On 6d “Gauge” Theories with Irrational Theta Angle

Barak Kol

Department of Physics
Stanford University
Stanford, CA 94305, USA
barak@leland.stanford.edu

Abstract

A recent proposal for 6d “gauge” theories with rational theta angle is discussed. These theories were constructed from $n_C$ coinciding $(p, q)$ 5-branes of IIB in the limit of vanishing string constant. They have $(1, 1)$ supersymmetry and the low energy theory is 6d Yang-Mills with gauge group $SU(n_C)$ and a rational theta angle $\theta_6/2\pi = f/q$, where $fp = 1 \pmod{q}$. By changing the point of view, considering the $(p, q)$ 5-branes to be D5-branes, the 6d theta angle is identified with a 10d theta angle. This point of view together with some assumptions suggests a generalization of the previous limit to arbitrary theta. This limit seems to define a decoupled 6d theory even though the 10d theory does not become free in general.
Six dimensional Theories without gravity were introduced in [1,2]. Related ideas were discussed in studies of black holes, BPS charges and strings inside 5-branes [3–5]. These theories were found to play a role in matrix M theory [6,7]. Consider those with (1,1) supersymmetry. Such theories reduce in low energies to 6d Yang-Mills, and their Lagrangian includes the following terms
\[ \mathcal{L} = \frac{1}{g_6^2} F^2 + \frac{\theta_6}{2\pi} F \wedge F \wedge F + \ldots \] (1)

where \( F \) is the field strength, \( g_6 \) is the 6d coupling constant, and \( \theta_6 \) is a 6d theta angle, related to the \( \pi_5 \) homotopy of the gauge group. The energy scale in the 5-brane is set by \( 1/g_6^2 \), the tension of the instanton inside the 5-brane. Some constructions of these theories in terms of the 5-branes of Type IIB [8,9] will be discussed here.

Let us set the notations to discuss Type IIB. Denote the complex scalar of IIB by \( \tau = \frac{\theta_B}{2\pi} + i/\lambda \), where \( \lambda \) is the string coupling and \( \theta_B \) is the IIB theta angle. Type IIB on a large circle of size \( L_B \) with a complex scalar \( \tau \) is the limit of M theory compactified on a small torus with base \( L_t \) and modular parameter \( \tau \) [10,11] (figure (1)). Let \( A_r \) denote the torus area \( A_r = L_t^2 \text{Im}(\tau) \). The compactification sizes are related through \( 2\pi/L_B = A_r T_M = L_t^2 \text{Im}(\tau) M_{11}^3/(2\pi)^2 \), where \( T_M \) is the M-membrane tension \(^1\), and \( M_{11} \) is the 11d Planck length. The 10d Planck length \( M_p \) is given by \( L_B M_p^8 = A_r M_{11}^9 \). An expression that depends on the M theory variables survives the IIB limit if when translated to IIB variables it is independent of \( L_B \). A \((p,q)\) 5-brane \(^2\) can have an instantonic string inside it. This

\(^1\)The conventions used for \( 2\pi \) factors are: \( T_M = M_{11}^3/(2\pi)^2, T_{M5} = M_{11}^6/(2\pi)^5, T_{Dp} = 1/[(\lambda(2\pi)^p\alpha'(p+1)/2)] \).

\(^2\) Wrapping M-branes around \((p,q)\) cycles produces branes of IIB. A \((1,0)\) string is the fundamental string with tension \( T_s = L_t T_M \), whereas a \((0,1)\) string is the D-string. The tension of a general \((p,q)\) string is given by \( T_{p,q} = |p + q\tau| L_t T_M \). A general \((p,q)\) 5-brane has tension \( T_{p,q}^{5-brane} = |p + q\tau| \text{Im}(\tau) T_s^3/(2\pi)^2 = |p + q\tau| L_B M_{11}^6/(2\pi)^5 \), where \((1,0)\) is the D5-brane and \((0,1)\) is the NS5-brane.
instanton was called a “strip” in [12] since its M theory origin is a membrane stretched over an interval with endpoints on the M5-brane (figure (1)). The strip tension in a general \((p, q)\) 5-brane with a general \(\tau\) is given by

\[
T_{p,q}^{\text{strip}} = \frac{A_{\tau}}{|p + \tau q|} L_t = \frac{Im(\tau)}{|p + \tau q|} T_s. \tag{2}
\]

In [8] Seiberg introduced a number of new 6d constructions. The relevant one to us is constructed from

\[
n_C \text{ coinciding NS5-branes}
\]

taking the limit \(\lambda \to 0\). \tag{3}

The low energy gauge group is \(SU(n_C)\). For the NS5-brane the tension of the instanton inside the 5-brane, the only energy scale, is \((2) T_{\text{NS5}}^{\text{strip}} = 1/g_6^2 = T_s\). Since this tension is independent of \(\lambda\), it was argued that for \(\lambda \to 0\), the bulk becomes free and a non-trivial 6d theory decouples from it.

To describe this theory in the IIB limit of M theory, we should perform the limit \(\lambda \to 0\) while fixing the compactification size in 6d units \(L_B \sqrt{T_{\text{strip}}} = \epsilon^{-1}\). For a theory on an NS5-brane the limit translates to \(L_t \to 0\), \(\tau = i \epsilon/L^{3/2}(2\pi/T_M^{3/2})\). After taking this limit we can take the decompactification limit \(\epsilon \to 0\).

The appearance of fractional instantons of tension \(T \sim T_{\text{strip}}/n_C\) [13] is natural in this picture\(^3\). In 11d the circle transverse to the M5-branes is small and the M5-branes could be distributed around it in various ways, while still coinciding in the 10d sense. A general configuration, roughly equally spaced around the circle, results in \(n_C\) fractional strips. These correspond to membranes ending on adjacent M5-branes. The existence of fractional strips can be ignored in the following discussion since \(n_C\) will be fixed.

The discussion was generalized by Witten [9] to describe theories parametrized by integers \(f, q, n_C\) where \(f\) and \(q\) are relatively prime and \(0 \leq f \leq q - 1\). The low energy description

\(^3\)The author thanks I. Klebanov, J. Maldacena and A. Rajaraman for discussing this point.
of these theories has $SU(n_C)$ gauge group, and a rational theta angle $\theta_6/2\pi = f/q$. Define integers $p, e$ such that

$$M = \begin{bmatrix} p & e \\ q & f \end{bmatrix} \in SL(2, \mathbb{Z}).$$

(4)

Explicitly, $p$ is given by $pf = 1 \pmod{q}$, and we have the freedom $(p, e) \to (p, e) + n(q, f), n \in \mathbb{Z}$. (Note: the notation used here is slightly different than in [9]). These theories are constructed in type IIB as the theory of

$n_C$ coinciding $(p, q)$ 5-branes

taking the limit $\tau \to i\infty$,

or $L_t \to 0, \tau \sim \frac{i\epsilon}{L_t^{3/2}\sqrt{q}} \cdot \frac{2\pi}{T_M^{3/2}}$.

(5)

As in the previous case the bulk theory becomes free, and the 6d is decoupled and non-trivial.

Let us change the point of view and consider the $n_C$ 5-branes to be $n_C$ D5-branes, through an $SL(2, \mathbb{Z})$ transformation. Explicitly, the required transformation is $M^{-1}(4)$. Then the construction consists of

$n_C$ coinciding D5-branes

taking the limit $\tau \to 0$ on the imaginary axis, for the first case (3),

and $\tau \to f/q$ with fixed real part, for the second case (5),

or $L_t \to \infty, \tau = Re(\tau) + i\frac{\epsilon^2}{L_t^3} \cdot \frac{(2\pi)^2}{T_M}.$

(6)

The advantage in this point of view is the direct relation

$$\theta_B = \theta_6$$

(7)

which can be read off the D5-brane action. Note that in this limit $\lambda \to \infty$ which is surprising at first.

Let us analyze the conditions for the 10d theory to be free and for the 6d theory to decouple. The 10d theory is free if there is some $(l, k)$ string with a small string coupling, so its tension satisfies
\[ T_{i,k}^{\text{string}} \ll M_p^2. \] (8)

In the limit \( \tau \to f/q \) it is the \((f, -q)\) string that becomes light. Its tension, as \( L_t \to \infty \) is

\[ T_{f,-q}^{\text{string}} \sim q \text{Im}(\tau)L_tT_M = \frac{q e^2}{L_t^2} \cdot \frac{(2\pi)^2}{T_M}. \] (9)

The tension of the strip, \( T_{\text{strip}} \), and the tension of the 5-brane, \( T_5 \), are related through

\[ T_{\text{strip}} \cdot T_5 = M_p^8. \] (10)

This relation is a result of Dirac quantization in 11d between the M-membrane and its magnetic dual, the M5-brane. So the energy scales in the 6d theory are characterized by a single scale, and it seems that the condition for the 6d theory to decouple is

\[ T_{\text{strip}} = 1/g_6^2 = \frac{A_r}{|p + \tau q|L_t}T_M \ll M_p^2. \] (11)

From this point of view a generalization to irrational theta angles suggests itself. Consider taking \( n_C \) coinciding D5-branes

and the limit \( \tau \to \frac{\theta_6}{2\pi} \)

or \( L_t \to \infty, \tau = \frac{\theta_6}{2\pi} + i \frac{e^2}{L_t^2} \cdot \frac{(2\pi)^2}{T_M}. \) (12)

\( \theta_6 \) is arbitrary now and \( p, q \) are not required. In order to consider the conditions (8,11) we should measure distances on the torus plane divided by \( M_p^2 \). In these units the limit turns out to be \( L_t \to \infty, \text{Im}(\tau) = 1/L_t \) (suppressing \( M_{11} \)), so that the area of the torus is fixed in these units. We see that the condition for the 6d theory to decouple (11) is satisfied. Assuming the mentioned procedures for defining 6d theories are correct, and that (11) is indeed the correct condition for a 6d theory to decouple, we have a theory with an arbitrary theta angle. This became possible by considering tori that do not necessarily generate a square lattice.

This theory does not allow to define (in general) a limit in which the bulk is free. It turns out that taking the limit for \( \tau \) while measuring tensions in Planck units corresponds
to the torus degenerating while keeping its area fixed. To have a free limit we should look for light strings. Consider an \((l, k)\) string. Its tension is \(\left| (l + k \frac{\theta_6}{2\pi}) L_t + i k / L_t \right|\) in 10d Planck units. As a function of \(L_t\) it has a minimum value of

\[
T_{l,k} = \sqrt{(k(l + k \frac{\theta_6}{2\pi}) \sqrt{2}}.
\]

(13)

An irrational number, \(x\), is said to be approximated by rationals to order \(n\) if \(\exists c > 0, \text{ and infinitely many integers } k\) such that

\[
|kx \pmod{1}| < \frac{c}{k^{(n-1)}}.
\]

(14)

We see that only if the degree of rational approximation for \(\theta_6/2\pi\) is greater than two, \(n > 2\), a limit can be defined such that the condition for the bulk theory to be free (8) is satisfied. It turns out that for a general irrational \(n = 2\) with a constant \(c = 1/\sqrt{5}\), and so in general a free limit cannot be defined. Special irrationals with higher \(n\) should allow a free bulk. Does an M theory construction [9] exist for them?

Let us discuss the role of rationals in these theories. It seems that there is a continuous transition between theories of different (rational) \(\theta_6/2\pi\) thus answering a question raised in [9]. From a physical point of view note that the transition from rationals to irrationals is smooth. From (9) we see that as the rational denominator, \(q\), grows, the free bulk limit is reached later (for higher \(L_t\)). Finally, when the number is irrational this limit cannot be reached anymore. It is interesting to note another “rational” physical phenomena - only for rational \(\theta_6/2\pi\) it is possible for a numbers of strips to join, become a string and leave the 5-brane.

ACKNOWLEDGEMENTS

The author thanks O. Aharony, E. Halyo, A. Rajaraman, L. Susskind and E. Witten. BK is supported by NSF grant PHY-9219345.
FIG. 1. The small torus of IIB with base $L_t$ and a theta angle. The shown dimensions are proportional to the D5-brane tension $T_{D5}$, the strip tension $T_{\text{strip}}$ (dashed), and the D-string tension $T_{D1}$. The limit $\text{Im}(\tau) \to 0$ is demonstrated by the dotted arrow.

REFERENCES

[1] E. Witten, “Some comments on string dynamics,” hep-th/9507121.

[2] A. Strominger, “Open p-branes,” Phys. Lett. B383 (1996) 44–47, hep-th/9512059.

[3] R. Dijkgraaf, E. Verlinde, and H. Verlinde, “BPS spectrum of the five-brane and black hole entropy,” Nucl. Phys. B486 (1997) 77–88, hep-th/9603126.

[4] R. Dijkgraaf, E. Verlinde, and H. Verlinde, “BPS quantization of the five-brane,” Nucl. Phys. B486 (1997) 89–113, hep-th/9604055.

[5] J. M. Maldacena, “Statistical entropy of near extremal five-branes,” Nucl. Phys. B477 (1996) 168–174, hep-th/9605016.

[6] M. Rozali, “Matrix theory and U duality in seven-dimensions,” Phys. Lett. B400 (1997) 260–264, hep-th/9702136.

[7] M. Berkooz, M. Rozali, and N. Seiberg, “Matrix description of M theory on $T^4$ and
$T^5$, “Phys. Lett. B408 (1997) 105–110, hep-th/9704089.

[8] N. Seiberg, “New theories in six-dimensions and matrix description of M theory on $T^5$ and $T^5/\mathbb{Z}_2$,” hep-th/9705221.

[9] E. Witten, “New 'gauge' theories in six-dimensions,” hep-th/9710065.

[10] J. H. Schwarz, “An SL(2,\mathbb{Z}) multiplet of Type IIB superstrings,” Phys. Lett. B360 (1995) 13–18, hep-th/9508143.

[11] P. S. Aspinwall, “Some relationships between dualities in string theory,” Nucl. Phys. Proc. Suppl. 46 (1995) 30, hep-th/9508154.

[12] O. Aharony, A. Hanany, and B. Kol, “Webs of (p,q) five-branes, five-dimensional field theories and grid diagrams,” hep-th/9710116.

[13] J. M. Maldacena and L. Susskind, “D-branes and fat black holes,” Nucl. Phys. B475 (1996) 679–690, hep-th/9604042.

[14] I. M. Niven, “Irrational numbers,” John Wiley (1956).