FIELD THEORIES OF TOPOLOGICAL RANDOM WALKS

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In this work we derive certain topological theories of transverse vector fields whose amplitudes reproduce topological invariants involving the interactions among the trajectories of three and four random walks. This result is applied to the construction of a field theoretical model which describes the statistical mechanics of an arbitrary number of topologically linked polymers in the context of the analytical approach of Edwards. With respect to previous attempts, our approach is very general, as it can treat a system involving an arbitrary number of polymers and the topological states are not only specified by the Gauss linking number, but also by higher order topological invariants.

I. INTRODUCTION

Topologically linked random chains are studied in connection with physical systems in which the topological constraints of one-dimensional objects play an essential role [1], [2]. Examples are vortex rings in fluids and dislocation lines in solids. To be concrete, the theory of random chains is applied here to the description of polymers in a good solvent [3]–[5], where the random chains are subjected to excluded volume interactions and topological interactions [6]–[7]. The former take into account the effective repulsions experienced by the polymers, while the latter arise due to presence of stable topological constraints in the system.

In a recent publication [8], the so-called analytical approach of Edwards [9] has been extended to the case of \( N \) entangling polymers, mapping the statistical mechanical problem...
of computing their free energy to a field theoretical problem. The model obtained in this
way is a $O(n)$ field theory coupled to Chern-Simons (C-S) terms $[10]$ in the limit $n \to 0$. At
one loop approximation, one finds that the topological interactions tend to counterbalance
the repulsive effects of the excluded volume forces. This effect has been indeed observed in
nature, for instance in the DNA of some bacteria, which forms topologically entangled rings
$[11]$. Moreover, one can compute the second topological momentum of two polymers exactly.
The analytical approach of Edwards is however limited by the fact that the topological
states of the system are distinguished using the Gauss linking number. The latter is a
relatively poor topological invariant and describes topological interactions in which only the
trajectories of two polymers are involved. Despite many efforts $[12]–[21]$, the inclusion in the
treatment of topological random walks of more sophisticated link invariants, like for instance
the Alexander and Jones polynomials $[22]$, $[23]$, has not yet been achieved. The reason is
that higher order link invariants have no immediate relation to the physical conformations
of the polymers in the space $[17]$. A remedy invoked by many authors is the introduction
of C-S field theories coupled to the polymer trajectories $[1]$, $[7]$. Unfortunately, this is not a
simple task. First of all, the amplitudes of non-Abelian C-S field theories contain topological
invariants $[24]$, but it is not clear how to use them to impose constraints on the configurations
of the system. On the other side, it is not easy to find a regularization or a mechanism that
suitably removes the spurious non-topological contributions arising in C-S field theories. For
example, the introduction of a framing $[24]$ complicates the integrations over the polymer
trajectories to the extent that the mapping of the statistical mechanics of the system to a
field theory is no longer possible $[23]$. To solve the above difficulties, we propose here interacting Abelian field theories contain-
ing transverse vector fields. Such theories are topological but not gauge invariant. Of course,
since the radiative corrections are not protected by a gauge principle, counter-terms may
arise which are of a non-topological nature. To exclude this possibility, the couplings among
the vector fields are chosen so that every quantum contribution disappears. In this way
one is able to construct theories whose amplitudes are purely classical and, in principle, can
generate topological terms describing interactions among an arbitrary number of random chains. Apart from a constant factor, each of these terms can be identified with the contribution of a tree-level Feynman diagram appearing in a non-Abelian C-S field theory [26]–[27]. However, it is exactly the freedom of choosing these factors that allows the fixing of the topological constraints. The above findings are used to build a model of entangling polymers in which the topological interactions induced by the Gauss linking invariant in the standard Edward approach are corrected by higher order topological interactions among three trajectories. The corresponding topological invariant has a simple physical interpretation which is expressed in terms of magnetic fields generated by fictitious charged particles moving along the trajectories of the polymers. A generalization to the case of four trajectories is also outlined.

The material presented in this paper is divided as follows. In the next Section we explain the treatment of the $N$ polymers problem of [8] based on the use of the Gauss linking number to distinguish the topological states. With respect to [8] some new investigations are made. For instance, the role of the C-S fields as propagators of the collective modes which are relevant in the topological entanglement [19] is explored in details. In Section three that approach is extended to include also higher order topological interactions. To this purpose, abelian theories are defined that generate topological interactions among three and four loops. Finally, the Conclusions are drawn in section four.

II. THE $N$–POLYMERS PROBLEM

Let $P_1, \ldots, P_N$ be a set of topologically entangling random chains at thermal equilibrium. If the step length $a$ of the segments composing the chains is very small, one can describe the chains as trajectories in the space parametrized by vectors $\mathbf{r}_i(s_i), i = 1, \ldots, N$ and continuous parameters $s_1, \ldots, s_N$ such that:

$$0 \leq s_i \leq L_i \quad \mathbf{r}_i(0) = \mathbf{r}_i', \mathbf{r}_i(L_i) = \mathbf{r}_i$$  \hspace{1cm} (1)
$L_i$ coincides with the contour length of $C_i$. In first approximation the topological constraints will be imposed exploiting the Gauss linking number:

$$\chi(C_i, C_j) \equiv \frac{1}{4\pi} \int_0^{L_i} \int_0^{L_j} d\mathbf{r}_i(s_i) \times d\mathbf{r}_j(s_j) \cdot \frac{(\mathbf{r}_i(s_i) - \mathbf{r}_j(s_j))}{|\mathbf{r}_i(s_i) - \mathbf{r}_j(s_j)|^3}$$

(2)

and requiring the conditions:

$$\chi(C_i, C_j) = m_{ij}$$

(3)

where $m_{ij} = m_{ji}, m_{ii} = 0$ are a set of topological numbers.

To describe the statistical mechanics of the chains we define the configuration probability $G_{\{m\}}(\{r\}, \{L\}; \{r'\}, 0)$. This function measures the probability that the trajectories $C_i$ have extrema (in the open case) at the points $r'_i$ and $r_i$ or a fixed point (in the closed case) in $r'_i = r_i$ for $i = 1, \ldots, N$. Moreover, they should fulfill the topological conditions (3). In our notations $\{m\}$ denotes the $n \times n$ symmetric matrix of topological numbers, while $\{r\} = r_1, \ldots, r_N, \{L\} = L_1, \ldots, L_N$ etc. In the path integral approach one obtains the following expression of $G_{\{m\}}(\{r\}, \{L\}; \{r'\}, 0)$:

$$G_{\{m\}}(\{r\}, \{L\}; \{r'\}, 0) = \int_{r'_1}^{r_1} \cdots \int_{r'_N}^{r_N} \exp \left\{ -(A_0 + A_{ev}) \right\} \prod_{i=1}^{N-1} \prod_{j>i}^{N} \delta(\chi(C_i, C_j) - m_{ij})$$

(4)

where

$$A_0 = \frac{3}{2a} \sum_{i=1}^{N} \int_0^{L_i} r_i^2(s_i)$$

(5)

is the action of a free random walk and

$$A_{ev} = \frac{1}{2a^2} \sum_{i,j=1}^{N} \int_0^{L_i} ds_i \int_0^{L_j} ds_j' v_{ij}^0 \delta(3)(\mathbf{r}_i(s_i) - \mathbf{r}_j(s_j'))$$

(6)

takes into account the excluded volume interactions. For convenience, we have put:

$$v_{ij}^0 = \begin{cases} \tilde{v}_{ij}^0 & \text{for } i = j \\ \tilde{v}_{ij}^0/2 & \text{for } i \neq j \end{cases} \quad \tilde{v}_{ij}^0 = \tilde{v}_{ji}^0$$

(7)

where the $\tilde{v}_{ij}^0$ are coupling constants with the dimension of a volume. The Fourier transformed of $G_{\{m\}}(\{r\}, \{L\}; \{r'\}, 0)$ with respect to the parameters $m_{ij}$ is:
\begin{align*}
G_{(\lambda)}(\{r\}, \{L\}; \{r'\}, 0) &= \int_{\mathbf{r}_1}^{\mathbf{r}} \mathcal{D}\mathbf{r}_1(s_1) \ldots \int_{\mathbf{r}_N}^{\mathbf{r}} \mathcal{D}\mathbf{r}_N(s_N) \exp \left\{ - (\mathcal{A}_0 + \mathcal{A}_{ev} + \mathcal{A}_{2L}) \right\} \quad (8)
\end{align*}

with

\begin{align*}
\mathcal{A}_{2L} &= i \sum_{i=1}^{N-1} \sum_{j=2 \atop j > i}^N \chi(C_i, C_j) \lambda_{ij} \quad (9)
\end{align*}

Following the approach of Edwards, one would like to transform the above path integral into a field theoretical problem. First of all, we treat the excluded volume interactions. To this purpose, we introduce \( N \) Gaussian scalar fields \( \phi_1(\mathbf{r}), \ldots, \phi_N(\mathbf{r}) \), with action

\begin{equation}
\mathcal{A}_\phi = \frac{a^2}{2} \sum_{i,j=1}^N \int d^3\mathbf{r} \phi_i[(\phi^0)^{-1}]^{ij} \phi_j \quad (10)
\end{equation}

The fundamental identity which relates the excluded volume term to a field theory amplitude is:

\begin{equation}
e^{-\mathcal{A}_{ev}} = \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_N \exp \left\{ - \mathcal{A}_{(\phi)} - i \sum_{i=1}^N \oint_{C_i} ds_i \phi_i(\mathbf{r}_i(s_i)) \right\} \quad (11)
\end{equation}

In the case of topological interactions one needs instead Chern-Simons fields \( A_{(i)}^{(j)} \) and \( B_{(i)}^{(j)} \)
with action:

\begin{equation}
S_{CS} = \frac{\kappa}{4\pi} \int d^3\mathbf{r} \sum_{i=1}^{N-1} \sum_{j=2 \atop j > i}^N A_{(i)}^{(j)} \cdot (\nabla \times B_{(j)}^{(i)}) \quad (12)
\end{equation}

The C-S theory will be quantized in the Landau gauge, where the fields are completely transverse. In the coupling with the random chains only the following linear combinations of fields are relevant:

\begin{equation}
C_{(1)} = \sum_{j=2}^N \frac{k}{4\pi} A_{(j)}^{(1)} \quad (13)
\end{equation}

\begin{equation}
C_{(i)} = \sum_{j=3 \atop j > i}^N \frac{k}{4\pi} A_{(j)}^{(i)} + \sum_{j=1 \atop j < i}^{N-2} \lambda_{ji} B_{(i)}^{(j)} \quad i = 2, \ldots, N - 1 \quad (14)
\end{equation}

and

\begin{equation}
C_{(N)} = \sum_{i=1}^{N-1} \lambda_{iN} B_{(N)}^{(i)} \quad (15)
\end{equation}
The topological term appearing in the configurational probability (8) can be rewritten as an amplitude of the above C-S field theory as follows:

\[
\int \mathcal{D}A \mathcal{D}B \exp \left\{ -iS_{CS} - i \sum_{i=1}^{N} \int_{0}^{L_i} C^{(i)}(r(s_i)) dr(s_i) \right\} = e^{-A_{2L}}
\]

(16)

where

\[
\int \mathcal{D}A \mathcal{D}B \equiv \int \prod_{i<j=1}^{N} \mathcal{D}A^{(i)}_{(j)} \mathcal{D}B^{(i)}_{(j)}
\]

(17)

![FIG. 1. Graphical interpretations of the topological interactions between two random walks \(C_i\) and \(C_j\) mediated by the Chern-Simons fields \(A^{(i)}_{(j)}, B^{(i)}_{(j)}\).](image)

Exploiting the identities (11) and (16) the configurational probability (8) becomes:

\[
G_{\{\lambda\}}(\{\mathbf{r}\}, \{L\}; \{\mathbf{r}'\}, 0) = \langle \prod_{i=1}^{N} G(\mathbf{r}_i, L_i; \mathbf{r}'_i, 0|\phi_i, C^{(i)})_{\{\phi\},\{A\},\{B\}} \rangle
\]

(18)

where

\[
G(\mathbf{r}_i, L_i; \mathbf{r}'_i, 0|\phi_i, C^{(i)}) = \int_{\mathbf{r}_i}^{\mathbf{r}'_i} \mathcal{D}\mathbf{r}_i(s_i) e^{-\int_{0}^{L_i} \mathcal{L}_i ds_i}
\]

(19)

and

\[
\mathcal{L}_i = \frac{3}{2a} \ddot{r}_i^2(s_i) + i\dot{\phi}_i(\mathbf{r}_i) - i\dot{r}_i(s_i) \cdot C^{(i)}(\mathbf{r}_i(s_i))
\]

(20)

As we see from eqs. (18) the trajectories \(C_1, \ldots, C_N\) are completely decoupled before averaging over the auxiliary fields \(\{\phi\}, \{A\}, \{B\}\). Formally, \(G(\mathbf{r}_i, L_i; \mathbf{r}'_i, 0|\phi_i, C^{(i)})\) is the evolution kernel of the random walk of a particle in an electromagnetic field \((i\phi_i, C^{(i)})\). Thus, it satisfies the pseudo-Schrödinger equation (14):
\[
\left[ \frac{\partial}{\partial L_i} - \frac{a}{6} D_i^2 + i \phi_i \right] G(r_i, L_i; r_i', 0 | \phi_i, C^{(i)}) = \delta(L_i) \delta(r_i - r_i') \tag{21}
\]

The covariant derivatives \( D_i \) appearing in the above equation are given by:

\[
D_i = \nabla + i C^{(i)} \quad i = 1, \ldots, N \tag{22}
\]

It is now convenient to perform a Laplace transformation of \( G(r_i, L_i; r_i', 0 | \phi_i, C^{(i)}) \) with respect to the length \( L_i \):

\[
G(r_i, r_i'; z_i | \phi_i, C^{(i)}) = \int_0^\infty dL_i e^{-z_i L_i} G(r_i, L_i; r_i', 0 | \phi_i, C^{(i)}) \tag{23}
\]

Accordingly, we are now considering the Laplace transformed configurational probability:

\[
G_{\{\lambda\}}(\{r\}, \{r'\}, \{z\}) = \int_{L_1}^{+\infty} dL_1 \ldots \int_{L_N}^{+\infty} dL_N \exp \left\{ - \sum_{i=1}^N z_i L_i \right\} G_{\{\lambda\}}(\{r\}, \{L\}; \{r'\}, 0) \tag{24}
\]

Since the order of the integrations over the auxiliary fields and the lengths \( L_i \) of the trajectories can be permuted, we have:

\[
G_{\{\lambda\}}(\{r\}, \{r'\}, \{z\}) = \langle \prod_{i=1}^N G(r_i, r_i'; z_i | \phi_i, C^{(i)}) \rangle_{\{\phi\}, \{A\}, \{B\}} \tag{25}
\]

The new variables \( z_i \) play the role of Boltzmann-like factors which govern the distribution lengths of the random chains. The advantage of having performed the Laplace transformations is that \( G(r_i, r_i'; z_i | \phi_i, C^{(i)}) \) satisfies a stationary pseudo Schrödinger equation:

\[
[z_i - H_i] G(r_i, r_i'; z_i | \phi_i, C^{(i)}) = \delta(r_i - r_i') \tag{26}
\]

Here the Hamiltonian \( H_i \) is given by:

\[
H_i = \frac{a}{6} D_i^2 - i \phi_i \tag{27}
\]

The solution of eq. (26) can now be expressed in terms of second quantized fields fields \( \psi_i^*, \psi_i, i = 1, \ldots, N \):

\[
G(r_i, r_i'; z_i | \phi_i, C^{(i)}) = Z_i^{-1} \int D\psi_i D\psi_i^* \psi_i(r_i) \psi_i^*(r_i') e^{-\mathcal{F}[\psi]} \tag{28}
\]
where $F[\psi_i]$ represents the Ginzburg-Landau free energy of a superconductor in a fluctuating magnetic field:

$$F[\psi_i] = \int d^3 r \left[ \frac{a}{6} |D_i \psi_i|^2 + (z_i + i \phi_i) |\psi_i|^2 \right]$$  \hspace{1cm} (29)

and $Z_i$ is the partition function of the system:

$$Z_i = \int D \psi_i D \psi_i^* e^{-F[\psi_i]}$$  \hspace{1cm} (30)

The auxiliary fields $\phi_i$ can be eliminated from the configurational probability (25) integrating them out. The integration over these fields is non-trivial due to the presence of the factors $Z_i^{-1}$ in the second quantized expression of $G(r_i, r'_i; z_i | \phi_i, C^{(i)})$ in (28), but can be made Gaussian by exploiting the identity (28):

$$Z_i^{-1} = \lim_{n_i \to 0} Z_i^{n_i-1}$$  \hspace{1cm} (31)

In this way

$$G(r_i, r'_i; z_i | \phi_i, C^{(i)}) = \lim_{n_i \to 0} \int D \psi_i D \psi_i^* \psi_i(r_i) \psi_i^*(r'_i) e^{-F[\psi_i]}$$  \hspace{1cm} (32)

The above equation should be understood as follows: the right hand side is first computed supposing that the replica index $n_i$ is an arbitrary positive integer and then one performs the analytic continuation of the result to the point $n_i = 0$. Now $G(r_i, r'_i; z_i | \phi_i, C^{(i)})$ is a product of $n_i$ path integrals. To each one we associate a set of replica fields $\psi_i^{a_i}, a_i = 1, \ldots, n_i$, so that it will be convenient to introduce the multiplets:

$$\Psi_i = (\psi_i^1, \ldots, \psi_i^{n_i})$$  \hspace{1cm} (33)

$$\Psi_i^* = (\psi_i^{*1}, \ldots, \psi_i^{*n_i})$$  \hspace{1cm} (34)

and to rewrite $G(r_i, r'_i; z_i | \phi_i, C^{(i)})$ as follows:

$$G(r_i, r'_i; z_i | \phi_i, C^{(i)}) = \lim_{n_i \to 0} \int D \Psi_i D \Psi_i^* \psi_i^1(r_i) \psi_i^{*1}(r'_i) e^{-F[\Psi_i]}$$  \hspace{1cm} (35)
where

\[
F[\Psi_i] = \sum_{a_i=2}^{n_i} \int d^3 r \left[ \frac{a_i}{6} |D\psi_i|_i^2 + (z_i + i\phi_i)|\psi_i^*|_i^2 \right]
\]

\[
\equiv \int d^3 r \left[ \frac{a_i}{6} |D\Psi_i|^2 + (z_i + i\phi_i)|\Psi_i|^2 \right]
\]  

(36)

and

\[
\int D\Psi_i D\Psi_i^* \equiv \prod_{a_i=1}^{n_i} D\psi_i^{a_i} D\psi_i^{*a_i}
\]

(37)

According to the replica method, one supposes that it is possible to commute the limit of vanishing replica index with the path integrations over the auxiliary fields. Thus, substituting eq. (35) in (25) and performing the integration over the scalars \(\phi_i\), which is now Gaussian, one obtains the final expression of the configurational probability \(G_{\{\lambda\}}(\{r\}, \{r^\prime\}, \{z\})\):

\[
G_{\{\lambda\}}(\{r\}, \{r^\prime\}, \{z\}) = \lim_{n_1, \ldots, n_N \to 0} \int D\Psi D\Psi^* \int D\mathcal{A} D\mathcal{B} \prod_{j=1}^{N} [\psi_j^1(r_j)\psi_j^{*1}(r_j^\prime)] \exp \{-A_{\text{Gauss}}\}
\]

(38)

\(A_{\text{Gauss}}\) is the free energy of the topologically linked random walk written in terms of C-S and second quantized fields. The subscript refers to the fact that the topological constraints have been imposed using the Gauss linking number. After a rescaling the complex scalar fields of the kind

\[
\Psi_i \to \sqrt{\frac{M}{2}} \Psi_i \quad \Psi_i^* \to \sqrt{\frac{M}{2}} \Psi_i^*
\]

(39)

where \(M\) is a mass parameter, the explicit expression of \(A_{\text{Gauss}}\) is:

\[
A_{\text{Gauss}} = i S_{C-S} + \sum_{i=1}^{N} \int d^3 r \left[ |D_i \Psi_i|^2 + m_i^2 |\Psi_i|^2 \right] + \sum_{i,j=1}^{N} \frac{2M^2\varphi_{ij}^0}{a^2} \int d^3 r |\Psi_i|^2 |\Psi_j|^2
\]

(40)

where
The above action describes a $O(n)$ model coupled to Chern-Simons fields in the limit $n = 0$. The analogous of the Planck constant is here the constant $\hbar = \frac{4\pi}{3}$ and has been set equal to one in (40). The topological fields are not just auxiliary, but play a physical role, since they propagate the long-range interactions that impose the topological constraints (3). One may argue at this point that the number of C-S fields used in the present approach, which is $N(N-1)$, is highly redundant with respect to the physical number of degrees of freedom involved if $N > 3$. This number can be computed exploiting refs. [19], [20], where a set of collective modes which are relevant in the topological interactions has been constructed in terms of the so-called bond vector densities:

$$m_i^2 = 2Mz_i \quad (41)$$

More precisely, the collective modes are linear combinations of the $N$ bond vector densities in the Fourier space, where

$$u_i(r) = \oint_{C_i} dr_i \delta(r - r_i) \quad i = 1, \ldots, N \quad (42)$$

Indeed, the Gauss linking invariant (2) can be expressed as follows:

$$\chi(C_i, C_j) = \int \frac{d^3q}{q^2} \ q \cdot (u^i(q) \times u^j(-q)) \quad (44)$$

As a consequence, if there is a number $M \leq N$ of random chains which have non-trivial topological relations with the others, i.e. the maximum rank of the matrices $\{m_{ij}\}$ and $\{\lambda_{ij}\}$ is $M$, there are at most $M$ degrees of freedom to be propagated. Of course, this is not in contradiction with our result. As a matter of fact it is possible to see that, exploiting the equations of motions, the number of C-S fields in the action (40) can be reduced to $N$. However, the C-S theory obtained in this way is not universal, since it cannot describe all the topological states of the system. The reason is that after the reduction the C-S propagators depend on the parameters $\lambda_{ij}$ and become singular in the limit in which some of them vanish. In general, it has not been possible to build a suitable Abelian C-S field theory with less
than $N(N - 1)$ fields without encountering the problem of diverging propagators wherever $\text{rank}[\lambda] < N$ or without resorting to a complicated parameterization of the matrix $\{\lambda\}$ provided for instance by the solution of the following algebraic system of equations:

$$\lambda_{ij} = \sum_{k=1}^{N} \eta_{ik} \eta_{kj} \quad (45)$$

**III. INCLUDING HIGHER LOOP INTERACTIONS**

The Gauss linking number describes a topological interaction between two loops and it is quite a poor topological invariants. Thus it would be interesting to include in the above approach also higher order topological interactions. To begin, we consider an interaction $\Gamma_3(C_i, C_j, C_k)$ among three loops, where $i < j < k$. To determine $\Gamma_3(C_i, C_j, C_k)$ we construct a suitable topological field theory with action:

$$S_3(i, j, k) = \epsilon^{\mu\nu\rho} \int d^3 x a^{(i)}_{\mu(jk)} \partial_\nu a^{(i)}_{\rho(jk)} + \epsilon^{\mu\nu\rho} \int d^3 x b^{(j)}_{\mu(ik)} \partial_\nu b^{(j)}_{\rho(ik)}$$

$$\epsilon^{\mu\nu\rho} \int d^3 x c^{(k)}_{\mu(ij)} \partial_\nu c^{(k)}_{\rho(ij)} + \Lambda_{ij}^k \epsilon^{\mu\nu\rho} \int d^3 x a^{(i)}_{\mu(jk)} b^{(j)}_{\nu(ik)} c^{(k)}_{\rho(ij)} \quad (46)$$

where the fields $a^{(i)}_{\mu(jk)}, \ldots, c^{(k)}_{\mu(ij)}$ are purely transverse. $S_3(i, j, k)$ describes at the classical level a topological field theory which is not gauge invariant. Moreover, it is easy to convince oneself that the theory has no radiative corrections that could spoil its topological properties. Despite of this fact, there are nontrivial amplitudes as for instance the following correlator:

$$G_{\Lambda_{jk}}(C_i, C_j, C_k) = \langle e^{i k \int_{C_i} dr_i a^{(i)}_{\alpha(\mu(ik))(r_i)}} e^{i k \int_{C_j} dr_j a^{(j)}_{\beta(\nu(ik))(r_j)}} e^{i \gamma k \int_{C_k} dr_k a^{(k)}_{\gamma(\nu(ik))(r_k)}} \rangle \quad (47)$$

The above amplitude can be exactly computed and one obtains:

$$G_{\Lambda_{jk}}(C_i, C_j, C_k) = \exp \left\{ \Lambda_{jk}^i \Gamma(C_i, C_j, C_k) \right\} \quad (48)$$

where

$$\Gamma(C_i, C_j, C_k) = \int_{C_i} dr_i^\alpha \int_{C_j} dr_j^\beta \int_{C_k} dr_k^\gamma I_{\alpha\beta\gamma}(r_i, r_j, r_k) \quad (49)$$
\[ I_{\alpha\beta\gamma}(r_i, r_j, r_k) = \epsilon^{\mu\nu\rho} \int d^3 r G_{\mu\alpha}(r - r_i) G_{\nu\beta}(r - r_j) G_{\rho\gamma}(r - r_k) \]  

(50)

In the above equation

\[ G_{\mu\nu}(r_1 - r_2) = -\epsilon_{\mu\nu\rho} \frac{(r_1 - r_2)^\rho}{|r_1 - r_2|^3} \]  

(51)

\( r^\mu \) being the components of the vector \( r \). An analogous of \( \Gamma(C_i, C_j, C_k) \) in the case \( C_i = C_j = C_k \) has been studied in connection with perturbative calculations of self-linking invariants in non-Abelian C-S field theories. After a volume integration in (50), the right hand side of eq. (49) has a complicated expression, which has been evaluated in [27]. With respect to [27], however, one does not need path ordering of the trajectories, so that it is possible to perform the volume integration using a different strategy. To this purpose, let us define the currents:

\[ j^\mu_{(l)}(r) \equiv \oint_{C_l} d r^\mu \delta(r - r_l) \]  

(52)

and the magnetic fields:

\[ B_{(l)\mu}(r) = \epsilon_{\mu\nu\rho} \oint_{C_l} d r^\nu \frac{(r - r_l)^\rho}{|r - r_l|^3} \]  

(53)

with \( \nabla \cdot \vec{B}_{(l)} = 0 \) and \( \nabla \times \vec{B}_{(l)} = 4\pi \vec{j}_{(l)} \). The vector potentials corresponding to these magnetic fields are:

\[ \vec{A}_{(l)}(r) = \oint_{C_l} \frac{d r_l}{|r - r_l|^3} = \int d^3 r' \frac{\vec{j}_{(l)}(r')}{|r - r'|} \]  

(54)

ad satisfy the relations

\[ \nabla \cdot \vec{A}_{(l)} = 0 \quad \nabla \times \vec{A}_{(l)} = \vec{B}_{(l)} \]  

(55)

\[ \triangle \vec{A}_{(l)} = -4\pi \vec{j}_{(l)} \]  

(56)

In terms of the magnetic fields \( B_{(l)\mu}^\mu \) we have:
\[ \Gamma(C_i, C_j, C_k) = \int d^3r \vec{B}_{(i)}(r) \cdot \vec{B}_{(j)}(r) \times \vec{B}_{(k)}^0(r) \] (57)

The space integral can be eliminated using the Stokes theorem and after some calculations one obtains:

\[
\frac{1}{4\pi} \Gamma(C_i, C_j, C_k) = - \int_{\Sigma C_k} dS_k \cdot \vec{B}_{(i)}(r_k) \times \vec{B}_{(j)}(r_k) + \int_{\Sigma C_j} dS_j \cdot \vec{B}_{(i)}(r_j) \times \vec{B}_{(k)}(r_j)
- \int_{\Sigma C_i} dS_i \cdot \vec{B}_{(j)}(r_i) \times \vec{B}_{(k)}(r_i) \] (58)

where the \( \Sigma C_i, \Sigma C_j, \Sigma C_k \) are arbitrary surfaces having respectively \( C_i, C_j, C_k \) as borders and infinitesimal surface elements \( dS_i, dS_j, dS_k \). To include the interactions among the loop trajectories, we extend the previous configurational probability of eq. (4) as follows:

\[
G_{\{m,M\}}(\{r\}, \{L\}; \{r'\}, 0) = \int_{r_1}^{r_1} \ldots \int_{r_N}^{r_N} \exp \left\{ -(A_0 + A_{ev}) \right\}
\prod_{i=1}^{N-1} \prod_{j=i+2}^{N} \delta(\chi(C_i, C_j) - m_{ij}) \prod_{i=1}^{N-2} \prod_{j=i+1}^{N} \prod_{k=j+1}^{N} \delta(\Gamma(C_i, C_j, C_k) - M_{ijk}) \] (59)

After performing the Fourier transformations with respect to the topological numbers \( m_{ij} \) and \( M_{ijk} \) one obtains:

\[
G_{\{\lambda\Lambda\}}(\{r\}, \{L\}; \{r'\}, 0) = \int_{r_1}^{r_1} Dr_1(s_1) \ldots \int_{r_N}^{r_N} Ds_N(s_N) \exp \left\{ -(A_0 + A_{ev} + A_{2L} + A_{3L}) \right\}
\] (60)

with

\[
A_{3L} = i \sum_{i,j,k=1 \atop i<j<k}^N \left\{ \Lambda^i_{jk} \Gamma(C_i, C_j, C_k) \right\} \] (61)

At this point we rewrite the higher loop topological interactions appearing in (60) as a field theory amplitude using eqs. (48) and (49):

\[
\int D[a] D[b] D[c] D[\bar{a}] D[\bar{b}] D[\bar{c}] e^{-iS_{3L}}
\]
\[
\left[ \prod_{\substack{i,j,k=1 \\
i<j<k}}^N e^{i \oint_{C_i} d r \alpha(i) (r_i)} e^{i \oint_{C_j} d r \beta(j) (r_j)} e^{i \oint_{C_k} d r \gamma(k) (r_k)} \right] = e^{-i A_{3L}} \quad (62)
\]

\( \mathcal{D}[a], \ldots, \mathcal{D}[c] \) denotes the measure over the C-S fields \( a^{(i)}_{\mu(jk)}, \ldots, c^{(k)}_{\mu(ij)} \) and

\[
S_{3L} = \sum_{\substack{i,j,k=1 \\
i<j<k}}^N S_3(i, j, k) \quad (63)
\]

Again, the C-S fields decouple the topological interactions. The integration over the trajectories of the random chains can now be performed following the same strategy of the previous section. In this way one obtains the following expression of the configurational probability in terms of the Laplace variables \( z_1, \ldots, z_N \):

\[
G_{\{\lambda, \Lambda\}}(\{r\}, \{r'\}, \{z\}) = 
\lim_{n_1, \ldots, n_N \to 0} \mathcal{D}[\Psi] \mathcal{D}[\Psi^*] \int \mathcal{D}[A] \mathcal{D}[B] \mathcal{D}[a] \ldots \mathcal{D}[c] \prod_{i=1}^N \left[ \psi_{i}^1(r_i) \psi_{i}^{*1}(r_i) \right] e^{-A_h} \quad (64)
\]

where

\[
A_h = i S_{CS} + i S_{3L} + \sum_{s=1}^N \int d^3 r \left[ |D^e s \Psi_s|^2 + m_s^2 |\Psi_s|^2 \right] + \sum_{i,j=1}^N \frac{2 M^2 v_{ij}^0}{a^2} \int d^3 r |\Psi_i|^2 |\Psi_j|^2 \quad (65)
\]

The covariant derivatives that include the three loop interactions are given by:

\[
D^e_s = \nabla + i (C^{(s)} + \tilde{d}^{(s)}) \quad (66)
\]

with

\[
\tilde{d}^{(s)} = \sum_{j,k=1}^N a^{(s)}_{(jk)} + \sum_{i,k=1}^N b^{(s)}_{(ik)} + \sum_{i,j=1}^N \tilde{c}^{(s)}_{(ij)} \quad (67)
\]

We notice that within the above approach it is possible to include also topological interactions among four or more trajectories. Let us consider for instance a topological four loop interaction \( \Gamma(C_1, C_2, C_3, C_4) \). To generate such interaction we define the following action:
\begin{align*}
S_4(1, 2, 3, 4) &= \frac{\kappa}{4\pi} \sum_{i=1}^{5} \int d^3x \epsilon^{\mu\nu\rho} a^{(i)}_{\mu} \partial_{\nu} b^{(i)}_{\rho} + \\
\Lambda_1 \int d^3x \epsilon^{\mu\nu\rho} a^{(1)}_{\mu} a^{(5)}_{\nu} a^{(2)}_{\rho} + \Lambda_2 \int d^3x \epsilon^{\mu\nu\rho} a^{(3)}_{\mu} b^{(5)}_{\nu} a^{(4)}_{\rho}
\end{align*}

As in the previous case, the theory is topological but has no gauge invariance. This can be dangerous, since radiative corrections may arise which spoil the topological properties of the theory, but it is easy to see that the above theory has no quantum contributions.

The relevant correlation function to be considered here is:

\begin{equation}
G_{\Lambda_1\Lambda_2}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \langle \prod_{i=1}^{4} e^{i\gamma_i \oint_{C_i} dx^{\alpha_i} b^{(i)}_{\alpha_i}} \rangle
\end{equation}

The above amplitude can be exactly computed and the result is:

\begin{align*}
G_{\Lambda_1\Lambda_2}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) &= \exp \left\{ -i\Lambda_1\Lambda_2 \left[ \prod_{i=1}^{4} \gamma_i \right] \Gamma(C_1, C_2, C_3, C_4) \right\} 
\end{align*}

where

\begin{equation}
\Gamma(C_1, C_2, C_3, C_4) = \prod_{i=1}^{4} \left[ \oint_{C_i} dx^{\alpha_i} \right] \int d^3x \int d^3y \epsilon^{\lambda\mu\nu} \epsilon^{\rho\sigma\tau} \\
\times G_{\mu\nu}(x - y) G_{\lambda\alpha_1}(x - x_1) G_{\nu\alpha_2}(x - x_2) G_{\rho\alpha_3}(y - x_3) G_{\tau\alpha_4}(y - x_4)
\end{equation}

Eq. (71) describes a topological interaction among four loops. Unfortunately, the elimination of the double volume integral has not been possible following the strategy of Section II. For this reason, an expression of \( \Gamma(C_1, C_2, C_3, C_4) \) in terms of the magnetic fields (53) could not be derived.

**IV. CONCLUSIONS**

In the first part of this work the statistical mechanical problem of a system of \( N \) polymers whose topological interactions are governed by the Gauss linking number has been mapped to a field theory following ref. [8]. In the model obtained in this way the C-S fields play a physical role, since they mediate the topological forces which impose the constraints (5). We
notice that the relation (16), which expresses the topological contributions appearing in the first quantized version of the configurational probability (8) in terms of field amplitudes, can be reproduced also by means of other Abelian C-S field theories. Each of these theories differs from the other by the number of fields. However, the requirement that their propagators should not diverge when \( \text{rank}[\lambda] < N \) implies that the independent fields should be at least \( N(N - 1) \). On the other side, if one starts with \( \tilde{N} > N(N - 1) \) C-S fields, it is always possible to reduce their number to \( N(N - 1) \) by exploiting the equations of motion. For this reason, the model of topologically entangling polymers given by eqs. (38)–(41) is unique.

In Section II the field theories are essential in order to decouple the random chains and to rewrite the configurational probability in a second quantized form. However, they start to play an even more active role in Section III. In fact, here the field amplitudes are fundamental to provide the explicit expression of the topological interactions among three or more polymer trajectories. In the case of a three loop interaction, the topological invariant \( \Gamma(C_i, C_j, C_k) \) has also a nice physical interpretation in terms of magnetic fields given by eq. (58) as the Gauss linking invariant. With respect to non-Abelian C-S field theories, the advantage of the theories of transverse vector fields defined here is the possibility of generating single topological invariants \( \Gamma(C_i, C_j, C_k) \) without the problem of spurious non-topological contributions or of higher order corrections. Moreover, the freedom to choose the parameters \( \Lambda_{jk}^i \) is crucial in order to impose constraints on the \( \Gamma(C_i, C_j, C_k) \) as shown by eqs. (59) and (60).

Concluding, the field theoretical approach illustrated in this paper solves in principle the most serious drawback present in the analytical approach of Edwards, i.e. the use of the Gauss linking invariant to specify the topological status of a system of random chains. However, more work and, possibly, numerical simulations are still needed in order to evaluate the phenomenological implications of the new topological terms introduced in the configurational probability (64) and to make contact with experiments.
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