On the generalized membership problem in relatively hyperbolic groups

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Abstract

The aim of this short note is to provide a proof of the decidability of the generalized membership problem for relatively quasi-convex subgroups of finitely presented relatively hyperbolic groups, under some reasonably mild conditions on the peripheral structure of these groups. These hypotheses are satisfied, in particular, by toral relatively hyperbolic groups.

Let \( G = \langle A \mid R \rangle \) be a finitely presented group and let \( \mathcal{P} \) be a finite collection of finitely generated subgroups of \( G \), called the peripheral subgroups of \( G \). There are several definitions of \( G \) being relatively hyperbolic with respect to the peripheral structure \( \mathcal{P} \), due to Gromov [7], Farb [6], Bowditch [2], Drutu and Sapir [5], Osin [12], which turn out to be equivalent (see Bumagin [3], Dahmani [4], Hruska [8, Theorem 5.1]). We refer to the literature for details [12, 8].

Let \( H \) be a subgroup of \( G \). There are also several definitions of relative quasi-convexity for \( H \), in terms of natural geometries on \( G \). Again, these are equivalent (Hruska [8]) and we refer to the literature for precise definitions.

Certain subclasses of relatively quasi-convex subgroups have been considered in the literature, that are defined in terms of the parabolic subgroups of \( H \), that is, the subgroups that are contained in a conjugate of a peripheral subgroup \( P \in \mathcal{P} \): \( H \) is peripherally finite if every \( H \cap P^x \) \( (P \in \mathcal{P}, x \in G) \) is finite [1]. \( H \) has peripherally finite index if every infinite \( H \cap P^x \) has finite index in \( P^x \) [2]. Such subgroups are always finitely generated (Osin [12], Thms 4.13 and 4.16) for the peripherally finite case, Kharlampovich et al. [9] for the peripherally finite index case.

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1These subgroups are called strongly quasi-convex in [12], and differ from the strongly quasi-convex subgroups of Tran [13].
2These subgroups are called fully quasi-convex in [10].
The problem we consider here is the so-called generalized membership problem: given a tuple $h_1, \ldots, h_k \in F(A)$ and letting $H$ be the subgroup they generate in $G$ (that is: $H$ is the subgroup of $G$ generated by the images of the $h_i$ in $G$), given an additional element $g \in G$ (in the form of a word in $F(A)$), decide whether $g \in H$.

Stated as above, the generalized membership problem is known to be undecidable without strong assumptions on the group $G$. Even in the relatively simple case of the direct product $F_2 \times F_2$, there are finitely generated subgroups with undecidable membership problem (see Mihailova’s subgroup [11]). So we are looking for a partial algorithm in the following sense: an algorithm which may not stop on all instances but which will stop at least on the instances when $H$ is relatively quasi-convex, and which will decide whether $g \in H$ when it stops.

Even so, we need to impose some conditions on $G$.

**Assumptions (Hyp)**

(H1) Each group $P \in \mathcal{P}$ satisfies the following: we are given a geodesically bi-automatic structure for $P$, on an alphabet $X_P$ and with language of representatives $L_P$, and we can compute a geodesically bi-automatic structure on every finite generating set of $P$ (given as a subset of $F(X_P)$).

(H2) The groups in $\mathcal{P}$ are slender (a.k.a. noetherian: every one of their subgroups is finitely generated) and LERF.

(H3) For each $P \in \mathcal{P}$, the set of tuples of words in $L_P$ that generate a finite index subgroup of $P$ is recursively enumerable.

(H4) We can solve the generalized membership problem in each $P \in \mathcal{P}$.

**Remark 1.** Hruska showed that every relatively quasi-convex subgroup of $G$ is finitely generated, if and only if every group in $\mathcal{P}$ is slender [8, Cor. 9.2], so (H2) is a reasonable hypothesis to make in this algorithmic context.

We note that (Hyp) is satisfied in particular if the peripheral structure consists of finitely generated abelian groups, and notably, if $G$ is toral relatively hyperbolic (that is: $G$ is torsion free and the peripheral structure $\mathcal{P}$ consists of non-cyclic free abelian groups).

We can now state the central result of this note.

**Theorem 2.** Let $G = \langle A \mid R \rangle$ be a finitely presented group, relatively hyperbolic with respect to the peripheral structure $\mathcal{P}$, and satisfying (Hyp). There is a partial algorithm which, given $g, h_1, \ldots, h_k \in F(A)$,

- halts at least if $g \in H$ or if the subgroup $H$ of $G$ generated by the $h_i$ is relatively quasi-convex and $g \not\in H$;
- when it halts, decides whether $g \in H$.

The algorithm in Theorem 2 is “impractical” in the following sense: there is no function bounding the time required for the algorithm to stop (if it will stop). It consists in two semi-algorithms, meant to be run concurrently, until one of them halts: one trying to witness the fact that $g \in H$ and the other trying to witness the opposite fact.

The rest of this note consists in the description of both these semi-algorithms.
Semi-algorithm to verify that $g \in H$ It is a classical result that, given the presentation $\langle A \mid R \rangle$ for $G$ and given a word $g \in F(A)$, there is a partial algorithm which will halt exactly if $g = 1$ in $G$. Indeed, $g = 1$ in $G$ if and only if a sequence of $R$-rewritings eventually leads to the empty word. A systematic exploration of the $R$-rewritings of $g$ will eventually uncover this sequence if $g = 1$ in $G$.

This semi-algorithm is naturally extended to the problem at hand (does $g$ belong to $H$?) as follows. One starts with the Stallings graph $\Gamma$ of the subgroup of $F(A)$ generated by the $h_i$ (see [14]), and iteratively:

- modify $\Gamma$ by gluing at every vertex a loop labeled by $r$ for every relator $r \in R$;
- fold $\Gamma$ (this is the central step of the construction of Stallings graphs: it consists in identifying vertices $p$ and $q$ each time that there are edges labeled by a letter $a \in A$ from some vertex $s$ to both $p$ and $q$, or edges labeled by a letter $a$ from both $p$ and $q$ to some vertex $s$);
- check whether $g$ labels a loop at the base vertex of $\Gamma$. If that is the case, then $g \in H$ and we stop the algorithm. If not, repeat.

A detailed discussion of this semi-algorithm can be found in [9, Section 4.1].

Semi-algorithm to verify that $g \notin H$ We call a subgroup of the form $H \cap P^x$ ($P \in \mathcal{P}$, $x \in G$) which is infinite, a maximal infinite parabolic subgroup of $H$. Our semi-algorithm relies on the following results.

[H] Hruska shows [8, Theorem 9.1] that, if $H$ is relatively quasi-convex, then there exists a finite collection of maximal infinite parabolic subgroups $\{K_i\}_{1 \leq i \leq \ell}$ such that every infinite maximal parabolic subgroup of $H$ is conjugated in $H$ to one of the $K_i$.

[MMP] Manning and Martínez-Pedroza show the following, under Hypothesis (H2) [10, Theorem 1.7]. Suppose that $H \leq G$ is relatively quasi-convex, $\{K_i\}_{1 \leq i \leq \ell}$ is a collection of subgroups as in [H], say with $K_i = H \cap P_{x_i}$, $1 \leq i \leq \ell$, $P_i \in \mathcal{P}$, $x_i \in G$ and $g \notin H$. Then there exist subgroups $R_i \leq P_{x_i}$ such that $R_i$ has finite index in $P_{x_i}$, $K_i \leq R_i$ and, if $K$ is generated by $H$ and the $R_i$, then $g \notin K$ and $K$ has peripherally finite index. Note that [10, Theorem 1.7] is a little more concise than this statement, which is extracted from the proof in that paper ([10, p. 319]).

[AC] Antolin and Ciobanu [1] show that, under Hypothesis (H1), one can compute an automatic structure for $G$, with alphabet $X$ containing $A$ and the $X_P$ ($P \in \mathcal{P}$), whose language $L$ of representatives consists only of geodesics (on alphabet $X$) and contains the $L_P$ ($P \in \mathcal{P}$), and satisfying additional properties.

[KhMW] Kharlampovich et al. [9, Sec. 7] build on [AC] to show that, if $H \leq G$ is relatively quasi-convex (with respect to alphabet $A$) and has peripherally finite index, then it is $L$-quasi-convex with respect to alphabet $X$ [9, Thm 7.5]. The proof of that theorem uses Hypothesis (H4). Moreover, they give a partial algorithm which, on input $h_1, \ldots, h_k \in F(A)$, (a) halts in particular on

\[3\text{That is: there exists } \delta > 0 \text{ such that every } L\text{-representative of an element of } H \text{ stays within distance } \delta \text{ of } H, \text{ in the } X\text{-Cayley graph of } G.\]
inputs such that the subgroup generated by the $h_i$ is relatively quasi-convex and has peripherally finite index (it may halt in other cases as well); and (b) when it stops, outputs a Stallings graph $(\Gamma, 1)$ for $H$ on alphabet $X$ (Cor. 7.9). Here, a Stallings graph for $H$ is a finite $X$-labeled graph $\Gamma$ with a distinguished base vertex $v_0$ such that a word $w$ in $L$ (the language of representatives for the elements of $G$) labels a loop in $\Gamma$ at $v_0$ if and only if $w$ represents an element of $H$.

Note that this solves the membership problem for $H$: given $g \in G$ (in the form of a word in $F(A)$), we use the automatic structure produced by [AC] to find an $L$-representative $w_g$ of $g$, and then verify whether $w_g$ labels a loop at vertex $v_0$ in $\Gamma$.

We can now give our semi-algorithm. For clarity, we give it as a non-deterministic partial algorithm. Such a non-deterministic algorithm can be turned into a deterministic one by standard methods (see, e.g., [13, Thm 3.16]).

(1) We first apply [AC] to compute an automatic structure for $G$ on generator set $X$ (using Hypothesis (H1)). Then we compute a finite presentation of $G$ on $X$, say $\langle X | R_X \rangle$. For instance, $R_X$ consists of $R$, the relators $xu_x^{-1}$, where $x \in X \setminus A$ and $u_x$ is a fixed element of $F(A)$ such that $x = u_x$ in $G$, and all the cyclic permutations of these relators and their inverses.

The words $u_x$ can be computed as follows. Since the automatic structure for $G$ allows us to solve the word problem, one systematically checks whether $xu_x^{-1}$ is trivial, when $u$ runs through $F(A)$. As $G$ is $A$-generated, some $u \in F(A)$ is equal to $x$ in $G$.

(2) Choose non-deterministically a tuple $\vec{x} = (x_1, \ldots, x_{\ell})$ of elements of $F(A)$; for each $1 \leq i \leq \ell$, choose non-deterministically an element $P_i \in \mathcal{P}$ and a tuple $\vec{g}_i$ of elements of $F(X_{P_i})$ generating a finite index subgroup of $P_i$ (this is possible under Hypothesis (H3)).

(3) For this choice of $\vec{x}$ and the $\vec{g}_i$ ($1 \leq i \leq \ell$), let $H_1 = \langle H \cup \bigcup_{i=1}^{\ell} \vec{g}_i X \rangle$. Run the partial algorithm [KhMW] to decide whether $g \in H_1$ (using Hypothesis (H4)).

Result [MMP] (which assumes Hypothesis (H2)), shows that, if $g \notin H$ and $H$ is relatively quasi-convex, then for an appropriate choice of $\vec{x}$ and the $\vec{g}_i$, $H_1$ is relatively quasi-convex and has peripherally finite index, and $g \notin H_1$. As $H_1$ has peripherally finite index, the partial algorithm in Step (3) will halt and certify that $g \notin H_1$, and hence that $g \notin H$ since $H \leq H_1$.

Summarizing: if $g \notin H$ and $H$ is relatively quasi-convex, then one of the non-deterministic choices in Step (2) will be such that the partial algorithm halts and states that $g \notin H$. This completes the proof of Theorem 2.

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