Mesons in the massive Schwinger model
on the light-cone

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Abstract

We investigate mesons in the bosonized massive Schwinger model in the light-front Tamm-Dancoff approximation in the strong coupling region. We confirm that the three-meson bound state has a few percent fermion six-body component in the strong coupling region when expressed in terms of fermion variables, consistent with our previous calculations. We also discuss some qualitative features of the three-meson bound state based on the information about the wave function.

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I. INTRODUCTION

Recently there has been increasing interest in light-front field theory [1]. In particular the light-front Tamm-Dancoff (LFTD) approximation [2] has proven to be very powerful, as a nonperturbative approach to the relativistic bound state problem. It has been successfully applied to two-dimensional models [3] as well as to four-dimensional Yukawa theory [4]. It is important to note that this new approach not only reproduced known results correctly, but also brought us new results, which have never been obtained by other methods [5,6]. (See also Refs. [7] for some of such “new” results in the discretized light-cone quantization (DLCQ) approach [3].)

In a previous paper [9], we studied the massive Schwinger model in the framework of the LFTD approximation up to including fermion six-body states. We showed that (1) the two-meson bound state has a negligibly small six-body component, (2) the three-meson bound state does exist, and (3) the two-meson bound state is well described in terms of the wave function of the relative motion. Surprisingly, however, the six-body component of the three-meson bound state is quite small, though it is large in comparison with those of other states below the three-meson threshold. Typically it is at most a few percent for a small value of the fermion mass. Despite this small six-body component, we identified it as the three-meson bound state based on the following reasons: (1) Since the meson creation operator contains the fermion annihilation operators (see Ref. [9] for the notation),

$$A^\dagger(p) = \int_0^p \frac{dk}{(2\pi)\sqrt{k(p-k)}} \psi(k, p-k)b^\dagger(k)d^\dagger(p-k) + \int_{\infty}^p \frac{dk}{(2\pi)\sqrt{k(p+k)}} \varphi(p+k,k) [b^\dagger(p+k)b(k) - d^\dagger(p+k)d(k)],$$

the three-meson bound state ($\sim A^\dagger|^3\rangle$) naturally contains fermion two- and four-body components, beside the six-body component. (2) This state is charge conjugation odd, while a two-meson state should be charge conjugation even.

Unfortunately, we were not able to explain why the six-body component of the three-meson bound state is so small. It is not the aim of this paper to do it, but to further confirm
our conclusion and to give some arguments by using bosonization.

How can we further justify the identification despite the smallness of the six-body component of the state? As emphasized in the previous paper, we have a simple picture of the massive Schwinger model in the strong coupling region. Because the massless theory (the strong coupling limit) is a free massive boson (meson) theory, we expect that it becomes a weakly interacting massive boson theory once a small mass term is included (the strong coupling region). In addition, light-front field theory provides us with the simple vacuum. Actually, these two allowed us to describe the two-meson bound state in terms of the wave function of the relative motion. One may think, therefore, of constructing the wave function of the relative motion for the three-meson bound state in a similar way. If such a description is a good approximation of the state, one may justify that it is a three-meson state. It turns out, however, that it is almost infeasible in terms of the fermion variables because it is difficult to find a simple set of basis functions which satisfies all the symmetry properties.

A simpler way we will follow in this paper is to consider the bosonized theory.

Is the Tamm-Dancoff approximation good for the bosonized theory too? The meson has an internal structure in the sense that the wave function, $\psi(k, p - k)$ in eq. (1.1), has nontrivial momentum dependence, which becomes negligible in the strong coupling limit, i.e., the meson becomes “structureless.” The Fock states in the bosonized theory is that of this “structureless” meson. The “structureless” meson state is a good approximation in the strong coupling region. It means that the Tamm-Dancoff approximation is good for the bosonized theory in the strong coupling region. (The wave function gets the momentum dependence from many-body states.) In the bosonized theory, it is easy to find a set of basis functions which satisfies all the symmetry properties.

In this paper, we consider the bosonized massive Schwinger model in the LFTD approximation up to including three-boson states. We show that almost all of our previous calculations are consistent with the results obtained in the bosonized theory in the strong coupling region. In particular, we show that the three-meson bound state, which is almost
100% a three-boson state, has a few percent fermion six-body component.

We are aware of the limitation of our analysis in this paper. The limitation comes from (1) the normal-ordering problem in light-front field theory, and (2) the unboundedness of the Hamiltonian for the charge conjugation even sector. We think that we can avoid these problems if we are confined in the very strong coupling region. We will discuss these problems in the final section.

II. BOSONIZED MASSIVE SCHWINGER MODEL

A. LFTD for the bosonized massive Schwinger model

It is well known that the massive Schwinger model \[11,12\] (QED_{1+1} with massive fermion),

\[ L = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (\gamma^\mu (i\partial_\mu - eA_\mu) - m) \psi, \]  

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]  

has an equivalent bosonic form

\[ \mathcal{L}_b = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\mu^2}{2} \phi^2 + cm \phi : \cos(2\sqrt{\pi} \phi) : , \]  

where \( c \) is numerical constant (\( c = e^\gamma/2\pi \), with \( \gamma \) being the Euler constant), and \( \mu = e/\sqrt{\pi} \).

In the following we set \( \mu = 1 \), so the strong coupling region corresponds to small fermion masses. We quantize this model on the light cone.

In the equal-time quantization, the normal-ordering is well-defined in the interaction picture. (The above normal-ordering is with respect to the free boson with mass \( \mu \).) It is known, on the other hand, that a naive normal-ordering fails in the light-front quantization, because one usually neglects generalized tadpole diagrams [13]. The effects of the generalized tadpoles amount to the renormalization of the coupling constant [14]. In this paper, however, we do not consider this coupling constant renormalization, simply assuming that the effects
of it are negligibly small for the very strong coupling region. We will discuss it in the final section.

Expanding $\phi$ (in the Schrödinger picture) in terms of creation and annihilation operators,

$$\phi(x^-) = \int_0^\infty \frac{dp^+}{2\sqrt{\pi}p^+} [a(p^+)e^{-ip^+x^-} + a^\dagger(p^+)e^{ip^+x^-}],$$  \hspace{1cm} (2.3)

$$[a(p^+), a^\dagger(q^+)] = p^+\delta(p^+ - q^+), \quad [a(p^+), a(q^+)] = [a^\dagger(p^+), a^\dagger(q^+)] = 0,$$  \hspace{1cm} (2.4)

we obtain the light-cone Hamiltonian:

$$P^- = H = H_0 + V,$$

$$H_0 = \frac{\bar{\mu}^2}{2} \int_0^\infty \frac{dp}{p^2} a^\dagger(p)a(p),$$

$$V = V_4 + V_6 + \cdots,$$

where $\bar{\mu}^2 = 1 + 4\pi cm$. Note that because we are going to consider the Tamm-Dancoff (TD) truncation up to including three-boson states, the interaction terms containing more than six creation and/or annihilation operators are irrelevant, and will be ignored hereafter. The interaction terms are expressed in terms of $a^\dagger$ and $a$ in the following way,

$$V_4 = \frac{cm}{4!}(2\pi) \int_0^\infty \prod_{i=1}^4 \frac{dp_i}{p_i} [4\delta(p_1 + p_2 + p_3 - p_4)a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger$$

$$+ 6\delta(p_1 + p_2 - p_3 - p_4)a_1^\dagger a_2^\dagger a_3 a_4 + 4\delta(p_1 - p_2 - p_3 - p_4)a_1^\dagger a_2 a_3 a_4],$$

$$V_6 = \frac{cm}{6!}(2\pi) \int_0^\infty \prod_{i=1}^6 \frac{dp_i}{p_i} [6\delta(p_1 + p_2 + p_3 + p_4 + p_5 - p_6)a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5 a_6$$

$$+ 15\delta(p_1 + p_2 + p_3 + p_4 - p_5 - p_6)a_1^\dagger a_2^\dagger a_3 a_4 a_5 a_6$$

$$+ 20\delta(p_1 + p_2 + p_3 - p_4 - p_5 - p_6)a_1^\dagger a_2 a_3 a_4 a_5 a_6$$

$$+ 15\delta(p_1 + p_2 - p_3 - p_4 - p_5 - p_6)a_1^\dagger a_2 a_3 a_4 a_5 a_6$$

$$+ 6\delta(p_1 - p_2 - p_3 - p_4 - p_5 - p_6)a_1^\dagger a_2 a_3 a_4 a_5 a_6],$$

where we use the abbreviation $a_i = a(p_i)$. The spectrum of the model is obtained by solving the Einstein-Schrödinger equation,

$$2P^+P^- |\phi\rangle = M^2 |\phi\rangle.$$  \hspace{1cm} (2.5)

We truncate the Fock space up to including three-boson states.
\[ |\phi\rangle_P = |\phi_1\rangle_P + |\phi_2\rangle_P + |\phi_3\rangle_P , \] (2.8)

\[ |\phi_1\rangle_P = \phi_1 a^\dagger(P) |0\rangle , \]

\[ |\phi_2\rangle_P = \frac{1}{\sqrt{2}} \int_P \prod_{i=1}^2 \frac{dp_i}{\sqrt{p_i}} \delta(\sum_{i=1}^2 p_i - P) \phi_2(p_1, p_2) a^\dagger(p_1) a^\dagger(p_2) |0\rangle , \]

\[ |\phi_3\rangle_P = \frac{1}{\sqrt{3!}} \int_P \prod_{i=1}^3 \frac{dp_i}{\sqrt{p_i}} \delta(\sum_{i=1}^3 p_i - P) \phi_3(p_1, p_2, p_3) a^\dagger(p_1) a^\dagger(p_2) a^\dagger(p_3) |0\rangle , \]

where \( P \) is the eigenvalue of the momentum operator \( P^+ \). Note that \( \phi_2(p_1, p_2) \) and \( \phi_3(p_1, p_2, p_3) \) are totally symmetric functions of the arguments.

The boson creation operator can be expressed in terms of the fermion and antifermion creation and annihilation operators (see Ref. [9] for the notation),

\[
a^\dagger(p) = \int_0^p \frac{dk}{(2\pi)^{1/2}k} b^\dagger(k) d^\dagger(p - k) \\
+ \int_0^\infty \frac{dk}{(2\pi)^{1/2}k(p + k)} [b^\dagger(p + k) b(k) - d^\dagger(p + k) d(k)]. \quad (2.9)
\]

It is clear that the symmetry under \( \phi \to -\phi \) (or \( a^\dagger \to -a^\dagger \)) is the charge conjugation symmetry of the fermionic theory. Because of this symmetry, a state has a definite transformation property,

\[
|\phi\rangle_P^e \to |\phi\rangle_P^e , \quad |\phi\rangle_P^o \to -|\phi\rangle_P^o . \quad (2.10)
\]

It is easy to see that \( |\phi_1\rangle_P \) and \( |\phi_3\rangle_P \) are odd while \( |\phi_2\rangle_P \) is even.

We rescale momenta as \( p_i \to x_i = p_i/P \), and the wave functions, \( \phi_2(p_1, p_2) \) and \( \phi_3(p_1, p_2, p_3) \), are replaced by \( \phi_2(x_1, x_2) \) and \( \phi_3(x_1, x_2, x_3)/\sqrt{P} \), respectively. The Einstein-Schrödinger equation leads to two sets of eigenvalue equations for the wave functions, according to the transformation property under the charge conjugation. The even one involves only \( \phi_2 \):

\[
M^2 \phi_2(x_1, x_2) = \tilde{\mu}^2 \left( \frac{1}{x_1} + \frac{1}{x_2} \right) \phi_2(x_1, x_2) - cm(2\pi) \int_0^1 \frac{dy_1 dy_2}{\sqrt{y_1 y_2}} \delta(y_1 + y_2 - 1) \frac{\phi_2(y_1, y_2)}{\sqrt{x_1 x_2}} , \quad (2.11)
\]

with \( x_1 + x_2 = 1 \), while the odd one is the coupled equations for \( \phi_1 \) and \( \phi_3 \):
\[ M^2 \phi_1 = \bar{\mu}^2 \phi_1 - cm(2\pi) \frac{2}{\sqrt{3!}} \int_0^1 \prod_{i=1}^3 \frac{dy_i}{\sqrt{y_i}} \delta \left( \sum_{i=1}^3 y_i - 1 \right) \phi_3(y_1, y_2, y_3) , \quad (2.12) \]

\[ M^2 \phi_3(x_1, x_2, x_3) = \bar{\mu}^2 \sum_{i=1}^3 \frac{1}{x_i} \phi_3(x_1, x_2, x_3) \]

\[-cm(2\pi) \int_0^1 \frac{dy_1 dy_2}{\sqrt{y_1 y_2}} \left[ \delta(y_1 + y_2 + x_1 - 1) \frac{\phi_3(y_1, y_2, x_1)}{\sqrt{x_2 x_3}} + \delta(y_1 + y_2 + x_2 - 1) \frac{\phi_3(y_1, y_2, x_2)}{\sqrt{x_3 x_1}} + \delta(y_1 + y_2 + x_3 - 1) \frac{\phi_3(y_1, y_2, x_3)}{\sqrt{x_1 x_2}} \right] \]

\[ + cm(2\pi) \frac{2}{3!} \int_0^1 \prod_{i=1}^3 \frac{dy_i}{\sqrt{y_i}} \delta \left( \sum_{i=1}^3 y_i - 1 \right) \frac{\phi_3(y_1, y_2, y_3)}{\sqrt{x_1 x_2 x_3}} \]

\[-cm(2\pi) \frac{2}{\sqrt{3!}} \frac{\phi_1}{\sqrt{x_1 x_2 x_3}} , \quad (2.13)\]

with \( x_1 + x_2 + x_3 = 1 \). Note that because the even sector \((2.11)\) does not depend on \( V_6 \), the Hamiltonian is not bounded from below. We will see, however, that we get reasonable results if we do not employ many basis functions (see below) and are confined in the very strong coupling region.

These complicated equations are converted to two matrix eigenvalue equations by expanding the wave functions in terms of basis functions. The choice of the basis functions is very important for efficient numerical work. We choose the following basis functions so that we can calculate the matrix elements analytically.

\[ \phi_2(x_1, x_2) = \sum_l b_l (x_1 x_2)^{l+1/2} , \quad (2.14) \]

\[ \phi_3(x_1, x_2, x_3) = \sum_l c_l (x_1 x_2 x_3)^{l_1+1/2} (x_1^{l_2} + x_2^{l_2} + x_3^{l_2}) , \quad (2.15) \]

where \( l = 0, 1, 2, \cdots, N_1 \), \( l_1 = 0, 1, 2, \cdots, N_2 \), and \( l_2 = 0, 2, \cdots, 2N_3 \). It is easy to see that any symmetric polynomial in \( x_1, x_2, x_3 \) with the constraint \( x_1 + x_2 + x_3 = 1 \) can be expressed by using the above basis functions (up to \( (x_1 x_2 x_3)^{1/2} \) \( (2.15) \)).

**B. States in terms of fermion variables**

How are those states expressed in terms of fermion operators? By using \( (2.9) \), it is straightforward to express \( |\phi_2\rangle_P \) as follows,
\[ |\phi_2\rangle_p = |\psi_2^{(2)}\rangle_p + |\psi_4^{(2)}\rangle_p , \]
\[ |\psi_2^{(2)}\rangle_p = \int_0^P \frac{dk_1 dk_2}{2\pi \sqrt{k_1 k_2}} \delta(k_1 + k_2 - \mathcal{P}) \psi_2^{(2)}(k_1, k_2) b_1^\dagger b_2^\dagger |0\rangle , \tag{2.16} \]
\[ |\psi_4^{(2)}\rangle_p = \frac{1}{2} \int_0^P \prod_{i=1}^4 \frac{dk_i}{\sqrt{2\pi k_i}} \delta\left(\sum_{i=1}^4 k_i - \mathcal{P}\right) \psi_4^{(2)}(k_1, k_2; k_3, k_4) b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger |0\rangle , \]
where
\[ \psi_2^{(2)}(k_1, k_2) = \frac{1}{\sqrt{2}} \left[ \int_0^{k_1} dq \frac{\phi_2(k_1 - q, k_2 + q)}{\sqrt{(k_1 - q)(k_2 + q)}} - (k_1 \leftrightarrow k_2) \right] , \tag{2.17} \]
\[ \psi_4^{(2)}(k_1, k_2; k_3, k_4) = \frac{1}{\sqrt{2}} \left[ \frac{\phi_2(k_1 + k_3, k_2 + k_4)}{\sqrt{(k_1 + k_3)(k_2 + k_4)}} - (k_1 \leftrightarrow k_2) \right] . \tag{2.18} \]

In a similar way, \( |\phi_3\rangle_p \) is expressed as
\[ |\phi_3\rangle_p = |\psi_2^{(3)}\rangle_p + |\psi_4^{(3)}\rangle_p + |\psi_6^{(3)}\rangle_p , \]
\[ |\psi_2^{(3)}\rangle_p = \int_0^P \frac{dk_1 dk_2}{2\pi \sqrt{k_1 k_2}} \delta(k_1 + k_2 - \mathcal{P}) \psi_2^{(3)}(k_1, k_2) b_1^\dagger d_2^\dagger |0\rangle , \tag{2.19} \]
\[ |\psi_4^{(3)}\rangle_p = \frac{1}{2} \int_0^P \prod_{i=1}^4 \frac{dk_i}{\sqrt{2\pi k_i}} \delta\left(\sum_{i=1}^4 k_i - \mathcal{P}\right) \psi_4^{(3)}(k_1, k_2; k_3, k_4) b_1^\dagger b_2^\dagger d_3^\dagger d_4^\dagger |0\rangle , \]
\[ |\psi_6^{(3)}\rangle_p = \frac{1}{3!} \int_0^P \prod_{i=1}^6 \frac{dk_i}{\sqrt{2\pi k_i}} \delta\left(\sum_{i=1}^6 k_i - \mathcal{P}\right) \psi_6^{(3)}(k_1, k_2; k_3; k_1, k_5, k_6) b_1^\dagger b_2^\dagger b_3^\dagger d_4^\dagger d_5^\dagger d_6^\dagger |0\rangle , \]
where
\[ \psi_2^{(3)}(k_1, k_2) = \frac{1}{\sqrt{3!}} \left[ \left( \int_0^{k_1} dl_1 \int_0^{l_1} dl_2 \frac{\phi_3(k_1 - l_1, l_1 - l_2, k_2 + l_2)}{\sqrt{(k_1 - l_1)(l_1 - l_2)(k_2 + l_2)}} \right. \right. \]
\[ \left. - \int_0^{k_1} dl_1 \int_0^{k_2} dl_2 \frac{\phi_3(k_1 - l_1, k_2 - l_2, l_1 + l_2)}{\sqrt{(k_1 - l_1)(k_2 - l_2)(l_1 + l_2)}} \right] + (k_1 \leftrightarrow k_2) \right] , \tag{2.20} \]
\[ \psi_4^{(3)}(k_1, k_2; k_3, k_4) = -\frac{\sqrt{3!}}{4} \left[ \int_{k_3}^{k_1} dt_1 \frac{\phi_3(l, k_1 + k_3 - l, k_2 + k_4)}{\sqrt{l(l + 1)(k_1 + k_3 - l)(k_2 + k_4)}} - (k_1 \leftrightarrow k_2) - (k_3 \leftrightarrow k_4) \right. \]
\[ \left. + (k_1 \leftrightarrow k_2, k_3 \leftrightarrow k_4) \right] , \tag{2.21} \]
and
\[ \psi_6^{(3)}(k_1, k_2, k_3; k_4, k_5, k_6) \]
\[ = -\frac{1}{\sqrt{3!}} \left[ \left( \frac{\phi_3(k_1 + k_4, k_2 + k_5, k_3 + k_6)}{\sqrt{(k_1 + k_4)(k_2 + k_5)(k_3 + k_6)}} - (k_2 \leftrightarrow k_3) \right) + \begin{pmatrix} k_1, k_2, k_3 \\ \text{cyclic} \end{pmatrix} \right] . \tag{2.22} \]
The even state $|\phi^e_p\rangle$ contains the fermion two- and four-body components:

$$
e_{p'}\langle \phi | \phi \rangle_p^e = \mathcal{P} \delta(p' - p) [W_2^e + W_4^e] ,$$

$$W_2^e = \int_0^1 dx_1 dx_2 \delta(x_1 + x_2 - 1) |\psi_2^{(2)}(x_1, x_2)|^2 ,$$

$$W_4^e = \int_0^1 \prod_{i=1}^4 dx_i \delta(\sum_{i=1}^4 x_i - 1) |\psi_4^{(2)}(x_1, x_2, x_3, x_4)|^2 ,$$

and the odd state $|\phi^o_p\rangle$ contains the fermion two-, four- and six-body components:

$$\mathcal{P} \langle \phi | \phi \rangle_p^o = \mathcal{P} \delta(p' - p) [W_2^o + W_4^o + W_6^o] ,$$

$$W_2^o = |\phi_1|^2 + \int_0^1 dx_1 dx_2 \delta(x_1 + x_2 - 1) |\psi_2^{(3)}(x_1, x_2)|^2 ,$$

$$W_4^o = \int_0^1 \prod_{i=1}^4 dx_i \delta(\sum_{i=1}^4 x_i - 1) |\psi_4^{(3)}(x_1, x_2, x_3, x_4)|^2 ,$$

$$W_6^o = \int_0^1 \prod_{i=1}^6 dx_6 \delta(\sum_{i=1}^6 x_i - 1) |\psi_6^{(3)}(x_1, x_2, x_3, x_4, x_5, x_6)|^2 .$$

Once we numerically obtain the eigenstates of the Einstein-Schrödinger equation, i.e., the basis function expansion coefficients of the boson wave functions, (2.14) and (2.15), we can calculate all of these integrals analytically so that we can obtain fermion wave functions. This virtue comes from our clever choice of the simple set of basis functions.

We normalize states in a Lorentz invariant way, $\mathcal{P} \langle \phi | \phi \rangle_p = \mathcal{P} \delta(p' - p)$, so that $W_2^e + W_4^o = W_2^o + W_4^o + W_6^o = 1$. By saying that a state has a 50.000% fermion two-body component, we mean that the state has $W_2 = 0.50000$.

C. Numerical Results

1. mass spectrum

We calculate the invariant masses $M$ for various values of the fermion mass $m$. We find that convergence is good enough for $N_1 = 5, N_2 = 4,$ and $N_3 = 1$ (total number of basis functions is 16). The mass spectrum for $0.001 \leq m \leq 0.05$ is shown in Fig. [4]. In the following, we concentrate on the case $m = 0.01$.  

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The lowest state is the meson state. It is charge conjugation odd. It has 99.997% one-boson component and 0.003% three-boson component. Its mass is 1.0174.

The second lowest one should be the two-meson bound state, though its mass appears slightly above the threshold, \( M = 2.0468 \). It is charge conjugation even. Because of the present approximation, its two-boson component is 100%. It may contain a four-boson component if we include four-boson states, but we expect that it is negligibly small in the strong coupling region.

The third and fourth states come from the charge conjugation even sector. These are regarded as two-meson scattering states. They are completely two-boson states because of the present approximation.

The fifth one is the three-meson bound state, though its mass also appears slightly above the threshold, \( M = 3.0766 \). It is charge conjugation odd. It has 0.000% one-boson component and 100.000% three-boson component.

The two-meson and three-meson bound states appear below the thresholds when the fermion mass gets larger. In the strong coupling region, they should appear just below the threshold. (In the strong coupling limit, they are just on the thresholds, not being bound.) It is therefore difficult to get them below the thresholds numerically.

2. Fermion components

How do these states look like if they are expressed in terms of the fermion variables? By using (2.23) and (2.24), we get the following results: (1) The meson state is perfectly a fermion two-body state (100.00%). (2) The two-meson bound state contains 45.92% fermion two-body component and 54.08% fermion four-body component. (Its fermion six-body component is 0% due to the present approximation.) (3) The three-meson bound state, which is of our main interest, contains 28.38% fermion two-body component and 63.87% fermion four-body component and 7.75% fermion six-body component.

For the comparison, we give our previous results [9]: (1) The meson state has \( M = \).
1.01813 and is perfectly a fermion two-body state (100.000%). (2) The two-meson bound state has $M = 2.05612$. It has 54.408% fermion two-body component and 45.592% fermion four-body component, with little fermion six-body component. (3) The three-meson bound state has $M = 3.10814$. It has 44.285% fermion two-body component, 53.123% fermion four-body component, and 2.592% fermion six-body component. Obviously these are consistent with the results obtained by using bosonization, except for the differences in the fermion two-body and four-body components of the three-meson bound state.

The most important result is the demonstration that the three-meson bound state, which is almost perfectly (100.000%) a three-boson state, has only 7.75% fermion six-body component. We think that this is a very strong support that the identification we made in the previous paper is correct.

3. Wave functions

It is interesting to see the wave function of the “relative motion” of the two-meson bound state (Fig. 2). It looks very similar to the one we obtained in the previous paper (Fig. 6 of Ref. [9]). We can also calculate the fermion two-body wave function of the two-meson bound state $\psi^{(2)}_2$ (Fig. 3) which should be compared with Fig. 7 of Ref. [9]. The results are completely consistent with the previous calculations.

We next try to figure out how the three-meson bound state looks like by examining the wave function. We show $|\phi_3|^2$ of the three-meson bound state in Fig. 4 (for $m = 0.001$) and Fig. 5 (for $m = 0.05$). Because the wave function in momentum space is spread for larger fermion masses, and has a sharp peak for smaller fermion masses, we have an intuitive picture that for strong couplings the three-meson bound state is loosely bound while for weak couplings it has a relatively compact form. We look for asymmetry which indicates that two of the three mesons are more closely bound than the third, but we are not able to find any.
III. SUMMARY AND DISCUSSIONS

We calculate the mass spectrum of the bosonized massive Schwinger model by using the LFTD approximation up to including three-boson states. We showed that the three-meson bound state, which is almost 100% a three-boson state, has only a few percent fermion six-body component, and the result is consistent with our previous LFTD calculations in terms of the fermion variables. We also show that the other quantities are also consistent with the previous calculations. By obtaining the wave function of the three-meson bound state we are able to have an intuitive picture of the three-meson bound state, namely, it is loosely bound for strong couplings while it has a relatively compact shape for small couplings.

In our present approximation, the two-meson bound state cannot have a non-zero fermion six-body component. Of course, if we include four-boson states, it can have a non-zero six-body component. We however expect that it will be negligibly small.

The LFTD approximation for the bosonized theory is good only for strong couplings. There are three reasons: (1) The Fock space of the bosonized theory is that of a structureless boson, so that we neglect the internal structure of the meson as the first approximation. As is known from previous calculations, it is not a good description of the meson to ignore the internal structure already at $m = 0.1$. Note that the nontrivial momentum dependence of the fermion two-body wave function of the meson, $\psi_2$, comes from the many-boson components, when expanded in terms of fermion variables. (2) The Hamiltonian for the charge conjugation even sector is not bounded from below. (Of course this is because of the Tamm-Dancoff truncation.) Because of that we observed instabilities when we increase the number of basis functions or when we increase the fermion mass. For small number of basis functions and small values of the fermion mass, our method does not scan the ‘high energy’ states and the low energy states are insensitive to the unboundedness of the upside-down double well potential. Our choice of basis functions might also suppress the ‘decay’ of the states. (3) The normal-ordering problem becomes serious for weaker couplings. Our previous calculation [9] shows that the meson mass depends on the fermion mass almost
linearly for a wide range of the fermion mass. But the equation (2.12) shows, according to the variational principle, that the meson mass must be smaller than \( \tilde{\mu} = \sqrt{1 + 4\pi cm} \), which has the linear dependence on the fermion mass only in the strong coupling regions. For weaker fermion masses, this appears to put a stringent “upper bound.” Of course this should not be true. (The structureless one-boson approximation to the meson cannot be better than that including up to fermion six-body states, especially for weak couplings.) The “coupling” \( c \) must be renormalized and must have a nontrivial fermion mass dependence. The investigation in this direction is now in progress [15]. For the purpose of the present paper, however, it is sufficient to notice that the renormalization effects are negligibly small for small fermion masses. The fermion mass dependence is almost identical to that of our previous calculation.

Another pathology of the bosonized model can be seen in the wave function of the meson state. It is known that the fermion two-body wave function of the meson state, \( \psi_2(x, 1-x) \equiv \phi_1 + \psi_2^{(3)}(x, 1-x) \), must vanish at \( x = 0 \) and \( x = 1 \) in the massive theory [16]. The calculated wave function shown in Fig. 6 does not satisfy this requirement. Furthermore, it is not a concave function of \( x \). (See Fig. 5 of Ref. [9], for example.) We do not know why it does not have the correct behavior.

It would be interesting to use the bosonized theory to investigate the effects of the theta vacuum [12], though the investigation in the original fermionic theory requires an intensive study [17].

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FIG. 1. The mass spectrum for $0.001 \leq m \leq 0.05$, at $N_1 = 5, N_2 = 4$, and $N_3 = 1$. The dashed and dotted line stand for the two-meson and three-meson thresholds respectively.
FIG. 2. Squared wave function of the relative motion of the two-meson bound state, $|\phi_2(x, 1-x)|^2$, is shown for various values of the fermion mass. This should be compared with Fig. 6 of our previous work.

FIG. 3. Fermion two-body wave function for the two-meson bound state, $\psi_2^{(2)}(x, 1-x)$, is shown (solid line) with that obtained by the previous calculation in terms of fermion variables (dashed line) for $m = 0.01$. 
FIG. 4. Squared wave function of the three-meson bound state, $|\phi_3(x_1, x_2, x_3)|^2$, for $m = 0.001$ projected on the $x_1$-$x_2$ plane. Only the region $(x_1 > 0, x_2 > 0, x_1 + x_2 < 1)$ is the support.

FIG. 5. Squared wave function of the three-meson bound state, $|\phi_3(x_1, x_2, x_3)|^2$, for $m = 0.05$ projected on the $x_1$-$x_2$ plane. Only the region $(x_1 > 0, x_2 > 0, x_1 + x_2 < 1)$ is the support.
FIG. 6. Fermion two-body wave function of the meson, $|\psi_2(x, 1-x)|^2$ is shown for various values of the fermion mass, where $\psi_2 \equiv \phi_1 + \psi_2^{(3)}$. This should be compared with Fig. 5 of our previous work.