THE CLASSIFICATION OF SURFACES WITH $p_g = 0$, $K^2 = 6$ AND NON BIRATIONAL BICANONICAL MAP

MARGARIDA MENDES LOPES, RITA PARDINI

Abstract. Let $S$ be a minimal surface of general type with $p_g = 0$ and $K^2 = 6$, such that its bicanonical map $\varphi : S \rightarrow \mathbb{P}^6$ is not birational. The map $\varphi$ is a morphism of degree $\leq 4$ onto a surface. The case of $\deg \varphi = 4$ is completely characterized in [Topology, 40 (5) (2001), 977–991] and the present paper completes the classification of these surfaces. It is proven that the degree of $\varphi$ cannot be equal to 3, and the geometry of surfaces with $\deg \varphi = 2$ is analysed in detail. The last section contains three examples of such surfaces, two of which appear to be new.

2000 Mathematics Subject Classification: 14J29.

1. Introduction

A minimal surface of general type with $p_g = 0$ satisfies the inequalities $1 \leq K^2 \leq 9$. It is known that for $K^2 \geq 2$ the image of the bicanonical map $\varphi$ is a surface ([X1]) and that for $K^2 \geq 5$ the bicanonical map is always a morphism of degree $\leq 4$ ([Ml2]). Furthermore, if $K^2 = 9$ then $\varphi$ is birational, whilst if $K^2 = 7, 8$ then $\deg \varphi \leq 2$ and those surfaces for which $\deg \varphi = 2$ can be characterized (see [MP2], [MP3], [Pa2]).

A complete classification of the minimal surfaces with $p_g = 0$, $K^2 = 6$ and bicanonical map of degree 4 was given in [MP1], namely showing that all such surfaces are Burniat surfaces (see [P2]).

The results in the present paper complete the classification of the minimal surfaces of general type with $p_g = 0$, $K^2 = 6$ for which the bicanonical map is not birational. We prove the following:

Theorem 1.1. Let $S$ be a minimal surface of general type with $p_g(S) = 0$ and $K^2_S = 6$ for which the bicanonical map $\varphi$ is not birational. Then the degree of $\varphi$ is either 2 or 4 and the image of $\varphi$ is a rational surface.

Theorem 1.2. Let $S$ be a minimal surface of general type with $p_g(S) = 0$ and $K^2_S = 6$ for which the bicanonical map $\varphi$ has degree 2. Then:

(i) there is a fibration $f : S \rightarrow \mathbb{P}^1$ such that the general fibre $F$ of $f$ is hyperelliptic of genus 3 and $f$ has 4 or 5 double fibres;
(ii) the bicanonical involution of $S$ induces the hyperelliptic involution on $F$.

In Section 4 it is also shown that both possibilities in (i) do occur. The first example is due to Inoue ([In]) whilst the two others were, to our knowledge,
not known previously. These examples are constructed as bidouble covers of rational surfaces.

We want to point out the striking similarity of the results of Theorem 1.2 to the case of surfaces with $p_g = 0, K^2 = 7, 8$, for which also the non-birationality of the bicanonical map implies the existence of a genus 3 fibration with multiple fibres (see [MP3]).

Section 2 is devoted to proving Theorem 1.1 whilst in Section 3 we prove Theorem 1.2. Finally, in Section 4 we describe the examples.

Acknowledgements. The present collaboration takes place in the framework of the European contract EAGER, No. HPRN-CT-2000-00099. It was partially supported by the 1999 Italian P.I.N. “Geometria sulle Varietà Algebriche” and by the “Finanziamento Pluriannuale” of CMAF. The first author is a member of CMAF and of the Departamento de Matemática da Faculdade de Ciências da Universidade de Lisboa and the second author is a member of GNSAGA of CNR.

Notation and conventions. We work over the complex numbers; all varieties are assumed to be compact and algebraic.

We do not distinguish between line bundles and divisors on a smooth variety, using the additive and the multiplicative notation interchangeably. Linear equivalence of divisors is denoted by $\equiv$ and numerical equivalence by $\sim$. The remaining notation is standard in algebraic geometry.

2. The degree

In this section we prove Theorem 1.1. We start by stating some general facts:

**Lemma 2.1.** Let $S$ be a minimal surface of general type with $p_g(S) = 0$ and $K^2_S = 6$ for which the bicanonical map $\varphi$ is not birational. Then the degree of $\varphi$ is less than or equal to 4 and the image of $\varphi$ is a rational surface.

**Proof.** Denote by $\Upsilon$ the image of $\varphi$, by $r$ the degree of $\Upsilon$ and by $d$ the degree of the bicanonical map. Recall that $K^2_S = 6$ implies that $h^0(S, 2K_S) = 7$ and $(2K_S)^2 = 24$. Since $|2K_S|$ is base point free by Reider’s theorem ([Re]) and $\Upsilon \subset \mathbb{P}^6$ is a non-degenerate surface, one has the following possibilities for the pair $(d, r)$: $(4, 6)$, $(3, 8)$ and $(2, 12)$ and therefore $d \leq 4$. In the two first cases $\Upsilon$, being a surface of degree $\leq 10$ in $\mathbb{P}^6$ with $p_g = q = 0$, is of course a rational surface (cf. Lemma 1.4 and Remark 1.5 of [Re1]). For the last case $\Upsilon$ is a rational surface by Theorem 3 of [X2].

By Lemma 2.1 to prove Theorem 1.1 we have to exclude the case $\deg \varphi = 3$. In this case, as seen in the proof of Lemma 2.1, the bicanonical image is a linearly normal rational surface $\Upsilon$ of degree 8 in $\mathbb{P}^6$. We now establish some properties of $\Upsilon$. Notice that surfaces of degree 8 in $\mathbb{P}^6$ have been studied
classically by Castelnuovo and later by P. Ionescu [Io], in the smooth case, and by E. Halanay [Hl], in the normal case.

**Proposition 2.2.** Let \( \mathcal{Y} \) be a linearly normal rational surface of degree 8 in \( \mathbb{P}^6 \), let \( \rho: X \to \mathcal{Y} \) be the minimal desingularization of \( \mathcal{Y} \) and let \( H := \rho^*O_{\mathcal{Y}}(1) \).

Then \( \mathcal{Y} \) has isolated singularities and one of the following occurs:

(i) \(-K_X\) is nef and big, \( H = -2K_X \) and \( h^0(X, -K_X) = 3 \);

(ii) \( X \) has a pencil \( |C| \) of rational curves such that \( HC = 2 \);

(iii) \( X \) has a pencil \( |C| \) of rational curves such that \( HC = 3 \).

**Proof.** For the reader’s convenience we break the proof into steps:

**Step 1:** The general hyperplane section of \( \mathcal{Y} \) is smooth of genus 3.

Let \( H \in |H| \) be general and set \( H' := \rho(H) \), so that \( H \to H' \) is the normalization map. The curve \( H' \subset \mathbb{P}^5 \) has degree 8, hence by Castelnuovo’s theorem (cf. [Ci]) its geometrical genus \( g(H) \) is less than or equal to 3. Since \( q(X) = 0 \), the restriction map \( H^0(X, H) \to H^0(H, H) \) is surjective and therefore \( h^0(H, H) = 6 \). On the other hand, Riemann–Roch gives \( 6 = h^2(H, H) \geq 8 + 1 - g(H) \), and so \( g(H) \geq 3 \). Therefore \( g(H) = 3 \) and by Castelnuovo’s theorem, \( H' \) is smooth and \( \mathcal{Y} \) has only isolated singularities.

**Step 2:** The divisor \( K_X + H \) is nef.

Since \( q(X) = p_g(X) = 0 \), the restriction map \( H^0(X, K_X + H) \to H^0(H, K_H) \) is an isomorphism. Since \( g(H) = 3 \) by Step 1, it follows \( h^0(X, K_X + H) = 3 \). Write \( |K_X + H| = |M| + D \), where \( |M| \) is the moving part and \( D \) is the fixed part. Since \( |K_X + H| \) cuts out the complete linear system \( |K_H| \), which is free for general \( H \), necessarily we have \( HC = 0 \) for every component \( \Gamma \) of \( D \). Then by the index theorem one has \( \Delta^2 < 0 \) for every divisor \( \Delta \) whose support is contained in the support of \( D \).

Assume that \( K_X + H \) is not nef and let \( \theta \) be an irreducible curve such that \( (K_X + H)\theta < 0 \). Since \( \theta M \geq 0 \), necessarily \( \theta D < 0 \) and so \( \theta \) is a component of \( D \). Therefore \( H\theta = 0 \) and \( \theta^2 < 0 \). The conditions \( \theta(K_X + H) < 0 \) and \( \theta H = 0 \) imply \( \theta K_X < 0 \), so that \( \theta \) is a \(-1\)-curve. Now, because \( H\theta = 0 \), this is a contradiction to the assumption that \( \rho: X \to \mathcal{Y} \) is the minimal desingularization of \( \mathcal{Y} \).

Since \( H^2 = 8 \), by the adjunction formula we conclude that \( K_X H = -4 \).

**Step 3:** If \( M^2 = 2 \), then \( D = 0 \) and \( H = -2K_X \). In particular \(-K_X\) is nef and \( h^0(X, -K_X) = 3 \).

In this case we have \( 2M \sim H \) by the index theorem. Thus \( DM = 0 \) and by the above remark it follows that \( D = 0 \). Furthermore, because \( X \) is a
rational surface, $2M \sim H$ implies $2M \equiv H$ and so the equality $K_X + H = M$ yields $M = -K_X$, $H = -2K_X$.

**Step 4:** If $|M|$ is composite with a pencil $|C|$, then the general curve $C$ is rational and $HC = 2$.

Since $h^0(X, M) = 3$, in this case we have $|M| = |2C|$. Thus $2 \geq M^2 = 4C^2$, hence $M^2 = C^2 = 0$. Since $HM = 4$, one has $HC = 2$. Since $K_X + H$ is nef by Step 1, we get $2 \geq (K_X + H)^2 \geq 2C(K_X + H)$, namely $K_XC \leq -1$. Since the general $C$ is irreducible and $C^2 = 0$, one has $K_XC = -2$ and $C$ is a smooth rational curve.

**Step 5:** If $M^2 = 1$, then there is a pencil $|C|$ on $X$ such that the general $C$ is rational and $HC = 3$.

In this case $\phi_M$ is a birational morphism $X \rightarrow \mathbb{P}^2$ by the proof of Step 4. Since a general curve $M$ in $|M|$ is smooth rational, we have $-2 = K_XM + M^2$ and so $K_XM = -3$. From $MH = 4$ we conclude $M(K_X + H) = 1 = M^2$ and so $MD = 0$, implying $D = 0$. Now the equalities $1 = (K_X + H)^2 = K_X^2 - 8 + 8$ mean that $K_X^2 = 1$ and thus $X$ is $\mathbb{P}^2$ blown-up in 8 points, possibly infinitely near. Let $E_1, ..., E_8$ be the corresponding exceptional divisors. Since $K_X = -3M + E_1 + \cdots + E_8$, we have $H = 4M - E_1 - \cdots - E_8$ and there is at least a pencil $|C|$ of rational curves on $X$ (corresponding to the lines in $\mathbb{P}^2$ passing through one of the blown-up points of $\mathbb{P}^2$) such that $HC = 3$. □

In order to prove Theorem [14] we need also the following technical result.

**Lemma 2.3.** Let $S$ be a minimal surface of general type with $p_g(S) = 0$, $K_S^2 = 6$ and let $|F|$ be a rational pencil on $S$ such that $F^2 = 1$, $K_SF = 3$. Then $h^1(S, 2K_S - F) = 0$.

**Proof.** We argue by contradiction. Assume that $h^1(S, 2K_S - F) \neq 0$. Then by the Riemann-Roch theorem we conclude that $h^0(S, 2K_S - F) \geq 4$. Considering now the long exact sequence obtained from

\[(2.1) \quad 0 \rightarrow \mathcal{O}_S(2K_S - 2F) \rightarrow \mathcal{O}_S(2K_S - F) \rightarrow \mathcal{O}_F(2K_S - F) \rightarrow 0\]

and using the fact that, by the Riemann-Roch theorem for curves, one has $h^0(F, \mathcal{O}_F(2K_S - F)) = 3$, we see that $h^0(S, 2K_S - 2F) \geq 1$.

Since $(K_S - F)^2 = 1$ and we are assuming that $h^1(S, 2K_S - F) \neq 0$, by the Kawamata-Viehweg’s vanishing theorem we conclude that $K_S - F$ is not nef. Let $\theta$ be an irreducible curve such that $(K_S - F)\theta < 0$. Notice that in particular $F\theta > 0$. Then for any effective divisor $G \in |2K_S - 2F|$ the curve $\theta$ is a component of $G$ and $G\theta \leq -2$. Write $G = \theta + A$. Since $F$ is nef and $FG = 4$, we have $F\theta \leq 4$ and $FA = 4 - F\theta \leq 3$. Furthermore, because $4 = G^2 = \theta G + AG$, we have $AG = 4 - \theta G \geq 6$.

We now show that this does not occur by examining the various possibilities for $F\theta$.

If $F\theta = 4$, then $K_S\theta \leq 3$ and so, because $\theta$ is an irreducible curve, $\theta^2 \geq -5$. On the other hand by the index theorem $A^2 < 0$, because $F^2 = 1$
and $FA = 0$. Since $6 \leq AG = A^2 + A\theta$, one has $A\theta \geq 7$. Then $-2 \geq \theta G = \theta^2 + A\theta$ implies \(\theta^2 \leq -9\), a contradiction.

If $F\theta = 3$, then $K\theta \leq 2$, and as above $\theta^2 \geq -4$. Since $FA = 1$ by the index theorem $A^2 \leq 1$ and as above we obtain a contradiction to $\theta^2 \geq -4$.

If $F\theta = 2$, as before we conclude that $\theta^2 \geq -3$, $A^2 \leq 4$, implying that $A\theta \geq 2$ which leads us to the same contradiction.

Finally, if $F\theta = 1$ then $K_S\theta = 0$ and $\theta$ is a $-2$-curve. In this case an easy calculation shows that $A^2 = 6$ and that $\theta \sim K_S$. We can write the equality of $\mathbb{Q}$-divisors: $K_S - F = \frac{1}{3}A + \frac{1}{2}\theta$. Since $\theta$ is a normal crossings divisor and $\frac{1}{2}A = \frac{1}{2}K_S$ is nef and big, by the vanishing theorem of Kawamata-Viehweg we obtain $h^1(S, K_S + K_S - F) = 0$, contradicting our assumption. $\Box$

We are finally ready to prove Theorem 1.1.

Proof of Theorem 1.1. In view of Lemma 2.1 we need to show that the case $\deg \varphi = 3$ does not occur. By Lemma 2.1 and its proof, in this case the bicanonical image $\Upsilon$ is a rational surface of degree 8 in $\mathbb{P}^6$. We prove the theorem by excluding all the possibilities for $\Upsilon$ described in Proposition 2.2.

In the first place notice that case (iii) is trivially impossible. In fact if $\Upsilon$ contains a pencil of rational curves of degree 3, then the pull back $|F|$ of this pencil satisfies $2K_S F = 9$, which is impossible.

Now we consider case (ii), i.e. $\Upsilon$ contains a pencil of rational curves of degree 2. This pencil gives rise to a pencil $|F|$ in $S$ such that $2K_S F = 6$, i.e. $K_S F = 3$. Hence $F^2 \geq 0$ is odd, and so by the index theorem $F^2 = 1$ and $g(F) = 3$. Since $2K_S F = 6$ and the image of $F$ is a conic, we conclude that the restriction map $H^0(S, 2K_S) \rightarrow H^0(F, \mathcal{O}_F(2K_S))$ is not surjective, hence $h^1(S, 2K_S - F) \neq 0$, contradicting Lemma 2.3.

So we are left with case (i), namely $H = -2K_X$. Consider the Stein factorization $X \xrightarrow{\eta} \overline{X} \xrightarrow{\varphi} \Upsilon$ of $\rho$; $X \rightarrow \Upsilon$. Since $-K_X = \frac{1}{2}H$ is nef, the map $\eta$: $X \rightarrow \overline{X}$ contracts only $-2$-curves. Hence $\overline{X}$ is a normal surface whose singularities are rational double points. In particular $\overline{X}$ is Gorenstein and $K_X = \eta^*K_{\overline{X}}$. By the normality of $\overline{X}$, the bicanonical map $\varphi$: $S \rightarrow \Upsilon$ induces a morphism $\overline{\varphi}$: $S \rightarrow \overline{X}$ such that $2K_S = \overline{\varphi}^*(-2K_{\overline{X}})$. Hence $\xi := \overline{\varphi}^*(-K_{\overline{X}}) - K_S$ is a non trivial $2$-torsion element of $\text{Pic}(S)$ and $h^0(S, K_S + \xi) \geq 3$ by Proposition 2.2 (i). Let $Y \rightarrow S$ be the étale double cover given by $\xi$. The standard formulae for double covers yield:

$$\chi(Y) = 2, \quad K_Y^2 = 12, \quad q(Y) \geq 2.$$  

On the other hand by Corollary 2.2 of [MP1] we have $K_Y^2 \geq 16(q(Y) - 1) \geq 16$, a contradiction. This completes the proof. $\Box$

3. The case of degree 2

This section is devoted to the proof of Theorem 1.2.

Throughout all the section we assume that $S$ is a minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 6$ such that the bicanonical map
\( \varphi: S \to \mathbb{P}^6 \) has degree 2 onto its image. We denote by \( \sigma \) the involution of \( S \) induced by \( \varphi \) and by \( \pi: S \to \Sigma := S/\sigma \) the quotient map.

### 3.1 Preliminaries.

Given a smooth projective surface \( Y \) and \( k \) disjoint nodal curves \( C_1, \ldots, C_k \) of \( Y \), we define the binary code \( V \) associated to \( C_1, \ldots, C_k \). Consider the map \( \psi: \mathbb{Z}_2^k \to \text{Pic}(Y)/2\text{Pic}(Y) \) defined by \( (x_1, \ldots, x_k) \mapsto \sum x_i[C_i] \), where \([D]\) denotes the class of a divisor \( D \) in \( \text{Pic}(Y)/2\text{Pic}(Y) \). We define \( V \) to be the kernel of \( \psi \) and we denote by \( r \) its dimension. We say that a curve \( C_i \) appears in \( V \) if there exists \( v = (x_1, \ldots, x_k) \in V \) with \( x_i \neq 0 \). We denote by \( m \) the number of curves \( C_i \) appearing in \( V \). The weight of an element \( v = (x_1, \ldots, x_k) \in V \) is the number of indices \( i \) such that the coordinate \( x_i \) is non zero. It is easy to show that the weights of the elements of \( V \) are divisible by 4.

This situation has been studied in detail in [DMP]. In particular, if we let \( G \) be the abelian group \( \text{Hom}(V, \mathbb{C}^*) \) then there exists a \( G \)--cover \( p: Z \to \Sigma \) branched precisely over the nodes of \( \Sigma \) corresponding to the curves that appear in \( V \). The numerical invariants of \( Z \) can be computed explicitly in terms of \( r, m \) and of the numerical invariants of \( V \). It is sometimes possible to determine \( V \) by studying the properties of \( Z \), and viceversa. For instance, if \( Y \) is a rational surface with \( b_2(Y) \geq 5 \) and the number \( k \) of disjoint \(-2\)--curves is the maximum possible (\( = b_2(Y) - 2 \)), then this technique is used in [DMP] to show that the code \( V \) is the code of \( \text{"doubly even" vectors} DE(s) \), where \( k = 2s \). We recall the definition of \( DE(s) \). Given the code of even vectors \( W = \{ \sum x_i = 0 \} \subset \mathbb{Z}_2^s \), \( DE(s) \) is the image of \( W \) via the injection \( \mathbb{Z}_2^s \to \mathbb{Z}_2^{2s} \) defined by \( (x_1, \ldots, x_s) \mapsto (x_1, x_1, \ldots, x_s, x_s) \).

We are going to study \( V \) in the case in which \( Y \) is the minimal resolution of the quotient surface \( \Sigma \) of \( S \) by the bicanonical involution and the \( C_i \) are the exceptional curves of \( Y \to \Sigma \). We will need the following auxiliary result:

**Lemma 3.1.** Let \( G \) be a finite abelian group, let \( Y \) be a smooth projective variety and let \( p: X \to Y \) be a flat \( G \)--cover with building data \( L_\chi, D_{(H, \psi)} \) (cf. [Pa1]). Let \( \gamma \in \text{Aut}(Y) \) and let \( \tau \in \text{Aut}(G) \) be such that for every pair \((H, \psi)\) one has \( \gamma^*D_{(H, \psi)} = D_{(\tau(H), \psi \circ \tau^{-1})} \) and for every \( \chi \in G^* \) one has \( \gamma^*L_\chi \cong L_{\chi \circ \tau^{-1}} \).

Then there exists an automorphism \( \gamma': X \to X \) that lifts \( \gamma \).

**Proof.** Consider the \( G \)--cover \( p': X' \to Y \) obtained from \( p: X \to Y \) by taking base change with \( \gamma: Y \to Y \):

\[
\begin{array}{ccc}
X' & \xrightarrow{\gamma_1} & X \\
p' \downarrow & & \downarrow p \\
Y & \xrightarrow{\gamma} & Y.
\end{array}
\]

If we define a new action of \( G \) on \( X' \) by letting \( g \in G \) act as \( \tau(g) \) and we denote by \( p'': X'' \to Y \) the \( G \)--cover thus obtained, then by the assumptions \( p'' \) and \( p \) have the same building data. By the main result of [Pa1] there
exists an isomorphism of $G$–covers $\Psi: X \to X''$. The map $\Psi$ can also be regarded as an isomorphism $\Psi: X \to X'$ preserving the covering maps to $Y$ (but not the $G$–action), and the automorphism $\gamma': X \to X$ can be taken to be the composition $\gamma_1 \circ \Psi$.

We return to our surface $S$ and consider the involution $\sigma$ of $S$ induced by the bicanonical map $\varphi$ and the quotient map $\pi: S \to \Sigma := S/\sigma$.

The fixed locus of $\sigma$ is the union of a smooth curve $R$ and of 10 isolated points $P_1, \ldots, P_{10}$ (cf. [MP3, §2]). We set $B := \pi(R)$ and $Q_i := \pi(P_i)$, $i = 1, \ldots, 10$. The surface $\Sigma$ is normal and $Q_1, \ldots, Q_{10}$ are ordinary double points, which are the only singularities of $\Sigma$. Let $Y$ be the bicanonical map $\epsilon: S \to S$ at $P_1, \ldots, P_{10}$ and let $E_i$ be the exceptional curve over $P_i$, $i = 1, \ldots, 10$. It is easy to check that $\epsilon$ induces an involution $\hat{\epsilon}$ of $S$ whose fixed locus is the union of $R_0 := \epsilon^{-1} R$ and of $E_1, \ldots, E_{10}$. Denote by $\hat{\pi}: \tilde{S} \to Y := \hat{S}/\hat{\epsilon}$ the projection onto the quotient and set $B_0 := \pi(R_0), C_i := \pi(E_i), i = 1, \ldots, 10$. The surface $Y$ is smooth and the $C_i$ are disjoint $-2$–curves. Denote by $\eta: Y \to \Sigma$ the morphism induced by $\epsilon$. The map $\eta$ is the minimal resolution of the singularities of $\Sigma$ and there is a commutative diagram:

$$
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\epsilon} & S \\
\downarrow \hat{\pi} & & \downarrow \pi \\
Y & \xrightarrow{\eta} & \Sigma
\end{array}
$$

We recall that by Lemma 2.1 $Y$ and $\Sigma$ are rational surfaces. The map $\hat{\pi}: \tilde{S} \to Y$ is a flat double cover branched on $B_0$ and on the $C_i$, hence it is given by a relation $2L \equiv B_0 + C_1 + \cdots + C_{10}$. The moving part of the bicanonical system $|2K_S|$ is equal to $\hat{\pi}^*|2K_Y + B_0| = \epsilon^*|2K_S|$. In particular, since $|2K_S|$ is free, $|2K_Y + B_0|$ is also free.

**Lemma 3.2.** One has:

(i) $K_Y^2 \geq -4$;

(ii) $L^2 + K_Y L = -2, K_Y^2 + K_Y L = 0$;

(iii) if $K_Y^2 = -4$, then $B_0^2 = -4, K_Y B_0 = 8$.

**Proof.** Let $\Gamma \subset H^2(Y, \mathbb{C})$ be the subspace generated by the classes of the curves $C_1, \ldots, C_{10}$. Since the intersection matrix $(C_i C_j)$ is negative definite, $\Gamma$ has dimension 10 and the orthogonal subspace $\Gamma^\perp$ has dimension $h^2(Y) - 10 = K_Y^2$. Since there is an injection $\Gamma^\perp \hookrightarrow H^2(S, \mathbb{C})$, we get $4 = h^2(S) \geq -K_Y^2$, namely $K_Y^2 \geq -4$.

By Riemann–Roch and by the standard formulae for double covers, the condition $L^2 + K_Y L = -2$ is equivalent to $\chi(S) = 1$. The condition $K_Y^2 + K_Y L = 0$ expresses the fact that the bicanonical map $\varphi$ factorizes through the cover $\pi: S \to \Sigma$ (cf. [MP3, Proof of Prop. 2.1]). Finally, (iii) follows from (ii) and from the relation $2L \equiv B_0 + C_1 + \cdots + C_{10}$. \qed
3.2. The proof of Theorem 1.2

We start by analysing the code $V$ associated to the curves $C_1, \ldots, C_{10}$ on $Y$.

**Lemma 3.3.** In the above setting:

(i) $r := \dim V \geq 3$, and if equality holds then $K_Y^2 = -4$;
(ii) $V$ is the code of doubly even vectors (cf. §3.1).

**Proof.** Consider the map $\psi: \mathcal{Z}^{10} \to \text{Pic}(Y)/2\text{Pic}(Y)$ introduced in §3.1. The intersection form on $\text{Pic}(Y)$ induces a non degenerate $\mathbb{Z}_2$-valued bilinear form on $\text{Pic}(Y)/2\text{Pic}(Y)$ and the image of $\psi$ is a totally isotropic subspace. Hence:

$$2 \dim \text{Im}\psi \leq \dim_{\mathbb{Z}_2} \text{Pic}(Y)/2\text{Pic}(Y) = 10 - K_Y^2 \leq 14,$$

where the last inequality follows by Lemma 3.2, (i). Thus the dimension of $V = \ker \psi$ is at least 3, and if it is equal to 3 then $K_Y^2 = -4$. This proves (i).

If the number $m$ of curves appearing in $V$ is $\geq 8$, then statement (ii) follows by [DMP, Theorem 3.2].

So assume that $m < 8$. Since $\dim V \geq 3$ and the weights of $V$ are divisible by 4, necessarily $\dim V = 3$ and $m = 7$ (in fact $V$ is the so-called Hamming code).

Set $G := \text{Hom}(V, \mathbb{C}^*) \cong \mathbb{Z}_2^3$ and consider the $G$–cover $p: Z \to \Sigma$ branched on the nodes of $\Sigma$ corresponding to the curves appearing in $V$ (cf. [DMP, §2]). By the formulae in [DMP, Proposition 2.3], $Z$ is a rational surface with $K_Z^2 = -32$ and therefore $b_2(Z) = 42$. The surface $Z$ has 24 nodes. Let $Z' \to Z$ be the minimal desingularization and denote by $C'_1, \ldots, C'_{24}$ the exceptional curves. Arguing as in the proof of (i), one sees that the code $V'$ associated to the $C'_i$ has dimension $\geq 3$. Since $G$ acts freely on the set of nodes of $Z$, the number of curves appearing in $V'$ is divisible by 8. Consider the cover $q: W \to Z$ associated to $V'$. Again by the formulae of [DMP], $W$ is an irregular ruled surface. Consider now the composite map $p' := p \circ q: W \to \Sigma$.

**Claim:** $p'$ is a Galois cover.

**Proof of claim.** We wish to show that any given $\gamma \in G$ can be lifted to $W$. This is easier to prove if one works with flat maps, hence we consider the following diagram, obtained by base change and normalization:

$$
\begin{array}{ccc}
W' & \longrightarrow & W \\
q' \downarrow & & \downarrow q \\
Z' & \longrightarrow & Z.
\end{array}
$$

The surface $W'$ is obtained from $W$ by taking the minimal resolution of its singularities and blowing up the ramification points of $q$. The map $q'$ is flat.

We denote again by $\gamma$ the induced automorphism of $Z'$. Clearly, $\gamma$ induces by pull-back an automorphism of the code $V'$ and, dually, an automorphism $\tau$ of the Galois group $G' := \text{Hom}(V', \mathbb{C}^*)$ of $q'$. If we denote by $D_{(H,\phi)}, L_x$
the building data of \( p' \), then we have \( \gamma^* D_{\langle \tau(H), \psi \rangle} = D_{\langle \tau(H), \psi \circ \tau^{-1} \rangle} \) (cf. Proof of \([\text{DMP}, \text{Prop. 2.1}]\)). Since the surface \( Z' \) is simply connected, the line bundles \( L_{\chi} \) associated to the cover \( q' \) are determined uniquely by the fundamental relations of the cover (cf. \([\text{Pa5}]\)) and thus \( \gamma^* L_{\chi} \cong L_{\chi \circ \tau^{-1}} \). So Lemma 3.1 applies, \( \gamma \) can be lifted to an automorphism \( \gamma' \) of \( W' \) and it is easy to see that \( \gamma' \) induces the required automorphism of \( W \), that we denote again by \( \gamma' \). \( \square \)

Now, let \( \alpha: W \rightarrow B \) be the Albanese pencil. The fibration \( \alpha \) induces a pencil \( g: Z \rightarrow \mathbb{P}^1 \) and, by the Claim, it induces also a pencil of rational curves \( h: \Sigma \rightarrow \mathbb{P}^1 \). We have a commutative diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{p} & \Sigma \\
\downarrow g & & \downarrow h \\
\mathbb{P}^1 & \xrightarrow{\bar{p}} & \mathbb{P}^1.
\end{array}
\]

The map \( \bar{p} \) is a Galois cover with group \( G \), since every element of \( G \), having 0-dimensional fixed locus, acts non-trivially on the set of fibres of \( g \). On the other hand this is impossible, since it is well known that \( \text{Aut}(\mathbb{P}^1) \) has no finite subgroup isomorphic to \( \mathbb{Z}_3 \). So the case in which \( \dim V = 3 \) and \( m = 7 \) cannot occur. This concludes the proof of Lemma 3.3. \( \square \)

**Proposition 3.4.** There exists a fibration \( h: \Sigma \rightarrow \mathbb{P}^1 \) such that:

(i) the general fibre of \( h \) is isomorphic to \( \mathbb{P}^1 \);

(ii) \( h \) has at least \( r + 1 \) double fibres.

Furthermore, \( h \) is uniquely determined by property (i).

**Proof.** The existence of the fibration \( h \) follows by \([\text{DMP}, \text{Theorem 3.2}]\), in view of Lemma 3.3.

To see that \( h \) is unique, consider any fibration \( h': \Sigma \rightarrow \mathbb{P}^1 \) such that the general fibre \( H \) of \( h' \) is isomorphic to \( \mathbb{P}^1 \). As explained in the proof of \([\text{DMP}, \text{Theorem 3.2}]\), if \( p: Z \rightarrow \Sigma \) is the cover associated to the code \( V \), then \( Z \) is an irrational ruled surface. The inverse image of \( H \) in \( Z \) is a disjoint union of smooth rational curves, since \( p \) is unramified in codimension 1. It follows that \( h' \) induces on \( Z \) a fibration in rational curves, which necessarily coincides with the Albanese fibration. Since the cover \( p: Z \rightarrow \Sigma \) is canonically associated to \( \Sigma \), this shows that \( h = h' \). \( \square \)

We denote by \( f: S \rightarrow \mathbb{P}^1 \) the fibration induced by \( h: \Sigma \rightarrow \mathbb{P}^1 \). The general fibre \( F \) of \( f \) is an hyperelliptic curve and \( \sigma \) induces on \( F \) the hyperelliptic involution. Furthermore the fibration \( f \) has at least \( r + 1 \) double fibres corresponding to the \( r + 1 \) double fibres of \( h \).

**Proposition 3.5.** If \( r \geq 4 \) then \( g(F) = 3 \) and the multiple fibres of \( f \) are 5 double fibres.
Proof. This proof is very similar to the proof of Theorem 3.2 of [MP3], but we include it for the reader’s convenience.

Let \( r \geq 4 \). By Proposition 3.4 and the above remark, the fibration \( f \) has at least 5 double fibres.

Suppose that \( f \) has at least 6 double fibres. Let \( \psi : C \to \mathbb{P}^1 \) be the double cover branched over the 6 image points of these double fibres of \( f \). Note that \( C \) is a genus 2 curve. Taking fibre product and normalization, one gets a commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\tilde{\psi}} & S \\
\downarrow f & & \downarrow f \\
C & \xrightarrow{\psi} & \mathbb{P}^1
\end{array}
\]  

(3.2)

The map \( \tilde{\psi} \) is étale, hence \( X \) is smooth and we have:

\[
\chi(O_X) = 2\chi(O_S) = 2, \quad K_X^2 = 2K_S^2 = 12.
\]

The fibrations \( f \) and \( \tilde{f} \) have the same general fibre, which we still denote by \( F \). Notice that the genus \( g(F) \) of \( F \) is odd, since \( f \) has double fibres, and \( > 1 \), since \( S \) is of general type. By [Be2], we have

\[
12 = K_X^2 \geq 8(g(C) - 1)(g(F) - 1) \geq 16,
\]

a contradiction.

Hence \( f \) has exactly 5 double fibres. Then there exists a \( \mathbb{Z}_2 \)-cover \( \psi : C \to \mathbb{P}^1 \) branched on the 5 image points of the double fibres of \( f \) (cf. [Pa1], Proposition 2.1) and again \( g(C) = 2 \). Taking base change and normalization, we obtain a commutative diagram similar to (3.2). In this case \( \tilde{\psi} \) is an étale \( \mathbb{Z}_2 \)-cover, hence \( K_X^2 = 24 \) and \( \chi(O_X) = 4 \). Again [Be2] gives

\[
24 = K_X^2 \geq 8(g(C) - 1)(g(F) - 1) = 8(g(F) - 1),
\]

namely \( g(F) = 3 \), again because \( g(F) \) is odd and \( S \) is of general type.

\[\square\]

Proof of Theorem 1.2. By Proposition 3.5 to finish the proof we have to study the case \( r = \dim V = 3 \). In this case the argument used in the proof of Proposition 3.5 does not give a bound on the genus of \( F \). We obtain such a bound by showing that the system \(|4K_Y + B_0|\) is composed with a pencil \( |C| \) of rational curves that satisfy \( CB_0 = 8, C^2 = 0 \), so that the pull back of \(|C| \) on \( S \) is a genus 3 pencil. Furthermore, we show that \( CC_i = 0 \) for \( i = 1, \ldots, 10 \), and thus \(|C| \) induces a fibration \( h' : \Sigma \to \mathbb{P}^1 \) with rational fibres. By Proposition 3.4, we have \( h = h' \).

By Lemma 3.3 \( r = 3 \) implies \( K_Y^2 = -4 \). We denote by \( H \) a general element of the linear system \(|2K_Y + B_0|\). Note that, since \(|2K_Y + B_0|\) is free, \( H \) is a smooth curve. By Lemma 3.2 (iii), \( H^2 = 12, K_YH = 0 \) and so \( g(H) = 7 \). We analyse the system \(|4K_Y + B_0| = |2K_Y + H| \) by repeated
use of adjunction. For the reader’s convenience we break this analysis into steps:

**Step 1:** The divisor $K_Y + H$ is nef, $h^0(Y, K_Y + H) = 7$ and every curve in $|K_Y + H|$ is 1-connected.

Since $g(H) = 7$, $g_Y(Y) = q(Y) = 0$, the map $H^0(Y, K_Y + H) 	o H^0(H, K_H)$ is an isomorphism, and so $h^0(Y, K_Y + H) = 7$. Write $|K_Y + H| = |M| + D$, where $|M|$ is the moving part and $D$ is the fixed part. Since $|K_Y + H|$ cuts out the complete linear system $|K_H|$, which is free for general $H$, necessarily we have $H^0 = 0$ for every component $\Gamma$ of $D$. Assume that $K_Y + H$ is not nef and let $\theta$ be an irreducible curve such that $(K_Y + H)\theta < 0$. Since $\theta M \geq 0$, necessarily $\theta D < 0$ and so $\theta$ is a component of $D$. Therefore $H\theta = 0$ and by the index theorem $\theta^2 < 0$. The conditions $(K_Y + H) < 0$ and $(\theta M) = 0$ imply $\theta K_Y < 0$, so that $\theta$ is a $-1$-curve. We have $\theta B_0 = 2$, hence $\theta$ is not contained in the smooth curve $B_0$ and the inverse image $\theta'$ of $\theta$ in $S$ is connected and reduced. Then $H\theta = 0$ implies $0 = \theta'((\pi^* H) = \theta'((\pi^* K_S))$ (see diagram 3.1 for the notation). It follows that $p_a(\theta') = 0$. Now the Hurwitz formula applied to the double cover $\theta' \to \theta$ gives $\theta(B_0 + C_1 + \cdots + C_{10}) = 2$, i.e. $\theta C_i = 0$ for $i = 1, \ldots, 10$. Let $Y'$ be the surface obtained from $Y$ by contracting $\theta$ and let $C_1', \ldots, C_{10}'$ be the images of $C_1, \ldots, C_{10}$ in $Y'$. The curves $C_1', \ldots, C_{10}'$ are disjoint $-2$-curves and $K_{X'}^2 = -3$. Arguing as in the proof of Lemma 2.6 we see that the corresponding code has dimension at least 4. It is immediate to check that the code $V$ associated to $C_1, \ldots, C_{10}$ has the same dimension, contradicting the assumption $r = 3$.

So $K_Y + H$ is nef. Furthermore $K_Y + H$ nef and $(K_Y + H)^2 = 8 > 0$ imply that every curve in $|K_Y + H|$ is 1-connected (see, e.g., [Ml1, Lemma 2.6]).

**Step 2:** The general curve $M$ of the linear system $|K_Y + H|$ is irreducible, smooth and $g(M) = 3$.

As above write $|K_Y + H| = |M| + D$, where $|M|$ is the moving part and $D$ the fixed part. Suppose the general curve in $|M|$ is not irreducible. Then, since $h^0(Y, K_Y + H) = 7$ and $Y$ is regular, we can write $|K_Y + H| = |G| + D$, where $|G|$ is a pencil with connected fibres such that $GH = 2$. Since $K_Y + H$ is nef and $(K_Y + H)^2 = 8$, one has $\frac{2}{5} \geq G(K_Y + H) = 6G^2 + GD$. Hence we must have $G^2 = 0$ and $GD = 1$ (by 1-connectedness). So $G(K_Y + H) = 1$. Since $GH = 2$, we have $GK_Y = -1$, contradicting $G^2 = 0$.

So the general curve in $|M|$ is irreducible. Then the image of the map defined by $|M|$ is a non-degenerate surface in $\mathbb{P}^6$, hence of degree $\geq 5$. Since $M^2 \leq (K_Y + H)^2 = 8$, the linear system $|M|$ can not have multiple base points, hence the general curve in $|M|$ is smooth.

Now we want to show that $D = 0$. Since $DH = 0$ and so $K_Y D = MD + D^2$, the number $MD$ is even. In addition, we have $MD \geq 0$ and equality holds if and only if $D = 0$, by the 1-connectedness of the curves in $|K_Y + H|$.
Now $K_Y + H$ nef and $DH = 0$ yield $K_Y D \geq 0$, and so $K_Y M \leq -4$, because $K_Y (K_Y + H) = -4$. So $M^2 \geq 2g(M) + 2$ and so $h^1(M, M) = 0$. We have $K_Y M + M^2 = (K_Y + H)M - HM + M^2 = 2M^2 + MD - 12$, hence $1 - g(M) = -M^2 - \frac{1}{2}MD + 6$. Since $h^0(M, M) = 6$ and $h^1(M, M) = 0$, we obtain by the Riemann-Roch theorem on curves $6 = M^2 - M^2 - \frac{1}{2}MD + 6$, hence $MD = 0$, and so $D = 0$. So $|K_Y + H| = |M|$, implying also that $g(M) = 3$.

**Step 3:** The system $|2K_Y + H| = |K_Y + M|$ is composed with a pencil $|G|$.

Since $M$ is nef and big by Step 1 and Step 2, Kawamata-Viehweg vanishing and Riemann–Roch give $h^0(Y, K_Y + M) = 3$.

By Lemma (ii), $V$ is the code of doubly even vectors. Since dim $V = 3$ by assumption, we may assume that the nodal curves $C_1$, $C_2$ do not appear in the code $V$ and that $C_3 + \cdots + C_{10}$ is divisible by 2 in Pic$(Y)$. Hence the divisors $B_0 - C_1 - C_2$ and $H - C_1 - C_2 = 2K_Y + B_0 - C_1 - C_2$ are also divisible by 2 in Pic$(Y)$ and we may write $H - C_1 - C_2 = 2\Delta$ for some $\Delta \in$ Pic$(Y)$. Then $\Delta^2 = 2$, $K_Y \Delta = 0$ and thus $\chi(\Delta) = 2$ by Riemann–Roch. We have $(K_Y - \Delta)H = -6$, hence $h^2(Y, \Delta) = 0$ and $h^0(Y, \Delta) \geq 2$. Since $Y$ is rational, the adjunction sequence for $\Delta$ gives $h^0(Y, K_Y + \Delta) = h^0(\Delta, K_\Delta) \geq 2$. There is an inclusion $|2(\Delta + \Delta)| \hookrightarrow |K_Y + M|$. Since dim $|K_Y + M| = 2$, the moving part $|G|$ of $|K_Y + \Delta|$ is a pencil and $|K_Y + M|$ is composed with $|G|$.

**Step 4:** The general $G \in |G|$ is isomorphic to $\mathbb{P}^1$, $G^2 = 0$ and $GH = 4$.

By Step 3 and its proof, we can write $|K_Y + M| = |2G| + D$, where $D$ is the fixed part. As in the proof of Step 1, we see that $MG = 0$ for every component $\Gamma$ of $D$, hence $MG = 2$, since $M(K_Y + M) = 4$. Since $M^2 = 8$ and $G$ is nef, the index theorem yields $G^2 = 0$. Thus $K_Y G = MG - HG = 2 - HG$ is even by the adjunction formula and it is $\leq 0$ since the system $|H|$ is birational and $G$ moves. If $K_Y G = 0$ and $HG = 2$, then the general $G$ is smooth of genus 1 and it is mapped by $|H|$ birationally onto a conic, a contradiction. So we have $GH = 4$, $GK_Y = -2$ and the general $G$ is smooth rational.

**Step 5:** $|G|$ induces on $\Sigma$ the fibration $h: \Sigma \to \mathbb{P}^1$ of Proposition 3.4.

By Step 4 and the uniqueness statement in Proposition 3.4, it is enough to show that $|G|$ induces a fibration $\Sigma \to \mathbb{P}^1$, namely that $GC_i = 0$ for $i = 1, \ldots, 10$. Notice that, since $G(K_Y + M) = G(2G + D) = 0$ and $G$ is nef, we have $GT = 0$ for every component $\Gamma$ of $D$. Now assume by contradiction that $GC_i > 0$ for some $i$. Since $0 = C_i(K_Y + M) = 2C_i G + C_i D$, this implies that $C_i$ is contained in $D$ and thus $GC_i = 0$ by the above remark.

**Step 6:** The general fibre $F$ of the fibration $f: S \to \mathbb{P}^1$ induced by $h: \Sigma \to \mathbb{P}^1$ has genus 2.

This follows by applying the Hurwitz formula to a general $G$, since $GB_0 = G(H - 2K_Y) = 8$. \hfill $\Box$
4. THE EXAMPLES

Here we show that Theorem 1.2 is effective, by presenting three examples of surfaces satisfying $K^2 = 6$, $p_g = 0$ and bicanonical map of degree 2. For one of the examples the genus 3 fibration of Theorem 1.2 has 4 double fibres and for the other two it has 5 double fibres. The first example, due to Inoue ([12], Remark 6), is a specialization of a construction of surfaces with $K^2 = 7$ and birational map of degree 2. The construction described here, as a $\mathbb{Z}_2^2$-cover of a singular rational surface, is different from the original one, but it has the advantage of enabling us to compute the degree of the bicanonical map (cf. [MP2], Example 1). The other examples are also built as $\mathbb{Z}_2^2$-covers of a singular rational surface and are, to our knowledge, new examples of surfaces of general type with these invariants.

We start by recalling that, given a smooth surface $Y$, to define a $\mathbb{Z}_2^2$-cover $\pi: X \to Y$ one must specify:

1) divisors $D_1$, $D_2$, $D_3$ of $Y$;

2) line bundles $L_1$, $L_2$ such that $2L_1 \equiv D_2 + D_3$, $2L_2 \equiv D_1 + D_3$.

Notice that if Pic($Y$) has no $2$-torsion then $L_1$ and $L_2$ are uniquely determined by $D_1$, $D_2$ and $D_3$. We set $L_3 := L_1 + L_2 - D_3$. The divisor $D := D_1 + D_2 + D_3$ is the (reduced) branch divisor of $\pi$. There is a decomposition $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus L_1^{-1} \oplus L_2^{-1} \oplus L_3^{-1}$ and the $L_i^{-1}$ are the eigenspaces corresponding to the 3 non trivial characters of $\mathbb{Z}_2^2$. We denote by $\chi_i$ the character corresponding to $L_i^{-1}$ and by $\gamma_i$ the generator of $\ker \chi_i$. The divisor $D_i$ is the image of the divisorial part of the fixed locus of $\gamma_i$. The surface $X$ is smooth if and only if $D_1$, $D_2$, $D_3$ are smooth and $D$ has normal crossings.

Now consider in $\mathbb{P}^2$ a quadrilateral $P_1P_2P_3P_4$ (see Figure 1). We let $P_i$ be the intersection point of the lines $P_1P_2$ and $P_3P_4$ and $P_i$ the intersection point of $P_1P_4$ and $P_2P_3$. Write $\Sigma \to \mathbb{P}^2$ for the blowup of $P_1, \ldots, P_6$, and $e_i$ for the exceptional curves of $\Sigma$ over $P_i$. Denote by $l$ the pull back of a line.

Write $S_1, \ldots, S_4$ for the strict transforms on $\Sigma$ of the sides $P_iP_{i+1}$ of the quadrilateral $P_1P_2P_3P_4$ (we take subscripts modulo 4); these are the only $-2$-curves of $\Sigma$. The anticanonical system $|-K_\Sigma|$ gives a birational morphism onto the symmetric cubic with 4 nodes $V \subset \mathbb{P}^3$. This morphism is precisely the contraction of the $-2$-curves of $\Sigma$ to canonical singularities.

If $A \subset \{P_1, \ldots, P_6\}$ consists of 4 points no three of which are collinear, then the linear system of conics through the points of $A$ gives rise to a free pencil on $\Sigma$; we denote by $f_1$ the strict transform of a general conic through $P_2P_4P_3P_6$, by $f_2$ that of a general conic through $P_1P_3P_5P_6$ and by $f_3$ that of a general conic through $P_1P_2P_3P_4$.

Finally, we write $\Delta_1, \Delta_2, \Delta_3$ for the strict transform of the lines $P_1P_3$, $P_2P_4$ and $P_3P_6$.

The divisors we have introduced satisfy the following relations:

(i) $f_i \equiv \Delta_{i+1} + \Delta_{i+2}$ for all $i \in \mathbb{Z}_3$;
Example 1: Using the above notation, Example 1 of \([\text{MP}2]\) is obtained by setting:

1) \(D_1 = \Delta_1 + f_2 + S_1 + S_2\),
   \(D_2 = \Delta_2 + f_3\),
   \(D_3 = \Delta_3 + f_1 + f'_1 + S_3 + S_4\);

where \(f_1, f'_1 \in |f_1|, f_2 \in |f_2|, f_3 \in |f_3|\) are general curves, and:

2) \(L_1 = 5l - e_1 - 2e_2 - e_3 - 3e_4 - 2e_5 - 2e_6\),
   \(L_2 = 6l - 2e_1 - 2e_2 - 2e_3 - 2e_4 - 3e_5 - 3e_6\)

and we obtain: \(L_3 = 4l - 2e_1 - 2e_2 - 2e_3 - e_4 - e_5 - e_6\).

For every \(i = 1, \ldots, 4\), the (set–theoretic) inverse image of \(S_i\) in \(X\) is the disjoint union of two \(-1\) curves \(E_{i1}, E_{i2}\): contracting these 8 exceptional curves on \(X\) and contracting the \(S_i\) on \(\Sigma\), one obtains a smooth \(\mathbb{Z}^2_2\)–cover \(p: S \to V\) such that \(S\) is of general type with \(p_g(S) = 0\) and \(K^2_S = 7\). The bicanonical map of \(S\) is of degree 2 and the bicanonical involution coincides with \(\gamma_1\). The linear system \(|f_1|\) induces a free pencil \(F\) of hyperelliptic curves of genus 3. The bicanonical involution restricts to the hyperelliptic involution on the general \(F\). The pencil \(|F|\) has 5 double fibres, corresponding to the pull backs of \(f_1, f'_1, \Delta_2 + \Delta_3\) and of the two fibres of \(|f_1|\) containing the \(-2\)–curves.

Now assume that \(f_1, f_2\) and \(f_3\) all pass through a general point \(P\) and that \(f_i\) and \(f_j\) intersect transversely at \(P\) for \(i \neq j\). In other words, in the terminology of \([\text{Ca}]\) we let the branch locus \(D\) acquire a \((1, 1, 1)\) point. Denote by \(p_0: S_0 \to V\) the corresponding \(\mathbb{Z}^2_2\)–cover. The surface \(S_0\) has a

**Figure 1.** The quadrilateral \(P_1P_2P_3P_4\) in \(\mathbb{P}^2\)

(ii) \(-K_S \equiv \Delta_1 + \Delta_2 + \Delta_3 \equiv f_1 + \Delta_1 \equiv f_2 + \Delta_2 \equiv f_3 + \Delta_3\);

(iii) \(\Delta_i S_j = 0\) for all \(i, j\);

(iv) \(\Delta_i f_j = 2\delta_{ij}\) for \(1 \leq i, j \leq 3\).
singularity of type $\frac{1}{4}(1, 1)$ over the image $P'$ of $P$ in $V$. This singularity can be solved by taking base change with the blow up $\tilde{V} \to V$ at $P'$ and normalizing. Let $p: S \to \tilde{V}$ be the cover thus obtained. The exceptional divisor of $S \to S_0$ is a smooth rational curve with self-intersection $-4$. The surface $S$ is smooth of general type with $p_g(S) = 0$ and $K_S^2 = 6$ (see [Ca]).

A computation very similar to the one in Example 1 of [MP2] shows that the bicanonical map of $S$ has degree 2 and that the bicanonical involution coincides with $\gamma_1$. As before, the linear system $|f_1|$ induces on $S$ a free pencil $|F|$ of hyperelliptic curves of genus 3 such that the bicanonical involution restricts to the hyperelliptic involution on the general $F$. Now the pencil $|F|$ has 4 double fibres, corresponding to the pull backs of $f'_1$, of $\Delta_2 + \Delta_3$ and of the two fibres of $|f_1|$ containing the $-2$--curves. Notice that in this case the pull back of $f_1$ contains with multiplicity 1 the exceptional curve of the resolution $S \to S_0$, hence it is not a multiple fibre.

**Example 2:** With the above notation, consider the point $P_7 = \Delta_2 \cap \Delta_3$ and denote by $\Sigma'$ the blow-up of $\Sigma$ at $P_7$ and by $e_7$ the corresponding exceptional divisor.

We denote by the same letter the line bundles/divisors on $\Sigma$ and their pull backs to $\Sigma'$. Denote by $\overline{\Delta}_2$ and $\overline{\Delta}_3$ the strict transforms of $\Delta_2$ and $\Delta_3$ and set:

1) $D_1 = C + S_1 + S_2$, 

$D_2 = f_3$, 

$D_3 = f_1 + f'_1 + \overline{\Delta}_2 + \overline{\Delta}_3 + S_3 + S_4$;

where $f_1, f'_1 \in |f_1|$, $f_3 \in |f_3|$ are general curves and $C \in |f_2 + f_3 - 2e_7|$ is also general;

2) $L_1 = 5l - e_1 - 2e_2 - e_3 - 3e_4 - 2e_5 - 2e_6 - e_7$, and

$L_2 = 7l - 2e_1 - 3e_2 - 2e_3 - 3e_4 - 3e_5 - 3e_6 - 2e_7$

and we obtain $L_3 = 4l - 2e_1 - 2e_2 - 2e_3 - e_4 - e_5 - e_6 - e_7$.

Since $f_2 + f_3 = 4l - 2e_1 - 2e_2 - 2e_3 - e_4 - e_5 - e_6$, it is not difficult to show, for instance by applying a Cremona transformation centered at $P_1, P_3, P_7$, that the general $C \in |f_2 + f_3 - 2e_7|$ is irreducible. Since $p_n(C) = 0$, the general $C$ is also smooth. Thus we obtain a smooth $\mathbb{Z}_2^2$--cover $\pi: X \to \Sigma'$. To compute the geometric genus of $X$, recall that $p_g(X) = p_g(\Sigma') + \sum h^0(\Sigma', K_{\Sigma'} + L_i)$ (cf. [Pa, Lemma 4.2]). We have

$$
K_{\Sigma'} + L_1 = 2l - e_2 - 2e_4 - e_5 - e_6,$$

$$
K_{\Sigma'} + L_2 = 4l - e_1 - 2e_2 - e_3 - 2e_4 - 2e_5 - 2e_6 - e_7,$$

$$
K_{\Sigma'} + L_3 = l - e_1 - e_2 - e_3.
$$

Clearly both $h^0(\Sigma', K_{\Sigma'} + L_1)$ and $h^0(\Sigma', K_{\Sigma'} + L_3)$ vanish.

Now we show that $h^0(\Sigma', K_{\Sigma'} + L_2) = 0$. Assume otherwise and let $\Gamma' \in |K_{\Sigma'} + L_2|$. The image $\Gamma$ of $\Gamma'$ in $\mathbb{P}^2$ is a quartic containing $P_1, \ldots, P_6, P_7$ which has double points at $P_2, P_3, P_5, P_6$. By Bezout’s theorem, the lines in $\mathbb{P}^2$ corresponding to $S_1$ and $S_2$ are contained in $\Gamma$ and thus $\Gamma' = S_1 + S_2 + Q'$,
where $Q'$ is the strict transform of a conic $Q$ containing $P_4, P_5, P_6, P_7$ and having a double point at $P_4$. But obviously there is no such $Q$ because $P_5, P_6, P_7$ all lie on the line $\Delta_3$, which does not contain $P_4$. Hence $p_9(X) = 0$.

For every $i = 1, \ldots, 4$, the (set–theoretic) inverse image of $S_i$ in $X$ is the disjoint union of two $-1$–curves $E_{i1}, E_{i2}$. Also the inverse image of $\Delta_2$ is the disjoint union of two $-1$–curves $E_1, E_2$. The bicanonical divisor $2K_X$ is equal to $\pi^*(2K_{\Sigma'}^0 + D) = \pi^*(-K_{\Sigma'}^0 + f_1 + \Delta_2 + S_1 + S_2 + S_3 + S_4) = \pi^*(-K_{\Sigma'}^0 + f_1) + 2E_1 + 2E_2 + 2\sum E_{ij}.

The system $|-K_{\Sigma'}|$ gives a degree 2 morphism $\Sigma' \to \mathbb{P}^2$. Hence $-K_{\Sigma'} + f_1$ is nef and big and it is easy to check that the linear system $|-K_{\Sigma'} + f_1|$ is birational of (projective) dimension 5. It follows that the surface $S$ obtained from $X$ by contracting $E_1, E_2$ and the $E_{ij}$ is minimal of general type and the rational map $S \to \Sigma'$ is composed with the bicanonical map $\varphi$ of $S$. We denote by the same letter the involutions of $S$ induced by $\gamma_1, \gamma_2, \gamma_3$.

Since $2K_X = \pi^*(-K_{\Sigma'} + f_1) + 2E_1 + 2E_2 + 2\sum E_{ij}$ one has $K_S^2 = \frac{1}{4}(2K_S)^2 = \frac{1}{4}(2K_{\Sigma'})^2 + 16(\pi^*(K_{\Sigma'} + f_1))^2 = 6$.

By the projection formulae for finite flat morphisms, the space $H^0(X, 2K_X)$ decomposes as:

$$H^0(\Sigma', -K_{\Sigma'} + f_1 + \Delta_2 + \sum S_j) \oplus (\oplus_i H^0(\Sigma', -K_{\Sigma'} + f_1 + \Delta_2 + \sum S_j - L_i)),$$

where $\mathbb{Z}^2_2$ acts on $H^0(\Sigma', -K_{\Sigma'} + f_1 + \Delta_2 + \sum S_j - L_i)$ via the character $\chi_i$. Since $P_2(S) = 7$ and $h^0(\Sigma', -K_{\Sigma'} + f_1 + \Delta_2 + \sum S_j) = h^0(\Sigma', -K_{\Sigma'} + f_1) = 6$, it follows that $h^0(\Sigma', -K_{\Sigma'} + f_1 + \Delta_2 + \sum S_j - L_i)$ is equal to 1 for one of the indices $i_0 \in \{1, 2, 3\}$ and it is equal to 0 for the remaining two, so that the bicanonical map has degree 2 and the bicanonical involution is $\gamma_{i_0}$. A computation shows $h^0(\Sigma', -K_{\Sigma'} + f_1 + \Delta_2 + \sum S_j - L_1)) = h^0(\Sigma', e_4 + \Delta_2 + S_1 + S_2 + S_3 + S_4) = 1$, hence the bicanonical involution of $S$ coincides with $\gamma_1$. The linear system $|f_1|$ induces on $S$ a free pencil $|F|$ of hyperelliptic curves of genus 3 such that the bicanonical involution restricts to the hyperelliptic involution on the general $F$. Now the pencil $|F|$ has 5 double fibres, corresponding to the pull backs of $f_1, f'_1, f_2, f_3, f_4$ of the two fibres of $|f_1|$ containing the curves $S_1, \ldots, S_4$ and of the fibre $\Delta_2 + \Delta_3 + 2e_7$. Let $2A$ be this last fibre of $|F|$ on $S$. The support of $A$ is the union of an elliptic curve with self–intersection -2, corresponding to $e_7$, and of a $-2$–curve, corresponding to $\Delta_3$ (recall that the inverse image of $\Delta_2$ in $X$ has been contracted in $S$). The two components of $A$ meet at two points.

**Example 3:** This is in fact a specialization of Example 2, obtained by letting $D_2$ contain the curve $\Delta_2$. In this case the cover $\pi: X \to \Sigma'$ is not normal. The normalization $X'$ of $X$ is again a $\mathbb{Z}^2_2$–cover of $\Sigma'$ with branch divisors:

$$D_1 = C + \Delta_2 + S_1 + S_2,$$

$$D_2 = \Delta_1 + e_7,$$

$$D_3 = f_1 + f'_1 + \Delta_3 + S_3 + S_4 \text{ (cf. Ca).}$$
SURFACES WITH \( p_g = 0 \) AND \( K^2 = 6 \)

The minimal model \( S \) of \( X' \) is a surface with the same properties as before, but the strict transform of \( \Delta_2 \) is now a \(-2\)-curve. Furthermore, if we denote again by \( 2A \) the reducible double fibre of the pencil \( |F| \) on \( S \), then \( A = \theta_1 + \theta_2 + 2E \), where \( \theta_1 \) and \( \theta_2 \) are disjoint \(-2\)-curves, corresponding to \( \Delta_2 \) and \( \Delta_3 \) and \( E \) is an elliptic curve with \( E^2 = -1 \), corresponding to \( e_7 \). One has \( \theta_1 E = \theta_2 E = 1 \).

**Remark:** Notice that for Examples 2 and 3 the divisor \( K_S \) is not ample, in contrast with the case of Burniat surfaces (cf. [MP1]) and of surfaces with \( \text{deg} \varphi = 2 \) and \( K^2_S \geq 7 \) (cf. [MP3]).

**References**

[Be1] A. Beauville, *L’application canonique pour les surfaces de type général*, Invent. Math., **55** (1979), 121–140.

[Be2] A. Beauville, *L’inégalité \( p_g \geq 2q - 4 \) pour les surfaces de type général*, Bull. Soc. Math. de France, **110** (1982), 343–346.

[Ca] F. Catanese, *Singular bidouble covers and the construction of interesting algebraic surfaces*, in Algebraic Geometry: Hirzebruch 70, A.M.S. Contemporary Mathematics, vol. 241 (1999), 97–120.

[Ci] C. Ciliberto, *Alcuni aspetti della classificazione delle famiglie di curve in uno spazio proiettivo*, Boll. Un. Mat. Ital., A (7) **10** (1996), no. 3, 491–535.

[DMP] I. Dolgachev, M. Mendes Lopes, R. Pardini, *Rational surfaces with many nodes*, Compositio Math., **132** (2002), 349–363.

[Fl] E. Halanay, *Normal surfaces of degree 8 in \( P^n \)*, Ann. Univ. Ferrara Sez. VII (N.S.), **41** (1995), 131–145.

[In] M. Inoue, *Some new surfaces of general type*, Tokyo J. of Math., Vol. 17 No. 2 (1994), 295–319.

[Io] P. Ionescu, *Embedded projective varieties of small invariants*, in Algebraic geometry, Bucharest 1982, Lecture Notes in Math., **1056**, Springer, Berlin, (1984), 142–186.

[Ml1] M. Mendes Lopes, *Adjoint systems on surfaces*, Boll. Un. Mat. Ital., A (7) **10** (1996), no. 1, 169–179.

[Ml2] M. Mendes Lopes, *The degree of the generators of the canonical ring of surfaces of general type with \( p_g = 0 \)*, Arch. Math., **69** (1997), 435–440.

[MP1] M. Mendes Lopes, R. Pardini, *A connected component of the moduli space of surfaces of general type with \( p_g = 0 \)*, Topology, **40** (5) (2001), 977–981.

[MP2] M. Mendes Lopes, R. Pardini, *The bicanonical map of surfaces with \( p_g = 0 \) and \( K^2 \geq 7 \)*, Bull. London Math. Soc., **33** (2001), 265–274.

[MP3] M. Mendes Lopes, R. Pardini, *The bicanonical map of surfaces with \( p_g = 0 \) and \( K^2 \geq 7 \)*, II, Bull. London Math. Soc., (to appear).

[Pa1] R. Pardini, *Abelian covers of algebraic varieties*, J. reine angew. Math., 417 (1991), 191–213.

[Pa2] R. Pardini, *The classification of double planes of general type with \( p_g = 0 \) and \( K^2 = 8 \)*, J. of Algebra, (to appear), (math.AG/0107100).

[Pe] C. Peters, *On certain examples of surfaces with \( p_g = 0 \) due to Burniat*, Nagoya Math. J., Vol. **166** (1977), 109–119.

[Re] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math., **127** (1988), 309–316.

[X1] G. Xiao, *Finitude de l’application bicanonique des surfaces de type général*, Bull. Soc. Math. France, **113** (1985), 23-51.
[X2] G. Xiao, Degree of the bicanonical map of a surface of general type, Amer. J. of Math., 112 (5) (1990), 713–737.