MINISUPERSPACE MODEL FOR REVISED CANONICAL QUANTUM GRAVITY

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We present a reformulation of the canonical quantization of gravity, as referred to the minisuperspace; the new approach is based on fixing a Gaussian (or synchronous) reference frame and then quantizing the system via the reconstruction of a suitable constraint; then the quantum dynamics is re-stated in a generic coordinates system and it becomes dependent on the lapse function.

The analysis follows a parallelism with the case of the non-relativistic particle and leads to the minisuperspace implementation of the so-called kinematical action as proposed in (here almost coinciding also with the approach presented in 2). The new constraint leads to a Schrödinger equation for the system, i.e. to non-vanishing eigenvalues for the super-Hamiltonian operator; the physical interpretation of this feature relies on the appearance of a “dust fluid” (non-positive definite) energy density, i.e. a kind of “materialization” of the reference frame.

As an example of minisuperspace model, we consider a Bianchi type IX Universe, for which some dynamical implications of the revised canonical quantum gravity are discussed. We also show how, on the classical limit, the presence of the dust fluid can have relevant cosmological issues.

Finally we upgrade our analysis by its extension to the generic cosmological solution, which is performed in the so-called long-wavelength approximation. In fact, near the Big-Bang, we can neglect the spatial gradients of the dynamical variables and arrive to implement, in each space point, the same minisuperspace paradigm valid for the Bianchi IX model.

1 Introduction

One of the most peculiar and puzzling features that the canonical quantum method has to face when it is applied to the gravitational field consists of a vanishing Hamiltonian function. Such a dynamical constraint reflects the invariance of the theory under infinitesimal time displacements and results into the non evolutive character singled out by the Wheeler-DeWitt equation (WDE) (see also 3). Indeed, via the Arnowitt-Deser-Misner (ADM) (3+1)-splitting of the space-time, the quantum information about the gravitational field is provided by a wave functional taken on a whole class of 3-geometries; in fact, the theory should be invariant under spatial coordinates re-parameterizations which are equivalent to the gauge symmetry observed in non-Abelian theories and is ensured requiring that the wave functional be annihilated by the super-momentum operator, i.e. $\hat{H}_i \Psi = 0$). Such equations restricts to a class of 3-geometries the dynamical variable on which the wave function is taken and then the canonical quantum dynamics is provided by requiring that also the super-Hamiltonian operator annihilates the system states, i.e. $\hat{H} \Psi = 0$; with respect to such a WDE equation we stress the following shortcomings features:

i) The WDE is a covariant quantum theory, but $\Psi$ does not depend on the lapse function $N$ and the shift vector $N^i$ (because of the two constraints associated with
the vanishing of their conjugate momenta) and we have information only on the 3-geometries.

ii) The wave functional takes the same value on each spatial hypersurface of the slicing and, therefore, no real evolution takes place (for a review about the problem of “time” in canonical quantum gravity see [8,9]).

iii) The quantum dynamical equation, in view of its hyperbolic structure, prevents a general prescription to arrange the space of the solutions into an Hilbert space and consequently no probability notion is naturally defined.

Over the years, many approaches have been presented in order to construct an internal physical clock for the quantum system and then to achieve the Hilbert space structure. In a recent work [1], it has been proposed a correlation between the “frozen formalism” of the WDE and the ambiguity in developing a (3+1)-slicing of a quantum space-time; in fact, it makes no precise sense, in a quantum picture, to speak of spatial hypersurfaces (or of a time-like normal) because such notions can be recognized, at most in terms of expectations values. Thus we infer that to implement a straightforward canonical quantization of the slicing can be responsible for a dynamics without evolution.

As a solution to the above ambiguity, in [1] was proposed to include the so-called kinematical action in treating the gravitational problem, as done for a quantum field on a fixed metric background.

Such a procedure is essentially equivalent to fix the reference frame before quantizing the gravitational field; This new approach finds its physical interpretation in the appearance of a real clock, consisting of a dust fluid; indeed the equations of motion associated to the kinematical action can be rewritten, under suitable hypotheses, as a dust fluid dynamics [1].

The aim of [1] is to argue that quantizing via a canonical method, the 3-geometries, it requires the existence of a “clock fluid” that makes physical the slicing.

Here we consider a minisuperspace cosmological model in a Gaussian (or synchronous) reference and quantize it in close analogy with the non-relativistic particle [4], as a result, we get the same quantum dynamics provided for this case by the kinematical term (as well as, by the, here almost overlapping, approach presented in [2]).

The quantization is then referred to a generic reference frame and we achieve a dynamical picture at all equivalent to that one proposed in [1] such a coincidence of two different approaches confirms the necessity of fixing the reference frame before the quantization of a (3 + 1)-slicing.

Then we show how it is possible for our system, to construct a natural Hilbert space and how, the evolution of the wave function becomes interpretable via a “dust fluid” of reference; finally we outline some relevant features concerning the semiclassical limit of the model.

As minisuperspace model we consider the Bianchi type IX cosmology and then we extend the analysis to a generic inhomogeneous cosmological solution; in fact we show how the generic case, close enough to the cosmological singularity, can
is reduced to a point-like minisuperspace dynamics. This feature is due to the dynamical decoupling of the space points which takes place when the Universe volume approaches zero near the Big-Bang. From a quantum dynamical point of view, we will deal with the generic cosmological solution in the limit of the so-called long-wavelength approximation; such an inhomogeneous extension provides with degree of generality to all the results derived for the Bianchi IX model.

In section 2 is discussed the quantization of the non-relativistic parameterized particle, regarded as the prototype for building up in section 3 an appropriate analysis of the Bianchi IX quantum dynamics, as viewed in a Gaussian frame. In section 4 is developed the classical limit of the revised minisuperspace quantization and some relevant cosmological implications of the outcoming picture are presented.

Section 5 is devoted to upgrading our previous analysis, by showing how, under well-grounded assumptions, it can be extended to the generic cosmological solution; by other words, we outline that, in each space point of a generic inhomogeneous Universe, takes place, independently, the same minisuperspace picture characterizing a Bianchi IX cosmology. In section 6 concluding remarks follows.

2 Parameterized particle

We start by reviewing the case of the one-dimensional non-relativistic (parametrized) particle, whose action reads

\[ S = \int \{ p \dot{q} - h(p, q) \} dt , \]  

(1)

where \( t \) denotes the Newton time and \( h \) the Hamiltonian function. In order to quantize this system, we parameterize the Newton time as \( t = t(\tau) \), which leads to have

\[ S = \int \{ p \frac{dq}{d\tau} - h(p, q) \frac{dt}{d\tau} \} d\tau . \]  

(2)

Now we set \( p_0 \equiv -\dot{h} \) and add this relation to the new action by a Lagrangian multiplier \( \lambda \), i.e.

\[ S = \int \{ p \frac{dq}{d\tau} + p_0 \frac{dt}{d\tau} - \bar{h}(p, q, p_0, \lambda) \} d\tau \quad \bar{h} \equiv \lambda(h + p_0) . \]  

(3)

By varying this action with respect to \( p \) and \( q \), we get the Hamilton equations

\[ \frac{dq}{d\tau} = \lambda \frac{\partial h}{\partial p} , \quad \frac{dp}{d\tau} = -\lambda \frac{\partial h}{\partial q} , \]  

(4)

while the variations of \( p_0 \) and \( t \) yield

\[ \frac{dt}{d\tau} = \lambda , \quad \frac{dp_0}{d\tau} = 0 . \]  

(5)
All together, these equations describe the same Newton dynamics with the energy as constant of the motion. But now, by varying $\lambda$, we get the (desired) constraint $h + p_0 = 0$, which, in terms of the operators $\hat{p}_0 = -i\hbar \partial_t$ and $\hat{h}$, provides the Schrödinger equation

$$i\hbar \partial_t \psi = \hat{h} \psi$$

for the system state function $\psi(t, q)$. Finally we remark that, when retaining the relation $dt/d\tau = \lambda$, we are able to write the wave equation in the parametric time as

$$i\hbar \partial_\tau \psi(\tau, q) = \lambda(\tau) \hat{h} \psi(\tau, q)$$

where the function $\lambda(\tau)$ is to be specified for completing the dynamical scheme.

3 Quantization of the Bianchi IX model

Now we implement this same method of quantization in the minisuperspace associated with an homogeneous cosmological model of the type IX. The Bianchi IX model is the most general one (together with type VIII) allowed by the homogeneity constraint and is described via a line element of the form

$$ds^2 = -N(t)^2 dt^2 + \frac{R(t)^2}{6\pi} \left( e^{2(\beta(t))} \right)_{ij} \sigma^i(x^l) \sigma^j(x^l) \quad i, j, l = 1, 2, 3,$$

where we take the diagonal form

$$\left( e^{2\beta(t)} \right)_{ij} = \text{diag.}\left\{ e^{2(\beta_+ + \sqrt{3}\beta_-)}, e^{2(\beta_+ - \sqrt{3}\beta_-)}, e^{-2\beta_+} \right\}$$

and the 1-forms $\sigma^i(x^l)$, to which is associated the Lie algebra of the isometries, read explicitly as

$$\sigma^1 = \cos \chi d\theta + \sin \chi \sin \theta d\varphi, \quad \sigma^2 = \sin \chi d\theta - \cos \chi \sin \theta d\varphi, \quad \sigma^3 = d\chi + \cos \theta d\varphi,$$

with $0 \leq \varphi < 2\pi$, $0 \leq \theta < \pi$ and $0 \leq \chi < 4\pi$.

By adopting these Misner-like variables, we separate the isotropic (volume) expansion of the Universe, represented by the function $R(t)$, from its anisotropies, which are described through the degrees of freedom $\beta_+(t)$ and $\beta_-(t)$.

The very early Universe evolution was characterized by a thermal bath containing all the fundamental particles and since most of the species were described by an ultrarelativistic equation of state, then we include into the problem a phenomenological energy density $\rho_{ur} = \mu^2/R^4$, $\mu = \text{const.}$. Furthermore, the idea that the Universe underwent an inflationary scenario, leads us to involve in the dynamics a real self-interacting scalar field $\phi$; the associated “finite temperature potential” $V_T(\phi)$ (here $T$ denotes the Universe temperature) can be taken in the Coleman-Weinberg form
\[
V_T(\phi) = \frac{B\sigma^4}{2h^3c^3} + B\frac{\phi^4}{hc} \left[ \ln \left( \frac{lp\phi^2}{\sigma^2} \right) - \frac{1}{2} \right] + \frac{1}{2}m_T^2\phi^2 
\]

\[
m_T = \sqrt{\lambda T^2 - m^2} \quad (m, \lambda) = \text{const.};
\]

(11)

here \( B \sim \mathcal{O}(10^{-3}) \) and \( \sigma \sim \mathcal{O}(10^{14}) \text{ GeV} \) denotes the scale of the transition phase.

In what follows, we regard the Universe temperature as a function of the variable \( R \), i.e. \( T(R) = T^* / R \), \( T^* = \text{const.} \).

The evolution of the cosmological model so obtained, is summarized, in a Gaussian reference \( (N = c, t \to T) \), by the action

\[
S_{IX} = \int \left\{ p_R \frac{dR}{dT} + p_+ \frac{d\beta_+}{dT} + p_- \frac{d\beta_-}{dT} + p_\phi \frac{d\phi}{dT} - H(R, \beta_\pm, \phi, p_R, p_\pm, p_\phi) \right\} dT,
\]

(12)

where all the \( p \)'s denote the conjugate momenta to the respective dynamical variables, and the Hamiltonian function takes the form

\[
H = \sqrt{\frac{3\pi}{2}} \left\{ \frac{l_p^2}{2h} \left[ -\frac{p_R^2}{R} + \frac{1}{R^4} \left( p_+^2 + p_-^2 \right) \right] + \frac{3c^2}{8\pi} p_\phi^2 + U(R, \phi) \right\},
\]

(13)

with the potential term

\[
U(R, \phi) = \frac{\mu^2}{R} + \frac{hc}{l_p} R \left( V(\beta_\pm) - 1 \right) + \frac{32}{18} R^3 V_T(\phi).
\]

(14)

The characterization of this dynamical system is completed by specifying the form of \( V(\beta_\pm) \) as [1]

\[
V(\beta_\pm) = \frac{1}{3} e^{-8\beta_+} - \frac{4}{3} e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + 1 + \frac{2}{3} e^{4\beta_+} \left( \cosh(4\sqrt{3}\beta_-) - 1 \right) \quad V(0, 0) = 0.
\]

(15)

Now we quantize this system in close analogy with the case of the non-relativistic particle, by re-parameterizing the Gaussian time as follows \( T = T(t) \); then, by the position \( H = -p_T \) (added to the action via a Lagrangian multiplier \( \Lambda(t) \)), we get

\[
S_{IX} = \int \left\{ p_R \frac{dR}{dt} + p_+ \frac{d\beta_+}{dt} + p_- \frac{d\beta_-}{dt} + p_\phi \frac{d\phi}{dt} + p_T \frac{dT}{dt} - \Lambda(t)(p_T + \mathcal{H}) \right\} dt.
\]

(16)

We stress how the variation of this action with respect to \( T \) yields the key relation \( dT/dt = \Lambda(t) \) and, by comparing it with [8], it comes out that \( \Lambda(t) = \pm N(t) \); hence, the choice of the positive root for \( \Lambda \), allows to rewrite [10] in the same form as we would have obtained applying, to the present case, the general method discussed in [11] (which is based on the use of the kinematical action), i.e.
\[ S_{IX} = \int \left\{ p_R \frac{dR}{dt} + p_+ \frac{d\beta_+}{dt} + p_- \frac{d\beta_-}{dt} + p_\phi \frac{d\phi}{dt} + pr \frac{dT}{dt} - N(t)(p_T + \mathcal{H}) \right\} dt. \]  

(17)

In both the cases (16) and (17), as soon as we implement the Hamiltonian constraint on a quantum level, we are lead to a Schrödinger equation of the form

(18)

It is worth stressing that, in analogy to equation (7) for the parameterized particle and, in view of the relation \( \partial_t T = N \), the Schrödinger equation, as written in a generic time variable reads

\[ i\hbar \partial_t \psi = \hat{H} \psi. \]  

(19)

This result is completely equivalent to apply, for our system, the quantum dynamics proposed in (11). Though the considerations developed up to the end of this sections holds in a generic time variable, now we come back to a synchronous frame.

The choice of the normal ordering \( p_R^2/R \rightarrow -\hbar^2 \partial_R(1/R)\partial_R \) is obliged by the requirement to turn the space of the solutions of equation (18) into an Hilbert one. In fact, when taken in this form, the Hamiltonian \( \mathcal{H} \) results to be Hermitian and therefore, by adopting the usual Dirac bra-ket notation, we get

\[ \partial_T \left( \langle \psi_1 | \psi_2 \rangle \right) = \frac{i}{\hbar} \left( \langle \psi_1 \mathcal{H} | \psi_2 \rangle - \frac{i}{\hbar} \langle \psi_1 | \mathcal{H} \psi_2 \rangle \right) = 0, \]  

(20)

Being \( \psi_1 \) and \( \psi_2 \) two generic state functions.

Indeed we can introduce a probability density, associated to the state function, defined as \( \rho \equiv \psi^* \psi \); it is easy to recognize that it satisfies a continuity equation of the form \( \partial_T \rho + \partial_a J^a = 0 \), \( a = R, \pm, \phi \), being, for instance

\[ J^R = \sqrt{\frac{3\pi l_p^3 h}{2}} \left( \frac{\psi^* R}{\rho} \partial_R \psi - \frac{\psi R}{\rho} \partial_R \psi^* \right) \]  

(21)

and analogous expressions for \( J^\pm \) and \( J^\phi \).

Such a continuity equation, in view of the Gauss theorem, as extended to the configuration space, implies that

\[ \partial_T \int_{-\infty}^{\infty} d^\pm \beta d\phi \int_0^\infty dR \rho(R, \beta, \phi) = 0 \quad \int_{-\infty}^{\infty} d^\pm \beta d\phi \int_0^\infty dR \rho = 1. \]  

(22)
An important step consists of observing that, if we expand the wave function as follows

\[ \psi(T, R, \beta, \phi) = \int_{-\infty}^{\infty} d\varepsilon C(\varepsilon) \chi(\varepsilon, R, \beta, \phi) \exp\left\{ -\frac{i}{\hbar} \varepsilon (T - T_0) \right\}, \]  

then we get, from the Schrödinger equation (18), the eigenvalue problem

\[ \hat{H} \chi = \sqrt{\frac{3\pi}{2}} \left\{ \frac{1}{R} \left( \frac{1}{R^3} (\partial_R^2 + \partial_R^2) \right) - \frac{3c^2}{8\pi} \partial_\phi^2 + U(R, \beta, \phi) \right\} \chi = \varepsilon \chi. \]

Here the function \( C(\varepsilon) \) is determined by assigning the initial wave function at the instant \( T = T_0 \), i.e.

\[ \psi(T = T_0, R, \beta, \phi) = \int_{-\infty}^{\infty} d\varepsilon C(\varepsilon) \chi(\varepsilon, R, \beta, \phi) \]  

where we have taken into account the orthonormality of the eigenfunctions \( \chi \)’s.

Thus (24) shows the following important issue: to fix the Gaussian reference leads to the appearance of a non-zero total quantum energy of the gravity-“matter” system.

4 Classical limit of the theory

Now, in order to understand the implications of this new quantum dynamics on the actual Universe, let us take the semiclassical expansion for the wave function, i.e.

\[ \psi = \exp\left\{ \frac{i}{\hbar} \sigma(R, \beta, \phi) \right\}, \quad \sigma = \sigma_0 + \frac{\hbar}{i} \sigma_1 + \left( \frac{\hbar}{i} \right)^2 \sigma_2 + ... \]  

and cutting it off, up to the zero-order of approximation, then the Schrödinger eigenvalues equation (24) rewrites as

\[ \frac{\hbar^2}{2m} \left[ -\frac{1}{R} \left( \frac{1}{R^3} (\partial_R^2 + \partial_R^2) \right) + \frac{3}{8\pi} (\partial_\phi^2 + U(R, \phi) - \frac{2}{3\pi} \varepsilon = 0. \]

Thus we get an Hamilton-Jacobi (H-J) equation in which appears a new term, coming from the, no longer zero, eigenvalue of the Hamiltonian operator; indeed, such a term is equivalent to the contribution given by a non-relativistic dust fluid and it can be interpreted as the energy density of the reference frame it-self (in this sense the non-relativistic nature comes from its “comoving state”). It is worth noting how the interpretation of \( -\varepsilon \) like density of energy must overcome the fact that it is not positive definite; as a solution to this problem, we propose the idea that the Universe spontaneously decays into the state of minimal energy, which, in general, corresponds to negative values of \( \varepsilon \). This fact is a consequence of being the super-Hamiltonian non-positive definite and it implies that the new term results into a positive definite energy density of the reference fluid.
Let us now take the following (general) expansion of the H-J function

$$\sigma_0 = \rho_0(R) + P_+ \beta_+ + P_- \beta_- + P \phi$$

(28)

being the P’s generic constants. If we assume that, near enough to the singularity ($R \to 0$), it is possible to neglect the potential term $U(R, \phi)$, then expression (28) reduces the above H-J equation (27) to the simple form

$$\frac{l_P^2}{\hbar} \left[ -\frac{1}{R} \left( \frac{d\rho_0}{dR} \right) + \frac{P^2}{R^3} \right] + \sqrt{\frac{2}{3\pi}} \varepsilon = 0 \quad P^2 \equiv P_+^2 + P_-^2 + \frac{3c^2 P \phi^2}{8\pi}. \quad (29)$$

Hence, we easily get:

$$\sigma_0 = \int dR \left( \frac{1}{R} \sqrt{P^2 - c R^3} \right) + P_+ \beta_+ + P_- \beta_- + P \phi \phi,$$

(30)

being $c \equiv \sqrt{\frac{2 \hbar}{3\pi l_P^4}}$. Now, in agreement with the H-J method, we differentiate with respect to the constants $P_i$’s ($i = \pm, \phi$) and (equating the results to certain constants $C_i$) arrive to the following expressions:

$$\beta_\pm = \int dR \left( \frac{P_\pm}{R \sqrt{P^2 - c R^3}} \right) + C_\pm, \quad \phi = \int dR \left( \frac{3c^2 P \phi}{8\pi R \sqrt{P^2 - c R^3}} \right) + C_\phi. \quad (31)$$

Now we explicit these solutions, respectively in the two asymptotic limits $R \to 0$ and $R \to \infty$, i.e.:

$$R \to 0 \Rightarrow \beta_\pm \sim \frac{P_\pm}{P} \ln R + C_\pm, \quad \phi \sim \frac{3c^2 P \phi}{8\pi P} \ln R + C_\phi \quad (32)$$

$$R \to \infty \Rightarrow \beta_\pm \sim -\frac{2P_\pm}{3 |q| R^{3/2}} + C_\pm, \quad \phi \sim -\frac{2c^2 P \phi}{8\pi |q| R^{3/2}} + C_\phi \quad (33)$$

where, in agreement with the above statement, we required that $c$ be a negative quantity, i.e. $-c = q^2$. On the classical limit, this assumption

Equation (32) shows that, near the singularity in $R = 0$, the solution takes, as expected, a Kasner-like form; in particular, by setting $\pi_i \equiv P_i / P$ ($i = \pm, \phi$), then we have $\sum_i \pi_i = 1$.

On the other hand, equation (33) implies that, far enough from the singularity, the anisotropy and scalar field degrees of freedom are frozen out of the dynamics and the Universe approaches an isotropic (the relic anisotropy, being no longer dynamical, can be ruled out by redefining the 1-forms $\sigma^\iota$’s) and scalar-field-free expansion.

Now, the validity of our interpolating solution requires the possibility to really neglect the whole potential term $U(R, \phi)$ near enough to the cosmological singularity. The presence of the scalar field potential term is surely crucial to generate the inflationary scenario, but, sufficiently close to the initial “Big-Bang”, its dynamical role is expected to be very limited; in fact, if we neglect the potential term, then,
remembering that for early time \( R \sim \sqrt{T} \Rightarrow H \equiv (1/R)dR/dT \sim 1/2T \), we get the classical free field solution \( \phi \propto \ln T \). Hence the kinetic term of the field reads of order \( O(1/T^2) \); therefore, having in the limit toward the “Big-Bang” \( (T \to 0) \) that \( T^2V_T(\phi(T)) \to 0 \), we can conclude that the Coleman-Weinberg potential is asymptotically negligible (this behavior remains valid for almost all inflationary potentials).

Taking into account such classical analysis, we assume that, during the Planck epoch, when the Universe performed its quantum evolution, the potential of the scalar field plies no significant role.

Instead, in order to neglect the ultrarelativistic energy density and the Bianchi IX potential, with respect to \( \varepsilon \), we have to require that, respectively, the following two conditions hold:

\[
R \gg \frac{\mu^2}{|\varepsilon|},
\]

\[
R(V(\beta_{\pm}(R)) \ll \frac{\ell_{Pl} |\varepsilon|}{\hbar c},
\]

Since, as shown in \ref{14}, for \( R \to 0 \) the term \( R \left| V - 1 \right| \) approaches zero, because of the scalar field presence, then the above inequality \ref{35} reduces to the simpler one \( R \ll R^* \) (being \( R^* = R^*(\pi_{\pm}, \varepsilon) \)). Thus, by \ref{35}, the validity of our approach is ensured by the inequality

\[
\mu \ll \sqrt{R^* |\varepsilon|}.
\]

5 The generic cosmological solution

In this section we show how the analysis above developed can be extended locally to a generic inhomogeneous cosmological model \cite{15} (see also \cite{16,17}). The leading idea in such an upgrading of our homogeneous picture consists of observing that, near the singularity, the generic cosmological solution can be approached as a long-wavelength one. From a quantum point of view, a long-wavelength evolution corresponds to neglect the spatial gradients of the dynamical variables, so reducing the Wheeler superspace to the direct product of \( \infty^3 \) minisuperspaces.

However, as we will see, from the potential structure it comes out that the terms containing the spatial gradients are of higher order near enough to the “Big-Bang”. Therefore below we will adopt the long-wavelength approximation because the quantum behavior of the real Universe had to be confined to the Planck era. Indeed to neglect the potential as a whole corresponds to take the quantum information as independent over causally disconnected regions.

We start by observing how, in the ADM formalism, the line element of a generic inhomogeneous cosmological model reads as

\[
ds^2 = -N^2dt^2 + \gamma_{\alpha\beta}(dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt),
\]
where the 3-metric tensor $\gamma_{\alpha \beta}$ is provided by

$$\gamma_{\alpha \beta} dx^\alpha dx^\beta = R^2(t, x^\sigma) \left( e^{2i(\beta(t,x^\sigma))} \right)_{ij} \sigma^i(x^\sigma) \sigma^j(x^\sigma), \quad (38)$$

Above by $\alpha, \beta = 1, 2, 3$ we denote the spatial indices and the 1-forms are now constructed as $\sigma^i = l^i_\alpha(x^\sigma) dx^\alpha$: the components of the vectors $l^i_\alpha$ correspond to arbitrary functions of the spatial coordinates and, therefore no special symmetry is assumed. The generality of this model implies also that the shift vector $N^\alpha$ can be taken no longer zero.

Then, by adopting the same parameterization (39), the gravity-“matter” action resembles, in a Gaussian reference ($N = 1$ and $N^\alpha = 0$), the form

$$S_{inh} = \int \left\{ pR \frac{\partial R}{\partial T} + \sum_r \left( p_r \frac{\partial \beta_r}{\partial T} - H(x^\sigma) \right) \right\} d^3x dT. \quad (39)$$

Here $r = \pm, \phi$ and, by $H(x^i)$, we denote the following point dependent Hamiltonian term

$$H(x^\sigma) = \frac{4\pi}{3J} \left\{ \frac{p^2_R}{R} + \frac{1}{R^3} \left( p_+^2 + p_-^2 + \frac{3}{8\pi} p_\phi^2 \right) + \mathcal{U} \right\}, \quad (40)$$

where $J \equiv l^1 \cdot l^2 \wedge l^3$ (the scalar and vector products are taken by treating the spatial coordinates as Euclidean ones) and the potential term $\mathcal{U}$ is defined by

$$\mathcal{U} = \frac{3R}{128\pi^2} \left\{ a_1^2(x^\sigma) e^{-8\beta_+} + a_2^2(x^\sigma) e^{4(\beta_+ + \sqrt{3}\beta_-)} + a_3^2(x^\sigma) e^{4(\beta_- - \sqrt{3}\beta_-)} + W(x^\sigma, R, \beta_\pm, \partial_\alpha R, \partial_\alpha \beta_\pm, \partial_\alpha \beta_\pm) \right\} + \frac{3R^3}{4\pi} V_\gamma(\phi) + \frac{\mu^2(x^\sigma)}{R} \quad (41)$$

Above, by $a_i (i = 1, 2, 3)$, we refer to the space quantities

$$a_i (x^\sigma) \equiv l^i \cdot rot l^i, \quad (42)$$

where we regard again the operations $\wedge$ and $rot$ in Euclidean sense.

To outline the relative behavior in the potential terms, as the singularity is approached for $R \to 0$, let us introduce the new variables

$$D \equiv R^3$$

$$H_1 \equiv \frac{1}{3} + \frac{\beta_+ + \sqrt{3}\beta_-}{3 \ln R}$$

$$H_2 \equiv \frac{2}{3} + \frac{\beta_- - \sqrt{3}\beta_-}{3 \ln R}$$

$$H_3 \equiv \frac{1}{3} - \frac{2\beta_-}{3 \ln R}$$

$$\sum_i H_i = 1. \quad (43)$$
Taking into account these definitions, the potential $U$ rewrites as follows

$$U = \sum_i (a_i^2 D^{4H_i}) + W \quad (44)$$

$$W \sim \sum_{j \neq k} \mathcal{O} (D^{2(H_j + H_k)}) ; \quad (45)$$

Now it is easy to realize that, near the cosmological singularity ($D \to 0$), the term $W$ becomes negligible. Indeed, this conclusion is supported by the classical behavior of the spatial gradients, which does not destroy the feature above outlined (see below for the classical solution of the model).

Summarizing, asymptotically, the system is described, in a Gaussian reference, by the action (39), of which super-Hamiltonian reads as follows:

$$H(x^\alpha) = -\frac{p_R^2}{R} + \frac{1}{R^3} \left( p_r^2 + \frac{3}{8\pi} p_\phi^2 \right) +$$

$$\frac{3R}{128\pi^2} \left\{ a_1^2(x^\alpha)e^{-8\beta_+} + a_2^2(x^\alpha)e^{4(\beta_+ + \sqrt{3}\beta_-)} + a_3^2(x^\alpha)e^{4(\beta_+ - \sqrt{3}\beta_-)} \right\} +$$

$$\frac{3R^3}{4\pi} V(\phi) + \frac{\mu^2}{R} . \quad (46)$$

We recognize that, in this asymptotic form, the dynamics of the generic cosmological solution corresponds to extend, in each space point, the same evolution discussed for the Bianchi IX model.

The absence of spatial gradients of the dynamical variables (the function $a_i(x^\alpha)$ specify the considered inhomogeneous model) implies that, if we parameterize the Gaussian time as $T = T(t, x^\alpha)$, then the new action takes the form

$$S_{Inh} = \int \left\{ p_R \frac{\partial R}{\partial t} + \sum_r \left( p_r \frac{\partial \beta_r}{\partial t} + p_T \frac{\partial T}{\partial t} - \Lambda(t, x^\alpha)(p_T + H(x^\alpha)) \right) \right\} d^3xdt . \quad (47)$$

Above we adopted the same procedure developed for the homogeneous case in section 3; here $p_T$ plays the role of conjugate momentum to the Gaussian time, while $\Lambda$ denotes a Lagrangian multiplier for which takes place the relation

$$\Lambda(t, x^\alpha) = \frac{\partial T}{\partial t} = N(t, x^\alpha) . \quad (48)$$

The canonical quantization of this dynamical system is achieved by implementing on operator level the super-Hamiltonian constraint, i.e., we have to require that it annihilates the state functional $\Psi(T(x^\alpha), R(x^\alpha), \beta_\pm(x^\alpha), \phi(x^\alpha))$; therefore, the quantum dynamics is described by the following $\hbar^3$ Schrödinger equations

$$i\hbar \frac{\delta \Psi}{\delta T(x^\alpha)} = \hat{H}(x^\alpha) \Psi . \quad (49)$$
where now the momentum operators are expressed in terms of functional derivatives (instead of ordinary ones like in the homogeneous case).

If we introduce the definition \( \partial_t \equiv \int_{\Sigma^3_t} (\delta \Psi / \delta T) \partial_t T d^3 x \) (being \( \Sigma^3_t \) the one parameter family of spatial hypersurfaces filling the space-time), then, taking into account the relation (48), the above system of \( \infty^3 \) (independent) equations (49) can be smeared as follows

\[
i \hbar \partial_t \Psi = \hat{H} \Psi \equiv \int_{\Sigma^3_t} d^3 x \left( N \hat{H}(x^\alpha) \right) \Psi.
\]  

(50)

Since here the lapse function must be regarded as assigned, then \( T \) is known by (48) and the wave functional depends directly on the label time \( t \), i.e. \( \Psi = \Psi(t, R, \beta_\pm, \phi) \).

It remains to require that the wave functional \( \Psi \) be invariant under space diffeomorphisms, i.e. transformations of the form \( x^\alpha \rightarrow x^\alpha + \xi^\alpha \), where \( \xi^\alpha \) denotes a generic infinitesimal displacement; as effect of this transformation \( \Psi \) changes by the amount:

\[
\delta \Psi = \int_{\Sigma^3_t} d^3 x \left\{ \left[ \frac{\delta \Psi}{\delta R} \frac{\partial}{\partial \beta_+} \frac{\partial}{\partial \alpha} \beta_+ + \frac{\delta \Psi}{\delta \beta_-} \frac{\partial}{\partial \alpha} \beta_- + \frac{\delta \Psi}{\delta \phi} \frac{\partial}{\partial \alpha} \phi \right] \xi^\alpha \right\}.
\]  

(51)

Since \( \xi^\alpha \) is a generic 3-vector, then \( \delta \Psi \) vanishes only if the following equation holds

\[
\frac{\delta \Psi}{\delta R} \frac{\partial}{\partial \alpha} R + \frac{\delta \Psi}{\delta \beta_+} \frac{\partial}{\partial \alpha} \beta_+ + \frac{\delta \Psi}{\delta \beta_-} \frac{\partial}{\partial \alpha} \beta_- + \frac{\delta \Psi}{\delta \phi} \frac{\partial}{\partial \alpha} \phi = 0.
\]  

(52)

This equation corresponds to the super-momentum constraint which has to appear because of the transformation \( T = T(t, x^\alpha) \).

It is worth noting how equations (50) and (52) coincides with the implementation to the present case of the analysis developed in [1]. The general nature of our cosmological model makes such a coincidence of physical interest; in fact, either the present analysis, as well as that one outlined in [1] lead to the issue that to fix a reference frame before quantizing the metric field implies that a time evolution is restored in the dynamics.

Equation (50) is reduced to an eigenvalues problem, as soon as we expand the wave functional in the form

\[
\Psi = \int_{\mathcal{F}} D\varphi \chi(\varphi, R, \beta_\pm, \phi) \exp \left\{ \frac{i}{\hbar} \int_{t_0}^t dt T\int_{\Sigma^3_t} d^3 x N(t, x^\alpha) \varphi(x^\alpha) \right\},
\]  

(53)

where \( D\varphi \) denotes the Lebesgue measure on the functional space \( \mathcal{F} \); in fact, substituting (53) into (50), we get

\[
\hat{H}(x^\alpha) \chi = \varphi(x^\alpha) \chi.
\]  

(54)

For the generic cosmological solution, this equation explicitly reads
\[
\left[ \frac{\hbar^2}{\delta R} \frac{1}{R} \frac{\delta}{\delta R} - \frac{\hbar^2}{R^3} \left( \frac{\delta^2}{\delta \beta^2_+} + \frac{\delta^2}{\delta \beta^2_-} + \frac{3h^2}{8\pi} \delta^2 \right) + U \right] \Psi = \varrho \Psi. \tag{55}
\]

Point by point in space, the above equation resembles the corresponding one for the Bianchi IX model, i.e. \([23]\); therefore all the results following in Section 3 and 4 hold as extended to the inhomogeneous case. In particular, it is worth noting that, even for the generic case, to quantize the system in a Gaussian frame, endow the quantum dynamics with a time evolution and allow to define an Hilbert space for the states of the theory. Furthermore, it can be shown, along the same lines of section 4, that in the semiclassical limit \(\varrho(x)\) induces the Universe isotropization. In fact, the ground state of the Universe has to correspond everywhere to a negative energy eigenvalue (i.e. \(\forall x^a : \varrho < 0\)), because of the super-Hamiltonian is not positive definite; such a ground state, if it is stable, (i.e. the negative spectrum of energies is bounded by below), corresponds, in the classical limit, to the phenomenology of a dust fluid filling the Universe and having energy density \(-\rho/R^3\).

We conclude this section by showing how the model here considered corresponds, in the classical limit, to a generic inhomogeneous model. We recall that the pure gravitational field, in the general picture, requires to be described four physically arbitrary functions of the spatial coordinates.

Once taken the following expansion for the wave functional

\[
\Psi = \exp \left\{ \frac{i}{\hbar} \Sigma(R, \beta, \phi) \right\} \quad \Sigma = \Sigma_0 + \frac{\hbar}{i} \Sigma_1 + \left( \frac{\hbar}{i} \right)^2 \Sigma_2 + \ldots, \tag{56}
\]

in the limit \(\hbar \to 0\), the zero order of approximation, to the eigenvalue problem \([22]\) and to the super-momentum constraint \([22]\), yields the following Hamilton-Jacobi equations

\[
- \frac{1}{R} \left( \frac{\delta \Sigma_0}{\delta R} \right)^2 + \frac{1}{R^3} \left( \frac{\delta \Sigma_0}{\delta \beta_+} \right)^2 + \frac{3}{8\pi} \left( \frac{\delta \Sigma_0}{\delta \phi} \right)^2 + U - \varrho = 0. \tag{57}
\]

\[
\frac{\delta \Sigma_0}{\delta R} \partial_\alpha R + \frac{\delta \Sigma_0}{\delta \beta_+} \partial_\alpha \beta_+ + \frac{\delta \Sigma_0}{\delta \beta_-} \partial_\alpha \beta_- + \frac{\delta \Sigma_0}{\delta \phi} \partial_\alpha \phi = 0. \tag{58}
\]

In the asymptotic limit toward the cosmological singularity \((R \to 0)\), the solution to these equations reads, for an expanding Universe, as

\[
\Sigma_0 = \int_{\Sigma_0^2} \left( -K(x^a) \ln R + K_+(x^a) \beta_+ + K_-(x^a) \beta_- + K_{\phi}(x^a) \phi \right) d^3x, \tag{59}
\]

with

\[
K^2 = K_+^2 + K_-^2 + \frac{3}{8\pi} K_{\phi}^2. \tag{60}
\]
Indeed if we take the functional derivatives of $\Sigma_0$ with respect to the $K$’s and equate them to space functions, then the anisotropies and the scalar field acquire the following dependence on $R$:

$$
\beta_{\pm}(R) = -\frac{K_{\pm}}{K} \ln R + \beta_{\pm}^*(x^\alpha) \quad \phi = -\frac{K_{\phi}}{K} \ln R + \phi^*(x^\alpha) .
$$

(61)

or, by an obvious position, the above expressions rewrite as

$$
\beta_{\pm}(R) = -\Pi_{\pm} \ln R + \beta_{\pm}^*(x^\alpha) \quad \phi = -\Pi_{\phi} \ln R + \phi^*(x^\alpha) .
$$

(62)

By (61), once taken into account the additional constraint

$$
\Pi_{\pm}^2 (x^\alpha) + \Pi_{\phi}^2 (x^\alpha) + \Pi_{\phi}^2 (x^\alpha) = 1 ,
$$

(63)

then equation (57) is automatically satisfied by (62), while the validity of equation (58) implies that

$$
\Pi_{\pm} \partial_\alpha \beta_{\pm}^* + \Pi_{\phi} \partial_\alpha \beta_{\pm}^* + \Pi_{\phi} \partial_\alpha \phi^* = 0 .
$$

(64)

In the considered approximation and in view of solution (62), the potential $U$ is of higher order with respect to the retained terms which behave like $O(1/R^3)$. The functions $\beta_{\pm}^*$ do not correspond to real new degrees of freedom because, by the metric tensor (38), we see that they can be included in the definition of the vectors $l^i_\alpha$; in this respect, we can think of these functions like two of the nine components of the 1-forms vectors.

Since the six functions $\Pi_{\pm}, \Pi_{\phi}, \beta_{\pm}^*, \phi^*$ have to satisfy the four equations (58) and (64), only two of them are really available for the Cauchy problem; adding these two free functions to the remaining seven components of the vectors $l^i_\alpha$ (two components were identified with $\beta_{\pm}^*$), we arrive to nine independent functions. However, taking into account the space diffeomorphisms $x^\alpha = x^{\alpha'}(x^\alpha)$ to kill other three degrees of freedom, we see that our solution contains the right number of physically arbitrary functions of spatial coordinates associated to the generality. In fact four functions correspond to the generic gravitational field and the remaining two are available for a generic scalar field; finally we observe that the functions $\mu$ and $\vartheta$ are not affected by any restriction. These considerations show how our model describes, on a classical level, a generic inhomogeneous cosmology and therefore, some of the features above outlined by its quantum dynamics have general validity.

Indeed the reduction of the superspace into the direct product of $\mathbb{R}^3$ minisuperspaces, each for each space point, is based on the long-wavelength approximation and therefore it is due to the structure of the Einstein theory near the cosmological singularity; but the appearance of a non-zero eigenvalue of the super-Hamiltonian as a consequence of the frame fixing constitutes a general picture.
6 Concluding remarks

In order to summarize the outcomes of this work, we can arrange them into the following main points.

i) By extending to the minisuperspace the same method for quantizing, via an Hamiltonian constraint, the non-relativistic particle, we could perform the canonical quantization of a Bianchi type IX model (in presence of ultrarelativistic matter and a real self-interacting scalar field), as viewed in a Gaussian reference frame. The issue of such a procedure consists of a quantum dynamics having a Schrödinger morphology, an appropriate Hilbert space of its states and a corresponding notion of probability density for the system configuration.

The main difference between such a canonical quantization in a fixed frame and the Wheeler-DeWitt one, consists of the super-Hamiltonian spectrum; in fact the “frame fixing” removes the time displacements invariance of the theory and restores non-zero super-Hamiltonian eigenvalues.

ii) When taking the semiclassical limit of this quantum theory, via a standard WKB approach, we get the H-J equation corresponding to the considered model, but with the appearance of a new energy contribution, which reflects the classical outcoming of the no longer zero eigenvalue of the super-Hamiltonian. If we argue that the Universe is forced to approach the (quantum) state of minimal energy and observe that the super-Hamiltonian is non-positive definite, then it is natural that the system settle down, in the classical limit, into a state of negative value of $\epsilon$.

iii) The negative nature of $\epsilon$ implies that the new term appearing in the dynamics resembles a non-relativistic matter component and therefore contributes to the total critical parameter of the Universe as “dark matter” component does.

Furthermore we have shown how this non-relativistic matter, under rather general conditions, becomes dominant and drives a process of isotropization which brings the Universe from a Kasner-like regime to a stage of frozen anisotropies and scalar field.

For a detailed discussion on the cosmological implications of such additional dark matter component, in the case of a closed Robertson-Walker model (i.e. the actual one with $\beta_+ = \beta_- = 0$), see 19.

iv) Near the cosmological singularity, the minisuperspace picture above outlined can be extended, point by point in space, to the generic cosmological solution; From a physical point of view this result is equivalent to claim that, within each causal region, the quantum behavior is completely equivalent to that one proper of a Bianchi IX model; therefore all the results, as described by the above three points, hold locally for a generic inhomogeneous cosmology.

In the quantum regime, the reduction of the superspace to the direct product of $\infty^3$ minisuperspaces, corresponds to adopting the long-wavelength approximation, i.e. to neglect the spatial gradients of the dynamical variables. We stress, in this respect, that the long-wavelength feature is dynamically induced by the asymptotic classical evolution and therefore we argue that it survives in the quantum dynamics which takes place just during the Planckian era toward the initial singularity.
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