On the complete classification of unitary $N = 2$
minimal superconformal field theories

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Abstract
Aiming at a complete classification of unitary $N = 2$ minimal models
(where the assumption of space-time supersymmetry has been dropped), it
is shown that each modular invariant candidate of a partition function for
such a theory is indeed the partition function of a minimal model. A family
of models constructed via orbifoldings of either the diagonal model or
of the space-time supersymmetric exceptional models demonstrates that
there exists a unitary $N = 2$ minimal model for every one of the allowed
partition functions in the list obtained from Gannon’s work [17].

Kreuzer and Schellekens’ conjecture that all simple current invariants
can be obtained as orbifolds of the diagonal model, even when the extra
assumption of higher-genus modular invariance is dropped, is confirmed
in the case of the unitary $N = 2$ minimal models by simple counting
arguments.

1 Introduction

Conformal field theories (CFTs) [3, 21, 7, 15, 14] have been a well-studied area
of research since the seminal paper of Belavin, Polyakov and Zamolodchikov
in 1984 and continue to be of interest today, e.g. [13]. Their solvability and
rich symmetries provides insight into both string theory [2, 24, 25, 36] and
microscopic statistical mechanical systems [6]. The $N = 2$ superconformal field
theories (SCFTs) in particular provide useful building blocks for Gepner models
in string theory [26].

Contrary to popular belief, to date the $N = 2$ unitary minimal models [4, 9, 8, 13, 34, 29, 39, 38, 37] have not been completely classified. It is commonly
stated that they fall into the famous $A$-$D$-$E$ meta-pattern, as in the $N = 0$
case [5], due to the work of [33, 32], in which those unitary $N = 2$ minimal
models which enjoy space-time supersymmetry are demonstrated to be in one-
to-one correspondence with the $A$-$D$-$E$ simple singularities. But when one quite
reasonably drops the condition of space-time supersymmetry, one finds a much
larger possible set of solutions.

Gannon [17] classified the possible partition functions of the unitary $N = 2$
minimal models, showing that in fact there is a much larger playground then
previously suspected: there are finitely many partition functions at each level $k$, but the number is unbounded as $k$ increases, in contrast with the $N = 0$ case. There are also many more “exceptional” cases: 10, 18 and 8 corresponding to what are somewhat misleadingly termed $E_6$, $E_7$ and $E_8$, respectively.

Two natural questions then arise: do all of these partition functions belong to genuine SCFTs, or are some just mathematical curiosities? And could there be more than one minimal model associated to each partition function? In this paper we answer the first of these questions. Perhaps surprisingly, it can be resolved using only orbifold-related arguments. It turns out that orbifoldings [10, 11] from every possible partition function to the partition function of one of a small list of well-known and fully understood models can be explicitly calculated, showing that each partition function is indeed that of a fully-fledged SCFT. This is an important step towards the full classification of the unitary $N = 2$ minimal models.

We note that Kreuzer and Schellekens [32] have proven a related result. They construct simple current modular invariant partition functions via orbifoldings of the diagonal model and use the further assumption of higher-genus modular invariance to show that all simple current modular invariant partition functions can be obtained this way. They hypothesise that this extra assumption is unnecessary, which we are able to confirm for the case of unitary $N = 2$ minimal models by simple counting arguments.

Section 2 is a review of Gannon’s program of classifying the possible partition functions of the $N = 2$ unitary minimal SCFTs, and the statement of the result, which did not appear explicitly in [17], with a few minor errors corrected.

Section 3 contains a brief review of orbifold techniques, and the statement and proof of the main theorem: every possible partition function in section 2.4 belongs to a fully-fledged SCFT. The proof is an explicit construction of orbifoldings from any given partition function to one of a few fixed and fully understood SCFTs.

Section 4 investigates the number of simple current physical invariants and confirms a hypothesis of Kreuzer and Schellekens for the special case of the unitary $N = 2$ minimal models; namely, every simple current invariant should be obtainable via an orbifold of the diagonal model.

Section 5 contains conclusions and further directions to be investigated.

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2 Gannon’s Classification of Partition Functions

2.1 Preliminaries

We will denote by $\mathcal{H}$ the underlying pre-Hilbert space of an $N = 2$ SCFT $\mathcal{C}$. $\mathcal{H}$ is a representation of two commuting copies of the $N = 2$ super Virasoro algebra (SVA) [1], whose ‘modes’ are $1, L_n, T_n, G^r_\pm$ with $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz (NS) sector and $r \in \mathbb{Z}$ in the Ramond (R) sector. The $L_n$
modes along with the central element $1$ form a Virasoro algebra with central charge $c \in \mathbb{C}$; the $J_n$ are the modes of a $U(1)$ current and the $G^\pm_r$ are modes of two fermionic super-partners. Together these elements span the left-hand copy of the SVA. The right-hand copy of the SVA is spanned by the elements $\{\mathbb{T}, L_n, J_n, G^\pm_r\}$ with the same commutator relations.

Unitary irreducible inequivalent representations of the SVA can be realised as lowest weight representations (LWR) which are characterised by a lowest weight vector $v$ with lowest weight $h$ and charge $Q$:

$$L_0v = hv, \quad J_0v = Qv,$$

$$L_nv = J_nv = G^\pm_r v = 0 \quad \forall n > 0, r > \frac{1}{2}.$$ 

Through calculation of the vanishing curves of the Kac determinant, Boucher, Friedan and Kent classified these irreducible unitary representations. They exist only when

$$c = \frac{3k}{k}, \quad k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$$

and where, throughout the paper, $\mathfrak{k} = k + 2$. Furthermore, at a given level $k \in \mathbb{N}_0$, irreducible unitary lowest weight representations only exist for a finite collection of possible lowest weights $h$ and charges $Q$. They are given by

$$h_{ac} = \frac{a(a+2)-c^2}{4k} + \frac{|a+c|^2}{8}, \quad \text{where} \quad a = 0, \ldots, k,$$

$$Q_{ac} = c - \frac{|a+c|}{4}, \quad |c - |a+c|| \leq a,$$

where we understand $[x]$ to be 0 if $x$ is even and 1 if $x$ is odd. Here $[a+c] = 0$ label LWRs of the NS sector and $[a+c] = 1$ label LWRs of the R sector. We will label the indexing set of $(a, c)$ satisfying $a = 0, \ldots, k$ and $|c - |a+c|| \leq a$ at level $k$ by $P_k$.

Di Vecchia et al. constructed explicit free fermion representations of each of the possible LWRs in 1986, via the coset construction of Goddard, Kent and Olive, while an alternative explicit construction using parafermions was found around the same time by Qiu. The characters of these representations

$$\text{ch}(\tau, z) = \text{Tr} \left( q^{L_0} \hat{\pi} y^{\mathfrak{k}_0} \right)$$

1Our normalisation of the $U(1)$ current agrees with that of e.g. [37]; as a consequence, $[J_0, G^\pm_r] = \pm \frac{1}{2} G^\pm_r$, and so the supersymmetry modes $G^\pm_r$ carry half integer charge.

2Lowest weight representations are frequently referred to as highest weight representations, a slightly perverse accident of history given that the ‘highest weight vector’ actually has the lowest weight of all states in the representation.

3It is hoped that the index $c$ will not be confused with the central charge $c$.

4We have actually made a choice here – choosing $[x] = -1$ for odd $x$ would give an equivalent realisation of the R sector.
were calculated shortly afterwards \[34, 30, 39\]. The trace is taken over the
states of an irreducible representation of one copy of the SVA, and we use the
standard convention that \( q = e^{2\pi i \tau}, y = e^{2\pi iz} \) for complex parameters \( \tau, z \) where
\( \tau \) is restricted to the upper half complex plane.

### 2.2 Modular Invariance

In an SCFT we demand that the bosonic part of the partition function be mod-
ular invariant. Consequently, the objects of interest to us are not the full char-
acters alluded to above, but rather the projections to the bosonic or fermionic
states in each irreducible LWR \[20\]:

\[
\chi_{ac}(\tau, z) = \text{Tr}_{\mathbb{H}_{ac}} \left( \left( 1 + (-1)^{2(J_0 - \tilde{Q}_{ac})} \right) q^{L_0 - \frac{c}{24}} y^{J_0} \right), \quad (a, c) \in P_k,
\]

is the trace over the representation \( \mathbb{H}_{ac} \) of the left-hand copy of the SVA and
\((-1)^{2(J_0 - \tilde{Q}_{ac})}\) is the chiral world-sheet fermion operator. It is well-defin ed since
\( J_0 \) has charge \( \tilde{Q}_{ac} \) on the lowest weight state \( |a, c\rangle \) of \( \mathbb{H}_{ac} \), and since the charge
of a descendent state differs from \( \tilde{Q}_{ac} \) by a half-integer or an integer. The
chiral world-sheet fermion operator commutes with the mode s \( L_n, J_n \) and anti-
commutes with the modes \( G_\pm^r \), so \((1 + (-1)^{2(J_0 - \tilde{Q}_{ac})})\) projects to those states
created from the lowest weight state \( |h_{ac}, \tilde{Q}_{ac}\rangle \) by the application of an even
number of fermionic modes \( G_\pm \), i.e. states of the form

\[
L_{-n_1} \ldots L_{-n_a} J_{-m_1} \ldots J_{-m_c} G_{l_1}^+ \ldots G_{l_\gamma}^+ G_{l_1}^- \ldots G_{l_\delta}^- |h, Q\rangle
\]

for which \( \gamma + \delta \) is even. Similarly we define

\[
\chi_{k-a,c+\mathbf{j}}(\tau, z) = \text{Tr}_{\mathbb{H}_{ac}} \left( \left( 1 - (-1)^{2(J_0 - \tilde{Q}_{ac})} \right) q^{L_0 - \frac{c}{24}} y^{J_0} \right), \quad (a, c) \in P_k,
\]

the character which counts only those states with \( \gamma + \delta \) odd. The notation
\( \chi_{k-a,c+\mathbf{j}} \) is chosen so that the state(s) with the lowest weight after projection
have weight \( h_{k-a,c+\mathbf{j}} \mod 1 \) and charge \( Q_{k-a,c+\mathbf{j}} \mod 1 \) where we have extended the definition of \( h \) and \( Q \) in equation \[2\] to \( P'_k = P_k \cup \mathbf{j} \cdot P_k = \{0, \ldots, k\} \times \mathbb{Z}_{2k} \), where \( \mathbf{j} \cdot (a, c) = (k - a, c + k) \).

These characters are the building blocks from which we can construct mod-
ular invariant partition functions of the minimal models:

\[
Z(\tau, z) = \sum_{(ac) \in P'_k} M_{a,c; a',c'} \chi_{ac}(\tau, z) \chi_{ac}(\tau, z)^*, \quad (a', c') \in P'_k
\]

where \( M \) is an non-negative integer matrix of multiplicities, and we insist that
the vacuum is unique: \( M_{0,0;0,0} = 1 \).

\[5\]Actually, the embedding diagrams conjectured in references \[34, 30\] were later shown to
be wrong, although the character formulae they produced were correct. See Dörzapf \[12\] for
a discussion and the correct embedding patterns.
The modular group $\text{SL}(2, \mathbb{Z})$ acts naturally on $\mathbb{H}^+ \times \mathbb{C}$ (where $\mathbb{H}^+$ is the upper half complex plane) via $S : (\tau, z) \mapsto \left( -\frac{1}{\tau}, \frac{z}{\tau} \right)$ and $T : (\tau, z) \mapsto (\tau + 1, z)$. This in turn gives a natural (right) action of $\text{SL}(2, \mathbb{Z})$ on the characters $\chi_{ac}$:

$$S \cdot \chi_{ac}(\tau, z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \chi_{ac}(\tau, z) = \chi_{ac}(\frac{1}{\tau}, \frac{z}{\tau}),$$

$$T \cdot \chi_{ac}(\tau, z) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \chi_{ac}(\tau, z) = \chi_{ac}(\tau + 1, z).$$

The characters $\{ \chi_{ac} \mid (ac) \in P_k^+ \}$ transform linearly among themselves under this action and hence span a representation of $\text{SL}(2, \mathbb{Z})$. The $S$- and $T$- matrices are given by

$$S_{a,c; a',c'} = 2S(k)_{a,a'} S'(2)_{[a+c],[a'+c']} S'(k)_{c,c'},$$

$$T_{a,c; a',c'} = T(k)_{a,a'} T'(2)_{[a+c],[a'+c']} T'(k)_{c,c'},$$

$$= e^{2\pi i (h_{ac} - \frac{c}{24})},$$

where $S(k)$ is the $S$-matrix of the $\mathfrak{su}(2)$ WZW model \[31\] at level $k$ and $S'(l)$ is the $S$-matrix of the $\mathfrak{u}(1)$ WZW model at level $l$, with similar notation for the $T$-matrices (see e.g. \[17\] for explicit formulae). $h_{ac}$ is given in equation (2).

Invariance of the partition function $Z(\tau, z)$ in (3) under the action of $\text{SL}(2, \mathbb{Z})$ is equivalent to

$$M = S M S^\dagger,$$

$$M = T M T^\dagger;$$

or, since $S$ and $T$ are unitary, equivalent to asking that $M$ commutes with both $S$ and $T$.

We note here one immediate consequence of modular invariance of a physical invariant: using equation (5), we deduce that $T$-invariance is equivalent to

$$M_{a,c; a',c'} \neq 0 \Rightarrow h_{ac} - h_{a',c'} \in \mathbb{Z}. \quad (7)$$

### 2.3 Gannon’s Classification

Gannon’s result \[17\] was to classify all the modular partition functions $Z$ of the form (3) with unique vacuum. The (non-negative) matrix of multiplicities $M$ of such a partition function is called a physical invariant. We briefly describe how this classification was achieved.

There are two key steps. The first is to observe that there is a connection between the minimal models, which as we mentioned earlier can be constructed via the coset representation $g/h$ with $g = \mathfrak{su}(2)_k \oplus \mathfrak{u}(1)_2$ and $h = \mathfrak{u}(1)_{\mathbb{T}}$, and the WZW model $g \oplus \mathfrak{h}$. Gannon had already shown \[18\] that the physical invariants of $g/h$ could be obtained from the physical invariants of $g \oplus \mathfrak{h}$ for various diagonal embeddings of $\mathfrak{h} \subset g$ at particular levels. This phenomenon occurs because of the similarity of the $S$-matrices of the two theories. In the present case we have
seen that the $S$-matrix is given by equation (4). The characters extend naturally to the indexing set $(a, b, c) \in \{0, \ldots, k\} \times \mathbb{Z}_4 \times \mathbb{Z}_2 =: P^0_k$ if we set
\[
\chi_{ac}^{(b)} := \chi_{ac} \quad \text{when } b = [a + c] \in \{0, 1\},
\]
\[
\chi_{k-a-c+\overline{k}}^{(b+2)} = \chi_{ac}^{(b)} \quad \forall (a, b, c) \in P^0_k,
\]
\[
\chi_{ac}^{(b)} = 0 \quad \text{when } a + b + c \not\equiv 0 \mod 2.
\]

With these definitions we find that the characters $\chi_{ac}^{(b)}$ transform under $S$ with $S$-matrix $S(k) \otimes S'(2) \otimes S'(\overline{k})^*$. Meanwhile the WZW model $\mathfrak{su}(2)_k \oplus \mathfrak{u}(1)_2 \oplus \mathfrak{u}(1)_{\overline{k}}$ has characters $\chi_a \chi_b \chi_c$ with $(a, b, c) \in P^0_k$ which transform under the action of $S$ with $S$-matrix $S(k) \otimes S'(2) \otimes S'(\overline{k})$. The crucial observation is that $\chi_a \chi_b \chi_c$ transforms under $S$ in exactly the same way as $\chi_{ac}^{(b)}$. Thus if $\sum M_{a, b, c; a', b', c'} \chi_{ac}^{(b')*}$ is a physical invariant of the coset $\mathfrak{g}/\mathfrak{h}$, then $\sum M_{a, b, c; a', b', c'} \chi_a \chi_b \chi_c \chi_{a'} \chi_{b'} \chi_{c'}$ is a physical invariant of the WZW model $\mathfrak{g} \oplus \mathfrak{h}$ (note the interchange of $c$ and $c'$). This correspondence is injective and thus every $\mathfrak{g}/\mathfrak{h}$ physical invariant is obtained from a $\mathfrak{g} \oplus \mathfrak{h}$ physical invariant, and the subset of $\mathfrak{g} \oplus \mathfrak{h}$ physical invariants corresponding to $\mathfrak{g}/\mathfrak{h}$ physical invariants are precisely those which respect the symmetry in (3), i.e.

\[
M_{k-a-b+2,c; a',b',c'+\overline{k}} = M_{a,b,c+\overline{k}; k-a',b'+2,c'} = M_{a,b,c; a',b',c'}.
\]

Gannon showed in Lemma 3.1 of [16] that it is enough to check this condition on the left- and right-hand vacua:

\[
M_{k,2,0,0,0,\overline{k}} = M_{0,0,\overline{k}; k,2,0} = 1.
\]

Thus the modular invariant partition functions of the minimal models at level $k$

\[
Z(\tau, z) = \sum_{(a,c) \in P^0_k} \tilde{M}_{a,c; a',c'} \chi_{ac}(\tau, z) \chi_{a',c'}(\tau, z)^*
\]

are obtained by

\[
\tilde{M}_{a,c; a',c'} = M_{a,[a+c],c'; a',[a'+c'],c}
\]

where $M$ is a physical invariant of $\mathfrak{su}(2)_k \oplus \mathfrak{u}(1)_2 \oplus \mathfrak{u}(1)_{\overline{k}}$ satisfying equation (9), and where, as before, $[x]$ is 0 or 1 depending on whether $x$ is even or odd.

The second step is to classify the physical invariants of $\mathfrak{su}(2)_k \oplus \mathfrak{u}(1)_2 \oplus \mathfrak{u}(1)_{\overline{k}}$ subject to equation (9). The crucial step is to note that the Verlinde formula of [43] implies that there is a Galois action on the $S$-matrix:

\[
\sigma \cdot S_{a,b,c; a',b',c'} = \epsilon(\sigma, a, b, c) S_{(a,b,c); a',b',c'} \quad \forall (a, b, c), (a', b', c') \in P^0_k
\]

where $\sigma \in \text{Gal}(K/\mathbb{Q})$ for some cyclotomic extension $K$ of $\mathbb{Q}$, for some $c: P^0_k \rightarrow \{\pm 1\}$ and a permutation $\lambda \mapsto \lambda^\sigma$ of $P^0_k$. From this we obtain a selection rule for the physical invariant $M$:

\[
M_{a,b,c; a',b',c'} \neq 0 \Rightarrow \epsilon(\sigma, a, b, c) = \epsilon(\sigma, a', b', c').
\]
This can be solved exactly: we find that either \( k \in \{4, 8, 10, 28\} \) or that whenever \( M_{0,0,0; a', b', c'} \neq 0 \) we have \( a' \in \{0, k\} \). The former case can be solved by brute force. The latter solutions comprise the so-called \( A-D-E_7 \)-invariants. The \( A-D-E_7 \)-invariants are defined by the condition

\[
M_{a,b,c;0,0,0} \neq 0 \Rightarrow (a, b, c) \in \mathcal{J}(0, 0, 0)
\]

\[
M_{0,0,0; a', b', c'} \neq 0 \Rightarrow (a', b', c') \in \mathcal{J}(0, 0, 0)
\]

where \( \mathcal{J} \) is the set of simple currents of the physical invariant \( M \) \([28, 40]\) (see also section \([32]\)). This is a generalisation of the notion of simple current invariant \([19]\), a physical invariant \( M \) satisfying

\[
M_{a,b,c; a', b', c'} \neq 0 \Rightarrow (a', b', c') \in \mathcal{J}(a, b, c).
\]

The classification of the physical invariants of \( su(2)_k \oplus u(1)_2 \oplus u(1)_\pi \) thus reduces to the classification of the \( A-D-E_7 \)-invariants of \( su(2)_k \oplus u(1)_2 \oplus u(1)_\pi \).

### 2.4 Explicit Classification of Minimal Partition Functions

We state the list of partition functions of the minimal models here for two reasons: firstly, it did not appear explicitly in Gannon’s paper, and deserves to be accessible in the literature; and secondly because there were a few minor errors in the ‘trivial’ (read: beneath contempt) application of the main theorem of that paper to the case of \( su(2)_k \oplus u(1)_2 \oplus u(1)_\pi \). The corrections are highlighted in footnotes.

Throughout this section and the rest of the paper \( J \) will denote the \( su(2)_k \) simple current \( J : a \mapsto k - a \) and we write \( \overline{k} = k + 2 \).

\( k \) odd:

- We have a physical invariant \( \tilde{M}^0 \) for each triple \((v, z, n)\) with \( v | \overline{k}, \overline{k} | v^2 \) and \( \overline{k}(4z^2 - 1)/v^2 \in \mathbb{Z} \) where \( z \in \{1, ..., v^2/\overline{k}\} \) and \( n \in \{0, 1\} \). Its non-zero entries are

\[
\tilde{M}^0_{a,v,\overline{k}; a', v', \overline{k}} = 1 \iff \begin{cases} a' = J^{n(a+c)}a \\ c' \equiv c + n(a + c) \pmod{2} \\ c' \equiv 2zc \pmod{v^2/\overline{k}} \end{cases} \tag{10}
\]

- \( 4 \) divides \( \overline{k} \):

\( k \) even:

- We have a physical invariant \( \tilde{M}^{2.0} \) for each triple \((v, z, n)\) with \( 2v | \overline{k}, \overline{k} | v^2 \) and \( y := \overline{k}(z^2 - 1)/2v^2 \in \mathbb{Z} \) where \( z \in \{1, ..., 2v^2/\overline{k}\} \) and \( n \in \{0, 1\} \). Its non-zero entries are

\[
\tilde{M}^{2.0}_{a, v, \overline{k}; a', v', \overline{k}} = 1 \iff \begin{cases} a' = J^{an + cy}a \\ c' \equiv cz + ayv^2/\overline{k} \pmod{2v^2/\overline{k}} \end{cases} \tag{11}
\]

\[\text{So called because in the classification of the } su(2)_k \text{ WZW models } [5], \text{ these are precisely the models } A-D \text{ and } E_7. \]

\[\text{Gannon’s words!} \]
• We have a physical invariant \( \tilde{M}^{2,1} \) for each triple \( (v, z, n) \) with \( 2v^2/k, 2k^2 \in 2\mathbb{Z} + 1 \) and \( k(z^2 - 1)/2v^2 \in \mathbb{Z} \) where \( z \in \{1, ..., 2v^2/k\} \) and \( n \in \{0, 1\} \). Its non-zero entries are

\[
\tilde{M}^{2,1}_{a,c,k/2v; a',c',k/2v} = 1 \iff \begin{cases} 
  a \equiv a' \equiv c \equiv c' \pmod 2 \\
  a' = J^{a+c} a \\
  c' \equiv cz \pmod {2v^2/k} 
\end{cases} \quad (12)
\]

• We have a physical invariant \( \tilde{M}^{2,2} \) for each quadruple \( (v, z, n, m) \) with \( k/v \) odd, \( v^2/k \in \mathbb{Z} \) and \( k(z^2 - 1)/4v^2 \in \mathbb{Z} \) where \( z \in \{1, ..., 2v^2/k\} \) and \( n, m \in \{0, 1\} \). Its non-zero entries are

\[
\tilde{M}^{2,2}_{a,c,k/2v; a',c',k/2v} = 1 \iff \begin{cases} 
  a' = J^{a+cm}a \\
  c' \equiv cz + (a + c)mv^2/k \pmod {2v^2/k} 
\end{cases} \quad (13)
\]

4 divides \( k \)

• If \( 8|k + 4 \) then we have a physical invariant \( \tilde{M}^{4,0} \) for each quadruple \( (v, z, n, m) \) with \( k/2v \in \mathbb{Z} \), \( x := (1/4 + v^2/k) \in \mathbb{Z} \) and \( k(z^2 - 1)/2v^2 \in \mathbb{Z} \) where \( z \in \{1, ..., 2v^2/k\} \) and \( m, n \in \{0, 1\} \). Its non-zero entries are

\[
\tilde{M}^{4,0}_{a,c,k/2v; a',c',k/2v} = 1 \iff \begin{cases} 
  c + c' \equiv a \equiv a' \pmod 2 \\
  a' = J^{ax+cn+c(1-c)/2}a \\
  c' \equiv cz \pmod {2v^2/k} \\
  2c'm + c'(1 - c') \equiv 2cn + c(1 - c) \pmod 4 
\end{cases} \quad (14)
\]

Note that \( \tilde{M}^{4,0} \) is only symmetric when \( m = n \). In fact \( (\tilde{M}^{[v,z,n,m]} T = \tilde{M}^{[v,z,m,n]} \). Note also that the condition that \( x \) be an integer follows directly from the conditions that \( 8|k + 4 \) and \( k/2v^2 \).

• If \( 8|k \) then we have a physical invariant \( \tilde{M}^{4,1} \) for each quadruple \( (v, z, x, y) \) with \( v|k, k|v^2, 2k(4z^2 - 1)/v^2 \equiv 7 \pmod 8 \) where \( z \in \{1, ..., v^2/k\} \) and \( x, y \in \{1, 3\} \). Its non-zero entries are

\[
\tilde{M}^{4,1}_{a,c,k/v; a',c',k/v} = 1 + \delta_{a,k/2} \iff \begin{cases} 
  a \equiv a' \equiv 0 \pmod 2 \\
  a' = J^l a \quad \text{for some } l \in \mathbb{Z} \\
  c' \equiv 2cz \pmod {2v^2/k} \\
  c(c - x) \equiv 2c'z \pmod 4 \\
  c'(c - y) \equiv 2cz \pmod 4 
\end{cases} \quad (15)
\]

Note that \( \tilde{M}^{4,1} \) is only symmetric when \( x = y \). In fact \( (M^{[v,z,x,y]} T = M^{[v,z,y,x]} \). Note also that the condition \( 2k(4z^2 - 1)/v^2 \equiv 7 \pmod 8 \) is equivalent to \( 2k(4z^2 - 1)/v^2 \in \mathbb{Z} \) and \( k/8 \equiv z \pmod 2 \).

\#In the original classification the modulo 8 condition was only given modulo 1

8
We have a physical invariant $\widetilde{M}^{4,2}$ for each triple $(v, z, x)$ with $2v|\mathbb{K}$, $\mathbb{K}|2v^2$ and $\mathbb{K}(z^2 - 1)/2v^2 \in \mathbb{Z}$ where $z \in \{1, ..., 2v^2/\mathbb{K}\}$ and $x \in \{1, 3\}$. Its non-zero entries are

$$\widetilde{M}^{4,2}_{a,c;v, a',c';v/2v} = 1 + \delta_{a,k/2} \iff \begin{cases} a \equiv a' \equiv 0 \pmod{2} \\ a' = J^l a \\ c' \equiv cz \pmod{2v^2/\mathbb{K}} \\ c' \equiv cx \pmod{4} \end{cases}$$ (16)

We have a physical invariant $\widetilde{M}^{4,3}$ for each triple $(v, z, n)$ with $2v|\mathbb{K}$, $\mathbb{K}|2v^2$ and $\mathbb{K}(z^2 - 1)/4v^2 \in \mathbb{Z}$ where $z \in \{1, ..., 8v^2/\mathbb{K}\}$ and $n \in \{0, 1\}$. Its non-zero entries are

$$\widetilde{M}^{4,3}_{a,c;v, a',c';v/2v} = 1 \iff \begin{cases} a' = J^{n(a+c)} a \\ c' \equiv cz \pmod{2v^2/\mathbb{K}} \\ c' \equiv cz + 2(a + c)n \pmod{4} \end{cases}$$ (17)

Exceptional Models

- When $k = 10$ we have a physical invariant $\widetilde{E}_1^{10}$ for the 2 pairs $(v = 6, z)$ with $z \in \{1, 5\}$. $\widetilde{E}_1^{10} = E^{10} \otimes \overline{M}$ where $E^{10}$ is the $\mathfrak{su}(2)$ exceptional physical invariant and $\overline{M}$ is the projection onto the $u(1)$ part of $M^{2,0}$: the non-zero entries of $\overline{M}$ are

$$\overline{M}_{2v; 2v'} = 1 \iff \{ c' \equiv cz \pmod{6} \}$$ (18)

- When $k = 10$ we have a physical invariant $\widetilde{E}_2^{10}$ for the 8 quadruples $(v = 12, z, n = 0, m)$ with $z \in \{1, 7, 17, 23\}$ and $m \in \{0, 1\}$. Let $E^{10}$ be the $\mathfrak{su}(2)$ exceptional physical invariant. Then $\overline{M}$ is given by

$$\left(\widetilde{E}_2^{10}\right)_{a,c; a',c'} = 1 \iff \begin{cases} E^{10}_{J^{m(a+c)} a, a'} = 1 \\ c' \equiv cz + 12(a + c)m \pmod{24} \end{cases}$$ (19)

- When $k = 16$ we have a physical invariant $\widetilde{E}_1^{16}$ for the 12 quadruples $(v, z, x, y)$ with either $v = 6, z = 2$ or $v = 18, z \in \{4, 5\}$, and $x, y \in \{1, 3\}$. $\widetilde{E}_1^{16} = E^{16} \otimes \overline{M}$ where $E^{16}$ is the $\mathfrak{su}(2)$ exceptional physical invariant and $\overline{M}$ is the projection onto the $u(1)$ part of $M^{4,1}$: the non-zero entries of $\overline{M}$ are

$$\overline{M}_{18c/v, 18c'/v} = 1 \iff \begin{cases} c' \equiv 2cz \pmod{v^2/36} \\ c(c-x) \equiv 0 \pmod{4} \\ c'(c'-y) \equiv 0 \pmod{4} \end{cases}$$ (20)

In the original classification of the $\mathfrak{su}(2)_k \oplus u(1)_2 \oplus u(1)_4$ invariants, the non-zero entries of $M^{4,3}$ should have read $M_{a,b,c;J^lu,v,2, cv+2l,v} = 1$ with $(c + b - v)w/\mathbb{K} \in \mathbb{Z}$ and $l \in \mathbb{Z}$, and $z$ should be allowed to run from 1 to $8v^2/\mathbb{K}$ rather than only up to $4v^2/\mathbb{K}$.
• When \( k = 16 \) we have a physical invariant \( \tilde{E}^{16}_2 \) for the 6 triples \((v, z, x)\) with either \( v = 3, z = 1 \) or \( v = 9, z \in \{1, 8\} \), and \( x \in \{1, 3\} \). \( \tilde{E}^{16}_2 = E^{16} \otimes M \) where \( E^{16} \) is the \( \mathfrak{su}(2) \) exceptional physical invariant and \( M \) is the projection onto the \( \mathfrak{u}(1) \) part of \( \tilde{M}^{4,2} \): the non-zero entries of \( M \) are:

\[
\begin{align*}
\tilde{M}_{9c/v; 9c'/v} = 1 & \iff \begin{cases}
  c' \equiv cz \pmod{v^2/9} \\
  c' \equiv cx \pmod{4}
\end{cases}
\end{align*}
\] (21)

• When \( k = 28 \) we have a physical invariant \( \tilde{E}^{28} \) for the 8 triples \((v, z, x)\) with \( z \in \{1, 4, 11, 14\} \) and \( x \in \{1, 3\} \). \( \tilde{E}^{28} = E^{28} \otimes \tilde{M} \) where \( E^{28} \) is the \( \mathfrak{su}(2) \) exceptional physical invariant and \( \tilde{M} \) is the projection onto the \( \mathfrak{u}(1) \) part of \( \tilde{M}^{4,2} \): the non-zero entries of \( M \) are

\[
\begin{align*}
\tilde{M}_{c; c'} = 1 & \iff \begin{cases}
  c' \equiv cz \pmod{15} \\
  c' \equiv cx \pmod{4}
\end{cases}
\end{align*}
\] (22)

2.5 Simple Examples

To illustrate the foregoing classification, and to demonstrate that, at least for the lowest levels, the partition functions often turn out to be given in terms of familiar functions, we will calculate the partition functions explicitly for levels \( k = 1 \) and \( k = 2 \).

2.5.1 \( k = 1 \)

Level \( k = 1 \) yields \( N = 2 \) super conformal unitary minimal models with central charge \( c = 1 \). We can express the characters in terms of familiar functions:

\[
\chi_{a,c}(\tau, z) = K_{2c-3a+c}^{(6)}(\tau, z)
\]

where \( K_{\frac{1}{2}}^{(6)} \) are the \( \mathfrak{u}(1)_6 \) characters defined by

\[
K_{\frac{1}{2}}^{(l)}(\tau, z) = \frac{1}{\eta(\tau)} \sum_{Q \in \Gamma_x^{(l)}} q^{Q^2} y^Q, \quad x \in \mathbb{Z}_{2l},
\] (23)

the lattice \( \Gamma_x^{(l)} \) is given by \( \Gamma_x^{(l)} = \{ (n + \frac{x}{2})| n \in \mathbb{Z} \} \) and \( \eta \) is the Dedekind \( \eta \)-function. We can then read off from section 2.4 the partition functions of the 4 minimal models with \( c = 1 \). We label the four partition functions by the parameters \([v, z, n]\) (see equation (10) for notation):

\[
\begin{align*}
Z[3,2,0](\tau, z) &= Z_{R=\sqrt{3}}(\tau, z); \\
Z[3,1,1](\tau, z) &= Z_{R=\frac{1}{\sqrt{3}}}(\tau, z),
\end{align*}
\]

\footnote{There are 16 models described as coming from \( M^{4,0} \) in the original classification, but no such models in fact exist.}

\footnote{The Kac-Moody algebra of \( \mathfrak{u}(1) \) does not have levels as such, since the generators can always be rescaled. We borrowed the notation \( \mathfrak{u}(1)_l \) from [7].}
\[ Z[3,2,1](\tau, z) = Z_{R=\sqrt{3}}(\tau, z); \]
\[ Z[3,1,0](\tau, z) = Z_{R=\sqrt{2}}(\tau, z); \]

where \( Z_R \) is the partition function of the boson on the circle at radius \( R \) (see e.g. [21]).

\[
Z_R(\tau, z) = \frac{1}{|\eta(\tau)|^2} \sum_{(Q,\overline{Q}) \in \Gamma_R} q^{Q^2} y^{\overline{Q}^2} \eta^{\overline{Q}Q},
\]

(24)

\[
\Gamma_R = \left\{ \frac{1}{\sqrt{2l}} \left( \frac{n}{R} + mR, \frac{n}{R} - mR \right) \bigg| n, m \in \mathbb{Z} \right\},
\]

(25)

where here \( l = 6 \). The pair \((Q, \overline{Q}) \in \Gamma_R\) labels a conformal primary state with \( U(1) \) charges \((Q, \overline{Q})\) and conformal weights \((h, \overline{h}) = (6Q^2, 6\overline{Q}^2)\).

The first and second partition functions, and the third and fourth partition functions are mirror symmetry pairs. Mirror symmetry is realised by acting by the charge conjugation matrix \( C = S^2 \) on one of the chiral sectors. At the level of primary states, mirror symmetry acting on the left-hand representations maps states with \( U(1) \) charges \((Q, \overline{Q})\) to states with charges \((-Q, -\overline{Q})\). This implies that one model can be obtained from the other by relabelling the generators of the left \( U(1) \) current:

\[
\{L_n, J_n, G^\pm_r, L_n, J_n, G^\mp_r\} \rightarrow \{L_n, -J_n, G^+_r, L_n, J_n, G^-_r\}.
\]

Thus the two mirror symmetric models describe identical physics, and we would normally consider them to be equivalent theories. However, since they give rise to different partition functions, it will be convenient to treat them as belonging to separate theories. The analogue is true for mirror symmetry acting on the right-hand states.

We note that combining both left- and right- mirror symmetry transformations yields the charge conjugation transformation, which acts on charges of states via \((Q, \overline{Q}) \rightarrow (-Q, -\overline{Q})\). Since the charge conjugation matrix \( C \) satisfies \( C^2 = S^4 = 1\), we see that this leaves the partition functions invariant. We will therefore consider charge conjugate theories to be identical.

In the current case, we note that mirror symmetry acts by \( T \)-duality, interchanging \( Z_R \) and \( Z_{\frac{1}{R}} \).

---

12 In our normalisation the self-dual radius is \( R = 1 \). Some authors use \( R = \sqrt{2} \).

13 It is perhaps more usual to re-scale the \( U(1) \) current for the boson on the circle by \( \sqrt{12} \) to obtain \( h = \frac{Q^2}{4} \). The price, of course, is that the \( N = 2 \) algebra, which is a symmetry of these \( c = 1 \) theories at the special radii \( R, R^{-1} \in \{\sqrt{2}, \sqrt{3}, \sqrt{4} \} \), will then differ from its usual form: e.g. we would find \([J_0, G^\pm_r] = \pm \sqrt{3} G^\pm_r \). See Waterson [15] for an explicit construction of the irreducible representations of the unitary \( N = 2 \) minimal models at \( c = 1 \).

14 We emphasise that acting with the charge conjugation matrix \( C \) on one chiral halve yields the mirror symmetry transformation; acting on both halves simultaneously yields the charge conjugation transformation.
2.5.2 \( k = 2 \)

The level \( k = 2 \) models correspond to the \( N = 2 \) super conformal unitary minimal models with central charge \( c = \frac{3}{2} \). Again, we can express the characters in terms of familiar functions:

\[
\chi_{a,c}(\tau, z) = \eta(\tau) c_{a,c-[a+c]}(\tau) K_{c-2[a+c]}^{(4)}(\tau, z)
\]

where \( K_{x}^{(4)}, x \in \mathbb{Z}_8 \) are the \( u(1) \) characters given in equation (23) for \( l = 4 \) and \( c_{a,c}^{(2)} \) are the level 2 \( su(2) \) string functions. The string functions can be written in terms of the Jacobi theta functions and the Dedekind eta function as follows:

\[
\eta(\tau)c_{a,c-b}^{(2)}(\tau) = \begin{cases} 
\frac{1}{2} \left( \sqrt{\frac{\theta_2(\tau, 0)}{\eta(\tau)}} + (-1)^a \sqrt{\frac{\theta_4(\tau, 0)}{\eta(\tau)}} \right) & \text{if } a = 1 \\
\frac{1}{2} \left( \sqrt{\frac{\theta_2(\tau, 0)}{\eta(\tau)}} + (-1)^{b-a} \sqrt{\frac{\theta_4(\tau, 0)}{\eta(\tau)}} \right) & \text{if } a \text{ is even.}
\end{cases}
\]

We can now evaluate the five modular invariant partition functions\(^{13}\) using the labels [0; v, z] for the unique \( \tilde{M}^{2,0} \), invariant (see equation (11) – we have dropped the label \( n \) since \( n = 0 \) or 1 give the same partition function for \( k = 2 \)) and labels [2; v, z, m] for the four partition functions in the family \( \tilde{M}^{2,2} \) (see equation (13) – again we have dropped the \( n \) label).

\[
\begin{align*}
Z[0; 2, 1](\tau, z) &= Z_{\text{Ising}}(\tau) Z_{R=1}(\tau, z); \\
Z[2; 4, 1, 0](\tau, z) &= Z_{\text{Ising}}(\tau) Z_{R=2}(\tau, z); \\
Z[2; 4, 7, 1](\tau, z) &= Z_{\text{Ising}}(\tau) Z_{R=4}(\tau, z); \\
Z[2; 4, 7, 0](\tau, z) &= \frac{1}{2} \sum_{c \in \mathbb{Z}_8} \left( \frac{\theta_3(\tau, 0)}{\eta(\tau)} \right) + (-1)^c \left( \frac{\theta_4(\tau, 0)}{\eta(\tau)} \right) \sum_{c \in \mathbb{Z}_8} K_c(\tau, z) K_{3c}(\tau, z)^* \\
&\quad + \frac{1}{2} \left( \frac{\theta_2(\tau, 0)}{\eta(\tau)} \right) \sum_{c \in \mathbb{Z}_8} K_c(\tau, z) K_{3c+4}(\tau, z)^*; \\
Z[2; 4, 1, 1](\tau, z) &= \frac{1}{2} \sum_{c \in \mathbb{Z}_8} \left( \frac{\theta_3(\tau, 0)}{\eta(\tau)} \right) + (-1)^c \left( \frac{\theta_4(\tau, 0)}{\eta(\tau)} \right) \sum_{c \in \mathbb{Z}_8} K_c(\tau, z) K_{5c}(\tau, z)^* \\
&\quad + \frac{1}{2} \left( \frac{\theta_2(\tau, 0)}{\eta(\tau)} \right) \sum_{c \in \mathbb{Z}_8} K_c(\tau, z) K_{5c+4}(\tau, z)^*,
\end{align*}
\]

where here

\[
Z_{\text{Ising}} = \frac{1}{2} \left( \frac{\theta_2(\tau, 0)}{\eta(\tau)} \right) + \frac{\theta_3(\tau, 0)}{\eta(\tau)} + \frac{\theta_4(\tau, 0)}{\eta(\tau)}
\]

is the partition function of the Ising model (see e.g. [21]), and \( Z_R \) is the partition function of the boson on the circle given in equation (24) with \( l = 4 \).

\(^{13}\)In a later section when we count the number of simple current invariants, we will see that there should be 10 partition functions at level 2. This discrepancy arises from the identity \( A_2 = D_2 \), which does not generalise to other levels \( k \).
We note that the second partition function is that of the diagonal model. The first model is self-mirror-symmetric, and the second and third, and the fourth and fifth partition functions belong to mirror symmetry pairs. Again the mirror symmetry is realised via $T$-duality, by interchanging $Z_R$ and $Z_{\bar{R}}$; on the level of primary states it acts on the left-hand representations by mapping the primary state $|\text{Ising}\rangle \otimes |Q, \bar{Q}\rangle$ to $|\text{Ising}\rangle \otimes | -Q, \bar{Q}\rangle$, and similarly on the right-hand representations.

2.6 Classification of Theories with Space-Time Supersymmetry

In this section we show that those partition functions belonging to space-time supersymmetric models fall into the well-known $A-D-E$ pattern in accordance with [33, 42]. Specifically we will find which of the partition functions satisfy the following condition: the $R \otimes R$ sector of the theory is obtained from $\text{NS} \otimes \text{NS}$ sector under simultaneous spectral flow by half a unit on both chiral halves of the theory, and the $\text{NS} \otimes R$ and $R \otimes \text{NS}$ sectors are similarly interchanged. The spectral flow is rather easy to describe in our notation: it simply maps between the NS sector and the R sector via $(a, c) \leftrightarrow (a, c + 1)$ where $a + c$ is even. One can check using equations (1) and (2) that for $a + c$ even we have

$$h_{ac} \rightarrow h_{a,c+1} = h_{ac} - Q_{ac} + \frac{c}{24},$$

as expected from e.g. [26]. The constraint that a theory should be invariant under the interchange of $\text{NS} \otimes \text{NS} \leftrightarrow R \otimes R$ and $\text{NS} \otimes R \leftrightarrow R \otimes \text{NS}$ is a very strong one. In particular, since the vacuum representation must be present in any theory, the representation obtained from the vacuum by spectral flow should be present in the $R \otimes R$ sector; that is, $M_{0,0,0,0} \neq 0$. One can read off from the explicit list in section 2.4 that the only space-time supersymmetric theories have the following partition functions:

\begin{align*}
\tilde{M}^0[v = 1, z = 1, n = 0] &= A_k \otimes I_{2k}, \quad k \text{ odd} \\
\tilde{M}^2.2[v = k, z = 1, n = 0, m = 0] &= A_k \otimes I_{2k} \quad 4 \text{ divides } k \\
\tilde{M}^2.2[v = k, z = 1, n = 1, m = 0] &= D_k \otimes I_{2k} \quad 4 \text{ divides } k \\
\tilde{M}^4.3[v = k/2, z = 1, n = 0] &= A_k \otimes I_{2k} \quad 4 \text{ divides } k \\
\tilde{M}^4.2[v = k/2, z = 1, x = 1] &= D'_k \otimes I_{2k} \quad 4 \text{ divides } k \\
\tilde{E}^{10}_2[v = 12, z = 1, n = 0, m = 0] &= \mathcal{E}_{10} \otimes I_{24}, \quad k = 10 \\
\tilde{E}^{16}_2[v = 9, z = 1, x = 1] &= \mathcal{E}_{16} \otimes I_{16}, \quad k = 16 \\
\tilde{E}^{28}_2[v = 15, z = 1, x = 1] &= \mathcal{E}_{28} \otimes I_{60}, \quad k = 28
\end{align*}
Here the $A_k, D_k, E_k$ are the $\mathfrak{su}(2)_k$ physical invariants of [5] and the $L_{\mathfrak{u}}^k$ are $\mathfrak{u}(1)_{\mathfrak{u}}$ diagonal invariants. These theories have no NS$\otimes$R or R$\otimes$NS sectors, and the NS$\otimes$NS sector can be recovered from the R$\otimes$R sector via spectral flow by half a unit in the opposite direction.

The familiar $A$-$D$-$E$ pattern has emerged. It is quite remarkable that the $A$-$D$-$E$ classification arises already at the level of partition functions.

We note here that there is (at least) one space-time supersymmetric minimal model in each “orbifold class” of the unitary $N = 2$ minimal models; that is, every model in Gannon’s list can be mapped to one of the space-time supersymmetric models by an orbifolding constructed in section 3.

3 Orbifold Construction of the $N = 2$ Unitary Minimal Models

The main result of this paper is the construction of a unitary $N = 2$ minimal models for each possible partition function. The main step is to prove the following theorem:

Theorem 3.1.

- Every non-exceptional partition function of a unitary $N = 2$ minimal model at level $k$ can be obtained by orbifoldings of the diagonal partition function at level $k$.

- Every exceptional partition function of a unitary $N = 2$ minimal model with level $k = 10, 16$ or $28$ can be obtained by orbifoldings of the $E_6 \otimes I_{24}, E_7 \otimes I_{36}$ or $E_8 \otimes I_{60}$ partition functions, respectively, where $E_6, E_7, E_8$ are the $\mathfrak{su}(2)_k$ exceptional physical invariants, and $L_{\mathfrak{u}}^k$ is the $\mathfrak{u}(1)_{\mathfrak{u}}$ diagonal invariant.

We must first explain what we mean by orbifolding.

3.1 Orbifolding

We first describe the orbifolding procedure in the case of a bosonic CFT, i.e. when no fermionic modes are present. Let $\mathbb{H}$ be the underlying pre-Hilbert space of a CFT $\mathcal{C}$ and let $\rho : G \to \text{End}(\mathbb{H})$ be an action of a discrete group on $\mathbb{H}$ such that

1. $\mathbb{H}$ is simultaneously diagonalisable with respect to $L_0, \overline{T}_0$ and $\rho(g)$ for every $g \in G$, where $L_0, \overline{T}_0$ are viewed as linear operators on $\mathbb{H};$

2. $\rho(g)$ commutes with $L_n$ and $\overline{T}_n$ for every $n$, where $L_n, \overline{T}_n$ are viewed as linear operators on $\mathbb{H}.$

3. The action of $G$ preserves the $n$-points functions of $\mathcal{C}$.

\footnote{We use the notation $I_{\mathfrak{u}}^k$ since they are $2\mathfrak{u} \times 2\mathfrak{u}$ matrices. Some authors use $I_{\mathfrak{u}}^k.$}
Decomposing $\mathbb{H} = \bigoplus_{a,b \in P} \mathbb{H}_a \otimes \mathbb{H}_b$ into a direct sum of irreducible components, we see that the above conditions imply that $\rho(g)$ must act by multiplication by a root of unity $\xi_{a,b}(g)$ on the lowest weight vector of $\mathbb{H}_a \otimes \mathbb{H}_b$, and therefore by multiplication by $\xi_{a,b}(g)$ on the whole of $\mathbb{H}_a \otimes \mathbb{H}_b$. It follows that the action of $G$ on the states of $\mathbb{H}$ is entirely described by its action on the characters $\rho(g)(\chi_{ac}\chi_{a'}^{*}c) = \xi_{a,b}(g)\chi_{ac}\chi_{a'}^{*}c$. For notational simplicity we shall now simply write $g$ in place of $\rho(g)$.

We want to construct a $G$-invariant CFT from $\mathcal{C}$, the $G$-orbifold of $\mathcal{C}$, denoted $\mathcal{C}/G$. We will restrict our attention to an abelian group $G$ for ease of notation, but one can generalise to non-abelian groups with a little care (see e.g. [21]).

We begin by projecting onto the $G$-invariant states of $\mathcal{C}$:

$$\mathbb{H}^{\text{inv}} := \mathcal{P} \cdot \mathbb{H}$$

where the projector $\mathcal{P}$ is given by $\frac{1}{|G|} \sum_{g \in G} g$. We use a notational shorthand

$$g_{\boxed{1}} := \text{Tr}_{\mathbb{H}}(gq^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}})$$

for the trace with $g$ inserted, which makes sense because of condition 1 above. This allows us to write the partition function of the $G$-invariant sector as

$$Z^{\text{inv}}(\tau) = \text{Tr}_{\mathbb{H}}(\mathcal{P}q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}) = \frac{1}{|G|} \sum_{g \in G} g_{\boxed{1}}.$$ 

Unless $G$ is trivial, $Z^{\text{inv}}(\tau)$ will not be modular invariant. In order to restore modular invariance we need to add in extra $G$-invariant states, the so called twisted states. Two problems arise here: how do we go about constructing the twisted sector? And how do we extend the action of $G$ to the twisted states?

The first question is difficult to answer in general, but we will only be interested in the case of the unitary $N = 2$ minimal models. In this case we can construct the twisted sector out of known representations, using the following arguments: by condition 2, the $L_n,\overline{L}_n$ modes commute with the $G$-action and so the central charge $c$ is left invariant, and since the action of $\text{SL}(2,\mathbb{Z})$ leaves $c$ invariant, the twisted sector should also be composed of irreducible representations at central charge $c$. But in the situation of interest to us, the collection of irreducible representations are explicitly known for fixed $c$. Thus the twisted sector can be constructed from these known representations. It is therefore sufficient to find the partition function of the twisted sector using standard tricks below.

The answer to the second question is that there may be no unique way to extend the action of $G$ to the twisted sector. The freedom we have in choosing an extension is called discrete torsion and is classified by the second group cohomology class $H^2(G, U(1))$ [41]. In this paper we will need to consider only the cases $G = \mathbb{Z}_k$ with discrete torsion $\mathbb{Z}_1$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$ with discrete torsion $\mathbb{Z}_2$. 

15
We now return to the construction of the partition function of the twisted sector. For each \( h \in G \) we denote by \( H_h \) the sector of states ‘twisted by \( h \)’ in the space direction; in the language of fields we make a cut from 0 to \( \tau \) along the world-sheet torus \( T = \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z}) \) and require that a field crossing the cut is acted on by \( h \):

\[
\phi(z + 1) = h\phi(z).
\]

Since we want to keep only \( G \)-invariant states, we project the partition function of \( H_h \) with \( \mathcal{P} \):

\[
\text{Tr}_{H_h}(P q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}) = \frac{1}{|G|} \sum_{g \in G} g \boxed{h}
\]

where we have introduced the notational shorthand

\[
g \boxed{h} := \text{Tr}_{H_h}(g q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}).
\]

Then the partition function of the orbifold theory is the sum of the contributions from each of the twisted sectors:

\[
Z^{\text{orb}} = \frac{1}{|G|} \sum_{g, h \in G} g \boxed{h}.
\]

We interpret the box \( g \boxed{h} \) as counting states whose fields live on the world-sheet torus with a cut along each cycle, such that cycling around once in the space-direction yields a factor of \( h \) and cycling around once in the time-direction yields a factor of \( g \):

\[
\phi(z + 1) = h\phi(z), \quad \phi(z + \tau) = g\phi(z).
\]

Then we find that the \( S \) and \( T \)-transformations act to permute the ‘boundary conditions’ in the following way:

\[
S \left( g \boxed{h} \right) = h^{-1} \boxed{g}, \quad T \left( g \boxed{h} \right) = gh \boxed{h},
\]

thus ensuring modular invariance of the orbifold partition function.

This completes the construction for bosonic CFTs. In order to extend the prescription to the SCFT case, we just replace the space of states \( H \) with the bosonic states, and add the \( z \)-dependence (via \( y^{h} \)) into the traces in the obvious manner.
3.2 Proof of Theorem 3.1

The proof is constructive: given any non-exceptional physical invariant \( M \) at level \( k \) in Gannon’s list, we construct a chain of orbifoldings (by cyclic groups) mapping \( M \) to a particular level \( k \) physical invariant. Since this also applies to \( A_k \), and since an orbifolding by a solvable group always has an orbifolding inverse (see e.g. [21]), we see that any non-exceptional partition function belongs to a model that can be obtained as the result of a chain of orbifoldings beginning at the diagonal model. Similarly, given an exceptional physical invariant at level \( k = 10, 16 \) or \( 28 \), we will construct a chain of orbifoldings to a particular level \( k \) physical invariant.

The proof will be broken down into several sections. In subsections 3.2.1 and 3.2.2 we will introduce some simple \( \mathbb{Z}_2 \) orbifolds which realise certain global symmetries discussed briefly in subsection 3.2.3. In subsection 3.2.4 we generalise a well-known \( \mathbb{Z}_2 \) orbifold from the \( \mathfrak{su}(2)_k \) models to the minimal models, and observe that we can construct an orbifolding between the minimal “families” listed in section 2.4.

In subsections 3.2.5-3.2.7 we state and prove a proposition that every physical invariant \( M \) can be mapped into either \( \tilde{M}_0, \tilde{M}_4, \tilde{M}_2, \tilde{E}_10, \tilde{E}_16 \) or \( \tilde{E}_28 \) depending on the level \( k \) and whether \( M \) is exceptional or not.

Lastly, in subsections 3.2.13-3.2.17 we try to control the parameter \( z \). We summarise these results in subsection 3.2.18, finally completing the proof.

In order to cut out pages of technical proofs, we will in general just write down the general ‘box’ \( \begin{array}{c} \hline \end{array} \) for \( g, h \in G \) for an orbifolding, observe that it gives the expected result when \( h = 0 \), and state the resulting orbifold partition function. The behaviour under modular transformations will be shown to be correct only for the first simple examples, since the proof is similar in the other cases. The reader who wants more detailed proofs should consult [23].

3.2.1 The Orbifolding \( \mathcal{O}_L^1, \mathcal{O}_R^1 \)

Let \( Z(\tau, z) \) be a physical invariant from the list in 2.4. We write

\[
Z = 1 \begin{array}{c} \hline \end{array} = \sum_{(a,c) \in P_k} M_{a,c; a'; c'} \chi_{ac} \chi^*_{a'c'}
\]

and let \( \mathbb{Z}_2 = \langle g \rangle \) act on the states via\(^{17}\)

\[
g \cdot \chi_{ac} \chi^*_{a'c'} = (-1)^{a+c} \chi_{ac} \chi^*_{a'c'}.
\]

\(^{17}\)Strictly speaking we begin the orbifolding process only knowing how \( G \) acts on those characters \( \chi_{ac} \chi^*_{a'c'} \) for which \( M_{a,c; a'; c'} \neq 0 \). But \( G = \mathbb{Z}_2 \) is cyclic, so there is no discrete torsion, and thus the action given in (26) must be the only consistent way to extend the action of \( G \) to the complete set of \( \chi_{ac} \chi^*_{a'c'} \).
Since the parity of \( a + c \) determines whether the states counted by \( \chi_{ac} \) are in the NS or R sectors, we see that this action leaves the NS sector invariant. The general box for \( m, n \in \{0, 1\} \) is given by

\[
g^m \begin{array}{c} \chi_{ac} \end{array} g^n = \sum_{(a, c) \in P'_k} \sum_{(a', c') \in P'_k} M_{j^m(a, c); a', c'}(-1)^{(a+c+n)m} \chi_{ac} \chi^*_{a', c'},
\]

where \( j(a, c) = (k - a, c + \bar{k}) \). This is clearly correct when \( n = 0 \), and since there is no discrete torsion, it remains to check that the general box transforms correctly under the \( S \)- and \( T \)-transformations.

The \( T \) commutator can be easily checked using equations (5) and (7): for the \( T \)-transformation we find

\[
T \cdot g^m \begin{array}{c} \chi_{ac} \end{array} g^n = \sum_{(a, c) \in P'_k} \sum_{(a', c') \in P'_k} M_{j^m(a, c); a', c'}(-1)^{(a+c+n)m} e^{2\pi i (h_{a,c} - h_{a',c'})} \chi_{a,c} \chi^*_{a', c'}
\]

\[
= \sum_{(a, c) \in P'_k} \sum_{(a', c') \in P'_k} M_{j^m(a, c); a', c'}(-1)^{(a+c+n)m} \times (-1)^{(a+c+1)n} e^{2\pi i (h_{a,c} - h_{a',c'})} \chi_{a,c} \chi^*_{a', c'}
\]

\[
= \sum_{(a, c) \in P'_k} \sum_{(a', c') \in P'_k} M_{j^m(a, c); a', c'}(-1)^{(a+c+n)(m+n)} \chi_{a,c} \chi^*_{a', c'}
\]

\[
= g^{m+n} \begin{array}{c} \chi_{ac} \end{array} g^n.
\]

For the \( S \)-matrix, we can simplify the calculation enormously if we consider simple currents \([28, 40]\). These are the primary fields which upon fusion with any other field yield precisely one conformal family. One reads off from the Verlinde formula \([43]\) for the unitary \( N = 2 \) minimal models \([44, 35]\) that the simple currents are

\[
\mathcal{J} = \{0, k\} \times \mathbb{Z}_{k^2}
\]

where we have identified the simple currents with their labels in \( P'_k \) for simplicity of notation. Each current acts naturally on the set of weights of the \( N = 2 \) minimal models: \( j \) maps the weight \( (a, c) \) to the weight labelling the field which appears in the OPE of \( \phi_1 \) and \( \phi_{a,c} \). Thus, writing \( J \) for the \( \mathfrak{su}(2) \) current \( J : a \mapsto k - a \) as before, we read off from the Verlinde formula

\[
(J^l 0, d) \cdot (a, c) = (J^l+(lk+d)(a+c))a, c + d + (lk+d)(a + c)k).
\]

The simple currents form a group under this action isomorphic to \( \mathbb{Z}_{4} \) when \( k \) is odd and isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_{k^2} \) when \( k \) is even, but the reason they are so
useful is that the $S$-matrix behaves well under the action of the currents on the weights. In fact

$$S_{J(a,c);a',c'} = e^{2\pi i Q_j(a',c')} S_{a,c;a',c'}$$

(27)

where $Q_j(p_0, d_j)(a', c') = \frac{\gamma_j}{2} + \frac{\epsilon_j}{2\pi} - \frac{|b + d| a' + c'}{4}$ and we have written $|b| \in \{0, 1\}$ for the value of $b$ modulo 2, as before. $Q_j$ is called the charge of the field $\phi_{a,c}$ with respect to the current $j$. The charges satisfy

$$Q_j(a, c) = h_j + h_{(a, c)} - h_{(a,c)}$$

so $Q_j(a, c)$ is also the monodromy of $\phi_{a,c}$ with $\phi_{j}$, as expected [21]. In particular, when $j = (J, \bar{K})$ equation (27) implies that

$$S_{k-a,c+\bar{K};a',c'} = (-1)^{a'+c'} S_{a,c;a',c'}.$$  

We find that

$$S \cdot g^m \square_{g^n} = \sum_{(a, c) \in \mathcal{P}'_g} \sum_{(a', c') \in \mathcal{P}'_g} S_{r,s; a,c} M_{j(a,c);a',c'} S^*_{a',c'}; t,u (-1)^{(a+c+n)m} \chi_{rs} \chi_{tu}^*$$

(27)

and thus the boxes transform correctly under the action of SL(2, $\mathbb{Z}$). We therefore obtain

$$Z^{\text{inv}} = \sum_{(a, c) \in \mathcal{P}'_g, (a', c') \in \mathcal{P}'_g} M_{a,c; a', c'} \chi_{ac} \chi_{a'c'}$$

$$Z^{\text{twist}} = \sum_{(a, c) \in \mathcal{P}'_g, (a', c') \in \mathcal{P}'_g} M_{J(a,c);a',c'} \chi_{ac} \chi_{a'c'}$$

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Again we start with a minimal model with partition function

\[ Z_{\text{orb}} = \sum_{(a,c) \in P_k'} M_{a,c; a',c'} \chi_{ac} \chi_{a',c'} + \sum_{(a,c) \in P_k'} M_{j(a,c); a',c'} \chi_{ac} \chi_{a',c'}^{*} \, . \]

This orbifolding defines a \( \mathbb{Z}_2 \) action on the set of physical invariants. We will refer to it as the orbifolding \( O_L^1 \) (where the L stands for left). Since \( S \) and \( T \) are symmetric, it is clear that we could equally well have let \( \mathbb{Z}_2 \) act on the right-hand representations, \( g \cdot \chi_{ac} \chi_{a',c'}^{*} = (-1)^a + c' \chi_{ac} \chi_{a',c'}^{*} \). The result would be

\[ Z_{\text{orb}} = \sum_{(a,c) \in P_k'} M_{a,c; a',c'} \chi_{ac} \chi_{a',c'}^{*} + \sum_{(a,c) \in P_k'} M_{a,c; j(a',c')} \chi_{ac} \chi_{a',c'}^{*} \, . \]

We will refer to this orbifolding as \( O_R^1 \).

### 3.2.2 The Orbifoldings \( O_L^2, O_R^2 \)

Again we start with a minimal model with partition function

\[ Z = 1 \begin{array}{c} a \end{array} = \sum_{(a,c) \in P_k'} M_{a,c; a',c'} \chi_{ac} \chi_{a',c'}^{*} \, . \]

and define a group action by \( g \cdot \chi_{ac} \chi_{a',c'}^{*} = e^{2\pi i m} \chi_{ac} \chi_{a',c'}^{*} \). This defines a \( \mathbb{Z}_2 \) action. We claim that the general box for \( m, n \in \{0, \ldots, k-1\} \) is given by

\[ g^m = \sum_{(a,c) \in P_k'} M_{a,c; a',c'} e^{2\pi i m(n-c)} \chi_{ac-2n} \chi_{a',c'}^{*} \, . \]

One easily checks that this is correct when \( n = 0 \). It remains to check that it transforms correctly under the \( S \) and \( T \) transformations. We find for the \( T \)-transformation

\[ T \cdot g^m = \sum_{(a,c) \in P_k'} M_{a,c; a',c'} e^{2\pi i m(n-c)} e^{2\pi i (a-c-n-h_{a',c'})} \chi_{ac-2n} \chi_{a',c'}^{*} \, . \]

\[ = g^{m+n} \begin{array}{c} a \end{array} \, . \]
and for the $S$-transformation we find
\[ S \cdot g^m = \sum_{(a,c) \in P'_k} \sum_{(a',c') \in P'_k} S_{r,s; a,c - 2n} M_{a,c; a',c'} S^*_{a',c'; t,u} e^{2\pi im(c-n)} \chi_{rs} \chi_{tu}^* \]
\[ \sum_{(a,c) \in P'_k} \sum_{(r,s) \in P'_k} \sum_{(a',c') \in P'_k} S_{r,s+2m; a,c} M_{a,c; a',c'} S^*_{a',c'; t,u} e^{2\pi i(-n(m+s))} \chi_{rs} \chi_{tu}^* \]
\[ \sum_{(a,c) \in P'_k} \sum_{(r,s) \in P'_k} M_{r,s+2m; t,u} e^{2\pi i(-n(s-m))} \chi_{rs} \chi_{tu}^* \]
\[ = \sum_{(a,c) \in P'_k} M_{a,c; a',c'} e^{2\pi i(-n(s-m))} \chi_{a,c - 2m} \chi_{a',c'} \]
\[ = g^{-n} \]

Thus the boxes span a representation of $SL_2(\mathbb{Z})$. To calculate the resulting orbifold we need
\[ Z_{g^n} := \frac{1}{k} \sum_{m=0, \ldots, k-1} g^m \]
\[ = \sum_{(a,c) \in P'_k} M_{a,c; a',c'} \left[ \frac{1}{k} \sum_{m=0, \ldots, k-1} e^{2\pi i m(c-n)} \right] \chi_{a,c - 2n} \chi_{a',c'}^* \]
\[ = \sum_{(a,c) \in P'_k} M_{a,c; a',c'} \delta(c \equiv n \mod k) \chi_{a,c - 2n} \chi_{a',c'}^* \]
\[ = \sum_{a=0, \ldots, k} \sum_{l=0, 1} M_{a,n+l; a',c'} \chi_{a,n+l} \chi_{a',c'}^* \]

Thus we see that
\[ Z^{\text{orb}} = \sum_{n=0, \ldots, k-1} Z_{g^n} \]
\[ = \sum_{a=0, \ldots, k} \sum_{n=0, \ldots, k-1} \sum_{l=0, 1} M_{a,n+l; a',c'} \chi_{a,n+l} \chi_{a',c'}^* \]
\[
\sum_{(a,c) \in P'_k} M_{a,-c; a', c'} \chi_{ac} \chi_{a', c'}^{*}.
\]

This orbifold is well-defined for all minimal models. In fact this symmetry is none other than the infamous mirror symmetry \[27\] (acting on the left-hand representations). We will refer to it by \(\mathcal{O}_L^2\). The group \(\mathbb{Z}_k\) could equally as well have acted upon the right-hand representations. In that case we would obtain

\[
Z_{\text{orb}} = \sum_{(a,c) \in P'_k} M_{a,c; a', c'} \chi_{ac} \chi_{a', c'}^{*}.
\]

We will refer to this orbifolding as \(\mathcal{O}_R^2\). Clearly these orbifolding give the same result if the initial minimal model is symmetric.

The reason we have done these relatively simple examples in such great detail is that the procedure for checking \(\text{SL}(2,\mathbb{Z})\)-invariance for all other orbifolding in this paper is very similar: one directly checks \(T\)-invariance with the help of equation (7) and then uses the simple current action on the \(S\)-matrix to check \(S\)-invariance. For an orbifolding with a cyclic group \(G\), there is no discrete torsion, so it is then enough to check that the general box agrees with the original partition function \((n = m = 0)\), and that it agrees with the action of \(G\) \((n = 0, m \neq 0)\).

### 3.2.3 Symmetries generated by \(\mathcal{O}_{L,R}^1\) and \(\mathcal{O}_{L,R}^2\)

Note that these orbifoldings are self-inverse, they are mutually commuting, and the effect of concatenating \(\mathcal{O}_L^1 \mathcal{O}_L^2\) or \(\mathcal{O}_R^1 \mathcal{O}_R^2\) is to perform the mirror symmetry transformation on the left- or right-chiral half of the theory, respectively:

\[
\mathcal{O}_L^1 \mathcal{O}_L^2 : M_{a,c; a', c'} \mapsto M_{j^{a+c}(a,-c); a', c'},
\]

\[
\mathcal{O}_R^1 \mathcal{O}_R^2 : M_{a,c; a', c'} \mapsto M_{a,c; j^{a'+c'}(a',-c')}.
\]

where left- or right-handed mirror symmetry is defined by performing charge conjugation on the left- or right-handed representations, respectively. In terms of the partition functions, it is realised by multiplication of the physical invariant \(M\) by the permutation matrix \(S^2\) on the left or right respectively. Using equation (2), one check that making the transformation \((a, c) \rightarrow j^{a+c}(a, -c)\) has the effect of sending

\[
(h_{ac}, Q_{ac}) \rightarrow (h_{ac}, -Q_{ac}) \mod 1
\]

as expected.

Performing charge conjugation on both sides simultaneously amounts to performing all 4 orbifoldings \(\mathcal{O}_L^1 \mathcal{O}_L^2 \mathcal{O}_R^1 \mathcal{O}_R^2\) in succession. Since \(S^4 = \text{Id}\) and physical invariants commute with \(S\), this has no overall effect on the partition function. As discussed in section 2.5.1 we consider two charge conjugate models (i.e. related by simultaneous charge conjugation on both chiral halves of the theory) to
be equivalent; indeed they have the same partition function. We will however not consider the mirror symmetry pairs to be equivalent in this paper, since they generally have distinct partition functions.

The results of applying $O^1_{L,R}, O^2_{L,R}$ to the minimal partition functions listed in section 2.4 are given in table 1. The third column lists the values of the defining parameters before any orbifolding is applied.

3.2.4 The generalised $A_k \leftrightarrow D_k$ Orbifolding

The family $\tilde{M}^{2.2}$ exists for any $k$ with $4|k$. Given such a $k$, we can always choose $v = \frac{k}{2}$ and $z = 1$. Then, from equation (13), we obtain a physical invariant $M$ with $M_{a,c;a',c'} = \delta(a' = J^{ac}a)\delta(c' = c)$. Thus

$$M = \begin{cases} A_k \otimes I_k & \text{if } n = 0 \\ D_k \otimes I_k & \text{if } n = 1, \end{cases}$$

where the $A$ and $D$ are the partition functions of the $su(2)_k$ models of the same name encountered in [5] and $I_k$ is the diagonal $u(1)_k$ invariant. Similarly, when $4$ divides $k$, the physical invariant $\tilde{M}^{4.3}$ with parameters $v = \frac{k}{2}$, $z = 1$ and $x = 1$ yields $A_k \otimes I_k$ and the physical invariant $\tilde{M}^{4.2}$ with $v = \frac{k}{2}$, $z = 1$ and $x = 1$ yields $D_k \otimes I_k$, where again the $A$ and $D$ are the partition functions of the $su(2)_k$ classification. Inspired by the well-known $Z_2$ orbifolding between the $A$- and $D$-models (see e.g. [7]), we define a $Z_2$ action on the states of an arbitrary physical invariant with even $k$ by

$$g \cdot \chi_{ac} \chi_{a'c'}^* := (-1)^n \chi_{ac} \chi_{a'c'}^*.$$ 

Then we find

$$Z^{\text{inv}} = \sum_{(a,c) \in P'_k} \sum_{(a',c') \in P'_k} M_{a,c;a',c'} \chi_{ac} \chi_{a'c'}^*,$$

$$Z^{\text{twist}} = \sum_{(a,c) \in P'_k} \sum_{(a',c') \in P'_k} M_{J_{a,c};a',c'} \chi_{ac} \chi_{a'c'}^*,$$

$$Z^{\text{orb}} = \begin{cases} \sum_{(a,c) \in P'_k} M_{J_{a,c};a',c'} \chi_{ac} \chi_{a'c'}^*, & \text{if } 4|k; \\ \sum_{(a,c) \in P'_k} (M_{a,c;a',c'} + M_{J_{a,c};a',c'}) \chi_{ac} \chi_{a'c'}^*, & \text{if } 4|k. \end{cases}$$
Table 1: Action of $\mathcal{O}_{L,R}^1$, $\mathcal{O}_{L,R}^2$ on minimal partition functions

| $k$ odd | $\bar{M}^0$ | $\mathcal{O}_{L}^1$ | $\mathcal{O}_{R}^1$ | $\mathcal{O}_{L}^2$ | $\mathcal{O}_{R}^2$ |
|---------|-------------|---------------------|---------------------|---------------------|---------------------|
| $k$ divides $\bar{k}$ | $\bar{M}^{2,0}$ | $[v, z, n]$ | $[v, z, n + 1]$ | $[v, -z, n]$ | $[v, -z, n]$ |
|        | $\bar{M}^{2,0}$ | $[v, z, n]$ | $[v, z, n + 1]$ | $[v, -z, n]$ | $[v, -z, n]$ |
|        | $\bar{M}^{2,1}$ | $[v, z, n]$ | $[v, z, n + 1]$ | $[v, -z, n + 1]$ | $[v, -z, n + 1]$ |
|        | $\bar{M}^{2,1}$ | $[v, z, n]$ | $[v, z, n + 1]$ | $[v, -z, n + 1]$ | $[v, -z, n + 1]$ |
| $k$ divides $k$ | $\bar{M}^{4,0}$ | $[v, z, n, m]$ | $[v, z, n, m + 1]$ | $[v, -z, n, m]$ | $[v, -z, n, m + 1]$ |
|        | $\bar{M}^{4,0}$ | $[v, z, n, m]$ | $[v, z, n, m + 1]$ | $[v, -z, n, m]$ | $[v, -z, n, m + 1]$ |
|        | $\bar{M}^{4,1}$ | $[v, z, x, y]$ | $[v, z, x + 2, y]$ | $[v, -z, x + 2, y]$ | $[v, -z, x + 2, y]$ |
|        | $\bar{M}^{4,1}$ | $[v, z, x, y]$ | $[v, z, x + 2, y]$ | $[v, -z, x + 2, y]$ | $[v, -z, x + 2, y]$ |
|        | $\bar{M}^{4,3}$ | $[v, z, n]$ | $[v, z, n + 1]$ | $[v, -z, n]$ | $[v, -z, n]$ |
|        | $\bar{M}^{4,3}$ | $[v, z, n]$ | $[v, z, n + 1]$ | $[v, -z, n]$ | $[v, -z, n]$ |
| $k = 10$ | $\bar{E}^{10}_1$ | $[6, z]$ | $[6, z]$ | $[6, -z]$ | $[6, -z]$ |
|        | $\bar{E}^{10}_1$ | $[12, z, 0, m]$ | $[12, z, 0, m + 1]$ | $[12, -z, 0, m]$ | $[12, -z, 0, m]$ |
| $k = 16$ | $\bar{E}^{16}_1$ | $[v, z, x, y]$ | $[v, z, x + 2, y]$ | $[v, -z, x + 2, y]$ | $[v, -z, x + 2, y]$ |
|        | $\bar{E}^{16}_1$ | $[v, z, x, y]$ | $[v, z, x + 2, y]$ | $[v, -z, x + 2, y]$ | $[v, -z, x + 2, y]$ |
| $k = 28$ | $\bar{E}^{28}$ | $[15, z, x]$ | $[15, z, x + 2]$ | $[15, -z, x + 2]$ | $[15, -z, x + 2]$ |
The action on the minimal partition functions with $4|k$ is given by Table 2.

For $\tilde{M}^{2,0}$ and $\tilde{M}^{2,1}$ the action coincides with that of $O^1$ (as we would expect since if $\tilde{M}_{a,c,a'c' \neq 0}$ then $c$ is even for these families). For $\tilde{M}^{2,2}$ we have obtained an additional $\mathbb{Z}_2$ symmetry, which along with $O^1$ and $O^2$ from the previous section allows us to construct an orbifolding between any two $\tilde{M}^{2,2}$ physical invariants with $v_1 = v_2$ and $z_1 = \pm z_2$. As one might expect, for the special case $v = \bar{k}$ and $z = 1$ this orbifolding manifests itself as $A_k \otimes \mathcal{I}_k \leftrightarrow D_k \otimes \mathcal{I}_k$ The exceptional physical invariants $\tilde{E}_{16}^1$ are left invariant.

The effect on the minimal models with $4|k$ is given in Table 3.

In particular, $\tilde{M}^{3,0}[\bar{k}, 1, 0] = A_k$ is mapped to $\tilde{M}^{4,2}[\bar{k}, 1, 1] = D_k^\prime$ as we might expect. The physical invariants in the families $\tilde{M}^{4,1}$ and $\tilde{M}^{4,2}$ are sent to $\tilde{M}^{4,2}$ under this orbifolding. This demonstrates that orbifoldings can map between, as well as within, families of minimal model partition functions. In the next subsection we will show that in fact all the non-exceptional families

---

**Table 2:**

| $\tilde{M}^{2,0}$ | $[v, z, n] \leftrightarrow [v, z, n + 1]$ |
| $\tilde{M}^{2,1}$ | $[v, z, n] \leftrightarrow [v, z, n + 1]$ |
| $\tilde{M}^{2,2}$ | $[v, z, n, m] \leftrightarrow [v, z, n + 1, m]$ |
| $\tilde{E}_{16}^1$ | $[6, z] \leftrightarrow [6, z]$ |
| $\tilde{E}_{28}^1$ | $[12, z, 0, m] \leftrightarrow [12, z, 0, m]$ |

---

**Table 3:**

| $\tilde{M}^{4,0}[v, z, n, m]$ | $\rightarrow \tilde{M}^{4,2}[v, z, 2m + 2n + 1]$ |
| $\tilde{M}^{4,1}[v, z, x, y]$ | $\rightarrow \tilde{M}^{4,1}[v, z, x, y]$ |
| $\tilde{M}^{4,2}[v, z, x]$ | $\rightarrow \tilde{M}^{4,2}[v, z, x]$ |
| $\tilde{M}^{4,3}[v, z, n]$ | $\rightarrow \tilde{M}^{4,2}[v, z, 2n + z]$ |
| $\tilde{E}_{16}^1[v, z, x, y]$ | $\rightarrow \tilde{E}_{16}^1[v, z, x, y]$ |
| $\tilde{E}_{28}^1[v, z, x]$ | $\rightarrow \tilde{E}_{28}^1[v, z, x]$ |
| $\tilde{E}_{28}^1[15, z, x]$ | $\rightarrow \tilde{E}_{28}^1[15, z, x]$ |

---

$18$ The parameter $z$ is defined modulo some number $\alpha$ in each case. $-z$ is to be understood as $-z \mod \alpha$.

$19$ Actually the formula given above for the $\mathbb{Z}_2$ orbifolding has to be divided through by 2 in order to get $\tilde{M}_{0,0,0,0} = 1$. This factor of 2 appears because $\mathbb{Z}_2$ acts trivially on all the states so $Z = Z^{twist} = Z^{twist}$ and so $Z^{orb} = 2Z$.  

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at a given level $k$ can be mapped into one another via orbifolding, and that the same holds true for the exceptional families.

### 3.2.5 Orbifoldings between Minimal Families

We shall prove the following proposition:

**Proposition 3.2.** 1. Let $k$ be odd. Then all simple current invariants at level $k$ can be mapped by an orbifolding to the family $\tilde{M}^0$.

2. Let $4|k$. Then all simple current invariants at level $k$ can be mapped by an orbifolding to the family $\tilde{M}^{4,2}$.

3. Let $4|k$. Then all simple current invariants at level $k$ can be mapped by an orbifolding to the family $\tilde{M}^{2,0}$.

4. Let $k = 10$. Then all exceptional invariants at level $k$ can be mapped by an orbifolding into $\tilde{E}^{10}$.

5. Let $k = 16$. Then all exceptional invariants at level $k$ can be mapped by an orbifolding into $\tilde{E}^{16}$.

6. Let $k = 28$. Then all exceptional invariants at level $k$ can be mapped by an orbifolding into $\tilde{E}^{28}$.

When $k$ is odd there is only one family of partition functions of minimal models and when $k = 28$ there is only one family of exceptionals, so parts 1 and 6 are trivial, but we include these statements for completeness. We construct the necessary orbifoldings to prove statements 2-5 in the following two subsections.

### 3.2.6 Orbifoldings between Minimal Families: $4|k$

In section 3.2.4 we saw that the generalised $A \leftrightarrow D$ orbifold $O^3$ mapped members of the family $\tilde{M}^{4,1}$ and $\tilde{M}^{4,3}$ into the family $\tilde{M}^{4,2}$. We will now show that $\tilde{M}^{4,2}$ contains an orbifold of every member of the family $\tilde{M}^{4,1}$, and that $\tilde{E}^{16}$ contains an orbifold of every member of $\tilde{E}^{16}$. This will prove parts 2 and 5.

Fix some $k \in 4\mathbb{Z}$. We want to construct an orbifold which in particular sends $\tilde{M}^{4,1}$ to $\tilde{M}^{4,2}$. The latter only has left-right couplings in the NS⊗NS and R⊗R sectors, but the former has couplings in all 4 possible sectors NS⊗NS, NS⊗R, R⊗NS and R⊗R. So we define a $\mathbb{Z}_2$ action by $g\chi_{ac}\chi_{a'c'} = (-1)^{a+c+a'+c'}\chi_{ac}\chi_{a'c'}$ in order to preserve the NS⊗NS and R⊗R sectors and remove the NS⊗R and R⊗NS sectors. For $m, n \in \{0, 1\}$ we find

$$g^n_{\gamma_\gamma} = \sum_{(a,c) \in \mathcal{P}'_R} (-1)^{a+c+a'+c'} m^{J^m a, c + n} J^{n a', c'}\chi_{ac}\chi_{a'c'}.$$
This transforms correctly under the $S$- and $T$-transformations, resulting in an orbifold

$$Z_{\text{orb}} = \sum_{a+c+a'+c' \equiv 0 \mod 2} (M_{a,c;a',c'} + M_{Ja,c+Ja',c'+Ja}) \chi_a \chi_{a'} \chi_c \chi_{c'}$$

which we call $\mathcal{O}^4$.

This orbifold acts trivially on those models which only have NS$\otimes$NS and R$\otimes$R sectors: $\tilde{M}^{4,2}$, $\tilde{M}^{4,3}$, $\tilde{E}_2^{16}$ and $\tilde{E}^{28}$. The action of $\mathcal{O}^4$ on the other minimal models which occur when $4|k$ is given in table 4.

| $\tilde{M}^{4,0}[v, z, n, m]$ | $\rightarrow$ | $\tilde{M}^{4,2}[v, z, 2m + 2n + 1]$ |
| $\tilde{M}^{4,1}[v, z, x, y]$ | $\rightarrow$ | $\tilde{M}^{4,2}[v, z, 2z - y + 1]$ |
| $\tilde{M}^{4,2}[v, z, x]$ | $\rightarrow$ | $\tilde{M}^{4,2}[v, z, x]$ |
| $\tilde{M}^{4,3}[v, z, n]$ | $\rightarrow$ | $\tilde{M}^{4,3}[v, z, 2n + z]$ |
| $\tilde{E}_1^{16}[v, z, x, y]$ | $\rightarrow$ | $\tilde{E}_2^{16}[v, z, 2z - y + 1]$ |
| $\tilde{E}_2^{16}[v, z, x]$ | $\rightarrow$ | $\tilde{E}_2^{16}[v, z, x]$ |
| $\tilde{E}^{28}[15, z, x]$ | $\rightarrow$ | $\tilde{E}^{28}[15, z, x]$ |

### 3.2.7 Orbifoldings between Minimal Families: $4|k$

In this section we shall show that all non-exceptional invariants with $4|k$ can be sent into $\tilde{M}^{2,0}$ by an orbifolding, and all exceptional invariants with $k = 10$ can be sent into $\tilde{E}_1^{10}$, proving parts 3 and 4 of proposition 3.2.

First we shall construct an orbifold $\mathcal{O}^5$ from $\tilde{M}^{2,1}$ to $\tilde{M}^{2,0}$. Fix a $k$ with $4|k$ and fix $(v, z, n)$ satisfying $\frac{v}{2k} \in \mathbb{Z}$, $\frac{z}{k} \in \mathbb{Z} + 1$ and $\frac{2n + 1}{2k} \in \mathbb{Z}$ where $z \in \{1, \ldots, \frac{2k}{16} \}$ and $n \in \{0, 1\}$. Then from section 2.4 there is a minimal partition function $\tilde{M}^{2,1}[v, z, n]$. We need to define a group action on the states of $M \equiv \tilde{M}^{2,1}[v, z, n]$. Note that $M_{a,d; a', d'} = 0 \Rightarrow d = \frac{v}{2k}$, $d' = \frac{v}{2k}$ and $c + c' \equiv 0 \mod 2$; thus there is a $\mathbb{Z}_2$ action on the states given by $g^* \chi_a \chi_{a'} = (-1)^{a+a'} \chi_{a-a'} \chi_{a+a'}$ and which for $n, m \in \{0, 1\}$ gives rise to

\[
g^* = \sum_{(a,c) \in P'_v, (a',c') \in P'_v} M_{a,c; a',c'} e^{\frac{\pi i (a+a')}{2}} \chi_{a,c-nv} \chi_{a',c'+nv}
\]

Note that in the RHS of the second and fifth lines the parameter $2z$ is to be understood modulo $\frac{1}{2}$. Recall that the $z$ parameter in each of the minimal partition functions given in section 4.6 is defined modulo some integer.
whence we conclude that

$$Z^{\text{orb}} = \sum_{(a,c) \in P'_k} \left( M_{a,c; a', c'} + M_{a,c+v; a', c'-v} \right) \delta(c + c' \equiv 0 \mod \frac{2k}{v}) \chi_{ac} \chi_{a', c'}.$$  

Inserting $M = M^{2.1}[v, z, n]$ from equation (11) one finds $Z^{\text{orb}} = \tilde{M}^{2.0}[v', z', n]$, where $v' = 2v$, $z' = \left( \frac{2z}{k} \right)^2 (3-z)$, where we understand $z'$ to be defined modulo $\frac{8v^2}{k}$. We have therefore demonstrated that every model with partition function in $\tilde{M}^{2.1}$ gives rise to a $\mathbb{Z}_2$ orbifold in $\tilde{M}^{2.0}$.

Constructing an orbifolding $O^6$ from $\tilde{M}^{2.2}$ to $\tilde{M}^{2.0}$ is similar: fixing some $k$ such that $4|k$, we define a $\mathbb{Z}_2$ action by $g \cdot \chi_{ac} \chi_{a', c'} = (-1)^{c} \chi_{ac} \chi_{a', c'}$. We claim that for $m, n \in \{0, 1\}$

$$g^n \begin{array}{c} a \\ g^n \\ a \end{array} = \sum_{(a,c) \in P'_k} (-1)^{cm} M_{a,c+n; a', c'} \chi_{ac} \chi_{a', c'}.$$  

This is evidently correct when $n = 0$ and it is not hard to check that it transforms correctly under the $S$ and $T$ transformations. It yields

$$Z^{\text{orb}} = \sum_{(a,c) \in P'_k} \left[ M_{a,c; a', c'} + M_{a,c+v; a', c'-v} \right] \delta(c \equiv 0 \mod 2) \chi_{ac} \chi_{a', c'}. \quad (28)$$

Choosing some $v, z$ such that $\frac{v}{k}$ is odd and $\frac{z}{k} \in \mathbb{Z}$, we can apply $O^6$ to the minimal model $M \equiv \tilde{M}^{2.2}[v, z, n, m]$. Using equation (13) and (11) we find

$$Z^{\text{orb}} = \tilde{M}^{2.0}[v', z, n]$$

where we have set $2v' = v$ and $z$ is now understood to be defined modulo $\frac{2v'^2}{k}$.

It remains to show that the family $\tilde{E}_1^{10}$ can be mapped via an orbifolding into the family $\tilde{E}_1^{10}$. We simply apply the orbifold $O^6$ from the previous subsection to the exceptional model $\tilde{E}_2^{10}[12, v, m]$: substituting (10) into (28) we obtain

$$Z^{\text{orb}} = \tilde{E}_1^{10}[6, z].$$

This completes the proof of proposition 3.2.

### 3.2.8 Orbifolds within Minimal Families – a Useful Formula

In order to complete the proof of theorem 3.1 we must find orbifolds within the families $\tilde{M}^{0}$, $\tilde{M}^{4.2}$, $\tilde{M}^{2.0}$, $\tilde{E}_1^{10}$, $\tilde{E}_2^{10}$ and $\tilde{E}_2^{28}$ which map all members down to a specific partition function. Since we already have control of the $\mathbb{Z}_2$ parameters (labelled by $n$ or $x$) via the orbifolds $O^1$ and $O^2$, in this section we concentrate on trying to control the parameters $v$ and $z$. 

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We begin by considering a general orbifold by a group $\mathbb{Z}_\beta$, acting on the $u(1)$ label $c$ on the left-hand side. Fix a physical invariant $M$ in one of the above families and take the largest integer $\alpha$ such that

$$M_{a,c}^{a',c'} \neq 0 \Rightarrow c, c' \in \alpha \mathbb{Z}.$$  

(29)

For these families, $\frac{\alpha}{\beta} \in \mathbb{Z}$. We will define a $\mathbb{Z}_\beta$-orbifold $\mathcal{O}_7$ for some integer $\beta$ satisfying $\beta | \frac{\alpha}{\beta}$. Let $\mathbb{Z}_\beta = \langle g \rangle$ act on the states of $M$ via

$$g \cdot \chi_{a,\alpha c}^{a',\alpha c'} = e^{2\pi i \frac{m}{\alpha \beta}} \chi_{a,\alpha c}^{a',\alpha c'}.$$  

We claim that the result is

$$g^m \chi_{a,\alpha c}^{a',\alpha c'} = \sum_{a, a' = 0, \ldots, k \atop c, c' \in \mathbb{Z} \frac{\alpha}{\beta}} M_{a,\alpha c}^{a',\alpha c'} e^{\frac{2\pi im}{\beta}} \chi_{a,\alpha c}^{a',\alpha c'}.$$  

It is easy to see this is correct when $n = 0$. We must check that it behaves correctly under the action of the $S$- and $T$-transformations. The line of attack is the usual one: for the $T$-transformation we use the integer-spin condition (equation (7)) to remove the otherwise unwieldy factor of $e^{2\pi i (h_{a,c} - h_{a',c'})}$:

$$T \cdot g^m \chi_{a,\alpha c}^{a',\alpha c'} = \sum_{a, a' = 0, \ldots, k \atop c, c' \in \mathbb{Z} \frac{\alpha}{\beta}} M_{a,\alpha c}^{a',\alpha c'} e^{\frac{2\pi im}{\beta} \frac{2\pi i}{\alpha \beta}} \chi_{a,\alpha c}^{a',\alpha c'} \times e^{2\pi i (h_{a,c} - h_{a',c'})} e^{2\pi i (h_{a,c} - h_{a',c'})} \chi_{a,\alpha c}^{a',\alpha c'}.$$  

and for the $S$-matrix we use the nice behaviour of the simple current action (equation (27)) to juggle unwanted factors on and off the $S$-matrices until one

Note that the remaining families are all symmetric, so it doesn’t matter whether we act on the left- or right-hand sides.
has something of the form $\text{SMS}^\dagger$, which can be replaced with $M$:

$$S \cdot g^n \square = \sum_{a,a'=0,\ldots,k} \sum_{c,c' \in \mathbb{Z} \frac{M}{N}} \sum_{(r,s) \in Q_k} \sum_{(t,u) \in Q_k} S_{r,s; a,a'} M_{a,\alpha; a',\alpha'} S^*_{a',c'; tu}$$

$$\times e^{\frac{2\pi im}{\alpha} \left( c - \frac{m}{\alpha^2} \right)} \chi_{rs} \chi_{tu}^*$$

$$\sum_{a,a'=0,\ldots,k} \sum_{c,c' \in \mathbb{Z} \frac{M}{N}} \sum_{(r,s) \in Q_k} \sum_{(t,u) \in Q_k} S_{r,s; a,a'} M_{a,\alpha; a',\alpha'} S^*_{a',c'; tu}$$

$$\times e^{\frac{2\pi im}{\alpha} \alpha^2} e^{-\frac{2\pi im}{\alpha} \alpha^2} \chi_{rs} \chi_{tu}^*$$

$$\sum_{a,a'=0,\ldots,k} \sum_{c,c' \in \mathbb{Z} \frac{M}{N}} M_{a,\alpha; a',\alpha'} e^{-\frac{2\pi im}{\alpha} \left( c + \frac{m}{\alpha^2} \right)} \chi_{a,\alpha} \chi_{a',\alpha'}^*$$

$$= \sum_{a,a'=0,\ldots,k} \sum_{c,c' \in \mathbb{Z} \frac{M}{N}} M_{a,\alpha; a',\alpha'} e^{-\frac{2\pi im}{\alpha} \left( c - \frac{m}{\alpha^2} \right)} \chi_{a,\alpha} \chi_{a',\alpha'}^*$$

$$= g^{-n} \square$$

as required, and thus the boxes span a representation of $\text{SL}_2(\mathbb{Z})$. We can now calculate the $\mathbb{Z}_\beta$-invariant $g^N$-twisted sectors for $N = 0, \ldots, \beta - 1$:

$$Z^N = \frac{1}{\beta} \sum_{M=0,\ldots,\beta-1} g^N \square$$

$$= \frac{1}{\beta} \sum_{M=0,\ldots,\beta-1} \sum_{a,a'=0,\ldots,k} \sum_{c,c' \in \mathbb{Z} \frac{M}{N}} M_{a,\alpha; a',\alpha'} e^{\frac{2\pi im}{\alpha} \left( c - \frac{N\beta}{\alpha^2} \right)} \chi_{a,\alpha} \chi_{a',\alpha'}^*$$

$$= \sum_{a,a'=0,\ldots,k} \sum_{c,c' \in \mathbb{Z} \frac{M}{N}} M_{a,\alpha; a',\alpha'} \left[ \frac{1}{\beta} \sum_{M=0,\ldots,\beta-1} e^{\frac{2\pi im}{\alpha} \left( c - \frac{N\beta}{\alpha^2} \right)} \right] \chi_{a,\alpha} \chi_{a',\alpha'}^*$$

$$= \sum_{a,a'=0,\ldots,k} \sum_{c,c' \in \mathbb{Z} \frac{M}{N}} M_{a,\alpha; a',\alpha'} \chi_{a,\alpha} \chi_{a',\alpha'}^*$$

$$= \chi_{a,\alpha} \chi_{a',\alpha'}^*$$

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\[
\begin{align*}
&= \sum_{a,d=0,\ldots,k} M_{a,\alpha c; a',\alpha c'} \delta \left( c \equiv \frac{N(a^2)}{\alpha^2 \beta} \mod \beta \right) \chi_{a,\alpha}(c - \frac{N(a^2)}{\alpha^2 \beta}) \chi_{a',\alpha c'}^* \\
&= \sum_{a=0,\ldots,k} \sum_{c,c' \in \mathbb{Z}_2^k} M_{a,\alpha}(\beta c + \frac{N(a^2)}{\alpha^2 \beta}; a',\alpha c') \chi_{a,\alpha}(\beta c - \frac{N(a^2)}{\alpha^2 \beta}) \chi_{a',\alpha c'}^* 
\end{align*}
\]

where in the last line we wrote \( c = \beta s + \frac{N(a^2)}{\alpha^2 \beta} \) where \( s \) is defined modulo \( \frac{2k}{\alpha^2 \beta} \). The partition function of the orbifold \( O^7 \) is then given by the sum over the twisted sectors:

\[
Z_{\text{orb}} = \sum_{N=0,\ldots,\beta - 1} \sum_{a=0,\ldots,k} \sum_{c,c' \in \mathbb{Z}_2^k} M_{a,\alpha}(\beta s + \frac{N(a^2)}{\alpha^2 \beta}; a',\alpha c') \chi_{a,\alpha}(\beta s - \frac{N(a^2)}{\alpha^2 \beta}) \chi_{a',\alpha c'}^*. \tag{30}
\]

If it happens that \( \frac{N(a^2)}{\alpha^2 \beta} \in \mathbb{Z} \) then the above simplifies to

\[
Z_{\text{orb}} = \sum_{N=0,\ldots,\beta - 1} \sum_{a=0,\ldots,k} \sum_{c,c' \in \mathbb{Z}_2^k} M_{a,\alpha}(s + \frac{N(a^2)}{\alpha^2 \beta}; a',\alpha c') \chi_{a,\alpha}(s - \frac{N(a^2)}{\alpha^2 \beta}) \chi_{a',\alpha c'}^*. \tag{31}
\]

3.2.9 Controlling the Parameter \( v \)

The aim of this subsection is to find an orbifolding which sends the parameter \( v \) to the smallest possible value it can take:

**Proposition 3.3.** Fix \( k \) and let \( M \) be a level \( k \) physical invariant in one of the families \( \tilde{M}^0, \tilde{M}^4, \tilde{M}^2, \tilde{E}_1^{10}, \tilde{E}_2^4, \tilde{E}_1^2, \tilde{E}_2^{28} \) with parameters \( (v, z, *) \) where \( * \) is either \( v \) or \( x \). Then we can orbifold \( M \) to a minimal model with partition function in the same family with parameters \( (v', z, *) \) where \( v' \) is the smallest possible value of \( v \) allowed.

In the exceptional cases \( \tilde{E}_1^{10} \) and \( \tilde{E}_2^{28} \) there is only one allowed value of \( v \), so the proposition is trivial in these cases; they are included for completeness.

We shall prove the claim using the orbifolds constructed in section 3.2.8. The idea is to orbifold by the largest possible value of \( \beta \) which satisfies \( \frac{N(a^2)}{\alpha^2 \beta} \in \mathbb{Z} \). We shall carry out the computation in detail for the odd \( k \) situation only, since the other cases are similar. For the full computations, see [23].
3.2.10 $k$ odd

Let $k$ be an odd integer and let $M$ be a physical invariant at level $k$ with parameters $(v, z, n)$ (see (10)). Write $\overline{k} = \prod_{i=1}^{l} p_i^{2a_i+\delta_i}$ where the $p_i$ are distinct odd primes and $\delta_i \in \{0, 1\}$ for each $i = 1, \ldots, l$. Similarly write $v = \prod_{i=1}^{l} p_i^{b_i}$ for some integers $b_i$. The conditions $\frac{v}{\beta}, \frac{\overline{k}}{\beta} \in \mathbb{Z}$ are equivalent to $a_i + \delta_i \leq b_i \leq 2a_i + \delta_i$, so we can define an integer $\beta = \prod_{i=1}^{l} p_i^{b_i-a_i-\delta_i}$.

As in the previous section we find the biggest integer $a$ such that $M_{a,c} a' c' \neq 0 \Rightarrow c, c' \in \alpha \mathbb{Z}$ here, $\alpha = \frac{v}{\beta} = \prod_{i=1}^{l} p_i^{2a_i-b_i+\delta_i}$. With these values we see that $\frac{v}{\alpha \beta} = \prod_{i=1}^{l} p_i^{\delta_i} \in \mathbb{Z}$, so we can perform $O^7$, the $\mathbb{Z}_\beta$ orbifold from the previous subsection, on $M$ using the simplified formula in equation (31).}

$$Z^{\text{orb}} = \sum_{N=0, \ldots, \beta-1} \sum_{a=0, \ldots, k} \sum_{c' \in \mathbb{Z}_{\overline{k}}} M_{a,c,b}(c + \frac{2N}{\alpha \beta})\delta(a') \delta(c' \equiv c + n(a+c) \mod 2)$$

$$\times \delta\left(c' \equiv 2z \beta \left(c + \frac{2N}{\alpha \beta} \right) \mod \frac{\overline{k}}{\alpha^2} \right) \lambda_{a,c,b}^* \left(a', c', a' \right)$$

$$= \sum_{N=0, \ldots, \beta-1} \sum_{a=0, \ldots, k} \sum_{c' \in \mathbb{Z}_{\overline{k}}} \delta(a') \delta(c' \equiv c + n(a+c) \mod 2)$$

$$\times \delta\left(c' \equiv 2z \left(c + \frac{2N}{\alpha \beta} \right) \mod \frac{\overline{k}}{\alpha^2} \right) \lambda_{a,c,b}^* \left(a', c', a' \right),$$

where in the last line we implement the fact that the summand vanishes unless $c' \equiv 0 \mod \beta$. Now let us evaluate $\sum_{N \in \mathbb{Z}_a} \delta(x \equiv 4zN \frac{\overline{k}}{\alpha^2 \beta} \mod \frac{\overline{k}}{\alpha^2})$. From the condition $(2z+1)(2z-1) \equiv 0 \mod \frac{\overline{k}}{\beta}$ and the fact that $\beta$ divides $\overline{k}$ we see that $\operatorname{hcf}(2z, \beta) = 1$. $\beta$ is odd, so in fact $\operatorname{hcf}(4z, \beta) = 1$. It follows that $4zN \mod \beta$ cycles over the values $1, \ldots, \beta$ as $N$ runs over $1, \ldots, \beta$. Thus

$$\sum_{N \in \mathbb{Z}_a} \delta\left(x \equiv 4zN \frac{\overline{k}}{\alpha^2 \beta} \mod \frac{\overline{k}}{\alpha^2} \right) = \sum_{N \in \mathbb{Z}_a} \delta\left(x \equiv N \frac{\overline{k}}{\alpha^2 \beta} \mod \frac{\overline{k}}{\alpha^2} \right) = \delta\left(x \equiv 0 \mod \frac{\overline{k}}{\alpha^2 \beta} \right).$$

Plugging this with $x = c' - 2zc$ into the main calculation gives

$$Z^{\text{orb}} = \sum_{a=0, \ldots, k} \sum_{a' \in \mathbb{Z}_{\overline{k}}} \sum_{c' \in \mathbb{Z}_{\overline{k}}} \delta(a') \delta(c' \equiv c + n(a+c) \mod 2)$$

$$\times \delta\left(c' \equiv 2z \left(c + \frac{2N}{\alpha \beta} \right) \mod \frac{\overline{k}}{\alpha^2} \right) \lambda_{a,c,b}^* \left(a', c', a' \right).$$
\[
\times \delta \left( c' \equiv 2zc \mod \frac{\overline{k}}{\alpha^2 \beta^2} \right) \chi_{\alpha, \beta, c} \tilde{\chi}_{\alpha', \beta', c'}
\]
\[
= \sum_{a=0, \ldots, k} \sum_{c' \in \mathbb{Z}_{2v'}} \delta(a' = J^{a(a+c)}a) \delta(c' \equiv c + n(a + c) \mod 2)
\]
\[
\times \delta \left( c' \equiv 2zc \mod \frac{\overline{k'}}{\overline{k}} \right) \chi_{\alpha, \beta, c} \tilde{\chi}_{\alpha', \beta', c'}
\]
\[
= \tilde{M}^{0}[v', z, n]
\]

where we have defined \( v' = \frac{\overline{\alpha \beta}}{\alpha \beta} = \prod_{i=1}^{l} p_i^{a_i + \delta_i} \). Note that this is the smallest divisor \( v' \) of \( \overline{k} \) satisfying \( \frac{v'^2}{k} \in \mathbb{Z} \). Thus we have successfully minimised the parameter \( v \).

### 3.2.11 4 divides \( k \)

The \( \tilde{M}^{4,2} \) case is similar. Fix \( k \) such that \( 4 | k \) and choose an \( \tilde{M}^{4,2} \) physical invariant with parameters \( (v, z, x) \). We write \( \overline{k} = 2 \prod_{i=1}^{l} p_i^{2a_i + \delta_i} \) with \( p_i \) distinct odd primes and \( \delta_i \in \{0, 1\} \) and write \( v = \prod_{i=1}^{l} p_i^{b_i} \) for some integers \( b_i \). This time \( \alpha = \frac{\overline{\alpha}}{\overline{\beta}} = \prod_{i=1}^{l} p_i^{2a_i - b_i + \delta_i} \) and we set \( \beta = \prod_{i=1}^{l} p_i^{b_i - a_i - \delta_i} \). Again we find that \( \frac{\overline{\alpha \beta}}{\alpha \beta} = \prod_{i=1}^{l} p_i^{b_i} \in \mathbb{Z} \) so we can apply equation (31) to the partition function given by equations (10) in order to calculate the \( \mathbb{Z}_\beta \) orbifold. The result is

\[
Z^{\text{orb}} = \tilde{M}^{4,2}[v', z, x],
\]

where we have defined \( v' = \frac{\overline{\alpha \beta}}{\alpha \beta} = \prod_{i=1}^{l} p_i^{a_i + \delta_i} \). This shows that for a fixed \( k \) we can always send \( v \) to its smallest possible value in the family \( \tilde{M}^{4,2} \).

### 3.2.12 4 divides \( \overline{k} \)

Finally we address the case when \( k \) satisfies \( 4 | \overline{k} \). Fix such a \( k \) and a \( \tilde{M}^{2,0} \) physical invariant \( M \) with parameters \( (v, z, n) \) (see equation (11)). As before write \( \overline{k} = \prod_{i=0}^{l} p_i^{2a_i + \delta_i} \) where \( p_0 = 2 \) and the \( p_i \) are distinct odd primes for \( i \geq 1 \), \( \delta_i \in \{0, 1\} \) for each \( i = 0, \ldots, l \) and \( a_0 \geq 1 \). For this partition function \( \alpha = \frac{\overline{\alpha}}{\overline{\beta}} = \prod_{i=0}^{l} p_i^{2a_i + \delta_i} \) and we set \( \beta = \prod_{i=0}^{l} p_i^{b_i - a_i - \delta_i} \), which is bound to be an integer by the condition \( \frac{\overline{\beta}}{\overline{\alpha}} \in \mathbb{Z} \). We find once again that \( \frac{\overline{\alpha \beta}}{\alpha \beta} = \prod_{i=0}^{l} p_i^{b_i} \in \mathbb{Z} \) and so we can use the formula (31) to calculate the \( \mathbb{Z}_\beta \) orbifold of \( M \). Substituting in equation (11) we find

\[
Z^{\text{orb}} = \tilde{M}^{2,0}[v', z, n],
\]

where we have defined \( v' = \frac{\overline{\alpha \beta}}{\alpha \beta} = \prod_{i=0}^{l} p_i^{a_i + \delta_i} \). This completes the proof of proposition 3.3 for the simple current invariants.
It remains to check the case $\tilde{E}_2^{16}$. Let $M$ be the physical invariant in $\tilde{E}_2^{16}$ with parameters $(v = 9, z, x)$. Then $\alpha = 1$ and we choose $\beta = 3$ so that \( \frac{x}{\alpha + \alpha x} = 2 \in \mathbb{Z} \). It is then straight-forward to apply equation (31) to find

\[ Z^{\text{orb}} = \tilde{E}_2^{16}[3, 1, x]. \]

### 3.2.13 Controlling the Parameter $z$

Now that we can orbifold any minimal model partition function into a particular family with a particular value of $v$, it remains to find an orbifold which lets us control the parameter $z$. We will prove

**Proposition 3.4.** Fix $k$ and let $M$ be a level $k$ physical invariant in one of the families $\tilde{M}^0$, $M^{k,2}$, $M^{2,0}$, $\tilde{E}_1^{10}$, $\tilde{E}_2^{16}$ or $\tilde{E}_2^{28}$ with parameters $(v, z, \ast)$ where $v$ is as small as possible and $\ast$ is either $n$ or $x$. Then we can orbifold $M$ to a minimal model with partition function in the same family with parameters $(v, z', \ast)$ where

\[ 2z \equiv 1 \mod \frac{v^2}{k} \text{ for odd } k, \]

\[ z \equiv 1 \mod \frac{2v^2}{k} \text{ otherwise}. \]

When $v$ is minimised in the family $\tilde{E}_2^{16}$ then $z$ is forced to be 1, so the statement is trivial in this case; it is included in the proposition only for completeness.

The proof is similar for each family of simple current invariants, so we do the odd $k$ case in detail and then go through the other two simple current cases a little more quickly. Finally we will tackle the exceptional cases.

### 3.2.14 $k$ odd

Let $k$ be odd and let $M$ be a level $k$ physical invariant with parameters $(v, z, n)$ where $v$ is as small as possible (see (10)). Write $k = \prod_{i=1}^{t} p_i^{a_i + 1} \prod_{j=1}^{m} q_j^{b_j}$, where the $p_i$ and $q_j$ are mutually distinct odd primes. Then since $v$ is the smallest solution to $\frac{v^2}{k} \in \mathbb{Z}$, we must have $v = \prod_{i=1}^{t} p_i^{a_i + 1} \prod_{j=1}^{m} q_j^{b_j}$ and therefore $\frac{v^2}{k} = \prod_{i=1}^{t} p_i$. Now $z$ is defined to be a solution to $4z^2 - 1 \equiv 0 \mod \frac{v^2}{k}$. So we have

\[ (2z + 1)(2z - 1) \equiv 0 \mod \prod_{i=1}^{t} p_i. \]

But since a given odd prime cannot divide both $2z + 1$ and $2z - 1$, is it equivalent to say that there must exist a partition \( \{p_{i_1}, \ldots, p_{i_t}\} \cup \{p_{j_1}, \ldots, p_{j_u}\} \) of the $p_i$ such that

\[
\begin{align*}
2z + 1 & \equiv 0 \mod \prod_{k=1}^{t} p_{i_k}, \\
2z - 1 & \equiv 0 \mod \prod_{k=1}^{u} p_{j_k}.
\end{align*}
\]
We are trying to orbifold this partition function to one where \( z \) is given by the choice of partition \( \{ \} \cup \{ p_1, \ldots, p_l \} \). So we set \( \beta = \prod_{k=1}^{l} p_k \) and try to make a \( \mathbb{Z}_\beta \) orbifold. Recall that the largest integer \( \alpha \) satisfying the condition

\[
M_{a,c, a', c'} \neq 0 \Rightarrow c, c' \in \alpha \mathbb{Z}
\]

is \( \alpha = \frac{\beta}{v} = \prod_{i=1}^{t} p_i^a \prod_{j=1}^{m} q_j^b \). Thus \( \frac{\beta}{\alpha^2 \beta} = \prod_{k=1}^{u} p_j \in \mathbb{Z} \) and we can apply the orbifold in equation (30). We obtain

\[
Z_{\text{orb}}(30) = \sum_{N=0, \ldots, \beta-1} \sum_{s \in \mathbb{Z}_{\frac{\alpha^2 \beta}{\beta}}} M_{a,a'}(s\beta + \frac{N \bar{k}}{\alpha^2 \beta}; a', a') \chi_{a,a}(s\beta - \frac{N \bar{k}}{\alpha^2 \beta}) \chi_{a', a'}^*.
\]

Note that hcf \( \left( \beta, \frac{\beta}{\alpha^2 \beta} \right) = 1 \) so we cannot pull out any common factor in the ‘\( c \)’ label as we did in equation (31). This is as it should be, as it was that mechanism that was used to change the value of \( v \) in the previous proposition. We now substitute in the defining equations of the physical invariant \( M \) to find

\[
Z_{\text{orb}}(30) = \sum_{N=0, \ldots, \beta-1} \sum_{s \in \mathbb{Z}_{\frac{\alpha^2 \beta}{\beta}}} M_{a,a'}(s\beta + \frac{N \bar{k}}{\alpha^2 \beta}; a', a') \chi_{a,a}(s\beta - \frac{N \bar{k}}{\alpha^2 \beta}) \chi_{a', a'}^*.
\]

We claim that \( 2z \left( s\beta + \frac{N \bar{k}}{\alpha^2 \beta} \right) \equiv s\beta - \frac{N \bar{k}}{\alpha^2 \beta} \mod \beta \). To prove this note that we have

\[
\frac{2z \left( s\beta + \frac{N \bar{k}}{\alpha^2 \beta} \right) - \left( s\beta - \frac{N \bar{k}}{\alpha^2 \beta} \right)}{\alpha^2 \beta} \equiv (2z + 1)N \frac{k}{\alpha^2 \beta} + (2z - 1)s \beta
\]

\[
\equiv (2z + 1)N \prod_{k=1}^{u} p_j + (2z - 1)s \prod_{k=1}^{t} p_k \equiv 0 \mod \prod_{k=1}^{t} p_k.
\]

Substituting this back in allows us to make a simple change of variables:

\[
Z_{\text{orb}} = \sum_{N=0, \ldots, \beta-1} \sum_{s \in \mathbb{Z}_{\frac{\alpha^2 \beta}{\beta}}} \delta(a' = J^{n(a+s+N)} a) \delta(c' \equiv s + N + n(a+s+N) \mod 2)
\]

\[
\times \delta \left( c' \equiv \left( s\beta - \frac{N \bar{k}}{\alpha^2 \beta} \right) \mod \frac{k}{\alpha^2} \right) \chi_{a,a}(s\beta - \frac{N \bar{k}}{\alpha^2 \beta}) \chi_{a', a'}^*.
\]
\[
\sum_{a=0, \ldots, k} \sum_{a'=0, \ldots, k} \delta(a') J^{n(a+c)} a \delta(c' \equiv c + n(a + c) \mod 2) \\
\times \delta(c' \equiv c \mod \frac{k}{2}) \chi_{a,ac} \chi_{a',a'c} \\
= \tilde{M}^0[v, z', n]
\]

where \(z'\) is the unique solution to \(2z \equiv 1 \mod \frac{v^2}{k}\) as required.

3.2.15 4 divides \(k\)

The proof of proposition 3.4 in the case where 4 divides \(k\) proceeds in a very similar way to the case where \(k\) is odd. Fix a physical invariant \(M \equiv \tilde{M}^{4,2}\) with parameters \((v, z, x)\) where \(v\) is minimal. We write \(k = 2^{r+\epsilon} \prod_{i=1}^{l} p_i^{a_i+1} \prod_{j=1}^{m} q_j^{b_j}\) with \(p_i, q_j\) mutually distinct odd primes, \(r \geq 1\) and \(\epsilon \in \{0, 1\}\). Note that since \(v\) is minimal (see (16)) we must have \(v = \prod_{i=1}^{l} p_i^{a_i+1} \prod_{j=1}^{m} q_j^{b_j}\) and \(2^{r+\epsilon} = \prod_{i=1}^{l} p_i\). The equation for \(z\) for \(\tilde{M}^{4,2}\) is \(z^2 - 1 \equiv 0 \mod \frac{v^2}{k}\) so we have \((z + 1)(z - 1) \equiv 0 \mod \prod_{i=1}^{l} p_i\).

Equivalently, there exists a \(t\) such that, after relabelling the \(p_i, q_j\),

\[
\begin{align*}
z + 1 & \equiv 0 \mod \prod_{i=1}^{l} p_i, \\
z - 1 & \equiv 0 \mod \prod_{i=t+1}^{l} p_i.
\end{align*}
\]  

This time \(\alpha = \frac{k}{2^{r+\epsilon}} = \prod_{i=1}^{l} p_i^{a_i} \prod_{j=1}^{m} q_j^{b_j}\) and again we set \(\beta = \prod_{i=1}^{l} p_i\). Then we can perform the \(\mathbb{Z}_\beta\) orbifold given in equation (30) on \(M\). This end result is

\[
\mathbb{Z}^{\text{orb}} = \tilde{M}^{4,2}[v, 1, x]
\]

as required.

3.2.16 4 divides \(\overline{k}\)

The case where 4 divides \(\overline{k}\) is again very similar. Fix a physical invariant \(M \equiv \tilde{M}[v, z, n]\) where \(v\) is minimal. We write \(\overline{k} = 2^{r+\epsilon} \prod_{i=1}^{l} p_i^{a_i+1} \prod_{j=1}^{m} q_j^{b_j}\) with \(p_i, q_j\) mutually distinct odd primes, \(r \geq 1\) and \(\epsilon \in \{0, 1\}\). Note that since \(v\) is minimal (see (11)) we must have \(v = \prod_{i=1}^{l} p_i^{a_i+1} \prod_{j=1}^{m} q_j^{b_j}\) and \(2^{r+\epsilon} = 2^{1+\epsilon} \prod_{i=1}^{l} p_i\). Since \(z\) satisfies \(z^2 - 1 \equiv 0 \mod \frac{2^{r+\epsilon}}{k}\) we must have \((z + 1)(z - 1) \equiv 0 \mod 2^{1+\epsilon} \prod_{i=1}^{l} p_i\). Equivalently, there exists a \(t\) such that, after relabelling the \(p_i, q_j\),

\[
\begin{align*}
z + 1 & \equiv 0 \mod 2 \prod_{i=1}^{l} p_i, \\
z - 1 & \equiv 0 \mod 2 \prod_{i=t+1}^{l} p_i.
\end{align*}
\]
We have \( \alpha = \frac{r}{v} = 2^r \prod_{i=1}^t p_i^{a_i} \prod_{j=1}^m q_j^{b_j} \) and we set \( \beta = 2^x \prod_{i=1}^t p_i \) where \( x \) is either 0 or 1 and will be specified later. Then \( \frac{7}{\alpha^2 \beta} = 2^r (1-x) \prod_{i=t+1}^1 p_i \) is an integer, so we may perform the \( \mathbb{Z}_\beta \) orbifold given in equation (30) on \( M \):

\[
Z_{\text{orb}} = \sum_{N=0,...,\beta-1} \sum_{a=0,...,k} \sum_{a'=0,...,k} \sum_{c' \in \mathbb{Z}_{\frac{\beta}{\alpha^2}} \mathbb{Z}} M_{a,a'}(s\beta + \frac{N\alpha}{\alpha^2}) ; (s\beta - \frac{N\alpha}{\alpha^2}) \chi_{a,a'}(s\beta - \frac{N\alpha}{\alpha^2})
\]

where we have utilised the fact that the parameter \( y \) must be even when \( v \) is minimal. In analogy with the previous two cases, we wish to conclude from the equation (41) that \( (s\beta + \frac{N\alpha}{\alpha^2}) \equiv s\beta - \frac{N\alpha}{\alpha^2} \mod \frac{2k}{\alpha^2} \). We have to be a little careful with the powers of 2: since \( z \) is odd, either \( z-1 \) or \( z+1 \) must be a multiple of 4. If the former we set \( x = 0 \) and if the latter, \( x = 1 \). With this definition, it is easy to check that the desired conclusion holds and we have

\[
Z_{\text{orb}} = \sum_{N=0,...,\beta-1} \sum_{a=0,...,k} \sum_{a'=0,...,k} \sum_{c' \in \mathbb{Z}_{\frac{\beta}{\alpha^2}} \mathbb{Z}} \delta(a' = J^a a) \delta \left( c' \equiv s\beta - \frac{N\alpha}{\alpha^2} \mod \frac{2k}{\alpha^2} \right) 
\]

\[
\times \chi_{a,a'}(s\beta - \frac{N\alpha}{\alpha^2}) \chi_{a,a'}^*(s\beta - \frac{N\alpha}{\alpha^2})
\]

\[
= \sum_{a=0,...,k} \sum_{c' \in \mathbb{Z}_{\frac{\beta}{\alpha^2}} \mathbb{Z}} \delta(a' = J^a a) \delta \left( c' \equiv c \mod \frac{2k}{\alpha^2} \right) \chi_{a,a} \chi_{a,a'}^*
\]

which completes the proof of proposition 3.4 for the simple current invariants.

### 3.2.17 The Exceptional Cases

When \( k = 10 \) we need to show that there is an orbifold connecting the \( \tilde{E}_1^{10} \) invariants with parameters \((v = 6, z = 5)\) and \((v = 6, z = 1)\). But we have already seen in table 1 that the orbifold \( O^2 \) acts on \( \tilde{E}_1^{10}[6, z] \) by \( z \leftrightarrow -z \mod 6 \).

When \( k = 28 \) we follow exactly the method we used for the simple current invariants when \( 4|k \): we have \( \tilde{k} = 30 = 2 \cdot 3 \cdot 5 \) and \( v = 15 \). The solutions to \( z^2 - 1 \equiv 0 \mod 15 \) are \( z \in \{1, 4, 11, 14\} \) (see equation (22)), corresponding
respectively to the situations
\[
\begin{align*}
z = 1, & \quad \left\{ \begin{array}{l}
z + 1 \equiv 0 \mod 1 \\
z - 1 \equiv 0 \mod 15
\end{array} \right\}, \quad \beta = 1 \\
z = 4, & \quad \left\{ \begin{array}{l}
z + 1 \equiv 0 \mod 5 \\
z - 1 \equiv 0 \mod 3
\end{array} \right\}, \quad \beta = 5 \\
z = 11, & \quad \left\{ \begin{array}{l}
z + 1 \equiv 0 \mod 3 \\
z - 1 \equiv 0 \mod 5
\end{array} \right\}, \quad \beta = 3 \\
z = 14, & \quad \left\{ \begin{array}{l}
z + 1 \equiv 0 \mod 15 \\
z - 1 \equiv 0 \mod 1
\end{array} \right\}, \quad \beta = 15.
\end{align*}
\]

In each case \( \alpha = 1 \) and so we apply orbifold \( O^7 \) to the invariants \( M \equiv E^{28}[15, z, x] \) using equation (30). The end result is
\[
Z_{\text{orb}} = \tilde{E}^{28}[15, 1, x].
\]

This completes the proof of proposition 3.4. We summarise the result in the next section.

### 3.2.18 Proof of the Theorem

We are now ready to prove theorem 3.1. We will restate the theorem here in a little more detail. For notation, see section 2.4.

**Theorem 3.5** (Reformulation of theorem 3.1).

- Let \( k \) be odd and let \( M \) be a simple current invariant at level \( k \). Then there exists a chain of orbifolds mapping \( M \) to \( A_k \otimes \overline{M} \) where \( A_k \) is the diagonal \( \mathfrak{su}(2) \) invariant at level \( k \) and the non-zero values of \( \overline{M} \) are given by

\[
\overline{M} \frac{c}{k} \frac{v}{k} = 1 \iff c' \equiv c \mod 2v^2 \frac{v}{k}
\]

where \( v \) is the smallest divisor of \( k \) satisfying \( \frac{v^2}{k} \in \mathbb{Z} \).

- Let \( 4 | k \) and let \( M \) be a simple current invariant at level \( k \). Then there exists a chain of orbifolds mapping \( M \) to \( A_k \otimes \overline{M} \) where \( A_k \) is the diagonal \( \mathfrak{su}(2) \) invariant at level \( k \) and the non-zero values of \( \overline{M} \) are given by

\[
\overline{M} \frac{c}{k} \frac{v}{k} = 1 \iff c' \equiv c \mod 2v^2 \frac{v}{k}
\]

where \( v \) is the smallest divisor of \( \frac{k}{2} \) satisfying \( \frac{v^2}{k} \in \mathbb{Z} \).
• Let $4|k$ and let $M$ be a simple current invariant at level $k$. Then there exists a chain of orbifolds mapping $M$ to $D_k \otimes \overline{M}$ where $D_k$ is the level $k$ $D$ invariant in the $\mathfrak{su}(2)$ $A$-$D$-$E$ classification, and the non-zero values of $\overline{M}$ are given by

$$\overline{M}^{c,c'} = 1 \iff c' \equiv c \mod \frac{8v^2}{k}$$

where $v$ is the smallest divisor of $\frac{k}{2}$ satisfying $\frac{2v^2}{k} \in \mathbb{Z}$.

• Let $M$ be an exceptional invariant at level $k = 10$. Then there exists a chain of orbifolds mapping $M$ to $E_{10} \otimes \overline{M}$ where $E_{10}$ is the exceptional $\mathfrak{su}(2)$ invariant at level 10 and the non-zero values of $\overline{M}$ are given by

$$\overline{M}_{2c,2c'} = 1 \iff c' \equiv c \mod 6.$$ 

• Let $M$ be an exceptional invariant at level $k = 16$. Then there exists a chain of orbifolds mapping $M$ to $E_{16} \otimes \overline{M}$ where $E_{16}$ is the exceptional $\mathfrak{su}(2)$ invariant at level 16 and the non-zero values of $\overline{M}$ are given by

$$\overline{M}_{3c,3c'} = 1 \iff c' \equiv c \mod 4.$$ 

• Let $M$ be an exceptional invariant at level $k = 28$. Then there exists a chain of orbifolds mapping $M$ to $E_{28} \otimes \overline{M}$ where $E_{28}$ is the exceptional $\mathfrak{su}(2)$ invariant at level 28 and $\overline{M}$ is given by

$$\overline{M}_{c,c'} = 1 \iff c' \equiv c \mod 60.$$ 

Proof. The requisite orbifolds were constructed in the previous sections. Given a physical invariant $M$ we use proposition 3.2 to map $M$ into one of the families $\tilde{M}_0, \tilde{M}_2, \tilde{E}_{10}, \tilde{E}_{16}, \tilde{E}_{28}$ depending on the value of $k$ and whether $M$ is a simple current invariant. We can then apply proposition 3.3 to map $v$ to the smallest possible value it can take in that family, while leaving the other parameters unchanged. Proposition 3.4 sends $z$ to 1 if $k$ is even or $2z \equiv 1$ if $k$ is odd. Finally, if necessary, we use the orbifold $O^1$ of subsection 3.2.1 to fix $n = 0$ when $k$ is odd or $4|k$; or to fix $x = 1$ when $4|k$. The resulting partition functions are given explicitly above using equations (10)–(22).

3.3 Construction of the Unitary $N = 2$ Minimal Models

We are now in a position to prove the existence of an $N = 2$ unitary minimal model for every one of Gannon’s partition functions.

Corollary 3.6. Each of the partition functions in Gannon’s list corresponds to a fully-fledged SCFT.
**Proof.** We need to show that there exist structure constants for the OPEs for each of the models in Gannon’s list. Theorem 3.5 shows that every one of Gannon’s partition functions can be obtained as the result of a chain of orbifoldings of either the diagonal partition function, or of one of the partition functions of $E_6 \otimes I_{24}$, $E_7 \otimes I_{36}$ or $E_8 \otimes I_{60}$. Each of these is a known SCFT, and since the orbifold of an SCFT is again an SCFT the theorem is complete. □

4 Analysis of the simple current invariants

4.1 The Kreuzer-Schellekens Construction

In [32] it is shown that all simple current invariants which obey both 1-loop and higher-genus modular invariance can be obtained as orbifolds of the diagonal physical invariant by a subgroup of the centre. It is conjectured that all simple current physical invariants can be obtained in this way; that is, it is conjectured that the higher-genus modular invariance is in fact superfluous. We will analyse the solutions of Gannon’s classification to show that this is indeed the case for the unitary $N = 2$ minimal models.

4.1.1 $k$ odd

One can easily read off from Gannon’s classification that every physical invariant with $k$ odd is a simple current invariant. Furthermore, following [32], precisely one physical invariant can be constructed via an orbifold for each subgroup of the effective centre $\mathcal{C} \cong \mathbb{Z}_{2k}$ (there is no discrete torsion in this case, since subgroups of $\mathbb{Z}_{2k}$ are cyclic).

One can check using induction on the number of prime factors that the number of subgroups of $\mathbb{Z}_q$, equal to the number of divisors of $q$, is $d(q) := \prod_{i=1}^{l}(1 + n_i)$ where $q$ is written $q = \prod_{i=1}^{l} p_i^{n_i}$ for distinct primes $p_i$. The following lemma establishes that the number of physical invariants at each odd level $k$ (see equation (10)) is precisely the number of subgroups of $\mathbb{Z}_{2k}$, showing that the Schellekens-Kreuzer orbifold construction does indeed give all physical invariants when the level $k$ is odd.

**Lemma 4.1.** Let $k$ be odd. Then the number of solutions $(v, z, n) \in \{1, \ldots, k\} \times \{1, \ldots, \frac{k^2}{k}\} \times \{0, 1\}$ to the equations

$$\frac{v^2}{k} \in \mathbb{Z}, \quad 4z^2 \equiv 1 \mod \frac{k^2}{k}$$

is equal to $d(2k)$.

The proof is a simple counting argument. The main step is counting the number of possible values of $z$ for a given $v$, and we partially solved this problem already in constructing the $z$-controlling orbifoldings of subsection 3.2.13. For a detailed proof, we refer the reader to the author’s PhD thesis [23].
4.1.2 4 divides k

We now turn our attention to the case when $4 \mid k$. Again we can immediately read off from Gannon’s classification that $\tilde{M}^{4,0}$, $\tilde{M}^{4,1}$, $\tilde{M}^{4,2}$ and $\tilde{M}^{4,3}$ are all simple current invariants.

The subgroups of the effective centre $C_k \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \mathbb{F}$ are given by

- $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_2 \mathbb{F}$, $2\parallel k$
- $\mathbb{Z}_2 \times \{0\} \cong \mathbb{Z}_2$, $\parallel 2\mathbb{F}$
- $\langle (J, \frac{\mathbb{F}}{2}) \rangle \cong \mathbb{Z}_2$, $\parallel \mathbb{F}$.

We can define an orbifold for each subgroup of the centre and for each choice of discrete torsion associated to that subgroup. For a cyclic group $\mathbb{Z}_q$ there is no choice to make; for a group $\mathbb{Z}_2 \times \mathbb{Z}_2 \mathbb{q}$ there are two degrees of freedom.

Writing $\tau(G)$ for the number of degrees of freedom coming from discrete torsion associated to the group $G$, we find the number of simple current invariants obtained via an orbifold of the diagonal invariant when $4 \mid k$ is

$$N = \sum_{G \leq \mathbb{Z}_2 \times \mathbb{Z}_2 \mathbb{F}} \tau(G) = 5d(\mathbb{F})$$

where $d(q)$, as above, is the number of divisors of $q$.

The following lemma shows that if $4 \mid k$ then the number of simple current physical invariants is equal to $N = 5d(\mathbb{F})$, the number of orbifolds of the diagonal invariant, so the Schellekens-Kreuzer construction does again find all simple currents invariants when $4 \mid k$.

**Lemma 4.2.** Let $8 \mid k+4$. Then the number of solutions $(v, z, n, m) \in \{1, \ldots, \frac{k}{2}\} \times \{1, \ldots, \frac{2v^2}{k}\} \times \{0, 1\}^2$ to the equations

$$\frac{2v^2}{k}, \frac{z}{\mathbb{F}} \in \mathbb{Z}, \quad z^2 \equiv 1 \mod \frac{2v^2}{k}$$

is equal to $2d(\mathbb{F})$.

Let $8\mid k$. Then the number of solutions $(v, z, x, y) \in \{1, \ldots, \frac{k}{2}\} \times \{1, \ldots, \frac{2v^2}{k}\} \times \{1, 3\}^2$ to the equations

$$\frac{x^2}{k}, \frac{z}{\mathbb{F}} \in \mathbb{Z}, \quad z \equiv \frac{k}{8} \mod 2, \quad 4z^2 \equiv 1 \mod \frac{4\mathbb{F}}{2k}$$

is equal to $2d(\mathbb{F})$.

Let $4\mid k$. Then the number of solutions $(v, z, x) \in \{1, \ldots, \frac{k}{2}\} \times \{1, \ldots, \frac{2v^2}{k}\} \times \{1, 3\}$ to the equations

$$\frac{2v^2}{k}, \frac{z}{\mathbb{F}} \in \mathbb{Z}, \quad z^2 \equiv 1 \mod \frac{2v^2}{k}$$

is equal to $d(\mathbb{F})$. 

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Let $4|k$. Then the number of solutions $(v, z, n) \in \{1, \ldots, \frac{7}{2}\} \times \{1, \ldots, \frac{8v^2}{k}\} \times \{0, 1\}$ to the equations
\[ \frac{2v^2}{k}, \frac{2v^2}{2k} \in \mathbb{Z}, \quad z^2 \equiv 1 \mod \frac{4v^2}{k} \]
is equal to $2d(\overline{k})$.

Again the details of the proof are to be found in \[23\].

### 4.1.3 4 divides $k + 2$

As in the previous cases, every physical invariant with $4|k + 2$ is a simple current invariant.

Write $\overline{k} = 2^m p$ where $p$ is odd and $m \geq 2$. Then the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_{\overline{k}}$ are given by
\[
\begin{align*}
\mathbb{Z}_2 \times \mathbb{Z}_l &\cong \mathbb{Z}_{2l}, \quad l | p \\
\mathbb{Z}_2 \times \mathbb{Z}_{2l} &\cong \mathbb{Z}_{2l}, \quad 2 | l | \overline{k} \\
\{0\} \times \mathbb{Z}_l &\cong \mathbb{Z}_l, \quad l | \overline{k} \\
\langle (J, \frac{\overline{k}}{2l}) \rangle &\cong \mathbb{Z}_{2l}, \quad 2 | l | \overline{k}.
\end{align*}
\]

Writing $\tau(G)$ for the number of degrees of freedom coming from discrete torsion of a subgroup $G$ of $\mathbb{Z}_2 \times \mathbb{Z}_{\overline{k}}$ we find that the number of possible orbifolds of the diagonal partition function is
\[
N = \sum_{G \leq \mathbb{Z}_2 \times \mathbb{Z}_{\overline{k}}} \tau(G) = 2 \left( d(\overline{k}) + d \left( \frac{\overline{k}}{2} \right) \right)
\]
The following lemma shows that this is precisely the number of simple current invariants when the level $k$ satisfies $4|k + 2$, proving that the Schellekens-Kreuzer orbifolds do indeed find all the physical invariants at these levels.

**Lemma 4.3.** Let $4|k + 2$ and write $\overline{k} = 2^{2r+\epsilon}p$ where $\epsilon \in \{0, 1\}, r > 0$ and $p$ is odd.

The number of solutions $(v, z, n) \in \{1, \ldots, \frac{7}{2}\} \times \{1, \ldots, \frac{2v^2}{k}\} \times \{0, 1\}$ to the equations
\[ \frac{v^2}{k}, \frac{2v^2}{2k} \in \mathbb{Z}, \quad z^2 \equiv 1 \mod \frac{2v^2}{k} \]
is equal to $2(4r + 3 + \epsilon)d(p)$.

The number of solutions $(v, z, n) \in \{1, \ldots, \frac{7}{2}\} \times \{1, \ldots, \frac{2v^2}{k}\} \times \{0, 1\}$ to the equations
\[ \frac{2v^2}{k} \in 2\mathbb{Z} + 1, \quad \frac{2v^2}{2k} \in \mathbb{Z}, \quad z^2 \equiv 1 \mod \frac{2v^2}{k} \]
is equal to $24d(p)$.

The number of solutions $(v, z, n, m) \in \{1, \ldots, \frac{7}{2}\} \times \{1, \ldots, \frac{2v^2}{k}\} \times \{0, 1\}^2$ to the equations
\[ \frac{v^2}{k} \in \mathbb{Z}, \quad \frac{2v^2}{k} \in 2\mathbb{Z} + 1, \quad z^2 \equiv 1 \mod \frac{4v^2}{k} \]
is equal to $8d(p)$.
4.1.4 Simple Current Invariant Classification

These counting results coupled with the explicit orbifolds given by Schellekens and Kreuzer \cite{32} can be summarised in the following theorem:

**Theorem 4.4.** Denote \( k + 2 = \overline{k} \). Then every simple current \( N = 2 \) unitary minimal partition function at level \( k \) is realised via an orbifold (possibly with discrete torsion) of the diagonal partition function by a subgroup of the effective centre

\[
C \cong \begin{cases} 
\mathbb{Z}_2 & \text{if } k \text{ is odd,} \\
\mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } 4 \text{ divides } k, \\
\mathbb{Z}_2 \times \mathbb{Z}_{\overline{k}} & \text{if } 4 \text{ divides } k + 2.
\end{cases}
\]

The number of simple current invariants at each level \( k \) is given by

\[
N(k) = \begin{cases} 
2d(\overline{k}) & \text{if } k \text{ is odd,} \\
5d(\overline{k}) & \text{if } 4 \text{ divides } k, \\
2d(\overline{k}) + 2d\left(\frac{k}{2}\right) & \text{if } 4 \text{ divides } k.
\end{cases}
\]

(35)

where \( d(n) \) is the number of divisors of \( n \).

5 Conclusion

In this paper we have reviewed Gannon’s classification of the partition functions of the unitary \( N = 2 \) minimal models and given the explicit results with a few minor errors corrected. It is hoped that by making this list explicit, the less studied theories therein may receive more attention.

The main result is to show that every one of these possible partition functions really does correspond to a full minimal SCFT. This is a large step towards completing the full classification of the unitary \( N = 2 \) minimal models.

We also showed that Kreuzer and Schellekens’ result that every simple current invariant is realised via an orbifolding of the diagonal partition function holds without the extra assumption of higher-genus modular invariant.

This paper brings us tantalisingly close to the complete classification of the unitary \( N = 2 \) minimal models. If one could show that there is just one SCFT belonging to each partition function the classification would be complete.

It would also be satisfying to find some geometric classification of the minimal models in terms of singularities, analogous to the classification of the space-time supersymmetric models in terms of simple singularities arising in their Landau-Ginzburg descriptions \cite{33,42}.

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