Do all pure entangled states violate Bell’s inequalities for correlation functions?

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Quantum mechanics violates Bell type inequalities \footnote{Gisin 3, 4, 5} which hold for any local realistic theory. In a realistic model the measurement results are determined by "hidden" properties the particles carry prior to and independent of observation. In a local model the results obtained at one location are independent of any measurements or actions performed at space-like separation. The theorem of Gisin \cite{Gisin} states that any pure non-product state of two particles violates a Clauser-Horne-Shimony-Holt (CHSH) \footnote{CHSH inequality} inequality, which involves only two-particle correlation functions, for two alternative dichotomic observables per local measuring station. We also find that Mermin-Ardehali-Belinskii-Klyshko inequalities may not always be optimal for refutation of local realistic description.

Any pure entangled state of two particles violates a Bell inequality for two-particle correlation functions (Gisin’s theorem). We show that there exist pure entangled \(N > 2\) qubit states that do not violate any Bell inequality for \(N\) particle correlation functions for experiments involving two dichotomic observables per local measurement station. We also find that Mermin-Ardehali-Belinskii-Klyshko inequalities may not always be optimal for refutation of local realistic description.

Theorem of Gisin \cite{Gisin} states that any pure non-product state of two particles violates a Clauser-Horne-Shimony-Holt (CHSH) inequality, which involves only two-particle correlation functions, for two alternative dichotomic measurements for each of the local observers.

Can Gisin’s theorem be generalized to all \(N\)-particle pure entangled states? We show here that this is not the case for Bell inequalities involving only correlation functions in experiments in which local observers can choose between two dichotomic observables. We find a family of pure entangled states of \(N\) qubits which do not violate any such Bell inequality. This family is a subset of a larger one of the generalized GHZ states given by

\[
|\psi\rangle = \cos \alpha |0, \ldots, 0\rangle + \sin \alpha |1, \ldots, 1\rangle,
\]

with \(0 \leq \alpha \leq \pi/4\). The GHZ states \footnote{GHZ states} are for \(\alpha = \pi/4\).

Scarani and Gisin \cite{Scarani} noticed a surprising feature of such states. They show that for \(\sin 2\alpha \leq 1/\sqrt{2}^{N-1}\) the states \footnote{states do not violate the Mermin-Ardehali-Belinskii-Klyshko (MABK) inequalities} do not violate the Mermin-Ardehali-Belinskii-Klyshko (MABK) inequalities \footnote{MABK inequalities}. This has been obtained numerically for \(N = 3, 4, 5\) and conjectured for \(N > 5\). Their result contrasts the case of \(N = 2\) of two qubits and is surprising as the states \footnote{are a generalization of the GHZ states} which violate maximally the MABK inequalities.

Scarani and Gisin write that "this analysis suggest that MK [here, MABK] inequalities, and more generally the family of Bell’s inequalities with two observables per qubit, may not be the ‘natural’ generalizations of the CHSH inequality to more than two qubits" \footnote{are a generalization of the GHZ states}. Concerning this question we find here an interesting discrepancy between the case of even and odd number of qubits. We prove that for all \(N\) odd and for \(\sin 2\alpha \leq 1/\sqrt{2}^{N-1}\) the generalized GHZ states satisfy all possible Bell inequalities for \(N\)-particle correlation functions, which involve two alternative dichotomic observables at each local measurement station. Note also, that since the reduced density matrices of all proper subsystems of the \(N\)-qubit system described by \footnote{are a generalization of the GHZ states} are separable, no Bell inequality for \(K < N\) particle correlation functions can be violated. Thus, Gisin’s theorem cannot be straightforwardly generalized in this case. We also find that for all \(N\) even the generalized GHZ state always violates a Bell inequality for \(N\)-particle correlation functions. Interestingly, this inequality, as conjectured in \footnote{are a generalization of the GHZ states}, is not a MABK one. This implies that MABK inequalities may not always be optimal for refutation of local realism.

It is important to notice that the measurements on qubits in the states \footnote{are a generalization of the GHZ states} can violate Bell inequalities if (i) one makes an additional postselection of measurements, and/or (ii) not all \(N\) observers are separated. Popescu and Rohrlich \footnote{showed that no local realist description is possible for any pure multipartite entangled state, provided additional manipulations are allowed.} showed that for all \(N\) even the generalized GHZ state always violates a Bell inequality for \(N\)-particle correlation functions. Interestingly, this inequality, as conjectured in \footnote{are a generalization of the GHZ states}, is not a MABK one. This implies that MABK inequalities may not always be optimal for refutation of local realism.

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entangled states of \( N \) qubits are distillable. However, in the present paper, as in the standard Bell-GHZ type situation, we are interested in the correlations between measurements on \( N \) \textit{separated} qubits without postselection.

We base our analysis on the results of Ref.\cite{10}. Here is a summary. A single generalized Bell inequality was derived, which is equivalent to the full set of \( 2^{2N} \) Bell inequalities for the correlation functions between measurements on \( N \) particles, which involve two alternative dichotomic observables at each local measurement station \cite{10,11,12}. The correlation functions for such measurements cannot be described by a local realistic model, if and only if, the generalized Bell inequality is violated. In parallel, in Ref. \cite{10} a \textit{necessary} and \textit{sufficient} condition was derived for the correlations of \( N \) qubits in an arbitrary state to violate the generalized Bell inequality. \textit{(Recently it was shown \cite{13}, that all states violating the generalized Bell inequality are distillable.)}

The condition is as follows. Consider \( N \) observers and allow each of them to choose a local coordinate system. An arbitrary quantum state \( \rho \) of \( N \) qubits can be represented by tensor products of local Pauli operators in the following way

\[
\rho = \frac{1}{2^N} \sum_{x_1,\ldots,x_N=0}^{3} T_{x_1\ldots x_N} \sigma_{x_1} \otimes \ldots \otimes \sigma_{x_N}. \tag{2}
\]

Here \( \sigma_0 \) is the identity operator and \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are three Pauli operators associated with mutually orthogonal directions. The coefficients \( T_{x_1\ldots x_N} = \text{Tr}[\rho(\sigma_{x_1} \otimes \ldots \otimes \sigma_{x_N})] \) for \( x_i = 1, 2, 3 \) are elements of the \( N \)-qubit correlation tensor \( \hat{T} \).

The \textit{necessary and sufficient condition for a quantum state to satisfy the generalized \( N \)-particle Bell inequality} is given by the following criterion: in \textit{any} set of local coordinate systems for the \( N \) observers and for \textit{any} set of unit vectors \( \vec{c} = (c_1^x, c_2^y, c_3^z) \) one has

\[
T^\text{mod}_{\vec{c}_1, \ldots, \vec{c}_N} \equiv \sum_{x_1,\ldots,x_N=1}^{2} c_{x_1}^1 \cdot c_{x_N}^N |T_{x_1\ldots x_N}| \leq 1. \tag{3}
\]

This can be restated as follows. Suppose, one replaces the components of the correlation tensor \( T_{x_1\ldots x_N} \) by their moduli \( |T_{x_1\ldots x_N}| \), and builds of such moduli a new tensor \( \hat{T}^\text{mod} \). Suppose moreover, that this new tensor is transformed to a new set of local coordinate systems. This transformation is for each local observer a rotation in the plane spanned by the orthogonal directions 1 and 2. If the coordinates of the transformed tensor \( \hat{T}^\text{mod}_{\vec{c}_1, \ldots, \vec{c}_N} \) always satisfy constraint (3), i.e. are not greater than 1, then, and only then, a local realistic description is possible.

By applying the Cauchy inequality to the middle term of expression (3) one obtains a \textit{simple sufficient condition for the generalized \( N \)-particle inequality to be satisfied}:

in \textit{any} set of local coordinate systems of \( N \) observers one must have

\[
\sum_{x_1,\ldots,x_N=1}^{2} T^2_{x_1\ldots x_N} \leq 1. \tag{4}
\]

In the conditions (3) and (4) the sums are taken over any two orthogonal axes 1 and 2 for all local coordinate systems. The two axes can be either \( x \) and \( y \), or \( x \) and \( z \), or \( x \) and \( z \). Since the full group of rotations contains within itself cyclic permutations of the coordinates (e.g. \( x \rightarrow y \rightarrow z \rightarrow x \)), it is enough to consider just one particular choice for a set pairs of local axes, and then consider \textit{arbitrary} local transformations of the full tensor. In other words, one can work with a particular sub-tensor of the correlation tensor, which is build by the components containing only, say, the \( x \) and \( y \) indices.

Once a sub-tensor is chosen, then on one hand, if the inequality (4) holds under \textit{arbitrary} changes of the local coordinate systems, the correlations described by the quantum state satisfy the generalized inequality. On the other hand, if the condition (3) is violated at least for one choice of local coordinate systems, no local realistic description is possible for the \( N \)-particle correlations.

When considering arbitrary rotation, we will use the Euler theorem. Any rotation of a local coordinate system can be expressed as a sequence of rotations around three axes \( x \), \( x' \) and \( z'' \). Note that rotations of coordinate systems of different observers commute.

Using a lengthy but otherwise straightforward algebra one can show that the correlation tensor for the states (1) has only the following nonvanishing components: (i) For \( N \) even \( T_{x-z} = 1 \) and \( T_{x-x} = \sin 2\alpha \), and all components with 2 \( k \)'s and otherwise only \( x \)'s (e.g., for \( N = 4 \), \( T_{xyz} \), etc.) are equal to \(-1\)^k sin 2\( \alpha \). (ii) For \( N \) odd \( T_{x-z} = \cos 2\alpha \), and \( T_{x-x} = \sin 2\alpha \), and all components with 2 \( k \)'s and otherwise only \( x \)'s (e.g., for \( N = 3 \), \( T_{xyy} \), etc.) are equal to \(-1\)^k sin 2\( \alpha \). It will be convenient to represent the correlation tensor as a sum of tensor products of unit three-dimensional vectors:

\[
\hat{T} = \sum_{x_1,\ldots,x_N=1}^{3} T_{x_1\ldots x_N} \vec{e}_1 \otimes \ldots \otimes \vec{e}_N. \tag{5}
\]

We now give the main technical results.

\textbf{Statement 1:} For \( \sin 2\alpha \leq 1/\sqrt{2^{N-1}} \) and \( N \) odd, the correlations between measurements on qubits in the \textit{generalized} GHZ state (1) satisfy all Bell inequalities for correlation functions, \textit{which involve two dichotomic observables per local measurement station}.

The main idea of the proof is to show that condition (4) is satisfied for the range of \( \alpha \) given above. We give the proof for \( N = 3 \). We show that, for this range of \( \alpha \), the value of \( \sum_{(i,j,k=x,y)} T_{ijk}^2 \) after an arbitrary set of local rotations is performed is never larger than one. The proof for general odd \( N \) is a straightforward generalization.
The correlation tensor of state (1), for \( N = 3 \), reads
\[
\hat{T} = \cos 2\alpha \bar{z}_1 \otimes \bar{z}_2 \otimes \bar{z}_3 + \sin 2\alpha \left[ x_1 \otimes x_2 \otimes \bar{x}_3 \right] \quad (6)
\]
\[ - \bar{x}_1 \otimes \bar{y}_2 \otimes \bar{y}_3 - \bar{y}_1 \otimes \bar{x}_2 \otimes \bar{y}_3 - \bar{y}_1 \otimes \bar{y}_2 \otimes x_3 , \]

We first rotate the local coordinate systems of each of the observers around the local \( \vec{x} \) axes. Such a set of rotations of course leaves \( \sum_{i,j,k=x,y,z} T_{ijk}^2 \) invariant, therefore it stays put at its initial value \( 4 \sin^2 2\alpha \leq 1 \). The correlation tensor in the new set of local coordinate systems is given by
\[
\hat{T}' = \cos 2\alpha \bar{z}_1 \otimes \bar{z}_2 \otimes \bar{z}_3 + \sin 2\alpha \sum_{i,j,k=x,y,z} T'_{ijk} \bar{y}_1 \otimes \bar{y}_2 \otimes \bar{y}_3 \quad (7)
\]
The values for \( T'_{ijk} \) can be obtained by replacing \( \bar{x}_1 \to \cos \theta \bar{x}_1' + \sin \theta \bar{y}_1' \) and \( \bar{y}_1 \to \cos \theta \bar{y}_1' - \sin \theta \bar{x}_1' \) in Eq. (1) for \( \hat{T} \). The components satisfy the following relations
\[
T'_{xxy} = T'_{yxy} = T'_{yyx} = T'_{xxx} \cos^2 \sum_{i=1}^{3} \phi_i , \quad (8)
\]
\[
T'_{yzz} = T'_{zzx} = T'_{xxz} = T'_{yyy} \sin^2 \sum_{i=1}^{3} \phi_i . \quad (9)
\]
Next, we rotate the local coordinate systems of each of the observers around the local \( \vec{y} \) by the angle \( \theta \). Now the specific values for \( T_{ijk}^2 \) can be obtained by replacing \( \bar{z}_i' \to \cos \theta \bar{z}_i'' + \sin \theta \bar{z}_i' ' \) and \( \bar{y}_i' \to \cos \theta \bar{y}_i'' - \sin \theta \bar{y}_i' ' \) in Eq. (6). The new components in the \( xy \) sector of the correlation tensor satisfy the following relations
\[
\sum_{i,j,k=x,y,z} T'_{ijk} \leq 1 \quad (11)
\]
and for all other cases \( \sum_{i,j,k=x,y,z} T'_{ijk} \leq 1 \) provided \( \sin 2\alpha \leq 1/2 \). Thus, for the considered range of \( \alpha \), after arbitrary subsequent local rotations along three local \( z \) axes and arbitrary local rotations along three \( x' \) axes, one has \( \sum_{i,j,k=x,y,z} T''_{ijk} \leq 1 \).

The final stage of our proof rest upon the observation, that the final Euler rotation of the local coordinate systems of each of the three observers around axes \( x'' \) leaves the sum of squares of the components in the \( xy \) sector of the tensor invariant.

The proof for an arbitrary odd \( N \) follows the same pattern. For \( \alpha \) satisfying \( \sin 2\alpha \leq 1/\sqrt{2N-1} \), the generalized Bell inequalities cannot be violated. The threshold value for \( \alpha \) decreases exponentially with the growing odd \( N \) because the number of nonvanishing components in the \( xy \) sector of the correlation tensor of the state (1) is \( 2^{N-1} \).

Thus the sufficient condition is met. All possible Bell inequalities for \( N \) particle correlation functions in tests involving two alternative dichotomic observables for each observer must hold, for the range of \( \alpha \) given above. Recalling that for (1) any subset of less than \( N \) qubits is in a separable state, we conclude that all possible Bell inequalities for correlation functions in tests involving two alternative observables for each observer must hold, for the range of \( \alpha \) given above. Outside of this range the MABK inequalities can always be violated (3), and consequently also the generalized Bell inequality.

Statement 2: The correlations between measurements for an even number of qubits in the generalized GHZ state (1) cannot be described within a local realistic model.

We shall show, that for all even \( N \) the generalized GHZ state violates the constraint (3). We give the proof only for \( N = 4 \). The generalization to arbitrary \( N \) is obvious. Of course the \( N = 2 \) is covered by Gisin’s theorem (1).

For \( N = 4 \) the sector of the correlation tensor \( \hat{T} \), which is limited to \( x \) and \( z \) components, is given by
\[
\hat{T}_{[xz]} = \bar{z}_1 \otimes \bar{z}_2 \otimes \bar{z}_3 \otimes \bar{z}_4 + \sin 2\alpha x_1 \otimes x_2 \otimes x_3 \otimes \bar{x}_4 . \quad (12)
\]

Let us first rotate the coordinate axes of the first three observers around the local directions \( \vec{y} \) by \( 45^\circ \), that is
\[
\bar{z}_i = \frac{1}{\sqrt{2}} \left( \bar{z}_i' + \bar{z}_i'' \right) \quad \text{and} \quad \bar{x}_i = \frac{1}{\sqrt{2}} \left( \bar{x}_i' - \bar{z}_i'' \right) \quad \text{for} \quad i = 1, 2, 3.
\]

The sub-tensor in the new coordinates reads
\[
\hat{T}'_{[xz]} = \frac{1}{2^{3/2}} \left( \bar{z}_1' + \bar{z}_2' \right) \otimes \left( \bar{x}_2' + \bar{z}_3' \right) \otimes \left( \bar{x}_3' + \bar{z}_3' \right) \otimes \left( \bar{x}_4' - \bar{z}_4' \right) \quad \text{for} \quad i = 1, 2, 3 ,
\]

Next, we build a new tensor \( \hat{T}'_{[xz]} \) by replacing all components of \( \hat{T}'_{[xz]} \) by their moduli:
\[
\hat{T}'_{[xz]} (mod) = \frac{1}{2^{3/2}} \left( \bar{z}_1' + \bar{z}_2' \right) \otimes \left( \bar{x}_2' + \bar{z}_3' \right) \otimes \left( \bar{x}_3' + \bar{z}_3' \right) \otimes \left( \bar{x}_4' + \sin 2\alpha \bar{x}_4' \right).
\]
Therefore we find local coordinate systems in which our modified tensor has a component of a value higher than 1 for all non-vanishing values of $\sin 2\alpha$. The criterion (4) is violated. No local realistic description is possible.

To generalize the proof to an arbitrary even $N$ it is enough to notice, that the sub-tensor built out of the $zx$ components of the full tensor has the characteristic form

$$
\tilde T_{[zx]} = \tilde z_1 \otimes \tilde z_2 \ldots \otimes \tilde z_N \pm \sin 2\alpha \tilde x_1 \otimes \tilde x_2 \ldots \otimes \tilde x_N,
$$

and proceed as in the case $N = 4$.

The correlations between the measurements on an even number of qubits in the GHZ generalized state do not allow any local realistic model. Since the generalized Bell inequality is violated, i.e., at least one out of the full set of $2^N$ inequalities implied by the generalized one must be violated. Surprisingly, the violated inequality is not a MABK one but a generalized CHSH inequality.

Consider a Bell experiment in which observers 1 and 2 choose between two dichotomic observables and the other $N-2$ ones keep their settings at $\vec z$ unchanged. The states $|0\rangle$ and $|1\rangle$ in Eq. (1) are eigenstates of $\vec z \cdot \vec \sigma$. Since $N-2$ is also even, the product of the local results of the $N-2$ observers for the case of the generalized GHZ state (i) is always 1. Thus their results effectively do not contribute to the value of the total correlation function. One has $E(\vec n_{k_1}, \vec n_{k_2}, \vec z, \ldots, \vec z) = E(\vec n_{k_1}, \vec n_{k_2})$. Therefore within the local realism these correlation functions have to satisfy the CHSH inequality. Of course the generalized GHZ state (i) violates it for the whole range of $\alpha \neq 0$.

What are the reasons for the completely different behavior for $N$ even and $N$ odd? The expression on the left-hand side of (4) can be understood as a ”total measure of the strength of correlations” in mutually complementary sets of local measurements (as defined by the summation over $x$ and $y$). Then the unity on the right-hand side of (4) is the classical limit for the amount of correlations. Specifically, pure product states cannot exceed the limit of 1, as they can show perfect correlations in one set of local measurements directions only. In contrast, entangled states can show perfect correlations for more than one such set (3). Now, only if $N$ is even, the states (i) already show perfect correlation between measurements along $z$-directions (as the product is then always $+1$) reaching therefore the classical limit. Yet, they also show additional correlations in other, complementary, directions. In the case of $N$ odd, however, there is no perfect correlation between measurements along $z$-directions and the amount of correlations in complementary directions are not enough to violate (4).

In summary, we have shown that Gisin’s theorem cannot be straightforwardly generalized to all multi-particle systems, for the case of Bell inequalities involving correlation functions, in which local observers can choose between two dichotomic observables (see Statement 1 and 2). However, the question posed in the title of this letter may find a different answer, if, e.g., more than two dichotomic observables per local measurement station are allowed.

Our results may shed a new light on the connection between the violation of Bell inequalities and the quantum information tasks (3, 4). In the problem of classification of entangled states our results reveal a new class of pure entangled states which do not violate any Bell inequalities for correlation functions.

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