Integral decompositions of varifolds

Hsin-Chuang Chou

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Abstract
This paper introduces a notion of decompositions of integral varifolds into countably many integral varifolds, and the existence of such decomposition of integral varifolds whose first variation is representable by integration is established. However, the decompositions may fail to be unique. Furthermore, this result can be generalized by replacing the class of integral varifolds with some classes of rectifiable varifolds whose density is uniformly bounded from below; for these classes, we also prove a general version of the compactness theorem for integral varifolds.

Keywords Varifold · Compactness · Decomposition · Indecomposability

Mathematics Subject Classification 49Q15

1 Introduction

General hypothesis In this section, we suppose $m, n, Q$ are positive integers, $U$ is an open subset of $\mathbb{R}^n$, $V \in V_m(U)$, $\|\delta V\|$ is a Radon measure, and $\delta_a$ denotes the Dirac measure at $a$.

Motivation To study the regularity of area-minimizing currents in higher codimension, Almgren introduced weakly differentiable $Q$-valued functions in [3], which was extended by De Lellis and Spadaro in an intrinsic way in [5]. To study the theory of PDEs on varifolds, Menne introduced weakly differentiable functions on varifolds in [8]; in this theory, a central element is the Poincaré inequality, a special case of which is the constancy theorem [8, 8.34]: If the weak derivative vanishes, then the function must be constant on some decomposition of the varifold. To study the convergence of pairs of varifolds and weakly differentiable functions thereon, a theory of weakly differentiable multiple-valued functions on varifolds is necessary. For instance, if $V_i \in IV_2(\mathbb{R}^2)$ is the union of two lines $y = i^{-1}$ and $y = -i^{-1}$ on which $f_i$ has values 1 and $-1$, respectively, whenever $i$ is a positive integer, then we expect that the sequence $(V_i, f_i)$ should converge to $(V, f)$ in some sense, where $V \in IV_2(\mathbb{R}^2)$ is the line $y = 0$ of multiplicity 2 and $f$ is the constant multiple-valued function whose value
is $2^{-1}(\delta_1 + \delta_{-1})$. Therefore, we intend to provide a notion of indecomposability adapted to multiple-valued functions on varifolds.

**Description of results** In [8, 6.2], the notion of indecomposability of $V$ was introduced by means of the distributional $V$ boundary of sets; more precisely, $V$ is termed *indecomposable* if and only if there exists no $\| V \| + \| \delta V \|$ measurable set $E$ such that $\| V \|(E) > 0$, $\| V \|(U \sim E) > 0$, and the distributional $V$ boundary $V \partial E$ of $E$ is identically zero. In this case, a plane of multiplicity 2 is indecomposable. For our purpose, we instead make the following definition.

**Definition** (see Definition 4.1) Suppose further that $V \in IV_m(U)$. Then $V$ is called *integrally indecomposable* if and only if there exists no $W \in IV_m(U)$ such that $W \leq V$, $W \neq 0$, $V - W \neq 0$, $\| V \| = \| W \| + \| V - W \|$, and $\| \delta V \| = \| \delta W \| + \| \delta (V - W) \|$.

Roughly speaking, we allow an integral varifold $V$ to be decomposed not only by restriction to subsets of zero distributional $V$ boundary but also by peeling off into several sheets without producing extra boundary. For instance, if $V \in IV_2(\mathbb{R}^2)$ satisfies $\| V \| = 2\mathbb{Z}^2$, then $\delta V = 0$; letting $W = 2^{-1}V$, we see that $V$ is integrally decomposed as $V = W + W$. It turns out that the two definitions of indecomposability are equivalent if $V \in IV_m(U)$ has density 1 for $\| V \|$ almost everywhere and $\| \delta V \|$ is absolutely continuous with respect to $\| V \|$ (see 4.2 and 4.6).

The aim of this paper is to show the following existence theorem of integral decomposition:

**Theorem** (see Theorem 5.12) Suppose $m, n$ are positive integers, $U$ is an open subset of $\mathbb{R}^n$, $V \in IV_m(U)$, and $\| \delta V \|$ is a Radon measure. Then, there exist a countable subset $H$ of $IV_m(U)$ and a function $\xi : H \to \mathcal{P}$ such that $W$ is nonzero and integrally indecomposable whenever $W \in H$, and such that

$$V(k) = \sum_{W \in H} \xi(W)W(k) \quad \text{whenever } k \in \mathcal{H}(U \times \mathcal{G}(n, m)),$$

and $\| \delta V \|(f) = \sum_{W \in H} \xi(W)\| \delta W \|(f)$ whenever $f \in \mathcal{H}(U)$.

Moreover, our result can be generalized to the more general classes of rectifiable varifolds, see 5.1; for these classes, we also prove a general version of the compactness theorem for integral varifolds, see 5.4.

The existence theorem of decompositions of rectifiable varifolds in the sense of [8, 6.2] is established in [8, 6.12]; a similar notion of indecomposability for integral currents and the existence theorem of such decompositions are established in [6, 4.2.25].

**Future goal** From [5, 2.12] and [8, 8.34], it seems conceivable to have the following constancy theorem: if $V$ is an integral varifold, $\| \delta V \|$ is a Radon measure, $\| \delta V \|$ is absolutely continuous with respect to $\| V \|$, $Y$ is a finite-dimensional normed space, and $f$ is a $Y$-valued weakly differentiable multiple-valued function on $V$ whose derivative vanishes $\| V \|$ almost everywhere, then there exist an integral decomposition $(H, \xi)$ and a function $v : (H \times \mathcal{P}) \cap \{(W, i) : 1 \leq i \leq \xi(W)\} \to Y$ such that

$$f(x) = \frac{1}{\Theta(\| V \|, x)} \sum_{W \in H} \Theta(\| W \|, x) \sum_{i=1}^{\xi(W)} \delta_{v(W, i)} \text{ for } \| V \| \text{ almost } x.$$

**Organization of this paper** In Sect. 2, we introduce the notation. In Sect. 3, to extend the main theorem for non-integral varifolds, we recall the strong topology on the class of all...
Daniell integrals. In Sect. 4, the integral indecomposability of integral varifolds is defined and its relation to the indecomposability employed in [8, 6.2] is established. In Sect. 5, we prove the existence theorem of integral decomposition of varifolds.

2 Notation

Mostly, the notation and the terminology agree with [6] and [1]. We also introduce additional notation and definitions.

**Basic notation and definitions** The set of positive integer is denoted by \( \mathcal{P} \) (see [6, 2.2.6]). The extended real number system is denoted by \( \mathbb{R}^\infty \) (see [6, 2.1.1]). The difference of sets \( A \) and \( B \) is denoted by \( A \sim B \). The domain and image of a function \( f \) are denoted by \( \text{dmn} \ f \) and \( \text{im} \ f \). The topological closure and interior of a set \( A \) are denoted by \( \text{Clos} \ A \) and \( \text{Int} \ A \), respectively. The open and closed balls with center \( a \) and radius \( r \) are denoted by \( U(a, r) \) and \( B(a, r) \), respectively (see [6, 2.8.1]). If \( (X, \rho) \) is a metric space, \( A \subset X \), and \( x \in X \), then the distance of \( x \) to \( A \) is defined by \( \text{dist}(x, A) = \inf \{\rho(x, a) : a \in A\} \).

For integration, the alternate notations \( \int f \, d\mu \), \( \int f(x) \, d\mu(x) \) and \( \mu(f) \) are employed; in this respect, \( \mu \) integrability of \( f \) means that \( \int f \, d\mu \) is defined in \( \mathbb{R} \) and \( \mu \) summability of \( f \) means that \( \int f \, d\mu \in \mathbb{R} \) (see [6, 2.4.2]). Inner products are denoted by a \( \cdot \) (see [6, 1.7.1]).

**Numerical summation** Whenever \( A \) is a set and \( f \) is an \( \mathbb{R} \)-valued function, \( \sum_{x \in A} f(x) \) denotes the numerical sum of \( f \) over \( A \) (see [6, 2.1.1]), and in case \( A = \text{dmn} \ f \), we abbreviate \( \sum f = \sum_{x \in A} f(x) \).

**The space of locally summable functions** Suppose \( \mu \) is a Borel measure over an open subset \( U \) of \( \mathbb{R}^n \). We denote by \( L^1_{\text{loc}}(\mu) \) the space of all real-valued functions \( f \) such that \( \int_{K} |f| \, d\mu < \infty \) whenever \( K \) is a compact subset of \( U \). Taking a sequence of compact subsets \( K_i \) of \( U \) with \( U = \bigcup_{i=1}^{\infty} K_i \), the space \( L^1_{\text{loc}}(\mu) \) endowed with the pseudo-metric

\[
\rho(f, g) = \sum_{i=1}^{\infty} 2^{-i} \inf \left\{ \int_{K_i} |f - g| \, d\mu, 1 \right\}
\]

becomes a complete pseudo-metric space.

**The space of continuous functions with compact support, its dual space, and Daniell integrals** Whenever \( X \) is a locally compact Hausdorff space, \( \mathcal{K}(X) \) denotes the vector space of continuous real-valued functions on \( X \) with compact support, and \( \mathcal{K}(X)^\ast \) denotes the vector space of all Daniell integrals on \( \mathcal{K}(X) \) (see [6, 2.5.14,2.5.19]). The topology of \( \mathcal{K}(X) \) is defined such that \( \mathcal{K}(X)^\ast \) becomes the dual topological vector space to \( \mathcal{K}(X) \) (see [8, 2.10,2.11]). For \( \mu \in \mathcal{K}(X)^\ast \), we define \( \mu^+, \mu^- \), \( |\mu| \in \mathcal{K}(X)^\ast \) such that

\[
\mu^+(f) = \sup \{\mu(g) : 0 \leq g \leq f, g \in \mathcal{K}(X)\} \quad \text{whenever } f \in \mathcal{K}(X)^+, \\
\mu^- = (-\mu)^+, \text{ and } |\mu| = \mu^+ + \mu^- \quad (\text{see [6, 2.5.5]}).
\]

If \( \mu \in \mathcal{K}(X)^\ast \) is monotone \( (\mu^- = 0) \), then the associated \( \mathcal{K}(X) \) regular measure of \( \mu \) is still denoted by \( \mu \) (see [6, 2.5.3]).

**Functions and submanifolds of class \( k \)** Functions that are \( k \) times continuously differentiable and submanifolds defined by such functions are termed “of class \( k \)” (see [6, 3.1.11,3.1.19]).

**Absolute continuity** Suppose \( X \) is a metric space, \( \phi \) and \( \psi \) are Borel regular measures on \( X \) such that every bounded subset of \( X \) has finite measure, and we define a Borel regular measure by

\[
\psi_{\phi}(A) = \inf \{\psi(B) : B \text{ is a Borel set and } \phi(A \sim B) = 0\}
\]
whenever $A \subset X$. In case $\psi_\phi = \psi$, we say $\psi$ is absolutely continuous with respect to $\phi$ (see [6, 2.9.1, 2.9.2]). In general, the definition of $\psi_\phi$ is also employed, while $X$ has a countable exhaustion by open sets on which $\psi$ and $\phi$ have finite measure.

**Approximate tangent cones** Whenever $m \in \mathcal{P}$, $\mu$ measures an open subset $U$ of a normed space $X$, $a \in U$, and $i: U \to X$ is the inclusion map, we abbreviate $\text{Tan}^m(i_\#\mu, a)$ as $\text{Tan}^m(\mu, x)$ (see [6, 3.2.16]).

**Distributions** Whenever $n \in \mathcal{P}$, $U$ is an open subset of $\mathbb{R}^n$, and $Y$ is a separable Banach space, $\mathcal{D}(U, Y)$ denotes the vector space of $Y$-valued functions of class $\infty$ with compact support, of which the topology is defined as in [8, 2.13] and $\mathcal{D}'(U, Y)$ denotes the dual topological vector space to $\mathcal{D}(U, Y)$. For $T \in \mathcal{D}'(U, Y)$, $\|T\|$ is defined to be the largest Borel regular measure over $U$ such that

$$\|T\|(G) = \sup \{T(g) : g \in \mathcal{D}(U, Y), \text{spt } g \subset G, |g| \leq 1\}$$

whenever $G$ is an open subset of $U$ (see [8, 2.18]); in case $\|T\|$ is a Radon measure, this concept agrees with [6, 4.1.5]; hence, we say $T$ is representable by integration and $T(g)$ continues to denote the value of the unique $T(\|\|_1)$ continuous extension of $T$ to $L_1(\|\|_1, Y)$ at $g \in L_1(\|\|_1, Y)$, and for every $\|T\|$ measurable set $A$, we define $T \downarrow A \in \mathcal{D}'(U, Y)$ by $(T \downarrow A)(g) = T(g_A)$, where $g_A(x) = g(x)$ for $x \in A$ and $g_A(x) = 0$ for $x \in U \sim A$ (see [8, 2.18, 2.19, 2.20]).

**Variolds and distributional boundary of sets** Whenever $n \in \mathcal{P}$, $U$ is an open subset of $\mathbb{R}^n$, and $m$ is a nonnegative integer, the space of varifolds, rectifiable varifolds, and integral varifolds of dimension $m$ are denoted by $V_m(U)$, $RV_m(U)$ and $IV_m(U)$, respectively; whenever $E$ is a $\mathcal{H}^m$ measurable subset of $\mathbb{R}^n$ which meets every compact subset of $\mathbb{R}^n$ in an $(\mathcal{H}^m, m)$ rectifiable subset of $\mathbb{R}^n$, we define $v_m(E) \in V_m(\mathbb{R}^n)$ by

$$v_m(A) = \mathcal{H}^m \{ x : (x, \text{Tan}^k(\mathcal{H}^m \downarrow E, x)) \in A \} \quad \text{whenever } A \subset \mathbb{R}^n \times G(n, m)$$

(see [1, 3.1, 3.5]). Whenever $V \in V_m(U)$ such that $\|\delta V\|$ is a Radon measure, there exists an $\mathbb{R}^n$-valued function $\eta(V, \cdot)$ defined by the requirement that, for $x \in U$,

$$\eta(V, x) \cdot u = \lim_{r \to 0^+} \frac{\delta V(b_{x,r} \cdot u)}{\|\delta V\|_{B(x, r)}} \quad \text{whenever } u \in \mathbb{R}^n,$$

where $b_{x,r}$ is the characteristic function of $B(x, r)$ on $U$; hence, $x \in \text{dmn } \eta(V, \cdot)$ if and only if the above limit exists. This definition adapts [1, 4.3] in the spirit of [6, 4.1.5]; in particular, $\eta(V, \cdot)$ is $\|\delta V\|$ measurable, $|\eta(V, \cdot)| = 1$ for $\|\delta V\|$ almost all $x$, and

$$(\delta V)(g) = \int \eta(V, x) \cdot g(x) \, d\|\delta V\|_x \quad \text{for } g \in L_1(\|\delta V\|, \mathbb{R}^n).$$

Similarly, we also define a $\|V\|$ measurable $\mathbb{R}^n$-valued function $h(V, \cdot)$ by the requirement that, for $x \in U$,

$$h(V, x) \cdot u = -\lim_{r \to 0^+} \frac{\delta V(b_{x,r} \cdot u)}{\|V\|_{B(x, r)}} \quad \text{whenever } u \in \mathbb{R}^n$$

which satisfies

$$\delta V(g) = -\int h(V, x) \cdot g(x) \, d\|V\|_x + \int \eta(V, x) \cdot g(x) \, d\left(\|\delta V\| - \|\delta V\|_{\|V\|}\right) \quad \text{whenever } g \in L_1(\|\delta V\|, \mathbb{R}^n).$$

If $E$ is $\|V\|$ + $\|\delta V\|$ measurable, then the distributional $V$ boundary of $E$ is given by

$$V \partial E = (\delta V) \downarrow E - (\delta(V \downarrow E \times G(n, m))) \in \mathcal{D}'(U, \mathbb{R}^n)$$

(see [8, 5.1]).
3 Topology

In this section, we present the necessary results about the strong topology.

Definition 3.1 (see [4, III.14, Example 4]) Suppose $X$ is a locally compact Hausdorff space. There exists a unique locally convex topology on $\mathcal{K}(X)^*$ termed strong topology such that the sets
\[ \mathcal{K}(X)^* \cap \{ \mu : |\mu(f)| < r \text{ for all } f \in B \} \]
corresponding to $r \in \mathbb{R}$, $r > 0$ and bounded subsets $B$ of $\mathcal{K}(X)$ give a local base at 0.

Remark 3.2 (see [4, III.12, Examples 1, 3] and [4, III.23, Corollary 1]) The space $\mathcal{K}(X)^*$ equipped with the strong topology is complete.

Remark 3.3 (see [8, 2.11, 2.12] and [4, III.5, Proposition 6]) Suppose $X$ has a sequence of compact subsets $K_i$ such that $K_i \subset \text{Int } K_{i+1}$ for $i \in \mathcal{P}$ and $X = \bigcup_{i=1}^{\infty} K_i$. Then, the strong topology on $\mathcal{K}(X)^*$ is metrizable; in fact, it is generated by the family of semi-norms, with value
\[ \sup\{ \mu(f) : f \in \mathcal{K}(X), \text{spt } f \subset K_i, |f| \leq 1 \} \]
at $\mu \in \mathcal{K}(X)^*$, corresponding to $i \in \mathcal{P}$.

4 Integral indecomposability

Definition 4.1 Suppose $m, n \in \mathcal{P}$, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in \mathbf{IV}_m(U)$ and $\|\delta V\|$ is a Radon measure. Then $V$ is called integrally indecomposable if and only if there exists no $W \in \mathbf{IV}_m(U)$ such that $W \leq V$, $W \neq 0$, $V - W \neq 0$, $\|V\| = \|W\| + \|V - W\|$, and $\|\delta V\| = \|\delta W\| + \|\delta(V - W)\|$.

The rest of this section contributes to the relation between the integral indecomposability and the distributional boundary of sets.

Lemma 4.2 Suppose that $m, n \in \mathcal{P}$ with $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, $E$ is $\|V\| + \|\delta V\|$ measurable with $V \partial E = 0$, and $W = V \setminus E \times G(n, m)$. Then, we have
\[ \|\delta V\| = \|\delta W\| + \|\delta(V - W)\| . \]

Proof It follows from [12, 5.3].

Remark 4.3 In contrast to 4.2, it may happen that $V$ is an integral varifold, $\|V\|$ has density 1 everywhere, and there exists an integral varifold $W$ such that $W$ and $V - W$ are integrally indecomposable and $\|\delta V\| = \|\delta W\| + \|\delta(V - W)\|$, but there exists no Borel set $E$ such that $W = V \setminus E \times G(n, m)$ and $V \partial E \neq 0$. Let
\[ R_1 = \mathbb{R}^2 \cap \{(x, y) : x \geq 0, y = 0\}, \]
\[ R_2 = \mathbb{R}^2 \cap \{(x, y) : |y| = \sqrt{3}x\}. \]
Define $V, W \in \mathbf{IV}_1(\mathbb{R}^2)$ to satisfy $\|V\| = \mathcal{H}^1(R_1 \cup R_2)$ and $\|W\| = \mathcal{H}^1 \setminus R_1$. Therefore, $W$ and $V - W$ are integrally indecomposable, $\|\delta V\| = 2\delta_{(0,0)}$, $\|\delta W\| = \delta_{(0,0)} = \|\delta(V - W)\|$, and if $W = V \setminus E \times G(2, 1)$ for some Borel set $E$, then
\[ \|V \partial E\| = \delta_{(0,0)}. \]
Next, we will show that if $V \in \mathbf{I}V_m(U)$ and $\|\delta V\| = \|\delta V\|_{\|V\|}$, then the converse of 4.2 holds.

**Theorem 4.4** Suppose that $m, n \in \mathcal{P}, m \leq n, U$ is an open subset of $\mathbb{R}^n$, $V, W \in \mathbf{I}V_m(U)$, $\|\delta V\| + \|\delta W\|$ is a Radon measure and

$$A = \{x : \Theta^m(\|V\|, x) > 0 \text{ and } \Theta^m(\|W\|, x) > 0\}.$$

Then,

$$h(V, x) = h(W, x) \text{ for } \mathcal{H}^m \text{ almost all } x \in A.$$

**Proof** In view of [1, 3.5(1b)], the theorem [7, 4.8] reduces the problem to the case of submanifolds of class 2. 

**Remark 4.5** In case $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, we have $V$ is integrally indecomposable if and only if there exists no $W \in \mathbf{I}V_m(U)$ such that $W \leq V, W \neq 0, V - W \neq 0$, and

$$\|\delta W\| \text{ is absolutely continuous with respect to } \|W\|,$$

$$\|\delta(V - W)\| \text{ is absolutely continuous with respect to } \|V - W\|.$$

In fact, from 4.4, we have

$$\|\delta V\|_{\|V\|} = \|\delta W\|_{\|W\|} + \|\delta(V - W)\|_{\|V - W\|},$$

and the assertion follows. Therefore, the present definition of indecomposability extends [10, 2.15] when $\text{spt } \|V\|$ is compact and $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$.

**Corollary 4.6** Suppose that $m, n \in \mathcal{P}, m \leq n, U$ is an open subset of $\mathbb{R}^n$, $V \in \mathbf{I}V_m(U)$, $\|\delta V\|$ is a Radon measure, $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, $E$ is $\|V\| + \|\delta V\|$ measurable, and $W = V \downarrow E \times G(n, m)$ satisfies $\|\delta V\| = \|\delta W\| + \|\delta(V - W)\|$. Then, $V \partial E = 0$.

**Proof** In view of 4.4, we have

$$\|\delta V\|_{\|V\|} = \|\delta W\|_{\|W\|} + \|\delta(V - W)\|_{\|V - W\|},$$

and hence

$$\|\delta W\| - \|\delta W\|_{\|W\|} = 0 = \|\delta(V - W)\| - \|\delta(V - W)\|_{\|V - W\|}.$$

Therefore, whenever $g \in \mathcal{D}(U, \mathbb{R}^n)$, we have

$$(\delta V \downarrow E)(g) = -\int_U h(V, x) \cdot g(x) \, d\|W\| \, x$$

$$= -\int_U h(W, x) \cdot g(x) \, d\|W\| \, x$$

$$= (\delta W)(g)$$

This shows $V \partial E = 0$. 

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5 Integral decomposition

Definition 5.1 Suppose $m, n \in \mathcal{P}, m \leq n$, $U$ is an open subset of $\mathbb{R}^n$ and $P \subset \mathbf{RV}_m(U)$. Then $P$ is called appropriate if and only if

1. If $V, W \in P$, then $V + W \in P$.
2. If $V \in P$, then $\Theta^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all $x$.
3. $P$ is closed with respect to the strong topology.

Now, we aim to provide examples of appropriate classes; for this purpose, the following lemma is a powerful tool to verify the closedness of a class with respect to the strong topology.

Lemma 5.2 Suppose $V_i$ forms a sequence in $\mathbf{RV}_m(U)$. Then the following statements are equivalent.

1. $V_i$ is Cauchy with respect to the strong topology.
2. $\|V_i\|$ is Cauchy with respect to the strong topology.
3. $\Theta^m(\|V\|, \cdot)$ is Cauchy in $L^1_{\text{loc}}(\mathcal{H}^m)$.

In this case, the limit $V$ of $V_i$ satisfies $V \in \mathbf{RV}_m(U)$ and $\Theta^m(\|V\|, \cdot)$ is the limit of $\Theta^m(\|V_i\|, \cdot)$ in $L^1_{\text{loc}}(\mathcal{H}^m)$. In particular, $\mathbf{RV}_m(U)$ and $\mathbf{IV}_m(U)$ are closed subsets of $\mathcal{V}_m(U)$ with respect to the strong topology.

Proof Note that if $W_1, W_2 \in \mathbf{RV}_m(U)$ and $G$ is an open set with compact closure in $U$, then applying [12, 3.3] with

$$T = \|W_1\| - \|W_2\|, \quad \phi = \mathcal{H}^m \cup |\Theta^m(\|W_1\|, \cdot) - \Theta^m(\|W_2\|, \cdot)|,$$

$$k = \text{sign}(\Theta^m(\|W_1\|, \cdot) - \Theta^m(\|W_2\|, \cdot)),$$

we have $\|T\| = \phi$ and

$$\int_G |\Theta^m(\|W_1\|, x) - \Theta^m(\|W_2\|, x)| \, d\mathcal{H}^m x$$

$$= \sup \{\|W_1\|(g) - \|W_2\|(g) : g \in \mathcal{H}(U), \text{spt } g \subset G, |g| \leq 1\}.$$ 

On the other hand, we have

$$W_1(f) - W_2(f) = \int f(x, \text{Tan}^m(\|W_1 + W_2\|, x)) \, d\|W_1\| x$$

$$- \int f(x, \text{Tan}^m(\|W_1 + W_2\|, x)) \, d\|W_2\| x$$

$$\leq \sup \text{im } f \int_K |\Theta^m(\|W_1\|, x) - \Theta^m(\|W_2\|, x)| \, d\mathcal{H}^m x$$

whenever $f \in \mathcal{H}(U \times G(n, m))$ with $\text{sup } f \subset K \times G(n, m)$ and $K$ is a compact subset of $U$, and note that $|p_\#(W_1 - W_2)| \leq p_\#|W_1 - W_2|$, where $p : U \times G(n, m) \to U$ is the projection map. Now, the main assertion is obvious.

Note that the subset $D$ of $L^1_{\text{loc}}(\mathcal{H}^m)$, which consists of all nonnegative functions $\theta$ such that $\{x : \theta(x) > 0\}$ is countably $(\mathcal{H}^m, m)$ rectifiable, is closed. Denote by $\theta$ the limit of $\Theta^m(\|V_i\|, \cdot)$ in $L^1_{\text{loc}}(\mathcal{H}^m)$. Since $\theta \in D$, it follows from [6, 2.19.13] and [11, 3.25] that $\mathcal{H}^m \cup \theta$ is the weight of some member $V$ of $\mathbf{RV}_m(U)$. Finally, applying the results in the previous paragraph with $W_1 = V_i$ and $W_2 = V$, we deduce that $V$ is the limit of $V_i$ with respect to the strong topology, and that if $V_i$ is integral whenever $i \in \mathcal{P}$, then so is $V$. \(\Box\)

Lemma 5.3 Suppose $m, n \in \mathcal{P}$ and $U$ is an open subset of $\mathbb{R}^n$. Then the following statements hold.

\(\Box\)
1. If $C$ is a closed subset of $\mathbb{R}$ satisfying $\inf C \geq 1$ and $c + d \in C$ whenever $c, d \in C$, then

$$P = \mathbf{R}V_m(U) \cap \{V : \Theta_m(\|V\|, x) \in C \text{ for } \|V\| \text{ almost all } x\}$$

is appropriate.

2. If $n' \in \mathcal{P}$, $U'$ is an open subset of $\mathbb{R}^{n'}$, $f : U \to U'$ is of class $\infty$, and $f|\text{spt } V$ is proper, then we have

$$\Theta_m(\|f#V\|, y) \in C \text{ for } \|f#V\| \text{ almost all } y$$

whenever $V \in P$.

3. If $m = n$, $U = \mathbb{R}^m$, $0 < r < \infty$, $0 < d < \infty$, and the sequence $W_i \in P$ satisfies that

$$W_i \subseteq U(0, r) \times G(m, m) \to dV_m(U(0, r)) \quad \text{as } i \to \infty$$

with respect to the weak topology, and that

$$\lim_{i \to \infty} \sup\{\langle\delta W_i(\psi), \psi \in \mathcal{D}_{B(0, r)}(\mathbb{R}^m, \mathbb{R}^m)\rangle \leq 1\} = 0, \quad (\dagger)$$

then $d \in C$.

**Proof** Clearly, (1) follows from 5.2, and (2) follows from [1, 3.5(1b),3.5(3)]. To prove (3), we denote by $e_1, e_2, \ldots, e_m$ the standard base of $\mathbb{R}^m$ and let $T_i \in \mathcal{P}'(\mathbb{R}^m, \mathbb{R})$ be such that

$$T_i(\phi) = (\|W_i\| \cup \mathbf{U}(0, r)) (\phi) \quad \text{whenever } \phi \in \mathcal{D}(\mathbb{R}^m, \mathbb{R}).$$

Then, we have

$$(D_j T_i)(\phi) = (\delta W_i)(\phi e_j) \quad \text{whenever } \phi \in \mathcal{D}_{B(0, r)}(\mathbb{R}^m, \mathbb{R}) \text{ and } j = 1, 2, \ldots, m.$$ 

It follows from $(\dagger)$ that there exist numbers $0 < \kappa_i < \infty$ such that $\lim_{i \to \infty} \kappa_i = 0$ and such that

$$|(D_j T_i)(\phi)| \leq \kappa_i \sup \|D \phi\| \quad \text{whenever } \phi \in \mathcal{D}_{B(0, r)}(\mathbb{R}^m, \mathbb{R}) \text{ and } j = 1, 2, \ldots, m.$$ 

Letting $0 < \lambda < 1$ and applying Allard’s strong constancy lemma [9, 3.7] (for the original formulation, see [2, Section 1]), there exists $0 \leq c_i < \infty$ such that

$$\lim_{i \to \infty} \sup\{|T_i(\phi) - c_i \int \phi d\mathcal{L}^m| : \phi \in \mathcal{D}_{B(0, \lambda r)}(\mathbb{R}^m, \mathbb{R})\}, \sup \|\phi\| \leq 1\} = 0.$$ 

It follows from [9, 2.34(3)] and 5.2 that $W_i \subseteq U(0, \lambda r) \times G(m, m) - c_i V_m(U(0, \lambda r)) \to 0$ as $i \to \infty$ with respect to the strong topology. Therefore, we conclude that $\lim_{i \to \infty} c_i = d$ and that $W_i \subseteq U(0, \lambda r) \times G(m, m) \to dV_m(U(0, \lambda r))$ as $i \to \infty$ with respect to the strong topology. Finally, it follows from (1) that $dV_m(U(0, \lambda r)) \in P$; that is, $d \in C$. \hfill \Box

**Theorem 5.4** (General compactness theorem) Suppose $m, n, U, C$, and $P$ are as in 5.3, $G_i$ is a sequence of open subsets of $U$ such that $U = \bigcup_{i=1}^{\infty} G_i$, and $M_i$ is a sequence of non-negative real numbers. Then, the class

$$P \cap \{V : \|V\| + \|\delta V\|(G_i) \leq M_i \text{ whenever } i \in \mathcal{P}\}$$

is compact with respect to the weak topology.
Proof From the proof of [1, 6.4], it is enough to show the following statement: if $U = \mathbb{R}^n$, $T \in G(n, m)$, $V_i$ is a sequence in $P$ such that $V_i \to d\nu_m(T)$ as $i \to \infty$ with respect to the weak topology and such that

$$\lim_{i \to \infty} \|\delta V_i\|U(0, 4r) = 0 \quad \text{for some } 0 < r < \infty,$$

then $d \in C$. Denote by $T^\perp \in G(n, n - m)$ the orthogonal complement of $T$ in $\mathbb{R}^n$. Let $X = \mathbb{R}^n \cap \{x : \sup(|Tx|, |T^\perp x|) < 2r\}$. From 5.3(2), we have

$$W_i = T_u(V_i \cap X \times G(n, m)) \in RV_m(T),$$

$$\Theta^m(\|W_i\|, z) \in C \quad \text{for all } W_i$$

whenever $i \in \mathcal{P}$. Choose $\gamma \in B_{0,2r}(T^\perp, \mathbb{R})$ such that $\gamma|U(0, r) = 1$. Because $V_i \cap X \times G(n, m) \to \nu_m(X \cap T)$ as $i \to \infty$ with respect to the weak topology by [1, 2.6(2d)], we conclude that

$$W_i \to d\nu_m(T \cap U(0, r)) \quad \text{as } i \to \infty$$

with respect to the weak topology, and observe that

$$\sup(|(\delta V_i)((\gamma \circ T^\perp)(\psi \circ T)) - (\delta W_i)(\psi)| : \psi \in B_{0, r}(T, T), \sup \|D \psi\| \leq 1)$$

tends to 0 as $i \to \infty$. By 5.3(3), the assertion follows. \hfill \Box

Remark 5.5 The set $C$ mentioned in 5.3(1) could be $\mathcal{P}, \mathbb{R} \cap \{t : 1 \leq t\}$, or

$$\{1\} \cup (\mathbb{R} \cap \{t : 2 \leq t\});$$

the last class also occurs in [13].

Definition 5.6 Suppose $m, n \in \mathcal{P}, m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $P$ is an appropriate subset of $RV_m(U)$, $V \in P$ and $\|\delta V\|$ is a Radon measure. Then $V$ is called indecomposable with respect to $P$ if and only if there exists no $W \in P \sim \{0\}$ such that $V - W \in P \sim \{0\}$ and $\|\delta V\| = \|\delta W\| + \|\delta(V - W)\|$.

Definition 5.7 Suppose $m, n \in \mathcal{P}, m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $P$ is an appropriate subset of $RV_m(U)$, $W, V \in P$ and $\|\delta V\|$ is a Radon measure. Then $W$ is called a component of $V$ with respect to $P$ if and only if $W \neq 0, W \leq V, \|\delta V\| = \|\delta W\| + \|\delta(V - W)\|$ and $W$ is indecomposable with respect to $P$.

Definition 5.8 Suppose $m, n \in \mathcal{P}, m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $P$ is an appropriate subset of $RV_m(U)$, $V \in P$ and $\|\delta V\|$ is a Radon measure. Then a countable subfamily $H$ of $P$ together with a map $\xi : H \to \mathcal{P}$ is called a decomposition of $V$ with respect to $P$ if and only if

1. Each member of $H$ is a component of $V$ with respect to $P$.
2. $\sum_{W \in H} \xi(W)W(k) = V(k)$ whenever $k \in \mathcal{K}(U \times G(n, m)).$
3. $\sum_{W \in H} \xi(W)\|\delta W\|(f) = \|\delta V\|(f)$ whenever $f \in \mathcal{K}(U)$.

Example 5.9 Let $0 < \theta < \pi$ be such that $\cos \theta = 1/4$. Consider the six rays

$$R_1 = \{t(1, 0) : 0 < t < \infty\},$$

$$R_2 = \{t(\cos \theta, \sin \theta) : 0 < t < \infty\},$$

$$R_3 = \{t(\cos(\pi - \theta), \sin \theta) : 0 < t < \infty\},$$
in $\mathbb{R}^2$ and the associated varifolds $V_i \in \mathbf{IV}_1(\mathbb{R}^n)$ with $\|V_i\| = \mathscr{H}^1 \setminus R_i$. Note that the integral varifold defined by

$$V = 2(V_1 + V_2 + V_3 + V_4 + V_5 + V_6)$$

is stationary. Let

$$H_1 = \{V_1 + V_4, V_2 + V_5, V_3 + V_6\},$$

$$H_2 = \{V_1 + 2(V_3 + V_5), V_4 + 2(V_2 + V_6), V_1 + V_4\}$$

and define $\xi_i : H_i \to \mathcal{P}$ for $i = 1, 2$ by

$$\text{im } \xi_1 = \{2\} \quad \text{and} \quad \text{im } \xi_2 = \{1\}.$$

Then, $(H_i, \xi_i)$ for $i = 1, 2$ are distinct decompositions of $V$ with respect to $\mathbf{IV}_1(\mathbb{R}^2)$, see Fig. 1. It shows that there may exist different types of decompositions for a varifold and the components need not have constant density. Furthermore, the decompositions may fail to be unique even if $\|V\|$ has density 1, see also [8, 6.13].

To prove the main theorem, the following a priori estimate is a key observation: under smallness conditions on the first variation, the weight measure on a ball has a positive lower bound. This will provide, locally, an upper bound of the number of varifolds in a decomposition; moreover, it also suggests a way to construct a decomposition.

**Lemma 5.10** (a priori estimate) Suppose $0 < c < \infty$, $0 < d < \infty$, $m, n \in \mathcal{P}$, $m \leq n$, $U$ is an open subset of $\mathbb{R}^n$, $a \in U$, $r > 0$, $B(a, r) \subset U$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure and

$$\Theta^m(\|V\|, a) \geq d,$$

$$\|\delta V\|_{B(a, t)} \leq c\alpha(m)t^m \quad \text{for } 0 < t < r.$$

Then, there holds

$$\|V\|_{B(a, r)} \geq \alpha(m)(d - cr)r^m.$$
**Proof** From [8, 4.5, 4.6], we have
\[ s^{-m} ||V||B(a, s) \leq r^{-m} ||V||B(a, r) \]
\[ + \int_s^r t^{-m} \int_{B(a,t)} (x-a) \cdot \eta(V, x) d\|V\| x d\mathcal{L}^1 t \]
whenever \( 0 < s \leq r \); note that the last term is less than
\[ \int_s^r t^{-m} \|V\|B(a, t) d\mathcal{L}^1 t \leq c\alpha(m)(r - s) \]
hence, letting \( s \to 0^+ \),
\[ r^{-m} ||V||B(a, r) \geq \alpha(m)\Theta^m(||V||, a) - c\alpha(m)(d - cr) \]
which means \( ||V||B(a, r) \geq \alpha(m)(d - cr)r^m \).

**Definition 5.11** Suppose \( m, n \in \mathcal{P}, m \leq n, U \) is an open subset of \( \mathbb{R}^n \), and \( P \) is an appropriate subset of \( RV_m(U) \). Then, \( \Xi \) denotes the class of all functions \( \xi \) such that
\[ \text{dmn } \xi \text{ is a finite subset of } P \sim [0], \quad \text{im } \xi \subset \mathcal{P}, \]
\[ \sum_{W \in \text{dmn } \xi} \xi(W) \|\delta W\| = \|\delta v(\xi)\|, \]
where the map \( v : \Xi \to P \) is defined by
\[ v(\xi)(f) = \sum_{W \in \text{dmn } \xi} \xi(W)W(f) \text{ whenever } f \in \mathcal{H}(U \times G(n, m)). \]
Furthermore, \( \xi \in \Xi \) is called maximal with respect to a Borel set \( B \) if and only if \( ||W||B(\xi) > 0 \) for all \( W \in \text{dmn } \xi \) and
\[ \sum \xi \geq \sum \rho \]
whenever \( \rho \in \Xi \) satisfies
\[ v(\rho) = v(\xi) \]
and \( ||X||B(\rho) > 0 \) for all \( X \in \text{dmn } \rho \). We say \( W \) splits \( V \) in \( P \) if and only if \( W \in P, V \in P, V - W \in P, \) and \( \|\delta W\| + \|\delta(V - W)\| = \|\delta V\| \).

**Theorem 5.12** Suppose \( m, n \in \mathcal{P}, m \leq n, U \) is an open subset of \( \mathbb{R}^n \), \( P \subset RV_m(U) \) is appropriate, \( V \in P \) and \( \|\delta V\| \) is a Radon measure. Then, there exists a decomposition of \( V \) with respect to \( P \).

**Proof** Assume \( V \neq 0 \). Define \( \delta_i = \alpha(m)2^{-m-1}i^{-m}, \varepsilon_i = 2^{-1}i^{-1} \) for \( i \in \mathcal{P} \) and let \( A_i \) denote the Borel set of \( a \in \mathbb{R}^n \) satisfying
\[ |a| \leq i, \quad U(a, 2\varepsilon_i) \subset U, \quad 1 \leq \Theta^m(||V||, a) < \infty, \]
\[ \|\delta V\|B(a, r) \leq \alpha(m)ir^m \quad \text{for } 0 < r < \varepsilon_i \]
whenever \( i \in \mathcal{P} \). Clearly, \( A_i \subset A_{i+1} \) for \( i \in \mathcal{P} \) and \( ||V||B(U \sim \bigcup_{i=1}^\infty A_i) = 0 \) by [1, 3.5 (1a)] and [6, 2.8.18, 2.9.5]. For each \( i \in \mathcal{P} \), we infer from 5.10 that
\[ \|W\|B(a, \varepsilon_i) \geq \delta_i \text{ whenever } a \in A_i \text{ and } W \in P \text{ satisfy } \]
\[ W \leq V, ||\delta W|| \leq ||\delta V||, \text{ and } \Theta^m(||W||, a) \geq 1 \]
\[ (1) \]
and hence
\[
\delta_i \sum_{x \in \text{dmn} \xi} \xi(W) \|x : \text{dist}(x, A_i) \leq \varepsilon_i\| \\
= \|v(\xi)\| [x : \text{dist}(x, A_i) \leq \varepsilon_i] < \infty
\]
whenever \( \xi \in \Xi \) satisfies \( v(\xi) \leq V, \|\delta v(\xi)\| \leq \|\delta V\|, \) and \( \|W\|(A_i) > 0 \) for each \( W \in \text{dmn} \xi. \)

Since \( V \neq 0 \), there exists \( \lambda \in \mathcal{P} \) such that \( \|V\|(A_\lambda) > 0 \). From now on, we replace \( A_i \) by \( A_{i+\lambda} \) for \( i \in \mathcal{P} \). Let
\[
R = P \cap \{W : W \leq V, \|\delta W\| \leq \|\delta V\|,\}
\]
\[
P_i = R \cap \{W : \|W\|(A_i) > 0\} \quad \text{whenever } i \in \mathcal{P}.
\]
Then, we may select functions \( c_i : P_i \rightarrow \Xi \) such that \( v(c_i(W)) = W \) and \( c_i(W) \) is maximal with respect to \( A_i \); in particular, \( \text{dmn} c_i(W) \subset P_i \) whenever \( W \in P_i \).

From now on, we will use the convention that \( \infty \cdot 0 = 0 \). Let \( \Sigma \) be the class of all sequences \( Z_1, Z_2, Z_3, \ldots \) in \( P \) satisfying \( Z_1 = V \) and \( Z_{i+1} \in \text{dmn} c_i(Z_i) \) and abbreviate \( \lim Z = \lim_{i \to \infty} Z_i \in P \) for \( Z \in \Sigma \), where the limit is taken with respect to the strong topology. Let \( C = \Sigma \cap \{Z : \text{lim } Z \neq 0\} \) and define \( v : \Sigma \to \mathcal{P} \cup \{\infty\} \) by
\[
v(Z) = \prod_{i=1}^{\infty} c_i(Z_i)(Z_{i+1}).
\]
Note that for \( Z \in \Sigma, \) and \( i, j \in \mathcal{P} \) with \( i \leq j \), we have
\[
\prod_{k=i}^{j} c_k(Z_k)(Z_{k+1})Z_{j+1} \leq Z_i,
\]
hence
\[
v(Z) \lim Z \leq \prod_{k=1}^{j} c_k(Z_k)(Z_{k+1})Z_{j+1} \leq V;
\]
thus, \( \text{im}(v|C) \subset \mathcal{P}. \)

Now, we aim to prove \( C \) is countable,
\[
\sum_{Z \in C} v(Z) \lim Z \leq V, \quad \text{and} \quad \sum_{Z \in C} v(Z)\|\delta \lim Z\| \leq \|\delta V\|.
\]
For \( i \in \mathcal{P}, \) let \( F_i \) be the set of all finite sequences \( Z_1, Z_2, \ldots, Z_i \) such that \( Z_1 = V \) and \( Z_{j+1} \in \text{dmn} c_j(Z_j) \) for \( 1 \leq j \leq i - 1 \). We define, for \( i \in \mathcal{P}, \) the restriction map
\[
R_i : \Sigma \cup \bigcup_{i < j \in \mathcal{P}} F_j \to F_i, \quad R_i(Z) = Z|\mathcal{P} \cap \{k : 1 \leq k \leq i\}.
\]
Observe that whenever \( i \in \mathcal{P}, Y \in F_i, \) and \( F \) is a finite subset of \( \Sigma \) satisfying \( R_i[F] = \{Y\}, \) there exists \( j \in \mathcal{P} \) such that \( j > i \) and \( R_j|F \) is injective and that
\[
\sum_{X \in F_j, R_i(X) = Y} \prod_{k=1}^{j-1} c_k(X_k)(X_{k+1})X_j = \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})Y_i \quad (2)
\]
whenever \( i, j \in \mathcal{P} \) with \( i < j \). Therefore, we have
\[
\sum_{Z \in F} v(Z) \lim_{Z \to F} Z \leq \sum_{Z \in F} \prod_{k=1}^{j-1} c_k(Z_k)(Z_{k+1})Z_j
\]
\[
\leq \sum_{X \in F_j, R_t(X) = Y} \prod_{k=1}^{j-1} c_k(X_k)(X_{k+1})X_j
\]
\[
= \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})Y_i;
\]

similarly, we have
\[
\sum_{Z \in F} v(Z)\| \delta \lim_{Z \to F} Z \| \leq \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})\| \delta Y_i \|.
\]

Choosing by [8, 2.2, 2.23] countable dense subsets of \( \mathcal{K}(U \times G(n, m))^+ \) and \( \mathcal{K}(U)^+ \), we conclude from (3), (4) with \( i = 1 \) and [6, 2.1.1(3)] that
\[
\sum_{Z \in \Sigma} v(Z) \lim_{Z \to F} Z(f) \leq V(f) \quad \text{whenever } f \in \mathcal{K}(U \times G(n, m))^+,
\]
\[
\sum_{Z \in \Sigma} v(Z)\| \delta \lim_{Z \to F} Z \|(g) \leq \| \delta V \|(g) \quad \text{whenever } g \in \mathcal{K}(U)^+;
\]

in particular, \( C \) is countable by [6, 2.1.1(12)].

Next, to show the equalities in (5) hold, we shall first prove that, for \( \mathcal{K}^m \) almost all \( x \),
\[
\sum_{Z \in \mathcal{C}} v(Z)\Theta^m(\| \lim_{Z \to F} Z \|, x) = \Theta^m(\| V \|, x).
\]

If \( x \in U \) satisfies \( \Theta^m(\| V \|, x) = 0 \), then (6) is trivial. To the other case, let \( B \) consist of all \( x \in \bigcup_{i=1}^{\infty} A_i \) such that
\[
\Theta^m(\| W \|, x) \in \{0\} \cup (\mathbb{R} \cap \{ t : 1 \leq t \}),
\]
\[
\Theta^m(\| \lim_{Z \to F} Z \|, x) = \lim_{i \to \infty} \Theta^m(\| Z_i \|, x)
\]

whenever \( W \in \bigcup_{Z \in \Sigma} \text{im } Z \) and \( Z \in C \); in particular,
\[
\Theta^m(\| Z_i \|, x) = \sum_{W \in \text{dimg } c_i(Z_i)} c_i(Z_i)(W)\Theta^m(\| W \|, x) \quad \text{for all } x \in B
\]

whenever \( Z \in \Sigma \) and \( i \in \mathcal{P} \). For \( x \in B \) and \( Z \in \Sigma \), we abbreviate
\[
\Theta_Z(x) = \lim_{i \to \infty} \Theta^m(\| Z_i \|, x)
\]

and the crucial observation is that
\[
\Theta_Z(x) > 0 \quad \text{implies } Z \in C \quad \text{and } \Theta_Z(x) = \Theta^m(\| \lim_{Z \to F} Z \|, x);
\]

in fact, if \( x \in A_i \), then by [1, 2.6(2c)] and (1), we have \( \lim_{Z \to B(x, \delta_i)} Z \geq \delta_i \) and the assertion follows. Therefore, \( E_x = \Sigma \cap \{ Z : \Theta_Z(x) > 0 \} \subset C \) whenever \( x \in B \). Similarly as (5), we derive from (7) that
\[
\sum_{Z \in \Sigma} v(Z)\Theta_Z(x) \leq \Theta^m(\| V \|, x) < \infty,
\]

\( \Theta \) Springer
and it follows that $E_x$ is finite whenever $x \in B$. Accordingly, letting $x \in B$, there exists $i \in \mathcal{P}$ such that $R_j|E_x$ is injective for $i < j \in \mathcal{P}$; furthermore, we observe that

$$R_j[E_x] = F_j \cap \{Z : \Theta^m(\|Z_j\|, x) > 0\} \text{ whenever } i < j \in \mathcal{P}. \quad (8)$$

By (7) and (8), we have

$$\Theta^m(\|V\|, x) = \sum_{Z \in F_j} \prod_{k=1}^{j-1} c_k(Z_k)(Z_{k+1})\Theta^m(\|Z_j\|, x)$$

$$= \sum_{Z \in E_x} \prod_{k=1}^{j-1} c_k(Z_k)(Z_{k+1})\Theta^m(\|Z_j\|, x)$$

whenever $i < j \in \mathcal{P}$; letting $j \to \infty$, we conclude

$$\sum_{Z \in E_x} v(Z)\Theta^m(\|Z\|, x) = \Theta^m(\|V\|, x).$$

Therefore, (6) holds for all $x \in B$, and it remains to prove

$$\mathcal{H}^m(\{x : \Theta^m(\|V\|, x) > 0\} \sim B) = 0.$$ 

Note that $\mathcal{H}^m(\{x : \Theta^m(\|V\|, x) > 0\} \sim \bigcup_{i=1}^{\infty} A_j) = 0$, since $\|V\|(U \sim \bigcup_{i=1}^{\infty} A_j) = 0$. Thus, it is enough to show $\mathcal{H}^m(\bigcup_{i=1}^{\infty} A_i \sim B) = 0$. From [1, 3.5(1b)], we write $\|\lim Z\| = \mathcal{H}^m \setminus \Theta^m(\|Z\|, \cdot)$ and $\|Z_i\| = \mathcal{H}^m \setminus \Theta^m(\|Z_i\|, \cdot)$ whenever $Z \in C$ and $i \in \mathcal{P}$. Since $\|Z_i\|$ converges to $\|\lim Z\|$ with respect to the strong topology and $\|Z_i\|$ is non-increasing, it follows from 5.2 that

$$\Theta^m(\|\lim Z\|, x) = \lim_{i \to \infty} \Theta^m(\|Z_i\|, x) \text{ for } \mathcal{H}^m \text{ almost all } x.$$

Then, the assertion follows from that $C \cup \bigcup_{Z \in C} \text{im } Z$ is countable. From (6), we deduce

$$\sum_{Z \in C} v(Z)\lim Z = \|V\|;$$

it follows that $Tan^m(\|\lim Z\|, x) = Tan^m(\|V\|, x)$ for $\|\lim Z\|$ almost all $x$ whenever $Z \in C$, hence

$$\sum_{Z \in C} v(Z)\lim Z = V,$$

$$\sum_{Z \in C} v(Z)\delta \lim Z = \|\delta V\|. \quad (9)$$

Furthermore, it follows from (3) that for $i \in \mathcal{P}$ and $Y \in F_i$,

$$\sum_{Z \in C, R_i(Z) = Y} v(Z)\lim Z \leq \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})Y_i.$$

hence from [6, 2.1.1(9)] that

$$\sum_{Z \in C} v(Z)\lim Z = \sum_{Y \in F_i} \sum_{Z \in C, R_i(Z) = Y} v(Z)\lim Z$$

$$\leq \sum_{Y \in F_i} \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})Y_i$$
which forces
\[ \sum_{Z \in C, R_i(Z) = Y} v(Z) \lim Z = \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})Y_i \quad \text{for } i \in \mathcal{P} \text{ and } Y \in F_i; \]
similarly, we have
\[ \sum_{Z \in C, R_i(Z) = Y} v(Z)\| \delta \lim Z \| = \prod_{k=1}^{i-1} c_k(Y_k)(Y_{k+1})\| \delta Y_i \| \quad \text{for } i \in \mathcal{P} \text{ and } Y \in F_i. \]

In particular, \( \lim Z \) splits \( Z_i \) in \( P \) whenever \( Z \in C \) and \( i \in \mathcal{P} \).

Finally, we will prove \( \lim Z \) is indecomposable with respect to \( P \) whenever \( Z \in C \). If it were not the case, there would exist \( W \in P \sim \{0\} \) such that
\[
W \leq \lim Z, \quad \lim Z - W \in P \sim \{0\}
\]
and \( \| \delta \lim Z \| = \| \delta W \| + \| \delta (\lim Z - W) \| \).

We could choose \( i \) large such that \( \| W \| (A_i) > 0 \) and \( \| \lim Z - W \| (A_i) > 0 \), since \( \| W \| + \| \lim Z - W \| \leq \| V \| \). Then, \( W \) would split \( Z_{i+1} \), \( \| W \| (A_i) > 0 \) and
\[
\| Z_{i+1} - W \| (A_i) \geq \| \lim Z - W \| (A_i) > 0,
\]
in contradiction to the maximality of \( c_i(Z_i) \) with respect to \( A_i \).

Let \( H = \{ \lim Z : Z \in C \} \) and define \( \xi : H \to \mathcal{P} \) by \( \xi(W) = \sum_{C \in Z, \lim Z = W} v(Z) \), then \( (H, \xi) \) is a decomposition of \( V \) with respect to \( P \).

\[ \square \]

**Remark 5.13** The structure of the proof of 5.12 is similar to the one of [8, 6.12]. However, since we allow to decompose varifolds not just by restriction to subsets, we should take the multiplicity of varifolds into account which makes it much more complicated to verify the condition 5.8 (2).

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**Declarations**

**Conflict of interest** The authors declare no competing interests.

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