We extend a new finite element code, Einstein PHG (iPHG), to solve the evolution part of Einstein equations in first-order GH formalism. This paper is the third one of a systematic investigation of applying adaptive finite element method to the Einstein equations, especially binary compact objects simulations. The main motivation of this work is to evolve black holes for the first time by means of a continuous Galerkin finite element method on unstructured (tetrahedral) mesh. We test our code by evolving a nonlinear scalar wave equation. It works well and runs stably with both reflect and radiative boundary conditions. Then we use iPHG to simulate the full three-dimensional spacetime of a single black hole. We find that the filter used to dealt with aliasing error is a crucial ingredient for numerical stability. For simplicity, we impose the “freezing” ingoing characteristic fields condition in weak form at the outer boundary. Our simulations show both the convergence and stability.

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I. INTRODUCTION

There exist three different kinds of numerical methods in the world: finite difference method, spectral method and finite element method. In the numerical relativity community, the first two methods are much more popular than the last one. In [1–3], finite difference method has been used to simulate coalescing of compact binaries. Spectral method has also been successfully used in [4–9] to study gravitational collapse and binary black hole dynamics. However, a full three-dimensional binary black hole simulation using finite element method is still missing (but see [10,11]).

Even though we are now capable of studying all kinds of physical phenomena related to strong gravitational field and highly dynamical spacetime using numerical relativity [18,19], there are still some challenges. For example, the gravitational waves calculated using existing finite difference or spectral codes are already accurate enough to make a detection in the network of laser interferometric detectors, such as LIGO and VIRGO. But the highest mass ratio of a binary black hole system, which possibly makes finite element method admit higher parallel scalability, has to be transferred between different mesh levels, all elements in finite element method are treated uniformly, (see [10,11]).

The reason why finite difference or spectral codes are not capable of simulate intermediate mass ratio inspirals (IMRIs) is their pool parallel scalability. Huge mass ratio means huge scale difference and requires huge scale parallel computing. So strong parallel scalability is essential for evolving the kind of binary systems. For finite difference codes, especially for those who use moving-box mesh refinement technique, it is their hierarchical structure that limits the parallel scalability. Furthermore, the size of the buffer zone due to the structure of finite difference’s stencil also sets an up limit for its ability of parallel scaling. For (multi-domain) spectral codes, in principle they can have strong parallel scalability. But it requires a lot of fine tuning and complicated grid structures, which make IMRIs simulations using spectral method quit challenging.

For finite element method, its discretization admits a local property similar to finite difference case, so its robustness is possibly as good as finite difference method. While in each element, high order polynomial function basis and/or spectral function basis can be used to achieve high accuracy, just as spectral method. Furthermore, in contrast to finite difference method where data has to be transferred between different mesh levels, all elements in finite element method are treated uniformly, which possibly makes finite element method admit higher strong parallel scalability.

The application of finite element method to general relativity is just in beginning and still needs a lot of exploration. Discontinuous Galerkin finite element method has been implemented in [20] to deal with general relativistic hydrodynamics and in [17] to evolve a Kerr black hole and a neutron star. Scott et al [13,14] used local discontinuous Galerkin finite element method to solve spherically reduced Baumgarte-Shapiro-Shibata-Nakamura (BSSN) system with first-order and second-order operators. In [21], we also used local discontinuous Galerkin finite element method to solve spherically reduced first order general harmonic (GH) system.

Previously we have developed a new finite element code, iPHG [22], for solving the constraint part of the
Einstein equations in full three dimensions. In the current work, we extend our code such that it can solve the evolution part of the Einstein equations. Our code is based on one recently developed adaptive finite element library—Parallel Hierarchical Grid (PHG) [22, 23]. PHG is a toolbox for writing scalable parallel adaptive finite element programs and provide functions which perform common and difficult tasks in parallel adaptive finite element programs, such as management of unstructured parallel (distributed) meshes, parallel adaptive mesh refinement and coarsening, dynamic load balancing via mesh repartitioning and redistribution and so on.

Throughout this work the geometry units with $G = c = 1$ are used. The rest of the paper is arranged as follows. In the next section, we will review the GH formulation treating them with the finite element method. Then in Sec. III, we present the numerical results. Firstly, we show results for some simple tests where we evolve nonlinear scalar waves. Following that, the results for evolving, in 3D, a Schwarzschild black hole are presented. We conclude in Sec. IV.

II. NUMERICAL ALGORITHM

In the case of first-order reduction of GH formulation [7], the state vector is denoted as $u^\alpha = \{g_{ab}, \Pi_{ab}, \phi_{ab}\}$, where $g_{ab}$ is the spacetime metric, $\Pi_{ab} = -\nabla^c g_{acb}$ and $\phi_{ab} = \partial_t g_{ab}$ are two auxiliary variables. Then the dynamical equations which are reduced from the Einstein equations can be written as

$$\partial_t u^\alpha + A^{\alpha\beta} \partial^\beta u^\beta = S^\alpha,$$  \hspace{1cm} (1)

with

$$A^{\alpha\beta} = \begin{pmatrix} -(1 + \gamma_1)\beta^k & 0 & 0 \\ -\gamma_1\gamma_2\beta^k & -\beta^k & \alpha\gamma^k \\ -\gamma_2\alpha^k & \alpha\beta^k & -\beta^k \end{pmatrix},$$  \hspace{1cm} (2)

$$S^\alpha = \begin{pmatrix} -\alpha \Pi_{ab} - \gamma_1\beta^i \phi_{ab} \\ \alpha \left[ \frac{1}{2} x^d \phi_{ed} \Pi_{ab} + \gamma_2 t^c \phi_{ajc} \phi_{abcd} - \gamma_2 \phi_{ab} \right] \end{pmatrix},$$  \hspace{1cm} (3)

and

$$S^{(\Pi)}_{ab} = 2\alpha g^{cd} (\gamma_1 \phi_{ed} \phi_{jdb} - \Pi_{ed} \Pi_{db} - g^{ef} \Gamma_{abc} \Gamma_{bdef}) - 2\alpha \nabla_a (H_b) - \frac{1}{2} \alpha t^c \Pi_{ed} \Pi_{ab} - \alpha t^c \gamma_2 \Pi_{ed} \phi_{ab} + \alpha \gamma_0 \left[ 2\beta^c t^b - g_{ab} t^c \right] C_c - \gamma_1 \gamma_2 \beta^i \phi_{ijab},$$  \hspace{1cm} (4)

where $H_a$ is the source function for generalized harmonic formalism, $t^a$ is the unit vector normal to the spatial slices of constant coordinate time $t$, $\Gamma_a = g^{bc} \Gamma_{abc}$ are the contracted Christoffel symbol and $C_c = \Gamma_c + H_c$ is the constraint that ensures the coordinates satisfy the GH coordinate condition. The lapse $\alpha$, shift $\beta^i$ and spatial metric $\gamma_{ij}$ are defined by

$$ds^2 = g_{ab} dx^a dx^b = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt).$$  \hspace{1cm} (5)

The terms multiplied with $\gamma_{0,1,2}$ are the additional constraint terms beyond original Einstein equations. In the simulations, we set $\gamma_0 = \gamma_2 = 1$, $\gamma_1 = -1$, as in [7]. Throughout this paper, we use the Latin $a, b, c, ...$ for four-dimensional indices, while $i, j, k, ...$ for spatial indices.

In the following we will use the continuous Galerkin (CG) finite element method to solve Eq. (4). We multiply it by a test function $v$ and integrate over the whole computational domain $\Omega$. Using integration by parts, we get the following weak form of the original equation, (here we omit the state index for simplicity;)

$$\int_{\Omega} \partial_t u^i v_i dx = - \int_{\Omega} \partial_a A_{ab} v_i dx - \int_{\Omega} A_{ab} u_i \partial_t v_i dx + \int_{\partial \Omega} A_{ab} u_i \gamma_{ik} \frac{1}{\sqrt{\gamma}} d^2 \Sigma + \int_{\Omega} S_i v_i dx,$$  \hspace{1cm} (6)

where $\gamma_{ik}$ is the outward directed unit normal to the boundary of $\Omega$, $d^2 \Sigma$ is the invariant surface element [25].

Denoting the basis functions of the finite element $\phi_i$, we can expand the state vector $u$ as $u = \sum_i u_i \phi_i$. The set of test functions $v$ is also chosen to be the same with the set of basis functions (Galerkin method). Then the above weak form equations can be written as

$$M_{ij} \partial_t u_i = - \int_{\Omega} \partial_a A_{ab} \phi_j d^3 x - \int_{\Omega} A_{ab} u_i \partial_t \phi_j d^3 x + \int_{\partial \Omega} A_{ab} u_i \gamma_{ik} \frac{1}{\sqrt{\gamma}} d^2 \Sigma + \int_{\Omega} S_i \phi_j d^3 x,$$  \hspace{1cm} (7)

where the mass matrix $M_{ij}$ is defined as

$$M_{ij} = \int_{\Omega} \phi_i \phi_j d^3 x.$$  \hspace{1cm} (8)

We solve Eq. (7) to get $\partial_t u_i$, and use total variational diminishing (TVD) third order Runge-Kutta method [26] to update $u_i$ in time. Following [25], we will work with a nodal expansion (in this work, we use Lagrange interpolating polynomials as our basis and test functions), which is an interpolation for some choice of grid points $x_r$. For simplicity, we choose the grid nodes $x_r$ to be the quadrature nodes and evaluate the integrals with quadrature rules on triangles and tetrahedra from [24]. For a nonlinear term like $A^k u$, it is expanded as

$$A^k u = \sum_j A^k_j u_j \phi_j,$$  \hspace{1cm} (9)

where $A^k_j$ and $u_j$ are the values at the grid nodes of function $A^k$ and $u$. This expression is not exact and will introduce aliasing error, which can make our simulation unstable. To get rid of aliasing error, filtering is required. We
can construct Legendre polynomials and Vandermonde matrix for each tetrahedral element using the method introduced by [27] and filter the higher modes in the solution’s modal representation. However, the filtered solution will be discontinuous at the boundaries between each two elements. This discontinuity might not be a problem for discontinuous Galerkin (DG) finite element method, but it will invalidate our continuous Galerkin (CG) method. Instead, we use the filter developed by Fischer and Mullen [28, 29].

\[ F_\alpha = \alpha F_{N-1} + (1 - \alpha) \text{Id}, \]  

(10)

where \( F_{N-1} \) is the interpolation operator from the space of the polynomials of maximum degree \( N \) to the space of the polynomials of maximum degree \( N - 1 \), \( \text{Id} \) is the identity operator, and \( \alpha \in (0, 1] \) is the relaxation parameter which allows us to filter only a fraction of the highest mode. Since this filter is based on interpolations in physical space, the filtered solution will still be continuous at the boundaries between each two elements.

### III. NUMERICAL RESULTS

#### A. Nonlinear scalar waves

For code test, we investigate a nonlinear wave equation first. The nonlinear term that we choose is the same with the one in [30]. The evolution equations in Cartesian coordinate can be written as

\[ \partial_t \psi = \Pi \]  

(11)

\[ \partial_t \Pi = \partial_t \Phi_k - \lambda \frac{\psi^3}{1 + \psi^2} \]  

(12)

\[ \partial_t \Phi_k = \partial_t \Pi \]  

(13)

where \( \lambda \) parametrizes the nonlinearity. The characteristic fields for the system, which are associated with the outward directed unit normal \( n_i \) to the boundary, are

\[ u^\hat{0} = \psi, \quad \text{speed 0}, \]  

(14)

\[ u^\pm = \Pi \mp n_k \Phi_k, \quad \text{speed } \pm 1, \]  

(15)

\[ u^2 = \Phi_k - n_i n_k \Phi_k, \quad \text{speed 0}. \]  

(16)

It is well known that this system is well-posed if boundary condition is of the following form [30, 31]

\[ u^{1^-} = \kappa u^{1^+} + f, \]  

(17)

where \( |\kappa| \leq 1 \) and \( f \) is a given function. We set \( f = 0 \) for convenience and consider two different cases: \( \kappa = -1 \) and \( \kappa = 0 \), which correspond to reflect and radiative boundary conditions, respectively.

The initial data is chosen to be a Gaussian wave package,

\[ \psi = \frac{c}{r} \exp \left\{ - \frac{\tan^2 \left( \frac{\pi (r - R_{\text{out}})}{R} \right) - \frac{\pi}{2}}{\left( \sigma/R \right)^2} \right\}, \]  

(18)

\[ \Pi = 0, \]  

(19)

where \( R = R_{\text{out}} - R_{\text{in}}, \ R_{\text{out}} \) and \( R_{\text{in}} \) are the outer and inner boundaries of our computational domain, \( c \) and \( \sigma \) are two parameters which characterize the amplitude and width of the wave package correspondingly. In the simulations, we set \( c = 0.5, \sigma = 4, R_{\text{in}} = 1.8, R_{\text{out}} = 11.8 \).

We use NetGen to generate a hollow spherical polyhedron shell grid with radius \( r \in [R_{\text{in}}, R_{\text{out}}] \), which is made up of 1235 vertices and 6444 simplest tetrahedral elements, see Fig. 1. Then we uniformly refine the grid \( N \) times. After that we refine the boundary elements three times. At both the inner and outer boundaries, we set the boundary conditions according to Eq. (17). For the radiative boundary condition or \( f = \kappa = 0 \) case, we are basically “freezing” the ingoing mode \( u^{1^-} \) to its initial value, which is zero.

Fig. 2 shows the time evolution of scalar field energy \( E_\psi = \int_{\Omega} \rho d^3x \) with the energy density

\[ \rho = \frac{\Pi^2}{2} + \frac{\Phi^2}{2} + \frac{\lambda}{2} \left( \psi^2 - \log(1 + \psi^2) \right). \]  

(20)

Three cases with different boundary conditions and nonlinearity are presented. As we can see, the energy is conserved for the \( \lambda = 0 \) case with reflect boundary condition. While for the \( \lambda = 0 \) case with radiative boundary condition, the energy drops when the scalar field leaves the grid. The qualitative behavior of the \( \lambda = 100 \) case with radiative boundary condition is the same with the corresponding \( \lambda = 0 \) case, which means that the radiative boundary condition can also propagate nonlinear waves off the grid.
FIG. 2: The scalar field energy of three different cases. No energy can escape the computation domain if we implement a reflect boundary condition (dotted line). If we use the radiative boundary condition, the scalar leaves the domain and the energy drops. The linear ($\lambda = 0$, broken line) and nonlinear ($\lambda = 100$, solid line) cases have the same qualitative behavior. We have used first order polynomials in all three cases.

There are two kinds of refinements that we can do to improve the accuracy of solutions: $h$-refinement (by further splitting each element) and $p$-refinement (by increasing the order of polynomial basis). We have tried both methods and obtained convergence under both refinements. In Fig. 3 (a) we show the convergence under $h$-refinement. The error in energy is defined as

$$\delta_{N,h}(E^{(h)}_\psi) = |E^{(h)}_{\psi,N} - E^{(h)}_{\psi,8}|,$$  \hspace{1cm} (21)

where $E^{(h)}_{\psi,N}$ is the scalar field energy measured at $t = 40$ for the nonlinear wave equation with radiative boundary condition, using grid that was uniformly refined $N$ times from the initial grid. We use $E^{(h)}_{\psi,8}$ as the reference energy. The basis functions used here are first order Lagrange polynomials (P1). As we can see, the errors decrease as we split each element.

Fig. 3(b) illustrates the convergence under $p$-refinement, as we increase the order of polynomial basis. Similarly, the error in energy is defined as

$$\delta_{k,p}(E^{(p)}_\psi) = |E^{(p)}_{\psi,k} - E^{(p)}_{\psi,5}|,$$  \hspace{1cm} (22)

where $E^{(p)}_{\psi,k}$ represents scalar field energy obtained at $t = 40$, using $k$-th order Lagrange polynomials as the basis functions. The reference energy is $E^{(p)}_{\psi,5}$. The grid used in this subplot is obtained from the initial grid after three times uniformly refinement. As expected, the errors decrease exponentially with the order $k$.

FIG. 3: The error in energy at $t = 40$ for the nonlinear wave equation with radiative boundary condition. It decreases in both cases as we uniformly refine the grid more times (upper panel) or use higher order polynomials (lower panel).

B. Schwarzschild black hole in Kerr-Schild coordinate

Now we turn to evolving the spacetime of a single black hole. As initial data, we use the metric of a Schwarzschild black hole in Kerr-Schild coordinates \textsuperscript{32},

$$g_{ab} = \eta_{ab} + \frac{2M}{r}l_\alpha l_\beta,$$  \hspace{1cm} (23)

where $\eta_{ab}$ is the Minkowski metric, $M$ is the mass of the black hole. In Cartesian coordinates, $r = (x^2 + y^2 + z^2)^{1/2}$ and $l_\alpha = (1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r})$. We use the units where $M = 1$.

We numerically evolve Eq. (1) (or Eq. (7)) using continuous Galerkin finite element method that we described in Sec. [1]. The gauge source function is initialized based on the metric (23), which is left constant during the simulation \textsuperscript{33},

$$H_a(t = 0) = -\Gamma_a(t = 0), \quad \partial_t H_a = 0.$$  \hspace{1cm} (24)

The characteristic fields for the first-order GH system, Eq. (1), are given by (c.f., Eq. (32-34) of \textsuperscript{7})

$$u^0_{ab} = g_{ab}, \quad \text{speed} - (1 + \gamma_1)n_k \beta^k,$$  \hspace{1cm} (25)

$$u^\pm_{1ab} = \Pi_{ab} \pm n^i \Phi_{iab} - \gamma_2 g_{ab}, \quad \text{speed} - n_k \beta^k \pm \alpha,$$  \hspace{1cm} (26)

$$u^2_{iab} = P_i^k \Phi_{kab}, \quad \text{speed} - n_k \beta^k.$$  \hspace{1cm} (27)
which is associated with the outward directed unit normal $n_i$ to the boundary. The projection operator is defined as $P_i^k = \delta_i^k - n_i n^k$. Our computational domain consists of a hollow spherical polyhedron shell that extends from $r_{\text{min}} = 1.8$ to $r_{\text{max}} = 11.8$, see Fig. 1. The inner boundary is lightly inside the horizon which is located at $r_{\text{EH}} = 2$. At inner boundary, all the characteristic modes are outgoing (relative to the computational domain), so no boundary condition needs to be imposed. At the outer boundary, we “freeze” the values of incoming characteristic fields to their initial values [7, 33].

Again, we generate the initial mesh with the simplest tetrahedral element decomposition with 1235 vertices and 6444 elements using NetGen. Then we use the dimensionless constraint $L_2$ norm over each element

$$\|C_i\|/\|\partial\mathbf{U}_i\| = \sqrt{\int_{i\text{-th element}} C_i^2 \sqrt{g} d^3x \int_{j\text{-th element}} (\partial\mathbf{U}_j)^2 \sqrt{g} d^3x}.$$  \hfill (28)

as the refinement indicator and let PHG do the adaptive refinement, until some preset threshold for the dimensionless constraint $L_2$ norm over the whole computational domain $\|C_0\|/\|\partial\mathbf{U}_0\|$ is reached. Here $C$ is a measure of the constraint violations and $\partial\mathbf{U}$ is a measure of the first order derivatives (c.f., Eq. (A.2) and Eq. (A.3) of [34]).

1. Filtering

As we have mentioned in Sec. [1] we used a filter [10] to control the aliasing error. To understand this error, let’s consider, for example, the integral of two functions, $\int_{\Omega} f(x)g(x)dx$, which are both defined on the domain $\Omega$. Suppose that the quadrature rule we use is $k$-th order, then the polynomials used to expand $f$ and $g$ are also $k$-th order, since we have chosen the grid nodes to be the quadrature nodes for convenience. However, when the functions are expanded using $k$-th order basis, the proper quadrature rule should be $2k$-th order. If we still integral the product of these two functions using $k$-th order quadrature rule, it will not be exact. Those modes with order higher than $k$-th will not be resolved and be ‘alised’ into lower order modes. To ameliorate this error, we can of course prepare another $2k$-th order quadrature rule. But the algorithm will lose the convenience and be very expensive for any realistic, long-term simulations.

Instead, we address the aliasing error by filtering a fraction of the highest modes in physical space using Eq. (10), where $\alpha$ controls the strength of the filter’s effect. The filter is applied after every full time step. And it turns out to be a crucial ingredient for numerical stability. In Fig. 3, we plot the dimensionless constraint violations of two simulations with and without filtering. If the solution is not filtered after each time step, we see that the constraint violation grows up after a three stages evolution: after a initial increase which is stage one, it settles down for a little while in its second stage, and finally it starts to grow exponentially without bound at $t \simeq 20$. However, with filtering, the dimensionless constraint violation becomes flat after an initial increase, which indicates that the system become stable after filtering a fraction of the highest modes in physical space. Here we used 4-th order Lagrange polynomials (P4) as basis functions.

Boundary condition is also a very important ingredient for numerical stability. There exists a number of sophisticated and complicated outer boundary condition for the GH system, see [7, 33, 34]. However, since our focus here is on exploring the finite element as a mean of solving the Einstein equations on unstructured (tetrahedral) grids, we ignore these boundary conditions and use the simplest condition that is successful for the single black hole test case: “freezing” the incoming characteristic fields [7, 33].

$$\partial_t u^a |_{\text{boundary}} = 0 \quad \text{for} \quad v_{(\alpha)} < 0,$$  \hfill (29)

where $v_{(\alpha)}$ is the characteristic speed.

We can transform Eq. (29) back to condition in terms of primitive valuables and impose them on each boundary grid node,

$$\partial_t u_i = B_i,$$  \hfill (30)

where $B_i$ represent the boundary condition evaluated at the boundary node $x_i$. At the outer boundary, the state vector $u^a$ is integrated in time using Eq. (29), instead of using Eq. (7). Unfortunately, the runs which impose the outer boundary condition in this way are not stable. The reason we suspect for the instability is the inconsistency between the weak form evolution in the bulk [7] and the strong form evolution at the outer boundary [30].
Therefore, we modify the form of boundary condition (29) by integrating it in time and obtain

\[ u^\alpha |_{\text{boundary}} = C^\alpha \text{ for } v(\alpha) < 0, \tag{31} \]

where \( C^\alpha \) is constant in time and determined by initial data. We again transform Eq. (31) back to condition in terms of primitive valuables

\[ u_i = B_i^{(C)}, \tag{32} \]

where \( B_i^{(C)} \) is the boundary condition in terms of state vector. Then the “freezing” incoming characteristic fields boundary condition can be imposed in weak form as follows: at each time step, we replace the primitive valuables \( u^\alpha \) present in the surface integral terms in Eq. (7) with \( B_i^{(C)} \). This weak form boundary condition works well and removes the instability present in the strong form boundary condition cases.

In Fig. 5 we show two simulations with their outer boundary conditions imposed using strong and weak forms. They share the same behavior before \( t \simeq 20 \); after an initial increase, the dimensionless constraint violations become flat for a while. Then the strong form case diverges from the weak form case and increase exponentially until the run fails.

Our code is still stable if we “freeze” all the modes on the outer boundary, or in other word, fix the boundary value to the analytic solution, just as we found in [21]. We plot in Fig. 6 the result of two cases where we “freeze” all the modes and “freeze” only the ingoing modes. As we can see, the behavior of the dimensionless constraint violations are almost the same for these two cases, except that the “freezing” all modes case settles down to a little bit larger value of constraint violation.

3. Convergence

In Fig. 7 we show the convergence and stability of the Schwarzschild black hole evolution. The simulations are carried out using 4-th order Lagrange polynomials (P4) as basis functions. The top panel displays the dimensionless constraint violations over the whole computational domain, \( ||C||/||\partial U|| \), with different resolutions. The grids used in these three cases are generated with different preset threshold \( ||C_0||/||\partial U_0|| \), which labels the resolutions. As we can see, they share the same qualitative behavior: after an initial increase, \( ||C||/||\partial U|| \) settles down to a constant. In particular, \( ||C||/||\partial U|| \) settles down to 4.0 × 10\(^{-3}\), 6.8 × 10\(^{-3}\) and 1.1 × 10\(^{-2}\), corresponding to the preset threshold \( ||C_0||/||\partial U_0|| \) equals 1.0 × 10\(^{-3}\), 2.0 × 10\(^{-3}\) and 4.0 × 10\(^{-3}\). The dimensionless constraint violations decrease as we increase the resolution.

The last case where \( ||C_0||/||\partial U_0|| = 4.0 \times 10^{-3} \) has been evolved to \( t = 1000 \) in the bottom panel of Fig. 7 and we see no sign of instability. We conclude that our code is convergent and stable up to at least \( t = 1000 \), and, we presume, forever.

IV. SUMMARY

A new finite element code, Einstein PHG (iPHG), has been extended to solve the evolution part of Einstein equations in first-order GH formalism. It has two main features: first, thanks to PHG, we expect iPHG to have good parallel scalability after some modulation, which is very crucial for the simulations of IMRIs; second, it is equipped with unstructured mesh and can do parallel adaptive mesh refinement and coarsening, which we believe will benefit a lot when we apply it to simulation...
FIG. 7: Evolution of dimensionless constraint violations for Schwarzschild initial data. In the upper panel, we show the evolutions using three different numerical resolutions. The lower panel shows the long timescale evolution for $||C_0||/||\partial U_0|| = 4 \times 10^{-3}$. 

the binary neutron star coalescence or black hole-neutron star coalescence.

In this paper, we focused on exploring the finite element as a mean of solving the Einstein equations’ evolution part on unstructured (tetrahedral) mesh and applied iPHG to evolve the spacetime of a single black hole. Before going to the Einstein equations, we tested our code by solving nonlinear wave equations. Our code worked well with both reflect and radiative boundary conditions and exhibited both convergence with h-refinement and p-refinement. For the single black hole case, we found that filter and boundary conditions were both crucial ingredients for numerical stability. We armed iPHG with the filter (10) developed by Fischer and Mullen. For simplicity, the “freezing” incoming characteristic fields condition was impose in weak form at the outer boundary. We showed that the algorithm is convergent and stable for long-timescale spacetime evolution.

In future work we intend to combine our algorithm with some discontinuity capturing schemes [37–41] to suppress oscillations that may occur near shocks, such that it can handle coupled Einstein equations with hydrodynamics equations. We would also like to explore the discontinuous Galerkin method as a mean of solving Einstein and hydrodynamics equations, just as in [17], but using unstructured grid.

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