NEWMA: a new method for scalable model-free online change-point detection

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Abstract

We consider the problem of detecting abrupt changes in the distribution of a multi-dimensional time series, with limited computing power and memory. In this paper, we propose a new method for model-free online change-point detection that relies only on fast and light recursive statistics, inspired by the classical Exponential Weighted Moving Average algorithm (EWMA). The proposed idea is to compute two EWMA statistics on the stream of data with different forgetting factors, and to compare them. By doing so, we show that we implicitly compare recent samples with older ones, without the need to explicitly store them. Additionally, we leverage Random Features to efficiently use the Maximum Mean Discrepancy as a distance between distributions. We show that our method is orders of magnitude faster than usual non-parametric methods for a given accuracy.

1 Introduction

The goal of online change-point detection is to detect abrupt changes in the distribution of samples in a data stream. One seeks to detect a change as soon as it occurs, while minimizing the number of false alarms. Online change-point detection has numerous practical applications, for instance medical monitoring via the segmentation of EEG, ECG and fMRI signals \cite{28, 36, 6}, or detections of changes in audio \cite{5} or video \cite{23, 1} streams. We refer to \cite{30} for a more thorough review. In recent applications, the need arises to perform such methods on embedded devices, for instance in video streams from body-worn or surveillance video cameras \cite{2}, or on data collected by smart phones \cite{22}. In addition to being constrained by limited power and memory, such personal devices collect data that can be potentially sensitive. Hence the need to process the stream locally without storing the data and sharing it with other machines.

Setting. We consider a stream of samples \((x_t)_{t \in \mathbb{N}}\) with values in a measurable space \(\mathcal{X}\). The goal of change-point detection is to detect changes in the distribution of the samples \(x_t\). In this paper, we will aim at detecting changes in the value of \(E\Psi(x)\), where \(\Psi : \mathcal{X} \rightarrow \mathcal{H}\) is a mapping to a normed space \((\mathcal{H}, \| \cdot \|)\). It covers most of the usual cases found in the literature: for instance, when \(\mathcal{X} = \mathbb{R}^d\) and \(\Psi = \text{Id}\), we aim at detecting variations in the mean of the distribution of samples, and when \(\Psi : x \mapsto x^2\), changes in the variance. As a slightly more involved example, when \(\mathcal{X}\) is partitioned into regions \(B_1, \ldots, B_r\) and \(\Psi(x) = (1_{x \in B_i})^T\), then \(E\Psi\) is a histogram, potentially sensitive to a wide variety of changes. While it is well known that histograms suffer from the curse...
of dimensionality, we will see in Sec. 3 that more efficient embeddings can be obtained through reproducing kernels. In the course of time, multiple changes can happen, think for instance of a continuous video feed. Hence, unlike usual algorithms for online change-point detection which stop as soon as a change is detected, in our case we want to “flag” time-points when a change occurred recently, then continue running the algorithm. Therefore, during a single run, the \textit{in-control} situation (that is, the situation before the change) can be different at each change-point.

\textbf{Methods with prior knowledge.} Historically, many methods for online change-point detection assume knowledge of the distributions of samples before and after the change, and rely on a (generalized) likelihood ratio test. See the textbook \cite{3} and references therein. In parallel, so-called “distribution-free” methods \cite{9,39} do not assume complete knowledge of the distributions of samples, but monitor changes in a quantity related to it, for instance the mean or the variance \cite{10}. These methods require prior knowledge of this quantity in the in-control situation, and raise an alarm when it deviates too much from this value.

The method proposed in this paper takes its inspiration in a classical approach called Exponential Weighted Moving Average (EWMA) \cite{32}, see Alg. 1. Recall that our goal is to monitor the quantity $E\Psi(x)$: to emphasize the dependency on $\Psi$, we write EWMA-$\Psi$. Assuming that we know the in-control value of $E\Psi(x)$, denoted by $\theta^\star$, EWMA-$\Psi$ computes recursively a weighted average of $\Psi(x_t)$, with exponential weights that favor the more recent samples. When this average deviates too much from $\theta^\star$, an alarm is raised. The exponential weights (instead of, say, uniform weights) reduce the detection delay, and make the statistic more robust to potentially irrelevant data in the past.

Parametric methods such as EWMA-$\Psi$ have two main advantages. They are extremely fast and have low memory footprint, due to their recursive nature: when a new sample arrives, the cost of the update is only that of computing $\Psi(x)$ once. Moreover, they preserve data privacy, in the sense that they never store raw samples but only averaged statistics. Nevertheless, they require prior knowledge on the in-control situation, which severely limits their use.

\textbf{Methods without prior knowledge.} To solve these problems, methods without prior knowledge were recently proposed, that do not make any assumption on the in-control situation. Some \cite{20,41} are two-steps adaptation of the previous approaches, with an in-control estimation phase followed by the detection itself. Others perform online estimation of a time-varying model and raise an alarm when the data deviates from it too abruptly \cite{40,8}. Finally, some methods are said \textit{model-free}: they compare a pool of recent samples with samples that came before, using a metric that measures the difference between the distributions of the two batches of samples \cite{21,26,25}. A popular, recent choice \cite{19,25} is the metric referred to as Maximum Mean Discrepancy (MMD) \cite{16}, that depends on a positive semi definite kernel $\kappa$ defined on $X$, and which can be estimated efficiently. It is used for instance in the kernel Scan-B algorithm \cite{25}, which computes an unbiased estimator of the MMD between a fixed window of recent samples and one or several windows of samples that came just before, and raise an alarm if it exceeds a threshold.

Model-free methods are useful in a wide class of problems. Moreover, they can adapt to changing in-control situation. Despite these advantages, they are often computationally intensive and have a high memory footprint, since they store raw data that may be high dimensional.

\textbf{Contributions and outline.} The main goal of this paper is to get the best of both worlds, that is, to propose a model-free method that does not store any raw data and enjoys the speed of recursive methods. To this end, in Sec. 2 we introduce “No-prior-knowledge” EWMA (NEWMA), a model-free algorithm that has the same low complexity as EWMA. We show that NEWMA approximately computes an intuitive pseudometric between the distributions of recent and old samples \textit{without having to keep them in memory}. We derive its mean time between false alarms based on a Markov Chain approach. In Sec. 3, we focus on the case where $\Psi$ is a kernel feature map, and we show how it can be approximated with Random Features (RFs) \cite{31}. In the experiments presented in Sec. 4, we also take advantage of a sublinear construction of RFs \cite{24}. We show that our algorithm retrieves change-points at a given precision orders of magnitude faster than competing approaches.
2 Proposed algorithm and theoretical analysis

In this section we introduce and analyze the proposed NEWMA algorithm (Alg. 2). It is based on the following idea: compute two EWMA statistics with different forgetting factors $\lambda < \Lambda$, and raise an alarm when these two statistics are too far apart. Intuitively, the statistic with the larger forgetting factor $\Lambda$ gives “more importance” to recent samples than the one that uses $\lambda$, so the distance between them should increase when recent samples are different from older ones. We are indeed going to show that NEWMA compares implicitly samples in a recent time interval with samples that came before, not unlike classical model-free methods, but without storing the raw data. We write NEWMA-Ψ when we want to pinpoint the mapping choice $\Psi$.

Multiple forgetting factors. The idea of using several forgetting factors in recursive updates has been proposed before, for instance in the so-called Multivariate EWMA (MEWMA) [27, 22], which uses a different factor for each coordinate of multivariate data, or in the context of Recursive Least Squares [12]. However, in these methods, each forgetting factor is used on a different variable, in the eventualty that these variables do not have the same relevant time-scales. The statistic computed by NEWMA-Ψ is fundamentally different: it compares two recursive averages computed on the same data, but with two distinct forgetting factors. To the best of our knowledge, the key idea behind NEWMA-Ψ has not been proposed before.

In the rest of this section we analyze the NEWMA-Ψ algorithm. We start by proving a natural “pointwise” detection result at a given time. Then, we focus on the case where no change occurs (“under the null”), and study the convergence in law of the proposed statistic. Finally, we investigate the mean time between false alarms of the method. In this section, we assume that $H$ is a Hilbert space with norm $\|\cdot\|$. All proofs can be found in the Appendix.

2.1 Detection at a given time

In this section, we place ourselves at a given time point $t$, and show that if a change occurred recently then $t$ is flagged with high probability. Conversely, if no change occurs, then the detection statistic is bounded with high probability. Note that such pointwise guarantees are fairly different from typical results that focus on the delay before detecting a change under various scenarios such as the mean time between false alarms, that we shall examine later. We start by reformulating $z_t’ - z_t$ as a difference between weighted averages of $\Psi(x_t)$ in different time intervals.

**Proposition 1 (Rewriting the detection statistic).** Define $B \defeq \left\lceil \frac{\log(\Lambda/\lambda)}{\log((1-\lambda)/(1-\Lambda))} \right\rceil$. Then, for any $t > B$, 

$$z_t’ - z_t = C \left( \sum_{i=t-B+1}^{t} b_i \Psi(x_i) - \left( a_0 z_0 + \sum_{i=1}^{t-B} a_i \Psi(x_i) \right) \right),$$

where $C = (1 - \lambda)^B - (1 - \Lambda)^B \in (0, 1)$, and $a_i, b_i$ are positive numbers such that $\sum_{i=0}^{t-B} a_i = 1$ and $\sum_{i=t-B+1}^{t} b_i = 1$.

A crucial quantity for analyzing the NEWMA-Ψ algorithm is therefore the window size $B$, that is, the number of recent samples that are compared against older ones, using only $z_t’ - z_t$. Intuitively, the bigger $B$ is, the “smoother” the algorithm will be, resulting in a lower rate of false alarms but an
increased detection delay. Compared to classical model-free algorithms that store pools of samples in memory, a major advantage of NEWMA-$\Psi$ is that its complexity in time and memory is independent from $B$ (see Sec. 3). It can be convenient for the practitioner to directly choose the value of $B$ instead of the forgetting factors $\lambda$ and $\Lambda$. Given a constant $c > 1$, this can be done by setting $\Lambda = c \lambda$ and $\lambda = \left( c^B - 1 \right) / \left( c^{B+1} - 1 \right)$: an alternate parametrization for NEWMA-$\Psi$ is thus through $B$ and $c$.

Note that when $\lambda \to 0$ and $\Lambda = c \lambda$, we have $C \to c^{-1} \pi^{-1} - c^{-1} - \pi$. The exact expressions of the weight coefficients $a_t$ and $b_t$ can be found in App. A.1. They depend only on $\lambda$ and $\Lambda$.

Recall that we aim at detecting changes in the distribution of the samples through the lens of a function $\Psi$. Hence, for any $\pi, \pi'$ probability distributions on $\mathcal{X}$, we define the pseudometric

\[ d_{\Psi}(\pi, \pi') \overset{\text{def.}}{=} \| \mathbb{E}_{\pi} \Psi(x) - \mathbb{E}_{\pi'} \Psi(x) \| . \]

A straightforward consequence of Prop. 1 is that, if the last $B$ samples are drawn i.i.d. according to $\pi'$ and all $t - B$ samples that came before are drawn i.i.d. from $\pi$, then $\| z_t - z_t' \| \approx C d_{\Psi}(\pi, \pi')$. Hence, if $d_{\Psi}(\pi, \pi')$ is large enough, NEWMA-$\Psi$ flags $t$ with high probability. We formalize this in the next proposition, where we also prove robustness to having a slight mismatch between the ideal window sizes $B$ and $t - B$, and the windows in which we actually sample the samples to have different distributions.

**Proposition 2 (Pointwise detection).** Suppose that $M \overset{\text{def.}}{=} \sup_{x \in \mathcal{X}} \| \Psi(x) \| < \infty$. Let $t > B$ be a fixed time point, and $\rho \in (0, 1)$ be some probability of failure. Let $D_1, D_2$ be positive numbers such that the last $B_1 \overset{\text{def.}}{=} D_1 B$ samples are drawn i.i.d. from a distribution $\pi'$ and the $B_2 \overset{\text{def.}}{=} D_2 (t - B)$ samples that came immediately before are drawn i.i.d. from $\pi$. Namely, $x_{t - B_1 + 1}, \ldots, x_t \overset{i.i.d.}{\sim} \pi'$ and $x_{t - B_1 - B_2 + 1}, \ldots, x_{t - B_1} \overset{i.i.d.}{\sim} \pi$. If

\[ d_{\Psi}(\pi, \pi') \geq C^{-1} \left( \tau + E_{\text{conc.}} + E_{\text{init.}} + E_{\text{assum.}} \right), \tag{1} \]

where $E_{\text{init.}} \leq (1-\lambda)^t \| z_0 - \mathbb{E}_{\pi} \Psi(x) \|$, $E_{\text{assum.}} \lesssim M \min\left(1, |D_1 - 1| + |D_2 - 1|\right)$, and $E_{\text{conc.}} = \mathcal{O} \left( \sqrt{(\lambda + \Lambda) \log \frac{1}{\rho}} \right) \quad \text{when} \quad \lambda, \Lambda \to 0$, then with probability at least $1 - \rho$ on the samples, we have $\| z_t - z_t' \| \geq \tau$ and NEWMA-$\Psi$ flags $t$. The exact expressions of $E_{\text{conc.}}, E_{\text{init.}}$, and $E_{\text{assum.}}$ can be found in Sec. A.2 of the Appendix.

We comment further on the error terms appearing in Eq. (1).

- The term $E_{\text{init.}}$ is due to inexactitude in the initialization of the algorithm. It is 0 when $z_0 = \mathbb{E}_{\pi} \Psi(x)$, and otherwise decreases exponentially as $(1 - \lambda)^t$.

- The term $E_{\text{assum.}}$ is due to the difference between the window sizes $B_1, B_2$ appearing in the assumptions of Prop. 2 and the “ideal” case where $B_1 = B$ and $B_2 = t - B$, that is, $D_1 = D_2 = 1$.

- Finally, the term $E_{\text{conc.}}$ stems from concentration inequalities.

We can apply the same reasoning under the null, i.e. in the case where all the sample have the same distribution. We then obtain to obtain an upper bound on our statistic that holds with high probability.

**Proposition 3 (Pointwise bound under the null).** Let $\rho \in (0, 1)$. Assume that all samples $x_1, \ldots, x_t$ are drawn i.i.d. from a distribution $\pi$ and $\sup_{x \in \mathcal{X}} \| \Psi(x) \| < \infty$. Then, with probability at least $1 - \rho$, we have $\| z_t - z_t' \| \leq E_{\text{conc.}} + E_{\text{init.}}$.

Note that Prop. 3 is a worst-case result that doesn’t take into account the “shape” of the null distribution $\pi$. While this type of bound is therefore adapted to all situations, in practice we have found it to be too pessimistic to set the threshold $\tau$. In the experiments (Sec. 4), we use a numerical procedure to dynamically adapt this threshold instead.

### 2.2 Convergence under the null

In this section, we consider that all samples are drawn from $\pi$ and derive an asymptotic result more precise than Prop. 3. Namely, a properly normalized version of our statistic converges to an infinite sum of independent chi-squared random variables, weighted by the eigenvalues of the centered kernel operator associated to $\Psi$, when $\lambda \to 0$. Unlike Prop. 2, where $\Psi$ is assumed uniformly bounded, it relies on the slightly weaker assumption that $\Psi$ has a finite fourth order moment.
Theorem 4 (Convergence under the null). Let $c > 1$ be a constant, set $\Lambda = c\lambda$ and $t \geq \frac{2}{\lambda} \log \frac{1}{\Lambda}$. Assume that all samples $x_i$ are drawn i.i.d. from $\pi$. Suppose that $\mathbb{E}_\pi \| \Psi(x) \|^4 < +\infty$. Set $\mu = \mathbb{E}_\pi \Psi(x)$, and $\kappa(x, x') = \langle \Psi(x) - \mu, \Psi(x') - \mu \rangle$. Define the eigenvalues and eigenvectors of $\kappa$ in $L^2(\pi)$, i.e., define $\lambda_\ell \geq 0$ and $\psi_\ell \in L^2(\pi)$ such that $\kappa(x, x') = \sum_{\ell \geq 1} \lambda_\ell \psi_\ell(x) \psi_\ell(x')$ and $(\psi_\ell, \psi_\ell')_{L^2(\pi)} = 1_{\ell = \ell'}$. Then,

$$\frac{1}{\Lambda} \| z'_\ell - z_\ell \|^2 \xrightarrow{\ell \to 0} Y \overset{\text{def.}}{=} \frac{(1 - c)^2}{2(1 + c)} \sum_{\ell \geq 1} \lambda_\ell W_\ell^2,$$

where $(W_\ell)_{\ell \geq 1}$ is an infinite sequence of independent standard normal random variables.

Note that, with our choice of $t$, $t \to +\infty$ when $\lambda \to 0$. The proof follows closely [14, Sec. 5.5.2] adapted to our setting, with the use of a multivariate version of Lindeberg’s central limit theorem (Th. 12 in the Appendix) instead of the classical Central Limit Theorem. Th. 4 gives additional information on the distribution of our statistic under the null. It may be used to set the threshold $\tau$ if the eigenvalues $\lambda_\ell$ are approximately known, for instance they can be estimated using the Gram matrix of $\kappa$ on a sufficient number of samples [17], which we leave for future work. In Fig. 1 (left), we illustrate the result on a toy example where the eigenvalues are known.

2.3 Mean time between false alarms

An important question for any online procedure is that of false alarms. Assuming that no change occurs in the distribution of the signal, how long before our algorithm raises an alarm? Ideally, this quantity should be as large as possible. It is formally defined as $T \overset{\text{def.}}{=} \mathbb{E} [\inf \{ t \mid t \text{ is flagged} \}]$, where the expectation is over the samples under the null. In the literature, this $T$ is sometime referred to as the Average Run Length (ARL) under control. In the case of NEWMA-$\Psi$, when $\mathcal{H} = \mathbb{R}^m$ is finite dimensional, we show in App. B that it is possible to adapt the Markov chain-based proof developed for classical EWMA in [13]. Unlike the results from the previous sections, our analysis is valid without any boundedness assumption on $\Psi$ or its moments.

The statement of the result (Th. 8) and its proof, rather technical, are deferred to App. B. Let us give instead a brief overview of the approach in [13] and our modifications. The original proof is developed for unidimensional EWMA, and relies on the observation that $z_t$ is a Markov chain with domain $[\theta^* - \tau, \theta^* + \tau]$ (since EWMA stops when $\| z_t - \theta^* \| > \tau$). The authors then discretize this domain with precision $\varepsilon$, compute $T$ for the discrete Markov chain formed by EWMA projected on the grid, then prove that it converges towards the desired quantity when $\varepsilon \to 0$. We adapt this proof for NEWMA-$\Psi$ by taking $\{ z_t, z'_t \}$ as the Markov chain of interest, and $\varepsilon$-coverings instead of unidimensional grids. However, unlike what happens for EWMA, the norm of $z_t$ and $z'_t$ is not necessarily bounded. We overcome this difficulty by letting the compact domain on which we take the $\varepsilon$-coverings grow simultaneously as the coverings become finer.

In simple cases, one can use Th. 8 to approximately compute $T$. For instance, as described in App. B, it can be done when $m = 1$ and one has a closed-form expression for the c.d.f. of $\Psi(x)$. In Fig. 1 (right panel) we compare $T$ for NEWMA-$\Psi$ and EWMA-$\Psi$ on such an example, using respectively Th. 8 and the original approach in [13], as well as numerical simulations. For both algorithms we see that the theoretical expression closely matches the empirical observations. On this toy example, we also see that NEWMA-$\Psi$ has a longer mean time between false alarms than EWMA-$\Psi$. In the multidimensional case, the computation (or estimation) of the transition matrix of $\{ z_t, z'_t \}$ quickly becomes intractable as its size is exponential in $m$. We leave for future work the development of a more efficient procedure to estimate $T$, e.g., with a more tractable asymptotic expression as in [25].

3 Maximum Mean Discrepancy, Random Features

In this section, we turn to the question of a good choice for the embedding $\Psi$, which is crucial since NEWMA-$\Psi$ aims at detecting changes in the expectation of $\Psi(x)$. For most $\Psi$, $d_\Phi$ is only a pseudometric on probability distributions. For instance, when $\Psi(x) = x$, it can only distinguish distributions that have different means. Ideally, we would like $d_\Phi$ to be a true metric, that is, we want $d_\Phi(\pi, \pi') = 0$ if, and only if, $\pi = \pi'$. In this way, our algorithm would be sensitive to “any” change in the distribution of the data.
Then, with probability at least $\frac{1}{1000}$ with Th. 8 (resp. [13]) with a grid of precision $\varepsilon = 2 \cdot 10^{-2}$ (see Sec. B of the Appendix).

**Maximum Mean Discrepancy.** A possible choice for such a metric is the Maximum Mean Discrepancy (MMD, [16]). Given a positive definite kernel $\kappa$ on $\mathcal{X}$, take $\mathcal{H}$ as the reproducing kernel Hilbert space associated to $\kappa$. Set $\Psi(x) = \kappa(x, \cdot)$ and $||\cdot|| = ||\cdot||_{\mathcal{H}}$. Then, in our notation, $d_{\Psi}(\pi, \pi')$ is the MMD between $\pi$ and $\pi'$, that we denote by $\text{MMD}(\pi, \pi')$. When the kernel $\kappa$ is characteristic, it is a true metric. Many conditions have been formulated over the years for $\kappa$ to be characteristic [35], and for instance the Gaussian and the Laplace kernels are characteristic. First introduced in the context of two-sample test, the MMD appears quite naturally in the context of kernel change-point detection [19] [15] [25].

**Random Features.** Empirical estimates of the MMD between two distributions usually require the computation of a $U$-statistic depending on populations drawn from both distributions. Since we do not want to store samples when NEWMA-$\Psi$ is running, least of all perform costly computations on these samples, we resort to kernel Random Features (RF, [31]) to approximate the MMD. RFs and MMD have been combined together before, for accelerating the estimation of the MMD [37] or as a mean to design random projections of distributions in an inverse-problem context [18].

Let us briefly describe the RF machinery. Assume that the kernel $\kappa$ can be written as $\kappa(x, x') = \mathbb{E}_{\omega \sim \Gamma} \phi_\omega(x) \phi_\omega(x')$ for a family of functions $\phi_\omega : \mathcal{X} \rightarrow \mathbb{C}$ parameterized by $\omega \in \mathbb{R}^q$, and a probability distribution $\Gamma$ on $\mathbb{R}^q$. This is for instance the case for all translation-invariant kernels [4].

Drawing $m$ parameters $\omega_1, \ldots, \omega_m \overset{i.i.d.}{\sim} \Gamma$, the RF paradigm consists in defining $\Psi : \mathbb{R}^d \rightarrow \mathbb{C}^m$ as

$$\Psi(x) \overset{\text{def.}}{=} \frac{1}{\sqrt{m}} (\phi_{\omega_1}(x))_{j=1}^m,$$  

(3)

and taking $||\cdot||$ as the classical Hermitian norm on $\mathbb{C}^m$ to run NEWMA-$\Psi$. We have then the following pointwise result, similar to Prop. 2 but with an assumption on the MMD instead of $d_{\Psi}(\pi, \pi')$.

**Proposition 5.** Assume that there exist $M > 0$ such that $\sup_{x \in \mathcal{X}} |\phi_\omega(x)| \leq M$. Define $\Psi(\cdot)$ as in Eq. (3). Let $\rho \in (0, 1)$. Suppose that the assumptions of Prop. 2 on $B_1$ and $B_2$ hold, and that

$$\text{MMD}(\pi, \pi') \geq \left( \frac{1}{\sqrt{m}} (\tau + E_{\text{conc.}} + E_{\text{init.}} + E_{\text{assum.}}) \right)^2 + 2 \sqrt{2 \rho M^2} \sqrt{\log \frac{1}{\rho}} \right)^{1/2}.$$  

(4)

Then, with probability at least $1 - 2\rho$ on both samples $x_i$ and parameters $\omega_j$, it holds that $||z_i - z_i'|| \geq \tau$ and NEWMA-$\Psi$ flags $t$.

Hence the lower bound in (4) is of the same order than that in Prop. 2 if $m = O ((\lambda + \Lambda)^{-2})$.

**Comparison with Scan-B.** When it is used with RFs, NEWMA-$\Psi$ approximates the MMD between a window of recent samples and samples that came before. It is reminiscent of the Scan-B algorithm [25], which estimates the MMD between a window of size $B$ of recent samples and $N$
such windows of older samples, with a U-statistic. Unlike Scan-\(B\), NEWMA-\(\Psi\) does not store the samples, but perform the estimation of the MMD implicitly by comparing two weighted averages of RFs. However, a drawback of NEWMA-\(\Psi\) is that it cannot be applied when no approximate finite embedding for the kernel \(\kappa\) (such as RFs) is available: since we do not keep any samples in memory, we cannot leverage the “kernel trick” as in the traditional U-statistic for the MMD.

Fast random features and computational cost. In Table 1 we compare the computational cost of NEWMA-\(\Psi\) with RF and that of the Scan-\(B\) algorithm. For the latter, both computing cost and memory requirement are dominated by the storage and the processing of the fixed-size windows of samples. For NEWMA-\(\Psi\), the complexity is largely dominated by the RF computation, which often amounts to a dense matrix-vector product in \(O(md)\), with a memory requirement to store the matrix of features in \(O(md)\). However, a large body of work is dedicated to accelerate the computation of such random features. For instance, the Fastfood approximation \cite{ahoc14} reduces the time complexity to \(O(m\log d)\) and memory to \(O(m)\). Another example particularly relevant to our work is \cite{mohri17}, where the authors demonstrate how to build an optical device that computes random features in \(O(1)\) and thus eliminates the need to store the random matrix.

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
 & NEWMA & Scan-B \cite{ahoc14} \\
\hline
Time & \(O(C_{\Psi} + m)\) & \(O(NBC_{\kappa})\) \\
Memory & \(O(M_{\Psi} + m)\) & \(O(NBd)\) \\
\hline
\end{tabular}
\caption{Time (of one iteration) and memory for NEWMA-\(\Psi\) and Scan-\(B\) algorithm \cite{ahoc14}. The cost and memory for the RFF operator \(\Psi\) are respectively denoted by \(C_{\Psi}\) and \(M_{\Psi}\): they are in \(O(md)\) in the naive way, but sublinear in \(d\) with \cite{ahoc14} or even constant with \cite{mohri17}. The cost of computing \(\kappa\) (usually \(O(d)\)) is denoted by \(C_{\kappa}\), and \(N\) is the number of windows in the Scan-\(B\) algorithm.}
\end{table}

4 Experiments

In this experimental section we compare three approaches.\(^1\) NEWMA-\(\Psi\) with classical Random Fourier Features \cite{rahimi08}, denoted by NEWMA-RF, NEWMA-\(\Psi\) with RF computed with FastFood \cite{ahoc14}, denoted by NEWMA-FF, and the Scan-\(B\) algorithm \cite{ahoc14}, all implemented with a Gaussian kernel or the corresponding RFs. For all experiments we use a window size \(B = 250\) for both Scan-\(B\) and NEWMA, \(c = 2\) for NEWMA (the forgetting factors \(\lambda\) and \(\Lambda\) are then set according to Sec. 2.1 in function of \(c\) and \(B\)), and \(N = 3\) windows for Scan-\(B\). We choose a Gaussian kernel with a dimension-dependent bandwidth \(\sigma^2 = d\), which we found to be empirically a good choice for the considered data. By default, NEWMA is performed with \(m = 3000\) features, and the synthetic data are in dimension \(d = 100\). The experiments run on a laptop with an Intel Core i7-7600U 2.80GHz.

4.1 Adaptive threshold

Consider any online change-point detection method based on a statistic \(S_t \geq 0\) maintained in a streaming fashion, such that \(t\) is flagged if \(S_t\) exceeds a threshold \(\tau > 0\). Generally, the detection threshold \(\tau\) is set by examining the properties of \(S_t\) under the null, that is, in our case, Prop. \(5\) Th. \(4\) and Th. \(8\) in App. \(B\). However, such an approach may be only feasible with strong prior on the in-control situation, and may not be suited to multiple changes in a stream, where the in-control situation before each change might be different.

We propose here to let the threshold \(\tau\) be also time-varying. More precisely, we make it follow the values of the statistic \(S_t\) “by above,” such that \(S_t\) exceeds the threshold only if it increases abruptly. This is done by maintaining a weighted average \(S_t\) of the previous values of \(S_t\) recursively:

\[ S_t = (1 - \eta)S_{t-1} + \eta S_t, \]

and defining the threshold above this average by scaling it:

\[ \tau_t = aS_t \]

with \(a > 1\). A time-point is then flagged when \(S_t \geq \tau_t\). In the experiments we set \(\eta = \frac{1}{2}\lambda\). Empirically, we found this adaptive threshold procedure to be far more flexible than a fixed threshold. The analysis of such a procedure may however prove challenging and is left for future work.

4.2 Experimental results

Time of execution. In Fig. 2\(a\) we examine the time of execution of the three algorithms with respect to the dimension \(d\) of the data and window size \(B\). NEWMA-RF is seen to be approximately 12 times faster than Scan-\(B\). As expected, NEWMA-FF is sublinear in the dimension and much faster.

\(^1\)The code is available on the first author’s webpage.
than the other approaches in high dimension. The results also confirm that NEWMA’s complexity is independent of $B$, while that of Scan-$B$ increases linearly with $B$.

**Results on synthetic data.** Next we examine the detection performance of the three algorithms on synthetic data. We generate the data as follows: $10^6$ samples are drawn from Gaussian Mixture Models (GMM) with $k = 10$ components, and the GMM changes every 2000 samples (at each change, we draw $k$ new vector means according to a centered Gaussian distribution, $k$ new covariance matrices according to an inverse-Wishart distribution, and $k$ new mixing weights according to a Dirichlet distribution). In total, there are 500 changes to detect. We say that a change is detected when a non-flagged sample is followed by a flagged one. For a continuous run, our goal is to evaluate the trade-off between speed of detection and in-control false alarm rate. To do so, we divide each window of 2000 samples between two ground-truth changes in two. In the first half (i.e., in the first 1000 samples after the true change) we record the time until the first detected change as detection delay, later averaged over the whole run to obtain the Expected Detection Delay (EDD). If no change is detected, we record a missed detection. In the second half of the window (i.e., the last 1000 samples before the next true change), we record every detected change as a false alarm. We plot ratios EDD-to-False Alarms or Missed Detections-to-False Alarms by varying the threshold $\tau$ when using a fixed threshold, or varying the multiplicative constant $a$ when using the adaptive threshold procedure.

In Fig. 2b we examine the effect of using the adaptive threshold instead of a fixed threshold. The adaptive procedure is far superior to the fixed one, which cannot adapt to several in-control situations in a single run. In the rest of the experiments we always use the adaptive procedure.

In Fig. 2c we compare the three algorithms. On this particular example, Scan-$B$ exhibits slightly less false alarms than NEWMA for the same number of missed detections. On the contrary, NEWMA has a lower EDD than Scan-$B$ for the same window size $B$. This can be attributed to the fact that the estimator of the MMD used in Scan-$B$ has a lower variance, and that on the contrary the exponential weights used in NEWMA decrease the detection delay.

**Results on real data.** We apply our method to a Voice Activity Detection (VAD) task on audio data. We consider real environments background noise from the QUT-NOISE dataset and add, every 10s, a 3s speech extract from the TIMIT dataset, with $-7.5$dB Signal-to-Noise Ratio. Our goal is to detect the onset of the speech segments. We use the Short Time Fourier Transform of the signal (which can also be computed online). At the end of the day, the data is $d = 120$-dimensional, with a ground truth change every 1250 samples, for 300 changes in total. We take a window size $B = 150$, and again evaluate performance by dividing every intervals between two ground truth changes in two. We display the results in Fig. 2d. Similar to the results on synthetic data, Scan-$B$ yields slightly less missed detections, but has a higher detection delay.
5 Conclusion and outlooks

We introduced NEWMA-Ψ, a new method for online change-point detection that is orders of magnitude faster and lighter than other existing model-free methods. The key idea behind NEWMA-Ψ is to compare recursive averages computed with different forgetting factors on the same data, in order to implicitly extract time-varying information without keeping in memory the raw data.

In the future, we plan to use the constant-time random features developed by [33] to apply NEWMA-Ψ on high-dimensional real-time data, such as video monitoring. Another direction for our research is the study of mappings Ψ for graph data, which, combined with NEWMA-Ψ, would allow to detect changes in large-scale social networks [29]. Finally, the use of recent literature on density estimation from random feature moments [18] would allow extracting more information from $z_t - z'_t$ than mere occurence of a change, such as the region of space in which the change occurred.

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Appendix

In this Appendix we collect the proofs of all the theoretical results of the paper and some additional results. The proofs themselves are in Sec. A. A statement about the mean time between false alarms of NEWMA-Ψ can be found in Sec. B. Finally, Sec. C contains technical results used throughout this Appendix.

A Proofs of the theoretical results

In this section we collect the proofs of all the theoretical results of the paper. We start with some elementary computations that are used throughout the rest of the proofs.

Set $0 < \lambda < \Lambda < 1$. For any $i, t \geq 1$, define $\alpha_0 = (1 - \lambda)^t$, $\beta_0 = (1 - \Lambda)^t$,

\[ \alpha_i = \lambda(1 - \lambda)^{t-i} \quad \text{and} \quad \beta_i = \Lambda(1 - \Lambda)^{t-i}. \]

Then, for any $1 \leq t_1 < t_2 \leq t$,

\[ \sum_{i=0}^{t_1} \alpha_i = (1 - \lambda)^{t-t_1}, \quad \sum_{i=0}^{t_2} \beta_i = (1 - \Lambda)^{t-t_1}, \]

\[ \sum_{i=t_1}^{t_2} \alpha_i = (1 - \lambda)^{t-t_1} - (1 - \lambda)^{t-t_1+1}, \quad \sum_{i=t_1}^{t_2} \beta_i = (1 - \Lambda)^{t-t_1} - (1 - \Lambda)^{t-t_1+1}, \]

and

\[ \sum_{i=1}^{t} \alpha_i^r \beta_i^s = \frac{\lambda^r \Lambda^s}{1 - A} (1 - A^t), \quad \text{with} \quad A = (1 - \lambda)^{t_1} (1 - \Lambda)^{t_2}. \]

A.1 Proof of Prop. 1

Recall that we defined

\[ B = \left\lceil \frac{\log (\Lambda/\lambda)}{\log ((1 - \lambda)/(1 - \Lambda))} \right\rceil. \]

Let $t > B \geq 1$. By construction of definition of $z_t$ and $z'_t$ and by definition $\alpha_i, \beta_i$, we have

\[ \begin{align*}
  z_t &= \alpha_0 z_0 + \sum_{i=1}^{t} \alpha_i \Psi (x_i) \\
  z'_t &= \beta_0 z_0 + \sum_{i=1}^{t} \beta_i \Psi (x_i).
\end{align*} \]

A straightforward computation yields that $t - B$ is the “shifting” point for the weight coefficients. Namely, for $i = 1, \ldots, t - B$, we have $\alpha_i \geq \beta_i$, and for $i = t - B + 1, \ldots, t$, we have $\beta_i \geq \alpha_i$. According to Eq. (5) and Eq. (6), $\sum_{i=0}^{t-B} (\alpha_i - \beta_i) = \sum_{i=t-B+1}^{t} (\beta_i - \alpha_i) = C$, where $C = (1 - \lambda)^B - (1 - \Lambda)^B$. Hence, if we define $a_i \equiv (\alpha_i - \beta_i)/C$ for $i = 0, \ldots, t - B$ and $b_i \equiv (\beta_i - \alpha_i)/C$ for $i = t - B + 1, \ldots, t$, we have

\[ z'_t - z_t = \sum_{i=1}^{t} (\beta_i - \alpha_i) \Psi (x_i) + (\beta_0 - \alpha_0) z_0 \]

\[ = \sum_{i=t-B+1}^{t} (\beta_i - \alpha_i) \Psi (x_i) - \left( \sum_{i=1}^{t-B} (\alpha_i - \beta_i) \Psi (x_i) + (\alpha_0 - \beta_0) z_0 \right) \]

\[ z'_t - z_t = C \cdot \left( \sum_{i=t-B+1}^{t} b_i \Psi (x_i) - \left( a_0 z_0 + \sum_{i=1}^{t-B} a_i \Psi (x_i) \right) \right). \]

\[ \square \]
A.2 Proof of Prop. 2 and 3

The proofs of Prop. 2 and 3 are a direct consequence from the following result.

Lemma 6 (Concentration of the detection statistic). Suppose that $M = \sup_{x \in X} \|\Psi(x)\| < +\infty$. At time $t$, under the assumptions of Prop. 2, with probability at least $1 - \rho$ on the samples $x_i$, we have

$$\left\|z'_i - z_i\right\| - C_\phi(p', \pi) \leq E_{\text{conc.}} + E_{\text{init.}} + E_{\text{assum.}},$$

where

$$E_{\text{conc.}} = 2\sqrt{2}M \left(1 + \sqrt{\log \frac{2}{\rho}}\right) \sqrt{\varphi(\lambda, \lambda) + \varphi(\Lambda, \Lambda) - 2\varphi(\lambda, \Lambda)},$$

with $\varphi(\lambda, \Lambda) = \frac{\lambda(1 - (1 - \lambda)^t)(1 - \Lambda)^t}{\lambda + \Lambda(1 - \lambda)^t},$

$$E_{\text{init.}} = \left((1 - \lambda)^t - (1 - \Lambda)^t\right) \|z_0 - E_\pi \Psi(x)\|,$$

$$E_{\text{assum.}} = 2M \left((1 - \lambda)^t - (1 - \Lambda)^t - \left[(1 - \Lambda)^{B_1 + B_2} - (1 - \Lambda)^t\right]\right) + \left((1 - \lambda)^{\min(B, B_1)} - (1 - \lambda)^{\min(B, B_1)}\right) - \left[(1 - \Lambda)^{\min(B, B_1)} - (1 - \Lambda)^{\max(B, B_1)}\right]$$

\[\lesssim M \min(1, |D_1 - 1| + |D_2 - 1|).\]

Proof. We have seen that, ideally, the last $B$ samples are drawn from $\pi'$, and all the samples that came before are drawn from $\pi$, but that we have assumed existence of window sizes $B_1 = D_1 B$ and $B_2 = D_2(t - B)$ that can slightly deviate from these values. Let us call $I$ the time interval of samples that are not drawn from the “correct” distribution:

$$I = [t - B_1 - B_2, t - \max(B, B_1) + 1, t - \min(B_1, B)].$$

Let us introduce “ghost samples” $y_1, \ldots, y_t$ drawn from the “correct” distributions, i.e. such that $y_1, \ldots, y_{t - B} \overset{i.i.d.}{\sim} \pi, y_{t - B + 1}, \ldots, y_t \overset{i.i.d.}{\sim} \pi'$, and such that $y_i = x_i$ for $i \notin I$. The idea of the proof is to introduce the analogous of $\|z'_i - z_i\|$ for the ghost samples in the left-hand side of Eq. (8), to use the triangle inequality, and then to bound the resulting error terms. Thus, with the help of Prop. 1, we first write

$$\left\|z'_i - z_i\right\| - C_\phi(p, \pi') = \left\|(\alpha_0 - \beta_0)z_0 + \sum_{i=1}^t (\alpha_i - \beta_i)\Psi(x_i)\right\| - C \left\|E_\pi \Psi(y) - E_{\pi'} \Psi(y)\right\|$$

$$\leq \left\|(\alpha_0 - \beta_0)z_0 + \sum_{i=1}^t (\alpha_i - \beta_i)\Psi(x_i)\right\| - C \left\|E_\pi \Psi(y) - E_{\pi'} \Psi(y)\right\|$$

$$+ \left\|(\alpha_0 - \beta_0)z_0 + \sum_{i=1}^t (\alpha_i - \beta_i)\Psi(y_i)\right\| - \left\|(\alpha_0 - \beta_0)z_0 + \sum_{i=1}^t (\alpha_i - \beta_i)\Psi(x_i)\right\|$$

$$\overset{\text{def}}{=} (I) + (II).$$

since $|x - y| \leq |z - x| + |z - y|.$

We first show that (II) is upper bounded by $E_{\text{assum.}}$. Since $y_i = x_i$ for any $i \notin I$,

$$\text{(II)} \leq 2M \sum_{i \in I} |\alpha_i - \beta_i|.$$

By definition of the integer interval $I$,

$$\sum_{i \in I} |\alpha_i - \beta_i| = \sum_{i=1}^{t - B_1 - B_2} |\alpha_i - \beta_i| + \sum_{i=t - \max(B, B_1) + 1}^{t - \min(B, B_1)} |\alpha_i - \beta_i|$$

$$= \sum_{i=1}^{t - B_1 - B_2} (\alpha_i - \beta_i) + \sum_{i=t - \max(B, B_1) + 1}^{t - \min(B, B_1)} (\alpha_i - \beta_i).$$
since \(\alpha_i - \beta_i\) has constant sign in the considered intervals. Using Eq. (5) and (6), we obtain the desired expression for \(E_a\). For simplicity, we then show that \(E_a\) is roughly bounded by \(M \min(1, |D_1 - 1| + |D_2 - 1|)\) up to a multiplicative constant. First, it is at least bounded by \(4M\), and then, we have for instance

\[
(1 - \lambda)^{B_1 + B_2} - (1 - \lambda)^t = (1 - \lambda)^{B_1 + B_2} (1 - (1 - \lambda)^{t - B_1 - B_2}) \\
\leq (1 - \lambda)^{D_2 t + (D_1 - D_2) B} (1 - (1 - \lambda)^{(1 - D_2) t + (D_2 - D_1) B}) \\
\leq (1 - \lambda)^{D_2 t + (D_1 - D_2) B} (t + B) \log \frac{1}{1 - \lambda} \cdot (|D_2 - 1| + |D_1 - D_2|)
\]

and it is easy to show that the left multiplicative term is bounded. We proceed similarly for the other terms in \(E_a\).

We now prove that (I) is upper bounded by \(E_{init} + E_{conc}\). By the triangle inequality and the definition of \(a_i\) and \(b_i\),

\[
(I) \leq \left\| (\alpha_0 - \beta_0)z_0 + \sum_{i=1}^{t-B} (\alpha_i - \beta_i) \Psi(y_i) - C\mathbb{E}_{\pi}\Psi(y) - \left( \sum_{i=t-B+1}^t (\beta_i - \alpha_i) \Psi(y_i) - C\mathbb{E}_{\pi}\Psi(y) \right) \right\| \\
\leq C|a_0| \left\| z_0 - \mathbb{E}_{\pi}\Psi(y) \right\| + C \left\| \sum_{i=1}^{t-B} a_i (\Psi(y_i) - \mathbb{E}_{\pi}\Psi(y)) \right\| + C \left\| \sum_{i=t-B+1}^t b_i (\Psi(y_i) - \mathbb{E}_{\pi}\Psi(y)) \right\|
\]

since \(\sum_{i=1}^t (\alpha_i - \beta_i) = \sum_{i=t-B+1}^t (\beta_i - \alpha_i) = C\).

We now apply McDiarmid’s inequality (Lemma[11]) to bound the right-hand side of the last display with high probability. Define \(\Delta : \mathcal{X}^{t-B} \rightarrow \mathbb{R}\) by

\[
\Delta(y_1, \ldots, y_{t-B}) = \left\| \sum_{i=1}^{t-B} a_i (\Psi(y_i) - \mathbb{E}_{\pi}\Psi(y)) \right\|.
\]

This function satisfies the bounded difference property, that is,

\[
|\Delta(y_1, \ldots, y_{t-B}) - \Delta(y_1, \ldots, y_{t-B}^\prime)| \leq 2Ma_i.
\]

We then apply Lemma[11] with \(f = \Delta\) and \(c_i = 2Ma_i\) to obtain

\[
\mathbb{P}(\Delta \geq \mathbb{E}\Delta + \varepsilon) \leq \exp\left( -\frac{\varepsilon^2}{4M^2 \sum_{i=1}^{t-B} a_i^2} \right).
\]

We now bound \(\mathbb{E}\Delta\) by a symmetrization argument. Let us introduce the random variables \(y_i^\prime\) that have the same law as the \(y_i\) and are independent from the \(y_i\), and the \(\sigma_i\), Rademacher random variables independent from both \(y_i\) and \(y_i^\prime\). We write

\[
\mathbb{E}\left\| \sum_{i=1}^{t-B} a_i (\Psi(y_i) - \mathbb{E}\Psi(y)) \right\| = \mathbb{E}_{y, y^\prime}\left\| \sum_{i=1}^{t-B} a_i (\Psi(y_i) - \mathbb{E}\Psi(y_i^\prime)) \right\| \\
= \mathbb{E}_{y, y^\prime, \sigma}\left\| \sum_{i=1}^{t-B} a_i \sigma_i (\Psi(y_i) - \mathbb{E}\Psi(y_i^\prime)) \right\| \\
\leq 2\mathbb{E}_{y, \sigma}\left\| \sum_{i=1}^{t-B} a_i \sigma_i (\Psi(y_i)) \right\| \\
= 2\sqrt{\mathbb{E}_y \sum_{i=1}^{t-B} a_i^2 \mathbb{E}_{\sigma} \left\| \sum_{j=1}^{t-B} a_j \sigma_j (\Psi(y_j)) \right\|} \leq 2M \sqrt{\sum_{i=1}^{t-B} a_i^2}
\]

By applying the same reasoning to \(\Delta' \overset{def}{=} \left\| \sum_{i=t-B+1}^t b_i (\Psi(y_i) - \mathbb{E}_{\pi}\Psi(y)) \right\|\) and a union bound, we obtain that, with probability at least \(1 - \rho\),

\[
(I) \leq E_{init} + 2MC \left( 1 + \sqrt{\log \frac{2}{\rho}} \right) \left( \sqrt{\sum_{i=t-B+1}^t b_i^2} + \sqrt{\sum_{i=1}^{t-B} a_i^2} \right).
\]
Since
\[
\sqrt{\sum_{i=t-B+1}^{t} b_i^2} + \sqrt{\sum_{i=1}^{t-B} a_i^2} \leq \sqrt{2} \left( \sum_{i=t-B+1}^{t} b_i^2 + \sum_{i=1}^{t-B} a_i^2 \right) = \frac{\sqrt{2}}{C} \sum_{i=1}^{t} (\alpha_i^2 + \beta_i^2 - 2\alpha_i\beta_i),
\]
we recover the expression of \( E_{\text{conc.}} \) with the help of Eq. (7). Therefore, we showed that, with probability at least \( 1 - \rho \),
\[
\|\| z_i^\prime - z_i \| - C d\psi (\pi, \pi^\prime) \| \leq E_{\text{conc.}} + E_{\text{init.}} + E_{\text{assum.}}.
\]
\[\square\]

### A.3 Proof of Th. 4

Our proof follows closely [34], Sec. 5.5.2. with some modifications.

In the following, we let \( \lambda \to 0 \) with \( \Lambda = c\lambda \) and \( t \geq \frac{1}{\lambda} \log (1/\lambda) \) (which goes to \( +\infty \) when \( \lambda \) goes to \( 0 \)), such that \( (1 - \lambda)^t = O (\lambda^t) \). At time \( t \), we denote \( \gamma_i = \alpha_i - \beta_i \), with \( \alpha \) and \( \beta \) defined as in the proof of Prop. [4]. Note that \( \alpha \) and \( \beta \) also depend on \( \lambda \), and that by Eq. (1), we have
\[
\frac{1}{\lambda} \sum_{i=1}^{t} \gamma_i^2 \xrightarrow{\lambda\to 0} G \overset{\text{def.}}{=} \frac{(1 - c)^2}{2(1 + c)},
\]
and \( \sum_{i=1}^{t} \gamma_i^2 = O (\lambda^{t-1}) \).

Define \( \mu \overset{\text{def.}}{=} E \Psi (x) \). At time \( t \) we have
\[
\frac{1}{\lambda} \| z_t^\prime - z_t \|^2 = \frac{1}{\lambda} \| \gamma_0 z_0 + \sum_{i=1}^{t} \gamma_i \Psi (x_i) \|^2
\[
= \frac{1}{\lambda} \| \gamma_0 (z_0 - \mu) + \sum_{i=1}^{t} \gamma_i (\Psi (x_i) - \mu) \|^2 \quad \text{since} \sum_{i=0}^{t} \gamma_i = 0
\[
= \frac{1}{\lambda} \sum_{i,j=1}^{t} \gamma_i \gamma_j \kappa (x_i, x_j) + \frac{2}{\lambda \gamma_0} \sum_{i=1}^{t} \gamma_i \langle z_0 - \mu, \Psi (x_i) - \mu \rangle_{\mathcal{H}} + \frac{1}{\lambda} \gamma_0^2 \| z_0 - \mu \|^2,
\]
where \( \kappa \) is a positive semi-definite kernel on \( \mathcal{H} \) defined by \( \kappa (x, x^\prime) = \langle \Psi (x) - \mu, \Psi (x^\prime) - \mu \rangle_{\mathcal{H}} \).

The last term of Eq. (10) is deterministic and goes to 0 with \( \lambda \) since \( \gamma_0^2 = O (\lambda^4) \). Let us now prove that the second term converges in probability to 0. By Cauchy-Schwarz’s and Jensen’s inequalities we have
\[
E (z_0 - \mu, \Psi (x_i) - \mu)_{\mathcal{H}} \langle z_0 - \mu, \Psi (x_j) - \mu \rangle_{\mathcal{H}} \leq \| z_0 - \mu \|^2 \mathbb{E} (\kappa (x_i, x_j)) \mathbb{E} (\kappa (x_j, x_j))
\]
\[
\leq \| z_0 - \mu \|^2 \mathbb{E} (\kappa (x, x) < \infty), \quad \text{and}
\]
\[
\frac{1}{\lambda} \gamma_0 = O (\lambda) \xrightarrow{\lambda\to 0} 0.
\]

Hence \( \frac{2}{\lambda} \gamma_0 \sum_{i=1}^{t} \gamma_i \langle z_0 - \mu, \Psi (x_i) - \mu \rangle_{\mathcal{H}} \) has a second order moment that converges to 0, and by Markov’s inequality it converges in probability to 0.

Let us now prove that the first term in Eq. (10) converges in law, and conclude with Slutsky’s Lemma (Lemma [13]). We start by using Mercer’s theorem on \( \kappa \), within the ambient space \( L^2 (\pi) \); we write
\[
\kappa (x, x^\prime) = \sum_{\ell \geq 1} \lambda_{\ell} \psi_{\ell} (x) \psi_{\ell} (x^\prime),
\]
with $\lambda_t \geq 0$ and $\langle \psi_t, \psi_t \rangle_{L^2(x)} = 1_{\ell = \ell'}$, such that $\langle \kappa(x, \cdot), \psi_t \rangle_{L^2(x)} = \lambda_t \psi_t(x)$. Note that, since $E_x \kappa(x, x') = 0$, for any $\lambda_t \neq 0$ we have $E \psi_t(x) = \frac{1}{\lambda_t} \langle E \kappa(x, \cdot), \psi_t \rangle_{L^2(x)} = 0$. Finally, we have $\sum_{\ell \geq 1} \lambda^2 \mathbb{E} \psi^2_t(x) \leq \mathbb{E} \kappa^2(x, x) < \infty$ since $E \| \Psi(x) \|^4 < +\infty$.

Our goal is to show that $T_\lambda \overset{\text{def}}{=} \frac{1}{\lambda} \sum_{i,j=1}^t \gamma_i \gamma_j k(x_i, x_j)$ converges in law to $Y = G \sum_{\ell \geq 1} \lambda_t W^2_{\ell}$ where $W_\ell$ are independent centered normal variable. We are going to use the characteristic function, i.e., we are going to prove that:

$$\forall u \in \mathbb{R}, \quad E e^{iuT_\lambda} \overset{\lambda \to 0}{\longrightarrow} E e^{iuY}.$$

Fix any $u \in \mathbb{R}$ and $\varepsilon > 0$. Our goal is to prove that, for $\lambda$ sufficiently small, we have $|E e^{iuT_\lambda} - E e^{iuY}| \leq \varepsilon$. We decompose the bound in three parts.

**Step 1.** For an integer $k \geq 0$, define $T^{(k)}_\lambda = \frac{1}{\lambda} \sum_{i,j=1}^t \gamma_i \gamma_j \left[ \sum_{\ell = 1}^k \lambda_t \psi_t(x_i) \psi_t(x_j) \right]$. We are first going to approach $E e^{iuT_\lambda}$ by $E e^{iuT^{(k)}_\lambda}$ for $k$ sufficiently big. We write

$$|E e^{iuT_\lambda} - E e^{iuT^{(k)}_\lambda}| \leq E \left| e^{iuT_\lambda} - e^{iuT^{(k)}_\lambda} \right| \leq |u| E \left| T_\lambda - T^{(k)}_\lambda \right| \leq |u| \sqrt{E \left( T_\lambda - T^{(k)}_\lambda \right)^2}.$$

Denote $f_k(x, x') = \kappa(x, x') - \sum_{\ell=1}^k \lambda_t \psi_t(x) \psi_t(x') = \sum_{\ell \geq k+1} \lambda_t \psi_t(x) \psi_t(x')$, such that $T_\lambda - T^{(k)}_\lambda = \frac{1}{\lambda} \sum_{i,j=1}^t \gamma_i \gamma_j f_k(x_i, x_j)$. We have $E \left[ f_k(x_1, x'_1) f_k(x_2, x'_2) \right] \neq 0$ if and only if both $x_1 = x_2$ and $x'_1 = x'_2$ (or permuted since $f_k$ is symmetric), and we have

$$E_{x,x'} f_k(x, x')^2 = \sum_{\ell \geq k+1} \lambda^2 \mathbb{E} \psi^2_t(x),$$

$$E f_k(x, x)^2 = \sum_{\ell \geq k+1} \lambda^2 \mathbb{E} \psi^2_t(x),$$

where the last expression is summable since $E \kappa^2(x, x) < \infty$. Then we have

$$E \left( T_\lambda - T^{(k)}_\lambda \right)^2 = \frac{1}{\lambda^2} E \left[ \sum_{i_1, j_1=1}^t \sum_{i_2, j_2=1}^t \gamma_{i_1} \gamma_{j_1} \gamma_{i_2} \gamma_{j_2} f_k(x_{i_1}, x_{j_1}) f_k(x_{i_2}, x_{j_2}) \right]$$

$$\leq \frac{2}{\lambda^2} E \left[ \sum_{i,j=1}^t \gamma_i^2 \gamma_j^2 f_k(x_i, x_j)^2 \right]$$

$$\leq 2 \left( \frac{1}{\lambda} \sum_{i=1}^t \gamma^2_i \right)^2 \max \left( E f_k^2(x, x'), E f_k^2(x, x) \right) \leq C \left( \sum_{\ell \geq k+1} \lambda^2 \max \left( 1, \mathbb{E} \psi_t^4(x) \right) \right),$$

for some constant $C$, since $\sum_{i=1}^t \gamma^2_i = O(\lambda)$. Hence for $k$ sufficiently big we have:

$$\forall \lambda \in (0, 1), \quad |E e^{iuT_\lambda} - E e^{iuT^{(k)}_\lambda}| \leq \frac{\varepsilon}{3}. \quad (11)$$

**Step 2.** Let us now temporarily consider a fixed $k$, and prove that $T^{(k)}_\lambda$ converges in law to $Y^{(k)} = B \sum_{\ell=1}^k \lambda_t W^2_{\ell}$. We write

$$T^{(k)}_\lambda = \frac{1}{\lambda} \sum_{i,j=1}^t \gamma_i \gamma_j \left[ \sum_{\ell = 1}^k \lambda_t \psi_t(x_i) \psi_t(x_j) \right] = \sum_{\ell = 1}^k \lambda_t \left( \frac{1}{\sqrt{\lambda}} \sum_{i=1}^t \gamma_i \psi_t(x_i) \right)^2.$$

We now use Lindeberg’s theorem (Th. 12) on the random vectors

$$X^{(i,t)} = \left( \frac{1}{\sqrt{\lambda}} \gamma_i \psi_t(x_i) \right)^k \overset{i.i.d.}{\longrightarrow} X.$$
They are centered and their covariance is such that
\[ \sum_{i=1}^{t} \text{Cov}(X^{(i,t)}) = \left( \frac{1}{\lambda} \sum_{i=1}^{t} \gamma_{i}^{2} \right) \text{Id} \xrightarrow{\lambda \to 0} B \cdot \text{Id}. \]

We now check Lindeberg’s condition (18). By Cauchy-Schwartz and Markov’s inequality, for all \( \delta > 0 \) we have
\[ \mathbb{E} \left[ \left\| X^{(i,t)} \right\|^{2} I_{\{ \left\| X^{(i,t)} \right\| \geq \delta \}} \right] \leq \sqrt{\mathbb{E} \left[ \left\| X^{(i,t)} \right\|^{4} \right]} \cdot \mathbb{P} \left[ \left\| X^{(i,t)} \right\| \geq \delta \right] \]
\[ \leq \sqrt{\mathbb{E} \left[ \left\| X^{(i,t)} \right\|^{4} \right]} \cdot \delta^{-2} \mathbb{E} \left[ \left\| X^{(i,t)} \right\|^{2} \right] \leq C \delta^{-2} \gamma_{i}^{4} \]
where \( C \) is a constant, since \( \psi_{l}(x) \) has finite second and fourth order moment. Using the fact that \( \sum_{i=1}^{t} \gamma_{i}^{2} = O(\lambda) \), Lindeberg’s condition is satisfied. Hence, applying theorem 12, \( \sum_{i=1}^{t} X^{(i,t)} \) converges in law to \( \mathcal{N}(0, G \cdot \text{Id}) \), and \( T_{\lambda}^{(k)} \) converges in law to \( Y^{(k)} \). Hence for a sufficiently small \( \lambda \)
\[ \| \mathbb{E} e^{iuX^{(k)}} - \mathbb{E} e^{iuY^{(k)}} \| \leq \varepsilon. \]

**Step 3.** Finally, similar to Step 1 we have
\[ \left\| \mathbb{E} e^{iuY^{(k)}} - \mathbb{E} e^{iuY} \right\| \leq \left| u \right| \mathbb{E} \left| Y^{(k)} - Y \right| \leq \left| u \right| \sqrt{\mathbb{E} \left( Y^{(k)} - Y \right)^{2}} \]
\[ \leq \left| u \right| \left( \sum_{\ell \geq k+1} \lambda_{\ell} \right)^{2} \max(1, \mathbb{E} W^{4}) \text{ where } W \sim \mathcal{N}(0, 1), \]
and therefore for a sufficiently big \( k \)
\[ \left\| \mathbb{E} e^{iuY^{(k)}} - \mathbb{E} e^{iuY} \right\| \leq \varepsilon. \]

To conclude, we fix \( k \) large enough such that Eq. (11) and (13) are satisfied, then \( \lambda \) small enough and Eq. (12) is satisfied, which concludes the proof.

**A.4 Proof of Prop. 5**

The proof is a combination of Prop. 2 and the following lemma, which is a consequence of Hoeffding’s inequality.

**Lemma 7 (Concentration of \( d_{\phi} \)).** Define \( \Psi \) as (3) and let \( \rho \in (0, 1) \). For any distributions \( \pi, \pi' \), with probability at least \( 1 - \rho \) on the \( \omega_{j} \)’s, we have
\[ d_{\phi}(\pi, \pi')^{2} \geq \text{MMD}(\pi, \pi')^{2} - \frac{2\sqrt{2M^{2}}}{\sqrt{m}} \sqrt{\log \frac{1}{\rho}}. \]

**Proof.** One can show (35) that the MMD can also be expressed as
\[ \text{MMD}(\pi, \pi')^{2} = \int |\phi_{\omega}(\pi) - \phi_{\omega}(\pi')|^{2} d\Gamma(\omega), \]
where \( \phi_{\omega}(\pi) \overset{\text{def}}{=} \int \phi_{\omega}(x) d\pi(x) \). By the definition of \( \Psi(x) = \frac{1}{\sqrt{m}} (\varphi_{j}(x))_{j=1}^{m}, \)
\[ d_{\phi}(\pi, \pi')^{2} = \frac{1}{m} \sum_{j=1}^{m} |\phi_{\omega_{j}}(\pi) - \phi_{\omega_{j}}(\pi')|^{2}. \]
Since \( \sup_{x} |\phi_{\omega}(x)| \leq M \), we deduce that \( |\phi_{\omega_{j}}(\pi) - \phi_{\omega_{j}}(\pi')|^{2} \leq 4M^{2} \) we can apply Hoeffding’s inequality (Lemma 10) to \( d_{\phi} \). Thus, with probability \( 1 - \rho \), it holds that
\[ \text{MMD}(\pi, \pi')^{2} - d_{\phi}(\pi, \pi')^{2} \leq \frac{2\sqrt{2M^{2}}}{\sqrt{m}} \sqrt{\log \frac{1}{\rho}}. \]

\[ \square \]
B Mean time between false alarm

Consider an online change-point detection algorithm that produces a statistic $S_t$ that must not exceed a threshold $\tau$. The mean time between false alarms, also called Average Run Length (ARL) under control, is the expected runtime of the algorithm when no change occurs. Assuming all samples $x_t$ are drawn i.i.d. from a distribution $\tau$, recall that we define it as

$$T = E \left[ \inf \{ t \mid S_t \geq \tau \} \right].$$

(14)

In the case of NEWMA-$\Psi$, we have $S_t = \|z_t - z'_t\|$. In this section, we derive a more tractable expression for $T$.

Our proof strategy relies on the observation that $(z_t, z'_t)$ is a Markov chain in $\mathcal{H}^2$, and thus it is possible to apply a method similar to [13] for classical EWMA, with non-trivial modifications. We show that their approach can be adapted to NEWMA-$\Psi$ when $\mathcal{H} \overset{\text{def.}}{=} \mathbb{R}^m$ is finite-dimensional. We consider $v_t \overset{\text{def.}}{=} (z_t, z'_t)$ as the Markov chain of interest. Intuitively, we must handle several key points for NEWMA-$\Psi$:

— **Multidimensionality**: we replace unidimensional grids by $\varepsilon$-coverings in $\mathbb{R}^m$.

— **Stopping condition**: the stopping condition for NEWMA involves both components $z_t$ and $z'_t$ of $v_t$. We define the set $V_\tau \subset \mathcal{H}^2$ as the domain in which the algorithm continues:

$$V_\tau \overset{\text{def.}}{=} \{ v = (z, z') \mid \|z - z'\| < \tau \}. $$

(15)

— **Boundedness**: finally, here we do not assume that $\Psi$ is bounded like in Prop. [2] for instance. As a consequence, neither $z_t$ nor $z'_t$ is bounded during the run. Nevertheless, we will show that we can handle this fact by discretizing a compact domain whose size grows simultaneously as the net becomes finer, which is sufficient for the convergence properties of interest to us.

With these notations, when we run NEWMA-$\Psi$ and stop as soon as an alarm is raised, we produce a Markov chain $v_t \in \mathcal{H}^2$ defined as: $v_0 = (z_0, z_0)$, and

$$v_t = \begin{cases} (1 - \lambda)v_{t-1,1} + \lambda \Psi (x_t) & \text{if } v_{t-1} \in V_\tau, \\ (1 - \Lambda)v_{t-1,2} + \Lambda \Psi (x_t) & \text{otherwise.} \end{cases}$$

In other words, the chain is stationary as soon as an alarm is raised.

**Coverings.** Let us now define the discretization that we will use for our Markov chain-based strategy. Consider the space $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$, equipped with the norm $\| (x, x') \| = \| x \| + \| x' \|$. For $\varepsilon > 0$, consider the ball of radius $\varepsilon^{-1}$ in $\mathcal{H}^2$, $B_\varepsilon = \{ v \mid \| v \| \leq 1/\varepsilon \}$, which is compact since $\mathcal{H}^2$ has finite dimension. Define $C_\varepsilon = \{ u_1, \ldots, u_{N_\varepsilon} \} \subset B_\varepsilon$ any $\varepsilon$-net of $B_\varepsilon$ (we will see that its choice does not matter) such that $u_1 = (z_0, z_0)$, where $N_\varepsilon$ is the $\varepsilon$-covering number of $B_\varepsilon$. Without lost of generality, assume they are ordered such that $u_1, \ldots, u_{N_\varepsilon} \in V_\tau$ and $u_{M_\varepsilon + 1}, \ldots, u_{N_\varepsilon} \in V_\tau'$ for some $M_\varepsilon$. Denote $P_\varepsilon : \mathcal{H}^2 \to C_\varepsilon$ the projection operator onto $C_\varepsilon$ (i.e., that returns the $u_i$ closest to its input).

Define the following Markov chain $v_{\varepsilon}^t \in C_\varepsilon$: initialize $v_{\varepsilon}^0 = (z_0, z_0) = u_1$, and

$$v_{\varepsilon}^t = \begin{cases} P_\varepsilon \left( (1 - \lambda)v_{\varepsilon}^{t-1,1} + \lambda \Psi (x_t) \right) & \text{if } v_{\varepsilon}^{t-1} \in V_\tau, \\ v_{\varepsilon}^{t-1} & \text{otherwise.} \end{cases}$$

It is a projected and bounded version of the output of NEWMA-$\Psi$, which is stationary as soon as it gets out of $V_\tau$. A key observation is that the radius of $B_\varepsilon$ grows to $+\infty$ as $\varepsilon \to 0$.

For $1 \leq i, j \leq N_\varepsilon$, define $p_{ij} = \mathbb{P} (v_{\varepsilon}^{t+1} = u_j \mid v_{\varepsilon}^t = u_i)$ the transition probabilities of the markov chain $v_{\varepsilon}^t$. Define $A = \left[ p_{ij} \right]_{1 \leq i, j \leq M_\varepsilon}$ that corresponds to the states $u_i \in V_\tau$, all other states being absorbant, and $a_{ij}^{(t)}$ such that $A^t = \left[ a_{ij}^{(t)} \right]_{1 \leq i, j \leq M_\varepsilon}$.

Our result on the mean time between false alarms is the following.
Theorem 8 (Mean time between false alarms of NEWMA-$\Psi$). Assume that $\pi$ is such that $\Psi(x) \in \mathbb{R}^m$ has a density with respect to the Lebesgue measure when $x \sim \pi$. Then, the quantity $\gamma_\ell = \lim_{\varepsilon \to 0} \left( \sum_{j=1}^{M_\ell} a^{(j)}_\ell \right)$ does not depend on the choice of the nets $C_\varepsilon$, and the mean time between false alarms of NEWMA-$\Psi$ is given by

$$T = \mathbb{E} \left[ \inf \{ t \mid \| z_t - z'_t \| \geq \tau \} \right] = 1 + \sum_{\ell \geq 1} \gamma_\ell . \quad (16)$$

Our proof relies on the key lemma:

Lemma 9 (Almost sure convergence of $v_\ell^\varepsilon$). For any fixed $t$, when $\varepsilon$ goes to 0, $v_\ell^\varepsilon$ converges to $v_\ell$ almost surely.

Proof. Let us first note that, since by assumption $\Psi(x)$ has a density, it is easy to prove by recurrence on $t$ that $v_\ell$ has also a density. Therefore, for all $t$, $\mathbb{P}(v_\ell \not\in \partial V_\tau) = 0$ (it is trivial that the boundary of $V_\tau$ has zero Lebesgue measure, and by a countable union of zero-measure sets:

$$\mathbb{P}(\exists \ell \mid v_\ell \in \partial V_\tau) = 0 . \quad (17)$$

Since we want to prove an almost sure convergence, we explicitly denote the set of all events such that (17) is satisfied by $\Omega$ (which has probability 1), and the events in $\Omega$ by $\omega$. A draw of a r.v. $X$ will be denoted by $X(\omega)$.

Fix any $t \geq 1$. Consider any $\omega \in \Omega$, corresponding to a draw of samples $x_\ell(\omega)$, $\ell = 1, \ldots, t$. Remember that, by the definition of $\Omega$, $v_\ell(\omega) \not\in \partial V_\tau$. Note that, when $\varepsilon$ varies, the $v_\ell^\varepsilon(\omega)$ change, but in a deterministic fashion. Our goal is to show that $v_\ell^\varepsilon(\omega) \to v_\ell(\omega)$ when $\varepsilon \to 0$.

We are going to show by induction that $\|v_\ell^\varepsilon(\omega) - v_\ell(\omega)\| \xrightarrow{\varepsilon \to 0} 0$ for all $\ell = 1, \ldots, t$. Since $v_0 = v_0^\varepsilon$, it is obviously true for $\ell = 0$. Then, for any $\ell$, suppose that $\|v_{\ell-1}^\varepsilon(\omega) - v_{\ell-1}(\omega)\| \xrightarrow{\varepsilon \to 0} 0$. By (17), we have either $v_{\ell-1}(\omega) \in V_\tau$ or $v_{\ell-1}(\omega) \in \partial V_\tau$ since it does not belong to the boundary. We study separately these two cases.

$v_{\ell-1}(\omega)$ inside of $V_\tau$. Since by inductive hypothesis $\|v_{\ell-1}^\varepsilon(\omega) - v_{\ell-1}(\omega)\| \xrightarrow{\varepsilon \to 0} 0$, and since $V_\tau$ is an open set of $\mathbb{R}^m$, for all $\varepsilon$ sufficiently small we have that $v_{\ell-1}^\varepsilon(\omega) \in V_\tau$ and the Markov chain $v^\varepsilon$ is updated at step $\ell$. Furthermore, since the radius of $B_\varepsilon$ goes to $\infty$ when $\varepsilon \to 0$, for all $\varepsilon$ sufficiently small, $v_\ell(\omega) \in B_\varepsilon$, and $\|P_\varepsilon(v_\ell(\omega)) - v_\ell(\omega)\| \leq \varepsilon$. Hence, in that case,

$$\|v_\ell^\varepsilon(\omega) - v_\ell(\omega)\| \leq \|v_\ell^\varepsilon(\omega) - P_\varepsilon(v_\ell(\omega))\| + \varepsilon \quad \text{(triangle inequality)}$$

(definition of $v_\ell$ and $v_\ell^\varepsilon$)

$$\leq \left\| \left( 1 - \lambda \right)v_{\ell-1}^\varepsilon(\omega) + \lambda \Psi(x_\ell(\omega)) \right\| + \varepsilon \quad \text{(projections are contracting)}$$

$$\leq \left\| (1 - \lambda)v_{\ell-1}(\omega) + \lambda \Psi(x_\ell(\omega)) \right\| + \varepsilon \quad \text{(definition of the norm)}$$

Therefore $\|v_\ell^\varepsilon(\omega) - v_\ell(\omega)\| \xrightarrow{\varepsilon \to 0} 0$. 


\(v_{\ell-1}(\omega)\) outside \(V_r\). We have \(v_\ell(\omega) = v_{\ell-1}(\omega)\) by definition of the Markov chain \(v_\ell\). Since \(V_r\) is an open set, by inductive hypothesis for all \(\varepsilon\) sufficiently small we have \(v_{\ell-1}^\varepsilon(\omega) \in V_r\) and \(v_\ell^\varepsilon(\omega) = v_{\ell-1}^\varepsilon(\omega)\), from which \(\|v_\ell^\varepsilon(\omega) - v_\ell(\omega)\| = \|v_{\ell-1}^\varepsilon(\omega) - v_{\ell-1}(\omega)\| \xrightarrow{\varepsilon \to 0} 0\), which concludes the proof. 

We can now turn to proving the theorem itself.

**Proof of Th. 8.** We start by a reformulation:

\[
\overline{T} = \mathbb{E} \left[ \inf \{ t \mid \|z_t - z'_t\| \geq \tau \} \right] = \mathbb{E} \left[ \inf \{ t \mid v_t \notin V_r \} \right]
\]

\[
= \sum_{\ell \geq 0} \mathbb{P}(\inf \{ t \mid v_t \notin V_r \} > \ell) = 1 + \sum_{\ell \geq 1} \mathbb{P}(v_\ell \in V_r),
\]

since the first time \(v_\ell\) exits \(V_r\) is strictly greater than \(\ell\) if, and only if, \(v_\ell \in V_r\).

Since almost sure convergence implies weak convergence, by Lemma 9 we have

\[
\overline{T} = 1 + \sum_{\ell \geq 1} \lim_{\varepsilon \to 0} \mathbb{P}(v_\ell^\varepsilon \in V_r).
\]

Note that the convergence in \(\varepsilon\) is not necessarily uniform: in general, one cannot exchange the limit operator with the infinite sum in the last display.

To conclude the proof, we just have to compute \(\mathbb{P}(v_\ell^\varepsilon \in V_r)\). For \(1 \leq i, j \leq N_r\), recall that we defined the transition probabilities of the Markov chain \(v_\ell^\varepsilon\) by

\[
p_{ij} = \mathbb{P}\left( v_{\ell+1}^\varepsilon = u_j \mid v_\ell^\varepsilon = u_i \right).
\]

The transition matrix of this Markov chain has the form:

\[
P = [p_{ij}]_{1 \leq i, j \leq N_r} = \begin{bmatrix} A & B \\ 0 & \text{Id} \end{bmatrix},
\]

where \(A = [p_{ij}]_{1 \leq i, j \leq M_r}\) corresponds to the states \(u_i \in V_r\). Then, if we define \(a_{ij}^{(t)}\) such that \(A^t = \begin{bmatrix} a_{ij}^{(t)} \\ \text{Id} \end{bmatrix}_{1 \leq i, j \leq M_r}\), it is possible to show by induction [23] that:

\[
P^t = \begin{bmatrix} A^t \\ 0_{(N_r - M_r) \times M_r} \end{bmatrix} \begin{bmatrix} 1_{M_r} \\ \text{Id}_{N_r - M_r} \end{bmatrix},
\]

and therefore

\[
\mathbb{P}(v_\ell^\varepsilon \in V_r) = [1, 0, \ldots, 0] P^t \begin{bmatrix} 1_{M_r} \\ 0_{N_r - M_r} \end{bmatrix} = [1, 0, \ldots, 0] A^t \begin{bmatrix} 1_{M_r} \\ \vdots \\ 1_{M_r} \end{bmatrix} = \sum_{j=1}^{M_r} a_{ij}^{(t)}
\]

which concludes the proof.

**Application to the unidimensional case.** We conclude this section by describing approximate closed-form expression for Th. 8 when \(m = 1\) and the c.d.f. of \(\Psi(x)\) is known.

Fix a small \(\varepsilon > 0\) that we will use for our approximation, a grid \(G \triangleq \{a_1, \ldots, a_N\} \subset [-1/\varepsilon, 1/\varepsilon]\) of \(\varepsilon\)-separated points (one can replace \(1/\varepsilon\) by a smaller value to save some computation time at the expense of losing in accuracy), and \(B_\varepsilon = G \times G\). For any two states \(u_i = (a_i, b_i)\) and \(u_j = (a_j, b_j)\), both in \(V_r\), we have

\[
p_{ij} = \mathbb{P}\left( v_\ell^\varepsilon = u_j \mid v_{\ell-1}^\varepsilon = u_i \right)
\]

(definition of \(p_{ij}\))

\[
= \mathbb{P}\left( (1 - \lambda)a_i + \lambda \Psi(x) \in [a_j \pm \varepsilon/2] \text{ and } (1 - \Lambda)b_i + \Lambda \Psi(x) \in [b_j \pm \varepsilon/2] \right)
\]

(definition of the detection statistic)

\[
= \mathbb{P}\left( \Psi(x) \in [\ell, u] \right)
\]
Then we start the Appendix by recalling some technical results that will be used in the proofs. We illustrate this principle in the paper (see the right panel of Fig. 1) with a Gaussian distribution, \( \sum \) where we introduced

\[
\begin{align*}
\ell &= \max \left( \frac{1}{X}(a_j - (1 - \lambda)a_i - \varepsilon/2), \frac{1}{X}(b_j - (1 - \Lambda)b_i - \varepsilon/2) \right), \\
u &= \min \left( \frac{1}{X}(a_j - (1 - \lambda)a_i + \varepsilon/2), \frac{1}{X}(b_j - (1 - \Lambda)b_i + \varepsilon/2) \right).
\end{align*}
\]

Hence

\[
p_{ij} = \begin{cases} 
\mathbb{P}(\Psi(x) \leq u) - \mathbb{P}(\Psi(x) \leq \ell) & \text{if } \ell < u \\
0 & \text{otherwise}.
\end{cases}
\]

Using this expression, one can compute the matrix \( A = [p_{ij}] \).

Then, since we are reasoning at a fixed \( \varepsilon \), we can use a Von Neumann series on \( A \) in Eq. 16, that is, \( \sum_{\ell \geq 0} A^\ell = (\text{Id} - A)^{-1} \). Note that one cannot exchange the limit \( \lim_{\varepsilon \to 0} \) and the infinite sum \( \sum_{\ell \geq 1} \) in Eq. 16 since the convergence in \( \varepsilon \) is not uniform. Nevertheless, for the sake of approximate numerical computations with a fixed \( \varepsilon \), we write

\[
T \approx 1 + \sum_{\ell \geq 1} (e_1^\top A^\ell 1) = e_1^\top (\text{Id} - A)^{-1} 1
\]

where \( 1 = [1, \ldots, 1]^\top \) and \( e_1 = [1, 0, \ldots, 0]^\top \).

We illustrate this principle in the paper (see the right panel of Fig. 1) with a Gaussian distribution, and find the above approximation to be quite accurate, even for moderate values of \( \varepsilon \).

C Third-party technical results

We start the Appendix by recalling some technical results that will be used in the proofs.

Lemma 10 (Hoeffding’s inequality ([7], Th. 2.8)). Let \( X_i \) be independent, bounded random variables such that \( X_i \in [a_i, b_i] \) a.s. Then, for any \( t > 0 \),

\[
\mathbb{P}\left( \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}\left( \frac{1}{n} \sum_{i=1}^n X_i \right) \geq t \right) \leq \exp\left( -\frac{2n^2t^2}{\sum_{i=1}^n (a_i - b_i)^2} \right).
\]

Lemma 11 (McDiarmid’s inequality ([7], Th. 6.2)). Let \( f : E^n \to \mathbb{R} \) be a measurable function that satisfies the bounded difference property, that is, there exist positive numbers \( c_1, \ldots, c_n \) such that

\[
\sup_{x_1, \ldots, x_n, x'_1, \ldots, x'_n \in E} |f(x_1, \ldots, x_n) - f(x_1, \ldots, x'_i, \ldots, x_n)| \leq c_i.
\]

Let \( X_1, \ldots, X_n \) be independent random variables with values in \( E \) and set \( Z = f(X_1, \ldots, X_n) \). Then

\[
\mathbb{P}(Z - \mathbb{E}Z \geq t) \leq \exp\left( -\frac{2t^2}{\sum_{i=1}^n c_i} \right).
\]

Theorem 12 (Multivariate Lindeberg’s Theorem ([38], Th. 2.27)). For each \( n \), let \( X^{(i,n)}, 1 \leq i \leq n \), be independent, \( \mathbb{R}^d \)-valued random vectors with zero mean and covariance \( \Sigma^{(i,n)} \) such that \( \sum_{i=1}^n \Sigma^{(i,n)} \to \Sigma \) (for the Frobenius norm), and Lindeberg’s condition is satisfied:

\[
\forall \varepsilon > 0, \sum_{i=1}^n \mathbb{E}\left[ \left\| X^{(i,n)} \right\|^2 I\left\{ \left\| X^{(i,n)} \right\| > \varepsilon \right\} \right] \to 0 \text{ when } n \to \infty.
\]

Then \( S_n = \sum_{i=1}^n X^{(i,n)} \) converges in law toward a centered Gaussian with covariance \( \Sigma \).

Lemma 13 (Slutsky’s Lemma ([38], Th. 2.7)). Let \( X_n, X, Y_n \) be random variables. If \( X_n \) converges in law to \( X \) and \( Y_n \) converges in probability to a constant \( c \), then \( (X_n, Y_n) \) converges in law to \( (X, c) \).