Collision Drag Effect on Propagation of Sound in Liquid $^3$He in Aerogel

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Sound propagation in a Fermi liquid with impurities is studied using the Landau-Boltzmann equation and the result is compared with a recent experiment in liquid $^3$He in aerogel by Northwestern University group. The sound absorption calculated using the fixed impurity model is a few orders of magnitude larger than the experiment. We take into account the simultaneous motion of aerogel molecules and propose a model in which the momentum loss of $^3$He quasi-particles during the collisions with the aerogel is converted to a drag force that acts on the aerogel molecules. This collision drag model gives a reasonable description for the temperature and the pressure dependence of observed sound velocity and absorption.

KEYWORDS: Liquid $^3$He, Aerogel, Sound, Impurity Scattering, Transport Relaxation Time, Collision Drag Effect, Fermi Liquid

Liquid $^3$He in aerogel has recently attracted much attention as a model system for studying impurity scattering effect in $p$-wave pairing superfluid. When a homogeneous impurity model is assumed, theoretical calculations are rather straightforward\textsuperscript{1–8)} The reduction of the transition temperature is determined by the ratio of the coherence length $\xi$ to the so-called transport mean free path $\ell_{tr}$\textsuperscript{1)} The parameters in the Ginzburg-Landau theory\textsuperscript{4)} and the superfluid density\textsuperscript{8)} are written in terms of the mean free path and the scattering cross section, when the ABM state or the BW state is assumed. Sound propagation in dirty $p$-wave superfluid has been also discussed\textsuperscript{5, 9)}

Since the discovery of superfluid transition of liquid $^3$He in aerogel\textsuperscript{10)} a lot of experiments on superfluidity have been performed, using torsion oscillator\textsuperscript{11–13)} NMR\textsuperscript{14–18)} acoustics\textsuperscript{19–22)} and vibrating wire technique\textsuperscript{23)} However even the identification of the superfluid phase is not at satisfactory level.

Recently, Nomura et al\textsuperscript{21)} and Gervais et al\textsuperscript{22)} reported high frequency acoustic experiment in the normal and the superfluid phase of liquid $^3$He in 98% porous silica aerogel. The attenuation of sound is of the same order as in pure liquid $^3$He but neither first-to-zero sound transition in the normal phase nor collective mode attenuation peak which is characteristic to pure liquid $^3$He was observed. These authors analyzed their data using the visco-elastic model\textsuperscript{24)} They suggested that the normal state attenuation can be obtained by choosing the impurity scattering mean free path in the range of 200–300nm which is a bit larger than but of the same order as previously estimated values.

The aim of this letter is to discuss the sound propagation taking into account the simultaneous motion of aerogel silica molecules. The impurity scattering effect cannot be fully treated by the
visco-elastic model. This model is essentially a modification of the stress tensor such that can allow for the high frequency collisionless behavior, but still assumes that the momentum density is conserved. Since the impurity scattering does not conserve the momentum density, the sound dispersion relation has a different form from that can be obtained by the visco-elastic model. If the aerogel molecules are fixed, the sound attenuation will become a few orders of magnitude larger than the observed value. To remedy the situation we employ the idea of the collision drag effect.

The momentum of $^3$He quasi-particles lost in the scattering processes plays a role of drag force that acts on aerogel silica molecules. Since the mass density of aerogel molecules is smaller than that of liquid $^3$He, the aerogel molecules move almost coupled with $^3$He oscillation. The deviation in the motion between these two systems gives rise to a damping of sound, in addition to the damping in liquid $^3$He that can be accounted for by the visco-elastic model.

Let us consider the Landau-Boltzmann equation for the linearized distribution function $\delta f_{\vec{p}}(\vec{q},\tau - \omega t)$ of the quasi-particle of liquid $^3$He in the normal phase

$$\omega \delta f_{\vec{p}} - \vec{v}_F \cdot \vec{q} (\delta f_{\vec{p}} - f'\delta\epsilon_{\vec{p}}) = iI[\delta f_{\vec{p}}],$$

(1)

where $\omega, \vec{q}$ are the frequency and the wave vector of the disturbance, $\vec{v}_F = \vec{p}_F/m^*$ is the Fermi velocity, $f'$ is the derivative of the equilibrium Fermi distribution function $f^0(\epsilon_{\vec{p}})$ with respect to the quasi-particle energy $\epsilon_{\vec{p}}$ and $\delta\epsilon_{\vec{p}}$ is the additional quasi-particle energy due to the Fermi liquid interaction. The right-hand-side is the collision integral that consists of an impurity scattering term $I_{\text{imp}}$ and a collision term due to the mutual collisions between the quasi-particles.

\begin{equation}
I = I_{\text{imp}} + I_{\text{coll}}.
\end{equation}

(2)

In the longitudinal sound propagation, the angle dependence of $\delta f_{\vec{p}}$ is only through $\cos \theta = \vec{p} \cdot \vec{q}/pq$. The solution of Eq. (1) is known to have a form

$$\delta f_{\vec{p}} = (-f') \sum_{\ell} y_{\ell} P_{\ell}(\cos \theta),$$

(3)

where $P_{\ell}$ is the $\ell$-th Legendre polynomial. Using this notation, Fermi liquid effect is given by

$$\delta\epsilon_{\vec{p}} = \sum_{\ell} \frac{F_{\ell}^S}{2\ell + 1} y_{\ell} P_{\ell}(\cos \theta),$$

(4)

where $F_{\ell}^S$'s are the symmetric part of the Landau parameters.

The impurity collision term due to the scattering by aerogel has a form

$$I_{\text{imp}} = -\sum_{\vec{p}'} W_{\vec{p}\vec{p}'} 2\pi \delta(\epsilon_{\vec{p}} - \epsilon_{\vec{p}'}) (\delta f_{\vec{p}} - \delta f_{\vec{p}'}),$$

(5)

where $W_{\vec{p}\vec{p}'} 2\pi \delta(\epsilon_{\vec{p}} - \epsilon_{\vec{p}'})$ is the scattering probability and

$$\epsilon_{\vec{p}'} = \epsilon_{\vec{p}} + \delta\epsilon_{\vec{p}} - \vec{p} \cdot \vec{v}.$$
In the energy conservation, we have taken into account the Fermi liquid effect and also the local velocity \( \vec{v} \) of the aerogel. We assume that the aerogel motion is described by a longitudinal displacement field

\[
\vec{u}(\vec{r}, t) = \vec{u}_{\vec{q}} e^{i(\vec{q} \cdot \vec{r} - \omega t)}, \quad \vec{u}_{\vec{q}} = u_q \vec{q}/q, \tag{7}
\]

therefore, \( \vec{v} = -i\omega \vec{u}_{\vec{q}} \).

The local equilibrium is achieved when the \(^3\)He quasi-particles move together with the scatterers, therefore the local equilibrium distribution function \( \delta f_{1,e} \) is given by

\[
\delta f_{1,e} = f' (\delta \epsilon_{\vec{p}} - \vec{p} \cdot (-i\omega) \vec{u}_{\vec{q}}). \tag{8}
\]

The collision term is a functional of \( \delta f' = \delta f_{\vec{p}} - \delta f_{1,e} \). Bearing this into mind and defining the impurity scattering time \( \tau_i \) by

\[
\frac{1}{\tau_i} = \sum_{\vec{p}} W_{\vec{p}\vec{p}'} 2\pi \delta(\epsilon_{\vec{p}} - \epsilon_{\vec{p}'}, \tag{9}
\]

we find that \( I_{\text{imp}} \) is parametrized as follows:

\[
I_{\text{imp}} = -\frac{1}{\tau_i} \left( \delta f' - \sum_{\ell} (2\ell + 1) \lambda^i_{\ell} \langle \delta f' \hat{P}_\ell(\cos \theta) \rangle \hat{P}_\ell(\cos \theta) \right), \tag{10}
\]

where \( \langle \cdots \rangle \) stands for the angle average. The number conservation at the impurity scattering requires that

\[
\lambda^i_0 = 1. \tag{11}
\]

Note that \( \delta f_{1,e} \) of Eq. (8) is also a local equilibrium solution of the mutual collision term. It follows that \( I_{\text{coll}} \) is quite satisfactorily approximated by

\[
I_{\text{coll}} = -\frac{1}{\tau_c} \left( \delta f' - \sum_{\ell} (2\ell + 1) \lambda^c_{\ell} \langle \delta f' \hat{P}_\ell(\cos \theta) \rangle \hat{P}_\ell(\cos \theta) \right), \tag{12}
\]

where \( \tau_c \) is the mutual collision time which is well known to be proportional to \( 1/T^2 \). In the mutual collision process, the momentum as well as the quasi-particle number is conserved. Hence

\[
\lambda^c_0 = \lambda^c_1 = 1. \tag{13}
\]

Substituting these expressions into Eq. (4), and summing up both the hand sides over \( \vec{p} \) and the spin degrees of freedom, we find the number conservation law

\[
\omega y_0 = \frac{1}{3} v_F q \left( 1 + \frac{F_S}{3} \right) y_1. \tag{14}
\]
Multiplying \( \cos \theta \) on both the hand sides of Eq. (1) and summing up, we obtain the momentum conservation law

\[
\omega y_1 - v_F q \left( (1 + F_0^S) y_0 + \frac{2}{5} \left( 1 + \frac{F_2^S}{5} \right) y_2 \right) = -\frac{i}{\tau_{tr}} \left( \left( 1 + \frac{F_1^S}{3} \right) y_1 + i\omega p_F u_q \right),
\]

(15)

where \( \tau_{tr} \) is the so-called transport relaxation time defined by

\[
\frac{1}{\tau_{tr}} = \frac{1 - \lambda_1^i}{\tau_1}.
\]

(16)

Since \( F_0^S \gg 1 \) in liquid \( ^3\text{He} \), we have only to estimate \( y_2/y_0 \) up to order \( 1/F_0^S \sim (v_F q/\omega)^2 \). In the same way as that in pure liquid \( ^3\text{He} \), we obtain

\[
\frac{y_2}{y_0} = \frac{2\omega}{\omega + \frac{i}{\tau_2} \left( 1 + \frac{F_2^S}{5} \right)},
\]

(17)

where

\[
\frac{1}{\tau_2} = \frac{1 - \lambda_2^i}{\tau_1} + \frac{1 - \lambda_2^c}{\tau_c}.
\]

(18)

This \( d \)-wave contribution is that can be taken care of by the visco-elastic model.

Let us first consider the case where the aerogel molecules are fixed, i.e., \( u_q = 0 \). From Eqs. (14), (15) and (18), we obtain the sound dispersion relation

\[
\omega^2 + \frac{i\omega}{\tau_{tr}} \left( 1 + \frac{F_1^S}{3} \right) = c_1^2 q^2 \left( 1 + \frac{4}{5} \left( 1 + \frac{F_1^S}{3} \right) \right),
\]

(19)

where \( c_1 \) is the first sound velocity of pure liquid \( ^3\text{He} \). The damping term due to \( \tau_{tr} \) is characteristic to the fixed impurity model and dominates the damping of sound. In the collisionless regime \( \omega \tau_i, \omega \tau_c \gg 1 \), the sound velocity is equal to the zero sound velocity \( c_0 \) of pure \( ^3\text{He} \) liquid, while the absorption is given by

\[
\alpha = \frac{1}{2c_0 \tau_{tr}} (1 + \frac{F_1^S}{3}).
\]

(20)

The absorption is much larger than that in the pure liquid, because the prefactor \( (c_0^2 - c_1^2)/c_1^2 \) that appears in the absorption of pure liquid is missing. In the hydrodynamic regime \( \omega \tau_i \ll 1 \), the sound dispersion becomes

\[
\frac{i\omega}{\tau_{tr}} \left( 1 + \frac{F_1^S}{3} \right) = c_1^2 q^2.
\]

(21)

The frequency used by Northwestern group \( ^{21, 22} \) is 15MHz. If one takes the mean free path \( \ell \sim v_F \tau_{tr} \sim 270\text{nm} \), then \( \omega \tau_{tr} \sim 0.7 \). Thus the system is rather in the hydrodynamic regime and
the sound is hard to propagate in contrast to the experimental observation. It is not possible to interpret the experiment using the fixed impurity model.

Now we consider the effect of the motion of aerogel molecules. We assume that in the skeleton aerogel the displacement field \( u(\vec{r}, t) \) obeys the wave equation

\[
\rho_a \ddot{u} = \rho_a c_a^2 \Delta u,
\]

where \( \rho_a \) is the aerogel mass density and \( c_a \sim 50 \text{m/sec} \) is the sound velocity of the skeleton aerogel. We can easily calculate from Eq. (15) the momentum density transfer per unit time from \(^3\text{He}\) to aerogel:

\[
\frac{1}{\tau_{tr}} \frac{N_F p_F}{3} \left( \left( 1 + \frac{F_1}{3} \right) y_1 + i \omega p_F u_q \right),
\]

where \( N_F \) is the density of states of \(^3\text{He}\) quasi-particles for both spin projections. This is just the drag force density exerted to the aerogel molecules. We have for the equation of motion of the aerogel

\[
\rho_a \left( -\omega^2 + \omega_q^2 \right) u_q = \frac{1}{\tau_{tr}} \frac{N_F p_F}{3} \left( \left( 1 + \frac{F_1}{3} \right) y_1 + i \omega p_F u_q \right),
\]

from which we find

\[
u_q = \frac{-1}{\rho_a} \frac{N_F p_F}{3} \left( 1 + \frac{F_1}{3} \right) y_1 \rho \left( \omega^2 - \omega_q^2 + \frac{i \omega}{\tau_{tr}} \frac{N_F p_F}{2} \right).
\]

Substituting this into Eq. (21), we find that the dispersion relation of sound has the same form as Eq. (19), but \( \tau_{tr} \) is replaced by \( \tau_{eff} \)

\[
\frac{1}{\tau_{eff}} = \frac{1}{\tau_{tr}} \frac{\omega^2 - \omega_q^2}{\omega^2 - \omega_q^2 + \frac{i \omega}{\tau_{tr}} \frac{N_F p_F}{3 \rho_a}}.
\]

In the experiment by Northwestern group \(21, 22\) the observed sound velocity is of order the first sound velocity. In this case, \( \omega_q^2 = c_a^2 q^2 \) can be neglected when compared with \( \omega^2 \). Hence \( \tau_{eff} \) is much simplified to

\[
\frac{1}{\tau_{eff}} = \frac{1}{\tau_{tr}} \frac{\omega}{\omega + \frac{\omega_q}{\tau_{tr}}},
\]

where we have used the fact that \( N_F p_F^2 \) is related to the \(^3\text{He}\) mass density \( \rho_3 \) via \( N_F p_F^2 = 3(1 + F_1^S/3)\rho_3 \). The aerogel density \( \rho_a \) is 0.04g/cm\(^3\) \(21, 22\) while \(^3\text{He}\) density is about 0.1g/cm\(^3\) at 15bar \(29\).

Since \( \omega \tau_{tr} \) is small compared with unity, we can further approximate

\[
\frac{1}{\tau_{eff}} = -i \omega \frac{\tau'}{\tau_{tr}} (1 + i \omega \tau') = -i \omega \frac{\rho_a 1 + i \omega \tau'}{\rho_3 1 + F_1^S/3}.
\]
We finally find the dispersion relation
\[
\left(1 + \frac{\rho_a}{\rho_3}\right) \omega^2 + i\omega^3 \left(\frac{\rho_a}{\rho_3}\right)^2 \frac{1}{1 + \frac{\tau_{tr}}{\tau_2}} = c_1^2 q^2 \left(1 + \frac{4}{5} \frac{F_S}{F_0} \frac{\omega}{\tau_2} \left(1 + \frac{F_S}{F_0}\right)^2\right). \tag{29}
\]
Thus the sound velocity is
\[
c^2 = c_1^2 / (1 + \frac{\rho_a}{\rho_3}) \tag{30}
\]
and the absorption is
\[
\alpha = \frac{\omega^2}{c} \left(\frac{1}{2} \left(\frac{\rho_a}{\rho_3}\right)^2 \frac{\tau_{tr}}{1 + \frac{\rho_a}{\rho_3} + \frac{2}{5} \frac{\tau_2}{1 + F_S}}\right). \tag{31}
\]
The first term gives the absorption in the high temperature limit. The second term explains the temperature dependence of \(\alpha\) reported by Nomura et al.\[21\] because
\[
\tau_2 = \begin{cases} 
\tau_i / (1 - \lambda_i^1) & \text{for } T \to 0, \\
\propto 1/T^2 \to 0 & \text{for } T \to \infty.
\end{cases} \tag{32}
\]
The present result reproduces quite well the pressure dependence of the sound velocity and the absorption reported by Northwestern group\[21, 22\]. To fit the reported value of the absorption \(\alpha\), however, we have to take \(\ell_{tr} = v_F \tau_{tr}\) or \(\ell_1 = v_F \tau_i\) around 50nm, which is about 3 times smaller than the previous estimates\[5, 13\]. This discrepancy is still to be examined.

The collision drag effect is prominent when the condition \(\omega \tau_i < 1\) is satisfied. As is evident from Eq. (29), the dragged aerogel gives an extra inertia in the sound oscillation\[26\]. The low frequency experiment by Golov et al.\[19\] in the normal phase can be also interpreted by the present theory. In the high frequency limit \(\omega \tau_i \gg 1\), the aerogel molecules cannot move and the result is the same as that of the fixed impurity model. In this sense, the frequency 15MHz used by Northwestern group is not a high frequency but a low frequency.

In conclusion, we have presented a microscopic theory of collision drag effect on the sound propagation in liquid \(^3\)He aerogel system. Extension of the theory to superfluid phase shall be reported elsewhere.

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