When is the Bloch–Okounkov $q$-bracket modular?

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Abstract
We obtain a condition describing when the quasimodular forms given by the Bloch–Okounkov theorem as $q$-brackets of certain functions on partitions are actually modular. This condition involves the kernel of an operator $\Delta_1$. We describe an explicit basis for this kernel, which is very similar to the space of classical harmonic polynomials.

Keywords Modular forms · Partitions · Harmonic polynomials

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1 Introduction

Given a family of quasimodular forms, the question which of its members are modular often has an interesting answer. For example, consider the family of theta series

$$\theta_P(\tau) = \sum_{x \in \mathbb{Z}^r} P(x)q^{x_1^2 + \ldots + x_r^2} \quad (q = e^{2\pi i \tau})$$

given by all homogeneous polynomials $P \in \mathbb{Z}[x_1, \ldots, x_r]$. The quasimodular form $\theta_P$ is modular if and only if $P$ is harmonic (i.e. $P \in \ker \sum_{i=1}^r \frac{\partial^2}{\partial x_i^2}$) [10]. (As quasimodular forms were not yet defined, Schoeneberg only showed that $\theta_P$ is modular if $P$ is harmonic. However, for every polynomial $P$ it follows that $\theta_P$ is quasimodular by decomposing $P$ as in Formula (1).) Also, for every two modular forms $f, g$, one can consider the linear combination of products of derivatives of $f$ and $g$ given by

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This linear combination is a quasimodal form which is modular precisely if it is a multiple of the Rankin–Cohen bracket \([f, g]_n\) [4, 9]. In this paper, we provide a condition to decide which member of the family of quasimodular forms provided by the Bloch–Okounkov theorem is modular. Let \(\mathcal{P}\) denote the set of all partitions of integers and \(|\lambda|\) denote the integer that \(\lambda\) is a partition of. Given a function \(f : \mathcal{P} \to \mathbb{Q}\), define the \(q\)-bracket of \(f\) by

\[
\langle f \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}}.
\]

The celebrated Bloch–Okounkov theorem states that for a certain family of functions \(f : \mathcal{P} \to \mathbb{Q}\) (called shifted symmetric polynomials and defined in Sect. 2) the \(q\)-brackets \(\langle f \rangle_q\) are the \(q\)-expansions of quasimodular forms [2].

Besides being a wonderful result, the Bloch–Okounkov theorem has many applications in enumerative geometry. For example, a special case of the Bloch–Okounkov theorem was discovered by Dijkgraaf and provided with a mathematically rigorous proof by Kaneko and Zagier, implying that the generating series of simple Hurwitz numbers over a torus are quasimodular [5, 7]. Also, in the computation of asymptotics of geometrical invariants, such as volumes of moduli spaces of holomorphic differentials and Siegel–Veech constants, the Bloch–Okounkov theorem is applied [3, 6].

Zagier gave a surprisingly short and elementary proof of the Bloch–Okounkov theorem [13]. A corollary of his work, which we discuss in Sect. 3, is the following proposition:

**Proposition 1** There exists actions of the Lie algebra \(\mathfrak{sl}_2\) on both the algebra of shifted symmetric polynomials \(\Lambda^*\) and the algebra of quasimodular forms \(\tilde{M}\) such that the \(q\)-bracket \(\langle \cdot \rangle_q : \Lambda^* \to \tilde{M}\) is \(\mathfrak{sl}_2\)-equivariant.

The answer to the question in the title is provided by one of the operators \(\Delta\) which defines this \(\mathfrak{sl}_2\)-action on \(\Lambda^*\). Namely letting \(\mathcal{H} = \ker \Delta|_{\Lambda^*}\), we prove the following theorem:

**Theorem 1** Let \(f \in \Lambda^*\). Then \(\langle f \rangle_q\) is modular if and only if \(f = h + k\) with \(h \in \mathcal{H}\) and \(k \in \ker \langle \cdot \rangle_q\).

The last section of this article is devoted to describing the graded algebra \(\mathcal{H}\). We call \(\mathcal{H}\) the space of *shifted symmetric harmonic polynomials*, as the description of this space turns out to be very similar to the space of classical harmonic polynomials. Let \(\mathcal{P}_d\) be the space of polynomials of degree \(d\) in \(m \geq 3\) variables \(x_1, \ldots, x_m\), let \(||x||^2 = \sum_i x_i^2\), and recall that the space \(\mathcal{H}_d\) of degree \(d\) harmonic polynomials is given by \(\ker \sum_{i=1}^r \frac{\partial^2}{\partial x_i^2}\). The main theorem of harmonic polynomials states that every polynomial \(P \in \mathcal{P}_d\) can uniquely be written in the form

\[
P = h_0 + ||x||^2 h_1 + \ldots + ||x||^{2d} h_d
\] (1)
with \( h_i \in \mathcal{H}_{d-2i} \) and \( d' = \lfloor d/2 \rfloor \). Define \( K \), the Kelvin transform, and \( D^\alpha \) for \( \alpha \) an \( m \)-tuple of non-negative integers by

\[
 f(x) \mapsto ||x||^{2-m} f \left( \frac{x}{||x||^2} \right) \quad \text{and} \quad D^\alpha = \prod_i \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}.
\]

An explicit basis for \( \mathcal{H}_d \) is given by

\[
 \{ KD^\alpha K(1) \mid \alpha \in \mathbb{Z}^m_{\geq 0}, \sum_i \alpha_i = d, \alpha_1 \leq 1 \},
\]

see for example [1]. We prove the following analogous results for the space of shifted symmetric polynomials:

**Theorem 2** For every \( f \in \Lambda^* \) there exists unique \( h_i \in \mathcal{H}_{n-2i} \) (\( i = 0, 1, \ldots, n' \) and \( n' = \lfloor n/2 \rfloor \)) such that

\[
 f = h_0 + Q_2 h_1 + \ldots + Q_{n'} h_{n'},
\]

where \( Q_2 \) is an element of \( \Lambda^* \) given by \( Q_2(\lambda) = |\lambda| - \frac{1}{24} \).

**Theorem 3** The set

\[
 \{ \text{pr}_K \Delta_\lambda K(1) \mid \lambda \in \mathcal{P}(n), \text{all parts are } \geq 3 \}
\]

is a vector space basis of \( \mathcal{H}_n \), where \( \text{pr} \), \( K \), and \( \Delta_\lambda \) are defined by (4), Definition 4, respectively, Definition 6.

The action of \( \mathfrak{sl}_2 \) given by Proposition 1 makes \( \Lambda^* \) into an infinite-dimensional \( \mathfrak{sl}_2 \)-representation for which the elements of \( \mathcal{H} \) are the lowest weight vectors. Theorem 2 is equivalent to the statement that \( \Lambda^* \) is a direct sum of the (not necessarily irreducible) lowest weight modules

\[
 V_n = \bigoplus_{m=0}^{\infty} Q_m^2 \mathcal{H}_n \quad (n \in \mathbb{Z}).
\]

### 2 Shifted symmetric polynomials

Shifted symmetric polynomials were introduced by Okounkov and Olshanski as the following analogue of symmetric polynomials [8]. Let \( \Lambda^*(m) \) be the space of rational polynomials in \( m \) variables \( x_1, \ldots, x_m \) which are shifted symmetric, i.e. invariant under the action of all \( \sigma \in \mathfrak{S}_m \) given by \( x_i \mapsto x_{\sigma(i)} + i - \sigma(i) \) (or more symmetrically \( x_i - i \mapsto x_{\sigma(i)} - \sigma(i) \)). Note that \( \Lambda^*(m) \) is filtered by the degree of the polynomials. We have forgetful maps \( \Lambda^*(m) \to \Lambda^*(m-1) \) given by \( x_m \mapsto 0 \), so that we can define the space of shifted symmetric polynomials \( \Lambda^* \) as \( \varprojlim \Lambda^*(m) \) in the category of
filtered algebras. Considering a partition $\lambda$ as a non-increasing sequence $(\lambda_1, \lambda_2, \ldots)$ of non-negative integers $\lambda_i$, we can interpret $\Lambda^*$ as being a subspace of all functions $\mathcal{P} \to \mathbb{Q}$.

One can find a concrete basis for this abstractly defined space by considering the generating series

$$w_\lambda(T) := \sum_{i=1}^{\infty} T^{\lambda_1-i+\frac{1}{2}} \in T^{1/2} \mathbb{Z}[T][[T^{-1}]]$$

for every $\lambda \in \mathcal{P}$ (the constant $\frac{1}{2}$ turns out to be convenient for defining a grading on $\Lambda^*$). As $w_\lambda(T)$ converges for $T > 1$ and equals

$$\frac{1}{T^{1/2} - T^{-1/2}} + \sum_{i=1}^{\ell(\lambda)} \left( T^{\lambda_i-i+\frac{1}{2}} - T^{-i+\frac{1}{2}} \right)$$

one can define shifted symmetric polynomials $Q_i(\lambda)$ for $i \geq 0$ by

$$\sum_{i=0}^{\infty} Q_i(\lambda) z^{i-1} := w_\lambda(e^z) \quad (0 < |z| < 2\pi).$$

The first few shifted symmetric polynomials $Q_i$ are given by

$$Q_0(\lambda) = 1, \quad Q_1(\lambda) = 0, \quad Q_2(\lambda) = |\lambda| - \frac{1}{4}.$$

The $Q_i$ freely generate the algebra of shifted symmetric polynomials, i.e. $\Lambda^* = \mathbb{Q}[Q_2, Q_3, \ldots]$. It is believed that $\Lambda^*$ is maximal in the sense that for all $Q : \mathcal{P} \to \mathbb{Q}$ with $Q / \Lambda^*$ it holds that $\langle \Lambda^*[Q] \rangle_q \not\subseteq \tilde{M}$.

**Remark 1** The space $\Lambda^*$ can equally well be defined in terms of the Frobenius coordinates. Given a partition with Frobenius coordinates $(a_1, \ldots, a_r, b_1, \ldots, b_r)$, where $a_i$ and $b_j$ are the arm and leg lengths of the cells on the main diagonal, let

$$C_\lambda = \left\{-b_1 - \frac{1}{2}, \ldots, -b_r - \frac{1}{2}, a_r + \frac{1}{2}, \ldots, a_1 + \frac{1}{2}\right\}.$$

Then

$$Q_k(\lambda) = \beta_k + \frac{1}{(k-1)!} \sum_{c \in C_\lambda} \text{sgn}(c) c^{k-1},$$

where $\beta_k$ is the constant given by

$$\sum_{k \geq 0} \beta_k z^{k-1} = \frac{1}{2 \sinh(z/2)} = w_\emptyset(e^z).$$
We extend $\Lambda^*$ to an algebra where $Q_1 \neq 0$. Observe that a non-increasing sequence $(\lambda_1, \lambda_2, \ldots)$ of integers corresponds to a partition precisely if it converges to 0. If, however, it converges to an integer $n$, Eqs. (2) and (3) still define $Q_k(\lambda)$. In fact, in this case

$$Q_k(\lambda) = (e^{n\partial}) Q_k(\lambda - n)$$

by [13, Proposition 1] where $\partial Q_0 = 0$, $\partial Q_k = Q_{k-1}$ for $k \geq 1$, and $\lambda - n = (\lambda_1 - n, \lambda_2 - n, \ldots)$ corresponds to a partition (i.e. converges to 0). In particular, $Q_1(\lambda) = n$ equals the number the sequence $\lambda$ converges to. We now define the Bloch–Okounkov ring $\mathcal{R}$ to be $\Lambda^*[Q_1]$, considered as a subspace of all functions from non-increasing eventually constant sequences of integers to $\mathbb{Q}$. It is convenient to work with $\mathcal{R}$ instead of $\Lambda^*$ to define the differential operators $\Delta$ and more generally $\Delta_{\lambda}$ later. Both on $\Lambda^*$ and $\mathcal{R}$, we define a weight grading by assigning to $Q_i$ weight $i$.

Denote the projection map by

$$\text{pr} : \mathcal{R} \to \Lambda^*.$$  \hspace{1cm} (4)

We extend $\langle \cdot \rangle_q$ to $\mathcal{R}$.

The operator $E = \sum_{m=0}^{\infty} Q_m \frac{\partial}{\partial Q_m}$ on $\mathcal{R}$ multiplies an element of $\mathcal{R}$ by its weight. Moreover, we consider the differential operators

$$\partial = \sum_{m=0}^{\infty} Q_m \frac{\partial}{\partial Q_{m+1}} \quad \text{and} \quad \mathcal{D} = \sum_{k,\ell \geq 0} \binom{k+\ell}{k} Q_{k+\ell} \frac{\partial^2}{\partial Q_{k+1} \partial Q_{\ell+1}}.$$  

Let $\Delta = \frac{1}{2}(\mathcal{D} - \partial^2)$, i.e.

$$2\Delta = \sum_{k,\ell \geq 0} \binom{k+\ell}{k} Q_{k+\ell} - Q_k Q_{\ell} \frac{\partial^2}{\partial Q_{k+1} \partial Q_{\ell+1}} - \sum_{k \geq 0} Q_k \frac{\partial}{\partial Q_{k+2}}.$$  

In the following (antisymmetric) table, the entry in the row of operator $A$ and column of operator $B$ denotes the commutator $[A, B]$, for proofs see [13, Lemma 3].

| $\Delta$ | $\partial$ | $E$ | $Q_1$ | $Q_2$ |
|----------|------------|-----|-------|-------|
| $\Delta$ | 0          | 0   | $2\Delta$ | 0   |
| $\partial$ | 0         | 0   | $\partial$ | 1   |
| $E$ | $-2\Delta$ | $-\partial$ | 0   | $Q_1$ |
| $Q_1$ | 0          | $-1$ | $-Q_1$ | 0   |
| $Q_2$ | $-E + Q_1 \partial + \frac{1}{2}$ | $-Q_1$ | $-2Q_2$ | 0   |

**Definition 1** A triple $(X, Y, H)$ of operators is called an $sl_2$-triple if

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [Y, X] = H.$$
Let \( \hat{Q}_2 := Q_2 - \frac{1}{2} Q_1^2 \) and \( \hat{E} := E - Q_1 \partial - \frac{1}{2} \). The following result follows by a direct computation using the above table:

**Proposition 2** The operators \((\hat{Q}_2, \Delta, \hat{E})\) form an \(\mathfrak{sl}_2\)-triple. \(\Box\)

For later reference, we compute \([\Delta, Q^n_2]\). This could be done inductively by noting that \([\Delta, Q^n_2] = Q^n_2 - [\Delta, Q_2] + [\Delta, Q_2^{n-1}]Q_2\) and using the commutation relations in the above table. The proof below is a direct computation from the definition of \(\Delta\).

**Lemma 1** For all \(n \in \mathbb{N}\), the following relation holds

\[
[\Delta, Q^n_2] = -\frac{n(n-1)}{2} Q^n_2 Q_2^{n-2} - n Q_1 Q^n_2 \partial + n Q^n_2 (E + n - \frac{3}{2}).
\]

**Proof** Let \(f \in \mathbb{Q}[Q_1, Q_2], g \in \mathcal{R}, \) and \(n \in \mathbb{N}\). Then

\[
[\Delta, Q^n_2] = \frac{\partial f}{\partial Q_2} (Eg - Q_1 \partial g) + f \Delta(g),
\]

(5)

\[
[\Delta, Q^n_2] = n(n - \frac{3}{2}) Q^n_2 Q_2^{n-2} - \frac{n(n-1)}{2} Q^n_2 Q_2^{n-2}. \quad (6)
\]

By (5) and (6), we find

\[
[\Delta, Q^n_2] = (n(n - \frac{3}{2}) Q^n_2 Q_2^{n-2} - \frac{n(n-1)}{2} Q^n_2 Q_2^{n-2}) g
\]

\[
+ n Q^n_2 (Eg - Q_1 \partial g) + Q^n_2 \Delta(g). \quad \Box
\]

### 3 An \(\mathfrak{sl}_2\)-equivariant mapping

The space of quasimodular forms for \(\text{SL}_2(\mathbb{Z})\) is given by \(\tilde{M} = \mathbb{Q}[P, Q, R]\), where \(P, Q, \) and \(R\) are the Eisenstein series of weight 2, 4, and 6, respectively (in Ramanujan’s notation). We let \(\tilde{M}_k^{(\leq p)}\) be the space of quasimodular forms of weight \(k\) and depth \(\leq p\) (the depth of a quasimodular form written as a polynomial in \(P, Q, \) and \(R\) is the degree of this polynomial in \(P\)). See [12, Section 5.3] or [13, Section 2] for an introduction into quasimodular forms.

The space of quasimodular forms is closed under differentiation, more precisely the operators \(D = q \frac{d}{dq}, \partial = 12 \frac{d}{dP}, \) and the weight operator \(W\) given by \(Wf = k f\) for \(f \in \tilde{M}_k\) preserve \(\tilde{M}\) and form an \(\mathfrak{sl}_2\)-triple. In order to compute the action of \(D\) in terms of the generators \(P, Q, \) and \(R,\) one uses the Ramanujan identities

\[
D(P) = \frac{P^2 - Q}{12}, \quad D(Q) = \frac{PQ - R}{3}, \quad D(R) = \frac{PR - Q^2}{2}.
\]

In the context of the Bloch–Okounkov theorem, it is more natural to work with \(\hat{D} := D - \frac{P}{24},\) as for all \(f \in \Lambda^*\) one has \(\langle Q_2 f \rangle_q = \hat{D}(f)_q\). Moreover, \(\hat{D}\) has the property that it increases the depth of a quasimodular form by 1, in contrast to \(D\) for which \(D(1) = 0\) does not have depth 1:
Lemma 2  Let \( f \in \tilde{M} \) be of depth \( r \). Then \( \hat{D} f \) is of depth \( r + 1 \).

**Proof** Consider a monomial \( P^a Q^b R^c \) with \( a, b, c \in \mathbb{Z}_{\geq 0} \). By the Ramanujan identities, we find

\[
D(P^a Q^b R^c) = \left( \frac{a}{12} + \frac{b}{3} + \frac{c}{2} \right) P^{a+1} Q^b R^c + O(P^a),
\]

where \( O(P^a) \) denotes a quasimodular form of depth at most \( a \). The lemma follows by noting that \( \frac{a}{12} + \frac{b}{3} + \frac{c}{2} - \frac{1}{24} \) is non-zero for \( a, b, c \in \mathbb{Z} \).

Moreover, letting \( \hat{W} = W - \frac{1}{2} \), the triple \((\hat{D}, \partial, \hat{W})\) forms an \( \mathfrak{sl}_2 \)-triple as well.

Proposition 3 (The \( \mathfrak{sl}_2 \)-equivariant Bloch–Okounkov theorem) The mapping \( \langle \cdot \rangle_q : \mathcal{R} \to \tilde{M} \) is \( \mathfrak{sl}_2 \)-equivariant with respect to the \( \mathfrak{sl}_2 \)-triple \((\hat{Q}_2, \Delta, \hat{E})\) on \( \mathcal{R} \) and the \( \mathfrak{sl}_2 \)-triple \((\hat{D}, \partial, \hat{W})\) on \( \tilde{M} \), i.e. for all \( f \in \mathcal{R} \), one has

\[
\hat{D} \langle f \rangle_q = \langle \hat{Q}_2 f \rangle_q, \quad \partial \langle f \rangle_q = \langle \Delta f \rangle_q, \quad \hat{W} \langle f \rangle_q = \langle \hat{E} f \rangle_q.
\]

**Proof** This follows directly from [13, Equation (37)] and the fact that for all \( f \in \mathcal{R} \) one has \( \langle Q_1 f \rangle_q = 0 \).

§ 4 Describing the space of shifted symmetric harmonic polynomials

In this section, we study the kernel of \( \Delta \). As \( [\Delta, Q_1] = 0 \), we restrict ourselves without loss of generality to \( \Lambda^* \). Note, however, that \( \Delta \) does not act on \( \Lambda^* \) as, for example, \( \Delta(Q_3) = -\frac{1}{2} Q_1 \). However, \( \text{pr} \Delta \) does act on \( \Lambda^* \).

**Definition 2** Let

\[
\mathcal{H} = \{ f \in \Lambda^* \mid \Delta f \in Q_1 \mathcal{R} \} = \ker \text{pr} \Delta,
\]

be the space of shifted symmetric harmonic polynomials.

**Proposition 4** If \( f \in Q_2 \Lambda^* \) is non-zero, then \( f \not\in \mathcal{H} \).

**Proof** Write \( f = Q_2^n f' \) with \( f' \in \Lambda^* \) and \( f' \not\in Q_2 \Lambda^* \). Then

\[
\text{pr} \Delta(f) = Q_2^{n-1}(n(n+k-\frac{3}{2}) f' + Q_2 \text{pr} \Delta f')
\]

by Lemma 1. As \( f' \) is not divisible by \( Q_2 \), it follows that \( \text{pr} \Delta(f) = 0 \) precisely if \( f' = 0 \).

**Proposition 5** For all \( n \in \mathbb{Z} \), one has

\[
\Lambda_n^* = \mathcal{H}_n \oplus Q_2 \Lambda_{n-2}^*.
\]
Proof For uniqueness, suppose \( f = Q_2g + h \) and \( f = Q_2g' + h' \) with \( g, g' \in \Lambda_{n-2}^* \) and \( h, h' \in \mathcal{H}_n \). Then, \( Q_2(g - g') = h' - h \in \mathcal{H} \). By Proposition 4 we find \( g = g' \) and hence \( h = h' \).

Now, define the linear map \( T : \Lambda_n^* \rightarrow \Lambda_n^* \) by \( f \mapsto \text{pr}\Delta(Q_2f) \). By Proposition 4 we find that \( T \) is injective, which by finite dimensionality of \( \Lambda_n^* \) implies that \( T \) is surjective. Hence, given \( f \in \Lambda_n^* \) let \( g \in \Lambda_{n-2}^* \) be such that \( T(g) = \text{pr}\Delta(f) \in \Lambda_{n-2}^* \). Let \( h = f - Q_2g \). As \( f = Q_2g + h \), it suffices to show that \( h \in \mathcal{H} \). That holds true because \( \text{pr}\Delta(h) = \text{pr}\Delta(f) - \text{pr}\Delta(Q_2g) = 0 \).

Proposition 5 implies Theorem 2 and the following corollary. Denote by \( p(n) \) the number of partitions of \( n \).

Corollary 1 The dimension of \( \mathcal{H}_n \) equals the number of partitions of \( n \) in parts of size at least 3, i.e.

\[
\dim \mathcal{H}_n = p(n) - p(n-1) - p(n-2) + p(n-3).
\]

Proof Observe that \( \dim \Lambda_n^* \) equals the number of partitions of \( n \) in parts of size at least 2. Hence, \( \dim \Lambda_n^* = p(n) - p(n-1) \) and the Corollary follows from Proposition 5.

Proof of Theorem 1 If \( \langle f \rangle_q \) is modular, then \( \langle \Delta f \rangle_q = \partial \langle f \rangle_q = 0 \). Write \( f = \sum_{r=0}^{n'} Q_2^r h_r \) as in Theorem 2 with \( n' = \lfloor \frac{n}{2} \rfloor \). Then by Lemma 1 it follows that \( \text{pr}\Delta f = \sum_{r=0}^{n'} r(n - r - \frac{3}{2}) Q_2^{r-1} h_r \). Hence,

\[
\sum_{r=1}^{n'} r(n - r - \frac{3}{2}) \hat{D}^{r-1} \langle h_r \rangle_q = 0. \tag{7}
\]

As \( \langle h_r \rangle_q \) is modular, either it is equal to 0 or it has depth 0. Suppose the maximum \( m \) of all \( r \geq 1 \) such that \( \langle h_r \rangle_q \) is non-zero exists. Then, by Lemma 2 it follows that the left-hand side of (7) has depth \( m - 1 \), in particular is not equal to 0. So, \( h_1, \ldots, h_{n'} \in \ker \langle \cdot \rangle_q \). Note that \( f \in \ker \langle \cdot \rangle_q \) implies that \( Q_2 f \in \ker \langle \cdot \rangle_q \). Therefore, \( k := \sum_{r=1}^{n'} Q_2^r h_r \in \ker \langle \cdot \rangle_q \) and \( f = h + k \) with \( h = h_0 \) harmonic.

The converse follows directly as \( \partial \langle h + k \rangle_q = \partial \langle h \rangle_q = \langle \Delta h \rangle_q = 0 \).

Remark 2 A description of the kernel of \( \langle \cdot \rangle_q \) is not known.

Another corollary of Proposition 5 is the notion of depth of shifted symmetric polynomials which corresponds to the depth of quasimodular forms:

Definition 3 The space \( \Lambda_k^* \) of shifted symmetric polynomials of depth \( \leq p \) is the space of \( f \in \Lambda_k^* \) such that one can write

\[
f = \sum_{r=0}^{p} Q_2^r h_r,
\]

with \( h_r \in \mathcal{H}_{k-2r} \).
Theorem 4  If \( f \in \Lambda^*_{(\leq p)} \), then \( \langle f \rangle_q \in \tilde{M}_{(\leq p)} \).

**Proof** Expanding \( f \) as in Definition 3 we find

\[
\langle f \rangle_q = \sum_{k=0}^{p} \langle Q_k^2 h_k \rangle_q = \sum_{k=0}^{p} \hat{D}^k \langle h_k \rangle_q.
\]

By Lemma 2, we find that the depth of \( \langle f \rangle_q \) is at most \( p \).

Next, we set up notation to determine the basis of \( \mathcal{H} \) given by Theorem 3.

Let \( \tilde{R} = R[Q_2^{-1/2}] \) and \( \tilde{\Lambda} = \Lambda^*[Q_2^{-1/2}] \) be the formal polynomial algebras graded by assigning to \( Q_k \) weight \( k \) (note that the weights are—possibly negative—integers). Extend \( \Delta \) to \( \tilde{\Lambda} \) and observe that \( \Delta(\tilde{\Lambda}) \subset \tilde{\Lambda} \). Also extend \( \mathcal{H} \) by setting

\[ \tilde{\mathcal{H}} = \{ f \in \tilde{\Lambda} \mid \Delta f \in Q_1\tilde{R} \} = \ker \text{pr}\Delta|_{\tilde{\Lambda}}. \]

**Definition 4** Define the *partition-Kelvin transform* \( K : \tilde{\Lambda}_n \to \tilde{\Lambda}_{3-n} \) by

\[ K(f) = Q_2^{3/2-n} f. \]

Note that \( K \) is an involution. Moreover, \( f \) is harmonic if and only if \( K(f) \) is harmonic, which follows directly from the computation

\[ \Delta K(f) = Q_2^{3/2-n} \Delta f - \left( \frac{3}{2} - n \right) Q_1 Q_2^{1-n} \partial f - \frac{1}{4} \left( \frac{3}{2} - n \right) \frac{1}{2} Q_1^2 Q_2^{-1-n} f. \]

**Example 1** As \( K(1) = Q_2^{3/2} \), it follows that \( Q_2^{3/2} \in \tilde{\mathcal{H}} \).

**Definition 5** Given \( \underline{i} \in \mathbb{Z}_{\geq 0}^n \), let

\[ |\underline{i}| = i_1 + i_2 + \ldots + i_n, \quad \partial_\underline{i} = \frac{\partial^n}{\partial Q_{i_1+1} \partial Q_{i_2+1} \ldots \partial Q_{i_n+1}}. \]

Define the \( n \)th order differential operators \( \mathcal{D}_n \) on \( \tilde{\mathcal{R}} \) by

\[ \mathcal{D}_n = \sum_{\underline{i} \in \mathbb{Z}_{\geq 0}^n} \left( \begin{array}{c} |\underline{i}| \\ i_1, i_2, \ldots, i_n \end{array} \right) Q_{|\underline{i}|} \partial_\underline{i}, \]

where the coefficient is a multinomial coefficient.

This definition generalises the operators \( \partial \) and \( \mathcal{D} \) to higher weights: \( \mathcal{D}_1 = \partial \), \( \mathcal{D}_2 = \mathcal{D} \), and \( \mathcal{D}_n \) reduces the weight by \( n \).
Lemma 3 The operators \( \{D_n\}_{n \in \mathbb{N}} \) commute pairwise.

Proof Set \( I = |i| \) and \( J = |j| \). Let \( \hat{a}^k = (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) \). Then

\[
\begin{align*}
\left[ \begin{pmatrix} I \\ i_1, i_2, \ldots, i_n \end{pmatrix} Q_I \partial_{i}, \begin{pmatrix} J \\ j_1, j_2, \ldots, j_m \end{pmatrix} Q_J \partial_{j} \right] & = \sum_{k=1}^{n} \delta_{i_k, J-k} \begin{pmatrix} I \\ i_1, i_2, \ldots, \hat{i}_k, \ldots, i_n \end{pmatrix} Q_I \partial_{\hat{i}_k} \partial_{j} J \\
- \sum_{l=1}^{m} \delta_{j_l, I-l} \begin{pmatrix} J \\ j_1, j_2, \ldots, \hat{j}_l, \ldots, j_m \end{pmatrix} Q_J \partial_{\hat{j}_l} \partial_{i} I.
\end{align*}
\]

Hence, \( [D_n, D_m] \) is a linear combination of terms of the form \( Q_{|a|+1} \partial_a \), where \( a \in \mathbb{Z}_{\geq 0}^{n+m-1} \). We collect all terms for different vectors \( a \) which consists of the same parts (i.e. we group all vectors \( a \) which correspond to the same partition). Then, the coefficient of such a term equals

\[
\sum_{k=1}^{n} \sum_{\sigma \in S_{m+n-1}} (a_{\sigma(1)} + \ldots + a_{\sigma(m)}) \binom{|a| + 1}{a_1, a_2, \ldots, a_{n+m-1}} \\
- \sum_{l=1}^{m} \sum_{\sigma \in S_{m+n-1}} (a_{\sigma(1)} + \ldots + a_{\sigma(n)}) \binom{|a| + 1}{a_1, a_2, \ldots, a_{n+m-1}} \\
= (mn - mn) \sum_{\sigma \in S_{m+n-1}} a_{\sigma(1)} \binom{|a| + 1}{a_1, a_2, \ldots, a_{n+m-1}} = 0.
\]

Hence, \( [D_n, D_m] = 0 \). \( \square \)

It does not hold true that \( [D_n, Q_1] = 0 \) for all \( n \in \mathbb{N} \). Therefore, we introduce the following operators:

Definition 6 Let

\[
\Delta_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} D_{n-i} \partial^i.
\]

For \( \lambda \in \mathcal{P} \) let

\[
\Delta_\lambda = \binom{|\lambda|}{\lambda_1, \ldots, \lambda_{\ell(\lambda)}} \prod_{i=1}^{\infty} \Delta_{\lambda_i}.
\]

(Note that \( \Delta_0 = D_0 = 1 \), so this is in fact a finite product.)
Remark 3 By Möbius inversion

\[ \mathcal{D}_n = \sum_{i=0}^{n} \binom{n}{i} \Delta_{n-i} \partial^i. \]

The first three operators are given by

\[ \Delta_0 = 1, \quad \Delta_1 = 0, \quad \Delta_2 = \mathcal{D} - \partial^2 = 2\Delta. \]

Proposition 6 The operators \( \Delta_\lambda \) satisfy the following properties: for all partitions \( \lambda, \lambda' \)

(a) the order of \( \Delta_{|\lambda|} \) is \( |\lambda| \);
(b) \([\Delta_\lambda, \Delta_{\lambda'}] = 0\);
(c) \([\Delta_\lambda, Q_1] = 0\).

Proof Property (a) follows by construction and (b) is a direct consequence of Lemma 3. For property (c), let \( f \in \Lambda \) be given. Then

\[ \Delta_n(Q_1 f) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathcal{D}_{n-i} \partial^i (Q_1 f) \]

\[ = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left( (n-i) \mathcal{D}_{n-i-1} \partial^i f + Q_1 \mathcal{D}_{n-i} \partial^i f + i \mathcal{D}_{n-i} \partial^{i-1} f \right) \]

\[ = Q_1 \Delta_n(f) + \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left( (n-i) \mathcal{D}_{n-i-1} \partial^i f + i \mathcal{D}_{n-i} \partial^{i-1} f \right). \]

Observe that by the identity

\[ (n - i) \binom{n}{i} = (i + 1) (n + 1), \]

the sum in the last line is a telescoping sum, equal to zero. Hence \( \Delta_n(Q_1 f) = Q_1 \Delta_n(f) \) as desired. \( \square \)

In particular, the above proposition yields \([\Delta_\lambda, \Delta] = 0\) and \([\Delta_\lambda, \text{pr}] = 0\).

Denote by \((x)_n\) the falling factorial power \((x)_n = \prod_{i=0}^{n-1} (x - i)\) and for \( \lambda \in \mathcal{P}_n \) define \( Q_\lambda = \prod_{i=1}^{\infty} Q_{\lambda_i} \). Let

\[ h_\lambda = \text{pr} K \Delta_\lambda K(1). \]

Observe that \( h_\lambda \) is harmonic, as \( \text{pr} \Delta \) commutes with \( \text{pr} \) and \( \Delta_\lambda \).
Proposition 7  For all \( \lambda \in \mathcal{P}_n \) there exists an \( f \in \Lambda^*_n \) such that

\[
h_{\lambda} = \left( \frac{3}{2} \right) n! Q_{\lambda} + Q_2 f.
\]

Proof  Note that the left-hand side is an element of \( \Lambda^* \) of which the monomials divisible by \( Q^2_2 \) correspond precisely to terms in \( \Delta_{\lambda} \) involving precisely \( n - i \) derivatives of \( K(1) \) to \( Q_2 \). Hence, as \( \Delta_{\lambda} \) has order \( n \) all terms not divisible by \( Q_2 \) correspond to terms in \( \Delta_{\lambda} \) which equal \( \frac{\partial^n}{\partial Q^2_2} \) up to a coefficient. There is only one such term in \( \Delta_{\lambda} \) with coefficient \( \left( \frac{\left| \lambda \right|}{\lambda_1! \ldots \lambda_r!} \right) Q_{\lambda} \).

For \( f \in \mathcal{R} \), we let \( f^\vee \) be the operator where every occurrence of \( Q_i \) in \( f \) is replaced by \( \Delta_i \). We get the following unusual identity:

Corollary 2  If \( h \in \mathcal{H}_n \), then

\[
h = \frac{\text{pr} K h^\vee K(1)}{n! \left( \frac{3}{2} \right)_n}. \tag{9}
\]

Proof  By Proposition 7, we know that the statement holds true up to adding \( Q_2 f \) on the right-hand side for some \( f \in \Lambda^*_n \). However, as both sides of (9) are harmonic and the shifted symmetric polynomial \( Q_2 f \) is harmonic precisely if \( f = 0 \) by Proposition 4, it follows that \( f = 0 \) and (9) holds true.

Proof of Theorem 3  Let \( B_n = \{ h_{\lambda} \mid \lambda \in \mathcal{P}_n \text{ all parts are } \geq 3 \} \). First of all, observe that by Corollary 1 the number of elements in \( B_n \) is precisely the dimension of \( \mathcal{H}_n \). Moreover, the weight of an element in \( B_n \) equals \( |\lambda| = n \). By Proposition 7 it follows that the elements of \( B_n \) are linearly independent harmonic shifted symmetric polynomials.

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Appendix: Tables of shifted symmetric harmonic polynomials up to weight 10

We list all harmonic polynomials \( h_{\lambda} \) of even weight at most 10. The corresponding \( q \)-brackets \( \langle h_{\lambda} \rangle_q \) are computed by the algorithm prescribed by Zagier [13] using SageMath [11].
When is the Bloch–Okounkov $q$-bracket modular?

| $\lambda$  | $h_\lambda$ | $(h_\lambda)_q$ |
|------------|-------------|----------------|
| 0          | $1$         | $1$            |
| 4          | $\frac{27}{4} (Q_6^2 + 2Q_4)$ | $\frac{9}{3}Q$ |
| 6          | $\frac{225}{4} (63Q_6 + 9Q_2Q_4 + Q_2^3)$ | $-55 \frac{394}{5} R$ |
| (3,3)      | $\frac{225}{4} (63Q_3^2 - 108Q_2Q_4 + 2Q_2^2)$ | $\frac{115}{5} R$ |
| 8          | $\frac{19845}{16} (3960Q_8 + 360Q_2Q_6 + 20Q_2^2Q_4 + Q_2^4)$ | $19173 \frac{4096}{Q^2}$ |
| (5,3)      | $\frac{19845}{2} (495Q_3Q_5 + 45Q_2Q_3^2 - 135Q_2Q_6 - 50Q_2^2Q_4 + 2Q_2^4)$ | $-2415 \frac{125}{Q^2}$ |
| (4,4)      | $\frac{297675}{8} (132Q_4^2 + 24Q_2Q_3^2 - 440Q_2Q_6 - 28Q_2^2Q_4 + Q_2^4)$ | $-38241 \frac{2088}{Q^2}$ |
| (10)       | $\frac{382725}{8} (450450Q_{10} + 30030Q_2Q_8 + 1155Q_2^2Q_6 + 35Q_2^3Q_4 + Q_2^5)$ | $-2053485 \frac{Q}{4096} R$ |
| (7,3)      | $\frac{1913625}{8} (90090Q_3Q_7 + 6006Q_2Q_3Q_5 - 336336Q_2Q_8 + 231Q_2Q_3^2 + 12936Q_3^2Q_6 - 112Q_2^2Q_4 + 10Q_2^4)$ | $11975985 \frac{4096}{Q R}$ |
| (6,4)      | $\frac{13395375}{8} (12870Q_4Q_6 + 1716Q_2Q_3Q_5 + 858Q_2^2Q_4^2 - 96096Q_2Q_8 + 132Q_2^3Q_6^2 - 6501Q_2^2Q_6 - 89Q_2^3Q_4 + 5Q_4^2))$ | $21255885 \frac{4096}{Q R}$ |
| (5,5)      | $\frac{8037225}{4} (10725Q_5^2 + 1430Q_2Q_3Q_5 + 1430Q_2Q_3^2 - 10010Q_2Q_8 + 165Q_2^3Q_6 - 770Q_2^2Q_6 - 120Q_2^3Q_4 + 6Q_4^2)$ | $7759395 \frac{1024}{Q R}$ |
| (4,3,3)    | $\frac{13395375}{8} (12870Q_3^2Q_4 - 34320Q_2Q_3Q_5 + 10296Q_2Q_3^2Q_4 + 363Q_2^2Q_5^2 + 55440Q_2^2Q_6 - 376Q_2^3Q_4 + 10Q_2^4)$ | $-16583805 \frac{4096}{Q R}$ |

In case $|\lambda|$ is odd, the harmonic polynomials $h_\lambda$ up to weight 9 are given in the following table. The $q$-bracket of odd degree (harmonic) polynomials is zero, hence trivially modular.

| $\lambda$ | $h_\lambda$ |
|-----------|-------------|
| 3         | $-\frac{9}{4}Q_3$ |
| 5         | $-\frac{135}{4}(5Q_5 + Q_2Q_3)$ |
| 7         | $-\frac{14175}{16}(126Q_7 + 14Q_2Q_5 + Q_2^2Q_3)$ |
| (4, 3)    | $-\frac{99225}{16}(18Q_3Q_4 - 40Q_2Q_5 + Q_2^2Q_3)$ |
| (9)       | $-\frac{297675}{8}(7722Q_9 + 594Q_2Q_7 + 27Q_2^2Q_5 + Q_2^3Q_3)$ |
| (6, 3)    | $-\frac{893025}{4}(1287Q_3Q_6 + 99Q_2Q_3Q_4 - 4158Q_2Q_7 - 162Q_2^2Q_5 + 5Q_3Q_3)$ |
| (5, 4)    | $-\frac{8037225}{8}(286Q_4Q_5 + 66Q_2Q_3Q_4 - 1540Q_2Q_7 - 117Q_2^2Q_5 + 3Q_2Q_3Q_3)$ |
| (3, 3, 3) | $-\frac{893025}{4}(1287Q_3^3 - 3564Q_2Q_3Q_4 + 3240Q_2^2Q_5 + 10Q_2^3Q_3)$ |
References

1. Axler, S., Bourdon, P., Wade, R.: Harmonic Function Theory. Graduate Texts in Mathematics, vol. 137, 2nd edn. Springer, New York (2011)
2. Bloch, S., Okounkov, A.: The character of the infinite wedge representation. Adv. Math. 149(1), 1–60 (2000)
3. Chen, D., Möller, M., Zagier, D.: Quasimodularity and large genus limits of Siegel–Veech constants. J. Am. Math. Soc. 31(4), 1059–1163 (2018)
4. Cohen, H.: Sums involving the values at negative integers of $L$-functions of quadratic characters. Math. Ann. 217(3), 271–285 (1975)
5. Dijkgraaf, R.: Mirror symmetry and elliptic curves. In: Dijkgraaf, R., Faber, C., van der Geer, G. (eds.) The Moduli Space of Curves (Texel Island, 1994), volume 129 of Progress-Mathematics, pp. 149–163. Birkhäuser Boston (1995)
6. Eskin, A., Okounkov, A.: Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials. Invent. Math. 145(1), 59–103 (2001)
7. Kaneko, M., Zagier, D.: A generalized Jacobi theta function and quasimodular forms. In: Dijkgraaf, R., Faber, C., van der Geer, G. (eds.) The Moduli Space of Curves (Texel Island, 1994), volume 129 of Progress-Mathematics, pp. 165–172. Birkhäuser Boston, Boston (1995)
8. Okounkov, A., Olshanski, G.: Shifted Schur functions. Algebra i Analiz 9(2), 73–146 (1997)
9. Rankin, R.A.: The construction of automorphic forms from the derivatives of a given form. J. Indian Math. Soc. 20, 103–116 (1956)
10. Schoeneberg, B.: Das verhalten von mehrfachen thetareihen bei modulsubstitutionen. Math. Ann. 116(1), 511–523 (1939)
11. The Sage Developers.: SageMath, the Sage Mathematics Software System (Version 8.0) (2017). http://www.sagemath.org
12. Zagier, D.: Elliptic modular forms and their applications. In: Ranestad, K. (ed.) The 1-2-3 of Modular Forms, Universitext, pp. 1–103. Springer, Berlin (2008)
13. Zagier, D.: Partitions, quasimodular forms, and the Bloch–Okounkov theorem. Ramanujan J. 41(1–3), 345–368 (2016)

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