Infinite matrix product states, boundary conformal field theory, and the open Haldane-Shastry model

Hong-Hao Tu\textsuperscript{1} and Germán Sierra\textsuperscript{2,3}

\textsuperscript{1}Max-Planck Institut für Quantenoptik, Hans-Kopfermann-Str. 1, D-85748 Garching, Germany
\textsuperscript{2}Instituto de Física Teórica, UAM-CSIC, Madrid, Spain
\textsuperscript{3}Department of Physics, Princeton University, Princeton, NJ 08544, USA

We show that infinite Matrix Product States (MPS) constructed from conformal field theories can describe ground states of one-dimensional critical systems with open boundary conditions. To illustrate this, we consider a simple infinite MPS for a spin-1/2 chain and derive an inhomogeneous open Haldane-Shastry model. For the spin-1/2 open Haldane-Shastry model, we derive an exact expression for the two-point spin correlation function. We also provide an SU(n) generalization of the open Haldane-Shastry model and determine its twisted Yangian generators responsible for the highly degenerate multiplets in the energy spectrum.

PACS numbers: 11.25.Hf, 75.10.Pq, 02.30.Ik

\textbf{Introduction.}— For a long time, it has been known that the main curse of quantum many-body theory is the exponential growth of the Hilbert space dimension with respect to the number of constituting particles. In the last decades, the study of entanglement has significantly alleviated this curse, at least to some extent, by recognizing the fact that only a tiny corner of the Hilbert space, with small amount of entanglement, is pertinent for the low-energy sector of Hamiltonians with local interactions. This deep insight lies at the heart of tensor network states [1], a family of trial wave functions designed for efficiently representing the physically relevant states in the tiny corner. The best known instance among them is the Matrix Product States (MPS) in one spatial dimension, described in terms of local matrices with finite dimensions. Their entanglement entropies are bounded by the local matrix dimensions, which are nevertheless sufficient for accurately approximating gapped ground states of one-dimensional (1D) local Hamiltonians [2,3]. This discovery not only provides a transparent theoretical picture for real-space renormalization group methods [4,5], but also leads to a recent complete classification of all possible 1D gapped phases [6,8].

For 1D critical systems, the low-energy physics is usually described by conformal field theories (CFT). Their state entanglement entropies exhibit unbounded logarithmic growth [9,11] with respect to the subsystem size, indicating the deficiency of a usual MPS description. To overcome this difficulty, infinite MPS, whose local matrices are conformal fields living in an infinite-dimensional Hilbert space, have been introduced in Ref. [12]. The lattice sites for the infinite MPS locate on a unit circle, embedded in a complex plane. This construction shares conceptual similarity to Moore and Read’s approach [13] of writing 2D trial fractional quantum Hall states in terms of conformal blocks. For a variety of examples [12,14,20], the infinite MPS (as well as their parent Hamiltonians) have been shown to describe critical chains with periodic boundary conditions (PBC) and, furthermore, their critical behaviors are often related to the CFT whose fields are used for constructing the wave functions [21]. In this sense, the infinite MPS introduced in Ref. [12] provide a systematic way of finding lattice discretizations of CFT.

In this Rapid Communication, we show that the infinite MPS ansatz can describe ground states of 1D critical systems with open boundary conditions (OBC), thus complementing the PBC case in Ref. [12]. Unlike bulk CFT for periodic chains, open critical chains are instead described by boundary CFT. Taking a spin-1/2 chain as an example, we show how the infinite MPS with an image prescription allows us to derive an inhomogeneous open Haldane-Shastry model, including the original spin-1/2 open Haldane-Shastry models [22,23] as special cases. Within the new formalism, an exact expression for the two-point spin correlator of the spin-1/2 open Haldane-Shastry model is obtained. This, together with numerical results for the entanglement entropy, is in perfect agreement with the theoretical predictions based on boundary CFT, which thus confirms that our infinite MPS with the image prescription is suitable for describing open critical chains. The open infinite MPS construction is readily applicable to any boundary CFT for finding their lattice discretizations. As a further example, we derive an SU(n) generalization of the open Haldane-Shastry model.

We characterize its full spectrum and also determine the twisted Yangian generators responsible for the highly degenerate multiplets in the energy spectrum.

\textbf{Infinite MPS and parent Hamiltonian.}— Let us consider a spin-1/2 chain located on the upper unit circle in the complex plane, with \(L\) lattice sites and complex lattice coordinates \(z_j = e^{i\theta_j} (j = 1, \ldots, L\) and \(\theta_j \in [0, \pi] \ \forall j\), see Fig. 1(a). We denote by \(S_j^a (a = 1, 2, 3)\) the spin-1/2 operators at site \(j\). The local spin basis is defined by \(|s_j\rangle\), where \(s_j = \pm 1\) (twice of the \(S_j^z\) projection value). For each site, we introduce its \textit{mirror image} in the lower unit circle, e.g., site \(j\) has an image \(\bar{j}\), with complex coordinate \(z_{\bar{j}} = z_j^*\). Following Ref. [12], the wave function...
\(w_{ij} = (z_i + z_j)/(z_i - z_j)\) and which also annihilate the wave function \(|\Psi\rangle\), since \(\Lambda_i^a = (z_i - z_i^*)C_i^a\).

The parent Hamiltonian for (2) is then defined as
\[
H = \frac{1}{8} \sum_{i,a} (\Lambda_i^a)^4 \Lambda_i^a + \frac{1}{2} S^2 + E,
\]
where \(S^2 = \sum_j \vec{S}_j \cdot \vec{S}_j\) is the total spin operator and \(E = \frac{1}{16} \sum_{i,j} (w_{ij}^2 + w_{ji}^2) - \frac{1}{2} L^2\).

After some algebra [20], we arrive at a long-range Heisenberg model
\[
H = \sum_{i \neq j} \left[ \frac{1}{2} \left( \frac{1}{|z_i - z_j|} - \frac{1}{|z_i - z_j|} \right) \right] \left( \vec{S}_i \cdot \vec{S}_j \right)

\]
with ground-state energy \(E\), where \(c_j = w_{jj}^2 + \sum_{l \neq j} (w_{lj} + w_{jl})\).

Three choices of the lattice coordinates deserve special attention (see Fig. 1): (i) type-I: \(\theta_j = \frac{\pi}{2} (j - \frac{1}{2})\); (ii) type-II: \(\theta_j = \frac{\pi}{2} (\frac{1}{2} + j)\); (iii) type-III: \(\theta_j = \frac{\pi}{2} (\frac{1}{2} + j)\). For these three cases (termed as uniform cases afterwards), one obtains \(w_{ij}(c_i - c_j) + w_{ji}(c_i + c_j) = 0, 4, \) and 2, respectively. Accordingly, the parent Hamiltonians, after removing the (unimportant) total spin operator \(S^2\) and constant terms in (3), have purely inverse-square exchange interactions (between the spins and also their images), which coincide with the open Haldane-Shastry models first introduced in Refs. [22, 23]. These uniform models are integrable and have highly degenerate multiplets in their energy spectrum [22, 23], similar to their periodic counterpart [28], see Fig. 2 for the full spectrum of the open and periodic Haldane-Shastry models with \(L = 6\). We postpone the discussion of this degeneracy until presenting the SU(\(n\)) generalization of these models, where a unified treatment is possible. The Hamiltonian (3) with lattice coordinates other than the three uniform cases is an inhomogeneous generalization of the open Haldane-Shastry models and does not exhibit the huge degeneracy in the spectrum.

**Spin correlator.** — A nontrivial application of the infinite MPS formulation is that, for the wave function (2), the spin correlation functions can be computed easily. Since \(C_i^a |\Psi\rangle = 0\), one has \(\langle \Psi | \sum_a S_i^a C_j^a |\Psi\rangle = 0\) and \(\langle \Psi | \sum_a (C_j^a)^\dagger S_i^a |\Psi\rangle = 0\) \(\forall i, j\), which lead to a set of linear equations relating two-point correlators \(C_{ij} + \sum_{l \neq i,j} \frac{u_i^*}{u_i - u_l} C_{il} = -\frac{1}{4}\) [14], where \(C_{ij} \equiv \langle \Psi | \vec{S}_i \cdot \vec{S}_j |\Psi\rangle / |\Psi\rangle\). These equations are sufficient for computing the two-point spin correlators for arbitrary choices of \(\theta_j\) (both inhomogeneous and uniform cases). The generalization to arbitrary higher-order spin correlators is rather straightforward.

Most remarkably, for the type-I uniform case, these linear equations allow us to find an analytical expression for the two-point spin correlator [20]
\[
C_{ij} = \frac{3(-1)^{i-j} \sin \theta_i \sin \theta_j}{L \cos \theta_i - \cos \theta_j} \sum_{p=1}^{L/2} \sum_{q=0}^{p-1} g_{pq} \cos(2p - 1) \theta_i \times \cos 2q \theta_j - \cos 2q \theta_i \cos(2p - 1) \theta_j,
\]
Two free spin-1/2 spinons), the open models do not have this degeneracy, indicating the importance of the boundary effect.

![Energy spectrum of three types of spin-1/2 open Haldane-Shastry models and the spin-1/2 periodic Haldane-Shastry model](image)

**FIG. 2:** (Color online) The energy spectrum of the three types of spin-1/2 open Haldane-Shastry models and the spin-1/2 periodic Haldane-Shastry model ($H = \sum_{\langle i,j \rangle} S_i \cdot S_j$) with $L = 6$. All four models have highly degenerate multiplets in their energy spectrum. While the first excited states of the periodic model are degenerate singlet and triplet (due to two free spin-1/2 spinons), the open models do not have this degeneracy, indicating the importance of the boundary effect.

![Two-point spin correlators of the wave function](image)

**FIG. 3:** (Color online) Two-point spin correlators of the wave function \(\Psi\) in the type-I uniform case with $L = 100$. The blue circles are the exact results from [4], and the red crosses are fits in cases (a) and (b). For (a) and (b), the first four points are excluded when computing the fits, since the theoretical predictions are far from the boundary. (c) Two free spin-1/2 spinons), the open models do not have this degeneracy, indicating the importance of the boundary effect.
The blue circles (with errorbars) are fits based on the theoretical prediction of the SU(2) Hamiltonian with PBC, which already shows up in the SU(2) case (see Fig. 2), making it possible to determine the SU(n) open Haldane-Shastry models [7]. As an outlook, we expect that the infinite MPS with open boundaries is readily applicable here. This explains the appearance of degenerate eigenstates with different SU(n) representations. As the monodromy matrix relevant for these models (open boundaries) satisfies the reflection equation [36], the algebraic structure of the SU(n) open Haldane-Shastry models [7] is the twisted Yangian [37]. Thus, the conserved charges $Q^a$ and $T^a$ form the lowest twisted Yangian generators.

**Conclusions.**—In this Rapid Communication, we have shown that infinite MPS with the image prescription are relevant for 1D critical chains with OBC, by presenting a spin-1/2 model, as well as its SU(n) generalization. We have constructed inhomogeneous open Haldane-Shastry models as their parent Hamiltonians, including the three open Haldane-Shastry models as special uniform cases. For the type-I spin-1/2 open Haldane-Shastry model, an exact expression for the two-point spin correlator has been derived and compared with theoretical predictions, supporting that the low-energy effective theory is the SU(2) WZW model with free boundary condition. We also characterize the full spectrum of the SU(n) open Haldane-Shastry models and determine the twisted Yangian generators responsible for the highly degenerate multiplets in the energy spectrum. The present infinite MPS with open boundaries is readily applicable to any boundary CFT for finding their lattice discretizations. As an outlook, we expect that the infinite MPS with OBC could be very useful for proposing trial wave functions for single-impurity Kondo problems, where boundary CFT are known [38, 39] to play an important role.

**Acknowledgment.**—We acknowledge J. I. Cirac and A. E. B. Nielsen for helpful discussions. This work has been supported by the EU project SIQS, FIS2012-33642, QUITEMAD (CAM), the Severo Ochoa Program, and the Fulbright grant PRX14/00352.
Note however that exceptional cases exist, for which the connection of the critical behaviors of the infinite MPS and the CFT for constructing them is unclear, see, e.g., the SU(n) states with alternating fundamental and conjugate representations in Refs. [17,18].
Supplemental Material

Inhomogeneous open Haldane-Shastry models

In this Section, we provide details on the derivation of the spin-1/2 inhomogeneous open Haldane-Shastry model and its SU(n) generalization.

To construct the spin-1/2 inhomogeneous open Haldane-Shastry model, we use the operators annihilating the spin-1/2 open infinite MPS

$$\Lambda^a_i = \frac{2}{3} \sum_{j \neq i}(w_{ij} + w_{ij}^*)(S_j^a + i\varepsilon_{abc}S^b_jS^c_j),$$

(1)

to build a positive semidefinite operator

$$\sum_a (\Lambda^a_i)^\dagger \Lambda^a_i = \frac{4}{9} \sum_{j,k(\neq i)}(w_{ij} + w_{ij}^*)(w_{ik} + \bar{w}_{ik})(S_j^a - i\varepsilon_{abc}S^b_jS^c_j)(S_k^a + i\varepsilon_{ade}S^d_kS^e_k)$$

$$= -\frac{4}{9} \sum_{j,k(\neq i)}(w_{ij} + w_{ij}^*)(w_{ik} + \bar{w}_{ik})(\bar{S}_j \cdot \bar{S}_k - 2i\varepsilon_{abc}S_j^aS_k^b\bar{S}_j^c + \varepsilon_{abc}\varepsilon_{ade}S_j^aS^d_k\bar{S}_j^e)$$

$$= -\frac{2}{3} \sum_{j,k(\neq i)}(w_{ij} + w_{ij}^*)(w_{ik} + \bar{w}_{ik})(\bar{S}_j \cdot \bar{S}_k - i\varepsilon_{abc}S^a_jS^b_k\bar{S}_j^c)$$

$$= -\frac{2}{3} \sum_{j(\neq i)}(w_{ij} + w_{ij}^*)^2\left(\frac{3}{4} - S_j \cdot S_j\right) - \frac{2}{3} \sum_{j(\neq i)}(w_{ij} + w_{ij}^*)(w_{ik} + \bar{w}_{ik})(\bar{S}_j \cdot \bar{S}_k),$$

(2)

where we have used $w_{ij}^* = -w_{ij}$, $S_j^aS^d_k = \frac{1}{4}\delta_{ab} + \frac{i}{2}\varepsilon_{abc}S^c$, $\varepsilon_{abc}\varepsilon_{abd} = 2\delta_{cd}$, and $\varepsilon_{abc}\varepsilon_{ade}\varepsilon_{bdf} = \varepsilon_{cde}$. Then, we obtain

$$\sum_{i,a}(\Lambda^a_i)^\dagger \Lambda^a_i = -\frac{2}{3} \sum_{i,j}(w_{ij} + w_{ij}^*)^2\left(\frac{3}{4} + S_j \cdot S_j\right) - \frac{2}{3} \sum_{3,j \neq k}(w_{ij} + w_{ij}^*)(w_{ik} + \bar{w}_{ik})(\bar{S}_j \cdot \bar{S}_k).$$

(3)

The following cyclic identity is the key for simplifying (3):

$$(w_{ij} + w_{ij}^*)(w_{ik} + \bar{w}_{ik}) + (w_{ji} + w_{ji}^*)(w_{jk} + \bar{w}_{jk}) + (w_{ki} + w_{ki}^*)(w_{kj} + \bar{w}_{kj}) = 4.$$  

(4)

By using this identity, we obtain

$$\sum_{i(\neq j,k)}(w_{ij} + w_{ij}^*)(w_{ik} + \bar{w}_{ik}) = \sum_{i(\neq j,k)}[4 - (w_{ji} + w_{ji}^*)(w_{jk} + \bar{w}_{jk}) - (w_{ki} + w_{ki}^*)(w_{kj} + \bar{w}_{kj})]$$

$$= 4(L - 2) - (w_{jk} + \bar{w}_{jk})\sum_{i(\neq j,k)}(w_{ji} + \bar{w}_{ji}) - (w_{kj} + \bar{w}_{kj})\sum_{i(\neq j,k)}(w_{ki} + \bar{w}_{ki})$$

$$= 4(L - 2) + 2(w_{jk}^2 + \bar{w}_{jk}^2) - (w_{jk} + \bar{w}_{jk})\left[w_{jj} + \sum_{i(\neq j)}(w_{ji} + \bar{w}_{ji})\right]$$

$$- (w_{kj} + \bar{w}_{kj})\left[w_{kk} + \sum_{i(\neq k)}(w_{ki} + \bar{w}_{ki})\right] + w_{jj}(w_{jk} + \bar{w}_{jk}) + w_{kk}(w_{kj} + \bar{w}_{kj})$$

$$= (4L - 6) + 2(w_{jk}^2 + \bar{w}_{jk}^2) + w_{jk}(c_j - c_k) + w_{jk}(c_j + c_k),$$

(5)

where we have defined $c_j \equiv w_{jj} + \sum_{i(\neq j)}(w_{ij} + \bar{w}_{ij})$ and have used $w_{j\bar{j}}(w_{jk} + \bar{w}_{jk}) + w_{j\bar{k}}(w_{kj} + \bar{w}_{kj}) = 2$ (the latter can be easily proved by using the cyclic identity $w_{ij}w_{ik} + w_{ji}w_{jk} + w_{ki}w_{kj} = 1$).
By substituting (5) into (3), we arrive at

\[
\sum_{i,a}(\Lambda^a_i)^\dagger \Lambda^a_i = \frac{2}{3} \sum_{i\neq j} (w_{ij} + w_{ij})^2 (\frac{3}{4} + \vec{S}_i \cdot \vec{S}_j)
\]

\[
= \frac{2}{3} \sum_{j \neq k} \left[ (4L - 6) + 2(w_{jk}^2 + w_{jk}^2) + w_{jk}(c_j - c_k) + w_{jk}(c_j + c_k) \right] (\vec{S}_j \cdot \vec{S}_k)
\]

\[
= 8 \sum_{i \neq j} \left[ \frac{1}{|z_i - z_j|^2} + \frac{1}{|z_i - z_j|^2} - \frac{w_{ij}(c_i - c_j) + w_{ij}(c_i + c_j)}{12} \right] (\vec{S}_i \cdot \vec{S}_j)
\]

\[
- \frac{8L}{3} S^2 - \frac{1}{2} \sum_{i \neq j} (w_{ij} + w_{ij})^2 + \frac{2L^2}{4},
\]

where we have used \(w_{ii}^2 = 1 - \frac{4}{|z_i - z_j|^2}\).

Then, the spin-1/2 inhomogeneous open Haldane-Shastry model is defined by

\[
H = \frac{1}{8} \sum_{i,a}(\Lambda^a_i)^\dagger \Lambda^a_i + \frac{L}{3} \vec{S}^2 + E
\]

\[
= \sum_{i \neq j} \left[ \frac{1}{|z_i - z_j|^2} + \frac{1}{|z_i - z_j|^2} - \frac{w_{ij}(c_i - c_j) + w_{ij}(c_i + c_j)}{12} \right] (\vec{S}_i \cdot \vec{S}_j),
\]

whose ground-state energy \(E\) is given by

\[
E = \frac{1}{16} \sum_{i \neq j} (w_{ij} + w_{ij})^2 - \frac{1}{4} L^2.
\]

The derivation of the SU(\(n\)) inhomogeneous open Haldane-Shastry model follows the similar steps for the spin-1/2 case. The operators annihilating the SU(\(n\)) infinite MPS are given by

\[
\Lambda^a_i = \frac{n + 2}{2(n + 1)} \sum_{j(k \neq i)} (w_{ij} + w_{ij})[\vec{t}^a_i + \frac{n}{n + 2} d_{abc} + i f_{abc} \vec{t}^b \vec{t}^c],
\]

where \(d_{abc}\) and \(f_{abc}\) are the SU(\(n\)) totally symmetry tensor and the totally antisymmetric structure constant, respectively.

Similar to the spin-1/2 case, we consider the positive semidefinite operator

\[
\sum_a (\Lambda^a_i)^\dagger \Lambda^a_i = \frac{(n + 2)^2}{4(n + 1)^2} \sum_{j,k(\neq i)} (w_{ij}^2 + w_{ij}^2)(w_{ik}^2 + w_{ik}^2)[\vec{t}^a_i + \frac{n}{n + 2} d_{abc} + i f_{abc} \vec{t}^b \vec{t}^c][\vec{t}^a_k + \frac{n}{n + 2} d_{abc} + i f_{abc} \vec{t}^b \vec{t}^c]
\]

\[
= \sum_{j,k(\neq i)} (w_{ij}^2 + w_{ij}^2)(w_{ik}^2 + w_{ik}^2)[\frac{n + 2}{2(n + 1)} (\vec{t}^a_i \cdot \vec{t}^a_k) + \frac{n}{2(n + 1)} d_{abc} t^b_i t^c_j t^d_k] - \frac{n + 2}{2(n + 1)} i f_{abc} t^b_i t^c_k
\]

\[
= - \sum_{j(\neq i)} (w_{ij}^2 + w_{ij}^2)^2 \left[ \frac{(n - 1)(n + 2)}{4n} + \frac{(n - 1)(n + 2)}{2(n + 1)} (\vec{t}^a_i \cdot \vec{t}^a_k) \right]
\]

\[
- \sum_{j \neq k(\neq i)} (w_{ij} + w_{ij})(w_{ik} + w_{ik}) \left[ \frac{n + 2}{2(n + 1)} (\vec{t}^a_i \cdot \vec{t}^a_k) + \frac{n}{2(n + 1)} d_{abc} t^b_i t^c_j t^d_k \right],
\]

where we have extensively used the identities listed in the Appendix A in Ref. \[1\]. Notice that

\[
\sum_{i \neq j \neq k} (w_{ij} + w_{ij})(w_{ik} + w_{ik}) d_{abc} t^a_i t^b_j t^c_k
\]

\[
= \frac{1}{3} \sum_{i \neq j \neq k} [(w_{ij} + w_{ij})(w_{ik} + w_{ik}) + (w_{ji} + w_{ji})(w_{jk} + w_{jk}) + (w_{ki} + w_{ki})(w_{kj} + w_{kj})] d_{abc} t^a_i t^b_j t^c_k
\]

\[
= \frac{4}{3} \sum_{i \neq j \neq k} d_{abc} t^a_i t^b_j t^c_k
\]

\[
= \frac{4}{3} d_{abc} T^a T^b T^c - \frac{2(n^2 - 4)}{n} T^a T^a + \frac{2(n^2 - 1)(n^2 - 4)}{3n^2} L,
\]
where \( T^a = \sum_j t^a_j \). Together with (15), we obtain

\[
\sum_{i,a} (\Lambda_i^a)^\dagger \Lambda_i^a = - \sum_{i \neq j} (w_{ij} + w_{ij}) \left[ \frac{(n-1)(n+2)}{4n} + \frac{(n-1)(n+2)}{2(n+1)} (\bar{t}_i \cdot \bar{t}_j) \right] \\
- \sum_{i \neq j \neq k} (w_{ij} + w_{ik}) (w_{ik} + w_{ik}) \left[ \frac{n+2}{2(n+1)} (\bar{t}_k \cdot \bar{t}_k) + \frac{n}{2(n+1)} d_{abc} t_i^a t_j^b t_k^c \right]
\]

\[
= 2(n+2) \sum_{i \neq j} \left[ \frac{1}{|z_i - z_j|^2} + \frac{1}{|z_i - z_j|^2} - \frac{w_{ij}(c_i - c_j) + w_{ij}(c_j + c_i)}{4(n+1)} \right] (\bar{t}_i \cdot \bar{t}_j)
\]

\[
- \frac{2n}{3(n+1)} d_{abc} T^a T^b T^c - \frac{2(n+2)L}{n+1} T^a T^a - \frac{(n-1)(n+2)}{4n} \sum_{i \neq j} (w_{ij}^2 + w_{ij}^2)
\]

(11)

Then, the SU(\( n \)) inhomogeneous open Haldane-Shastry model can be defined as

\[
H = \frac{1}{2(n+2)} \sum_{i,a} (\Lambda_i^a)^\dagger \Lambda_i^a + \frac{n}{3(n+1)(n+2)} d_{abc} T^a T^b T^c + \frac{L}{n+1} T^a T^a + E
\]

(12)

whose ground-state energy \( E \) is given by

\[
E = \frac{n-1}{8n} \sum_{i \neq j} (w_{ij}^2 + w_{ij}^2) - \frac{n-1}{12n} L(6L + n - 2).
\]

**Two-point spin correlation function for the type-I spin-1/2 open Haldane-Shastry model**

In this Section, we derive the exact expression of the two-point spin correlation function for the **type-I** spin-1/2 open Haldane-Shastry model.

As we mentioned in the main text, the two-point spin correlation function \( C_{ij} = \langle \Psi | \vec{S}_i \cdot \vec{S}_j | \Psi \rangle / \langle \Psi | \Psi \rangle \) satisfies the following linear equations:

\[
\frac{1}{u_i - u_j} C_{ij} + \sum_{l(\neq i,j)} \frac{1}{u_l - u_j} C_{jl} = - \frac{3}{4} \frac{1}{u_i - u_j}, \quad \forall i, j
\]

(13)

where \( u_j = \cos \theta_j \). Since \( |\Psi\rangle \) is a spin singlet, \( \sum_{j=1}^L \vec{S}_j |\Psi\rangle = 0 \), the correlator also satisfies

\[
\sum_{j(\neq i)} C_{ij} = - \frac{3}{4}.
\]

(14)

For instance, if one wants to determine the correlators involving the first spin, one could write down the \( L-1 \) linear equations (relating \( C_{1j}, j = 2, \ldots, L \)) in a matrix form:

\[
\begin{pmatrix}
-\frac{1}{u_1 - u_2} & \frac{1}{u_2 - u_3} & \frac{1}{u_2 - u_4} & \cdots & \frac{1}{u_2 - u_L} \\
\frac{1}{u_1 - u_3} & -\frac{1}{u_3 - u_4} & \frac{1}{u_3 - u_5} & \cdots & \frac{1}{u_3 - u_L} \\
\frac{1}{u_4 - u_2} & \frac{1}{u_4 - u_3} & -\frac{1}{u_4 - u_5} & \cdots & \frac{1}{u_4 - u_L} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{u_L - u_2} & \frac{1}{u_L - u_3} & \frac{1}{u_L - u_4} & \cdots & -\frac{1}{u_L - u_L}
\end{pmatrix} \begin{pmatrix}
C_{12} \\
C_{13} \\
C_{14} \\
\vdots \\
C_{1L}
\end{pmatrix} = - \frac{3}{4} \begin{pmatrix}
\frac{1}{u_2 - u_1} \\
\frac{1}{u_3 - u_1} \\
\frac{1}{u_4 - u_1} \\
\vdots \\
\frac{1}{u_L - u_1}
\end{pmatrix}.
\]

(15)

The correlators involving other spins can be solved in a similar fashion. For the moment, we carry out the derivations based on (15) for ease of notation and, in the end, extend the final result to the most general case.

For the type-I case with \( \theta_j = \frac{\pi}{2}(j - \frac{1}{2}) \), the following sum identity is very useful:

\[
\sum_{j(\neq i)} \frac{1}{u_j - u_i} \cos m \theta_j = \frac{2(L - m) \sin \theta_i \sin m \theta_i - \cos \theta_i \cos m \theta_i}{2 \sin^2 \theta_i},
\]

(16)
where \( m \) is an integer and \( m \in [0, 2L] \).

For the \( l \)-th row in (15), we multiply \( \cos m\theta_{l+1} \) and then sum over all the linear equations. By using (16), we obtain

\[
\sum_{j=2}^{L} \left[ \frac{\cos m\theta_j + \cos m\theta_j}{\cos \theta_1 - \cos \theta_j} - \frac{2(L-m) \sin \theta_j \sin m\theta_j - \cos \theta_j \cos m\theta_j}{2 \sin^2 \theta_j} \right] C_{1j} = \frac{3}{8} \frac{(L-m) \sin \theta_1 \sin m\theta_1 - \cos \theta_1 \cos m\theta_1}{\sin^2 \theta_1},
\]

where \( m \in [0, 2L] \). For \( m = 0 \), this yields

\[
\sum_{j=2}^{L} \left( \frac{2}{\cos \theta_1 - \cos \theta_j} + \frac{\cos \theta_j}{2 \sin^2 \theta_j} \right) C_{1j} = -\frac{3 \cos \theta_1}{8 \sin^2 \theta_1}.
\]

When multiplying (18) by \( \cos m\theta_1 \) and then subtracting with (17), we obtain

\[
\sum_{j=2}^{L} \left[ \frac{\cos m\theta_j - \cos m\theta_1}{\cos \theta_1 - \cos \theta_j} - \frac{2(L-m) \sin \theta_j \sin m\theta_1 + \cos \theta_j (\cos m\theta_1 - \cos m\theta_j)}{2 \sin^2 \theta_j} \right] C_{1j} = \frac{3}{4} \frac{(L-m) \sin m\theta_1}{\sin \theta_1}.
\]

Manipulating three consecutive linear equations [taking \( m - 1, m, \text{and} m + 1 \) in (19)], we arrive at

\[
\sum_{j \neq 1} \left[ (2L-2m+1) \frac{\cos(m+1)\theta_j}{\sin^2 \theta_j} - (2L-2m-1) \frac{\cos(m-1)\theta_j}{\sin^2 \theta_j} \right] (\cos \theta_1 - \cos \theta_j) C_{1j} = 3 \cos m\theta_1,
\]

which we have verified to hold for \( m \in [0, 2L] \).

In general, the two-point spin correlator satisfies the following equation:

\[
\sum_{j \neq i} \left[ (2L-2m+1) \frac{\cos(m+1)\theta_j}{\sin^2 \theta_j} - (2L-2m-1) \frac{\cos(m-1)\theta_j}{\sin^2 \theta_j} \right] (\cos \theta_i - \cos \theta_j) C_{ij} = 3 \cos m\theta_i,
\]

where \( m \in [0, 2L] \).

In practice, finding the analytical form of \( C_{ij} \) directly from (21) does not seem to be a simple task. Here we adopt an approach used in Ref. 3 to determine the analytical form of \( C_{ij} \) for a few finite-size chains, from which a well-educated guess helps to solve (21).

In the hardcore boson basis, the type-I open Haldane-Shastry ground state is written as

\[
|\Psi \rangle = \sum_{x_1 < \ldots < x_{L/2}} \Psi(x_1, \ldots, x_{L/2}) S_{x_1}^+ \cdots S_{x_{L/2}}^+ |0\rangle,
\]

where

\[
\Psi(x_1, \ldots, x_{L/2}) = (-1)^{\sum_{i=1}^{L/2} x_i} \prod_{i=1}^{L/2} \sin \theta_{x_i} \prod_{1 \leq i < j \leq L/2} (\cos \theta_{x_i} - \cos \theta_{x_j})^2.
\]

Here \( x_1, \ldots, x_{L/2} \) denote the positions of the hardcore bosons (up spins).

The norm of (22) is given by

\[
\langle \Psi | \Psi \rangle = \sum_{x_1 < \ldots < x_{L/2}} |\Psi(x_1, \ldots, x_{L/2})|^2
\]

\[
= \frac{1}{(L/2)!} \sum_{x_1, \ldots, x_{L/2}} \prod_{i=1}^{L/2} \sin^2 \theta_{x_i} \prod_{1 \leq i < j \leq L/2} (\cos \theta_{x_i} - \cos \theta_{x_j})^4
\]

\[
= \frac{1}{(L/2)!} \sum_{x_1, \ldots, x_{L/2}} \prod_{i=1}^{L/2} \sin^2 \theta_{x_i} \det
\begin{pmatrix}
1 & \cos \theta_{x_1} & \cos^2 \theta_{x_1} & \cdots & \cos^{L-1} \theta_{x_1} \\
0 & 1 & 2 \cos \theta_{x_1} & \cdots & (L-1) \cos^{L-2} \theta_{x_1} \\
1 & \cos \theta_{x_2} & \cos^2 \theta_{x_2} & \cdots & \cos^{L-1} \theta_{x_2} \\
0 & 1 & 2 \cos \theta_{x_2} & \cdots & (L-1) \cos^{L-2} \theta_{x_2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \cos \theta_{x_{L/2}} & \cos^2 \theta_{x_{L/2}} & \cdots & \cos^{L-1} \theta_{x_{L/2}} \\
0 & 1 & 2 \cos \theta_{x_{L/2}} & \cdots & (L-1) \cos^{L-2} \theta_{x_{L/2}}
\end{pmatrix},
\]

(24)
where in the last step we have used the **Confluent Alternant identity** \[3\]

\[
\prod_{1 \leq i < j \leq M} (y_i - y_j)^k = \det \begin{pmatrix}
1 & y_1 & y_1^2 & \cdots & y_1^{M-1} \\
0 & 1 & 2y_1 & \cdots & (M-1)y_1^{M-2} \\
1 & y_2 & y_2^2 & \cdots & y_2^{M-1} \\
0 & 1 & 2y_2 & \cdots & (M-1)y_2^{M-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & y_M & y_M^2 & \cdots & y_M^{M-1} \\
0 & 1 & 2y_M & \cdots & (M-1)y_M^{M-2}
\end{pmatrix}.
\] (25)

Similarly, the unnormalized transverse spin correlator (for \(i \neq j\)) can be expressed as

\[
\langle \Psi | S_i^+ S_j^- | \Psi \rangle = \frac{1}{(L/2 - 1)!} \sum_{x_1, \ldots, x_{L/2-1}} \Psi^*(i, x_1, \ldots, x_{L/2-1}) \Psi(j, x_1, \ldots, x_{L/2-1})
\]

\[
= \frac{(-1)^{i-j} \sin \theta_i \sin \theta_j}{(L/2 - 1)! \cos \theta_i - \cos \theta_j} \sum_{x_1, \ldots, x_{L/2-1}} \prod_{i=1}^{L/2-1} \sin^2 \theta_{x_i}
\]

\[
\times \det \begin{pmatrix}
1 & \cos \theta_i & \cos^2 \theta_i & \cos^3 \theta_i & \cdots & \cos^{L-1} \theta_i \\
1 & \cos \theta_j & \cos^2 \theta_j & \cos^3 \theta_j & \cdots & \cos^{L-1} \theta_j \\
1 & \cos \theta_{x_1} & \cos^2 \theta_{x_1} & \cos^3 \theta_{x_1} & \cdots & \cos^{L-1} \theta_{x_1} \\
0 & 1 & 2 \cos \theta_{x_1} & 3 \cos^2 \theta_{x_1} & \cdots & (L-1) \cos^{L-2} \theta_{x_1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \cos \theta_{x_{L/2-1}} & \cos^2 \theta_{x_{L/2-1}} & \cos^3 \theta_{x_{L/2-1}} & \cdots & \cos^{L-1} \theta_{x_{L/2-1}} \\
0 & 1 & 2 \cos \theta_{x_{L/2-1}} & 3 \cos^2 \theta_{x_{L/2-1}} & \cdots & (L-1) \cos^{L-2} \theta_{x_{L/2-1}}
\end{pmatrix}.
\] (26)

For small \(L\), \[24\] and \[26\] can be computed by expanding the determinants (with Laplace’s formula). After the expansion, the discrete sums over the coordinates can be carried out by using the following identities:

\[
\sum_{x=1}^{L} \sin^2 \theta_x \cos^{2r} \theta_x = \frac{1}{r + 1} \frac{1}{2^{2r+1}} \binom{2r}{r} L,
\] (27)

and

\[
\sum_{x=1}^{L} \sin^2 \theta_x \cos^{2r+1} \theta_x = 0,
\] (28)

which are valid for the type-I case and \(r = 0, \ldots, L/2 - 1\).

Following this procedure, we obtain for \(L = 4\)

\[
\langle \Psi | S_i^+ S_j^- | \Psi \rangle = \frac{(-1)^{i-j} \sin \theta_i \sin \theta_j}{L \cos \theta_i - \cos \theta_j} \left[ 2(\cos \theta_i - \cos \theta_j) + \frac{6}{5} (\cos 3\theta_i - \cos 3\theta_j) \right]
\]

\[
- \frac{4}{5} (\cos 2\theta_i \cos 3\theta_j - \cos 3\theta_i \cos 2\theta_j) \right].
\] (29)

For \(L = 6\), we obtain

\[
\langle \Psi | S_i^+ S_j^- | \Psi \rangle = \frac{(-1)^{i-j} \sin \theta_i \sin \theta_j}{L \cos \theta_i - \cos \theta_j} \left[ 2(\cos \theta_i - \cos \theta_j) + \frac{6}{5} (\cos 3\theta_i - \cos 3\theta_j) + \frac{14}{15} (\cos 5\theta_i - \cos 5\theta_j) \right]
\]

\[
- \frac{4}{5} (\cos 2\theta_i \cos 3\theta_j - \cos 3\theta_i \cos 2\theta_j) - \frac{28}{45} (\cos 2\theta_i \cos 5\theta_j - \cos 5\theta_i \cos 2\theta_j)
\]

\[
- \frac{4}{9} (\cos 4\theta_i \cos 5\theta_j - \cos 5\theta_i \cos 4\theta_j) \right].
\] (30)
For $L = 8$, we obtain

$$
\frac{\langle \Psi | S_i^+ S_j^- | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{(-1)^{i-j}}{L} \sin \theta_i \sin \theta_j \left[ 2(\cos \theta_i - \cos \theta_j) + \frac{6}{5} (\cos 3\theta_i - \cos 3\theta_j) + \frac{14}{15} (\cos 5\theta_i - \cos 5\theta_j) \right]
$$

\begin{align*}
&+ \frac{154}{195} (\cos 7\theta_i - \cos 7\theta_j) - \frac{4}{5} (\cos 2\theta_i \cos 3\theta_j - \cos 3\theta_i \cos 2\theta_j) - \frac{28}{45} (\cos 2\theta_i \cos 5\theta_j - \cos 5\theta_i \cos 2\theta_j) \\
&- \frac{308}{585} (\cos 2\theta_i \cos 7\theta_j - \cos 7\theta_i \cos 2\theta_j) - \frac{4}{9} (\cos 4\theta_i \cos 5\theta_j - \cos 5\theta_i \cos 4\theta_j) \\
&- \frac{44}{117} (\cos 4\theta_i \cos 7\theta_j - \cos 7\theta_i \cos 4\theta_j) - \frac{4}{13} (\cos 6\theta_i \cos 7\theta_j - \cos 7\theta_i \cos 6\theta_j). \tag{31}
\end{align*}

Since $\langle \Psi | S_i^+ S_j^- | \Psi \rangle = \langle \Psi | S_i^- S_j^+ | \Psi \rangle$ and $| \Psi \rangle$ is a spin singlet, we have

$$
C_{ij} = \frac{\langle \Psi | S_i^+ S_j^- | \Psi \rangle}{\langle \Psi | \Psi \rangle} = 3 \frac{\langle \Psi | S_i^+ S_j^- | \Psi \rangle}{\langle \Psi | \Psi \rangle}. \tag{32}
$$

For larger $L$, a direct computation of (24) and (26) becomes quickly involved. However, from the finite-size results (29)–(31), there is an indication that, for general $L$, the analytical form of the two-point spin correlator $C_{ij}$ is given by

$$
C_{ij} = 3(-1)^{i-j} \sin \theta_i \sin \theta_j \sum_{p=1}^{L/2} \sum_{q=0}^{p-1} g_{pq} \cos(2p-1)\theta_i \cos 2q\theta_j - \cos 2q\theta_i \cos(2p-1)\theta_j, \tag{33}
$$

where $g_{pq}$ has no $L$ dependence and its initial values are readily available from (31).

By substituting (33) into (21), the well-educated guess (33) indeed solves the linear equation and the general expression for $g_{pq}$ is found to be

$$
g_{pq} = \begin{cases} 
1 & p = 1, q = 0 \\
\prod_{m=1}^{p-1} \frac{4m-1}{4m+1} & p > 1, q = 0 \\
2 \prod_{m=1}^{p-1} \frac{4m-1}{4m+1} \prod_{n=1}^{q} \frac{4n-3}{4n-1} & p > 1, q > 0
\end{cases}. \tag{34}
$$

Twisted Yangian generators for the SU($n$) open Haldane-Shastry model

In this Section, we provide details on the derivation of the twisted Yangian generators for the SU($n$) open Haldane-Shastry model.

For the SU(2) open Haldane-Shastry model, such formalism has already been developed in Ref. [4]. Although its SU($n$) generalization is rather straightforward, we present the derivation below for the purpose of being self-contained.

Following Ref. [4], we introduce an unprojected Hamiltonian

$$
\hat{H} = -\sum_{i \neq j} \left[ \frac{z_i z_j}{(z_i - z_j)^2} (K_{ij} - 1) + \frac{z_i z_j^{-1}}{(z_i - z_j^{-1})^2} (\bar{K}_{ij} - 1) \right] - \sum_{i=1}^{L} \left[ b_1 \frac{z_i}{(z_i - 1)^2} + 2b_2 \frac{1}{(z_i - z_i^{-1})^2} \right] (K_i - 1), \tag{35}
$$

where the coordinates $z_i$ are viewed as dynamical variables, the coordinate permutation operators $K_{ij}$, $\bar{K}_{ij}$, and $K_i$, when acting on the coordinates, yield $K_{ij} z_i = z_j K_{ij}$, $\bar{K}_{ij} z_i = z_j^{-1} \bar{K}_{ij}$, and $K_i z_i = z_i^{-1} K_i$, and the constants $b_1$ and $b_2$ will be specified below.

We also define a projection operation $\pi$ which replaces the operators $K_{ij}$ and $\bar{K}_{ij}$ by the SU($n$) spin permutation operator $P_{ij} = 2\ell_j \cdot e_j^{\parallel} + \frac{1}{2}$, and $K_i$ by the identity operator once they have been moved to the right of an expression. In the simplest case with only one of these operators, we have

$$
\pi(K_{ij}) = \pi(K_{ij}) = P_{ij}, \quad \pi(K_i) = 1. \tag{36} \tag{37}
$$

If there are multiply coordinate permutation operators $K_{ij}$ and $\bar{K}_{ij}$ present, the rule of the projection operation is to insert a designed product of SU($n$) spin permutation operators (which itself should be an identity, e.g., $P_{ik} P_{ij} P_{ij} P_{ik} = \ldots$).
\[ 1 \) into the expression and then replace each combined product \( P_{ij} K_{ij} \) \( (\text{appearing to the right of an expression}) \) by an identity, e.g.,
\[
\pi(K_{ij} K_{ik}) = \pi(P_{ik} P_{ij} K_{ij} P_{ik} K_{ik}) = P_{ik} P_{ij}.
\]

After the projection operation, the coordinates are not dynamical any more. Then, the projected Hamiltonian is a pure SU(\( n \)) spin model
\[
H = \pi(H) = -\sum_{i \neq j} \left[ \frac{z_{ij} z_{ij}}{(z_i - z_j)^2} (P_{ij} - 1) + \frac{z_i z_j^{-1}}{(z_i - z_j)^2} (P_{ij} - 1) \right]
\]
\[
= \sum_{i \neq j} \left[ \frac{1}{|z_i - z_j|^2} + \frac{1}{|z_i - z_j|^2} \right] (P_{ij} - 1).
\]

In Ref. [4], it has been shown that the projected Hamiltonian is integrable, if the lattice coordinates correspond to the three uniform cases (see Fig. 1 in the main text) and the constants \( b_1 \) and \( b_2 \) in (35) are given by (i) type-I: \( b_1 = 0 \) and \( b_2 = 1 \); (ii) type-II: \( b_1 = 0 \) and \( b_2 = 3 \); (iii) type-III: \( b_1 = b_2 = 1 \). Notice that the three projected Hamiltonians [38], after subtracting a constant, just correspond to the open SU(\( n \)) Haldane-Shastry model [Eq. (7) in the main text].

The integrability becomes manifest by introducing the Dunkl operators
\[
d_i = \sum_{j(\neq i)} \frac{z_i}{z_i - z_j} K_{ij} - \sum_{j(< i)} \frac{z_j}{z_i - z_j} K_{ij} + \sum_{j(\neq i)} \frac{z_i}{z_i - z_j} \bar{K}_{ij} + \left( \frac{b_1}{|z_i - z_i - 1|} + \frac{b_2}{|z_i - z_i - 1|} \right) K_i,
\]
which are mutually commuting, \( [d_i, d_j] = 0 \) \( \forall i, j \), and all commute with the unprojected Hamiltonian, \( [d_i, \hat{H}] = 0 \) \( \forall i \).

After introducing an extra \( n \)-dimensional auxiliary Hilbert space (denoted by “0”), the SU(\( n \)) monodromy matrix \( T(u) \) can be defined as
\[
T(u) = \pi \left[ \prod_{i=1}^{L} \left( 1 + \frac{P_{00}}{u - d_i} \right) \left( 1 + \frac{b_1 + b_2}{2} \frac{1}{u} \right) \prod_{i=1}^{L} \left( 1 + \frac{P_{00}}{u + d_i} \right) \right],
\]
which is a \( n \times n \) operator-valued matrix function of the spectral parameter \( u \). Actually, it is a generating function of conserved charges, \( [T(u), \hat{H}] = 0 \) \( \forall u \). By using the Taylor expansion \( \frac{1}{u - d_i} = \frac{1}{u} + \frac{d_i}{u^2} + \frac{d_i^2}{u^3} + O(1/u^4) \) and implementing the projection, one obtains formally the following expression:
\[
T(u) = 1 + \frac{1}{u} \left( t_0^0 \otimes J_0^0 + \sum_{a=1}^{n^2-1} t_0^a \otimes J_1^a \right) + \frac{1}{u^2} \left( t_0^0 \otimes J_0^0 + \sum_{a=1}^{n^2-1} t_0^a \otimes J_1^a \right) + \cdots,
\]
where \( J_0^a \) and \( J_1^a \) \( (a = 1, \ldots, n^2 - 1) \) and \( \mu = 1, \ldots, \infty \) are conserved charges for the SU(\( n \)) open Haldane-Shastry model, \( [J_0^a, \hat{H}] = [J_1^a, \hat{H}] = 0 \). For the monodromy matrix (41), the conserved charges in the first- and second-order expansions in \( 1/u \) are trivial (such as \( T^a, d_{abc} T^b T^c, T^a T^a, \text{etc} \)). In the third-order expansion, we obtain, after a tedious but straightforward calculation, the following nontrivial conserved charge:
\[
Q^a = \sum_k t_k^a (w_{kk}^2 + \gamma_1 w_{kk}^2) - \gamma_2 \sum_{i \neq j \neq k} (w_{ij} + w_{kj} + w_{jk}) (w_{ij} - w_{ij}) t_k^a P_{ij} P_{ij},
\]
where \( \gamma_1 \) and \( \gamma_2 \), for the three uniform cases, are given by (i) type-I: \( \gamma_1 = 0, \gamma_2 = \frac{1}{2} \); (ii) type-II: \( \gamma_1 = 0, \gamma_2 = \frac{1}{10} \); (iii) type-III: \( \gamma_1 = 1, \gamma_2 = \frac{1}{2} \), respectively.

[1] H.-H. Tu, A. E. B. Nielsen, and G. Sierra, Quantum spin models for the SU(\( n \)) Wess-Zumino-Witten model, Nucl. Phys. B 886, 328 (2014).
[2] R. Bondesan and T. Quella, Infinite matrix product states for long-range SU(N) spin models, Nucl. Phys. B 886, 483 (2014).
[3] Y. Kuramoto and Y. Kato, Dynamics of one-dimensional quantum systems: inverse-square interaction models (Cambridge University Press, New York, 2009).
[4] D. Bernard, V. Pasquier, and D. Serban, Exact solution of long-range interacting spin chains with boundaries, Europhys. Lett. 30, 301 (1995).