Optimized Memory Reveals a Survival Strategy for a Multi-agent Competition

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(Dated: August 19, 2019)

We present a fractional model to clarify the dynamical evolution of how and under what circumstances—in a multi-agent economic area—newly founded ventures prolong their existence throughout the market. Since the increase in the number of newly-established firms in a market may generally lead to the reduction of the market share of players in the contest, after a while some may be faded out in the market. In this regard, considering a long term evolution, all firms are exposed to be eliminated and to give up their market share. That is why managers have the concern of attracting more customers to satisfy at least a minimum market share, and ideally prolong their firm’s lifetime. On the other side, due to the establishment of more newly founded ventures in a certain industry, it is vital to be flexible and run new strategies. The triggered strategies not only must be on time but also must be led to lengthening the additional survival time. In the present study, it is shown that the existence of memory or lack of memory in the evolution may not be the case, but also having a strategy plays a significant role in survival. Hence, in spite of exposure to the risk of missing market share, the recalling of the past offered services and products of the weaker firm—compared with the whole market—may prolong the time-length of surviving in the market. In this regard, managers corresponding to the weaker firm will be able to quantitatively make decisions toward two main concepts, firstly the strategical usage of their firms’ reputation and secondly, launching new features and services. The aim behind this optimization is to delay the time of reaching a minimum market share.

PACS numbers:

I. INTRODUCTION

Considering scarce resources, two growing economic sectors in a selfish interaction [1] contribute to a competition of gaining the possible maximum market share and customers. Throughout a certain real-world network of competing agents, in spite of cumulative growth [2–5], there may exist some frictions and drivers which affect the growth [6–9]. Following this train of thought, there exist internal and external dynamics which create cost of growth. Accordingly, the states of failure to possess a definite market share, and, ever-growing market share, or even a trade-off between further growth or failure in a temporal behavior will emerge [8]. Considering the memory of systems as a decaying factor against sudden alterations [7, 10–12], besides with probable strategies [13] as a temporal game-changer, in this study, we apply the memory created by an individual firm—in statue quo—in the customers’ viewpoint or launching new strategies in the firms as an advantage to compete against the whole market.

Our results will build a bridge connecting a rivalry of possessing market share and fractional calculus. We will further discuss on:

I. the temporal properties of this multi-agent contest;
II. the memory effects of one firm on the evolution of the whole system;
III. by changing the strategy, the extent which our individual firm can sustain in the temporal contest to possess at least a minimum ad hoc market share for a longer time;
IV. the phase spaces of $\alpha$, $\Delta \tau$, $\gamma$.

The notation $\alpha$ is a tunable memory factor which determines the state of “how much the memory is stimulated in the weaker firm customers’ point of view”. Also, $\Delta \tau$ denotes the added lifetime after launching the strategy. $0 < \gamma < 1$ refers to the relative growth rate of the market share of our individual firm in respect to the relative growth rate of the market share of the other side of the competition (the whole market except our individual firm). Further, we will reveal the critical time, which the whole potential market is occupied by the competitors and achieving more market share for one firm, yields to giving up the market share for another firm in the contest. Accordingly, a zero-sum gain [14, 15] will emerge.

In the present study, we introduce a simple dynamical model to compare the behavior of a multi-agent competing market containing two sides: our individual firm on one side, and the whole market except the so-called individual firm, on the other side (see Fig. 1). In order to trace the patterns of dynamical growth of such contest, we suggest a master equation which predicts the future payoff of the mentioned contest. After a specific time, the counter-side market with higher relative growth rate, will occupy the whole market and maintain their growing market share influenced by advertisements, financial investments [14, 15], hub-connections and united competitors [13] and etc. On the other side of the rivalry,
the one with lower relative growth rate (our individual firm with $\gamma < 1$) is vulnerable to its market share extinction. Further, by taking into account the memory effects [7, 10, 11] on the growth evolution of the weaker firm, it is able to rise the time interval, $\Delta \tau$, of maintaining its minimum market share. In the demand for demonstrating the competitors’ behavior, some scholars considered restricted areas which are exposed to overcrowding [16]. In this context, over time the systems increasingly grow [6–8, 17]. As soon as the accessible area reduces, newer agents may locate in the territory of others, or their territories squeeze. Due to lack of resources—the density of locating in the spatial area around agents—the involving agents are eliminated. This phenomenon will be amplified when the space of the contest reduces. Indeed, after a critical time the systems are vulnerable to some effects against growth, say lack of space in a rivalry and squeezed territories [16] or cost of growth [8], or agents extinction [18]. Nevertheless, in a limited space, the process of squeezing continues to the extent that a saturation regime [16], or characteristic time [6, 7, 19] emerges. Passing the aforementioned time-stamp, the duopoly contest encounter with a zero-sum game, that is, customers as scarce resources are distributed among the firms and it may cause some advertising [19, 20] and competition costs. The more scarcity of the customers, the higher the possibility of the zero-sum game. When it comes to the role of strategy in a contest, it stems from some different internal and external aspects [21, 22]. As proof of this concept, the underdogs may be united—in a cooperative process—to overcome the stronger side [13]. In addition, for the sake of achieving a winning position in the market share, marketing strategies may be changed by the managers in some time-stamps. In this perspective, the firms upon their ability to invest and their internal and environmental situations apply defensive or offensive marketing strategies to boost their market share [23–26]. Nevertheless, the background of any economic firm in the market and the history of interactions with the customers and recalling it to customers and also previous potential consumers [27] may be an indication of memory. For now, motivated by the mentioned references, throughout this study, we will further investigate the influence of memory existence and discuss its possible pay-offs. At the heart of this approach, it should be highly emphasized that exploring a new strategy and also other striking actions take time to propagate in society and this time-lag must be considered [28]. After the above overview of memory and strategy, it is worthy to shed light upon possible applications of our proposed model in the industries and lay beyond the reach of theoretical aspects, namely competitive financial interactions [13, 29], social marketing events [30, 31], sales promotion which may be applied in a saturated market [32], and the new phenomenon so-called crowdfunding and financing state-of-the-art technologies [33]. As well, the proposed idea is not only limited to economics but also extended to other fields of study involving an analogous model. In the following, section II deals with introducing the master equation with integer order and analyzing its dynamic behavior. In section III, the differential equation associated with the weaker firm—the one with lower relative growth rate—is incorporated into the concept of memory by applying Caputo approach. To optimize the memory effects, a strategy will be suggested in section IV, and its quality will be checked in section V. In section VI, the conclusions and future directions are taken.

II. DEFINITIONS AND METHODOLOGY

Let us denote the two new companies’ shares of a market at time $t$ as $I_1(t)$ and $I_2(t)$. We consider $S(t) \geq 0$ as a measure of the size of the rest of the market (the number of potential customers) at time $t$. We define a
constant coefficient $\gamma$ reflecting the relative growth rates of the market share.

Since the size of the whole market is assumed to be constant, the summation over the numbers of customers of the two sides, $I_1(t)$, $I_2(t)$ and the number of potential customers $S(t)$ are not independent, so we consider the normalized form satisfying:

$$1 = S(t) + I_1(t) + I_2(t) - (I_1(t) \cap I_2(t)). \quad (1)$$

By defining the potential customers might convert to each of the two companies’ customers through time, the customers of the two companies could be exchanged. Due to all the above assumptions, we define the dynamical behavior of the potential customers $S(t)$ with the following master equation,

$$\frac{dS}{dt} = -(I_1 + \gamma I_2)S. \quad (2)$$

Since the potential customers, $S$, might choose one or both of the two firms, the growth of $I_1$ and $I_2$ leads to the reduction of $S$.

The negative sign shows that any growth in the values of $I_1$ or $I_2$ reduces the value of $S$. The conversion rate of the potential customers to the customers of the two companies depends on their growth rate and the number of potential customers. On the other hand, the growth of our individual firm, the side $I_2$, reduces the growth of the other side, $I_1$, and vice versa. Let’s assume that the growth rate of side 1 is higher than side 2 when $0 < \gamma < 1$. Therefore, one can formulate the dynamics of each company as:

$$\frac{dI_1}{dt} = (1 - \gamma)I_1I_2 + I_1S, \quad (3)$$

$$\frac{dI_2}{dt} = (\gamma - 1)I_1I_2 + \gamma I_2S, \quad (4)$$

Under the condition of $\gamma = 1$, the two dynamical equations turn into two equal coupled differential equations. For the same initial values of $I_1$ and $I_2$, the two companies will grow symmetrically as long as half of the market is occupied.

In Fig. 2, the dynamical of growth and failure of two sides of simultaneously founded ventures $I_1(0) = 0.1 = I_2(0) = 0.1$ with the relative growth rate $\gamma = 0.995$ show the emerging pattern of the two companies and their competitions to earn a greater share of the market. $I_2(t)$ reaches a maximum value at a critical time $t_c$ where $I_1(t_c) + I_2(t_c) \simeq 1$ and $S(t_c) \simeq 0$. In the case of memoryless, Fig. 2(a), the competition between the two companies begins at $t_c$. At this time, companies on the other side, side 1, start growing faster than our company, side 2, and obtains a greater part of the share market. However, our company follows a decreasing trend and loses its share of the market. Interestingly, a small difference between the growth rate coefficients of the companies causes two totally diverse destinies for the two start-ups. The development of the company with a higher growth rate saturates at a high value while the other company ends up with a total loss. That is, the more powerful the company will monopolize the market. It shows that the relative growth rate plays a significant role in the success and failure of ventures so that relatively smaller businesses have no chance to survive under competition with the bigger ones.

All the above discussion are based on the defined set of dynamical equations 2 to 4. Further in Sec. V, we will discuss on the future states of the temporal contest while the relative growth rate $\gamma$ changes from 0 through 1. The main question is that under which condition the newly founded enterprise has a chance to survive? Is there any modification for the master equation to indicate a strategy to raise the chance of success of the weaker competitor?

In order to address this question, we consider the effect of memory in the growth pattern of the weaker company in the market.

III. FRACTIONAL CALCULUS AND MEMORY

A market as a system which includes intelligent elements is affected by memory. Start-up experience provides tacit knowledge of organizing routines (for example, routines for coordinating the activity of organization members) and skills (for instance, a choice of alliance partners) that have already been learned from their prior activities. Thus, customers may interfere with their previous experiences whereby the process of decision making is influenced by recalling past events. Therefore, the positive or negative experiences of the market caused by the products or services of the companies could influence the trends of the market. Hence, it suggests us to consider the effect of memory in the evolution of the weaker company ($I_2$); e.g. the way the customers select their vendors [34].

However, the proposed model 2-4 described by integer order derivatives cannot perfectly describe non-Markovian processes (processes with memory) [10, 35], due to the fact that such derivatives are determined by only a very small neighborhood around each point of time. To overcome this shortcoming, we incorporate the concept of fractional calculus into the system as a kernel of the differential operator—that is, substituting a fractional order derivative. Indeed, it is shown that fractional derivatives can appropriately represent the effects of power-law memory [36]. As a result, we formulate the memory effects and intellectual behaviors by an integral equation with a time-dependent kernel $\kappa(t-t')$ [10]. This enables us to take the effect of previous time steps into account:
gives the value of \( I \) and is a generalized form of the with Eq. 40 at the weaker company, a "stronger" (long-lasting) memory of customers of the dynamical equations. There are different types of fractional differential \( \alpha \) where 0 \( < \alpha \leq 1 \) and \( \Gamma \) denotes the Gamma function.

There are different types of fractional differential operators that are suggested by Riemann, Liouville, Grunwald, Letnikov, Sonine, Marchaud, Weyl, Riesz, Caputo, Fabrizio, Atangana and other scientists [35–39]. But, in this paper, we consider the Caputo fractional time derivative of order \( \alpha \) which can describe physical meanings of real-world phenomena [35]:

\[
\frac{\partial D_\alpha}{} y(t) = \frac{1}{\Gamma(\alpha - 1)} \int_0^t \frac{y'(\tau)d\tau}{(t-\tau)^{\alpha-1}}.
\]

A lower degree of the fractional derivative \( \alpha \) indicates a "stronger" (long-lasting) memory of customers of the weaker company, \( I_2 \). Hence, the dynamical equation of \( I_2 \) will follow a fractional differential while the two other dynamical equations 2 and 3 will remain unchanged:

\[
\frac{dS}{dt} = -(I_1 + \gamma I_2)S,
\]

\[
\frac{dI_2}{dt} = \int_{t_0}^{t} \kappa(t-t')Hdt',
\]

where

\[
H = ((\gamma - 1) I_1(t') I_2(t')) + \gamma I_2(t') S(t'),
\]

and we set the kernel as:

\[
\kappa(t-t') = \frac{1}{\Gamma(\alpha - 1)(t-t')^{\alpha-2}},
\]

where 0 \( < \alpha \leq 1 \) and \( \Gamma \) denotes the Gamma function.

The numerical solution of such equations comes from the discretization of an equivalent Volterra integral equation which is extensively presented in [7, 40, 41]:

\[
\frac{\partial y(t)}{\partial t} = \gamma y(t) + \int_0^t \kappa(t-t')Hdt',
\]

\[
y_n = y_0 + h^\alpha \sum_{k=0}^{n-1} b_{n-k-1} f_k.
\]

In the numerical solution, the time is discretized as \( T = t_0, \ldots, t_n \) where \( t_n = nh \) and \( h \) is the step size. The recursive Eq. 13 gives the value of \( y \) at time \( n \) based on the initial states \( y_0 \) and the solutions of the Eq. 12 at the prior time steps functions \( f_k \) with weight \( b_{n-k-1} \). The weight coefficient is given by:
IV. STRATEGY FOR OPTIMIZING THE MEMORY EFFECT

“Elephants can remember, but we are human beings and mercifully human beings can forget.”—Agatha Christie. Managers of the newly founded firms usually try to overcome a suffering liability of newness with different strategies. An idea that may strike the mind would be optimizing the behavior of the company by considering a dynamical memory that varies through time or renewing the memory at a particular moment. This strategy may lead the growth curve to the highest level of curves based on different memory stages. Initiating the memory from different spots of the functional history timeline of the company, and drawing the corresponding curves enables us to compare the growth patterns depending on the memory start point. Such selective reminding could be considered as an approach to maximize the efficiency of newly founded ventures and raise their chance to survive in the market.

Fig. 3 illustrates a comparison of the behavior of the system including memory and strategy (red dashed and dotted line), only memory (blue dashed line), without memory (black solid line), which lead to different growth dynamical curves. The black diagram shows the evolution of $I_2(t)$ with the relative growth rate $\gamma = 0.995$ with the initial value $I_2(0) = 0.01$ and $\alpha = 1$. The non-fractional value of $\alpha$ does not guarantee long-standing survival time, due to the absence of the memory effects in the growth process of the company 2. The blue curve indicates the growth of $I_2(t)$ with a similar relative growth rate, initial values, and with the memory factor $\alpha = 0.5$. In this case, the market share proportion of the memory-less process lower than the process with memory, however, it achieves a local success after the peak time-stamp (or at the conflict time-stamp). The red curve corresponds to the growth process of the company with a new memory starting from the peak of the memory process. In fact, to optimize the efficiency of a company with a lower growth rate, it must start by recalling the past until the peak point is achieved and

\[ b_{n-k-1}^i = \frac{(n-1-k)^{\alpha_i} - (n-k)^{\alpha_i}}{\Gamma(\alpha_i + 1)}. \]  

For simplicity, we assume that the memory of Eq. (11) is constant through time. Thus, by considering $\alpha = 0.5$, as it is illustrated in Fig. 2, the emerging firms start developing with almost similar rate and an equal number of potential customers converting to each of the customers of the two businesses' sides by considering the effect of memory. Interestingly, the influential memory affects the contest before the time-stamp $t_c$, when the total market is divided into the shares of the two companies. In fact, it reduces the negative slope of the curve and slows down the loss rate of the weaker company, and reduces the growth rate of the more powerful side of the market. Nevertheless, it is not capable to alter the final destiny of the weaker company. That is, the benefits of experience that permit survival may not be sufficient to generate remarkable results. Therefore, after a comparatively longer time, the weaker company inevitably loses its whole market share and the more powerful side of the competition earns the whole market.
the past experiences must be forgotten, and the process be continued with a new memory starting from the last peak. To predict such critical points in the real-world and provoke managers to start the strategy, the early-warning signs [42] of peaks, in a plausible way, can warn an approaching threshold. We can call this strategy as a “selective recalling-forgetting strategy” which may be an indicative of some well-known intelligent reactions in the context of business or other possible aspects. Furthermore, in spite of the maximum value of \( I_2 \), examining this strategy for two other moments are interesting; firstly, at the inflection of the curve \( S \), when the customers’ behaviors are changing, and secondly, at the intersection of \( I_1 \) and \( I_2 \), when the market is saturated and the market shares of both sides of the contest are equal.

V. VERIFICATION AND VALIDATION

The proposed system can be validated by a well-known biological model with a similar concept; in fact, equations 3 and 4 are analogous to Lotka-Volterra model [43] which states that, in the competition among two species (an individual firm and the whole market) that use the same scarce resource (say customers), the superior competitor (whole competitors in the market) will at last overcome the other whereby the inferior competitor (an individual firm) will suffer a decline in population (market share) overtime.

In this section, in order to compare the total number of achieved customers of the weaker company, \( I_2 \), for three different cases—that is, the model without memory (\( NMI_2 \)), with memory (\( MI_2 \)), and with memory and strategy (\( SMI_2 \)), we suggest using cumulative market share through the time. In addition, to clarify the effectiveness of the proposed model for various relative growth rates, we provide heatmaps of some proportions of cumulative market share for various competition ranges, \( 0 < \gamma < 1 \), versus time. We also denote cumulative function by “\( \int \)”, and use the notation \( C = \frac{\int SMI_2}{\int MI_2} \) in order to show a proportion of the cumulative market share of \( I_2 \) including strategy and memory to cumulative market share of \( I_2 \) with memory. Let’s suppose the model for initial conditions \( S(0) = 0.98, I_1(0) = 0.01, \) and \( I_2(0) = 0.01 \) with the fractional order \( \alpha = 0.5 \), to the timestamp 1000.

Fig. 4 shows the cumulative market share of the company \( I_2 \) for three aforementioned cases with the relative growth rate \( \gamma = 0.995 \). We can easily see the evolution process involving the strategy (red dashed line) performs better than two other cases, as well the memory influences the system (blue solid line) after a time-stamp near to 500. It confirms that, for such a big \( \gamma \), it is necessary to use strategy because the impact of using strategy and memory is more than the effects of exclusive memory. Thus, it can be concluded when the competition between two firms is tight (e.g. for \( \gamma = 0.995 \)), it is plausible to introduce a selective recalling-forgetting strategy.

![Fig. 4: A comparison of cumulative market shares of \( I_2 \) for three different cases; involving memory and strategy, including memory, and without memory, when \( \gamma = 0.995 \).](image)

Fig. 5 illustrates the proportion of the cumulative market shares of \( I_2 \) with strategy and memory to cumulative market shares of \( I_2 \) with memory. One can find out for the range of \( 0.6 < \gamma < 0.7 \) and \( \gamma \approx 1 \) (when the contest of two sides is so close) using a selective recalling-forgetting strategy is highly recommended.

Considering a predefined minimum market share, the Fig. 6 predicts the effect of triggering the new strategy on the lengthening the additional survival time (\( \Delta \tau \)) of the weaker side (our individual firm). When it comes to a lower ratio of relative growth (\( \gamma \to 0 \)), the managers may be indifferent toward running the strategy. Because, when \( \gamma \to 0 \), it results in too small additional survival time (\( \Delta \tau \to 0 \)). For larger \( \gamma \), managers can provide an trade-off analysis [19] to evaluate the probable profitability.

![Fig. 5: (color online) Proportions of cumulative market shares of \( I_2 \), for the system including memory and strategy to the system with memory, in a range of relative growth rates \( 0 < \gamma < 1 \) through the time-stamp 1000.](image)

![Fig. 6: Predicts the effect of triggering the new strategy on the lengthening the additional survival time (\( \Delta \tau \)) of the weaker side (our individual firm).](image)
VI. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper, by incorporating the concept of fractional calculus, we have introduced a model to interfere the existence of memory and running a strategy to study the process of the prolong life cycle of an individual firm. The model has been proposed in the two distinctive processes, the memory-less case, and the memory case, then we have determined a novel strategy to help the survival of the individual firm.

In the memory-less process, in which the model was described by integer orders, both sides reached a maximum value when the conflict began. After this time, the number of clients of the market was diverging exponentially so that the more powerful side, even for the relative growth rate $\gamma \approx 1$, would be dominant the whole share market. Thus, it was shown that the individual firm has no chance to survive under competition with the whole market on the bigger side. However, there are some factors in real intelligent interactions which deteriorate such intensive divergence dynamic. In this regard, we have considered addressing this issue by imposing memory into the model and so the fractional Caputo derivative as an appropriate candidate for representing memory effects of an individual firm has been used.

It is illustrated that to some time lack of memory leads to higher achievements, and on the other hand, after some time memory existence leads to more sustainability of the firm. In this regard, when the firm is decaying (and also) growing, the memory will have a slowing down effect on the processes—a conservative action. However, one deterministic criterion is having a strategy to trigger on time. This phenomenon makes the firm prolong its existence in the market.

In the “selective recalling-forgetting strategy,” we have presented a novel strategy to maximize the efficiency of an individual weaker venture (relative to the whole market) by recalling the past until the peak point is achieved and the past experiences must be forgotten, and the process is continued with a new memory starting from the last peak. Here, we have utilized the same memory, that is, the same fractional derivative order, for both starting points, the initial time and the peak. Nonetheless, for further interpretation, we can exploit the selective recalling-forgetting strategy with variable fractional order $\alpha(t)$ for a different position.

For the future investigation, it would be interesting to expand the meaning of growth rates and the concept of memory (or the fractional derivative order) of the proposed model in the business context. Here, we have suggested that the relative growth rate coefficients can play the role of trade-off effects between value and cost of individual customers [19] and it is plausible that the memory [7, 10, 11] represents the characteristics of the value-cost trade-off and provides the customers to satisfy their utility [1]. Furthermore, more realistic modelings can be studied through networks so that it encourages us to extend this model into complex models representing the competitions of more start-ups with various initial times on structured networks. This is a direction we plan to explore in the future.

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