**U(1) spin Chern-Simons theory and Arf invariants in two dimensions**

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Abstract

The level-\(k\) \(U(1)\) Chern-Simons theory is a spin topological quantum field theory for \(k\) odd. Its dynamics is captured by the 2d CFT of a compact boson with a certain radius. Recently it was recognized that a dependence on the 2d spin structure can be given to the CFT by modifying it using the so-called Arf invariant. We demonstrate that one can reorganize the torus partition function of the modified CFT into a finite sum involving a finite number of conformal blocks. This allows us to reproduce the modular matrices of the spin theory. We use the modular matrices to calculate the partition function of the spin Chern-Simons theory on the lens space \(L(a, \pm1)\), and demonstrate the expected dependence on the 3d spin structure.
1 Introduction

It is well-known that the 2d conformal field theory (CFT) of a free compact boson $\phi$ of radius $R$ is rational when $R^2$ is rational. (We use the normalization such that T-duality acts as $R \rightarrow 2/R$.) For simplicity let $k$ be a positive integer and set $R = \sqrt{k}$. By the bulk-boundary correspondence, this “edge state” CFT characterizes the dynamics of the $U(1)$ level-$k$ Chern-Simons (CS) theory \cite{1}. We consider the extended chiral algebra generated by

$$e^{\pm i \sqrt{k} \phi_L}$$

(1.1)

in addition to $i\partial \phi_L$, where $\phi_L$ is the left-moving part of $\phi$. See Appendix C.1 of \cite{2} for a modern discussion. For $k$ even, the chiral operators (1.1) are in the physical spectrum, and the usual torus partition function can be written as a finite sum in terms of a finite number of characters of the algebra \cite{3}. (See Appendix A)

The case of $k$ odd is somewhat more subtle. As we will see, the chiral operators (1.1) are not in the physical spectrum of the boson theory. Also, the $U(1)_k$ Chern-Simons theory is a spin topological quantum field theory (TQFT) \cite{5} although the CFT is bosonic and does not depend on the 2d spin structure. We will see that these issues are nicely resolved by

\begin{footnote}
The chiral operators $e^{\pm 2i \sqrt{k} \phi_L}$ corresponding to $U(1)_{4k}$ are in the physical spectrum, and the boson torus partition function can be written as a finite sum in terms of a finite number of characters for the algebra they generate. The $U(1)_{4k}$ CS theory is obtained from the $U(1)_k$ CS theory by gauging the fermionic parity, \textit{i.e.}, by summing over spin structures \cite{4}. The former is called the “shadow” of the latter.
\end{footnote}
modifying the bosonic CFT into a spin CFT according to the recently proposed procedure involving the so-called Arf invariant \[6, 7\]

In this note we demonstrate that one can rewrite the torus partition function of the modified theory as a finite sum in terms of a finite number of spin structure dependent conformal blocks of the extended chiral algebra. We use the conformal blocks to compute the modular matrices of the modified compact boson CFT. They coincide with the matrices obtained from the Chern-Simons theory \[11, 12\] up to conjugation. We then use the modular matrices to compute the partition function of the \(U(1)_k\) spin Chern-Simons theory on the lens space \(L(a, \pm 1)\). For \(k\) odd and \(a\) even, we obtain the expected dependence on the spin structure on \(L(a, \pm 1)\).

This paper is organized as follows. In Section 2 we study the free boson theory modified by the Arf invariant. We first review the modification procedure as described in \[7\]. We then expand the modified torus partition function in terms of a finite number of conformal blocks. Using the conformal blocks we compute the modular matrices. In Section 3 we use the modular matrices to compute the partition functions on \(L(a, \pm 1)\). In Appendix A we summarize the modular matrices and the \(L(a, \pm 1)\) partition functions for \(k\) even. In Appendix B we review the notions of quadratic refinements and their Arf invariants.

2 Modular matrices of the compact boson spin CFT

We consider the theory \(\mathcal{T}_\phi\) of a free boson \(\phi\) parametrizing the circle of radius \(R\) \((\phi \sim \phi + 2\pi R)\) with an action

\[
S = \frac{1}{8\pi} \int d^2 x \partial_{\mu} \phi \partial^{\mu} \phi. \tag{2.1}
\]

In this normalization T-duality acts as \(R \rightarrow 2/R\). Let \(\tau\) be the modulus of the torus and set \(q = e^{2\pi i \tau}\). The torus partition function is given as the sum over the physical spectrum

\[
Z[\mathcal{T}_\phi] = \frac{1}{|\eta(\tau)|^2} \sum_{n, w \in \mathbb{Z}} q^{n^2/4 + wR^2/2} \bar{q}^{n^2/4 - wR^2/2}. \tag{2.2}
\]

Each term corresponds to the local operator

\[
e^{ip_L \phi_L + ip_R \phi_R} \tag{2.3}
\]

with \(p_L = \frac{n}{R} + \frac{wR}{2}\), \(p_R = \frac{n}{R} - \frac{wR}{2}\) for \(n, w \in \mathbb{Z}\). For generic \(R\) the chiral algebra is generated by \(i\partial L\).

We are interested in the so-called rational boson, for which the radius-squared is a rational number. We set \(R = 2(p/p'')^{1/2}\) with \(p\) and \(p''\) relatively prime positive integers. We also set \(\sqrt{k} = pp''\).

\[\text{For the earlier related literature see the references mentioned in \[8\]. The procedure was recently applied to minimal models to obtain fermionic minimal models \[9, 10\].}\]

\[\text{The } R = \sqrt{k} \text{ case in the introduction corresponds to the T-dual with } p = 1, p'' = k.\]
The chiral operators $e^{\pm i\sqrt{k}\phi_L}$ with $(p_L, p_R) = (\pm \sqrt{k}, 0)$ correspond to $(n, w) = \pm (p, p''/2)$. For $p''$ even, hence with $k$ even, these are part of the physical spectrum, and extend the chiral algebra. For $p''$ odd, and especially when $k$ is odd, however, they are not in the spectrum.

\section*{2.1 Modification by the Arf invariant}

We now introduce the spin structure dependence into the 2d CFT following \cite{7}.

The crucial ingredient is the topological theory that we call $\mathcal{T}_{\text{Arf}}$. It is the low-energy limit of the Kitaev Majorana chain \cite{13} and has the partition function given as

$$Z[\mathcal{T}_{\text{Arf}}; \rho] = e^{\pi i \text{Arf}[\rho]},$$

(2.5)

where $\text{Arf}[\rho]$ is the so-called Arf invariant\footnote{More precisely, this is the Arf invariant of a quadratic refinement of the intersection pairing on $H_1(\Sigma, \mathbb{Z}_2)$, which is in one-to-one correspondence with a spin structure on the surface $\Sigma$ \cite{14}. The Arf invariant also appears in \cite{11}, where it is called the mod 2 index, but it plays different roles.} determined by the spin structure $\rho$ on a 2d surface. It is identified with the index mod 2 of the Dirac operator given by the spin structure $\rho$. In particular we have

$$\text{Arf}[\rho] = \begin{cases} 
1 & \text{if } \rho = PP, \\
0 & \text{if } \rho = PA, AP, AA 
\end{cases}$$

(2.6)

for the four spin structures on the torus, where $P$ and $A$ denote periodic and anti-periodic boundary conditions, respectively.\footnote{For example, $PA$ corresponds to the periodic boundary condition in the space (horizontal) direction and the anti-periodic boundary condition in the time (vertical) direction.}

On a general surface $\Sigma$, a $\mathbb{Z}_2$ gauge field $S$ is an element of $H^1(\Sigma, \mathbb{Z}_2)$ and acts on the spin structure: $\rho \rightarrow S \cdot \rho$. Indeed a $\mathbb{Z}_2$ gauge field can be regarded as a $\mathbb{Z}_2$ holonomy and it modifies the boundary condition of a fictitious spinor along a closed path dictated by $\rho$. The holonomy along the horizontal (resp. vertical) direction can be regarded as the insertion of a topological defect along the vertical (resp. horizontal) direction. The defect is the symmetry generator of the 0-form (i.e., ordinary) $\mathbb{Z}_2$ global symmetry \cite{13}. Thus the topological theory $\mathcal{T}_{\text{Arf}}$ has a global $\mathbb{Z}_2$ symmetry, even though it has no local operators on which the symmetry acts.

The boson theory $\mathcal{T}_\phi$ also has a symmetry $\mathbb{Z}_2^S$ generated by the shift $\phi \rightarrow \phi + \pi R$\footnote{Another symmetry $\mathbb{Z}_2^S$ generated by $\phi \rightarrow -\phi$ would replace $\mathbb{Z}_2^S$ if we work in the T-dual frame.} We now consider the theory

$$(\mathcal{T}_\phi \times \mathcal{T}_{\text{Arf}})/\mathbb{Z}_2^{\text{diag}},$$

(2.7)

obtained by gauging a $\mathbb{Z}_2$ symmetry of the product theory $\mathcal{T}_\phi \times \mathcal{T}_{\text{Arf}}$. The group $\mathbb{Z}_2^{\text{diag}}$ is the diagonal subgroup of the product of $\mathbb{Z}_2^S$ and the symmetry group $\mathbb{Z}_2$ of $\mathcal{T}_{\text{Arf}}$.

We wish to study the torus partition function of $(\mathcal{T}_\phi \times \mathcal{T}_{\text{Arf}})/\mathbb{Z}_2^{\text{diag}}$. To write down formulas succinctly, we choose a reference spin structure on the torus, say $\rho_0 = AA$\footnote{This choice is motivated by the labeling (characteristics) of the theta functions $\vartheta_{ab}(\nu; \tau)$. See \cite{12}.}. We

\footnotetext[6]{More precisely, this is the Arf invariant of a quadratic refinement of the intersection pairing on $H_1(\Sigma, \mathbb{Z}_2)$, which is in one-to-one correspondence with a spin structure on the surface $\Sigma$ \cite{14}. The Arf invariant also appears in \cite{11}, where it is called the mod 2 index, but it plays different roles.}

\footnotetext[7]{For example, $PA$ corresponds to the periodic boundary condition in the space (horizontal) direction and the anti-periodic boundary condition in the time (vertical) direction.}

\footnotetext[8]{Another symmetry $\mathbb{Z}_2^S$ generated by $\phi \rightarrow -\phi$ would replace $\mathbb{Z}_2^S$ if we work in the T-dual frame.}

\footnotetext[9]{This choice is motivated by the labeling (characteristics) of the theta functions $\vartheta_{ab}(\nu; \tau)$. See \cite{12}.}
identify general spin structures $\rho$ with $\mathbb{Z}_2$ gauge fields $S = (S_1, S_2)$ via $\rho = S \cdot \rho_0$:

$$AA \leftrightarrow S = (0, 0), \quad AP \leftrightarrow (0, 1), \quad PA \leftrightarrow (1, 0), \quad PP \leftrightarrow (1, 1). \quad (2.8)$$

To compute the torus partition function we sum over dynamical $\mathbb{Z}_2^{\text{diag}}$ gauge fields $s = (s_1, s_2)$ and divide by $|\mathbb{Z}_2^{\text{diag}}| = 2$. We get

$$Z[(\mathcal{T}_\phi \times \mathcal{T}_{\text{Art}})/\mathbb{Z}_2^{\text{diag}}, R; \rho = S \cdot \rho_0] = \frac{1}{2|\eta(\tau)|^2} \sum_{s=(s_1, s_2)} (-1)^{(S_1+s_1)(S_2+s_2)} \sum_{n \in \mathbb{Z}} \sum_{w \in \mathbb{Z} + \frac{s_1}{2}} (-1)^{ns_2} q^{\frac{1}{2}(\frac{n}{R} + \frac{wR}{2})^2} q^{\frac{1}{2}(\frac{n}{R} - \frac{wR}{2})^2}. \quad (2.9)$$

The sign $(-1)^{(S_1+s_1)(S_2+s_2)}$ is the partition function (2.5) of $\mathcal{T}_{\text{Art}}$ for the spin structure $(S + s) \cdot \rho_0$. The winding number $w$ is a half-odd integer in the twisted sector $(s_1 = 1)$. The momentum $n$ induces a sign $(-1)^n$ under the shift $\phi \to \phi + \pi R$. Explicitly, we have

$$Z[(\mathcal{T}_\phi \times \mathcal{T}_{\text{Art}})/\mathbb{Z}_2^{\text{diag}}, R; \rho] = \frac{1}{|\eta(\tau)|^2} \times$$

$$\sum_{m, \bar{m} \in \mathbb{Z}} \begin{cases} q^\frac{1}{2}(m(\frac{R}{2} + \bar{R}) + \bar{m}(\frac{R}{2} - \frac{R}{2}))^2 q^\frac{1}{2}(\bar{m}(\frac{R}{2} + \frac{R}{2}) + m(m - \frac{R}{2}))^2 & \rho = AA, \\
(-1)^{m+\bar{m}} q^\frac{1}{2}(m(\frac{R}{2} + \bar{R}) + \bar{m}(\frac{R}{2} - \frac{R}{2}))^2 q^\frac{1}{2}(\bar{m}(\frac{R}{2} + \frac{R}{2}) + m(m - \frac{R}{2}))^2 & \rho = AP, \\
q^\frac{1}{2}(m(\frac{R}{2} + \bar{R}) + \bar{m}(\frac{R}{2} - \frac{R}{2}))^2 q^\frac{1}{2}(\bar{m}(\frac{R}{2} + \frac{R}{2}) + m(m - \frac{R}{2}))^2 & \rho = PA, \\
(-1)^{m+\bar{m}+1} q^\frac{1}{2}(m(\frac{R}{2} + \bar{R}) + \bar{m}(\frac{R}{2} - \frac{R}{2}) + \frac{R}{2}))^2 q^\frac{1}{2}(\bar{m}(\frac{R}{2} + \frac{R}{2}) + m(m - \frac{R}{2}) + \frac{R}{2})^2 & \rho = PP. 
\end{cases} \quad (2.10)$$

These partition functions should be interpreted as $\text{Tr}_{\mathcal{H}_L} q^{\frac{1}{24}(\bar{R}-R)}$ for $\rho = *A$ and as $\text{Tr}_{\mathcal{H}_F} q^{\frac{1}{24}(\bar{R}+R)}$ for $\rho = *P$, where $*$ is either $A$ or $P$.

In the NS sector ($* = A$ in the spatial direction), the physical spectrum is given by

$$p_L = \frac{1}{2} \left[ m \left( \frac{2}{R} + \frac{R}{2} \right) + \bar{m} \left( \frac{2}{R} - \frac{R}{2} \right) \right], \quad p_R = \frac{1}{2} \left[ \bar{m} \left( \frac{2}{R} + \frac{R}{2} \right) + m \left( \frac{2}{R} - \frac{R}{2} \right) \right]. \quad (2.11)$$

with $m, \bar{m} \in \mathbb{Z}$. The Fermion number is $(-1)^F = (-1)^{m+\bar{m}}$.

In the R sector ($* = P$ in the spatial direction), the physical spectrum is given by

$$p_L = \frac{1}{2} \left[ m \left( \frac{2}{R} + \frac{R}{2} \right) + \bar{m} \left( \frac{2}{R} - \frac{R}{2} \right) + \frac{2}{R} \right], \quad p_R = \frac{1}{2} \left[ \bar{m} \left( \frac{2}{R} + \frac{R}{2} \right) + m \left( \frac{2}{R} - \frac{R}{2} \right) + \frac{2}{R} \right] \quad (2.12)$$

with $m, \bar{m} \in \mathbb{Z}$. The Fermion number is $(-1)^F = (-1)^{m+\bar{m}+1}$.

For $R = 2(p/p'')^{1/2}$ and $k = p p''$, the generators $e^{\pm i\sqrt{k}p_L}$ of the extended chiral algebra have $(p_L, p_R) = (\pm \sqrt{k}, 0)$. For $p$ and $p''$ both odd, they are physical and are in the NS sector, with $(m, \bar{m}) = (\frac{p+p''}{2}, \frac{p-p''}{2})$.

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8This is equivalent to (3.28) of [7] with $S_{\text{there}} = C = 0$. 

2.2 Torus conformal blocks

We can expand the spin structure dependent torus partition function (2.10), or its generalization by $U(1)$ symmetries, as a finite sum in terms of a finite number of conformal blocks.

We need some preparation. We define

$$p_1 := \frac{p'' + p}{2}, \quad p_2 := \frac{p'' - p}{2},$$

(2.13)

which are relatively prime integers because $p$ and $p''$ are odd and relatively prime. Let us choose $m_0, \bar{m}_0 \in \mathbb{Z}$ such that $m_0 p_1 + \bar{m}_0 p_2 = 1$ and set

$$\omega := m_0 p_2 + \bar{m}_0 p_1.$$  

(2.14)

It can be shown that

$$\omega (mp_1 + \bar{m} p_2) = mp_2 + \bar{m} p_1 \mod k.$$  

(2.15)

The compact boson has two $U(1)$ symmetries whose charges are the momentum and the winding number. Their linear combinations give left- and right-moving $U(1)$ symmetries.

We let $z := e^{2\pi i \nu}$ and $\bar{z} := e^{-2\pi i \bar{\nu}}$ be their corresponding fugacities.

Let us define the function

$$K_\lambda(z, \tau) := \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{(kn + \lambda)^2} z^n,$$

(2.16)

and the conformal blocks $L^{(S_1, S_2)}_\lambda (\nu, \tau)$ ($S_{1, 2} \in \{0, 1\}$)

$$L^{(S_1, S_2)}_\lambda (\nu, \tau) = i^{S_1 S_2} z^{\lambda/k} K_\lambda ((-1)^{S_2} z, \tau), \quad \lambda \in \mathbb{Z} + S_2 \frac{1}{2}. $$

(2.17)

The blocks $L^{(S_1, 0)}_\lambda (S_1 = 0, 1)$ coincide with the characters of the extended chiral algebra

$$L^{(S_1, 0)}_\lambda (\nu, \tau) = \text{Tr}_{V_\lambda} q^{L_0 - \frac{c}{24} z J_0},$$

(2.18)

where $J_0 = p_L / \sqrt{k}$, and $V_\lambda$ is the representation that contains the state corresponding to the chiral operator $e^{i \sqrt{k} \phi_L}$.

\textsuperscript{9}Our method is brute force. The orbifold consideration in [9] should yield the same results.

\textsuperscript{10}These are essentially the specialization of the conformal blocks (the physical wave functions) considered in [11] to the case with genus one and gauge group $U(1)$.

\textsuperscript{11}The functions $L^{(S_1, 1)}_\lambda (S_1 = 0, 1)$ are roughly the “supercharacters” $\text{Tr}_{V_\lambda} (\chi^{(1)} F_L) q^{L_0 - \frac{c}{24} z J_0}$, where $F_L$ is a would-be left-moving fermion number. We do not try to make this precise. Our normalization is motivated by the relation [2223].
For $\rho = AA$, the torus partition function with $U(1)$ fugacities is

$$Z[(T_\phi \times T_{\text{Art}})/\mathbb{Z}_2^\text{diag}; R = 2(p/p'')^{1/2}; \rho = AA] = \frac{1}{|\eta(\tau)|^2} \sum_{m,n \in \mathbb{Z}} q^{\frac{1}{2}(m(\frac{2}{n} + \frac{1}{2}) + m(\frac{2}{n} - \frac{1}{2}))^2 \oint_{2k/2} \frac{1}{2} (m(\frac{2}{n} + \frac{1}{2}) + m(\frac{2}{n} - \frac{1}{2}))}$$

$$\times q^{\frac{1}{2}(\bar{m}(\frac{2}{n} + \frac{1}{2}) + \bar{m}(\frac{2}{n} - \frac{1}{2}))^2 \oint_{2k/2} \frac{1}{2} (\bar{m}(\frac{2}{n} + \frac{1}{2}) + \bar{m}(\frac{2}{n} - \frac{1}{2}))}$$

$$= \frac{1}{|\eta(\tau)|^2} \sum_{m,n \in \mathbb{Z}} q^{\frac{1}{2}(m(\frac{2}{n} + \frac{1}{2}) + m(\frac{2}{n} - \frac{1}{2}))^2 \oint_{2k/2} \frac{1}{2} (m(\frac{2}{n} + \frac{1}{2}) + m(\frac{2}{n} - \frac{1}{2}))}$$

$$\times q^{\frac{1}{2}(\bar{m}(\frac{2}{n} + \frac{1}{2}) + \bar{m}(\frac{2}{n} - \frac{1}{2}))^2 \oint_{2k/2} \frac{1}{2} (\bar{m}(\frac{2}{n} + \frac{1}{2}) + \bar{m}(\frac{2}{n} - \frac{1}{2}))}$$

$$= \sum_{\lambda=0}^{k-1} L_{\lambda}^{(0,0)}(\nu, \tau) \bar{L}_{\omega \lambda}^{(0,0)}(\bar{\nu}, \bar{\tau}).$$

Similarly for $AP$,

$$Z[(T_\phi \times T_{\text{Art}})/\mathbb{Z}_2^\text{diag}; R = 2(p/p'')^{1/2}; \rho = AP] = \frac{1}{|\eta(\tau)|^2} \sum_{m,n \in \mathbb{Z}} (-1)^{m+n+m} q^{\frac{1}{2}(m(\frac{2}{n} + \frac{1}{2}) + m(\frac{2}{n} - \frac{1}{2}))^2 \oint_{2k/2} \frac{1}{2} (m(\frac{2}{n} + \frac{1}{2}) + m(\frac{2}{n} - \frac{1}{2}))}$$

$$\times q^{\frac{1}{2}(\bar{m}(\frac{2}{n} + \frac{1}{2}) + \bar{m}(\frac{2}{n} - \frac{1}{2}))^2 \oint_{2k/2} \frac{1}{2} (\bar{m}(\frac{2}{n} + \frac{1}{2}) + \bar{m}(\frac{2}{n} - \frac{1}{2}))}$$

$$= \sum_{\lambda=0}^{k-1} (-1)^{(1+\omega)\lambda} L_{\lambda}^{(0,1)}(\nu, \tau) \bar{L}_{\omega \lambda}^{(0,1)}(\bar{\nu}, \bar{\tau}).$$

For $\rho = PA$,

$$Z[(T_\phi \times T_{\text{Art}})/\mathbb{Z}_2^\text{diag}; R = 2(p/p'')^{1/2}; \rho = PA] = \frac{1}{|\eta(\tau)|^2} \sum_{m,n \in \mathbb{Z}} q^{\frac{1}{2}(m(\frac{2}{n} + \frac{1}{2}) + m(\frac{2}{n} - \frac{1}{2}))^2 \oint_{2k/2} \frac{1}{2} (m(\frac{2}{n} + \frac{1}{2}) + m(\frac{2}{n} - \frac{1}{2}) + \frac{2}{n})}$$

$$\times q^{\frac{1}{2}(\bar{m}(\frac{2}{n} + \frac{1}{2}) + \bar{m}(\frac{2}{n} - \frac{1}{2}) + \frac{2}{n})^2 \oint_{2k/2} \frac{1}{2} (\bar{m}(\frac{2}{n} + \frac{1}{2}) + \bar{m}(\frac{2}{n} - \frac{1}{2}) + \frac{2}{n})}$$

$$= \sum_{\lambda=0}^{k-1} L_{\lambda+p''/2}^{(1,0)}(\nu, \tau) \bar{L}_{\omega \lambda+p''/2}^{(1,0)}(\bar{\nu}, \bar{\tau}).$$
Finally for \( \rho = PP \),
\[
Z[\mathcal{T}_\phi \times \mathcal{T}_\lambda]/\mathbb{Z}_2; R = 2(p/p'')^{1/2}; \rho = PP
= \frac{-1}{|\eta(\tau)|^2} \sum_{m, n \in \mathbb{Z}} (-1)^{m+n} \frac{1}{q} \eta^2(m(\frac{m}{2} + \frac{1}{2}) + (\frac{m}{2} - \frac{1}{2})^2) \frac{1}{2} \zeta_{2k/2} \left[m(\frac{m}{2} + \frac{1}{2}) + (\frac{m}{2} - \frac{1}{2})^2\right] \times \frac{1}{q} \eta^2(m(\frac{m}{2} + \frac{1}{2}) + (\frac{m}{2} - \frac{1}{2})^2) \frac{1}{2} \zeta_{2k/2} \left[m(\frac{m}{2} + \frac{1}{2}) + (\frac{m}{2} - \frac{1}{2})^2\right]
\]
\[
= \sum_{\lambda = 0}^{k-1} (-1)^{(1+\omega)\lambda+1} L^{(1,1)}_{\lambda+p''/2}(\nu, \tau) L^{(1,1)}_{\omega+\omega+p''/2}(\nu, \tau).
\]

For \( k = 1 \) our conformal blocks are related to Jacobi’s theta functions \( \vartheta_0(\nu, \tau) \) and the Dedekind eta function as
\[
L^{(S_1, S_2)}_{\lambda=S_1/2}(\nu, \tau) = \frac{\vartheta_0(\nu, \tau)}{\eta(\tau)}.
\]

### 2.3 Modular matrices

We note the relation
\[
K_{\lambda+1/2}(z, \tau) = q^{\frac{\lambda}{2}\tau + \frac{1}{4}} K_{\lambda}(q^{1/2}z, \tau).
\]

For \( \lambda \in \frac{1}{2} \mathbb{Z} \), the function \( K_{\lambda}(z, \tau) \) transforms as
\[
K_{\lambda}(z, \tau + 1) = e^{-\pi i \frac{1}{2}} e^{\frac{\pi i}{8} \lambda^2} K_{\lambda}((-1)^{2\lambda+1} z, \tau),
\]
\[
K_{\lambda}(e^{2\pi i \nu/\tau}, -1/\tau) = e^{\pi i \nu^2} \prod_{\mu=0}^{k-1} e^{-2\pi i \mu \nu} e^{2\pi i \frac{\lambda^2}{8}} K_{\lambda}((-1)^{2\lambda} e^{2\pi i \nu}, \tau).
\]

These can be used to show, for \( \lambda \in \mathbb{Z} + \frac{1}{2} \), that
\[
L^{(S_1, S_2)}_\lambda(\nu, \tau + 1) = e^{-\pi i \frac{\lambda^2}{8}} e^{\frac{\pi i}{8} \lambda^2} L^{(S_1, S_2)}_\lambda(\nu, \tau),
\]
\[
L^{(S_1, S_2)}_\lambda(\nu/\tau, -1/\tau) = e^{\pi i \nu^2} \prod_{\mu=0}^{k-1} e^{-2\pi i \mu \nu} L^{(S_1, S_2)}_\mu(\nu, \tau).
\]

In terms of the functions \( K^{(S_1, S_2)}_{\lambda}(\nu, \tau, \rho) := e^{-2\pi i \rho/k} L^{(S_1, S_2)}_\lambda(\nu, \tau) \) that depend on an extra parameter \( \rho \in \mathbb{C} \), we can write these transformations as \(^{12}\)
\[
K^{(S_1, S_2)}_{\lambda}(\nu, \tau + 1, \rho) = \sum_{T_1, T_2, \mu} T^{(S_1, S_2; T_1, T_2; \mu)}(T_1, T_2; \rho) K^{(T_1, T_2)}_{\lambda}(\nu, \tau, \rho),
\]
\[
K^{(S_1, S_2)}_{\lambda}(\frac{\nu}{\tau}, -1/\tau, \rho + \frac{\nu^2}{2\tau}) = \sum_{T_1, T_2, \mu} S^{(S_1, S_2; \lambda; T_1, T_2; \mu)}(T_1, T_2; \rho) K^{(T_1, T_2)}_{\lambda}(\nu, \tau, \rho),
\]

\(^{12}\)The following is a well-defined action of the group \( SL(2, \mathbb{Z}) \).
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\nu; \tau; \rho) \rightarrow \left( \frac{\nu}{c\tau + d}, \frac{a\nu + b}{c\tau + d}, \rho + \frac{c|\nu|^2}{2(c\tau + d)} \right).
\]
with the matrices $S$ and $T$ given as

$$S(S_1,S_2;\lambda)(T_1,T_2;\mu) = \frac{1}{\sqrt{k}} \delta_{S_1T_2} \delta_{S_2T_1} e^{-\frac{2\pi}{k} \lambda \mu},$$

(2.32)

$$T(S_1,S_2;\lambda)(T_1,T_2;\mu) = \delta_{S_1T_1} \delta_{S_2T_2} e^{-\frac{2\pi}{k} \lambda \mu} e^{\frac{2\pi}{k} \lambda^2}.$$  

(2.33)

Here $S_i, T_i \in \{0,1\}$ and we have $\lambda \in \{0, 1, \ldots, k-1\}$, $\mu \in \{0, 1, \ldots, k-1\}$.

The matrices $S$ and $T$ respectively represent the generators of $SL(2, \mathbb{Z})$.

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$  

(2.34)

3. **$U(1)_k$-odd Chern-Simons theory on $L(a, \pm 1)$**

As an application of (2.32) and (2.33), we compute the partition function of the $U(1)_k$ Chern-Simons theory, with $k$ odd, on the lens space $L(a, \pm 1)$.

3.1 Gluing matrix for $L(a,b)$

We begin, rather pedantically, by reviewing how the general lens space $L(a,b)$ ($a > 0$, $a$ and $b$ relatively prime) is obtained by gluing two copies of solid torus. Let us view the three-sphere $S^3$ as a subset $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}$ of $\mathbb{C}^2$. We take two integers $a \geq 1$ and $b$ that are relatively prime. There exists a non-unique pair of integers $a'$ and $b'$ such that $bb' - aa' = 1$. The lens space $L(a,b)$ is defined as a $\mathbb{Z}_a$ quotient

$$L(a,b) := S^3 / \mathbb{Z}_a$$  

(3.1)

where the $\mathbb{Z}_a$-action depends on $b$ and is specified by the action of the generator

$$(z_1, z_2) \mapsto (e^{\frac{2\pi i}{a} z_1}, e^{\frac{2\pi i}{a} z_2}).$$  

(3.2)

The lens space $L(a,b)$ can be obtained by gluing two copies of solid torus $D^2 \times S^1$:

$$L(a,b) \simeq \left( (D^2 \times S^1)_1 \cup (D^2 \times S^1)_2 \right) / \sim,$$  

(3.3)

where the boundaries of the first copy

$$(D^2 \times S^1)_1 := \{(re^{i\phi_1}, e^{i\phi_2}) \in \mathbb{C}^2 | 0 \leq r \leq 1, \phi_1, \phi_2 \in [0,2\pi)\}$$  

(3.4)

and the second copy

$$(D^2 \times S^1)_2 := \{(re^{i\tilde{\phi}_1}, e^{i\tilde{\phi}_2}) \in \mathbb{C}^2 | 0 \leq \tilde{r} \leq 1, \tilde{\phi}_1, \tilde{\phi}_2 \in [0,2\pi)\}$$  

(3.5)

---

13 The modular matrices in [11] were corrected in [12], and are related to ours by conjugation.
14 The matrices satisfy the defining relations, $S^4 = 1$ and $(ST)^3 = S^2$ of $SL(2, \mathbb{Z})$. 

8
at \( r = \tilde{r} = 1 \) are identified via the relation\(^{15}\)

\[
\begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} = U \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mod 2\pi, \quad U = \begin{pmatrix} b & a' \\ a & b' \end{pmatrix} \in SL(2, \mathbb{Z}).
\] (3.7)

We note that the shift \( b \to b + ja \) corresponds to multiplying \( U \) by \( T^j \) from the left. Similarly the shift \( (a', b') \to (a' + jb, b' + ja) \) corresponds to the multiplication by \( T^j \) from the right.

The transformation

\[
\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto M \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})
\] (3.8)
on the angular coordinates on the two-dimensional torus\(^{16}\) induces the transformation

\[
\tau \mapsto \frac{\alpha \tau + \beta}{\gamma \tau + \delta}
\] (3.9)
of the modulus \( \tau \) defined as the ratio \( \tau = \omega_2/\omega_1 \) of two periods \( \omega_1, \omega_2 \in \mathbb{C}\setminus\{0\} \) such that \( \text{Im}(\omega_2/\omega_1) > 0 \) if we adopt the convention where \((\phi_1, \phi_2)\) corresponds to a point \( \phi_2\omega_1 - \phi_1\omega_2 \) on the complex plane \( \mathbb{C} \).

For a bosonic TQFT such as the \( SU(2)_k \) Chern-Simons theory considered in [19], the partition function on \( L(a, b) \) is computed as follows. Let \( \mathcal{F}_i(\tau) \) be the torus conformal blocks (characters) of the edge state CFT. Under the transformation \((3.9)\), the blocks behave as

\[
\mathcal{F}_i \left( \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \right) = \sum_j M_{ij} \mathcal{F}_j(\tau),
\] (3.10)

where the matrix \( M = (M_{ij}) \) represents \( M \) on the space spanned by the conformal blocks. In [19] a formula was given in terms of the matrix \( U \) for the \( L(a, b) \) partition function of the Chern-Simons theory. It reads

\[
Z[L(a, b)] = U_{00},
\] (3.11)

where 0 denotes the identity state, and \( U \) represents the gluing matrix \( U \) in \((3.7)\). In this note we choose to ignore framing dependence [19].

\(^{15}\)In terms of the variables \((\psi_1, \psi_2)\) such that \( z_1 = \cos \frac{a}{2} e^{i\psi_1}, z_2 = \sin \frac{a}{2} e^{i\psi_2}, \) we have the relations

\[
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 1 & b'/a \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1/a \\ -1 & b/a \end{pmatrix} \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix}.
\] (3.6)

\(^{16}\)The meridian \( \partial D^2 \times pt \) and the longitude \( pt \times S^1 \) are parametrized by \( \phi_1 \) and \( \phi_2 \), respectively. This implies that our matrix convention in \((3.8)\) is consistent with those of [16] [17] [18]. In addition, the transformation \((3.9)\) of \( \tau \) is consistent with [19] and the CFT literature.
3.2 Spin structures in 2d and 3d

For a spin TQFT, the partition function depends on the choice of a spin structure.

An orientable manifold admits a spin structure if and only if its second Stiefel-Whitney
class $w_2$ vanishes. Any orientable 3-manifold admits a spin structure because its tangent
bundle is trivial (and hence $w_2 = 0$). The cohomology $H^1(X, \mathbb{Z}_2)$ of any manifold $X$ acts
on the spin structures on $X$ transitively and freely, and thus classifies them. (We stated
this fact in terms of $\mathbb{Z}_2$ gauge fields in Section 2.1.) The cohomology $H^1(L(a, b), \mathbb{Z}_2)$ is 0
for $a$ odd and $\mathbb{Z}_2$ for $a$ even.

Thus $L(a, b)$ admits a single spin structure for $a$ odd and two spin structures for $a$ even.
This can also be seen in terms of spin structures on the 2d torus as follows.

The lens space $L(a, b)$ is obtained by gluing two copies of the solid tori as in (3.3).
Their boundaries $(T^2)_1 := \partial(D^2 \times S^1)_1$ and $(T^2)_2 := \partial(D^2 \times S^1)_2$ carry 2d spin structures.
A spin structure on $T^2$ is specified by periodic ($P \leftrightarrow 1$) or anti-periodic ($A \leftrightarrow 0$) boundary
conditions (for an auxiliary spinor) along the two circles. The 2d spin structures on the
two tori that are compatible with gluing give rise to a 3d spin structure $\rho_{3d}$ on $L(a, b)$. Let
us study the boundary conditions on the auxiliary spinor

$$
\Psi(\phi_1, \phi_2) = \tilde{\Psi}(\tilde{\phi}_1, \tilde{\phi}_2) \quad (3.12)
$$

with coordinates related via (3.7). We have

$$
\Psi(\phi_1, \phi_2) = (-1)^{1+S_1} \Psi(\phi_1 + 2\pi, \phi_2) = (-1)^{1+S_2} \Psi(\phi_1, \phi_2 + 2\pi),
\tilde{\Psi}(\tilde{\phi}_1, \tilde{\phi}_2) = (-1)^{1+T_1} \tilde{\Psi}(\tilde{\phi}_1 + 2\pi, \tilde{\phi}_2) = (-1)^{1+T_2} \tilde{\Psi}(\tilde{\phi}_1, \tilde{\phi}_2 + 2\pi). 
$$

We see that the spin structure $\rho = (S_1, S_2) \cdot \rho_0$ on $(T^2)_1$ and another $\sigma = (T_1, T_2) \cdot \rho_0$ on
$(T^2)_2$ are related as

$$
S_1 = 1 + (1 - T_1)b + (1 - T_2)a \mod 2, \\
S_2 = 1 + (1 - T_1)a' + (1 - T_2)b' \mod 2. \quad (3.14)
$$

It is well known that the boundary condition along the boundary of a disk is necessarily
anti-periodic ($A \leftrightarrow 0$) corresponding to the NSNS sector [20]. Thus we necessarily have
$S_1 = T_1 = 0$. For a given $U$ in (3.7) with $bb' - aa' = 1$, the equations (3.14) then admit
two solutions for $S_2$ and $T_2$ if $a$ is even, and a unique solution if $a$ odd:

$$
\begin{align*}
a = 0 & \Rightarrow b = b' = 1 \Rightarrow S_2 = 0 \text{ or } 1, \quad T_2 = a' + S_2 \\
a = 1 & \Rightarrow a' = bb' + 1 \Rightarrow S_2 = b', \quad T_2 = b
\end{align*} \quad \text{all mod } 2. \quad (3.15)
$$

Thus the lens space $L(a, b)$ admits two spin structures for $a$ even and a single spin structure
for $a$ odd, as expected.

We expect that for a general spin TQFT, the edge state CFT with a chiral algebra
symmetry is a spin CFT whose conformal blocks depend on the spin structure $\rho$ on the
surface, as in [22]. Let $F_{\rho,i}(\tau)$ denote the torus conformal blocks that depend on the spin
structure $\rho$, in addition to the representation $i$ of the algebra and the modulus $\tau$. The modular matrices are introduced by

$$F_{\rho,i}(-1/\tau) = \sum_{\sigma} \sum_{j} S_{(\rho,i)(\sigma,j)} F_{\sigma,j}(\tau), \quad F_{\rho,i}(\tau + 1) = \sum_{\sigma} \sum_{j} T_{(\rho,i)(\sigma,j)} F_{\sigma,j}(\tau), \quad (3.16)$$

$$F_{\rho,i}(b'\tau + a' + b) = \sum_{\sigma} \sum_{j} U_{(\rho,i)(\sigma,j)} F_{\sigma,j}(\tau). \quad (3.17)$$

The matrix $U$ represents the gluing matrix $U$ in (3.1) for the lens space. We propose that the lens space partition function of the spin TQFT for a given 3d spin structure $\rho_{3d}$ is given by

$$Z\left[ L(a,b); \rho_{3d} \right] = U_{(\rho,0)(\sigma,0)}, \quad (3.18)$$

where $0$ denotes the representation that contains the ground state in the given sector, and the pair $(\rho, \sigma)$ of 2d spin structures corresponds to the 3d spin structure $\rho_{3d}$.

### 3.3 Partition function on lens space $L(a, \pm 1)$

For the lens space $L(a, \epsilon = \pm 1)$, the gluing matrix is

$$U = \begin{pmatrix} \epsilon & 0 \\ a & \epsilon \end{pmatrix} = S^{\epsilon}T^{-\epsilon a}S^{-1}. \quad (3.19)$$

Using the formulas (2.32) and (2.33) we obtain, for $U = S^{\epsilon}T^{-\epsilon a}S^{-1},$

$$U_{(S_1, S_2; \lambda)(T_1, T_2; \mu)} = e^{\frac{4\pi i}{k} T_{S_1 T_1} \delta_{S_2 T_2}} \frac{1}{k} \sum_{\nu = S_2 / 2}^{k - 1 + S_2 / 2} e^{\frac{2\pi i}{k} \nu(T_{\lambda + \mu}) e^{-\frac{\epsilon\pi i}{k} \nu^2}}. \quad (3.20)$$

Here $S_i, T_i \in \{0, 1\}$ and we have $\lambda \in \{0, 1, \ldots, k - 1\} + S_1 / 2$, $\mu \in \{0, 1, \ldots, k - 1\} + T_1 / 2$.

Let us recall from (3.15) that the possible spin structures on the 2d torus depend on $a$. For both $a$ even and odd, there exists a 3d spin structure corresponding to the 2d spin structures $\rho = \sigma = (0, 1) \cdot \rho_0 = AP$ for which we obtain via the reciprocity formula for the Gauss sum:

$$Z = U_{(0,1;0)(0,1;0)} = \frac{e^{\frac{\pi i}{4} \epsilon} e^{-\frac{\pi i}{4} \epsilon}}{\sqrt{ka}} \sum_{h = 0}^{a - 1} e^{\pi i \epsilon h^2} e^{-\pi i h} = \frac{e^{\frac{\pi i}{4} \epsilon} e^{-\frac{\pi i}{4} \epsilon}}{\sqrt{ka}} \sum_{h = 0}^{a - 1} e^{\pi i \epsilon h^2 / (1 - a)h^2}. \quad (3.22)$$

Let $\alpha, \beta, \gamma$ be integers with $\alpha \gamma \neq 0$ and $\alpha \gamma + \beta$ even. Then (see for example Section 1.2 of [21])

$$\sum_{n = 0}^{\left|\gamma\right|-1} e^{\pi i \epsilon \frac{\gamma^2 + \beta n}{\gamma}} = \left|\gamma\right|^{1/2} e^{\frac{\pi i}{2} (\text{sgn}(\alpha \gamma) - \frac{\beta}{\gamma})} \sum_{n = 0}^{\left|\alpha\right|-1} e^{-\pi i \frac{\gamma \alpha^2 + \beta n}{\alpha}}. \quad (3.21)$$
We interpret $h$ as the $U(1)$ holonomy around the generator of $\pi_1(L(a, \pm 1)) = \mathbb{Z}_a$, or equivalently the first Chern class $c_1(L) \in H^2(L(a, \pm 1), \mathbb{Z})$ of the line bundle $L$. For $a$ even, there is another 3d spin structure corresponding to $\rho = \rho' = (0, 0) \cdot \rho_0 = AA$, which gives

$$Z = U_{(0,0;0)(0,0;0)} = e^{\frac{\pi i}{k} e^{-\frac{\pi i}{4}} \epsilon a \sum_{h=0}^{a-1} e^{\epsilon \pi i \frac{h^2}{a}}}. \tag{3.23}$$

We see that $\epsilon = 1$ and $\epsilon = -1$ are related by complex conjugation as expected; a change of orientation should replace the partition function by its complex conjugate \cite{23}.

We note that the sums in (3.22) and (3.23) involve the so-called quadratic refinements of the linking pairing on $H^2(L(a, \pm 1), \mathbb{Z}) = \mathbb{Z}_a$. Let us consider the bilinear pairing $B : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $B(x, y) = \epsilon a x y$. Correspondingly \cite{24} we have the bilinear pairing $L : \mathbb{Z}_a \times \mathbb{Z}_a \to \mathbb{Q}/\mathbb{Z}$ given by $L(h, h') = hh'/\epsilon a \mod \mathbb{Z}$. For $v \in \{0, 1\}$, let us try to define the map $\psi_v : \mathbb{Z}_a \to \mathbb{Q}/\mathbb{Z}$ defined by

$$\psi_v(h) := \frac{1}{2} h \left( \frac{1}{\epsilon a} h + v \right) \mod \mathbb{Z}. \tag{3.24}$$

When $a$ is odd, $\psi_0$ is well-defined and is the unique quadratic refinement of $L$. When $a$ is even, both $\psi_0$ and $\psi_1$ are well-defined and are the only possible quadratic refinements of $L$. The integer $v$ is called a Wu class for $B$ in \cite{24}. The exponentials in (3.22) and (3.23) involve $\psi_1$ and $\psi_0$, respectively.

4 Discussion

It will be interesting to apply the analysis of this paper to more general 2d CFTs such as WZW models and compute the modular matrices and the partition functions of spin Chern-Simons theories. It also seems worthwhile to revisit the general $U(1)^N$ spin Chern-Simons theory \cite{11} and compare with the boson CFT with the target space $T^N$ modified by the Arf invariant. Another future direction is the computation of the partition functions on more general lens spaces \cite{25}.

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\footnote{The difference between $e^{\epsilon \pi i \frac{h^2}{a}}$ in (3.22) and $(3.23)$ is identical to the difference noted in footnote 2 of \cite{22}.}
A  \( k \) even

Recall that \( R = 2(p/p'')^{1/2} \) and \( k = pp'' \). Let us consider the case \( p'' = 2p' \) even (hence \( k \) even). Since \( p \) and \( p' \) are relatively prime there exist integers \((r_0, s_0)\) such that \( pr_0 - p's_0 = 1 \). We set \( \omega_0 = pr_0 + p's_0 \). Then we can expand the original torus partition function (2.2) as

\[
Z[T_0] = \sum_{\lambda=0}^{k-1} K_{\lambda}(\tau) K_{\omega_0 \lambda}(\bar{\tau}) ,
\]

where

\[
K_{\lambda}(\tau) := \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{(kn+\lambda)^2/2k} .
\]

See Exercise 10.21 of [26]. We have

\[
K_{\lambda}(-1/\tau) = \sum_{\mu=0}^{k-1} S_{\lambda \mu} K_{\mu}(\tau) , \quad K_{\lambda}(\tau + 1) = \sum_{\mu=0}^{k-1} T_{\lambda \mu} K_{\mu}(\tau) ,
\]

where

\[
S_{\lambda \mu} = \frac{1}{\sqrt{k}} e^{-\frac{2\pi i}{k} \lambda \mu} , \quad T_{\lambda \mu} := e^{\frac{\pi i}{k}\lambda^2} e^{-\frac{\pi i}{12} \delta_{\lambda \mu}} .
\]

See (2.25) and (2.26). Using the reciprocity formula (3.21) one can check that these give rise to a genuine (rather than projective) representation of \( SL(2, \mathbb{Z}) \).

For \( L(a, \epsilon = \pm 1) \) the partition function is computed via (3.11) and (3.21) as

\[
Z = (S^T - e^a S^{-1})_{00} = \frac{e^{\frac{\pi i}{12} a}}{k} \sum_{\nu=0}^{k-1} e^{-\epsilon \frac{\pi i}{k} a \nu^2} = \frac{e^{\frac{\pi i}{12} a}}{\sqrt{k a}} \sum_{h=0}^{a-1} e^{\epsilon \frac{\pi i}{a} kh^2} .
\]

This is the same expression as (3.23). There is no dependence on the 3d spin structure.

B  Quadratic refinements and their Arf invariants

Let \( K \) be a finite abelian group, and \( L : K \times K \to \mathbb{Q}/\mathbb{Z} \) a symmetric bilinear pairing. Here \( \mathbb{Q} \) is the additive group of rational numbers. We assume that \( L \) is non-degenerate, meaning that if \( L(x, y) = 0 \) for all \( x \in K \) then \( y = 0 \).

In this paper we define a quadratic refinement over \( L \) to be a map \( \psi : K \to \mathbb{R}/\mathbb{Z} \) such that

\[
\psi(x + y) - \psi(x) - \psi(y) = L(x, y) \quad \text{for } x, y \in K \quad (B.1)
\]

19In some references such as [11] the term “quadratic refinement” refers to a more general map \( \psi' : K \to \mathbb{R}/\mathbb{Z} \) satisfying \( \psi'(x + y) - \psi'(x) - \psi'(y) + \psi'(0) = L(x, y) \) for \( x, y \in K \). A quadratic refinement in the sense of this paper is called a quadratic function over \( L \) in [24]. In the terminology used in nLab our \( \psi \) is the quadratic refinement of \( L \) by a quadratic form. Our terminology coincides with that of [27].
and
\[ \psi(nx) = n^2 \psi(x) , \quad n \in \mathbb{Z} \quad x \in K . \quad (B.2) \]
Given (B.1), the condition (B.2) is equivalent to \( 2\psi(x) = L(x, x) \).

Following [24], we define the Arf invariant \( \text{Arf}(\psi) \) of a quadratic refinement \( \psi \) by
\[ e^{2\pi i \text{Arf}(\psi)} := \sum_{x \in K} e^{2\pi i \psi(x)} . \quad (B.3) \]
It is known that \( \text{Arf}(\psi) \) takes values in \( \mathbb{Q}/\mathbb{Z} \).

The Arf invariant \( \text{Arf}[\rho] \) for 2d spin structures \( \rho \), discussed in Section 2.1, fits the definition (B.3) up to normalization. Indeed the spin structures \( \rho \) on a closed surface \( \Sigma \) are in one-to-one correspondence with the quadratic refinements \( \psi_\rho \) of the pairing \( L = (1/2)\phi \) on \( K = H_1(\Sigma, \mathbb{Z}_2) \), where \( \phi \) is the \( \mathbb{Z}_2 \)-valued intersection form \( (a, b) \mapsto \phi(a, b) = \#(a \cap b) \mod \mathbb{Z}_2 \) [14]. We have the relation
\[ \text{Arf}[\rho] = \frac{1}{2} \text{Arf}(\psi_\rho) \quad (B.4) \]
under the correspondence.

References

[1] G. W. Moore and N. Seiberg, “Taming the Conformal Zoo,” Phys. Lett. B 220 (1989) 422–430.
[2] N. Seiberg and E. Witten, “Gapped Boundary Phases of Topological Insulators via Weak Coupling,” PTEP 2016 (2016), no. 12 12C101, 1602.04251.
[3] G. W. Moore and N. Seiberg, “Naturality in Conformal Field Theory,” Nucl. Phys. B 313 (1989) 16–40.
[4] L. Bhardwaj, D. Gaiotto, and A. Kapustin, “State sum constructions of spin-TFTs and string net constructions of fermionic phases of matter,” JHEP 04 (2017) 096, 1605.01640.
[5] R. Dijkgraaf and E. Witten, “Topological Gauge Theories and Group Cohomology,” Commun. Math. Phys. 129 (1990) 393.
[6] Y. Tachikawa, “Topological phases and relativistic QFTs.” Notes of the lectures given in the CERN winter school, February 2018.
[7] A. Karch, D. Tong, and C. Turner, “A Web of 2d Dualities: \( \mathbb{Z}_2 \) Gauge Fields and Arf Invariants,” SciPost Phys. 7 (2019) 007, 1902.05550.
[8] J. A. Harvey and G. W. Moore, “Moonshine, Superconformal Symmetry, and Quantum Error Correction,” 2003.13700.
[9] C.-T. Hsieh, Y. Nakayama, and Y. Tachikawa, “On fermionic minimal models,” 2002.12283
[10] J. Kulp, “Two More Fermionic Minimal Models,” 2003.04278
[11] D. Belov and G. W. Moore, “Classification of Abelian spin Chern-Simons theories,” hep-th/0505235.
[12] S. D. Stirling, *Abelian Chern-Simons theory with toral gauge group, modular tensor categories, and group categories*. PhD thesis, Texas U., Math Dept., 2008. 0807.2857.
[13] A. Y. Kitaev, “Unpaired Majorana fermions in quantum wires,” *Phys. Usp.* **44** (2001), no. 10S 131–136. cond-mat/0010440.
[14] D. Johnson, “Spin structures and quadratic forms on surfaces,” *J. London Math. Soc. (2)* **22** (1980), no. 2 365–373.
[15] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, “Generalized Global Symmetries,” *JHEP* **02** (2015) 172. 1412.5148.
[16] D. S. Freed and R. E. Gompf, “Computer calculation of Witten’s three manifold invariant,” *Commun. Math. Phys.* **141** (1991) 79–117.
[17] L. C. Jeffrey, “Chern-Simons-Witten invariants of lens spaces and torus bundles, and the semiclassical approximation,” *Comm. Math. Phys.* **147** (1992), no. 3 563–604.
[18] S. K. Hansen and T. Takata, “Reshetikhin-Turaev invariants of Seifert 3-manifolds for classical simple Lie algebras, and their asymptotic expansions,” arXiv Mathematics e-prints (Sep, 2002) math/0209403, math/0209403.
[19] E. Witten, “Quantum Field Theory and the Jones Polynomial,” *Commun. Math. Phys.* **121** (1989) 351–399. [,233(1988)].
[20] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007.
[21] B. C. Berndt, R. J. Evans, and K. S. Williams, *Gauss and Jacobi sums*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1998. A Wiley-Interscience Publication.
[22] Y. Imamura, H. Matsuno, and D. Yokoyama, “Factorization of the $S^3/Z_n$ partition function,” *Phys. Rev. D* **89** (2014), no. 8 085003. 1311.2371.
[23] M. Atiyah, “Topological quantum field theories,” *Inst. Hautes Études Sci. Publ. Math.* (1988), no. 68 175–186 (1989).
[24] G. W. Brumfiel and J. W. Morgan, “Quadratic functions, the index modulo 8, and a \( \mathbb{Z}/4 \)-Hirzebruch formula,” *Topology* **12** (1973) 105–122.

[25] T. Okuda, K. Saito, and S. Yokoyama. In progress.

[26] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[27] L. I. Nicolaescu, *The Reidemeister torsion of 3-manifolds*, vol. 30 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2003.