Combinatorial approach to Mathieu and Lamé equations

Wei He

Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China

Abstract

Based on some recent progress on a relation between four dimensional super Yang-Mills gauge theory and quantum integrable system, we give a combinatorial method to derive the asymptotic expansions for eigenvalues of the Mathieu and Lamé equations. The WKB method is also used to extend the expansions in one asymptotic region to other asymptotic regions.
1 Introduction

Nonrelativistic quantum mechanics is naturally related to the second order differential equations. Exactly solvable quantum mechanics models play a very fundamental role in demonstrating the basic concepts and methods of quantum theory. In the textbook fashion, it is often the classical results of these differential equations, i.e. the Shrödinger equations, give complete solutions to the corresponding quantum mechanics problem. However, in this paper we will present two examples where things go in the opposite direction: we can first solve the quantum mechanics problems using a combinatorial method, without direct referring to the associated differential equations, and then we can say something new about these equations based on the combinatorial results. The two ordinary differential equations are the Mathieu equation and Lamé equation. They are respectively associated to two mathematicians who studied them in the 19th century\cite{1,2}. They arose in classical mechanics problems then, and now also arise in simple quantum models. The two equations and their solutions are related to each other: the Mathieu equation is a particular limit of the Lamé equation.

The method we use to study the two equations are basically combinatorial, namely we will relate the equations to some Young diagrams, and give counting data to each Young diagram. With these data we can obtain the eigenvalues following a well defined algorithm. This algorithm is rooted in quantum field theory, some beautiful results about four dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills(SYM) gauge theory that are developed since 1990s. More specifically, we are referring to:

(1). Seiberg-Witten theory of $\mathcal{N} = 2$ SYM\cite{19,20}, and instanton counting\cite{21},

(2). The correspondence between $\mathcal{N} = 2$ SYM and classical\cite{30,31,32,33,34,35}/quantum\cite{36,37} integrable models.

This particular class of quantum gauge theory is solved by utilizing its symmetry, duality and integrability properties. The intriguing relation between the quantum gauge theory and integrable system indicates the two theories share some very basic physical properties despite they look very different. In this paper, using this relation, we obtain information about the mechanical system from the dual gauge theory.

The instanton counting deals with $\mathcal{N} = 2$ quantum gauge theories of general type, with various gauge groups and matter multiplets. The gauge theories relevant to the Mathieu and Lamé equations are:

(i) The Mathieu equation is related to $\mathcal{N} = 2$ pure gauge theory with SU(2) gauge group.

(ii) The Lamé equation is related to $\mathcal{N} = 2$ theory with SU(2) gauge group coupled to an adjoint matter multiplet. This is the mass deformed maximally supersymmetric Yang-Mills theory($\mathcal{N} = 4$ SYM), denoted as $\mathcal{N} = 2^*$ theory.
More general gauge theories are related to some higher order differential equations, or their algebraic form—the Baxter equation.

The method used to solve the quantum gauge theory/integrable model is in vogue in quantum field theory, although it is as yet not fully understood. It is hard to understand this method solely in the context of quantum mechanics/differential equation, because the quantum mechanics Shrödinger equation does not give us any hint about the hidden symmetry and duality properties which are crucial for the solvability of gauge theory. Further clarification of the meaning of the combinatorial nature of these differential equations, and possible extension to other equations, will be an interesting problem.

The eigenvalue expansions for the Mathieu equation have been well studied in the literatures, our combinatorial approach gives exactly the same results and provides some new perspective on this equation. For the Lamé equation, its expansions are more complicated, some analytical results are only obtained recently. We will give an asymptotic expansion for the eigenvalue, through the combinatorial method, which coincide with results obtained by different method. Some classical literatures containing results of the Mathieu and the Lamé equations are \cite{3, 4, 5, 6, 7, 8, 9, 10, 11}, several useful original papers are \cite{12, 13, 14, 15}.

2 Differential equations and gauge theory

The equations we discuss are the 2-particle Schrödinger equations of two kinds of famous integrable models: the periodic Toda chain and the elliptic Calogero-Moser model. The relation between these classical integrable models and supersymmetric gauge theory is a well known result\cite{30, 31, 32, 33, 34, 35}. The quantum gauge theory method can be used to obtain some results of the two differential equations, this is based on the following fact: According to a recent study due to Nekrasov and Shatashvili\cite{36}, the vacua of $\Omega$ deformed supersymmetric gauge theories are in one to one correspondence with the spectrum of certain quantum integrable systems. The work of Seiberg and Witten\cite{19, 20} establishes the basic notion of four dimensional $\mathcal{N} = 2$ SYM, especially the electric-magnetic duality in the moduli space. The instanton counting program\cite{21} provides a combinatorial approach to the theory. Among a wide class of $\mathcal{N} = 2$ theories, two theories are simple but typical: the SU(2) pure gauge theory\cite{19} and the $\mathcal{N} = 2^*$ theory\cite{20}. The two theories are exactly related to the Mathieu and Lamé equations, respectively, through the gauge theory/quantum integrable system correspondence.

The gauge theories we study carry the simplest nonAbelian gauge group SU(2), the corresponding quantum mechanical models are simply two body systems. Remember that for an integrable many body system, the building block of its scattering process is the two
body scattering data due to the factorization of S-matrix. Therefore, the two body problem plays a basic role. After reducing the motion of the center of mass, it reduces the problem of a single particle in external potential.

The gauge /integrable correspondence indicates there is a well defined algorithm to evaluate the eigenvalues of the Mathieu and Lamé equations in a particular asymptotic expansion region, up to arbitrary higher order of the expansion parameter, by counting certain Young diagrams with weights. With this information we can extend the eigenvalues to other asymptotic expansion regions by utilizing their duality relation.

### 2.1 Mathieu equation

The Mathieu differential equation is:

\[
\frac{d^2 \Psi}{dz^2} + (\lambda - 2h \cos 2z) \Psi = 0.
\]  

(1)

The related modified Mathieu equation is obtained by \( z \to iz \):

\[
\frac{d^2 \Psi}{dz^2} - (\lambda - 2h \cosh 2z) \Psi = 0.
\]  

(2)

The (modified) Mathieu equation is useful in various mathematics and physics problems. That includes the separation of variables for the wave equation in the elliptical coordinates, a quantum particle moving in periodic potential. It also appears in problems such as wave scattering by D-brane of string theory[17], reheating process in some inflation models[18]. In the problem we concern, it is the Shrödinger equation of the two body Toda system.

The potential \( \cos 2z \) is periodic along the real axes, therefore the Mathieu equation is a Floquet(or Floquet-Bloch) differential equation. A Floquet function with period \( \pi \) should shift a phase as

\[
\Psi(z + \pi) = e^{i\nu \pi} \Psi(z),
\]  

(3)

the quantity \( \nu = \nu(\lambda, h) \) is the Floquet index. For periodic integrable models, the Floquet indexes of wave function are quasimomenta of quasiparticles. Our focus in this paper is the eigenvalue \( \lambda \), as a function of \( \nu, h \).

The eigenvalue can not be written in elementary functions, only asymptotic expansions are obtained when a small expansion parameter is available. And for the Mathieu equation there are three asymptotic expansion regions for \( \lambda \), they are located at:

\[
\begin{align*}
\lambda & \sim \nu^2 + h^2 \text{-corrections}, \quad h \ll 1, \nu \gg 1 \\
\lambda & \sim \pm 2h + (\frac{1}{\sqrt{\nu h}}) \text{-corrections}, \quad h \gg 1.
\end{align*}
\]  

(4)
These are well known results about the Mathieu equation, they can be found in handbooks such as [8, 9, 11]. In its relation to gauge theory, each region corresponds to a weak coupling description of the gauge theory, they are related by electric-magnetic duality.

### 2.2 Lamé equation

The Lamé differential equation can be written in several forms. The Jacobi form is:

\[
\frac{d^2 \Phi}{d \kappa^2} - [A + n(n-1)k^2 \text{sn}^2 \kappa] \Phi = 0, \tag{5}
\]

where \( \text{sn} \kappa = \text{sn}(\kappa|k^2) \) is the Jacobi \text{sn}-function. The \( n(n-1) \) is in accordance with usual literature, but we will not discuss whether \( n \) is an integer or not which is crucial for the classification of the solution. The Weierstrass form is:

\[
\frac{d^2 \Phi}{dz^2} - [B + n(n-1)\wp(z)] \Phi = 0, \tag{6}
\]

where \( \wp(z) \) is the Weierstrass elliptic function. The Lamé equation is obtained from separation of variables for the Laplace equation in the ellipsoidal coordinates. For our interest, it is also the Shrödinger equation of two body elliptic Calogero-Moser system.

The two equations are related by a change of variables [7]. If we change the variables as

\[
\frac{\kappa - i K'}{\sqrt{e_1 - e_2}} = z, \tag{7}
\]

with \( K' = K(k') \) the complete elliptic integrals of the first kind and \( k' = \sqrt{1 - k^2} \) the complementary module. Then Eq. (6) changes to Eq. (5). We have the relation

\[
B = (e_1 - e_2)A - e_2n(n-1). \tag{8}
\]

with

\[
k^2 = \frac{e_3 - e_2}{e_1 - e_2}, \tag{9}
\]

the elliptic module. Therefore

\[
A = \frac{B}{e_1 - e_2} - \frac{1 + k^2}{3}n(n-1). \tag{10}
\]

The two equations are equivalent provided the change of variables is smooth. However, the change is actually singular in the limit \( k \to 0 \), as in this limit \( K' \to \infty \). Therefore, the two equations may reduce to different equations in limits involving \( k \to 0 \).

The elliptic functions \( \text{sn} \kappa \) and \( \wp(z) \) are double periodic. The periods of \( \text{sn} \kappa \) are \( 4K \) and \( 2iK' \); the periods of \( \wp(z) \) are \( 2\omega_1 \) and \( 2\omega_2 \). The periods are related by

\[
\omega_1 = \frac{K}{\sqrt{e_1 - e_2}}, \quad \omega_2 = \frac{iK'}{\sqrt{e_1 - e_2}}. \tag{11}
\]
However, note

\[ \text{sn}(x + 2K) = -\text{sn}(x), \quad \text{sn}(x + iK') = \frac{1}{k\text{sn}(x)} \]  

(12)

therefore, the periods of the potential \( \text{sn}^2 x \) are \( 2K \) and \( 2iK' \).

Thus, the Lamé equation also falls into the equations of Floquet type. However, its Floquet analysis is not very complete. For example, when the coordinate \( x \) shifts a period \( 2K \), or coordinate \( z \) shifts a period \( 2\omega_1 \), the phase shift of the function \( \Phi \) is

\[ \Phi(x + 2K) = e^{i2K\mu}\Phi(x), \]  

(13)
or

\[ \Phi(z + 2\omega_1) = e^{i2\omega_1\nu}\Phi(z), \]  

(14)

Where \( \mu \) and \( \nu \) are the Floquet indexes for equation (5) and (6). The phase shifts should be the same, then we have

\[ \frac{\nu}{\mu} = \frac{K}{\omega_1} = \sqrt{e_1 - e_2}. \]  

(15)

We use different letters \( \mu \) and \( \nu \) to denote the index for the Lamé equation, depending its appearance in \( A \) or \( B \). In the limit \( k \to 0 \), \( \mu \) and \( \nu \) coincide. This limit is involved when we reduce the Lamé equation to the Mathieu equation, and indeed their eigenvalue expansions share some common features, see formulae (34) and (57), (37) and (82).

The main goal of this paper is to derive the eigenvalue of Lamé equation, \( A \) (or \( B \)), as a function of \( \mu \) (or \( \nu \)) and \( n, k \). The Lamé eigenvalue also has three asymptotic expansion regions, they are located at:

\[ -(A + \frac{\kappa^2}{2}) \sim \mu^2 + k^4\text{-corrections}, \quad \kappa^2 = n(n - 1)k^2 \ll 1, \]

\[ -(A + \frac{\kappa^2}{2}) \sim -\frac{\kappa^2}{2} + \frac{1}{\kappa}\text{-corrections}, \quad \kappa^2 = n(n - 1)k^2 \gg 1. \]  

(16)

The second expansion has been given by E. L. Ince, and later by H. J. W. Müller, see the book[11]. The first expansion was obtained recently by E. Langmann as a special case in his algorithm[15].

Often, quantities \( z, \lambda, h \) and \( x, A, n \) (and \( z, B, n \)) are taken to be real, however, in this paper we extend them to the complex domain. The reason to do so is that the \( N = 2 \) gauge theories are related to algebraic integrable systems whose spectral curves are defined over the field of complex numbers. Then the abelian variety of the curve, which governs the linearized motion of corresponding mechanical system, is the Jacobian variety which carries a complex structure, i.e., a complex tori. When we compare the magnitude of these parameters with some numbers, we mean taking their absolute values or restoring their real values.
2.3 QFT method of deriving the eigenvalue

If there is a small expansion parameter, then the WKB method is a useful way to derive the eigenvalue as an asymptotic expansion in the region where the expansion is valid. Although the WKB method is a well studied technique in differential equation theory, we note that in the literatures about the Mathieu and Lamé equations the WKB analysis is not used to derive the eigenvalue. Recently, the higher order WKB analysis was carried out in [38] due to the interest about the \( \mathcal{N} = 2 \)/quantum integrable correspondence. The new ingredient is that the WKB analysis is carried out in the complex plane, the phase is a contour integral along the closed path, therefore a trick of exchanging order of integration and differentiation can be applied to obtain higher order expansions. In its relation to gauge theory, the integral contours are homology cycles of the Seiberg-Witten curve. Following this development we derive all the asymptotic expansions of the Mathieu eigenvalue in [40, 41]. However, this method needs some higher order differential operators that become very hard to determine when the expansion order grows, even though their general structure is conjectured and tested up to the 4th order.

The standard way to solve an integrable system is to derive the eigenvalue of Hamiltonians as functions of quasimomenta, and the Bethe equations that constraint quasimomenta. The gauge theory/integrable system correspondence, proposed in [36, 37] by Nekrasov and Shatashvili, provides a new way to the problem. According to their work, there is a precise correspondence between certain quantum integrable models and vacuum sector of gauge theory in the Coulomb branch. The complex scalar condensations \( a_i \ (i = 1, 2 \cdots \text{rank} G) \) in the gauge theory are identified with the quasimomenta of integrable model, and the vacuum expectation value of gauge invariant operators \( \text{Tr} \phi^k \) are identified with eigenvalues of \( H_k \). The prepotential of gauge theory is identified with Yang’s potential that gives the Bethe equation. For \( G = SU(2) \) theory, the only nontrivial \( \text{Tr} \phi^k \) is for \( k = 2 \), the corresponding Hamiltonian \( H_2 \) leads to the Schrödinger equations. Therefore, for our purpose we need the following relation in the gauge theory context,

\[
< \text{Tr} \phi^2 > = u(a_i, q_{in}, \epsilon_1, \epsilon_2).
\] (17)

where \( q_{in} \) is the instanton expansion parameter, related to some scale of the integrable model, \( \epsilon_1, \epsilon_2 \) are the equivalent rotation parameters. A further limit \( \epsilon_2 \to 0 \) is needed, and then \( \epsilon_1 \) plays the role of Plank constant.

We noted [42] that the Matone’s relation of \( \mathcal{N} = 2 \) SYM gives a very direct way to derive \( u(a_i, q_{in}, \epsilon_1, \epsilon_2) \) from gauge theory partition function. It states a relation between the prepotential \( \mathcal{F} \) and \( u \) [28, 29]:

\[
2u = q_{in} \frac{\partial}{\partial q_{in}} \mathcal{F}.
\] (18)
According to the instanton counting program, the prepotential of $\mathcal{N} = 2$ theory can be obtained through a combinatorial method. We will show in this paper how to utilize this relation to derive the eigenvalues. To fully understand the combinatorial approach, the reader should understand some related quantum field theory background, especially the instanton counting of $\mathcal{N} = 2$ gauge theory\cite{21,22}, based on techniques developed in \cite{23,24,25}.

So first we need the partition function of $\mathcal{N} = 2$ gauge theory in the $\Omega$ background. The most nontrivial part is the instanton contribution. Our problem is concerned with gauge theories with SU(2) gauge group, the basic combinatorial objects are the Young diagram doublet ($Y_1,Y_2$). The total number of boxes of the Young diagram doublet is the instanton number, $|Y_1| + |Y_2| = k \in \mathbb{Z}^+$. For each Young diagram $Y_\alpha, \alpha = 1, 2$, denote the coordinates of a box $s$ as $(i_\alpha, j_\alpha)$, and define its leg-length and arm-length in the diagram $Y_\beta, \beta = 1, 2$, as

$$h_\beta(s) = v_{\beta,i_\alpha} - j_\alpha, \quad v_\beta(s) = \bar{v}_{\beta,j_\alpha} - i_\alpha,$$

where $v_{\beta,i_\alpha}$ is the number of boxes in the $i_\alpha$-th row of $Y_\beta$, while $\bar{v}_{\beta,j_\alpha}$ is the number of boxes in the $j_\alpha$-th column of $Y_\beta$. Indexes $\alpha$ and $\beta$ may be not the same, when this happens, some boxes $s(i_\alpha, j_\alpha)$ are outside of the diagram $Y_\beta$, $h_\beta(s)$ and $v_\beta(s)$ can be negative numbers. See Fig.(1)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A Young diagram $Y_\alpha$, with $s$ inside it and $s'$ outside.}
\end{figure}

The partition function of the four dimensional $\mathcal{N} = 2$ SYM theory consists the perturbative(classical and 1-loop) part and the instanton part,

$$Z = Z^{pert} \cdot Z^{inst} = Z^{pert} \cdot \sum_{k=0}^{\infty} Z_k q_{in}^k,$$

where $q_{in}$ is the instanton expansion parameter.

The instanton counting program gives an elegant way to evaluate the instanton partition
function. For example, for the SU(2) pure SYM, we have $Z_0 = 1$ and for $k \geq 1$,\[26, 27\]

$$Z_k = \sum_{s' \in \mathcal{Y}_\alpha} \prod_{s' \in \mathcal{Y}_\beta} \frac{1}{E_{\alpha\beta}(s)(\epsilon_+ - E_{\alpha\beta}(s'))} \quad \text{with}$$

$$E_{\alpha\beta}(s) = a_{\alpha\beta} - h_{\beta}(s)\epsilon_1 + (v_{\alpha}(s) + 1)\epsilon_2. \quad (22)$$

defined by the Young diagram doublets. Here $\epsilon_+ = \epsilon_1 + \epsilon_2$, $a_{\alpha\beta} = a_{\alpha} - a_{\beta}$. For SU(2), we have $a_1 = -a_2 = a$. Only some basic doublets are needed, the contribution of $(Y_{\beta}, Y_{\alpha})$ is obtained from the contribution of $(Y_{\alpha}, Y_{\beta})$ by change $a_{\alpha} \leftrightarrow a_{\beta}$, the contribution of transpose doublet $(Y^t_{\alpha}, Y^t_{\beta})$ is obtained from the contribution of $(Y_{\alpha}, Y_{\beta})$ by change $\epsilon_1 \leftrightarrow \epsilon_2$.

The prepotential $F$ of the gauge theory is defined by

$$F = \exp(\frac{\mathcal{F}}{\epsilon_1\epsilon_2}) = \exp(\frac{1}{\epsilon_1\epsilon_2}(F_{\text{pert}} + \sum_{k=1} F_k q_{in}^k)), \quad (23)$$

The instanton contributions to the prepotential $F_k, k = 1, 2, \cdots$ can be obtained through $Z_k$, the first few instanton contributions are

$$F_1 = -\epsilon_1\epsilon_2 Z_1,$$

$$F_2 = -\epsilon_1\epsilon_2 (Z_2 - \frac{1}{2}Z_1^2),$$

$$F_3 = -\epsilon_1\epsilon_2 (Z_3 - Z_1Z_2 + \frac{1}{3}Z_1^3),$$

$$F_4 = -\epsilon_1\epsilon_2 (Z_4 - Z_1Z_3 + \frac{1}{2}Z_2^2 + Z_1^2Z_2 - \frac{1}{4}Z_1^4),$$

$$F_5 = -\epsilon_1\epsilon_2 (Z_5 - Z_1Z_4 - Z_2Z_3 + Z_1^2Z_3 + Z_1Z_2^2 - Z_1^3Z_2 + \frac{1}{5}Z_1^5), \quad (24)$$

The last ingredient we need is the Matone’s relation (18). As $u$ also receives instanton corrections, $u = \sum_{k=0} u_k q_{in}^k$, this leads to

$$2u_0 = \frac{\partial}{\partial \ln q_{in}} F_{\text{pert}}, \quad 2u_k = kF_k, \quad k \geq 1. \quad (25)$$

As a consequence, if we choose a “good” expansion parameter, denoted as $q$, then the $k$-th coefficients of $q$-expansion of the eigenvalue is proportional to the $k$-th instanton contribution of the SU(2) gauge theory prepotential. The criterion for “good” will be clear when we deal
with the Lamé equation. This relation is a quantum field theory result, it is fairly novel from
the point view of differential equation.

According to the proposal of [36], the quantization of corresponding integrable system
is achieved when \( \epsilon_2 \to 0 \). Therefore, after we obtain the prepotential, we take the limit
\( \epsilon_1 \neq 0, \epsilon_2 \to 0 \).

3 Review results of Mathieu equation

As the Mathieu and Lamé equations are closely related to each other, the procedure of deriving
their eigenvalues is largely parallel. The results in this section have already been given in earlier works, see [26, 38, 40, 41, 42]. Here we give the results of the Mathieu eigenvalue as an
educational example to illustrate the basic method and logic of our combinatorial approach.
After understanding this, it is straightforward to apply this method to the Lamé eigenvalue
problem which is technically more involved.

3.1 Combinatorial evaluating in the electric region

First we state a relation between singularities in the moduli space of the gauge theory and
asymptotic expansion regions of the Mathieu eigenvalue. The moduli space of the \( \mathcal{N} = 2 \)
SU(2) pure gauge theory is parameterized by the complex scalar field vacuum expectation
value \( u = \frac{1}{2} \text{Tr} \phi^2 \) which breaks SU(2) symmetry to abelian U(1). The physical theory is sin-
gular at \( u = \pm \Lambda^2, \infty \) in the sense that new massless particles appear. These singularities are
labeled by the U(1) charges of the massless particles, denoted as “electric”, “magnetic” and
“dyonic”, respectively. Near each singularity the gauge theory has an unique weak coupling
description, therefore perturbative expansion is valid. Our calculation in [40, 41] shows that
there is a one to one correspondence between the perturbative expansion of gauge theory
near a singularity in the moduli space and the asymptotic expansion for the corresponding
Mathieu eigenvalue. For the eigenvalue, in each expansion region the intervals between ad-
jacent energy levels are very small compared to the eigenvalues themselves, therefore the
eigenstates are dense there. See Fig(2) and Fig(3).

The combinatorial method is used to derive eigenvalue at the electric point, at \( u \sim \infty \),
where electric coupling is weak: \( \frac{\Lambda}{q} \ll 1 \). More explicitly, the parameters of gauge theory and
Mathieu eigenvalue are identified as

\[
\lambda = \frac{8u}{\epsilon_1^2}, \quad \nu = \frac{a_{12}}{\epsilon_1}, \quad h = \frac{4\Lambda^2}{\epsilon_1^2} = \frac{4\sqrt{d_{1n}}}{\epsilon_1^2}.
\]  

(26)

The exact expression of \( Z_{\text{pert}} \) is given in [22]. In the limit \( \epsilon_2 \to 0 \), we get the perturbative
part of the prepotential \[36\]:

\[
\mathcal{F}^{\text{pert}} = \frac{1}{2} \ln \left( \frac{\Lambda}{\epsilon_1} \right)^4 \sum_{i=1}^{N} a_i^2 + \epsilon_1 \sum_{i,j=1}^{N} \varpi_{\epsilon_1}(a_{ij}),
\]  

(27)

with \(\varpi_{\epsilon}(x)\) satisfying

\[
\frac{d}{dx} \varpi_{\epsilon}(x) = \ln \Gamma(1 + \frac{x}{\epsilon}).
\]

(28)

For our problem \(N = 2\) and \(a_{12} = 2a\).

As an exercise and experiment, we work out the \(h^{10}\) order of \(\lambda\) for \(h \ll 1\), i.e. the five instanton contribution. The instanton counting formula is \[21\]. For the 1-instanton contribution, we need only one Young diagram doublet:

\[
\begin{pmatrix}
\square \\
\end{pmatrix}
\]

\(\emptyset\)

where \(\emptyset\) is the empty diagram. Then we get

\[
Z_1 = \frac{-2}{\epsilon_1 \epsilon_2 (a_{12}^2 - (\epsilon_1 + \epsilon_2)^2)},
\]

(29)

For the 2-instanton contribution, we need two Young diagram doublets:

\[
\begin{pmatrix}
\square \\
\end{pmatrix}
\begin{pmatrix}
\emptyset
\end{pmatrix}
\]

\[
\begin{pmatrix}
\square \\
\square
\end{pmatrix}
\]

and we get

\[
Z_2 = \frac{2a_{12}^2 - 8\epsilon_1^2 - 17\epsilon_1 \epsilon_2 - 8\epsilon_2^2}{\epsilon_1^2 \epsilon_2 (a_{12}^2 - (\epsilon_1 + \epsilon_2)^2)(a_{12}^2 - (2\epsilon_1 + \epsilon_2)^2)(a_{12}^2 - (\epsilon_1 + 2\epsilon_2)^2)},
\]

(30)

Similarly, we need three Young diagram doublets to derive \(Z_3\):

\[
\begin{pmatrix}
\square \\
\emptyset
\end{pmatrix}
\begin{pmatrix}
\square \\
\end{pmatrix}
\begin{pmatrix}
\emptyset
\end{pmatrix}
\]

\[
\begin{pmatrix}
\square \\
\square \\
\emptyset
\end{pmatrix}
\]

\[
\begin{pmatrix}
\square \\
\square \\
\square
\end{pmatrix}
\]
\[ Z_3 = \frac{\text{numerator}}{\text{denominator}} \]
\[
\text{num} = -2(2a_{12}^4 - 26a_{12}^2\epsilon_1^2 + 72\epsilon_1^4 - 47a_{12}^2\epsilon_1\epsilon_2 + 363\epsilon_1^3\epsilon_2 - 26a_{12}^2\epsilon_2^2
+ 594\epsilon_1^2\epsilon_2^2 + 363\epsilon_1\epsilon_2^3 + 72\epsilon_2^4),
\]
\[
\text{den} = 3\epsilon_1^2\epsilon_2^2(a_{12}^2 - (\epsilon_1 + \epsilon_2)^2)(a_{12}^2 - (\epsilon_1 + 2\epsilon_2)^2)(a_{12}^2 - (\epsilon_1 + 3\epsilon_2)^2)
(a_{12}^2 - (2\epsilon_1 + \epsilon_2)^2)(a_{12}^2 - (3\epsilon_1 + \epsilon_2)^2),
\] (31)

For \( Z_4 \) there are seven Young diagram doublets:

\[
(\begin{array}{ccc}
\hline
& & \\
\hline
\end{array}) (\begin{array}{c}
0
\end{array}) \quad (\begin{array}{c}
0
\end{array}) \quad (\begin{array}{c}
0
\end{array})
\]

\[
(\begin{array}{c}
\hline
& \\
\hline
\end{array}) (\begin{array}{c}
\hline
& \\
\hline
\end{array}) \quad (\begin{array}{c}
\hline
& \\
\hline
\end{array})
\]

\[
Z_4 = \frac{\text{numerator}}{\text{denominator}} \]
\[
\text{num} = 4a_{12}^8 - 132a_{12}^6\epsilon_1^2 + 1440a_{12}^4\epsilon_1^4 - 6208a_{12}^2\epsilon_1^6 + 9216\epsilon_1^8
-212a_{12}^6\epsilon_1\epsilon_2 + 5644a_{12}^4\epsilon_1^3\epsilon_2 - 44336a_{12}^2\epsilon_1^5\epsilon_2 + 100608\epsilon_1^7\epsilon_2 - 132a_{12}^6\epsilon_2^2
+8651a_{12}^4\epsilon_1^2\epsilon_2^2 - 124139a_{12}^2\epsilon_1^4\epsilon_2^2 + 440688\epsilon_1^6\epsilon_2^2 + 5644a_{12}^4\epsilon_1\epsilon_2^3 - 171845a_{12}^2\epsilon_1^3\epsilon_2^3
+1009131\epsilon_1^5\epsilon_2^3 + 1440a_{12}^4\epsilon_1^4 - 124139a_{12}^2\epsilon_1^2\epsilon_2^4 + 1319994\epsilon_1^6\epsilon_2^4 - 44336a_{12}^2\epsilon_1\epsilon_2^5
+1009131\epsilon_1^3\epsilon_2^5 - 6208a_{12}^6\epsilon_2^6 + 440688\epsilon_1^2\epsilon_2^6 + 100608\epsilon_1^7\epsilon_2 + 9216\epsilon_2^8,
\]
\[
\text{den} = 6\epsilon_1^2a_{12}^2(a_{12}^2 - (\epsilon_1 + \epsilon_2)^2)(a_{12}^2 - (\epsilon_1 + 2\epsilon_2)^2)(a_{12}^2 - (\epsilon_1 + 3\epsilon_2)^2)
(a_{12}^2 - (\epsilon_1 + 4\epsilon_2)^2)(a_{12}^2 - (2\epsilon_1 + \epsilon_2)^2)(a_{12}^2 - (3\epsilon_1 + \epsilon_2)^2)
(a_{12}^2 - (4\epsilon_1 + \epsilon_2)^2)(a_{12}^2 - (2\epsilon_1 + 2\epsilon_2)^2),
\] (32)

With \( Z_k \) in the hand, it is easy to derive \( F_k \) using (24). The complete form of the first three \( F_1, F_2, F_3 \) are presented in (26). As observed in (42), according to the Matone’s relation (25), and taking in to account (26), the Mathieu eigenvalue is

\[
\lambda = \nu^2 + \sum_{k=1}^{\infty} \frac{\epsilon_1^{4k-2}}{16^k} 4k F_k(a_{12}, \epsilon_1, \epsilon_2 = 0) h^{2k}
= \nu^2 + \sum_{k=1}^{\infty} 4k F_k(\nu, 1, 0) \left(\frac{h}{4}\right)^{2k}.
\] (33)

The leading term \( \nu^2 \) comes from the perturbative contribution. We have used the fact that the numerator and denominator of \( F_k(a_{12}, \epsilon_1) \) are homogeneous polynomials of degree \( 2k + 2 \)
and $6k$, respectively. Up to $k = 4$, we get
\[
\lambda = \nu^2 + \frac{1}{2(\nu^2 - 1)} h^2 + \frac{5\nu^2 + 7}{32(\nu^2 - 1)^3(\nu^2 - 4)} h^4 + \frac{9\nu^4 + 58\nu^2 + 29}{64(\nu^2 - 1)^5(\nu^2 - 4)(\nu^2 - 9)} h^6 + \frac{1469\nu^{10} + 9144\nu^8 - 140354\nu^6 + 64228\nu^4 + 827565\nu^2 + 274748}{8192(\nu^2 - 1)^7(\nu^2 - 4)^3(\nu^2 - 9)(\nu^2 - 16)} h^8 + \ldots
\]
for $h \ll 1, \nu \gg 1$ and/or $h \gg 1, h/\nu^2 \ll 1$.

3.2 Extension to other regions

A complete solution of the eigenvalue problem requires extending the asymptotic expansion from $u \sim \infty$ to other two regions near $u = \pm \Lambda^2$. This is achieved by utilizing the duality relation between the three regions, a crucial result of Seiberg-Witten theory. Physical quantities at $u \sim \infty$ have their dual counterparts at $u = \pm \Lambda^2$. For undeformed gauge theory, $\epsilon_1 = \epsilon_2 = 0$, the scalar condensation $a$ and its dual $a_D$ are given by integrals along the homology circles of the Seiberg-Witten curve which is a torus for SU(2) gauge theory. As functions of $u$, they can be expanded in the three regions. The leading order WKB expansion of the Mathieu equation gives solution of the undeformed gauge theory [38, 39, 40].

Theory with $\epsilon_1 \neq 0, \epsilon_2 = 0$ is the “quantized” version of theory with $\epsilon_1 = \epsilon_2 = 0$. As first studied in [38], the WKB method, widely used in quantum mechanics problems, provides an efficient method to obtain higher order quantum corrections. Actually, the work of [38] shows that the $\epsilon_1$-corrected scalar condensation and its dual can be generated from the classical one by action of certain differential operators with respect to $u$ and $\Lambda$. (In practice, setting $\Lambda = 1$ will further simplify the procedure. It can be restored by dimensional consideration.)

\[
a(u, q_m, \epsilon_1) = D(u, \partial_u, \epsilon_1) \int_a \lambda_{SW}.
\]

where $\lambda_{SW}$ is the Seiberg-Witten form, in the context of gauge theory/integrable system, the integral $\int \lambda_{SW}$ coincides with the classical action $\int p(z)dz$ of a particle. $D(u, \partial_u, \epsilon_1)$ can be expanded as $D(u, \partial_u, \epsilon_1) = 1 + \sum_{n=1}^{\infty} \epsilon^n D_n(u, \partial_u)$. The differential operators $D_n(u, \partial_u)$ are polynomials of $u$ and $\partial_u$, the first few of them can be found in [38, 40, 41]. Reverse the function $a(u, q_m, \epsilon_1)$ we can get $u(a, q_m, \epsilon_1)$ which gives the eigenvalue through parameters identification (26).

The WKB quantization method is independent of the instanton counting approach, in principle it serves as a self-consistent way to solve the eigenvalue problem. Nevertheless, the WKB method gives a nontrivial check for the combinatorial method. It turns out that at $u \sim \infty$, series expansion of $a(u, \Lambda, \epsilon_1)$ from (35) exactly gives eigenvalue (34).
The dual $a_D$ is given by integral along the dual homology circle,

$$ a_D(u, q_{in}, \epsilon_1) = D(u, \partial_u, \epsilon_1) \int_{\beta} \lambda_{SW}. \quad (36) $$

$D(u, \partial_u, \epsilon_1)$ remains the same, and a reverse of the function $a_D(u, q_{in}, \epsilon_1)$ to $u = (a_D, q_{in}, \epsilon_1)$ is needed. Near $u = \Lambda^2$ where the magnetic coupling is small, $\frac{a_D}{\Lambda} \ll 1$, we found that the series expansion of $a_D$ gives another well known eigenvalue expansion:

$$ \lambda = 2h - 4\nu\sqrt{h} + \frac{4\nu^2 - 1}{2^3} + \frac{4\nu^3 - 3\nu}{2^6\sqrt{h}} + \frac{80\nu^4 - 136\nu^2 + 9}{2^{12}h} $$

$$ + \frac{528\nu^5 - 1640\nu^3 + 405\nu}{2^{16}h^{\frac{5}{2}}} + \cdots \quad (37) $$

for $h \gg 1, \frac{\hbar}{\nu^2} \gg 1$, with identification

$$ \lambda = \frac{8u}{\epsilon_1}, \quad \nu = \frac{2ia_D}{\epsilon_1}, \quad h = \frac{4\Lambda^2}{\epsilon_1}. \quad (38) $$

The details are given in [41], see there for result up to the order $h^{-\frac{7}{2}}$. Moreover, the third expansion region near $u = -\Lambda^2$, the dyonic region, is the mirror of region near $u = \Lambda^2$. The eigenvalue there is obtained through $\nu \to i\nu, h \to -h$ [40]:

$$ \lambda = -2h + 4\nu\sqrt{h} - \frac{4\nu^2 + 1}{2^3} - \frac{4\nu^3 + 3\nu}{2^6\sqrt{h}} - \frac{80\nu^4 + 136\nu^2 + 9}{2^{12}h} $$

$$ - \frac{528\nu^5 + 1640\nu^3 + 405\nu}{2^{16}h^{\frac{5}{2}}} + \cdots \quad (39) $$

We should stress that we can combine the combinatorial and WKB method to make a calculable algorithm to get all eigenvalues to higher order. As we have said, the higher order differential operators used in [38, 41] become very hard to determine. With the combinatorial results of higher order electric expansion, according to the **Claim 1** of [41], we can easily extend the WKB differential operators to even higher order. It becomes quite straightforward if some simple computer codes are used properly. For example, only a bit more labor is needed to get the next order results, continuing [41]:

$$ 4471\nu^{12} + 69361\nu^{10} - 1039598\nu^8 - 2844430\nu^6 + 13541915\nu^4 + 20651309\nu^2 + 4453452 $$

$$ 16384(\nu^2 - 1)^2(\nu^2 - 4)^3(\nu^2 - 9)(\nu^2 - 16)(\nu^2 - 25)h^{10}, \quad (40) $$

for $h \ll 1, \nu \gg 1$ and/or $h \gg 1, \frac{\hbar}{\nu^2} \ll 1$. Now we can determine $D_n(u, \partial_u)$ up to $n = 10$, and

---

[2] However, see the Appendix [11] for an algorithm we recently find
then continue to determine higher order magnetic expansion. With this strategy, we get

\[
\frac{5}{2^{38}h^4}(21635328\nu^{10} - 388041984\nu^8 + 1537405408\nu^6 - 1708613456\nu^4 + 449869257\nu^2
- 11917692) + \frac{1}{2^{14}h^2}(4295427072\nu^{11} - 97098594560\nu^9 + 523957083264\nu^7
- 888479287968\nu^5 + 440967453876\nu^3 - 41540033277\nu).
\]

(41)

for \( h \gg 1, \frac{h}{\nu^2} \gg 1 \).

4 Combinatorial approach to Lamé eigenvalue

The Lamé equation is closely related to the Mathieu equation. In the limit \( n \to \infty, k \to 0 \) while keep \( n(n - 1)k^2 = \kappa^2 \) fixed, the Jacobi \( sn \)-function becomes the trigonometric function \( \sin \), and Eq.(5) becomes equivalent to Eq.(11). This fact indicates each asymptotic expansion of the Mathieu eigenvalue presented above should be the limit of a Lamé eigenvalue expansion. That means the Lamé eigenvalue has \textit{at least} three asymptotic expansion regions, located approximately at three point as in (16). In this section, based on the quantum field theory, we argue that there are \textit{only} three asymptotic expansion regions. We also derive the eigenvalue at the electric region using the combinatorial method, and write it in a proper form.

4.1 Identify parameters

First, we have to make a identification between the module parameters in the Lamé equation and in the \( \mathcal{N} = 2^* \) gauge theory. The parameter \( k \) appearing in Eq.(5) is the elliptic module associated to the periods \( 4K(k) + 2iK'(k) \) of the Jacobi elliptic function \( sn \). The Jacobi theta constants \( \theta_i \) are often expanded in another module parameter, the nome \( q \). The two parameters are related to each other, their precise relation can be found in Appendix 8. Especially, we notice

\[
k^2 = 16q^{\frac{1}{2}} - 128q + 704q^3 - 3072q^2 + 11488q^\frac{5}{2} - 38400q^3 + \cdots = \frac{\theta_2(q)^4}{\theta_3(q)^4}.
\]

(42)

Our claim is that \( q \) is exactly the instanton expansion parameter of the \( \mathcal{N} = 2^* \) theory, and is the “good” expansion parameter for the eigenvalue \( B \) in the electric region.

Another thing we have to identify is the relation between the parameter \( n \) in the Lamé equation and the adjoint mass of the gauge theory. The equivalent parameter \( m \) appears in the instanton partition function of \( \mathcal{N} = 2^* \) theory as\,[27],

\[
Z_k = \sum_{|Y| = k} \prod_{a, \alpha = 1}^2 \prod_{s, s' \in Y} \left( \frac{E_{\alpha \beta}(s) - m)(\epsilon_+ - E_{\beta \alpha}(s') - m)}{E_{\alpha \beta}(s)(\epsilon_+ - E_{\beta \alpha}(s'))} \right),
\]

(43)
As clarified in [43], the parameter $m$ differs from the physical mass $m^*$ by $\epsilon_1, \epsilon_2$ shift: $m = m^* + (\epsilon_1 + \epsilon_2)/2$. We identify $n = \epsilon_1^{-1}m$.

Lastly, we have to identify the relation between the eigenvalue $A$, or $B$, and the scalar condensation $u$. The Seiberg-Witten curve of the $\mathcal{N} = 2^*$ theory is

$$y^2 = (x - e_1\tilde{u} - \frac{1}{8}e_1^2m^2)(x - e_2\tilde{u} - \frac{1}{8}e_2^2m^2)(x - e_3\tilde{u} - \frac{1}{8}e_3^2m^2),$$

(44)

with

$$u = \langle \text{Tr}\phi^2 \rangle = \tilde{u} + m^2\left(\frac{1}{24} + c_1q + c_2q^2 + \cdots\right).$$

(45)

We have rescaled the mass in [20] by $m^2 \rightarrow \frac{m^2}{2}$ in order to match the mass parameter that appears in the instanton counting formula [43]. Again this is an elliptic curve, with two conjugate homology circles $\alpha$ and $\beta$, and the Seiberg-Witten differential one form. $\tilde{u}$ and $u$ differ by terms caused by mass deformation. It turns out that $\tilde{u}$ is directly related to the eigenvalue $B$.

In summary, we identify

$$n = \frac{m}{\epsilon_1}, \quad B = -8\tilde{u}, \quad \nu = a_{12}/\epsilon_1, \quad q = q_m.$$  

(46)

\[\text{Figure 4: Singularities in the } \tilde{u}\text{-plane for } N = 2^* \text{ theory.}\]

The moduli space has three singularities in the $\tilde{u}$ plane, as in Fig. (4), they are located at

$$\tilde{u} = \frac{1}{8}e_1m^2 = m^2\left(\frac{1}{12} + 2q + 2q^2 + \cdots\right),$$

$$\tilde{u} = \frac{1}{8}e_2m^2 = m^2\left(-\frac{1}{24} - q^\frac{1}{2} - q - 4q^\frac{3}{2} - q^2 + \cdots\right),$$

$$\tilde{u} = \frac{1}{8}e_3m^2 = m^2\left(-\frac{1}{24} + q^\frac{1}{2} - q + 4q^\frac{3}{2} - q^2 + \cdots\right).$$

(47)

The relation proposed in [20] is $u = \langle \text{Tr}\phi^2 \rangle = \tilde{u} + \frac{1}{8}e_1m^2$. But as noted in [44], the one instanton calculation for the coefficient $c_1$ indicates this relation is not correct. The correct relation is pointed out in [45] as $u = \tilde{u} - \frac{m^2}{24} + \frac{m^2}{12}E_2(\tau)$, with $E_2(\tau)$ the second Eisenstein series. Some scaling of parameters in that paper are needed to match ours.
Note that the stationary points of the potential $\varphi(z)$ are exactly at $\varphi(z^*) = e_1, e_2, e_3$ where $\partial_z \varphi(z) = 0$, $\partial_z^2 \varphi(z) > 0$. In the $u$ plane, the singularities are located at

$$u = m^2 \left( \frac{1}{8} - 4q^2 - 12q^4 + 12q^5 + \cdots \right),$$

$$u = \mp m^2 (\sqrt{q} - 3q + 4q^3 - 7q^2 + \cdots).$$

As in the case for the pure SU(2) SYM (Mathieu equation), there are only three singularities in the moduli space, therefore, there are only three asymptotic expansion regions for the eigenvalue. While the Mathieu equation has two free parameters $h$ and $\nu$, the Lamé equation has three free parameters, $n, k$ and $\nu$, it is not a prior fact the number of expansion regions remains to be three. However, the physical argument indicates there is no other expansion region.

### 4.2 Combinatorial evaluating in the electric region

We derive the eigenvalue near the electric point. Let us look at the perturbative part, in the limit $\epsilon_2 = 0$, the prepotential is given by

$$F_{\text{pert}} = \pi i \tau \sum_{i=1}^{N} a_i^2 + 2\pi i \epsilon_1 \sum_{i,j=1}^{N} (\omega_{\epsilon_1} (a_{ij}) - \omega_{\epsilon_1} (a_{ij} - m - \epsilon_1)).$$

(50)

Here we have $N = 2$, and $q_{in} = \exp(2\pi i \tau)$. Only the first term $a^2 \ln q_{in}$ contributes to the eigenvalue.

Then we come to the instanton part of the gauge theory, or the $q$ correction part of the Lamé equation. Following the line of instanton counting, we derive the prepotential $F_k(a_{12}, \epsilon_1, \epsilon_2, m), k = 1, 2, 3, \cdots$, and set $\epsilon_2 = 0$, the eigenvalue $B$ can be obtained via the Matone’s relation. The whole procedure is the same as that for the Mathieu eigenvalue, but with new counting formula (43) and identification (46).

Without presenting the results of $Z_k, k = 1, 2, 3, \cdots$, which are fairly lengthy, we only give the final results of the prepotential. The 1-instanton contribution is

$$F_{1q_{in}} = \frac{2 [m^4 - 2m^3 \epsilon_1 - m^2 (a_{12}^2 - 2 \epsilon_1^2) + m (a_{12}^2 \epsilon_1 - \epsilon_3^2)]}{a_{12}^2 - \epsilon_1^2} q_{in} - \frac{2m(m - \epsilon_1) [m(m - \epsilon_1) - a_{12}^2 + \epsilon_1^2]}{a_{12}^2 - \epsilon_1^2} q_{in},$$

(51)

Writing in terms of $n, \nu, q$, it is

$$F_{1q_{in}} = \epsilon_1^2 \frac{2m(n - 1)q}{\nu^2 - 1} \times [n(n - 1) - (\nu^2 - 1)],$$

(52)
Similarly, the 2-instanton contribution gives

\[ F_{2q}^2 = \varepsilon_1^2 \frac{n(n-1)q^2}{(\nu^2 - 1)^3(\nu^2 - 4)} \times \left[ n^3(n-1)^3(5\nu^2 + 7) - 12n^2(n-1)^2(\nu^2 - 1)^2 + 6n(n-1)(\nu^2 - 1)^2(\nu^2 - 2) - 3(\nu^2 - 1)^3(\nu^2 - 4) \right] \] (53)

The 3-instanton contribution gives

\[ F_{3q}^3 = \varepsilon_1^2 \frac{8n(n-1)q^3}{3(\nu^2 - 1)^5(\nu^2 - 4)(\nu^2 - 9)} \times \left[ 2n^5(n-1)^5(9\nu^4 + 58\nu^2 + 29) - 8n^4(n-1)^4(\nu^2 - 1)^2(7\nu^2 + 17) + 4n^3(n-1)^3(\nu^2 - 1)^2(15\nu^4 - 37\nu^2 - 2) - 24n^2(n-1)^2(\nu^2 - 1)^4(\nu^2 - 4) + 3n(n-1)(\nu^2 - 1)^4(\nu^2 - 3)(\nu^2 - 4) - (\nu^2 - 1)^5(\nu^2 - 4)(\nu^2 - 9) \right] \] (54)

A notable fact is that in the final results of instanton counting the parameter \( n \) always appears in the form \( n(n-1) \), consistent with its appearance in the Lamé equation. This is already visible from the numerator of counting formula (43) where terms involving \( m \) combine to \( m(m - \varepsilon_1) \). The full prepotential is

\[ F = F^{\text{pert}} + \sum_{k=1}^{\infty} F_k q^k = F^{\text{pert}} + F_1 q + F_2 q^2 + F_3 q^3 + \cdots \] (55)

According to the Matone’s relation, the scalar condensation \( u \) is given by \( 2u = q \frac{\partial}{\partial q} F \), and according to (45), \( \tilde{u} \) and \( u \) are related by

\[ \tilde{u} = u + \frac{m(m - \varepsilon_1)}{24} - \frac{m(m - \varepsilon_1)}{12} E_2(\tau). \] (56)

where we have turned on the \( \varepsilon_1 \) (quantum) correction to the “classical” relation, and \( E_2(\tau) \)
is the second Eisenstein series. Now we are in the position to get the eigenvalue $B$:

$$B = -\frac{8\ddot{u}}{\epsilon_1^2} = -\frac{4}{\epsilon_1^2} q \frac{\partial}{\partial q} \mathcal{F} - \frac{n(n-1)}{3} (1 - 2E_2(\tau))$$

$$= -\nu^2 - \frac{n(n-1)}{3} (1 - 2E_2(q)) - \frac{8n(n-1)q}{\nu^2 - 1} [n(n-1) - (\nu^2 - 1)]$$

$$- \frac{8n(n-1)q^2}{(\nu^2 - 1)^3(\nu^2 - 4)} \times [n^3(n-1)^3(5\nu^2 + 7) - 12n^2(n-1)^2(\nu^2 - 1)^2$$

$$+ 6n(n-1)(\nu^2 - 1)^2(\nu^2 - 2) - 3(\nu^2 - 1)^3(\nu^2 - 4)]$$

$$- \frac{32n(n-1)q^3}{(\nu^2 - 1)^5(\nu^2 - 4)(\nu^2 - 9)} \times [2n^5(n-1)^5(9\nu^4 + 58\nu^2 + 29)$$

$$- 8n^4(n-1)^4(\nu^2 - 1)^2(7\nu^2 + 17)$$

$$+ 4n^3(n-1)^3(\nu^2 - 1)^2(15\nu^4 - 37\nu^2 - 2)$$

$$- 24n^2(n-1)^2(\nu^2 - 1)^4(\nu^2 - 4)$$

$$+ 3n(n-1)(\nu^2 - 1)^4(\nu^2 - 3)(\nu^2 - 4)$$

$$- (\nu^2 - 1)^5(\nu^2 - 4)(\nu^2 - 9)] + \cdots$$

(57)

We have used the perturbative result $q \partial_q \mathcal{F}_{\text{pert}} = \epsilon_1^2 \nu^2 / 4$. In the electric region, we have the leading order behavior $8u \sim m^2$, that is $\nu^2 \sim n(n-1)$. Therefore, $B \sim -\frac{2}{3} n(n-1) + \mathcal{O}(q)$.

The expansion of $B$ is valid when

$$q \ll 1, \quad n \sim \nu \gg 1, \quad nq^{\frac{1}{2}} \ll 1,$$

(58)

and it is also valid for

$$q \ll 1, \quad n \sim \nu \gg 1, \quad 1 \ll nq^{\frac{1}{2}} \ll \nu.$$

(59)

An equivalent expansion was also obtained by E. Langmann in [15]. His method is different from ours, we have shown the equivalence of his eigenvalue expansion $\mathcal{E}$ and our expansion $B$, see Appendix [10].

In order to compare to the WKB results, we do some further works to derive the expansion of $A$ from $B$. $B$ can be further expanded as

$$B = -\nu^2 + \frac{1}{3} n(n-1)(1 - 24q - 72q^2 - 96q^3 - \cdots)$$

$$- \frac{8n^2(n-1)^2}{\nu^2} (q + 6q^2 + 12q^3 + \cdots)$$

$$- \frac{8n^2(n-1)^2}{\nu^4} [q + (18 - 12n(n-1))q^2 + (84 - 96n(n-1))q^3 + \cdots]$$

$$- \frac{8n^2(n-1)^2}{\nu^6} [q + (66 - 60n(n-1) + 5n^2(n-1)^2)q^2$$

$$+ (732 - 960n(n-1) + 240n^2(n-1)^2)q^3 + \cdots]$$

$$+ \cdots$$

(60)
Then it is easy to get the expansion for eigenvalue $A$, taking into account the relation between $A$ and $B$ (10), and the relation between $\nu$ and $\mu$ (15). As we have worked out terms of order up to $q^3$, taking into account relation (42), we keep terms up to the order $\mu^{-6}$ and $k^{12}$. We finally get

$$A = -\mu^2 - \frac{1}{2} n(n-1)k^2 - n(n-1)\left(\frac{1}{16}k^4 + \frac{1}{32}k^6 + \frac{41}{2048}k^8 + \frac{59}{4096}k^{10} + \frac{727}{65536}k^{12} + \cdots\right)$$

$$- \frac{n^2(n-1)^2}{\mu^2} \left[\frac{1}{32}k^4 - \frac{1}{4096}k^8 - \frac{1}{4096}k^{10} - \frac{29}{131072}k^{12} + \cdots\right]$$

$$- \frac{n^2(n-1)^2}{\mu^4} \left[\frac{1}{32}k^4 - \frac{1}{64}k^6 - \left(\frac{7 + 6n(n-1)}{4096}\right)k^8 - \left(\frac{7 + 6n(n-1)}{8192}\right)k^{10}\right]$$

$$- \left(\frac{73 + 60n(n-1)}{131072}\right)k^{12} + \cdots$$

$$- \frac{n^2(n-1)^2}{\mu^6} \left[\frac{1}{32}k^4 - \frac{1}{32}k^6 + \left(\frac{74 - 60n(n-1) + 5n^2(n-1)^2}{8192}\right)k^8\right]$$

$$- \left(\frac{29 - 10n^2(n-1)^2}{131072}\right)k^{12} + \cdots$$

$$+ \cdots$$  \quad (61)

Before doing the WKB check, we first give a simple consistent check using a limit of the Lamé equation. The $\mathcal{N} = 2^*$ theory has a decoupling limit. When the energy scale is very small, or the mass is very large, the adjoint mass decouples and the theory flows to the pure gauge theory. In this limit, every singularity in the moduli space of the $\mathcal{N} = 2^*$ theory becomes a singularity of the pure gauge theory. The limit is

$$q \to 0, \quad m \to \infty, \quad m^2\sqrt{q} \to -\Lambda^2, \quad (62)$$

or

$$k \to 0, \quad m \to \infty, \quad m^2k^2 \to -16\Lambda^2. \quad (63)$$

Writing in the parameters of the differential equation, the limit is

$$q \to 0, \quad n \to \infty, \quad n(n-1)\sqrt{q} \to -\frac{h}{4}, \quad (64)$$

or

$$k \to 0, \quad n \to \infty, \quad n(n-1)k^2 \to -4h. \quad (65)$$

Accordingly, in this limit the Lamé equation reduces to the Mathieu equation, and the Lamé eigenvalue (57) and (61) reduces to the Mathieu eigenvalue (34):

$$-B \to \lambda, \quad -A \to \lambda - 2h. \quad (66)$$
5 Check through the WKB method

When there is a small expansion parameter, the WKB method can be applied. This is well studied technique in some differential equations, as we have said, the new ingredient contributed in [38] is doing the WKB analysis in the complex domain. In this section, we do the same thing for the Lamé equation, as have done for the Mathieu equation in [38, 40, 41].

Rewriting the Eq. (5) as

\[ \frac{\epsilon^2}{2} \Phi'' - [\omega + \text{sn}^2 \kappa] \Phi = 0, \]

where

\[ \epsilon^2 = \frac{2n(n-1)}{k^2}, \quad \omega = \frac{A}{n(n-1)k^2}. \]

Note that \( \epsilon_1 \) is contained in \( \epsilon \) and \( \omega \). Suppose \( \epsilon \ll 1 \), i.e., \( nq_0^2 \gg 1 \), then \( \epsilon \) is a valid small WKB expansion parameter. We choose Eq. (5) to do the WKB analysis because the readily available integrals formulae are suitable to deal with the Jacobi elliptic functions. It is legal to choose Eq. (6) instead, as in [50, 51] where the leading order expansion is performed. Expanding the function \( \Phi \) as WKB series:

\[ \Phi(\kappa) = \exp i \int_{\kappa_0}^{\kappa} d\kappa' p(\kappa') \]

\[ = \exp i \int_{\kappa_0}^{\kappa} d\kappa' \left( \frac{p_0(\kappa')}{\epsilon} + p_1(\kappa') + \epsilon p_2(\kappa') + \epsilon^2 p_3(\kappa') + \cdots \right), \]

then we have

\[ p_0 = i\sqrt{2} \sqrt{\omega + \text{sn}^2 \kappa}, \quad p_1 = \frac{i}{2} (\text{ln}(p_0))', \]

\[ p_2 = \frac{1}{4p_0} \left( \frac{3}{2} \left( \frac{p_0'}{p_0} \right)^2 - \frac{p_0''}{p_0} \right), \quad p_3 = \frac{i}{2} \left( \frac{p_2}{p_0} \right)', \]

\[ p_4 = \frac{i}{2} \left( \frac{p_3}{p_0} \right)' - \frac{p_2^2}{2p_0}, \quad \cdots \]

(70)

where the prime denotes \( \frac{\partial}{\partial \kappa} \).

We have to evaluate the contour integral \( \oint_\alpha p_0 d\kappa \), where \( \alpha \) is the homology cycle of the curve (44) relevant to electric scalar condensation. Introduce the amplitude,

\[ \varphi = a \text{m} \kappa, \]

then we have

\[ \text{sn} \kappa = \sin \varphi, \quad d\kappa = \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \]

(72)
The integral becomes
\[ \int_{\alpha}^{\beta} \sqrt{\omega + \sin^2 \varphi} d\varphi = 2 \int_{0}^{K} \sqrt{\omega + \sin^2 \varphi} d\varphi = 2 \int_{0}^{\pi/2} \sqrt{\omega + \sin^2 \varphi} d\varphi. \] (73)

In the electric region, \( A \gg \kappa^2 \), therefore \( \omega \gg 1 \), we have expansion
\[ \sqrt{\omega + \sin^2 \varphi} = \sqrt{\omega} (1 + \frac{\sin^2 \varphi}{2\omega} - \frac{\sin^4 \varphi}{8\omega^2} + \frac{\sin^6 \varphi}{16\omega^3} - \frac{5\sin^8 \varphi}{128\omega^4} + \cdots), \] (74)

All the integrals we need to do are of the form
\[ \int_{0}^{\pi/2} \frac{\sin^{2n} \varphi}{\Delta} d\varphi, \quad \text{with} \quad \Delta = \sqrt{1 - \kappa^2 \sin^2 \varphi}, \quad n = 0, 1, 2, 3, \cdots \] (75)

They are given in Chapter 2 of [10]. After collecting all terms together, we get
\[ \int_{\alpha}^{\beta} \sqrt{\omega + \sin^2 \varphi} d\varphi = 2K\sqrt{\omega} + [K - E] \omega^{-1/2} \frac{(2 + k^2)K - 2(1 + k^2)E}{12k^4} \omega^{-3/2} \]
\[ + \frac{(8 + 3k^2 + 4k^4)K - (8 + 7k^2 + 8k^4)E}{120k^6} \omega^{-5/2} \]
\[ - \frac{(48 + 16k^2 + 17k^4 + 24k^6)K - 8(6 + 5k^2 + 5k^4 + 6k^6)E}{1344k^8} \omega^{-7/2} \]
\[ + \cdots. \] (76)

where \( K \) and \( E \) are the complete elliptic integrals of the first and second kind, respectively.

For higher order contour integrals, as usual the odd order integrals do not contribute,
\[ \int_{\alpha}^{\beta} p_{2l+1} d\varphi = 0, \quad l = 0, 1, 2 \cdots \] (77)

Hence, all odd differential operators \( D_{2l+1}(\omega, \partial_\omega, k) = 0 \). While for the even order integrals, using the trick of [38], we get the operator \( D_2(\omega, \partial_\omega, k) \) from
\[ \int_{\alpha}^{\beta} \frac{1}{\sqrt{1 - k^2 \sin^2 \varphi}} = 1 + \frac{k^2}{2} \sin^2 \varphi + \frac{3k^4}{8} \sin^4 \varphi + \frac{5k^6}{16} \sin^6 \varphi + \frac{35k^8}{128} \sin^8 \varphi + \cdots. \]

The integrals we need to do become
\[ \frac{2}{\Delta} \sin^{2n} \varphi d\varphi, \quad \text{with} \quad \Delta = \sqrt{1 - \kappa^2 \sin^2 \varphi}, \quad \kappa^2 = -\frac{1}{\omega}. \]

The formulae are also given in Chapter 2 of [10]. This gives the same final result (76).
and get $D_4(\omega, \partial_\omega, k)$ from
\[
\int_{\alpha,\beta} p_4 d\zeta = \frac{1}{64\sqrt{2i}} \int_{\alpha,\beta} \frac{G(\zeta)}{(\omega + \text{sn}^2 \zeta)^{7/4}} - \frac{5}{2} \left(\text{sn} \text{cn} \text{dn}^2\right)^2 d\zeta
\]
\[
= \frac{1}{64\sqrt{2i}} \int_{\alpha,\beta} \left[ \frac{7}{12} (G(\zeta))^2 - \frac{1}{12} G''(\zeta) \right] d\zeta
\]
\[
= \frac{1}{64} \frac{14}{45} \left[ 1 + 4(1 + k^2) \omega + 2(2 + 7k^2 + 2k^4) \omega^2 + 12k^2(1 + k^2) \omega^3 + 9k^4 \omega^4 \right] \partial_\omega^4
\]
\[
+ \frac{8}{3} \left[ (1 + k^2) + 2(1 + 3k^2 + k^4) \omega + 8k^2(1 + k^2) \omega^2 + 8k^4 \omega^3 \right] \partial_\omega^3
\]
\[
+ \frac{1}{3} \left[ (10 + 23k^2 + 10k^4) + 66k^2(1 + k^2) \omega + 99k^4 \omega^2 \right] \partial_\omega^2
\]
\[
+ 2[k^2(1 + k^2) + 3k^4 \omega] \partial_\omega - \frac{1}{8} k^4 \right] \int_{\alpha,\beta} p_0 d\zeta. \quad (79)
\]
where $G(\zeta) = 1 - 2(1 + k^2) \text{sn}^2 \zeta + 3k^2 \text{sn}^4 \zeta$, and $G''(\zeta) = \partial_\zeta^2 G(\zeta)$. In the Appendix (11), we give a systematic method to obtain higher order differential operators.

The contour integral along the $\alpha$ circle is actually the phase shift of the function $\Phi(\zeta)$ in (13), i.e.
\[
2K\mu = \oint_\alpha d\zeta \rho(\zeta) = (1 + \sum_{n=1}^\infty \epsilon^n D_n(\omega, \partial_\omega, k)) \oint_\alpha d\zeta \frac{p_0(\zeta)}{\epsilon}, \quad (80)
\]
Therefore the Floquet index is given by
\[
\mu = \frac{1}{2K} \oint_\alpha d\zeta \rho(\zeta). \quad (81)
\]

The remaining work is straightforward but tedious: expand the $p_0$ integral (76) as power series of $\omega$ and $k$, use the differential operators from (78) and (79) to generate the next two integrals for $p_2, p_4$, we get the series expansion of $\mu = \mu(\omega, k)$. Then inverse the series $\mu(\omega, k)$ to get a series $\omega = \omega(\mu, k)$, we finally get the series expansion for the eigenvalue $A$ using $A = n(n - 1)k^2 \omega$. We have checked it indeed gives the expansion (61), up to order $\mu^{-6}$ and $k^8$. In our WKB analysis, the $k$-expansion is truncated at order $k^8$ because in (76) we have only worked out terms up to the order $\omega^{-7/2}$, with the accuracy of order $k^8$. At this order the WKB check matches sixteen coefficients appearing in (61).

6 Extension to other expansion regions

6.1 Magnetic expansion

The last task we have to do is extending the asymptotic expansion in the electric region to the magnetic and dyonic regions, as what we have done for the Mathieu equation in [41].
We stress that the asymptotic expansion in the magnetic region has been worked out in the literature, using purely mathematical technique. The explicit formula is given in the book by H. J. W. Müller-Kirsten[11], which cites results from earlier original papers of E. L. Ince and H. J. W. Müller. It is also presented in Chapter 29 of[9]. It is

\[ A = -i2\kappa\mu - \frac{1}{2^3}(1 + k^2)(4\mu^2 - 1) \]

\[ -\frac{i}{2^5\kappa^2}[(1 + k^2)^2(4\mu^3 - 3\mu) - 4k^2(4\mu^3 - 5\mu)] \]

\[ + \frac{1}{2^{10}\kappa^2}(1 + k^2)(1 - k^2)^2(80\mu^4 - 136\mu^2 + 9) \]

\[ + \frac{i}{2^{13}\kappa^3}[(1 + k^2)^4(528\mu^5 - 1640\mu^3 + 405\mu) - 24k^2(1 + k^2)^2(112\mu^5 - 360\mu^3 + 95\mu) \]

\[ + 16k^4(144\mu^5 - 520\mu^3 + 173\mu)] \]

\[ + \cdots \] (82)

The expansion is valid for

\[ nk \gg 1, \quad nk \gg \mu. \] (83)

Note \( k \sim q^\frac{1}{4} \). We remind the readers that the Floquet index in[11] is denoted by \( q \), its relation to \( \mu \) is \( q = 2i\mu \). Also their eigenvalue is denoted by \( \Lambda \), related to ours by \( \Lambda = -A \). Again we use the letter \( \mu \) to denote the Floquet index of \( A \). In the limit \( n \to \infty, k \to 0 \) while keep \( \kappa^2 = n(n - 1)k^2 \to -4h \) fixed, the eigenvalue (82) reduces to the corresponding Mathieu eigenvalue, see formula (37).

Now, let us look at how to extend the Lamé eigenvalue (61) in the electric region to the magnetic region. The tool is the WKB data obtained in the last section, and apply them to the \( \beta \)-contour integral. In the magnetic region located near \( A \sim 0 \), where \( \omega \ll 1 \), we can expand \( \sqrt{\omega + \sin^2 \varphi} \) as:

\[ \sqrt{\omega + \sin^2 \varphi} = \sin \varphi + \omega \frac{1}{2 \sin \varphi} - \frac{\omega^2}{8 \sin^3 \varphi} + \frac{\omega^3}{16 \sin^5 \varphi} - \frac{5\omega^4}{128 \sin^7 \varphi} + \cdots . \] (84)

All the integrals now we need to do are

\[ \int_\beta \frac{1}{\Delta \sin^{2n-1} \varphi} d\varphi = 2 \int_{\varphi_0}^{\varphi_0} \frac{1}{\Delta \sin^{2n-1} \varphi} d\varphi, \] (85)

with

\[ \sin \varphi_0 = \frac{1}{k}, \quad \Delta = \sqrt{1 - k^2 \sin^2 \varphi}, \quad n = 0, 1, 2, 3, \cdots \] (86)

They are also given in Chapter 2 of [10]. The first few orders of the integral \( \int_\beta \rho_0 d\kappa \) give

\[ \int_\beta \sqrt{\omega + \sin^2 \varphi} d\varphi = \frac{i\pi}{2} \omega - \frac{1 + k^2}{16} \omega^2 + \frac{3k^4 + 2k^2 + 3}{128} \omega^3 - \frac{5(1 + k^2)(15k^4 - 6k^2 + 15)}{6144} \omega^4 + \cdots . \] (87)

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Using the differential operators obtained in (78) and (79) we have obtained in the last section, we can generate integrals $\int_\beta p_2 d\kappa$ and $\int_\beta p_4 d\kappa$.

For the Floquet index in the magnetic region, our claim is

$$\mu = \frac{1}{i\pi} \int_\beta d\kappa p(\kappa).$$  (88)

There is a point a bit puzzling to us. When we go through the circle $\beta$, the coordinate $\kappa$ shifts $2iK' = 2iK(k')$. Therefore the phase $\int_\beta pd\kappa$ should be divided by $2iK'$ to get the index $\mu$. But we find the correct number to divide by is $i\pi$.

The remaining work is the same as that in the last section: use the differential operators in (78) and (79) to generate the next two integrals about $p_2, p_4$, so we get the series expansion of $\mu = \mu(\omega, k)$. Then inverse the series $\mu(\omega, k)$ to get a series $\omega = \omega(\mu, k)$. Finally we get the series expansion for the eigenvalue $A$ using $A = n(n - 1)k^2\omega = \kappa^2\omega$. Or we can substitute (82) into the series $\mu = \mu(\omega, k)$, and check if the relation holds. We have checked the first few terms in (82), up to terms of order $\kappa^{-2}$.

From the magnetic expansion $A(\mu, k)$ given above, using the relation (10) and changing $\mu, k$ to $\nu, q$, we get the magnetic expansion of $B(\nu, q)$. Of course, $B(\nu, q)$ is not economic for the magnetic expansion.

### 6.2 Dyonic expansion

The eigenvalues near the dyonic point are mirror to the ones at the magnetic point. This is most obvious for $B(\nu, q) \sim \tilde{u}$. From (47) we know that the magnetic point $\tilde{u} = e_2m^2/8$ is mapped to the dyonic point $\tilde{u} = e_3m^2/8$ by $e_2 \rightarrow e_3$, or $q^{\frac{1}{2}} \rightarrow -q^{\frac{1}{2}}$. Also from the story of Mathieu equation [10], we know that the index of magnetic region is mapped to the index of dyonic region by $\nu \rightarrow i\nu$. Therefore by the mirror map $q^{\frac{1}{2}} \rightarrow -q^{\frac{1}{2}}, \nu \rightarrow i\nu$ we get the dyonic expansion $B_d$ from its magnetic expansion $B_m$.

It is also interesting to look at how the dyonic expansion of $A$ is mapped from its magnetic expansion. For this purpose, we need to know how the other parameters are changed by the map. It simple to see that under $q^{\frac{1}{2}} \rightarrow -q^{\frac{1}{2}}$, we have $e_2$ and $e_3$ interchanged while $e_1$ unchanged. Furthermore, since

$$k = 4q^{\frac{1}{4}} \prod_{n=1}^{\infty} \left( \frac{1 + q^n}{1 + q^n - 2} \right)^4, \quad k' = \prod_{n=1}^{\infty} \left( \frac{1 - q^{n-\frac{1}{2}}}{1 + q^{n-\frac{1}{2}} - 2} \right)^4.$$  (89)

The map leads to a simple change for $k$ and $k'$

$$k' \rightarrow \frac{1}{k'}, \quad k \rightarrow i\frac{k}{k'}.$$  (90)
As for the magnetic expansion, $A$ and $B$ satisfy
\begin{align}
B_m &= (e_1 - e_2)A_m(\mu, k) - e_2 n(n - 1) \\
&= (e_1 - e_2)A_m \left( \frac{i\nu}{\sqrt{e_1 - e_2}}, k(q) \right) - e_2 n(n - 1),
\end{align}
(91)
The mirror map changes it to
\begin{align}
B_d &= (e_1 - e_3)A_m \left( \frac{i\nu}{\sqrt{e_1 - e_3}}, i \frac{k}{k'} \right) - e_3 n(n - 1).
\end{align}
(92)
Then substituent this into the dyonic relation $B_d = (e_1 - e_2)A_d(\mu, k) - e_2 n(n - 1)$, we get
\begin{align}
A_d(\mu, k) &= \frac{e_1 - e_3}{e_1 - e_2} A_m \left( \frac{i\nu}{\sqrt{e_1 - e_3}}, i \frac{k}{k'} \right) - \frac{e_3 - e_2}{e_1 - e_2} n(n - 1) \\
&= \frac{e_1 - e_3}{e_1 - e_2} A(i\mu \sqrt{\frac{e_1 - e_3}{e_1 - e_2}}, i \frac{k}{k'}) - \frac{e_3 - e_2}{e_1 - e_2} n(n - 1).
\end{align}
(93)
where the function $A$ is the magnetic expansion [82] given in the previous subsection. Using
\begin{align}
\frac{e_1 - e_3}{e_1 - e_2} = k'^2, \quad \frac{e_3 - e_2}{e_1 - e_2} = k^2,
\end{align}
(94)
it simplifies to
\begin{align}
A_d = k'^2 A \left( \frac{i\mu}{k'}, \frac{ik}{k'} \right) - k^2 n(n - 1),
\end{align}
(95)
with the first few terms expanded as
\begin{align}
A_d &= -\kappa^2 + i2\kappa\mu + \frac{1}{2^3} (1 - 2k^2)(\frac{4\mu^2}{k'^2} + 1) \\
&\quad + i \left( \frac{1 - 2k^2}{k'} \right)^2 \left( \frac{4\mu^2}{k'^2} + \frac{3\mu}{k'} \right) + i k^2 k' \left( \frac{4\mu^3}{k'^3} + \frac{5\mu}{k'} \right) + \cdots.
\end{align}
(96)
A WKB analysis can be carried out for this expansion. Set $A = -\kappa^2 + \tilde{A}$, i.e., $\omega = -1 + \tilde{\omega}$, then the Lamé equation (67) becomes
\begin{align}
\frac{\epsilon^2}{2} \Phi'' - [\tilde{\omega} - cn^2 \alpha]\Phi = 0,
\end{align}
(97)
The WKB solution for this equation is very similar as in the magnetic case, now with the leading order solution $\tilde{\rho}_0 = i\sqrt{2}\sqrt{\tilde{\omega} - cn^2 \alpha}$, and the integral contour is $\gamma = -(\alpha + \beta)$,
\begin{align}
(i\sqrt{2})^{-1} \int_{\gamma} \tilde{\rho}_0 d\alpha = -2 \int_{0}^{\kappa + i\kappa'} \sqrt{\tilde{\omega} - cn^2 \alpha} d\alpha = -2 \int_{0}^{\kappa} \sqrt{\tilde{\omega} - \cos^2 \varphi} d\varphi.
\end{align}
(98)
In the dyonic region, the eigenvalue is expanded for $\kappa \gg 1, \kappa \gg \mu$, i.e. $\tilde{\omega} \ll 1$, therefore using
\begin{align}
\sqrt{\tilde{\omega} - \cos^2 \varphi} = i(\cos \varphi - \frac{\tilde{\omega}}{2\cos \varphi} - \frac{\tilde{\omega}^2}{8\cos^3 \varphi} - \frac{\tilde{\omega}^3}{16\cos^5 \varphi} - \frac{5\tilde{\omega}^4}{128\cos^7 \varphi} + \cdots),
\end{align}
(99)
the integrals we need to do are
\[ \oint_\gamma \frac{1}{\Delta \cos^{2n-1} \varphi} d\varphi = 2 \int_0^{\varphi_0} \frac{1}{\Delta \cos^{2n-1} \varphi} d\varphi, \quad n = 0, 1, 2, 3, \cdots. \] (100)

The first few of them gives
\[ \oint_\gamma \sqrt{\omega - \text{cn}^2 \varphi} d\varphi = -\pi \left( \frac{1}{2k^2} \sqrt{\omega + 1 - 2k^2} \varphi + \frac{8k^2 - 8k^2 + 3}{128k^5} \varphi^3 + \frac{5(1 - 2k^2)(24k^4 - 24k^2 + 15)}{6144k^7} \varphi^4 + \cdots \right). \] (101)

For the higher order WKB integrals, as in the previous cases, for odd order integrals \( \oint_\gamma \tilde{p}_{2l+1} d\kappa = 0 \). For even order integrals, they are generated from \( \oint_\gamma \tilde{p}_0 d\kappa \) by some differential operators. We get \( \tilde{D}_2(\omega, \partial_\omega, k) \) from
\[ \oint_\gamma \tilde{p}_2 d\kappa = \oint_\gamma \frac{i}{8\sqrt{2}} \frac{\text{sn}^2 \omega \text{cn}^2 \omega \text{dn}^2 \omega}{(\omega - \text{cn}^2 \omega)^{3/2}} d\kappa \]
\[ = -\frac{1}{12} \left[ (-1 + k^2 + 2\omega - 4k^2\omega + 3k^2\omega^2) \partial_\omega^2 \right. \\
\left. + (1 - 2k^2 + 3k^2\omega) \partial_\omega - \frac{3}{4} k^2 \right] \oint_\gamma \tilde{p}_0 d\kappa. \] (102)

and \( \tilde{D}_4(\omega, \partial_\omega, k) \) from
\[ \oint_\gamma \tilde{p}_4 d\kappa = \oint_\gamma \frac{1}{64\sqrt{2}k^4} \left[ \frac{7}{12} \frac{(G(\kappa))^2}{(\omega - \text{cn}^2 \omega)^{7/2}} - \frac{1}{12} \frac{G''(\kappa)}{(\omega - \text{cn}^2 \omega)^{5/2}} \right] d\kappa \]
\[ = \frac{1}{64} \left\{ \frac{14}{45} \left[ (1 - k^2)^2 - 4(1 - 3k^2 + 2k^4)\omega + 2(2 - 11k^2 + 11k^4)\omega^2 \\
+ 12k^2(1 - 2k^2)\omega^3 + 9k^4\omega^4 \right] \partial_\omega^4 \\
- \frac{8}{3} \left[ (1 - 3k^2 + 2k^4) - 2(1 - 5k^2 + 5k^4)\omega - 8k^2(1 - 2k^2)\omega^2 - 8k^4\omega^3 \right] \partial_\omega^3 \\
+ \frac{1}{3} \left[ (10 - 4k^2 + 3k^4) + 6k^2(1 - 2k^2)\omega + 99k^4\omega^2 \right] \partial_\omega^2 \\
+ 2[k^2(1 - 2k^2) + 3k^4\omega] \partial_\omega - \frac{1}{8} k^4 \right\} \oint_\gamma \tilde{p}_0 d\kappa. \] (103)

We find that for the dyonic expansion, the Floquet index is given by
\[ \mu = \frac{k'}{\pi} \oint_\gamma d\kappa \tilde{p}(\kappa). \] (104)

At the moment we do not have a satisfying explanation for the factor \( k'/\pi \), but the relation is indeed satisfied. We check this up to terms of order \( \kappa^{-2} \) in \( \{95\} \).

However, some interesting phenomena are observed. Look at the leading order contour integrals for the magnetic and dyonic case, formulae \( \{87\} \) and \( \{101\} \), they are related in a simple way:
\[ k \to i \frac{k}{k'}, \quad \omega \to -\omega, \quad \text{leads to:} \quad \frac{1}{\epsilon} \oint_\beta \sqrt{\omega + \text{sn}^2 \omega} d\kappa \to -\frac{1}{\epsilon} \oint_\gamma \sqrt{\omega - \text{cn}^2 \omega} d\kappa. \] (105)
Note that $\epsilon \sim \kappa^{-1} \sim k^{-1}$ also changes as $\epsilon \to -ik'\epsilon$. Then look at the differential operators, formulae (78,79) and (102,103), they are also related in a simple way:

$$k \to ik, \quad \omega \to -\bar{\omega}, \quad \text{leads to:} \quad \epsilon^nD_n(\omega, \partial_\omega, k) \to \epsilon^n\bar{D}_n(\bar{\omega}, \partial_{\bar{\omega}}, k), \quad n = 2, 4.$$  

Combining the two facts together, under the mirror map we have

$$\int_\beta d\epsilon p(\epsilon) \to -\int_\gamma d\bar{\epsilon} p(\bar{\epsilon}), \quad \text{i.e.} \quad \mu \to i\frac{\mu}{k}.$$  

This is consistent with (95) where the index appears as $i\mu/k'$.

### 7 Conclusion

Combinatorics is a very interesting and fruitful subject of mathematics, its relation to integrable models is not new\[46\]. In recent years, some progresses continue to reveal its fascinating connection to quantum field theory, integrable model, and string/M theory\[22, 36, 47, 48\]. Based on a relation between $\mathcal{N} = 2$ theory and quantum integrable system\[36, 37\], we provide a quantum field theory approach to the eigenvalues of Mathieu and Lamé equations. The approach is combinatorial, based on the instanton counting program\[21\].

Our main result is the expansion formula (57) and (95). At the moment we know some other independent methods that can derive the same results. The first one is E. Langmann’s results\[15\] that give an expansion equivalent to (57), we have shown the equivalence in Appendix 10. The second is the WKB analysis we give in section 5 and 6, giving all three expansions. Also in the decoupling limit the expansions correctly reduce to expansion of the Mathieu eigenvalues.

We can achieve a relatively simple expansion formula of (57) partly because we have chosen $q$ as the right expansion parameter. A similar lesson comes form the recently discovered AGT relation(Alday-Gaiotto-Tachikawa)\[48\], which connects four dimensional $\mathcal{N} = 2$ gauge theory to two dimensional Liouville type conformal field theory(CFT). There, expanding the CFT conformal block according to the gauge theory partition function has shielded new light on nonrational CFTs. On the other hand, studying the Lamé equation of the form (5) in the 6th section reveals interesting duality properties of the $\mathcal{N} = 2^*$ theory.

In this paper we do not discuss the combinatorial connection to the eigenfunctions: the Mathieu function and Lamé function/polynomial. There are examples where a full combinatorial approach to the eigenvalue and eigenstate exists. The Calogero-Sutherland model is a famous example, see a very recent discussion in [52], and relevant references therein. In fact, the Calogero-Sutherland model is the trigonometric limit of the elliptic Calogero-Moser
model. In another recent paper\cite{49}, it is argued that the AGT relation with full surface operator, in the semiclassical limit $\epsilon_2 \to 0$, gives the eigenfunction of the corresponding quantized Hamiltonian. See also\cite{51}. This maybe helpful for an explicit combinatorial construction of the eigenfunction.

While the basic tools of the paper, such as Seiberg-Witten theory, instanton calculation, have been extensively studied by mathematical physicists and have a solid mathematical foundation, the knowledge in this field is still expanding. Some points of our results lack rigorous mathematical proof. This includes the general form of $D_n(u, \partial_u)$ conjectured in\cite{41}(however see the Appendix (11)), the puzzling factors $(i\pi)^{-1}$ and $k' (\pi)^{-1}$, the convergence of the eigenvalue expansions, the precise meaning of the potential $F$ for the equations, the relation to Langmann’s method. We hope to clarify some of them in the future. Therefore, we do not present a mathematically rigorous treatment of the Mathieu and Lamé equation in this single paper, but just to show how the quantum field theories are related to these classical differential equation theory and can be used to obtain some results of them.

Our conclusion can be summarized as the following points:

(A): There are only three asymptotic expansion regions for both the Mathieu and Lamé eigenvalues, they are in one to one correspondence.

(B): The Lamé eigenvalue correctly reduces to the Mathieu eigenvalue in the limit $n \to \infty, q \to 0$ while $n^2(n - 1)^2 q \to \hbar^2/16$ or $\kappa \to i\hbar^{\frac{1}{2}}$ keep fixed.

(C): The WKB method gives a nontrivial check for the combinatorial approach. This also supports the gauge theory/integrable model proposal of\cite{36}.

(D): The asymptotic expansion can be consistently extended from the electric region to the magnetic and dyonic regions.

(E): Eq. (5) is more suitable for studying its behavior in the magnetic and dyonic regions, while Eq. (6) is more suitable for studying its behavior in the electric region. The information of dyonic expansion can be obtained from the magnetic expansion by a simple map.

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8 Appendix: Modular forms, Theta constants

We collect some basic facts about modular forms and theta constants we have used. Our main references are [6, 53, 54].

Theta constant \( \theta_i \) are theta function \( \theta_i(z; q) \) at \( z = 0 \):

\[
\begin{align*}
\theta_1 &= 0, \\
\theta_2 &= 2q^{\frac{1}{12}} \sum_{n=1}^{\infty} q^{\frac{n(n+1)}{2}} = 2q^{\frac{1}{12}}(1 + q + q^2 + q^3 + q^6 + q^{10} + q^{15} + \cdots), \\
\theta_3 &= 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = 1 + 2q^{\frac{1}{2^2}} + 2q^2 + 2q^3 + 2q^4 + 2q^5 + \cdots, \\
\theta_4 &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} = 1 - 2q^{\frac{1}{2^2}} + 2q^2 - 2q^3 + 2q^4 - 2q^5 + \cdots. \quad (108)
\end{align*}
\]

satisfying \( \theta_4^4 = \theta_2^4 + \theta_3^4 \), with the nome \( q = e^{2\pi i \tau} \), \( \tau \) defined by

\[
\tau = \frac{i K'}{K}. \quad (109)
\]

\( K \) and \( K' \) are the complete elliptic integrals of the first kind \( k \ll 1 \):

\[
K = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = \frac{\pi}{2} F\left( \frac{1}{2}, \frac{1}{2}; 1; k^2 \right) = \frac{\pi}{2} \theta_3^2
\]

\[
= \frac{\pi}{2} \left( 1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256} k^6 + \cdots \right). \quad (110)
\]

\[
K' = -i \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = \frac{\pi}{2} F\left( \frac{1}{2}, \frac{1}{2}; 1; 1-k^2 \right) = K(k')
\]

\[
= (1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256} k^6 + \cdots) \ln \frac{4}{k} + O(k^2). \quad (111)
\]

where \( \sin \varphi_0 = \frac{1}{k} \). The complete elliptic integrals of the second kind is

\[
E = \int_0^{\frac{\pi}{2}} d\varphi \sqrt{1-k^2 \sin^2 \varphi} = \frac{\pi}{2} F\left( -\frac{1}{2}, \frac{1}{2}; 1; k^2 \right)
\]

\[
= \frac{\pi}{2} \left( 1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256} k^6 - \cdots \right). \quad (112)
\]

The relation between the module \( k \), the complementary module \( k' \), and \( q \) are

\[
k = 4q^{\frac{1}{12}} \prod_{n=1}^{\infty} \left( 1 + \frac{q^n}{1 + q^{n-\frac{1}{2}}} \right)^4,
\]

\[
= 4q^{\frac{1}{12}} - 16q^{\frac{5}{12}} + 56q^{\frac{5}{6}} - 160q^{\frac{7}{6}} + 404q^{\frac{11}{6}} - 944q^{\frac{13}{6}} + \cdots.
\]

\[
k' = \sqrt{1-k^2} = \prod_{n=1}^{\infty} \left( \frac{1 - q^{n-\frac{1}{2}}}{1 + q^{n-\frac{1}{2}}} \right)^4 = \frac{\theta_4^2}{\theta_3^2}
\]

\[
q = \frac{1}{256} k^4 + \frac{1}{256} k^6 + \frac{29}{8192} k^8 + \frac{13}{4096} k^{10} + \frac{11989}{4194304} k^{12} + \frac{10879}{4194304} k^{14} + \cdots. \quad (113)
\]
The roots $e_i = \phi(z_i)$ of $\psi^2(z) = 4\psi^3(z) - g_2\psi(z) - g_3 = 0$ are related to the theta constants as
\begin{align*}
e_1 &= \frac{2}{3} \theta_3^4 - \frac{1}{3} \theta_2^4 = \frac{2}{3} + 16q + 16q^2 + \cdots, \\
e_2 &= \frac{1}{3} (\theta_3^4 + \theta_2^4) = \frac{1}{3} - 8q^{1/2} - 8q - 32q^{3/2} - 8q^2 + \cdots, \\
e_3 &= \frac{1}{3} \theta_3^4 + \frac{2}{3} \theta_2^4 = \frac{1}{3} + 8q^{1/2} - 8q + 32q^{3/2} - 8q^2 + \cdots, \tag{114}
\end{align*}
satisfying $e_1 + e_2 + e_3 = 0$. With this we have
\begin{align*}
k^2 &= \frac{e_3 - e_2}{e_1 - e_2} = \frac{\theta_2^4}{\theta_3^4}. \tag{115}
\end{align*}

When we derive expansion of $A$ from $B$, we have used
\begin{align*}
\frac{1}{e_1 - e_2} &= 1 - \frac{1}{2} k^2 - \frac{3}{32} k^4 - \frac{3}{64} k^6 - \frac{243}{8192} k^8 - \frac{345}{16384} k^{10} - \frac{4197}{262144} k^{12} - \cdots \tag{116}
\end{align*}
The second Eisenstein series is represented by
\begin{align*}
E_2(\tau) &= 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} = 1 - 24 \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \\
&= 1 - 24q - 72q^2 - 96q^3 - 168q^4 - 144q^5 - \cdots \tag{117}
\end{align*}
The equality of the two summation would be obvious from the following,
\begin{align*}
\sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} nq^{n(m+1)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} nq^{nm}, \\
\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} &= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(1-q^n)} = \sum_{n=1}^{\infty} \left( \sum_{r=1}^{\infty} q^r \right) \left( \sum_{s=0}^{\infty} q^s \right) \\
&= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \sum_{s=0}^{\infty} q^{r+s} = \sum_{n=1}^{\infty} \sum_{t \geq 1} t q^{tn}, \tag{118}
\end{align*}
the last step uses the fact for a fixed $t = r + s$, there are exactly $t$ pairs of $(r \geq 1, s \geq 0)$ satisfy it. $E_2$ is related to the Dedekind Eta function by
\begin{align*}
\frac{d}{d\tau} \ln \eta(\tau) = \frac{i\pi}{12} E_2(\tau). \tag{119}
\end{align*}

9 Appendix: Jacobi elliptic functions

The Jacobi elliptic functions satisfy
\begin{align*}
\text{sn}^2 \varphi + \text{cn}^2 \varphi &= 1, \\
\text{dn}^2 \varphi + k^2 \text{sn}^2 \varphi &= 1, \\
k^2 \text{cn}^2 \varphi + k^2 &= \text{dn}^2 \varphi, \\
\text{cn}^2 \varphi + k^2 \text{sn}^2 \varphi &= \text{dn}^2 \varphi. \tag{120}
\end{align*}
their derivative is
\[
\frac{d}{d\kappa} \text{sn}\kappa = \text{cn}\kappa \text{dn}\kappa,
\frac{d}{d\kappa} \text{cn}\kappa = -\text{sn}\kappa \text{dn}\kappa,
\frac{d}{d\kappa} \text{dn}\kappa = -k^2 \text{sn}\kappa \text{cn}\kappa.
\] (121)

Especially, in the WKB analysis we have used
\[
\frac{d}{d\kappa} (\text{sn}\kappa \text{cn}\kappa \text{dn}\kappa) = 1 - 2(1 + k^2) \text{sn}^2\kappa + 3k^2 \text{sn}^4\kappa.
\] (122)

The limit of $k = 0$ and $k = 1$:
\[
\text{sn}\kappa|_{k=0} = \sin\kappa, \quad \text{cn}\kappa|_{k=0} = \cos\kappa, \quad \text{dn}\kappa|_{k=0} = 1.
\]
\[
\text{sn}\kappa|_{k=1} = \tanh\kappa, \quad \text{cn}\kappa|_{k=1} = \frac{1}{\sinh\kappa}, \quad \text{dn}\kappa|_{k=0} = \text{cn}\kappa|_{k=1} = \frac{1}{\sinh\kappa}.
\] (123)

10 Appendix: Compare to Langmann’s expansion

E. Langmann developed an algorithm to derive the eigenvalue and eigenfunction of elliptic Calogero-Moser(-Sutherland) model for general particle number\[16, 15\]. In his paper\[15\] a series expansion for the 2-body eigenvalue when $q << 1$ is given(formulae 26a-d). Although his method is different method form combinatorial and WKB methods we present here, we will show that his expansion is equivalent to our expansion (57). Langmann considered the 2-particle Shrödinger equation
\[
\left[-\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + 2\lambda(\lambda - 1)V(x_1 - x_2)\right]\psi(x) = E\psi(x).
\] (124)

with potential $V(x) = -\varphi(x) + c_0$,
\[
c_0 = \frac{1}{12} - 2 \sum_{m=1}^{\infty} \frac{1}{(q^{m/2} - q^{-m/2})^2} = \frac{1}{12} E_2(q)
= \frac{1}{12} - 2q - 6q^2 - 8q^3 - 14q^4 + \cdots
\] (125)

After changing variables, the equation becomes the Lamé equation (6) describing the relative motion. The eigenvalue $E$ is explicitly given as an expansion in $q$: $E = E_0 + E_1q + E_2q^2 + E_3q^3 + \cdots$. Especially notice that, in the notation of paper[15], the relation
\[
2E_0 = P^2 + (n_1 + n_2)^2
\] (126)
The term \((n_1 + n_2)^2\) is the kinetic energy of center-of-mass motion.

The relation between the eigenvalues \(E\) and \(B\) is

\[
B = -2E + (n_1 + n_2)^2 + 4\lambda(\lambda - 1)c_0. \tag{127}
\]

if we identify notations as \(P = \nu, \lambda = n\). Then using the relation (127) to Langmann’s expansion of \(E\), we can write our expansion (57) in a slightly different form

\[
B = -\frac{n(n-1)}{3}E_2(q) - \frac{8n^2(n-1)^2q}{\nu^2 - 1}
\]

\[-\frac{8n^2(n-1)^2q^2}{(\nu^2 - 1)^3(\nu^2 - 4)} \times [n^2(n-1)^2(5\nu^2 + 7) - 12n(n-1)(\nu^2 - 1)^2
\]

\[+6(\nu^2 - 1)^2(\nu^2 - 2)]

\[-\frac{32n^2(n-1)^2q^3}{(\nu^2 - 1)^5(\nu^2 - 4)(\nu^2 - 9)} \times [2n^4(n-1)^4(9\nu^4 + 58\nu^2 + 29)
\]

\[-8n^3(n-1)^3(\nu^2 - 1)^2(7\nu^2 + 17)
\]

\[+4n^2(n-1)^2(\nu^2 - 1)^2(15\nu^4 - 37\nu^2 - 2)
\]

\[-24n(n-1)(\nu^2 - 1)^4(\nu^2 - 4)
\]

\[+3(\nu^2 - 1)^4(\nu^2 - 3)(\nu^2 - 4)] + \cdots \tag{128}
\]

Compare (57) and (128), the differences between them is that we have collected some terms coming from instanton contribution in (57) and they, together with \(-\frac{1}{3}n(n-1)(1 - 2E_2(q))\), equal to \(\frac{1}{3}n(n-1)E_2(q)\). Although we only give the \(q\)-expansion up to order \(q^3\), this fact can be checked further by calculating higher order instanton effects in gauge theory.

11 Appendix: A method to obtain the differential operators

When we derive the differential operators in (78, 79), we need to separate from the integrand some terms of total derivative. By discarding these total derivative terms from the contour integral, we can simplify the differential operators as far as possible. However, there is no general principle of how to do this. We have obtained a few leading order differential operators for the Mathieu and Lamé equations, it is necessary to find a systematic method to deal with higher orders to make the program complete. After some trial and error, we find a workable method.

The method is carried out as the following.

The first step, we solve the WKB relation for \(p_n(\kappa)\), explicitly carry out the derivatives and the final expressions are of the form

\[
p_{2l+1}(\kappa) = \frac{\text{sn}_n \text{cn}_n \text{dn}_n}{(\omega + \text{sn}_n \kappa)^{3l+1}} \times (\text{polynomial of sn}_n \kappa), \quad l = 0, 1, 2, \cdots, \tag{129}
\]
\[ p_{2l}(\tau) = \frac{1}{(\omega + \text{sn}^2 \tau)^{\frac{3l-\frac{1}{2}}{2}}} \times \text{(polynomial of sn}^2 \tau), \quad l = 0, 1, 2, \cdots, \quad (130) \]

The \( p_{2l+1}(\tau) \) can be written as a total derivative term, so its contour integrals is zero. \( p_1 \sim \partial_{\tau}(\ln p_0) \) is special. The polynomial in the numerator of \( p_{2l+1} \) for \( l \geq 1 \) are of the form \( c_s \text{sn}^{6l-2} \tau + c_s \text{sn}^{6l-4} \tau + \cdots + c_s \text{sn}^2 \tau + c_s \), where we use \( c_s \) to represent all of the coefficients whose explicit form is not important here, they are some polynomials of \( k^2, \omega \). Keep in mind in the following formulae all \( c_s \) are different from each other. Set \( \omega + \text{sn}^2 \tau = t \), then \( p_{2l+1} \) can be written as

\[ p_{2l+1}(\tau) = (\text{sn} \tau \text{cn} \tau \text{dn} \tau)(c_s t^{-(3l+1)} + c_s t^{-3l} + c_s t^{-(3l-1)} + \cdots + c_s t^{-3} + c_s t^{-2}), \quad (131) \]

Every term in \((131)\) is a total derivative given by

\[ c_s \partial_{\tau}(\omega + \text{sn}^2 \tau)^{-l'}, \quad l' = 1, 2, \cdots, 3l, \quad l \geq 1 \quad (132) \]

with proper choice of the coefficient. Therefore, \( \oint d\tau p_{2l+1}(\tau) = 0 \) as expected.

So we only need to deal with \( p_{2l} \) for \( l \geq 0 \). The polynomial in the numerator of \( p_{2l} \) are of the form \( c_s \text{sn}^{6l} \tau + c_s \text{sn}^{6l-2} \tau + \cdots + c_s \text{sn}^2 \tau + c_s \). Then \( p_{2l} \) can be expanded as a polynomial of \( t \),

\[ p_{2l}(\tau) = c_s t^{-(3l-\frac{1}{2})} + c_s t^{-(3l-\frac{3}{2})} + c_s t^{-(3l-\frac{5}{2})} + \cdots + c_s t^{-\frac{1}{2}} + c_s t^{\frac{1}{2}}, \quad (133) \]

It is generated by a differential operator acting on \( p_0 \sim \sqrt{t} \),

\[ \oint d\tau p_{2l}(\tau) = (c_s \partial_\omega^{3l} + c_s \partial_\omega^{3l-1} + c_s \partial_\omega^{3l-2} + \cdots + c_s \partial_\omega + c_s) \oint d\tau p_0(\tau), \quad (134) \]

The second step is to reduce the order of the derivative operator for \( p_{2l} \). The method is to find proper total derivative terms like \( \partial_{\tau}(* \cdots *) \) that can substitute terms of \( c_s t^{-(3l-\frac{1}{2})} + c_s t^{-(3l-\frac{3}{2})} + \cdots + c_s t^{-(2l+\frac{1}{2})} \) in \((133)\), then abandon them in the contour integral \( \oint p_{2l} \). Therefore, the order of polynomial integrand \((133)\) is increased and the corresponding differential operator for \( p_{2l} \) is also simplified,

\[ \oint d\tau p_{2l}(\tau) = \oint d\tau (c_s t^{-(2l-\frac{1}{2})} + c_s t^{-(2l-\frac{3}{2})} + c_s t^{-(2l-\frac{5}{2})} + \cdots + c_s t^{-\frac{1}{2}} + c_s t^{\frac{1}{2}}) \]

\[ = (c_s \partial_\omega^{2l} + c_s \partial_\omega^{2l-1} + c_s \partial_\omega^{2l-2} + \cdots + c_s \partial_\omega + c_s) \oint d\tau p_0(\tau), \quad (135) \]

We find the total derivative terms are generated by

\[ c_s \partial_\omega^{2l}(\omega + \text{sn}^2 \tau)^{-\left(3l-\frac{5}{2}-l'\right)}, \quad l' = 1, 2, \cdots, 2l - 1. \quad (136) \]
A nice property of this term is that its final expression can be expressed in terms of $\text{sn}^2 \kappa$, or in terms of $t$, as

$$c_s t^{-(3l + \frac{1}{2} - l')} + c_s t^{-(3l - \frac{1}{2} - l')} + c_s t^{-(3l - \frac{3}{2} - l')} + c_s t^{-(3l - \frac{5}{2} - l')}.$$  \hspace{1cm} (137)

Now we can discard some terms in (133) using (136). For $l' = 1$, choosing proper coefficient in (136), we can make the coefficient of the first term in (137) equals to the coefficient of the first term in (133), hence

$$p_{2l}(\kappa) = \left[ c_s \partial_\kappa \left( \frac{\text{sn} \text{cn} \text{dn}}{\omega + \text{sn}^2 \kappa} \right)^{3l - \frac{1}{2}} - \left( c_s t^{-(3l - \frac{3}{2})} + c_s t^{-(3l - \frac{5}{2})} + c_s t^{-(3l - \frac{7}{2})} \right) \right]$$

$$+ \left( c_s t^{-(3l - \frac{3}{2})} + c_s t^{-(3l - \frac{5}{2})} + \cdots c_s t^{\frac{1}{2}} + c_s t^{\frac{3}{2}} \right),$$  \hspace{1cm} (138)

Now the total derivative term can be abandoned, and the order $p_{2l}$ is increased to $t^{-(3l - \frac{3}{2})}$, meanwhile the coefficients of other three terms are changed. Repeat this process for $l' = 2$, we can increase the order of $p_{2l}$ to $t^{-(3l - \frac{5}{2})}$, and so on. For every $l' \in \{1, 2, \cdots, l\}$, the process of removing (136) is carried out once.

The third step is to minimize the differential operator. After carrying out the process for $l' = 1, 2, \cdots, l$, we can increase the order of $p_{2l}$ and the differential operator to the form in (135). But we notice its coefficients can be further simplified. The reason is the following: we assume the differential operators for the Mathieu equation, conjectured in [41], is the simplest form we can get through the simplification process. We say this kind of differential operators minimal. The differential operators of the Lamé equation, if they are minimal, should reduce to the minimal differential operators of the Mathieu equation in the limit $k \to 0, \omega \to (\omega - 1)/2$. Then we find the differential operator obtained after performing the second step for $l' = 1, 2, \cdots, l$ is not minimal, some redundant terms can be further discarded. Continue the process for $l' = l + 1, l + 2, \cdots, 2l - 1$, for every $l' \in \{l + 1, l + 2, \cdots, 2l - 1\}$, the process of removing (136) is repeated for a few times. The simplification does not increase the order of $p_{2l}$, but simplifies the remaining coefficients to their minimal form.

Using this method, we successfully derive the first few differential operators for $p_{2l}$ of the Lamé equation, for the first few $l$. In the limit $k \to 0, \omega \to (\omega - 1)/2$, these differential operators correctly reduce to the minimal differential operators of the Mathieu equation derived in [41].

A few comments here.

When we derive formulae (78) and (79), during the process we discard any kind of total derivative terms if there is a chance to do so at any stage. This introduces some arbitrariness, the final differential operator $D_{2l}(\omega, \partial_\omega, k)$ may take different forms if it is done by different people. The arbitrariness also makes it difficult to work out the result for higher order $l$. 
This kind of arbitrariness is cured by the systematic method, we get a unique differential
operator for every $p_{2l}$. Following our algorithm, we obtain the same $D_2(\omega, \partial_\omega, k)$ as in (78). 
But the $D_4(\omega, \partial_\omega, k)$ is different form (79), we have

$$\oint p_4 \, d\kappa = \frac{1}{64} \left\{ \frac{2}{135} \left[ 21 + (84 + 359k^2)\omega + (84 + 1394k^2 + 359k^4)\omega^2 
+ k^2(1077 + 1352k^2)\omega^3 + 1014k^4\omega^4 \right] \right\} p_0 \, d\kappa. \hspace{1cm} (139)$$

The differential operators in (79) and (139) are the same form but differ by numerical coefficients. The difference is caused by some total derivative terms, therefore they give the same series expansion when applied to $\oint p_0 \, d\kappa$.

Apparently, the minimal differential operators of the Lamé equation in the form (97) can be obtained in the same way, with the total derivative terms for $p_{2l}$ generated by

$$c_s \partial_x^2 (\tilde{\omega} - cn^2 \x) - (\frac{3l}{2} - \frac{3}{2} - l'), \hspace{1cm} l' = 1, 2, \cdots, 2l - 1. \hspace{1cm} (140)$$

The relation between $D_n(\omega, \partial_\omega, k)$ and $\tilde{D}_n(\tilde{\omega}, \partial_{\tilde{\omega}}, k)$ in (100) still holds. And the differential operators of the Mathieu equation can be obtained following the same steps, with the total derivative terms for $p_{2l}$ generated by

$$c_s \partial_x^2 (w - \cos 2\x) - (\frac{3l}{2} - \frac{3}{2} - l'), \hspace{1cm} l' = 1, 2, \cdots, 2l - 1. \hspace{1cm} (141)$$

with the notation used in [41]. This also gives an explanation for the conjectural form of $D_n(u, \partial_u)$ in that paper.
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