Robust Product-line Pricing under Generalized Extreme Value Models

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We study robust versions of pricing problems where customers choose products according to a generalized extreme value (GEV) choice model, and the choice parameters are not known exactly but lie in an uncertainty set. We show that, when the robust problem is unconstrained and the price sensitivity parameters are homogeneous, the robust optimal prices have a constant markup over products and we provide formulas that allow to compute this constant markup by bisection. We further show that, in the case that the price sensitivity parameters are only homogeneous in each partition of the products, under the assumption that the choice probability generating function and the uncertainty set are partition-wise separable, a robust solution will have a constant markup in each subset, and this constant-markup vector can be found efficiently by convex optimization. We provide numerical results to illustrate the advantages of our robust approach in protecting from bad scenarios. Our results generally hold for convex and bounded uncertainty sets, and for any arbitrary GEV model, including the multinomial logit, nested or cross-nested logit.

Key words: Robust optimization, multi-product pricing, generalized extreme value model

1. Introduction

In revenue management, pricing is an important problem that refers to the selection of prices for a set of products in order to maximize an expected revenue. This is motivated by the fact that prices are key features that may significantly affect demand for products. The literature of multi-product pricing has seen a large number of papers focusing on how to set prices when customers purchase products according to a discrete choice model (e.g. Talluri and Van Ryzin 2004, Gallego and Wang 2014, Zhang et al. 2018). To the best of our knowledge, prior work all assumes that the parameters of the choice models are known in advance or can be estimated exactly from data. Thus, the corresponding pricing optimization models are built based on pre-determined parameters and ignore any uncertainty in case the parameters are estimated. Nevertheless, in practice, the parameter estimates may vary significantly for different customer types or in different purchasing periods of the year. Thus, ignoring such uncertainties may lead to bad pricing decisions. To deal with the uncertainty issue, one may consider a stochastic approach, i.e., a model aiming at maximizing
an average expected revenue over a finite number of scenarios of the choice parameters. This would require of course a trusted assumption and/or a solid optimization of these parameters in each of these scenarios. Moreover, such a stochastic optimization model would be computationally difficult to handle, as the objective function does not have nice properties to derive tractable solutions as in the deterministic case; e.g., a stochastic objective function would be non-unimodal and non-concave when defined in terms of purchase probabilities (Li et al. 2018).

In this paper, we formulate and solve pricing optimization problems under uncertainty in a robust manner. That is, we assume customers’ behavior is driven by any choice model in the Generalized Extreme Value (GEV) family such as the Multinomial Logit (MNL) or nested logit model, and the parameters of the choice model are not known exactly but belong to an uncertainty set. The goal here is to maximize the worst-case expected revenue when the choice parameters vary in their support set. We consider problems where the price sensitivity parameters (PSP) are homogeneous or partition-wise homogeneous, i.e., the set of products can be separated into disjoint subsets and the PSP are the same in each subset but can be different over subsets. For the latter, we assume that the choice probability generating function (Fosgerau et al. 2013) has a separable structure and the uncertainty set is partition-wise separable. We also look at expected-sale requirements in pricing decisions and argue that the model with expected-sale constraints is not appropriate in our robust setting. Therefore, we propose an alternative formulation by adding a penalty term to the objective function for violated expected-sale constraints. We are able to show that the models can then be solved in a tractable way. Our results generally hold for any convex and bounded uncertainty set, and for any choice model in the GEV family.

From now on, when saying “a GEV model”, we refer to any choice model in the GEV family. Each GEV model can be represented by a choice probability generating function (CPGF) \( G(\cdot) \) (see our detailed definition in the next section). To relax the homogeneity of the PSP, we need to assume that the CPGF has a separable structure, which means that \( G(\cdot) \) can be written as a sum of sub-CPGFs, each corresponding to a subset of products.

**Our contributions:** We consider robust versions of the standard pricing optimization problem under GEV models. The setting here is to assume that the parameters of the choice model are not known with certainty and the aim is to find optimal prices associated with products, which maximize the worst-case expected revenue when the choice parameters vary in an uncertainty set. For the unconstrained problem with homogeneous PSP, we show that if the uncertainty set is convex and compact, the robust optimal prices have a constant markup with respect to the products costs, i.e., the robust optimal price of a product is equal to its unit cost plus a constant that is the same over all products. We also provide formulas that allow efficient computation of that constant markup by binary search. This finding generalizes the results for the deterministic
unconstrained problem with homogeneous PSP considered in Zhang et al. (2018). We also provide comparative insights showing how the robust optimal revenue and the robust optimal constant markups change as functions of the uncertainty level (i.e., the size of the uncertainty set).

For the pricing problem with non-homogeneous PSP, we assume that the CPGF is partition-wise separable and in each partition, the PSP are homogeneous. Moreover, the uncertainty set is also assumed to be partition-wise separable. We show that the robust problem can be converted equivalently into a reduced optimization problem, which can be conveniently solved by convex optimization. As a result, the robust optimal prices have partition-wise constant markups, i.e., in each partition, the robust optimal prices have a constant markup with respect to their costs, and these constant markups can be obtained by convex optimization. We also provide comparative insights for the robust optimal prices and solutions when the size of the uncertainty set varies.

For both cases (i.e., homogeneous PSP and partition-wise homogeneous PSP), we further show that the robust optimal solutions form saddle points of the robust problems, leading to an equality between the objective functions of the max-min problem and its min-max counterpart.

Previous studies (Zhang et al. 2018, Song and Xue 2007, Zhang and Lu 2013) have been looking at constraints on the expected sales, as motivated by applications with inventory considerations (Gallego and Van Ryzin 1997). In this context, the aim is to select prices that maximize the expected revenue while requiring that the expected sales of products lie in a convex set. The advantage of such constraints is that the pricing problem can be reformulated equivalently as a convex program where the decision variables are the purchase probabilities. However, the final decision is a vector of prices and there may be no fixed prices under which the resulting purchase probabilities always satisfy the expected sale constraints when the choice parameters vary. For this reason, the use of the constrained formulation is not appropriate in our robust setting. Thus, we propose an alternative formulation in which, instead of requiring that the expected sale constraints be satisfied, we add a penalty cost to the objective function for violated constraints. Our formulation, called pricing with over-expected-sale penalties, is more general than the constrained formulation, in the sense that if the penalty parameters increase to infinity, then the corresponding optimal solutions will converge to those from the constrained problem, and with zero penalty parameters, the pricing problem becomes the unconstrained one. We show that if the CPGF and the uncertainty set are partition-wise separable, then the robust problem can be converted into a reduced optimization problem, which can be conveniently solved by convex optimization.

In summary, we show that the robust versions of the pricing problem with homogeneous PSP and partition-wise PSP, with and without over-expected-sale penalties, can be solved in tractable ways by bisection and convex optimization. Our results generally holds for convex and compact
uncertainty sets, and for any choice model in the GEV family. In Table 1 below we give a summary and comparison of the solution methods used to solve the robust pricing problems and their deterministic counterparts, under different settings. The solution methods proposed in this paper are highlighted in bold.

| Settings                          | Deterministic pricing | Robust pricing                        |
|----------------------------------|-----------------------|---------------------------------------|
| Unconstrained and homogeneous PSP | Closed-form solutions | Bisection and convex optimization      |
| Unconstrained and partition-wise homogeneous PSP | Bisection | Convex optimization |
| Expected-sale constraints        | Convex optimization   | Not appropriate                        |
| Over-expected-sale penalties     | Convex optimization   | Convex optimization                   |

Table 1 Solution methods for deterministic pricing and robust pricing problems under different settings.

Literature review: The GEV family includes most of the parametric discrete choice models in the demand modeling and operations research literatures. The simplest and most popular member is the MNL (McFadden 1978, 1980) and it is well-known that the MNL model retains the independence from irrelevant alternatives (IIA) property, which does not hold in many contexts. There are a number of GEV models that relax this property and provide flexibility in modeling the correlation between alternatives, for example, the nested logit model (Ben-Akiva et al. 1985, Ben-Akiva 1973), the cross-nested logit (Vovsha and Bekhor 1998), the generalized nested logit (Wen and Koppelman 2001), the paired combinatorial logit (Koppelman and Wen 2000), the ordered generalized extreme value (Small 1987), the specialized compound generalized extreme value models (Bhat 1998, Whelan et al. 2002) and network-based GEV (Daly and Bierlaire 2006, Mai et al. 2017) models. Fosgerau et al. (2013) show that the cross-nested logit model and its generalized version (i.e. network-based GEV) are fully flexible in the sense that they can approximate arbitrarily close any random utility maximization model. Beside the GEV family, it is worth noting that the mixed logit model (McFadden and Train 2000) is also popular due to its flexibility in capturing utility correlation. There is a fundamental trade-off between the flexibility and the generality of the choice models and the complexity of their estimation and application in operational problems. For the case of GEV models, even being flexible in modeling choice behavior, the resulting operational problems (e.g., product assortment or pricing) are often nonlinear and non-convex, leading to difficulties solving them in practice.

There is a large amount of research on unconstrained pricing under different discrete choice models. For example, Hopp and Xu (2005) and Dong et al. (2009) consider the pricing problem under the MNL model, Li and Huh (2011) consider the nested logit model, Li et al. (2015) consider the pricing problem under the paired combinatorial logit model, and Zhang et al. (2018) consider
the pricing problem under any choice model in the GEV family. Under the assumption that the PSP are the same over product, these authors show that the prices have a constant markup with respect to the product costs and provide formulas to explicitly compute this constant markup.

There are some papers trying to get over the assumption that the PSP are homogeneous over products. Li and Huh (2011) study the pricing problem under the nested logit model and assume that the PSP are homogeneous only in each nest and can be different over nests. They then show that the PSP in each nest have a constant markup. Zhang et al. (2018) generalize these results by considering the pricing problem under GEV models, in which the CPGF is partition-wise separable and the PSP are assumed to be homogeneous in each partition. The authors also show that, in this case, the optimal prices have a constant markup in each partition.

There are also publications considering the pricing problem with arbitrary PSP. Gallego and Hu (2014) show that the pricing optimization problem under the nested logit model can have multiple local optimal solutions if the PSP are arbitrarily heterogeneous and provide sufficient conditions to ensure unimodality of the expected revenue function. Li et al. (2015) and Huh and Li (2015) consider the pricing problem under the $d$-nested and paired combinatorial logit models and also provide sufficient conditions on the PSP to ensure unimodality of the expected revenue function.

The constrained pricing problem where the prices are required to lie in a feasible set is difficult to solve as the expected revenue function is nonlinear and non-concave in the prices. Motivated by applications with inventory considerations (Gallego and Van Ryzin 1997) and the observation that the expected revenue function is concave in the purchase probabilities, researchers have considered the pricing problem with constraints on the expected sales. For example, Song and Xue (2007), Zhang and Lu (2013) consider the pricing problem under the MNL model and show that the expected revenue is concave in the purchase probabilities if the PSP are homogeneous. Keller (2013) consider the pricing problem under the MNL and nested logit models and show that the expected revenue function is concave in the purchase probabilities under the MNL and arbitrary PSP, and establish sufficient conditions on the PSP to ensure that the expected revenue under the nested logit model is concave. Zhang et al. (2018) also generalize all these results by showing that, under any GEV model, if the PSP are homogeneous or partition-wise homogeneous, then the expected revenue is concave in purchasing probabilities, making the pricing problem with expected sale constraints tractable.

All above publications assume that the parameters of the choice model is given in advance and ignore any uncertainty associated with such parameters in the pricing problem. However, the choice parameters typically need to be inferred from data and uncertainties may occur, for instance, due to the heterogeneity of the market. In this work, we explicitly take into consider this issue by considering robust versions of the unconstrained and constrained pricing problems, with
homogeneous and partition-wise homogeneous PSP. Our results directly generalize the results for deterministic pricing from Zhang et al. (2018), which already covers most of the pricing optimization studies in the literature.

Our work is concerned with robust solutions for the pricing problem under uncertainty, so it is directly related to the concept of robust optimization, an important research area in operations research which has received a growing attention over the past two decades. Robust optimization is motivated by the fact that many real-world decision problems arising in engineering and management science have uncertain parameters due to limited data or noisy measurements. The literature on robust optimization includes a larger number of excellent studies (see Ben-Tal and Nemirovski 1998, 2000, Ben-Tal et al. 2006, for instance). Most of the studies in the literature of robust optimization focus on linear, piece-wise linear or convex objective functions. In our context, the expected revenue is nonlinear and non-convex/non-concave in the prices, implying that existing robust optimization results do not apply (except the part where we consider the constrained pricing problem under uncertain expected-sale constraints in Section B.1), and making our robust problem challenging to solve in a tractable way. It is worth noting that our work is relevant to Rusmevichientong and Topaloglu (2012) where the authors consider robust versions of the assortment planning problem. The main difference is that the decision variables in Rusmevichientong and Topaloglu (2012) are discrete (i.e., a set of products).

**Paper outline:** We organize the paper as follows. In Section 2, we present the deterministic pricing problem under GEV models and recall some results from previous work. In Section 3 and 4, we present our results for the robust pricing problem under homogeneous PSP and partition-wise homogeneous PSP. In Section 5 we provide some experimental results and in Section 6 we conclude. In the appendix, Section A provides detailed proofs for our main claims and Section B investigates the robust pricing problem with over-expected-sale penalties.

**Notation:** Boldface characters represent matrices (or vectors), and \( a_i \) denotes the \( i \)-th element of vector \( a \). We use \([m]\), for any \( m \in \mathbb{N} \), to denote the set \( \{1, \ldots, m\} \). For any vector \( b \) with all equal elements, we use \( \langle b \rangle \) to denote the value of one element of the vector. Given two vectors of the same size \( a, b \in \mathbb{R}^m \), \( a \succeq b \) is equivalent to \( a - b \in \mathbb{R}^m_+ \), and \( a \preceq b \) is equivalent to \( b \succeq a \).

2. **Background: Deterministic Pricing under Generalized Extreme Value Models**

We denote by \( \mathcal{V} = \{1, \ldots, m\} \) the set of \( m \) available products. There is a *non-purchase item* indexed by 0, so the set of all possible *products* is \( \mathcal{V} \cup \{0\} \). We also denote by \( x_i \) and \( c_i \) the price and the cost of product \( i \), respectively. The random utility maximization (RUM) framework (McFadden 1978) is the most popular approach to model discrete choice behavior. Under this framework, each
product \( i \in \mathcal{V} \) is assigned with a random utility \( U_i \) and the additive RUM framework (Fosgerau et al. 2013, McFadden 1978) assumes that each random utility can be expressed as a sum of two part \( U_i = u_i + \varepsilon_i \), where the term \( u_i \) is deterministic and can include values representing characteristics of the product, and the term \( \varepsilon_i \) is unknown to the analyst. The RUM principle then assume that the selections are made by maximizing these utilities and the probability that a product \( i \) (including the non-purchase item) is selected can be computed as \( P(U_i \geq U_j, \forall j \in \mathcal{V} \cup \{0\}) \).

In our context, we are interested in the effect of the prices on the expected revenue. So we assume that the deterministic terms \( u_i, \forall i \in \mathcal{V} \), can be expressed as \( u_i = a_i - b_i x_i \), where \( b_i \) is the PSP associated with product \( i \) and \( a_i \) can include other information that may affect customer’s demand such as the brand, size or color of the items. These values can be obtained by fitting the choice model with observation data.

A GEV model can be represented by a choice probability generating function (CPGF) \( G(Y) \), where \( Y \) is a vector of size \( m \) with entries \( Y_i = e^{u_i} \), for all \( i \in \mathcal{V} \). Given \( i_1, \ldots, i_k \in [m] \), let \( \partial G_{i_1,\ldots,i_k}(Y) \) be the mixed partial derivatives of \( G \) with respect to \( Y_{i_1}, \ldots, Y_{i_k} \). It is well-known that the CPGF \( G(\cdot) \) and the mixed partial derivatives have the the following properties (McFadden 1978, Ben-Akiva et al. 1985).

**Remark 1 (Properties of GEV-CPGF).** A GEV-CPGF \( G(Y) \) has the following properties.

(i) \( G(Y) \geq 0, \forall Y \in \mathbb{R}^m \),

(ii) \( G \) is homogeneous of degree one, i.e., \( G(\lambda Y) = \lambda G(Y) \)

(iii) \( G(Y) \to \infty \) if \( Y_i \to \infty \)

(iv) Given \( i_1, \ldots, i_k \in [m] \) distinct from each other, \( \partial G_{i_1,\ldots,i_k}(Y) > 0 \) if \( k \) is odd, and \( \leq \) if \( k \) is even

(v) \( G(Y) = \sum_{i \in \mathcal{V}} Y_i \partial G_i(Y) \)

(vi) \( \sum_{j \in \mathcal{V}} Y_j \partial G_{ij}(Y) = 0, \forall i \in \mathcal{V} \).

Here we note that (i)-(iv) are basic properties of the CPGF to ensure that the choice model is consistent with the RUM principle (McFadden 1980). Properties (v) and (vi) are direct results from the homogeneity property (Zhang et al. 2018).

Under a GEV model specified by a CPGF \( G \), given any vector \( Y \in \mathbb{R}^m \), the choice probability of product \( i \in \mathcal{V} \) is given by

\[
P_i(Y|G) = \frac{Y_i \partial G_i(Y)}{1 + G(Y)}.
\]

Note that the above formulation also implies that the choice probability of the non-purchase item is \( P_0(Y|G) = 1/(1 + G(Y)) \). The GEV becomes the MNL model if \( G(Y) = \sum_{i=1}^m Y_i \), and it becomes the nested logit model if \( G(Y) = \sum_{n \in \mathcal{N}} \left( \sum_{i \in C_n} (\sigma_{in} Y_i)^{\mu_n} \right)^{\mu/\mu_n} \), where \( \mathcal{N} \) is the set of nests, \( C_n \) is the set of items in nest \( n \) and \( \sigma_{in}, \mu > 0, \mu_n > 0 \) are the parameters of the nested logit model. In
the generalized version of the nested logit model proposed by Daly and Bierlaire (2006), called the network GEV, the corresponding CPGF can be computed recursively based on a rooted and cycle-free graph representing the correlation structure of the items.

Under a GEV model specified by a CPGF $G(\cdot)$, the deterministic version of the pricing problem is stated as

$$\max_{x \in \mathbb{R}^m} \left\{ R(x) = \sum_{i=1}^{m} (x_i - c_i) P_i(Y(x,a,b)|G) \right\}, \quad (P1)$$

where $Y(x,a,b) \in \mathbb{R}^m$ with entries $Y_i(x,a,b) = \exp(a_i - b_i x_i)$. The expected revenue $R(x)$ becomes more difficult to handle as the GEV model becomes more complicated. By leveraging the properties of GEV models stated in Remark 1, Zhang et al. (2018) manage to show that if the PSP are homogeneous, i.e., $b_i = b_j$ for all $i,j \in V$ and if $x^*$ is an optimal solution to (P1), then

$$x^*_i - c_i = \frac{1}{\langle b \rangle} + R(x^*), \forall i \in V \text{ and } R(x^*) = \frac{W(\gamma e^{-1})}{\langle b \rangle}, \quad (1)$$

where $\gamma = G(Y_1(c_1), \ldots, Y_m(c_m))$ and $W(\cdot)$ is the Lambert-W function. The results in (1) indeed imply that a constant markup solution is optimal to (P1) and this constant markup can be computed explicitly. Moreover, if the PSP are partition-wise homogeneous and $G$ is separable, then Zhang et al. (2018) show that the optimal prices have a constant markup in each partition. These results also provide an explicit way to compute optimal prices for the pricing problem under the MNL with arbitrary PSP. Zhang et al. (2018) also show that the expected revenue function is concave in the purchasing probabilities under any GEV model, making the pricing problem with expected sale constraints tractable.

### 3. Robust Pricing under Homogeneous Price Sensitivity Parameters

In this section, we study a robust version of the unconstrained pricing problem, under the setting that the choice parameters $(a, b)$ are not known exactly but belong to an uncertainty set. We focus here on the case of homogeneous PSP. In our robust model, we aim at maximizing the worst-case expected revenue over all parameters in the uncertainty set. The robust unconstrained pricing problem can be formulated as

$$\max_{x \in \mathbb{R}^m} \left\{ g(x) = \min_{(a,b) \in \mathcal{A}} \sum_{i=1}^{m} (x_i - c_i) P_i(Y(x,a,b)|G) \right\}, \quad (RO)$$

where $\mathcal{A}$ is the uncertainty set of the parameters $(a, b)$. We denote $\Phi(x,a,b) = \sum_{i=1}^{m} (x_i - c_i) P_i(Y(x,a,b)|G)$ for notational simplicity. We assume that $\mathcal{A}$ is convex and bounded. The convexity and boundedness assumptions are useful later in the section, as we need to show that, under a constant-markup style vector of prices, the objective function of the adversary’s problem is convex on $\mathcal{A}$, which in turn helps identify a saddle point of the robust problem. The boundedness
assumption is realistic in the context, as the choice parameters are often inferred from data and it is expected that they are finite. We also assume that the PSP are positive, i.e., \( b > 0 \) for any \((a, b) \in \mathcal{A}\), which is conventional from a behavior point of view.

When the PSP are the same over all the products, we will show that the robust optimal prices have a constant markup and this constant markup can be computed efficiently by binary search. The idea is motivated by the observation that if we consider the \( \min\max \) counterpart of the robust problem \( \min_{(a,b) \in \mathcal{A}} \max_{x \in \mathbb{R}^m} \left\{ \Phi(x, a, b) \right\} \), then we know that the adversary problem always yields a constant-markup optimal solution for any fixed choice parameters \((a, b)\) (Zhang et al. 2018). So, the \( \min\max \) counterpart is equivalent to

\[
\min_{(a,b) \in \mathcal{A}} \max_{x \in \mathcal{X}} \left\{ \Phi(x, a, b) \right\},
\]

where \( \mathcal{X} \) is the set of constant-markup solutions, i.e., \( \mathcal{X} = \{ x \in \mathbb{R}^m \mid x_i - c_i = x_j - c_j, \forall i, j \in [m] \} \).

This suggests that if there is a saddle point of the \( \max\min \) problem \( \text{(RO)} \), then it should have a constant-markup form.

To prove the result, we will consider the robust unconstrained pricing problem with constant-markup prices, i.e., we only look at prices \( x \in \mathcal{X} \). Then we show that there exist constant-markup prices \( x^* \) such that if \((a^*, b^*)\) is an optimal solution to the adversary’s problem under prices \( x^* \), then \( x^* \) is also optimal to the deterministic unconstrained problem with choice parameters \((a^*, b^*)\).

In other words, \((x^*, a^*, b^*)\) is a saddle point of \( \text{(2)} \) and \( x^* \) is also an optimal solution to the robust problem.

Given constant-markup prices \( x \in \mathcal{X} \) and choice parameters \((a, b) \in \mathcal{A}\), the expected revenue becomes

\[
\sum_{i=1}^m (x_i - c_i) p_i(Y(x, a, b)|G) = z \sum_{i \in V} Y_i(x, a, b) \partial G_i(Y(x, a, b)) \left[ 1 + G(Y(x, a, b)) \right]^{-1} = z \left[ 1 + G(Y(x, a, b)) \right]^{-1},
\]

where \( z = x_i - c_i, \forall i \in V \) and \( Y \) is a vector with entries \( Y_i = \exp(a_i - b_i(z + c_i)) \) for all \( i \in V \). For the sake of simplicity, from now on we will write \( Y \) instead of \( Y(x, a, b) \). The expected revenue is a function of \( z \) and \((a, b)\), and if \((a^*(z), b^*(z))\) is an optimal solution to the adversary’s problem, then we also have

\[
(a^*(z), b^*(z)) = \arg\min_{a, b \in \mathcal{A}} G(Y|z, a, b), \tag{3}
\]

where \( G(Y|z, a, b) = G(Y_1, \ldots, Y_m) \) with \( Y_i = e^{a_i - b_i(z + c_i)} \). In Proposition 1 below, we first show that \( G(Y|z, a, b) \) is strictly convex in \((a, b)\). As a result, \((a^*(z), b^*(z))\) is always uniquely determined. This result is important to identify a saddle point of the robust problem.
Proposition 1. Given any \( z \in \mathbb{R}_+ \), \( G(Y|z, \mathbf{a}, \mathbf{b}) \) is strictly convex on \( A \), Problem 3 always has a unique solution, and \((\mathbf{a}^*(z),\mathbf{b}^*(z))\) determined in (3) is continuous in \( z \in \mathbb{R}_+ \).

The proof is given in Appendix A.1. The proposition plays an important role in our main claim, as in the theorem below we will show that a solution to the robust problem can be found by solving a 1-dimensional fixed-point problem. The continuity of \((\mathbf{a}^*(z),\mathbf{b}^*(z))\) guarantees that this fixed-point problem always has a solution that can be found efficiently by bisection.

**Theorem 1 (Constant markup is optimal to the robust problem)).** There always exists a unique solution \( z^* \in \mathbb{R} \) to the fixed point problem

\[
z = \frac{1 + W(G(Y|0, \mathbf{a}^*(z), \mathbf{b}^*(z))e^{-1})}{\mathbf{b}^*(z)}, \tag{4}
\]

where \( W(\cdot) \) is the Lambert-W function and the constant-markup prices \( x^* \) defined as \( x^*_i = z^* + c_i, \forall i \in [m] \), is the unique robust solution of the robust problem (RO). Moreover, \((x^*,\mathbf{a}^*(z^*),\mathbf{b}^*(z^*))\) is a saddle point of (RO) and the minimax equality holds, i.e.,

\[
\max_{\mathbf{x} \in \mathbb{R}^m} \min_{(\mathbf{a}, \mathbf{b}) \in A} \Phi(x, \mathbf{a}, \mathbf{b}) = \min_{(\mathbf{a}, \mathbf{b}) \in A} \max_{\mathbf{x} \in \mathbb{R}^m} \Phi(x, \mathbf{a}, \mathbf{b}).
\]

We highlight two important claims from Theorem 1. First, the robustness preserves the constant-markup property of the solutions to the deterministic pricing problem, and second, the minimax equality holds. We note that the minimax equality is not straightforward to see at first sight, as the objective function \( \Phi(x, \mathbf{a}, \mathbf{b}) \) is not (quasi) concave in \( \mathbf{x} \) nor convex in \((\mathbf{a}, \mathbf{b})\).

We will make use of Lemmas 1 -2 below to prove the theorem. In Lemma 1 we show that function \( G(Y|z, \mathbf{a}, \mathbf{b}) \) is always bounded. Together with the results established in Proposition 1, it then becomes clear that there always exists a fixed point solution to (4). As a result, if \( z^* \) is a solution (4), then it will form optimal constant-markup prices for the robust problem. Now, let us go into details of the lemmas and proofs.

**Lemma 1.** If there are \((\mathbf{a}, \mathbf{b}), (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2m} \) such that \( \mathbf{a} \leq \mathbf{a} \leq \mathbf{a} \) and \( \mathbf{b} \leq \mathbf{b} \leq \mathbf{b} \) for all \((\mathbf{a}, \mathbf{b}) \in A\), then \( G(Y|z, \mathbf{a}, \mathbf{b}) \leq G(Y|z, \mathbf{a}, \mathbf{b}) \leq G(Y|z, \mathbf{a}, \mathbf{b}), \forall z \in \mathbb{R}_+, (\mathbf{a}, \mathbf{b}) \in A.\)

The proof can be done quite easily using the properties of function \( G(\cdot) \) and we refer the reader to Appendix A.2 for details. We are now ready to show that there is a solution to the fixed point problem (4). In Lemma 2 below we show this by making use of the continuity of \( \mathbf{a}^*(z), \mathbf{b}^*(z) \) (showed above) and the boundedness assumption on \( A \) to identify an interval where we can find \( z^* \). Without this assumption, one can simply choose 0 as a lower bound, as \( f(0) \) is always less than 0. However, to identify an upper bound, one needs some limits from the uncertainty set. This is because even in the deterministic case, if the choice parameters \( \mathbf{b} \) approach zero, or \( \mathbf{a} \) increase to infinity, then the optimal constant markup will go to infinity (see Equation 1).
Lemma 2. For any \( i \in V \), there exists \( z^* \in \mathbb{R}_+ \) such that
\[
z^* = \frac{1+W(\tau(z^*))}{\langle b^*(z^*) \rangle} \in \left[ Z^0, Z^0 \right]
\]
where
\[
Z^0 = \frac{1+W(G(Y|0,\tilde{a},\tilde{b})e^{-1})}{\langle \tilde{b} \rangle}
\]
\[
Z^0 = \frac{1+W(G(Y|0,\overline{a},\overline{b})e^{-1})}{\langle \overline{b} \rangle}
\]
\[
\tau(z^*) = G(Y|0,a^*(z^*),b^*(z^*))e^{-1}
\]
and \( a,\overline{a},b,\overline{b} \in \mathbb{R}^m \) such that \( a \leq \overline{a} \) and \( b \leq \overline{b} \) for all \( (a,b) \in A \), and \( W(\cdot) \) is the Lambert-W function.

Proof: Let \( f(z) = z - (1+W(\tau(z)))/(\langle b^*(z) \rangle) \). From Lemma 1, we have
\[
Z^0 \leq \frac{1+W(\tau(z))}{\langle b^*(z) \rangle} \leq Z^0, \, \forall z \in \mathbb{R}_+.
\]
Which means
\[
f(Z^0) \leq 0; \quad f(Z^0) \geq 0
\]
Since \( f(z) \) is continuous in \( z \) (Proposition 1), equation \( f(z) = 0 \) always has a solution in the interval \( \left[ Z^0, Z^0 \right] \).

We are now ready for the proof of Theorem 1. Basically, we will show that a \( z^* \) determined in Lemma 2 and \( (a^*(z^*),b^*(z^*)) \) will form a saddle point to the robust problem.

Proof of Theorem 1: We know that there always exists \( z^* \) being a fixed point solution to (4) (Lemma 2). Given \( x^* \) and \( z^* \), we first remark that \( (a^*(z^*),b^*(z^*)) \) is also the unique solution of the adversary’s problem
\[
\arg\min_{(a,b) \in A} \Phi(x^*,a,b)
\]
Moreover, according to the way \( x^* \) is computed and Theorem 3.1 of Zhang et al. (2018), \( x^* \) is optimal to the following problem
\[
\max_{x \in \mathbb{R}^m} \{ \Phi(x,a^*(z^*),b^*(z^*)) \}.
\]
This leads to the fact that \( (x^*,a^*(z^*),b^*(z^*)) \) is a saddle point to the robust max-min problem (RO). In other words, \( x^* \) is an optimal solution to the robust problem.

Note that the deterministic version of the unconstrained pricing problem always has a unique solution, which is a constant markup one. So, for any \( x \neq x^* \) we have
\[
g(x^*) = \Phi(x^*,a^*(z^*),b^*(z^*)) \\
> \Phi(x,a^*(z^*),b^*(z^*)) \\
\geq g(x).
\]
Thus, there is only one solution to the robust pricing problem (RO) and there is only one solution to the equation (4), as required. Since there is a saddle point to the \textit{max-min} problem (RO), the minimax equality holds, i.e., \( \min_{(a,b) \in A} \max_{x \in \mathbb{R}^m} \{ \Phi(x,a,b) \} = \max_{x \in \mathbb{R}^m} \min_{(a,b) \in A} \{ \Phi(x,a,b) \} \).

Note that the existence of a saddle point directly implies the minimax equality (a.k.a minimax equality), but the opposite does not always hold. □

Theorem 1 implies that a solution to the robust problem can be found by solving the equation

\[
 f(z) = z - \frac{1 + W(G(Y|0,a^*(z),b^*(z))e^{-1})}{\langle b^*(z) \rangle} = 0, \tag{6}
\]

in the interval \([\underline{Z}, \overline{Z}]\), in which \(\underline{Z}, \overline{Z}\) are defined in Lemma 2. This is a one-dimensional problem which could be solved efficiently via bisection and convex optimization. That is, we use convex optimization to compute \(f(x)\) for any given \(z \in [\underline{Z}, \overline{Z}]\) and use bisection to find \(z^*\) such that \(f(z^*) = 0\). In comparison with its deterministic counterpart, the robust problem requires an extra computing cost of \(\delta O(\ln(1/\epsilon))\) to obtain a constant markup that is in the \(\epsilon\)-neighbourhood of the optimal solution, where \(\delta\) is the computation cost to solve the adversary problem.

4. Robust Pricing under Partially Heterogeneous Price Sensitivity Parameters

We relax the assumption that the PSP are homogeneous. Completely relaxing this assumption makes the pricing problem challenging, even for its deterministic version (Gallego and Wang 2014). Thus, we assume that the products can be separated into partitions, and the PSP can be different over partitions. More specifically, we assume that the products can be partitioned into disjoint subsets and the products in each partition share the same PSP, and the CPGF is also partition-wise separable. This assumption has been used in previous work to derive tractable solutions to the deterministic pricing problems (Zhang et al. 2018). More precisely, we partition the set of all products \(\mathcal{V}\) into \(N\) non-empty subsets \(\mathcal{V}_1, \ldots, \mathcal{V}_N\) such that \(\mathcal{V} = \bigcup_{n=1}^N \mathcal{V}_n\) and \(\mathcal{V}_i \cap \mathcal{V}_j = \emptyset, \forall i \neq j, i, j \in [N]\). Moreover, we separate the vector \(Y\) into sub-vectors \(Y^1, \ldots, Y^N\) such that \(Y^n = \{Y_i| i \in \mathcal{V}_n\}\) for all \(n \in [N]\). We assume that the GEV-CPGF \(G(Y)\) can be separated into \(N\) GEV-CPGFs as

\[
 G(Y) = \sum_{n=1}^N G^n(Y^n).
\]

Note that the nested logit model (Ben-Akiva 1973), one of the most widely-used GEV models in the literature, also has this separating structure. For notational convenience, we also separate \((a,b) \in \mathbb{R}_+^{2m}\) into sub-vectors \((a^1,b^1), \ldots, (a^N,b^N)\) such that \((a^n,b^n) = \{(a_i,b_i)| i \in \mathcal{V}_n\}, \forall n \in [N]\).

To deal with the robust problem, we further assume that the uncertainty set \(A\) is also partition-wise separable, i.e., \(A = \bigotimes_{n \in [N]} A^n\), where \(\bigotimes\) is the Cartesian operation, \(A^n \subset \mathbb{R}^{2|\mathcal{V}_n|}\) is the uncertainty set for the sub-vector \((a^n,b^n)\), and \(A^n\) are \textit{convex and bounded} for all \(n \in [N]\). In other
words, we assume that the vector of choice parameters can vary independently across partitions. This assumption is a bit restrictive, but important to maintain the tractability of the robust problem. The reason is that if we use a general uncertainty set that allows for dependency between the choice parameters from different partitions, the adversary problem itself is generally not convex or quasi-convex in \((a, b)\), even under constant-markup prices, thus not tractable to solve. On the other hand, the assumption will allow us to handle each function \(G^n(Y^n)\) independently, thus making it possible to convert the robust optimization problem into a convex one. In fact, one can construct a partition-wise separable uncertainty set by collecting some samples of choice parameter estimates from each partition. We will discuss this in more detail in Section 5.

In this context, the difficulty lies in the fact that the optimal prices to the deterministic pricing problem do not have a single constant markup over all products. As a consequence, the robust optimal prices to \((\text{RO})\) would generally not have a single constant markup over all the products and the corresponding adversary’s objective function would not be quasi-convex and solutions to the adversary’s problem may not be unique. For this reason, we can not apply the techniques used in the previous section to identify a saddle point of the robust problem.

In the rest of the section, we will show that the robust problem can be converted equivalently into a convex optimization.

To start our exposition, we note that, in analogy to the analysis in the case of homogeneous PSP, we also see that the \(\min\)-\(\max\) counterpart always yields a partition-wise constant-markup solution (Zhang et al. 2018). Thus, if there is a saddle point in the robust problem, then it should have a partition-wise constant-markup form. This motivates us to find such a saddle point of the \(\max\)-\(\min\) problem.

First, let us look at the robust problem where we only seek prices that have a constant markup in each partition, i.e., \(x \in X^N\), where \(X^N = \{x \in \mathbb{R}^m \mid x_i - c_i = x_j - c_j, \forall i, j \in V_n, n \in [N]\}\). Let \(z_n = x_i - c_i\) for all \(i \in V_n\) and \(n \in [N]\). The robust problem becomes

\[
\max_{z \in \mathbb{R}^N} \left\{ \min_{(a, b) \in A} \sum_{n \in [N]} \sum_{i \in V_n} z_n P_i(\mathcal{Y}_n^i(z_n + c, a, b) \mid G^n) \right\},
\]

or equivalently

\[
\max_{z \in \mathbb{R}^N} \left\{ \min_{(a, b) \in A} \frac{\sum_{n \in [N]} z_n G^n(\mathcal{Y}_n^i(z_n + c, a, b) \mid G^n)}{1 + \sum_{n \in [N]} G^n(\mathcal{Y}_n^i(z_n + c, a, b) \mid G^n)} \right\},
\]

(7)

where \(G^n(\mathcal{Y}_n^i \mid z, a^n, b^n) = G^n(Y_i, i \in V_n)\) with \(Y_i = e^{a_i - b_i(z_n + c_i)}\) for all \(i \in V_n\). For notational brevity, let

\[
\rho(z, a, b) = \frac{\sum_{n \in [N]} z_n G^n(\mathcal{Y}_n^i(z_n + c, a, b) \mid G^n)}{1 + \sum_{n \in [N]} G^n(\mathcal{Y}_n^i(z_n + c, a, b) \mid G^n)}.
\]
Let us also denote
\[ G^n(z_n) = \min_{(a^n, b^n) \in A^n} \left\{ G^n(Y^n | z_n, a^n, b^n) \right\} . \]

Since \( G^n(Y^n | z_n, a^n, b^n) \) is strictly convex in \((a^n, b^n)\) (Proposition 1), we see that \( G^n(z_n) \) is continuous and differentiable in \( z_n \). Let \((a^n_*, b^n_*(z_n)) = \arg \min_{(a^n, b^n) \in A^n} \left\{ G^n(Y^n | z_n, a^n, b^n) \right\} \), which are always uniquely determined given any \( z_n \in \mathbb{R} \) (Proposition 1). To handle the robust problem (7), let us consider the following reduced optimization problem, which is obtained by forcing each component \( G^n(\cdot) \) to its minimum value over \( A^n \).

\[
\max_{z \in \mathbb{R}^N} \left\{ W(z) = \frac{\sum_{n \in [N]} z_n G^n(z_n)}{1 + \sum_{n \in [N]} G^n(z_n)} \right\}. \tag{8}
\]

In the rest of the section, we will focus on solving the robust problem (RO) by making use of Problems (7) and (8). More specifically, we will prove the following chain of results.

(i) The reduced problem (8) always yields a unique solution, and this solution can be found by convex optimization (Theorem 2).

(ii) Any optimal solution to (8) is also a robust solution to (7) and vice-versa (Theorem 3).

(iii) A solution to (7) forms an optimal solution to robust problem (RO) (Theorem 4).

To make the technical results easier to follow, we separate the rest of the section into two sub-sections, where Section 4.1 will focus on the reduced problem, and Section 4.2 shows how to convert the original robust problem (7) into the reduced one, which eventually leads to the result that (7) and (RO) can be solved by convex optimization.

### 4.1. Convexity of the Reduced Problem

The reduced problem is indeed not convex if it is defined in terms of the prices \( x \). Nevertheless, we can show that it becomes convex if we view it under purchase probabilities. More precisely, we will do some change of variables. Let use denote a vector \( p^G \in \mathbb{R}^N \) with entries
\[
p_n^G = \frac{G^n(z_n)}{1 + \sum_{n \in [N]} G^n(z_n)}, \quad \forall n \in [N], \tag{9}
\]
then the objective function in (8) can be written as \( W(z) = \sum_{n \in [N]} z_n p_n^G \). This vector \( p^G \) can be interpreted as an aggregated purchase probabilities for the partitions, i.e., \( p_n = \sum_{i \in V_n} p_i \), where \( p_i \) is the purchase probability of item \( i \in V \). In Theorem 2 below, we show that, given any \( p^G \in \mathcal{P}^G = \{ p^G \in \mathbb{R}_+^N | \sum_{n \in [N]} p_n^G < 1 \} \), there is a unique \( z(p^G) \in \mathbb{R}^N \) satisfying (9). Moreover, Problem (8) can be formulated as a convex optimization program of variables \( p^G \). Note that a similar result has been shown previously (Zhang et al. 2018) for the case that the choice parameters \((a, b)\) are fixed. In our setting, \((a, b)\) are a solution to convex optimization problems parameterized by \( z(p^G) \), thus requiring a new and more complicated proof.
Theorem 2 (Convexity of the reduced problem). Given any $\p^G \in \mathcal{P}^G$, there is a unique vector $\z(\p^G) \in \mathbb{R}^N$ satisfying (9), and this vector can be found by bisection. Moreover, $W(\z(\p^G))$ is strictly concave in $\p^G$.

To prove the result, we first show that each function $G^n(z_n)$ is invertible. That is, for any $\alpha > 0$ there is a unique $z_n \in \mathbb{R}_+$ such that $G^n(z_n) = \alpha$. This allows us to define the inverse function $(G^n)^{-1}$ such that $(G^n)^{-1}(G^n(z_n)) = z_n$. This inverse function can be computed by bisection. The existence of the inverse function is necessary for the claim that there is always a unique vector $\z$ that yields a given purchase probability vector $\p^G \in \mathcal{P}^G$ and this vector can be computed as

$$
\z(\p^G)_n = (G^n)^{-1}\left(\frac{p^n_n}{1 - \sum_{l \in [N]} p^n_l}\right).
$$

To show the convexity of $W(\z(\p^G))$ in $\p^G$, we first validate convexity of its deterministic counterpart, i.e., the version in which all the choice parameters are given $W(\z(\p^G|a, b)) = \sum_{n \in [N]} \z(\p^G|a, b)_n p^n_G$, where $\z(\p^G|a, b) \in \mathbb{R}^N$ are a vector of constant-markups that archive vector $\p^G$ as

$$
p^n_G = \frac{G^n(Y^n|z_n, a^n, b^n)}{1 + \sum_{l \in [N]} G^l(Y^l|z_l, a^l, b^l)}, \forall n \in [N]. \tag{10}
$$

Once the convexity of $\z(\p^G|a, b)$ is validated, we can further take the derivatives of $\z(\p^G)$ with respect to $\p^G$ and show that they are equal to the derivative values of a deterministic function. This is the key result to show that the second-order derivative of $W(\z(\p^G))$ is positive-definite, leading to the convexity of $W(\z(\p^G))$. We provide the detailed proof in Appendix A.3.

We further characterize a solution to (8). In Proposition 2 below we show that (8) always has a unique local optimal $\z^*$ (i.e., $W(\z)$ is unimodal), and this solution will satisfy a fixed point system that is an extended version of the one shown in Theorem 1. Note that the uniqueness of a local optimal solution of (8) defined in terms of $\p^G$ is straightforward due to the concavity of $W(\z(\p^G))$.

It is however not trivial when the objective function is defined in terms of $\z$.

**Proposition 2.** Problem 8 always yields a unique local optimal solution $\z^*$ (i.e., $W(\z)$ is unimodal) and this solution satisfies the following fixed point system

$$
z_n = \frac{1}{\langle b^n(z_n) \rangle} + \sum_{l \in [N]} \frac{G^l(z_l)}{\langle b^l(z_l) \rangle}, \forall n \in [N]. \tag{11}
$$

Proposition 2 implies that solving the fixed-point problem (11) will yield a solution to (2). However, directly solving (11) would be not tractable. Instead, Theorem 2 show that it can be solved conveniently by convex optimization. The fixed-point system in Proposition 2 is however important to establish the saddle point result in the next section (Proposition 3).
4.2. Solving the Robust Problem

We know from the previous section that the reduced problem is tractable to solve. We now move to the second part showing that the original robust optimization problem can be converted into the reduced problem, for which a solution can be found by convex optimization. We first state the following result connecting the reduced problem and (7).

**Theorem 3 (Equivalence between (7) and the reduced problem).** Any optimal solution to (7) is also optimal to (8) and vice-versa.

The general idea to prove the theorem is to show that, under the optimal price solution, the adversary will force each component $G^n(Y_n|z_n,a^n,b^n)$ of the objective function to its minimum values. We refer the reader to Appendix A.4 for a detailed proof.

We now come back to the original robust problem (RO) with partition-wise homogeneous PSP. We will gather all the results established above to show how we can get an optimal solution of (RO) by convex optimization. Before stating the main theorem, let us introduce the following result saying that a solution obtained by solving the reduced problem (8) forms a saddle point to (7), thus the minimax equality holds.

**Proposition 3 (Saddle point of (7)).** If $z^*$ is a solution to (7), then $(z^*,a^*(z^*),b^*(z^*))$ is a saddle point of the max-min problem (7). As a result, the minimax equality holds, i.e.,

$$\max_{x \in \mathbb{R}^m} \min_{(a,b) \in A} \rho(z,a,b) = \min_{(a,b) \in A} \max_{x \in \mathbb{R}^m} \rho(z,a,b).$$

**Proof:** It is clear from Theorem 3 that if $z^*$ to a solution to (7), then $(a^*(z^*),b^*(z^*))$ is a solution to the corresponding adversary’s problem. Moreover, from Proposition 2 and Theorem C1 of Zhang et al. (2018), we also see that $z^*$ is a solution to the pricing problem under fixed parameters $(a^*(z^*),b^*(z^*))$, i.e., $\max_{x \in \mathbb{R}^N} \rho(z,a^*(z^*),b^*(z^*))$. Thus, $(z^*,a^*(z^*),b^*(z^*))$ is clearly a saddle point of (7). The minimax equality follows directly from the existence of a saddle point.

We now gather all the previous results to establish our main theorem. Theorem 4 below states that a solution to the robust problem (RO) will have a constant-markup style and this constant-markup vector can be found be convex optimization. The proof can be done easily given all the claims we have in Section 4.1 and Theorem 3 above.

**Theorem 4. (A partition-wise constant-markup solution is optimal to the robust problem).** Under partition-wise homogeneous PSP and partition-wise decomposable uncertainty sets, the robust problem (RO) yields a unique partition-wise constant-markup solution $x^*$ such that $x^*_i = z^*_n + c_i$, $\forall n \in [N], i \in V_i$, where $z^*$ is a unique solution to Problem (7), which can be solved by convex optimization. Moreover, $(x^*,a^*(z^*),b^*(z^*))$ is a saddle point of (RO) and the minimax equality holds, i.e.,

$$\max_{x \in \mathbb{R}^m} \min_{(a,b) \in A} \Phi(x,a,b) = \min_{(a,b) \in A} \max_{x \in \mathbb{R}^m} \Phi(x,a,b).$$
when the objective function is defined in terms of the purchase probability \( p \).

Proof: We first prove the minimax equality property by the chain

\[
\max_{x \in \mathbb{R}^m} \min_{(a,b) \in A} \Phi(x, a, b) \overset{(a)}{=} \min_{(a,b) \in A} \max_{x \in \mathbb{R}^m} \Phi(x, a, b) \overset{(b)}{=} \min_{(a,b) \in A} \max_{z \in \mathbb{R}^N} \rho(z, a, b) \overset{(c)}{=} \max_{z \in \mathbb{R}^N} \min_{(a,b) \in A} \rho(z, a, b) \overset{(d)}{=} \max_{x \in \mathbb{R}^m} \min_{(a,b) \in A} \Phi(x, a, b),
\]

where \((a)\) is from the well-known max-min inequality, \((b)\) is from the property that any deterministic pricing problem (with fixed \((a, b)\)) always yields a partition-wise constant-markup solution, \((c)\) is due to the minimax equality of \((7)\) shown in Proposition 3 above. This chain of \((in)\)equalities leads to the minimax equality property of \((RO)\) and the result that \( x^* \) defined by a constant-markup solution \( z^* \) of \((7)\) is a robust solution to \((RO)\) and \((x^*, a^*(z^*), b^*(z^*))\) forms a saddle point of the max-min problem \((RO)\).

We now discuss in detail how to solve the reduced problem \((8)\). Since the problem is convex when the objective function is defined in terms of the purchase probability \( p \), we show how to compute \( \mathcal{W}(z(p^G)) \) and its gradients, which are crucial for the optimization process. Given a purchase probability \( p^G \in \mathcal{P}^G \), from Lemma 4, we can compute \( \mathcal{W}(z(p^G)) \) as

\[
\mathcal{W}(z(p^G)) = \sum_{n \in [N]} z(p^G)_n p^G_n = \sum_{n \in [N]} (G^n)^{-1} \left( \frac{p^G_n}{1 - \sum_{l \in [N]} p^G_l} \right) p^G_n
\]

where the inverse function \( (G^n)^{-1} \cdot \cdot \cdot \) can be computed efficiently by \textit{bisection}. The gradients of \( \mathcal{W}(z(p^G)) \) are more difficult to get and we show how to do it in Proposition 4 below.

**Proposition 4 (Gradients of \( \mathcal{W}(z(p^G)) \)).** For any \( p^G \in \mathcal{P}^G \), we have

\[
\frac{\partial \mathcal{W}(z(p^G))}{\partial p^G_n} = z(p^G)_n - \frac{1}{(b^*(z_k))} - \frac{1}{(1 - e^p p^G)} \sum_{k \in [N]} \frac{p^G_k}{(b^*(z_k))}, \forall n \in [N],
\]

where \( (a^k(z_k), b^k(z_k)) = \arg\min_{(a^k, b^k) \in A} \{ G^k(Y^k|z_k, a^k, b^k) \} \).

The proof (details in Appendix A.6) can be done by directly taking the derivatives of \( \mathcal{W}(z(p^G)) \) with respect to \( p^G \) and using \((17)\). The computation of \( \mathcal{W}(z(p^G)) \) for a given \( p^G \in \mathcal{P}^G \) can be done by performing the following steps: (i) compute \( z(p^G) \) using Lemma 4, (ii) compute \( (a^*(z), a^*(z)) \) as (unique) optimal solutions of the problems \( \min_{(a^n, b^n) \in A^n} \{ G^n(Y^n|z_n, a^n, b^n) \} \), \( n \in [N] \), (iii) compute \( \mathcal{W}(z(p^G)) = z(p^G)^T p^G \) and its gradients by \((4)\). Since the objective function is strictly concave, we know that the optimization problem can be solved efficiently by a convex optimization solver. When the uncertainty set is rectangular, the reduced optimization problem can be further simplified, as a solution to \( \min_{(a^n, b^n) \in A^n} \{ G^n(Y^n|z_n, a^n, b^n) \} \) can be identified, thus the reduced
problem can be transformed equivalently to a deterministic pricing problem with fixed choice parameters and it is known that such a deterministic pricing problem yields closed form solutions Zhang et al. (2018). We state this result in the following corollary.

**Corollary 1 (Rectangular uncertainty sets).** If the uncertainty is rectangular, i.e.,

\[ A = \{ (a, b) | a \in [a, \overline{a}], \ b \in [\overline{b}, b], \ b_i = b_j, \ \forall i, j \in V_n, \ \forall n \}, \]

then the robust problem (RO) is equivalent to the deterministic pricing problem \( \max_{x \in \mathbb{R}^m} \Phi(x, a, b) \).

The result is easy to validate, as from Lemma (1) we see that \((a^n, \overline{b}^n) = \arg\min_{(a, \overline{b}) \in A} G^n(Y^n|z^n, a^n, b^n)\) for any \(z^n \in \mathbb{R}\).

### 5. Numerical experiments

We provide experimental results to show how the robust model considered above (i.e., robust unconstrained pricing with homogeneous and partition-wise homogeneous PSP) protect us from choice parameter uncertainties. We first discuss our approach to construct uncertainty sets and different baseline approaches for the sake of comparison.

#### 5.1. Constructing Uncertainty Sets

Inspired by Rusmevichientong and Topaloglu (2012) in the context of robust assortment optimization, such an uncertainty set can be created for the situation that the market is heterogeneous, i.e., the market has several costumer types and the choice parameters would vary across them, but the proportion of each customer type is not known with certainty. To be more precise, let assume that the true parameters for the underlying GEV choice model can be one of \(K\) vectors \(\{(a^{(1)}, b^{(1)}), \ldots, (a^{(K)}, b^{(K)})\}\), representing \(K\) types of customers. For ease of notation, let \(w^k = (a^{(k)}, b^{(k)})\) for all \(k \in [K]\). Let \(\tau_1, \ldots, \tau_K \in [0, 1]\) be the proportion of each customer type with \(\sum_{k \in [K]} \tau_k = 1\). We are interested in the situation that the proportions can be estimated somehow using historical data, but estimation may have errors and the proportion estimates may not make good representation to the “true” ones. In this situation, an uncertainty set can be constructed around the proportion estimates as

\[
A = \left\{ w = \sum_{k \in [K]} \lambda_k w^k \left| \sum_{k \in [K]} \lambda_k = 1 \text{ and } \max_{k \in [K]} |\lambda_k - \tau_k| \leq \epsilon \right. \right\}.
\]

In the partially homogeneous case, as our results require partition-wise separable uncertainty sets, such an uncertainty set can be constructed in a similar way as follows. For each customer type \(k\), let \(w^{k,n}\) be the vector of choice parameters of partition \(n \in [N]\). The uncertainty set for each partition can be defined as

\[
A_n = \left\{ w^n = \sum_{k \in [K]} \lambda_k^n w^{k,n} \left| \sum_{k \in [K]} \lambda_k^n = 1 \text{ and } \max_{k \in [K]} |\lambda_k^n - \tau_k| \leq \epsilon \right. \right\}, \forall n \in [N].
\]
Here $\epsilon \in [0, 1]$ reflects an “uncertainty level” of the uncertainty set. Larger $\epsilon$ values provide larger uncertainty sets, corresponding to more conservative models that may help protect well against worst-case scenarios, but may lead to low average performance. On the other hand, smaller $\epsilon$ values provide smaller uncertainty sets and would lead to less conservative robust solutions, which may perform well in terms of average performance but would be worse in protecting bad scenarios of the choice parameters. Adjusting $\epsilon$ would help the firm balance the worst-case protection and average performance. Clearly, $\epsilon = 0$ corresponds to the deterministic case, i.e., the proportion of each customer type are given with certainty, and $\epsilon = 1$ reflects the situation that we are totally uncertain about how likely the proportion of each customer type is, and have to ignore the predefined proportions $\{\tau_1, \ldots, \tau_K\}$.

### 5.2. Baseline Approaches

We discuss tractable baseline approaches that would be used to solve the pricing problem when facing the issue of choice parameter uncertainty. A straightforward approach would be to employ the mean values of the choice parameters and solve the deterministic version. In this context, we know that the pricing problem is computationally tractable. Alternatively, one may look at different possibilities of the choice parameters and define a mixed formulation where the market is divided into a finite number of market segments and each segment is governed by a scenario of the choice parameters. However, one can show that the expected revenue in this context is no longer unimodal and the constant-markup property identified for the GEV pricing problem no-longer holds, even if there are only two market segments (Li et al. 2019). As a result, this mixed version is not computationally tractable.

Another baseline approach is to sample some choice parameters from the uncertainty set and use simulation to select a solution that provides best protection from worst-case scenarios. More precisely, let assume that the firm needs to make a pricing decision while being aware that the choice parameters may vary in an uncertainty set. In this context, the firm can sample some points from the uncertainty set and compute the corresponding optimal prices for each selection, using the deterministic approach. Then, for each price vector, the firm can sample a sufficiently large number of vector of choice parameters from the uncertainty set, in order to evaluate how each price vector obtained performs when the choice parameters vary in the uncertainty set. This can be done by simply selecting the solution that gives the best worst-case profit among the samples. This approach may be computationally tractable with a reasonable number of samples, but would be much more computationally expensive than the robust and deterministic approaches. We refer to this as the sampling-based approach. One can show that solutions given by this approach will converge to those from the robust counterpart when the sample sizes grow to infinity.
In these experiments, we will compare our robust models (denoted as RO), which are computationally tractable, against the sampling-based approach (denoted as SA) and the deterministic one with mean-value choice parameters (denoted as DET). We will employ two popular GEV models in the literature, i.e., the MNL and nested logit models. For the SA approach, we sample points uniformly from the uncertainty set since we do not make any assumption about the distribution of the choice parameters. One can argue that the uniform distribution may not be the best choice in the case that the firm believes that it has some ideas (perhaps via estimation) about the distribution of the choice parameters. Nevertheless, estimating such a distribution is not easy in practice. A common approach in choice modeling is to assume that the parameters follow some distributions (e.g., normal distribution) with unknown coefficients and try to estimate these coefficients by maximum likelihood estimation (McFadden and Train 2000). This approach, even though popular, does not guarantee that the distribution obtained is the true distribution of the choice parameters, assuming that there exists a true distribution. As such, the distribution of the choice parameters is typically only known ambiguously. Distributionally robust optimization is a robust approach that is explicitly designed to handle this ambiguity (Shapiro 2018), which we keep for future research.

5.3. Experimental Settings
We choose $m = 50$, and $K = 5$ (i.e., there are 5 customer types) and randomly choose the proportions and the underlying choice parameter vectors $\{w^1, \ldots, w^5\}$. For each $\epsilon > 0$ we define the uncertainty set as in (13). The comparison is done as follows. For each $\epsilon$, we solve the corresponding robust problem and obtain a robust solution $x^{\text{RO}}$. For the DET, we solve the deterministic model with the weighted average parameters $\bar{w} = \sum_{k \in [K]} \tau_k w^k$ and obtain an optimal solution $x^{\text{DET}}$. For the SA approach, we sample randomly and uniformly $s_1$ points from the uncertainty set, and for each point compute the corresponding optimal prices, which have a constant markup over products. For each pricing solution, we again sample randomly and uniformly 1000 choice parameters from $\mathcal{A}$, and compute and pick a pricing solution with the largest worst-case expected revenue among the 1000 samples. We test this approach with $s_1 = 10$ and $s_1 = 50$ and denote the corresponding solutions as $x^{\text{SA10}}$, $x^{\text{SA50}}$, respectively. Larger $s_1$ can be chosen, but it would mean that the SA becomes way more expensive as compared to the RO and DET approaches. For example, if we choose $s_1 = 100$, the SA requires to solve 100 deterministic problems and compute $10^5$ expected revenues to obtain a pricing solution.

5.4. Comparison Results
We provide experimental results for the robust model under the nested logit model. The CPGF of the nested logit model is given as $G(Y) = \sum_{n \in [N]} \left( \sum_{i \in C_n} Y_i^{\mu_n} \right)^{\mu/\mu_n}$, where $[N]$ is the set of nests and for each $n \in [N]$, $C_n$ is the corresponding subset of the items, $\mu$ and $\mu_n$, $n \in \mathcal{N}$ are the positive
parameters of the nested logit model. In this experiment, we separate the whole item set into 5 nests of the same size (10 items per each nest), i.e. \( N = 5 \) and \( |C_n| = 10 \) for all \( n \in [N] \). To evaluate the performance of the three approaches when the choice parameters vary, given the uncertainty set defined above, we randomly and uniformly sample 1000 parameters \((a, b)\) from the set \( \mathcal{A} \), and compute the expected revenues given by \( x^{\text{RO}}, x^{\text{SA10}}, x^{\text{SA50}}, \) and \( x^{\text{DET}} \). So, for each solution, we get a distribution of expected revenues over 1000 samples. We then draw the histograms of of the distributions to compare. We first provide experiments for the case of homogeneous PSP and then move to the case of partition-wise homogeneous PSP.

5.4.1. Homogeneous PSP. The histograms of the distributions obtained in Figure 1 for \( \epsilon \in \{0.02, 0.04, 0.06\} \). We see that the distributions given by the RO approach always have higher peaks, lower variances and shorter tails, as compared to the other approaches. The difference becomes clearer with larger \( \epsilon \). This demonstrates the capability of the RO approach in giving not-too-low revenues. In addition, the sampling-based approach (SA10 and SA50) perform better then the DET in terms of protecting us against too low revenues. In this aspect, the SA50 also performs better than the SA10, especially when \( \epsilon \) increases.

**Figure 1** Comparison between revenue distributions given by optimal price vectors given by the robust (RO) and deterministic (DET) and sampling-based (SA10 and SA50) approaches, under the nested logit model and different uncertainty levels \( \epsilon \).

In Table 2, we provide more details about the average and worst-case values of the distributions given by the three approach. In particular, we compute the “percentile ranks” of the RO worst-case revenues, which indicates the percentages that the expected revenues given by the baseline approaches (DET, SA10 and SA50) are lower than the corresponding worst-case expected revenues given by the RO. For example, for \( \epsilon = 0.1 \), there are 23% of the revenues given by the DET (over 1000 sampled revenues) are less than the corresponding RO worst-case revenue. Over \( \epsilon \in \{0.02, \ldots, 0.4\} \), the average percentile ranks of the RO worst-case revenues are 26.7%, 9.5% and 5.3% for the DET, SA10 and SA50 approaches, respectively, which clearly indicates gains from
the use of the RO approach. It can be seen that in terms of average revenue, the DET approach performs the best, followed by the the SA10, SA50 and RO approaches. In general, the baseline approaches (DET, SA10, SA50) always give higher average revenues, but lower worst-case revenues, which clearly indicates that the RO approach does a better job in protecting us from worst-case situations, but also show the trade-off of being robust. Moreover, the results in Table 2 also tell us that if the firm cares more about the worst cases, a large $\epsilon$ can be chosen to have better protection against too low expected revenues. On the other hand, if average performance is of concern, then by choosing a small $\epsilon$, one can still get a protection from the robust solutions, but also get an average performance that is comparable to that of the solutions by the deterministic approach. This observation is also consistent with those from other robust work in the revenue management literature (Li and Ke 2019, Rusmevichientong and Topaloglu 2012).

| $\epsilon$ | Average |  |  | Percentile rank |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | DET | SA10 | SA50 | RO | DET | SA10 | SA50 | RO |
| 0.02 | 9.4 | 9.4 | 9.3 | 9.2 | 8.7 | 8.7 | 8.7 | 8.7 | 4 | 8 | 4 |
| 0.04 | 9.3 | 9.1 | 9.0 | 8.8 | 7.5 | 8.1 | 8.1 | 8.2 | 13 | 8 | 4 |
| 0.06 | 9.2 | 9.0 | 8.7 | 8.3 | 6.0 | 7.1 | 7.4 | 7.5 | 17 | 8 | 5 |
| 0.08 | 9.1 | 8.1 | 8.8 | 7.8 | 4.2 | 7.1 | 6.9 | 7.0 | 18 | 7 | 7 |
| 0.10 | 8.9 | 8.7 | 7.8 | 7.3 | 2.6 | 4.7 | 5.6 | 6.4 | 23 | 12 | 6 |
| 0.12 | 8.8 | 7.7 | 7.6 | 6.7 | 1.4 | 5.8 | 5.9 | 5.9 | 26 | 7 | 6 |
| 0.14 | 8.6 | 8.0 | 7.2 | 6.2 | 1.1 | 4.6 | 5.0 | 5.3 | 26 | 9 | 4 |
| 0.16 | 8.3 | 6.6 | 7.2 | 5.8 | 0.9 | 4.9 | 4.9 | 5.1 | 34 | 7 | 4 |
| 0.18 | 8.4 | 7.3 | 6.6 | 5.3 | 0.5 | 3.7 | 4.3 | 4.7 | 30 | 9 | 7 |
| 0.20 | 8.3 | 7.4 | 6.7 | 4.9 | 0.2 | 2.0 | 2.8 | 4.1 | 31 | 10 | 6 |
| 0.22 | 8.2 | 7.1 | 6.4 | 4.5 | 0.1 | 1.4 | 2.0 | 3.9 | 33 | 11 | 6 |
| 0.24 | 8.0 | 6.6 | 5.8 | 4.2 | 0.1 | 2.7 | 2.7 | 3.4 | 32 | 8 | 7 |
| 0.26 | 8.1 | 7.9 | 6.0 | 3.9 | 0.0 | 0.1 | 0.8 | 3.1 | 31 | 28 | 5 |
| 0.28 | 8.0 | 5.4 | 5.5 | 3.7 | 0.0 | 2.3 | 2.1 | 2.9 | 32 | 8 | 6 |
| 0.30 | 7.7 | 5.1 | 5.1 | 3.5 | 0.0 | 2.1 | 2.2 | 2.3 | 37 | 7 | 5 |
| 0.32 | 7.9 | 4.4 | 3.8 | 3.3 | 0.0 | 1.9 | 2.0 | 2.2 | 31 | 7 | 4 |
| 0.34 | 8.1 | 5.4 | 3.9 | 3.1 | 0.0 | 1.2 | 1.5 | 1.8 | 27 | 9 | 6 |
| 0.36 | 7.8 | 4.8 | 4.2 | 3.0 | 0.0 | 1.0 | 1.0 | 1.3 | 29 | 7 | 5 |
| 0.38 | 7.9 | 5.3 | 4.2 | 2.8 | 0.0 | 0.7 | 0.9 | 1.3 | 26 | 9 | 5 |
| 0.40 | 7.7 | 5.6 | 5.2 | 2.7 | 0.0 | 0.3 | 0.6 | 1.1 | 33 | 11 | 4 |

Table 2 Comparison results for unconstrained robust (RO), deterministic (DET), and sampling-based (SA10 and SA50) pricing under the nested logit model with homogeneous PSP.

5.4.2. Partition-wise Homogeneous PSP. We provide comparison results for the case of partition-wise homogeneous PSP considered in Section 4. We use the same nested logit model with partition-wise decomposable CPGF specified above, i.e., $G(Y) = \sum_{n \in [N]} \left( \sum_{i \in C_n} Y_i^{\mu_n} \right)^{1/\mu_n}$, but the PSP are the same in each nest but different across nests. In this context, we know that the robust
problem can be converted equivalently into a convex optimization problem. On the other hand, for the SA approach, if we select \( s_1 \) vectors of choice parameters from the uncertainty set, we need to solve \( s_1 \) convex optimization problems.

We select \( N = 5 \) partitions of the same size. For each uncertainty level \( \epsilon > 0 \) and for each partition (or nest) \( n \in [N] \), we define a polyhedron uncertainty set as in Section 5.1 above. Similarly to the previous section, we first solve the deterministic problem by bisection with the weighted average parameters \( \bar{w} = \sum_{k \in [K]} \tau_k w^k \) to obtain a solution \( x^{DET} \). Then, for each set \( \mathcal{A}' \) we solve the RO problem by convex optimization to obtain a robust solution \( x^{RO} \). We also sample \( s_1 = 10 \) and \( s_1 = 50 \) points from the uncertainty set for the SA approach.

To evaluate the performance of the solutions obtained, we also sample 1000 points randomly and uniformly from \( \mathcal{A}'^n, n \in [N] \), and compute the expected revenues given by \( x^{RO}, x^{SA10}, x^{SA50}, \) and \( x^{DET} \). The distributions of the expected revenue over 1000 samples with \( \epsilon \in \{0.02, 0.04, 0.06\} \) are plotted in Figure 2. There is nothing surprising, as similarly to the previous experiments, distributions given by \( x^{RO} \) have small variances, higher peaks, shorter tails and higher worst-case revenues, as compared to those from \( x^{SA10}, x^{SA50} \) and \( x^{DET} \). In Table 3, we report in detail the average, maximum and worst-case revenues when \( \epsilon \) increases from 0.02 to 0.4. We also see that the RO approach always gives higher worst-case revenues but lower average revenues, and the SA approaches also provide some protections against low revenues. However, in this case, the percentile ranks for the DET and SA approaches are significantly lower (3.45 on average). In particular, we see that there are some instances where the percentile ranks are only 3-th, which means that only 3\% of the revenues are lower than the corresponding RO worst-case revenues. Nevertheless, the average revenues given by the SA50 are remarkably higher than those from the RO, especially when \( \epsilon \) is large. From this view point, the RO seems too conservative.

![Figure 2](image.png)

**Figure 2** Distributions of the expected revenues under partition-wise homogeneous PSP.

In summary, our experiments show gains from our robust models in protecting us from revenues that would be too low. The histograms given by the robust models have higher peaks, smaller
variances, higher worst-case revenues, but lower averages, as compared to their deterministic and sampling-based counterparts. This observation also shows the trade-off in being robust in making pricing decisions when the choice parameters are uncertain, and also consistent with observations from other relevant studies in the revenue management literature (Rusmevichientong and Topaloglu 2012, Li and Ke 2019).

6. Conclusion

In this paper, we have considered robust versions of the pricing problem under GEV choice models, in which the choice parameters are not given in advance but lie in an uncertainty set. These robust models are motivated by the fact that uncertainties may occur in the estimation procedure of the choice parameters. We have shown that when the problem is unconstrained and the PSP are the same over all the products, the robust optimal prices have a constant markup with respect to the product costs and we have shown how to efficiently compute this constant markup by bisection. When the PSP are partition-wise homogeneous and the CPGF and the uncertainty set are also partition-wise separable, we have shown that the robust problem can be converted equivalently into a reduced optimization program, and the reduce problem can be solved conveniently by convex optimization.
We have also considered the pricing problem with over-expected-revenue-penalties as an alternative to the constrained pricing problem. We have shown that under the same assumptions as in the case of partition-wise homogeneous PSP, the robust problem can be converted equivalently into a reduced one, which can be further solved by convex optimization. Experimental results based on the nested logit model have shown the advantages of our robust model in providing protection against bad-case revenues. In future research, it would be interesting to look at distributionally robust versions of the pricing problem, which may help provide less conservative robust solutions as compared to the standard robust optimization approaches. We are also interested in robust approaches for the joint assortment and pricing problem under GEV choice models.

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Appendix A: Proofs

This section provides some detailed proofs of the claims presented in the main part of the paper.

A.1. Proof of Proposition 1

First, we consider function $f^G(s): \mathbb{R}^m \rightarrow \mathbb{R}_+$

$$f^G(s) = G(Y_1, \ldots, Y_m), \text{ where } Y_i = e^{s_i}, \forall i \in \mathcal{V}$$

We will prove that $f^G(s)$ is convex. Taking the first and second derivatives of $f^G(s)$ we obtain

$$\frac{\partial f^G(s)}{\partial s_i} = \partial G_i(Y)Y_i,$$

and

$$\frac{\partial^2 f^G(s)}{\partial s_i \partial s_j} = \partial G_{ij}(Y)Y_i Y_j. $$

So we have

$$\nabla^2 f^G(s) = \text{diag}(Y)\nabla^2 G(Y)\text{diag}(Y) + \text{diag}(\nabla G(Y) \circ Y),$$

where $\text{diag}(Y)$ is the square diagonal matrix with the elements of vector $Y$ on the main diagonal. The second term $\text{diag}(\nabla G(Y) \circ Y)$ is always positive definite, where $\circ$ is the element-by-element operator. Moreover, $\text{diag}(Y)\nabla^2 G(Y)\text{diag}(Y)$ is symmetric and its $(i,j)$-th component is given by $Y_i \partial G_{ij}(Y)Y_j$. For $i \neq j$, we have $\partial G_{ij}(Y) \leq 0$ by the property of the GEV-CPGF $G$, so all off-diagonal entries of the matrix are non-positive. In addition, $\sum_{j \in \mathcal{V}} Y_j \partial G_{ij}(Y) = 0$, so that each row of the matrix sums to zero. Thus, $\text{diag}(Y)\nabla^2 G(Y)\text{diag}(Y)$ is positive semi-definite (see Theorem A.6 in De Klerk 2006). So, $\nabla^2 f^G(s)$ is positive definite, or equivalently, $f^G(s)$ is strictly convex in $s$. This lead to the following inequality, for all $s^1, s^2 \in \mathbb{R}^m$ and $\lambda \in (0, 1)$

$$\lambda f^G(s^1) + (1 - \lambda) f^G(s^2) > f^G(\lambda s^1 + (1 - \lambda)s^2).$$

For all $(a^1, b^1), (a^2, b^2) \in \mathcal{A}$, replace $s^1_i$ by $a^1_i - b^1_i(z + c_i)$ and $s^2_i$ by $a^2_i - b^2_i(z + c_i)$ we have

$$\lambda G(Y|z, a^1, b^1) + \lambda G(Y|z, a^2, b^2) > G(Y|z, \lambda a^1 + (1 - \lambda) a^2, \lambda b^1 + (1 - \lambda) b^2), \forall \lambda \in (0, 1)$$

which means that $G(Y|z, a, b)$ is strictly convex in $a, b$.

The continuity of $(a^*(z), b^*(z))$ is a direct result from the convexity of $G(Y|z, a, b)$ and the Corollary 8.2 of Hogan (1973). This completes the proof.

A.2. Proof of Lemma 1

Consider $f^G(s) = G(Y_1, \ldots, Y_m)$, where $Y_i = e^{s_i}$, $\forall i = 1, \ldots, m$. Taking the derivative of $f^G(s)$ w.r.t. $s_i$ we have

$$\frac{\partial f^G(s)}{\partial s_i} = \partial G_i(Y)Y_i \geq 0$$

So, $f^G(s)$ is monotonic in every coordinate, meaning that given any $s, s_0 \in \mathbb{R}^m$, $s \geq s_0$, we have $f^G(s) \geq f^G(s_0)$. Moreover, it is clear that

$$a - b \circ (c + ze) \preceq a - b \circ (c + ze) \preceq a - b \circ (c + ze), \forall z \in \mathbb{R}_+, a, b \in \mathcal{A}.$$ 

So, we obtain the following inequality

$$G(Y|z, a, b) \preceq G(Y|z, \tilde{a}, b) \leq G(Y|z, \tilde{a}, b), \forall (a, b) \in \mathcal{A},$$

which completes the proof.
A.3. Proof of Theorem 2

The first lemma shows that $G^a$ is invertible.

**Lemma 3.** Given $n \in [N]$, for any $\alpha > 0$, there is a unique $z_n \in \mathbb{R}$ such that $G^a(z_n) = \alpha$.

**Proof:** From the properties of CPGF (Remark 1), we see that function $G^a(Y^n|z_n,a^n,b^n)$ is strictly monotonic-decreasing, so $G^a(z_n)$ is also strictly monotonic-decreasing. Moreover, we have $\lim_{z_n \to +\infty} G^a(Y^n|z_n,a^n,b^n) = 0$ and $\lim_{z_n \to -\infty} G^a(Y^n|z_n,a^n,b^n) = \infty$. Thus, $G^a(Y^n|z_n,a^n,b^n)$ spans all over the set $\mathbb{R}_+$ when $z_n$ varies. On the other hand, $\mathbb{A}^n$ is bounded, we will also have $\lim_{z_n \to +\infty} G^a(z_n) = 0$ and $\lim_{z_n \to -\infty} G^a(z_n) = \infty$. Since $G^a(z_n) = 0$ is continuous and strictly monotonic-decreasing, we easily obtain the desired result. \qed

The above lemma allows us to define the inverse function of $G^a(\cdot)$ as $(G^a)^{-1}(\alpha): \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $G^a((G^a)^{-1}(\alpha)) = \alpha$. As shown above, this function can be computed by bisection. Lemma 4 below shows how to identify $z(p^G)$.

**Lemma 4.** Given any $p^G \in P^G$, $z(p^G)$ can be uniquely computed as

$$ z(p^G)_n = (G^a)^{-1}\left(\frac{p^G_n}{1 - \sum_{i \in [N]} p^G_i}\right) $$

**Proof:** From (9) we see that

$$ \sum_{i \in [N]} p^G_i = \frac{\sum_{i \in [N]} G^a_i(z_i)}{1 + \sum_{i \in [N]} G^a_i(z_i)}, $$

so we have

$$ 1 + \sum_{i \in [N]} G^a_i(z_i) = \frac{1}{1 - \sum_{i \in [N]} p^G_i}. $$

Thus, for any $n \in [N]$

$$ G^a_i(z_n) = \frac{p^G_n}{1 + \sum_{i \in [N]} G^a_i(z_i)} = \frac{p^G_n}{1 - \sum_{i \in [N]} p^G_i}, $$

which directly leads to the desired result. \qed

We now move to the second claim of Theorem 2, i.e., the convexity of $\mathcal{W}(z(p^G))$. To support the proof, let use consider a deterministic version of (8) in which all the choice parameter are given $\hat{W}(\hat{z}(p^G|a,b)) = \sum_{n \in [N]} \hat{z}(p^G|a,b)_n p^G_n$, where $\hat{z}(p^G|a,b) \in \mathbb{R}^N$ are a vector of constant-markups that archive vector $p^G$ as

$$ p^G_n = \frac{G^a(Y^n|\hat{z}_n,a^n,b^n)}{1 + \sum_{j \in [N]} G^a(Y^j|\hat{z}_j,a^j,b^j)}, \forall n \in [N]. $$

(15)

**Lemma 5.** Given any $p^G \in P^G$, $\hat{z}(p^G|a,b)$ can be uniquely determined by solving a strictly convex optimization problem. Moreover, $\hat{W}(\hat{z}(p^G|a,b))$ is strictly concave in $p^G$.

**Proof:** Let $\Theta(z): \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $\Theta(z)_n = G^a(Y^n|z_n) / \left(1 + \sum_{j \in [N]} G^a(Y^j|z_j)\right)$, where $G^a(Y^n|z_n) = G^a(Y^n|z_n,a,b)$ but we omit the choice parameters $(a,b)$ for notational simplicity. We also denote by $\tilde{b}$ a vector of size $N$ with entries $\tilde{b}_n = (b^n)$. Consider the problem

$$ \min_{z \in \mathbb{R}^N} \left\{ \ln \left(1 + \sum_{n \in [N]} G^a(Y^n|z_n)\right) + \sum_{n \in [N]} p^G_n \tilde{b}_n z_n \right\} $$

(16)
and we now show that (16) is a strictly convex optimization problem and solving it will yield a solution \( z^* \) such that \( \Theta(z^*) = p^G \). Note that the structure of the problem presented in this lemma is slightly different with those considered in Theorem 4.1 in Zhang et al. (2018), so even though the proof of the lemma is quite similar, we provide its own proof for the sake of self-contained. To prove that (16) is a strictly convex optimization problem, we will show that \( \nabla^2 Q(z) \) is a positive definite matrix, where \( Q(z) = \ln \left( 1 + \sum_{n \in [N]} G^n(Y^n|z_n) \right) \).

To simplify the proof and make use of previous results, let us denote \( u(z) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( u(z)_n = -\ln G^n(Y^n|z_n)/\tilde{b}_n \), with this definition we have

\[
\frac{\partial u(z)_n}{\partial z_n} = \frac{\sum_{i \in V_n} \partial G^i_n(Y^i|z_n)Y_ib_n}{G^n(Y^n|z_n)\tilde{b}_n} = 1.
\]

The objective function now can be written as

\[
Q(u(z)) = \ln \left( 1 + \sum_{n \in [N]} \exp(-u(z)_n) \right).
\]

Taking the derivative of \( Q \) with respect to \( z_n \) we obtain

\[
\frac{\partial Q(u(z))}{\partial z_n} = \frac{\partial Q(u)}{\partial u_n} \bigg|_{u=u(z)} \frac{\partial u(z)_n}{\partial z_n} = \frac{\partial Q(u)}{\partial u_n} \bigg|_{u=u(z)}.
\]

And if we take the second derivative with respective to \( z_n, z_k, n, k \in [N] \) we get

\[
\frac{\partial^2 Q(u(z))}{\partial z_n \partial z_k} = \frac{\partial^2 Q(u)}{\partial u_n u_k} \bigg|_{u=u(z)}
\]

or equivalently \( \nabla^2 Q(z) = \nabla^2_{u} Q(u) \), where \( u = u(z) \). Moreover, \( Q(u) \) is just a special objective function under the MNL model with \( N \) products and all the PSP are equal to 1. As a result, \( \nabla^2_{u} Q(u) \) is positive definite (see Theorem 4.1 Zhang et al. 2018), so \( \nabla^2 Q(z) \) is also positive definite, as desired.

Now we know that (16) is strictly convex, so it yields a unique solution. Moreover, one can show that (16) have finite optimal solutions. For any \( n \in [N] \), taking the derivative of \( Q(z) \) with respect to \( z_n \) and set it to zero we obtain

\[
\frac{\sum_{n \in [N]} \sum_{i \in V_n} -\partial G^i_n(Y^i|z_n)Y_i\tilde{b}_n}{1 + G(Y)} = p_n^G \tilde{b}_n,
\]

or equivalently, \( p^G = \Theta(z) \). So, if \( z(p^G) \) is the unique solution to (16), we always have \( p^G = \Theta(z(p^G)) \) as desired.

Next, we will show that \( \widehat{W}(\tilde{z}(p^G(a, b))) \) is a strictly concave function of \( p^G \). We also omit the choice parameters for notational convenience and denote \( \widehat{W}(\tilde{z}) = \sum_{n \in [N]} \tilde{z}_n p^G_n \). We first see that \( p^G \) is also a choice probability vector given by a MNL model with \( N \) products with the utility vector \( -u(\tilde{z}(p^G)) \cdot b \). So, if we denote \( u'(p^G) \) be a mapping from \( \mathbb{R}^N \) to \( \mathbb{R}^N \) such that \( p^G_n = \exp(-\tilde{b}_n u'(p^G)_n) \left/ \left( \sum_{n \in [N]} \exp(-\tilde{b}_n u'(p^G)_n) \right) \right. \), then we have \( u(\tilde{z}(p^G)) = u'(p^G) \)

\[
\frac{\partial \widehat{W}(\tilde{z}(p^G))}{\partial p^G_n} = \tilde{z}(p^G)_n + \sum_{j \in [n]} p^G_j \frac{\partial \tilde{z}(p^G)_j}{\partial p^G_n} = \tilde{z}(p^G)_n + \sum_{j \in [n]} p^G_j \frac{\partial \tilde{z}(p^G)_j}{\partial u(p^G)_j} \frac{\partial u(p^G)_j}{\partial p^G_n} = \tilde{z}(p^G)_n + \sum_{j \in [n]} p^G_j \frac{\partial u'(p^G)_j}{\partial p^G_n}
\]
and
\[ \frac{\partial \tilde{W}(\bar{z}(\bar{p}^G))}{\partial p^G_n \partial p^G_k} = \frac{\partial \tilde{z}(\bar{p}^G_n)}{\partial p^G_n} + \sum_{j \in [N]} p^G_j \frac{\partial^2 u(\bar{p}^G_j)}{\partial p^G_n \partial p^G_k} \]
\[ = \frac{\partial \tilde{z}(\bar{p}^G_n)}{\partial u(\bar{z}(\bar{p}^G_n))} \frac{\partial u(\bar{z}(\bar{p}^G_n))}{\partial p^G_n} + \sum_{j \in [N]} p^G_j \frac{\partial^2 u(\bar{p}^G_j)}{\partial p^G_n \partial p^G_k} \]
\[ = \frac{\partial u(\bar{p}^G_n)}{\partial p^G_n} + \sum_{j \in [N]} p^G_j \frac{\partial^2 u(\bar{p}^G_j)}{\partial p^G_n \partial p^G_k} \]

Moreover, if we denote \( \tilde{W}(\bar{p}^G) = \sum_{n \in [N]} u(\bar{p}^G_n) p^G_n \), we also have
\[ \frac{\partial^2 \tilde{W}(\bar{p}^G)}{\partial p^G_n \partial p^G_k} = \frac{\partial u'(\bar{p}^G_n)}{\partial p^G_n} + \sum_{j \in [N]} p^G_j \frac{\partial^2 u'(\bar{p}^G_j)}{\partial p^G_n \partial p^G_k}, \forall n, k \in [N]. \]

So, \( \nabla^2 \tilde{W}(\bar{z}(\bar{p}^G)) = \nabla^2 \tilde{W}'(\bar{p}^G) \). We also see that \( \tilde{W}'(\bar{p}^G) \) is the expected revenue function (as a function of the purchase probabilities \( \bar{p}^G \)) where there are \( N \) products, the choice model is MNL, the PSP are \( \bar{b} \) and the utility vector is \( -u'(\bar{p}^G) \circ \bar{b} \) (\( \circ \) is the dot product). So we know that \( \nabla^2 \tilde{W}'(\bar{p}^G) \) is negative definite (Zhang et al. 2018), so \( \tilde{W}(\bar{z}(\bar{p}^G)) \) is strictly concave in \( \bar{p}^G \). This completes the proof. \( \Box \)

We now make a connection between \( z(\bar{p}^G) \) defined in (9) and (15). This is crucial to show the concavity of \( \tilde{W}(z(\bar{p}^G)) \). We have the following lemma.

**Lemma 6.** Given any \( p^G \in P^G \), we have the following equalities

(i) \( z(\bar{p}^G) = \bar{z}(\bar{p}^G) | a^*(z), b^*(z) \)

(ii) The first and second-order derivatives of \( z(\bar{p}^G)_n \), \( n \in [N] \)
\[ \frac{\partial z(\bar{p}^G)_n}{\partial p^G_l} = \frac{\partial \bar{z}(\bar{p}^G)_n}{\partial p^G_l} \bigg|_{a=a^*(z), b=b^*(z)} \quad \forall n \in [N] \]
\[ \frac{\partial^2 z(\bar{p}^G)_n}{\partial p^G_l \partial p^G_k} = \frac{\partial^2 \bar{z}(\bar{p}^G)_n}{\partial p^G_l \partial p^G_k} \bigg|_{a=a^*(z), b=b^*(z)} \quad \forall l, k \in [N] \]

where \( (a^*(z), b^*(z)) = \{(a^{n*}(z), b^{n*}(z)) | n \in [N]\} \) and \( (a^{n*}(z_n), b^{n*}(z_n)) = \arg\min_{(a^n, b^n) \in A^n G^n(Y^n|z_n, a^n, b^n)}. \)

**Proof:** First, we see that since the uncertainty set \( A^n \) does not depend on \( z_n \), the derivatives of \( G^n(z_n) \) can be computed as
\[ \frac{\partial G^n(z_n)}{\partial z_n} = \frac{\partial G^n(Y^n|z_n, a^n, b^n)}{\partial z_n} \bigg|_{a^n=a^{n*}(z_n), b^n=b^{n*}(z_n)} \]
\[ \frac{\partial^2 G^n(z_n)}{\partial^2 z_n} = \frac{\partial^2 G^n(Y^n|z_n, a^n, b^n)}{\partial^2 z_n} \bigg|_{a^n=a^{n*}(z_n), b^n=b^{n*}(z_n)}. \]

For (i), we know that \( z(\bar{p}^G) \) is a unique solution to the following system
\[ \bar{G}^n(z_n) = \frac{p^G_n}{1 - \sum_{i \in [N]} p^G_i}, \forall n \in [N], \]
and \( \bar{z}(\bar{p}^G|a^*(z), b^*(z)) \) is a unique solution to
\[ G^n(Y^n|z_n, a^*(z), b^*(z)) = \frac{p^G_n}{1 - \sum_{i \in [N]} p^G_i}, \forall n \in [N], \]
and note that \( G^n(Y^n|z_n, a^*(z), b^*(z)) = \tilde{G}^n(z_n) \). This leads to the desired equality.

For (ii), we take the derivative of (18) with respect to \( p^G_j \), \( j \in [N] \), and obtain

\[
\frac{\partial h^n(p^G)}{\partial p^G_j} = \frac{\partial G^n(z(p^G))}{\partial z(p^G)} \frac{\partial z(p^G)}{\partial p^G_j} = G^n(Y^n|z(p^G)_n, a, b) \left|_{a=a^*(z)} \right. \frac{\partial z(p^G)}{\partial p^G_j} \tag{20}
\]

where \( h^n(p^G) = (p^G_j) / (1 - \sum_{i \in [N]} p^G_i) \). We also take the first derivatives of (19) and obtain

\[
\frac{\partial h^n(p^G)}{\partial p^G_i} = G^n(Y^n|\tilde{z}(p^G)_n, a^*(z), b^*(z)) \frac{\partial \tilde{z}(p^G)}{\partial p^G_i}. \tag{21}
\]

Now we just combine (20) and (21) and the result that \( z(p^G) = \tilde{z}(p^G) \) to have the first equation of (ii). The second equation of (ii) can be verified similarly, as we just need to take the second-order derivatives of (18) and (19) and use the results from (i) and (21) to obtain the desired equality.

We are now to provide a complete proof for Theorem 2.

**Proof of Theorem 2:** The first claim is already validated in Lemma 4. To prove that \( W(z(p^G)) \) is strictly concave, we will show that its second derivative \( W(z(p^G)) \) is negative definite. This can be easily seem as

\[
\frac{\partial W(z(p^G))}{\partial p^G_i} = z(p^G)_n + \sum_{i \in [N]} p^G_i \frac{\partial z(p^G)_i}{\partial p^G_i}
\]

\[
\frac{\partial^2 W(z(p^G))}{\partial p^G_i \partial p^G_k} = \frac{\partial z(p^G)_i}{\partial p^G_k} + \frac{\partial z(p^G)_k}{\partial p^G_i} + \sum_{i \in [N]} p^G_i \frac{\partial^2 z(p^G)_i}{\partial p^G_i \partial p^G_k}
\]

Then using Lemma 6 we have

\[
\frac{\partial^2 W(z(p^G))}{\partial p^G_i \partial p^G_k} = \left. \left( \frac{\partial \tilde{z}(p^G|a, b)}{\partial p^G_k} + \frac{\partial \tilde{z}(p^G|a, b)}{\partial p^G_i} \right) + \sum_{i \in [N]} p^G_i \frac{\partial^2 \tilde{z}(p^G|a, b)}{\partial p^G_i \partial p^G_k} \right|_{a=a^*(z)} \frac{\partial \tilde{z}(p^G|a, b)}{\partial p^G_k} \tag{22}
\]

It not difficult to see that the left hand side of (22) is equal to \( \partial^2 \tilde{W}(\tilde{z}(p^G|a^*(z), b^*(z)))/(\partial p^G_i \partial p^G_k) \), leading to

\[
\nabla^2 W(z(p^G)) = \nabla^2 \tilde{W}(\tilde{z}(p^G|a^*(z), b^*(z))).
\]

Since \( \nabla^2 \tilde{W}(\tilde{z}(p^G|a^*(z), b^*(z))) \) is always negative definite (Lemma 5), so is \( \nabla^2 W(z(p^G)) \). This completes the proof.

**A.4. Proof of Theorem 3**

We will make use of Lemmas 7-9 below to prove the claim. Lemma 7 shows that under any prices \( z \in \mathbb{R}^N \), the adversary will force \( G^n(Y^n|z, a^*, b^*) \) to either its maximum or minimum value, for any \( a \in [N] \), with a note that both cases can occur. We further, in Lemmas 8 and 9, show that, under an optimal price solution, the adversary will always force \( G^n(Y^n|z_n, a^*, b^*) \) to their minimums. This is an essential claim to show the equivalence between the robust problem and the reduced one.

First, let

\[
\bar{G}^n(z_n) = \max_{(a^*, b^*) \in A^n} \left\{ G^n(Y^n|z_n, a^*, b^*) \right\}.
\]
Lemma 7. Give a markup vector \( z \in \mathbb{R}^N \), let \( (a^*, b^*) = \{(a^{n*}, b^{n*}), \ n \in [N] \} \) be a solution to the adversary’s problem of (7), then for any \( n \in [N] \), we have
\[
G^n(Y^n|z, a^{n*}, b^{n*}) = \begin{cases} 
G^n(z_n) & \text{if } \rho(z, a^*, b^*) < z_n \\
\overline{G}^n(z_n) & \text{if } \rho(z, a^*, b^*) > z_n 
\end{cases} \forall n \in [N].
\]

Moreover, if there are a set of indexes \( \mathcal{N} \subset [N] \) such that \( \rho(z, a^*, b^*) = z_n, \forall n \in \mathcal{N} \), then all the solutions in the following set are optimal to the adversary’s problem
\[
S^* = \{(a, b) \in A | G^n(Y^n|z, a^*, b^*) = G^n(Y^n|z, a^{n*}, b^{n*}), \forall l \in [N], n \notin \mathcal{N} \}.
\]

Proof: Given \( n \in [N] \), let us denote
\[
A = \sum_{l \in [N], l \neq n} z_l G^l(Y^l|z_l, a^l, b^l) \quad B = 1 + \sum_{l \in [N], l \neq n} G^l(Y^l|z_l, a^l, b^l).
\]

We can write
\[
\rho(z, a^*, b^*) = \frac{A + z_n G^n(Y^n|z_n, a^{n*}, b^{n*})}{B + G^n(Y^n|z_n, a^{n*}, b^{n*})}
\]

We prove the lemma by considering the following three cases:

(i) If \( \rho(z, a^*, b^*) < z_n \), then for any \( \gamma < G^n(Y^n|z_n, a^{n*}, b^{n*}) \) one can easily show the following inequality
\[
\rho(z, a^*, b^*) = \frac{A + z_n G^n(Y^n|z_n, a^{n*}, b^{n*})}{B + G^n(Y^n|z_n, a^{n*}, b^{n*})} > \frac{A + z_n \gamma}{B + \gamma}
\]

Since \( (a^*, b^*) \) is a solution to the adversary problem (7), \( G^n(Y^n|z_n, a^{n*}, b^{n*}) \) must be equal to its minimum, i.e., \( G^n(Y^n|z_n, a^{n*}, b^{n*}) = \overline{G}^n(z_n) \).

(ii) If \( \rho(z, a^*, b^*) > z_n \), then similarly to the previous case, we can show that, for any \( \gamma > G^n(Y^n|z_n, a^{n*}, b^{n*}) \) one can easily show the following inequality
\[
\rho(z, a^*, b^*) > \frac{A + z_n \gamma}{B + \gamma}.
\]

Thus, \( G^n(Y^n|z_n, a^{n*}, b^{n*}) \) must be equal to its maximum, i.e., \( G^n(Y^n|z_n, a^{n*}, b^{n*}) = \overline{G}^n(z_n) \).

(iii) If \( \rho(z, a^*, b^*) = z_n \), then for any \( \gamma \in \mathbb{R} \) we have
\[
\rho(z, a^*, b^*) = \frac{A + z_n \gamma}{B + \gamma},
\]

meaning that any solution \( (a^*, b^*) \in A^n \) would be chosen minimize the adversary’s objective function.

Combining the above three cases, we obtain the desired result. \( \square \)

Here we remark that the behavior of the adversary showed in Lemma 7 depends on the values of \( z \) and both cases can occur (Remark 2).

Remark 2. Given a constant-markup vector \( z \in \mathbb{R}^N \), let \( n_1 = \arg \max_n \{z_n\} \), the adversary would need to force \( G^{n_1}(Y^{n_1}|z_{n_1}, a^{n_1}, b^{n_1}) \) to its minimum over \( A^{n_1} \). Moreover, if there is \( n_2 \in [N] \) such that the markup \( z_{n_2} = 0 \) and \( z_n > 0 \) for all \( n \neq n_2 \), then the adversary would need to force \( G^{n_2}(Y^{n_2}|z_{n_2}, a^{n_2}, b^{n_2}) \) to its maximum over \( A^{n_2} \). So, in general, the adversary would force \( G^n(Y^n|z_n, a^n, b^n) \) to either its minimum or its maximum value, and both cases can occur.
The remark is easy to verify, as we see that if \( n_1 = \arg \max_{n \in [N]} \{ z_n \} \), then
\[
\rho(z, a^*, b^*) \leq z_{n_1} \sum_{n \in [N]} \frac{G^n(Y^n|z_n, a^n, b^n)}{1 + \sum_{n \in [N]} G^n(Y^n|z_n, a^n, b^n)} < z_{n_1}
\]
Thus, according to Lemma 7, the adversary would need to force \( G^{n_1}(Y^{n_1}|z_{n_1}, a^{n_1}, b^{n_1}) \) to its minimum over \( A^{n_1} \). On other hand, if \( z_{n_2} = 0 \) and \( z_n > 0 \) for all \( n \neq n_2 \), then \( z_{n_2} < \rho(z, a^*, b^*) \), meaning that adversary would need to force \( G^{n_2}(Y^{n_2}|z_{n_2}, a^{n_2}, b^{n_2}) \) to its maximum over \( A^{n_2} \).

We now further characterize the adversary problem under an optimal prices \( z^* \) by showing that \( \rho(z^*, a^*, b^*) \leq z_n^* \) for all \( n \in [N] \), meaning that the adversary always force \( G^n(Y^n|z_n^*, a^n, b^n) \) to its minimum. Before doing this, let us define a function \( \psi(z|u) : \mathbb{R}^N \rightarrow \mathbb{R} \) parameterized by a binary vector \( u \in \{0,1\}^N \), in such a way that \( \psi(z|u) \) has the following form
\[
\psi(z|u) = \frac{\sum_{a \in [N]} z_a \theta^n(z_a)}{1 + \sum_{a \in [N]} \theta^n(z_a)},
\]
where \( \theta^n(z_a) = g^n(z_a) \) if \( u_a = 0 \) or \( \theta^n(z_a) = \overline{g}^n(z_a) \) if \( u_a = 1 \). A binary vector \( u \) can be referred to as a configuration of the adversary’s objective function and Lemma 7 tells us that, for any \( z \in \mathbb{R}^N \), there is \( u \in \{0,1\}^N \) such that the adversary’s objective function can be written as \( \min_{(a,b) \in A} \rho(z, a, b) = \psi(z|u) \). We also denote \( e^k \) as a vector of size \( N \) with zero elements except the \( k \)-element that is equal to 1, for any \( k \in [N] \).

We need Lemma 8 below to support the main claim.

**Lemma 8.** Given \( z \in \mathbb{R}^N \) and \( u \in \{0,1\}^N \), if there is \( k \in [N] \) such that \( \psi(z|u) \geq z_k \), then given any \( \epsilon > 0 \) we have \( \psi(z|u) < \psi(z + \epsilon e^k|u) \).

**Proof:** For notational brevity, let \( A = \sum_{a \in [N], a \neq k} z_a \theta^n(z_a) \) and \( B = 1 + \sum_{a \in [N], a \neq k} \theta^n(z_a) \). We write
\[
\psi(z|u) = \frac{A + z_k \theta^k(z_k)}{B + \theta^k(z_k)},
\]
\[
\psi(z + \epsilon e^k|u) = \frac{A + (z_k + \epsilon) \theta^k(z_k + \epsilon)}{B + \theta^k(z_k + \epsilon)}
\]
Thus we have
\[
\psi(z + \epsilon e^k|u) - \psi(z|u) = \frac{A \theta^k(z_k) + B(z_k + \epsilon) \theta^k(z_k + \epsilon) + \epsilon \theta^k(z_k) \theta^k(z_k + \epsilon) - A \theta^k(z_k + \epsilon) - B z_k \theta^k(z_k)}{(B + \theta^k(z_k))(B + \theta^k(z_k + \epsilon))}
\]
\[
= \frac{(A - B z_k)(\theta^k(z_k) - \theta^k(z_k + \epsilon)) + B \epsilon \theta^k(z_k + \epsilon) + \epsilon \theta^k(z_k) \theta^k(z_k + \epsilon)}{(B + \theta^k(z_k))(B + \theta^k(z_k + \epsilon))}
\]
\[
> (A - B z_k)(\theta^k(z_k) - \theta^k(z_k + \epsilon)).
\]
Moreover, from the assumption \( \psi(z|u) \geq z_k \), we can easily see that \( A \geq B z_k \). On the other hand, we know that \( g^k(z_k) \) and \( \overline{g}^k(z_k) \) are monotonic decreasing in \( z_k \), so \( \theta^k(z_k) \) is also monotonic-decreasing in \( z_k \). Thus \( (A - B z_k)(\theta^k(z_k) - \theta^k(z_k + \epsilon)) \geq 0 \). Combine this with (24) we have \( \psi(z + \epsilon e^k|u) > \psi(z|u) \) as desired. \( \square \)

We are now to show that under an optimal price vector \( z^* \), the adversary needs to force each component \( G^n(\cdot) \) to its minimum. The proof idea is to show that if it is not the case, then we can always find another price solution \( z' \) that yields a better worst-case profit.
Lemma 9. Under robust optimal prices \( z^* \in \mathbb{R}^N \), let \( (a^*, b^*) = \{(a^{*n}, b^{*n}), \ n \in [N]\} \) be a solution to the adversary’s problem of (7), then for any \( n \in [N] \), we have

\[
G^*(Y^*|z_n^*, a^{*n}, b^{*n}) = G^*(z_n^*).
\]

Proof: Let \( f(z) = \arg\min_{(a, b) \in \mathcal{A}} \rho(z, a, b) \) and \( \mathcal{U}^* \) be the set of parameter \( u \) such that \( f(z^*) = \psi(z|u) \). To prove the equality, we just need to show that \( f(z^*) < z_n^* \) for all \( n \in [N] \). By contradiction, assume that there exists \( n \in [N] \) such that \( \rho(x^*, a^*, b^*) \geq z_n^* \). Let

\[
k = \arg\max\{z_n^* | n \in [N], f(z^*) > z_n^*\}
\]

\[
h = \arg\min\{z_n^* | n \in [N], f(z^*) < z_n^*\}.
\]

Indeed, \( h \) always exists because \( \rho(z^*, a^*, b^*) < \max_{n \in [N]} z_n^* \). We consider two following cases

(i) If such \( k \) exists. We have \( z_h^* > \rho(z^*, a^*, b^*) > z_l^* \) and for any \( l \neq k \) and \( l \neq k \) we have either \( \rho(z^*, a^*, b^*) = z_l^* \) or \( z_k^* \geq z_l^* \) \( \) or \( z_h^* < z_l^* \). Moreover, the function \( f(z) \) is continuous in \( x \) (Theorem 7, Hogan 1973). So, there is \( \delta > 0 \) such that

\[
z_h^* > f(z^* + te^k) > z_h^* + t, \ \forall t \in [0, \delta],
\]

As a result, for any \( l \in [N] \) such that \( z_l^* \geq z_h^* \) we have \( z_l^* > f(z^* + te^k) \) and if \( z_l^* \leq z_h^* \) we have \( z_l^* < f(z^* + te^k) \).

From Lemma (7), this means that there is a parameter \( \bar{u} \in \mathcal{U}^* \) and \( t \in (0, \delta) \) such that

\[
f(z^* + te^k) = \psi(z^* + te^k|\bar{u})
\]

\[
f(z^*) = \psi(z^*|\bar{u})
\]

Moreover, from Lemma 8, we see that \( \psi(z^* + te^k|\bar{u}) > \psi(z^*|\bar{u}) \), or equivalently, \( f(z^* + te^k) > f(z^*) \), which is contradictory to the assumption that \( z^* \) is a robust solution to (7).

(ii) If such \( k \) does not exist, then \( f(z^*) = z_n^* \) and for any \( l \in [N] \), either \( f(z^*) < z_l^* \) or \( f(z^*) = z_l^* \).

Similar to the previous case, we also have the result that there exists \( \delta > 0 \) such that \( z_l^* > f(z^* + te^o) \) for any \( l \in [N] \) and \( l \neq n \) and for all \( t \in (0, \delta) \), which also leads to the result that there is \( \bar{u} \in \mathcal{U}^* \) such that

\[
f(z^* + te^o) = \psi(z^* + te^o|\bar{u}).
\]

Using Lemma 8 and the fact that \( f(z^*) = z_n^* \), we have \( \psi(z^* + te^o|\bar{u}) > \psi(z^*|\bar{u}) \) for a \( t \in (0, \delta) \). Thus, \( f(z^* + te^o) > f(z^*) \), which is also contradictory to the assumption that \( z^* \) is a robust solution to (7).

So, in closing, we can claim that \( f(z^*) < z_n^* \), for all \( n \in [N] \). Thus, from Lemma 7 we obtain the desired result.

Now we know that under the optimal prices, the adversary will force each function \( G^*(Y^*|z_n^*, a^{*n}, b^{*n}) \) to its minimum over \( \mathcal{A}^n \), suggesting that we may be able to convert the robust problem into the maximization problem in (8). With all the lemmas above, we are ready to prove Theorem 3.

Proof of Theorem 3: We need to prove that if \( z^* \) is a robust optimal solution to (7), then it is also optimal to (8), and vice-versa. By contradiction, assume that \( z^* \) is optimal to (7) but \( z^* \notin \arg\max_{z^* \in \mathcal{R}^N} \mathcal{W}(z) \). Since the problem \( \max_{z^* \in \mathcal{R}^N} \mathcal{W}(z) \) has a unique local optimum (Proposition 2), \( \nabla z \mathcal{W}(z) \neq 0 \). Thus, there always exists a vector \( e \in \mathbb{R}^N \neq 0 \) and a constant \( \delta > 0 \) such that

\[
\mathcal{W}(z^*) < \mathcal{W}(z^* + te), \forall t \in (0, \delta).
\]
Moreover, according to Lemma 9, we know that \( W(z^*) < z_n^* \) for all \( n \in [N] \). Since \( W(z) \) and \( f(z) \) are continuous in \( z \) (recall that \( f(z) = \min_{a,b} \rho(z,a,b) \)) and \( f(z^*) = W(z^*) \), we can always choose \( \delta_1 \in (0, \delta) \) such that

\[
f(z^* + t\epsilon) < z_n^* < z_n^* + t\epsilon + 1, \quad \forall n \in [N], \quad t \in (0, \delta_1).
\]

Thus, using Lemma 7, we see that the adversary under prices \( z^* + t\epsilon \) will also force \( G^n(y^n | z^* + t\epsilon, a, b) \) to be equal to their minimum values. Hence, we have

\[
f(z^* + t\epsilon) = W(z^* + t\epsilon) \overset{(i)}{>} W(z^*) = f(z^*),
\]

where (i) is due to (25). This is contradictory against the assumption that \( z^* \) is optimal to (7). So, \( z^* \) needs to be optimal to (8) as well. For the opposite side, Proposition 2 already tells us that (8) always yields a unique optimal solution \( z^* \). Thus, this solution is also optimal to (7). \( \square \)

### A.5. Proof of Proposition 2

We first prove the equality (11). Taking the first-order derivatives of the objective function \( W(z) \), we have

\[
\frac{\partial W(z)}{\partial z_n} = \left( \sum_{i \in [N]} \frac{G^i(z_i)}{G^n(z_n)} + z_n \frac{\partial G^n(z_n)}{\partial z_n} \right) = \left( \sum_{i \in [N]} \frac{G^i(z_i)}{G^n(z_n)} \right) + z_n \frac{\partial G^n(z_n)}{\partial z_n}.
\]

Now, since (8) is an unconstrained problem, if \( z^* \) is an optimal solution to (8), we have

\[
\frac{\partial W(z^*)}{\partial z_n} = 0, \quad \forall n \in [N],
\]

Moreover, From Lemma (6) - (i), we see that \( \tilde{W}(p^G | a^*(z^*), b^*(z^*)) = z^* \), so \( z^* \) is also a solution to the system

\[
\frac{\partial \tilde{W}(\tilde{z}(p^G | a^*(z^*), b^*(z^*)))}{\partial z_n} = 0, \quad \forall n \in [N].
\]

Note that \( \tilde{W}(p^G | a^*(z^*), b^*(z^*)) \) is the objective function of the deterministic pricing problem with partition-wise homogeneous PSP considered in Zhang et al. (2018). Thus, using Theorem C1 of Zhang et al. (2018) we see that \( z^* \) has to satisfy the system of equalities

\[
\begin{aligned}
R(z^*) &= \sum_{n \in [N]} \frac{1}{\{b^*(z^*)\}} G^n(y^n | z_n, a^*(z^*), b^*(z^*)) \\
z_n &= \frac{1}{\{b^*(z^*)\}} + R(z^*), \quad \forall n \in [N].
\end{aligned}
\]

Thus, we have

\[
\begin{aligned}
z^*_n &= \frac{1}{\{b^*(z^*)\}} + \sum_{i \in [N]} \frac{1}{\{b^*(z^*)\}} G^i(y^i | z_i^*, a^*(z^*), b^*(z^*)) \\
&= \frac{1}{\{b^*(z^*)\}} + \sum_{n \in [N]} \frac{1}{\{b^*(z^*)\}} G^n(z_n^*),
\end{aligned}
\]
which is also the desired equality (11).

We now prove that (11) always yields a unique local optimum. By contradiction, assume that there are to points $z_1, z_2 \in \mathbb{R}^N$ such that $z_1 \neq z_2$ and $\frac{\partial \psi(a_1)}{\partial \alpha_n} = \frac{\partial \psi(a_2)}{\partial \alpha_n} = 0$ for all $n \in [N]$. Let $p_G^1$ and $p_G^2$ be two vectors of purchase probabilities defined by (9) under $z_1, z_2$, respectively. From Lemma 4 we have $p_G^1 \neq p_G^2$. Moreover, taking the derivatives of $\mathcal{W}(z(p_G^i))$ with respect to $p_G^i, n \in [N]$ we get

$$\frac{\partial \mathcal{W}(z(p_G^i))}{\partial p_G^i} = \sum_{l \in [N]} \frac{\partial \mathcal{W}(z)}{\partial z_l} \bigg|_{z=z_1} \times \frac{\partial z(p_G^i)}{\partial p_G^i} \bigg|_{p_G^i=p_G^l} = 0.$$  

Similarly, we also have $\partial \mathcal{W}(z(p_G^1))/\partial p_G^1 = 0$, implying that both $p_G^1$ and $p_G^2$ are local optimal solutions to the problem $\max_{p_G} \mathcal{W}(z(p_G))$, which is contrary to the claim that $\mathcal{W}(z(p_G))$ is strictly concave in $p_G$ (Theorem 2). This completes the proof.

A.6. Proof of Proposition 4

Taking the gradient of $\mathcal{W}(z(p_G^i))$ with respect to $p_G^i, n \in [N]$, we get

$$\frac{\partial \mathcal{W}(z(p_G^i))}{\partial p_G^i} = z(p_G^i)_n + \sum_{k \in [N]} \frac{\partial z(p_G^i)_k}{\partial p_G^i} p_G^i.$$  

Now, from Lemma 4, for any $k \in [N]$, we have $G^k(z(p_G^i)_k) = p_G^i/(1 - e^T p_G)$. Taking the first-derivatives of the both sides with respect to $p_G^i$ we have

$$\frac{p_G^i}{(1 - e^T p_G)^2} + \frac{\mathbb{I}[k = n]}{(1 - e^T p_G)} = \frac{\partial G^k(z(p_G^i)_k)}{\partial z_k} \frac{z(p_G^i)_k}{\partial p_G^i}.$$  

Moreover, we can write

$$\frac{\partial G^k(z_k)}{\partial z_k} = \frac{\partial G^k(Y_k|z_k, a^k, b^k)}{\partial z_k} \bigg|_{a^k=a^k(z_k), b^k=b^k(z_k)} = -\sum_{i \in V_k} \partial G^k(Y_i|z_k, a^k, b^k) \bigg|_{a^k=a^k(z_k), b^k=b^k(z_k)} = -\frac{(b^{k^*}(z_k)) p_G^k}{1 - e^T p_G}.$$  

Combine (28) and (29) we have

$$\frac{z(p_G^i)_k}{\partial p_G^i} = -\frac{\mathbb{I}[k = n]}{(b^{k^*}(z_k)) p_G^k} - \frac{1}{(b^{k^*}(z_k))(1 - e^T p_G)}.$$  

We substitute (30) into (27) and get

$$\frac{\partial \mathcal{W}(z(p_G^i))}{\partial p_G^i} = z(p_G^i)_n - \frac{1}{(b^{k^*}(z_n))} - \frac{1}{(1 - e^T p_G)} \sum_{k \in [N]} \frac{p_G^k}{(b^{k^*}(z_k))},$$

as desired.

Appendix B: Robust Pricing with Over-expected-sale Penalties

Motivated by applications in inventory considerations (Gallego and Hu 2014), we study a robust model for the pricing problem with expected sale requirements under uncertain choice parameters $(a, b)$. We first show in Section B.1 below that there may be no fixed prices such that the corresponding expected sale constraints are always satisfied when the choice parameters vary in the uncertainty set. It motivates us to consider a new robust model with over-expected-sales penalties in Section B.2. We provide the proofs of the results in this section in Section B.3.
B.1. Robust Pricing with Expected Sale Constraints

Motivated by the fact that the expected profit is concave in the purchase probabilities, previous studies (Zhang et al. 2018, Keller 2013) show that it is convenient to consider the pricing problem with expected sale constraints. Technically speaking, given a GEV-CPGF \( G(Y) \), price vector \( x \in \mathbb{R}^m \) and parameters \((a, b) \in \mathbb{R}^{2m}\), let us define the vector of purchase probabilities of products \( p \) with entries \( p_i = P_i(x, a, b|G) \). We also let \( x(p|a, b, G) \) be the denote the prices that achieve the purchase probabilities \( p \). The deterministic version of the constrained pricing problem can be formulated as

\[
\max_{p \in P} \left( \sum_{i \in V} x(p|a, b)_i - c_i \right) p_i.
\]

(31)

where \( P \in \mathbb{R}^m \) is a convex set such that for all \( p \in P \), \( \sum_{i \in V} p_i \leq 1 \). One the optimal purchase probabilities \( p \) is specified, we can obtain the optimal prices \( x(p|a, b, G) \) by solving a convex optimization problem. A natural robust version of the constrained pricing problem can be formulated as

\[
\max_{p \in P} \left\{ \phi(p) = \min_{(a, b) \in \mathcal{A}} \sum_{i \in V} (x(p|a, b, G)_i - c_i) p_i \right\}.
\]

(32)

Even though it is not difficult to show (32) is computationally tractable under rectangular or some polyhedrons uncertainty sets, the issue here is that the final decision is a price vector, not purchase probabilities. So even if we get an optimal purchase probabilities \( p \) from the robust model, it is not clear how to compute the corresponding optimal prices under \((a, b)\) uncertainty. On the other hand, one can show that given any prices \( x \), there may be \((a, b) \in \mathcal{A}\) such the resulting purchase probability vector \( p = P(x, a, b|G) \) that does not belong to the feasible set (i.e. the expected sale constraints are not satisfied). All these make the robust version in (32) inappropriate to use. This is the reason we propose an alternative robust model in Section B, in which instead of requiring the purchasing probabilities to satisfy some constraints, we add a penalty cost to the objective function.

Alternatively, in some situations the firm may face uncertainties occurring in the inventory, leading to uncertain expected sale constraints. A robust model may require the expected sales constraints to be satisfied for all the scenarios that may occur, i.e., \( p \in P(\xi) \), for all \( \xi \in \Xi \). Such a robust model can be formulated as

\[
\max_{p} \left( \sum_{i \in V} x(p|a, b)_i - c_i \right) p_i
\]

subject to \( (a^t(\xi))^T p \leq r_t(\xi) \quad \forall \xi \in \Xi \)

\[
\sum_{i \in V} p_i \leq 1, \quad p \geq 0
\]

where \((a^t(\xi), r_t(\xi))\), \( \forall t \), are the parameters of the expected sale constraints, which are not certain in the context and depend on a random vector \( \xi \in \Xi \). Since the objective function is concave and all the constraints are linear in \( p \), the above problem is generally tractable (Ben-Tal and Nemirovski 1998). A simple but useful setting is that the parameter of the expected sale constraints vary in a rectangular uncertainty set, i.e., \( a^t \leq a^t \leq a^t \) and \( r^t \leq r^t \leq r^t \) for all \( t \in [T] \). In this context, one can show that (33) is equivalent to the following convex optimization problem

\[
\max_{p} \left( \sum_{i \in V} x(p|a, b)_i - c_i \right) p_i
\]

(34)
subject to \((\alpha^t)^T p \leq r_t\)
\[\sum_{i \in V} p_i \leq 1, \ p \geq 0\]

Other uncertainty sets may be considered, i.e., polyhedron or ellipsoidal ones, and we refer the reader to Ben-Tal and Nemirovski (1998) for details.

**B.2. Robust Pricing with Over-expected-sale Penalties**

We propose a version with over-expected-sale penalties, which allows us to handle both the expected sale requirements and the uncertainty issue. Our idea is to put the expected sale constraints to the objective function, i.e., we do not force the purchase probabilities to be in a feasible set, but instead we add penalties for purchase probabilities violating the constraints. More precisely, we consider the objective function \(\Phi(x, a, b) - \sum_{t=1}^{T} \lambda_t \max\{0, (\alpha^t)^T p - r_t\}\), where \(\lambda_t \geq 0, \ t = 1, \ldots, T\), are penalty parameters and \(p \in \mathbb{R}^m\) is a vector of purchase probabilities with entries \(p_i = P_i(Y(x, a, b), \forall i \in V). In this objective function, if a constraint is violated, i.e., \((\alpha^t)^T p > r_t\), then a cost \(-\lambda_t \max\{0, (\alpha^t)^T p - r_t\}\) is added to the expected revenue. In general, if we choose \(\lambda_t\) large enough, we will need a vector of purchase probabilities satisfying all the expected sale constraints to obtain high objective values. The deterministic pricing problem under the above objective function is

\[
\max_{x \in \mathbb{R}^m} \left\{ \Phi(x, a, b) - \sum_{t=1}^{T} \lambda_t \max\{0, (\alpha^t)^T p - r_t\} \right\}. \tag{35}
\]

In general a solution to (35) does not have a constant markup over products, even when the PSP are all homogeneous.

**Proposition 5.** The problem with penalties (35) does not have a constant markup over products, even when the PSP are homogeneous.

For this reason, the results presented in this section are not a generalized version of those shown in Sections 3 and 4 when the penalty parameters \(\lambda\) equals zero. Since we consider the pricing problem with over-expected-sale penalties, we do not face the issue of violating the expected sale constraints when the choice parameters vary in the uncertainty set. We consider the robust version of (35) under \((a, b)\) uncertainty

\[
\max_{x \in \mathbb{R}^m} \min_{(a, b) \in A} \left\{ \mathcal{K}(x, a, b) = \Phi(x, a, b) - \sum_{t=1}^{T} \lambda_t \max\{0, (\alpha^t)^T p - r_t\} \right\} \tag{36}
\]

In this version, we consider the settings that the PSP are partition-wise homogeneous, and the CPGF and uncertainty set are partition-wise separable, as in Section 4. The adversarial problem of (36) is way more difficult to solve as compared to the robust versions considered in the previous sections, as the objective function now is not differentiable. Moreover, a solution to the deterministic problem (35) would not have a constant-markup style, thus a robust solution to (36) would not have either. The adversary’s problem under such a non-constant-markup solution seems not possible to handle tractably. For that reason, we consider a robust version in which we only seek constant-markup solutions. Moreover, to have a tractable structure, we also need to further assume that the expected-sale parameters \(\alpha^t, \ t \in [T]\) are partition-wise homogeneous,
i.e., for any partition \( n \in [N], \alpha_n^i = \alpha_n^j , \forall i, j \in V_n, t \in [T] \). Let denote by \( d^i \) a vector in \( \mathbb{R}^N \) such that \( d^i_n = \alpha_n^i \), for any \( j \in V_n, n \in [N], t \in [T] \). The robust problem with all the above settings becomes

\[
\max_{x} \min_{(a,b) \in \mathcal{A}} \left\{ \mathcal{L}(z, a, b) = \frac{\sum_{n \in [N]} z_n G^n(z_n, a^n, b^n)}{1 + \sum_{n \in [N]} G^n(z_n, a^n, b^n)} - \sum_{t=1}^T \lambda_t \max\{0, (d^i)^T \tilde{p}^G(z, a, b) - r_t \} \right\}, \tag{37}
\]

where \( \tilde{p}^G(z, a, b) \in \mathbb{R}^N \) with entries

\[
\tilde{p}^G(z, a, b)_n = \frac{G^n(z_n, a^n, b^n)}{1 + \sum_{t \in [N]} G^l(z_l, a^l, b^l)}.
\]

Theorem 5 below states that (37) can be solved by convex optimization.

**Theorem 5.** (Robust solutions for the robust pricing problem with over-expected-sale penalties). If \( z^* \) is optimal to the problem

\[
\max_{x \in \mathbb{R}^N} \left\{ \mathcal{H}(z) = \frac{\sum_{n \in [N]} z_n G^n(z_n)}{1 + \sum_{n \in [N]} G^n(z_n)} - \sum_{t=1}^T \lambda_t \max\{0, (d^i)^T p^G - r_t \} \right\}, \tag{38}
\]

where \( p^G(z) \) is of size \( N \) with entries \( \tilde{p}^G(z)_n = \mathcal{G}^n(z_n) / \left( 1 + \sum_{t \in [N]} \mathcal{G}^l(z_l) \right) \), then the prices \( x^* \in \mathbb{R}^n \) such that \( x^*_n = c_n + z^*_n, \forall n \in [n], i \in V_n \) is optimal to the robust problem \( \max_{x \in \mathbb{R}^N} \min_{(a,b) \in \mathcal{A}} \mathcal{K}(x, a, b) \). Moreover, the objective function of (38) is concave in \( \tilde{p}^G \).

We provide the proof in Appendix B.3.2. In general, the robust problem (37) is challenging to handle because of the term \( \sum_{t=1}^T \lambda_t \max\{0, (d^i)^T \tilde{p}^G - r_t \} \), which makes the objective function no-longer differentiable in \( z \). However, if we look at the subset \( T \in [T] \) such that the constraints are violated only in \( T \), we can write the objective function as

\[
\mathcal{H}(z) = z^T p^G - \sum_{t \in T} \lambda_t (d^i)^T p + \sum_{t \in T} \lambda_t r_t = \sum_{n \in [N]} (z_n - \sum_{t \in T} \lambda_t d_n^i) p^G_n + \sum_{t \in T} \lambda_t r_t \tag{39}
\]

and note that \( \sum_{n \in [N]} (z_n - \sum_{t \in T} \lambda_t d_n^i) p^G_n \) is also an expected revenue with shifted item costs \( c_n = \sum_{t \in T} \lambda_t d_n^i + c_n, n \in [N], i \in V_n \). We leverage this observation and follow the spirit of the proof of Theorem 4 to prove the results. The main idea is to show that, under an optimal prices of (37), the adversary will also force the each component \( G^l(\cdot) \) to its minimums. From this, we can show an equivalence between the reduced problem (38) and the robust one (37). The proof is indeed more complicated as we have to handle the term \( \max\{0, (d^i)^T \tilde{p}^G - r_t \} \).

The limitation of Theorem 5 is that it only returns best solutions among those that have a constant markup in each partition, and all the expected sale parameters in each partition need to be homogeneous. Relaxing these assumption would make the robust problem challenging to handle (see the discussion before (37)). Moreover, we believe that the theorem is still useful in contexts where the firm only wants to make pricing decisions for each group of products and only impose expected sale requirements for the whole groups instead of each single product in the groups.
An interesting and important question here is how the robust optimal value and optimal solutions change when the penalty parameters \( \lambda \) increase. To answer this, let us consider the following constrained problem

\[
\begin{align*}
\max_{\mathbf{p}^G} & \quad W(\mathbf{z}(\mathbf{p}^G)) \\
\text{subject to} & \quad (\mathbf{d}_i^T)^T \mathbf{p}^G \leq r_i \\
& \quad \sum_{n \in [N]} p_{ni}^G \leq 1 \\
& \quad \mathbf{p}^G \geq 0.
\end{align*}
\]

We also define \( \phi^{RO,\lambda} \) as the optimal value of the robust problem in (36) under penalty parameters \( \lambda \), \( \varphi \) as the optimal value of the constrained problem (40), \( \mathbf{x}^{RO,\lambda} \) is an robust solution to (36) and \( \mathbf{p}^{G,\lambda} \) is the purchase probabilities given by the robust solution \( \mathbf{x}^* \) in the worst-case. Proposition 6 below tells us how the optimal value of the robust problem with over-expected-sale penalties (56) when the parameters \( \lambda \) increase.

**Theorem 6. (Convergence of the robust optimal value when the penalty parameters \( \lambda \) increase).** Given any \( \epsilon > 0 \), we have

(i) If we select \( \lambda \) such that \( \min_i \lambda_i \geq \frac{(\Delta^* - \varphi)}{\epsilon} \) then \( \sum_i \max\{0, (\mathbf{d}_i^T)^T \mathbf{p}^{G,\lambda} - r_i\} \leq \epsilon \), where \( \Delta^* = \max_{\mathbf{z} \in \mathcal{R}^N} W(\mathbf{z}) \).

(ii) Assume that there are positive constant \( L_i, l_i, i \in \mathcal{V} \) such that \( Y_i \partial G_i(\mathbf{Y}) \) is bounded from above by \( L_i Y_i^{l_i} \) for all prices \( \mathbf{x} \geq 0 \), then if we select \( \epsilon \) such that

\[
\epsilon \leq \min_{t \in [T], n \in [N]} \{d_{nt}^i | d_i^i > 0\} \min_i \left\{ \frac{r_{nt}}{(\mathbf{d}_i^T)^T \mathbf{1}} \right\},
\]

then \( \phi^{RO,\lambda} - \varphi \) can be bounded as

\[
0 \leq \phi^{RO,\lambda} - \varphi \leq \max \left\{ \frac{a_i - b_i c_i}{b_i} - \frac{1}{b_i |\mathcal{V}|} \log \frac{\delta(\epsilon)}{L_i} \right\} \frac{N \epsilon}{\min_{n \in [N]} \{d_{nt}^i | d_i^i > 0\}}
\]

where \( \delta(\epsilon) = \min_i \left\{ \frac{d_{nt}^i}{(\mathbf{d}_i^T)^T \mathbf{1}} \right\} - \min_{n \in [T], n \in [N]} \frac{x}{d_{nt}^i |\mathcal{V}|} \). This upper bound converges to zero linearly when \( \epsilon \) tends to zero (i.e., \( \min_i \{\lambda_i\} \) goes to infinity).

The proof can be found in Appendix B.3.3. Here, it is not difficult to validate that the assumption in Theorem 6–(ii) holds for all the well-known GEV models in the literatures. For examples, for the MNL, \( Y_i \partial G_i(\mathbf{Y}) = Y_i \).

For a nested logit model specified by \( G(\mathbf{Y}) = \sum_{n \in \mathcal{N}} (\sum_{i \in \mathcal{C}_n} \sigma_{in} Y_i^{\mu_n})^{1/\mu_n} \), where \( \mathcal{N} \) is the set of nests, \( C_n \) is the corresponding nest and \( \mu, \mu_n \) are some parameters, we have \( Y_i \partial G_i(\mathbf{Y}) = Y_i^{\mu_n} (\sum_{j \in C_n} \sigma_{jn} Y_j^{\mu_n})^{1/\mu_n-1} \). If \( \mu_n > 1 \) then \( Y_i \partial G_i(\mathbf{Y}) \leq Y_i^{1/\mu_n-1} Y_i \) and if \( \mu_n < 1 \) then \( Y_i \partial G_i(\mathbf{Y}) \leq L_i Y_i^{\mu_n} \), where \( L_i \) is an upper bound of \( (\sum_{j \in C_n} \sigma_{jn} Y_j^{\mu_n})^{1/\mu_n-1} \) for all \( \mathbf{x} \in \mathbb{R}^m \), which always exists. For a more general GEV model, we note that \( \partial G_i(\mathbf{Y}) \leq 0 \) and \( Y_j \geq 0 \) for all \( i, j \in \mathcal{V}, i \neq j \). As a result, we have \( \partial G_i(\mathbf{Y}) \leq \partial G_i(\tilde{\mathbf{Y}}^i) \), where \( \tilde{\mathbf{Y}}^i \) is a vector of size \( m \) with entries \( \tilde{Y}^i_i = Y_i \) and \( \tilde{Y}^i_j = 0 \) for all \( j \neq i \). Thus, \( \partial G_i(\tilde{\mathbf{Y}}^i) \) is a function of only \( Y_i \). For a more complicated GEV model such as the network GEV model (Daly and Bierlaire 2006, Mai et al. 2017), we can easily upper-bound \( Y_i \partial G_i(\tilde{\mathbf{Y}}^i) \) by a function of form \( L_i Y_i^{l_i} \), where \( L_i, l_i > 0 \).

In Theorem 6, (ii) tells us explicitly that the penalty term will converge to zero when the parameters \( \lambda \) are large enough. It also provides an estimate for \( \min_i \{\lambda_i\} \) to get arbitrarily small penalty costs. The
second bound (ii) provides an upper-bound for the gap between the optimal expected revenues given by the constrained pricing problem and the pricing problem with over-expected-sale penalties, and this upper bound converges to zero linearly when $\epsilon$ goes to zero. So in general, Problem 6 can be viewed as a generalized version of the constrained pricing problem, in the sense that if we select the penalty parameters $\lambda$ large enough, then we will get a solution that is similar to the one from the constrained problem, and if we set $\lambda = 0$ then we come back to the unconstrained problem. Thus, the formulation in (37) provides a more flexible way to handle expected sale requirements.

B.3. Proofs of the Results in Section B.2

B.3.1. Proof of Proposition 5. We will give a counter example to illustrate the claim. For the sake of illustration, we only consider a pricing problem under the MNL model with 2 products and homogeneous PSP. We also consider only one expected sale constraint as $\alpha_1 p_1 \leq r_t$, where $r_t/\alpha_1$ is very small. Let $p^\lambda = (p_1^\lambda, p_2^\lambda)$ be a solution to (31) under penalty parameter $\lambda$. When $\lambda$ goes to infinity, Theorem 7 tells us that a solution to the pricing problem with penalties converge to a solution to the constrained pricing problem. Thus, for any $\epsilon > 0$ arbitrarily small, we can chose $\lambda$ large enough such that $\alpha_1 p_1^\lambda \leq r_t + \epsilon$. So, if we choose $\epsilon$ and $r_t/\alpha_1$ to be very small, then $p_1^\lambda$ would be very close to zero. Since $p_1^\lambda = \exp(a_1 - bx_1^\lambda) / (1 + \exp(a_1 - bx_1^\lambda) + \exp(a_2 - bx_2^\lambda))$ (where $b$ is the PSP of the two products, $x^\lambda$ is an optimal price solution to the pricing problem under penalty parameter $\lambda$), we have $\lim_{p_1^\lambda \to 0} x_1^\lambda(p) = +\infty$, meaning that to have an arbitrarily small probability $p_1^\lambda$, we need to increase the price of Product 1 to infinity. On the other hand, $p_2^\lambda$ does not affect the penalty term and we have $\lim_{x_2 \to +\infty} (x_2 - c_2) \exp(a_2 - bx_2) / (1 + \exp(a_1 - bx_1) + \exp(a_2 - bx_2)) = 0$. Thus, to maximize the objective function, the solution $x_2^\lambda$ needs to be finite. So, in summary, we can create an example yielding a solution $(x_1^\lambda, x_2^\lambda)$ such that $x_1^\lambda$ can be arbitrarily large and $x_2^\lambda$ is bounded from above. Thus, $(x_1^\lambda, x_2^\lambda)$ would not have the constant-markup style.

B.3.2. Proof of Theorem 5. First, let $f(z) = \min_{(a,b) \in A} \mathcal{L}(z,a,b)$. In Lemma 10 below, we show that given any prices $x \in \mathbb{R}^n$, the adviser’s problem will force each component $G^n(Y^n|z_n, a^n, b^n)$ to either its minimums or maximums. This result is similar to the case of partition-wise PSP without penalties considered in Section 4. The proof is however more complicated as it involves the term $\sum_{t=1}^T \lambda_t \max\{0,(d^t)^T P_{\tilde{G}} - r_t\}$.

Lemma 10. Given any $z \in \mathbb{R}^N$, there is a solution $(a^*, b^*) = \{(a^{n*}, b^{n*})| n \in [N]\}$ to the corresponding adversary’s problem (37) such that  

$$G^n(Y^n|z_n, a^{n*}, b^{n*}) \in \left\{ \tilde{G}^n(z_n), \tilde{G}^n(z_n) \right\}. $$

Proof: We denote by $T$ a subset of $[T]$ such that $(d^t)^T \tilde{P}^G(z,a^*,b^*) \geq r_t$ for all $t \in T$ and $(d^t)^T \tilde{P}^G(z,a^*,b^*) < r_t$ if $t \notin T$. The adversary’s optimal value at $z$ becomes 

$$\mathcal{L}(z,a^*,b^*) = \sum_{n \in [N]} z_n \tilde{P}^G(z,a^*,b^*)_n - \sum_{t \in T} \lambda_t (d^t)^T \tilde{P}^G(z,a^*,b^*) + \sum_{t \in T} \lambda_t r_t = \frac{\sum_{n \in [N]} (z_n - \sum_{t \in T} \lambda_t d^t_n) G^n(Y^n|z_n, a^{n*}, b^{n*})}{1 + \sum_{n} G^n(Y^n|z_n, a^{n*}, b^{n*})} + \sum_{t \in T} \lambda_t r_t,$$
For notational brevity, let
\[
\rho^* = \frac{\sum_{n \in [N]} \left( z_n - \sum_{t \in T} \lambda_t d_{t,n}^* \right) G^a(Y^n|z_n,a^{n*},b^{n*})}{1 + \sum_{n} G^a(Y^n|z_n,a^{n*},b^{n*})}
\]
\[
\mathcal{I}_1 = \{ n \in [N] | \rho^* < z_n - \sum_{t \in T} \lambda_t d_{t,n} \}
\]
\[
\mathcal{I}_2 = \{ n \in [N] | \rho^* > z_n - \sum_{t \in T} \lambda_t d_{t,n} \}
\]
\[
\mathcal{A}^a = \{ (a,b) \in \mathcal{A} | G^a(Y^n|z_n,a^{n*},b^{n*}) = \mathcal{G}^a(z_n) \text{ if } n \in \mathcal{I}_1, \quad G^a(Y^n|z_n,a^{n*},b^{n*}) = \mathcal{G}^a(z_n) \text{ if } n \in \mathcal{I}_2 \}
\]
From Lemma 7, if \((a^*,b^*) \notin \mathcal{A}^a\), then for any \((a,b) \in \mathcal{A}^a\) we have
\[
\mathcal{L}(z,a^*,b^*) > \sum_n \left( z_n - \sum_{t \in T} \lambda_t d_{t,n} \right) \mathcal{G}(z,a,b) + \sum_{t \in T} \lambda_t r_t
\]
which is contradictory to the assumption that \((a^*,b^*)\) is optimal to the adversary’s problem. So we have \((a^*,b^*) \in \mathcal{A}^a\). On the other hand, Lemma 7 tells us that if we take any point \((a,b) \in \mathcal{A}^a\) such that \(G^a(Y^n|z_n,a^{n*},b^{n*}) \in \{ \mathcal{G}^a(z_n), \mathcal{G}(z_n) \}\) for all \(n \notin \mathcal{I}_1 \cup \mathcal{I}_2\), we also have
\[
\mathcal{L}(z,a^*,b^*) = \sum_n \left( z_n - \sum_{t \in T} \lambda_t d_{t,n} \right) \mathcal{G}(z,a,b) + \sum_{t \in T} \lambda_t r_t
\]
\[
\geq \sum_n \left( z_n \mathcal{G}(z,a,b) - \sum_{t \in T} \lambda_t \max \{ 0, (d^t)^T \mathcal{G}(z,a,b) - r_t ) \right) = \mathcal{L}(z,a,b).
\]
Since \(\mathcal{L}(z,a^*,b^*) = \min_{(a,b) \in \mathcal{A}} \mathcal{L}(z,a,b)\), we have \(\mathcal{L}(z,a^*,b^*) = \mathcal{L}(z,a,b)\), meaning that \((a,b)\) is also optimal to the adversary’s problem under prices \(x\). This completes the proof. 

The lemma above tells us that a optimal solution to the adversary’s problem for which the adversary will force \(G^a(Y^n|z_n,a,b)\) to their minimum or maximum values. The next lemma further characterizes an important property of the robust optimal prices, which states that, under a robustly optimal solution, the adversary will always force \(G^a(Y^n|z_n,a,b)\) to their minimums. This claim is similar to the claim in Lemma 9. The proof is however more challenging due to the term \(\sum_{t=1}^{T} \lambda_t \max \{ 0, (d^t)^T \mathcal{G}(z,a,b) - r_t \}\).

First, let \(z^*\) be a robust optimal solution to the robust problem and \((a^*,b^*)\) be an optimal solution to the adversary problem such that \(G^a(Y^n|z^*_n,a^{n*},b^{n*}) \in \{ \mathcal{G}^a(z^*_n), \mathcal{G}(z^*_n) \}\). We also denote by \(T^*\) a subset of \([T]\) such that \((d^t)^T \mathcal{G}(z^*,a^*,b^*) \geq r_t\) for all \(t \in T^*\) and \((d^t)^T \mathcal{G}(z^*,a^*,b^*) < r_t\) if \(t \notin T^*\), and let
\[
\rho^* = \frac{\sum_{n \in [N]} \left( z_n - \sum_{t \in T} \lambda_t d_{t,n} \right) G^a(Y^n|z^*_n,a^{n*},b^{n*})}{1 + \sum_{n} G^a(Y^n|z^*_n,a^{n*},b^{n*})}
\]

**Lemma 11.** \(\rho^* < z_n^* - \sum_{t \in T^*} \lambda_t d_{t,n}^*\), for all \(n \in [N]\).

**Proof:** Let \(k = \arg \max_{k \in [N]} z_k^* - \sum_{t \in T^*} \lambda_t d_{t,k}^*\). By contradiction, assume that \(\rho^* \geq z_k^* - \sum_{t \in T^*} \lambda_t d_{t,k}^*\). According to Lemma 10 and to facilitate the exposition, we also parameterize the adversary’s objective function by a vector \(u \in \{0,1\}^N\) as
\[
\mathcal{L}(z|u) = \psi(z|u) - \sum_{t=1}^{T} \lambda_t \max \{ 0, (d^t)^T \mathcal{G}(z|u) - r_t \}
\]
where $\psi(z|u)$ is defined in (23) and $\tilde{p}^G(z|u)$ is defined as
\[
\tilde{p}^G(z|u)_n = \frac{\theta_n(z_n)}{1 + \sum_{n \in [N]} \theta_n(z_n)},
\]
where $\theta_n(z_n) = G_n(z_n)$ if $u_n = 0$ and $\theta_n(z_n) = \overline{G}_n(z_n)$ otherwise. We also let
\[
\delta = \min_{u \in \{0, 1\}^N} \left\{ \mathcal{L}(z^*|u) - f(z^*) \mid \mathcal{L}(z^*|u) > f(z^*) \right\},
\]
with a note that we set $\delta = +\infty$ if the corresponding searching set is empty. Let $U^*$ be the set of parameter $u^*$ such that $\mathcal{L}(z^*|u^*) = f(z^*)$. For any $u^* \in U^*$ and any $\epsilon > 0$ we easily have $\tilde{p}^G(z^* + \epsilon e^k|u^*)_n > \tilde{p}^G(z^*|u^*)_n$. Moreover, from Lemma 8 we have $\psi(z^* + \epsilon e^k|u^*) > \psi(z^* + \epsilon e^k|u^*)$, leading to the fact that, for any $\epsilon > 0$ we have $\mathcal{L}(z^* + \epsilon e^k|u^*) > \mathcal{L}(z^*|u^*)$. Moreover, since $\mathcal{L}(z|u^*)$ and $f(z)$ are continuous in $z$, we always can select $\epsilon > 0$ small enough such that
\[
\mathcal{L}(z^*|u^*) < \mathcal{L}(x + \epsilon e^k|u^*)
\]
\[
|f(z^*) - f(z^* + \epsilon e^k)| < \delta/2
\]
\[
\left| \mathcal{L}(z^* + \epsilon e^k|u^*) - \mathcal{L}(z^*|u^*) \right| < \delta/2,
\]
where $u^*$ is a configuration of the adversary’s problem under prices $z + \epsilon e^k$. Applying the triangular inequality with (43) and (44) we have
\[
|f(z^*) - \mathcal{L}(z^*|u^*)| \leq |f(z^*) - f(z^* + \epsilon e^k)| + |f(z^* + \epsilon e^k) - \mathcal{L}(z^*|u^*)| < \delta.
\]
So, according to the definition of $\delta$ in (41), we have $f(z^*) = \mathcal{L}(z^*|u^*)$, meaning that $u^* \in U^*$. So, we can always choose a configuration $\hat{u} \in U^*$ such that $\hat{u}$ is also a configuration for the adversary’s problem under $z^* + \epsilon e^k$, i.e., $\mathcal{L}(z^*|\hat{u}) = f(z^*)$ and $\mathcal{L}(z^* + \epsilon e^k|\hat{u}) = f(z^* + \epsilon e^k)$. Together with (42), we have
\[
f(z^*) = \mathcal{L}(z^*, \hat{u}) < \mathcal{L}(z^* + \epsilon e^k|\hat{u}) = f(z^* + \epsilon e^k),
\]
which is contradictory to our initial assumption that $z^*$ is a robust optimal solution. So our contradiction hypothesis is untrue and this completes the proof.

We are now ready to show the proof of Theorem 5.

Proof of Theorem 5. From Lemma 10 and 11, we have that if $z^*$ is a robust optimal solution, then the adversary will force all $G^n(Y^n|z^n, a^n, b^n)$ to their minimum values, i.e., $f(z^*) = \mathcal{L}(z^*|u^0)$, where $u^0$ is a vector of size $N$ with all zero entries. Moreover, $f(z^*) < \mathcal{L}(z^*|u)$ for any $u \in \{0, 1\}^N$, $u \neq u^0$. We now need to prove that $z^*$ is also optimal to the maximization problem $\max_u \mathcal{L}(z|u^0)$. In this case, the function $\mathcal{L}(z|u^0)$ is not differentiable in $z$, so we cannot use the techniques in the proof of Theorem 3 above. Fortunately, if we consider the objective function $\mathcal{L}(z|u^0)$ as a function of the purchase probabilities $p^G$, then we can show that this function is strictly concave in $p^G$. To facilitate this point, let’s us define
\[
\mathcal{F}(\tilde{p}^G|u^0) = \mathcal{L}(z|u^0) = z(p^G|u^0)^T p^G - \sum_{t=1}^T \lambda_t \max\{0, (d^t)^T p^G - r_t\}
\]
We know that the first term $z(p^G|u^0)^T p^G$ is strictly concave in $p^G$ (Theorem 2) and it is not difficult to show that $-\sum_{t=1}^T \lambda_t \max\{0, (d^t)^T p^G - r_t\}$ is concave in $p^G$. As a result, $\mathcal{F}(\tilde{p}^G|u^0)$ is strictly concave in $p^G$. 

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Now, let \( p^{G^*} \) be the purchase probabilities given by prices \( z^* \). We will prove that \( p^{G^*} = \arg\max_{p \in \mathcal{P}} F(p^G|u^0) \). We omit \( u^0 \) for notational simplicity. By contradiction, assume that \( \tilde{p} = \arg\max_{p \in \mathcal{P}} F(p^G) \) and \( F(\tilde{p}) > F(p^{G^*}) \). Since \( F(p^G) \) is strictly concave in \( p^G \), we have, for any \( t \in (0, 1) \),
\[
 tF(\tilde{p}) + (1-t)F(p^{G^*}) < F(t\tilde{p} + (1-t)p^{G^*})
\]
Since \( F(\tilde{p}) \geq K((t\tilde{p} + (1-t)p^{G^*}) \), we have \( F(p^{G^*}) < F((t\tilde{p} + (1-t)p^{G^*}) \) for all \( t \in (0, 1) \). This also mean that for any \( \epsilon > 0 \), we always can find a point \( p \in \mathcal{P}^N \) such that \( ||p^{G^*} - p|| \leq \epsilon \) and \( F(p^{G^*}) < F(p) \). Since \( p^G(z|u^0) \) (\( p^G \) as a function of \( z \)) is continuous in \( z \), this also means that given any \( \epsilon > 0 \), there always exists \( x \in \mathbb{R}^m \) such that \( ||x - x^*|| \leq \epsilon \) and \( \mathcal{L}(x^*|u^0) < \mathcal{L}(x|u^0) \).

Now, similarly to the proof of Lemma 11, let
\[
\delta = \min_{u \in (0,1)^N} \left\{ \frac{\mathcal{L}(z^*|u) - \mathcal{L}(z^*|u^0)}{\mathcal{L}(z^*|u) > \mathcal{L}(z^*|u^0)} \right\}
\]
Since \( f(z) \) and \( \mathcal{L}(z|u) \) are continuous in \( z \), there is an \( \epsilon > 0 \) such that, for all \( z \in \mathbb{R}^m \), \( ||z^* - z|| \leq \epsilon \)
\[
\begin{cases}
|f(z^*) - f(z)| < \delta/2 \\
|\mathcal{L}(z^*|u^*) - \mathcal{L}(z|u^*)| < \delta/2,
\end{cases}
\]
where \( (u^*) \) is a configuration vector of the adversary’s problem (36) under prices \( z \). As a result, for all \( z \) such that \( ||z^* - z|| \leq \epsilon \)
\[
|\mathcal{L}(z^*|u^0) - \mathcal{L}(z^*|u^0)| < \tau.
\]
Combine this with (45), since \( u^0 \) is the unique configuration for the adversary’s problem under prices \( z^* \), we have that for all \( z \) such that \( ||z^* - z|| \leq \epsilon \), \( u^* = u^0 \). Moreover, we have shown that given any \( \epsilon > 0 \), there exists \( z \) such that \( ||z - z^*|| \leq \epsilon \) and \( \mathcal{L}(z|u^0) > \mathcal{L}(z^*|u^0) \). So, if we choose \( \epsilon > 0 \) and small enough, we have
\[
f(z^*) = \mathcal{L}(z^*|u^0) = \mathcal{L}(z|u^0) > \mathcal{L}(z^*|u^0) = f(z^*).
\]
This is contradictory to the assumption that \( z^* \) is a robust optimal solution. So, our contradiction hypothesis that \( p^{G^*} \) is not optimal to \( \max_{p \in \mathcal{P}} F(p^G|u^0) \) is untrue, meaning \( z^* \) is optimal to \( \max_z \mathcal{L}(z|u^0) \). Moreover, \( \max_{p \in \mathcal{P}} F(p^G|u^0) \) always yields a unique solution as the objective function is strictly concave, \( \max_z \mathcal{L}(z|u^0) \) also yields a unique solution and this solution is also a unique robust optimal solution to (37).

We need a final step to complete the proof. We can easily see that Problem (38) can be converted equivalently as
\[
\begin{align*}
\max_{p^G, y} & \quad \mathcal{W}(z(p^G)) - \sum_{t=1}^{T} \lambda_t y_t \\
\text{subject to} & \quad (d^t)^T p^G - y_t \leq r_t \\
& \quad \sum_{n \in [N]} p_{nt}^G \leq 1 \\
& \quad p^G, y \geq 0.
\end{align*}
\]
which is a convex optimization problem, as \( \mathcal{W}(z(p^G)) \) is strictly concave (2).
B.3.3. Proof of Theorem 6  First, let us consider the deterministic version of the pricing problem with penalties, which can be formulated as the convex optimization problem

\[
\max_{p,y} \sum_{t \in V} (x(p|a, b, G) - c_t) p_t - \sum_{t=1}^{T} \lambda_t y_t \quad \text{subject to} \quad (\alpha^t)^T p - y_t \leq r_t \sum_{t \in V} p_t \leq 1 \quad p, y \geq 0.
\] (47)

Before moving to a robust version, we investigate some characteristics of the deterministic pricing problem with penalties (47). First, let us denote by \( v^* \) and \( p^* \) the optimal value and optimal solution of the standard pricing problem under expected sale constraints and \( v^0 \) and \( p^0 \) the optimal value and optimal solution to the pricing problem with over-expected-sale penalties (47). Theorem 7 below shows that the expected value given by (47) will converges to the optimal expected revenue given by the constrained pricing problem when \( \lambda_t, \forall t \in [T] \), increase to infinity.

Theorem 7 (Convergence of the optimal value when the penalty parameters increase). For any \( \epsilon > 0 \), we have

(i) For any \( \lambda^1, \lambda^2 \in \mathbb{R}_+^T \) such that \( \lambda^1 - \lambda^2 = \epsilon \mathbf{1} \),

\[
\sum_t \max\{0, (\alpha^t)^T p^\lambda^1 - r_t\} \leq \sum_t \max\{0, (\alpha^t)^T p^\lambda^2 - r_t\},
\]

where \( \mathbf{1} \) is a unit vector of appropriate size.

(ii) \( v^\lambda \geq v^* \) for all \( \lambda \in \mathbb{R}_+^T \) and if \( \lambda_0 = \min_{t \in [T]} \lambda_t \geq (\Delta^* - v^*)/\epsilon \) then \( \sum_t \max\{0, (\alpha^t)^T p^\lambda - r_t\} \leq \epsilon \), where \( \Delta^* = \max_x \Phi(x, a, b) \).

(iii) Assume that there are positive constant \( L_t, l_t, i \in V \) such that \( Y_t \partial G_i(Y) \) is bounded from above by \( L_t Y_t^{\max} \) for all prices \( x \geq 0 \), then for any \( \epsilon \) such that

\[
\epsilon \leq \min_{t,i} \{ a_i^t | \alpha_i^t > 0 \} \min_{t} \left\{ \frac{r_t}{(\alpha^t)^T \mathbf{1}} \right\},
\]

then if we choose \( \lambda_0 \geq (\Delta^* - v^*)/\epsilon \), we can upper-bound \( |v^\lambda - v^*| \) as

\[
|v^\lambda - v^*| \leq \max \left\{ \max_i \left\{ \frac{a_i}{b_i} - \frac{1}{b_i} \log \left( \frac{\delta(\epsilon)}{L_i} \right) \right\}, 0 \right\} \frac{m \epsilon}{\min_{t,i} \{ a_i^t \alpha_i^t > 0 \}} ,
\]

where \( \delta(\epsilon) = \min_i \left\{ \frac{r_i}{(\alpha^t)^T \mathbf{1}} \right\} - \frac{\epsilon}{\min_{t,i} \{ a_i^t \alpha_i^t > 0 \}} \), and this upper bound converges to zero linearly when \( \epsilon \) tends to zero.

Proof: First, for notational simplicity we denote \( R(p) = \sum_{t \in V} (x(p|a, b, G) - c_t) p_t \). For (i), we have the following inequalities

\[
\begin{align*}
R(p^{\lambda^1}) - \sum_t \lambda_t^1 \max\{0, (\alpha^t)^T p^{\lambda^1} - r_t\} &\geq R(p^{\lambda^2}) - \sum_t \lambda_t^1 \max\{0, (\alpha^t)^T p^{\lambda^2} - r_t\} \\
&= R(p^{\lambda^2}) - \sum_t \lambda_t^2 \max\{0, (\alpha^t)^T p^{\lambda^2} - r_t\} - \epsilon \sum_t \max\{0, (\alpha^t)^T p^{\lambda^2} - r_t\} \\
&\geq R(p^{\lambda^1}) - \sum_t \lambda_t^2 \max\{0, (\alpha^t)^T p^{\lambda^2} - r_t\} - \epsilon \sum_t \max\{0, (\alpha^t)^T p^{\lambda^2} - r_t\}
\end{align*}
\]
So, we have
\[ \epsilon \sum_i \max \{0, (\alpha^i)^T p^\lambda - r_t \} \geq \sum_i (\lambda_i^1 - \lambda_i^2) \max \{0, (\alpha^i)^T p^\lambda - r_t \}, \]
which leads to the desired inequality.

For (ii), since \((\alpha^i)^T p^\star \leq r_t\) for all \(i\), given \(\lambda \in \mathbb{R}^T_+\), we have
\[ v^\lambda \geq R(p^\star) - \sum_i \lambda_i \max \{0, (\alpha^i)^T p^\star - r_t \} = R(p^\star) = v^\star. \]

Moreover, since \(v^\lambda - v^\star = R(p^\lambda) - v^\star - \sum_i \lambda_i \max \{0, (\alpha^i)^T p^\lambda - r_t \}\), we have
\[ R(p^\lambda) - v^\star \geq \sum_i \lambda_i \max \{0, (\alpha^i)^T p^\lambda - r_t \} \geq \lambda_0 \sum_i \max \{0, (\alpha^i)^T p^\lambda - r_t \}. \] (48)

The left hand side of (48) is less than \(\Delta^\star - v^\star\), so if we choose \(\lambda_0 \geq (\Delta^\star - v^\star)/\epsilon\) then \(\sum_i \max \{0, (\alpha^i)^T p^\lambda - r_t \} \leq \epsilon\) as desired.

We move to (iii). As shown previously, we can choose \(\lambda_0\) such that \(\max \{0, (\alpha^i)^T p^\lambda - r_t \} \leq \epsilon\) or \((\alpha^i)^T p^\lambda \leq r_t + \epsilon\) for all \(t \in [T]\). We now consider the following problem
\[ \max_{\sum_i p_i \geq 1} \left\{ R(p) \left| (\alpha^i)^T p \leq r_t + \epsilon, \forall t \right. \right\} \] (49)
and denote by \(p^\star\) as an optimal solution to (49). Since \(p^\lambda\) is feasible to (49) we have \(R(p^\star) \geq R(p^\lambda)\). Moreover, if we define \(\mathcal{P} := \{p \in \mathbb{R}^m | p_i \geq 0, \sum_i p_i \leq 1, (\alpha^i)^T p^\lambda \leq r_t, \forall t \in [T]\}\), then \(v^\star \geq R(p)\) for all \(p \in \mathcal{P}\). Therefore, we have
\[ |v^\lambda - v^\star| \leq R(p^\star) - R(p), \forall p \in \mathcal{P}. \] (50)

We will show that there is \(p \in \mathcal{P}\) such that \(\|p^\star - p\|\) can be arbitrarily small when \(\epsilon\) decreases, which allows us to use the Mean Value Theorem to bound \(|R(p^\star) - R(p)|\). If \(p^\star \notin \mathcal{P}\), let \(\mathcal{T} := \{t \in [T] | (\alpha^i)^T p^\star > r_t\}\) and for any \(t \in \mathcal{T}\) we select \(i_t = \arg\max_{i \in \mathcal{V}} \{p^\star_i | \alpha_t^i > 0\}\). Then we denote \(\mathcal{I} = \{i_t | t \in \mathcal{T}\}\). We pick a \(\tilde{p}\) such that
\[ \begin{cases} \tilde{p}_i = p^\star_i - \epsilon/(\min_{i \in \mathcal{I}} \{\alpha_t^i | \alpha_t^i > 0\}), & \forall i \in \mathcal{I} \\ \tilde{p}_j = p^\star_j, & \forall j \notin \mathcal{I}. \end{cases} \] (51)

With this selection, we see that, for any \(t \in \mathcal{T}\)
\[ (\alpha^i)^T \tilde{p} \leq (\alpha^i)^T p^\star - \alpha_t^i \epsilon/(\min_{i \in \mathcal{I}} \{\alpha_t^i | \alpha_t^i > 0\}) \leq (\alpha^i)^T p^\star - \epsilon \leq r_t. \] (52)

And indeed for any \(t \notin \mathcal{T}\) we have \((\alpha^i)^T \tilde{p} \leq (\alpha^i)^T p^\star \leq r_t\). Furthermore, for any \(t \in \mathcal{T}\), we have \((\alpha^i)^T p^\star > r_t\). Combine this with the fact that \(i_t = \arg\max_{i \in \mathcal{V}} \{p^\star_i | \alpha_t^i > 0\}\) we have
\[ \left( \sum_i \alpha_t^i \right) p^\star_{i_t} \geq (\alpha^i)^T p^\star > r_t. \]

So, under the assumption on the selection of \(\epsilon\), we have the chain of inequalities
\[ p^\star_{i_t} > \frac{r_t}{(\alpha^i)^T 1} \geq \min_t \left\{ \frac{r_t}{(\alpha^i)^T 1} \right\} \geq \frac{\epsilon}{\min_{t \in \mathcal{I}} \{\alpha_t^i | \alpha_t^i > 0\}}, \]
meaning that \(\tilde{p} > 0\). So, combine with (52) we have \(\tilde{p} \in \mathcal{P}\).
Moreover, for any point \( p' \in [\bar{p}, p'] \) and any \( i \in \mathcal{I} \), we have
\[
 p'_i \geq \bar{p}_i = p'_i - \frac{\epsilon}{\min_{i, i'} \{ \kappa_i^t | \kappa_i^t > 0 \}} > \min_{i} \left\{ \frac{r_i}{(\kappa_i^t)^T\kappa_i^t} \right\} - \frac{\epsilon}{\min_{i, i'} \{ \kappa_i^t | \kappa_i^t > 0 \}} := \delta(\epsilon).
\] (53)

So, if we denote \( x' = x(p', G) \) (i.e., the prices that result in purchase probabilities \( p' \)). For any \( i \in \mathcal{I} \), under the assumption that \( Y_i \partial G_i(Y) \leq L_i Y_i^t \), we have
\[
 L_i Y_i(x')^t_i \geq p'_i (1 + G(Y(x')))
\]
\[
 \geq p'_i \geq \delta(\epsilon),
\]
where \( Y(x') \) is a vector of size \( m \) with entries \( Y_j(x') = \exp(a_j - b_j x'_j) \), \( \forall j \in \mathcal{V} \). So we have
\[
 x'_i \leq \frac{a_i}{b_i} - \frac{1}{b_i L_i} \log \frac{\delta(\epsilon)}{L_i}.
\]
Moreover, if we look at the gradient of \( R(p) \) at \( p'_i \). According to Theorem 4.3 in Zhang et al. (2018) we have
\[
 \nabla_p R(p'_i) \leq x'_i \leq \max_j \left\{ \frac{a_j}{b_j} - \frac{1}{b_j L_j} \log \frac{\delta(\epsilon)}{L_j} \right\}.
\] (54)

Now, we look at \( |R(p') - R(\bar{p})| \) and by combining (51), (54), the Mean Value Theorem tells us that there is \( p' \in [\bar{p}, p'] \)
\[
 |R(p') - R(\bar{p})| = \sum_{i \in \mathcal{I}} \nabla_p R(p'_i) |\bar{p}_i - p'_i|
\]
\[
 \leq \max_i \left\{ \frac{a_i}{b_i} - \frac{1}{b_i L_i} \log \frac{\delta(\epsilon)}{L_i} \right\} \frac{m\epsilon}{\min_{i, i'} \{ \kappa_i^t | \kappa_i^t > 0 \}}.
\] (55)

Combine (55) with (50) and recall that \( \bar{p} \in \mathcal{P} \), we have
\[
 |\varphi^\Lambda - \varphi^*| \leq \max_i \left\{ \frac{a_i}{b_i L_i} - \frac{1}{b_i L_i} \log \frac{\delta(\epsilon)}{L_i} \right\} \frac{m\epsilon}{\min_{i, i'} \{ \kappa_i^t | \kappa_i^t > 0 \}}.
\]

Combine with the case \( p' \in \mathcal{P} \), we obtain the desired bound, which definitely converge to zero when \( \epsilon \) tends to zero, as desired. Q.E.D.

Now we are ready for the main proof.

**Proof of Theorem 6:** Using a similar evaluation as in (48) we can have
\[
 W(z(p^{G,\Lambda})) - \bar{\varphi} \geq \sum_t \lambda_t \max \{ 0, (d^t)^T p^{G,\Lambda} - r_t \} \geq \lambda_0 \sum_t \max \{ 0, (d^t)^T p^{G,\Lambda} - r_t \}
\]
and we also see that the left hand side of the above is less than \( \Delta^* - \bar{\varphi} \). Thus, if we choose \( \lambda_0 \geq (\Delta^* - \bar{\varphi})/\epsilon \), the we have \( \sum_t \lambda_t \max \{ 0, (d^t)^T p^{G,\Lambda} - r_t \} \leq \epsilon \).

To prove the second claim of the corollary, we can choose \( \lambda_0 \) such that \( \sum_t \lambda_t \max \{ 0, (d^t)^T p^{G,\Lambda} - r_t \} \leq \epsilon \) and consider \( p^{G,\epsilon} \) as an optimal solution to the following problem
\[
 \max_{p^{G} \in \mathcal{P}^{G}} \left\{ W(z(p^{G})) \left| (d^t)^T p^{G} \leq r_t + \epsilon, \forall t \right. \right\}.
\] (56)

Let us define \( \bar{\mathcal{P}}^G = \{ p^{G} \in \mathcal{P}^{G} | (d^t)^T p^{G} \leq r_t, \forall t \} \). Then, we have
\[
 |\varphi^{RO,\Lambda} - \bar{\varphi}| \leq W(z(p^{G,\epsilon})) - W(z(p^{G})), \forall p^{G} \in \bar{\mathcal{P}}^G.
\] (57)
We now try to bound $W(z(p^{G,c})) - W(z(p^G))$ using the Mean Value Theorem. We see that if $p^{G,c} \in \tilde{P}^G$ then $\varphi^{RO,A} = \varphi$. Otherwise, assume that $p^{G,c} \notin \tilde{P}^G$, let $T := \{t \in [T] \mid (d^t)^{T}p^{G,c} > r_t\}$ and for any $t \in T$ we select $n_t = \arg \max_{n \in [N]} \{p_n^{G,c} \mid d^n_t > 0\}$. We also denote $I = \{n_t \mid t \in T\}$. We pick a vector $\tilde{p}^G$ such that

\[
\begin{aligned}
\tilde{p}_n^{G,c} &= p_n^{G,c} - \epsilon/(\min_{t,k \in [N]} \{d^n_t \mid \alpha^n_k > 0\}), \quad \forall k \notin I \\
\tilde{p}_k^{G,c} &= p_k^{G,c}, \quad \forall k \notin I.
\end{aligned}
\]

Then for any $t \in T$ we have

\[
(d^t)^{T}p^G \leq (d^t)^{T}p^{G,c} - d^n_t \epsilon/(\min_{t,k \in [N]} \{d^n_t \mid d^n_k > 0\}) \leq (d^t)^{T}p^{G,c} - \epsilon \leq r_t.
\]

Now, for any $t \notin T$ we have $(d^t)^{T}p^G \leq (d^t)^{T}p^{G,c} \leq r_t$. Furthermore, for any $t \in T$, we have $(d^t)^{T}p^{G,c} > r_t$. Combine this with the selection of $n_t$ as $n_t = \arg \max_{n \in [N]} \{p_n^{G,c} \mid d^n_t > 0\}$ we have

\[
\left( \sum_{n} d^n_t \right) p^{G,c} > (d^t)^{T}p^{G,c} > r_t.
\]

So, from the selection of $\epsilon$, we have

\[
p_n^{G,c} > \frac{r_t}{(d^t)^{T}1} \geq \min_{t} \left\{ \frac{r_t}{(d^t)^{T}1} \right\} \geq \frac{\epsilon}{\min_{t,n} \{d^n_t \mid d^n_t > 0\}}.
\]

Thus, $\tilde{p}^G \in \tilde{P}^G$. Moreover, for any for any point $p' \in [\tilde{p}^G, p^{G,c}]$ and any $n \in I$, we have

\[
p'_n \geq \tilde{p}_n = p_n^{G,c} - \frac{\epsilon}{\min_{t,n} \{d^n_t \mid d^n_t > 0\}} \geq \min_{t} \left\{ \frac{r_t}{(d^t)^{T}1} \right\} - \frac{\epsilon}{\min_{t,n} \{d^n_t \mid d^n_t > 0\}} := \delta(\epsilon).
\]

Hence, if we denote $z' = z(p')$ (i.e., the prices that result in the purchase probabilities $p'$), then for any $n \in I$, under the assumption that $Y_n \partial G_t(Y) \leq L_i Y_i$, we have

\[
G^n(Y^n) = \sum_{i \in V_n} Y_n \partial G_i(Y) \leq \sum_{i \in V_n} L_i Y_i \leq |V_n| \sum_{i \in V_n} Y_i \leq |V_n|
\]

where $i_n \in V_n$ is chosen such that $L_{i_n} = \max_{i \in V_n} L_i Y_i$. We have

\[
L_{i_n} Y_{i_n} (z'^{i_n}) \geq p_n' / |V_n|.
\]

Moreover, we can write $Y(z')_{i_n} = \exp(a_{i_n} - b_{i_n} z_{i_n} - b_{i_n} c_{i_n})$, for a given $(a, b) \in A$. So we have

\[
z'_n \leq \max_{(a, b) \in A} \max_{i \in V_n} \left\{ \frac{a_{i_n} - b_{i_n} c_{i_n}}{b_{i_n}} - \log \frac{\delta(\epsilon)}{L_i} \right\}.
\]

Moreover, looking at the gradient of $W(z(p))$ at $p'$, we also see that there is a vector of parameters $(a, b) \in A$ such that

\[
\nabla_p W(z(p')) \leq z'_n \leq \max_{(a, b) \in A} \max_{i \in V_n} \left\{ \frac{a_{i_n} - b_{i_n} c_{i_n}}{b_{i_n}} - \log \frac{\delta(\epsilon)}{L_i} \right\}.
\]

Now, using the Mean Value Theorem, there is $p' \in [\tilde{p}^G, p^{G,c}]$ such that

\[
|W(z(p^{G,c})) - W(z(p^G))| = \sum_{n \in I} \nabla_p W(z(p')) \leq \sum_{n \in I} \left| p_n^{G,c} - \tilde{p}_n^{G,c} \right|
\]

\[
\leq \max_{(a, b) \in A} \max_{n \in [N]} \left\{ \frac{a_{i_n} - b_{i_n} c_{i_n}}{b_{i_n}} - \log \frac{\delta(\epsilon)}{L_i} \right\} \sum_{n \in I} \min_{n \in [N]} \{d^n_t \mid d^n_t > 0\}.
\]

Combine (61) with (57) and recall that $\tilde{p} \in \tilde{P}^G$, we have

\[
|\varphi^{RO,A} - \varphi| \leq \max_{(a, b) \in A} \max_{n \in [N]} \left\{ \frac{a_{i_n} - b_{i_n} c_{i_n}}{b_{i_n}} - \log \frac{\delta(\epsilon)}{L_i} \right\} \sum_{n \in I} \min_{n \in [N]} \{d^n_t \mid d^n_t > 0\}.
\]

Combine the case $p^{G,c} \in \tilde{P}^G$, we obtain the desired bound. Since $A$ is bounded, the left hand side of (62) will always converges to zero when $\epsilon$ tends to zero, as desired. \[\square\]
B.4. Experiments

Our goal here is to illustrate how the robust model with over-expected-sale penalties performs, as compared to other baseline approaches, i.e., deterministic and sampling-based counterparts. We employ the same nested logit model with partition-wise homogeneous PSP considered above. We create one expected sale constraint (i.e., $T = 1$) in such a way that the optimal prices from the unconstrained problem do not satisfy the expected sale constraint. We solve the deterministic problem with the weighted average parameters $\bar{w} = \sum_{k \in [K]} \tau_k w^k$ to obtain a solution $x^{DET}$. Then, for each uncertainty level $\epsilon > 0$ we solve the RO problem by convex optimization to obtain a robust solution $x^{RO}$. For the sampling-based approach, we also sample 10 and 50 points from the uncertainty set to get solutions $x^{SA10}$ and $x^{SA50}$, respectively. We do not select a large sample size for the sampling-based approach due to the fact that the number of points $s_1$ is also the number of convex optimization problems to be solved, and these optimization problems, even-though computationally tractable, are still expensive to be done.

To evaluate the performance of the solutions obtained, similarly to the other cases, we sample randomly and uniformly 1000 points from $\mathcal{A}$ and compute the corresponding expected revenues given by the four solutions $x^{DET}$, $x^{SA10}$, $x^{SA50}$ and $x^{RO}$. The distributions of the profit values (the expected revenue minus the penalty cost) for different $\lambda$ and $\epsilon$ are plotted in Figure 3, where similar observations apply. The histograms given by $x^{RO}$ always have higher peaks, smaller variances, shorter tails and get tighter as $\epsilon$ increases, as compared to the other solutions. The histograms given by $x^{SA50}$ are quite similar to those from $x^{SA10}$ and also have higher peaks and smaller variances, as compared to those from $x^{DET}$. In general, we also see that the RO approach always gives higher worst-case but lower average profits. The SA10 and SA50 also provide some protections against worst-case scenarios. The protection becomes better when the same size increases, which is rational given the fact that the minimax equality holds, thus a solution to the SA will approach a robust solution when the same size grows.
Figure 3  Distributions of the profit values under over-expected-sale penalties given by $x_{RO}^*, x_{DE}^*, x_{SA10}^*, x_{SA50}^*$. 