Fixed Point Actions for Lattice Fermions

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The fixed point actions for Wilson and staggered lattice fermions are determined by iterating renormalization group transformations. In both cases a line of fixed points is found. Some points have very local fixed point actions. They can be used to construct perfect lattice actions for asymptotically free fermionic theories like QCD or the Gross-Neveu model. The local fixed point actions for Wilson fermions break chiral symmetry, while in the staggered case the remnant $U(1)_c \otimes U(1)_o$ symmetry is preserved. In addition, for Wilson fermions a nonlocal fixed point is found that corresponds to free chiral fermions. The vicinity of this fixed point is studied in the Gross-Neveu model using perturbation theory.

1. Wilson Fermions

The continuum limit of a lattice field theory is defined at a fixed point of the renormalization group. The lattice models on a renormalized trajectory emanating from the fixed point are free of cut-off effects and hence have perfect lattice actions. Recently, Hasenfratz and Niedermayer realized that perfect actions can be constructed explicitly for asymptotically free theories [1]. In addition, in the 2-d nonlinear $\sigma$-model the renormalization group transformation can be optimized such that the fixed point action is extremely local. This is essential for numerical simulations. The question arises if fixed point actions for fermionic theories are local as well [2]. Since the fixed point of an asymptotically free theory is close to the Gaussian fixed point this question can be studied perturbatively, to lowest order even in the free theory. The corresponding calculation for a free scalar field was done long ago by Bell and Wilson [3].

Let us consider free Wilson fermion fields $\Psi$ and $\bar{\Psi}$ with the action $S(\bar{\Psi}, \Psi)$ on a hypercubic lattice $\Lambda$, which is then blocked to a lattice $\Lambda'$ of doubled lattice spacing. Then each point $x' \in \Lambda'$ corresponds to a hypercubic block of $2^d$ points $x \in \Lambda$ and each point $x$ belongs to exactly one block $x'$ (we denote this by $x \in x'$). The block transform-

*Based on two talks presented by the authors

\[ \exp(-S'(\bar{\Psi}', \Psi')) = \int D\bar{\Psi}' D\Psi \exp(-S(\bar{\Psi}, \Psi)) \times \]

Figure 1. Blocking of a 2-d lattice.
At this point it is convenient to introduce lattices that may converge to a fixed point action one generates actions on coarser and coarser by iterating the renormalization group transformation corresponds to a chirally invariant Grassmann $\delta$-function, whereas for finite $a$ the $\eta_{x'}$ term breaks chiral symmetry explicitly. The parameter $b$ is needed to renormalize the blocked fermion field.

Starting from a point on the critical surface and iterating the renormalization group transformation one generates actions on coarser and coarser lattices that may converge to a fixed point action $S[\bar{\Psi}, \Psi]$. To find the fixed point we make the ansatz

$$S[\bar{\Psi}, \Psi] = i \sum_{x,y} \rho_\mu(x-y)\bar{\Psi}_x \gamma_\mu \Psi_y + \sum_{x,y} \lambda(x-y)\bar{\Psi}_x \Psi_y. \quad (2)$$

We go to momentum space and integrate out $\bar{\Psi}, \Psi$ and $\eta, \eta'$ to obtain the blocked action $S'[\bar{\Psi}', \Psi']$. At this point it is convenient to introduce

$$\alpha_\mu(k) = \frac{\rho_\mu(k)}{\rho(k)^2 + \lambda(k)^2}, \beta(k) = \frac{\lambda(k)}{\rho(k)^2 + \lambda(k)^2}. \quad (3)$$

After $n$ renormalization group steps one finds

$$\alpha_\mu^{(n)}(k) = (b^2/2^d)^n \sum_l \alpha_\mu\left(\frac{k + 2\pi l}{2^n}\right) \times \prod_\nu \left(\frac{\sin(k_\nu/2)}{2^n \sin((k_\nu + 2\pi l_\nu)/2^{n+1})}\right)^2,$$

$$\beta^{(n)}(k) = (b^2/2^d)^n \sum_l \beta\left(\frac{k + 2\pi l}{2^n}\right) \times \prod_\nu \left(\frac{\sin(k_\nu/2)}{2^n \sin((k_\nu + 2\pi l_\nu)/2^{n+1})}\right)^2 + \frac{1 - (b^2/2^d)^n}{a(1 - b^2/2^d)}. \quad (4)$$

The exponent $(d-1)/2$ is the dimension of a free fermion field in $d$ dimensions, and the blocked fermion field gets renormalized appropriately. At the fixed point one finds

$$\alpha_\mu^*(k) = \sum_{l_\mu \in \{1, 2, 3, ..., 2^n\}} k_\mu + 2\pi l_\mu \prod_\nu \left(\frac{\sin(k_\nu/2)}{k_\nu + 2\pi l_\nu}\right)^2, \quad \beta^*(k) = 2/a. \quad (6)$$

Note that the result is independent of the parameter $r$ in the original action, as it should since the Wilson term is irrelevant. One finds a whole line of fixed points parametrized by $a$, and one may optimize $a$ such that the fixed point action is as local as possible. In $d = 1$ the fixed point action has only nearest neighbor couplings when $a = 4$. This value is close to optimal also in $d = 2$. In fig.2 the corresponding function $\rho_1^*(z)$ is displayed in coordinate space. Like in the scalar field case it is extremely local and very promising for numerical simulations of perfect fermion actions.

For finite $a$ both the renormalization group transformations and the fixed point actions break...
chiral symmetry and the fixed point actions are local. For $a = \infty$, on the other hand, the renormalization group transformation is chirally symmetric. As a consequence, there exists a chirally invariant fixed point with $\rho_1^{\mu}(k) = \alpha^\mu(k)/\alpha^\mu(k)^2$, $\lambda^*(k) = 0$, for which the Nielsen-Ninomiya theorem would suggest that the fermion spectrum is doubled. Fortunately, this is not the case because the corresponding fixed point action is non-local and the theorem does not apply. Hence, the resulting continuum theory describes free chiral fermions.

The poles at the boundary of the Brillouin zone give rise to the action’s nonlocality. SLAC fermions also have a nonlocal action which is known to cause problems in perturbation theory. For fixed point fermions, however, the nonlocality should be acceptable, because it arises naturally due the integration over the high momentum modes of the fermion field.

The real issue is to find a fixed point that describes interacting chiral fermions. In particular, one should switch on a small gauge coupling and investigate the vicinity of the nonlocal fixed point of the free theory. Some time ago Rebbi suggested a lattice action for chiral fermions with a similar nonlocality, which unfortunately suffered from spurious ghost states. In fact, there is a no-go theorem due to Pelissetto that excludes lattice chiral fermion constructions using certain nonlocal actions. Still, we are optimistic that in case of the nonlocal fixed point action the renormalization group is clever enough to circumvent these arguments, although they certainly apply to generic man-made actions.

So far we have studied the vicinity of the nonlocal fixed point in the Gross-Neveu model. The 4-Fermi interaction is linearized by an auxiliary scalar field $\Phi$ with a Yukawa coupling $y$. In momentum space the Yukawa coupling takes the form

$$\int d^2k_1d^2k_2 \overline{\psi}(k_1)\sigma(k_1,k_2)\psi(k_2)\Phi(-k_1 - k_2).$$

The coupling $\sigma(k_1,k_2)$ is a matrix in Dirac space. Now the renormalization group is iterated by also blocking the field $\Phi$ to coarser and coarser lattices. At the fixed point of the coupled system one finds to leading order in $y$

$$\sigma^*(k_1,k_2) = y \rho_1^\mu(k_1)\gamma^\mu \sum_{l_1,l_2} \frac{k_{1\nu} + 2\pi l_{1\nu}}{(k_1 + 2\pi l_1)^2} \gamma^\nu \times$$

$$\frac{k_{2\nu} + 2\pi l_{2\nu}}{(k_2 + 2\pi l_2)^2} \gamma^\nu \rho_1^\nu(k_2)\gamma^\nu \times$$

$$\prod_{\lambda} \frac{\sin((k_{1\lambda} + k_{2\lambda})/2)}{(k_{1\lambda} + 2\pi l_{1\lambda} + k_{2\lambda} + 2\pi l_{2\lambda})/2} \times$$

$$\frac{\sin((k_{1\lambda}/2)}{(k_{1\lambda} + 2\pi l_{1\lambda}/2)} \times \frac{\sin((k_{2\lambda}/2)}{(k_{2\lambda} + 2\pi l_{2\lambda}/2)}.$$  

Before attacking the gauge theory we will use this vertex function to check if ghost states spoil the continuum limit in this nonlocal chirally invariant formulation of the Gross-Neveu model. This investigation is in progress.

2. Staggered Fermions

An alternative formulation of lattice fermions uses staggered fermion fields. This has the advantage that the cut-off theory has a remnant chiral symmetry. The class of lattice fermions which
is described by the ansatz (12) does not include staggered fermions, since their action is invariant only under translations by an even number of lattice spacings. Staggered fermions can be treated analogously, but the renormalization group transformations are different and new properties of the fixed point actions arise.

As usual we view the staggered fermion variables $\bar{\chi}^i_x$, $\chi^i_x$ as fields with pseudoflavors $i = 1 + n_1 + 2n_2 + \ldots + 2^{d-1}n_d$ ($n_\mu \in \{0, 1\}$) located at the $2^d$ corners of the hypercubes centered at points $x$ which form a lattice with spacing 2. Staggered fermions have various symmetries, among them a $U(1)_c \otimes U(1)_\rho$ remainder of chiral invariance, an analog of charge conjugation $C_0$, and shift symmetries containing translations by one lattice spacing. To be consistent with these symmetries the renormalization group transformation has to be modified. At the end, one wants to reconstruct Dirac spinors from the pseudoflavors. Therefore it is important not to mix the corners of the hypercubes in the renormalization group transformation.

Kalkreuter, Mack and Speh have proposed a suitable blocking scheme with block factor 3 (in general it must be odd) (3). One among $3^d$ block centers $x$ remains a block center after the renormalization group step. We denote it by $x'$.

The $x'$ form a lattice of spacing 6. Each pseudoflavor builds its individual block variable, and every $\chi^i_{x'}$ on the fine lattice contributes to exactly one $\chi^i_x$ on the coarse lattice. The block transformation is illustrated in fig.4. First we apply a $\delta$-function renormalization group transformation using this blocking scheme

$$\exp(-S'[\bar{\chi}', \chi']) = \int D\bar{\chi}D\chi \exp(-S[\bar{\chi}, \chi]) \times$$

$$\prod_{x',i} \delta(\chi'^i_{x'} - \frac{b}{3^d} \sum_{x \in x'} \bar{\chi}^i_x) \delta(\bar{\chi}^i_{x'} - \frac{b}{3^d} \sum_{x \in x'} \chi^i_x)$$

$$= \int D\bar{\chi}D\chi D\eta D\bar{\eta} \exp(-S[\bar{\chi}, \chi]) \times$$

$$\exp\{ \sum_{x' \in \Lambda'} \sum_i [(\bar{\chi}^i_{x'} - \frac{b}{3^d} \sum_{x \in x'} \bar{\chi}^i_x) \bar{\eta}^i_{x'} +$$

$$\bar{\eta}^i_{x'} (\chi^i_{x'} - \frac{b}{3^d} \sum_{x \in x'} \chi^i_x)\} \}. \quad (9)$$

For each pseudoflavor we have defined coarse grained auxiliary Grassmann fields $\bar{\eta}^i_{x'}$, $\eta^i_{x'}$. Clearly this is analogous to the chirally invariant transformation eq.(12) for Wilson fermions with $a = \infty$.

Now we look for a general ansatz, analogous to (12) which describes the action after a number of renormalization group steps

$$S[\bar{\chi}, \chi] = i \sum_{x,y} \sum_{i,j} \bar{\chi}^i_x \rho_{ij} (x-y) \chi^j_y. \quad (10)$$

Here $\rho$ depends only on the distance $x-y$ between two hypercube centers, because the action is invariant against translations by an even number of lattice spacings.

The standard action as well as the renormalization group transformation (12) respect various symmetries, which therefore also hold for (10). This specifies the form of the matrix $\rho$ as follows: the $U(1)_c \otimes U(1)_\rho$ symmetry implies $\rho_{ij} \neq 0$ only if the pseudoflavors $i$ and $j$ belong to different sublattices (for one pseudoflavor the sum of lattice point coordinates is even and for the other one it is odd), and $C_0$ invariance requires $\rho_{ij}(x-y) = -\rho_{ji}(y-x)$. Shifts by one lattice spacing yield further relations among the remain-
ing degrees of freedom; they relate sets of $2^{d-1}$ matrix elements. As for Wilson fermions it is natural to go to momentum space and to work with the inverse matrix $\alpha(k) = \rho(k)^{-1}$. In particular, for $d = 2$ the various symmetries imply

$$\alpha(k) = \begin{pmatrix}
0 & \alpha_1(k) & \alpha_2(k) & 0 \\
\alpha_1(k) & 0 & 0 & -\alpha_2(k) \\
\alpha_2(k) & 0 & 0 & \alpha_1(k) \\
0 & -\alpha_2(k) & \alpha_1(k) & 0
\end{pmatrix}. \quad (11)$$

In higher dimensions also couplings along certain space diagonals are permitted by the symmetries; however, they are absent in the standard action and it turns out that the renormalization group transformation does not activate them either. Hence, in general one can parametrize the inverse of $\rho(k)$ by $d$ functions $\alpha_{\mu}(k)$ with $\alpha_{\mu}(-k) = -\alpha_{\mu}(k)$.

After $n$ renormalization group steps we obtain

$$\alpha_{\mu}^{(n)}(k) = (b^2/3^d)^n \sum_l \alpha_{\mu}(\frac{k+2\pi l}{3^n})(-1)^l \times$$

$$\prod_{\nu} \left( \frac{\sin(k_{\nu}/2)}{3^n \sin((k_{\nu} + 2\pi l_{\nu})/3^n/2)} \right)^2, \quad (12)$$

with $l_{\mu} \in \{1, 2, 3, ..., 3^n\}$. The sign factor $(-1)^l_{\mu}$ arises because certain terms are antiperiodic with respect to $k_{\mu}$. As for Wilson fermions a nontrivial fixed point is reached only for one particular value of $b$; in this case $b = 3^{(d-1)/2}$ in accordance with dimensional considerations.

It turns out that the $U(1)_c \otimes U(1)_a$ invariant fixed point action is local as one infers from figs.5 and 6. This is an important qualitative difference to the chirally invariant nonlocal fixed point for Wilson fermions. In the staggered case there is no contradiction to the Nielsen-Ninomiya theorem since the $U(1)_c \otimes U(1)_a$ symmetry does not imply full chiral invariance of the Dirac spinors that one can reconstruct. Adding a mass term of the auxiliary fields (as it was done to optimize the locality of the fixed point action for Wilson fermions) would break the remnant chiral symmetry and would therefore destroy an essential advantage of the staggered fermion formulation. Still, we can improve the locality without loss of

Figure 5. The function $i\rho_{12}^*(z)$ for the staggered fermion fixed point in coordinate space ($d = 2$).

Figure 6. The function $\rho_{12}^*(k)$ for the staggered fermion fixed point action ($d = 2$).
Figure 7. The function $i\rho_{12}^*(z)$ for the optimally local fixed point ($d = 2, a = 9/4$).

If we add a kinetic term of the auxiliary fields, again suppressed by a parameter $1/a$. At the fixed point one obtains

$$\alpha^*_\mu(k) = 2 \sum_{l \in \mathbb{Z}} \frac{k_\mu + 2\pi l_\mu}{(k + 2\pi l)^2} (-1)^{l_\nu} \times \prod_\nu \left( \frac{\sin(k_\nu/2)}{k_\nu/2 + \pi l_\nu} \right)^2 + \frac{9}{8a} \sin(k_\mu/2).$$

As long as no gauge fields are present one can still show that the partition function remains invariant under renormalization group transformations. In $d = 1$ the fixed point action is an optimally local nearest neighbor interaction when $a = 9/4$. Numerically it turns out that this choice is also successful for $d = 2$ as one sees in fig.7. However, if we switch on a gauge field this transformation is not allowed any more and we are restricted to the $\delta$-function transformation ($a = \infty$) from above.

In summary we repeat that the fixed point actions we obtain for staggered fermions are local and $U(1)_e \otimes U(1)_\sigma$ invariant, and are hence well suited for numerical simulations.

Next we intend to search for the fixed point action in the Gross-Neveu model using staggered fermions. Here the kinetic term of the auxiliary field can be used and the location of the critical surface is obvious. Since the Gross-Neveu model is asymptotically free, there is one (weakly) relevant direction (to lowest order it is marginal). When one determines this direction one obtains the analog of the line of classical perfect actions in the bosonic case [1]. Then one may adopt the concept of Hasenfratz and Niedermayer and follow this (straight) line far enough to small correlation lengths where numerical simulations become feasible. If the simulated action is close to a perfect quantum action one can extract continuum physics to a good approximation with only little numerical effort. However, in the fermionic case it remains to be seen if an (almost) perfect action can be simulated with the Hybrid-Monte-Carlo algorithm, or if one is restricted to less efficient numerical methods.

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