Three-dimensional strings
I. Classical theory

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Abstract

I consider a three-dimensional string theory whose action, besides the standard area term, contains one of the form \( \int_\Sigma \epsilon_{\mu\nu\sigma} X^\mu dX^\nu \wedge dX^\sigma \). In the case of closed strings this extra term has a simple geometrical interpretation as the volume enclosed by the surface. The associated variational problem yields as solutions constant mean curvature surfaces. One may then show the equivalence of this equation of motion to that of an \( SU(2) \) principal chiral model coupled to gravity. It is also possible by means of the Kemmotsu representation theorem, restricted to constant curvature surfaces, to map the solution space of the string model into the one of the \( \mathbb{CP}^1 \) nonlinear sigma model. I also show how a description of the Gauss map of the surface in terms of \( SU(2) \) spinors allows for yet a different description of this result by means of a Gross-Neveu spinorial model coupled to 2-D gravity. The standard three-dimensional string equations can also be recovered by setting the current-current coupling to zero.

1 Introduction

The geometry of surfaces have found several application in physics specially since the advent of string theory as a candidate to describe QCD in four dimensions or as a “theory of everything” [1]. Nevertheless, in spite of some spectacular successes from the purely technical point of view, there is a general consensus that we are still far away from a phenomenologically realistic theory. In particular for the case of QCD, the fact that the Nambu-Goto action only seems to make sense, due to anomalies, in 26 dimensions induced Polyakov [9]
to consider an alternative approach based on the coupling of conformal matter to two-dimensional gravity. Although classically both approaches are easily seen to be equivalent for the case of $D$ bosons coupled to two-dimensional gravity (where $D$ represents the dimension of the target space), Polyakov approach permitted, through a careful treatment of the Weyl anomaly, to extend the analysis to the non-critical case. But unfortunately the existence of the “infamous” $c = 1$ barrier [2] has not allowed us to study the physically interesting dimensions, arguably 3 and 4.

The purpose of this paper is to study an alternative three-dimensional string model with the hope that it will bring new insight into this difficult subject. The model under consideration, besides the standard area term contains another one that, for the particular case of closed surfaces, can be interpreted as the volume enclosed by it. This implies, among other things, that this action can be useful to describe the statistical mechanics of interfaces in three dimensions. This follows from the fact that the volume term may describe a bulk contribution whenever the energy density of one of the phases is different from the other.

The plan of the paper is as follows. First I will remind the reader of some basic notions about the geometry of immersed surfaces in $\mathbb{R}^3$ that will be used in the following. I will then continue by introducing the three-dimensional string model action under consideration. Its equations of motion turn to be the condition of constant mean curvature for a surface immersed in $\mathbb{R}^3$. I will then show how it is possible to map this classical problem into the one of a principal chiral $SU(2)$ model coupled to 2-D gravity.

In the fourth section I pass to introduce the Gauss map of a surface in $\mathbb{R}^3$, in order to take advantage of all the machinery already developed by mathematicians in the subject. In particular it is possible to map the solution space of our string model into the one of the nonlinear $\mathbb{CP}^1$ model, by means of the Kemmotsu representation theorem [3] restricted to surfaces of constant mean curvature. Although the relationship between surfaces of constant mean curvature and the $\mathbb{CP}^1$ nonlinear sigma model has already been used in the physics literature [4][6], I believe that completeness, as well as a slightly different presentation that will be of later use for my specific purposes, justify a detailed presentation. I will also comment about the relationship with affine $SL(2)$ Toda theory [7], and the geometrical interpretation of its affine Toda fields [5].

In the fifth section I will use the covariant spinorial construction, developed to introduce the Gauss map [8], to show the equivalence of the string model with a spinorial Gross-Neveu model. I will then show that in the limit when the current-current interaction is set to zero one recovers the equation of motion of the standard three-dimensional string action. As a simple exercise, I will
recover the Weierstrass-Enneper representation for minimal surfaces inside this formalism.

I will finish by making some considerations about the quantization of this model and remarking some relationships with the Polyakov rigid string approach [9][4].

2 A very brief course about surface theory in $\mathbb{R}^3$

The purpose of this introductory section is to present in a simple manner the most important geometrical constructions to be used in the sequel, as well as to set up my notations. For a comprehensive introduction to this fascinating subject I refer the reader to the excellent book of M. Spivak [10].

Let $\Sigma$ be an oriented two-dimensional connected Riemannian manifold and $X : \Sigma \to \mathbb{R}^3$ an isometric immersion of $\Sigma$ into $\mathbb{R}^3$. At any point $p$ of $\Sigma$ a basis for the tangent plane is provided by $\partial_\alpha X^i$. The induced metric, or first fundamental form of the immersion, is then given by

$$g_{\alpha\beta} = \partial_\alpha X^i \cdot \partial_\beta X^i. \quad (1)$$

It is now possible to obtain a basis for $T\mathbb{R}^3$ at $p$ by adding a unitary perpendicular vector $n$, which explicit coordinate expression may be given by

$$n^i = \frac{1}{2\sqrt{g}} \epsilon^{ijk} \epsilon^{\alpha\beta} \partial_\alpha X^j \partial_\beta X^k, \quad (2)$$

with $g$ being the determinant of the induced metric.

One may now write down the structural equations of the immersion as

$$\partial_\beta \partial_\alpha X = \Gamma^\rho_{\beta\alpha} \partial_\rho X + K_{\beta\alpha} n \quad (3)$$

$$\partial_\alpha n = -g^{\beta\rho} K_{\alpha\beta} \partial_\rho X. \quad (4)$$

The first of this equation may be taken as the definition of the extrinsic curvature $K$, or second fundamental form of the immersion, while the second follows from consistency with the relations $n \cdot n = 1$ and $\partial_\alpha X \cdot n = 0$. Notice that multiplying the first of this equations by $\partial_\gamma X$ one readily obtains that the connection coefficients $\Gamma$ are the ones of the Levi-Civita connection associated with the induced metric; multiplication by $n$ implies that $K$ is a symmetric tensor.
The Codazzi-Mainardi equation is obtained from
\[ \partial_\gamma X \cdot (\epsilon^{\alpha\beta} \partial_\alpha \partial_\beta n) = 0, \] (5)
which yields that \( \nabla_{[\alpha} K_{\beta]\gamma} = 0 \). And finally the Gauss equation is obtained from
\[ \partial_\gamma X \cdot (\epsilon^{\rho\beta} \partial_\rho \partial_\beta X) = 0, \] (6)
which implies that \( R_{\gamma\alpha\rho\beta} = K_{\gamma\rho\beta} - K_{\gamma\rho\alpha} \), where \( R \) is the Riemann curvature tensor associated with the induced metric.

It is now intuitively clear that given two symmetric tensors \( g \) and \( K \) obeying the integrability condition one may recover, up to Euclidean motions\(^2\), the associated surface by integrating the structural equations.

One may define now the mean curvature, \( H \), and the Gaussian curvature, \( K \), by
\[ H = \frac{1}{2} g^{\alpha\beta} K_{\alpha\beta}, \quad \text{and} \quad K = \frac{1}{2} \epsilon^{\alpha\rho\beta\gamma} K_{\alpha\beta} K_{\rho\gamma}. \] (7)

With all of this in mind we now may pass to study the string model at hand.

3 The action principle

As already commented in the introduction I will consider in what follows a string action that besides the standard area term contains a contribution of the form
\[ S_I = \frac{\Omega}{3} \int_{\Sigma} d^2 x \epsilon_{ijk} \epsilon^{\alpha\beta} X^i \partial_\alpha X^j \partial_\beta X^k. \] (8)

Despite its appearance \( S_I \) its invariant under Euclidean motions in the target space. While the rotational invariance is explicit, translational invariance is only achieved up to total derivatives, which of course do not change the dynamical properties of the action.

\(^2\) This due to the fact that the first and second fundamental forms, as defined above, are invariant under global translations and rotations in \( \mathbb{R}^3 \).
Does \( S_I \) have a simple geometrical interpretation? The answer turns out to be positive. This can be most easily seen by rewriting the extra term in the action as

\[
S_I = \frac{2\Omega}{3} \int_{\Sigma} d^2x \sqrt{g} \, \mathbf{X} \cdot \mathbf{n},
\]

which, in the case of a closed surface, is proportional to the enclosed volume. This can be easily checked by taking the origin to be an interior point of the surface and considering the volume element enclosed by an infinitesimally small solid angle bounded by the surface. This term has already been considered in the mathematical literature as a Lagrange multiplier to determine closed minimal surfaces subject to a constant volume constraint [11].

A straightforward computation shows that the equation of motion associated with the full action \( S(\Sigma) = 1/\tau \text{Area}(\Sigma) + S_I(\Sigma) \) also has a simple geometrical interpretation: the solution of the associated variational problem is given by surfaces of constant mean curvature. Explicitly

\[
\square \mathbf{X} = \tau \Omega \mathbf{n},
\]

which corresponds to a constant mean curvature \( H = \tau \Omega /2 \), as follows from the definition of \( H \). Notice that this result can also be achieved by choosing the Nambu-Goto or Polyakov prescription for the area term in the action.

A parenthetical comment: one may choose a representation where Euclidean invariance is manifest by introducing a vector valued auxiliary field \( \varphi \). Let me consider the following action

\[
\tilde{S} = \frac{1}{\tau} \text{Area}(\Sigma) + \frac{2}{3} \Omega \int d^2x \epsilon_{ijk} \epsilon^{\alpha\beta} \left( \varphi^i \partial_\alpha X^j \partial_\beta X^k - \frac{1}{2} \varphi^i \partial_\alpha \varphi^j \partial_\beta X^k \right). \tag{11}
\]

Now one may eliminate the auxiliary field from the equation of motion and recover the constant mean curvature condition, while keeping explicit translational invariance.

Interestingly enough equation (10) can be written as a zero curvature condition associated with a \( SU(2) \) gauge connection. The construction goes as follows: let me define a one-form \( A \) taking values in the Lie algebra \(^3\) of \( SU(2) \).

\[
A = i(\tau \Omega) \star d\mathbf{X} \cdot \mathbf{\sigma}, \tag{12}
\]

\(^3\) I take the adjoint representation of \( SU(2) \) to be spanned by the antihermitian matrices \( i\sigma^j \).
where the star stands for the two-dimensional Hodge dual with respect the induced metric, while the $\sigma_j$ are the standard Pauli matrices. If one works within the Polyakov approach, because the Hodge star for a one-form is blind to Weyl rescalings of the metric one could have also used the Polyakov metric.

It follows directly from the definition that $\star d \star A = 0$, which is tantamount to saying that $A$ is in the Lorenz gauge. It is a now a straightforward computation to check that the zero curvature condition

$$dA + A \wedge A = 0 $$

reproduces the constant mean curvature condition when written in terms of $X$. Therefore, there is a one-to-one relationship between flat $SU(2)$ connections in the Lorenz gauge and constant curvature surfaces.

One may use now the flatness of $A$ to parametrize it in the usual way using group variables: $A = h^{-1} dh$. But notice now that the Lorentz condition becomes in terms of $h$ nothing but the equations of motion of the principal chiral $SU(2)$ model coupled to 2-D gravity, i.e.

$$\star d \star h^{-1} dh = 0, $$

which follow from the action

$$\text{Tr} \int d^2x \sqrt{g} g^{\alpha \beta} h^{-1} \partial_\alpha h h^{-1} \partial_\beta h. $$

A little more of work shows that the equation of motion for the 2-D metric implies that $g_{ab}$ is conformally equivalent, through the map (12), to the induced one; thus proving the equivalence between the solution spaces of both theories.

The geometrical data necessary to encode the geometrical properties of constant curvature surfaces is supplied by the Gauss map of the surface. Although a thorough introduction to the subject can be found in a plethora of good textbooks, I will pass now to introduce the necessary concepts in a way that will show particularly useful for my purposes.

4 The generalized Gauss map

One of the main geometrical tools in the study of immersed surfaces in $n$-dimensional Euclidean space is provided by the generalized Gauss map. This map is most simply defined as the map assigning to any single point in the immersed surface $\Sigma$ its tangent plane, i.e. it is a map from $\Sigma$ into $G_{(2,n)}$ (the
grassmannian of two planes in $\mathbb{R}^n$). In the case the surface is immersed in $\mathbb{R}^3$ is easy to convince oneself that this map is equivalent to the classical Gauss map which associates to every point in $\Sigma$ its unit normal vector $\mathbf{v}$.

There are traditionally several ways to parametrize this map. For our purposes it will prove convenient to parametrize a tangent plane by a null complex vector $\mathbf{v}$ modulo the multiplication by a nonzero complex number. It is clear that

$$\mathbf{v}_0 = \text{Re}(\mathbf{v}) \quad \text{and} \quad \mathbf{v}_1 = \text{Im}(\mathbf{v}),$$

form an orthogonal frame with $|\mathbf{v}_0| = |\mathbf{v}_1|$ by virtue of the condition $\mathbf{v} \cdot \mathbf{v} = 0$. Notice also that multiplication of $\mathbf{v}$ by a complex number simply amounts to a rotation and dilatation of the frame thus corresponding to the same tangent plane.

The connection of $G_{(2,3)}$ with $\mathbb{CP}^1$ can be most elegantly made by using the two-to-one homomorphism between $SU(2) \cong Spin(3)$ and $SO(3)$. To each vector in $\mathbb{R}^3$ one can associate a $2 \times 2$ matrix in the algebra of $SU(2)$ as follows

$$\mathbf{y}^i \rightarrow Y^{AB} = y^i \sigma_i^{AB} = \begin{pmatrix} y^3 & y^1 - iy^2 \\ y^1 + iy^2 & -y^3 \end{pmatrix},$$

where the $\sigma_i$ are the standard Pauli matrices; our convention for them can be easily read from the above formula. From here it follows that

$$|\mathbf{y}|^2 = -\det Y = \frac{1}{2} \text{Tr} YY,$$

thus making explicit the above homomorphism between Lie algebras, with the equivalent of the Euclidean metric in $\mathbb{R}^3$ being provided by $\delta^{AB}$.

From the fact that $\mathbf{v} = \mathbf{v}_0 + i\mathbf{v}_1$ is a null complex vector is easy to convince oneself that it could be represented in terms of a complex two-spinor $\xi^A$ by

$$\mathbf{v} = i\xi^A (\sigma_2 \sigma)_{AB} \xi^B,$$

while $\bar{\mathbf{v}}$ stands for its complex conjugate. In order to obtain the above relation we have used in a crucial way that

$$\sigma_{AB} \sigma_{CD} = \delta_{AD} \delta_{BC} - \epsilon_{AC} \epsilon_{BD}.$$ 

This requires, of course, the choice of an orientation.
From this we have that

$$|v_a| = \frac{1}{2} vv = (\xi \bar{\xi})^2,$$  \hspace{1em} (21)

for $a = 0, 1$. Moreover, it also directly follows that

$$(\xi \bar{\xi}) n = \bar{\xi} \sigma \xi,$$  \hspace{1em} (22)

where $n$ is a real unit vector everywhere perpendicular to the plane determined by the $v$’s.

From the fact that $v$ is defined up to multiplication by an arbitrary nonzero complex number it follows that $\xi^A$ is also defined modulo multiplication by $a \in \mathbb{C}^\times$, thus being naturally identified with homogeneous coordinates in $\mathbb{CP}^1$. We can now make contact with standard parametrizations (which otherwise hide the Euclidean invariance of the target space) by choosing standard inhomogeneous coordinates. Explicitly by setting $\xi^0 = \omega$ and $\xi^1 = 1$ one obtains

$$n = \frac{1}{1 + \omega \bar{\omega}} (2\text{Re}(\omega), -2\text{Im}(\omega), \omega \bar{\omega} - 1).$$  \hspace{1em} (23)

The coordinate $\omega$ has otherwise a simple geometrical interpretation as the complex coordinate associated by stereographic projection with the Riemann sphere. One should of course be careful with the fact that these coordinates cannot cover the whole of the Riemann sphere and should work instead with the standard atlas. I will at any rate obviate here the details and leave the reader to fill the gaps [3].

The reason for the above exercise is to show how the constant curvature condition can be neatly written in terms of the Gauss map and its different parametrizations. Let me start by the classical Kemmotsu representation of a three-dimensional surfaces in terms of the mean curvature and the Gauss map.

In order that $\omega(z, \bar{z})$ describes the Gauss map of a surface some integrability conditions must be fulfilled. They are easily stated: if the $v_i$, with $i = 0, 1$, are to define an orthogonal frame to the surface there must exist a zweibein field such that

$$\partial_\alpha X = e^i_\alpha v_i.$$  \hspace{1em} (24)

Then the integrability condition reduces to

$$0 = \varepsilon^{\alpha\beta} \partial_\alpha \partial_\beta X = \varepsilon^{\alpha\beta} \partial_\beta (e^i_\alpha v_i).$$  \hspace{1em} (25)
It will now prove convenient to arrive at a simpler expression via a judicious use of the symmetries at hand. The fact that $\mathbf{v}$ is defined up to multiplication by a nonzero complex number is translated into local Weyl and Lorentz invariance of the zweibein. One may use these symmetries together with reparametrization invariance to write

$$\partial \mathbf{X} = e \mathbf{v},$$

(26)

or equivalently

$$\partial \mathbf{X} = \frac{1}{\sqrt{2} \sqrt{1 + \omega \bar{\omega}}} (\omega^2 - 1, i(1 + \omega^2), -2\omega),$$

(27)

where we have fixed the normalization of $e$ such that $\mathbf{v}$ is complex unitary, and $\partial$ stands for $\partial/\partial z$. From here it follows that $\partial \mathbf{X} \cdot \partial \mathbf{X} = \bar{\partial} \mathbf{X} \cdot \bar{\partial} \mathbf{X} = 0$, while $g_{\bar{z}z} = \partial \mathbf{X} \cdot \bar{\partial} \mathbf{X} = e \bar{e}$.

If one now takes the derivative with respect $\bar{z}$ in equation (26) and projects into the tangent component one obtains:

$$\bar{\partial} e + \bar{\eta} e = 0,$$

(28)

where $\eta$ is nothing but the spin connection in the conformal gauge, i.e. $\bar{\eta} = \bar{\mathbf{v}} \bar{\partial} \mathbf{v}$. The projection into the normal component simply yields an expression of $e$ in terms of $H$ and $\omega$. Indeed, from the definition of the mean curvature one obtains that

$$\bar{\partial} \partial \mathbf{X} \cdot \mathbf{n} = H e \bar{e}$$

(29)

which directly implies that

$$\bar{e} = \frac{\sqrt{2}}{H} \frac{\bar{\partial} \omega}{1 + \omega \bar{\omega}},$$

(30)

where we are assuming that $H$ is everywhere a nonvanishing function. Therefore one obtains that

$$\partial \mathbf{X} = \frac{\bar{\partial} \omega}{H} \frac{1}{(1 + \omega \bar{\omega})^2} (\omega^2 - 1, i(1 + \omega^2), -2\omega).$$

(31)

Now the integrability condition of Kemmotsu in terms of $H$ and $\omega$ follows from (28). A direct computation yields

$$H \left( \bar{\partial} \partial \omega - 2 \bar{\omega} \partial \omega \bar{\omega} \right) = \partial H \bar{\partial} \omega.$$

(32)
That this is the only integrability condition can be checked by a simple counting argument. In three dimensions one needs three real coordinate functions to define uniquely a surface, but in the conformal gauge one has two extra conditions coming from $\partial X \cdot \partial X = 0$, which leaves us with one degree of freedom, exactly the ones obtained from $H$ and $\omega$ subject to the integrability condition (32) above.

All of this allows us to introduce the Kemmotsu representation theorem:

**Theorem 1** Let $\Sigma$ be a simply-connected two-dimensional smooth manifold and $H : \Sigma \to \mathbb{R}$ be a nonzero and differentiable function. Let $\omega : \Sigma \to S^2$ be a smooth map from the surface into the Riemann sphere. If $\omega$ satisfies the integrability condition (32) for the above $H$, then $\omega$ is the Gauss map of some surface. More precisely, if we define the vector valued differential form

$$\theta = \text{Re} \left\{ \frac{1}{H(1 + \omega \bar{\omega})^2} (\omega^2 - 1, i(1 + \omega^2), -2\omega) dz \right\},$$

(33)

with $\bar{\partial}\omega \neq 0$ everywhere on $\Sigma$, then

$$X = \left( \int \theta_1, \int \theta_2, \int \theta_3 \right)$$

(34)

describes a regular surface such that its mean curvature is $H$ and its Gauss map is $\omega$.

The fact that $X$ is well defined follows from the closedness of $\theta$, i.e. it is oblivious to the integration path used in the definition, which is itself a direct consequence of the integrability condition, as a straightforward computation shows.

After this somehow long detour we may come back to our original problem. The equations of motion of our string action implied that the mean curvature was constant. In this case the integrability condition takes a particularly simple form

$$\bar{\partial}\partial\omega - \frac{\bar{\omega}\partial\omega\bar{\omega}}{1 + \omega\bar{\omega}} = 0.$$  

(35)

which is nothing but the equation of motion of the $\mathbb{CP}^1$ nonlinear sigma model in stereographic coordinates. Now through the Kemmotsu representation theorem we may obtain a regular surface which is solution of the string equation by setting $H = \tau\Omega/2$ and choosing a nonholomorphic solution of the $\mathbb{CP}^1$ model. The reason to exclude the holomorphic solutions, which correspond to
instantons of the associated nonlinear sigma model, come from the fact that those surfaces are minimal and therefore have vanishing mean curvature.

It is also of independent interest to notice that by a slight modification of standard procedures [7] one can show that the integrability equation takes the form of the affine $SL(2)$ Toda field equations in the variables

$$e^{\phi_1} = \frac{\partial \omega \bar{\partial} \omega}{(1 + \omega \bar{\omega})^2}, \quad e^{\phi_2} = \frac{\partial \omega \bar{\partial} \omega}{(1 + \omega \bar{\omega})^2}. \quad (36)$$

The Toda variables have now a direct geometrical interpretation [5]. The $\phi_1$ field corresponds to the conformal factor of the induced metric, as can be easily seen by computing $\partial X \cdot \bar{\partial} X$. One readily obtains

$$g_{zz} = e^\phi_2 = \frac{2}{H^2} e^{\phi_1}, \quad (37)$$

which is up to the constant $2/H^2$ the conformal factor. The geometrical interpretation for $\phi_2$ is slightly more involved, but simple as well. The required geometrical data is provided by the so called skew curvature; which is nothing but the $K_{zz}$ component of the extrinsic curvature. If one computes the norm squared of the skew curvature one obtains

$$K_{zz} K_{\bar{z}\bar{z}} = \frac{4}{H^2} \frac{\partial \omega \bar{\partial} \omega}{(1 + \omega \bar{\omega})^2} \frac{\partial \omega \bar{\partial} \omega}{(1 + \omega \bar{\omega})^2} = 2g_{zz} e^{\phi_2}, \quad (38)$$

from where directly follows an expression of $\phi_2$ exclusively in terms of the geometrical data associated with the constant curvature surface.

5 The covariant spinorial description

From the previous results it may seem that the relationship between the string model and the nonlinear $\mathbb{CP}^1$ sigma model requires a non Euclidean covariant choice. Therefore it will be interesting to test how far one can arrive keeping explicitly Euclidean covariance of the target. As I would like to show now this is indeed possible and the final result will be provided by a spinorial Gross-Neveu model coupled to 2-D gravity.

Let me first show how to recover the covariant description of the $\mathbb{CP}^1$ model from the structural equations of the surface. If one computes the covariant Laplacian, acting on $n$ one obtains

$$\Box n = g^{\alpha \beta} \nabla_\alpha \partial_\beta n = g^{\alpha \rho} g^{\beta \alpha} (\nabla_\beta K_{\alpha \rho}) \partial_\rho X + K_{\alpha \beta} K^{\alpha \beta} n. \quad (39)$$
From the Codazzi-Mainardi integrability condition follows that
\[ g^{\beta\alpha} \nabla_{\beta} K_{\alpha\pi} - 2 \nabla_{\pi} H = 0, \]  
(40)
thus in the case of constant curvature surfaces one gets that
\[ g^{\beta\alpha} \nabla_{\beta} K_{\alpha\pi} = 0, \]  
(41)
from where the equation (39) reduces to
\[ \Box \mathbf{n} = K_{\alpha\beta} K^{\alpha\beta} \mathbf{n}. \]  
(42)
A moment’s thought reveals that \( K_{\alpha\beta} K^{\alpha\beta} = \mathbf{n} \cdot \Box \mathbf{n} \), and one recovers the
standard field equation for the \( O(3) \) nonlinear sigma model, which is well
known to be equivalent to \( \mathbb{C}P^1 \) through the map (22). Notice that to deduce
this result one only need to use the fact that \( H \) is constant, therefore it includes
the case of minimal surfaces for which \( H = 0 \). It is a simple exercise to check
that consistency in that case with the structural equations imply the instanton
condition [9].

Yet a different, and more interesting, formulation is obtained if one chooses
to work in a Euclidean covariant manner within the spinor formulation. Let
me come back to the covariant expression of the tangent frame in terms of
spinors. The integrability condition may now be derived from
\[ \partial X = i e \xi \sigma_2 \sigma \xi \]  
(43)
by setting the imaginary part of \( \partial \bar{\partial} X \) to zero, i.e.
\[ \partial \bar{J} + \bar{\partial} J = 0, \]  
(44)
where \( J = e \xi \sigma_2 \sigma \xi \). Therefore the integrability condition can be simply stated
as the conservation of the \( J \) current.

All of this suggest to look for Lagrangian densities where the current de-
defined above is a conserved quantity. Let me consider the Gross-Neveu type
Lagrangian coupled to 2-D gravity
\[ \mathcal{L} = i e \xi \sigma_2 \bar{\partial} \xi + \text{c.c.} + \frac{1}{2} \beta J \bar{J}. \]  
(45)
Invariance under global Lorentz transformations on the target,
\[ \delta \rho \xi = i \rho \sigma \xi, \]  
(46)
imply the conservation of $\mathbf{J}$ as desired. The associated Euler-Lagrange equation for $\xi$ is given by:

$$\bar{\partial}\xi - \frac{1}{2}\bar{\eta}\xi = i\beta\bar{e}\sigma_2\xi(\xi\bar{\xi}),$$  \hspace{1cm} (47)$$

where $\eta$ is the spin connection associated with $e$. The equation of motion for the zweibein yields

$$\bar{e} = -\frac{i}{2\beta} \frac{\xi\sigma_2\bar{\partial}\xi}{(\xi\bar{\xi})^2}.$$  \hspace{1cm} (48)$$

Notice that the first hint that we are on the right track comes from this equation. If one goes to the gauge where $\xi^0 = \omega$ and $\xi^1 = 1$ the above equation reduces to

$$\bar{e} = \frac{1}{\beta} \frac{\bar{\partial}\omega}{(1 + \omega\bar{\omega})^2}$$  \hspace{1cm} (49)$$

which turns out to be the expression of the induced zweibein in terms of the Gauss map for constant curvature surfaces if we set $H = \beta$. Notice that the extra factor of $\sqrt{2}(1 + \omega\bar{\omega})$ comes from the different parametrization of $e$; if we use (43) to define $g_{z\bar{z}}$ one gets

$$g_{z\bar{z}} = 2e\bar{e}(\xi\bar{\xi})^2,$$  \hspace{1cm} (50)$$

and not $ee$ as before. As I will pass to show this turns to be more than a coincidence.

From the equation of motion one gets that

$$\bar{\partial}\mathbf{J} = (\bar{\partial}e + \bar{\eta}e)\xi\sigma_2\sigma\xi - 2i\beta\bar{e}e(\xi\bar{\xi})\xi\sigma\xi.$$  \hspace{1cm} (51)$$

The first term is identically zero because of the definition of $\bar{\eta}$. If one now plugs back the expression of $\partial\mathbf{X}$ in terms of $\mathbf{J}$ one recovers the constant curvature condition if one sets

$$\beta = \frac{\tau\Omega}{2}.$$  \hspace{1cm} (52)$$

It is clear from all of this that by setting $\beta$ equal to zero one recovers the usual string equation. In particular in the gauge $\xi^0 = \omega$ and $\xi^1 = 1$ one obtains that $\omega$ and $e$ are holomorphic functions. From this one may recover.
the Weierstrass-Enneper representation of minimal surfaces as follows. From equation (43) one gets that

$$X = \text{Re} \int dz \left( \omega^2 - 1, i(1 + \omega^2), -2\omega \right),$$

which is well defined by the holomorphicity of $\omega$ and the zweibein. If one now implements the conformal reparametrization $z \rightarrow \omega^{-1}(z)$ one may rewrite the equation above as

$$X = \text{Re} \int dz \left( z^2 - 1, i(1 + z^2), -2z \right) \zeta(z),$$

with $\zeta(z) = e/\partial \omega$ an holomorphic function, thus reproducing the celebrated Weierstrass-Enneper formula.

6 Some final comments

I believe that this trip through the geometry associated with the string model under study has revealed its intrinsic interest. Of course, the final test should be provided by the quantum properties of the string model. Nevertheless, it is to be expected that its interconnections with classical integrable systems should pave the way to quantization in the covariant phase space approach. I hope to come back to this subject in a future publication.

I would not like to finish without commenting on certain results in the field of three-dimensional rigid strings which seem to establish some links with the results obtained here. Although the Polyakov rigid string does not have as general solution constant mean curvature surfaces, it was noted by Viswanathan and Parthasarathy [4] that its action reduces to the one of the $\mathbb{CP}^1$ model for constant curvature surfaces. The equivalence of both approaches only being complete for the anti-instanton solutions of the $\mathbb{CP}^1$ model. How all of this fits into our case is something which for the time being escapes my understanding, but which I believe is worth of further study.

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