THE ASYMPTOTIC PROFILE OF AN ETA-THETA QUOTIENT RELATED TO ENTANGLEMENT ENTROPY IN STRING THEORY

JOSHUA MALES

Abstract. In this paper we investigate a certain eta-theta quotient which appears in the partition function of entanglement entropy. Employing Wright’s circle method, we give its bivariate asymptotic profile.

1. Introduction and Statement of Results

Modern mathematical physics in the direction of string theory and black holes is intricately linked to number theory - for example work of Dabholkar, Murthy, and Zagier in relation to quantum black holes and wall crossing relates certain mock modular forms to the physical phenomena [6]. Similarly, the connections between automorphic forms and a second quantised string theory are described in [7], and modular forms for certain elliptic curves and their realisation in string theory is discussed in [13]. Further, the recent paper [10] discusses in depth the links between work of the enigmatic Ramanujan in relation to modular forms and their generalisations, and string theoretic objects (and indeed, why such links should be expected).

Knowledge of the behaviour of the modular objects aid the descriptions of physical phenomena. For instance, in [9], the authors use the classical number-theoretic Jacobi triple product identity to demonstrate the supersymmetry of the open-string spectrum using RNS fermions in light-cone gauge (see also [20]). In particular, parts of physical partition functions are often modular or mock modular objects. For example, the partition functions of the Melvin model [17] and the conical entropy of both the open and closed superstring [11] both involve the weight $-3$ and index $0$ meromorphic Jacobi form

$$f(z; \tau) := \frac{\vartheta(z; \tau)^4}{\eta(\tau)^9 \vartheta(2z; \tau)},$$

where $\eta$ is the Dedekind eta function given by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n),$$

and

$$\vartheta(z; \tau) := i^{\frac{1}{2}} q^{\frac{1}{8}} \prod_{n \geq 1} ((1 - q^n)(1 - \zeta q^n)(1 - \zeta^{-1} q^{n-1})$$

is the Jacobi theta function, with $\zeta := e^{2\pi iz}$ for $z \in \mathbb{C}$, and $q := e^{2\pi i \tau}$ with $\tau \in \mathbb{H}$, the upper half-plane.
We are particularly interested in the coefficients of the $q$-expansion of $f$ where $0 \leq z \leq 1$, away from the pole at $z = 1/2$, where the residue of $f$ is calculated in [20] - the other residues may be calculated using the elliptic transformation formulae for $f$. For instance, the asymptotic behaviour of the coefficients is required in order to investigate the UV limit. For a fixed value of $z$ the problem of finding the asymptotics of the coefficients is elementary, as [11] notes. In particular, fixing $z = \frac{h}{k}$ a rational number with gcd$(h, k) = 1$ and $0 \leq h < \frac{k}{2}$, then classical results in the theory of modular forms (see Theorem 15.10 of [3] for example) give that the coefficients of $f(\frac{h}{k}; \tau) = \sum_{n \geq 0} a_{h,k}(n)q^n$ behave asymptotically as

$$a_{h,k}(n) \sim \frac{(\frac{h}{k})^{\frac{7}{4}}}{2\sqrt{\pi n}} n^{-\frac{9}{4}} e^{\frac{4\pi}{\sqrt{2}} \frac{n}{k}}.$$

In the present paper, we let

$$f(z; \tau) = \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} b(m, n)\zeta^m q^n.$$

and investigate the coefficients $b(m, n)$; in particular we want to compute the bivariate asymptotic profile of $b(m, n)$ for a certain range of $m$.

In [2], the authors introduce techniques in order to compute the bivariate asymptotic behaviour of coefficients for a Jacobi form, in order to answer Dyson’s conjecture on the bivariate asymptotic behaviour of the partition crank. This method is used in numerous other papers - for example, in relation to the rank of a partition [8], ranks and cranks of cubic partitions [12], and certain genera of Hilbert schemes [14], along with many other partition-related statistics.

Using Wright’s circle method [21, 22] and following the same approach as [2] we show the following theorem.

**Theorem 1.1.** For $\beta := \pi \sqrt{\frac{2}{n}}$ and $|m| \leq \frac{1}{6\beta} \log(n)$ we have that

$$b(m, n) = (-1)^{m+1+\delta} \frac{\beta^5}{\pi^3|m|^\frac{1}{2}(2\pi)^{\frac{1}{2}}(2n)^{\frac{1}{2}}} e^{2\pi \sqrt{2n}} + O \left( |m|^{-\frac{2}{3}} n^{-\frac{1}{4}} e^{2\pi \sqrt{2n}} \right)$$

as $n \to \infty$. Here, $\delta := 1$ if $m < 0$ and $\delta = 0$ otherwise.

**Remark.** Although our approach is similar to [2, 8], in some places we require a little more care, since finding the Fourier coefficients require taking an integral over a path where $f$ has a pole. In this case, we turn to the framework of [6] - this is explained explicitly in Section 3.

We begin in Section 2 by recalling relevant results that are pertinent to the rest of the paper. In Section 3.1 we investigate the behaviour of $f$ toward the dominant pole $q = 1$. We follow this is Section 3.2 by bounding the contribution away from the pole at $q = 1$. We finish in Section 4 by applying Wright’s circle method to find the asymptotic behaviour of $b(m, n)$ and hence prove Theorem 1.1.
The author would like to thank Kathrin Bringmann for initially suggesting the project, as well as insightful conversations and useful comments on the contents of the paper.

2. Preliminaries

2.1. Properties of \( \vartheta \) and \( \eta \). When determining the asymptotic behaviour of \( f \) we will require the modularity behaviour of both \( \vartheta \) and \( \eta \). It is well-known that \( \vartheta \) satisfies the following lemma (see e.g. [15]).

**Lemma 2.1.** The function \( \vartheta \) satisfies the following transformation properties.

\[
\begin{align*}
(1) \quad \vartheta(-z; \tau) &= -\vartheta(z; \tau) \\
(2) \quad \vartheta(z + 1; \tau) &= -\vartheta(z; \tau) \\
(3) \quad \vartheta(z; \tau) &= \frac{i}{\sqrt{-4\tau}} e^{-\frac{\pi iz^2}{\tau}} \vartheta \left( \frac{z}{\tau}; -\frac{1}{\tau} \right)
\end{align*}
\]

Further, we have the following well-known modular transformation formula of \( \eta \) (see e.g. [18]).

**Lemma 2.2.** We have that

\[
\eta \left( \frac{-1}{\tau} \right) = \sqrt{-i\tau} \eta(\tau).
\]

2.2. Euler Polynomials. We will also make use properties of the Euler polynomials \( E_r \), defined by the generating function

\[
\frac{2e^{xt}}{1 + e^t} = \sum_{r \geq 0} E_r(t) \frac{t^r}{r!}.
\]

Lemma 2.2 of [2] shows that the following lemma holds.

**Lemma 2.3.** We have

\[
-\frac{1}{2} \mathrm{sech}^2 \left( \frac{t}{2} \right) = \sum_{r \geq 0} E_{2r+1}(0) \frac{t^{2r}}{(2r)!}.
\]

Further, Lemma 2.3 of [2] gives the following integral representation for the Euler polynomials.

**Lemma 2.4.** We have that

\[
\mathcal{E}_j := \int_0^{\infty} \frac{w^{2j+1}}{\sinh(\pi w)} dw = \frac{(-1)^{j+1} E_{2j+1}(0)}{2}.
\]
2.3. A particular bound. In Section 3.2 we require a bound on the size of

\[ P(q) := \frac{q^{\frac{1}{2\pi}}}{\eta(\tau)}, \]

away from the pole at \( q = 1 \). For this we use the following lemma which is shown to hold in Lemma 3.5 of [2].

**Lemma 2.5.** Let \( \tau = u + iv \in \mathbb{H} \) with \( Mv \leq u \leq \frac{1}{2} \) for \( u > 0 \) and \( v \to 0 \). Then

\[ |P(q)| \ll \sqrt{v} \exp \left[ \frac{1}{v} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1 + M^2}} \right) \right) \right]. \]

In particular, with \( v = \frac{\beta}{2\pi}, \ u = \frac{\beta m - \frac{1}{2} \pi}{2\pi} \) and \( M = m^{-\frac{1}{4}} \) this gives for \( 1 \leq x \leq \frac{\pi m}{\beta} \) the bound

\[ |P(q)| \ll n^{-\frac{1}{4}} \exp \left[ \frac{2\pi}{\beta} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-\frac{1}{4}}}} \right) \right) \right]. \] (3.1)

2.4. \( I \)-Bessel functions. Here we recall relevant results on the \( I \)-Bessel function defined by

\[ I_\ell(x) := \frac{1}{2\pi i} \int_{\Gamma} t^{\ell-1} e^{\frac{t}{2}(x+\frac{1}{x})} dt, \]

where \( \Gamma \) is a contour which starts in the lower half plane at \( -\infty \), surrounds the origin counterclockwise and returns to \( -\infty \) in the upper half-plane. We are particularly interested in the asymptotic behaviour of \( I_\ell \), given in the following lemma (see e.g. (4.12.7) of [1]).

**Lemma 2.6.** For fixed \( \ell \) we have

\[ I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O \left( \frac{e^x}{x^{\frac{3}{2}}} \right) \]

as \( x \to \infty \).

We also require the behaviour of an integral related to the \( I \)-Bessel function. Define

\[ P_s := \frac{1}{2\pi i} \int_{1-im^{\frac{1}{4}}}^{1+im^{\frac{1}{4}}} v^s e^{\pi \sqrt{2n(v+\frac{1}{v})}} dv. \]

Then Lemma 4.2 of [2] reads as follows.

**Lemma 2.7.** For \( |m| \leq \frac{1}{6\beta} \log(n) \) we have

\[ P_s = I_s \left( 2\pi \sqrt{2n} \right) + O \left( \exp \left( \pi \sqrt{2n} \left( 1 + \frac{1}{1 + |m|^{-\frac{1}{2}}} \right) \right) \right) \]

as \( n \to \infty \).
3. Asymptotic behaviour of $f$

The aim of this Section is to determine the asymptotic behaviour of $f$. To do so we consider two separate cases; when $q$ tends toward the pole $q = 1$, and when $q$ is away from this pole. It is shown that the behaviour toward the pole at $q = 1$ gives the dominant contribution when applying the circle method in Section 4.

First note that Lemma 2.1 implies that $f(-z; \tau) = -f(z; \tau)$, and so $b(-m, n) = -b(m, n)$. We now restrict our attention to the case of $m \geq 0$.

We next define the Fourier coefficient of $\zeta_m$ of $f$, following the framework of [6]. Since there is a pole of $f$ at $z = \frac{1}{2}$, we define

$$f_m^\pm(\tau) := \int_{0}^{\frac{1}{2}-a} f(z; \tau)e^{-2\pi i mz}dz + \int_{\frac{1}{2}+a}^{1} f(z; \tau)e^{-2\pi i mz}dz + G^\pm$$

where $a > 0$ is small, and

$$G^\pm := \int_{\frac{1}{2}-a}^{\frac{1}{2}+a} f(z; \tau)e^{-2\pi i mz}dz.$$

For $G^+$ the integral is taken over a semi-circular path passing above the pole. Similarly, $G^-$ is taken over a semi-circular path passing below the pole. Then the Fourier coefficient of $\zeta_m$ of $f$ is

$$f_m(\tau) := \frac{f_m^+ + f_m^-}{2} = \int_{0}^{\frac{1}{2}-a} f(z; \tau)\sin(2\pi mz)dz + \frac{G^+ + G^-}{2}. \quad (3.1)$$

Shifting $z \mapsto z - \frac{1}{2}$ and parameterising the semi-circle we see that

$$G^+ = \lim_{a \to 0^+} \int_{-\pi}^{0} ae^{i\theta}f \left( ae^{i\theta} + \frac{1}{2}; \tau \right) e^{-2\pi im(\theta+\frac{1}{2})}d\theta$$

$$= (-1)^m \int_{-\pi}^{0} a f \left( \frac{a e^{i\theta} + 1/2}{2}; \tau \right) e^{i\theta}d\theta.$$

We next note that using L’Hôpital’s rule and Lemma 2.1 gives

$$\lim_{a \to 0^+} \left( a f \left( \frac{a e^{i\theta} + 1/2}{2}; \tau \right) \right) = -e^{-i\theta} \frac{\vartheta \left( \frac{1}{2}; \tau \right)}{\eta(\tau)\vartheta'(0; \tau)}.$$

Therefore we see that

$$G^+ = (-1)^m \frac{\vartheta \left( \frac{1}{2}; \tau \right)}{\eta(\tau)\vartheta'(0; \tau)} = 8(-1)^{m+1} \frac{\eta(2\tau)^8}{\eta(\tau)^{16}},$$

where we have used the well-known facts that

$$\vartheta \left( \frac{1}{2}; \tau \right) = -2 \frac{\eta(2\tau)^2}{\eta(\tau)},$$
A similar calculation shows that \( G^- = G^+ \). Hence the contribution of the final term in (3.1) is given by

\[
\frac{G^+ + G^-}{2} = 8(-1)^{\ell+1} \frac{\eta(2\tau)^8}{\eta(\tau)^{16}}.
\]

Remark. The residue term \( \frac{\eta(2\tau)^8}{\eta(\tau)^{16}} \) is the generating function for 8-tuple partitions [19]. Various number-theoretic properties of similar overpartition tuple functions are studied in e.g. [4, 5]. The physical interpretation of the residue term is discussed in Section 3.2 of [20], and it is an interesting question as to whether further number-theoretic properties (aside from asymptotics) of \( \frac{\eta(2\tau)^8}{\eta(\tau)^{16}} \) also have a physical interpretation.

### 3.1. Bounds towards the dominant pole

Here we find the asymptotic behaviour of \( f \) toward the dominant pole at \( q = 1 \), shown in the following lemma.

**Lemma 3.1.** Let \( \tau = \frac{i\varepsilon}{2\pi} \), with \( 0 < \text{Re}(\varepsilon) \ll 1 \), and \( 0 < z < \frac{1}{2} \). Then we have that

\[
f\left( z; \frac{i\varepsilon}{2\pi} \right) = -\frac{\varepsilon^3}{8\pi^3} \frac{\sinh\left( \frac{2\pi^2 z}{\varepsilon} \right)}{\sinh\left( \frac{4\pi^2 z}{\varepsilon} \right)} \left( 1 + e^{-4\pi^2 \text{Re}(\frac{1}{\tau})(1-z)} + O\left( e^{-4\pi^2 \text{Re}(\frac{1}{\tau})(1-z)} \right) \right).
\]

**Proof.** Using the modularity of \( f \) (following from Lemmas 2.1 and 2.2) and setting \( q_0 := e^{-\frac{2\pi i}{\tau}} \), we have that

\[
f(z; \tau) = \frac{\tau^3 \zeta^2 \prod_{n \geq 1} \left( 1 - \zeta^{\frac{1}{n}} q_0^n \right)^4 \left( 1 - \zeta^{-\frac{1}{n}} q_0^{-n} \right)^4}{i \zeta^{\frac{1}{n}} \prod_{n \geq 1} \left( 1 - q_0^n \right)^6 \left( 1 - \zeta^{\frac{1}{n}} q_0^{n} \right) \left( 1 - \zeta^{-\frac{1}{n}} q_0^{-n} \right)^4} = \frac{\tau^3 \left( \zeta^{\frac{1}{n}} - \zeta^{-\frac{1}{n}} \right)^{\frac{1}{n}} \prod_{n \geq 1} \left( 1 - \zeta^{\frac{1}{n}} q_0^{n} \right)^4 \left( 1 - \zeta^{-\frac{1}{n}} q_0^{-n} \right)^4}{i \left( \zeta^{\frac{1}{n}} - \zeta^{-\frac{1}{n}} \right) \prod_{n \geq 1} \left( 1 - \zeta^{\frac{1}{n}} q_0^{n} \right)^4 \left( 1 - \zeta^{-\frac{1}{n}} q_0^{-n} \right)^4}.
\]

This gives

\[
-\frac{\varepsilon^3}{8\pi^3} \frac{\sinh\left( \frac{2\pi^2 z}{\varepsilon} \right)}{\sinh\left( \frac{4\pi^2 z}{\varepsilon} \right)} \prod_{n \geq 1} \left( 1 - e^{-\frac{4\pi^2}{\varepsilon}(z-n)} \right)^4 \left( 1 - e^{-\frac{4\pi^2}{\varepsilon}(-z-n)} \right)^4.
\]

In order to find a bound we expand the denominator using geometric series. For \( 0 < z < \frac{1}{2} \) and \( 0 < \text{Re}(\varepsilon) \ll 1 \) we see that \( | e^{-\frac{4\pi^2}{\varepsilon}(\pm 2z-n)} | < 1 \) for all \( n \geq 1 \), and so we expand the denominator to obtain the product as

\[
\prod_{n \geq 1} \left( 1 - e^{-\frac{4\pi^2}{\varepsilon}(z-n)} \right)^4 \left( 1 - e^{-\frac{4\pi^2}{\varepsilon}(z+n)} \right)^4 \sum_{j \geq 0} e^{-\frac{4\pi^2}{\varepsilon}(2z-n)} \sum_{k \geq 0} e^{-\frac{4\pi^2}{\varepsilon}(2z+n)} \sum_{\ell \geq 0} e^{-\frac{4\pi^2}{\varepsilon}(\pm 2n)}.
\]
which is of order
\[ 1 + e^{-4\pi^2 \text{Re}(\frac{i}{\tau})/2}(1-2z) + O\left(e^{-4\pi^2 \text{Re}(\frac{i}{\tau})/2}(1-z)\right). \]

Hence overall we find that
\[ f\left(z; \frac{i\varepsilon}{2\pi}\right) = -\frac{\varepsilon^3}{8\pi^3} \frac{\sinh\left(\frac{2\pi^2 z}{\varepsilon}\right)}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} \left(1 + e^{-4\pi^2 \text{Re}(\frac{i}{\tau})/2}(1-2z) + O\left(e^{-4\pi^2 \text{Re}(\frac{i}{\tau})/2}(1-z)\right)\right), \]
yielding the claim. \(\square\)

Remark. It is easy to see that this gives the same main term as noted in Section 4.5 of [11] (up to sign, which the authors there do not make use of).

Since \( f(z; \tau) = -f(1-z; \tau) \) we see this immediately also implies the following lemma.

Lemma 3.2. Let \( \tau = \frac{i\varepsilon}{2\pi} \), with \( 0 < \text{Re}(\varepsilon) \ll 1 \), and \( \frac{1}{2} < z < 1 \). Then we have that
\[ f\left(z; \frac{i\varepsilon}{2\pi}\right) = -\frac{\varepsilon^3}{8\pi^3} \frac{\sinh\left(\frac{2\pi^2 (1-z)}{\varepsilon}\right)}{\sinh\left(\frac{4\pi^2 (1-z)}{\varepsilon}\right)} \left(1 + e^{-4\pi^2 \text{Re}(\frac{i}{\tau})/(2z-1)} + O\left(e^{-4\pi^2 \text{Re}(\frac{i}{\tau})/(z)}\right)\right). \]

We now look to find the behaviour of \( f_m \) toward the pole at \( q = 1 \). We begin with the contribution from the residue term
\[ 8(-1)^{m+1} \frac{\eta(2\tau)^8}{\eta(\tau)^{16}}. \]

Lemma 3.3. As \( n \to \infty \) we have
\[ 8(-1)^{m+1} \frac{\eta(2\tau)^8}{\eta(\tau)^{16}} = (-1)^{m+1} \frac{\varepsilon^4}{2\pi^4} \left(e^{\frac{2\pi^2}{\varepsilon}} + O(1)\right). \]

Proof. Using the modularity of \( \eta \) given in Lemma 2.2 we see that
\[ \frac{\eta(2\tau)^8}{\eta(\tau)^{16}} = (-i\tau)^4 \frac{\eta(-\frac{1}{2\pi})^8}{\eta(-\frac{1}{\tau})^16} = \tau^4 \left(e^{\frac{n}{16\pi^2}} + O(1)\right). \]

As \( \tau = \frac{i\varepsilon}{2\pi} \) this yields
\[ \frac{\varepsilon^4}{2\pi^4} \left(e^{\frac{2\pi^2}{\varepsilon}} + O(1)\right). \]

To estimate the contribution from the first integral in (3.1), we follow the approach of [2,8], and define three further integrals
\[ g_{m,1} := -\frac{\varepsilon^3}{8\pi^3} \int_0^{\frac{1}{z}} \frac{\sinh\left(\frac{2\pi^2 z}{\varepsilon}\right)}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} \sin(2\pi mz)dz, \quad (3.2) \]
along with
\[
g_{m,2} := -\frac{\varepsilon^3}{8\pi^3} \int_0^{\frac{1}{2}-a} \sinh \left( \frac{2\pi^2 z}{\varepsilon} \right)^4 \frac{e^{-4\pi^2 \text{Re}(\frac{1}{2})(1-2z)}}{\sinh \left( \frac{4\pi^2 z}{\varepsilon} \right)} \sin(2\pi mz) \, dz,
\]
(3.3)
and
\[
g_{m,3} := \int_0^{\frac{1}{2}-a} \left( f(z; \tau) + \frac{\varepsilon^3}{8\pi^3} \frac{\sinh \left( \frac{2\pi^2 z}{\varepsilon} \right)^4}{\sinh \left( \frac{4\pi^2 z}{\varepsilon} \right)} \frac{e^{-4\pi^2 \text{Re}(\frac{1}{2})(1-2z)}}{1 + e^{-4\pi^2 \text{Re}(\frac{1}{2})(1-2z)}} \right) \sin(2\pi mz) \, dz.
\]

We now investigate the contribution from \( g_{m,1} \) and show the following proposition.

**Proposition 3.4.** Assume that \(|x| \leq 1\). Then for \(a \to 0^+\), and with \(\tau = \frac{i\varepsilon}{2\pi}\) with \(\varepsilon \to 0^+\) we have that
\[
g_{m,1} = O \left( \beta^4 \right).
\]

**Proof.** We first use the Taylor series representation of \(\sinh(x)^4\) and \(\sin(x)\) which are given by
\[
\sinh(x)^4 = \sum_{n \geq 0} \frac{(4^{n+2}(4^{n+2}-4)) x^{2n+4}}{8 \cdot (2n + 4)!},
\]
\[
\sin(x) = \sum_{n \geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.
\]
Thus we see that
\[
\sinh \left( \frac{2\pi^2 z}{\varepsilon} \right)^4 \sin(2\pi mz) = \sum_{k \geq 0} \frac{(4^{k+2}(4^{k+2}-4)) \left( \frac{2\pi^2 z}{\varepsilon} \right)^{2k+4}}{8 \cdot (2k + 4)!} \sum_{r \geq 0} \frac{(-1)^r (2\pi mz)^{2r+1}}{(2r + 1)!}.
\]
Substituting this into (3.2) we find that
\[
g_{m,1}(\tau) = \frac{\varepsilon^3}{8\pi} \sum_{k,r \geq 0} \frac{(-1)^r (4^{k+2}(4^{k+2}-4)) (2\pi m)^{2r+1} \left( \frac{2\pi^2 z}{\varepsilon} \right)^{2k+4}}{8 \cdot (2k + 4)!(2r + 1)!} I_{k+r+2},
\]
where
\[
I_{\ell} := \lim_{a \to 0^+} \int_0^{\frac{1}{2}-a} \frac{z^{2\ell+1}}{\sinh \left( \frac{4\pi^2 z}{\varepsilon} \right)} \, dz = \int_0^{\frac{1}{2}} \frac{z^{2\ell+1}}{\sinh \left( \frac{4\pi^2 z}{\varepsilon} \right)} \, dz.
\]
Following the ideas of [2] we further define \(I_{\ell}'\) by
\[
I_{\ell}' := \int_0^{\infty} \frac{z^{2\ell+1}}{\sinh \left( \frac{4\pi^2 z}{\varepsilon} \right)} \, dz - I_{\ell}.
\]
Then
\[ I'_\ell = \int_\frac{1}{2}^{\infty} \frac{z^{2\ell+1}}{\sinh \left( \frac{4\pi z}{\varepsilon} \right)} dz \ll \int_\frac{1}{2}^{\infty} z^{2\ell+1} e^{-4\pi z \text{Re} \left( \frac{1}{\varepsilon} \right)} \]

where we use the incomplete gamma function \( \Gamma(\alpha; x) := \int_x^{\infty} e^{-w} w^{\alpha-1} dw \). Since we have the asymptotic behaviour of
\[ \Gamma(\ell; x) \sim x^{\ell-1} e^{-x} \]
as \( x \to \infty \) we find that
\[ \Gamma \left( 2\ell + 2; 2\pi^2 \text{Re} \left( \frac{1}{\varepsilon} \right) \right) \sim \left( 2\pi^2 \text{Re} \left( \frac{1}{\varepsilon} \right) \right)^{2\ell+1} e^{-2\pi^2 \text{Re} \left( \frac{1}{\varepsilon} \right)}. \]

Hence we may conclude that
\[ I'_\ell \ll \text{Re} \left( \frac{1}{\varepsilon} \right)^{-1} e^{-2\pi^2 \text{Re} \left( \frac{1}{\varepsilon} \right)} \leq e^{-2\pi^2 \text{Re} \left( \frac{1}{\varepsilon} \right)}. \]

Now under the substitution \( z \mapsto \frac{z}{4\pi} \) we find that
\[ \int_0^{\infty} \frac{z^{2\ell+1}}{\sinh \left( \frac{4\pi z}{\varepsilon} \right)} dz = \left( \frac{\varepsilon}{4\pi} \right)^{2\ell+2} \int_0^{\infty} \frac{z^{2\ell+1}}{\sinh(\pi z)} dz = \left( \frac{\varepsilon}{4\pi} \right)^{2\ell+2} E_\ell = \left( \frac{\varepsilon}{4\pi} \right)^{2\ell+2} \frac{(-1)^{\ell+1} E_{2\ell+1}(0)}{2}. \]

Then we obtain that
\[ g_{m,1} = \frac{\varepsilon^3}{16\pi} \sum_{k,r \geq 0} \frac{(-1)^r (4^{k+2} (4^{k+2} - 4)) (2\pi m)^{2r+1} \left( \frac{2\pi^2}{\varepsilon} \right)^{2k+4}}{8 \cdot (2k + 4)! (2r + 1)!} \left( \frac{\varepsilon}{4\pi} \right)^{2r+2k+6} \times \left[ E_{2r+2k+5}(0) + O \left( |\varepsilon|^{-2r-2k-6} e^{-2\pi^2 \text{Re} \left( \frac{1}{\varepsilon} \right)} \right) \right]. \]

Letting \( m' := m/2 \) we see that
\[ g_{m,1} = -\frac{\varepsilon^3 \pi^3}{2^{10}} \sum_{r \geq 0} \frac{(m' \varepsilon)^{2r+1}}{(2r + 1)!} \left[ E_{2r+5}(0) + O \left( |\varepsilon|^2 \right) \right]. \]

Next, using Lemma 2.3 we recognise that
\[ \sum_{r \geq 0} \frac{(m' \varepsilon)^{2r+1}}{(2r + 1)!} E_{2r+5}(0) = \frac{1}{m'^3} \frac{\partial^3}{\partial \varepsilon^3} \sum_{r \geq 0} \frac{(m' \varepsilon)^{2r+4}}{(2r + 4)!} E_{2r+5}(0) = -\frac{1}{2m'^3} \frac{\partial^3}{\partial \varepsilon^3} \text{sech}^2 \left( \frac{m' \varepsilon}{2} \right) \]
\[ = -\frac{4}{m'^3} \frac{\partial^3}{\partial \varepsilon^3} \text{sech}^2 \left( \frac{m \varepsilon}{4} \right). \]

We therefore obtain
\[ g_{m,1} = \frac{\varepsilon^3 \pi^3}{2^{8} m'^3} \left( \frac{\partial^3}{\partial \varepsilon^3} \text{sech}^2 \left( \frac{m \varepsilon}{4} \right) + O \left( |\varepsilon|^2 \cosh (m \varepsilon) \right) \right). \]
Further, we have that
\[ \frac{\partial^3}{\partial \varepsilon^3} \text{sech}^2 \left( \frac{m \varepsilon}{4} \right) = -m^3 \left( \cosh^2 \left( \frac{m \varepsilon}{4} \right) - 3 \right) \frac{\sinh \left( \frac{m \varepsilon}{4} \right)}{8 \cosh^5 \left( \frac{m \varepsilon}{4} \right)} . \]

It is clear that
\[ \frac{\left( \cosh^2 \left( \frac{m \varepsilon}{4} \right) - 3 \right) \sinh \left( \frac{m \varepsilon}{4} \right)}{8 \cosh^5 \left( \frac{m \varepsilon}{4} \right)} = O(1) . \]

Therefore, we see that
\[ g_{m,1} = \frac{\varepsilon^3 \pi^3}{2^8} \left( O(1) + O \left( \varepsilon^2 m^{-3} \cosh(m \varepsilon) \right) \right) . \]

Further, we have that
\[ \cosh(m \varepsilon) = \cosh \left( \beta m + i \beta m \frac{8}{2} x \right) = \cosh(\beta m) \left( 1 + O(\beta \frac{4}{3}) \right) . \]

Hence we obtain
\[ g_{m,1} = \frac{\varepsilon^3 \pi^3}{2^8} \left( O(1) + O(\varepsilon^2 m^{-3} \cosh(m \beta)) = O(\varepsilon^4) = O(\beta^4) \right) , \]
where for the last equality we use that \( \varepsilon \ll \beta \). \( \square \)

To bound the contribution of \( g_{m,2} \) we note the following trivial lemma.

**Lemma 3.5.** For \( |x| \leq 1 \) we have that
\[ |g_{m,2}| \ll g_{m,1} . \]

Next we bound the contribution from \( g_{m,3} \).

**Proposition 3.6.** For \( |x| \leq 1 \), we have that
\[ |g_{m,3}| \ll \frac{\varepsilon^3}{8 \pi^3} . \]

**Proof.** We see that
\[ |g_{m,3}| = \int_0^{1 - a} \left| f(z; \tau) + \frac{\varepsilon^3}{8 \pi^3} \frac{\sinh \left( \frac{2 \pi^2 z}{\varepsilon} \right)}{\sinh \left( \frac{4 \pi^2 z}{\varepsilon} \right)} \left( 1 + e^{-4 \pi^2 \text{Re} \left( \frac{1}{2} \right) (1 - 2z)} \right) \sin(2 \pi m z) dz \right| . \]

We estimate the right-hand side using Lemma 3.1, to find that
\[ |g_{m,3}| \ll \int_0^{1 - a} \frac{\varepsilon^3}{8 \pi^3} \left| \frac{\sin(2 \pi m z)}{1 - e^{-8 \pi^2 z}} \right| e^{4 \pi^2 \text{Re} \left( \frac{1}{2} \right) (2z - 1)} dz . \]

We see that
\[ \frac{\sin(2 \pi m z)}{1 - e^{-8 \pi^2 z}} \ll 1 , \]
and so it follows that \( |g_{m,3}| \ll \frac{\varepsilon^3}{8 \pi^3} e^{-4 \pi^2 \text{Re} \left( \frac{1}{2} \right)} \ll \frac{\varepsilon^3}{8 \pi^3} . \) \( \square \)
Combining Lemmas 3.3 and 3.5, and Propositions 3.4 and 3.6, we obtain the following theorem.

**Theorem 3.7.** For $|x| \leq 1$ we have that

$$f_m(\tau) = (-1)^{m+1} \frac{\varepsilon^4}{2\pi^4} e^{\frac{2\pi^2}{x}} + O(\beta^3).$$

### 3.2. Bounds away from the dominant pole.

We next investigate the behaviour of $f_m$ away from the pole $q = 1$, by assuming that $1 \leq x \leq \frac{\pi m}{\beta}$. In the following lemma we bound the residue term

$$\frac{\eta(2\tau)^8}{\eta(\tau)^{16}},$$

away from the pole $q = 1$.

**Lemma 3.8.** For $1 \leq x \leq \frac{\pi m}{\beta}$ we have that

$$\frac{\eta(2\tau)^8}{\eta(\tau)^{16}} \ll n^{-4} \exp \left[ \frac{4\pi \sqrt{2n}}{3} - \frac{8\sqrt{2n}}{\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-2}}} \right) \right].$$

**Proof.** We use equation (3.1) directly to find that

$$\frac{\eta(2\tau)^8}{\eta(\tau)^{16}} = \frac{P(q)^{16}}{P(q^2)^8} \ll P(q)^{16} \ll n^{-4} \exp \left[ \frac{4\pi \sqrt{2n}}{3} - \frac{8\sqrt{2n}}{\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-2}}} \right) \right].$$

Next, we investigate the contribution of

$$\left| \int_0^{\frac{1}{2} - a} f(z; \tau) \sin(2\pi mz) dz \right| \ll \int_0^{\frac{1}{2} - a} |f(z; \tau) \sin(2\pi mz)| dz.$$

Then we want to bound

$$|f(z; \tau)| = \left| \frac{\sin(2\pi mz) \theta(z; \tau)^4}{\eta(\tau)^9 \theta(2z; \tau)} \right|$$

away from the dominant pole. For $0 < b < \frac{1}{2}$ far from $\frac{1}{2}$ we see that we may bound the integrand in modulus by

$$|f(b; \tau) \sin(2\pi mb)| \ll |P(q)|^9 \left| q^{-\frac{3}{4}} \frac{\partial(b; \tau)^4}{\partial(2b; \tau)} \right| \ll |P(q)|^9 \sum_{n \in \mathbb{Z}} |q^{\frac{n^2}{2} + \tau} \ll |P(q)|^9 \sum_{n \in \mathbb{Z}} e^{-\beta n^2}.$$

As $z \to \frac{1}{2}$ we apply L’Hôpital’s rule to the integrand $|f(z; \tau) \sin(2\pi mz)|$ which yields the bound

$$\left| \int_0^{\frac{1}{2} - a} f(z; \tau) \sin(2\pi mz) dz \right| \ll \frac{\eta(2\tau)^8}{\eta(\tau)^{16}}.$$

Hence, away from the dominant pole in $q$, we have shown the following proposition.
Proposition 3.9. For $1 \leq x \leq \frac{2m}{\beta}$ we have that

$$|f(z; \tau)| \ll n^{-4} \exp \left[ \frac{4\pi \sqrt{2n}}{3} - \frac{8\sqrt{2n}}{\pi} \left(1 - \frac{1}{\sqrt{1 + m^{-\frac{2}{3}}}}\right) \right].$$

4. THE CIRCLE METHOD

In this section we use Wright’s variant of the Circle Method to complete the proof of Theorem 1.1. We start by noting that Cauchy’s theorem implies that

$$b(m, n) = \frac{1}{2\pi i} \int_C \frac{f_m(\tau)}{q^{n+1}} dq,$$

where $C := \{q \in \mathbb{C} \mid |q| = e^{-\beta}\}$ is a circle centred at the origin of radius less than 1. Making a change of variables and recalling that $\varepsilon = \beta(1 + ixm^{-\frac{1}{3}})$ we have

$$b(m, n) = \frac{\beta}{2\pi m^{\frac{1}{3}}} \int_{|x| \leq \frac{2m}{n^{\frac{1}{3}}}} f_m(e^{-\varepsilon}) e^{\varepsilon n} dx.$$

Splitting this integral into two pieces, we have $b(m, n) = M + E$ where

$$M := \frac{\beta}{2\pi m^{\frac{1}{3}}} \int_{|x| \leq 1} f_m(e^{-\varepsilon}) e^{\varepsilon n} dx,$$

and

$$E := \frac{\beta}{2\pi m^{\frac{1}{3}}} \int_{1 \leq |x| \leq \frac{2m}{n^{\frac{1}{3}}}} f_m(e^{-\varepsilon}) e^{\varepsilon n} dx.$$

Next we determine the contributions of each of the integrals $M$ and $E$, and see that $M$ contributes to the main asymptotic term, while $E$ is part of the error term.

4.1. The major arc. First we concentrate on the contribution $M$. Then we obtain the following proposition.

Proposition 4.1. We have that

$$M = (-1)^{m+1} \frac{\beta^5}{2\pi^4 m^{\frac{2}{3}} (2\pi)^{\frac{3}{2}} (2n)^{\frac{1}{4}}} e^{2\pi \sqrt{2n}} + O \left( m^{-\frac{2}{3}} n^{-\frac{13}{4}} e^{2\pi \sqrt{2n}} \right)$$

as $n \to \infty$.

Proof. By Theorem 3.7 and making the change of variables $v = 1 + ixm^{-\frac{1}{3}}$ we obtain

$$M = (-1)^{m+1} \frac{\beta^5}{2\pi^4 m^{\frac{2}{3}}} P_4 + O \left( \beta^4 e^{\pi \sqrt{2n}} \right).$$

Now we rewrite $P_4$ in terms of the $I$-Bessel function using Lemma 2.7, yielding

$$M = (-1)^{m+1} \frac{\beta^5}{2\pi^4 m^{\frac{2}{3}}} I_{\frac{5}{2}}(2\pi \sqrt{2n}) + O \left( \beta^5 e^{\pi \sqrt{2n}} \left( \frac{1}{1 + m^{-\frac{2}{3}}} \right) \right) + O \left( \beta^4 e^{\pi \sqrt{2n}} \right).$$
The asymptotic behaviour of the $I$-Bessel function given in Lemma 2.6 gives that

$$M = (-1)^{n+1} \frac{\beta^5}{2\pi^4 m^2 (2\pi)^{3/2} (2n)^{3/4}} e^{2\pi\sqrt{2n}} + O \left( m^{-2/3} n^{-11/4} e^{2\pi\sqrt{2n}} \right) + O \left( \beta^5 e^{\pi \sqrt{2n} (1 + \frac{1}{1 + m^{-2/3}})} \right) + O \left( \beta^4 e^{\pi \sqrt{2n}} \right).$$

It is clear that the first error term is the dominant one, and the result follows.

4.2. The error arc. Now we bound $E$ as follows.

**Proposition 4.2.** As $n \to \infty$

$$E \ll n^{-4} \exp \left[ \frac{\pi \sqrt{2n}}{6} - 2\pi \sqrt{n} \left( 1 - \frac{1}{\sqrt{1 + m^{-2/3}}} \right) \right].$$

**Proof.** By Proposition 3.9 we see that the main term in the error arc is given by the residue. Hence we may bound

$$E \ll \int_{1 \leq x \leq \frac{n+1}{\beta}} n^{-4} \exp \left[ \frac{4\pi \sqrt{2n}}{3} - \frac{8\sqrt{2n}}{\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-2/3}}} \right) \right] e^{\epsilon n} dx \ll n^{-4} \exp \left[ \frac{5\pi \sqrt{2n}}{3} - \frac{8\sqrt{2n}}{\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-2/3}}} \right) \right].$$

Noting that this is exponentially smaller than $M$ finishes the proof of Theorem 1.1.

5. Open questions

We end by commenting on some questions related to the results presented above.

1. Here we discuss the asymptotic profile of the coefficients $b(m, n)$ for $|m| \leq \frac{1}{6\beta} \log(n)$. We are also interested in the profile when $m$ is larger than this bound, and so in future it would be instructive to investigate the asymptotic profile of $b(m, n)$ for large $|m|$. For example, similar results in this direction for the crank of a partition are given in [16].

2. In the present paper, we provide a framework for investigating the profile of eta-theta quotients. In particular, we deal with the case of a function with a single simple pole on the path of integration. Future research is planned in order to expand this framework for a family of meromorphic eta-theta quotients with a finite number of (not necessarily single) poles on the path of integration. This should include similar eta-theta quotients that appear in other physical partition functions.

3. In showing Theorem 1.1 we see that the main asymptotic term arises from the pole at $z = 1/2$, and in turn from the residue term $\frac{\eta(2\pi)^8}{\eta'(\tau) \tau}$; is there a physical interpretation for the fact that these terms give the largest contribution to the asymptotic behaviour of $b(m, n)$?
References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1999.

[2] K. Bringmann and J. Dousse, *On Dyson’s crank conjecture and the uniform asymptotic behavior of certain inverse theta functions*, Transactions of the American Mathematical Society 368 (2016), no. 5, 3141–3155.

[3] K. Bringmann, A. Folsom, K. Ono, and L. Rolen, *Harmonic Maass forms and mock modular forms: theory and applications*, Vol. 64, American Mathematical Soc., 2017.

[4] K. Bringmann and J. Lovejoy, *Rank and congruences for overpartition pairs*, International Journal of Number Theory 4 (2008), no. 02, 303–322.

[5] A. Ciolan, *Ranks of overpartitions: asymptotics and inequalities*, arXiv preprint arXiv:1904.07055 (2019).

[6] A. Dabholkar, S. Murthy, and D. Zagier, *Quantum black holes, wall crossing, and mock modular forms*, arXiv preprint arXiv:1208.4074 (2012).

[7] R. Dijkgraaf, G. Moore, E. Verlinde, and H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, Communications in Mathematical Physics 185 (1997), no. 1, 197–209.

[8] J. Dousse and M. Mertens, *Asymptotic formulae for partition ranks*, Acta Arithmetica 168 (2015), no. 1, 83–100.

[9] F. Gliozzi, J. Scherk, and D. Olive, *Supersymmetry, supergravity theories and the dual spinor model*, Nuclear Physics B 122 (1977), no. 2, 253–290.

[10] J. A. Harvey, *Ramanujan’s influence on string theory, black holes and moonshine*, 2019.

[11] S. He, T. Numasawa, T. Takayanagi, and K. Watanabe, *Notes on entanglement entropy in string theory*, Journal of High Energy Physics 2015 (2015), no. 5, 106.

[12] B. Kim, E. Kim, and H. Nam, *On the asymptotic distribution of cranks and ranks of cubic partitions*, Journal of Mathematical Analysis and Applications 443 (2016), no. 2, 1095–1109.

[13] S. Kondo and T. Watari, *String-theory realization of modular forms for elliptic curves with complex multiplication*, Communications in Mathematical Physics 367 (2019), no. 1, 89–126.

[14] J. Manschot and J. M. Z. Rolon, *The asymptotic profile of $\chi_y$-genera of hilbert schemes of points on k3 surfaces*, Commun. Num. Theor. Phys. 9 (2014), 413–435.

[15] D. Mumford, M. Nori, and P. Norman, *Tata lectures on theta III*, Vol. 43, Springer, 2007.

[16] D. Parry and R. Rhoades, *On Dyson’s crank distribution conjecture and its generalizations*, Proceedings of the American Mathematical Society 145 (2017), no. 1, 101–108.

[17] J. G. Russo and A. A. Tseytlin, *Magnetic flux tube models in superstring theory*, Nuclear Physics B 461 (1996), no. 1, 131–154.

[18] C. L. Siegel, *A simple proof of $\eta(-1/\tau) = \eta(\tau)\sqrt{\tau/\pi}$*, Mathematika 1 (1954), no. 1, 4.

[19] L. Q. Wang, *Arithmetic properties of overpartition triples*, Acta Mathematica Sinica, English Series 33 (2017), no. 1, 37–50.

[20] E. Witten, *Open strings on the Rindler horizon*, Journal of High Energy Physics 2019 (2019), no. 1, 126.

[21] E. M. Wright, *Asymptotic partition formulae:(II) weighted partitions*, Proceedings of the London Mathematical Society 2 (1934), no. 1, 117–141.

[22] ________, *Stacks (II)*, The Quarterly Journal of Mathematics 22 (1971), no. 1, 107–116.

University of Cologne, Department of Mathematics and Computer Science, Weyertal 86-90, 50931 Cologne, Germany

E-mail address: jmales@math.uni-koeln.de