Twisted $\mathcal{N} = 2$ exact SUSY on the lattice for BF and Wess-Zumino$^{\ast \dagger}$

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We formulate exact supersymmetric models on a lattice. We introduce noncommutativity to ensure the Leibniz rule. With the help of superspace formalism, we give supertransformations which keep the $\mathcal{N} = 2$ twisted SUSY algebra exactly. The action is given as a product of (anti)chiral superfields on the lattice. We present BF and Wess-Zumino models as explicit examples of our formulation. Both models have exact $\mathcal{N} = 2$ twisted SUSY in 2 dimensional space at a finite lattice spacing. In component fields, the action has supercharge exact form.

1. Introduction

In order to formulate supersymmetric theories nonpertubatively, a lattice formulation is required in which supersymmetry(SUSY) is exactly reserved. It is well-known, however, that there are some difficulties to keep the SUSY exact. One of the difficulties comes from the breakdown of the Leibniz rule. Since the SUSY algebra contains derivatives, its breakdown is crucial.

Keeping this point in mind, we adapt following approach [1]:

- Discretization of SUSY algebra itself
- Twisted SUSY
- Superspace with “mild noncommutativity”

The discretization breaks the Leibniz rule, but the noncommutativity restores it. We construct supersymmetric abelian BF and Wess-Zumino model in 2 dimensions as explicit examples of our approach. In this talk we mainly concentrate on free cases. The basic idea of our formulation is given by the talk of N. Kawamoto, some other models with interactions in the poster of K. Nagata [2]. The continuum version of our model can be found in [4].

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2. Twisted Supersymmetry

$\mathcal{N} = 2$ twisted SUSY algebra is,

$\{Q, Q_\mu\} = i\partial_\mu, \quad \{\bar{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\partial^\nu,$

$Q^2 = \bar{Q}^2 = \{\bar{Q}, Q\} = \{Q_\mu, Q_\nu\} = 0,$

where $Q$’s are twisted supercharges and related to usual supercharges of Majorana spinor by

$Q_{\alpha i} = (1Q + \gamma_\mu Q_\mu + \gamma_5 \bar{Q})_{\alpha i}.$

The $\gamma$ matrices are given by $\gamma_1 = \sigma_3, \gamma_2 = \sigma_1$ and $\gamma_5 = \gamma_1 \gamma_2$. The original charge $Q_{\alpha i}$ has two kinds of suffices, spinor($\alpha$) and extended SUSY($i$). We regard $i$ as a spinor suffix and mix both suffices through the twist.

The twisted supercharges are no longer spinors. $Q$ is a scalar, $Q_\mu$ a vector and $\bar{Q}$ a pseudo scalar. These quantities are much easier to treat on the lattice than spinors. This is an advantage of twisted SUSY. Another advantage is that the action is given by supercharge-exact form so the invariance is manifest because of the nilpotency of $Q$’s. Some authors use this advantage to formulate twisted SUSY on lattice [5][6].

3. Superspace with Noncommutativity

Next we discretise the algebra and show how the “mild noncommutative” superspace arises.

Consider for instance a commutator:

$[\theta Q, \theta_\mu Q_\mu] = i\theta \theta_\mu \partial_{\mu \phi},$
where we use Grassmann parameters $\theta$ and $\theta_{\mu}$, and a forward difference $\partial_{+\mu} f(x) = f(x + 2\hat{n}_{\mu}) - f(x)$ as a derivative.

Operating the r.h.s of eq. (4) on a product of functions,
\[
\theta_{\mu} \partial_{+\mu} (f(x)g(x)) = (\theta_{\mu} \partial_{+\mu} f(x))g(x) + \theta_{\mu} f(x + 2\hat{n}_{\mu})\partial_{+\mu} g(x),
\]
we obtain the breakdown of Leibniz rule, where the argument of $f$ is shifted. Introducing following "mild noncommutativity",
\[
\theta_{\mu} f(x + 2\hat{n}_{\mu}) = f(x)\theta_{\mu},
\]
we can recover the usual Leibniz rule. Thus in order to compensate the breakdown of Leibniz rule, we need noncommutative Grassmann parameters. And these parameters naturally lead us to super-space formulation. Note this kind of noncommutativity is discussed in the context of differential forms on the lattice and gauge theories [7,8].

We use the following symmetric choice:
\[
\Psi(x) = \theta_{\mu} \partial_{+\mu} x = \theta_{\mu} x + \partial_{+\mu} \theta_{\mu},
\]
where we use symmetric choice of $\hat{n}$.

Here $\cdots$ contains only derivative terms. The component of $\theta^A \theta^B \cdots$ lives on the site of $x + \hat{a}_x + \hat{a}_y + \cdots$ due to the noncommutativity of $\hat{a}_x$. Thus the superfield has semi-local structure which goes to local in the naive continuum limit because $\hat{a}_x$ vanishes in this limit. The transformation of each component field can be read from using eq. (7):
\[
Q_A \Psi(x) = s_A \Psi(x), \quad Q_A \bar{\Psi}(x) = s_A \bar{\Psi}(x).
\]
We use $s_A$ as transformations of the components and distinguish from $Q_A$.

4. Actions

Having defined almost the same tools as in the continuum, we can straightforwardly construct actions on the lattice. The most simple one is a product of chiral and anti-chiral superfields:
\[
S = \sum_x \bar{\Psi}(x) \Psi(x) \bigg|_{\theta_4}
\]
\[
= \sum_x is_1 s_2 s c(x) + b(x - \hat{a}) \partial_{+\mu} \omega_\mu(x + \hat{a})
\]
\[
+ i\bar{c}(x) \partial_{+\mu} \bar{\phi}(x + \hat{a} + \hat{a})
\]
\[
- i\bar{c}(x) \partial_{+\mu} \bar{\phi}(x + \hat{a} + \hat{a})
\]
\[
= \sum_x \left[ \phi(x + a) \epsilon_{\mu\nu} \partial_{+\mu} \omega_{\nu}(x + \hat{a}) + b(x - \hat{a}) \partial_{-\mu} \omega_{\mu}(x + \hat{a}) 
\]
\[
- i\bar{c}(x) \partial_{+\mu} \bar{\phi}(x + \hat{a} + \hat{a}) - i\bar{c}(x) \partial_{+\mu} \bar{\phi}(x + \hat{a} + \hat{a}) \right],
\]
where superfield $\Psi$ and $\bar{\Psi}$ (and their lowest components $c$ and $\bar{c}$) are fermionic. This action is supersymmetric BF with gauge fixing term, ghost($c, \bar{c}$) and fermionic auxiliary fields($\lambda, \rho$). Note it is supercharge exact as in eq. (10), and all supercharges are nilpotent, thus the SUSY invariance is manifest for all the four charges.

The action (17) has a kind of dual structure with the symmetric choice of $\hat{a}_x$. Since we adapted double size convention for difference operator, we first set $x = (\text{even}, \text{even})$. Then the noncommutativity shifts the argument by $2\hat{a}_x$, for example,
\[
s(\bar{c}(x)c(x)) = (s\bar{c}(x))c(x) - \bar{c}(x + 2\hat{a})sc(x),
\]
where $x + 2\hat{a}_x = (\text{odd}, \text{odd})$. Thus we need $c$ on both (even, even) and (odd, odd). And bosons live on the dual site while fermions on the original (Fig. 1).
Next, let us change statistics of the superfields. Using bosonic ($\varphi, \phi, \tilde{\varphi}, \tilde{\phi}$) and fermionic ($\chi, \tilde{\chi}, \psi_\mu$) components, we obtain

$$S = \sum_x \left[ \varphi(x) \partial^+ \mu \partial^- \mu \phi(x) + \varphi(x) \tilde{\varphi}(x) - i (\partial^+ \mu \chi(x - \hat{a}) - \epsilon_{\mu\nu} \partial^- \nu \tilde{\chi}(x + \hat{a})) \psi^\mu(x + \hat{a}_\mu) \right].$$

(19)

After untwisting the fermions

$$\xi_{\alpha i}(x) = \frac{1}{2} (1 \chi(x - \hat{a}) + \gamma_\mu \psi^\mu(x + \hat{a}_\mu)$$

$$+ \gamma_5 \tilde{\chi}(x + \hat{a}))_{\alpha i},$$

(20)

and suitable redefinition of bosons, we obtain $\mathcal{N} = 2$ Wess-Zumino model:

$$S = \sum_x \left[ -\phi_i(x) \partial^+ \mu \partial^- \mu \phi_i(x) + F_i(x) F_i(x)$$

$$+ \frac{i}{2} \xi_{\alpha i}(x) (\gamma_\mu)_{\alpha \beta} (\partial^+ \mu + \partial^- \mu) \xi_{\beta i}(x)$$

$$- \frac{i}{2} \xi_{\alpha i}(x) (\gamma_5)_{\alpha \beta} (\partial^+ \mu - \partial^- \mu) \xi_{\beta j}(x) (\gamma_5 \gamma_\mu)_{ji} \right].$$

(21)

The fermion part is nothing but staggered fermion in K-S form. Note that the flavor(or taste) index of staggered fermion $i$ now becomes that of extended SUSY. We call this relation “Dirac-Kähler mechanism” since staggered fermion is a lattice version of Dirac-Kähler fermion. See Fig. 2. We would like to comment that a different use of staggered fermion is found in [9].

### 5. Discussion

We have defined lattice models which keep exact $\mathcal{N} = 2$ SUSY for all charges. We use mild noncommutativity to compensate the breakdown of Leibniz rule. Since supercharges also have noncommutativity, our definition of SUSY may be a new one in the same sense that Ginsparg-Wilson relation defines a new chiral symmetry.

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