A Hopf bundle over a quantum four-sphere from the symplectic group

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Abstract

We construct a quantum version of the $SU(2)$ Hopf bundle $S^7 \rightarrow S^4$. The quantum sphere $S^7_q$ arises from the symplectic group $Sp_q(2)$ and a quantum 4-sphere $S^4_q$ is obtained via a suitable self-adjoint idempotent $p$ whose entries generate the algebra $A(S^4_q)$ of polynomial functions over it. This projection determines a deformation of an (anti-)instanton bundle over the classical sphere $S^4$. We compute the fundamental $K$-homology class of $S^4_q$ and pair it with the class of $p$ in the $K$-theory getting the value $-1$ for the topological charge. There is a right coaction of $SU_q(2)$ on $S^7_q$ such that the algebra $A(S^7_q)$ is a non trivial quantum principal bundle over $A(S^4_q)$ with structure quantum group $A(SU_q(2))$.
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1 Introduction

In this paper we study yet another example of how “quantization removes degeneracy” by constructing a new quantum version of the Hopf bundle $S^7 \to S^4$. This is the first outcome of our attempt to generalize to the quantum case the ADHM construction of $SU(2)$ instantons together with their moduli spaces.

The $q$-monopole on two dimensional quantum spheres has been constructed in [8] more than a decade ago. There it was also introduced the general notion of a quantum principal bundle with quantum differential calculi, from a geometrical point of view. With universal differential calculi, this notion was later realised to be equivalent to the one of Hopf-Galois extension (see e.g. [14]). An analogous construction for $q$-instantons and their principal bundles has been an open problem ever since. A step in this direction was taken in [3] resulting in a bundle which is only a coalgebra extension [4]. Here we present a quantum principal instanton bundle which is a honest Hopf-Galois extension. One advantage is that non-universal calculi may be constructed on the bundle, as opposite to the case of a coalgebra bundle where there is not such a possibility.

In analogy with the classical case [1], it is natural to start with the quantum version of the (compact) symplectic groups $A(Sp_q(n))$, i.e. the Hopf algebras generated by matrix elements $T_{ij}$’s with commutation rules coming from the $R$ matrix of the $C$-series [25]. These quantum groups have comodule-subalgebras $A(S_q^{4n-1})$ yielding deformations of the algebras of polynomials over the spheres $S_q^{4n-1}$, which give more examples of the general construction of quantum homogeneous spaces [8].

The relevant case for us is $n = 2$, i.e. the symplectic quantum 7-sphere $A(S_q^7)$, which is generated by the matrix elements of the first and the last column of $T$. Indeed, as we will see, $T^1_i \propto T^1_{4-i}$. A similar conjugation occurs for the elements of the middle columns, but contrary to what happens at $q = 1$, they do not generate a subalgebra. The algebra $A(S_q^7)$ is the quantum version of the homogeneous space $Sp(2)/Sp(1)$ and the injection $A(S_q^7) \hookrightarrow A(Sp_q(2))$ is a quantum principal bundle with “structure Hopf algebra” $A(Sp_q(1))$.

Most importantly, we show that $S_q^7$ is the total space of a quantum $SU_q(2)$ principal bundle over a quantum 4-sphere $S_q^4$. Unlike the previous construction, this is obviously not a quantum homogeneous structure. The algebra $A(S_q^4)$ is constructed as the subalgebra of $A(S_q^7)$ generated by the matrix elements of a self-adjoint projection $p$ which generalizes the anti-instanton of charge $-1$. This projection will be of the form $vv^*$ with $v$ a $4 \times 2$ matrix whose entries are made out of generators of $A(S_q^7)$. The naive generalization of the classical case produces a subalgebra with extra generators which vanish at $q = 1$. Luckily enough, there is just one alternative choice of $v$ which gives the right number of generators of an algebra which deforms the algebra of polynomial functions of $S^4$. At $q = 1$ this gives a projection which is gauge equivalent to the standard one.

This good choice becomes even better because there is a natural coaction of $SU_q(2)$ on $A(S_q^7)$ with coinvariant algebra $A(S_q^4)$ and the injection $A(S_q^4) \hookrightarrow A(S_q^7)$ turns out to be a faithfully flat $A(SU_q(2))$-Hopf-Galois extension.

Finally, we set up the stage to compute the charge of our projection and to prove the non triviality of our principal bundle. Following a general strategy of noncommutative index theorem [10], we construct representations of the algebra $A(S_q^4)$ and the corresponding $K$-homology. The analogue of the fundamental class of $S^4$ is given by a non
trivial Fredholm module $\mu$. The natural coupling between $\mu$ and the projection $p$ is computed via the pairing of the corresponding Chern characters $\text{ch}^*(\mu) \in HC^*[A(S_q^4)]$ and $\text{ch}_*(p) \in HC_*[A(S_q^4)]$ in cyclic cohomology and homology respectively [10]. As expected the result of this pairing, which is an integer by principle being the index of a Fredholm operator, is actually $-1$ and therefore the bundle is non trivial.

Clearly the example presented in this paper is very special and limited, since it is just a particular anti-instanton of charge $-1$. Indeed our construction is based on the requirement that the matrix $v$ giving the projection is linear in the generators of $A(S_q^7)$ and such that $v^*v = 1$. This is false even classically at generic moduli and generic charge, except for the case considered here (and for a similar construction for the case of charge 1). A more elaborate strategy is needed to tackle the general case.

2 Odd spheres from quantum symplectic groups

We recall the construction of quantum spheres associated with the compact real form of the quantum symplectic groups $Sp_q(N, \mathbb{C})$ ($N = 2n$), the latter being given in [25]. Later we shall specialize to the case $N = 4$ and the corresponding 7-sphere will provide the ‘total space’ of our quantum Hopf bundle.

2.1 The quantum groups $Sp_q(N, \mathbb{C})$ and $Sp_q(n)$

The algebra $A(Sp_q(N, \mathbb{C}))$ is the associative noncommutative algebra generated over the ring of Laurent polynomials $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$ by the entries $T_{ij}^k$ of a matrix $T$ which satisfy RTT equations:

$$RTT = T_{ij}^{kl} T_{kl}^{ij}$$

In components $(T \otimes 1)_{ij}^{kl} = T_{ij}^{kl} \delta_j^l$. Here the relevant $N^2 \times N^2$ matrix $R$ is the one for the $C_N$ series and has the form [25],

$$R = q \sum_{i=1}^{N} e_{i}^{i} \otimes e_{i}^{i} + \sum_{i,j=1}^{N} e_{i}^{i} \otimes e_{j}^{j} + q^{-1} \sum_{i=1}^{N} e_{i}^{i} \otimes e_{i}^{i}$$

$$+(q - q^{-1}) \sum_{i,j=1}^{N} e_{i}^{j} \otimes e_{j}^{i} - (q - q^{-1}) \sum_{i,j=1}^{N} q^{\rho_i - \rho_j} \varepsilon_i \varepsilon_j e_{i}^{j} \otimes e_{i}^{j}, \quad(1)$$

where

$$i' = N + 1 - i;$$
$$e_{i}^{j} \in M_n(\mathbb{C}) \text{ are the elementary matrices, i.e. } (e_{j}^{i})_{l}^{k} = \delta_{jl} \delta^{ik};$$
$$\varepsilon_i = 1, \text{ for } i = 1, \ldots, n;$$
$$\varepsilon_i = -1, \text{ for } i = n + 1, \ldots, N;$$
$$(\rho_1, \ldots, \rho_N) = (n, n - 1, \ldots, 1, -1, \ldots, -n).$$

The symplectic group structure comes from the matrix $C_{i}^{j} = q^{\rho_j} \varepsilon_i \delta_{ij'}$ by imposing the additional relations

$$TCTC^{-1} = CT'C^{-1}T = 1.$$
The Hopf algebra co-structures \((\Delta, \varepsilon, S)\) of the quantum group \(Sp_q(N, \mathbb{C})\) are given by
\[
\Delta(T) = T \otimes T, \quad \varepsilon(T) = I, \quad S(T) = CT^tC^{-1}.
\]
In components the antipode explicitly reads
\[
S(T)_i^j = -q^{\rho_i^+ + \rho_j^+} \varepsilon_i \varepsilon_j T_{j'}^{i'}.
\] (2)
At \(q = 1\) the Hopf algebra \(Sp_q(N, \mathbb{C})\) reduces to the algebra of polynomial functions over the symplectic group \(Sp(N, \mathbb{C})\).

The compact real form \(A(\mathfrak{sp}_q(n))\) of the quantum group \(A(\mathfrak{sp}_q(N, \mathbb{C}))\) is given by taking \(q \in \mathbb{R}\) and the anti-involution \([25]\)
\[
\mathbf{T} = S(T)^t = C^t T (C^{-1})^t.
\] (3)

### 2.2 The odd symplectic spheres

Let us denote
\[
x_i = T_{i}^{N}, \quad v^j = S(T)_N^j, \quad i, j = 1, \ldots, N.
\]
As we will show, these generators give subalgebras of \(A(\mathfrak{sp}_q(N, \mathbb{C}))\). With the natural involution \([3]\), the algebra generated by the \(\{x_i, v^j\}\) can be thought of as the algebra \(A(S_q^{4n-1})\) of polynomial functions on a quantum sphere of 'dimension' \(4n - 1\).

From here on, whenever no confusion arises, the sum over repeated indexes is understood. In components the RTT equations are given by
\[
R_{ij}^{kp} T_{k}^{r} T_{p}^{s} = T_{j}^{p} T_{i}^{m} R_{mp}^{rs}.
\] (4)
Hence
\[
R_{ij}^{kl} T_{k}^{r} = T_{j}^{p} T_{i}^{m} R_{mp}^{rs} S(T)_s^l,
\]
and in turn
\[
S(T)_p^j R_{ij}^{kl} = T_{i}^{a} R_{ap}^{rs} S(T)_s^l S(T)_r^k,
\]
so that
\[
S(T)_a^i S(T)_p^j R_{ij}^{kl} = R_{ap}^{rs} S(T)_s^l S(T)_r^k.
\] (5)
Conversely, if we multiply \(R_{ij}^{kp} T_{k}^{r} = T_{j}^{l} T_{i}^{m} R_{ml}^{rs} S(T)_s^p\) on the left by \(S(T)\) we have
\[
S(T)_l^j R_{ij}^{kp} T_{k}^{r} = T_{i}^{m} R_{ml}^{rs} S(T)_s^p.
\] (6)
We shall use equations (4), (5) and (6) to describe the algebra generated by the \(x_i\)'s and by the \(v^i\)'s.

**The algebra** \(\mathbb{C}_q[x_i]\)

From (4) with \(r = s = N\) we have
\[
R_{ij}^{kp} x_k x_p = T_{j}^{p} T_{i}^{m} R_{mp}^{NN}.
\] (7)
Since the only element \( R_{mp}^{NN} \propto e_m^N \otimes e_p^N \) \((m, p \leq N)\) which is different from zero is \( R_{NN}^{NN} = q \), it follows that
\[
R_{ij}^{kp} x_k x_p = q x_j x_i , \tag{8}
\]
and the elements \( x_i \)'s give an algebra with commutation relations
\[
x_i x_j = q x_j x_i , \quad i < j, \quad i \neq j' ,
x_{i'} x_i = q^{-2} x_i x_{i'} + (q^{-2} - 1) \sum_{k=1}^{i-1} q^{p_i - p_k} \varepsilon_i \varepsilon_k x_k x_{i'} , \quad i < i' . \tag{9}
\]

The algebra \( \mathbb{C}_q[v^i] \)

Putting \( a = p = N \) in equation (5), we get
\[
v^j v^i R_{ij}^{kl} = R_{NN}^{rs} S(T)_s^l S(T)_r^k .
\]
The sum on the r.h.s. reduces to \( R_{NN}^{NN} S(T)_N^l S(T)_N^k \) and the \( v^i \)'s give an algebra with commutation relations
\[
v^i v^k R_{tk}^{ji} = q v^i v^j . \tag{10}
\]
Explicitly
\[
v^i v^j = q^{-1} v^j v^i , \quad i < j, \quad i \neq j' ,
v^i v^j = q^2 v^i v^j + (q^2 - 1) \sum_{k=v'+1}^{N} q^{p_i - p_{v'}} \varepsilon_i \varepsilon_{v'} v_{v'} v^k v^k , \quad i < v' . \tag{11}
\]

The algebra \( \mathbb{C}_q[x_i, v^j] \)

Finally, for \( l = r = N \) the equation (6) reads:
\[
v^j R_{ij}^{kp} x_k = T_{im}^s R_{mp}^{Ns} S(T)_s^l .
\]
Once more, the only term in \( R \) of the form \( e_m^N \otimes e_N^s \) \((m \leq N)\) is \( e_N^N \otimes e_N^N \) and therefore
\[
v^j R_{ij}^{kp} x_k = q x_i v^p . \tag{12}
\]
Explicitly the mixed commutation rules for the algebra \( \mathbb{C}_q[x_i, v^j] \) read,
\[
x_i v^j = v^j x_i + (1 - q^{-2}) \sum_{k=1}^{i-1} v^k x_k + \left(1 - q^{-2}\right) q^{p_i - p_{v'}} v^i v_{i'} ,
x_i v^j = q^{-2} v^j x_i ,
x_i v^j = q^{-1} v^j x_i , \quad i \neq j \quad \text{and} \quad i < j' ,
x_i v^j = q^{-1} v^j x_i + (q^{-2} - 1) q^{p_i - p_{v'}} \varepsilon_i \varepsilon_{j'} v^i v_{j'} , \quad i \neq j \quad \text{and} \quad i > j' . \tag{13}
\]
The quantum spheres $S_{q}^{4n-1}$

Let us observe that with the anti-involution (3) we have the identification $v^{i} = S(T)_{N}^{i} = \bar{x}^{i}$. The subalgebra $A(S_{q}^{4n-1})$ of $A(Sp_{q}(n))$ generated by \{ $x_{i}, v^{i} = \bar{x}^{i}, i = 1, \ldots, 2n$ \} is the algebra of polynomial functions on a sphere. Indeed

\[ S(T)T = I \Rightarrow \sum S(T)_{N}^{i}T_{i}^{N} = \delta_{N}^{i} = 1 \]

i.e.

\[ \sum_{i} \bar{x}^{i}x_{i} = 1. \] (14)

Furthermore, the restriction of the comultiplication is a natural left coaction

\[ \Delta_{L} : A(S_{q}^{4n-1}) \rightarrow A(Sp_{q}(n)) \otimes A(S_{q}^{4n-1}). \]

The fact that $\Delta_{L}$ is an algebra map then implies that $A(S_{q}^{4n-1})$ is a comodule algebra over $A(Sp_{q}(n))$.

At $q = 1$ this algebra reduces to the algebra of polynomial functions over the spheres $S_{4n-1}$ as homogeneous spaces of the symplectic group $Sp(n) : S^{4n-1} = Sp(n)/Sp(n-1)$.

2.3 The symplectic 7-sphere $S_{7}^{7}$

The algebra $A(S^{7})$ is generated by the elements $x_{i} = T_{i}^{4}$ and $\bar{x}^{i} = S(T)_{4}^{i} = q^{2+\rho_{i}}\varepsilon_{i}T_{i}^{4}$, for $i = 1, \ldots, 4$. From $S(T) \ T = 1$ we have the sphere relation $\sum_{i=1}^{4} \bar{x}^{i}x_{i} = 1$. Since we shall systematically use them in the following, we shall explicitly give the commutation relations among the generators.

From (3), the algebra of the $x_{i}$'s is given by

\[
\begin{align*}
x_{1}x_{2} &= qx_{2}x_{1}, & x_{1}x_{3} &= qx_{3}x_{1}, \\
x_{2}x_{4} &= qx_{4}x_{2}, & x_{3}x_{4} &= qx_{4}x_{3}, \\
x_{4}x_{1} &= q^{-2}x_{1}x_{4}, & x_{3}x_{2} &= q^{-2}x_{2}x_{3} + q^{-2}(q^{-1} - q)x_{1}x_{4}, \\
\end{align*}
\]

(15)

together with their conjugates (given in (11)).

We have also the commutation relations between the $x_{i}$ and the $\bar{x}^{j}$ deduced from (12):

\[
\begin{align*}
x_{1}\bar{x}^{1} &= \bar{x}^{1}x_{1}, & x_{1}\bar{x}^{2} &= q^{-1}\bar{x}^{2}x_{1}, \\
x_{1}\bar{x}^{3} &= q^{-1}\bar{x}^{3}x_{1}, & x_{1}\bar{x}^{4} &= q^{-2}\bar{x}^{4}x_{1}, \\
x_{2}\bar{x}^{2} &= \bar{x}^{2}x_{2} + (1 - q^{-2})\bar{x}^{1}x_{1}, \\
x_{2}\bar{x}^{3} &= q^{-2}\bar{x}^{3}x_{2}, \\
x_{2}\bar{x}^{4} &= q^{-1}\bar{x}^{4}x_{2} + q^{-1}(q^{-2} - 1)\bar{x}^{3}x_{1}, \\
x_{3}\bar{x}^{3} &= \bar{x}^{3}x_{3} + (1 - q^{-2})[\bar{x}^{1}x_{1} + (1 + q^{-2})\bar{x}^{2}x_{2}], \\
x_{3}\bar{x}^{4} &= q^{-1}\bar{x}^{4}x_{3} + (1 - q^{-2})q^{-3}\bar{x}^{2}x_{1}, \\
x_{4}\bar{x}^{4} &= \bar{x}^{4}x_{4} + (1 - q^{-2})[(1 + q^{-4})\bar{x}^{1}x_{1} + \bar{x}^{2}x_{2} + \bar{x}^{3}x_{3}], \\
\end{align*}
\]

again with their conjugates.
Next we show that the algebra $A(S^7_q)$ can be realized as the subalgebra of $A(Sp_q(2))$ generated by the coinvariants under the right-coaction of $A(Sp_q(1))$, in complete analogy with the classical homogeneous space $Sp(2)/Sp(1) \simeq S^7$.

**Lemma 1.** The two-sided *-ideal in $A(Sp_q(2))$ generated as

$$I_q = \{ T_1^4 - 1, T_4^4 - 1, T_1^2 T_3^2, T_1^4, T_2^4, T_3^4, T_4^1, T_4^2, T_4^3 \}$$

with the involution (3) is a Hopf ideal.

**Proof.** Since $S(T)^j_i \propto T^{j'}_i$, $S(I_q) \subseteq I_q$ which also proves that $I_q$ is a *-ideal. One easily shows that $\varepsilon(I_q) = 0$ and $\Delta(I_q) \subseteq I_q \otimes A(Sp_q(2)) + A(Sp_q(2)) \otimes I_q$. □

**Proposition 1.** The Hopf algebra $B_q := A(Sp_q(2))/I_q$ is isomorphic to the coordinate algebra $A(SU_{q^2}(2)) \cong A(Sp_q(1))$.

**Proof.** Using $\overline{T} = S(T)^i_j$ and setting $T_2^2 = \alpha$, $T_3^2 = \gamma$, the algebra $B_q$ can be described as the algebra generated by the entries of the matrix

$$T' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & -q^2\gamma & 0 \\ 0 & \gamma & \bar{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  \hfill (17)

The commutation relations deduced from RTT equations (4) read:

$$\alpha\gamma = q^2\gamma\alpha, \quad \alpha\gamma = q^2\gamma\alpha, \quad \gamma\gamma = \gamma\gamma,$$

$$\bar{\alpha}\alpha + \gamma\gamma = 1; \quad \bar{\alpha}\alpha + q^4\gamma\gamma = 1.$$  \hfill (18)

Hence, as an algebra $B_q$ is isomorphic to the algebra $A(SU_{q^2}(2))$. Furthermore, the restriction of the coproduct of $A(Sp_q(2))$ to $B_q$ endows the latter with a coalgebra structure, $\Delta(T') = T' \otimes T'$, which is the same as the one of $A(SU_{q^2}(2))$. We can conclude that also as a Hopf algebra, $B_q$ is isomorphic to the Hopf algebra $A(SU_{q^2}(2)) \cong A(Sp_q(1))$. □

**Proposition 2.** The algebra $A(S^7_q) \subset A(Sp_q(2))$ is the algebra of coinvariants with respect to the natural right coaction

$$\Delta_R : A(Sp_q(2)) \rightarrow A(Sp_q(2)) \otimes A(Sp_q(1)) ; \quad \Delta_R(T) = T \otimes T'.$$

**Proof.** It is straightforward to show that the generators of the algebra $A(S^7_q)$ are coinvariants:

$$\Delta_R(x_i) = \Delta_R(T_i^4) = x_i \otimes 1 ; \quad \Delta_R(\bar{x}^i) = -q^{2+\rho_i} \varepsilon_i \Delta_R(T_i^1) = \bar{x}^i \otimes 1$$

thus the algebra $A(S^7_q)$ is made of coinvariants. There are no other coinvariants of degree one since each row of the submatrix of $T$ made out of the two central columns is a fundamental comodule under the coaction of $SU_{q^2}(2)$. Other coinvariants arising at higher even degree are of the form $(T_{i2}T_{i3} - q^2T_{i3}T_{i2})^n$; thanks to the commutation relations of $A(Sp_q(2))$, one checks these belong to $A(S^7_q)$ as well. It is an easy computation to check
that similar expressions involving elements from different rows cannot be coinvariant.

The previous construction is one more example of the general construction of a quantum principal bundle over a quantum homogeneous space \([8]\). The latter is the datum of a Hopf quotient \(\pi : A(G) \rightarrow A(K)\) with the right coaction of \(A(K)\) on \(A(G)\) given by the reduced coproduct \(\Delta_R := (id \otimes \pi)\Delta\) where \(\Delta\) is the coproduct of \(A(G)\). The subalgebra \(B \subset A(G)\) made of the coinvariants with respect to \(\Delta_R\) is called a quantum homogeneous space. To prove that it is a quantum principal bundle one needs some more assumptions (see Lemma 5.2 of \([8]\)). In our case \(A(G) = A(Sp_q(2))\), \(A(K) = A(Sp_q(1))\) with \(\pi(T) = T'\). We will prove in Sec. 6 that the resulting inclusion \(B = A(S^4_q) \hookrightarrow A(Sp_q(2))\) is indeed a Hopf Galois extension and hence a quantum principal bundle.

3 The principal bundle \(A(S^4_q) \hookrightarrow A(S^7_q)\)

The fundamental step of this paper is to make the sphere \(S^7_q\) itself into the total space of a quantum principal bundle over a deformed 4-sphere. Unlike what we saw in the previous section, this is not a quantum homogeneous space construction and it is not obvious that such a bundle exists at all. Nonetheless the notion of quantum bundle is more general and one only needs that the total space algebra is a comodule algebra over a Hopf algebra with additional suitable properties.

The notion of quantum principle bundle, as said, is encoded in the one of Hopf-Galois extension (see e.g. \([8]\), \([14]\)). Let us recall some relevant definitions \([20]\) (see also \([22]\)). Recall that we work over the field \(k = \mathbb{C}\).

**Definition 1.** Let \(H\) be a Hopf algebra and \(P\) a right \(H\)-comodule algebra with multiplication \(m : P \otimes P \rightarrow P\) and coaction \(\Delta_R : P \rightarrow P \otimes H\). Let \(B \subseteq P\) be the subalgebra of coinvariants, i.e. \(B = \{ p \in P | \Delta_R(p) = p \otimes 1\}\). The extension \(B \subseteq P\) is called an \(H\) Hopf-Galois extension if the canonical map

\[
\chi : P \otimes_B P \longrightarrow P \otimes H, \quad \chi := (m \otimes id) \circ (id \otimes_B \Delta_R), \quad p' \otimes_B p \mapsto \chi(p' \otimes_B p) = p'p_{(0)} \otimes p_{(1)}
\]

is bijective.

We use Sweedler-like notation \(\Delta_R p = p_{(0)} \otimes p_{(1)}\). The canonical map is left \(P\)-linear and right \(H\)-colinear and is a morphism (an isomorphism for Hopf-Galois extensions) of left \(P\)-modules and right \(H\)-comodules. It is also clear that \(P\) is both a left and a right \(B\)-module.

The injectivity of the canonical map dualizes the condition of a group action \(X \times G \rightarrow X\) to be free: if \(\alpha\) is the map \(\alpha : X \times G \rightarrow X \times_M X, \ (x, g) \mapsto (x, x \cdot g)\) then \(\alpha^* = \chi\) with \(P, H\) the algebras of functions on \(X, G\) respectively and the action is free if and only if \(\alpha\) is injective. Here \(M := X/G\) is the space of orbits with projection map \(\pi : X \rightarrow M, \ \pi(x \cdot g) = \pi(x),\) for all \(x \in X, g \in G\). Furthermore, \(\alpha\) is surjective if and only if for all \(x \in X\), the fibre \(\pi^{-1}(\pi(x))\) of \(\pi(x)\) is equal to the residue class \(x \cdot G\), that is, if and only if \(G\) acts transitively on the fibres of \(\pi\).
In differential geometry a principle bundle is more than just a free and effective action of a Lie group. In our example, thanks to the fact that the “structure group” is \(SU_q(2)\), from Th. I of [28] further nice properties can be established. We shall elaborate more on these points later on in Sect. 6.

The first natural step would be to construct a map from \(S^7\) into a deformation of the Stieffel variety of unitary frames of 2-planes in \(\mathbb{C}^4\) to parallel the classical construction as recalled in the Appendix A. The naive choice we have is to take as generators the elements of two (conjugate) columns of the matrix \(T\). We are actually forced to take the first and the last columns of the matrix \(T\) because the other choice (i.e. the second and the third columns) does not yield a subalgebra since commutation relations of their elements will involve elements from the other two columns. If we set

\[
v = \begin{pmatrix}
\bar{x}^4 & x_1 \\
q^{-1}\bar{x}^3 & x_2 \\
-q^{-3}\bar{x}^2 & x_3 \\
-q^{-4}\bar{x}^1 & x_4
\end{pmatrix},
\]

we have \(v^*v = \mathbb{I}_2\) and the matrix \(p = vv^*\) is a self-adjoint idempotent, i.e. \(p = p^* = p^2\). At \(q = 1\) the entries of \(p\) are invariant for the natural action of \(SU(2)\) on \(S^7\) and generate the algebra of polynomials on \(S^4\). This fails to be the case at generic \(q\) due to the occurrence of extra generators e.g.

\[
p_{14} = (1 - q^{-2})x_1\bar{x}^4, \quad p_{23} = (1 - q^{-2})x_2\bar{x}^3,
\]

which vanish at \(q = 1\).

### 3.1 The quantum sphere \(S^4_q\)

These facts indicate that the naive quantum analogue of the quaternionic projective line as a homogeneous space of \(Sp_q(2)\) has not the right number of generators. Rather surprisingly, we shall anyhow be able to select another subalgebra of \(A(S^7_q)\) which is a deformation of the algebra of polynomials on \(S^4\) having the same number of generators. These generators come from a better choice of a projection.

On the free module \(\mathcal{E} := \mathbb{C}^4 \otimes A(S^7_q)\) we consider the hermitean structure given by

\[
h(\xi_1, \xi_2) = \sum_{j=1}^4 \xi_1^j \bar{\xi}_2^j.
\]

To every element \(|\xi\rangle \in \mathcal{E}\) one associates an element \(\langle \xi|\) in the dual module \(\mathcal{E}^*\) by the pairing

\[
\langle \xi| (\eta) := \langle \xi |\eta\rangle = h(|\xi\rangle , |\eta\rangle).
\]

Guided the classical construction which we present in Appendix A we shall look for two elements \(|\phi_1\rangle, |\phi_2\rangle\) in \(\mathcal{E}\) with the property that

\[
\langle \phi_1 |\phi_1\rangle = 1, \quad \langle \phi_2 |\phi_2\rangle = 1, \quad \langle \phi_1 |\phi_2\rangle = 0.
\]
As a consequence, the matrix valued function defined by

\[ p := |\phi_1\rangle \langle \phi_1| + |\phi_2\rangle \langle \phi_2| , \]  

(22)
is a self-adjoint idempotent (a projection).

In principle, \( p \in \text{Mat}_4(A(S^7_q)) \), but we can choose \( |\phi_1\rangle, |\phi_2\rangle \) in such a way that the entries of \( p \) will generate a subalgebra \( A(S^4_q) \) of \( A(S^7_q) \) which is a deformation of the algebra of polynomial functions on the 4-sphere \( S^4 \). The two elements \( |\phi_1\rangle, |\phi_2\rangle \) will be obtained in two steps as follows.

Firstly we write the relation \( 1 = \sum \bar{x}^i x_i \) in terms of the quadratic elements \( \bar{x}^1 x_1, x_2 \bar{x}^2, \bar{x}^3 x_3, x_4 \bar{x}^4 \) by using the commutation relations of Sect. 2.3. We have that

\[ 1 = q^{-6} \bar{x}^1 x_1 + q^{-2} x_2 \bar{x}^2 + q^{-2} \bar{x}^3 x_3 + x_4 \bar{x}^4 . \]

Then we take,

\[ |\phi_1\rangle = (q^{-3} x_1, -q^{-1} \bar{x}^2, q^{-1} x_3, -\bar{x}^4) , \]

(23)

(\( t \) denoting transposition) which is such that \( \langle \phi_1 | \phi_1 \rangle = 1 \).

Next, we write \( 1 = \sum \bar{x}^i x_i \) as a function of the quadratic elements \( x_1 \bar{x}^1, \bar{x}^2 x_2, x_3 \bar{x}^3, \bar{x}^4 x_4 ; \)

\[ 1 = q^{-2} x_1 \bar{x}^1 + q^{-4} \bar{x}^2 x_2 + x_3 \bar{x}^3 + \bar{x}^4 x_4 . \]

By taking,

\[ |\phi_2\rangle = (\pm q^{-2} x_2, \pm q^{-1} \bar{x}^1, \pm x_4, \pm \bar{x}^3) , \]

we get \( \langle \phi_2 | \phi_2 \rangle = 1 \). The signs will be chosen in order to have also the orthogonality \( \langle \phi_1 | \phi_2 \rangle = 0 \); for

\[ |\phi_2\rangle = (q^{-2} x_2, q^{-1} \bar{x}^1, -x_4, -\bar{x}^3) , \]

(24)

this is satisfied.

The matrix

\[ v = (|\phi_1\rangle, |\phi_2\rangle) = \begin{pmatrix} q^{-3} x_1 & q^{-2} x_2 \\ -q^{-1} \bar{x}^2 & q^{-1} \bar{x}^1 \\ q^{-1} x_3 & -x_4 \\ -\bar{x}^4 & -\bar{x}^3 \end{pmatrix} , \]

(25)
is such that \( v^* v = 1 \) and hence \( p = vv^* \) is a self-adjoint projection.

**Proposition 3.** The entries of the projection \( p = vv^* \), with \( v \) given in (25), generate a subalgebra of \( A(S^4_q) \) which is a deformation of the algebra of polynomial functions on the 4-sphere \( S^4 \).

**Proof.** Let us compute explicitly the components of the projection \( p \) and their commutation relations.

1. The diagonal elements are given by

\[ p_{11} = q^{-6} x_1 \bar{x}^1 + q^{-4} x_2 \bar{x}^2 , \quad p_{22} = q^{-2} \bar{x}^2 x_2 + q^{-2} \bar{x}^1 x_1 , \]

\[ p_{33} = q^{-2} x_3 \bar{x}^3 + x_4 \bar{x}^4 , \quad p_{44} = \bar{x}^4 x_4 + \bar{x}^3 x_3 , \]
and satisfy the relation
\[ q^{-2}p_{11} + q^2p_{22} + p_{33} + p_{44} = 2 . \] (26)

Only one of the \( p_{ii} \)'s is independent; indeed by using the commutation relations and the equation \( \sum \bar{x}^i x_i = 1 \), we can rewrite the \( p_{ii} \)'s in terms of
\[ t := p_{22} , \] (27)
as
\[ p_{11} = q^{-2}t , \quad p_{22} = t , \quad p_{33} = 1 - q^{-4}t , \quad p_{44} = 1 - q^2t . \]

Equation (26) is easily verified. Notice that \( t \) is self-adjoint: \( \bar{t} = t \).

2. As in the classical case, the elements \( p_{12}, p_{34} \) (and their conjugates) vanish:
\[ p_{12} = -q^{-4} x_1 x_2 + q^{-3} x_2 x_1 = 0 , \quad p_{34} = -q^{-1} x_3 x_4 + x_4 x_3 = 0 . \]

3. The remaining elements are given by
\[ p_{13} = q^{-4} x_1 \bar{x}^3 - q^{-2} x_2 \bar{x}^4 , \quad p_{14} = -q^{-3} x_1 x_4 - q^{-2} x_2 x_3 , \]
\[ p_{23} = -q^{-2} \bar{x}^2 \bar{x}^3 - q^{-1} \bar{x}^1 \bar{x}^4 , \quad p_{24} = q^{-1} \bar{x}^2 x_4 - q^{-1} \bar{x}^1 x_3 , \]

with \( p_{ji} = \bar{p}_{ij} \) when \( j > i \).

By using the commutation relations of \( A(S^7) \), one finds that only two of these are independent. We take them to be \( p_{13} \) and \( p_{14} \); one finds \( p_{23} = q^{-2} \bar{p}_{14} \) and \( p_{24} = -q^2 \bar{p}_{13} \).

Finally, we also have the sphere relation,
\[ (q^6 - q^8)p_{11}^2 + p_{22}^2 + p_{44}^2 + q^4(p_{13}p_{31} + p_{14}p_{41}) + q^2(p_{24}p_{42} + p_{23}p_{32}) = \left( \sum \bar{x}^i x_i \right)^2 = 1 . \] (28)

Summing up, together with \( t = p_{22} \), we set \( a := p_{13} \) and \( b := p_{14} \). Then the projection \( p \) takes the following form
\[ p = \begin{pmatrix}
q^{-2}t & 0 & a & b \\
0 & t & q^{-2}b & -q^2a \\
\bar{a} & q^{-2}b & 1 - q^{-4}t & 0 \\
\bar{b} & -q^2a & 0 & 1 - q^2t
\end{pmatrix} . \] (29)

By construction \( p^* = p \) and this means that \( \bar{t} = t \), as observed, and that \( \bar{a}, \bar{b} \) are conjugate to \( a, b \) respectively. Also, by construction \( p^2 = p \); this property gives the easiest way to compute the commutation relations between the generators. One finds,
\[ ab = q^4 ba , \quad \bar{a}b = b\bar{a} , \]
\[ ta = q^{-2} at , \quad tb = q^4 bt , \] (30)

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together with their conjugates, and sphere relations
\[
\begin{align*}
 a\bar{a} + b\bar{b} &= q^{-2}t(1 - q^{-2}t) , \\
 b\bar{b} - q^{-4}bb &= (1 - q^{-3})t^2 .
\end{align*}
\] (31)

It is straightforward to check also the relation (28). \(\blacksquare\)

We define the algebra \(A(S_q^4)\) to be the algebra generated by the elements \(a, \bar{a}, b, \bar{b}, t\) with the commutation relations (30) and (31). For \(q = 1\) it reduces to the algebra of polynomial functions on the sphere \(S^4\). Otherwise, we can limit ourselves to \(|q| < 1\), because the map
\[
q \mapsto q^{-1}, \quad a \mapsto q^2\bar{a}, \quad b \mapsto q^{-2}\bar{b}, \quad t \mapsto q^{-2}t
\]
yields an isomorphic algebra.

At \(q = 1\), the projection \(p\) in (29) is conjugate to the classical one given in Appendix A by the matrix \(\text{diag}[1, -1, 1, 1]\) (up to a renaming of the generators).

Our sphere \(S_q^4\) seems to be different from the one constructed in [3]. Two of our generators commute and most importantly, it does not come from a deformation of a subgroup (let alone coisotropic) of \(Sp(2)\). However, at the continuous level these two quantum spheres are the same since the \(C^*\)-algebra completion of both polynomial algebras is the minimal unitization \(K \oplus \mathbb{C} I\) of the compact operators on an infinite dimensional separable Hilbert space, a property shared with Podleš standard sphere as well [24]. This fact will be derived in Sect. 4 when we study the representations of the algebra \(A(S_q^4)\).

### 3.2 The \(SU_q(2)\)-coaction

We now give a coaction of the quantum group \(SU_q(2)\) on the sphere \(S_q^7\). This coaction will be used later in Sect. 6 when analyzing the quantum principle bundle structure. Let us observe that the two pairs of generators \((x_1, x_2), (x_3, x_4)\) both yield a quantum plane,
\[
egin{align*}
x_1x_2 &= qx_2x_1 , \\
x_3x_4 &= qx_4x_3 ,
\end{align*}
\]
\[
\bar{x}^1\bar{x}^2 = q^{-1}\bar{x}^2\bar{x}^1 , \\
\bar{x}^3\bar{x}^4 = q^{-1}\bar{x}^4\bar{x}^3 .
\]
Then we shall look for a right-coaction of \(SU_q(2)\) on the rows of the matrix \(v\) in (25). Other pairs of generators yield quantum planes but the only choice which gives a projection with the right number of generators is the one given above.

The defining matrix of the quantum group \(SU_q(2)\) reads
\[
\begin{pmatrix}
\alpha & -q\gamma \\
\gamma & \bar{\alpha}
\end{pmatrix}
\] (32)

with commutation relations [30],
\[
\begin{align*}
\alpha\gamma &= q\gamma\alpha , \\
\alpha\bar{\gamma} &= q\bar{\gamma}\alpha , \\
\gamma\bar{\gamma} &= \bar{\gamma}\gamma , \\
\alpha\bar{\alpha} + q^2\bar{\gamma}\gamma &= 1 , \\
\bar{\alpha}\alpha + \bar{\gamma}\gamma &= 1 .
\end{align*}
\] (33)
We define a coaction of $SU_q(2)$ on the matrix (34) by,

$$
\delta_R(v) := \begin{pmatrix}
q^{-3}x_1 & q^{-2}x_2 \\
-q^{-1}x_2 & -q^{-1}x_1 \\
q^{-1}x_3 & -x_4 \\
-x^i & -\bar{x}^i
\end{pmatrix} \otimes \begin{pmatrix}
\alpha & -q\gamma \\
\gamma & \bar{\alpha}
\end{pmatrix}.
$$

(34)

We shall prove presently that this coaction comes from a coaction of $A(S^7_q)$ on the sphere algebra $A(S^7_q)$. For the moment we remark that, by its form in (34) the entries of the projection $\rho = vv^*$ are automatically coinvariants, a fact that we shall also prove explicitly in the following.

On the generators, the coaction (34) is given explicitly by

$$
\delta_R(x_1) = x_1 \otimes \alpha + q \cdot x_2 \otimes \gamma, \quad \delta_R(\bar{x}^i) = q\bar{x}^2 \otimes \bar{\gamma} + \bar{x}^1 \otimes \bar{\alpha} = \delta_R(x_1),
$$

$$
\delta_R(x_2) = -x_1 \otimes \bar{\gamma} + x_2 \otimes \alpha, \quad \delta_R(\bar{x}^2) = \bar{x}^2 \otimes \alpha - \bar{x}^1 \otimes \gamma = \delta_R(x_2),
$$

$$
\delta_R(x_3) = x_3 \otimes \alpha - q \cdot x_4 \otimes \bar{\gamma}, \quad \delta_R(\bar{x}^3) = -q\bar{x}^4 \otimes \bar{\gamma} + \bar{x}^3 \otimes \bar{\alpha} = \delta_R(x_3),
$$

$$
\delta_R(x_4) = x_3 \otimes \bar{\gamma} + x_4 \otimes \alpha, \quad \delta_R(\bar{x}^4) = \bar{x}^4 \otimes \alpha + \bar{x}^3 \otimes \gamma = \delta_R(x_4),
$$

(35)

from which it is also clear its compatibility with the anti-involution, i.e. $\delta_R(\bar{x}^i) = \bar{\delta}(x_i)$. The map $\delta_R$ in (35) extends as an algebra homomorphism to the whole of $A(S^7_q)$. Then, as alluded to before, we have the following

**Proposition 4.** The coaction (35) is a right coaction of the quantum group $SU_q(2)$ on the 7-sphere $S^7_q$,

$$
\delta_R : A(S^7_q) \longrightarrow A(S^7_q) \otimes A(SU_q(2)).
$$

(36)

**Proof.** By using the commutation relations of $A(SU_q(2))$ in (33), a lengthy but easy computation gives that the commutation relations of $A(S^7_q)$ are preserved. This fact also shows that extending $\delta_R$ as an algebra homomorphism yields a consistent coaction. □

**Proposition 5.** The algebra $A(S^4_q)$ is the algebra of coinvariants under the coaction defined in (35).

**Proof.** We have to show that $A(S^4_q) = \{ f \in A(S^7_q) \mid \delta_R(f) = f \otimes 1 \}$. By using the commutation relations of $A(S^7_q)$ and those of $A(SU_q(2))$, we first prove explicitly that the generators of $A(S^4_q)$ are coinvariants:

$$
\delta_R(a) = q^{-4}\delta_R(x_1)\delta_R(\bar{x}^3) - q^{-2}\delta_R(x_2)\delta_R(\bar{x}^4) = q^{-4}x_1\bar{x}^3 \otimes (\alpha\bar{\alpha} + q^2\bar{\gamma}\gamma) - q^{-2}x_2\bar{x}^4 \otimes (\gamma\bar{\gamma} + \bar{\alpha}\alpha) = (q^{-4}x_1\bar{x}^3 - q^{-2}x_2\bar{x}^4) \otimes 1 = a \otimes 1
$$

$$
\delta_R(b) = -q^{-3}\delta_R(x_1)\delta_R(x_4) - q^{-2}\delta_R(x_2)\delta_R(x_3) = -q^{-3}x_1x_4 \otimes (\alpha\bar{\alpha} + q^2\bar{\gamma}\gamma) - q^{-2}x_2x_3 \otimes (\gamma\bar{\gamma} + \bar{\alpha}\alpha) = -(q^{-3}x_1x_4 + q^{-2}x_2x_3) \otimes 1 = b \otimes 1
$$

$$
\delta_R(t) = q^{-2}\delta_R(\bar{x}^2)\delta_R(x_2) + q^{-2}\delta_R(\bar{x}^1)\delta_R(x_1) = q^{-2}\bar{x}^x \otimes (\alpha\bar{\alpha} + q^2\bar{\gamma}\gamma) + q^{-2}\bar{x}^i \otimes (\gamma\bar{\gamma} + \bar{\alpha}\alpha) = (q^{-2}\bar{x}^x \otimes x_2 + q^{-2}\bar{x}^i \otimes x_1) \otimes 1 = t \otimes 1
$$
By construction the coaction is compatible with the anti-involution so that
\[ \delta_R(\bar{a}) = \delta_R(a) = \bar{a} \otimes 1, \quad \delta_R(\bar{b}) = \delta_R(b) = \bar{b} \otimes 1. \]

In fact, this only shows that \( A(S^4_q) \) is made of coinvariants but does not rule out the possibility of other coinvariants not in \( A(S^4_q) \). However this does not happen for the following reason. From eq. (35) it is clear that \( w_1 \in \{ x_1, x_3, \bar{x}^2, \bar{x}^3 \} \) (respectively \( w_{-1} \in \{ x_2, x_4, \bar{x}^1, \bar{x}^3 \} \)) are weight vectors of weight 1 (resp. \(-1\)) in the fundamental comodule of \( SU_q(2) \). It follows that the only possible coinvariants are of the form \( (w_1 w_{-1} - qw_{-1} w_1)^n \). When \( n = 1 \) these are just the generators of \( A(S^4_q) \).

**Remark 1.** The last part of the proof above is also related to the quantum Plücker coordinates. For every \( 2 \times 2 \) matrix of \( SU_q(2) \), let us define the determinant by
\begin{equation}
\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := a_{11}a_{22} - qa_{12}a_{21}.
\end{equation}
(Note that \( a_{12}, a_{21} \) do not commute and so in the previous formula the ordering between them is fixed.) Let \( m_{ij} \) be the minors of \( \begin{pmatrix} q^2p_{11} & t \\ -q \alpha & b \end{pmatrix} \). Then
\begin{align}
m_{12} &= q^2p_{11} = t, \quad m_{13} = p_{14} = b, \\
m_{14} &= -q \alpha = -qa, \quad m_{23} = p_{24} = -q^2\bar{a}, \\
m_{24} &= -q \alpha = -q^{-1}\bar{b}, \quad m_{34} = -q^{3}p_{33} = q^{-3}t - q.
\end{align}

At \( q = 1 \), these give the classical Plücker coordinates \[1\].

The right coaction of \( SU_q(2) \) on the 7-sphere \( S^7_q \) can be written as
\[ \delta_R(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) \otimes \begin{pmatrix} \alpha & -\bar{\gamma} & 0 & 0 \\ q\gamma & \hat{\alpha} & 0 & 0 \\ 0 & 0 & \alpha & -\bar{\gamma} \\ 0 & 0 & -q\gamma & \hat{\alpha} \end{pmatrix}, \]

Together with \( \delta_R(\bar{x}_i) = \delta_R(x_i) \).

In the block-diagonal matrix which appears in \(39\) the second copy is given by \( SU_q(2) \) while the first one is twisted as
\[ \begin{pmatrix} \alpha & -\bar{\gamma} \\ q\gamma & \hat{\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha & \bar{\gamma} \\ -q\gamma & \hat{\alpha} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

A similar phenomenon occurs in \[3\].

**Remark 2.** It is also interesting to observe that
\[ \delta_R(v^*v) = v^*v \otimes 1 = 1 \otimes 1. \]
According to the values of the eigenvalue $q$, by evaluating
there exists a vector $\langle \phi_0 | \phi_1 \rangle$
the commutation relations $t_a$, $t_b$
of $t$
We will consider the representations which are
separable Hilbert space
be of the form
$\langle \phi_2 | \phi_1 \rangle \otimes \gamma \alpha + \langle \phi_1 | \phi_2 \rangle \otimes \bar{\alpha} \bar{\gamma}$
$= 1 \otimes (\bar{\gamma} \gamma + \bar{\alpha} \alpha) = 1 \otimes 1$

\[
\delta_R (\langle \phi_2 | \phi_2 \rangle ) = \begin{cases} 
\delta_R (q^{-2} x_i \bar{x}^i + q^{-4} x^2 x_x + x_x \bar{x}^i + \bar{x}^i x_x) \\
(q^{-4} x^2 x_x - q^{-1} x_i \bar{x}^i - \bar{x}^i x_x + q x \bar{x}^i) \otimes \alpha \bar{\gamma}
\end{cases}
\]

\[
\delta_R (\langle \phi_0 | \phi_2 \rangle ) = q^{-5} \delta_R (\bar{x}^i) \delta_R (x_x) - q^{-2} \delta_R (x_x) \delta_R (\bar{x}^i)
\]

Since $\delta_R$ defines a coaction on $S_q^4$ and so preserves its commutation relations.

4 Representations of the algebra $A(S_q^4)$

Let us now construct irreducible $*$-representations of $A(S_q^4)$ as bounded operators on a separable Hilbert space $H$. For the moment, we denote in the same way the elements of the algebra and their images as operators in the given representation. As mentioned before, since $q \rightarrow q^{-1}$ gives an isomorphic algebra, we can restrict ourselves to $|q| < 1$. We will consider the representations which are $t$-finite \[19\], i.e. such that the eigenvectors of $t$ span $H$.

Since the self-adjoint operator $t$ must be bounded due to the spherical relations, from the commutation relations $ta = q^{-2} at, \ t\bar{b} = q^{-4} b t$, it follows that the spectrum should be of the form $\lambda q^{2k}$ and $a, \bar{b}$ (resp. $\bar{a}, b$) act as rising (resp. lowering) operators on the eigenvectors of $t$. Then boundedness implies the existence of a highest weight vector, i.e. there exists a vector $|0,0\rangle$ such that

\[
|0,0\rangle = t_{00} |0,0\rangle, \quad a |0,0\rangle = 0, \quad \bar{b} |0,0\rangle = 0.
\]

By evaluating $q^4 \bar{a}a + \bar{b}b = (1 - q^{-4} t) t$ on $|0,0\rangle$ we have

\[
(1 - q^{-4} t_{00}) t_{00} = 0
\]

According to the values of the eigenvalue $t_{00}$ we have two representations.
4.1 The representation $\beta$

The first representation, that we call $\beta$, is obtained for $t_{00} = 0$. Then, $t |0,0\rangle = 0$ implies $t = 0$. Moreover, using the commutation relations (30) and (31), it follows that this representation is the trivial one

$$t = 0, \quad a = 0, \quad b = 0,$$

the representation Hilbert space being just $\mathbb{C}$; of course, $\beta(1) = 1$.

4.2 The representation $\sigma$

The second representation, that we call $\sigma$, is obtained for $t_{00} = q^4$. This is infinite dimensional. We take the set $|m,n\rangle = N_{mn} a^m b^n |0,0\rangle$ with $n, m \in \mathbb{N}$, to be an orthonormal basis of the representation Hilbert space $\mathcal{H}$, with $N_{00} = 1$ and $N_{mn} \in \mathbb{R}$ the normalizations, to be computed below.

Then

$$t |m,n\rangle = t_{mn} |m,n\rangle,$$

$$\bar{a} |m,n\rangle = a_{mn} |m+1,n\rangle,$$

$$b |m,n\rangle = b_{mn} |m,n + 1\rangle.$$

By requiring that we have a $*$-representation we have also that

$$a |m,n\rangle = a_{m-1,n} |m-1,n\rangle, \quad \bar{b} |m,n\rangle = b_{m,n-1} |m,n-1\rangle,$$

with the following recursion relations

$$a_{m,n+1} = q^{\pm 2} a_{m,n}, \quad b_{m+1,n} = q^{\pm 2} b_{m,n}, \quad b_{m,n} = q^2 a_{2n+1,m}.$$  

By explicit computation, we find

$$t_{m,n} = q^{2m+4n+4},$$

$$a_{m,n} = N_{mn} N_{m+1,n}^{-1} = (1 - q^{2m+2})^{\frac{1}{2}} q^{m+2n+1},$$

$$b_{m,n} = N_{mn} N_{m+1,n}^{-1} = (1 - q^{4n+4})^{\frac{1}{2}} q^{2(m+n+2)}.$$  

In conclusion we have the following action

$$t |m,n\rangle = q^{2m+4n+4} |m,n\rangle,$$

$$\bar{a} |m,n\rangle = (1 - q^{2m+2})^{\frac{1}{2}} q^{m+2n+1} |m+1,n\rangle,$$

$$a |m,n\rangle = (1 - q^{2m})^{\frac{1}{2}} q^{m+2n} |m-1,n\rangle,$$

$$b |m,n\rangle = (1 - q^{4n+4})^{\frac{1}{4}} q^{2(m+n+2)} |m,n+1\rangle,$$

$$\bar{b} |m,n\rangle = (1 - q^{4n})^{\frac{1}{4}} q^{2(m+n+1)} |m,n-1\rangle.$$  

It is straightforward to check that all the defining relations (30) and (31) are satisfied.
In this representation the algebra generators are all trace class:

\[
\text{Tr}(t) = q^4 \sum_m q^{2m} \sum_n q^{4n} = \frac{q^4}{(1-q^2)(1-q^4)},
\]

\[
\text{Tr}(|a|) = q \sum_{m,n} (1-q^{2m+2})^{\frac{1}{2}} q^{m+2n} = \frac{q}{1-q^2} \sum_m (1-q^{2m+2})^{\frac{1}{2}} q^m 
\leq \frac{q}{1-q^2} \sum_m q^m = \frac{q}{(1-q)(1-q^2)}, \tag{44}
\]

\[
\text{Tr}(|b|) = q^4 \sum_{m,n} (1-q^{4n+4})^{\frac{1}{2}} q^{2(n+m)} = \frac{q^4}{1-q^2} \sum_n (1-q^{4n+4})^{\frac{1}{2}} q^{2n} 
\leq \frac{q^4}{1-q^2} \sum_n q^{2n} = \frac{q^4}{(1-q^2)^2}.
\]

From the sequence of Schatten ideals in the algebra of compact operators one knows \cite{29} that the norm closure of trace class operators gives the ideal of compact operators \(\mathcal{K}\). As a consequence, the closure of \(A(S_4^q)\) is the \(C^*\)-algebra \(\mathcal{C}(S_4^q) = \mathcal{K} \oplus \mathbb{C}\).

\section{The index pairings}

The ‘defining’ self-adjoint idempotent \(p\) in \cite{29} determines a class in the \(K\)-theory of \(S_4^q\), i.e. \([p] \in K_0[\mathcal{C}(S_4^q)]\). A way to prove its nontriviality is by pairing it with a nontrivial element in the dual \(K\)-homology, i.e. with (the class of) a nontrivial Fredholm module \([\mu] \in K^0[\mathcal{C}(S_4^q)]\). In fact, in order to compute the pairing of \(K\)-theory with \(K\)-homology, it is more convenient to first compute the corresponding Chern characters in the cyclic homology \(\text{ch}^*_s(p) \in HC_*[A(S_4^q)]\) and cyclic cohomology \(\text{ch}^*_s(\mu) \in HC^*[A(S_4^q)]\) respectively, and then use the pairing between cyclic homology and cohomology \cite{10}.

Like it happens for the \(q\)-monopole \cite{14}, to compute the pairing and to prove the nontriviality of the bundle it is enough to consider \(HC_0[A(S_4^q)]\) and dually to take a suitable trace of the projector.

The Chern character of the projection \(p\) in \cite{29} has a component in degree zero \(\text{ch}_0(p) \in HC_0[A(S_4^q)]\) simply given by the matrix trace,

\[
\text{ch}_0(p) := \text{tr}(p) = 2 - q^{-4}(1-q^2)(1-q^4) \ t \in A(S_4^q). \tag{45}
\]

The higher degree parts of \(\text{ch}_s(p)\) are obtained via the periodicity operator \(S\); not needing them here we shall not dwell more upon this point and refer to \cite{10} for the relevant details.

As mentioned, the \(K\)-homology of an involutive algebra \(\mathcal{A}\) is given in terms of homotopy classes of Fredholm modules. In the present situation we are dealing with a 1-summable Fredholm module \([\mu] \in K^0[\mathcal{C}(S_4^q)]\). This is in contrast to the fact that the analogous element of \(K_0(S^4)\) for the undeformed sphere is given by a 4-summable Fredholm module, being the fundamental class of \(S^4\).

The Fredholm module \(\mu := (\mathcal{H}, \Psi, \gamma)\) is constructed as follows. The Hilbert space is \(\mathcal{H} = \mathcal{H}_\sigma \oplus \mathcal{H}_\sigma\) and the representation is \(\Psi = \sigma \oplus \beta\). Here \(\sigma\) is the representation of \(A(S_4^q)\)
introduced in (13) and \( \beta \) given in (11) is trivially extended to \( \mathcal{H}_\sigma \). The grading operator is

\[
\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The corresponding Chern character \( \text{ch}^*(\mu) \) of the class of this Fredholm module has a component in degree 0, \( \text{ch}^0(\mu) \in HC^0[A(S^q_{2n})] \). From the general construction [10], the element \( \text{ch}^0(\mu_{ev}) \) is the trace

\[
\tau^1(x) := \text{Tr} (\gamma \Psi(x)) = \text{Tr} (\sigma(x) - \beta(x)).
\]  \( \tag{46} \)

The operator \( \sigma(x) - \beta(x) \) is always trace class. Obviously \( \tau^1(1) = 0 \). The higher degree parts of \( \text{ch}^*(\mu_{ev}) \) can again be obtained via a periodicity operator.

A similar construction of the class \([\mu]\) and the corresponding Chern character were given in [21] for quantum two and three dimensional spheres.

We are ready to compute the pairing:

\[
\langle [\mu], [p] \rangle := \langle \text{ch}^0(\mu), \text{ch}_0(p) \rangle = -q^{-4}(1 - q^2)(1 - q^4)^3 \tau^1(t) \\
= -q^{-4}(1 - q^2)(1 - q^4)^4 \text{Tr}(t) = -q^{-4}(1 - q^2)(1 - q^4)^4 \text{Tr}(t) = -1.
\]  \( \tag{47} \)

This result shows also that the right \( A(S^q_4) \)-module \( p[A(S^q_4)^4] \) is not free. Indeed, any free module is represented in \( K_0[\mathcal{C}(S^4_q)] \) by the idempotent 1, and since \( \langle [\mu], [1] \rangle = 0 \), the evaluation of \([\mu]\) on any free module always gives zero.

We can extract the ‘trivial’ element in the \( K \)-homology \( K^0[\mathcal{C}(S^4_q)] \) of the quantum sphere \( S^4_q \) and use it to measure the ‘rank’ of the idempotent \( p \). This generator corresponds to the trivial generator of the \( K \)-homology \( K_0(S^4) \) of the classical sphere \( S^4 \). The latter (classical) generator is the image of the generator of the \( K \)-homology of a point by the functorial map \( K_*(\iota) : K_0(*) \to K_0(S^N) \), where \( \iota : * \to S^N \) is the inclusion of a point into the sphere. Now, the quantum sphere \( S^4_q \) has just one ‘classical point’, i.e. the 1-dimensional representation \( \beta \) constructed in Sect. 4.1. The corresponding 1-summable Fredholm module \([\varepsilon] \in K^0[\mathcal{C}(S^4_q)]\) is easily described: the Hilbert space is \( \mathbb{C} \) with representation \( \beta \); the grading operator is \( \gamma = 1 \). Then the degree 0 component \( \text{ch}^0(\varepsilon) \in HC^0[A(S^q_{2n})] \) of the corresponding Chern character is the trace given by the representation itself (since it is a homomorphism to a commutative algebra),

\[
\tau^0(x) = \beta(x),
\]  \( \tag{48} \)

and vanishes on all the generators whereas \( \tau^0(1) = 1 \).

Not surprisingly, the pairing with the class of the idempotent \( p \) is

\[
\langle [\varepsilon], [p] \rangle := \tau^0(\text{ch}_0(p)) = \beta(2) = 2.
\]  \( \tag{49} \)

### 6 Quantum principal bundle structure

Recall that if \( H \) is a Hopf algebra and \( P \) a right \( H \)-comodule algebra with multiplication \( m : P \otimes P \to P \) and coaction \( \Delta_R : P \to P \otimes H \) and \( B \subseteq P \) is the subalgebra of
coinvariants, the extension $B \subseteq P$ is $H$ Hopf-Galois if the canonical map

$$\chi : P \otimes_B P \longrightarrow P \otimes H, p' \otimes_B p \mapsto \chi(p' \otimes_B p) = p'p_{(0)} \otimes p_{(1)},$$

is bijective. As mentioned, for us a quantum principle bundle will be the same as a Hopf-Galois extension. For quantum structure groups which are cosemisimple and have bijective antipodes, as is the case for $SU_q(2)$, Th. I of [26] grants further nice properties. In particular the surjectivity of the canonical map implies bijectivity and faithfully flatness.

Moreover, an additional useful result [26] is that the map $\chi$ is surjective whenever, for any generator $h$ of $H$, the element $1 \otimes h$ is in its image. This follows from the left $P$-linearity and right $H$-colinearity of the map $\chi$. Indeed, let $h$, $k$ be two elements of $H$ and $\sum p_i \otimes p_i$, $\sum q_j \otimes q_j \in P \otimes P$ be such that $\chi(\sum p_i \otimes_B p_i) = 1 \otimes h$, $\chi(\sum q_j \otimes_B q_j) = 1 \otimes k$. Then $\chi(\sum p_i \otimes_B p_i) = 1 \otimes kh$, that is $1 \otimes kh$ is in the image of $\chi$. But, since the map $\chi$ is left $P$-linear, this implies its surjectivity.

**Definition 2.** Let $P$ be a bimodule over the ring $B$. Given any two elements $|\xi_1\rangle$ and $|\xi_2\rangle$ in the free module $\mathcal{E} = \mathbb{C}^m \otimes P$, we shall define $\langle \xi_1 \hat{\otimes}_B \xi_2 \rangle \in P \otimes_B P$ by

$$\langle \xi_1 \hat{\otimes}_B \xi_2 \rangle := \sum_{j=1}^{m} \xi_1^j \otimes_B \xi_2^j.$$  

(51)

Analogously, one can define quantities $\langle \xi_1 \hat{\otimes} \xi_2 \rangle \in P \otimes P$ with the same formula as above and tensor products taken over the ground field $\mathbb{C}$.

**Proposition 6.** The extension $A(S_q^7) \subset A(Sp_q(2))$ is a faithfully flat $A(Sp_q(1))$-Hopf-Galois extension.

**Proof.** Now $P = A(Sp_q(2))$, $H = A(Sp_q(1))$ and $B = A(S_q^7)$ and the coaction $\Delta_R$ of $H$ is given just before Prop. 2. Since $A(Sp_q(1)) \simeq A(SU_q(2))$ has a bijective antipode and is cosemisimple ([19], Chapter 11), from the general considerations given above in order to show the bijectivity of the canonical map

$$\chi : A(Sp_q(2)) \otimes_{A(S_q^7)} A(Sp_q(2)) \longrightarrow A(Sp_q(2)) \otimes A(Sp_q(1)),$$

it is enough to show that all generators $\alpha, \gamma, \bar{\alpha}, \bar{\gamma}$ of $A(Sp_q(1))$ in (11) are in its image. Let $|T^2\rangle, |T^3\rangle$ be the second and third columns of the defining matrix $T$ of $Sp_q(2)$. We shall think of them as elements of the free module $\mathbb{C}^4 \otimes A(Sp_q(2))$. Obviously, $\langle T^2 | T^3 \rangle = \delta^{ij}$. Recalling that $A(Sp_q(2))$ is both a left and right $A(S_q^7)$-module and using Def. 2 we have that

$$\chi \left( \begin{array}{c} \langle T^2 \hat{\otimes}_{A(S_q^7)} T^2 \rangle \\ \langle T^3 \hat{\otimes}_{A(S_q^7)} T^2 \rangle \\ \langle T^3 \hat{\otimes}_{A(S_q^7)} T^3 \rangle \end{array} \right) = 1 \otimes \begin{pmatrix} \alpha & -q^2 \gamma \\ \gamma & \bar{\alpha} \end{pmatrix}.$$  

Indeed,

$$\chi\langle T^2 \hat{\otimes}_{A(S_q^7)} T^2 \rangle = \overline{T^2_i} \Delta_R T^2_i = \langle T^2 | T^2 \rangle \otimes \alpha + \langle T^2 | T^3 \rangle \otimes \gamma = 1 \otimes \alpha,$$

$$\chi\langle T^3 \hat{\otimes}_{A(S_q^7)} T^2 \rangle = \overline{T^3_i} \Delta_R T^3_i = \langle T^3 | T^2 \rangle \otimes \alpha + \langle T^3 | T^3 \rangle \otimes \gamma = 1 \otimes \gamma;$$

a similar computation giving the other two generators. 

\[\square\]
**Proposition 7.** The extension $A(S_q^4) \subset A(S_q^7)$ is a faithfully flat $A(SU_q(2))$-Hopf-Galois extension.

**Proof.** Now $P = A(S_q^7)$, $H = A(SU_q(2))$ and $B = A(S_q^4)$ and the coaction $\delta_R$ of $H$ is given in Prop. 4. As already mentioned $A(SU_q(2))$ has a bijective antipode and is cosemisimple, then as before in order to show the bijectivity of the canonical map

\[ \chi : A(S_q^7) \otimes A(S_q^4) \to A(S_q^7) \otimes A(SU_q(2)) , \]

we have to show that all generators $\alpha, \gamma, \bar{\alpha}, \bar{\gamma}$ of $A(SU_q(2))$ in (32) are in its image. Recalling that $A(S_q^4)$ is both a left and right $A(S_q^4)$-module and using Def. 2, we have that

\[ \chi \left( \left\langle \phi_1 \otimes_{A(S_q^4)} \phi_1 \right\rangle \right) = 1 \otimes \left( \begin{array}{cc} \alpha & -q\bar{\gamma} \\ \gamma & \bar{\alpha} \end{array} \right) , \]

where $|\phi_1\rangle, |\phi_2\rangle$ are the two vectors introduced in eqs. (23) and (24). Indeed

\[ \chi(\left\langle \phi_1 \otimes_{A(S_q^4)} \phi_1 \right\rangle) = \chi \left( q^{-6}\bar{x} \otimes_{A(S_q^4)} x_1 + q^{-2}x_2 \otimes_{A(S_q^4)} \bar{x}^2 \right) \]
\[ + q^{-2}\bar{x}^3 \otimes_{A(S_q^4)} x_3 + x_4 \otimes_{A(S_q^4)} \bar{x}^4 \right) \]
\[ = q^{-6}\bar{x}^3 \delta_R(x_1) + q^{-2}x_2 \delta_R(\bar{x}^2) + q^{-2}\bar{x}^3 \delta_R(x_3) + x_4 \delta_R(\bar{x}^4) \]
\[ = q^{-6}\bar{x}^3 x_1 \otimes \alpha + q^{-4}\bar{x}^2 x_2 \otimes \gamma + q^{-2}x_2 \bar{x}^2 \otimes \alpha - q^{-2}x_1 \bar{x}^2 \otimes \gamma \]
\[ + q^{-2}\bar{x}^3 x_3 \otimes \alpha - q^{-1}\bar{x}^3 x_4 \otimes \gamma + x_4 \bar{x}^4 \otimes \alpha + x_4 \bar{x}^3 \otimes \gamma \]
\[ = \langle \phi_1|\phi_1 \rangle \otimes \alpha = 1 \otimes \alpha , \]

with similar computations for the other generators. \qed

It was proven in [4] that the bundle constructed in [3] is a coalgebra Galois extension [4, 3]. The fact that our bundle $A(S_q^4) \subset A(S_q^7)$ is Hopf-Galois shows also that these two bundles cannot be the same.

On our extension $A(S_q^4) \subset A(S_q^7)$ there is a strong connection. Indeed a $H$-Hopf-Galois extension $B \subseteq P$ for which $H$ is cosemisimple and has a bijective antipode is also equivariantly projective, that is there exists a left $B$-linear right $H$-colinear splitting $s : P \to B \otimes P$ of the multiplication map $m : B \otimes P \to P$, $m \circ s = id_P$ [27]. Such a map characterizes the so called strong connection. Constructing a strong connection is an alternative way to prove that one has a Hopf Galois extension [12, 13].

In particular, if $H$ has an invertible antipode $S$, an equivalent description of a strong connection can be given in terms of a map $\ell : H \to P \otimes P$ satisfying a list of conditions
We denote by $\Delta$ the coproduct on $H$ with Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$, by $\delta : P \to P \otimes H$ the right-comodule structure on $P$ with notation $\delta p = p_{(0)} \otimes p_{(1)}$, and $\delta_l : P \to H \otimes P$ is the induced left $H$-comodule structure of $P$ defined by $\delta_l(p) = S^{-1}(p_{(1)}) \otimes p_{(0)}$. Then, for the map $\ell$ one requires that $\ell(1) = 1 \otimes 1$ and that for all $h \in H$,

$$\chi(\ell(h)) = 1 \otimes h,$$
$$\ell(h_{(1)}) \otimes h_{(2)} = (id \otimes \delta) \circ \ell(h),$$
$$h_{(1)} \otimes \ell(h_{(2)}) = (\delta_l \otimes id) \circ \ell(h).$$

The splitting $s$ of the multiplication map is then given by

$$s : P \to B \otimes P, \quad p \mapsto p_{(0)} \ell(p_{(1)}).$$

Now, if $g, h \in H$ are such that $\ell(g) = g^1 \otimes g^2$ and $\ell(h) = h^1 \otimes h^2$ satisfy condition \[52\], so does $\ell(gh)$ defined by

$$\ell(gh) := h^1 g^1 \otimes g^2 h^2.$$ 

If $H$ has a PBW basis \[15\], this fact can be used to iteratively construct $\ell$ once one knows its value on the generators of $H$.

For $H = A(SU_q(2))$, with generators, $\alpha, \gamma, \bar{\alpha}$ and $\bar{\gamma}$, the PBW basis is given by $\alpha^{k_1} \gamma^{l_1} \bar{\alpha}^{m_1}$, with $k, l, m \in \{0, 1, 2, \ldots\}$ and $\gamma^{k_2} \bar{\gamma}^{l_2} \bar{\alpha}^{m_2}$, with $k, l \in \{0, 1, 2, \ldots\}$ and $m \in \{1, 2, \ldots\}$ \[30\]. Then, for our extension $A(S^+_q) \subset A(S^+_q)$ the map $\ell$ can be constructed as follows. Firstly, we put $\ell(1) = 1 \otimes 1$. Then, on the generators we set

$$\ell(\alpha) := \langle \phi_1 \otimes \phi_1 \rangle, \quad \ell(\bar{\alpha}) := \langle \phi_2 \otimes \phi_2 \rangle,$$
$$\ell(\gamma) := \langle \phi_2 \otimes \phi_1 \rangle, \quad \ell(\bar{\gamma}) := -q^{-1} \langle \phi_1 \otimes \phi_2 \rangle.$$

These expressions for $\ell$ satisfy all the properties \[52\]: Firstly, $\chi(\ell(\alpha)) = 1 \otimes \alpha$ follows from the proof of Prop. \[7\]. Then,

$$\delta_l \circ \ell(\alpha) = q^{-6} x^1 \otimes \delta x_1 + q^{-2} x_2 \otimes \delta \bar{x}^2 + q^{-2} \bar{x}^3 \otimes \delta x_3 + x_4 \otimes \delta \bar{x}^4$$
$$= \langle \phi_1 \otimes \phi_1 \rangle \otimes \alpha + \langle \phi_1 \otimes \phi_2 \rangle \otimes \gamma$$
$$= \ell(\alpha) \otimes \alpha - q\ell(\bar{\gamma}) \otimes \gamma = \ell(\alpha_{(1)}) \otimes \alpha_{(2)}.$$

Moreover

$$\delta_l \circ \ell(\alpha) = q^{-6} (\alpha \otimes \bar{x}^1 - q^2 \bar{\gamma} \otimes \bar{x}^2) \otimes x_1 + q^{-2} (q \bar{\gamma} \otimes x_1 + \alpha \otimes x_2) \otimes \bar{x}^2$$
$$+ q^{-2} (q^2 \bar{\gamma} \otimes \bar{x}^3 + \alpha \otimes \bar{x}^4) \otimes x_3 + (-q \bar{\gamma} \otimes x_3 + \alpha \otimes x_4) \otimes \bar{x}^4$$
$$= \alpha \otimes \langle \phi_1 \otimes \phi_1 \rangle - q \bar{\gamma} \otimes \langle \phi_2 \otimes \phi_1 \rangle$$
$$= \alpha \otimes \ell(\alpha) - q \bar{\gamma} \otimes \ell(\gamma) = \alpha_{(1)} \otimes \ell(\alpha_{(2)}).$$

Similar computations can be carried for $\gamma, \bar{\alpha}$ and $\bar{\gamma}$.

That an iterative procedure constructed by using \[53\] on the PBW basis leads to a well defined $\ell$ on the whole of $H = A(SU_q(2))$ will be proven in the forthcoming paper \[23\] where other elaborations coming from the existence of a strong connection will be presented as well.
6.1 The associated bundle and the coequivariant maps

We now give some elements of the theory of associated quantum vector bundles \( \mathcal{E} \) (see also [11]). Let \( B \subset P \) be a \( H \)-Galois extension with \( \Delta_R \) the coaction of \( H \) on \( P \). Let \( \rho : V \to H \otimes V \) be a corepresentation of \( H \) with \( V \) a finite dimensional vector space. A coequivariant map is an element \( \varphi \) in \( P \otimes V \) with the property that

\[
(\Delta_R \otimes \text{id}) \varphi = (\text{id} \otimes (S \otimes \text{id} \circ \rho)) \varphi ;
\]

where \( S \) is the antipode of \( H \). The collection \( \Gamma_{\rho}(P,V) \) of coequivariant maps is a right and left \( B \)-module.

The algebraic analogue of bundle nontriviality is translated in the fact that the Hopf-Galois extension \( B \subset P \) is not cleft. On the other hand, it is known that for a cleft Hopf-Galois extension, the module of coequivariant maps \( \Gamma_{\rho}(P,V) \) is isomorphic to the free module of coinvariant maps \( \Gamma_0(P,V) = B \otimes V \) \([8, 14]\).

For our \( A(SU_q(2)) \)-Hopf-Galois extension \( A(S^4_q) \subset A(S^7_q) \), let \( \rho_1 : \mathbb{C}^2 \to \mathbb{C}^2 \otimes A(SU_q(2)) \) be the fundamental corepresentation of \( A(SU_q(2)) \) with \( \Gamma_1(A(S^4_q), \mathbb{C}^2) \) the right \( A(S^4_q) \)-module of corresponding coequivariant maps.

Now, the projection \( p \) in (29) determines a quantum vector bundle over \( S^4_q \) whose module of section is \( p[A(S^4_q)] \), which is clearly a right \( A(S^4_q) \)-module. The following proposition in straightforward

**Proposition 8.** The modules \( \mathcal{E} := p[A(S^4_q)] \) and \( \Gamma_1(A(S^7_q), \mathbb{C}^2) \) are isomorphic as right \( A(S^4_q) \)-modules.

**Proof.** Remember that \( p = vv^* \) with \( v \) in (25). The element \( p(F) \in \mathcal{E} \), with \( F = (f_1, f_2, f_3, f_4)^t \), corresponds to the equivariant map \( v^*F \in \Gamma_1(A(S^7_q), \mathbb{C}^2) \). \( \square \)

We expect that a similar construction extends to every irreducible corepresentation of \( A(SU_q(2)) \) by means of suitable projections giving the corresponding associated bundles \([23]\).

**Proposition 9.** The Hopf-Galois extension \( A(S^4_q) \subset A(S^7_q) \) is not cleft.

**Proof.** As mentioned, the cleftness of the extension does imply that all modules of coequivariant maps are free. On the other hand, the nontriviality of the pairing (47) between the defining projection \( p \) in (29) and the Fredholm module \( \mu \) constructed in Sect. 5 also shows that the module \( p[A(S^4_q)] \simeq \Gamma_{\rho}(A(S^7_q), \mathbb{C}^2) \) is not free. \( \square \)

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A The classical Hopf fibration $S^7 \to S^4$

We shall review the classical construction of the basic anti-instanton bundle over the four dimensional sphere $S^4$ in a 'noncommutative parlance' following [16]. This has been useful in the main text for our construction of the quantum deformation of the Hopf bundle.

We write the generic element of the group $SU(2)$ as

$$w = \begin{pmatrix} w_1 & w_2 \\ -\bar{w}_2 & \bar{w}_1 \end{pmatrix}.$$  \hfill (55)

The $SU(2)$ principal fibration $SU(2) \to S^7 \to S^4$ over the sphere $S^4$ is explicitly realized as follows. The total space is $S^7 = \{ z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid \sum_{i=1}^4 |z_i|^2 = 1 \}$, with right diagonal action

$$S^7 \times SU(2) \to S^7, \quad z \cdot w := (z_1, z_2, z_3, z_4) \begin{pmatrix} w_1 & w_2 & 0 & 0 \\ -\bar{w}_2 & \bar{w}_1 & 0 & 0 \\ 0 & 0 & w_1 & w_2 \\ 0 & 0 & -\bar{w}_2 & \bar{w}_1 \end{pmatrix}. \hfill (56)$$

The bundle projection $\pi : S^7 \to S^4$ is just the Hopf projection and it can be explicitly given as $\pi(z_1, z_2, z_3, z_4) := (x, \alpha, \beta)$ with

$$x = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = -1 + 2(|z_1|^2 + |z_2|^2) = 1 - 2(|z_3|^2 + |z_4|^2),$$

$$\alpha = 2(z_1\bar{z}_3 + z_2\bar{z}_4), \quad \beta = 2(-z_1z_4 + z_2z_3). \hfill (57)$$

One checks that $|\alpha|^2 + |\beta|^2 + x^2 = (\sum_{i=1}^4 |z_i|^2)^2 = 1$.

We need the rank 2 complex vector bundle $E$ associated with the defining left representation $\rho$ of $SU(2)$ on $\mathbb{C}^2$. The quickest way to get this is to identify $S^7$ with the unit sphere in the 2-dimensional quaternionic (right) $\mathbb{H}$-module $\mathbb{H}^2$ and $S^4$ with the projective line $\mathbb{P}^1(\mathbb{H})$, i.e. the set of equivalence classes $(w_1, w_2)^t \simeq (w_1, w_2)^t\lambda$ with $(w_1, w_2) \in S^7$ and $\lambda \in Sp(1) \simeq SU(2)$. Identifying $\mathbb{H} \simeq \mathbb{C}^2$, the vector $(w_1, w_2)^t \in S^7$ reads

$$v = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \\ z_3 & z_4 \\ -\bar{z}_4 & \bar{z}_3 \end{pmatrix}. \hfill (58)$$

This is actually a map from $S^7$ to the Stieffel variety of frames for $E$. In particular, notice that the two vectors $|\psi_1\rangle, |\psi_2\rangle$ given by the columns of $v$ are orthonormal, indeed $v^*v = I_2$. As a consequence, $p := vv^* = |\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2|$ is a self-adjoint idempotent (a projector), $p^2 = p$, $p^* = p$. Of course $p$ is $SU(2)$ invariant and hence its entries are functions on $S^4$ rather than $S^7$. An explicit computation yields

$$p = \frac{1}{2} \begin{pmatrix} 1 + x & 0 & \alpha & \beta \\ 0 & 1 + x & -\bar{\beta} & \bar{\alpha} \\ \bar{\alpha} & -\beta & 1 - x & 0 \\ \beta & \alpha & 0 & 1 - x \end{pmatrix}, \hfill (59)$$

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where \((x, \alpha, \beta)\) are the coordinates on \(S^4\). Then \(p \in \text{Mat}_4(C^\infty(S^4, \mathbb{C}))\) is of rank 2 by construction.

The matrix \(v\) in \(58\) is a particular example of the matrices \(v\) given in \(1\), for \(n = 1, k = 1, C_0 = 0, C_1 = 1, D_0 = 1, D_1 = 0\). This gives the (anti-)instanton of charge \(-1\) centered at the origin and with unit scale. The only difference is that here we identify \(C^4\) with \(\mathbb{H}^2\) as a right \(\mathbb{H}\)-module. This notwithstanding, the projections constructed in the two formalisms actually coincide. Finally recall that, as mentioned already, the classical limit of our quantum projection \((29)\) is conjugate to \((39)\).

The canonical connection associated with the projector,

\[
\nabla := p \circ d : \Gamma^\infty(S^4, E) \rightarrow \Gamma^\infty(S^4, E) \otimes_{C^\infty(S^4, \mathbb{C})} \Omega^1(S^4, \mathbb{C}),
\]

corresponds to a Lie-algebra valued \((su(2))\) 1-form \(A\) on \(S^7\) whose matrix components are given by

\[
A_{ij} = \langle \psi_i | d\psi_j \rangle, \quad i, j = 1, 2.
\]

This connection can be used to compute the Chern character of the bundle. Out of the curvature of the connection \(\nabla^2 = p(dp)^2\) one has the Chern 2-form and 4-form given respectively by

\[
C_1(p) := -\frac{1}{2\pi i} \text{tr}(p(dp)^2),
\]

\[
C_2(p) := -\frac{1}{8\pi^2} \left[ \text{tr}(p(dp)^4) - C_1(p)C_1(p) \right],
\]

with the trace \(\text{tr}\) just an ordinary matrix trace. It turns out that the 2-form \(p(dp)^2\) has vanishing trace so that \(C_1(p) = 0\). As for the second Chern class, a straightforward calculation shows that,

\[
C_2(p) = -\frac{1}{32\pi^2} [(x_0 dx_4 - x_4 dx_0)(d\xi)^3 + 3d x_0 dx_4 \xi (d\xi)^2]
\]

\[
= -\frac{3}{8\pi^2} [x_0 dx_1 dx_2 dx_3 dx_4 + \text{cyclic permutations}]
\]

\[
= -\frac{3}{8\pi^2} d(\text{vol}(S^4)).
\]

The second Chern number is then given by

\[
c_2(p) = \int_{S^4} C_2(p) = -\frac{3}{8\pi^2} \int_{S^4} d(\text{vol}(S^4)) = -\frac{3}{8\pi^2} \frac{8}{3} \pi^2 = -1.
\]

The connection \(A\) in \(61\) is (anti-)self-dual, i.e. its curvature \(F_A := dA + A \wedge A\) satisfies (anti-)self-duality equations, \(\ast_H F_A = -F_A\), with \(\ast_H\) the Hodge map of the canonical (round) metric on the sphere \(S^4\). It is indeed the basic Yang-Mills anti-instanton found in \(2\).
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