Polynomial identities of the Rogers–Ramanujan type

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ABSTRACT

Presented are polynomial identities which imply generalizations of Euler and Rogers–Ramanujan identities. Both sides of the identities can be interpreted as generating functions of certain restricted partitions. We prove the identities by establishing a graphical one-to-one correspondence between those two kinds of restricted partitions.

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1 Introduction

The Rogers–Ramanujan identities appear in combinatorial problems of number theory \cite{1}, lattice statistical mechanics \cite{2}, and identities among Virasoro characters \cite{3, 4, 5, 6}. The analytic form of the Rogers–Ramanujan identities are stated as follows \cite{7}:

\[
\prod_{n=1}^{\infty} (1 - q^n)^{-1} = \sum_{r=0}^{\infty} \frac{q^{r^2 + jr}}{(q)_r}, \quad j = 0, 1,
\]

where \(|q| < 1 \) and

\[
(a)_n \equiv (a; q)_n = \frac{(a; q)_\infty}{(aq^n, q)_\infty}, \quad (a)_\infty \equiv (a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).
\]

It is natural to define for positive integer \(n\)

\[
\frac{1}{(q)_n} = \frac{(q^{1-n})_\infty}{(q)_\infty} = 0.
\]

Consequently, if we introduce the following symbol for a non negative integer \(N\) and an integer \(M\)

\[
\left[ \begin{array}{c} N \\ M \end{array} \right]_q = \frac{(q)_N}{(q)_M(q)_{N-M}},
\]

then it becomes polynomial when \(0 \leq M \leq N\), otherwise it vanishes.

Gordon’s generalization of the Rogers–Ramanujan identities has the following analytic form \cite{7} for \(|q| < 1 \) and \(1 \leq i \leq k\)

\[
\prod_{n=0, \pm i \text{(mod2k+1)}}^{\infty} (1 - q^n)^{-1} = \sum_{n_1 \geq \cdots \geq n_{k-1} \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1}}}{(q)_{n_1-n_2-\cdots-(q)_{n_{k-2}-n_{k-1}}}(q)_{n_{k-1}}}. \tag{1.3}
\]

Thanks to Jacobi’s triple product formula we can recast \eqref{1.3} as

\[
\frac{1}{(q)_\infty} \sum_{r=-\infty}^{\infty} (-1)^r q^{r((2k+1)r + 2k-2i+1)/2} = \sum_{n_1 \geq \cdots \geq n_{k-1} \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1}}}{(q)_{n_1-n_2-\cdots-(q)_{n_{k-2}-n_{k-1}}}(q)_{n_{k-1}}}. \tag{1.4}
\]

In the process of a trial to prove several conjectures obtained in \cite{6}, we encounter the following polynomial identities:

**Theorem 1.1** Let \(n, k, i\) be fixed non negative integers such that \(1 \leq i \leq k\). Then the following
polynomial identity holds.

\[ \sum_{r=-\infty}^{\infty} (-1)^r q^{r((2k+1)r+2k-2i+1)/2} \left[ \frac{n}{n-k+i-(2k+1)r} \right] q \]

\[ = \sum_{n_1 \geq \cdots \geq n_{k-1} \geq n_k = 0} q^{n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1} \times} \times \]

\[ \prod_{j=1}^{k-1} \left[ \frac{n - 2(n_1 + \cdots + n_{j-1}) - n_j - n_{j+1} - \alpha_{ij}}{n_j - n_{j+1}} \right] q. \]

Here \([x]\) denotes the largest integer part of \(x\), and \(\alpha_{ij}^{(k)}\) is the \((i,j)\)th entry of the following \(k \times (k-1)\) matrix

\[ A^{(k)} = \begin{pmatrix} 1 & 2 & \cdots & k-1 \\ 0 & 1 & \cdots & k-2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \]

Note that both sides of (1.5) are polynomials because all but finitely many terms vanish in the sums and all the nonzero terms are polynomials. The first special cases \(k = 1, 2\) of Theorem 1.1 are due to Andrews [7, Ch.9, Ex.4]. Thus Theorem 1.1 is a generalization of Andrews’ polynomial identities. You can reproduce (1.4) by taking the limit \(n \to \infty\).

Comments on historical matter. The case \(k = 2\) in [7, Ch.9, Ex.4] reads as follows:

\[ \sum_{r=-\infty}^{\infty} (-1)^r q^{r(5r+1)/2-2ar} \left[ \frac{n}{n-5r} \right] + a \left[ \frac{n-5r}{2} \right] q \]

\[ = \sum_{j=0}^{n-a} q^{2j + aj} + a \left[ \frac{n - j - a}{j} \right] q, \]

where \(a = 0, 1\). Proving \(q\)-series identities in terms of polynomials with a finite parameter \(n\) was initiated by Schur [8], who studied the LHS of (1.7) in order to prove the Rogers–Ramanujan identities; whereas the RHS of (1.7) was found by MacMahon [9]. As an identity, (1.7) appeared for the first time in [10].

There exists another combinatorial identity by Euler [11, §1, Prob.23]

\[ \prod_{n=1}^{\infty} (1 + q^n) = \sum_{r=0}^{\infty} q^{r^2 + jr} = \sum_{r=0}^{\infty} \frac{q^{r^2 + jr}}{(q^2; q^2)_r}, \quad j = 0, 1. \]

\[ \text{1 Here we replace } n+1 \text{ by } n. \text{ The } a = 0 \text{ and } a = 1 \text{ correspond to } k = i = 2 \text{ and } k = 2, i = 1, \text{ respectively. The LHS of (1.7) for } a = 1 \text{ looks different from ours for } k = 2, i = 1, \text{ but they actually coincide.} \]
A generalization of (1.8) is given by [12]

\[
\frac{1}{(q)_{\infty}} \prod_{n=0,\pm \mod(2k)}^{\infty} (1 - q^n) = \sum_{n_1 \geq \cdots \geq n_{k-1} \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1}}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-2}-n_{k-1}} (q^2; q^2)_{n_{k-1}}},
\]

where \(1 \leq i \leq k\). It can be also recasted as

\[
\frac{1}{(q)_{\infty}} \sum_{r=-\infty}^{\infty} (-1)^r q^{r(kr+k-i)} = \sum_{n_1 \geq \cdots \geq n_{k-1} \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1}}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-2}-n_{k-1}} (q^a; q^a)_{n_{k-1}}},
\]

where \(1 \leq i \leq k\).

It may be useful to express (1.4) and (1.10) in a unified form. Let \(L \geq 3\), \(k = \lfloor L/2 \rfloor\), and \(1 \leq i \leq k\). Then the following holds:

\[
\frac{1}{(q)_{\infty}} \sum_{r=-\infty}^{\infty} (-1)^r q^{r(Lr+L-2i)/2} = \sum_{n_1 \geq \cdots \geq n_{k-1} \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1}}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-2}-n_{k-1}} (q^a; q^a)_{n_{k-1}}},
\]

where \(a = 1\) (resp. \(a = 2\)) when \(L\) is odd (resp. even).

We would like to also present the following polynomial identity which reduces to (1.10) in the limit \(n \to \infty\):

**Theorem 1.2** Let \(n, k, i\) be fixed non negative integers such that \(1 \leq i \leq k\) and \(k \geq 2\). Then the following polynomial identity holds.

\[
\sum_{r=-\infty}^{\infty} (-1)^r q^{r(kr+k-i)} \frac{[2n + k - i]}{n - kr} \sum_{n_1 \geq \cdots \geq n_{k-1} \geq 0} q^{n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1}} \times \prod_{j=1}^{k-2} \frac{2n - 2(n_1 + \cdots + n_{j-1}) - n_j - n_{j+1} + \beta^{(k)}_{ij}}{n_j - n_{j+1}} q^{n - (n_1 + \cdots + n_{k-2})} \times q^{n_{k-1}},
\]

where \(\beta^{(k)}_{ij}\) is the \((i,j)\)th entry of the following \(k \times (k - 2)\) matrix

\[
B^{(k)} = \begin{pmatrix}
    k - 2 & k - 3 & k - 4 & \cdots & 1 \\
    k - 2 & k - 3 & k - 4 & \cdots & 1 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    3 & 2 & 1 & \cdots & 1 \\
    2 & 1 & 1 & \cdots & 1 \\
    1 & 1 & 1 & \cdots & 1 \\
    0 & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

(1.13)
Note that both sides of (1.12) are again polynomials from the same reason as for (1.5).

Now we would like to mention some related works on Bose–Fermi correspondence of Virasoro characters. As was discussed in [6], (1.4) is the simplest example of $q$-series identities between bosonic and fermionic representations (LHS and RHS, respectively) of Virasoro characters. In [13, 14] character identities of the types (1.4) and (1.10) were considered on the basis of path space representations. Theorems 1.1 and 1.2 exhibit "finitized" forms of (1.4) and (1.10), respectively. In [15] polynomial identities for characters $\chi_{r,s}^{(\nu,\nu+1)}(q)$ of unitary minimal models $M(\nu,\nu+1)$ were conjectured (and proved for $\nu = 3,4$). It was Berkovich [17] who proved these conjectures for arbitrary $\nu$ and $s = 1$.

The rest of this paper is organized as follows. In section 2 we introduce a certain restricted partition function and we evaluate the associated generating function by sieve technique [7, 18]. We thus obtain the LHS’s of (1.5) and (1.12). In section 3 we introduce another kind of restricted partition functions whose generating functions give the RHS’s of (1.5) and (1.12). We also present the main Propositions of this paper, which shall be proved by the induction with respect to $(k,i)$. In section 4 we prove the first special cases $k = 2$. In section 5 we prove the main Propositions by establishing a graphical one-to-one correspondence between two kinds of partitions introduced in sections 2 and 3. In section 6 we give discussion and remarks.

## 2 Interpretation of LHS

Let us begin by fixing several terms [6, 18]. Let the rank of a partition be the largest part minus the number of parts. For instance, the rank of the partition $18 = 7 + 4 + 3 + 2 + 2$ is $7 - 5 = 2$. In general, let $N = a_1 + \cdots + a_s$ be a partition of $N$ such that $a_1 \geq \cdots \geq a_s$. Then we construct the Ferrers graph of the partition by putting $a_l$ dots on the $l$th row, starting from the left. Fig. 1 represents the Ferrers graph of the above partition.

![Fig.1: An example of partition of 18.](image)

The subgraph of a Ferrers graph is a portion of the Ferrers graph which lies below a given row and to the right of a given column. The $l$th proper subgraph is the subgraph lying below the $l$th row.

\footnote{This terminology is due to Melzer [6, 16].}
and to the right of the \(l\)th column.

The \(l\)th right angle refers to a portion of the \((l-1)\)st proper subgraph minus the \(l\)th proper subgraph. In the above example there are three right angles.

The length of the \(l\)th row (resp. column) refers to the number of dots on the \(l\)th row (resp. column). The direction parallel to rows (resp. column) is horizontal (resp. vertical). Let \(\mu_l\) and \(\nu_l\) be the lengths of the \(l\)th row and column, respectively. The \(l\)th longer side (resp. shorter side) relative to \(c\) refers to the \(l\)th row (resp. column) if \(\mu_l - \nu_l > c + 1\). The \(l\)th longer side (resp. shorter side) relative to \(c\) refers to the \(l\)th column (resp. row) if \(\mu_l - \nu_l < c\). For the above example, the length and the direction of the second longer side relative to 1 is 5 and vertical, respectively.

Let \(r_l(\pi)\) denote the \(l\)th successive rank of a partition \(\pi\), the rank of the \((l-1)\)st proper subgraph of the corresponding Ferrers graph. For the above partition \(\pi\) we have \(r_1(\pi) = 2, r_2(\pi) = -1, r_3(\pi) = 0\). The \(l\)th successive rank is larger than \(c + 1\) (resp. less than \(c\)) if and only if the \(l\)th row is the \(l\)th longer (resp. shorter) side relative to \(c\).

For a given partition \(\pi\) and positive integers \(a, b\), let \(\lambda\) be the largest integer for which there exists a sequence \(l_1 < \cdots < l_\lambda\) such that \(r_{l_1}(\pi) \geq a - 1, r_{l_2}(\pi) \leq -b + 1, r_{l_3}(\pi) \geq a - 1, r_{l_4}(\pi) \leq -b + 1\), and so on. Then \(\pi\) has an \((a, b)\)-positive oscillation of length \(\lambda\). For a given partition \(\pi\) let \(\lambda\) be the largest integer for which there exists a sequence \(l_1 < \cdots < l_\lambda\) such that \(r_{l_1}(\pi) \leq -b + 1, r_{l_2}(\pi) \geq a - 1, r_{l_3}(\pi) \leq -b + 1, r_{l_4}(\pi) \geq a - 1\), and so on. Then \(\pi\) has an \((a, b)\)-negative oscillation of length \(\lambda\).

In what follows we will often consider partitions into at most \(\nu\) parts, and with the largest part at most \(\mu\). A partition has the maximal size \((\mu, \nu)\) if each part does not exceed \(\mu\), and the number of parts does not exceed \(\nu\).

Let \(p_{a,b}(\mu, \nu; \lambda; N)\) (resp. \(m_{a,b}(\mu, \nu; \lambda; N)\)) stand for the number of partitions of \(N\) with the maximal size \((\mu, \nu)\), and with \((a, b)\)-positive (resp. negative) oscillation of length at least \(\lambda\). To such types partition functions we associate the generating functions

\[
P_{a,b}(\mu, \nu; \lambda; q) = \sum_{N \geq 0} p_{a,b}(\mu, \nu; \lambda; N)q^N, \\
M_{a,b}(\mu, \nu; \lambda; q) = \sum_{N \geq 0} m_{a,b}(\mu, \nu; \lambda; N)q^N.
\]

The main diagonal line refers to the diagonal line starting from the top-left dot of the Ferrers graph. For a given Ferrers graph, we can obtain another Ferrers graph by reflecting the original one with respect to the main diagonal line. Let us call the graph thus obtained the dual Ferrers graph of the original one. Note that the dual Ferrers graph of any partition counted by \(p_{a,b}(\mu, \nu; \lambda; N)\) gives a partition counted by \(m_{b,a}(\nu, \mu; \lambda; N)\), and vice versa. Hence we obtain

\[
p_{a,b}(\mu, \nu; \lambda; N) = m_{b,a}(\nu, \mu; \lambda; N), \\
P_{a,b}(\mu, \nu; \lambda; q) = M_{b,a}(\nu, \mu; \lambda; q).
\] (2.1)
We cite a number of results from [7] in extended forms. You will know that those results permit the following extension by reexamining proofs given in [7]:

**Lemma 2.1** Let \( p(\mu, \nu; N) \) be the number of partitions of \( N \) with the maximal size \((\mu, \nu)\). Then the generating function associated with this partition function is given as follows:

\[
P(\mu, \nu; q) \equiv \sum_{N \geq 0} p(\mu, \nu; N)q^N = \begin{bmatrix} \mu + \nu \\ \mu \end{bmatrix}_q = \begin{bmatrix} \mu + \nu \\ \nu \end{bmatrix}_q.
\] (2.2)

Since all partitions have \((a, b)\)-positive and negative oscillation of length more than or equal to 0, both \( P_{a,b}(\mu, \nu; 0; q) \) and \( M_{a,b}(\mu, \nu; 0; q) \) coincide with \( P(\mu, \nu; q) \). Consequently we obtain

**Lemma 2.2**

\[
P_{a,b}(\mu, \nu; 0; q) = M_{a,b}(\mu, \nu; 0; q) = \begin{bmatrix} \mu + \nu \\ \mu \end{bmatrix}_q = \begin{bmatrix} \mu + \nu \\ \nu \end{bmatrix}_q.
\] (2.3)

The definitions immediately derive the following lemma:

**Lemma 2.3** For \( \lambda \geq 1 \), the following hold:

\[
P_{a,b}(0, \nu; \lambda; q) = P_{a,b}(\mu, 0; \lambda; q) = M_{a,b}(0, \nu; \lambda; q) = M_{a,b}(\mu, 0; \lambda; q) = 0.
\] (2.4)

**Lemma 2.4** The following recursion relation holds for \( \lambda \geq 1 \):

\[
p_{a,b}(\mu, \nu; \lambda; N) - p_{a,b}(\mu - 1, \nu; \lambda; N) - p_{a,b}(\mu, \nu - 1; \lambda; N) + p_{a,b}(\mu - 1, \nu - 1; \lambda; N)
\]
\[
= \begin{cases} 
p_{a,b}(\mu - 1, \nu - 1; \lambda; N - \mu - \nu + 1), & \text{if } \mu - \nu \leq a - 2, \\
m_{a,b}(\mu - 1, \nu - 1; \lambda - 1; N - \mu - \nu + 1), & \text{if } \mu - \nu \geq a - 1,
\end{cases}
\] (2.5)

\[
m_{a,b}(\mu, \nu; \lambda; N) - m_{a,b}(\mu - 1, \nu; \lambda; N) - m_{a,b}(\mu, \nu - 1; \lambda; N) + m_{a,b}(\mu - 1, \nu - 1; \lambda; N)
\]
\[
= \begin{cases} 
p_{a,b}(\mu - 1, \nu - 1; \lambda - 1; N - \mu - \nu + 1), & \text{if } \mu - \nu \leq -b + 1, \\
m_{a,b}(\mu - 1, \nu - 1; \lambda; N - \mu - \nu + 1), & \text{if } \mu - \nu \geq -b + 2,
\end{cases}
\] (2.6)

As a Corollary of Lemma 2.4, we have
Corollary 2.5

\[ P_{a,b}(\mu, \nu; \lambda; q) - P_{a,b}(\mu - 1, \nu; \lambda; q) - P_{a,b}(\mu, \nu - 1; \lambda; q) + P_{a,b}(\mu - 1, \nu - 1; \lambda; q) = \]
\[ q^{\mu + \nu - 1} \times \begin{cases} 
P_{a,b}(\mu - 1, \nu - 1; \lambda; q), & \text{if } \mu - \nu \leq a - 2, \\
M_{a,b}(\mu - 1, \nu - 1; \lambda - 1; q), & \text{if } \mu - \nu \geq a - 1,
\end{cases} \quad (2.7)
\]

\[ M_{a,b}(\mu, \nu; \lambda; q) - M_{a,b}(\mu - 1, \nu; \lambda; q) - M_{a,b}(\mu, \nu - 1; \lambda; q) + M_{a,b}(\mu - 1, \nu - 1; \lambda; q) = \]
\[ q^{\mu + \nu - 1} \times \begin{cases} 
P_{a,b}(\mu - 1, \nu - 1; \lambda - 1; q), & \text{if } \mu - \nu \leq -b + 1, \\
M_{a,b}(\mu - 1, \nu - 1; \lambda; q), & \text{if } \mu - \nu \geq -b + 2,
\end{cases} \quad (2.8)
\]

The initial conditions (2.3, 2.4) and recursion relations (2.7, 2.8) uniquely determine \( P_{a,b}(\mu, \nu; \lambda; q) \) and \( M_{a,b}(\mu, \nu; \lambda; q) \). When the set \((a, \mu, \nu)\) (resp. \((b, \mu, \nu)\)) satisfies a certain relation, \( P_{a,b}(\mu, \nu; \lambda; q) \) (resp. \( M_{a,b}(\mu, \nu; \lambda; q) \)) can be expressed in a simple form.

Lemma 2.6 For positive integers \( a, b \), let \( \mu, \nu \) be non negative integers such that \( \mu - \nu \leq a - 1 \). Then \( P_{a,b}(\mu, \nu; \lambda; q) \) has the following expressions:

\[ P_{a,b}(\mu, \nu; 2\lambda; q) = q^{2(\mu + \nu)} \left( \begin{array}{c} \mu + \nu \\ \mu - (a + b)\lambda \\ q \end{array} \right) \]
\[ \quad \left( \begin{array}{c} \mu + \nu \\ \mu - (a + b)\lambda \\ q \end{array} \right) \]
\[ P_{a,b}(\mu, \nu; 2\lambda - 1; q) = q^{(2\lambda - 1)((a + b)\lambda - b + \mu + \nu)} \left( \begin{array}{c} \mu + \nu \\ \mu - (a + b)\lambda \\ q \end{array} \right) \]
\[ \quad \left( \begin{array}{c} \mu + \nu \\ \mu - (a + b)\lambda \\ q \end{array} \right) \]
\[ (2.9) \]

For positive integers \( a, b \), let \( \mu, \nu \) be non negative integers such that \( \mu - \nu \geq -b + 1 \). Then \( M_{a,b}(\mu, \nu; \lambda; q) \) has the following expressions:

\[ M_{a,b}(\mu, \nu; 2\lambda; q) = q^{2(\mu + \nu)} \left( \begin{array}{c} \mu + \nu \\ \nu - (a + b)\lambda \\ q \end{array} \right) \]
\[ \quad \left( \begin{array}{c} \mu + \nu \\ \nu - (a + b)\lambda \\ q \end{array} \right) \]
\[ M_{a,b}(\mu, \nu; 2\lambda - 1; q) = q^{(2\lambda - 1)((a + b)\lambda - a + \mu + \nu)} \left( \begin{array}{c} \mu + \nu \\ \nu - (a + b)\lambda \\ q \end{array} \right) \]
\[ \quad \left( \begin{array}{c} \mu + \nu \\ \nu - (a + b)\lambda \\ q \end{array} \right) \]
\[ (2.10) \]

Remark. From (2.1), one of (2.9) and (2.10) implies the other.

In order to state the last lemma we cite from [1], we introduce another kind of partition function. Let \( Q_{a,b}(\mu, \nu; N) \) be the number of partitions \( \pi \) of \( N \geq 0 \) with the maximal size \( \mu, \nu \), and \(-b + 2 \leq r_1(\pi) \leq a - 2 \) for any successive ranks of \( \pi \).

Lemma 2.7 Three kinds of partition functions \( Q_{a,b}(\mu, \nu; N) \), \( p_{a,b}(\mu, \nu; \lambda; N) \) and \( m_{a,b}(\mu, \nu; \lambda; N) \) satisfy the following relation:

\[ Q_{a,b}(\mu, \nu; N) = p_{a,b}(\mu, \nu; 0; N) + \sum_{\lambda=1}^{\infty} (-1)^\lambda p_{a,b}(\mu, \nu; \lambda; N) + \sum_{\lambda=1}^{\infty} (-1)^\lambda m_{a,b}(\mu, \nu; \lambda; N). \]
\[ (2.11) \]
**Proof.** This can be shown by using sieve technique.

The LHS of (2.11) counts all partitions of \( N \) with the maximal size \((\mu, \nu)\), and with \((a, b)\)-positive and negative oscillation of length 0. On the other hand, the first term in the RHS refers to the number of partitions of \( N \) with the maximal size \((\mu, \nu)\). Subtract the number of partitions which have \((a, b)\)-positive or negative oscillation of length at least 1 form the first term:

\[
p_{a,b}(\mu, \nu; 0; N) - p_{a,b}(\mu, \nu; 1; N) - m_{a,b}(\mu, \nu; 1; N).
\]

In this way, however, partitions which have both \((a, b)\)-positive and negative oscillation of length at least 1, are subtracted twice. Hence we have to add the number of partitions which have \((a, b)\)-positive or negative oscillation of length at least 2:

\[
p_{a,b}(\mu, \nu; 0; N) - p_{a,b}(\mu, \nu; 1; N) - m_{a,b}(\mu, \nu; 1; N) + p_{a,b}(\mu, \nu; 2; N) + m_{a,b}(\mu, \nu; 2; N).
\]

Note that \( 2 \sum_{i=1}^{\lambda-1} (-1)^i = 0 \) (resp. \(-2\)) when \( \lambda \) is odd (resp. even). Thus in general, partitions which have \((a, b)\)-positive or negative oscillation of length at least \( 2\lambda - 1 \) should be subtracted once more, and those which have \((a, b)\)-positive or negative oscillation of length at least \( 2\lambda \) should be added once more.

Since the alternating sums in the RHS of (2.11) are actually finite sums for fixed \( \mu, \nu \) and \( N \), after repeating this procedure finitely many times, we obtain (2.11).

Let us introduce the generating function of \( Q_{a,b}(\mu, \nu; N) \) as follows:

\[
Q_{a,b}(\mu, \nu; q) = \sum_{N \geq 0} Q_{a,b}(\mu, \nu; N) q^N.
\]

Now we wish to show that for appropriate sets \((a, b, \mu, \nu)\), \( Q_{a,b}(\mu, \nu; q) \) coincides with the LHS’s of (1.5) and (1.12):

**Proposition 2.8** Let \( a = 2k + 1 - i, b = i \) and \( \mu = \left[ \frac{n+1+k-i}{2} \right], \nu = \left[ \frac{n-k+i}{2} \right], \) where \( n \) is a non-negative integer. Then \( Q_{a,b}(\mu, \nu; q) \) gives the LHS of (1.5).

**Proof.** First we notice that \( a + b = n \). Furthermore, \(-b + 1 \leq \mu - \nu \leq a - 1\) holds because \( \mu - \nu = k - i \) when \( n \equiv k - i \) mod 2, and \( \mu - \nu = k - i + 1 \) when \( n \not\equiv k - i \) mod 2. Hence from
Lemma 2.2, 2.6 and 2.7 we have

\[ Q_{a,b}(\mu, \nu; N) \]

\[
= \sum_{N \geq 0} p_{a,b}(\mu, \nu; 0; N)q^N + \sum_{N \geq 0} \sum_{\lambda=1}^{\infty} (-1)^{\lambda}p_{a,b}(\mu, \nu; \lambda; N)q^N + \sum_{N \geq 0} \sum_{\lambda=1}^{\infty} (-1)^{\lambda}m_{a,b}(\mu, \nu; \lambda; N)q^N
\]

\[
= P_{a,b}(\mu, \nu; 0; q) + \sum_{\lambda=1}^{\infty} (-1)^{\lambda}P_{a,b}(\mu, \nu; \lambda; q) + \sum_{\lambda=1}^{\infty} (-1)^{\lambda}M_{a,b}(\mu, \nu; \lambda; q)
\]

\[
= \left[ \begin{array}{cccc}
\mu + \nu \\
\nu
\end{array} \right] q + \sum_{\lambda=1}^{\infty} q^{(2\lambda-1)((a+b)\lambda-b)} \left[ \begin{array}{cccc}
\mu + \nu \\
\mu - (a+b)\lambda + b
\end{array} \right] q
\]

\[
= \sum_{\lambda=1}^{\infty} q^{(2\lambda-1)((a+b)\lambda-a)} \left[ \begin{array}{cccc}
\mu + \nu \\
\nu - (a+b)\lambda + a
\end{array} \right] q
\]

\[
= \sum_{\lambda=-\infty}^{\infty} q^{\lambda(2(a+b)\lambda-a-b)} \left[ \begin{array}{cccc}
\mu + \nu \\
\nu - (a+b)\lambda
\end{array} \right] q - \sum_{\lambda=1}^{\infty} q^{(2\lambda-1)((a+b)\lambda-b)} \left[ \begin{array}{cccc}
\mu + \nu \\
\mu - (a+b)\lambda + b
\end{array} \right] q
\]

\[
= \sum_{r=-\infty}^{\infty} (-1)^r q^{r((2k+1)r+2k-2i+1)/2} \left[ \begin{array}{cccc}
\frac{n}{2} \\
\frac{n-3r}{-2r}
\end{array} \right] q
\]

Thus the claim of this proposition was verified. □

Now it is very easy to prove Theorem 1.1 for \( i = k = 1 \). In this case the LHS of (1.7) is equal to \( Q_{2,1}([n+1]/2, [n]/2); q) \). On the other hand, \( Q_{2,1}(\mu, \nu; N) \) counts the number of partitions \( \pi \) of \( N \) with the maximal size \( (\mu, \nu) \), and \( 1 \leq r_l(\pi) \leq 0 \) for all \( l \). Hence by the definition of the generating function, we have \( Q_{2,1}([n+1]/2, [n]/2); q) = 1 \). Thus we obtain

\[
\sum_{r=-\infty}^{\infty} (-1)^r q^{r(3r+1)/2} \left[ \begin{array}{cccc}
\frac{n}{2} \\
\frac{n-3r}{-2r}
\end{array} \right] q = 1, \quad (2.12)
\]

which is nothing but Theorem 1.1 for \( i = k = 1 \).

The analogue of Proposition 1.1 for Theorem 1.2 is given as follows:

**Proposition 2.9** Let \( a = 2k-i, b = i \) and \( \mu = n + k - i, \nu = n \), where \( n \) is a non negative integer. Then \( Q_{a,b}(\mu, \nu; q) \) gives the LHS of (1.12).

**Proof.** First we notice that \(-b + 1 \leq \mu - \nu \leq a - 1\). Hence from the same calculation as in the
proof of Proposition 2.8 we have
\[
Q_{a,b}(\mu, \nu; q) = \sum_{\lambda=1}^{\infty} \left( q^{N(2(a+b)\lambda+a-b)} \left[ \begin{array}{c} \mu + \nu \\ \nu - (a + b)\lambda \\
\end{array} \right]_q - q^{(2\lambda-1)((a+b)\lambda-b)} \left[ \begin{array}{c} \mu + \nu \\ \mu - (a+b)\lambda + b \\
\end{array} \right]_q \right)\]

Thus the claim of this proposition was verified.  \( \square \)

3 Main Propositions

For a fixed set \((k, i)\) such that \(1 \leq i \leq k\) and a non negative integer \(n\), let \(d(k, i; n; N)\) be the number of partitions of \(N\) of the form
\[
N = n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1} + \sum_{j=1}^{k-1} \sum_{p=1}^{n_j-n_{j+1}} a_p^{(j)},
\]
where
\[
n_1 \geq n_2 \geq \cdots \geq n_{k-1} \geq n_k = 0, \quad 2(n_1 + \cdots + n_{k-1}) \leq n - k + i,
\]
and
\[
n - 2(n_1 + \cdots + n_j) - \alpha_{ij}^{(k)} \geq a_1^{(i)} \geq \cdots \geq a_{n_j-n_{j+1}}^{(j)} \geq 0, \quad 1 \leq j \leq k - 1.
\]

Since \(\alpha_{ik-1}^{(k)} = k - i\), (3.3) implies the second condition of (3.2). We call (3.3) the finiteness condition for partitions of the form (3.1).

We notice that by taking into account Lemma 2.1, the RHS of (1.5) is the generating function of \(d(k, i; n; N)\). On the other hand, we showed in the last section that the LHS of (1.5) coincides with \(Q_{2k+1-i,i}([\frac{n+1+k-i}{2}], [\frac{n-k+i}{2}]; q)\). Thus what we should establish in order to show Theorem 1.1 is that there exists a one-to-one correspondence between restricted partitions counted by \(d(k, i; n; N)\) and those counted by \(Q_{2k+1-i,i}([\frac{n+1+k-i}{2}], [\frac{n-k+i}{2}]; n_1; N)\). Actually, more than this is true:

Proposition 3.1 Fix a set \((k, i)\) such that \(1 \leq i \leq k\) and non negative integers \(n, n_1, N\). Let \(d(k, i; n; n_1; N)\) denote the number of partitions of \(N\) of the form (3.1), and let \(Q_{2k+1-i,i}([\frac{n+1+k-i}{2}], [\frac{n-k+i}{2}]; n_1; N)\) denote the number of partitions of \(N\) with the maximal size \([\frac{n+1+k-i}{2}], [\frac{n-k+i}{2}]\), with \(n_1\) right angles, and all \(n_1\) successive ranks belong to the interval \([-i + 2, 2k - i - 1]\). Then there exists a one-to-one correspondence between restricted partitions counted by \(d(k, i; n; n_1; N)\) and those counted by \(Q_{2k+1-i,i}([\frac{n+1+k-i}{2}], [\frac{n-k+i}{2}]; n_1; N)\).
Let us introduce another restricted partition. For a fixed set \((k, i)\) such that \(k \geq 2, 1 \leq i \leq k\) and a non-negative integer \(n\), let \(\delta(k, i; n; N)\) be the number of partitions of \(N\) of the form

\[
N = n_1^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1} + \sum_{j=1}^{k-2} \sum_{p=1}^{n_j-n_{j+1}} a_p^{(j)} + 2 \sum_{p=1}^{n_{k-1}} b_p, \tag{3.4}
\]

where

\[
n_1 \geq n_2 \geq \cdots \geq n_{k-1} \geq 0, \quad n_1 + \cdots + n_{k-1} \leq n, \tag{3.5}
\]

and

\[
2n - 2(n_1 + \cdots + n_j) + \beta_{ij}^{(k)} \geq a_1^{(j)} \geq \cdots \geq a_{n_j-n_{j+1}}^{(j)} \geq 0, \quad 1 \leq j \leq k - 2,
\]

\[
n - (n_1 + \cdots + n_{k-1}) \geq b_1 \geq \cdots \geq b_{n_{k-1}} \geq 0. \tag{3.6}
\]

Note that the second condition of (3.6) implies the second one of (3.5). We call (3.6) the finiteness condition for partitions of the form (3.4).

By taking into account Lemma 2.1, the RHS of (1.12) is the generating function of \(\delta(k, i; n; N)\). On the other hand, we showed in the last section that the LHS of (1.12) gives \(Q_{2k-i,i}(n + k - i, n; q)\). Thus what we should establish in order to show Theorem 1.2 is that there exists a one-to-one correspondence between restricted partitions counted by \(\delta(k, i; n; N)\) and those counted by \(Q_{2k-i,i}(n + k - i, n; N)\). In this case, the following holds:

**Proposition 3.2** Fix a set \((k, i)\) such that \(k \geq 2, 1 \leq i \leq k\) and non-negative integers \(n, n_1, N\). Let \(\delta(k, i; n_1; N)\) denote the number of partitions of \(N\) of the form (3.4), and let \(Q_{2k-i,i}(n + k - i, n; n_1; N)\) denote the number of partitions of \(N\) with the maximal size \((n + k - i, n\), with \(n_1\) right angles, and all \(n_1\) successive ranks belong to the interval \([-i + 2, 2k - i - 2]\). Then there exists a one-to-one correspondence between restricted partitions counted by \(\delta(k, i; n; n_1; N)\) and those counted by \(Q_{2k-i,i}(n + k - i, n; n_1; N)\).

Propositions 3.1 and 3.2 are the main Propositions of the present paper. We prove these by induction with respect to \((k, i)\) in section 4 and 5. In section 4 we verify the first special cases \(k = 2\). In section 5 we establish a graphical one-to-one correspondence which implies the claims of the main Propositions. We notice that a graphical correspondence presented below can be obtained by translating Burge’s method [19, 20, 21] into language of partitions.

### 4 The first special cases

In this section we show the first special cases. The following is Proposition 3.1 for \(k = 2, i = 2\).
**Lemma 4.1** Fix non-negative integers $n, n_1, N$ such that $2n_1 \leq n$. Then there exists a one-to-one correspondence between a partition of $N$ of the form

$$N = n_1^2 + \sum_{p=1}^{n_1} a_p, \quad n - 2n_1 \geq a_1 \geq \cdots \geq a_{n_1} \geq 0,$$

and a partition counted by $Q_{3,2}([\frac{n+1}{2}],[\frac{n}{2}]; n_1; N)$.

**Proof.** Let us recall that $Q_{3,2}([\frac{n+1}{2}],[\frac{n}{2}]; n_1; N)$ counts partitions of $N$ with the maximal size $([\frac{n+1}{2}],[\frac{n}{2}])$, with $n_1$ right angles, and all $n_1$ successive ranks are equal to 0 or 1. Note that the $p$th successive rank should be equal to 0 (resp. 1) when the number of dots of the $p$th right angle is odd (resp. even). Thus we can construct a Ferrers graph of partition counted by $Q_{3,2}([\frac{n+1}{2}],[\frac{n}{2}]; n_1; N)$ as follows. If the number of dots of the $p$th right angle is odd, we array dots of the $p$th right angle in a symmetric manner with respect to the main diagonal line. If the number of dots of $p$th right angle is even, we array dots on the $p$th row one more than dots on the $p$th column. Since the numbers of dots of adjacent right angles differ by at least 2, the graph thus obtained always becomes a Ferrers graph.

Let us count the number of dots as follows. First remove the $n_1 \times n_1$ square in the top-left corner, and count the number of dots on the $p$th row to the right of the square plus that on the $p$th column below the square. In this way, we get a partition of $N$ into $n_1^2$ and at most $n_1$ positive integers less than or equal to $n - 2n_1$. Inversely, for any given partition of $N$ of the form (4.1), we can construct a Ferrers graph with the maximal size $([\frac{n+1}{2}],[\frac{n}{2}])$, and all $n_1$ successive ranks are equal to 0 or 1, by putting $n_1 \times n_1$ dots, and after that by putting $[\frac{n+1}{2}]$ dots on the $p$th row to the right of the square and $[\frac{n}{2}]$ dots on the $p$th column below the square, respectively, where $1 \leq p \leq n_1$. Thus the claim was established. □

![Fig.2: An example of partition counted by $Q_{3,2}(6, 6 : 19)$.](image)

The following is Proposition 3.1 for $k = 2, i = 1$.

**Lemma 4.2** Fix non-negative integers $n, n_1, N$ such that $2n_1 \leq n - 1$. Then there exists a one-to-one
correspondence between a partition of \( N \) of the form

\[
N = n_1(n_1 + 1) + \sum_{p=1}^{n_1} a_p, \quad n - 2n_1 - 1 \geq a_1 \geq \cdots \geq a_{n_1} \geq 0, \quad (4.2)
\]

and a partition counted by \( Q_{4,1}([\frac{n+2}{2}], [\frac{n-1}{2}]; n_1; N) \).

**Proof.** Let us recall that all \( n_1 \) successive ranks of partition counted by \( Q_{4,1}([\frac{n+2}{2}], [\frac{n-1}{2}]; n_1; N) \) are equal to 1 or 2. Hence the number of each right angle is at least 2. In this case we can construct a Ferrers graph of such a partition as follows. If the \( p \)th right angle has \( d_p - 1 \) \( (\geq 2) \) dots, we put \( \lceil \frac{d_p+2}{2} \rceil \) dots on the horizontal part of the \( p \)th right angle including the dot on the main diagonal line, and \( \lceil \frac{d_p-1}{2} \rceil \) dots on the vertical part of the \( p \)th right angle including the dot on the main diagonal line. Now we can count the number of dots by removing the \( n_1 \times (n_1 + 1) \) rectangle in the top-left corner, and by counting the number of dots on the \( p \)th row to the right of the rectangle plus that on the \( p \)th column below the rectangle. Then we obtain a partition of \( N \) into \( n_1(n_1 + 1) \) and at most \( n_1 \) positive integers less than or equal to \( n - 2n_1 - 1 \). Thus the claim was verified. \( \square \)

![Fig.3: An example of partition counted by \( Q_{4,1}(7, 6 : 20) \).](image)

Next we consider the first special cases of Proposition 3.2. The following is Proposition 3.2 for \( k = 2, i = 2 \).

**Lemma 4.3** Fix non negative integers \( n, n_1, N \) such that \( n_1 \leq n \). Then there exists a one-to-one correspondence between a partition of \( N \) of the form

\[
N = n_1^2 + 2 \sum_{p=1}^{n_1} b_p, \quad n - n_1 \geq b_1 \geq \cdots \geq b_{n_1} \geq 0, \quad (4.3)
\]

and a partition counted by \( Q_{2,2}(n, n; n_1; N) \).

**Proof.** Let us recall that \( Q_{2,2}(n, n; n_1; N) \) counts partitions of \( N \) with the maximal size \( (n, n) \), with \( n_1 \) right angles, and all \( n_1 \) successive ranks are equal to 0. Any Ferrers graph corresponding to such a partition is symmetric with respect to the main diagonal line. Let us count the number of dots as follows. First remove the \( n_1 \times n_1 \) square in the top-left corner, and count the number of
dots on the $p$th row to the right of the square plus that on the $p$th column below the square. In this way, we get a partition of $N$ into $n_1^2$ and at most $n_1$ even positive integers less than or equal to $2(n-n_1)$. Inversely, for any given partition of $N$ of the form (4.3), we can construct a symmetric Ferrers graph by putting $n_1 \times n_1$ dots, and after that for $1 \leq p \leq n_1$ by putting $b_p$ dots on the $p$th row to the right of the square and $b_p$ dots on the $p$th column below the square, respectively. Thus the claim was established. ☐

![Ferrers Graph Example](image)

Fig.4: An example of partition counted by $Q_{2,2}(6,6:17)$.

Remark. For a symmetric Ferrers graph, every right angle has odd number of dots. Hence $Q_{2,2}(n,n;q)$ gives the generating function of partitions into distinct odd positive integers less than or equal to $2n-1$:

$$Q_{2,2}(n,n;q) = \prod_{j=1}^{n}(1 + q^{2j-1}).$$

The following is Proposition 3.2 for $k = 2, i = 1$.

**Lemma 4.4** Fix non negative integers $n,n_1,N$ such that $n_1 \leq n$. Then there exists a one-to-one correspondence between a partition of $N$ of the form

$$N = n_1(n_1 + 1) + 2 \sum_{p=1}^{n_1} b_p, \quad n - n_1 \geq b_1 \geq \cdots \geq b_{n_1} \geq 0,$$

and a partition counted by $Q_{3,1}(n+1,n;n_1;N)$.

**Proof.** Let us recall that all the successive ranks of a partition counted by $Q_{3,1}(n+1,n;n_1;N)$ are equal to 1. Hence in this case, we can count the number of dots by removing the $n_1 \times (n_1 + 1)$ rectangle in the top-left corner, instead of the $n_1 \times n_1$ square. Thus the claim was verified. ☐
5 Proof of Main Propositions

Proof of Proposition 3.1. We prove Proposition 3.1 by induction. We already showed the case \( k = 1 \) by (2.12), and the case \( k = 2 \) by Lemmas 4.1 and 4.2. Now, fix a set \((k, i)\) such that \( k \geq 3, 2 \leq i \leq k \), and suppose that the claim of Proposition 3.1 holds for \((k - 1, i - 1)\): i.e., for any \( n', n_2, N' \) there exists one-to-one correspondence between partition of \( N' \) counted by \( Q_{2k-i,i-1}(\lfloor n' + 1 + k - i \rfloor, \lfloor n' - k + i \rfloor; n_2; N') \) and that of the form

\[
N' = n_2^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1} + \sum_{j=2}^{k-1} \sum_{p=1}^{a_{j-1}^{(i-1)}} a_p^{(j)},
\]

(5.1)

where

\[
n_2 \geq \cdots \geq n_{k-1} \geq n_k = 0, \quad 2(n_2 + \cdots + n_{k-1}) \leq n' - k + i,
\]

and

\[
n' - 2(n_2 + \cdots + n_j) + a_{i-1}^{(k-1)} \geq a_1^{(j)} \geq \cdots \geq a_{n_j-n_{j+1}}^{(j)} \geq 0, \quad 2 \leq j \leq k - 1.
\]

(5.2)

Fix one of graphs \( G \) of partition of \( N' \) with the maximal size \( (\lfloor n' + 1 + k - i \rfloor, \lfloor n' - k + i \rfloor) \), with \( n_2 \) right angles, and all \( n_2 \) successive ranks belong to the interval \([-i + 3, 2k - i - 2]\). In what follows we consider two cases, separately. Let us distinguish them, say, the Case A and the Case B. For the Case A, we wish to transform \( G \) into \( G^{(3)} \) of a partition of

\[
N = N' + n_1^2 + a_1^{(1)} + \cdots + a_{n_1-n_2}^{(1)}, \quad a_1^{(1)} \geq \cdots \geq a_{n_1-n_2}^{(1)} \geq 0,
\]

(5.3)
with the maximal size \( ([n_{1}+k-1], [n_{1}+k-1]) \), with \( n_{1} \geq n_{2} \) right angles, and all \( n_{1} \) successive ranks belong to the interval \([-i+2, 2k-i-1]\), where \( 2 \leq i \leq k \). When \( i = 2 \), we have another possibility: i.e., we can transform \( G \) into a partition of

\[
N = N' + n_{1}(n_{1} + 1) + a^{(1)}_{1} + \cdots + a^{(1)}_{n_{1}-n_{2}}, \quad a^{(1)}_{1} \geq \cdots \geq a^{(1)}_{n_{1}-n_{2}} \geq 0,
\]

with the maximal size \( ([n_{1}+k-1], [n_{1}+k-1]) \), with \( n_{1} \geq n_{2} \) right angles, and all \( n_{1} \) successive ranks belong to the interval \([1, 2k-2]\). This is the case B. (See 2nd step below in detail.)

We notice that now \( n, n_{1} \) and \( N \) are fixed while \( n', n_{2} \) and \( N' \) should be regarded as variables. In fact, we will show \( n' = n - 2n_{1} \) (resp. \( n' = n - 2n_{1} - 1 \)) for the Case A (resp. Case B), later.

The transformation from \( G \) to \( G^{(3)} \) consists of three steps. The first two steps will increase \( N' \) by \( n_{2}^{2} \) (resp. \( n_{1}(n_{1} + 1) \)) for the Case A (resp. Case B), in the last step we will corporate \( n_{1} - n_{2} \) non negative integers.

1st step. In this step, we transform \( G \) to \( G^{(1)} \) as follows. If \( r_{1}(G) \geq k - i + 1 \) (resp. \( \leq k - i \)), add \( n_{2} \) (resp. \( n_{2} - 1 \)) dots to the right of the first row, and add \( n_{2} - 1 \) (resp. \( n_{2} \)) dots below the first column. If \( r_{2}(G) \geq k - i + 1 \) (resp. \( \leq k - i \)), add \( n_{2} - 1 \) (resp. \( n_{2} - 2 \)) dots to the right of the second row, and add \( n_{2} - 2 \) (resp. \( n_{2} - 1 \)) dots below the second column. In general, If \( r_{l}(G) \geq k - i + 1 \) (resp. \( \leq k - i \)), add \( n_{2} - (l-1) \) (resp. \( n_{2} - l \)) dots to the right of the \( l^{th} \) row, and add \( n_{2} - l \) (resp. \( n_{2} - (l-1) \)) dots below the \( l^{th} \) column. The graph thus obtained is \( G^{(1)} \). This is also a Ferrers graph, because by the construction the number of dots on the \( l^{th} \) row (resp. column) of \( G^{(1)} \) are not larger than that of the \((l-1)^{st}\) row (resp. column).

Note that the number of dots increase by \((2n_{2} - 1) + (2n_{2} - 3) + \cdots + 1 = n_{2}^{2}\), and \(-i + 2 \leq r_{l}(G^{(1)}) \leq k - i - 1\), or \( k - i + 2 \leq r_{l}(G^{(1)}) \leq 2k - i - 1\), for \( 1 \leq l \leq n_{2} \).

Fig. 6 is an example for \((k - 1, i - 1) = (2, 1)\). In this case Fig. 6 gives \( G^{(1)} \).

![Graph](image)

Fig.6: The resulting graph after 1st step obtained from Fig. 3

Remark. Suppose that \(-i + 2 \leq r_{l}(G^{(1)}) \leq k - i - 1\) or \( k - i + 2 \leq r_{l}(G^{(1)}) \leq 2k - i - 1\), for \( 1 \leq l \leq n_{2} \). Then if \( r_{l}(G^{(1)}) \geq k - i + 2 \) (resp. \( k - i - 1 \)), remove \( n_{2} - (l-1) \) (resp. \( n_{2} - l \)) dots on the \( l^{th} \) row, and remove \( n_{2} - l \) (resp. \( n_{2} - (l-1) \)) dots on the \( l^{th} \) column. This is the original \( G \). Thus 1st step is invertible.
2nd step. From this step, we have to consider the Case A and B separately.

Case A. Now we wish to add additional \((n_1 - n_2) \times (n_1 - n_2)\) square to this graph. For that purpose, add \(n_1 - n_2\) dots to the right of the \(l\)th row and below the \(l\)th column, where \(1 \leq l \leq n_2\). Then we can add an \((n_1 - n_2) \times (n_1 - n_2)\) square in the top-left corner of the \(n_2\)th proper subgraph. This resulting graph is \(G^{(2)}\). Now the total increase of the number of dots is

\[
n_2^2 + 2n_2(n_1 - n_2) + (n_1 - n_2)^2 = n_1^2.
\]

This accounts the additional \(n_1^2\) in (5.3).

Note that at this stage \(-i + 2 \leq r_l(G^{(2)}) \leq k - i - 1\) or \(k - i + 2 \leq r_l(G^{(2)}) \leq 2k - i - 1\) for \(1 \leq l \leq n_2\), and \(r_l(G^{(2)}) = 0\) for \(n_2 + 1 \leq l \leq n_1\).

Fig.7 is an example of \(G^{(2)}\) obtained from Fig.6, where we set \(n_1 - n_2 = 2\).

Case B. When \(i = 2\), we have two choices to add an \((n_1 - n_2) \times (n_1 - n_2)\) square or an \((n_1 - n_2) \times (n_1 - n_2 + 1)\) rectangle. The latter one corresponds to the Case B. In this case for \(1 \leq l \leq n_2\), we add \(n_1 - n_2 + 1\) dots to the right of the \(l\)th row and add \(n_1 - n_2\) dots below the \(l\)th column. After that we add an \((n_1 - n_2) \times (n_1 - n_2 + 1)\) rectangle at the \(n_2\)th proper subgraph. This resulting graph is \(G^{(2)}\). Now the total increase of the number of dots is

\[
n_2^2 + n_2(n_1 - n_2) + n_2(n_1 - n_2 + 1) + (n_1 - n_2)(n_1 - n_2 + 1) = n_1^2 + n_1.
\]

This accounts the additional \(n_1(n_1 + 1)\) in (5.4).

Note that at this stage \(1 \leq r_l(G^{(2)}) \leq k - 2\) or \(k + 1 \leq r_l(G^{(2)}) \leq 2k - 2\) for \(1 \leq l \leq n_2\), and \(r_l(G^{(2)}) = 1\) for \(n_2 + 1 \leq l \leq n_1\).

Fig.8 represents \(G^{(2)}\) obtained from Fig.6, where we set \(n_1 - n_2 = 2\).

Remark. By the construction, \(G^{(2)}\) has \(n_1\) right angles. Since we added at least one dots for the first \(n_2\) right angles at 1st step, the smallest number \(l\) such that \(l\)th subgraph is an \((n_1 - l) \times (n_1 - l)\) square (resp. \((n_1 - l) \times (n_1 - l + 1)\) rectangle) is equal to \(n_2\), for the Case A (resp. Case B). Remove
$n_2$th proper subgraph of $G^{(2)}$, and then remove $n_1-n_2$ dots on each column and $n_1-n_2$ (resp. $n_1-n_2+1$) dots on each row for the Case A (resp. Case B). Then we obtain $G^{(1)}$. Thus 2nd step is invertible.

3rd step. Now we are in a position to corporate $n_1-n_2$ non negative parts $a_1^{(1)} \geq \cdots \geq a_{n_1-n_2}^{(1)} \geq 0$. For simplicity let us denote them as $a_1 \geq \cdots \geq a_{n_1-n_2} \geq 0$. This step consists of $n_1-n_2$ substeps, each of which corresponds to the procedure of adding $a_p$ dots, where $1 \leq p \leq n_1-n_2$.

In the argument presented below, $c$ denotes $k-i$ (resp. $k-1$) for the Case A (resp. Case B). For a Ferrers graph corresponding to a certain partition, if the graph obtained by adding or removing a dot is a still Ferrers graph, then such a manipulation is called admissible.

1st substep. We add $a_1$ dots to $G^{(2)}$ following the Rules. First we set $l = n_2+1$.

Rule I($l$). If $r_l(G^{(2)}) < c$ (resp. $> c+1$), we add dots to the right of the $l$th row (resp. below the $l$th column) whenever admissible, until the total number of dots reaches $a_1$ or the $l$th successive rank reaches $c$. If we can add $a_1$ dots in this way, we go to the next substep.

Rule II($l$). Starting from the $l$th row we add a dot on the $l$th row and column in turn, whenever admissible. In other words, we add a dot to the right of the $l$th row if admissible. Then we add a dot below the $l$th column, if admissible. After that we add a dot to the right of the $l$th row, if admissible, and so on. If we can add $a_1$ dots in this way, we go to the next substep.

Note that at this stage, the $l$th successive rank takes the values $c+1$ and $c$, in turn.

Rule III($l$). When $r_{l-1}(G^{(2)}) < c$ (resp. $> c+1$), it may happen that the manipulation of adding a dot to the right of the $l$th row (resp. below the $l$th column) is not admissible. In such a case we add dots below the $l$th column (resp. to the right of $l$th row) whenever admissible. If we can add $a_1$ dots in this way, we go to the next substep.

Unfortunately, however, maybe we cannot add a dot below the $l$th column (resp. to the right of the $l$th row) any more without breaking the admissibility, before finishing to add $a_1$ dots. Then we reset $l = n_2$ and repeat the above manipulations. In general when we cannot add a dot any more following the Rule III($l$), without breaking the admissibility, we replace $l$ by $l-1$ and repeat the above manipulations. In this way, eventually we can add $a_1$ dots. Note that we will have no chance to follow the Rule III(1).

The following figure gives the way how to add $a_1 (=16)$ dots to Fig.3.
2nd substep. The aim of this substep is to add $a_2$ dots. Replace $n_2$ by $n_2 + 1$ and $a_1$ by $a_2$. Then repeat the above procedure. Note that we can add up to $a_1$ by the construction. Graphically speaking, $p$th added dot in this substep locates below or to the right of $p$th added one in the last substep.

$p$th substep. In general, after $(p-1)$th substep, we repeat this procedure under the replacement of $n_2$ by $n_2 + 1$ and $a_{p-1}$ by $a_p$.

It is evident from the construction that all the successive ranks of the resulting graph $G^{(3)}$ lie in the interval $[-i + 2, 2k - i - 1]$ (resp. $[1, 2k - 2]$) for the Case A (resp. Case B).

Now we would like to consider the maximal size of the original graph $G$ and the maximal value of $a_1$ for fixed $k, i, n_1$ and $n$.

Case A. Let $\mu_1$ and $\nu_1$ be the length of the first row and column of the original graph $G$, respectively. In 1st step, these length increase by $n_2$ and $n_2 - 1$ (resp. $n_2 - 1$ and $n_2$) if $\mu_1 - \nu_1 \geq c + 1$ (resp. $\mu_1 - \nu_1 \leq c$). In 2nd step, they both increase by $n_1 - n_2$. Thus after 2nd step the length of the first row and column increase by $n_1$ and $n_1 - 1$ (resp. $n_1 - 1$ and $n_1$) if $\mu_1 - \nu_1 \geq c + 1$ (resp. $\mu_1 - \nu_1 \leq c$). Hence we have $\mu_1 + n_1 \leq \left\lfloor \frac{n+1+k-i}{2} \right\rfloor$ and $\nu_1 + n_1 \leq \left\lfloor \frac{n-k+i}{2} \right\rfloor$. The maximal values of $\mu_1$ and $\nu_1$ are $\left\lceil \frac{n'+1+k-i}{2} \right\rceil$ and $\left\lceil \frac{n'-k+i}{2} \right\rceil$, respectively. In other words, $n'$ is the possible maximal number that satisfies both $\left\lceil \frac{n'+1+k-i}{2} \right\rceil + n_1 \leq \left\lfloor \frac{n+1+k-i}{2} \right\rfloor$ and $\left\lceil \frac{n'-k+i}{2} \right\rceil + n_1 \leq \left\lfloor \frac{n-k+i}{2} \right\rfloor$. Therefore we conclude that $n' = n - 2n_1$. Consequently (5.2) reads as

\[ n - 2n_1 - 2(n_2 + \cdots + n_j) - a_{i-1j-1}^{(k-1)} \geq a_1^{(j)} \geq \cdots \geq a_{n_j-n_{j+1}}^{(j)} \geq 0, \tag{5.5} \]

which implies $a_{ij}^{(k)} = a_{i-1j-1}^{(k-1)}$ for $2 \leq i \leq k, 2 \leq j \leq k - 1$.

Next let us determine the maximal value of $a_1$. Every time we add a dot, we project each dot to the first row (resp. column) orthogonally if we added it on the $l$th row (resp. column) following

\[
\begin{array}{c@{\qquad}c@{\qquad}c@{\qquad}c@{\qquad}c}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & 1 \\
\bullet & \bullet & \bullet & 2 \\
\bullet & 5 \\
8 \\
9 \\
12 \\
13 \\
15 \\
\end{array}
\]

Fig.9: The order of addition of $a_1 (= 16)$ dots.
one of the Rules I(l), II(l) and III(l). The lattice site \((l_1, l_2)\) refers to the intersection point of \(l_1\)th column and \(l_2\)th row. Following one of the Rule I(l), II(l) and III(l), we can put a dot at \((m, l)\) (resp. \((l, m)\)) if and only if \((m, l)\) (resp. \((l, m)\)) is not occupied yet, all \((l_1, l)\) with \(l_1 < m\) (resp. \((l, l_2)\) with \(l_2 < m\) are already occupied, and \(m \leq r\), where \(r\) is the number of dots on the \((l-1)\)st row (resp. column). Thus \(m > n_1\) because \(G^{(2)}\) has an \(n_1 \times n_1\) square in the top-left corner. When we put dots on the \((l-1)\)st row (resp. column) following one of the Rules I(l – 1), II(l – 1) and III(l – 1), we begin by adding a dot at \((r+1, l-1)\) (resp. \((l-1, r+1)\)). Consequently, projected dots occupy the first row (resp. column) starting from \((n_1 + 1, 1)\) (resp. \((1, n_1 + 1)\)) to outside. (See Fig. 8) This observation immediately derive that

\[ n - 2n_1 \geq a_1 \geq \cdots \geq a_{n_1-n_2} \geq 0, \]  

which implies \(a^{(k)}_{l_1} = 0\) for \(2 \leq i \leq k\). Two relations \((5.5)\) reproduce the finiteness condition \((3.3)\) with \((1.6)\) for \(2 \leq i \leq k\).

**Case B.** Let \(\mu_1\) and \(\nu_1\) be the length of the first row and column, respectively. By the parallel argument above, after 2nd step the length of the first row and column increase by \(n_1 + 1\) and \(n_1 - 1\) (resp. \(n_1\) and \(n_1 - 1\)) if \(\mu_1 - \nu_1 \geq k - 1\) (resp. \(\mu_1 - \nu_1 \leq k - 2\)). In this case, \(n'\) is the maximal number that satisfies both \(\left[\frac{n'+k-1}{2}\right] + n_1 + 1 \leq \left[\frac{n+k}{2}\right]\) and \(\left[\frac{n'-k+2}{2}\right] + n_1 \leq \left[\frac{n-k+1}{2}\right]\). Therefore we conclude that \(n' = n - 2n_1 - 1\). Consequently \((5.2)\) becomes

\[ n - 2n_1 - 1 - 2(n_2 + \cdots + n_j) - a^{(k-1)}_{1j-1} \geq a^{(j)}_{1j} \geq \cdots \geq a^{(j)}_{n_j-n_j+1} \geq 0, \]  

which implies \(a^{(k)}_{1j} = a^{(k-1)}_{1j-1} + 1\) for \(2 \leq j \leq k - 1\). As for the maximal value of \(a_1\), from the parallel argument above, we obtain

\[ n - (2n_1 + 1) \geq a_1 \geq \cdots \geq a_{n_1-n_2} \geq 0, \]  

which implies \(a^{(k)}_{11} = 1\). Two relations \((5.6)\) reproduce the finiteness condition \((3.3)\) with \((1.6)\) for \(i = 1\).

Thus the remaining work is to show that for any graph \(G^{(3)}\) with the maximal size \([\left[\frac{n+1+k-i}{2}\right], \left[\frac{n-k+i}{2}\right]\]) with \(n_1\) right angles, and all \(n_1\) successive ranks lie in the interval \([-i+2, 2k - i - 1]\), we can extract information of \(n_1 - n_2\) and \(a_1 \geq \cdots \geq a_{n_1-n_2} \geq 0\) by reversing 3rd step. Because by taking into account remarks at the end of 1st step and 2nd step, we can reconstruct \(G\), which reduces to the assumption of the induction.

Suppose that we add a dot on the \(l\)th right angle following the Rule II(l), the successive rank of the \(l\)th right angle takes the values \(c + 1\) and \(c\), in turn. This will break when the length of the \(l\)th shorter side relative to \(c\) reaches that of the \((l-1)\)st shorter side relative to \(c\). In that case, we have to begin to add dots on the \(l\)th right angle following the Rule III(l). When we add dots on the
the common length and direction, and the \((l-1)\)st shorter side relative to \(c\) is longer than the \(l\)th one. Special consideration is required when we begin to add dots following the Rule I\((n_2 + p)\) in \(p\)th substep. In this case we always add dots to the right of the \((n_2 + p)\)th row because \(r_{n_2+p}(G^{(2)}) = 0\) (resp. = 1) \(\leq c\) for the Case A (resp. Case B), and thus the length of the \((n_2 + p)\)th column is equal to a given number \(n_1\).

Therefore we conclude that the last added dot among \(a_p\) \((1 \leq p \leq n_1 - n_2)\) dots is located at the \(l\)th right angle, if \(l\) satisfies at least one of the following four conditions:

1. \(r_l(G^{(3)}) = c\), or \(c + 1\);
2. The \(l\)th shorter side relative to \(c\) has the same length and direction as those of the \((l-1)\)st shorter side relative to \(c\);
3. The \(l\)th longer side relative to \(c\) has the same length and direction as those of the \((l+1)\)st longer side relative to \(c\), and the \(l\)th shorter side relative to \(c\) is longer than the \((l+1)\)st shorter side relative to \(c\);
4. The length of the \(l\)th column is equal to \(n_1\).

From this observation, we can reverse 3rd step as follows. For any given graph \(G^{(3)}\) counted by \(Q_{2k+1-i,i}(\frac{n+1+k-i}{2}, \frac{n-k+i}{2}; n_1; N)\), we scan right angles from the most outer one. Whenever we encounter a right angle which satisfies at least one of the above four conditions, then we mark that right angle. The first marked right angle signifies the position of the last added dot of \(a_1\) dots. The second marked right angle signifies the position of the last added dot of \(a_2\) dots. In general, The \(p\)th marked right angle signifies the position of the last added dot of \(a_p\) dots. The total number of right angles we marked is equal to \(n_1 - n_2\).

Let us denote the \(p\)th marked right angle by the \(l_p\)th right angle. We begin by removing \(a_{n_1-n_2}\) dots. Set \(p = n_1 - n_2\). If the \(l_p\)th successive rank is less than \(c\) (resp. larger than \(c + 1\)), then we remove dots on \(l_p\)th column (resp. row) until the \(l_p\)th successive rank reaches \(c\), whenever admissible. This is the reverse manipulation of the Rule III\((l_p)\). After that we remove a dot on the \(l_p\)th column and row in turn, such that the \(l_p\)th successive rank takes values \(c + 1\) and \(c\), in turn. This is the reverse manipulation of the Rule II\((l_p)\). In the case we cannot remove a dot on the \(l_p\)th row (resp. column) any more without breaking the admissibility, we remove dots on the \(l_p\)th column (resp. row) whenever admissible. This is the reverse manipulation of the Rule I\((l_p)\). When we cannot remove a dot on the \(l_p\)th column (resp. row) any more without breaking the admissibility, we repeat the above manipulations by replacing \(l_p\) by \(l_p + 1\). We continue this procedure until we remove all dots on the \((n_2 + p)\)th right angle except the ones within the \(n_1 \times n_1\) square (resp. \(n_1 \times (n_1 + 1)\) rectangle) in the top-left corner for the Case A (resp. Case B). The total number of dots we remove is equal to \(a_p\). After that we repeat this procedure under the replacement of \(p\) by \(p - 1\). In this way, we can
determine $0 \leq a_{n_1-n_2} \leq \cdots \leq a_1$. Thus this step is invertible.

Therefore, the claim of this Proposition was established. \hfill \square

We obtain Theorem \ref{thm:1.1} as a Corollary of Propositions \ref{prop:2.8} and \ref{prop:3.1}.

**Proof of Proposition \ref{prop:3.2}**. We already showed $k = 2$ by Lemmas \ref{lem:4.3} and \ref{lem:4.4}. Fix a set $(k, i)$ such that $k \geq 3, 2 \leq i \leq k$. Suppose that for any non negative integers $n', n_2, N'$, there exists a one-to-one correspondence between partition of $N'$ counted by $Q_{2k-i-1,i-1}(n'+k-i, n'; n_2; N')$ and that of the form

$$N' = n_2^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1} + \sum_{j=2}^{k-2} \sum_{p=1}^{n_j-n_{j+1}} a_p^{(j)} + 2 \sum_{p=1}^{n_{k-1}} b_p, \quad (5.9)$$

where

$$n_2 \geq \cdots \geq n_{k-1} \geq 0, \quad n_2 + \cdots + n_{k-1} \leq n',$$

and

$$2n' - 2(n_2 + \cdots + n_j) + \beta_{i-1,j-1}^{(k-1)} \geq a_1^{(j)} \geq \cdots \geq a_{n_j-n_{j+1}}^{(j)} \geq 0, \quad 2 \leq j \leq k-2,$$

$$n' - (n_2 + \cdots + n_{k-1}) \geq b_1 \geq \cdots \geq b_{n_{k-1}} \geq 0. \quad (5.10)$$

We can prove this Proposition in a perfectly parallel way as we proved Proposition \ref{prop:3.1}. The transformation rules for $(k, i)$ in the present case is exactly the same as the one for $(k, i)$ given in the proof of Proposition \ref{prop:3.1}. The only difference is the evaluation of the finiteness conditions.

Here we also discuss them for the Case A and the Case B, separately.

**Case A**. By the parallel argument to before, after 2nd step the length of the first row and column increase by $n_1$ and $n_1 - 1$ (resp. $n_1 - 1$ and $n_1$) if the first successive rank is larger or equal to $c + 1$ (resp. less than or equal to $c$). Hence $n'$ is the maximal value that satisfies both $n' + k - i + n_1 \leq n + k - i$ and $n' + n_1 \leq n$. Therefore we conclude that $n' = n - n_1$. Consequently (5.10) reads as

$$2(n - n_1) - 2(n_2 + \cdots + n_j) + \beta_{i-1,j-1}^{(k-1)} \geq a_1^{(j)} \geq \cdots \geq a_{n_j-n_{j+1}}^{(j)} \geq 0,$$

$$n - n_1 - (n_2 + \cdots + n_{k-1}) \geq b_1 \geq \cdots \geq b_{n_{k-1}} \geq 0, \quad (5.11)$$

which implies $\beta_{i-1,j-1}^{(k)} = \beta_{i-1,j-1}^{(k-1)}$, for $2 \leq i \leq k, 2 \leq j \leq k-2$.

As for the maximal value of $a_1$, from the parallel argument to before we have

$$n + (n + k - i) - 2n_1 \geq a_1 \geq \cdots \geq a_{n_1-n_2} \geq 0,$$

which implies $\beta_{i-1}^{(k)} = k - i$. Two relations (5.11), (5.12) reproduce the finiteness condition (3.6) with (1.13) for $2 \leq i \leq k$. 

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**Case B.** By the parallel argument for the Case A, we again obtain \( n = n' + n_1 \). Consequently \((5.10)\) becomes \((5.11)\), which implies \( \rho_{1j}^{(k)} = \rho_{1j-1}^{(k-1)} \).

As for the maximal value of \( a_1 \), from the parallel argument above, we obtain

\[
n + (n + k - 1) - (2n_1 + 1) \geq a_1 \geq \cdots \geq a_{n_1-n_2} \geq 0,
\]

which implies \( \rho_{11}^{(k)} = k - 2 \). Two relations \((5.11,5.13)\) reproduce the finiteness condition \((3.6)\) with \((1.13)\) for \( i = 1 \).

Therefore, the claim of this Proposition was established. \( \square \)

We obtain Theorem 1.2 as a Corollary of Propositions 2.9 and 3.2.

### 6 Discussion

Some groups [4, 5, 6] found expressions of Virasoro characters in terms of fermionic sum representation, by using Bethe ansatz. As a byproduct, they obtained the Rogers–Ramanujan type identities including conjectures. Such intimate connection between physics and the Rogers–Ramanujan type identities is really worth surprising. As was mentioned in Introduction, our original aim is to prove several conjectures appeared in [6]. We will discuss this matter and wish to prove those mathematically in a separate paper.

We obtain the graphical proofs for Propositions 3.1 and 3.2 by translating Burge’s correspondence [19, 20, 21] into language of partitions. Burge’s interpretation for the multiple sums on the RHS’s of \((1.3,1.9)\) reminds us the space of states of the CTM Hamiltonian of the generalized Hard Hexagon model [22, 23]. It is also interesting to study relation among our graphical method, Burge’s correspondence, the theory of the crystal base [26], etc.

We wish to add a few words to conclude the present paper. Lemma 2.6 was one of key lemmas to evaluate the LHS’s of the polynomial identities. This lemma implies that for \( \mu - \nu \leq a - 1 \)

\[
p_{a,b}(\mu, \nu; \lambda; N) = p(\mu - \sum_{i=1}^{\lambda} a_i, \nu + \sum_{i=1}^{\lambda} a_i; N - \sum_{i=1}^{\lambda} (2i - 1)a_i),
\]

where \( a_i = a \) (resp. \( b \)) when \( i \) is odd (resp. even). We notice that the RHS of \((6.1)\) for any \( \mu, \nu \) satisfies exactly the same recursion relation as the top half of \((2.3)\).

Let us introduce the integrated partition functions

\[
p(\mu; N) = \sum_{\nu \geq 0} p(\mu, \nu; N),
\]

\[
p_{a,b}(\mu; \lambda; N) = \sum_{\nu \geq 0} p_{a,b}(\mu, \nu; \lambda; N).
\]
Bressoud’s map given in [18] establishes a graphical one-to-one correspondence between partitions counted by $p_{a,b}(\mu; \lambda; N)$ and those counted by $p(\mu - \sum_{i=1}^{\lambda} a_i; N - \sum_{i=1}^{\lambda} (2i-1)a_i)$. Unfortunately, his map does not give a one-to-one correspondence between partitions counted by $p_{a,b}(\mu, \nu; \lambda; N)$ and those counted by $p(\mu - \sum_{i=1}^{\lambda} a_i, \nu + \sum_{i=1}^{\lambda} a_i; N - \sum_{i=1}^{\lambda} (2i-1)a_i)$, under the condition $\mu - \nu \leq a - 1$. Thus it is still an open problem to show (6.1) graphically.

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Note Added

After finishing this work, G. E. Andrews informed us that Theorem 1 in [25] includes Propositions 2.8 and 2.9 as special cases. We have also received [26] that contains an independent proof of Theorem 1.1.

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