Newtonian limit of Induced Gravity

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Abstract

We discuss the weak-field limit of induced gravity and show that results directly depend on the coupling and self–interaction potential of the scalar field. A static spherically symmetric exact solution is found and its conformal properties are studied. As an application, it is shown that the light deflection angle and the microlensing quantities can be parametrized by the coupling of the theory.

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1 Introduction

Extended theories of gravity have become a sort of paradigm in the study of gravitational interaction since several motivations push for enlarging the traditional scheme of Einstein general relativity. Such issues come, essentially, from cosmology and quantum field theory.

In the first case, it is well known that higher–derivative theories \[1\] and scalar–tensor theories \[2\] furnish inflationary cosmological solutions capable, in principle, of solving the shortcomings of standard cosmological model \[3\]. Besides, they have relevant features also from the quantum cosmology point of view since they give interesting solutions to the initial condition problem, at least in the restricted context of minisuperspaces \[4\].

In the second case, every unification scheme as superstrings, supergravity or great unified theories, takes into account effective actions where nonminimal coupling to the geometry or higher order terms in the curvature invariants come out. Such contributions are due to one–loop or higher–loop corrections in the high curvature regimes near the full (not yet available) quantum gravity regime \[5\]. However, in the weak–limit approximation, all these classes of theories should be expected to reproduce the Einstein general relativity which, in any case, is experimentally tested only in this limit \[6\].

This fact is matter of debate since several relativistic theories do not reproduce Einstein results at the Newtonian approximation but, in some sense, generalize them. In fact, as it was firstly noticed by Stelle \[7\], a \(R^2\)–theory gives rise to Yukawa–like corrections to the Newtonian potential which could have interesting physical consequences.

For example, some authors claim for explaining the flat rotation curves of galaxies by using such terms \[8\]. Others \[9\] have shown that a conformal theory of gravity is nothing else but a fourth–order theory containing such terms in the Newtonian limit and, by invoking these results, it could be possible to explain the missing matter problem without dark matter.

Besides, indications of an apparent, anomalous, long–range acceleration revealed from the data analysis of Pioneer 10/11, Galileo, and Ulysses spacecrafs could be framed in a general theoretical scheme by taking into account Yukawa–like or higher order corrections to the Newtonian potential \[11\].

In general, any relativistic theory of gravitation can yield corrections to the Newton potential (see for example \[12\]) which in the post-Newtonian (PPN) formalism, could furnish tests for the same theory \[1\].

Futhermore the newborn gravitational lensing astronomy \[13\] is giving rise to additional tests for general relativity over small, large, and very large scales which very soon will provide direct measurements for the variation of Newton coupling \(G_N\) \[14\], the potential of galaxies and clusters of galaxies \[15\] and several other features of gravitating systems. Such data will be very likely capable of confirming or ruling out the physical consistency of extended theories of gravity.

In this paper, we want to discuss the Newtonian limit of a class of scalar–tensor theories of gravity, the induced gravity theories, which are invoked in order to solve
several problems in cosmology (e.g. extended inflation [2]) and in fundamental physics (e.g. the tree-level-string-dilaton effective action can be recast as an induced gravity theory [16]). In particular, we show that the Newtonian limit of an induced gravity theory depends on the parameters of the nonminimal coupling between scalar field and Ricci scalar and the self–interaction potential of such a scalar field. Furthermore, a quadratic correction of the Newtonian potential strictly depends on the presence of the scalar–field potential which acts, in the low energy limit, as a cosmological constant (Sec.2).

In Sec.3, we obtain a static, spherically symmetric, asymptotically flat exact solution which is the generalization, for the nonminimally coupled theory with coupling of the form $F(\phi) = \xi \phi^2$, of the well–known Janis–Newman–Winicour solution to the Einstein–massless scalar equations [17]. The deflection angle and some microlensing observables are deduced in Sec.4. Conclusions are drawn in Sec.5.

\section{The field equations and the linearized solutions}

The most general action for a theory of gravity where a scalar–field is nonminimally coupled to the geometry is, in four dimensions,

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[ F(\phi) R + \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) + \mathcal{L}_m \right], \quad (2.1)$$

where $F(\phi), V(\phi)$ are generic functions of $\phi$ and $\mathcal{L}_m$ is the ordinary matter Lagrangian density [18],[19].

We obtain the field equations by varying the action with respect to the metric tensor field $g_{\mu\nu}$

$$G_{\mu\nu} = T_{\mu\nu}^{(\text{eff})}, \quad (2.2)$$

where $G_{\mu\nu}$ is the Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad (2.3)$$

and $T_{\mu\nu}^{(\text{eff})}$, whose expression is found to be

$$T_{\mu\nu}^{(\text{eff})} = \frac{1}{F(\phi)} \left\{-\frac{1}{2} \phi_{,\mu} \phi_{,\nu} + \frac{1}{4} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} g_{\mu\nu} V(\phi) + - g_{\mu\nu} \Box F(\phi) + F(\phi)_{,\mu\nu} - 4 \pi \tilde{G} T_{\mu\nu} \right\}, \quad (2.4)$$

is the effective stress-energy tensor containing terms of nonminimal coupling, kinetic terms, potential of the scalar field $\phi$ and the usual stress-energy tensor of matter, $T_{\mu\nu}$, calculated from $\mathcal{L}_m$. $\tilde{G}$ is a dimensional, strictly positive, constant. In our units, with $c = 1$, Einstein general relativity is obtained when the scalar field coupling $F(\phi)$ is a constant and $\tilde{G}$ reduces to the Newton gravitational constant $G_N$ [19].
Equations for the scalar field are found by varying the action with respect to the same field
\[ \Box \phi - R F'(\phi) + V'(\phi) = 0, \tag{2.5} \]
where \( F'(\phi) = dF(\phi)/d\phi \), \( V'(\phi) = dV(\phi)/d\phi \) and \( \Box \phi \equiv g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \). This Klein-Gordon equation can also be obtained from the contracted Bianchi identity \[19\].

Field equations derived from the action (2.1) can be recast in a Brans–Dicke equivalent form by choosing
\[ \varphi = F(\phi), \ \omega(\phi) = \frac{F(\phi)}{2 F'(\phi)^2}, \quad W(\varphi) = V(\phi(\varphi)). \tag{2.6} \]

Action (2.1) now reads
\[ A = \int d^4x \sqrt{-g} \left[ \varphi R + \frac{\omega(\varphi)}{\varphi} g^{\mu\nu} \varphi_{;\mu} \varphi_{;\nu} - W(\varphi) + L_m \right], \tag{2.7} \]
that is nothing else but the extension of the original Brans–Dicke proposal (where \( \omega \) is a constant), with \( \omega = \omega(\varphi) \) plus a potential term \( W(\varphi) \). By varying the new action with respect to \( g_{\mu\nu} \) and the new scalar field \( \varphi \), we obtain again the field equations. The effective stress energy tensor in Eq.(2.2) and the Klein–Gordon equation (2.5), in the same units as before, become
\[ T^{(eff)}_{\mu\nu} = -\frac{4 \pi \hat{G}}{\varphi} T_{\mu\nu} - \frac{\omega(\varphi)}{\varphi^2} \left( \varphi_{;\mu} \varphi_{;\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \varphi_{;\alpha} \varphi_{;\beta} \right) + \]
\[ + \frac{1}{\varphi} (\varphi_{;\mu\nu} - g_{\mu\nu} \Box \varphi) - \frac{1}{2} \varphi g_{\mu\nu} W(\varphi), \tag{2.8} \]
\[ 2 \omega(\varphi) \Box \varphi - \frac{\omega(\varphi)}{\varphi} g^{\alpha\beta} \varphi_{;\alpha} \varphi_{;\beta} - \frac{d\omega(\varphi)}{d\varphi} g^{\alpha\beta} \varphi_{;\alpha} \varphi_{;\beta} - \varphi R + \varphi W''(\varphi) = 0. \tag{2.9} \]
This last equation is usually rewritten eliminating the scalar curvature term \( R \) with the help of Eq.(2.8), so that one obtains
\[ \Box \varphi = \frac{1}{3 - 2 \omega(\varphi)} \left( -4 \pi \hat{G} T - 2 W(\varphi) + \varphi W''(\varphi) - \frac{d\omega(\varphi)}{d\varphi} g^{\alpha\beta} \varphi_{;\alpha} \varphi_{;\beta} \right). \tag{2.10} \]
The minus sign in the denominator comes from the sign chosen in our action \[2.1\]. Our aim now is to study the linearized equations derived from the action \[2.1\] or, equivalently, from \[2.7\].

Before starting, we need a choice for the up to now arbitrary functions \( F(\phi) \) and \( V(\phi) \). A rather general choice is given by
\[ F(\phi) = \xi \phi^m, \tag{2.11} \]
\[ V(\phi) = \lambda \phi^n, \tag{2.12} \]
where $\xi$ is a coupling constant, $\lambda$ gives the self-interaction potential strength, $m$ and $n$ are arbitrary, for the moment, parameters. This choice is in agreement with the existence of a Noether symmetry in the action (2.1) as discussed in [18, 19]. Furthermore, several scalar–tensor physical theories (e.g. induced gravity) admit such a form for $F(\phi)$ and $V(\phi)$.

In order to recover the Newtonian limit, we write, as usual, the metric tensor as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

(2.13)

where $\eta_{\mu\nu}$ is the Minkoski metric and $h_{\mu\nu}$ is a small correction to it. In the same way, we define the scalar field $\psi$ as a perturbation, of the same order of the components of $h_{\mu\nu}$, of the original field $\phi$, that is

$$\phi = \phi_0 + \psi,$$

(2.14)

where $\phi_0$ is a constant of order unit. It is clear that for $\phi_0 = 1$ and $\psi = 0$ Einstein general relativity is recovered.

To write in an appropriate form the Einstein tensor $G_{\mu\nu}$, we define the auxiliary fields

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h,$$

(2.15)

and

$$\sigma_\alpha \equiv \overline{h}_{\alpha\beta,\gamma} \eta^{\beta\gamma}.$$

(2.16)

Given these definitions, to the first order in $h_{\mu\nu}$, we obtain

$$G_{\mu\nu} = -\frac{1}{2} \left\{ \Box_\eta \overline{h}_{\mu\nu} + \eta_{\mu\nu} \sigma_{\alpha,\beta} \eta^{\alpha\beta} - \sigma_{\mu,\nu} - \sigma_{\nu,\mu} \right\},$$

(2.17)

where $\Box_\eta \equiv \eta^{\mu\nu} \phi_{,\mu} \phi_{,\nu}$. We have not fixed the gauge yet.

We pass now to the right hand side of Eq.(2.2), namely to the effective stress energy tensor. Up to the second order in $\psi$, the coupling function $F(\phi)$ and the potential $V(\phi)$, by using Eqs.(2.11) and (2.12), become

$$F(\phi) \simeq \xi \left( \phi_0^m + m \phi_0^{m-1} \psi + \frac{m(m-1)}{2} \phi_0^{m-2} \psi^2 \right),$$

(2.18)

$$V(\phi) \simeq \lambda \left( \phi_0^n + n \phi_0^{n-1} \psi + \frac{n(n-1)}{2} \phi_0^{n-2} \psi^2 \right).$$

(2.19)

To the first order, the effective stress–energy tensor becomes

$$\tilde{T}_{\mu\nu} = -m \phi_0^{2m-1} \eta_{\mu\nu} \Box_\eta \psi + m \phi_0^{2m-1} \psi_{,\mu\nu} - \frac{\lambda \phi_0^{m+n}}{2\xi} \eta_{\mu\nu} - \left( \frac{4\pi \tilde{G}}{\xi} \right) \frac{\phi_0^m}{\xi} T_{\mu\nu},$$

(2.20)
and then the field equations are
\[
\frac{1}{2} \left\{ \Box_{\eta} \eta_{\mu \nu} + \eta_{\mu \sigma} \eta_{\alpha \beta} \eta_{\nu} - \eta_{\mu \nu} \right\} = m \varphi_0^{2m-1} \eta_{\mu \nu} \Box \eta \psi - m \varphi_0^{2m-1} \psi_{\mu \nu} +
\]
\[
+ \frac{\lambda \varphi_0^{m+n}}{2 \xi} \eta_{\mu \nu} + (4 \pi \tilde{G}) \frac{\varphi_0^m}{\xi} T_{\mu \nu}.
\]
We can eliminate the term proportional to \( \psi_{\mu \nu} \) by choosing an appropriate gauge. In fact, by writing the auxiliary field \( \sigma_\alpha \), given by Eq. (2.16), as
\[
\sigma_\alpha = m \varphi_0^{2m-1} \psi_{\alpha},
\]
field equations (2.21) read
\[
\Box_{\eta} \eta_{\mu \nu} - m \varphi_0^{2m-1} \eta_{\mu \nu} \Box_{\eta} \psi \simeq \frac{\lambda \varphi_0^{m+n}}{\xi} \eta_{\mu \nu} + (8 \pi \tilde{G}) \frac{\varphi_0^m}{\xi} T_{\mu \nu}
\]
By defining the auxiliary field with components \( \tilde{h}_{\mu \nu} \) as
\[
\tilde{h}_{\mu \nu} \equiv \eta_{\mu \nu} - m \varphi_0^{2m-1} \eta_{\mu \nu} \psi,
\]
the field equations take the simpler form
\[
\Box_{\eta} \tilde{h}_{\mu \nu} = \frac{\lambda \varphi_0^{m+n}}{\xi} \eta_{\mu \nu} + (8 \pi \tilde{G}) \frac{\varphi_0^m}{\xi} T_{\mu \nu}
\]
The original perturbation field \( h_{\mu \nu} \) can be written in terms of the new field as (with \( \tilde{h} \equiv \eta_{\mu \nu} \tilde{h}_{\mu \nu} \))
\[
h_{\mu \nu} = \tilde{h}_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} h - m \varphi_0^{2m-1} \eta_{\mu \nu} \psi.
\]
We turn now to the Klein-Gordon Eq. (2.23). \( \Box_{\eta} \psi \) can be written, from the linearized Klein-Gordon equation, in terms of the matter stress energy tensor and of the potential term. If we calculate the scalar invariant of curvature \( R = g^{\mu \nu} R_{\mu \nu} \) from Eq. (2.7), we find
\[
\Box \phi + \frac{F'(\phi)}{F(\phi)} \left( \frac{1}{2} g^{\alpha \beta} \phi_{\alpha} \phi_{\beta} - 2 V(\phi) - 3 F(\phi) - 4 \pi \tilde{G} T \right) + V'(\phi) = 0,
\]
and, to the first order, it reads
\[
\Box_{\eta} \psi + \frac{\lambda (n-2m)(n-1)\varphi_0^{n-2}}{1 - 3 \xi m^2 \varphi_0^{n-2}} \psi = \frac{\lambda (2m-n)\varphi_0^{n-1}}{1 - 3 \xi m^2 \varphi_0^{n-2}} + \frac{4 \pi \tilde{G} m^2}{(1 - 3 \xi m^2 \varphi_0^{n-2}) \varphi_0} T.
\]
We work in the weak-field and slow motion limits, namely we assume that the matter stress-energy tensor \( T_{\mu \nu} \) is dominated by the mass density term and we neglect time derivatives with respect to the space derivatives, so that \( \Box_{\eta} \rightarrow -\Delta \), where \( \Delta \) is the ordinary Laplacian operator in flat spacetime. The linearized field equations (2.23) and
(2.28) have, for point–like distribution of matter, which is \( \rho(r) = M\delta(r) \), the following solutions:

for \( n \neq 2m, n \neq 1 \), we get

\[
h_{00} \approx \left[ (4\pi\tilde{G}) \frac{\varphi_0^m}{\xi} \right] \frac{M}{r} - \left[ \frac{4\pi\lambda \varphi_0^{m+n}}{\xi} \right] r^2 - \left[ (4\pi\tilde{G}) \frac{m^2 \varphi_0^{2m-2}}{1 - 3 \xi m^2 \varphi_0^{m-2}} \right] \frac{e^{-pr}}{r} + \left[ \frac{4\pi m \varphi_0^{2m}}{n-1} \right] \cosh(pr),
\]

\[
h_{il} \approx \delta_{il} \left\{ \left[ (4\pi\tilde{G}) \frac{\varphi_0^m}{\xi} \right] \frac{M}{r} + \left[ \frac{4\pi\lambda \varphi_0^{m+n}}{\xi} \right] r^2 + \left[ (4\pi\tilde{G}) \frac{m^2 \varphi_0^{2m-2}}{1 - 3 \xi m^2 \varphi_0^{m-2}} \right] \frac{e^{-pr}}{r} \right\} + \left[ \frac{4\pi \varphi_0^m}{n-1} \right] \cosh(pr),
\]

\[
\psi \approx \left[ (4\pi\tilde{G}) \frac{m M}{1 - 3 \xi m^2 \varphi_0} \right] \frac{e^{-pr}}{r} - \left[ \frac{4\pi \varphi_0}{n-1} \right] \cosh(pr),
\]

where the parameter \( p \) is given by

\[
p^2 = \frac{\lambda(n - 2m)(n - 1)\varphi_0^{m-2}}{1 - 3 \xi m^2 \varphi_0^{m-2}}.
\]

For \( n = 2m \), we obtain

\[
h_{00} \approx \left[ (4\pi\tilde{G}) \frac{\varphi_0^m(1 - 4 \xi m^2 \varphi_0^{m-2})}{\xi(1 - 3 \xi m^2 \varphi_0^{m-2})} \right] \frac{M}{r} - \left[ \frac{4\pi\lambda \varphi_0^{m+n}}{\xi} \right] r^2 - \Lambda,
\]

\[
h_{il} \approx \delta_{il} \left\{ \left[ (4\pi\tilde{G}) \frac{\varphi_0^m(1 - 2 \xi m^2 \varphi_0^{m-2})}{\xi(1 - 3 \xi m^2 \varphi_0^{m-2})} \right] \frac{M}{r} + \left[ \frac{4\pi\lambda \varphi_0^{m+n}}{\xi} \right] r^2 + \Lambda \right\},
\]

\[
\psi \equiv \left[ (4\pi\tilde{G}) \frac{m M}{(1 - 3 \xi m^2 \varphi_0^{m-2})\varphi_0} \right] \frac{M}{r} + \psi_0
\]

where \( \Lambda = m \varphi_0^{2m-1} \psi_0 \) and \( \psi_0 \) are arbitrary integration constants. Let us note that the values \( m = 1, n = 2 \) and \( m = 2, n = 4 \) correspond to the well known couplings and potentials, i.e. \( F \sim \phi, V \sim \phi^2 \) and \( F \sim \phi^2, V \sim \phi^4 \), respectively.

Finally for \( n = 1 \), we obtain

\[
h_{00} \approx \left[ (4\pi\tilde{G}) \frac{\varphi_0^m(1 - 4 \xi m^2 \varphi_0^{m-2})}{\xi(1 - 3 \xi m^2 \varphi_0^{m-2})} \right] \frac{M}{r} - \left[ \frac{4\pi\lambda \varphi_0^{m+n}[1 - \xi m(m + 1)\varphi_0^{m-2}]}{\xi(1 - 3 \xi m^2 \varphi_0^{m-2})} \right] r^2,
\]

\(^1\)To be precise, we can define a Schwarzschild mass of the form

\[
M = \int (2T^0_0 - T^\mu_\mu) \sqrt{-g} d^3x.
\]
\[ h_{il} \simeq \delta_{il} \left\{ \frac{(4\pi\tilde{G})\varphi_0^m(1 - 2\xi m^2\varphi_0^{-m-2})}{\xi(1 - 3\xi m^2\varphi_0^{-m-2})} \frac{M}{r} + \left[ \frac{4\pi\lambda\varphi_0^{m+n}[1 - \xi m(m + 1)\varphi_0^{-m-2}]}{\xi(1 - 3\xi m^2\varphi_0^{-m-2})} \right] r^2 \right\}, \tag{2.37} \]

\[ \psi \simeq \left[ \frac{(4\pi\tilde{G})m}{(1 - 3\xi m^2)\varphi_0} \right] \frac{M}{r} + \left[ \frac{4\pi\lambda(2m - 1)\varphi_0^{-m+1}}{1 - 3\xi m^2\varphi_0^{-m-2}} \right] r^2. \tag{2.38} \]

If we demand the \((0, 0)\)-component of the field Eq.\((2.23)\), when \(\lambda = 0\), to read as the usual Poisson equation (that is nothing else but a definition of the mass)

\[ \Delta \Phi = 4\pi G_N \rho, \tag{2.39} \]

where \(\Phi\), linked with the metric tensor by the relation \(h_{00} = 2\Phi\), is the Newtonian potential, we have to put

\[ G_N = -\frac{\varphi_0^m}{2\xi} \left( 1 - 4\xi m^2\varphi_0^{-m-2} \right) \tilde{G}. \tag{2.40} \]

We may now rewrite the nonzero components of \(h_{\mu\nu}\) and the scalar perturbed field. Let us take into account, for example, Eqs. \((2.33)\)–\((2.35)\). We get

\[ h_{00} \simeq -\frac{2G_NM}{r} - \frac{4\pi\lambda\varphi_0^{m+n}}{\xi} r^2 - \varphi_0^{-2m-1}\psi_0 \tag{2.41} \]

and

\[ h_{il} \simeq \delta_{il} \left\{ -\frac{2G_NM}{r} \left( 1 - 2\xi m^2\varphi_0^{-m-2} \right) \left( \frac{1 - 4\xi m^2\varphi_0^{-m-2}}{1 - 4}\xi m^2\varphi_0^{-m-2} \right) + \frac{4\pi\lambda\varphi_0^{m+n}}{\xi} r^2 + m\varphi_0^{-2m-1}\psi_0 \right\}, \tag{2.42} \]

and

\[ \psi = -\frac{2G_NM}{r} \left( \frac{\xi m\varphi_0^{-m-1}}{1 - 4\xi m^2} \right) + \psi_0, \tag{2.43} \]

where the Newton constant explicitly appears. Similar considerations hold in the other cases.

What we have obtained are solutions of the linearized field equations, starting from the action of a scalar field nonminimally coupled to the geometry, and minimally coupled to the ordinary matter. Such solutions depend on the parameters which characterize the theory: \(\xi, m, n, \lambda\). The results of Einstein general relativity are obtained for \(F(\phi) = F_0\), with \(F_0\) negatively–defined due to the sign choice in the action \((2.1)\). As we can easily see from above, in particular from Eqs.\((2.41)\)–\((2.43)\), we have the usual Newtonian potential and a sort of cosmological term ruled by \(\lambda\) which, from the Poisson equation, gives a quadratic contribution.

We consider now the Brans-Dicke-like action \((2.7)\) where \(\omega = \omega(\phi)\). It is actually simple to see, from the field Eqs.\((2.8)\) and \((2.10)\), that, if we want to limit ourselves to the linear approximation, we may consider as well \(\omega = \text{constant}\). The link with the results that follows from the action \((2.7)\) are given by the transformation laws \((2.6)\), that is

\[ \omega(\varphi(\phi)) = \frac{1}{2\xi m^2} \phi^{2-m}. \tag{2.44} \]

The potential term \(W(\varphi)\) in the linear approximation, behaves as \(V(\phi)\). Results in the approaches given by the actions \((2.7)\) and \((2.7)\) are completely equivalent.
3 An exact solution

We are going now to consider a particular choice for the coupling field \( F(\phi) \) and the potential \( V(\phi) \) and to obtain an exact solution for the field equations in the case of the external field of a distribution of matter endowed with spherical symmetry. Namely, we consider

\[
F(\phi) = \xi \phi^2, \quad (3.1)
\]

\[
V(\phi) = 0. \quad (3.2)
\]

The physical relevance of the above assumption is widely discussed in literature (see e.g. [19] and references therein). Here we want to show that it is possible to find an exact solution of the same form of those (more general) discussed in previous section. This is nothing else but a generalization to the nonminimal coupling case of the well known solution of Janis–Newman–Winicour [17], [20].

The exact solution that we will find is linked, of course, with the one obtained in Brans-Dicke theory with \( \omega = \text{constant} \) (from (2.44) it follows that it is just for the value \( m = 2 \), that we find \( \omega = \text{constant} \)).

We look for a solution for a static scalar field, so that, given the symmetry of the problem, the Birkhoff theorem holds, and we can write the line element as

\[
ds^2 = a(r) \, dt^2 - b(r) \, dr^2 - c(r) \, r^2 d\Omega^2,
\]

where \( a(r), b(r) \) and \( c(r) \) are strictly positive functions of the radial coordinate \( r \), \( d\Omega^2 \) is the usual spherical element.

We can write down the Einstein field equations (2.2) as

\[
- \frac{a}{r^2 b} + \frac{a}{r^2 c} + \frac{ab'}{rb} - \frac{3ac'}{rbc} + \frac{abc'}{2b'^2} + \frac{ac^2}{4b^2c} - \frac{ac''}{bc} =
\]

\[
= \frac{4a\phi'}{rb\phi} - \frac{ab'\phi'}{b^2\phi} + \frac{2ac'\phi'}{bc\phi} + \frac{2a\phi'^2}{b\phi^2} - \frac{a\phi'^2}{4\xi b\phi^2} + \frac{2a\phi''}{b\phi}, \quad (3.4)
\]

\[
\frac{1}{r^2} - \frac{b}{r^2 c} + \frac{a'}{ra} + \frac{ac'}{2ac} + \frac{c'^2}{4c^2} = - \frac{4\phi'}{r\phi} - \frac{a'\phi'}{a\phi} - \frac{2c'\phi'}{c\phi} - \frac{\phi'^2}{4\xi \phi^2},
\]

\[
\frac{rca'}{2ab} - \frac{r^2ca'd^2}{4a^2b} - \frac{rcb'}{2b^2} - \frac{r^2ca'b'}{4ab^2} + \frac{rc'}{b} + \frac{r^2a'c'}{4ab} - \frac{r^2b'c'}{4b^2} - \frac{r^2c'^2}{4bc} + \frac{r^2ca''}{2ab} + \frac{r^2c''}{2b} =
\]

\[
= - \frac{2rca'}{b\phi} - \frac{r^2ca'\phi'}{ab\phi} + \frac{r^2cb'\phi'}{b^2\phi} - \frac{r^2c'\phi'}{b\phi} - \frac{2r^2c\phi'^2}{b\phi^2} + \frac{r^2c\phi'^2}{4\xi b\phi^2} - \frac{2r^2c\phi''}{b\phi}, \quad (3.5)
\]

where the prime now indicates the derivative with respect to \( r \). The Klein-Gordon equation is

\[
(1 - 12\xi) \left( \Box \phi - \frac{\phi'^2}{b\phi} \right) = 0. \quad (3.7)
\]
The case $\xi = 1/12$ is the conformal case. Of course, the usual Schwarzschild solution of Einstein general relativity is recovered for $\phi =$ constant.

We look for a solution of the form

$$ds^2 = \left(1 - \frac{\chi}{r}\right)\alpha dt^2 - \left(1 - \frac{\chi}{r}\right)\beta dr^2 - \left(1 - \frac{\chi}{r}\right)\nu r^2 d\Omega^2,$$

$$\phi(r) = \left(1 - \frac{\chi}{r}\right)^\delta,$$  \hspace{1cm} (3.8)

where $\alpha$, $\beta$ and $\nu$ are the parameters that specify the solution (and we expect to find $\alpha = -\beta = 1$ and $\nu = 0$ as a particular solution), $\chi$ is a constant related to the mass of the system which generates the gravitational field.

Eq. (3.7) is satisfied when the algebraic relation

$$4 \delta + \alpha - \beta + 2 \nu - 2 = 0,$$  \hspace{1cm} (3.10)

holds. By studying Einstein equations, we find that a solution of the form (3.8) does exist for

$$\alpha = -2 \delta + \varepsilon,$$  \hspace{1cm} (3.11)

$$\beta = -2 \delta - \varepsilon,$$  \hspace{1cm} (3.12)

$$\gamma = -2 \delta - \varepsilon + 1,$$  \hspace{1cm} (3.13)

where $\varepsilon$ is an auxiliary parameter which we discuss below. The link between them and the coupling constant $\xi$ is the quadratic algebraic equation

$$\delta^2 (1 - 12 \xi) + \xi \left(1 - \varepsilon^2\right) = 0.$$  \hspace{1cm} (3.14)

We notice that, in order to specify completely the solution, we have to give, besides $\chi$, the value of two of the three parameters $\delta$, $\varepsilon$ and $\xi$. It is only to recover the Schwarzschild solution that we need to specify $\varepsilon^2 = 1$. We have also to note that, in the weak field limit, that we are now going to study, the parameters $\delta$ and $\varepsilon$ can be expressed in term of $\xi$.

We may study the range of applicability of the found solution writing the Ricci scalar of curvature whose expression is given by

$$R = \left[\frac{1 - \varepsilon^2}{1 - 12 \xi}\right] \left[\frac{\chi^2 (1 - \frac{\chi}{r})^{2\delta+\varepsilon}}{2 r^4 \left(1 - \frac{\chi}{r}\right)^2}\right].$$  \hspace{1cm} (3.15)

We find, beside the usual singularity in $r = 0$, a singularity in $r = \chi$, which is a null surface (see also [20]). Actually $\chi$ can take both positive and negative values, so that the solution is defined respectively for $r > \chi$ and $r > 0$.

We now expand the solution found at the first order in $\chi/r$ and we identify it with the result of the linearized field equations for a point-like distribution of matter, with
mass $M$. We demand the Poisson equation to hold, that is, we are taking into account a particular case of what we considered in the last section. We find

$$\chi = 2\varepsilon G_N M,$$

(3.16)

and from Eq. (2.40), we have (for $\varphi_0 = 1$, and $m = 2$)

$$G_N = -\frac{1}{2\varepsilon} \left( \frac{1 - 16\xi}{1 - 12\xi} \right) \tilde{G}. $$

(3.17)

$\tilde{G}$, as above, is a dimensional strictly positive constant. Moreover we find, from the $(0,0)$–component of the field Eq. (2.25) and Eq.(2.28) for the scalar perturbation field $\psi$, where we put $m = 2$ and $\lambda = 0$, a second equation for $\delta$, $\varepsilon$ and $\xi$,

$$(1 - 16\xi) \delta = 2 (\varepsilon - 2\delta) \xi.$$ 

(3.18)

Together with Eq. (3.16), this equation allows us to express the solution as a function of the parameter $\xi$ alone. It results

$$\varepsilon = \sqrt{\frac{1 - 12\xi}{1 - 16\xi}}, $$

(3.19)

and

$$\delta = \frac{2\varepsilon \xi}{1 - 12\xi} = \frac{2\xi}{\sqrt{(1 - 12\xi)(1 - 16\xi)}}.$$ 

(3.20)

As it can be seen, the allowed range for $\xi$ is $\xi < 1/16$, $\xi > 1/12$ (from which one should exclude the value $\xi = 0$, the value that would give $\delta = 0$ and $\varepsilon^2 = 1$). The role of the parameter $\varepsilon$ is now clarified: it tells us how much the solution differs from the usual Schwarzschild solution of general relativity since we are considering the nonminimal coupling $F(\phi) = \xi \phi^2$.

The linearized solution can then be written as

$$ds^2 = \left( 1 - \frac{2G_NM}{r} \right) dt^2 - \left[ 1 - \frac{2G_NM}{r} \left( \frac{1 - 8\xi}{1 - 16\xi} \right) \right] dr^2 +$$

$$ - \left[ 1 - \frac{2G_NM}{r} \left( \frac{1 - 8\xi}{1 - 16\xi} - \sqrt{\frac{1 - 12\xi}{1 - 16\xi}} \right) \right] r^2 d\Omega^2,$$

(3.21)

$$\phi (r) = 1 - \frac{2G_NM}{r} \left( \frac{2\xi}{1 - 16\xi} \right),$$

(3.22)

which, by a linear transformation for the radial coordinate, can be put in the same form of the solution (2.41), (2.42) and (2.43) with the choice $m = 2$ and $\lambda = 0$.

In order to recover the corresponding Brans-Dicke solution, with $\omega =$constant and $\phi \rightarrow \varphi = \xi \phi^2$, let us introduce the coordinate transformation

$$\left( \frac{dr}{dr'} \right)^2 = \left( \frac{r}{r'} \right)^2 \frac{c(r)}{b(r)} = \left( \frac{r}{r'} \right)^2 \left( 1 - \frac{X}{r} \right)^{\nu - \beta} = \left( \frac{r}{r'} \right)^2 \left( 1 - \frac{X}{r} \right),$$

(3.23)
where, as already noted, $\beta = -1$ and $\nu = 0$. The new variable

$$r = r' \left(1 + \frac{\chi}{4r'}\right)^2,$$  \hspace{1cm} (3.24)

is consistent with (3.23). The line element in the new coordinate system is

$$ds^2 = \left(\frac{1 - \chi/r'}{1 + \chi/r'}\right)^{2\alpha} dt^2 - \left(1 + \frac{\chi}{r'}\right)^4 \left(\frac{1 - \chi/r'}{1 + \chi/r'}\right)^{2(\beta + 2)} \left(dr'^2 + r'^2d\Omega^2\right),$$  \hspace{1cm} (3.25)

while the scalar field is

$$\phi (r') = \left(\frac{1 - \chi/r'}{1 + \chi/r'}\right)^{2\delta}.$$  \hspace{1cm} (3.26)

The Brans-Dicke solution is then immediately found (see also [21] for the asymptotic discussion). The line element is the same as (3.25), whereas the scalar field is equal to

$$\varphi (r') = \frac{1}{8\omega} \left(\frac{1 - \chi/r'}{1 + \chi/r'}\right)^{4\delta}.$$  \hspace{1cm} (3.27)

If we introduce the new parameters $\varepsilon'$, $\delta'$ and $\nu'$

$$2\alpha \equiv \alpha' \equiv \frac{2}{\varepsilon'}, \hspace{0.5cm} 2\beta + 2 \equiv \beta' \equiv \frac{2}{\varepsilon'} (\varepsilon' - \delta' - 1), \hspace{0.5cm} 2\nu \equiv \nu' \equiv 4\delta,$$  \hspace{1cm} (3.28)

we find

$$\nu' = \frac{\delta'}{\varepsilon'}.$$  \hspace{1cm} (3.29)

As it has been done for $\xi$, it is possible to express everything as a function of the parameter $\omega$.

It is worth noticing that the above solutions are equivalent, up to a conformal transformation, to the Schwarzschild-like solution obtained in the contest of minimally coupled theory of gravitation with a scalar field given by the action

$$\mathcal{A} = \int d^4x \sqrt{-\tilde{g}} \left[k\tilde{R} + \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\phi}_{,\mu}\tilde{\phi}_{,\nu}\right],$$  \hspace{1cm} (3.30)

where $k$ is a dimensional, strictly negative, constant that fixes units and $\sqrt{-g}$ is the square root of the determinant of conformally transformed metric. This solution is known as the Janis–Newman–Winicour one of the Einstein–massless scalar field theory [17].

However, we have to stress that, when we look for the linearized equations, such a conformal transformation has to be performed, if we want to express the parameters that specify the solution as a function of the physical quantities of the system. For example, to define the mass, we must choose the coupling with the ordinary matter. That is, we must decide whether the Jordan frame description assumed for the action (2.1) or (2.7) is the physical one, or if, by coupling minimally to the scalar field also the Lagrangian
for ordinary matter in action (3.30), the Einstein frame description is the physical one. Of course, in the two case we get different results. For a detailed discussion, see e.g. [22]. Furthermore, deviations from the usual expression for the bending angle of light in Einstein general relativity, are already present at first order for solutions in the Jordan frame, but only at second order in the Einstein frame [23]. In this sense the conformal equivalence of Schwarzschild-like solutions in the two frames is broken when we introduce ordinary matter.

4 Deflection Angle and Microlensing Observables

In the previous sections, we dealt with the problem of the determination of the geometry of a static and spherically symmetric spacetime in a given nonminimally coupled theory of gravity. From this solution it is possible to calculate, in a straightforward way, the deflection of light. We start with a line element written in the form

\[ ds^2 = a(r) dt^2 - b(r) \left( dr^2 + r^2 d\Omega^2 \right) , \]  

(4.1)

from which the geodesic equation of motion for photons is found to be

\[ u^2 (u_{\phi\phi} + u) + \frac{1}{2} \left( b^{-1} b' - a^{-1} a' \right) \left( u^2 + u_{\phi}^2 \right) = 0 , \]  

(4.2)

where \( u \equiv 1/r \) and \( u_{\phi} \equiv du/d\phi \) (here \( \phi \) is the azimuthal angle). If we consider this equation in the same limit where our solution (2.41), (2.42) is valid, we get (putting \( \psi_0 = 0 \))

\[ a(r) = 1 - 2 \frac{G_NM}{r} - \frac{4\pi \lambda \varphi_0^{m+n}}{\xi} r^2 , \]  

(4.3)

\[ b(r) = 1 + \frac{2}{4} \frac{G_NM}{r} \gamma + \frac{4\pi \lambda \varphi_0^{m+n}}{\xi} r^2 , \]  

(4.4)

where \( \lambda \) and \( \gamma = \gamma(\xi, m) \), which is given by

\[ \gamma = \frac{1 - 2 \xi m^2 \varphi_0^{m-2}}{1 - 4 \xi m^2 \varphi_0^{m-2}} , \]  

(4.5)

parametrize the scalar–tensor theory. We can recast Eq.(1.2) as

\[ u_{\phi\phi} + u = G_NM \left( 1 + \gamma \right) \left( u^2 + u_{\phi}^2 \right) . \]  

(4.6)

It is worth noticing that the potential term \( \lambda \) does not appear in this equation, so that the deflection angle, \( \hat{\alpha} \), does not depend from it either (at least in the approximation order which we have considered). We have

\[ \hat{\alpha} = 2 \frac{G_NM}{r_0 c^2} (1 + \gamma) , \]  

(4.7)
where, as usual, \( r_0 \) is the minimum distance from the deflector, and, to this order, it is nothing else but an impact parameter. We have reintroduced the dimensional constant \( c \). The result is similar to that given in [24] where also the singular isothermal sphere model has been studied. These arguments specify how deflection of light depends on the parameters \( \xi \) and \( m \) of our scalar–tensor theory. This gives us the chance of looking at the well known effect of gravitational lensing, as a possible test for these theories. We are now going to give the expression for some quantities relevant for this phenomenon.

We consider the simplest situation, that is, the so called Schwarzschild lens. With this geometry usually two images are formed, but when they are no longer separables, and the source brightness is magnified, one speaks about microlensing. In this context the relevant quantity is the Einstein radius, \( R_E \), that is given by

\[
R_E = \sqrt{\frac{2 G_N M}{c^2} \frac{D_l D_{ls}}{D_s}} \tag{4.8}
\]

where \( D_l, D_{ls} \) and \( D_s \) stand respectively for the distances between observer and lens, lens and source, observer and source. The Einstein angle, \( \theta_E \) is given by \( R_E D_l \).

The maximum amplification is equal to

\[
A_{\text{max}} = \frac{r_0^2 + 2 R_E^2}{r_0 \sqrt{r_0^2 + 4 R_E^2}} \tag{4.9}
\]

Clearly, also \( A_{\text{max}} \) depends on the parameters of the given scalar–tensor theory and, in general, all the quantities of gravitational lensing can be parametrized by them. In conclusion, beside classical tests [6], gravitational lensing could be useful, in general, to test gravity. On the other hand, anomalies in lensing features could be explained by enlarging the available set of gravitational theories.

It is interesting to observe that Eq. (4.5) allows to determine the relation between \( \xi \) and \( m \). In fact, inverting (4.5) one gets

\[
\xi m^2 = \frac{1 - \gamma}{2(1 - 2\gamma)} \tag{4.10}
\]

considering, for simplicity, \( \phi_0 = 1 \). A measurement of \( \gamma \), i.e. the deviation from the expected value of the gravitational mass \( G_N M \), estimated for example by Keplerian motions, could roughly indicate the form of the relativistic theory of gravity which holds at a given scale. On the other hand, the result strictly depends on the accuracy by which \( \hat{\alpha} \) is measured.

5 Discussion and Conclusions

The Newtonian limit of induced–gravity theories, where the scalar field is nonminimally coupled to the geometry, strictly depends on the parameters of the coupling and the self–interaction potential as it has to be by a straightforward PPN parametrization [3].
In this paper, we have constructed the weak limit solution of induced–gravity assuming, as it is quite natural to do, power law couplings and potentials.

As we have seen, the role of the self–interaction potential is essential to obtain corrections to the Newtonian potential. Such corrections are, in any case, constant, quadratic or Yukawa–like as for other generalized theories of gravity [7, 9, 10], so also induced gravity could be a candidate to solve the problems of flat rotation curves of spiral galaxies [see e.g. the solution (2.29)]. Essentially the corrections depends on the strength of the coupling and the “mass” of the scalar field $\phi$ given by $\lambda$. Besides, we have always scale lengths where Einstein general relativity can be recovered. This fact could account why measurements inside the Solar System confirm such theory while outside of it there are probable deviations [11]. Furthermore, it is possible to show that the induced-gravity picture and the Brans-Dicke picture are completely equivalent in the Newtonian limit as it trivially has to be.

The so called Janis–Newman–Winicour solution of minimally coupled scalar–tensor gravity can be easily extended to nonminimally coupled theories (at least without self–interaction potential) and it appears as a particular case of the general Newtonian solution (2.41)–(2.43) discussed above.

Finally all the lensing observables, first of all the deflection angle, are affected if we introduce a nonminimal coupling and, consequently, gravitational lensing could constitute a further tool to investigate relativistic theories of gravity. Vice versa, relativistic theories could explain anomalous effects in gravitational lensing.

In the case which we have analysed, it is interesting to stress the fact that the self–interaction potential of the scalar field does not intervene while deviations and then the “parametrization” strictly depend on the coupling.

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