Sequence Nets

Jie Sun,† Takashi Nishikawa,‡ and Daniel ben-Avraham§

†Department of Mathematics & Computer Science, Clarkson University Potsdam, NY 13699-5815
‡Department of Physics, Clarkson University, Potsdam, NY 13699-5820
§Electronic address: sunj@clarkson.edu

We study a new class of networks, generated by sequences of letters taken from a finite alphabet consisting of \( m \) letters (corresponding to \( m \) types of nodes) and a fixed set of connectivity rules. Recently, it was shown how a binary alphabet might generate threshold nets in a similar fashion [Hagberg et al., Phys. Rev. E 74, 056116 (2006)]. Just like threshold nets, sequence nets in general possess a modular structure reminiscent of everyday life nets, and are easy to handle analytically (i.e., calculate degree distribution, shortest paths, betweenness centrality, etc.). Exploiting symmetry, we make a full classification of two- and three-letter sequence nets, discovering two new classes of two-letter sequence nets. The new sequence nets retain many of the desirable analytical properties of threshold nets while yielding richer possibilities for the modeling of everyday life complex networks more faithfully.

PACS numbers: 89.75.Hc 02.10.Ox, 89.75.Fb, 05.10.-a

I. INTRODUCTION

Threshold nets are obtained by assigning a weight \( x \), from a distribution \( \rho(x) \), to each of \( N \) nodes and connecting any two nodes \( i \) and \( j \) whose combined weights exceed a certain threshold, \( \theta \): \( x_i + x_j > \theta \). Threshold nets can be produced of (almost) arbitrary degree distributions, including scale-free, by judiciously choosing the weight distribution \( \rho(x) \) and the threshold \( \theta \), and they encompass an astonishingly wide variety of important architectures: from the star graph (a simple “cartoon” model of scale-free graphs — consisting of a single hub) with its low density of links, \( 2/N \), to the complete graph. Studied extensively in the graph-theoretical literature \([3, 4, 5, 6]\), they have recently come to the attention of statistical and non-linear physicists due to the beautiful work of Hagberg, Swart, and Schult \([7]\).

Hagberg et al., exploit the fact that threshold graphs may be more elegantly encoded by a two-letter sequence, corresponding to two types of nodes, \( A \) and \( B \). As new nodes are introduced, according to a prescribed sequence, nodes of type \( A \) connect to none of the existing nodes, while nodes of type \( B \) connect to all of the nodes, of either type: \( B \rightarrow A \) and \( B \rightarrow B \). In Fig. 1(a) we show an example of the threshold graph obtained from the sequence \((A, A, A, B, B, A, A, B)\). Note the modular structure of threshold graphs: a subsequence of \( n \) consecutive \( B \)-s gives rise to a \( K_n \)-clique, while nodes in a subsequence of \( A \)-s connect to \( B \)-nodes thereafter, but not among one another. We highlight this modularity with a diagram of boxes (similar to \([8]\)): oval boxes enclose nodes of type \( A \), that are not connected among themselves, while rectangular boxes enclose \( K \)-cliques of \( B \)-nodes \([9]\). A link between two boxes means that all of the nodes in one box are connected to all of the nodes in the other, Fig. 1(b).

Given the sequence of a threshold net, there exist fast algorithms to compute important structural benchmarks, besides its modularity, such as degree distribution, triangles, betweenness centrality, and the spectrum and eigenvectors of the graph Laplacian \([9]\). The latter are a crucial determinant of dynamics and synchronization and have applications to graph partitioning and mesh processing \([10, 11, 12, 13, 14, 15, 16, 17]\). Perhaps more importantly, it becomes thus possible to design threshold nets with a particular degree distribution, spectrum of eigenvalues, etc., \([11]\).

Despite their malleability, threshold nets are limited in some obvious ways, for example their diameter is 1 or 2, regardless of the number of nodes \( N \). Our idea consists of studying the broader class of nets that can be constructed from a sequence (formed from two or more letters) by deterministic rules of connectivity on their own right. It is truly this property that gives the nets all their desired attributes: modularity (as in everyday life complex nets), easily computable structural measures — including the possibility of design — and a high degree of compressibility. Roughly speaking, each additional letter to the...
alphabet allows for an increase of one link in the nets’ diameter, so that the three-letter nets possess diameter 3 or 4 (some of the new types of two-letter nets have diameter 3). This modest increase is very significant, however, in view of the fact that the diameter of many everyday life complex nets is not much larger than that \( R \). Sequence nets gain us much latitude in the types of nets that can be described in this elegant fashion, while retaining much of the analytical appeal of threshold nets. Another unusual property of sequence nets is that any ensemble of sequence nets admits a natural ordering; simply list them alphabetically according to their sequences. One may use this ordering for exploring eigenvalues and other structural properties of sequence nets.

In this paper, we make a first stab at the general class of \textit{sequence nets}. In Section II we explore systematically all of the possible rules for creating connected sequence nets from a two-letter alphabet. Applying symmetry arguments, we find that threshold nets are only one of three equivalence classes, characterized by the highest level of symmetry. We then discuss the remaining two classes, showing that also then there is a high degree of modularity and that various structural properties can be computed easily. Curiously, the new classes of two-letter sequence nets can be related to a generalized form of threshold nets, where the difference of the others, there are \( 2^4 = 16 \) possible rules. We shall disregard, however, the four rules that fail to connect between \( A \) and \( B \),

\[
R_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

for they yield simple \textit{disjoint} graphs of the two types of nodes: \( R_0 \) yields isolated nodes only, \( R_3 \) yields one complete graph of type \( A \) and one of type \( B \), \( R_1 \) yields a complete graph of type \( A \) and isolated nodes of type \( B \), etc.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2}
\caption{Combined time reversal and permutation symmetry: The graphs resulting from \( R_4 \) applied to the sequence \((A, A, A, B, B, A, A, B)\) \( (a) \), and from \( R_5 \) applied to the reverse-inverted sequence \((A, B, B, A, A, B, B, B)\) \( (b) \), are identical.}
\end{figure}

The list of remaining rules can be shortened further by considering two kinds of symmetries: (a) permutation, and (b) time reversal. \textit{Permutation} is the symmetry obtained by permuting between the two types of nodes, \( A \leftrightarrow B \). Thus, a permuted rule \((R_{11} \leftrightarrow R_{22}\text{ and }R_{12} \leftrightarrow R_{21})\) acting on a permuted sequence \((S_1, S_2, \ldots, S_N)\) yields back the original graph \([10]\). \textit{Time reversal} is the symmetry obtained by reversing the arrows (“time”) in the connectivity rules, or taking the transpose of \( R \). The transposed rule acting on the reversed sequence \((S_N, S_{N-1}, \ldots, S_1)\) yields back the original graph. The two symmetry operations are their own inverse and they form a symmetry group. In particular, one may combine the two symmetries: a rule with \( R_{11} \leftrightarrow R_{22}\) applied on a reversed sequence with inverted types \((S_N, S_{N-1}, \ldots, S_1)\) yields back the original graph, see Fig. 2.

All of the four rules

\[
R_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad R_5 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad R_6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_7 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},
\]

are equivalent and generate threshold graphs. \( R_4 \) is the rule for threshold graphs exploited by Hagberg et al., [3], and \( R_5 \) is equivalent to it by permutation. \( R_6 \) is obtained from \( R_4 \) by time reversal and permutation (Fig. 2), and \( R_7 \) is obtained from \( R_4 \) by time reversal.

The two rules

\[
R_8 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad R_9 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]
are equivalent, by either permutation or time reversal, and generate non-trivial bipartite graphs that are different from threshold nets (Fig. 3).

The rule $R_{10} = \binom{1}{1}^T$ generates complete bipartite graphs. However, the complete bipartite graph $K_{p,q}$ can also be produced by applying $R_8$ to the sequence $(A, A, ..., A, B, B, ..., B)$ of $p$ $A$'s followed by $q$ $B$'s, so the rule $R_{10}$ is a “degenerate” form of $R_8$. One could see that this is the case at the outset, because of the symmetrical relations $A \rightarrow B, B \rightarrow A$: these render the ordering of the $A$'s and $B$'s in the graph’s sequence irrelevant. By the same principle, $R_{11} = \binom{0}{1}^T$ and $R_{12} = \binom{1}{0}^T$ are degenerate forms of $R_4$ and $R_5$, respectively. They yield threshold graphs with segregated sequences of $A$'s and $B$'s.

The two rules

$$R_{13} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R_{14} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

are equivalent, by either permutation or time reversal, and generate non-trivial graphs different from threshold graphs and graphs produced by $R_8$ (Fig. 3). Finally, the rule $R_{15} = \binom{1}{1}^T$ is a degenerate form of $R_{13}$ (or $R_{14}$) and yields only complete graphs (which are threshold graphs, so $R_{15}$ is subsumed also in $R_4$).

FIG. 3: Distinct types of connected non-trivial two-letter sequential graphs: All three graphs are generated from the same sequence, \((A, A, A, B, B, A, A, B)\), applying rules $R_8$ (a), $R_4$ (b), and $R_5$ (c). Note the figure-background symmetry of (a) and (c): the graphs are the inverse, or complement of one another (see text). The inverse of the threshold graph (b) is also a (two-component) threshold graph, obtained from the same sequence and applying the rule $R_5$ ($R_4$’s complement).

To summarize, $R_4$, $R_8$, and $R_{13}$ are the only two-letter rules that generate different classes of non-trivial connected graphs. There is yet another amusing type of symmetry: applying $R_8$ and $R_{13}$ to the same sequence yields complement, or inverse graphs — nodes are adjacent in the inverse graph if and only if they are not connected in the original graph. The figure-background symmetry manifest in the rules $R_8$ and $R_{13}$ ($0 \leftrightarrow 1$) is also manifest in the graphs they produce (Fig. 3b,c). On the other hand, the inverse of threshold graphs are also threshold graphs. Also, the complement of a threshold rule applied to the complement (inverted) sequence yields back the original graph. In this sense, threshold graphs have maximal symmetry. $R_8$-graphs are typically less dense, and $R_{13}$-graphs are typically denser than threshold graphs.

FIG. 4: Diagrammatic representation of rules for two-letter sequence nets: (a) All of the \(2^4\) possible connections between nodes of type $A$ and $B$. (b) Three equivalent representations of the threshold rule $R_4$. The second and third diagram are obtained by label permutation and time-reversal, respectively. (c) Diagrams for $R_8$ and $R_{13}$. Note how they complement one another to the full set of connections in part (a).

The connectivity rules have an additional useful interpretation as directed graphs, where the nodes represent the letters of the sequence alphabet, a directed link, e.g., from $A$ to $B$ indicates the rule $A \rightarrow B$, and a connection of a type to itself is denoted by a self-loop (Fig. 1). Because the rules are the same under permutation of types, there is no need to actually label the nodes: all graph isomorphs represent the same rule. Likewise, time-reversal symmetry means that graphs with inverted arrows are equivalent as well. Note that the direction of self-loops is irrelevant in this respect, so we simply take them as undirected. We shall make use of this notation, extensively, for the analysis of 3-letter sequence nets in Section 11.

B. Alphabetic ordering

A very special property of sequence nets is the fact that any arbitrary ensemble of such nets possesses a natural ordering, simply listing the nets alphabetically according to their sequences. In contrast, think for example of the ensemble of Erdős-Rényi random graphs of $N$ nodes, where links are present with probability $p$: there is no natural way to order the $2^N$ graphs in the ensemble [20].

Plotting a structural property against the alphabetical ordering of the ensemble reveals some inner structure of the ensemble itself, yielding new insights into the nature of the nets. As an example, in Fig. 5 we show $\lambda_2$, the second smallest eigenvalue, for the ensemble of connected threshold nets containing $N = 8$ nodes (there are $2^7 = 128$ graphs in the ensemble, since their sequences must all start with the letter $A$). Notice the beautiful pattern followed by the eigenvalues plotted in this way, which resembles a fractal, or a Cayley tree: the values within the first half of the graphs in the $x$-axis repeat in the second half, and the pattern iterates as we zoom further into the picture.

C. The new classes of two-letter sequence nets

Structural properties of the new classes of two-letter sequence nets, $R_8$ and $R_{13}$, are as easily derived as for
threshold nets. Here we focus on $\mathbf{R}_8$ alone, which forms a subset of bipartite graphs. The analysis for $\mathbf{R}_{13}$ is very similar and often can be trivially obtained from the complementary symmetry of the two classes.

All connected sequence nets in the $\mathbf{R}_8$ class must begin with the letter $A$ and end with the letter $B$. A sequence of this sort may be represented more compactly by the numbers of $A$'s and $B$'s in the alternating layers, $(N_{A_1}, N_{B_2}, \ldots, N_{B_n})$. We assume that there are $N$ nodes and $n$ layers ($n$ is even). We also use the notation $N_A = \sum N_{A_i}$, and $N_B = \sum N_{B_i}$ for the total number of $A$'s and $B$'s, as well as

$$N_{B_i}^- = \sum_{i<j} N_{A_i} ; \quad N_{A_j}^+ = \sum_{i>j} N_{A_i} ,$$

and likewise for $N_{B_i}^+$. Finally, since all the nodes in a layer have identical properties, we denote any $A$ in the $i$-th layer by $A_i$ and any $B$ in the $j$-th layer by $B_j$. With this notation in mind we proceed to discuss several structural properties.

**Degree distribution:** Since $A$'s connect only to subsequent $B$'s (and $B$'s only to preceding $A$’s) the degree $k$ of the nodes is given by

$$k(A_j) = N_{B_j}^+ ; \quad k(B_j) = N_{A_j}^- .$$

**Clustering:** There are no triangles in $\mathbf{R}_8$ nets so the clustering of all nodes is zero.

**Distance:** Every $A$ is connected to the last $B$, so the distance between any two $A$’s is 2. Every $B$ is connected to the first $A$ in the sequence, so the distance between any two $B$’s is also 2. The distance between $B_j$ and $A_j$ is 1 if $j < i$ (they connect directly), and 3 if $j > i$ ($B_i$ links to $A_1$, that links to $B_{A_1}$, that links to $A_j$).

**Betweenness centrality:** Because of the time-reversal symmetry between $A$ and $B$, it suffices to analyze $B$ nodes only. The result for $A$ can then be obtained by simply reversing the creation sequence and permuting the letters.

The vertex betweenness $b(v)$ of a node $v$ is defined as:

$$b(v) = \frac{1}{2} \sum_{s \neq t \neq v} \frac{\sigma_{st}(v)}{\sigma_{st}} \quad (7)$$

where $\sigma_{st}$ is the number of shortest paths from node $s$ to $t$ ($s \neq t$), excluding the cases that $s = v$ or $t = v$. $\sigma_{st}(v)$ is the number of shortest paths from $s$ to $t$ that goes through $v$. The factor $\frac{1}{2}$ appears for undirected graphs since each pair is counted twice in the summation.

The betweenness of $B$’s can be calculated from lower layers to higher layers recursively. In the first $B$-layer

$$b(B_2) = \frac{1}{2} \frac{N_{A_1}(N_{A_1} - 1)}{N_B} ,$$

for $j > 2$. The second term on the rhs accounts for the shortest paths from layer $A_{j-1}$ to itself and all previous layers of $A$, and the third term corresponds to paths from $A_{j-1}$ to $B_j$ to $A_i$ ($i < j - 1$) to $B_{j-2}$. Although this recursion can be solved explicitly it is best left in this form, as it thus highlights the fact that the betweenness centrality increases from one layer to the next. In other words, the networks are modular, where each additional $B$-layer dominates all the layers below.

**Laplacian spectrum:** Unlike threshold nets, for $\mathbf{R}_8$ nets the eigenvalues are not integer, and there seems to be no easy way to compute them. Instead, we focus on the second smallest and largest eigenvalues, $\lambda_2$ and $\lambda_N$, alone, for their important dynamical role: the smaller the ratio $\tau \equiv \lambda_N / \lambda_2$, the more susceptible the network is to synchronization [12].

Consider first $\lambda_2$. For $\mathbf{R}_8$ it is easy to show that both the vertex and edge connectivity are equal to $\min(N_{A_1}, N_{B_n})$. Then, following an inequality in [21],

$$2(1 - \cos(\frac{\pi}{N})) \min(N_{A_1}, N_{B_n}) \leq \lambda_2 \leq \min(N_{A_1}, N_{B_n}) .$$

The upper bound seems stricter and is a reasonable approximation to $\lambda_2$ (see Fig. 6).

For $\lambda_N$, using Theorem 2.2 of [21] one can derive the bounds

$$\frac{N}{N - 1} \max(N_A, N_B) \leq \lambda_N \leq N ,$$

but they do not seem very useful, numerically. Playing with various structural properties of the nets, plotted against their alphabetical ordering, we have stumbled upon the approximation

$$\lambda_N \approx N - \left( 2 \frac{N_A N_B}{N} - \langle k \rangle \right) ,$$

where $\langle k \rangle$ is the average degree of the network.
where \( \langle k \rangle \) is the average degree of the graph, see Fig. 4.

The approximation is exact for bipartite complete graphs \((n = 1)\) and the relative error increases slowly with \( N \); it is roughly at 10\% for \( N = 60 \).

**FIG. 6:** Plot of second smallest eigenvalues of all connected \( R_8 \) nets with \( N = 8 \) against their alphabetical ordering (solid curve), and their upper and lower bounds (broken lines).

**FIG. 7:** Plot of largest eigenvalue of all connected \( R_8 \) nets with \( N = 8 \) against their alphabetical ordering (solid curve), and its approximated value (broken line).

**D. Relation to threshold nets**

In [9] it was shown that threshold graphs have a mapping to a sequence net, with a unique sequence (under the “threshold rule” \( R_4 \)); and conversely, for any \( R_4 \)-sequence net there exists a set of weights \( x_i \) of the nodes (not necessarily unique), such that connecting any two nodes that satisfy \( x_i + x_j > \theta \) reproduces the sequence net. Here we establish a similar relation between \( R_8 \) (or \( R_{13} \)) sequence nets and a different kind of threshold net, where connectivity is decided by the difference \( |x_i - x_j| \) rather than the sum of the weights.

We begin with the mapping of a weighted set of nodes to a \( R_8 \)-sequence net. Let a set of \( N \) nodes have weights \( x_i (i = 1, 2, \ldots, N) \), taken from some probability density, and we assume \( 0 < x_i < 2\theta \), without loss of generality. Denote nodes with \( x_i < \theta \) as type A and nodes with \( x_i > \theta \) as type B. Finally, connect any two nodes \( i \) and \( j \) that satisfy \( |x_i - x_j| > \theta \). The resulting graph can be constructed by a unique sequence under the rule \( R_8 \), obtained as follows.

For convenience, rewrite the set of weights as

\[
0 < u_1 < u_2 < \cdots < u_{N_A} < \theta < v_1 < \cdots < v_{N_B} < 2\theta,
\]

where the first \( N_A \) weights correspond to A-nodes and the rest to B-nodes. Denote the creation sequence by \( (S_1, S_2, \ldots, S_N) \) and determine the \( S_i \) by the algorithm (in pseudo-code):

1. Set \( i = 1, j = 1 \)
2. For \( k = 1, 2, \ldots, N \), do:
   - If \( |u_i - v_j| > \theta \)
     - set \( S_k = A \) and \( i = i + 1 \);
   - Else
     - set \( S_k = B \) and \( j = j + 1 \).
3. End.

It is understood that if the \( u_i \) are exhausted before the end of the loop, the remainder B-nodes are automatically affixed to the end of the sequence (and similarly for the \( v_j \)). For example, using this algorithm we find that the “difference-threshold” graph resulting from the set of weights \( \{1, 2, 3, 5, 7, 16, 17, 20\} \) and \( \theta = 12 \), can be reproduced from the sequence \( (A, A, A, B, B, A, A, B) \), with the rule \( R_8 \).

Consider now the converse problem: given a graph created from the sequence \( (S_1, S_2, \ldots, S_N) \) with the rule \( R_8 \), we derive a (non-unique) set of weights \( \{x_i\} \) such that connecting any two nodes with \( |x_i - x_j| > \theta \) results in the same graph. Rewrite first the creation sequence into its compact form \((N_{A_1}, N_{B_1}, \ldots, N_{A_4}, N_{B_4})\), and assign weights \( l \) for nodes A in layer \( l \), weights \( n + m \) for nodes B in layer \( m \), and set the threshold at \( \theta = n \). For example, the sequence \( (A, A, A, B, B, A, A, B) \) has a compact representation \( (3, 2, 2, 1) \), with \( n = 4 \) layers, so the three A’s in layer 1 have weights 1, the two B’s in layer 2 have weights 6, the two A’s in layer 3 have weights 3, and the single B in layer 4 has weight 8. The weights \( \{1, 1, 1, 6, 6, 3, 3, 8\} \), with connection threshold \( \theta = 4 \), reproduce the original graph.

Sequence graphs obtained from the rule \( R_{13} \) can be also mapped to difference threshold graphs in exactly the same way, only that the criterion for connecting two nodes is then \( |x_i - x_j| < \theta \), instead of \( |x_i - x_j| > \theta \), as for \( R_8 \). The mapping of sequence nets to generalized threshold graphs may be helpful in the analysis of some of their properties, for example, for finding the isoperimetric number of a sequence graph [21, 22].
III. THREE-LETTER SEQUENCE NETS

A. Classification

With a three-letter alphabet, \{A, B, C\}, there are at the outset \(2^3 = 8\) possible rules. Again, these can be reduced considerably, due to symmetry. Because the rule matrix has 9 entries (an odd number) no rule can be identical to its complement. Thus, we can limit ourselves to rules with no more than 4 non-zero entries and apply symmetry arguments to reduce their space — at the very end we can then add the complements of the remaining rules.

In Fig. 8 we list all possible three-letter rules with two, three, and four interactions. Rules that lead to disconnected graphs, and symmetric rules (by label permutation or time-reversal) have been omitted from the figure.

![Figure 8: Rules for three-letter sequence nets: Shown are rules with (a) two, (b) three, and (c) four interactions. All label permutations and time reversals are omitted. In addition, rules 2 and 7 degenerate to two-letter rules (identifying A and C), and rules 3, 12, 13, and 14 are degenerate cases of rules 2, 6, 7, and 6, respectively. This leaves us with fifteen distinct three-letter rules (underlined), and their fifteen complements, for a total of 30 different classes of three-letter sequence nets.](image)

Rule \(R_3^{(3)}\) is in fact not new: identifying nodes of type A and C (as marked in rule 1 of the figure) we can easily see that the rule is identical to the two-letter rule \(R_8^{(2)}\). In the same fashion, rule \(R_5^{(3)}\) is the same as the two-letter threshold rule \(R_4^{(2)}\).

Rule \(R_3^{(3)}\) is a degenerate form of \(R_2^{(2)}\): Because of the double connection \(B \rightarrow C\) and \(C \rightarrow B\), the order at which \(B\) and \(C\) appear in the sequence relative to one another is inconsequential. (On the other hand, the order of the \(B\)’s relative to \(A\)’s is important, since \(A\)’s connect only to those \(B\)’s that appear earlier in the sequence.)

Then, given a sequence one can rearrange it by moving all the \(C\)’s to the end of the list. If we now apply \(R_2^{(3)}\), \(A \rightarrow B\) and \(C \rightarrow B\), then we get the same graph as from the original sequence under the rule \(R_3^{(3)}\). The same consideration applies to rules \(R_1^{(3)}\), \(R_3^{(3)}\) and \(R_4^{(3)}\), that are degenerate forms of \(R_6^{(3)}\), \(R_7^{(3)}\) and \(R_8^{(3)}\) (or \(R_6^{(3)}\)), respectively. We are thus left with only 15 distinct rules with fewer than 5 connections. To these one should add their complements, for a total of 30 distinct three-letter rules.

Note the resemblance of \(R_6^{(3)}\), \(R_8^{(3)}\), and \(R_9^{(3)}\) to two-letter threshold nets. \(R_1^{(3)}\) seems like a particularly symmetrical generalization and we will focus on it in much of our discussion below.

B. Connectedness

While one can easily establish whether a graph is connected or not, a posteriori, with a burning algorithm that requires \(O(N)\) steps, it is useful to have shortcut rules that tell us how to avoid bad sequences at the outset: knowing that two-letter threshold graphs are connected if and only if their sequence ends with \(B\), deals with the question most effectively. Analogous criteria exist for three-letter sequence graphs but they are a bit more complicated. For example, three-letter sequences interpreted as an example, a few basic attributes of \(R_1^{(3)}\) are degenerate forms of \(R_1^{(3)}\). Because of \(R_1^{(3)}\), the order \(C\) must appear after the first \(A\) and the first \(B\) in the sequence.

C. Structural properties

Structural properties of three-letter sequence nets are analyzed as easily as those of two-letter nets. Here we list, as an example, a few basic attributes of \(R_1^{(3)}\) sequence nets. We use a notation similar to that of Section II C.

Degree distribution: A and C nodes form complete subgraphs, while B nodes connect to all preceding A’s and C’s. Thus the degree of the nodes are:

\[
k(A_i) = N_A - 1 + N^+_B, \quad k(B_i) = N^+_A + N^+_C, \quad k(C_i) = N_C - 1 + N^+_B.
\]

Distance: Since the A nodes make a subset complete graph \(d(A_i, A_j) = 1\), and likewise for \(C\), \(d(C_i, C_j) = 1\).
The $B$'s do not connect among themselves, but they all connect to the nodes in the first layer (which does not consist of $B$'s), so $d(B_i, B_j) = 2$. For the distance of $A$ nodes from $B$, we have

$$d(A_i, B_j) = \begin{cases} 
1 & i < j, \\
2 & i > j, a_1 < j, \\
3 & i > j, a_1 > j, i < b_n, \\
4 & i > j, a_1 > j, i > b_n, 
\end{cases} \quad (15)$$

where $a_1$ is the index of the first $A$-layer and $b_n$ is the index of the last $B$-layer. The first line follows since $B$'s are directly connected to preceding $A$'s and $C$'s. The second, and third and fourth lines are illustrated in Fig. 9a and b, respectively. The distance $d(C_i, B_j)$ follows the very same pattern. Finally, inspecting all different cases one finds

$$d(A_i, C_j) = \begin{cases} 
2 & i, j < b_n, \\
3 & i < b_n < j, \text{ or } j < b_n < i, \\
4 & i > b_n. 
\end{cases} \quad (16)$$

FIG. 9: The distance $d(A_i, B_j)$ in $R_{18}^{(3)}$ nets. (a) If $i > j$ and the first $A$ is below $B_j$, the distance is 2. (b) If the first $A$ is above $B_j$, then the first $C$ must be below ($B$ can’t start the sequence); in that case if $A_i$ is below the last $B$ the distance is 3, and otherwise the distance is 4. Only the relevant parts of the complete net are shown.

**Eigenvalues:** We have found no obvious way to compute the eigenvalues, despite the similarities between $R_{18}^{(3)}$ nets and two-letter threshold nets. However, plots of the eigenvalues against the alphabetical ordering of the nets once again reveals intriguing fractal patterns, and one can hope that these might be exploited at the very least to produce good bounds and approximations.

In Fig 10 we plot the ratio $r = \lambda_N / \lambda_2$ for $R_{18}^{(3)}$ nets with $N = 7$ against their alphabetical ordering. The $x$-axis includes sequences of nets that are not connected: In this case $\lambda_2 = 0$ and synchronization is not possible. These cases show as gaps in the plot, for example, the big gap in the center corresponds to disconnected sequences that start with the letter $B$ (see Section III B).

D. Multi-threshold nets

Some of the three-letter sequence nets can be mapped to generalized forms of threshold nets. For example, the following scheme yields a two-threshold net, equivalent to three-letter sequence nets generated by the rule $R_{20}^{(3)}$. Let the nodes be assigned weights $0 < x_i < 3\theta/2$, from a random distribution, and connect any two nodes $i$ and $j$ that satisfy $x_i + x_j < \theta \equiv \theta_1$ or $x_i + x_j > 2\theta \equiv \theta_2$. Identifying nodes with weight $0 < x_i < \theta/2$ with $A$, nodes with $\theta/2 < x_i < \theta$ with $B$, and nodes with $\theta < x_i < 3\theta/2$ with $C$, we see that all $A$'s connect to one another and all $C$'s connect to one another but the $B$'s do not, and $A$'s and $C$'s do not connect; nodes of type $A$ and $B$ may or may not connect, and likewise for nodes of type $C$ and $B$. To reflect the actual connections, the nodes of type $A$ and $B$ may be arranged in sequence according to the algorithm in [9], for the threshold rule $R_{5}^{(2)}$. Also the nodes of type $C$ and $B$ may be arranged in a sequence, to reflect the actual connections, with the very same algorithm. Because there are no connections between $A$ and $C$ the two results may be trivially merged. Note, however, that once the $A$-$B$ sequence is established the order of the $B$'s is set, so the direction of connections between $C$ and $B$ ($C \rightarrow A$ or $A \rightarrow C$) is not arbitrary.

In our example, the mapping is possible to $R_{20}^{(3)}$ but not to $R_{18}^{(3)}$.

IV. SUMMARY AND DISCUSSION

We have introduced a new class of nets, sequence nets, obtained from a sequence of letters and fixed rules of connectivity. Two-letter sequence nets contain threshold nets, and in addition two newly discovered classes. The $R_{13}^{(2)}$ class can be mapped to a “difference-threshold” net,
where nodes $i$ and $j$ are connected if their weights difference satisfies $|x_i - x_j| < \theta$. This type of net may be a particularly good model for social nets, where the weights might measure political leaning, economical status, number of offspring, etc., and agents tend to associate when they are closer in these measures. We have shown that the structural properties of the new classes of two-letter sequence nets can be analyzed with ease, and we have introduced an ordering in ensembles of sequence nets that is useful in visualizing and studying their various attributes.

We have fully classified 3-letter sequence nets, and looked at a few examples, showing that they too can be analyzed simply. The diameter of sequence nets grows linearly with the number of letters in the alphabet, and agents tend to associate when they are closer in these measures. We have shown how to map sequence nets to generalized types of threshold nets, in some cases — Is such a mapping always possible? Is there a systematic way to find such mappings for any sequence rule? What kinds of nets would result if the connectivity rules applied only to the $q$ preceding letters, instead of to all preceding letters? etc. We hope to tackle some of these questions in future work.

There remain numerous open questions: Applying symmetry arguments we have managed to reduce the class of 3-letter nets to just 30 types, but we have not ruled out the possibility that some overlooked symmetry might reduce the list further; The question of which sequences lead to connected nets can be studied by inspection for small alphabets, but we have no comprehensive approach to solve the problem in general; We have shown how to map sequence nets to generalized types of threshold nets, in some cases — Is such a mapping always possible? Is there a systematic way to find such mappings for any sequence rule? What kinds of nets would result if the connectivity rules applied only to the $q$ preceding letters, instead of to all preceding letters? etc. We hope to tackle some of these questions in future work.

Acknowledgments

Partial funding from the NSF (DbA) and ARO (JS) is gratefully acknowledged.

[1] G. Caldarelli, A. Capocci, P. De Los Rios, and M. A. Muñoz, Phys. Rev. Lett. 89, 258702 (2002).
[2] M. Boguñá and R. Pastor-Satorras, Phys. Rev. E 68, 036112 (2003).
[3] N. Masuda, H. Miwa, and N. Konno, Phys. Rev. E 70, 036124 (2004).
[4] N. Konno, N. Masuda, R. Roy, and A. Sarkar, J. Phys. A 38, 6277 (2005).
[5] M. C. Golumbic, Algorithmic Graph Theory and Perfect Graphs. (Academic Press, New York, 1980).
[6] N. V. R. Mahadev and U. N. Peled, si Threshold Graphs and Related Topics, Vol. 56 of Annals of Discrete Mathematics (Elsevier, New York, 2005).
[7] P. L. Hammer, T. Ibaraki, and B. Simeone, SIAM J. Algebraic Discrete Methods 2, 39 (1981).
[8] R. Merris, Linear Algebra. Appl. 198, 143 (1994); Eur. J. Comb. 24, 413 (2003).
[9] A. Hagberg, P. J. Swart, and D. A. Schult, Phys. Rev. E 74, 056116 (2006).
[10] In [2] the two types of nodes are 0 and 1. We change the notation to avoid confusion with the entries of the rule matrix $R$.
[11] We prefer the more “boxy” look of rectangular boxes to denote the tight connections in complete graph cliques.
[12] M. Barahona and L. M. Pecora, Phys. Rev. Lett. 89, 054101 (2002).
[13] T. Nishikawa, A. E. Motter, Y. C. Lai, and F. C. Hoppensteadt, Phys. Rev. Lett. 91, 014101 (2003).
[14] H. Hong, B. J. Kim, M. Y. Choi, and H. Park, Phys. Rev. E 69, 067105 (2004).
[15] D. U. Hwang, M. Chavez, A. Amann, and S. Boccaletti, Phys. Rev. Lett. 94, 138701 (2005).
[16] A. E. Motter, C. Zhou, and J. Kurths, Phys. Rev. E 71, 016116 (2005).
[17] C. Gotsman, in Proceedings of the Shape Modeling International 2003, (IEEE Computer Society, Washington DC, 2003) p. 156.
[18] R. Albert, A.-L. Barabási, Rev. of Mod. Phys. 74, 47 (2002).
[19] Here $\overline{S}_n$ stands for the inverted type: $\overline{S}_n = A$ if $S_n = B$, and vice versa.
[20] One could order the graphs partially, according to their size (the number of links), however, it is less clear how to order the $\binom{n}{m}$ “degenerate” graphs with $m$ links.
[21] B. Mohar, Graph Theory, Combinatorics, and Applications, Vol. 2, Wiley (1991) pp. 871-898.
[22] The isoperimetric number of a graph $G$ is $i(G) = \inf x |\partial X|/|X|$, where the infimum is taken over all subsets $X$ of $V(G)$ satisfying $|X| \leq \frac{1}{2}|V(G)|$.
[23] We use the superscript to distinguish between two- and three-letter rules when doubt may arise otherwise.