Generalized (2+1) dimensional black hole by Noether symmetry

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Abstract

We use the Noether symmetry approach to find \( f(R) \) theory of (2 + 1) dimensional gravity and (2 + 1) dimensional black hole solution consistent with this \( f(R) \) gravity and the associated symmetry. We obtain \( f(R) = D_1R(n/n + 1)(R/K)^{1/n} + D_2R + D_3 \), where the constant term \( D_3 \) plays no dynamical role. Then, we find general spherically symmetric solution for this \( f(R) \) gravity which is potentially capable of being as a black hole. Moreover, in the special case \( D_1 = 0, D_2 = 1 \), namely \( f(R) = R + D_3 \), we obtain a generalized BTZ black hole which, other than common conserved charges \( m \) and \( J \), contains a new conserved charge \( Q \). It is shown that this conserved charge corresponds to the freedom in the choice of the constant term \( D_3 \) and represents symmetry of the action under the transformation \( R \rightarrow R' = R + D_3 \) along the killing vector \( \partial_R \). The ordinary BTZ black hole is obtained as the special case where \( D_3 \) is fixed to be proportional to the infinitesimal cosmological constant and consequently the symmetry is broken via \( Q = 0 \). We study the thermodynamics of the generalized BTZ black hole and show that its entropy can be described by the Cardy-Verlinde formula.

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1 Introduction

General relativity in (2 + 1)-dimensional spacetime becomes a topological field theory with only a few nonpropagating degrees of freedom [1]. The vacuum solution of (2+1)-dimensional gravity is necessarily flat when the cosmological constant is zero, and it can be shown that no black hole solutions exist [2]. Moreover, the black hole thermodynamics, accounted by quantum states, is ill-defined in this low dimensional model, because of few degrees of freedom. However, it came as a great surprise when (2 + 1)-dimensional BTZ black hole solutions for a negative cosmological constant were shown to exist which can have an...
arbitrarily high entropy [3]. Indeed, Bañados, Teitelboim and Zanelli [3] have shown that (2 + 1)-dimensional gravity with a negative cosmological constant has a black hole solution, so called BTZ black hole. The BTZ black hole solution in (2 + 1) dimensional spacetime is derived from a three dimensional action of gravity

\[ I = \frac{1}{2} \int dx^3 \sqrt{-\hat{g}} (R - 2\Lambda) \]  

where \( \Lambda = -\ell^{-2} \) is a negative cosmological constant characterized by a typical length \( \ell \). The line element in the Schwarzschild coordinates is taken as

\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 \left( d\phi - \frac{J}{2r^2} dt \right)^2 \]  

where

\[ f(r) = \left( -m + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} \right). \]  

This metric is stationary and axially symmetric having just two Killing vectors \( \partial_t \) and \( \partial_\phi \), corresponding to time displacement and rotational symmetry and generically has no other symmetries. Therefore, it is described by two parameters, mass \( m \) and angular momentum (spin) \( J \). BTZ black holes are asymptotically anti-de-Sitter (AdS) spacetime with no curvature singularity at the origin, and differ from Schwarzschild and Kerr solutions which are asymptotically flat spacetimes with curvature singularity at the origin. They describe a spacetime of constant negative curvature, with outer and inner horizons, i.e. \( r_+ \) (event horizon) and \( r_- \) (Cauchy horizon) subject to \( J \neq 0 \), respectively, given by

\[ r_{\pm}^2 = \frac{\ell^2}{2} \left( m \pm \sqrt{m^2 - \frac{J^2}{\ell^2}} \right). \]  

The two-parametric family of BTZ black holes, as AdS black holes, play a central role in AdS/CFT conjecture [4] and in brane-world scenarios [5, 6]. The AdS/CFT correspondence has also been generalized for BTZ black holes in higher curvature gravity [7].

Recently, \( f(R) \) gravity as a modified theory of gravity has received considerable attention concerning the current acceleration of the universe [8]. On the other hand, Noether symmetry is a physical criterion which allows one to select \( f(R) \) gravity models which are compatible with this symmetry [9]. This approach has also been used to obtain \( f(T) \) gravity models respecting the Noether symmetry [10]. On the other hand, new spherically symmetric solutions in \( f(R) \) gravity have been obtained by Noether Symmetries [11]. In the present paper, we apply the Noether symmetry approach to obtain (2 + 1) dimensional black hole solutions in \( f(R) \) gravity which are consistent with the Noether symmetry.

2 (2 + 1)-dimensional \( f(R) \) gravity with spherical symmetry

The action in the metric formalism for (2 + 1) \( f(R) \) gravity takes the form

\[ I = \frac{1}{2} \int d^3 x \sqrt{-g} f(R). \]
This action describes a theory of \((2 + 1)\) gravity where \(f(R)\) is a typical function of the Ricci scalar \(R\). In order to study the spherical solutions we take the metric in the following form

\[
 ds^2 = [-N^2(r) + r^2M^2(r)]dt^2 + N^{-2}(r)dr^2 + 2r^2M(r)dtd\phi + r^2d\phi^2, \tag{6}
\]

where the radial functions \(N(r)\) and \(M(r)\) are to be determined as the degrees of freedom. The corresponding Ricci scalar is calculated as

\[
 R = -\frac{1}{2r}(4rN'^2 + 4rNN'' - r^3M'^2 + 8NN'), \tag{7}
\]

where \('\) denotes the derivative with respect to \(r\). In order to derive the field equations in the \(f(R)\) gravity, we generalize the degrees of freedom and define a canonical (point like) Lagrangian \(\mathcal{L} = \mathcal{L}(N, M, R, N', M', R')\) so that \(Q = \{N, M, R\}\) is the configuration space and \(TQ = \{N, M, R, N', M', R'\}\) is the related tangent bundle on which \(\mathcal{L}\) is defined \cite{9}. Now, we use the method of Lagrange multipliers to set \(R\) as a constraint of the dynamics. To this end, by taking a suitable Lagrange multiplier \(\lambda\) and integrating by parts, the Lagrangian becomes canonical and the action takes on the following form

\[
 S = \int d^3x \sqrt{-g}[f(R) - \lambda(R + \frac{1}{2r}(4rN'^2 + 4rNN'' - r^3M'^2 + 8NN'))]. \tag{8}
\]

The variation of action with respect to \(R\) gives \(\lambda = f_R \equiv df/dR\), so the action can be rewritten as

\[
 S = \int d^3x \sqrt{-g}[f(R) - f_R(R + \frac{1}{2r}(4rN'^2 + 4rNN'' - r^3M'^2 + 8NN'))]. \tag{9}
\]

Integrating by parts results in the following point-like Lagrangian

\[
 \mathcal{L} = r(f - Rf_R) + \frac{r^3}{2}f_RM'^2 - 2f_RNN' + 2rf_{RR}R'NN', \tag{10}
\]

where \(f_{RR} \equiv d^2f/dR^2\). The equations of motion for \(N, M\) and \(R\) are obtained respectively as

\[
 N(f_{RRR}R'^2 + f_{RR}R''') = 0, \tag{11}
\]

\[
 (r^3f_RM')' = 0, \tag{12}
\]

\[
 -Rf_{RR} + \frac{r^3}{2}f_{RR}M'^2 - 4f_{RR}NN' - 2rf_{RR}N'^2 - 2rf_{RR}NN'' = 0. \tag{13}
\]

### 3 Noether Symmetry

Solutions for the dynamics given by the point-like canonical Lagrangian \cite{10} can be obtained by choosing cyclic variables which are related to some Noether symmetries. In general, a non-degenerate point-like canonical Lagrangian \(\mathcal{L}\) depends on the variables \(q^i(x^\mu)\) and on their...
derivatives $\partial_\nu q^j(x^\mu)$. Using the Euler-Lagrange equations and after some simple calculations we obtain
\[ \partial_\mu \left( \alpha^j \frac{\partial L}{\partial q^j} \right) = \alpha^j \frac{\partial L}{\partial q^j} + (\partial_\mu \alpha^j) \frac{\partial L}{\partial \partial_\mu q^j} = L_\mathbf{X} L, \tag{14} \]
where $L_\mathbf{X}$ denotes the Lie derivative along the vector field $\mathbf{X}$ defined by
\[ \mathbf{X} = \alpha^j \frac{\partial}{\partial q^j} + (\partial_\mu \alpha^j) \frac{\partial}{\partial \partial_\mu q^j}, \tag{15} \]
which is the generator of symmetry for the dynamics derived by $L$. This is a statement of Noether theorem which asserts that if $L_\mathbf{X} L = 0$, then the Lagrangian $L$ is invariant along the vector field $\mathbf{X}$. As a consequence, we can define the current
\[ j^\mu = \alpha^j \frac{\partial L}{\partial \partial_\mu q^j}, \tag{16} \]
which is conserved as
\[ \partial_\mu j^\mu = 0. \tag{17} \]

The presence of Noether symmetries allows one to reduce the dynamics and find out exact solutions as well as the analytic form of $f(R)$ \[9, 11\].

4 $f(R)$ gravity consistent with Noether symmetry

Following \[9\], we define the Noether symmetry in the present model by a vector field $\mathbf{X}$ on the tangent space $TQ = (M, N, R, M', N', R')$ of the configuration space $Q = (M, N, R)$
\[ \mathbf{X} = \alpha \frac{\partial}{\partial N} + \beta \frac{\partial}{\partial M} + \gamma \frac{\partial}{\partial R} + \alpha' \frac{\partial}{\partial N'} + \beta' \frac{\partial}{\partial M'} + \gamma' \frac{\partial}{\partial R'}, \tag{18} \]
such that
\[ L_\mathbf{X} L = 0. \tag{19} \]
Therefore, a symmetry exists if one finds solutions of the equation $L_\mathbf{X} L = 0$ for the functions $\alpha$, $\beta$ and $\gamma$, where at least one of them is different from zero. Imposing (19), we obtain the following system of partial differential equations
\[ R'N' \left( \alpha f_{RR} + f_{RRN} \frac{\partial \alpha}{\partial N} + \gamma f_{RRR} N + f_{RRN} \frac{\partial \gamma}{\partial R} \right) = 0, \tag{20} \]
\[ R'M' \left( 2f_{RRN} \frac{\partial \alpha}{\partial M} + r^2 f_R \frac{\partial \beta}{\partial R} \right) = 0, \tag{21} \]
\[ N'M' \left( 2f_{RRN} \frac{\partial \gamma}{\partial M} + r^2 f_R \frac{\partial \beta}{\partial N} \right) = 0, \tag{22} \]
\[ N' \left( \alpha f_R + \gamma f_{RRN} + f_R \frac{\partial \alpha}{\partial N} N \right) = 0, \tag{23} \]
\[ M' \left( \frac{1}{2} \gamma f_{RR} + f_R \frac{\partial \beta}{\partial M} \right) = 0, \quad (24) \]
\[ N' \left( f_{RR} N \frac{\partial \gamma}{\partial N} \right) = 0, \quad (25) \]
\[ R' \left( f_{RR} N \frac{\partial \alpha}{\partial R} \right) = 0, \quad (26) \]
\[ M' \left( f_R N \frac{\partial \alpha}{\partial M} \right) = 0, \quad (27) \]
\[ R' \left( f_R N \frac{\partial \alpha}{\partial R} \right) = 0, \quad (28) \]

which is subject to the following constraint

\[ \gamma R f_{RR} = 0. \quad (29) \]

The case \( f_{RR} = 0 \) leads to the common Einstein-Hilbert action which is not of our interest. Moreover, we take \( f_{RR} \neq 0, f_R \neq 0 \) and also \( M' \neq 0, N' \neq 0, R' \neq 0 \). So, (29) gives rise to

\[ \gamma = 0. \quad (30) \]

Moreover, (26) and (27) leads to

\[ \alpha = \alpha(N). \quad (31) \]

Using (24) and (30) we obtain

\[ \frac{\partial \beta}{\partial M} = 0 \implies \beta = \beta(N, R). \quad (32) \]

On the other hand, Eq.(22) results in

\[ \frac{\partial \beta}{\partial N} = 0 \implies \beta = \beta(R). \quad (33) \]

Finally, using (21) and (31) we have

\[ \beta = \beta_0 = \text{Const.} \quad (34) \]

Imposing (20) in (30), we obtain the following result

\[ \alpha = \frac{A}{N}, \quad (35) \]

which solves Eq.(23), \( A \) being a constant. Using the above results in (18), the vector field \( X \) becomes

\[ X = \frac{A}{N} \frac{\partial}{\partial N} + \beta_0 \frac{\partial}{\partial M} - \frac{AN'}{N^2} \frac{\partial}{\partial N'}. \quad (36) \]
The conserved current (16) is written as
\[ j^r = A \frac{\partial \mathcal{L}}{\partial N'} + \beta_0 \frac{\partial \mathcal{L}}{\partial M'} = -2A(f_R + rf_{RR}R') + \beta_0 r^3 f_{RM}', \]
(37)
whose conservation through (17) results in
\[ -2A(f_R + rf_{RR}R') + \beta_0 r^3 f_{RM}' = C_1, \]
(38)
where \( C_1 \) is a constant. So, we have
\[ -2A(f_R + R'f_{RR}) = C_1 - \beta_0 C_2, \]
(39)
where according to (12)
\[ C_2 = r^3 f_{RM}' = \text{Const.} \]
(40)
The dynamical equation (11) can be written as
\[ (R'f_{RR})' = 0, \]
(41)
which gives rise to
\[ f_R = D_1 r + D_2, \]
(42)
where \( D_1, D_2 \) are the constants of integration. Putting (42) into (40) results in
\[ M(r) = -\frac{C_2D_1^2 \ln (D_1r + D_2)}{D_2^3} + \frac{C_2D_1^2 \ln (r)}{D_2^3} - \frac{1}{2} \frac{C_2}{D_2^2 r^2} + \frac{C_2D_1}{D_2^2 r}. \]
(43)
In order to find \( M \) and \( R \) as functions of \( r \), we follow the procedure as is explained below. Using the fact that we are looking for the spherical solutions, one may choose the following ansatz
\[ R(r) = Kr^n, \]
(44)
from which we find
\[ r = \left( \frac{R}{K} \right)^{1/n}, \]
(45)
where \( K \) is a constant with appropriate dimension. Putting this into (12) leads to
\[ f_R = D_1 \left( \frac{R}{K} \right)^{1/n} + D_2. \]
(46)
Integration with respect to \( R \) yields
\[ f(R) = D_1 R \left( \frac{n}{n + 1} \right) \left( \frac{R}{K} \right)^{1/n} + D_2 R + D_3, \]
(47)
where \( D_3 \) is a constant of integration. It is important to note that since \( f(R) \) is not appeared in the field equation of \( N(r) \), namely (13), the constant term \( D_3 \) will not appear in the solution for \( N(r) \), as a direct consequence of imposing the Noether symmetry. Therefore, the solutions \( N(r) \) and \( M(r) \) have symmetry in changing the value of \( D_3 \).
5 Generalized (2+1) dimensional black hole

Using (43), (44) and (47) in the equation of motion (13) we obtain

\[ N^2(r) = \frac{1}{4D_2^2 r^2} \left[ D_2 \left( C_2^2 (6D_1 r + D_2) + 8D_2^2 r (Pr + Q) \right) \right] + \frac{1}{4D_4^2 r^2} \left[ 2C_2^2 D_1 r \ln \frac{r}{D_1 r + D_2} (2D_2 + 3D_1 r) \right] - \frac{Kr^{n+2}}{n^2 + 5n + 6}, \]

where \( P \) and \( Q \) are constants of integrations. Now, (43) and (48) determine the spherical solutions for the metric (6) subject to a specific spherically symmetric Ricci scalar (44). To explore the black hole solutions, the metric (6) can be written in the following convenient form

\[ ds^2 = -N^2(r) dt^2 + N^{-2}(r) dr^2 + r^2 [M^2(r) dt + d\phi]^2. \]

For given constants, \( D_1, D_2, C_2, P, Q \) and given values for \( n \), the shift function \( N^2(r) \) may vanish and so the horizons may exist for those values of \( r \) satisfying the following equation

\[ D_2 \left( C_2^2 (6D_1 r + D_2) + 8D_2^2 r (Pr + Q) \right) + 2C_2^2 D_1 r \ln \frac{r}{D_1 r + D_2} (2D_2 + 3D_1 r) - \frac{4D_3^2 r^2 K r^{n+2}}{n^2 + 5n + 6} = 0. \]

Therefore, we have found spherical solutions (43) and (48) capable of being as a black hole solution for \( f(R) \) gravity (47) subject to the Noether symmetry.

6 Generalized BTZ black hole

For the special case \( D_1 = 0, D_2 = 1 \), namely \( f(R) = R + D_3 \), we obtain

\[ N^2(r) = \frac{C_2^2}{4r^2} - \frac{K}{n^2 + 5n + 6} r^{n+2} + \frac{2Q}{r} + 2P, \]

where, as was expected before, \( D_3 \) is not shown up in the solution \( N(r) \). It is appealing to investigate whether one can recover the (2+1)-dimensional BTZ black hole from \( N^2(r) \) and \( M(r) \). At first, it seems impossible because contrary to the BTZ solution, \( D_3 \) which can potentially play the role of a cosmological constant does not appear in the solution \( N(r) \). However, this conflict may be avoided if we use freedom in taking some arbitrary parameters appearing in the solution \( N(r) \). One such appropriate arbitrary parameter is the constant \( K \) through which the quantity \( D_3 \) can appear in the solution \( N(r) \) by setting a relation between \( K \) and \( D_3 \).

To set this relation, we use the following procedure. We assume \( n = 0, Q = 0 \) and use the identifications \( 2P = -m, C_2 = J, \) and \( K = -6l^{-2} \) such that \( N^2(r) \) and \( M(r) \) are identified with the well known solutions of the BTZ black hole [3]

\[ N^2(r) = -m + \frac{r^2}{l^2} + \frac{J^2}{4l^2}, \]
\[
M(r) = -\frac{J}{2r^2},
\]
where \(m\) and \(J\), respectively are the mass and angular momentum of the black hole, and \(l^{-2}\) accounts for the cosmological constant \(\Lambda\). Note that, according to (44), we have \(R = -6l^{-2}\) which is in exact agreement with that of BTZ solution [3]. We know that the BTZ solution is obtained for \(f(R) = R + 2l^{-2}\), whereas here we have recovered the BTZ solution as the special case \(n = 0, Q = 0\) of the generalized solution (51) obtained for \(f(R) = R + D_3\). Hence, we have a freedom to match the two actions \(f(R) = R + 2l^{-2}\) and \(f(R) = R + D_3\). In doing so, we assume \(D_3 = 2l^{-2}\), and use \(R = K = -6l^{-2}\) to obtain a desired relation \(D_3 = -\frac{K}{2}\), or \(K = -3D_3\) through which the constant \(D_3 = 2l^{-2}\) is appeared in the BTZ solution (52).

Now, let us assume \(n = 0, Q \neq 0\). In this generalized case, the quantity \(D_3\) is no longer fixed by \(K\) or \(R\), rather it is left as a continuous parameter of the symmetry. Therefore, \(N^2(r)\) becomes independent of \(D_3\) as follows
\[
N^2(r) = -m - \frac{R}{6}r^2 + \frac{J^2}{4r^2} + \frac{2Q}{r},
\]
\[
M(r) = -\frac{J}{2r^2}.
\]
The horizons of this generalized BTZ black hole are given by four real roots of the following equation
\[
-2R r^4 - 12 mr^2 + 24Qr + 3J^2 = 0.
\]
The location of surface of infinite red shift \(r_{\text{erg}}\), is also obtained as the solution of the following equation
\[
\frac{2Q}{r} - \frac{R}{6}r^2 - m = 0.
\]
Considering the above results, one may conclude that the BTZ black hole is a solution corresponding to the action \(f(R) = R + D_3\) equipped with a symmetry whose freedom in choosing the constant term \(D_3\) is fixed as \(D_3 = 2l^{-2}\). It is then reasonable that, similar to \(m\) and \(J\), we consider the constant \(Q\) as the conserved charge corresponding to the symmetry of solutions under the infinitesimal displacement of \(R\) by \(D_3\), as is appeared in the action \(R + D_3\). In other words, it seems we are dealing with a symmetry of the action under the transformation \(R \rightarrow R + D_3\). Then, we may interpret \(Q = 0\) (BTZ solutions [52], [53]) as an indication for the broken symmetry caused by fixing \(D_3\).

Now, we show that such a symmetry does really exist. Although according to (18), \(\gamma = 0\) indicates that no symmetry of \(\partial/\partial R\) exist explicitly, however we can show that the killing vector \(X\) is indeed proportional to \(\partial/\partial R\). To this end, we first use (54) and (55) to insert for \(N(r)\), \(M(r)\), and \(N(r)'\) in (36) and express all partial derivatives with respect to \(r\). Then, we use (45) to replace \(r\) by \(R\) and \(\partial/\partial r\) by \(\partial/\partial R\), respectively, so that (36) casts in the following form
\[
X \sim \frac{\partial}{\partial R},
\]
This shows that \(X\) is a killing vector field along which we have a symmetry under an infinitesimal transformation \(R \rightarrow R + D_3\).
7 Black hole Thermodynamics

In this section, we consider the metric (6) with the functions given by (54) and (55) where \( m > 0 \).

7.1 Thermodynamical quantities

The mass, angular momentum and area of the black hole are given respectively by

\[
m = \frac{J^2}{4r_+^2} + \frac{2Q}{r_+} - \frac{R}{6} r_+^2,
\]

\[
J = 2r_+ \sqrt{m + \frac{R}{6} r_+^2 - \frac{2Q}{r_+}},
\]

\[
A_H = \frac{r_+}{2},
\]

where suitable units has been used. By employing the well-known Bekenstein-Hawking area formula [14], the entropy of black hole is given by

\[
S = r_+.
\]

We can express the mass \( m \) in terms of \( S, J, R, \) and \( Q \) as

\[
m = \frac{J^2}{4S^2} + \frac{2Q}{S} - \frac{R}{6} S^2.
\]

The Hawking temperature, angular velocity and heat capacity of the black hole are given respectively by

\[
T_H = \left[ \frac{\partial m}{\partial S} \right]_{J,Q,R} = -\frac{R}{3} S - \frac{J^2}{2S^3} - \frac{2Q}{S^2},
\]

\[
\Omega = \left[ \frac{\partial m}{\partial J} \right]_{S,Q,R} = \frac{J}{2S^2},
\]

\[
C = T_H \left[ \frac{\partial T_H}{\partial S} \right]_{J,Q,R}^{-1} = T_H \left( -\frac{R}{3} + \frac{3J^2}{2S^4} + \frac{4Q}{S^3} \right)^{-1}.
\]

The thermodynamic potential conjugate to \( Q \) is also obtained as

\[
\Phi_c = \left[ \frac{\partial m}{\partial Q} \right]_{S,J} = \frac{2}{S} = A_H^{-1}.
\]
7.2 Cardy-Verlinde Formula

Verlinde has proposed a generalization of the Cardy formula from \((1 + 1)\) dimensional conformal field theory (CFT) to \((n + 1)\)-dimensional one \([15]\). The Cardy-Verlinde formula is given by

\[
S_{\text{CFT}} = \frac{2\pi R_0}{\sqrt{ab}} \sqrt{E_C(2E - E_C)},
\]

(68)

where \(E\) is the total energy, \(E_C\) is the Casimir energy, \(R_0\) is the radius of the system, and \(a\) and \(b\) are arbitrary positive coefficients independent of \(R_0\) and \(S\). The Casimir energy is defined by the violation of Euler relation

\[
E_C = n(E + PV - T_H S - \Phi_c Q - \Omega J),
\]

(69)

where the pressure of the CFT is defined as \(P = E/nV\). The total energy is given by the following sum

\[
E = E_E + \frac{1}{2} E_C,
\]

(70)

where \(E_E\) is the purely extensive part of the total energy \(E\). Also, the Casimir energy \(E_C\) and the purely extensive part of energy \(E_E\) expressed in terms of the \(R_0\) and \(S\) are given by

\[
E_C = \frac{b}{2\pi R_0} S^{1-1/n},
\]

(71)

\[
E_E = \frac{a}{4\pi R_0} S^{1+1/n}.
\]

(72)

Using Witten’s work on AdS/CFT correspondence \([13]\), the Cardy-Verlinde formula \((68)\) can be derived by use of the thermodynamics of various black holes with AdS asymptotic, in arbitrary dimension \([16]\).

7.3 Entropy of the generalized BTZ black hole by Cardy-Verlinde formula

The entropy of generalized BTZ black hole with AdS asymptotic described by \((54)\) and \((55)\) can be derived by the Cardy-Verlinde formula \((68)\). We obtain the Casimir energy \(E_C\) using \((69)\) where \(n = 1\). In so doing, we evaluate the following terms

\[
T_H S = -\frac{R}{3} S^2 - \frac{J^2}{2S^2} - \frac{2Q}{S},
\]

(73)

\[
\Phi_c Q = \frac{2Q}{S},
\]

(74)

\[
\Omega J = \frac{J^2}{2S^2},
\]

(75)

Since the generalized black hole is asymptotically anti-de-Sitter, the total energy is \(E = m\) and the Casimir energy is obtained

\[
E_C = 2 \left( \frac{J^2}{4S^2} + \frac{2Q}{S} \right).
\]

(76)
On the other hand, putting \( n = 1 \) in (71) leads to

\[ E_C = \frac{b}{2\pi R_0}. \]  

(77)

By equating the right hand sides of (76) and (77), the radius \( R_0 \) is obtained as

\[ R_0 = \frac{b}{4\pi} \left( \frac{J^2}{4S^2} + \frac{2Q}{S} \right)^{-1}. \]  

(78)

Moreover, by using \( PV = E, (\ref{73}), (\ref{74}), \) and (\ref{75}) in (69), the quantity \((2E - E_C)\) is evaluated as

\[ 2E - E_C = -\frac{R}{3}S^2. \]  

(79)

The purely extensive part of the total energy \( E_E \) is then obtained by substitution of (79) in (70)

\[ E_E = -\frac{R}{6}S^2. \]  

(80)

On the other hand, putting \( n = 1 \) in (72) gives

\[ E_E = \frac{a}{4\pi R_0}S^2. \]  

(81)

By equating the right hand sides of (80) and (81), the radius \( R_0 \) is obtained again as

\[ R_0 = -\frac{3a}{2\pi R}. \]  

(82)

By using (78) and (82), the radius expressed in terms of the arbitrary positive coefficients \( a \) and \( b \) is obtained

\[ R_0 = \frac{1}{\pi} \sqrt{-\frac{3ab}{8R} \left( \frac{J^2}{4S^2} + \frac{2Q}{S} \right)^{-1/2}}. \]  

(83)

Substitution of (76), (79) and (83) in the Cardy-Verlinde formula (68) gives the following result

\[ S_{\text{CFT}} = S, \]  

(84)

which asserts that the entropy of the generalized BTZ black hole can be expressed in the form of Cardy-Verlinde formula.

8 Conclusions

In the present paper, we have obtained generalized \((2 + 1)\) dimensional spherical symmetric solution in \( f(R) \) gravity by using the Noether symmetry. This solution has capability of being \((2 + 1)\) dimensional black hole. In a special case, this solution casts in the form of generalized \((2 + 1)\) dimensional BTZ black hole. This black hole has three conserved charges as mass \( m \), angular momentum \( J \) and a new conserved charge \( Q \) corresponding respectively to the invariance of the solution under time translation, rotation, and continuous displacement
of the Ricci scalar in the action. In the same way that the killing vectors \( \partial_t \) and \( \partial_\phi \) with continuous space-time parameters \( t \) and \( \phi \) cause continuous symmetry under \( t \to t' = t + \epsilon \) and \( \phi \to \phi' = \phi + \delta \) (\( \epsilon \) and \( \delta \) being infinitesimal constants) and yield the conserved charges \( m \) and \( J \) respectively, the appearance of conserved charge \( Q \) in this black hole is a necessary consequence of a killing vector \( \partial_R \) (considering \( R \) as a continuous quantity) which causes continuous symmetry under \( R \to R' = R + D_3 \) \( (D_3 \) being infinitesimal constant)\(^1\).

We have shown that the ordinary anti-de Sitter BTZ black hole within Einstein-Hilbert theory of gravity with a negative cosmological constant is the special case \( Q = 0 \) of this generalized BTZ black hole, where the continuous symmetry along the vector field \( \partial_R \) is broken by \textit{fixing} the constant term \( D_3 \) to be an infinitesimal cosmological constant \( 2l^{-2} \), for ever. In other words, it seems that what we know as the ordinary BTZ black hole, is nothing but the reduction of generalized BTZ black hole to the \textit{fixed} constant hypersurface \( R' = R + 2l^{-2} = -4l^{-2} \).

The present study may also have important impact on the AdS/CFT correspondence from the black hole and its thermodynamics point of view. Witten has argued that the thermodynamics of a certain conformal field theory can be identified with the thermodynamics of black holes in anti-de Sitter space \([13]\). Here, we have obtained a new class of anti-de Sitter BTZ black holes in modified \( f(R) \) theory of gravity subject to the Noether symmetry which has a conserved charge playing the role of a geometric mass. Hence, one may think that the thermodynamics of a certain conformal field theory can be identified with the thermodynamics of a black hole solution obtained in the modified \( f(R) \) theory of gravity consistent with such a Noether symmetry. This is an evidence for the validity of Ads/CFT correspondence in the \( f(R) \) theory of gravity subject to the Noether symmetry. The rigorous study of such Ads/CFT correspondence is very appealing and needs another work.

**References**

[1] E. J. Martinec, Phys. Rev. D30 (1984) 1198; A. Ach´ucarro and P. K. Townsend, Phys. Lett. B180 (1986) 89; E. Witten, Nucl. Phys. B311 (1988) 46; E. Witten, Commun. Math. Phys. 121 (1989) 351.

[2] D. Ida, Phys. Rev. Lett. 85, (2000) 3758,

[3] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992); M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D48 (1993) 1506.

[4] J. Maldacena, Adv. Theo. Math. Phys. 2 (1998) 231.

[5] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370.

[6] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690.

\(^1\)This may shed light on the cosmological constant problem in that why the cosmological constant is so extremely small.
[7] H. Saida and J. Soda, Phys. Lett. B 471, 358 (2000); H. Saida and J. Soda, arXiv:gr-qc/0011095.

[8] S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007); S. Nojiri and S. D. Odintsov, Phys. Rep. 505, 59 (2011); T. P. Sotiriou and V. Faraoni, Rev. Modern. Phys. 82, 451 (2010); A. De Felice and S. Tsujikawa, Living Reviews in Relativity 13: 3, (2010); S. Tsujikawa, Lectures on Cosmology: Accelerated expansion of the universe, Lectures Notes in Physics 800 (Springer, N.Y., 2010) pp. 99-145; S. Capozziello and M. De Laurentis, Phys. Rep. 509, 167-321 (2011).

[9] M. Demianski, R. de Ritis, C. Rubano and P. Scudellaro, Phys. Rev. D 46, (1992) 1391; S. Capozziello, A. De Felice, JCAP 0808, (2008) 016; B. Vakili, Phys. Lett. B 669, (2008) 206.

[10] H. Wei, X.-J. Guo, L.-F. Wang, Phys. Lett. B 707, (2012) 298; K. Atazadeh, F. Darabi, Eur. Phys. J. C 72, (2012) 2016.

[11] S. Capozziello, N. Frusciante, D. Vernieri, New Spherically Symmetric Solutions in f(R)-gravity by Noether Symmetries, arXiv:1204.4650.

[12] M. R. Setare, E. C. Vagenas, Phys. Rev. D 68, (2003) 064014.

[13] E. Witten, Adv. Theor. Math. Phys. 2, (1998) 505.

[14] J. D. Bekenstein, Phys. Rev. D 7, (1973) 2333; J. D. Bekenstein, Phys. Rev. D 9, (1974) 3292; S. W. Hawking, Phys. Rev. D 13, (1976) 191.

[15] E. Verlinde, On the holographic principle in a radiation dominated universe, hep-th/0008140; J. L. Cardy, Nucl. Phys. B 270, (1986), 186.

[16] D. Klemm, A. C. Petkou and G. Siopsis, Nucl. Phys. B 601, (2001) 380; R-G. Cai, Phys. Rev. D 63, (2001) 124018.