Asynchronous Policy Evaluation in Distributed Reinforcement Learning over Networks

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Abstract

This paper proposes a fully asynchronous scheme for policy evaluation of distributed reinforcement learning (DisRL) over peer-to-peer networks. Without any form of coordination, nodes can communicate with neighbors and compute their local variables using (possibly) delayed information at any time, which is in sharp contrast to the asynchronous gossip. Thus, the proposed scheme fully takes advantage of the distributed setting. We prove that our method converges at a linear rate $O(c^k)$ where $c \in (0, 1)$ and $k$ increases by one no matter on which node updates, showing the computational advantage by reducing the amount of synchronization. Numerical experiments show that our method speeds up linearly w.r.t. the number of nodes, and is robust to straggler nodes. To the best of our knowledge, our work is the first theoretical analysis for asynchronous update in DisRL, including the parallel RL domain advocated by A3C.

1 Introduction

Reinforcement learning (RL) aims to guide decision makers (a.k.a. agents) to learn optimal policies/strategies by interacting with the environment, which has achieved super-human performance in many tasks [12, 15]. This work considers the distributed reinforcement learning (DisRL) problem over a directed peer-to-peer (p2p) network, which includes two distinct RL setups depending on the role of the network node. One is the so-called the multi-agent RL (MARL) in which network nodes are the acting agents of the MARL [19, 38, 45], and has proved its capability in e-sports [37], swarm robotics [20], traffic control [36] and resource allocation [24], etc. The other is the parallel RL where network nodes are designed to jointly solve a large-scale RL problem over decentralized sets of experiences [9, 25].

As the MARL in [7, 19, 28, 45, 46], each agent of the p2p network has access to a local dataset and exchanges information only with a subset of agents to learn globally. For the parallel RL, the approach with a central node to communicate with all the other nodes has been already adopted, which is well-known as the master/slave configuration, see e.g. the very famous A3C in [25]. However, this thought is practically restrained by the vulnerability/high communication cost of the central node [21] and can also be considered as a special case of the star p2p network.

In the DisRL, all the nodes over the p2p network participate in the decision learning process either synchronously or asynchronously. For a synchronous model, all nodes communicate with neighbors and compute their local variables in synchronized rounds [30, 33]. To this end, they need a global clock/counter for synchronization, which may be hard to implement in large-scale systems [4, 39], and may suffer from deadlocks [1, 22]. While synchronous algorithms are easier for design and analysis, their efficiency is greatly dragged by the slowest node in heterogenous cases. To facilitate the implementation and improve the
computational efficiency, we adopt the *fully asynchronous* model in [10, Section 2] where every node can communicate with its neighbors and compute its local variables using (possibly) delayed information at any time. Since we also aim for *directed* communications, each node does not need any coordination with other nodes. This is in sharp contrast to the random asynchronous gossip where only a pair of nodes concurrently update via pairwise averaging in each round [3, 22, 40] and may create deadlocks in practice, as well as vulnerable to information delays. It can also be problematic if a node is unable to response or has only access to its local dataset. In fact, the advantages of the fully asynchronous model has been well documented in the context of distributed learning and optimization both empirically and theoretically, see e.g. [1, 2, 34, 43]. In this work, we are the first to study the fully asynchronous policy evaluation task in the DisRL over directed p2p networks, which is a pivotal subproblem of the actor-critic RL algorithms [13, 31].

1.1 Related works

The MARL has already demonstrated its powerfulness in several challenging tasks, see e.g. [11, 37] and [14], although a solid theoretical foundation remains to be established. [45] propose two multi-agent actor-critic methods and provide an almost sure convergence result with linear approximation. As an important problem, the policy evaluation of the MARL has been studied in [5, 7, 29, 38], among which [7, 38] and [5] are closely related to this work in the sense that we also use the primal-dual reformulation of the mean square projected Bellman error (MSPBE) objective function and obtain a linear convergence guarantee. Striking differences from this work are that [38] only consider the synchronous case and require each agent to select the same sample per learning iteration. Thus, they are unable to handle the decentralized datasets and the space complexity increases linearly with respect to the data amount. This has be resolved in [5] by designing consensus-based distributed (but synchronous) algorithms over undirected p2p networks. More general works on the MARL setting can be found in [44]. Overall, we are the first to exploit the fully asynchronous model in the MARL.

Interestingly, our fully asynchronous algorithm for MARL can also be easily used to accelerate the policy evaluation of the parallel RL over a directed p2p network. The abundant theoretical analysis for the RL can be extended to the parallel RL [6] which has registered great success in complex control tasks by employing massive parallel RL workers [9, 18]. The A3C with the master/slave configuration [25] is the first asynchronous parallel RL, where the asynchronism has stabilizing advantages in the training process. In GALA [3], the authors replace the central learner node of A3C with a group of learners organized over a p2p network. Thus each learner only coordinates with its subordinate worker to perceive RL experiences. It is further shown that GALA outperforms A3C in both data efficiency and robustness, and the learners of GALA have policies keeping within an $\epsilon$-ball of one-another when being trained in the random asynchronous model. In our work, we can explicitly give the convergence rate of the fully asynchronous model in the worst-case for the parallel RL over directed p2p networks.

It should also be noted that most of synchronous algorithms can be easily implemented with random asynchronous gossip. This is not the case for the fully asynchronous model. Particularly, [2] shows that a naive extension of the synchronous gradient-push optimization algorithm to the fully asynchronous model may even diverge due to uneven update frequencies or possible communication delays among agents. This issue has been solved by designing a novel adaptive learning rate [43] or leveraging the gradient tracking method [34, 41], which inspires the design of our algorithm. There are also works on asynchronous settings in different contexts. For example, [16, 23] present asynchronous *coordinate descent* algorithms which are not amenable to p2p networks.

1.2 Contributions

Overall, our main contribution can be summarized as follows: (1) We propose a unified policy evaluation formulation for two DisRL setups, namely, collaborative MARL and parallel RL. (2) We propose a novel asynchronous policy evaluation algorithm for agents connected via a p2p network with *directed* communications, which achieves a linear convergence rate within a certain level of tolerance. (3) We develop a novel augmented graph approach to handle full asynchrony and delays for solving the distributed
saddle-point problems of this work. (4) We validate through simulations that our approach achieves a linear speedup with respect to the number of computational nodes and is robust to straggler nodes than synchronous counterparts.

2 Problem formulation

2.1 The DisRL over networks

Let \( G = (V, E) \) be a directed p2p network, where \( V = \{1, \cdots, n\} \) is the set of network nodes and \( E \subseteq V \times V \) is the set of edges. Depending on the specific DisRL setting, a node can be either a learner in the parallel RL, or an acting agent in the MARL. A directed edge \( (i, j) \in E \) means that node \( i \) can directly send information to node \( j \). Denote \( N^i_{\text{in}} = \{j|(j, i) \in E \} \cup \{i\} \) and \( N^i_{\text{out}} = \{j|(i, j) \in E \} \cup \{i\} \) as the sets of in-neighbors and out-neighbors of node \( i \), respectively.

The DisRL model over a p2p network is based on a discrete Markov decision process (MDP) with a tuple \((S, \otimes_{i=1}^n A_i, \mathcal{P}, \{R_i\}_{i=1}^n, \gamma, \mathcal{G})\), where the finite sets \( S \) and \( A_i \) denote the state space and the action space of node \( i \) in \( G \), respectively. \( \mathcal{P}(s'|s,a) \) is the probability of transitioning from \( s \) to \( s' \), after an action \( a \in A_i \). \( R_i = R_i(s,a) \) is the reward perceived by a single node \( i \). \( \gamma \in (0,1) \) is the discount factor. \( G \) is used to model the interactions among nodes.

Each node follows a policy, the decisional component of MDP. A stochastic policy \( \pi(a|s) \) is the conditional probability of taking action \( a \) given a specific state \( s \). If we focus on a target policy \( \pi \), the transition probabilities between states are fixed. We denote the fixed transition probability matrix by \( \mathcal{P}^\pi \), whose \((s,s')\)-th entries are given by \( [\mathcal{P}^\pi]_{s,s'} = \mathcal{P}(s'|s,a) = \sum_{a \in A} \pi(a|s) \cdot \mathcal{P}(s'|s,a) \). Let \( R^\pi(s) = \frac{1}{n} \sum_{i=1}^n R^\pi_i(s) \), where \( R^\pi_i(s) = \mathbb{E}_{a \sim \pi(\cdot|s)}[R_i(s,a)] \) is denoted as the expected reward at state \( s \) if the group follows the policy \( \pi \).

2.2 Policy evaluation

To improve the capability of RL algorithms, one should incessantly evaluate the performance of a fixed policy \( \pi \) in the training process. The value function \( V^\pi(s) \) is defined as the function of a state \( s \) under a given policy \( \pi \), implying the expected discounted cumulative reward the nodes can receive with the given initial state following policy \( \pi \), i.e.,

\[
V^\pi(s) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t R^\pi(s(t)) | s(0) = s, \pi \right].
\]

Then we denote the vector \( V^\pi \in \mathbb{R}^{|S|} \) as the value function vector of \( \pi \), where the \( k \)-th entry is \([V^\pi]_k = V^\pi(s_k), s_k \in S, k = 1, \cdots, |S|\). The vector satisfies the Bellman function \([31], i.e., V^\pi = R^\pi + \gamma \mathcal{P}^\pi V^\pi, \) where \( R^\pi \) can be obtained by stacking up \( R^\pi(s_k) \).

In practice, an approximator is required to learn an estimate of the value function. To simplify analysis, this work adopts a linear approximator: \( V^\pi(s) \approx \Phi^T(s) \theta \), where \( \Phi(s) \in \mathbb{R}^d \) is a feature vector corresponding to \( s \). By defining \( \Phi = [\phi(s_1)^T, \phi(s_2)^T, \cdots, \phi(s_{|S|})^T]^T \), we can rewrite the approximation function as \( V^\pi \approx \Phi \theta \).

Then, our problem is reduced to one of finding a \( \theta \in \mathbb{R}^d \) such that \( \Phi^\pi \approx V^\pi \). As in \([32], an appropriate way is to minimize the mean squared projected Bellman error (MSPBE), i.e.,

\[
J(\theta) = \frac{1}{2} \| \Pi_\Phi (V^\theta - \gamma D^\pi V^\theta - R^\pi) \|_D^2 + \frac{\rho}{2} \| \theta \|^2
\]

(1)

where \( D = \text{diag}(\mu^\pi(s)) \) is a diagonal matrix and \( \mu^\pi(s) \) is the stationary distribution of states by following the policy \( \pi \), \( \Pi_\Phi \) is the projection onto subspace \( \{ \Phi \theta \} \), and \( \frac{\rho}{2} \| \theta \|^2 \) is the regularization term. \( \| \cdot \|_D \) denotes the \( D \)-norm, i.e., \( \| y \|_D = \sqrt{y^T Dy} \) if the matrix \( D \) is semi-positive definite. By \([32], \) the loss function in (1) is equivalent to

\[
J(\theta) = \frac{1}{2} \| A \theta - b \|_{C^{-1}}^2 + \frac{\rho}{2} \| \theta \|^2
\]

(2)
where the matrices are defined as $A = \mathbb{E}_{\mu^*}[\phi_t(\phi_t - \gamma \phi_{t+1})^T], b = \mathbb{E}_{\mu^*}[R^\pi(s(t))\phi_t], C = \mathbb{E}_{\mu^*}[\phi_t\phi_t^T]$.

(\phi_i is the short for $\phi(s(t)))$.

In the DisRL, we estimate the matrices $A, b$ and $C$ from batches of experiences. We further discuss how to solve the MSPBE in (2) in both MARL and parallel RL setups next.

### 2.2.1 Parallel RL

The parallel RL adopts multiple computing nodes to learn an optimal policy $\pi$ for a specific MDP problem with high efficiency [25]. Thus, the MDP tuple of each node $i$ is $(S_i, A_i, \mathcal{P}_i, R_i, \gamma)$, where $S_i = S, A_i = A, \mathcal{P}_i = \mathcal{P}$ since the MDP model is identical. The local rewards may differ among nodes, but follow the same distribution and the expectation

$$\mathbb{E}_{\mu^*}[R_i(s, a)] = R(s, a), \forall (s, a) \in S \times A.$$ 

In the learning process, each parallel worker generates $\{s_{i,p}, a_{i,p}, R_i(s_{i,p}, a_{i,p})\}_{p=1}^m$, a finite sequence of trajectories, under the policy $\pi$. We construct the estimates for the matrices in (2) by fusing local estimates, i.e., $\hat{A}_i = \frac{1}{m} \sum_{p=1}^m \hat{A}_{i,p}, \hat{b}_i = \frac{1}{m} \sum_{p=1}^m \hat{b}_{i,p}, \hat{C}_i = \frac{1}{m} \sum_{p=1}^m \hat{C}_{i,p}$, where the local estimates are given by

$$\hat{A}_{i,p} = \frac{1}{n} \sum_{i=1}^n \hat{A}_{i,p}, \hat{b}_{i,p} = \frac{1}{n} \sum_{i=1}^n \hat{b}_{i,p}, \hat{C}_{i,p} = \frac{1}{n} \sum_{i=1}^n \hat{C}_{i,p}. \quad (3)$$

Here $\phi_{i,p}$ is the short for $\phi(s_{i,p}), \hat{A}_{i,p} = \phi_{i,p}(\phi_{i,p} - \gamma \phi_{i,p+1})^T, \hat{b}_{i,p} = R_i(s_{i,p}, a_{i,p})\phi_{i,p}$ and $\hat{C}_{i,p} = \phi_{i,p}\phi_{i,p}^T$ can be derived by one transition tuple $(s_{i,p}, s_{i,p+1}, R_i(s_{i,p}, a_{i,p}))$.

To use the distributed setting, we create local copies $\{\theta_i\}$ of $\theta$ for each node and obtain the following form:

$$\min_{\theta} \frac{1}{2} \left\| \frac{1}{n} \sum_{i=1}^n (\hat{A}_i\theta_i - \hat{b}_i) \right\|_C^2 \quad \text{s.t. } \theta_1 = \theta_2 = \cdots = \theta_n \quad (4)$$

### 2.2.2 Collaborative MARL

In the MARL, the nodes of $\mathcal{G}$ are heterogeneous agents which communicate and collaborate with each other [44]. In the fully observed setting, each agent observes a global state $s$ shared by all the agents and then chooses an action $a_i \in A_i$ according to its local policy $\pi_i(a_i|s)$. The joint action $a = (a_1, \cdots, a_n)$ is then executed by the group. Thus the state transition directly depends on the joint policy $\pi = (\pi_1, \cdots, \pi_n)$.

Under the policy $\pi$, the distributed agents together generate a joint trajectory $\{s_p, a_p\}_{p=1}^m$, but with different reward samples $\{R_i(s_p, a_p)\}_{p=1}^m$ for every node $i \in \mathcal{V}$. The agents aim to collaboratively maximize the group mean reward $R(s, a) = \frac{1}{m} \sum_{i=1}^n R_i(s, a)$. The empirical MSPBE of MARL then is identical to (3) and (4), with the constraint that $\hat{C}_i = \hat{C}$ and $\hat{A}_i = \hat{A}$ for all $i \in \mathcal{V}$.

### 2.3 A saddle-point reformulation for the MSPBE

We aim to design a fully asynchronous SGD method over the directed p2p network to solve the policy evaluation problem of the DisRL. To this end, we reformulate it into a summation of functions corresponding to each samples. However, (4) cannot be categorized into this type. From [8, 38], we can obtain the conjugate form of the original $C^{-1}$-norm, i.e., $\frac{1}{2}\|A\theta - b\|_{C^{-1}}^2 = \max_{\omega} (\omega^T (A\theta - b) - \frac{1}{2}\omega^TC\omega)$, and rewrite the first term in (4) as:

$$\max_{\omega \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n \omega^T(\hat{A}_i\theta - \hat{b}_i) - \frac{1}{2}\omega^T\hat{C}_i\omega$$
Algorithm 1 The PD-APP from the view of node $i$

Require:

Node $i$ arbitrarily sets $z_i = \left[\theta_i^T \quad \omega_i^T\right]^T$, initializes $s_i \leftarrow 0, g_i \leftarrow 0$, and creates local buffers $Z_i^{rec}, S_i^{rec}$. Let stepsizes $\eta_1, \eta_2 > 0$.

1: **First broadcast**: Node $i$ broadcasts $\tilde{z}_i \leftarrow z_i, \tilde{s}_i \leftarrow s_i/|N^i_{out}|$ to all out-neighbors of $i$

2: repeat

3: Keep receiving $\tilde{z}_j, \tilde{s}_j$ from in-neighbors of node $i$ and save to $Z_i^{rec}$ and $S_i^{rec}$, respectively, until node $i$ is ready to update.

4: To start a new update, node $i$ uniformly picks a random sample indexed by $p_i \in \{1, \cdots, m\}$.

5: Update the gradient surrogates by

\[
\begin{align*}
\hat{z}_i & \leftarrow \text{avg}(Z_i^{rec}), \\
g_i^- & \leftarrow g_i, \\
g_i & \leftarrow \left[\nabla_{\theta_i} J_{i,p_i}(\theta_i, \omega_i)^T - \nabla_{\omega_i} J_{i,p_i}(\theta_i, \omega_i)^T\right]^T, \\
s_i & \leftarrow \text{sum}(S_i^{rec}) + g_i - g_i^-.
\end{align*}
\]

6: Perform local primal-dual updates using gradient surrogates $s_i$ and $z_i$

\[
\tilde{z}_i \leftarrow z_i - \left[\eta_1 I_d \quad 0 \\
0 \quad \eta_2 I_d\right] s_i.
\]

7: Broadcast $\tilde{z}_i \leftarrow z_i$ and $\tilde{s}_i \leftarrow s_i/|N^i_{out}|$ to all out-neighbors of node $i$.

8: Empty $Z_i^{rec}$ and $S_i^{rec}$.

9: **until** a stopping criteria is satisfied, e.g., $\|s_i\| < \epsilon$ for a predefined $\epsilon > 0$.

Finally, we introduce the local copies for both the primal and dual variables and rewrite (4) as

\[
\min_{\theta} \max_{\omega} \frac{1}{mn} \sum_{i=1}^{n} \sum_{p=1}^{m} J_{i,p}(\theta_i, \omega_i)
\]

\[
J_{i,p}(\theta_i, \omega_i) = \omega_i^T (\hat{A}_{i,p} \theta_i - \hat{b}_{i,p}) - \frac{1}{2} \omega_i^T \hat{C}_{i,p} \omega_i + \frac{\rho}{2} \|\theta_i\|^2
\]

s.t. $\theta_1 = \theta_2 = \cdots = \theta_n, \omega_1 = \omega_2 = \cdots = \omega_n$

where $\hat{A}_{i,p}, \hat{b}_{i,p}$ and $\hat{C}_{i,p}$ are defined in (3).

Let $J_i(\theta_i, \omega_i) = \sum_{p=1}^{m} J_{i,p}(\theta_i, \omega_i)$ and $J(\theta, \omega) = \sum_{i=1}^{n} \sum_{p=1}^{m} J_{i,p}(\theta, \omega)$. The primal and dual stochastic gradients in (7) are then explicitly given by

\[
\nabla_{\theta_i} J_{i,p}(\theta_i, \omega_i) = \hat{A}_{i,p} \omega_i + \rho \theta_i, \quad \nabla_{\omega_i} J_{i,p}(\theta_i, \omega_i) = \hat{A}_{i,p} \theta_i - \hat{C}_{i,p} \omega_i - \hat{b}_{i,p}.
\]

3 Algorithm and convergence analysis

3.1 The PD-APP

Motivated by the synchronous push-pull algorithm in [27], we propose the primal-dual asynchronous push-pull (PD-APP) algorithm to solve the saddle-point problem in (7) in Algorithm 1.

The PD-APP is implemented in a fully asynchronous way over the directed $G$. Particularly, every node keeps receiving information from its in-neighbors and copying to its local buffers $Z_i^{rec}$ and $S_i^{rec}$, both of which possibly contain multiple receptions from a single in-neighbor, respectively. If node $i$ starts to update at any time, it simply picks a (or a batch of) random sample(s) from its local sample set.
and queries its buffers to perform computations in (5)-(6), where \( \text{avg}(\cdot) \) and \( \text{sum}(\cdot) \) return the average and sum over their arguments respectively. Then, it broadcasts the updated local variables \( \tilde{z}_i \) and \( \tilde{s}_i \) to all out-neighbors, which may also be subject to unpredictable delays, and empties both buffers. It is clear that the above process does not need any coordination with other nodes and each node is fully autonomous, all of which render our asynchronous model sharp contrast to the gossip-based asynchronous model [3, 22, 40] as the later involves a pair of neighbors to concurrently update with some coordination.

Unlike the distributed algorithm in [38], they require a global clock for synchronization and all nodes use the same sample per iteration, e.g., \( p_i = p_j \) in Step 4 of Algorithm 1. That is, every node should access to all the samples and cannot handle decentralized sets [33], which is an important issue in the parallel RL setting.

To get a global full view of the PD-APP, we shall design a novel augmented system to convert the asynchronous update pattern into an equivalent delay-free synchronous version in Subsection 3.3, based on which the convergence of the PD-APP is established in the worst-case point of view.

### 3.2 Convergence result

Our analytical result is based on the following assumptions.

**Assumption 1.** (a) The digraph \( G \) is strongly connected, i.e. any pair of two nodes can be connected via a sequence of consecutive directed edges.

(b) (Bounded activation time interval) Let \( t_i \) and \( t_i^+ \) be any two consecutive activation times of node \( i \). Then, \( |t_i^+ - t_i| \) is bounded, i.e. \( \exists \tau > 0, \bar{\tau} > 0 \) such that \( \tau \leq |t_i^+ - t_i| \leq \bar{\tau}, \forall i \in V \).

(c) (Bounded transmission delays) For any \( (i, j) \in E \), the transmission time for node \( i \) to send information to node \( j \) is bounded by a constant \( \tau > 0 \).

**Assumption 2.** (a) The sample amount is sufficient so that the sampled correlation matrix \( \hat{A} \) is full rank, and the sampled covariance \( \hat{C} \) is non-singular.

(b) All local stochastic gradients have bounded variances , i.e. there exist \( \sigma > 0 \) such that for all \( i, \theta_i \), and \( \omega_i \),

\[
E[\|\nabla_{\theta_i} J_{i}(\theta_i, \omega_i) - \nabla_{\theta_i} J_{i}(\theta_i, \omega_i)\|^2] \leq \sigma^2, \quad E[\|\nabla_{\omega_i} J_{i,p}(\theta_i, \omega_i) - \nabla_{\omega_i} J_{i}(\theta_i, \omega_i)\|^2] \leq \sigma^2.
\]

Assumption 1(a) is necessary, since otherwise there exists nodes unable to access information from other nodes. Assumption 1(b) is easy to satisfy in practice, since the computation of each update cannot be completed instantaneously and takes a finite time. If there is an infinite time interval between two consecutive updates, it implies that such a node is dead and cannot participate in the learning process. Assumption 1(c) imposes a uniform upper bound on the unpredictable and possibly time-varying transmission delays, meaning that all the transmitted messages will be eventually successfully delivered no matter on how long it takes. It should be noted that the design and implementation of the PD-APP does not depend on such an upper bound.

Assumption 2(a) ensures the existence of a unique optimal solution for (7), while Assumption 2(b) is commonly used when analyzing the convergence rate of SGD methods, see e.g. [28].

Let \( T = \{t(k)\}_{k \geq 1} \) be an increasing sequence to record the updating time instants of all the nodes, e.g., \( t(k) \in T \) if and only if some node starts a new update at time instant \( t(k) \). Moreover, let \( \theta_i(k) \) and \( \omega_i(k) \) denote the latest value of the primal and dual local variables of node \( i \) just before \( t(k) \), respectively. Then, we have the following result on the information flow of the PD-APP, the proof of which can be found in Appendix B.1.

**Lemma 1.** Under Assumption 1, let \( b = (n-1)[\tau/\bar{\tau}] + n[\tau/\bar{\tau}] + 1 \), where \( [\cdot] \) is the standard floor function that returns the greatest integer less than or equal to its input. Then, there is at least a new update for every node \( i \) during the time interval \( [t(k), t(k+b)] \) and the broadcast variables of such a node \( i \) can be received by each of its out-neighbor node \( j \) for any \( j \in \mathcal{N}_i^{\text{out}} \).
In the subsequent discussion, we use the constant $b$ to measure the updating frequency of each node, which clearly is evaluated from the worst-case point of view by jointly accounting the mixed effect of network topology, asynchronous level and the delays. Thus, our theoretical convergence rate of the PD-APP appears to be much slower than its practical performance. For the random gossip-based asynchronous algorithms, the theoretical convergence rate essentially is evaluated in the mean sense [3,22,40]. The main theoretical result of this paper is then given below.

**Theorem 1.** Under Assumptions 1 and 2, we define $\{\theta^*, \omega^*\}$ as the saddle-point solution to the problem in (7) and set the step sizes as
\[
\eta_1 = \eta, \eta_2 = \zeta \eta, \zeta = \frac{8(\eta + \lambda_{\text{max}}(A^T \hat{C}^{-1} \hat{A}))}{\lambda_{\text{min}}(C)}
\]

If the step size $\eta$ is sufficiently small, the expected optimality gap for each local copies in the PD-PPG,
\[
E\|\theta_i(k) - \theta^*\| + E\|\omega_i(k) - \omega^*\|, \forall i \in V, k \in N,
\]
converges to a neighborhood of origin at the rate of $O(c^k), 0 < c < 1$, where the size of the neighborhood is $O(br)$.

Though the linear convergence rate seems mediocre in the synchronous context, we emphasize that the virtual counter $k$ plays an extraordinary part in our result. Unlike that synchronous works commonly use $k$ to represent how many times the global clock clicks, in our case each update from local computing units will make a contribution to $k$’s increasing by 1. For the most ideal situation, $n$ nodes with the same update frequency can increase $k$ by $n \times 1$ while each node only performs local update once. Note that [1] have studied this kind of linear speedup elaborately.

The non-zero neighborhood in Theorem 1 is caused by the asynchronous nature of our scheme, where each node performs autonomous update without coordination. Thus, the radius of the neighborhood is related to $b$, the measurement of the asynchronous degree and transmission delays of the system. Since heterogenous nodes can contribute to the counter $k$ in the same way, our theoretical result guarantees the linear performance even if in the aforementioned worst case, e.g., (a) some nodes seldom participate in the update for its poor computational ability, (b) some nodes are in bad communication status.

The detailed proof for Theorem 1 is quite lengthy. We provide a roadmap in the following two subsections. In Subsection 3.3 we advocate our augmented system and in Subsection 3.4 we provide a sketch of the proof with some crucial inequality conclusions. The complete proof can be found in Appendix B.

### 3.3 The augmented system

As in lemma 1, we prove that within the interval $(t(k), t(k+b)]$, each node is activated at least once. We note that there is no other interaction between $z_i$ and $s_i$ except for in (6), and thus we can consider two individual nodes listening to incoming $\tilde{z}_i, \tilde{s}_i$ independently, which are denoted by $v_{z,i}^{(0)}, v_{s,i}^{(0)}$, respectively.

We further set up $b$ virtual nodes for each $v_{z,i}^{(0)}$ node. We use $\tilde{G}_z = (\tilde{V}_z, \tilde{E}_z(k))$ to represent the (virtual) communication network of virtual nodes $\{v_{z,i}^{(u)}| 1 \leq i \leq n, 0 \leq u \leq b\}$, the overall $n(b+1)$ virtual nodes. In $\tilde{G}_z$, the virtual nodes always send the $\tilde{z}_i$ message to its following virtual nodes, i.e., $v_{z,i}^{(u)} v_{z,i}^{(u+1)} \in \tilde{E}_z(k), \forall 0 \leq u \leq b-1, \forall i$. Then, $v_{z,i}^{(u)} (u \geq 1)$ can be regarded as the future replica of the original node. If a message is sent by $i$ at the time $t(k) - t(u)$ and received by $j$ at time $t(k)$, then in the augmented system, we view its future surrogate $v_{z,i}^{(u-1)}$ as the sender and $v_{z,j}^{(0)}$ as the receiver, i.e., $(v_{z,i}^{(u-1)}, v_{z,j}^{(0)}) \in \tilde{E}_z$.

Then we move on to $s_i$’s part. Similarly, we create $\tilde{G}_s = (\tilde{V}_s, \tilde{E}_s(k))$, where $\tilde{V}_s = \{v_{s,i}^{(u)}| 1 \leq i \leq n, 0 \leq u \leq b\}$. However, the communication mode is exactly reversed, i.e., $(v_{s,i}^{(u+1)}, v_{s,j}^{(u)}) \in \tilde{E}_s(k), \forall 0 \leq u \leq b-1, \forall i$. Meanwhile, in the $s$-buffer, a message should wait in $j$’s virtual node queue and move forward before
The stochastic gradient is sampled only when the node is activated, i.e., $k$ where matrix $I_n$, Lemma 2. the original asynchronous algorithm in the following lemma.

Finally being used by $j$. If a message is sent by $i$ at the time $t(k) - t(u)$ and received by $j$ at time $t(k)$, then in the augmented system, we use $v_{s,j}^{(u-1)}$ to save the message from $v_{s,i}^{(0)}$, i.e., $(v_{s,i}^{(0)}, v_{s,j}^{(u-1)}) \in \tilde{E}_s(k)$. Then, it takes $u - 1$ time steps for the message in $v_{s,j}^{(u-1)}$ to reach $v_{s,j}^{(0)}$ and participate in the computation.

We provide simple, but illuminating examples to illustrate how the augmented graphs handle delays. Suppose that node $i$ sends message to node $j$ at $t(k)$, and due to delay $j$ receives it in $(t(k+1), t(k+2)]$, which is then used in update at $t(k+3)$. Then, $(v_{z,i}^{(0)}, v_{z,j}^{(1)}), (v_{z,i}^{(1)}, v_{z,j}^{(2)}), (v_{z,i}^{(2)}, v_{z,j}^{(0)}) \in \tilde{E}_z(k)$, where $v_{z,i}$ is the future surrogate. Also, $(v_{s,i}^{(0)}, v_{s,j}^{(1)}), (v_{s,i}^{(1)}, v_{s,j}^{(2)}), (v_{s,i}^{(2)}, v_{s,j}^{(0)}) \in \tilde{E}_s(k)$, where $v_{s,i}$ and $v_{s,j}$ play as the local queue of $j$. Specially, if $u = 1$ in both situations, then $(v_{z,i}^{(0)}, v_{z,j}^{(0)}) \in \tilde{E}_z(k)$, and $(v_{s,i}^{(0)}, v_{s,j}^{(0)}) \in \tilde{E}_s(k)$, respectively. The nodes are connected directly in the augmented systems as in the original one in this case.

With the foundation of the augmented digraph, we now introduce the equivalent synchronous form of the original asynchronous algorithm in the following lemma.

**Lemma 2. Synchronous form of PD-APP in augmented system:** Let $z_i^{(u)}(k), s_i^{(u)}(k), 1 \leq i \leq n, 0 \leq u \leq b$ denote the value in virtual nodes $v_{z,i}^{(u)}, v_{s,i}^{(u)}$ after time $t(k)$, while $\tilde{n} = n(b + 1)$. Then, for each $k \in \mathbb{N}$, there exists a row-stochastic matrix $\tilde{A}(k)$, a column-stochastic matrix $\tilde{B}(k)$ and a diagonal matrix $I^*(k)$ that can compress our asynchronous Algorithm 1 as:

$$
\begin{align*}
\tilde{z}(k+1) &= \tilde{A}(k)\tilde{z}(k) - \eta I^*(k) \tilde{s}(k) \Lambda, \\
\tilde{s}(k+1) &= \tilde{B}(k)\tilde{s}(k) + \partial(k+1) - \partial(k),
\end{align*}
\tag{10}
$$

where

$$
\begin{align*}
\tilde{z}(k) &= [Z_i^{(0)}(k) \cdots Z_i^{(b)}(k)]^T \in \mathbb{R}^{\tilde{n} \times 2d}, \\
\tilde{s}(k) &= [S_i^{(0)}(k) \cdots S_i^{(b)}(k)]^T \in \mathbb{R}^{\tilde{n} \times 2d}, \\
Z_i^{(u)}(k) &= [z_i^{(u)}(k) \cdots z_i^{(u)}(k)]^T \in \mathbb{R}^{n \times 2d}, \\
S_i^{(u)}(k) &= [s_i^{(u)}(k) \cdots s_i^{(u)}(k)]^T \in \mathbb{R}^{n \times 2d}, \\
\partial(k) &= [\partial J_i(k) \cdots \partial J_n(k)]^T \in \mathbb{R}^{n \times 2d}, \\
\Lambda &= \begin{bmatrix} I_d & 0 \\ 0 & \zeta I_d \end{bmatrix}.
\end{align*}
\tag{11}
$$

The stochastic gradient is sampled only when the node is activated, i.e.,

$$
\begin{align*}
\partial J_i(k) &= \begin{bmatrix} \nabla_{\theta_i} J_i(p_i(k)(\theta_i(k), \omega_i(k))) \\ -\nabla_{\omega_i} J_i(p_i(k)(\theta_i(k), \omega_i(k))) \end{bmatrix}, & \text{if } t(k) \in T_i \\
\partial J_i(k) &= \partial J_i(k-1), & \text{if } t(k) \notin T_i
\end{align*}
\tag{12}
$$
with $\mathbf{Z}(0) = [\mathbf{Z}^{(0)}(0)^T \mathbf{0}_{(n-n)\times2m}]^T, \mathbf{S}(0) = [\partial J(k)(0)^T \mathbf{0}_{(n-n)\times2m}]^T, \partial J(k)(0) = \mathbf{0}$, and $\mathcal{T}_i$ is an increasing sequence to record the updating time instants of node $i$.

We shall define $\tilde{\mathbf{A}}(k), \tilde{\mathbf{B}}(k)$ and $\mathbf{I}^a(k)$ in Appendix A.1. To make our idea concrete, we also illustrate an example of $\mathcal{G}$ and its corresponding $\mathcal{G}_z, \mathcal{G}_s$ and $\tilde{\mathbf{A}}(k), \tilde{\mathbf{B}}(k)$ at a specific time $k$, in Appendix A.2.

3.4 Proof sketch of Theorem 1

First, we introduce an important concept called absolute probability sequence from [35].

Lemma 3. For any sequence of row-stochastic matrices $\{\mathbf{A}(k)\}$, e.g., $\mathbf{A}(k)\mathbf{1} = \mathbf{1}, \forall k \in \mathbb{N}$, there exists a sequence of nonnegative stochastic vectors $\{\mathbf{p}(k)\}$ that satisfies the following equations,

$$\mathbf{p}(k + 1)^T \mathbf{A}(k) = \mathbf{p}(k)^T, \forall k \in \mathbb{N}.$$ 

Our approach relies on the following quantities:

Definition 1. Let $\| \cdot \|_F$ denote an appropriate Frobenius norm and define

1. $\| \mathbf{Z}(k) \|_F = \| \mathbf{T}_A(k)\tilde{\mathbf{Z}}(k)\tilde{\mathbf{A}} \|_F$, where $\mathbf{T}_A(k) = I_n - \mathbf{1}_n \mathbf{p}(k - 1)^T$, $\mathbf{p}(k + 1)^T \tilde{\mathbf{A}}(k) = \mathbf{p}(k)^T$, and $\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{-1/2}$ is a diagonal matrix. $\| \mathbf{Z}(k) \|_F$ is the weighted consensus error of $\tilde{\mathbf{Z}}(k)$ in the augmented network.

2. $\| \mathbf{S}_V(k) \|_F = \| \mathbf{T}_B(k)\tilde{\mathbf{S}}_V(k)\tilde{\mathbf{A}}^{-1} \|_F$, where $\mathbf{T}_B = \mathbf{I}_V(k) - \frac{1}{\eta} \mathbf{1}_V(k)^T \mathbf{v}(k)^T$ (see also: Definition 2 in Appendix B.2), which is an error estimate corresponding to the gradient surrogates of distributed nodes.

3. $\| \mathbf{z}_\pi(k) \|_F$, where $\mathbf{z}_\pi(k) = \mathbf{A}(\mathbf{z}_\pi(k) - \mathbf{z}^*) = \mathbf{A}(\tilde{\mathbf{Z}}(k)^T \mathbf{p}(k) - \mathbf{z}^*)$, which is the optimality gap between the weighted average and the saddle point.

Then, we introduce the error vector

$$\mathbf{e}(k) = [\mathbb{E}\| \mathbf{Z}(k) \|_F \quad \mathbb{E}\| \mathbf{S}_V(k) \|_F \quad \mathbb{E}\| \mathbf{z}_\pi(k) \|_F]^T$$

and demonstrate that it converges to a neighborhood of zero as $k \to \infty$ by employing the following lemma.

Lemma 4. With a sufficiently small $\eta$ in (8), the following inequality holds,

$$[\mathbf{e}^T(k + \tilde{t}) \cdots \mathbf{e}^T(k + 1)]^T \preceq \mathbf{M}(\eta) [\mathbf{e}^T(k + \tilde{t} - 1) \cdots \mathbf{e}^T(k)]^T + \mathbf{w}(\sigma) \quad (13)$$

where $\preceq$ denotes entry-wise inequality, $\tilde{t} \in \mathbb{N}^+$ can be found in Corollary 1, Appendix B.3, $\mathbf{M}(\eta) \in \mathbb{R}^{3\tilde{t} \times 3\tilde{t}}$, whose spectral radius $\varrho(\mathbf{M}(\eta)) < 1$, is the upper-bounded state transition matrix and $\mathbf{w}(\sigma)$ is the error term caused by the stochasticity of the gradients and the asynchronous nature.

Let $[\mathbf{e}(k + \tilde{t} - 1)^T \cdots \mathbf{e}(k)]^T = \mathbf{\hat{e}}(k)$. Note from (13),

$$\mathbf{\hat{e}}(k) \preceq \mathbf{M}^k(\eta)\mathbf{\hat{e}}(0) + \sum_{j=0}^{k-1} \mathbf{M}^j(\eta)\mathbf{w}(\sigma).$$

Since a sufficiently small $\eta$ guarantees that $\varrho(\mathbf{M}(\eta)) < 1$, then $\mathbf{e}(k)$ converges to a neighborhood of zero linearly. Clearly, since $\mathbf{e}(k)$ is bounded, then

$$\mathbb{E}\| \mathbf{z}_i(k) - \mathbf{z}^* \| \leq 2\mathbb{E}\| \tilde{\mathbf{Z}}(k) \|_F + \mathbb{E}\| \mathbf{z}_\pi(k) - \mathbf{z}^* \|, \forall i \in \mathcal{V},$$

is bounded. The result of Theorem 1 then follows.
Figure 2: (a) Network topology with eight nodes. (b) Convergence performance of PD-APP and PD-distIAG with similar cores. (c) Convergence performance of PD-APP and PD-distIAG with one core slowed down.

Figure 3: Speedup of PD-APP and FDPE when one core slows down. In the simulation, the computational capabilities of different CPU cores are similar. We manually slow down one core to simulate different computational speeds.

4 Numerical results

4.1 Experiment for MARL

To compare our work with the existing works, we conducted an experiment on mountaincar reinforcement learning task, which was first introduced in [31]. We adopt a simple Sarsa algorithm with feature number $d = 300$ to learn a fixed policy. With the given policy, we obtain trajectory experiences of states, actions and rewards of sample size $M = 5000$. To satisfy the MARL setting, we randomly divide the reward into different portions for separated nodes and make sure that we can retrieve the total reward by averaging the local rewards.

We compare our proposal to PD-distIAG [38]. In this experiment, we have eight nodes served by eight CPU cores. The network topology is the digraph counterpart of [22], i.e., each node $i$ is able to send information to node $\text{mod}(2^j + i, n)$, where $0 \leq j < \lfloor \log_2(n) \rfloor$. The connectivity is $O(\log(n))$. An eight-node example is demonstrated in Fig.2a. The learning rates for primal variables of PD-APP and PD-distIAG are $5 \times 10^{-5}$ and $8 \times 10^{-6}$, which are manually selected for the best performance, and we choose $\zeta = 0.5$ for both algorithms.

In Fig. 2b, the PD-APP shows its performance in accordance with our theoretical result and confirms the converge to a neighborhood of the optimal solution at a linear rate. Then we randomly slow down one core for both algorithms. Fig.1c demonstrates that PD-APP maintains its performance while PD-distIAG
suffers greatly from the slow core.

4.2 Experiment for the parallel RL

For the parallel RL, we consider the speedup effect, i.e., the more nodes the better convergence speed. On mountaincar task, the data \( \{s_p, a_p, s_{p+1}, R_i(s_p, a_p)\}_{p=1}^m \) with \( d = 300, M = 24000 \) are evenly divided to \( n \)-node networks. The network topology is the same as in Subsection 4.1. We compare the speedup result with FDPE [5]. We set the local batch size as \( 1024/n \), so that the problem has a fixed computational workload for networks with different numbers of nodes. We set \( \eta = 2 \times 10^{-5}, \zeta = 0.5 \) for both methods. The algorithms stop if \( \|\theta(t) - \theta^*\| \leq 10^{-3} \). We also test the robustness of both algorithms by slowing one node down. We manually add a time delay (2ms) after the node finishes each local update to simulate either poor communication situation or slow computation. Fig. 3 shows the speedup of the running time for PD-APP and FDPE, which demonstrates that PD-APP maintains its linear speedup in both situations, while FDPE, as a synchronous method, is decelerated even if only one node is disturbed.

5 Conclusion

This work has proposed a novel fully asynchronous PD-APP for policy evaluation in the disRL over a directed p2p network, which covers the MARL and parallel RL. The striking feature allows each node to communicate with its neighbors and compute its local variables at any time, without any form of coordination. In the worst-case view, we show that the PD-APP converges linearly with respect to an newly introduced virtual counter. Simulation results on the mountaincar have illustrated the significance of the PD-APP.
Appendix A and B

A Details of the augmented system

We list the following notations and definitions for the appendix.

- $a, a$ and $A$ are used to denote a scalar, column vector and matrix, respectively.
- $a^T$ and $A^T$ are transposes of $a$ and $A$, respectively.
- $\| \cdot \|_2$ denotes the $l_2$-norm of a vector or matrix. $\| \cdot \|_F$ denotes the matrix Frobenius norm.
- $A$ is called a row-stochastic matrix if each element of $A$ is nonnegative and $A1 = 1$. $A$ is column-stochastic if $A^T$ is row-stochastic.
- $[A]_{ij}$ denotes the element in row $i$ and column $j$ of $A$.
- $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$ denote the largest and smallest eigenvalues of $A$, respectively.
- $[x]$ denotes the largest integer less than or equal to $x$.
- $1_n$ and $0_n$ denote the $n$-dimensional vector with all ones and all zeros, respectively.
- The distance between nodes $i$ and $j$ in a graph $G$ is the smallest number of edges among all paths from $i$ to $j$. We use $d_g \leq n$ to denote the diameter of a strongly connected graph $G$, which is the largest distance between any pair of nodes.
- $\otimes$ denotes the matrix Kronecker product.

A.1 The explicit expression of system matrices

Recall the augmented system transition in Lemma 2,

$$
\tilde{Z}(k+1) = \tilde{A}(k)(\tilde{Z}(k) - \eta I^a(k)\tilde{S}(k)\Lambda),
$$

$$
\tilde{S}(k+1) = \tilde{B}(k)\tilde{S}(k) + \partial(k+1) - \partial(k),
$$

where

$$
\tilde{Z}(k) = \begin{bmatrix} Z^{(0)}(k)^T & \cdots & Z^{(b)}(k)^T \end{bmatrix}^T \in \mathbb{R}^{n \times 2d},
$$

$$
Z^{(u)}(k) = \begin{bmatrix} z_1^{(u)}(k) & \cdots & z_n^{(u)}(k) \end{bmatrix}^T \in \mathbb{R}^{n \times 2d},
$$

$$
\tilde{S}(k) = \begin{bmatrix} S^{(0)}(k)^T & \cdots & S^{(b)}(k)^T \end{bmatrix}^T \in \mathbb{R}^{n \times 2d},
$$

$$
S^{(u)}(k) = \begin{bmatrix} s_1^{(u)}(k) & \cdots & s_n^{(u)}(k) \end{bmatrix}^T \in \mathbb{R}^{n \times 2d},
$$

$$
\partial(k) = \begin{bmatrix} \partial J_1(k)^T & \cdots & \partial J_n(k)^T \end{bmatrix}^T \in \mathbb{R}^{n \times 2d},
$$

$$
\Lambda = \begin{bmatrix} I_d & 0 \\ 0 & \zeta I_d \end{bmatrix}.
$$
We now explicitly provide $\tilde{A}(k), \tilde{B}(k), I^o(k)$ as follows,

$$\left[\tilde{A}(k)\right]_{ij} = \begin{cases} \frac{1}{|Z_i^{rec}(k)|}, & \text{if } i, v \in \mathcal{V}, j = nu + v, t(k+1) \in T_i, \\
1, & \text{if } i \in \mathcal{V}, t(k+1) \notin T_i \text{ and } j = i, \\
1, & \text{if } i \notin \mathcal{V} \text{ and } j = i-n, \\
0, & \text{otherwise}, \end{cases}$$

$$\left[\tilde{B}(k)\right]_{ji} = \begin{cases} \frac{1}{|\mathcal{V}^{out}|}, & \text{if } i, v \in \mathcal{V}, j = nu + v, t(k+1) \in T_i, \\
1, & \text{if } i \in \mathcal{V}, t(k+1) \notin T_i \text{ and } j = i, \\
1, & \text{if } i \notin \mathcal{V} \text{ and } j = i-n, \\
0, & \text{otherwise}, \end{cases}$$

where $|Z_i^{rec}(k)|$ is the number of elements in the buffer $Z_i^{rec}$ at time $t(k+1)$. Moreover,

$$[I^o(k)]_{ij} = \begin{cases} 1, & \text{if } i = j, i \in \mathcal{V}, \text{ and } t(k+1) \in T_i, \\
0, & \text{otherwise}. \end{cases}$$

The stochastic gradient is sampled only when the node is activated, i.e.,

$$\begin{cases} \partial J_i(k) = \left[\nabla_{\theta_i} J_i, p_{i,0}(k) \right] \omega_i(k), & \text{if } t(k) \in T_i \\
\partial J_i(k) = \partial J_i(k-1), & \text{if } t(k) \notin T_i \end{cases}$$

with $\tilde{Z}(0) = \mathbf{Z}^{(0)}(0)^T \mathbf{0}_{(n-n) \times 2m}^T$, $\tilde{S}(0) = \mathbf{\partial J}(k)(0)^T \mathbf{0}_{(n-n) \times 2m}^T$, $\mathbf{\partial J}(k)(0) = \mathbf{0}$, and $T_i$ is an increasing sequence to record the updating time instants of node $i$.

We provide an example of $\tilde{A}(k)$ and $\tilde{B}(k)$ in the next subsection. $I^o(k)$ is a diagonal matrix whose $i$-th diagonal entry is 1 if and only if node $i$ is activated at time $t(k+1)$. Noticing that $\partial J_i(k) = \partial J_i(k+1)$ for any $i \in \{i[[I^o(k)]_{ii} = 0\}$, the second equality in (10) is also equivalent to $\tilde{S}(k+1) = B(k)\tilde{S}(k) + I^o(k)(\partial(k+1) - \partial(k))$. Recall that $\tilde{A}(k)$ and $\tilde{B}(k)$ are row-stochastic and column-stochastic, respectively, which results from the use of two different types of virtual nodes. By the column-stochasticity of $\tilde{B}(k)$, we left multiply the second equation of (10) with $1_n^T$ and obtain the following important property of $\tilde{S}(k)$:

$$1_n^T \tilde{S}(k) = 1_n^T \mathbf{\partial}(k) = 1_n^T \partial J(k), \quad \forall k \geq 0,$$

which implies that $s_i$, each row vector in $\tilde{S}(k)$, serves as an asymptotically unbiased estimate of the global gradients.

The matrices $\tilde{A}$ and $\tilde{A}^{-1}$ introduced in Definition 1 of the main body are used to study the convergence of the global average variables in Lemma 8 and Lemma 9, which are

$$\tilde{A} = \begin{bmatrix} I_d & 0 \\ 0 & \zeta^{-1/2} I_d \end{bmatrix}, \quad \tilde{A}^{-1} = \begin{bmatrix} I_d & 0 \\ 0 & \zeta^{1/2} I_d \end{bmatrix}.$$  

With $\tilde{A}$, we rewrite (10) into the following form:

$$\tilde{Z}(k+1)\tilde{A} = \tilde{A}(k)(\tilde{Z}(k)\tilde{A} - \eta I^o(k)\tilde{S}(k)\tilde{A}^{-1}),$$

$$\tilde{S}(k+1)\tilde{A}^{-1} = \tilde{B}(k)\tilde{S}(k)\tilde{A}^{-1} + (\partial(k+1) - \partial(k))\tilde{A}^{-1}.$$  

A.2 A simple example for the augmented system

Consider an example with three nodes, the original graph of which is illustrated in Fig. 4a. We let $b = 2$ to illustrate the example. Based on the construction of the virtual graph in Section 3.3, there are
1 \times (2 + 1) = 9 \text{ nodes in the virtual } z\text{-graph, and another 9 nodes in the virtual } s\text{-graph, as depicted in Fig. 4c and Fig. 4d.}

Suppose that at a specific time } t(k) \text{, nodes 1, 2 and 3 are all activated as in Fig. 4b. Node 1 uses the latest information } z_1(k - 1) \text{ from itself, } z_2(k - 2) \text{ from node 2 and } z_3(k - 3) \text{ from node 3, with 0-step, 1-step and 2-step time delays, respectively, to compute } z_1(k). \text{Node 2 receives the information } z_1(k - 1) \text{ and adds its own stock } z_2(k - 1) \text{ to compute } z_2(k). \text{Node 3 computes } z_3(k) \text{ from } z_2(k - 2) \text{ and } z_3(k - 1). \mathcal{G}_z(k) \text{ is in Fig. 4c and the corresponding } \bar{A}(k) \text{ is }

\begin{pmatrix}
1 & 0 & 0 \\
\frac{1}{3} & 0 & 0 \\
\frac{1}{3} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}

\begin{pmatrix}
\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}

\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}

Meanwhile, nodes 1, 2 and 3 send and receive the s vectors together with the z vectors described above. However, } \mathcal{G}_s(k) \text{ and } \bar{B}(k) \text{ are inconsistent with those corresponding to } \mathcal{G}_z(k) \text{ and } \bar{A}(k), \text{ since the former ones follow the future-surrogate way and the latter ones follow the waiting-queue way. } \mathcal{G}_s(k) \text{ is in Fig. 4d and } \bar{B}(k) \text{ is depicted above.}
B Proof of Theorem 1

B.1 Proof of Lemma 1

The proof of Lemma 1 is fully based on the two statements below.

1. Under Assumption 1(b), let \( b_1 = (n-1)[\bar{\tau}/\tau] + 1 \). Then, within the time interval \((t(k), t(k+b_1)]\), each node performs its local update at least once.

2. Under Assumptions 1(a) and 1(c), let \( b_2 = n[\tau/\tau] \) and \( b = b_1 + b_2 \). Then, the information sent from node \( i \) at time \( t(k) \) can be received by node \( j \) before time \( t(k+b_2) \) and used for computing an update before time \( t(k+b) \) for any \( k \) and \( (i,j) \in \mathcal{E} \).

Proof. 1. Suppose that node \( i \) is not activated during the time interval \((t(p), t(q)], p, q \in \mathbb{N} \) but is activated at \( t(q+1) \). Under Assumption 1(b), then, \( t(q) - t(p) \leq \bar{\tau} \). Thus, any other node can be activated at most \( \lfloor (t(q) - t(p))/\tau \rfloor \) times during the time interval \((t(p), t(q)], \) i.e., \( q - p \leq (n-1)[\bar{\tau}/\tau] \). Then the first part of the result follows.

2. Suppose that node \( i \) sends message at time \( t(p), p \in \mathbb{N} \), which is received by node \( j \) in the time interval \((t(q), t(q+1)], q \in \mathbb{N} \). From Assumption 1(c), then, \( t(q) - t(p) \leq \tau \). In addition, Assumption 1(c) implies that any node can be activated at most \( \lfloor \tau/\tau \rfloor \) times during \([t(p), t(q)], \) i.e., \( q - p + 1 \leq n[\tau/\tau] \), and thus, \( q + 1 \leq p + n[\tau/\tau] \). Let \( p = k \) and the second result follows immediately.

In view of the two statements above, Lemma 1 follows immediately.

B.2 Proof of Lemma 4

We have introduced \( \bar{Z}(k), \bar{S}_\nu(k) \) and \( \bar{z}_\nu(k) \) in Definition 1 and used that the Frobenius norms of the above matrices are bounded in a linear inequality system in Lemma 4, the derivation of which depends on the following four claims.

Claim 1. Let \( \|\bar{Z}(k)\|_F = \|T_\check{\lambda}(k)\bar{Z}(k)\bar{A}\|_F \), where \( T_\check{\lambda}(k) = I_{\check{\lambda}} - \lambda_1 \pi(k-1)^T \) and \( \pi(k+1)^T \bar{A}(k) = \pi(k)^T \) is the absolute probability sequence of \( \bar{A}(k) \) in Lemma 3. Then, \( \|\bar{Z}(k)\|_F \) is bounded as follows:

\[
\mathbb{E}\|\bar{Z}(k+t)\|_F \leq 2\mu \mathbb{E}\|\bar{Z}(k)\|_F + \eta \sqrt{2\check{\lambda}} \sum_{t=0}^{T-1} \mathbb{E}\|\bar{S}(k+t)\bar{A}^{-1}\|_F, \quad \forall k \geq 0, \tag{17}
\]

where \( \mu \) is a constant defined in Corollary 1 of Appendix B.3.

Proof: See Appendix B.4.1.

Noticing that the update in (16), \( \bar{S}(k+1)\bar{A}^{-1} = \bar{B}(k)\bar{S}(k)\bar{A}^{-1} + (\partial(k+1) - \partial(k))\bar{A}^{-1} \), only happens if \( i \) is active at time \( k \), we define \( \nu(k) \) as follows and then focus on \( \bar{S}_\nu(k) = V^\dagger(k)\bar{S}(k) \).

Definition 2. Let \( \nu(k+1) = \bar{B}(k)\nu(k) \) and \( \nu(0) = [1_n^T, 0_{n-n}^T]^T \). Let \( V(k) = \text{diag}(\nu(k)) \) and \( V(k)^\dagger \) be the Moore-Penrose inverse of \( V(k), \) i.e.,

\[
[V(k)]_{ij} = \begin{cases} 
1/[V(k)]_{ii}, & \text{if } i = j \leq n \text{ and } [V(k)]_{ii} > 0, \\
0, & \text{otherwise.}
\end{cases}
\]

Then, we define

\[
\bar{S}_\nu(k) = V^\dagger(k)\bar{S}(k), \tag{18}
\]

where \( \bar{S}(k) \) is from (16).
Claim 2. Let $\|\mathbf{S}_k\|_F = \|\mathbf{T}_k\mathbf{S}_k\|_F$, where $\mathbf{T}_k = \mathbf{I}_V(k) - \frac{1}{\eta} \mathbf{1}_V(k)\mathbf{v}(k)^T$. Then,

$$
\mathbb{E}\|\mathbf{S}_k^{(k + l)}\|_F \leq 2\theta^{-1}\mu_n \mathbb{E}\|\mathbf{S}_k\|_F + 2\beta\theta^{-1}\sqrt{n} \sum_{t=0}^{l-1} 2\sqrt{n}\mathbb{E}\|\mathbf{Z}(k)\|_F
$$

(19)

$$
+ 2\beta\theta^{-1} \sum_{t=0}^{l-1} \eta \mathbb{E}\|\mathbf{S}(k + t)\|_F + 2\sqrt{2}\theta^{-1}\sqrt{n\eta t}\sigma, \quad \forall k \geq 0,
$$

where $\theta$ and $\beta$ are as introduced in Lemma 7 and Lemma 9, respectively.

Proof: See Appendix B.4.2.

Claim 3. Let $\mathbf{z}_n(k) = \mathbf{A}(\mathbf{z}_n(k) - \mathbf{z}^*) = \mathbf{A} \left( \mathbf{Z}(k)^T\mathbf{p}(k) - \mathbf{z}^* \right)$, the expectation of which is bounded as follows:

$$
\mathbb{E}\|\mathbf{z}_n(k + b)\|_F \leq (1 - \eta \alpha b n) \mathbb{E}\|\mathbf{z}_n(k)\|_F
$$

(20)

$$
+ \eta(2n + \sqrt{n})\beta \sum_{t=0}^{b-1} \mathbb{E}\|\mathbf{Z}(k + t)\|_F
$$

$$
+ \eta \beta n \sqrt{n}\mathbb{E}\|\mathbf{S}_k\|_F + \eta \beta n \sigma, \quad \forall k \geq 0,
$$

where $\alpha$ is as defined in Lemma 8.

Proof: See Appendix B.4.3.

Claim 4. The following inequality holds:

$$
\mathbb{E}\|\mathbf{S}(k)\|_F \leq \beta \sqrt{n\eta} (\sqrt{n} + 1) \mathbb{E}\|\mathbf{Z}(k)\|_F + \mu \mathbb{E}\|\mathbf{S}_k\|_F
$$

(21)

$$
+ \beta n \sqrt{n}\mathbb{E}\|\mathbf{z}_n(k)\|_F + \mu \beta n \sigma, \quad \forall k \geq 0.
$$

Proof: See Appendix B.4.4.

Then we provide the explicit form of the linear inequality system in Lemma 4. Summarizing the results of (17), (19), (20) and (21), we bound $\mathbf{e}(k + l)$ by

$$
\mathbf{e}(k + l) \leq \left[ \begin{array}{ccc}
\eta \sqrt{2n} c_1 & \eta \sqrt{2n} \beta & \eta \sqrt{2n} c_2 \\
\eta n & 2\beta^{-1} n & 0
\end{array} \right] (\mathbf{e}(k + l - 1) + \cdots + \mathbf{e}(k + l - b + 1))
$$

$$
+ \left[ \begin{array}{ccc}
\eta \sqrt{2n} c_1 & \eta \sqrt{2n} \beta & \eta \sqrt{2n} c_2 \\
\eta n & 2\beta^{-1} n & 0
\end{array} \right] \mathbf{e}(k + l - b)
$$

$$
+ \left[ \begin{array}{ccc}
\eta \sqrt{2n} c_1 & \eta \sqrt{2n} \beta & \eta \sqrt{2n} c_2 \\
0 & \eta n & 0
\end{array} \right] \mathbf{e}(k + l - b - 1) + \cdots + \mathbf{e}(k + l - b + 1)
$$

$$
+ \left[ \begin{array}{ccc}
\eta \sqrt{2n} c_1 + 2\mu & \eta \sqrt{2n} \beta & \eta \sqrt{2n} c_2 \\
0 & 2\beta^{-1} n + 2\mu n & 0
\end{array} \right] \mathbf{e}(k + l) + \mathbf{w}(\sigma),
$$

(22)
where \( c_1 = \beta \sqrt{n+1}, c_2 = \beta n \sqrt{n} \) and \( c_3 = (2n + \sqrt{n}) \beta \).

The matrix form of (22) can also be written as:

\[
\begin{pmatrix}
M_1 & \cdots & M_1 & M_2 & M_3 & \cdots & M_3 & M_4 \\
I & & & & & & & \\
\vdots & & & & & & & \\
e(k + \hat{i}) & & & & & & & \vdots \\
\vdots & & & & & & & \\
e(k + 1) & & & & & & & I
\end{pmatrix}
\begin{pmatrix}
e(k + \hat{i} - 1) \\
\vdots \\
e(k)
\end{pmatrix}
+ w(\sigma), \quad (23)
\]

where

\[
w(\sigma) = \left[ \sqrt{2\eta n c_2}, 2\beta^{-1} n c_2, \sqrt{2\eta n c_2}, \eta c_3, 0, \ldots, 0 \right]^{T} \in \mathbb{R}^M.
\]

The matrix \( M(\eta) \) in (23) is exactly the one in Lemma 4 and can be further decomposed into \( M(\eta) = M^0 + \eta M^E \), i.e.,

\[
M(\eta) = \begin{pmatrix}
M_1^0 & \cdots & M_1^0 & M_2^0 & M_3^0 & \cdots & M_3^0 & M_4^0 \\
I & & & & & & & I \\
\vdots & & & & & & & \vdots \\
0 & & & & & & & 0 \\
\vdots & & & & & & & \vdots \\
0 & & & & & & & I
\end{pmatrix}
+ \eta \begin{pmatrix}
M_4^E & \cdots & M_4^E & M_3^E & M_2^E & \cdots & M_3^E & M_2^E \\
0 & & & & & & & 0 \\
\vdots & & & & & & & \vdots \\
0 & & & & & & & 0
\end{pmatrix}, \quad (24)
\]

where

\[
M_1^0 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
4\beta^{-1} \sqrt{n c_1} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, 
M_1^E = \begin{bmatrix}
\sqrt{2n c_1} & \sqrt{2n c_1} & \sqrt{2n c_2} \\
\eta c_3 & 2\beta^{-1} \sqrt{n c_1} & 2\beta^{-1} \sqrt{n c_1} & 2\beta^{-1} \sqrt{n c_2} \\
\eta c_3 & \eta c_3 & \eta c_3 & \eta c_3
\end{bmatrix},
\]

\[
M_2^0 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, 
M_2^E = \begin{bmatrix}
\sqrt{2n c_1} & \sqrt{2n c_1} & \sqrt{2n c_2} \\
\eta c_3 & 2\beta^{-1} \sqrt{n c_1} & 2\beta^{-1} \sqrt{n c_1} & 2\beta^{-1} \sqrt{n c_2} \\
\eta c_3 & \eta c_3 & \eta c_3 & \eta c_3
\end{bmatrix},
\]

\[
M_3^0 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, 
M_3^E = \begin{bmatrix}
\sqrt{2n c_1} & \sqrt{2n c_1} & \sqrt{2n c_2} \\
\eta c_3 & 2\beta^{-1} \sqrt{n c_1} & 2\beta^{-1} \sqrt{n c_1} & 2\beta^{-1} \sqrt{n c_2} \\
\eta c_3 & \eta c_3 & \eta c_3 & \eta c_3
\end{bmatrix},
\]

\[
M_4^0 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, 
M_4^E = \begin{bmatrix}
\sqrt{2n c_1} & \sqrt{2n c_1} & \sqrt{2n c_2} \\
\eta c_3 & 2\beta^{-1} \sqrt{n c_1} & 2\beta^{-1} \sqrt{n c_1} & 2\beta^{-1} \sqrt{n c_2} \\
\eta c_3 & \eta c_3 & \eta c_3 & \eta c_3
\end{bmatrix}.
\]

To prove that the spectral radius of \( M(\eta) \) is strictly smaller than 1, we introduce the following matrix eigenvalue perturbation lemma in [17, Theorem 6.3.12].
Lemma 5. Let \( A, E \in \mathbb{R}^{n \times n} \) and suppose that \( \lambda \) is a simple eigenvalue of \( A \). Let \( x, y \) be, respectively, the right and left eigenvectors of \( A \) corresponding to \( \lambda \). Then, there is a unique eigenvalue \( \lambda(t) \) such that \( \lambda(t) \) is continuous and differentiable at \( t = 0 \), and

\[
\frac{d\lambda(t)}{dt} \bigg|_{t=0} = \frac{y^T Ex}{y^Tx}.
\]

We next have a lemma describing the unit eigenvalue and corresponding eigenvectors of matrix \( M_0 \).

Lemma 6. The spectral radius of \( M_0 \) is 1, which is also a simple eigenvalue of \( M_0 \). One pair of right and left eigenvectors \( u, v \) such that \( M_0 u = u \), \( v^T M_0 = v^T \), are given by

\[
\begin{align*}
\mathbf{u} &= 1_e \otimes [0 \ 0 \ 1]^T, \\
\mathbf{v}^T &= [0, 0, 1, 0, \ldots, 0, 0, 0, 0, 1, 0, \ldots, 0].
\end{align*}
\]

(26)

Proof. The characteristic polynomial of \( M_0 \) is \((-1)^{(n-1)}(\lambda-1)(2\mu-2\phi+\lambda\mu)\lambda^{n-1}\), where \( 2\mu, 2\phi-\mu \in (0,1) \), given in Corollary 1 and Corollary 2, respectively. Then, it is obvious that \( \varrho(M_0) = 1 \) and \( \lambda = 1 \) is a simple root of the characteristic polynomial. By direct calculation, we can verify that \( u, v \) are, respectively, the corresponding right and left eigenvectors of \( M_0 \).

Now we discuss the spectral radius of \( M(\eta) \) based on Lemma 5 and Lemma 6. Let \( q(\eta) \) be one simple eigenvalue of \( M(\eta) \) satisfying that \( q(0) = 1 \). By Lemma 5, we obtain the derivative of \( q(\eta) \), i.e.,

\[
\left. \frac{dq(\eta)}{d\eta} \right|_{\eta=0} = \frac{v^T M_E u}{v^T u},
\]

where \( u, v \) are right and left eigenvectors of \( M_0 \) corresponding to the simple eigenvalue, \( q(0) \). By direct computation, it can be verified that

\[
v^T u = 2, v^T M_E u = -\alpha bn < 0,
\]

which implies that

\[
\left. \frac{dq(\eta)}{d\eta} \right|_{\eta=0} < 0,
\]

i.e., the spectral radius \( \varrho(M(\eta)) < 1 \) if \( \eta \) is sufficiently small, which is in accordance with the result in Lemma 4.

B.3 Some preliminaries

Before proving the four claims in B.2, we first introduce some important properties of \( \widetilde{A}(k) \) and \( \widetilde{B}(k) \). We start with the accumulative products of \( \widetilde{A}(k) \) and \( \widetilde{B}(k) \), which converge to rank-one matrices linearly.

Lemma 7. ([42, Lemma 2] and [26, Lemma 5]) Define the accumulative product of \( \widetilde{A}(k) \) and \( \widetilde{B}(k) \) as

\[
\Phi_A(k, k + t) = \widetilde{A}(k + t - 1) \cdots \widetilde{A}(k + 1) \widetilde{A}(k),
\]

\[
\Phi_B(k, k + t) = \widetilde{B}(k + t - 1) \cdots \widetilde{B}(k + 1) \widetilde{B}(k),
\]

where \( k \geq 0, t > 0 \). If \( t = 0 \), then define \( \Phi_A(k, k) = \Phi_B(k, k) = I_n \).

Under Assumption 1, the following statements hold.
(a) For all \(k,t \geq 0\), there exist stochastic vectors \(\phi_A(k, k + t), \phi_B(k, k + t) \in \mathbb{R}^n\) such that

\[
\|\phi_A(k, k + t) - \mathbf{1}_n \phi_A(k, k + t)\|_F \leq 2\kappa t,
\]
\[
\|\phi_B(k, k + t) - \phi_B(k, k + t)\|_F \leq 2\kappa t,
\]

where \(\kappa = (1 - \theta)^{1/d_b}\) and \(\theta = (1/n)^{d_b} \in (0, 1)\), \(b\) is as stated in Lemma 1 and \(d_b\) denotes the diameter of \(\mathcal{G}\).

(b) For all \(i \in \mathcal{V}, j \in \tilde{\mathcal{V}}, t \geq 0\), the following inequality is in force:

\[
\sum_{j=1}^n [\phi_B(0, t)]_{ij} \geq n\theta.
\]

Then the following corollary follows directly, by which we introduce the constant \(\tilde{t}\) and \(\mu\) to characterize the long-term convergence property of our method.

**Corollary 1.** Under the conditions in Lemma 7, define \(\mu < \theta/2n \in (0, 1)\). There exists a constant \(\tilde{t} \geq 0\) such that \(\kappa^\tilde{t} \leq \mu/2\), and

\[
\|\phi_A(k, k + \tilde{t}) - \mathbf{1}_n \phi_A(k, k + \tilde{t})\|_F \leq \mu, \quad \forall k \geq 0,
\]
\[
\|\phi_B(k, k + \tilde{t}) - \phi_B(k, k + \tilde{t})\|_F \leq \mu, \quad \forall k \geq 0.
\]

We use the following two lemmas to study the properties of \(z\) and its corresponding full gradients.

**Lemma 8.** Let

\[
G = \begin{bmatrix}
\eta I - \sqrt{\zeta} \hat{A}^T \\
\sqrt{\zeta} \hat{A} \\
\zeta C
\end{bmatrix}, \quad \nabla j(z) = \begin{bmatrix}
\nabla_{\theta j}(\theta, \omega) \\
-\nabla_{\omega j}(\theta, \omega)
\end{bmatrix} = \begin{bmatrix}
\hat{A}^T \omega + \rho \theta \\
\hat{A} \theta - \hat{C} \omega - \tilde{b}
\end{bmatrix},
\]

where \(z = [\theta^T \quad \omega^T]^T\).

If \(\zeta = \eta_1/\eta_2, \eta_1 = \eta\) satisfies that

\[
\zeta = \frac{8(\eta + \lambda_{\max}(\hat{A}^T \hat{C}^{-1} \hat{A}))}{\lambda_{\min}(\hat{C})},
\]

\[
0 < \eta < \left| \frac{\lambda_{\min}(\hat{C})}{\lambda_{\max}(\hat{C})} \right| \frac{1}{\lambda_{\max}(\eta I + \hat{A}^T \hat{C}^{-1} \hat{A})},
\]

then there exists \(\alpha = \lambda_{\min}(G) > 0\), such that

\[
\|\hat{A}z - \eta \hat{A}^{-1} \nabla j(z) - \hat{A}z^*\|_2 \leq (1 - \alpha\eta)\|\hat{A}z - \hat{A}z^*\|_2, \quad \forall z \in \mathbb{R}^{2d}.
\]

**Proof.** According to [8, Appendix A], the eigenvalues of \(G\) satisfies that

\[
\lambda_{\max}(G) \leq \left| \frac{\lambda_{\max}(\hat{C})}{\lambda_{\min}(\hat{C})} \right| \lambda_{\max}(\eta I + \hat{A}^T \hat{C}^{-1} \hat{A})), \quad \lambda_{\min}(G) \geq \frac{8}{9} \lambda_{\min}(\hat{A}^T \hat{C}^{-1} \hat{A}) > 0.
\]

By direct computation, we can verify that

\[
\hat{A}z - \eta \hat{A}^{-1} \nabla j(z) - \hat{A}z^* = (I - \eta G)\hat{A}(z - z^*),
\]

where \(\phi(I - \eta G) \leq 1 - \eta\lambda_{\min}(G)\). Define \(\alpha = \lambda_{\min}(G)\) and the result follows.
**Lemma 9.** Let 
\[ G_i = \begin{bmatrix} \eta I & -\sqrt{\zeta} A_i \\ \sqrt{\zeta} A_i & \zeta C_i \end{bmatrix}, \quad \nabla j_i(z) = \begin{bmatrix} \nabla \theta J_i(\theta, \omega) \\ -\nabla \omega J_i(\theta, \omega) \end{bmatrix} = \begin{bmatrix} \hat{A}_i^T \omega + \rho \theta \\ \hat{A}_i \theta - \hat{C}_i \omega - \hat{b}_i \end{bmatrix}, \]
where \( z = [\theta^T \quad \omega^T]^T \). Then,
\[ \| \hat{A}^{-1}(\nabla J_i(z_1) - \nabla J_i(z_2)) \|_2 \leq \beta \| \hat{A}(z_1 - z_2) \|_2, \quad \forall z_1, z_2 \in \mathbb{R}^{2d}, \forall 1 \leq i \leq n, \]
where \( \beta = \max_{1 \leq i \leq n} \lambda_{\max}(G_i) \).

**Proof.** By direct computation, we can verify that
\[ \hat{A}^{-1}(\nabla j_i(z_1) - \nabla j_i(z_2)) = G_i \hat{A}(z_1 - z_2), \quad \forall z_1, z_2 \in \mathbb{R}^{2d}, \forall 1 \leq i \leq n, \]
and the result follows. \( \blacksquare \)

### B.4 Proof of the four claims

#### B.4.1 Proof of Claim 1

Recalling Definition 1 and \( T_{\lambda}(k) = I_n - 1_\lambda \pi^T(k - 1) \), we have
\[ \| \tilde{Z}(k + \tilde{t}) \|_F = \| T_{\lambda}(k + \tilde{t}) \tilde{Z}(k + \tilde{t}) \hat{A} \|_F \]
\[ \leq \| T_{\lambda}(k + \tilde{t}) \phi_{\lambda}(k, k + \tilde{t}) \tilde{Z}(k) \hat{A} \|_F \\
+ \eta \sum_{t=0}^{\tilde{t}-1} \| T_{\lambda}(k + \tilde{t}) \phi_{\lambda}(k + t, k + \tilde{t}) I^a(k) \tilde{S}(k + t) \hat{A}^{-1} \|_F. \tag{28} \]

For the first term in (28), we use the property that \( \pi(k) \) is row-stochastic and get
\[ T_{\lambda}(k + \tilde{t}) \phi_{\lambda}(k, k + \tilde{t}) = (I_n - 1_\lambda \pi(k + \tilde{t} - 1)^T) \phi_{\lambda}(k, k + \tilde{t}) \\
= (I_n - 1_\lambda \pi(k + \tilde{t} - 1)^T)(\phi_{\lambda}(k, k + \tilde{t}) - 1_\lambda \pi^T(k - 1)) \\
= (I_n - 1_\lambda \pi(k + \tilde{t} - 1)^T)(\phi_{\lambda}(k, k + \tilde{t}) - 1_\lambda \pi_{\lambda}(k, k + \tilde{t})^T)(I_n - 1_\lambda \pi^T(k - 1)) \\
= T_{\lambda}(k + \tilde{t})(\phi_{\lambda}(k, k + \tilde{t}) - 1_\lambda \phi_{\lambda}(k, k + \tilde{t})^T)T_{\lambda}(k), \tag{29} \]
Apply two important properties of Frobenius norm, \( \| AB \|_F \leq \| A \|_2 \| B \|_F \) and \( \| AB \|_F \leq \| A \|_F \| B \|_F \), and obtain
\[ \| T_{\lambda}(k + \tilde{t}) \phi_{\lambda}(k, k + \tilde{t}) \tilde{Z}(k) \hat{A} \|_F \\
\leq \| T_{\lambda}(k + \tilde{t}) \phi_{\lambda}(k, k + \tilde{t}) - I_n \phi_{\lambda}(k, k + \tilde{t}) \|_F \| T_{\lambda}(k) \tilde{Z}(k) \hat{A} \|_F \\
\leq 2\mu \| \tilde{Z}(k) \|_F, \tag{30} \]
where the last inequality is on the basis of \( \| T_{\lambda}(k + \tilde{t}) \|_2 = \| I_n - 1_\lambda \pi(k + \tilde{t} - 1)^T \|_2 < 2 \) and Corollary 1.

For the second term in (28),
\[ \eta \sum_{t=0}^{\tilde{t}-1} \| T_{\lambda}(k + \tilde{t}) \phi_{\lambda}(k + t) I^a(k) \tilde{S}(k) \hat{A}^{-1} \|_F \\
\leq \eta \sum_{t=0}^{\tilde{t}-1} \| T_{\lambda}(k + \tilde{t}) \phi_{\lambda}(k + t) I^a(k) \tilde{S}(k) \hat{A}^{-1} \|_F \\
\leq 2\eta \sum_{t=0}^{\tilde{t}-1} \| \phi_{\lambda}(k + t) \|_2 \| \tilde{S}(k) \hat{A}^{-1} \|_F \leq \eta \sqrt{2\eta} \sum_{t=0}^{\tilde{t}-1} \| \tilde{S}(k) \hat{A}^{-1} \|_F, \tag{31} \]

---

20
where \( \| \phi_A(k, k + t) \|_2 \leq \sqrt{n/2}, \forall t \geq 0. \)

Combining (30) and (31), we obtain

\[
\| \tilde{Z}(k + \hat{t}) \|_F \leq 2\mu \| \tilde{Z}(k) \|_F + \eta \sqrt{2\hat{n}} \sum_{t=0}^{\hat{t}-1} \| S(k) \tilde{\Lambda}^{-1} \|_F. \tag{32}
\]

The desired result then follows by taking the full expectation on both sides of (32).

### B.4.2 Proof of Claim 2

To study \( \tilde{S}_V(k) \) defined in (18), recall Definition 2 and let

\[
\begin{align*}
\mathbf{I}_V(k) &= V(k) V(k)^\dagger = V(k)^\dagger V(k), \\
1_V(k) &= \mathbf{I}_V(k) 1_n, \\
\tilde{B}_V(k) &= V(k + 1)^\dagger \tilde{B}(k) V(k).
\end{align*}
\tag{33}
\]

Furthermore, we define \( \Phi_V(k, k + t) \) as

\[
\Phi_V(k, k + t) = \prod_{j=k}^{k+t-1} \tilde{B}_V(j),
\tag{34}
\]

and state a lemma that \( \Phi_V(k, k + t) \) converges to a rank-one matrix at the same rate as \( \Phi_B(k, k + t) \).

**Lemma 10.** For all \( k \geq 0, t > 0, \)

\[
\left\| \Phi_V(k, k + t) - \frac{1}{n} 1_V(k) v(k)^\dagger \right\|_F < \theta^{-1} \cdot 2\kappa^t \cdot n.
\]

**Proof.** First,

\[
\begin{align*}
\Phi_V(k, k + t) &= \tilde{B}_V(k + t - 1) \cdots \tilde{B}_V(k) \\
&= V(k + t)^\dagger \tilde{B}(k + t - 1) \cdots \tilde{B}(k) V(k) \\
&= V(k + t)^\dagger \Phi_B(k, k + t) V(k).
\end{align*}
\tag{35}
\]

The last equality is tenable becasue \( \mathbf{I}_V(k + 1) \tilde{B}(k) V(k) = \tilde{B}(k) V(k), \forall k \geq 0, \) which can be verified by computing each row on both sides, noticing \( v(k + 1) = \tilde{B}(k) v(k). \)

By Lemma 7(a), there exists \( \Delta \Phi_B(k, k + t) \in \mathbb{R}^{n \times n} \) such that \( \| \Delta \Phi_B(k, k + t) \|_F \leq 2\kappa^t, \) and

\[
\phi_B(k, k + t) 1^\dagger + \Delta \Phi_B(k, k + t) = \Phi_B(k, k + t). \tag{36}
\]

Right multiply \( v(k) \) on both sides,

\[
\phi_B(k, k + t) 1^\dagger v(k) + \Delta \Phi_B(k, k + t) v(k) = \Phi_B(k, k + t) v(k) = v(k + t). \tag{37}
\]

Since \( \phi_B(k, k + t) 1^\dagger v(k) = \phi_B(k, k + t) 1^\dagger v(0) = n \phi_B(k, k + t), \) we have

\[
n \phi_B(k, k + t) + \Delta \Phi_B(k, k + t) v(k) = v(k + t), \tag{38}
\]

i.e.,

\[
\phi_B(k, k + t) = \frac{1}{n} (v(k + t) - \Delta \Phi_B(k, t) v(k)). \tag{39}
\]
It follows (35)-(39) that

\[
\left\| \Phi(k, k + t) - \frac{1}{n} 1_V(k) v(k)^T \right\|_F
= \left\| \Phi(k, k + t) - \frac{1}{n} V(k + t)^T v(k + t) 1_n^T V(k) \right\|_F
= \left\| V(k + t)^T \Delta \Phi_B(k, t) \left( I_n - \frac{1}{n} v(k) 1_n^T \right) \right\|_F
\leq \| V(k + t)^T \|_2 \| \Delta \Phi_B(k, t) \|_F \left\| \left( I_n - \frac{1}{n} v(k) 1_n^T \right) \right\|_F
< \theta^{-1} \cdot 2n^t \cdot n,
\]

where we have used the fact that all the entries of \( V^T(k) \) are less than \( \theta^{-1} \) from Lemma 7(b).}

The next corollary follows Lemma 10 by substituting \( \hat{t} \) into (27).

**Corollary 2.** If \( t = \hat{t} \) in Lemma 1, then,

\[
\left\| \Phi(k, k + \hat{t}) - \frac{1}{n} 1_V(k) v(k)^T \right\|_F < \theta^{-1} \mu n \mu < \frac{1}{2},
\]  (40)

Define \( D(k) = \partial J(k + 1) - \partial J(k) \) and rewrite the gradient update in (16) as

\[
\tilde{S}_V(k + 1) \tilde{A}^{-1} = \tilde{B}_V(k) \tilde{S}_V(k) \tilde{A}^{-1} + V(k + 1)^T D(k) \tilde{A}^{-1}.
\]  (41)

We use Lemma 11 to study the upper bound of \( \mathbb{E}\| D(k) \tilde{A}^{-1} \|_F \).

**Lemma 11.** For all \( k \geq 0 \), the following inequality holds,

\[
\mathbb{E}\| D(k) \tilde{A}^{-1} \|_F \leq 2\beta \sqrt{n} \mathbb{E}\| \tilde{Z}(k) \|_F + \beta \eta \mathbb{E}\| \tilde{S}(k) \tilde{A}^{-1} \|_F + \sqrt{2n} \zeta \sigma,
\]

where \( \beta \) is from Lemma 9.

**Proof.** Define the full gradient matrix \( \nabla(k) \) in the same form as \( \partial(k) \), i.e.,

\[
\nabla(k) = \begin{bmatrix} \nabla J(k)^T & 0_{2d \times bn} \end{bmatrix}^T \in \mathbb{R}^{n \times 2d},
\]

\[
\nabla J(k) = \begin{bmatrix} \nabla J_1(k) & \cdots & \nabla J_n(k) \end{bmatrix}^T \in \mathbb{R}^{n \times 2d},
\]

\[
\nabla J_i(k) = \begin{bmatrix} \nabla \theta_i J_i(\theta_i, \omega_i(k)) & -\nabla \omega_i J_i(\theta_i(\omega_i, \omega_i(k)) \cdot 1 \leq i \leq n,
\]

and let \( \nabla J(k)(0) = 0 \).

First, using triangle inequality, we obtain that

\[
\mathbb{E}\| D(k) \tilde{A}^{-1} \|_F = \mathbb{E}\| (\partial J(k + 1) - \partial J(k)) \tilde{A}^{-1} \|_F
= \mathbb{E}\| \nabla \theta_i(k) \partial J_i(k) - \nabla \theta_i J_i(k) \|_F + \mathbb{E}\| \nabla J(k + 1) \|_F \leq n(1 + \zeta) \sigma^2.
\]  (43)

Combining (15) and the condition that \( \mathbb{E}\| \nabla \theta_i(k) \partial J_i(k) - \nabla \theta_i J_i(k) \|_F^2 \leq n \sigma^2 \) from Assumption 2(b), we get

\[
\mathbb{E}\| \nabla \theta_i(k) \partial J_i(k) - \nabla \theta_i J_i(k) \|_F^2 \leq n(1 + \zeta) \sigma^2.
\]

Thus, for the first term in (43), we get

\[
\mathbb{E}\| \nabla \theta_i(k) \partial J_i(k) - \nabla \theta_i J_i(k) \|_F^2 = \mathbb{E}\| \nabla \theta_i(k) \partial J_i(k) - \nabla \theta_i J_i(k) \|_F^2 + \mathbb{E}\| \nabla \theta_i(k) \partial J_i(k) - \nabla J(k) \|_F^2
\]

\[
+ 2\mathbb{E}\| \nabla \theta_i(k) \partial J_i(k) - \nabla J(k) \|_F \mathbb{E}\| \nabla \theta_i(k) \partial J_i(k) - \nabla J(k) \|_F
\]

\[
\leq 2n \zeta^2 \sigma^2,
\]  (44)
where $\zeta = \sqrt{1+\zeta}$, and the last inequality follows from the independency between $I^a(k)\partial J(k+1)$ and $I^n(k)\partial J(k)$.

Combining (42) and Lemma 9, the second term can be bounded as follows,

$$\|\langle \nabla J(k+1) - \nabla J(k) \rangle \tilde{A}^{-1} \|_F \leq \beta \|\langle \tilde{Z}(k+1) - \tilde{Z}(k) \rangle \tilde{A} \|_F.$$  \hfill (45)

Meanwhile, by (16),

$$\|\langle \tilde{Z}(k+1) - \tilde{Z}(k) \rangle \tilde{A} \|_F = \| \tilde{A}(k)\tilde{Z}(k)\tilde{A} - \eta I^a(k)\tilde{S}(k)\tilde{A}^{-1} - \tilde{Z}(k)\tilde{A} \|_F$$

$$\leq \| \langle \tilde{A}(k) - I \rangle T_A(k)\tilde{Z}(k)\tilde{A} \|_F + \eta \|\tilde{S}(k)\tilde{A}^{-1}\|_F$$

$$\leq 2\sqrt{n}\|\tilde{Z}(k)\|_F + \eta \|\tilde{S}(k)\tilde{A}^{-1}\|_F,$$

where the second inequality follows from the row-stochasticity of $\tilde{A}(k)$, and the third one comes from

$$\|\tilde{A}(k) - I\|_2 \leq \|\tilde{A}(k)\|_2 + \|I\|_2$$

$$\leq \sqrt{\|\tilde{A}(k)\|_1 + 1} \leq \sqrt{n} + 1 \leq 2\sqrt{n}.  \hfill (47)$$

We finally complete the proof by combining (44), (45) and (46), and taking expectation on both side.

With the two lemmas above, we prove Claim 2 in a similar way as in the previous subsection, Appendix B.4.1.

**Proof.** Applying triangle inequality and the update in (16), we get

$$\| \tilde{S}(k + \bar{t}) \|_F = \| T_B(k + \bar{t}) \tilde{S}_V(k + \bar{t}) \tilde{A}^{-1} \|_F$$

$$\leq \| T_B(k + \bar{t}) \Phi_V(k, k + \bar{t}) \tilde{S}_V(k) \tilde{A}^{-1} \|_F$$

$$+ \sum_{t=0}^{\bar{t}-1} \left\| \left( I_V(k + \bar{t}) - \frac{1}{n} 1_V(k + \bar{t}) V(k + \bar{t})^T \right) \Phi_V(k + t + 1, k + \bar{t}) V(k + t + 1)^T D(k + t) \tilde{A}^{-1} \right\|_F$$

$$\leq 2\theta^{-1} \mu n \| \tilde{S}_V(k) \|_F + \sum_{t=0}^{\bar{t}-1} 2 \cdot \sqrt{n} \cdot \theta^{-1} \| D(k + t) \tilde{A}^{-1} \|_F,$$

where the last inequality is from Corollary 2. For the first term of the second row in (48),

$$T_B(k + \bar{t}) \Phi_V(k + \bar{t}) = \left( I_V(k + \bar{t}) - \frac{1}{n} 1_V(k + \bar{t}) V(k + \bar{t})^T \right) \Phi_V(k, k + \bar{t})$$

$$= \left( I_V(k + \bar{t}) - \frac{1}{n} 1_V(k + \bar{t}) V(k + \bar{t})^T \right) \left( \Phi_V(k, k + \bar{t}) - \frac{1}{n} 1_V(k + \bar{t}) V(k)^T \right) \left( I_V(k) - \frac{1}{n} 1_V(k) V(k)^T \right)$$

$$= T_B(k + \bar{t}) \left( \Phi_V(k, k + \bar{t}) - \frac{1}{n} 1_V(k + \bar{t}) V(k)^T \right) T_B(k),$$

which can be verified in the same way as in (29).

The upper bound in (48) along with Lemma 11 implies the desired result (19).
B.4.3 Proof of Claim 3

By definition, \( \tilde{z}_T^T(k) = \left( z_\pi^T(k) - (z^*)^T \right) \tilde{A} = \left( \pi^T(k) \tilde{Z}(k) - (z^*)^T \right) \tilde{A} \). Repeating the first step in the above two proofs, we get

\[
\begin{align*}
    z_\pi^T(k + b) \tilde{A} &= \pi^T(k + b) \tilde{Z}(k + b) \tilde{A} \\
    &= \pi^T(k + b) \Phi_A(k, k + b) \tilde{Z}(k + b) \tilde{A} \\
    &- \eta \pi^T(k + b) \sum_{t=0}^{b-1} \Phi_A(k + t + 1, k + b) I^\pi(k + t) V(k + t) \hat{S}_V(k + t) \\
    &= z_\pi^T(k) \tilde{A} - \eta \pi^T(k + b) \sum_{t=0}^{b-1} \Phi_A(k + t + 1, k + b) I^\pi(k + t) V(k + t) \hat{S}_V(k + t).
\end{align*}
\]

(49)

Then we try to extract the full gradient ingredient in (49) by defining

\[
\begin{align*}
    \eta(k, t) &= \frac{\eta}{2} \pi^T(k + b) \Phi_A(k + t + 1, k + b) I^\pi(k + t) V(k + t), \\
    \eta(k) &= \sum_{t=0}^{b-1} \eta(k, t),
\end{align*}
\]

where \( V(k) \) and \( 1_V(k) \) are defined in Definition 2 and (33), respectively. By introducing \( \nabla j(z_\pi(k)) \), the global full gradient of \( z_\pi(k) \), the last row in (49) is decomposed into

\[
\left( z_\pi^T(k + b) - (z^*)^T \right) \tilde{A} = \left( z_\pi^T(k) - (z^*)^T \right) \tilde{A} - \frac{\eta}{2} \left( \nabla j(z_\pi(k)) \right)^T \tilde{A}^{-1} \\
+ \sum_{t=0}^{b-1} \eta(k, t) \left( \nabla j(z_\pi(k)) \right)^T \tilde{A}^{-1} \\
- \sum_{t=0}^{b-1} \eta(k, t) \left( \nabla j(z_\pi(k)) \right)^T \tilde{A}^{-1} \\
+ \sum_{t=0}^{b-1} \eta(k, t) \left[ \frac{1}{n} 1^T \nabla(k + t) - \partial(k + t) \right] \tilde{A}^{-1}.
\]

(50)

The first row of (50), \( \left( z_\pi^T(k) - (z^*)^T \right) \tilde{A} - \frac{\eta}{2} \left( \nabla j(z_\pi(k)) \right)^T \tilde{A}^{-1} \), depicts the full gradient ascent/descent of the weight average variables \( z_\pi^T(k) \), while the second to the fourth rows are error terms that are bounded, as in (52)-(54).

First, notice that for all \( k \geq 0, t \geq 0 \), we have \( \eta(k, t) \leq \eta n \) and \( \frac{\eta}{2} \leq \eta bn \). Applying Lemma 8, we have

\[
\| \left( z_\pi^T(k) - (z^*)^T \right) \tilde{A} - \frac{\eta}{2} \left( \nabla j(z_\pi(k)) \right)^T \tilde{A}^{-1} \|_F \leq (1 - \eta nb n) \| \left( z_\pi^T(k) - (z^*)^T \right) \tilde{A} \|_F.
\]

(51)

Second, let \( I_{n \times \tilde{n}} = \left[ I_n^T, 0_{b-n}^T \right] \in \mathbb{R}^{n} \), and \( I_{b\times \tilde{n}} \in \mathbb{R}^{b \times \tilde{n}} \):

\[
[I_{n \times \tilde{n}}]_{ij} = \begin{cases}
1, & \text{if } i = j \leq n, \\
0, & \text{otherwise}.
\end{cases}
\]

By direct computation, \( \nabla j(z)^T = n^{-1} I_n^T \nabla J(1_n z^T), \forall z \in \mathbb{R}^{2d} \). We can also derive that for all \( k \),
\[ 1_n^T \nabla (k) = 1_n^T \nabla (Z^{(0)}(k)) \] from (42). Then,
\[
\left\| \sum_{t=0}^{b-1} \eta(k,t) \left( \nabla j(z_{n}(k)) - \frac{1}{n} 1_n^T \nabla (k+t) \right) \right\|_F \leq \eta \sqrt{n} \beta \sum_{t=0}^{b-1} \left\| (1_n^T z_{n}(k) - 1_n^T z_n^{(0)}(k)) \right\|_F
\]
\[
\leq \eta \sqrt{n} \beta \sum_{t=0}^{b-1} \left\| (1_n^T z_{n}(k) - 1_n^T z_n^{(0)}(k)) \right\|_F
\]
where the first inequality follows from Lemma 9.

Third, by (14), (16) and (33), then,
\[
\left\| \sum_{t=0}^{b-1} \eta(k,t) \left( \frac{1}{n} 1_n^T \nabla (k+t) \right) \Lambda^{-1} \right\|_F
\]
\[
\leq \eta \sqrt{n} \beta \sum_{t=0}^{b-1} \left\| (1_n^T z_{n}(k) - 1_n^T z_n^{(0)}(k)) \right\|_F
\]
\[
\leq \eta \sqrt{n} \beta \sum_{t=0}^{b-1} \left\| (1_n^T z_{n}(k) - 1_n^T z_n^{(0)}(k)) \right\|_F
\]
\[
\leq \eta(2n + \sqrt{n}) \beta \sum_{t=0}^{b-1} \| \tilde{Z}(k+t) \|_F,
\]
So far, one can summary (50)-(54) and take expectation to get the final result of (20).
B.4.4 Proof of Claim 4

By (14) and (18), \( 1_n^T \partial(k) \tilde{\Lambda}^{-1} = v(k)^T S_v(k) \tilde{\Lambda}^{-1} \). Then,
\[
\| \tilde{S}(k) \tilde{\Lambda}^{-1} \|_F = \| V(k) T_b(k) S_v(k) \tilde{\Lambda}^{-1} + \frac{1}{n} V(k) 1_v(k) v(k)^T S_v(k) \tilde{\Lambda}^{-1} \|_F \\
\leq \| V(k) \|_2 \| T_b(k) S_v(k) \tilde{\Lambda}^{-1} \|_F + \left( \frac{1}{n} V(k) 1_v(k) 1_n^T \partial(k) \tilde{\Lambda}^{-1} \right)_F \tag{55}
\]
\[
\leq n \| \tilde{S}_v(k) \|_F + \frac{1}{n} \cdot n \| 1_v(k) 1_n^T \partial(k) \tilde{\Lambda}^{-1} \|_F.
\]

For the second term in (55), we have
\[
\| 1_v(k) 1_n^T \partial(k) \tilde{\Lambda}^{-1} \|_F
\leq \| 1_v(k) (1_n^T \partial(k) - 1_n^T \nabla J(1_n(z^*)^T) \tilde{\Lambda}^{-1}) \|_F
\leq \| 1_v(k) 1_n^T \|_2 \left( \| \nabla J(k) - \nabla J(1_n(z^*)^T) \tilde{\Lambda}^{-1} \|_F + \| (\partial J(k) - \nabla J(k)) \tilde{\Lambda}^{-1} \|_F \right) \tag{56}
\]
\[
\leq \sqrt{n/m} \left( \| \nabla J(k) - \nabla J(1_n(z^*)^T) \tilde{\Lambda}^{-1} \|_F + \| (\partial J(k) - \nabla J(k)) \tilde{\Lambda}^{-1} \|_F \right).
\]

Following from (44), the second term of (56) is bounded. For the first term,
\[
\| (\nabla J(k) - \nabla J(1_n(z^*)^T)) \tilde{\Lambda}^{-1} \|_F
\leq \beta \| (I_{n|\tilde{z}} \tilde{Z}(k) - 1_{n|\tilde{z}} (z^*)^T) \tilde{\Lambda} \|_F
\leq \beta \left( \| I_{n|\tilde{z}} \tilde{Z}(k) - 1_{n|\tilde{z}} \pi^T(k-1) \| \tilde{Z}(k) \tilde{\Lambda} \|_F + \beta \| 1_{n|\tilde{z}} (\pi^T(k-1) \tilde{Z}(k) - (z^*)^T) \tilde{\Lambda} \|_F \right) \tag{57}
\]
\[
\leq \beta \left( \| I_{n|\tilde{z}} - 1_{n|\tilde{z}} \pi^T(k-1) \| T_\lambda(k) \tilde{Z}(k) \tilde{\Lambda} \|_F + \beta \| 1_{n|\tilde{z}} (z_\pi(k) - (z^*)^T) \tilde{\Lambda} \|_F \right)
\leq \beta \cdot (\sqrt{n} + 1) \| T_\lambda(k) \tilde{Z}(k) \tilde{\Lambda} \|_F + \beta \cdot \sqrt{n} \| (z_\pi(k) - z^*)^T \tilde{\Lambda} \|_F.
\]

Thus, we obtain (21) by using (56) and (57).

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