Quadratic forms connected with Fourier coefficients of holomorphic and Maass cusp forms

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Abstract

In this work we prove a prime number type theorem involving the normalised Fourier coefficients of holomorphic and Maass cusp forms, using the classical circle method. A key point is in a recent paper of Fouvry and Ganguly, based on Hoffstein-Ramakrishnan’s result about the non-existence of the Siegel zeros for $GL(2)$ $L$-functions, which allows us to improve preceding estimates.

1 Introduction

The aim of this work is to prove a prime number type theorem involving the normalised Fourier coefficients of holomorphic and Maass cusp forms. Our result improves a recent bound obtained by Hu in [3] (which could have been reached with a simpler argument). The main tool used here is, as in Hu’s paper, the classical circle method, but we give a different definition of the major arcs, as we shall see. Another key ingredient in our work is a result due to Fouvry and Ganguly (1), based on Hoffstein-Ramakrishnan’s result 2, Section 5 about the non-existence of the Siegel zeros for $GL(2)$ $L$-functions. Some of the estimates we need have been proved in detail by Hu, so we refer to 3 for the complete arguments. We first discuss a few preliminary results, underlining the differences with preceding estimates, and then we focus on the proof of the main theorem.

In the following, we will assume $f$ to be either a holomorphic or a Maass cusp form, with normalised Fourier coefficients given by the sequence $(a(n))$, $n \geq 1$. We consider a definition of Maass forms general enough to include holomorphic cusp forms, in order to proceed with a unified argument (see 1, Section 2.1 for details). Moreover, we can suppose that $f$ is a Hecke-Maass cusp form, that is an element of the so-called Hecke basis of the space of Maass forms. In particular, we will focus on Maass cusp forms for the full modular group $SL_2(\mathbb{Z})$, which are trivially primitive forms (again according to the notation of 1, Section 2.3)) and these include holomorphic cusp forms.

We recall that for Maass forms the Ramanujan’s conjecture, that is $|a(p)| \leq 2$ for $p$ prime, is still open, but we know that

$$|a(p)| \leq 2p^\theta \quad \text{with} \quad \theta = \frac{7}{64}.$$ 

Hence, the coefficients of Maass cusp forms satisfy, for all $\varepsilon > 0$,

$$a(n) = O(n^{\theta + \varepsilon}),$$

while it is well known that the bound

$$a(n) = O(n^\varepsilon)$$

holds for holomorphic cusp forms (this result is due to Deligne).
Moreover, let $\Lambda$ be the classical von Mangoldt function,

$$
\Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^m \text{ with } m \geq 1 \\
0 & \text{otherwise}
\end{cases}
$$

and define

$$
\pi_{a,\Lambda}(x) = \sum_{m_1^2 + m_2^2 + m_3^2 \leq x} a(m_1^2 + m_2^2 + m_3^2) \Lambda(m_1^2 + m_2^2 + m_3^2).
$$

In his paper, Hu proves a bound of the form

$$
\pi_{a,\Lambda}(x) = O(x^{3/2} \log^c x),
$$

(1.1)

where $c$ is a suitable positive constant. We claim that, taking into account the absence of exceptional zeros proved by Hoffstein-Ramakrishnan and a consequent estimate of [1], a stronger bound can be obtained. More precisely, our purpose is to prove the following statement.

**Theorem 1.** Let $a(n)$ be as above. There exists a constant $c > 0$ such that

$$
\pi_{a,\Lambda}(x) = O(x^{3/2} \exp(-c \sqrt{\log x})).
$$

2 Notation and outline of the method

Since we follow Hu’s approach to the problem and mainly refer to [3] for the details, we introduce the same notation. For $\alpha \in \mathbb{R}$ and $y > 1$, define

$$
S_1(\alpha, y) = \sum_{1 \leq m \leq y} e(m^2 \alpha), \quad S_2(\alpha, y) = \sum_{|m| \leq y} e(m^2 \alpha)
$$

and the exponential sum

$$
T(\alpha, y) = \sum_{1 \leq n \leq y} a(n) \Lambda(n) e(n\alpha).
$$

As a first step, we have

$$
\pi_{a,\Lambda}(x) = \int_0^1 S_3^2(\alpha, \sqrt{x}) T(-\alpha, x) d\alpha.
$$

Moreover, by $S_2(\alpha, y) = 2S_1(\alpha, y) + 1$, we get, for all $\varepsilon > 0$,

$$
\pi_{a,\Lambda}(x) = 8 \int_0^1 S_3^3(\alpha, \sqrt{x}) T(-\alpha, x) d\alpha + O(x^{1+\theta+\varepsilon}).
$$

Let now $P = \exp(C \sqrt{\log x})$, where $C$ is a positive constant, and let $Q = xP^{-1} = x \exp(-C \sqrt{\log x})$. Thanks to the periodicity of the integrand,

$$
\pi_{a,\Lambda}(x) = 8 \int_0^1 S_3^1(\alpha, \sqrt{x}) T(-\alpha, x) d\alpha + O(x^{1+\theta+\varepsilon})
$$

$$
= 8 \int_{1/Q}^{1+1/Q} S_3^1(\alpha, \sqrt{x}) T(-\alpha, x) d\alpha + O(x^{1+\theta+\varepsilon}).
$$

(2.1)

Dirichlet’s theorem on rational approximation assures that for any real $\alpha$ and any $Q \geq 1$ there exists a rational number $a/q$, with $(a, q) = 1$, $1 \leq q \leq Q$ such that

$$
\alpha = \frac{a}{q} + \beta \quad \text{with} \quad |\beta| \leq \frac{1}{qQ}.
$$

For $1 \leq q \leq P$ and $1 \leq a \leq q$, $(a, q) = 1$ let

$$
M(a, q) = \left[ \frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ} \right].
$$
The set $M$ of the major arcs is given by the union of the above defined $M(a,q)$, as $a,q$ run in the proper ranges. As usual, the set $m$ of the minor arcs is instead

$$m = \left[ \frac{1}{Q}, 1 + \frac{1}{Q} \right] \setminus M.$$ 

Hence, by equation (2.1)

$$\pi_{a,\Lambda}(x) = 8 \int_M S_1^3(\alpha, \sqrt{x}) T(-\alpha, x) d\alpha + 8 \int_m S_1^3(\alpha, \sqrt{x}) T(-\alpha, x) d\alpha + O(x^{1+\theta+\varepsilon}). \quad (2.2)$$

Our goal is now to give an estimate of the integrand on major and minor arcs.

### 3 First estimates

In this section we first state a couple of results about $S_1^3(\alpha, \sqrt{x})$ on major and minor arcs. For the proof of these facts we refer to [3], Lemma 4.1 and 4.2 respectively. Then, we will discuss a key result that will allow us to improve the estimate of $T(\alpha, x)$ on major arcs.

We start introducing the Gauss sum. For $a, b \in \mathbb{Z}, q \in \mathbb{N}$ let

$$G(a,b,q) = \sum_{r=1}^{q} e\left(\frac{ar^2 + br}{q}\right).$$

Suppose that $q \geq 1$ and $(a,q) = 1$, then the above sum satisfies (see [3, Lemma 3.5])

$$G(a,b,q) \ll \sqrt{q}.$$

**Lemma 1.** *(Estimate of $S_1(\alpha, \sqrt{x})$ on major arcs)*

Let $1 \leq q \leq P, 1 \leq a \leq q, (a,q) = 1$ and $\alpha = \frac{\alpha}{q} + \beta \in M(a,q)$ with $|\beta| \leq \frac{1}{qQ}$, then

$$S_1(\alpha, \sqrt{x}) = \frac{G(a,0,q)}{q} \sqrt{x} \int_0^1 e(x\beta v^2) dv + O(\sqrt{q \log(q+1)}).$$

**Lemma 2.** *(Estimate of $S_1(\alpha, \sqrt{x})$ on minor arcs)*

Let $\alpha = \frac{\alpha}{q} + \beta \in m$ with $1 \leq a \leq q, (a,q) = 1$ and $\beta \leq \frac{1}{qQ}$. Hence,

$$S_1(\alpha, \sqrt{x}) \ll x^{1/2} q^{-1/2} + q^{1/2} (\log q)^{1/2} + x^{1/4} (\log q)^{1/2}.$$ 

**Remark 1.** By Lemma 2 we have

$$S_1(\alpha, \sqrt{x}) \ll x^{1/2} P^{-1/2} = x^{1/2} \exp\left(-\frac{C}{2 \sqrt{\log x}}\right),$$

recalling that, on minor arcs, $P \leq q \leq Q$ and the value of $P$.

Let now $\chi$ be a Dirichlet character modulo $q$ and define

$$\psi_f(x,\chi) = \sum_{n \leq x} a(n)\chi(n)\Lambda(n).$$

**Lemma 3.** There exists a constant $A > 0$ such that

$$\psi_f(x,\chi) \ll \sqrt{x} \exp(-A\sqrt{\log x}).$$

**Proof.** We will not give the details of the proof. The bound can be easily deduced from [1, Theorem 4.1]. In particular, it follows with standard methods by formula (28), which is essentially based on the zero-free region proved by Hoffstein-Ramakrishnan (see [2, Theorem C, part (3)]) and the consequent absence of exceptional zeros.
Remark 2. In the below lemma, we will use the following identity. For \((a,m) = 1\),

\[
e^\left(\frac{a}{m}\right) = \frac{1}{\phi(m)} \sum_{\chi \pmod{m}} \bar{\chi}(a)\tau(\chi),
\]

where \(\tau(\chi)\) is the Gauss sum defined as

\[
\tau(\chi) = \sum_{b \pmod{m}} \chi(b)e\left(\frac{b}{m}\right).
\]

Moreover, note that

\[
\chi(a)\tau(\bar{\chi}) = \sum_{b \pmod{m}} \bar{\chi}(b)e\left(\frac{ab}{m}\right).
\]

Proof. We apply equations (3.1), (3.2), (3.3), partial summation and Lemma 3, getting

\[
T(a,x) = \sum_{n \leq x} a(n)\Lambda(n)e\left(\frac{an}{q}\right)e(\frac{n\beta}{q}) = \sum_{n \leq x} a(n)\Lambda(n)e\left(\frac{an}{q}\right)e(\frac{n\beta}{q}) + \sum_{n \leq x} a(n)\Lambda(n)e\left(\frac{an}{q}\right)e(\frac{n\beta}{q})
\]

\[
= \sum_{n \leq x} a(n)\Lambda(n)\left(\frac{1}{\varphi(q)}\right) \sum_{\chi \pmod{q}} \bar{\chi}(an)\tau(\chi)e(\frac{n\beta}{q}) + O(x^{\theta+\varepsilon}\log^2 x)
\]

\[
= \sum_{n \leq x} \sum_{\chi \pmod{q}} \sum_{b \pmod{q}} \chi(b)e\left(\frac{ab}{q}\right)\left(\frac{1}{\varphi(q)}\right)\sum_{\chi \pmod{q}} \bar{\chi}(an)\Lambda(n)e(\beta n) + O(x^{\theta+\varepsilon}\log^2 x)
\]

\[
= \sum_{n \leq x} a(n)\chi(n)\Lambda(n)e(\beta n) + 2\pi i \sum_{\chi \pmod{q}} \sum_{n \leq x} \chi(b)e(\beta n)\left(\psi_j(x,\bar{\chi})e(\beta n) - 2\pi i \sum_{u \leq 1} \psi_j(u,\bar{\chi})e(\beta n)du\right)
\]

\[
\ll \sum_{b \pmod{q}} (1 + |\beta|)\sqrt{q} \exp(-A\sqrt{\log x}) \ll x^{3/2} \exp(-A\sqrt{\log x}) \ll x \exp(-B\sqrt{\log x}),
\]

where \(B = A - \frac{3}{2}C\) and \(A\) is the constant of Lemma 3. Hence, as a first restriction on \(C\) we impose

\[
A - \frac{3}{2}C > 0 \iff C < \frac{2}{3}A.
\]

Moreover, observe that the term \(O(x^{\theta+\varepsilon}\log^2 x)\) is due to the values of \(n\) such that \((n,q) > 1\). In fact

\[
\sum_{n \leq x} a(n)\Lambda(n)e(na) \ll \sum_{n \leq x} a(n)\Lambda(n) \ll x^{\theta+\varepsilon}\sum_{n \leq x} \Lambda(n) \ll x^{\theta+\varepsilon}\log^2 x.
\]

Remark 3. In [3], the author claims that, simply normalising the coefficients, it is possible to generalise [3 Theorem 1] to the case of Maass forms. However, the proof of that result is strongly based on Ramanujan’s conjecture, so we are not sure of the estimate stated in [3 Lemma 5.1]. For this reason, our argument for \(T(a,x)\) on major arcs follows a different approach.
4 Proof of the main result

We are now ready to prove Theorem 1. We start considering the major arcs. By construction
\[
\int_M S_3^1(\alpha, \sqrt{x})T(\alpha, x) d\alpha = \sum_{1 \leq q \leq P} \sum_{a \equiv \pm 1} \sum_{(a, q) = 1} \int_{\frac{\pi}{q}}^{\frac{\pi}{q} + \frac{1}{q}} S_3^1(\alpha, \sqrt{x})T(\alpha, x) d\alpha.
\]

Assume now, with the usual meaning, \(\alpha \in M(a, q)\). By Lemma 1 developing the cube and observing that a priori two of the terms dominate the others,
\[
S_3^1(\alpha, \sqrt{x}) \ll \frac{G^3(a, 0, q)}{q^3} x^{3/2} + q^{3/2} \log^3(q + 1) \ll x^{3/2} q^{-3/2}.
\]

Hence, the bound obtained in Lemma 1 gives
\[
S_3^1(\alpha, \sqrt{x})T(\alpha, x) \ll x^{3/2} q^{-3/2} \exp(-B \sqrt{\log x}) = x^{5/2} q^{-3/2} \exp(-B \sqrt{\log x}).
\]

Now, recalling that the length of the interval \(M(a, q)\) is \(2(qQ)^{-1}\),
\[
\int_{\frac{\pi}{q}}^{\frac{\pi}{q} + \frac{1}{q}} S_3^1(\alpha, \sqrt{x})T(\alpha, x) d\alpha \ll (qQ)^{-1} x^{5/2} q^{-3/2} \exp(-B \sqrt{\log x}) = q^{-1} x^{3/2} q^{-3/2} P \exp(-B \sqrt{\log x}),
\]

having \(Q = xP^{-1}\), so \(Q^{-1} = x^{-1}P\). We sum over the residue classes modulo \(q\), obtaining
\[
\sum_{q=1}^{q^{-1}} \int_{\frac{\pi}{q}}^{\frac{\pi}{q} + \frac{1}{q}} S_3^1(\alpha, \sqrt{x})T(\alpha, x) d\alpha \ll x^{3/2} q^{-3/2} P \exp(-B \sqrt{\log x}),
\]

and finally over \(1 \leq q \leq P\). Observing that \(\sum_{q=1}^{P} q^{-3/2} = O(1)\),
\[
\int_M S_3^1(\alpha, \sqrt{x})T(\alpha, x) d\alpha \ll x^{3/2} P \exp(-B \sqrt{\log x}) = x^{3/2} \exp(-(B - C) \sqrt{\log x}), \quad (4.1)
\]

since \(P = \exp(C \sqrt{\log x})\). Note that, to our purpose we have to impose \(B - C > 0\), so
\[
C < B = A - \frac{3}{2} C \iff C < \frac{5}{2} A. \quad (4.2)
\]

We still have to find an estimate for
\[
\int_m S_3^1(\alpha, \sqrt{x})T(\alpha, x) d\alpha.
\]

Using Cauchy inequality,
\[
\int_m S_3^1(\alpha, \sqrt{x})T(\alpha, x) d\alpha \ll \max_m |S_1(\alpha, \sqrt{x})| \int_0^1 |S_1(\alpha, \sqrt{x})| |T(\alpha, x)| d\alpha \ll \max_m |S_1(\alpha, \sqrt{x})| \left( \int_0^1 |S_1(\alpha, \sqrt{x})|^4 d\alpha \right)^{1/2} \left( \int_0^1 |T(\alpha, x)|^2 d\alpha \right)^{1/2}.
\]

Now, as observed in Lemma 2,
\[
\max_m |S_1(\alpha, \sqrt{x})| \ll x^{1/2} P^{-1/2} = x^{1/2} \exp(-\frac{C}{2} \sqrt{\log x}), \quad (4.3)
\]

Moreover, since for Maass forms the following average bound holds
\[
\sum_{n \leq x} |a(n)|^2 \ll x,
\]
we have
\[
\int_{0}^{1} |T(-\alpha, x)|^2 \, d\alpha \ll \sum_{n \leq x} a^2(n)\Lambda^2(n) \ll x \log^2 x. \tag{4.4}
\]

Finally,
\[
\int_{0}^{1} |S_1(\alpha, \sqrt{x})|^4 \, d\alpha = \int_{0}^{1} \sum_{1 \leq m_1, m_2 \leq \sqrt{x}} e((m_1^2 - m_2^2)\alpha) \sum_{1 \leq m_3, m_4 \leq \sqrt{x}} e((m_3^2 - m_4^2)\alpha) \, d\alpha
\]
\[
= \sum_{1 \leq m_i \leq \sqrt{x}} \int_{0}^{1} e((m_1^2 + m_3^2 - m_2^2 - m_4^2)\alpha) \, d\alpha
\]
\[
= \sum_{1 \leq m_i \leq \sqrt{x}} \sum_{i=1,2,3,4, \ m_1^2 - m_2^2 = m_3^2 - m_4^2} 1 \ll \sum_{n \leq x} d^2(n) \ll x \log^3 x, \tag{4.5}
\]
where \(d(n)\) is the number of divisors of \(n\). Then, by the above estimates,
\[
\int_{m} S_1^2(\alpha, \sqrt{x})T(-\alpha, \sqrt{x}) \, d\alpha \ll x^{1/2} P^{-1/2} x^{1/2} \log x \log^{1/2} x
\]
\[
= x^{3/2} \log^{5/2} x \exp(-C' \sqrt{\log x}) \ll x^{3/2} \exp(-C' \sqrt{\log x}), \tag{4.6}
\]
where \(C'\) is a positive constant such that \(C' < \frac{C}{2}\).

Now, equations (4.1) and (4.6) give
\[
\pi_{a, \Lambda}(x) \ll x^{3/2} \exp(-(B - C) \sqrt{\log x}) + x^{3/2} \exp(-C' \sqrt{\log x}).
\]

Hence, combining (5.3) and (12), we choose \(P = \exp(C \sqrt{\log x})\), with \(C < \frac{2}{3}A\), getting
\[
\pi_{a, \Lambda}(x) \ll x^{3/2} \exp(-C \sqrt{\log x}),
\]
where \(c = \min(B - C, C') = \min(A - \frac{4}{3}C, C')\).

To conclude, note that a similar argument can also be applied to improve the estimate in [5, Theorem 2]. In fact, the statement can be reformulated as follows, according to the notation of [5].

**Theorem 2.** There exists a positive constant \(c'\) such that
\[
r_{\tau}(N) = \sum_{n_1 + n_2 + n_3 = N} \tau(n_1)\Lambda(n_1)\tau(n_2)\Lambda(n_2)\tau(n_3)\Lambda(n_3) \ll N^{37/2} \exp(-c' \sqrt{\log N}).
\]

The key point is again Hoffstein-Ramakrishnan’s result, since Ramanujan’s \(\tau\) function is a holomorphic cusp form of level 12. This assures the non-existence of exceptional zeros and then an analogous version of Lemma 3 even holds in this case. We can apply the circle method choosing \(P = \exp(A' \sqrt{\log N})\), where \(A' > 0\) depends on the constant \(A\) of Lemma 8. As a consequence, the Corollary in [3] gives
\[
S_{\tau}(\alpha) \ll N^{13/2} \exp(-C \sqrt{\log N}),
\]
where again the positive constant \(C\) depends on \(A\) and
\[
S_{\tau}(\alpha) = \sum_{n \leq N} \tau(n)\Lambda(n)e(n\alpha).
\]
The result then follows as in [5] (the constant \(c'\) of the statement will depend on \(A\)).
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