BOULIGAND-SEVERI TANGENTS IN MV-ALGEBRAS

MANUELA BUSANICHE AND DANIELE MUNDICI

Abstract. In their recent seminal paper published in the Annals of Pure and Applied Logic, Dubuc and Poveda call an MV-algebra $A$ strongly semisimple if all principal quotients of $A$ are semisimple. All boolean algebras are strongly semisimple, and so are all finitely presented MV-algebras. We show that for any 1-generator MV-algebra semisimplicity is equivalent to strong semisimplicity. Further, a semisimple 2-generator MV-algebra $A$ is strongly semisimple iff its maximal spectral space $\mu(A) \subseteq [0, 1]^2$ does not have any rational Bouligand-Severi tangents at its rational points. In general, when $A$ is finitely generated and $\mu(A) \subseteq [0, 1]^n$ has a Bouligand-Severi tangent then $A$ is not strongly semisimple.

1. Introduction: stable consequence

We refer to [3] and [6] for background on MV-algebras. Following Dubuc and Poveda [4], we say that an MV-algebra $A$ is strongly semisimple if for every principal ideal $I$ of $A$ the quotient $A/I$ is semisimple. Since $\{0\}$ is a principal ideal of $A$, every strongly semisimple MV-algebra is semisimple.

From a classical result by Hay [5] and Wójcicki [11] (also see [3, 4.6.7] and [6, 1.6]), it follows that every finitely presented MV-algebra is strongly semisimple. Trivially, all hyperarchimedean MV-algebras, whence in particular all boolean algebras, are strongly semisimple, and so are all simple and all finite MV-algebras, [3, 3.5 and 3.6.5].

Our paper is devoted to characterizing $n$-generator strongly semisimple MV-algebras for $n = 1, 2$. In Theorem 2.1 we show that when $n = 1$ strong semisimplicity is equivalent to semisimplicity.

As the reader will recall ([3, 9.1.5]), the free $n$-generator MV-algebra is the MV-algebra $M([0, 1]^n)$ of all McNaughton functions $f : [0, 1]^n \to [0, 1]$, with pointwise operations of negation $\neg x = 1 - x$ and truncated addition $x \oplus y = \min(1, x + y)$.

For any nonempty closed set $X \subseteq [0, 1]^n$ we let $\mathcal{M}(X)$ denote the MV-algebra of restrictions to $X$ of the functions in $\mathcal{M}([0, 1]^n)$, in symbols,

$$\mathcal{M}(X) = \{ f \upharpoonright X \mid f \in \mathcal{M}([0, 1]^n) \}.$$ 

By [3, 3.6.7], $\mathcal{M}(X)$ is a semisimple MV-algebra—actually, up to isomorphism, $\mathcal{M}(X)$ is the most general possible $n$-generator semisimple MV-algebra $A$: to see this, pick generators $\{a_1, \ldots, a_n\}$ of $A$. Let $\pi_i : [0, 1]^n \to [0, 1]$ be the projection functions in the free MV-algebra $\mathcal{M}([0, 1]^n)$ for $i = 1, \ldots, n$. Then the assignment that maps $\pi_i \mapsto a_i$ for each $i = 1, \ldots, n$ uniquely extends to a homomorphism $\eta_a : \mathcal{M}([0, 1]^n) \to A$ of the free $n$-generator MV-algebra onto $A$. Let $\mathfrak{h}_a = \ker(\eta_a)$ be the kernel of this homomorphism and

$$Z_a = \bigcap \{ Zf \mid f \in \mathfrak{h}_a \} \quad (1)$$
the intersection of the zeroesets of the McNaughton functions in $h_a$. Then
\[ A \cong \mathcal{M}(\mathbb{Z}_a). \] (2)

In Theorem 3.4 we prove that a 2-generator MV-algebra $A = \mathcal{M}(X)$ with $X \subseteq [0, 1]^2$ is strongly semisimple iff $X$ has no rational outgoing Bouligand-Severi tangent vector at any of its rational points, [1, 9, 7]. Having such a tangent is a sufficient condition for $\mathcal{M}(X)$ not to be strongly semisimple, for any $X \subseteq [0, 1]^n$, (Theorem 3.3). Here, as usual, a point $x \in \mathbb{R}^n$ is said to be rational if so are all its coordinates.

By a rational vector we mean a nonzero vector $w \in \mathbb{R}^n$ such that the line $\mathbb{R}w \subseteq \mathbb{R}^n$ contains at least two rational points.

**Notation:** Given $g \in \mathcal{M}([0, 1]^n)$ let $Zg = \{ x \in [0, 1]^n \mid g(x) = 0 \}$. Following [3, p.33] or [6, p.21], for $k \in \mathbb{N}$, $k \cdot g$ stands for $k$-fold pointwise truncated addition of $g$.

## 2. One-generator MV-algebras

**Theorem 2.1.** Every one-generator semisimple MV-algebra $A$ is strongly semisimple.

**Proof.** As in (1)-(2), let $X \subseteq [0, 1]$ be a nonempty closed set such that $A \cong \mathcal{M}(X)$. For some $g \in \mathcal{M}([0, 1])$ let $J$ be the principal ideal of $\mathcal{M}([0, 1])$ generated by $g$, and $J'$ be the principal ideal of $\mathcal{M}(X)$ generated by $g' = g|X$. Observe that $J' = \{ f|X \mid f \in J \}$. For every $f \in \mathcal{M}([0, 1])$, letting $f' = f|X$ we must prove: if $f'$ belongs to all maximal ideals of $\mathcal{M}(X)$ to which $g'$ belongs, then $f'$ belongs to $J'$. In the light of [3, 3.6.6] and [6, 4.19], this amounts to proving

\[ \text{if } f = 0 \text{ on } Zg \cap X \text{ then } f|X \in J'. \] (3)

Let $\Delta$ be a triangulation of $[0, 1]$ such that $f$ and $g$ are linear over every simplex of $\Delta$. The existence of $\Delta$ follows from the piecewise linearity of $f$ and $g$, [10]. In view of the compactness of $X$ and $[0, 1]$, it is sufficient to settle the following

**Claim.** Suppose $f \in \mathcal{M}([0, 1])$ vanishes over $Zg \cap X$. Then for all $x \in X$ there is an open neighbourhood $N_x \ni x$ in $[0, 1]$ together with an integer $m_x \geq 0$ such that $m_x \cdot g \geq f$ on $N_x \cap X$.

We proceed by cases:

**Case 1:** $g(x) > 0$. Then for some integer $r$ and open neighbourhood $N_x \ni x$ we have $g > 1/r$ over $N_x$. Letting $m_x = r$ we have $1 = m_x \cdot g \geq f$ over $N_x$, whence a fortiori, $m_x \cdot g \geq f$ over $N_x \cap X$.

**Case 2:** $g(x) = 0$. Since $f$ vanishes over $Zg \cap X$, then $f(x) = 0$. Let $T$ be a 1-simplex of $\Delta$ such that $x \in T$. Let $T_x$ be the smallest face of $T$ containing $x$.

**Subcase 2.1:** $T_x = T$. Then $x \in \text{int}(T)$. Since $g$ is linear over $T$ then $g$ vanishes over $T$. By our hypotheses on $f$ and $\Delta$, $f$ vanishes over $T$, whence and $0 = g \geq f = 0$ on $T$. Letting $N_x = \text{int}(T)$ and $m_x = 1$, we get $m_x \cdot g \geq f$ over $N_x$ whence a fortiori, the inequality holds over $N_x \cap X$.

**Subcase 2.2:** $T_x = \{x\}$. Then $T = \text{conv}(x, y)$ for some $y \neq x$. Without loss of generality, $y > x$. We will exhibit a right open neighbourhood $R_x \ni x$ and an integer $r_x \geq 0$ such that $r_x \cdot g \geq f$ on $R_x \cap X$. The same argument yields a left neighbourhood $L_x \ni x$ and an integer $l_x \geq 0$ such that $l_x \cdot g \geq f$ on $L_x \cap X$. One then takes $N_x = R_x \cup L_x$ and $m_x = \max(r_x, l_x)$.

**Subsubcase 2.2.1:** If both $g$ and $f$ vanish at $y$, then they vanish over $T$ (because they are linear over $T$). Upon defining $R_x = \text{int}(T) \cup \{x\}$ and $r_x = 1$ we get $r_x \cdot g \geq f$ over $R_x$, whence in particular, over $R_x \cap X$.
Subsubcase 2.2.2: If both \( g \) and \( f \) are \( > 0 \) at \( y \) then for all suitably large \( m \) we have \( 1 = m \cdot g \geq f \) on \( T \). Letting \( r_x \) the smallest such \( m \) and \( \mathcal{R}_x = \text{int}(T) \cup \{x\} \) we have the desired inequality over \( \mathcal{R}_x \) and a fortiori over \( \mathcal{R}_x \cap X \).

Subsubcase 2.2.3: \( g(y) = 0, f(y) > 0 \). By our hypotheses on \( \Delta \), \( g \) is linear over \( T \) and hence \( g = 0 \) over \( T \). It follows that \( X \cap T = \{x\} \); for otherwise, our assumption \( Z \cap X \supset Zg \cap X \) together with the linearity of \( f \) over \( T \) would imply \( f(y) = 0 \), against our current hypothesis. Letting \( \mathcal{R}_x = \text{int}(T) \cup \{x\} \) and \( r_x = 1 \) we have \( r_x \cdot g \geq f \) over \( \mathcal{R}_x \cap X \). \( \square \)

3. Strong semisimplicity and Bouligand-Severi tangents

Severi [8, §53, p.59 and p.392], [9, §1, p.99] and independently, Bouligand [1, p.32] called a half-line \( H \subset \mathbb{R}^n \) tangent to a set \( X \subset \mathbb{R}^n \) at an accumulation point \( x \) of \( X \) if for all \( \epsilon, \delta > 0 \) there is \( y \in X \) other than \( x \) such that \( ||y - x|| < \epsilon \), and the angle between \( H \) and the half-line through \( y \) originating at \( x \) is \( > \delta \).

Here as usual, \( ||v|| \) is the length of vector \( v \in \mathbb{R}^n \).

Severi [9, §2, p. 100 and §4, p.102] noted that for any accumulation point \( x \) of a closed set \( X \) there is a half-line \( H \) tangent to \( X \) at \( x \).

Today (see, e.g., [2, p.16], [7, p.1376]), Bouligand-Severi tangents are routinely introduced as follows:

**Definition 3.1.** Let \( x \) be an element of a closed subset \( X \) of \( \mathbb{R}^n \), and \( u \) a unit vector in \( \mathbb{R}^n \). We then say that \( u \) is a Bouligand-Severi tangent (unit) vector to \( X \) at \( x \) if \( X \) contains a sequence \( x_0, x_1, \ldots \) of elements, all different from \( x \), such that

\[
\lim_{i \to \infty} x_i = x \quad \text{and} \quad \lim_{i \to \infty} (x_i - x)/||x_i - x|| = u.
\]

Observe that \( x \) is an accumulation point of \( X \). We further say that \( u \) is outgoing if for some \( \lambda > 0 \) the segment \( \text{conv}(x, x + \lambda u) \) intersects \( X \) only at \( x \).

Already Severi noted that his definition of tangent half-line \( H = \mathbb{R}_{\geq 0}u \) is equivalent to Definition 3.1:

**Proposition 3.2.** ([9, §5, p.103]). For any nonempty closed subset \( X \) of \( \mathbb{R}^n \), point \( x \in X \), and unit vector \( u \in \mathbb{R}^n \) the following conditions are equivalent:

(i) For all \( \epsilon, \delta > 0 \), the cone \( \text{cone}_{x, u, \epsilon, \delta} \) with apex \( x \), axis parallel to \( u \), vertex angle \( 2\delta \) and height \( \epsilon \) contains infinitely many points of \( X \).

(ii) \( u \) is a Bouligand-Severi tangent vector to \( X \) at \( x \).

When \( n = 1 \), \( \text{cone}_{x, u, \epsilon, \delta} \) is the segment \( \text{conv}(x, x + \epsilon u) \). When \( n = 2 \), \( \text{cone}_{x, u, \epsilon, \delta} \) is the isosceles triangle \( \text{conv}(x, a, b) \) with vertex \( x \), basis \( \text{conv}(a, b) \), height equal to \( \epsilon \) and vertex angle \( \arctan a/b = 2\delta \).

The next two results provide geometric necessary and sufficient conditions on \( X \) for the semisimple MV-algebra \( \mathcal{M}(X) \) to be strongly semisimple. These conditions are stated in terms of the non-existence of Bouligand-Severi tangent vectors having certain rationality properties.

**Theorem 3.3.** Let \( X \) be a nonempty closed set in \([0, 1]^n\). Suppose \( X \) has a Bouligand-Severi rational outgoing tangent vector \( u \) at some rational point \( x \in X \). Then \( \mathcal{M}(X) \) is not strongly semisimple.

*Proof.* Since \( u \) is outgoing, let \( \lambda > 0 \) satisfy \( X \cap \text{conv}(x, x + \lambda u) = \{x\} \). Without loss of generality \( x + \lambda u \in \mathbb{Q}^n \). Our hypothesis together with Proposition 3.2 yields a sequence \( w_1, w_2, \ldots \) of distinct points of \( X \), all distinct from \( x \), accumulating at \( x \), at strictly decreasing distances from \( x \), in such a way that the sequence of unit vectors \( u_i \) given by \( (w_i - x)/||w_i - x|| \) tends to \( u \) as \( i \) tends to \( \infty \). Let \( y = x + \lambda u \).
Since \( X \cap \text{conv}(x, y) = \{x\} \), no point \( w_i \) lies on the segment \( \text{conv}(x, y) \), and we can further assume that the sequence of angles \( \omega \) is strictly decreasing and tends to zero as \( i \) tends to \( \infty \).

Since both points \( x \) and \( y \) are rational, then by [6, 2.10] for some \( g \in \mathcal{M}([0, 1]^n) \) the zeroset

\[
Z g = \{ z \in [0, 1]^n \mid g(z) = 0 \}
\]
coinsides with the segment \( \text{conv}(x, y) \). Thus,

\[
\frac{\partial g(x)}{\partial (u)} = 0.
\]

Let \( J \) be the ideal of \( \mathcal{M}([0, 1]^n) \) generated by \( g \),

\[
J = \{ f \in \mathcal{M}([0, 1]^n) \mid f \leq k \cdot g \text{ for some } k = 0, 1, 2, \ldots \}.
\]

Then for each \( f \in J \),

\[
\frac{\partial f(x)}{\partial (u)} = 0.
\]

Since the directional derivatives of \( f \) at \( x \) are continuous, (meaning that the map \( t \mapsto \partial f(x)/\partial t \) is continuous) it follows that

\[
\lim_{t \to u} \frac{\partial f(x)}{\partial t} = \frac{\partial f(x)}{\partial (u)} = 0.
\] (4)

Let \( g' = g|X \) and

\[
J' = \{ f' \in \mathcal{M}(X) \mid f' \leq k \cdot g' \text{ for some } k = 0, 1, 2, \ldots \}
\]

be the ideal of \( \mathcal{M}(X) \) generated by \( g' \). A moment’s reflection shows that

\[
J' = \{ l|X \mid l \in J \}.
\] (5)

One inclusion is trivial. For the converse inclusion, if \( f|X \leq (k \cdot g)|X \) then letting \( l = f \land k \cdot g \) we get \( l \leq k \cdot g \). So \( l \in J \) and \( l|X = f|X \), showing that \( f|X \) is extendible to some \( l \in J \).

For any \( f \in \mathcal{M}([0, 1]^n) \), the piecewise linearity of \( f \) ensures that for all large \( i \) the value of the incremental ratio \( (f(w_i) - f(x))/||w_i - x|| \) coincides with the directional derivative \( \partial f(x)/\partial u \) along the unit vector \( u_i = (w_i - x)/||w_i - x|| \). Thus in particular, if \( f|X = f'| \in J' \), from (4)-(5) it follows that

\[
\lim_{i \to \infty} \frac{f(w_i) - f(x)}{||w_i - x||} = 0.
\]

Since \( x \) is rational, again by [6, 2.10] there is \( j \in \mathcal{M}([0, 1]^n) \) with \( Z j = \{x\} \). For some \( \omega > 0 \) we have \( \partial j(x)/\partial (u) = \omega \), whence

\[
\lim_{i \to \infty} \frac{j(w_i) - j(x)}{||w_i - x||} = \omega.
\]

Therefore, \( j' \notin J' \). Since \( Z g \cap X = \{x\} \), recalling [6, 4.19] we see that the only maximal of \( \mathcal{M}(X) \) containing \( J' \) is the set of all functions in \( \mathcal{M}(X) \) that vanish at \( x \). Thus, \( j' \) belongs to all maximal ideals of \( \mathcal{M}(X) \) containing \( J' \). By [3, 3.6.6], \( \mathcal{M}(X) \) is not strongly semisimple: specifically, \( j'/J' \) is infinitesimal in the principal quotient \( \mathcal{M}(X)/J' \).

As a partial converse we have:

**Theorem 3.4.** Let \( X \subseteq [0, 1]^n \) be a nonempty closed set. Suppose the MV-algebra \( \mathcal{M}(X) \) is not strongly semisimple.
(i) Then $X$ has a Bouligand-Severi tangent vector $u$ at some point $x \in X$ satisfying the following nonalignment condition: there is a sequence of distinct $w_i \in X$, all distinct from $x$ such that
\[
\lim_{i \to \infty} w_i = x, \quad \lim_{i \to \infty} \frac{w_i - x}{||w_i - x||} = u, \quad w_i \notin \text{conv}(x, x + u) \text{ for all } i.
\]

(ii) In particular, if $n = 2$, then $X$ has a Bouligand-Severi outgoing rational tangent vector $u$ at some rational point $x \in X$.

Proof. (i) The hypothesis yields a function $g \in \mathcal{M}([0,1]^n)$, with its restriction $g' = g|X \in \mathcal{M}(X)$, in such a way that the principal ideal $J'$ of $\mathcal{M}(X)$ generated by $g'$,
\[
J' = \{ l' \in \mathcal{M}(X) \mid l' \leq k \cdot g' \text{ for some } k = 1, 2, \ldots \}
\]
is strictly contained in the intersection $I$ of all maximal ideals of $\mathcal{M}(X)$ containing $J'$. Thus for some $j \in \mathcal{M}([0,1]^n)$ letting $j' = j|X$ we have $j' \in I \setminus J'$. By [3, 3.6.6] and [6, 4.19],
\[
j' = 0 \text{ on } Zg, \quad \text{i.e., } X \cap Zj \supseteq X \cap Zg \quad (6)
\]
and
\[
\forall m = 0, 1, \ldots \exists z_m \in X, \ j'(z_m) > m \cdot g'(z_m). \quad (7)
\]
There is a sequence of integers $0 < m_0 < m_1 < \ldots$ and a subsequence $y_0, y_1, \ldots$ of $\{z_1, z_2, \ldots\}$ such that $y_i \neq y_l$ for $i \neq l$ and
\[
\forall t = 0, 1, \ldots, j'(y_t) > m_t \cdot g'(y_t). \quad (8)
\]
The compactness of $X$ yields an accumulation point $x \in X$ of the $y_t$. Without loss of generality (taking a subsequence, if necessary) we can further assume
\[
||y_0 - x|| > ||y_1 - x|| > \cdots, \text{ whence } \lim_{i \to \infty} y_i = x. \quad (9)
\]
By (8), for all $t$, $j'(y_t) > 0$. Then by (6), $g'(y_t) > 0$. For each $i = 0, 1, \ldots$, letting the unit vector $u_i \in \mathbb{R}^n$ be defined by $u_i = (y_i - x)/||y_i - x||$, we obtain a sequence of (possibly repeated) unit vectors $u_i \in \mathbb{R}^n$. Since the boundary of the unit ball in $\mathbb{R}^n$ is compact, some unit vector $u \in \mathbb{R}^n$ satisfies
\[
\forall \epsilon > 0 \text{ there are infinitely many } i \text{ such that } ||u_i - u|| < \epsilon.
\]
Some subsequence $w_0, w_1, \ldots$ of the $y_t$ will satisfy the condition
\[
\forall \epsilon, \delta > 0 \text{ there is } k \text{ such that for all } i > k, \quad w_i \in \text{cone}_{x, u, \epsilon, \delta}. \quad (10)
\]
Correspondingly, the sequence $v_0, v_1, \ldots$ given by $v_k = (w_k - x)/||w_k - x||$ will satisfy
\[
\lim_{i \to \infty} v_i = u. \quad (11)
\]
We have just proved that $u$ is a Bouligand-Severi tangent to $X$ at $x$.

To complete the proof of (i) we prepare:

**Fact 1.** $g'(x) = 0$.

Otherwise, from the continuity of $g$, for some real $\rho > 0$ and suitably small $\epsilon > 0$, we have the inequality $g(z) > \rho$ for all $z$ in the open ball $B_{x, \epsilon}$ of radius $\epsilon$ centered at $x$. By (10), $B_{x, \epsilon}$ contains infinitely many $w_i$. There is a fixed integer $\bar{m} > 0$ such that $1 = \bar{m} \cdot g' \geq j'$ for all these $w_i$, which contradicts (8).

**Fact 2.** $j'(x) = 0$.

This immediately follows from (6) and Fact 1.
Fact 3. $\partial g(x)/\partial u = 0$.

By way of contradiction, suppose $\partial g(x)/\partial u = \theta > 0$. In view of the continuity of the map $t \mapsto \partial g(x)/\partial t$, let $\delta > 0$ be such that $\partial g(x)/\partial r > \theta/2$, for any unit vector $r$ such that $\hat{r}u < \delta$. Since by Fact 2 $j(x) = 0$ and both $g$ and $j$ are piecewise linear, there is an $\epsilon > 0$ together with an integer $\bar{k} > 0$ such that $\partial g(x)/\partial r > \theta/2$, for any unit vector $r$ such that $\hat{r}u < \delta$.

Since by Fact 2 $j(x) = 0$ and both $g$ and $j$ are piecewise linear, there is an $\epsilon > 0$ together with an integer $\bar{k} > 0$ such that $\bar{k}/squaresmallsolid g \geq j$ over the cone $C = \text{cone}_{x,u,\epsilon,\delta}$. By (10), $C$ contains infinitely many $w_i$, in contradiction with (8).

To conclude the proof of the nonalignment condition in (i), it is sufficient to settle the following:

Fact 4. There is $\lambda > 0$ such that for all large $i$ the segment $\text{conv}(x, x+\lambda u)$ contains no $w_i$.

For otherwise, from Fact 3 $\partial g(x)/\partial u = 0$, whence the piecewise linearity of $g$ ensures that $g$ vanishes on infinitely many $w_i$ of $\text{conv}(x, x+\lambda u)$ arbitrarily near $x$. Any such $w_i$ belongs to $X$, whence by (6), $j(w_i) = 0$, in contradiction with (8).

The proof of (i) is now complete.

(ii) Let $H^\pm$ be the two closed half-spaces of $\mathbb{R}^2$ determined by the line passing through $x$ and $x + u$. By (10), infinitely many $w_i$ lie in the same closed half-space, say, $H^+$. Without loss of generality, $H^+ \cap \text{int}([0, 1]^2) \neq \emptyset$. Let $u^\perp$ be the orthogonal vector to $u$ such that $x + u^\perp \in H^+$.

Fact 5. For all small $\epsilon > 0$,

$$\frac{\partial g(x + \epsilon u)}{\partial u^\perp} > 0.$$

By way of contradiction, assume $\partial g(x + \epsilon u)/\partial u^\perp = 0$. Since $g$ is piecewise linear, by Facts 1 and 3, for suitably small $\eta, \omega > 0$, the function $g$ vanishes over the triangle $T = \text{conv}(x, x + \eta u, x + \eta u + \omega u^\perp)$. By (10), $T$ contains infinitely many $w_i$. By (6), $g(w_i) = j(w_i) = 0$ against (8).

Fact 6.

$$\frac{\partial j(x)}{\partial u} > 0.$$

Otherwise, $\partial j(x)/\partial u = 0$. Fact 5 yields a fixed integer $\bar{h}$ such that, on a suitably small triangle of the form $T = \text{conv}(x, x + \epsilon u, x + \epsilon u + \omega u^\perp)$, we have $\bar{h} \cdot g \geq j$. By (10), $T$ contains infinitely many $w_i$, again contradicting (8).

We now prove a strong form of Fact 4, showing that $u$ is an outgoing tangent vector:

Fact 7. For some $\lambda > 0$ the segment $\text{conv}(x, x+\lambda u)$ intersects $X$ only at $x$.

Otherwise, from Facts 1 and 3 it follows that $g$ vanishes on infinitely many points of $X \cap \text{conv}(x, x + \lambda u)$ converging to $x$. By (6), $j$ vanishes on all these points. Since $j$ is piecewise linear, $\partial j(x)/\partial u = 0$, against Fact 6.

By a rational line in $\mathbb{R}^n$ we mean a line passing through at least two distinct rational points.

Fact 8. $x$ is a rational point, and $u$ is a rational vector.
As a matter of fact, Facts 6 and 2 yield a rational line $L$ through $x$. On the other hand, Facts 3 and 5 show that the line passing through $x$ and $x+u$ is rational and different from $L$. Thus $x$ is rational, whence so is the vector $u$.

We conclude that $X$ has $u$ as a Bouligand-Severi outgoing rational tangent vector at the rational point $x$. □

Recalling Theorem 3.3 we now obtain:

**Corollary 3.5.** Let $X \subseteq [0,1]^2$ be a nonempty closed set. Then $\mathcal{M}(X)$ is not strongly semisimple iff $X$ has a Bouligand-Severi outgoing rational tangent vector $u$ at some rational point $x \in X$.

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