Bell Gems: the Bell basis generalized

Gregg Jaeger,\textsuperscript{1,2}

\textsuperscript{1}College of General Studies, Boston University
\textsuperscript{2}Quantum Imaging Laboratory, Boston University

871 Commonwealth Ave., Boston MA 02215

email: jaeger@bu.edu
telephone: +1 617 353-3251

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Abstract

A class of self-similar sets of entangled quantum states is introduced, for which a recursive definition is provided. These sets, the “Bell gems,” are defined by the subsystem exchange symmetry characteristic of the Bell states. Each Bell gem is shown to be an orthonormal basis of maximally entangled elements.
I. INTRODUCTION.

The Bell basis states, the two-qubit state-vectors introduced by Bohm [1], have proven to be the most fruitful objects of study in the foundations of quantum mechanics and quantum information science, due to their extreme entanglement properties [2]. The non-factorizability that defines entanglement arises in these states from their symmetry or antisymmetry under the binary exchange of their qubit subsystems, each half the size of the composite system. Here, we use these properties to define a new, broader class of sets of quantum states, which we call “Bell gems,” generalizing the Bell basis to larger qubit numbers. We recursively define the members of this class and exhibit maps through which they can be formally constructed. We show that the elements of these sets form orthonormal bases for the multiple-qubit Hilbert spaces in which they reside and possess maximal entanglement according to the $N$-tangle measure. Finally, we identify Bell gems and elements that have proven useful in quantum information processing applications and provide quantum circuits for their creation from computational basis elements.

The Bell basis for the two-qubit quantum state space $\mathcal{H}_4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ is the set consisting of the following two-qubit state vectors, the Bell states:

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \quad (1)$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) . \quad (2)$$

These states possess subsystem exchange symmetry, being of either of the two basic forms

$$\frac{1}{\sqrt{2}}(|i\rangle|i\rangle \pm |j\rangle|j\rangle) \quad (3)$$

$$\frac{1}{\sqrt{2}}(|i\rangle|j\rangle \pm |j\rangle|i\rangle) , \quad (4)$$

where $|i\rangle$ and $|j\rangle$ are orthogonal and normalized quantum state vectors of the same dimensionality (the two orthogonal single-qubit states $|0\rangle$ and $|1\rangle$ of the quantum computational basis). The forms, given in expressions (3) and (4), of the Bell states guarantee their non-factorizability and maximal entanglement as measured by the concurrence or equivalently by its square, the tangle $\tau$ [3].
Here, we construct bases generalizing the Bell basis for higher multiparticle systems by retaining these forms as qubit number is scaled up.

II. DEFINITIONS AND THEOREMS.

The binary subsystem exchange symmetry inherent in the forms given by expressions (3) and (4) is now incorporated in the definition of the sets of states generalizing the Bell basis, which are thereby retained under changes of scale. The Bell gems are recursively defined, as follows.

**Definition:** A Bell gem, $G_d$, is a set of mutually orthogonal normalized quantum state vectors, lying in the $d = 2^{2^n}$-dimensional Hilbert space $\mathcal{C}^{2^{2^n}}$, possessing the form

$$\frac{1}{\sqrt{2}}(|i\rangle |i\rangle \pm |j\rangle |j\rangle)$$

$$\frac{1}{\sqrt{2}}(|i\rangle |j\rangle \pm |j\rangle |i\rangle),$$

where $|i\rangle \neq |j\rangle$ are elements of a Bell gem $G_{d'}$ of dimensionality $d' = 2^{2^{n-1}}$, $n \in \mathbb{N}, n \geq 2$, the simplest Bell gem being the Bell basis, $G_4$, for $\mathcal{C}^4$:

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).$$

With this definition, the following theorem is seen to hold.

**Theorem 1:** The Bell gem $G_{2^n}$ is an orthonormal basis for the $2^{2^n}$-dimensional Hilbert space of quantum state vectors, $\mathcal{H}_{2^n} = \mathcal{C}^{2^{2^n}}$, that is, for the Hilbert space of $2^n$ qubits.

**Proof.** A Bell gem is, by the above definition, a set of normalized linearly independent quantum state vectors of equal qubit number. To prove this set is an orthonormal basis, it therefore remains only show that $G_{2^n}$ spans the Hilbert space of $2^n$ qubits. That is, we now need only show that it consists of $2^{2^n}$ elements. This can be done by counting the number of elements of $G_{2^n}$ of each of the two available forms, given by expressions (5) and (6). There are $2^{2^{(n-1)}}$ such linearly independent vectors of the form (5) since each
product $|k⟩|k⟩$ of two identical copies of any one of the $2^{2(2^{n-1})}$ state vectors $|k⟩ ∈ G_{2^{2(2^{n-1})}}$ from which they are constructed can appear only once in expression (5) (recalling that, by definition, they are mutually orthogonal). There are, similarly, $2^{2(2^{n-1})} (2^{2^{(n-1)}} - 1)$ vectors of the form of expression (6), since there are $2^{2(n-1)}$ choices in $G_{2^{n-1}}$ available for $|i⟩$ and $2^{2^{(n-1)}} - 1$ choices for $|j⟩$ remaining after the choice of $|i⟩$ has been made, due to the constraint that $|j⟩ ≠ |i⟩$. Summing the total number of state vectors of the two forms (5) and (6), the two forms themselves being orthogonal to one another by construction, we have $2^{2^{(n-1)}} + 2^{2^{(n-1)}} (2^{2^{(n-1)}} - 1) = 2^{2n}$ linearly independent vectors. Thus, we see that the $2^n$-qubit state space $H_{2^{2n}}$ is spanned by the Bell gem $G_{2^{2n}}$. The Bell gem $G_{2^{2n}}$ is, therefore, an orthonormal basis for the space of $2^n$ qubits. □

One can view the Bell gems larger than the simplest Bell gem ($G_4$) as arising through the action of simple maps on products of elements of the predecessor gem. Assume we are given two subsystems, A and B, each described by elements of a Bell gem $G_{2^{2(2^{n-1})}}$. To construct elements of a Bell gem $G_{2^{2n}}$ of the form (5) from these, one can use the two following sorts of maps:

$$P : |α⟩|β⟩ → (|α⟩|α⟩ + |β⟩|β⟩)$$

$$N : |α⟩|β⟩ → (|α⟩|α⟩ - |β⟩|β⟩),$$

where $|α⟩$ and $|β⟩$ are distinct elements of $G_{2^{2(2^{n-1})}}$. To construct elements of a Bell gem $G_{2^{2n}}$ of the form (6) from these, one can similarly use the maps

$$S : |α⟩|β⟩ → (|α⟩|β⟩ + |β⟩|α⟩)$$

$$A : |α⟩|β⟩ → (|α⟩|β⟩ - |β⟩|α⟩),$$

such as have been considered in the study of quantum entanglers. The second simplest gem, $G_{16}$, which unambiguously exhibits the multiplicities described above, is shown explicitly in Section III below.

Like their simplest examplar, the Bell states, the elements of all Bell gems are maximally
entangled according to the \( n \)-tangle measure, \( \tau_n \equiv |\langle \psi|\tilde{\psi}\rangle|^2 \), where \( |\tilde{\psi}\rangle = \sigma_2^{\otimes n}|\psi^*\rangle \) [4,5]. This is shown in the following theorem.

**Theorem 2:** The elements of \( G_{2^2^n} \) have maximal \( 2^n \)-tangle.

*Proof.* The effect of \( |k\rangle \rightarrow |\tilde{k}\rangle \) on each element of the simplest Bell gem, \( G_4 \), is simply to produce a negative sign in the case it is one of the even parity states, \( |\Phi^\pm\rangle \), and to have no effect in the case it is one of the odd parity states, \( |\Psi^\pm\rangle \). For every successor gem, in the case of elements of the form (5), there are identical signs produced by the transformation \( |k\rangle \rightarrow |\tilde{k}\rangle \) on the (predecessor Bell gem element) factors of each of the two addends both of which are, therefore, unchanged. In the case of the form (6), the sign of the addends after the transformation is the same, leaving the element itself unchanged up to a sign. Thus \( \tau_{2^n} = |\langle k|\tilde{k}\rangle|^2 = 1 \), *i.e.* all elements of \( G_{2^2^n} \) have maximal \( 2^n \)-tangle. \( \Box \)

## III. EXAMPLE STATES AND THEIR APPLICATION.

The simplest Bell gem is the Bell basis, the importance of which for quantum information science (*e.g.* in entangled photon quantum cryptography [6,7]) and the foundations of quantum mechanics (*e.g.* [1]) is well known. Here, as a further example, the next simplest Bell gem, \( G_{16} \) for \( 2^{2^2} = 16 \) qubits is written explicitly and quantum circuits for creating them are given. The significance of its elements for quantum information processing is mentioned and a specific application in quantum communication is pointed out.

The 16 elements \( |e_i\rangle \) of the 4-qubit Bell gem \( G_{2^2^2} \) for \( \mathcal{H}_{16} = \mathcal{C}^{2 \otimes 2} \) can be written

\[
\mathcal{G}_{16} = \left\{ \frac{1}{\sqrt{2}}(|\Phi^+\rangle|\Phi^+\rangle \pm |\Phi^-\rangle|\Phi^-\rangle) \right\}, \quad (13)
\]

\[
\frac{1}{\sqrt{2}}(|\Psi^+\rangle|\Psi^+\rangle \pm |\Psi^-\rangle|\Psi^-\rangle), \quad (14)
\]

\[
\frac{1}{\sqrt{2}}(|\Phi^+\rangle|\Phi^-\rangle \pm |\Phi^-\rangle|\Phi^+\rangle), \quad (15)
\]

\[
\frac{1}{\sqrt{2}}(|\Psi^+\rangle|\Psi^-\rangle \pm |\Psi^-\rangle|\Psi^+\rangle), \quad (16)
\]

\[
\frac{1}{\sqrt{2}}(|\Psi^+\rangle|\Phi^-\rangle \pm |\Phi^-\rangle|\Psi^+\rangle), \quad (17)
\]

5
\[
\frac{1}{\sqrt{2}}(|\Phi^+\rangle|\Psi^-\rangle \pm |\Psi^-\rangle|\Phi^+\rangle),
\]
(18)
\[
\frac{1}{\sqrt{2}}(|\Phi^+\rangle|\Psi^+\rangle \pm |\Psi^+\rangle|\Phi^+\rangle),
\]
(19)
\[
\frac{1}{\sqrt{2}}(|\Psi^-\rangle|\Phi^-\rangle \pm |\Phi^-\rangle|\Psi^-\rangle)\}.
\]
(20)

It is known that simple quantum logic circuits involving only two quantum logic gates, the c-NOT and Hadamard gates, allow one to obtain the Bell states - forming the archetypical Bell gem - from two unentangled qubits (see, e.g [8]). Simple quantum logic circuits will now be exhibited that similarly create the above Bell gem elements from four unentangled qubits - that is, from elements of the computational basis. The states \(|e_1\rangle, |e_2\rangle, |e_3\rangle, |e_4\rangle, |e_5\rangle, |e_6\rangle, |e_7\rangle\), and \(|e_8\rangle\), are obtained using the circuit shown in Figure 1 with input qubits \(|x_1\rangle, |x_2\rangle, |x_3\rangle, |x_4\rangle\), where \((x_1, x_2, x_3, x_4)\) are specifically \((0, 0, 0, 0), (0, 0, 1, 1), (0, 0, 0, 1), (0, 0, 1, 0), (1, 0, 1, 0), (1, 0, 1, 0)\) and \((1, 0, 0, 1)\), respectively. Similarly, the states \(|e_{13}\rangle, |e_{14}\rangle, |e_{15}\rangle, \text{ and } |e_{16}\rangle\) are obtained using the circuit shown in Figure 2 with input qubits \(|x_1\rangle, |x_2\rangle, |x_3\rangle, |x_4\rangle\), where \((x_1, x_2, x_3, x_4)\) are \((0, 0, 0, 1), (1, 1, 0, 1), (1, 0, 1, 1)\) and \((0, 1, 1, 1)\). Finally, the states \(|e_9\rangle, |e_{10}\rangle, |e_{11}\rangle\), and \(|e_{12}\rangle\) are obtained using the circuit shown in Figure 3 with input qubits \(|x_1\rangle, |x_2\rangle, |x_{11}\rangle, |x_{12}\rangle\), where \((x_1, x_2, x_3, x_4)\) are \((0, 0, 0, 1), (1, 1, 0, 1), (1, 0, 1, 1)\) and \((0, 1, 1, 1)\). While the first of these circuits is somewhat similar to that used to produce the Bell basis from the computational basis - like the Bell circuit, it contains only one Hadamard gate and c-NOT gates - the other two appear significantly different from it.

The elements of the form given in expression (5), namely the first four of the above elements, \(|e_1\rangle, |e_2\rangle, |e_3\rangle, \text{ and } |e_4\rangle\), of the forms (13) and (14), are also the codes states of the (extended) quantum erasure channel [9]. \(|e_2\rangle, |e_3\rangle, \text{ and } |e_4\rangle\) are codes states comprising a one-error correcting detected-jump quantum code and are basis states for a decoherence-free subspace in which universal 4-qubit quantum computing can be carried out [10]. As a specific application of some of these Bell gem elements, note that the states \(|e_1\rangle, |e_2\rangle, |e_3\rangle, \text{ and } |e_4\rangle\) can be used for performing error correction in the context of quantum communication.

In particular, the above-mentioned states provide a two-qubit to four-qubit error-
correction code capable of recovery from the loss of one photon without the loss of a qubit [11], which is highly beneficial since photon loss is a primary factor in the performance of simple optical quantum memories. Specifically, there exists an error correction protocol based on these states that provides for a more efficient quantum memory based on optical fiber delay-line loops [12]. Successfully carrying out this error correction process within an optical fiber loop (see [11] to view this circuit and [12] for a description of an optical fiber loop memory) allows quantum errors to be continually corrected on captured flying data qubits, improving the performance of quantum communication lines.

IV. CONCLUSIONS.

We introduced a class of self-similar sets of quantum states, which we call the “Bell gems,” and provided a recursive definition of these sets. Every Bell gem of finite dimensionality was shown to be an orthonormal basis for the Hilbert space in which it naturally resides. Furthermore, the elements of all Bell gems of $2^n$ qubits were shown to have maximal $2^n$-tangle. A nontrivial Bell gem was exhibited and the utility of its elements for quantum information processing was pointed out. Quantum circuits for the creation of its elements were provided and a specific application of some of these for quantum communication was pointed out. Given this utility, it seems likely that the Bell gems of yet greater complexity will be similarly useful for quantum information applications, such as quantum coding and quantum error correction subspaces, given the importance of entanglement and multipartite symmetry in such applications.
FIGURE CAPTIONS

Figure 1. Quantum circuit for creating states $|e_1\rangle$, $|e_2\rangle$, $|e_3\rangle$, $|e_4\rangle$, $|e_5\rangle$, $|e_6\rangle$, $|e_7\rangle$, and $|e_8\rangle$ from $|0, 0, 0, 1\rangle$, $|0, 0, 1, 0\rangle$, $|0, 0, 1, 1\rangle$, $|0, 1, 0, 0\rangle$, $|1, 0, 0, 1\rangle$, $|1, 0, 1, 0\rangle$ and $|1, 0, 1, 0\rangle$ utilizing Hadamard and c-NOT gates.

Figure 2. Quantum circuit for creating states $|e_{13}\rangle$, $|e_{14}\rangle$, $|e_{15}\rangle$, and $|e_{16}\rangle$ from input states $|0, 0, 0, 1\rangle$, $|1, 1, 0, 1\rangle$, $|1, 0, 1, 1\rangle$ and $|0, 1, 1, 1\rangle$ utilizing Hadamard and c-NOT gates.

Figure 3. Quantum circuit for creating states $|e_9\rangle$, $|e_{10}\rangle$, $|e_{11}\rangle$, and $|e_{12}\rangle$ from input states $|0, 0, 0, 1\rangle$, $|1, 1, 0, 1\rangle$, $|1, 0, 1, 1\rangle$ and $|0, 1, 1, 1\rangle$ utilizing Hadamard, c-NOT and controlled-phase gates introducing a controlled phase shift of $\pi/2$. 
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