On the Coefficients of Quasi-Convex Functions

Osman Altıntaş$^{1,a}$ and Melike Aydoğan$^{2,b}$
$^1$ Department of Mathematics Education, Başkent University, Ankara, Turkey
$^2$ Department of Mathematics, Istanbul Technical University, İstanbul, Turkey
E-mail: $^a$ oaltintas@baskent.edu.tr
$^b$ aydogans@itu.edu.tr

Abstract. In this paper we introduce and investigate the class of $T(\lambda, \beta, A, B)$ that we call the class of quasi-starlike and quasi-convex functions with respect to the values of the parameter $\lambda$. Also we define the class $K(\lambda, \beta, \mu, m, A, B)$ which satisfy non-homogeneous Cauchy-Euler differential equation. We obtain coefficient bounds for the functions belonging the above classes.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of analytic functions in the open unit disc $U = \{z \in C : |z| < 1 \}$ as the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1)

We say that the function $f(z)$ is subordinate to $g(z)$ and can be represented as $f < g$, if there exists a function $w(z)$ such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$. If $g(z)$ is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [1]).

Remark 1.1. We have the symbols for following classes.

i. $ST(\beta)$ is the class of starlike functions of order $\beta$.

ii. $CV(\beta)$ is the class of convex functions of order $\beta$.

iii. $CST(\alpha)$ is the class of close-to-starlike functions of order $\alpha$.

iv. $CCV(\alpha)$ is the class of close-to-convex functions of order $\alpha$.

v. $QST(\alpha)$ is the class of quasi-starlike functions of order $\alpha$.

vi. $QCV(\alpha)$ is the class of quasi-convex functions of order $\alpha$.

Remark 1.2. We have the definitions of the following classes.

i. $ST(\alpha) = S^+(\alpha) = \{f \in A : Re \frac{zf^\prime(z)}{g(z)} > \alpha, 0 \leq \alpha < 1 \}$.

ii. $CV(\alpha) = C(\alpha) = \{f \in A : Re \left(1 + \frac{zf^\prime(z)}{g(z)}\right) > \alpha, 0 \leq \alpha < 1 \}$.

iii. $CST(\alpha) = \{f \in A : Re \frac{f(z)}{g(z)} > \alpha, g \in ST(0), 0 \leq \alpha < 1 \}$.

iv. $CCV(\alpha) = \{f \in A : Re \frac{f^\prime(z)}{g^\prime(z)} > \alpha, g \in CV(0), 0 \leq \alpha < 1 \}$.

v. $QST(\alpha) = \{f \in A : Re \frac{zf^\prime(z)}{g(z)} > \alpha, g \in ST(0), 0 \leq \alpha < 1 \}$. 


vi. \( QCV(a) = \{ f \in A : \text{Re}\left(\frac{(zf'(z))^r}{g'(z)}\right) > \alpha, g \in CV(0), 0 \leq \alpha < 1 \} \).

We have the following properties:

**Remark 1.3.**

i. \( f \in CV(\alpha) \Rightarrow zf'(z) \in ST(\alpha), (0 \leq \alpha < 1) \).

ii. \( f \in CCV(\alpha) \Rightarrow zf'(z) \in CST(\alpha), (0 \leq \alpha < 1) \).

iii. \( f \in QCV(\alpha) \Rightarrow zf'(z) \in QST(\alpha), (0 \leq \alpha < 1) \).

**2. Main Results**

**Definition 2.1.** A function \( g \in \mathcal{A} \) is said to be in the class \( T(\lambda, \beta, A, B) \) if the following condition is satisfied.

\[
g'(z) + \lambda zg'''(z) > 1 + Az \frac{1}{1 + Bz},
\]

where \( h(z) \in CV(\beta), 0 \leq \lambda \leq 1, -1 \leq B < A \leq 1 \).

If \( \lambda = 0 \) then \( g(z) \in CCV(\beta, A, B) \) and if \( \lambda = 1 \) then \( g(z) \in QCV(\beta, A, B) \).

**Definition 2.2.** A function \( f \in \mathcal{A} \) is said to be in the class \( K(\lambda, \beta, \mu, m, A, B) \) if the following condition is satisfied

\[
z^m \frac{d^m w}{dz^m} + \left( \frac{m}{1} \right) (\mu + m - 1) 2^{m-1} \frac{d^{m-1} w}{dz^{m-1}} + \cdots + \left( \frac{m}{m} \right) \prod_{j=0}^{m-1} (m + j) w = \prod_{j=0}^{m-1} (\mu + j) g(z)
\]

where \( w = f(z) \in A, g \in T(\lambda, \beta, A, B), 0 \leq \lambda \leq 1, 0 \leq \beta < 1, -1 \leq B < A \leq 1, \mu > -1, m = \{2, 3, 4, \cdots \} \).

The class \( K(\lambda, \mu, 2, A, B) \) is studied in [2] and [3]. The class \( K(\lambda, \mu, m, A, B) \) is defined and studied in [4].

**Lemma 2.3.** \( p(z) = \frac{1+Az}{1+Bz} \) \((-1 \leq B < A \leq 1, z \in U) \) is convex in \( U \).

**Proof.** We have

\[
1 + \frac{zp''(z)}{p'(z)} = \frac{1 - Bz}{1 + Bz} = w(z) = u + iv
\]

and

\[
|z| = \left| \frac{1 - w(z)}{B(1 + w(z))} \right| < 1,
\]

\[
|1 - u - iv|^2 < B^2 |1 + u + iv|^2.
\]

Since \( B < 1 \), we obtain

\[
(1 - u)^2 + v^2 < B^2 ((1 + u)^2 + v^2) < (1 + u)^2 + v^2
\]

or

\[
u = \text{Re} \left( 1 + \frac{zp''(z)}{p'(z)} \right) > 0.
\]

Hence we have \( p(z) \) is convex in \( U \).

Note: \( h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) and \( k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n \) are analytic in \( U \), \( k(z) \) is convex in \( U \) and \( h(z) < k(z) \) then we know that \( |c_n| \leq |k_1| \) (see [5], p 50, Prob. 5).

Hence if \( h(z) < p(z) \) then we obtain \( |c_n| \leq A - B \). However in the previous works is used the relation

\[
|c_n| \leq \frac{2(A - B)}{1 - B}.
\]
It is clear that,
\[ A - B \leq \frac{2(A - B)}{1 - B}. \]

\[ \square \]

**Lemma 2.4.** If \( f \in K(\lambda, \beta, \mu, m, A, B) \) then
\[ a_n = \frac{\prod_{j=0}^{m-1}(\mu + j + 1)}{\prod_{j=0}^{m-1}(\mu + j + n)} b_n \]

(see [4]) where \( f(z) = z + \sum_{n=1}^{\infty} a_n z^n \), \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \).

The class of close-to-convex functions was introduced by Kaplan in [6] and the class of close-to-star functions was introduced by Read in [7]. For the class of close-to-convex functions of order \( \alpha \) and the class of close-to-star functions of order \( \alpha \), \( 0 \leq \alpha < 1 \) introduced by Goodman in [1].

The aim of this paper, firstly to obtain coefficient bounds for the functions \( g(z) \) in Definition 2.2. Secondly, using this result (Theorem 2.5), the coefficient bounds for the functions \( f(z) \) satisfying the differential equation in Definition 2.2.

**Theorem 2.5.** If \( g(z) = z + \sum_{n=1}^{\infty} b_n z^n \) and \( g(z) \in A \) belongs to the class \( T(\lambda, \beta, A, B) \), then we have
\[ |b_n| \leq \frac{(A - B)[2 + (1 - \beta)(n^2 - n - 2)] + 2n(1 - \beta)}{2n[1 + \lambda(n - 1)]} \] (2)

where \( 0 \leq \lambda \leq 1, 0 \leq \beta < 1, -1 \leq B < A \leq 1. \)

**Proof.** Suppose that \( g \in T(\lambda, \beta, A, B) \) then from Definition 2.1 we have
\[ \frac{g'(z) + \lambda zg''(z)}{h'(z)} = k(z) < \frac{1 + Az}{1 + Bz} \] (3)

where \( h(z) = z + c_2 z^2 + c_3 z^3 + \cdots, k(z) = 1 + k_1 z + k_2 z^2 + \cdots \) and \( h \in CV(\beta) \). We know that \(|c_n| \leq 1 - \beta\) (see [1]), since in Lemma 2.3 \(|k_n| \leq A - B\). From (3) we have
\[ g'(z) + \lambda zg''(z) = h'(z), k(z) = (1 + 2c_2 z + 3c_3 z^2 + \cdots)(1 + k_1 z + k_2 z^2 + \cdots) \]

and
\[ n[1 + \lambda(n - 1)] b_n = k_{n-1} + 2c_2 k_{n-2} + 3c_3 k_{n-3} + \cdots + nc_n \]

or
\[ n[1 + \lambda(n - 1)] b_n \leq (A - B)[1 + 2(1 - \beta) + 3(1 - \beta) + \cdots + (n - 1)(1 - \beta)] + n(1 - \beta). \]

\[ n[1 + \lambda(n - 1)] b_n \leq (A - B) \left[ 2 + (1 - \beta) \left( \frac{n(n - 1)}{2} \right) - 1 \right] + n(1 - \beta). \]

Hence we have (2) and the proof of the theorem is completed. \( \square \)

If we choose special values for \( \lambda, \beta, A, B \) in Theorem 2.5, we can find the following corollaries. The extremal function \( f(z) \) is satisfying the inequality (2) is given below
\[ g'(z) + \lambda zg''(z) = 1 + (A - B) \frac{z}{1 - z (1 - z)^2}. \]
Corollary 2.6. If \( g \in T(0, \beta, A, B) \) then we have
\[
|b_n| \leq \frac{(A - B)[2 + (1 - \beta)(n^2 - n) - 2] + 2n(1 - \beta)}{2n}
\]
In this case \( g \in CCV(\beta, A, B) \).

Corollary 2.7. If \( g \in T(0, 0, A, B) \) then we have
\[
|b_n| \leq 1 + \frac{(n - 1)(A - B)}{2}.
\]
In this case \( g \in CCV(A, B) \).

Corollary 2.8. If \( g \in T(0, 0, 1 - 2\alpha, -1) \) then we have
\[
|b_n| \leq (1 - \alpha)n + \alpha
\]
where \( 0 \leq \alpha < 1 \) [8]. In this case
\[
\frac{g'(z)}{h'(z)} < \frac{1 + (1 - 2\alpha)z}{1 - z}
\]
or
\[
\text{Re}\frac{g'(z)}{h'(z)} > \alpha,
\]
h \( \in CV(0) \) and \( g(z) \) is the class of close-to-convex function of order \( \alpha \).

Corollary 2.9. If \( g \in T(1, \beta, A, B) \) then we have
\[
|b_n| \leq \frac{(A - B)[2 + (1 - B)(n^2 - n) - 2] + 2n(1 - \beta)}{2n^2}
\]
In this case
\[
\frac{(zg'(z))^s}{h'(z)} < \frac{1 + Az}{1 + Bz}
\]
h \( \in CV(\beta) \) and \( g(z) \in QCV(\beta, A, B) \). For \( g(z) \in QCV(0, A, B) \) the bounds of the coefficients is obtained in [8], Corollary 2.7).

Remark 2.10. If \( g(z) \in CC(\beta, A, B) \) then
\[
\frac{g'(z)}{h'(z)} = \frac{zg'(z)}{zh'(z)} = \frac{zg'(z)}{s(z)} < \frac{1 + Az}{1 + Bz} \quad \text{and} \quad s(z) \in ST(\beta)
\]
Hence \( g(z) \in QST(\beta, A, B) \) and \( CCV(\beta, A, B) \subset QST(\beta, A, B) \).

Corollary 2.11. If \( g \in T(1, 0, A, B) \) then we have
\[
\frac{(zg'(z))^s}{h'(z)} < \frac{1 + Az}{1 + Bz}, \quad h(z) \in CV(0)
\]
\[
|b_n| \leq \frac{(A - B)(n - 1) + 2}{2n}
\]
In this case \( g(z) \in QCV(A, B) \).

Remark 2.12. If \( g(z) \in QCV(\beta, A, B) \) then we have
\[
\frac{(zg'(z))^s}{h'(z)} = \frac{z(zg'(z))^s}{zh'(z)} = \frac{z(zg'(z))^s}{s(z)} < \frac{1 + Az}{1 + Bz}
\]
and \( s(z) \in ST(\beta) \). Hence \( g(z) \in QST(\beta, A, B) \) and we obtain if \( g(z) \in QCV(\beta, A, B) \) then \( zg'(z) \in QST(\beta, A, B) \).
Corollary 2.13. If \( g(z) \in T(1,0,1,-1) \) then we have

\[
|b_n| \leq 1
\]

In this case \( g(z) \in QCV(1,-1) \) and this result was obtained in [8] and in [9].

Remark 2.14. The bounds in Theorem 2.5 and its corollaries are smaller than the bounds in [[8], Theorem 2.5].

Theorem 2.15. If \( f \in A \) is in the class \( K(\lambda, \beta, \mu, A, B) \) then we have

\[
|a_n| \leq \frac{\prod_{j=0}^{n-1} (\mu + j + 1)(A - B)[2 + (1 - \beta)(n^2 - n - 2)] + 2n(1 - \beta)}{\prod_{j=0}^{n-1} (\mu + j + n)2n[1 + \lambda(n - 1)]}
\]

where \( 0 \leq \lambda \leq 1, \ 0 \leq \beta < 1, \ -1 \leq B < A \leq 1, \ \mu \in R \setminus (-\infty, -1], \ m = \{2,3,4,\cdots\} \).

Proof. Using Theorem 2.5 and Lemma 2.4 we complete the proof of Theorem 2.15.

\[\square\]

Hence, we obtain the coefficient bounds satisfying the Cauchy-Euler type differential equation in Definition 2.2.

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