On special partition of metric space

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Abstract

The main result of this paper is to show (under assumption that continuum is real-valued measurable) the existence of complete metric space of small cardinality admitting Kuratowski partition.

1 Introduction

In 1935 K. Kuratowski in [7], followed the results of Lusin from 1912, ([9]), posed the problem whether a function $f : X \to Y$, (where $X$ is completely metrizable and $Y$ is metrizable), such that each preimage of an open set of $Y$ has the Baire property, is continuous apart from a meager set.

At the same time the research concerning partitions of the interval $[0, 1]$ was carried out. R. H. Solovay (unpublished result) gave the proof (with using forcing methods and the generic ultrapower) of the result that any partition of $[0, 1]$ into Lebesgue measure zero sets produces a non-measurable set. Later, Bukovsky in [1] showed shorter and less complicated proof the Solovay’s.

In [2] there is shown that Kuratowski’s problem is equivalent to the problem of the existence of partitions of completely metrizable spaces into meager sets with the property that the union of each subfamily of this partition has

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the Baire property. Such a partition is called Kuratowski partition, shortly \( K \)-partition, (see the next section for the formal definition).

Our paper gives another perspective on this problem than was presented in [4], where the authors proved among others that ZFC + "there exists a \( K \)-partition of a Baire metric space" is consistent, then ZFC + "there exists measurable cardinal" is consistent as well, with using forcing methods and localisation properties.

The main result of this paper is to show (under assumption that continuum is real-valued measurable) the existence of complete metric space of small cardinality admitting Kuratowski partition.

We previously thought that for this purpose a special kind of an ideal associated with \( K \)-partition of a given space which is in [6] called \( K \)-ideal could be used, but it could not be for two reasons. First of all: it can be supposed that from a structure of such \( K \)-ideal one can "decode" complete information about \( K \)-partition of a given space. Unfortunately, this is not the case because, as shown in [6], the structure of such an ideal can be almost arbitrary, i.e. it can be the Fréchet ideal, (in the proof of this result there is used in [6] direct sum of spaces), so by [3] Lemma 22.20, p. 425] it is not precipitous, whenever \( \kappa \) is regular. Moreover, as demonstrated in [6], for measurable cardinal \( \kappa \), a \( \kappa \)-complete ideal can be represented by some \( K \)-ideal. However if \( \kappa = |\mathcal{F}| \) is not measurable cardinal, where \( \mathcal{F} \) is \( K \)-partition of a given space, then one can obtain an \( |\mathcal{F}| \)-complete ideal which can be the Fréchet ideal or a \( \kappa \)-complete ideal representing some \( K \)-ideal or can be a proper ideal of such \( K \)-ideal and contains the Fréchet ideal. Thus, for obtaining \( K \)-partition from \( K \)-ideal we need to have complete information about the space in which the ideal is considered. Secondly, wanting to use the "idea" of \( K \)-ideal, for proving this result, we must additionally assume that the space \( X(I) \), where \( I \) is \( K \)-ideal, is complete, (see [4] or the next section for formal definitions) because, as will be shown in Proposition 1 and Theorem 1 (in Section 3) only if \( X(I) \) is complete we have that \( I \) is maximal. Thus, the assumption used in Theorem 2 is the only one under which we can show the existence of complete metric space with \( K \)-partition.

From [2] follows that the restriction "\( \leq 2^{\omega} \)" of cardinality of a space with \( K \)-partition used in the sentence of Theorem 2 is the real restriction, (for simplifying we assume that \( c \) is real-valued measurable). Moreover, if we assume a space with smaller cardinality it admits \( K \)-partition but the existence of this partition does not enlarges for completion of this space, (compare [6]).
Summing up, Theorem 2 (which also works for arbitrary metric space) was a missing result in considerations on $K$-partitions and completes considerations around Kuratowski’ problem.

It is worth to add that the subject taken up in this paper has applications among others in measurable selectors theory and related topics.

This paper consists of two sections. Section 2 contains definitions and previous results concerning among others $K$-partition, precipitous ideal and measurable cardinal. Main results are given in Section 3.

Although a number of well-known definitions are given in Section 2, for definitions and facts not cited here we refer to e.g. [8] (topology) and [5] (set theory).

2 Definitions and previous results

2.1. Let $X$ be a topological space. A set $U \subseteq X$ has the Baire property iff there exist an open set $V \subset X$ and a meager set $M \subset X$ such that $U = V \Delta M$, where $\Delta$ means the symmetric difference of sets.

2.2. A partition $\mathcal{F}$ of $X$ into meager subsets of $X$ is called Kuratowski partition, (shortly $K$-partition), iff $\bigcup \mathcal{F}'$ has the Baire property for all $\mathcal{F}' \subseteq \mathcal{F}$. If there exists a $K$-partition of $X$ we always denote by $\mathcal{F}$ with the smallest cardinality $\kappa$. Moreover, we enumerate

$$\mathcal{F} = \{F_\alpha : \alpha < \kappa\}.$$ 

Obviously, $\kappa$ is regular. If $\kappa$ was singular, then $cf(\kappa)$ would be the minimal one. By Baire Theorem $\kappa$ is uncountable.

For a given set $U \subseteq X$ the family

$$\mathcal{F} \cap U = \{F \cap U : F \in \mathcal{F}\}$$

is $K$-partition of $U$ as a subspace of $X$.

2.3. With any $K$-partition $\mathcal{F} = \{F_\alpha : \alpha < \kappa\}$, indexed by a cardinal $\kappa$, one may associate an ideal

$$I_\mathcal{F} = \{A \subset \kappa : \bigcup_{\alpha \in A} F_\alpha \text{ is meager}\}$$
which is called K-ideal, (see [3]).
Note, that $I_F$ is a non-principal ideal. Moreover, $[\kappa]^{<\kappa} \subseteq I_F$ because $\kappa = \min\{|F| : F$ is K-partition of $X\}$.

2.4. Let $I$ be an ideal on $\kappa$ and let $S$ be a set with positive measure, i.e. $S \in P(\kappa) \setminus I$. (For our convenience we use $I^+$ instead of $P(\kappa) \setminus I$).
An I-partition of $S$ is a maximal family $W$ of subsets of $S$ of positive measure such that $A \cap B \in I$ for all distinct $A, B \in W$.

An I-partition $W_1$ of $S$ is a refinement of an I-partition $W_2$ of $S$, ($W_1 \leq W_2$), iff each $A \in W_1$ is a subset of some $B \in W_2$.

If $I$ is a $\kappa$-complete ideal on $\kappa$ containing singletons, then $I$ is precipitous iff whenever $S \in I^+$ and $\{W_n : n < \omega\}$ is a sequence of I-partitions of $S$ such that $W_0 \geq W_1 \geq \ldots \geq W_n \geq \ldots$, then there exists a sequence of sets $X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n \supseteq \ldots$ such that $X_n \in W_n$ for each $n \in \omega$ and $\bigcap_{n=0}^{\omega} X_n \neq \emptyset$, (see also [3] p. 424-425).

The ideal $I_F$ is an everywhere precipitous ideal if $I_{F \cap U}$ is precipitous for each non-empty open set $U \subseteq X$.

Fact 1 ([3]) Let $X$ be a Baire metric space with K-partition $F$ of cardinality $\kappa = \min\{|G| : G$ is K-partition for $X\}$. Then there exists an open set $U \subset X$ such that the K-ideal $I_{F \cap U}$ on $\kappa$ associated with $F \cap U$ is precipitous.

2.5. An uncountable cardinal $\kappa$ is measurable iff there exists a non-principal maximal and $\kappa$-complete ideal on $\kappa$.

Fact 2 ([4]) ZFC + "there exists measurable cardinal" is equiconsistent with ZFC + "there exists a Baire metric space with K-partition of cardinality $\kappa"$.

Fact 3 ([5]) (a) If $\kappa$ is a regular uncountable cardinal that carries a precipitous ideal, then $\kappa$ is measurable in some transitive model of ZFC.
(b) If $\kappa$ is measurable cardinal, then there exists a generic extension in which $\kappa = \aleph_1$, and $\kappa$ carries a precipitous ideal.

2.6. Let $I$ be an ideal over a cardinal $\kappa$ and let

$$X(I) = \{x \in (I^+)\omega : \bigcap\{x(n) : n \in \omega\} \neq \emptyset \text{ and } \forall_{n \in \omega} \bigcap\{x(m) : m < n\} \in I^+\}.$$ 

The set $X(I)$ is considered as a subset of a complete metric space $(I^+)\omega$,
where the set $I^+$ is equipped with the discrete topology, (see also [4]).

**Fact 4 ([4])** $X(I)$ is a Baire space iff $I$ is a precipitous ideal.

**Fact 5 ([4])** Let $I$ be a precipitous ideal over some regular cardinal. Then there is $K$-partition of $X(I)$.

2.7. A **nontrivial measure** on $X$ is a map $\mu : P(X) \to [0, 1]$ such that $\mu$ is countably additive measure vanishing on points with $\mu(X) = 1$.

A measure $\mu$ is $\kappa$-additive whenever $\{A_\xi : \xi < \lambda\}$ is a family of sets of measure zero and $\lambda < \kappa$ then $\bigcup_{\xi < \lambda} A_\xi$ is measure zero. There is the largest $\kappa$ such that $\mu$ is $\kappa$-additive. Then

$$add(\mu) = \min\{\kappa : \mu(\bigcup_{\xi < \kappa} A_\xi) > 0, \mu(A_\xi) = 0\}.$$

A cardinal $\kappa$ is **real-valued** iff $\kappa$ carries a nontrivial $\kappa$-additive measure.

**Fact 6 ([12], [11])** Let $\kappa$ be real-valued measurable. If $\kappa \leq 2^{\aleph_0}$ then there is an extension $\mu$ of Lebesque measure defined on all subsets of $\mathbb{R}$ with $add(\mu) = \kappa$.

**Fact 7 ([11])** The following theories are equiconsistent.
(1) ZFC+ “there is a measurable cardinal”.
(2) ZFC+”Lebesgue measure has a countably additive extension $\mu$ defined on every set of reals”.

**Fact 8 ([11])** Let $\kappa$ be a real-valued measurable cardinal. Let $\mu$ be a nontrivial $\kappa$-additive real-valued measure on $\kappa$. Then $I = \{A \subseteq \kappa : \mu(A) = 0\}$ is a nontrivial ideal in $P(\kappa)$.

**Fact 9 (Ulam, [11])** Let $\kappa$ be a real-valued measurable cardinal. Let $\mu$ be a nontrivial measure on $\kappa$. Then $I = \{A \subseteq \kappa : \mu(A) = 0\}$ is $\aleph_1$-saturated.

Obviously, $I$ defined in Fact 7 and Fact 8 is precipitous, (compare [5, Lemma 22.22]).

2.8. Let $\mu$ be a nontrivial measure on $X$. For all $\mu$-measurable sets of $X$
we define the density of a set $A$ at point $x \in X$ by

$$d(x, A) = \limsup_{\mu(U) \to 0} \frac{\mu(A \cap U)}{\mu(U)} > 0,$$

where $U$ is open subset of $X$ and $x \in U$.

**Fact 10** ([10], p. 129) Almost all points of any set $A$ are points of density for $A$.

### 3 Main results

**Proposition 1** Let $I$ be an ideal on $\kappa$. If $X(I)$ is complete then $I$ is maximal.

**Proof.** Suppose that $I$ is not maximal. Then there is a set $A_0 \in I^+$ such that $\kappa \setminus A_0 \in I^+$. Next, there is $A_1 \subset A_0$ such that $A_1 \in I^+$. Obviously $\kappa \setminus A_1 \in I^+$ because $\kappa \setminus A_0 \subset \kappa \setminus A_1$. Continuing we construct a decreasing sequence $(A_n)_{n \in \omega}$ such that

1. $\forall_{n,m \in \omega}$ if $m < n$ then $\bigcap_{k=m}^{n} A_k \in I^+$
2. $\bigcap_{n \in \omega} A_n = \emptyset$.

By (2), $\bigcap_{n \in \omega} A_n \in I$, hence does not belong to $X(I)$. Thus, $X(I)$ is not complete. 

**Theorem 1** Let $X$ be a complete metric space with $K$-partition $F$ of cardinality $\kappa$, where $\kappa = \min \{|G| : G \text{ is } K\text{-partition of } X\}$ and let $I_F$ be $K$-ideal associated with $F$. If $X(I_F)$ is complete, then $\kappa$ is measurable.

**Proof.** Let $F$ be the $K$-partition of $X$ of cardinality $\kappa$. Let $I_F$ be the $K$-ideal associated with $F$. By Fact 1, there exists a non-empty open set $U \subseteq X$ such that $I_{F \cap U}$ is a precipitous ideal. Without the loss of generality we can assume that $I_{F \cap U}$ is everywhere precipitous, hence is $\kappa$-complete. By remark in Section 2.3, $I_{F \cap U}$ is non-principal. By Proposition 1, $I_{F \cap U}$ is maximal. Hence it $\kappa$ is measurable.

**Theorem 2** Let $c$ be the smallest real-valued measurable cardinal. Then there exists a complete metric space of cardinality $\leq 2^c$ which admits $K$-partition.
**Proof.** Let $\mu: P(\mathfrak{c}) \rightarrow [0,1]$ be a nontrivial $\mathfrak{c}$-additive measure. Then, by Fact 6, $\mu$ extends the Lebesgue measure on $\mathbb{R}$. Let $A, B \in P(\mathfrak{c})$ be $\mu$-measurable sets. Define a relation

$$A \sim B \text{ iff } \mu(A \triangle B) = 0,$$

where $\triangle$ means the symmetric difference of sets. Notice that $\sim$ defined above is equivalence relation. Let $[A]$ means the equivalence class determined by $\mu$-measurable $A \in P(\mathfrak{c})$. Define a metric

$$\rho([A],[B]) = \mu(A \triangle B).$$

Since $A, B \in P(\mathfrak{c})$ are $\mu$-measurable, $\rho$ is well defined.

Let

$$Y = \{[A]: A \in P(\mathfrak{c}), A \text{ are } \mu - \text{measurable}\}.$$ 

The space $(X, \rho)$ is complete. Indeed. Let $([A_n])_{n \in \omega}$ be a sequence fulfilling Cauchy condition. Then $\bigcup_{n \in \omega} A_n$ is its limit point.

Now, divide $Y$ into $\{F_\alpha: \alpha < \mathfrak{c}\}$, where

$$F_\alpha = \{[A] \in X: \alpha = \min\{\beta < \mathfrak{c}: d(x_\beta, A) > 0\}\},$$

and $\{x_\beta : \beta < \mathfrak{c}\} = \mathfrak{c}$. It is obvious that $F_\alpha \cap F_\beta = \emptyset$ for any $\alpha, \beta < \mathfrak{c}$, $\alpha \neq \beta$.

For each $\alpha < \mathfrak{c}$, $F_\alpha$ is nowhere dense because it contains the set

$$V = \{[A \setminus U_\alpha]: \mu(A) > 0, U_\alpha \text{ is a neighbourhood of } x_\alpha\}$$

which is open and dense. To see this take $[A \setminus U_\alpha] \in V$ then $d(x_\alpha, A \setminus U_\alpha) > 0$, hence $[A \setminus U_\alpha] \in F_\alpha$. Moreover, $\bigcup_{\alpha \in B} F_\alpha$ has the Baire property because it contains

$$V(B) = \{[A \setminus U_\alpha]: \alpha \in B, \mu(A) > 0, U_\alpha \text{ is a neighbourhood of } x_\alpha\}$$

which is open and dense. Indeed. For arbitrary $[A \setminus U_\alpha]$, $d(x_\alpha, A \setminus U_\alpha) > 0$, thus $[A \setminus U_\alpha] \in F_\alpha$, $\{\alpha < \mathfrak{c}: [A \setminus U_\alpha] \in V(B)\} = B$ and $\min\{\alpha < \mathfrak{c}: [A \setminus U_\alpha] \in V(B)\} \in B$. Thus $Y$ is the required space. ■

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