Coupling Between the Spin and Gravitational Field and the Equation of Motion of the Spin

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Abstract

In general relativity, the equation of motion of the spin is given by the equation of parallel transport, which is a result of the space-time geometry. Any result of the space-time geometry can not be directly applied to gauge theory of gravity. In gauge theory of gravity, based on the viewpoint of the coupling between the spin and gravitational field, an equation of motion of the spin is deduced. In the post Newtonian approximation, it is proved that this equation gives out the same result as that of the equation of parallel transport. So, in the post Newtonian approximation, gauge theory of gravity gives out the same prediction on the precession of orbiting gyroscope as that of general relativity.

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1 Introduction

It is known that any theory on gravitational interactions should first pass classical tests of gravity. General Relativity (GR) has passed several classical tests[1, 2], including the deflection of light by the sun[1, 3], the precession of the perihelia of the orbits of the inner planets[1, 4], and the time delay of radar echoes passing the sun[5, 6]. Another new test, the precession of orbiting gyroscopes[7, 8, 9], is going on the way. It is reported that, with an accuracy of about 20%, the observed precession of the orbiting gyroscopes is consistent with the predictions of GR[10, 11]. Basis of the calculation of these classical tests in GR is Schwarzschild solution, geodesic equation, and the equation of parallel transport[12, 13].

The precession of orbiting gyroscopes is a gravitomagnetic effect. The classical effects of gravitomagnetism were studied for more than one hundred years. The close analogy between Newton’s gravitation law and Coulomb’s law of electricity led to the birth of the concept of gravitomagnetism in the nineteenth century[14, 15, 16, 17]. Later, in the framework of GR, gravitomagnetism was extensively explored[18, 19, 20]. Some recently reviews on gravitomagnetism from different viewpoints can be found in literatures [21, 22, 23].

Quantum Gauge Theory of Gravity (QGTG) is first proposed in 2001[24, 25, 26, 27]. The motivation to propose QGTG is try to unify general relativity with quantum theory in the framework of gauge field theory. This goal does not reached until 2003, when Quantum Gauge General Relativity (QGGR) is proposed in the framework of QGTG[28, 29]. QGGR is a perturbatively renormalizable quantum theory, so based on it, quantum effects of gravity[30, 31, 32, 33] and gravitational interactions of some basic quantum fields [34, 35] can be explored. Unification of fundamental interactions including gravity can be fulfilled in a simple and beautiful way[36, 37, 38]. If we use the mass generation mechanism reported in the literatures [39, 40], we can propose a new theory on gravity which contains massive graviton and the introduction of massive graviton does not affect the strict local gravitational gauge symmetry of the Action and does not affect the traditional long-range gravitational force[41]. The existence of massive graviton will help us to understand the possible origin of dark matter. In QGGR, the field equation of gravitational gauge field is just Einstein’s field equation, so in classical level, we can set up its geometrical formulation[42], and QGGR returns to Einstein’s general relativity in classical level. In QGGR, the field equation of gravitational gauge field is the same as Einstein’s field equation in GR, so two equations have the same solutions, though mathematical expressions of the two equations are completely different[43]. For three classical tests, the deflection of light by the sun, the precession of the perihelia of the orbits of the inner planets, and the time delay of radar echoes passing the sun, QGGR gives out the same predictions as those of GR[43].
In this paper, the equation of motion of the spin is discussed in the framework of QGGR. In QGGR, gravity is treated as a kind of physical interactions, which is transmitted by graviton. So in QGGR, the equation of parallel transport can not be directly used, for it is a result of space-time geometry. Based on the coupling between the spin and gravitational field, the equation of motion of the spin can be obtained. Then we prove that this equation and the equation of parallel transport give out the same results in the post Newtonian approximation.

2 Gravitomagnetic and Gravitoelectric Field

For the sake of integrity, we give a simple introduction to QGGR and introduce some notations which is used in this paper. Details on QGGR can be found in literatures [24, 25, 26, 27, 28, 29]. In gauge theory of gravity, the most fundamental quantity is gravitational gauge field $C_\mu(x)$, which is the gauge potential corresponding to gravitational gauge symmetry. Gauge field $C_\mu(x)$ is a vector in the corresponding Lie algebra, which, for the sake of convenience, will be called gravitational Lie algebra. So $C_\mu(x)$ can be expanded as

$$C_\mu(x) = C_\mu^\alpha(x)\hat{P}_\alpha, \quad (\mu, \alpha = 0, 1, 2, 3) \quad (2.1)$$

where $C_\mu^\alpha(x)$ is the component field and $\hat{P}_\alpha = -i\partial/\partial x^\alpha$ is the generator of global gravitational gauge group. The gravitational gauge covariant derivative is given by

$$D_\mu = \partial_\mu - igC_\mu(x) = G_\mu^\alpha \partial_\alpha, \quad (2.2)$$

where $g$ is the gravitational coupling constant and matrix $G = (G_\mu^\alpha) = (\delta_\mu^\alpha - gC_\mu^\alpha)$. Its inverse matrix is $G^{-1} = \frac{1}{-g^2} = (G^{-1})_{\alpha\mu}$. Using matrix $G$ and $G^{-1}$, we can define two important composite operators

$$g^{\alpha\beta} = \eta^{\mu\nu}G_\mu^\alpha G_\nu^\beta, \quad (2.3)$$

$$g_{\alpha\beta} = \eta_{\mu\nu}G^{-1}_\alpha^\mu G^{-1}_\beta^\nu, \quad (2.4)$$

which are widely used in QGGR. In QGGR, space-time is always flat and space-time metric is always Minkowski metric, so $g^{\alpha\beta}$ and $g_{\alpha\beta}$ are no longer space-time metric. They are only two composite operators which consist of gravitational gauge field.

The field strength of gravitational gauge field is defined by

$$F_{\mu\nu}(x) \triangleq \frac{1}{-ig}[D_\mu, D_\nu] = F_{\mu\nu}^\alpha(x) \cdot \hat{P}_\alpha \quad (2.5)$$

where

$$F_{\mu\nu}^\alpha = G_\mu^\beta \partial_\beta C_\nu^\alpha - G_\nu^\beta \partial_\beta C_\mu^\alpha. \quad (2.6)$$
Define
\[ F_{ij}^\alpha = -\varepsilon_{ijk} B_k^\alpha, \quad F_{0i}^\alpha = E_i^\alpha. \] (2.7)
Then field strength of gravitational gauge field can be expressed as
\[ F^\alpha = \begin{cases} 
0 & E_1^\alpha & E_2^\alpha & E_3^\alpha \\
-E_1^\alpha & 0 & -B_3^\alpha & B_2^\alpha \\
-E_2^\alpha & B_3^\alpha & 0 & -B_1^\alpha \\
-E_3^\alpha & -B_2^\alpha & B_1^\alpha & 0 
\end{cases}. \] (2.8)

This form is quite similar to that of field strength in electrodynamics, but with an extra group index \( \alpha \). The component \( E_\alpha^i \) of field strength is called gravitoelectric field, and \( B_\alpha^i \) is called gravitomagnetic field. Traditional Newtonian gravity is transmitted by gravitoelectric field \( E_1^0 \), and the gravitational Lorentz force is transmitted by gravitomagnetic field \( B_\alpha^1 \)[44]. Effects of gravitomagnetic field in astrophysical processes maybe observable[33]. The equation of motion of the spin discussed in this paper is dominated by the coupling between the spin of the particle and gravitomagnetic field. The \( \alpha = 0 \) components \( B_0^i \) and \( E_0^i \) respectively correspond to the gravitomagnetic field and gravitoelectric field defined in literature [18, 19, 20, 21, 22, 23].

### 3 The Equation of Motion of the Spin in Gravitational Field

First, we need to determine the coupling term between the spin of a particle and gravitomagnetic field. In electromagnetic interactions, the coupling energy between the spin \( \vec{J} \) of a particle and magnetic field \( \vec{B}_e \) is
\[ -\frac{q}{m} \vec{J} \cdot \vec{B}_e, \] (3.1)
where \( q \) is the electric charge of the particle, and \( m \) is its mass. We can conjecture that the coupling between the spin \( \vec{J} \) and gravitomagnetic field \( \vec{B}^\alpha \) should be proportional to \( \vec{J} \cdot \vec{B}_e^\alpha \). This is true. In literature [35, 32], in the non-relativistic limit and weak gravitational field approximation, we have deduced that the coupling between the spin of a Dirac particle and gravitomagnetic field \( \vec{B}^\alpha \) is
\[ -\frac{g}{2} \vec{\sigma} \cdot \vec{B} \] (3.2)
where \( \sigma_i \) is the Pauli matrix. Because the spin operator of the Dirac particle is \( \frac{\vec{\sigma}}{2} \), for a general particle with spin \( \vec{S} \), we can generate the equation (3.2) to the following form
\[ \Delta H = -g \vec{S} \cdot \vec{B}^0 = -gSB^0 \cos \theta, \] (3.3)
where $\Delta H$ is the energy shift caused by this coupling, and $\theta$ is the angle between directions of spin and gravitomagnetic field. Then the magnitude of the torque acting on the spin is

$$L = -\frac{\partial \Delta H}{\partial \theta} = -gS B^0 \sin \theta. \quad (3.4)$$

Its direction along the direction of enlarging the $\theta$ angle. So, the torque vector $\vec{L}$ should be

$$\vec{L} = g \vec{S} \times \vec{B}^0. \quad (3.5)$$

From definition (2.7) of gravitomagnetic field, we have

$$L_i = -g F^0_{i\gamma} S_{\gamma}. \quad (3.6)$$

The above relation gives out the leading contribution of the torque caused by the gravitomagnetic interaction of the spin, which is in a non-relativistic form. Indeed, it is obtained in the non-relativistic limit and weak gravitational field approximation. According to the gauge principle and the principle of relativity, any fundamental equation of motion should be Lorentz covariant and gauge covariant. Denote the final required torque as $L_\mu$. Then under Lorentz transformations, it should be transformed as

$$L_\mu \rightarrow L'_\mu = \Lambda_\mu^\nu L_\nu, \quad (3.7)$$

and under gravitational gauge transformations, it should be transformed as

$$L_\mu \rightarrow L'_\mu = (\tilde{U}_\mu L_\nu), \quad (3.8)$$

and it should return to equation (3.6) in non-relativistic limit and weak field approximation. We can prove that the following torque satisfies all above requirements

$$L_\mu = g \eta^{\lambda \nu} g_{\alpha \beta} F^\alpha_{\mu \nu} S_{\lambda} U^\beta, \quad (3.9)$$

where $S_\mu$ is the gravitogauge canonical spin, and $U^\beta$ is the velocity of the particle. Therefore, the equation of motion of the spin in gravitational field is

$$\frac{dS_\mu}{d\tau} = g \eta^{\lambda \nu} g_{\alpha \beta} F^\alpha_{\mu \nu} S_{\lambda} U^\beta. \quad (3.10)$$

The relation between the gravitogauge canonical spin $S_\mu$ and its ordinary spin four-vector $S_\alpha$ is

$$S_\mu = G_\mu^\alpha S_\alpha. \quad (3.11)$$

Therefore, we have

$$G^{-1}_\mu \frac{dS_\mu}{d\tau} = \frac{dS_\alpha}{d\tau} + G^{-1}_\mu (\partial_\gamma G_\mu^\beta) S_\beta U^\gamma. \quad (3.12)$$
Then equation (3.10) can be changed into
\[
\frac{dS_\alpha}{d\tau} = G_\beta^\gamma (\partial_\gamma G_\alpha^{-1\mu}) S_\beta U^\gamma + g\eta^{\lambda\nu} g_\beta_\gamma G_\alpha^{-1\mu} G_\lambda^\delta F_{\mu\nu}^\beta S_\delta U^\gamma.
\] (3.13)
This is the equation of motion of the spin $S_\alpha$.

4 Post Newtonian Approximation

Now, let's discuss the post Newtonian approximation of the equation (3.10). In post Newtonian approximation, we have

\[
\begin{align*}
g_{00} &= \eta_{00} - 2\phi + o(\bar{v}^4) \\
g_{ij} &= \eta_{ij} - 2\delta_{ij}\phi + o(\bar{v}^4) \\
g_{0i} &= g_{0i} = \zeta_i + o(\bar{v}^5) \\
U^0 &= 1 + o(\bar{v}^2) \\
U^i &= \bar{v}^i + o(\bar{v}^3) \\
S_i &= S_i + o(\bar{v}^2 S) \\
S_0 &= -\bar{v} \cdot \bar{S} + 0(\bar{v}^3 S)
\end{align*}
\] (4.1)

where $\phi$ and $\zeta_i$ are given by the following relations

\[
\begin{align*}
\phi &= -\frac{GM}{r}, \quad (4.2) \\
\zeta &= \frac{2G}{r^3}(\vec{x} \times \vec{J}). \quad (4.3)
\end{align*}
\]

In above relations, $M$ and $\vec{J}$ are mass and spin angular momentum of the source. In post Newtonian approximation, we have

\[
\frac{dS_i}{d\tau} = \frac{dS_i}{dt} + S_i \frac{\partial \phi}{\partial t} + S_i (\bar{v} \cdot \nabla \phi) + o(\bar{v}^4),
\] (4.4)

and

\[
g\eta^{\lambda\nu} g_{\alpha\beta} F_{\mu\nu} S_\chi U^\beta = -2(\partial_i \phi)(\bar{v} \cdot \bar{S}) + \frac{1}{2}(\partial_i \zeta_j) S_j - \frac{1}{2}(\bar{S} \cdot \nabla) \zeta_i + v_i (\bar{S} \cdot \nabla \phi) + o(\bar{v}^4). \quad (4.5)
\]
In equation (3.10), set $\nu = i$ and apply the above results, we obtain the following equation of motion

$$\frac{d}{dt} \vec{S} = \frac{1}{2} \vec{S} \times (\nabla \times \vec{\zeta}) - \vec{S} \frac{\partial \phi}{\partial t} - 2(\vec{v} \cdot \vec{S}) \nabla \phi - \vec{S} (\vec{v} \cdot \nabla \phi) + \vec{v} (\vec{S} \cdot \nabla \phi) + o(\vec{v}^4). \quad (4.6)$$

It is found that this equation is completely the same as the equation (9.6.5) in literature [12], it is also the same as the equation (5.5.12) in literature [13] if we neglect the influence of Thomas precession. Follow the deduction in literature[12, 13], and define

$$\vec{S}' = (1 + \phi) \vec{S} - \frac{1}{2} \vec{v} (\vec{v} \cdot \vec{S}), \quad (4.7)$$

we can deduce the following equation from (4.6)

$$\frac{d}{dt} \vec{S}' = \vec{\Omega} \times \vec{S}' \quad (4.8)$$

with the precession angular velocity

$$\vec{\Omega} = -\frac{1}{2} \nabla \times \vec{\zeta} - \frac{3}{2} \vec{v} \times \nabla \phi. \quad (4.9)$$

The first term gives the Lense-Thirring precession, and the second term gives the geodesic precession. Applying (4.2) and (4.3), we get

$$\vec{\Omega} = 3G \vec{x} (\vec{x} \cdot \vec{J})r^{-5} - G \vec{J} r^{-3} + \frac{3GM(\vec{x} \times \vec{v})}{2r^2}. \quad (4.10)$$

This result is completely the same as those predicted by general relativity[12, 13].

5 Summary and Discussions

In this paper, based on the concept of the coupling between the spin of the particle and gravitomagnetic field, the equation of motion of the spin is obtained. In post Newtonian approximation, this equation of motion of the spin gives out completely the same results as those given by GR, whose calculation is based on the equation parallel transport. So, for the precession of orbiting gyroscope, QGGR gives out the same theoretical prediction as that of GR.

Other three classical tests of gravity, including the deflection of light by the sun, the precession of the perihelia of the orbits of the inner planets and the time delay of radar echoes passing the sun, are discussed in the literature [43]. Now, we know that, for all classical tests of gravity, QGGR gives out the same theoretical predictions as those of GR. As we have stated before[28, 29], in classical level, QGGR returns
to GR. The word "return" means that QGGR gives out the same results for all classical phenomenon as those of GR. Because QGGR is a perturbatively renormalizable quantum theory of gravity and can be used to explore quantum phenomenon of gravitational interactions, QGGR can be regarded as a fundamental theory that was developed from GR and the unification of quantum theory and GR.

In QGGR, gravity is treated as a kind of fundamental interactions in Minkowski space-time. The self-consistent treatment of the classical phenomenon of gravity in QGGR is systematically formulated in this paper and the literature [43]. For a long time, we use the language of space-time geometry to study classical problems of gravitational interactions, and do not know how to study classical problems of gravitational interactions by using language of pure physics. Now, in QGGR, we propose a systematic method to study classical problems of gravitational interactions by using the language of pure physics, which can provide us many useful information on the nature of gravity.

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