Instability Proof for Einstein-Yang-Mills Solitons and Black Holes with arbitrary Gauge Groups

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We prove that static, spherically symmetric, asymptotically flat soliton and black hole solutions of the Einstein-Yang-Mills equations are unstable for arbitrary gauge groups, at least for the “generic” case. This conclusion is derived without explicit knowledge of the possible equilibrium solutions.
1 Introduction

In several recent papers [1, 2, 3, 4], we have studied important aspects of the Einstein-Yang-Mills (EYM) system for arbitrary gauge groups. In particular, we investigated the classification and properties of spherically symmetric EYM solitons (magnetic structure, Chern-Simons numbers) and a generalization of the Birkhoff theorem for the non-Abelian case. We also worked out the generalization of the first law of black hole physics (Bardeen-Carter-Hawking formula), allowing for additional Higgs and dilaton fields [5, 6]. For other studies of these and related topics we refer to [7, 8, 9, 10].

In the present paper, we prove that static, spherically symmetric, asymptotically flat solutions of the EYM equations are unstable for any gauge group, if they are “generic” (defined in Sec. 2). In a recent letter [11], we have already sketched how we arrived at this result for solitons. Here we present details of the proof and extend it to black holes. We discuss also some further mathematical issues involved.

This general instability was expected since the Bartnik-McKinnon solutions [12] for the gauge group SU(2) and the related black hole solutions [13, 14, 15] are unstable [16, 17, 18, 19]. A mathematical proof of this expectation presents, however, quite a challenge, since one cannot rely on any knowledge of the possible solutions (apart from regularity and boundary conditions).

Our strategy is based on the study of the pulsation equations, describing linear radial perturbations of the equilibrium solutions and involves the following main steps: First we show, that the frequency spectrum of a class of radial perturbations is determined by a coupled system of radial, respectively one-dimensional “Schrödinger equations”. Negative parts in the spectrum of the effective Hamiltonian imply linear instability. With the help of suitably constructed trial functions, it is then proven, that the spectrum contains always a negative part (for “generic” solutions).

We have recently used a similar procedure to establish the instability of the gravitating, regular sphaleron solutions of the SU(2) EYM-Higgs system with a SU(2) Higgs doublet [20], which have been constructed numerically in [21]. Our results contain, as a special case, the conclusion of Ref. [22] for the gauge group SU(2). Here, we analyze the regular SU(2) case further. We show that the effective Hamiltonian for “sphaleron-like” perturbations has the form of a “deuteron” Hamiltonian.

The paper is organized as follows: In Sec. 2 we recall some basic facts and equations of our previous work [2, 4], which will be needed in the present analysis. In Sec. 3 we then derive the linearized perturbation equations for solitons and black holes and bring them into a convenient, partially decoupled form. The resulting eigenvalue problem is discussed in Sec. 4 and in Sec. 5 we show the existence of unstable perturbations. The “deuteron” interpretation for the unstable modes of a SU(2) soliton is presented in Sec. 6. In the appendix, we elaborate on mathematical issues, related to the self-adjointness of the effective
Instability of EYM Solutions

Hamiltonian and the connection between the negative part in its spectrum and unstable solutions of the perturbation equations.

2 Spherically symmetric EYM fields

We begin with a convenient description of gauge fields with spherical symmetry (for derivations see [2]).

Let us fix a maximal torus $T$ of the gauge group $G$ with corresponding integral lattice $I$ ($= \text{kernel of the exponential map restricted to the Lie algebra } LT$ of the torus $T$). In addition, we choose a basis $S$ of the root system $R$ of real roots. The corresponding fundamental Weyl chamber

$$K(S) = \{ H \in LT \mid \alpha(H) > 0 \text{ for all } \alpha \in S \}$$

plays an important role in what follows.

To a spherically symmetric gauge field there belongs a canonical element $H_\lambda \in I \cap K(S)$, which characterizes the corresponding principal bundle $P(M, G)$ over the spacetime manifold $M$, admitting a SU(2) action. If the configuration is also regular at the origin, $H_\lambda$ is restricted to a small, finite subset of $I \cap K(S)$, which is described in [4]. In the present discussion, we exclude (for technical reasons) the possibility that $H_\lambda$ lies on the boundary of the fundamental Weyl chamber. The term “generic” always refers to fields, for which the classifying element $H_\lambda$ is contained in the open Weyl chamber $K(S)$.

The SU(2) action on $P(M, G)$ by bundle automorphisms induces an action on the base manifold $M$. A SU(2) invariant connection in $P(M, G)$ defines an invariant connection in each subbundle over a single orbit of the action on $M$. By Wangs theorem, the induced connections are described by a linear map $\Lambda : \text{LSU}(2) \to \text{LG}$, which depends locally smoothly on the orbit and satisfies

$$\Lambda = [\Lambda_2, \Lambda_3], \quad \Lambda_2 = [\Lambda_3, \Lambda_1], \quad \Lambda_3 = -H_\lambda/4\pi, \quad (2.2)$$

where $\Lambda_k := \Lambda(\tau_k)$ and $2i\tau_k$ are the Pauli matrices. These equations imply that $\Lambda_+ := \Lambda_1 + i\Lambda_2$ lies in the following direct sum of root spaces $L_\alpha$ of $\text{LG}$:

$$\Lambda_+ \in \bigoplus_{\alpha \in S(\lambda)} L_\alpha,$$

$$S(\lambda) := \{ \alpha \in R_+ \mid \alpha(H_\lambda) = 2 \}, \quad (2.3)$$

where $R_+$ denotes the set of positive roots in $R$ (relative to the basis $S$). In the generic case $S(\lambda)$ turns out to be a basis of a root system contained in $R$ (see appendix A of Ref. [4]).

The $\text{LG}$-valued functions $\Lambda_\pm$ on the orbit space determine part of the connection on $P(M, G)$. Before we give a parametrization of the YM potential in a convenient gauge, we fix our conventions in parametrizing the Lorentz metric.
on $M$ and introduce some further notation. We use standard Schwarzschild-like coordinates and set

$$ds^2 = -N S^2 dt^2 + N^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta \, d\phi^2), \quad (2.4)$$

where the metric functions $N = 1 - 2m/r$ and $S$ depend only on $r$ and $t$.

A suitably normalized $\text{Ad}(G)$-invariant scalar product on $LG$ will be denoted by $\langle \cdot, \cdot \rangle$. We use the same symbol for the hermetian extension to $LG_C$ (linear in the second argument), and $| \cdot |$ means the corresponding norm. Note that the original $\text{Ad}(G)$-invariance extends on $LG_C$ to

$$\langle X, [Z,Y] \rangle + \langle c(Z), X \rangle, Y \rangle = 0, \quad (2.5)$$

where $c$ is the conjugation in $LG_C$.

In Ref. [2], it is shown the gauge potential $A$ can be chosen to have the form

$$A = \tilde{A} + \hat{A}, \quad (2.6)$$

with

$$\hat{A} = \Lambda_2 \, d\vartheta + (\Lambda_3 \cos \vartheta - \Lambda_1 \sin \vartheta) \, d\varphi \quad (2.7)$$

and

$$\tilde{A} = \hat{A} dt + \hat{B} dr, \quad (2.8)$$

where $\hat{A}$ and $\hat{B}$ commute with $H_\lambda$ (i.e. with $\Lambda_3$). Since $H_\lambda$ is assumed to be generic, its centralizer is the infinitesimal torus $LT$. Hence, $\hat{A}$ and $\hat{B}$ are $LT$-valued and $\hat{A}$ is thus abelian.

For the example of the gauge group $SU(2)$, $H_\lambda$ is an integer multiple of $4\pi \tau_3$: $H_\lambda = 4\pi k \tau_3$ with $k \in \mathbb{Z}$, and the only solutions of (2.2) are $\Lambda_1 = \Lambda_2 = 0$, $\Lambda_3 = k \tau_3$ and

$$\Lambda_1 = w \tau_1 + \bar{w} \tau_2, \quad \Lambda_2 = \mp \bar{w} \tau_1 \pm w \tau_2, \quad \Lambda_3 = \pm \tau_3. \quad (2.9)$$

The gauge potential $A$ contains a “trivial”, abelian part, which decouples from the EYM equations. To demonstrate this, let us first construct a convenient decomposition of $LT$. For a given potential we restrict the sum in Eq. (2.3) to the smallest subset $\Sigma$ of $S(\lambda)$ for which

$$\Lambda_+ \in \bigoplus_{\alpha \in \Sigma \subset S(\lambda)} L_\alpha. \quad (2.10)$$

Since every rootspace $L_\alpha$ is $\text{Ad}(T)$-invariant and since the residual gauge group of the potential $A$ is just the torus $T$, the subset $\Sigma$ is unique and depends only on the invariant connection. With the help of $\Sigma$ we now split $LT$:

$$LT = \langle \Sigma \rangle \oplus \langle \Sigma \rangle^\perp, \quad (2.11)$$
where $\langle \Sigma \rangle$ denotes the linear span of $\Sigma$. The decomposition (2.11) is *independent* of the chosen $\text{Ad}(G)$-invariant scalar product $\parallel \parallel$ and satisfies
\[
\langle \Sigma \rangle^\perp, \Lambda_+ \parallel = 0 .
\] (2.12)

This property motivates to set
\[
\hat{A} = a + A ,
\]
\[
\hat{B} = b + B ,
\]
\[
\Lambda_3 = \Lambda_{3\perp} + \Lambda_{3\parallel} ,
\] (2.13)

with $a, b, \Lambda_{3\perp} \in \langle \Sigma \rangle^\perp$ and $A, B, \Lambda_{3\parallel} \in \langle \Sigma \rangle$. For our instability proof we adopt the following (mixed) gauge:
\[
A \equiv 0 , \quad b \equiv 0 , \quad \lim_{r \to \infty} a = 0 .
\] (2.14)

If we now insert the parametrizations (2.4), (2.6) – (2.8), (2.13), (2.14) into the EYM equations, we obtain a system of partial differential equations for the metric functions $N, S$ and the YM amplitudes $\Lambda_{\pm}, B$. As noted above and as Eq. (2.12) indicates, the equation for $a$ decouples. Specializing the results of $[2]$ (and using slightly different notation), they read as follows:

The Einstein equations give two constraint equations for the $r$ derivative (denoted by a dash) and the $t$ derivative (denoted by a dot) of $m$
\[
m' = \frac{\kappa}{2} \left\{ NG + r^2 p_\theta \right\} , \quad \dot{m} = \frac{\kappa}{2} NH ,
\] (2.15)

$(\kappa := 8\pi G)$, and the $(rr)$-equation reduces to
\[
\frac{S'}{S} = \frac{\kappa}{r} G ,
\] (2.16)

where
\[
G = \frac{1}{2} \left\{ (NS)^{-2} \left| \Lambda_+ \right|^2 + \left| \Lambda_{\pm} \right|^2 + \left| \Lambda_{\pm} \right|^2 \right\} ,
\] (2.17)
\[
H = \text{Re} \left\{ \dot{\Lambda}_+ , \Lambda_{\pm} + [B, \Lambda_{\pm}] \right\} ,
\] (2.18)
\[
p_\theta = \frac{1}{2r^2} \left\{ \left| \vec{F}_\parallel \right|^2 + \left| \vec{F}_\parallel \right|^2 + \left| P_\perp \right|^2 + \left| Q_\perp \right|^2 \right\}
\] (2.19)

with
\[
\vec{F}_\parallel = \frac{i}{2} [\Lambda_+, \Lambda_-] - \Lambda_{3\parallel} , \quad \vec{F}_\parallel = \frac{r^2}{S} \hat{B}
\] (2.20)

and
\[
P_\perp = \Lambda_{3\perp} , \quad Q_\perp = -\frac{r^2}{S} a'.
\] (2.21)
The YM equations decompose into

\begin{align}
\frac{2}{NS} \left( \frac{r^2}{S} \dot{B} \right)' + [\Lambda_+ - N S [B, \Lambda_+] + [\Lambda_-, \Lambda_+ + [B, \Lambda_+]] = 0, \\
\frac{1}{S} \left( \frac{1}{NS} \dot{\Lambda}_+ \right)' - \frac{1}{S} \left( S \left( \Lambda'_+ + [B, \Lambda_+] \right) \right)' - N [B, \Lambda'_+ + [B, \Lambda_+]] + \frac{i}{r^2} [\tilde{F}_\parallel, \Lambda_+] = 0, \\
2 \left( \frac{r^2}{S} \dot{B} \right)' + \frac{1}{NS} \left\{ [\Lambda_+, \dot{\Lambda}_+] + [\Lambda_-, \dot{\Lambda}_+] \right\} = 0.
\end{align}

(2.22)

(2.23)

(2.24)

The abelian electric part of the potential satisfies

\[ Q_\perp = -\frac{r^2}{S} a' = \text{constant} \quad (\in \langle \Sigma \rangle^\perp) \]

(2.25)

and hence decouples.

Eqn. (2.24) is the Gauss constraint. For static solutions all time derivatives disappear, \( B \) can be gauged away and the basic equations simplify considerably. (For the Bartnik-McKinnon solutions \( \Lambda \) is of the form (2.9) with \( \tilde{w} = 0, \Lambda_3 = \tau_3 \) and \( A = 0 \) in (2.6).)

3 Perturbation equations

In this section we study time-dependent perturbations of a given static, asymptotically flat solution of the coupled EYM equations (2.15), (2.16), (2.22) – (2.25). Regular solutions are “purely magnetic” (\( \dot{A} = 0 \) in (2.9)) with vanishing YM charge (\( P_\perp = Q_\perp = 0 \) and \( \lim_{r \to \infty} \tilde{F}_\parallel = 0 \)). Unfortunately, this is not yet proven with satisfactory weak fall-off conditions, but there is strong evidence for this (see [4, 23] for partial results.) The perturbation equations we derive hold also for black holes, if their gauge potentials \( A \) have the form

\[ A = a \, dt + \dot{A} \]

(3.1)

with

\[ a(r) = Q_\perp \int_r^\infty \frac{S}{y^2} dy \]

(3.2)

for a constant vector \( Q_\perp \) in \( \langle \Sigma \rangle^\perp \) (i.e. \( \mathcal{A} = \mathcal{B} = 0 \) in Eq. (2.13)). We call such gauge fields “essentially magnetic”.

From now on \( \Lambda_\pm, N, S, \) etc. refer to an essentially magnetic equilibrium solution and time-dependent perturbations are denoted by \( \delta \Lambda_\pm, \delta \mathcal{B}, \) etc.. All
basic equations are linearized around the equilibrium solution. In order to
decouple the perturbation $\delta a$, we impose the additional constraint $\delta Q_\perp = 0$.

First, we linearize the right hand sides of the Einstein equations (2.15) and
(2.16). Since $B$ and $\dot{\Lambda}_\pm$ vanish for the equilibrium solution, the first order
variation of the source $G$ is

$$
\delta G = \text{Re} \left\langle \Lambda'_+ , \delta \Lambda'_+ \right\rangle - \text{Re} \left\langle \Lambda'_+ , [\Lambda_+ , \delta B] \right\rangle .
$$

(3.3)

Here, the last term vanishes, because the property (2.7) of the scalar product
implies

$$
-2 \text{Re} \left\langle \Lambda'_+ , [\Lambda_+ , \delta B] \right\rangle = \left\langle [\Lambda_+ , \Lambda'_+] + [\Lambda_- , \Lambda'_+] , \delta B \right\rangle ,
$$

(3.4)

and the YM equation (2.22) for the equilibrium solution shows that

$$
[\Lambda_+ , \Lambda'_+] + [\Lambda_- , \Lambda'_+] = 0 .
$$

(3.5)

Thus,

$$
\delta G = \text{Re} \left\langle \Lambda'_+ , \delta \Lambda'_+ \right\rangle .
$$

(3.6)

The only first order variation for $p_\theta$ comes from $\delta |\mathcal{F}_\parallel|^2 = 2\langle \mathcal{F}_\parallel, \delta \mathcal{F}_\parallel \rangle$. Using

$$
\delta \mathcal{F}_\parallel = \frac{i}{2} [\Lambda_+ , \delta \Lambda_-] - \frac{i}{2} [\Lambda_- , \delta \Lambda_+] ,
$$

(3.7)

(see Eq. (2.20)), we have

$$
\delta p_\theta = \frac{1}{r^4} \text{Re} \left\langle i [\mathcal{F}_\parallel, \Lambda_+] , \delta \Lambda_+ \right\rangle .
$$

(3.8)

Now we can work out the variation of the first Einstein equation in (2.15).

With (3.6), (3.8) and (2.16) for the equilibrium solution, we find

$$
\delta m' = - \frac{S'}{S} \delta m + \frac{\kappa}{2} \left\{ N \text{Re} \left\langle \Lambda'_+ , \delta \Lambda'_+ \right\rangle + \text{Re} \left\langle \frac{i}{r^2} [\mathcal{F}_\parallel, \Lambda_+] , \delta \Lambda_+ \right\rangle \right\} .
$$

(3.9)

For the commutator in the last term we use the unperturbed YM equation
(2.23), i.e.

$$
\frac{i}{r^2} [\mathcal{F}_\parallel, \Lambda_+] = N \frac{S'}{S} \Lambda'_+ + N' \Lambda'_+ + N \Lambda''_+ ,
$$

(3.10)

whence

$$
\delta m' = - \frac{S'}{S} \delta m + \frac{S'}{S} \left\{ \frac{\kappa}{2} N \text{Re} \left\langle \Lambda'_+ , \delta \Lambda_+ \right\rangle \right\} ' + \left\{ \frac{\kappa}{2} N \text{Re} \left\langle \Lambda'_+ , \delta \Lambda_+ \right\rangle \right\} ' .
$$

(3.11)

or

$$
(\delta m S)' = \left\{ \frac{\kappa}{2} N S \text{Re} \left\langle \Lambda'_+ , \delta \Lambda_+ \right\rangle \right\} ' .
$$

(3.12)
Therefore, \( \delta m \) must be of the form
\[
\delta m = \frac{\kappa}{2} N \Re \langle \Lambda'_{+}, \delta \Lambda_{+} \rangle + \frac{f(t)}{S'},
\] (3.13)
where \( f(t) \) is a function of \( t \) alone. This function is determined by considering the variation of the second Einstein equation in (2.15), which reads
\[
\delta \dot{m} = \frac{\kappa}{2} N \Re \langle \Lambda'_{+}, \delta \dot{\Lambda}_{+} \rangle.
\] (3.14)
Thus, we have also
\[
\delta m = \frac{\kappa}{2} N \Re \langle \Lambda'_{+}, \delta \Lambda_{+} \rangle + g(r),
\] (3.15)
with a function \( g(r) \) of \( r \) alone. By comparing (3.13) and (3.15), we arrive at the remarkably simple result
\[
\delta m = \frac{\kappa}{2} N \Re \langle \Lambda'_{+}, \delta \Lambda_{+} \rangle,
\] (3.16)
which generalizes an observation already made in [16].

The variation of the Einstein equation (2.16) is immediately obtained with (3.6)
\[
\delta \left( \frac{S'}{S} \right) = \frac{\kappa}{r} N \Re \langle \Lambda'_{+}, \delta \Lambda_{+} \rangle.
\] (3.17)
Before also linearizing the YM equations, we introduce a suitable decomposition of \( \Lambda_{+} \) and \( \delta \Lambda_{+} \). To do so, we choose a base element \( e_{\alpha} \) of the root spaces \( L_{\alpha} \) and expand the unperturbed \( \Lambda_{+} \) as well as its perturbation \( \delta \Lambda_{+} \):
\[
\Lambda_{+} = \sum_{\alpha \in \Sigma} w^{\alpha} e_{\alpha}, \quad \delta \Lambda_{+} = \sum_{\alpha \in \Sigma} \delta w^{\alpha} e_{\alpha}.
\] (3.18)
Then, we have
\[
\delta \Lambda_{\pm} = \delta X_{\pm} \pm i \delta Y_{\pm}
\] (3.19)
with
\[
\delta X_{\pm} = \sum_{\alpha \in \Sigma} \Re (\delta w^{\alpha}) e_{\alpha}, \quad \delta Y_{\pm} = \sum_{\alpha \in \Sigma} \Im (\delta w^{\alpha}) e_{\alpha}
\] (3.20)
and the corresponding expansion for \( \delta X_{-} \) and \( \delta Y_{-} \) with \( e_{\alpha} \) replaced by \( c(e_{\alpha}) \in L_{-\alpha} \), because \( \delta \Lambda_{-} = c(\delta \Lambda_{+}) \) and thus
\[
\delta X_{-} = c(\delta X_{+}), \quad \delta Y_{-} = c(\delta Y_{+}).
\] (3.21)
We call \( \delta X_{\pm}, \delta Y_{\pm} \) the “real” (or “gravitational”) and “imaginary” (or “sphaleron-like”) parts of the perturbations \( \delta \Lambda_{\pm} \). It was shown in [3] that the unperturbed \( \Lambda_{+} \) can be chosen to have only a real part.
Instability of EYM Solutions

This decomposition will lead to a significant decoupling of the perturbation equations. Note in particular, that the variations $\delta m$ and $\delta p_\theta$ in (3.8) and (3.10) depend only on the real part $\delta X_+:

$$
\delta m = \frac{\kappa}{2} N \langle \Lambda_+', \delta X_+ \rangle, \quad (3.22)
$$

$$
\delta p_\theta = \frac{1}{r^2} (i [\hat{F}_{||}, \Lambda_+], \delta X_+). \quad (3.23)
$$

We consider now the first variation of the YM equation (2.23). Its decomposition into real and imaginary parts yields

$$
- \frac{1}{NS^2} \delta \ddot{X}_+ = -N \delta X_+'' - \frac{(NS)'}{S} \delta X_+' - \frac{i}{r^2} [\Lambda_+, \delta \hat{F}_{||}] + \frac{i}{r^2} [\hat{F}_{||}, \delta \Lambda_+]
$$

$$
- \delta N \Lambda_+'' - \delta \left( \frac{(NS)'}{S} \right) \Lambda_+'. \quad (3.24)
$$

and

$$
- \frac{1}{NS^2} \delta \ddot{Y}_+ = -N \left\{ \delta Y_+'' + i [\Lambda_+, \delta B]' + i [\Lambda_+', \delta B] \right\}
$$

$$
- \frac{(NS)'}{S} \left\{ \delta Y_+' + i [\Lambda_+, \delta B] \right\} + \frac{i}{r^2} [\hat{F}_{||}, \delta Y_+]. \quad (3.25)
$$

The third term on the right hand side of (3.24) is indeed real and can be written, using (3.7), as

$$
\frac{i}{r^2} [\Lambda_+, \delta \hat{F}_{||}] = \frac{1}{r^2} \text{ad}(\Lambda_+) \text{ad}(\Lambda_-) \delta X_+. \quad (3.26)
$$

Equation (3.24) can be simplified further. From (3.22) and the equilibrium equation (3.10), we deduce

$$
- \delta N \Lambda_+'' = \frac{2}{r} \delta m \Lambda_+''
$$

$$
= \kappa N \text{Re} \langle \Lambda_+', \delta X_+ \rangle \Lambda_+''
$$

$$
= \kappa \text{Re} \langle \Lambda_+', \delta X_+ \rangle \left\{ -\frac{(NS)'}{S} \Lambda_+ + \frac{i}{r^2} [\hat{F}_{||}, \Lambda_+] \right\},
$$

and the Einstein equations (2.13), (2.14) give

$$
- \delta \left( \frac{(NS)'}{S} \right) = -\frac{2}{r^2} \delta m + \kappa r \delta p_\theta. \quad (3.27)
$$
If we use here (3.22) and (3.23), we see that the last two terms in (3.24) can be expressed as follows:

\[-\delta N \Lambda_r' - \delta \left(\frac{(NS)'}{S}\right) = \frac{1}{NS^2} \left\{ -\Lambda_r' \kappa r \mu^2 \left\{ \frac{(NS)'}{S} + \frac{N}{r} \right\} \langle \Lambda_r', \delta X_+ \rangle 
\right.\]

\[+ \Lambda_r' \kappa \frac{\mu^2}{r} \left( [i\hat{F}_\parallel, \Lambda_+], \delta X_+ \right) + [i\hat{F}_\parallel, \Lambda_+] \kappa \frac{\mu^2}{r} \langle \Lambda_r', \delta X_+ \rangle \right\}, \quad (3.28)\]

where

\[\mu^2 := \frac{NS^2}{r^2}. \quad (3.29)\]

Inserting these expressions into (3.24) gives the following pulsation equation for the real amplitude \(\delta X_+\) of the YM field:

\[\delta \ddot{X}_+ + U_{XX} \delta X_+ = 0, \quad (3.30)\]

where the operator \(U_{XX}\) is given by

\[U_{XX} = p_*^2 + \mu^2 \text{ad}(i\hat{F}_\parallel) - \mu^2 \text{ad}(\Lambda_+) \text{ad}(\Lambda_-)\]

\[- \Lambda_r' \kappa \mu^2 \left\{ 1 - \kappa r^2 p_0 \right\} \langle \Lambda_r', \cdot \rangle\]

\[+ \Lambda_r' \kappa \frac{\mu^2}{r} \left( [i\hat{F}_\parallel, \Lambda_+], \cdot \right) + [i\hat{F}_\parallel, \Lambda_+] \kappa \frac{\mu^2}{r} \langle \Lambda_r', \cdot \rangle, \quad (3.31)\]

and \(p_*\) denotes the differential operator

\[p_* = -iNS \frac{\partial}{\partial r}. \quad (3.32)\]

It is remarkable that the perturbations \(\delta Y_\pm\) and \(\delta B\) do not appear in (3.30) and that the back reaction of gravitation on \(\delta X_+\) can be described by an effective potential (last three terms in (3.31)).

Equation (3.24) can easily be brought into the form

\[\delta \ddot{Y}_+ + U_{YY} \delta Y_+ + U_{YB} \sqrt{N r} \delta B = 0, \quad (3.33)\]

where

\[U_{YY} = p_*^2 + \mu^2 \text{ad}(i\hat{F}_\parallel), \quad (3.34)\]

\[U_{YB} = p_* \mu \text{ad}(\Lambda_+) + \mu \text{ad}(p_* \Lambda_+). \quad (3.35)\]

We have thus achieved a partial decoupling, because neither \(\delta X_+\), nor the metric perturbations, appear in (3.33).
We proceed with the linearization of the YM equation (2.22). The variation of the last two terms is

$$\left[ \Lambda_+, [\Lambda_-, \delta B] \right] + [\Lambda_+, \delta \Lambda_-] - [\Lambda'_-, \delta \Lambda_+] + \text{conjugate},$$

which leads (with $\delta \Lambda_\pm = \delta X_\pm \pm i \delta Y_\pm$) to

$$\left\{ \left[ \Lambda_+, [\Lambda_-, \delta B] \right] + i \left[ \Lambda_+, \delta Y'_- \right] + i \left[ \Lambda'_-, \delta Y_+ \right] \right\}$$

$$+ \left\{ [\Lambda_+, \delta X'_-] - [\Lambda'_-, \delta X_+] \right\} + \text{conjugate}.$$  

Here, the terms in the first curly bracket are in $LT$, while those in the second are in $iLT$. The latter are compensated by their conjugates and we find

$$\sqrt{N r} \delta \dot{B} + U_{BB} \sqrt{N r} \delta B + U_{BY} \delta Y_+ = 0$$  

(3.37)

with

$$U_{BB} = -\mu^2 \text{ad}(\Lambda_+) \text{ad}(\Lambda_-),$$

$$U_{BY} = -\mu \text{ad}(\Lambda_-) p_s + \mu \text{ad}(p_s \Lambda_-).$$

At this point, we collect the results obtained so far as follows: Let

$$\Phi = \begin{pmatrix} \phi_Y \\ \phi_B \end{pmatrix} = \begin{pmatrix} \delta Y_+ \\ \sqrt{N r} \delta B \end{pmatrix},$$

then (3.33) and (3.37) can be written as a $2 \times 2$ matrix equation

$$\ddot{\Phi} + U \Phi = 0$$

(3.41)

with

$$U = \begin{pmatrix} U_{YY} & U_{YB} \\ U_{BY} & U_{BB} \end{pmatrix}.$$  

(3.42)

The operators in this matrix are given in Eq. (3.34), (3.35), (3.38) and (3.39).

The perturbation equations (3.33) and (3.41) do not include the Gauss constraint (2.24), whose linearization is easily found to be

$$\partial_t \left\{ p_s \frac{1}{\mu} \phi_B + \text{ad}(\Lambda_-) \phi_Y \right\} = 0.$$  

(3.43)

The role of this constraint will be discussed below.

In concluding this section, we emphasize once more, that the perturbation equations hold also for black holes, if these are assumed to be of essentially magnetic type (see Eq. (3.1)). We also would like to note that a comprehensive discussion of the pulsation equations for the SU(2) YM-Higgs sphaleron can be found in Ref. [24].
4 Transformation to a hyperbolic system

A look at the second order differential operator $U$ shows that it is not elliptic and, thus, the system (3.41) of partial differential equations is not hyperbolic. With the help of the Gauss constraint (3.43) it is, however, possible to derive a hyperbolic system for the subspace of physical perturbations orthogonal to a space of pure gauge modes. This reformulation of the perturbation equations will turn out to be very useful for several purposes.

We need first some notation. It is natural to introduce the following scalar product for $LG_C$-valued functions on $(r_0, \infty) \subset \mathbb{R}^+$:

$$\langle \phi | \psi \rangle = \int_{r_0}^{\infty} \langle \phi, \psi \rangle \, dr_*$$  \hspace{1cm} (4.1)

with the weighted measure

$$dr_* = \frac{dr}{NS}.$$

For a black hole, the lower limit $r_0$ is the radius of the horizon and for a regular solution it is zero. The operators $U_{XX}$ and $U$ are symmetric with respect to this scalar product on a dense domain of $L^2$-functions. This can be seen easily, using

$$\langle \phi | p_* \psi \rangle = \langle p_* \phi | \psi \rangle$$  \hspace{1cm} (4.2)

for smooth functions, which vanish at $r_0$, and

$$\langle \phi | \text{ad}(Z) \psi \rangle = - \langle \text{ad}(c(Z)) \phi | \psi \rangle$$  \hspace{1cm} (4.3)

for arbitrary $LG_C$-valued functions $\phi, \psi, Z$ in $L^2$ (see (2.5)).

A "gauge mode" $\Phi_G$ is by definition a perturbation of the form

$$\Phi_G = -i \mathcal{G} \chi,$$  \hspace{1cm} (4.4)

where $\mathcal{G}$ is the linear operator

$$\mathcal{G} \chi = \begin{pmatrix} -\text{ad}(\Lambda_+) \chi \\
1 \\
\mu \end{pmatrix}$$  \hspace{1cm} (4.5)

and $\chi$ is a $\langle \Sigma \rangle_C$-valued function. Note, that such variations arise if (2.10) is subjected to $(T$-valued) gauge transformations $g = \exp(-c \chi)$. Eqn. (2.7) and (2.8) show that this induces the infinitesimal transformation

$$\Lambda_+ \rightarrow \Lambda_+ - \text{ad}(\Lambda_+) \chi, \quad \sqrt{N r} \mathcal{B} \rightarrow \sqrt{N r} \mathcal{B} - i \frac{1}{\mu} p_* \chi.$$  \hspace{1cm} (4.6)

It is not surprising that the following identity holds

$$U \mathcal{G} = 0,$$  \hspace{1cm} (4.7)
Instability of EYM Solutions

whence

\[ U \Phi_G = 0 \]  \hspace{1cm} (4.8)

“Physical perturbations” \( \Phi_P \) satisfy by definition

\[ \tilde{G} \Phi_P = 0 \]  \hspace{1cm} (4.9)

where \( \tilde{G} \) is the linear operator

\[ \tilde{G} \Phi = p_\mu \frac{1}{\mu} \phi_B + \text{ad}(\Lambda_-) \phi_Y . \]  \hspace{1cm} (4.10)

The component \( \phi_Y \) is assumed to have values in the subspace (2.10) of \( LG_C \) and \( \phi_B \) has to be \( \langle \Sigma \rangle_C \)-valued. Hence, physical perturbations are by definition those, for which the curly bracket in (3.43) vanishes.

Roughly speaking, a physical perturbation is orthogonal to all gauge modes. More precisely, modulo boundary terms we have

\[ i \langle \Phi_P | \Phi_G \rangle = \langle \Phi_P | G \chi \rangle = \langle \tilde{G} \Phi_P | \chi \rangle = 0 , \]  \hspace{1cm} (4.11)

which follows easily with Eq. (4.2) and (4.3).

The identity

\[ \tilde{G} U = 0 , \]  \hspace{1cm} (4.12)

which can be verified by direct calculation, is related to the Gauss constraint

\[ \partial_t \tilde{G} \Phi = 0 \]  \hspace{1cm} (4.13)

in the following way: Assume Eq. (4.13) is satisfied for \( t = t_0 \), then the dynamical equation (3.41) implies that (4.13) is satisfied for all times. Indeed, we conclude with (4.12) that

\[ \partial_t^2 (\tilde{G} \Phi) = \tilde{G} (\partial_t^2 \Phi) = -\tilde{G} (U \Phi) = 0 . \]  \hspace{1cm} (4.14)

As a corollary we have: A solution of (3.41), which lies initially in the physical subspace (4.10) and satisfies initially the Gauss constraint (4.13), will satisfy the “strong” Gauss constraint (4.9) for all times. For physical perturbations we can thus use this strong form to bring Eq. (3.41) to a hyperbolic form. After some manipulations, one finds

\[ U = \left\{ p_\mu^2 + V \right\} - \mathcal{G} \mu^2 \tilde{G} , \]  \hspace{1cm} (4.15)

where

\[ V = \begin{pmatrix} V_{YY} & V_{YB} \\ V_{BY} & V_{BB} \end{pmatrix} \]  \hspace{1cm} (4.16)
is the following (matrix-valued) potential
\begin{align*}
V_{YY} & = \mu^2 K^2 + \mu^2 \text{ad}(i\hat{\mathcal{F}}), \\
V_{YB} & = 2(p^*\mu)K_+ + 2\mu \text{ad}(p\Lambda_+), \\
V_{BY} & = -2(p^*\mu)K_- + 2\mu \text{ad}(p\Lambda_-), \\
V_{BB} & = \mu^2 K^2 - \frac{(p^*\mu)^2}{\mu},
\end{align*}
with
\begin{align*}
K^2 & = -\text{ad}(\Lambda_+) \text{ad}(\Lambda_-), \\
K_\pm & = \pm \text{ad}(\Lambda_\pm).
\end{align*}
Modulo the strong Gauss constraint \( \tilde{G}\Phi = 0 \), Eq. (3.41) is thus equivalent to
\[ \partial_t^2 \Phi = -\{p^* + V\} \Phi. \]
This system is clearly hyperbolic. We emphasize that this new system implies the strong Gauss constraint for all times, if it is satisfied initially: \( \tilde{G}\Phi|_{t=0} = \tilde{G} \partial_t \Phi|_{t=0} = 0 \). The argument runs as follows: As a result of (4.8), (4.15) and (4.23), \( \tilde{G}\Phi \) satisfies the hyperbolic equation
\[ \partial_t^2 (\tilde{G}\Phi) = -\tilde{G} \mu^2 (\tilde{G}\Phi) = -\{p^* \frac{1}{\mu^2} p + K^2\} \mu^2 (\tilde{G}\Phi). \]
Uniqueness of the Cauchy problem for the hyperbolic system (4.24), with appropriate boundary conditions at \( r_0 \), then implies our claim.

We specialize now to harmonic perturbations proportional to \( e^{-i\omega t} \) and obtain for the amplitude of \( \Phi \), denoted by the same letter, the two eigenvalue problems
\[ U \Phi = \omega^2 \Phi \]
and
\[ \{p^* + V\} \Phi = \omega^2 \Phi. \]
The second equation has the form of a (vector-valued) Schrödinger equation.

In the next section, we prove that the spectrum of \( U \) has a nonempty negative part (which is presumably discrete), by constructing a smooth trial function \( \delta \Phi \) for which \( \langle \delta \Phi | U | \delta \Phi \rangle \) is strictly negative. This implies that the operator \( p^* + V \) has also a negative part in the spectrum. This can be seen as follows:

If we can show that there exists a smooth function \( \chi \), such that
\[ i\tilde{G}\delta \Phi = \tilde{G} \chi = \left\{ p^* \frac{1}{\mu^2} p + K^2 \right\} \chi, \]
then we have a decomposition

\[ \delta \Phi = \delta \Phi_P - i \tilde{G} \chi \quad (4.28) \]

into smooth physical and gauge components. Using also (4.8), we have

\[ \langle \delta \Phi | U | \delta \Phi \rangle = \langle \delta \Phi_P | U | \delta \Phi_P \rangle = \langle \delta \Phi_P | p_s^2 + V | \delta \Phi_P \rangle < 0, \quad (4.29) \]

which would imply our claim.

Since \( \tilde{G} G \) is a positive operator and since \( i \tilde{G} \delta \Phi \) is smooth, we expect on the basis of elliptic existence and regularity theorems that (4.27) has indeed a smooth solution. This is one of several mathematical points which will be discussed in the appendix. Another issue will be, whether the operator \( p_s^2 + V \) is essentially self-adjoint on a dense domain of smooth functions, which satisfy the boundary conditions implied by the physics of the problem. This will be analyzed in section 6 and in the appendix.

The relation between the operators \( U \) and \( Q := p_s^2 + V \), given explicitly in (4.15), can be summarized (on a formal level) as follows: As a result of (4.7) and (4.15), both operators split relative to the decomposition of the \( L^2 \) space of perturbations into physical and gauge degrees of freedom, \( L^2 = \mathcal{H}_P \oplus \mathcal{H}_G \), and their restrictions satisfy

\[ Q|_{\mathcal{H}_P} = U|_{\mathcal{H}_P}, \]

\[ U|_{\mathcal{H}_G} = 0, \]

\[ Q|_{\mathcal{H}_G} = \mathcal{G} \mu^2 \tilde{G}|_{\mathcal{H}_G} \geq 0. \]

The last inequality follow from

\[ \langle \Phi_G | \mathcal{G} \mu^2 \tilde{G} | \Phi_G \rangle = \langle \tilde{G} \Phi_G | \mu^2 | \tilde{G} \Phi_G \rangle \]

for \( \Phi_G \in \mathcal{H}_G \). In particular, the negative part of the spectra of \( U \) is contained in that of \( Q \) and the discrete spectra of the two operators coincide.

5 Instability of generic EYM solutions

We are now ready to establish the main point of this paper: For a given solution with \( \Lambda_+ = \sum_{a \in \Sigma} w_a e_a \), we construct a one-parameter family of field configurations \( \Lambda_{(\tau)} +, B_{(\tau)} \) such that \( \langle \delta \Phi | U | \delta \Phi \rangle < 0 \) for the variation

\[ \delta \Phi = \begin{pmatrix} \delta \phi_Y \\ \delta \phi_B \end{pmatrix} = \begin{pmatrix} -i \partial_{\tau} \Lambda_{(\tau)}^{+} |_{\tau=0} \\ \sqrt{Nr} \partial_{r} B_{(\tau)} |_{\tau=0} \end{pmatrix}. \quad (5.1) \]
The families we consider are of the form
\[
\Lambda(\tau) = \text{Ad}(\exp(\tau Z))\left\{ \Lambda_+ \cos(\tau) + iT_+ \sin(\tau) \right\},
\]
(5.2)
\[
B(\tau) = -\tau Z',
\]
(5.3)
where \(T_+\) is a real element in the subspace (2.10), satisfying
\[
[T_+, T_-] = -2i\Lambda_3,
\]
(5.4)
and \(Z\) is a \(\langle \Sigma \rangle\)-valued function of \(r\) with
\[
\lim_{r \to r_0, \infty} \text{ad}(\Lambda_+) Z = i T_+,
\]
(5.5)
for \(\Sigma\) is not empty such an element \(T_+\) always exists (see appendix A of ref. [4]). A function \(Z\) with the required properties can be found if
\[
\lim_{r \to r_0, \infty} w_\alpha \neq 0 \quad \text{for all} \quad \alpha \in \Sigma.
\]
(5.6)
This can be seen as follows: Let \(\{h_\alpha\}_{\alpha \in \Sigma}\) be the dual basis of \(2\pi \Sigma\) and put
\[
Z = \sum_{\alpha \in \Sigma} Z_\alpha h_\alpha, \quad T_+ = \sum_{\alpha \in \Sigma} T_\alpha c_\alpha
\]
(5.7)
and
\[
Z_\alpha = \left\{ \begin{array}{ll}
-T_\alpha/w_\alpha(r_0) & \text{for} \quad r < r_0 + (1 - \epsilon) \\
-T_\alpha/w_\alpha(\infty) & \text{for} \quad r > r_0 + (1 + \epsilon)
\end{array} \right.
\]
(5.8)
for an \(\epsilon > 0\). Then, both conditions in (5.5) are satisfied.

For a regular (uncharged) solution, condition (5.6) is fulfilled and \(\Sigma\) is not empty [4]. Thus, a family (5.2), (5.3) always exists for solitons [4].

We note some properties of the families above. For the gauge group SU(2), these are closely related to families studied by other authors [25]. The equilibrium solution is clearly obtained for \(\tau = 0\). Applying a gauge transformation with \(g = \exp(\tau Z)\), we obtain
\[
\Lambda(\tau)_+ \to \Lambda_+ \cos(\tau) + iT_+ \sin(\tau), \quad B(\tau) \to 0.
\]
(5.9)
The first variations of (5.2) and (5.3) are
\[
\delta \phi_Y = i \text{ad}(\Lambda_+) Z + T_+, \quad \delta \phi_B = -i \frac{1}{\mu} p_* Z,
\]
(5.10)
and these satisfy by construction the desired boundary conditions
\[
\lim_{r \to r_0, \infty} \delta \phi_Y = 0, \quad \lim_{r \to r_0, \infty} \delta \phi_B = 0.
\]
(5.11)
Instability of EYM Solutions

\( \delta \phi_B \) has even compact support in \((r_0, \infty)\). Since an equilibrium solution satisfies

\[ p_+ \Lambda_+ |_{r_0} = p_+ \Lambda_+ |_{\infty} = 0, \]

we also have

\[ \lim_{r \to r_0, \infty} p_+ \delta \phi_Y = 0. \]  

This choice of trial functions fulfills our goal: \( \delta \Phi \) is normalizable and \( \langle \delta \Phi | U | \delta \Phi \rangle \) is finite and turns out to be strictly negative.

The first of these two points is simple. Since \( \delta \phi_{B} \) in (5.10) has compact support, we have to check only whether

\[ \int_{r_0}^{\infty} |\delta \phi_Y|^2 \frac{dr}{NS} < \infty. \]  

By construction,

\[ \delta \phi_Y = \left\{ \begin{array}{ll}
\sum_{\alpha \in \Sigma} T_\alpha \left( w_\alpha / w_\alpha (r_0) - 1 \right) e_\alpha & \text{for } r < r_0 + (1 - \epsilon), \\
\sum_{\alpha \in \Sigma} T_\alpha \left( w_\alpha / w_\alpha (\infty) - 1 \right) e_\alpha & \text{for } r > r_0 + (1 - \epsilon).
\end{array} \right. \]  

Hence, the integrand has a finite limit for \( r \to r_0 \) (even for extreme black hole solutions). Since \( N \) and \( S \) both approach one at infinity, the integral is finite if \( \Lambda_+ - \Lambda_+ (\infty) \) converges to zero faster than \( r^{-1/2} \).

The calculation of \( \langle \delta \Phi | U | \delta \Phi \rangle \) is somewhat tedious. Considerable simplifications occur by separating a gauge mode in \( \delta \Phi \):

\[ \delta \Phi = \delta \tilde{\Phi} - i G Z \]  

with

\[ \delta \tilde{\Phi} = \begin{pmatrix} T_+ \\ 0 \end{pmatrix}, \quad G Z = \begin{pmatrix} -\text{ad}(\Lambda_+) Z \\ 1/\mu p_+ Z \end{pmatrix}. \]  

We stress, that neither \( \delta \tilde{\Phi} \) nor \( G Z \) are normalizable. Nevertheless, we have \( U G Z = 0 \) and thus (5.16) and (5.17) give (with a slight abuse of notation)

\[ \langle \delta \Phi | U | \delta \Phi \rangle = \langle \delta \tilde{\Phi} | U | \delta \tilde{\Phi} \rangle + i \langle G Z | U \delta \Phi \rangle 
\]

\[ = \left( \delta \tilde{\Phi} | U | \delta \tilde{\Phi} \right) + i \left( U G Z | \delta \tilde{\Phi} \right) + \langle \text{ad}(p_+ \Lambda_+) Z, T_+ \rangle \bigg|_{r_0}^{\infty} 
\]

\[ = \langle \delta \tilde{\Phi} | U | \delta \tilde{\Phi} \rangle. \]  

The boundary term does not contribute because of Eq. (5.12). From this, we obtain the intermediate result

\[ \langle \delta \Phi | U | \delta \Phi \rangle = \int_{r_0}^{\infty} \mu^2 \langle T_+, \text{ad}(i F_{||}) T_+ \rangle dr_+ \]
Instability of EYM Solutions

\[ = 2 \int_{r_0}^{\infty} \mu^2 \langle \mathcal{F}_\parallel, \Lambda_3 \rangle \, dr_\ast, \]  

(5.19)

where we have used (2.3) and the property (5.4) of \( T_+ \).

Finally, we show that the last term has a definite sign:

\[ 2 \int_{r_0}^{\infty} \mu^2 \langle \mathcal{F}_\parallel, \Lambda_3 \rangle \, dr_\ast = - \int_{r_0}^{\infty} |p_\ast \Lambda_+|^2 + 2 \mu^2 |\mathcal{F}_\parallel|^2 \, dr_\ast. \]  

(5.20)

After a partial integration, we find with the unperturbed YM equation (2.23)

\[ \int_{r_0}^{\infty} |p_\ast \Lambda_+|^2 \, dr_\ast = -i \langle p_\ast \Lambda_+, \Lambda_+ \rangle \bigg|_{r_0}^{\infty} - \int_{r_0}^{\infty} \mu^2 (\Lambda_+, \text{ad}(i\mathcal{F}_\parallel)\Lambda_+) \, dr_\ast. \]  

(5.21)

The boundary term vanishes because of Eq. (5.12), and since

\[ 2 |\mathcal{F}_\parallel|^2 = \langle \mathcal{F}_\parallel, i[\Lambda_+, \Lambda_-] - 2 \Lambda_3 \rangle = \langle \Lambda_+, \text{ad}(i\mathcal{F}_\parallel)\Lambda_+ \rangle - 2 \langle \mathcal{F}_\parallel, \Lambda_3 \rangle, \]  

(5.22)

we have established the crucial result

\[ \langle \delta \Phi | U | \delta \Phi \rangle = -\langle p_\ast \Lambda_+ | p_\ast \Lambda_+ \rangle - 2 \langle \mu \mathcal{F}_\parallel | \mu \mathcal{F}_\parallel \rangle \]  

\[ = - \int_{r_0}^{\infty} \left\{ N |\Lambda_+'|^2 + \frac{2}{r^2} |\mathcal{F}_\parallel|^2 \right\} S \, dr. \]  

(5.23)

This expression is clearly finite and strictly negative.

One can show that expression (5.23) is also equal to the second variation of the Schwarzschild mass for the one-parameter family (5.2), (5.3). (This is the way we arrived originally at the variation (5.10)). For a systematic discussion of the relation between variational principles for the spectra of radial pulsations and second variations of the total mass, we refer to [26].

In summary, we have proven (apart from technical subtleties) that static, spherically symmetric, asymptotically flat solutions of the EYM equations are unstable. More precisely, we have established:

**Theorem 1** A generic, regular solution is unstable, if the (magnetic) YM charge vanishes (i.e., if \( \lim_{r \to \infty} \Lambda(r) \) is a homomorphism from \( \text{LSU}(2) \) to \( \text{LG} \)) and if asymptotically \( \Lambda_+ - \Lambda_+(\infty) \sim r^{-\alpha} \) with \( \alpha > 1/2 \).

For a black hole (with horizon at \( r_h \) and \( \Lambda_+ = \sum_{\alpha \in \Sigma} \omega_\alpha e_\alpha \)), the assumptions are somewhat more restrictive and “trivial” solutions have to be excluded. We call a generic, essentially magnetic solution “trivial” if either \( \Sigma \) is empty or each amplitude \( \omega_\alpha \) is constant. These are clearly just the Reissner-Nordström solutions.
Theorem 2 A generic, essentially magnetic, non-trivial black hole solution is unstable, if \( \lim_{r \to r_h, \infty} w_\alpha \neq 0 \) for all \( \alpha \in \Sigma \) and if asymptotically \( \Lambda_+ - \Lambda_+ (\infty) \sim r^{-\alpha} \) with \( \alpha > 1/2 \).

We would like to stress that we were able to draw this conclusion, assuming only weak asymptotic conditions for the solutions. In particular, the fall-off condition is mild and is certainly fulfilled for the Bartnik-McKinnon and the related black hole solutions, as was shown rigorously in [27]. The same is true for the regular solutions, which have been found numerically by H.P. Künzle for the group SU(3) [23]. (For both types, the exponent \( \alpha \) is equal to one.)

6 Sphaleron-like instabilities as bound states of a fictitious deuteron problem

We address now the question, whether the operator \( p^2 + V \) in the eigenvalue problem (4.26) is essentially self-adjoint on a dense domain of smooth functions, which satisfy the boundary conditions implied by the physics of the problem. That this is indeed the case, will be shown in the present section for SU(2) solitons. The discussion of the general case is deferred to the appendix.

For regular SU(2) solutions, it turns out, that the eigenvalue equation (4.26) can be interpreted as a fictitious deuteron problem for a neutron-proton potential, consisting of a central part, a tensor force and a spin-orbit coupling. All parts are determined by the unperturbed soliton and can be shown to be bounded. The corresponding Schrödinger operator is thus essentially self-adjoint on the subspace of smooth functions with compact support and self-adjoint on the Sobolev space \( H^2(\mathbb{R}^3) \). These facts will be used later in an analysis of the instabilities, implied by the existence of bound states (see the appendix).

In order to bring the operator \( p^2 + V \) to a standard Schrödinger form, we introduce the new radial coordinate

\[
\rho(r) = \int_0^r \frac{d y}{N S}
\]

in terms of which \( p_\rho = -id/d\rho \). Since \( \mu^2 \) behaves like \( 1/\rho^2 \) near the origin, we separate from the potential (4.16) the singular term

\[
V(\rho) = \frac{J^2}{\rho^2} + \tilde{V}(\rho),
\]

where

\[
J^2 = \begin{pmatrix}
K^2(0) & 2iK_+(0) \\
-2iK_-(0) & K^2(0) + 2
\end{pmatrix}
\]

and the remainder \( \tilde{V} \) is bounded.
For a generic soliton, the eigenvalues of $J^2$ are equal to $j(j+1)$ with $j = k \pm 1$, whereby the integer $k$ runs through a (strictly) positive, finite set. This set always contains $k = 1$ and is uniquely determined by $\Lambda_3 = \Lambda_{3\parallel}$.

In a representation in which $J^2$ is diagonal, $J^2/\rho^2$ thus describes the central barriers of a finite set of partial waves.

We now discuss in detail the equations for the gauge group SU(2). For this group, only S and D waves occur in Eq. (4.26). In the variable $\rho$ and with the parametrization

$$\Lambda_+ = \rho \tau_+ = \rho (\tau_1 + i\tau_2),$$

$$\Lambda_3 = \tau_3$$

and

$$ \begin{pmatrix} \phi_Y \\ \phi_B \end{pmatrix} = \frac{u_S}{\sqrt{3}} \begin{pmatrix} \tau_+ \\ \tau_3 \end{pmatrix} + \frac{u_D}{\sqrt{3}} \begin{pmatrix} -\tau_+ \\ \tau_3 \end{pmatrix}, \quad (6.4)$$

the eigenvalue equation (4.26) takes the form

$$\left\{ -\frac{d^2}{d\rho^2} + \frac{1}{\rho^2} \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} + \hat{V} \right\} \begin{pmatrix} u_S \\ u_D \end{pmatrix} = \omega^2 \begin{pmatrix} u_S \\ u_D \end{pmatrix}. \quad (6.5)$$

For the potential $\hat{V}$ we find

$$\hat{V} = \frac{1}{3} \begin{pmatrix} \hat{V}_{SS} & \hat{V}_{SD} \\ \hat{V}_{DS} & \hat{V}_{DD} \end{pmatrix}, \quad (6.6)$$

where

$$\hat{V}_{SS} = \left\{ \mu''/\mu + 8(\mu w)' + 6\mu^2 w^2 \right\} + 2\mu^2 (1 - w^2), \quad (6.7)$$

$$\hat{V}_{SD} = \sqrt{2} \left\{ \mu''/\mu + 2(\mu w)' \right\} - \sqrt{2}\mu^2 (1 - w^2), \quad (6.8)$$

$$\hat{V}_{DS} = \hat{V}_{SD}, \quad (6.9)$$

$$\hat{V}_{DD} = 2 \left\{ \mu''/\mu - 4(\mu w)' + 3\mu^2 w^2 - 9/\rho^2 \right\} + \mu^2 (1 - w^2) \quad (6.10)$$

and a dash denotes a derivative with respect to $\rho$.

It is amusing and helpful to note, that this coupled eigenvalue problem has the same form as the Schrödinger equation for the relative motion of a two-body proton-neutron system with the three standard terms $V_C(r)$ (central potential),
Instability of EYM Solutions

$V_T(r) S_{12}$ (tensor interaction) and $V_{LS}(r) L \cdot S$ (spin-orbit interaction). For total angular momentum $J = 1$ and total spin $S = 1$, the possible orbital angular momenta are $L = 1$ and $L = 0, 2$. Because of parity conservation, the P wave decouples from the S and D waves. The remaining equation, describing coupled S and D waves, reads in suitable units

\[
\left\{-\frac{d^2}{dr^2} + \frac{1}{r^2} \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} \right\} \begin{pmatrix} u_S \\ u_D \end{pmatrix} = E \begin{pmatrix} u_S \\ u_D \end{pmatrix}.
\] (6.11)

These equations have first been derived by Rarita and Schwinger [28]. Our eigenvalue problem (6.5) is clearly just a special case of (6.11) and we can, by identification, express the three potentials in terms of the functions $N, S, w$ of the Bartnik-McKinnon solutions.

We present numerical results elsewhere (see also Ref. [29]) and emphasize here only, that with this interpretation the mathematical nature of our eigenvalue problem is automatically settled, because the perturbation $\tilde{V}$ is completely harmless. We come back to this in the appendix, where we discuss also the operator corresponding to the strong Gauss constraint.

Acknowledgments

Special thanks go to Dieter Maison for very useful comments. Interesting discussions with members of our theory group, especially with George Lavrelishvili and Michael Volkov, are gratefully acknowledged. We would also like to thank Markus Heusler for discussions at an earlier stage of our work. Finally, we wish to thank the Swiss National Science Foundation for financial support.

Appendix

In the main body of the text, we deferred on several occasions some of the mathematical subtleties to this appendix.

A Essential self-adjointness of the effective hamiltonian

For black holes, the operator $Q = p^2 + V$ in (4.15), with the expressions (4.17) – (4.20) for the matrix-valued potential $V$, is effectively a standard Schrödinger
operator on the whole real line (see [17]) and is thus essentially self-adjoint on $C^\infty$ functions with compact support. (The potential $V$ is bounded for black holes.) For solitons, we can use Weyl’s limit point – limit circle criterion (see [30], Sec. X.1 or [31]) for the first two terms of the operator

$$Q = -\frac{d^2}{d\rho^2} + \frac{J^2}{\rho^2} + \tilde{V}(\rho) \quad (A1)$$

(see (6.2), (6.3)). Since $\tilde{V}$ is bounded, the Rellich-Kato theorem implies, that the domains of (essential) self-adjointness are not changed by this additive term.

Another method which will be used later, is to lift $Q$ to a Schrödinger operator $H_Q$ on $\mathbb{R}^3$ and to use powerful results for this kind of operators. In Sec. 6 we showed how this can be achieved, if the gauge group is SU(2): $H_Q$ can then be chosen to be of the standard form for a deuteron problem. This operator is essentially self-adjoint on $C^0_0(\mathbb{R}^3) \otimes C^4$ and self-adjoint on the Sobolev space $H^2(\mathbb{R}^3) \otimes C^4$ (see, e.g., [30], Sec. X.2). Restricting these domains to the subspace of S and D waves, provides the domains we are interested in for the original operator $Q$. For instance, $Q$ is essentially self-adjoint on

$$\mathcal{D}(Q) = \left\{ (u_S, u_D) \left| \begin{array}{c} u_S \in C^\infty_0[0, \infty), u_S(0) = 0; u_D \in C^\infty_0(0, \infty) \end{array} \right. \right\}. \quad (A2)$$

Although, we have not yet generalized this construction to arbitrary gauge groups, the generalization of $\mathcal{D}(Q)$ is obvious: The S waves have to be restricted as in (A2) and the higher waves have to lie in $C^\infty_0(0, \infty)$. We also note at this point that the variation (5.10) lies in the domain of definition of the self-adjoint extension of $(Q, \mathcal{D}(Q))$.

If we would restrict the S waves also to $C^\infty_0(0, \infty)$, the operator $Q$ would not be essentially self-adjoint. For each S wave sector, it would actually have a one-parameter family of self-adjoint extensions. The self-adjoint extension, given above, is just the Friedrichs extension, and one can show that it is the only one which is compatible with the strong Gauss constraint (4.9).

The existence and smoothness problems in connection with Eq. (4.27) can also be solved by lifting the equation to $\mathbb{R}^3$ and using standard existence and regularity theorems for elliptic operators on $\mathbb{R}^3$. (The details can easily be worked out for $G = \text{SU}(2).$)

**B Spectral properties and unstable perturbations**

In Sec. 4 it was shown that the perturbation equations for even parity perturbations are equivalent to the hyperbolic system

$$\partial^2_t \Phi = -Q \Phi. \quad (B1)$$
We recall also that these equations imply the propagation of the strong Gauss constraint. As a main point of this paper we proved that the self-adjoint operator $Q$, restricted to the subspace of physical states, satisfying the strong Gauss constraint, has a non-empty negative spectral part. This fact implies, of course, that there are unstable Hilbert space solutions of (E1). We just have to choose the initial data $\Phi_0$ such that $E_Q(-\infty, 0) \Phi_0 \neq 0$, where $E_Q(\cdot)$ denotes the projection valued measure belonging to $Q$ (see below). It is even possible to choose $\Phi_0 \in C_0^\infty$, because the smooth functions with compact support are dense in the Hilbert space $L^2$.

The question now arises, whether such a Hilbert space solution is even a (classical) solution of the system of partial differential equations (E1), in other words, whether the Hilbert space solutions with $\Phi_0 \in C_0^\infty$ are automatically smooth. For black holes, the positive answer to this question is contained in a paper by Wald [32]. His analysis does, however, not directly apply to solitons, because he assumed, that space is a complete Riemannian manifold.

A direct attack of the problem on the half-line $(0, \infty)$ is again difficult. Once more, a way out is lifting the problem to $\mathbb{R}^3$, where

$$\partial_t^2 \Phi = -H_Q \Phi,$$

where the analysis of [32] applies. We showed earlier, how this can be done for SU(2). Since the required smoothness properties certainly do not depend on the gauge group, it is not worthwhile to elaborate further on this. We would like, however, to present here a simplified version of Wald’s argument.

Consider a hyperbolic system on $\mathcal{R} \times \mathbb{R}^n$ of the form (E2) with a smooth elliptic operator $A$ (instead of $H_Q$), which is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. For systems of this kind, a lot is known about the Cauchy problem (a standard reference is [33]). In particular, one knows (see Theorem 23.2.2 in Vol. III of [33]), that the Cauchy problem with initial data $\Phi_0, \dot{\Phi}_0 \in C_0^\infty(\mathbb{R}^n)$ has a unique solution in $C^\infty(\mathcal{R} \times \mathbb{R}^n)$, which for any fixed time $t$ is in $C_0^\infty(\mathbb{R}^n)$. This smooth solution must agree with the Hilbert space solution of the Cauchy problem because the latter is also unique.

Let us now assume that the spectrum $\sigma(A)$ of $A$ has a non-empty intersection $\sigma(A)_-$ with $(-\infty, 0)$. We also assume that $\sigma(A)$ is bounded from below (this is the case for $A = H_Q$). The Hilbert space solution of the Cauchy problem can easily be expressed in terms of the projection valued measure $E(\cdot)$ belonging to $A$. It suffices, for what follows, to take as initial data $\Phi|_{t=0} = \Phi_0, \partial_t \Phi|_{t=0} = 0$. Then, the corresponding Hilbert space solution of

$$\dot{\Phi}_t = -A \Phi_t$$

is

$$\Phi_t = E(\{0\}) \Phi_0 + \int_{(0, \infty)} \cos(t \sqrt{\lambda}) \ d E(\lambda) \Phi_0.$$
Instability of EYM Solutions

\[ + \int_{\sigma(A)_-} \cosh(t \sqrt{-\lambda}) \ dE(\lambda)\Phi_0 , \]  
(B4)

as can easily be verified. Note in particular, that \( \Phi_{t=0} = E(\mathcal{R})\Phi_0 = \Phi_0 \), as required.

With standard rules (see, e.g., [34] Chap. 13), we obtain from this

\[ \langle \Phi_0 | \Phi_t \rangle = \| E(\{0\})\Phi_0 \|^2 + \int_{(0,\infty)} \cos(t \sqrt{\lambda}) \ d\mu_{\Phi_0}(\lambda) \]

\[ + \int_{\sigma(A)_-} \cosh(t \sqrt{-\lambda}) \ d\mu_{\Phi_0}(\lambda) \]  
(B5)

and

\[ \| \Phi_t \|^2 \geq \int_{\sigma(A)_-} \cosh(t \sqrt{-\lambda}) \ d\mu_{\Phi_0}(\lambda) , \]  
(B6)

where \( \mu_{\Phi_0} \) is the finite measure \( \langle \Phi_0 | E(\cdot)\Phi_0 \rangle \) on \( \mathcal{R} \), whose support is contained in \( \sigma(A) \). As emphasized above, we can choose \( \Phi_0 \) such that \( \text{supp} \mu_{\Phi_0} \cap \sigma(A)_- \) is non-empty. Then (B5) and (B6) imply that both quantities on the left diverge exponentially. This exponential grows translates to an average exponential grows of the classical solution of the hyperbolic system for smooth initial data with compact support.

These considerations conclude our instability proof.

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