Experimentally probing topological order and its breakdown through modular matrices

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The modern concept of phases of matter has undergone tremendous developments since the first observation of topologically ordered states in fractional quantum Hall systems in the 1980s. In this paper, we explore the following question: in principle, how much detail of the physics of topological orders can be observed using state of the art technologies? We find that using surprisingly little data, namely the toric code Hamiltonian in the presence of generic disorders and detuning from its exactly solvable point, the modular matrices—characterizing anyonic statistics that are some of the most fundamental fingerprints of topological orders—can be reconstructed with very good accuracy solely by experimental means. This is an experimental realization of these fundamental signatures of a topological order, a test of their robustness against perturbations, and a proof of principle—that current technologies have attained the precision to identify phases of matter and, as such, probe an extended region of phase space around the soluble point before its breakdown. Given the special role of anyonic statistics in quantum computation, our work promises myriad applications both in probing and realistically harnessing these exotic phases of matter.

Landau's theory fails to describe many exotic phases of matter such as topological orders, where no symmetry breaking is involved4–7. Topological orders are characterized by their robust ground-state degeneracy and long-range entanglement8. The fractional quantum Hall states are among the best known examples of such states7. It is well known that topological orders do not admit any local order parameters. Characterization of topological orders based on quantities such as topological entanglement entropy9–11 has also proved inadequate, as they often take the same value for different phases. It is thus a profound and paramount quest to single out the minimal set of topological observables that would uniquely identify a topological order.

Recent theoretical works have shown that modular matrices might help achieve precisely this goal12–17. More specifically, imagine placing a two-dimensional system on a torus, which practically corresponds to periodic boundary conditions along two independent cycles. The S and T modular matrices record the Berry phases accumulated when one adiabatically deforms the system geometrically. The S matrix corresponds to a π/2 rotation, and T to a shear—often called a Dehn twist—of the fundamental region of the torus, as illustrated in Fig. 1. Generically, a topological order has degenerate ground states on a torus, and the modular matrices furnish a non-trivial representation of the modular group $SL(2,\mathbb{Z})$, which are thus instances of non-Abelian geometric phases. Like a fingerprint, modular matrices can be used to distinguish different topological orders, in fact uniquely for bosonic non-chiral phases. For example, the $\mathbb{Z}_2$ toric code order and the doubled semion order have different modular matrices18, which also encode the anyonic statistics in these states. We review this in the Supplementary Information, along with some further details on the theoretical description of the modular matrices. We note that there is a very recent related work on identifying non-Abelian topological phases of matter based on the Levin–Wen model through measuring the relevant $ST^{1/2}$ matrices19.

In principle, modular matrices can be obtained in a discrete model from ground-state wavefunction overlap after appropriate transformations on the lattice. This has been employed as a numerical test to identify topological orders12,15–18. Another recent progress was also achieved by constructing string operators, which is highly non-trivial even numerically when the model is not exactly solvable20. A concrete demonstration that these quantities can in fact be measured to sufficient accuracy using current experimental techniques would turn these purely theoretical discussions into realistic physical observables, opening up entirely new possibilities in experimental studies of these orders. In this paper, we are more ambitious than that: we study a family of spin Hamiltonians at different points in their phase space and measure the corresponding modular matrices. We show that without further theoretical input other than identifying approximate point-group symmetry of the Hamiltonian, the experiment proves itself capable of identifying the underlying topological order, in this case the $\mathbb{Z}_2$ order, by accurately recovering the modular matrices that stay exactly stationary before the analogue of phase transition—more appropriately level crossing where higher levels cross the supposedly degenerate ground-state
subspace. We clarify our meaning of a phase transition in detail in the Supplementary Information. We note that information about point-group symmetry is necessary because the modular transformations were implemented by permuting the lattice sites which would preserve the degenerate ground-state subspace only if they were symmetries of the Hamiltonian. It is remarkable, however, that an approximate symmetry suffices, where the symmetry breaking scale is smaller than the topological mass gap. The experiment can also probe finite regions of the phase diagram, and locate the maximal value of the magnetic field where a transition occurs, beyond which the topological properties change abruptly, exhibiting in action the drastic phenomenology of topological orders that has a Hall-plateau-like flavour.

Let us first describe the target Hamiltonian studied in this experiment, which reads

$$\hat{H}_T = \hat{H}_{Z_2} - \hbar \sum_i \hat{\sigma}_i^z - \sum_i \epsilon_i \hat{\sigma}_i^x$$  

(1)

where on each site $i$ of the $N \times N$ square lattice resides a spin $1/2$, and $\hat{\sigma}_i^{x,y,z}$ are the Pauli matrices acting on site $i$. Let us explain the special role of each term in equation (1) separately. The first term is the Hamiltonian of the Kitaev $Z_2$ toric code model,$^{24}$ which in its simplest version reads

$$\hat{H}_{Z_2} = \sum_{\text{white plaquettes}} X_p + \sum_{\text{yellow plaquettes}} Z_p$$  

(2)

Here $X_p = \prod_{i \in p} \hat{\sigma}_i^x$, $Z_p = \prod_{i \in p} \hat{\sigma}_i^z$ are the plaquette operators that act on four spins surrounding a plaquette of $p$. This is illustrated in Fig. 2.

It is well known that $H_{Z_2}$ is exactly solvable because $[X_p, Z_p] = 0$ for all $p$ and $p'$. The ground-state subspace is given by

$$\mathcal{L} = \{ |\psi_x \rangle \in \mathcal{H} : X_p |\psi_x \rangle = Z_p |\psi_x \rangle = -|\psi_x \rangle \text{ for all } p \}$$  

(3)

The ground-state degeneracy is $D = 2^{g}$, where $g$ is the genus number of the Riemann surface where the system resides. If the Riemann surface has genus $g$, we can define $2g$ non-contractible loops that connect to different ground states in $\mathcal{L}$. Particularly, on a torus the model has four-fold degenerate ground states. The Kitaev model is the infrared fixed point model that describes the $Z_2$ toric code order. The $Z_2$ order, however, exists in a finite region of the phase space of the model. Particularly, a homogeneous magnetic field, corresponding to the second term in equation (1), can be added while preserving the order. A sufficiently large magnetic field would cause the system to undergo a transition to a trivial phase. This has been studied fairly extensively in the literature.$^{25,26}$ The modular matrices are indeed invariant throughout the entire phase space before the transition occurs. To showcase the robustness of the order stemming from its topological nature, we would like to break all translation symmetry in the system, by introducing a random magnetic field at each site, which yields the third term of equation (1).

In the experiment, we consider a unit cell (that is, a $2 \times 2$ square lattice) of the torus as our test system. The ground states of the Kitaev toric code model are fixed point wavefunctions, whose

Figure 1 | Geometrical diagrams of the modular $S$ and $T$ transformations. They correspond to $\pi/2$ rotation and Dehn twist on a torus, respectively.

Figure 2 | Kitaev toric code model on a torus. When $N$ is even, there exist two sublattices denoted by white and yellow. The blue solid strings ($\gamma_1 \sim \gamma_2$) and their dual dashed strings ($\gamma_1' \sim \gamma_2'$) are defined in the yellow sublattice and in the white sublattice, respectively. $e$ and $m$ represent the elementary excitations (anyons): electric charge and magnetic vortex, which are in pairs generated by open string operators. The black dashed box is a unit cell of the square lattice and forms a torus under the periodic boundary condition. The red spheres on the torus represent spins. Topological properties are independent of lattice size.$^{27}$ Under the periodic boundary condition, the total Hamiltonian reduces to

$$\hat{H}_T = 2(\hat{\sigma}_1^+ \hat{\sigma}_2^- \hat{\sigma}_3^+ \hat{\sigma}_4^- + \hat{\sigma}_1^- \hat{\sigma}_2^+ \hat{\sigma}_3^- \hat{\sigma}_4^+) - \hbar \sum_{i=1}^4 \hat{\sigma}_i^- - \sum_{i=1}^4 \epsilon_i \hat{\sigma}_i^+$$  

(4)

The energy levels of the system as a function of $\hbar$ are plotted later in Fig. 5a. The level crossing point is located at $\hbar_{\text{c}} = 2\sqrt{10}/3$. The disorders are randomly generated, with a strength $\epsilon$, in the range $[-0.05, 0.05]$. There is a significant hierarchy in energy gaps between the splittings among the four lowest states in the ground-state subspace and the ‘topological’ gap. This feature is supposedly more pronounced in the thermodynamic limit. Therefore, the above definition of transitions for such a small system asymptotes to the true phase transition in the thermodynamic limit. Let us emphasize here again that phase transition for a small system is not an entirely well-defined concept. The best analogue of a phase transition in a small system is level crossing, which the experiment is essentially detecting. A thorough discussion of this delicate issue is relegated to the Supplementary Information. We are going to make measurements over four different choices of $\hbar$ in the range $0 \leq \hbar < 2.5$.

Figure 3a,b clearly illustrates that the $\pi/2$ rotation and Dehn twist are equivalent to the experimental operations of SWAP$^{13}$ and SWAP$^{12}$, respectively, which keep the toric code Hamiltonian...
in a homogeneous magnetic field invariant. Here SWAP_i is the operation of swapping spin i and spin j. It is also necessary to emphasize that although we only consider a unit cell as our testing system, the proposal for measuring S and T matrices is scalable to larger systems. The details on experimentally implementing S and T transformations and their complexity analysis are given in the Supplementary Information. From the analysis, one can see that the measurement proposal acquires only polynomial complexity and is thus efficient.

Now we turn to the experimental realization to directly measure the modular matrices of the $Z_2$ topological order. With the well-developed control technology\textsuperscript{27}, nuclear magnetic resonance (NMR) has been widely utilized for many of the first demonstrations in quantum simulation\textsuperscript{28}. In the experiment we employed three $^{19}$F spins and two $^1$H spins of 1-bromo-2,4,5-trifluorobenzene partially oriented in liquid crystal N-(4-methoxybenzaldehyde)-4-butylaniline (MBBA) as a 5-qubit NMR simulator\textsuperscript{26}. The molecular structure and the labelled qubits are shown in Fig. 4a. The first $^{19}$F spin (labelled by 0) is used as a probe qubit, and the rest of the spins (labelled by $1 \sim 4$) constitute the 4-qubit quantum register to simulate the system of a 2 $\times$ 2 spin lattice. The experiment was carried out at 303 K on a Bruker AV-400 spectrometer. In our molecule, the effective couplings between nuclear spins originate from partially averaged dipolar interactions $D_{ij}$ (DD-couplings) and scalar interactions $J_{ij}$ (I-couplings). Since the chemical shift difference in each pair of spins is much higher than the effective coupling strength, the $x$ and $y$ components in DD-coupling interaction can be ignored by secular approximation\textsuperscript{31}. Therefore, the effective Hamiltonian of this 5-qubit system in a doubly rotating frame is

$$\hat{H}_{\text{NMR}} = \sum_{j=0}^{4} \pi \nu_j \hat{\sigma}_j^z + \sum_{j<k=0}^{4} \frac{\pi}{2} (J_{jk} + 2D_{jk}) \hat{\sigma}_j^+ \hat{\sigma}_k^- \quad (5)$$

with the related parameters shown in Fig. 4b.

It is one of the main purposes in developing quantum simulators to obtain and subsequently measure the ground states of some given Hamiltonian dynamically, thereby solving otherwise (classically) computationally challenging problems that often do not have analytic solutions. To ask the simulator to solve our problem at hand (obtaining the degenerate ground states of the given Hamiltonian without prior analytical input), we come up with the method—random adiabatic evolution. To put it simply, we randomly generate simple Hamiltonians as starting point and adiabatically evolve the system to the target Hamiltonian. This would generically prepare dynamically all the linearly independent ground states with only a few trials. This can be contrasted with prior numerical or experimental work that makes use of the string operators\textsuperscript{23,32}. The current strategy highlights the strengths of the NMR system, substituting as much analytical insight as possible by experimental manoeuvre.

The quantum circuit for randomly preparing linear-independent ground states is shown in Fig. 4c. Using the line-selective approach\textsuperscript{33}, the quantum system was first prepared in the initial pseudo-pure state (PPS): $\rho_{\text{PPS}} = (1 - \epsilon)/32I + \epsilon|00000\rangle\langle00000|$, with $I$ representing a 32 $\times$ 32 identity operator and $\epsilon \approx 10^{-5}$ the polarization. Here we introduced a probe qubit for the interferometry, which is initialized in the superposition state (that is, $1/\sqrt{2}(|0\rangle + |1\rangle)$) by a Hadamard gate. In Fig. 4c, APGSi: $|0000\rangle_{1234} \mapsto |\psi^s\rangle$ and APGSj: $|0000\rangle_{1234} \mapsto |\psi^t\rangle$, where APGS stands for random adiabatic evolution. Consider the following time-dependent Hamiltonian:

$$\hat{H}(t) = [1 - s(t)]\hat{H}_d + s(t)\hat{H}_r \quad (6)$$

where $\hat{H}_d = \sum_{i,j}^N C_i^* \hat{\sigma}_i^z \hat{\sigma}_j^z$ and $C_i^*$ are randomly generated coefficients between $[-1, 1]$. To obtain the ground state of our target Hamiltonian $\hat{H}_r$, we first prepare the ground state of a simpler local Hamiltonian $\hat{H}_d$, which can be readily implemented using only single-qubit rotations. The system is then evolved adiabatically by varying the parameter function $s(t)$ slowly enough from 0 at $t = 0$ to 1 at $t = T$, where $s(t)$ was interpolated linearly with $M = 100$ discretized steps, and the duration of each step is $\tau = T/M$. According to the adiabatic theorem\textsuperscript{34}, the total evolution time obeys $T = O(1/\Delta_{\text{min}})$, where $\Delta_{\text{min}}$ is the minimum energy gap between four almost degenerate ground states and the first excited state encountered along the adiabatic evolution. The rate of change of $s(t)$ is so chosen that there is certain probability of excitation in the entire ground-state subspace. Let us pause here and comment that the possibility of adiabatic evolution as a means to ground-state preparation in topological orders was first proposed and studied in ref. 35, particularly for the detuned toric code model, as in the current paper. One crucial observation is that even in the thermodynamic limit and for couplings close to the topological phase transition, the adiabatic timescale $T$ needs only to scale at worst as $\sqrt{n}$, where $n$ is the total number of spins in the system, to avoid jumping between the ground-state subspace which is shown to be optimal. Moreover, error following from excitations above the topological gap $\Delta$ can be made to be smaller than an arbitrary polynomial $(T \Delta)^{\gamma}$. Starting from a different random Hamiltonian $\hat{H}_d$, we can search for four linearly independent ground states by measurement results on the probe qubit, that is, $\text{Tr}(\hat{\rho}_0 \hat{\sigma}_j^z) = 1/2(|\psi_s^s\rangle |\psi_s^s\rangle)$, where $\hat{\sigma}_j^z = |0\rangle_j \langle 1|$ is the observable operator of the probe qubit (see Methods in detail). Let us remark that the fact we can sensibly select four states out of the entire spectrum despite generically unavoidable finite size splittings among the ground-state subspace is not an accident. There is a hierarchy between the finite size splitting and the topological gap controlled by the couplings, as shown in Fig. 5a. This feature would only become more pronounced as the system size gets bigger, which is demonstrated in ref. 25. The experimental details can be seen in the Supplementary Information.

Therefore, all elements of the modular matrices in four randomly generated linearly independent ground states (that is, $|\psi_s^s\rangle [S |\psi_s^s\rangle$ or $|\psi_t^t\rangle [T |\psi_t^t\rangle$) can be subsequently obtained after the quantum circuit in Fig. 4d, directly from measuring the probe qubit. Note that the $S$ and $T$ matrices obtained in the random basis are not in

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**Figure 3** | Physical realizations of the modular $S$ and $T$ transformations on a $2 \times 2$ torus. a: $\pi/2$ rotation equivalent to a SWAP$_{12}$ operation. b: $T$: a shear equivalent to a SWAP$_{12}$ operation.
Figure 4 | Physical system and quantum circuit for measuring the modular matrices. a, Molecular structure of 1-bromo-2,4,5-trifluorobenzene. b, Relevant parameters measured. The diagonal and off-diagonal elements represent the chemical shifts $\nu_i$ and effective coupling constants $(J_{ik} + 2D_{ik})$ in units of Hz, respectively. c, Quantum circuit for randomly preparing linear-independent ground states. $H$ stands for the Hadamard gate acting on the probe qubit. The system (spins $1 \sim 4$) is adiabatically prepared into a random ground state $|\psi_G\rangle$, labelled by APGS.$\dagger$. d, Quantum circuit for measuring the modular $S$ and $T$ matrices. Each element of $S$ or $T$ can be extracted by measuring the probe qubit.

In summary, we presented the first proof-in-principle experimental identification of $Z_2$ topological order by measuring the modular $S$ and $T$ matrices for a given Hamiltonian with minimal analytical input from state preparation to measurements. The only analytical input is in the approximate lattice symmetry of the system. This is a major improvement that put the experiment to full use in finding its own ground state, thereby solving the model, without requiring string operators that are not easily acquired for a generic model not exactly solvable. Before we end, let us emphasize the promise of eventually applying our method to larger and more generic systems. As discussed in the main text, the adiabatic method is shown to be optimal, with an adiabatic timescale $T$ that scales at worst only as $\sqrt{n}$ for a system of $n$ spins.$^9$ Another feature of topological orders that works to our advantage is their robustness, which is made even more pronounced for larger system size. As system size grows, the energy splitting within the ground-state subspace decreases exponentially with system size, while the topological gap remains constant in the limit. Such robustness is underlined in ref 25, where it is demonstrated that the energy splitting among the ground-state subspace diminishes quickly as system size increases, and that thermal excitation above the topological gap remains exponentially suppressed all the way up to phase transition as the detuning parameter $h$ is increased in the $Z_2$ order. Therefore the choice of $T$ need not be fine-tuned using detailed knowledge of the energy spectrum which is unavailable in general. Rather, we need only a rough estimate of the size of the topological gap, which can be estimated from the values of the coupling. In this respect, the finite size splitting effect of the topological ground-state subspace may in fact be put to good use—by picking a sufficiently large $T$ so that $1/T$ is smaller than all the energy gaps in the system, we will find the adiabatic method outputting to us the same state independently of the random starting Hamiltonian—simulations beyond level crossing where the energy splitting gets significantly larger indeed confirm such an expectation. In other words, the random method itself is...
implementing size system. At the same time, that complexity of the quantum gates ground-state subspace despite unavoidable level splitting in a finite us to make the case for selecting the lowest four states as part of the capable of discovering hierarchies in the energy gaps, and allowing us to make the case for selecting the lowest four states as part of the ground-state subspace despite unavoidable level splitting in a finite size system. At the same time, that complexity of the quantum gates implementing $S$ and $T$ transformation scales only polynomially with system size, and that our numerical method recovering the standard basis has no dependence on system size, all works to our advantage against small local disorders in the form of an inhomogeneous magnetic field that breaks all accidental symmetries. The method also allows us to explore regions of the phase diagram in which a finite homogeneous magnetic field varies, and locate the phase transition across which the modular matrices jump discontinuously. Given the utility of the modular matrices in uniquely characterizing at least non–chiral bosonic $2+1$ dimensional topological orders, the success of our NMR systems opens up future experimental avenues towards identifying topological orders whose Hamiltonians may not be exactly solvable. Our method is suitable not only for NMR systems, but will also work well in physical systems for quantum computers, such as superconductors and trapped ions. It will be interesting to generalize our measurement method to characterize other topological phases and their phase transitions.

Methods

Methods, including statements of data availability and any associated accession codes and references, are available in the online version of this paper.

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Author contributions

X.P. initiated the project. L.-Y.H. and Y.W. formulated the theory. X.P. and Z.L. designed the experiments. Z.L. and Y.W. performed the calculation. Z.L. carried out the experiment and analyzed the data. X.P. and J.D. supervised the experiment. Z.L. and L.-Y.H. wrote the draft. All authors contributed to discussing the results and writing the manuscript.

Additional information

Supplementary information is available in the online version of the paper. Reprints and permissions information is available online at www.nature.com/reprints. Publisher’s note: Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations. Correspondence and requests for materials should be addressed to L.-Y.H., Y.W. or X.P.

Competing financial interests

The authors declare no competing financial interests.
Methods

Randomly preparing linearly independent ground states. Starting from different initial Hamiltonians $H_\omega$ in random adiabatic evolution, we can sequentially prepare a series of ground states denoted as $\{|\psi_1^{rd}\rangle, |\psi_2^{rd}\rangle, \ldots, |\psi_n^{rd}\rangle\}$, which can always be rewritten in the following form:

$$|\psi_i^{rd}\rangle = |\phi_i\rangle$$

$$|\psi_j^{rd}\rangle = a_i^1|\phi_i\rangle + a_i^2|\phi_j\rangle$$

$$\cdots$$

$$|\psi_k^{rd}\rangle = a_i^n|\phi_i\rangle + a_j^1|\phi_j\rangle + \cdots + a_k^n|\phi_k\rangle$$

where $\{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\}$ is an orthogonal basis and all $a_i^k$ are defined in the real number interval by removing the phase into $|\phi_i\rangle$. Using the quantum circuit in Fig. 4c, we measure the overlaps of these randomly generated ground states on the probe qubit, namely, $\langle \psi_i^{rd}|\psi_i^{rd}\rangle$, $i < j \in [2, n]$. The coefficients $a_i^k$ can be easily solved only from these overlaps. If

$$a_i^k \gg |k_{ij}|$$

$|\psi_i^{rd}\rangle$ is linearly independent of $|\psi_i^{rd}\rangle$ for $K < k$; Otherwise, repeat the random adiabatic evolution until all linearly independent ground states have been found. Here $k_i$ admits two contributions: $k_i = |x_i^1| + |k_{ij}|$. The first is the readout error in the experiment, $x_i$, and the second is $\epsilon$, which is the overlap of the prepared states with the excited states beyond the supposed ground-state subspace. The presence of non-vanishing $\epsilon$ means that

$$|\psi_i^{rd}\rangle = \sum_i b_i|\tilde{\psi}_i^{rd}\rangle + \epsilon|\phi_m\rangle$$

where $|\tilde{\psi}_i^{rd}\rangle$ corresponds to states in the ground-state subspace for the detuned model, and $|\phi_m\rangle$ are states beyond the ground-state subspace. Following the procedure using (7), it would suggest that $|\phi_i\rangle$ also takes a form like (9), provided that (8) is satisfied— that is, $|\phi_i\rangle$ remains mostly within the ground-state subspace, and the contribution of the excited states remain linear in $\epsilon$. Far away from the transition point $h_c$, $\epsilon \leq 10^{-4}$, and so $\phi_i$ remains safely within the ground-state subspace. As $h_c$ is approached, $\epsilon$ can become greater than 0.2. As a result equation (8) cannot in general be satisfied, and a drastic change in the modular matrix would be observed. A proper adiabatic timescale $T$ should thus be chosen such that $\epsilon$ remains sufficiently small as $h$ approaches $h_c$. An optimal $T$ can be chosen when the finite size splitting among the ground states is much smaller than the topological gap.

There are four linearly independent ground states in our experiment, which can usually be achieved within random 100 times. The experimental overlaps and coefficients are included in the Supplementary Information.

Recovering standard modular matrices. According to equation (7), we have

$$\langle \psi_i^{rd}|(O)|\psi_j^{rd}\rangle = \sum A_{ij} (|\phi_i\rangle |O\rangle |\phi_j\rangle)$$

where $O = S$ or $T$, and $A_{ij}$ is the transformation matrix that has been obtained by the procedure above. Once all elements $\langle \psi_i^{rd}|(O)|\psi_j^{rd}\rangle$ are measured directly from Fig. 4d, we can construct $(|\phi_i\rangle |O\rangle |\phi_j\rangle)$ for all $i, j$. The results of the $S$ and $T$ matrices in the random basis and in the orthogonal basis are shown in the Supplementary Information. Note that the experimental errors inevitably lead to $S$ and $T$ matrices being non-normal. To diagonalize the $T$ matrix for recovering the standard form in the next step, it is necessary to constrain the matrices in the orthogonal basis into normal matrices, which can be realized by an optimal searching algorithm—that is, to find the normal matrices that are the closest to experimental matrices. Actually, the normal constraint condition also makes the experimental data more physical. Then, in principle, one can follow an algorithm proposed in ref. 18 to recover the standard basis. Here we simplified the procedure slightly given the specific problem at hand. First, like in ref. 18 we first diagonalize the $T$ matrix. As we will see in the detailed discussion of the experimental data, the diagonalization of $T$, whose eigenvalues are the self-statistics of the anyons in a topological order, immediately suggests that the phase carries three bosonic anyons and one fermion. One can isolate the fermionic basis corresponding to the eigenvector of $T$ with eigenvalue $-1$ and recover the correct basis for the bosonic anyons by requiring the following: there exists a boson that corresponds to the trivial sector and which has trivial statistics with all other anyons; and they are orthogonal. A numerical optimization quickly converges, recovering the pair of $S$ and $T$ matrices in their standard form of the $\mathbb{Z}_2$ phase before level crossing occurs. Across the phase transition, one readily sees a jump in the modular matrices, which also cease to be unitary. This is because in the absence of an invariant ground-state subspace under the $S$ and $T$ transformations, which is the case of a trivial phase or a different order where a different subspace is preserved, an arbitrary projection to some subspace does not preserve unitarity. This allows a precise identification of the phase transition.

Data availability. The data that support the plots within this paper and other findings of this study are available from the corresponding author upon reasonable request.