The SBP Algorithm for Maximizing Revenue in Online Dial-a-Ride

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Abstract

In the Online-Dial-a-Ride Problem (OLDARP) a server travels through a metric space to serve requests for rides. We consider a variant where each request specifies a source, destination, release time, and revenue that is earned for serving the request. The goal is to maximize the total revenue earned within a given time limit. We prove that no non-preemptive deterministic online algorithm for OLDARP can be guaranteed to earn more than twice the revenue earned by an optimal offline solution. We then investigate the segmented best path (SBP) algorithm of [6] for the general case of weighted graphs. The previously-established lower and upper bounds for the competitive ratio of SBP are 4 and 6, respectively, under reasonable assumptions about the input instance. We eliminate the gap by proving that the competitive ratio is 5 (under the same reasonable assumptions). We also prove that when revenues are uniform, SBP has competitive ratio 4. Finally, we provide a competitive analysis of SBP on complete bipartite graphs.

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1 Introduction

In the On-Line Dial-a-Ride Problem (OLDARP), a server travels in some metric space to serve requests for rides. Each request specifies a source, which is the pick-up (or start) location of the ride, a destination, which is the delivery (or end) location, and the release time of the request, which is the earliest time the request may be served. Requests arrive over time; specifically, each arrives at its release time and the server must decide whether to serve the request and at what time, with the goal of meeting some optimality criterion. The server has a capacity that specifies the maximum number of requests it can serve at any time. Common optimality criteria include minimizing the total travel time (i.e. makespan) to satisfy all requests, minimizing the average completion time (i.e. latency), or maximizing the number of served requests within a specified time limit. In many variants preemption is not allowed, so if the server begins to serve a request, it must do so until completion. On-Line Dial-a-Ride Problems have many practical applications in settings where a vehicle is dispatched to satisfy requests involving pick-up and delivery of people or goods. Important examples include ambulance routing, transportation for the elderly and disabled, taxi services including Ride-for-Hire systems (such as Uber and Lyft), and courier services.

We study a variation of OLDARP where in addition to the source, destination and release time, each request also has a priority and there is a time limit within which requests must be served. The server has unit capacity and the goal for the server is to serve requests within the time limit so as to maximize the total priority. A request’s priority may simply represent the importance of serving the request in settings such as courier services. In more time-sensitive settings such as ambulance routing, the priority may represent the urgency of a request. In profit-based settings, such as taxi and ride-sharing services, a request’s priority may represent the revenue earned from serving the request. For the remainder of this paper, we will refer to the priority as “revenue,” and to this variant of the problem as ROLDARP. Note that if revenues are uniform the problem is equivalent to maximizing the number of served requests.

1.1 Related work

The Online Dial-a-Ride problem was introduced by Feuerstein and Stougie [8] and several variations of the problem have been studied since. For a comprehensive survey on these and many other problems in the general area of vehicle routing see [10] and [14]. Feuerstein and Stougie studied the problem for two different objectives: minimizing completion time and minimizing latency. For minimizing completion time, they showed that any deterministic algorithm must have competitive ratio of at least 2 regardless of the server capacity. They presented algorithms for the cases of finite and infinite capacity with competitive ratios of 2.5 and 2, respectively. For minimizing latency, they proved that any algorithm must have a competitive ratio of at least 3 and presented a 15-competitive algorithm on the real line when the server has infinite capacity. Ascheuer et al. [2] studied OLDARP with multiple servers with the goal of minimizing completion time and presented a 2-competitive algorithm. More recently, Birx and Disser [4] studied OLDARP on the real line and presented a new upper bound of 2.94 for the smartstart algorithm [4], which improves the previous bound of 3.41 [12]. For OLDARP on the real line, Bjelde et al. [5] present a preemptive algorithm with competitive ratio 2.41.

The Online Traveling Salesperson Problem (OLTSP), introduced by Ausiello et al. [3] and also studied by Krumke [13], is a special case of OLDARP where for each request the source and destination are the same location. There are many studies of variants of OLDARP and
OLTSP \[3, 9, 11, 13\] that differ from the variant that we study which we omit here due to space limitations.

In this paper, we study OLDARP where each request has a revenue that is earned if the request is served and the goal is to maximize the total revenue earned within a specified time limit; the offline version of the problem was shown to be NP-hard in \[6\]. More recently, it was shown that even the special case of the offline version with uniform revenues and uniform weights is NP-hard \[1\]. Christman and Forcier \[7\] presented a 2-competitive algorithm for OLDARP on graphs with uniform edge weights. Christman et al. \[6\] showed that the lack of a competitive algorithm for OLDARP with nonuniform edge weights is due to arbitrarily large edge weights alone, i.e. if edge weights may be arbitrarily large, then regardless of revenue values, no deterministic algorithm can be competitive. They therefore considered graphs where edge weights are bounded by \(T/f\), where \(T\) is the time limit, for some \(1 < f < T\), and gave a 6-competitive algorithm for this problem. Note that this is a natural subclass of inputs since in real-world dial-a-ride systems, drivers would be unlikely to spend a large fraction of their day moving to or serving a single request.

### 1.2 Our results

In this work we begin with improved lower and upper bounds for the competitive ratio of the segmented best path (SBP) algorithm that was presented in \[6\]. In \[6\], it was shown that SBP’s competitive ratio has lower bound 4 and upper bound 6, provided that the edge weights are bounded by \(T/f\) where \(T\) is the time limit and \(1 < f < T\), and that the revenue earned by the optimal offline solution in the last \(2T/f\) time units is bounded by a constant. This assumption is imposed because, as we show in Lemma 1, no non-preemptive deterministic online algorithm can be guaranteed to earn this revenue. We also show that no non-preemptive deterministic online algorithm for OLDARP can be guaranteed to earn more than twice the revenue earned by an optimal offline solution in the first \(T - 2T/f\) time units. We then close the gap between the upper and lower bounds of SBP by providing an instance where the lower bound is 5 (Section 3.1) and a proof for an upper bound of 5 (Section 3.2). We note that another interpretation of our result is that under a weakened-adversary model where OPT has two fewer time segments available, while SBP has the full time limit \(T\), SBP is 5-competitive. We then investigate the problem for uniform revenues (so the objective is to maximize the total number of requests served) and prove that SBP earns at least \(1/4\) the revenue of the optimal solution, minus an additive term linear in \(f\), the number of time segments (Section 4). This variant is useful for settings where all requests have equal priorities such as not-for-profit services that provide transportation to elderly and disabled passengers and courier services where deliveries are not prioritized.

We then consider the problem for complete bipartite graphs; for these graphs every source is from the left-hand side and every destination is from the right-hand side (Section 5). These graphs model the scenario where only a subset of locations may be source nodes and a disjoint subset may be destinations, e.g. in the delivery of goods from commercial warehouses only the warehouses may be sources and only customer locations may be destinations. We refer to this problem as ROLDARP-B. We first show that if edge weights are not bounded by a minimum value, then ROLDARP on general graphs reduces to ROLDARP-B. We therefore impose a minimum edge weight of \(kT/f\) for some constant \(k\) such that \(0 < k \leq 1\). We show that if revenues are uniform, SBP has competitive ratio \(\lceil 1/k \rceil\). Finally, we show that if revenues are nonuniform SBP has competitive ratio \(1/k\), provided that the revenue earned by the optimal offline solution in the last \(2T/f\) time units is bounded by a constant. (This assumption is justified by Lemma 1 which says no deterministic algorithm can be guaranteed
to earn any fraction of what is earned by the optimal solution in the last $2T/f$ time units.)

## 2 Preliminaries

The Revenue-Online-Dial-a-Ride Problem (ROLDARP) is formally defined as follows. The input is an undirected complete graph $G = (V, E)$ where $V$ is the set of vertices (or nodes) and $E = \{(u, v) : u, v \in V, u \neq v\}$ is the set of edges. For every edge $(u, v) \in E$, there is a weight $w_{u,v} > 0$, which represents the amount of time it takes to traverse $(u,v)$. We note that any simple, undirected, connected, weighted graph is allowed as input, with the simple pre-processing step of adding an edge wherever one is not present whose weight is the length of the shortest path between its two endpoints. We further note that the input can be regarded as a metric space if the weights on the edges are expected to satisfy the triangle-inequality. One node in the graph, $o$, is designated as the origin and is where the server is initially located (i.e. at time 0). The input also includes a time limit $T$ and a sequence of requests, $\sigma$, that are dynamically issued to the server.

Each request is of the form $(s,d,t,p)$ where $s$ is the source node, $d$ is the destination, $t$ is the time the request is released, and $p$ is the revenue (or priority) earned by the server for serving the request. The server does not know about a request until its release time $t$. To serve a request, the server must move from its current location $x$ to $s$, then from $s$ to $d$. The total time for serving the request is equal to the length of the path from $x$ to $d$ and the earliest time a request may be released is at $t = 0$. For each request, the server must decide whether to serve the request and if so, at what time. A request may not be served earlier than its release time and at most one request may be served at any given time. Once the server decides to serve a request, it must do so until completion. The goal for the server is to serve requests within the time limit so as to maximize the total earned revenue.

The authors of [6] showed that if edge weights may be arbitrarily large then no deterministic algorithm can be competitive. They therefore considered graphs where edge weights are bounded by $T/f$ where $T$ is the time limit, for some $1 < f < T$, and presented the segmented best path (sbp) algorithm for this problem (please refer to Algorithm 1).

### Algorithm 1: Algorithm SEGMENTED BEST PATH (SBP)

**Input** is complete graph $G$ with time limit $T$ and maximum edge weight $T/f$.  

1. Let $t_1,t_2,\ldots,t_f$ denote the time segments ending at times $T/f, 2T/f, \ldots, T$, resp.
2. Let $i = 1$.
3. **if** $f$ is odd **then**
4. At $t_1$, do nothing. Increment $i = 2$.
5. **end if**
6. **while** $i < f$ **do**
7. At the start of $t_i$, find the max-revenue-request-set, $R$.
8. **if** $R$ is non-empty **then**
9. Move to the source location of the first request in $R$.
10. At the start of $t_{i+1}$, serve request-set $R$.
11. **else**
12. Remain idle for $t_i$ and $t_{i+1}$
13. **end if**
14. Let $i = i + 2$.
15. **end while**
The algorithm SBP starts by splitting the total time $T$ into $f$ segments each of length $T/f$. At the start of a time segment, the server determines the max-revenue-request-set, i.e. the maximum revenue set of requests that can be served within one time segment, and moves to the source of the first request in this set. During the next time segment, it serves the requests in this set. It continues this way, alternating between moving to the source of first request in the max-revenue-request-set during one time segment, and serving this request-set in the next time segment. To find the max-revenue-request-set, the algorithm maintains a directed auxiliary graph, $G'$ to keep track of unserved requests (an edge between two vertices $u,v$ represents a request with source $u$ and destination $v$). It finds all paths of length at most $T/f$ between every pair of nodes in $G'$ and returns the path that yields the maximum total revenue (please refer to [6] for full details). Finding all paths of length at most $T/f$ in $G'$ requires enumeration of all paths in $G'$ and the number of possible paths is exponential in the size of $G'$, which is determined directly by the number of outstanding requests in the current time segment. However, in many real world settings, the size of $G'$ will be small relative to the size of $G$ and in settings where $T/f$ is small, the run time is further minimized. Therefore it should be feasible to execute the algorithm efficiently in many realistic settings.

It was observed in [6] that no deterministic online algorithm can be guaranteed to serve the requests served by OPT during the last time segment and the authors proved that SBP is 6-competitive barring an additive factor equal to the revenue earned by OPT during the last two time segments. More formally, let $\text{rev}(\text{SBP}(t_j))$ and $\text{rev}(\text{OPT}(t_j))$ denote the revenue earned by SBP and OPT respectively during the $j$-th time segment. Then if $\text{rev}(\text{OPT}(t_f)) + \text{rev}(\text{OPT}(t_{f-1})) \leq c$ for some constant $c$, then $\sum_{j=1}^{f} \text{rev}(\text{OPT}(t_j)) \leq 6 \sum_{j=1}^{f} \text{rev}(\text{SBP}(t_j)) + c$. It was also shown in [6] that as $T$ grows, the competitive ratio of SBP is at best 4 (again with the additive term equal to $\text{rev}(\text{OPT}(t_f)) + \text{rev}(\text{OPT}(t_{f-1}))$), resulting in a gap between the upper and lower bounds.

### 2.1 General lower bound

We first show that no non-preemptive deterministic online algorithm (e.g. SBP) can be competitive with the revenue earned by an optimal offline solution in the last two segments of time. We note that this claim applies to a stronger notion of non-preemption where, as in real-world systems like Uber/Lyft, once the server decides to serve a request, it must move there and serve it to completion.

**Lemma 1.** No non-preemptive deterministic online algorithm can be guaranteed to earn any fraction of the revenue earned by an optimal offline solution in the last $2T/f$ time units. This is the case whether revenues are uniform or nonuniform.

**Proof idea.** The adversary releases a request in the last two time segments and if the online algorithm chooses not to serve it no other requests will be released. If the algorithm chooses to serve it, another batch of requests will be released elsewhere that the algorithm cannot serve in time. Please see Appendix 6.1 for details.

We now present a general lower bound for our problem and show that no non-preemptive deterministic online algorithm (e.g. SBP) can be better than 2-competitive with respect to the revenue earned by the offline optimal schedule (ignoring the last two time segments, due to Lemma 1 above).

**Theorem 2.** No non-preemptive deterministic online algorithm for OLDARP can be guaranteed to earn more than twice the revenue earned by an optimal offline solution in the first $T-2T/f$ time units. This is the case whether revenues are uniform or nonuniform.
Figure 1 An instance where opt (whose path is shown in green below) earns $5 - 4/(f - 2)$ times the revenue of sbp (shown in yellow above). In this instance, $T = 2hf$, and edges that represent requests are shown as solid edges. For each such edge the release time followed by revenue of the corresponding request is shown in parenthesis above the edge. The weight of an edge is shown below the edge. Dashed edges represent empty moves.

Proof idea. The adversary releases requests within the first $T - 2T/f$ time segments such that depending on which request(s) the algorithm serves, another set of request(s) with twice as much revenue is released elsewhere that the algorithm cannot serve in time. Please see Appendix 6.1 for details.

3 Nonuniform revenues

In this section we improve the lower and upper bounds for the competitive ratio of the segmented best path algorithm [6]. In particular, we eliminate the gap between the lower and upper bounds of 4 and 6, respectively, from [6], by providing an instance where the lower bound is 5 and a proof for an upper bound of 5. Note that throughout this section we assume the revenue earned by opt in the last two time segments is bounded by some constant. We must impose this restriction on the opt revenue of the last two time segments because, as we show in Lemma 1 no non-preemptive deterministic online algorithm can be guaranteed to earn any constant fraction of this revenue.

3.1 Lower bound on SBP

Theorem 3. If the revenue earned by opt in the last two time segments is bounded by some constant, and sbp is $\gamma$-competitive, then $\gamma \geq 5$.

Proof idea. For the formal details, please refer to the proof of Theorem 3 in Appendix 6.2. Consider the instance depicted in Figure 1. Since $T = 2hf$ in this instance, $h$ represents “half” the length of one time segment, so only one request of length $h + 1$ fits within a single time segment for sbp. The general idea of the instance is that while sbp is serving every other request across the top row of requests (since the other half across the top are not released until after sbp has already passed them by), opt is serving the entire bottom row in one long chain, then also has time to serve the top row as one long chain.
### 3.2 Upper bound on SBP

We now show that SBP is 5-competitive by creating a modified, hypothetical SBP schedule that has additional copies of requests. First, we note that SBP loses a factor of 2 due to the fact that it serves requests during only every other time segment. Then, we lose another factor of two to cover requests in OPT that overlap between time segments. Finally, by adding at most one more copy of the requests served by SBP to make up for requests that SBP “incorrectly” serves prior to when they are served by OPT, we end up with 5 copies of SBP being sufficient for bounding the total revenue of OPT. Note that while this proof uses some of the techniques of the proof of the 6-competitive upper bound in [6], it reduces the competitive ratio from 6 to 5 by cleverly extracting the set of requests that SBP serves prior to OPT before making the additional copies.

Let \( \text{rev}(\text{OPT}) \) and \( \text{rev}(\text{SBP}) \) denote the total revenue earned by OPT and SBP over all time segments \( t_j \) from \( j = 1 \ldots f \).

**Theorem 4.** If the revenue earned by OPT in the last two time segments is bounded by some constant, then SBP is 5-competitive, i.e., if \( \text{rev}(\text{OPT}(t_f)) + \text{rev}(\text{OPT}(t_{f-1})) \leq c \) for some constant \( c \), then \( \sum_{f=1}^{\lfloor f/2 \rfloor} \text{rev}(\text{OPT}(t_j)) \leq 5 \sum_{j=1}^{f} \text{rev}(\text{SBP}(t_j)) + c \). We note that another interpretation of this result is that under a resource augmentation model where SBP has two more time segments available than OPT, SBP is 5-competitive.

**Proof.** We analyze the revenue earned by SBP by considering the time segments in pairs (recall that the length of a time segment is \( T/f \) for some \( 1 < f < T \)). We refer to each pair of consecutive time segments as a time window, so if there are \( f \) time segments, there are \( \lceil f/2 \rceil \) time windows. Note that the last time window may have only one time segment.

For notational convenience we consider a modified version of the SBP schedule, that we refer to as SBP', which serves exactly the same set of requests as SBP, but does so one time window earlier. Specifically, if SBP serves a set of requests during time window \( i \geq 2 \), SBP' serves this set during time window \( i - 1 \) (so SBP' ignores the set served by SBP in window 1). We note that the schedule of requests served by SBP' may be infeasible, and that it will earn at most the amount of revenue earned by SBP.

Let \( B_i \) denote the set of requests served by OPT in window \( i \) that SBP' already served before in some window \( j < i \). And let \( B \) be the set of all requests that have already been served by SBP' in a previous window by the time they are served in the OPT schedule. Formally, \( B = \bigcup_{i=2}^{\lfloor f/2 \rfloor} B_i \). Consider a schedule OP\( \text{T} \) that contains all of the requests in the OPT schedule minus the requests in \( B \). So OPT earns total revenue \( \text{rev}(\text{OPT}) - \text{rev}(B) \), where \( \text{rev}(B) \) denotes the total revenue of the set \( B \).

Let \( \text{OPT}(t_j) \) denote the set of requests served by OPT in time segment \( t_j \). Let \( \text{OPT}_i \) denote the set of requests served by OPT in the time segment of window \( i \) with greater revenue, i.e. \( \text{OPT}_i = \arg \max \{ \text{rev}(\text{OPT}(t_{2i-1})), \text{rev}(\text{OPT}(t_{2i})) \} \). Note this set may include a request that was started in the prior time segment, as long as it was completed in the time segment of OPT\( _i \). Let \( \text{rev}(\text{OPT}_i) \) denote the revenue earned in OPT\( _i \).

Let SBP\( _i \) denote the set of requests served by SBP' in window \( i \) and let \( \text{rev}(\text{SBP}_i) \) denote the revenue earned by SBP\( _i \). Let \( H \) denote the chronologically ordered set of time windows \( w \) where \( \text{rev}(\text{OPT}(w)) > \text{rev}(\text{SBP}_w') \), and let \( h_j \) denote the \( j \)th time window in \( H \). We refer to each window of \( H \) as a window with a “hole,” in reference to the fact that SBP' does not earn as much revenue as OPT in these windows. In each window \( h_j \) there is some amount of revenue that OPT earns that SBP' does not. In particular, there must be a set of requests that OPT serves in window \( h_j \) that SBP' does not serve in \( h_j \). Note that this set must be available for SBP' in \( h_j \) since OPT does not include the set \( B \).
Let $\text{OPT}_{h_j} = A_j \cup C_j^*$, where $A_j$ is the subset of requests served by both $\text{OPT}$ and $\text{SBP}'$ in $h_j$ and $C_j^*$ is the subset of $\text{OPT}$ requests available for $\text{SBP}'$ to serve in $h_j$ but $\text{SBP}'$ chooses not to serve. Let us refer to the set of requests served by $\text{SBP}'$ in $h_j$ as $\text{SBP}'_{h_j} = A_j \cup C_j$ for some set of requests $C_j$. Note that if $\text{OPT}_{h_j} = A_j \cup C_j^*$ can be executed within a single time segment, then $\text{rev}(C_j) \geq \text{rev}(C_j^*)$ by the greediness of $\text{SBP}'$. However, since $h_j$ is a hole we know that the set $\text{OPT}_{h_j}$ cannot be served within one time segment.

Our plan is to build an infeasible schedule $\text{SBF}$ that will be similar to $\text{SBP}'$ but contain additional “copies” of some requests such that no windows of $\text{SBF}$ contain holes. We first initialize $\text{SBF}$ to have the same schedule of requests as $\text{SBP}'$. We then add additional requests to $h_j$ for each $j = 1 \ldots |H|$, based on $\text{OPT}_{h_j}$.

Consider one such window with a hole $h_j$, and let $k$ be the index of the time segment corresponding to $\text{OPT}_{h_j}$. We know $\text{OPT}$ must have begun serving a request of $\text{OPT}_{h_j}$ in time segment $h_{k-1}$ and completed this request in time segment $h_k$. Let us use $r^*$ to denote this request that “straddles” the two time segments.

After the initialization of $\text{SBF} = \text{SBP}'$, recall that the set of requests served by $\text{SBF}$ in $h_j$ is $\text{SBF}_{h_j} = A_j \cup C_j$ for some set of requests $C_j$. We add to $\text{SBF}$ a copy of a set of requests. There are two sub-cases depending on whether $r^* \in C_j^*$ or not.

Case $r^* \in C_j^*$. In this case, by the greediness of $\text{SBF}$, and the fact that both $r^*$ alone and $C_j^* \setminus \{r^*\}$ can separately be completed within a single time segment, we have: $\text{rev}(C_j) \geq \max\{\text{rev}(r^*), \text{rev}(C_j^* \setminus \{r^*\})\} \geq \frac{1}{2} \text{rev}(C_j^*)$. We then add a copy of the set $C_j$ to the $\text{SBF}$ schedule, so there are two copies of $C_j$ in $h_j$. Note that for $\text{SBF}$, $h_j$ will no longer be a hole since: $\text{rev}(\text{OPT}_{h_j}) = \text{rev}(A_j) + \text{rev}(C_j^*) \leq \text{rev}(A_j) + 2 \cdot \text{rev}(C_j) = \text{rev}(\text{SBF}_{h_j})$.

Case $r^* \notin C_j^*$. In this case $C_j^*$ can be served within one time segment but $\text{SBP}'$ chooses to serve $A_j \cup C_j$ instead. So we have $\text{rev}(A_j) + \text{rev}(C_j) \geq \text{rev}(C_j^*)$, therefore we know either $\text{rev}(A_j) \geq \frac{1}{2} \text{rev}(C_j^*)$ or $\text{rev}(C_j) \geq \frac{1}{2} \text{rev}(C_j^*)$. In the latter case, we can do as we did in the first case above and add a copy of the set $C_j$ to the $\text{SBF}$ schedule in window $h_j$, to get $\text{rev}(\text{OPT}_{h_j}) \leq \text{rev}(\text{SBF}_{h_j})$, as above. In the former case, we instead add a copy of $A_j$ to the $\text{SBF}$ schedule in window $h_j$. Then again, for $\text{SBF}$, $h_j$ will no longer be a hole, since this time: $\text{rev}(\text{OPT}_{h_j}) = \text{rev}(A_j) + \text{rev}(C_j^*) \leq 2 \cdot \text{rev}(A_j) + \text{rev}(C_j) = \text{rev}(\text{SBF}_{h_j})$.

Note that for all windows $w \notin H$ that are not holes, we already have $\text{rev}(\text{SBF}_w) \geq \text{rev}(\text{OPT}_w)$. So we have

$$\sum_{i=1}^{\lfloor f/2 \rfloor - 1} \text{rev}(\text{OPT}_i) \leq \sum_{i=1}^{\lfloor f/2 \rfloor - 1} \text{rev}(\text{SBF}_i) \leq 2 \sum_{i=1}^{\lfloor f/2 \rfloor - 1} \text{rev}(\text{SBF}'_i).$$

(1)

where the second inequality is because $\text{SBF}$ contains no more than two instances of every request in $\text{SBP}'$. Combining (1) with the fact that $\text{SBP}'$ earns at most what $\text{SBF}$ does yields

$$\sum_{i=1}^{\lfloor f/2 \rfloor} \text{rev}(\text{OPT}_i) \leq 2 \sum_{i=1}^{\lfloor f/2 \rfloor} \text{rev}(\text{SBF}_i) + \text{rev}(\text{OPT}(t_{f-1})) + \text{rev}(\text{OPT}(t_f)).$$

(2)

Since $\text{SBP}$ serves in only one of two time segments per window, we have $\sum_{i=1}^{\lfloor f/2 \rfloor} \text{rev}(\text{SBF}_i) = \sum_{j=1}^{f} \text{rev}(\text{SBP}(t_j))$. Hence, by the definition of $\text{OPT}$, and by (2) we can say

$$\sum_{j=1}^{f} \text{rev}(\text{OPT}(t_j)) \leq 2 \sum_{i=1}^{\lfloor f/2 \rfloor} \text{rev}(\text{OPT}_i) \leq 4 \sum_{j=1}^{f} \text{rev}(\text{SBP}(t_j)) + \text{rev}(\text{OPT}(t_{f-1})) + \text{rev}(\text{OPT}(t_f)).$$

(3)

Now we must add in any request in $B$, such that $\text{OPT}$ serves the request in a time window after $\text{SBP}'$ serves that request. By definition of $B$ (as the set of all requests that have been served by $\text{SBP}'$ in a previous window) $B$ may contain at most the same set of requests served
by SBP'. Therefore \( rev(B) \leq rev(SBP') \), so \( rev(B) \leq rev(SBP) \). By the definition of OPT, \( OPT = \overline{OPT} + B \), so

\[
\sum_{j=1}^{f} rev(OPT(t_j)) = rev(B) + \sum_{j=1}^{f} rev(\overline{OPT}(t_j))
\]

(4)

And by combining (3)–(4) with the fact that \( rev(B) \leq rev(SBP) \), we have

\[
\sum_{j=1}^{f} rev(OPT(t_j)) \leq \sum_{j=1}^{f} rev(SBP(t_j)) + 4 \sum_{j=1}^{f} rev(SBP(t_j)) + rev(OPT(t_{f-1})) + rev(OPT(t_f))
\]

\[
\leq 5 \sum_{j=1}^{f} rev(SBP(t_j)) + rev(OPT(t_{f-1})) + rev(OPT(t_f)).
\]

\[\square\]

4 Uniform revenues

We now consider the setting where revenues are uniform among all requests, so the goal is to maximize the total number of requests served. This variant is useful for settings where all requests have equal priorities, for example for not-for-profit services that provide transportation to elderly and disabled passengers. The proof strategy is to carefully consider the requests served by SBP in each window and track how they differ from that of OPT. The final result is achieved through a clever accounting of the differences between the two schedules, and bounding the revenue of the requests that are “missing” from SBP.

We note that the lower bound instance of Theorem 3 can be modified to become a uniform-revenue instance that has ratio \( 5 - 14/f \). On the other hand, we also show that OPT earns at most 4 times the revenue of SBP in this setting if we assume the revenue earned by OPT in the last two time segments is bounded by a constant, and allow SBP an additive bonus of \( f \). Note that when revenues are uniform, no non-preemptive deterministic online algorithm can earn \( rev(OPT(t_f)) + rev(OPT(t_{f-1})) \) (see Lemma 1). We begin with several definitions and lemmas.

As in the proof of Theorem 1 we consider a modified version of the SBP schedule, that we refer to as SBP', which serves exactly the same set of requests as SBP, but does so one time window earlier. For all windows \( i = 1, 2, ..., m \), where \( m = \lceil f/2 \rceil - 1 \), we let \( S'_i \) denote the set of requests served by SBP' in window \( i \) and \( S^*_i \) denote the set of requests served by OPT during the time segment of window \( i \) with greater revenue, i.e. \( S^*_i = \arg \max \{ rev(OPT(t_{2i-1}), rev(OPT(t_{2i})) \} \) where \( rev(OPT(t_j)) \) denotes the revenue earned by OPT in time segment \( t_j \). We define a new set \( J^*_i \) as the set of requests served by OPT during the time segment of window \( i \) with less revenue, i.e. \( J^*_i = \arg \min \{ rev(OPT(t_{2i-1}), rev(OPT(t_{2i})) \} \} \).

Let \( S^*_i = A_i \cup X^+_i \cup Y^+_i \), and \( S'_i = A_i \cup X_i \cup Y_i \), where: (1) \( A_i \) is the set of requests that appear in both \( S^*_i \) and \( S'_i \); (2) \( X^+_i \) is the set of requests that appear in \( S'_w \) for some \( w = 1, 2, ..., i - 1 \). Note there is only one possible \( w \) for each individual request \( r \in X^+_i \), because each request can be served only once; (3) \( Y^+_i \) is the set of requests such that no request from \( Y^+_i \) appears in \( S'_w \) for any \( w = 1, 2, ..., i - 1, i \); (4) \( X_i \) is the set of requests that appear in \( S'_w \) for some \( w = 1, 2, ..., i - 1 \). Note there is only one possible \( w \) for each individual request \( r \in X_i \), because each request can be served only once; (5) \( Y_i \) is the set of requests such that no request from \( Y_i \) appears in \( S'_w \) for any \( w = 1, 2, ..., i - 1, i \).
Note that elements in $Y_i$ can appear in a previous $J_m^*$ for any $w = 1, 2, \ldots, i-1, i$ or in a future $S_i^*$ or $J_v^*$ for any $v = i+1, i+2, \ldots, m$, or may not appear in any other sets. Also note that since each request can be served at most once, we have: $A_1 \cap X_1^* \cap Y_1^* \cap A_2 \cap X_2^* \cap Y_2^* \cap \ldots \cap A_m \cap X_m^* \cap Y_m^* = \emptyset$ and $A_1 \cap X_1^* \cap Y_1^* \cap A_2 \cap X_2^* \cap Y_2^* \cap \ldots \cap A_m \cap X_m^* \cap Y_m^* = \emptyset$.

Given the above definitions, we have the following lemma whose proof has been deferred to Appendix 6.3. It states that at any given time window, the cumulative requests of $S$ in the last two time segments is bounded by $\frac{f}{2}$.

Lemma 5. $|X_1^*| + |X_2^*| + \ldots + |X_t^*| \leq |Y_1| + |Y_2| + \ldots + |Y_{i-1}| + |Y_i|$ for all $i = 1, 2, \ldots, m$.

We are now ready to prove our main theorem of this section.

Theorem 6. If the revenue earned by OPT in the last two time segments is bounded by some constant, $c$, then SBP earns at least $\frac{1}{4}$ the revenue of the optimal solution, minus an additive term linear in $f$, where $T/f$ is the length of one time segment. (So if $f$ is bounded by some constant, then SBP is 4-competitive). I.e., $\sum_{i=1}^{f} rev(OPT(t_i)) \leq 4 \sum_{i=1}^{f} rev(SBP(t_i)) + 2\lceil f/2 \rceil + c$.

Proof. Note that since revenues are uniform, the revenue of a request-set $U$ is equal to the size of the set $U$, i.e., $rev(U) = |U|$. Consider each window $i$ where $rev(S_i^*) > rev(S_i^*)$. Note that the set $S_i^*$ may not fit within a single time segment. We consider two cases based on $S_i^*$:

1. The set $S_i^*$ can be served within one time segment. Note that within $S_i^* = A_i \cup X_i^* \cup Y_i^*$, $X_i^*$ is not available for SBP to serve because SBP has served the requests in $X_i^*$ prior to window $i$. Among requests that are available to SBP, SBP greedily chooses to serve the maximum revenue set that can be served within one time segment. Therefore, we have $rev(X_i) + rev(Y_i) \geq rev(Y_i^*)$. Since revenues are uniform, we also have $|X_i| + |Y_i| \geq |Y_i^*|$. If this is not the case, then SBP would have chosen to serve $Y_i^*$ instead of $X_i \cup Y_i$ since it is feasible for SBP to do so because the entire $S_i^*$ can be served within one time segment.

2. The set $S_i^*$ cannot be served within one time segment. This means there must be one request in $S_i^*$ that OPT started serving in the previous time segment. We refer to this straddling request as $r^*$. There are three sub-cases based on where $r^*$ appears.

a. If $r^* \in Y_i^*$, then due to the greediness of SBP, we know that $rev(X_i) + rev(Y_i) \geq rev(r^*)$.

b. If $r^* \in X_i^*$, then $r^*$ is not available to SBP and only $A_i, X_i, Y_i, Y_i^*$ are available to SBP. Therefore we know that $rev(X_i) + rev(Y_i) \geq rev(Y_i^*)$ since otherwise, by its greediness, SBP would have chosen to serve $A_i$ and $Y_i^*$ instead of $A_i, X_i$ and $Y_i^*$ because $A_i$ and $Y_i^*$ can be served within one time segment. Therefore, we have $|X_i| + |Y_i| \geq |Y_i^*|$. From (5), we have $|X_i| + |Y_i| \geq 1$ and from (6), we have $|X_i| + |Y_i| \geq |Y_i^*| - 1$.

c. If $r^* \in A_i$. Then $r^*$ is served by both OPT and SBP. We know that $A_i \cup Y_i^* \setminus \{r^*\}$ can be served within one time segment since $r^*$ is the only request that causes $S_i^*$ to straddle between two time segments. Again by the greediness of SBP, we have $rev(A_i) + rev(X_i) + rev(Y_i) \geq rev(A_i) + rev(Y_i^*) - rev(r^*)$ which means $rev(X_i) + rev(Y_i) \geq rev(Y_i^*) - rev(r^*)$ and $|X_i| + |Y_i| \geq |Y_i^*| - 1$. 
Therefore, for all cases, for window $i$, we have $|X_i| + |Y_i| \geq |Y_i^*| - 1$, which means $|Y_i^*| - |X_i| \leq 1 + |Y_i|$, and with $m = \lceil f/2 \rceil - 1$,

$$\sum_{i=1}^{m} (|Y_i^*| - |X_i|) \leq m + \sum_{i=1}^{m} |Y_i|.$$  \hspace{1cm} (7)

Now we will build an infeasible schedule $\text{SBP}'$ that will be similar to $\text{SBP}'$ but contain additional “copies” of some requests such that no windows of $\text{SBP}'$ contain holes, i.e. such that $\text{rev}(\text{SBP}) \geq \sum_{i=1}^{m} \text{rev}(S_i^*)$.

We define a modified $\text{OPT}$ schedule which we refer to as $\text{OPT}'$ such that $\text{OPT}' = \cup_{i=1}^{m} S_i^*$ and observe that $\text{rev}(\text{OPT}') = \sum_{i=1}^{m} |A_i| + \sum_{i=1}^{m} |X_i^*| + \sum_{i=1}^{m} |Y_i^*|$, while $\text{rev}(\text{SBP}') = \sum_{i=1}^{m} |A_i| + \sum_{i=1}^{m} |X_i| + \sum_{i=1}^{m} |Y_i|$.

By Lemma 5 and equation (7), we can say

$$\text{rev}(\text{OPT}') - \text{rev}(\text{SBP}') = \sum_{i=1}^{m} |Y_i^*| - \sum_{i=1}^{m} |X_i| + \sum_{i=1}^{m} |X_i^*| - \sum_{i=1}^{m} |Y_i|$$

$$\leq \sum_{i=1}^{m} |Y_i^*| - \sum_{i=1}^{m} |X_i| \leq m + \sum_{i=1}^{m} |Y_i|.$$  \hspace{1cm} (8)

Inequality (9) tells us that to form an $\text{SBP}'$ whose revenue is at least that of $\text{OPT}'$, we must “compensate” $\text{SBP}'$ by adding to it at most copies of all requests in the set $Y_i$ for all $i = 1, 2, ..., m$, plus $m$ “dummy requests.” In other words,

$$\text{rev}(\text{SBP}) = \text{rev}(\text{SBP}') + m + \sum_{i=1}^{m} |Y_i| \geq \text{rev}(\text{OPT}).$$  \hspace{1cm} (10)

We know the total revenue of all $Y_i$ can not exceed the total revenue of $\text{SBP}'$, hence we have

$$\text{rev}(\text{SBP}) = \text{rev}(\text{SBP}') + m + \sum_{i=1}^{m} |Y_i| \leq 2\text{rev}(\text{SBP}') + m.$$  \hspace{1cm} (11)

Combining (10) and (11), we get $\text{rev}(\text{OPT}') \leq 2\text{rev}(\text{SBP}') + m$, which means

$$\sum_{i=1}^{m} \text{rev}(S_i^*) \leq 2 \sum_{i=1}^{m} \text{rev}(S_i^*) + m.$$  \hspace{1cm} (12)

Recall that $S_i^*$ is the set of requests served by $\text{OPT}$ during the time segment of window $i$ with greater revenue. In other words, $\sum_{j=1}^{2m} \text{rev}(S^*(t_j)) \leq 2 \sum_{i=1}^{m} \text{rev}(S_i^*)$, which, combined with (12), gives us

$$\sum_{j=1}^{2m} \text{rev}(S^*(t_j)) \leq 4 \sum_{i=1}^{m} \text{rev}(S_i^*) + 2m.$$  \hspace{1cm} (13)

We assumed that the total revenue of requests served in the last two time segments by $\text{OPT}$ is bounded by $c$. From (13), we get

$$\sum_{j=1}^{f} \text{rev}(S^*(t_j)) \leq \sum_{j=1}^{2m} \text{rev}(S^*(t_j))+\text{rev}(S^*(t_{f-1}))+\text{rev}(S^*(t_f)) \leq 4 \sum_{i=1}^{m} \text{rev}(S_i^*)+2m+c.$$  \hspace{1cm} (14)

We also know that the total revenue of requests served by $\text{SBP}'$ during the first $m$ windows is less than or equal to the total revenue of $\text{SBP}$. Therefore, from (14), we have $\sum_{j=1}^{f} \text{rev}(S^*(t_j)) \leq 4 \sum_{j=1}^{f} \text{rev}(S(t_j)) + 2m + c$. 

\hspace{2cm} □
5 Bipartite graphs

In this section, we consider ROLDARP for complete bipartite graphs \( G = (V = V_1 \cup V_2, E) \), where only nodes in \( V_1 \) may be source nodes and only nodes in \( V_2 \) may be destination nodes. One node is designated as the origin and there is an edge from this node to every node in \( V_1 \) (so the origin is a node in \( V_2 \)). Due strictly to space limitations, most proofs of theorems in this section are deferred to Appendix 6.4.

We refer to this problem as ROLDARP-B and the offline version as RDARP-B. We first show that if edge weights of the bipartite graph are not bounded by a minimum value, then the offline version of ROLDARP on general graphs, which we refer to as RDARP, reduces to RDARP-B. Since RDARP has been show in [6, 1] to be NP-hard (even if revenues are uniform), this means RDARP-B is NP-hard as well.

\[ \text{Theorem 7.} \quad \text{The problem RDARP is poly-time reducible to RDARP-B. Also, RDARP with uniform revenues is poly-time reducible to RDARP-B with uniform revenues.} \]

\[ \text{Proof idea.} \quad \text{The idea of the reduction is to split each node into two nodes connected by an edge in the bipartite graph with a distance of } \epsilon. \text{ Then we turn each edge in the original graph into two edges in the bipartite graph. Please see Appendix for details.} \]

5.1 Uniform revenue bipartite

We show that for bipartite graph instances, if revenues are uniform, we can guarantee that SBP earns a fraction of OPT equal to the ratio between the minimum and maximum edge-length.

\[ \text{Theorem 8.} \quad \text{For any instance of ROLDARP-B where the revenues are uniform for all requests, if edge weights are upper and lower bounded by } T/f \text{ and } kT/f, \text{ respectively, for some constant } 0 < k \leq 1, \text{ then} \]

\[ \text{rev}(\text{OPT}) \leq \left\lceil \frac{1}{k} \right\rceil \cdot \text{rev}(\text{SBP}) + \left\lceil \frac{1}{k} \right\rceil. \]

\[ \text{Proof idea.} \quad \text{The proof idea is akin to that of Theorem 9 below. Please see Appendix for details.} \]

5.2 Nonuniform revenue bipartite

In this section we show that even if revenues are nonuniform, we can still guarantee that SBP earns a fraction of OPT equal to the ratio between the minimum and maximum edge-length, minus the revenue earned by OPT in the last window. Recall that we refer to each pair of consecutive time segments as a time window. Note that no non-preemptive deterministic online algorithm can be competitive with any fraction of the revenue earned by an optimal offline solution in the last \( 2T/f \) time units (i.e. Lemma 1 also holds for ROLDARP-B with nonuniform revenues).

\[ \text{Theorem 9.} \quad \text{For any instance of ROLDARP-B where the revenues of requests are nonuniform, if edge weights are upper and lower bounded by } T/f \text{ and } kT/f, \text{ respectively, for some constant } 0 < k \leq 1, \text{ and if the revenue earned by OPT in the last time window is bounded by some constant } c, \text{ then} \]

\[ \text{rev}(\text{OPT}) \leq \left\lceil \frac{1}{k} \right\rceil \cdot \text{rev}(\text{SBP}) + c. \]

\[ \text{Proof of Theorem 9} \quad \text{Again, we refer to each pair of consecutive time segments as a time window. We consider a hypothetical schedule which we refer to as SBP' that proceeds as} \]
follows. In the first time window, SBP’ does nothing. In the \(i^{th}\) window \(2 \leq i \leq \lfloor f/2 \rfloor\), SBP’ serves exactly one request: the maximum revenue request served by OPT in the \((i-1)^{th}\) window. (In Lemmas 14 and 15 of Appendix 6.4 we show that the revenue earned by SBP’ is no greater than the revenue earned by SBP.) Let \(Q_i, Q_i^c, \text{ and } Q_i^s\) denote the sets of requests served by SBP, SBP′, and OPT, respectively, in window \(i\). There are two cases based on the performance of SBP.

Case 1: SBP serves at least one request per window. Again let \(\mu = \lfloor f/2 \rfloor\) denote the total number of time windows. Let \(r = \lfloor 1/k \rfloor\). We know from Theorem 8 that OPT can serve at most \(r\) requests per window. We assume without loss of generality that OPT serves exactly \(r\) requests per window and let \(\rho_1, \rho_1, 1, \rho_1, 2, \ldots, \rho_1, r\) denote the \(r\) revenues earned by OPT in window \(i\). Consider the first window of OPT and the second window of SBP’. In the first window, OPT earns revenues \(\rho_1, \rho_1, 1, \rho_1, 2, \ldots, \rho_1, r\). In the second window, SBP’ serves the maximum revenue request from \(\rho_1\). Therefore, \(\text{rev}(Q_1^i) = \sum_{k=1}^{\rho_1} \rho_1, k \leq r \cdot \max\{\rho_1\}\) and \(\text{rev}(Q_1^s) = \max\{\rho_1\}\). So we have \(\text{rev}(Q_1^i) \leq r \cdot \text{rev}(Q_1^s)\). Similarly, we have \(\text{rev}(Q_1^i) \leq r \cdot \text{rev}(Q_1^s)\) for all \(i = 1, 2, \ldots, \mu - 1\). Summing up for all \(i = 1, 2, \ldots, \mu - 1\), we know

\[
\sum_{i=1}^{\mu-1} \text{rev}(Q_i^s) \leq r \sum_{i=2}^{\mu} \text{rev}(Q_i^s).
\]  

(15)

From Lemmas 14 and 15 of Appendix 6.4 we know the right-hand-side of (15) is no more than the total revenue earned by SBP during all \(\mu\) windows, therefore \(\sum_{i=1}^{\mu} \text{rev}(Q_i^s) \leq r \sum_{i=1}^{\mu} \text{rev}(Q_i^s)\). Since the revenue earned by OPT in the last (i.e. \(\mu^{th}\)) window is bounded by a constant \(c\), \(\sum_{i=1}^{\mu} \text{rev}(Q_i^s) \leq r \sum_{i=1}^{\mu} \text{rev}(Q_i^s) + c\). In other words, \(\text{rev(OPT)} \leq r \cdot \text{rev(SBP)} + c = \lfloor 1/k \rfloor \cdot \text{rev(SBP)} + c\).

Case 2: There may be empty windows (i.e. windows where SBP serves nothing). Let \(w\) denote the last empty window that occurred during the entire time limit and let \(\tau\) denote the start time of window \(w\). We analyze the requests served before, during, and after \(w\).

- Before window \(w\): since SBP serves nothing during window \(w\), we know that all requests released before time \(\tau\) have been served by SBP. Let \(b\) denote the total revenue of these requests. We know that before \(\tau\), OPT could have earned revenue at most \(b\).
- During window \(w\): OPT earns revenue \(\rho_{w,1}, \rho_{w,2}, \ldots, \rho_{w,r}\) and SBP earns nothing.
- After window \(w\): now we proceed by running SBP’ which serves the maximum revenue request served in the previous window in the OPT schedule. Similar to (15), we have \(\text{rev}(Q_i^s) \leq r \cdot \text{rev}(Q_i^s)\) for all \(i = w, w+1, \ldots, \mu - 1\). Summing up for all \(i = w, w+1, \ldots, \mu - 1\) yields \(\sum_{i=w}^{\mu-1} \text{rev}(Q_i^s) \leq r \sum_{i=w+1}^{\mu} \text{rev}(Q_i^s)\). From Lemma 14 (in the appendix) we know \(r \sum_{i=w+1}^{\mu} \text{rev}(Q_i^s)\) is no more than the total revenue earned by SBP during all windows after window \(w\), therefore \(\sum_{i=w}^{\mu-1} \text{rev}(Q_i^s) \leq r \sum_{i=w+1}^{\mu} \text{rev}(Q_i^s)\). Since the revenue earned by OPT in the last (i.e. \(\mu^{th}\)) window is bounded by a constant \(c\), we have

\[
\sum_{i=w}^{\mu-1} \text{rev}(Q_i^s) \leq r \sum_{i=w+1}^{\mu} \text{rev}(Q_i^s) + c.
\]  

(16)

So

\[
\text{rev(OPT)} = \sum_{i=1}^{\mu} \text{rev}(Q_i^s) \leq b + \sum_{i=w}^{\mu} \text{rev}(Q_i^s) \leq b + r \sum_{i=w+1}^{\mu} \text{rev}(Q_i^s) + c.
\]  

(17)

and

\[
\text{rev(SBP)} = \sum_{i=1}^{\mu} \text{rev}(Q_i) = b + 0 + \sum_{i=w+1}^{\mu} \text{rev}(Q_i).
\]  

(18)
Combining 17 and 18 we have:

\[
\frac{\text{rev}(\text{OPT}) - c}{\text{rev}(\text{SBP})} \leq \frac{b + r \sum_{i=w+1}^{\mu} \text{rev}(Q_i)}{b + \sum_{i=w+1}^{\mu} \text{rev}(Q_i)} \leq \frac{rb + r \sum_{i=w+1}^{\mu} \text{rev}(Q_i)}{b + \sum_{i=w+1}^{\mu} \text{rev}(Q_i)} = r.
\]

(19)

Which means \(\text{rev}(\text{OPT}) \leq \lceil 1/k \rceil \cdot \text{rev}(\text{SBP}) + c.\)
6 Appendix

In this section we provide all proofs missing from the main body of the paper. Most of these proofs were deferred to this section due strictly to space limitations.

6.1 Proof from Preliminaries Section

Proof of Lemma [1] Consider the following instance for $f \geq 2$ for some non-preemptive deterministic algorithm $\text{alg}$. The adversary releases a request $(s, d, T - 2T/f, 1)$ where the distance between $s$ and $d$ is $T/f$ and it takes $T/f$ time to travel between $\text{alg}$’s server location at time $T - 2T/f$ and $s$. If $\text{alg}$ chooses not to serve the request, then the adversary releases no more requests, so $\text{alg}$ earns 0 while $\text{opt}$ serves the request and earns 1. If $\text{alg}$ serves the request, it moves to $s$ for $T/f$ time units, then serves the request until $T$, and earns revenue 1. During this time, the adversary releases a request $(a, b, T - 2T/f + \delta, k)$, for some small $\delta$ and an arbitrarily large $k$, where $a$ is the location of an optimal server, $\text{opt}$, at time $T - 2T/f$. The $\text{opt}$ solution serves this request earning revenue $k$, so $\text{opt}_{\text{alg}} = k$.

For the case of uniform revenues, we simply modify the above instance so that at time $T - 2T/f + \delta$ for some small $\delta$ (i.e. while $\text{alg}$ is serving the $(s, d, T - 2T/f, 1)$ request), the adversary releases $k$ requests $r_1, r_2, \ldots, r_k$ such that the source of $r_1$ is the location of $\text{opt}$ at time $T - 2T/f$, the distance between the source and destination of each request is $\epsilon$ for some small $\epsilon$, each request has revenue 1, and the sequence of requests can be served in a chain (i.e. with no intermediary moves in between). The $\text{opt}$ solution serves the $k$ requests earning revenue $k$, so $\text{opt}_{\text{alg}} = k$.

Proof of Theorem [2] for nonuniform revenues. Consider the following instance with $f = 5$ (so there are 5 time segments of length $T/f$). For simplicity, we let $X = T/f \geq 3$ denote the length of a time segment and therefore the maximum distance between two locations, so $T = 5X$. All distances are $X$ unless otherwise stated. Let $\text{opt}$ denote an optimal schedule, let $\text{on}$ denote a deterministic online algorithm and let $a_0$ denote the origin, i.e. the location of $\text{on}$ and $\text{opt}$ at time 0. The adversary releases requests $r_1 = (a_1, b_1, X, \epsilon)$ and $r_2 = (a_2, b_2, X, \epsilon)$. Let $d(u, v)$ denote the distance between locations $u$ and $v$.

1. Case: $\text{on}$ does not ever visit $a_1$ or $a_2$. Then the adversary releases no more requests, so $\text{on}$ earns 0 while $\text{opt}$ serves one of the requests and earns $\epsilon$ within the first $T - 2T/f$.

2. Case: $\text{on}$ moves from its location at time $t_1 > X$ to either $a_1$ or $a_2$. Note $\text{on}$ has not earned any revenue yet.

a. Case: $X < t_1 < 2X$. Since the following release time is after $t_1$, we may assume w.l.o.g. that $\text{on}$ is at $a_1$ at time $t_1 + X$. Then the adversary releases request $r_3 = (b_2, c_2, 2X + 1, \epsilon)$. When $\text{on}$ arrives at $a_1$ there is fewer than $3X$ units of time remaining so there is insufficient time for $\text{on}$ to serve more than one request.

b. Case: $t_1 = 2X$. Since the following release time is after $t_1$, we may assume w.l.o.g. that $\text{on}$ is at $a_1$ at time $t_1 + X$. Then the adversary releases request $r_3 = (c_2, d_2, 2X + 1, \epsilon)$, where $d(b_2, c_2) = 1$ and $d(c_2, d_2) = X - 1$. When $\text{on}$ arrives at $a_1$, there is $2X$ time remaining. There is insufficient time for $\text{on}$ to serve more than one request.

c. Case: $t_1 > 2X$. When $\text{on}$ arrives at $a_1$ or $a_2$ at time $t_1 + X > 3X$, there is $< 2X$ time remaining. The adversary releases request $r_3 = (c_2, d_2, 2X + 1, \epsilon)$, where $d(b_2, c_2) = 1$ and $d(c_2, d_2) = X - 1$. It takes at least $2X$ time for $\text{on}$ to serve two requests from either $a_1$ or $a_2$ so there is insufficient time for $\text{on}$ to serve more than one request.
In cases 2(a)-2(c), there is insufficient time for ON to serve two or more of \( r_1, r_2 \) and \( r_3 \) so ON earns revenue at most \( \epsilon \). On the other hand, \( \text{OPT} \) serves \( r_2 \) and \( r_3 \) by traversing \( a_0, a_2, b_2, c_2 \) or \( a_0, a_2, b_2, c_2, d_2 \) in time \( 3X \) earning revenue \( 2\epsilon \), so \( \frac{\text{OPT}}{\text{ON}} = 2 \).

3. Case: ON moves from its location at time \( t_1 = X \) to either \( a_1 \) or \( a_2 \). Since all future requests are released after time \( X \), we can assume w.l.o.g. ON moves to \( a_1 \) and arrives there at time \( 2X \). Then the adversary releases the requests: \( r_3 = (a_3, b_3, X + 1, 1) \), \( r_4 = (a_4, b_4, X + 1, 1) \), and \( r_5 = (c_3, d_3, 3X - 1, 1) \) or \( r_5 = (c_4, d_4, 3X - 1, 1) \), depending on when and where ON moves. Let \( d(b_3, c_3) = d(b_4, c_4) = X - 2 \), and \( d(c_3, d_3) = d(c_4, d_4) = 1 \).

a. Case: ON serves \( r_1 \) or \( r_2 \). Then there will be at most \( 2X \) time remaining. So ON can serve at most one additional request of revenue \( 1 \) or \( \epsilon \). So ON earns at most \( 1 + \epsilon \).

b. Case: ON does not serve \( r_1 \) or \( r_2 \) but moves from \( a_1 \) at time \( t_2 \). Note that if ON does not eventually move to one of \( a_3, a_4, c_3, c_4 \), then ON would earn 0.

i. Case: \( 2X \leq t_2 < 3X - 1 \). Since \( r_5 \) is released after \( t_2 \), we may assume w.l.o.g. that ON will move to \( a_3 \) and \( r_5 = (c_4, d_4, 3X - 1, 1) \). When ON arrives at \( a_3 \) at time \( t_2 + X \geq 3X \), there is \( \leq 2X \) time remaining. But it takes at least time \( 2X + 1 \) for ON to earn revenue 2 (e.g. by traversing \( a_3, b_3, c_4, d_4 \) with other paths taking longer). Hence ON earns at most revenue 1.

ii. Case: \( t_2 \geq 3X - 1 \). Then ON sees which \( r_5 \) request was released and can choose to head towards any of the locations such as \( a_3, a_4, c_3, c_4 \), and arrives there at time \( t_2 + X \geq 4X - 1 \), so there is \( X + 1 \) remaining. W.l.o.g let \( r_5 = (c_4, d_4, 3X - 1, 1) \). From any location, it will take ON at least time \( 2X - 1 \) to earn revenue 2 (e.g. by traversing \( a_4, b_4, c_4, d_4 \) with other paths taking longer). Since \( X \geq 3 \), we have \( 2X - 1 > X + 1 \), and so ON can earn at most 1.

In all subcases of Case 3, ON earns at most revenue \( 1 + \epsilon \). On the other hand, \( \text{OPT} \) serves \( r_4 \) and \( r_5 \) by traversing \( a_0, a_4 \), waiting time 1, and then traversing \( b_4, c_4, d_4 \), in total time \( 3X \) earning revenue 2, so \( \frac{\text{OPT}}{\text{ON}} = 2/(1 + \epsilon) \).

\[ \square \]

**Proof of Theorem 2 for uniform revenues.** Consider the following instance with \( f = 5 \) (so there are 5 time segments of length \( T/f \)). For simplicity, we let \( X = T/f \) denote the length of a time segment and therefore the maximum distance between two locations, so \( T = 5X \).

All distances are \( X \) unless otherwise stated. We let the uniform revenue be 1. Fix a positive integer \( k \) and let \( 0 < \delta < X/(2k) \).

Let \( \text{OPT} \) denote an optimal schedule, let \( \text{ON} \) denote a deterministic online algorithm and let \( a_0 \) denote the origin, i.e. the location of ON and OPT at time 0. The adversary releases requests \( r_1 = (a_1, a_2, X, 1) \) and \( r_2 = (b_1, b_2, X, 1) \). Let \( d(u, v) \) denote the distance between locations \( u \) and \( v \).

1. Case: ON does not ever visit \( a_1 \) or \( b_1 \). Then the adversary releases no more requests, so ON earns 0 while \( \text{OPT} \) serves one of the requests and earns 1 within the first \( T - 2T/f \).

2. Case: ON moves from its location at time \( t_1 > X \) to either \( a_1 \) or \( b_1 \). Note ON has not earned any revenue yet.

a. Case: \( X < t_1 < 2X \). Since the following release time is after \( t_1 \), we may assume w.l.o.g. that ON is at \( a_1 \) at time \( t_1 + X \); the adversary releases request \( r_3 = (b_2, b_3, 2X, 1) \). When ON arrives at \( a_1 \) there is \( < 3X \) time remaining so there is insufficient time for ON to serve more than one request.

b. Case: \( t_1 = 2X \). Since the following release time is after \( t_1 \), we may assume w.l.o.g. that ON is at \( a_1 \) at time \( t_1 + X \); the adversary releases request \( r_3 = (b_3, b_4, 2X + \delta, 1) \), where \( d(b_2, b_3) = \delta \) and \( d(b_3, b_4) = X - \delta \). When ON arrives at \( a_1 \), there is \( 2X \) time remaining. There is insufficient time for ON to serve more than one request.
c. Case: \( t_1 > 2X \). When \( \text{ON} \) arrives at \( a_1 \) or \( b_1 \) at time \( t_1 + X > 3X \), there is < 2X time remaining. The adversary releases request \( r_3 = (b_3, b_4, 2X + \delta, 1) \), where \( d(b_2, b_3) = \delta \) and \( d(b_3, b_4) = X - \delta \). It takes at least 2X time for \( \text{ON} \) to serve two requests from either \( a_1 \) or \( b_1 \) so there is insufficient time for \( \text{ON} \) to serve more than one request.

In cases 2(a)-2(c), there is insufficient time for \( \text{ON} \) to serve two or more of \( r_1, r_2 \) and \( r_3 \) so \( \text{ON} \) earns revenue at most 1. On the other hand, \( \text{OPT} \) serves \( r_2 \) and \( r_3 \) by traversing \( a_0, b_1, b_2, b_3 \) (case 2(a)) or \( a_0, b_1, b_2, b_3, b_4 \) (cases 2(b) and 2(c)) in time \( 3X \) earning revenue 2. So \( \frac{\text{OPT}}{\text{ON}} = 2 \).

3. Case: \( \text{ON} \) moves from its location at time \( t_1 = X \) to either \( a_1 \) or \( b_1 \). Since all future releases are released after time \( X \), we can assume w.l.o.g. \( \text{ON} \) moves to \( a_1 \) and arrives there at time \( 2X \). Then the adversary releases the requests: \( r_i' = (c_{i-1}, c_i, X + \delta, 1) \), \( r_i'' = (d_{i-1}, d_i, X + \delta, 1) \), for \( i = 1, \ldots, k \), where \( d(c_i, c_j) = d(d_i, d_j) = |i - j|X/k \). Depending on what \( \text{ON} \) does, the adversary will also choose \( w \) with \( 0 < w \leq X - \delta \) and release \( \bar{r}_i = (e_{i-1}, e_i, 3X - w, 1) \) for \( i = 1, \ldots, k \), where \( d(e_i, e_j) = |i - j|w/k \), and one of \( d(e_0, c_k) = X - \delta - w \) or \( d(e_0, d_k) = X - \delta - w \) will be chosen.

a. Case: \( \text{ON} \) serves \( r_1 \) or \( r_2 \). Then there will be at most 2X time remaining. So \( \text{ON} \) can serve at most \( k \) additional requests from the \( r', r'', \bar{r} \) family. \( \text{ON} \) earns at most 1 + \( k \).

b. Case: \( \text{ON} \) does not serve \( r_1 \) or \( r_2 \) but moves from \( a_1 \) at time \( t_2 \). Note that if \( \text{ON} \) does not eventually move to one of \( c_i, d_i, e_i \), then \( \text{ON} \) would earn 0.

i. Case: \( 2X \leq t_2 < 3X - \delta \). The adversary chooses \( w = 3X - t_2 - \delta > 0 \) so that \( 3X - w = t_2 + \delta \). Since \( \bar{r}_i \) are released after \( t_2 \), we may assume w.l.o.g. that \( \text{ON} \) will move to some \( c_j \) and set \( d(e_0, d_k) = X - \delta - w \) and \( d(e_0, c_k) = X \). When \( \text{ON} \) arrives at \( c_j \) at time \( t_2 + X \), there is \( 4X - t_2 \) time remaining. But it takes at least \( X/k + X + w = 4X - t_2 - \delta + X/k > 4X - t_2 \) for \( \text{ON} \) to earn revenue \( k + 1 \) (e.g. by traversing \( c_{k-1}, c_k, e_0, e_1, \ldots, e_k \) with other paths taking longer). Hence \( \text{ON} \) earns at most revenue \( k \).

ii. Case: \( t_2 \geq 3X - \delta \). In this case, the adversary picks \( w = \delta \) and releases the \( \bar{r}_i \) requests at time \( 3X - \delta \) and set \( d(e_0, d_k) = X - \delta - w \) and \( d(e_0, c_k) = X \). Then \( \text{ON} \) can choose to head towards any of the locations such as the sources of \( r', r'', \bar{r} \), and arrives there at time \( t_2 + X \geq 4X - \delta \), so there is \( X + \delta \) remaining. From any location, it will take \( \text{ON} \) at least time \( X/k + X - \delta - w + w \) to earn revenue \( k + 1 \) (e.g. by traversing \( d_{k-1}, d_k, e_0, e_1, \ldots, e_k \) with other paths taking longer). But \( X/k + X - \delta - w + w > X + \delta \) because we chose \( 2\delta < X/k \). So \( \text{ON} \) can earn at most \( k \).

In all subcases of Case 3, \( \text{ON} \) earns at most revenue \( 1 + k \). On the other hand, \( \text{OPT} \) can traverse \( a_0, d_0, d_1, \ldots, d_k, e_0, e_1, \ldots, e_k \), with pausing at \( d_0 \) until time \( X + \delta \), in total time \( 3X \) to earn \( 2k \). So \( \frac{\text{OPT}}{\text{ON}} = 2k/(1 + k) \).

### 6.2 Proof of SBP lower bound

We now present the formal proof of Theorem 3 in Section 3.1

**Proof of Theorem 3** Consider the instance depicted in Figure 1. Fix \( f > 2 \) to be an even integer. Fix \( T > 0 \), such that the time segment length \( T/f > 1 \), where distances are discretized so 1 is the smallest possible unit of distance, i.e. all distances are integer-valued. Let \( h = T/(2f) \). Assume further \( h > 1 \). Let \( 0 < \epsilon < 1 \) be vanishingly small and let \( B > 0 \).

Let \( o \) be the origin, with other points in the metric space being \( u_i \) for \( i = 1, 2, \ldots, f \) and \( v_i \) for \( i = 1, 2, \ldots, m \), where \( m \) will be determined below.
The idea is that SBP will take the path $o, u_1, \ldots, u_f$ in time $T$ serving a single request of revenue $B + \epsilon$ every other time segment as prescribed by the algorithm. Meanwhile, discounting the revenue earned in the last two time segments, OPT will take the path $o, v_1, \ldots, v_m, u_2, \ldots, u_{f-2}$ in time $T - 2T/f = T - 4h$. The distances are shown below each edge in the figure: $d(o, u_1) = 1$, $d(u_i, u_{i+1}) = h$ for $i$ odd, $d(u_i, u_{i+1}) = h + 1$ for $i$ even, and $d(v_m, u_2) = h + 1$, and all other distances (not shown) are $h$.

The requests are depicted as directed edges in the figure. They are: $(u_1, u_2, 0, \epsilon)$, $(u_{2i+1}, u_{2i+2}, 4kh, B + \epsilon)$ for $i = 1, \ldots, f/2 - 1$, $(u_{2i+1}, u_{2i+2}, 4kh + 1, B)$ for $i = 1, \ldots, f/2 - 1$, $(v_i, v_{i+1}, 1, B)$ for $i = 1, \ldots, m - 1$, and $(v_m, u_2, 1, B)$.

Note that SBP will take the path $o, u_1, \ldots, u_f$:

1. At time $t = 0$, SBP will choose drive $(o, u_1)$ followed by request $(u_1, u_2)$ because that is all that is available.
2. For $k = 1, \ldots, f/2 - 1$, at time $t = 4kh$ (the start time of a pair of time segments of length $2h = T/f$ each), SBP is at vertex $u_{2k}$. The available requests (that have not yet been served) are: $(u_{2k+1}, u_{2k+2}, 4kh, B + \epsilon)$, $(u_{2i+1}, u_{2i+2}, 4kh + 1, B)$ for $i = 1, \ldots, k - 1$, $(v_i, v_{i+1}, 1, B)$ for $i = 1, \ldots, m - 1$, and $(v_m, u_2, 1, B)$. Note that none of the requests of revenue $B$ along the top path arrive in time for SBP to serve more than a single request at a time. Further, since we are looking for a revenue set that has path length at most $2h$, we cannot put together a path of length at most $2h$ that has 2 or more of these requests (since an edge from either $u_{2k+1}$ or $u_{2k+2}$ to any of the other vertices listed above has weight $h$ by design). Thus a maximum revenue set chosen by SBP using a path of length at most $2h$ has only one request. And a maximum revenue request can clearly be the request $(u_{2k+1}, u_{2k+2}, 4kh, B + \epsilon)$. Thus SBP would drive $(u_{2k}, u_{2k+1})$ at time $t = 4kh$ followed by the request $(u_{2k+1}, u_{2k+2})$ at time $t = 4kh + 2h$. And at time $t = 4(k + 1)h$, SBP would be at vertex $u_{2k+2}$.

Thus SBP earns revenue $\epsilon + (f/2 - 1)(B + \epsilon) = B(f/2 - 2) + f \cdot \epsilon/2$.

We now consider OPT, but we disregard the last two time segments. That is, we consider only the revenue earned up until time $T - 2T/f = 2fh - 4h = (2f - 4)h$. Consider the following path, call it OPT’ for convenience: $o, v_1, \ldots, v_m, u_2, \ldots, u_{f-2}$. Note we stop at node $u_{f-2}$ because the requests from $u_{f-2}$ to $u_{f-1}$ and $u_f$ have not yet been released at time $(2f - 4)h$. OPT’ takes $1 + m \cdot (h + 1)$ time to get to $u_2$ and takes time $(1 + h)(f/2 - 2)$ to go from $u_2$ to $u_{f-2}$. The total is $1 + m(h + 1) + (1 + h) \cdot (f/2 - 2)$. The largest $m$ for which OPT’ is completed before time $(2f - 4)h$ is

$$m = \lceil (3fh - 4h - f + 2)/(2(h + 1)) \rceil.$$ 

Observe that for this value of $m$, we have $1 + m(h + 1) + (f/2 - 2) \cdot (1 + h) \leq (2f - 4)h$ as needed. Clearly, OPT’ can serve all the requests on the path $v_1, \ldots, v_m$, $u_2$ because these requests were all released at time $t = 1$. Now, for each $k = 1, \ldots, f/2 - 2$, OPT’ arrives at vertex $u_{2k}$ at time $\tau_k = 1 + m(h + 1) + (1 + h)(k - 1)$. By Lemma 10 in Appendix 6.2, $\tau_k \geq 4kh + 1$ for each $k = 1, \ldots, f/2 - 2$. Therefore the requests $(u_{2k}, u_{2k+1}, 4kh + 1, B)$ and $(u_{2k+1}, u_{2k+2}, 4kh, B + \epsilon)$ are released on or before $\tau_k$, allowing OPT’ to serve these two requests when it reaches $u_{2k}$ at time $\tau_k$. Therefore all the drives starting at $v_1$ are revenue generating requests for OPT’.

Now, OPT’ has revenue $mB$ from $v_1$ up to $u_2$ and revenue $(2B + \epsilon)(f - 4)/2$ from $u_2$ to $u_{f-2}$. Let rec(ALG) denote the total revenue earned by a schedule, ALG. Then we can write

$$\text{rec(OPT')} = mB + (2B + \epsilon)(f - 4)/2 = mB + Bf - 4B + \epsilon(f - 4)/2 = \left(\frac{3fh - 4h - f + 2}{2(h + 1)}\right)B + fB - 4B + \epsilon(f - 4)/2.$$
We then rewrite as
\[ \frac{rev(OPT')}{rev(SBP)} = \frac{(3hf - 4h - f + 2)/(2(h + 1))B + fB - 4B + \epsilon(f - 4)/2}{B \cdot (f - 2)/2 + f \cdot \epsilon/2}. \]

Taking the limit as \( \epsilon \) approaches 0,
\[ \lim_{\epsilon \to 0} \frac{rev(OPT')}{rev(SBP)} = \frac{(3hf - 4h - f + 2)/(2(h + 1))B + fB - 4B}{(f - 2)/2}. \]

Next we take the limit as \( T \) approaches infinity, which is the same as taking \( h \) to infinity, because \( f \) is fixed. The inside of the floor can be rewritten as
\[ (3hf - 4h - f + 2)/(2(h + 1)) = 3f/2 - 2 - (2f - 3)/(h + 1). \]

Note \( 3f/2 - 2 \) is an integer since \( f \) is even. So when \( h + 1 > 2f - 3 \), we have \( 0 < (2f - 3)/(h + 1) < 1 \). Thus \( 3f/2 - 2 - (2f - 3)/(h + 1) = 3f/2 - 2 - 1 = 3f/2 - 3 \) when \( h > 2f - 4 \). Then \( \lim_{T \to \infty} [3f/2 - 2 - (2f - 3)/(h + 1)] = 3f/2 - 2 - 1 = 3f/2 - 3 \). Therefore
\[ \lim_{T \to \infty} \lim_{\epsilon \to 0} \frac{rev(OPT')}{rev(SBP)} = \frac{3f}{2} - 3 + \frac{f - 4}{(f - 2)/2} = \frac{5f/2 - 7}{(f - 2)/2} = \frac{(5f - 14)}{(f - 2)} = 5 - 4/(f - 2). \]

Summarizing, for a fixed \( f > 2 \), this instance gives a lower bound of \( 5 - 4/(f - 2) \) as \( \epsilon \) approaches 0 and \( T \) approaches infinity.

For definitions and context for the following lemma, please refer to the proof of Theorem 3 above, from which the following lemma is referenced. Recall that in the analysis of the lower bound instance, \( \tau_k \) was used to denote the time at which \( OPT' \) arrives at vertex \( u_{2k} \).

**Lemma 10.** \( \tau_k \geq 4kh + 1 \) for \( k = 1 \ldots f/2 - 2 \).

**Proof.** Recall that for each \( k = 1 \ldots f/2 - 2 \), \( OPT' \) arrives at vertex \( u_{2k} \) at time \( \tau_k = 1 + m(h + 1) + (1 + h)(k - 1) \). By definition of \( m = [(3hf - 4h - f + 2)/(2(h + 1))] \), we know
\[ m \geq (3hf - 4h - f + 2)/(2(h + 1)) - 1. \]

So
\[ m(h + 1) \geq (3hf - 4h - f + 2)/2 - (h + 1). \]

Then
\[ \tau_k \geq 1 + (3hf - 4h - f + 2)/2 - (h + 1) + (1 + h)(k - 1). \]

We then rewrite as
\[ \tau_k \geq 1 + (3f/2 - 2)h - f/2 + 1 - h - 1 + k - 1 + h(k - 1) - 4kh + 4kh \] (20)
\[ \geq 1 + 4kh + (3f/2 - 2 + k - 1 - 1 - 4k)h - f/2 - 1 + k \] (21)
\[ \geq 1 + 4kh + (3f/2 - 4 - 3k)h - f/2 + 4/3 + k - 1 - 4/3 \] (22)
\[ \geq 1 + 4kh + (3f/2 - 4 - 3k)(h - 1/3) - 1 - 4/3 \] (23)
\[ \geq 1 + 4kh + (3f/2 - 4 - 3k)(h - 1/3) - 1 - 4/3 \] (24)

Since \( k \leq f/2 - 2 \), then \( 3k \leq 3f/2 - 6 \). Then \( 3f/2 - 6 - 3k \geq 0 \). Hence \( 3f/2 - 4 - 3k \geq 2 \). So
\[ (3f/2 - 4 - 3k)(h - 1/3) - 1 - 4/3 \geq 2(h - 1/3) - 2(h - 1) - 4/3 \geq 2h - 2/3 - 2h + 2 - 4/3 \geq 0 \]
Thus we have shown \( \tau_k \geq 1 + 4kh \). \[ \nabla \]
6.3 Proofs for the uniform revenue case

We begin by reiterating several definitions that will aid us in proving Lemma 5 which is needed for Theorem 3 of Section 4.

For all windows \( i = 1, 2, ..., m \), where \( m = \lfloor f/2 \rfloor - 1 \), we let \( S'_i \) denote the set of requests served by \( SBP' \) in window \( i \) and \( S^*_i \) denote the set of requests served by \( \text{OPT} \) during the time segment of window \( i \) with greater revenue, i.e. \( S^*_i = \arg \max \{ \text{rev}(\text{OPT}(t_{2i-1})), \text{rev}(\text{OPT}(t_{2i})) \} \) where \( \text{rev}(\text{OPT}(t_j)) \) denotes the revenue earned by \( \text{OPT} \) in time segment \( t_j \).

We define a new set \( J^*_i \) as the set of requests served by \( \text{OPT} \) during the time segment of window \( i \) with less revenue, i.e. \( J^*_i = \arg \min \{ \text{rev}(\text{OPT}(t_{2i-1})), \text{rev}(\text{OPT}(t_{2i})) \} \).

Let \( S^*_i = A_i \cup X^*_i \cup Y^*_i \), and \( S'_i = A_i \cup X_i \cup Y_i \), where:
1. \( A_i \) is the set of requests that appear in both \( S^*_i \) and \( S'_i \).
2. \( X^*_i \) is the set of requests that appear in \( S^*_w \) for some \( w = 1, 2, ..., i - 1 \). Note there is only one possible \( w \) for each individual request \( r \in X^*_i \), because each request can only be served once.
3. \( Y^*_i \) is the set of requests such that no request from \( Y^*_i \) appears in \( S^*_w \) for any \( w = 1, 2, ..., i - 1, i \).
4. \( X_i \) is the set of requests that appear in \( S^*_w \) for some \( w = 1, 2, ..., i - 1 \). Note there is only one possible \( w \) for each individual request \( r \in X_i \), because each request can only be served once.
5. \( Y_i \) is the set of requests such that no request from \( Y_i \) appears in \( S^*_w \) for any \( w = 1, 2, ..., i - 1, i \).

Note that elements in \( Y_i \) can appear in a previous \( J^*_w \) for any \( w = 1, 2, ..., i - 1, i \) or in a future \( S^*_v \) or \( J^*_v \) for any \( v = i + 1, i + 2, ..., m \), or may not appear in any other sets.

Also note that since each request can be served at most once, we have:
1. \( A_i \cap X^*_i \cap Y^*_i \cap A_2 \cap X^*_2 \cap Y^*_2 \cap ... \cap A_m \cap X^*_m \cap Y^*_m = \emptyset \)
2. \( A_i \cap X_i \cap Y_i \cap A_2 \cap X_2 \cap Y_2 \cap ... \cap A_m \cap X_m \cap Y_m = \emptyset \)

Given the above definitions, we have the following lemmas:

▶ **Lemma 11.** All requests \( r \in X^*_i \) must satisfy that \( r \in Y_w \) for some \( w = 1, 2, ..., i - 1 \), and there is only one possible value of \( w \).

**Proof.** By definition, each request of \( X^*_i \) must appear in \( S^*_w \) for some \( w = 1, 2, ..., i - 1 \), and there is only one possible value of \( w \). Let \( r \) be a request of \( X^*_i \). We know that \( r \) must appear in either \( A_w \), or \( X_w \), or \( Y_w \). However, \( r \) cannot appear in \( A_w \), since otherwise \( r \) would have been served in \( S^*_w \), where \( w < i \), which is a contradiction since \( r \) is served in \( S^*_i \). Similarly, \( r \) cannot appear in \( X_w \), since otherwise \( r \) would have been served in \( S^*_w \) for some \( v = 1, 2, ..., w - 1 \), where \( v < w < i \), which is a contradiction since we know \( r \) is served in \( S^*_i \).

By elimination, \( r \) must be a request of \( Y_w \).

▶ **Lemma 12.** \( X^*_1 \cup X^*_2 \cup ... \cup X^*_i \subseteq Y_1 \cup Y_2 \cup ... \cup Y_{i-1} \) for all \( i = 2, 3, ..., m \).

**Proof.** We prove this by induction. For the base case, by Lemma 11, \( X^*_1 \) must be a subset of \( Y_0 \), where \( Y_0 \) is the empty set; \( X^*_2 \) must be a subset of \( Y_1 \). Therefore, \( X^*_1 \cup X^*_2 = \emptyset \cup X^*_2 \subseteq Y_1 \). For the inductive case, assume

\[
X^*_1 \cup X^*_2 \cup ... \cup X^*_k \subseteq Y_1 \cup Y_2 \cup ... \cup Y_{k-1}.
\]

Consider \( X^*_k \) and \( Y_k \). By Lemma 11, elements of \( X^*_k \) can come from only two sources: \( Y_k \) and \( Y_1 \cup Y_2 \cup ... \cup Y_{k-1} \). Therefore,

\[
X^*_k \subseteq Y_1 \cup Y_2 \cup ... \cup Y_{k-1} \cup Y_k.
\]
Combining (25) and (26), we have
\[ X_1^i \cup X_2^i \cup ... \cup X_k^i \subseteq Y_1 \cup Y_2 \cup ... \cup Y_{k-1} \cup Y_k. \]

We now restate Lemma 5 originally presented in Section 4 for the reader’s convenience before giving its proof.

Lemma 13. \(|X_1^i| + |X_2^i| + ... + |X_k^i| \leq |Y_1| + |Y_2| + ... + |Y_{k-1}| + |Y_i| \) for all \(i = 1, 2, ..., m\)

Proof. The size of a set \(A\) must be at least the size of any subset of \(A\) so from Lemma 12 we have for all \(i = 1, 2, ..., m\),
\[ |X_1^i \cup X_2^i \cup ... X_k^i| \leq |Y_1 \cup Y_2 \cup ... Y_{k-1}| \]

Recall we observed above that \(X_1^i \cap X_2^i \cap ... \cap X_k^i = \emptyset\) and \(Y_1 \cap Y_2 \cap ... \cap Y_{k-1} = \emptyset\). Hence, we can rewrite the inequality as
\[ |X_1^i| + |X_2^i| + ... + |X_k^i| \leq |Y_1| + |Y_2| + ... + |Y_{k-1}| \]

By adding \(|Y_i|\) on the right-hand-side, we have proven the claim.

6.4 Proofs for bipartite graphs

Proof of Theorem 7. For every node \(u\) in the RDARP graph, make two nodes \(u_1\) and \(u_2\) for the bipartite graph such that \(u_1 \in V_1\) (the source side) and \(u_2 \in V_2\) (the destination side), then set the weights \(w(u_1, u_2) = w(u_2, u_1)\) in the bipartite graph to a diminishingly small \(\epsilon > 0\).

For every edge \((u, v)\) in the RDARP graph with weight \(w(u, v)\), make two edges in the bipartite graph: one from \(u_1 \in V_1\) to \(v_2 \in V_2\) and a second from \(u_2 \in V_2\) to \(v_1 \in V_1\), both with weight \(w(u, v)\). Finally, for each request from source \(s\) to destination \(d\) in the RDARP instance, create an equivalent-revenue request in the RDARP-B instance from \(s_1 \in V_1\) to \(d_2 \in V_2\). Set the time limit in the RDARP-B instance to be \(T + \delta\), where \(T\) is the time limit of the RDARP instance and \(\delta = T*\) in order to accommodate any time needed to travel the \(\epsilon\)-weight edges. We also choose \(\epsilon\) so that \(\delta\) is smaller than the smallest-weight edge of the original graph.

Observe that a schedule \(\sigma\) with revenue \(R\) exists in the RDARP instance if and only if a schedule \(\sigma'\) with revenue \(R\) exists in the RDARP-B instance:

1. Let the path \(P = (a, b, c, ... )\) be the path followed in the execution of the schedule \(\sigma\) in the RDARP instance. The equivalent path in the RDARP-B instance would be \(P' = (a_1, b_1, c_1, c_2, c_1, ... )\). Notice that by the construction above, \(P'\) has the same revenue and execution time (to within an additive \(\delta\)) in the bipartite instance as \(P\) does in the original instance.

2. Any path \(P'\) followed in the execution of a schedule \(\sigma'\) in the bipartite graph must alternate from one side of the bipartition to the other. Empty moves in \(P'\) from a vertex \(x_1 \in V_1\) to another vertex \(y_2 \in V_2\) in the bipartite graph represent empty moves from \(x\) to \(y\) in \(P\). Moves in \(P'\) from any node \(y_2 \in V_2\) to another node \(z_1 \in V_1\) also represent empty moves from vertices \(y\) to \(z\) in \(P\). Any moves in \(P'\) along \(\epsilon\)-weight edges are not moves at all along the corresponding path \(P\) in the original graph, as they take virtually no time and represent the server staying put in \(P\). Finally, by construction, any requests served in \(P'\) will be along an edge from some vertex \(x_1 \in V_1\) to another vertex \(y_2 \in V_2\), which will correspond

to a request with source $x$ and destination $y$ in the RDARP instance. In sum, $P$ will take essentially the same amount of time (minus at most $\delta$ time units) and earn the same revenue as $P'$.

**Proof of Theorem 8** We define a $k$-unit as a length of time of duration $kT/f$. As before, we refer to each pair of consecutive time segments as a time window. We say a $k$-unit belongs to time window $i$ if it ends within time window $i$. Note that a $k$-unit may straddle two windows by starting at one window and ending in the next.

If $2/k$ is an integer, then there are strictly $\frac{kT/f}{kT/f} = 2/k$ $k$-units in one time window. Note that this is true even if the window contains a straddling $k$-unit, since this $k$-unit will force another one to straddle into the next time window.

If $2/k$ is not an integer, then there are $\left\lfloor \frac{kT/f}{kT/f} \right\rfloor = \lfloor 2/k \rfloor$ non-straddling $k$-units within one window. If the window contains a straddling $k$-unit, the number of $k$-units will be $\lfloor 2/k \rfloor + 1 = \lceil 2/k \rceil$. Among those $\lceil 2/k \rceil$ $k$-units, at most $\left\lfloor \lfloor 2/k \rfloor / 2 \right\rfloor = \lceil 1/k \rceil$ of them can be used in an OPT schedule to serve requests since no algorithm can serve two or more requests consecutively without a move in between. Therefore, whether or not $2/k$ is an integer, the maximum number of requests that can be served in each window is $\lceil 1/k \rceil$.

Let $\mu = \lfloor f/2 \rfloor$ denote the total number of windows. There are two cases based on the performance of SBP:

1. SBP serves at least 1 request per window. In this case, SBP serves at least $\mu$ requests, and OPT serves at most $\mu \cdot \lceil 1/k \rceil$. Therefore, $\frac{rev(OPT)}{rev(SBP)} \leq \frac{\mu \lceil 1/k \rceil}{\mu} \leq \lceil 1/k \rceil$.

2. There exists at least one window where SBP serves no requests. We refer to such a window as an “empty window.” Consider the last empty window that occurred within the entire time limit, and denote this window as $w$. Let $\tau$ denote the start time of window $w$. We analyze the requests served (1) before, (2) during, and (3) after $w$:

   - Before window $w$: Since SBP serves nothing during window $w$, we know that all requests released before time $\tau$ have been served by SBP. Let $b$ denote this number of requests. So before $\tau$, OPT could have served at most $b$ requests.
   - During window $w$: OPT can serve at most $\lceil 1/k \rceil$ requests and SBP serves no requests.
   - After window $w$: Suppose there are $x$ windows after $w$ (excluding window $w$). By definition, window $w$ is the last window where SBP serves no requests, hence during the $x$ windows afterwards, SBP serves at least one request per window. On the other hand, OPT serves at most $\lceil 1/k \rceil$ requests per window.

We know that $\lceil 1/k \rceil \geq 1$. Therefore we have $rev(OPT) \leq b + \lceil 1/k \rceil + x \cdot \lceil 1/k \rceil$ and $rev(SBP) \geq b + 0 + x$. Hence

$$rev(OPT) \leq \lceil 1/k \rceil \cdot rev(SBP) + \lceil 1/k \rceil$$

The proof of Theorem 9 (in Section 5.2) relies on the fact that $rev(SBP) \geq rev(SBP')$ which we now prove using the following two lemmas. Recall that within the $i^{th}$ window $SBP'$ serves exactly one request: the maximum revenue request served by OPT during the $(i - 1)^{th}$ window. We now create another subroutine, called $SBP''$, that, for every every window, serves the highest revenue available request. Lemma 14 shows why $rev(SBP'') \geq rev(SBP')$, and Lemma 15 shows why $rev(SBP) \geq rev(SBP'')$, so we know $rev(SBP) \geq rev(SBP')$. 

\[\square\]
Lemma 14. Let $O'$ and $O''$ denote the set of revenues of the $\mu = \lceil f/2 \rceil$ requests served by SBP$'$ and SBP$''$ by the end of window $\mu$, sorted in descending order. Let $O'[z]$ and $O''[z]$ denote the $z^{th}$ element of $O'$ and $O''$, respectively, where $z = 1, 2, ..., \mu$.

Then, $O''[z] \geq O'[z]$ for all $z = 1, 2, ..., \mu$.

Proof. We proceed by strong induction. Recall $Q'_i$ is the set of revenues of all requests served by SBP$'$ during window $i$. Let $Q''_i$ and $Q'_i$ denote the set of revenues of all requests served by SBP$''$ and OPT, respectively, during window $i$.

Base case. $z = 1$. We know that $O'[1] = \max\{Q'_1 \cup Q'_2 \cup ... \cup Q'_{\mu-1}\}$.

Consider SBP$''$ during the last window; there are two cases: 1. SBP$''$ has served a request with revenue equal to or larger than $O'[1] = \max\{Q'_1 \cup Q'_2 \cup ... \cup Q'_{\mu-1}\}$; 2. SBP$''$ has not served such a request.

In the first case, no matter which request SBP$''$ serves in the last window, $O''[1] \geq O'[1]$.

In the second case, SBP$''$ will choose one available request with the maximum revenue to serve in the last window, so $Q''_i$ will have revenue either equal to $\max\{Q'_1 \cup Q'_2 \cup ... \cup Q'_{\mu-1}\}$ or larger than $\max\{Q'_1 \cup Q'_2 \cup ... \cup Q'_{\mu-1}\}$. Thus, when SBP$''$ is done, $O''[1] \geq O'[1]$.

Inductive case. Suppose $O''[z] \geq O'[z]$ is true for $z = 1, 2, ..., l$. Consider $z = l + 1$. We will show by contradiction that $O''[l + 1] \geq O'[l + 1]$. Suppose $O''[l + 1] < O'[l + 1]$.

By the definition of $O''$, we know $O''[1], O''[2], ..., O''[l]$ are the $l$ largest revenues served by SBP$''$, and

$$O''[1] \geq O''[2] \geq ... \geq O''[l]$$ (27)

From the inductive hypothesis,

$$O''[l] \geq O'[l]$$ (28)

Given the ordered nature of $O'$, we have

$$O'[l] \geq O'[l + 1]$$ (29)

Given $O''[l + 1] < O'[l + 1]$, (28), and (29), we have

$$O''[l] \geq O'[l] \geq O'[l + 1] > O''[l + 1]$$ (30)

The general approach of this proof is that, if the request that corresponds to the revenue $O'[l + 1]$ is not the request that corresponds to any of $O''[1], O''[2], ..., O''[l]$, then the $(l + 1)^{th}$ largest request selected by SBP$''$ would have been $O'[l + 1]$ instead of $O''[l + 1]$, since $O''[l + 1] < O'[l + 1]$. This is a contradiction. Now we must affirm that the request corresponding to $O'[l + 1]$ does not correspond to any of $O''[1], O''[2], ..., O''[l]$. To verify this precondition, we consider the two possible cases of (30).

Case 1. If $O''[l] > O'[l]$ or $O'[l] > O'[l + 1]$, then given (28) and (29), the consequence of this case will be $O[l] > O'[l + 1]$. More generally, taking into account (27) and (30), we have

$$O''[1] \geq O''[2] \geq ... \geq O''[l] > O'[l + 1] > O''[l + 1]$$ (31)

which tells us that revenue $O'[l + 1]$ does not correspond to any of $O''[1], O''[2], ..., O''[l]$. Therefore $O'[l + 1]$ should have been the $(l + 1)^{th}$ largest value of the set $O''$ instead of $O''[l + 1]$, which is a contradiction.

Case 2a. If $O''[l] = O'[l] = O'[l + 1] > O''[l + 1]$ and the request that corresponds to $O'[l + 1]$ never corresponds to any revenue among $O''[1], O''[2], ..., O''[l]$, then $O'[l + 1]$
should also have been the \((l + 1)^{th}\) largest value of the set \(O''\) instead of \(O''[l + 1]\), which is a contradiction.

**Case 2b.** If \(O''[l] = O'[l] = O'[l + 1] > O''[l + 1]\), and the request that corresponds to \(O'[l + 1]\) also corresponds to \(O''[g]\) for some \(1 \leq g \leq l\), which indicates that

\[
O''[g] = O''[g + 1] = \ldots = O''[l] > O''[l + 1]
\]

(32)

From the ordered nature of \(O'\) and the inductive hypothesis, we know

\[
O''[g] \geq O'[g] \geq O'[g + 1] \geq \ldots \geq O'[l + 1]
\]

(33)

Combined with the fact that \(O''[g] = O'[l + 1]\), it must be the case that

\[
O''[g] = O'[g] = O'[g + 1] = \ldots = O'[l + 1]
\]

(34)

Given that there could exist elements in \(O''\) prior to \(O''[g]\) that are equal to \(O'[l + 1]\), we let \(O''[x]\) be the first element in \(O''\) that is equal to \(O'[l + 1]\), where \(1 \leq x \leq g\). Similar to (33), we have

\[
O''[x] \geq O'[x] \geq O'[x + 1] \geq \ldots \geq O'[l + 1]
\]

(35)

Combining (35) and \(O[x] = O'[l + 1]\), we know that

\[
O''[x] = O'[x] = O'[x + 1] = \ldots = O'[l + 1]
\]

(36)

Now we know that in \(O''\), there are \((l - x + 1)\) elements that have revenue \(O'[l + 1]\), while in \(O'\), there are \(((l + 1) - x + 1) = (l - x + 2)\) elements with revenue \(O'[l + 1]\). The set \(O'\) has one more element with revenue \(O'[l + 1]\) than the set \(O''\).

This is a contradiction because then \(\text{SBP}''\) would have chosen the extra request in \(O'\) that has revenue \(O'[l + 1]\) to serve instead of directly serving \(O''[l + 1]\).

The direct consequence of Lemma \([14]\) is that

\[
\text{rev}(\text{SBP}'' ) = \sum_{z=1}^{n} O''[z] \geq \sum_{z=1}^{n} O'[z] = \text{rev}(\text{SBP}' ).
\]

**Lemma 15.** Let \(\text{SBP}''[i]\) and \(\text{SBP}[i]\) denote the \(i^{th}\) window served by \(\text{SBP}''\) and \(\text{SBP}\) respectively. Define \(U = \{1, 2, \ldots, \mu\}\) as the set of all possible indices of windows. Define \(A_j\) and \(B_j\) as any two subsets of \(U\) that satisfy the following criteria:

1. Both \(A_j\) and \(B_j\) are of size \(j\) and are in increasing order.
2. \(A_j[m] \leq B_j[m]\) for all \(m = 1, 2, \ldots, j\), where \(A_j[m]\) is the \(m^{th}\) element of \(A_j\) and \(B_j[m]\) is the \(m^{th}\) element of \(B_j\).
3. For requests served in \(\{\text{SBP}''[A_j[1]], \text{SBP}''[A_j[2]], \ldots, \text{SBP}''[A_j[j]]\}\), if they are ever served by \(\text{SBP}\), then they are served only in \(\{\text{SBP}[B_j[1]], \text{SBP}[B_j[2]], \ldots, \text{SBP}[B_j[j]]\}\).

Then, for all such possible \(A_j\) and \(B_j\), we have

\[
\sum_{m=1}^{j} \text{rev}(\text{SBP}''[A_j[m]]) \leq \sum_{m=1}^{j} \text{rev}(\text{SBP}[B_j[m]])
\]
Proof. Base case. When \( j = 1 \), we assume \( A_1[1] \leq B_1[1] \) and that if the request served by \( \text{SBP}''[A_1[1]] \) is served by \( \text{SBP} \), it is served only in \( \text{SBP}[B_1[1]] \). This implies that the request served in \( \text{SBP}''[A_1[1]] \) is available to \( \text{SBP}[B_1[1]] \), so given the greedy nature of \( \text{SBP} \), \( \text{rev}(\text{SBP}[B_1[1]]) \geq \text{rev}(\text{SBP}''[A_1[1]]) \). Consequently, we can say

\[
\sum_{m=1}^{1} \text{rev}(\text{SBP}''[A_1[m]]) \leq \sum_{m=1}^{1} \text{rev}(\text{SBP}[B_1[m]])
\]

Inductive Case. We assume that \( \sum_{m=1}^{j} \text{rev}(\text{SBP}''[A_j[m]]) \leq \sum_{m=1}^{j} \text{rev}(\text{SBP}[B_j[m]]) \) for all \( j \) where \( 1 \leq j \leq k \). Then we want to prove for all possible \( A_{k+1} \) and \( B_{k+1} \),

\[
\sum_{m=1}^{k+1} \text{rev}(\text{SBP}''[A_{k+1}[m]]) \leq \sum_{m=1}^{k+1} \text{rev}(\text{SBP}[B_{k+1}[m]])
\]

(37)

There are two cases:

1. \( \text{rev}(\text{SBP}''[A_{k+1}[k+1]]) \leq \text{rev}(\text{SBP}[B_{k+1}[k+1]]) \)
2. \( \text{rev}(\text{SBP}''[A_{k+1}[k+1]]) > \text{rev}(\text{SBP}[B_{k+1}[k+1]]) \)

If the first case is true, then combining it with the inductive hypothesis, (37) is clearly true.

If the second case is true, we denote \( r \) as the request served in window \( \text{SBP}''[A_{k+1}[k+1]] \). Then the equation of case 2 can be rewritten as

\[
\text{rev}(\text{SBP}[B_{k+1}[k+1]]) < \text{rev}(r)
\]

(38)

Eqn. (38) implies that \( r \) is not available to \( \text{SBP}[B_{k+1}[k+1]] \), because otherwise in window \( B_{k+1}[k+1] \) \( \text{SBP} \) would have served some request(s) whose total revenue is at least the revenue of \( r \). Suppose it is at window \( \text{SBP}[B_{k+1}[h]] \) that \( \text{SBP} \) serves \( r \) where \( 1 \leq h < k+1 \).

Define new subsets \( A_k \) and \( C_k \) where \( A_k = A_{k+1} \setminus \{A_{k+1}[k+1]\} \) and \( C_k = B_{k+1} \setminus \{B_{k+1}[1]\} \).

It is evident that both \( A_k \) and \( C_k \) are of size \( k \). Define \( A_k[m] \) as the \( m^\text{th} \) element of the newly defined \( A_k \), and \( C_k[m] \) as the \( m^\text{th} \) element of the newly defined \( C_k \).

Here we declare a shortcut of notation. For any set \( V \) and integers \( s \leq t \), \( V[s : t] = \{V[s], V[s + 1], \ldots, V[t - 1], V[t]\} \).

One observation is that for each element \( C_k[m] \) of \( C_k \), we have \( C_k[m] \geq A_k[m] \). This is because each element of \( C_k \) will be minimized when \( C_k[1] = B_{k+1}[1], C_k[2] = B_{k+1}[2], \ldots, C_k[k] = B_{k+1}[k] \) respectively. Since we know \( B_{k+1}[m] \geq A_{k+1}[m] \), we have \( C_k[m] \geq A_k[m] \). There are two sub-cases:

1. If \( \text{SBP}[B_{k+1}[h]] \) does not contain any of the requests of \( \text{SBP}''[A_k[1 : k]] \): Then \( A_k \) and \( C_k \) satisfy the three criteria listed in the lemma, so according to the inductive hypothesis,

\[
\sum_{m=1}^{k} \text{rev}(\text{SBP}''[A_k[m]]) \leq \sum_{m=1}^{k} \text{rev}(\text{SBP}[C_k[m]])
\]

(39)

Adding \( \text{rev}(r) \) on both sides of (39), we have

\[
\sum_{m=1}^{k+1} \text{rev}(\text{SBP}''[A_{k+1}[m]]) \leq \sum_{m=1}^{k+1} \text{rev}(\text{SBP}[C_{k}[m]]) + \text{rev}(r) \leq \sum_{m=1}^{k+1} \text{rev}(\text{SBP}[B_{k+1}[m]])
\]

(40)

Note the second \( \leq \) sign is valid since all the requests corresponding to \( \sum_{m=1}^{k} \text{rev}(\text{SBP}[C_k[m]]) \)

\( \text{rev}(r) \) are served in \( \text{SBP} \) (and \( \text{SBP} \) may serve additional requests as well).
2. If \( \text{SBP}[B_{k+1}[h]] \) contains any of the requests of \( \text{SBP}'[A_k[1 : k]] \): Then \( A_k \) and \( C_k \) violate criterion (3) (since \( B_{k+1}[h] \) is not in \( C_k \)) so the inductive hypothesis cannot be applied directly. Suppose there are \( n \) such requests where \( n \leq k \), and denote the total revenue of those \( n \) requests as \( N \). Then we define \( A_{k-n} = A_k \setminus \{ \text{indices of windows where the } n \text{ requests reside in } \text{SBP}' \} \). We also shrink \( C_k \) to \( C_{k-n} \) by removing \( n \) windows that do not contain any of the requests of \( A_{k-n} \). Then we can follow the same reasoning above to deduce \( C_{k-n}[m] \geq A_{k-n}[m] \) for all \( 1 \leq m \leq k - n \).

So according to the inductive hypothesis,

\[
\sum_{m=1}^{k-n} \text{rev}(\text{SBP}'[A_{k-n}[m]]) \leq \sum_{m=1}^{k-n} \text{rev}(\text{SBP}[C_{k-n}[m]])
\]  

Adding \( N \) and \( \text{rev}(r) \) on both sides, we have

\[
\sum_{m=1}^{k+1} \text{rev}(\text{SBP}'[A_{k+1}[m]]) \leq \sum_{m=1}^{k-n} \text{rev}(\text{SBP}[C_{k-n}[m]]) + N + \text{rev}(r) \leq \sum_{m=1}^{k+1} \text{rev}(\text{SBP}[B_{k+1}[m]])
\]

Note the second \( \leq \) sign of (42) is valid since all the requests corresponding to \( \sum_{m=1}^{k-n} \text{rev}(\text{SBP}[C_{k-n}[m]]) + N + \text{rev}(r) \) are served in \( \text{SBP} \) (and \( \text{SBP} \) may serve additional requests as well).

Finally, to prove \( \text{rev}(\text{SBP}) \geq \text{rev}(\text{SBP}') \), we let \( j = \mu \), \( A_\mu = \{1, 2, \ldots, \mu\} \), \( B_\mu = \{1, 2, \ldots, \mu\} \). This satisfies (1) \( A_\mu \) and \( B_\mu \) are in increasing order, (2) \( A_\mu[m] \leq B_\mu[m] \) for all \( m = 1, 2, \ldots, \mu \), and (3) for requests served in \( \text{SBP}'[A_\mu[1]], \text{SBP}'[A_\mu[2]] \ldots \text{SBP}'[A_\mu[\mu]] \), if they are ever served by \( \text{SBP} \), are served only in windows \( \{\text{SBP}[B_\mu[1]], \text{SBP}[B_\mu[2]] \ldots \text{SBP}[B_\mu[\mu]]\} \). Therefore, \( \text{rev}(\text{SBP}) \geq \text{rev}(\text{SBP}') \) is simply a specific case of \( \sum_{m=1}^{\mu} \text{rev}(\text{SBP}'[A_j[m]]) \leq \sum_{m=1}^{\mu} \text{rev}(\text{SBP}[B_j[m]]) \) where \( j = \mu \).

\section*{References}

1. Barbara Anthony, Sara Boyd, Ricky Birnbaum, Ananya Christman, Christine Chung, Patrick Davis, and Jigar Dhimar. Maximizing the number of rides served for dial-a-ride. In 19th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2019). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2019.

2. Norbert Ascheuer, Sven O Krumke, and Jörg Rambau. Online dial-a-ride problems: Minimizing the completion time. In Annual Symposium on Theoretical Aspects of Computer Science, pages 639–650. Springer, 2000.

3. Giorgio Ausiello, Esteban Feuerstein, Stefano Leonardi, Leen Stougie, and Maurizio Talamo. Algorithms for the on-line travelling salesman 1. Algorithmica, 29(4):560–581, 2001.

4. Alexander Birx and Yann Disser. Tight analysis of the smartstart algorithm for online dial-a-ride on the line. arXiv preprint arXiv:1901.04272, 2019.

5. Antje Bjelde, Yann Disser, Jan Hackfeld, Christoph Hansknecht, Maarten Lipmann, Julie Meißner, Kevin Schewior, Miriam Schlöter, and Leen Stougie. Tight bounds for online tsp on the line. In Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 994–1005. Society for Industrial and Applied Mathematics, 2017.

6. Ananya Christman, Christine Chung, Nicholas Jazcko, Marina Milan, Anna Vasilchenko, and Scott Westvold. Revenue maximization in online dial-a-ride. In 17th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2017). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017.

7. Ananya Christman and William Forcier. Maximizing revenues for on-line dial-a-ride. In International Conference on Combinatorial Optimization and Applications, pages 522–534. Springer, 2014.

8. Esteban Feuerstein and Leen Stougie. On-line single-server dial-a-ride problems. Theoretical Computer Science, 268(1):91–105, 2001.
9 Patrick Jaillet and Michael R Wagner. Generalized online routing: New competitive ratios, augmentation, and asymptotic analyses. *Operations Research*, 56(3):745–757, 2008.

10 Patrick Jaillet and Michael R Wagner. Online vehicle routing problems: A survey. *The Vehicle Routing Problem: Latest Advances and New Challenges*, pages 221–237, 2008.

11 Vinay A Jawgal, VN Muralidhara, and PS Srinivasan. Online travelling salesman problem on a circle. In *International Conference on Theory and Applications of Models of Computation*, pages 325–336. Springer, 2019.

12 Sven O Krumke. Online optimization: Competitive analysis and beyond. 2002.

13 Sven O Krumke, Willem E de Paepe, Diana Poensgen, Maarten Lipmann, Alberto Marchetti-Spaccamela, and Leen Stougie. On minimizing the maximum flow time in the online dial-a-ride problem. In *International Workshop on Approximation and Online Algorithms*, pages 258–269. Springer, 2005.

14 Yves Molenbruch, Kris Braekers, and An Caris. Typology and literature review for dial-a-ride problems. *Annals of Operations Research*, 259(1-2):295–325, 2017.