Heat flow within convex sets

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Abstract

Solutions to the heat equation between Riemannian manifolds, where the domain is compact and possibly has boundary, will not leave a compact and locally convex set before the image of the boundary does.

1 Acknowledgements

These results are sketched in my doctoral dissertation [D]. They were proved while I was a graduate student at Rutgers University–New Brunswick, during which time I benefitted greatly from the guidance of my dissertation advisor, Xiaochun Rong, as well as from conversations with Penny Smith and Armin Schikorra. Part of this work was completed while I was visiting Capital Normal University in Beijing.

2 Introduction

In their foundational paper [ES], Eells–Sampson invented the harmonic map heat flow for maps between Riemannian manifolds. If $M$ is a compact Riemannian manifold without boundary, $N$ a Riemannian manifold, and $u_0 : M \to N$ a $C^1$ map, this flow is a solution $u : M \times [0, \varepsilon) \to N$ to their heat equation

$$\frac{\partial u}{\partial t} = \tau_u$$

$$u(\cdot, 0) = u_0$$

(1)

Here, $\tau_u$ denotes the tension field of $u$, which is the trace of the second fundamental form of its positive time-slices. Eells–Sampson proved short-term existence and uniqueness of solutions to (1) for any $C^1$ initial data. The case where $\partial M \neq \emptyset$ was handled by Hamilton [H], who proved short-term existence and uniqueness of solutions to the corresponding Dirichlet problem. Hamilton also proved that solutions to (1) will not leave a compact and locally convex subset of $N$ with codimension zero and smooth boundary before the image of $\partial M$ does. The goal of this paper is to generalize that result to arbitrary compact and locally convex sets.

**Theorem 1.** Let $M$ and $N$ be Riemannian manifolds, where $M$ is compact and possibly has boundary $\partial M \neq \emptyset$. Let $Y \subseteq N$ be a compact and locally convex set. Suppose $u : M \times [a, b] \to N$ is a continuous function that, in the interior of $M \times [a, b]$, is smooth and satisfies $\frac{\partial u}{\partial t} = \tau_u$. If $u(M \times \{a\}) \subseteq Y$ and, in the case that $\partial M \neq \emptyset$, $u(\partial M \times [a, b]) \subseteq Y$, then $u(M \times [a, b]) \subseteq Y$.

The proof combines ideas of Hamilton [H] and L. Christopher Evans [E]. Hamilton’s idea was to apply a maximum principle to the composition of the flow with the signed distance to $\partial Y$. Evans proved a similar result for solutions to certain reaction-diffusion systems that map into convex subsets of $\mathbb{R}^n$. His argument
resembled Hamilton's, with the additional insight that, when $\partial Y$ is not smooth, that composition is still a viscosity solution to the differential inequality needed to apply the corresponding maximum principle. Combining these arguments allows one to remove the assumption that $\partial Y$ is smooth. To handle the case where $Y$ has arbitrary codimension, one need only change the function that's composed with the flow; rather than work with the signed distance to $\partial Y$, one may work with the distance to $Y$ itself.

3 Convexity

A subset $Y \subseteq N$ is strongly convex if, given any $p,q \in Y$, there exists a unique minimal geodesic $\gamma : [0,1] \to N$ such that $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma([0,1]) \subseteq Y$. The convexity radius of $N$ will be denoted $r : N \to (0,\infty]$. This is the continuous function characterized by the fact that, for each $y \in N$, $r(y) = \max\{\varepsilon | B(x,\delta) \text{ is strongly convex for all } 0 < \delta < \varepsilon\}$. A subset $Y \subseteq N$ is locally convex if, for each $y \in Y$, there exists $0 < \varepsilon(y) < r(y)$ such that $Y \cap B(y,\varepsilon(y))$ is strongly convex. The following theorem about the structure of locally convex sets was proved by Ozols [O] and, independently, Cheeger–Gromoll [CG].

Theorem 2. (Ozols, Cheeger–Gromoll) Let $N$ be a Riemannian manifold. If $Y \subseteq N$ is a closed and locally convex set, then $Y$ is an embedded submanifold of $N$ with smooth and totally geodesic interior and possibly non-smooth boundary.

It's also shown in [CG] that, at each $p \in Y$, $Y$ has a unique tangent cone $C_p = \{t \cdot \exp_p^{-1}(y) | y \in Y \cap B(p, r(p)), t \geq 0\}$. It follows from Theorem 2 that, when $p$ lies in the interior of $Y$, $C_p = T_p Y$.

The following theorem about metric projection onto locally convex sets was proved by Walter [W].

Theorem 3. (Walter) Let $N$ be a Riemannian manifold. If $Y \subseteq N$ is a closed and locally convex set, then there exists an open set $U \subseteq N$ containing $Y$ such that the following hold:

(i) For each $y \in U$, there exists a unique $\pi(y) \in Y$ such that $d(y, \pi(y)) = d(y, Y)$;

(ii) The geodesic $t \mapsto \exp_y \left( t \cdot \exp_y^{-1}(\pi(y)) \right)$, defined from $[0,1]$ into $N$, remains in $U$ and is the unique minimal geodesic connecting $y$ to $\pi(y)$;

(iii) The map $y \mapsto \pi(y)$ is locally Lipschitz and, therefore, differentiable almost everywhere;

(iv) The map $h : U \times [0,1] \to U$ defined by $h(y,t) = \exp_y \left( t \cdot \exp_y^{-1}(\pi(y)) \right)$ is a locally Lipschitz strong deformation retraction of $U$ onto $Y$.

If $Y$ in the above is compact, one may take $U = B(Y, \varepsilon)$ for some $\varepsilon > 0$.

4 Proof of the main theorem

Fix everything as in the statement of Theorem 1. By Theorem 2, there exists $\varepsilon > 0$ such that the projection $\pi : B(Y, \varepsilon) \to Y$ is well-defined and continuous. Because $B(Y, 2\varepsilon)$ is compact, standard curvature comparison arguments imply the existence of a lower bound $0 < R < r(B(Y, \varepsilon))$ for the focal radius $r_f(B(Y, \varepsilon)) = \inf_{y \in B(Y, \varepsilon)} r_f(y)$, where by definition

$$r_f(p) = \min\{T > 0 | \exists \text{ a non-trivial normal Jacobi field } J \text{ along a unit-speed geodesic } \gamma$$

$$\text{ with } \gamma(0) = p, J(0) = 0, \text{ and } ||J'||(T) = 0\}.$$
dim(N). Let \( v \in H^1 \) have unit length, so that \( H^1 = \{ tv \mid t \in \mathbb{R} \} \). For each \( 0 \leq t < R \), the exponential map restricted to the normal bundle of the embedded submanifold \( \exp_q(H \cap B(0,R)) \) is a local diffeomorphism around the vector \(-tv\). It follows that there exists \( 0 < \delta_H < R \) such that, for \( S_H = \exp_q(H \cap B(0,\delta_H)) \) and \( P_H = \{ w \in S_H^2 \mid \|w\| < \epsilon \} \), the map \( \exp|_{P_H} \) is a diffeomorphism onto its image. Similarly, there exists an open set \( U_v \) containing \( \exp(-tv) \) such that, for each \( z \in U_v \), the minimal geodesic \( \gamma_z \) connecting \( z \) to \( q \) remains inside \( \exp(P_H) \). Without loss of generality, one may take \( U_v \) to be small enough that \( U_v \cap S_H = \emptyset \).

Let \( SY = \{ w_y \in TN \mid y \in Y, \| y \| = 1 \} \). Denote by \( \mathcal{S}_{G(n-1)}(Y(e)) \) the space of bilinear forms on hyperplanes in \( G(n-1, \mathbb{B}(Y,e)) \). Define a function \( \mu : \mathcal{S}_{G(n-1)}(Y(e)) \times [0, \epsilon) \to \mathcal{S}_{G(n-1)}(Y(e)) \) by setting \( \mu(w_y, t) \) equal to the second fundamental form of the level set of \( d_y \) through \( \exp_y(-tv) \); equivalently, \( \mu(w_y, t) \) is the Hessian of \( d_y \) at \( \exp_y(tv) \). With respect to the usual smooth structure on \( \mathcal{S}_{G(n-1,N)} \) inherited from its structure as a vector bundle over \( G(n-1,N) \), the map \( \mu \) is smooth. Let \( \mu : \mathcal{S}_{G(n-1)}(Y(e)) \times [0, \epsilon) \to \mathbb{R} \) be the function that takes \((y, t)\) to the minimum eigenvalue of \( \mu(y, t) \).

**Lemma 4.** The function \( \mu \) is Lipschitz continuous.

*Proof.* Let \( V \) be a open subset of \( N \) that’s small enough that its closure is compact and admits an orthonormal frame \( \{ e_1, \ldots, e_{n-1} \} \). For each \( A \in \mathcal{S}_{G(n-1)}(Y(e)) \) and \( 1 \leq i, j \leq n-1 \), let \( \varsigma_{ij}(A) = A(e_i, e_j) \). Write \( A_c = [\varsigma_{ij}(A)]_{1 \leq i, j \leq n-1} \). Then the eigenvalues of \( A_c \) are equal to the eigenvalues of \( A_c \). In particular, for the minimum eigenvalue function \( \nu \), one has that \( \nu(A) = \nu(A_c) \) on \( G(n-1, V) \). Following Hamilton, one computes

\[
\| \nu(A) - \nu(B) \| = \| \nu(A_c) - \nu(B_c) \| \leq \| A_c - B_c \| \leq C \sum_{i,j=1}^{n-1} |\varsigma_{ij}(A) - \varsigma_{ij}(B)|,
\]

where \( \| \cdot \| \) denotes the usual matrix norm and the constant \( C \) exists because all norms on a finite-dimensional space are equivalent. Since the \( \varsigma_{ij} \) vary smoothly, the term on the right is a Lipschitz continuous function on \( \mathcal{S}_{G(n-1)} \times \mathcal{S}_{G(n-1)} \), i.e., \( \| \varsigma_{ij}(A) - \varsigma_{ij}(B) \| \leq D \| A(B, B) \| \), that latter distance being measured with respect to the natural metric on \( \mathcal{S}_{G(n-1,N)} \). Thus \( \| \nu(A) - \nu(B) \| \leq CD \| A(B, B) \| \) and \( \nu \) is locally Lipschitz. Since a locally Lipschitz function on a compact set is in fact Lipschitz, the restriction \( \nu|_{\mathcal{S}_{G(n-1)}(Y(e))} \) is Lipschitz. Because \( \mu \) can be written as the composition of \( \nu|_{\mathcal{S}_{G(n-1)}(Y(e))} \) with a smooth function defined on a compact set, \( \mu \) is Lipschitz.

\[\square\]

For any \( y \in B(Y,e) \setminus Y \), let \( \gamma_y : [0, 2d(y, Y)] \to B(Y,e) \) the unique minimal geodesic with \( \gamma_y(0) = y \) and \( \gamma_y(d(y, Y)) = \pi(y) \), and set \( v = \gamma_y'(d(y, Y)) \) and \( H = H_v = v^\perp \) in the above construction. For simplicity of notation, write \( I_y = I_y(\gamma_y) \); since this is a bilinear form on the tangent space to the level set of \( d_{S_{H_v}} \) through \( y \), it accepts pairs of vectors, i.e., \( I_y = \mathbb{B}(\gamma_y, \gamma_y) \). Denote by \( \sigma_y \) the projection from \( T_y N \) onto the tangent space of the level set of \( d_{S_{H_v}} \) through \( y \). Denote by \( u_* \) the spatial derivative of \( u \), i.e., the restriction of \( Du \) to the tangent space of \( M \).

**Lemma 5.** Suppose \( u : M \times [a,b] \to B(Y,e) \) is a continuous function that, in the interior of \( M \times [a,b] \), is smooth and satisfies the heat equation \( \frac{D}{dt} u = \tau_u \). Let \((x, t)\) be a point in the interior of \( M \times [a,b] \) such that \( y = u(x, t) \in B(Y,e) \setminus Y \). Then \( \rho = d_{S_{H_v}} u \) satisfies \( D\rho = \frac{D\rho}{dt} + \text{trace}(I_y(\sigma_y \circ u_*, \sigma_y \circ u_*)) \) at \((x, t)\).

*Proof.* On a small neighborhood of \((x, t)\), \( u \) remains within \( \exp(P_{H_v}) \). In any local coordinates \((x_1, \ldots, x_m)\) for \( M \) around \( x \) and \((y_1, \ldots, y_n)\) for \( N \) around \( y \), Hamilton computes

\[
g^{ij} \left[ \frac{\partial^2 d_{S_{H_v}}}{\partial y^i \partial y^j} - \frac{\partial d_{S_{H_v}}}{\partial y^i} \Gamma^i_{jk} \right] \frac{\partial u^k}{\partial x^j} \frac{\partial u^j}{\partial x^i} = \Delta \rho - \frac{D\rho}{dt}.
\]
By the definition of viscosity solution, if \( f \) is a viscosity solution on \( M \) and \( \Gamma_{\beta\gamma}^\alpha \) are the Christoffel symbols of the coordinates on \( N \). The matrix \( [\sigma_{\beta\gamma}]_{1 \leq \beta, \gamma \leq n-1} \), where \( \sigma_{\beta\gamma} = \frac{\partial^2 d_{S,y}}{\partial y^\beta \partial y^\gamma} - \frac{\partial d_{S,y}}{\partial y^\beta} \Gamma_{\beta\gamma}^\alpha \frac{\partial}{\partial x^\alpha} \), is the coordinate representative of \( \Pi_y \). In exponential normal coordinates for \( M \) around \( x \), \( g^{ij} \) becomes the Kronecker delta; in normal coordinates with respect to \( \exp|S,y| \) within \( \exp(P_{H_y}) \), \( \sigma_{\beta\gamma} = \delta_{\beta\gamma} = 0 \) for all \( 1 \leq \beta, \gamma \leq n \). Writing \( m = \dim(M) \), one has that

\[
g^{ij} \left[ \frac{\partial^2 d_{S,y}}{\partial y^\beta \partial y^\gamma} - \frac{\partial d_{S,y}}{\partial y^\beta} \Gamma_{\beta\gamma}^\alpha \frac{\partial}{\partial x^\alpha} \right] \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} = \sum_{i=1}^m \left[ \frac{\partial^2 d_{S,y}}{\partial y^\beta \partial y^\gamma} - \frac{\partial d_{S,y}}{\partial y^\beta} \Gamma_{\beta\gamma}^\alpha \frac{\partial}{\partial x^\alpha} \right] \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j}
\]

\[
= \sum_{i=1}^m \Pi_y \left( \sigma_y \circ u_* \left( \frac{\partial}{\partial x^i} \right), \sigma_y \circ u_* \left( \frac{\partial}{\partial x^j} \right) \right)
\]

\[
= \text{trace} \left( \Pi_y (\sigma_y \circ u_*, \sigma_y \circ u_*) \right).
\]

\[\square\]

**Lemma 6.** The subspace \( H_y \) is a supporting hyperplane to \( Y \), i.e., the closure \( \overline{C}_{\Pi(y)} \) of the tangent cone at \( \pi(y) \) is contained in a closed half-space \( \overline{H}_y \) with boundary \( H_y \). Moreover, \( \exp_{\Pi(y)}^{-1}(y) \notin \overline{H}_y \).

**Proof.** This is a consequence of the first variation formula for length.

\[\square\]

**Lemma 7.** The function \( d_{S,y} \) touches \( d_y \) from below at \( y \), i.e., \( d_{S,y}(y) = d_y(y) \) and \( d_{S,y} \leq d_y \) near \( y \).

**Proof.** Because \( \exp \mid_P \) is a diffeomorphism, \( d_{S,y}(y) = d_y(\pi(y)) = d_y(y) \). By Lemma 6, for any \( z \in U_y \), the geodesic \( \gamma \) must hit \( S_y \) before it hits \( Y \). This shows that \( d_{S,y} \leq d_y \) within \( U_y \).

\[\square\]

I will also use the following maximum principle for viscosity solutions, which generalizes the maximum principle of Hamilton.

**Theorem 8.** Let \( M \) be a compact Riemannian manifold, possibly with boundary. Suppose that \( f : M \times [a,b] \to \mathbb{R} \) is a continuous function such that \( f \leq 0 \) on \( M \times [a] \) and, in the case that \( \partial M \neq \emptyset \), on \( \partial M \times [a,b] \). If there exists \( C \in \mathbb{R} \) such that, at any point in the interior of \( M \times [a,b] \) where \( f > 0 \), \( f \) is a viscosity solution to \( \frac{\partial f}{\partial t} - \Delta f - C f \leq 0 \), then \( f \leq 0 \) on \( M \times [a,b] \).

**Proof.** The trick is to define a function \( h : M \times [a,b] \to \mathbb{R} \) by \( h(x,t) = e^{-Ct}f(x,t) \). Then \( h > 0 \) if and only if \( f > 0 \). Fix \( a < T < b \). Assume \( h \) is positive somewhere on \( M \times [a,T] \). Then, by compactness, \( h \) achieves a positive maximum on \( M \times [a,T] \), say at \((x_0,t_0)\). By assumption, \( x_0 \notin \partial M \) and \( 0 < t_0 < T < b \), so \((x_0,t_0)\) lies in the interior of \( M \times [a,b] \), and \( h \) satisfies \( \frac{\partial h}{\partial t} - \Delta h + h \leq 0 \) in the viscosity sense at \( (x_0,t_0) \). Because \( x_0 \) is a global maximum of \( h \) on \( M \times \{t_0\} \), one has that \( \Delta h \leq 0 \) in the viscosity sense there. Let \( \phi \) be a smooth function that touches \( h \) from above at \( (x_0,t_0) \), i.e., \( \phi(x_0,t_0) = h(x_0,t_0) \) and \( \phi \geq h \) on an open set around \( (x_0,t_0) \). By the definition of viscosity solution, \( \frac{\partial \phi}{\partial t}|_{(x_0,t_0)} - \Delta \phi|_{(x_0,t_0)} + \phi(x_0,t_0) \leq 0 \). Since \( \Delta \phi|_{(x_0,t_0)} \leq 0 \) and \( \phi(x_0,t_0) > 0 \), \( \frac{\partial \phi}{\partial t}|_{(x_0,t_0)} < 0 \). But this implies that the constant function \( \psi(x,t) = h(x_0,t_0) \) touches \( h \) from above at \( (x_0,t_0) \), which means that \( \frac{\partial \psi}{\partial t}|_{(x_0,t_0)} - \Delta \psi|_{(x_0,t_0)} + \psi(x_0,t_0) = \psi(x_0,t_0) = h(x_0,t_0) \leq 0 \). This is a contradiction. Thus \( h \leq 0 \) on \( M \times [a,T] \), and, letting \( T \to b \), the result follows by continuity.

\[\square\]

It is now possible to prove the main theorem.
Proof. (Theorem [1]) Let $d_Y : N \to [0, \infty)$ denote the distance to $Y$. The idea is to show that, wherever the composition $\sigma = d_Y \circ u$ is positive and sufficiently small, it is a viscosity solution to an equation of the form $\frac{\partial \sigma}{\partial t} - \Delta \sigma - C \sigma \leq 0$. The result will then follow from Theorem 8.

By Lemma 4, $\mu$ is Lipschitz continuous. Let $C_0 \geq 0$ be a Lipschitz constant for $\mu$. Since the exponential image of a hyperplane is totally geodesic at the image of the origin, $\mu(w, 0) = 0$ for all $w$. Therefore, $|\mu(w, t) - \mu(w, 0)| \leq C_0 |t - 0| = C_0 t$ and, consequently, $\mu(w, t) \geq -C_0 t$. By compactness, $\|u_*\|$ is bounded above on $M \times [a, b]$, say by $D_0 \geq 0$. Let $C = mD_0C_0$. For all $y \in B(Y, \varepsilon)$ and $v = \gamma'_r (d(y, Y))$, one has that

$$\text{trace}(Y, (\sigma_y \circ u_*, \sigma_y \circ u_*)) \geq mD_0\mu\left(\frac{v}{\|v\|}, d(y, Y)\right) \geq -mD_0C_0 d(y, Y) = -C d(y, Y). \quad (2)$$

Assume that, somewhere in $M \times [a, b]$, $u$ maps outside of $Y$. This is equivalent to the statement that $\sigma > 0$ somewhere in $M \times [a, b]$. Because $u(M \times [a]) \subset Y$ and $\|u_*\| \leq D_0$, one may, without loss of generality, shrink $b$ so that $u(M \times [a, b]) \subset B(Y, \varepsilon)$ and still have that $\sigma > 0$ somewhere in $M \times [a, b]$. Then $\sigma = 0$ on $M \times [a]$ and, in the case that $\partial M \neq 0$, on $\partial M \times [a, b]$. It will now be shown that, at any interior point $(x, t)$ of $M \times [a, b]$ such that $\sigma(x, t) > 0$, $\sigma$ is a viscosity solution to $\frac{\partial \sigma}{\partial t} - \Delta \sigma - C \sigma \leq 0$. By (2), $\frac{\partial \sigma}{\partial t} = \Delta \rho - \text{trace}(Y, (\sigma_y \circ u_*, \sigma_y \circ u_*))) \leq \Delta \rho + C \rho$ at $(x, t)$, so at that point $\rho$ satisfies $\frac{\partial \rho}{\partial t} - \Delta \rho - C \rho \leq 0$. Let $\phi$ be any smooth function that touches $\sigma$ from below at $(x, t)$, i.e., $\phi(x, t) = \sigma(x, t)$ and $\phi \geq \sigma$ on a neighborhood of $(x, t)$. By Lemma 7, $\rho$ touches $\sigma$ from below at $(x, t)$, which implies that $\phi$ touches $\rho$ from above at $(x, t)$. Thus $\frac{\partial \rho}{\partial t} = \frac{\partial \sigma}{\partial t}$, $\Delta \phi \geq \Delta \rho$, and $\phi \geq \rho$ at $(x, t)$. So $\frac{\partial \rho}{\partial t} - \Delta \phi - C \phi \leq \frac{\partial \rho}{\partial t} - \Delta \rho - C \rho \leq 0$ at $(x, t)$. This shows that $\sigma$ is a viscosity solution to $\frac{\partial \sigma}{\partial t} - \Delta \sigma - C \sigma \leq 0$ at $(x, t)$. Theorem 8 implies that $\sigma = 0$ on $M \times [a, b]$, a contradiction.

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