PROPAGATION OF CHAOS FOR BRANCHING STOCHASTIC PARTICLES MODEL WITH CHEMOTAXIS

RADOSLAW WIECZOREK

University of Silesia, ul. Bankowa 14, 40-007 Katowice, Poland.

Abstract. A hybrid stochastic individual-based model of proliferating cells with chemotaxis is presented. The model is expressed by a branching diffusion process coupled to a partial differential equation describing concentration of a chemotactic factor. It is shown that in the hydrodynamic limit when number of cells goes to infinity the model converges to the solution of nonconservative Patlak-Keller-Segel-type system. A nonlinear mean-field stochastic model is defined and it is proven that the movement of descendants of a single cell in the individual model converges to this mean-field process.

1. Introduction

This paper is concerned with a stochastic model of biological cells that undergo chemotaxis and proliferate. The hydrodynamic limit of this model is shown to solve the equation of Patlak-Keller-Segel type with cell proliferation of the form

\begin{align}
\partial_t p(t, x) &= \frac{1}{2} \Delta p(t, x) + \nabla \cdot (p(t, x) \mathbf{b}(x, \nabla \varrho)) + \lambda(x, \nabla \varrho(t, x)) p(t, x) \\
\partial_t \varrho(t, x) &= D \Delta \varrho(t, x) - r \varrho(t, x) + \alpha [\kappa \ast p(t, \cdot)](x)
\end{align}

Mathematical description of biological cells undergoing chemotaxis, i.e. moving in response to gradients of chemical factors, has a long tradition and still attracts great interest. The seminal papers by Patlak [30] and Keller and Segel [20] stimulated the whole branch of applied mathematics (one can find extensive reviews [17, 16] with many citations). The limit passages between microscopic, kinetic models of collective cell behaviour to the macroscopic description with PDE’s has received great attention in recent four decades [28, 33, 26, 31, 14, 37, 7, 12, 3]. The first rigorous proof of the convergence of a stochastic particle system to chemotaxis equations system was given in [32] following the work of [29]. Later many authors continued these ideas [18, 14, 7, 5]. Macroscopic mean-field limits of weakly interacting particles where also investigated in [25, 4, 22, 23]. Most particle approximations of PKS type equations consider cells and chemical particles in similar way [32, 18, 14], while they have completely different scale. It is natural to consider cells as stochastic particles and chemical factor as continuous field described by diffusion equation [10, 7, 9, 12]. Although usually the equation or system describing the cell population is conservative, i.e. it preserves total mass (the number of particles), in many models it is natural to assume that the cell population is not constant and the cells proliferate [38, 36, 35, 1]. It is crucial for angiogenesis models [11, 24, 10, 12], where the proliferation is responsible for vessel branching.

In this paper we use the pathwise propagation of chaos approach similar to that of [7], but we allow for the proliferation of cells. Therefore, the number of cells (or total mass) is not conserved and, moreover, the individual process has noncontinuous trajectories. To retain the pathwise description and convergence, we write the birth and death process as a solution of

\textit{E-mail address: radoslaw.wieczorek@us.edu.pl.}

2020 Mathematics Subject Classification. 92C17; 60F99; 60C35.

Key words and phrases. stochastic particles system; branching diffusion; chemotaxis; hydrodynamic limit; propagation of chaos; mean field approximation, Patlak-Keller-Segel equation.
stochastic equations (cf. [15]). Such models, where continuous part is coupled to discreet part (often with jumps) are sometimes refered as hybrid [8, 6, 24, 10].

We construct a sequence of processes indexed by the initial number of particles \( n_0 \). The description of our process can be divided into three componentes:

**Movement of cells.** The cells move according to the following SDE

\[
\alpha \cdot dX_{i,j}^{(n_0)}(t) = b(X_{i,j}^{(n_0)}(t), \nabla \rho_{n_0}(t, X_{i,j}^{(n_0)}(t)))dt + \sigma \cdot dW_{i,j}(t),
\]

where \( W_{i,j}(t) \) are independent Brownian motions, \( \sigma \) is the diffusion coefficient and \( b \) is the (chemotactic) drift that depends on the position of a cell and the gradient of the concentration \( \rho_{n_0} \) of some chemical factor. One can also allow for the dependence of \( b \) on the concentration \( \rho_{n_0} \) itself, not only on its gradient, and all facts and proofs of the paper remain true, but for the sake of shortness and simplicity of the notation we neglect this dependence. The indexing \( i, j \) will be explained later.

**Equation for chemoreactant.** The concentration \( \rho_{n_0} \) of chemotactic factor satisies the following PDE

\[
\frac{\partial \rho_{n_0}(t, x)}{\partial t} = D \Delta \rho_{n_0}(t, x) - r \rho_{n_0}(t, x) + \alpha \kappa \ast \xi_t^{n_0}(x),
\]

where \( D, r \) and \( \alpha \) are diffusion, degradation and production rates. The measure

\[
\xi_t^{n_0} = \frac{1}{n_0} \sum_{i,j} \delta_{X_{i,j}^{(n_0)}(t)},
\]

is the empirical measure of all cells and function \( \kappa \) is a mollifying kernel that represents the fact that cells are actually not points, but have spatial size. The spatial convolution

\[
\kappa \ast \xi_t^{n_0}(x) = \int_{\mathbb{R}^d} \kappa(x - y) \xi_t^{n_0}(dy) = \frac{1}{n_0} \sum_{i,j} \int_{\mathbb{R}^d} \kappa(x - y) \delta_{X_{i,j}^{(n_0)}(t)}(dy)
\]

\[
= \frac{1}{n_0} \sum_{i,j} \kappa(x - X_{i,j}^{(n_0)}(t))
\]

is a mollified version of the empirical measure describing spatial positions of cells, responsible for the production of the chemoreactant. From the mathematical point of view this allows us to consider classical solutions to equation (3).

**Cell population dynamics.** We assume that cells may die or proliferate with rates depending on the chemoreactant. Death means that a cell disappears and proliferation means that a cell dies leaving two new daughter cells at the same place as the mother cell. The birth rate of a cell placed at \( x \) at time \( t \) depend on the position and on the the concentration of chemoreactant and is given by \( \lambda_b(x, \rho_{n_0}(t, x)) \) and the mortality rate is \( \lambda_d(x, \rho_{n_0}(t, x)) \).

The main goal of the paper is to rigorously define the described individual processes and show wellposedness, then to prove that in hydrodynamic limit they converge to the solutions of (1) (Theorem 5), and, most importantly, to prove that, if the initial number of cells tends to infinity, the trajectories of the descendants of a single cell converge to the trajectories of the nonlinear mean-field model defined in section 2.2 (Theorem 6).

The article is organised as follows. In the next section we introduce the notation and present the rigorous definitions of the microscopic model and two ‘hybrid’ mean field models and write up the macroscopic equations. Section 3 is devoted to the presentation of assumptions and results concerning the wellposedness. Section 4 contains the convergence results. The proofs are presented in section 5.

2. Definitions of the processes

In this section we formally define the considered processes. We start with the individual, fully microscopic model.
2.1. Definition of the microscopic processes. Note that if we use the empirical measure approach even for a simple two Brownian particles case, then giving the initial condition $\delta_{X_1^0} + \delta_{X_2^0}$ and two Brownian motions $W_1$ and $W_2$ does not guarantee pathwise uniqueness — we need to know the order of particles. The problem gets harder if the number of particles varies in time. Since our goal is to obtain a pathwise propagation of chaos result, we need to define a process in a more direct way.

Therefore, we construct the process in the following way. We mark particles by means of a subtree of Ulam-Harris tree, namely let

$$\mathcal{J} = \bigcup_{n \in \mathbb{N}_0} \{0, 1\}^n$$

with a convention that $\{0, 1\}^0 = \emptyset$ means the root. Elements of $\mathcal{J}$ will be written as blackboard bold lowercase letters such as $i, j, k$. If we write $k = j0$, we mean that $k$ is longer by one then $j$ and is created from $j$ by adding 0 at the end. Moreover, we will denote by $j^{-}$ element of $\mathcal{J}$ obtained from $j$ by removing last number, eg. if $j = \emptyset 0101$, then $j^{-} = \emptyset 10$. Moreover, let us assume that

(A.1) $(W_{i,j})_{i \in \mathbb{N}, j \in \mathcal{J}}$ is an infinite array of independent $d$-dimensional standard Wiener processes, and $(\mathcal{N}_{i,j})_{i \in \mathbb{N}, j \in \mathcal{J}}$ is an array of independent standard (i.e. such that intensity is Lebesgue measure) Poisson point processes on $[0, \infty) \times [0, \infty)$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

These processes will be used as a source of randomness in all processes, in particular in Eq. (7a) and (9a).

We actually define a sequence of processes indexed by the initial number of cell $n_0$. Let us assume that we start with $n_0$ cells and let each initial cell be described by its position in $\mathbb{R}^d$, the number $i$, and $j = \emptyset$ denoting that it is the first cell in its own tree of inheritance. The initial cells are located at $X_{i,\emptyset}(0) \in \mathbb{R}^d$, $i = 1, \ldots, n_0$. At each branching event, two new cells appear as daughters of a cell described by $X_{i,j}^{n_0}$ at the same place as the mother cell. The daughter cells inherit the position, the cell line number $i$ and obtain new subsequent indices $k_1 = j0$ and $k_2 = j1$. So, each cell is described by a triple $(x, i, j) \in \mathbb{R}^d \times \{1, \ldots, n_0\} \times \mathcal{J}$ and possesses its own Brownian motion $W_{i,j}$ and a Poisson clock $\mathcal{N}_{i,j}$. Let $\tau_{i,j}^{n_0}$ denote the moment when $i,j$-th cell appears — that is $\tau_{i,j}^{n_0}$ is always 0, and let $\sigma_{i,j}^{n_0}$ be the moment when $i,j$-th cell dies (the production of daughter cells also means death of the mother). The time of death $\sigma_{i,j}^{n_0}$ of an $(i,j)$-th cell is defined as a minimal $\sigma$ for which

$$\mathcal{N}_{i,j}\left(t, z : z \in [0, \lambda_b(X_{i,j}^{n_0}(t), g^{n_0}(t, X_{i,j}^{n_0}(t))) + \lambda_d(X_{i,j}^{n_0}(t), g^{n_0}(t, X_{i,j}^{n_0}(t)))\right], t \in [\tau_{i,j}^{n_0}, \sigma]\right) = 1$$

with a convention that $\min \emptyset = \infty$. The times of birth $\tau_{i,j}^{n_0} = \tau_{i,j}^{n_0}$ of daughters of $i,j$-th cell are defined as a minimal $\tau$ for which

$$\mathcal{N}_{i,j}\left(t, z : z \in [0, \lambda_b(X_{i,j}^{n_0}(t), g^{n_0}(t, X_{i,j}^{n_0}(t)))\right), t \in [\tau_{i,j}^{n_0}, \tau]\right) = 1.$$  

Clearly, not for every $j \in \mathcal{J}$ a cell will exist, e.g. if $k$-th cell dies, no cell with index created from $k$ by appending any zeros or ones cannot be born. In this case we have $\tau_{i,j}^{n_0} = \sigma_{i,j}^{n_0} = \infty$.

The movement of a $i,j$-th cell between time $\tau_{i,j}^{n_0}$ and $\sigma_{i,j}^{n_0}$ is given by

$$dX_{i,j}^{n_0}(t) = \mathbf{b}(X_{i,j}^{n_0}(t), \nabla g^{n_0}(t, X_{i,j}^{n_0}(t)))dt + \sigma dW_{i,j}(t).$$

with initial condition $X_{i,j}^{n_0}(\sigma_{i,j}^{n_0}) = X_{i,j}^{n_0}$ coupled with the equation for nutrient field

$$\frac{\partial g^{n_0}(t, x)}{\partial t} = D\Delta g^{n_0}(t, x) - r g^{n_0}(t, x) + \alpha[K * \xi_{i,j}^{n_0}](x)$$
with \( \varrho^{n_0}(0, \cdot) = \varrho_0 \), where \( \xi_t^{n_0} \), given by

\[
(8) \quad \xi_t^{n_0} = \frac{1}{n_0} \sum_{i,j} \mathbb{1}_{[\sigma_{i,j}, \tau_{i,j}]}(t) \delta_X(t),
\]
is an empirical measure of all cells alive at time \( t \).

Note that the generating processes \( W_{i,j} \) and \( \mathcal{N}_{i,j} \) have intentionally no index \( n_0 \). In order to obtain pathwise convergence, they are shared by processes with all \( n_0 \). They are defined for all \( i \in \mathbb{N} \), but the definition of \( n_0 \)-th process uses only those with \( i = 1, \ldots, n_0 \).

**Remark 1.** Note, that equation (7a) can be written in a form

\[
X_{i,j}^{n_0}(t) = X_{i,j}^{n_0}(\sigma_{i,j}^{n_0}) + \int_{\sigma_{i,j}^{n_0}}^t b(X_{i,j}^{n_0}(s), \nabla \varrho^{n_0}(s, X_{i,j}^{n_0}(s))) \, ds
+ \sigma(W_{i,j}(t) - W_{i,j}(\sigma_{i,j}^{n_0})), \quad \text{for } t \in [\sigma_{i,j}^{n_0}, \tau_{i,j}^{n_0}).
\]

It does not demand using Itô integral and has a pathwise unique solution.

2.2. Hybrid mean-field model. The next model considered will be the limit of the individual-based model (cf. Theorems 5 and 6). We consider one initial cell at position \( \bar{X}_0(0) = \bar{X}_{1,0}(0) \) with the same population dynamics as before. Its descendants will be denoted by \( \bar{X}_j \) with \( j \in \mathbb{J} \) and their birth and death times are \( \bar{\sigma}_j \) and \( \bar{\tau}_j \), respectively, defined in analogous way to (5)-(6) with \( \varrho^{n_0} \) replaced by \( \varrho \) and the same Poisson clocks \( \mathcal{N}_{1,j} \), as for the first cell line in each microscopic model. The movement of the \( j \)-th cell during its life is given by

\[
(9a) \quad d\bar{X}_j(t) = F(\bar{X}_j, \nabla \varrho(t, \bar{X}_j)) \, dt + \sigma dW_{i,j}(t),
\]
coupled with the mean-field chemotactrant equation

\[
(9b) \quad \frac{\partial \varrho(t, x)}{\partial t} = D\Delta \varrho(t, x) - r \varrho(t, x) + \alpha(K \ast \bar{\mu}_t)(x),
\]
with \( \varrho(0, \cdot) = \varrho_0 \), where \( \bar{\mu}_t \) is the mean of the empirical measure of \( (\bar{X}_j(t))_{j \in \mathbb{J}} \), namely

\[
(10) \quad \bar{\mu}_t(A) = \mathbb{E} \bar{\xi}_t(A) = \mathbb{E} \left[ \sum_{i,j} \mathbb{1}_{[\bar{\sigma}_j, \bar{\tau}_j]}(t) \delta_X(t) \right], \quad \text{for } A \in \mathcal{B}(\mathbb{R}^d),
\]
where \( \bar{\xi}_t, \bar{\tau}_j \) and \( \bar{\sigma}_j \) are defined analogously to \( \xi_t^{n_0}, \tau_{i,j}^{n_0} \) and \( \sigma_{i,j}^{n_0} \).

**Remark 2.** Note that (9b) differs from (7b) only by replacing \( \xi_t^{n_0} \) by \( \bar{\mu}_t \).

2.3. Macroscopic model: Patlak-Keller-Segel type equation with proliferation. The limit macroscopic model is given by the system of equations

\[
(11a) \quad \partial_t p(t, x) = \frac{1}{2} \Delta p(t, x) + \nabla \cdot (p(t, x)b(x, \nabla \varrho)) + \lambda(x, \nabla \varrho(t, x))p(t, x)
\]
\[
(11b) \quad \partial_t \varrho(t, x) = D \Delta \varrho(t, x) - r \varrho(t, x) + \alpha[\kappa \ast p(t, \cdot)](x)
\]
with \( \lambda = \lambda_b - \lambda_d \) where \( [\kappa \ast p(t, \cdot)](x) = \int_{\mathbb{R}^d} \kappa(x - y)p(t, y) \, dy \).

2.4. Second hybrid model. One of the motivations of the convergence result of this paper is to show the possibility of replacing in simulations a multiparticle model (5)-(7b) by a one with smaller number cells. However, the hybrid model (9a)-(9b) still includes the number of proliferating cells. Therefore, we present here the alternative, much simpler version of a hybrid model that is related to the limit of the ibm. Namely, let us consider a single cell which which
moves according to the same rule as the particles in previous models, and endow it with a variable $M(t)$ denoting its mass:

\begin{align}
(12a) & \quad dX(t) = b(X(t), \nabla \rho(t, X(t))) dt + \sigma dW(t), \\
(12b) & \quad dM(t) = \lambda(X(t), \rho(t, X(t))) M(t), \\
(12c) & \quad \frac{\partial \rho(t, x)}{\partial t} = D \Delta \rho(t, x) - r \rho(t, x) + \alpha(K \ast \mu_t)(x),
\end{align}

where

\begin{equation}
\mu_t(A) = \mathbb{E}[M(t) \mathbb{1}_A(X(t))] = \int_{A \times [0, \infty)} m \mathbb{P}_t(dx, dm) \text{ for } A \in \mathcal{B}(\mathbb{R}^d)
\end{equation}
is the average mass in the area $A$.

**Remark 3.** The measure $\mu_t$ is equal to $\bar{\mu}_t$ given by (10). Note that equations (9b) and (12c) are then the same. These facts will be proven and used in the proof of wellposedness of the hybrid model.

**Remark 4.** Note moreover, that if $\mu_0$ is absolutely continuous, then the density of $\mu_t$ (and therefore $\bar{\mu}_t$) satisfies (11a). In that case all equations (9b), (11b) and (12c) coincide.

### 3. Assumptions and wellposedness

One of main goals of this paper is to make the definition of the individual model as strict as possible while keeping it readable. To that aim, besides the description in section 2.1, we need to define the state space of the process, which can be done in various ways. We add to the space of positions $\mathbb{R}^d$ an additional state $\phi$ denoting a nonexisting cell and we describe the state of all particles as an infinite array of points from $\mathbb{R}^d \cup \{\phi\}$ indexed by $(i, j) \in \mathbb{N} \times \mathbb{J}$ such that only finite numbers of elements are different then $\phi$, that is

\begin{equation}
X = \left\{(x_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{J}} : x_{i,j} \in \mathbb{R}^d \cup \{\phi\}, \ \text{such that} \ #\{x_{i,j} : x_{i,j} \neq \phi\} \text{ is finite}\right\}
\end{equation}

with a natural metrics

\begin{equation}
d_X(x, y) = \max_{(i,j) \in \mathbb{N} \times \mathbb{J}} |x_{i,j} - y_{i,j}|, \quad \text{for } x = (x_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{J}}, \ y = (y_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{J}} \in X,
\end{equation}

with a convention that $|x - \phi| = 1$ for any $x \in \mathbb{R}^d$. Because of the birth and death process the trajectories of the microscopic process are not continuous. We apply the standard convention to use càdlàg paths, so the space of trajectories will be the Skorokhod space $D_{\mathbb{R}}[0, \infty)$.

Now, we can formally describe the branching diffusion component of the microscopic model as the solution to the following system of SDEs

\begin{equation}
\begin{aligned}
X_{i,j}^{n_0}(t) &= X_{i,j}^{n_0}(0) + \int_0^t b(X_{i,j}^{n_0}(s), \nabla \rho(t, X_{i,j}^{n_0}(s))) ds + \sigma \int_0^t \mathbb{1}_{\mathbb{R}^d}(X_{i,j}^{n_0}(s)) \ dW_{i,j}(s), \\
&\quad + \int_0^t \chi_b^n(X_{i,j}^{n_0}(s), s) \mathcal{N}i,j-(ds, dz) + \int_0^t \chi_d^n(X_{i,j}^{n_0}(s), s) \mathcal{N}i,j-(ds, dz)
\end{aligned}
\end{equation}

for $i \in \{1, \ldots, n_0\}$, $j \in \mathbb{J}$ with

\begin{align}
\chi_b^n(x, z) &= \begin{cases} 
-\phi + x, & \text{if } x \neq \phi, \ z \leq \lambda_b(x, \rho^n(x)), \\
0, & \text{otherwise},
\end{cases} \\
\chi_d^n(x, z) &= \begin{cases} 
\phi, & \text{if } x \neq \phi, \ z \leq \lambda_b(x, \rho^n(x)) + \lambda_d(x, \rho^n(x)), \\
0, & \text{otherwise},
\end{cases}
\end{align}

where we use a convention that $\phi - \phi = 0 \in \mathbb{R}^d$ and $x + \phi = \phi$ for any $x \in \mathbb{R}^d$ and $b(\phi, x) = 0$ for $x \in \mathbb{R}$. 
In a similar way, to fully describe the branching diffusion of the hybrid model we add birth and death events to the equation (9a) obtaining

\[
\bar{X}_j(t) = \bar{X}_j(0) + \int_0^t b(\bar{X}_j(s), \nabla \varrho(t, \bar{X}_j(s))) \, ds + \sigma \int_0^t 1_{\mathbb{R}^d} (\bar{X}_j(s)) \, dW_{1,j}(s),
\]

and, analogously, equation (9b) as

\[
\varrho_t^{\alpha}(x) = S_t \varrho_0(x) + \alpha \int_0^t S_{t-s} [K \ast \xi^{\alpha}_s](x) \, ds,
\]

and similarly (12c) with $\mu_s$ replaced by $\bar{\mu}_s$.

We will use the following assumptions:

(A.2) let $\sigma, D, r, \alpha$ be positive constants.

(A.3) let $\lambda_0, \lambda_2 \in C_0^1(\mathbb{R}^d \times \mathbb{R}_+)$ be nonnegative functions and $\lambda_b + \lambda_d < \bar{\lambda}$ for some constant $\bar{\lambda} > 0$; let $b : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be bounded and boundedly differentiable; let $\kappa \in C_0^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \kappa(x) \, dx = 1$ and $\Delta \kappa$ is Lipschitz and let $L_\kappa$ be maximum of Lipschitz coefficients for $\kappa$ and $\Delta \kappa$.

Now we can state the well-posedness theorems. Their proofs will be given in section 5.1.

**Theorem 1.** Suppose that $\varrho_0 \in C_b^2(\mathbb{R}^d)$ and $X_i^{\alpha_0}(0)$ for $i = 1, \ldots, n_0$ are independent random variables with probability law $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. Let assumptions (A.1)-(A.3) be satisfied. For any $T > 0$ there exists a process $(x_t^{\alpha_0}, \varrho_t^{\alpha_0}) = \left(\left(X_{i,j}^{\alpha_0}\right)_{i \in \{1, \ldots, n_0\}, j \in J^*}, \varrho_t^{\alpha_0}\right) \in D_\mathcal{X}[0,T] \times C_b^2(\mathbb{R}^d)[0,T]$ described by (7a) and (7b) with population dynamics given by (5) and (6) and it is pathwise uniquely defined.

**Theorem 2.** Suppose that $\varrho_0 \in C_b^2(\mathbb{R}^d)$ and $X_0(0)$ is a random variable with probability law $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. Let assumptions (A.1)-(A.3) be satisfied. For any $T > 0$ there exists a unique function $\varrho \in C_b(\mathbb{R}^d)[0,T]$ and the hybrid mean-field process $\bar{z} = (\bar{X}_j)_{j \in J^*}$ with trajectories in $D_\mathcal{X}[0,T]$ described by (9) with (17) and it is pathwise uniquely defined.

For completeness we state also the existence-uniqueness theorem for the macroscopic model. We do not present its proof, which is straightforward thanks to the regularization by $\kappa$. It goes by simple fixed point argument.

**Theorem 3.** Let assumptions (A.1)-(A.3) be satisfied. If $\varrho_0$ and $p_0$ are in $C_b^2(\mathbb{R}^d)$, then there exists a unique classical solution to the system (11).

Now, we have the wellposedness theorem for the second mean-field model.

**Theorem 4.** Let assumptions (A.2)-(A.3) be satisfied and $W$ be a $d$-dimensional standard Wiener process. Suppose that $\varrho(0, \cdot) = \varrho_0 \in C_b^2(\mathbb{R}^d)$, $X(0)$ is a random variable with probability law $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ and $M(0) = 1$. Then system (12)-(13) has a unique solution.
4. Convergence results

The approach to the convergence of particle systems is based on the propagation of chaos results from [7] and [34, 13], namely, the proof of Theorem 6 is based on the fact that processes for all $n_0$ and the limit are coupled by use of the same probability space and the same processes $(W_{i,j})_{i \in N, j \in J}$ and $(N_{i,j})_{i \in N, j \in J}$. Since the processes have not continuous trajectories it demands more delicate approach. We will use the following notation:

- denote by $z_i^{n_0}(t) = (X_{i,j}^{n_0}(t))_{i \in \{1, \ldots, n_0\}, j \in J}$ coupled with $\xi_i^{n_0}$, $t \in [0, T]$, the solution of microscopic model defined by (5)-(8).
- let $Z_i^{n_0}(t) = (X_{i,j}^{n_0}(t))_{j \in J}$, $t \in [0, T]$ denote the branch of process $z_i^{n_0}$ starting from the first cell $X_{i,j}^{n_0}(0)$ driven by processes $(W_{i,j})_{j \in J}$ and $(N_{i,j})_{j \in J}$.
- now we define a mean field processes $\bar{X}_i^{n_0}(t)$ driven by processes $(W_{i,j})_{j \in J}$ and $\xi_i$, $t \in [0, T]$, defined by (17)-(9b) driven by processes $(W_{i,j})_{j \in J}$ and $(N_{i,j})_{j \in J}$.
- $\xi_i^{n_0}$ is a process given by (8) which can be written as

$$\xi_i^{n_0} = \frac{1}{n_0} \sum_{i=1}^{n_0} \sum_{j \in J} 1_{\mathbb{R}^d}(X_{i,j}^{n_0}(t)) \delta_{X_{i,j}^{n_0}(t)}.$$ 

Now we can state two convergence theorems.

**Theorem 5.** Let assumptions (A.1)-(A.3) be satisfied. Then for each $T > 0$

(i) the sequence of processes $\xi_i^{n_0}$ converges in distribution to $\tilde{\mu}_t$ defined by (10) on $D_{\mathcal{M}}[0, T]$ with Skorokhod topology. The space $\mathcal{M}$ is considered here with a topology of vague convergence.

(ii) the sequence of processes $\xi_i^{n_0}$ converges to $q_t$ given by (19) in distribution on $C_{C_b(\mathbb{R}^d)}[0, T]$.

Since the limit

**Theorem 6.** Let assumptions (A.1)-(A.3) be satisfied. Then for each $T > 0$

$$\sup_{t \in [0, T]} d_{\mathcal{X}}(Z_i^{n_0}(t), \bar{X}_i(t))$$

converges to 0 in probability.

5. Proofs

5.1. Wellposedness. In this section we prove firstly Theorem 1, then Theorem 4 and eventually Theorem 2, since its proof depends on fragments of two previous proofs.

We will repeatedly use the following fact

**Lemma 1.** For any $T > 0$ and all $n_0 \in \mathbb{N}$

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \langle 1, \xi_i^{n_0} \rangle \right] = \mathbb{E} \left[ \sup_{t \in [0, T]} \frac{1}{n_0} \langle 1, X_i^{n_0}(t) \rangle \right] < e^{\lambda t}$$

**Proof.** Since the birth rate is $\lambda_b \leq \bar{\lambda}$, we knot that $\langle 1, X_i^{n_0}(t) \rangle$ (which is the number of cells) is at any $t$ less (in the sense of distribution) than the number of particles in a simple birth only (Yule) process with branching rate $\bar{\lambda}$. \hfill \Box

**Proof of Theorem 1.** For notational convenience we assume $n_0 = 1$ and we omit indices $i$ and $n_0$, besides $\xi^{n_0}$ to avoid confusion with $\xi$ from eqn. (9). The prove for any other $n_0$ follows analogously with $j$ replaced by $(i, j)$.

Fix $T > 0$. We will prove the existence and uniqueness in $[0, T]$. Since we think about the solution to (7) pathwise, the prove will be done for fixed $\omega$. But firstly, we can neglect the set of probability 0. By Lemma 1, the number of cells at any finite $t$ is this number is finite with probability one. Therefore, if $\Omega_0$ is the set of those $\omega \in \Omega$, that the microscopic model
has infinite number of cells before time $T$, then $\mathbb{P}(\Omega_0) = 0$. Let $\Omega_1$ be the zero measure set for which any of the Poisson point processes $\mathcal{N}^j$ has infinite number of points in $[0, T] \times [0, \bar{\lambda}]$. 

Let us now fix $\omega \in \Omega \setminus (\Omega_0 \cup \Omega_1)$ and consider fixed trajectories of $(W^j)_{j \in J}$ and $(\mathcal{N}^j)_{j \in J}$ for this $\omega$. For $\omega \not\in (\Omega_0 \cup \Omega_1)$ there is a finite number of particles born before time $T$ and for any of those particles its Poisson point process has finite number of points in $[0, T] \times [0, \bar{\lambda}]$, so there is a finite number, say $n$, of points in all those Poisson point processes. Let us denote those times by $(t_k, z_k)_{k=1, \ldots, n}$ in the order of increasing times and $t_0 = 0$. Inbetween times $t_k$ the number of cells is constant, so we solve recursively in the intervals $[t_k, t_{k+1})$, $k = 0, 1, 2, \ldots$ a deterministic system

$$
\begin{align*}
\begin{cases}
X_j(t) = X_j(t_k) + \int_{t_k}^t b(X_j(s), \nabla g^{\alpha}(s, X_j(s))) ds + \sigma(W_j(t) - W_j(t_k)), \quad &\text{for } j \in J,
\frac{\partial g^{\alpha}(t, x)}{\partial t} = D\Delta g^{\alpha}(t, x) - r g^{\alpha}(t, x) + \alpha[K * \xi_t](x),
\end{cases}
\end{align*}
$$

with $\xi_t = \sum_{j \in J} \delta_{X_j(t)}$ where $J$ is the set of indices of cells alive in interval $(t_k, t_{k+1})$, i.e.

$$J_k = J_{k-1} \cup \left\{ j : j \in J_{k-1}, i = 0, 1, z_k < \lambda_b(X_j(t^{-}_k), g^{\alpha}(t^{-}_k, X_j(t^{-}_k))) \right\} \setminus \left\{ j : j \in J_{k-1}, z_k < \lambda_b(X_j(t^-_k), g^{\alpha}(t^-_k, X_j(t^-_k))) + \lambda_d(X_j(t_k), g^{\alpha}(t_k, X_j(t_k))) \right\}.
$$

The initial conditions at zero are $X_j(t_0) = X_{j,0}(0)$ and $g^{\alpha}(0, \cdot) = g_0$, and for $t_k > 0$ recursively $g^{\alpha}(t_k, \cdot) = g^{\alpha}(t^{-}_k, \cdot)$ and for $j \in J_k$

$$X_j(t_k) = \begin{cases}
X_j(t^-_k), \quad &\text{if } j \in J_{k-1},
X_j(t^-_k) \text{ otherwise.}
\end{cases}
$$

The proof of existence and uniqueness of solutions to (21) on $(t_k, t_{k+1})$ is straightforward and goes e.g. by Picard type argument.

The next proof uses classical methods from [34] and is similar to the proof of Proposition 2.3. in [7].

**Proof of Theorem 4.** We prove the existence on the interval $[0, T]$. The scheme of the proof is the following: given a fixed function $g : [0, T] \to C^2(\mathbb{R}^d)$ we solve the SDE (12a)-(12b). Let $\tilde{g}$ be a solution to (19) with $\mu$ given by (13). Then we show that operator $P : g \mapsto \tilde{g}$ is a contraction, so there is a unique $g$ satisfying (12), and therefore a unique $X$ and $M$. We consider the operator $P$ on the space $E = \{ g \in C_{[0,T]}(C^2(\mathbb{R}^d)) : \text{Lip } g_t \wedge \text{Lip } \nabla g_t \leq L \text{ for } t \in [0,T]\}$ where $L$ is the maximum of $L_n$ and the Lipschitz coefficient of $\varphi_0$, with the Bielecki norm $\| g \|_\varphi = \max_{t \in [0,T]} e^{-\gamma t} (\| \varphi_t \|_\infty + \| \nabla \varphi_t \|_\infty)$. Note that, thanks to properties of heat kernel, $\varphi_t$ given by (19) and $\nabla \varphi_t$ are Lipschitz with coefficient $L$, so $P(E) \subset E$.

To prove that $P$ is contractive, take $g^{(1)}, g^{(2)} \in C_{[0,T]}(C^2(\mathbb{R}^d))$ and let $(X^{(i)}, M^{(i)})$ be the pathwise unique solutions to

$$
\begin{align*}
\begin{cases}
X^{(i)}(t) = X(0) + \int_0^t b(X^{(i)}(s), \nabla g^{(i)}(s, X^{(i)}(s))) ds + \sigma W(t),
M^{(i)}(t) = M(0) + \int_0^t \lambda_s(X^{(i)}(s), g^{(i)}(s, X^{(i)}(s))) M^{(i)}(s) ds,
\end{cases}
\end{align*}
$$

for $i = 1, 2$. Note that $M^{(i)}(t) \leq M(0) e^{\lambda t} = e^{\lambda t}$. Thus we have

$$
\begin{align*}
| M^{(1)}(t) - M^{(2)}(t) | &\leq \int_0^t | \lambda | M^{(1)}(s) - M^{(2)}(s) | + e^{\lambda t} L_\lambda (1 + L) \| X^{(1)}(s) - X^{(2)}(s) \| \\
&\qquad + e^{\lambda t} L_\lambda \| g^{(1)}_{s} - g^{(2)}_{s} \| \| ds,
\end{align*}
$$

$$
\begin{align*}
\| X^{(1)}(t) - X^{(2)}(t) \| &\leq \int_0^t [ e^{\lambda t} L_b (1 + L) \| X^{(1)}(s) - X^{(2)}(s) \| + L_b \| \nabla g^{(1)}_{s} - \nabla g^{(2)}_{s} \| \| \| ds.
\end{align*}
$$
So by Gronwall’s Lemma we have
\[(23) \quad |M^{(1)}(t) - M^{(2)}(t)| + \|X^{(1)}(t) - X^{(2)}(t)\| \leq c_1 \int_0^t \left( \|q_s^{(1)} - q_s^{(2)}\|_\infty + \|\nabla q_s^{(1)} - \nabla q_s^{(2)}\|_\infty \right) ds,\]
where \(c_1\) (and similarly \(c_2\) to \(c_4\) below) is a constant depend only on the bounds and Lipschitz coefficients of the functions \(K, \nabla K, \Delta K, \lambda\) and \(b\) and time \(T\). Moreover
\[\left| \kappa(y - X^{(1)}(t))M^{(1)}(t) - \kappa(y - X^{(2)}(t))M^{(2)}(t) \right| \leq c_2(|M^{(1)}(t) - M^{(2)}(t)| + \|X^{(1)}(t) - X^{(2)}(t)\|).\]
If \(\mu^{(i)}\) is given by (13) for \((X^{(i)}, M^{(i)})\) then
\[\kappa \ast \mu^{(i)}_t(y) = \mathbb{E} \left[ \kappa(y - X^{(i)}(t))M^{(i)}(t) \right].\]
Therefore, if \(\tilde{q}_t^{(1)}(x) = S_t q_0(x) + \alpha \int_0^t S_{t-s}[K \ast \mu^{(1)}_s](x) ds\) then
\[|\tilde{q}_t^{(1)}(x) - \tilde{q}_t^{(2)}(x)| = \left| \alpha \int_0^t \int_{\mathbb{R}^d} p(t-s, x, y)[K \ast \mu^{(1)}_s - K \ast \mu^{(2)}_s](y) dy ds \right| \leq c_3 \int_0^t \int_0^r \left( \|q_s^{(1)} - q_s^{(2)}\|_\infty + \|\nabla q_s^{(1)} - \nabla q_s^{(2)}\|_\infty \right) ds dr \]
and likewise
\[|\nabla \tilde{q}_t^{(1)}(x) - \nabla \tilde{q}_t^{(2)}(x)| \leq c_4 \int_0^t \int_0^r \left( \|q_s^{(1)} - q_s^{(2)}\|_\infty + \|\nabla q_s^{(1)} - \nabla q_s^{(2)}\|_\infty \right) ds dr.\]
Therefore,
\[\left\| \tilde{q}^{(1)} - \tilde{q}^{(2)} \right\| \leq (c_3 + c_4) \int_0^t \int_0^r e^{-\gamma(t-s)} \left\| \tilde{q}^{(1)}(s) - \tilde{q}^{(2)}(s) \right\| ds dr = (c_3 + c_4) \frac{e^{-\gamma t}(-\gamma t + e^{\gamma t} - 1)}{\gamma^2} \left\| \tilde{q}^{(1)} - \tilde{q}^{(2)} \right\|,\]
so, for sufficiently large \(\gamma\), \(P\) is contractive in \(\|\cdot\|_\gamma\).

Now we are ready to prove the well-posedness of the mean-field model.

**Proof of Theorem 2.** The scheme will be the following. For any function \(q \in C_{C_b^2(\mathbb{R}^d)}[0, T]\) we notice the existence and uniqueness of branching diffusion process given by (17) (a proof can be done as in the proof of Theorem 1). Then we show that \(\tilde{\mu}\) given by (10) is equal to \(\mu\) given by (13) for \((X, M)\) obtained as solution to (22) with the same \(q\). That means that, by Theorem 4, there exists a unique \(q\) such that (9) is satisfied.

To this aim, fix \(q : [0, T] \to C_{C_b^2(\mathbb{R}^d)}(\mathbb{R}^d)\) continuous in time and let process \(\tilde{z}(t)(X_j(t))_{j \in J}, t \in [0, T]\) be the solution to (17) with this fixed \(q\). For \(\varphi : \mathbb{R}^d \cup \{\phi\} \to \mathbb{R}\) such that \(\varphi(\phi) = 0\) and \(\varphi|_{\mathbb{R}^d} \in C_{C_b^2(\mathbb{R}^d)}\) denote
\[\langle \varphi, \tilde{z}(t) \rangle = \sum_{j \in J} \varphi(X_j(t)).\]

Note that this sum is finite. By Itô’s Lemma we have
\[\langle \varphi, \tilde{z}(t) \rangle = \langle \varphi, \tilde{z}(0) \rangle + \int_0^t \left\langle b(\cdot, \nabla \varphi(\tilde{z}(\cdot)), \nabla \varphi(\cdot) + \frac{\mu^2}{2} \Delta \varphi(\cdot), \tilde{z}(s) \right\rangle ds + \sigma \sum_{j \in J} \int_0^t \nabla \varphi(X_j(s)) dW_{1,j}(s) - \sum_{j \in J} \int_0^t \varphi(X_j(s^-)) \mathbf{1}_{[0, \lambda_0(X_j(s^-), \rho(s, X_j(s^-))) + \lambda_1(X_j(s^-), \rho(s, X_j(s^-)))]}(z) N_{1,j}(ds, dz)\]
\[+ \sum_{j \in J} \int_0^t 2\varphi(X_j(s^-)) \mathbf{1}_{[0, \lambda_0(X_j(s^-), \rho(s, X_j(s^-)))]}(z) N_{1,j}(ds, dz)\]
and thus
\[ \mathbb{E}\langle \varphi, x(t) \rangle = \mathbb{E}\langle \varphi, x(0) \rangle + \mathbb{E} \int_0^t \langle B_{\varphi, \varphi}, x(s) \rangle + \langle \lambda(\cdot, \varphi(s)), \varphi(\cdot), x(s) \rangle \, ds, \]
where \( B_{\varphi, \varphi}(x) = \frac{\sigma^2}{2} \Delta \varphi(x) + b(x, \nabla \varphi(s, x)) \nabla \varphi(x) \) for \( x \in \mathbb{R}^d \) and \( B_{\varphi, \varphi}(\phi) = 0 \), and \( \lambda = \lambda_b - \lambda_d \).

Let \( \xi_t = \sum_{j \in J} 1 \mathbb{E}(X_j(t)) \delta_{X_j(t)} \) and let \( \langle \varphi, \xi_t \rangle = \int_{\mathbb{R}^d} \varphi(x) \xi_t(dx) \). Then \( \langle \varphi, \xi_t \rangle = \langle \varphi, x(t) \rangle \) and
\[ \langle \varphi, \mathbb{E} \xi_t \rangle = \langle \varphi, \xi_t \rangle + \int_0^t \langle B_{\varphi, \varphi}, \mathbb{E} \xi_s \rangle + \langle \lambda(\cdot, \varphi(s)), \varphi(\cdot), \mathbb{E} \xi_s \rangle \, ds, \]
which means that
\[ \langle \varphi, \bar{\mu}_t \rangle = \langle \varphi, \bar{\mu}_0 \rangle + \int_0^t \langle B_{\varphi, \varphi}, \bar{\mu}_s \rangle + \langle \lambda(\cdot, \varphi(s)), \varphi(\cdot), \bar{\mu}_s \rangle \, ds. \]

This is the weak version of Equation (11a) and it is well known that it admits a unique solution which is absolutely continuous with respect to Lebesgue measure for \( t > 0 \) even if \( \bar{\mu}_0 \) is not.

Let now \((X, M)\) be a process obtained as a solution to (22) with \( g \) and \( \varphi \in C^2_b(\mathbb{R}^d) \). Then, by Itô formula we have
\[ \varphi(X(t))M(t) = \varphi(X(0))M(0) + \int_0^t \left[ M(s) \nabla \varphi(X(s)) \cdot b(X(s), \nabla g(s, X(s))) 
+ \varphi(X(s)) \lambda(X(s), g(s, X(s)))M(s) + \frac{\sigma^2}{2} \Delta \varphi(X(s)) \right] \, ds 
- \int_0^t \nabla \varphi(X(s)) \cdot b(X(s), \nabla g(s, X(s))) \, dW(s). \]

Note that for \( \mu_t \) given by (13) we have \( \langle \varphi, \mu_t \rangle = \int_{\mathbb{R}^d} \varphi(x) \mu_t(dx) = \mathbb{E}[\varphi(X(t))M(t)] \). Taking expectation on both sides of the equation above we get
\[ \langle \varphi, \mu_t \rangle = \int_0^t \langle B_{\varphi, \varphi}(\cdot) + \lambda(\cdot, g(s, \cdot)) \varphi(\cdot), \mu_s \rangle \, ds. \]
which is exactly the same as (24).

\[ \square \]

5.2. Proof of convergence.

Proof of Theorem 5. In order to prove the point (i) we check that the sequence \( \xi^n_t \) is tight on \( D_{\mathcal{M}|[0, t]} \) and then we check that the limit of any subsequence has to coincide with \( \bar{\mu}_t \). Similarly, we prove tightness of \( \bar{\xi}_t^n \) in \( C^2_{C_c(\mathbb{R}^d)}[0, T] \) and check that the limit has to satisfy (19).

Tightness of \( \{\xi^n_t\}_{n \in \mathbb{N}} \). The process \( \xi^n_t \) has values in the space \( \mathcal{M} \) of finite positive Radon measures on \( \mathbb{R}^d \). Note that the \( \mathcal{M} \) with the vague convergence topology can be metrizable, eg.

with metric
\[ d_{\mathcal{M}}(\mu, \nu) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, \langle \varphi_k, \mu - \nu \rangle \} \]

with some sequence \( \varphi_k \in C_c(\mathbb{R}^d) \) (cf. [2, Section §31]) in such a way that \( \mathcal{M} \) is complete. Moreover, set \( H \subset \mathcal{M} \) is vaguely relatively compact if and only if
\[ \sup_{\mu \in H} \langle f, \mu \rangle \text{ for all } f \in C_c(\mathbb{R}^d), \]
where one can take \( \langle \varphi_k \rangle_{k \in \mathbb{N}} \) instead of all \( f \in C_c(\mathbb{R}^d) \). Therefore, Proposition 1.7 from [21, Chapter 4] holds for processes with values in \( (\mathcal{M}, d_{\mathcal{M}}) \). Now, thanks to Aldous criterion (see, eg. [19, Chapter VI, Theorem 4.5]) for the relative compactness of \( \{\xi^n_t\} \) it suffices to check for all \( \varphi_k \) that for any \( \varepsilon > 0 \) there exists \( M > 0 \) such that
\[ \mathbb{P} \left( \langle \varphi_k, \xi^n_t \rangle > M \right) < \varepsilon, \text{ for all } t \in [0, T] \text{ and } n_0 \in \mathbb{N} \]
By Itô’s Lemma,\n
\[
\lim_{\gamma \to 0} \lim_{n_0 \to \infty} \sup_{\tau \in T, \theta < \gamma} \mathbb{P} \left( \left| \langle \varphi_k, \xi_{n_0}^{\tau + \theta} \rangle - \langle \varphi_k, \xi_{n_0}^{\tau} \rangle \right| > \varepsilon \right) = 0,
\]

where $T$ is the set of all stopping times bounded by $T$. Note that (26) follows by Markov’s inequality from Lemma 1. To prove (27), using Itô’s Lemma we calculate

\[
\langle \varphi_k, \xi_{n_0}^{\tau} \rangle = \langle \varphi_k, \frac{1}{n_0} \tilde{X}^{n_0}_i (t) \rangle = \langle \varphi_k, \frac{1}{n_0} \xi_{n_0}^{0} (0) \rangle + \frac{1}{n_0} \int_0^t \left( \sum_{i=1}^{n_0} \sum_{j \in J} b(X_{i,j}^{n_0}(s), \nabla \varphi_k(t, X_{i,j}^{n_0}(s))) \nabla \varphi_k(X_{i,j}^{n_0}(s)) + \frac{\sigma^2}{2} \Delta \varphi_k(X_{i,j}^{n_0}(s)) \right) ds
\]

\[
+ \frac{1}{n_0} \int_0^t \sum_{i=1}^{n_0} \sum_{j \in J} \nabla \varphi_k(X_{i,j}^{n_0}(s)) dW_{i,j}(s) + \frac{1}{n_0} \sum_{i=1}^{n_0} \sum_{s \leq t} \left( \langle \varphi_k(X_{i,j}^{n_0}(s)) - \varphi_k(X_{i,j}^{n_0}(\cdot)) \rangle \right)
\]

\[
= \langle \varphi_k, \xi_{n_0}^{\tau} \rangle + \int_0^t \left( \nabla \varphi_k(\cdot) + \frac{\sigma^2}{2} \Delta \varphi_k(\cdot) + \lambda(\cdot, g^{n_0}(\cdot, \cdot)) \varphi_k(\cdot), \xi_{n_0}^{\tau} \right) ds
\]

\[
M_{1,k}(t) \left\{ \begin{array}{l}
\frac{\sigma^2}{n_0} \sum_{i=1}^{n_0} \sum_{j \in J} \int_0^t \nabla \varphi_k(X_{i,j}^{n_0}(s)) dW_{i,j}(s)
\end{array} \right.
\]

\[
M_{2,k}(t) \left\{ \begin{array}{l}
\frac{1}{n_0} \sum_{i=1}^{n_0} \sum_{j \in J} \int_0^t \varphi_k(X_{i,j}^{n_0}(s)) - \varphi_k(X_{i,j}^{n_0}(s)) ds
\end{array} \right.
\]

where $M_{1,k}(t)$ and $M_{1,k}(t)$ are matringales. Therefore,

\[
\langle \varphi_k, \xi_{n_0}^{\tau + \theta} - \xi_{n_0}^{\tau} \rangle = \int_0^{\tau + \theta} \left( \nabla \varphi_k(\cdot) + \frac{\sigma^2}{2} \Delta \varphi_k(\cdot) + \lambda(\cdot, g^{n_0}(\cdot, \cdot)) \varphi_k(\cdot), \xi_{n_0}^{\tau} \right) ds
\]

\[
+ M_{1,k}^{n_0}(\tau + \theta) - M_{1,k}^{n_0}(\tau) + M_{2,k}^{n_0}(\tau + \theta) - M_{2,k}^{n_0}(\tau)
\]

The integral over $ds$ can be estimated by a constant times $\sup_{s \in [0,T]} \langle 1, \xi_{n_0}^{s} \rangle$, so, since $\theta \leq \gamma$, by Lemma 1 and Markov inequality, probability that it is greater then $\varepsilon$ goes to zero as $\gamma \to 0$. By Itô’s Lemma,

\[
(M_{1,k}^{n_0}(t))^2 = \frac{\sigma^2}{n_0} \sum_{i=1}^{n_0} \sum_{j \in J} \int_0^t \left( \nabla \varphi_k(X_{i,j}^{n_0}(s)) \right)^2 ds + \tilde{M}_{1,k}^{n_0}(t),
\]

where $\tilde{M}_{1,k}^{n_0}(t)$ is a martingale, so

\[
\mathbb{E} \left[ (M_{1,k}^{n_0}(\tau + \theta) - M_{1,k}^{n_0}(\tau))^2 \right] = \mathbb{E} \left[ (M_{1,k}^{n_0}(\tau + \theta))^2 \right] - \mathbb{E} \left[ (M_{1,k}^{n_0}(\tau))^2 \right] =
\]

\[
\mathbb{E} \left[ \frac{\sigma^2}{n_0} \sum_{i=1}^{n_0} \sum_{j \in J} \int_0^{\tau + \theta} \left( \nabla \varphi_k(X_{i,j}^{n_0}(s)) \right)^2 ds \right] \leq \theta \frac{\sigma^2}{n_0} \left\| \nabla \varphi_k \right\|_{\infty} \mathbb{E} \left[ \sup_{s \in [0,T]} \langle 1, \xi_{n_0}^{s} \rangle \right].
\]
Similarly, using Itô’s formula again we get

\[
(M_{2,k}^n(t))^2 - 2M_{2,k}^n(s) \langle \lambda(\cdot, \vartheta^n(s, \cdot)) \varphi_k(\cdot), \xi^n_s \rangle \, ds
- \sum_{i=1}^{n_0} \sum_{j \in J} \int_0^t \left( \frac{1}{n_0} \varphi_k(X_{i,j}^n(s^-))^2 - \frac{2}{n_0} M_{2,k}^n(s) \varphi_k(X_{i,j}^n(s^-)) \right)
\times \mathbb{I}_{[0,\lambda_0(t,s), \vartheta^n(s-s^-), \varphi_k(s-s^-)]}(z) \mathcal{N}_{i,j}(ds, dz)
+ \sum_{i=1}^{n_0} \sum_{j \in J} \int_0^t \left( \frac{4}{n_0} \varphi_k(X_{i,j}^n(s^-))^2 + \frac{4}{n_0} M_{2,k}^n(s) \varphi_k(X_{i,j}^n(s^-)) \right)
\times \mathbb{I}_{[0,\lambda_0(t,s), \vartheta^n(s-s^-), \varphi_k(s-s^-)]}(z) \mathcal{N}_{i,j}(ds, dz)
\]

\[= \frac{1}{n_0} \int_0^t \langle [3 \lambda_0(\cdot, \vartheta^n(s-s^-)) - \lambda_0(\cdot, \vartheta^n(s-s^-)) \varphi_k(s-s^-)]^2, \xi^n_s \rangle \, ds + \tilde{M}_{2,k}^n(t).
\]

where \(\tilde{M}_{2,k}^n(t)\) is a martingale, thus again

\[
\mathbb{E} \left[ (M_{2,k}^n(t + \theta) - M_{2,k}^n(t))^2 \right] \leq \theta \frac{3\lambda}{n_0} \| \varphi_k \|_\infty \mathbb{E} \left[ \sup_{s \in [0,T]} \{1, \xi^n_s \} \right],
\]

which completes the proof of \((27)\).

**Tightness of \(\{\vartheta^n\}_{n \in \mathbb{N}}.** Although it would be sufficient to use the topology of locally uniform convergence, we need a stronger convergence in the next proof, so let us consider \(C_b^1(\mathbb{R}^d)\) with the topology of locally uniform convergence of function and its derivative, that is we use a norm

\[
\| f \|_{C_b^1} = \sum_{R=1}^{\infty} \frac{1}{\gamma R} \left( \sup_{x \in B(0, R)} |f(x)| + \sup_{x \in B(0, R)} \| \nabla f(x) \| \right).
\]

Note that a set \(K_M = \{ f \in C_b^1(\mathbb{R}^d) : \| f \|_{C_b^1} \leq M, \| \nabla f \|_{C_b^1} \leq M, |\text{Hess} f|_{C_b^1} \leq M \}, \) where \(|\text{Hess} f|_{C_b^1} = \sup_{x \in \mathbb{R}^d} \max_{1 \leq i, j \leq d} |\partial_i \partial_j f(x)|\), is relatively compact in this norm. Using the version of Ascoli Theorem (see eg. \([27, \text{Theorem } 47.1]\)) we know that a family \(\mathcal{K}_M \subset C_T = C_{C_b^1(\mathbb{R}^d)} [0, T] \) of functions \(g\), which are equicontinuous in \(t\) and such that \(\{g(t) : g \in \mathcal{K}_M\} \subset \mathcal{K}_M \) for each \(t \in [0, T]\), is relatively compact. Now, in order to prove prove tightness of \(\vartheta^n_t\), we need to check that for any \(\varepsilon > 0\) there exists \(M > 0\) such that \(\mathbb{P}(\vartheta^n \in \mathcal{K}_M) > 1 - \varepsilon\). To that end, recall that \(\vartheta^n_t = S_t \rho_0 + \alpha \int_t^{t+\theta} S_{t-s} [\kappa * \xi^n_s](x) \, ds\) where the first summand is continuous in \(C_T\) and the latter is Lipschitzian with probability \(1 - \varepsilon\), because

\[
|\alpha \int_t^{t+\theta} S_{t-s} [\kappa * \xi^n_s](x) \, ds| \leq \theta \alpha \| \kappa \|_{\infty} \sup_{s \in [0,T]} \{1, \xi^n_s\}
\]

and

\[
|\alpha \nabla \int_t^{t+\theta} S_{t-s} [\kappa * \xi^n_s](x) \, ds| \leq \theta \alpha \| \nabla \kappa \|_{\infty} \sup_{s \in [0,T]} \{1, \xi^n_s\}.
\]

Moreover, we have

\[
|\vartheta^n_t(x)| \leq |S_t \rho_0(x)| + \sup_{s \in [0,T]} \| \kappa * \xi^n_s \|_{\infty} \leq \| \rho_0 \|_{\infty} + t \| \kappa \|_{\infty} \sup_{s \in [0,T]} \{1, \xi^n_s\},
\]

similarly

\[
|\nabla \vartheta^n_t(x)| \leq \| \nabla \rho_0 \|_{\infty} + t \| \nabla \kappa \|_{\infty} \sup_{s \in [0,T]} \{1, \xi^n_s\},
\]

and further

\[
|\partial_i \partial_j \vartheta^n_t(x)| \leq \| \partial_i \partial_j \rho_0 \|_{\infty} + t \| \partial_i \partial_j \kappa \|_{\infty} \sup_{s \in [0,T]} \{1, \xi^n_s\}.
\]
These estimates with Lemma 1 and Markov’s inequality complete the proof of tightness.

**Identification of the limit.** By the similar estimates as in (28) and (29) we get that the limit has to satisfy (24) for any \( \varphi \in C^2_c(\mathbb{R}^d) \) and (19) which admit a unique solution. □

5.3. **Proof of Theorem 6.** Fix \( T > 0 \) and \( \varepsilon > 0 \). We use here the coupling of \( \tilde{x}_1(t) \) and \( \tilde{z}(t) \) obtained by using the same processes \( W_{1,j} \) and \( N_{1,j} \), and the fact from Theorem 5 that \( g^{n_0} \) converges in probability to \( g \).

Let \( \tilde{\sigma}_j \) and \( \tilde{\tau}_j \) denote times of birth and death, respectively, of the \( j \)-th particle of \( \tilde{z}(t) \) and let us construct such a process \( \tilde{x}^{n_0}_1 \) that its \( j \)-th particle lives from \( \tilde{\sigma}_j \) to \( \tilde{\tau}_j \) and moves during this time according to the equation

\[
\tilde{X}^{n_0}_{1,j}(t) = \tilde{X}^{n_0}_{1,j}(\tilde{\sigma}_j) + \int_{\tilde{\sigma}_j}^t b(\tilde{X}^{n_0}_{1,j}(s), \nabla g^{n_0}(s, \tilde{X}^{n_0}_{1,j}(s))) \, ds + \sigma (W_{1,j}(t) - W_{1,j}(\tilde{\sigma}_j)),
\]

for \( t \in [\tilde{\sigma}_j, \tilde{\tau}_j] \). It means that the particles of \( \tilde{x}_1^{n_0} \) die and are born in the same times as particles of \( \tilde{z} \) but their dynamics is the same as the dynamics of \( x_1^{n_0} \). The idea is to prove, that for \( n_0 \) large enough with high probability \( \tilde{x}_1^{n_0} \) is close to \( \tilde{z} \) and equal to \( x_1^{n_0} \).

**Lemma 2.** Fix \( T > 0 \). For any \( \eta > 0 \) we can find a set \( \Omega_\eta \) such that \( \mathbb{P}(\Omega_\eta) > 1 - \eta \) and

(i) there exists \( \tilde{N} > 0 \) such that \( \sup_{t \in [0,T]} \langle 1, \tilde{z}(t) \rangle \leq \tilde{N} \) i.e. there is at most \( \tilde{N} \) particles of \( \tilde{z}(t) \) alive to time \( T \)

(ii) there exists \( R > 0 \) such that all particles of \( \tilde{z}(t) \) live in the ball of radius \( R \), i.e.

\[
\sup_{t \in [0,T]} \max_{j \in J} \| \tilde{X}^{n_0}_j \| \leq R
\]

for \( \omega \in \Omega_\eta \).

**Proof.** The first point is a simple consequence of Lemma 1. Once we have finite number of particles, their positions are described by a finite number of Itô equations with bounded drift \( b \), so (ii) obviously follows. □

Let us denote by \( J_\tilde{N} \) the (finite) set of all indices of the length at most \( \tilde{N} \). Apparently, if there were not more than \( \tilde{N} \) particles of \( \tilde{z}(t) \) up to time \( T \), then their indices are in \( J_\tilde{N} \).

**Lemma 3.** For any \( \eta > 0 \) there exists \( \delta > 0 \) such that if

\[
\sup_{t \in [0,T]} \left( \sup_{x \in B(0,R)} \| g^{n_0}(x) - g(x) \| + \sup_{x \in B(0,R)} \| \nabla g^{n_0}(x) - \nabla g(x) \| \right) < \delta,
\]

where \( R \) is from Lemma 2, then the probability that process \( \tilde{x}_1^{n_0} \) is different than \( x_1^{n_0} \) is less than \( 3\eta \).

**Proof.** Throughout the proof we assume we are in \( \Omega_\eta \) from Lemma 2, \( \tilde{N} \) and \( R \) are as in Lemma 2 and every particle of \( \tilde{z} \) alive during \( [0,T] \) has index \( j \in J_\tilde{N} \). The movement of particles of \( \tilde{x}_1^{n_0} \) and \( x_1^{n_0} \) is given by the same equation, so the processes are different if and only if any time of birth or death is different. Recall that \( \bar{\sigma}_{1,j} \) and \( \bar{\tau}_{1,j} \) given by (5) and (6), and \( \tilde{\sigma}_j \) and \( \tilde{\tau}_j \) analogously with \( x_1^{n_0,j} \) and \( g^{n_0,j} \) replaced by \( \tilde{X}_j \) and \( g \). Therefore, if for every \( j \in J_\tilde{N} \) there are no points of \( N_{1,j} \) inbetween \( \lambda(\tilde{X}_j(t), g^{n_0}(t, \tilde{X}_j(t))) \) and \( \lambda(\tilde{X}_j(t), g(t, \tilde{X}_j(t))) \) nor between \( \lambda_b(\tilde{X}_j(t), g^{n_0}(t, \tilde{X}_j(t))) \) and \( \lambda_b(\tilde{X}_j(t), g(t, \tilde{X}_j(t))) \) for \( t \in [\tilde{\sigma}_j, \tilde{\tau}_j] \), then for all \( j \in J_\tilde{N} \) we have

\[
\bar{\sigma}_{1,j} = \tilde{\sigma}_j, \quad \bar{\tau}_{1,j} = \tilde{\tau}_j, \quad \bar{\lambda}_{1,j} = \tilde{\lambda}_{1,j}.
\]

Note that for \( t \in [\tilde{\sigma}_j, \tilde{\tau}_j] \) we have

\[
\tilde{X}^{n_0}_{1,j}(t) - \tilde{X}_j(t) = \tilde{X}^{n_0}_{1,j}(\tilde{\sigma}_j) - \tilde{X}_j(\tilde{\sigma}_j)
\]

\[
+ \int_{\tilde{\sigma}_j}^t \left[ b(\tilde{X}^{n_0}_{1,j}(s), \nabla g^{n_0}(s, \tilde{X}^{n_0}_{1,j}(s))) - b(\tilde{X}_j(s), \nabla g(s, \tilde{X}_j(s))) \right] \, ds.
\]
where
\[ \left| b(\tilde{X}_{i,j}^n(t), \nabla g^n(s, \tilde{X}_{i,j}^n(s))) - b(\tilde{X}_j(s), \nabla g(s, \tilde{X}_j(s))) \right| \leq L_b \left( \left| \tilde{X}_{i,j}^n(s) - \tilde{X}_j(s) \right| + \left| \nabla g^n(s, \tilde{X}_{i,j}^n(s)) - \nabla g^n(s, \tilde{X}_j(s)) \right| + \left| \nabla g^n(s, \tilde{X}_j(s)) - \nabla g(s, \tilde{X}_j(s)) \right| \right). \]

By (31) and (32), functions \( g^n \) and \( \nabla g^n \) are Lipschitzian with some constant \( L_{g,\eta} \) with probability greater than \( 1 - \eta \), so if we denote \( \Delta_1(t) = \max_{j \in J_N} \left| \tilde{X}_{i,j}^n(t) - \tilde{X}_j(t) \right| \) then by (33)
\[ \Delta_1(t) \leq L_b \int_0^t \left( (1 + L_{g,\eta}) \Delta_1(s) + \sup_{t \in [0,T]} \sup_{x \in B(0,R)} \| \nabla g^n(t, x) - \nabla g^n(t, \tilde{X}_j(s)) \| \right) ds.
\]

By Gronwall’s inequality we have \( \Delta_1(t) \leq \delta t L_{e} e^{L_{g,\eta} t} \), so
\[ \sup_{t \in [0,T]} \Delta_1(t) \leq c^{(1)}_{T,\eta} \delta. \]

and thus
\[ \left| \lambda(X_{i,j}^n(0, t), g^n(t, X_{i,j}^n(0, t))) - \lambda(X_{i,j}(t), g(t, \tilde{X}_j(t))) \right| \leq L_\lambda((1 + L_{g,\eta}) \Delta_1(t)) \leq c^{(2)}_{T,\eta} \delta, \]
where \( c^{(i)}_{T,\eta}, i = 1, 2 \) depend only on \( T \) and \( L_{g,\eta} \). Similar estimates hold for \( \lambda_b \). Let
\[ A_j = \left\{ (t, z) \in [0, T] \times [0, \tilde{\lambda}] : t \in [\tilde{\sigma}_j, \tilde{\tau}_j], z \in [\tilde{\lambda}^n_b(t), \tilde{\lambda}^n_b(t)] \text{ or } z \in [\tilde{\lambda}^n(t), \tilde{\lambda}^n(t)] \right\}, \]
where \( \tilde{\lambda}^n(t) = \min\{\lambda(X_{i,j}^n(t), g^n(t, X_{i,j}^n(t))), \lambda(X_{i,j}(t), g(t, \tilde{X}_j(t)))\} \) and \( \tilde{\lambda}^n(t) = \max\{\lambda(X_{i,j}^n(t), g^n(t, X_{i,j}^n(t))), \lambda(X_{i,j}(t), g(t, \tilde{X}_j(t)))\} \) and analogously for \( \lambda_b \). By (35) the area of \( A_j \) is less than \( T \delta \) for any \( j \in J_N \). Therefore, taking \( \delta \) sufficiently small we have
\[ \mathbb{P} \left( \mathcal{N}_{i,j}(A_j) > 0 \text{ for any } j \in J_N \right) < \eta. \]

Now we are ready to prove Theorem 6.

**Proof of Theorem 6.** Fix \( \epsilon > 0 \). We have to prove that \( \mathbb{P} \left( \sup_{t \in [0,T]} d_X(\tilde{X}_{i,j}^n(t), \tilde{X}(t)) > \epsilon \right) \) tends to 0 as \( n_0 \to \infty \). To that end fix \( \eta > 0 \), and take \( R > 0 \) from Lemma 2 for this \( \eta \). Take \( \delta > 0 \) small enough for Lemma 3 to be satisfied and such that \( c^{(1)}_{T,\eta} \delta \leq \epsilon \) in (34). Since, by Theorem 5, \( g^n \) converges to \( g \) in probability on \( C_T \) in the norm \( \| f(\cdot, \cdot) \|_{C_T} = \sup_{t \in [0,T]} \| f(t, \cdot) \|_{C_T} \), for sufficiently big \( n_0 \) we have (33) with probability \( 1 - \eta \). Using Lemma 3 we know that with probability at least \( 1 - 3\eta \) we have \( \tilde{X}_{i,j}^n = \tilde{X}_{i,j}^n \) and \( \sup_{t \in [0,T]} d_X(\tilde{X}_{i,j}^n(t), \tilde{X}(t)) > \epsilon \) by (34). \( \square \)

**References**

[1] Vincent Bansaye and Florian Simatos, *On the scaling limits of Galton-Watson processes in varying environments*, Electron. J. Probab. 20 (2015), no. 75, 36. MR 3371434

[2] Heinz Bauer, *Measure and integration theory*, De Gruyter Studies in Mathematics, vol. 26, Walter de Gruyter & Co., Berlin, 2001, Translated from the German by Robert B. Burckel. MR 1897176

[3] Silvia Boi, Vincenzo Capasso, and Daniela Morale, *Modeling the aggregative behavior of ants of the species Polyergus rufescens*, vol. 1, 2000, Spatial heterogeneity in ecological models (Alcalá de Henares, 1998), pp. 163–176. MR 1794944

[4] François Bolley, Arnaud Guillin, and Cédric Villani, *Quantitative concentration inequalities for empirical measures on non-compact spaces*, Probab. Theory Related Fields 137 (2007), no. 3-4, 541–593. MR 2280433
Federica Bubba, Tommaso Lorenzi, and Fiona R. Macfarlane, \textit{From a discrete model of chemotaxis with volume-filling to a generalized Patlak-Keller-Segel model}, Proc. A. \textbf{476} (2020), no. 2237, 20190871, 19. MR 4111971

Evelyn Buckwar and Martin G. Riedler, \textit{An exact stochastic hybrid model of excitable membranes including spatio-temporal evolution}, J. Math. Biol. \textbf{63} (2011), no. 6, 1051–1093. MR 2855804

Amarjit Budhiraja and John Lygeros, \textit{Toward a general theory of stochastic hybrid systems}, Stochastic hybrid systems, Lect. Notes Control Inf. Sci., vol. 337, Springer, Berlin, 2006, pp. 3–30. MR 2246646

Vincenzo Capasso and Franco Flandoli, \textit{On the mean field approximation of a stochastic model of tumor-induced angiogenesis}, European Journal of Applied Mathematics \textbf{30} (2019), no. 4, 619–658.

Vincenzo Capasso and Daniela Morale, \textit{A multiscale approach leading to hybrid mathematical models for angiogenesis: the role of randomness}, Mathematical methods and models in biomedicine, Lect. Notes Math. Model. Life Sci., Springer, New York, 2013, pp. 87–115. MR 3013068

Vincenzo Capasso, Daniela Morale, and Giuseppe Facchetti, \textit{The role of stochasticity in a model of retinal angiogenesis}, IMA J. Appl. Math. \textbf{77} (2012), no. 6, 729–747. MR 2999135

Vincenzo Capasso and Radoslaw Wieczorek, \textit{A hybrid stochastic model of retinal angiogenesis}, Math. Methods Appl. Sci. \textbf{43} (2020), no. 18, 10578–10592. MR 4177212

P. Cattiaux, A. Guillin, and F. Malrieu, \textit{Probabilistic approach for granular media equations in the non-uniformly convex case}, Probab. Theory Related Fields \textbf{140} (2008), no. 1-2, 19–40. MR 2357669

Nicolas Fournier and Benjamin Jourdain, \textit{Stochastic particle approximation of the Keller-Segel equation and two-dimensional generalization of Bessel processes}, Ann. Appl. Probab. \textbf{27} (2017), no. 5, 2807–2861. MR 3719947

Nancy L. Garcia and Thomas G. Kurtz, \textit{Spatial birth and death processes as solutions of stochastic equations}, ALEA Lat. Am. J. Probab. Math. Stat. \textbf{1} (2006), 281–303. MR 2249658

T. Hillen and K. J. Painter, \textit{A user’s guide to PDE models for chemotaxis}, J. Math. Biol. \textbf{58} (2009), no. 1-2, 183–217. MR 2448428

Dirk Horstmann, \textit{From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I.}, Jahresber. Deutsch. Math.-Verein. \textbf{105} (2003), no. 3, 103–165. MR 2013508

H. J. Hwang, K. Kang, and A. Stevens, \textit{Drift-diffusion limits of kinetic models for chemotaxis: a generalization}, Discrete Contin. Dyn. Syst. Ser. B \textbf{15} (2010), no. 2, 319–334. MR 2129381

Jean Jacod and Albert N. Shiryaev, \textit{Limit theorems for stochastic processes}, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 288, Springer-Verlag, Berlin, 1987. MR 959133

Evelyn F. Keller and Lee A. Segel, \textit{Initiation of slime mold aggregation viewed as an instability}, J. Theoret. Biol. \textbf{26} (1970), no. 3, 399–415. MR 3925816

Claude Kipnis and Claudio Landim, \textit{Scaling limits of interacting particle systems}, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 320, Springer-Verlag, Berlin, 1999. MR 1707314

Peter M. Kotelenez and Thomas G. Kurtz, \textit{Macroscopic limits for stochastic partial differential equations of McKean-Vlasov type}, Probab. Theory Related Fields \textbf{146} (2010), no. 1-2, 189–222. MR 2550362

Lei Li, Jian-Guo Liu, and Pu Yu, \textit{On the mean field limit for Brownian particles with Coulomb interaction in 3D}, J. Math. Phys. \textbf{60} (2019), no. 11, 115101, 34. MR 4026330

S. R. McDougall, M. G. Watson, and Fiona R. Macfarlane, \textit{A hybrid discrete-continuum mathematical model of pattern prediction in the developing retinal vasculature}, Bull. Math. Biol. \textbf{74} (2012), no. 10, 2272–2314. MR 2978781

Sylvie Méléard, \textit{Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models}, Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995), Lecture Notes in Math., vol. 1627, Springer, Berlin, 1996, pp. 42–95. MR 1431299

D. Morale, V. Capasso, and K Oelschläger, \textit{An interacting particle system modelling aggregation behavior: from individuals to populations}, J. Math. Biol. \textbf{50} (2005), 49–66.

James R. Munkres, \textit{Topology}, Prentice Hall, Inc., Upper Saddle River, NJ, 2000, Second edition of [ MR0464128]. MR 3728284

K. Oelschläger, \textit{A martingale approach to the law of large numbers for weakly interacting stochastic processes.}, Ann. Probab. \textbf{12} (1984), 458–479.

Karl Oelschläger, \textit{On the derivation of reaction-di ffusion equations as limit dynamics of systems of moderately interacting stochastic processes}, Probab. Theory Related Fields \textbf{82} (1989), no. 4, 565–586. MR 1002901

Clifford S. Patlak, \textit{Random walk with persistence and external bias}, Bull. Math. Biophys. \textbf{15} (1953), 311–338. MR 81586
[31] Ryszard Rudnicki and Radoslaw Wieczorek, Phytoplankton dynamics: from the behaviour of cells to a transport equation, Math. Model. Nat. Phenom. 1 (2006), no. 1, 83–100. MR 2318468
[32] Angela Stevens, The derivation of chemotaxis equations as limit dynamics of moderately interacting stochastic many-particle systems, SIAM J. Appl. Math. 61 (2000), no. 1, 183–212. MR 1776393
[33] Alain-Sol Sznitman, Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated, J. Funct. Anal. 56 (1984), no. 3, 311–336. MR 743844
[34] ———, Topics in propagation of chaos, École d’Été de Probabilités de Saint-Flour XIX-1989, Lecture Notes in Math., vol. 1464, Springer, Berlin, 1991, pp. 165–251. MR 1108185
[35] Meng Wang, Roseanne M. Ford, and Ronald W. Harvey, Coupled Effect of Chemotaxis and Growth on Microbial Distributions in Organic-Amended Aquifer Sediments: Observations from Laboratory and Field Studies, Environmental Science & Technology 42 (2008), no. 10, 3556–3562.
[36] Xuefeng Wang, Qualitative behavior of solutions of chemotactic diffusion systems: effects of motility and chemotaxis and dynamics, SIAM J. Math. Anal. 31 (2000), no. 3, 535–560. MR 1740723
[37] Radoslaw Wieczorek, A stochastic particles model of fragmentation process with shattering, Electron. J. Probab. 20 (2015), no. 86, 1–17.
[38] D.E. Woodward, R. Tyson, M.R. Myerscough, J.D. Murray, E.O. Budrene, and H.C. Berg, Spatio-temporal patterns generated by Salmonella typhimurium, Biophysical Journal 68 (1995), no. 5, 2181–2189.