and the speed of the portable movement of the center of mass of the wheelset, allows to avoid all kinds of errors. In this paper, we describe an algorithm for constructing dynamic equations describing the joint motion of a wheel pair and a rail track, as well as an algorithm for solving the dynamic contact problem. The equations of translational and rotational dynamics of a wheel pair of a railway rolling stock (locomotive), the equations of motion of interacting bodies and communications are constructed. Based on the described mathematical algorithms, programs have been developed for calculating the forces of dynamic interaction in the wheel – rail system.

1. Introduction

The interaction of the wheel and the rail is the physical basis for the movement of rolling stock on railways. The safety of the movement and the main technical and economic indicators of the track economy and rolling stock largely depend on the parameters of this interaction. So, in particular, energy losses due to wear in the wheel-rail system make up 10-30\% of fuel and energy resources spent on traction of trains. In addition, the costs of renovating rails and wheelsets make up a considerable part of the total costs of track distances and locomotive and car depots, respectively. Locomotive depots suffer especially large costs in connection with these costs, since over the past half century, the average service life of a locomotive wheelset has significantly decreased. In order to reduce the wear rate of wheel sets to acceptable values, in recent years a number of measures have been taken of a technical, organizational and technological nature (lubrication, improving track design and rolling stock, improving the geometry of the rolling surface profile of wheel sets and rails, improving the quality of their metal, etc.). Unfortunately, none of these measures solved the problem in full. A radical solution to the problem can be found only on the basis of the use of scientific knowledge and the creation of a mathematical model of the dynamic interaction of the pair “wheelset rolling stock – railway track”.

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When constructing the equations of dynamics of individual elements of the car in [1–5], local coordinate systems that are non-inertial are used. In this case, the loss of inertial members (centrifugal, Coriolis, etc.) is possible. This is especially dangerous when used to construct the dynamic equations of the Lagrangian formalism, when the terms of the effective potential energy and the effective dissipative function [1–3] are not included in the Lagrange function [4]. In order to avoid these shortcomings, in our opinion, it is advisable to construct the initial equations of the dynamics of systems with many degrees of freedom in a basic fixed coordinate system, and then introduce a transformation into local coordinate systems. Such an algorithm for constructing dynamic equations avoids the above errors.

2. Coordinate systems and their transformation
Suppose that the profile of the path is given by an arbitrary flat curve (configuration of the profile of the inner rail), the parametric equation of which has the form [6]:

\[ L \]

\[ \begin{align*}
L &= \frac{\alpha}{\alpha_X} \left( \begin{array}{c}
\alpha_X \\
\alpha_Y 
\end{array} \right) \\
L &= \begin{pmatrix}
\alpha_X \\
\alpha_Y 
\end{pmatrix}
\end{align*} \]  

(1)

where \( L \) is an element of the length of the curve. We introduce the angle of orientation of the tangent to the curve \( \phi(L) \).

\[ \varphi(L) = \atan \left( \frac{\alpha_X}{\alpha_Y} \right) = \atan \left( \begin{pmatrix}
\alpha_X \\
\alpha_Y 
\end{pmatrix} \right) \]  

(2)

We introduce two coordinate systems: fixed (laboratory) and moving, connected with the point of intersection of the line \( L \) and the normal to it passing through the center of mass of the wheelset, rotated relative to the laboratory by an angle \( \phi \) (Figure 1).

![Figure 1. Transition from laboratory to moving coordinate system (lab-move).](image)

The transformation of an arbitrary radius vector during the transition from a laboratory to a moving coordinate system can be represented as a superposition of translation and rotation [7]:

\[ \begin{align*}
\vec{R}^{(\text{lab})} &= \vec{R}^{(L)} + \vec{\Phi} \cdot \vec{R}^{(\text{move})} \\
\vec{\Phi} &= \begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix} \\
\end{align*} \]  

(3)

Assuming that the railroad track can change the slope [8] (for example, when entering the curve), we introduce the rotation of the coordinate system by the angle \( \gamma(L) \) – the elevation angle of the outer rail (Figure 2). The matrix of the transition to a new coordinate system, which we will call the track (railway), can be represented as [9]:

\[ \begin{align*}
\vec{R}^{(\text{rail})} &= \vec{\Gamma} \cdot \vec{R}^{(\text{move})} \\
\vec{\Gamma} &= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \gamma & -\sin \gamma \\
0 & \sin \gamma & \cos \gamma
\end{pmatrix}
\end{align*} \]  

(4)
The superposition of transformations (3) and (4) determines the transition from a laboratory to a track coordinate system (lab-rail):

\[ \tilde{R}^{(lab)} = \tilde{R}^{(L)} + \tilde{T} \cdot \tilde{R}^{(rail)}, \quad \tilde{T} = \tilde{\Phi} \cdot \tilde{\Gamma}. \]  

(5)

Note that when converting vector physical quantities that are not radius vectors (forces, moments of forces, etc.), only rotational transformation should be performed, without translation [10, 11].

In the above transformation from the laboratory to the track coordinate system, only two dynamic variables of the system appear: the coordinate \( L(t) \) and the speed of the portable movement of the center of mass of the wheelset [12]

\[ V^{(L)}(t) = \dot{L}(t) \]  

(6)

To a first approximation, a wheel pair can be considered as a symmetric top performing a rotational movement around the main axis of inertia, which coincides with the axis of symmetry of the wheel pair, and the precession of this axis around a certain fixed direction [13]. In the general case, the instantaneous orientation of the axis of symmetry of the wheelset may not coincide with the orientation of the axis \( OY^{(rail)} \) of the track coordinate system. We introduce the Euler angles describing the instantaneous orientation of the axis of symmetry of the wheelset \( \tilde{\Phi} \) and \( \tilde{\Gamma} \) in the laboratory coordinate system. Then the transformation from the laboratory to the coordinate system associated with the axis of symmetry of the wheelset (lab-wheel) can be represented as:

\[ \tilde{R}^{(lab)} = \tilde{R}^{(lab)}_0 + \tilde{T} \cdot \tilde{R}^{(wheel)} \]  

\[ \tilde{T} = \tilde{\Phi} \cdot \tilde{\Gamma} \]  

(7)

where \( \tilde{R}^{(lab)}_0 \) is the radius vector of the center of mass of the wheelset in the laboratory coordinate system.

Transformations (5) and (7) will be further used in constructing the equations of dynamics of the wheelset.

3. The equation of translational dynamics of a wheelset
Consider the equation of translational motion of the center of mass of a loaded pair of wheels in a laboratory coordinate system. We assume that the wheelset is in the Earth's gravitational field, and also forces act on it from the side frames of the trolley, which we denote \( \tilde{F}^{(side)}(t) \). The result of the interaction of the wheelset with the rails is the action of the reaction forces applied at the contact points [14, 15]. Given the above forces, the equation of translational dynamics of a wheelset can be represented as:
\[ M_0 \cdot \ddot{\mathbf{R}}^{(lab)}_0 = M_0 \cdot \mathbf{g}^{(lab)} + \dot{\mathbf{F}}^{(lab)}(t) + \sum_{i=1}^{kl} \dot{\mathbf{F}}^{(lab)}_{0i} + \sum_{i=1}^{kr} \dot{\mathbf{F}}^{(lab)}_{ni} \]  

(8)

where \( M_0 \) is the mass of the wheelset; \( \mathbf{g}^{(lab)} \) – acceleration of gravity in the laboratory coordinate system; \( \dot{\mathbf{F}}^{(lab)}(t) \) – force acting on the wheelset from the side of the side frames of the trolley (not specified in the laboratory coordinate system); \( \dot{\mathbf{F}}^{(lab)}_0 \) and \( \dot{\mathbf{F}}^{(lab)}_n \) – reaction forces acting from the side of the left and right rail to the wheelset (not specified in the laboratory coordinate system); \( kl \) and \( kr \) – the number of points of contact of the wheelset with the left and right rail.

Dividing equation (8) by mass and introducing specific forces

\[ \ddot{\mathbf{f}}^{(lab)}(t) = \frac{\dot{\mathbf{F}}^{(lab)}(t)}{M_0}, \quad \ddot{\mathbf{g}}^{(lab)}_{0i} = \frac{\dot{\mathbf{F}}^{(lab)}_{0i}}{M_0} \quad \ddot{\mathbf{g}}^{(lab)}_{ni} = \frac{\dot{\mathbf{F}}^{(lab)}_{ni}}{M_0} \]  

(9)

we obtain the equation of translational dynamics in the following form:

\[ \ddot{\mathbf{K}}^{(lab)}_0 = \mathbf{g}^{(lab)} + \ddot{\mathbf{f}}^{(lab)}(t) + \sum_{i=1}^{kl} \ddot{\mathbf{g}}^{(lab)}_{0i} + \sum_{i=1}^{kr} \ddot{\mathbf{g}}^{(lab)}_{ni} \]  

(10)

Applying transformation (5), we transform equation (10) into the track coordinate system (11). The first three terms on the right side of equation (11) are inertial terms that appear due to the transition to a non-inertial reference frame. The generality of the statement of the problem has led to the fact that inertial terms have a somewhat unusual form [16]. After simple transformations of the considered terms, they acquire a known physical meaning.

\[ \ddot{\mathbf{K}}^{(int)}_0 = -\ddot{\mathbf{T}}^{-1} \cdot \dot{\mathbf{R}}^{(c)} - \ddot{\mathbf{T}}^{-1} \cdot \dot{\mathbf{R}}^{(int)} - \]  

\[ -2 \ddot{\mathbf{T}}^{-1} \cdot \dot{\mathbf{R}}^{(int)} + \ddot{\mathbf{T}}^{-1} \cdot \mathbf{g}^{(lab)} + \ddot{\mathbf{T}}^{-1} \cdot \dot{\mathbf{f}}^{(lab)} + \]  

\[ + \sum_{i=1}^{kl} \ddot{\mathbf{g}}^{(lab)}_{0i} + \sum_{i=1}^{kr} \ddot{\mathbf{g}}^{(lab)}_{ni} \]  

(11)

In the first term of the right-hand side of the equation after the transformation, we can distinguish the translational and centrifugal inertial terms (associated with the portable movement):

\[ -\ddot{\mathbf{T}}^{-1} \cdot \dot{\mathbf{R}}^{(c)} = \left[ \begin{array}{c} -\frac{\rho L}{\rho(L)} \cos \gamma \\ \frac{\rho L}{\rho(L)} \sin \gamma \end{array} \right] \]  

(12)

where \( \rho(L) \) is the local radius of curvature of the path line (in this case, negative).

The second term in the right-hand side of equation (13) represents the generalized centrifugal forces and, after transformation, can be represented as:

\[ -\ddot{\mathbf{T}}^{-1} \cdot \dot{\mathbf{R}}^{(int)} = \left[ \begin{array}{c} -\frac{1}{\rho \cdot \eta} \cdot \dot{\mathbf{L}} \cdot (Y_0 \cdot \sin \gamma + Z_0 \cdot \cos \gamma) + \frac{\rho L}{\rho(L)} \cdot \dot{\mathbf{L}} \cdot \left( -Y_0 \cdot \cos \gamma + Z_0 \cdot \sin \gamma \right) \\ 0 \end{array} \right] \]  

\[ -\frac{1}{\rho \cdot \eta} \cdot \dot{\mathbf{L}} \cdot (Y_0 \cdot \cos \gamma - Z_0 \cdot \sin \gamma) \]  

(13)
where \( \eta \) is the torsion radius.

\[
\eta(L) = \frac{1}{\gamma_z} \tag{14}
\]

In the future, the index (rail) of the track coordinate system is omitted. The third term on the right side of equation (11) represents the generalized Coriolis forces:

\[
-2 \cdot \hat{T} \cdot \hat{r} \cdot \hat{R}_0 = -2 \cdot \frac{1}{\eta} \cdot \hat{L} \cdot \begin{bmatrix} 0 \\ \hat{Y}_0 \\ -\hat{Z}_0 \end{bmatrix} + 2 \cdot \frac{1}{\rho} \cdot \hat{L} \cdot (\hat{Y}_0 \cdot \cos \gamma - \hat{Z}_0 \cdot \sin \gamma) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{15}
\]

Substituting (12) – (15) into (11), we obtain a system of equations describing the portable motion of a wheelset. When substituting the results of calculating the inertial terms in the equations of motion, we will rearrange the terms, the meaning of which will be visible in the final equations.

First of all, we consider the equation of motion of a pair of wheels along the axis \( OX^{\text{rail}} \) of the track coordinate system:

\[
\ddot{L} + \frac{1}{\rho \cdot \eta} \dot{L} \cdot (\dot{Y}_0 \cdot \sin \gamma + \dot{Z}_0 \cdot \cos \gamma) + \left( \frac{\partial_k}{\partial_l} - \frac{1}{\rho} \right) \left( \dot{Y}_0 \cdot \cos \gamma - \dot{Z}_0 \cdot \sin \gamma \right) - \frac{1}{\rho} \cdot \dot{L} \cdot (\dot{Y}_0 \cdot \cos \gamma - \dot{Z}_0 \cdot \sin \gamma) + f_x + \sum_{i=1}^{k_{li}} g_{x_l} + \sum_{i=1}^{k_{li}} g_{x_i} \tag{16}
\]

Since the curvature of the trajectory along which the center of mass of the wheelset moves is different from the curvature of the line \( L \) [17], we introduce a coefficient that takes into account changes in the metric:

\[
\tau(L) = 1 + \frac{1}{\rho} \left( \dot{Y}_0 \cdot \cos \gamma - \dot{Z}_0 \cdot \sin \gamma \right) \tag{17}
\]

Then the speed of the center of mass of the wheelset can be represented as:

\[
\dot{L} = L \cdot \tau. \tag{18}
\]

It is easy to show that, taking into account (17) and (18), equation (16) takes the form:

\[
\ddot{L} = \frac{1}{\rho} \cdot L \cdot (\dot{Y}_0 \cdot \cos \gamma - \dot{Z}_0 \cdot \sin \gamma) + f_x + \sum_{i=1}^{k_{li}} g_{x_l} + \sum_{i=1}^{k_{li}} g_{x_i} \tag{19}
\]

The first term on the right side of equation (19) is the residual Coriolis accelerations. Equation (19) is a consequence of the exact transformation of the equations of dynamics into a non-inertial coordinate system and does not contain approximations.

We proceed to the analysis of the equations of motion of the center of mass of the wheelset in the plane \( YOZ^{\text{rail}} \) of the track coordinate system. In accordance with formulas (11) – (15) we obtain:

\[
\begin{bmatrix} \dot{Y}_0 \\ \dot{Z}_0 \end{bmatrix} = \frac{1}{\eta} \cdot \dot{L} \cdot \begin{bmatrix} Y_0 \\ Z_0 \end{bmatrix} + 2 \cdot \frac{1}{\eta} \cdot \dot{L} \cdot \begin{bmatrix} -Z_0 \\ Y_0 \end{bmatrix} - \frac{\eta_k}{\eta} \cdot \dot{L} \cdot \begin{bmatrix} -Y_0 \\ Z_0 \end{bmatrix} + \sum_{i=1}^{k_{li}} g_{x_l} + \sum_{i=1}^{k_{li}} g_{x_i} \tag{20}
\]

where the influence of the metric on the radius of curvature of the trajectory is taken into account

\[
\rho = \rho \cdot \tau \tag{21}
\]
After the simple transformations, the first three terms on the right-hand side of equation (20) can be reduced to a well-known form that describes non-inertial effects when the coordinate system rotates with an angular velocity $\dot{\phi}$:

$$\frac{1}{\eta^2} \cdot \ddot{L} \cdot [Y_0] + 2 \frac{1}{\eta} \cdot \ddot{L} \cdot [Z_0] - \left( \frac{\eta}{\eta^2} \cdot \ddot{L} \cdot \frac{1}{\eta} \cdot \ddot{L} \right) [Z_0] =$$

$$= \left[ \dot{R} \times \dot{\gamma} \right] + 2 \left[ \dot{R} \times \dot{\gamma} \right] + \left[ \ddot{\gamma} \times \dot{\dot{R}} \times \dot{\gamma} \right].$$

The fourth term of equation (20) describes the centrifugal effects caused by the rotation of the coordinate system with angular velocity $\dot{\phi}$. In the particular case, when the elevation of the outer rail remains constant, the non-inertial effects (22) disappear, and the equations of motion of the wheelset (20) take the form:

$$\begin{bmatrix} \ddot{X}_0 \\ \ddot{Z}_0 \end{bmatrix} = \frac{\dot{L}}{\rho} \begin{bmatrix} -\cos \gamma \\ \sin \gamma \end{bmatrix} - g \begin{bmatrix} \sin \gamma \\ \cos \gamma \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} g_{y_i} \\ f_{z_i} \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} g_{z_i} \\ f_{y_i} \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} g_{r_i} \end{bmatrix}. \quad (23)$$

If the metric coefficient is close to unity, then the system of equations (19) and (23) is reduced to the form:

$$\ddot{L} = f_x + \sum_{i=1}^{m} g_{x_i} + \sum_{i=1}^{n} g_{x_i}, \quad (24)$$

$$\begin{bmatrix} \ddot{X}_0 \\ \ddot{Z}_0 \end{bmatrix} = \frac{\dot{L}}{\rho} \begin{bmatrix} -\cos \gamma \\ \sin \gamma \end{bmatrix} - g \begin{bmatrix} \sin \gamma \\ \cos \gamma \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} g_{y_i} \\ f_{z_i} \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} g_{z_i} \\ f_{y_i} \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} g_{r_i} \end{bmatrix}. \quad (23)$$

In this case, the motion of the center of mass of the wheelset can be considered as flat in the track coordinate system [18]. We proceed to the analysis of the rotational motion of the wheelset.

4. The equation of rotational dynamics of the wheelset

In deriving the equations of rotational dynamics of a wheelset, one could use the well-known Euler equations. But, preserving the general algorithm for constructing equations, we consider the rotational motion of a pair of wheels in a laboratory coordinate system:

$$\dot{\mathbf{K}}^{(lab)} = \mathbf{K}^{(lab)}(t) + \sum_{i=1}^{m} \mathbf{K}_{ri}^{(lab)} + \sum_{i=1}^{n} \mathbf{K}_{ni}^{(lab)}, \quad (25)$$

where $\mathbf{K}^{(lab)}$ is the angular momentum of the wheelset in the laboratory coordinate system; $\mathbf{K}_{ri}^{(lab)}$ – the moment of external forces acting on the wheelset from the side of the side frames of the trolley; $\mathbf{K}_{ni}^{(lab)}$ – the moment of reaction forces acting on the wheelset from the side of the left and right rail.

The transition from the laboratory to its own coordinate system is carried out by transformation (7).

$$\dot{\mathbf{K}}^{(lab)} = \mathbf{T} \cdot \dot{\mathbf{K}}^{(wheel)} + \mathbf{T} \cdot \mathbf{K}^{(wheel)}. \quad (26)$$

After substituting (26) in (25), we obtain the equation of rotational motion of the wheelset in its own coordinate system:

$$\dot{\mathbf{K}}^{(wheel)} = \mathbf{T} \cdot \mathbf{K}^{(wheel)}(t) + \sum_{i=1}^{n} \mathbf{K}_{i}^{(wheel)} + \sum_{i=1}^{n} \mathbf{K}_{ni}^{(wheel)}. \quad (27)$$

After fairly simple calculations, equation (27) is reduced to the form:
where $I_{II}$ and $I_\perp$ are the moments of inertia of the wheelset relative to the main axes of inertia (it is assumed that the wheelset is a symmetric top); $\Omega_x$, $\Omega_y$, $\Omega_z$ are the angular velocities of rotation around the principal axes of inertia; $\dot{\omega}$ – the angular frequency of rotation of the wheelset around the axis of symmetry.

When deriving equation (28), it was taken into account that the following relations are satisfied:

$$\Omega_x = \dot{\gamma}, \quad \Omega_y = \dot{\phi} \cdot \sin \dot{\gamma} + \dot{\theta}, \quad \Omega_z = \dot{\phi} \cdot \cos \dot{\gamma} .$$

As expected, equations (28) are well known Euler equations. In the previous subsection, when analyzing the motion of the center of mass of the wheelset, the track coordinate system was used. Consider the equation of rotational motion of the wheelset in a track coordinate system. The derivation of the equations completely repeats the above procedure, but transformation (5) is used.

The equation of the rotational movement of the wheelset in the track coordinate system has the form (here, as in the previous subsection, the index (rail) is omitted):

$$\ddot{\mathbf{3}} = \mathbf{\alpha} \cdot \mathbf{3} + \mathbf{K}(t) + \sum_{i=1}^{i=I} \tilde{K}_i + \sum_{i=I}^{n} \tilde{K}_n , \quad \mathbf{\alpha} = \begin{bmatrix} 0 & \dot{\phi} \sin \gamma & -\dot{\phi} \sin \gamma \\ -\dot{\phi} \cos \gamma & 0 & \gamma \\ \sin \gamma & -\gamma & 0 \end{bmatrix} ,$$

where $\mathbf{3}$ is the angular momentum in the directional coordinate system.

The rotational movement of the wheelset in the track coordinate system can be represented as a superposition [19]:

a) rotation around the axis of symmetry with angular velocity $\dot{\theta}$;

b) rotation around the axis $OZ^{(rail)}$ by an angle $\Delta \phi$;

c) rotation around the axis $OX^{(rail)}$ by an angle $\Delta \gamma$.

Assuming that the angles $\Delta \phi$ and $\Delta \gamma$ are small, the matrix of transformation of the angular momentum of the wheelset from the track to its own coordinate system can be represented as:

$$\dot{\mathbf{I}} + \Delta \mathbf{\tilde{I}} = \begin{bmatrix} 0 & -\Delta \phi & 0 \\ \Delta \phi & 0 & -\Delta \gamma \\ 0 & \Delta \gamma & 0 \end{bmatrix} + G(\Delta \phi, \Delta \gamma) .$$

The angular momentum in the directional coordinate system is expressed in terms of the angular momentum with its own coordinate system with the following formula:

$$\mathbf{3} = (\mathbf{I} + \Delta \mathbf{\tilde{I}}) \cdot \mathbf{\dot{\omega}}$$

where $\mathbf{\dot{\omega}}$ is the angular velocity in its own coordinate system.

Substituting (32) into (30), we obtain the equation of rotational motion of the wheelset with accuracy to linear terms in $\Delta \mathbf{\tilde{I}}$:

$$\dot{\mathbf{I}} \cdot \mathbf{\dot{\omega}} = \left(\mathbf{\alpha} - \Delta \mathbf{\tilde{I}} + \mathbf{\alpha} \Delta \mathbf{\tilde{I}} \right) \cdot \mathbf{I} \cdot \mathbf{\dot{\omega}} + \left(\mathbf{\dot{\mathbf{I}}(t)} + \sum_{i=I}^{i=I} \tilde{K}_i + \sum_{i=I}^{n} \tilde{K}_n \right) , \quad \mathbf{\dot{I}} = \begin{bmatrix} I_\perp & 0 & 0 \\ 0 & I_{II} & 0 \\ 0 & 0 & I_\perp \end{bmatrix} .$$

When integrating Eq. (33), we use the small-term perturbation theory $\Delta \mathbf{\tilde{I}}$.
\[
\ddot{\Omega} = \sum_{m=1}^{n} \omega_m \cdot \dot{\Omega}^{(m)}, \quad \Delta \bar{\Omega} = \sum_{m=1}^{n} \omega_m \cdot \Delta \dot{\Omega}^{(m)}, \quad \bar{K}_{i\alpha \beta} = \sum_{m=1}^{n} \omega_m \cdot \bar{K}_{i\alpha \beta}^{(m)} \tag{34}
\]

Substituting (34) into (33), we obtain the final equations of rotational motion of the wheelset in the zeroth and first approximation:

\[
\bar{I} \cdot \dot{\Omega}^{(0)} = \bar{\omega} \cdot \bar{I} \cdot \dot{\Omega}^{(0)} + \bar{K} (t) + \sum_{i=1}^{k_f} \bar{K}_i^{(0)} + \sum_{i=1}^{k_s} \bar{K}_s^{(0)}
\]

\[
\bar{I} \cdot \dot{\Omega}^{(1)} = \bar{\omega} \cdot \bar{I} \cdot \dot{\Omega}^{(1)} - \left( \Delta \dot{\Omega}^{(0)} - \left[ \bar{\omega} \Delta \dot{\Omega}^{(0)} \right] \right) \bar{I} \cdot \dot{\Omega}^{(0)} - \Delta \dot{\Omega}^{(0)} \cdot \bar{K} (t) - \Delta \dot{\Omega}^{(0)} \cdot \left( \sum_{i=1}^{k_f} \bar{K}_i^{(0)} + \sum_{i=1}^{k_s} \bar{K}_s^{(0)} \right) \tag{35}
\]

The dynamic equations obtained in this subsection must be supplemented by solving the dynamic contact problem. An algorithm for solving the dynamic contact problem is given below.

For each of the contacting bodies, we introduce a moving coordinate system associated with the center of inertia and the main axes of inertia. The number of points of contact with the rail is considered arbitrary [6]. The movement is considered in a fixed coordinate system (track coordinate system).

We give the notation used in the model.

For the main body – the wheelset: \( M_0 \) – mass of the wheelset; \( I_0 \) – central inertia tensor of the wheelset in the main axles; \( \bar{R}_0 \) – the radius of the vector of the center of inertia of the wheelset in a fixed coordinate system (hereinafter dynamic variables); \( \bar{v}_0 = \dot{\bar{R}}_0 \) – speed of the center of inertia of the wheelset (dynamic variables); \( \varphi_0 \) – Euler angles for the main body (dynamic variables); \( \bar{\omega}_0 \) – the angular velocity of the wheelset in its own coordinate system (dynamic variables); \( \bar{\epsilon}_0 \) – the result of the forces of interaction with external bodies; \( \bar{k}_0 \) – the moment of forces of interaction with external bodies; \( n \) – the number of bodies in contact with the main body (wheelset); \( \alpha, \beta = 1, \ldots, n \) is the number of the body in the system (for example: 1 – left rail; 2 – right rail); \( m \) – the number of contact points; \( \alpha \)-body (hereinafter referred to as a countable set); \( m = \sum_{\alpha=1}^{\alpha} m_\alpha \) – total number of contact points; \( \bar{r}_{\alpha,i} \) – radius vectors of contact points in the coordinate system associated with a pair of wheels \( (i = 1, \ldots, m_\alpha) \); \( \bar{v}_{\alpha,i} \) – the speed of the contact points of the main body in a fixed coordinate system; \( \bar{R}_{\alpha,i} \) – reaction forces acting at the contact points; \( \bar{\mu}_{\alpha,i} \) – the orientation vectors of the reaction forces; \( \bar{\eta}_{\alpha,i} \) – the unit normal vectors at the contact points (for definiteness, we consider directed inward to the main body); \( \bar{\tau}_{\alpha,i} \) – tangential unit vector at the point of contact; \( \kappa_{\alpha,i} \) – friction coefficients (possibly depending on the relative speed of the contacting bodies, for example, in the case of the so-called “creep”).

For a rail, we will consider elements of the rail length as concentrated masses. In the future, the dynamics of concentrated masses is used to construct equations of motion of rails in partial derivatives (equations of hyperbolic type).

For rail: \( M_\alpha \) – mass of the \( \alpha \)-body (rail); \( \bar{I}_\alpha \) – the central inertia tensor of the \( \alpha \)-body in the main axles; \( \bar{R}_\alpha \) is the radius of the vector of the center of inertia of the \( \alpha \)-body in a fixed coordinate system (hereinafter dynamic variables); \( \bar{v}_\alpha = \dot{\bar{R}}_\alpha \) – speed of the center of inertia of the \( \alpha \)-body (dynamic variables); \( \bar{\varphi}_\alpha \) – Euler angles for the \( \alpha \)-body (dynamic variables); \( \bar{\omega}_\alpha \) – angular velocities of the \( \alpha \)-body in its own coordinate system (dynamic variables); \( \bar{k}_\alpha \) – the resultant interaction forces of the \( \alpha \)-body with external bodies; \( \bar{\epsilon}_\alpha \) – the moment of forces of interaction of the \( \alpha \)-body with external bodies; \( \bar{\rho}_\alpha \) – the radius vectors of the points of contact of the \( \alpha \)-body with the principal in the

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It seems there might be a typographical error in the equation. The correct form should be:

\[
\ddot{\Omega} = \sum_{m=1}^{n} \omega_m \cdot \dot{\Omega}^{(m)}, \quad \Delta \bar{\Omega} = \sum_{m=1}^{n} \omega_m \cdot \Delta \dot{\Omega}^{(m)}, \quad \bar{K}_{i\alpha \beta} = \sum_{m=1}^{n} \omega_m \cdot \bar{K}_{i\alpha \beta}^{(m)} \tag{34}
\]
coordinate system associated with the \( i \)-body (\( i = 1, \ldots, m \)); \( \dot{v}_{a,j} \) – the velocity of the points of contact of the \( \alpha \)-body with the main one in a fixed coordinate system.

5. The equations of motion of interacting bodies

The equations of motion of the main body – the wheels follow from the previous subsection of the article.

\[
\dot{\mathbf{R}}_0 = \mathbf{V}_0, \quad \dot{\mathbf{V}}_0 = M_0^{-1} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{m_0} y_{a,i} \dot{\mu}_{a,i} \right\}
\]

(37)

\[
\dot{\mathbf{V}}_0 = \dot{\Omega}_0, \quad \dot{\Omega}_0 = \dot{R}_0^{-1} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{m_0} y_{a,i} \left[ \mathbf{r}_{a,i} \times \dot{\mu}_{a,i} \right] \right\}
\]

(38)

The equations of motion of the \( \alpha \)-body have the form (\( \alpha = 1, \ldots, n \)).

\[
\dot{\mathbf{R}}_\alpha = \mathbf{V}_\alpha, \quad \dot{\mathbf{V}}_\alpha = M_\alpha^{-1} \left\{ \sum_{j=1}^{m_0} y_{a,j} \dot{\mu}_{a,j} \right\}
\]

\[
\dot{\mathbf{V}}_\alpha = \dot{\Omega}_\alpha, \quad \dot{\Omega}_\alpha = \dot{R}_\alpha^{-1} \left\{ \sum_{j=1}^{m_0} y_{a,j} \left[ \mathbf{r}_{a,j} \times \dot{\mu}_{a,j} \right] \right\}
\]

(39)

Note that the vectors \( \mathbf{r}_{a,j} \) are not independent and are defined as follows:

\[
\mathbf{r}_{a,j} = \mathbf{r}_{0,j} + \mathbf{R}_0 - \mathbf{R}_\alpha
\]

With this in mind, the last equation in (38) takes the form:

\[
\dot{\mathbf{V}}_\alpha = \dot{\Omega}_\alpha \left( \mathbf{r}_{a,j} \times \mathbf{V}_{a,j} - \mathbf{r}_{a,j} \times \mathbf{V}_0 \right)
\]

(40)

The vector that determines the orientation of the reaction forces can be represented as follows:

\[
\mathbf{\tilde{K}}_\alpha = \mathbf{\tilde{K}}_\alpha - \sum_{j=1}^{m_0} y_{a,j} \left[ \mathbf{r}_{a,j} \times \mathbf{\tilde{K}}_{a,j} \right]
\]

(41)

where \( \Delta \mathbf{v}_{a,j} = \mathbf{v}_{a,j} - \mathbf{\tilde{v}}_{a,j} \) is the relative velocity of the bodies at the contact points.

The velocities of bodies at points of contact are expressed in terms of dynamic variables in a known manner:

\[
\mathbf{v}_{a,j} = \mathbf{V}_0 + \left[ \mathbf{\tilde{K}}_0 \times \mathbf{r}_{a,j} \right], \quad \mathbf{\tilde{v}}_{a,j} = \mathbf{\tilde{V}}_\alpha + \left[ \mathbf{\tilde{K}}_\alpha \times \mathbf{\tilde{r}}_{a,j} \right]
\]

(42)

In view of (39) and (41), the expression for the relative velocity of the contact points takes the form:

\[
\Delta \mathbf{v}_{a,j} = \mathbf{V}_0 - \mathbf{\tilde{V}}_\alpha + \left[ \mathbf{\tilde{K}}_0 \times \mathbf{r}_{a,j} \right] - \left[ \mathbf{\tilde{K}}_\alpha \times \left( \mathbf{r}_{a,j} + \mathbf{R}_0 - \mathbf{R}_\alpha \right) \right]
\]

(43)

Thus, the above ones determine the dynamics of the \( n + 1 \)st body of the system. In these equations, the reaction forces determined \( y_{a,j} \) by the boundary conditions (or by the coupling equations) are not defined.
6. Communication Equations

We assume that at the initial time \( t = 0 \), there is a non-penetrating contact of the bodies, i.e., the normal components of the relative velocities are equal to zero [19, 20]:

\[
(\Delta \dot{\nu}_{\alpha,i}(t = 0) \cdot \vec{n}_{\alpha,i}) = 0
\]  

(43)

The integration problem is considered correctly posed if conditions (43) are satisfied at the initial instant of time. The same equations can be used to determine the initial conditions (for example, for \( n = 2 \) and \( m_1 = m_2 = 1 \), we obtain a system of two equations to further determine the initial values of the velocities \( \vec{v}_{\alpha,i} \) and \( \vec{\epsilon}_{\alpha,i} \)). Then the condition for the remaining contact will be the equality to zero of the normal component of the relative acceleration at an arbitrary point in time:

\[
(\Delta \ddot{\nu}_{\alpha,i} \cdot \vec{n}_{\alpha,i}) = 0
\]  

(44)

Since the bonds are nonholonomic [21], the equations of coupling are not integrable and can be considered as equations for determining unknown reaction forces. We proceed to the derivation of equations for determining the reaction forces. Differentiating (42), we obtain:

\[
\Delta \ddot{\nu}_{\alpha,i} = \dot{v}_a - V_a + \sum_{\beta=1}^{m_0} \sum_{j=1}^{n} \mathcal{J}_{(\alpha,i), (\beta,j)} \cdot \delta_{\beta,j} = \mathcal{N}_{\alpha,i},
\]  

(45)

We take into account that \( \dot{\nu}_{\alpha,i} = \dot{v}_a \), \( -V_a = [\Omega \times \vec{r}_{\alpha,i}] \), we substitute accelerations in (45). After simple transformations, we obtain the communication equations in the form of a system of \( m \) linear equations for determining the reaction forces \( \mathcal{R}_{\alpha,i} \):

\[
\alpha = 1 \ldots n, i = 1 \ldots m_a, \sum_{\beta=1}^{m_0} \sum_{j=1}^{n} \mathcal{J}_{(\alpha,i), (\beta,j)} \cdot \delta_{\beta,j} = \mathcal{N}_{\alpha,i},
\]  

(46)

The matrix \( \mathcal{J}_{(\alpha,i), (\beta,j)} \) and vector \( \mathcal{N}_{\alpha,i} \) are defined by the following expressions:

\[
\mathcal{J}_{(\alpha,i), (\beta,j)} = \left( M_0^{-1} + \partial_{\alpha,\beta} \cdot M_0^{-1} \right) \left( \vec{\eta}_{\alpha,i} \cdot \vec{\mu}_{\beta,j} \right) - \left( \vec{\eta}_{\alpha,i} \times \vec{t}_0^{-1} \cdot \left( \vec{f}_{\beta,j} \times \vec{\mu}_{\beta,j} \right) \right) - \delta_{\alpha,\beta} \left( \vec{\eta}_{\alpha,i} \times \vec{r}_{\beta,j} + \vec{\epsilon}_{\alpha,i} + \vec{\epsilon}_{\beta,j} \right) \times \vec{t}_0^{-1} \left( \vec{f}_{\beta,j} + \vec{\epsilon}_{\beta,j} + \vec{\epsilon}_{\beta,j} \right) \times \vec{\mu}_{\beta,j},
\]  

(47)

\[
\mathcal{N}_{\alpha,i} = -\left( \vec{\eta}_{\alpha,i} \cdot \left( M_0^{-1} \cdot \vec{f}_{\beta,j} + \vec{\epsilon}_{\alpha,i} \times \vec{t}_0^{-1} \cdot \vec{F}_a \right) \right) + \left( \vec{\eta}_{\alpha,i} \times \vec{t}_0^{-1} \cdot \vec{F}_a \right) \left( \vec{f}_{\beta,j} + \vec{\epsilon}_{\beta,j} \right) \times \vec{t}_0^{-1} \cdot \vec{K}_a - \left( \vec{\epsilon}_{\alpha,i} \times \vec{\mu}_{\beta,j} \right) \times \vec{t}_0^{-1} \cdot \vec{F}_a \right) \times \vec{t}_0^{-1} \cdot \vec{K}_a - \left[ \vec{\epsilon}_{\alpha,i} \times \left( \vec{\epsilon}_{\alpha,i} \times \vec{r}_{\beta,j} \right) \right] - \left[ \vec{\epsilon}_{\alpha,i} \times \left( \vec{\epsilon}_{\alpha,i} \times \vec{r}_{\beta,j} \right) \right] + \vec{V}_a - \vec{V}_a.
\]

In the particular case \( n = 2, m_1 = m_2 = 1 \) (contact at two points with two rails), the problem reduces to solving two equations with two unknown reaction forces.

It can be proved that if the initial conditions are correctly specified for the case \( n = 2 \) and \( m_1 = m_2 = 1 \) (two-point contact with two bodies), the solutions of system (46) are non-negative \( \mathcal{R}_{\alpha,i} \geq 0 \). In more complex cases, it must be assumed that the appearance of negative solutions means a change in the number of contact points.

7. Conclusions

Thus, the dynamic problem of the motion of the \( n + 1 \)st contacting bodies is mathematically formulated, the description of the algorithms is given:

a) the construction of dynamic equations describing the joint movement of a pair of wheels and a rail track;

b) solving the dynamic contact problem
Based on the described mathematical algorithms, programs have been developed for calculating the forces of dynamic interaction in the wheel – rail system.

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