Word problems and ceers

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This note addresses the issue as to which ceers can be realized by word problems of computably enumerable (or, simply, c.e.) structures (such as c.e. semigroups, groups, and rings), where being realized means to fall in the same reducibility degree (under the notion of reducibility for equivalence relations usually called “computable reducibility”), or in the same isomorphism type (with the isomorphism induced by a computable function), or in the same strong isomorphism type (with the isomorphism induced by a computable permutation of the natural numbers). We observe, e.g., that every ceer is isomorphic to the word problem of some c.e. semigroup, but (answering a question of Gao and Gerdes) not every ceer is in the same reducibility degree of the word problem of some finitely presented semigroup, nor is it in the same reducibility degree of some non-periodic semigroup. We also show that the ceer provided by provable equivalence of Peano Arithmetic is in the same strong isomorphism type as the word problem of some non-commutative and non-Boolean c.e. ring.

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1 Introduction

Computably enumerable equivalence relations, or ceers, have been an active field of research in recent years. A great deal of the interest in ceers certainly is due to the fact that they appear quite often in mathematical logic (where they appear, e.g., as the relations of provable equivalence in formal systems), and in general mathematics and computer science where they appear as word problems of effectively presented familiar algebraic structures. An important example in this sense is the word problem for finitely presented (or, f.p.) groups. If \( \langle X; R \rangle \) is a f.p. group and one codes the universe of the free group \( F_X \) on \( X \) with \( \omega \), then the word problem of the group is the ceer that identifies two elements \( x, y \in F_X \) if \( xy^{-1} \) lies in the normal subgroup of \( F_X \) generated by the relators appearing in the relation \( R \) of the presentation of the group. The word problem of a f.p. group can be decidable (i.e., the corresponding ceer is decidable), but also undecidable, and in fact can be of any c.e. Turing degree, or even m-degree: this was obtained independently by Fridman, Clapham, and Boone (cf. [9–11, 13, 18]; despite the difference in publication dates, the work of these authors was essentially simultaneous).

Of course not every ceer can be the word problem of a f.p. group, or even of a computably enumerable (c.e.) group, cf. Definition 1.1 below. For instance, the equivalence classes of the word problem of a c.e. group are uniformly computably isomorphic with each other: to show that the equivalence class of \( u \) is isomorphic to the equivalence class of \( v \), just use the mapping \( x \mapsto xu^{-1}v \). Therefore no ceer having both finite classes and infinite classes, or even having at least two classes of different m-degree, can be the word problem of a group. Therefore the question naturally arises as to which ceers can be identified as word problems not only of groups, but of other familiar computably enumerable structures, modulo several ways of “identifying” equivalence relations, based on natural measures of their relative complexity. The present paper is meant to be a contribution to this line of research.

We first need of course to specify what we mean by “computably enumerable structures” and their “word problems”, and how we intend to measure the relative complexity of equivalence relations.

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1.1 C.e. algebras

Following the tradition of Mal’cev and Rabin, it is common to postulate that the complexity of the problem of presenting the particular copy of a structure is captured by its atomic diagram. Yet, in algebra one naturally deals with structures whose algebraic structure is easy to describe but it is hard to know whether two terms represent the same element. The paradigmatic example of this phenomenon is the construction, independently due to Boone and Novikov, of a finitely presented group with an undecidable word problem [8, 30]. Moreover, the first homomorphism theorem ensures that every countable algebra arises as a quotient of the term algebra on countably many generators. So a countable algebra can always be represented in a way in which the complexity of the structure is entirely encoded in its word problem. This motivates the idea, often recurring in the literature, of looking at c.e. structures as given by quotienting \( \omega \) modulo a ceer. In this paper, we shall only be concerned with structures that are algebras.

We recall that a type of algebras is a set \( \tau \) of function symbols, such that each member \( f \in \tau \) is assigned a natural number \( n \), called the arity of \( f \). An algebra of type \( \tau \) is a pair \( A = \langle A, F \rangle \), where \( A \) is a nonempty set, and \( F \) is a set of operations on \( A \) interpreting the type, i.e., in one-to-one correspondence with the function symbols in \( \tau \), so that \( n \)-ary function symbols of \( \tau \) correspond to \( n \)-ary operations in \( F \).

Definition 1.1 An algebra \( A \) of decidable type \( \tau \), is computably enumerable (or, simply, c.e.) if there is a triple \( A^− = \langle \omega, F, E \rangle \) (called a positive presentation of \( A \)) such that: (1) \( F \) consists of uniformly computable operations on \( \omega \) interpreting the type \( \tau \); (2) \( E \) is a ceer, which is also a congruence with respect to the operations in \( F \); (3) the quotient \( A_E = \langle \omega_E, F_E \rangle \) (called a positive copy of \( A \)) is isomorphic with \( A \), where \( F_E = \{ f_E : f \in F \} \), with \( f_E([x]_E) = [f(x)]_E \).

For a thorough and clear introduction to c.e. structures, cf. [32] (where they are called positive structures) and [24].

We shall consider c.e. algebras \( A = \langle A, F \rangle \) given by some positive presentation \( A^− = \langle \omega, F^−, E \rangle \), and we shall work directly with the positive presentation rather than the algebra itself. Thus, if we say \( a \in A \) we in fact mean any \( a^− \in \omega \) such that \( [a^−]_E = a \). The ceer \( E \) will be often denoted also by \( =_A \), as it yields equality in the quotient algebra.

Definition 1.2 The word problem of a c.e. algebra \( A \) is the ceer \( =_A \). (Up to isomorphism of ceers, as in Definition 1.4, the word problem is independent of the choice of the positive presentation).

Given a ceer \( E \) (having possibly some interesting computational property) it is natural to ask which algebras \( A \) can be positively presented having \( E \) as their equality relation \( =_A \) (cf., e.g., [19, 22]). Surprisingly, much less is known about the reverse problem, namely, given a class of structures \( \mathcal{C} \), which ceers are “realized” by members of \( \mathcal{C} \)? This is the main topic of our paper. But, of course, we still need to give a rigorous definition of what we mean by a structure “realizing” a ceer.

1.2 Measures of the relative complexity of equivalence relations

The most useful and popular way of measuring the relative complexity of ceers has been (at least in recent years: cf., e.g., [3, 4, 20, 21]) via the following notion of reducibility.

Definition 1.3 Given a pair of equivalence relations \( R, S \) on \( \omega \) we say that \( R \) is computably reducible to \( S \) (\( R \leq S \)) if there exists a computable function \( f \) such that

\[
(\forall x, y)[x R y \iff f(x) S f(y)].
\]

In the rest of the paper “computable reducibility” will be simply referred to as “reducibility”. This leads to identifying two equivalence relations \( R, S \), if they both belong to the same reducibility degree, i.e., \( R \leq S \) and \( S \leq R \). In this paper, we shall consider two additional ways of comparing equivalence relations based on the notion of “isomorphism”. If \( R \) is an equivalence relation on \( \omega \) then for every number \( x \) we denote by \([x]_R \) the \( R \)-equivalence class of \( x \); the collection of all \( R \)-equivalence classes is denoted by \( \omega_R \).
Definition 1.4 Given ceers \( R, S \), we say that \( R \) and \( S \) are isomorphic (notation \( R \simeq S \)) if there is a reduction \( f : R \to S \) such that the range of \( f \) intersects all \( S \)-equivalence classes. We say in this case that \( f \) induces an isomorphism from \( R \) to \( S \).

The choice of the name “isomorphism” is justified by Lemma 1.5 below. Following the category theoretic approach to numberings proposed by Ershov [17], equivalence relations on \( \omega \) can be structured as objects of a category (cf. also [16]). The lemma shows in fact that, when restricting attention only to equivalence relations that are ceers, two objects are isomorphic in the category theoretic sense if and only if they are isomorphic in the sense of our Definition 1.4.

Lemma 1.5 (Inversion Lemma) If \( R, S \) are ceers then \( f \) induces an isomorphism from \( R \) to \( S \) if and only if \( f \) has an equivalence inverse, i.e., there is a reduction \( g : S \leq R \) such that \( g(f(x)) \) \( R \) \( x \), and \( f(g(x)) \) \( S \) \( x \), for all \( x \in \omega \).

Proof. By [4, Lemma 1.1]. □

Definition 1.6 We say that \( R \) and \( S \) are strongly isomorphic if there is a computable permutation \( f \) of \( \omega \) providing a reduction \( f : R \to S \). We say in this case that \( f \) induces the strong isomorphism.

Trivially, if \( f \) induces a strong isomorphism from \( R \) to \( S \) then it also induces an isomorphism from \( R \) to \( S \). It is also clear that if \( R \) has at least one finite class and \( S \) has only infinite classes then \( R \) and \( S \) cannot be strongly isomorphic.

Lemma 1.7 For every ceer \( R \) there exists a ceer \( S \) having only infinite classes and such that \( R \simeq S \).

Proof. Let \( \langle \_,\_ \rangle \) be the Cantor pairing function, and let \( \langle \_ \rangle_0 \) be its first projection. Given \( R \), let \( S \) be such that \( x \leq R y \iff \langle x \rangle_0 R \langle y \rangle_0 \). As is immediate to see, the computable function \( \langle \_ \rangle_0 \) induces an isomorphism from \( S \) to \( R \), since it provides a reduction whose range intersects all \( R \)-equivalence classes. □

We summarize the various definitions, and introduce suitable notations for them.

Definition 1.8 If \( R, S \) are ceers, we say that

1. \( R \) is bi-reducible with \( S \) (notation \( R \equiv S \)) if \( R \leq S \) and \( S \leq R \);
2. \( R \) is isomorphic to \( S \) (notation: \( R \simeq S \)) if there is a reduction \( f : R \to S \) such that \( \text{range}(f) \cap [x]_S \neq \emptyset \), for all \( x \);
3. \( R \) is strongly isomorphic to \( S \) (notation: \( R \simeq_S S \)) if there is a computable permutation of \( \omega \) reducing \( R \) to \( S \).

The following is a useful observation:

Fact 1.9 If \( R, S \) are ceers such that all \( R \)-classes and all \( S \)-classes are infinite then \( R \simeq_S S \) if and only if \( R \simeq S \).

Proof. The nontrivial implication \( R \simeq S \implies R \simeq_S S \) follows by a straightforward back-and-forth argument similar to the one used in the proof of the Myhill Isomorphism Theorem; cf., e.g., [2, Remark 1.2] and [1, Lemma 2.3]. □

Fact 1.10 The following proper implications hold on ceers \( R, S \):

\[ R \simeq_S S \implies R \simeq S \implies R \equiv S. \]

Proof. The proof follows easily from known facts in the literature, and a few other obvious observations. We have already observed that any computable function inducing a strong isomorphism induces also an isomorphism. So \( \simeq_S \) implies \( \simeq \). Next, suppose that \( f : R \to S \) induces an isomorphism. Then \( f \) is already a reduction from \( R \) to \( S \) giving \( R \leq S \). But \( f \) has an equivalence-inverse reduction \( g : S \leq R \), so \( S \leq R \) as well. In conclusion \( R \equiv S \), and thus \( \simeq \) implies \( \equiv \).

Let us now show through a few examples that the implications are proper. It is known that there are universal ceers \( E \) (which therefore are reducible to each other: we recall that a ceer \( E \) is universal, if \( S \leq E \) for every ceer \( S \)), whose equivalence classes are all undecidable. This is the case, e.g., of precomplete and u.f.p. ceers (for these notions and their properties, cf., e.g., [2]): a concrete example of a universal ceer with undecidable equivalence classes (cf. [6, Example 2]) is the ceer \( \sim_{\text{PE}} \) induced by provable equivalence of Peano Arithmetic; cf. § 4. Now if \( E \) is such a universal ceer then \( E \oplus \text{Id}_1 \) (that is, the ceer \( \{(2x, 2y) : x E y \} \cup \{(2x + 1, 2y + 1) : x, y \in \omega \} \)) is universal too, thus \( E \equiv E \oplus \text{Id}_1 \), but \( E \ncong E \oplus \text{Id}_1 \) since that latter ceer has only decidable equivalence class.

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Finally, take $R$ to be any ceer with at least one finite equivalence class, and let $S$ be the ceer built from $R$ as in the proof of Lemma 1.7. Thus $R \cong S$, but $R \not\cong S$, as $R$ has at least a finite equivalence class whereas all $S$-equivalence classes are infinite.

**Definition 1.11** If $R$ is a ceer, $A$ is a c.e. algebra, and $\approx \in \{ \equiv, \simeq, \cong \}$ then we say that $R$ is $\approx$-realized by $A$ if $R \cong A$. (Recall that $\equiv$ denotes the word problem of $A$.)

**Definition 1.12** A class $\mathcal{C}$ of algebras of the same type is $\approx$-complete for a class $\mathcal{C}$ of ceers (where $\approx \in \{ \equiv, \simeq, \cong \}$) if every ceer in $\mathcal{C}$ is $\approx$-realized by some c.e. copy of an algebra from $\mathcal{C}$. We simply say that $\mathcal{C}$ is $\approx$-complete for the ceers if $\mathcal{C}$ is $\approx$-complete for the class of all ceers.

**Corollary 1.13** If $\mathcal{C}$ is a class of ceers all of whose members have no finite equivalence classes, and $\mathcal{C}$ is a class of algebras then $\mathcal{C}$ is $\simeq$-complete for $\mathcal{C}$ if and only if $\mathcal{C}$ is $\simeq$-complete for $\mathcal{C}$.

**Proof.** It follows from Fact 1.9.

Moreover, it is trivial to observe:

**Fact 1.14** If $\mathcal{C}$ is a class of ceers and $\mathcal{C}$ is a class of algebras then if $\mathcal{C}$ is $\simeq$-complete for $\mathcal{C}$, then it is $\equiv$-complete for $\mathcal{C}$; if $\mathcal{C}$ is $\simeq$-complete for $\mathcal{C}$, then it is $\cong$-complete for $\mathcal{C}$.

**Proof.** The proof follows from Fact 1.10.

## 2 Classes of algebras that are complete for the ceers

We now begin to look at some natural classes of c.e. algebras in relation to the problem of $\simeq$-completeness for ceers, with $\approx \in \{ \equiv, \simeq, \cong \}$. Our examples of c.e. algebras will be more conveniently introduced via the notion of a computably enumerable presentation. In a variety of algebras with finite or countable type, if the term algebra $T(X)$ on a finite or countable set $X$ (cf., e.g., [12, §10]) exists (existence is guaranteed if, as in our future examples, $X$ is nonempty) then, up to isomorphisms, $T(X)$ can be presented as a computable algebra: we may assume that $X$ is decidable, $T(X)$ has decidable universe (which is infinite in all our examples), computable operations, and equality is syntactic equality. If, in addition the identities of the variety form a c.e. binary relation on $T(X)$, then we have the following definition.

**Definition 2.1** In a variety as above, a c.e. presentation is a pair $\mathcal{A} = \langle X; R \rangle$ where $X$ is a set, $R$ is a binary relation on $T(X)$, and $\mathcal{A}$ denotes the quotient algebra $T(X)/N_R$, where $N_R$ is the c.e. congruence on $T(X)$ generated by $R$ together with the identities of the variety. An algebra $\mathcal{A}$ of the variety is c.e. presented (c.e.p.), if it is of the form $\langle X; R \rangle$ as just described.

A special case is provided by finite presentations, where both $X$ and $R$ are finite.

The following fact is well known:

**Lemma 2.2** In a variety as above, an algebra is c.e. if and only if it is isomorphic to some c.e.p. algebra.

**Proof.** We sketch the proof. If $\mathcal{A} = \langle X; R \rangle$ is a c.e. presentation, then there is a computable isomorphism $f$ of $T(X)$ with an algebra having $\omega$ as universe, and equipped with a set $F$ of suitable computable functions corresponding, via the isomorphism, to the operations of $\mathcal{A}$. Then $\mathcal{B} = (\omega, F, E)$ is a positive presentation of $\mathcal{A}$, where $E$ is the ceer corresponding under the isomorphism to the c.e. relation $N_R$ on $T(X)$. Notice that according to Definition 1.2, equality $=_{E}$ of $\mathcal{B}$ coincides with $E$. For the converse, assume that $\mathcal{A} = (\omega, F, E)$ is a positive presentation. By the universal property of $T(\omega)$ (namely, the term algebra on the set $\omega$ of generators), there is a unique epimorphism $\nu: T(\omega) \rightarrow \mathcal{A}_E$ which commutes with the mapping $x \mapsto [x]_E$ from $\omega$ to $\mathcal{A}_E$, and the insertion of generators $x \mapsto x$ from $\omega$ to $T(\omega)$. Namely, if $p(x_1, \ldots, x_k) \in T(\omega)$ is a term, and $p_F$ interprets $p$ using the operations in $F$, then $\nu(p(x_1, \ldots, x_k)) = [p_F(x_1, \ldots, x_k)]_E$, by the properties of $E$. It follows that the kernel $R$ of $\nu$ is a c.e. binary relation on $T(\omega)$, and by universal algebra, the c.e. presentation $\langle \omega; R \rangle$ is isomorphic with $\mathcal{A}_E$.

To describe some of the consequences of Lemma 2.2 which are relevant to our later examples, we first generalize Definition 1.4 to partial ceers, i.e., c.e. equivalence relations having as domains c.e. subsets of $\omega$. If $R, S$ are partial
ceers with domains $X, Y$ respectively, we say that $R$ and $S$ are isomorphic ($R \simeq S$; we use the same symbol as in Definition 1.4) if there is partial computable function $g$ whose domain contains $X$ and $g(X) \subseteq Y$, such that $x R y$ if and only if $g(x) S g(y)$ for all $x, y \in X$, and range($g$) intersects all $S$-equivalence classes.

One direction of the proof of the previous lemma actually shows that every c.e. presentation $(x; R)$ has a positive presentation $\langle \omega, F, E \rangle$ such that $N_R \simeq E$ as partial ceers, as witnessed by the computable isomorphism $f : T(X) \to \omega$. The other direction of the proof shows in fact that for every positive presentation $A = \langle \omega, F, E \rangle$ there is a c.e. presentation $\langle \omega; R \rangle$ which is isomorphic to $A/E$, and $R \simeq E$ as partial ceers. This follows from the fact that $\nu$ is onto, and therefore the computable mapping $p(x_1, \ldots, x_k) \mapsto p_F(x_1, \ldots, x_k)$ provides a reduction from $R$ to $E$ whose range intersects all $E$-equivalence classes.

### 2.1 The word problem as a ceer on terms, or as a ceer on the free algebra

When trying to show that some ceer $S$ is $\simeq$-realized by a c.e. presentation $(X; R)$, the above remarks suggest, in accordance to many algebra textbooks (cf., e.g., [12, p. 252]), to consider $N_R$ as the word problem of the c.e. presentation, and show that $S \simeq N_R$ as partial ceers. This is fully consistent with Definition 1.2, since, as we have seen, $S \simeq E$, where $E$ is the ceer of the positive presentation assigned to $(X; R)$ in the proof of Lemma 2.2.

In fact, our examples of c.e. algebras will come from varieties (such as semigroups, monoids, groups, rings) in which the identities of the variety generate a decidable congruence $I$ on $T(X)$. By decidability of $I$, we mean that we can fix a computable mapping $p \mapsto \overline{p} : T(X) \to T(X)$ with decidable range, picking up exactly one element in each $I$-equivalence class, so that the free algebra $F(X)$, taken to be $T(X)/I$, can be presented as a computable algebra having this range as universe. Let now $(X; R)$ be a c.e. presentation. By universal algebra, there is a c.e. congruence $\overline{R}$ on $F(X)$ (namely, $\overline{R} = N_R/I$, using common notation in universal algebra) such that $T(X)/N_R$ is isomorphic with $F(X)/\overline{R}$ and

$$p\Nm{N_R}q \iff p\Nm{R}q,$$

for every $p, q \in T(X)$. This gives an isomorphism of partial ceers between $N_R$ and $\overline{R}$. Conversely, given a binary c.e. relation $\overline{R}$ on $\overline{X} = \{\overline{x} : x \in X\}$, then one can find a c.e. congruence $R$ on $T(X)$ such that $T(X)/R$ is isomorphic with $F(X)/N_R$, where $N_R$ is the c.e. congruence on $F(X)$ generated by $\overline{R}$. Moreover, $R$ and $N_R$ are isomorphic as partial ceers.

This suggests to adopt, in these varieties, even a more simplified, yet equivalent, approach to word problems of c.e. algebras, and agree that a c.e. presentation is a pair $(X; R)$ where $R$ is a binary c.e. relation on $F(X)$ and, in this case, $(X; R)$ denotes the quotient $F(X)/N_R$, where $N_R$ is the congruence generated on $F(X)$ by $R$, and we take $N_R$ as the word problem of the c.e. algebra so presented. Of course, in general the elements of $F(X)$ will not be presented directly as certain elements of $T(X)$ but in some simplified “normal form”, obtaining in any case a computably isomorphic copy of the free algebra, and up to isomorphism of partial ceers, the same word problem.

### 2.2 Semigroups

Throughout the paper our references for terminology about semigroups and monoids are the textbooks [14] and [23]. In view of Definition 2.1 (and the subsequent adjustment in Subsection 2.1), towards an explicit description of a c.e.p. semigroup it is sufficient to describe what the free semigroup $F(X)$ on $X$ and the c.e. binary relation $R$ on $F(X)$ are. Hence, we recall that the free semigroup on a set $X$ can be taken to be $(X^* \setminus \{\lambda\}, \cdot)$, where in general $Y^*$ denotes the collection of finite words of letters from a set $Y$, $\lambda$ is the empty string, and $\cdot$ is concatenation of words.

**Definition 2.3** A semigroup $S$ is a right-zero band if $ab = b$ for all $a, b \in S$.

**Theorem 2.4** The class of right-zero bands is $\simeq$-complete for the ceers.

**Proof.** Let $R$ be a given ceer, and fix a computable set $X = \{x_i : i \in \omega\}$ of generators. Consider the c.e. binary relation $\overline{R} := \{(x_i, x_j) : i R j\} \cup \{(x_i, x_j) : i, j \in \omega\}$ on $F(X)$. Let $S = (X; \overline{R})$ be the c.e.p. semigroup so presented. In particular, notice that $ux_i =_S x_i$, for any word $u$ and any generator $x_i$. It is easy to see that $i R j$ if and only if $x_i =_S x_j$, so that $R \leq_S$ by the reduction $f(i) =_S x_i$. On the other hand, as the range of $f$ intersects
all \(=_S\)-equivalence classes (since \(u =_S x_i\) where \(x_i\) is the last bit of \(u\), as follows from the relations), we have that \(=_S \simeq R\).

Of course the same result holds if we replace right-zero bands with left-zero bands, i.e., semigroups in which \(ab = a\) for all pairs \(a, b\).

### 2.3 Monoids

Next, we consider the case of monoids. Recall in this case that the free monoid \(F(X)\) on \(X\) can be taken to be \(\langle X^*, \cdot \rangle\), where again \(\cdot\) is concatenation.

**Definition 2.5** A monoid \(M = \langle M, \cdot \rangle\) is right-zero band-like if \(ab = b\) for every \(a, b \in M\{1\}\) (where 1 denotes the identity element).

**Theorem 2.6** The class of right-zero band-like monoids is \(\simeq\)-complete for the ceers.

**Proof.** Let \(R\) be a given ceer, and fix again a computable set \(X = \{x_i : i \in \omega\}\) of generators. Consider the c.e. binary relation \(\hat{R} := \{(x_i, x_j) : i R j\} \cup \{(x_i, x_j) : i \in \omega \setminus \{0\}, j \in \omega\} \cup \{(x_0, \lambda)\}\) on \(F(X)\). Let \(M = \langle X; \hat{R}\rangle\) be the c.e.p. monoid so presented. The proof that \(R \simeq =_M\) is as in the proof of Theorem 2.4.

Again, right-zero band-like monoids can be replaced by left-zero band-like monoids in the result above.

### 3 Classes of c.e. algebras that are not complete for the ceers

We try in this section to identify algebraic properties that prevent classes of algebras sharing these properties to be \(\approx\)-complete for the ceers, with \(\approx \in \{\equiv, \simeq, \simeq_s\}\).

#### 3.1 Semigroups

For our first observation, we need the following definition.

**Definition 3.1** A semigroup \(S\) is periodic if, for all \(a \in S\), there are numbers \(1 \leq n < m\) such that \(a^n = a^m\).

Recall that a ceer \(R\) is dark if \(R\) has infinitely many equivalence classes but it does not admit any infinite c.e. transversal, i.e., an infinite c.e. set \(W\) such that if \(x, y \in W\) and \(x \neq y\) then \(x R y\). For the existence and properties of dark ceers, cf. [4].

**Theorem 3.2** The class of semigroups which are not periodic is not \(\equiv\)-complete for the ceers.

**Proof.** Let \(S\) be a non-periodic c.e. semigroup. Then there exists an element \(a \in S\) such that \(d^m \neq S d^n\) if \(m \neq n\). Thus \(\{d^n : n \in \omega\}\) is an infinite c.e transversal, implying that \(=_S\) cannot \(\equiv\)-realize any dark ceer, as the property of having an infinite c.e. transversal is invariant under bi-reducibility.

Recall that a diagonal function for an equivalence relation \(R\) is a computable function \(d\) such that \(d(x) R x\), for every \(x\). The next theorem identifies a natural class of semigroups which are not \(\simeq\)-complete for the ceers. Examples of semigroups filling the description in the statement of the theorem are, e.g., the semigroups without idempotent elements.

**Theorem 3.3** The class of semigroups \(S\) for which there exists a number \(n\) such that \(x^n \neq x\) for every \(x\) is not \(\simeq\)-complete for the ceers.

**Proof.** Suppose \(S\) is a c.e. semigroup as in the statement of the theorem. Take any \(x\), and define \(d(x) = x^n\). Then \(d\) is a diagonal function for \(=_S\), and thus \(S\) cannot \(\simeq\)-realize any ceer which does not possess a diagonal function, such as, e.g., the weakly precomplete ceers (including the precomplete ones). For these notions and their properties, cf. again [2].
3.2 Monoids

We now take a quick look at monoids.

Definition 3.4 Let $M$ be a monoid. A non-unit element $x \in M$ is a torsion element if there exists a number $n > 0$ such that $x^n = 1$; otherwise $x$ is non-torsion. Moreover, a monoid is said to be torsion if every element is a torsion element, non-torsion otherwise.

We observe:

Theorem 3.5 The class of non-torsion monoids is not $\equiv$-complete for the ceers.

Proof. Let $x \in M$ be a non-torsion element, with $M$ c.e. Then $\{x, x^2, (x^2)^2, \ldots \}$ is an infinite c.e. transversal for $=_M$. The proof is now similar to the proof of Theorem 3.2. □

3.3 On finitely presented semigroups and a question of Gao and Gerdes

We recall the following observation of Gao and Gerdes (where the statement refers to finitely presented groups, but it is obviously extendable to all groups) [21, p.58].

Fact 3.6 The class of groups is not $\equiv$-complete for the ceers.

Proof. If $R$ is an undecidable ceer with only finitely many undecidable equivalence classes (it is easy to see that there are even undecidable ceers with only finite equivalence classes: e.g., there are dark ceers with only finite classes, cf. [4, Corollary 4.15]) then there cannot be any c.e. group $G$ such that $=_G \equiv R$: for otherwise, by the reduction $=_G \leq R$ we would have that either $=_G$ is finite, and thus $R \not\leq =_G$, or there are decidable $=_G$-classes, but as observed in the introduction all $=_G$-equivalence classes are computably isomorphic with each other, which would imply that $[1]_G$ is decidable and thus $=_G$ is decidable ($u =_G v$ if and only if $uv^{-1} \in [1]_{=_G}$), giving that $R$ is decidable by the reduction $R \leq =_G$.

For this reason, Gao and Gerdes ask whether the class of f.p. semigroups is $\equiv$-complete for the ceers (cf. [21, Problem 10.3]). This is an interesting question, motivated by a celebrated theorem due to Shepherdson stating that if $\{A_i : i \in \omega\}$ is a uniformly c.e. sequence of c.e. sets (meaning that the relation “$x \in A_i$,” in i, x, is c.e.), $B$ is a c.e. set, and the relation “$x \in A_i$” is $\leq_T B$, then there is a f.p. semigroup $S$ with the following three properties:
(1) there is an effective correspondence $w_i \leftrightarrow A_i$ between a c.e. set $\{w_i : i \in \omega\}$ of words and $\{A_i : i \in \omega\}$ so that, effectively in i, one can find Turing reductions establishing $[w_i]_{=_{\omega}} \equiv_T A_i$; (2) the Turing degrees of the various classes $[w_i]_{=_{\omega}}$ consist of the least Turing degree, together with all finite joins of the various degrees $\text{deg}_e(A_i)$; (3) $\equiv_{\omega} \equiv_T B$ [34].

The next theorem will provide a negative answer to the question by Gao and Gerdes.

Theorem 3.7 Suppose that $\{E_i : i \in \omega\}$ is a uniformly c.e. sequence of ceers such that the set $\{i : E_i \text{ is finite}\}$ is c.e. Then there exists an infinite ceer $E$ such that for every i, $E_i \not\leq E$ or $E_i$ is finite (i.e., $E_i$ has finitely many classes). In particular, for every i, $E_i \not\equiv E$.

Proof. Let $V = \{i : E_i \text{ is finite}\}$ be c.e., and let $\{V_s : s \in \omega\}$ be a c.e. approximation to $V$, that is, a strong array of finite sets, with $V_s \subseteq V_{s+1}$ for every $s$, and $V = \bigcup_s V_s$. Let also $\{\varphi_j : j \in \omega\}$ be an acceptable indexing of the partial computable functions.

Our desired ceer must satisfy the following requirements:

- $P_n$: $E$ has at least $n + 1$ classes,
- $Q_{(i,j)}$: $E_i \not\leq E$ via $\varphi_j$, or $E_i$ is finite.

We order the requirements according to the priority ordering:

$P_0 < Q_0 < \ldots < P_n < Q_n < \ldots .$

We say that $R$ has higher priority than $R'$ (or $R'$ has lower priority than $R$) if $R < R'$. We construct $E$ in stages. At stage $s$ we define an equivalence relation $E_s$, so that: $E_0 = \text{Id}$ (the identity ceer); for every $s$, $E_s \subseteq E_{s+1}$. $E_s$
is a finite extension of $\text{Id}$ (the identity cease) and, uniformly in $s$, $E_i \setminus \text{Id}$ can be presented by its canonical index; and finally $E = \bigcup_i E_i$ is our desired equivalence relation. $E_{i+1}$ will be generated by $E_i$ plus finitely many pairs of numbers which are, as we say, $E$-collapsed at $s + 1$.

The strategy to satisfy $P_n$ consists in picking $n + 1$ numbers which are still pairwise $E$-non-equivalent, and restraining their equivalence classes from future $E$-collapses.

The strategy to satisfy $Q_{(i, j)}$ goes as follows. At a given stage $s$, we say that \textit{evidence appears that } $\varphi_j$ \textit{is not a reduction from } $E_i$ \textit{to } $E$ \textit{if one of the following happens:}

(A) $\varphi_j$ does not look total, i.e., we see some witness $x$ such that $\varphi_j$ is still undefined on $x$;

(B) we see two witnesses $y$, $w$ such that $\varphi_j(y)$ and $\varphi_j(w)$ both converge, and at the given stage $x \not\in E_i$ $y$, but already $\varphi_j(x) \not\in E \varphi_j(y)$.

Notice that, contrary to what one may expect, we do not bother to seek evidence given by two witnesses $x, y$ such that $\varphi_j(x)$ and $\varphi_j(y)$ both converge, and at the given stage $x \not\in E_i$ $y$, but $\varphi_j(x) \not\in E \varphi_j(y)$. Indeed, our action on trying to meet $Q_{(i, j)}$ will force the opponent to give up on totality of $\varphi_i$, or leave non-$E_i$-equivalent two numbers whose $\varphi_j$-images we have already $E$-collapsed.

Notice that, independently of our will, evidence due to (A) may be lost at a later stage $t$, if $\varphi_{j,t}(v) \not\downarrow$; evidence due to (B) may be lost at a later stage $t$ if $x \not\in E_{i,t}$ $y$.

Here is the description of our strategy in isolation:

(1) we wait to see $i \in V$; if $i$ gets enumerated into $V$ then the requirement is satisfied, so we stop worrying about it, and definitively move on to satisfy the lower priority requirements;

(2) while waiting to see $i \in V$ or for evidence to appear that $\varphi_j$ is not a reduction as in (B), we threaten to make $E$ finite by $E$-collapsing all the elements $> m$, where $m$ is a threshold indicated to $Q_{(i, j)}$ by the restraint placed by higher priority requirements;

(3) while waiting to see $i \in V$, if evidence has appeared that $\varphi_j$ is not a reduction as in (B) then

(a) while this evidence persists, we move on to satisfy the lower priority requirements;

(b) when this evidence gets lost, we loop back to (2).

The outcomes of the strategy are evident: moving out of (1) is a finitary outcome satisfying the requirement, as $E_i$ is finite.

If (1) does not show up, then we claim that we cannot loop between (3b) and (2) infinitely often. For otherwise $\varphi_j$ would be total, $E$ finite (as we $E$-collapse all $x, y > m$), but then $\varphi_j$ cannot be an injective reduction from the equivalence classes of $E_i$ (which is infinite) to the equivalence classes of $E$ (which would be finite). Therefore our strategy eventually stops at (3b) because of (A) (outcome: $\varphi_j$ is not total), or because of (B) (outcome: $x \not\in E_i$ $y$ but $\varphi_j(x) \not\in E \varphi_j(y)$ for some $x, y$).

Since all strategies have finite outcomes, the conflicts between different strategies are resolved by a straightforward finite priority argument.

\textbf{The construction.} At each stage, requirements may be initialized, and they are so at stage 0; or, in case of $Q$-requirements, they may be declared permanently satisfied in which case they are met once and for all.

The construction makes use at each stage of the following parameters for every requirement $R$: if $R$ is initialized, then these parameters are undefined. The parameter $m^R(s)$ denotes the restraint imposed at stage $s$ by $R$, with $R \in \{P, Q\}$, to lower priority requirements, so that they can only $E$-collapse pairs of elements $x, y > m^R(s)$.

The parameter $M((i, j), s)$ (if $Q_{(i, j)}$ is not initialized, and thus we may suppose $s > 0$), is defined as follows: if there is $v \leq s$ such that either

1. $\varphi_{j,v}(v) \not\downarrow,$ or
2. $v = (x, y)$ and $\varphi_{j,v}(x)$ and $\varphi_{j,v}(y)$ both converge and $x \not\in E_{i,v}$ $y$, but $\varphi_{j,v}(x) E_{i,v-1} \varphi_{j,v}(y)$.

then let $M((i, j), s) = (v, 0)$ in the former case, otherwise $M((i, j), s) = (v, 1)$. Let $M((i, j), s) = (0, 2)$ if there exists no such $v$.

If not otherwise specified, at each stage $s > 0$ each parameter maintains the same value as at the previous stage, or stays undefined if it was undefined at the previous stage.
We say that \( P_n \) requires attention at \( s + 1 \) if it is initialized. We say that \( R = Q_{l,j} \) requires attention at \( s + 1 \) if \( Q_{l,j} \) has not as yet been declared permanently satisfied and (in order):

1. \( Q_{l,j} \) is initialized; or
2. \( i \in V; \) or
3. \( M(0, j, s + 1) \neq M(0, j, s). \)

**Stage 0:** Initialize all requirements, and set \( m^R(k, 0) \) and \( M(k, 0) \) undefined for all \( R \in \{P, Q\} \), and \( k \in \omega \). Let \( E_0 = \text{Id.} \)

**Stage \( s + 1 \):** Let \( R \) be the least requirement that requires attention; there is such a least requirement since almost all requirements are initialized when we begin stage \( s + 1 \).

**Case 1:** If \( R = P_n \) then \( R \) is initialized: pick the least \( n + 1 \) numbers bigger than any number so far used in the construction (thus these numbers are still non-\( E \)-equivalent) and let \( m^R(s + 1) \) be the greatest one of the numbers which have been picked; \( R \) stops being initialized.

**Case 2:** Suppose that \( R = Q_{l,j} \). We refer to the various cases for which \( R \) may require attention:

1. (Case (1) of requiring attention) let \( m^R(s + 1) = \max\{m^R(s) : R < R \} \) (notice that no \( R' < R \) is initialized), so that \( R \) stops being initialized;
2. (Case (2) of requiring attention) declare \( R \) permanently satisfied (and will stay so forever);
3. (Case (3) of requiring attention) \( E \)-collapse all \( x, y \) such that \( m^R(s + 1) < x, y \leq s \);

Whatever the case, initialize all \( R' > R \), by setting \( m^R(s + 1) \uparrow \), and \( M(0, j, s) \uparrow \) if \( Q_{l,j} > R \).

Let \( E_{s+1} \) be the equivalence relation generated by \( E_s \) plus the pairs of numbers which have been \( E \)-collapsed at \( s + 1 \).

**Verification.** The verification is based on the following lemma.

**Lemma 3.8** For every requirement \( R \), \( R \) is initialized only finitely many times, \( m^R = \lim, m^R(s) \) exists, \( R \) eventually stops requiring attention, and \( R \) is met.

**Proof.** Suppose that the claim is true of every \( R' < R \), and let \( s_0 \) be the greatest stage at which some \( R' < R \) has received attention, with \( s_0 = 0 \) if \( R = P_0 \). Let \( m = \max\{m^R : R' < R \} \).

At the beginning of stage \( s_0 + 1 \), \( R \) is initialized, and thus requires attention, acts through Case 1 or Case 2a, and after this stage it will never be initialized again.

**Case 1:** Suppose that \( R = P_n \). If \( R = P_n \), then \( R \) acts, picks \( n + 1 \) unused numbers. These numbers are still \( E \)-non-equivalent, \( R \) defines a value of \( m^R(s_0 + 1) \) which will never change hereafter, and thus is the limit value of \( m^R(s) \). This limit value sets a restraint on lower priority requirements which therefore can never \( E \)-collapse any pair of these \( n + 1 \) numbers.

**Case 2:** Suppose that \( R = Q_{l,j} \). At stage \( s_1 + 1 \), \( Q_{l,j} \) defines the last value \( m^R = m^R(s_0 + 1) \) of its parameter \( m^R \); notice that this value will never change again, and is in fact the same as \( m^R \), where \( R' \) is the \( P \)-requirement immediately preceding \( R \) in the priority ordering. If \( R \) receives attention at some stage \( s_1 + 1 > s_0 + 1 \) and acts through Case (2b), then the action declares \( R \) permanently satisfied, \( R \) will never receive attention again, \( E_i \) is finite then \( R \) is met.

If we exclude action Case (2b) after \( s_0 + 1 \), then \( E_i \) is infinite. We claim that still \( R \) requires attention finitely many times after \( s_0 + 1 \). For otherwise, at infinitely many stages \( s \) we \( E \)-collapse all numbers \( m^R < x, s \leq s \), and therefore \( E \) is finite since we \( E \)-collapse all \( x, y > m^R \). On the other hand \( E_i \) is infinite, so \( \varphi_i \) cannot induce a 1-1 mapping from \( E_i \)-equivalence classes to \( E \)-equivalence classes, thus eventually \( M((0, 1, j), s) \) stabilizes on a value \( (v, k) \) with \( k \in [0, 1] \) and stops receiving attention again: contradiction. So (if we never act through Case (2b)) we are forced to conclude that \( M((0, 1, j), s) \) stabilizes on some \( \langle y, 0 \rangle \), and thus \( \varphi_i \) is not total, \( R \) is satisfied, and \( \lim, m^R(s) = m^R \); or it stabilizes on some \( \langle x, y \rangle, 1 \), in which case \( x E y \) and \( \varphi_i(x) E \varphi_i(y) \), and \( R \) is met.

**Corollary 3.9** No class \( \mathfrak{A} \) of finitely generated semigroups is \( \equiv \)-complete for the ceers.

**Proof.** Up to computable isomorphisms, we can assume that a finitely generated c.e.p. semigroup is of the form \( \langle \{0, 1, \ldots, n\}, R \rangle \) where \( R \) is a c.e. subset of \( \langle \{0, 1, \ldots, n\}^* \rangle \). Let \( f \) be a computable function such that \( \{V_{j,1}, \ldots, V_{j,n} : j \in \omega \} \) computably lists all c.e. subsets of \( \langle \{0, 1, \ldots, n\}^* \rangle \). From this we get a computable listing \( \langle S_{j,1} : n, i \in \omega \rangle \) (where \( S_{j,1} = \langle \{0, 1, \ldots, n\}^*, V_{j,1} \rangle \)) of all finitely generated c.e.p. semigroups, and a corresponding computable listing \( \langle E_{j,1} : n, i \in \omega \rangle \) of their word problems.
In view of the previous theorem it suffices to show that \( \{ \langle n, i \rangle : E_{(n,i)} \text{ is finite} \} \) is c.e. Let \( X = \{ 0, 1, \ldots, n \} \). We claim that \( E_{(n,i)} \) is finite if and only if

\[
\exists m > 0 \forall \sigma \in X^+ [ |\sigma| = m \implies \exists \tau \in X^+ [ |\tau| < |\sigma| \& \tau E_{(n,i)} \sigma ]],
\]

which is a c.e. expression (in which for a given string \( \varrho \), the symbol \( |\varrho| \) denotes the length of \( \varrho \)). On the one hand, if \( E_{(n,i)} \) is finite, one can fix a finite transversal \( A \) which meets all the equivalence classes of \( E_{(n,i)} \). Since each word of \( S_{(n,i)} \) is equivalent to a word from \( A \), we have that (\( \ast \)) holds for \( m \), where \( m = \max |\sigma| : \sigma \in A \).

On the other hand, assume that (\( \ast \)) holds, and fix such an \( m \). We claim in this case that every word is \( E_{(n,i)} \)-equivalent to some word of length \( \leq m \). Towards a contradiction, let \( n > m \) be the least number such that there exists \( \sigma \) with \( |\sigma| = n \), and \( \{\sigma\}_{E_{(n,i)}} \) contains no word of length \( \leq m \). Now, let \( \sigma_0 = \sigma | m \) (i.e., the initial segment of \( \sigma \) of length \( m \)), and let \( \sigma_1 \) be such that \( \sigma = \sigma_0 \sigma_1 \). Then \( \sigma_0 \) is \( E_{(n,i)} \)-equivalent to some \( \varrho \) with \( |\varrho| < m \). Therefore, by definition of \( \langle X; E_{(n,i)} \rangle \), we have that \( \sigma = \sigma_0 \sigma_1 \) \( E_{(n,i)} \) \( \varrho \sigma_1 \), but \( |\varrho \sigma_1| < n \), contradicting the minimality of \( n \).

As a particular case, this provides a negative solution to the question by Gao and Gerdes:

**Corollary 3.10** The class of f.p. semigroups is not \( \equiv \)-complete for the ceers.

**Proof.** Immediate.

Next, we observe that, given a f.p. semigroup \( S \), the number of finite and infinite equivalence classes of the word problem \( \equiv_S \) gives us some information about the ceers realized by \( S \). We basically owe the following arguments to [7]; cf. also [25].

**Lemma 3.11** If \( S \) is a f.p. semigroup then there is a partial computable function \( \psi \) such that for every word \( w \), \( \psi(w) \downarrow \) if and only if the \( \equiv_S \)-equivalence class of \( w \) is finite, and, when convergent, \( \psi(w) \) outputs the canonical index of the equivalence class \( \{ w \} \equiv_S \) of \( w \).

**Proof.** Given a word \( w \), we can effectively generate its \( \equiv_S \)-equivalence class in a treelike fashion as follows. The root of the tree is \( w \). Each node \( u \) has as children the words that can be obtained from \( u \) using the relations and which have not yet appeared as a node in the path from the root to the present node. Note that one relation produces only finitely many children, and there are only finitely many relations: hence, this is a finitely branching tree. By König’s Lemma if the equivalence class of \( w \) is finite, we eventually stop generating new nodes on any branch of the tree: when this happen we have generated the entire equivalence class of \( w \), and we can compute the canonical index of this class.

**Theorem 3.12** Let \( S \) be a f.p. semigroup.

(i) If \( S \) has finitely many infinite equivalence classes, then \( \equiv_S \) is decidable. Therefore no undecidable ceer can be \( \equiv \)-realized by such an \( S \).

(ii) If \( S \) has infinitely many finite equivalence classes, then \( \equiv_S \) is light. Therefore, neither finite nor dark ceers can be \( \equiv \)-realized by such an \( S \).

**Proof.** Suppose that \( \equiv_S \) has only finitely many infinite equivalence classes. Assume that \( \{ v_i : i \in I \} \) is a finite set of words, with \( v_i \neq S v_j \) if \( i \neq j \), spanning these infinite equivalence classes. Given words \( x, y \), generate the equivalence classes of \( x \) and \( y \) in a tree-like fashion as in the proof of Lemma 3.11, until one of the following happens: (1) \( \{ x \} \equiv_S \) and \( \{ y \} \equiv_S \) cannot grow any more (and we can decide this, as explained in the proof Lemma 3.11); (2) some \( v_i \) is generated in one equivalence class, and some \( v_j \) is generated in the other one; (3) some \( v_i \) is generated in one of the two equivalence classes and the other one has stopped (again, we can decide this latter outcome). In any case we can decide if the two words are equal. This proves statement (i).

Now, we prove (ii). Let \( S \) be a f.p. semigroup with infinitely many finite equivalence classes and let \( \psi \) be the partial computable function of Lemma 3.11. Using \( \psi \) we can build in stages an infinite c.e. transversal \( \{ a_0, a_1, \ldots \} \) for \( \equiv_S \).

**Step 0.** Let \( w_0 \) be the first word such that \( \psi(w_0) \downarrow \) and define \( a_0 \) to be the least element of the finite set \( D_{\psi(w_0)} \).

**Step \( n + 1 \).** Let \( w_{n+1} \) be the first word such that \( \psi(w_{n+1}) \downarrow \) and \( D_{\psi(w_{n+1})} \cap ( \bigcup_{j \leq n} D_{\psi(w_j)} ) = \emptyset \) and let \( a_{n+1} \) be the least element of \( D_{\psi(w_{n+1})} \).
4 Classes of algebras $\simeq_s$-realizing provable equivalence of Peano Arithmetic

Although by Fact 3.6 there are ceers $R$ such that $R \not\cong =_G$, for every c.e. group $G$, it is known that there are f.p. groups $G$ such that $=_G$ is universal. This was first proved by Miller III [26]. Another example, due to [29] refers to the computability theoretic notion of effective inseparability. We recall that a disjoint pair $(U, V)$ of sets of numbers is **effectively inseparable** (e.i.) if there exists a partial computable function $\psi$ such that for each pair $(u, v)$, if $U \subseteq W_u$ and $V \subseteq W_v$ and $W_u \cap W_v = \emptyset$ then $\psi(u, v)$ converges and $\psi(u, v) \notin W_u \cup W_v$. A f.p. group $G$ is built in [29] such that $=_G$ is **uniformly effectively inseparable**, i.e., uniformly in $x, y$ one can find an index of a partial recursive function $\psi$ witnessing that the pair of sets $([x]_{=_G}, [y]_{=_G})$ is e.i., if $[x]_{=_G} \cap [y]_{=_G} = \emptyset$. Such a f.p. group has universal word problem, since it is known that every uniformly effectively inseparable ceer is universal [3].

An important $\simeq_s$-type among the universal ceers is given by the $\simeq_s$-type of the relation $\sim_T$ of provable equivalence of any consistent formal system $T$ extending Robinson’s systems Q or R (cf., e.g., [35] for an introduction to formal systems of arithmetic), i.e., $x \sim_T y$ if (identifying sentences with numbers through a suitable Gödel numbering) $T \vdash x \leftrightarrow y$. For example, let us take $T$ to be Peano Arithmetic.

The question naturally arises as to which algebras $\simeq_s$-realize $\sim_T$. Notice that by Fact 1.9, “$\simeq_s$-realizing $\sim_T$” is equivalent to “$\simeq_s$-realizing $\sim_T$”. Here are some initial remarks about this question:

1. As far as we know, the question of whether there are f.p. semigroups, or f.p. groups, having word problems strongly isomorphic to $\sim_T$ is still open.
2. On the other hand, by Theorem 2.4 there exist c.e. semigroups whose word problem is strongly isomorphic to $\sim_T$. We do not know if there are c.e. groups $\simeq_s$-realizing $\sim_T$.
3. If one computably identifies with numbers the sentences of our chosen formal system $T$, and considers the computable operations provided by the connectives $\land, \lor, \neg, \top, \bot$ (where $\bot$ and $\top$ denote any contradiction and any theorem, respectively), then $\langle \omega, \land, \lor, \neg, \top, \bot, E \rangle$ (where $x E y$ if $T \vdash x \leftrightarrow y$) is a positive presentation of the Lindenbaum algebra of the sentences of $T$, which is therefore a c.e. Boolean algebra. It is known that the word problem of this c.e. Boolean algebra is strongly isomorphic to $\sim_T$: cf. [31] (cf. also [28]).

The above item (3) identifies a very special class of rings which $\simeq_s$-realize $\sim_T$, namely Boolean rings, i.e., rings satisfying $x^2 = x$ for all $x$. Is that all? Can we find non-Boolean rings $\simeq_s$-realizing $\sim_T$? We shall identify in the following a c.e. ring $R$ which is neither Boolean nor commutative, such that $=_R \simeq_s \sim_T$.

The $\simeq_s$-type of $\sim_T$ can be characterized through the already given notion of a diagonal function, and the notion of **uniformly finite precompleteness**, originating from [27] (cf. also [33]).

**Definition 4.1** A nontrivial ceer $S$ is **uniformly finitely precomplete** (abbreviated as u.f.p.) if there exists a computable function of three variables $f(D, e, x)$ (where $D$ is a finite set given by its canonical index) such that

$$(\forall D, e, x)[\varphi_e(x) \downarrow \land (\exists y)[y \in D \land \varphi_e(x) S y]] \implies \varphi_e(x) S f(D, e, x)].$$

Then we have:

**Fact 4.2** For every ceer $S$, $S \simeq_s \sim_T$ if and only if $S$ is u.f.p. and possesses a diagonal function.

The rest of the section is devoted to seeing that there is a non-commutative and non-Boolean c.e. ring whose word problem is strongly isomorphic to $\sim_T$. The following result is essentially a rephrasing of Theorem 4.1 of [5].

**Lemma 4.3** Let $A$ be a c.e. algebra whose type contains two binary operations $\cdot, \cdot, \cdot$ and two constants $0, 1$ such that $\cdot$ is associative, the pair $(U_0, U_1)$ is e.i., where $U_i = \{x : x =_A i\}$, and, for every $a, a + 0 =_A a$, $\cdot 0 =_A 0$, and $a \cdot 1 =_A a$. Then $=_A$ is a u.f.p. ceer.

**Proof.** For the convenience of the reader, we recall the argument in [5], adapting it to our context and notations. We look for a computable function $f(D, e, x)$ such that if $\varphi_e(x) \downarrow$, and $\varphi_e(x) =_A d$ for some $d \in D$ then $f(D, e, x) =_A \varphi_e(x)$.

Let $p$ be a productive function for the pair $(U_0, U_1)$; it is well known that we may assume that $p$ is total. Let $\{u_d : d \in D\}$ be a computable set of indices we control by
the Recursion Theorem. For a pair \((u_d, d, e, x, v_d, d, e, x)\) in this set let \(c_{d, d, e, x} = p(u_d, d, e, x, v_d, d, e, x)\) and \(a_{d, d, e, x} = d \cdot c_{d, d, e, x}\). Define \(f(D, e, x) = \sum_{d \in D} a_{d, d, e, x}\).

Let us define two c.e. sets \(W_{u_d, d, e, x}\) and \(W_{v_d, d, e, x}\), for each \(d \in D\), which are computably enumerated as follows. Wait for \(\psi_e(x)\) to converge to some \(y\) which is \(\rightarrow\) to some element in \(D\), and while waiting, we let \(W_{u_d, d, e, x}\) and \(W_{v_d, d, e, x}\) enumerate \(U_0\) and \(U_1\), respectively. If we wait forever then for all \(d \in D\) we end up with \(W_{u_d, d, e, x} = U_0\) and \(W_{v_d, d, e, x} = U_1\). If the wait terminates, let \(d_0 \in D\) be the first seen so that \(\psi_e(x) = \rightarrow d_0\), enumerate also \(c_{d_0, d, e, x}\) into \(W_{u_d, d, e, x}\); this ends up with \(W_{u_d, d, e, x} = U_0 \cup \{c_{d_0, d, e, x}\}\) and \(W_{v_d, d, e, x} = U_1\), thus forcing \(c_{d_0, d, e, x} = 1\) (since it must be that \(W_{u_d, d, e, x} \cap W_{v_d, d, e, x} \neq \emptyset\), for otherwise \(c_{d_0, d, e, x} = p(u_d, d, e, x, v_d, d, e, x) \in W_{u_d, d, e, x} \cup W_{v_d, d, e, x}\), a contradiction) and thus \(a_{d_0, d, e, x} = \rightarrow d_0\). For all \(d \in D\) with \(d \neq d_0\), we let \(W_{u_d, d, e, x} = U_0\) and \(W_{v_d, d, e, x} = U_1\) \(\cup \{c_{d, d, e, x}\}\); this forces \(c_{d_0, d, e, x} = \rightarrow 0\) and thus \(a_{d, d, e, x} = \rightarrow 0\) for each such \(d\). Therefore \(f(D, e, x) = \sum_{d \in D} a_{d, d, e, x} = \rightarrow d_0 = \rightarrow \psi_e(x)\).

In order to prove the existence of a ring with the desired properties, let us first recall the notion of free ring. For more details on the following construction, cf., e.g., [15, IV.2].

Let \(R\) be a ring and \(M\) be a monoid. The monoid ring of \(M\) over \(R\), denoted \(RM\), is the set \(\{\psi : M \to R : \text{supp}(\psi) \text{ is finite}\}\), where \(\text{supp}(\psi) = \{m \in M : \psi(m) \neq 0\}\), equipped with the following operations. Given \(\psi, \psi' \in RM\), their sum is the function \(\psi + \psi' : M \to R\) given by \((\psi + \psi')(m) = \psi(m) + \psi(m)\), and their product is the function \(\psi \psi' : M \to R\) given by \((\psi \psi')(m) = \sum \psi(h) \psi(k)\).

**Remark 4.4** Equivalently, as is easily seen, \(RM\) is the set of formal sums \(\sum_{m \in M} r_m m\), where \(r_m \in R, m \in M\) and \(r_m = 0\) for all but finitely many \(m\), equipped with coefficient-wise sum, and product in which the elements of \(R\) commute with the elements of \(M\).

**Definition 4.5** The free ring on a set \(X\) (denoted \(\mathbb{Z}X^*\)) is the monoid ring of the free monoid \(X^*\) over the ring \(\mathbb{Z}\) of the integers.

**Theorem 4.6** There exist non-commutative (and hence non-Boolean) c.e. rings \(R\) satisfying that \(\approx R\mathbb{Z}X^*\).

**Proof.** Assume that \(X = \{x_i : i \in \omega\}\) is a decidable set and consider the free ring \(R = \mathbb{Z}X^*\). Notice that, up to coding, we can identify the universe of \(R^*\) with \(\omega\) and assume that its operations are computable and equality is decidable.

Let \(U, V \subseteq \mathbb{N}\) be an e.i. pair of c.e. sets, and consider the ideal \(K\) of \(\mathbb{Z}X^*\) generated by \(\{x_j : i \in U\} \cup \{1 - x_j : j \in V\}\). Thus, any element of \(K\) is of the form

\[
\sum_{i \in U} r_i x_i^V q_i + \sum_{j \in J} s_j x_j^V (1 - x_j^V) v_j.
\]

(\dagger)

where each of \(r_i, s_j\) is in \(\mathbb{Z}\), and each of \(\tau_i, q_i, \mu_j, v_j\) is in \(X^*\), and finally \(I \subseteq U\) and \(J \subseteq V\) are finite sets. Up to shrinking the sets of indices, we can suppose that no further simplification can be made in either sum.

The ideal \(K\) gives rise to a congruence, which we still denote with \(K\), such that \([0]_K = K\). We claim that \(1 \notin K\), which implies \([0]_K \cap [1]_K = \emptyset\). To see that our claim is true, we show in fact that no nonzero integer can be written as in (2). Calculating we get

\[
\sum_{i \in U} r_i x_i^V q_i + \sum_{j \in J} s_j x_j^V v_j - \sum_{j \in J} s_j x_j^V v_j - \sum_{j \in J} s_j x_j^V v_j,
\]

(\dagger\dagger)

where \(J_0 = \{j \in J : \mu_j v_j \neq \lambda\}\) and \(J_1 = \{j \in J : \mu_j v_j = \lambda\}\). By our assumptions, neither the first, nor the third, nor the last sum of (3) contain any pair of like monomials, so that in these sums no further simplification can be made. In order to get a nonzero integer \(s\) from this sum we must have that

\[
0 = \sum_{i \in U} r_i x_i^V q_i + \sum_{j \in J_0} s_j x_j^V v_j - \sum_{j \in J_1} s_j x_j^V v_j.
\]

(\dagger\dagger)

\[
\sum_{j \in J_1} s_j = s, \text{ and } J_1 \neq \emptyset.
\]

We are going to see that the assumption \(J_1 \neq \emptyset\) leads to a contradiction, by showing that there would be an infinite sequence \(\alpha_n = s \alpha_n (n \geq 1)\), with \(\alpha_n \in \{x_j : j \in V\}^*\) of length \(n\), and \(s \in \mathbb{Z}\setminus\{0\}\), such that each \(\alpha_n\) occurs as a summand in the sum of (4).

Take \(j \in J_0\) and let \(s = s_j\). So \(-s_j x_j^V\) occurs in the fourth sum of (4). To cancel the monomial \(s_j x_j^V\) in (4), there must be a monomial of the form \(s_j x_j^V\) (hence from the second sum) such that \(\mu_j v_j = x_j^V\). Let \(s_1 = x_j^V\), and
\(\alpha_1 = s\sigma_1\). So \(\alpha_1\) satisfies the claim. Now suppose that we have found already \(\alpha_n = s\sigma_n\) in the second sum and satisfying the claim. Then \(\sigma_n\) is of the form \(\mu_j, v_j\) which (via multiplication \(s\mu_j, (1 - x_j) v_j\)) in (2) corresponds to an summand in the third sum \(-s\mu_j, x_j v_j\), so that \(\sigma_{n+1} = \mu_j, x_j v_j\) has length \(n + 1\), and lies in \([x_j : j \in V]^*\).

Again, this cannot cancel with anything in the first sum, for each summand in the first sum contains an element indexed from \(U\); it cannot cancel with anything in the fourth sum, nor can it cancel with anything in the third sum, because we have assumed that it does not contain like monomials; so it must cancel with something in the second sum, which therefore contains \(\alpha_{n+1} = s\sigma_{n+1}\) satisfying the claim.

**Lemma 4.7** \((U, V) \leq_m ([0]_K, [1]_K), \) hence the pair \(([0]_K, [1]_K)\) is e.i.

**Proof.** We want to show that \((U, V) \leq_m ([0]_K, [1]_K)\) via \(f(i) = x_i\). Thus we must verify that \(x_i \in [0]_K\) if and only if \(i \in U\) and \(x_j \in [1]_K\) if and only if \(j \in V\). The facts that \(i \in U\) implies \(x_i \in [0]_K\) and \(j \in V\) implies \(x_j \in [1]_K\) are obvious.

On the other hand, if \(x_i \in [0]_K\), then \(x_i\) must be of the form \((\dagger)\), from which we obtain again the expression \((\dagger\dagger)\), with the same assumptions on already done simplifications. Assume that there is an \(x_j \in [0]_K\) with \(i \notin U\). Since no nonzero integer must appear, either \(J_1 = \varnothing\) or \(J_1\) has at least two elements. Assume the latter. Then in the last sum there is a monomial \(-sx_i^j\) which must cancel with a like monominal, which can be nowhere but in the second sum. But the existence of such a monomial implies that there is a monomial of the form \(sx_i^j x_j^1\) or \(sx_i^j x_j^1 - j\) which in turn leads to a contradiction, by an argument similar to the one above. Thus \(J_1\) must be empty. Now assume \(J_0\) is non-empty, so that there is \(j \in J_0\) with \(\mu_j, v_j = x_j\), where \(i \notin U\). But then in the third sum there must be a corresponding monomial \((-sx_j^i\) or \(-x_j^1 x_i\), whose existence, by reasoning as in the argument used to see that no nonzero integer lies in \(K\), leads again to a contradiction.

Since \(x_j \in [1]_K\) if and only if \(1 - x_j \in [0]_K\), a completely similar argument shows that \(x_j \in [1]_K\) implies \(j \in V\).

Effective inseparability of the pair \(([0]_K, [1]_K)\) follows from the fact that \((U, V)\) is e.i., and effective inseparability is a \(\leq_m\)-upwards closed property. \(\square\)

Consider the ring \(R\) obtained by dividing \(R^-\) by the congruence \(K\). \(R\) is a c.e. ring according to Definition 1.1, as it can be positively presented as \((\omega, F, E)\) where we effectively identify modulo coding \(R^-\) with \(\omega\), \(F\) is the set of computable operations on \(\omega\) which correspond via coding to the operations of \(R^-\), and \(E\) is the cee induced on \(\omega\) by the congruence \(K\).

Moreover, \(R\) is equipped with two binary operations \(+, \cdot\) (which are its ring binary operations) and two constants \(0, 1\) (again, its ring zero-ary operations). Therefore \(=_{R}\) is a u.f.p. cee according to Lemma 4.3. To conclude that \(=_{R}\) is strongly isomorphic to \(\sim_{F}\) is then enough by Fact 4.2 that we find a diagonal function for \(=_{R}\). For this, just take any \(v \neq 0\), and consider the function \(d'(u) = u + v\). It immediately follows that \(d(u) \neq_{R} u\), for otherwise \(v =_{R} 0\). \(\square\)

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