A new nonlinear turbulence model based on Partially-Averaged Navier-Stokes Equations

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Abstract. Partially-averaged Navier-Stokes (PANS) Model was recognized as a Reynolds-averaged Navier-Stokes (RANS) to direct numerical simulation (DNS) bridging method. PANS model was purported for any filter width—from RANS to DNS. PANS method also shared some similarities with the currently popular URANS (unsteady RANS) method. In this paper, a new PANS model was proposed, which was based on RNG k-ε turbulence model. The Standard and RNG k-ε turbulence model were both isotropic models, as well as PANS models. The sheer stress in those PANS models was solved by linear equation. The linear hypothesis was not accurate in the simulation of complex flow, such as stall phenomenon. The shear stress here was solved by nonlinear method proposed by Ehrhard. Then, the nonlinear PANS model was set up. The pressure coefficient of the suction side of the NACA0015 hydrofoil was predicted. The result of pressure coefficient agrees well with experimental result, which proves that the nonlinear PANS model can capture the high pressure gradient flow. A low specific centrifugal pump was used to verify the capacity of the nonlinear PANS model. The comparison between the simulation results of the centrifugal pump and Particle Image Velocimetry (PIV) results proves that the nonlinear PANS model can be used in the prediction of complex flow field.

1. Introduction

Speziale[1] proposed a new turbulence model that combines the advantages of RANS (Reynolds-averaged Navier-Stokes) method with those of LES. Inspiring from the model proposed by Speziali, Girimaji[2] developed a bridging method[1], which was purported for any filter width[3]. The model was given the name partially averaged Navier-Stokes (PANS) model.

During the past seven years, PANS model has been found to be better than the LES with less mesh in the near-wall region[4]. Wang[5] used PANS model to investigate the cavitating flow around a hydrofoil. The revolution of cavitating flow based on PANS model was in good agreement with experimental result. Ji[6] studied the unsteady cavitating turbulence flow around a twist hydrofoil based on the PANS method. The PANS models mentioned above were all modified from standard k-ε turbulence model, and the standard k-ε turbulence model was poor in the simulation of strong swirling flows[7]. Most of the RANS turbulence models solved the shear stress by linear difference scheme and
they were isotropic models[8], they can’t be used to predict strong turbulent flow with nonlinear characteristic.

In this paper, a new nonlinear PANS turbulence model was proposed, which was modified from RNG $k$-$\varepsilon$ turbulence model. The shear stresses were solved by Ehrhard's nonlinear methods. The new nonlinear PANS turbulence model was used to simulate the flow field of a Naca 0015 hydrofoil and a low specific centrifugal pump. The results based on the new nonlinear PANS turbulence model were compared with the experimental results. RNG $k$-$\varepsilon$ turbulence model and LES model were also used in the simulation of the flow field of the pump case. All their results were compared with the results performed by the nonlinear PANS model.

2. Nonlinear PANS model development

The parameter of velocity-$V_i$ can be partitioned into resolved and unresolved parts in the instantaneous velocity field, which is shown as follows,

$$ V_i = U_i + u_i $$ (1)

where $U_i$ is the resolved velocity field; $u_i$ is the unresolved field. The symbol $<>$ is used to denote the filtering operation.

$$ U_i = \langle V_i \rangle $$ (2)

For incompressible flow, the governing equations for the resolved velocity field $U_i$ can be calculated.

$$ \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + \frac{\partial \tau(V_i, V_j)}{\partial x_j} = \frac{\partial \langle p \rangle}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_i \partial x_i} $$ (3)

$$ \frac{\partial^2 \langle p \rangle}{\partial x_i \partial x_i} = -\frac{\partial U_i}{\partial x_i} \frac{\partial U_i}{\partial x_i} + \frac{\partial \tau(V_i, V_j)}{\partial x_i \partial x_j} $$ (4)

The additional non-linear term $\tau(V_i, V_j)$, which is the generalized central second moment, is defined as:

$$ \tau(V_i, V_j) = \langle (V_i V_j) - \langle V_i \rangle \langle V_j \rangle \rangle $$ (5)

It is sub-filter stress (SFS) term in PANS equations. The evolution equation for the SFS term is defined as [9]:

$$ \frac{\partial \tau(V_i, V_j)}{\partial t} + U_i \frac{\partial \tau(V_i, V_j)}{\partial x_i} = P_{ij} + \phi_{ij} - D_{ij} + T_{ij} $$ (6)

where $P_{ij}$ is the production term of SFS stress, $\phi_{ij}$ is the pressure-correlation term of SFS stress, $D_{ij}$ is the dissipation term of SFS stress, $T_{ij}$ is the transport term of SFS stress.

$$ P_{ij} = -\tau(V_i, V_j) \frac{\partial U_i}{\partial x_j} - \tau(V_j, V_i) \frac{\partial U_i}{\partial x_j} $$ (7)

$$ \phi_{ij} = -2\tau (p, SF_{ij}) $$ (8)

$$ SF_{ij} = -\frac{1}{2} \left( \frac{\partial V_i}{\partial x_j} \frac{\partial V_j}{\partial x_i} \right) $$ (9)

$$ D_{ij} = -2\nu \tau \left( \frac{\partial V_i}{\partial x_j} \frac{\partial V_j}{\partial x_i} \right) $$ (10)

$$ T_{ij} = -\frac{\partial}{\partial x_i} \left( \tau(V_i, V_j, V_k) + \tau(p, V_i) \delta_{jk} + \tau(p, V_j) \delta_{ik} - \frac{\partial \tau(V_i, V_j)}{\partial x_k} \right) $$ (11)

where,

$$ \tau(V_i, V_j, V_k) = \langle (V_i V_j V_k) - \langle V_i \rangle \langle V_j \rangle \langle V_k \rangle \rangle $$ (12)

The PANS models by the modification of RNG $k$-$\varepsilon$ turbulence model are,
\[
\frac{\partial (\rho k)}{\partial t} + \frac{\partial (\rho U_j k)}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ \alpha_k \left( \mu + \frac{\mu_\alpha}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] + P_{\text{in}} - \rho \varepsilon \tag{13}
\]

\[
\frac{\partial (\rho \varepsilon)}{\partial t} + \frac{\partial (\rho U_j \varepsilon)}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ \alpha_\varepsilon \left( \mu + \frac{\mu_\alpha}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right] + C_{\varepsilon 1}^* P_{\text{in}} - C_{\varepsilon 2}^* \frac{\varepsilon^2}{k} \tag{14}
\]

where,

\[
\mu_\varepsilon = \rho c_p \frac{k^2}{\varepsilon} \tag{15}
\]

\[
\sigma_\varepsilon = \frac{f_k^2}{f_\varepsilon} \tag{16}
\]

\[
C_{\varepsilon 2}^* = C_{\varepsilon 2} + \frac{f_k}{f_\varepsilon} (C_{\varepsilon 2} - C_{\varepsilon 1}^*) \tag{17}
\]

\[
C_{\varepsilon 1}^* = C_{\varepsilon 1} - \eta \frac{(1-\eta / \eta_b)}{1 + \beta \eta^3} \tag{18}
\]

\[
\eta = (2S_y \cdot S_y)^{1/2} \frac{k}{\varepsilon} \tag{19}
\]

\[
S_y = \frac{1}{2} \left( \frac{\partial U_j}{\partial x_j} + \frac{\partial U_j}{\partial x_j} \right) \tag{20}
\]

where \( C_\mu = 0.0845 \), \( \alpha_k = \alpha_\varepsilon = 1.39 \), \( C_{\varepsilon 1} = 1.42 \), \( \eta_b = 4.377 \), \( \beta = 0.012 \), \( \alpha_k = \alpha_\varepsilon = 1.39 \), \( C_{\varepsilon 1} = 1.42 \), \( C_{\varepsilon 2} = 1.68 \), \( \eta_b = 4.377 \), \( \beta = 0.012 \), \( P_k \) denotes the production terms of turbulence kinetic energy, \( U \) is the mean velocity.

For PANS methods, the ratio of unresolved-to-total kinetic energy \( f_k \) and the ratio of unresolved-to-total kinetic energy dissipation \( f_\varepsilon \) have relationships with the kinetic energy \( k \) and the kinetic energy dissipation \( \varepsilon \),

\[
f_k = \frac{k_u}{k} \tag{21}
\]

\[
f_\varepsilon = \frac{\varepsilon_u}{\varepsilon} \tag{22}
\]

\[
k_u = \frac{1}{2} \tau (V, V) \tag{23}
\]

\[
\varepsilon_u = \nu \tau \left( \frac{\partial V_j}{\partial x_j}, \frac{\partial V_j}{\partial x_j} \right) \tag{24}
\]

Equating the source terms, \( P_k \) in RNG \( k-\varepsilon \) turbulence mode has a relationship with \( P_{\text{in}} \) in PANS model,

\[
P_k = \frac{1}{f_k} (P_{\text{in}} - \varepsilon_u) + \frac{\varepsilon_u}{f_\varepsilon} \tag{25}
\]

Considering the nonlinear turbulence flow in the pump-turbine, the Reynolds stress was solved by nonlinear turbulence model which was proposed by Ehrhard \[^{10}\],

\[
P_k = -\rho U_j U_j' \frac{\partial U_j}{\partial x_j} \tag{26}
\]
\[
U_i'U_j' = \frac{2}{3} k \delta_{ij} - 2C_{mu} \nu T S_{ij} + C_{cs} \nu T^2 \left( S_{ij} \delta_{ij} - \frac{1}{3} S_{ij} \delta_{ij} \right) \\
+ C_{cm} \nu T \left( \Omega_{ij} + \Omega_{ji} \right) + C_{cs} \nu T^2 \left( \Omega_{ij} + \Omega_{ji} \right) \\
+ C_{cm} \nu T \left( S_{ij} \delta_{ij} - \frac{1}{3} \Omega_{ij} \delta_{ij} \right) \\
C_{\mu_F} = \min \left( \frac{1}{0.9S^{1.4} + 0.4\Omega^{1.4} + 3.5}, 0.15 \right) \\
\Omega_{ij} = \frac{1}{2} \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \\
S = \frac{k}{\varepsilon} \sqrt{2 \Omega_{ij} \delta_{ij}} \\
\Omega = \frac{k}{\varepsilon} \sqrt{2 \Omega_{ij} \delta_{ij}}
\]

(27)

where \(C_{\mu_F} = \beta \frac{k^2}{\varepsilon}, \ C_1 = -0.2, \ C_2 = 0.4, \ C_3 = 2.0 - \exp \left( -\left(S - \Omega \right)^2 \right), \ C_4 = -32.0C_{\mu_F}, \ C_5 = -16.0C_{\mu_F}^2, \ C_6 = 16.0C_{\mu_F}^2, T \) is turbulence time scale, \( u \) is the turbulence velocity scale.

3. Test cases

The nonlinear PANS method was performed by the commercial software Fluent by user define function (UDF). Couple method was used to correct the pressure during the calculation. Second order upwind scheme was used to resolve the Navies-Stocks equations. The rate of unresolved-to-total kinetic energy \( f_k = 0.2 \), and the rate of unresolved-to-total kinetic energy dissipation \( f_\epsilon = 1 \) were used in the calculation[6]. Convergence was determined by the residual error less than 0.0001.

3.1. 2-D NACA 0015 hydrofoil

The computational domain of the two-dimensional NACA 0015 hydrofoil was set according to Cervone’s model [11]. The local structure of the computational domain was shown in Fig.1. The angle of attack for NACA 0015 hydrofoil was 8°, and the inlet velocity \( u_0 \) was 8 m/s. The Reynolds number of the flow was 5x10^5. Structured meshes were used in the computational domain, and there had 12 layers for the pressure side and suction side of the NACA 0015 hydrofoil.

Calculations results based on RNG \( k-\epsilon \) turbulence model and the nonlinear PANS model were shown in Fig.2. \( C_p \) denotes the pressure coefficient.

\[
C_p = \frac{p_r - p_s}{\rho u_0^2} / 2
\]

(32)

where \( p_r \) is the reference pressure, \( \rho \) is the density of the fluid, \( p \) is static pressure.

The result of nonlinear PANS model agrees well with the experimental data, especially in the low pressure region. The result based on RNG \( k-\epsilon \) turbulence model has large errors compared with experimental result. The nonlinear PANS model can be used to predict the flow with large pressure gradient.

Figure1. Local structure of NACA0015 hydrofoil
3.2. Low specific centrifugal pump

A closed-loop test rig of Pedersen \cite{12} was investigated by numerical simulation. The closed-loop test rig was shown in Fig.3. Fluid was cycling in the cylindrical tank, pipes and the low specific centrifugal pump. Pedersen \cite{12} investigated the flow field of the centrifugal pump by particle image velocimetry (PIV) method. The velocity distributions on the measuring surface were calculated and compared with experimental results.

The parameters of the centrifugal pump used by Pedersen during the experiment were shown in Tab.1. The hydraulic region of the impeller of the centrifugal pump was shown in Fig.4.

**Table 1.** Geometry parameters of the low specific centrifugal pump

| Parameter                  | Value  |
|----------------------------|--------|
| Inlet diameter $D_{in}$ (mm)| 71.0   |
| Outlet diameter $D_{out}$ (mm)| 190.0  |
| Inlet height $b_{in}$ (mm)  | 13.8   |
| Outlet height $b_{out}$ (mm)| 5.8    |
| Blade curvature radius $R_b$ (mm)| 70.0   |
| Number of blades $Z_p$      | 6      |
| Blade thickness $t_p$ (mm)  | 3.0    |
| Inlet angle $\beta_1$ (°)   | 19.7   |
| Outlet angle $\beta_2$ (°)  | 18.4   |
| Specific speed $N_s$        | 16.3   |

**Figure 3.** Closed-loop test rig

**Figure 4.** The impeller of the pump

The hydraulic region of the closed-loop test rig was discretized by ICEM software using structured hexahedron. The total cells used for the calculation were 5,217,684, and the number of cells in runner was 1,799,964.

The rotation of runner was performed by moving mesh method. Interface was used to exchange data between moving part and stationary part. The enhanced wall treatment was used to calculate the flow near the wall.

The performance of the nonlinear PANS model, LES model and RNG $k$-$\varepsilon$ turbulence model are shown in Fig.5. $q_v$ denotes the flow rate of calculation point, $H_p$ denotes the head of the pump. When the flow rate is 3 L/s, the pump runs at the rated condition. The performances of the pump calculated by the three models are all in good agreement with experimental results. When the pump runs at small flow rate condition, the performances of LES model and the nonlinear PANS model are in accordance with experimental data, but the result base on RNG $k$-$\varepsilon$ turbulence model has large error. This may be caused by the different capacities for capturing complex flow phenomena of the three models.
Flow fields in the impeller obtained by unsteady flow calculation under different working conditions on the measuring surface were analyzed. Velocities at different diameters in the impeller on the measuring surface at rated condition are shown in Fig. 6. Four kinds of diameters, which are 0.5 $D_{out}$, 0.65 $D_{out}$, 0.75 $D_{out}$ and 0.9 $D_{out}$, were investigated. All the calculation results are in accordance with the experimental result at the Survey curve.

![Figure 5. Performance of the pump](image)

**Figure 5.** Performance of the pump

Velocities at different diameters in the impeller on the measuring surface at the small flow rate condition are shown in Fig. 7. The velocity distributions performed by the nonlinear PANS model agree well with experimental result. The nonlinear PANS model can capture the large scale vortex in the impeller accurately. The vortexes in the impeller performed by LES model have some difference compared with experimental data. The velocity at the impeller inlet is larger than the result of the nonlinear PANS model. The result of RNG $k$-$\varepsilon$ turbulence model varies greatly from the experimental result. The closed-loop test rig has a periodic distribution in the hydraulic region. The flow in the cycling system should have periodic distribution, so the result of the nonlinear PANS model is more realistic.

![Figure 6. Velocity distributions in the impeller at different diameters (Unit: m/s)](image)

**Figure 6.** Velocity distributions in the impeller at different diameters (Unit: m/s)
4. Conclusions

Most of the PANS models are proposed according to linear hypothesis for solving the Reynolds stress. For turbomachaneries, strong reverse flow and large curvature flow can be seen in the unit. This paper proposes a nonlinear PANS model based on the RNG $k$-$\varepsilon$ turbulence model. The nonlinear PANS model involves the prediction of large curvature flow, high pressure gradient flow and the nonlinear characteristic of the Reynolds stress. The results of the low specific centrifugal pump show that the nonlinear PANS model can capture complex vortexes in a turbomachinery.

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