Space Efficient Deterministic Approximation of String Measures

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Abstract

We study approximation algorithms for the following three string measures that are widely used in practice: edit distance, longest common subsequence (LCS), and longest increasing sequence (LIS). All three problems can be solved exactly by standard algorithms that run in polynomial time with roughly $O(n)$ space, where $n$ is the input length, and our goal is to design deterministic approximation algorithms that run in polynomial time with significantly smaller space. Towards this, we design several algorithms that achieve $1 + \varepsilon$ or $1 - \varepsilon$ approximation for all three problems, where $\varepsilon > 0$ can be any constant. Our algorithms use space $n^{\delta}$ for any constant $\delta > 0$ and have running time essentially the same as or slightly more than the standard algorithms. Our algorithms significantly improve previous results in terms of space complexity, where all known results need to use space at least $\Omega(\sqrt{n})$. Some of our algorithms can also be adapted to work in the asymmetric streaming model [SS13], and output the corresponding sequence.

Our algorithms are based on the idea of using recursion as in Savitch’s theorem [Sav70], and a careful modification of previous techniques to make the recursion work. Along the way we also give a new logspace reduction from longest common subsequence to longest increasing sequence, which may be of independent interest.

1 Introduction

Strings are fundamental objects in computer science, and problems related to strings are among the most well studied problems in the literature. In this paper, we consider the problem of approximating the following three classical string measures:

**Edit distance**: given two strings, the edit distance (ED) between these strings is the minimum number of insertions, deletions, and substitutions to transform one string into another.

**Longest common subsequence**: given two strings, the longest common subsequence (LCS) between these strings is the longest subsequence that appears in both strings.

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Longest increasing subsequence: given one string and a total order over the alphabet, the longest increasing subsequence (LIS) is the longest sequence in the string that is in an increasing order.

These problems have found applications in a wide range of applications, including bioinformatics, text processing, compilers, data analysis and so on. As a result, all of them have been studied extensively. Specifically, suppose the length of each string is \( n \), then both ED and LCS can be computed in time \( O(n^2) \) and space \( O(n) \) using standard dynamic programming. For LIS, it is known that it can be computed exactly in time \( O(n \log n) \) with space \( O(n \log n) \). However, in practical applications these problems often arise in situations of huge data sets, where the magnitude of \( n \) can be in the order of billions (for example, when one studies human gene sequences). Thus, even a running time of \( \Theta(n^2) \) can be too costly. Similarly, even a \( \Theta(n) \) memory consumption can be infeasible in many applications, especially for basic tasks such as ED, LCS, and LIS.

Motivated by this, there have been many attempts at reducing the time of computing ED and LCS, however none of these attempts succeeded significantly. Recent advances in fine grained complexity provide a justification for these failures, where the work of Backurs and Indyk [BI15] and the work of Abboud, Backurs, and Williams [ABW15] show that no algorithm can compute ED or LCS in time \( O(n^{1.99}) \) unless the strong Exponential time hypothesis [IPZ01] is false. Since then, the focus has been on developing approximation algorithms for ED and LCS with significantly better running time, and there has been much success here. Specifically, currently the best known randomized algorithms can achieve a polylogarithmic approximation of ED in near linear time [AKO10], and a constant factor approximation in truly sub-quadratic time [CDG+19]. Furthermore, when the edit distance is near-linear, two recent works [BR20] [KS19] achieve a constant factor approximation in near-linear time. For LCS the situation appears to be harder, and the best known randomized algorithm [HSSS19] only achieves an \( O(n^{0.998}) \) approximation using linear time, which slightly beats the trivial \( O(\sqrt{n}) \) approximation obtained by sampling. Additionally, there is a trivial linear time algorithm that can approximate LCS within a factor of \( \frac{1}{|\Sigma|} \) where \( \Sigma \) is the alphabet of the strings. A recent work [RSSS19] further provides a randomized algorithm in truly sub-quadratic time that achieves an approximation factor of \( O(\lambda^3) \), where \( \lambda \) is the ratio of the optimal solution size over the input size. Another recent work by Rubinstein and Song [RS20] shows how to reduce LCS to ED for binary strings, and uses the reduction to achieve a \( \frac{1}{\lambda} + \varepsilon \) approximation algorithm for LCS, where \( \varepsilon > 0 \) is some constant and the algorithms runs in near linear time.

Despite these success, the equally important question of approximating ED and LCS using small space has not been studied in depth. Only a few previous works have touched on this topic, but with a different focus. For example, assume the edit distance between two strings is at most \( k \), the work of Chakraborty et. al. [CGK16] provides a randomized streaming algorithm that obtains an \( O(k) \) approximation to ED, using linear time and \( O(\log n) \) space. Based on this, the work of Belazzougui and Zhang [BQ16] provides a randomized streaming algorithm for computing ED and LCS exactly using polynomial time and \( poly(k \log n) \) space. More generally, inspired by the work of Andoni et. al. [AKO10], Saks et. al. [SS13] studied the asymmetric data streaming model. This model allows one to have one way streaming access to one string (say \( x \), but random access to the other string (say \( y \)). [SS13] gives a \( 1 + \varepsilon \) deterministic approximation of ED in this model using space \( O(\sqrt{n \log n} / \varepsilon) \), as well as a randomized algorithm that achieves an \( \varepsilon n \) additive approximation to LCS in this model, using space \( O(k \log^2 n / \varepsilon) \) where \( k \) is the maximum number of times any symbol appears in \( y \).

For LIS the situation is slightly better. In particular, the work of Gopalan et. al. [GJKK07] provides a deterministic streaming algorithm that approximates LIS to within a \( 1 - \varepsilon \) factor, using time \( O(n \log n) \) and space \( O(\sqrt{n / \varepsilon \log n}) \); while a very recent work by Kiyomi et. al.
KOO+18 obtains a deterministic algorithm that computes LIS exactly using $O(n^{1.5} \log n)$ time and $O(\sqrt{n} \log n)$ space.

In this paper we seek to better understand the space complexity of these problems, while at the same time maintaining a polynomial running time. The first and most natural goal would be to see if we can compute for example ED and LCS exactly using significantly smaller space (i.e., truly sub-linear space of $n^{1-\alpha}$ for some constant $\alpha > 0$). However, this again appears to be hard as no success has been achieved so far. Thus, we turn to a more realistic goal — to approximate ED and LCS using significantly smaller space. For LIS, our goal is to use approximation to further reduce the space complexity in CJKK07 and KOO+18.

More broadly, the questions studied in this paper are closely related to the general question of non-deterministic small space computation vs. deterministic small space computation. Specifically, the decision versions of all three problems (ED, LCS, LIS) can be easily shown to be in the class NL (i.e., non-deterministic log-space), and the question of whether NL = L (i.e., if non-deterministic log-space computation is equivalent to deterministic log-space computation) is a major open question in complexity theory. Note that if NL = L, this would trivially imply polynomial-time algorithms for exactly computing ED, LCS, and LIS in logspace. However, although we know that NL ⊆ P and NL ⊆ SPACE($\log^2 n$) (by Savitch’s theorem [Sav70]), it is not known if every problem in NL can be solved simultaneously in polynomial time and polylog space. In fact, it is not known if an NL-complete language (e.g., directed s-t connectivity) can be solved simultaneously in polynomial time and strongly sub linear space (i.e., space $n^{1-\alpha}$ for some fixed constant $\alpha > 0$). Thus, studying special problems in NL such as ED, LCS, and LIS, and the relaxed version of approximation is a reasonable first step towards the major open problems.

We show that we can indeed achieve our goals. Specifically, for all three problems ED, LCS, and LIS, we give efficient deterministic approximation algorithms that can achieve $1 + \varepsilon$ or $1 - \varepsilon$ approximation, using significantly smaller space than all previous works. At the same time, the running time of our algorithms is essentially the same or only slightly more than the standard dynamic programming approach. This is in contrast to the time complexity of ED and LCS, where we only know how to beat the standard dynamic programming significantly by using randomized algorithms.

We have the following theorems.

**Theorem 1.** Given any strings $x$ and $y$ each of length $n$, and any constants $\varepsilon, \delta \in (0, 1)$, there is a deterministic algorithm that computes a $(1 + \varepsilon)$-approximation of ED($x, y$) in $O_{\varepsilon,\delta}(n^2)$ time with $O_{\varepsilon,\delta}(n^5)$ bits of space. The algorithm can be adapted to work in the asymmetric streaming model with $O_{\varepsilon}(\sqrt{n})$ bits of space.

Note that our algorithm for ED uses roughly the same running time as the standard dynamic programming, but much smaller space. Indeed, we can use space $n^5$ for any constant $\delta > 0$. This also significantly improves the previous result of [SS13], which needs to use space $\Omega(\sqrt{n} \log n)$.

Next we have the following theorem for LCS.

**Theorem 2.** Given any strings $x$ and $y$ each of length $n$, and any constant $\varepsilon \in (0, 1)$, for any integer $d \geq 2$, there is a deterministic algorithm that computes a $(1 - \varepsilon)$-approximation of LCS($x, y$) in $O_{\varepsilon,d}(n^{3-\frac{1}{d}} \log^{d-1} n)$ time with $O_{\varepsilon,d}(n^{\frac{1}{d}} \log n)$ bits of space. Furthermore, the algorithm can output a common subsequence whose length is at least $(1 - \varepsilon)$LCS($x, y$) in $O_{\varepsilon,d}(n^{3} \log^{d-1} n)$ time with $O_{\varepsilon,d}(n^{\frac{1}{d}} \log n)$ bits of space. In addition, the special case of $d = 2$ can be be adapted to work in the asymmetric streaming model, with $O_{\varepsilon}(n^{\frac{1}{2}} \log n)$ time and $O_{\varepsilon}(\sqrt{n} \log n)$ bits of space.

\[1\] However, in the asymmetric streaming model, our result does not improve the result of [SS13].
To the best of our knowledge, Theorem 2 is the first $1 - \varepsilon$ approximation of LCS using truly sub-linear space, and in fact we can achieve space $n^\delta$ for any constant $\delta > 0$. Meanwhile the running time is only slightly larger than the standard dynamic programming approach.

For LIS, we also give an efficient deterministic approximation algorithms that can achieve $1 - \varepsilon$ approximation, with better space complexity than that of [GJKK07] and [KOO+18]. In particular, we can achieve space $n^\delta$ for any constant $\delta > 0$. We have the following theorem.

**Theorem 3.** Given any string $x$ of length $n$ and any constant $\varepsilon \in (0,1)$, for any integer $d \geq 2$, there is a deterministic algorithm that computes a $(1 - \varepsilon)$-approximation of $\text{LIS}(x)$ in $O_{\varepsilon,d}(n^{2-\frac{1}{d}}\log^{d-1} n)$ time with $O_{\varepsilon,d}(n^{\frac{1}{d}} \log n)$ bits of space. Furthermore, the algorithm can output an increasing subsequence of $x$ whose length is at least $(1 - \varepsilon)\text{LIS}(x)$ in $O_{\varepsilon,d}(n^{2-\frac{1}{d}}\log^{d-1} n)$ time with $O_{\varepsilon,d}(n^{\frac{1}{d}} \log n)$ bits of space.

**Remark.** In all our theorems, the parameter $\varepsilon$ can actually be slightly sub-constant, i.e., $o(1)$.

### 1.1 Technique Overview

The starting point of all our space efficient approximation algorithms is the well known Savitch’s theorem [Sav70], which roughly shows that any non-deterministic algorithm running in space $s \geq \log n$ can be turned into a deterministic algorithm running in space $O(s^2)$ by using recursion. Since all three problems of ED, LCS, and LIS can be computed exactly in non-deterministic logspace, this trivially gives deterministic algorithms that compute all of them exactly in space $O(\log^2 n)$. However in the naive way, the running time of these algorithms become quasi-polynomial.

To reduce the running time, we turn to approximation. Here we use two different sets of ideas. The first set of ideas applies to ED. Note that the reason that the above algorithm for computing ED runs in quasi-polynomial time, is that in each recursion we are computing the ED between all possible substrings of the two input strings. To avoid this, we use an idea from [HSS19], which shows that to achieve a good approximation, we only need to compute the ED between some carefully chosen substrings of the two input strings. Using this idea in each level of recursion gives us the space efficient approximation algorithms for ED.

The second set of ideas applies to LCS and LIS. Here, we first give a small space reduction from LCS to LIS, and then we can focus on approximating LIS. Again, the reason that the naive $O(\log^2 n)$ space algorithm for LIS runs in quasi-polynomial time, is that in each recursion we are breaking the input string into all possible cases of two substrings, computing the LIS which ends and starts at the break point, and taking the maximum of the sums. To get an approximation, we use the patience sorting algorithm for computing LIS exactly [AD99], and the modification in [GJKK07] which gives an approximation of LIS using smaller space by equivalently looking at only some carefully chosen cases of breaking the input string into substrings. The rough idea is to use the algorithm in [GJKK07] recursively, but achieving this requires careful modification of the algorithm in [GJKK07], both to make the recursion work and to make it work under the reduction from LCS to LIS.

We now give more details below.

#### 1.1.1 Edit Distance

As discussed before, our approximation algorithm for ED is based on recursion. In each level of recursion, we use an idea from [HSS19] to approximate the edit distance between certain pairs of substrings. We start by giving a brief description of the algorithm in [HSS19].
Let $x$ and $y$ be two input strings each of length $n$. Assume we want to get a $(1 + \varepsilon)$-approximation of $\text{ED}(x, y)$ where $\varepsilon$ is any constant in $(0, 1)$. Let $\delta \in (0, 1)$ be a constant which we choose later, and $\varepsilon' = \varepsilon/10$. The algorithm guesses a value $\Delta \leq n$ which is supposed to be a $(1 + \varepsilon')$-approximation of $\text{ED}(x, y)$. If this is true then the algorithm will output a $(1 + \varepsilon)$-approximation of $\text{ED}(x, y)$. To get rid of guessing, we can try every $\Delta \leq n$ such that $\Delta = \lceil (1 + \varepsilon')^i \rceil$ for some integer $i$ and take the minimum. This does not affect the space required, and only increases the time complexity by a logarithmic factor.

The idea is to first divide $x$ into $N = n^\delta$ blocks each of length $n^{1-\delta}$. For simplicity, we fix an optimal alignment between $x$ and $y$ such that $x[i, r]$ is matched to the substring $y[\alpha_i, \beta_i]$ and the intervals $[\alpha_i, \beta_i]$ are disjoint and span the entire length of $y$. We say that an interval $[\alpha', \beta']$ is an $(\varepsilon, \Delta)$-approximately optimal candidate of the block $x^i = x[i, r_i]$ if the following two conditions hold:

$$\alpha_i \leq \alpha' \leq \alpha_i + \delta \Delta \frac{N}{\varepsilon}$$

$$\beta_i - \varepsilon \frac{\Delta}{N} - \varepsilon \text{ED}(x[i, r_i], y[\alpha_i, \beta_i]) \leq \beta' \leq \beta_i$$

[HSS19] showed that, for each block of $x$ that is not matched to a too large or too small interval in $y$, there is a way to choose $O_\varepsilon(n^{6\delta} \log n)$ intervals such that one of them is an $(\varepsilon', \Delta)$-approximately optimal candidate. Then we can compute the edit distance between each block and all of its corresponding candidate intervals, which gives $O_\varepsilon(n^{2\delta} \log n)$ values. After this, we can use dynamic programming to find a $(1 + \varepsilon)$-approximation of the edit distance if $\Delta$ is a $(1 + \varepsilon')$-approximation of $\text{ED}(x, y)$.

Computing the edit distance of each block in $x$ with one of its candidate intervals in $y$ takes $O_\varepsilon(n^{1-\delta} \log n)$ bits of space (we assume each symbol can be stored with space $O(\log n)$). We can run this algorithm sequentially and reuse the space for each computation. Storing the edit distance of each pair takes $O_\varepsilon(n^{2\delta})$ space. Thus, if we take $\delta = 1/3$, the above algorithm uses a total of $O_\varepsilon(n^{2\delta})$ bits of space.

We now run the above algorithm recursively to further reduce the space required. Let $\delta$ be a small constant in $(0, 1)$. Our algorithm takes four inputs: two strings $x$ and $y$ each of length $n$, $N = n^\delta$, and $\varepsilon \in (0, 1)$. The goal is to output a $(1 + \varepsilon)$ approximation of $\text{ED}(x, y)$ with $O_\varepsilon(N^2)$ space. Similarly, we first divide $x$ into $n^\delta$ blocks. We try every $\Delta$ that is equal to $\lceil (1 + \varepsilon')^i \rceil$ for some integer $i$, and for each $\Delta$ there is a set of candidate intervals depending on $\Delta$. Then, for each block of $x$ and each of its $O_\varepsilon(n^\delta \log n)$ candidate intervals, instead of computing the edit distance exactly, we recursively call our space efficient approximation algorithm with this pair as input, while keeping $N$ unchanged and decreasing $\varepsilon$ by a factor of $2$. We argue that if the recursive call outputs a $(1 + \varepsilon/2)$-approximation of the actual edit distance, the output of the dynamic programming increases by at most a $(1 + \varepsilon/2)$ factor. Thus if $\Delta$ is a $(1 + \varepsilon')$-approximation of $\text{ED}(x, y)$, the output of the dynamic programming is guaranteed to be a $(1 + \varepsilon)$-approximation. The recursion stops whenever the input string has length at most $N$. In this case, we compute the edit distance exactly with $O(n^\delta \log n)$ space.

Notice that at each level of the recursion, the first input string is divided into $N$ blocks if it has length larger than $N$. Thus the length of first input string at the $i$-th level of recursion is at most $n^{1-\delta}$. Hence, the depth of recursion is bounded by a constant $d = \lceil \frac{\log n}{2\delta} \rceil$. We denote the $\varepsilon$ at the $i$-th level by $\varepsilon_i$, thus we have $\varepsilon_i = \frac{\varepsilon_i}{2\delta}$. Similarly we set $\varepsilon_i' = \varepsilon_i/10$. We show that the output of the $i$-th level of recursion is a $1 + \varepsilon_i$ approximation of the edit distance by induction on $i$ from $d$ to 1. Thus, the output in the first level is guaranteed to be a $(1 + \varepsilon)$-approximation. At the $i$-th level, we either invoke one more level of recursion and maintains $O_{\varepsilon_i, \delta}(N^2)$ values, or do an exact computation of edit distance which takes $O(n^\delta \log n)$ space. Hence, the space used at each level is bounded by $O_{\varepsilon_i, \delta}(N^2)$. There are at most $d = \lceil \frac{\log n}{2\delta} \rceil + 1$ levels. The aggregated
space used by our recursive algorithm is still $O_{\varepsilon,\delta}(N^2)$. For time complexity, we can bound the number of times the algorithm enters the $i$-th level of recursion by $O_{\varepsilon,\delta}(n^{2(i-1)})$. At the $d$-th level, an exact computation of edit distance takes $O_{\varepsilon,\delta}(n^{d})$ time. For $i < d$, the computation at the $i$-th level uses a dynamic programming that takes $O_{\varepsilon,\delta}(n^{d})$ time. Thus, the total time is bounded by $O_{\varepsilon,\delta}(n^2)$.

Our algorithm can also be modified to work in the asymmetric model in [SS13] to get a matching result. In this model, one has streaming access to one string $x$ and random access to the other string $y$. To see this, notice that the block decomposition of the string $x$ can be viewed as a tree, and for a fixed sequence of $\Delta$ in each level of recursion, the algorithm we discussed above is essentially doing a depth first search on the tree, which implies a streaming computation on $x$. However, the requirement to try all possible $\Delta$ and all candidate intervals may ruin this property since we need to traverse the tree multiple times. To avoid this, our idea is to simultaneously keep track of all possible $\Delta$ and candidate intervals in the depth first search tree on $x$. We stop the recursion and do exact computation whenever each block of $x$ is no larger than $\sqrt{n}$. By doing so, we can still bound the space usage by $O(\sqrt{n})$.

### 1.1.2 Longest Increasing Subsequence

We now consider the problem of approximating the LIS of a string $x \in \Sigma^n$ over the alphabet $\Sigma$ which has a total order. We assume each symbol in $\Sigma$ can be stored with $O(\log n)$ bits of space. For our discussion, we let $\infty = -\infty = \sigma$ be two special symbols such that for any symbol $\sigma \in \Sigma$, $-\infty < \sigma < \infty$. We denote the length of the longest increasing subsequence of $x$ by $\text{LIS}(x)$.

Again our algorithm is a recursive one, and in each recursion we use a deterministic streaming algorithm from [GJKK07] that gives a $1 - \varepsilon$ approximation of $\text{LIS}(x)$ with $O(\sqrt{n}/\varepsilon \log n)$ space. Before describing their approach, we first give a brief introduction to a classic algorithm for LIS, known as PatienceSorting. The algorithm initializes a list $P$ with $n$ elements such that $P[i] = \infty$ for all $i \in [n]$, and then scans the input sequence $x$ from left to right. When reading a new symbol $x_i$, we find the smallest index $l$ such that $P[l] \geq x_i$ and set $P[l] = x_i$. After processing the string $x$, for each $i$ such that $P[i] < \infty$, we know $\sigma = P[i]$ is the smallest possible character such there is an increasing subsequence in $x$ of length $i$ ending with $\sigma$. We give the pseudocode in algorithm 1 and refer readers to [AD99] for more details about this algorithm.

**Algorithm 1: PatienceSorting**

**Input:** A string $x \in \Sigma^n$

1. initialize a list $P$ with $n$ elements such that $P[i] = \infty$ for all $i \in [n]$
2. for $i = 1$ to $n$ do
   3. let $l$ be the smallest index such that $P[l] \geq x_i$
   4. $P[l] \leftarrow x_i$
3. end
4. let $l$ be the largest index such that $P[l] < \infty$
5. return $l$

We have the following result.

**Lemma 1.1.** Given a string $x$ of length $n$, PatienceSorting computes $\text{LIS}(x)$ in $O(n \log n)$ time with $O(l \log n)$ bits of space where $l = \text{LIS}(x)$.

In the streaming algorithm from [GJKK07], we maintain a set $S$ and a list $Q$, such that, $Q[i]$ is stored only for $i \in S$ and $S$ is a set of size $O(\sqrt{n}/\varepsilon)$. We can use $S$ and $Q$ as an approximation to the list $P$ in PatienceSorting in the sense that for each $s \in S$, there is an increasing subsequence in $x$ of length $s$ ending with $Q[s]$. More specifically, we can generate a list $P'$ from $S$ and $Q$ such
that \( P[i] = Q[j] \) for the smallest \( j \geq i \) that lies in \( S \). For \( i \) larger than the maximum element in \( S \), we set \( P[i] = \infty \). Each time we read a new element from the data stream, we update \( Q \) and \( S \) accordingly. The update is equivalent to doing \textit{PatienceSorting} on the list \( P' \). When \( S \) gets larger than \( 2\sqrt{n}/\varepsilon \), we do a cleanup to \( S \) by only keeping \( \sqrt{n}/\varepsilon \) evenly picked values from 1 to \( \max S \) and storing \( Q[s] \) for \( s \in S \). Each time we do a “cleanup”, we lose at most \( \sqrt{\varepsilon} n \text{LIS}(x) \) in the length of the longest increasing subsequence detected. Since we only do \( \sqrt{\varepsilon} n \text{LIS}(x) \), we are guaranteed to detect an increasing subsequence of length at least \((1 - \varepsilon)\text{LIS}(x)\).

We now modify the above algorithm into another form. This time we first divide the input string \( x \) into many small blocks. Meanwhile, we also maintain a set \( S \) and a list \( Q \). We now process \( x \) from left to right, and update \( S \) and \( Q \) each time we have processed one block of \( x \). If the number of blocks in \( x \) is small, we can get the same approximation as in \cite{GJKK07} with \( S \) and \( Q \) having smaller size. For example, we can divide \( x \) into \( O_s(n^{1/3}) \) blocks each of size \( O_x(n^{2/3}) \), and we update \( S \) and \( Q \) once after processing each block. If we do exact computation within each block, we only need to maintain the set \( S \) and the list \( Q \) of size \( O(\sqrt{n}/\varepsilon) \). We can still get a \((1 - \varepsilon)\) approximation, because we do \( O(\sqrt{n^{1/3}}) \) cleansups and for each cleanup, we lose at most \( \varepsilon/n \) in the length of the longest increasing subsequence detected.

This almost already gives us an \( O_s(n^{1/3}) \) space algorithm, except the exact computation within each block needs \( \Omega_x(n^{2/3}) \) space. A natural idea to reduce the space complexity is to replace the exact computation with an approximation. When each block \( x' \) has size \( O_x(n^{2/3}) \), running the approximation algorithm from \cite{GJKK07} takes \( O_l(n^{1/3}) \) space and thus we can hope to reduce the total space required to \( O_x(n^{1/3}) \). However, a problem with this is that by running the approximation algorithm on each block \( x' \), we only get an approximation of \( \text{LIS}(x') \). This alone does not give us enough information on how to update \( S \) and \( Q \). Also, for a longest increasing subsequence of \( x' \), say \( s \), \( \tau \) is the subsequence of \( \tau \) that lies in \( x' \). \( \tau \) may be much shorter than \( \text{LIS}(x') \). This subsequence of \( \tau \) may be ignored if we run the approximation algorithm instead of using exact computation.

We now give some intuition of our approach to fix these issues. Let us consider a longest increasing subsequence \( \tau \) of \( x \) such that \( \tau \) can be divided into many blocks, where \( \tau \) lies in \( x' \). We denote the length of \( \tau \) by \( d_\tau \). Let the first symbol of \( \tau \) be \( \alpha_\tau \) and the last symbol be \( \beta_\tau \). When we process the block \( x' \), we want to make sure that our algorithm can detect an increasing subsequence of length \( (1 - \varepsilon)d_\tau \) in \( x_i \), where the first symbol is at least \( \alpha_\tau \) and the last symbol is at most \( \beta_\tau \). We can achieve this by running a bounded version of the approximation algorithm which only considers increasing subsequences no longer than \( d_\tau \). Since we do not know \( \alpha_\tau \) or \( d_\tau \) in advance, we can guess \( \alpha_\tau \) by trying every symbol in \( Q[s] \) and argue that one of them is close enough to \( \alpha_\tau \). For \( d_\tau \), we can try \( O(\log n) \) different values \( l \) such one \( l \) is close enough to \( d_\tau \). In this way, we are guaranteed to detect a good approximation of \( \tau \).

Based on this, our approach is to build a sequence of algorithms called \textit{ApproxLIS} for each integer \( i \geq 2 \). The first algorithm \textit{ApproxLIS} is exactly the same as the algorithm from \cite{GJKK07}. Then, we build \textit{ApproxLIS} using \textit{ApproxLIS}. We will show that \textit{ApproxLIS} uses only \( O_{\varepsilon,i}(n^{3/4}\log n) \) space.

For each \textit{ApproxLIS} of \textit{ApproxLIS}, we also introduce a slightly modified version called \textit{ApproxLISBound}. \textit{ApproxLISBound} takes an additional input \( l \), which is an integer at most \( n \). We want to guarantee that if \( x \) has an increasing subsequence of length \( l \) ending with \( \alpha \in \Sigma \), then \textit{ApproxLISBound} can detect an increasing subsequence of length at least \((1 - \varepsilon)l \) ending with some symbol \( \beta \in \Sigma \) such that \( \beta \leq \alpha \) (recall that \( \Sigma \) has a total order). \textit{ApproxLISBound} has the same space and time complexity as \textit{ApproxLIS}. The difference is that \textit{ApproxLISBound} only considers increasing subsequences of length at most \( l \).

Assume we are given \textit{ApproxLIS} and \textit{ApproxLISBound} such that \textit{ApproxLIS} outputs a \((1 - \varepsilon)\) approximation of \( \text{LIS}(x) \) with \( O_{\varepsilon,d}(n^{3/4}\log n) \) space. We now describe how
ApproxLIS^{d+1} works. We maintain a set $S$ and $Q$ of size $O_{\varepsilon}(\sqrt{n} \pi^n)$ as an approximation of the list $P$ we get when running PatientSorting. It is enough because we only update the set $S$ and the list $Q$ for $O(\sqrt{n} \pi^n)$ times and we lose about $O(\sqrt{n} \pi^n \varepsilon) \text{LIS}(x)$ after each update. To achieve this, we first divide $x$ into $N = O(\sqrt{n} \pi^n)$ blocks each of size $O(n \pi^n / \sqrt{n})$. We denote the $i$-th block by $x^i$. Initially, $S$ contains only one element $0$ and $Q[0] = -\infty$. We then update $S$ and $Q$ after processing each block of $x$.

We denote $S$ and $Q$ after processing the $t$-th block by $S_t$ and $Q_t$. To see how $S$ and $Q$ are updated, we take the $t$-th update as an example. Assume we are given $S_{t-1}$ and $Q_{t-1}$, we first determine the length of LIS in $x^1 \cdots x^t$ that can be detected based on $S_{t-1}$ and $Q_{t-1}$. Let the length be $k_t$ and it is computed as follows. Since each block is of size $O_{\varepsilon}(\sqrt{n} \pi^n)$, we cannot afford to do exact computation, thus we use ApproxLIS instead. For each $s \in S_{t-1}$, we run ApproxLIS^{d}(z^s, \varepsilon/3) where $z^s$ is the subsequence of $x^t$ with only symbols larger than $Q_{t-1}[\varepsilon]$.

Finally let $k_t = \max_{s \in S_{t-1}} (s + \text{ApproxLIS}^{d}(z^s, \varepsilon/3))$. Given $k_t$, we set $S_t$ to be the $n \pi^n / \sqrt{n}$ evenly picked integers from 0 to $k_t$.

The next step is to compute $Q_t$. We first set $Q_t[0] = \infty$ for all $s \in S_t$ except $Q_t[0] = -\infty$. Then, for each $s \in S_{t-1}$ and $t = 1, 1+\varepsilon/3, (1+\varepsilon/3)^2, \ldots, k_t-s$, we run ApproxLISBound^{d}(z^s, \varepsilon/3, l). For each $s' \in S_t$ such that $s \leq s' \leq s+l$, we update $Q_t[s']$ if ApproxLISBound^{d}(z^s, \varepsilon/3, l) detects an increasing subsequence of length $s' - s$ ending with a symbol smaller than the old $Q_t[s']$.

Continue doing this, we get $S_N$ and $Q_N$. ApproxLIS^{d+1} outputs the largest element in $S_N$.

To see the correctness of our algorithm, let us consider a longest increasing subsequence $\tau$ of $\varepsilon$ of $x$. $\tau$ can be divided into $N$ parts such that $\tau^i$ lies in $x^i$ although some part may be empty. For our analysis, let $P_t^i$ be the list generated by $S_t$ and $Q_t$ in the following way: for every $i$ let $P_t^i[i] = Q_t[j]$ for the smallest $j \geq i$ that lies in $S_t$. If no such $j$ exists, set $P_t^i[i] = \infty$. Correspondingly, $P_i$ is the list $P$ after running PatientSorting with input $x^1 \cdot x^2 \cdot \cdots \cdot x^t$.

Let $h_t = \sum_{j=1}^{i} |\tau^j|$ and $k_t = \max S_t$, our main observation is the following inequality:

$$P_t^i[1-\frac{2\varepsilon}{3}h_t - 2\varepsilon' n^\pi / \sqrt{n} k_t] \leq P_t[h_t]$$

When $t = N$, $h_N = \text{LIS}(x)$. By the correctness of PatientSorting, we have $P_t[h_t] < \infty$. Since $(1 - \frac{2\varepsilon}{3})h_N - 2\varepsilon' n^\pi / \sqrt{n} k_N \geq (1 - \varepsilon)h_N$, there is an element in $S_N$ larger than $(1 - \varepsilon)\text{LIS}(x)$ which directly shows the correctness of ApproxLIS^{d+1}.

We prove this inequality by induction on $t$. The intuition is that by trying $l = 1, 1+\varepsilon/3, (1+\varepsilon/3)^2, \ldots, k_t-s$, one $l$ is close enough to $|\tau^i|$. Thus we are guaranteed to detect a good approximation of $\tau_i$ in $\tau^i$.

For the space complexity, running ApproxLIS^{d} and ApproxLISBound^{d} on each block of size $O_{\varepsilon}(\sqrt{n} \pi^n)$ takes $O_{\varepsilon}(\sqrt{n} \pi^n)$ space, and storing $S$, $Q$ takes an additional $O_{\varepsilon}(\sqrt{n} \pi^n)$ space. Thus the total space used by ApproxLIS^{d+1} is still $O_{\varepsilon}(\sqrt{n} \pi^n)$.

Our algorithm for approximating the length of LIS can be modified to output an increasing subsequence. Again, the idea is to build a sequence of algorithms called LISSequence^{i} for each integer $i \geq 1$ such that LISSequence^{i}(x, \varepsilon) can output an increasing subsequence of $x$ with length at least $(1 - \varepsilon)\text{LIS}(x)$, using $O_{\varepsilon,i}(n^\pi \log n)$ space. For the first algorithm LISSequence^{1}, we can output the LIS exactly with $O(n \log n)$ space, see [AD99] for example. Now, assume we are given algorithm LISSequence^{i}, we show how LISSequence^{i+1} works. Let $\rho$ be the longest increasing subsequence detected by ApproxLIS^{i+1}(x, \varepsilon/2), thus $\rho$ has length $(1 - \varepsilon/2)\text{LIS}(x)$. We divide $\rho$ into $N$ parts such that the $i$-th part $\rho^i$ lies in $x^i$, thus $\rho^i$ has length at most $|x^i| = n^\pi / \sqrt{n}$. If we know the first and last symbol of $\rho^i$, we can output an increasing subsequence of length at least $(1 - \varepsilon/2)|\rho^i|$ by running LISSequence^{i}(x^i, \varepsilon/2) while ignoring all symbols.
in $z^i$ that is smaller than the first symbol of $\rho^i$ or larger than the last symbol of $\rho^i$. This can be done with $O_{\varepsilon,d}(n^{3/2}\log n)$ space. To determine the range of $\rho$ and $B[|B|]$ is the last symbol of $\rho$. Then, we compute the list $B$ from right to left. Once we know $B[i] = Q_i[s_i]$ for some $s_i \in S_i$, we compute $S_{i-1}$ and $Q_{i-1}$ by running $\text{ApproxLIS}^{d+1}(x, \varepsilon/2)$ again. Then, for each $s \in S_{i-1}$ and $s \leq s_i$, if we can find an increasing subsequence in $x^{-1}$ of length $s_i - s$ with first symbol larger than $Q_{i-1}[s]$ and last symbol at most $B[i]$, we set $s_{i-1} = s$, $B[i-1] = Q_{i-1}[s_{i-1}]$ and continue to compute $B[i - 2]$. After we have computed $B$, we can use $B[i - 1]$ and $B[i]$ as the range of $\rho^i$. Computing $B$ needs to run $\text{ApproxLIS}^{d+1}$ for $N$ times sequentially and $B$ is of size $N = O_{\varepsilon}(n^{3/2}\log n)$, thus the space used is bounded by $O_{\varepsilon,d}(n^{3/2}\log n)$.

### 1.1.3 Longest Common Subsequence

For longest common subsequence, our algorithm is based on a reduction from LCS to LIS. We assume the inputs are two strings $x \in \Sigma^n$ and $y \in \Sigma^m$. Our goal is to output a $(1 - \varepsilon)$-approximation of the LCS of $x$ and $y$.

We first introduce the following reduction from LCS to LIS. Given the strings $x$ and $y$, for each $i \in [n]$ let $b^i \in [m]^*$ be the sequence consisting of all distinct indices $j$ in $[m]$ such that $x_i = y_j$, arranged in descending order. Note that $b^i$ may be empty. Let $z$ be the sequence such that $z = b^{s_1} \cdot b^{s_2} \cdot \cdots \cdot b^{s_n}$, which has length $O(nm)$ since each sequence $b^{s_i}$ is of length at most $m$. We claim that $\text{LIS}(z) = \text{LCS}(x, y)$. This is because for every increasing subsequence of $z$, say $t = t_1 t_2 \cdots t_d$, the corresponding subsequence $y_{t_1} y_{t_2} \cdots y_{t_d}$ of $y$ also appears in $x$. Conversely, for every common subsequence of $x$ and $y$, we can find an increasing subsequence in $z$ with the same length. We call this procedure $\text{ReduceLCSToLIS}$. Note that in our algorithms, $z$ need not be stored, since we can compute each element in $z$ as necessary in logspace by querying $x$ and $y$.

Once we reduced the LCS problem to an LIS problem, we can use similar techniques as we used for LIS. We build a sequence of algorithms called $\text{ApproxLCS}^d$ for each integer $i \geq 1$ such that $\text{ApproxLCS}^i(x, y, \varepsilon)$ computes a $(1 - \varepsilon)$-approximation of LCS$(x, y)$ with $O_{\varepsilon,d}(n^{3/2}\log n)$ space. For the first algorithm $\text{ApproxLCS}^1$, we run $\text{PatienceSorting}$ on $z$ to compute $\text{LIS}(z)$ exactly. Since $\text{LIS}(z) \leq n$, it can be done with $O(n \log n)$ space.

For $\text{ApproxLCS}^{d+1}$, the goal is to compute an approximation of $\text{LIS}(z)$. We first divide $x$ evenly into $N = O_{\varepsilon}(n^{3/2}\log n)$ blocks $x^1, x^2, \ldots, x^N$. Correspondingly the string $z = \text{ReduceLCSToLIS}(x, y)$ can be divided into $N$ blocks with $z^i = \text{ReduceLCSToLIS}(x^i, y)$. We know $\text{LIS}(z^i)$ is at most $O_{\varepsilon}(n^{3/2})$ since the length of $x^i$ is $O_{\varepsilon}(n^{3/2})$. Then we can compute an approximation of $\text{LIS}(z^i)$ with $\text{ApproxLCS}^d$, which takes only $O_{\varepsilon,d}(n^{3/2}\log n)$ space. We can now build $\text{ApproxLCS}^{d+1}$ based on $\text{ApproxLCS}^d$ with the same approach as in the construction of $\text{ApproxLIS}^{d+1}$. Since we divide $z$ into $N$ blocks with $z^i = \text{ReduceLCSToLIS}(x^i, y)$, this approach also gives us a slight improvement on running time over the naïve approach of running our LIS algorithm after the reduction.

We note that the algorithm $\text{ApproxLCS}^2$ can be adapted to work in the asymmetric model. With random access to $y$, we only need to query $x^i$ to know $z^i$. Thus, when processing $z^i$, we can keep the corresponding $O(\sqrt{n}/\varepsilon)$ symbols of $x^i$ in the memory. Since we only process $z^i$ from $i = 1$ to $\sqrt{n}/\varepsilon$ once, we only need to read $x$ from left to right once. Thus, our algorithm is a streaming algorithm that only queries $x$ in one pass.

**Independent work.** Our results in the asymmetric streaming model are also achieved in a recent independent work by Farhadi et. al. [FHR20]. Furthermore, [FHR20] gives an algorithm...
for ED in the asymmetric streaming model that achieves a $O(2^{1/\delta})$ approximation using $\tilde{O}(n^{3/\delta})$ space, at the price of using exponential running time. However, [HRS20] does not give our main results in the non streaming model, where we can achieve $1 + \varepsilon$ or $1 - \varepsilon$ approximation for all of ED, LCS, LIS using space $n^\delta$.

### 1.2 Organization of this paper

The rest of the paper is organized as follows. In Section 2, we introduce some notation and give a formal description of the problems studied. In Section 3, we present our algorithms for edit distance. In Section 4, we present our algorithms for LIS. Then in Section 5, we present our algorithms for LCS. Finally in section 6, we conclude with discussion on our results and open problems.

## 2 Preliminaries

We use the following conventional notations. Let $x \in \Sigma^n$ be a string of length $n$ over alphabet $\Sigma$. By $|x|$, we mean the length of $x$. We denote the $i$-th character of $x$ by $x_i$ and the substring from the $i$-th character to the $j$-th character by $x_{i:j}$. We denote the concatenation of two strings $x$ and $y$ by $x \circ y$. By $[n]$, we mean the set of positive integers no larger than $n$.

**Edit Distance** The edit distance (or Levenshtein distance) between two strings $x, y \in \Sigma^*$, denoted by $\text{ED}(x, y)$, is the smallest number of edit operations (insertion, deletion, and substitution) needed to transform one into another. The insertion (deletion) operation adds (removes) a character at some position. The substitution operation replace a character with another character from the alphabet set $\Sigma$.

**Longest Common Subsequence** We say the string $s \in \Sigma^t$ is a subsequence of $x \in \Sigma^n$ if there exists indices $1 \leq i_1 < i_2 < \cdots < i_t \leq n$ such that $s = x_{i_1}x_{i_2} \cdots x_{i_t}$. A string $s$ is called a common subsequence of strings $x$ and $y$ if $s$ is a subsequence of both $x$ and $y$. Given two strings $x$ and $y$, we denote the length of the longest common subsequence (LCS) of $x$ and $y$ by $\text{LCS}(x, y)$.

**Longest Increasing Subsequence** In the longest increasing subsequence problem, we assume there is a given total order on the alphabet set $\Sigma$. We say the string $s \in \Sigma^t$ is an increasing subsequence of $x \in \Sigma^n$ if there exists indices $1 \leq i_1 < i_2 < \cdots < i_t \leq n$ such that $s = x_{i_1}x_{i_2} \cdots x_{i_t}$ and $x_{i_1} < x_{i_2} < \cdots < x_{i_t}$. We denote the length of the longest increasing (LIS) subsequence of string $x$ by $\text{LIS}(x)$. In our analysis, for a string $x$ of length $n$, we assume each element in the string can be stored with space $O(\log n)$. For analysis, we introduce two special symbols $\infty$ and $-\infty$ with $\infty > i$ and $-\infty < i$ for any character $i \in \Sigma$. In our discussion, we let $\infty$ and $-\infty$ to be two imaginary characters such that $-\infty < \alpha < \infty$ for all $\alpha \in \Sigma$.

## 3 Edit Distance

In this section, we prove Theorem 1

### 3.1 $(1 + \varepsilon)$-approximation for Edit Distance with $\tilde{O}(\frac{1}{\delta})$ space

We first present a space-efficient algorithm for approximating edit distance that outputs a $(1 + \varepsilon)$-approximation with only $\tilde{O}(n^\delta)$ space for any $\varepsilon \in (0, 1)$, and constant $\delta \in (0, 1)$.

Our algorithm utilizes ideas from the massively parallel approximation algorithm by [HSS19]. We first briefly describe their approach. Let $x$ and $y$ be the two input strings with $|x| = n$, $|y| = m$. $\delta \in (0, 1)$ is a fixed constant up to our choice and $(1 + \varepsilon)$ is the desired approximation.
ratio. Let \( \varepsilon' = \varepsilon/10 \) be a relatively smaller constant. We also assume a value \( \Delta \leq \max\{n, m\} \) is given to us, the algorithm will output a \((1 + \varepsilon)\)-approximation of \( \text{ED}(x, y) \) if \( \Delta \) is a \((1 + \varepsilon')\)-approximation of \( \text{ED}(x, y) \). Since we can try every \( \Delta \leq \max\{n, m\} \) such that \( \Delta = [1/(1 + \varepsilon')] \) for some integer \( i \), this assumption will only increase the time complexity by a logarithmic factor.

We first divide \( x \) into \( N = n^8 \) blocks each of size at most \([n/N]\) and write \( x \) as \( x^1 \circ x^2 \circ \cdots \circ x^N \) where \( x^i = x|_{[i,r_i]} \). That is, \( x^i \) starts at the \( l_i \)-th position and ends at the \( r_i \)-th position. In the following discussion, we fix an optimal alignment such that \( x|_{[l_r, r]} \) is mapped to a block \( y|_{[\alpha, \beta]} \) and \( [\alpha, \beta] \)'s are disjoint and span the entire length of \( y \). This optimal alignment minimizes the operations needed to transform \( x \) into \( y \). Then, we give the following definition of \((\varepsilon, \Delta)\)-approximately optimal candidate.

**Definition 3.1 (HSS19).** We say an interval \([\alpha', \beta']\) is an \((\varepsilon, \Delta)\)-approximately optimal candidate of the block \( x^i = x|_{[l_i, r_i]} \) if the following two conditions hold:

\[
\alpha_i \leq \alpha' \leq \alpha_i + \varepsilon \frac{\Delta}{N},
\]

\[
\beta_i - \frac{\Delta}{N} - \varepsilon \text{ED}(x|_{[l_i, r_i]}, y|_{[\alpha, \beta]}) \leq \beta' \leq \beta_i
\]

We first show that we can get a good approximation of \( \text{ED}(x, y) \) if \( \Delta \) is a good approximation of \( \text{ED}(x, y) \) and for each block that is not matched to a too large or a too small interval, we know the edit distance between it and one of its approximately optimal candidate. We put it formally in the following Lemma.

**Lemma 3.1 (Implicit from HSS19).** Let \( \varepsilon \in (0, 1) \) be a constant and \( \varepsilon' = \varepsilon/10 \). Assume \( \text{ED}(x, y) \leq \Delta \leq (1 + \varepsilon')\text{ED}(x, y) \). Fix an optimal alignment such that \( x|_{[l_i, r_i]} \) is matched to substring \( y|_{[\alpha_i, \beta_i]} \) and interval \([\alpha_i, \beta_i] \)'s are disjoint and span the entire length of \( y \). For each \( i \in [N] \), let \((\alpha'_i, \beta'_i)\) be any \((\varepsilon', \delta)\)-approximately optimal candidate of \( x^i \). If \( \varepsilon'|\alpha_i - \beta_i + 1| \leq |x^i| \leq 1/\varepsilon'|\alpha_i - \beta_i + 1| \), let \( D'_i = |\alpha_i - \alpha'_i| + \text{ED}(x^i, y|_{[\alpha'_i, \beta'_i]}) + |\beta_i - \beta'_i| \). Otherwise, let \( D'_i = |x^i| + |\alpha_i - \beta_i + 1| \).

Then

\[
\text{ED}(x, y) \leq \sum_{i=1}^{N} D'_i \leq (1 + \varepsilon)\text{ED}(x, y).
\]

To make our work self-contained, we provide a proof here.

**Proof of Lemma 3.1.** Let \( D_i = \text{ED}(x^i, y|_{[\alpha_i, \beta_i]}) \). Since we assume the alignment is optimal and \([\alpha_i, \beta_i] \) are disjoint and span the entire length of \( y \), we know \( \text{ED}(x, y) = \sum_{i=1}^{N} D_i \).

For each \( i \in [N] \), if \( \varepsilon'|\alpha_i - \beta_i + 1| \leq |x^i| \leq 1/\varepsilon'|\alpha_i - \beta_i + 1| \), by the definition of \((\varepsilon', \Delta)\)-approximately optimal candidate, we know,

\[
|\alpha_i - \alpha'_i| \leq \varepsilon' \frac{\Delta}{N} \tag{1}
\]

and

\[
|\beta_i - \beta'_i| \leq \frac{\Delta}{N} + \varepsilon' \text{ED}(x_i, y|_{[\alpha_i, \beta_i]}) \tag{2}
\]

Also notice that we can transform \( y|_{[\alpha'_i, \beta'_i]} \) to \( y|_{[\alpha_i, \beta_i]} \) with \( |\alpha_i - \alpha'_i| + |\beta_i - \beta'_i| \) insertions and then transform \( y|_{[\alpha_i, \beta_i]} \) to \( x^i \) with \( \text{ED}(x^i, y|_{[\alpha_i, \beta_i]}) \) edit operations. We have

\[
\text{ED}(x^i, y|_{[\alpha'_i, \beta'_i]}) \leq \text{ED}(x^i, y|_{[\alpha_i, \beta_i]}) + |\alpha_i - \alpha'_i| + |\beta_i - \beta'_i| \tag{3}
\]
Meanwhile, we can always transform \( y_{[\alpha_1, \beta_1]} \) to \( y_{[\alpha_1', \beta_1']} \) with \( |\alpha_1 - \alpha_1'| + |\beta_1 - \beta_1'| \) deletions and then transform \( y_{[\alpha_1', \beta_1']} \) to \( x^i \) with \( \text{ED}(x^i, y_{[\alpha_1', \beta_1']}) \). We have
\[
D'_i \geq D_i. \tag{4}
\]

Combining (1), (2), and (3) we have
\[
D_i \leq D'_i \leq \text{ED}(x^i, y_{[\alpha_1, \beta_1]}) + 2|\alpha_1 - \alpha_1'| + 2|\beta_1 - \beta_1'| \leq (1 + 2\varepsilon')D_i + 4\varepsilon' \frac{\Delta}{N}. \tag{5}
\]

For those \( i \) such that \( |x^i| > (1/\varepsilon')[\alpha_1 - \beta_1 + 1] \) or \( |x^i| < \varepsilon'|\alpha_1 - \beta_1 + 1| \), to transform \( x^i \) to \( y_{[\alpha_1, \beta_1]} \), we need to insert (or delete) \( |\alpha_1 - \beta_1 + 1| - |x^i| \) characters to make sure the length of \( x^i \) equals to the length of \( y_{[\alpha_1, \beta_1]} \). Thus, \( D_i = \text{ED}(x^i, y_{[\alpha_1, \beta_1]}) \) is at least \( ||\alpha_1 - \beta_1| - |l_i - r_i|| \). Since \( D'_i = |\alpha_1 - \beta_1| + |l_i - r_i| \), we have
\[
D'_i \leq \frac{1 + \varepsilon'}{1 - \varepsilon'}D_i \leq (1 + 3\varepsilon')D_i \quad \text{Since we set } \varepsilon = \varepsilon/10 \leq 1/10
\]

Also notice that we can turn \( x^i \) into \( y_{[\alpha_1, \beta_1]} \) with \( |l_i - r_i| \) deletions and \( |\alpha_1 - \beta_1| \) insertions, we know \( D'_i \geq D_i \). It gives us
\[
D_i \leq D'_i \leq (1 + 3\varepsilon')D_i \quad \text{Thus for each } i \in [N], \text{ by (7) and (5) we have} \tag{7}
\]
\[
D_i \leq D'_i \leq (1 + 3\varepsilon')D_i + 4\varepsilon' \frac{\Delta}{N}. \tag{8}
\]

Since we assume \( \Delta \leq (1 + \varepsilon')\text{ED}(x, y) \), we have \( \varepsilon'\Delta \leq 1.1\varepsilon'\text{ED}(x, y) \), this gives us
\[
\text{ED}(x, y) \leq \sum_{i=1}^{N} D'_i \leq (1 + 3\varepsilon')\text{ED}(x, y) + 4\varepsilon' \Delta \leq (1 + 10\varepsilon')\text{ED}(x, y) = (1 + \varepsilon)\text{ED}(x, y). \tag{9}
\]

For each \( i \) and \( \varepsilon, \Delta \), there exist a set of intervals \( C_{\varepsilon, \Delta}^i \) such that \( C_{\varepsilon, \Delta}^i \) is of size \( O(N \log n/\varepsilon) \) and one of the intervals in \( C_{\varepsilon, \Delta}^i \) is an \((\varepsilon, \Delta)\)-approximately optimal candidate for \( x^i \). The set \( C_{\varepsilon, \Delta}^i \) can be find with the algorithm \textit{CandidateSet} which is implicit from [HSS10]. The algorithm takes six inputs : three integers \( n, m \), and \( N \), an interval \((l_i, r_i), \varepsilon \in (0, 1) \), and \( \Delta \leq n \) and outputs set \( C_{\varepsilon, \Delta}^i \). Here, \( n \) and \( m \) are the lengths of string \( x \) and \( y \) correspondingly. The pseudocode is
Algorithm 2: CandidateSet

Input: three integers $n, m$, and $N$, an interval $(l_i, r_i)$, $\varepsilon \in (0, 1)$, and $\Delta \leq n$

1. $|x| = r_i - l_i + 1$
2. Initialize $C$ to be an empty set
3. foreach $i' \in [l_i - \Delta - \varepsilon \frac{\Delta}{N}, l_i + \Delta + \varepsilon \frac{\Delta}{N}] \cap [m]$ such that $i'$ is a multiple of $[\varepsilon \frac{\Delta}{N}]$
   a. if $i' = 0$
   b. foreach $j' = 0$ or $j' = \lceil (1 + \varepsilon) i \rceil$ for some integer $i \leq \lceil \log_{1 + \varepsilon}(m) \rceil$
      a. pick $O_{\varepsilon}(\log n)$ ending points
      b. if $|i'| - j' \geq \varepsilon |x||\$ then
         c. add $(i', i' + |x| - 1 - j')$ to $C$
5. end
6. if $|i'| + j' \leq 1/\varepsilon |x|$ then
7.   add $(i', i' + |x| - 1 + j')$ to $C$
8. end
9. end
10. end
11. return $C$

Lemma 3.2 (Implicit from [HST19]). If $\varepsilon m \leq n \leq \frac{1}{2} m$, then $C_{\varepsilon, \Delta}^i = \text{CandidateSet}(n, m, N, (l_i, r_i), \varepsilon, \Delta)$ is of size $O(\frac{N \log n}{\varepsilon})$. For $x^i = x_{[l_i, r_i]}$, if $\varepsilon |\alpha_i - \beta_i + 1| \leq |x^i| \leq 1/\varepsilon |\alpha_i - \beta_i + 1| ~ \text{and} \Delta \geq \text{ED}(x, y)$, then one of the elements in $C^i_{\varepsilon, \Delta}$ is an $(\varepsilon, \Delta)$-approximately optimal candidate of $x^i$.

Proof of Lemma 3.2. Let $C^i_{\varepsilon, \Delta}$ be the outputed set of CandidateSet$(n, m, N, (l_i, r_i), \varepsilon, \Delta)$. For the starting point $i'$, we only choose multiples of $\varepsilon \frac{\Delta}{N}$ from $[l_i - \Delta - \varepsilon \frac{\Delta}{N}, l_i + \Delta + \varepsilon \frac{\Delta}{N}]$. At most $O(\Delta/(\varepsilon \frac{\Delta}{N})) = O(N/\varepsilon)$ starting points will be chosen. For each starting point, we consider $O(\log_{1 + \varepsilon} m) = O(\log_{1 + \varepsilon} m) = O(\log n)$ ending point since we assume $\varepsilon m \leq n \leq \frac{1}{2} m$. Thus, the size of set $C^i_{\varepsilon, \Delta}$ is at most $O(\frac{N \log n}{\varepsilon})$.

We now show there is an element in $C^i_{\varepsilon, \Delta}$ that is an $(\varepsilon, \Delta)$-approximately optimal candidate for $x^i$ if $\varepsilon |\alpha_i - \beta_i + 1| \leq |x^i| \leq 1/\varepsilon |\alpha_i - \beta_i + 1|$. Since we assume $\Delta \geq \text{ED}(x, y)$, we are guaranteed that $l_i - \Delta \leq \alpha_i \leq l_i + \Delta$. Thus, there is a multiple of $[\varepsilon \frac{\Delta}{N}]$, denoted by $\alpha'$, such that

$$l_i - \Delta - \varepsilon \frac{\Delta}{N} \leq \alpha' \leq \alpha' \leq \alpha_i + \frac{\Delta}{N} \leq l_i + \Delta + \varepsilon \frac{\Delta}{N},$$

since we try every multiple of $[\varepsilon \frac{\Delta}{N}]$ between $l_i - \Delta - \varepsilon \frac{\Delta}{N}$ and $l_i + \Delta + \varepsilon \frac{\Delta}{N}$, one of them equals to $\alpha'$.

For the ending point, we first consider the case when the length of $y_{[\alpha_i, \beta_i]}$ is larger than the length of $x^i$, that is $\beta_i - \alpha_i + 1 \geq r_i - l_i + 1$. We know $\text{ED}(x^i, y_{[\alpha_i, \beta_i]}) \geq \beta_i - \alpha_i + 1 - |x^i|$. Let $j$ be the largest element in $\{0, 1, [1 + \varepsilon], [1 + \varepsilon]^2], \ldots, [1 + \varepsilon]^{\log_{1 + \varepsilon}(m)}\}$ such that $\alpha' + |x^i| - 1 + j \leq \beta_i$. We set $\beta' = \alpha' + |x^i| - 1 + j$. Since $j \geq (\beta_i - (\alpha' + |x^i| - 1))/1 + \varepsilon$, we have

$$\beta' \geq \alpha' + |x^i| - 1 + (\beta_i - (\alpha' + |x^i| - 1))/1 + \varepsilon) \geq \frac{\beta_i}{1 + \varepsilon} + \frac{\varepsilon}{1 + \varepsilon}(\alpha' + |x^i| - 1) \geq \frac{\beta_i - \varepsilon}{1 + \varepsilon} (\beta_i - \alpha' + 1 - |x^i|) \geq \beta_i - \varepsilon \text{ED}(x^i, y_{[\alpha_i, \beta_i]})$$

(10)
The last inequality is because $\mathbf{ED}(x^i, y_{[\alpha, \beta]}) \geq \beta_i - \alpha_i + 1 - |x^i| \geq \beta_i - \alpha' + 1 - |x^i|$ and $\varepsilon \geq \frac{\Delta}{1+\varepsilon}$. Thus, $(\alpha', \beta') \in C^i_{\varepsilon, \Delta}$ is an $(\varepsilon, \Delta)$-approximately optimal candidate of $x^i$.

For the case when $|\alpha_i - \beta_i| + 1 < |x^i|$, similarly, we know $\mathbf{ED}(x^i, y_{[\alpha, \beta]}) \geq |x^i| - (\beta_i - \alpha_i + 1).$ We pick $j$ to be the smallest element in \{0, 1, \lceil 1 + \varepsilon \rceil, \lceil (1 + \varepsilon)^2 \rceil, \ldots, \lceil (1 + \varepsilon)^{\log_{1+\varepsilon}(m)} \rceil \} such that $\alpha' + |x^i| - 1 - j \leq \beta_i$. We know $j \leq (1 + \varepsilon)(\alpha' + |x^i| - 1 - \beta_i)$. We set $\beta' = \alpha' + |x^i| - j$. Then
\[
\begin{align*}
\beta' & \geq \alpha' + |x^i| - 1 - (1 + \varepsilon)(\alpha' + |x^i| - 1 - \beta_i) \\
& \geq \beta_i - \varepsilon(\alpha' - \beta_i + |x^i| - 1) \\
& \geq \beta_i - \varepsilon(\alpha_i + \varepsilon \frac{\Delta}{N} - \beta_i + |x^i| - 1) \\
& \geq \beta_i - \varepsilon \mathbf{ED}(x^i, y_{[\alpha, \beta]}) - \varepsilon^2 \frac{\Delta}{N} \\
& \geq \beta_i - \varepsilon \mathbf{ED}(x^i, y_{[\alpha, \beta]}) - \varepsilon \frac{\Delta}{N}
\end{align*}
\]

Thus, $(\alpha', \beta') \in C^i_{\varepsilon, \Delta}$ is an $(\varepsilon, \Delta)$-approximately optimal candidate of $x^i$. \(\square\)

We now describe a dynamic programming algorithm $\text{DPEditDistance}$ that computes an approximation of $\mathbf{ED}(x, y)$ with the information of edit distances between $x^i$ and each of intervals in $C^i_{\varepsilon, \Delta}$. $\text{DPEditDistance}$ takes six inputs, $n$, $m$, $N$, $\Delta$, $\varepsilon$, and a two dimensional list $M$ such that $M(i, (\alpha, \beta)) = \mathbf{ED}(x^i, y_{[\alpha, \beta]})$ for each $i$ and $(\alpha, \beta) \in C^i_{\varepsilon, \Delta}$. The pseudocode is given in algorithm 3.

**Algorithm 3: DPEditDistance**

**Input:** three integers $n$, $m$, $N$, $\Delta \leq n$, $\varepsilon \in (0, 1)$, and a two dimensional list $M$ such that $M(i, (\alpha, \beta)) = \mathbf{ED}(x^i, y_{[\alpha, \beta]})$ for each $i$ and $(\alpha, \beta) \in C^i_{\varepsilon, \Delta}$.

1. Let $C^i$ be the set of starting points of intervals in $C^i_{\varepsilon, \Delta}$ with no repetition for each $i \in [N]$.
2. For each $\alpha \in C^i$ do
   3. $A(0, \alpha - 1) = \alpha - 1 \triangleright A$ is a two dimensional array for storing the intermediate results of the dynamic programming.
4. End
5. For $i = 1$ to $N - 1$ do
   6. For each $\alpha \in C^{i+1}$ do
      7. $A(i, \alpha - 1) = \min_{\alpha' \preceq \alpha} \left\{ \min_{\alpha' \in C^{i}, \alpha' \preceq \alpha} [A(i - 1, \alpha' - 1) + |\alpha - \alpha'| + |x^i|], \min_{(\alpha', \beta') \in C^i_{\varepsilon, \Delta}, \beta' \preceq \alpha - 1} [A(i - 1, \alpha' - 1) + M(i, (\alpha', \beta')) + \alpha - \beta' - 1] \right\}$
     8. End
   9. End
10. $d = \min_{\alpha' \in C^{N}} \left\{ \min_{(\alpha', \beta') \in C^i_{\varepsilon, \Delta}, \beta' \preceq m} [A(N - 1, \alpha' - 1) + |m - \alpha'| + |x^N|], \min_{(\alpha', \beta') \in C^i_{\varepsilon, \Delta}, \beta' \preceq m} [A(N - 1, \alpha' - 1) + M(N, (\alpha', \beta')) + m - \beta'] \right\}$
11. Return $d$
Lemma 3.3. Let $\varepsilon \in (0,1)$ and $\varepsilon' = \varepsilon/10$, $\ED(x,y) \leq \Delta \leq (1+\varepsilon')\ED(x,y)$ and $\ED(x^i,y_{[\alpha,\beta]}) = M(i,(\alpha,\beta))$, then $\text{DPEditDistance}(n,m,N,\varepsilon',\Delta,M)$ outputs a $(1+\varepsilon)$-approximation of $\ED(x,y)$ in $O(N^2\log n)$ time with $O(N \Delta \log n)$ bits of space.

Also, in the input, if we replace $M(i,(\alpha,\beta))$ with a $(1+\varepsilon)$ approximation of $\ED(x^i,y_{[\alpha,\beta]})$, that is
\[
\ED(x^i,y_{[\alpha,\beta]}) \leq M(i,(\alpha,\beta)) \leq (1+\varepsilon)\ED(x^i,y_{[\alpha,\beta]})
\]
then $\text{DPEditDistance}(n,m,N,\varepsilon',\Delta,M)$ outputs a $(1+2\varepsilon)$-approximation of $\ED(x,y)$.

Proof of Lemma 3.3. We start by explaining the dynamic programming. Let $f$ be a function such that $f(i) \in C^\varepsilon_{\varepsilon',\Delta} \cup \{0\}$. We say an interval $x^i$ is matched if $f(i) \in C^\varepsilon_{\varepsilon',\Delta}$ and it is unmatched if $f(i) = 0$. Let $S^i_f$ be the set of indices of matched blocks under function $f$ and $S^i_f = [N] \setminus S^i_f$ be the set of indices of unmatched blocks. We let $f(i) = (\alpha^i_f,\beta^i_f)$ for each $i \in S^i_f$. We also require that, for any $i,j \in S^i_f$ with $i < j$, $(\alpha^i_f,\beta^i_f)$ and $(\alpha^j_f,\beta^j_f)$ are disjoint and $\beta^f < \alpha^f$. Let $u_f$ be the number of unmatched characters under $f$ in $x$ and $y$. That is, $u_f$ equals to the number of indices in $[n]$ that is not in any matched block plus the number of indices in $[m]$ that is not in $f(i)$ for any $i \in S^i_f$. Then we define the edit distance under match $f$ by
\[
\ED_f := \sum_{i \in S^i_f} \ED(x^i,y_{[\alpha^i_f,\beta^i_f]}) + u_f
\]
Since we can always transform $x$ to $y$ by deleting (inserting) every unmatched characters in $x$ ($y$), and transforming each matched block $x^i$ into $y_{[\alpha^i_f,\beta^i_f]}$ with $\ED(x^i,y_{[\alpha^i_f,\beta^i_f]})$ edit operations. We know $\ED_f \geq \ED(x,y)$.

Let $F^i$ be the set of all matchings. Also, given $i \in [N]$ and $\alpha \in [m]$, we let $F_{i,\alpha}$ be the set of matching such that $f(i')$ is within $(1,\alpha)$ for all $i' \leq i$. Similarly, for each $f \in F_{i,\alpha}$, let $u_{i,\alpha}$ be the number of unmatched characters in $x_{[1,i]}$ and $y_{[1,\alpha]}$ under $f$. We can also define $\ED_{i,\alpha} = \sum_{i \in S^i_f} \ED(x^i,y_{[\alpha^i_f,\beta^i_f]}) + u_{i,\alpha}$. We now show that in algorithm 3, for each $i \in [N]$ and $\alpha \in C^\varepsilon_{\varepsilon',\Delta}$, we have
\[
A(i,\alpha-1) = \min_{f \in F_{i,\alpha-1}} \ED_{i,\alpha-1}^{f-1}
\]
We can prove this by induction on $i$. For simplicity, let $C^\varepsilon$ be the set of starting points of all intervals in $C^\varepsilon_{\varepsilon',\Delta}$. For the case base $i = 1$, we fix an $\alpha \in C^2$. For each $f \in F_{1,\alpha-1}$, if $f(1) = \emptyset$, then every character in $x^1$ and $y_{[1,\alpha-1]}$ are unmatched. In this case, $\ED_{1,\alpha-1}^f = \|x^1\| + \alpha - 1 = A(0,\alpha'-1) + \alpha - \alpha' + \|x^1\|$ for every $\alpha' \in C^1$ such that $\alpha' \leq \alpha$. When $f(1) \neq \emptyset$, we assume $f(1) \geq (\alpha^1_f,\beta^1_f)$, then $\ED_{1,\alpha-1}^f = \alpha^1_f - 1 + M(1,\alpha^1_f,\beta^1_f) + \alpha - \beta = A(0,\alpha') + M(1,\alpha^1_f,\beta^1_f) + \alpha - \beta$. By the updating rule of $A(1,\alpha-1)$ at line 8, we know $A(1,\alpha-1) = \min_{f \in F_{1,\alpha-1}} \ED_{1,\alpha-1}^f$ for every $\alpha \in C^2$.

Now assume $A(t-1,\alpha-1) = \min_{f \in F_{t-1,\alpha-1}} \ED_{t-1,\alpha-1}^f$ for any $\alpha \in C^t$ for $1 < t \leq N - 1$. Fix an $\alpha_0 \in C^{t+1}$, we show $A(t,\alpha_0-1) = \min_{f \in F_{t,\alpha_0-1}} \ED_{t,\alpha_0-1}^f$. For each matching $f$, if $f(t) = \emptyset$, we know
\[
\ED_{t,\alpha_0-1}^f = \ED_{t-1,\alpha_0-1}^f + \|x^t\| \geq \min_{\alpha' \in C^1,\alpha' \leq \alpha} A(t-1,\alpha'-1) + \alpha_0 - \alpha' \geq A(t,\alpha_0-1)
\]
If \( f(t) \neq \emptyset \), we assume \( f(t) = (\alpha_t', \beta_t') \). Then

\[
ED_{f_{t-1}} = ED_{f_{t-1}} + M(t, (\alpha_t', \beta_t')) + \alpha_0 - \beta_0'.
\]

For every \( (\alpha', \beta') \in C_{i, \Delta} \), we calculate at most

\[
O(\frac{N}{\varepsilon \log n})
\]

For the time complexity, since the size of \( C_i \) is \( O(\frac{N}{\varepsilon}) \), we calculate at most \( O(\frac{N^2}{\varepsilon}) \) number of \( A(i, \alpha - 1) \). For computing each \( A(i, \alpha - 1) \), we take the minimum over every \( (\alpha', \beta') \in C_{i, \Delta} \). By Lemma 3.2, the size of \( C_{i, \Delta} \) is at most \( O(\frac{N^2 \log n}{\varepsilon^2}) \). This takes at most \( O(\frac{N \log n}{\varepsilon^2}) \) time. Thus, everything before line 11 takes \( O(\frac{N_3 \log n}{\varepsilon^2}) \) time. For line 11, we take the minimum over every \( (\alpha', \beta') \in C_{i, \Delta} \); this again takes \( O(\frac{N \log n}{\varepsilon^2}) \) time. The total time required is \( O(\frac{N^3 \log n}{\varepsilon^2}) \).

For the space complexity, notice when updating \( A(i, \alpha) \), we only need the information of \( A(i - 1, \alpha' - 1) \) for every \( \alpha' \in C_i \). Thus, we can release the space used to store \( A(i - 2, \alpha'' - 1) \) for every \( \alpha'' \in C_{i - 1} \). And for line 11, we only need the information of \( A(i - 1, \alpha - 1) \) for every \( \alpha \in C_N \). From algorithm 2, we know that for each \( i \), we pick at most \( N/\varepsilon \) points as the starting point of the candidate intervals. The size of \( C_{iN} \) is at most \( N/\varepsilon \). Since each element in \( A \) is a number at most \( n \), it can be stored with \( O(\log n) \) bits of space. Thus, the space required is \( O(\frac{N \log n}{\varepsilon^2}) \).

Now, if we replace \( M(i, (\alpha, \beta)) \) with a \( (1 + \varepsilon) \) approximation of \( ED(x, y_{[\alpha, \beta]}) \). Each \( M(i, (\alpha, \beta)) \) will add at most an \( \varepsilon ED(x, y_{[\alpha, \beta]}) \) additive error. The amount of error added is bounded by \( \varepsilon ED(x, y) \). Thus, \( DPEditDistance(n, m, N, \varepsilon', \Delta, M) \) outputs a \( (1 + 2\varepsilon) \)-approximation of \( ED(x, y) \). The time and space complexity is not affected.

If we set \( N = O(n^\delta) \) for some constant \( \delta \) up to our choice and divide \( x \) into \( N \) blocks each of size at most \( n^{1-\delta} \). For each block \( x^i \), the approximation algorithm from [HSS19] computes the edit distance between \( x^i \) and every candidate intervals in \( C_{iN}^\Delta \) exactly in parallel. This gives us \( O_*(n^{\Delta}) \) values. Calculating the edit distance of one block \( x^i \) and one of its candidates
takes \( O_\varepsilon(n^{2(1-\delta)}) \) time with \( O_\varepsilon(n^{1-\delta}) \) space. For the next step, we run \( DPEditDistance \) on these \( O_\varepsilon(n^{2\delta} \log n) \) values, we are guaranteed to get a \( 1 + \varepsilon \) approximation of \( ED(x, y) \).

An easy analysis shows the above parallel algorithm can be turned into a sequential one with space \( O_\varepsilon(n^{2/3}) \). We can first calculate the edit distance between each \( x^i \) and its candidates sequentially one by one and store the results. Calculating each edit distance takes \( O_\varepsilon(n^{1-\delta}) \) space and the space can be reused. Storing all these edit distances takes \( O_\varepsilon(n^{2\delta} \log n) \). By taking \( \delta = 1/3 \), the aggregated space required is \( O_\varepsilon(n^{2/3} \log n) \).

We now describe our algorithm that uses only \( O_{\varepsilon, \delta}(n^\delta \log n) \) space for any constant \( \delta \in (0, 1) \). Let \( N = O(n^\delta) \) be a fixed number. The high-level idea is to run the parallel algorithm from [HSS19] recursively.

We call our space-efficient approximation algorithm for edit distance \( SpaceEfficientApproxED \) and give the pseudocode in algorithm 4.

**Algorithm 4: SpaceEfficientApproxED**

\[
\text{Input: Two strings } x \text{ and } y, \text{ integer } N, \text{ and a small constant } \varepsilon \in (0, 1) \\
1. \text{if } |x| \leq N \text{ then} \\
2. \quad \text{compute } ED(x, y) \text{ exactly} \\
3. \quad \text{return } ED(x, y) \\
4. \text{end} \\
5. \text{ed} \leftarrow \infty \\
6. \text{set } n = |x| \text{ and } m = |y| \\
7. \text{divide } x \text{ into } N \text{ block each of length at most } \lfloor n/N \rfloor \text{ such that } x = x^1 \circ x^2 \circ \cdots \circ x^N \\
8. \text{set } \varepsilon' = \varepsilon/2 \text{ and } \varepsilon'' = \varepsilon'/10 \\
9. \text{foreach } \Delta = (1 + \varepsilon'')^j \text{ for some integer } j \text{ and } \Delta \leq \max\{n, m\} \text{ do} \\
10. \quad \text{foreach } (a, b) \in \text{CandidateSet}(n, m, \lfloor l_j \rfloor, \varepsilon'', \Delta) \text{ do} \\
11. \quad \quad M(i, (a, b)) \leftarrow \text{SpaceEfficientApproxED}(x^i, y[a:b], N, \varepsilon') \\
12. \quad \text{end} \\
13. \text{end} \\
14. \text{end} \\
15. \text{return } \min\{ed, \text{DEditDistance}(n, m, N, \Delta, \varepsilon'', M)\} \\
16. \text{end} \\
17. \text{return } ed
\]

We have the following result.

**Lemma 3.4.** For any constant \( \varepsilon \in (0, 1) \) and \( \delta \in (0, 1) \) such that \( \frac{1-\delta}{\varepsilon} \) is an integer, let \( N = 2[\varepsilon^\delta] \) be an integer, then \( SpaceEfficientApproxED(x, y, N, \varepsilon) \) outputs a \( (1+\varepsilon) \)-approximation of \( ED(x, y) \) with \( O(1/\varepsilon^{1/\delta}) \) \( 2^{\log^2 n} \) bits of space in \( O_{\varepsilon, \delta}(n^\delta) \) time.

**Proof of Lemma 3.4.** Algorithm 4 is recursive. We start from level one and when \( SpaceEfficientApproxED \) is called, we enter the next level. We say the largest level we will reach is the maximum depth of recursion. In the following, \( n \) is the length of input \( x \) at the first level.

Let \( d = \frac{1-\delta}{\varepsilon} + 1 \) be a constant. We first show the maximum depth of recursion is at most \( d \). Notice that on the first level, the input string \( x \) is divided into small blocks each of length at most \( \lfloor n/N \rfloor \leq n^{1-\delta} \). Then, each block together with one of its candidate substring in \( y \) is sent to the second level. In the second level, again, divide \( x \) into \( N \) blocks each of size at most \( n^{1-2\delta} \). Continue doing this, if the recursion reaches \( d \)-th level, the input \( x \) has length at most \( n^{(1-(d-1)\delta)} \leq n^{\delta} < N \). Algorithm 4 will calculate the edit distance exactly. Thus, the maximum depth of recursion is at most \( d \).

We now prove the correctness of our algorithm. Notice that each time we enter a new level, \( \varepsilon \) is decreased by a factor of 2. For our analysis, we denote the input \( \varepsilon \) to the \( i \)-level of recursion by...
\(\varepsilon_i\). Thus, \(\varepsilon'_i = \frac{\varepsilon_i}{2}\). Similarly, we let \(\varepsilon''_i = \varepsilon'_i / 10\). We need to show the following claim.

**Claim 3.1.** At the \(i\)-th level, the output is a \((1 + \varepsilon_i)\) approximation of the edit distance of its input strings.

**Proof.** We prove this by induction on \(i\) from \(d\) to 1. For the case \(i = d\), we output the exact edit distance between its input strings. Thus the claim holds.

Now, we assume the claim holds for the \(i + 1\)-th level. In the level \(i\), if the input string \(x\) has length no larger than \(N\), we output the exact edit distance. The claim holds for level \(i\). Otherwise, since we tried every \(\Delta = \left\lfloor (1 + \varepsilon''_i) \right\rfloor\) for some integer \(j\) and \(\Delta \leq n + m\), one of \(\Delta\) satisfies \(ED(x, y) \leq \Delta \leq (1 + \varepsilon''_i) ED(x, y)\). Denote such a \(\Delta\) by \(\Delta_0\). Notice that \(M(i, (a, b))\) is a \((1 + \varepsilon_{i+1})\) approximation of \(ED(x', y_{[a, b]}\). Since \(\varepsilon''_i = \varepsilon_{i+1}/20\) By lemma 3.3 DPEditDistance\((n, m, N, \varepsilon_{i+1}/20, (M))\) output a \(1 + \varepsilon_{i+1} = 1 + \varepsilon_i\) approximation of \(ED(x, y)\) when \(\Delta = \Delta_0\). Also notice that DPEditDistance\((n, m, N, \varepsilon_{i+1}/20, (M))\) is always at least \(ED(x, y)\). This proves our claim.

Thus, for the first level, our algorithm always output a \(1 + \varepsilon_1 = 1 + \varepsilon\) approximation of \(ED(x, y)\).

We now turn to the space and time complexity. In a recursion, except the last level, we maintain a list \(M\) of size \(O\left(\frac{N^2 \log^2 n}{\varepsilon_i^2}\right)\) for level \(i\) with each element taking \(O(\log n)\) bits. We need \(O\left(\frac{N^2 \log n}{\varepsilon_i^2}\right)\) bits of space to store \(M\) at each level. Running DPEditDistance take an additional \(O\left(\frac{N \log n}{\varepsilon_i^2}\right)\) bits of space by lemma 3.3. Since the depth of recursion is at most a constant \(d\), space used for these levels is \(O\left(\frac{1}{\delta} \frac{n^{2(d-1)}}{\varepsilon_{d-1}^2}\log^2 n\right) = O\left(\frac{1}{\delta} \frac{n^{2d-1}}{\varepsilon_d^2}\log^2 n\right) = 2^{O(1/\delta)} \frac{n^{2d}}{\varepsilon_d^2} \log^2 n\). For the last level, we calculate the exact edit distance between two strings with one of them has length no larger than \(N = n^d\). This can be done with space \(O(n^d \log n)\). The aggregated space required is \(O\left(\frac{1}{\delta} \frac{n^{2d}}{\varepsilon_d^2}\right) \log^2 n\).

For time complexity, we denote the time used for computing at the \(i\)-th level by \(T_i\) (exclude the time used for running SpaceEfficientApproxED at the \(i\)-level). Each time SpaceEfficientApproxED is called at the \((i-1)\)-th level, we enter the \(i\)-th level. If \(i < d\), there are two possible cases, first, the operation at the \(i\)-th level is calculating the exact edit distance with one of the input strings has length at most \(N\) and the other string has length \(O(N/\varepsilon_i)\). It takes \(O(n^{2d}/\varepsilon_i)\) time. In the other case, we run DPEditDistance for \(O(\log_{1+\varepsilon_i} N) = O\left(\frac{\log n}{\varepsilon_i}\right)\) times with \(M\) of size \(O\left(\frac{N^2 \log n}{\varepsilon_i^2}\right)\) which takes \(O\left(\frac{N^2 \log n}{\varepsilon_i^2}\right) = O\left(2^{\frac{N^2 \log n}{\varepsilon_i^2}}\right)\) time. If \(i = d\), we always do the exact edit distance computation which takes \(O\left(\frac{n^{2d}}{\varepsilon_d}\right) = O\left(2^d n^{2d}\right)\) time. Thus, the time required in the \(i\)-th level is \(O\left(2^{3i-1} N^{2 \log n}/\varepsilon_d^2\right)\) each time SpaceEfficientApproxED is called at the \((i-1)\)-th level if \(i < d\) or \(O\left(2^d n^{2d}\right)\) if \(i = d\).

We now bound the number of times SpaceEfficientApproxED is called at the \(i\)-th level for each \(i \leq d - 1\). At the \(i\)-th level, if the input string has length larger than \(N\), we divide it into \(N\) blocks and for each block, we pick \(O\left(\frac{N \log n}{\varepsilon_i^2}\right)\) candidates. Thus, once we enter the \(i\)-th level, we call SpaceEfficientApproxED at most \(O\left(\frac{N^2 \log n}{\varepsilon_i^2}\right)\) times. The total number of times we will enter the \(i\)-th level is bounded by \(\prod_{j=2}^{i-1} O\left(\frac{N^2 \log n}{\varepsilon_j^2}\right) = O\left(2^{(i-1) \frac{N^2 \log n}{\varepsilon_i^2}}\right) = O\left(2^{2d}\right)\). For \(i < d\), we have

\[
T_i = \tilde{O}_{\varepsilon, \delta}(2^{2(i-1)} N^3) = \tilde{O}_{\varepsilon, \delta}(n^{2d(4 - d + 3\delta)}) = \tilde{O}_{\varepsilon, \delta}(n^{2 - \delta})
\]
Notice that there are at most $d$ levels, which is a constant, $\sum_{i=1}^{d-1} t_i$ is $\tilde{O}_{\varepsilon, \delta}(n^{2-\delta})$. For $i = d = \frac{1}{\delta} + 1$, 

$$T_d = O(2^{d(d-1)} \frac{n^{2-\delta} \log^{d-1} n}{\varepsilon^{2(d-1)}} \cdot 2^{d} n^{2\delta}) = O(2^{\frac{1}{\delta} n^{2} \log^{\frac{1}{\delta}} n})$$

Since we assume $\varepsilon$ and $\delta$ are both constants, we have $T_d = \tilde{O}_{\varepsilon, \delta}(n^2)$. Thus, the aggregated time is $\tilde{O}_{\varepsilon, \delta}(n^2)$.

### 3.2 Asymmetric Model

Asymmetric model has been considered in [SS13]. In this model, we have streaming access to one string and random access to the other string. We now show that our recursive algorithm can be adapted to this model to get a result comparable to [SS13]. In this section, we show that our method can be used to achieve an algorithm that is comparable to the results of [SS13]. We now present our algorithm in Algorithm 5.

**Algorithm 5: SpaceEfficientApproxEDAsymmetricModel**

**Input:** Two strings $x$ and $y$, integer $N$, and a small constant $\varepsilon \in (0, 1)$

1. if $|x| \leq \sqrt{n_0}$ then
   1.1. $n_0$ is a constant that equals to the length of input string at the first recursive level
   1.2. compute $ED(x, y)$ exactly
   1.3. return $ED(x, y)$
2. end
3. $ed \leftarrow \infty$
4. set $n = |x|$ and $m = |y|$
5. divide $x$ into $N$ block each of length at most $\lceil \frac{n}{N} \rceil$ such that $x = x^1 \circ x^2 \circ \cdots \circ x^N$
6. set $\varepsilon' = \varepsilon/2$ and $\varepsilon'' = \varepsilon'/10$
7. for $i = 1$ to $N$ do
6.1. try each $\Delta = 0$ or $\lfloor (1 + \varepsilon'')^j \rfloor$ for some integer $j$ and $\Delta \leq n/\varepsilon$ in parallel
6.2. try each $(a, b) \in \text{CandidateSet}(n, m, (l_i, r_i), \varepsilon'', \Delta)$ in parallel
6.3. $M(i, (a, b)) \leftarrow \text{SpaceEfficientApproxED}(x^i, y_{[a,b]}, N, \varepsilon'')$
7. end
8. $ed^\Delta = \text{DPEditDistance}(n, m, N, \Delta, \varepsilon'', M)$
9. $ed \leftarrow \min_{\Delta} ed^\Delta$
10. return $ed$

**Lemma 3.5.** Assume we have streaming access to string $x \in \Sigma^n$ and random access to the other string $y \in \Sigma^n$, then, for any constant $\varepsilon \in (0, 1)$ and $\delta \in (0, 1)$, there is a deterministic algorithm that makes one pass through $x$ and outputs a $(1 + \varepsilon)$-approximation of $ED(x, y)$ with $\tilde{O}_{\varepsilon}(\sqrt{n})$ bits of space in $\tilde{O}_{\varepsilon}(n^2)$ time.

**Proof of Lemma 3.5.** Let $\delta \in (0, 1)$ such that $1/\delta = 2d$ where $d \geq 1$ is an integer and $\varepsilon$ is any constant in $(0, 1)$. We run algorithm 5 with inputs $x \in \Sigma^n$, $y \in \Sigma^n$, $N = 2\lceil n^\delta \rceil$, and $\varepsilon \in (0, 1)$. Algorithm 5 is recursive for convenience, we let $n_0$ be the length of input string $x$ at the first recursive level.
Algorithm 5 is similar to the algorithm 4. The idea is to try each $\Delta$ and each candidate interval in parallel to make sure our algorithm access $x$ through one pass.

We make three changes to algorithm 4 to get algorithm 5. First, we compute the edit distance exactly whenever the length of the first input string is no longer than $\sqrt{n}$. Second, we try every $\Delta$ in parallel. Third, we approximate edit distance to each candidate interval in parallel.

We first show algorithm 5 is indeed a streaming algorithm that access $x$ through one pass and can be run with $\tilde{O}(\sqrt{n})$ space.

In the following discussion, to avoid ambiguity, we denote the input string $x$ at the 1st recursive level by $\tilde{x}$. Notice that we only need to access $\tilde{x}$ when we need to calculate the edit distance exactly at line 2. At the $d$-th recursive level, the length of the first input string is at most $n/N^d$, which is at most $\sqrt{n}$. Thus, the depth of recursion is at most $d$. For simplicity, assume that we only access $\tilde{x}$ at the $d$-th level.

The block decomposition of string $x$ can be viewed as a tree. Since we try all possible $\Delta$ and candidate intervals in parallel, our algorithm runs in the same order as doing depth first search to the tree in terms of how we access $x$. Notice that at each level, there are $O_{\varepsilon, \delta}(\log n)$ choices of $\Delta$ and $\tilde{O}_{\varepsilon, \delta}(n^d)$ candidate interval, we will create $\tilde{O}_{\varepsilon, \delta}(n^d)$ parallel instances at each node of the tree. For each instance at the $i$-th recursive level with $i < d$, the job is to invoke algorithm 5 $\tilde{O}_{\varepsilon, \delta}(n^d)$ times and then run a dynamic programming. Notice that we can run the dynamic programming $DPEditDistance$ while we compute the approximation of edit distance between each block and each of its candidate intervals. More specifically, we can first compute the approximation of edit distance between block $x^j$ and all of its candidate intervals, store them and then run the dynamic programming with those information. Then, we discard the stored edit distances and continue to compute the approximation between block $x^{j+1}$ and its candidate intervals. The order of computation is inline with the depth first search. Since the number of candidate intervals is $\tilde{O}_{\varepsilon, \delta}(n^d)$ and we only need to store $\tilde{O}_{\varepsilon, \delta}(n^d)$ values for the dynamic programming. The space required is $\tilde{O}_{\varepsilon, \delta}(n^d)$. At the $i$-th level, there are $\tilde{O}_{\varepsilon, \delta}(n^{di})$ instances running in parallel, the total space used at the $i$-th level is bounded by $\tilde{O}_{\varepsilon, \delta}(n^{d(i+1)})$ which is at most $\tilde{O}_{\varepsilon, \delta}(n^{i+1})$.

At the $d$-th recursive level, we calculate the edit distance exactly with $O(\sqrt{n} \log) \text{ bits of space.}$ We will invoke $\tilde{O}_{\varepsilon, \delta}(n^{bd}) = \tilde{O}_{\varepsilon, \delta}(n^{1/2})$ instances at each time. These instance all compute the edit distance between some block $\tilde{x}^j$ of size at most $\sqrt{n}$ and some interval in $y$. Thus, we can first store $\tilde{x}^j$ in the memory and run these instances in a sequential order so that the total space required is bounded by $\tilde{O}_{\varepsilon, \delta}(n^d d) = \tilde{O}_{\varepsilon, \delta}(n^{1/2})$. Since the order of computation is the same as depth first search, algorithm 5 is indeed a streaming algorithm with one pass.

Notice the depth of recursion is a constant $d$, the total memory required is $\tilde{O}_{\varepsilon, \delta}(n^{1/2})$. The order of computation is the same as depth first search, algorithm 5 is a streaming algorithm that access $\tilde{x}$ with one pass.

For the time complexity, the analysis is similar to that of algorithm 4. The only difference is that we stop the recursion whenever the size of block in $x$ is no larger than $\sqrt{n}$. Thus, the total time is still $\tilde{O}_{\varepsilon, \delta}(n^d)$

4 Longest Increasing Subsequence

In this section, we present our space-efficient algorithm for LIS. Let $x \in \Sigma^n$ be a sequence of length $n$ over alphabet $\Sigma$ (we assume each symbol in $\Sigma$ can be stored with $O(\log n)$ bits) and $\varepsilon$ be any constant in $(0, 1)$. We will show that for any integer $d$, there is an algorithm $\text{ApproxLIS}^d$ computes a $1 - \varepsilon$ approximation of $\text{LIS}(x)$ with only $O_{\varepsilon, \delta}(n^{d^2} \log n)$ bits of space.
Our approach is to build a sequence of algorithms ApproxLIS\(i\) for each integer \(i \geq 2\) where the algorithm ApproxLIS\(i+1\) is based on ApproxLIS\(i\). For each ApproxLIS\(i\), we introduce a slightly modified version of it called ApproxLISBound\(i\). ApproxLISBound\(i\) takes an additional input \(l\), which is an integer at most \(n\). We want to guarantee that if \(x\) has an increasing subsequence of length \(l\) ending with \(\alpha \in \Sigma\), then ApproxLISBound\(i\)(\(x, \varepsilon, l\)) can detect an increasing subsequence of length \((1 - \varepsilon)l\) ending with some symbol in \(\Sigma\) at most \(\alpha\).

The first algorithm ApproxLIS\(2\) is essentially the same as the streaming algorithm from [GJKK07]. We give the pseudocode of ApproxLIS\(2\) in algorithm \ref{alg:approxlis2}.

**Algorithm 6: ApproxLIS\(2\)**

\begin{algorithm}
\begin{algorithmic}[1]
\Input{A string \(x \in \Sigma^n\) and a constant \(\varepsilon \in (0, 1)\)}
\State let \(|x^i| = \sqrt{n}\) and divide \(x\) evenly into \(N\) blocks such that \(x = x^1 \circ x^2 \circ \cdots \circ x^N\)
\EndLine
\State initialize \(S = \{0\}\) and \(Q[0] = -\infty\)
\For {\(i = 1\) to \(N\)}
\For {\(j = 1\) to \(|x^i|\)}
\State find the largest \(s \in S\) such that \(Q[s] \leq x^i_j\) and set \(Q[s + 1] = x^i_j\)
\EndLine
\If {\(s + 1 \not\in S\)}
\State add \(s + 1\) to \(S\)
\EndLine
\EndLine
\State \(k = \max S\)
\EndLine
\State \(S' = \{0, 1, 2, \ldots, k\}\) if \(k \leq \sqrt{n}\), otherwise let \(S' = \{0, \sqrt{n}k, 2\sqrt{n}k, \ldots, k\}\)
\EndLine
\State \(Q' = Q\) for each \(s' \in S'\), let \(Q'[s'] = Q[s]\) where \(s\) is the smallest element in \(S\) such that \(s \geq s'\)
\State \(S' \leftarrow S', Q' \leftarrow Q'\)
\EndLine
\EndLine
\State return \(\max S\)
\EndLine
\end{algorithmic}
\end{algorithm}

**Lemma 4.1.** On input \(x \in \Sigma^n\) and constant \(\varepsilon \in (0, 1)\), ApproxLIS\(2\) outputs a \(1 - \varepsilon\) approximation of \(\text{LIS}(x)\) in \(O(n \log n)\) time with \(O(n \log n)\) bits of space.

The algorithm ApproxLISBound\(2\) is similar to ApproxLIS\(2\) with only two differences: first, at line \([6]\) if we find the largest \(s \in S\) such that \(Q[s] \leq x^i_j\) is larger or equal to \(l\), we do not add \(s + 1\) to \(S\) and continue. Second, ApproxLISBound\(2\) outputs set \(S\) and list \(Q\) but not only \(\max S\).

**Lemma 4.2.** On input \(x \in \Sigma^n\), constant \(\varepsilon \in (0, 1)\), and \(1 \leq l \leq n\), ApproxLISBound\(2\) outputs a set \(S\) and a list \(Q\), such that, if \(x\) has an increasing subsequence of length \(l\) ending with \(\alpha \in \Sigma\), then, there is an \(s \in S\) with \(Q[s] \leq \alpha\) and \(s \geq (1 - \varepsilon)l\). ApproxLISBound\(2\)(\(x, \varepsilon, l\)) runs in \(O(n \log n)\) time with \(O(n \log n)\) bits of space.

Since ApproxLIS\(2\) and ApproxLISBound\(2\) is essentially the same as the streaming algorithm from [GJKK07]. We omit the proof of Lemma 4.1 and Lemma 4.2.

Assume we are given ApproxLIS\(2\) and ApproxLISBound\(2\). We give the pseudocode of
Proof of Lemma 4.3. For our analysis, let \( \tau \) be one of the longest increasing subsequence of \( x \). \( \tau \) can be divided into \( N(=n^{\frac{\varepsilon}{1+\varepsilon/3}}) \) parts such that \( \tau^1 \circ \tau^2 \circ \cdots \circ \tau^N \) and \( \tau^i \) lies in \( x^i \). We define the following variables.

- \( \alpha_i \) is the first symbol of \( \tau^i \) (if \( \tau^i \) is not empty).
- \( \beta_i \) is the last symbol of \( \tau^i \) (if \( \tau^i \) is not empty).
- \( d_i = |\tau^i| \) is the length of \( \tau_i \).
- \( \gamma^i = \tau^1 \circ \tau^2 \circ \cdots \circ \tau^i \) is the concatenation of the first \( i \) blocks in \( \tau \).
- \( h_i = \sum_{j=1}^{i} d_j = |\gamma^i| \) is the length of \( \gamma^i \).

In the following, we let \( P \) be the list we get after running \( \text{PatienceSorting} \) with input \( x \). \( P' \) is the list “interpolated” by \( Q \) such that \( P'[i] = Q[j] \) for the smallest \( j \geq i \) that lies in \( S \). If no such \( j \) exist, set \( P'[i] = \infty \). We denote the set \( S \) and list \( Q \) after processing the block \( x^i \) (the
t-th outer loop) by $S_t$ and $Q_t$ and the largest element in $S_t$ by $k_t$. Correspondingly, $P'_t$ is the list $P''_t$ after processing the t-th block $x^t$ and $P_t$ is the list after running PatienceSorting with input $x^1 \circ x^2 \circ \cdots \circ x^t$.

Since $\tau$ is a longest increasing subsequence, without loss of generality, we can assume $P_t[h_t] = \beta_t$ (if $\tau^t$ is not empty) for each $t$ from 1 to $N$. This is because, if $P_t[h_t] < \beta_t$, we can replace $\gamma^t$ with another increasing subsequence of $x^1 \circ x^2 \circ \cdots \circ x^t$ with length $h_t$ and ends with $P_t[h_t]$. On the other hand, we must have $P_t[h_t] \leq \beta_t$ since $\gamma^t$ is an increasing subsequence of $x^1 \circ x^2 \circ \cdots \circ x^t$ with length $h_t$.

We also assume that $P_t[h_t] = P_{t+1}[h_t]$ if $\tau^t$ is an empty string ($h_t = h_{t+1}$). This is because if $P_t[h_t] > P_{t+1}[h_t]$, we can replace $\gamma^{t+1}$ with another increasing subsequence of $x^1 \circ x^2 \circ \cdots \circ x^{t+1}$ with length $h_t$ and ends with $P_{t+1}[h_t]$.

We first show the following claim.

**Claim 4.1.** For each $t \in [N]$, we have

$$P'_t[(1 - \frac{2\varepsilon}{3})h_t - 2t\varepsilon' n^{-\frac{1}{3\tau}} k_t] \leq P_t[h_t]$$

**Proof of Claim 4.1.** We prove this by induction on $t$. For the base case $t = 1$, if $d_1 = 0$, then $h_1 = 0$, $P'_1[-2t\varepsilon' n^{-\frac{1}{3\tau}} k_1]$ is not defined, we assume without loss of generality that $P'_1[2t\varepsilon' n^{-\frac{1}{3\tau}} k_1] \leq P'_1[0]$, since $P'_1[0]$ and $P'_1[0]$ are both special symbol $-\infty$, the claim holds.

If $d_1 > 0$, let $l$ be the largest number such that $l = (1 + \varepsilon/3)^a$ for some integer $a$ and $l \leq d_1$. We have $\frac{d_1}{\varepsilon} \leq l \leq \frac{d_1}{\varepsilon}$. Let $S, Q$ be the output of ApproxLISBound$(x^1, \varepsilon/3, l)$.

By our assumption on the correctness of ApproxLISBound, there exist an $s \in S$ such that $s \geq (1 - \varepsilon/3)l$ and $Q[s] \leq P_t[l] \leq P_t[h_t] = \beta_t$. By the choice of $S_1$, we know there is an $s \in S_1$ such that $s - \varepsilon' n^{-\frac{1}{3\tau}} k_1 \leq s \leq \hat{s}$ and $Q_1[s] \leq Q[s]$. Notice that

$$\hat{s} \geq (1 - \varepsilon/3)l \geq (1 - \varepsilon/3)^2d_1 \geq (1 - \varepsilon/3)d_1$$

We have $P'_t[(1 - 2\varepsilon/3)h_1 - 2t\varepsilon' n^{-\frac{1}{3\tau}} k_1] \leq P'_t[\hat{s} - \varepsilon' n^{-\frac{1}{3\tau}} k_1] \leq P_t[s] \leq Q[s] \leq P_t[h_t]$. This proved the base case.

Now we assume the claim holds for some fixed integer $t - 1 < N$, we show it also holds for $t$. If $\tau^t$ is an empty string, we have $h_t = h_{t-1}$ and $P_{t-1}[h_t] = P_t[h_t]$. Since $k_t \geq k_{t-1}$, we have

$$P'_t[(1 - \frac{2\varepsilon}{3})h_t - 2t\varepsilon' n^{-\frac{1}{3\tau}} k_t] \leq P'_t[(1 - \frac{2\varepsilon}{3})h_t - 2t\varepsilon' n^{-\frac{1}{3\tau}} k_{t-1}]$$

$$\leq P'_{t-1}[(1 - \frac{2\varepsilon}{3})h_t - 2t\varepsilon' n^{-\frac{1}{3\tau}} k_{t-1}]$$

$$\leq P_1[h_t]$$

Thus, the claim holds for the case when $\tau^t$ is an empty string. If $d_t > 0$ ($\tau^t$ is not empty), we know there is an $s_t \in S_{t-1}$ such that

$$(1 - \frac{2\varepsilon}{3})h_{t-1} - 2(t - 1)\varepsilon' n^{-\frac{1}{3\tau}} k_{t-1} - \varepsilon' n^{-\frac{1}{3\tau}} k_{t-1} \leq s_t \leq (1 - \frac{2\varepsilon}{3})h_{t-1} - 2(t - 1)\varepsilon' n^{-\frac{1}{3\tau}} k_{t-1}$$

and

$$Q_{t-1}[s_t] \leq P'_{t-1}[(1 - \frac{2\varepsilon}{3})h_{t-1} - (t - 1)\varepsilon' n^{-\frac{1}{3\tau}} k_{t-1}]$$

Let $z$ be the subsequence of $x^t$ by only considering the elements larger than $Q[s]$. Similarly, we let $l$ be the largest number such that $l = (1 + \varepsilon/3)^a$ for some integer $a$ and $l \leq d_t$. We
run ApproxLISBound\textsuperscript{d}(x, ε/3, l) to get \( \tilde{S} \) and \( \tilde{Q} \). By our assumption on the correctness of ApproxLISBound\textsuperscript{d}, there exist an \( \tilde{s} \in \tilde{S} \) such that \( \tilde{s} \geq (1 - \varepsilon/3)l \) and \( \tilde{Q}[\tilde{s}] \leq P_t[\sigma_a + l] \leq P_t[h_t] = \beta_t \). Let \( s_b \) be the largest element in \( S_t \) such that \( s_b \leq s_a + \tilde{s} \) and \( Q[s_b] \leq \tilde{Q}[\tilde{s}] \leq P_t[h_t] \). By the choice of set \( S_t \), we have

\[
\begin{align*}
s_b &\geq s_a + \tilde{s} - \varepsilon' n^{-\frac{1}{3d_l}} k_t \\
&\geq (1 - \frac{2\varepsilon}{3}) h_{t-1} - 2((t-1)\varepsilon' n^{-\frac{1}{3d_l}} k_{t-1} - \varepsilon' n^{-\frac{1}{3d_l}} k_t) + (1 - \varepsilon/3)^2 d_t - \varepsilon' n^{-\frac{1}{3d_l}} k_t \\
&\geq (1 - \frac{2\varepsilon}{3}) h_t - 2\varepsilon' n^{-\frac{1}{3d_l}} k_t
\end{align*}
\]

Here, the last inequality is from the fact that \( h_t = h_{t-1} + d_t \) and \( k_t \geq k_{t-1} \). Since \( P_t'[s_b] \leq P_t[h_t] \), we have shown that \( P_t'[(1 - \frac{2\varepsilon}{3}) h_t - \varepsilon' n^{-\frac{1}{3d_l}} k_t] \leq P_t[h_t] \). This finishes our proof of Claim 4.1. \( \square \)

Lemma 4.3 is a direct result of Claim 4.1. When \( t = N \), we have \( P'_N[(1 - \frac{2\varepsilon}{3}) h_N - 2N \varepsilon' n^{-\frac{1}{3d_l}} k_N] \leq P_N[h_N] \). Notice that \( \varepsilon' = \sqrt{\frac{\varepsilon}{N}} \) and \( N = \varepsilon' n^{-\frac{1}{3d_l}} \), we have \( 2N \varepsilon' n^{-\frac{1}{3d_l}} k_N = \varepsilon/3k_N \leq \varepsilon/3 \text{LIS}(x) \). Since \( h_N = \text{LIS}(x) \), we know

\[
P'_N[(1 - \varepsilon) \text{LIS}(x)] \leq P'_N[(1 - \frac{2\varepsilon}{3}) h_N - \varepsilon/3k_N] \leq P_N[\text{LIS}(x)] \leq \infty
\]

Thus, the output of \( \text{ApproxLIS}^{d+1}(x, \varepsilon) \) is at least \((1 - \varepsilon)\text{LIS}(x)\). \( \square \)

We now show the correctness of \( \text{ApproxLISBound}^{d+1} \).

**Lemma 4.4.** On input \( x \in \Sigma^n \), constant \( \varepsilon \in (0, 1) \), and \( 1 \leq l \leq n \), \( \text{ApproxLISBound}^{d+1} \) outputs a set \( S \) and a list \( Q \), such that, if \( x \) has an increasing subsequence of length \( l \) ending with \( \alpha \in \Sigma \), then, there is an \( s \in S \) with \( Q[s] \leq \alpha \) and \( s \geq (1 - \varepsilon)l \).

**Proof of Lemma 4.4.** The proof is similar to that of Lemma 4.3. Let \( P \) be the list we get after running PatienceSorting on \( x \). We only need to consider the case when \( \text{LIS}(x) \geq l \). Let \( \tau \) be an increasing subsequence of length \( l \) ending with \( \alpha = P[l] \). Notice that \( \alpha \) is the smallest symbol in \( \Sigma \) such that there is an increasing subsequence of length \( l \) ending with it.

We use the same notation as in the proof Lemma 4.3. Similarly, since \( \tau \) is the longest increasing subsequence ending with \( \alpha \), we can assume without loss of generality that \( P_t[h_t] = \beta_t \) and \( P_t[h_t] = P_{t+1}[h_t] \) if \( \tau^t \) is an empty string (\( h_t = h_{t+1} \)). We have the following claim.

**Claim 4.2.** For each \( t \in [N] \), we have

\[
P'_t[(1 - \frac{2\varepsilon}{3}) h_t - 2\varepsilon' n^{-\frac{1}{3d_l}} k_t] \leq P_t[h_t]
\]

The proof of Claim 4.2 directly follows from the proof of Claim 4.1.

Notice that in ApproxLISBound, we require \( k_t \leq l \) and \( h_N = l \), when \( t = N \), we have \( P'(1 - \varepsilon)l \leq P[l] \). Let \( S \) and \( Q \) be the output of \( \text{ApproxLISBound}^{d+1}(x, \varepsilon, l) \). There must exist an \( s \in S \) such that \( Q[s] \leq P[l] \) and \( s \geq (1 - \varepsilon)l \). \( \square \)

We now give the analysis of time and space complexity of \( \text{ApproxLIS}^d \) and \( \text{ApproxLISBound}^d \). We have the following results.
Lemma 4.5. On input $x \in \Sigma^n$ and $\varepsilon \in (0, 1)$, ApproxLIS$^d$ and ApproxLISBound$^d$ runs in $O_{\varepsilon,d}(n^{2-\frac{d}{\varepsilon}} \log^{d-1} n)$ time with $(\frac{\varepsilon}{\varepsilon'})^{O(d)} n^{\frac{d}{\varepsilon}} \log n$ bits of space.

Proof of Lemma 4.5. We only show the analysis for ApproxLIS$^d$ here. The analysis of ApproxLISBound$^d$ is exactly the same.

ApproxLIS$^d$ invokes a chain of algorithms. That is, for each $d \geq i \geq 3$, ApproxLIS$^i$ invokes ApproxLIS$^{i-1}$ and ApproxLISBound$^{i-1}$. Thus, the algorithm can be viewed as having $d-1$ levels. At the $i$-th level ($1 \leq i \leq d-1$), we run ApproxLIS$^{d-i+1}$ and ApproxLISBound$^{d-i+1}$. We denote the input length at the $i$-th level by $n_i$ and the $\varepsilon$ at the $i$-th level by $\varepsilon_i$. Similarly, we let $\varepsilon_i = \sqrt{\frac{\varepsilon}{\varepsilon'}}$. At the first level, $n_1 = n$ and $\varepsilon_1 = \varepsilon$.

Notice that every time we enter the next level, $\varepsilon$ is decreased by a factor of 3. Thus, $\varepsilon_i = \frac{\varepsilon}{3^{i-1}}$.

At the $i$-th level for $1 \leq i \leq d-2$, the input has length $n_i$, and the input is divided into $\frac{n_i}{\varepsilon_i}$ parts of size $n_i \frac{\varepsilon_i}{\varepsilon_i'} / \varepsilon_i'$, which gives $n_{i+1} = n_i \frac{\varepsilon_i}{\varepsilon_i'} / \varepsilon_i'$. Thus, we have $n_i = \frac{n_{i+1}}{\varepsilon_i} \frac{\varepsilon_i}{\varepsilon_i'}$ for $i > 1$.

We start with the space complexity. At the $i$-th level ($i < d-1$), the space used in part: storing $S, Q, S', Q'$ and running ApproxLIS$^{d-i}$ and ApproxLISBound$^{d-i}$. We denote the space used within the $i$-th level by $f_i$. That is, $f_i$ is the space used at the $i$-th level ignoring the space used for computing ApproxLIS$^{d-i}$ and ApproxLISBound$^{d-i}$. Since the size of $S, S'$ is bounded by $n_i \frac{\varepsilon_i}{\varepsilon_i'} / \varepsilon_i'$, we have $f_i = O(n_i \frac{\varepsilon_i}{\varepsilon_i'} / \varepsilon_i') = O(\frac{\varepsilon_i}{\varepsilon_i'} \log n)$. For the $(d-1)$-th level, we run ApproxLIS$^2$ and ApproxLISBound$^2$, the space used is $O(\sqrt{n_{d-1}/\varepsilon_{d-1}} \log n)$. The total space is equal to $\sum_{i=1}^{d-1} f_i$. Since $f_1 \leq f_{d-1}$, the total space used is bounded by $O(d \cdot f_{d-1}) = (\frac{\varepsilon_i}{\varepsilon_i'})^{O(d)} n^{\frac{d}{\varepsilon_i}} \log n$.

For the time complexity, we first consider the time used within one level. Each time we enter the $i$-th level, for $i < d-1$, we update $S$ and $Q$ $\varepsilon_i n_i \frac{\varepsilon_i}{\varepsilon_i'}$ times. Since the size of $S$ and $Q$ is bounded by $n_i \frac{\varepsilon_i}{\varepsilon_i'} \frac{\varepsilon_i}{\varepsilon_i'}$. Updating $S$ and $Q$ takes $O(n_i \frac{\varepsilon_i}{\varepsilon_i'}) = O_{\varepsilon,d}(n^{\frac{d}{\varepsilon}})$ time. When $i = d$, we run ApproxLIS$^2$ and ApproxLISBound$^2$ with input string of length $n_{d-1}$. It takes $O(n_{d-1} \log n) = O_{\varepsilon,d}(n^{\frac{d}{\varepsilon}} \log n)$ time.

We now bound the number of times we enter the $i$-th level. At the $i$-th level, we divide the input string into $N_i = \varepsilon_i n_i \frac{\varepsilon_i}{\varepsilon_i'}$ blocks and the set $S$ at that level is of size $n_i \frac{\varepsilon_i}{\varepsilon_i'} / \varepsilon_i'$. Thus, each time we enter the $i$-th level, we invoke ApproxLIS$^{d-i}$ $O(n_i \frac{\varepsilon_i}{\varepsilon_i'})$ times and ApproxLISBound$^{d-i}$ $O(n_i \frac{\varepsilon_i}{\varepsilon_i'} \log_{\varepsilon_i} n)$ times. It means, we enter the $(i+1)$-th level $(n_i \frac{\varepsilon_i}{\varepsilon_i'})^2$ times each time we enter the $i$-th level. Starting from the first level, let $g_i$ be the number of times we enter the $i$-th level. We have

$$g_i = O(\prod_{j=1}^{i-1} n_j^{\frac{d}{\varepsilon_i'}} \log_{\varepsilon_i} n)$$

Since $n_i = \frac{n_{i+1}}{\varepsilon_i} \frac{\varepsilon_i}{\varepsilon_i'}$, we have $g_i = O_{\varepsilon,d}(n^{(i-1)\frac{d}{\varepsilon}} \log^{i-1} n)$. $\varepsilon$ and $d$ are constants, the aggregated time is dominated by the time spend at $(d-1)$-th level. Thus, the aggregated time is $O(g_{d-1} \log n) = O((\prod_{j=1}^{d-2} n_j^{\frac{d}{\varepsilon_j}} \log_{\varepsilon_j} n_{d-1} \log n) = O_{\varepsilon,d}(n^{2-\frac{d}{\varepsilon}} \log^{d-1} n)$.

4.1 Output the Approximated Longest Increasing subsequence

We now show how our space-efficient algorithm for approximating the length of LIS can be modified to output the increasing subsequence it detected but not only the length.
Our goal is to build a sequence of algorithms called \textit{LISSequence}\(^d\) for each integer \(d \geq 1\) such that, \textit{LISSequence}\(^d\)(\(x, \varepsilon\)) outputs an increasing subsequence of \(x\) with length at least \((1 - \varepsilon)\text{LIS}(x)\) using only \(O_{\epsilon, d}(n^{d/2} \log d)\) bits of space. We build these algorithms one by one such that \textit{LISSequence}\(^{d + 1}\) is based on \textit{LISSequence}\(^d\). For \textit{LISSequence}\(^1\), we directly use the classical \textit{PatienceSorting} algorithm to output an LIS of \(x\) with \(O(n \log d)\) space \cite{AD99} (slightly different from the algorithm \cite{8} we presented in this work).

We give the pseudocode of \textit{LISSequence}\(^d\) for \(d > 1\) in algorithm \cite{8}. In the following discussion, the same notation is used as in the analysis of algorithm \cite{7} let \(S_t\) and \(Q_t\) be the set \(S\) and list \(Q\) after processing the \(i\)-th block of \(x\) and \(S_0 = \{0\}\), \(Q_0 = -\infty\).

**Algorithm 8: LISSequence\(^d\)**

\begin{verbatim}
Input: A string \(x \in \Sigma^n\) and a small constant \(\varepsilon \in (0, 1)\)
1 set \(\varepsilon_1 = \varepsilon/2\) and \(\varepsilon' = \sqrt{\varepsilon_1/6}\)
2 let \(N = \varepsilon' n^{1/3}\) and divide \(x\) evenly into \(N\) blocks \(x^1 \circ x^2 \circ \cdots \circ x^N\) compute \(S_N\) and \(Q_N\)
   by running \textbf{ApproxLIS}\(^d\)(\(x, \varepsilon_1\)) \(\triangleright S_i\) and \(Q_i\) are the set \(S\) and list \(Q\) after \(i\)-th outer loop of \textbf{ApproxLIS}\(^d\)
3 set \(B\) to be a list with \(B[0] = -\infty\) and \(B[N] = Q_N[S_N]\) where \(S_N = \max\{s \in S_N\}\)
4 for \(i = N - 1 \text{ to } 1\) do
   5 release the space used to store \(S_{i+1}\) and \(Q_{i+1}\)
   6 compute \(S_i\), \(Q_i\) by running \textbf{ApproxLIS}\(^d\)(\(x, \varepsilon_1\))
   7 foreach \(s \in S_i\) \textbf{such that} \(s \leq s_{i+1}\) do
      8 let \(z\) be the subsequence of \(x^i\) by only considering the elements larger than \(Q[s]\)
      9 foreach \(l = 1, 1 + \varepsilon_1/3, (1 + \varepsilon_1/3)^2, \ldots, k - s\) do
         10 \(\tilde{S}, \tilde{Q} \leftarrow \textbf{ApproxLISBound}\(^{d-1}\)(z, \varepsilon_1/3, l)\)
         11 if there is an \(\tilde{s} \in \tilde{S}\) such that \(\tilde{s} + \tilde{s} \geq s_{i+1}\) and \(B[i + 1] = \tilde{Q}[\tilde{s}]\), we set
            \(B[i] = Q_i[s], s_i = s\) and \textbf{continue}
      12 end
   13 end
14 end
15 for \(i = 1 \text{ to } N\) do
16 \(\text{let } z\) be the subsequence of \(x^i\) ignoring every element larger than \(B[i]\) or less or equal to \(B[i - 1]\)
17 \(\text{LISSequence}\(^{d-1}\)(z, \varepsilon_1)\)
18 end
\end{verbatim}

**Lemma 4.6.** For any constant \(d \geq 1\), there is an algorithm \textit{LISSequence}\(^d\), such that, on input string \(x \in \Sigma^n\) and any constant \(\varepsilon \in (0, 1)\), \textit{LISSequence}\(^d\)(\(x, \varepsilon\)) outputs an increasing subsequence of \(x\) in \(O_{\varepsilon, d}(n^{d/2} \log^{d-1} n)\) time with \(O_{\varepsilon, d}(n^{d/2} \log n)\) bits of space. The length of the output sequence is at least \((1 - \varepsilon)\text{LIS}(x)\).

**Proof of Lemma 4.6.** We proof this lemma by induction on \(d\). For the base case \(d = 1\), we use the exact algorithm as \textit{LISSequence}\(^1\) to output a longest increasing sequence in \(O(n \log n)\) time with \(O(n \log n)\) space (see \cite{AD99} for example).

Now, assume the lemma holds for \(d - 1\). We can construct \textit{LISSequence}\(^d\) as described in algorithm \cite{8}. We first show the correctness of \textit{LISSequence}\(^d\). We use the same notations as in algorithm \cite{8}.

Notice that \textbf{ApproxLIS}\(^d\)(\(x, \varepsilon_1\)) detects an increasing subsequence \(t\) such that \((1 - \varepsilon_1)\text{LIS}(x) \leq |t| \leq \text{LIS}(x)\). Our goal is to output a sequence that has length at least \((1 - \varepsilon_1)|t|\).
Same as in the pseudocode, we let \( \varepsilon_1 = \varepsilon / 2 \) and \( \varepsilon' = \sqrt{\varepsilon_1 / 6} \). We divide \( x \) into \( N = \varepsilon' n^{1/d} \) blocks such that \( x = x_1 \circ x_2 \circ \cdots \circ x_N \) where each block \( x_i \) has length \( n / N = O_\varepsilon(n^{2/3}) \).

**Claim 4.3.** After running \( \text{LISSequence}^d(x, \varepsilon) \), \( B \) is a list of \( N + 1 \) elements with \( B[0] = -\infty \), such that, there is an increasing subsequence \( t = t^1 \circ t^2 \circ \cdots \circ t^N \) of \( x \) with length at least \((1 - \varepsilon_1) \text{LIS}(x) \) and \( t^i \) lies in \( x_i \), and the last element of \( t^i \) (if not empty) is at most \( B[i] \) for each \( 1 \leq i \leq N \).

**Proof of Claim 4.3.** We know that for each \( 1 \leq i \leq N \) and \( s \in S_i \), there is an increasing subsequence of \( x^i \circ x^2 \circ \cdots \circ x^i \) of length \( s \) and the last element of the subsequence is at most \( Q_i[s] \).

We first set \( s_N = \max \{ S_N \) and \( B[N] = Q_N[s_N] \). We know there is a increasing subsequence of \( x \) with length \( s_N \) and the last element is \( B[N] \). As we have proved in the correctness of \( \text{ApproxLIS}^d \), \((1 - \varepsilon_1) \text{LIS}(x) \leq s_N \leq \text{LIS}(x) \). Thus the claim holds for \( i = N \).

We prove the claim by induction on \( i \) from \( N - 1 \) to 1. In the \( i \)-th loop, we are given a value \( s_{i+1} \in S_{i+1} \) such that \( Q_{i+1}[s_{i+1}] = B[i+1] \). We first compute \( S_i \) and \( Q_i \). Let \( z \) be the subsequence of \( x^i \) ignoring all elements no larger than \( Q_i[s_i] = B[i] \). By our choice of \( s_i \), we know that there is some \( l \) such that \( \text{ApproxLISBound}^{d-1}(z, \varepsilon_1/3, l) \) detects an increasing subsequence of length \( s_{i+1} - s_i \) with first symbol larger than \( Q_i[s_i] = B[i] \) and last symbol at most \( Q_{i+1}[s_{i+1}] = B[i+1] \). We denote this increasing subsequence by \( t^i \).

Thus, we are able to find a sequence \( t = t^1 \circ t^2 \circ \cdots \circ t^N \). The length of \( t \) is \(|t| = \sum_{i=1}^{N} s_i - s_{i-1} = s_N \) (we let \( s_0 = 0 \)). This has proved the claim 4.3.

\[ \square \]

Let \( t \) be the increasing subsequence in the proof of Claim 4.3. Given such a list \( B \), for each block \( x_i \), let \( x' \) be the subsequence of \( x^i \) ignoring every element larger than \( B[i] \) or less or equal to \( B[i-1] \), we run \( \text{LISSequence}^{d-1}(x, \varepsilon_1) \). Notice that we do not need to store the sequence \( z \), instead, when algorithm \( \text{LISSequence}^{d-1} \) reads a symbol from \( x^i \), say \( x^i_j \), if \( x^i_j \leq B[i] \) or \( x^i_j > B[i+1] \), we replace it with \( \infty \). By our assumption on the correctness of \( \text{LISSequence}^{d-1} \), we are able to output a sequence of length \((1 - \varepsilon_1)|t^i| \). Since we set \( \varepsilon_1 = \varepsilon / 2 \), we are able to output a sequence at least \((1 - \varepsilon) \text{LIS}(x) \)

For the space complexity, \( B \) is a list of size \( N = \varepsilon' n^{1/d} \). \( S \) and \( Q \) are both of size \( O_\varepsilon(n^{1/d}) \). Storing \( B, S, Q \) takes \( O_\varepsilon(n^{2/d} \log n) \). We first fix \( s_N \) and \( B[N] \) by running \( \text{LISSequence}^d(x, \varepsilon_1) \). This can be done with \( O_{\varepsilon,d}(n^{1/d} \log n) \). In the loop starting from line 4 for each \( i \) from \( N - 1 \) to 1, we compute \( S_i \) and \( Q_i \) by running \( \text{LISSequence}^d(x, \varepsilon_1) \), this also can be done with \( O_{\varepsilon,d}(n^{1/d} \log n) \) space and the space can be reused. Once we computed \( S_{i-1} \) and \( Q_{i-1} \), we run \( \text{ApproxLISBound}^{d-1}(O_{\varepsilon,d}(\log n) \) times on input \( z \), which has length at most \(|x^i| = n^{2/d}/\varepsilon' \) since \( z \) is a subsequence of \( x^i \). Thus, by Lemma 4.1, computing \( \text{ApproxLISBound}^{d-1}(z, \varepsilon_1/3, l) \) and storing \( S \), \( Q \) takes \( O_{\varepsilon,d}(n^{1/d} \log n) \) space. Also, the space can be reused. Thus, everything before line 15 uses only \( O_{\varepsilon,d}(n^{1/d} \log n) \) space. For the loop starting from line 15 we compute \( \text{LISSequence}^{d-1} \) \( \text{LISSequence}^{d-1} \) \( n \) times with input string length at most \( n^{2/d}/\varepsilon' \). Since the space can be reused, by the inductive hypothesis, this can be done with \( O_{\varepsilon,d}(n^{1/d} \log n) \). Thus, the space complexity is still \( O_{\varepsilon,d}(n^{1/d} \log n) \).

For the time complexity, first, we need to run \( \text{ApproxLIS} \) \( N \) times to compute \( S_i \) and \( Q_i \) for each \( i \in [N] \). By Lemma 4.1, this can be done in \( O_{\varepsilon,d}(Nn^{2-2/d} \log^{d-1} n) = O_{\varepsilon,d}(n^{2-1/d} \log^{d-1} n) \) time. Then for each \( i \in [N] \), the size of \( S_i \) is \( O_{\varepsilon,d}(n^{1/d}) \) and there are \( O(\log_2(n)) \) choices of \( l \). We need to run \( \text{ApproxLISBound} \) \( O_{\varepsilon,d}(n^{1/d} \log n) \) times. Since the length of \( z \) is at most \( n^{2/d}/\varepsilon' \). Running \( \text{ApproxLISBound} \) one time takes \( O_{\varepsilon,d}(n^{4-1/d} \log^{2-1/d-1} n) = n^{2-1/d} \log^{d-2} n \).
$O_{ε,d}(n^{2-4/d} \log^{d-2} n)$. The total time used for computing ApproxLISBound$^{d-1}$ is $O_{ε,d}(n^{2-2/d} \log^{d-1} n)$. Thus everything before line 15. We compute LISSequence line 15. We compute LISSequence$^{d-1} \times$ LIS times with input string length at most $n^{2-2/d}/ε’$. Running LISSequence$^{d-1}$ each time takes $O_{ε,d}(n^{2-2/d-1} \log^{d-2} n) = O_{ε,d}(n^{2-3/d} \log^{d-2} n)$. Thus, the loop starting from line 15 takes $O_{ε,d}(n^{2-2/d} \log^{d-2} n)$ time. Combining this with the time used before this loop, the total time is $O_{ε,d}(n^{2-1/d} \log^{d-1} n)$.

\[ \square \]

5 Longest Common Subsequence

In this section, we describe our algorithm for approximating LCS($x$) with small space. Before introducing our algorithm, we introduce the following reduction from LCS to LIS.

5.1 Reducing LCS to LIS

Our space efficient algorithm for LCS is based on a reduction (algorithm 9) from LCS to LIS.

**Algorithm 9: ReduceLCS to LIS**

| Input: Two strings $x \in \Sigma^n$ and $y \in \Sigma^m$. |
| Output: An integer sequence $z \in [m]^*$ |
| 1 initialize $z$ to be an empty string |
| 2 for $i = 1$ to $n$ do |
| 3 \quad for $j = m$ to 1 do |
| 4 \quad \quad if $x_i = y_j$, add $j$ to the end of $z$. |
| 5 end |
| 6 end |
| 7 return $z$ |

**Lemma 5.1.** Given two strings $x \in \Sigma^n$ and $y \in \Sigma^m$ as input to algorithm 9, let $z = \text{ReduceLCS to LIS}(x,y) \in [m]^*$ be the output, then the length of $z$ is $O(mn)$ and $\text{LIS}(z) = \text{LCS}(x,y)$.

**Proof of Lemma 5.1.** $z$ can be viewed as the concatenation of $n$ blocks such that $z = b^1ob^2\cdots ob^n$ ($b^i$‘s can be empty). For each $i$, the length of $b^i$ is equal to the number of times the character $x_i$ appeared in $y$. The elements of $b^i$ are the indices of characters in $y$ that are equal to $x_i$. These indices in $b^i$ are sorted in descending order. Since the length of $b^i$ for each $i$ is at most $m$, the length of $z$ is at most $mn$.

Assuming $\text{LIS}(z) = l$, we show $\text{LCS}(x,y) \geq l$. By the assumption, there exists a subsequence of $z$ with length $l$. We denote this subsequence by $t \in [m]^l$. Let $t = t_1t_2\cdots t_l$. Since $b^i$’s are strictly descending, each element in $t$ is picked from a distinct block. We assume for each $i \in [l]$, $t_i$ is picked from the block $b^i$. Then by the algorithm, we know $x_{t_i} = y_{t_i}$. For $1 \leq i < j \leq l$, $t_i$ appears before $t_j$. The block $b^j$ also appears before $b^j$. We have $1 \leq t_1 < t_2 < \cdots < t_l \leq n$. Thus, $x_{t_1}x_{t_2}\cdots x_{t_l}$ is a subsequence of $x$ with length $l$ and it is equal to $y_{t_1}y_{t_2}\cdots y_{t_l}$. Hence, LCS($x,y$) is at least $l$.

On the other direction, assuming LCS($x,y$) = $l$, we show $\text{LIS}(z) \geq l$. By the assumption, let $x’ = x_{t_1}x_{t_2}\cdots x_{t_l}$ be a subsequence of $x$ and $y’ = y_{t_1}y_{t_2}\cdots y_{t_l}$ be a subsequence of $y$ such that $x’ = y’$. Let $z’ = b^{t_1}ob^{t_2}\cdots ob^{t_l}$, which is a subsequence of $z$. For each $i \in [l]$, since $x_{t_i} = y_{t_i}$, $t_i$ appears in the block $b^i$. By $1 \leq t_1 < t_2 < \cdots < t_l \leq m$, we know $t = t_1t_2\cdots t_l$ is an increasing subsequence of $z’$ and thus also an increasing subsequence of $z$. \[ \square \]
5.2 \((1 - \varepsilon)\)-approximation using \(O(\sqrt{n/\varepsilon} \log n)\) space

In the following, we assume both \(x\) and \(y\) are strings over alphabet \(\Sigma\) with length \(n\).

We first describe a \((1 - \varepsilon)\)-approximation algorithm for LCS using \(O(\sqrt{n} \log n)\) space. The idea is to first reduce calculating LCS\((x, y)\) to an LIS problem. Then, we can apply a deterministic algorithm that is similar to the streaming algorithm described in [GJKK07]. However, storing \(z = \text{ReduceLCSToLIS}(x, y)\) already takes \(O(n^2 \log n)\) bits of space. We will show that this is not required for our algorithm.

In the following discussion, \(z\) can be viewed as a string consists of \(n\) blocks such that \(z = b^1 \circ b^2 \circ \cdots \circ b^n\). According to algorithm [4], \(b^i\) is a string consisting all the indices \(j\) such that \(y_j = x_i\) and these indices in \(b^i\) are sorted in descending order.

We now present a space-efficient algorithm for approximating the length of longest common subsequence that gives a \((1 - \varepsilon)\) approximation of LCS\((x, y)\) with only \(O(\sqrt{n/\varepsilon})\) space. We call this algorithm \textbf{ApproxLCS}^2 and give the pseudocode in algorithm [10].

Algorithm 10: \textbf{ApproxLCS}^2

\begin{verbatim}
Input: Two strings \(x \in \Sigma^n\) and \(y \in \Sigma^m\), a small constant \(\varepsilon \in (0, 1)\)
1. initialize \(S = \{0\}\) and \(Q[0] = 0\)
2. \(\varepsilon' = \varepsilon/2\), \(N = \sqrt{\varepsilon' n}\)
3. divide \(x\) evenly into \(N\) blocks such that \(x = x^1 \circ x^2 \circ \cdots \circ x^N\)
4. for \(i = 1\) to \(N\) do
5. \(z^i = \text{ReduceLCSToLIS}(x^i, y)\)  \(\triangleright\) we use the notation \(z^i\) for convenience. it is not required to store \(z^i\). we will discuss it in the analysis.
6. \(k = \text{LIS}(z^i)\)  \(\triangleright\) calculate \(\text{LIS}(z^i)\) with \textbf{PatienceSorting}
7. foreach \(s \in S\) do  \(\triangleright\) loop1
8. \(D\) be the list we get after running \textbf{PatienceSorting} on input \(z^i\) ignoring all elements no larger than \(Q[s]\)
9. let \(l\) be the largest index such that \(D[l] < \infty\)
10. \(k \leftarrow \max\{s + l, k\}\)
11. end
12. if \(k \leq \sqrt{\varepsilon}\), let \(S' = \{0, 1, 2, \ldots, k\}\), otherwise let \(S' = \{0, \sqrt{\varepsilon} k, 2\sqrt{\varepsilon} k, \ldots, k\}\)
\(\triangleright\) evenly pick \(\sqrt{\varepsilon}\) integers from 0 to \(k\) (including 0 and \(k\))
13. \(Q'[s'] \leftarrow \infty\) for all \(s' \in S'\) except \(Q'[0] = 0\)
14. foreach \(s \in S\) do  \(\triangleright\) loop2
15. \(D\) be the list we get after running \textbf{PatienceSorting} on input \(z^i\) ignoring all elements no larger than \(Q[s]\)
16. foreach \(s' \in S'\) such that \(s' \geq s\) do
17. if \(s' = s\) then
18. \(Q'(s') = \min\{Q'[s'], Q[s]\}\)
19. else
20. \(Q'(s') \leftarrow \min\{Q'[s'], D[s' - s]\}\)
21. end
22. end
23. end
24. \(S \leftarrow S'\)
25. \(Q \leftarrow Q'\)
26. end
27. return \(\max\{s \in S\}\)
\end{verbatim}
Algorithm\textsuperscript{10} can be viewed as a simulation of the streaming algorithm for LIS from [GJKK07]. It computes the longest increasing subsequence of \( z = \text{ReduceLCStoLIS}(x, y) \). However, storing \( z \) takes too much space and we cannot afford it. The idea is to divide \( z \) into \( \sqrt{\varepsilon n} \) parts. We can compute LIS exactly in each part with \( O(\sqrt{n/\varepsilon \log n}) \) space. Then, by maintaining a small size approximation of \( P \) from the \textit{PatienceSorting} algorithm, we can compute a good approximation of \( \text{LIS}(z) \).

**Lemma 5.2.** Given two input strings \( x \in \Sigma^n \) and \( y \in \Sigma^m \) with \( n \leq m \). For any \( \varepsilon \in (0, 1) \), algorithm\textsuperscript{11} \((\text{ApproxLCS}^2(x, y, \varepsilon))\) outputs a \((1 - \varepsilon)\)-approximation of \( \text{LCS}(x, y) \) in time \( O(\frac{n^{3/2}}{\varepsilon^{2/3}} \log n) \) with \( O(\sqrt{n/\varepsilon \log m}) \) bits of space.

**Proof of Lemma 5.2** We start by showing the correctness of \text{ApproxLCS}^2. For our analysis, let \( z = \text{ReduceLCStoLIS}(x, y) \) and \( \varepsilon' = 1/2\varepsilon \), note that \( z \) need not to be fully stored. Similar to the previous analysis, \( z \) is the concatenation of \( n \) blocks. That is \( z = b^n \circ b \circ \cdots \circ b \). We divide these \( n \) blocks evenly into \( \sqrt{\varepsilon n} \) parts such that \( z = z^1 \circ z^2 \circ \cdots \circ z^{\sqrt{\varepsilon n}} \), where \( z^i = \text{ReduceLCStoLIS}(\sqrt{\varepsilon n/\varepsilon} \cdot \langle i \rangle) \) for \( i \in [\sqrt{\varepsilon n}] \), i.e. each \( z^i \) consists \( \sqrt{n/\varepsilon} \) consecutive blocks in \( z \). In other words, \( z^i = \text{ReduceLCStoLIS}(x^i, y) \).

Notice that \( \text{LIS}(z^i) = \text{LCS}(x^i, y) \) and the length of \( x^i \) is \( \sqrt{n/\varepsilon} \), we have \( \text{LIS}(z^i) \leq \sqrt{n/\varepsilon} \). Thus, we only need a list of size \( \sqrt{n/\varepsilon} \) to run \textit{Patience Sorting} on \( z^i \). Also, we do not need to store \( z^i \). Since \textit{PatienceSorting} scans \( z^i \) from left to right once, this can be done by scanning \( y \) from right to left \( \sqrt{n/\varepsilon} \) times.

The proof of approximation ratio is similar to the analysis in [GJKK07]. We compare our algorithm with patience sorting. Let \( P \) be the list we get after running \textit{PatienceSorting} with input \( z \). Algorithm\textsuperscript{10} maintains an approximated version of \( P \) while only store no more than \( \sqrt{n/\varepsilon} \) values. It is achieved by only store values \( Q[i] \) for \( i \in S \). When running the algorithm, we make sure the size of \( S \) is no larger than \( \sqrt{n/\varepsilon} \). For our analysis, we let \( P' \) be the list “interpolated” by \( S \) \( Q \) such that \( P'[i] = Q[j] \) for the smallest \( j \geq i \) that lies in \( S \). If no such \( j \) exist, set \( P'[i] = \infty \).

Consider the outer for-loop starting from line 4 we denote \( S \) and \( Q \) after the \( t \)-th loop by \( S_t \) and \( Q_t \). Correspondingly, let \( P'_t \) be the list generated by \( S_t \) and \( Q_t \) and \( P_t \) be the list after running \textit{PatienceSorting} on \( z^1 \circ z^2 \circ \cdots \circ z^t \). Let the largest element in \( S_t \) be \( k_t \). Then, \( S_t = \{ \lfloor \sqrt{\varepsilon/\varepsilon k_t} \rfloor, \lfloor 2\sqrt{\varepsilon/\varepsilon k_t} \rfloor, \ldots, k_t \} \).

We call the inner loop starting from line 7 \textit{loop1} and the inner loop starting from line 13 \textit{loop2}. \textit{loop1} is used to determine the length \( (k_t) \) of the longest increasing subsequence that can be extended with values in \( Q_{t-1} \). With \( k_t \), we are able to construct the set \( S_t \). For every \( s \in S_t \), \textit{loop2} calculates \( Q_t(s) \) which is the smallest value such that there is an increasing subsequence of length \( s \) ending with \( Q_t(s) \).

In the \( t \)-th loop, if we replace \( P_{t-1} \) with \( P'_{t-1} \) and run \textit{PatienceSorting} on \( z^t \), we get a list \( P_{t-1} \). In the following claim, we show that running \textit{loop2} is equivalent to running \textit{PatienceSorting} on \( z_t \) with \( P_{t-1} = P'_{t-1} \).

**Claim 5.1.** Let \( l_1 \) be the largest index such that \( P'_t[l_1] < \infty \) and \( l_2 \) the largest index such that \( P'_t[l_2] < \infty \). Then \( l_1 = l_2 \). Also, for each \( i \in S_t \), we have \( P'_t[i] = P_t[i] \).

**Proof.** Let the longest increasing subsequence detected by running \textit{PatienceSorting} be \( \tau \). Then, \( \tau \) is of length \( l_2 \). Divide \( \tau \) into two parts \( \tau_1 \) and \( \tau_2 \) such that \( \tau_1 \) is an increasing subsequence of \( z^1 \circ z^2 \circ \cdots \circ z^{i-1} \) and \( \tau_2 \) is an increasing subsequence of \( z^i \). Let the length of \( \tau_1 \) and \( \tau_2 \) be \( i_1 \) and \( i_2 \) respectively. If \( i_1 = 0 \), then we can detect \( \tau_2 \) by running \textit{PatienceSorting} on \( z^i \) with only elements larger than \( Q[0] = 0 \) at line 8 (since all elements in \( z^i \) are positive integers, this is equal to running \textit{PatienceSorting} on \( z^i \)). We have \( l_1 \geq i_2 = l_2 \). If \( i_1 > 0 \), assume \( \tau_1 \) ends
with $a$ and $\tau_2$ starts with $b$ so that $a < b \leq P_t'[i_2]$. We know $P_{t-1}'[i_1] = a$, since otherwise we can replace $\tau_1$ with a string of the same length ending with $P_{t-1}'[i_1]$. Let $i_1'$ be the smallest element of $S_{t-1}$ with $i_1' \geq i_1$. Then we have $Q_{t-1}[i_1'] = P_{t-1}'[i_1]$ by the definition of $P_{t-1}'$. At line 8 running PatienceSorting on $z'$ with only elements larger than $Q[i_1']$ will detect the longest increasing subsequence in $z'$ with first element larger than $P_{t-1}'[i_1]$. Since $\tau_2$ is such a sequence with length $i_2$. We can detect an increasing subsequence with length at least $i_2$. Thus, $i_1 \geq i_1' + i_2 \geq i_1 + i_2 = i_2$. Also, any subsequence detected by loop1 can be detected by PatienceSorting, thus $i_1 \leq i_2$.

Similarly, for any $i \in S_t$, PatienceSorting detects an increasing subsequence $\tau$ that has length $i$ and ending with $P_t'[i]$. We divide $\tau$ into two parts $\tau_1$ and $\tau_2$. Similarly, let $\tau_1$ ends with $a$ if $i_1 > 0$. Let $i_1'$ be the smallest element in $S_{t-1}$ with $i_1' \geq i_1$. We know $Q_{t-1}[i_1'] = a$. Since $\tau_2$ is the longest increasing substring in $z'$ with first element larger than $a$ and last element equal to $P_t'[i]$. Running PatienceSorting on $z'$ with only elements larger than $Q_{t-1}[i_1']$ will detect $\tau$. $P_t'[i_1' + i_2] \leq P_t'[i_1' + i_2] \leq P_t'[i]$. Since any subsequence detected by loop1 can be detected by PatienceSorting, we have $P_t'[i] \geq P_t'[i]$. □

We now prove the following claim which mostly follows from lemma 2.2 of [CJJK07].

**Claim 5.2.** For each $t \in [\sqrt{\varepsilon/n}]$, we have

$$P_t'[i - 2(t-1)\lfloor \sqrt{\varepsilon/nk_t} \rfloor] \leq P_t[i]$$

Here, we assume $P_t'[i - 2(t-1)\lfloor \sqrt{\varepsilon/nk_t} \rfloor] = 0$ if $i \leq 2(t-1)\lfloor \sqrt{\varepsilon/nk_t} \rfloor$.

**Proof.** We prove the claim by induction. For the base case when $t = 1$, notice that LIS($z_1$) = $k_1 = \lfloor \sqrt{n/\varepsilon} \rfloor$, we have $S_1 = [k_1]$. Thus we have stored every value in $P$ which means $P = P'$. We have $P_t'[i - 2(t-1)\lfloor \sqrt{\varepsilon/nk_t} \rfloor] \leq P_t'[i] = P_t[i]$.

Now consider $t > 1$, we assume the claim holds for $t-1$. For simplicity, we let $P_t[i] = c$ (if $P_t[i] = \infty$, the statement holds).

Then there is an increasing subsequence $\tau$ in $z^1 \circ z^2 \circ \cdots \circ z^t$ of length $i$ and ending with $c$. Again, we divide $\tau$ into two parts $\tau_1$ and $\tau_2$ such that $\tau_1$ is a subsequence of $z^1 \circ z^2 \circ \cdots \circ z^{t-1}$ and $\tau_2$ is a subsequence of $z_t$. Let the length of $\tau_1$ and $\tau_2$ be $i_1$ and $i_2$ respectively. Assume $\tau_1$ ends with $a$ and $\tau_2$ starting with $b$ so that $a < b < c$.

We can assume $P_{t-1}'[i_1] = a$ since if not, we can replace $\tau_1$ with another string of length $i_1$ ends with $P_{t-1}'[i_1] \geq a$. By the induction assumption, we have

$$P_{t-1}'[i - 2(t-2)\lfloor \sqrt{\varepsilon/nk_{t-1}} \rfloor] \leq P_{t-1}'[i] = a$$

If $i - 2(t-2)\lfloor \sqrt{\varepsilon/nk_{t-1}} \rfloor < 0$, the inner loop starting from line 14 (loop2) is guaranteed to find an increasing subsequence of length $i_2$ with last element at most $c$. Let $i'$ be the largest element in $S_t$ such that $i' \leq i_2$. We have $P_t'[i'] \leq P_t[i_2] = c$. We also have

$$i' \geq i_2 - \lfloor \sqrt{\varepsilon/nk_t} \rfloor$$

$$\geq i_1 - 2(t-2)\lfloor \sqrt{\varepsilon/nk_{t-1}} \rfloor - \lfloor \sqrt{\varepsilon/nk_t} \rfloor + i_2 - \lfloor \sqrt{\varepsilon/nk_t} \rfloor$$

$$\geq i_1 - 2(t-1)\lfloor \sqrt{\varepsilon/nk_t} \rfloor$$

The last inequality is due to the fact that $k_t \geq k_{t-1}$.

If $i - 2(t-2)\lfloor \sqrt{\varepsilon/nk_{t-1}} \rfloor \geq 0$, notice that $S$ contains one integer in every $\lfloor \sqrt{\varepsilon/nk_{t-1}} \rfloor$ consecutive numbers, there is an $i'_1 \in S_{t-1}$ such that

$$i_1 - 2(t-2)\lfloor \sqrt{\varepsilon/nk_{t-1}} \rfloor - \lfloor \sqrt{\varepsilon/nk_{t-1}} \rfloor \leq i'_1 \leq i_1 - 2(t-2)\lfloor \sqrt{\varepsilon/nk_{t-1}} \rfloor$$

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Thus, the inner loop starting from line 14 (loop2) is guaranteed to find a increasing subsequence of $z^i \circ z^{i+1} \circ \cdots \circ z^t$ with length at least $i_1 + i_2$ with last element at most $c$. Let $i'$ be the largest element in $S_t$ such that $i' \leq i_1 + i_2$. Then
\[
P'_t[i'] \leq P_t[i] = c
\]
We also have
\[
i' \geq i_1 + i_2 - \lfloor \sqrt{e'/nk_t} \rfloor \\
\geq i_1 - 2(t-2)\lfloor \sqrt{e'/nk_{t-1}} \rfloor - \lfloor \sqrt{e'/nk_t} \rfloor + i_2 - \lfloor \sqrt{e'/nk_t} \rfloor \\
\geq i - 2(t-1)\lfloor \sqrt{e'/nk_t} \rfloor
\]
The last inequality is due to the fact that $k_t \geq k_{t-1}$. Since $P'_t$ is nondecreasing, we have
\[
P'_t[i - 2(t-1)\lfloor \sqrt{e'/nk_t} \rfloor] \leq P_t[i]
\]
This finishes the induction. \hfill \square

Now we assume LIS($z$) = $l$. By claim 5.2, we have
\[
P'[l - 2(\sqrt{e/n} - 1)\lfloor \sqrt{e/m} \rfloor] \leq P[l] \leq \infty
\]
Since $l - 2(\sqrt{e/n} - 1)\lfloor \sqrt{e/m} \rfloor \geq (1-\varepsilon)l$, ApproxLCS$^2(x, y)$ must have detected an increasing substring of length at least $(1-\varepsilon)LIS(z) = (1-\varepsilon)LCS(x, y)$. We now turn to the time and space complexity. We first analyse the two inner loops. Notice that the size of $S$ is at most $\sqrt{n/e'} = O(\sqrt{n/e})$. Both loop1 and loop2 runs PatienceSorting on $z^i$ (although each time, we ignore some elements in $z^i$) $O(\sqrt{n/e'})$ times with $z^i \leq \sqrt{n/e'm}$. The total time used to run PatienceSorting on $z^i$ is $O(\frac{nm}{e} \log n)$. For the remaining part in loop1, finding $l$ takes $O(\sqrt{n/e} \log n)$ time and set $k$ takes $O(\sqrt{n/e})$ time. For loop2, updating $Q$ takes $O(\sqrt{n/e})$ time. Thus, running both loop1 and loop2 once takes time $O(\frac{nm}{e} \log n)$. For the outer loop, initializing $S'$ and $Q'$ and updating $S$ and $Q$ both takes $O(\sqrt{n/e})$ time. Since the outer loop executes loop1 and loop2 $O(\sqrt{n/m})$ times. The total time running time is $O(\frac{nm^2}{e} \log n)$.

For the space complexity, storing $S$, $S'$, $Q$, and $Q'$ all takes $O(\sqrt{n/e} \log m)$. Since the LIS($z^i$) $\leq |z^i| = \sqrt{n/e}$, running PatienceSorting on $z^i$ takes an additional $O(\sqrt{n/e} \log m)$ bits of space. The list $P$ we get after running PatienceSorting on $z^i$ is also of size $O(\sqrt{n/e} \log m)$. Thus, the total space required is $O(\sqrt{n/e} \log m)$ bits. \hfill \square

### 5.3 $(1-\varepsilon)$-approximation using $O_{\varepsilon,d}(n^{1.5} \log n)$ space
Similar to our approach for LIS, we can build a sequence of algorithms called ApproxLCS$^i$ for $i \geq 2$.

We first introduce a slightly modified version of ApproxLCS$^2$ called ApproxLCSBound$^2$. ApproxLCSBound$^2$ takes an additional input $l \leq m$ such that if there is an increasing subsequence in $z$ of length $l$ ending with $\alpha \in [m]$, ApproxLCSBound$^2(x, y, \varepsilon, l)$ can detect an increasing subsequence in $z$ of length at least $(1-\varepsilon)l$ with last element no larger than $\alpha$.

ApproxLCSBound$^2$ is essentially the same as ApproxLCS$^2$. It takes one additional input: $1 \leq l \leq n$. The only difference is that at line 12 of algorithm 10 we require $k$ to be no larger
than \( l \). That is, we set \( k = \min\{k, l\} \) before line 12. We also require \( \text{ApproxLCSBound}^d \) to output set \( S \) and \( Q \) but not only the largest value in \( S \).

Now, we assume we are given algorithm \( \text{ApproxLCS}^d \) and \( \text{ApproxLCSBound}^d \). We give the pseudocode of algorithm \( \text{ApproxLCS}^{d+1} \) in algorithm 11.

Algorithm 11: \( \text{ApproxLCS}^{d+1} \)

| Input: Two strings \( x \in \Sigma^n \) and \( y \in \Sigma^m \), a constant \( \epsilon \in (0, 1) \) |
| --- |
| 1. \( \epsilon' = \sqrt{\frac{n}{\epsilon}} \) |
| 2. let \( N = \epsilon'n^{\frac{1}{1-d}} \) and divide \( x \) evenly into \( N \) blocks \( x^1 \circ x^2 \circ \cdots \circ x^N \) \( \triangleright \) \( |x^i| \leq n^{\frac{1}{1-d}}/\epsilon' \) |
| 3. initialize \( S = \{0\} \) and \( Q[0] = -\infty \) |
| 4. for \( i = 1 \) to \( N \) do |
| 5. \( k = 0 \) |
| 6. foreach \( s \in S \) do |
| 7. \( d = \text{ApproxLCS}^d(x^i, y^*(Q[s]), \epsilon/3) \) \( \triangleright \) by \( y^*(Q[s]) \), we mean the string we get by replacing the first \( Q[s] \) elements of \( y \) with a special symbol \( \ast \) that does not appear in \( x \) |
| 8. \( k = \max\{k, s + d\} \) |
| 9. if \( k \leq n^{\frac{1}{1-d}}/\epsilon' \), let \( S' = \{0, 1, 2, \ldots, k\} \), otherwise let \( S' = \{0, \sqrt{n/2}, \sqrt{n}, \ldots, k\} \) |
| 10. \( \triangleright \) evenly pick \( n^{\frac{1}{1-d}}/\epsilon' \) integers from 0 to \( k \) (including 0 and \( k \)) |
| 11. \( Q'[s] \leftarrow \infty \) for all \( s \in S' \) except \( Q'[0] = -\infty \) |
| 12. foreach \( s \in S' \) do |
| 13. \( \triangleright \) \( s' \in S' \) such that \( s \leq s' \leq s + l \), let \( \tilde{s} \) be the smallest element in \( \tilde{S} \) that is larger than \( s' - s \) and set \( Q'[s'] = \min\{\tilde{Q}[\tilde{s}], Q'[s']\} \) |
| 14. \( S \leftarrow S', Q \leftarrow Q' \) |
| 15. end |
| 16. end |
| 17. \( \text{return} \ max S \) |

We now describe the algorithm \( \text{ApproxLCSBound}^{d+1} \). Compared to \( \text{ApproxLCS}^{d+1} \), it takes an additional input: a number \( 1 \leq l \leq n \). \( \text{ApproxLCSBound}^{d+1} \) is similar to \( \text{ApproxLCS}^{d+1} \) except that at line 10 of algorithm 11 we require \( k \) to be no larger than \( l \). That is, we set \( k = \min\{k, l\} \) before line 11. We also require \( \text{ApproxLCSBound}^{d+1} \) to output set \( S \) and \( Q \) but not only the largest value in \( S \).

We have the following lemma that gives the correctness of \( \text{ApproxLCS}^{d+1} \).

Lemma 5.3. On input \( x \in \Sigma^n, y \in \Sigma^m \), and a constant \( \epsilon \in (0, 1) \), \( \text{ApproxLCS}^{d+1}(x, y, \epsilon) \) outputs a \( 1 - \epsilon \) approximation of \( \text{LIS}(x, y) \).

Proof of Lemma 5.3. Let \( z = \text{ReduceLCSToLIS} \in [m]^* \), our goal is to compute a \( 1 - \epsilon \) approximation of \( \text{LIS}(z) \). We first divide \( x \) into \( N = \epsilon'n^{\frac{1}{1-d}} \) blocks such that \( x = x^1 \circ x^2 \circ \cdots \circ x^N \). Correspondingly, we can divide \( z \) into \( N \) parts. With the \( i \)-th part \( z^i = \text{ReduceLCSToLIS}(x^i, y) \).

The proof is similar to that of Lemma 4.3. We use \( \tau \) be a longest increasing subsequence of \( z \). \( \tau \) can be divided into \( N(= \epsilon'n^{\frac{1}{1-d}}) \) parts such that \( \tau^1 \circ \tau^2 \circ \cdots \circ \tau^N \) and \( \tau^i \) lies in \( z^i \). Let \( \alpha_i, \beta_i, d_i, \gamma_i \) and \( h_i \) be the same as in the proof of Lemma 4.3.

In the following, we let \( P \) be the list we get after running \( \text{PatienceSorting} \) with input \( z \). \( P' \)
is the list generated by \( Q \) such that \( P'[i] = Q[j] \) for the smallest \( j \geq i \) that lies in \( S \). If no such \( j \) exist, set \( P'[i] = \infty \). We denote the set \( S \) and list \( Q \) after processing the block \( z^t \) (the \( t \)-th outer loop) by \( S_t \) and \( Q_t \) and the largest element in \( S_t \) by \( k_t \). Correspondingly, \( P'_t \) is the list \( P' \) after processing the \( t \)-th block \( z^t \) and \( P_t \) is the list after running \textit{PatienceSorting} with input \( z^1 \circ z^2 \circ \ldots \circ z^t \).

We can assume \( P_t[h_t] = \beta_t \) and \( P_t[h_t] = P_{t+1}[h_t] \) if \( z^t \) is an empty string for the same reason as in the proof of Lemma 4.3.

We can show the following claim.

**Claim 5.3.** For each \( t \in [N] \), we have

\[
P'_t\left((1 - \frac{2\epsilon}{3})h_t - 2\epsilon^\prime n^{-\frac{1}{\max_3}}k_t\right) \leq P_t[h_t]
\]

**Proof of Claim 5.3.** The proof directly following from our proof for Claim 4.1. Except that \( x \) in that proof is now \( z \), \textbf{ApproxLIS} and \textbf{ApproxLISBound} is now replaced by \textbf{ApproxLCS} and \textbf{ApproxLCSBound}.

**Lemma 5.4.** On input \( x \in \Sigma^n, y \in \Sigma^m \), constant \( \epsilon \in (0,1) \), and \( 1 \leq l \leq n \), \textbf{ApproxLISBound} \( l+1 \) outputs a set \( S \) and a list \( Q \), such that, if \( z = \text{ReduceLCS} \circ \text{LIS}(x, y) \) has an increasing subsequence of length \( l \) ending with \( \alpha \in [m] \), then, there is an \( s \in S \) with \( Q[s] \leq \alpha \) and \( s \geq (1 - \epsilon)l \).

**Lemma 5.5.** On input string \( x \in \Sigma^n, y \in \Sigma^m \) and constant \( \epsilon \in (0,1) \), \textbf{ApproxLCS} \( d \) and \textbf{ApproxLCSBound} \( d \) runs in \( O_{d, \epsilon}(n^{2 - \frac{4}{d}}m \log^{d-1} n) \) time with \((\frac{3N}{\epsilon})^{O(\epsilon)}n^\frac{1}{d} \log m \) bits of space.

**Proof of Lemma 5.5.** We only show the analysis for \textbf{ApproxLCS} \( d \) here. The analysis of \textbf{ApproxLCSBound} \( d \) is exactly the same.

We assume the input string \( x \) has length \( n \) and the other input string has length \( m \).

This proof follows mostly from the proof of Lemma 4.3. \textbf{ApproxLCS} \( d \) can be viewed as having \( d - 1 \) levels. At the \( i \)-th level \((1 \leq i \leq d - 1)\), we run \textbf{ApproxLCS} \( d - i + 1 \) and \textbf{ApproxLCSBound} \( d - i + 1 \). We denote the length of the first input string at the \( i \)-th level by \( n_i \) and the third input \( \epsilon_i \) (the length of the second input string is always \( m \)). We let \( \epsilon'_i = \sqrt{\frac{\epsilon}{3}} \). At the first level, \( n_1 = n \) and \( \epsilon_1 = \epsilon \).

Notice that every time we enter the next level, \( \epsilon \) is decreased by a factor of 3. Thus, \( \epsilon_i = \frac{\epsilon}{3^{i-1}} \).

At the \( i \)-th level for \( i \leq d - 2 \), the input has length \( n_i \), we then divides it into \( \frac{n_i}{\epsilon_i} \) parts each of size \( \epsilon_i \), which gives us \( n_{i+1} = \frac{n_i}{\epsilon_i^2} \). Thus, we have \( n_{i} = \frac{n_{i-1}}{\epsilon_{i-1}^2} \) for \( i > 1 \).
We start with the space complexity. At the $i$-th level ($i < d - 1$), the space is used in two parts: storing $S, Q, S', Q'$ and running $\text{ApproxLCS}^{d-i}$ and $\text{ApproxLCSBound}^{d-i}$. We denote the space used within the $i$-th level by $f_i$. That is, $f_i$ is the space used at the $i$-th level ignoring the space used for computing $\text{ApproxLCS}^{d-i}$ and $\text{ApproxLCSBound}^{d-i}$. Since the size of $S$, $S'$ is bounded by $n_{i+1}^{d-i+1}/\varepsilon^2_i$, we have $f_i = O(n_{i+1}^{d-i+1}/\varepsilon^2_i) = O(n_{i+1}^{d} \log m)$. For the $(d - 1)$-th level, we run $\text{ApproxLCS}^2$ and $\text{ApproxLCSBound}^2$, the space used is $O(\sqrt{n_{d-1}/\varepsilon_{d-1}} \log m)$. The total space used is equal to $\sum_{i=1}^{d-1} f_i$. Since $f_i \leq f_{i+1}$, the total space used is bounded by $O(d \cdot f_{d-1}) = O(d \cdot n^d \log m).

For the time complexity, we first consider the time used within one level. Each time we enter the $i$-th level, for $i < d - 1$, we update $S$ and $Q$ with $n_{i+1}^{d-i+1}$ times. Since the size of $S$ and $Q$ is bounded by $n_{i+1}^{d-i+1}/\varepsilon^2_i$. Updating $S$ and $Q$ takes $O_{\varepsilon,i}(n_{i+1}^{d-i+1}) = O_{\varepsilon,i}(n^d \log m)$ time. When $i = d$, we run $\text{ApproxLCS}^2$ and $\text{ApproxLCSBound}^2$ with input string of length $n_d$. By Lemma 5.2, it takes $O(n_{d-1}^{3/2} \log n) = O_{\varepsilon,d}(n^d \log m)$ time.

We now bound the number of times we enter the $i$-th level. At the $i$-th level, we divide the input string into $N_i = \varepsilon_i n_{i+1}^{d-i+1}$ blocks and the set $S$ at that level is of size $n_{i+1}^{d-i+1}/\varepsilon^2_i$. Thus, each time we enter the $i$-th level, we invoke $\text{ApproxLIS}^{d-i} O(n_{i+1}^{d-i+1})$ time and $\text{ApproxLISBound}^{d-i} O(n_{i+1}^{d-i+1} \log \varepsilon_i) n$ times, which means we enter the $(i+1)$-th level $O(n_{i+1}^{d-i+1} \log \varepsilon_i) n$ times. Starting from the first level, let $g_i$ be the number of times we enter the $i$-th level. We have

$$g_i = O(\prod_{j=1}^{i-1} n_j^{d-j+1} \log \varepsilon_j) n$$

Since $n_j = n_{i+1}^{d-i+1}/\Pi_{k=1}^{j-1} \varepsilon_k$, we have $g_i = O_{\varepsilon,i}(n^{(i-1)d} \log^{i-1} n)$. $\varepsilon$ and $d$ are constants, the aggregated time is dominated by the time used at $(d-1)$-th level. Thus, the aggregated time is $O(n_{d-1}^{3/2} \log n) = O(n^{(d-2)d} \log^{d-2} \Pi \log m) = O_{\varepsilon,d}(n^{2d-2} m \log^{d-1} n)$.

### 5.4 Output the Approximated Longest Common subsequence
We now show how to output a common subsequence of length $(1 - \varepsilon)\text{LCS}(x, y)$ with small space. The idea is similar to our approach on how to output LIS.

We build a sequence of algorithms called $\text{LCSSequence}^d$ for each integer $d \geq 1$, such that, on input $x \in \Sigma^n$, $y \in \Sigma^m$, and constant $\varepsilon \in (0, 1)$. $\text{LISSequence}^d(x, y)$ outputs a common subsequence of $x$ and $y$ with length at least $(1 - \varepsilon)\text{LCS}(x, y)$ using only $O_{\varepsilon,d}(n^d \log m)$ bits of space. We build these algorithms one by one such that $\text{LISSequence}^{d+1}$ is based on $\text{LISSequence}^d$.

For the first algorithm $\text{LISSequence}^1$, linear space algorithm from [Hir75] that output a LCS of $x$ and $y$ with $O(\min(n, m) \log n)$ space (we assume each symbol in the alphabet set $\Sigma$ can be stored with $O(\log n)$ bits of space). Then, assume we are given $\text{LISSequence}^{d-1}$, we show
Lemma 5.6. For any constant \(d \geq 1\), there is an algorithm \(\text{LCSSequence}^d\), such that, on input strings \(x \in \Sigma^n\), \(y \in \Sigma^m\), and any constant \(\varepsilon \in (0, 1)\), \(\text{LCSSequence}^d(x, \varepsilon)\) outputs a common sequence of \(x\) and \(y\) in \(O_{\varepsilon, d}(n^2 \varepsilon d^{-1} \log d)\) time with \(O_{\varepsilon, d}(n^2 \varepsilon d^{-1} \log n)\) bits of space. The length of the output sequence is at least \((1 - \varepsilon)\text{LCS}(x, y)\).

Proof of Lemma 5.6 This proof follows mostly from our proof of Lemma 4.6. We proof this lemma by induction on \(d\). For the base case \(d = 1\), we use the linear space algorithm from [Hir75] that output a LCS of \(x\) and \(y\) with \(O(n \log n)\) space in \(O(nm)\) time.

Now, assume the lemma holds for \(d - 1\). We can construct \(\text{LCSSequence}^d\) as described in algorithm [12]. We first show the correctness of \(\text{LCSSequence}^d\). We use the same notations as in algorithm [12].

The correctness follows directly from the proof of the correctness of Lemma 4.6. Except that \(x\) in that proof is now \(z\), \(\text{LISSequence}\), \(\text{ApproxLIS}\) and \(\text{ApproxLISBound}\) is now replaced by \(\text{LCSSequence}, \text{ApproxLCS}\) and \(\text{ApproxLCSBound}\)

For the space complexity, \(B\) is a list of size \(N = \varepsilon’ n^{1 + \varepsilon}\). \(S\) and \(Q\) are both of size \(O_\varepsilon(n^{1 + \varepsilon})\). Storing \(B, S, Q\) takes \(O_\varepsilon(n^{1 + \varepsilon})\) log \(n\). We first fix \(S_N\) and \(Q_N\) by running \(\text{LCSSequence}^d(x, y, \varepsilon_1)\). This can be done with \(O_\varepsilon(n^{1 + \varepsilon})\) space. In the loop starting from line 4 for each \(i\) from \(N - 1\) to 1, we compute \(S_i\) and \(Q_i\) by running \(\text{LCSSequence}^d(x, y, \varepsilon_1)\), this also can be done with \(O_\varepsilon(n^{1 + \varepsilon})\) log \(n\) space and the space can be reused. Once we computed \(S_i\) and \(Q_i\), for each \(s \in S_{i-1}\), we run \(\text{ApproxLCSBound}^d\) \(O_\varepsilon(\log n)\) times on input \(x'\), \(y*(Q[s])\). We have \(|x'| = n^{1+\varepsilon}/\varepsilon’\). Thus, by Lemma 5.5 computing \(\text{ApproxLCSBound}^d(x', y*(Q[s]), \varepsilon_1/3, l)\)
and storing $\tilde{S}$, $\tilde{Q}$ takes $O_{\varepsilon,d}(n^{\frac{d}{2}} \log n)$ space. Again, the space can be reused. Thus, everything before line 15 uses only $O_{\varepsilon,d}(n^2 \log n)$ space. For the loop starting from line 15, we compute $\text{LCSSequence}^{d-1} \text{ LISSequence}^{d-1} N$ times with input strings $x^i$ and $y_{B[i-1]+1:B[i]}$. $x^i$ has length at most $n^{d-1}/\varepsilon'$. Since the space can be reused, by the inductive hypothesis, this can be done with $O_{\varepsilon,d}(n^2 \log n)$. Thus, the space complexity is still $O_{\varepsilon,d}(n^2 \log n)$.

For the time complexity, we need to run $\text{ApproxLCS}$ $N$ times to compute $S_i$ and $Q_i$ for each $i \in [N]$. By Lemma 5.5 this can be done in $O_{\varepsilon,d}(Nn^{2-1/d}m \log^{d-1} n) = O_{\varepsilon,d}(n^2m \log^{d-1} n)$ time. Then for each $i \in [N]$, the size of $S_i$ is $O_{\varepsilon,d}(n^{\frac{d}{2}})$ and there are $O(\log \varepsilon (n))$ choices of $l$. We need to run $\text{ApproxLCSBound}$ $O_{\varepsilon,d}(n^2 \log n)$ times. Since the length of $x^i$ is $n^{\frac{d}{2}}/\varepsilon'$. Running $\text{ApproxLCSBound}$ one time takes $O_{\varepsilon,d}(n^{2-1/d}m \log^{d-2} n) = O_{\varepsilon,d}(n^{2-3/d}m \log^{d-2} n)$. The total time used before this loop, the total time is $O_{\varepsilon,d}(n^{2/d} \log n)$ times. For the loop starting from line 15, we compute $\text{LISSequence}^{d-1} \text{ LISSequence}^{d-1} N$ times with input string length at most $n^{d-1}/\varepsilon'$. Running $\text{LISSequence}^{d-1}$ each time takes $O_{\varepsilon,d}(n^{2-1/d}m \log^{d-2} n) = O_{\varepsilon,d}(n^{2d-3}m \log^{d-2} n)$. Thus, the loop starting from line 15 takes $O_{\varepsilon,d}(n^{2/d} \log n)$ time. Combining this with the time used before this loop, the total time is $O_{\varepsilon,d}(n^2m \log^{d-1} n)$.

5.5 Asymmetric Model

Algorithm 10 can be directly used for the asymmetric setting. That is, we have streaming access to one string and random access to the other. We have the following result.

**Lemma 5.7.** Given two strings $x \in \Sigma^n$ and $y \in \Sigma^m$. Suppose we have streaming access to string $x$ and random access to string $y$. Then, there is a deterministic algorithm that, makes one pass through $x$, outputs a $(1-\varepsilon)$-approximation of LCS$(x, y)$ in time $O(\sqrt{n/\varepsilon} \log m)$ with $O(\sqrt{n/\varepsilon} \log m)$ bits of space.

**Proof of Lemma 5.7.** We will show $\text{ApproxLCS}^2$ is such a streaming algorithm. The only place we need to query $x$ is when running $\text{PatienceSorting}$ on $x^i$. Also notice that to read $z^i$, we only need to query the $i$-th block $x^i$. Since the length of $x^i$ is $\sqrt{n/\varepsilon}$, in the streaming model, we can first store $x^i$ in our memory with an additional $O(\sqrt{n/\varepsilon} \log m)$ bits of space.

Thus, we can do the following, when $i = i_0$, we read and store the block $x^{i_0}$ and then do the computation inside that loop. After finishing it, $i$ is incremented. We release the memory used for storing $x^{i_0}$ and then read and store the next block $x^{i_0+1}$ of $x$. Storing one block of $x$ requires an additional $O(\sqrt{n/\varepsilon} \log m)$ bits of space. The aggregated space is still $O(\sqrt{n/\varepsilon} \log m)$. Since $\text{ApproxLCS}^2$ makes only one pass through $x$, this proves the lemma.

6 Discussion and Open Problems

In this paper we designed several space efficient approximation algorithms for three string problems that are widely used in practice: edit distance, longest common subsequence, and longest increasing sequence. All our algorithms are deterministic and can use space $n^\delta$ for any constant $\delta > 0$, while achieving $1 + \varepsilon$ or $1 - \varepsilon$ approximation for any constant $\varepsilon > 0$. The running time of our algorithms are essentially the same as, or only slightly larger than the standard algorithms which solve these problems exactly. Our work leaves many interesting open problems, and we list them below.
1. Can we achieve better space complexity? For example, is it possible to reduce the space complexity to sub-polynomial, poly-logarithmic, or even logarithmic while still maintaining polynomial running time? What kind of approximations can we achieve in these cases? For example, can we keep the approximation to be $1 + \varepsilon$ or $1 - \varepsilon$, or at least a constant approximation? We note that our algorithms can actually be pushed to work in slightly sub-polynomial space, but going beyond that seems to require new ideas.

2. So far all our algorithms are deterministic. How does randomness help here? Can we design randomized algorithms that achieve $1 + \varepsilon$ or $1 - \varepsilon$ approximation, but with better space complexity?

3. Can we design approximation algorithms that simultaneously have significantly better running time and significantly better space? For example, can we design an algorithm that achieves constant approximation to edit distance, with truly sub-quadratic running time and $n^\delta$ space for any constant $\delta > 0$? One possible approach towards this is to see if we can combine the techniques in our paper with the techniques in other papers (e.g., [AKO10], [CDG+10]).

4. Finally, is there a good reason for the lack of progress on computing edit distance and longest common subsequence exactly using polynomial time and strongly sub linear space? In other words, it would be nice if one can provide justification like the SETH-hardness of computing edit distance and longest common subsequence exactly in truly sub-quadratic time. We note that a recent work of Yamakami [Yam17] proposes a so called Linear Space Hypothesis, which conjectures that some NL-complete problems cannot be solved simultaneously in polynomial time and strongly sub linear space. Thus it would be nice to show reductions from these problems to edit distance and longest common subsequence. We note that here we need a small space and polynomial time reduction.

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