Bounce scenarios in the Sotiriou–Visser–Weinfurtner generalization of the projectable Horava–Lifshitz gravity

E Czuchry
Instytut Problemów Jądrowych, ul. Hoża 69, 00-681 Warszawa, Poland
E-mail: eczuchry@fuw.edu.pl

Received 16 September 2010, in final form 5 April 2011
Published 11 May 2011
Online at stacks.iop.org/CQG/28/125013

Abstract
The occurrence of a bounce in the FRW cosmology requires modifications of general relativity. An example of such a modification is the recently proposed Hořava–Lifshitz (HL) theory of gravity, which includes a ‘dark radiation’ term with a negative coefficient in the analog of the Friedmann equation. A modification of the HL gravity, relaxing the ‘detailed-balance’ condition, brings additional terms to the equations of motion, corresponding to stiff matter. This paper presents a comparison of the phase structure of the original and modified Hořava cosmology. Special attention is paid to the analysis of a wide range of the bouncing solution, appearing in both versions of the Hořava theory.

PACS numbers: 98.80.Cq, 04.50.Kd

(Some figures in this article are in colour only in the electronic version)

1. Introduction
There have been many attempts to modify Einstein’s theory of gravity to avoid an initial singularity. Some were made at the classical level, some involve quantum effects. Examples include the ekpyrotic/cyclic model ([1–6]) and loop quantum cosmology ([7–9]), which replace the big bang with a big bounce. Attempts to address these issues at the classical level include braneworld scenarios ([10, 11]), where the universe goes from an era of accelerated collapse to an expanding era without any divergences or singular behavior. There are also higher order gravitational theories and theories with scalar fields (see [12] for a review of bouncing cosmologies). However, it is fair to say that the issue of the initial singularity still remains one of the key questions of the early universe cosmology.

Recently, much effort has been devoted to studies of a proposal for a UV complete theory of gravity due to Hořava [13–15] and modifications of the theory [14, 16–19] (for a recent review see [20]). Because in the UV the theory possesses a fixed point with an anisotropic, Lifshitz scaling between time and space, this theory is referred to as the Hořava–Lifshitz...
(HL) gravity. From the time at which the Hořava theory was presented, there is also quite much discussion of possible problems and instabilities of HL gravity [21–24]. Numerous sophisticated versions contain new terms added to the original Lagrangian with attempt to make the proposal more general [19] and to solve the so-called strong coupling problem [21, 25–28]. Even so, it is still tempting to investigate issues opened by this theory and its modifications.

Soon after this theory was proposed, many specific solutions have been found, including cosmological ones ([16, 29–37]). It was also realized that the analog of the Friedmann equation in HL gravity contains a term which scales in the same way as dark radiation in braneworld scenarios [29–31] and gives a negative contribution to the energy density. Thus, at least in principle it is possible to obtain non-singular cosmological evolution within Hořava theory, as was pointed out in [29, 31, 32, 39]. Such possibility may have dramatic consequences for potential histories of the universe—other than avoiding the initial singularity. New imaginable scenarios of cosmological evolution include contraction from the infinite size, bounce and then expansion to infinite size again, or eternal cycles of contraction, bounce and expansion.

Additional possibilities are brought by some interesting modifications of HL gravity, either by softly breaking a detailed-balance condition [14, 16, 17] or relaxing it completely [18, 19]. In the latter work, Sotiriou, Visser and Weinfurtner (SVW) in search for the more general renormalizable gravitational theory took a gravitational action containing terms not only up to quadratic in curvature, like in the original HL formulation, but also cubic ones, as suggested earlier in [29, 30]. The generalized Friedmann equation of this model includes a modified dark radiation term proportional to $\sim 1/a^4$ ($a$ is a scale factor) from the original HL formulation, and also an additional term $1/a^6$. This new term, negligible at large scales, becomes significant at small scales and modifies bounce solutions. In particular, with the opposite sign (the value of a coupling constant is arbitrary) to that of the $1/a^4$ term, it may compensate the dark radiation term at small distances and cancel the possibility of avoiding singularity, like in HL gravity with the softly broken detailed-balance condition and negative spatial curvature [38]. Thus, one of the questions to be answered is how the additional terms in the generalized Friedmann equations of the SVW HL gravity influence the existence and stability of a cosmological bounce.

In this work, we are going to investigate, with the help of the phase portrait techniques, how relaxing the detailed-balance condition affects the dynamics of the system, and then compare the results to those in the standard HL theory. We will focus on non-flat cosmologies, with the space curvature $k = \pm 1$, allowing non-singular solutions. Unlike in our previous paper [40], we are going to describe matter by a cosmological stress–energy tensor added to the gravitational field equations. Such analysis is more effective and avoids unneeded approximations and simplifications. In this hydrodynamical approach two quantities, density $\rho$ and pressure $p$, describe the matter properties.

Nonetheless, the constant parameter $w$ of the equation of state is of course an idealization, hard to avoid at this level of research. It would be better to have the history of the SVW HL universe constructed in a similar way as in the standard $\Lambda$CDM model, with phases and epochs of different matter/radiation contents. Yet unless a rich structure of the original and the generalized HL theory, with additional coupling constants whose range of values is not fully understood thus far, is investigated deeper, we shall use simpler tools. Thus in this paper, within limited physical understanding of the theory and its parameters, we would rather present lists of possibilities than likely physical solutions. With the progress of research in this field and better understanding of the nature of these parameters, it will be possible to assign more physical interpretation to a set of solutions/scenarios found.
Related analysis of generalized HL cosmology has recently appeared in [41] and [42], which we became aware of while this work was being typed. Those papers address a somewhat different set of issues, i.e. the static solutions of the HL universe. Here, we are interested in stable and unstable solutions leading to the cosmological bounce, and focus on both cases of the non-flat universe \((k = -1\) and \(k = 1\)) with the value of a HL constant \(\lambda\) arbitrary. We agree on the regions of overlap. General discussion of the full phase space of the original HL cosmology is contained e.g. in [43, 44].

The structure of this paper is as follows. In section 2, we briefly sketch the HL gravity and cosmology. In section 3, the possibility of the bounce in theory with the detailed-balance condition is discussed. In section 4, we discuss phase portraits of the HL universe with the condition of the detailed balance relaxed.

2. Hořava–Lifshitz cosmology

The metric of the HL theory, due to anisotropy in UV, is written in the \((3 + 1)\)-dimensional ADM formalism:

\[
d s^2 = -N^2 d t^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt),
\]

where \(N, N_i, \) and \(g_{ij}\) are the dynamical variables.

2.1. Detailed balance

The action of the HL theory is [14]

\[
I = \int dt d^3x (\mathcal{L}_0 + \mathcal{L}_1),
\]

\[
\mathcal{L}_0 = \sqrt{-g} \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_w R - 3 \Lambda_{\bar{W}})}{8(1 - 3 \lambda)} \right\},
\]

\[
\mathcal{L}_1 = \sqrt{-g} \left\{ \frac{\kappa^2 \mu^2 (1 - 4 \lambda)}{32(1 - 3 \lambda)} R^2 - \frac{\kappa^2}{2 \omega^2} Z_{ij} Z^{ij} \right\},
\]

where \(K_{ij} = \frac{1}{N} \left[ \frac{1}{2} \dot{g}_{ij} - \nabla_i N_j \right] \) is the extrinsic curvature of a space-like hypersurface with a fixed time, the dot denotes a derivative with respect to the time \(t\) and covariant derivatives are defined with respect to the spatial metric \(g_{ij}\). Moreover,

\[
Z_{ij} = C_{ij} - \frac{\mu \omega^2}{2} R_{ij}.
\]

\(\kappa^2, \lambda, \mu, \omega\) and \(\Lambda_w\) are the constant parameters and the Cotton tensor, \(C_{ij}\), is defined by

\[
C^{ij} = \epsilon^{ikl} \nabla_k \left( R^l_j - \frac{1}{4} R \delta^l_j \right) = \epsilon^{ikl} \nabla_k R^l_j - \frac{1}{4} \epsilon^{ijkl} \partial_k R.
\]

In (2), \(\mathcal{L}_0\) is the kinetic part of the action, while \(\mathcal{L}_1\) gives the potential of the theory in the so-called ‘detailed-balance’ form.

Matter may be added by inserting a cosmological stress–energy tensor in the gravitational field equation. Within such framework we approximate the stress–energy tensor by two quantities: density \(\rho\) and pressure \(p\), then simply add them to the vacuum equations by demanding the correct limit as one approaches general relativity—the low-energy limit of the HL theory. Relation between \(\rho\) and \(p\) is given by the equation \(p = w \rho\), with \(w\) being the equation of state parameter.
Comparing the action of the HL theory in the IR limit to the Einstein–Hilbert action of
general relativity, one can see that the speed of light $c$, Newton’s constant $G$ and the effective
cosmological constant $\Lambda$ are
\[ c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda W}{1 - 3\lambda}}, \quad G = \frac{\kappa^2 c}{32\pi}, \quad \Lambda = -\frac{3\kappa^4 \mu^2}{\Lambda W} = \frac{3c^2}{2} \Lambda W, \tag{5} \]
respectively. To have a real value of the speed of light $c$ emerging, the HL cosmological
constant $\Lambda_W$ has to be negative for $\lambda > 1/3$ and positive for $\lambda < 1/3$. It is possible to obtain
a positive cosmological constant $\Lambda_W$ in the IR limit $\lambda = 1$ if one performs in (2) an analytic
continuation of the constant parameters $\mu \mapsto i\mu$ and $\omega^2 \mapsto -i\omega^2$.

The equations for HL cosmology are obtained by imposing conditions of homogeneity
and isotropy of the metric. The associated ansatz is $N = N(t), N_i = 0, g_{ij} = a^2(t)\gamma_{ij}$,
where $a(t)$ is a scale factor and $\gamma_{ij}$ is a maximally symmetric constant curvature metric, with
a curvature $k = (-1, 0, 1)$. On this background
\[ K_{ij} = \frac{H}{N} g_{ij}, \quad R_{ij} = 2\kappa \frac{k}{a^2} g_{ij}, \quad C_{ij} = 0, \tag{6} \]
where $H = \dot{a}/a$ is the Hubble parameter.

The gravitational action (2) becomes
\[ S_{\text{FRW}} = \int dt d^3x N a^3 \left\{ \frac{3(1 - 3\lambda)}{2\kappa^2} \frac{H^2}{N^2} + \frac{3\kappa^2 \mu^2 \Lambda W}{4(1 - 3\lambda)} \left( \frac{k a^2}{2} - \frac{\Lambda W}{6} \right) - \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \frac{k^2}{a^4} \right\}. \tag{7} \]
The equations of motion are obtained by varying the action (7) with respect to $N, a$ and $\varphi$,
setting $N = 1$ at the end of the calculations and adding terms with density $\rho$ and pressure $p$,
leading to
\[ H^2 = \frac{\kappa^2 \rho}{6(3\lambda - 1)} \pm \frac{\kappa^2 \mu^2}{8(3\lambda - 1)^2} \left( \frac{k a^2}{2} - \frac{\Lambda W}{6} - \frac{k^2}{2a^4} \right), \tag{8} \]
\[ \dot{H} = -\frac{\kappa^2 (\rho + p)}{4(3\lambda - 1)} \pm \frac{\kappa^2 \mu^2}{8(3\lambda - 1)^2} \left( \frac{k a^2}{2} + \frac{k^2}{4a^4} \right), \tag{9} \]
and the continuity equation
\[ \dot{\rho} + 3H(\rho + p) = 0. \tag{10} \]
The upper sign denotes the $\Lambda_W < 0$ case, the lower sign denotes the analytic continuation
$\mu \mapsto i\mu$ with a positive $\Lambda_W$.

The significant new terms in the above equations of motion are the $(1/a^4)$-terms on
the right-hand sides of (8) and (9). They are reminiscent of the dark radiation term
in the braneworld cosmology [45] and are present only if the spatial curvature of the metric is
non-vanishing.

Values of constant parameters $\kappa^2$ and $\mu$ may be expressed in terms of cosmological
constants according to (5). We will also work in units such that $8\pi G = 1$ and $c = 1$. Then
\[ \kappa^2 = 32\pi G c = 4, \quad \Lambda = \frac{3}{2} \Lambda_W, \tag{11} \]
and
\[ \frac{\mu^2}{1 - 3\lambda} = \pm \frac{3}{2\lambda}. \tag{12} \]
Substituting the above expressions and the equation of state $p = w\rho$ to (8–9) leads to
\[ H^2 = \frac{2}{3\lambda - 1} \left[ \frac{\rho}{3} \pm \left( \frac{\Lambda}{3} - \frac{k}{a^2} + \frac{3}{4\lambda} \frac{k^2}{a^4} \right) \right], \tag{13} \]
\[ \dot{H} = \frac{2}{3\lambda - 1} \left[ -\frac{(1 + w)}{2} \rho \pm \left( \frac{k}{a^2} - \frac{3}{2\Lambda} \frac{k^2}{a^4} \right) \right]. \] (14)

2.2. Beyond detailed balance

The gravitational action written in the ‘detailed-balance’ form (2) [14] contains terms up to quadratic in the curvature. However, the most general renormalizable theory also contains the cubic terms, as was pointed out in [29, 30]. Thus, SVW [18, 19] built a theory with the projectability condition \( N = N(t) \), as in the original Hořava theory, but without the detailed-balance condition. This led to the Friedmann equations with an additional term \( \sim 1/a^6 \) and uncoupled coefficients:

\[ H^2 = \frac{2}{(3\lambda - 1)} \left( \frac{\rho + \sigma_1 + \sigma_2 k}{a^2} + \sigma_3 k^2 \frac{k}{a^4} + \sigma_4 k \frac{k}{a^6} \right), \] (15)

\[ \dot{H} = \frac{2}{(3\lambda - 1)} \left[ -\frac{p}{2} - \frac{\rho}{2} - \sigma_1 k \frac{k}{a^2} + 2\sigma_2 k^2 \frac{k}{a^4} + 3\sigma_3 k^3 \frac{k}{a^6} \right]. \] (16)

Values of the constants \( \sigma_3, \sigma_4 \) are arbitrary. In order to coincide with the Friedmann equations in the IR limit \( \lambda = 1 \) and for large \( a \) (terms proportional to \( 1/a^4 \) and to \( 1/a^6 \) are then negligible) one has to set \( \sigma_1 = \Lambda/3 \) and \( \sigma_2 = -1 \). Thus, the above equations take the following forms:

\[ H^2 = \frac{2}{(3\lambda - 1)} \left( \frac{\rho + \Lambda}{3} - \frac{k}{a^2} + \sigma_3 k^2 \frac{k}{a^4} + \sigma_4 k \frac{k}{a^6} \right), \] (17)

\[ \dot{H} = \frac{2}{(3\lambda - 1)} \left[ -\frac{\rho(1 + w)}{2} + \frac{k}{a^2} - 2\sigma_2 k^2 \frac{k}{a^4} - 3\sigma_3 k^3 \frac{k}{a^6} \right]. \] (18)

where we have used the equation of state \( p = w\rho \). The density parameter follows the standard evolution equation (10). New terms, proportional to \( 1/a^6 \), appearing in the analogs of the Friedmann equations, mimic stiff matter (e.g. such that \( \rho = p \) and \( \rho_{\text{stiff}} \sim 1/a^6 \)). These terms are negligibly small at large scales, but may play a significant role at small values of a scale parameter.

3. Bounce stability in the original HL theory

In order to investigate the appearance of a bounce in the original HL gravity, we are first going to simplify the equations of motion (8–9) and then to reduce them with respect to (8). In this way, we will obtain the two-dimensional dynamical system describing the evolution of \( a \) and \( H \).

Solving equation (13) for \( \rho \) gives

\[ \rho = \frac{3(3\lambda - 1)}{2} H^2 \mp \left( \Lambda - 3 \frac{k}{a^2} + \frac{9}{4\Lambda} \frac{k^2}{a^4} \right). \] (19)

Inserting the above formula to (14) leads to

\[ \dot{H} = \frac{\pm 1}{3\lambda - 1} \left[ (1 + w) \Lambda - (3w + 1) \frac{k}{a^2} + \frac{3}{4\Lambda} \frac{(3w - 1) k^2}{a^4} \right] - \frac{3}{2} (1 + w) H^2. \] (20)

Equation (20) and the definition of the Hubble parameter,

\[ \dot{a} = a H, \] (21)

provide the two-dimensional dynamical system for the variables \( a \) and \( H \).
To find the finite critical points, we set all the rhs of equations (20)–(21) to zero. This gives two points

\[ P_1 : a^2 = \frac{3k}{2\Lambda}, \quad H = 0, \]  

\[ P_2 : a^2 = \frac{(3w - 1)k}{(1 + w)2\Lambda}, \quad H = 0. \]  

These points are finite, unless \( w = -1 \). In the latter case, point \( P_2 \) is moved to infinity. Point \( P_1 \) exists for \( k/\Lambda > 0 \). Point \( P_2 \) exists for \( w > 1/3 \) and \( k/\Lambda > 0 \) or \( w < 1/3 \) and \( k/\Lambda < 0 \). Thus, these two points exist both at the same time for \( w > 1/3 \).

Stability properties of the critical points are determined by the eigenvalues of the Jacobian of the system (20)–(21). More precisely, one has to linearize transformed equations (20)–(21) at each point. Inserting \( \vec{x} = \vec{x}_0 + \delta \vec{x} \), where \( \vec{x} = (a, H) \), and keeping terms up to first order in \( \delta \vec{x} \) leads to an evolution equation of the form \( \delta \dot{\vec{x}} = A\delta \vec{x} \). The eigenvalues of \( A \) describe the stability properties at the given point. Critical points at which all the eigenvalues have real parts different from zero are called hyperbolic. Among them one can distinguish sources (unstable) for positive real parts, saddle for real parts of a different sign and sinks (stable) for negative real parts. If at least one eigenvalue has a zero real part (non-hyperbolic critical point) it is not possible to obtain conclusive information about the stability from just linearization and needs to resort to other tools like e.g. numerical simulation [46].

The eigenvalues at \( P_1 \) are as follows:

\[ \left( -2 \sqrt{\frac{\mp 2\Lambda}{3(1 - 3\lambda)}}, 2 \sqrt{\frac{\mp 2\Lambda}{3(1 - 3\lambda)}} \right). \]

For all admitted values of \( \Lambda \) and \( \lambda \), expression \( \mp \Lambda/(1 - 3\lambda) \) is negative; thus, \( P_1 \) is a center. (Both eigenvalues of \( A \) are purely imaginary at this point.)

The eigenvalues at the second finite critical point \( P_2 \) read

\[ \left( -2 \sqrt{\frac{\mp 2\Lambda(1 + w)}{(1 - 3\lambda)(1 - 3w)}}, 2 \sqrt{\frac{\mp 2\Lambda(1 + w)}{(1 - 3\lambda)(1 - 3w)}} \right). \]

Depending on the value of parameter \( w \) point \( P_2 \) may be a center or a saddle (two real numbers with opposite signs). Precisely, \( P_2 \) is a linear center (non-hyperbolic center with one eigenvector) for \( w = -1 \) \( (k/\Lambda < 0) \), a center for \( -1 < w < 1/3 \) \( (k/\Lambda < 0) \) and a saddle for \( w > 1/3 \) \( (k/\Lambda > 0) \).

Properties of the critical points \( P_1 \) and \( P_2 \) in dependence on the values of \( \Lambda, k, w \) are gathered in table 1. \( \Lambda < 0 \) corresponds to the solutions of (20)–(21) with the upper sign, the case of \( \Lambda > 0 \) to the lower sign in (20)–(21).

To find critical points that occur at infinite values of the parameters, we rescale the infinite space \( (a, H) \) into a finite Poincaré sphere (as in [47, 48]) in such a way that the new coordinates \( (\tilde{a}, \tilde{H}) \) are written in the polar coordinates \( r, \phi \): \( \tilde{a} = r \cos \phi \) and \( \tilde{H} = r \sin \phi \) and

\[ a = \frac{r}{1 - r} \cos \phi, \]

\[ H = \frac{r}{1 - r} \sin \phi. \]

We also rescale the time parameter \( t \) by defining the new time parameter \( T \) such that \( dT = dt/(1 - r) \). In these coordinates, our phase space is contained within a sphere of
Table 1. Properties of the finite critical points in the HL theory. The plus sign stands for ‘exists’ and the minus sign stands for ‘does not exist’.

| $k/\Lambda$ | $w$ | $P_1$ Stability | $P_2$ Stability |
|------------|-----|-----------------|-----------------|
| $> \frac{1}{3}$ | + Center | + Saddle |
| $> 0$ | $-1 < w < \frac{1}{3}$ | + Center | - |
| $-1$ | + Center | - |
| $> \frac{1}{4}$ | - | - |
| $< 0$ | $-1 < w < \frac{1}{4}$ | + Center |
| $-1$ | - | Moves to $\infty$ Linear center |

radius 1—infinity corresponds to $r = 1$. More precisely, semi-sphere, as a scale factor $a$, may take only non-negative values.

This leads to the dynamical equations in terms of $r, \phi$ and their derivatives with respect to new time $T$. Taking limit $r = 1$ we obtain

$$r'(T) = 0,$$

$$\phi'(T) = -\frac{5 + 3w}{2} \cos \phi \sin^2 \phi.$$  \hspace{1cm} (26)

(27)

Putting the rhs of the above equations to zero, we find four solutions,

$$P_3 = (1, 0),$$
$$P_4 = (1, \pi/2),$$
$$P_5 = (1, \pi),$$
$$P_6 = (1, 3\pi/2),$$

in polar coordinates $(r, \phi)$. Point $P_3$ is non-physical (a negative $a$) and shall be removed from further discussions. The eigenvalues of the Jacobian matrix at the above points are as follows:

$$(0, 0) \text{ at } P_3 :$$
$$\begin{pmatrix} 5 + 3w/2 & 1 + w/2 \\ -5 + 3w/2 & -3 + 1 + w/2 \end{pmatrix} \text{ at } P_4,$$

$$(5 + 3w/2, 3 + 1 + w/2) \text{ at } P_5,$$

$$(5 + 3w/2, -3 + 1 + w/2) \text{ at } P_6.$$

The point $P_3$ is non-hyperbolic and we determine its properties by numerical simulations for each set of parameters. Unless $w = -1$ points $P_4$ and $P_6$ are respectively a repelling and an attracting node. For $w = -1$, the finite fixed point $P_3$ is moved to $(\infty, 0)$ becoming $P_3$, which is then a linear center. For this value of $w$, points $P_4$ and $P_6$ are non-hyperbolic. It follows from the numerical simulations that they are saddles then ends of a separatrix.

Figure 1 shows the phase portrait of the HL universe containing matter with the equation of state parameter $w > 1/3, k/\Lambda > 0$. figure 2 shows the phase portrait for $-1 < w < 1/3, k/\Lambda < 0$ and figure 3 shows the phase portrait for $w = -1, k/\Lambda < 0$. One has to note that these figures contain the deformed phase space, scaled to fit on the finite Poincaré sphere. One may have the impression that they describe the regions in which e.g. the scale factor $a$ increases although the Hubble parameter $H$ is negative. However, it is the parameter $\tilde{a}$ that is increasing in the diagram, not the scale factor $a$.

Bounce scenarios are thus possible when the critical points exist. If these points are centers then there are closed orbits around them and the universe goes through eternal oscillations:
expansion, collapse to a finite size, expansion etc. Point $P_1$ may be a center (for certain values of parameters), but then $\rho = 0$, which is physically not interesting. A more interesting case is when $P_2$ is a center, then there are closed orbits with a non-zero density $\rho$. The third bounce scenario is around the linear center $P_2$ (moved to $\infty$ and coinciding with $P_3$). In this case, there are onefold closed trajectories: the universe starts from a static one ($H = 0$) and infinite ($a = \infty$), goes through a period of collapsing to a finite size, then after a bounce starts expanding and finishes as a static infinite universe.

4. Bounce stability in the SVW generalization

Relaxing detailed-balance condition leads to the generalized Friedmann equations (17)–(18) with the additional term $\sim 1/a^6$ and uncoupled coefficients.

We may solve equation (17) for $\rho$ obtaining

$$\rho = 3 \left(3\lambda - 1\right) \frac{H^2}{2} - \Lambda - 3 \frac{k}{a^2} - 3 \frac{\sigma_3 k^2}{a^4} - 3 \frac{\sigma_4 k}{a^6}. \tag{28}$$

Substituting this expression on $\rho$ into (18) and using the equation of state $p = w\rho$ lead to

$$\dot{H} = \frac{2}{3\lambda - 1} \left( \frac{(1 + w)}{2} - \frac{k(1 + 3w)}{2a^2} + \frac{\sigma_3(1 + 3w)k^2}{2a^4} + \frac{3\sigma_4(1 + w)k}{2a^6} \right) = \frac{3(1 + w)}{2} H^2. \tag{29}$$

Figure 1. Projected phase space of the HL universe with $k \Lambda > 0$ and $w > 1/3$. 
Figure 2. Projected phase space of the HL universe with $k/\Lambda < 0$ and $-1 < w < 1/3$.

The above equation, together with the definition of the Hubble parameter, provides the two-dimensional dynamical system for the variables $a$ and $H$.

Finite critical points are the solutions of equations (21) and (29) with the rhs set to zero. Hence, these points fulfill $H = 0$ and

$$\Lambda(1 + w)a^6 - k(1 + 3w)a^4 + \sigma_3(-1 + 3w)k^2a^2 + 3\sigma_4(-1 + w)k = 0$$

(30)

The latter one is a bicubic equation, which may be simplified in few special cases.

4.1. Cosmological constant $w = -1$

For $w = -1$, equation (30) reduces to a biquadratic equation:

$$ka^4 - 2\sigma_3k^2a^2 - 3\sigma_4k = 0.$$  

(31)

The solutions of (31) are as follows:

$$P_1 : a^2 = k\sigma_3 - \sqrt{\sigma_3^2 + 3\sigma_4},$$

(32)

$$P_2 : a^2 = k\sigma_3 + \sqrt{\sigma_3^2 + 3\sigma_4}.$$  

(33)
Figure 3. Projected phase space of the HL universe with $k/\Lambda < 0$ and $w = -1$.

Point $P_1$ exists for \{(k\sigma_3 > 0, \sigma_4 < 0); (|\sigma_4| < \sigma_3^2/3)\}. Point $P_2$ exists for \{(k\sigma_3 > 0, \sigma_4 > 0); (k\sigma_3 > 0, \sigma_4 < 0, |\sigma_4| < \sigma_3^2/3); (k\sigma_3 < 0, \sigma_4 > 0)\}. Stability properties of the critical points found are given by the eigenvalues of the Jacobian matrix $A$ of the system (21), (29). The eigenvalues of $A$ at $P_1$ are as follows:

$$
\left( \pm \sqrt{-\frac{kC_1}{3\Lambda - 1}}, \mp \sqrt{-\frac{kC_1}{3\Lambda - 1}} \right),
$$

where $C_1$ denotes the expression in $k, \sigma_3, \sigma_4$, being positive when point $P_1$ exists. Similarly, the eigenvalues of $A$ at point $P_2$ read

$$
\left( \pm \sqrt{\frac{kC_2}{3\Lambda - 1}}, \mp \sqrt{\frac{kC_2}{3\Lambda - 1}} \right),
$$

where $C_2$ denotes the expression in $k, \sigma_3, \sigma_4$ being positive when point $P_2$ exists.

Thus, for $k/(3\Lambda - 1) > 0$ point $P_1$ is a center and $P_2$ is an unstable saddle, for $k/(3\Lambda - 1) < 0$ $P_1$ is a saddle and $P_2$ is a center—provided that the values of $k, \sigma_3, \sigma_4$ allow their physical existence ($a^2 > 0$).
Density \( \rho \) at these points may be written as

\[
\rho = -\frac{2\sigma_3^3 + 9\sigma_3\sigma_4 + 9L\sigma_4^2 \pm 2k\sigma_3^2 \sqrt{\sigma_3^2 + 3\sigma_4 \pm 6k\sigma_4 \sqrt{\sigma_3^2 + 3\sigma_4}}}{9\sigma_4^2},
\]

(37)

where ‘+’ corresponds to point \( P_1 \) and ‘−’ to \( P_2 \). Thus, at point \( P_1 \) density \( \rho > 0 \) if this point exists and \((k < 0, \sigma_3 < 0, 0 > \Lambda > 1/\sigma_3)\). At point \( P_2 \) density is positive if \((k < 0, \sigma_3 < 0, \Lambda < 0)\).

4.2. Radiation \( w = 1/3 \)

When \( w = 1/3 \) equation (30) reduces to the following:

\[
\frac{2\Lambda}{3} x^3 - kx^2 - k\sigma_4 = 0,
\]

(38)

where \( x = a^2 \). The discriminant of the cubic polynomial \( a_3 y^3 + a_2 y^2 + a_1 y + a_0 \) is of the following form: \( \Delta = 18a_0a_1a_2 - 4a_2^3a_0 + a_2^3a_3^3 - 4a_3a_1^3 - 27a_2^3a_0^2 \). The discriminant of equation (38) reads

\[
\Delta = -4k^2\sigma_4(k^2 + L^2\sigma_4).
\]

(39)
For $\Delta > 0$, the cubic equation (38) has three real solutions. Condition $\Delta > 0$ is fulfilled for a non-flat universe ($k \neq 0$) when $\sigma_4 < 0$ and $|\sigma_4| < 1/(3\Lambda^2)$. Otherwise (38)—the equation with real coefficients has one real solution and two non-real complex conjugate roots ($\Delta < 0$) or multiple real roots ($\Delta = 0$).

Physical points exist if the real solutions $x = a^2 > 0$. Equation (38) cannot have three real positive roots, as is implied by Viète’s formulas—precisely, by the second formula of the following:

\[
\frac{3k}{2\Lambda} = x_1 + x_2 + x_3, \tag{40}
\]

\[
0 = x_1x_2 + x_2x_3 + x_3x_1, \tag{41}
\]

\[
\frac{3k\sigma_4}{2\Lambda} = x_1x_2x_3. \tag{42}
\]

If there are three real solutions ($\Delta > 0$), one or two of them may also be positive. There is one positive solution if $k/\Lambda < 0$ and two positive solutions if $k/\Lambda > 0$.

Multiple real solutions exist when $\Delta = 0$ hence when $\sigma_4 = 0$ or $\sigma_4 = -k^2/(3\Lambda^2)$. The former case corresponds to the HL theory with the detailed-balance condition; the latter case implies the solutions

\[
Q_1 : a^2 = -k/(2\Lambda), \tag{43}
\]

\[
Q_2 : a^2 = k/\Lambda \quad \text{(double root),} \tag{44}
\]

and $H = 0$. Depending on the sign of $k/\Lambda$, one of the two solutions has a physical meaning.

One real solution ($\Delta < 0$), i.e. when $\sigma_4 > 0$ or $\sigma_4 < -1/(3\Lambda^2)$, has a positive value if $k\sigma_4/\Lambda > 0$.

The eigenvalues of the Jacobian matrix at the critical points $a_i$ are as follows:

\[
\left( \pm \frac{2}{a_i^2} \sqrt{\frac{k(3\sigma_4 + a_i^4)}{3\Lambda - 1}}, \pm \frac{2}{a_i^2} \sqrt{\frac{k(3\sigma_4 + a_i^4)}{3\Lambda - 1}} \right) \tag{45}\]

Thus, they may be saddles or centers, depending upon the sign of $k/(3\Lambda - 1)$ and $3\sigma_4 + a_i^4$.

More precisely we can describe the case when $\sigma_4 = -k^2/(3\Lambda^2)$ (the 0 case is described within the original HL cosmology) and the critical points are $a_1^2 = -k/(2\Lambda)$ ($Q_1$) or $a_2^2 = k/\Lambda$ ($Q_2$). Figure 4 shows the projected phase space in this case. Then the eigenvalues of the Jacobian matrix read

\[
\left( 2\sqrt{6} \sqrt{\frac{\Lambda}{3\Lambda - 1}}, -2\sqrt{6} \sqrt{\frac{\Lambda}{3\Lambda - 1}} \right) \text{ at } Q_1, \tag{46}
\]

and

\[
(0, 0) \text{ at } Q_2. \tag{47}
\]

Therefore, the point $Q_1$ may be a saddle or a center, depending upon the sign of $\Lambda/(3\Lambda - 1)$. Point $Q_2$ is non-hyperbolic; numerical simulations (figure 5) show that it is a cusp. Comparing the neighborhood of $Q_2$ in figures 4 and 5, one can see the difference between the deformed phase space and the non-deformed one. At the former one, there are regions in which the parameter $\tilde{a}$ increases although the parameter $\tilde{H}$ is negative, whereas this behavior is absent at the latter one. This is due to the fact that parameters $\tilde{a}$ and $\tilde{H}$ are geometric objects without the same physical meaning as the scale factor $a$ and the Hubble parameter $H$. 

12
Figure 5. Phase trajectories around the non-hyperbolic critical point $Q_2$.

Density $\rho$ is as follows:

\[
\rho = -3\Lambda(5 + 4\Lambda\sigma_3) \quad \text{at} \quad Q_1, \tag{48}
\]

\[
\rho = -3\Lambda(-1 + \Lambda\sigma_3) \quad \text{at} \quad Q_2. \tag{49}
\]

Density at $Q_1$ is positive if

\[
\{(\Lambda < 0, \sigma_3 < 0); (\Lambda > 0, 5/(4\Lambda) > \sigma_3)\}.
\]

At $Q_2$, $\rho$ is positive for

\[
\{(\Lambda < 0, \Lambda\sigma_3 > 1); (\Lambda > 0, \sigma_3 < 1/\Lambda)\},
\]

plus conditions for existence: $Q_1$ has the physical meaning if $k/\Lambda < 0$ and $Q_2$ has the physical meaning if $k/\Lambda > 0$.

4.3. General case

In general, the critical points of the systems (21) and (29) are of the following form: $(a_x, 0)$, where $a_x^2$ is a root of the equation

\[
\Lambda(1 + w)x^3 - k(1 + 3w)x^2 + \sigma_3(-1 + 3w)k^2x + 3\sigma_4(-1 + w)k = 0. \tag{50}
\]

Depending on the sign of the discriminant $\Delta = 18a_0a_1a_2a_3 - 4a_2^2a_0 + a_2^2a_1^2 - 4a_3a_1^3 - 27a_3^2a_0$, where $a_0 = \Lambda(1 + w)$, $a_1 = -k(1 + 3w)$, $a_2 = \sigma_3(-1 + 3w)k^2$, $a_2 = 3\sigma_4(-1 + w)k$, the above equation has one, two or three real solutions. Namely, for $\Delta > 0$, there are three real roots, for $\Delta < 0$ there is one real root and two complex conjugates, for $\Delta = 0$ those conjugates become a real double root.
and it is always positive for $w > 1/3$. For $w < 1/3$ and $-\sigma_4 > -\sigma_3^2/3$, expression $D_1$ is always negative and it is always positive for $w > -1/3$ and $\sigma_4 < -\sigma_3^2/(1-3w)/((3w+1)(w-1))$. Thus, depending on the sign of $k/(3\lambda - 1)$, the critical points, if exist, are either always stable or always unstable. Nature of their stability depends on the values of $\alpha_3, \Lambda, \sigma_3$ and $\sigma_4$.

Stability properties of the critical points at infinity are the same as for the detailed-balance case. After the Poincaré transformation (24) and (25) the whole phase space is contained within a semi-circle ($a \geq 0$) of radius 1.

Points at $r = 1$ and $\phi = \pi/2$, $3\pi/2$ are repelling and attracting node, respectively. Point $r = 1$, $\phi = 0$ is non-hyperbolic, and its stability properties can be obtained e.g. from the numerical simulations.

Figure 6 shows the phase space of the system with three finite critical points. Points $S_1$ and $S_3$ are the centers, point $S_2$ is a saddle.

In table 2, we have gathered properties of the finite critical points in the SVW generalization of the Hořava cosmology.

| Point | $\sigma$ | Existence | Stability | $\rho$ positive |
|-------|----------|-----------|-----------|----------------|
| $P_1$ | $-1$     | $(k\sigma_1 > 0, \sigma_4 < 0)$ | Center for $k/(3\lambda - 1) > 0$ | $\sigma_3 < 0$ |
|       |          |           | Saddle for $k/(3\lambda - 1) < 0$ | $0 > \Lambda > 1/\sigma_3$ |
| $P_2$ | $-1$     | $(k\sigma_1 > 0, 0 > \sigma_4 > -\sigma_3^2/3)$ | Center for $k/(3\lambda - 1) < 0$ | $\sigma_3 < 0$ |
|       |          |           | Saddle for $k/(3\lambda - 1) > 0$ | $\Lambda < 0$ |
| $Q_1$ | $1/3$    | $(\sigma_4 = -k^2/3\Lambda^2, k/\Lambda < 0)$ | Center for $\Lambda/(3\lambda - 1) < 0$ | $\sigma_3 < 0$ |
|       |          |           | Saddle for $\Lambda/(3\lambda - 1) > 0$ | $\sigma_3 > -5/(4\Lambda)$ |
| $Q_2$ | $1/3$    | $(\sigma_4 = -k^2/3\Lambda^2, k/\Lambda > 0)$ | Cusp | $\sigma_3 < 1/\Lambda$ |

The eigenvalues of the Jacobian matrix at these critical points read

\[
\begin{pmatrix}
-\sqrt{\frac{2D_1k}{a_9^2(3\lambda - 1)}} & \sqrt{\frac{2D_1k}{a_9^2(3\lambda - 1)}}
\end{pmatrix},
\]

(51)

where

\[
D_1 = (1 + 3w)a^4 + 2k\sigma_3(1 - 3w)a^2 - 9\sigma_4(w - 1).
\]

(52)

For $w < -1/3$ and $-\sigma_4 > \sigma_3^2(1 - 3w)/(3w + 1)(w - 1)$, expression $D_1$ is always negative and it is always positive for $w > -1/3$ and $\sigma_4 > \sigma_3^2(1 - 3w)/(3w + 1)(1 - w)$. Thus, depending on the sign of $k/(3\lambda - 1)$, the critical points, if exist, are either always stable or always unstable. Nature of their stability depends on the values of $\alpha_3, \Lambda, \sigma_3$ and $\sigma_4$. 

Table 2. Properties of the finite critical points in the SVW HL theory.
5. Conclusions

In this work, we have performed a detailed analysis of a phase structure of HL cosmology with and without detailed-balance condition. Both models contain a dark radiation term $1/a^4$ in the analogs of the Friedmann equations. Thus, it is possible for a non-flat universe ($k \neq 0$) that the Hubble parameter $H = 0$ at some moment of time, which is a necessary condition for the realization of the bounce. Comparing phase trajectories obtained in those models we have attempted to answer the question how the generalization of Hořava gravity (breaking the detailed-balance condition) impacts the occurrence and behavior of bouncing solutions. The additional term $1/a^6$ that appears in the Friedmann equations of the SVW model is of either sign, and thus it may possibly compensate the $1/a^4$ term (generic for HL gravity) leading to the singular solution.

Indeed, it occurred that the biggest difference between the Hořava theory and its generalization arrives for the small values of a scale parameter $a$ and a Hubble parameter $H$. This is not surprising as the SVW gravity term $1/a^6$ plays a role only for the small values of $a$ and becomes insignificant for the bigger ones.

In the original Hořava formulation there may be two finite critical points, one of them is a center and another is a saddle. They are of the type $(a_c, 0)$ in the $(a, H)$ space, thus strictly connected to bounce solutions. These pairs of points exist both only for matter with $w > 1/3$. Around a center there are closed orbits corresponding to the oscillating universe, i.e. going through eternal cycles of contraction, bounce and expansion. These orbits resemble the bounce solution described by Brandenberger [32] or quasi-stationary solutions presented in our previous work [40] based on a field approach. Such solutions are physically interesting (density $\rho > 0$) for $w < 1/3$ and $k/\Lambda < 0$—so either for a closed universe with a positive cosmological constant, or an open universe with $k = 1$ and a negative cosmological constant $\Lambda$. The second class of oscillating solutions, with the vanishing density $\rho = 0$, appears when $k/\Lambda > 0$. Additionally, there is a third bounce scenario around a linear center $P_2$ (moved to $\infty$ and coinciding with $P_3$). Here, there are onefold closed trajectories: the universe starts from a static one ($H = 0$) and infinite ($a = \infty$) goes through a period of contraction to a finite size, then after a bounce starts expanding and again ends as a static infinite universe. Moreover, for some values of parameters, i.e. $k/\Lambda < 0$ and $w > 1/3$, there are no finite critical points, thus no bouncing solutions.

In SVW HL cosmology, with an additional term appearing in the analogs of the Friedmann equations, there may exist zero, one, two or three finite critical points. They are also of the type $(a_c, 0)$ in $(a, H)$ space. Here, $a_c$ is the solution of the bicubic equation. In general, there exists at least one real solution of the cubic equation with real coefficients, but physical points correspond only to positive values of these roots. Critical points might be stable centers—surrounded by closed orbits, describing oscillating universes, or unstable saddles. There also exist solutions with orbits around a linear center at $(\infty, 0)$, where similarly as in the original HL theory, a universe starts from a static infinite, collapses to a finite size, undergoes a bounce and then expands to a static infinite state. Thus, there is one cycle only without further oscillations. There are also sets of parameters, much wider than in the original HL theory, that do not allow the existence of finite critical points, leading only to singular solutions.

The most significant feature of oscillating (and bouncing) solutions in the SVW formulation is the existence of two centers, with a saddle between them (three finite critical points) for some values of parameters. We present such a solution in figure 6. In a more realistic situation, that includes dynamical change of the state parameter, it would be possible to go from one oscillating bouncing solution to another. The present framework does not allow such evolution as it describes matter as hydrodynamical fluid with a constant $w$. We
expect that the field approach, with a more complete dynamics, may be suitable for further investigation of this interesting scenario.

The phase structure at infinity is the same at both formulations. Except bouncing solutions around finite critical points, there are also solutions leading to big bang, big crunch or eternal expansion. It is worth stressing that in both models, the original HL gravity and the SVW generalization, there are classes of parameters that do not allow a non-singular evolution. Physical interpretation of some of these parameters (coupling constants $\sigma_3$ and $\sigma_4$ in SVW model) still remains an open question.

Acknowledgments

This work has been supported by the Polish Ministry of Science and Higher Education grant PBZ/MNiSW/07/2006/37.

References

[1] Khoury J, Ovrut B A, Steinhardt P J and Turok N 2001 Phys. Rev. D 64 123522 (arXiv:hep-th/0103239)
[2] Buchbinder E I, Khoury J and Ovrut B A 2007 Phys. Rev. D 76 123503 (arXiv:hep-th/0702154)
