EXPLICIT SUBCONVEXITY FOR GL₂ AND SOME APPLICATIONS

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Abstract. We make the subconvex exponent for GL₂ cuspidal representation in the work of Michel & Venkatesh explicit. The result depends on an effective dependence on the “fixed” GL₂ representation in our former work on the subconvex bounds for twists by Hecke characters, which in turn depends on the L₄-norm of the test function. We also give some applications of our results, including a new bound of the error term in the expansion of the partition function due to Rademacher.

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1. Introduction

1.1. Main Result. Let \( F \) be a number field with ring of adeles \( \mathbb{A} \). Let \( \pi \) be an automorphic cuspidal representation of GL₂(\( \mathbb{A} \)). This is the natural generalization of Hecke characters \( \chi \) of \( F^\times \backslash \mathbb{A}^\times \) to the GL₂ setting, hence generalization of Dirichlet L-functions (for \( F = \mathbb{Q} \)) in particular. Similarly, we also have the associated L-function \( L(s, \pi) \). While good and uniform bounds for Hecke L-functions \( L(s, \chi) \) are so far available in the literature, in particular in the case \( F = \mathbb{Q} \), no bounds for \( L(s, \pi) \) of similar quality is
known. In particular, if \( \omega \) denotes the central character of \( \pi \) and if \( \omega \) varies with \( \pi \), the known subconvex bounds for \( L(1/2, \pi) \) are of poor quality especially for the level aspect. For example:

1. Over \( F = \mathbb{Q} \), for a Maass form \( f \) of level \( q \) and (necessarily even) primitive central character \( \omega \), subconvexity in the \( q \)-aspect was solved by Duke, Friedlander and Iwaniec \([15, \text{Theorem 2.4}]\) with a power saving \( q^{1/2+\epsilon} \) from the convex bound \( q^{1/4} \).

2. In the same setting, the above method was simplified and generalized by Blomer, Harcos and Michel \([2] \) with an improvement on the power saving unspecified. However, it is believed that the method of Michel and Venkatesh goes beyond the method of Duke, Friedlander and Iwaniec. We re-confirm this opinion and make their result effective in this paper. Our main tool is a further development & adaptation to the triple product case of an improvement of the theory of regularized integral due to Zagier \([40]\), developed in our previous paper \([39, \S 2]\). Precisely, we shall make the following assumption:

- For \( \pi' \) cuspidal representation of \( \text{GL}_2(\mathbb{A}) \) with trivial central character, spherical at all infinite places and Hecke character \( \chi \) such that \( \pi_{\text{fin}}', \chi_{\text{fin}} \) have disjoint ramification, assume

\[
L(1/2, \pi' \otimes \chi) \ll_F, \epsilon \left( C(\pi_{\text{fin}}') C(\chi) \right)^{\epsilon} C(\pi_{\infty}')^{B} C(\pi_{\text{fin}}')^{A} C(\chi)^{2-\delta'}
\]

for some constants \( A, B > 0, 0 < \delta' < 1/2 \).

The main result of this paper is:

**Theorem 1.1.** Assuming \( \delta' \leq (1-2\theta)/8 \) and \( A \geq 1/4 \) in the assumption \([1, 2]\), we have for any \( \epsilon > 0 \)

\[
\left| L\left(\frac{1}{2}, \pi\right) \right| \ll_F, \epsilon C(\pi)^{\frac{1}{2} + \epsilon} \left( \frac{C(\pi)}{C(\omega)} \right)^{-\frac{1-2\theta}{12+16A}} C(\omega)^{-\frac{\delta'}{12+16A}}.
\]

**Remark 1.2.** The above bound separates \( C(\pi)/C(\omega) \) from the problematic part \( C(\omega) \). In fact, in some applications, one does not need to vary \( \omega \) but does need uniform bound.\(^1\) For such applications the following Corollary is more suitable.

**Remark 1.3.** The assumption \([1, 2]\) should not be regarded as a condition, since an effective value of \( A \) is obtained in \([36, \text{Theorem 2.1}]\). In particular, it implies that \( \delta' = (1-2\theta)/8, A = 5/4 \) is admissible (note that \( \pi_{\text{fin}}', \chi_{\text{fin}} \) have disjoint ramification) implies \( C_{\text{fin}}(\pi', \chi) = C_{\text{fin}}[\pi', \chi] = 1 \). We record the numerical subconvex saving for these values:

\[
\frac{1 - 2\theta}{24 + 32A} = \frac{1 - 2\theta}{64} > \frac{1}{128}, \quad \frac{\delta'}{12 + 16A} = \frac{1 - 2\theta}{256} > \frac{1}{1889}.
\]

It should even be possible to improve to \( A = 3/4 \) once the relevant sup-norm result becomes available, see the discussion in \([36, \S 1.3] \).

Some immediate consequences are as follows.

**Corollary 1.4.** If \( F = \mathbb{Q} \), then we have for any \( \epsilon > 0 \)

\[
\left| L\left(\frac{1}{2}, \pi\right) \right| \ll_F, \epsilon C(\pi)^{\frac{1}{2} + \epsilon} \left( \frac{C(\pi)}{C(\omega)} \right)^{-\frac{1-2\theta}{12+16A}} C(\omega)^{-\frac{\delta'}{12+16A}}.
\]

**Proof.** Over \( \mathbb{Q} \), the assumption \([1, 2]\) together with \( \delta' = 1/8, A = 1/2 \) is admissible by \([3, \text{Theorem 2}]\). \( \Box \)

\(^1\)Such potential applications were communicated to the author by Prof. Soumya and Dr. Andersen.
Remark 1.5. For the case $\text{F} = \mathbb{Q}$, there is a recent uniform bound due to Blomer and Khan [10]. Although their result seems to give better bound in some aspects, its effectiveness depends on the unspecified polynomial dependence of the usual conductor in Ivić’s bound [24, Corollary 2]. In any case, our bound is valid over any number field.

Corollary 1.6. If the central character $\omega$ of $\pi$ is fixed, then we have for any $\epsilon > 0$

$$\left| L\left(\frac{1}{2}, \pi\right) \right| \ll_{\text{F}, \epsilon} C(\pi)^{\frac{1}{2} - \frac{1-2\theta}{2\pi^2} + \epsilon}. $$

Proof. The convex bound ($A = 1/4, \delta' = 0$) is in any case valid for [17].

1.2. Geometric Intuition of the Method: Recall and Adaptation. We recall the geometric intuition of the method, which imitates the description given just after [32, Proposition 4.1]. We adapt it using our extension of regularized integral [34, §2].

For simplicity, we assume $\text{F} = \mathbb{Q}$, the central character $\omega$ of $\pi$ remains trivial, $\pi_{\infty}$ remains spherical and the usual conductor $C(\pi_{\infty}) = p$ is a large varying prime. Recall the standard notations

$$\Gamma_0(p) := \left\{ \begin{array}{lcl} (a & b & \gamma) & \in & \text{GL}_2(\mathbb{Z}) \mid p \mid c \end{array} \right\}, \quad Y(p) := \Gamma_0(p) \backslash \text{PGL}_2(\mathbb{R}) / \text{PSO}_2(\mathbb{R}) = \Gamma_0(p) \backslash \mathbb{H}. $$

A ($L^2$-normalized) new form in $\pi$ can be regarded as a function $\varphi$ on $Y(p)$. Let

$$E(s, z) := \sum_{\gamma \in \Gamma_0(1) \backslash \Gamma_0(1)} \Theta(\gamma, z)^s, \quad E^*(s, z) := \Lambda(2s)E(s, z)$$

be the standard spherical analytic Eisenstein series and its completion, where $\Lambda(s)$ is the complete Riemann zeta function. The following integral represents $L(1/2, \pi)^2$

$$I(\varphi, p) := \int_{Y(p)} \varphi(z)E^*(1/2, z)E^*(1/2, pz) d\mu(z), \quad d\mu(z) := \frac{dx dy}{y^2}, z = x + iy$$

It turns out that the product of the local terms of $I(\varphi, p)$ compensate the convex bound. It suffices to bound $I(\varphi, p)$. We regard $Y(p)$ as the graph of the $p$-th Hecke correspondence via

$$Y(p) \to Y(1) \times Y(1), \quad z \mapsto (z, pz), $$

thus the function

$$\phi_p(z) := E^*(1/2, z)E^*(1/2, pz)$$

can be regarded as the restriction to $Y(p)$ of the fixed function on $Y(1) \times Y(1)$

$$\phi(z_1, z_2) := E^*(1/2, z_1)E^*(1/2, z_2).$$

$I(\varphi, p)$ is thus expected to be bounded as

$$|I(\varphi, p)|^2 \leq \int_{Y(p)} |\varphi(z)|^2 d\mu(z) \cdot \int_{Y(p)} |\phi_p(z)|^2 d\mu(z) \to \int_{Y(1) \times Y(1)} |\phi(z_1, z_2)|^2 d\mu(z_1) d\mu(z_2),$$

and one shall apply the method of amplification to deal with the non-decreasing limit. This argument has a technical name, namely $\phi_p(z)$ is not $L^2$-integrable.

However, $\phi_p$ is finitely regularizable in the sense of [39, Definition 2.14]. In order not to let the complication of multiple cusps of $Y(p)$ obscure the idea, we pretend $p = 1$. The skeptic reader is invited to gain the necessary information of rigorous computation from [38, §6]. We thus propose to regularize

$$I(\varphi, 1) = \int_{Y(1)} \varphi(z)\phi_1(z) d\mu(z), \quad \phi_1(z) := E^*(1/2, z)^2. $$

The essential constant term [39, Definition 2.14] of $\phi_1(z)$ is equal to

$$\phi_1^*N(z) = E_N^*(1/2, z)^2 = 4\gamma^2y + 4\gamma\Lambda^*y log y + (\Lambda^*)^2y(log y)^2,$$

where the constants $\gamma, \Lambda^*$ appear in the Laurent expansion

$$\Lambda(s) = \frac{\Lambda^*}{s-1} + \gamma + O((s-1)),$$
We thus need to take the $L^2$-residue \[^{[39}, \text{Definition 2.20}\] of $\phi_1$ as
\[
E_1 = E(\phi_1) = 4\gamma_2^2E_{\text{reg}}(1, z) + 4\gamma_4^2\Lambda^sE_{\text{reg}}(1, z) + (\Lambda^s)^2E_{\text{reg}}(2)(1, z),
\]
where the regularizing Eisenstein series and its derivatives are defined as
\[
E_{\text{reg}}(s, z) := E(s, z) - \frac{3}{\pi(s-1)}, \quad E_{\text{reg}}^{(n)}(1, z) := \frac{d^n}{ds^n}|_{s=1}E_{\text{reg}}(s, z).
\]
Choose a small prime $p_0 \ll \log p$, and let $T_0(1)$ be the level one normalized Hecke operator in Definition \[^{2.1}\]. We have
\[
T_0(1)E(s, z) = \lambda_0(s)E(s, z), \quad \lambda_0(s) = \frac{p_0^{s-1/2} + p_0^{1/2-s}}{p_0^{1/2} + p_0^{-1/2}}.
\]
Writing $\lambda_k^{(k)}(s)$ for the $k$-th derivative of $\lambda_0(s)$ and making $n$-th derivative on both sides, we get
\[
T_0(1)E_{\text{reg}}^{(n)}(s, z) = \lambda_0(s)E_{\text{reg}}^{(n)}(s, z) + \sum_{k=1}^{n} \binom{n}{k} \lambda_k^{(k)}(s)E_{\text{reg}}^{(n-k)}(s, z) + \frac{3}{\pi} \frac{d^n}{ds^n} \left( \frac{\lambda_0(s) - 1}{s-1} \right).
\]
Thus in the space spanned by $1, E_{\text{reg}}(1, z), E_{\text{reg}}^{(1)}(1, z), \ldots, E_{\text{reg}}^{(n)}(1, z)$, the operator $T_0(1)$ corresponds to a unipotent matrix with diagonal entries constant equal to $\lambda_0(1) = 1$. In particular, we deduce\[^{2}\]
\[
(T_0(1) - 1)^{n+2}E_{\text{reg}}^{(n)}(1, z) = 0, \quad \Rightarrow \quad (T_0(1) - 1)^4E_1 = 0.
\]
It follows that
\[
|I(\varphi, 1)| = \frac{1}{|\lambda_\varphi(p_0) - 1|^4} \left| \int_{Y(1)} (T_0(1) - 1)^4\varphi(z) \cdot \phi_1(z) d\mu(z) \right| = \frac{1}{|\lambda_\varphi(p_0) - 1|^4} \left| \int_{Y(1)} \varphi(z) \cdot (T_0(1) - 1)^4\phi_1(z) d\mu(z) \right| \\
\leq \frac{1}{|\lambda_\varphi(p_0) - 1|^4} \left| \int_{Y(1)} |\varphi(z)|^2 d\mu(z) \right|^{'2} \cdot \left( \int_{Y(1)} |(T_0(1) - 1)^4\phi_1(z)|^2 d\mu(z) \right)^{'2},
\]
where $\lambda_\varphi(p_0)$ is the eigenvalue of $\varphi$ for $T_0(1)$. Since we have a non-trivial estimate of the constant $\theta \leq 7/64$ towards the Ramanujan-Petersson conjecture \[^{[4, 22]}\], $|\lambda_\varphi(p_0) - 1|^4$ is bounded from above and below by some absolute constants. The function
\[
(T_0(1) - 1)^4\phi_1(z)
\]
is now of rapid decay, hence a fortiori $L^2$-integrable. We then apply the Plancherel formula for the $L^2$-norm of the above function (with amplification) to get a non trivial bound of $I(\varphi, 1)$.

1.3. Organization of the Paper. As \[^{[30]}\], instead of a linear exhibition according to the logical order, we decide to regroup the ingredients according to their natures. Each “proof” of a global result in the proof of Theorem \[^{1.1}\] serves as a pointer to the relevant global or local results at a more fundamental level. In fact, the number of period formulas contained in the current paper is much more than those (even the sum of) of our previous works \[^{[35, 36, 38]}\]. The transitions between local and global computations occur so often that it is too difficult to write down the argument in a linear logical way. We can only encourage the reader, who really want to understand every detail of the proof, to linearize the argument by him/herself. Moreover, the current regroupment of arguments has the advantage to facilitate the possible future improvements if one seeks a better test function in our method.

Precisely, we will fix the notations & conventions, set up the precise measure/operator of regularization and amplification in §2.1. We then recall our extension of the theory of regularized integrals as well as its first development to triple product case in §2.2. After these preparations, we give a formal proof of the main result in §2.3, reducing/pointing the task to the relevant local and global estimations scattered in §3 and §4. This part is the adelization of the description given above in §1.2 in the general case and

\[^{2}\]In a simpler way, we have $(T_0(1) - 1)^4\phi_1^*|_{N^2} = 0$, which immediately implies the rapid decay of $(T_0(1) - 1)^4\phi_1$.\[\]
makes that subsection rigorous. Since a big number of different triple product periods come into play in this paper, we standardize their Euler product decompositions in §2.4.

In §3, we recollect all the local estimations. They serve either directly for the “compensation of convex bound” in §2.3, or for the estimation of global periods in §4. Along the way, we also specify the test functions via their local data in the Kirillov model.

§4 contains all the relevant global estimations. Note that many proofs given there are again pointers to the local estimations given in §3.

In §5, we give some technical complements, which seem to be useful for the analytic theory of automorphic representation in general.

Once again, this paper is NOT organized linearly. For the first reading, we highly recommend the following order of “linearisation”:

1. §2.3 with “return jumps” to §3 indicated by pointers;
2. §4 with “return jumps” to §3 indicated by pointers.

§2.1 and §2.4 should be consulted constantly whenever a notation or convention is not clear. A linear reading of §3 would make sense only for a second reading when the reader gets sufficiently familiar with the global steps.

2. Preliminaries and First Reductions

2.1. Setup. A list of basic notations and conventions is given as follows.

- \( \textbf{F} \): base number field with ring of adeles \( \mathbb{A} \), recall \( \lambda_\mathbb{F}(s) := \Lambda_\mathbb{F}(-2s)/\Lambda_\mathbb{F}(2+2s) \) \([39, (2.2)]\) and \( \Lambda_\mathbb{F}(s) \) is the complete Dedekind zeta function;
- generally, \( L(\cdot) \) denotes \( L \)-functions without factors at infinity. \( \Lambda(\cdot) \) denotes complete \( L \)-functions;
- \( \pi \): varying cuspidal representation of \( \text{GL}_2(\mathbb{A}) \);
- \( \omega \): central character of \( \pi \), varying with \( \pi \);
- for \( f \in \pi(\chi_1, \chi_2) \) in the induced model with \( \chi_1, \chi_2 \) unitary Hecke characters of \( \mathbb{F}^\times \backslash \mathbb{A}^\times \), denote \( E^*(s,f) = \Lambda(1+2s,\chi_1 \chi_2^{-1})E(s,f) \) resp. \( E^2(s,f) = L(1 + 2s, \chi_1 \chi_2^{-1})E(s,f) \);
- Whittaker functions are taken with respect to the fixed standard additive character \( \psi \) or \( \psi_v \) à la Tate.

For other notations, we import those in \([35, \S 2.1]\), with the following differences or emphasis:

1. The number field is written in bold character \( \textbf{F} \), with ring of algebraic integers \( \mathfrak{o} \). \( v \) denotes a place of \( \textbf{F} \). If \( v < \infty \) is finite, we usually write \( v = p \), which is identified with a prime ideal \( p \) of \( \mathfrak{o} \). A uniformizer in \( \mathfrak{o}_p \) is written as \( \varpi_p \).
2. We write the algebraic groups defined over \( \textbf{F} \) in bold characters such as \( \textbf{G}, \textbf{N}, \textbf{B}, \textbf{Z} \) etc, where \( \textbf{G} = \text{GL}_2 \), \( \textbf{B} \) is the upper triangular subgroup of \( \textbf{G} \), \( \textbf{N} \triangleleft \textbf{B} \) is the unipotent upper triangular subgroup, and \( \textbf{Z} \) is the center of \( \textbf{G} \).
3. \( \textbf{K} = \prod_v \textbf{K}_v \) is the standard maximal compact subgroup of \( \text{GL}_2(\mathbb{A}) \), i.e.

\[
\textbf{K}_v = \begin{cases} 
\text{SO}_2(\mathbb{R}) & \text{if } \textbf{F}_v = \mathbb{R} \\
\text{SU}_2(\mathbb{C}) & \text{if } \textbf{F}_v = \mathbb{C} \\
\text{GL}_2(\mathfrak{o}_p) & \text{if } v = p < \infty
\end{cases}
\]

4. In \( \text{GL}_2 \), for local or global variables \( x \in \textbf{F}_v \) or \( \mathbb{A} \), \( y \in \textbf{F}_v^\times \) or \( \mathbb{A}^\times \), we write

\[
n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a(y) = \begin{pmatrix} y \\ 0 \end{pmatrix}.
\]

5. We use the abbreviation

\[
[\text{GL}_2] = \text{GL}_2(\textbf{F})Z(\mathbb{A})\backslash \text{GL}_2(\mathbb{A}) = [\text{PGL}_2].
\]

Due to the varying central character \( \omega \), we need to choose the measure formed by Hecke operators which regularizes the product of Eisenstein series in a way different from the one described in \([1.2]\) Precisely,
choose a finite place \( p_0 \) at which \( F, \pi \) are unramified. Write \( \varpi_0 = \varpi_{p_0}, q_0 = q_{p_0}, K_0 = K_{p_0}, \omega_0 = \omega_{p_0} \) for simplicity and denote \( \alpha_0 = \omega_{p_0}(\varpi_0) \). We can assume
\[
(2.1) \quad q_0 \ll \log C(\pi).
\]

**Definition 2.1.** Define the Hecke operators for \( n \in \mathbb{N} \)
\[
T_0(n) := \int_{K_0^2} \kappa_1 a(\varpi^n) \kappa_2 d\kappa_1 d\kappa_2.
\]

\( T_0(1) \) acts on the spherical vector in \( \pi(|q_0^{1/2+s}|^{-1} \cdot |q_0^{-1}|^{-1/2+s}) \) resp. \( \pi(|q_0^{-1} q_0^{1/2+s}|) \) as multiplication by
\[
\lambda_0(s) := \frac{\frac{q_0^{-1/2}}{1 + q_0^{-1}} \left( q_0^{-(1/2+s)} \right) \alpha_0^{-1} q_0^{1/2+s}}{\frac{q_0^{-1/2}}{1 + q_0^{-1}} \left( \alpha_0^{-1} q_0^{-(1/2+s)} + q_0^{1/2+s} \right)}, \quad \text{resp.} \quad \tilde{\lambda}_0(s) := \frac{\frac{q_0^{-1/2}}{1 + q_0^{-1}} \left( \alpha_0^{-1} q_0^{(1/2+s)} \right)}{\frac{q_0^{-1/2}}{1 + q_0^{-1}} \left( \alpha_0^{-1} q_0^{-(1/2+s)} + q_0^{1/2+s} \right)}.
\]

Define the operator/measure of regularization
\[
\sigma_0 := (T_0(1) - \lambda_0(0))^2(T_0(1) - \tilde{\lambda}_0(0))^2.
\]

Finally, we construct the amplifiers as follows. For some \( K > 0 \) to be optimized later, let
\[
(2.2) \quad S = S(K) = \{ p : p \neq p_0, K < \text{Nr}(p) \leq 2K, \pi \text{ and } F \text{ are unramified at } p \}, \quad S^* = S \cup \{p_0\}.
\]

For \( p \in S \), let \( \lambda_\pi(p^n) \) be the eigenvalue of the \( n \)-th Hecke operator \( T(p^n) \) on the spherical vector in \( \pi_p \), and define an operator/measure (amplifier)
\[
(2.3) \quad \sigma = \sigma(S, \pi) = \sum_{p \in S} a_p T(p^{n_p}),
\]
where \( n_p \in \{1, 2\}, |a_p| = 1 \) are such that
\[
|\lambda_\pi(p^{n_p})| = \max(|\lambda_\pi(p)|, |\lambda_\pi(p^2)|) \Rightarrow \sum_{p \in S} |\lambda_\pi(p^{n_p})| \gg |S| \gg K^{1-\epsilon},
\]
\[
a_p = \text{sgn}(\lambda_\pi(p^{n_p})) = \frac{\lambda_\pi(p^{n_p})}{|\lambda_\pi(p^{n_p})|}.
\]

#### 2.2. Extension of Zagier’s Regularized Integral

This subsection is a recall and summarize our extension of the theory of regularized integrals in [33 §5 & §6] without proofs. The first part is identical with [33 §2.1]. This extension fits well in the context of the Rankin-Selberg trace formula. It could not be well understood in the framework of the subconvexity problem. Hence we encourage the interested reader to read [33 §2 & §3] as well as its preliminary extension to triple product case in [33 §2.2] for a better understanding. We include this subsection to make the current paper self-contained.

We begin with the recall on the following space of functions on the automorphic quotient of \( \text{GL}_2 \) over a general number field \( F \) with the ring of adeles \( \mathbb{A} \).

**Definition 2.2.** ([33, Definition 2.14]) Let \( \omega \) be a unitary character of \( F^x \backslash \mathbb{A}^x \). Let \( \varphi \) be a smooth function on \( \text{GL}_2(F) \backslash \text{GL}_2(\mathbb{A}) \) with central character \( \omega \). We call \( \varphi \) finitely regularizable if there exist characters \( \chi_i : F^x \backslash \mathbb{A}^x \to \mathbb{C}(1) \), \( \alpha_i \in \mathbb{C}, n_i \in \mathbb{N} \) and smooth functions \( f_i \in \text{Ind}_B^{K}(\text{GL}_2(\mathbb{A}))(\chi_i, \omega^{n_i}) \) for \( 1 \leq i \leq l \), such that
\[
(1) \quad \text{for any } M \gg 1 \quad \varphi(n(x)a(y))k = \varphi_N(n(x)a(y))k + O(|y|^{-M}) \quad \text{as } |y|_\mathbb{A} \to \infty,
\]
\[
(2) \quad \text{we can differentiate the above equality with respect to the universal enveloping algebra of the lie algebra of } \text{GL}_2(\mathbb{A}_\infty).
\]
Here we have written/defined the essential constant term

\[ \varphi_N(n(x)a(y)k) = \varphi_N(a(y)k) = \sum_{i=1}^{l} \chi_i(y) |y|_{\lambda}^{\frac{1}{2} + \alpha_i} \log^{\alpha_i} |y|_{\lambda} \cdot f_i(k). \]

In this case, we call \( E(x)(\varphi) = \{ \chi_i|^{1/2+\alpha_i} : 1 \leq i \leq l \} \) the exponent set of \( \varphi \), and define

\[ E^+(\varphi) = \{ \chi_i|^{1/2+\alpha_i} \in E(x)(\varphi) : \Re \alpha_i \geq 0 \}; \quad E^-(\varphi) = \{ \chi_i|^{1/2+\alpha_i} \in E(x)(\varphi) : \Re \alpha_i \leq 0 \}. \]

The space of finitely regularizable functions with central character \( \omega \) is denoted by \( A^f(\GL_2, \omega) \).

Obviously \( A^f(\GL_2, \omega) \) is stable under the right regular translation of \( \GL_2(\A) \) and contains the Schwartz space with central character \( \omega \), hence the space of smooth cusp forms. It also contains any finite product of Eisenstein series (39, Remark 2.19)). In the case \( \omega = 1 \) and for any \( \varphi \in A^f(\GL_2, 1) \), the integral

\[ R(s, \varphi) := \int_{A^1 \times K} (\varphi_N - \varphi_N^*)(a(y)k)|y|_{\lambda}^{s-1/2} d^\times ydk \]

is convergent for any \( s \in \C \). We use it to define the regularized integral as

\[ A^f(\GL_2, 1) \to \C, \quad \varphi \mapsto \int_{[\PGL_2]} \varphi(g)dg := \frac{1}{\text{Vol}([\PGL_2])} \left( \text{Res}_{s=1/2} R(s, \varphi) + \sum_{n_0=1}^{\infty} f_i(1) \right). \]

If \( f \in \text{Ind}_{B(\A)}^{\GL_2(\A)}(\chi_1, \chi_2) \) such that \( \chi_1\chi_2^{-1} = |^{1/2} \mu \) for some \( \mu \in \R \), we introduce the regularizing Eisenstein series as (39, Definition 2.16)

\[ E^{reg}(s, f)(g) = E(s, f)(g) - \frac{\Lambda_F(1 - 2s - i\mu_j)}{\Lambda_F(1 + 2s + i\mu_j)} \int_K f(k)dk \cdot \chi_1^{-1}(\det g)|^{1/2} \mu \].

For any \( \varphi \in A^f(\GL_2, \omega) \) with auxiliary data given in Definition 2.2 we define (39, (2.3))

\[ E(\varphi) = \sum \frac{\partial^{\nu_j}}{\partial s^{\nu_j}} E(\alpha_j, f_j) + \sum \frac{\partial^{\nu_j}}{\partial s^{\nu_j}} E^{reg}(\alpha_j, f_j), \]

where \( \mu_j \in \R \) is defined only if \( \omega^{-1} \chi_2^2(y) = |^{1/2} \mu \). This defines a linear map

\[ A^f(\GL_2, \omega) \to A^f(\GL_2, \omega), \quad \varphi \mapsto E(\varphi), \]

such that \( \varphi - E(\varphi) \in L^1(\GL_2, \omega) \), which is \( \GL_2(\A) \)-intertwining when \( E(x)(\varphi) \) does not contain \( |^{1/2} \). We denote the image by \( E(\GL_2, \omega) \). Moreover if \( E^+(\varphi) \cap E^-(\varphi) = \emptyset \) then \( E(\varphi) \) is the unique element in \( E(\GL_2, \omega) \) such that \( \varphi - E(\varphi) \in L^2(\GL_2, \omega) \) (39, Proposition 2.25), and we call it the \( L^2 \)-residue of \( \varphi \) (39, Definition 2.26). In the case \( \omega = 1, A^f(\GL_2, 1) \) is in the range of applicability of the regularized integral and (39, Proposition 2.27)

\[ \int_{[\PGL_2]} \varphi(g)dg = \int_{[\PGL_2]} (\varphi(g) - E(\varphi)(g))dg. \]

In particular the above equation proves the \( \GL_2(\A) \)-invariance of the regularized integral as a functional on \( A^f(\GL_2, 1) \), when \( E(x)(\varphi) \) does not contain \( |^{1/2} \). In this case the above equality was originally due to Zagier (40). We carefully generalized in (39, Theorem 2.12 & Definition 2.13) this theory into the adelic setting and proved the above equality without constraint on \( E(x)(\varphi) \).

In view of the inclusion (39, Remark 2.19)

\[ A^f(\GL_2, \omega_1) \cdot A^f(\GL_2, \omega_2) \subset A^f(\GL_2, \omega_1\omega_2), \]

we can consider the following bilinear form. Let \( \pi_j, j = 1, 2 \) be two principal series representations with central character \( \omega_j \) satisfying \( \omega_1\omega_2 = 1 \). Let \( V_j \) be the vector space of \( \pi_j \) realized in the induced model from \( B(\A) \) with subspace of smooth vectors \( V_j^\infty \). We then get a \( \GL_2(\A) \)-invariant bilinear form

\[ V_1^\infty \times V_2^\infty \to \C, \quad (f_1, f_2) \mapsto \int_{[\PGL_2]} E(f_1)(g)E(f_2)(g)dg, \]

where \( E(f_j) \) should be suitably regularized if \( \pi_j \) is at a position which creates a pole/zero for the relevant Eisenstein series. We succeeded in (39, Theorem 3.5) to identify this bilinear form in the induced model.
In order to present the result, we need to introduce some extra notations. Precisely, if we identify for any $s \in \mathbb{C}$ the space of functions $\pi_s$ with $H$, where

$$\pi_s := \text{Ind}_{\text{B}}^{\text{GL}_{2}}(1, |x|^{-s} \cdot |y|^{-s})$$

and $H := \text{Ind}_{\text{B}}^{\text{GL}_{2}}(1)$, then we can regard the intertwining operator $M_s: \pi_s \to \pi_{-s}$ as a map from $H$ to itself. Using the flat section map $H \to H$, $f \mapsto f_{-s}$, we mean

$$(M_s f_s)(a(y)\kappa) := |y|^{-s} \cdot (M_s f)\kappa, \quad \text{i.e., } M_s f_s = (M_s f)_{-s}.$$ 

Let $c_0 \in H$ be the constant function taking value 1. Define

$$\text{P}_K: H \to \mathbb{C}, \quad f \mapsto \int_H f(\kappa) d\kappa,$$

where $d\kappa$ is the probability Haar measure on $K$. We obtain a map from $H$ to itself

$$\tilde{M}_s := M_s \circ (I - \text{P}_K c_0),$$

where $I$ is the identity map. Since $M_s$ is “diagonalizable”, we obtain the Taylor expansion as operators

$$M_s f = \sum_{n=0}^{\infty} s^n M_s^{(n)} f, \quad \text{resp.} \quad \tilde{M}_{1/2+s} f = \sum_{n=0}^{\infty} s^n \tilde{M}_{1/2}^{(n)} f.$$

**Theorem 2.3.** (33, Theorem 3.5) The regularized integral of the product of two unitary Eisenstein series is computed as:

1. If $\pi_1 = \pi(\xi_1, \xi_2)$, $\pi_2 = \pi(\xi_1^{-1}, \xi_2^{-1})$ resp. $\pi_2 = \pi(\xi_2^{-1}, \xi_1^{-1})$ and $\xi_1 \neq \xi_2$, then

$$\int_{[PGL_2]}^{\text{reg}} E(0, f_1) E(0, f_2) = \frac{2\lambda_{\text{P}}^{(0)}(0)}{\lambda_{\text{P}}^{(-1)}(0)} \text{P}_K(f_1 f_2) - \text{P}_K(M_0^{(1)} f_1 \cdot M_0 f_2),$$

resp.

$$\frac{\lambda_{\text{P}}^{(0)}(0)}{\lambda_{\text{P}}^{(-1)}(0)}(\text{P}_K(f_1 M_0 f_2) + \text{P}_K(f_2 M_0 f_1)) - \text{P}_K(M_0^{(1)} f_1 \cdot f_2).$$

2. If $\pi_1 = \pi(\xi, \xi)$, $\pi_2 = \pi(\xi^{-1}, \xi^{-1})$, then

$$\int_{[PGL_2]}^{\text{reg}} E^{(1)}(0, f_1) E^{(1)}(0, f_2) = \frac{4 \lambda_{\text{P}}^{(2)}(0)}{\lambda_{\text{P}}^{(-1)}(0)} \text{P}_K(f_1 f_2) + \frac{4 \lambda_{\text{P}}^{(2)}(0)}{\lambda_{\text{P}}^{(-1)}(0)} \text{P}_K(f_1 \cdot M_0^{(1)} f_2)$$

+ $\frac{\lambda_{\text{P}}^{(0)}(0)}{\lambda_{\text{P}}^{(-1)}(0)} \text{P}_K(M_0^{(1)} f_1 \cdot M_0^{(1)} f_2) - \frac{1}{3} \text{P}_K(M_0^{(3)} f_1 \cdot f_2) - \text{P}_K(M_0^{(2)} f_1 \cdot M_0^{(1)} f_2).$

Here we have written (33, (2.2))

$$\lambda_{\text{P}}(s) := \frac{\lambda_{\text{P}}(-2s)}{\lambda_{\text{P}}(2 + 2s)} = \frac{\lambda_{\text{P}}(-1)(0)}{s} + \sum_{n=0}^{\infty} \frac{s^n \lambda_{\text{P}}^{(n)}(0)}{n!}.$$ 

In 38, we have used and extended the above theory to a special case of regularized triple product of Eisenstein series. This will be further extended to other relevant cases in this paper in 5.3. For the moment, we simply record 38, Theorem 2.7 for the convenience of the reader.

**Theorem 2.4.** Let $f_j \in \pi(1, 1), j = 1, 2, 3$. Then for any $n \in \mathbb{N}$

$$\int_{[PGL_2]}^{\text{reg}} E^*(0, f_1) \cdot E^*(0, f_2) \cdot E^{\text{reg}, (n)}(\frac{1}{2}, f_3)$$

is the sum of

$$\left(\frac{\partial^n R}{\partial s^n}\right)^{\text{hol}} \left(\frac{1}{2} E^*(0, f_1) \cdot E^*(0, f_2); f_3\right)$$

and a weighted sum with coefficients depending only on $\lambda_{\text{P}}(s)$ of

$$\text{P}_K(M_0^{(l)} f_1 \cdot f_2) \text{P}_K(f_3), \quad 0 \leq l \leq 3;$$
Proposition 2.5. \[ \begin{align*}
P_K(f_1 \cdot f_2 \cdot \widehat{M}^{(4)}_{1/2} f_3), & \quad 0 \leq l \leq \max(2, n) \quad \& \quad l = n + 3; \\
P_K((f_1 M_0 f_2 + f_2 M_0 f_1) \cdot \widehat{M}^{(4)}_{1/2} f_3), & \quad 0 \leq l \leq \max(1, n) \quad \& \quad l = n + 2; \\
P_K(M_0 f_1 \cdot M_0 f_2 \cdot \widehat{M}^{(4)}_{1/2} f_3), & \quad 0 \leq l \leq n \quad \& \quad l = n + 1. \end{align*} \]

2.3. Proof of Main Result: First Reduction. Take \( \varphi \in \pi, f_2 \in \pi(1, 1) \) and \( f_3 \in \pi(1, \omega^{-1}) \), which will be specified in Section 3.1.1 & 3.2.1 and adjusted such that \( \| \varphi \| = 1. \)

**Proposition 2.5.** With our choice of test vectors and \( \ell_v \), defined in 2.4.6, we have

\[ \prod_v \ell_v(W_{\varphi, v}, W_{f_2, v}, f_{3, v}) \gg_{\epsilon} C(\pi)\beta^{-\frac{1}{2} - \epsilon}. \]

**Proof.** This follows from Lemma 3.1 and 3.9

Write \( E_2 = E^*(0, f_2) \) resp. \( E_3^s = E^s(0, f_3) \).

**Lemma 2.6.** We have some basic properties concerning \( \sigma_0 \).

1. There exist rational functions \( h_k \in \mathbb{Q}(Y)[X] \subset \mathbb{Q}(X, Y) \) such that

\[
\sigma_0 = \sum_{k=0}^{4} h_k(\alpha_0, q_0^{1/2})T_0(k).
\]

2. \( \varphi \) is an eigenvector of the dual operator \( \sigma_0^\vee = (\alpha_0^{-1}T_0(1) - \lambda_0(0))^2(\alpha_0^{-1}T_0(1) - \tilde{\lambda}_0(0))^2 \) with eigenvalue \( R_0 \) satisfying \( |R_0| \approx 1 \).

3. \( \sigma_0(E_2^s E_3^s) \in L^2(\text{GL}_2, \omega^{-1}) \).

**Proof.** (1) follows from basic relations among the Hecke operators, which are summarized formally as

\[
\left( \sum_{n=0}^{\infty} T_0(n)X^n \right) \cdot \left( 1 - q_0^{1/2}(1 + q_0^{-1})T_0(1)X + \alpha_0 X^2 \right) = q_0^{-1/2} - q_0^{-1}T_0(1)X.
\]

(2) follows trivially from MacDonald’s formula [11, Theorem 4.6.6]. For (3), we only treat the case \( \omega \neq 1 \). Note that \( \sigma_0 \) annihilates

\[
(E_2^s \cdot E_3^s)(a(y)) = \frac{\lambda_0}{\alpha_0} L(1, \omega) \cdot \left\{ \frac{\alpha_0}{\lambda_0} y \log|y|_{\omega} f_3(\kappa) + 2^{-1}\lambda_0^{(1)}(-1/2)|y|_{\omega} f_3(\kappa) \right.
\]

\[ + \omega(y)|y|_{\omega} \log|y|_{\omega} M_0 f_3(\kappa) + 2^{-1}\lambda_0^{(1)}(-1/2)|y|_{\omega} y|_{\omega} M_0 f_3(\kappa) \right\}, \]

since we have, if we denote by \( f_\chi \) resp. \( \hat{f}_\chi \) a flat spherical section in \( \pi(|p_0|^{1/2 + s}, \omega^{-1}q_0^{-1/2 + s}) \) resp. \( \sigma_0(\chi, \omega^{-1}q_0^{-1/2 + s}) \) and \( f_0 := (\partial f_\chi/\partial s) |_{s=0}, \)

\[
T_0(1)(f_0, f_0') = (f_0, f_0') \left( \frac{\lambda_0(0)}{\alpha_0}, \frac{\chi_0(0)}{\lambda_0(0)} \right) \quad \text{resp.} \quad T_0(1)(\hat{f}_0, \hat{f}_0') = (\hat{f}_0, \hat{f}_0') \left( \frac{\lambda_0(0)}{\alpha_0}, \frac{\chi_0(0)}{\lambda_0(0)} \right).
\]

Hence \( \sigma_0(E_2^s E_3^s) \) is finitely regularizable [39, Definition 2.14] with essential constant term equal to 0, thus of rapid decay, a fortiori square integrable.

By 2.11 and Proposition 2.5, we are reduced to bounding

\[
\int_{[PGL_2]} \varphi \cdot E_2^s \cdot E_3^s = \left( \frac{R_0}{S} \sum_{p \in S} |\lambda_p(p^{n_p})| \right)^{-1} \int_{[PGL_2]} \sigma_0^\vee \sigma(\varphi) \cdot E_2^s \cdot E_3^s
\]

\[ \approx \left( \frac{R_0}{S} \sum_{p \in S} |\lambda_p(p^{n_p})| \right)^{-1} \int_{[PGL_2]} \varphi \cdot \sum_{p \in S} a_p a(\varphi^{-n_p}) \cdot \sigma_0(E_2^s E_3^s).
\]
We apply C-S and unfold the square as

\[ \left( \int_{[\mathrm{PGL}_2]} \varphi \cdot \sum_{p \in S} a_p(\varpi_{p}^{-n_{p}}) \cdot \sigma_0(E_{p}^{1}E_{3}^{1}) \right)^2 \leq \left( \int_{[\mathrm{PGL}_2]} \sum_{p \in S} a_p(\varpi_{p}^{-n_{p}}) \cdot \sigma_0(E_{p}^{1}E_{3}^{1}) \right)^2 \]

\[ = \left( \sum_{p_1, p_2 \in S} a_{p_1}a_{p_2} \int_{[\mathrm{PGL}_2]} a(\varpi_{p_1}^{-n_{p_1}}, \varpi_{p_2}^{-n_{p_2}}) \cdot \sigma_0(E_{p_1}^{1}E_{3}^{1}) \right)^2 \]

\[ = \sum_{k_1, k_2 = 0}^{4} \sum_{p_1, p_2 \in S} h_{k_1}h_{k_2}a_{p_1}a_{p_2} \int_{[\mathrm{PGL}_2]} a(\varpi_{p_1}^{-k_1}, \varpi_{p_2}^{-n_{p_2}}) \cdot \sigma_0(E_{p_1}^{1}E_{3}^{1}) \cdot \sigma_0(E_{p_2}^{1}E_{3}^{1}), \]

where we have applied \[34\], Proposition 2.27 (2)] in the last step. Writing \( \bar{t} = \varpi_{0}^{-k_1} \varpi_{p_1}^{-n_{p_1}} \varpi_{p_2}^{-n_{p_2}} \), we re-arrange the last integral as

\[ \int_{[\mathrm{PGL}_2]} a(\tilde{t}).(E_{2}^{1}E_{3}^{1}) \cdot \sigma_0(E_{3}^{1}E_{3}^{1}) \]

\[ = \int_{[\mathrm{PGL}_2]} (a(\tilde{t}).E_{2}^{1} \cdot \sigma_0(E_{3}^{1}E_{3}^{1})) \cdot (a(\tilde{t}).E_{3}^{1} \cdot \sigma_0(E_{3}^{1}E_{3}^{1})) \]

\[ + \int_{[\mathrm{PGL}_2]} \int_{[\mathrm{PGL}_2]} a(\tilde{t}).E_{2}^{1} \cdot \sigma_0(E_{3}^{1}E_{3}^{1}) \cdot \sigma_0(a(\tilde{t}).E_{3}^{1}E_{3}^{1} \cdot \sigma_0(E_{3}^{1}E_{3}^{1})) \]

\[ - \int_{[\mathrm{PGL}_2]} \sigma_0(a(\tilde{t}).E_{2}^{1}E_{3}^{1} \cdot \sigma_0(E_{3}^{1}E_{3}^{1})). \]

**Definition 2.7.** For \( \tilde{t} \) as above, we define

\[ ||\tilde{t}|| = \begin{cases} n_{p_1} + n_{p_2} & \text{if } p_1 \neq p_2, \\ 0 & \text{if } p_1 = p_2; \end{cases} \]

\[ n(\tilde{t}) = (m_{p})_{p < \infty} \text{ with } m_{p} = \begin{cases} |k_1 - k_2| & \text{if } p = p_0, \\ -n_{p_1} & \text{if } p = p_1, \\ n_{p_2} & \text{if } p = p_2, \\ 0 & \text{otherwise}. \end{cases} \]

For \( \pi' \) an automorphic representation, we write \( \pi' \leq \tilde{t} \) if \( \epsilon(\pi'_{p_j}) \leq n_{p_j} \), for \( j = 0, 1, 2 \).

**Remark 2.8.** We will refer to \[2.7\] resp. \[2.9\] resp. \[2.10\] as the regular term resp. regularized term resp. degenerate term. The effect of the measure of regularization and the one of amplification will be treated in the same way, explaining why we put them together into \( \tilde{t} \). The smallness of \( q_0 \) \[2.4\], implying that any power of it is \( \ll \epsilon \), hence negligible, allows us to basically ignore the contribution at \( p_0 \) in the estimation.

In the sum \[2.7\], we call the terms with \( ||\tilde{t}|| = 0 \) the **diagonal terms**, while the ones with \( ||\tilde{t}|| \neq 0 \) the **off-diagonal terms**. The number of diagonal terms is \( O(|S|) \). The number of off-diagonal terms is \( O(|S|^2) \). By Lemma \[4.1\] \[3.2\] \[1.4\] and \[1.5\] their contribution to \[2.6\] is bounded by

\[ (C(\pi)K)^{\epsilon} \cdot \max \left( C(\pi)C(\omega)^{\frac{2}{3} \frac{1}{\theta} + \frac{2}{3}} K^{2(A + \frac{2}{3})}, C(\pi)C(\omega)^{-\frac{2}{3} \frac{1}{\theta} - \frac{2}{3}} K^{-1 - \frac{2}{3}}, K^{-\frac{2}{3}} \right). \]

Assuming \( \delta' \leq (1 - 2\theta)/8 \) and \( A \geq 1/4 \), we deduce that \[2.6\] is bounded by

\[ \left( C(\pi)C(\omega)^{\frac{1}{12 + 16A}} \right)^{\frac{1 - 2\theta}{6 + 8A}} C(\omega)^{-\frac{2}{3} \frac{1}{\theta} \frac{1}{\theta + \frac{1}{2}}}, \text{ with } K = \left( C(\pi)C(\omega)^{\frac{1 - 2\theta}{6 + 8A}} C(\omega)^{-\frac{2}{3} \frac{1}{\theta} \frac{1}{\theta + \frac{1}{2}}}. \right. \]

2.4. **Explicit Decomposition of Periods.**
2.4.1. On $\pi \times \pi(1, 1) \times \pi(1, \omega^{-1})$. For $\varphi \in \pi$, $f_2 \in \pi(1, 1)$, $f_3 \in \pi(1, \omega^{-1})$, we take the explicit decomposition of period:

$$
(2.11) \quad \int_{[\text{PGL}_2]} \varphi \cdot E^\pi(0, f_2) \cdot E^\pi(0, f_3) = L \left( \frac{1}{2}, \pi \right)^2 \cdot \prod_v \ell_v(W_{\varphi,v}, f_{2,v}, f_{3,v})
$$

where the local trilinear forms are defined by

$$
\ell_v(\cdots) = \int_{F_v^\times \times K_v} W_{\varphi_v}(a(y)\kappa)W_{f_{2,v}}(a(-y)\kappa)f_{3,v}(\kappa)|y_v|^{-\frac{1}{2}} d^v y d\kappa, \quad v \mid \infty;
$$

$$
\ell_p(\cdots) = \frac{L(1, \omega_p)}{L(1/2, \pi_p)} \int_{F_p^\times \times K_p} W_{\varphi_p}(a(y)\kappa)W_{f_{2,p}}(a(-y)\kappa)f_{3,p}(\kappa)|y_p|^{-\frac{1}{2}} d^p y d\kappa, \quad p < \infty;
$$

so that $\ell_p = 1$ for all but finitely many $p$.

2.4.2. On $\pi' \times \bar{\pi}(1, \omega^{-1}) \times \pi(1, \omega^{-1})$. Let $\pi'$ be a cuspidal representation of $\text{PGL}_2$. For $\varphi \in \pi'$, $f_3, f'_3 \in \pi(1, \omega^{-1})$, we take the explicit decomposition of period:

$$
(2.12) \quad \int_{[\text{PGL}_2]} \varphi \cdot E^\pi(0, f_3) \cdot E^\pi(0, f'_3) = L \left( \frac{1}{2}, \pi' \otimes \omega \right) \cdot \prod_v \ell_v(W_{\varphi,v}, f_{3,v}, f'_{3,v})
$$

where the local trilinear forms are defined by

$$
\ell_v(\cdots) = \int_{F_v^\times \times K_v} W_{\varphi_v}(a(y)\kappa)W_{f_{3,v}}(a(y)\kappa)f'_{3,v}(\kappa)|y_v|^{-\frac{1}{2}} d^v y d\kappa, \quad v \mid \infty;
$$

$$
\ell_p(\cdots) = \frac{L(1, \omega_p)}{L(1/2, \pi_p)\cdot L(1/2, \pi_p \otimes \omega_p)} \int_{F_p^\times \times K_p} W_{\varphi_p}(a(y)\kappa)W_{f_{3,p}}(a(y)\kappa)f'_{3,p}(\kappa)|y_p|^{-\frac{1}{2}} d^p y d\kappa, \quad p < \infty;
$$

so that $\ell_p = 1$ for all but finitely many $p$.

2.4.3. On $\pi(\xi|_\A^{1}, \xi|_\A^{-s}) \times \pi(1, \omega^{-1}) \times \pi(1, \omega^{-1})$. Let $\xi$ be a character of $F^\times \backslash \A^{1}$, trivially extended to a Hecke character. Let $\Phi \in \pi(\xi, \xi^{-1})$, to which is associated a flat section $\Phi_s \in \pi(\xi|_\A^{1}, \xi|_\A^{-s})$. For $f_3, f'_3 \in \pi(1, \omega^{-1})$ and $\Re s \in (-1/2, 1/2)$, we have by [38, Proposition 2.5]

$$
(2.13) \quad \int_{[\text{PGL}_2]} E(s, \Phi) \cdot \left( E^\pi(0, f_3) \cdot E^\pi(0, f'_3) - E^\pi(0, f_3)E^\pi(0, f'_3) \right) = L(\frac{1}{2} + s, \xi^2) L(\frac{1}{2} + s, \xi \omega) \prod_v \ell_v(s; \Phi_v, f_{3,v}, f'_{3,v})
$$

where the local factors are defined by

$$
\ell_v(\cdots) = \int_{F_v^\times \times K_v} W_{f_{3,v}}(a(y)\kappa)\Phi_v(\kappa)\xi_v(y)|y_v|^{-\frac{1}{2}} d^v y d\kappa, \quad v \mid \infty;
$$

$$
\ell_p(\cdots) = \frac{L(1 + 2s, \xi_p^2)}{L(\frac{1}{2} + s, \xi_p \omega_p)\cdot L(\frac{1}{2} + s, \xi_p \omega^{-1}_p)} \int_{F_p^\times \times K_p} W_{f_{3,p}}(a(y)\kappa)\Phi_p(\kappa)\xi_p(y)|y_p|^{-\frac{1}{2}} d^p y d\kappa, \quad p < \infty;
$$

so that $\ell_p = 1$ for all but finitely many $p$. As a $\text{GL}_2(F_v)$-invariant trilinear form, $\ell_v$ is not always convenient for our purpose of estimation. We shall also need $\hat{\ell}_v(\Phi_v, f_{3,v}, f'_{3,v})$ defined by

$$
\hat{\ell}_v(\cdots) = \int_{F_v^\times \times K_v} W_{\Phi_v}(a(y)\kappa)W_{f_{3,v}}(a(y)\kappa)f'_{3,v}(\kappa)|y_v|^{-\frac{1}{2}} d^v y d\kappa, \quad v \mid \infty;
$$

$$
\hat{\ell}_p(\cdots) = \frac{L(1, \omega_p)}{L(\frac{1}{2} + s, \xi_p^2)\cdot L(\frac{1}{2} - s, \xi_p^{-1})\cdot L(\frac{1}{2} + s, \xi_p \omega_p)\cdot L(\frac{1}{2} - s, \xi_p^{-1} \omega_p)} \int_{F_p^\times \times K_p} W_{\Phi_p}(a(y)\kappa)W_{f_{3,p}}(a(y)\kappa)f'_{3,p}(\kappa)|y_p|^{-\frac{1}{2}} d^p y d\kappa, \quad p < \infty;
$$

so that $\hat{\ell}_p = 1$ for all but finitely many $p$. As a $\text{GL}_2(F_v)$-invariant trilinear form, $\hat{\ell}_v$ is not always convenient for our purpose of estimation. We shall also need $\hat{\ell}_v(\Phi_v, f_{3,v}, f'_{3,v})$ defined by
so that \( \tilde{\ell}_p = 1 \) for all but finitely many \( p \).

2.4.4. **One Dimensional Projection of** \( \pi(1, \omega^{-1}) \times \pi(1, \omega^{-1}) \). Let \( \chi \) be a quadratic Hecke character unramified at every finite place (i.e., a quadratic class group character). For \( f_3, f'_3 \in \pi(1, \omega^{-1}) \), we take the explicit decomposition of the regularized integral (c.f. [39, §2.3])

\[
\int_{\text{reg}}^{\text{reg}} E^2(0, f_3) \cdot E^2(0, f'_3) \cdot \chi \circ \det
\]

\[
= \text{Res}_{s=\frac{1}{2}} L(\frac{1}{2} + s, \chi^2)L(\frac{1}{2} + s, \chi \omega)L(\frac{1}{2} + s, \chi \omega^{-1}) \prod_v \ell_v(s; f_3, v, f'_3, v, \chi, v)
\]

where the local factors are defined by

\[
\ell_v(\cdots) = \int_{F_v \times K_v} W_{f_3, v}(a(y)\kappa) W_{f'_3, v}(a(y)\kappa) \chi_v(y) |y_v|_v^{s - \frac{1}{2}} d^\kappa y \kappa, \quad v \mid \infty;
\]

\[
\ell_p(\cdots) = \text{Res}_{s=\frac{1}{2}} L(\frac{1}{2} + s, \chi_p)L(\frac{1}{2} + s, \chi_p \omega_p)L(\frac{1}{2} + s, \chi_p \omega_p^{-1})
\]

\[
= \text{Res}_{s=\frac{1}{2}} L(\frac{1}{2} + s, \chi_p)L(\frac{1}{2} + s, \chi_p \omega_p)L(\frac{1}{2} + s, \chi_p \omega_p^{-1})
\]

so that \( \ell_p = 1 \) for all but finitely many \( p \), and all \( \ell_v \) are holomorphic at \( s = 1/2 \).

**Remark 2.9.** The pole at \( s = 1/2 \) of the global \( L \)-factor in (2.14) has order equal to 0 if \( \chi \neq 1, \omega \neq \chi \); 2 if \( \chi \neq 1, \omega = \chi \) or \( \chi = 1, \omega \neq \chi \); 4 if \( \chi = \omega = 1 \). Hence if \( C(\omega) \) is sufficiently large (depending only on \( F \)), (2.14) is non-vanishing only if \( \chi = 1 \). We also note \( L(k,1, \chi^{-1}) \ll_{F, \pi} C(\omega) \) for \( 0 \leq k \leq 4 \).

2.4.5. **On Regularized** \( \pi(1, 1) \times \pi(1, 1) \times \pi(1, \omega^{-1}) \text{ respectively} \). For \( \bar{f}_2, f_2 \in \pi(1, 1), \bar{f}_3, f_3 \in \pi(1, \omega^{-1}) \), let \( f \in \{(\bar{f}_3 \cdot f_3) |K, (R_0 f_3 \cdot f_3) |K\} \), regarded as an element in \( \pi(1, 1) \). Write \( \bar{E}^2_2 = E^2(0, \bar{f}_2) \) and \( \bar{E}^2_3 = E^2(0, \bar{f}_3) \). We take the explicit decomposition of the extended Rankin-Selberg integral (c.f. [38, Proposition 2.5])

\[
R \left( \frac{1}{2} + s, \bar{E}^2_2 \cdot f_2; f \right) = \frac{\xi_F(1 + s)}{\xi_F(2 + 2s)} \prod_v \ell_v(s; f_2, v, f_3, v; f)
\]

where the local factors are defined by

\[
\ell_v(\cdots) = \int_{F_v \times K_v} W_{f_2, v}(a(y)\kappa) W_{f_3, v}(a(y)\kappa) \chi_v(y) |y_v|_v^{s - \frac{1}{2}} d^\kappa y \kappa, \quad v \mid \infty;
\]

\[
\ell_p(\cdots) = \frac{\xi_p(2 + 2s)}{\xi_p(1 + s)} \frac{1}{\xi_p(1 + s)} \frac{1}{\xi_p(1 + s)} \int_{F_p \times K_p} W_{f_2, p}(a(y)\kappa) W_{f_3, p}(a(y)\kappa) f_p(\kappa) |y_p|_p^{s - \frac{1}{2}} d^\kappa y \kappa, \quad p < \infty;
\]

so that \( \ell_p = 1 \) for all but finitely many \( p \), and all \( \ell_v \) are holomorphic at \( s = 0 \).

2.4.6. **On Regularized** \( \pi(1, 1) \times \pi(1, 1) \times \pi(\omega^{-1}) \text{ respectively} \). \( \pi(\omega^{-1}) \text{ respectively} \). For \( f \in \{(R_0 f_3 \cdot f_3) |K, (f_3 \cdot R_0 f_3) |K\}, \) regarded as an element in \( \pi(1, 1) \) resp. \( \pi(\omega^{-1}) \text{ resp.} \). Let notations be as in the previous case. Consider \( f = (R_0 f_3 \cdot f_3) |K\) resp. \( (f_3 \cdot R_0 f_3) |K\), regarded as an element in \( \pi(1, 1) \) resp. \( \pi(\omega^{-1}) \). We take

\[
R \left( \frac{1}{2} + s, \bar{E}^2_2 \cdot f; f \right) = \frac{L(1 + s, \omega^\mp)}{L(2 + 2s, \omega^\mp)} \prod_v \ell_v(s; f_2, v, f_3, v; f)
\]

where the local factors are defined by

\[
\ell_v(\cdots) = \int_{F_v \times K_v} W_{f_2, v}(a(y)\kappa) W_{f_3, v}(a(y)\kappa) \chi_v(y) |y_v|_v^{s - \frac{1}{2}} d^\kappa y \kappa, \quad v \mid \infty;
\]

\[
\ell_p(\cdots) = \frac{L_p(2 + 2s, \omega_p^\mp)}{L_p(1 + s, \omega_p^\mp)} \frac{1}{\xi_p(1 + s)} \frac{1}{\xi_p(1 + s)} \int_{F_p \times K_p} W_{f_2, p}(a(y)\kappa) W_{f_3, p}(a(y)\kappa) f_p(\kappa) |y_p|_p^{s - \frac{1}{2}} d^\kappa y \kappa, \quad p < \infty;
\]

so that \( \ell_p = 1 \) for all but finitely many \( p \), and all \( \ell_v \) are holomorphic at \( s = 0 \).
2.4.7. On Regularized $\pi(1, \omega^{-1}) \times \pi(1, \omega^{-1}) \times \pi(1, \omega^{-1})$. For $f_2, f_3 \in \pi(1, 1)$, $f_2, f_3 \in \pi(1, \omega^{-1})$, let $f = (f_2 \cdot f_2) |_{K}$, regarded as an element in $\pi(1, 1)$. Write $E_3^3 = E(0, f_3)$. We take the explicit decomposition of the extended Rankin-Selberg integral (c.f. [38, Proposition 2.5])

$$R(1, \epsilon, E_3^3; f) = \frac{\zeta_F(1 + s) L(1 + s, \omega^{-1}) L(1 + s, \omega)}{\zeta_F(2 + 2s)} \prod_v \ell_v(s; f_3, f_3; f_v)$$

where the local factors are defined by

$$\ell_v(\cdots) = \int_{F_v \times K_v} W_{f_3, v}^\times (a(y)\kappa) W_{f_2, v}^\times (a(y)\kappa) f_v(\kappa) |y|_v^s d^\times \kappa,$$

$$\ell_p(\cdots) = \frac{\zeta_p(2 + 2s)}{\zeta_p(1 + s)^2 L_p(1 + s, \omega_p^{-1}) L_p(1 + s, \omega_p)} \int_{F_p \times K_p} W_{f_3, p}^\times (a(y)\kappa) W_{f_2, p}^\times (a(y)\kappa) f_p(\kappa) |y|_p^s d^\times \kappa,$$

so that $\ell_p = 1$ for all but finitely many $p$, and all $\ell_v$ are holomorphic at $s = 0$.

3. Local Choices and Estimations

3.1. Non Archimedean Places.

3.1.1. Choices and Main Bounds. At $p < \infty$, we choose test vectors as [27, §3.6.2]. Precisely, choose $W_{\varphi, p}$ to be a new vector of $\pi_p$ in the Whittaker model; $f_2, f_3$ to be the spherical function taking value 1 at 1 in the induced model of $\pi(1, 1)$; $f_3, f_3$ whose restriction to $K_p$ is

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \mapsto \text{Vol}(K_0[p^{\ell(\tau_p)}])^{-1/2} \omega_p^{-1}(d) 1_{K_0[p^{\ell(\tau_p)}]}.$$

Lemma 3.1. If $\ell(\pi_p) > 0$, then we have, with $\ell_p$ defined in (2.11) and absolute implicit constant,

$$|\ell_p(W_{\varphi, p}, f_2, f_3, f_3)| \gg C(\pi_p)^{1/2} \cdot \frac{||W_{\varphi, p}||}{\sqrt{L(1, \pi_p \times \pi_p)}}.$$

Proof. The proof is exactly the same as [27, §3.6.2], except that we take into account various $L$-factors. We also remark that the necessary formula of $W_{\varphi, p}$ can be found in [17, Table 1], and that we are following the style of [35], i.e., without specific normalization for $W_{\varphi, p}$.

Lemma 3.2. (1) Let $\pi_p^\prime$ be a unitary spherical representation with trivial central character and spectral parameter $\leq \theta$. Let $W' = W_{\varphi, p}^\prime$ be the Whittaker function of a spherical vector in $\pi_p^\prime$. Let $\ell_p$ be defined in (2.12). With absolute implicit constant, we have

$$|\ell_p(W', f_3, f_3, f_3)| \ll C(\pi_p)^{-1/2} (C(\pi_p)/C(\omega_p))^{\theta} \cdot \frac{||W'||}{\sqrt{L(1, \pi_p^\prime \times \pi_p^\prime)}}$$

(2) Let $\Phi_p$ be the spherical function of $\pi(\xi_p, \xi_p^{-1})$ taking value 1 on $K_p$ and $\tau \in \mathbb{R}$. Let $\ell_p$ be defined in (2.13). With absolute implicit constant, we have for any $\epsilon > 0$

$$|\ell_p(\tau, \Phi_p, f_3, f_3)| \ll \epsilon C(\pi_p)^{-1/2 + \epsilon}.$$

(3) Let $\ell_p$ be defined in (2.14). With absolute implicit constant, we have for any $k \in \mathbb{N}$ and $\epsilon > 0$

$$|\ell_p^{(k)}(1/2, f_3, f_3, 1)| \ll_{k, \epsilon} C(\pi_p)^{\epsilon}.$$
Proof. (1) We drop the subscript $p$ for simplicity of notations. Let $e_0$ be a unitary new vector in $\pi(1, \omega^{-1})$. We also write $W_{\pi', s}^*$ resp. $W_{\pi', s}^*$ for $W_{\pi', s}^*$ resp. $W_{\pi', s}^*$. As [27, §3.6.2], the integral part of $\ell_p$ is equal to

$$\text{Vol}(K_0[p^{(\pi')}]^{1/2}) \int_{F^*} W'(a(y))W_{\pi', s}^*(a(y))|y|^{-1/2}d^*y.$$ 

If $c(\omega) > 0$, then $f_3 = a(\omega^{-n}).e_0$ for $n = c(\pi) - c(\omega)$ by Proposition [5.1] (3). Using [17, Table 1], we can evaluate the above integral as, with $c(\alpha, \alpha^{-1})$ the Satake parameter of $\pi'$,

$$\text{Vol}(K_0[p^{(\pi')}])^{1/2}W'(1)L(1/2, \pi') \left(\frac{\alpha^{n+1} - \alpha^{-(n+1)}}{\alpha - \alpha^{-1}} - \frac{\alpha^n - \alpha^{-n}}{\alpha - \alpha^{-1}}q^{-\frac{s}{2}}\right),$$

and conclude by $q^{-\theta} \leq |\alpha| \leq q^{\theta}$.

If $c(\omega) = 0$ with $\alpha_1 = \omega(\underline{\omega})$ (we can assume $n > 0$ since the case $n = 0$ is easy), then $f_3 \simeq L(1, \omega)(a(\omega^{-n}).e_0 - \alpha_1 q^{-1/2}a(\omega^{-n-1}).e_0)$ by Proposition [5.1] (4). $a(\omega^{-n}).e_0$ contributes to the integral as the product of

$$L(1/2, \pi')L(1/2, \pi' \otimes \omega) \cdot \text{Vol}(K_0[p^{(\pi')}])^{1/2}W'(1)$$

and

$$\{\frac{\alpha^{n+1} - \alpha^{-(n+1)}}{\alpha - \alpha^{-1}} - (1 + \alpha_1)q^{-\frac{s}{2}}\alpha^n - \alpha^{-n} + \alpha_1 q^{-1}\alpha^{n-1} - \alpha^{-(n-1)}\alpha - \alpha^{-1}\},$$

while the second term contributes less. We conclude.

(2) Since $s = i\tau \in i\mathbb{R}$, Proposition [5.3] tells us that $\ell_p$ and $\ell_p'$ have the same size. But $\ell_p'$ is of the same shape as $\ell_p$ in the cuspidal case above. Hence our bounds is the same as (1) with $\theta = 0$.

(3) By Proposition [5.3], $\ell_p^{(k)}$ is of size $C(\omega_p)^{1/2+\epsilon}$ times $\ell_p^{(k)}$. Arguing as in (1), replacing $\alpha$ with $\alpha q^s$ ($s$ around $1/2$), where $\alpha = \xi(\omega)$, we obtain and conclude by

$$\text{Vol}(K_0[p^{(\pi')}])^{1/2+\epsilon}.$$

3.1.2. Bounds for Regularization and Amplification. We restrict to a finite place $p \in S^*$ defined in [2.2]. Let $n_p \in \mathbb{Z}$ and $1 \leq |n_p| \leq 2$ (NOT the same as in [2.2]) if $p \in S$; $1 \leq |n_p| \leq 4$ if $p = p_0$. Recall $f_{3,p}$ is $K_p$-invariant.

Lemma 3.3. (1) Let $\pi'_{p}$ be a unitary representation with trivial central character and $c(\pi'_{p}) \leq |n_p|$. Let $W' = W_{\pi', p}^*$ run over an orthogonal basis of $K_p \cap a(\varpi_p^{n_p})K_p a(\varpi_p^{-n_p})$ invariant vectors in the Whittaker model of $\pi'_{p}$, with different $K_p$-types. For $\ell_p$ defined in [2.12] we have the estimation

$$||\ell_p(W', f_{3,p}, a(\varpi_p^{n_p})f_{3,p})|| \ll q^{-\frac{|n_p|}{2}}L(1, \pi'_{p} \times \pi_{p}).$$

(2) Let $\xi_p$ be a character of $F_{p}^*$ with $c(\xi_p) \leq |n_p|/2$. Let $\Phi_p$ run over an orthogonal basis of $K_p \cap a(\varpi_p^{n_p})K_p a(\varpi_p^{-n_p})$ invariant vectors in $\pi(\xi, \xi^{-1})$, with different $K_p$-types. For $\ell_p$ defined in [2.13] and $\tau \in \mathbb{R}$ we have the estimation

$$\ell_p(i\tau; \Phi_p, f_{3,p}, a(\varpi_p^{n_p}).f_{3,p}) \ll q^{-\frac{|n_p|}{2}}.$$
By Proposition 5.1 (4-2), \( a(\varpi^n) f_3 \) is a linear combination of \( q^{-\frac{k}{2}} D_k \) for \( 0 \leq k \leq n \), with coefficients of size \( \ll 1 \). If \( W' \) is of level \( m \leq n \), then we have

\[
\int_K W'(a(y)\kappa) W_3^*(a(y)\kappa) q^{-\frac{k}{2}} D_k(\kappa) d\kappa \approx q^{-\frac{k}{2}} W'(a(y)) W_3^*(a(y)) 1_{m \geq k},
\]

since (L^1-normalized) \( D_k \) induces the orthogonal projection onto the subspace of \( K_0[p^k]\)-invariant vectors. By Proposition 5.1 (1)+(2)+(3)+(4-2), \( W' \) is a linear combination of \( a(\varpi^{-i}) W_0' \) for \( m - 2 \leq l + c(\pi') \leq m \), with coefficients of absolute value \( \leq 1 \), where \( W_0' \) is an L^2-normalized new vector in the Whittaker model of \( \pi' \). We are reduced to computing

\[
\int_{F^\times} a(\varpi^{-l}) W_0'(a(y)) W_3^*(a(y)) |y|^{-\frac{k}{2}} d^\times y.
\]

We write \( \alpha = \omega(\varpi) \) and use [17, Table 1] distinguishing several cases: (i) \( \pi' \) is spherical with Satake parameters \( \alpha_1, \alpha_2 \) \((\alpha_1\alpha_2 = 1)\). (3.1) is equal to \((l \in \{0, 1, 2\})

\[
\frac{L(1/2, \pi') L(1/2, \pi' \otimes \omega)}{\sqrt{L(1, \pi' \times \pi')}} \cdot \left( \frac{1 - \alpha'^{-1}}{1 - \alpha} - (\alpha_1 + \alpha_2) \alpha q^{-\frac{1}{2}} \frac{1 - \alpha'}{1 - \alpha} + \alpha'^2 q^{-1} \frac{1 - \alpha'^{-1}}{1 - \alpha} \right).
\]

(ii) \( \pi' \simeq St_\chi \) with \( \alpha' = \chi(\omega) \in \{\pm 1\} \). (3.1) is equal to \((l = 0)\)

\[
\frac{L(1/2, \pi') L(1/2, \pi' \otimes \omega)}{\sqrt{L(1, \pi' \times \pi')}} \cdot \left( \frac{1 - \alpha'^{-1}}{1 - \alpha} - \alpha' \alpha q^{-1} \frac{1 - \alpha'}{1 - \alpha} \right).
\]

(iii) \( c(\pi') = 2 \), which in our case implies \( L(s, \pi') = 1 \). (3.1) is equal to \((l = 0)\).

In conclusion, (3.1) does not create increase or decrease in terms of \( q^n \) and we are done.

(2) Proposition 5.3 tells us that \( \ell_p \) is of the same size as \( \hat{\ell_p} \), which can be bounded the same way as in the cuspidal case above.

**3.1.3. Main Bounds in Regularized Term.** We restrict to a finite place \( p \notin S \).

**Lemma 3.4.**

1. For \( \ell_p \) in (2.15) and \( f_p \in \{(f_3, f_3, f_3) |_{K_p}, (R_0 f_3, f_3, f_3) |_{K_p} \}, \) we have

\[
\ell_p(s; f_2, f_2; f_p) = 1.
\]

2. For \( \ell_p \) in (2.17) and \( f_p \in \{(R_0 f_3, f_3, f_3) |_{K_p}, (f_3, f_3, f_3) |_{K_p} \}, \) \( \ell_p(s; f_2, f_2; f_p) \) is a constant

\[
\left| \ell_p(s; f_2, f_2; f_p) \right| = 0 \quad \text{if } c(\omega_p) \neq 0,
\]

\[
\approx C(\pi_p)^{-1} \quad \text{if } c(\omega_p) = 0.
\]

3. For \( \ell_p \) in (2.17) and \( f_p = (f_2, f_2, f_2) |_{K_p} = 1 \), we have for any \( k \in \mathbb{N} \) and \( \epsilon > 0 \)

\[
\left| \ell_p^{(k)}(0; f_3, f_3, f_3; 1) \right| \ll_{k, \epsilon} C(\pi_p)^{\epsilon}.
\]

**Proof.** (1) \( f_2, f_2 \) being \( K_p \)-invariant, we get

\[
\ell_p(s; f_2, f_2, f_2; f_p) = \frac{c_p(1 + s)^4}{c_p(2 + 2s)} \cdot \int_{F_p^\times} |W^*_2(a(y))|^2 |y|^s d^\times y \cdot \int_{K_p} f_p(\kappa) d\kappa = \int_{K_p} f_p(\kappa) d\kappa.
\]

We get the desired equality since \( f_3, f_3 \) is a unitary vector and \( R_0 \) is unitary.

(2) Similar argument as in (1) gives

\[
\ell_p(s; f_2, f_2, f_2; f_p) = \begin{cases} 0 & \text{if } c(\omega_p) \neq 0, \\ \int_{K_p} f_p(\kappa) d\kappa & \text{if } c(\omega_p) = 0. \end{cases}
\]
Drop the subscript \( p \) for simplicity. Assume \( c(\omega) = 0 \). Take the case \( f = (R_0 f_3 \cdot f_3) |_K \) for example. Write \( n = c(\pi) \). By choice, \( \int_K f(\kappa) d\kappa = \text{Vol}(K_0[p^n])^{1/2} \) times the orthogonal projection onto the \( K_0[p^n] \)-invariant subspace, hence
\[
\int_K f(\kappa) d\kappa = \text{Vol}(K_0[p^n])^{1/2} R_0 f_3(1).
\]

Proposition 5.1 (4-2), together with the observation \( e_k(1) = 0 \) for \( k \geq 1 \) with notations in that proposition, then gives
\[
R_0 f_3(1) = q^{n/2}(1 + q^{-1})^{-1/2} R_0 e_0(1) = q^{n/2}(1 + q^{-1})^{-1/2},
\]
where \( e_0 \) is the spherical function in \( \pi(1, \omega^{-1}) \) taking value 1 on \( K \).

(3) This is in fact the same as Lemma 5.2 (3). \( \square \)

3.1.4. Bounds in Regularized Term for Regularization and Amplification. We restrict to a finite place \( p \in S^* \) and write \( t_p = \omega_p^{-n} \) with \( n \in \{n_p, -n_p\} \) or \( \{k_1 - k_2, k_2 - k_1\} \).

**Lemma 3.5.**

1. For \( \ell_p \) in (2.10) and \( k \in \mathbb{N} \), we have
\[
|\ell_p^{(k)}(0; a(t_p), f_{2,p}, \overline{f_{2,p}}; a(t_p), f_3, f_{3,p}, f_{3,p})| \ll_k (|n| + 1)^k q^{-n}(\log q)^k.
\]

2. For \( \ell_p \) in (2.10) and \( k \in \mathbb{N} \), we have
\[
|\ell_p^{(k)}(0; a(t_p), f_{2,p}, \overline{f_{2,p}}; R_0(a(t_p), f_3, f_{3,p}, f_{3,p})| \ll_k (|n| + 1)^k q^{-n}(\log q)^k.
\]

3. For \( \ell_p \) in (2.17) and \( k \in \mathbb{N} \), we have
\[
|\ell_p^{(k)}(0; a(t_p), f_3, f_{3,p}, \overline{f_{3,p}}; a(t_p), f_{2,p}, \overline{f_{2,p}})| \ll_k (|n| + 1)^k q^{-n}(\log q)^k,
\]
\[
|\ell_p^{(k)}(0; a(t_p), f_3, f_{3,p}, \overline{f_{3,p}}; R_0(a(t_p), f_2, f_{2,p})| \ll_k (|n| + 1)^k q^{-n}(\log q)^k,
\]
\[
|\ell_p^{(k)}(0; a(t_p), f_3, f_{3,p}, \overline{f_{3,p}}; R_0^{(1)}(a(t_p), f_{2,p}, f_{2,p})| \ll_k (|n| + 1)^k q^{-n}(\log q)^k.
\]

**Proof.** We drop the subscript \( p \) for simplicity of notations.

(1) The second inequality essentially follows from the first by replacing \( \omega \) with \( \omega^{-1} \), since \( R_0 f_3 \cdot \omega \circ \det \) is the corresponding \( f_3 \). By \( K \)-invariance of \( \ell \) and \( f_2, f_3 \) we have
\[
\ell(s; a(t), f_2, \overline{f_2}; a(t), f_3, f_{3}) = \ell(s; wa(t)w, f_2, \overline{w}f_2; wa(t)w, f_3, f_{3}) = \omega^{-1}(t)\ell(s; a(t^{-1}), f_2, f_3),
\]
hence we may assume \( n > 0 \). The integral part of \( \ell \) has the form
\[
\int_{F^*} W_2^*(a(y))|y|^dK \int_K a(t), f_3 (\kappa) \cdot (a(t), W_2^*) |(a(y))| dK.
\]

We enter into the setting of Section 5.1, distinguishing elements related to \( f_2 \) from those to \( f_3 \) by putting a “\( \ast \)”. (For example, \( e_0 = f_3, W_0^* = W_2^* \).) Recall the projectors \( P_n \) defined in Corollary 5.2. We have the relations
\[
P_n = \int_K q^\frac{1}{2} (1 + q^{-1})^\frac{1}{2} D_n(\kappa) d\kappa, \text{ if } n \geq 1; \quad P_0 = D_0.
\]

By Proposition 5.1 (4-2), writing \( \alpha = \omega(\varpi) \), we get
\[
\int_K a(t), e_0(\kappa) d\kappa = \alpha^n q^{-\frac{n}{2}} \left\{ P_0 + \frac{1 - \alpha q^{-1}}{1 + q^{-1}} \sum_{\ell=1}^n \alpha^{-\ell} P_\ell \right\}.
\]
Together with Corollary \textbf{5.2} we obtain
\[
\int_{\mathbf{K}} a(t).e_0(\kappa)a(t).e_0^*d\kappa = \alpha^n q^{-n}. \left\{ (n + 1 - \frac{2n}{q + 1})e_0^* + \frac{1 - \alpha q^{-1}}{1 + q^{-1}} \right\}
\]
\[
\sum_{l=1}^{n} \alpha^{-l} \left( (n - l + 1)q^{\frac{l}{2}}a(\varpi_l).e_0^* - (n - l)q^{\frac{l}{2}}a(\varpi_l^{-1}).e_0^* \right).
\]
Hence we are reduced to computing
\[
q^{\frac{l}{2}} \int_{\mathbf{F}_v^\times} a(\varpi^{-l}).W_0^*(a(y))|W_0^*(a(y))|^sd^y = q^{-ls} \left\{ 1 + \frac{q^{-l(1+s)}}{(1 - q^{-l(1+s)})^2} + \frac{l}{1 - q^{-l(1+s)}} \right\}.
\]

If we put \( A_l(s) := q^{-ls}(1 + l(1 - \alpha q^{-1}(1+s)))^{-1} \) for \( l \geq 1 \) and \( A_0(s) = 1 \), then we get
\[
\ell(s; a(t), f_2; a(t)f_3 f_4) = \alpha^n q^{-n} . \left\{ (n + 1 - \frac{2n}{q + 1}) + \frac{1 - \alpha q^{-1}}{1 + q^{-1}} \right\}
\]
\[
\sum_{l=1}^{n} \alpha^{-l} ((n - l + 1)A_l(s) - (n - l)A_{l-1}(s))\}
\]
from which we easily deduce the desired bound.

(2) The argument is quite similar to (1) above. For example for the case \( a(t)f_3 R_0 f_3 \), we only need to replace \( A_l(s) \) with \( A''_l(s) := \alpha l^{-ls}(1 + l(1 - \alpha q^{-1}(1+s)))^{-1} \).

(3) The argument is again similar to (1) above. For example for the case \( a(t)f_2 f_2 \), we need to replace \( A_l(s) \) resp. \( \ell(s; \cdot ; \cdot) \) with
\[
A''_l(s) = q^{-ls} \left\{ 1 + \frac{(1 - \alpha l) - \alpha q^{-1}}{1 - q^{-1}} \right\}, \text{ resp.}
\]
\[
\ell(s; a(t), f_3; a(t)f_3, f_4) = q^{-n} \left\{ \frac{1 - \alpha l + 1}{1 - \alpha} \cdot \frac{q^{-n+1} + \alpha - 1}{q + 1}, \right\} + \frac{1 - q^{-1}}{1 + q^{-1}}
\]
\[
\sum_{l=1}^{n} \left\{ \frac{1 - \alpha l + 1}{1 - \alpha} A''_l(s) - \frac{1 - \alpha l}{1 - \alpha} A''_{l-1}(s) \right\}.
\]

\[\square\]

3.2. Archimedean Places.

3.2.1. \textit{Choices and Lower Bounds.} The choice of the local test functions at the archimedean places is the subtest construction in [27]. We find it convenient if we specify them in two steps with some non-vanishing condition:

(1) Our test function \( f_{3,v} = a(C).f_0 \) where \( C \in \mathbf{F}_v^\times \) with \( |C| = \mathbf{C}(\pi_v)^{1+\epsilon} \) and \( f_0 \in \pi(1, \omega_v^{-1}) \) is a smooth unitary vector such that
\[
f_0 : \mathbf{K}_v \rightarrow \mathbf{C}, \left( \begin{array}{c} a \\ c \\ b \\ d \end{array} \right) \mapsto \kappa \mapsto \omega_v(d).f_0(\kappa)
\]
is a fixed (depending only on \( \mathbf{F}_v = \mathbb{R} \) or \( \mathbb{C} \)) smooth unitary vector in \( \text{Res}_{\mathbf{G}}^{\mathbf{GL}_2(\mathbf{F}_v)} \pi(1,1) \) with support contained in a small neighborhood \( U \) of \( \mathbf{B}_v \cap \mathbf{K}_v \) in \( \mathbf{K}_v \). It can be easily verified that for any Sobolev norm \( S \) defined with differential operator on \( \mathbf{G} \)
\[
S_d(f_0) \ll \mathbf{C}(\omega_v)^d.
\]

(2) There is a non-negative bump function \( \phi \) on \( \mathbf{F}_v \) with support contained in a small compact neighborhood of 0, say in \( \{ x \in \mathbf{F}_v : |x| \leq \delta_0 \} \) such that
\[
f_0(\kappa) = \int_{\mathbf{F}_v^\times} \Psi_0((0,t).\kappa) \omega_v(t)|t|_v d^\times_t, \text{ i.e. } \tilde{f}_0(\kappa) = \int_{\mathbf{F}_v^\times} \phi(ct) \phi(dt-1)|t|_v d^\times_t,
\]
where $\Psi_0 \in S(F_v^\times)$ is defined via

$$\Psi_0(x, y) = \phi(x)\phi(y - 1)\omega^{-1}_v(y),$$

and $\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

**Remark 3.6.** Note that the Kirillov norm

$$\|W_{3, v}\|^2 = \int_{F_v^\times} |W_{3, v}(a(y))|^2 d^\times y = \frac{\zeta_v(1)^2}{\zeta_v(2)} \int_{K_v} |f_{3, v}(\kappa)|^2 d\kappa \asymp \|f_{3, v}\|^2$$

is essentially the same as the induced norm. Hence we may regard $W_{3, v}$ as unitary.

**Lemma 3.7.** If $U \subset K$ is a small neighborhood of $K \cap B$, then for $C \in F_v^\times$

$$U_C := \{ \kappa \in K : \kappa a(C) \in BU \}$$

is a small neighborhood in $K$ which shrinks to $K \cap B$ as $|C| \to \infty$. Moreover, we have

$$H_t v(\kappa a(C)) \asymp_U |C|, \quad \forall \kappa \in U_C.$$

**Proof.** Let $\alpha, \beta \in F_v$ such that $|\alpha|^2 + |\beta|^2 = 1$, we have

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} C \\ 1 \end{pmatrix} = \begin{pmatrix} C \sqrt{|\alpha|^2 + |\beta|^2} & * \\ \sqrt{|\alpha|^2 + |\beta|^2} \end{pmatrix} \begin{pmatrix} \alpha & \bar{C} \beta \\ -C \bar{\beta} & \bar{\alpha} \end{pmatrix} \cdot \frac{1}{\sqrt{|\alpha|^2 + |\beta|^2}}.$$

The smallness of $U$ implies $|C\beta| \leq \delta_0 |\alpha|$ for some small $\delta_0 > 0$, which implies

$$|\beta|^2 \leq |\delta_0|^2(|C|^2 + |\delta_0|^2)^{-1} \leq (|\delta_0|/|C|)^2;$$

$$1 \leq |\alpha|^2 + |C\beta|^2 = 1 + ([C]^2 - 1)|\beta|^2 \leq 1 + |\delta_0|^2([C]_v^2 - 1)(|C|^2 + |\delta_0|^2)^{-1} \leq 1 + |\delta_0|^2.$$

We conclude both assertions. $\square$

**Lemma 3.8.** ([27, (3.43)]) We have as $C(\pi_v) \to \infty$

$$\left| \int_{U_C} f_{3, v}(\kappa) d\kappa \right| \geq \frac{1}{2} \int_{U_C} |f_{3, v}(\kappa)| d\kappa \asymp |C|^{-1/2}. $$

**Proof.** The first inequality was explained just after [27, (3.43)]. For the second, we apply the second assertion of Lemma 3.7 and get

$$\int_{U_C} |f_{3, v}(\kappa)| d\kappa \asymp |C|^{-1/2} \int_{U_C} |f_0(\kappa')| H_{t v}(\kappa a(C)) d\kappa = |C|^{-1/2} \int_K |f_0(\kappa)| d\kappa,$$

if we write $\kappa a(C) = b' \kappa'$. $\square$

Since $\pi_{2, v} = \pi(1, 1)$ is unitary and spherical, we can make choice of $f_{2, v}$ simpler than [27, §3.6.4 & 3.6.5], i.e., let $f_{2, v}$ be the spherical function in $\pi(1, 1)$ taking value 1 at 1. Specify $W_{\varphi, v}$ by taking $W_{\varphi, v}(a(y))$ to be a fixed smooth function $\delta_v(y)$ with support in a compact neighborhood of 1 in $F_v^\times$, invariant by $C^1$ if $F_v = C$, such that

$$\int_{F_v^\times} \delta_v(y) W_{\varphi, v}^2(a(-y)) |y|^{-\frac{1}{2}} d^\times y \gg 1, \quad \int_{F_v^\times} |\delta_v(y)|^2 d^\times y = 1.$$

**Lemma 3.9.** With the above choices, we have as $C(\pi_v) \to \infty$

$$|\ell_v(W_{\varphi, v}, f_{2, v}, f_{3, v})| \asymp |C|^{-1/2} = C(\pi_v)^{-\frac{1}{2} - \frac{1}{2}}.$$
Proof. The proof is similar to [27, §3.6.5]. We drop the subscript $v$ and write $W = W_{v,v}$ for simplicity. Defining a bilinear form

$$L(\tilde{W}, \tilde{W}_2) = \int_{F_x^*} \tilde{W}(a(y))\tilde{W}_2(a(-y))|y|^{-\frac{1}{2}}d^xy, \ \ \ \ \ \ \ \forall \tilde{W} \in \mathcal{W}(\pi, \psi), \ \tilde{W}_2 \in \mathcal{W}(\pi(1,1), \psi),$$

we have for $\varepsilon > 0$ small enough

$$|L(\kappa, W, \kappa, W_2^*) - L(W, W_2^*)| = |L(\kappa W - W, W_2^*)| \leq \left( \int_{F_x^*} |(\kappa W - W)(a(y))|^2|y|^{-\varepsilon}d^xy \right)^{1/2} \left( \int_{F_x^*} |W_2^*(a(y))|^2|y|^{-1+\varepsilon}d^xy \right)^{1/2}.$$

For $\kappa \in U_C$ arguing as in [27, §3.6.4], i.e., using [27, Proposition 3.2.3] and for any $\tilde{W} \in \mathcal{W}(\pi^\infty)$

$$\int_{|y|_v \leq 1} |\tilde{W}(a(y))|^2|y|^{-\varepsilon}d^xy \leq S(\tilde{W})^{\frac{1}{\gamma}} \int_{|y|_v \leq 1} |\tilde{W}(a(y))|^{2-\frac{2}{\gamma}}d^xy \leq S(\tilde{W})^{\frac{1}{\gamma}} \cdot \|\tilde{W}\|^{2-\frac{2}{\gamma}},$$

we see the above is bounded as (c.f. [35, §2.7])

$$\ll_{v} (C(\pi)/C)^{(1-d\varepsilon)/d}\|F^\circ\|C(\pi)^{d\varepsilon} + C(\pi)/C = C(\pi)^{-\varepsilon(1-d\varepsilon)/d} + C(\pi)^{-\varepsilon}$$

for some absolute $d, d' \in \mathbb{N}$. If $\varepsilon$ is sufficiently small (depending on $\varepsilon, d, d'$), the above tends to 0 as $C(\pi) \to \infty$. Thus

$$|\ell_v(W_{\pi,v}, f_{2,v}, f_{3,v})| \geq L(W, W_2^*) \cdot \left| \int_{K\cap B\setminus K} f_3(\kappa) d\kappa \right| - o(1) \int_{K\cap B\setminus K} |f_3(\kappa)| d\kappa.$$

We conclude by Lemma 3.10.

3.2.2. Upper Bounds.

Lemma 3.10. Let $\Phi \in S(F_v^\times)$ and $\chi$ be a (unitary) character of $F_v^\times$ with analytic conductor $C = C(\chi)$. Then for any $1/2 \leq \alpha < 1$, we have

$$\left| \int_{F_v} \Phi(x)\psi_v(tx)\chi(x)dx \right| \ll_{\Phi, N, \alpha} \min(CN(1 + |t|_v)^{-N}, C^{1/2-\alpha}(1 + |t|_v)^{\alpha-1})$$

where the dependence on $\Phi$ involves only some Schwartz norms of $\Phi$ of order depending on $N$.

Proof. This is part of [27, Lemma 3.1.14] or [35, Lemma 4.1]. Incidentally, we find our previous proofs not clearly written for the second bound. We give a clearer version as follows. Writing $\Phi_\alpha(x) = \Phi(x)|x|_v^\alpha$ and $\bar{\Phi}_\alpha$ for the Fourier transform of $\Phi_\alpha$ w.r.t. $\psi_v$, we have

$$\int_{F_v} \Phi_\alpha(x)\psi_v(tx)\chi(x)dx = \gamma(\chi, \psi_v, 1 - \alpha)^{-1} \int_{F_v} \bar{\Phi}_\alpha(x + t)\chi^{-1}(x)|x|_v^{\alpha-1}dx.$$}

The (inverse) $\gamma$ factor is bounded as $\ll_{\varepsilon} C^{1/2-\alpha}$ uniformly for $\alpha \in [1/2, 1 - \varepsilon]$. If $|t|_v \leq 1$, the integral at the RHS is bounded as

$$\left| \int_{F_v} \bar{\Phi}_\alpha(x + t)\chi^{-1}(x)|x|_v^{\alpha-1}dx \right| \leq \left| \int_{|x|_v \leq 1} |x|_v^{\alpha-1}dx \cdot \|\bar{\Phi}_\alpha\|_\infty + \|\bar{\Phi}_\alpha\|_1;$$

while if $|t|_v \geq 1$, the integral at the RHS is bounded as

$$\left| \int_{F_v} \bar{\Phi}_\alpha(x + t)\chi^{-1}(x)|x|_v^{\alpha-1}dx \right| \leq \int_{|x|_v \leq |t|_v/2} |x|_v^{\alpha-1}dx \cdot \max_{|x|_v \geq |t|_v/2} |\bar{\Phi}_\alpha(x)| + (|t|_v/2)^{\alpha-1}\|\bar{\Phi}_\alpha\|_1.$$
Lemma 3.11. Write $W_0$ for $W_{f_0}$. Uniformly in $\kappa \in U$ and $y \in F_v^\times$, we have
\[
|W_0(a(y)\kappa)| \ll_{U,\epsilon,N} \left( \frac{|y|}{C(\omega_v)} \right)^{\frac{1}{2} - \epsilon} \left( 1 + \frac{C(\omega_v)}{|y|} \right)^{-N}.
\]

Proof. Introducing the variables
\[
\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U, \quad X = by + dx,
\]
we can write the LHS as
\[
W_0(a(y)\kappa) = \psi_v(-\frac{by}{d}) |y|^{1/2} \int_{F_v^\times} d' \int F_v d' \int F_v d' \int F_v d' \int F_v \phi(c/d X + t/d y) \phi(X - 1) \psi_v(X/d - 1)(X) dX.
\]
The inner integral is non-vanishing only if $X$ is close to 1 hence $|t|_v \leq \delta$ for some $\delta$ depending only on $\delta_0, U$. Applying Lemma 3.10 to $\Phi(X) := \phi(c/d X + t) \phi(X - 1)$, $\alpha = 1 - \epsilon$, the inner integral is bounded by
\[
\left( \frac{C(\omega_v)}{1 + |y/(td^2)|} \right)^N \ll_{U,N} \left( \frac{C(\omega_v)}{|y|} \right)^N, \quad \text{resp.} \quad C(\omega_v)^{-1/2+\epsilon} \left| \frac{y}{d^2 t_v} \right|^{-\epsilon} \ll_{U} C(\omega_v)^{-1/2+\epsilon} \left| \frac{t}{y} \right|^{\epsilon}
\]
with implied constant depending only on the Schwartz norms of $\phi$. We conclude. \(\square\)

Remark 3.12. The reason for which $W_0$ satisfies a “better” bound than the general one satisfied by a smooth vector shows some finer aspects of the integral representation of Whittaker functions than the smooth structures.

Corollary 3.13. With assumptions as in Lemma 3.11 and $|C|_v$ sufficiently large (depending on $U$), we have uniformly in $\kappa \in U_C$ and $y$
\[
|W_0(a(y)\kappa a(C))| \ll_{U,\epsilon,M} \left( \frac{|C| |y|}{C(\omega_v)} \right)^{\frac{1}{2} - \epsilon} \left( 1 + \frac{|C| |y|}{C(\omega_v)} \right)^{-M}.
\]

Proof. The general case follows from the case $C = 1$ and the “moreover” part of Lemma 3.11. For $C = 1$, we apply Lemma 3.11. \(\square\)

Lemma 3.14. (1) Let $\pi'_\theta$ be a unitary irreducible representation with trivial central character and spectral parameter $\leq \theta$. Let $W' = W_{\pi'_\theta}$ be the Whittaker function of a unitary vector in $\pi'_\theta$. Let $\ell_v$ be defined in [2.14]. With absolute implicit constant, we have
\[
|\ell_v(W', f_{3, v}, f_{3, v})| \ll_\epsilon C(\pi'_\theta)^{-\frac{1}{2}} (C(\pi'_\theta) / C(\omega_v))^{\theta+\epsilon} \cdot S_d(W')
\]
for some Sobolev norm of an absolute order $d$.

(2) Let $\xi_v$ be a unitary character of $F_v^\times$. Let $\Phi_v$ be a smooth function in $\pi(\xi_v, \xi_v^{-1})$. Let $\ell_v$ be defined in [2.13]. With absolute implicit constant, we have for $\tau \in \mathbb{R}$
\[
|\ell_v(\tau; \Phi_v, f_{3, v}, f_{3, v})| \ll_\epsilon C(\pi_v)^{-\frac{1}{2}+\epsilon} \cdot S_d(\Phi_v, \tau)
\]
for some Sobolev norm of an absolute order $d$.

(3) Let $\ell_v$ be defined in [2.14]. With absolute implicit constant, we have for any $k \in \mathbb{N}$ and $\epsilon > 0$
\[
|\ell_v^{(k)}(1/2, f_{3, v}, f_{3, v}, 1)| \ll_{k, \epsilon} C(\pi_v)^{\epsilon}.
\]
Proof. (1) The proof is the same as the one given in [27, Corollary 3.7.1], using Lemma 3.8, Proposition 3.2.3] (which gives d) and the inequality
\[
\int_{F_v} \frac{(X | y | v)^{1-\epsilon}}{1 + X | y | v} | y | v^{-\theta} d^\infty y = X^{\theta + \epsilon} \int_{F_v} (X | y | v)^{1-\theta-2\epsilon} (1 + X | y | v)^{\frac{1}{2}} | y | v^{\frac{1}{2}} d^\infty y \leq X^{\theta + \epsilon} \left( \int_{F_v} \frac{|y|^{1-2\epsilon}}{1 + |y|^{2\epsilon}} d^\infty y \right)^{\frac{1}{2}} \left( \int_{F_v} \frac{|y|^{2\epsilon}}{(1 + |y|^{2\epsilon})^{2\epsilon}} d^\infty y \right)^{\frac{1}{2}}.
\]
(2) Since \( \tau \in \mathbb{R} \), Proposition 5.3 tells us that \( \ell_v \) and \( \tilde{\ell}_v \) are of the same size. We argue as above for \( \tilde{\ell}_v \).
(3) Drop subscript \( v \). Recall \( f_0 = a(C) \cdot f_0 \) for some \( C \in F_v, |C| \approx C(\pi_v)^{1+\epsilon} \). Let \( \Phi \in \pi(1,1) \) be the spherical function taking value 1 on \( K \). We find
\[
\ell_v(a; f_3, f_3, v, 1) = \int_{K \times K} |W_0(a(y)\kappa a(C))|^2 \Phi_v(a(y)\kappa) |y| v^{-1} d^\infty y d\kappa.
\]
Writing \( \kappa a(C) = b' \kappa' \), we get
\[
\ell_v(s; \cdot \cdot \cdot) = \int_{K \times K} |W_0(a(y)\kappa')|^2 |y| v^{-1/2} d^\infty y d\kappa \Rightarrow (\kappa a(C))^{1/2-s} d\kappa.
\]
Note that uniformly in \( \kappa \in K_v \)
\[
C^{-1} \leq \text{Ht}(\kappa a(C)) \leq C \Rightarrow |\log(\cdot \cdot \cdot)| \leq \log C.
\]
Arguing as in the proof of Lemma 3.9, we also have for any \( k \in \mathbb{N}, \epsilon > 0 
\[
\left| \int_{|y| v \leq 1} \frac{|W_0(a(y)\kappa')|^2}{2} \log |y| v^k d^\infty y \right| \ll S(\kappa', W_0)^{2\epsilon} \int_{|y| v \leq 1} \left| \frac{|W_0(a(y)\kappa')|^2}{2} \log |y| v^k d^\infty y \right| \leq S(\kappa', W_0)^{\epsilon} \cdot \|W_0\|^2 - 2\epsilon \ll C(\omega_v)^{d\epsilon},
\]
for some absolute constants \( d, d' \). We deduce and conclude by
\[
\left| \ell_v^{(1)}(1/2; \cdot \cdot \cdot) \right| \ll_{k, \epsilon} C(\omega_v)^{d\epsilon}. \]
\]
3.2.3. Main Bounds in Regularized Term.

**Lemma 3.15.** (1) For \( \ell_v \) in [2.15], \( f_v \in \{(f_3, v, f_3, v) | K_v, (R_0, f_3, v, \overline{R_0} f_3, v) | K_v \} \) and any \( k \in \mathbb{N} \), we have
\[
\left| \ell_v^{(1)}(0; f_2, v, f_2, v; f_v) \right| \ll_{k} 1.
\]
(2) For \( \ell_v \) in [2.10] and \( f_v \in \{(R_0, f_3, v, \overline{R_0} f_3, v) | K_v, (f_3, v, \overline{R_0} f_3, v) | K_v \}, \ell_v(s; f_2, v, f_2, v; f_v) \) is non-vanishing only if \( \omega_v \) is trivial on \( v_v' \) (i.e., \( \{ \pm 1 \} \) if \( F_v = \mathbb{R}, = C(1) \) if \( F_v = \mathbb{C} \)). In this case we have for any \( k \in \mathbb{N} \)
\[
\left| \ell_v^{(1)}(0; f_2, v, f_2, v; f_v) \right| \ll_{k} 1.
\]
(3) For \( \ell_v \) in [2.17] and \( f_v = 1 \), we have for any \( k \in \mathbb{N} \) and \( \epsilon > 0 \)
\[
\left| \ell_v^{(1)}(0; f_3, v, f_3, v; 1) \right| \ll_{k, \epsilon} C(\pi_v)^{\epsilon}.
\]
Proof. Drop the subscript $v$ for simplicity. Since $f_2$ is $K$-invariant, the integral part of $\ell$ in (1) resp. (2) has the form
\[
\int_{F^\times} |W^\times_2(a(y))|^2 \cdot |y|^d y \int_K f(\kappa) d\kappa, \quad \text{ resp. } \int_{F^\times} |W^\times_2(a(y))|^2 \cdot \omega(\varphi)(y)|y|^d y \int_K f(\kappa) d\kappa.
\]
All assertions follow taking into account
\[
\int_{F^\times} |W^\times_2(a(y))|^2 \cdot |\log^k |y|| d^k y \ll_k 1, \quad \int_K |f_3(\kappa)|^2 d\kappa = \|a(C).f_0\|^2 = \|f_0\|^2 = 1,
\]
\[
\int_K |R_0 f_3(\kappa)|^2 d\kappa = \|R_0 f_3\| = \|f_3\| = 1, \quad \left| \int_K f_3(\kappa) \overline{R_0 f_3(\kappa)} d\kappa \right|^2 \leq \int_K |f_3(\kappa)|^2 d\kappa \cdot \int_K |R_0 f_3(\kappa)|^2 d\kappa.
\]
(3) is the same as Lemma 3.14 (3). \qed

4. Global Estimations

4.1. Regular Term. Applying the Fourier inversion to the first factor of the integrand in (2.5), we get
\[
(2.5) \quad \sum_{\pi' \leq \varphi' \in B(\pi)} C(\varphi') \int_{[PGL_2]} \varphi' \cdot E_3^f \cdot a(\tilde{t}).E_3^t + \sum_{\xi} \sum_{\Phi \in \pi, \xi^{-1}} \left( \int_{[PGL_2]} C(i\tau, \Phi) \int_{[\pi]} E(i\tau, \Phi) \cdot \left( a(\tilde{t}).E_3^f \cdot \overline{E_3^t} - E(a(\tilde{t}).E_3^f E_3^t) \right) + \sum_{\chi, \chi^2 = 1} \int_{[PGL_2]} \left( a(\tilde{t}).E_3^f \cdot \overline{E_3^t} - E(a(\tilde{t}).E_3^f E_3^t) \right) \cdot \chi \circ \det \right)
\]
with the Fourier coefficients
\[
C(\varphi') := \langle a(\tilde{t}).E_3^f \cdot \overline{E_3^t}, \varphi' \rangle_{[PGL_2]}, \quad C(i\tau, \Phi) = \langle a(\tilde{t}).E_3^f \cdot \overline{E_3^t} - E(a(\tilde{t}).E_3^f E_3^t), E(i\tau, \Phi) \rangle_{[PGL_2]}.
\]
We estimate the cuspidal contribution (4.1), the Eisenstein contribution (4.2) and the one-dimensional contribution (4.3) one by one and get

Lemma 4.1. The contribution of the regular term (2.5) is bounded as
\[
\langle C(\pi)K \rangle^{\varepsilon} \max \left( \frac{C(\pi)}{C(\omega)} - \frac{1}{2} \log K \|\ell\| (A + 1/2) \right), \frac{C(\pi)}{C(\omega)} - \frac{1}{2} \log K \|\ell\| (A + 1/2), -\frac{1}{2} \log K \|\ell\| (A + 1/2)
\].

Proof. The estimation follows from the last line of each subsequent subsection. \qed

4.1.1. Cuspidal Contribution. By (2.12), Lemma 3.2 (1), Lemma 3.3 (1), Lemma 3.11 (1), together with [22] and [8, Lemma 3] giving bounds for $L(1, \pi' \times \pi)$, we get
\[
\left| \int_{[PGL_2]} \varphi' \cdot \overline{E_3^t} \cdot a(\tilde{t}).E_3^f \right| \ll_{\varepsilon} \frac{C(\pi)}{C(\omega)} - \frac{1}{2} \log K \|\ell\| (A + 1/2) \cdot L(\frac{1}{2}, \pi') L(\frac{1}{2}, \pi' \otimes \omega) \cdot K^{-1/4 + \varepsilon}.
\]
Inserting (11), summing over $\varphi'$ and $\pi'$ like in [35] (6.16), applying [35, Corollary 6.7] and [38, Lemma 4.11] (to obtain $S_{d'}(a(\tilde{t}).E_3^f \cdot \overline{E_3^t} - E(a(\tilde{t}).E_3^f E_3^t)) \ll \log^3 K$ for some $d' > d$), we get
\[
11 \ll_{\varepsilon} \left( \frac{C(\pi)K}{C(\omega)} \right)^{1/2 + \theta} \frac{C(\pi)}{C(\omega)} - \frac{1}{2} \log K \|\ell\| (A + 1/2).
\]
4.1.2. Eisenstein Contribution. By [23, Lemma 3.2 (2), Lemma 3.4 (2), Lemma 3.14 (2), together with Siegel’s lower bound, we get

\[
\int \left[ PGL_2 \right] E(i\tau, \Phi) \cdot \left( a(\bar{t}).E^3_2 \cdot E^3_2 - \mathcal{E}(a(\bar{t}).E^3_2 E^3_2) \right) \ll_{c} C(\pi)^{-\frac{1}{2} + \varepsilon} \cdot (1 + |\tau|)^{c} S_{d}(\Phi_{i\tau}) \cdot \left| L\left( \frac{1}{2} + i\tau, \xi \right)^{2} L\left( \frac{1}{2} + i\tau, \xi\omega \right) L\left( \frac{1}{2} + i\tau, \xi\omega^{-1} \right) \right| \cdot K^{-\||\tau||/2 + \varepsilon}.
\]

Inserting \[38, \text{Theorem 1.1}\], summing over \( \Phi \) and \( \xi \) like in \[37, \text{§6.4}\], applying the analogue of \[35, \text{Corollary 6.7}\] for the fourth moment bound of Hecke \( L \)-functions and \[38, \text{Lemma 4.11}\], we get

\[
\left\langle 4.2 \right\rangle \ll_{c} (C(\pi)K)^{c} (C(\pi)/C(\omega))^{-1/2} C(\omega)^{-1/2 - (1 - 2\delta)/8} K^{-\||\tau||/2 + (1 - 2\delta)/8}.
\]

4.1.3. One Dimensional Contribution. Applying Remark 2.29 to \( \omega = 1 \), we see that

\[
\int \left[ PGL_2 \right] (a(\bar{t}).E^3_2 \cdot E^3_2 - \mathcal{E}(a(\bar{t}).E^3_2 E^3_2)) \cdot \chi \circ \det = \int \left[ PGL_2 \right] a(\bar{t}).E^3_2 \cdot E^3_2 \cdot \chi \circ \det
\]

is non-vanishing only if \( \chi = 1 \) (this is different from the cuspidal case where \( \pi_2 \otimes \chi \simeq \pi_2 \) is possible for non-trivial \( \chi \)). If \( \bar{t} = \omega_{p_1}^{-1} \omega_{p_2} \), write \( T(\bar{t}) = T(p_1^a)T(p_2^p) \) to be the corresponding product of Hecke operators, then we see [39, Proposition 2.27]

\[
\left\langle 4.2 \right\rangle \ll_{c} K^{-\frac{1}{2} + \varepsilon} \left\langle 4.2 \right\rangle \ll_{c} K^{-\frac{1}{2} + \varepsilon}.
\]

Similar argument leads to

\[
\left\langle 4.3 \right\rangle \ll_{c} K^{-\||\tau||} C(\pi)^{c}.
\]

4.2. Regularized Term.

Lemma 4.2. In \[28\], we have

\[
\left\langle 4.4 \right\rangle \ll_{c} (C(\pi)K)^{c} K^{-\||\tau||}.
\]

Proof. (1) First suppose \( \omega \neq 1 \). It is easy to write explicitly the \( L^2 \)-residue, taking \( |u_{\infty}| = 1 \) into account where we denote \( u_{\infty} = L(1, \omega^{-1})/L(1, \omega) \),

\[
\mathcal{E}(a(\bar{t}).E^3_2 E^3_2) = |L(1, \omega)^2 E_{reg}(\frac{1}{2}, a(\bar{t}), f_{3} f_{3}) + |L(1, \omega^{-1})|^2 E_{reg}(\frac{1}{2}, a(\bar{t}), R_{0} f_{3} R_{0} f_{3})
\]

\[
+ \left\{ \begin{array}{ll}
\frac{u_{\infty} L(1, \omega^{-1})^2 E(\frac{1}{2}, a(\bar{t}), R_{0} f_{3} R_{0} f_{3})}{2} + \frac{u_{\infty} L(1, \omega)^2 E(\frac{1}{2}, a(\bar{t}), f_{3} R_{0} f_{3})}{2} & \text{if } \omega^2 \neq 1 \\
\frac{u_{\infty} L(1, \omega^{-1})^2 E_{reg}(\frac{1}{2}, a(\bar{t}), R_{0} f_{3} R_{0} f_{3})}{2} + \frac{u_{\infty} L(1, \omega)^2 E_{reg}(\frac{1}{2}, a(\bar{t}), f_{3} R_{0} f_{3})}{2} & \text{if } \omega^2 = 1
\end{array} \right.
\]

We shall apply Proposition 5.3 to treat

\[
\left\langle 4.5 \right\rangle \ll_{c} (C(\pi)K)^{c} K^{-\||\tau||}.
\]

In fact, \[21\] , Lemma 3.4 (2), Lemma 3.14 (2) and Lemma 3.15 (2) imply

\[
|R_{h}^0(1/2, a(\bar{t}).E^3_2 \cdot E^3_2; a(\bar{t}), R_{0} f_{3})| \ll_{c} (C(\pi)K)^{c} K^{-\||\tau||}.\]
We shall apply \cite{38} Theorem 2.7 to treat
\[ \int_{[\mathcal{PGL}_2]}^{\text{reg}} a(\bar{t}), E_2^* \cdot E_2^- \cdot \mathcal{M}^{\text{reg}}(\frac{1}{2}, a(\bar{t}), f_3 f_3) \text{ resp. } \int_{[\mathcal{PGL}_2]}^{\text{reg}} a(\bar{t}), E_2^* \cdot E_2^- \cdot \mathcal{M}^{\text{reg}}(\frac{1}{2}, a(\bar{t}), f_3 f_3) \]

Combining \cite{24, 15} Lemma 3.18 (4) and Lemma \cite{38} (1) and Lemma \cite{38} (1) we get
\[ |R^{\text{hol}}(1/2, a(\bar{t}), E_2^* \cdot E_2^-; a(\bar{t}), f_3 f_3)| \text{ resp. } |R^{\text{hol}}(1/2, a(\bar{t}), E_2^* \cdot E_2^-; a(\bar{t}), R_0 f_3 R_0 f_3)| \ll_{F, e} K^{-\|\bar{t}\|^{+}}. \]

The bounds of the remaining terms corresponding to \cite{38} Theorem 2.7 follow from
\[ |P_K(a(\bar{t}), f_2 f_2)| \ll_e K^{-\|\bar{t}\|^{+}}, \quad |P_K(a(\bar{t}), f_3 f_3)| \ll_e K^{-\|\bar{t}\|^{+}}, \quad |P_K(a(\bar{t}), R_0 f_3 R_0 f_3)| \ll_{e} K^{-\|\bar{t}\|^{+}}; \]

where \( 0 \leq k \leq 3 \). The bounds in the first line are easy consequences of the general matrix coefficients decay \cite{14} Theorem 2], MacDonald’s formula \cite{11} Theorem 4.6.6] and the unitarity of \( R_0 \). For \( 4.4 \), we first note that we can assume \( n(\bar{t}) \leq 0 \), since with \( w_p \) the Weyl element at \( p \in S \) we have
\[ P_K(\mathcal{M}^{(k)}_{\frac{1}{2}} (a(\bar{t}), f_2) \cdot a(\bar{t}), f_3 \cdot f_3) = P_K(\mathcal{M}^{(k)}_{\frac{1}{2}} (w_p a(\bar{t}) w_p^{-1}, f_2) \cdot w_p a(\bar{t}) w_p^{-1}, f_3 \cdot f_3). \]

Extracting the components at \( p \in S^* \) \cite{22} and distinguishing elements related to \( f_2 \) from those to \( f_3 \) by putting a “*” (for example, \( f_3^* \otimes e_0 = f_3, f_3^* \otimes e_0^* = f_2 \)), Proposition \cite{51} (4-1) shows that
\[ a(\bar{t}), f_2 = f_3 \otimes a(\bar{t}), e_0 = \sum_{n(\bar{t}) \leq n(\bar{t})} O_{\epsilon}(K^{\frac{\|\bar{t}\|^{+}}{4} + \epsilon}) f_3 \otimes e_0^{\epsilon}, \]
\[ a(\bar{t}), f_3 = f_3^* \otimes a(\bar{t}), e_0 = \sum_{n(\bar{t}) \leq n(\bar{t})} O_{\epsilon}(K^{\frac{\|\bar{t}\|^{+}}{4} + \epsilon}) f_3^* \otimes e_0^{\epsilon}. \]

By \cite{37} Lemma 3.18 (4) or simply \cite{38} Lemma 4.4 (1), we have
\[ \mathcal{M}^{(k)}_{\frac{1}{2}} (e_0^S \otimes e_0^S) = O_{F, e}(K^{\|\bar{t}\|^{+}}) \mathcal{M}^{(k)}_{\frac{1}{2}} (e_0^S \otimes e_0^S) \]

Although the \( K_{S^*} \)-isotypic vectors \( e_0^S \) and \( e_0^S \) belong to different representations, Proposition \cite{51} (4-2) implies that their restriction to \( K_{S^*} \) are the same real function. We deduce that
\[ P_K(\mathcal{M}^{(k)}_{\frac{1}{2}} (a(\bar{t}), f_2) \cdot a(\bar{t}), f_3 \cdot f_3) = \sum_{n(\bar{t}) \leq n(\bar{t})} O_{\epsilon}(K^{\|\bar{t}\|^{+}}) P_K(\mathcal{M}^{(k)}_{\frac{1}{2}} (f_3^S \otimes f_3^S) = O_{F, e}(K^{\|\bar{t}\|^{+}}). \]

Similar argument applies to \( P_K(\mathcal{M}^{(k)}_{\frac{1}{2}} (a(\bar{t}), f_2) \cdot a(\bar{t}), R_0 f_3 R_0 f_3) \) and we obtain \cite{4.4].

(2) For the case \( \omega = 1 \), we have
\[ \mathcal{E}(a(\bar{t}), E_2^* \cdot E_2^-) = (\mathcal{E}(\mathcal{E}(2)) \frac{1}{2}, a(\bar{t}), f_3 f_3) + \frac{1}{4} \mathcal{E}(\mathcal{E}(1)) \frac{1}{2}, M_0^{(1)} a(\bar{t}), f_3 M_0^{(1)} f_3) \]
\[ + \frac{1}{2} \mathcal{E}(\mathcal{E}(1)) \frac{1}{2}, a(\bar{t}), f_3 M_0^{(1)} f_3) + \frac{1}{2} \mathcal{E}(\mathcal{E}(1)) \frac{1}{2}, M_0^{(1)} a(\bar{t}), f_3 f_3) \).

Hence, the extra difficulty is the analysis of \( M_0^{(1)} \), given in the next lemma. It follows that
\[ \|R_0^{(1)} f_3\| \ll_{e} C(\pi)^{\epsilon}, \quad R_0^{(1)} (f_3^S \otimes e_0^S) = O_{\epsilon}(K^{\epsilon}) \cdot R_0^{(1)} f_3^S \otimes e_0^S. \]

The argument of (1) can thus be easily adapted, using inequalities like
\[ \|P_K^{(1)} (R_0^{(1)} f_3^S \otimes f_3^S)\| \leq \|R_0^{(1)} f_3^S\| \cdot \|f_3^S\|, \quad \|P_K^{(1)} (R_0^{(1)} f_3^S \otimes R_0^{(1)} f_3^S)\| \leq \|R_0^{(1)} f_3^S\| \cdot \|f_3^S\|^2. \]

**Lemma 4.3.** Decompose \( \pi(1, 1) = \pi_{\infty} \otimes (\otimes_{p < \infty} \pi_p) \) and let \( S_{\pi} \) be a Sobolev norm system involving only the differential operators of \( K_{\infty} \). Write \( C_{K, \infty} \) for the Casimir element of \( K_{\infty} \).

1. For any \( f_\infty \in \pi_{\infty}, C \in (C_v)_v \in F_{\infty}^{\infty} \) we have
\[ \|R_0^{(1)} f_\infty\| \ll_{e} \|C_{K, \infty} f_\infty\| \cdot \|f_\infty\|^{1-\epsilon}, \quad \|C_{K, \infty} a(C), f_\infty\| \ll F \prod_{v \mid \infty} \max\{|C_v|^{2}, |C_v|^{-2}\} \cdot S_2(f_\infty); \]
(2) For \( \bar{n} = (n_p)_p \) with \( n_p \geq 0 \) \( \forall p \neq 0 \) for only finitely many \( p \), and \( e_{\bar{n}} = \otimes_p e_{n_p} \) with local elements defined in Section [5.1] we have
\[
\mathcal{R}_{0, \text{fin}}^{(1)} e_{\bar{n}} = O(\sum_p n_p \log q_p) \cdot e_{\bar{n}}.
\]

Proof. (2) is a direct consequence of [38, Lemma 4.4 (1)], which admits a similar version at \( \infty \). Namely, if we write \( e_{\bar{m}} \) for \( K_\infty \)-isotypic unitary vectors with the natural numeration given in [38, §4.2], then we have
\[
\mathcal{R}_{0, \infty}^{(1)} e_{\bar{m}} = O(\log \lambda_{\bar{m}}) \cdot e_{\bar{m}},
\]
where \( \lambda_{\bar{m}} \) is the eigenvalue of \( e_{\bar{m}} \) w.r.t. \( C_{K, \infty} \). Writing \( f_\infty = \sum_{\bar{m}} a_{\bar{m}} e_{\bar{m}} \) with \( a_{\bar{m}} \in \mathbb{C} \), we get
\[
\left\| \mathcal{R}_{0, \infty}^{(1)} f_\infty \right\|^2 = \left\| \sum a_{\bar{m}} \mathcal{R}_{0, \text{fin}}^{(1)} e_{\bar{m}} \right\|^2 = \sum |a_{\bar{m}}|^2 \left\| \mathcal{R}_{0, \text{fin}}^{(1)} e_{\bar{m}} \right\|^2 \lesssim \epsilon \sum |a_{\bar{m}}|^2 \lambda_{\bar{m}}^2 \leq \left( \sum |a_{\bar{m}}|^2 \lambda_{\bar{m}}^2 \right) \epsilon \left( \sum |a_{\bar{m}}|^2 \right)^{1-\epsilon},
\]
proving the first inequality in (1). The second inequality is elementary. \( \square \)

Lemma 4.4. In (6.4), we have
\[
\left| \int_{[PGL_2]} \mathcal{E}(a(\bar{t}).E^2_3 \cdot \overline{E^3_2}) \right| \lesssim_{F, \epsilon} (C(\pi)K)^\epsilon K^{-||\bar{t}||}.
\]

Proof. The proof is similar to that of Lemma 4.2. By \( \mathcal{M}_s f_2 = \lambda_F (s - 1/2) \), we easily obtain
\[
\mathcal{E}(a(\bar{t}).E^2_3 \cdot \overline{E^3_2}) = |A^s_F|^2 \cdot \left\{ E^{\text{reg},(2)}(\frac{1}{2}, a(\bar{t}).f_2) + \frac{1}{2} \lambda_F^{(1)} (\frac{1}{2}) E^{\text{reg},(1)}(\frac{1}{2}, a(\bar{t}).f_2)
\right.
\]
\[
+ \frac{1}{2} E^{\text{reg},(1)}(\frac{1}{2}, \mathcal{M}_0^{(1)} a(\bar{t}).f_2) + \frac{1}{4} \lambda_F^{(1)} (\frac{1}{2}) E^{\text{reg},(1)}(\frac{1}{2}, \mathcal{M}_0^{(1)} a(\bar{t}).f_2) \left. \right\}.
\]

We then apply Proposition 5.5 to treat each term of
\[
\left| \int_{[PGL_2]} \mathcal{E}(a(\bar{t}).E^2_3 \cdot \overline{E^3_2}) \right|, \quad \text{resp. } \left| \int_{[PGL_2]} \mathcal{E}(a(\bar{t}).E^2_3 \cdot \overline{E^3_2}, \mathcal{M}_0^{(1)} a(\bar{t}).f_2) \right| \lesssim_{F, \epsilon} (C(\pi)K)^\epsilon K^{-||\bar{t}||/2}.
\]

Most of the remaining terms have already been treated in the proof of Lemma 4.2 except
\[
\left| P_{K_0} \mathcal{M}_0^{(1)} a(\bar{t}).f_3 \mathcal{M}_0^{(1)} f_3 \right| \lesssim_{F, \epsilon} (C(\pi)K)^\epsilon K^{-||\bar{t}||/2},
\]
which follows from an analogue of Lemma 4.3 for \( \pi(1, \omega^{-1}) \) together with the argument given in Lemma 4.2 (2). One may find such technical analysis of \( \mathcal{M}_0^{(1)} \) (in fact explicit formula for \( \mathcal{M}_s \)) in [37]. \( \square \)

4.3. Degenerate Term.

Lemma 4.5. The contribution of (2.10) is
\[
\left| \int_{[PGL_2]} \mathcal{E}(a(\bar{t}).E^2_3 \cdot \overline{E^3_2}) \cdot \mathcal{E}(a(\bar{t}).E^2_3 \cdot \overline{E^3_2}) \right| \lesssim_{\epsilon} K^{-||\bar{t}|| + \epsilon}.
\]

Proof. By [38, Theorem 2.4], the desired bound follows from (4.4), which is already proved. \( \square \)
5. Complements

5.1. Base for Generalized New Vectors. We restrict to a local \( p \)-adic field \( F \) in this subsection. We assume the cardinality of the residue field is \( q \), and fix a uniformizer \( \varpi \). Recall that the subspace of “generalized new vectors” in a (unitary) admissible irreducible representation \( \pi \) of \( GL_2(F) \) consists of the vectors invariant by \( B_1(\mathfrak{p}) \); the level \( n \) subspace of generalized new vectors consists of the vectors invariant by \( K_1[p^n] \), where

\[
B_1(\mathfrak{p}) = \begin{pmatrix} \alpha^+ & 0 \\ 0 & 1 \end{pmatrix}, \quad K_1[p^n] = \begin{pmatrix} \alpha^+ & 0 \\ p^n & 1 + p^n \end{pmatrix}.
\]

Three base of the subspace of generalized new vectors arise naturally. Their mutual relations are our concern in this subsection.

**Basis 1:** Let \( e_0 \) be a unitary new vector of \( \pi \). \( \{ e_0, a(\varpi^{-1}) e_0, \cdots, a(\varpi^{-k}) e_0 \} \) is a (normal) basis of the level \( c(\pi) + k \) subspace for \( k \in \mathbb{N} \).

**Basis 2:** Applying Gramm-Schmidt to Basis 1, we get an ortho-normal basis of the level \( c(\pi) + k \) subspace, denoted by \( \{ e_0, e_1, \cdots, e_k \} \).

**Basis 3:** In the case \( \pi = \pi(1, \omega) \) with \( \omega \) unitary (hence \( c(\pi) = c(\omega) \)) resp. \( \pi \) is principal spherical, realized in the induced model, we denote by \( D_k \) the function upon restriction to \( K \) supported in \( K_0[p^{c(\pi)+k}] \), invariant by \( K_1[p^{c(\pi)+k}] \), taking value \( Vol(K_0[p^{c(\pi)+k}])^{1/2} \) at 1. \( \{ D_0, D_1, \cdots, D_k \} \) is also a (normal) basis of the level \( c(\pi) + k \) subspace for \( k \in \mathbb{N} \).

**Proposition 5.1.** (1) If \( \pi \) is such that \( L(s, \pi) = 1 \), then basis 1 and basis 2 coincide with each other. 
(2) If \( \pi \simeq St_\chi \) is Steinberg with unramified twist, then basis 1 and basis 2 are related by

\[
e_n = (1 - q^{-2})^{-1/2} \cdot \{ a(\varpi^{-n}) e_0 - \chi(\varpi) q^{-1} a(\varpi^{-(n-1)}) e_0 \},
\]

(3) If \( c(\omega) > 0, \pi = \pi(1, \omega) \), then basis 1 and basis 3 coincide with each other. Their relation to basis 2 is given by

\[
e_0 = D_0, \quad e_n = (1 - q^{-1})^{-1/2} (D_n - q^{-1/2} D_{n-1}), \forall n \geq 1;
\]

\[
D_n = (1 - q^{-1})^{1/2} \sum_{k=0}^{n} q^{-k/2} e_{n-k}, \forall n \geq 1.
\]

Moreover, the dimension \( d_n \) of the \( K \)-representation generated by \( e_n \) (which is irreducible) is

\[
d_0 = q^{(1 + q^{-1})}, \quad d_n = q^{n+c} (1 - q^{-2}), n \geq 1 \quad \text{where} \quad c = c(\omega).
\]

(4-1) If \( \pi \) is spherical with Satake parameters \( \alpha_1, \alpha_2 \), then basis 1 and basis 2 are related by

\[
e_1 = \alpha_1^{-1/2} \cdot \left\{ a(\varpi^{-1}) e_0 - q^{-1/2} (\alpha_1 + \alpha_2) a(\varpi^{-(n-1)}) e_0 \right\},
\]

\[
e_n = \alpha_n^{-1/2} \left\{ a(\varpi^{-n}) e_0 - q^{-1/2} (\alpha_1 + \alpha_2) a(\varpi^{-(n-1)}) e_0 + q^{-1} \alpha_1 \alpha_2 a(\varpi^{-2}) e_0 \right\}, \forall n \geq 2,
\]

with \( c_1 = 1 - q^{-1} |\alpha_1 + \alpha_2|^2 (1 + q^{-1})^2 \simeq 1 \), \( \iota = 1 - q^{-2} - \frac{q^{-1} - q^{-2} - q^{-3}}{(1 + q^{-1})^2} |\alpha_1 + \alpha_2|^2 \simeq 1 \);

\[
a(\varpi^{-1}) e_0 = \alpha_1^{1/2} e_1 + \frac{q^{-1/2}}{1 + q^{-1}} (\alpha_1 + \alpha_2) e_0,
\]

\[
a(\varpi^{-1}) e_0 = \sum_{k=0}^{n-2} q^{\frac{k}{2}} \frac{\alpha_1^{k+1} - \alpha_2^{k+1}}{\alpha_1 - \alpha_2} e_k + q^{\frac{n-1}{2}} \alpha_1^{n} e_n + q^{\frac{n-1}{2}} \alpha_2^{n} e_n + q^{\frac{n-1}{2}} \alpha_1^{n} e_n + q^{\frac{n-1}{2}} \alpha_2^{n} e_n + e_0.
\]

(4-2) If \( \pi \) is moreover principal, then their relations to basis 3 are given by
• **Basis 1 ⇔ Basis 3:**

\[ e_0 = D_0, \quad a(\varpi^{-n}).e_0 = \alpha_2^{-n} q^{-n/2} D_0 + \frac{1 - \alpha_1 \alpha_2^{-1} q^{-1}}{(1 + q^{-1})^{1/2}} \sum_{k=1}^{n} \alpha_1^{-k} \alpha_2^{-n} q^{-(n-k)/2} D_k, \forall n \geq 1; \]

\[ D_n = \frac{\alpha_2^{n} (1 + q^{-1})^{1/2}}{1 - \alpha_1 \alpha_2^{-1} q^{-1}} \cdot \{a(\varpi^{-n}).e_0 - \alpha_2^{-1} q^{-1/2} a(\varpi^{-(n-1)}).e_0\}, \forall n \geq 1. \]

• **Basis 2 ⇔ Basis 3:**

\[ e_1 = (1 + q^{-1})^{1/2} \cdot \{D_1 - (q + 1)^{-1/2} D_0\}, \quad e_n = (1 - q^{-1})^{-1/2} \cdot \{D_n - q^{-1/2} D_{n-1}\}, \forall n \geq 2; \]

\[ D_n = (1 - q^{-1})^{1/2} \sum_{k=0}^{n-2} q^{-k/2} e_{n-k} + q^{-(n-1)/2} (1 + q^{-1})^{-1/2} e_1 + q^{-(n-1)/2} (q + 1)^{-1/2} e_0, \forall n \geq 1. \]

Moreover, the dimension \( d_n \) of the \( K \)-representation generated by \( e_n \) is

\[ d_0 = 1, \quad d_1 = q, \quad d_n = q^n (1 - q^{-2}), n \geq 2. \]

**Proof.** The proof is very computational. We only give hints for the fastest way we have found.

1 & 2 Use the description of the Kirillov new vector given in [17, Table 1].

3 The first assertion follows by direct computation. The second uses again [17, Table 1] or direct computation of \( \langle D_n, D_m \rangle \) below in (4-2). The dimension formula follows by noticing and evaluating at 1, up to a complex number with norm 1, that

\[ e_n(\kappa) = d_n^{1/2}(\kappa, e_n, e_n), \forall \kappa \in K. \]

(4-1) We first play with the MacDonald’s formula [11, Proposition 4.6.6], from which we easily deduce that

\[ e'_n = a(\varpi^{-n}).e_0 - q^{-1/2}(\bar{\alpha}_1 + \bar{\alpha}_2)a(\varpi^{-(n-1)}).e_0 + q^{-1} \bar{\alpha}_1 \bar{\alpha}_2 a(\varpi^{-(n-2)}).e_0, n \geq 2 \]

is orthogonal to \( a(\varpi^{-k}).e_0 \) for \( 0 \leq k \leq n - 2 \), since \( \langle a(\varpi^{-n}).e_0, e_0 \rangle \) is of the form \( C_1 \bar{\alpha}_1^2 q^{-k/2} + C_2 \bar{\alpha}_2^2 q^{-k/2} \)

with \( C_1, C_2 \) constants. The verification that it is also orthogonal to \( a(\varpi^{-(n-1)}).e_0 \) uses the fact that \( \pi \) is unitary, i.e. either \( |\alpha_1| = |\alpha_2| = 1 \) or \( \alpha_1 \bar{\alpha}_2 = 1 \). Hence \( e'_n \) is proportional to \( e_n \) and the formula for \( c \) follows easily from \( \|e'_n\|^2 = \langle e'_n, a(\varpi^{-n}).e_0 \rangle \).

In order to invert the relations, we write \( f_n = a(\varpi^{-n}).e_0, \sigma_1 = q^{-1/2}(\bar{\alpha}_1 + \bar{\alpha}_2), \sigma_2 = q^{-1} \bar{\alpha}_1 \bar{\alpha}_2 \) and introduce the formal series

\[ \sum_{n=0}^{\infty} e'_n X^n = f_0 + \left( f_1 - \frac{\sigma_1}{1 + q^{-1}} f_0 \right) X + \sum_{n=2}^{\infty} \left( f_n - \sigma_1 f_{n-1} + \sigma_2 f_{n-2} \right) X^n \]

\[ = \left( \sum_{n=0}^{\infty} f_n X^n \right) \cdot \left( 1 - \sigma_1 X + \sigma_2 X^2 \right) + \frac{\sigma_1}{q + 1} f_0 X, \]

from which we get and conclude by

\[ \sum_{n=0}^{\infty} f_n X^n = \left( \sum_{n=0}^{\infty} (\bar{\alpha}_1 q^{-1/2} X)^n \right) \left( \sum_{n=0}^{\infty} (\bar{\alpha}_2 q^{-1/2} X)^n \right) \left( \sum_{n=0}^{\infty} e'_n X^n - \frac{\sigma_1}{q + 1} e_0 X \right). \]

(4-2) For the first relation, we evaluate \( a(\varpi^{-n}).e_0 \) at \( n_-(\varpi^k), k = 0, \ldots, n \) and use

\[ \begin{pmatrix} 1 & k \varpi^{-n} \\ \bar{\varpi}^{-n} & 1 \end{pmatrix} = \begin{pmatrix} \varpi^{-k} & \varpi^{-n} \\ \varpi^{k-n} & \varpi^{-n} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \varpi^{-k} \end{pmatrix}. \]

For the second, we apply Gramm-Schmidt to \( D_0, D_1, \ldots \) using

\[ \langle D_m, D_n \rangle = \left( \frac{\text{Vol}(K_0[p^{(\sigma)} + \max(m,n)])}{\text{Vol}(K_0[p^{(\sigma)} + \min(m,n)])} \right)^{1/2}. \]

The dimension formula follows the same way as in the proof of (3). \( \square \)
Corollary 5.2. Let $\pi$ be unitary spherical with Satake parameter $\alpha_1, \alpha_2$. Let $P_n$ denote the orthogonal projection onto the $K_0[p^n]$-invariant subspace of $\pi$. Then we have

$$P_{n-k}(a(\varpi^n).e_0) = q^{-\frac{k}{2}} \frac{\alpha_1^{k+1} - \alpha_2^{k+1}}{\alpha_1 - \alpha_2} a(\varpi^{-(n-k)}).e_0 - q^{-\frac{k}{2}} \frac{\alpha_1^{k} - \alpha_2^{k}}{\alpha_1 - \alpha_2} a(\varpi^{-(n-k-1)}).e_0, 0 \leq k \leq n-1;$$

$$P_0(a(\varpi^n).e_0) = q^{-\frac{n}{2}} \left\{ \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} - \frac{\alpha_1 + \alpha_2}{q+1} \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \right\} e_0.$$

Proof. Proposition 5.1 (4-1) gives

$$P_{n-1}(a(\varpi^n).e_0) = q^{-1/2}(\alpha_1 + \alpha_2) a(\varpi^{-(n-1)}).e_0 - q^{-1} \alpha_1 \alpha_2 a(\varpi^{-(n-2)}).e_0, n \geq 2;$$

$$P_1(a(\varpi^{-1}).e_0) = \frac{q^{-1/2}}{1+q^{-1}}(\alpha_1 + \alpha_2).e_0.$$

Suppose we have

$$P_{n-k}(a(\varpi^n).e_0) = q^{-k/2} A_k(\alpha_1, \alpha_2) a(\varpi^{-(n-k)}).e_0 - q^{-(k+1)/2} B_k(\alpha_1, \alpha_2) a(\varpi^{-(n-k-1)}).e_0, 0 \leq k \leq n - 1.$$ 

Since $P_{n-k-1} \circ P_{n-k} = P_{n-k-1}$, we get

$$\begin{pmatrix} A_{k+1} \\ B_{k+1} \end{pmatrix} = \begin{pmatrix} A_k \\ B_k \end{pmatrix}.$$

Diagonalizing the matrix, we easily get the desired formula. \(\square\)

5.2. Transposition Formula for Local Rankin-Selberg. Consider a local field $F$, a generic representation $\pi$ of $G = GL_2(F)$ with central character $\omega$, two induced representations $\pi_j = \pi(\chi_j, \chi_j')$ with $\omega \chi_1 \chi_2 \chi_2' = 1$. There are two ways of realizing the $GL_2(F)$-invariant trilinear form on $\pi^\infty \times \pi_1^\infty \times \pi_2^\infty$. Namely, we have

$$\ell_1: \pi^\infty \times \pi_1^\infty \times \pi_2^\infty \rightarrow \mathbb{C}, (e, f_1, f_2) \mapsto \int_{\mathbb{Z}N\backslash G} W_e(g)W_1(g)f_2(g)dg;$$

$$\ell_2: \pi^\infty \times \pi_1^\infty \times \pi_2^\infty \rightarrow \mathbb{C}, (e, f_1, f_2) \mapsto \int_{\mathbb{Z}N\backslash G} W_e(g)W_2(g)f_1(g)dg;$$

where $W_e$ resp. $W_j$ is the Whittaker function with respect to $\psi$ resp. $\psi_j$ of $e$ resp. $f_j$ for $j = 1, 2$.

Proposition 5.3. The two trilinear forms are related by

$$\ell_1 = \chi_1 \chi_2' (-1) \gamma(\frac{1}{2}, \pi, \chi_2'; \psi)^{-1} \ell_2,$$

where the gamma factor is the one appearing in the theory of $GL_2 \times GL_1$ (Hecke-Jacquet-Langlands).

Proof. Write $W$ for $W_e$. Taking $\Phi_j \in S(F^2)$ such that

$$f_j(g) = f_{\Phi_j}(g) = \chi_j(\det g) |\det g|^{\frac{1}{2}} \int_{F^\times} \Phi_j((0, t)g) \chi_j(\chi_j')^{-1}(t)|t|d^\times t,$$

we can proceed as [23, §8.2]

$$\ell_1 = \int_{\mathbb{Z}N\backslash G} W(g)W_1(g)\Phi_2((0, 1)g)\chi_2(\det g)|\det g|^{\frac{1}{2}}dg$$

$$= \int_{G} W(g)f_{\Phi_1}(wg)\Phi_2((0, 1)g)\chi_2(\det g)|\det g|^{\frac{1}{2}}dg$$

$$= \int_{F^\times G} \Phi_1((1, 0)g)W(a(t^{-1})g)\Phi_2((0, 1)g)\chi_1\chi_2(\det g)|\det g|((\chi_1'\chi_2)^{-1}(t))dg d^\times t$$

$$= \int_{G} \left( \int_{F^\times} W(a(t)g)\chi_1'\chi_2(t)d^\times t \right) \Phi_1((1, 0)g)\Phi_2((0, 1)g)\chi_1\chi_2(\det g)|\det g|dg.$$

The expression of $\ell_2$ is similar. Applying local functional equation to the inner integral and making variable change $g \mapsto a(-1)w^{-1}g$, we conclude. \(\square\)
5.3. Some Regularized Triple Product Formulas. All our regularized triple products are in the singular case, so that neither \[10\] nor \[23\] \S4.4] (especially \[23\] \S4.4.3] apply). \[38\] Theorem 2.7 has set an example of such analysis at the singular points. We need two more variants of it. We only give the proof of the first proposition as a recall on the technics of \[38, 39\] and omit the other one.

**Proposition 5.4.** Let \(1 \neq \omega\) be a non-trivial Hecke character. Let \(f_1, f_2 \in \pi(1, 1)\) and \(f_3 \in \pi(\omega, \omega^{-1})\). For any \(n \in \mathbb{N}\) and \(\omega^2 = 1\) resp. \(\omega^2 \neq 1\),

\[
\int_{[PGL_2]}^\text{reg} E^*(0, f_1) \cdot E^*(0, f_2) \cdot E^\text{reg.}(n)^{(1/2)} f_3 \quad \text{resp.} \quad \int_{[PGL_2]}^\text{reg} E^*(0, f_1) \cdot E^*(0, f_2) \cdot E^\text{reg.}(n)^{(1/2)} f_3
\]

is equal to the generalized Rankin-Selberg value

\[
\left(\frac{\partial^n R}{\partial s^n}\right)^\text{hol} \left(\frac{1}{2} E^*(0, f_1) \cdot E^*(0, f_2); f_3\right).
\]

**Proof.** Let \(E(f_1, f_2)\) be the \(L^2\)-residue of \(E^*(0, f_1) \cdot E^*(0, f_2)\) and \(\varphi := E^*(0, f_1) \cdot E^*(0, f_2) - E(f_1, f_2)\). In the case \(\omega^2 = 1\), we are reduced to computing

\[
\int_{[PGL_2]}^\text{reg} \varphi \cdot E^\text{reg.}(n)^{(1/2)} f_3 + \int_{[PGL_2]}^\text{reg} E(f_1, f_2) \cdot E^\text{reg.}(n)^{(1/2)} f_3.
\]

By \[38\] Proposition 2.6 (2)], the first term is equal to the generalized Rankin-Selberg value plus

\[
\lambda_F^{(n)}(0) \cdot \text{Pr}(f_3 \otimes \omega^{-1}) \cdot \int_{[PGL_2]}^\text{reg} \varphi \otimes \omega = \lambda_F^{(n)}(0) \cdot \text{Pr}(f_3 \otimes \omega^{-1}) \cdot \int_{[PGL_2]}^\text{reg} E^*(0, f_1) \cdot E^*(0, f_2 \otimes \omega),
\]

which is vanishing by \[39\] Lemma 3.1], while other terms are 0. The second term is also vanishing by \[38\] Theorem 2.4 (1)]. In the case \(\omega^2 \neq 1\), we proceed similarly using \[38\] Proposition 2.6 (1)].

**Proposition 5.5.** Let \(1 \neq \omega\) be a non-trivial Hecke character. Let \(f_1 \in \pi(1, \omega^{-1}), f_2 \in \pi(1, \omega)\) and \(f_3 \in \pi(1, 1)\). For any \(n \in \mathbb{N}\)

\[
\int_{[PGL_2]}^\text{reg} E(0, f_1) \cdot E(0, f_2) \cdot E^\text{reg.}(n)^{(1/2)} f_3
\]

is equal to the sum of the generalized Rankin-Selberg value

\[
\left(\frac{\partial^n R}{\partial s^n}\right)^\text{hol} \left(\frac{1}{2} E(0, f_1) \cdot E(0, f_2); f_3\right)
\]

and a weighted sum of the following terms with weights depending only on \(F\) (\(\lambda_F(s)\))

- \(\text{Pr}(f_1 f_2 \cdot f_3), \text{Pr}(M_{0, f_1} M_{0, f_2} \cdot f_3), \text{Pr}(M_{0, f_1} f_2 \cdot M_{0, f_2} \cdot f_3)\)
- \(\text{Pr}(f_1 f_2 \cdot M_{1/2} f_3), \text{Pr}(M_{0, f_1} M_{0, f_2} \cdot M_{1/2} f_3)\) for \(0 \leq l \leq n + 1\).

6. Appendix (with Nickolas Andersen): An Application to the Partition Function

6.1. An Explicit Waldspurger Formula. Explicit formula for the (square of the norm of the) Fourier coefficients of modular forms of half integral weights attract attention of many people since the work of Waldspurger \[34\]. Among others, there are a series of works of Baruch-Mao leading to \[12, 13\], establishing a Kohnen-Zagier type formula for the Kohnen plus space. The local difficulty at a complex place in the works of Baruch-Mao was recently solved by Chai-Qi \[12\]. For our purpose, we will need to work with a space slightly larger than the Kohnen plus one. We find the version of Waldspurger formula due to Qiu \[28\] the most convenient. We shall translate Qiu’s formula from the adelic setting into the classical setting over \(\mathbb{Q}\) in this subsection with complements.
6.1.1. Notations in Classical Setting. Let $N > 0$ be an integer divisible by 4. Let $\Gamma_0(N)$ be the subgroup of $SL_2(\mathbb{Z})$ with lower-left entry divisible by $N$. Let $\chi$ be a Dirichlet character of modulus $N$. The space of cusp forms of weight $k/2$ for $k \in \mathbb{Z}, 2 \nmid k$ with respect to the $\theta$-multiplier system $\mathfrak{S}_{k/2}(N, \chi)$, consists of real analytic functions on the upper half plane $f : \mathbb{H} \to \mathbb{C}$ such that:

1. $f \left( \frac{az + b}{cz + d} \right) = \chi(d) \cdot \left( \frac{c}{d} \right) \cdot \left( \frac{cz + d}{cz + d} \right)^{\frac{k}{2}} \cdot f(z)$, where $\left( \frac{a}{c} \right) \in \Gamma_0(N)$, $\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4} \\ i & \text{if } d \equiv -1 \pmod{4} \end{cases}$, and where the extended quadratic residue symbol is defined to be the Jacobi symbol if 0 < $d \equiv 1 \pmod{2}$, extended via $\frac{c}{d} = \frac{c}{|c|} \left( \frac{c}{d} \right)$ if $c \neq 0$, $\left( \frac{0}{d} \right) = \begin{cases} 1 & \text{if } d = \pm 1 \\ 0 & \text{otherwise} \end{cases}$;

2. $f(z)$ vanishes at the cusps for $\Gamma_0(N)$;

3. $\Delta_{k/2}f = \frac{s^2 - 1}{4}f$ for some spectral parameter $s \in \mathbb{C}$, where the $k/2$-th Laplacian operator is defined in the coordinates $z = x + iy$ by

$$\Delta_{k/2} := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{k}{2} y \frac{\partial}{\partial x}.$$

We will be particularly interested in the case $k = 1$. Forms $f \in S_{1/2}(N, \chi)$ with spectral parameter $s = \pm 1/2$ are obtained from holomorphic forms (by \cite{31}, (1.9))). Their ortho-complement in $S_{1/2}(N, \chi)$, denoted by $\tilde{S}_{1/2}(N, \chi)$, is the subspace of Maass forms. For Maass forms, the spectral parameter $s \in i\mathbb{R} \cup (-1/2, 0) \cup (0, 1/2)$. We will be only interested in forms with $s \in i\mathbb{R}$. Any $f \in \tilde{S}_{1/2}(N, \chi)$ admits a Fourier expansion at $\infty$ as (see for example \cite{31}, (2.1) & (2.3)) or \cite{5}, (1.1))

$$f(x + iy) = \sum_{n \in \mathbb{Z}, n \neq 0} c_f(n)e(nx)W_{\frac{1}{2} + \frac{s}{2}}(4\pi|n|y), \quad e(x) := e^{2\pi i x},$$

where $W_{\nu, \mu}(\cdot)$ is the classical Whittaker function defined in \cite{28}, §7.4.2] by

$$W_{\nu, \mu}(z) := \frac{\pi^{\mu - \nu}e^{-\frac{1}{2}z}}{\Gamma(\frac{1}{2} - \nu + \mu)} \int_0^\infty t^{\nu - \frac{1}{2} - \mu} \left( 1 + \frac{t}{z} \right)^{\frac{1}{2} - \nu + \mu} e^{-t} dt.$$

Introducing the Petersson inner product by

$$\langle f, f \rangle := \int_{\Gamma_0(N) \backslash \mathbb{H}} |f(z)|^2 \frac{dx dy}{y^2},$$

our central objects of interest are the normalized Fourier coefficients for square-free $n$

$$\frac{|c_f(n)|^2}{\langle f, f \rangle}.$$

6.1.2. Notations in Adelic Setting & Adelization. We essentially follow the notations in \cite{34} for the adelic setting. For a place $v$ of $\mathbb{Q}$, $\widehat{SL}_2(\mathbb{Q}_v)$ denotes the metaplectic double cover of $SL_2(\mathbb{Q}_v)$. Similarly, $\widehat{SL}_2(\mathbb{A})$ denotes the metaplectic double cover of $SL_2(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles for $\mathbb{Q}$. As sets, $\widehat{SL}_2(\mathbb{Q}_v)$ resp. $\widehat{SL}_2(\mathbb{A})$ is $SL_2(\mathbb{Q}_v) \times \mathbb{Z}_2$ resp. $SL_2(\mathbb{A}) \times \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{ \pm 1 \}$. The multiplication is defined via a cocycle $\beta_v$ resp. $\beta_\mathbb{A}$ by

$$(\sigma, \epsilon)(\sigma', \epsilon') = (\sigma \sigma', \epsilon \epsilon' \beta(\sigma, \sigma')),$$

$\beta = \beta_v$ resp. $\beta_\mathbb{A}$.

The local cocycles are defined by

$$\beta_v(\sigma, \sigma') = (x(\sigma), x(\sigma')), \quad (-x(\sigma)x(\sigma'), x(\sigma x(\sigma'))_v \cdot s_v(\sigma)s_v(\sigma')s_v(\sigma').$$

\footnote{We omit the dependence on the multiplier system in the notations, because only the theta one has been adelized in the literature and is relevant in this subsection. In the next subsection, dependence on the multiplier systems will appear in the relevant notations.}
where $(\cdot,\cdot)_v$ is the Hilbert symbol and the functions $x(\cdot)$ and $s_v$ are defined by

$$
x(A) = \begin{cases} 
c & \text{if } c \neq 0 \\
d & \text{if } c = 0 \end{cases}, \quad s_v(A) = \begin{cases} 
(c,d)_p & \text{if } v = p < \infty, cd \neq 0 \text{ and } 2 \nmid v_p(c) \\
1 & \text{otherwise}
\end{cases}.
$$

If $\sigma = (\sigma_v)$ and $\sigma' = (\sigma'_v) \in \SL_2(\mathbb{A})$, we put

$$s_{\mathbb{A}}(\sigma) = \prod_v s_v(\sigma_v), \quad \beta_{\mathbb{A}}(\sigma,\sigma') = \prod_v \beta_v(\sigma_v,\sigma'_v).$$

The map $\sigma \mapsto (\sigma, s_{\mathbb{A}}(\sigma))$ is a homomorphism from $\SL_2(\mathbb{Q})$ to $\SL_2(\mathbb{A})$. We write its image as $\SL_2(\mathbb{Q})$ by abuse of notations.

If $\sigma \in \SL_2(\mathbb{Q}_v)$ resp. $\SL_2(\mathbb{A})$, we write $[\sigma]$ for the element $(\sigma,1)$ in $\tilde{\SL}_2(\mathbb{Q}_v)$ resp. $\tilde{\SL}_2(\mathbb{A})$. We use the matrix notation

$$\begin{bmatrix} a & b \\
c & d \end{bmatrix} = \begin{bmatrix} a & b \\
c & d \end{bmatrix}, 1 \in \tilde{\SL}_2(\mathbb{Q}_v) \text{ or } \tilde{\SL}_2(\mathbb{A}).$$

For $\sigma \in \SL_2(\mathbb{Q}_v)$, we write $\sigma |_v$ for the element $(\sigma_v')_v' \in \SL_2(\mathbb{A})$ such that $\sigma_v = \sigma$, $\sigma_v' = 1$ if $v' \neq v$. If $U$ is a subset of $\SL_2(\mathbb{Q}_v)$ resp. $\SL_2(\mathbb{A})$, we write $U$ for its inverse image in $\tilde{\SL}_2(\mathbb{Q}_v)$ resp. $\tilde{\SL}_2(\mathbb{A})$. In particular, $\SO_2(\mathbb{R})$ is a group isomorphic to $\mathbb{R}/4\pi\mathbb{Z}$ with an isomorphism given by

$$\tilde{\kappa} : \mathbb{R}/4\pi\mathbb{Z} \to \tilde{\SO}_2(\mathbb{R}), \quad \theta \mapsto \begin{cases} 
(\kappa(\theta),1) & \text{if } \theta \in (-\pi,\pi] + 4\pi\mathbb{Z} \\
(\kappa(\theta),-1) & \text{if } \theta \in (\pi,3\pi] + 4\pi\mathbb{Z}
\end{cases},$$

where we have written

$$\kappa(\theta) := \begin{pmatrix} \cos \theta & \sin \theta \\
-\sin \theta & \cos \theta \end{pmatrix}.$$

We put $\Gamma_\infty = \SO_2(\mathbb{R}), \Gamma_p = \SL_2(\mathbb{Z}_p)$. For $n \in \mathbb{N}$, we denote by $\Gamma_p(n)$ the subgroup of elements in $\Gamma_p$ whose lower-left entry $c$ satisfies $v_p(c) \geq v_p(n)$. For any prime $p \neq 2$, $\beta_p(\cdot,\cdot)$ is trivial over $\Gamma_p \times \Gamma_p$. If $\Gamma_2^1(4) < \Gamma_2(4)$ is the subgroup of matrices whose upper-left element lies in $1 + 4\mathbb{Z}_2$, then $\beta_2(\cdot,\cdot)$ is trivial over $\Gamma_2^1(4) \times \Gamma_2^1(4)$, but not trivial over $\Gamma_2(4) \times \Gamma_2(4)$ (see [19, Proposition 2.8]).

A function $\varphi$ on $\tilde{\SL}_2(\mathbb{A})$ is called genuine if

$$\varphi(\gamma(\sigma,\epsilon)) = \epsilon \varphi(\sigma,1), \quad \forall \gamma \in \SL_2(\mathbb{Q}).$$

We equip $\SL_2(\mathbb{A})$ with the Tamagawa measure $d\sigma$ such that $\Vol(\SL_2(\mathbb{Q}) \setminus \SL_2(\mathbb{A})) = 1$. We denote by $L^2(\SL_2, -)$ the space of genuine functions such that

$$\langle \varphi, \varphi \rangle = \int_{\SL_2(\mathbb{Q}) \setminus \SL_2(\mathbb{A})} |\varphi([\sigma])|^2 d\sigma < \infty.$$

$\tilde{\SL}_2(\mathbb{A})$ acts on $L^2(\tilde{\SL}_2, -)$ via right translation. The resulted representation is denoted by $\tilde{\rho}$. The local component at $v$, i.e., the associated representation of $\tilde{\SL}_2(\mathbb{Q}_v)$, is denoted by $\tilde{\rho}_v$.

Recall that $\psi$ is the unitary character of $\mathbb{Q}\setminus \mathbb{A}$, with local component $\psi_v$ such that for $t \in \mathbb{R}$, $\psi_\infty(t) = e^{2\pi it}$. For $t \in \mathbb{Q}_v$, we have the Weil index $\gamma_v(t)$ associated with the character $\psi_v$ and the quadratic form $tx^2$. We write

$$\tilde{\gamma}_v(t) = (t,t)v \gamma_v(t) \gamma_v(1)^{-1}.$$

Then we have the relations

$$\tilde{\gamma}_v(tt') = (t,t')v \gamma_v(t') \gamma_v(t), \quad \tilde{\gamma}_v(t^2) = 1.$$

Let $p$ be a prime number. If $p \neq 2$ and $t \in \mathbb{Z}_p^\times$, then $\tilde{\gamma}_p(t) = 1$. If $p = 2$, then

$$\tilde{\gamma}_2(1 + 4\mathbb{Z}_2) = 1, \quad \tilde{\gamma}_2(-1 + 4\mathbb{Z}_2) = -i.$$

We define a map

$$\tilde{\varepsilon}_2 : \tilde{\Gamma}_2(4) \to \mathbb{C}^1, \quad (\sigma,\epsilon) \mapsto \begin{cases} 
\epsilon \tilde{\gamma}_2(1)^{-1}(c,d) s_2(\sigma) & \text{if } c \neq 0 \\
\epsilon \tilde{\gamma}_2(d) & \text{if } c = 0 \quad \text{for } \sigma = \begin{pmatrix} a & b \\
c & d \end{pmatrix}.
\end{cases}$$

It can be verified that $\tilde{\varepsilon}_2$ is a character.
If \( \chi \) is a Dirichlet character of modulus \( N_\chi \), we denote by \( \chi \) the idele class group character of \( \mathbb{Q}^x / \mathbb{A}^x \) associated with \( \chi \). We have a decomposition \( \check{\chi} = \otimes_v \chi_v \) such that \( \check{\chi}_\infty(t) = 1 \) for \( t \in \mathbb{R}_{>0} \), and for primes \( p \nmid N_\chi, \check{\chi}_p(p) = \chi(p) \). \( \check{\chi} \) is the *adelization* of \( \chi \).

If \( f \in \widetilde{S}_{1/2}(N, \chi) \), then there is a smooth function \( \varphi = \varphi_f \in L^2(\widetilde{\text{SL}}_2, -) \) uniquely determined by

\[
\varphi_f \left( \left[ \begin{array}{cc} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{array} \right] \kappa(\theta) \right) = e^{i\theta} f(x + yi), \quad \forall x, y \in \mathbb{R}, y > 0, \theta \in (-\pi, \pi],
\]

\( \varphi \) is the *adelization* of \( f \). It satisfies:

1. If \( p \nmid N \) and \( \sigma \in \Gamma_p \), then \( \check{\rho}_p(\sigma) \varphi = \varphi \);
2. If \( p \mid N, p \neq 2 \) and \( \sigma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_p(N) \), then \( \check{\rho}_p([\sigma]) \varphi = \chi(N,d) \varphi \);
3. If \( N, \chi \nmid N, \chi = 2 \) and \( \sigma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(2,N,\chi) \), then \( \check{\rho}_2([\sigma]) \varphi = \check{\varepsilon}_2([\sigma]) \chi(N,d) \varphi \);
4. For \( \theta \in \mathbb{R} \), \( \check{\rho}_\infty(\kappa(\theta)) \varphi = e^{i\theta/2} \varphi \);
5. \( \check{\rho}_\infty(\Delta) \varphi = \frac{s^2 - 1}{4} \varphi \) for the Casimir element of the Lie group \( \widetilde{\text{SL}}_2(\mathbb{R}) \)

\[
\Delta := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial^2}{\partial x \partial \theta}
\]

defined with respect to the following coordinates in the Iwasawa decomposition of \( \widetilde{\text{SL}}_2(\mathbb{R}) \)

\[
\left( \begin{array}{cc} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{array} \right) \kappa(\theta) , \quad x, \theta \in \mathbb{R}, y > 0.
\]

The existence and uniqueness of \( \varphi \) is proved in the same manner as \(^3\) \[ III.B.1 \] Proposition 3] and left as an exercise to the reader.

### 6.1.3. Qiú’s Formula & Complements.

It is possible to find an orthonormal basis of \( \widetilde{S}_{1/2}(N, \chi) \) such that every element \( f \) in the basis corresponds to \( \varphi = \varphi_f \), which generates a single genuine irreducible cuspidal automorphic sub-representation \( \tilde{\pi} = \otimes_v \tilde{\pi}_v \) of \( \tilde{\rho} \). For any \( \alpha \in \mathbb{Q}^x \), we introduce the global Whittaker period functional on \( \tilde{\rho} \) hence on \( \tilde{\pi} \)

\[
\ell_\alpha(\varphi) := \int_{\mathbb{Q}^A} \varphi \left( \left[ \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right] \right) \psi(-\alpha x) dx, \quad \varphi \in \tilde{\pi}.
\]

In the above integral, the measure on \( \mathbb{A} \) is the usual Tamagawa measure whose local component is the self-dual measure with respect to \( \psi_v \). Choosing any inner product \( \langle \cdot, \cdot \rangle \) on \( \tilde{\pi}_v \) and taking a pair of vectors \( \varphi_{1,v}, \varphi_{2,v} \in \tilde{\pi}_v \), the following function

\[
C_{\varphi_{1,v},\varphi_{2,v}} : \mathbb{Q}_v \to \mathbb{C}, \quad x_v \mapsto \langle \tilde{\pi}_v([n(x_v)]), \varphi_{1,v}, \varphi_{2,v} \rangle_v
\]

defines a tempered distribution on \( \mathcal{S}(\mathbb{Q}_v) \). We denote its Fourier transform for the hermitian pairing by

\[
W_{\varphi_{1,v},\varphi_{2,v}}(y_v) := \int_{\mathbb{Q}_v} C_{\varphi_{1,v},\varphi_{2,v}}(x_v) \psi_v(-x_v y_v) dx_v,
\]

where the integral is interpreted in a certain sense of regularization. Let \( \chi_\alpha \) be the quadratic Hecke character associated with the quadratic extension \( \mathbb{Q}(\sqrt{\alpha})/\mathbb{Q} \). Let \( \pi = \otimes_v \pi_v = \Theta_{\text{PGL}_2(\mathbb{A})}(\tilde{\pi}, \psi) \) be the global theta lift of \( \tilde{\pi} \) to \( \text{PGL}_2(\mathbb{A}) \) with respect to \( \psi \). We may assume that \( \varphi_f = \varphi = \otimes_v \varphi_v \) is a (abstractly) decomposable vector. Then Qiú’s formula \(^8\) specialized to our setting reads

\[
|\ell_\alpha(\varphi)|^2 \langle \varphi, \varphi \rangle = \frac{1}{2} \frac{L(\frac{1}{2}, \pi \otimes \chi_\alpha)}{L(1, \pi, \text{Ad})} \cdot W_{\varphi_{\infty,v},\varphi_{\infty,v}}(\alpha) \cdot \prod_p \frac{L_p(1, \pi, \text{Ad})}{L_p(\frac{1}{2}, \pi \otimes \chi_\alpha)} W_{\varphi_{p},\varphi_{p}}(\alpha),
\]

\(^5\)Note the difference on the conventions for \( L \)-functions: the ours are without factors at infinite places.
We shall relate the LHS of (6.5) to the classical counterpart (6.4) and compute/bound the RHS. If we identify $\mathbb{H}$ diffeomorphically with a subgroup of the Borel subgroup of $SL_2(\mathbb{R})$, equip $\Gamma_\infty$ and $\Gamma_p$ with the probability Haar measure, this gives $SL_2(\mathbb{A})$ another Haar measure, called the hyperbolic measure and denoted by $d_h \sigma$. It is easy to see

$$\text{Vol}(SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}), d_h \sigma) = \text{Vol}(SL_2(\mathbb{Z}) \backslash \mathbb{H}) = \pi/3.$$ 

Since $f$ is invariant by $\Gamma_0(N)$, $\varphi = \varphi_f$ is invariant by $\Gamma_p(N)$ for all primes $p$. By the strong approximation theorem, we get

$$\int_{SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A})} |\varphi([\sigma])|^2 d_h \sigma = \frac{1}{[SL_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N) \backslash \mathbb{H}} |f(z)|^2 \frac{dx dy}{y^2}.$$

It follows that

$$\langle \varphi, \varphi \rangle = \frac{3}{\pi} \cdot \frac{1}{[SL_2(\mathbb{Z}) : \Gamma_0(N)]} \langle f, f \rangle.$$ 

In particular, $\varphi$ is invariant by $[n(\mathbb{Z}_p)]$ for all primes $p$. Hence the non-vanishing of $\ell_n(\varphi)$ implies $\psi_p(\alpha \mathbb{Z}_p) = 1$, thus $\alpha \in \mathbb{Z}_p$ for all $p$, or equivalently $\alpha = n \in \mathbb{Z} - \{0\}$. Now that $Q \backslash A/\mathbb{Z} \cong Z \backslash R$ and $\tilde{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ has total mass 1 for our measure normalization, we get by (6.4)

$$\ell_n(\varphi) = \int_{\mathbb{Z} \backslash R} \varphi \left( \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \right) \psi_{\infty}(-nu) du = \int_{\mathbb{Z} \backslash R} f(u + i)e(-nu) du = c_f(n)W_{\text{sgn}(n)}(4\pi|n|).$$

It can be inferred from the classification of unitary irreducible representations of $\mathbb{SL}_2(\mathbb{R})$ that $\tilde{\pi}_\infty = \tilde{\pi}(-s, 1/2)$ $\mathbb{SL}_2(\mathbb{R})$ consisting of functions $\phi$ on $\mathbb{SL}_2(\mathbb{R})$ satisfying:

1. $\phi \left( \left( \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix}, \epsilon \right) \left( \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \right) g \right) = \epsilon y^{1-s}\phi(g)$ for all $y > 0, u \in \mathbb{R}$ and $g \in \mathbb{SL}_2(\mathbb{R})$;
2. $\phi(\tilde{n}(\pi) g) = i\phi(g)$ for all $g \in \mathbb{SL}_2(\mathbb{R})$.

Moreover, $\varphi_\infty \in \tilde{\pi}(-s, 1/2)$ satisfies

$$\varphi_\infty(\tilde{n}(\pi)g) = e^{\theta/2}\varphi_\infty(g), \quad \forall \theta \in \mathbb{R}, g \in \mathbb{SL}_2(\mathbb{R}).$$

We shall choose the inner product in the “line model” by putting

$$\langle \phi, \phi \rangle_\infty := \int_{\mathbb{R}} \left| \phi \left( \left( \begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix} \right) \right) \right|^2 du, \quad \phi \in \tilde{\pi}(-s, 1/2).$$

We normalize $\varphi_\infty(1) = 1$ and deduce

$$\langle \varphi_\infty, \varphi_\infty \rangle_\infty = \int_{\mathbb{R}} \frac{1}{1 + u^2} du = \pi.$$ 

Define the Fourier transform of $\varphi_\infty$ in the line model by

$$\xi_\infty(x) := \int_{\mathbb{R}} \varphi_\infty \left( \left( \begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix} \right) \right) e(-ux) du.$$ 

This Fourier transform is in the sense of $L^2$ functions and the above integral should be interpreted in a certain sense of regularization, by analytic continuation in $s$ or Cauchy principal value for example. It is intimately related to the Whittaker model of $\tilde{\pi}_\infty$ and can be studied more directly via a deformation of contour in the complex plane as in the proof of [33, §III.2 Proposition 7]. In particular, $\xi_\infty(x)$ is rapidly decreasing at $\pm\infty$.

**Lemma 6.1.** $|\xi_\infty(x)|^2$ is the Fourier transform of $u \mapsto \langle \tilde{\pi}_\infty([n(u)] \varphi_\infty, \varphi_\infty \rangle$ in the sense of tempered distributions. Consequently, we have

$$\pi^{-1}|\xi_\infty(x)|^2 = W_{\varphi_\infty, \varphi_\infty}(x).$$
Proof. By definition, it is easy to see that
\[ \tilde{\pi}_{\infty}([n(u)]), \xi_{\infty}(x) = e(ux)\xi_{\infty}(x). \]
Hence for \( P \in \mathcal{S}(\mathbb{R}) \), we can apply the Plancherel formula over \( \mathbb{R} \) and Fubini to get
\[
\int_\mathbb{R} \langle \tilde{\pi}_{\infty}([n(u)]), \varphi_{\infty} \rangle_{\varphi_{\infty}} \cdot P(u) \, du = \int_\mathbb{R} \int_\mathbb{R} e(ux)\xi_{\infty}(x)\overline{\xi_{\infty}(x)} \cdot P(u) \, dx \, du
= \int_\mathbb{R} |\xi_{\infty}(x)|^2 \cdot P(x) \, dx.
\]
This proves the first assertion and the second one follows readily. \( \square \)

For \( \epsilon > 0 \) small, let \( C^+_{\epsilon} \) be the contour consisting of three parts: (i) the directed semi-line linking \( i\infty \) to \( i(1+\epsilon) \); (ii) the anti-clockwise circumference of the circle centered at \( i \) with radius \( \epsilon \); (iii) the directed semi-line linking \( i(1+\epsilon) \) to \( i\infty \). Let \( C^-_{\epsilon} \) be the mirror image of \( C^+_{\epsilon} \) with respect to the real axis. For \( x > 0 \), we have (recall the classical Whittaker function (6.2))
\[
\xi_{\infty}(x) = \int_\mathbb{R} \frac{1}{(1+u^2)^{1/4}} \cdot \left( \frac{u-i}{|u-i|} \right)^{1/2} \cdot e^{-2\pi iux} \, du
= \lim_{\epsilon \to 0} \int_{C^-_{\epsilon}} \frac{1}{(u+i)^{1/4} - (u-i)^{1/4}} \cdot e^{-2\pi iux} \, du
= (ie^{-i\pi s} + e^{i\pi s}) \cdot \int_0^\infty y^{-1/4} (2 + y)^{-1/4} \cdot e^{-2\pi(y+1)x} \, dy
= (ie^{-i\pi s} + e^{i\pi s}) \cdot 2^{-1/4} \cdot (2\pi x)^{-1/4} \cdot \Gamma(\frac{3}{4} + \frac{s}{2}) \cdot W_{\frac{1}{4}, \frac{1}{2}}(4\pi x).
\]
Similarly, for \( x < 0 \) we have
\[
\xi_{\infty}(x) = \lim_{\epsilon \to 0} \int_{C^+_{\epsilon}} \frac{1}{(u+i)^{1/4} - (u-i)^{1/4}} \cdot e^{-2\pi iux} \, du
= (ie^{i\pi s} - e^{-i\pi s}) \cdot \int_0^\infty y^{-1/4} (2 + y)^{-1/4} \cdot e^{-2\pi(y+1)|x|} \, dy
= (ie^{i\pi s} - e^{-i\pi s}) \cdot 2^{-1/4} \cdot (2\pi|x|)^{-1/4} \cdot \Gamma(\frac{3}{4} + \frac{s}{2}) \cdot W_{-\frac{1}{4}, \frac{1}{2}}(4\pi|x|).
\]
Summarizing the formulas, we have for \( x \neq 0 \) (recall \( s \in i\mathbb{R} \))
\[
\left| \frac{\xi_{\infty}(x)}{\pi} \right|^2 = \frac{e^{-is} + e^{is}}{\pi} \cdot \left| \Gamma \left( \frac{1+s}{2} \right) \right|^2 \cdot \left| W_{\frac{1}{2}, \frac{3}{2}}(4\pi|x|) \right|^2.
\]
At a place \( p \nmid nN \), the local factor in (6.5) is 1 (see the paragraph just below (3.22)). Remember that we are only interested in square-free \( n \). Hence \( p \nmid N, p \nmid n \) implies \( p \neq 2, p \parallel n \). At such a place, \( \varphi_p \) is spherical, hence lies in a spherical representation
\[ \tilde{\pi}_p \simeq \tilde{\pi}(\mu) = \text{Ind}_{B(p)}^{\text{SL}_2(Q_p)}(\mu, \chi \psi_p) = \left\{ \phi : \text{SL}_2(Q_p) \to \mathbb{C} \mid \phi \left( \begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix} \right) = e^{\chi \psi_p(y)(y)(y)} \phi(g) \right\}, \]
where we recall the notations (see (3.1.3)): (1) \( \mu \) is an unramified, i.e., trivial on \( \mathbb{Z}_p^\times \), character of \( Q_p^\times \), which is either unitary or equal to \( \chi |_{p}^{a} \) with \( \chi \) unramified quadratic and \( 0 \neq a \in (-1/2, 1/2) \); (2) \( \chi \psi_p \) is defined via \( \gamma(t, \psi_p) \), the Weil index associated with \( \psi_p \) and the quadratic form \( tx^2 \), by
\[ \chi_{\psi_p}(t) := (-1, t)_p \cdot \gamma(t, \psi_p) \cdot \gamma(1, \psi_p)^{-1}, \]
which satisfies
\[ \chi_{\psi_p}(t_1t_2) = (t_1, t_2)_p \chi_{\psi_p}(t_1) \chi_{\psi_p}(t_2), \quad \chi_{\psi_p}(t^2) = 1. \]
Recall the formula for the normalized spherical matrix coefficient \( \bar{C}_p(t) := \frac{\langle \tilde{Y}_p(x), \varphi_p \rangle}{\langle \varphi_p, \varphi_p \rangle} = \frac{|t|_p \chi_p(t)}{1 + p^{-1}} \cdot \left( \mu(t) \cdot \frac{1 - \mu^{-2}(p)p^{-1}}{1 - \mu^{-2}(p)} + \mu^{-1}(t) \cdot \frac{1 - \mu^2(p)p^{-1}}{1 - \mu^2(p)} \right), \)

where \( |t|_p \leq 1, \xi := \left[ \begin{smallmatrix} t \\ 0 \\ 0 \end{smallmatrix} \right] \).

Also recall the formula \( 28 \) (3.18), the precise meaning of regularization

\[
W_{\varphi_p,\varphi_p}(n) = \lim_{m \to \infty} \int_{p^{-m}Z_p} C_{\varphi_p,\varphi_p}(x)\psi_p(-nx)dx.
\]

Write \( C_p(\cdot) = C_{\varphi_p,\varphi_p}(\cdot) \) for simplicity of notations. By the invariance of \( \varphi_p \) by \( [n(Z_p)] \), we have

\[
\int_{Z_p} C_p(x)\psi_p(-nx)dx = \text{Vol}(Z_p) = 1.
\]

\( \varphi_p \) is also invariant by \( \xi^{-1} \) for \( t \in Z_p^\times \), which implies \( C_p(\cdot t^2) = C_p(\cdot) \) since

\[
\begin{bmatrix} 1 \\ x \\ 0 \end{bmatrix} = \xi \cdot \begin{bmatrix} 1 \\ x \end{bmatrix} \cdot \xi^{-1}.
\]

Hence \( C_p \) is invariant by multiplication by \( (Z_p^\times)^2 \), in particular by \( 1 + pZ_p \). Thus for \( m \geq 3 \), we have

\[
\int_{p^{-m}Z_p} C_p(x)\psi_p(-nx)dx = \sum_{b \in Z_p^\times/(1+pZ_p)} C_p(p^{-m}b)\psi_p(-np^{-m}b) \int_{p^{-m}Z_p} \psi_p(-nx)dx = 0.
\]

For \( m = 1, 2 \), we need the following equation for \( |x|_p < 1 \)

\[
\begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & x^{-1} \end{bmatrix} \cdot \begin{bmatrix} x^{-1} & 0 \\ 0 & x \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ x^{-1} & 1 \end{bmatrix} \cdot (1, (-1, x)_p) \Rightarrow C_p(x) = (-1, x)_p \cdot \bar{C}_p(x^{-1}).
\]

For \( m = 1 \), we have the computation

\[
\int_{p^{-1}Z_p} (-1, x)_p\chi_p(x^{-1})\psi_p(-nx)dx = p \int_{Z_p^\times} (-1, p^{-1}x)_p\psi_p(px^{-1})dx = p(-1, p)_p\chi_p(p) \int_{Z_p^\times} (p, x^{-1})dx = 0,
\]

where we have used \( \chi_p(Z_p^\times) = 1 \), which can be deduced from \( 12 \), Corollary 3.2]. It follows that

\[
\int_{p^{-1}Z_p} C_p(x)\psi_p(-nx)dx = 0.
\]

Similarly for \( m = 2 \), we have

\[
\int_{p^{-2}Z_p} (-1, x)_p\chi_p(x^{-1})\psi_p(-nx)dx = p^2 \int_{Z_p^\times} \psi_p(-np^{-2}x)dx = -p.
\]

It follows that

\[
\int_{p^{-1}Z_p} C_p(x)\psi_p(-nx)dx = \frac{-p^{-1}}{1 + p^{-1}} \cdot \left( \mu(p^2) \cdot \frac{1 - \mu^{-2}(p)p^{-1}}{1 - \mu^{-2}(p)} + \mu^{-1}(p^2) \cdot \frac{1 - \mu^2(p)p^{-1}}{1 - \mu^2(p)} \right)
\]

is \( \leq 1 \) in absolute value. We deduce the bound

\[
W_{\varphi_p,\varphi_p}(n) \leq 1.
\]

At primes \( p \mid N \), the general bound of Whittaker functions \( 28 \), Lemma 3.3] applies, whose proof uses the idea in the above case \( m \geq 3 \) plus the decay of matrix coefficients. It implies

\[
W_{\varphi_p,\varphi_p}(n) \ll 1.
\]
Lemma 6.2. Let $f \in S_{1/2}(N, \chi)$ with spectral parameter $s \in i\mathbb{R}$, Fourier coefficients $c_f(n)$ defined in (6.1), Petersson norm defined in (6.3). Suppose $f$ correspond to the cuspidal automorphic representation $\pi$ of $GL_2(\mathbb{A})$ under Shimura correspondence with respect to the standard additive character $\psi$. Then for square-free $n$ and any $\epsilon > 0$, we have

$$|n| \cdot \frac{|c_f(n)|^2}{(f, f)} \ll_{N, \epsilon} e^{\frac{|n|}{2}} \cdot (1 + |s|)^{-\frac{1}{4} \text{sgn}(n) + \epsilon} \cdot \left| L\left(\frac{1}{2}, \pi \otimes \chi_{\varepsilon}\right) \right|.$$  

Proof. Inserting (6.9), (6.10) and Lemma 6.1 + 6.8 into (6.3), we get

$$|n| \cdot \frac{|c_f(n)|^2}{(f, f)} = \frac{3}{8\pi} \cdot \frac{1}{[SL_2(\mathbb{Z}) : \Gamma_0(N)]} \cdot e^{-\pi s} \cdot \left| \frac{1 + s}{2} - \frac{1}{4} \text{sgn}(n) \right|^2 \cdot \frac{L\left(\frac{1}{2}, \pi \otimes \chi_{\varepsilon}\right)}{L(1, \pi, Ad)} \cdot \prod_{p|n} L_p\left(1, \pi_p, Ad\right) \cdot L_p\left(\frac{1}{2}, \pi_p \otimes \chi_{\varepsilon}\right) \zeta(p(2)) W_{\phi, \varphi}(n).$$

The estimations (6.9), (6.10), Stirling’s estimation [22, (B.8)] and the lower bound of $L(1, \pi, Ad)$ due to Hoffstein-Lockhart [20, Theorem 0.2] concludes the proof.

6.2. New Bound of Error Term in Rademacher’s Formula. There are many applications of the main result of this paper. Besides the obvious ones, such as making the first sign change of the Fourier coefficients of the Yoshida lift associated with a pair of elliptic cusp forms explicit [1], we mention here another one of improving the error term in the expansion of the partition function $p(n)$ due to Rademacher.

We recall the problem. Let $\eta(z)$ denote the Dedekind eta-function and let $\chi : SL_2(\mathbb{Z}) \rightarrow \mathbb{C}$ denote the associated multiplier system, i.e.

$$\eta(\gamma z) = \chi(\gamma)(cz + d)^{1/2} \eta(z), \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$  

The Kloosterman sums with this multiplier system are given by

$$S(m, n, c, \chi) := \sum_{0 \leq a, d < c} \chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) e\left(\frac{ma + nd}{c}\right), \quad \tilde{m} := m - \frac{23}{24}.$$  

Rademacher [29, 30] proved the exact formula

$$p(n) = \frac{2\pi}{(24n - 1)^{1/4}} \sum_{c=1}^{\infty} A_c(n) \frac{1}{c} I_{3/2}\left(\pi \sqrt{24n - 1}/6c\right), \quad A_c(n) = \sqrt{-1} S(1, 1 - n, c, \chi),$$

where $I_{3/2}(\cdot)$ denotes the $I$-Bessel function. Rademacher [29], Lehmer [25], and several others (for example, see [1, 2, 18]) were interested in estimating the error $R(n, N)$ which results from truncating the series after the $N$-th term:

$$p(n) = \frac{2\pi}{(24n - 1)^{1/4}} \sum_{c=1}^{N} A_c(n) \frac{1}{c} I_{3/2}\left(\pi \sqrt{24n - 1}/6c\right) + R(n, N).$$

The best estimate thus far is due to Ahlgren and Dunn, who proved in [3, Theorem 1.5] that if $24n - 23$ is squarefree, then for any $\alpha, \epsilon > 0$ we have

$$R(n, \sqrt{n}) \ll_{\alpha, \epsilon} n^{-\frac{1}{2} - \frac{1}{4}\alpha + \epsilon}.$$  

A slightly weaker bound was obtained in [1] under the assumption that $24n - 23$ is not divisible by $5^4$ or $7^4$. Here we apply Corollary 1.6 to obtain the following improvement.

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\*Preprint of Soumya Das & Ritwik Pal “The first negative eigenvalue of Yoshida lifts” on the webpage of Soumya.
Theorem 6.3. For any \( n \geq 1 \) and for any \( \alpha, \epsilon > 0 \) we have

\[
R(n, \alpha \sqrt{n}) \ll_{\alpha, \epsilon} n^{-\frac{28}{25} + \epsilon},
\]

where \( \theta \) is any constant toward the Ramanujan-Petersson conjecture for \( GL_2 \) over \( \mathbb{Q} \). In particular, with \( \theta = \frac{7}{64} \) and \( \epsilon = \frac{4}{125457} \) we have

\[
R(n, \alpha \sqrt{n}) \ll_{\alpha} n^{-\frac{1}{2} - \frac{1}{125457}}.
\]

The proof of Theorem 6.3 follows the same basic outline as in [1, §9]. For any \( n < 0 \), we apply the Kuznetsov trace formula

\[
\sum_{c=1}^{\infty} \frac{S(1,n,c \chi)}{c} \phi \left( \frac{n}{c} \right) = 8\sqrt{i} \sqrt{|n|} \sum_{r_j} \frac{\rho_j(1)\rho_j(n)}{\cosh \pi r_j} \phi(r_j),
\]

with the same test function \( \phi \) as in [1], where \( a = 4\pi \sqrt{|n|} \). Here \( \rho_j(\cdot) \) denotes the Fourier coefficients of an orthonormal basis \( \{u_j\} \) for \( S_{1/2}(1, \chi) \), the space of half-integral weight Maass cusp forms on \( SL_2(\mathbb{Z}) \) with multiplier \( \chi \), and \( r_j \) is the spectral parameter of \( u_j \). Breaking the spectral sum into three ranges as in the proof of [1, Proposition 9.2] and using the bounds from that paper for ranges (i) and (iii), we find

\[
\sum_{x \leq c \leq 2x} \frac{S(1,n,c \chi)}{c} \ll_{\delta} |n|^\frac{3}{4} x^{\frac{1}{2} + \frac{\delta}{2}} + x^{\frac{1}{2} + \frac{\delta}{4}} \log x + |n|^\frac{1}{2} \sum_{r_j < \frac{1}{8}} \left| \frac{\rho_j(1)\rho_j(n)}{\cosh \pi r_j} \phi(r_j) \right|
\]

for any fixed \( \delta \in (0,1/2) \). By Theorem 6.1 of [1] we can replace \( \phi(r_j) \) by \( r_j^{-1} \) in the latter sum. By Cauchy-Schwarz and [1, (9.7)] we obtain

\[
\sqrt{|n|} \sum_{r_j < \frac{1}{8}} \left| \frac{\rho_j(1)\rho_j(n)}{\cosh \pi r_j} \phi(r_j) \right| \ll |n|^{-\frac{1}{2}} x^{\frac{1}{2}} \left( |n| \sum_{r_j < \frac{1}{8}} \left| \frac{\rho_j(n)}{\cosh \pi r_j} \right|^2 \right)^{\frac{1}{2}}.
\]

Let \( v_j(z) := u_j(24z) \). Then, up to multiplication by a fixed normalizing constant, the set \( \{v_j\} \) is an orthonormal subset of \( S_{1/2}(576, \left( \frac{12}{7} \right) \nu_\theta) \), where \( \nu_\theta \) is the multiplier system associated to the theta function \( \theta(z) = \sum_{n \in \mathbb{Z}} e(n^2z) \). Let \( \mu_j(\cdot) \) denote the Fourier coefficients of \( v_j(z) \), so that

\[
(6.11) \quad \rho_j(n) = \mu_j(24n - 1).
\]

Let \( f_j(z) \) denote the Shimura lift of \( v_j(z) \) and write \( 24n - 1 = dw^2 \), where \( d \) is a fundamental discriminant. We will relate the coefficients \( \mu_j(dw^2) \) to the twisted \( L \)-functions \( L(s, f_j \times \chi_d) \) by following the argument given in Section 6 of [3] and using Lemma 6.2. If \( a_j(n) \) denotes the \( n \)-th Fourier coefficient of \( f_j \), then

\[
(6.12) \quad w \mu_j(dw^2) = \mu_j(d) \sum_{\ell | w} \ell^{-1} \mu(\ell) \chi_d(\ell) a_j(w/\ell) \ll w^{\theta + \epsilon} |\mu_j(d)|,
\]

using the bound \( a_j(w) \ll w^{\theta + \epsilon} \) due to Kim and Sarnak [24]. Then Lemma 6.2 gives

\[
|d| \sum_{r_j \leq x} \left| \frac{\mu_j(d)}{\cosh \pi r_j} \right|^2 \ll \left(1 + |r_j|\right)^{\frac{5}{4} + \epsilon} |L\left(\frac{1}{2}, f_j \times \chi_d\right)|.
\]

Our Corollary 1.4 implies the bound

\[
|L\left(\frac{1}{2}, f_j \times \chi_d\right)| \ll \left((1 + |r_j|)^2 |d|^2\right)^{\frac{5}{4} - \frac{1}{2} + \epsilon},
\]

which, together with (6.11), (6.12), and Weyl’s law, gives

\[
|n| \sum_{0 < r_j \leq x} \left| \frac{\rho_j(n)}{\cosh \pi r_j} \right|^2 \ll |d|^{\frac{5}{4} - \frac{1}{2} + \epsilon} w^{2\theta + \epsilon} x^{3 - \frac{1-2\theta}{2} + \epsilon} \ll |n|^{\frac{5}{4} - \frac{1-2\theta}{2} + \epsilon} x^{3 - \frac{1-2\theta}{2} + \epsilon}.
\]
It follows that
\[
\sum_{x \leq c \leq 2x} \frac{S(1, n, c, \chi)}{c} \ll \delta |n|^{\frac{13}{70} + \varepsilon} x^{\frac{3}{4} + \varepsilon} + x^\frac{1}{2} \log x + |n|^{\frac{3}{2} \left(1 - \frac{1}{2n}\right)^{1 + \varepsilon}} x^{-\frac{1}{4} + \frac{1}{10} + \varepsilon}.
\]

Following the proof of [1, Proposition 9.2] we conclude that
\[
\sum_{c \leq X} \frac{S(1, n, c, \chi)}{c} \ll \delta \left(|n|^{\frac{13}{70} + \varepsilon} + |n| X^{\frac{3}{4}} + X^{\frac{1}{2} - \delta}\right) (|n| X)^\varepsilon,
\]
from which it follows that [1, (10.5)] can be improved to
\[
R(1 - n, \alpha |n|^{\frac{1}{2}}) \ll_{\delta, \alpha} \left(|n|^{\frac{3}{2} \delta - \frac{29}{56}} + |n|^{-\frac{29}{56}} + |n|^{-\frac{1 + 4}{2}}\right) |n|^{\varepsilon}.
\]
Choosing \(\delta = \frac{1}{49}\), we obtain
\[
R(1 - n, \alpha |n|^{\frac{1}{2}}) \ll_{\alpha} |n|^{-\frac{29}{56} + \varepsilon}.
\]

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