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On Representation of Riesz–space–valued Functions by Fourier Series on Multiplicative Systems

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Abstract. The result that was established in [1] for Walsh series with coefficients belonging to a Riesz–space is generalized for multiplicative systems. Conditions for recovering the coefficients of a convergent series on the multiplicative systems from its sum by Fourier formulae are considered. For these formulae the Henstock–Kurzweil integral for Riesz–space–valued functions defined with respect to the special basis \(B_P\) will be used.

In this work the result that was established in [1] for Walsh series [2] with coefficients belonging to a Riesz–space [3] is generalized for multiplicative systems. These systems are very popular for solving applied tasks nowadays [4],[5]. In particular, we consider conditions, for which such series are some generalized Fourier series, where in order to write down Fourier’s formulae we use the Henstock–Kurzweil type integral [1, 6]. This integral is defined with respect to the basis \(B_P\) generated by intervals \(\{\delta_r^{(k)}\}\) of the special form [2]:

\[
\delta_r^{(k)} = \left[ \frac{r}{m_k}, \frac{r + 1}{m_k} \right], \quad k = 0, 1, 2, \ldots, \quad 0 \leq r \leq m_k - 1.
\]

We recall the definition of the Henstock–Kurzweil type integral with respect to a basis for Riesz–space–valued functions following [1].

Let \([a, b]\) be a fixed interval of \(\mathbb{R}\) and \(I\) be the family of all subintervals of \([a, b]\). A basis on \([a, b]\) is any subset \(B\) of \(I \times [a, b]\) such that \((I, x) \in B\) implies \(x \in I\).

An interval \(I\) for a given basis \(B\) is called a \(B\)–interval if \((I, x) \in B\), for some \(x \in I\). Furthermore, assume that \([a, b]\) is a \(B\)–interval. For a set \(E\) such that \(\emptyset \neq E \subset [a, b]\), put

\[
B[E] = \{(I, x) \in B : x \in E\}.
\]

For \(\emptyset \neq E \subset [a, b]\) by \(\Delta_E\) denote the directed set of all positive real valued functions defined on \(E\) and endowed with the natural ordering: for given two functions \(\delta_1\) and \(\delta_2\), \(\delta_1 \leq \delta_2\) iff \(\delta_1(x) \leq \delta_2(x)\) for all \(x \in E\).

A function \(\delta \in \Delta_E\) is often said to be a gage on \(E\).

For a given gage \(\delta \in \Delta\), put

\[
B_\delta = \{(I, x) \in B : I \subset (x - \delta(x), x + \delta(x))\}.
\]
Let \( \emptyset \neq E \subset [a, b] \). A finite subset \( \Pi \) of \( \mathcal{B}[E] \) is called a \( \mathcal{B} \)-decomposition on \( E \) if for every distinct elements \( (I', x') \) and \( (I'', x'') \) of \( \Pi \), the corresponding intervals \( I' \) and \( I'' \) are nonoverlapping, that is, their interiors are pairwise disjoint. If else \( \bigcap_{(I, x) \in \Pi} I = [a, b] \), for \( \Pi \subset \mathcal{B} \), then we say that \( \Pi \) is a \( \mathcal{B} \)-partition of \([a, b]\).

Note that \( \mathcal{B}_\delta \) is also a basis on \([a, b]\), so it can define the set \( \mathcal{B}_\delta[E] \). Thus a \( \mathcal{B}_\delta \)-partition can be well defined.

Let \( \mathcal{B} \) be a fixed basis on \([a, b]\). A function \( f : [a, b] \to R \) is Henstock–Kurzweil integrable on a \( \mathcal{B} \)-interval \( E \subset [a, b] \) with respect to \( \mathcal{B} \) if there exists an element \( Y \in R \) such that

\[
\inf_{\delta \in \Delta_{[a, b]}} \left( \sup \left\{ \left| \sum_{(I, x) \in \Pi} f(x)|I| - Y \right| : \Pi \text{ is a } \mathcal{B}_\delta \text{-partition of } E \right\} \right) = 0.
\]

This element \( Y \) is called \( \mathcal{H}_\delta \)-integral and is denoted by \( (\mathcal{H}_\delta \int_E f = Y) \).

Now we recall the definition of the multiplicative systems using the \( \mathbb{P} \)-digit expansion for the real numbers; by definition, put

\[
m_0 = 1, \quad m_j = \prod_{s=1}^{j} p_s,
\]

where \( p_s \) is a member of a sequence of the set of natural numbers

\[
\mathbb{P} = \{p_1, p_2, \ldots, p_j, \ldots\}, \quad p_j \geq 2, \quad j \geq 1,
\]

and any integer \( n \geq 0 \) has the form

\[
n = \sum_{i=1}^{k} \alpha_j m_{j-1}, \quad 0 \leq \alpha_j \leq p_j, \quad j = 1, 2, \ldots, k.
\]

Take the point \( x \in [0, 1) \) and consider the series \( x = \sum_{j=1}^{\infty} \frac{x_j}{m_j} \), where \( 0 \leq x_j \leq p_{j-1}, \quad j \geq 1 \).

Here for \( \mathbb{P} \)-digit rational \( x \) we use only the finite expansion. The point \( x \) is called the \( \mathbb{P} \)-digit rational point if there exists a finite representation \( x = \sum_{j=1}^{K} \frac{x_j}{m_j} \), \( K \in \mathbb{N} \). Using the introduced notations, by definition, put

\[
\chi_n(x) = \exp \left( 2\pi i \sum_{j=1}^{k} \frac{\alpha_j x_j}{p_j} \right).
\]

**Proposition 1.** Under the condition of \( n < m_k \), the functions \( \chi_n(x) \) are constant on the intervals

\[
delta_n^{(k)} = \left[ \frac{r}{m_k}, \frac{r+1}{m_k} \right), \quad k = 0, 1, 2, \ldots, \quad 0 \leq r \leq m_k - 1.
\]

For our purpose, with pointwise \( (o) \)-converges (order converges) we use \( (u) \)-converges on a set (see [1]), according to the definition:

Let \( \Lambda \) be any nonempty set, \( R \) be Dedekind complete Riesz space and \( D = N^\Lambda \). The sequence of \( R \)-valued functions \( (S_n : \Lambda \to R)_n \) \( (u) \)-converges to the function \( S : \Lambda \to R \) (with respect to order convergence) if there exists an \( (o) \)-net \( (a_\nu)_{\nu \in D} \) such that for any \( \nu \in D \) the following condition holds:

\[
\sup \{|S_n(x) - S(x)| : x \in \Lambda, \quad n \leq \nu(x)\} \leq a_\nu.
\]
Proposition 2. It is evident that \((u)\)-convergence of a sequence on a set implies \((a)\)-convergence of this sequence at each point of the set.

Let \(S_n = \sum_{j=0}^{n-1} a_j \chi_j\) be the partial sum of a series in the multiplicative system \(\{\chi_j\}_j\), where coefficients \(a_j\) belong to a Riesz-space \(R\).

Hereafter in this paper, by \(\mathcal{P}\) we denote the basis \(\mathcal{B}_\mathcal{P}\) consisting of all pairs \((I, x)\), where \(x \in I = \left(\frac{i}{m_k} \frac{+1}{m_k}\right)\) and \(k \in N \cup \{0\}\), and \(i = 0, 1, \ldots, m_{k-1}\).

The Fourier series of a \(H_{\mathcal{P}}\)-integrable function \(f\) is a series \(S_n = \sum_{j=0}^{n-1} f(n) \chi_j(x)\), where \(\hat{f}(n) = \int_0^1 f(t) \overline{\chi_n(t)} \, dt\).

Since the system \(\{\chi_n\}_n\) is an orthonormal system (see [2]), it follows that \(\hat{S}_{m_k}(n) = \int_0^1 \sum_{j=0}^{m_k-1} a_j \chi_j(t) \chi_n(t) \, dt = \sum_{j=0}^{m_k-1} a_j \int_0^1 \chi_j(t) \overline{\chi_n(t)} \, dt = \sum_{j=0}^{m_k-1} a_j \int_{\chi_j=\chi_n} dt = a_n\). Thus we have

Proposition 3. For a number \(n, n < m_k\), the coefficients \(a_j\) of a series \(S_n(x) = \sum_{j=0}^{n-1} a_j \chi_j(x)\) on the system \(\{\chi_n\}_n\) are Fourier coefficients of the partial sums \(S_{m_k}(x)\).

Proof. Denoting by the \(\chi_m\) value of the function \(\chi_n\) on \(\delta_k^{(n)}\), we have

\[a_n = \int_0^1 S_{m_k}(t) \chi_n(t) \, dt = \sum_{j=0}^{m_k-1} \int S_{m_k}(t) \chi_n(t) \, dt = \sum_{j=0}^{m_k-1} \chi_n \int S_{m_k}(t) \, dt = \]

\[\sum_{j=0}^{m_k-1} \chi_n \int \psi(\delta_k^{(j)}) = \sum_{j=0}^{m_k-1} \chi_n (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \]

\[\sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \]

Since for any \(n\) there exists \(k\) such that \(n < m_k\), it follows that the first equality in the chain of the formulae above is true and the proposition is proved.

Our main result is as follows.

Theorem 1. If \(R\) is a regular Riesz-space and the series

\[\sum_{j} a_j \chi_j, \quad a_j \in R \quad (*)\]

\((u)\)-converges to a function \(f\) on a set \([0, 1] \setminus E\), where \(E\) is a countable subset of the interval \([0, 1]\), then \(f\) is \(H_{\mathcal{B}}\)-integrable on \([0, 1]\), and the series \((*)\) is the Fourier series of \(f\) in the sense of the \(H_{\mathcal{B}}\)–integral.

For proving this theorem the function \(\psi(\delta_i^{(k)}) = \int_0^1 \delta_i^{(k)} S_{m_k}\) of the intervals \(\delta_i^{(k)}\) is used, where \(S_{m_k}\) is the partial sum of the series \((*)\). This function is additive with respect to the measure algebra, generated by the intervals \(\{\delta_i^{(k)}\}\). This follows from the equality

\[\psi(\delta_i^{(k)}) = \psi(\delta_{p_k+1}^{(k+1)}) + \psi(\delta_{p_k+1}^{(k+1)}) + \cdots + \psi(\delta_{p_k+1}^{(k+1)}) = \sum_{s=p_k+1}^{(r+1)p_{k+1}-1} \psi(\delta_{s}^{(k+1)}), \quad (**),\]
\[a_n = \int_0^1 S_{m_k}(t) \chi_n(t) \, dt = \sum_{j=0}^{m_k-1} \int S_{m_k}(t) \chi_n(t) \, dt = \sum_{j=0}^{m_k-1} \chi_n \int S_{m_k}(t) \, dt = \]
\[\sum_{j=0}^{m_k-1} \chi_n \int \psi(\delta_k^{(j)}) = \sum_{j=0}^{m_k-1} \chi_n (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \]
\[\sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \]
\[\sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \]
\[\sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \]
\[\sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \]
\[\sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \]
\[\sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \]
\[\sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \]
\[\sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \sum_{j=0}^{m_k-1} (H_{\mathcal{P}}) \int f(t) \overline{\chi_n(t)} \, dt = \]
where \( \delta'^{(k)} = \bigcup_{s=r_{p_k+1}}^{(r+1)p_k+1} \delta^{(k+1)}_s \).

Theorem 1 follows as a result of applying the statement of Theorem 2 (see [1]) to the function \( \psi \). First, we recall some definitions.

The function \( \tau \) is called \((o)-continuous\) at a point \( x_0 \in [a,b] \) with respect to the basis \( \mathcal{B} \) if

\[
\inf_{\delta} \sup \{ |\tau(I)| : (I, x_0) \in \mathcal{B}_\delta([x_0]) \} = 0.
\]

The function \( \tau \) is \((o)-continuous\) on \( E \) with respect to the basis \( \mathcal{B} \) if it is \((o)-continuous\) at every point \( x_0 \in E \).

The function \( \tau \) is \((u)-differentiable\) on \( E \) with respect to the basis \( \mathcal{B} \) if there exists a function \( g : E \to R \) such that

\[
\inf_{\delta} \left[ \sup \left\{ \frac{|\tau(I)|}{|I|} - g(x) : (I, x) \in \mathcal{B}_\delta[E] \right\} \right] = 0.
\]

This function \( g \) is called the \((u)-derivative\) with respect to the basis \( \mathcal{B} \) of \( \tau \) on \( E \).

**Theorem 2.** Let \( R \) be a regular Riesz–space, \( \mathcal{B} \) a fixed basis, \( f : [a,b] \to R \) and let \( \tau \) be the \( R \)-value \( \mathcal{B} \)-interval function, such that for some countable set \( Q \subseteq [a,b] \) the function \( f \) is the \((u)-derivative\) of \( \tau \) with respect to the basis \( \mathcal{B} \) on the set \([a,b] \setminus Q\) and \( \tau \) is \((o)-continuous\) with respect to the basis \( \mathcal{B} \) on \( Q \). Then \( f \) is \( H_\mathcal{B} \)-integrable over \([a,b]\), and

\[
(H_\mathcal{B}) \int_a^b f = \tau([a,b]).
\]

From analysis of the proof of Theorem 4.2 from the work [1], it follows that to prove this theorem, in our case it is necessary and sufficient to have \( \tau([0,1]) = \sum_{i=1}^{q} \tau(I_i) \) for any \( \mathcal{B}_\delta \)-partition \( \Pi \) on \([0,1]\), where \( \Pi = \{(I_i, x_i), \ i = 1, \ldots, q\} \). According our notations we here have \( \tau = \psi \), \( \mathcal{B} = \mathcal{P} \).

We claim that \( \tau([0,1]) = \sum_{i=1}^{q} \tau(I_i) \) for any partition \( \Pi \) on \([0,1]\). Indeed, this follows from the fact that a \( \mathcal{P}_\delta \)-partition \( \Pi \) on \([0,1]\) consists of intervals \( \delta' \) and from the equality (**), where the function \( \psi \) is replaced by the function \( \tau \).

Thus, Theorem 2 can be proved similarly to Theorem 4.2 from [1] and Theorem 1 can be proved similarly to Theorem 5.4, using Proposition 3.

It is important to note that the elements of a Riesz–space can multiply by real numbers only, to preserve the axioms of an order.

In our case, the functions \( \chi_n(x) \) can be complex–valued. Thus, we must give some definitions.

We define the product of two numbers \( z \in \mathbb{C} \) and \( r \in R \) as

\[
 r \cdot z = r(a + ib) = ra + i rb,
\]

where \( r \) is a element of the Riesz–space and \( z = a + ib \), \( a, b \in \mathbb{R} \). Here, addition is applied separately to each component as it is usually done for complex numbers.

By definition, the numbers \( r_1 \cdot z_1 \) and \( r_1 \cdot z_2 \) are equal if \( Re r_1 \cdot z_1 = Re r_2 \cdot z_2 \) and \( Im r_1 \cdot z_1 = Im r_2 \cdot z_2 \), where \( z_1, z_2 \in \mathbb{C} \) and \( r_1, r_2 \in R \).

If \( f(x) = f_1(x) + i f_2(x) \), where \( f_1(x) \) and \( f_2(x) \) are Riesz–space–valued functions, \( x \in R \), then \( f_1(x) = Re f(x) \) and \( f_2(x) = Im f(x) \) and by definition, put

\[
\int f(x) dx = \int f_1(x) dx + \int f_2(x) dx.
\]

**Hypothesis.** If a series \( \sum_{j=1}^{\infty} a_j \chi_j \), where \( a_j \in R \), \((u)-converge\) to a Riesz–space–valued function (not complex–valued), then \( S_{m_k}(x) \in R \), \( k = 0, 1, \ldots \).

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