Abstract. We use Busemann functions to construct volume preserving mappings in an asymptotically harmonic manifold. If the asymptotically harmonic manifold satisfies the visibility condition, we construct mappings which preserve distances in some directions. We also prove that some integrals on the intersection of horospheres are independent of the differences between the values of the corresponding Busemann functions and we establish an upper bound of the volume of the intersection of two horospheres which is independent of the difference between values of corresponding Busemann functions.

1. Introduction

Let \( (M, g) \) be a Hadamard manifold, that is, simply connected, complete Riemannian manifold of non-positive sectional curvatures. Let \( d(p, q) \) be the distance between \( p, q \in M \). For each unit tangent vector \( v \) to \( M \), the Busemann function \( b_v : M \to \mathbb{R} \) on \( (M, g) \) is defined by

\[
b_v(x) = \lim_{t \to \infty} (d(x, \gamma_v(t)) - t),
\]

where \( \gamma_v : [0, \infty) \to M \) is a geodesic ray such that \( \gamma_v'(0) = v \). Busemann functions on a Hadamard manifold are convex \([7]\) and \( C^2 \) \([10]\). The level hypersurfaces of Busemann functions are called horospheres. A Riemannian manifold is called harmonic, if, about any point, the geodesic spheres of sufficiently small radii are of constant mean curvature. A Hadamard manifold \( (M, g) \) is called asymptotically harmonic if the mean curvatures of its horospheres are constant \([9]\). Asymptotically harmonic manifolds belong to a special class of Riemannian manifolds. All harmonic manifolds are asymptotically harmonic, and all known asymptotically harmonic manifolds are harmonic and homogeneous \([9]\). It is also remarkable that the following question is remained open: Is a harmonic manifold homogeneous?

To consider the homogeneousness of a harmonic manifold and, more generally, of an asymptotically harmonic manifold, we construct a volume preserving mapping which maps one point to another point. For this construction, we use the mean curvature of horospheres and the variation of the volume density for the flow in the orthogonal direction to horospheres.

Let \( C_c(M) \) be the set of continuous functions with compact support and \( T^1M \) be a unit tangent bundle. We obtain the following theorem.

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Theorem 1.1. Let $(M, g)$ be an asymptotically harmonic manifold. Then, for all points $p \neq q \in M$, there exists a diffeomorphism $F : M \rightarrow M$ such that $F(p) = q$ and
\[
\int_{M} f(x) d\mu(x) = \int_{M} f(F(x)) d\mu(x),
\]
for all $f \in C_c(M)$, where $d\mu$ is the Riemannian measure on $M$.

Szabó [14] proved that a Riemannian manifold is a harmonic manifold if and only if the volume of the intersection of geodesic spheres depends only on the radii and the distance between the centers of the geodesic spheres. Csikós and M. Horváth [3, 5] proved that the intersections can be restricted to the cases with same radii, and they also proved that the volume of a tubular neighborhood about a geodesic depends only on the length of the geodesic and the radius if and only if the Riemannian manifold is harmonic. In addition, some relations between integrals and measures on a harmonic manifold and its ideal boundary were found by Itoh and Satoh [11] and by Rouvière [13]. Knieper and Peyerimhoff [12] also considered the integrals and measures on harmonic manifolds to find a solution of the Dirichlet problem at infinity, and Biswas, Knieper, and Peyerimhoff [2] proved that there exists a Fourier transform between harmonic manifolds and its ideal boundary.

A Hadamard manifold $(M, g)$ is called a visibility manifold, or satisfies the visibility condition, if, for any two different points $v_1$, $v_2$ at infinity, there is a geodesic $\gamma : \mathbb{R} \rightarrow M$ with $\gamma(\infty) = v_1$, $\gamma(-\infty) = v_2$ (see section 2). All intersections of two horospheres in a Hadamard manifold are bounded if and only if the Hadamard manifold is a visibility manifold (see Definition 2.6 and Lemma 2.7). We note that the volume of an intersection of two horospheres in a visibility manifold is finite. In particular, for every harmonic manifold which satisfies the visibility condition, the volume of the intersection $b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2)$ of two horospheres is independent of $c_1 - c_2$. For asymptotically harmonic manifolds which satisfy the visibility condition (i.e., asymptotically harmonic, visibility manifolds), in this article, we prove that some integrals on the intersection of two horospheres are independent of the difference between values of corresponding Busemann functions. We also obtain an upper bound of the volume of the intersection of two horospheres. Throughout this paper, we assume that the dimension of the manifold is $n \geq 2$. Our main theorem is the following:

Theorem 1.2. Let $(M, g)$ be an asymptotically harmonic, visibility manifold. Let $p \in M$, $v_1 \neq v_2 \in T_p^u M$, and $c \in \mathbb{R}$. Then there exists a constant $C > 0$ such that the $(n - 2)$-dimensional volume of the intersection $b_{v_1}^{-1}(c - t) \cap b_{v_2}^{-1}(c + t)$ is less than $C$, for all $t \in \mathbb{R}$.

2. Preliminary

The integration of a function on a connected Riemannian manifold $(M, g)$ can be computed in terms of the integrals on hypersurfaces:

Proposition 2.1. [6] Let $(M, g)$ be a connected Riemannian manifold. Let $\varphi : M \rightarrow \mathbb{R}$ be a $C^1$ function such that $\nabla \varphi$ is non-vanishing on $M$, and let $S_t$ be the hypersurface defined by $S_t = \{x \in M : \varphi(x) = t\}$, for all $t \in \mathbb{R}$. Then, for all $f \in C_c(M)$ and $t \in \mathbb{R}$,
\[
\int_{M} f(x) d\mu(x) = \int_{\mathbb{R}} \int_{S_t} \frac{f(x)}{\|\nabla \varphi(x)\|} d\mu_t(x) dt.
\]
where $d\mu$ is the Riemannian measure on $M$ and $d\mu_t$ is the induced Riemannian measure on $S_t$.

We introduce an infinitesimal “volume preserving” mapping on $(M,g)$ in $\mathbb{R}$. We say that a vector field $X$ on $M$ is volume preserving if $L_X(d\mu) = 0$, where $L_X$ is the Lie derivative with respect to $X$. If $\phi_t$ is the flow generated by $X$, then we call $\phi_t$ volume preserving if $X$ is volume preserving. In that case, we have $(\phi_t)^*(d\mu) = d\mu$. This flows preserve integrals on a Riemannian manifold $(M,g)$ since it preserves the Riemannian measure if it is a diffeomorphism.

**Proposition 2.2.** Let $\phi_t$ be a flow on a Riemannian manifold $(M,g)$ and suppose that $\phi_t$ is a diffeomorphism on $M$ for all $t \in \mathbb{R}$. Then $\phi_t$ is volume preserving if and only if

$$\int_M f(x)d\mu(x) = \int_M f(\phi_t(x))d\mu(x),$$

for all $f \in C_c(M)$, where $d\mu$ is the Riemannian measure on $M$.

**Proof.** If $\phi_t$ is volume preserving, then, clearly, equation $[1]$ holds. Now, suppose that equation $[1]$ holds for all $f \in C_c(M)$. Then

$$\int_M f(\phi_t(x))d\mu(x) = \int_M f(x)d\mu(x) = \int_M f(\phi_t(x))(\phi_t)^*(d\mu)(x)$$

for all $f \in C_c(M)$. We denote $(\phi_t)^*(d\mu) = f_0 d\mu$ for some function $f_0$. Then, by equation $[2]$, $f_0 = 1$ and $(\phi_t)^*(d\mu) = d\mu$. $\square$

Now, we define asymptotic geodesic rays:

**Definition 2.3.** [3] Let $(M,g)$ be a Hadamard manifold, and $\gamma_1, \gamma_2 : [0,\infty) \to M$ geodesic rays. $\gamma_1, \gamma_2$ are said to be asymptotic if there exists a constant $C > 0$ such that

$$d(\gamma_1(t), \gamma_2(t)) \leq C,$$

for all $t \geq 0$.

This gives an equivalence relation on geodesic rays: two geodesic rays are equivalent if and only if they are asymptotic. The set $\partial_\infty M$ of points at infinity is the set of equivalence classes of this relation. This is also called the ideal boundary of $(M,g)$. Consequently, there exists a set of distance functions from each point and their limits. Let $C(M)$ be the space of continuous functions on $M$ equipped with the topology of uniform convergence on bounded subsets. Let $C_*(M)$ denote the quotient of $C(M)$ by the 1-dimensional subspace of constant functions. For all $v \in T^1 M$ and $t \in \mathbb{R}$,

$$\nabla b_v(\gamma(t)) = -\gamma'(t),$$

where $\gamma$ is a geodesic ray in $(M,g)$ asymptotic to $\gamma_v$. In particular, the image of Busemann functions in $C_*(M)$ can be associated to geodesic rays. The points at infinity also corresponds to the images of Busemann functions in $C_*(M)$.

**Proposition 2.4.** [3] Let $(M,g)$ be a Hadamard manifold. Then the Busemann functions associated to asymptotic geodesic rays in $M$ are equal up to addition of a constant.

**Proposition 2.5.** [3] Let $(M,g)$ be a Hadamard manifold, $\theta \in \partial_\infty M$, and $x \in M$. Then there exists a unique geodesic ray $\gamma \in \theta$ from $x$. 

By the propositions, points at infinity bijectively correspond to images of Busemann functions in \( C_c(M) \).

For two elements of \( \partial_\infty M \), there exists a geodesic from one direction to another direction. If such geodesic exists for all pairs of distinct elements of \( \partial_\infty M \), then we call such a Hadamard manifold a visibility manifold:

**Definition 2.6.** A Hadamard manifold \((M, g)\) is called a visibility manifold if, for all \( p \in M \) and \( v_1 \neq v_2 \in T_p^\infty M \), there exists a geodesic ray \( \gamma \) such that \( \gamma \) is asymptotic to \( \gamma_{v_1} \), and the geodesic ray \( t \mapsto \gamma(-t) \) is asymptotic to \( \gamma_{v_2} \).

Such geodesic ray \( \gamma \) is said to be bi-asymptotic to \( v_1, v_2 \). There exist several equivalent conditions for the visibility, one of which is as follows.

**Lemma 2.7.** Let \((M, g)\) be a Hadamard manifold. Then the following statements are equivalent:

1. \((M, g)\) is a visibility manifold.
2. \( b_{v_1}^{-1}((-\infty, c_1)) \cap b_{v_2}^{-1}((-\infty, c_2)) \) is bounded for all \( p \in M \), \( v_1 \neq v_2 \in T_p^\infty M \), and \( c_1, c_2 \in \mathbb{R} \).

For example, if a Hadamard manifold \((M, g)\) satisfies the curvature condition \( K \leq -a^2 < 0 \), for some \( a \in \mathbb{R} \), then \((M, g)\) is a visibility manifold [1]. In a Hadamard manifold, bi-asymptotic geodesics are normal geodesics of some intersection of horospheres of the form \( b_{v_1}^{-1}(0) \cap b_{v_2}^{-1}(0) \):

**Proposition 2.8.** Let \((M, g)\) be a Hadamard manifold. Then, for every \( v \in T^1 M \), \( b_{v_1}^{-1}(0) \cap b_{v_2}^{-1}(0) \) is connected,

\[
\nabla b_v(x) + \nabla b_{-v}(x) = 0,
\]

for all \( x \in b_{v_1}^{-1}(0) \cap b_{v_2}^{-1}(0) \), and the geodesics which is asymptotic to \( \gamma_v \) and intersects \( b_{v_1}^{-1}(0) \cap b_{v_2}^{-1}(0) \) orthogonally at a point are bi-asymptotic to \( v, -v \).

Consequently, for two distinct bi-asymptotic geodesics, there exists a 2-dimensional flat, totally geodesic embedding containing them:

**Theorem 2.9.** Let \((M, g)\) be a Hadamard manifold. Then, for all \( v \in T^1 M \), the Busemann function \( b_v \) is convex, and the set \( b_{v_1}^{-1}(0) \cap b_{v_2}^{-1}(0) \) is convex. If two geodesics \( \gamma_1, \gamma_2 \) are bi-asymptotic, there exists \( a > 0 \) and a totally geodesic, isometric embedding \( F : [0, a] \times \mathbb{R} \to M \) such that \( \gamma_1 = F|_{\{0\} \times \mathbb{R}} \) and \( \gamma_2 = F|_{\{a\} \times \mathbb{R}} \).

Let \( p \in M \), \( v_1 \neq v_2 \in T^1_p M \), and

\[
D = \{ x \in M : \nabla b_{v_1}(x) + \nabla b_{v_2}(x) = 0 \}.
\]

Note that \( D \) is closed and \( D \) is nonempty if \((M, g)\) is a visibility manifold. For every point \( x \in D \), by Proposition 2.8 there exists a unique geodesic ray from \( x \) which is asymptotic to \( v_1 \), so it is bi-asymptotic to \( v_1, v_2 \). In particular, by Theorem 2.9 \( D \) is connected. Suppose that \( D \) is nonempty, so \( D \) contains a bi-asymptotic geodesic to \( v_1, v_2 \). Let \( c_0 \) be the constant value of \( b_{v_1} + b_{v_2} \) on \( D \). If \( c_1, c_2 \in \mathbb{R} \) and \( c_1 + c_2 = c_0 \), then there exists a point \( x \) on each bi-asymptotic geodesic such that \( b_{v_1}(x) = c_1 \) and \( b_{v_2}(x) = c_2 \), and

\[
b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2) = b_{v_1}^{-1}(0) \cap b_{v_2}^{-1}(0) \subseteq D,
\]

where \( v = \nabla b_{v_1}(x) \).
Lemma 3.1. \[
\text{Lemma 3.1.} \]

Let \( p, q \in M \). Since \((M, g)\) is a Hadamard manifold, there exists a unit-speed geodesic \( \gamma \) such that \( \gamma(0) = p \), \( \gamma(t_0) = q \) for some \( t_0 > 0 \). Set \( v = \gamma'(0) \). Let \( S_t = b_v^{-1}(t) \) be the level set of \( b_v \) for all \( t \in \mathbb{R} \). Consider a diffeomorphism \( \phi : \mathbb{R} \times S_0 \to M \) defined by\[
\phi_t(x) = \phi(t, x) := \exp_x \left( t \nabla b_v(x) \right),
\]
for all \((t, x) \in \mathbb{R} \times S_0\).

For all \( w \in T^1 M \), we denote the space of orthogonal tangent vectors to \( w \) by \( w^\perp \). Define \( U = U(t) : \nabla b_v(\phi_t(x))^\perp \to \nabla b_v(\phi_t(x))^\perp \) by
\[
U(w) := \nabla_w \nabla b_v,
\]
for all \( t \in \mathbb{R} \) and \( w \in \nabla b_v(\phi_t(x))^\perp \). Since \((M, g)\) is asymptotically harmonic, \( \text{tr} U = -\Delta b_v = h \) for some constant \( h \in \mathbb{R} \). Since \( b_v \) is convex by Theorem 2.9, \( h \geq 0 \).

**Lemma 3.1.** For every \( t \in \mathbb{R} \),
\[
(\phi_t)^*(d\mu_t) = e^{ht}d\mu_0.
\]

**Proof.** For all \((t, x) \in \mathbb{R} \times S_0\),
\[
(\phi_t)_* x \left( \frac{\partial}{\partial t} \right) = \nabla b_v(\phi_t(x)).
\]
Then
\[
(4) \quad (\phi_t)_* x (w) = \exp_{\nabla b_v(x)} (w, tU(w)).
\]
Set \( x \in S_0 \) and \( w \in \nabla b_v(x)^\perp \). Define
\[
\Gamma(s, t) := \exp_{\sigma(s)}(t \nabla b_v(\sigma(s))),
\]
for all \( s \in (-\varepsilon, \varepsilon) \) and \( t \in \mathbb{R} \), where \( \sigma = \sigma(s) \) is the curve in \( S_0 \) such that \( \sigma(0) = x \) and \( \sigma'(0) = w \) and \( \varepsilon > 0 \).

We denote \( \partial_t \Gamma = \Gamma_*(\frac{\partial}{\partial t}) \), \( \partial_s \Gamma = \Gamma_*(\frac{\partial}{\partial s}) \), and \( J(t) = \partial_s \Gamma(0, t) \), for all \( t \in \mathbb{R} \). So, \((\phi_t)_* x \left( \frac{\partial}{\partial x} \right) = \nabla b_v(\phi_t(x)) \) and \((\phi_t)_* x (w) = J(t) \), for all \( t \in \mathbb{R} \). Then, from equation (4), we have
\[
U(J(t)) = \nabla_{\partial_t \Gamma(0, t) \partial_t \Gamma} \Gamma = \nabla_{\partial_t \Gamma(0, t) \partial_t \Gamma} = J'(t),
\]
for all \( t \in \mathbb{R} \). Thus, we get
\[
U = A' A^{-1},
\]
on \( \nabla b_v(\phi_t(x))^\perp \), where \( t \in \mathbb{R} \), \( R_t \) is the Jacobi operator along \( t \to \phi_t(x) \), and \( A = A(t) \) is the \((1, 1)\) tensor field on \( \nabla b_v(\phi_t(x))^\perp \) such that \( A'' + R_t A = 0 \). Since \((M, g)\) is asymptotically harmonic, \( \text{tr} U = h \) is constant. So,
\[
(\ln(\det A(t)))' = \text{tr} U(t) = h,
\]
and \( \det A(t) = e^{ht} \). Hence,
\[
(\phi_t)^*(d\mu_t(\phi_t(x))) = d\mu_t ((\phi_t)_* x (w_1), \ldots, (\phi_t)_* x (w_{n-1})) d\mu_0(x) = \det A(t) d\mu_0(x) = e^{ht}d\mu_0(x),
\]
where \( w_1, \ldots, w_{n-1} \) are orthonormal tangent vectors to \( S_0 \) at \( x \). \( \square \)
Define $F : M \to M$ by
$$F(\phi(t, x)) := \phi(\alpha(t), x),$$
for all $(t, x) \in \mathbb{R} \times S_0$, where $\alpha = \alpha(t)$ is a smooth function on $\mathbb{R}$ such that $\alpha(0) = t_0$.

**Corollary 3.2.** For every $t \in \mathbb{R}$,
$$F^*(d\mu_{\alpha(t)}) = e^{h\alpha(t) - ht}d\mu_t.$$

**Proof.** For all points $x \in S_t$,
$$F(x) = \exp_x((\alpha(t) - t)\nabla b_v(x)),$$
So, similarly to Lemma 3.1, Corollary 3.2 holds. \(\square\)

Therefore, by Proposition 2.1 and Corollary 3.2,
$$\int_M f(x)d\mu(x) = \int_{\mathbb{R}} \int_{S_t} f(F(x))d\mu_{\alpha(t)}(x)dt$$
$$= \int_{\mathbb{R}} \int_{F^{-1}(S_t)} f(F(x))F^*(d\mu_{\alpha(t)})(x)dt$$
$$= \int_{\mathbb{R}} \int_{S_t} f(F(x))\alpha'(t)F^*(d\mu_{\alpha(t)})(x)dt$$
$$= \int_{\mathbb{R}} \int_{S_t} f(F(x))\alpha'(t)e^{h\alpha(t)} - ht d\mu_t(x)dt$$
$$= \int_{\mathbb{R}} \int_{S_t} f(F(x))d\mu_t(x)dt$$
$$= \int_M f(F(x))d\mu(x),$$
if $\alpha$ is strictly monotonic and
$$\alpha'(t)e^{h\alpha(t)} - ht = 1$$
for all $t \in \mathbb{R}$. Consider two cases. First assume that $h = 0$. Then, equation (5) holds if $\alpha(t) = t + t_0$. Thus $F = \phi_{t_0}$. Now assume that $h > 0$. Then equation (5) holds if and only if
$$\alpha'(t)e^{h\alpha(t)} = e^{ht},$$
or, equivalently,
$$\frac{e^{h\alpha(t)} - e^{ht_0}}{h} = \frac{e^{ht} - 1}{h},$$
so
$$\alpha(t) = \frac{1}{h} \ln \left( e^{ht} + e^{ht_0} - 1 \right).$$
Therefore,
$$F(\phi(t, x)) = \phi \left( \frac{1}{h} \ln \left( e^{ht} + e^{ht_0} - 1 \right), x \right).$$ \(\square\)

**Remark.** If $h = 0$, then $F = \phi_{t_0}$ is just a translation in $\mathbb{R}^n$, so it is an isometry. On the other hand, if $h > 0$, then
$$F_*(\nabla b_v(\phi_t(x))) = \alpha'(t)\nabla b_v(\phi_{\alpha(t)}(x)),$$
so $F$ is not an isometry since
\[ \alpha'(t) = \frac{he^{ht}}{h(e^{ht} + e^{ht_0} - 1)} = \frac{e^{ht}}{e^{ht} + e^{ht_0} - 1} < 1. \]
Also, $\alpha(t) - t$ is a strictly monotone decreasing function on $\mathbb{R}$, so $F$ is well-defined.

4. Proof of Theorem 1.2

For asymptotically harmonic manifolds, we can find a volume preserving mapping from two Busemann functions directly.

**Lemma 4.1.** Let $(M, g)$ be an asymptotically harmonic manifold, $p \in M$, and $v_1 \neq v_2 \in T^1_p M$. Then the vector field $X = \nabla v_1 - \nabla v_2$ is volume preserving.

**Proof.** Note that $X$ is non-vanishing since $v_1 \neq v_2$ and
\[ \text{div} X = -\Delta b_{v_1} + \Delta b_{v_2}. \]
In particular, since $(M, g)$ is asymptotically harmonic, $L_X(d\mu) = (\text{div} X)d\mu = 0$. \hfill \Box

Consequently, this vector field derives a flow such that it preserves $\nabla b_{v_1}$ and $\nabla b_{v_2}$, and the flow is not volume preserving in general.

**Lemma 4.2.** Let $(M, g)$ be an asymptotically harmonic manifold, $p \in M$, $v_1 \neq v_2 \in T^1_p M$, and
\[ X = \frac{1}{\|\nabla b_{v_1} - \nabla b_{v_2}\|^2}(\nabla b_{v_1} - \nabla b_{v_2}). \]
Then the flow $\phi_t$ of $X$ satisfies
\[ (\phi_t)_*(\nabla b_{v_1}(x)) = \nabla b_{v_1}(\phi_t(x)), \quad (\phi_t)_*(\nabla b_{v_2}(x)) = \nabla b_{v_2}(\phi_t(x)), \]
for all $t \in \mathbb{R}$ and $x \in M$.

**Proof.** We denote $\beta = g(\nabla b_{v_1}, \nabla b_{v_2})$ and set
\[ (6) \quad X := \frac{1}{\|\nabla b_{v_1} - \nabla b_{v_2}\|^2}(\nabla b_{v_1} - \nabla b_{v_2}) = \frac{1}{2 - 2\beta}(\nabla b_{v_1} - \nabla b_{v_2}). \]

Then, since $g(X, \nabla b_{v_1} - \nabla b_{v_2}) = 1$ and $g(X, \nabla b_{v_1} + \nabla b_{v_2}) = 0$, \hfill (7)
\[ b_{v_1}(\phi_t(x)) = b_{v_1}(x) + \frac{t}{2}, \quad b_{v_2}(\phi_t(x)) = b_{v_2}(x) - \frac{t}{2}, \]
for all $t \in \mathbb{R}$ and $x \in M$, where $\phi_t$ is the flow of $X$. In particular,
\[ (\phi_t)_*(\nabla b_{v_1}(x)) = \nabla b_{v_1}(\phi_t(x)), \quad (\phi_t)_*(\nabla b_{v_2}(x)) = \nabla b_{v_2}(\phi_t(x)), \]
for all $t \in \mathbb{R}$ and $x \in M$. \hfill \Box

The flow in Lemma 4.2 maps one intersection of two horospheres onto another intersection of two horospheres and the intersection of two horospheres is orthogonal to each gradient of the Busemann function. Similarly, we consider the flow of
\[ \frac{1}{2 + 2\beta}(\nabla b_{v_1} + \nabla b_{v_2}) \]:

**Proposition 4.3.** Let $(M, g)$ be an asymptotically harmonic, visibility manifold, $p \in M$, and $v_1 \neq v_2 \in T^1_p M$. Then there exists a constant $c_0 \in \mathbb{R}$ such that, for every $x \in M$,
\[ b_{v_1}(x) + b_{v_2}(x) \geq c_0. \]
In addition, if $b_{v_1}(x) + b_{v_2}(x) = c_0$, then $\nabla b_{v_1}(x) + \nabla b_{v_2}(x) = 0$, for any $x \in M$. 

Proof. Let \( v_1 \neq v_2 \in T_p^1 M \),
\[ D := \{ x \in M : \nabla b_{v_1}(x) + \nabla b_{v_2}(x) = 0 \}, \]
and let \( c_0 \) be the value of \( b_{v_1} + b_{v_2} \) on \( D \). Note that \( D \neq \emptyset \) since \((M, g)\) is a visibility manifold. Set
\[ Y := \frac{1}{\| \nabla b_{v_1} + \nabla b_{v_2} \|^2} (\nabla b_{v_1} + \nabla b_{v_2}) = \frac{1}{2 + 2\beta} (\nabla b_{v_1} + \nabla b_{v_2}), \]
on \( M - D \), where \( \beta = g(\nabla b_{v_1}, \nabla b_{v_2}) \). Then, since \( g(Y, \nabla b_{v_1} + \nabla b_{v_2}) = 1 \) and \( g(Y, \nabla b_{v_1} - \nabla b_{v_2}) = 0 \), for all \( s \in \mathbb{R} \) and \( x \in M \) satisfying \( (b_{v_1}(x) + b_{v_2}(x) - c_0)(b_{v_1}(x) + b_{v_2}(x) + s - c_0) > 0 \), we have
\[ b_{v_1}(\psi_s(x)) = b_{v_1}(x) + \frac{s}{2}, \quad b_{v_2}(\psi_s(x)) = b_{v_2}(x) + \frac{s}{2}, \]
where \( \psi_s \) is the flow of \( Y \).

Now, suppose that \( b_{v_1}(x) + b_{v_2}(x) = c < c_0 \) for some \( x \in M \). Let \( U_i \) be the \((0, 2)\)-tensor field on \( M \) defined by
\[ U_i(w_1, w_2) := g(\nabla w_1 \nabla b_{v_i}, w_2) \]
for all \( x' \in M \) and \( w_1, w_2 \in T_{x'} M \), where \( i = 1, 2 \). Since \( b_{v_i} \) is convex, \( U_i \) is positive semi-definite. Consequently,
\[ Y[\beta] = \frac{\nabla b_{v_1} + \nabla b_{v_2}}{2 + 2\beta} \left[ g(\nabla b_{v_1}, \nabla b_{v_2}) \right] = \frac{g(\nabla b_{v_2}, \nabla b_{v_1}) + g(\nabla b_{v_2}, \nabla b_{v_1})}{2 + 2\beta} = \frac{U_2(\nabla b_{v_2}, \nabla b_{v_1}) + U_1(\nabla b_{v_2}, \nabla b_{v_1})}{2 + 2\beta} \geq 0. \]

Hence, \( s \mapsto \beta(\psi_s(x)) \) is non-decreasing. By Lemma 2.7,
\[ (b_{v_1} + b_{v_2})^{-1}((-\infty, c_0]) \cap (b_{v_1} - b_{v_2})^{-1}(t) \]
is compact and it contains \( \psi_s(x) \) for all \( s \in (0, c_0 - c) \), where \( t = b_{v_1}(x) - b_{v_2}(x) \). Thus, there exists a sequence \( s_i \in (0, c_0 - c), i = 1, 2, \ldots \), such that \( \lim s_i = c_0 - c \) and the limit \( x_0 := \lim_{s \to \infty} \psi_s(x) \) exists. Since
\[ (b_{v_1} + b_{v_2})(x_0) = (b_{v_1} + b_{v_2})(x) + c_0 - c = c_0, \]
by equation (8), we have \( x_0 \in D \) so that \( \beta(x_0) = -1 \). Thus, since \( s \mapsto \beta(\psi_s(x)) \) is non-decreasing, we have
\[ -1 \leq \beta(x) \leq \beta(x_0) = -1, \]
which implies \( x \in D \) and \( c = c_0 \). It is a contradiction to \( c < c_0 \). \( \square \)

To prove Theorem 4.4. we need the following theorem:

**Theorem 4.4.** Let \((M, g)\) be an asymptotically harmonic, visibility manifold. Let \( p \in M, v_1 \neq v_2 \in T_p^1 M \), and \( c_1, c_2 \in \mathbb{R} \). Let \( S \) be the intersection \( b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2) \) of horospheres and \( \nabla b_{v_1}(x) + \nabla b_{v_2}(x) \neq 0 \) for all points \( x \in S \). Then the following integrals are independent of \( c_1 - c_2 \):
\[ \int_S \frac{1 - g(\nabla b_{v_1}, \nabla b_{v_2})}{1 + g(\nabla b_{v_1}, \nabla b_{v_2})} d\mu', \quad \int_S \frac{1 + g(\nabla b_{v_1}, \nabla b_{v_2})}{1 - g(\nabla b_{v_1}, \nabla b_{v_2})} d\mu', \]
where \( d\mu' \) is the induced measure on the submanifold \( S \) of \((M, g)\).
Proof. We adopt the notations

\[ \beta, X, Y, \phi_t, \psi_s, D, c_0, U_1, U_2 \]

in the proof of Lemma 4.2 and Proposition 4.3.

Note that \( D \) is closed. If there exists a neighborhood \( V \) of \( x \in D \) in \( M \) such that \( V \subseteq D \), then, by Theorem 2.9 the level set of \( b_v \) containing \( x \) is totally geodesic, thus \( h = -\Delta b_v = 0 \), for all \( v \in T^1 M \). Then, since a Busemann function is convex, by the Riccati equation for horospheres, \( (M, g) \) is flat. Hence, \( (M, g) \) is not a visibility manifold which contradicts the assumption of Theorem 4.4.

Suppose that \( h \neq 0 \). We note that \( b_{v_1} + b_{v_2} \geq c_0 \) by Proposition 4.3. By equations (7) and (8), the flows \( \phi_t, \psi_s \) map one intersection \( b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2) \) of horospheres onto another intersection, where \( c_1, c_2 \in \mathbb{R} \) and \( c_1 + c_2 > c_0 \). Let \( E_1, \ldots, E_n \) be a positively oriented, orthonormal frame on \( M - D \), and \( \theta_1, \ldots, \theta_n \) be the dual 1-forms of the frame field such that \( E_1, E_2 \) spans a subbundle containing \( \nabla b_{v_1}, \nabla b_{v_2} \), and \( db_{v_1} \wedge db_{v_2} = \sqrt{1 - \beta^2} \theta_1 \wedge \theta_2 \). Then \( db_{v_1} \wedge db_{v_2} \) is invariant under the pullback of \( \phi_t \) and \( \psi_s \), so that, for all \( s > 0 \) and \( t \in \mathbb{R} \),

\[
\begin{align*}
(\phi_t)^*(\theta_1 \wedge \theta_2) &= (\phi_t)^* \left( \frac{1}{\sqrt{1 - \beta^2}} db_{v_1} \wedge db_{v_2} \right) \\
&= \frac{1}{\sqrt{1 - \beta^2(\phi_t)}} (\phi_t)^*(db_{v_1} \wedge db_{v_2}) \\
&= \frac{1}{\sqrt{1 - \beta^2(\phi_t)}} db_{v_1} \wedge db_{v_2} \\
&= \sqrt{\frac{1 - \beta^2}{1 - \beta^2(\phi_t)}} \theta_1 \wedge \theta_2, \\
(10) \\
(\psi_s)^*(\theta_1 \wedge \theta_2) &= \sqrt{\frac{1 - \beta^2}{1 - \beta^2(\psi_s)}} \theta_1 \wedge \theta_2.
\end{align*}
\]

on \( M - D \). Since \( (M, g) \) is asymptotically harmonic,

\[
\text{div} (\nabla b_{v_1} - \nabla b_{v_2}) = 0, \quad \text{div} (\nabla b_{v_1} + \nabla b_{v_2}) = 2h.
\]

Thus, we obtain

\[
\begin{align*}
\text{div } X &= \text{div} \left( \frac{1}{2 - 2\beta} (\nabla b_{v_1} - \nabla b_{v_2}) \right) \\
&= (\nabla b_{v_1} - \nabla b_{v_2}) \left[ \frac{1}{2 - 2\beta} \right] + \frac{1}{2 - 2\beta} \text{div} (\nabla b_{v_1} - \nabla b_{v_2}) \\
&= - \frac{(\nabla b_{v_1} - \nabla b_{v_2}) [2 - 2\beta]}{(2 - 2\beta)^2} = \frac{X [2 - 2\beta]}{2 - 2\beta} \\
&= -X \ln(2 - 2\beta) \\
&= X \left[ \ln \left( \frac{1}{1 - \beta} \right) \right].
\end{align*}
\]

(12)
\[ \text{div } Y = \text{div} \left( \frac{1}{2 + 2\beta} (\nabla b_{v_1} + \nabla b_{v_2}) \right) \]

\[ = (\nabla b_{v_1} + \nabla b_{v_2}) \left[ \frac{1}{2 + 2\beta} \right] + \frac{1}{2 + 2\beta} \text{div} (\nabla b_{v_1} + \nabla b_{v_2}) \]

\[ = -\frac{(\nabla b_{v_1} + \nabla b_{v_2})[2 + 2\beta]}{(2 + 2\beta)^2} + \frac{2h}{2 + 2\beta} \]

\[ = -\frac{Y[2 + 2\beta]}{2 + 2\beta} + \frac{h}{1 + \beta} \]

\[\left( 13 \right) \]

Consequently, for the induced Riemannian measure \( d\mu \) on \( M - D \), the following equations hold:

\[ \frac{\partial}{\partial t} (\phi_t)^*(d\mu) = (\phi_t)^*(L_X (d\mu)) = (\phi_t)^*(\text{div} X d\mu) \]

\[ = (\text{div} X \circ \phi_t) (\phi_t)^*(d\mu), \]

\[ \frac{\partial}{\partial s} (\psi_s)^*(d\mu) = (\text{div} Y \circ \psi_s) (\psi_s)^*(d\mu). \]

Set \((\phi_t)^*(d\mu) = f_t \, d\mu\) where \(f_t \in C^\infty(M - D)\). Then

\[ \frac{\partial}{\partial t} f_t \, d\mu = (\text{div} X \circ \phi_t) f_t \, d\mu, \]

\[ \frac{\partial}{\partial t} \ln f_t = \text{div} X \circ \phi_t. \]

Note that \(X[f](\phi_t) = \frac{\partial}{\partial t} f(\phi_t)\) for all \(f \in C^\infty(M - D)\). Thus, since \(f_0 = 1\), from equation \(12\), we get

\[ f_t = \frac{1 - \beta}{1 - \beta(\phi_t)}. \]

Hence,

\[\left( 14 \right) \]

\[ (\phi_t)^*(d\mu) = \frac{1 - \beta}{1 - \beta(\phi_t)} d\mu. \]

Similarly, when \((\psi_s)^*(d\mu) = g_s \, d\mu\) for some \(g_s \in C^\infty(M - D)\), by equation \(13\),

\[ \ln g_s = \ln \left( \frac{1 + \beta}{1 + \beta(\psi_s)} \right) + \int_0^s \frac{h}{1 + \beta(\psi_k)} dk, \]

so

\[\left( 15 \right) \]

\[ (\psi_s)^*(d\mu) = \exp \left( \int_0^s \frac{h}{1 + \beta(\psi_k)} dk \right) \frac{1 + \beta}{1 + \beta(\psi_s)} d\mu. \]

Set \(d\mu' := \theta_3 \wedge \cdots \wedge \theta_n\). Then, by equations \(10\), \(11\), \(14\), and \(15\), we have

\[ (\phi_t)^*(d\mu') = \sqrt{\frac{(1 - \beta)(1 + \beta(\phi_t))}{(1 + \beta)(1 - \beta(\phi_t))}} d\mu', \]

\[ (\psi_s)^*(d\mu') = \exp \left( \int_0^s \frac{h}{1 + \beta(\psi_k)} dk \right) \sqrt{\frac{(1 + \beta)(1 - \beta(\psi_s))}{(1 - \beta)(1 + \beta(\psi_s))}} d\mu'. \]
or, equivalently,

\begin{align}
(16) \quad \sqrt{\frac{1 - \beta(\phi_t)}{1 + \beta(\phi_t)}} (\phi_t)^* (d\mu') &= \sqrt{\frac{1 - \beta}{1 + \beta}} d\mu', \\
(17) \quad \sqrt{\frac{1 + \beta(\psi_s)}{1 - \beta(\psi_s)}} (\psi_s)^* (d\mu') &= \exp\left( \int_0^s \frac{h}{1 + \beta(\psi_k)} \, dk \right) \sqrt{\frac{1 + \beta}{1 - \beta}} d\mu'.
\end{align}

Now, let

\[ S(s, t) := b_{v_1}^{-1} \left( \frac{s + c_0 + t}{2} \right) \cap b_{v_2}^{-1} \left( \frac{s + c_0 - t}{2} \right), \]

for all \( s \geq 0 \) and \( t \in \mathbb{R} \). Note that

\[ S(c_1 + c_2 - c_0, c_1 - c_2) = b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2), \]

for all \( c_1, c_2 \in \mathbb{R} \) such that \( c_1 + c_2 \geq c_0 \). Set

\[ V(s, t) := \int_{S(s, t)} \sqrt{\frac{1 - \beta}{1 + \beta}} d\mu', \]

\[ W(s, t) := \int_{S(s, t)} \sqrt{\frac{1 + \beta}{1 - \beta}} d\mu', \]

for all \( s > 0 \) and \( t \in \mathbb{R} \). Then \( \phi_t(S(s, t_0)) = S(s, t_0 + t) \) and \( \psi_s(S(\varepsilon, t)) = S(s + \varepsilon, t) \), for all \( s, \varepsilon > 0 \) and \( t_0, t \in \mathbb{R} \). By equation (16),

\[ V(s, t) = \int_{\phi_t(S(s, 0))} \sqrt{\frac{1 - \beta(\phi_t)}{1 + \beta(\phi_t)}} (\phi_t)^* (d\mu') = \int_{S(s, 0)} \sqrt{\frac{1 - \beta(\phi_t)}{1 + \beta(\phi_t)}} (\phi_t)^* (d\mu') = \int_{S(s, 0)} \sqrt{\frac{1 - \beta}{1 + \beta}} d\mu' = V(s, 0), \]

for all \( s > 0 \) and \( t \in \mathbb{R} \). For all \( s, \varepsilon > 0 \) and \( t \in \mathbb{R} \), from equation (17), we obtain

\[ W(s + \varepsilon, t) = \int_{\psi_s(S(\varepsilon, t))} \sqrt{\frac{1 + \beta(\psi_s)}{1 - \beta(\psi_s)}} (\psi_s)^* (d\mu') = \int_{S(\varepsilon, t)} \sqrt{\frac{1 + \beta(\psi_s)}{1 - \beta(\psi_s)}} (\psi_s)^* (d\mu') = \int_{S(\varepsilon, t)} \exp\left( \int_0^s \frac{h}{1 + \beta(\psi_k)} \, dk \right) \sqrt{\frac{1 + \beta}{1 - \beta}} d\mu'. \]
By differentiating it for $s$, we get
\[
\frac{\partial}{\partial s} W(s + \varepsilon, t) = \int_{S(\varepsilon, t)} \frac{h}{1 + \beta(\psi_s)} \exp \left( \int_0^s \frac{h}{1 + \beta(\psi_k)} dk \right) \sqrt{\frac{1 + \beta}{1 - \beta}} d\mu'
\]
\[
= \int_{S(\varepsilon, t)} \frac{h}{1 + \beta(\psi_s)} \sqrt{\frac{1 + \beta(\psi_s)}{1 - \beta(\psi_s)}} (\psi_s)^* (d\mu')
\]
\[
= \int_{S(s + \varepsilon, t)} \frac{h}{\sqrt{1 - \beta^2}} d\mu'.
\]

By equation (18), we have
\[
\begin{align*}
\frac{\partial}{\partial t} V(s, t) &= 0, \\
\frac{\partial}{\partial s} W(s, t) &= \int_{S(s, t)} \frac{h}{\sqrt{1 - \beta^2}} d\mu' \\
&= \frac{h}{2} W(s, t) + \frac{h}{2} V(s, t).
\end{align*}
\]

By differentiating equation (20) for $t$, from equation (19), the following equation holds:
\[
\frac{\partial^2}{\partial s \partial t} W(s, t) = \frac{h}{2} \frac{\partial}{\partial t} W(s, t),
\]
which implies, for all $s, \varepsilon > 0$ and $t \in \mathbb{R}$,
\[
\frac{\partial}{\partial t} W(s + \varepsilon, t) = e^{\frac{hs}{2}} \frac{\partial}{\partial t} W(\varepsilon, t).
\]

By equation (16),
\[
W(s, t) = \int_{\phi_t(S(s, 0))} \sqrt{\frac{1 + \beta}{1 - \beta}} d\mu'
\]
\[
= \int_{S(s, 0)} \sqrt{\frac{1 + \beta(\phi_t)}{1 - \beta(\phi_t)}} (\phi_t)^* (d\mu')
\]
\[
= \int_{S(s, 0)} \frac{1 + \beta(\phi_t)}{1 - \beta(\phi_t)} \sqrt{\frac{1 - \beta(\phi_t)}{1 + \beta(\phi_t)}} (\phi_t)^* (d\mu')
\]
\[
= \int_{S(s, 0)} \frac{1 + \beta(\phi_t)}{1 - \beta(\phi_t)} \sqrt{\frac{1 - \beta}{1 + \beta}} d\mu'
\]
\[
= \int_{S(s, 0)} \left( -1 + \frac{2}{1 - \beta(\phi_t)} \right) \sqrt{\frac{1 - \beta}{1 + \beta}} d\mu'.
\]
Thus we obtain

\[ \frac{\partial}{\partial t} W(s, t) = \int_{S(s,0)} \frac{2X[\beta](\phi_t)}{(1 - \beta(\phi_t))^2} \sqrt{\frac{1 - \beta}{1 + \beta}} d\mu' \]

\[ = \int_{S(s,0)} \frac{2X[\beta](\phi_t)}{(1 - \beta(\phi_t))^2} \left( \frac{1 - \beta(\phi_t)}{1 + \beta(\phi_t)} \right)^*(d\mu') \]

\[ = \int_{S(s,t)} \frac{2X[\beta]}{(1 - \beta)} \sqrt{1 - \beta^2} d\mu'. \]

Now, we get, from equation (22),

\[ X[\beta] = \frac{1}{2 - 2\beta} (\nabla b_{v_1} - \nabla b_{v_2}) [g(\nabla b_{v_1}, \nabla b_{v_2})] \]

(22)

\[ = \frac{1}{2 - 2\beta} (U_2(\nabla b_{v_1}, \nabla b_{v_1}) - U_1(\nabla b_{v_2}, \nabla b_{v_2})). \]

Consequently, we have

\[ \frac{\partial}{\partial t} W(s, t) = \int_{S(s,t)} \sqrt{1 - \beta^2} \frac{U_2(\nabla b_{v_1}, \nabla b_{v_1}) - U_1(\nabla b_{v_2}, \nabla b_{v_2})}{(1 - \beta)^2} \frac{d\mu'}{1 - g(\nabla b_{v_1}, \nabla b_{v_2})^2} \]

\[ = \int_{S(s,t)} \sqrt{1 - \beta^2} \frac{U_2(\nabla b_{v_1} - \beta \nabla b_{v_2}, \nabla b_{v_1} - \beta \nabla b_{v_2})}{(1 - \beta)^2} \frac{d\mu'}{\|\nabla b_{v_1} - \beta \nabla b_{v_2}\|^2} \]

\[ - \int_{S(s,t)} \sqrt{1 - \beta^2} \frac{U_1(\nabla b_{v_2} - \beta \nabla b_{v_1}, \nabla b_{v_2} - \beta \nabla b_{v_1})}{(1 - \beta)^2} \frac{d\mu'}{\|\nabla b_{v_2} - \beta \nabla b_{v_1}\|^2}. \]

Now, use the following lemma:

**Lemma 4.5.** Let \((M, g)\) be an asymptotically harmonic manifold. Then, for all \(v, w \in T^1M\),

\[ g(\nabla_w \nabla b_v, w) \leq h, \]

where \(h = -\Delta b_v\).

**Proof.** Let \(U\) be the \((0, 2)\)-tensor field defined by \(U(x_1, x_2) := g(\nabla_{x_1} \nabla b_{x_2}, x_2)\) for all \(x \in M\) and \(x_1, x_2 \in T_x M\). Then \(U\) is symmetric, positive semi-definite, and \(\text{tr} U = h\). Thus, every eigenvalue of \(U\) is real, non-negative, and less than or equal to \(h\). In particular, equation \(22\) holds. \(\square\)

By Lemma 4.5,

\[ 0 \leq \frac{U_1(\nabla b_{v_2}, \nabla b_{v_2})}{1 - \beta^2} = \frac{\nabla b_{v_2} - \beta \nabla b_{v_1}, \nabla b_{v_1} - \beta \nabla b_{v_2}}{\|\nabla b_{v_2} - \beta \nabla b_{v_1}\|^2} \leq h, \]

(24)

\[ 0 \leq \frac{U_2(\nabla b_{v_1}, \nabla b_{v_1})}{1 - \beta^2} = \frac{U_2(\nabla b_{v_1} - \beta \nabla b_{v_2}, \nabla b_{v_1} - \beta \nabla b_{v_2})}{\|\nabla b_{v_1} - \beta \nabla b_{v_2}\|^2} \leq h. \]

Hence, by equations (22) and (24), \(X[\beta]/\sqrt{1 - \beta^2}\) is bounded. By Lemma 4.5 \(\bigcup_{S(s,0) S(\varepsilon, t)}\) is compact for all \(s > 0\). Fix \(s > 0\). Then \(\beta\) has a maximum value on the compact set \(\bigcup_{S(\varepsilon, s)}\). In particular, there exists a constant \(C > 0\) such that, for all \(\varepsilon \in (0, s)\),

\[ \left| \frac{\partial}{\partial t} W(\varepsilon, t) \right| \leq C \int_{S(\varepsilon, t)} \sqrt{1 + \beta} d\mu'. \]

(25)
Now, we use the following lemma:

**Lemma 4.6.** The \((n - 2)\)-dimensional volume \(\text{vol}_{n-2} S(s, t)\) of \(S(s, t)\) is non-decreasing for \(s > 0\) where \(t \in \mathbb{R}\) is fixed, and, for all \(s > 0\) and \(x \in S(s, t)\),

\[
\beta(x) \leq -2e^{-hs} + 1.
\]

**Proof.** For all \(s, \varepsilon > 0\) and \(t \in \mathbb{R}\),

\[
\text{vol}_{n-2} S(s + \varepsilon, t) = \int_{\psi_s(S(\varepsilon, t))} \text{d}\mu' = \int_{S(\varepsilon, t)} (\psi_s)^*(d\mu')
\]

Thus, we obtain

\[
\frac{\partial}{\partial s} \text{vol}_{n-2} S(s + \varepsilon, t) = \int_{S(\varepsilon, t)} \left( \frac{h}{1 + \beta(\psi_s)} - \frac{Y[\beta](\psi_s)}{1 - \beta^2(\psi_s)} \right) (\psi_s)^*(d\mu')
\]

and

\[
\ln \left( \frac{1 - \beta(x)}{1 - \beta(\psi_s(x))} \right) \leq hs
\]

for all \(s > 0\) and \(x \in M - D\). Since, \(\lim_{k \to s-} \beta(\psi_k(x)) = -1\) for all \(s \in S(s, t)\), by considering \(\beta(x) = \beta(\psi_s(\psi_s(x)))\), we obtain, for all \(s > 0\) and \(x \in S(s, t)\),

\[
\beta(x) \leq -2e^{-hs} + 1.
\]

Thus, by equations (26), (27), and (28),

\[
\frac{\partial}{\partial s} \text{vol}_{n-2} S(s, t) \geq \frac{1}{2(1 - e^{-hs})} (h - h) \text{vol}_{n-2} S(s, t) = 0.
\]

Thus, \(\text{vol}_{n-2} S(s, t)\) is non-decreasing for \(s > 0\).

By equation (25) and Lemma 4.6, for small \(\varepsilon > 0\), \(\text{vol}_{n-2} S(\varepsilon, t)\) is bounded, and for some constant \(C > 0\)

\[
\left| \frac{\partial}{\partial t} W(\varepsilon, t) \right| \leq C \sqrt{2 - 2e^{-h\varepsilon}}.
\]
Then we obtain

\[(30) \lim_{\varepsilon \to 0^+} \frac{\partial}{\partial t} W(\varepsilon, t) = 0,\]

and, by equations (21) and (30), we have

\[(31) \frac{\partial}{\partial t} W(s, t) = 0\]

By equations (19) and (31), Theorem 4.4 is proved.

Proof of Theorem 1.2. Since $\beta = -1$ on $D$, the notations $X$, $\phi_t$ can be extended to $M$, and, by equation (14), we have

$$\operatorname{vol}_{n-2} S(0, t) = \int_{S(0,0)} \phi_t^*(d\mu') = \int_{S(0,0)} d\mu' = \operatorname{vol}_{n-2} S(0, 0).$$

Now, suppose that $c_1 + c_2 \neq c_0$. The $(n - 2)$-dimensional volume of the intersection $S = b_{c_1}^{-1}(c_1) \cap b_{c_2}^{-1}(c_2)$ satisfies

$$\operatorname{vol}_{n-2} S = \int_S d\mu' \leq \int_S \frac{1}{\sqrt{1 - \beta^2}} d\mu' = \frac{1}{2}(V(s, t) + W(s, t)),$$

where $s = c_1 + c_2 - c_0$, $t = c_1 - c_2$, and $V(s, t) + W(s, t)$ is independent of $t$ by Theorem 4.4.

Now, we give an example which supports the main theorem.

Example. Consider the Poincaré upper-half plane model $\{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ with the metric:

$$g = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

The distance between two points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ is

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = 2 \arcsinh \left( \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}{2z_1z_2} \right),$$

We consider the volume of the intersections of horospheres for the case: $v_1 = \frac{\partial}{\partial x}|_{(0,0,1)}$ and $v_2 = -\frac{\partial}{\partial z}|_{(0,0,1)}$. The geodesic $\gamma_{v_1}$ is the unit circle centered at $(0,0,0)$ and the equation of the geodesic sphere centered at $(\cos r, 0, \sin r)$ containing $(0, 0, 2)$ is

$$\frac{(x - \cos r)^2 + y^2 + (z - \sin r)^2}{4 \sin r z} = \frac{5 - 4 \sin r}{8 \sin r},$$

or equivalently,

$$\frac{(x - \cos r)^2 + y^2 + \left( z - \frac{5}{4} \right)^2}{\frac{9}{16}} = 1.$$

Every point $(x, y, z)$ of the intersection of the geodesic spheres centered at $(\cos r, 0, \sin r)$ and $(-\cos r, 0, \sin r)$ containing $(0, 0, 2)$ satisfies

$$x = 0, \quad y^2 + \left( z - \frac{5}{4} \right)^2 = \frac{9}{16}.$$
It is also the equation of the intersection of horospheres. Write \( y = \frac{3}{4} \cos t \) and \( z = \frac{3}{4} \sin t + \frac{5}{4} \). Then the volume of the intersection of horospheres is

\[
\int_0^{2\pi} \frac{3}{3 \sin t + 5} dt < 3\pi.
\]

**Remark.** Let \((M, g)\) be an asymptotically harmonic, visibility manifold. Let \(c_1, c_2 \in \mathbb{R}\) such that \(c_1 + c_2 > c_0\) and \(r > 0\). The set \( b_{v_1}^{-1}([c_1, c_1 + r]) \cap b_{v_2}^{-1}([c_2, c_2 + r]) \) is the union of countably many sets of the form

\[
S = \{x \in M : a_1 + c_0 \leq b_{v_1} + b_{v_2} \leq a_2 + c_0, \ a_3 \leq b_{v_1} - b_{v_2} \leq a_4\},
\]

where \(a_1, a_2 \geq 0\) and \(a_3, a_4 \in \mathbb{R}\). It can be obtained by taking the middle points of each sides of the square \([c_1, c_1 + r] \times [c_2, c_2 + r]\) repeatedly. The \(n\)-dimensional volume of a set of the form equals

\[
\int_S d\mu = \int_S \theta_1 \wedge \theta_2 \wedge d\mu' = \frac{1}{2} \int_{a_1}^{a_2} \int_{a_3}^{a_4} \int_{S(s,t)} \frac{1}{\sqrt{1 - \beta^2}} d\mu' dt ds.
\]

By Theorem 4.4

\[
\int_S d\mu = \frac{a_4 - a_3}{2} \int_{a_1}^{a_2} \int_{S(s,0)} \frac{1}{\sqrt{1 - \beta^2}} d\mu' ds.
\]

Therefore, the \(n\)-dimensional volume of \( b_{v_1}^{-1}([c_1, c_1 + r]) \cap b_{v_2}^{-1}([c_2, c_2 + r]) \) is independent of \(c_1 - c_2\).

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