Universal effective couplings of the three-dimensional $n$-vector model and field theory

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Abstract

We calculate the universal ratios $R_{2k}$ of renormalized coupling constants $g_{2k}$ entering the critical equation of state for the generalized Heisenberg (three-dimensional $n$-vector) model. Renormalization group (RG) expansions of $R_8$ and $R_{10}$ for arbitrary $n$ are found in the four-loop and three-loop approximations respectively. Universal octic coupling $R_8^*$ is estimated for physical values of spin dimensionality $n = 0, 1, 2, 3$ and for $n = 4, ... 64$ to get an idea about asymptotic behavior of $R_8^*$. Its numerical values are obtained by means of the resummation of the RG series and within the pseudo-$\varepsilon$ expansion approach. Regarding $R_{10}$ our calculations show that three-loop RG and pseudo-$\varepsilon$ expansions possess big and rapidly growing coefficients for physical values of $n$ what prevents getting fair numerical estimates.

Keywords: renormalization group, multi-loop calculations, effective coupling constants, universal ratios.

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1. Introduction

The thermodynamics of O($n$)-symmetric systems near the Curie point is characterized by universal parameters that depend only on the ordering field dimensionality $n$ and the number of space dimensions $D$. The renormalized effective coupling constants $g_{2k}$ and the ratios $R_{2k} = g_{2k}/g_4^{k-1}$ entering the small-magnetization expansion of the free energy (the effective potential)

$$F(z, m) - F(0, m) = \frac{m^4}{g_4} \left( \frac{z^2}{2} + z^4 + R_6 z^6 + R_8 z^8 + R_{10} z^{10} + ... \right),$$

are among them. They determine, along with effective potential, the nonlinear susceptibilities $\chi_{2k} = \partial^{2k-1} M_\alpha / \partial H_\alpha^{2k-1}$ above $T_c$:

$$\begin{align*}
\chi_4 &= -24 \frac{\chi^2}{m^2} g_4, \\
\chi_6 &= -64 \frac{\chi^2}{m^6} (R_6 - 8), \\
\chi_8 &= -8 \frac{\chi^2}{m^8} (R_8 - 24R_6 + 96), \\
\chi_{10} &= -10 \frac{\chi^2}{m^{10}} (R_{10} - 32R_8 - 18R_6^2 + 528R_6 - 1408).
\end{align*}$$

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Here \( z = M \sqrt{g_4/m^{1+\eta}} \) is a dimensionless magnetization, \( m \sim (T - T_c)^\nu \) being an inverse correlation length, \( \chi \) is a linear susceptibility and \( \chi_4, \chi_6, \chi_8, \chi_{10} \) are nonlinear susceptibilities of corresponding orders.

For the three-dimensional (3D) \( n \)-vector and related models, the effective potential and nonlinear susceptibilities are intensively studied during several decades. Renormalized coupling constants \( g_{2k} \) and universal ratios \( R_{2k} \) were evaluated by a variety of analytical and numerical methods [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50].

Estimating universal critical values of quartic and sextic effective couplings by means of the field-theoretical renormalization group (RG) approach has shown that RG technique enables one to get rather accurate numerical results for these quantities. Multi-loop RG calculations of \( g_4 \) and \( g_6(R_6) \) combined with proper resummation procedures were found to yield numerical estimates that have three-four decimals level of accuracy. For example, advanced RG estimates of the Wilson fixed point location \( g = g_4(n + 8)/2\pi \) for \( n = 1 \) lie between 1.411 and 1.417 [2, 3, 4, 19] while field-theoretical results \( R_6 = 1.648 \) and \( R_6 = 1.649 \) [16, 25, 48] perfectly agree with the value \( 1.649 \pm 0.002 \) given by advanced lattice calculations [41].

The attempts to get accurate enough field-theoretical estimates for higher-order coupling \( g_8 \) and universal ratio \( R_8 \) were, however, much less successful. The numbers obtained by means of resummation of the three-loop RG series for \( g_8 \) [25] and extracted from corresponding pseudo-\( \epsilon \) expansions for \( g_8 \) and \( R_8 \) turn out to differ markedly from each other and from known lattice estimates (see, e. g. Table X in Ref. [42] where the relevant data are collected). In principle, this is not surprising since the RG expansion for octic coupling found in Ref. [25] is shorter than those for \( g_8 \) (four-loop) and for \( \beta \) function (six-loop) resulting in numerical values of \( g_4 \). Moreover, it is stronger divergent. For \( g_8 \), however, there is an extra reason – the last but not least – making existing numerical estimates unexpectedly crude. The point is that the RG series for \( g_8 \) has an unusual feature: the first two terms in this series tend to compensate each other both for physical values of \( n \) and for \( n \gg 1 \) [25]. This makes their mutual contribution small and increases the role of the higher-order terms. The same is true for corresponding pseudo-\( \epsilon \) expansions [42]. So, finding of higher-order terms in the RG series appears to be extremely important for getting proper numerical estimates of the octic coupling.

In this paper, we extend the RG expansion of effective coupling constants \( g_8 \) up to four-loop order and calculate three-loop RG series for \( g_{10} \) under arbitrary \( n \). Since the pseudo-\( \epsilon \) expansion approach was shown to be highly efficient when used to evaluate critical exponents and other universal quantities [4, 42, 44, 46, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67] we calculate the pseudo-\( \epsilon \) expansions for both couplings as well. To get reliable numerical estimates for higher-order couplings and universal ratios the Wilson fixed point location is refined using the resummation procedures that allow optimization by adjusting free parameters involved. The RG series and corresponding pseudo-\( \epsilon \) expansion for the octic coupling are then resummed by means of Borel-transformation-based techniques, and the numbers obtained are compared with their counterparts found within the alternative approaches. The structure of RG series and pseudo-\( \epsilon \) expansions for \( g_{10} \) and \( R_{10} \) is discussed and the conclusion concerning the capability of the field theory to give proper numerical estimates for this coupling is made.

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2. Perturbative, RG and pseudo-ε expansions

The critical behavior of 3D n-vector model is described by Euclidean field theory with the Hamiltonian:

\[ H = \int d^3x \left[ \frac{1}{2} \left( m_0^2 \phi^2 + (\nabla \phi)^2 \right) + \frac{\lambda}{24} (\phi^4) \right]. \tag{3} \]

where \( \phi \) is a real vector field, \( \alpha = 1, \ldots, n \), bare mass squared \( m_0^2 \) being proportional to \( T - T_c^{(0)} \) and \( T_c^{(0)} \) – mean field transition temperature.

To derive RG expansions for \( g_8 \) and \( g_{10} \) we employ the following – straightforward – method. The renormalized perturbative series are found within the massive theory from conventional Feynman graph expansions for eight-point and ten-point vertices in terms of the quartic coupling constant \( \lambda \). In the course of such calculations the tensor structure of these vertices is taken into account:

\[ \Gamma_{\alpha\beta\gamma\delta\mu\nu\xi\zeta} = \frac{1}{105} (\delta_{\alpha\beta} \delta_{\gamma\delta} \delta_{\mu\nu} \delta_{\xi\zeta} + 104 \text{ permutations}) \Gamma_8, \tag{4} \]

\[ \Gamma_{\alpha\beta\gamma\delta\mu\nu\xi\zeta} = \frac{1}{945} (\delta_{\alpha\beta} \delta_{\gamma\delta} \delta_{\mu\nu} \delta_{\xi\zeta} + 944 \text{ permutations}) \Gamma_{10}, \tag{5} \]

where \( \Gamma_{2k} = g_{2k} m^{2-k(1+\eta)} \), \( m \) and \( \eta \) are renormalized mass and Fisher exponent respectively. The bare coupling constant \( \lambda \), in its turn, is expressed perturbatively as a function of the renormalized dimensionless \( g_4 \). Substituting corresponding power series for \( \lambda \) into original expansions we obtain the RG series for \( g_8 \) and \( g_{10} \).

The one-, two-, three- and four-loop contributions to \( g_8 \) are formed by 1, 5, 36 and 268 one-particle reducible Feynman graphs, respectively. Corresponding integrals, symmetry and tensor factors are presented in Table 1 of Supplementary materials (see Appendix A) where Nickel’s notations describing graphs topology are used. Summing up all the calculated graphs we obtain:

\[
g_8 = -\frac{81}{2\pi} \left( \frac{\lambda Z^2}{m} \right)^4 \left[ \frac{n + 80}{81} \frac{405n^2 + 35626n + 342320}{13122\pi} \left( \frac{\lambda Z^2}{m} \right)^2 + (0.0046907955n^3 + 0.463650683n^2 + 8.86811653n + 45.4769028) \left( \frac{\lambda Z^2}{m} \right)^2 + 0.00174198n^4 + 0.194893055n^3 + 5.58218793n^2 + 59.2588362n + 209.398270 \left( \frac{\lambda Z^2}{m} \right)^{1.4} \right]. \tag{6} \]

The expansion for \( \lambda \) in terms of renormalized dimensionless effective coupling \( g_4 \) emerges directly from the normalizing condition \( \lambda = mZ^2 g_4 \) and the known series for \( Z_4 \) \[11]:

\[
Z_4 = 1 + \frac{n + 8}{2\pi} g_4 + \frac{3n^2 + 38n + 148}{12\pi^2} g_4^2 + (0.0040314418 n^3 + 0.0679416657 n^2 + 0.466356233 n + 1.240338484) g_4^3. \tag{7} \]

Combining these expressions we obtain:

\[
g_8 = -\frac{81}{2\pi} \left( \frac{\lambda Z^2}{m} \right)^4 \left[ \frac{n + 80}{81} \frac{81n^2 + 7114n + 134960}{13122\pi} g_4 + (0.00943497n^2 + 0.60941312n + 7.156515323) g_4^2 + (0.00013078n^3 + 0.04703841n^2 + 1.97176517n + 16.56483375) g_4^3 \right]. \tag{8} \]
The coefficients of the series in square brackets are seen to grow progressively with $n$. On the other hand, the Wilson fixed point coordinate $g_4^*$ is known to diminish as a function of this variable. That is why in order to make the numerical structure of RG expansions for higher-order couplings more transparent it is natural to use, instead of $g_4$, the rescaled quartic coupling constant

$$g = \frac{n + 8}{2\pi} g_4,$$  

(9)

whose critical (fixed point) value only weakly depends on $n$ lying between 1.42 and 1 for any $n$. The RG series for $g_8$ in terms of $g$ is as follows:

$$g_8 = -\frac{648\pi^3}{(n + 8)^2} g^4 \left[ \frac{n + 80}{81} - \frac{g}{(n + 8)} \frac{81n^2 + 7114n + 134960}{651} + \frac{g^2}{(n + 8)^3} \left( 0.372477560n^2 
+ 24.0586651n + 282.513606 \right) - \frac{g^3}{(n + 8)^3} \left( 0.0324416663n^3 + 11.6678970n^2 
+ 489.096788n + 4108.91056 \right) \right].$$  

(10)

In the case of $g_{10}$, the one-, two-, and three-loop contributions are given by 1, 6 and 64 Feynman graphs, respectively. They are listed in Table 2 of Supplementary materials (see Appendix A). Corresponding "bare" and renormalized perturbative expansions are found to be:

$$g_{10} = \frac{243}{\pi} \left( \frac{\lambda Z^2}{m} \right)^5 \left[ \frac{n + 242}{243} - \frac{567n^2 + 128108n + 1544380}{393666} \right] + \left( 0.0029187172n^3 
+ 0.6439281n^2 + 15.53330379n + 94.77350036 \right) \left( \frac{\lambda Z^2}{m} \right)^2 \right],$$  

(11)

$$g_{10} = \frac{243}{\pi} g_4^4 \left[ \frac{n + 242}{243} - \frac{81n^2 + 13429n + 380150}{19683} \right] + \left( 0.0001042399n^3 + 0.02139208n^2 + 1.42119791n + 21.74148152 \right) g_4^2.$$

(12)

The latter series in terms of $g$ reads:

$$g_{10} = \frac{7776\pi^4}{(n + 8)^3} g^8 \left[ \frac{n + 242}{243} - \frac{162n^2 + 26858n + 760300}{19683(n + 8)} g^4 
+ \left( 0.00411522634n^3 + 0.844525548n^2 + 56.1066447n + 858.319287 \right) \frac{g^2}{(n + 8)^2} \right].$$  

(13)

The RG series derived turn out to be strongly divergent. To make them more suitable for getting numerical estimates the pseudo-$\varepsilon$ expansion technique may be employed. Pseudo-$\varepsilon$ expansions for the critical values of $g_8$ and $g_{10}$ can be derived from (8) and (12) substituting the
pseudo-\( \varepsilon \) expansion for the Wilson fixed point coordinate which is as follows \[42\]:

\[
\begin{align*}
g^*_8 &= \frac{2\pi}{n+8} \left[ \tau + \frac{\tau^2}{(n+8)^2} \left( 6.074074074\ n + 28.14814815 \right) \\
+ \frac{\tau^3}{(n+8)^4} \left( -1.34894276\ n^3 + 8.056832799\ n^2 + 44.73231547\ n - 12.48684745 \right) \\
+ \frac{\tau^4}{(n+8)^6} \left( -0.15564589\ n^5 - 7.63802173\ n^4 + 100.0250844\ n^3 + 679.8756744\ n^2 \\
+ 1604.099837\ n + 3992.366079 \right) \right].
\end{align*}
\]

(14)

Combining (8), (12) and (14) we obtain:

\[
\begin{align*}
g^*_8 &= -\frac{8\pi^3}{(n+8)^4} \left[ n + 80 + \frac{\tau}{(n+8)^2} \left( 248.05021n^3 + 17743.246n^2 + 77514.161n + 1072066.9 \right) \\
+ \frac{\tau^2}{(n+8)^4} \left( 1387.9548n^4 + 197852.87n^3 + 1715306.9n^2 + 15922971n + 8711449.0 \right) \\
+ \frac{\tau^3}{(n+8)^6} \left( -866.7700n^5 - 37159.01n^4 + 1.204930 \cdot 10^6n^3 + 2.880413 \cdot 10^7n^2 \\
+ 4.461346 \cdot 10^9n + 2.163345 \cdot 10^9n + 4.901591 \cdot 10^9 \right) \right].
\end{align*}
\]

(15)

\[
\begin{align*}
g^*_{10} &= \frac{32\pi^4}{(8 + n)^5} \left[ n + 242 - \frac{\tau}{(n+8)^2} \left( 6234.18183n^3 + 988772.024n^2 + 14178684.5n + 127900782 \right) \\
+ \frac{\tau^2}{(n+8)^4} \left( 3117.09091n^5 + 441336.763n^4 + 8582256.44n^3 + 140472393n^2 \\
+ 1.02642700 \cdot 10^9n + 8.00714134 \cdot 10^9 \right) \right]
\end{align*}
\]

(16)

Since the equation of state and expressions for nonlinear susceptibilities contain the universal ratios rather than effective coupling constants themselves it is reasonable to have the pseudo-\( \varepsilon \) expansions for the critical values of these ratios. They are as follows:

\[
\begin{align*}
R^*_8 &= -\frac{n+80}{n+8} \tau + \frac{\tau^2}{(n+8)^3} (n^3 + 89.753086n^2 + 1854.7160n + 11077.531) \\
- \frac{\tau^3}{(n+8)^5} (2.874679n^4 + 1111.2054n^3 + 22512.707n^2 + 199142.42n + 713156.70) \\
+ \frac{\tau^4}{(n+8)^7} (0.0855557n^5 + 272.9566n^4 + 15580.31n^3 + 363790.3n^2 \\
+ 4151127n^2 + 2.348384 \cdot 10^7n + 5.664042 \cdot 10^7),
\end{align*}
\]

(17)

\[
\begin{align*}
R^*_{10} &= \frac{2(n + 242)}{n+8} \tau - \frac{\tau^2}{(n+8)^3} (4n^3 + 683.012346n^2 + 21081.9753n + 136559.012) \\
+ \frac{\tau^3}{(n+8)^5} (2n^5 + 391.148939n^4 + 24655.9875n^3 + 552045.964n^2 \\
+ 5261106.26n + 1823638.8).
\end{align*}
\]

(18)
These RG expansions and τ-series will be used to estimate higher-order effective couplings near the Curie point.

3. Wilson fixed point coordinate

Before going further to the numerical analysis of the universal ratios we would like to perform some refinement of the Wilson fixed point coordinate for different \( n \) which was calculated earlier in a number of papers (see, e. g. \([2, 3, 4, 11, 19, 24, 25, 42]\)). For this purpose the authors developed certain procedures based on both Padé–Borel–Leroy (PBL) and conform-Borel (CB) resummation techniques. Here we briefly describe the steps of the algorithms with the specific attention paid to the choice of the fitting parameters involved.

Let us start with the first – PBL – technique which is a transparent and concise procedure. As a starting point we take the \( \beta \) function of the \( O(n) \)-symmetric model obtained within the RG perturbation theory in three dimensions. Knowing the asymptotic behaviour of the expansion coefficients for \( \beta \) function we can apply the Borel transformation which factorially diminishes these coefficients making the transformed series - the Borel image - convergent with finite radius of convergence. This expansion when being complete, i. e. having infinite number of perturbative terms, would converge to some analytical function which may be considered as an analytical continuation of the series sum beyond the convergence radius. In practice, however, only truncated series are in hand, so we have to perform the analytical continuation approximately, in some reasonable way using all the information available. Within the PBL resummation approach Padé approximants being an example of classical method of rational approximation are employed for the analytical continuation. Schematically, this procedure looks as follows:

\[
\beta^{\text{N}}_{\text{PBL}}(g) = \int_0^\infty dt e^{-t}[L/M][B^N_{\text{PBL}}(\beta)](g), \quad B^N_{\text{PBL}}(\beta)(g) = \sum_{k=0}^N \frac{\beta_k}{\Gamma(k + b + 1)} g^k, \quad \beta(g) = \sum_{k=0}^\infty \beta_k g^k, \quad (19)
\]

\[
[B^N_{\text{PBL}}(\beta)](g) = \frac{a_0([\beta]^N) + a_1([\beta]^N)g + a_2([\beta]^N)g^2 + \cdots + a_N([\beta]^N)g^N}{1 + b_1([\beta]^N)g + b_2([\beta]^N)g^2 + \cdots + b_N([\beta]^N)g^M}, \quad L + M = N.
\]

In order to evaluate the Wilson fixed point coordinate \( g^* \) in the six-loop (highest-order available) approximation, we have to find the non-trivial root of the following equation:

\[
\beta(g^*) = 0, \quad \beta^{\text{N}}_{\text{PBL}}(g(b, L, M)) = 0, \quad g^* = g(b^*, M^*, N^*). \quad (20)
\]

Obviously, with this set of parameters – \( \{L, M, b\} \) – we have some ambiguity in the course of finding \( g^* \). How to choose properly the values of these parameters? As for fitting parameter \( b \), it accelerates the convergence of the estimates. However, since it has no physical meaning, when choosing its specific numerical value it is natural to require the least sensitivity of the function \( g(b, L, M) \) in the vicinity of \( b^* \) with respect to \( L \) and \( M \). As is well known the diagonal – \([L/L] \) – and near-diagonal – \([L/L - 1], [L - 1/L] \) – approximants possess the best approximating properties. However, there are no obvious preferences between them except for the fact that the approximant should have no poles on positive real axis which would cause divergences in integration \([19]\). That is why here we require also a minimal variation of the estimate of the fixed point location when using certain approximant. So, our way of doing is as follows.

We define reasonable interval of variation of parameter \( b \), choose a step with which we will analyze it and calculate a Padé triangle for each point \( b \), the example of which is presented.
Table 1: Example of Padé triangle with PBL estimates of fixed-point coordinate for \( n = 6 \). The optimal value of shift parameter \( b \) is 0.01. '-' indicates that corresponding approximant is not relevant.

|   | 0   | 1  | 2  | 3  | 4  | 5  | 6  |
|---|-----|----|----|----|----|----|----|
| 0 | -   | 1.18 | -  | -  | 1.098 | -  |
| 1 | 1.426 | 1.331 | 1.338 | 1.340 | 1.337 |
| 2 | 1.294 | 1.338 | 1.341 | 1.339 |
| 3 | 1.359 | 1.340 | 1.339 |
| 4 | 1.348 | 1.336 |
| 5 | 1.339 |
| 6 | -   |

In Table 1. With these triangles in hand we construct some sample which is formed by the most reliable approximants. What we have to include into this set? In addition to the highest-order approximants, in order to somehow take into account fluctuations of convergence which are believed to decrease with increasing order, we have to include into consideration the lower-order ones. Then, as was already said, we exclude approximants spoiled by the positive axis poles. Since boundary approximants \([L/0]\) and \([0/L]\) usually contribute big errors they are also excluded. The results of such calculations for various \( n \) are presented in Table 2 and in Figure 1.

Also, for clarity, we demonstrate in Figure 2 the dependence of \( g^* \) given by three most stable PBL approximants on fitting parameter \( b \).

If we have the information about the asymptotic behavior of the series coefficients we can resort to the CB resummation technique. For \( \varphi^4 \) theory, the Lipatov’s asymptotics is well
known \[68,69\]:

\[
\beta_k \propto a^k k! b^k, \quad k \to \infty.
\] (21)

It allows to construct an analytical continuation of the Borel image that is expected to have efficient approximating properties. The detailed description of CB resummation procedure was given in many works (see, e.g., \[4, 16, 19, 31, 36, 37\]). Here we restrict ourselves with the working formulas and description of the algorithm of choosing the resummation parameters. The machinery used by authors is the following. First, we separate in series for \(\beta\) function its “loop” (perturbative) part. Within the normalization we use, in the critical point this part has to coincide with negative value of the Wilson fixed point coordinate (\(-g^*\)):

\[
\beta(g) = \sum_{k=0}^{\infty} \beta_k g^k, \quad \beta_{\text{loop}}(g) = \beta(g) - g.
\] (22)

Next, we resum the loop part of the \(\beta\) function taking into account the asymptotic behavior of its coefficients. The resummation parameter \(b\), that is necessary for a better fitting of the \(\beta\)-function coefficients behaviour according to (21), varies within \(b_{\text{range}} = [0, 25, 0.5]\). \(\lambda\) – within \(\lambda_{\text{range}} = [0, 4.5, 0.02]\). The parameter \(\lambda\) allows us to also adjust the growth \(B_n^\lambda[\beta](g)\) from (19) as \(\sim g^*\) that markedly improves the convergence of numerical estimates.

We take the first point from \(b_{\text{range}} \times \lambda_{\text{range}}\) and calculate resummed \(\beta_{\text{loop}}(g)\) by means of the
Table 2: The Wilson fixed point location obtained from six–loop RG expansion for $\beta$ function. For $n > 5$ the values of $g^*$ given by CB and PBL resummations practically coincide to each other. The estimates found earlier using Padé–Borel (PB), Padé–Borel–Leroy (PBL) and Borel–Leroy + conformal mapping (CB) resummation procedures are also presented for comparison.

| n  | CB ($n \leq 5$), PBL ($n > 5$), this work | PB [11] | PBL [3] | CB [12] | CB [19] | PBL [24] | CB [39] |
|----|------------------------------------------|---------|---------|---------|---------|---------|---------|
| 0  | 1.4185                                   | 1.402   | 1.421   | 1.417   | 1.413(6)|         |         |
| 1  | 1.4166                                   | 1.401   | 1.416   | 1.414   | 1.411(4)| 1.419   |         |
| 2  | 1.4061                                   | 1.394   | 1.406   | 1.405   | 1.403(3)| 1.4075  |         |
| 3  | 1.3914                                   | 1.383   | 1.392   | 1.391   | 1.390(4)| 1.392   |         |
| 4  | 1.3745                                   | 1.369   |         |         | 1.377(5)| 1.3745  |         |
| 5  | 1.3566                                   | 1.353   |         |         | 1.3565  | 1.3569  |         |
| 6  | 1.3388                                   | 1.336   |         |         | 1.3385  | 1.3397  |         |
| 8  | 1.3050                                   | 1.303   |         |         | 1.3045  |         |         |
| 10 | 1.2754                                   | 1.274   |         |         | 1.2745  |         |         |
| 12 | 1.2491                                   | 1.248   |         |         | 1.2487  |         |         |
| 14 | 1.2270                                   | 1.226   |         |         | 1.2266  |         |         |
| 16 | 1.2080                                   | 1.207   |         |         | 1.2077  |         |         |
| 18 | 1.1918                                   | 1.191   |         |         | 1.1914  |         |         |
| 20 | 1.1776                                   | 1.177   |         |         | 1.1773  |         |         |
| 24 | 1.1545                                   | 1.154   |         |         | 1.1542  |         |         |
| 28 | 1.1365                                   | 1.136   |         |         | 1.1361  |         |         |
| 32 | 1.1222                                   | 1.122   |         |         | 1.1218  | 1.1219  |         |
| 48 | 1.0858                                   |         |         |         |         |         | 1.0656  |
| 64 | 1.0659                                   |         |         |         |         |         |         |

According to the asymptotic analysis, $a$ defines the closest to the origin singularity as $-1/a$ of Borel-transformed function $B_N^\beta[g]$ in (19).

Having obtained $\beta_{b,\text{loop}}^N$ as function of $g$, we have to find a minimal value of the following functional on some a priori known grid, for example, from the PBL analysis with the step $\Delta g$:

$$F[g] = |\beta_{b,\text{loop}}^N(g) + g|, \quad g^* : F[g^*] = \min_g F[g].$$

(24)

If this minimal value is bigger than some marginal one – $\varepsilon_\beta$, which we set at the beginning of
the procedure we discard currently considered point \((b, \lambda)\) and move to the next one, otherwise we begin to analyze the stability of the result obtained. In fact, the found candidate for the fixed point coordinate is a function of the resummation parameters – \(g^*(b, \lambda)\). As was already claimed, these parameters have no physical meaning, therefore it looks reasonable to adopt as optimal such their values in the vicinity of which the estimate would be least sensitive to their variations.

To do this, we use proposed in [49] measure of spreading (error bar) – \(\Delta_s(b, \lambda)\). Having ended the first cycle, we add all information related to this point – \(((b, \lambda), F[g^*], g^*, \Delta_s(b, \lambda))\) – to the database and then repeat all the above steps for the whole grid of resummation parameters. The minimal value of the \(\Delta_s(b, \lambda)\) is, in fact, an indicator that analyzed function \(\beta_{model}(g^*)\) achieved a plateau. After analyzing the whole grid, we find point through collected data with the minimal value of \(\Delta_s(b^*, \lambda^*)\). It is natural to take as a final estimate of the fixed point coordinate the corresponding value of \(g^*(b^*, \lambda^*)\). In order to exclude the possibility of accidentally extreme deviations we perform the following steps. We construct the set of points, the spreading of which would not exceed the minimal one – \(\Delta_s(b^*, \lambda^*)\) – by more than three times. Then we take the weighted average of the obtained sample and tripled standard deviation is accepted as an error bar. To illustrate the work of this algorithm, the histograms for \(n = 0\) and \(n = 5\) with distribution of \(g^*(b, \lambda)\) constructed on the base of the whole parameter grid are presented in Figure 3.

The values of \(g^*\) corresponding to picks of histograms are accepted as final estimates of the Wilson fixed point location given by CB resummation procedure. They are collected, along with PBL estimates, in Table 2.

4. Octic effective interaction at criticality

To get an idea about numerical structure of the expansions for the octic effective coupling let us consider the series obtained under physically most interesting values \(n = 1\) and \(n = 3\) corresponding to simple fluids, binary mixtures, uniaxial and Heisenberg ferromagnets etc. It is instructive also to address the cases \(n = 10\) (superfluid neutron liquid [70, 71]), \(n = 18\) (superfluid helium-3 [72, 74]) and \(n = 64\) that shed light on the behavior of the RG and pseudo-\(\epsilon\) expansions in the limit \(n \gg 1\). These series read:

\[1\text{If the functional takes smaller values than } \varepsilon, \text{ we assume that we found a zero of } \beta \text{ function.}\]
The value of the parameter $b$ means of PBL resummation technique. By analogy with the calculation of the fixed point we apply resummation procedures. Let us first process the RG expansion and $\tau$ this universal ratio.

determining important physical quantities. That is why further we will work with the series for $R$ than that for the coupling constant. Indeed, although the coefficients of the series (26) are bigger than those of the series (25), these series are alternating, i.e. have a regular structure. Moreover, the universal ratio $R_8$ itself enters the expressions for free energy and nonlinear susceptibilities determining important physical quantities. That is why further we will work with the series for this universal ratio.

Since the series (24), (26) are divergent, to get proper numerical estimates as usual we have to apply resummation procedures. Let us first process the RG expansion and $\tau$-series for $R_8^\tau$ by means of PBL resummation technique. By analogy with the calculation of the fixed point the value of the parameter $b$ will be referred to as optimal, $b_{opt}$, if it provides the fastest convergence of the series, i.e. minimizes the differences between the estimates given by the most stable (diagonal or near-diagonal) Padé approximants for the Borel-Leroy image. Looking for more advanced resummation technique we address the CB machinery. The procedure is similar to that

$$R_8 = -9g\left(1 - 2.4074074g + 3.7894413g^2 - 6.3233302g^3\right), \ n = 1$$

$$R_8 = -\frac{83}{11}g\left(1 - 2.1233892g + 2.8877222g^2 - 4.1661684g^3\right), \ n = 3$$

$$R_8 = -5g\left(1 - 1.6323731g + 1.5565223g^2 - 1.5739367g^3\right), \ n = 10$$

$$R_8 = -\frac{49}{13}g\left(1 - 1.4015156g + 1.0224677g^2 - 0.7939063g^3\right), \ n = 18$$

$$R_8 = -2g\left(1 - 1.0979081g + 0.3632743g^2 - 0.1382065g^3\right), \ n = 64,$$

$$g^*_8 = \frac{8\pi^3}{81}r^4\left(1 - 0.7174211r - 0.2013970r^2 - 0.706239r^3\right), \ n = 1$$

$$g^*_8 = \frac{664\pi^3}{14641}r^4\left(1 - 0.5904844r - 0.2566839r^2 - 0.446149r^3\right), \ n = 3$$

$$g^*_8 = \frac{5\pi^3}{729}r^4\left(1 - 0.5349794r - 0.2352017r^2 - 0.141469r^3\right), \ n = 10$$

$$g^*_8 = \frac{49\pi^3}{28561}r^4\left(1 - 0.5880157r - 0.1936037r^2 - 0.051025r^3\right), \ n = 18$$

$$g^*_8 = -\frac{\pi^3}{23328}r^4\left(1 - 0.7762346r - 0.0866804r^2 + 0.014012r^3\right), \ n = 64,$$

$$R'_8 = -9r\left(1 - 1.9849108r + 1.7611357r^2 - 1.966585r^3\right), \ n = 1$$

$$R'_8 = -\frac{83}{11}r\left(1 - 1.7401630r + 1.2710234r^2 - 1.194256r^3\right), \ n = 3$$

$$R'_8 = -5r\left(1 - 1.3580247r + 0.6598114r^2 - 0.409513r^3\right), \ n = 10$$

$$R'_8 = -\frac{49}{13}r\left(1 - 1.1981406r + 0.4426329r^2 - 0.201501r^3\right), \ n = 18$$

$$R'_8 = -2r\left(1 - 1.0174897r + 0.1748660r^2 - 0.033612r^3\right), \ n = 64.$$
used for evaluating the Wilson fixed point location with an important simplification – there is no necessity in the coordinate space probing. The main working formula reads

$$f(g) \approx f_{b,\lambda}^{(N)}(g) = \int_{0}^{\infty} dt \, t^b e^{-t} \left( \frac{g t}{w(gt)} \right)^{\lambda} \sum_{k=0}^{N} W_{b,\lambda}(w(gt))^k,$$

(28)

where coefficients are defined in the same way as in (23). The choice of fitting parameters is determined by the region of greatest stability, which, in turn, is determined by the special error function suggested in (49). The probing range for $b$ and $\lambda$ is the same as in the case of the fixed point coordinate search. A couple of histograms with the distribution of the universal ratio values under the whole grid of the resummation parameters are shown in Figures 4 and 5 for 3D RG and pseudo-$\varepsilon$ expansions respectively.

Four sets of $R_8^*$ estimates obtained from four-loop RG and pseudo-$\varepsilon$ expansions within PBL and CB resummation approaches are collected in Table 3 along with the low-order RG, $\varepsilon$ expansion and lattice estimates found earlier. PBL estimates are also depicted in Figure 6 to give a general view of $R_8^*$ as function of $n$.

At the end of this section let us make a few remarks. It is worthy to note that errors of PBL estimates extracted from the corresponding pseudo-$\varepsilon$ expansions within the chosen resummation strategy have been severely underestimated, especially for small $n$. Generally speaking, the estimating of errors for quantities obtained from divergent series is a tricky problem solution of which is often based on some empirical judgments and experience. Therefore these error estimates can not be treated as fully reliable. As for the values of $R_8^*$ themselves, one can see that with growing $n$ the numerical estimates become more and more stable with respect to the resummation approach used. At the same time, for physically interesting cases $n = 0, 1, 2, 3$ the situation remains somewhat dramatic even in the four-loop approximation. Indeed, the values of $R_8^*$ given by 3D RG and pseudo-$\varepsilon$ expansions are scattered within big (15-25%) intervals and differ considerably from the three-loop RG estimates. This reflects the unfavorable structure of the RG series and pseudo-$\varepsilon$ expansion for $R_8^*$ discussed in Introduction and signals that higher-
(a) For \( n = 0 \): \( R_8 = 1.6484 \).

(b) For \( n = 12 \): \( R_8 = -0.3430 \).

Figure 5: Distributions of CB estimates resulting from pseudo-\( \epsilon \) expansion for universal ratio \( R_8^\ast \) generated by the variation of resummation parameters. The results for \( n = 0 \) and \( n = 12 \) are presented. Others – for different values of \( n \) – can be found in Supplementary materials. The vertical axis represents the frequency of hits in a particular bin. The difference in the quality of pictures is caused by peculiar behavior of \( \tau \)-series at a specific value of \( n \) and also by different sizes of sample each element of which has to meet certain requirements.

Figure 6: PBL estimates resulting from pseudo-\( \epsilon \) and 3D RG expansions of universal ratio \( R_8^\ast \) for different values of \( n \).

Order RG calculations of this quantity are certainly desirable. Such calculations, however, are expected to be rather complicated since finding of the next, five-loop contribution to the universal eight-order coupling will require evaluation of 2319 Feynman graphs.
Table 3: The values of $R_{10}^*$ for various $n$ found by resummation of the four-loop (4l) RG series and of corresponding pseudo-$\epsilon$ expansion (P$\epsilon$E). The estimates of $R_{10}^*$ resulting from the three-loop (3l) RG series in 3 dimensions [16, 25, 39], obtained within the $\epsilon$ expansion ($\epsilon$ exp) approach [16, 27, 39] and extracted from lattice calculations (LC) are also presented for comparison.

| $n$ | 4l RG | 4l P$\epsilon$E | 3l RG | $\epsilon$ exp | LC |
|-----|--------|----------------|-------|---------------|----|
|     | PBL | CB | PBL | CB | PBL [25] | [27] |
| 0   | 1.548 | 1.382 | 1.370 | 1.648 | 1.1(2) |
| 1   | 1.109 | 0.968 | 0.967 | 1.159 | 0.856 | 0.94(14) | 0.871(14) [41] |
|     |       |       |       |       | 0.857(86) [16] |       |
| 2   | 0.770 | 0.646 | 0.656 | 0.786 | 0.563 | 0.71(16) | 0.494(34) [33] |
| 3   | 0.506 | 0.397 | 0.412 | 0.503 | 0.334 | 0.33(10) | 0.21(7) [34] |
| 4   | 0.299 | 0.206 | 0.222 | 0.286 | 0.15  | 0.065(80) |              |
| 5   | 0.137 | 0.058 | 0.063 | 0.123 | -0.3(9) [39] | -0.1(2) [39] |
| 6   | 0.010 | -0.056 | -0.051 | -0.004 | -0.09 | -0.2(1) [39] |
| 7   | -0.090 | -0.144 | -0.140 | -0.101 |      |              |
| 8   | -0.169 | -0.212 | -0.213 | -0.177 | -0.25 | -0.405(31) |
| 10  | -0.280 | -0.306 | -0.313 | -0.281 |      |              |
| 16  | -0.412 | -0.412 | -0.428 | -0.400 | -0.44 | -0.528(14) |
| 24  | -0.424 | -0.413 | -0.431 | -0.408 |      |              |
| 32  | -0.394 | -0.380 | -0.396 | -0.377 | -0.42 | -0.425(7) |
| 48  | -0.324 | -0.311 | -0.324 | -0.310 | -0.35 | -0.322(2) |
| 64  | -0.270 | -0.259 | -0.269 | -0.259 | -0.29(3) [39] | -0.269(3) [39] |

5. Tenth-order coupling in the critical region

To analyze the structure of RG expansions for $R_{10}$ and $\tau$-series for $R_{10}^*$, we proceed in the same way as in the case of the eight-order coupling. For selected values of $n$ the series of interest are:

\[
R_{10} = 54g \left(1 - 4.4444444g + 11.299686g^2\right), \quad n = 1
\]
\[
R_{10} = \frac{490}{11} g \left(1 - 3.8586866g + 8.4785734g^2\right), \quad n = 3
\]
\[
R_{10} = 56g \left(1 - 5.6888105g + 17.951828g^2\right), \quad n = 10
\]
\[
R_{10} = \frac{260}{13} g \left(1 - 2.3672876031g + 2.994454005g^2\right), \quad n = 18
\]
\[
R_{10} = \frac{17}{2} g \left(1 - 1.7610546276g + 1.376700840908g^2\right), \quad n = 64,
\]
\[ g_{10}^* = \frac{32\pi^4}{243} \tau^5 \left( 1 - 2.3319616\tau + 1.84782991\tau^2 \right), \quad n = 1 \]
\[ g_{10}^* = \frac{7840\pi^4}{161051} \tau^5 \left( 1 - 1.9425556\tau + 1.1285875\tau^2 \right), \quad n = 3 \]
\[ g_{10}^* = \frac{28\pi^4}{6561} \tau^5 \left( 1 - 1.4726631\tau + 0.55331470\tau^2 \right), \quad n = 10 \]
\[ g_{10}^* = \frac{20\pi^4}{28561} \tau^5 \left( 1 - 1.3504127\tau + 0.47054348\tau^2 \right), \quad n = 18 \]
\[ g_{10}^* = \frac{17\pi^4}{3359232} \tau^5 \left( 1 - 1.3589627\tau + 0.53252324\tau^2 \right), \quad n = 64. \]

\[ R_{10}^* = 54\tau \left( 1 - 4.0219479\tau + 7.5500981\tau^2 \right), \quad n = 1 \]
\[ R_{10}^* = 490\frac{\tau}{11} \left( 1 - 3.4754604\tau + 5.5318517\tau^2 \right), \quad n = 3 \]
\[ R_{10}^* = 28\tau \left( 1 - 2.5700568\tau + 2.92620786\tau^2 \right), \quad n = 10 \]
\[ R_{10}^* = 20\tau \left( 1 - 2.1639126\tau + 2.0217915\tau^2 \right), \quad n = 18 \]
\[ R_{10}^* = \frac{17\tau}{2} \left( 1 - 1.6806362\tau + 1.0816342\tau^2 \right), \quad n = 64. \]

These series possess big coefficients even for \( n \gg 1 \), to say nothing about those for physical values \( n = 1, 2, 3 \). It is not surprising, therefore, that application of all the resummation techniques having used above leads to quite chaotic estimates for \( R_{10}^* \) covering huge (several dozens) intervals. Of course, the series found are too short to provide accurate estimates but their small length is not the only reason of such a failure. Rather strong divergence of 3D RG expansion for universal tenth-order coupling seems to play more important role. Considerable scattering of numerical estimates for \( R_{10}^* \) obtained in the five-loop RG approximation for 3D Ising model \( (n = 1) \) \(^{48} \) confirms this conclusion. This implies that calculation of higher-order terms in corresponding RG expansion may turn out to be insufficient for getting proper numerical results. Moreover, such a calculation would be really complicated since it includes evaluation of thousands of graphs. On the other hand, working with lengthy enough RG series may help to overcome the problem of its dramatic divergence and result in fair numerical estimates.

### 6. Conclusion

To summarize, for 3D \( O(n) \)-symmetric \( \lambda \phi^4 \) field theory we have calculated the four-loop RG contribution to the universal ratio \( R_8 \) and three-loop RG series for \( R_{10} \). This has required to evaluate in total 339 Feynman diagrams. We have also found corresponding four-loop and three-loop pseudo-\( \epsilon \) expansions. The series for \( R_8 \) have been resummed by means Pade–Borel–Leroy and conform–Borel resummation techniques and calculated additive has been found to appreciably shift the numerical estimates for this coupling under the physical values of \( n \). On the other hand,

\(^{48} \)For example, in order to obtain four-loop contribution to \( R_{10} \) one should calculate 684 Feynman diagrams.
for these $n$ the numerical estimates for $R_8$ have turned out to notably depend on the resummation procedure signaling that higher-order RG calculations of this quantity remain desirable. Three-loop RG and pseudo-$\epsilon$ expansions for $R_{10}$ have been found to strongly diverge both for physical $n$ and $n \gg 1$: they possess big coefficients $C_k$ that grow or, at least, do not regularly diminish with growing $k$. This prevents extracting numerical values of this universal ratio from the series found. Perhaps, the calculation of rather lengthy RG expansions for $R_{10}$ would soften this problem and pave the way to getting proper numerical estimates.

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Appendix A. Supplementary materials

In Supplementary materials we present one-particle irreducible Feynman graphs forming RG expansions for $g_8$ and $g_{10}$ couplings (rg_expansions_couplings_g8_g10.pdf). It includes integrals, symmetry and tensor factors. The Nickel’s notations describing graphs topology are used there.
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