*On geometric infinite divisibility, p-thinning and Cox processes*

Sandhya E
Department of Statistics, University of Kerala
Thiruvananthapuram - 695 581, India.
email: esandhya@hotmail.com

Abstract

The connections among geometric infinite divisibility, p-thinning and Cox processes are established in this paper. Some results on subordination of stochastic processes with stationary independent increments, connected with geometric infinite divisibility are derived. A characterization of the renewal process with semi-Mittag-Leffler as inter arrival time distribution is obtained in the context of p-thinning, which is an extension of a result due to Renyi (1956). It is identified that only geometrically infinitely divisible distributions can define a Cox and renewal process. An analogue of a theorem in Feller (1966) is given.

1 Introduction

The concept of geometric infinite divisibility (g.i.d) was introduced by Klebanov et al. (1984). A random variable (r.v.) \( X \) is said to be g.i.d if for every \( p \in (0, 1) \) it can be expressed as \( X \overset{d}{=} \sum_{j=1}^{N_p} X_j^{(p)} \), where \( N_p, X_1^{(p)}, X_2^{(p)} \ldots \) are independent, the \( X_j^{(p)} \) are i.i.d. and \( P\{N_p = n\} = p(1 - p)^{n-1}, \ n = 1, 2, \ldots \). Equivalently in terms of characteristic functions, definition is the following. Let \( g(t) = E(e^{itX}), \phi_p(t) = E\left(e^{itX_j^{(p)}}\right) \). If for all \( p \in (0, 1) \), \( g(t) = \frac{p\phi_p(t)}{1-q\phi_p(t)} \), \( q = 1 - p \), then \( X \) is said to be g.i.d. They proved that a r.v. with characteristic function \( g(t) \) is g.i.d if and only if \( g(t) = \frac{1}{1+\psi(t)} \), where \( e^{-\psi(t)} \) is infinitely divisible (i.d). Pillai and Sandhya (1990) studied the class of g.i.d distributions more deeply.

The idea of \( p \)-thinning goes back to Renyi (1956). Let \( 0 = t_0 < t_1 < \ldots < t_n \) denote the random epochs corresponding to a renewal process. If we retain every epoch \( t_n, n = 1, 2, \ldots \) with probability ‘\( p \)’ and delete it with probability \( q = (1 - p) \), independent of every other points and the process itself, the resulting process is called the \( p \)-thinned process of the original process. Renyi used the term ‘rarification’ for \( p \)-thinning. The process obtained by replacing each epoch \( t_n, n = 1, 2, \ldots \) by ‘\( pt_n \)’, \( p \in (0, 1) \) is called the contraction of the original process.
Renyi (1956) proved that Poisson process is the only one that is invariant under contraction and \( p \)-thinning applied together.

An ordinary renewal process with inter arrival time distribution \( G \) is said to be a Cox process if there exists a process with inter arrival time distribution \( F_p \) such that the process corresponding to \( G \) in the \( p \)-thinning of the process corresponding to \( F_p \), for all \( p \in (0, 1) \). That is, the process corresponding to \( F_p \) is the \( p \)-inverse of the process corresponding to \( G \) for all \( p \in (0, 1) \). For relevant work see Yannaros (1988, 1989).

Pillai (1990) introduced the class of distributions called the Mittag-Leffler distributions which are defined by their Laplace transform given by \( \frac{1}{1+\lambda a} \), \( 0 < a \leq 1 \). Obviously, \( a = 1 \) corresponds to the exponential distribution. This is also a subclass of the semi-\( \alpha \)-Laplace distribution introduced in Pillai (1985). The semi-Mittag-Leffler distributions which we will be defining in the sequel contains the Mittag-Leffler distributions and is contained in the semi-\( \alpha \)-Laplace distributions.

In section 1, we present some results regarding the subordination of infinitely divisible processes (Feller (1966, p.335) directed by gamma, exponential and Mittag-Leffler processes such that the increments of the subordinated process are g.i.d. The main result in the next section is a characterization of the renewal process with semi-Mittag-Leffler as the inter arrival time distribution in the context of \( p \)-thinning. In the third section it is observed that only g.i.d. distributions can define a Cox and renewal process. An analogue of a Theorem in Feller (1966, p.294) is also established combining the main ideas of this paper.

# Geometric Infinite Divisibility and Subordination

**Theorem 2.1.** Let \( X \) be an i.d. r.v. with positive support. Let \( X(t), t \geq 0 \) be the associated process with stationary independent increments having Laplace transform \( e^{-t\psi(\lambda)} \). Let \( Y(t) \) be the process subordinated to \( X(t) \) by the gamma operational time with distribution function \( G_t(x) = \frac{1}{\Gamma(t)} \int_0^x \frac{y^{t-1}e^{-y}dy}{\lambda} \). Then the distribution of the increments of the subordinated process is g.i.d. for \( t \leq 1 \).

**Proof.** Let \( F_s(x) \) be the distribution corresponding to the stochastic process \( X(s), s \geq 0 \) and \( H_t(x) \) that of \( Y(t), t \geq 0 \). By assumption

\[
H_t(x) = \int_0^\infty F_s(x)G_t(ds).
\]
Taking Laplace transforms on both sides we get that of $H_t(x)$ as

\[
\hat{H}_t(x) = \int_0^\infty e^{-s\psi(\lambda)}G_t\{ds\} = \frac{1}{(1 + \psi(\lambda))^t} = \frac{1}{1 + (1 + \psi(\lambda))^t - 1} = \frac{1}{1 + h(\lambda)}.
\]

Since $\psi(\lambda)$ has complete monotone derivative and $\psi(0) = 0$, $h(\lambda)$ also has complete monotone derivative for $t \leq 1$ and $h(0) = 0$ (Feller (1966, p.417). Now the result follows from Lemma 2.1 of Pillai and Sandhya (1990).

**Theorem 2.2.** Let $X$ be an i.d. r.v. with positive support. Let $X(t)$, $t \geq 0$ be the associated process with stationary independent increments having Laplace transform $e^{-t\psi(\lambda)}$. Let $Y(t)$ be the process subordinated to $X(t)$ by the directing exponential operational time with distribution function $G_t(x) = \frac{1}{t} \int_0^x e^{-u/t} \, du$. Then the distribution of the increments of the subordinated process is g.i.d for all $t > 0$.

**Theorem 2.3.** With the above set up if the operational time has a Mittag-Leffler distribution with Laplace transform $\frac{1}{1+\lambda^t}$, $0 < t \leq 1$, then the distribution of the increments of the subordinated process is g.i.d.

The proof of the above two theorems follow along the same lines as that of Theorem 2.1.

### 3 \( p \)-thinning of Renewal Processes

Let $X_1, X_2, \ldots$ denote a sequence of i.i.d. r.v.s with distribution function $F$. Then $S_n = X_1 + \ldots + X_n$, $n = 1, 2, \ldots$ denotes a renewal process. Here $X_1, X_2, \ldots$ are the inter arrival times of the renewal process. Then $X \stackrel{d}{=} \sum_{j=1}^{N_0} X_j^{(p)}$ denotes the inter arrival time of the corresponding $p$-thinned process with thinning probability $q = (1 - p)$. The inter arrival time distribution $G$ of $X$ is a geometric convolution of $F$, i.e.,

\[
G(x) = \sum_{n=1}^\infty pq^{n-1}F^{*n}(x),
\]

where $F^{*n}$ is the $n$-fold convolution of $F$. 

3
Taking Laplace transforms on both sides of the above equation we have

\[ g(\lambda) = \frac{p\phi_p(\lambda)}{1 - q\phi_p(\lambda)} \]

or

\[ \phi_p(\lambda) = \frac{g(\lambda)}{p + qg(\lambda)}. \]

Thus a Laplace transform \( g(\lambda) \) corresponds to a thinned renewal process if and only if there exists a Laplace transform \( \phi_p(\lambda) \) such that

\[ g(\lambda) = \frac{p\phi_p(\lambda)}{1 - q\phi_p(\lambda)} \]

or if

\[ \phi_p(\lambda) = \frac{g(\lambda)}{p + qg(\lambda)} \]

is a Laplace transform.

**Definition 3.1.** A distribution with positive support is said to be semi-Mittag-Leffler of exponent \( \alpha, 0 < \alpha \leq 1 \), if its Laplace transform is of the form \( \frac{1}{1 + \psi(\lambda)} \), where \( \psi(\lambda) \) satisfies \( \psi(\lambda) = a\psi(b\lambda), 0 < b < a \) and \( \alpha \) is the unique solution of \( ab^{\alpha} = 1 \).

**Theorem 3.1.** The semi-Mittag-Leffler distribution is the only inter arrival time distribution such that the corresponding renewal process is invariant under \( p \)-thinning (up to a scale change, \( 0 < c < 1 \)).

**Proof.** Let \( \phi(\lambda) \) be the Laplace transform of the inter arrival time distribution of some renewal process. Applying \( p \)-thinning and allowing a scale change \( c, 0 < c < 1 \), we get

\[ g(c\lambda) = \phi(\lambda), \text{ where } g(\lambda) = \frac{p\phi(\lambda)}{1 - q\phi(\lambda)} \]

i.e., \( \phi(\lambda) = \frac{p\phi(c\lambda)}{1 - q\phi(c\lambda)}, 0 < c < 1 \).

Let \( \phi(\lambda) = \frac{1}{1 + \psi(\lambda)} \), where \( \psi(\lambda) = \frac{1}{\phi(\lambda)} - 1 \) and then \( \psi(\lambda) \) satisfies

\[ \psi(\lambda) = \frac{1}{p}\psi(c\lambda). \]

This implies that the distribution is semi-Mittag-Leffler of exponent \( \alpha, 0 < \alpha \leq 1 \), choosing \( c^{\alpha} = p \).

The converse follows by choosing \( c^{\alpha} = p \) and retracing the steps. \( \square \)
**Corollary 3.1.** If for two values of ‘\(p\)’, say \(p_1\) and \(p_2\), \(\frac{\log p_1}{\log p_2}\) is irrational, and (3.1) is satisfied, then \(\psi(\lambda) = A\lambda^\alpha, 0 < \alpha \leq 1, A > 0\) a constant, which implies that the distribution is Mittag-Leffler of exponent \(\alpha\).

Proof follows from Kagan, Linnik and Rao (1973, p.324).

### 4 Cox and Renewal Processes

An ordinary renewal process with inter arrival time distribution \(G\) is said to be a Cox process if there exists a process with inter arrival time distribution \(F_p\), such that \(G\) is the \(p\)-thinned process of \(F_p\), for all \(p \in (0, 1)\). That is, \(F_p\) is the \(p\)-inverse of \(G\) for all \(p \in (0, 1)\). Yannaros (1989) proved that if a renewal process \(N\) is a Cox process, than it cannot be the thinning of some non-renewal process, and all the possible original processes are \(p\)-thinnings of other renewal processes for every thinning parameter \(p \in (0, 1)\) and this properly characterises the processes which are both Cox and renewal.

Stated in the form of Laplace transforms \(g(\lambda)\) corresponds to a Cox and renewal process if and only if \(\phi_p(\lambda) = \frac{g(\lambda)}{p + g(\lambda)}\) is a Laplace transform for all \(p \in (0, 1)\), as \(\phi_p(\lambda)\) need not be a Laplace transform always. Yannaros (1988) has proved that this is possible if and only if \(g(\lambda) = \frac{1}{1+\psi(\lambda)}, \psi(0) = 0\) and \(\psi(\lambda)\) has a complete monotone derivative i.e., when \(g(\lambda)\) is g.i.d. Therefore, we have,

**Theorem 4.1.** An ordinary renewal process defines a Cox process if and only if its inter arrival time distribution is g.i.d.

Now we examine the behaviour of a \(p\)-thinned renewal process with \(p = 1/n\) and \(\phi_p(\lambda) = e^{-(1/n)\psi(\lambda)}\) as \(n \to \infty\).

**Theorem 4.2.** The \((1/n)\)-thinning of an ordinary renewal process whose inter arrival time distribution is i.d. with Laplace transform \(e^{-(1/n)\psi}\), as \(n \to \infty\) defines a Cox and renewal process.

**Proof.** The Laplace transform of the \((1/n)\) thinned process is given by

\[
\frac{(1/n)e^{-(1/n)\psi}}{1 - (1 - 1/n)e^{-(1/n)\psi}}.
\]

Taking the limit as \(n \to \infty\), the Laplace transform reduces to \(\frac{1}{1+\psi(\lambda)}\), which is g.i.d. Hence the result. \(\square\)
Remark 4.1. We saw that Laplace transforms of the form \( \frac{1}{1 + t\psi(\lambda)} \) for \( t \geq 0 \), \( \frac{1}{(1 + \psi(\lambda))^r} \) and \( \frac{1}{1 + \psi^2(\lambda)} \) for \( t \leq 1 \) are g.i.d. This means that they can serve as the inter arrival times of Cox and renewal processes. Or in other words subordination of i.d. processes (with positive support) with exponential, gamma and Mittag-Leffler operational times generates Cox and renewal processes.

An analogue of a theorem in Feller (1966, p.294) can be proved combining the main ideas.

Theorem 4.3. The following classes of probability distribution are identical.

(i) The set of all g.i.d. distributions with positive support and the limits of such distributions.

(ii) Distributions with positive support whose characteristic function is

\[
g(t) = \lim_{n \to \infty} \frac{1}{1 + \sum_{k=1}^{k_n} \lambda_{n,k} (1 - e^{i\beta_{n,k} t})}, \lambda_n > 0
\]

(iii) Limit distributions of \( \sum_{j=1}^{N_n} X_j \) where \( N_n \) follows a geometric distribution with mean \( n \), independent of \( X_j \)'s and \( X_j \)'s are uniformly asymptotically negligible r.vs with a common distribution function having positive support and Laplace transform \( e^{-(1/n)\psi} \).

(iv) The set \( T = \{ F : F \text{ is the inter arrival time distribution of a Cox and renewal process} \} \).

Proof.

(i) Follows from Klebanov et al. (1984).

(ii) Follows from Laha and Rohatgi(1979, p.237) and from the connection between i.d. and g.i.d. in terms of characteristic function (Klebanov et al. (1984).

(iii) Follows from Theorem 4.2.

(iv) Follows from Theorem 4.1.

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