Star Chromatic Index of Halin Graphs

MARZIEH VAHID DASTJERDI

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Abstract

A star edge coloring of a graph $G$ is a proper edge coloring of $G$ such that every path and cycle of length four in $G$ uses at least three different colors. The star chromatic index of $G$, is the smallest integer $k$ for which $G$ admits a star edge coloring with $k$ colors. In this paper, we obtain tight upper bound $\left\lceil \frac{\Delta^2}{2} \right\rceil + 2$ for the star chromatic index of every Halin graph, that proves the conjecture of Dvořák et al. (J Graph Theory, 72 (2013), 313–326) for cubic Halin graphs.

Keywords: Star edge coloring, star chromatic index, Halin graphs.

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1 Introduction

All graphs considered in this paper are finite and simple. For a graph $G$, we denote by $V(G)$ and $E(G)$, its vertex set and edge set, respectively. For every vertex $v \in V(G)$, a vertex $u \in V(G)$ is a neighbor of $v$ or adjacent to $v$ if $uv \in E(G)$. We denote the set of neighbors of $v$ in $G$, by $N_G(v)$. Two edges are said to be adjacent if they are incident to a common vertex. A proper edge coloring of $G$ is a function assigning colors to the edges of $G$ in such a way that no two adjacent edges receive the same color. A star edge coloring of $G$ is a proper edge coloring of $G$ with no bicolored path nor cycle on four edges. The star chromatic index of $G$, denoted by $\chi'_s(G)$, is the minimum $k$ such that $G$ admits a star edge coloring with $k$ colors.

In 2008, Liu and Deng [10] introduced the concept of star edge coloring motivated by the vertex version [3, 6]. Liu and Deng [10] presented upper bound $16(\Delta - 1)^\frac{2}{3}$ on the star chromatic index of graphs with maximum degree $\Delta \geq 7$. In [4], Dvořák et al. obtained a near-linear upper bound $\Delta 2^{O(1) \sqrt{\log(\Delta)}}$ for graphs with maximum degree $\Delta$ and provided some upper and lower bounds for complete graphs (see also [8]). They also considered cubic graphs and showed that the star chromatic index of such graphs lies between 4 and 7. Dvořák et al. proposed the following
conjecture for subcubic graphs (graphs with $\Delta \leq 3$) that is also opened. For more results on the star chromatic index of subcubic graph see [9, 11, 12].

**Conjecture 1.** [4] If $G$ is a subcubic graph, then $\chi'_s(G) \leq 6$.

In [1], Bezegová et al. obtained some upper bounds on the star chromatic index of subcubic outerplanar graphs, trees and outerplanar graphs, as follows.

**Theorem 1.** [4] Let $G$ be a graph with maximum degree $\Delta$.

(i) If $G$ is tree, then $\chi'_s(G) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor$.

(ii) If $G$ is subcubic outerplanar, then $\chi'_s(G) \leq 5$.

(iii) If $G$ is outerplanar, then $\chi'_s(G) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 12$.

They also conjectured that the star chromatic index of every outerplanar graph is at most $\left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$.

Wang et al. in [13, 14] proved bounds for graphs $G$ under different planarity conditions and when maximum degree $G$ is at most four.

**Theorem 2.** [13, 14]

- Let $G$ be a planar graph with girth $g$ (length of shortest cycle).

  (i) $\chi'_s(G) \leq \frac{11}{4} \Delta + 18$.

  (ii) If $G$ is $K_4$-minor free, then $\chi'_s(G) \leq \frac{9}{4} \Delta + 6$.

  (iii) If $G$ has no 4-cycle, then $\chi'_s(G) \leq \left\lfloor \frac{15}{4} \Delta \right\rfloor + 18$.

  (iv) If $g \geq 5$, then $\chi'_s(G) \leq \left\lfloor \frac{15}{4} \Delta \right\rfloor + 13$.

  (v) If $g \geq 8$, then $\chi'_s(G) \leq \left\lfloor \frac{9}{4} \Delta \right\rfloor + 3$.

  (vi) If $G$ is outerplanar, then $\chi'_s(G) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 5$.

- Let $G$ be a graph with maximum degree $\Delta \leq 4$.

  (i) $\chi'_s(G) \leq 14$.

  (ii) If $G$ is bipartite, then $\chi'_s(G) \leq 13$.

In this paper, we investigate the star edge coloring of Halin graphs as a family of planar graphs and find upper bound $\left\lfloor \frac{3\Delta}{2} \right\rfloor + 2$ for the star chromatic index of this family of planar graphs. This upper bound proves Conjecture [1] for the cubic Halin graphs. A *Halin graph* $G$ is a plane graph consisting of a tree $T$ and cycle $C$ that each vertex of $T$ is either of degree 1, called *leaf*, or of degree at least 3, and $C$ connecting the leaves of $T$ such that $C$ is the boundary of the exterior face. The
tree $T$ and the cycle $C$ are called the characteristic tree and the adjoint cycle of $G$, respectively. We usually write $G = T \cup C$ to make the characteristic tree and the adjoint cycle of $G$ explicit.

In this paper, the most important point in a star edge coloring of a Halin graph is the coloring of its characteristic tree. For this purpose, we consider the characteristic tree as a rooted tree, and in a specific order on its edges, we present the star edge coloring of the tree. A rooted tree is a tree in which one vertex has been designated as the root. Suppose that $T$ is a rooted tree. If $v$ is a vertex in $T$ other than the root, the parent of $v$ is the unique vertex that is the eventual predecessor of $v$, regarding the root. We use $p(v)$ to denote the parent of $v$.

## 2 Star edge coloring of cubic Halin graphs

In this section, we consider cubic Halin graphs and find tight upper bound 6 for the star chromatic index of these graph. The following theorem proves Conjecture \[\text{H}\] for cubic Halin graphs.

**Theorem 3.** If $G = T \cup C$ is a cubic Halin graph, then $\chi'_s(G) \leq 6$.

**Proof.** Let $G = T \cup C$ be a cubic Halin graph and $P : u_0, \ldots, u_\ell$ be a longest path in $T$. It is easy to see that

$$\chi'_s(G) = \begin{cases} 5 & \text{if } \ell = 2, \\ 6 & \text{if } \ell = 3. \end{cases}$$

Therefore, we assume that $P$ is of length at least four and prove the theorem by induction on the length of $C$. By our assumption that $P$ is a longest path, all neighbors of $u_1$, except $u_2$, must be leaves. We may change notations to let $v = u_1$, $u := u_2$, $w := v_3$, and $v_1, v_2$, be the neighbors of $v$ on $C$ as demonstrated in Figure \[\text{H}\]. Let $x_1, x_2, y_1, y_2$ be the vertices on $C$, where $x_1$ is adjacent to $v_1$ and $x_2$, $y_1$ is adjacent to $v_2$ and $y_2$. Let $x_3$ and $y_3$ be vertices not on $C$, where $x_1x_3$ and $y_1y_3$ are edges in $T$.

Since $u$ is a vertex of degree 3, there exists a path $P'$ in $T$ from $u$ to $x_1$ or from $u$ to $y_1$ with $P \cap P' = \{u\}$. Without loss of generality, we assume that $P'$ is from $u$ to $y_1$. Since $P$ is a longest path, $|P'| \leq 2$. Thus, either $u = y_3$ or $u$ is adjacent to $y_3$.

Let $G'$ be the graph obtained from $G$ by deleting $v, v_1, v_2, y_1$ and adding two new edges $ux_1$ and $uz$, where $z \in V(C)$ is an adjacent to $y_1$, and $z \neq v_2$. Since \(\ell \geq 4\), $G'$ is a Halin graph. We write $G' = T' \cup C'$.

By the inductive hypothesis as $|C'| < |C|$, $\chi'_s(G') \leq 6$. Let $C = \{1, \ldots, 6\}$, and $\phi'$ is a star edge coloring of $G'$ with color set $C$. Without loss of generality, assume that $\phi'(ux_1) = 1$, $\phi'(ux) = 2$, and $\phi'(uvw) = 3$. We obtain a 6-star edge coloring $\phi$ of $G$ as follows.

For every edge that belongs to $E(G) \cap E(G')$, we set $\phi(e) = \phi'(e')$. For edges in $E(G) \setminus E(G')$,
we consider different cases, and in each case, we give star edge coloring \( \phi \) for \( G \) with at most 6 colors. Suppose that \( \phi'(w) = \{3, r_1, r_2\} \), \( \phi'(x_1) = \{s_1, s_2\} \), and \( \phi'(z) = \{t_1, t_2\} \).

**Case 1:** \( u = y_3 \).
In this case, we set \( \phi(v_1x) = 1, \phi(v_1v_2) = 3, \phi(vv_1) = 2, \phi(uy_1) = 1, \) and \( \phi(uz) = 2. \) Let 
\( a \in C \setminus \{1, 2, 3, r_1, r_2\} \) and \( b \in C \setminus \{1, 2, 3, t_1, t_2\} \). We set \( \phi(uv) = a, \phi(v_2y_1) = b, \) and choose an arbitrary color in \( C \setminus \{1, 2, 3, a, b\} \) for edge \( vv_2 \) (see Case 1 in Figure 1).

**Case 2:** \( u \neq y_3 \).
We set \( \phi(y_3) = 1, \phi(uv) = 2, \phi(v_1x_1) = 1, \phi(y_1y_2) = 3, \) and \( \phi(y_2z) = 2. \) Then, we set \( \phi(y_2y_3) = a, \phi(vv_1) = b, \phi(v_2y_1) = c, \) and \( \phi(y_3y_1) = e, \) and \( \phi(vv_2) = d, \) where
\[
a \in C \setminus \{1, 2, 3, t_1, t_2\}, \quad b \in C \setminus \{1, 2, 3, s_1, s_2\}, \quad c \in C \setminus \{1, 2, 3, a, b\}, \quad d \in C \setminus \{1, 2, 3, b, c\}, \quad e \in C \setminus \{1, 2, 3, a, c\}.
\]
Note that this colors are choosen in ordering \( a, b, c, d, e. \)

![Figure 1: The neighborhood around one end of the longest path \( P \).](image)

### 3 Star edge coloring of Halin graphs with \( \Delta \geq 4 \)

In every Halin graph \( G = T \cup C \), we partite the edges of cycle \( C \) in to two subsets, as follow.
\[
C_s = \{uv \in E(C) : u \text{ and } v \text{ have the same neighbor in } T\},
\]
\[
C_d = E(C) \setminus C_s
\]
The first set contains all edges of $C$ connecting two leaves of $T$ with the same neighbor, and the second one with different neighbors.

For every non-leaf vertex $v$ of $T$, we define $LN_d(v)$ as the subset of all leaf neighbors of $v$ that at least an edge of $C_d$ incident to each of them. It is easy to see that $|LN_d(v)| \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$ (see Figure 2).

Figure 2: Two Halin graphs with maximum size of $LN_d(v)$ for $\Delta = 5$ and $\Delta = 6$.

**Theorem 4.** If $G = T \cup C$ is a Halin graph with maximum degree $\Delta$, then

$$\chi'_s(G) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 2.$$  

**Proof.** By Theorem 3 if $\Delta = 3$, then $\chi'_s(G) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 2 = 6$. We now assume that $\Delta \geq 4$ and find a star edge coloring of $G$, as follows. We first color some special edges of $C$, then give a star edge coloring of $T$ with at most $\left\lfloor \frac{3\Delta}{2} \right\rfloor$ colors, and finally complete the partial edge coloring of $C$.

The partial edge coloring of $C = e_1, \ldots, e_n$ uses colors $\{a, b\}$, with the following patterns. Let $k = n \mod 3 \times 4$, and use repetative coloring patterns $(a, b, a, *)$ and $(a, b, *)$ for edges of paths $e_1, \ldots, e_k$ and $e_{k+1}, \ldots, e_n$, respectively. In these patterns, the colors of edges that are shown with the notation $*$ is determined after coloring $T$. For every vertex $v$ of $T$, we denote the number of uncolored edges of $C_d$ that are incident $v$, with $UC_d(v)$. Note that if $|LN_d(v)| = \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$, then there are two leaves $v_1$ and $v_2$ adjacent to $v$ such that $C$ contains path $v_0v_1v_2v_3$, where edges $v_0v_1$ and $v_2v_3$ belong to $C_d$. Considering the partial edge coloring of $C$, it is clear that $v_0v_1$ or $v_2v_3$ are colored before with a color in $\{a, b\}$; therefore, $UC_d(v_1) = 0$ or $UC_d(v_2) = 0$. Therefore, there exist at most $\left\lfloor \frac{\Delta}{2} \right\rfloor$ uncolored edges of $C_d$ incident to a vertex in $LN_d(v)$.

We now give a star edge coloring of $T$ with the color set $C = \{1, \ldots, \left\lfloor \frac{3\Delta}{2} \right\rfloor \}$. For this purpose, we first choose an arbitrary vertex $r \in V(T)$ of degree $\Delta$ as the root. Let $\ell = 0$, and $r$ is a vertex in level 0 of $T$. Suppose that for each vertex $v \neq r$ of $T$, $v_0, \ldots, v_{\deg(v)-1}$ are the neighbors of $v$, such that $v_0$ is the parent of $v$, and for every $i$, $1 \leq i < \deg(v) - 1$, $UC_d(v_i) \geq UC_d(v_{i+1})$. Now we do the following steps to provide a star edge coloring of $T$. 

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Algorithm 1. Coloring characteristic tree $T$ of Halin graph $G = T \cup C$.

Step 1. Properly color the incident edges to $r$ with colors $1, 2, \ldots, \Delta$, and let $\ell = 1$.

Step 2. For each vertex $v$ at level $\ell$ which has at least an uncolored incident edge do

Step 2.1 Let $m = \min\{\deg(v) - 1, \left\lfloor \frac{\Delta}{2} \right\rfloor\}$, and properly color edges of $\{vv_i : i = 1, \ldots, m\}$ with the colors that were not used in the coloring of the edges incident to $v_0$.

Step 2.2 Properly color the remaining edges incident to $v$ by using any color from $C$ not yet used in step 7 such that any path of length four in $T$ is not bi-colored.

Step 3. Stop if all edges are already colored. Otherwise, Let $\ell = \ell + 1$ and go to step 2.

We now ready to complete the partial edge coloring of cycle $C$. Let $e = uv$ is an uncolored edges in $C$. If $e \in C_s$, then we choose a color of $C$ which is not used for the edges incident to $u_0$. Hence, suppose that $e \in C_d$. Note that if $e = uv$ is an uncolored edge in $C_d$, then $UC_d(v), UC_d(u) > 0$. Thus, in the provided star edge coloring of $T$, the color of $uu_0$ (resp. $v_0v$) is not used for any edges incident to $p(u_0)$ (resp. $p(v_0)$) if there exists. Moreover, the color of edge $uu_0$ (resp. $v_0v$) is used for at most $\left\lfloor \frac{\Delta}{2} \right\rfloor - 1$ incident edges to the neighbors of $u_0$ (resp. $v_0$). If $\{u_1, \ldots, u_k\}$ (rash. $\{v_1, \ldots, v_l\}$) is the set of neighbors of $u_0$ (resp. $v_0$) that there is an incident edge to every one with the color of $uu_0$ (resp. color of $v_0v$), the colors of edges in $E' = \{uu_0, u_1u_0, \ldots, u_ku_0\} \cup \{v_0v, v_1v_0, \ldots, v_\ell v_0\}$ are forbidden for $e$. Let $C'$ be the set of colors used for the edges in $E' \cup \{uu_0, v_0v\}$. Thus, the number of the allowed colors for $e$ is

$$|C \setminus C'| \geq \left\lfloor \frac{3\Delta}{2} \right\rfloor - (2 + k + \ell) \geq \left\lfloor \frac{3\Delta}{2} \right\rfloor - 2 \times \left\lfloor \frac{\Delta}{2} \right\rfloor. \quad (1)$$

Since $\Delta \geq 4$, $|C \setminus C'| \geq 1$ and we can choose a color for $e$.

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