Note on Viscosity Solution of Path-Dependent PDE and $G$-Martingales — 2nd version

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February 19, 2012

Abstract
In this note we introduce the notion of viscosity solution for a type of fully nonlinear parabolic path-dependent partial differential equations (P-PDE). We obtain new maximum principles, (or comparison theorem) for smooth solutions as well as for viscosity solutions. A solution of a backward stochastic differential equation and a $G$-martingale under a $G$-expectation are typical examples of such type of solutions of P-PDE.

Keyword: backward stochastic differential equation, nonlinear expectation, $G$-Brownian motion, Path-dependent PDE, viscosity solution, $g$-expectation, $G$-expectation, $g$-martingale, $G$-martingale, Itô integral and Itô’s calculus.

Mathematics Subject Classification (2010). 60H, 60E, 62C, 62D, 35J, 35K

1 Introduction
In general, a solution of a backward stochastic differential equation (BSDE in short) is an adapted process, namely, the value of this process at each time $t$ is a functional of the corresponding continuous path on $[0, t]$. When this value depends only on the current state of the path $\omega(t)$, we have proved (see [Peng1991], [Peng1992a,b], and [Peng-Pardoux1992]) that the solution of the BSDE is in fact a solution of a quasi-linear parabolic PDE. This relation was given by introducing what we called nonlinear Feynman-Kac formula. Recently we have introduced a new notion of $G$-martingale under a fully nonlinear expectation called $G$-expectation. A $G$-martingale can be also regarded as a solution of fully nonlinear BSDE if the solution is state dependent. We also refer to the 2BSDE formulation for such second order nonlinear BSDE (see [1] and [33]). It is then a very interesting problem, which was proposed in my lecture of ICM2010 [Peng2010b], whether a path dependent solution of a BSDE and/or a $G$-martingale can be considered as a nonlinear path-dependent PDE (PPDE) of parabolic and/or elliptic types.
Facing this challenge, in this note we will introduce a notion of viscosity solution for the above mentioned types of quasi-linear or fully nonlinear PPDE. Just as in the case of the classical PDE, an important advantage of a viscosity solution of PPDE is that we only need it to be a continuous functional of paths. Here a crucially important task is to prove the corresponding comparison (also called maximum) principle, or comparison theorem, which is the main objective of this note.

Smooth solutions of linear path-dependent PDE of parabolic types were initially introduced in [Dupire2009] in which a new type of functional Itô formula was proved and then used to find a \(C^{1,2}\) solution of the PDE. We also refer to [Cont-Fournié2010a,b] for further developments of this new calculus of Itô’s type with applications to finance. Recently we have obtained the existence and uniqueness of systems of smooth solutions of quasi-linear path-dependent PDE by a BSDE approach, see [Peng-Wang2011]. These methods are mainly based on stochastic calculus. Our approach in this note is based on techniques of PDE and can be directly applied to treat fully nonlinear path-dependent PDE. The advantage of this PDE approach is that, one can treat the solution locally (path by path), whereas BSDE and \(G\)-expectation are mainly a global approach.

In this 2nd version of the paper, the main improvement is as follows: It is known that the proof of maximum principles often involves a maximization procedure of the difference of a subsolution and a supersolution. Here a main difficulty is how to find a path which maximizes this difference, for the situation that the space of the path is not compact. In the 1st version, in order to get ride of this difficulty, we introduced an approach of “frozen-ness” of the main course of the paths where the maximization takes place.

But to apply this frozen procedure, we need to modify the definition of time-derivative (or horizontal derivative) which is a heavy cost. In this new version, we have improved this “frozen method” to a “left frozen” one. Using this we can find a desired maximum path without changing the original Dupire’s definition of the horizontal derivative. This method can be applied to obtain a comparison principle for smooth solutions as well as for viscosity solutions of 1st and 2nd order fully nonlinear PPDE. But for the case of the viscosity solution of 2nd order PPDE, in order to get the comparison principle, we need solutions to satisfy Condition (10), which is still to be improved.

This paper uses PDE methods and the results can have direct applications to stochastic analysis, e.g., martingales under a fully nonlinear expectation, called \(G\)-expectation, stochastic optimal controls, stochastic games, nonlinear pricing and risk measuring, and backward SDE. Recently many people are very interested in this new theory of path dependent PDE. [11] introduced a different stochastic approach to derive a maximum principle for a type of quasilinear PPDE, and the corresponding Perron’s approach to get the existence.

The note is organized as follows: in the next section we mainly recall the notion of space and time (or vertical and horizontal) derivatives of functional of paths, borrowed from [Dupire2009]. In section 3 we will introduce the “left frozen maximization” approach to obtain the maximum principle for \(C^{1,2}\)-solutions of fully nonlinear PDE. In Section 4, we introduce the notion of viscosity solution of fully nonlinear path-dependent
PDE. Section 5 is devoted to prove the maximum principle of these new PDE for viscosity solutions. Many important properties of this PPDE, such as uniqueness, monotonicity, positive homogeneity and convexity can be derived from this new maximum principle. It also provides a new PDE formulation of $G$-expectations with random coefficients $G$ (see [Nutz2010] for a formulation of stochastic calculus).

After the 1st version of this paper, Xiangdong Li told me about a different formulation of a type of loop-dependent PDE introduced by [Polyakov1980] in Gauge theory (see also Li’s paper). It’s relation with the present path dependent PDE is also an interesting problem.

2 Notations

For vectors $x, y \in \mathbb{R}^n$, we denote their scalar product by $\langle x, y \rangle$ and the Euclidean norm $\langle x, x \rangle^{1/2}$ by $|x|$. We also denote the linear space of $n \times n$ symmetric matrices by $\mathbb{S}(n)$.

The following notations are mainly from [Dupire2009]. Let $T > 0$ be fixed. For each $t \in [0, T]$, we denote by $\Lambda_t$ the set of right continuous, $\mathbb{R}^d$-valued functions on $[0, t]$, namely, $s_i \downarrow s$ implies $\omega(s_i) \to \omega(s)$, for each $\omega \in \Lambda_t$.

For each $\omega \in \Lambda_T$ the value of $\omega$ at time $s \in [0, T]$ is denoted by $\omega(s)$. Thus $\omega = (\omega(s))_{0 \leq s \leq T}$ is a right continuous process on $[0, T]$ and its value at time $s$ is $\omega(s)$. The path of $\omega$ up to time $t$ is denoted by $\omega_t$, i.e., $\omega_t = (\omega(s))_{0 \leq s \leq t} \in \Lambda_t$. We denote $\Lambda = \bigcup_{t \in [0, T]} \Lambda_t$. We also specifically write

$$\omega_t = (\omega(s))_{0 \leq s \leq t} = (\omega(s)_{0 \leq s \leq t}, \omega(t))$$

to indicate the terminal position $\omega(t)$ of $\omega_t$ which plays a special role in this framework. For each $\omega_t \in \Lambda$ and $x \in \mathbb{R}^d$ we denote

$$\omega_t^x(s) = \begin{cases} \omega(s), & \text{if } 0 \leq s < t, \\ \omega(t) + x, & \text{if } s = t, \end{cases} \quad \omega_{t, \delta}(s) = \begin{cases} \omega(s), & \text{if } 0 \leq s < t, \\ \omega(t), & \text{if } t \leq s < t + \delta. \end{cases}$$

Sometimes we denote $(\omega_t)^x = \omega_t^x$, and $(\omega_{t, \delta}) = \omega_{t, \delta}$. We also denote

$$\langle \omega_t^x \rangle_{t, \delta}(s) = \begin{cases} \omega(s), & \text{if } 0 \leq s < t, \\ \omega(t) + x, & \text{if } t \leq s < t + \delta. \end{cases}$$

Let $\bar{\omega}_t, \omega_t \in \Lambda_Q$ be given with $t \geq \bar{t}$, we denote $\bar{\omega}_t \otimes \omega_t \in \Lambda_t$ by

$$\bar{\omega}_t \otimes \omega_t := \begin{cases} \bar{\omega}(s), & \text{if } 0 \leq s < \bar{t}, \\ \omega(s), & \text{if } \bar{t} \leq s < t. \end{cases}$$

For a given open subset $Q \subset \mathbb{R}^d$, we denote it boundary by $\partial Q$ and $\bar{Q} = Q \cup \partial Q$.

$$\Lambda_{Q_t} := \{ \omega_t \in \Lambda_t : \omega(s) \in Q, s \in [0, t] \}, \quad \Lambda_Q := \bigcup_{t \in [0, T]} \Lambda_{Q_t},$$

$$\Lambda_{\bar{Q}_t} := \{ \omega_t \in \Lambda : \omega(s) \in Q, s \in [0, t), \omega(t) \in \bar{Q} \}, \quad \Lambda_{\bar{Q}} := \bigcup_{t \in [0, T]} \Lambda_{\bar{Q}_t},$$

$$\Lambda_{Q\bar{Q}} := \{ \omega_t \in \Lambda : \omega(s) \in Q, s \in [0, t), \omega(t) \in \partial Q \} \cup \Lambda_{\bar{Q}Q}.$$
We are interested in path functions. A path function \( u \) is a real function defined \( \Lambda \), i.e., \( u : \Lambda \rightarrow \mathbb{R} \). This function \( u = u(\omega_t) \), \( \omega_t \in \Lambda \) can be also regarded as a family of real valued functions:

\[
u(\omega_t) = u(t, \omega(s)_{0 \leq s \leq t}) = u(t, \omega(s)_{0 \leq s < t}, \omega(t)) : \omega_t \in \Lambda_{\Omega}, \quad t \in [0, T].\]

**Definition 1** We define

\( \text{USC}_* (\Lambda_{\Omega}) := \{ u : \Lambda \rightarrow \mathbb{R} : \text{such that} \)

(i) For each fixed \( \omega_t \in \Lambda u(t, x) := u((\omega_t)^i_{t, t}) \),

is a USC-function of \((t, x + \omega(t)) \in [0, T \times \ddot{Q}; \)

(ii) For each \( \omega_t \in \Lambda \) with \( t_i \uparrow t \), \( \lim sup_{i \rightarrow \infty} u(\omega_{t_i}) \leq sup u(\omega_t) \).

We also denote \( \text{LSC}_* (\Lambda_{\Omega}) := \{ -u | u \in \text{USC}_* (\Lambda_{\Omega}) \} \). A function \( u \in \text{USC}_* (\Lambda_{\Omega}) \) (resp. \( u \in \text{LSC}_* (\Lambda_{\Omega}) \)) is called an \( \Lambda \)-upper (resp. \( \Lambda \)-lower) semi continuous function. \( u \in C_\Lambda (\Lambda_{\Omega}) := \text{USC}_* (\Lambda_{\Omega}) \cap \text{LSC}_* (\Lambda_{\Omega}) \) is called an \( \Lambda \)-continuous function.

**Definition 2** Let \( u : \Lambda \Omega \rightarrow \mathbb{R} \) and \( \omega_t \in \Lambda_{\Omega} \) be given. If there exists \( p \in \mathbb{R}^d \) such that

\[
u(\omega_t^i) = u(\omega_t) + \langle p, x \rangle + o(|x|), \quad x + \omega(t) \in Q,
\]

then we say \( u \) is (vertically) differentiable (in \( x \)) at \( \omega_t \), and denote \( D_v u(\omega_t) = p \). If moreover, there exists \( A \in \mathbb{R}^d \) such that

\[
\nu(\omega_t^i) = u(\omega_t) + \langle p, x \rangle + \frac{1}{2} \langle Ax, x \rangle + o(|x|^2), \quad x + \omega(t) \in Q,
\]

then we say \( u \) is (vertically) twice differentiable (in \( x \)) at \( \omega_t \), and denote \( D_{v^2} u(\omega_t) = A \).

**Definition 3** Let \( u : \Lambda \Omega \rightarrow \mathbb{R} \) and \( \omega_t \in \Lambda_{\Omega} \) be given. If there exists \( a \in \mathbb{R} \) such that

\[
u(\omega_t, \delta) - u(\omega_t) = a \delta + o(\delta),
\]

then we say that \( u(\omega_t) \) is (horizontally) differentiable (in \( t \)) at \( \omega_t \), and denote \( D_h u(\omega_t) = a \).

**Definition 4** We define \( C^{1,0} (\Lambda_{\Omega}) \), the set of functions \( u := u(\omega_t), \omega_t \in \Lambda_{\Omega} \) for which \( D_t u(\omega_t) \) exists, for each \( \omega_t \in \Lambda_{\Omega} \), \( t \in [t, T] \) and \( u((\omega_t)^i_{t, t})_{t, s} \), \( D_t u(\omega_t) \) are continuous functions of \((s, x + \omega(t)) \in [t, T) \times Q \).

**Definition 5** We define \( C^{1,2} (\Lambda_{\Omega}) \) as the set of functions \( u := u(\omega_t) : \Lambda_{\Omega} \rightarrow \mathbb{R} \), for which \( \varphi(\omega_t) = D_t u(\omega_t), D_s u(\omega_t), D_{ss} u(\omega_t) \) exist for \( \omega_t \in \Lambda_{\Omega} \) and such that each \( \varphi((\omega_t)^i_{t, s+}) \) is a continuous function of \((s, x + \omega(t)) \in [t, T) \times Q \).

### 3 Comparison principle for \( C^{1,2} \)-solution of path-dependent PDE

In this paper we consider the following problem of path-dependent PDE of parabolic PDE. To find \( u \in C^{1,2} (\Lambda_{\Omega}) \) such that

\[
D_t u(\omega_t) + G(\omega_t, u(\omega_t), D_s u(\omega_t), D_{ss} u(\omega_t)) = 0, \quad \omega_t \in \Lambda_{\Omega},
\]

(1)
with a Cauchy condition:
\[ u(\omega_t) = \Phi(\omega_t), \quad \omega_t \in \Lambda_{Q}, \]
where \( G: \Lambda_Q \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d) \rightarrow \mathbb{R} \) and \( \Phi: \Lambda_{Q,R} \rightarrow \mathbb{R} \) are given functions.

We make the following assumption for \( G \):

\((H1)\) For each \( \omega_t \in \Lambda_Q, u, v \in \mathbb{R}, p \in \mathbb{R}^d \) and \( X, Y \in \mathbb{S}(d) \) such that \( u \geq v, X \leq Y \),
\[ G(\omega_t, u, p, X) \leq G(\omega_t, v, p, Y). \]

**Lemma 6 (Left frozen maximization)** Let \( Q \) be a bounded open subset of \( \mathbb{R}^d \) and let \( u \in USC_*(Q) \) be bounded from above. Then, for each \( \omega^{(0)}_t \in \Lambda_Q \), there exists \( \bar{\omega}_t \in \Lambda_Q \), \( \bar{t} \in [t_0, T] \), such that \( \bar{\omega}_t = \omega^{(0)}_t \otimes \bar{\omega}_t \), \( u(\bar{\omega}_t) \geq u(\omega^{(0)}_t) \), and
\[ u(\bar{\omega}_t) = \sup_{\gamma_t \in \Lambda_Q, t \geq \bar{t}} u(\omega_t \otimes \gamma_t). \]  

**Proof.** Without loss of generality, we can assume that \( u(\omega^{(0)}_t) \geq u(\omega^{(0)}_t, x) \), for all \( x \) such that \( x + \omega^{(0)}(t_0) \in Q \). We set \( m_0 := u(\omega^{(0)}_t) \) and
\[ \bar{m}_0 := \sup_{\gamma_t \in \Lambda_Q, t \geq t_0} u(\omega^{(0)}_t \otimes \gamma_t) \geq m_0. \]
If \( \bar{m}_0 = m_0 \) then we can take \( \bar{\omega}_t = \omega^{(0)}_t \) and finish the procedure. Otherwise there exists \( \omega^{(1)}_{t_1} \in \Lambda_Q \) with \( t_1 > t_0 \), such that \( \omega^{(1)}_{t_1} = \omega^{(0)}_t \otimes \omega^{(1)}_t \) and
\[ m_1 := u(\omega^{(1)}_{t_1}) \geq \frac{m_0 + \bar{m}_0}{2}. \]
We set
\[ \bar{m}_1 := \sup_{\gamma_t \in \Lambda_Q, t \geq t_1} u(\omega^{(1)}_{t_1} \otimes \gamma_t) \geq m_1. \]
If \( \bar{m}_1 = m_1 \) then we can take \( \bar{\omega}_t = \omega^{(1)}_{t_1} \) and finish the procedure. Otherwise we can find, for \( i = 2, 3, \cdots \), \( \omega^{(i)}_{t_i} \in \Lambda_Q \) with \( t_i > t_{i-1} \), such that \( \omega^{(i)}_{t_i} = \omega^{(i-1)}_{t_{i-1}} \otimes \omega^{(i)}_{t_{i-1}} \), \( u(\omega^{(i)}_{t_i}) \geq u(\omega^{(i-1)}_{t_{i-1}}) \), for all \( x \) such that \( x + \omega^{(i)}(t_i) \in Q \) and
\[ m_i := u(\omega^{(i)}_{t_i}) \geq \frac{m_{i-1} + \bar{m}_{i-1}}{2}, \]
\[ \bar{m}_i := \sup_{\gamma_t \in \Lambda_Q, t \geq t_i} u(\omega^{(i)}_{t_i} \otimes \gamma_t) \geq m_i. \]
and continue this procedure till the first time when \( \bar{m}_i = m_i \) and then finish the proof by setting \( \bar{\omega}_t = \omega^{(i)}_{t_i} \). For the last and “worst” case in which \( \bar{m}_i > m_i \), for all \( i = 0, 1, 2, \cdots \), we have \( t_i \uparrow \bar{t} \in [0, T] \). Then we can find \( \bar{\omega}_t \in \Lambda_Q \) such that \( \bar{\omega}_t = \omega^{(i)}_{t_i} \otimes \bar{\omega}_t \). Since \( u \in USC_*(\Lambda_Q) \), we can choose \( \bar{\omega}(\bar{t}) \in Q \) such that \( u(\bar{\omega}_t) \geq u(\bar{\omega})_t \), for all \( x \) such that \( x + \bar{\omega}(\bar{t}) \in \bar{Q} \). Since
\[ \bar{m}_{i+1} - m_{i+1} \leq \bar{m}_i - \frac{m_i + m_i}{2} = \frac{\bar{m}_i - m_i}{2}, \]
thus there exists $\bar{m} \in (m_0, \bar{m}_0)$, such that $m_i \downarrow \bar{m}$ and $m_i \uparrow \bar{m}$. Thus $\lim_{i \to \infty} u(\omega^{(i)}_t) = \lim_{i \to \infty} m_i \leq u(\bar{\omega}_t)$. We can claim that (3) holds for this $\bar{\omega}_t$. Indeed, otherwise there exist $\gamma_t \in \Lambda_Q$ and $\delta > 0$ with $t > \bar{t}$ and $\gamma_t = \bar{\omega}_t \cap \gamma_t$, such that $$u(\bar{\omega}_t \cap \gamma_t) \geq u(\bar{\omega}_t) + \delta = \bar{m} + \delta.$$ then the following contradiction is induced: $$u(\bar{\omega}_t \cap \gamma_t) = u(\omega^{(i)}_{\bar{t}} \cap \gamma_t) \leq \bar{m}_i \to \bar{m}.$$ The proof is complete. 

**Definition 7** A function $u \in C^{1,2}(\Lambda_Q)$ is called a $C^{1,2}$-solution of the path dependent PDE (1) if for each $\omega_t \in \Lambda_Q$, $t \in [0,T)$, the equality (1) is satisfied. $u$ is called a subsolution (resp. supersolution) of (1) if the "$\leq$" in (1) is replace by "$\geq$" (resp. "$\leq$").

**Remark 8** The solution of classical PDE is a special case when $u(\omega_t) = \bar{u}(t, \omega(t)), \bar{u} \in C^{1,2}([0,T] \times Q)$. Since, for each $\omega_t \in \Lambda_Q$ and $t \in (0,T)$, $\partial_t \bar{u}(t, \omega(t)) = D_t u(\omega_t), D_x \bar{u}(t, \omega(t)) = D_x u(\omega_t), D_{xx} \bar{u}(t, \omega(t)) = D_{xx} u(\omega_t),$ thus $\bar{u}(t,x)$ is a classical solution of PDE.

**Lemma 9** Let $u \in C^{1,2}(\Lambda_Q)$ and $\bar{\omega}_t \in \Lambda_Q$, $t \in [0,T)$ be given satisfying $u(\bar{\omega}_t) \geq u(\omega_t \cap \omega_t)$, for all $\omega_t \in \Lambda_Q$, $t \geq \bar{t}$.

Then we have $$D_t u(\bar{\omega}_t) \leq 0, \quad D_x u(\bar{\omega}_t) = 0, \quad D_{xx} u(\bar{\omega}_t) \leq 0.$$ 

**Proof.** We set $\omega(s) = x + \bar{\omega}(t) 1_{[\bar{t},t]}(s), s \in [0,T]$ and define $\bar{u}(t,x) := u(\bar{\omega}_t \cap \omega_t) = u((\bar{\omega}_t^2(t, \omega(t)) = u^2 \cap \omega_t)$.

For $x = 0$, we derive from (3) condition that $\bar{u}(t,0) \geq \bar{u}(t,0)$, for $t \geq \bar{t}$.

and thus $\frac{d}{dt} \bar{u}(t,0) \leq 0$, or $D_t \bar{u}(\bar{\omega}_t) \leq 0$. For $t = \bar{t}$ we derive from (4) $\bar{u}(\bar{t},0) \geq \bar{u}(\bar{t},x)$, for sufficiently small $x$, and thus $D_x \bar{u}(\bar{t},0) = 0, D_{xx} \bar{u}(\bar{t},0) \leq 0$, from which we have the second and third relations of (3).

The following result is the so called comparison principle, or comparison theorem, of PPDE for $C^{1,2}$-solutions of (1).

**Theorem 10** We make Assumption (H1). Let $Q$ be a bounded open subset of $\mathbb{R}^d$ and $u \in C^{1,2}(\Lambda_Q) \cap USC^*_t(\Lambda_Q)$ be a subsolution and $v \in C^{1,2}(\Lambda_Q) \cap LSC^*_t(\Lambda_Q)$ a supersolution of (1). We also assume that $u - v$ is bounded from the above. Then the maximum principle holds: if $u(\omega_t) \leq v(\omega_t)$ for all $\omega_t \in \Lambda_{t,0}$, then we also have $u(\omega_t) \leq v(\omega_t)$, $\forall \omega_t \in \Lambda_Q$. 

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Proof. We observe that for $\bar{\delta} > 0$, the function defined by $\bar{u} := u - \bar{\delta}/t$ is a subsolution of

$$D_t \bar{u}(\omega_t) + G(\omega_t, \bar{u}(\omega_t), D_x \bar{u}(\omega_t), D_{xx} \bar{u}(\omega_t)) \geq \frac{\bar{\delta}}{t^2}.$$  

Since $u \leq v$ follows from $\bar{u} \leq v$ in the limit $\bar{\delta} \downarrow 0$, it suffices to prove the theorem under the additional assumptions:

$$D_t u(\omega_t) + G(\omega_t, u(\omega_t), D_x u(\omega_t), D_{xx} u(\omega_t)) \geq c, \quad c := \frac{\bar{\delta}}{T^2},$$

and $\lim_{t \to 0} u(\omega_t) = -\infty$, uniformly on $[0, T]$.

(6)

Suppose by the contrary that there exists $\omega^{(0)}_{t_0} \in \Lambda_Q$, $t_0 < T$, such that

$$m_0 := u(\omega^{(0)}_{t_0}) - v(\omega^{(0)}_{t_0}) > 0.$$  

Then, by Lemma 8 there exists $\bar{\omega}_t \in \Lambda_Q$ such that

$$u(\bar{\omega}_t) - v(\bar{\omega}_t) \geq \sup_{\gamma_t \in \Lambda_Q, \, t \in [t, T]} u(\bar{\omega}_t \otimes \gamma_t) - v(\bar{\omega}_t \otimes \gamma_t) \geq m_0.$$  

(7)

But by Lemma 9

$$D_t u(\bar{\omega}_t) - D_t v(\bar{\omega}_t) \leq 0,$$

$$D_x u(\bar{\omega}_t) - D_x v(\bar{\omega}_t) = 0, \quad D_{xx}^2 u(\bar{\omega}_t) - D_{xx}^2 v(\bar{\omega}_t) \leq 0.$$  

It follows that

$$0 < c \leq D_t u(\bar{\omega}_t) + G(\bar{\omega}_t, u(\bar{\omega}_t), D_x u(\bar{\omega}_t), D_{xx}^2 u(\bar{\omega}_t))$$

$$\leq D_t v(\bar{\omega}_t) + G(\bar{\omega}_t, v(\bar{\omega}_t), D_x v(\bar{\omega}_t), D_{xx}^2 v(\bar{\omega}_t)) \leq 0.$$  

This induces a contradiction. The proof is complete. 

4 Viscosity solutions for path-dependent PDE

The notion viscosity solutions for classical (state-dependent) PDEs were firstly introduced by Crandall and Lions [1981]. For many important contributions in the developments of this powerful and elegant theory and rich literature, we refer to the well-known user’s guide by Crandall, Ishii and Lions [1992]. A parabolic version of this theory quite helpful to understand the present framework of path-dependent PDE can be found in the Appendix C of [Peng2010a].

Consider the following path-dependence parabolic PDE: to find a function $u = u(\omega_t) \in USC^*_c(\Lambda_Q)$ such that

$$\begin{cases}
D_t u(\omega_t) + G(u(\omega_t), D_x u(\omega_t), D_{xx} u(\omega_t)) = 0 \text{ on } \omega_t \in \Lambda_Q,
\quad u(\omega_T) = \Phi(\omega_T) \text{ for } \omega_T \in \Lambda_{Q_T},
\end{cases}$$

(8)

where $G : \mathbb{R} \times \mathbb{R}^d \times S(d) \mapsto \mathbb{R}$ and $\Phi \in USC^*_c(\Lambda_{Q_T}) : \Lambda_{Q_T} \mapsto \mathbb{R}$ are given functions. We make the following assumption:
Suppose that $G \in C(\mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d))$ is a continuous function satisfying the following condition:

$$G(u, p, X) \geq G(v, p, Y) \quad \text{whenever } X \geq Y, \quad u \leq v. \quad (9)$$

We now generalize the definition of viscosity solutions to the situation of PPDE.

Let $u \in USC(\Lambda_Q)$, we denote by $P^{2,+}u(\omega_t)$ (the ”parabolic superjet” of $u$ at a given $\omega_t \in \Lambda_Q$, $t \in (0, T)$) the set of triples $(a, p, X) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d)$ such that

$$u((\omega_t^x)_t, \delta) \leq u(\omega_t) + a\delta + \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle + o(|\delta + |x|^2),$$

for each $\delta \geq 0$ and $x \in \mathbb{R}^d$ such that $\omega(t) + x \in Q$. It is easy to check that if $\varphi \in \mathcal{C}^{1,2}(\Lambda_Q)$ satisfies

$$u(\omega_t) = \varphi(\omega_t), \quad u((\omega_t^x)_t, \delta) \leq \varphi((\omega_t^x)_t + \delta),$$

then $(D_t\varphi(\omega), D_t\varphi(\omega), D_{xx}\varphi(\omega)) \in P^{2,+}u(\omega_t)$.

We also denote

$$^\ast P^{2,+}u(\omega_t) = \{(a, p, X) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) : \exists (\omega^a_n, a_n, p_n, X_n)$$

$$\text{such that } (a_n, p_n, X_n) \in P^{2,+}u(\omega^a_n) \text{ and } (\omega^a_n, u(\omega^a_n), a_n, p_n, X_n) \to (\omega_t, u(\omega_t), a, p, X)\}.$$

We define $P^{2,-}u(\omega_t)$ (the “parabolic subjet” of $u$ at $\omega_t$) by $P^{2,-}u(\omega_t) = -P^{2,+}(-u)(\omega_t)$ and $P^{2,-}u(\omega_t)$ by $P^{2,-}u(\omega_t) = -^\ast P^{2,+}(-u)(\omega_t)$.

**Definition 11** A viscosity subsolution (resp. supersolution) of (8) on $\Lambda_Q$ is a function $u \in USC(\Lambda_Q)$ such that for each fixed $\omega_t \in \Lambda_Q$, $t \in (0, T)$ and for each $(a, p, X) \in P^{2,+}u(\omega_t)$, (resp. $(a, p, X) \in P^{2,-}u(\omega_t)$) we have

$$a + G(u(\omega_t), p, X) \geq 0 \quad (\text{resp. } \leq 0). \quad (10)$$

$u$ is a viscosity solution if it is both sub and supersolution.

**Remark 12** Since $G$ is a continuous function, thus (10) holds also for $(a, p, X) \in P^{2,+}u(\omega_t)$ (resp. $(a, p, X) \in P^{2,-}u(\omega_t)$).

**Remark 13** If $\psi \in \mathcal{C}^{1,2}(\Lambda_Q)$, $\hat{\omega}_i \in \Lambda$, $i \in (0, T)$, $\psi(\hat{\omega}_i) = u(\hat{\omega}_i)$ are such that

$$\psi(\hat{\omega}_i \otimes \gamma_t) \geq u(\hat{\omega}_i \otimes \gamma_t), \quad (\text{resp. } \psi(\hat{\omega}_i \otimes \gamma_t) \leq u(\hat{\omega}_i \otimes \gamma_t))$$

$$\forall \gamma_t \in \Lambda_Q, \ t \in [\hat{i}, T],$$

then we have $(D_t\psi(\hat{\omega}_i), D_t\psi(\hat{\omega}_i), D_{xx}\psi(\hat{\omega}_i)) \in P^{2,+}u(\hat{\omega}_i)$ (resp. $P^{2,-}u(\hat{\omega}_i)$).

**Remark 14** If $G = G(u, p)$ then (8) becomes a first order path-dependent PDE:

$$\begin{cases} D_tu(\omega_t) + G(u(\omega_t), D_xu(\omega_t)) = 0 \text{ on } \omega_t \in \Lambda_Q, \\
u(\omega_T) = \Phi(\omega_T) \text{ for } \omega_T \in \Lambda_{\Omega Q}, \end{cases} \quad (11)$$

In this case we can similarly define $P^{1,+}u(\omega_t)$ (resp. $P^{1,-}u(\omega_t)$), $^\ast P^{1,+}u(\omega_t)$ (resp. $^\ast P^{1,-}u(\omega_t)$) and use them to give the notion the corresponding viscosity solution.
5 Comparison principle for viscosity solution of 2nd order path-dependent PDE

In this section we show that the approach of left frozen maximization introduced in the proof of the comparison principle for smooth solutions of PPDE can be also applied for the situation of viscosity solutions.

The following lemma is from Theorem 8.2 of [CIL1992]. See also [Cradall1989].

Lemma 15 Let $u_i \in USC((0, T) \times Q)$ for $i = 1, \ldots, k$ be given. Let $\varphi$ be a function defined on $(0, T) \times Q^\otimes k$ such that $(t, x_1, \ldots, x_k) \mapsto \varphi(t, x_1, \ldots, x_k)$ is once continuously differentiable in $t$ and twice continuously differentiable in $(x_1, \ldots, x_k) \in Q^\otimes k$. Suppose that $t \in (0, T)$, $\hat{x}_i \in \mathbb{R}^d$ for $i = 1, \ldots, k$ and

$$w(t, x_1, \ldots, x_k) := u_1(t, x_1) + \cdots + u_k(t, x_k) - \varphi(t, x_1, \ldots, x_k)$$

$$w(t, x_1, \ldots, x_k) \leq w(\hat{t}, \hat{x}_1, \ldots, \hat{x}_k), \quad t \in [\hat{t}, T), \quad x_i \in Q \quad (12)$$

Assume, moreover, that there exists $r > 0$ such that for every $M > 0$ there exists constant $C$ such that for $i = 1, \ldots, k$,

$$b_i \leq C \text{ whenever } (b_i, q_i, X_i) \in \mathcal{P}^2 u_i(t, x_i), \quad |x_i - \hat{x}_i| + |t - \hat{t}| \leq r \text{ and } |u_i(t, x_i)| + |q_i| + \|X_i\| \leq M. \quad (13)$$

Then there exist $b_1, \ldots, b_k \in \mathbb{R}$, such that

(i) $(b_i, D_{x_i} \varphi(t, \hat{x}_1, \ldots, \hat{x}_k), X_i) \in \mathcal{P}^2 u_i(t, \hat{x}_i), \quad i = 1, \ldots, k,$

(ii) $b_1 + \cdots + b_k \leq \partial_t \varphi(t, \hat{x}_1, \ldots, \hat{x}_k),$

(iii) $$-\left(\frac{1}{\varepsilon} + \|A\|\right)I \leq \begin{bmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{bmatrix} \leq A + \varepsilon A^2,$$

where $A = D^2_{xy} \varphi(\hat{x}) \in \mathbb{S}(kd)$.

Theorem 16 (Comparison principle) We assume (H2). Let $Q$ be a bounded open subset of $\mathbb{R}^d$ and $u \in USC_*(\Lambda_Q)$ (resp. $v \in LSC_*(\Lambda_Q)$) be a viscosity subsolution (resp. supersolution) of

$$D_t u + G(u(\omega_t), D_x u(\omega_t), D_{xx}^2 u(\omega_t)) = 0. \quad (14)$$

Assume that $u - v$ is bounded from above by $C$ and they satisfy

$$|u(\omega_t) - u(\nu_s)| \vee |v(\omega_t) - v(\nu_s)| \leq \rho(d(\omega_t, \nu_s)),$$

where $\rho : [0, \infty) \to \mathbb{R}$ is a given continuous and increasing function with $\rho(0) = 0$ and

$$d(\omega_t, \nu_s) := |\omega(t) - v(s)|^2 + \int_0^{|\omega(t) - v(s)|} (d(r, \omega(t) - \hat{\omega}(r) \wedge s))^2 dr.$$

We also assume that, for each $\hat{\omega}_i \in \Lambda_Q$, the functions

$$\hat{u}(t, x) := u(\hat{\omega}_i(t, x)), \quad \hat{v}(t, x) := v(\hat{\omega}_i(t, x)), \quad t \in [\hat{t}, T), \quad x + \hat{\omega}(t) \in Q.$$
satisfy
\[
\bar{\mathbf{P}}^{2,+} \hat{u}(t,0) \subset \bar{\mathbf{P}}^{2,+} u(\hat{\omega}_t), \quad \bar{\mathbf{P}}^{2,-} \hat{v}(t,0) \subset \bar{\mathbf{P}}^{2,-} v(\hat{\omega}_t).
\]
(16)

Then the following comparison holds: if \( u(\omega_t) \leq v(\omega_t) \) for \( \omega_t \in \Lambda_{Q,T} \), then \( u(\omega_t) \leq v(\omega_t) \) for \( \omega_t \in \Lambda_{Q} \).

We first observe that for \( \bar{\delta} > 0 \), the functions defined by \( \bar{u}_t := u_t - \bar{\delta}/t \) is a subsolution of

\[
D_t \bar{u}(\omega_t) + G(\bar{u}(\omega_t), D_x \bar{u}(\omega_t), D_{xx} \bar{u}(\omega_t)) = \frac{\bar{\delta}}{t^2},
\]

Since \( u \leq v \) follows from \( \bar{u} \leq v \) in the limit \( \bar{\delta} \downarrow 0 \), it suffices to prove the theorem under the additional assumptions:

\[
D_t u(\omega_t) + G(u(\omega_t), D_x u(\omega_t), D_{xx} u(\omega_t)) \geq c, \quad c := \bar{\delta}/T^2.
\]
(17)

To prove this theorem, for \( \omega^1_t, \omega^2_t \in \Lambda \), we set

\[
w_a(\omega^1_t, \omega^2_t) := u(\omega^1_t) - v(\omega^2_t) - \varphi_a(\omega_t), \quad \omega = (\omega^1, \omega^2),
\]
\[
\varphi_a(\omega^1_t, \omega^2_t) := \frac{\alpha}{2} \|\omega^1_t - \omega^2_t\|^2
\]
\[
= \frac{\alpha}{2} |\omega^1(t) - \omega^2(t)|^2 + \frac{\alpha}{2} \int_0^t |\omega^1(s) - \omega^2(s)|^2 ds.
\]

**Proof of Theorem 16.** The proof is still based on our “left frozen maximization” approach. To prove the theorem, we assume to the contrary that there exists \( \hat{\omega}_t \in \Lambda_Q, \hat{t} \in (0,T) \), such that \( \bar{m} := u(\hat{\omega}_t) - v(\hat{\omega}_t) > 0 \).

Let \( \alpha \) be a large number such that

\[
\frac{1}{\alpha} + \rho(\frac{1}{\alpha} (\frac{1}{\alpha} + 2C - M_*)) \leq \frac{1}{2} (\bar{m} \wedge c).
\]

We set

\[
M_* := \sup_{\omega_t \in \Lambda_{Q \times Q}} w_a(\omega_t) \geq M_* := \sup_{\omega_t \in \Lambda_Q} [u(\hat{\omega}_t) - v(\hat{\omega}_t)] \geq \bar{m}.
\]

First let us check that if \( \omega_t = (\omega^1_t, \omega^2_t) \in \Lambda_{Q \times Q} \) satisfies \( w_a(\omega_t) + \frac{1}{\alpha} \geq M_a \).

Then we have

\[
\frac{\alpha}{2} \|\omega^1_t - \omega^2_t\|^2 \leq \frac{1}{\alpha} + u(\omega^1_t) - v(\omega^2_t) - M_a
\]
\[
\leq \frac{1}{\alpha} + u(\omega^1_t) - v(\omega^2_t) - M_*
\]
\[
\leq \frac{1}{\alpha} + 2C - M_*.
\]

But we also have

\[
M_* \leq \frac{1}{\alpha} + [u(\omega^1_t) - u(\omega^2_t)] + [u(\omega^2_t) - v(\omega^2_t)] - \frac{\alpha}{2} \|\omega^1_t - \omega^2_t\|^2
\]
\[
\leq \frac{1}{\alpha} + \rho(\|\omega^1_t - \omega^2_t\|^2) + [u(\omega^2_t) - v(\omega^2_t)] - \frac{\alpha}{2} \|\omega^1_t - \omega^2_t\|^2
\]
\[
\leq \frac{1}{\alpha} + \rho(\frac{1}{\alpha} (\frac{1}{\alpha} + 2C - M_*)) + M_* - \frac{\alpha}{2} \|\omega^1_t - \omega^2_t\|^2.
\]
Thus
\[
\frac{\alpha}{2} \|\omega_1^t - \omega_2^t\|^2 \leq \frac{1}{\alpha} + \rho\left(\frac{2}{\alpha} \left(\frac{1}{\alpha} + 2C - M_\alpha\right)\right) \leq \frac{c}{2}.
\]
Moreover \(\omega_1^t, \omega_2^t \in \Lambda_Q\). Indeed if, \(\omega_1^t\) or \(\omega_2^t\) \(\in \Lambda_{\partial Q}\), then we will deduce the following contradiction:
\[
\bar{m} \leq M_{\bar{m}} \leq M_\alpha \leq M_\alpha \leq M_\alpha \\leq \frac{1}{\alpha} + \frac{1}{\alpha} + [u(\omega_1^t) - u(\omega_2^t)] + u(\omega_2^t) - v(\omega_1^t) \leq \frac{1}{\alpha} + \rho\left(\frac{2}{\alpha} \left(\frac{1}{\alpha} + 2C - M_\alpha\right)\right) \leq \frac{c}{2}.
\]
Now we can apply Lemma 6 to find \(\tilde{\omega}_t = (\tilde{\omega}_1^t, \tilde{\omega}_2^t) \in \Lambda_{Q \times Q}\) satisfying \(\tilde{\omega}_t = \omega_t \otimes \omega_t, w_\alpha(\tilde{\omega}_t) \geq w_\alpha(\omega_t)\) such that
\[
w_\alpha(\tilde{\omega}_t) \geq w_\alpha(\omega_t \otimes \gamma_t), \text{ } \forall \gamma_t \in \Lambda_{Q \times Q}, \text{ } t \geq \bar{t}.
\]
(18)
Since \(w_\alpha(\tilde{\omega}_t) + \frac{1}{\alpha} \geq w_\alpha(\omega_t) + \frac{1}{\alpha} \geq M_\alpha\), we still have
\[
\frac{\alpha}{2} \|\omega_1^t - \omega_2^t\|^2 \leq \frac{c}{2}.
\]
We just need to take
\[
\gamma_1^t(s) = x + \bar{\omega}^1(t)1_{t \leq t}(s), \text{ } \gamma_2^t(s) = y + \bar{\omega}^2(t)1_{t \leq t}(s)
\]
in (18) and define
\[
\tilde{u}(t, x) = u(\tilde{\omega}_1^t \otimes \gamma_1^t) = u((\tilde{\omega}_1^t)^x)_{t \leq t}, \text{ } \tilde{v}(t, y) = v(\tilde{\omega}_2^t \otimes \gamma_2^t) = v((\tilde{\omega}_2^t)^y)_{t \leq t}.
\]
\[
\bar{\varphi}_\alpha(t, x, y) = \varphi_\alpha(\tilde{\omega}_t \otimes \gamma_t) = \varphi_\alpha((\tilde{\omega}_1^t)^x)_{t \leq t}, \text{ } (\tilde{\omega}_2^t)^y)_{t \leq t}.
\]
Inequality (18) becomes
\[
\tilde{u}(t, 0) - \tilde{v}(t, 0) - \bar{\varphi}_\alpha(t, 0, 0) \geq \tilde{u}(t, x) - \tilde{v}(t, y) - \bar{\varphi}_\alpha(t, x, y), \text{ } t \geq \bar{t}, \text{ } x + \bar{\omega}^1(t), \text{ } y + \bar{\omega}^2(t) \in Q.
\]
We then apply Lemma 15 and use Condition (16) to obtain that, for each \(\varepsilon > 0\), there exist \(X_1, X_2 \in S(d)\) such that
\[
(b_1, p, X_1) \in \hat{P}^{2,+} \tilde{u}(t, 0) \subset \hat{P}^{2,+} u(\tilde{\omega}_1^t),
\]
\[
(b_2, p, X_2) \in \hat{P}^{2,-} \tilde{v}(t, 0) \subset \hat{P}^{2,-} v(\tilde{\omega}_2^t),
\]
with \(p = \partial_x \bar{\varphi}_\alpha(t, 0, 0)\) and
\[
-(\frac{1}{\varepsilon} + \|A\|) I_{2d} \leq \begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq A + \varepsilon A^2, \text{ } b_1 - b_2 \leq \partial_t \bar{\varphi}(t, 0, 0), \text{ } (19)
\]
where \(A\) is explicitly given by
\[
A = \alpha J_{2d}, \text{ } \text{where } J_{2d} = \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\]
The second inequality of the first relation in (19) implies $X_1 \leq X_2$, the second relation implies

$$b_1 - b_2 \leq \partial_t \bar{\varphi}(t, 0, 0) = \frac{\alpha}{2} |\bar{\omega}_1(t) - \bar{\omega}_2(t)|^2 \quad (\leq \frac{c}{2}).$$

From these it follows that,

$$b_1 + G(u(\bar{\omega}_1), p, X_1) \geq c, \quad b_2 + G(v(\bar{\omega}_2), p, X_2) \leq 0.$$

This, together with condition (9) for $G$, it follows that

$$c \leq b_1 + G(u(\bar{\omega}_1), p, X_1) - [b_2 + G(v(\bar{\omega}_2), p, X_2)] + \frac{\alpha}{2} |\bar{\omega}_1(t) - \bar{\omega}_2(t)|^2$$

$$\leq \frac{\alpha}{2} |\bar{\omega}_1(t) - \bar{\omega}_2(t)|^2 \leq \frac{c}{2},$$

which induces a contradiction. 

**Remark 17** The left frozen maximization procedure introduced in this version can be also applied to obtain the corresponding domination theorem discussed in the 1st version of this note.

**Corollary 18** We assume (H2). Let $u \in USC_*(\Lambda_\omega) \cap C^{1,0}(\Lambda_\omega)$ (resp. $v \in LSC_*(\Lambda_\omega) \cap C^{1,0}(\Lambda_\omega)$) be a viscosity subsolution (resp. supersolution) of (19) satisfying (12). Then the comparison principle holds.

The proof of this corollary is based on the above theorem the following two lemmas.

**Lemma 19** Let $\bar{u} \in C^{1,0}([0, T) \times \mathbb{R}^d)$ and let $(p, X) \in J^{2,+} \bar{u}(\bar{t}, \bar{y})$ for a given $(\bar{t}, \bar{y}) \in [0, T)$, here $\bar{t}$ is regarded as a fixed variable. We have $(\partial_t \bar{u}(\bar{t}, \bar{y}), p, X) \in P^{2,+} \bar{u}(\bar{t}, \bar{y})$

**Proof.** $(p, X) \in J^{2,+} \bar{u}(\bar{t}, \bar{y})$ means that there exists $\bar{y}^{(j)} \to \bar{y}$ and $(p^{(j)}, X^{(j)}) \in J^{2,+} \bar{u}(\bar{t}, \bar{y}^{(j)})$ such that $\bar{u}(\bar{t}, \bar{y}^{(j)}) \to \bar{u}(\bar{t}, \bar{y})$ and $(p^{(j)}, X^{(j)}) \to (p, X)$.

From

$$\bar{u}(\bar{t}, y) \leq \bar{u}(\bar{t}, \bar{y}^{(j)}) + \left(p^{(j)}, y - \bar{y}^{(j)} \right) + \left(X^{(j)}(y - \bar{y}^{(j)}), y - \bar{y}^{(j)} \right) + o(|y - \bar{y}^{(j)}|^2)$$

we have

$$\bar{u}(t, y) - \bar{u}(\bar{t}, \bar{y}^{(j)})$$

$$= \left[\bar{u}(t, y) - \bar{u}(\bar{t}, y)\right] + \left[\bar{u}(\bar{t}, y) - \bar{u}(\bar{t}, \bar{y}^{(j)})\right]$$

$$\leq \int_0^1 \partial_t \bar{u}(\tilde{t}, \bar{y}^{(j)}) + |\alpha(t - \tilde{t}), y\rangle dt \cdot (t - \tilde{t})$$

$$+ \left(p^{(j)}, y - \bar{y}^{(j)} \right) + \left(X^{(j)}(y - \bar{y}^{(j)}), y - \bar{y}^{(j)} \right) + o(|y - \bar{y}^{(j)}|^2)$$

$$= \partial_t \bar{u}(\tilde{t}, \bar{y}^{(j)})(t - \tilde{t}) + \left(p^{(j)}, y - \bar{y}^{(j)} \right) + \left(X^{(j)}(y - \bar{y}^{(j)}), y - \bar{y}^{(j)} \right)$$

$$+ o(|y - \bar{y}^{(j)}|^2 + |t - \tilde{t}|^2).$$

It follows that $(\partial_t \bar{u}(\tilde{t}, \bar{y}^{(j)}), p^{(j)}, X^{(j)}) \in P^{2,+} \bar{u}(\tilde{t}, \bar{y}^{(j)})$, $j = 1, 2, \ldots$. Since this sequence converges to $(\partial_t \bar{u}(\tilde{t}, \bar{y}), p, X)$, we then get the result. 

We extend this result to the following case for path-dependent functions:
Lemma 20 For a given function \( u \in C^{1,0}(\Lambda \bar{Q}) \) and a fixed \( \hat{\omega}_t \in \Lambda, \hat{t} \in (0, T) \), we set \( \hat{u}(x) := u((\hat{\omega}_t)^t), x + \hat{\omega}(\hat{t}) \in Q \), and assume that

\[
(p, X) \in J^{2,+} \hat{u}(0).
\]

Then \( (D_t u(\hat{\omega}_t), p, X) \in \bar{P}^{2,+} u(\hat{\omega}_t) \).

Proof. \( 20 \) means that there exists a sequence \( \{\hat{y}^{(j)}\} \) in \( Q \) such that \( (p^{(j)}, X^{(j)}) \in J^{2,+} \hat{u}(\hat{y}^{(j)}) \) and \( \hat{y}^{(j)} \to 0 \), \( \hat{u}(\hat{y}^{(j)}) \to \hat{u}(0) \), \( (p^{(j)}, X^{(j)}) \to (p, X) \) as \( j \to \infty \). From

\[
\hat{u}(y) \leq \hat{u}(\hat{y}^{(j)}) + \left< p^{(j)}, y - \hat{y}^{(j)} \right> + \left< X^{(j)}(y - \hat{y}^{(j)}), y - \hat{y}^{(j)} \right> + o(|y - \hat{y}^{(j)}|^2),
\]

we have,

\[
u((\hat{\omega}_t)_{t, t-\hat{t}}) - u((\hat{\omega}_t)_{t, t-\hat{t}}) = [u((\hat{\omega}_t)_{t, t-\hat{t}}) - u((\hat{\omega}_t)_{0, t-\hat{t}})]
\leq \int_0^1 D_t u((\hat{\omega}_t)_{t, t-\hat{t}}) d\alpha \cdot (t - \hat{t})
\]

\[
+ \left< p^{(j)}, y - \hat{y}^{(j)} \right> + \left< X^{(j)}(y - \hat{y}^{(j)}), y - \hat{y}^{(j)} \right> + o(|y - \hat{y}^{(j)}|^2).
\]

It follows that \( (D_t u((\hat{\omega}_t)_{t, t-\hat{t}}), p^{(j)}, X^{(j)}) \) is \( \bar{P}^{2,+} u((\hat{\omega}_t)_{t, t-\hat{t}}) \). But also have, as \( j \to \infty \),

\[
(D_t u((\hat{\omega}_t)_{t, t-\hat{t}}), p^{(j)}, X^{(j)}) \to (D_t u((\hat{\omega}_t), p, X).
\]

Thus \( (D_t u((\hat{\omega}_t), p, X) \in \bar{P}^{2,+} u((\hat{\omega}_t)). \) The proof is complete. \( \square \)

6 Comparison principle for viscosity solution of 1st order path-dependent PDE

We consider the comparison principle for viscosity solutions of 1st order path-dependent PDE. The following lemma is a generalization of the left frozen maximization provided in Lemma 4.

Lemma 21 Let \( Q \) be a bounded open subset of \( \mathbb{R}^d \) and let \( u \in USC_*(\bar{Q} \times \bar{Q}) \) be bounded from above. Then for each given \( \omega^{(0)}_t, \nu^{(0)}_t \in \Lambda \bar{Q} \), there exist \( \hat{\omega}_t, \hat{\nu}_s \in \Lambda \bar{Q} \), satisfying \( \hat{t} \geq t, \hat{s} \geq s \), \( u(\omega^{(0)}_t, \nu^{(0)}_t) \leq u(\hat{\omega}_t, \hat{\nu}_s) \) and \( \hat{\omega}_t = \omega^{(0)}_t \otimes \hat{\nu}_t, \hat{\nu}_s = \nu^{(0)}_s \otimes \hat{\nu}_s \), such that

\[
u(\hat{\omega}_t, \hat{\nu}_s) = \sup_{\gamma_t \in \Lambda \bar{Q}, \tau \geq t, \eta_s \in \Lambda \bar{Q}, \alpha \geq \xi} u(\hat{\omega}_t \otimes \gamma_t, \hat{\nu}_s \otimes \eta_s).
\]

(21)
Proof. Clearly $u(\omega^{(0),x},v^{(0),y})$ is an USC-function of $x$ and $y$. Thus without loss of generality, we can assume that $u(\omega^{(0),x},v^{(0)}) \geq u(\omega^{(0),x},v^{(0),y})$, for all $x$ and $y$ such that $x + \omega^{(0)}(t_0), y + v^{(0)}(s_0) \in \bar{Q}$. We set $m_0 := u(\omega^{(0),x},v^{(0)})$ and

$$\tilde{m}_0 := \sup_{\gamma_t \in \Lambda_Q, t \geq t_0, \eta_s \in \Lambda_Q, s \geq s_0} u(\omega^{(0),x} \circ \gamma_t, v^{(0)} \circ \eta_s) \geq m_0.$$ 

If $\tilde{m}_0 = m_0$ then we take $\tilde{\omega} = \omega^{(0)}_t$, $\tilde{v} = v^{(0)}_t$ and finish the procedure. Otherwise there exists $\omega^{(1)}_{t_1} \in \Lambda_Q$ with $t_1 \geq t_0, s_1 \geq s_0$ and $t_1 + s_1 > t_0 + s_0,$

$$u(\omega^{(1)}_{t_1}, v^{(1)}_{s_1}) \geq u(\omega^{(1),x}_{t_1}, v^{(1),y}_{s_1}),$$

for all $x, y$ such that $x + \omega^{(1)}(t_1), y + v^{(1)}(s_1) \in \bar{Q}$, satisfying $\omega^{(1)}_{t_1} = \omega^{(0)}_{t_0} \circ \omega^{(1)}_{i_1}, v^{(1)}_{s_1} = v^{(0)}_{t_0} \circ v^{(1)}_{s_1}$ and

$$m_1 := u(\omega^{(1)}_{t_1}, v^{(1)}_{s_1}) \geq \frac{m_0 + \tilde{m}_0}{2}.$$

We set

$$\tilde{m}_1 := \sup_{\gamma_t \in \Lambda_Q, t \geq t_1, \eta_s \in \Lambda_Q, s \geq s_1} u(\omega^{(1)}_{t_1} \circ \gamma_t, v^{(1)}_{s_1} \circ \eta_s) \geq m_1,$$

If $\tilde{m}_1 = m_1$ then we can take $\tilde{\omega} = \omega^{(1)}_{t_1}, \tilde{v} = v^{(1)}_{s_1}$ and finish the procedure. Otherwise we can continue this procedure to find, for $i = 2, 3, \cdots$, $\omega^{(i)}_{t_i}, v^{(i)}_{s_i} \in \Lambda_Q$ with, $t_i \geq t_{i-1}, s_i \geq s_{i-1}$ and $u(\omega^{(i)}_{t_i}, v^{(i)}_{s_i}) \geq u(\omega^{(i),x}_{t_i}, v^{(i),y}_{s_i})$, for all $x, y$ such that $x + \omega^{(i)}(t_i), y + v^{(i)}(s_i) \in \bar{Q}$, such that $\omega^{(i)}_{t_i} = \omega^{(i-1)}_{t_{i-1}} \circ \omega^{(i)}_{t_i}, v^{(i)}_{s_i} = v^{(i-1)}_{t_{i-1}} \circ v^{(i)}_{s_i}$ and

$$m_i := u(\omega^{(i)}_{t_i}, v^{(i)}_{s_i}) \geq \frac{m_{i-1} + \tilde{m}_{i-1}}{2},$$

$$\tilde{m}_i := \sup_{\gamma_t \in \Lambda_Q, t \geq t_i, \eta_s \in \Lambda_Q, s \geq s_i} u(\omega^{(i)}_{t_i} \circ \gamma_t, v^{(i)}_{s_i} \circ \eta_s) \geq m_i,$$

and continue this procedure till the first time when $\tilde{m}_i = m_i$ and then finish the proof by setting $\tilde{\omega}_t = \omega^{(i)}_{t_i}$. For the last and “worst” case in which $\tilde{m}_i > m_i$, for all $i = 0, 1, 2, \cdots$, we have $s_i \uparrow \bar{s}, t_i \uparrow \bar{t}$, such that $\bar{s}, \bar{t} \in [0,T]$ Then we can find $\tilde{\omega}_t, \tilde{v}_t \in \Lambda_Q$ such that $\tilde{\omega}_t = \omega^{(i)}_{t_i} \circ \omega_t, \tilde{v}_t = v^{(i)}_{s_i} \circ \tilde{v}_t$ and $u(\tilde{\omega}_t, \tilde{v}_t) \geq u(\omega^{(i)}_{t_i}, v^{(i)}_{s_i})$, for all $x, y$ such that $x + \tilde{\omega}(\bar{t}), y + \tilde{v}(\bar{t}) \in \bar{Q}$. Since $u \in \text{USC}_\ast(\Lambda_Q)$. Since

$$m_{i+1} - \tilde{m}_{i+1} \leq m_i - \frac{\tilde{m}_i + m_i}{2} = \frac{\tilde{m}_i - m_i}{2},$$

thus there exists $\tilde{m} \in (m_0, \tilde{m}_0)$, such that $\tilde{m}_i \downarrow \tilde{m}$ and $m_i \uparrow \tilde{m}$. We can claim that (21) holds for this $(\tilde{\omega}_t, \tilde{v}_t)$. Indeed, otherwise there exists a $\eta_\bar{s}$, $\gamma_t \in \Lambda_Q$ with $s \geq \bar{s}, \bar{t} \geq \bar{t}$ and $\gamma_t = \tilde{\omega}_t \circ \gamma_t, \eta_s = \tilde{v}_s \circ \eta_s$ and a $\delta > 0$ such that

$$u(\tilde{\omega}_t \circ \gamma_t, \tilde{v}_s \circ \eta_s) \geq u(\tilde{\omega}_t, \tilde{v}_s) + \delta = \tilde{m} + \delta,$$
then the following contradiction is induced:
\[ u(\hat{\omega}_1 \otimes \gamma_1, \bar{v}_s \otimes \eta_s) = u(\omega^{(i)}_t \otimes \gamma_t, v^{(i)}_s \otimes \eta_s) \leq \bar{m}_i \rightarrow \bar{m}. \]

The proof is complete. \(\blacksquare\)

(H3) We make the following assumption for each \(u, v \in \mathbb{R}, p \in \mathbb{R}^d\) such that \(u \geq v\),
\[ G(u, p) \leq G(v, p). \]

**Theorem 22 (Comparison principle)** We assume (H3). Let \(Q\) be a bounded open subset of \(\mathbb{R}^d\) and \(u \in USC_*(\Lambda_Q)\) (resp. \(v \in LSC_*(\Lambda_Q)\)) be a viscosity subsolution (resp. supersolution) of
\[ Du + G(u(\omega_t), D_x u(\omega_t)) = 0. \] (22)

Assume that \(u, -v\) are bounded from above by \(C\) and they are continuous in the following sense: there exists a constant \(\bar{a} > 0\), such that, for each \(\omega_t \in \Lambda_Q, \gamma_s = \omega_t \otimes \gamma_s, \bar{\gamma}_t = \omega_t \otimes \bar{\gamma}_t \in \Lambda_Q\) such that \(s, \bar{s} \in [t, (t + \bar{a}) \land T]\),
\[ |u(\omega_t \otimes \gamma_s) - u(\omega_t \otimes \bar{\gamma}_t)| \vee |v(\omega_t \otimes \gamma_s) - v(\omega_t \otimes \bar{\gamma}_t)| \leq \rho(d(\gamma_s, \bar{\gamma}_t)), \]
(23)

where
\[ d(\gamma_s, \bar{\gamma}_t) = |s - \gamma(s) - \bar{\gamma}(s)|^2 + |s - \bar{s}|^2 + \int_0^{s \land \bar{s}} (s \land \bar{s} - r)|\gamma(r) - \bar{\gamma}(r)|^2 dr \]
and \(\rho : [0, \infty) \rightarrow \mathbb{R}\) is a given continuous and increasing function with \(\rho(0) = 0\). Then the following comparison principle holds: if \(u(\omega_t) \leq v(\omega_t)\) for \(\omega_t \in \Lambda_{\partial Q_T}\), then \(u(\omega_t) \leq v(\omega_t)\) for \(\omega_t \in \Lambda_Q\).

We first observe that for \(\bar{\delta} > 0\), the functions defined by \(\tilde{u}_i := u_i - \bar{\delta}/t\) is a subsolution of
\[ D_t \tilde{u}(\omega_t) + G(\tilde{u}(\omega_t), D_x \tilde{u}(\omega_t)) = \frac{\bar{\delta}}{T^2}. \]

Since \(u \leq v\) follows from \(\tilde{u} \leq v\) in the limit \(\bar{\delta} \downarrow 0\), it suffices to prove the theorem under the additional assumptions:
\[ D_t u(\omega_t) + G(u(\omega_t), D_x u(\omega_t)) \geq c, \quad c := \bar{\delta}/T^2. \] (24)

To prove this theorem, for \(\omega_t, v_s \in \Lambda_Q\), we set
\[ w_\alpha(\omega_t, v_s) := u(\omega_t) - v(v_s) - \frac{\alpha}{2} d(\omega_t, v_s). \]

**Proof of Theorem 22** The proof is still based on our “left frozen maximization” approach. We only need to prove that \(u(\omega_t) \leq v(\omega_t)\) for \(\omega_t \in \Lambda_Q, t \in [T - \bar{a}, T)\), then repeat the same procedure for cases \([T - 2\bar{a}, T - (1 - \bar{a})]\). To this end we assume to the contrary that there exists \(\hat{\omega}_t \in \Lambda_Q, \hat{t} \in [T - \bar{a}, T)\), such that \(\bar{m} := u(\hat{\omega}_t) - v(\hat{\omega}_t) > 0\). Let \(\alpha\) be a large number such that
\[ \frac{1}{\alpha} + \rho(\frac{1}{\alpha} + 2C - M_s) \leq \frac{1}{2}(\bar{m} \wedge c). \]
We set
\[ M_\alpha := \sup_{\gamma_t, \eta_t \in \Lambda_Q} \omega_\alpha(\hat{\omega}_t \otimes \gamma_t, \hat{\omega}_s \otimes \eta_s) \geq M_* := \sup_{\gamma_t, \eta_t \in \Lambda_Q} \sup_{s \geq T-\bar{d}} [u(\hat{\omega}_t \otimes \gamma_t) - v(\hat{\omega}_s \otimes \eta_s)] \geq \bar{m}. \]

We fix \( \gamma_t, \eta_t \in \Lambda_Q \) satisfying \( \gamma_t = \hat{\omega}_t \otimes \gamma_t, \eta_t = \hat{\omega}_s \otimes \eta_s \) and \( w_\alpha(\gamma_t, \eta_t) + \frac{1}{\alpha} \geq M_\alpha \). We can check that
\[
\frac{\alpha}{2} d(\gamma_t, \eta_t) \leq \frac{1}{\alpha} + u(\gamma_t) - v(\eta_t) - M_\alpha \\
\leq \frac{1}{\alpha} + u(\gamma_t) - v(\eta_t) - M_* \\
\leq \frac{1}{\alpha} + 2C - M_*.
\]

But we also have
\[
M_* \leq \frac{1}{\alpha} + [u(\gamma_t) - u(\eta_t)] + [u(\eta_t) - v(\eta_t)] - \frac{\alpha}{2} d(\gamma_t, \eta_t) \\
\leq \frac{1}{\alpha} + \rho(d(\gamma_t, \eta_t)) + [u(\eta_t) - v(\eta_t)] - \frac{\alpha}{2} d(\gamma_t, \eta_t) \\
\leq \frac{1}{\alpha} + \rho(\frac{2}{\alpha}(1 + 2C - M_*)) + M_* - \frac{\alpha}{2} d(\gamma_t, \eta_t).
\]

Thus
\[
\frac{\alpha}{2} d(\gamma_t, \eta_t) \leq \frac{1}{\alpha} + \rho(\frac{2}{\alpha}(1 + 2C - M_*)) \leq \frac{c}{2},
\]

Moreover \( \gamma_t, \eta_t \in \Lambda_Q \). Indeed if, say \( \eta_t \in \Lambda_{\partial Q} \), then we will deduce the following contradiction:
\[
\bar{m} \leq M_* \leq M_\alpha \\
\leq \frac{1}{\alpha} + [u(\gamma_t) - u(\eta_t)] + u(\eta_t) - v(\eta_t) \\
\leq \frac{1}{\alpha} + [u(\gamma_t) - u(\eta_t)] \\
\leq \frac{1}{\alpha} + \rho(\frac{2}{\alpha}(1 + 2C - M_*)) \leq \frac{\bar{m}}{2}.
\]

Now we can apply Lemma \([21]\) to find \( \hat{\omega}_t, \hat{v}_s \in \Lambda_Q \) satisfying
\[
\hat{\omega}_t = \gamma_t \otimes \hat{\omega}_t \text{ and } \hat{v}_s = \eta_t \otimes \hat{v}_s \text{ with } w_\alpha(\hat{\omega}_t, \hat{v}_s) \geq w_\alpha(\gamma_t, \eta_t) \text{ such that}
\]
\[
w_\alpha(\hat{\omega}_t, \hat{v}_s) \geq w_\alpha(\gamma_t \otimes \hat{\omega}_t, \eta_s \otimes \hat{v}_s), \quad (25)
\]
\[\forall \gamma_t, \eta_t \in \Lambda_Q, \ t \geq \hat{t}, \ s \geq \hat{s}.\]

Since \( w_\alpha(\hat{\omega}_t, \hat{v}_s) + \frac{1}{\alpha} \geq w_\alpha(\gamma_t, \eta_t) + \frac{1}{\alpha} \geq M_\alpha \), we still have
\[
\frac{\alpha}{2} d(\hat{\omega}_t, \hat{v}_s) \leq \frac{c}{2},
\]
as well as
\[
u(\hat{\omega}_t) - v(\hat{v}_s) \geq M_\alpha - \frac{1}{\alpha} \geq 0.
\]

We just need to take
\[
\gamma_t(r) = x + \hat{\omega}(\hat{t})1_{[1, \hat{t})}(r), \ r \leq t, \\
\eta_t(r) = y + \hat{v}(\hat{s})1_{[1, \hat{s})}(r), \ r \leq s,
\]

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in (25) and define
\[ u(t, x) = u(\hat{\omega} \otimes \gamma_t) = u((\hat{\omega}_t)^\chi_{t,t-\hat{t}}), \]
\[ \bar{v}(s, y) = v(\hat{v}_s \otimes \eta_s) = v((\hat{v}_s)^\gamma_{s,s-\hat{y}}), \]
\[ \bar{\varphi}_\alpha(t, s, x, y) = \varphi_\alpha(\hat{\omega} \otimes \gamma_t, \hat{v}_s \otimes \eta_s) = \varphi_\alpha((\hat{\omega}_t)^\chi_{t,t-\hat{t}}, (\hat{v}_s)^\gamma_{s,s-\hat{y}}). \]

Inequality (25) becomes
\[ \bar{u}(t, 0) - \bar{v}(t, 0) - \bar{\varphi}_\alpha(t, 0, 0) \geq \bar{u}(t, x) - \bar{v}(s, y) - \bar{\varphi}_\alpha(t, s, x, y), \]
\[ t \geq \check{t}, \ s \geq \check{s}, \ x + \hat{\omega}^\prime, \ y + \hat{v}(s) \in Q. \]

Since
\[ d(\omega_t, v_s) := |t - s|^2 + |\omega(t) - v(s)|^2 + \int_0^{t \wedge s} (t \wedge s - r)|\omega(r) - v(r)|^2dr, \]
thus
\[ b := \partial_t \bar{\varphi}_\alpha(\check{t}, \check{s}, 0, 0) = \alpha(\check{t} - \check{s}) = -\partial_t \bar{\varphi}_\alpha(\check{t}, \check{s}, 0, 0), \]
\[ p := \partial_s \bar{\varphi}_\alpha(\check{t}, \check{s}, 0, 0) = \alpha(\hat{\omega}^\prime - \hat{v}(s)) = -\partial_s \bar{\varphi}_\alpha(\check{t}, \check{s}, 0, 0). \]

It follows that
\[ (b, p) \in P^{1,+} \bar{u}(\check{t}, 0) \subset P^{1,+} u(\hat{\omega}_\check{t}), \]
\[ (b, p) \in P^{1,-} \bar{v}(\check{s}, 0) \subset P^{1,-} v(\hat{v}_\check{s}). \]

Consequently
\[ b + G(u(\hat{\omega}_\check{t}), p) \geq c, \quad b + G(v(\hat{v}_\check{s}), p) \leq 0. \]

It follows that
\[ c \leq b + G(u(\hat{\omega}_\check{t}), p) - [b + G(v(\hat{v}_\check{s}), p)] \leq 0, \]
which induces a contradiction. ■

**Remark 23** The above method can be also applied to obtain a comparison principle for the case \( G = G(\omega_t, u, p), \omega_t \in \Lambda_Q \) under the following condition:
\[ G(\omega_t \otimes \gamma_s, u, p) - G(\omega_t \otimes \bar{\gamma}_s, v, p) \leq \rho(d(\gamma_s, \bar{\gamma}_s)), \]
for each \( u, v \in \mathbb{R}, \ p \in \mathbb{R}^d, \) such that \( u \geq v \) and for each \( \omega_t \in \Lambda_Q, \gamma_s = \omega_t \otimes \gamma_s, \bar{\gamma}_s = \omega_t \otimes \bar{\gamma}_s \in \Lambda_Q \) such that \( s, \check{s} \in [t, (t + \check{a}) \wedge T]. \)

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