A P-ADIC STRUCTURE WHICH DOES NOT INTERPRET AN INFINITE FIELD BUT WHOSE SHELAH COMPLETION DOES

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Abstract. We give a $p$-adic example of a structure whose Shelah completion interprets $\mathbb{Q}_p$ but which does not (provided an extremely plausible conjecture holds) interpret an infinite field. In the final section we discuss the significance of such examples for a possible future geometric theory of NIP structures.

1. Introduction

In [21, 20] we described NIP structures $H$ such that $H$ does not interpret an infinite field but the Shelah completion of $H$ does. Here we describe a more natural example, modulo a reasonable conjecture. Fix a prime $p$ and let $K$ be a $(2^{\aleph_0})^+$-saturated elementary extension of $\mathbb{Q}_p$. Let $\text{Val}_p: K^\times \to \Gamma$ be the $p$-adic valuation on $K$ and $V$ be the valuation ring of $\text{Val}_p$, recall that $\text{Val}_p$ is $K$-definable. Given $a \in K$ and $t \in \Gamma$ let $B(a, t)$ be the ball with center $a$ and radius $t$, i.e. the set of $a' \in K$ such that $\text{Val}_p(a - a') \geq t$. Let $\mathbb{B}$ be the set of balls in $V$. Let $\Gamma_\geq$ be the set of nonnegative elements of $\Gamma$. Let $\sim$ be the equivalence relation on $V \times \Gamma_\geq$ where $(a, t) \sim (a', t')$ if and only if $B(a, t) = B(a', t')$. Identify $\mathbb{B}$ with $(V \times \Gamma_\geq)/\sim$, consider $\mathbb{B}$ to be a $K$-definable set of imaginaries, and let $\mathcal{B}$ be the structure induced on $\mathbb{B}$ by $K$.

Theorem 1.1. The Shelah completion of $\mathcal{B}$ interprets $\mathbb{Q}_p$.

Conjecture 1 is a well-known and well-believed folklore conjecture. Conjecture 1 is beyond the reach of current techniques, but its failure would be a huge surprise. The analogue of Conjecture 1 for ACVF is a result of Hrushovski and Rideau [9].

Conjecture 1. Any infinite field interpretable in $K$ is $K$-definably isomorphic to some finite extension of $K$.

The proof of Theorem 1.2 is easy.

Theorem 1.2. If Conjecture 1 holds then $\mathcal{B}$ does not interpret an infinite field.

Suppose that $\mathcal{O}$ is a structure, $A$ is a subset of $M^n$, and $\tau: O \to A$ is a bijection. We say that $M$ trace defines $\mathcal{O}$ via $\tau$ if for every $\mathcal{O}$-definable $X \subseteq O^n$ there is an $M$-definable $Y \subseteq M^m$ such that

$$X = \{(a_1, \ldots, a_n) \in O^n : (\tau(a_1), \ldots, \tau(a_n)) \in Y\},$$

and $M$ trace defines $\mathcal{O}$ if $M$ trace defines $\mathcal{O}$ via some injection $\tau: O \to M^m$.

Theorem 1.3. $\mathcal{B}$ trace defines $\mathbb{Q}_p$.

Theorem 1.3 follows from Theorem 1.1 and Theorem 1.4.

Theorem 1.4. Let $\lambda$ be a cardinal. Suppose $M$ is $\lambda$-saturated and NIP. If $\mathcal{O}$ is interpretable in the Shelah completion of $M$ and $|O| < \lambda$ then $M$ trace defines $\mathcal{O}$.

Date: June 2, 2020.
2. NIP-theoretic background

2.1. Shelah completeness. Let $\mathcal{M}$ be a structure and $\mathcal{M} \prec \mathcal{N}$ be $|\mathcal{M}|^+\text{-saturated.}$ A subset $X$ of $\mathcal{M}^n$ is externally definable if $X = \mathcal{M}^n \cap Y$ for some $\mathcal{N}$-definable subset $Y$ of $\mathcal{N}^n$. An application of saturation shows that the collection of externally definable sets does not depend on choice of $\mathcal{N}$. Fact 2.1 is well-known and easy.

**Fact 2.1.** Suppose that $X$ is an $\mathcal{M}$-definable set and $<$ is an $\mathcal{M}$-definable linear order on $X$. Then any $<$-convex subset of $X$ is externally definable.

Fact 2.2 is arguably the most important result on externally definable sets at present. It is a theorem of Chernikov and Simon [2]. The right to left implication is a saturation exercise which does not require NIP.

**Fact 2.2.** Suppose that $\mathcal{M}$ is NIP and $X$ is a subset of $\mathcal{M}^n$. Then $X$ is externally definable if and only if there is an $\mathcal{M}$-definable family $(X_a : a \in \mathcal{M}^m)$ of subsets of $\mathcal{M}^n$ such that for every finite $A \subseteq X$ we have $A \subseteq X_a \subseteq X$ for some $a \in \mathcal{M}^m$.

We say that a structure is Shelah complete if every externally definable set is already definable. Note that $\mathcal{M}$ is Shelah complete if and only if all types over $\mathcal{M}$ are definable. It follows that a theory $T$ is stable if and only if every model of $T$ is Shelah complete. The Marker-Steinhorn theorem [12] shows that if $\mathcal{O}$ is an o-minimal expansion of a dense linear order $(\mathcal{O}, <)$ then $\mathcal{O}$ is Shelah complete if and only if $(\mathcal{O}, <)$ is a complete linear order. Results from dp-minimality show that $(\mathbb{Z}, +, <)$ is Shelah complete, see [19]. Fact 2.3 is a theorem of Delon [4].

**Fact 2.3.** $\mathbb{Q}_p$ is Shelah complete.

The Shelah completion $\mathcal{M}^{Sh}$ of $\mathcal{M}$ is the expansion of $\mathcal{M}$ by all externally definable sets. Fact 2.4 is a theorem of Shelah [16] and also a corollary to Fact 2.2.

**Fact 2.4.** If $\mathcal{M}$ is NIP then every $\mathcal{M}^{Sh}$-definable set is externally definable in $\mathcal{M}$.

Fact 2.5 is an easy corollary to Fact 2.4 we leave the proof to the reader.

**Fact 2.5.** The Shelah completion of a NIP structure is Shelah complete.

The Shelah completion is usually called the “Shelah expansion”. We believe that Fact 2.5 justifies our nonstandard terminology.

2.2. Dp-rank. We will use a few basic facts about dp-rank. See [18] for an overview of the dp-rank. We let $\text{dp-rk}_{\mathcal{M}} X$ be the dp-rank of an $\mathcal{M}$-definable set $X$. The dp-rank of $\mathcal{M}$ is defined to be $\text{dp-rk}_{\mathcal{M}} \mathcal{M}$. The first three claims of Fact 2.6 are immediate consequences of the definition of dp-rank. The fourth is essentially due to Onshuus and Usvyatsov [13].

**Fact 2.6.** Suppose $X, Y$ are $\mathcal{M}$-definable sets. Then

1. $\text{dp-rk}_{\mathcal{M}} < \infty$ if and only if $\mathcal{M}$ is NIP,
2. $\text{dp-rk}_{\mathcal{M}} X = 0$ if and only if $X$ is finite,
3. If $f : X \to Y$ is a $\mathcal{M}$-definable surjection then $\text{dp-rk}_{\mathcal{M}} Y \leq \text{dp-rk}_{\mathcal{M}} X$,
4. $\text{dp-rk}_{\mathcal{M}^{Sh}} X = \text{dp-rk}_{\mathcal{M}} X$ when $\mathcal{M}$ is NIP.

Fact 2.7 is a theorem of Dolich, Goodrick, and Lippel [5].

**Fact 2.7.** The dp-rank of $\mathbb{Q}_p$ is one.
3. \mathcal{B}

3.1. **Proof of Theorem 1.1.** We identify the minimal positive element of \( \Gamma \) with 1 so that \( \mathbb{Z} \) is the minimal non-trivial convex subgroup of \( \Gamma \). By Fact 2.1 \( \mathbb{Z} \) is a \( K^{\text{Sh}} \)-definable subset of \( \Gamma \). Let \( v : K^\times \to \Gamma / \mathbb{Z} \) be the composition of \( \text{Val}_p \) with the quotient \( \Gamma \to \Gamma / \mathbb{Z} \). We equip \( \Gamma / \mathbb{Z} \) with a group order by declaring \( a + \mathbb{Z} \leq b + \mathbb{Z} \) when \( a < b \), so \( v \) is a \( K^{\text{Sh}} \)-definable valuation on \( K \). Let \( W \) be the valuation ring of \( v \) and \( \mathfrak{m}_W \) be the maximal ideal of \( W \). So \( W \) is the set of \( a \in K \) such that \( \text{Val}_p(a) \geq m \) for some \( m \in \mathbb{Z} \) and \( \mathfrak{m}_W \) is the set of \( a \in K \) such that \( \text{Val}_p(a) > m \) for all \( m \in \mathbb{Z} \). It is easy to see that for every \( a \in W \) there is a unique \( a' \in \mathbb{Q}_p \) such that \( a - a' \in \mathfrak{m}_W \). We identify \( W / \mathfrak{m}_W \) with \( \mathbb{Q}_p \), so that the residue map \( W \to \mathbb{Q}_p \) is the usual standard part map \( \text{st} \). So \( \mathbb{Q}_p \) is a \( K^{\text{Sh}} \)-definable set of imaginaries.

Lemma 3.1 is routine and left to the reader.

**Lemma 3.1.** Suppose that \( X \) is a closed \( \mathbb{Q}_p \)-definable subset of \( \mathbb{Z}_p^n \) and \( X' \) is the subset of \( V^n \) defined by any formula defining \( X \). Then \( \text{st}(X') \) agrees with \( X \).

Given \( B \in \mathbb{B} \) such that \( B = B(a, t) \) we let \( \text{rad}(B) = t \). As \( \text{rad} : \mathbb{B} \to \Gamma_\geq \) is surjective and \( K \)-definable we consider \( \text{Rad} \) to be an imaginary sort of \( \mathbb{B} \) and \( \text{rad} \) to be an \( \mathcal{B} \)-definable function. Fix \( \gamma \in \Gamma \) such that \( \gamma > \mathbb{N} \). Let \( E \) be the set of \( B \in \mathbb{B} \) such that \( \text{rad}(B) = \gamma \), so \( E \) is \( \mathcal{B} \)-definable.

**Proof.** It suffices to show that \( \mathcal{B}^{\text{Sh}} \) interprets \((\mathbb{Z}_p, +, \times)\). Note that for any \( a \in V \) and \( b \in B(a, \gamma) \) we have \( \text{st}(a) = \text{st}(b) \). So we define a surjection \( f : E \to \mathbb{Z}_p \) by declaring \( f(B(a, \gamma)) := \text{st}(a) \) for all \( a \in V \). Let \( \approx \) be the equivalence relation on \( E \) where \( B_0 \approx B_1 \) if and only if \( f(B_0) = f(B_1) \). Note that for any \( B_0, B_1 \in \mathbb{B} \) we have \( B_0 \approx B_1 \) if and only if

\[
\{ B' \in \mathbb{B} : \text{rad}(B') \in \mathbb{N}, B_0 \subseteq B' \} = \{ B' \in \mathbb{B} : \text{rad}(B') \in \mathbb{N}, B_1 \subseteq B' \}.
\]

So \( \approx \) is \( \mathcal{B}^{\text{Sh}} \)-definable. Let \( f : E^n \to \mathbb{Z}_p^n \) be given by

\[
f(B_1, \ldots, B_n) = \left( f(B_1), \ldots, f(B_n) \right)
\]

for all \( B_1, \ldots, B_n \in E \).

Suppose that \( X \) is a \( \mathbb{Q}_p \)-definable subset of \( \mathbb{Z}_p^n \). We show that \( f^{-1}(X) \) is \( \mathcal{B}^{\text{Sh}} \)-definable. As \( X \) is \( \mathbb{Q}_p \)-definable it is a boolean combination of closed \( \mathbb{Q}_p \)-definable subsets of \( \mathbb{Z}_p^n \), so we may suppose that \( X \) is closed. Let \( X' \) be the subset of \( V^n \) defined by any formula defining \( X \). Let \( Y_0 \) be the set of \( B \in E \) such that \( B \cap X' \neq \emptyset \) and \( Y \) be the set of \( B \in E \) such that \( B \approx B_0 \) for some \( B_0 \in Y_0 \). Observe that \( Y_0 \) is \( \mathbb{B} \)-definable and \( Y \) is \( \mathcal{B}^{\text{Sh}} \)-definable. Note that \( Y_0 \) is the set of balls of the form \( B(a, \gamma) \) for \( a \in X' \).

We show that \( Y = f^{-1}(X) \). Suppose \( B(a, \gamma) \in f^{-1}(X) \). So \( \text{st}(a) \in X \). We have \( B(\text{st}(a), \gamma) \in Y_0 \) and \( B(\text{st}(a), \gamma) \approx B(a, \gamma) \), so \( B(a, \gamma) \in Y \). Now suppose that \( B(a, \gamma) \in Y \) and fix \( B(b, \gamma) \in Y_0 \) such that \( B \approx B_0 \). We may suppose that \( b \in X' \). As \( X \) is closed an application of Lemma 3.1 shows that \( \text{st}(b) \in X \). So \( \text{st}(a) = \text{st}(b) \in X \). So \( B(a, \gamma) \in f^{-1}(X) \). \( \square \)
3.2. **Proof of Theorem 1.2** We first prove an easy lemma.

**Lemma 3.2.** Any $K$-definable function $\mathcal{B} \to K^m$ has finite image.

*Proof.* If $f : \mathcal{B} \to K^m$ has infinite image then there is a coordinate projection $e : K^m \to K$ such that $e \circ f$ has infinite image. So we suppose $m = 1$. Recall that if $X$ is a $K$-definable subset of $V$ then $X$ is either finite or has interior. So it suffices to show that the image of any $K$-definable function $\mathcal{B} \to K$ has empty interior. Let $\mathcal{B}(\mathbb{Q}_p)$ be the set of $\text{Val}_{\mathbb{Q}_p}$-balls in $\mathbb{Z}_p$. It is enough to show that the image of any function $\mathcal{B}(\mathbb{Q}_p) \to \mathbb{Q}_p$ has empty interior. This holds as $\mathcal{B}(\mathbb{Q}_p)$ is countable and every nonempty open subset of $\mathbb{Z}_p$ is uncountable.

We now prove Theorem 1.2.

*Proof.* Suppose that Conjecture 1 holds and $\mathcal{B}$ interprets an infinite field $L$. By Conjecture 1 there is a $K$-definable bijection $L \to K^m$ for some $m$, so for some $k$ there is a $K$-definable surjection $\mathcal{B}^k \to K^m$. This contradicts Lemma 3.2.

Let $\mathcal{B}(\mathbb{Q}_p)$ be the structure induced on $\mathcal{B}(\mathbb{Q}_p)$ by $\mathbb{Q}_p$. Conjecture 1 implies that $\mathcal{B}(\mathbb{Q}_p)$ does not interpret an infinite field. By Fact 2.3 $\mathcal{B}(\mathbb{Q}_p)$ is Shelah complete. This is why we start with a proper elementary extension of $\mathbb{Q}_p$.

3.3. **Dp-rank of $E$.** The examples in [21, 20] are weakly o-minimal, hence dp-rank one. It is easy to see that the dp-rank of $\mathcal{B}$ is two, but $E$ is dp-rank one.

**Proposition 3.3.** Then the dp-rank of $E$ (considered as either a $K$-definable or $K^{|\mathbb{Q}_p|}$-definable set) is one.

*Proof.* We apply Fact 2.4. Note that dp-rk$_K E$ agrees with dp-rk$_{K^{|\mathbb{Q}_p|}} X$. As $E$ is infinite dp-rk$_K E \geq 1$. By Fact 2.7 we have dp-rk$_K V = 1$. The map $V \to E$ given by $a \mapsto B(a, \gamma)$ is $K$-definable and surjective so dp-rk$_K E = 1$.

Letting $\mathcal{E}$ be the structure induced on $E$ by $K$ we see that $\mathcal{E}$ has dp-rank one, $\mathcal{E}^{|\mathbb{Q}_p|}$ interprets $\mathbb{Q}_p$, and Conjecture 1 implies that $\mathcal{E}$ does not interpret an infinite field.

3.4. **The geometric sorts.** We first recall some well-known facts about the “geometric sorts” introduced in [7]. Let $L$ be a valued field with valuation ring $O$. An $n$-lattice is a free rank $n$ $O$-submodule of $K^n$. Let $S_n(L)$ be the set of $n$-lattices. It is well known that there is a canonical bijection $S_n(L) \to \text{Gl}_n(L)/\text{Gl}_n(O)$ so we take $S_n(L)$ to be an $L$-definable set of imaginaries.

It is shown in [8] that $K$ eliminates imaginaries down to certain “geometric sorts”. One of the two kinds of “geometric sorts” is $S_n(K)$. We describe the canonical injection $\mathcal{B} \to S_2(K)$. Fix $B \in \mathcal{B}$. Let $R$ be the $V$-submodule of $K^2$ generated by $\{1\} \times B$. It is easy to see that $R$ is a 2-lattice and $R \cap [\{1\} \times K] = B$.

Note that Val$_p : K^\times \to \Gamma$ gives an isomorphism $S_1(K) \to \Gamma$. Recall that the structure induced on $\Gamma$ by $K$ is interdefinable with $(\Gamma, +, <)$. We expect that $(\Gamma, +, <)^{|\mathbb{Q}_p|}$ cannot interpret an infinite field.

**Proposition 3.4.** Let $S_n$ be the structure induced on $S_n(K)$ by $K$. If $n \geq 2$ then $S_n^{|\mathbb{Q}_p|}$ interprets $\mathbb{Q}_p$. Conjecture 1 implies that $S_n$ does not interpret an infinite field.
Proof. We may suppose that \( B \) is an algebraically closed valued field then the induced structure on \( S_n(L) \) does not interpret an infinite field, but their techniques do not directly apply to the \( p \)-adic case.

Proof. Fix \( n \geq 2 \). As there is a \( K \)-definable injection \( \mathbb{B} \to S_2(K) \) and a natural injection \( S_2(K) \to S_n(K) \) the first claim follows by Theorem \([13]\). The second claim follows by the proof of Theorem \([12]\) as \( S_n(\mathbb{Q}_p) = \text{Gl}_n(\mathbb{Q}_p)/\text{Gl}_n(\mathbb{Z}_p) \) is countable. \( \square \)

4. Trace definibility

We discuss trace definibility and in particular prove Theorem \([1.4]\). The first two claims of Proposition \([4.1]\) are immediate. The third is an easy corollary to Fact \([2.1]\).

Proposition 4.1. Let \( \mathcal{M}, \mathcal{O} \) and \( \mathcal{P} \) be structures.

1. Suppose that \( \mathcal{O} \) is trace definable in \( \mathcal{M} \) and \( \mathcal{P} \) is trace definable in \( \mathcal{O} \). Then \( \mathcal{P} \) is trace definable in \( \mathcal{M} \).
2. If \( \mathcal{O} \prec \mathcal{M} \) then \( \mathcal{O} \) is trace definable in \( \mathcal{M} \).
3. If \( \mathcal{M} \) is NIP, \( \mathcal{O} \prec \mathcal{M} \), and \( \mathcal{M} \) is \( |O|^+ \)-saturated, then \( \mathcal{M} \) trace defines \( \mathcal{O}^{\text{Sh}} \).

We view trace definibility as a weak notion of interpretability.

Proposition 4.2. If \( \mathcal{O} \) is interpretable in \( \mathcal{M} \) then \( \mathcal{O} \) is trace definable in \( \mathcal{M} \).

In the proof below \( \pi \) will denote a certain map \( E \to O \) and we will also use \( \pi \) to denote the map \( E^n \to O^n \) given by \( \pi(a_1, \ldots, a_n) = (\pi(a_1), \ldots, \pi(a_n)) \).

Proof. Suppose \( \mathcal{O} \) is interpretable in \( \mathcal{M} \). Let \( E \subseteq M^m \) be \( M \)-definable, \( \approx \) be an \( M \)-definable equivalence relation on \( E \), and \( \pi : E \to O \) be a surjection such that

1. for all \( a, b \in E \) we have \( a \approx b \) if and only if \( \pi(a) = \pi(b) \), and
2. if \( X \subseteq O^n \) is \( O \)-definable then \( \pi^{-1}(X) \) is \( M^{\text{Sh}} \)-definable.

Let \( \tau : O \to E \) be a section of \( \pi \) and \( A = \tau(O) \). So \( A \) contains exactly one element from every \( \approx \)-class. If \( X \subseteq O^n \) is \( O \)-definable then \( \pi^{-1}(X) \) is \( M \)-definable and \( X \) is the set of \( a \in O^n \) such that \( \tau(a) \in \pi^{-1}(X) \). \( \square \)

Proposition 4.3. Let \( \lambda \) be an uncountable cardinal. Suppose that \( \mathcal{M} \) is \( \lambda \)-saturated and NIP. Suppose \( M^{\text{Sh}} \) trace defines \( \mathcal{O} \) and \( |O| < \lambda \). Then \( \mathcal{M} \) trace defines \( \mathcal{O} \).

Proof. We may suppose that \( O \subseteq M^m \) and that for every \( O \)-definable \( X \subseteq O^n \) there is an \( M^{\text{Sh}} \)-definable \( Y \subseteq M^{mn} \) such that \( O^n \cap Y = X \). Fix an \( O \)-definable subset \( X \subseteq O^n \). Let \( Y \) be an \( M^{\text{Sh}} \)-definable subset of \( M^{mn} \) such that \( O^n \cap Y = X \). Applying Facts \([2.2]\) and \([2.3]\) we obtain an \( M \)-definable family \( (Z_b : b \in M^k) \) such that for every finite \( B \subseteq Y \) we have \( B \subseteq Z_b \subseteq Y \). So for any finite \( B \subseteq X \) there is \( b \in M^k \) such that \( B \subseteq Z_b \) and \( Z_b \cap (O^n \setminus X) = \emptyset \). As \( |O^n| < \lambda \) an application of saturation yields \( b \in M^k \) such that \( X = O^n \cap Z_b \). \( \square \)

Theorem \([1.3]\) is a special case of Proposition \([4.3]\).

Proposition 4.4. Let \( \lambda \) be an uncountable cardinal. Suppose that \( \mathcal{M} \) is \( \lambda \)-saturated and NIP. Suppose that \( M^{\text{Sh}} \) interprets \( \mathcal{O} \). If \( \mathcal{P} \prec \mathcal{O} \) and \( |P| < \lambda \) then \( \mathcal{M} \) trace defines \( \mathcal{P} \). In particular if \( \mathcal{M} \) is \( R_1 \)-saturated and \( M^{\text{Sh}} \) interprets an infinite field then \( \mathcal{M} \) trace defines an infinite field.

In \([20]\) we described a \( (O^{\text{Sh}})^+ \)-saturated NIP structure \( \mathcal{K} \) such that \( \mathcal{K} \) does not interpret an infinite group but \( \mathcal{K}^{\text{Sh}} \) interprets \((\mathbb{R}, +, \times)\). So \( \mathcal{K} \) trace defines \((\mathbb{R}, +, \times)\). The example presented in \([21]\) also trace defines \((\mathbb{R}, +, \times)\) for the same reason.
Proof. The second claim follows from the first claim and Löwenheim-Skolem. Let \( \mathcal{P} \) be an elementary substructure of \( \mathcal{O} \) such that \( |\mathcal{P}| < \lambda \). By Proposition 4.2 \( \mathcal{M}^{\text{Sh}} \) trace defines \( \mathcal{O} \). The first two items of Proposition 4.1 show that \( \mathcal{M}^{\text{Sh}} \) trace defines \( \mathcal{P} \). So \( \mathcal{M} \) trace defines \( \mathcal{P} \) by Proposition 4.3. \( \square \)

Finally, we show that if \( \mathcal{M} \) is NIP then any structure which is interpretable in the Shelah completion of \( \mathcal{M} \) is trace definable in an elementary extension of \( \mathcal{M} \).

**Proposition 4.5.** Suppose that \( \mathcal{M} \) is NIP and that \( \mathcal{M}^{\text{Sh}} \) trace defines \( \mathcal{O} \). Suppose that \( \mathcal{N} \) is an \( |\mathcal{M}|^{+} \)-saturated elementary extension of \( \mathcal{M} \). Then \( \mathcal{N} \) trace defines \( \mathcal{O} \). In particular if \( \mathcal{M}^{\text{Sh}} \) interprets \( \mathcal{P} \) then \( \mathcal{N} \) trace defines \( \mathcal{P} \).

Proof. By Proposition 4.1(3) \( \mathcal{N} \) trace defines \( \mathcal{M}^{\text{Sh}} \), so by Proposition 4.1(1) \( \mathcal{N} \) trace defines \( \mathcal{O} \). The second claim follows from the first claim by Proposition 4.2. \( \square \)

In the remainder of this section we make a few more observations about trace definibility which are not directly connected to the main topic of this paper.

### 4.1. Trace definability and tameness properties.

A number of positive model-theoretic properties are equivalent to non-trace definibility of a particular countable homogeneous relational structure with a finite language. We discuss two cases here: stability and NIP.

It is easier to trace define relational structures with quantifier elimination. We leave the verification of Proposition 4.6 to the reader.

**Proposition 4.6.** Suppose that \( L \) is a relational language, \( T \) is an \( L \)-theory which admits quantifier elimination, \( \mathcal{O} \models T \), and \( \tau : \mathcal{O} \to M^m \) is an injection. Suppose that for every \( n \)-ary relation symbol \( R \in L \) there is a \( M \)-definable \( Y \subseteq M^{mn} \) such that for all \( (a_1, \ldots, a_n) \in O^n \) we have

\[
\mathcal{O} \models R(a_1, \ldots, a_n) \text{ if and only if } (\tau(a_1), \ldots, \tau(a_n)) \in X.
\]

Then \( \mathcal{M} \) trace defines \( \mathcal{O} \) via \( \tau \).

Proposition 4.6 is easy and left to the reader.

**Proposition 4.7.** If \( \mathcal{M} \) is stable then any structure which is trace definable in \( \mathcal{M} \) is stable. If \( \mathcal{M} \) is NIP then any structure which is trace definable in \( \mathcal{M} \) is NIP.

The random graph is as usual the Fraïssé limit of the class of finite (symmetric irreflexive) graphs. Recall that the theory of the random graph is IP and has quantifier elimination.

**Proposition 4.8.** Let \( \mathcal{M} \) be \( \aleph_1 \)-saturated. Then \( \mathcal{M} \) is unstable if and only if \( \mathcal{M} \) trace defines \( (\mathbb{Q}, <) \) and \( \mathcal{M} \) is IP if and only if \( \mathcal{M} \) trace defines the random graph.

Proof. The right to left direction of both claims follows by Proposition 4.7. Suppose that \( \mathcal{M} \) is unstable. Applying \( \aleph_1 \)-saturation we obtain a sequence \( (a_q : q \in \mathbb{Q}) \) of elements of some \( M^m \) and a formula \( \phi(x, y) \) such that for all \( p, q \in \mathbb{Q} \) we have \( \mathcal{M} \models \phi(a_p, a_q) \) if and only if \( p < q \). Let \( \tau : \mathbb{Q} \to M^m \) be given by declaring \( \tau(q) = a_q \) for all \( q \in \mathbb{Q} \). As \( (\mathbb{Q}, <) \) admits quantifier elimination an application of Proposition 4.6 shows that \( \mathcal{M} \) trace defines \( (\mathbb{Q}, <) \) via \( \tau \).

Suppose that \( \mathcal{M} \) is IP. Applying \cite{11} Lemma 2.2 and \( \aleph_1 \)-saturation there is an isomorphic copy \( (V, E) \) of the random graph such that \( V \subseteq M^m \) for some \( m \) and
a formula \( \phi(x, y) \) such that for all \( a, b \in V \) we have \( (a, b) \in E \) if and only if \( M \models \phi(a, b) \). An application of Proposition 4.6 shows that \( M \) trace defines \((V, E)\) via the identity \( V \to M^m \).

We conjecture that any infinite field trace definable in an o-minimal structure is either real closed or algebraically closed and any infinite field trace definable in \( K \) is a finite extension of a \( p \)-adically closed field. Proposition 4.9 is a tiny step in this direction (both o-minimal structures and \( p \)-adically closed fields are distal.)

**Proposition 4.9.** A distal structure cannot trace define an infinite field of positive characteristic.

Chernikov and Starchenko \([3]\) show that a distal structure cannot interpret an infinite field. Proposition 4.9 follows by the same proof. A structure \( \mathcal{O} \) satisfies the strong Erdős-Hajnal property if for every \( \mathcal{O} \)-definable subset \( X \) of \( O^m \times O^n \) there is a real number \( \delta > 0 \) such that for any finite \( A \subseteq M^m, B \subseteq M^n \) there are \( A' \subseteq A, B' \subseteq B \) such that \( |A'| \geq \delta |A|, |B'| \geq \delta B \), and \( A' \times B' \) is either contained in or disjoint from \( X \). Proposition 4.10 is clear from the definitions.

**Proposition 4.10.** If \( M \) has the strong Erdős-Hajnal property than any structure trace definable in \( M \) has the strong Erdős-Hajnal property.

Chernikov and Starchenko \([3]\) show that any distal structure has the strong Erdős-Hajnal property. They also observe that the failure of the strong Erdős-Hajnal property for infinite fields of positive characteristic is a direct consequence of well-known facts from incidence combinatorics over finite fields together with the theorem of Kaplan, Scanlon, and Wagner \([10]\) that an infinite NIP field of positive characteristic contains the algebraic closure of its prime subfield. Proposition 4.9 follows from these facts and Proposition 4.7.

Another natural conjecture is that a divisible ordered abelian group cannot trace define \((\mathbb{R}, <, +, t \mapsto \lambda t)\) for any \( \lambda \in \mathbb{R} \setminus \mathbb{Q} \).

5. **Modularity?**

This paper is motivated by the following question.

*Is there a good notion of modularity for NIP structures?*

There is a good notion of modularity for stable structures which we refer to as “one-basedness”. There is also a good notion of modularity for o-minimal structures, an o-minimal structure is modular if and only if algebraic closure is locally modular. Fact 5.1 follows from the Peterzil-Starchenko trichotomy \([13]\).

**Fact 5.1.** Suppose \( M \) is an o-minimal expansion of an ordered group. Then the following are equivalent:

1. algebraic closure in \( M \) is locally modular,
2. \( M \) is a reduct of an ordered vector space over an ordered division ring,
3. \( M \) does not interpret an infinite field.

Here (1) is an abstract modularity notion, (2) asserts that definable sets are “affine objects” in a reasonable sense, and (3) asserts the absence of a certain algebraic structure. Analogues of Fact 5.1 should hold for other well-behaved classes of NIP structures. Examples such as \( \mathcal{B} \) indicate that (3) should be replaced with “\( \mathcal{M}^{\text{Sh}} \)
does not interpret an infinite field” or “$M$ does not trace define an infinite field”.
In [21] we gave an example of a weakly o-minimal expansion $Q$ of $(\mathbb{Q},+,<)$ such
that algebraic closure in $Q$ agrees with algebraic closure in $(\mathbb{Q},+,<)$ and if $Q \prec N$ is $(2^{\aleph_0})^+$-saturated then $N^{\text{Sh}}$ interprets $(\mathbb{R},+,\times)$. So analogues of Fact 5.1 will require
a new notion of modularity which is not defined in terms of algebraic closure.

We record some reasonable conditions for a notion of modularity for NIP structures
(but we do not insist that all of these conditions be satisfied, see the remarks below).

(A1) Modularity implies NIP,
(A2) Modularity is preserved under elementary equivalence.
(A3) Infinite fields are not modular.
(A4) A structure is modular if and only if its Shelah completion is modular.
(A5) Ordered abelian groups and ordered vector spaces are modular.
(A6) An o-minimal structure is modular if and only if it is modular in the sense
of Peterzil-Starchenko.
(A7) Monadically NIP structures are modular (so trees are modular and the
expansion of a linear order $(O, <)$ by any collection of monotone functions
$O \to O$ is modular).
(A8) Reasonable valued abelian groups are modular.
(A9) Any structure which is trace definable in a modular structure is modular
(so in particular modularity is preserved under interpretations).

Note that (A4) follows from (A2), (A9), and Proposition 4.1. But it is possible
that (A9) is too strong, as one-basedness is not preserved under reducts, see [6].
However, we do not insist that modularity and one-basedness agree over stable theories,
so it may be the case that a reduct of a one-based structure is always modular.

A structure $M$ is monadically NIP if the expansion of $M$ by all subsets of $M$ is NIP.
See [17] for a proof that trees and expansions of linear orders by monotone functions are monadically NIP.

The “reasonable” in (A8) is necessary as any structure is interpretable in some
valued abelian group, see Schmitt [15]. It is not clear what “reasonable” should
mean, at a minimum the valued additive group of a valued field should be modular.
We also expect the expansion of $(\mathbb{Z},+)$ by all $p$-adic valuations to be modular (this
structure is NIP by [1]).

Finally, we record a question of Hrushovski. We do not believe that this question has appear in print. Question 5.2 is motivated by the theorem of Chernikov and Starchenko [3] that a distal structure cannot interpret a field of positive characteristic (see Proposition 4.9 above).

**Question 5.2.** Let $\mathbb{F}$ be a finite field, $V$ be an $\mathbb{F}$-vector space, and $V$ be a distal expansion of $V$. Must $V$ be (in some sense) modular?

A more precise question: Can $V$ trace define an infinite field?
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