Meissner Effect of Dirac Electrons in Superconducting State
due to Inter-band Effect

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Dirac electrons in solids show characteristic physical properties due to their linear dispersion relation and two-band nature. Although the transport phenomena of Dirac electrons in a normal state have intensively been studied, the transport phenomena in a superconducting state have not been fully understood. In particular, it is not clear whether Dirac electrons in a superconducting state show Meissner effect (ME), since a diamagnetic term of a current operator is absent as a result of the linear dispersion. We investigate the ME of three dimensional massive Dirac electrons in a superconducting state on the basis of Kubo formula, and clarify that Meissner kernel becomes finite by use of the inter-band contribution. This mechanism of the ME for Dirac electrons is completely different from that for the electrons in usual metals. We also derive the result for the electron gas by taking the non-relativistic limit of Dirac Hamiltonian, and clarify that the diamagnetic term of the Meissner kernel can be regarded as the inter-band contribution between electrons and positrons in terms of the Dirac model.

1. Introduction

Dirac electron systems have been attracting a great interest in recent years. One of the characteristic features of the Dirac electron systems is that its motion is described by Dirac equation instead of Schrödinger equation. Although the original Dirac equation describes the motion of relativistic (i.e., high-energy) electrons, \(^1\) it has recently been found that the electronic states of several materials with linear dispersion relation near the Fermi energy can be described by a low-energy effective model which has the same structure as Dirac equation. Examples of such materials are graphene, \(^2\) \(\alpha\)-\((BEDT-TTF)\)_2I_3, \(^3\) Bi, \(^4, 5\) Ca_3PbO_5, \(^6\) iron pnictide, \(^7, 8\) Na_3Bi, \(^9\) Cd_3As_2, \(^10\) and so on. It has been reported that the physical properties of these materials are qualitatively different from those in the usual metals which are described by Schrödinger equation and Fermi liquid theory. Hence, it is important to investigate the fundamental properties of the Dirac electrons.

Up to now, many works have been carried out to investigate the normal-state properties of Dirac electron systems, such as Hall conductivity, \(^11\) magnetoresistance, \(^12, 13\) orbital magnetism, \(^14-16\) Nernst coefficient, \(^17\) and spin Hall conductivity. \(^18, 19\) In these studies, it has been pointed out that the inter-band effect plays important roles, since the conduction and valence bands are close to each other. For instance, the giant orbital magnetism in Bi is due to the inter-band effect, and can not be explained by Landau-Periels’s theory \(^20, 21\) of diamagnetism in which the inter-band is neglected.

On the other hand, the transport properties of Dirac electrons in a superconducting state have not been understood completely. In particular, it is not trivial whether the Dirac electrons in the superconducting state show the Meissner effect (ME), since the current operator does not have the diamagnetic term; \(^22\) for usual metals with a parabolic dispersion relation (i.e., in electron gas), the ME arises from the diamagnetic term of the current operator. Note that the diamagnetic current term appears when the Hamiltonian has a kinetic energy proportional to \(k^2\) with \(k\) being the momentum. \(^23\) It is therefore necessary to discuss the ME of Dirac electron in a superconducting state on the basis of the treatment which correctly includes the inter-band effect.

In this paper, we discuss the ME of a \(4 \times 4\) massive Dirac electron system in the three-dimensional space assuming an s-wave Cooper pairing. The derivation of the Meissner kernel is based on Kubo formula. We reveal that the inter-band effect plays an important role and that the ME appears as an inter-band contribution in spite of the absence of the diamagnetic term of the current operator.

Kopnin and Sonin discussed the similar problem in the case of graphene. However, the derivations are not based on the linear response theory and are not clear. Furthermore, obtained results show that ME remains in the normal state. Here we study the superconductivity in the \(4 \times 4\) massive Dirac model based on the Kubo formula. Our results also show that the ME remains in the normal state. However, as we show in the present paper, this is due to the unbounded property of the Dirac-type models. We will show that this artifact can be remedied by considering a simple model which avoids the unboundedness of Dirac dispersion.

This paper is organized as follows. In the next section, we introduce a \(4 \times 4\) massive Dirac Hamiltonian in the three-dimensional space, and the definition of the current operator in that model. We also show the treatment of the superconducting order parameter in the mean field approximation, and derive the Meissner kernel on the basis of Kubo formula. In Sect. 3, we show the explicit form of the Meissner kernel in the present model, and give the result of the numerical analysis. Then we discuss how Dirac electrons become the Meissner state. It will be shown that the inter-band contribution plays an important role in obtaining the finite Meissner kernel. We also mention the relation between the Dirac electron and the non-relativistic electron gas by considering the large band gap limit case. We will show that our theory can reproduce the well-known results in the non-relativistic electron gas, and that the origin of what we call the paramagnetic and diamagnetic terms originate from the intra and inter band term, respectively. Finally, the brief summary is given in Sect. 4. Part of the present work has been published before.\(^{25}\)
2. Formulation

2.1 Hamiltonian

We consider the following $4 \times 4$ massive Dirac Hamiltonian in the three-dimensional space: \cite{11}

$$H_0(k) = \gamma_k \left( \begin{array}{cc} M \hat{I} & ivk \cdot \sigma \\ -ivk \cdot \sigma & -M \hat{I} \end{array} \right) \hat{c}_k,$$  \hspace{1cm} (1)

where $\hat{I}$ is the $2 \times 2$ unit matrix and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices. $M$ is the band gap at the expanding center in the Brillouin zone, and $v$ is the Fermi velocity. The basis used in (1) is $\hat{c}_k = (c_{k,\uparrow}, c_{k,\downarrow}, c_{k,\uparrow'}, c_{k,\downarrow'})^T$ where $1, 2$ denote the orbital, and $\uparrow, \downarrow$ denote the real spin.

Diagonalizing this Hamiltonian, we obtain the following Hamiltonian,

$$H_0(k) = \alpha_k \left( \begin{array}{cc} \varepsilon(k) \hat{I} & 0 \\ 0 & -\varepsilon(k) \hat{I} \end{array} \right) \hat{a}_k,$$  \hspace{1cm} (2)

where $\varepsilon(k) = \sqrt{M^2 + (vk)^2}$, and $\hat{a}_k$ represents the band-basis which is expressed as $\hat{a}_k = (a_{k,\uparrow}, a_{k,\downarrow}, a_{k,\uparrow'}, a_{k,\downarrow'})^T$.

Here, the indices $\uparrow$ and $\downarrow$ denote the upper and the lower bands respectively, and $\uparrow$ and $\downarrow$ denote psudo-spins corresponding to the two-fold degeneracy of each band [see Fig. 1]. The unitary matrix which is defined as $\hat{U}(k)\hat{c}_k$ is given by

$$\hat{U}(k) = \left( \begin{array}{cc} \hat{X}(k) \hat{I} & -i \hat{Y}(k) \cdot \sigma \\ -i \hat{Y}(k) \cdot \sigma & \hat{X}(k) \hat{I} \end{array} \right),$$  \hspace{1cm} (3)

where $\hat{X}(k) = \sqrt{\frac{\varepsilon(k) + M}{2\varepsilon(k) + M}}$ and $\hat{Y}(k) = \sqrt{\frac{1}{2\varepsilon(k) + M}} v k$.

2.2 Current operator

In this subsection, we discuss the current operator in the absence and presence of the vector potential. Without the vector potential, $A$, the current operator in the momentum space is

$$\hat{j}(q) = -e \sum_{k} \hat{c}_{k-q}^\dagger \hat{a}_{k} H_0(k) \hat{c}_{k} = -e \sum_{k} \hat{c}_{k-q}^\dagger \left( \begin{array}{cc} 0 & iv \sigma \\ -iv \sigma & 0 \end{array} \right) \hat{c}_{k},$$  \hspace{1cm} (4)

where $e > 0$ is the absolute value of the charge of an electron.

The vector potential $A$ is introduced in the Hamiltonian by replacing $k$ by $k + eA$. However, we can readily see that the current operator in Eq. (4) does not change even in the presence of $A$, since $H_0(k)$ contains only linear terms with respect to $k$.

This property of the current operator is essentially different from that in the electron gas in which the current operator is given by $\hat{j}(q) = \sum_{k, \sigma} \hat{c}_{k-q, \sigma} \gamma(k + eA) \hat{c}_{k, \sigma}$ where the last term is called as the diamagnetic term. Apparently, the current operator of the Dirac electron systems does not have diamagnetic current. As we will show shortly, the absence of the diamagnetic term is crucial for discussing the mechanism of the ME in Dirac electron in superconducting state.

2.3 Superconductivity

In this subsection, we introduce superconductivity in the Dirac electron system. We assume the following attractive interation,

$$H_{int} = -V \sum_{k} \sum_{\xi = \pm} a_{k, \eta, \xi}^\dagger a_{-k, \eta, \xi} a_{-k', \eta', \xi'} a_{k', \eta', \xi'}^\dagger \hspace{1cm} (5)$$

where $k_c$ is the cutoff momentum corresponding to the range of energy in which the attractive interaction works. When $vk_c \gg M$, a superconducting state is realized even if the chemical potential is located in the band gap (i.e., there is no density of states at the Fermi level) for sufficiently large $V$, since electrons with $k < k_c$ can still contribute to the Cooper pairing.

We define the superconducting order parameter in the s-wave symmetry (see Fig. 2(a)),

$$\Delta_{\pm} = V \sum_{|k'| < k_c} \langle a_{-k', \pm, \xi} a_{k', \pm, \xi} \rangle \hspace{1cm} (6)$$

By applying the mean field approximation to the Hamiltonian, $H = H_0 + H_{int}$, we obtain the BCS Hamiltonian,

$$H_{BCS} = \sum_{k} \sum_{\xi = \pm} \left\{ \sum_{\eta} \left[ \xi_{\xi} (k) a_{k, \eta, \xi}^\dagger a_{k, \eta, \xi} - \Delta a_{k, \eta, \xi}^\dagger a_{-k, \eta, \xi} + a_{-k, \eta, \xi} a_{k, \eta, \xi} \right] \right\}$$  \hspace{1cm} (7)

where $\Delta = \Delta_{+} + \Delta_{-}$ is the sum of the band-dependent order parameters and $\xi_{\xi}(k) = \pm e(k) - \mu$.

Diagonalizing $H_{BCS}$, we obtain the excitation energy $E_{\pm}(k) = \sqrt{\xi_{\xi}(k)^2 + \Delta^2}$ [see Fig. 2(b)]. Then, thermal Green’s function,

$$[G(k, \omega_n)]_{\alpha_1 \alpha_2} = -\int_0^\beta d\tau e^{i\omega_n \tau} \langle T_\tau [a_{k, \alpha_1} (\tau) a_{k, \alpha_2}^\dagger (0)] \rangle \hspace{1cm} (8)$$

and the anomalous Green’s functions,

$$[F(k, \omega_n)]_{\alpha_1 \alpha_2} = -\int_0^\beta d\tau e^{i\omega_n \tau} \langle T_\tau [a_{k, \alpha_1}^\dagger (\tau) a_{k, \alpha_2} (0)] \rangle \hspace{1cm} (9a)$$

$$[F^\dagger(k, \omega_n)]_{\alpha_1 \alpha_2} = -\int_0^\beta d\tau e^{i\omega_n \tau} \langle T_\tau [a_{-k, \alpha_1}^\dagger (\tau) a_{k, \alpha_2} (0)] \rangle \hspace{1cm} (9b)$$

are obtained in the form of $4 \times 4$ matrices as follows. (Note...
that $\alpha_1$ and $\alpha_2$ are the sets of band and pseudo spin indices.)

$$\mathcal{G}(k, i\omega_n) = \begin{pmatrix} \mathcal{G}_+(k, i\omega_n) & 0 \\ 0 & \mathcal{G}_-(k, i\omega_n) \end{pmatrix},$$

and

$$\mathcal{F}(k, i\omega_n) = -\mathcal{F}^\dagger(k, i\omega_n) = \begin{pmatrix} i\sigma_y \mathcal{F}_+(k, i\omega_n) & 0 \\ 0 & i\sigma_y \mathcal{F}_-(k, i\omega_n) \end{pmatrix},$$

where

$$\mathcal{G}_\pm(k, i\omega_n) = \frac{u^2_\pm(k)}{i\omega_n - E_\pm(k)} + \frac{v^2_\pm(k)}{i\omega_n + E_\pm(k)},$$

and

$$\mathcal{F}_\pm(k, i\omega_n) = -u_\pm(k)v_\pm(k)\left(\frac{1}{i\omega_n - E_\pm(k)} - \frac{1}{i\omega_n + E_\pm(k)}\right),$$

with $u^2_\pm(k) = \frac{1}{2}[1 + \frac{\xi(k)}{E_{\pm}(k)}]$, $v^2_\pm(k) = \frac{1}{2}[1 - \frac{\xi(k)}{E_{\pm}(k)}]$, and $u_\pm(k)v_\pm(k) = \frac{\Delta(k)}{2E_{\pm}(k)}$. It should be noted that the upper and lower parts of these Green’s functions are decoupled with each other.

3. **Meissner effect in Dirac electron systems**

3.1 **Small band gap case: Dirac electrons in solids**

Using the formulations derived in the previous section, we calculate Meissner kernel, $K_{xx}^{(S)}$, through Kubo formula. Since there is no diamagnetic term of the current operator, $K_{xx}^{(S)}(q, i\omega_n)$ is given only by the current-current correlation function,

$$K_{xx}^{(S)}(q, i\omega_n) = -\int_0^\beta d\tau e^{i\alpha_n \tau} \langle T_{\tau}(\hat{j}_x(q, \tau)\hat{j}_x(-q, 0)) \rangle,$$

where $\omega_n = \frac{2\pi}{\beta} n$ ($n = 0, \pm 1, \pm 2, \cdots$) denotes a bosonic Matsubara frequency and $\tau$ an imaginary time. The Meissner kernel for real frequency is given by the analytic continuation $K_{xx}^{(S)}(q, \omega) = K_{xx}^{(S)}(q, i\omega_n) |_{i\omega_n \rightarrow \omega}$.

Applying the Bloch-De Dominics theorem to Eq. (14), we obtain the following expression of $K_{xx}(q, i\omega_n)$:

$$K_{xx}^{(S)}(q, i\omega_n) = T \sum_{k, \omega_n} \text{Tr} \{\mathcal{G}^\dagger(k - q, \omega_n - \omega_n)\hat{j}_x(k, q)\mathcal{G}(k, \omega_n)\hat{j}_x(-q, -q) - \mathcal{F}^\dagger(k - q, \omega_n - \omega_n)\hat{j}_x(k, q)\mathcal{F}(k, \omega_n)\hat{j}_x(-q, -q)\}.$$  \hspace{1cm} (15)

where $\hat{j}_x(k, q)$ is given by $\hat{j}_x(k, q) = \sum \mathcal{F}_0^\dagger(k, \omega_n)U^\dagger(k)\mathcal{G}_0(k, \omega_n)$, and $\omega_n = \frac{(2n+1)\pi}{\beta}$ ($n = 0, \pm 1, \pm 2, \cdots$) denotes a fermionic Matsubara frequency. It should be noted that the limit $\omega \rightarrow 0$ has to be taken before the limit $q \rightarrow 0$.

By taking the trace in Eq. (15), we obtain the Meissner kernel which consists of two parts as

$$K_{xx}^{(S)}(q, 0) = K_{xx}^{\text{intra}(S)}(q, 0) + K_{xx}^{\text{inter}(S)}(q, 0).$$

$K_{xx}^{\text{intra}(S)}(q, 0)$ comes from the term in which the two Green’s functions in Eq. (15), ($\mathcal{G}^\dagger$ or $\mathcal{F}^\dagger$) have the same band indeces $(+, +)$ or $(-, -)$, and $K_{xx}^{\text{inter}(S)}(q, 0)$ from the Green’s functions with the opposite band indeces, i.e., $(+, -)$ or $(-, +)$.

The explicit forms of these are obtained as

$$K_{xx}^{\text{intra}(S)}(q, 0) = -\frac{e^2v^2}{2} \sum_k \sum_{\eta = \pm} g_{xx}^{\text{intra}}(k, q),$$

and

$$K_{xx}^{\text{inter}(S)}(q, 0) = -\frac{e^2v^2}{2} \sum_k \sum_{\eta = \pm} g_{xx}^{\text{inter}}(k, q),$$

with

$$g_{xx}^{\text{intra}}(k, q) = \frac{\nu^2[k(k_x - q_x) - k_y(k_y - q_y) - k_z(k_z - q_z)]}{\epsilon(k)\epsilon(k - q)} + \frac{\epsilon(k)\epsilon(k - q) - M^2}{\epsilon(k)\epsilon(k - q)},$$

and

$$g_{xx}^{\text{inter}}(k, q) = \frac{\nu^2[k(k_x - q_x) - k_y(k_y - q_y) - k_z(k_z - q_z)]}{\epsilon(k)\epsilon(k - q)} - \frac{\epsilon(k)\epsilon(k - q) + M^2}{\epsilon(k)\epsilon(k - q)}.$$}

In Eqs. (17) and (18), $B$’s are coherence factors,

$$B_{\eta}^{\text{intra}}(k, q) = 1 + \frac{\Delta^2}{E_{\eta}(k - q)E_{\eta}(k)} - \frac{\Delta^2}{E_{\eta}(k - q)E_{\eta}(k)}.$$  \hspace{1cm} (20a)

$$B_{\eta}^{\text{inter}}(k, q) = 1 - \frac{\Delta^2}{E_{\eta}(k - q)E_{\eta}(k)} - \frac{\Delta^2}{E_{\eta}(k - q)E_{\eta}(k)}.$$  \hspace{1cm} (20b)
The spectrum has a finite gap at the chemical potential. Hence, the Meissner kernel from the paramagnetic current vanishes at the inter-band contribution. Even if we put remains which leads to the ME. In contrast, in the Dirac case, the total Meissner kernel in the normal state vanishes. In the usual situations in solids with finite Brillouin zone, \( K_{xx}^{(S)} = 0 \) holds since the paramagnetic term exactly cancels with the diamagnetic contribution. Therefore we expect that, in the Dirac case, there must be some counter-contributions from the vicinity of the Brillouin zone boundary which exactly cancel the contributions from the vicinity of the Dirac point. In order to demonstrate this kind of cancellation, we develop a simple model in the Appendix, and show that the total Meissner kernel in the normal state vanishes. In the Appendix it is also shown that the correct Meissner kernel in the superconducting state is obtained by subtracting the kernel in the normal state. Therefore, in the following, we calculate the Meissner kernel in the superconducting state by

\[
K_{xx} = K_{xx}^{(S)} - K_{xx}^{(N)}. \tag{25}
\]

By using Eq. (25), we numerically calculate \( K_{xx}(0, 0) \) at \( T = 0 \). In the numerical calculation, we set \( \kappa_c = 10 \text{MeV}/v \).

Figures 3 shows \( \mu \) dependence of \( K_{xx}^{(S)}, K_{xx}^{(N)}, \) and \( K_{xx} \) for a fixed \( \Delta \), (a) \( \Delta = 0.1M \), (b) \( \Delta = 0.5M \), and (c) \( \Delta = 0.7M \). (The insets show \( K_{xx} \) in the extended scale.) We can see that although \( K_{xx}^{(S)} \) and \( K_{xx}^{(N)} \) are both negative, \( K_{xx}(0, 0) \) is always positive and finite, thus we obtain the Meissner state.

We can also see in Figs. 3, that \( K_{xx} \) strongly depends on \( \Delta \), in \(-M \leq \mu \leq M \). This is another sharp contrast to the conventional BCS case in which \( K_{xx} \) does not depend on \( \Delta \) and depends only on the carrier density.

### 3.2 Large band gap case: non-relativistic electron

Up to now, we consider the case in which the band gap is very small (\( \sim 10 \text{meV} \)). In this subsection, we consider the opposite case, i.e., the large band gap case. In particular, it is important to test whether the present theory reproduces the results for non-relativistic theory. To access to the non-relativistic limit, we replace \( M \) to \( m c^2 \), and \( v \) to \( c \), with \( m \) being the mass of an electron and \( c \) the speed of light. Then, we can regard the upper band as the real electron, and the lower band as the positron [see Fig. 4]. We further assume that the chemical potential is located in the upper band, and that \( \partial \epsilon / \partial \xi \) is

\[
\partial \epsilon / \partial \xi = \partial \epsilon / \partial K = \partial \epsilon / \partial E = \partial \epsilon / \partial \mu = \partial \epsilon / \partial B = \partial \epsilon / \partial q.
\]

For (a) \( \Delta = 0.1M \), (b) \( \Delta = 0.5M \), and (c) \( \Delta = 0.7M \). In the numerical calculation, we set \( \kappa_c = 10 \text{MeV}/v \).

Figures 3 shows \( \mu \) dependence of \( K_{xx}^{(S)}, K_{xx}^{(N)}, \) and \( K_{xx} \) for a fixed \( \Delta \), (a) \( \Delta = 0.1M \), (b) \( \Delta = 0.5M \), and (c) \( \Delta = 0.7M \). (The insets show \( K_{xx} \) in the extended scale.)
much smaller than $mc^2$. Then we obtain the Meissner kernel in the normal state as
\[
K_{xx}^{\text{intra}(N)}(0, 0) = -\frac{e^2c^2}{3\pi^2} \left( \frac{\mu^2 - m^2c^4}{\mu c^5} \right)^{3/2} \sim -\frac{e^2n}{m},
\]
and
\[
K_{xx}^{\text{inter}(N)}(0, 0) = -\frac{e^2c^2}{3\pi^2} \left[ \frac{k_F^3}{\epsilon(k_F)} - \frac{(\mu^2 - m^2c^4)^{3/2}}{\mu c^5} \right]
\sim \frac{e^2n}{m} \left( \frac{e^2c^2}{3\pi^2} \frac{k_F^3}{\epsilon(k_F)} \right)^{1/2},
\]
where $k_F = \sqrt{2m(\mu - mc^2)}$ is the Fermi momentum and $n = \frac{k_F^3}{3\pi^2}$ is the electron density. Here we use the approximation $\mu - mc^2 \ll mc^2$.

Comparing Eqs. (26) and (27) with the results for the electron gas,
\[
\frac{(\mu^2 - m^2c^4)^{3/2}}{\mu c^5} \sim mc
\]
we can see that $K_{xx}^{\text{intra}(N)}$ is exactly equal to the paramagnetic term in the electron gas and that $K_{xx}^{\text{inter}(N)}$ is exactly equal to the diamagnetic term apart from the divergent part $\left( -\frac{e^2c^2}{3\pi^2} \frac{k_F^3}{\epsilon(k_F)} \right)$. Note that this divergent term comes from the region far away from the Fermi surface, and that this is exactly what we deal with by the subtraction in order to avoid the unphysical result in the last subsection. We think that this kind of divergence is inevitable within the Dirac theory. Eqs. (26) and (27) indicate, first, that we can reproduce the result of the electron gas starting from the Dirac model; second, that the diamagnetic term of the kernel in the electron gas originates from the inter-band contribution between positrons and electrons.

Now we turn to the superconducting state. We assume that the range of the energy in which the attractive interaction works is much smaller than $mc^2$ or $\mu - mc^2$ as shown in Fig. 4. Thus, the pairing occurs in the very vicinity of the Fermi level on the upper band [see the yellow shade in Fig. 4]. In this case, $K_{xx}^{\text{intra}(S)}$ becomes 0 as we have discussed in the last subsection. On the other hand, $K_{xx}^{\text{inter}(S)}$ is almost same as $K_{xx}^{\text{inter}(N)}$, since the states below the Fermi level do not change by the pairing except for the vicinity of the Fermi surface. Therefore, we obtain the Meissner kernel in the superconducting state as
\[
K_{xx}^{(S)} \sim K_{xx}^{\text{inter}(N)} \sim \frac{e^2n}{m},
\]
which is completely consistent with the result of the superconducting state in electron gas.

### 4. Summary

In summary, we have investigated the ME of three-dimensional massive Dirac electron in superconducting state on the basis of Kubo formula. Although the diamagnetic term of the current operator is absent in Dirac electron system, the Meissner kernel finite for any value of the chemical potential, since the inter-band contribution remains finite. We have to care about the finite Meissner kernel in the normal state, but this can be eliminated by subtracting the normal component. This inter-band mechanism of the ME of Dirac electron is essentially different from that of usual metals, and ubiquitous in other Dirac electron systems such as graphene. We also have studied the large band-gap limit to discuss the non-relativistic case. We clarify the relation that the paramagnetic term of the kernel originates from the intra-band term and the diamagnetic term from the inter-band term.

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### Appendix: Meissner effect in Dirac electron systems with a quadratic term

In this appendix, we perform a simple model calculation in which the Meissner kernel in the normal state vanishes. In order to avoid the unboundedness of the Dirac dispersion, we add a kinetic energy term $\frac{1}{2}k^2$ in the Hamiltonian instead of introducing the Brillouin zone,
\[
\hat{H}_0(k) = \frac{c^2}{k^2} \left( M + \frac{1}{2}k^2 \right) \hat{I} - ivk \cdot \sigma \left( -M + \frac{1}{2}k^2 \right) \hat{I} \hat{c}_k, \tag{A-1}
\]
where parameter $a$ is assumed to be very small. The energy dispersion becomes $\epsilon(k) + \frac{1}{2}k^2$, and a new Fermi surface appears as shown in Fig. A-1. The matrix form of the current operator, Eq. (4) changes as
\[
\hat{j}(q) = -e \sum_k \hat{c}_{k+q} \left( a(k + eA) \hat{I} \hat{c}^\dagger_k + iv\sigma \hat{a}(k + eA) \hat{I} \right)
\]
\[
\hat{j}(q) \hat{c}_k. \tag{A-2}
\]
Although there appears a “diamagnetic part” of the current, we will show that the main part of the Meissner kernel in the superconducting state comes from the inter-band contribution as discussed in Sect. 3.

We can carry out a similar calculation of the Meissner kernel as in Sect. 3. Note that the unitary matrix in Eq. (3) does not

![Fig. 4. Dispersion relation in the non-relativistic limit. Yellow shade denotes the range where the attractive interaction works.](image4)

![Fig. A-1. The dispersion relation of the 4 × 4 Dirac Hamiltonian in normal state with the quadratic term.](imageA1)
not change. After some algebra, we obtain
\[ K_{xx}^{\text{intra},(S)}(0,0) = 2e^2 \sum_{k,\eta=\pm} \frac{(v^2k_x + \eta ak_x \varepsilon(k))^2}{\varepsilon^2(k)} \frac{\partial f(\varepsilon_\eta(k))}{\partial \varepsilon_\eta(k)}, \]
(A-3)
and
\[ K_{xx}^{\text{inter},(S)}(0,0) = -2e^2v^2 \sum_k \left( 1 - \frac{v^2k_x^2}{\varepsilon^2(k)} \right) \times \frac{1}{E_+(k) + E_-(k)} \left( 1 - \frac{\varepsilon_+(k) \varepsilon_-(k) - \Delta^2}{E_+(k) E_-(k)} \right), \]
(A-4)
at \( T = 0 \) where \( E_{\pm}(k) = \sqrt{\varepsilon_{\pm}(k)^2 + \Delta^2} \), and \( \varepsilon_{\pm}(k) \) is now changed to \( \varepsilon_{\pm}(k) = \pm \varepsilon(k) + \frac{\pm k^2}{2} - \mu \).

Let us consider the normal state with \( \mu > M \). There are several contributions to the Meissner effect. First is the contribution from the diamagnetic current which appeared in Eq. (A-2) artificially. Since this diamagnetic contribution is proportional to the electron density, we obtain
\[ K_{xx}^{\text{dia}} = K_{xx}^{\text{dia},-} + K_{xx}^{\text{dia},+}, \quad K_{xx}^{\text{dia},-} = \frac{ae^2}{3\pi^2} k_0^3, \quad K_{xx}^{\text{dia},+} = \frac{ae^2}{3\pi^2} k_F^3, \]
(A-5)
where \( k_0 \) is the artificial Fermi surface of the lower band shown in Fig. A-1, i.e., \( \varepsilon_-(k_0) = 0 \) is satisfied, and \( k_F \) is the Fermi surface of the upper band. Sign \( \pm \) of the subscript indicates the upper (\( \eta = + \)) and lower (\( \eta = - \)) band. The second contribution is from the intra-band (Fermi-surface) contribution from the artificial Fermi surface at \( k_0 \) (Eq. (A-3) with \( \Delta = 0 \)), which is given by
\[ K_{xx}^{\text{intra},(N)}(0,0) = -\frac{e^2}{3\pi^2} k_0^3 \left( \frac{v^2k_0^2 - ak_0 \varepsilon(k_0)}{\varepsilon^2(k_0)} \right) \left( \frac{1}{|ak_0 - \varepsilon_0(k_0)|} \right), \]
(A-6)
Note that \( \frac{\partial f}{\partial \varepsilon_-} \) becomes a delta function \( -\delta(\varepsilon_-(k)) \) when \( \Delta = 0 \), and \( \varepsilon(k_0) - \frac{v^2k_0^2}{\varepsilon(k)} > 0 \) holds since \( \varepsilon(k_0) \) is approximated as \( \varepsilon(k_0) \sim v k_0 \sim 2v^2/a \). Similarly, from the Fermi surface at \( k_F \), we obtain
\[ K_{xx}^{\text{intra},(N)}(0,0) = -\frac{e^2}{3\pi^2} k_F^3 \left( a + \frac{v^2}{\varepsilon(k_F)} \right). \]
(A-7)
Finally, the inter-band contribution \( K_{xx}^{\text{inter},(N)}(0,0) \) (Eq. (A-4)) can be calculated analytically. The integrand in Eq. (A-4) is nonzero in the region where \( \varepsilon_+(k) \varepsilon_-(k) < 0 \), i.e., in the region of \( k_F < k < k_0 \). Thus we obtain
\[ K_{xx}^{\text{inter},(N)}(0,0) = -\frac{e^2}{2\pi^2} \int_{k_F}^{k_0} k^2 dk \frac{2v^2}{\varepsilon(k)} \left( 1 - \frac{v^2k_x^2}{3\varepsilon(k)^2} \right) \]
\[ = -\frac{e^2}{3\pi^2} \varepsilon(k) k_F \]
\[ = -\frac{e^2}{3\pi^2} \left( \frac{v^2k_0^3}{\varepsilon(k_0)} - \frac{v^2k_F^3}{\varepsilon(k_F)} \right) \]
\[ \equiv K_{xx}^{\text{inter},(N)}(0,0) + K_{xx}^{\text{inter},(N)}(0,0). \]
(A-8)