Kaon Condensation in Dense Matter

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The kaon energy in neutron matter is calculated analytically with the Klein-Gordon equation, by making a Wigner-Seitz cell approximation and employing a $K^{-}N$ square well potential. The transition from the low density Lenz potential, proportional to scattering length, to the high density Hartree potential is found to begin at fairly low densities. Exact nonrelativistic calculations of the kaon energy in a simple cubic crystal of neutrons are used to test the Wigner-Seitz and the Ericson-Ericson approximation methods. All the calculations indicate that by $\sim 4$ times nuclear matter density the Hartree limit is reached, and as the Hartree potential is less attractive, density for kaon condensation appears to higher than previously estimated. Effects of a hypothetical repulsive core in the $K^{-}N$ potential are also studied.

Kaon condensation in dense matter was suggested by Kaplan and Nelson \cite{1}, and has been discussed in many recent publications \cite{2}. Due to the attraction between $K^{-}$ and nucleons its energy decreases with increasing density, and eventually if it drops below the electron chemical potential in neutron star matter in $\beta$-equilibrium, a Bose condensate of $K^{-}$ will appear.

The $K^{-}n$ interaction in vacuo have been described by Brown, Lee, Rho, and Thorsson \cite{2} by an effective Lagrangian based on chiral perturbation theory. The $K^{-}n$ interaction is well described and is fortunately not affected much by resonances as is the $K^{-}p$ interaction. Kaiser, Rho, Waas and Weise \cite{3} have used energy dependent $KN$ amplitudes calculated in a coupled-channel scheme starting from the chiral SU(3) effective Lagrangian. They correct for correlation effects in a nuclear medium and find that $K^{-}$‘s condense at densities above $\sim 4\rho_0$, where $\rho_0 = 0.16$ fm$^{-3}$ is normal nuclear matter density. This is to be compared to the central density of $\sim 4\rho_0$ for a neutron star of mass $1.4M_\odot$ according to the estimates of Wiringa, Fiks and Fabrocini \cite{4} using realistic models of nuclear forces. The condensate could change the structure and affect maximum masses and cooling rates of massive neutron stars.

In this letter we calculate the kaon energy in neutron matter using the Wigner-Seitz approximation for the Klein-Gordon equation of kaons in neutron matter. Our formulation is exact in both, the low density and the high density limits.

We assume that the kaon-nucleon interaction is via the Weinberg-Tomozawa vector potential $V(r)$. In the analysis of Ref.\textsuperscript{4} the $K^+N$ interaction was also found to be dominated by $\omega$ and $\rho$ vector mesons. The energy of the kaon-nucleon center-of-mass system with respect to the nucleon mass is then

$$\omega = \sqrt{k^2 + m_K^2 + V(r) + \frac{k^2}{2m_N}},$$

where $m_N = 939.5$ MeV is the neutron mass, $m_K = 494$ MeV the kaon mass, and $k$ is the kaon momentum (we use units in which $\hbar$ and $c$ are unity) in center-of-mass frame.

We have included the recoil kinetic energy of the nucleon assuming that terms of order $k^4/8m_N^2$ and higher can be neglected. For a relativistic description of the kaon in a vector potential we employ the following recoil corrected Klein-Gordon (RCKG) equation obtained by quantizing Eq. \textsuperscript{1} $(k = -i\nabla)$ \textsuperscript{1}

$$\left\{(\omega - V(r))^2 + \frac{m_N + \omega - V(r)}{m_N} \nabla^2 - m_K^2 \right\} \phi = 0. \,$$

The shape of the kaon-nucleon potential is not known. In most of our work we approximate it with a square well:

$$V(r) = -V_0 \Theta(R - r). \,$$

Yukawa and repulsive core shaped potentials are less favorable for kaon condensation. The range of the interaction $R$ and the potential depth $V_0$ are related through the $s$-wave scattering length:

$$a = R - \frac{\tan(\kappa_0 R)}{\kappa_0},$$

where $\kappa_0^2 = (2m_KV_0 + V_0^2)m_N/(m_N + m_K + V_0)$. If $V_0 \ll m_K$, this reduces to the nonrelativistic result $\kappa_0^2 = 2m_RV_0$, where $m_R = m_Km_N/(m_K + m_N)$ is the kaon-nucleon reduced mass. The $K^{-}n$ scattering length is negative corresponding to a positive $V_0$.

The kaon-nucleon scattering lengths are, to leading order in chiral meson-baryon perturbation theory, given by the Weinberg-Tomozawa vector term

$$a_{K\pi} = a_{K\pi}/2 = \pm \frac{m_R}{8\pi f^2} \simeq \pm 0.31 \text{fm}, \,$$

where $f \simeq 90$ MeV is the pion decay constant. Scalar and other higher order terms have been estimated and we shall use the value $a_{K\pi} = -0.41 \text{fm}$ from \cite{5}. This estimate does not include the $\Sigma\pi$ decay channels which account for the complex parts of the $K^{-}N$ scattering.
lengths [3,8]. The empirical scattering lengths extracted from scattering measurements as well as kaonic atoms are \( a_{K^-n} = (-0.37 - i0.57) \) fm and \( a_{K^-p} = (0.67 - i0.63) \) fm.

Effective Lagrangians predict \( a_{K^-p} \simeq -0.82 \) fm, in perturbation theory. However, it becomes positive due to the presence of the nonperturbative \( \Lambda(1405) \) resonance. Interestingly, we can estimate the range of our square well potential assuming that the \( \Lambda(1405) \) is a \( K^-p \) bound state. As in [3] we assume the \( K^-p \) potential to be twice as strong as the \( K^-n \) potential given by Eq. (3). In order to form a \( K^-p \) bound state with binding energy \( m_p + m_K - m_\Lambda(1405) = 27 \) MeV, and have \( a_{K^-n} = -0.41 \) fm we need \( R \simeq 0.7 \) fm. For this reason we present results for \( R = 0.7 \) fm; however our main conclusions are valid for reasonable values of \( R \).

An analytical calculation of the kaon energy which will result in a repulsive core. The repulsion is chosen such that the final kaon energy is shown in Fig. (1). Note that this equation is valid in both relativistic and nonrelativistic limits. The cell boundary condition contains the important lengths [7,8]. The empirical scattering lengths extracted from scattering measurements as well as kaonic atoms are \( a_{K^-n} = (-0.37 - i0.57) \) fm and \( a_{K^-p} = (0.67 - i0.63) \) fm.

In the WS approximation \( \phi'(r_0) = 0 \), where the core size \( r_0 \) is given by the density \( \rho = (4\pi r_0^3/3)^{-1} \). From Eq. (3) this implies

\[
k r_0 = \frac{e^{2k r_0} + \frac{\rho}{4}}{e^{2k r_0} - \frac{\rho}{4}}.
\]

Eliminating the coefficient \( B/A \) from Eqs. (11) and (12) gives

\[
k \left( \frac{\tan(\kappa R)}{\kappa} \right) = \frac{e^{2k(r_0 - R)} - (1 - k r_0)/(1 + k r_0)}{e^{2k(r_0 - R)} + (1 - k r_0)/(1 + k r_0)},
\]

which determines \( k \) and thus the kaon energy. The resulting kaon energy is shown in Fig. (1). 

At low densities, \( r_0 \gg R \), the kaon energy is \( \omega \simeq m_K \) and \( k \sim 0 \). Since also \( k r_0 \ll 1 \), we can expand the r.h.s. of Eq. (12) and find

\[
\frac{\tan(\kappa R)}{\kappa} = R + \frac{1}{3} k^2 r_0^3 + \frac{1}{5} k^4 r_0^5 + \ldots.
\]

Now we can extract the kaon energy

\[
\omega^2 - m_K^2 = -k^2 \frac{m_K}{m_R} \rho \left( 1 + \frac{9}{5} a_{K^-n} \left( 4\pi/3 \rho \right)^{1/3} + \ldots \right).
\]

The linear part of Eq. (14) is the Lenz potential, Eq. (3), since \( \omega + m_K \sim 2m_K \) at small \( \rho \). The next order scales with \( \rho^{2/3} \) and it becomes a quarter of the leading Lenz potential at \( \rho \sim \rho_0/16 \). This demonstrates that the Lenz potential of kaons is valid only at very small densities, and is of limited interest for kaon condensation. 

At the density where \( r_0 = R \) equations (8) and (9) are solved by a constant \( \phi \) implying \( \omega = m_K - V_0 \), the Hartree energy. At even higher densities the two-body potentials overlap with each other, and the Hartree approximation presumably becomes valid. It gives for \( r_0 \lesssim R \):

\[
\omega = m_K - V_0 \left( \frac{R}{r_0} \right)^3.
\]

Note that this equation is valid in both relativistic and non-relativistic mean field limits; only the value of \( V_0 \) is influenced by relativistic effects in the scattering process. The energies obtained with the Lenz and Hartree approximations are also shown in Fig. (1). The cross-over from the Lenz to the Hartree limit takes place at rather small densities.

If the kaon-nucleon potential has a short range repulsive core of radius \( R_c \), a stronger attractive potential \(-V_0 \) is needed at \( R > r > R_c \) to obtain the same scattering length. In Fig. (1) we show an example of the effect of a repulsive core. The repulsion is chosen such that the interaction has zero volume integral, i.e., the core potential is \( V_c = V_0(R^3/R_c^3 - 1) \). We choose \( R = 1 \) fm and \( R_c = R/2 \). In order to obtain the scattering length
\(a_{K-n} = -0.41\) fm, with non-relativistic kinematics, the attractive potential depth is \(V_0 = 153\) MeV. At very low densities the kaon energy calculated with the WS approximation, follows the Lenz potential and is not affected by the presence of a repulsive core. At intermediate densities, \(r_0 \sim R\), it is actually lower with than without a repulsive core. However, at higher densities the kaon energy approaches \(m_K\), the Hartree limit for an interaction with zero volume integral. The presence of a repulsive core will thus further reduce the possibility of kaon condensation.

\[
\Psi(\tau + \Delta \tau) = \exp[-H \tau] \Psi(\tau) \quad \text{until convergence to the ground state.}
\]

This can be done efficiently by starting with a coarse grid and then using a finer mesh as the iterations proceed. The WS results for non-relativistic kinematics and infinitely heavy nucleons are compared with the lattice results for a \(R = 0.7\) fm square well potential with \(a_{K-n} = -0.41\) fm. The structure of the lattice is not important; results for the bcc lattice, for example, fall between the WS and the exact results for the simple cubic lattice. In Fig. 2 we show \((\omega_K - m_K)\rho_0/\rho\) which equals \(2\pi a_{K-n}\rho_0/m_R\) in the Lenz and \(-V_0 R^4/4\pi\rho_0/3\) in the Hartree limits. We find little difference between the two sets of results throughout the range of densities considered.

![Figure 2](image_url)

**FIG. 2.** Comparison of Wigner-Seitz energies, scaled by the factor \(\rho_0/\rho\), with numerical exact results for a simple cubic crystal (dots), and Ericson-Ericson energies for \(\xi \rho r_0 = 1.54\) and 1.23. These non-relativistic calculations use static, infinitely massive nucleons.

In Ref. 1 the kaon energy is corrected for correlations in the medium. This effect is analogous to the Ericson-Ericson (EE) correction for pions in a nuclear medium 12 and the Lorentz-Lorenz effect in dielectric medium 13. It is interesting to compare the kaon energy \(\omega_{EE}\) obtained with the EE method for a cubic massive neutron lattice. It is given by 14:

\[
\omega_{EE} = m_K = \frac{1}{m_K} \frac{2\pi a}{1 - \a_{K-n}}, \quad \xi = - \int d^3r \frac{C(r)}{r}, \quad (16)
\]

where \(C(r)\) is the nucleon-nucleon correlation function and \(\xi\) is the inverse correlation length. In a simple cubic lattice \(\xi \rho r_0 = 1.54\), while realistic neutron matter wavefunctions calculated in 1 give \(\xi \rho r_0 \sim 1.23\) at densities \(\geq 2\rho_0\). For comparison \(\xi \rho r_0 = 0.92\) in neutron Fermi-gas.

At lower densities there is little difference between the EE results for \(\xi \rho r_0 = 1.54\) and the WS or the exact results. However, at higher densities, the \(\omega_{EE}\) is larger than the correct result. Note that the WS and the exact results depend upon the interaction range, while the \(\omega_{EE}\) depends only on the scattering length \(a\). For larger values of \(R\), same \(a\), the exact and WS energies are lower, further below the EE result, while for smaller \(R\) they would move up towards or even above the EE result. This is to be expected, since the EE approximation assumes that the kaon interacts with only one nucleon at a time, which is valid when \(r_0 \gg R\). At high densities the EE method is not expected to be useful, however, it does seem to
be qualitatively good for \( R \sim 0.7 \) fm. The EE results are not excessively sensitive to the value of \( \xi r_0 \). In Fig. \( \text{[2]} \) we also show the results obtained with \( \xi r_0 = 1.23 \) appropriate for realistic neutron matter. They cross the Hartree line at \( \rho \sim 4 \rho_0 \).

Self-energies in dilute systems can be expanded in terms of the scattering length (the so called Galitskii’s integral equations \( \text{[13]} \)). Analogous results are obtained in chiral perturbation theory \( \text{[14]} \). However, such expansions are valid only when the interparticle distance is much larger than the pair interaction range, so that only the scattering length matters. They generally do not have the correct high density limit, which depends upon the shape of the interaction in addition to the scattering length.

The lowest order constrained variational method \( \text{[15]} \) used to treat strong correlations in nuclear matter and liquid Helium is identical to the WS cell approximation employed here if the healing distance is chosen as \( r_0 \). These methods have the correct low and high density limits, and are meant to provide a good approximation over the entire density range. They are probably less accurate than the low-density expansions in the region where the expansions are valid. For example, consider the low density expansion of the WS energy (Eq. \( \text{[13]} \)). To second order in the scattering length it corresponds to \( \xi r_0 = 1.8 \), much larger than the realistic values quoted above. Thus it is likely that at densities \( \ll \rho_0 \) the EE results for neutron matter shown in Fig.\( \text{2} \) for \( \xi r_0 = 1.23 \) are more accurate than the WS. However, the possible error in the WS results at \( \rho \ll \rho_0 \) seems to be \( < 5\% \) by comparison. It is also possible to calculate corrections to the Hartree potential at high densities by coupling the kaon motion to phonons. Their estimate \( \text{[3]} \) at \( \rho = 4 \rho_0 \) is \( \sim -16 \) MeV for the square well potential with a 0.7 fm radius. This correction decreases as \( \rho \) increases. It is larger for Yukawa shaped potentials (same \( a \) and comparable radius) than the square well, however the Hartree potential is less attractive for the Yukawa shaped interaction.

In conclusion, it seems that we can use the Hartree limit to estimate the kaon energy in matter at densities \( \gtrsim 4 \rho_0 \) where kaon condensation may occur. On this basis it appears from Fig. 1 that, for the electron potential \( \mu_e (\rho) \) calculated from modern realistic \( NN \) interactions \( \text{[2]} \), without quark drops in matter, kaon condensation is unlikely up to \( \rho = 7 \rho_0 \), which is just above the estimated range of densities possible in neutron stars. With the \( \mu_e (\rho) \) used in Ref. \( \text{[4]} \), which is larger, the condensation could occur at \( \rho \gtrsim 6 \rho_0 \). If dense matter has \( \sim 10\% \) protons, whose interaction with the \( K^- \) is believed to be twice as strong, the Hartree potential will be more attractive by \( \sim 10\% \). This together with the realistic \( \mu_e \), could reduce the condensation density to \( \sim 6.5 \rho_0 \). If the \( K^-N \) interaction has the Yukawa shape, or a repulsive core, that would push the condensation density higher, and if quark drops or hyperons reduce the value of \( \mu_e (\rho) \) at higher densities as indicated in Fig.1, kaon condensation becomes unlikely.

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