A remark on extremal stability of configuration spaces
by
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Abstract

We explicitly study the extremal stability of configuration spaces of complex projective spaces of any dimension, and show that the homology groups are vanish in extremal stable range. As a consequence, we give an affirmative answer of the question of Knudsen, Miller and Tosteson.

Key Words: Configuration spaces, extremal stability, Hilbert function, Chevalley–Eilenberg complex

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1 Introduction

For any manifold \( M \), let

\[
F_k(M) := \{(x_1, \ldots, x_k) \in M^k | x_i \neq x_j \text{ for } i \neq j\}
\]

be the configuration space of \( k \) distinct ordered points in \( M \) with induced topology. The symmetric group \( S_k \) acts on \( F_k(M) \) by permuting the coordinates. The quotient

\[
C_k(M) := F_k(M)/S_k
\]

is the unordered configuration space with quotient topology. It is a classical problem in algebraic topology to understand the homology and cohomology of such spaces. Arnold proved integral cohomological stability for \( \mathbb{R}^2 \):

\[
H^i(C_{2i-2}(\mathbb{R}^2); \mathbb{Z}) \cong H^i(C_{2i-1}(\mathbb{R}^2); \mathbb{Z}) \cong H^i(C_{2i}(\mathbb{R}^2); \mathbb{Z}) \cong \ldots
\]

The isomorphisms (for \( k \) large depending on \( i \))

\[
H^i(C_k(M); \mathbb{Z}) \cong H^i(C_{k+1}(M); \mathbb{Z}) \cong H^i(C_{k+2}(M); \mathbb{Z}) \cong \ldots
\]

were generalized for open manifolds by McDuff [9] and Segal [10]. Using representation stability, Church [4] proved that

\[
H^i(C_k(M); \mathbb{Q}) \cong H^i(C_{k+1}(M); \mathbb{Q}) \cong H^i(C_{k+2}(M); \mathbb{Q}) \cong \ldots
\]

for \( k > i \) and \( M \) a connected oriented manifold of finite type. This result was extended by Kundsen [7]. More recently, Knudsen, Miller and Tosteson [8] study the extremal stability (the stability of top cohomology) of unordered configuration spaces of manifolds. They
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asked the following question:

**Question.** (see Question 4.10 of [8]) Suppose that \( H_{d-1}(M; \mathbb{Q}) = 0 \). For \( i \in \mathbb{N} \), is the Hilbert function
\[
    k \mapsto \dim H_{k(d-2)+i}(C_k(M); \mathbb{Q})
\]
eventually a quasi-polynomial?

Here, the dimension of \( M \) is \( d \). The above question has an affirmative answer under the further assumption that \( H_{d-2}(M; Q) = 0 \) by Theorem 4.9 of [8]. Here is our main result:

**Theorem 1.** For \( i, m \in \mathbb{N} \), the Hilbert function
\[
    k \mapsto \dim H_{k(2m-2)+i}(C_k(\mathbb{P}^m); \mathbb{Q})
\]
is eventually a quasi-polynomial.

### 2 Chevalley–Eilenberg complex

The cohomology and homology of configuration spaces have received a lot of attention, since the work by Arnold [1]. Bödigheimer–Cohen–Taylor [3] studied the homology of \( C_k(M) \) in the case of odd dimensional manifolds. Félix–Thomas [6] (see also [5]) constructed a Sullivan model for the rational cohomology of configuration spaces of closed oriented even dimensional manifolds. Furthermore, Knudsen [7] extended the result of Félix–Thomas for general even dimensional manifolds.

Let \( \dim(M) = 2m \). Throughout the paper, we will consider the homology and cohomology over \( \mathbb{Q} \). The symmetric algebra \( \text{Sym}(A^*) \) is the tensor product of a polynomial algebra and an exterior algebra:
\[
    \text{Sym}(A^*) = \bigoplus_{k \geq 0} \text{Sym}^k(A^*) = \text{Poly}(A^{even}) \otimes \text{Ext}(A^{odd}),
\]
where \( \text{Sym}^k \) is generated by the monomials of length \( k \). The \( n \)-th suspension of the graded vector space \( V \) is the graded vector space \( V[n] \) with \( V[n]_i = V_{i-n} \), and the element of \( V[n] \) corresponding to \( a \in V \) is denoted \( s^na \). We write \( H_* (M; \mathbb{Q}) \) for the graded vector space whose degree \(-i\) part is the \( i \)-th homology group of \( M \).

Consider two graded vector spaces
\[
    V^* = H_{-*}(M; \mathbb{Q})[2m], \quad W^* = H_{-*}(M; \mathbb{Q})[4m-1]
\]
and a degree 1 linear map \( \partial : \)
\[
    \partial|_{V^*} = 0, \quad \partial|_{W^*} : W^* \simeq H_*(M; \mathbb{Q}) \xrightarrow{\Delta} \text{Sym}^2(V^*) \simeq \text{Sym}^2(H_*(M; \mathbb{Q})),
\]
where \( \Delta \) is diagonal co-multiplication corresponding to cup product. We choose bases in \( V^i \) and \( W^j \) as
\[
    V^i = \mathbb{Q}\langle v_{i,1}, v_{i,2}, \ldots \rangle, \quad W^j = \mathbb{Q}\langle w_{j,1}, w_{j,2}, \ldots \rangle
\]
(the degree of an element is marked by the first lower index; \(x^q_i\) stands for the product \(x_i \land x_i \land \ldots \land x_i\) of \(q\)-factors). Always we take \(V^0 = \mathbb{Q}\langle v_0 \rangle\). Now we consider the graded algebra:

\[
\Omega^\ast_{k,\ast}(M) = \bigoplus_{i \geq 0} \bigoplus_{\omega = 0} ^\left\lfloor k \right\rfloor (\text{Sym}^{k-2\omega}(V^\ast) \otimes \text{Sym}^{\omega}(W^\ast))
\]

the (total) degree \(i\) is given by the grading of \(V^\ast\) and \(W^\ast\), where \(\omega\) is a weight grading.

The differential \(\partial\) extends over graded algebra by using Leibniz’s rule. By definition of differential, we have

\[
\partial : \Omega^\ast_{k,\ast}(M) \longrightarrow \Omega^\ast_{k+1,\ast-1}(M).
\]

**Theorem 2.** ([6] [5] [7]) For a connected closed oriented manifold \(M\) of even dimension, we have

\[
H^\ast(C_k(M)) \simeq H^\ast(\Omega^\ast_{k,\ast}(M), \partial).
\]

The complex \((\Omega^\ast_{k,\ast}(M), \partial)\) is Chevalley–Eilenberg type complex. The Chevalley–Eilenberg complex has been a ubiquitous presence in the study of the configuration spaces of manifolds. Prominent examples of its appearance include the work of Bödigheimer–Cohen–Taylor [3] and Félix–Thomas [6], building on McDuff’s foundational work [9]; the work of Félix–Tanrè [5] following Totaro [11]; and the work of Knudsen [7] building on work of Ayala–Francis [2].

### 3 Proof of main Theorem

The cohomology ring of \(\mathbb{CP}^m\) is the truncated polynomial ring with single generator:

\[
H^\ast(\mathbb{CP}^m; \mathbb{Q}) = \mathbb{Q}[x]/(x^m), \quad \text{where } \deg(x) = 2.
\]

The corresponding two graded vector spaces are

\[
V^\ast = \langle v_0, v_2, \ldots, v_{2m} \rangle, \quad W^\ast = \langle w_{2m-3}, w_{2m+3}, \ldots, w_{4m-3} \rangle.
\]

The differential \(\partial\) is define on \(V^\ast\) and \(W^\ast\) as

\[
\partial(v_{2i}) = 0, \quad 0 \leq i \leq m,
\]

\[
\partial(W_{2i-1}) = \sum_{a+b=i, 0 \leq a,b \leq m} v_{2a}v_{2b}, \quad m \leq i \leq 2m.
\]

**Lemma 1.** The subspace \(\Omega^\ast_{k-2}(\mathbb{CP}^m).v^2_{2m}, w_{4m-1}\) < \(\Omega^\ast_k(\mathbb{CP}^m)\) is acyclic for \(k \geq 2\).

**Proof.** An element in \(\Omega^\ast_{k-2}(\mathbb{CP}^m).v^2_{2m}, w_{4m-1}\) has a unique expansion \(v^2_{2m}A + Bw_{4m-1}\), where \(A\) and \(B\) have no monomial containing \(w_{4m-1}\). The operator

\[
h(v^2_{2m}A + Bw_{4m-1}) = w_{4m-1}A
\]

gives a homotopy \(id \simeq 0\).
We denote the reduced complex \((\Omega^r_k(\mathbb{CP}^m)/\Omega^r_{k-2}(\mathbb{CP}^m). (v_{2m}^2, w_{4m-1}), \partial_{\text{induced}})\) by
\((\Omega^r_k(\mathbb{CP}^m), \partial)\).

**Corollary 1.** For \(k \geq 2\), we have isomorphism \(H^*(\Omega^r_k(\mathbb{CP}^m), \partial) \cong H^*(C_k(\mathbb{CP}^m))\).

**Lemma 2.** The cohomology groups \(H^{k(2m-2)+i}(C_k(\mathbb{CP}^m))\) are eventually vanish for \(m \geq 1\) and \(i \geq 4\).

**Proof.** For \(k \geq 4\), the highest degree element in the reduce complex \((\Omega^r_k(\mathbb{CP}^m), \partial)\) is \(v_{2m-2}^{k-3}v_{2m}^2w_{4m-3}\). The degree of \(v_{2m-2}^{k-3}v_{2m}^2w_{4m-3}\) is \((2m - 2)k + 3\). This implies that the cohomology groups \(H^{k(2m-2)+i}(C_k(\mathbb{CP}^m))\) are vanish for \(m \geq 1\), \(i \geq 4\) and \(k \geq 4\).

**Proof of Theorem** If \(m = 1\), then we have \(H^{k(2m-2)+i}(C_k(\mathbb{CP}^m)) = H^i(C_k(\mathbb{CP}^1))\). The church homological stability of configuration spaces (see Corollary 3 of [4]) implies that the function
\[ k \mapsto \dim H^i(C_k(\mathbb{CP}^1)) \]
is eventually constant for each \(i \in \mathbb{N}\).

Let \(m \geq 2\) and \(k \geq 8\). From Lemma 2, we have vanishing:
\[ H^{k(2m-2)+i}(C_k(\mathbb{CP}^m)) = 0 \quad \text{for} \quad i \geq 4. \]
Now, we just focus on the cohomology groups \(H^{k(2m-2)+i}(C_k(\mathbb{CP}^m))\) for \(i = 1, 2, 3\). There is no element of degree higher than \(k(2m - 2) + 3\) in reduced complex \((\Omega^r_k(\mathbb{CP}^m), \partial)\).

The elements of degrees \(k(2m - 2)\), \(k(2m - 2) + 1\), \(k(2m - 2) + 2\) and \(k(2m - 2) + 3\) are concentrated in the following two sub-complex:
\[
0 \rightarrow r_{\Omega^r_k(\mathbb{CP}^m)}^{k(2m-2)+2,2} \rightarrow r_{\Omega^r_k(\mathbb{CP}^m)}^{k(2m-2)+3,1} \rightarrow 0
\]
\[
\cdots \rightarrow r_{\Omega^r_k(\mathbb{CP}^m)}^{k(2m-2),2} \rightarrow r_{\Omega^r_k(\mathbb{CP}^m)}^{k(2m-2)+1,1} \rightarrow r_{\Omega^r_k(\mathbb{CP}^m)}^{k(2m-2)+2,0} \rightarrow 0
\]
where
\[
\begin{align*}
\partial r_{\Omega^r_k(\mathbb{CP}^m)}^{k(2m-2)+3,1} &= \langle v_{2m-2}^{k-3}v_{2m}w_{4m-3} \rangle, \\
\partial r_{\Omega^r_k(\mathbb{CP}^m)}^{k(2m-2)+2,2} &= \langle v_{2m-2}^{k-5}v_{2m}^2w_{4m-5}w_{4m-3} \rangle, \\
\partial r_{\Omega^r_k(\mathbb{CP}^m)}^{k(2m-2)+2,0} &= \langle v_{2m-2}^{k-1}v_{2m}^2 \rangle, \\
\partial r_{\Omega^r_k(\mathbb{CP}^m)}^{k(2m-2)+1,1} &= \langle v_{2m-2}^{k-5}v_{2m-2}v_{2m}w_{4m-3}, v_{2m-2}^{k-3}v_{2m}^2w_{4m-5}, v_{2m-2}^{k-2}v_{2m}w_{4m-3} \rangle, \\
\partial r_{\Omega^r_k(\mathbb{CP}^m)}^{k(2m-2),2} &= \langle v_{2m-2}^{k-5}v_{2m-2}^2w_{4m-7}w_{4m-3}, v_{2m-2}^{k-4}v_{2m-2}v_{2m}w_{4m-5}w_{4m-3}, v_{2m-2}^{k-4}w_{4m-5}w_{4m-3} \rangle.
\end{align*}
\]

The first sub-complex is exact because we have non-trivial differential:
\[
\partial(v_{2m-2}^{k-5}v_{2m}w_{4m-3}) = v_{2m-2}^{k-3}v_{2m}w_{4m-3}.
\]
Now we investigate the second sub-complex. The differential $\partial$ is define on the bases of second sub-complex as:

$$
\partial(v_{2m-1}^k v_{2m}^m) = 0,
$$

$$
\partial(v_{2m-4}^k v_{2m-2}^m w_{4m-3}) = 0,
$$

$$
\partial(v_{2m-2}^k v_{2m}^m w_{4m-5}) = v_{2m-1}^{k-1} v_{2m}^m,
$$

$$
\partial(v_{2m-2}^k w_{4m-3}) = 2v_{2m-2}^{k-1} v_{2m}^m,
$$

$$
\partial(v_{2m-2}^k v_{2m}^m w_{4m-7} w_{4m-3}) = 2v_{2m-4}^k v_{2m-2}^m w_{4m-3},
$$

$$
\partial(v_{2m-4}^k v_{2m-2}^m w_{4m-5}) = v_{2m-4}^{k-1} v_{2m-2}^m w_{4m-3},
$$

$$
\partial(w_{4m-3}, v_{2m-2}^k w_{4m-5} w_{4m-3}) = 2v_{2m-4}^k v_{2m-2}^m w_{4m-3} + v_{2m-2}^{k-2} w_{4m-3} + 2v_{2m-2}^{k-3} v_{2m}^m w_{4m-5}.
$$

Note that the monomial $v_{2m}^a$ is zero in $r \Omega_k^{k(2m-2),2}(\mathbb{C}P^m)$ for $a > 2$. The dimensions of image and kernel of the map $\partial : r \Omega_k^{k(2m-2)+1,1}(\mathbb{C}P^m) \to r \Omega_k^{k(2m-2)+2,0}(\mathbb{C}P^m)$ is 1 and 2, respectively. Moreover, the dimension of the image of map $\partial : r \Omega_k^{k(2m-2),2}(\mathbb{C}P^m) \to r \Omega_k^{k(2m-2)+1,1}(\mathbb{C}P^m)$ is 2. From these computations, we conclude that the cohomology groups

$$
H^{k(2m-2)+i}(\mathbb{C}P^m)
$$

are vanish for $k \geq 8$ and $i = 1, 2, 3$. We know that $\dim H^i(M; \mathbb{Q}) \cong \dim H_i(M; \mathbb{Q})$. Hence, for $i, m \in \mathbb{N}$, the Hilbert function

$$
k \mapsto \dim H_{k(2m-2)+i}(\mathbb{C}P^m; \mathbb{Q})
$$

is eventually a quasi-polynomial. □

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**References**

[1] V. I. Arnold, *On some topological invariants of algebraic functions*, Trans. Moscow Math. Soc. Vol 21 (1970), 30-52.

[2] D. Ayala, J. Francis, *Factorization homology of topological manifolds*, J. Topol. 8 (2015) 1045-1084.

[3] C. F. Bödigheimer, F. Cohen, L. Taylor, *On the homology of configuration spaces*, Topology, Vol. 28 no. 2 (1989), 111-123.

[4] T. Church, *Homological stability for configuration spaces of manifolds*, Invent. Math, Vol. 188 no. 2 (2012), 465-504.
[5] Y. Félix, D. Tanrè, *The cohomology algebra of unordered configuration spaces*, J. London Math. Soc, **Vol. 72 no. 2** (2005), 525-544.

[6] Y. Félix, J. C. Thomas, *Rational Betti numbers of configuration spaces*, Topology and its Application, **Vol. 102 no. 2** (2000), 139-149.

[7] B. Knudsen, *Betti numbers and stability for configuration spaces via factorization homology*, Algebraic and Geometric Topology, **Vol. 17 no. 5** (2017), 3137-3187.

[8] B. Knudsen, J. Miller, P. Tosteson *EXTREMAL STABILITY FOR CONFIGURATION SPACES*, preprint arXiv:2109.03855v1[math AT], (2021).

[9] D. McDuff, *Configuration spaces of positive and negative particles*, Topology **14**, 1 (1975), 91-107.

[10] G. Segal, *The topology of spaces of rational function*, Acta Math., **143 (1-2)** (1979), 39-72.

[11] B. Totaro, *Configuration spaces of algebraic varieties*, Topology 35 (1996) 1057–1067.

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