Classification of preordered spaces in terms of monotones - Filling in the gaps

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Abstract

Following the recent introduction of new classes of monotones, like injective monotones or strict monotone multi-utilities, we present the classification of preordered spaces in terms of both the existence and cardinality of real-valued monotones and the cardinality of the quotient space. In particular, we take advantage of a characterization of real-valued monotones in terms of separating families of increasing sets in order to obtain a more complete classification consisting of classes that are strictly different from each other.

1 Introduction

The question of how well a preorder relation can be captured through real-valued functions is an ongoing research topic since the introduction of utility functions in the early days of mathematical economics. The key observation is that sometimes preferences can not only be measured locally to decide between two elements, but there might be a global real-valued preference function that fully captures the corresponding order relation. That is, in certain situations, one can not only choose a preferable item between any two items in a given set of options, but one can find a single function, or a family of functions, defined on the decision space whose function values quantify the preference relation, so that one can compare function values to decide about the order relation of the corresponding arguments. Since the existence of such functions only depends on the properties of the corresponding preorder, this idea can naturally be applied in many domains of science. In particular, instead of considering preference relations and utility functions on decision spaces, many systems of interest can be thought of as sets of possible states endowed with an order relation encapsulating the intrinsic tendency of the system to transition from one state to another. The fields where these ideas are relevant include thermodynamics [26, 18], general relativity [5, 29], quantum physics [30, 9] and economics [12, 31], among others.

The basic property of these real-valued functions \( f \) is that they have to be monotones with respect to the corresponding preorder \( \preceq \), that is, \( x \preceq y \) implies \( f(x) \leq f(y) \). There are mainly three types of monotones that appear in this context: strict monotones [2, 32, 34], injective monotones [20], and utility
functions [12, 14]. In particular, these different types of monotones are used to classify preordered spaces mostly in two different ways: either by whether a given type of monotone exists, or, by whether there exists a family of such monotones, known as a multi-utility, that characterizes the preorder completely [17, 1, 2, 20]. Even though the cardinality of such representing families plays an important role, so far mostly the two cases of countable multi-utilies and multi-utilies consisting of a single element, that is, utilities, have been considered.

Moreover, several connections between both types of classifications have been pointed out in the literature [2, 20, 1, 6], but certain gaps in these connections have prevented the presentation of a general classification of preordered spaces through real-valued monotones. One of the aims of this contribution is to reduce this gap, achieving, thus, a more complete classification (see Figure 1). In particular, we take advantage of a characterization of real-valued monotones in terms of families of increasing sets [1, 20] that allows to distinguish more classes of preorderd spaces than before, both in terms of the cardinality of the multi-utilies and the cardinality of the quotient space of the preorder. Importantly, by providing the corresponding counter examples, we show that certain classes of preorded spaces are in fact strictly contained in each other, which, to our knowledge, was not known before.

2 Classification of preorders through real-valued monotones

A preorder \( \preceq \) on a set \( X \) is a reflexive (\( x \preceq x \forall x \in X \)) and transitive (\( x \preceq y \text{ and } y \preceq z \implies x \preceq z \forall x, y, z \in X \)) binary relation. A tuple \((X, \preceq)\) is called a preordered space and \( X \) the ground set. An antisymmetric (\( x \preceq y \text{ and } y \preceq x \implies x = y \forall x, y \in X \)) preorder \( \preceq \) is called a partial order. The relation \( x \sim y \), defined by \( x \preceq y \text{ and } y \preceq x \), forms an equivalence relation on \( X \), that is, it fulfills the reflexive, transitive and symmetric (\( x \sim y \text{ if and only if } y \sim x \forall x, y \in X \)) properties. Notice, a preorder \( \preceq \) is a partial order on the quotient set \( X/\sim = \{[x] \mid x \in X\} \), consisting of all equivalence classes \([x] = \{y \in X \mid y \sim x\}\).

In case \( x \preceq y \text{ and } \neg(x \sim y) \) for some \( x, y \in X \) we say \( y \) is strictly preferred to \( x \), denoted by \( x \prec y \). If \( \neg(x \preceq y) \) and \( \neg(y \preceq x) \), we say \( x \text{ and } y \) are incomparable, denoted by \( x \bowtie y \). Whenever there are no incomparable elements a preordered space is called total. By the Szpilrajn extension theorem [35, 22], every partial order can be extended to a total order, that is, to a partial order that is total. Notice Szpilrajn extension theorem is a consequence of the axiom of choice, which we assume throughout this work. Equivalently, we assume \( I \times I \) and \( I \) are equinumerous for any infinite set \( I \) and, thus, both \( I \times \mathbb{N} \) and \( I \cup I \) are also equinumerous to \( I \).

In order to numerically characterize the relations established in a preordered space, one or several real-valued functions may be used. This results in a classification of preorders according to how well their information can be captured using these functions. We introduce now several classes that have been previ-
Figure 1: Classification of preordered spaces according to the existence of various real-valued monotones. A distinction between our contributions here and previously known results can be found in the discussion (Section 3). Moreover, our contributions can be visualized in Figure 7.

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A real-valued function $f : X \to \mathbb{R}$ is called a *monotone* if $x \preceq y$ implies $f(x) \leq f(y)$ [17]. If the converse is also true, then $f$ is called a *utility function* [12]. Furthermore, if $f$ is a monotone and $x < y$ implies $f(x) < f(y)$, then $f$ is called a *strict monotone* or a *Richter-Peleg function* [2]. Similarly, a monotone $f$ is called an *injective monotone* if $f(x) = f(y)$ implies $x \sim y$, that is, if $f$ is injective considered as a function on the quotient set $X/\sim$ [20]. Whenever a single function is insufficient to capture all the information in a preorder, for example when it is non-total (see [10, Theorem 1.4.8] for the total case), a family of functions may be used instead. A family $V$ of real-valued

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functions \( v : X \to \mathbb{R} \) is called a multi-utility (representation) of \( \preceq \) if
\[
x \preceq y \iff v(x) \leq v(y) \quad \forall v \in V.
\]
Whenever a multi-utility consists of strict monotones it is called a strict monotone (or Richter-Peleg [2]) multi-utility (representation) of \( \preceq \). Analogously, if the multi-utility consists of injective monotones, we call it an injective monotone multi-utility (representation) of \( \preceq \). Notice the cardinality plays a key role in the classification when we consider multi-utilities.

An example of a preorder with a finite multi-utility is the uncertainty preorder \( \preceq_U \), which is defined on the space \( \mathcal{P}_\Omega \) of probability distributions on a finite set \( \Omega \) by
\[
p \preceq_U q \iff u_i(p) \leq u_i(q) \quad \forall i \in \{1, \ldots, |\Omega| - 1\},
\]
where \( u_i(p) := -\sum_{n=1}^{i} p_n^i \) and \( p_i^i \) denotes the decreasing rearrangement of \( p \) (same components as \( p \) but ordered decreasingly). Notice, \( \preceq_U \) is known in mathematics, economics, and quantum physics as majorization [21, 27, 3, 9].

Although several connections between the existence of these real-valued monotones are known [17, 2, 20], we further clarify the relation between them throughout this section. Mainly, using a characterization of these classes in terms of families of increasing sets which separate the elements in a preordered space [24, 1, 8, 20], we introduce several counterexamples which allow us to distinguish the scope of the different classes.

### 2.1 Characterization of real-valued monotones by families of increasing sets

A subset \( A \subseteq X \) is called increasing, if for all \( x \in A \), \( x \preceq y \) implies that \( y \in A \) [28]. We say a family \( (A_i)_{i \in I} \) of subsets \( A_i \subseteq X \) separates \( x \) from \( y \), if there exists \( i \in I \) with \( x \notin A_i \) and \( y \in A_i \). Families of increasing sets have been used to characterize the existence of several classes of preorders in terms of real-valued representations in the literature. We state these results in Lemma 1 without proof.

**Lemma 1.** Let \((X, \preceq)\) be a preordered space.

(i) For any infinite set \( I \), there exists a multi-utility with the cardinality of \( I \) if and only if there exists a family of increasing subsets \((A_i)_{i \in I}\) that \( \forall x, y \in X \) with \( x \prec y \) separates \( x \) from \( y \), and \( \forall x, y \in X \) with \( x \preceq y \) separates both \( x \) from \( y \) and \( y \) from \( x \).

(ii) There exists a strict monotone if and only if there exists a countable family of increasing subsets that \( \forall x, y \in X \) with \( x \prec y \) separates \( x \) from \( y \).

(iii) There exists an injective monotone if and only if there exists a countable family of increasing subsets that \( \forall x, y \in X \) with \( x \prec y \) separates \( x \) from \( y \) and \( \forall x, y \in X \) with \( x \preceq y \) separates either \( x \) from \( y \) or \( y \) from \( x \).
The proof of (i) can be found in [8, 1] and that of (ii) and (iii) in [20].

The characterizations in Lemma 1 can be useful to distinguish certain classes of preorders in terms of real-valued monotones, as we showed in [20, Proposition 8], where we used them to build a preorder where injective monotones exist and countable multi-utilities do not. Note that (i) is not true for finite sets $I$, because there are preordered spaces that have a finite multi-utility but do not have a finite separating family of increasing subsets, for example majorization.

The statements in Lemma 1 can be complemented with a characterization of the existence of strict monotone multi-utilities with the cardinality of an infinite set $I$, which we include in the following proposition.

**Proposition 1.** If $(X, \preceq)$ is a preordered space and $I$ is a set of infinite cardinality, then the following are equal.

(i) There exists a strict monotone multi-utility with the cardinality of $I$.

(ii) There exist a strict monotone and a multi-utility with the cardinality of $I$.

(iii) There exists a family of increasing sets $(A_i)_{i \in I}$ which separates $x$ from $y$ if $x \triangledown y$ and a countable set $I' \subseteq I$ such that $(A_i)_{i \in I'}$ separates $x$ from $y$ if $x \prec y$.

**Proof.** Clearly, (i) implies (ii) by definition. To show (ii) implies (iii), notice, given there is a multi-utility with the cardinality of $I$, we can follow the proof of Lemma 1 (i) in [1] to show there exists a family of increasing sets with the cardinality of $I \times \mathbb{N}$, which is equinumerous to $I$, which separates $x$ from $y$ whenever $x \triangledown y$. We can similarly follow the proof of Lemma 1 (ii) in [20] to get, given there exists a strict monotone, there exists a countable family of increasing sets which separates $y$ from $x$ whenever $x \prec y$. Since $I \cup \mathbb{N}$ is equinumerous to $I$, we get the desired result. In order to show (iii) implies (i), we can again follow both [1] and [20, Proposition 7]. From the first one, we can construct a multi-utility $(u_i)_{i \in I}$ and, from the second one, we can construct a strict monotone $v$. Finally, we consider, as in [2, Theorem 3.1], the family of monotones $(v_{i,n})_{i \in I, n \in \mathbb{N}}$ where $v_{i,n} := u_i + \alpha_n v$, where $(\alpha_n)_{n \in \mathbb{N}}$ is a numeration of the rational numbers which are greater than zero. This family can be shown to be a strict monotone multi-utility and has the cardinality of $I \times \mathbb{N}$, which is the same as that of $I$.

Notice, in case the set $I$ is finite, the relation between (i) and (ii) is addressed in Proposition 10.

### 2.2 Improving the classification of preorders

Let us begin with the relation between preorders which have strict monotones and those which have injective monotones. Clearly, an injective monotone is also a strict monotone, since $x \prec y$ and $f(x) = f(y)$ contradicts injectivity. There are, however, preordered spaces with strict monotones and without injective monotones, as was shown in [20, Proposition 1]. The argument there is purely in
terms of cardinality, since, whenever injective monotones exist, we have \(|X/\sim| \leq \mathfrak{c}\) with \(\mathfrak{c}\) the cardinality of the continuum, but there are preordered spaces with strict monotones and \(|X/\sim| = |P(\mathbb{R})|\). We can, however, improve upon this by showing there are preordered spaces where \(X/\sim\) has the cardinality of the continuum and strict monotones exist while injective monotones do not. We include such a preordered space in Proposition 2.

**Proposition 2.** There are preordered spaces \((X, \preceq)\) where \(X/\sim\) has the cardinality of the continuum \(\mathfrak{c}\) and strict monotones exist while injective monotones do not.

**Proof.** Consider \(X := [0,1] \cup [2,3]\) equipped with \(\preceq\) where

\[
x \preceq y \iff \begin{cases} x, y \in [0,1] \text{ and } x \leq y, \\ x, y \in [2,3] \text{ and } x \leq y, \\ x \in [0,1], \ y \in [2,3] \text{ and } x + 2 < y, \\ x \in [2,3], \ y \in [0,1] \text{ and } x - 2 < y.
\]

\(\forall x, y \in X\) (see Figure 2 for a representation of \(\preceq\)). Notice \((X, \preceq)\) is a preordered space and \(v : X \to \mathbb{R}\) where \(x \mapsto x\) if \(x \in [0,1]\) and \(x \mapsto x - 2\) if \(x \in [2,3]\) is a strict monotone. We will show that any family \((A_i)_{i \in I}\), where \(A_i \subseteq X\) is increasing \(\forall i \in I\) and \(\forall x, y \in X\) such that \(x \triangleright y\) there exists some \(i \in I\) such that either \(x \not\in A_i\) and \(y \in A_i\) or \(y \not\in A_i\) and \(x \in A_i\), is uncountable. Since the existence of some \((A_i)_{i \in I}\) with these properties and countable \(I\) is implied by the existence of an injective monotone by Lemma 1 (iii), we obtain that there is no injective monotone for \(X\).

Let \((A_i)_{i \in I}\) be a family with the properties in the last paragraph and, for each \(x \in [0,1]\), define \(y_x := x + 2\). Since \(x \triangleleft y_x\) by definition, there exists some \(A_x \in (A_i)_{i \in I}\) such that either \(x \in A_x\) and \(y_x \not\in A_x\) or \(y_x \in A_x\) and \(x \not\in A_x\). We fix such an \(A_x\) for each \(x \in [0,1]\) and consider the map \(f : [0,1] \to (A_i)_{i \in I}, \ x \mapsto A_x\). Consider some \(x, z \in [0,1]\) such that \(A_x = A_z\) and assume \(x \neq z\). We show first the case where \(z < x\) leads to contradiction. Assume first \(x \in A_x\) and \(y_x \not\in A_x\). Then, since \(A_x = A_z\), we either have \(z \in A_z\) or \(y_x \in A_z\). Both cases lead to contradiction, since we get \(y_x \in A_x\) because \(A_x\) is increasing and we either have \(z < y_x\) with \(z \in A_x\) or \(y_x < y_x\) with \(y_x \in A_x\). We can proceed analogously if we assume \(y_x \in A_x\), relying on the fact both \(z < x\) and \(y_x < x\) hold. In case we assume \(x < z\), we also achieve a contradiction following the same argument but interchanging the role of \(x\) and \(z\). Thus, \(x \neq z\) leads to a contradiction and by injectivity of \(f\) we get \(|[0,1]| \leq |(A_i)_{i \in I}|\). As a consequence, \(X\) has no injective monotone.

Notice, the proof of Proposition 2 relies on the existence of connections between several elements, which allow us to assure the sets from Lemma 1 that separate the preorder differ when different elements inside certain sets are considered. This is the reason why a related preorder was used in [20, Proposition 8] to show countable multi-utilities and injective monotones are not equivalent. One may think the trivial preorder on the real line \((\mathbb{R}, =)\) would have
Figure 2: Graphical representation of a preordered space, defined in Proposition 2, with the cardinality of the continuum and where strict monotones exist while injective monotone do not. In particular, we show $A := [0, 1], B := [2, 3]$, and how $x, y, z \in A, x < y < z$, are related to $x + 2, y + 2, z + 2 \in B$. Notice, an arrow from an element $w$ to an element $t$ represents $w \prec t$.

an injective monotone, the identity, and no countable multi-utility. However, due to it being completely disconnected, $(\chi_{\leq q}, \chi_{\geq q})_{q \in \mathbb{Q}}$ is a countable multi-utility, where $\chi_{\leq q}(x) := 1$ if $x \leq q$ and $\chi_{\leq q}(x) := 0$ otherwise, and $\chi_{\geq q}(x) := 1$ if $x \geq q$ and $\chi_{\geq q}(x) := 0$ otherwise.

While the existence of injective monotones implies the existence of multi-utilities with the cardinality of the continuum (in particular, composed of injective monotones), as we showed in [20, Proposition 4], the converse was unknown up to now. The preordered space in Proposition 2 shows the converse in false. Actually, it shows the stronger statement that the existence of strict monotone multi-utilities with cardinality $\mathfrak{c}$ is still not sufficient for the existence of an injective monotone, as we state in Corollary 1.

**Corollary 1.** There are preordered spaces which have strict monotone multi-utilities with cardinality $\mathfrak{c}$ and no injective monotone.

**Proof.** We can use the counterexample from Proposition 2 which has no injective monotone. Moreover, it is straightforward to see that $(\chi_{i(x)})_{x \in X}$ is a multi-utility with cardinality $\mathfrak{c}$ [17], where $\chi_A$ is the indicator function of a set $A$ and $i(x) := \{y \in X | x \preceq y\} \forall x \in X$. Since there exist strict monotones, as we

\footnote{We say a binary relation $\preceq$ on a set $X$ is a trivial ordering if $x \preceq y \iff x = y \forall x, y \in X$.}
showed in the proof of Proposition 2, we can follow Proposition 1 and get that there exist strict monotone multi-utilities with cardinality \( \kappa \).

Notice, Corollary 1 implies the class of preorders with injective monotones is strictly contained inside the class where multi-utilities with cardinality \( \kappa \) exist. In fact, we can improve upon this modifying the preorder in Proposition 2 to show there are preordered spaces where multi-utilities with cardinality \( \kappa \) exist while strict monotones do not. We present such a preorder in Proposition 3, which is the same as the one in Proposition 2 with the exception that we have \( x \prec y \) instead of \( x \bowtie y \) \( \forall x \in [0,1] \).

**Proposition 3.** There are preordered spaces which have multi-utilities with cardinality \( \kappa \) and no strict monotone.

**Proof.** Consider \( X := [0,1] \cup [2,3] \) equipped with \( \succeq \) where

\[
x \preceq y \iff \begin{cases} x, y \in [0,1] \text{ and } x \leq y, \\ x, y \in [2,3] \text{ and } x \leq y, \\ x \in [0,1], y \in [2,3] \text{ and } x + 2 \leq y \\ x \in [2,3], y \in [0,1] \text{ and } x - 2 \leq y
\end{cases}
\]
\( \forall x, y \in X \) (see Figure 3 for a representation of \( \leq \)), which differs from (2) only in \( x + 2 \leq y \) instead of \( x + 2 < y \) for \( x \in [0, 1] \) and \( y \in [2, 3] \). Notice \((X, \preceq)\) is a preordered space and there is a multi-utility with cardinality \( \mathfrak{c} \) as in the proof of Corollary 1. We will show that any family \((A_i)_{i \in I}\), where \( A_i \subseteq X \) is increasing \( \forall i \in I \) and \( \forall x, y \in X \) such that \( x \prec y \) there exists some \( i \in I \) such that \( y \in A_i \), and \( x \not\in A_i \), is uncountable. Since the existence of some \((A_i)_{i \in I}\) with these properties and countable \( I \) is implied by the existence of a strict monotone by Lemma 1 (ii), we conclude that there is no strict monotone for \( X \).

Let \((A_i)_{i \in I}\) be a family with the properties in the last paragraph and, for each \( x \in [0, 1]\), define \( y_x := x + 2 \). Since \( x \prec y_x \) by definition, there exists some \( A_x \in (A_i)_{i \in I} \) such that both \( y_x \in A_x \) and \( x \notin A_x \) hold. We fix such an \( A_x \) for each \( x \in [0, 1]\) and consider the map \( f : [0, 1] \to (A_i)_{i \in I}, x \mapsto A_x \). Consider some \( x, z \in [0, 1]\) such that \( A_x = A_z \) and assume \( x \neq z \). We show first the case where \( z < x \) leads to contradiction. Since \( A_x = A_z \), we have \( y_z \in A_z \). Given the fact \( A_x \) is increasing and \( y_z < x \) by definition, we get \( x \in A_x \), a contradiction. In case we assume \( x < z \), we also achieve a contradiction following the same argument but interchanging the role of \( x \) and \( z \). Thus, \( x \neq z \) leads to contradiction and we get, by injectivity of \( f \), \(|[0, 1]| \leq |(A_i)_{i \in I}|\). As a consequence, \( X \) has no strict monotone.

Notice, essentially, we recover in Proposition 3 the lexicographic plane, the classical counterexample used by Debreu [12, 13] to show the existence of total preordered spaces without utility functions. Another counterexample, which relies on Szpilrajn extension theorem, can be found in [20, A.2.1]. Notice, in particular, Proposition 3 implies that the class of preordered spaces with strict monotone multi-utilities with cardinality \( \mathfrak{c} \) is strictly contained inside the class with multi-utilities of the same cardinality. This contrasts with the fact that countable multi-utilities and countable strict monotone multi-utilities coincide for any preordered space [2, Proposition 4.1]. In fact, they also coincide with countable injective monotone multi-utilities [20, Proposition 6]. Notice, also, the preordered space in Proposition 3 shows the stronger fact that strict monotones do not always exist when \( X/\sim \) has cardinality \( \mathfrak{c} \), as we state in Corollary 2.

**Corollary 2.** There are preordered spaces \((X, \preceq)\) where \( X/\sim \) has cardinality \( \mathfrak{c} \) and strict monotone multi-utilities with cardinality \( \mathfrak{c} \) do not exist.

**Proof.** Consider the preordered space in Proposition 3. Notice, since \((\chi_{i(x)})_{x \in X/\sim}\) is a multi-utility of cardinality \( \mathfrak{c} \), strict monotone multi-utilities of cardinality \( \mathfrak{c} \) and strict monotones are equivalent, by Proposition 1. Thus, they do not exist.

Notice, if \( X/\sim \) is countable, then it has countable multi-utilities (we can follow the proof in Corollary 1) and, by [2, Theorem 3.1], countable strict monotone multi-utilities. Furthermore, we can follow Corollary 1 and Proposition 1 to conclude that every preorder with strict monotones has strict monotone multi-utilities with the cardinality of some infinite set \( I \) if \( X/\sim \) has the cardinality of
Proof. (i) For the first statement, notice, by Lemma 1 (i), there exists a family of increasing sets \((A_i)_{i \in I}\) that \(\forall x, y \in X\) with \(x \prec y\) separates \(x\) from \(y\) and \(\forall x, y \in X\) with \(x \asymp y\) separates both \(x\) from \(y\) and \(y\) from \(x\). Consider the map \(f : X/\sim \to \mathcal{P}(I)\), \([x] \mapsto B_x\) where \(B_x := \{i \in I| [x] \subseteq A_i\}\). If \([x] \neq [y]\), then we either have \(x \asymp y, x \prec y\) or \(y \prec x\) \(\forall x \in [x], y \in [y]\). In any case, there exists some \(i \in I\) such that \(x \subseteq A_i\) and \(y \nsubseteq A_i\) and vice versa. Thus, \(B_x \neq B_y\) and \(f\) is injective. We get \(|X/\sim| \leq |\mathcal{P}(I)|\).

For the second statement, consider the set \(X := \mathcal{P}(I)\) equipped with the preorder \(\subseteq\), where \(\subseteq\) denotes set inclusion. One can see \((f_i)_{i \in I}\) is a multi-utility for \(X\), where \(f_i : \mathcal{P}(I) \to \mathbb{R}, U \mapsto 1\) if \(i \in U\) and \(U \mapsto 0\) otherwise. Notice we have \(|\mathcal{P}(I)/\sim| = |\mathcal{P}(I)|\). Thus, the bound in the first statement cannot be improved.

(ii) Consider \(X := \mathcal{P}(I)\) equipped with the trivial ordering \(\preceq\). Notice \((f_i)_{i \in I} \cup (g_i)_{i \in I}\) is a strict monotone multi-utility with the cardinality of \(I \cup I\), which is equinumerous to \(I\), where \(f_i : X \to \mathbb{R}, U \mapsto 1\) if \(i \in U\) and \(U \mapsto 0\) otherwise and \(g_i := -f_i\), and we also have \(|\mathcal{P}(I)/\sim| = |\mathcal{P}(I)|\). Thus, the bound in (i) cannot be improved.

(iii) Consider \(A\) an antichain of \(X\) and, for each \(x \in A\), \(I_x := \{i \in I| i \preceq x\}\). We will show the map \(f : A \to \mathcal{P}(I), x \mapsto I_x\) is injective, proving, thus, any antichain \(A\) fulfills \(|A| \leq |\mathcal{P}(I)|\) which leads to \(w(X, \preceq) \leq |\mathcal{P}(I)|\). Given \(x, y \in A, x \neq y\), we have \(x \asymp y\) and, by Debreu upper density of \(I\), there exists some \(i \in I\) such that \(x \asymp i \preceq y\). As a consequence, \(i \in I_y\) and \(i \notin I_x\). Resulting in \(f(x) = I_x \neq I_y = f(y)\) and, hence, in \(f\) being injective.

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2Any two elements in an antichain are incomparable.

3Notice, for a fixed pair \(x, y \in X\) where \(x \asymp y\) holds, there exist \(a_1, a_2 \in Z\) such that \(x \asymp a_1 \preceq y\) and \(y \asymp a_2 \preceq x\).
For the second statement, consider the preorder $(\Sigma^* \cup \Sigma^\omega, \preceq)$ where $\Sigma := \{0, 1\}$, $\Sigma^*$ is the set of finite sequences over $\Sigma$, $\Sigma^\omega$ is the set of infinite sequences over $\Sigma$, if $x \in \Sigma^*$ and $y \in \Sigma^\omega$ then $x \preceq_C y$ if $x$ is a prefix of $y$ and $\preceq$ is defined $\forall x, y \in \Sigma^* \cup \Sigma^\omega$ like

$$x \preceq y \iff \begin{cases} x = y \\ x \preceq_C y. \end{cases}$$

Notice $\Sigma^*$ is a countable Debreu upper dense subset and we have $w(X, \preceq) = |\Sigma^\omega| = |\mathcal{P}(\Sigma^*)|$. Thus, the bound in the first statement cannot be improved. □

Notice, while the analog of (iii) remains a question whenever $I$ is a finite set, the bounds in both (i) and (ii) do not hold. Although a trivial example supporting this assertion would be the real line with its usual order $(\mathbb{R}, \leq)$, since the identity is a strict monotone finite multi-utility, we conclude this paragraph including two, perhaps, more interesting counterexamples. Majorization also proves (i) is false when $I$ is finite, since, although it is defined through (1), it fulfills $|\mathcal{P}_I/\sim| = c$. We adapt from [7, Example 2] a preorder which also illustrates that (ii) is false when $I$ is finite. In particular, we take $X := A \cup B$, where $A$ and $B$ are two copies of $\mathbb{R}/\{0\}$, and equip them with $\preceq$ where

$$x \preceq y \iff \begin{cases} x, y \in A \text{ and } x \leq y, \\ x, y \in B \text{ and } x \leq y, \\ x \in A, x < 0, y \in B \text{ and } 0 < y, \\ x \in B, x < 0, y \in A \text{ and } 0 < y. \end{cases} \quad (4)$$

$\forall x, y \in X$ (see Figure 4 for a representation of $\preceq$). Note $|X/\sim| = c$ and $V := \{v_1, v_2\}$ is a finite strict monotone multi-utility, where $v_1(x) := x - 1$ if $x \in A$ and $x < 0$, $v_1(x) := e^x - 1$ if $x \in B$ and $x < 0$, $v_1(x) := 1 - e^{-x}$ if $x \in B$ and $x > 0$ and $v_2(x) := x + 1$ if $x \in A$ and $x > 0$, $v_2(x) := 1$ if $x \in B$ and $x < 0$, $v_2(x) := e^x - 1$ if $x \in A$ and $x < 0$, $v_2(x) := 1 - e^{-x}$ if $x \in A$ and $x > 0$ and $v_2(x) := x + 1$ if $x \in B$ and $x > 0$.

Proposition 4 improves the relation between the existence of multi-utilities and the cardinality of $X/\sim$. In particular, whenever we have $|X/\sim| \leq \zeta$, then there exist multi-utilities with cardinality $\zeta$ [17] (see Corollary 1) and, whenever injective monotones exist, we have $|X/\sim| \leq \zeta$. However, there are preorders where, although $|X/\sim| \leq \zeta$ holds, injective monotones do not exist, like the one in Proposition 2. Furthermore, there are preorders with $|X/\sim| \leq \zeta$ where strict monotones do not exist, like the example in Corollary 2. Finally, there exist preordered spaces with strict monotone multi-utilities with cardinality $\zeta$ and where $|X/\sim| > \zeta$, like the one in Proposition 4 (ii).

Returning to Proposition 3, notice its converse also holds, that is, there are preordered spaces where strict monotones exist and multi-utilities with cardinality $\zeta$ do not. In general, for any uncountable set $I$, there exist preordered spaces where strict monotones exist and multi-utilities with the cardinality of $I$ do not, as we show in Proposition 5. Notice the counterexample we present is,
Figure 4: Representation of a preordered space, defined by (4), where finite strict monotone multi-utilities exist and \(|X/\sim|=\mathfrak{c}\). In particular, we relate three different points \(x, y, z \in A\) with \(i_d(x), i_d(y), i_d(z) \in C\), where \(A, B := \mathbb{R}/\{0\}\), \(x < 0 < y < z\) and \(i_d : A \to B\) is the identity on \(\mathbb{R}/\{0\}\). Notice an arrow from an element \(w\) to an element \(t\) represents \(w \preceq t\).

essentially, the one we introduced in [20, Proposition 8], but for a larger ground set. Despite the large ground set, however, the proof is constructive.

**Proposition 5.** If \(I\) is an uncountable set, then there exist preordered spaces with strict monotones and without multi-utilities with the cardinality of \(I\).

**Proof.** Consider \(X := B \cup C\), where \(B\) and \(C\) are two copies of \(P(I)\), equipped with \(\preceq\) where

\[
  x \preceq y \iff \begin{cases} 
  x = y \\
  x \in B, \ y \in C \text{ and } y \neq i_d(x)
\end{cases}
\]  

(5)

\(\forall x, y \in X\) with \(i_d : B \to C\) the identity on \(P(I)\) (see Figure 5 for a representation of \(\preceq\)). Notice \((X, \preceq)\) is a preordered space and \(v : X \to \mathbb{R} \mapsto 0\) if \(x \in B\) and \(x \mapsto 1\) if \(x \in C\) is a strict monotone. By Lemma 1 (i) there exists a family \((A_j)_{j \in J}\) of increasing subsets of \(X\) such that whenever \(x \bowtie y\) there exists some \(j \in J\) such that \(x \in A_j\) and \(y \notin A_j\). It is enough to show that such a family has larger cardinality than \(I\) in order to see that there is no multi-utility for \(X\) with the cardinality of \(I\).

Notice, \(x \bowtie i_d(x) \ \forall x \in B\). There exists, thus, some \(A_x \in (A_j)_{j \in J}\) such that \(x \in A_x\) and \(i_d(x) \notin A_x\). We fix such an \(A_x\) for each \(x \in B\) and consider the map
Figure 5: Representation of a preordered space, defined in Proposition 5, where strict monotones exist and multi-utilities with the cardinality of $I$, an uncountable set, do not. In particular, we relate three different points $x, y, z \in B$ with $id(x), id(y), id(z) \in C$, where $B, C := \mathcal{P}(I)$ and $id : B \to C$ is the identity on $\mathcal{P}(I)$. Notice an arrow from an element $w$ to an element $t$ represents $w \prec t$. Notice, also, this preorder is, essentially, the one we introduced in [20, Proposition 8] with a larger ground set.

$f : B \to \{A_j\}_{j \in J}, x \mapsto A_x$. Consider a pair $x, z \in B$ such that $A_x = A_z$ and assume $x \neq z$. Since $A_x$ is increasing, $z \prec id(x)$ and $z \in A_z$, we get $id(x) \in A_z$, a contradiction. Thus, $A_x = A_z$ implies $x = z$ and we have, by injectivity of $f$, $|\mathcal{P}(I)| = |B| \leq |\{A_j\}_{j \in J}|$. As a consequence, $X$ has no multi-utility with the cardinality of $I$.

Notice, in particular, Proposition 5 shows there are preordered spaces with strict monotones and without strict monotone multi-utilities with cardinality $\kappa$, which is not true for injective monotones (see [20, Proposition 4]). It also shows that the class of preordered spaces where multi-utilities with cardinality $\kappa$ exist is strictly contained inside the class of preordered spaces with multi-utilities, which consists of all preordered spaces [17, Proposition 1]. In fact, as we show in Proposition 6 through a variation of the preorder in Proposition 3, given an uncountable set $A$, there exist preordered spaces where neither strict monotones nor multi-utilities with the cardinality of $I$ exist. Notice the proof of Proposition 6 relies on Szpilrajn extension theorem and, thus, is non-constructive.

**Proposition 6.** If $I$ is an uncountable set, then there exist preordered spaces where neither multi-utilities with the cardinality of $I$ nor strict monotones exist.
Proof. Consider $X := B \cup C$, where $B$ and $C$ are two copies of $\mathcal{P}(I)$ and consider on both $C$ and $B$ the total order $\preceq_S$ that results from applying Szpilrajn extension theorem [35] to the partial order defined by set inclusion on $\mathcal{P}(I)$. Furthermore, equip $X$ with $\preceq$ where

$$
x \preceq y \iff \begin{cases} 
  x \preceq_S y \text{ and } x, y \in B \\
  x \preceq_S y \text{ and } x, y \in C \\
  i_d(x) \preceq_S y, \ x \in B \text{ and } y \in C \\
  x \prec_S i_d(y), \ x \in C \text{ and } y \in B 
\end{cases}
$$

(6)

$\forall x, y \in X$ with $i_d : B \to C$ the identity on $\mathcal{P}(I)$ (see Figure 6 for a representation of $\preceq$). Notice $(X, \preceq)$ is a preordered space. In analogy to Propositions 3 and 5, one can show that any family $(A_j)_{j \in J}$ of increasing subsets $A_j \subseteq X$ that separates $x$ and $y$ whenever $x \prec y$ has larger cardinality than $I$. Since the existence of some $(A_j)_{j \in J}$ with those properties and $|J| \leq |I|$ is implied by both the existence of a multi-utility by Lemma 1 (i) and the existence of a strict monotone by Lemma 1 (ii), we obtain that there is no multi-utility with the cardinality of $I$ nor a strict monotone for $X$.

Note that the preorder we introduced in Proposition 6 actually supports a stronger statement, which we include in Corollary 3. In order to prove it, we simply follow Proposition 6 and add the fact that, as in Corollary 1, $(\chi_{i(x)})_{x \in X}$ is a multi-utility with the cardinality of $\mathcal{P}(I)$.

**Corollary 3.** If $I$ is an uncountable set, then there exist preordered spaces where multi-utilities with the cardinality of $\mathcal{P}(I)$ exist, although neither strict monotones nor multi-utilities with the cardinality of $I$ do.

To complement Propositions 5 and 6, we show, in Corollary 4, for any uncountable set $I$ there exist preorders which have multi-utilities with the cardinality of $I$ and no strict monotones. Notice, again, we follow the basic construction in Proposition 3, although we use the same non-constructive approach in Proposition 6.

**Corollary 4.** If $I$ is an uncountable set, then there exist preordered spaces which have multi-utilities with the cardinality of $I$ and no strict monotone.

Proof. Consider $X := C \cup B$, where $C$ and $B$ are two copies of $I$, and equip it with a preorder analogous to the one in Proposition 6. Notice $(\chi_{i(x)})_{x \in X}$ is a multi-utility for $X$. By slightly modifying the argument in Proposition 6, we conclude there are no strict monotones.

In fact, as we show in Corollary 5, we can put together the preorders from Proposition 4 (i) and Corollary 4 to improve the relation between multi-utilities and the cardinality of $X/\sim$ even more.

**Corollary 5.** If $I$ is an uncountable set, then there exist preordered spaces where $|X/\sim| > |I|$ and multi-utilities with the cardinality of $I$ exist, while strict monotones do not.
Figure 6: Representation of a preordered space, defined in Proposition 6, which has no multi-utility with the cardinality of an uncountable set $I$ and no strict monotone. In particular, we show how $x, y, z \in B, x \prec y \prec z$, are related to $\text{id}(x) \prec \text{id}(y) \prec \text{id}(z) \in C$ where $B, C := \mathcal{P}(I)$ and $\text{id} : B \rightarrow C$ is the identity on $\mathcal{P}(I)$. Notice an arrow from an element $w$ to an element $t$ represents $w \prec t$.

Notice, also, this preorder is, essentially, the same as the one in the proof of Proposition 3 (see Figure 3) with a larger ground set. As a result, we used a non-constructive argument relying on Szpilrajn extension theorem to define it.

**Proof.** Take $X := A \cup B$ where $A$ and is the ground sets of the preorder in Corollary 4 and $B$ is the ground set of the preorder in Proposition 4 (i) without the empty set. We equip $X$ with the preorder in Corollary 4 on $A$ and that of Proposition 4 (i) on $B$, leaving $x \triangleright y \forall x, y \in X$ such that $x \in A$ and $y \in B$ or vice versa. Since any strict monotone on $X$ would also be a strict monotone on $A$, they do not exist by Corollary 4. Notice we have $|X/\sim| > |I|$, since $|B/\sim| > |I|$. Notice, also, $(g_i)_{i \in I} \cup (h_y)_{y \in A}$ is a multi-utility with the cardinality of $I$ for $X$, where $\forall i \in I g_i(x) := f_i(x)$ if $x \in B$ and $g_i(x) := 0$ if $x \in A$, with $(f_i)_{i \in I}$ defined as in Proposition 4 (i), and $\forall y \in A h_y(x) := \chi(y) i(x)$ if $x \in A$ and $h_y(x) := 0$ if $x \in B$. \hfill \Box

To finish this section, since we have been mainly concerned with preordered spaces with infinite multi-utilities and uncountable $X/\sim$, we address both finite multi-utilities and countable $X/\sim$. The first thing to notice is the existence of finite multi-utilities does not imply $X/\sim$ is countable. This is exemplified by majorization [27, 3], since it is defined through a finite multi-utility (1) but the corresponding quotient space $\mathbb{F}_{\Omega}/\sim_U$ has the cardinality of the continuum. It is
straightforward to see, whenever $X/\sim$ is finite, there exists a finite multi-utility (see Corollary 1). However, as we show in Proposition 7, there exist preorders where $X/\sim$ is countably infinite and finite multi-utilities do not exist. Notice the preorder that supports this claim is, essentially, the same as the one in Proposition 5. However, in Proposition 7, we follow a simpler proof.

**Proposition 7.** There are preordered spaces $(X, \preceq)$ where $X/\sim$ is countably infinite and no finite multi-utilities exist.

*Proof.* Consider $X := \mathbb{Z}/\{0\}$ equipped with $\preceq$ where

$$n \preceq m \iff \begin{cases} n = m \\ n > 0, \ m < 0 \text{ and } n \neq -m. \end{cases}$$

Notice $(\mathbb{Z}/\{0\})/\sim$ is countable. Assume there exists a finite multi-utility $(u_i)_{i=1}^k$. Notice for any pair $n, -n$ we have $n \sim -n$ and there must be some $i_n$ such that $u_{i_n}(-n) < u_{i_n}(n)$ by definition of multi-utility. If we consider, however, some $m \neq n$, then we have $u_{i_n}(m) \leq u_{i_n}(-n) < u_{i_n}(n) \leq u_{i_n}(-m)$. Thus, $i_m \neq i_n$. Considering w.l.o.g. $i_n = n$, we get $u_i(k + 1) < u_i(-k + 1)$ for $i = 1, \ldots, k$. Thus, there is no multi-utility of cardinality $k$ for any $k < \infty$. 

As a result of Proposition 7, there are preorders where countably infinite multi-utilities exist while finite ones do not. The preorder we used had, however, a countable $X/\sim$. We therefore complement this statement by showing in Proposition 8 that there are preorders with the same characteristics but uncountable $X/\sim$.

**Proposition 8.** There are preordered spaces $(X, \preceq)$ where $X/\sim$ is uncountable and, although countable multi-utilities exist, finite multi-utilities do not.

*Proof.* Let $\mathcal{P}_{\text{inf}}(\mathbb{N})$ be the set of infinite subsets of $\mathbb{N}$. Consider $X := (\mathbb{N} \cup \mathcal{P}_{\text{inf}}(\mathbb{N}), \preceq)$ equipped with the preorder $\preceq$

$$x \preceq y \iff \begin{cases} x = y \\ x \in \mathbb{N}, \ y \in \mathcal{P}_{\text{inf}}(\mathbb{N}) \text{ and } x \in y \end{cases}$$

$\forall x, y \in X$. Clearly, $|X/\sim| = \mathfrak{c}$, thus uncountable.

One can see $U := (u_n, v_n)_{n \geq 0}$ is a countable multi-utility, where $u_n(x) := 1$ if $x = n$ or $n \in x \in \mathcal{P}_{\text{inf}}(\mathbb{N})$ and $u_n(x) := 0$ otherwise, and $v_n(x) := 1$ if $n \notin x$ and $x \in \mathcal{P}_{\text{inf}}(\mathbb{N})$ and $u_n(x) := 0$ otherwise. Notice if $x \preceq y$ and $x \neq y$ then $x \in \mathbb{N}$ and $x \in y$. Thus, $u(x) \leq u(y) \forall u \in U$. Assume now we have $-(x \preceq y)$. If $y \preceq x$, then $y \in \mathbb{N}$ and $x \in \mathcal{P}_{\text{inf}}(\mathbb{N})$. Thus, there exists $m \in x$ such that $m \neq y$ and $u_m(y) < u_m(x)$. If $x \preceq y$, then we consider four cases. If $x, y \in \mathbb{N}$, then $u_x(x) > u_x(y)$. If $x, y \in \mathcal{P}_{\text{inf}}(\mathbb{N})$, then, if there exists $n \in x/y$, we have $u_n(x) > u_n(y)$. Otherwise, there exists $n \in y/x$ and we have $v_n(x) > v_n(y)$. If $x \in \mathbb{N}$ and $y \in \mathcal{P}_{\text{inf}}(\mathbb{N})$, then $x \notin y$ and we have $u_x(x) > U_x(y)$. If $y \in \mathbb{N}$ and $x \in \mathcal{P}_{\text{inf}}(\mathbb{N})$, then $y \notin x$ and we have $v_y(x) > v_y(y)$.
To conclude, we show there is no finite multi-utility. Let \( A_0 \subseteq P_{inf}(\mathbb{N}) \), fix some \( k \in \mathbb{N} \) and consider \((b_i)_{i=k+1}^{k+1} \subseteq A_0\), where \( b_i \neq b_j \) if \( i \neq j \), and \((A_i)_{i=k+1}^{k+1}\), where \( A_i := A_0/b_i \) for \( i = 1, .., k + 1 \). Notice \((b_i, A_i)_{i=k+1}^{k+1}\) is a finite portion of the preorder in Proposition 7, since we have \( b_i \preceq A_j \) if and only if \( i \neq j \), and we can argue analogously as we did there that no multi-utility with cardinality \( k \) exists. Since \( k \) is arbitrary, we obtain there is no finite multi-utility.

The preorder we introduced in Proposition 8 can, in fact, be used to improve the relation between real-valued monotones characterization of preorders and order density properties. A subset \( Z \subseteq X \), such that \( x \prec y \) implies that there exists \( z \in Z \) with \( x \preceq z \preceq y \) is called order dense in the sense of Debreu (or Debreu dense for short) [31, 10]. Accordingly, we say that \((X, \preceq)\) is Debreu separable [28] if there exists a countable Debreu dense set in \((X, \preceq)\). Similarly, \((X, \preceq)\) is called Debreu upper separable if there exists a countable subset which is both Debreu dense and Debreu upper dense [20] (we defined Debreu upper dense subsets right before Proposition 4). As was shown in [20, Proposition 9], Debreu upper separable preorders have countable multi-utilities. However, there exist preorders which have countable multi-utilities but are not Debreu separable, like majorization for \(|\Omega| \geq 3\) (see [20, Lemma 5 (ii)]). In Proposition 9, we complement these results by showing a preorder where countable multi-utilities exist and countable Debreu upper dense subsets do not. In particular, we show the preorder we introduced in Proposition 8 has no countable Debreu upper dense subsets although, as we showed there, it has countable multi-utilities. Notice, a preorder where the weaker fact that injective monotones exist and countable Debreu upper dense subsets do not can be found in [20, Proposition 8]. There, an injective monotone was introduced and, although it was shown no countable multi-utility exists, it is easy to see any Debreu upper dense subset would be uncountable.

**Proposition 9.** There are preordered spaces where countable multi-utilities exist and every Debreu upper dense subset is uncountable.

**Proof.** Consider the preorder \( X := (\mathbb{N} \cup P_{inf}(\mathbb{N}), \preceq) \) from Proposition 8. As we showed there, countable multi-utilities exist. Assume there exists a Debreu upper dense subset \( D \subseteq X \). Consider \( y \in P_{inf}(\mathbb{N}), y \neq \mathbb{N} \). Notice there exists some \( n_y \in \mathbb{N}/y \) and \( y \cup \{n_y\} \preceq y \). Since \( D \) is Debreu upper dense, there exists some \( d \in D \) such that \( y \cup \{n_y\} \preceq d \preceq y \). Since \( d \preceq y \) implies either \( d \in y \) or \( d = y \), and \( d \in y \) implies \( d \in y \cup \{n_y\} \), thus \( d \preceq y \cup \{n_y\} \) contradicting the definition of \( d \), we have \( d = y \). As a result, \( P_{inf}(\mathbb{N})/\{\mathbb{N}\} \subseteq D \) and \( D \) is uncountable.

Notice, although they coincide when they are countable (see [2, Proposition 4.1] and [20, Proposition 6]), it remains an open question how the different sorts of multi-utilities relate to each other when they are finite. As a first result in this direction, we finish with a characterization of preordered spaces with finite injective monotones multi-utilities.
Proposition 10. If \((X, \preceq)\) is a preordered space, then the following are equivalent:

(i) There exists a finite multi-utility \((u_i)_{i\leq N}\) such that the image of the non-injective set

\[ I_{u_i} := \{ r \in \mathbb{R} \mid \exists x, y \in X \text{ such that } x, y \in u_i^{-1}(r) \text{ and } - (x \sim y) \} \quad (7) \]

is countable \(\forall i \leq N\).

(ii) There exists a finite injective monotone multi-utility \((v_i)_{i\leq N}\).

Proof. By definition, given an injective monotone multi-utility \((v_i)_{i\leq N}\), we have \(I_{v_i} = \emptyset \ \forall i \leq N\). Conversely, consider \(u \in (u_i)_{i\leq N}\) a monotone such that the image of its non-injective set \(I_u\) is countable. Take \((r_n)_{n \geq 0}\) a numeration of \(I_u\), \((y_n)_{n \geq 0} \subseteq X\) a set such that \(u(y_n) = r_n \ \forall n \geq 0\) and, w.l.o.g., an injective monotone \(c_0 : X \to (0, 1)\). Notice injective monotones exist under the hypotheses, as we showed in [20, Proposition 5]. Define, then,

\[
\begin{align*}
w_0(x) := & \begin{cases} u(x) & \text{if } u(x) < r_0 \\ u(x) + c_0(x) & \text{if } u(x) = r_0 \\ u(x) + 1 & \text{else.} \end{cases}
\end{align*}
\]

\(\forall x \in X\). Notice \(I_{w_0} \subset I_u\), since \(x_0 \notin I_{w_0}\), and we have both \(u(x) \leq u(y)\) implies \(w_0(x) \leq w_0(y)\) and \(u(x) < u(y)\) implies \(w_0(x) < w_0(y)\) \(\forall x, y \in X\). Similarly, consider a family of injective monotones \((c_n)_{n \geq 1}\) such that \(c_n : X \to (0, 2^{-n})\) for \(n \geq 1\) and define, also for \(n \geq 1\),

\[
\begin{align*}
w_n(x) := & \begin{cases} w_{n-1}(x) & \text{if } w_{n-1}(x) < w_{n-1}(y_n) \\ w_{n-1}(x) + c_n(x) & \text{if } w_{n-1}(x) = w_{n-1}(y_n) \\ w_{n-1}(x) + 2^{-n} & \text{else} \end{cases}
\end{align*}
\]

\(\forall x \in X\). Notice \(I_{w_{n-1}} \subset I_{w_n}\) holds \(\forall n \geq 1\), since \(x_n \notin I_{w_n}\), and we have both \(w_{n-1}(x) \leq w_{n-1}(y)\) implies \(w_n(x) \leq w_n(y)\) and \(w_{n-1}(x) < w_{n-1}(y)\) implies \(w_n(x) < w_n(y)\) \(\forall x, y \in X\). Lastly, consider the pointwise limit \(\nu(x) := \lim_{n\to\infty} w_n(x)\). Notice \(\nu\) is well-defined and, also, an injective monotone, since \(I_\nu = \emptyset\) by construction.

Following the same procedure for each monotone in \((u_i)_{i\leq N}\), we get a family of injective monotones \((v_i)_{i\leq N}\). To conclude it is a multi-utility, we need to show, \(\forall x, y \in X\) with \(-(x \preceq y)\), there exists some \(i \leq N\) such that \(v_i(x) > v_i(y)\). If \(y < x\), then \(v_i(x) > v_i(y)\) \(\forall i \leq N\) by definition of injective monotone. Otherwise, if \(x \preceq y\), there exists some \(i \leq N\) such that \(u_i(x) > u_i(y)\). Thus, we also have \(v_i(x) > v_i(y)\). Hence, \((v_i)_{i\leq N}\) is a multi-utility. \(\square\)

Notice we can weaken the hypothesis, assuming, instead of \((7)\), that

\[ \{ r \in \mathbb{R} \mid \exists x, y \in X \text{ such that } x, y \in u_i^{-1}(r) \text{ and } x \sim y \} \]
is countable $\forall i \leq N$, to conclude, analogously, that the existence of finite multi-utilities and that of finite strict monotone multi-utilities are equivalent. Notice, as a result, we obtain the existence of finite multi-utilities coincides with that of finite strict monotone multi-utilities and that of finite injective monotone multi-utilities whenever $X/\sim$ is countable. The general case where $X/\sim$ is uncountable (in particular, when $|X/\sim| \leq \epsilon$ since, otherwise, there are no injective monotones), remains open. This is due to the fact the technique in Proposition 10 cannot be used and $I_{u_i}$ is not necessarily countable $\forall i \leq N$, as one can see in majorization, for example. There, taking $u_i$ as in (1), we have $(\frac{1}{2^i}, 1) \subseteq I_{u_i}, \forall i \leq |\Omega| - 1$ and, thus, $I_{u_i}$ is uncountable $\forall i \leq |\Omega| - 1$. Notice, also, the technique in Proposition 10 is similar to the one we used in [20, Proposition 2], where we showed the existence of an injective monotone is equivalent to that of a strict monotone $f$ whose non-injective set

$$\{x \in X | \exists y \in X \text{ s.t. } f(x) = f(y) \text{ and } x \not\preceq y\}$$

is countable. Notice the hypothesis there is stronger, since the hypothesis that the image of the non-injective set $I_f$ is countable is insufficient, as one can see using the preorder in [20, Proposition 1 (i)].

The technique in Proposition 10 can actually be used to prove that countable multi-utilities and countable injective monotone multi-utilities always coincide (see [20, Proposition 6]). The only detail of importance is, whenever a countable multi-utility exists, there exists, by Lemma 1 (i), a countable family of increasing sets $(A_n)_{n \geq 0}$ that $\forall x, y \in X$ with $x \prec y$ separates $x$ from $y$ and $\forall x, y \in X$ with $x \not\preceq \not\succeq y$ separates both $x$ from $y$ and $y$ from $x$. In particular, $(\chi_{A_n})_{n \geq 0}$ is a countable multi-utility with the property that $I_{\chi_{A_n}}$ is finite $\forall n \geq 0$. Since injective monotones exist, we can follow Proposition 10 to construct a countable injective monotone multi-utility.

3 Discussion

In this work, we have improved the classification of preordered spaces through real-valued monotones in terms of the cardinality of multi-utilities and quotient spaces, c.f. Figure 1.

Classification of preordered spaces through real-valued monotones.

The state of the classification of preordered spaces in terms of real-valued monotones can be found in Figure 1, whereas our contributions are shown in Figure 7. In this paragraph, we summarize the relation between the different classes and distinguish between our results and the ones in the literature. We will begin from the innermost class, preorders with utility functions, and finish with the outermost class, which contains all preorder [17], that is, preorders with multi-utilities.

The relation between utility functions and the subsequent classes, finite multi-utilities and preorders with countable $X/\sim$ is as follows. A utility function is a finite multi-utility, although there are preordered spaces where finite
multi-utilities exist and utilities do not, like majorization [3, 27]. We can also use majorization to show there are preorders with a finite multi-utility where \( X/\sim \) is uncountable. By Proposition 7, a countable \( X/\sim \) does not imply there exists a finite multi-utility. Notice, also, preorders with utilities can have an uncountable \( X/\sim \), the easiest example being \((\mathbb{R}, \leq)\), and any non-total preorder with countable \( X/\sim \) has no utility function.

The next class of interest are preorders with countable multi-utilities, which are exactly those with countable strict monotone multi-utilities [2, Proposition 4.1] and countable injective monotone multi-utilities [20, Proposition 6]. By Proposition 8, there are preorders with countable multi-utilities where \( X/\sim \) is uncountable such that no finite multi-utility exists, although finite multi-utilities are, of course, countable. Also, whenever \( X/\sim \) is countable, there exists a countable multi-utility, namely, \((\chi_{(x)})|_{x} \in X/\sim \) [17].

The following wider category are preorders with injective monotonies, which are equivalent to those with injective monotone multi-utilities by [20, Proposition 4]. As we showed in [20, Proposition 5], injective monotones can be constructed from countable multi-utilities. However, again by [20, Proposition 8], the converse is false. Injective monotones are contained inside two classes: preorders with strict monotone multi-utilities of cardinality \( \mathfrak{c} \) and preorders where \(|X/\sim| \leq \mathfrak{c} \). It is straightforward to see \(|X/\sim| \leq \mathfrak{c} \) whenever injective monotones exist. Because of this, since it implies multi-utilities of cardinality \( \mathfrak{c} \) exist [17], and Proposition 1, strict monotone multi-utilities of cardinality \( \mathfrak{c} \) exist whenever injective monotones do. However, by Proposition 1 and Corollary 1, there are preordered spaces with strict monotone multi-utilities of cardinality \( \mathfrak{c} \) and without injective monotones. Similarly, as Proposition 2 shows, there are preorders where we have \(|X/\sim| \leq \mathfrak{c} \) and no injective monotones. Moreover, by Proposition 4 (ii), having strict monotone multi-utilities of cardinality \( \mathfrak{c} \) does not imply \(|X/\sim| \leq \mathfrak{c} \). Conversely, as noticed in Corollary 2, we also get a negative result if we interchange the role of both clauses, that is, there are preorders where \(|X/\sim| \leq \mathfrak{c} \) holds and no strict monotone multi-utility of cardinality \( \mathfrak{c} \) exists. Notice the preorder in Corollary 2 was, essentially, already introduced by Debreu in [12]. As we stated in Proposition 1, having a strict monotone and a multi-utility of cardinality \( \mathfrak{c} \), the following class of interest, is equivalent to having a strict monotone multi-utility of that cardinality. However, by Proposition 3, \(|X/\sim| \leq \mathfrak{c} \) does not imply there exists a strict monotone multi-utility of cardinality \( \mathfrak{c} \). If we relax the implication of the statement to multi-utility of cardinality \( \mathfrak{c} \), then it is indeed true (see [17] or Corollary 1). There are, actually, preorders with a multi-utility of cardinality \( \mathfrak{c} \) and no strict monotone multi-utility of that cardinality such that \(|X/\sim| > \mathfrak{c} \), as Corollary 5 shows. Finally, by Proposition 5, there are preorders where strict monotones exist a multi-utilities of cardinality \( \mathfrak{c} \) do not. In fact, by Proposition 6, there are preorders without both strict monotones and multi-utilities of cardinality \( \mathfrak{c} \). This completes the results which are needed to construct Figure 1. Notice, although we have focused on the case \( I = \mathbb{R} \), many of the results hold for a general uncountable set \( I \), as we stated them in Section 2.
Figure 7: Contributions of this work to the classification of preordered spaces. We reproduce here Figure 1, incorporating a point for each preorder we have introduced that has allowed us to distinguish between classes. In particular, $A$ stands for the preorder in Proposition 7, $B$ for the one in Proposition 8, $C$ for the preorder in both Proposition 2 and Corollary 1, $D$ for that of Proposition 4 (ii) taking $I = \mathbb{R}$, $E$ for the one in Corollary 2, $F$ for the preorder in Corollary 5 taking $I = \mathbb{R}$, $G$ for the space in Proposition 5 taking $I = \mathbb{R}$ and, lastly, $H$ for the one in Proposition 6 taking $I = \mathbb{R}$.

Aside from those in the last paragraph, there are four more results in Section 2. Proposition 10 shows the equivalence between finite multi-utilities and finite injective monotone multi-utilities in well-behaved cases. Notice the only finite case which appears in Figure 1 is that of multi-utilities, as the relation with the other types remains to be clarified. Proposition 9 improves upon [20], where it was shown Debreu upper separable preorders have countable multi-utilities.
20, Proposition 9] while there are preorders with countable multi-utilities which are not Debreu separable [20, Lemma 5], by showing there exist preorders with countable multi-utilities where every Debreu upper dense subset is uncountable. Lastly, Corollary 3 is slightly stronger than Proposition 6 and uses the same preorder, while Corollary 4 is weaker than Corollary 5.

Debreu dimension. There is a notion of dimension for partial orders which goes back to [15] and has remained somewhat disconnected from the more intuitive geometrical notion, which corresponds to multi-utilities. In fact, there exist preorders where the classical definition of dimension is finite while the geometrical one is uncountable. In [19], we propose a variation of the classical notion, called Debreu dimension, and, using results from this work, show that such a disconnection between this definition and the geometrical one does not occur. That is, we show that the geometrical dimension is countable if and only if the Debreu dimension also is.

Open questions. Several scientific disciplines rely on preordered spaces and their representation via real-valued monotones. Thus, refining the classification via the introduction of new classes and establishing more connections between separated classes in cases of interest would, potentially, improve several areas, like utility theory [12, 33] and the study of social welfare relation [4] in economics, statistical estimation [23] in statistics, equilibrium thermodynamics [26, 11], entanglement theory [30, 36] and general relativity [5, 29] in physics and, lastly, multicriteria optimization [25, 16]. Specific questions that remain to be solved include, for example, the relation between the different sorts of finite multi-utilities we have introduced. In particular, it is unclear whether Proposition 10 can be improved or preorders with finite multi-utilities and no finite injective multi-utilities exist.
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