THE “FUNDAMENTAL THEOREM”
FOR THE ALGEBRAIC K-THEORY OF SPACES.
III. THE NIL-TERM

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Abstract. In this paper we identify the “nil-terms” for Waldhausen’s algebraic $K$-theory of spaces functor as the reduced $K$-theory of a category of equivariant spaces equipped with a homotopically nilpotent endomorphism.

1. Introduction

This is the third in a series of papers which concerns the decomposition

$$A^d(X \times S^1) \simeq A^d(X) \times BA^d(X) \times N_- A^d(X) \times N_+ A^d(X).$$

Here, $A^d(X)$ is Waldhausen’s algebraic $K$-theory of the space $X$ and $BA^d(X)$ is a certain nonconnective delooping of it. The remaining factors on the right, called “nil-terms”, are homotopy equivalent $[H_1]$, $[H_2]$. They have not been given a $K$-theoretic description thus far.

In this installment, we will identify the nil-terms as a shifted copy of the reduced $K$-theory of a category whose objects are equivariant spaces equipped with a homotopically nilpotent endomorphism.

Let $X$ be a connected based space. Let $G$ denote the Kan loop group of the total singular complex of $X$, and let $G$ denote the geometric realization of $G$. Then the classifying space $BG$ has the weak homotopy type of $X$.

Define a category nil$(X)$ in which an object consists of a pair

$$(Y, f)$$

such that $Y$ is a based space with $G$-action and $f: Y \to Y$ is an equivariant map which is homotopically nilpotent under composition. Additionally, we assume that $Y$ admits the structure of a based $G$-cell complex in which the action of $G$ is free away from the basepoint. A morphism $(Y, f) \to (Z, g)$ is a based $G$-map $e: Y \to Z$ such that $g \circ e = e \circ f$.

There is a full subcategory nil$_d(X)$ of nil$(X)$ whose objects are those $Y$ which are finitely dominated in the sense that $Y$ is a retract up to homotopy of an object which is built up from a point by attaching a finite number of free $G$-cells. A morphism of nil$_d(X)$ is a weak equivalence if and only if its underlying map of topological spaces is a weak homotopy equivalence. It is a cofibration if its underlying map of spaces is obtained up to isomorphism by attaching free $G$-cells.
With the above structure, it turns out that \( \text{nil}_{fd}(X) \) is a category with cofibrations and weak equivalences. It therefore has a \( K \)-theory, which is denoted \( K_{fd}(\text{nil}(X)) \).

The forgetful functor \((Y, f) \mapsto Y\) gives rise to a map on \( K \)-theories
\[
K_{fd}(\text{nil}(X)) \to A_{fd}(X).
\]
Here we are using the model for \( A_{fd}(X) \) given by the algebraic \( K \)-theory of the category of finitely dominated based \( G \)-spaces ([W] §2.1, [H+ 1.5]). Let \( \tilde{K}_{fd}(\text{nil}(X)) \) denote the homotopy fiber of the map \( K_{fd}(\text{nil}(X)) \to A_{fd}(X) \).

Our main result establishes the other half of the “fundamental theorem” for \( A_{fd}(X) \):

**Main Theorem.** There is a homotopy equivalence of functors
\[
\tilde{K}_{fd}(\text{nil}(X)) \simeq \Omega N_{+} A_{fd}(X).
\]

**Remark.** The above result is used in the paper [GKM], where it is shown that the homotopy groups of \( N_{+} A_{fd}(X) \) are either trivial or infinitely generated. Another result of that paper determines the \( p \)-complete homotopy type of \( N_{+} A_{fd}(\ast) \) in degrees \( \leq 4p - 7 \), for \( p \) an odd prime.

2. Preliminaries

In what follows, we assume that the reader is familiar with the material of [H+].

The spaces in this paper are to be given the compactly generated topology. Products are taken in the compactly generated sense. Let \( M \) be a simplicial monoid, and let \( M = |M| \) denote its geometric realization. Let \( T(M) \) denote the category of based (left) \( M \)-spaces and based \( M \)-maps. We say that a based morphism \( Y \to Z \) of \( T(M) \) is weak equivalence if (and only if) it is a weak homotopy equivalence of underlying topological spaces. Similarly, we say that is a fibration if it is a Serre fibration after forgetting actions. A morphism is a cofibration if and only if it satisfies the left lifting property with respect to the acyclic fibrations (i.e., those fibrations which are weak equivalences). Then \( T(M) \) is a Quillen model category (see, e.g., [VS]).

Then every object of \( T(M) \) is fibrant, and the cofibrant objects are precisely the retracts of those objects which are built up from a point by cell attachments, where the cell of dimension \( n \) is given by
\[
D^n \times M
\]
with action defined by left translation.

Recall from [H+] that \( C(M) \) denotes the full subcategory of \( T(M) \) consisting of the cofibrant objects. Then \( C(M) \) is a category with cofibrations and weak equivalences in the sense of Waldhausen [W]. For objects \( Y \) and \( Z \) of \( C(M) \), we let
\[
[Y, Z]
\]
denote the homotopy classes of morphisms in \( C_{fd}(G) \), i.e., the based equivariant homotopy classes.

We next recall the various finiteness notions. An object of \( C(M) \) is finite if it is built up from a point by finitely many cell attachments (up to isomorphism). An object of \( C(M) \) is said to be homotopy finite if there exists a weak equivalence to a
finite object. An object of $\mathcal{C}(M)$ is said to be \textit{finitely dominated} if it is a retract of a homotopy finite object. Let $\mathcal{C}_{fd}(M)$ denote the full subcategory of $\mathcal{C}(M)$ whose objects are finitely dominated.

We let $h\mathcal{C}_{fd}(M)$ denote the subcategory of $\mathcal{C}_{fd}(M)$ defined by the weak equivalences. Then the associated $K$-theory space is given by

$$A^f(*; M) := \Omega h\mathcal{S} \mathcal{C}_{fd}(M),$$

where the right side is the based loop space of the geometric realization of Waldhausen’s $\mathcal{S}$-construction of $\mathcal{C}_{fd}(M)$ (\cite{W} p. 330). If $M$ is the realization of a simplicial group, then $A^f(*; M)$ is one of the definitions of $A^f(BM)$ (cf. \cite{W} p. 379), \cite{H}, 1.6).

The category $\text{nil}_{fd}(X)$ has \textit{objects} specified by pairs $(Y, f)$ with $Y \in \mathcal{C}_{fd}(G)$ and object $f: Y \to Y$ a morphism which is homotopically nilpotent under composition, i.e., the associated homotopy class

$$[f] \in [Y, Y]$$

is nilpotent in the sense that some iterate $[f^{\circ k}] = [f]^{\circ k}$ is trivial.

A \textit{morphism} $(Y, f) \to (Z, g)$ of $\text{nil}_{fd}(X)$ is a map $e: Y \to Z$ such that $g \circ e = e \circ f$. A \textit{cofibration} of $\text{nil}_{fd}(X)$ is a morphism $(Y, f) \to (Z, g)$ such that $Y \to Z$ is a cofibration of $\mathcal{C}_{fd}(G)$. A \textit{weak equivalence} is a morphism whose underlying map of spaces is a weak homotopy equivalence.

\textbf{Lemma 2.1.} With respect to the above conventions, $\text{nil}_{fd}(X)$ is a category with cofibrations and weak equivalences.

\textit{Proof.} The nontrivial thing to be verified is that the cobase change axiom holds. Given a diagram

$$(B, f_1) \leftarrow (A, f_0) \to (C, f_2)$$

we define the pushout to be $(B \cup_A C, f)$, where $f$ denotes $f_1 \cup f_0 f_2$. Choose a positive integer $k$ such that $[f_i]^{\circ k}$ is trivial, for $i = 0, 1, 2$. It will be sufficient to check that $[f]$ is nilpotent. Let us rename $g_i = f_i^{\circ k}$ and $g = f^{\circ k}$. Then, using the model structure, one has a Barratt-Puppe cofiber sequence

$$B \vee C \xrightarrow{j} B \cup_A C \xrightarrow{\delta} \Sigma A$$

in $T(M)$, where $\vee$ means wedge and $\Sigma$ is suspension. Consequently, there is an exact sequence of pointed sets

$$[\Sigma A, B \cup_A C] \xrightarrow{\delta^*} [B \cup_A C, B \cup_A C] \xrightarrow{j^*} [B \vee C, B \cup_A C].$$

Then

$$j^*([g]) = [g \circ j] = [g_1 \vee g_2] = 0,$$

so there is a homotopy class

$$\gamma \in [\Sigma A, B \cup_A C]$$

such that $[g] = \delta^*(\gamma) = \gamma \circ [\delta]$. Then

$$[g]^{\circ 2} = \gamma \circ [\delta] \circ \gamma \circ [\delta]$$

is trivial because $[\delta] \circ \gamma \circ [\delta] = [\delta] \circ [g]$ coincides with $[\Sigma g_0] \circ [\delta]$, and $[\Sigma g_0]$ is trivial. \qed
3. Another look at the projective line

Let \( \mathbb{N}_- \) denote the monoid of negative integers with generator \( t^{-1} \) and \( \mathbb{N}_+ \) denote the monoid of positive integers with generator \( t \). Let \( G \) be the realization of a simplicial group \( G \).

Recall that the mapping telescope of an object \( Y_+ \in \mathcal{C}_fd(G \times \mathbb{N}_+) \) is the object \( Y_+(t^{-1}) \in \mathcal{C}_fd(G \times \mathbb{Z}) \) defined by taking the categorical colimit of the sequence

\[
\cdots \rightarrow Y_+ \rightarrow t \cdot Y_+ \rightarrow \cdots.
\]

Similarly, if \( Y_- \in \mathcal{C}_fd(G \times \mathbb{N}_-) \) is an object, we have a mapping telescope \( Y_-(t) \) given by the colimit of

\[
\cdots \rightarrow Y_- \rightarrow t^{-1} \cdot Y_- \rightarrow \cdots.
\]

Define \( \mathcal{D}_{fd}(G \times \mathbb{Z}) \) to be the category whose objects are diagrams

\[
Y_- \rightarrow Y \leftarrow Y_+
\]

in which \( Y_- \in \mathcal{C}_fd(G \times \mathbb{N}_-) \), \( Y \in \mathcal{C}_fd(G \times \mathbb{Z}) \) and \( Y_+ \in \mathcal{C}_fd(G \times \mathbb{N}_+) \), and where the maps \( Y_- \rightarrow Y \) and \( Y_+ \rightarrow Y \) are required to be based and equivariant. Moreover, the induced morphisms

\[
Y_-(t) \rightarrow Y(t) \cong Y \quad \text{and} \quad Y_+(t^{-1}) \rightarrow Y(t^{-1}) \cong Y
\]

are required to be cofibrations of \( \mathcal{C}_fd(G \times \mathbb{Z}) \). We take the liberty of specifying the object as a diagram or as a triple \((Y_-, Y, Y_+)\).

A morphism \((Y_-, Y, Y_+) \rightarrow (Z_-, Z, Z_+)\) of \( \mathcal{D}_{fd}(G \times \mathbb{Z}) \) is a morphism \( Y_- \rightarrow Z_- \), a morphism \( Y \rightarrow Z \) and a morphism \( Y_+ \rightarrow Z_+ \) so that the evident diagram commutes. A cofibration is a morphism \((Y_-, Y, Y_+) \rightarrow (Z_-, Z, Z_+)\) in which

- each of the maps

\[
Y_- \rightarrow Z_-, \quad Y_+ \rightarrow Z_+ \quad \text{and} \quad Y \rightarrow Z
\]

is a cofibration (of \( \mathcal{C}_fd(G \times \mathbb{N}_-) \), \( \mathcal{C}_fd(G \times \mathbb{N}_+) \) resp. \( \mathcal{C}_fd(G \times \mathbb{Z}) \)), and

- the induced maps

\[
Y \cup_{Y_-(t)} Z_-(t) \rightarrow Z \quad \text{and} \quad Y \cup_{Y_+(t^{-1})} Z_+(t^{-1}) \rightarrow Z
\]

are cofibrations of \( \mathcal{C}_fd(G \times \mathbb{Z}) \).

The projective line \( \mathbb{P}_{fd}(G) \) of \( [\mathbb{H}^2] \) is given by the full subcategory of \( \mathcal{D}_{fd}(G \times \mathbb{Z}) \) whose objects \((Y_-, Y, Y_+)\) satisfy an auxiliary condition, viz., that the induced maps \( Y_-(t) \rightarrow Y \) and \( Y_+(t^{-1}) \rightarrow Y \) are weak homotopy equivalences. A cofibration is a morphism which is a cofibration of \( \mathcal{D}_{fd}(G \times \mathbb{Z}) \). A weak equivalence is a morphism in which \( Y_- \rightarrow Z_- \), \( Y \rightarrow Z \) and \( Y_+ \rightarrow Z_+ \) are weak homotopy equivalences of spaces.

Let \( \mathcal{D}_{fd}(G \times \mathbb{N}_-) \subset \mathcal{D}_{fd}(G \times \mathbb{Z}) \) denote the full subcategory whose objects \((Y_-, Y, Y_+)\) satisfy the condition that \( Y_-(t) \rightarrow Y \) is a weak equivalence. Similarly, define \( \mathcal{D}_{fd}(G \times \mathbb{N}_+) \) to be the full subcategory whose objects \((Y_-, Y, Y_+)\) satisfy the condition that \( Y_+(t^{-1}) \rightarrow Y \) is a weak equivalence.

A morphism \((Y_-, Y, Y_+) \rightarrow (Z_-, Z, Z_+)\) of \( \mathcal{D}_{fd}(G \times \mathbb{N}_+) \) is a weak equivalence if the map \( Y_+ \rightarrow Z_+ \) is a weak homotopy equivalence. It is a cofibration if it is so when considered in \( \mathcal{D}_{fd}(G \times \mathbb{Z}) \).

Let \( \mathbb{P}_{fd}^{\mathbb{N}_+}(G) \subset \mathbb{P}_{fd}(G) \) denote the full subcategory with objects \((Y_-, Y, Y_+)\) such that \( Y_+ \) is acyclic.
Proposition 3.1. There is a homotopy fiber sequence
\[ \Omega|hS\mathbb{P}^{|_{fd}}(G)| \to \Omega|hS\mathbb{P}_{fd}(G)| \to \Omega|hS\mathbb{D}_{fd}(G \times \mathbb{N}_+)|. \]

Proof. Define a coarser notion of weak equivalence on the projective line by specifying a morphism \((Y_-, Y, Y_+) \to (Z_-, Z, Z_+)) to be an \(h_{\mathbb{N}_-}\)-equivalence if (and only if) the map \(Y_+ \to Z_+\) is a weak equivalence. Application of the fibration theorem \(\text{[H1] 1.6.5}\) shows that the sequence
\[ \Omega|hS\mathbb{P}^{|_{fd}}(G)| \to \Omega|hS\mathbb{P}_{fd}(G)| \to \Omega|hS\mathbb{D}_{fd}(G)\]

is a fibration up to homotopy.

Let \(P_{fd}(G) \to \mathbb{D}_{fd}(G \times \mathbb{N}_+)\) denote the inclusion functor. By \(\text{[H1] 4}\) we have that the induced map
\[ |h_{\mathbb{N}_-}S\mathbb{P}_{fd}(G)| \to |hS\mathbb{D}_{fd}(G \times \mathbb{N}_+)| \]

induces an isomorphism on homotopy groups in degrees \(> 1\). Hence, the homotopy fiber of the induced map of loop spaces
\[ \Omega|h_{\mathbb{N}_-}S\mathbb{P}_{fd}(G)| \to \Omega|hS\mathbb{D}_{fd}(G \times \mathbb{N}_+)| \]
is homotopically trivial.

It follows that the homotopy fiber of the map
\[ \Omega|hS\mathbb{P}_{fd}(G)| \to \Omega|hS\mathbb{D}_{fd}(G \times \mathbb{N}_+)| \]
is identified with the homotopy fiber of the map
\[ \Omega|hS\mathbb{P}_{fd}(G)| \to \Omega|h_{\mathbb{N}_-}S\mathbb{P}_{fd}(G)|. \]
The result follows. \(\square\)

4. The “characteristic sequence”

Let \((Y, f) \in \text{nil}_{fd}(X)\) be an object, and let \(Y \otimes \mathbb{N}_- \in \mathbb{C}_{fd}(G)\) be the object given by
\[ (Y \times \mathbb{N}_-)/(*) \times \mathbb{N}_-. \]
Then \(f\) induces a self-map of \(Y \otimes \mathbb{N}_-\) which is given by \((y, r) \mapsto (f(y), r)\). We will denote this self-map also by \(f\).

Let \(Y_f\) be the homotopy coequalizer of the pair of maps
\[ Y \otimes \mathbb{N}_- \xrightarrow{f/t^{-1}} Y \otimes \mathbb{N}_-, \]
where \(t^{-1}\) denotes the map \((y, r) \mapsto (y, r-1)\). (Recall that the homotopy coequalizer of a pair of morphisms \(\alpha, \beta: U \to V\) is defined to be the quotient of the disjoint union \(V \coprod (U \times [0, 1])\) which is given by identifying \((u, 0)\) with \(\alpha(u)\), \((u, 1)\) with \(\beta(u)\) and \(* \times [0, 1]\) with the basepoint of \(V\).)

If we give \(Y\) the structure of a based \((G \times \mathbb{N}_-)-\)space by letting \(\mathbb{N}_-\) act by means of \(f\), then we also have a \((G \times \mathbb{N}_-)\)-equivariant map
\[ \pi_f: Y \otimes \mathbb{N}_- \to Y \]
which is given by \((y, r) \mapsto f^{-r}(y)\). Then \(\pi_f\) coequalizes \(f\) and \(t^{-1}\), so by the universal property of the homotopy coequalizer, there is an induced map
\[ Y_f \to Y, \]
which is \((G \times \mathbb{N}_-)\)-equivariant.
Lemma 4.1. The map $Y_f \to Y$ induces an isomorphism in reduced singular homology.

Proof. Let $p: S^1 \to S^1 \vee S^1$ be the pinch map, and let $\rho: S^1 \to S^1$ be the reflection map. Then the composite

$$S^1 \xrightarrow{p} S^1 \vee S^1 \xrightarrow{id \vee \rho} S^1 \vee S^1$$

will be denoted by $(1, -1)$.

The homotopy coequalizer induces a homotopy cofiber sequence

$$\Sigma(Y \otimes N_\ast) \xrightarrow{t^{-1}f} \Sigma(Y \otimes N_\ast) \to \Sigma Y_f$$

where the first map is defined to be the composite

$$\Sigma(Y \otimes N_\ast) \xrightarrow{(1, -1) \wedge id} \Sigma(Y \otimes N_\ast) \vee \Sigma(Y \otimes N_\ast) \xrightarrow{t^{-1}f} \Sigma(Y \otimes N_\ast).$$

Taking reduced singular chains, we get an induced homotopy cofiber sequence of chain complexes

$$C_\ast(Y_f) \otimes Z[t^{-1}] \xrightarrow{t^{-1}f_*} C_\ast(Y) \otimes Z[t^{-1}] \to C_\ast(Y_f).$$

(Recall that a sequence $A \xrightarrow{i} B \xrightarrow{j} C$ of chain complexes is a homotopy cofiber sequence when the composite $j \circ i: A \to C$ is equipped with a null homotopy such that the induced map from the mapping cone $T_{i\circ j}$ to $C$ is a quasi-isomorphism.) Now, for any $Z$-module $M$ equipped with a self-map $f: M \to M$, we have an exact sequence of $Z[t^{-1}]$-modules

$$0 \to M \otimes Z[t^{-1}] \xrightarrow{t^{-1}f} M \otimes Z[t^{-1}] \to M_f \to 0$$

in which $M_f$ denotes $M$ considered as a $Z[t^{-1}]$-module where $t^{-1}$ acts via $f$ (see [B, p. 630]). This implies that the sequence (1) becomes exact when $C_\ast(Y_f)$ is replaced by $C_\ast(Y)$ by means of the chain map $C_\ast(Y_f) \to C_\ast(Y)$ which is induced by the map $Y_f \to Y$. Consequently, the five lemma implies that the chain map $C_\ast(Y_f) \to C_\ast(Y)$ is a quasi-isomorphism. □

Remark 4.2. The sequence (1) is a chain complex version of the so-called, “characteristic sequence” (2) of the module $M$. Consequently, it is not inappropriate to think of the homotopy coequalizer diagram

$$Y \otimes N_\ast \xrightarrow{f} Y \otimes N_\ast \xrightarrow{t^{-1}} Y_f$$

as a kind of nonlinear version of the characteristic sequence (of the object $Y$).

Preliminary identification of $K(\text{nil}_{fd}(X))$. Define an exact functor

$$\text{nil}_{fd}(X) \xrightarrow{\Phi} \mathbb{P}_{fd}^{h\text{ss}}(G)$$

by

$$(Y, f) \mapsto (Y_f, Y_f(t), \ast),$$

where $Y_f$ is defined above.
In the other direction, define an exact functor
\[ P_{fd}^{h\pi_+} (G) \xrightarrow{\Psi} \text{nil}_{fd} (X) \]
by
\[ (Y_-, Y, Y_+) \mapsto (Y_-, t^{-1}) \, . \]

To see that \( \Psi \) is well-defined, let \((Y_-, Y, Y_+)\) be an object of \( P_{fd}^{h\pi_+} (G) \). Then \( Y_+ \) and \( Y \) are acyclic. Hence \( Y_- \) has an acyclic mapping telescope. This implies that there exists a \( k \in \mathbb{N}_- \) such that \( t^k : Y_- \to Y_- \) is (equivariantly) null homotopic (this follows for finite objects by the “small object” argument, and hence for finitely dominated ones since a retract of a null homotopic morphism is again null homotopic; compare [H.+, p. 40 bottom]).

Let \( Z \) denote the quotient \( Y_- / t^k (Y_-) \) considered as an object of \( \mathbb{C}(G) \). Then \( Z \) is finitely dominated. This is a consequence of a cell-by-cell induction when \( Y_- \) is a finite object of \( \mathbb{C}(G \times \mathbb{N}_+) \). It is true for homotopy finite objects because the functor \( Y_+ \mapsto Y_+ / t^k (Y_-) \) preserves weak equivalences. It is therefore also true when \( Y_- \) is finitely dominated since this functor also preserves retracts (cf. [H.+, p. 41 top]).

Since \( t^k \) is \( G \)-equivariantly null homotopic, the identity map \( Y_- \to Y_- \) factors through \( Z \) up to homotopy. It follows that \( Y_- \) is also finitely dominated when considered as an object of \( \mathbb{C}(G) \). This shows that \((Y_-, t^{-1})\) is an object of \( \text{nil}_{fd} (X) \).

Lemma 4.3. The functors \( \Psi \) and \( \Phi \) induce mutually inverse homotopy equivalences on \( K \)-theory.

Proof. The composite \( \Psi \circ \Phi \) is given by
\[ (Y, f) \mapsto (Y_f, t^{-1}) \]
and Lemma 4.1 implies that there is a morphism \((Y_f, t^{-1}) \to (Y, f)\) which is a weak equivalence after taking a suitable number of suspensions. Since suspension induces a homotopy equivalence on the level of \( K \)-theory [W 1.6.2], it follows that \( \Psi \circ \Phi \) induces a homotopy equivalence.

The composite \( \Phi \circ \Psi \) is given by
\[ (Y_-, Y, Y_+) \mapsto (Y_-, Y_- (t), *) \, . \]
This admits an evident equivalence to the identity functor. Consequently \( \Phi \circ \Psi \) induces a map which is homotopic to the identity on the level of \( K \)-theory. \( \square \)

5. Proof of the main theorem

By Lemma 4.3 we have a homotopy equivalence,
\[ \Omega [hS \text{nil}_{fd} (X)] \simeq \Omega [hS P_{fd}^{h\pi_+} (G)] \, . \]
Plugging this into Proposition 3.1 we obtain a homotopy fiber sequence
\[ \Omega [hS \text{nil}_{fd} (X)] \to \Omega [hS P_{fd} (G)] \to \Omega [hS D_{fd} (G \times \mathbb{N}_+)] \, . \]
Let \( \epsilon : \Omega [hS D_{fd} (G \times \mathbb{N}_+)] \to \Omega [hS C_{fd} (G)] \) denote the augmentation map of [H.+, 7.1], which is induced by
\[ (Y_-, Y, t^k) \mapsto Y / \mathbb{Z} \, . \]
where $Y/Z$ denotes the orbit space under the $Z$-action. Recall that the nil-term $N_+\mathcal{A}^{fd}(X)$ was defined to be the homotopy fiber of $\epsilon$. Similarly, $\epsilon$ restricts to a map on $\Omega\mathcal{S}\mathcal{P}_{fd}(G)$. Denote the homotopy fiber of this restriction by $\Omega\mathcal{S}\mathcal{P}_{fd}(G)\epsilon$. Consequently, we have an induced homotopy fiber sequence

$$
\Omega\mathcal{S}\mathcal{P}_{fd}(G)\epsilon \to N_+\mathcal{A}^{fd}(X).
$$

In was shown in [H$^+$, 7.6] that the second of these maps, $\Omega\mathcal{S}\mathcal{P}_{fd}(G)\epsilon \to N_+\mathcal{A}^{fd}(X)$, is null homotopic. Moreover, it was shown in [H$^+$, 7.5] that there is a homotopy equivalence $\Omega\mathcal{S}\mathcal{P}_{fd}(G)\epsilon \simeq \Omega\mathcal{S}\mathcal{C}_{fd}(G)$ induced by the global sections functor $\Gamma: \mathcal{P}_{fd}(G) \to \mathcal{C}_{fd}(G)$ defined by

$$
(Y_-,Y,Y_+) \mapsto CY_- \cup Y \cup CY_+,
$$

where $CY_-$ denotes the cone on $Y_-$. Assembling this information, we have a homotopy fiber sequence

$$
(3) \quad \Omega\mathcal{S}\mathcal{P}_{fd}(G)\epsilon \xrightarrow{\alpha} \Omega\mathcal{S}\mathcal{C}_{fd}(G) \xrightarrow{\beta} N_+\mathcal{A}^{fd}(X)
$$

where $\alpha$ is induced by the functor $(Z,f) \mapsto \Sigma Z$. Since the suspension functor $\Sigma: \mathcal{C}_{fd}(G) \to \mathcal{C}_{fd}(G)$ induces a homotopy equivalence (by [W, 1.6.2]), we see that the homotopy fiber of $\alpha$ is homotopy equivalent to the homotopy fiber of the map $\alpha'$ which is induced by the forgetful map $(Z,f) \mapsto Z$.

On the one hand, the homotopy fiber of $\alpha'$ is $\tilde{K}^{fd}(\text{nil}(X))$, by definition. On the other hand, the homotopy fiber sequence (3) implies that the homotopy fiber of $\alpha$ is homotopy equivalent to $\Omega N_+\mathcal{A}^{fd}(X)$. We conclude that there is a homotopy equivalence

$$
\tilde{K}^{fd}(\text{nil}(X)) \simeq \Omega N_+\mathcal{A}^{fd}(X).
$$

This completes the proof of the theorem.

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