A spatial stochastic epidemic model: law of large numbers and central limit theorem

S. Bowong · A. Emakoua · E. Pardoux

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Abstract
We consider an individual-based SIR stochastic epidemic model in continuous space. The evolution of the epidemic involves the rates of infection and recovering of individuals. We assume that individuals move randomly on the two-dimensional torus according to independent Brownian motions. We define the sequences of empirical measures, which describe the evolution of the positions of the susceptible, infected and removed individuals. We prove the convergence in probability, as the size of the population tends to infinity, of those sequences of measures towards the solution of a system of parabolic PDEs. We show that appropriately centered renormalized sequences of fluctuations around the above limit converge in law, as the size of the population tends to infinity, towards a Gaussian distribution valued process, solution of a system of linear PDEs with highly singular Gaussian driving processes. In the case where the individuals do not move we also define and study the law of large numbers and central limit theorem for the same sequence.

Keywords Measure-valued process · Spatial stochastic epidemic models · Law of large numbers · Central limit theorem

A. Emakoua emakouaal@gmail.com
S. Bowong sbowong@gmail.com
E. Pardoux etienne.pardoux@univ-amu.fr

1 University of Douala, Douala, Cameroon
2 Institut de Mathématiques de Marseille, Marseille, France
1 Introduction

It is by now well understood that standard deterministic models (e.g. the well-known SIR model) are law of large numbers limits, as the size of the population tends to infinity, of homogeneous stochastic epidemic models, see e.g. Anderson and Britton [3] or Chapter 2 of Part I of Britton and Pardoux [7] for a recent account, where a central limit theorem is established. Note also that not only the CLT, but also moderate and large deviations can be used to describe the gap between stochastic models and their deterministic law of large numbers limit, see [7, 21] for their study in the case homogeneous epidemic models. On the other hand, there is by now a vast literature on spatial epidemic models, where some papers concentrate on the mathematical properties of the models, see e.g. Allen et al. [2], and others on the description of the effects of the spatial distribution of a population on the propagation of the epidemic, see e.g. Roques et al. [23].

We are concerned in the present paper with the study of the convergence of stochastic spatial epidemic models towards deterministic spatial epidemic models, as the size of the population tends to infinity. One approach, which has been explored by N’zi et al. [17], is to first consider the model on a discrete spatial grid, and to combine the limit as the size of the population tends to infinity, with the limit as the mesh size tends to 0. One advantage is that we obtain in the limit a model with local infection, which allows interactions between an infected and a susceptible individual only when both are located at the same point \( x \) in space.

In this paper, we consider the finite population stochastic model directly in continuous space. This forces us to introduce an interaction kernel, which allows infection of a susceptible individual by an infected individual where both are at a certain distance one from the other. One may want to scale that kernel in such a way that only local interactions are allowed in the infinite population limit, which is the kind of deterministic model which is commonly considered in the literature. However, this raises difficulties and remains as far as we know an open problem. Note that the fact that infection is a result of non local interaction is natural in certain contexts, notably connected with models without movement. In particular, plants do not move, but viruses are transported by the wind or parasites. Also, concerning human epidemics, the propagation of the epidemic from one region to another can be due to infected individuals visiting some regions for vacation, and then returning home. It is difficult to model such movements, where the individuals return to their starting point after some time. An alternative is to model such infection by a kernel, which allows infection at distance, and disregards the movement itself. Those applications are our motivation for considering spatial epidemic models without displacement of the individuals as one particular case. On the other hand, we will simplify our model for mathematical convenience, in that we consider our population distributed on the two dimensional torus, and the movement of the individuals to follow mutually independent Brownian motions. We study in this paper the law of large numbers and the central limit theorem for a stochastic SIR epidemic model in the two cases where each individual moves, and where there is no movement.

There is already a good number of papers which consider deterministic spatial SIR or SIS epidemic models, and we have already quoted two of them. However, the
connection between stochastic spatial individual based models and their law of large numbers limit is almost new. Kaj has established in [11] a law of large numbers limit which is similar to our, together with a large deviations result. Note that Kaj considers a mean field interaction which impacts the change of status of individuals (as we do), but also impacts the drift, unlike in our work. The details of the proof are not given in [11], so that is it difficult to compare the two approaches. Note that in both [11] and the present paper, the law of large numbers is not the main result. Lalley, Perkins and Zhang also study in [15] a scaling limit of a spatial epidemic, but under a scaling which is quite different from our, and their limit is a stochastic equation. As far as we know, our central limit theorem is completely new. Note however, that in a different context (where the parameter $x$ is not a spatial location, but denotes time since detection), Clémençon, Chi Tran and de Arazoza establish in [8] a central limit theorem for an epidemic model in a Sobolev space with negative index, as we do.

Let us now describe our model. We consider a population of size $N$ distributed on the torus $(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2)$ such that at any time each individual is either susceptible(S), infectious(I) or recovered(R). Let $S(t), I(t)$ and $R(t)$ denote the number of individuals in the different states at time $t$.

During the epidemic an individual $i$ moves on the torus according to the processes $\{X^i_t = \Pi(\tilde{X}^i_t), \ t \geq 0\}$, where $\Pi$ is the canonical projection from $\mathbb{R}^2$ to $\mathbb{T}^2$, $\tilde{X}^i_t = X^i_t + \sqrt{2} \gamma B^i_t$, with $\{X^i, \ 1 \leq i \leq N\}$ an independent and identically distributed family of random variables, globally independent of $\{B^i, \ 1 \leq i \leq N\}$, which in turn is a family of independent standard 2–dimensional Brownian motions ($\gamma$ is a positive constant in the first model and is zero in the second). We assume that at time $t=0$ (for $\gamma \geq 0$), that the population consists of two classes $S(0)$ and $I(0)$ described as follows. Let $A$ denote an arbitrary Borel subset of $\mathbb{T}^2$ and $0 < p \leq 1$, each individual $i$ is placed in $\mathbb{T}^2$ independently of the others at the position $X^i$, according to a given probability distribution $\nu$. If $X^i \in A^c$ then the individual $i$ is susceptible and if $X^i \in A$, the individual $i$ is infected with probability $p$ and susceptible with probability $1 - p$. This situation is modeled by the empirical measures

$$\mu^{S,N}_0 = \frac{1}{N} \sum_{i=1}^{N} \{1_A(X^i)(1 - \xi_i) + 1_{A^c}(X^i)\} \delta_{X^i},$$

$$\mu^{I,N}_0 = \frac{1}{N} \sum_{i=1}^{N} 1_A(X^i) \xi_i \delta_{X^i},$$

$$\mu^N_0 = \mu^{S,N}_0 + \mu^{I,N}_0 = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i},$$

where $\{\xi_i, \ 1 \leq i \leq N\}$ is a mutually independent family of $\text{Ber}(p)$ random variables, globally independent of $\{X^i, \ 1 \leq i \leq N\}$.

Let $K$ be a function defined by

$$K : \mathbb{T}^2 \times \mathbb{T}^2 \rightarrow \mathbb{R}_+ \quad (x, y) \mapsto k(d^2_{\mathbb{T}_2}(x, y)),$$ (1.1)
where \( k : \mathbb{R}_+ \to \mathbb{R}_+ \) is a given function, and for any \( x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{T}^2 \),

\[
d_{T^2}(x, y) = \inf_{\alpha \in B} \sqrt{(x_1 - y_1 - a_1)^2 + (x_2 - y_2 - a_2)^2},
\]

with \( B = \{b_1 = (0, 0), b_2 = (-1, 0), b_3 = (-1, 1), b_4 = (0, 1), b_5 = (1, 0), b_6 = (1, 1), b_7 = (0, -1), b_8 = (-1, -1), b_9 = (1, -1)\} \).

Let \( E_i^t \) be the state of the individual \( i \) at time \( t \), \( E_i^t \in \{S, I, R\} \) and \( \beta \) a positive constant. In the homogeneous model [7], the rate of infectious contacts can be thought of as a product of a rate \( c' \) at which each infectious individual has contacts with others, and the probability \( p' \) that such a contact results in an infection given that the other person is susceptible, which happens with the probability \( S_i/N \), where \( N \) is the total population size, because the contacted individual is chosen uniformly in the total population. However, in our case the rate \( P_{j,i} \) at which an infectious \( j \) has a contact with a susceptible \( i \) depends upon the distance that separates them, so \( P_{j,i} \) is proportional to \( K(X_j^t, X_i^t) \), thus

- \( P_{j,i} = \frac{\beta K(X_j^t, X_i^t)}{\sum_{i=1}^N K(X_j^t, X_i^t)} \).
- An infectious individual \( j \) has an infectious contact with some susceptible individual at the rate \( \beta \frac{\sum_{i=1}^N K(X_j^t, X_i^t)1_{\{E_i^t=S\}}}{\sum_{i=1}^N K(X_j^t, X_i^t)} \).

Hence the various infectious individuals have infectious contacts with susceptible individuals at the rate \( \beta \sum_{j=1}^N \frac{\sum_{i=1}^N K(X_j^t, X_i^t)1_{\{E_i^t=S\}}}{\sum_{i=1}^N K(X_j^t, X_i^t)} 1_{\{E_j^t=I\}} \).

The epidemic evolves according to the following rules:

- A susceptible \( i \) becomes infected at time \( t \) at the rate \( \beta 1_{\{E_i^t=S\}} \sum_{j=1}^N \frac{K(X_j^t, X_i^t)}{\sum_{i=1}^N K(X_j^t, X_i^t)} 1_{\{E_j^t=I\}} \).
- Each infected individual recovers at rate \( \alpha \), independently of the others, of the number of infected individual and of the respective positions of the individuals.

The evolution of the numbers of susceptible, infected and removed individuals is described by the following equations.

\[
S(t) = S(0) - P_{inf} \left( \beta \int_0^t \sum_{j=1}^N \frac{K(X_j^r, X_i^r)1_{\{E_i^r=S\}}}{\sum_{i=1}^N K(X_j^r, X_i^r)} 1_{\{E_j^r=I\}} dr \right),
\]

\[
I(t) = I(0) + P_{inf} \left( \beta \int_0^t \sum_{j=1}^N \frac{K(X_j^r, X_i^r)1_{\{E_i^r=S\}}}{\sum_{i=1}^N K(X_j^r, X_i^r)} 1_{\{E_j^r=I\}} dr \right) - P_{cu} \left( \alpha \int_0^t \sum_{j=1}^N 1_{\{E_j^t=I\}} dr \right),
\]

\[
R(t) = R(0) + P_{cu} \left( \alpha \int_0^t \sum_{j=1}^N 1_{\{E_j^t=I\}} dr \right).
\]
where $P_{inf}$ and $P_{cu}$ are two independent standard Poisson processes.

We now define the renormalized point processes:

\[
\begin{align*}
\mu_t^{S,N} &= \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{E_i^t = S\}} \delta_{X_i^t}, \\
\mu_t^{I,N} &= \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{E_i^t = I\}} \delta_{X_i^t}, \\
\mu_t^{R,N} &= \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{E_i^t = R\}} \delta_{X_i^t}, \\
\mu_t^N &= \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i^t} = \mu_t^{S,N} + \mu_t^{I,N} + \mu_t^{R,N}.
\end{align*}
\]

We will first study the law of large numbers and the central limit theorem of the initial conditions $(\mu_0^{S,N}, \mu_0^{I,N}, \mu_0^{N})_{N \geq 1}$. We shall next write the equation of evolution of $\mu_t^N$ and $(\mu_t^{S,N}, \mu_t^{I,N})$ and study the law of large numbers and the central limit theorem of those sequences when $\gamma$, the diffusion coefficient, is a positive constant, and finally when $\gamma$ is zero. The law of large number result will be a convergence result in the space of measure valued processes. The convergence proof will start with tightness in the appropriate space, identification of the limit of any weakly converging subsequence with the unique deterministic solution of a system of PDEs, from which the convergence in probability of the whole sequence will follow. The central limit theorem is technically more involved. The approximating sequence lives in the space of signed measures valued processes. But the limit takes its values in a larger space. The limit solves a linear Gaussian processes valued stochastic partial differential equation (abbreviated below SPDE). The regularity of the driving noise dictates the regularity of the solution. In the case without displacement the driving noise is the limit of the noises due to the infection and recovering events. These noises are processes with values in the Sobolev space $H^{-s}(\mathbb{T}^2)$, provided $s > 1$ (the definitions of the Sobolev spaces are recalled in the next section). In the case with displacement however, the limit of the displacement noise takes its values in $H^{-s}(\mathbb{T}^2)$, provided $s > 2$. In the latter case, the solution of the SPDE belongs a.s. to $C(\mathbb{R}_+; H^{-s}(\mathbb{T}^2))$, for any $s > 2$. Due to the regularizing effect of the heat kernel, it also belongs to $L^2_{loc}(\mathbb{R}_+; H^{-s+1}(-\mathbb{T}^2))$, for the same values of $s$. Note that whenever we need to have $s > 2$, we assume that $2 < s \leq 3$, and when we need to have $s > 1$, we assume that $1 < s \leq 2$. Since the computation of the norm in $H^s$ for $s$ not an integer is delicate, when we need to prove that a certain function belongs to some $H^s$ for $1 < s \leq 2$ (resp. for $2 < s \leq 3$), we shall prove that it belongs to $H^2$ (resp. to $H^3$).

The paper is organized as follows. In Sect. 2 we recall some results that will be useful in the sequel. In Sect. 3 we study the law of large numbers and the central limit theorem of the sequence $(\mu_0^{S,N}, \mu_0^{I,N}, \mu_0^{N})_{N \geq 1}$. In Sect. 4, for $\gamma > 0$, we first establish the evolution equations of the measure-valued process $\mu_t^{S,N}$ and $\mu_t^{I,N}$ then we show that $\mu_t^N$ converges in probability as $N \to \infty$ towards the processes $\{\mu_t, t \geq 0\}$ where for
each \( t \geq 0 \), \( \mu_t \) is the law \( \mathbb{X}^t \) and finally we prove that \( \{(\mu^S_t, \mu^L_t), t \geq 0\} \) converges in probability as \( N \to \infty \) towards \( (\mu^S, \mu^L) \) solution of a system of parabolic PDEs. In Sect. 5 we study the convergence of the renormalized sequences \( Z^N = \sqrt{N}(\mu^N - \mu), U^N = \sqrt{N}(\mu^S_N - \mu) \) and \( V^N = \sqrt{N}(\mu^L_N - \mu^I) \). Finally in Sect. 6 we assume that the individuals do not move \( (\gamma = 0) \) and study the law of large numbers and the central limit theorem of the sequence \( \{(\mu^S_t, \mu^L_t), N \geq 1, t \geq 0\} \).

2 Preliminaries

Notation:

- \( \mathcal{M}_F(\mathbb{T}^2) \) denotes the space of finite measures on \( \mathbb{T}^2 \).
- For any integer \( k \geq 0 \), \( C^k(\mathbb{T}^2) \) denotes the space of continuous and \( k \) times continuously differentiable real valued functions defined on \( \mathbb{T}^2 \).
- For \( \mu \in \mathcal{M}_F(\mathbb{T}^2) \) and \( \varphi \in C(\mathbb{T}^2) \), we denote the integral \( \int_{\mathbb{T}^2} \varphi(x) \mu(dx) \) by \( (\mu, \varphi) \).
- We define the Fortet distance on \( \mathcal{M}_F(\mathbb{T}^2) \) by \( \forall \mu, \nu \in \mathcal{M}_F(\mathbb{T}^2) \)

\[
d_F(\mu, \nu) = \sup_{f \in C(\mathbb{T}^2), \|f\|_\infty \leq 1, \|f\|_L \leq 1} |(\mu, f) - (\nu, f)|,
\]

where \( \|f\|_L = \sup_{x \neq y} (|f(x) - f(y)|)/d_{\mathbb{T}^2}(x, y) \).

This distance induces the topology of weak convergence, in other words the sequence of measures \( \mu_n \) converges weakly towards \( \mu \) if and only if

\[
\lim_{n \to \infty} d_F(\mu_n, \mu) = 0
\]

- In the following, the letter \( C \) will denote a (constant) positive real number which can change from line to line.
- We equip \( \mathcal{M}_F(\mathbb{T}^2) \) with the topology of weak convergence.
- Let \( \mathbb{X} \) be a complete separable metric space, \( C(\mathbb{R}^+, \mathbb{X}) \) (resp. \( D(\mathbb{R}^+, \mathbb{X}) \)) is a space of continuous (resp. càdlàg) functions from \( \mathbb{R}^+ \) to \( \mathbb{X} \), equipped with the locally uniform (resp. Skorokhod) topology. We refer the reader to section 12 of [5] for a presentation of the Skorokhod topology and its associated metric.

Definition 2.1 Sobolev spaces (see [1])

1. Let \( m \in \mathbb{N}^+ \), \( p \in \mathbb{R}^+ \). The Sobolev space \( W^{m,p}(\mathbb{T}^2) \) is defined by:

\[
W^{m,p}(\mathbb{T}^2) = \{ \varphi \in L^p(\mathbb{T}^2) : D^n\varphi \in L^p(\mathbb{T}^2), \forall \eta = (\eta_1, \eta_2) \in \mathbb{N}^2 \text{ such that } |\eta| = \eta_1 + \eta_2 \leq m \}
\]

where \( D^n\varphi \) is the weak derivative of the function \( \varphi \) with respect to the multi-index \( \eta \).

2. Let \( s = m + \sigma \) with \( n \in \mathbb{N} \) and \( \sigma \in [0, 1[. \) The Sobolev space \( W^{s,p}(\mathbb{T}^2) \) is defined as follows:

\[
W^{s,p}(\mathbb{T}^2) = \{ \varphi \in W^{m,p}(\mathbb{T}^2) \mid \sum_{|\eta| = n} \int_{\mathbb{T}^2 \times \mathbb{T}^2} \frac{|D^n(\varphi(x) - D^n\varphi(x'))|^2}{(d_{\mathbb{T}^2}(x, x'))^{2(1+\sigma)}} dx dx' < \infty \}.
\]
Notice that for \( p = 2 \) and \( s \in \mathbb{R}_+ \), \( W^{s, 2}(\mathbb{T}^2) \) is denoted \( H^s(\mathbb{T}^2) \) and is a Hilbert space.

**Definition 2.2 (White noise)** White noise on \( \mathbb{T}^2 \) is a random distribution \( \mathcal{W} \) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) which is such that the mapping \( \varphi \mapsto \langle \mathcal{W}, \varphi \rangle \) is linear and continuous from \( L^2(\mathbb{T}^2) \) into \( L^2(\Omega) \) and \( \{\langle \mathcal{W}, \varphi \rangle, \varphi \in L^2(\mathbb{T}^2)\} \) is a centered Gaussian generalized process satisfying:

\[
\mathbb{E}(\langle \mathcal{W}, \varphi \rangle \langle \mathcal{W}, \phi \rangle) = (\varphi, \phi)_{L^2}, \text{ for any } \varphi, \phi \in L^2(\mathbb{T}^2).
\]

Where \((., .)_{L^2}\) denotes a scalar product on \( L^2(\mathbb{T}^2) \).

Space-time white noise is a white noise on \( \mathbb{R}_+ \times \mathbb{T}^2 \).

**Proposition 2.3** (see [6]) Consider the following sets of functions.

\[
\mathcal{A}_1 = \{ f_{n_1, n_2}^i(x_1, x_2) = 2\sin(\pi n_1 x_1)\cos(\pi n_2 x_2), n_1 > 0, n_2 > 0 \text{ even} \}
\]

\[
\mathcal{A}_2 = \{ f_{n_1, n_2}^i(x_1, x_2) = 2\sin(\pi n_1 x_1)\sin(\pi n_2 x_2), n_1 > 0, n_2 > 0 \text{ even} \}
\]

\[
\mathcal{A}_3 = \{ f_{n_1, n_2}^i(x_1, x_2) = 2\cos(\pi n_1 x_1)\cos(\pi n_2 x_2), n_1 > 0, n_2 > 0 \text{ even} \}
\]

\[
\mathcal{A}_4 = \{ f_{n_1, n_2}^i(x_1, x_2) = 2\cos(\pi n_1 x_1)\sin(\pi n_2 x_2), n_1 > 0, n_2 > 0 \text{ even} \}
\]

\[
\mathcal{A}_5 = \{ f_{n_1, 0}^i(x_1, x_2) = \sqrt{2}\cos(\pi n_1 x_1), f_{0, n_2}^i(x_1, x_2) = \sqrt{2}\sin(\pi n_1 x_1), n_1 > 0 \text{ even} \}
\]

\[
\mathcal{A}_6 = \{ f_{0, n_2}^i(x_1, x_2) = \sqrt{2}\cos(\pi n_2 x_2), f_{0, n_2}^i(x_1, x_2) = \sqrt{2}\sin(\pi n_2 x_2), n_2 > 0 \text{ even} \}.
\]

For any \( \gamma > 0 \),

1. \( \{ f^0 = 1, (\mathcal{A}_i, i \in \{1, 2, 3, 4\}), \mathcal{A}_5, \mathcal{A}_6 \} \) are eigenfunctions of the operator \( \gamma \Delta \) on \( \mathbb{T}^2 \), with eigenvalues \( \{\lambda_0 = 0, -\lambda_{n_1, n_2} = -\gamma \pi^2 (n_1^2 + n_2^2), -\lambda_{n_1} = -\gamma \pi^2 n_1^2, -\lambda_{n_2} = -\gamma \pi^2 n_2^2 \} \), respectively, they form an orthonormal basis of \( L^2(\mathbb{T}^2) \).

2. \[
\begin{align*}
\{ \rho^0 = 1, \rho_{n_1, n_2}^{i, s} = \frac{f_{n_1, n_2}^i}{(1 + \gamma \pi^2 (n_1^2 + n_2^2))^s}, i \in \{1, 2, 3, 4\}, \\
\rho_{n_1, 0}^{i, s} = \frac{f_{n_1, 0}^i}{(1 + \gamma \pi^2 n_1^2)^s}, i \in \{5, 6\}, \\
\rho_{0, n_2}^{i, s} = \frac{f_{0, n_2}^i}{(1 + \gamma \pi^2 n_2^2)^s}, i \in \{7, 8\} \end{align*}
\]

is an orthonormal basis of \( H^s(\mathbb{T}^2) \).

**Proposition 2.4** Parseval identity (see the lemma 6.52 [1])

For any \( s > 0 \), let \( \varphi \in H^s(\mathbb{T}^2) \) and \( \Lambda \in H^{-s}(\mathbb{T}^2) \). If \((., .)_{L^2}\) denotes the scalar product in \( L^2(\mathbb{T}^2) \) and \((., .)\) the duality product between \( H^s(\mathbb{T}^2) \) and \( H^{-s}(\mathbb{T}^2) \), we have the following identities.

\[
\| \varphi \|^2_{H^s} = (\varphi, f^0)_{L^2}^2 + \sum_{i \in \{5, 6\}, n_1 > 0, \text{even}} (1 + \gamma \pi^2 n_1^2)^s (\varphi, f_{n_1, 0}^i)_{L^2}^2
\]
\[ + \sum_{i \in \{7, 8\}} (1 + \gamma \pi^2 n_2^2)^s (\varphi, f_{0,n_2}^i)^2_{L^2} \]

\[ + \sum_{i \in \{1, 2, 3, 4\}} (1 + \gamma \pi^2 (n_1^2 + n_2^2))^s (\varphi, f_{n_1,n_2}^i)^2_{L^2} \]

\[ = \sum_{i,n_1,n_2} (1 + \gamma \pi^2 (n_1^2 + n_2^2))^s (\varphi, f_{n_1,n_2}^i)^2_{L^2}. \]

\[ \| A \|_{H^{-s}} = \sup_{\varphi \neq 0, \varphi \in H^s} \frac{|(A, \varphi)|}{\| \varphi \|_{H^s}} \]

\[ \| A \|_{H^{-s}}^2 = |(A, \rho^0)|^2 + \sum_{i \in \{5, 6\}} |(A, \rho_{n_1,0}^i)|^2 \]

\[ + \sum_{i \in \{7, 8\}} |(A, \rho_{0,n_2}^i)|^2 + \sum_{i \in \{1, 2, 3, 4\}} |(A, \rho_{n_1,n_2}^i)|^2 \]

\[ = \sum_{i,n_1,n_2} |(A, \rho_{n_1,n_2}^i)|^2. \]

**Remark 2.5** In the above Proposition, the norms squared in the Sobolev spaces \( H^s \) and \( H^{-s} \) are written as complicated sums over the three indices \( i, n_1 \) and \( n_2 \). We shall write similar sums in the sequel, and refer to the above detailed expansion into three distinct sums, plus the index 0 term for the meaning of \( \sum_{i,n_1,n_2} \).

Similarly, \( \sum_{i,k_1,k_2} \) means the same combination of three distinct sums, plus the index 0 term, where in the first and the third one (resp. in the second and third ones), the summation over \( k_1 \) (resp. over \( k_2 \)) is restricted to all even integer between 2 and \( n_1 \) (resp. between 2 and \( n_2 \)).

**Proposition 2.6** *Sobolev injection (see [24] page 22)*

1. If \( s > k + 1 \) with \( k \in \mathbb{N} \) then \( H^s (\mathbb{T}^2) \subset C^k (\mathbb{T}^2) \) and \( \forall \varphi \in H^s (\mathbb{T}^2), \exists C(s) > 0 \) such that

\[ \| D^\eta \varphi \|_\infty \leq C(s) \| \varphi \|_{H^s}, \forall |\eta| \leq k. \]

2. If \( s > 1 \) then \( H^s (\mathbb{T}^2) \) is a Banach algebra, i.e \( \exists C > 0; \forall u, v \in H^s (\mathbb{T}^2), uv \in H^s (\mathbb{T}^2) \) and

\[ \| uv \|_{H^s} \leq C \| u \|_{H^s} \| v \|_{H^s}. \]

**Proposition 2.7** *Description of the Contraction Semigroup (see [6,13]).* Let \( \gamma > 0 \),
The operator $\gamma \Delta$ is selfadjoint, unbounded on $L^2(\mathbb{T}^2)$ and it is the infinitesimal generator of the semigroup $\Upsilon(t) = e^{\gamma t \Delta}$. Furthermore for all $\varphi \in L^2(\mathbb{T}^2)$,

$$
\Upsilon(t)\varphi = \sum_{i,n_1,n_2} \exp(-\lambda_{n_1,n_2} t)(\varphi, f_{n_1,n_2}^i)_{L^2} f_{n_1,n_2}^i.
$$

(2) $\forall s < 0$ (resp. $s > 0$) $\Upsilon(t)$ and $\gamma \Delta$ have an extension (resp. a restriction) to $H^s(\mathbb{T}^2)$ such that $\Upsilon(t)$ is a strongly-continuous contraction semigroup which is bounded on $H^s(\mathbb{T}^2)$ with $|\Upsilon(t)|_{L^2(H^s(\mathbb{T}^2))} \leq 1$, where $L(H^s(\mathbb{T}^2))$ is the space of continuous linear operator on $H^s(\mathbb{T}^2)$.

**Lemma 2.8** (see [13]) For any $s > 0$ and $\varphi \in H^s(\mathbb{T}^2)$ (resp. $\varphi \in L^\infty(\mathbb{T}^2)$), we have

$$
\| \Upsilon(t)\varphi \|_{H^s} \leq \| \varphi \|_{H^s} (\text{resp.} \| \Upsilon(t)\varphi \|_{\infty} \leq \| \varphi \|_{\infty}).
$$

Train denoting as above the canonical projection from $\mathbb{R}^2$ onto $\mathbb{T}^2$, $X_0$ being a $\mathbb{T}^2$-valued random vector, $\{B_t, \ t \geq 0\}$ a standard two-dimensional Brownian motion, and $\tilde{X}_t = X_0 + \sqrt{2\gamma} B_t$, $X_t = \Pi(\tilde{X}_t)$ is our diffusion process moving on $\mathbb{T}^2$. The semigroup $\Upsilon(t) = e^{\gamma t \Delta}$ being self-adjoint, it will act both on functions and on measures as follows. If $f \in C(\mathbb{T}^2)$,

$$
(\Upsilon(t)f)(x) = \mathbb{E}[f(X_t)|X_0 = x],
$$

and if $\mu_0$ denotes the law of $X_0$ then the collection $\{\mu_t, \ t \geq 0\}$ of the laws of $\{X_t, \ t \geq 0\}$ are given as

$$
\mu_t = \Upsilon(t)\mu_0.
$$

**Lemma 2.9** (A special case of theorem 2.1 in [14])

Let $(H, \| \cdot \|_H)$ be a separable Hilbert space, $M$ be an $H$-valued locally square integrable càdlàg martingale and $T(t)$ a contraction semigroup operator of $\mathcal{L}(H)$. Then there exists a finite constant $C$ depending only on the Hilbert norm $\| \cdot \|_H$ such that for all $T > 0$,

$$
\mathbb{E}\left( \sup_{0 \leq t \leq T} \left\| \int_0^t T(t - r) dM_r \right\|^2_H \right) \leq C e^{4\sigma T} \mathbb{E}\left( \| M_T \|^2_H \right),
$$

where $\sigma$ is a real number such that $\| T(t) \|_{\mathcal{L}} \leq e^{\sigma t}$.

Let $T > 0$. Let $F$ be a separable and reflexive Banach space included (with density and continuous injection) in a Hilbert space $H$. We identify $H$ with its dual. So $F \subset H \subset F'$, where $F'$ is the dual of $F$. Let $(A(t, \cdot))_{t \in [0,T]}$ be a family of linear operators from $F$ to $F'$, such that:

1. $\theta \rightarrow (A(t, u + \theta v), w)$ is continuous from $\mathbb{R}$ to $\mathbb{R}$, $\forall u, v, w \in F$
2. $\exists \delta > 0$, $\| A(t, u) \|_{F'} \leq \delta \| u \|_F$, $\forall u \in F$
(3) \( \exists \sigma_1 > 0, \sigma_2 \in \mathbb{R}, (A(t, u), u) + \sigma_2 \|u\|_H^2 \geq \sigma_1 \|u\|_F^2, \forall u \in F \)

(4) \( \forall u \in F, t \rightarrow A(t, u) \) est Lebesgue-mesurable with values in \( F' \), where \((.,.)\) is a duality product between \( F' \) and \( F \).

**Proposition 2.10** (see Theorem 1.1 page 81 in [18]) Let \((\Omega, \mathcal{F}_0, \mathbb{P})\) be a probability space, let \((A(t, .))_{t \in [0,T]}\) be a family of operators from \( F \) to \( F' \) which satisfy (1), (2), (3) and (4). Given \( u(0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}), f = f_1 + f_2 \) with

- \( f_1 \in L^2(\Omega, L^1(0, T, H)) \), progressively measurable,
- \( f_2 \in L^2(\Omega \times [0, T], F) \), progressively measurable,

and \((M_t)_{0 \leq t \leq T} \) a continuous square-integrable martingale with values in \( H \), the equation

\[
du(t) + A(u(t))dt = f(t)dt + dM(t), t \in [0, T] \text{ with } u(0) = u_0,
\]

admits a unique solution \( u \in L^2(\Omega \times [0, T], F) \cap L^2(\Omega, C([0, T], H)) \).

### 3 Law of Large Numbers and Central Limit Theorem for the Initial Measures

Recall that \( \mu_0^{S,N} = \frac{1}{N} \sum_{i=1}^{N} (1_A(X^i)(1 - \xi_i) + 1_{A^c}(X^i)) \delta_{X^i} \), \( \mu_0^{I,N} = \frac{1}{N} \sum_{i=1}^{N} 1_A(X^i) \delta_{X^i} \) and \( \mu_0^N = \mu_0^{S,N} + \mu_0^{I,N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i} \) where \( \{X^i, 1 \leq i \leq N\} \) is an independent identically distributed family of random variables, globally independent of \( \{\xi_i, 1 \leq i \leq N\} \), which in turn is a mutually independent family of \( \text{Ber}(p) \) random variables.

The following is assumed to be hold throughout this paper.

**Assumption (H0)** The law \( v \) of \( X^1 \) is absolutely continuous with respect to the Lebesgue measure and its density \( g \) satisfies:

\[
\text{there exist } \delta_1 > 0, \delta_2 > 0 \text{ such that } \delta_1 \leq g(x) \leq \delta_2, \forall x \in \mathbb{T}^2.
\]

#### 3.1 Law of large numbers of the initial measures

**Theorem 3.1** The sequence \((\mu_0^{S,N}, \mu_0^{I,N}, \mu_0^N)_{N \geq 1}\) converges a.s towards \((\mu_0^S, \mu_0^I, \mu_0)\) in \((\mathcal{M}_F(\mathbb{T}^2))^3\), where \( \mu_0^S(dx) = ((1 - p)1_A(x) + 1_{A^c}(x))v(dx) \), \( \mu_0^I(dx) = p1_A(x)v(dx) \) and \( \mu_0(dx) = v(dx) \).

**Proof** All we need is to prove that for any \( \varphi, \psi, \phi \in C(\mathbb{T}^2) \) the sequence

\[
\left( (\mu_0^{S,N}, \varphi), (\mu_0^{I,N}, \psi), (\mu_0^N, \phi) \right)_{N \geq 1}
\]

converges a.s towards \(( (\mu_0^S, \varphi), (\mu_0^I, \psi), (\mu_0, \phi) ) \).

Let \( \varphi, \psi, \phi \in C(\mathbb{T}^2) \), we have

\[
(\mu_0^{S,N}, \varphi) = \frac{1}{N} \sum_{i=1}^{N} [1_A(X^i)(1 - \xi_i) + 1_{A^c}(X^i)]\varphi(X^i)
\]
\[
(\mu_{0}^{I,N}, \psi) = \frac{1}{N} \sum_{i=1}^{N} 1_{A}(X^{i}) \xi_{i} \psi(X^{i})
\]

Furthermore, according to the law of large numbers

\[
(\mu_{0}^{S,N}, \varphi) \xrightarrow{a.s.} \mathbb{E}((1_{A}(X^{1})(1 - \xi_{1}) + 1_{A^{c}}(X^{1}))\varphi(X^{1}))
\]

\[
= \int_{\mathbb{T}^{2}} (1 - p) 1_{A}(x) + 1_{A^{c}}(x) \varphi(x) v(dx),
\]

\[
(\mu_{0}^{I,N}, \psi) \xrightarrow{a.s.} \mathbb{E}(1_{A}(X^{1}) \xi_{1} \psi(X^{1})) = p \int_{A} \psi(x) v(dx),
\]

\[
(\mu_{0}^{N}, \phi) \xrightarrow{a.s.} \mathbb{E}(\phi(X^{1})) = \int_{\mathbb{T}^{2}} \phi(x) v(dx).
\]

Thus

\[
\left( (\mu_{0}^{S,N}, \varphi), (\mu_{0}^{I,N}, \psi), (\mu_{0}^{N}, \phi) \right) \xrightarrow{a.s.} \left( (1 - p) \int_{A} \varphi(x) v(dx) + \int_{A^{c}} \varphi(x) v(dx),
\right.
\]

\[
\left. p \int_{A} \psi(x) v(dx), \int_{\mathbb{T}^{2}} \phi(x) v(dx) \right).
\]

3.2 Central limit theorem of the initial measures

We define \( U_{0}^{N} = \sqrt{N}(\mu_{0}^{S,N} - \mu_{0}^{S}) \), \( V_{0}^{N} = \sqrt{N}(\mu_{0}^{I,N} - \mu_{0}) \), \( Z_{0}^{N} = \sqrt{N}(\mu_{0}^{N} - \mu_{0}) \). In this section we study the convergence of the sequence \((U_{0}^{N}, V_{0}^{N}, Z_{0}^{N})\) in \((H^{-s}(\mathbb{T}^{2}))^{3}\), as \( N \to \infty \), with \( s > 1 \).

**Proposition 3.2** For any \( s > 1 \), there exists \( C_{1}, C_{2}, C_{3} > 0 \) such that

\[
\sup_{N \geq 1} \mathbb{E}(\|Z_{0}^{N}\|_{H^{-s}}^{2}) < C_{1}; \quad \sup_{N \geq 1} \mathbb{E}(\|U_{0}^{N}\|_{H^{-s}}^{2}) < C_{2} \quad \text{and} \quad \sup_{N \geq 1} \mathbb{E}(\|V_{0}^{N}\|_{H^{-s}}^{2}) < C_{3}.
\]

**Proof** We only prove that \( \sup_{N \geq 1} \mathbb{E}(\|V_{0}^{N}\|_{H^{-s}}^{2}) < C_{3} \). The other estimates follow by a similar argument. Since \( 1_{A}(X_{j}) \xi_{j} \delta_{X_{j}} \) are i.i.d with law \( \mu_{0}^{I} \), from assumption (H0) and Lemma 7.1 in the “Appendix” below, we have, since \( s > 1 \),

\[
\mathbb{E}(\|V_{0}^{N}\|_{H^{-s}}^{2}) = \mathbb{E}(\sum_{i,n_{1},n_{2}} (V_{0}^{N}, \rho_{n_{1},n_{2}}^{i,s}))
\]
Let us now state the main result of this section.

**Theorem 3.3** For any \( s > 1 \), the sequence \((U_0^N, V_0^N, Z_0^N)_{N \geq 1}\) converges in law in \((H^{-s}(\mathbb{T}^2))^3\) towards \((U_0, V_0, Z_0)\) where \( \forall \varphi, \psi, \phi \in H^s(\mathbb{T}^2), ((U_0, \varphi), (V_0, \psi), (Z_0, \phi)) \) is a Gaussian vector which satisfies:

\[
(U_0, \varphi) = W_1[\varphi \sqrt{g} (1 - p) 1_A + 1_{A^c} ] - (1 - p) W_1(\sqrt{g}) \int_A \varphi(x) g(x) dx \\
- W_1(\sqrt{g}) \int_{A^c} \varphi(x) g(x) dx \\
+ W_2(1_A \varphi \sqrt{(p - p^2)g}), \tag{3.1}
\]

\[
(V_0, \psi) = p W_1(1_A \psi \sqrt{g}) - p W_1(\sqrt{g}) \int_A \psi(x) g(x) dx - W_2(1_A \psi \sqrt{(p - p^2)g}), \tag{3.2}
\]

\[
(Z_0, \phi) = W_1(\phi \sqrt{g}) - W_1(\sqrt{g}) \left( \int_{\mathbb{T}^2} \phi(x) g(x) dx \right), \tag{3.3}
\]

where \( W_1, W_2 \) are mutually independent two dimensional white noises.

### 3.3 Proof of Theorem 3.3

We first prove the tightness of the sequence \((U_0^N, V_0^N, Z_0^N)_{N \geq 1}\), then identify the limit.
1.69 page 47 of [4]), \(B_{H^{-s'}} = \{ \mu \in H^{-s'}; \| \mu \|_{H^{-s'}} \leq R \} \), which is a closed and bounded subset of \(H^{-s'}\), is a compact subset of \(H^{-s}\). Thus
\[
\mathbb{P}(U_0^N \notin B_{H^{-s'}}) = \mathbb{P}(\|U_0^N\|_{H^{-s'}} > R) \leq \frac{1}{R^2} \mathbb{E}(\|U_0^N\|^2_{H^{-s'}}) \leq \frac{C_1}{R^2},
\]
for any \(N \geq 1\). By choosing \(R\) arbitrarily large, we make the right hand side as small as we wish, which yields the result. The tightness of \((V_0^N)_{N \geq 1}\) and \((Z_0^N)_{N \geq 1}\) are obtained by similar arguments. \(\Box\)

**Remark 3.5** Note that, in order to obtain tightness in the space \(H^{-s}\), we need to exploit an a priori estimate on the second moment of the norm of \(U_0^N\) in a slightly smaller space, i.e. in \(H^{-s'}\) with \(s' < s\) (and we need to restrict to \(s' > 1\), for Proposition 3.2 to be applicable). We shall use again below in Sect. 5 the same method for the proof of tightness of the processes \(U_t^N, V_t^N\) and \(Z_t^N\).

From Proposition 3.4 we deduce that the sequence \((U_0^N, V_0^N, Z_0^N)_{N \geq 1}\) is tight in \((H^{-s})^3\), thus by Prokhorov’s theorem there exists a subsequence still denoted \((U_0^N, V_0^N, Z_0^N)_{N \geq 1}\) which converges in law towards \((U_0, V_0, Z_0)\) in \((H^{-s})^3\).

### 3.3.2 The laws of \((U_0, V_0, Z_0)\)

Let \(\varphi, \psi, \phi \in H^s(\mathbb{T}^2), s > 1\). It follows from the classical central limit theorem that the random vector \(((U_0^N, \varphi), (V_0^N, \psi), (Z_0^N, \phi))\) converges in law, as \(N \rightarrow \infty\), to a centered Gaussian random vector, whose covariance matrix we will now compute.

– Computation of \(\text{Var}((U_0, \varphi))\). We have
\[
(U_0^N, \varphi) = \sqrt{N} \left[ \frac{1}{N} \sum_{i=1}^N \{1_A(X^i)(1 - \xi_i) + 1_{A^c}(X^i)\}\varphi(X^i) \right.
\]
\[
- (1 - p) \int_A \varphi(x) dx - \int_{A^c} \varphi(x) dx \bigg],
\]
and
\[
\text{Var}[\{1_A(X^1)(1 - \xi_1) + 1_{A^c}(X^1)\}\varphi(X^1)]
\]
\[
= \text{Var}[1_A(X^1)(1 - \xi_1)\varphi(X^1)] + \text{Var}[1_{A^c}(X^1)\varphi(X^1)] - 2\mathbb{E}[1_A(X^1)(1 - \xi_1)\varphi(X^1)\mathbb{E}(1_{A^c}(X^1)\varphi(X^1))]
\]
\[
= (1 - p) \int_A \varphi^2(x)g(x)dx - (1 - p)^2 \left( \int_A \varphi(x)g(x)dx \right)^2
\]
\[
+ \int_{A^c} \varphi^2(x)g(x)dx - \left( \int_{A^c} \varphi(x)g(x)dx \right)^2
\]
\[-2(1 - p) \int_A \varphi(x)g(x)dx \int_{A^c} \varphi(x)g(x)dx \]

\[:= \alpha_p^2 = \text{Var}(U_0, \varphi).\]

- Computation of \(\text{Var}((V_0, \psi))\). We have

\[\text{Var}[1_A(X^1)\xi_1\psi(X^1)] = p \int_A \psi^2(x)g(x)dx - p^2 \left( \int_A \psi(x)g(x)dx \right)^2\]

\[\beta_p^2 = \text{Var}(V_0, \psi).\]

- It is plain that \(\text{Var}((Z_0, \phi)) = \int_{T^2} \phi^2(x)g(x)dx - \left( \int_{T^2} \phi(x)g(x)dx \right)^2 = \sigma^2 = \text{Var}(Z_0, \phi).\)

- The following formulas for the covariances are easy to justify.

\[\text{Cov}((U_0, \varphi), (V_0, \psi)) = \text{Cov} \left( [1_A(X^1)(1 - \xi_1) + 1_{A^c}(X^1)]\varphi(X^1), 1_A(X^1)\xi_1\psi(X^1) \right)\]

\[\gamma_p = \int_A \psi(x)g(x)dx\]

\[:= y_p,\]

\[\text{Cov}((U_0, \varphi), (Z_0, \phi)) = \text{Cov} \left( [1_A(X^1)(1 - \xi_1) + 1_{A^c}(X^1)]\varphi(X^1), \phi(X^1) \right)\]

\[= (1 - p) \left\{ \int_A \varphi(x)\phi(x)g(x)dx - \int_A \varphi(x)g(x)dx \int_{T^2} \phi(x)g(x)dx \right\}\]

\[+ \int_{A^c} \varphi(x)\phi(x)g(x)dx - \int_{A^c} \varphi(x)g(x)dx \int_{T^2} \phi(x)g(x)dx\]

\[:= \eta_p,\]

\[\text{Cov}((V_0, \psi), (Z_0, \phi)) = \text{Cov} \left( 1_A(X^1)\xi_1\psi(X^1), \phi(X^1) \right)\]

\[= p \int_A \psi(x)\phi(x)g(x)dx \]

\[+ p \int_A \psi(x)g(x)dx \int_{T^2} \phi(x)g(x)dx\]

\[:= \xi_p,\]
Thus the law of the Gaussian random variable \(((U_0, \varphi) + (V_0, \psi) + (Z_0, \phi))\) is the law \(N(0, \alpha_p^2 + \beta_p^2 + \sigma_p^2 + 2(\gamma_p + \lambda_p + \eta_p))\). From this, we conclude that for any \(\varphi, \psi, \phi \in H^s\), \(s > 1\), \(((U_0, \varphi), (V_0, \psi), (Z_0, \phi))\) is a Gaussian vector with the same law as the vector given by (3.1), (3.2) and (3.3).

### 3.3.3 Conclusion

Since the law of the limit of any converging subsequence is uniquely characterized, for any \(s > 1\), the whole sequence \((U_0^N, V_0^N, Z_0^N)_{N \geq 1}\) converge in law in \((H^{-s})^3\) towards \((U_0, V_0, Z_0)\).

### 4 Law of large numbers

The aim of this section is to study the convergence of \((\mu^{S,N}, \mu^{I,N})\) under Assumption (H1) below, and the convergence of \(\mu^N\) as \(N \to \infty\).

To this end we are going to:

- Write the system of evolution equations of \((\mu^{S,N}, \mu^{I,N})\).
- Study the convergence of \([\mu^N_t, t \geq 0]\) in \(C(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2))\).
- Study the tightness of \((\mu^{S,N}, \mu^{I,N})_{N \geq 1}\) in Skorokhod’s space \((D(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2)))^2\).
- Show that all limit points \(\mu^S\) and \(\mu^I\) of \((\mu^{S,N})_{N \geq 1}\) and \((\mu^{I,N})_{N \geq 1}\) are absolutely continuous with respect to the Lebesgue measure with density \(f_S\) and \(f_I\) bounded by \(\delta_2\) (\(\delta_2\) is defined in Sect. 3).
- Show that the system of PDEs verified by the pair \((f_S, f_I)\) admits a unique solution in

\[
\Lambda = \{(f_1, f_2)/0 \leq f_i \leq \delta_2, i \in \{1, 2\}\}.
\]

The following is assumed to hold throughout Sect. 4. Recall the function \(k\) which was first introduced in (1.1).

**Assumption (H1)** \(k\) is Lipschitz, with the Lipschitz constant \(C_k\).

#### 4.1 System of evolution equations of \([\mu^{S,N}_t, \mu^{I,N}_t], t \geq 0\)

In this subsection, we shall establish the following result.

**Proposition 4.1** For any \(\varphi \in C^2(\mathbb{T}^2)\), the pair \(((\mu^{S,N}_t, \varphi), (\mu^{I,N}_t, \varphi))\) satisfies

\[
(\mu^{S,N}_t, \varphi) = (\mu^{S,N}_0, \varphi) + \gamma \int_0^t (\mu^{S,N}_r, \Delta \varphi) dr - \beta \int_0^t \left(\mu^{S,N}_r, \varphi(\mu^{I,N}_r, \frac{K}{(\mu^{N}_r, K)})\right) dr + M^{N, \varphi}_t,
\]

(4.1)
\[(\mu_{t}^{I,N}, \varphi) = (\mu_{0}^{I,N}, \varphi) + \gamma \int_{0}^{t} (\mu_{r}^{I,N}, \Delta \varphi) dr + \beta \int_{0}^{t} (\mu_{r}^{S,N}, \varphi(\mu_{r}^{I,N}, \frac{K}{(\mu_{r}^{N}, K)})) dr - \alpha \int_{0}^{t} (\mu_{r}^{I,N}, \varphi) dr + L_{t}^{N,\varphi}, \quad (4.2)\]

where,

\[(\mu_{r}^{S,N}, \varphi(\mu_{r}^{I,N}, \frac{K}{(\mu_{r}^{N}, K)})) = \int_{\mathbb{T}^2} \varphi(x) \int_{\mathbb{T}^2} K(y, z) \mu_{r}^{N}(dz) \mu_{r}^{S,N}(dy) \mu_{r}^{I,N}(dx),\]

and with \{M^{i}\}_{1 \leq i \leq N}, \{Q^{i}\}_{1 \leq i \leq N} two collections of standard (i.e. with mean the Lebesgue measure) Poisson random measures (abbreviated below PRM) on \(\mathbb{R}^2\), which are such that \(B^{1}, M^{1}, Q^{1}, \ldots, B^{N}, M^{N}, Q^{N}\) are mutually independent, and denoting by \(\overline{M}^{i}\) and \(\overline{Q}^{i}\) the compensated PRMs associated to \(M^{i}\) and \(Q^{i}\), we have

\[M_{t}^{N,\varphi} = -\frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} \int_{0}^{\infty} 1_{\{E_{r}^{i} = S\}} \varphi(X_{r}^{i}) 1_{\{u \leq \beta \sum_{j=1}^{N} \frac{K(x, z)}{K(x, x_j)} \}} \overline{M}^{i}(dr, du)\]

\[+ \sqrt{2\gamma} \sum_{i=1}^{N} \int_{0}^{t} \nabla \varphi(X_{r}^{i}) dB_{r}^{i},\]

\[L_{t}^{N,\varphi} = \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} \int_{0}^{\infty} 1_{\{E_{r}^{i} = S\}} \varphi(X_{r}^{i}) 1_{\{u \leq \beta \sum_{j=1}^{N} \frac{K(x, z)}{K(x, x_j)} \}} \overline{M}^{i}(dr, du)\]

\[+ \sqrt{2\gamma} \sum_{i=1}^{N} \int_{0}^{t} \nabla \varphi(X_{r}^{i}) dB_{r}^{i}\]

\[- \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} \int_{0}^{\alpha} 1_{\{E_{r}^{i} = I\}} \varphi(X_{r}^{i}) \overline{Q}^{i}(dr, du).\]

**Proof** We recall that \(\Pi\) is the canonical projection from \(\mathbb{R}^2\) on \(\mathbb{T}^2\), \(\tilde{X}_{r}^{i} = X_{r}^{i} + \sqrt{2\gamma} B_{r}^{i}\) and \(X_{r}^{i} = \Pi(\tilde{X}_{r}^{i})\). Now Let \(\varphi \in C^{2}(\mathbb{T}^2)\), if we let \(\tilde{\varphi} = \varphi \circ \Pi\). According to the Itô formula, we have

\[\tilde{\varphi}(\tilde{X}_{r}^{i}) = \varphi(X_{r}^{i}) + \sqrt{2\gamma} \int_{0}^{t} \nabla \varphi(\tilde{X}_{r}^{i}) dB_{r}^{i} + \gamma \int_{0}^{t} \Delta \varphi(\tilde{X}_{r}^{i}) dr,\]

hence

\[\varphi(X_{r}^{i}) = \varphi(X_{r}^{i}) + \sqrt{2\gamma} \int_{0}^{t} \nabla \varphi(X_{r}^{i}) dB_{r}^{i} + \gamma \int_{0}^{t} \Delta \varphi(X_{r}^{i}) dr.\]
Thus \( \{1_{E_i=S}\} \varphi(X^i_I), \ t \geq 0 \) is a jump diffusion process satisfying

\[
1_{E_i=S} \varphi(X^i_I) = 1_{E_0=S} \varphi(X^i_0) + \sqrt{2\gamma} \int_0^t 1_{E_i=S} \nabla \varphi(X^i_r) dB^i_r
+ \gamma \int_0^t 1_{E_i=S} \Delta \varphi(X^i_r) dr
- \int_0^t \int_0^\infty \left\{ \sum_{u \leq \beta} N \sum_{j=1}^{N} \sum_{i=1}^{N} \kappa(X^i_j, X^j_i) 1_{E^j_i=I} \right\} 1_{E_i=S} \varphi(X^i_r) M^i(du, dr).
\]

Taking the sum over \( i \) and multiplying by \( \frac{1}{N} \) we obtain

\[
\frac{1}{N} \sum_{i=1}^{N} 1_{E_i=S} \varphi(X^i_I) = \frac{1}{N} \sum_{i=1}^{N} 1_{E_0=S} \varphi(X^i_0) + \sqrt{2\gamma} \frac{1}{N} \sum_{i=1}^{N} \int_0^t 1_{E_i=S} \nabla \varphi(X^i_r) dB^i_r
+ \frac{\gamma}{N} \sum_{i=1}^{N} \int_0^t 1_{E_i=S} \Delta \varphi(X^i_r) dr
- \frac{1}{N} \sum_{i=1}^{N} \int_0^t \int_0^\infty \left\{ \sum_{u \leq \beta} \sum_{j=1}^{N} \sum_{i=1}^{N} \kappa(X^i_j, X^j_i) 1_{E^j_i=I} \right\} 1_{E_i=S} \varphi(X^i_r) \overline{M}^i(du, dr)
- \frac{1}{N} \sum_{i=1}^{N} \int_0^t \frac{\beta}{N} \sum_{j=1}^{N} \frac{K(X^i_j, X^j_i)}{N} \sum_{i=1}^{N} K(X^i_i, X^i_i) 1_{E^i_j=I} 1_{E_i=S} \varphi(X^i_r) dr
\]

from which (4.1) follows. Similarly, with again \( \varphi \in C^2(\mathbb{T}^2) \), \( \{1_{E_i=I}\} \varphi(X^i_I), \ t \geq 0 \) is a jump diffusion process satisfying

\[
1_{E_i=I} \varphi(X^i_I) = 1_{E_0=I} \varphi(X^i_0) + \sqrt{2\gamma} \int_0^t 1_{E_i=I} \nabla \varphi(X^i_r) dB^i_r
+ \gamma \int_0^t 1_{E_i=I} \Delta \varphi(X^i_r) dr
+ \int_0^t \int_0^\infty \left\{ \sum_{u \leq \beta} \sum_{j=1}^{N} \sum_{i=1}^{N} \kappa(X^i_j, X^j_i) 1_{E^j_i=I} \right\} 1_{E_i=I} \varphi(X^i_r) M^i(du, dr)
- \int_0^t \int_0^\infty 1_{E_i=I} \varphi(X^i_r) Q^i(du, dr).
\]
Summing over $i$ and multiplying by $\frac{1}{N}$ we obtain

$$
\frac{1}{N} \sum_{i=1}^{N} 1_{E_i^1} \varphi(X_i^1) = \frac{1}{N} \sum_{i=1}^{N} 1_{E_0^1} \varphi(X_i^1) + \sqrt{2\gamma} \sum_{i=1}^{N} \int_{0}^{\tau} 1_{E_i^1} \nabla \varphi(X_r^i) dB_r^i
$$

$$
+ \frac{\gamma}{N} \sum_{i=1}^{N} \int_{0}^{\tau} 1_{E_i^1} \Delta \varphi(X_r^i) dr
$$

$$
+ \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{\tau} \int_{0}^{\infty} 1_{\mu_i^1,\varphi} \sum_{j=1}^{N} \frac{K(X_i^r, X_j^r)}{\sum_{j=1}^{N} K(X_i^r, X_j^r)} 1_{E_j^1} \varphi(X_r^i) M_{t}^j(du, dr)
$$

$$
+ \frac{\beta}{N} \sum_{i=1}^{N} \int_{0}^{\tau} \int_{1}^{N} \frac{K(X_i^r, X_j^r)}{\sum_{j=1}^{N} K(X_i^r, X_j^r)} 1_{E_j^1} \varphi(X_r^i) dr du
$$

$$
- \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{\tau} \int_{0}^{\alpha} 1_{\mu_i^1,\varphi} \varphi(X_r^i) \mathcal{Q}_{t}^i(du, dr) - \frac{\alpha}{N} \sum_{i=1}^{N} \int_{0}^{\tau} 1_{E_i^1} \varphi(X_r^i) dr,
$$

from which (4.2) follows. \qed

### 4.2 Convergence of $\{\mu_t^N, t \geq 0\}_{N \geq 1}$ in $C(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2))$

Recall that we equip $\mathcal{M}_F(\mathbb{T}^2)$ with the topology of weak convergence and the space of continuous functions from $\mathbb{R}_+$ to $\mathcal{M}_F(\mathbb{T}^2)$, denoted $C(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2))$, with the locally uniform topology.

It follows from the Itô formula that the processes $\{\mu_t^N, t \geq 0\}$ satisfies $(\mu_t^N, \varphi) = (\mu_0^N, \varphi) + \gamma \int_{0}^{\tau} (\mu_t^N, \Delta \varphi) dt + \mathcal{H}_{t, \varphi}^N$, where $\mathcal{H}_{t, \varphi}^N = \sqrt{2\gamma} \sum_{i=1}^{N} \int_{0}^{\tau} \nabla \varphi(X_r^i) dB_r^i$, is a continuous martingale and $<\mathcal{H}_{t, \varphi}^N>_t = 2\gamma \int_{0}^{\tau} (\mu_t^N, \nabla \varphi)^2 dt$. Note that, contrary to the processes $(\mu_t^S, \varphi)$ and $(\mu_t^I, \varphi)$, the process $(\mu_t^N, \varphi)$ is continuous, since that process does not take into account the status of the individuals.

**Proposition 4.2** The sequence $\{\mu_t^N, t \geq 0, N \geq 1\}$ converges in probability in $C(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2))$ towards $\{\mu_t, t \geq 0\}$, where for each $t \geq 0$, $\mu_t$ is the law of $X_t^1$ and for any $\varphi \in C^2(\mathbb{T}^2)$, $t \geq 0$, $(\mu_t, \varphi) = (\mu_0, \varphi) + \gamma \int_{0}^{\tau} (\mu_r, \Delta \varphi) dt$.

**Proof** We first prove that the sequence $\{\mu_t^N, t \geq 0; N \geq 1\}$ converges in law towards $\{\mu_t, t \geq 0\}$.

We refer to Theorem 2.2 and Remark page 58 of Roelly [22]. Let $\Xi$ be a dense subset of $C(\mathbb{T}^2)$. In order to prove that $(\mu_t^N)_{N \geq 1}$ converges in law in $C(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2))$ towards $\{\mu_t, t \geq 0\}$ it is enough to prove that:

1. $\forall \varphi \in \Xi, \{(\mu_t^N, \varphi), t \geq 0\}_{N \geq 1}$ is tight in $C(\mathbb{R}_+, \mathbb{R}_+)$.

\[ \Xi \] Springer
2- For any \( m \geq 1 \), any \( (t_1, t_2, \ldots, t_m) \in \mathbb{R}_+^m \), and any \( (\varphi_1, \varphi_2, \ldots, \varphi_m) \in (\Pi)^m \) the sequence \( (\mu_{t_1}^N \varphi_1, \ldots, \mu_{t_m}^N \varphi_m) \) converges in law in \( \mathbb{R}^m \) towards \( ((\mu_{t_1}, \varphi_1), \ldots, (\mu_{t_m}, \varphi_m)) \)

Proof of 1. We choose \( \Xi = C^2(\mathbb{T}^2) \). Let \( \varphi \in C^2(\mathbb{T}^2) \), due to Proposition 37 of Pardoux [20], a sufficient condition for the sequence \( (\mu_t^N, \varphi)_{N \geq 1} \) to be tight in \( C(\mathbb{R}_+, \mathbb{R}_+) \) is that both

- \( (\mu_0^N, \varphi) \) is tight in \( \mathbb{R} \),
- \( \forall T > 0, \sup_{0 \leq t \leq T} (| (\mu_t^N, \Delta \varphi) | + \frac{1}{N} (\mu_t^N, (\nabla \varphi)^2)) \) is tight in \( \mathbb{R} \).

Since for all \( N \geq 0, t \geq 0, \mu_t^N \) is a probability measure, these two points follow readily from the fact that \( \varphi, \Delta \varphi \) and \( (\nabla \varphi)^2 \) are bounded on \( \mathbb{T}^2 \).

Proof of 2. According to the law of large numbers, \( \forall t \geq 0, (\mu_t^N, \varphi) \xrightarrow{a.s.} \mathbb{E}(\varphi(X_t^1)) \) so \( (\mu_{t_1}^N \varphi_1, \ldots, \mu_{t_m}^N \varphi_m) \xrightarrow{a.s.} (\mathbb{E}(\varphi_1(X_t^1)), \ldots, \mathbb{E}(\varphi_m(X_t^1))) = ((\mu_{t_1}, \varphi_1), \ldots, (\mu_{t_m}, \varphi_m)) \).

Furthermore the fact that \( (\mu_t, t \geq 0) \) solves the PDE appearing in the statement follows readily from the Itô formula. Finally since \( \{\mu_t, t \geq 0\} \) is uniquely determined and deterministic, the whole sequence converges in Probability. \( \square \)

**Lemma 4.3** For any \( t \geq 0 \), the measure \( \mu_t \) is absolutely continuous with respect to the Lebesgue measure and its density \( f(t, \cdot) \) verifies \( \delta_1 \leq f(t, x) \leq \delta_2, \forall x \in \mathbb{T}^2 \), where \( \delta_1 \) and \( \delta_2 \) are defined in Sect. 3.

**Proof** Given that \( \mu_t = \mu_0 + \gamma \int_0^t \Delta \mu_r dr, \mu_t = \Upsilon(t) \mu_0 \).

Thus as from (2.1) in Proposition 2.7, for any measurable subset of \( \mathbb{T}^2 \), with zero Lebesgue measure \( \Upsilon(t)1_A = 0 \), the absolute continuity of \( \mu_t \) with respect to the Lebesgue measure follows from the fact that \( \mu_t = \Upsilon(t) \mu_0 \). Furthermore we notice that the law \( \mu_t \) of \( X_t^1 \) is absolutely continuous with respect to the Lebesgue measure, this being true whether the law \( \mu_0 \) of \( X_0^1 = X_1 \) has or not this property.

- Let us now show that \( \forall x \in \mathbb{T}^2, \delta_1 \leq f(t, x) \leq \delta_2 \).

We first recall that \( g \) is the density of the law \( \mu_0 \) of \( X_1 \).

Let \( P_t \) be the heat kernel on the two dimensional torus. As the solution of the heat equation with the initial condition \( \phi \), is the function defined on \( \mathbb{T}^2 \) by \( \Upsilon(t) \phi(x) = \int_{\mathbb{T}^2} P_t(x, y) \phi(y) dy, \Upsilon(t) \) is non decreasing in the sense that \( \forall \varphi, \psi \in L^2(\mathbb{T}^2) \) such that \( \varphi \leq \psi, \Upsilon(t) \varphi \leq \Upsilon(t) \psi \). So since for any \( C \in \mathbb{R}, \Upsilon(t) C = C \) (which follows from (2.1)) and \( f(t, \cdot) = \Upsilon(t) g \), the result follows from the facts that \( \delta_1 \leq g \leq \delta_2 \) and \( \Upsilon(t) \) is non decreasing. \( \square \)

### 4.3 Tightness and convergence of \((\mu_{S,N}, \mu_{I,N})_{N \geq 1}\)

#### 4.3.1 Preliminaries and statement of the main result

Recall that we equip \( \mathcal{M}_F(\mathbb{T}^2) \) with the topology of weak convergence and the Skorokhod space of càdlàg functions from \( \mathbb{R}_+ \) to \( \mathcal{M}_F(\mathbb{T}^2) \), denoted \( D(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2)) \) with the Skorokhod topology (we refer to page 63 of [12] for an explicit definition).
We first note that:

\[
\begin{align*}
(\mu^{S,N}_t, 1_{T^2}) &= \frac{1}{N} \sum_{i=1}^{N} 1_{\{E^i_t = S\}} \leq 1, \\
(\mu^{I,N}_t, 1_{T^2}) &= \frac{1}{N} \sum_{i=1}^{N} 1_{\{E^i_t = I\}} \leq 1,
\end{align*}
\]

and therefore, \(\forall \varphi \in C(T^2)\)

\[
\begin{align*}
|\varphi|_{(\mu^{S,N}_t, \varphi)} &\leq \|\varphi\|_{\infty}, \\
|\varphi|_{(\mu^{I,N}_t, \varphi)} &\leq \|\varphi\|_{\infty}.
\end{align*}
\]

**Lemma 4.4** Let \(H^1 = \{(\mu, \nu, \rho) \in (\mathcal{M}(T^2))^3 / (\nu, 1_{T^2}) \leq 1; (\mu, \varphi) \leq (\rho, \varphi), \forall \varphi \in C(T^2; \mathbb{R}_+)\}\)

For all \((\mu, \nu, \rho) \in H^1, \varphi \in C(T^2)\), we have

\[
\left| \left( \mu, \left( \nu, \frac{K}{(\rho, K)} \right) \varphi \right) \right| \leq \|\varphi\|_{\infty}.
\]

**Proof**

\[
\begin{align*}
\left| \left( \mu, \left( \nu, \frac{K}{(\rho, K)} \right) \varphi \right) \right| &= \left| \int_{T^2} \varphi(x) \int_{T^2} K(x, y) \nu(dy) \mu(dx) \right| \\
&\leq \|\varphi\|_{\infty} \left| \int_{T^2} \int_{T^2} K(x, y) \mu(dx) \nu(dy) \right| \\
&\leq \|\varphi\|_{\infty},
\end{align*}
\]

where we have exploited the symmetry of \(K\): \(K(x, y) = K(y, x)\) for the first inequality and the facts that \(\int_{T^2} K(x, y) \mu(dx) \nu(dy) \leq 1\) and \((\nu, 1_{T^2}) \leq 1\) for the last inequality. \(\square\)

We can now establish the wished tightness.

**Proposition 4.5** Both sequences \((\mu^{S,N}_t)_{N \geq 1}\) and \((\mu^{I,N}_t)_{N \geq 1}\) are tight in \(D(\mathbb{R}_+, \mathcal{M}_F(T^2))\).

**Proof** - Let us prove that \((\mu^{S,N}_t)_{N \geq 1}\) is tight in \(D(\mathbb{R}_+, \mathcal{M}_F(T^2))\).

As already stated in the proof of Proposition 4.2, it suffices to prove that

\(\forall \varphi \in C^2(T^2), ((\mu^{S,N}_t, \varphi), t \geq 0)_{N \geq 1}\) is tight in \(D(\mathbb{R}_+, \mathbb{R})\).

Let \(\varphi \in C^2(T^2)\), we have

\[
(\mu^{S,N}_t, \varphi) = (\mu^{S,N}_0, \varphi) + \gamma \int_0^t (\mu^{S,N}_r, \Delta \varphi)dr
\]
\[
-\beta \int_0^t \left( \mu_r^{S,N} - \frac{K}{\mu_r^{I,N}}, \varphi(\mu_r^{I,N}, \frac{K}{\mu_r^{I,N}}, \mu_r^{I,N}) \right) dr + M_t^{N,\varphi} = (\mu_0^{S,N}, \varphi) + \int_0^t \gamma(\mu_r^{S,N}, \Delta \varphi) - \beta \left( \mu_r^{S,N}, \varphi(\mu_r^{I,N}, \frac{K}{\mu_r^{I,N}}, K) \right) dr + M_t^{N,\varphi}.
\]

We notice that \{\(\mu_i^{S,N}, \varphi\), \(t \geq 0\)\} is a semi-martingale since \(M_t^{N,\varphi}\) is a square integrable martingale. Indeed, \(M_t^{N,\varphi}\) is a local martingale as the sum of local martingales, and from Lemma 4.4 we deduce that
\[
< M_t^{N,\varphi} >_t = \frac{\beta \| \varphi \|^2 \| \Delta \varphi \|}{N} + 2 \gamma t \| (\nabla \varphi) \|_{\infty}.\]

Hence \(E(|M_t^{N,\varphi}|^2) = E(< M_t^{N,\varphi} >_t) < \infty\).

Consequently
\[
(\mu_t^{S,N}, \varphi) = (\mu_0^{S,N}, \varphi) + \int_0^t \omega_r^{N,\varphi} dr + M_t^{N,\varphi} \text{ with } < M_t^{N,\varphi} >_t = \int_0^t \sigma_r^{N,\varphi} dr,
\]

and
\[
\omega_r^{N,\varphi} = \gamma(\mu_r^{S,N}, \Delta \varphi) - \beta \left( \mu_r^{S,N}, \varphi(\mu_r^{I,N}, \frac{K}{\mu_r^{I,N}}, K) \right),
\]
\[
\sigma_r^{N,\varphi} = \frac{\beta}{N} \left( \mu_r^{S,N}, \varphi(\mu_r^{I,N}, \frac{K}{\mu_r^{I,N}}, K) \right) + \frac{2 \gamma}{N} \left( \mu_r^{S,N}, (\nabla \varphi)^2 \right).
\]

Furthermore \(\omega_t^{N,\varphi}\) and \(\sigma_t^{N,\varphi}\) are progressively measurable since they are adapted and right continuous, so according to Proposition 37 of [20] a sufficient condition for \((\mu_t^{S,N}, \varphi)_{N \geq 1}\) to be tight in \(D(\mathbb{R}^+, \mathbb{R})\) is that both:

- \(|(\mu_0^{S,N}, \varphi), N \geq 1\) is tight in \(\mathbb{R}\),
- \(\forall T \geq 0, \sup_{0 \leq t \leq T} (|\omega_t^{N,\varphi}| + \sigma_t^{N,\varphi})\) is tight in \(\mathbb{R}\).

These follow readily from the facts that:
- \(|(\mu_0^{S,N}, \varphi)| \leq \| \varphi \|_{\infty}.
- From Lemma 4.4, \(|\omega_t^{N,\varphi}| \leq \gamma \| \Delta \varphi \|_{\infty} + \beta \| \varphi \|_{\infty} \text{ and } \sigma_t^{N,\varphi} \leq \frac{\beta \| \varphi \|^2 \| \Delta \varphi \|}{N} + \frac{2 \gamma}{N} \| (\nabla \varphi) \|_{\infty}.

The same arguments yields the tightness of \{\(\mu_t^{I,N}, t \geq 0, N \geq 1\)\} in \(D(\mathbb{R}^+, \mathcal{M}_F(\mathbb{T}^2)))\).

**Proposition 4.6** All limit points \((\mu^S, \mu^I)\) of the sequence \((\mu^{S,N}, \mu^{I,N})_{N \geq 1}\) are elements of \((C(\mathbb{R}^+, \mathcal{M}_F(\mathbb{T}^2)))^2\).
Proof Since we know that \( \varphi \in D(\mathbb{R}_+; \mathcal{M}_F(\mathbb{T}^2)) \), it is enough to prove that \( \forall \varphi \in C(\mathbb{T}^2) \), the processes \( \{ (\mu^{S}_t, \varphi), t \geq 0 \} \) is continuous. However according to Proposition 3.26 page 315 in [9], a sufficient condition for \( \{ (\mu^{S}_t, \varphi), t \geq 0 \} \) to be continuous is that:

\[
\forall T > 0, \forall \varepsilon > 0 \lim_{N \to \infty} \mathbb{P}( \sup_{0 \leq t \leq T} |(\mu^{S,N}_t, \varphi) - (\mu^{S,N}_{t-}, \varphi)| > \varepsilon) = 0
\]

Let \( T > 0, \varepsilon > 0 \), since the infections of two distinct individuals cannot occur at the same time, we have:

\[
|(\mu^{S,N}_t, \varphi) - (\mu^{S,N}_{t-}, \varphi)| \leq \frac{1}{N} \sum_{i=1}^{N} |\varphi(X^i_t)| \mathbb{I}_{|E^i_t| = S} - \mathbb{I}_{|E^i_{t-}| = S} | \leq \frac{\| \varphi \|_{\infty}}{N} \sum_{i=1}^{N} |\mathbb{I}_{|E^i_t| = S} - \mathbb{I}_{|E^i_{t-}| = S}| \leq \frac{\| \varphi \|_{\infty}}{N}.
\]

So for any \( \varepsilon > 0 \), \( \lim_{N \to \infty} \mathbb{P}( \sup_{0 \leq t \leq T} |(\mu^{S,N}_t, \varphi) - (\mu^{S,N}_{t-}, \varphi)| > \varepsilon) = 0 \).

By a similar argument we obtain the continuity of \( \{ \mu^{I}_t, t \geq 0 \} \).

Let us now state the main result of this section.

**Theorem 4.7** The sequence \( \{ (\mu^{S,N}_t, \mu^{I,N}_t) \}_{N \geq 1} \) converges in probability in \( (D(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2)))^2 \) to \( (\mu^{S}, \mu^{I}) \in (C(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2)))^2 \) which is the unique solution of the following system of equations. For all \( \varphi \in C^2(\mathbb{T}^2) \),

\[
(\mu^{S}_t, \varphi) = (\mu^{S}_0, \varphi) + \gamma \int_0^t (\mu^{S}_r, \Delta \varphi) dr - \beta \int_0^t \left( \mu^{S}_r, \varphi(\mu^{I}_r, \frac{K}{\mu^{I}_r, K}) \right) dr \quad (4.3)
\]

\[
(\mu^{I}_t, \varphi) = (\mu^{I}_0, \varphi) + \gamma \int_0^t (\mu^{I}_r, \Delta \varphi) dr + \beta \int_0^t \left( \mu^{S}_r, \varphi(\mu^{I}_r, \frac{K}{\mu^{I}_r, K}) \right) dr - \alpha \int_0^t (\mu^{I}_r, \varphi) dr \quad (4.4)
\]

**4.3.2 Proof of Theorem 4.7**

By Proposition 4.5, both sequence \( \{ (\mu^{S,N}_t, \mu^{I,N}_t) \}_{N \geq 1} \) are tight in \( D(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2)) \), so the sequence \( \{ (\mu^{S,N}_t, \mu^{I,N}_t) \}_{N \geq 1} \) is tight in \( (D(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2)))^2 \). Thus according to Prokhorov’s theorem there exists a subsequence of \( \{ (\mu^{S,N}_t, \mu^{I,N}_t) \}_{N \geq 1} \) still denoted \( \{ (\mu^{S,N}_t, \mu^{I,N}_t) \}_{N \geq 1} \) which converges in law towards \( (\mu^{S}, \mu^{I}) \). Hence to complete the proof of Theorem 4.7 it remains to:

- Find the system of PDEs satisfied by \( \{ (\mu^{S}_t, \mu^{I}_t), t \geq 0 \} \).
- Show that \( \forall t \geq 0 \), the measure \( \mu^{S}_t \) and \( \mu^{I}_t \) are absolutely continuous with respect to the Lebesgue measure with densities \( f_S(t) \) and \( f_I(t) \) bounded by \( \delta_2 \).

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The following map is continuous.

Hence the result.

We first prove the following Lemmas, which will be useful to establish the system of PDEs satisfied by \((\mu^S, \mu^I), t \geq 0\).

**Lemma 4.8** Under the assumption (H1), the function \(K\) is Lipschitz on \(\mathbb{T}^2 \times \mathbb{T}^2\), with the Lipschitz constant \(2\sqrt{2}C_k\).

**Proof** Let \(x, x', y, y' \in \mathbb{T}^2\), one has

\[
|K(x, y) - K(x', y')| = |k(d_{\mathbb{T}^2}(x, y)) - k(d_{\mathbb{T}^2}(x', y'))| \\
\leq C_k|d_{\mathbb{T}^2}(x, y) - d_{\mathbb{T}^2}(x', y')| \leq C_k|d_{\mathbb{T}^2}(x, y) - d_{\mathbb{T}^2}(x', y')| \leq C_k|d_{\mathbb{T}^2}(x, y) - d_{\mathbb{T}^2}(x', y')|.
\]

Furthermore \(d_{\mathbb{T}^2}(x, y) - d_{\mathbb{T}^2}(x', y')\) \(\leq d_{\mathbb{T}^2}(x, x') + d_{\mathbb{T}^2}(y, y')\).

Thus since \(\sqrt{2}\) is the maximal distance between two points in \(\mathbb{T}^2\), we conclude from the above results that

\[
\text{For any } x, x', y, y' \in \mathbb{T}^2, \quad |K(x, y) - K(x', y')| \leq 2\sqrt{2}C_k(d_{\mathbb{T}^2}(x, x') + d_{\mathbb{T}^2}(y, y')). \quad (4.5)
\]

\(\square\)

**Lemma 4.9** For all \(\mu, v \in \mathcal{M}(\mathbb{T}^2)\), we have

\[
\sup_y \left| \int_{\mathbb{T}^2} K(x', y)(\mu(dx') - v(dx')) \right| \leq (\|k\|_\infty + 2\sqrt{2}C_k)d_F(\mu, v).
\]

**Proof** Since from (4.5), for any \(y \in \mathbb{T}^2\), the function \(K(., y)\) is Lipschitz uniformly in \(y\) with the Lipschitz constant \(2\sqrt{2}C_k\), we have

\[
\left| \int_{\mathbb{T}^2} K(x', y)(\mu - v)(dz) \right| = (\|k\|_\infty + 2\sqrt{2}C_k) \left| \int_{\mathbb{T}^2} \frac{|K(x', y)|}{\|K(., y)\|_\infty + 2\sqrt{2}C_k} (\mu - v)(dx') \right| \\
\leq (\|k\|_\infty + 2\sqrt{2}C_k)d_F(\mu, v).
\]

Hence the result. \(\square\)

**Lemma 4.10** The following map is continuous.

\[
G : (\mathcal{M}_F(\mathbb{T}^2), d_F) \times (\mathcal{M}_F(\mathbb{T}^2), d_F) \rightarrow (\mathcal{M}_F(\mathbb{T}^2 \times \mathbb{T}^2), d_F) \\
(\mu, v) \mapsto \mu \otimes v
\]
where \( \forall \phi \in C(\mathbb{T}^2 \times \mathbb{T}^2), (\mu \otimes \nu, \phi) = \int_{\mathbb{T}^2 \times \mathbb{T}^2} \phi(x, y) v(dy) \mu(dx) \)

**Proof** Let \((\mu, \nu), (\mu^1, \nu^1) \in (\mathcal{M}_F(\mathbb{T}^2))^2; \phi \) a Lipschitz function on \( \mathbb{T}^2 \times \mathbb{T}^2, \|\phi\|_{\infty} \leq 1 \) and \( \|\phi\|_L \leq 1 \). We have

\[
\left| \int_{\mathbb{T}^2 \times \mathbb{T}^2} \phi(x, y)(\mu \otimes \nu - \mu^1 \otimes \nu^1)(dx, dy) \right|
\leq \int_{\mathbb{T}^2} \left| \int_{\mathbb{T}^2} \phi(x, y)(\mu - \mu^1)(dx) \right| v(dy) + \int_{\mathbb{T}^2} \left| \int_{\mathbb{T}^2} \phi(x, y)(v - v^1)(dy) \right| \mu^1(dx)
\leq v(\mathbb{T}^2) \sup_y \left| \int_{\mathbb{T}^2} \phi(x, y)(\mu - \mu^1)(dx) \right| + \mu^1(\mathbb{T}^2) \sup_x \left| \int_{\mathbb{T}^2} \phi(x, y)(v - v^1)(dy) \right|
\leq C(d_F(\mu, \mu^1) + d_F(\nu, \nu^1)),
\]

since \( \sup_y \|\phi(., \cdot)\|_L \vee \sup_x \|\phi(\cdot, \cdot)\|_L \leq \|\phi\|_L \leq 1 \).

We can now establish the system of equations satisfied by \((\mu^S, \mu^I)\).

**Proposition 4.11** The processes \((\mu^S, \mu^I)\) satisfies the Eqs. (4.3) and (4.4)

**Proof** We prove this Proposition by taking the limit in the Eqs. (4.1) and (4.2).

1- Let us prove that

\[
\int_0^t \left( \mu^{S,N}_r, \varphi(\mu^{I,N}_r, \frac{K}{(\mu^{N}_r, K)}) \right) dr \overset{L}{\rightarrow} \int_0^t \left( \mu^S_r, \varphi(\mu^I_r, \frac{K}{(\mu_r, K)}) \right) dr.
\]

One has

\[
\left( \mu^{S,N}_r, \varphi(\mu^{I,N}_r, \frac{K}{(\mu^{N}_r, K)}) \right) = \left( \mu^{I,N}_r, \frac{\varphi K}{(\mu^N_r, K)} \right)
\]

\[
= \left( \mu^{I,N}_r, \frac{\varphi K}{(\mu^N_r, K)} \right) (\mu^N_r - \mu^I_r, K) + \left( \mu^I_r, \frac{\varphi K}{(\mu^N_r, K)} \right) (\mu^N_r - \mu^I_r, K).
\]

Moreover:

1-1. Since from Lemma 4.3, \( f(t, \cdot) \) is lower bounded by a positive constant and \( \forall y \in \mathbb{T}^2, \int_{\mathbb{T}^2} K(x, y) dx \) is a positive constant independent of \( y \),

\[
\exists C > 0 \text{ such that } \forall y \in \mathbb{T}^2, \int_{\mathbb{T}^2} K(x, y) f(t, x) dx \geq C.
\]

On the other hand, since from Proposition 4.2 for any \( \varphi \in C(\mathbb{T}^2), (\mu^N_r, \varphi) \overset{P}{\rightarrow} (\mu_r, \varphi), d_F(\mu^N_r, \mu_r) \overset{P}{\rightarrow} 0 \). Thus as \( d_F(\mu^N_r, \mu_r) \leq 2 \), so from Lemmas 4.3, 4.4 and 4.9 and
the Lebesgue dominated convergence theorem, we have

\[
\mathbb{E}\left( \left| \int_0^t \left( \mu_{r}^{I,N} - \mu_{r}^{I} \right) \phi \right| \right) \leq \frac{1}{C} \mathbb{E}\left( \int_0^t \int_{\mathbb{T}^2} \left| \frac{\phi(x)K(x,y)\mu_{r}^{S,N}(dx)}{\mu_{r}^{N}(dx)} \right| dy \right) \leq C \left\| \phi \right\|_{\infty} + 2\sqrt{2}C_k \int_0^t \mathbb{E}(d_F(\mu_{r}^{N},\mu_{r}))dr \xrightarrow{N\to\infty} 0
\]

1-2. We have

\[
\left( \mu_{r}^{I,N}, \mu_{r}^{I} \right) = \int_{\mathbb{T}^2 \times \mathbb{T}^2} \frac{\phi(x)K(x,y)}{\mu_{r}^{N}(dx')} \mu_{r}^{I,N}(dy)\mu_{r}^{I}(dx).
\]

Moreover since \( \int_{\mathbb{T}^2} K(x',y)\mu_{r}(dx') \) is lower bounded by a positive constant independent of \( y \in \mathbb{T}^2 \) and from Lemma 4.8, the map \( y \in \mathbb{T}^2 \mapsto \int_{\mathbb{T}^2} K(x',y)\mu_{r}(dx') \) is continuous, from Lemma 4.8 again, the map \( (x,y) \in \mathbb{T}^2 \times \mathbb{T}^2 \mapsto \frac{\phi(x)K(x,y)}{\mu_{r}^{N}(dx')} \) is continuous and bounded on \( \mathbb{T}^2 \times \mathbb{T}^2 \). Thus from Lemma 4.10, we deduce that

\[
\int_{\mathbb{T}^2 \times \mathbb{T}^2} \frac{\phi(x)K(x,y)}{\mu_{r}^{N}(dx')} \mu_{r}^{I,N}(dy)\mu_{r}^{I}(dx) \xrightarrow{L} \int_{\mathbb{T}^2 \times \mathbb{T}^2} \frac{\phi(x)K(x,y)}{\mu_{r}^{N}(dx')} \mu_{r}^{I}(dy)\mu_{r}^{S}(dx).
\]

2- Since \( \Delta \phi \) and \( \phi \) are continuous and bounded,

\[
\int_0^t (\mu_{r}^{I,N}, \Delta \phi)dr \xrightarrow{L} \int_0^t (\mu_{r}^{I}, \Delta \phi)dr \quad \text{and} \quad \int_0^t (\mu_{r}^{I,N}, \phi)dr \xrightarrow{L} \int_0^t (\mu_{r}^{I}, \phi)dr.
\]

3- Let us prove that, \( M_{t}^{N,\phi} \xrightarrow{P} 0 \) and \( L_{t}^{N,\phi} \xrightarrow{P} 0 \).

From Lemma 4.4, we have

\[
\mathbb{E}(\left| M_{t}^{N,\phi} \right|^2) = \mathbb{E}(\left| M_{t}^{N,\phi} \right|) \leq \frac{\beta}{N} \int_0^t \mathbb{E}\left( \left( \mu_{r}^{S,N}, \phi^2(\mu_{r}^{I,N}, \frac{K}{(\mu_{r}^{N}, K)}) \right) \right) dr + \frac{1}{N} \int_0^t \mathbb{E}(\mu_{r}^{S,N},(\nabla \phi)^2)dr \leq t \frac{\beta}{N} \left\| \phi \right\|_{\infty}^2 + \frac{t}{N} \left\| (\nabla \phi)^2 \right\|_{\infty} \xrightarrow{N\to\infty} 0.
\]
obtained in Theorem 3.1 we conclude that $L^N_{t,:}$ converges to 0 in $L^2$, so also in probability. A similar argument yields the fact that $L^N_{t,:}$ converges in probability to 0.

Thus from the results 1-, 2-, 3-, and from the convergence of the initial measures obtained in Theorem 3.1 we conclude that $(\mu^S_t, \mu^I_t)$ satisfies the Eqs. (4.3) and (4.4).

\[ \square \]

**Proposition 4.12** For any $t \geq 0$, the measures $\mu^S_t$ and $\mu^I_t$ are absolutely continuous with respect to the Lebesgue measure and their densities $f_S(t,.)$ and $f_I(t,.)$ are bounded by $\delta_2$.

**Proof** It follows from the above convergences that $\mu^S_t + \mu^I_t \leq \mu_t$. Hence we conclude that $\mu^S_t$ and $\mu^I_t$ are absolutely continuous with respect to the Lebesgue measure since $\mu_t$ has this property, and their densities satisfy

$$f_S(t,.) + f_I(t,.) \leq f(t,.) \leq \delta_2.$$

\[ \square \]

**Proposition 4.13** The pair of densities $(f_S(t,.)$, $f_I(t,.)$) of the pair of measures $(\mu^S_t, \mu^I_t)$ satisfies

$$f_S(t) = \Upsilon(t)f_S(0) - \beta \int_0^t \Upsilon(t-r) \left[ f_S(r) \int_{\mathbb{T}^2} \frac{K(.,y)}{\int_{\mathbb{T}^2} K(x',y)f(r,x')dx'} f_I(r,y)dy \right] dr,$$

(4.6)

$$f_I(t) = \Upsilon(t)f_I(0) + \beta \int_0^t \Upsilon(t-r) \left[ f_S(r) \int_{\mathbb{T}^2} \frac{K(.,y)}{\int_{\mathbb{T}^2} K(x',y)f(r,x')dx'} f_I(r,y)dy \right] dr$$

$$- \alpha \int_0^t \Upsilon(t-r)f_I(r)dr.$$ (4.7)

Moreover the system formed by the Eqs. (4.6) and (4.7) admits a unique solution on the set $\Lambda = \{ (f_1, f_2) / 0 \leq f_i \leq \delta_2, i \in \{1, 2\} \}$.

**Proof** Recall that $\forall y \in \mathbb{T}^2$, $(\mu^\tau, K(.,y)) = \int_{\mathbb{T}^2} K(x',y)f(r,x')dx' \geq C$. From the Eqs. (4.3) and (4.4), it is easy to deduce that $(f_S(t,.)$, $f_I(t,.)$) satisfies the Eqs. (4.6) and (4.7).

Let $(f^1_S(t), f^1_I(t))$, $(f^2_S(t), f^2_I(t)) \in \Lambda$ be two solutions of the system formed by Eqs. (4.6) and (4.7) with the same initial condition. Noticing that for any $\varphi \in L^\infty(\mathbb{T}^2)$,

$$\|\Upsilon(t)\varphi\|_\infty \leq \|\varphi\|_\infty \quad (\text{see Lemma 2.8})$$

and

$$\left\| \int_{\mathbb{T}^2} K(.,y)\varphi(y)dy \right\|_\infty \leq C \|\varphi\|_\infty,$$

we have

$$\|f^2_S(t) - f^1_S(t)\|_\infty \leq \beta \int_0^t \|\Upsilon(t-r)\left[ (f^2_S(r) - f^1_S(r)) \right. \frac{K(.,y)}{\int_{\mathbb{T}^2} K(x',y)f(r,x')dx'} f^1_I(r,y)dy \right\| dr$$

$$+ \beta \int_0^t \|\Upsilon(t-r)\left[ f^2_S(t) \right.$$
In this section we will study the convergence as $N \to \infty$.

Central limit theorem

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Hence summing (4.8) and (4.9) and applying Gronwall’s lemma, we obtain $f^1_I(t) = f^2_S(t)$ and $f^1_I(t) = f^2_S(t)$. \qed

We can now conclude the proof of Theorem 4.7. Since $(\mu^{S,N}, \mu^{I,N})_{N \geq 1}$ is tight in $(D(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2)))^2$, and all converging subsequences of the sequence $(\mu^{S,N}, \mu^{I,N})_{N \geq 1}$ weakly converge to the same limit $(\mu^S, \mu^I)$, the whole sequence $(\mu^{S,N}, \mu^{I,N})_{N \geq 1}$ weakly converges in $(D(\mathbb{R}_+, \mathcal{M}_F(\mathbb{T}^2)))^2$ towards $(\mu^S, \mu^I)$; furthermore $(\mu^S, \mu^I)$ is deterministic, so we have convergence in probability.

5 Central limit theorem

In this section we will study the convergence as $N \to \infty$ of

$$U^N = \sqrt{N}(\mu^{S,N} - \mu^S), \quad V^N = \sqrt{N}(\mu^{I,N} - \mu^I), \quad Z^N = \sqrt{N}(\mu^N - \mu)$$

under the assumption (H2) to be stated below. Note that the trajectories of these processes belong to $(D(\mathbb{R}_+, \mathcal{E}(\mathbb{T}^2)))^2$ and $C(\mathbb{R}_+, \mathcal{E}(\mathbb{T}^2))$ respectively, where $\mathcal{E}(\mathbb{T}^2)$ is the space of signed measures on the torus, which can be seen as the dual of $C(\mathbb{T}^2)$. However, since the limit processes may be less regular than their approximations we will first:

- Establish the equations verified by the process $Z^N$ and by the pair $(U^N, V^N)$.
- Fix the space in which the convergence results will be established.

Then we will study the convergence of the above sequences.

The following is assumed to hold throughout Sect. 5.

Assumption (H2) $k \in C^3(\mathbb{R}_+)$. 
The following result follows from Assumption (H2), and the fact that the square of the distance $d_{\mathbb{T}^2}$ on $\mathbb{T}^2 \times \mathbb{T}^2$ is smooth (since the torus $\mathbb{T}^2$ is a smooth manifold).

**Lemma 5.1** Let $x \in \mathbb{T}^2$, under (H2), we have
- $\forall \eta \in \mathbb{N}^2$, $|\eta| \leq 2$, the map $y \in \mathbb{T}^2 \mapsto D^\eta K(x, y)$ is Lipschitz and bounded with the Lipschitz constant independent of $x$.
- $\forall \eta \in \mathbb{N}^2$, $|\eta| \leq 3$, the map $y \in \mathbb{T}^2 \mapsto D^\eta K(x, y)$ is continuous and bounded by $C \max_{0 \leq |\eta| \leq 3} \|k^{(|\eta|)}\|_\infty$.

**Lemma 5.2** Under the assumption (H2), we have $\sup_x \|K(x, \cdot)\|_{H^3} < \infty$.

**Proof** We have
\[
\|K(x, \cdot)\|_{H^3} = \sum_{|\eta| \leq 3} \int_{\mathbb{T}^2} |D^\eta K(x, y)|^2 dy \leq C,
\]
which follows from Lemma 5.1. \qed

## 5.1 Evolution equations of $Z^N$ and of the pair $(U^N, V^N)$

### 5.1.1 Evolution equation of $Z^N = \sqrt{N}(\mu^N - \mu)$

Let $\varphi \in C^2(\mathbb{T}^2)$, we have
\[
(\mu_t^N, \varphi) = (\mu_0^N, \varphi) + \gamma \int_0^t (\mu_r^N, \triangle \varphi) dt + H_t^{N, \varphi}, \quad \text{where}
\]
\[
H_t^{N, \varphi} = \frac{\sqrt{2\gamma}}{N} \sum_{i=1}^N \int_0^t \nabla \varphi(X_r^i) dB_r^i,
\]
\[
(\mu_r, \varphi) = (\mu_0, \varphi) + \gamma \int_0^r (\mu_r, \triangle \varphi) dr.
\]
Hence
\[
(Z_t^N, \varphi) = (Z_0^N, \varphi) + \gamma \int_0^t (Z_r^N, \triangle \varphi) dr + \widehat{H}_t^{N, \varphi}, \quad \text{where} \quad \widehat{H}_t^{N, \varphi} = \sqrt{N} H_t^{N, \varphi}. \tag{5.1}
\]

### 5.1.2 System of evolution equations of the pair $(U^N, V^N)$

Let $\varphi \in C^2(\mathbb{T}^2)$, we have
\[
(\mu_t^{S,N}, \varphi) = (\mu_0^{S,N}, \varphi) + \gamma \int_0^t (\mu_r^{S,N}, \triangle \varphi) dt
\]

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\[-\beta \int_0^t \left( \mu_r^{S,N}, \varphi \left( \mu_r^{I,N}, \frac{K}{(\mu_r^{N}, K)} \right) \right) dr + M_t^{N,\varphi}, \]
\[(\mu_t^S, \varphi) = (\mu_0^S, \varphi) + \gamma \int_0^t (\mu_r^S, \Delta \varphi) dr - \beta \int_0^t \left( \mu_r^S, \varphi \left( \mu_r^{I,N}, \frac{K}{(\mu_r, K)} \right) \right) dr.\]

Note first that
\[
\left( \mu_r^{S,N}, \varphi \left( \mu_r^{I,N}, \frac{K}{(\mu_r^{N}, K)} \right) \right) = \int_{\mathbb{T}^2} \varphi(x) \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} K(x, y) \mu_r^N(\delta x) \mu_r^{I,N}(\delta y) \mu_r^{S,N}(\delta x) dr
\]
\[
= \left( \mu_r^{I,N}, \varphi K \right) \left( \mu_r^{S,N}, \varphi K \right) \frac{1}{(\mu_r^{N}, K)}.
\]

Thus
\[(U_t^N, \varphi) = (U_0^N, \varphi) + \gamma \int_0^t (U_r^N, \Delta \varphi) dr - \beta \int_0^t \left( \sqrt{N} \mu_r^{I,N}, \left( \mu_r^{S,N}, \varphi K \right) \right) dr
\]
\[+ \beta \int_0^t \left( \sqrt{N} \mu_r^{I,N}, \varphi K \right) dr + \sqrt{N} M_t^{N,\varphi}
\]
\[= (U_0^N, \varphi) + \gamma \int_0^t (U_r^N, \Delta \varphi) dr + \beta \int_0^t \left( \mu_r^{I,N}, \left( \mu_r^{S,N}, \varphi K \right) \right) dr
\]
\[- \beta \int_0^t \left( \mu_r^{I,N}, \left( U_r^N, \varphi K \right) \right) dr - \beta \int_0^t \left( V_r^N, \left( \mu_r^{S,N}, \varphi K \right) \right) dr + \sqrt{N} M_t^{N,\varphi}.
\]

Hence if we define \(\widetilde{M}_t^{N,\varphi} = \sqrt{N} M_t^{N,\varphi}\), we have
\[(U_t^N, \varphi) = (U_0^N, \varphi) + \gamma \int_0^t (U_r^N, \Delta \varphi) dr
\]
\[+ \beta \int_0^t \left( Z_r^N, G_r^{S,1,N} \varphi \right) dr - \beta \int_0^t \left( U_r^N, G_r^{I,N} \varphi \right) dr
\]
\[- \beta \int_0^t \left( V_r^N, G_r^{S} \varphi \right) dr + \widetilde{M}_t^{N,\varphi}, \quad \text{and also (5.2)}
\]
\[(V_t^N, \varphi) = (V_0^N, \varphi) + \gamma \int_0^t (V_r^N, \Delta \varphi) dr
\]
\[- \beta \int_0^t \left( Z_r^N, G_r^{S,1,N} \varphi \right) dr + \beta \int_0^t \left( U_r^N, G_r^{I,N} \varphi \right) dr
\]
and it follows from Theorem 4.7 that

\begin{equation}
\tilde{M}^N_\varphi = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_0^t \int_0^\infty 1_{\{E^i_{r-} = S\}} \varphi(X^i_r) 1_{\{u \leq S, \ n = N_{(\mu^r, K)}(\mu^r, K)} \} \overline{M}(dr, du)
+ \sqrt{\frac{2\gamma N}{\beta}} \sum_{i=1}^{N} \int_0^t 1_{\{E^i_{r-} = S\}} \nabla \varphi(X^i_r) dB_r,
\end{equation}

where \( \forall x, y, x' \in \mathbb{T}^2, \)

\begin{align*}
G_r^{S,1,N} \varphi(x') &= \left( \mu_r^{I,N}, K(x', \cdot) \right) \left( \mu_r^{S,N}, \varphi K \right) \\
&= \int_{\mathbb{T}^2} K(x', y) \frac{\int_{\mathbb{T}^2} \varphi(x) K(x, y) \mu_{r}^{S, N}(dx)}{\int_{\mathbb{T}^2} K(y, y') \mu_{r}^{N}(dy')} \mu_{r}^{I,N}(dy), \\
G_r^{I,N} \varphi(x) &= \varphi(x) \left( \mu_r^{I,N}, K(x, \cdot) \right) = \varphi(x) \int_{\mathbb{T}^2} \frac{K(x, y) \mu_r^{I,N}(dy)}{\int_{\mathbb{T}^2} K(y, y') \mu_r(dy')}, \\
G_r^{S} \varphi(y) &= \left( \mu_r^{S}, \varphi K(\cdot, y) \right) = \frac{\int_{\mathbb{T}^2} \varphi(x) K(x, y) \mu_r^{S}(dx)}{\int_{\mathbb{T}^2} K(y, y') \mu_r(dy')}. \\
\end{align*}

5.2 The space where the sequences \( Z^N \) and \( (U^N, V^N) \) converge

We first recall that for any \( s > 0 \), the family \((\rho_{i,n_1,n_2})_{i,n_1,n_2}\) (as defined in Proposition 2.2) is an orthonormal basis of \( H^s(\mathbb{T}^2) \).

**Proposition 5.3** Every limit point \( W^1 \) of the sequence \((\tilde{M}^N)_{N \geq 1}\) satisfies

\[ \forall T \geq 0, \quad \sup_{0 \leq t \leq T} \mathbb{E}(\|W^1_t\|_{H^{-s}}^2) < \infty \quad \text{iff} \quad s > 2. \]

**Proof** We have

\begin{align*}
< \tilde{M}^N_\varphi >_t &= \beta \int_0^t \left( \mu_r^{S,N}, \varphi^2 (\mu_r^{I,N}, K)(\mu_r, K) \right) dr + 2\gamma \int_0^t (\mu_r^{S,N}, (\nabla \varphi)^2) dr,
\end{align*}

and it follows from Theorem 4.7 that

\begin{align*}
< \tilde{M}^N_\varphi >_t \overset{P}{\to} \int_0^t \left\{ \beta \left( \mu_r^{S,N}, \varphi^2 (\mu_r, K) \right) + 2\gamma (\mu_r^{S,N}, (\nabla \varphi)^2) \right\} dr,
\end{align*}

furthermore \( \int_0^t \left\{ \beta \left( \mu_r^{S,N}, \varphi^2 (\mu_r, K) \right) + 2\gamma (\mu_r^{S,N}, (\nabla \varphi)^2) \right\} dr \) being the quadratic variation of a Gaussian martingale (we refer to Proposition 5.20 below, for
a justification of the Gaussian property) of the form \((W^1, \varphi)\), our aim is to find the smallest value of \(s\) for which \(\mathbb{E}(\|W^1_t\|_{H^{-s}}^2) < \infty\). We have

\[
\mathbb{E}(\|W^1_t\|_{H^{-s}}^2) = \mathbb{E}\left( \sum_{i,n_1,n_2} |(W^1_t, \rho_{n_1,n_2}^{i,s})|^2 \right) \\
= \sum_{i,n_1,n_2} \mathbb{E}(<(W^1_t, \rho_{n_1,n_2}^{i,s})>_t).
\]

However as \(\int_{\mathbb{T}^2} \int_{\mathbb{T}^2} K(x, y)\mu_r^S(dx)\mu_r^I(dy) \leq 1\), then from Lemma 7.1 in the “Appendix” below, iff \(s > 2\), we will have (for the meaning of the summation \(\sum_{i,n_1,n_2}\), we refer the reader to Remark 2.5)

\[
\sum_{i,n_1,n_2} < W^1_t, \rho_{n_1,n_2}^{i,s}>_t = \sum_{i,n_1,n_2} \int_0^t \beta \left( \mu_r^S, (\rho_{n_1,n_2}^{i,s}) \right) \left( \mu_r^I, \frac{K}{(\mu_r, K)} \right) dr \\
+ 2\gamma (\mu_r^S, (\nabla \rho_{n_1,n_2}^{i,s})^2) dr \\
\leq \int_0^t \left\{ \beta \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} (\rho_{n_1,n_2}^{i,s})^2(x) \frac{K(x, y)}{\int_{\mathbb{T}^2} K(y, x')\mu_r(dx')} \mu_r^I(dy) \mu_r^S(dx) \right\} dr \\
+ 2\gamma \int_{\mathbb{T}^2} \sum_{i,n_1,n_2} (\nabla \rho_{n_1,n_2}^{i,s}(x))^2 \mu_r^S(dx) dr \\
\leq \beta C \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} K(x, y)\mu_r^S(dx)\mu_r^I(dy) dr + 2\gamma C \int_0^t \int_{\mathbb{T}^2} \mu_r^S(dx) dr \\
\leq Ct(\beta + 2\gamma).
\]

The result follows. \(\square\)

By Doob’s inequality and by calculations similar to those done above we obtain the following result.

**Corollary 5.4** \(\forall T > 0, s > 2, \exists \ C(T) > 0 \) such that:

\[
\sup_{N \geq 1} \mathbb{E}( \sup_{0 \leq t \leq T} \|\tilde{L}_t^N\|_{H^{-s}}^2) \leq C(T), \sup_{N \geq 1} \mathbb{E}( \sup_{0 \leq t \leq T} \|\tilde{M}_t^N\|_{H^{-s}}^2) \leq C(T), \\
\sup_{N \geq 1} \mathbb{E}( \sup_{0 \leq t \leq T} \|\tilde{T}_t^N\|_{H^{-s}}^2) \leq C(T).
\]

**In the rest of this section we arbitrarily choose** \(2 < s < 3\), and we prove that the sequences \((Z^N)_{N \geq 1}\) and \((U^N, V^N)_{N \geq 1}\) converge in law in \(C(\mathbb{R}^+, H^{-s})\) and in \((D(\mathbb{R}^+, H^{-s}))^2\) respectively, where we have equipped \(C(\mathbb{R}^+, H^{-s})\) with the locally uniform topology and \(D(\mathbb{R}^+, H^{-s})\) with the Skorokhod topology (we refer to [16] or [5] for the explicit definition of this topology).
5.3 Tightness and convergence of \((Z_N^N)_{N \geq 1}\)

Recall that we just assumed that \(2 < s < 3\).

5.3.1 Preliminaries and statement of the main result

Recall that the sequence \((Z_N^N)_{N \geq 1}\) satisfies (6.1). We first give an estimate for the norm of the fluctuations process \(Z^N\) which is not uniform in \(N\).

**Lemma 5.5** For all \(N \geq 1\), there exists \(C > 0\) such that

\[
\mathbb{E}\left(\sup_{t \geq 0} \|Z_t^N\|_{H^{-s}}^2\right) \leq CN.
\]

**Proof** Since \(s > 2\), \(H^s(\mathbb{T}_2) \hookrightarrow C(\mathbb{T}_2)\) (see Proposition 2.6). Thus

\[
|Z_t^N, \varphi| = \sqrt{N} \left| \frac{1}{N} \sum_{i=1}^{N} \varphi(X^i_t) - (\mu_t, \varphi) \right| \\
\leq \sqrt{N} \left( \frac{1}{N} \sum_{i=1}^{N} |\varphi(X^i_t)| + \|\varphi\|_{\infty} \right) \\
\leq 2\sqrt{N} \|\varphi\|_{\infty} \\
\leq 2C\sqrt{N} \|\varphi\|_{H^s}.
\]

This inequality combined with \(\|Z_t^N\|_{H^{-s}} = \sup_{\varphi \neq 0, \varphi \in H^s} \frac{|(Z_t^N, \varphi)|}{\|\varphi\|_{H^s}}\), yields \(\mathbb{E}(\sup_{t \geq 0} \|Z_t^N\|_{H^{-s}}^2) \leq 4CN\).

**Lemma 5.6** For every integer \(N \geq 1\), the processes \(Z^N\) has paths in \(C(\mathbb{R}_+, H^{-s})\) a.s.

**Proof** We have \(Z_t^N = \sqrt{\frac{N}{N}}(\mu_t^N - \mu_t)\) and \(\mu_t \in C(\mathbb{R}_+, L^2(\mathbb{T}_2))\) (since for every \(t \geq 0\), \(\mu_t\) is absolutely continuous with respect to the Lebesgue measure and it’s density is bounded).

Furthermore \(\mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i}\), with \(X^i_t \in C(\mathbb{R}_+, \mathbb{T}_2)\), thus to show \(Z^N\) has paths in \(C(\mathbb{R}_+, H^{-s})\) (with \(s > 2\)) it is enough to prove that that \(\delta_{X_t^i} \in C(\mathbb{R}_+, H^{-s})\) (with \(s > 1\)), which in turn is obtained by proving that the map \(x \mapsto \delta_x\) is continuous from \(\mathbb{T}^2\) into \(H^{-s}(\mathbb{T}^2)\). (See also Theorem 1.69 page 47 of [4]).

Consider a sequence \((x_n)_n \subset \mathbb{T}^2\), which converges in \(\mathbb{T}^2\) to \(x \in \mathbb{T}^2\) as \(n \to \infty\), and let us show that the sequence \((\delta_{x_n})_n\) converges in \(H^{-s}\) (with \(s > 1\)) to \(\delta_x\) as \(n \to \infty\).

Let \(1 < s' < s\). Since \(x_n \to x\) in \(\mathbb{T}^2\) as \(n \to +\infty\) and \(H^{s'} \hookrightarrow C(\mathbb{T}_2)\) (see Proposition 2.6), for any \(\varphi \in H^{s'}, \varphi(x_n) \to \varphi(x)\) as \(n \to +\infty\), thus \(\delta_{x_n}\) weakly converges to \(\delta_x\) in \(H^{-s'}\). Furthermore as the embedding \(H^{-s'} \hookrightarrow H^{-s}\) is compact (see Theorem 1.69 page 47 of [4]), \(\delta_{x_n}\) converges to \(\delta_x\) in \(H^{-s}\). Hence the result.

\(\square\)

The main result of this subsection is the next Theorem.

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Theorem 5.7 The sequence \( \{Z^N, N \geq 1\} \) converges in law in \( C(\mathbb{R}_+, H^{-s}) \) towards 
\( \{Z_t, t \geq 0\} \in C(\mathbb{R}_+, H^{-s}) \), where \( \forall t \geq 0, Z_t = Z_0 + \gamma \int_0^t \Delta Z_r dr + \mathcal{H}_t \) and 
\( \forall \varphi \in H^s, (\mathcal{H}, \varphi) \) is a centered Gaussian martingale whose predictable quadratic variation is given by \( <(\mathcal{H}, \varphi)>_t = 2\gamma \int_0^t (\mu_r, (\nabla \varphi)^2) dr \).

Before we prove this Theorem we first state a condition of Aldous type for the tightness of a sequence of \( H^{-s} \)-valued càdlàg processes, exploiting the fact that \( H^{-s} \) is a Hilbert space (see Definition 2.2.1 of [10]).

Proposition 5.8 Let \( (\tilde{\vartheta}^n)_n \) be a sequence of \( H^{-s} \)-valued càdlàg processes, their laws \( (\tilde{\vartheta}^n) \) form a tight sequence in \( D(\mathbb{R}_+, H^{-s}) \) if 
\( (T_1) \) for each \( t \) in a dense subset \( T \) of \( \mathbb{R}_+ \), the sequence \( (\vartheta^n) \) is tight in \( H^{-s} \); 
\( (T_2) \) for each \( T > 0, \varepsilon_1, \varepsilon_2 > 0 \), there exist \( \delta > 0, n_0 \geq 1 \) such that for any collection of stopping times \( \tau^n \leq T \), 
\[
\sup_{n \geq n_0, t \leq T} P(\|\vartheta^n(\tau^n + \theta) - \vartheta^n_{\tau^n}\|_{H^{-s}} > \varepsilon_1) \leq \varepsilon_2.
\]

Note that \( (T1) + (T2) \) is also a sufficient condition for tightness in \( C(\mathbb{R}_+, H^{-s}) \). This follows readily from the facts that \( C(\mathbb{R}_+, H^{-s}) \subset D(\mathbb{R}_+, H^{-s}) \), the limit of a sequence of continuous functions which converges for the Skorohod topology is continuous, and whenever the limit is continuous, convergence for the Skorohod topology is equivalent to locally uniform convergence.

Let us now prove the following results which are useful for the proof of Theorem 5.7. Recall that for any \( t \geq 0, f(t, \cdot) \) denotes the density of \( \mu_t \) (see Lemma 4.3).

Proposition 5.9 The sequence \( \tilde{\mathcal{H}}^N \) converges in law in \( C(\mathbb{R}_+, H^{-s}) \) towards \( \tilde{\mathcal{H}} \) where 
\( \forall \varphi \in H^s, (\tilde{\mathcal{H}}, \varphi) \) is a centered, continuous, Gaussian martingale having the same law as 
\[
(\tilde{\mathcal{H}}_t, \varphi) = \int_0^t \int_{\mathbb{T}^2} \sqrt{2\gamma f(r, x)} \frac{\partial \varphi}{\partial x_1}(x) \mathcal{W}_1^1(dr, dx) + \int_0^t \int_{\mathbb{T}^2} \sqrt{2\gamma f(r, x)} \frac{\partial \varphi}{\partial x_2}(x) \mathcal{W}_2^1(dr, dx)
\]

where \( \mathcal{W}_1^1 \) and \( \mathcal{W}_2^1 \) are independent spatio-temporal white noises.

Proof The proof consists in first establishing tightness of the sequence \( \tilde{\mathcal{H}}^N \), and then identifying the unique possible limit of all converging subsequences. The proof of tightness is similar to (and a bit simpler than) the proof of Proposition 5.17, so in order to avoid unnecessary repetitions, we refer to that proof for the argument. By adapting the proof of Lemma 5.19 below, we show that \( \forall \varphi \in H^s \), the processes \( (\tilde{\mathcal{H}}, \varphi) \) is a centered, continuous martingale. On the other hand \( \forall \varphi \in H^s \), 
\[
<(\tilde{\mathcal{H}}^N, \varphi)>_t = 2\gamma \int_0^t (\mu^N_r, (\nabla \varphi)^2) dr \to 2\gamma \int_0^t (\mu_r, (\nabla \varphi)^2) dr = <(\tilde{\mathcal{H}}, \varphi)>_t,
\]
thus the quadratic variation \(< (\tilde{H}, \varphi) >_t \) being deterministic, \((\tilde{H}, \varphi)\) is a centered, continuous, Gaussian martingale which can be expressed as in the statement. □

**Proposition 5.10** There exists \( C > 0 \), such that for any stopping times \( \tau < \infty \) a.s and \( \theta > 0 \),

\[
\mathbb{E} \left( \left\| \int_{\tau}^{\tau + \theta} \gamma(t) \left( t + \theta - r \right) d\tilde{H}_r \right\|_{H^{-s}}^2 \right) \leq C \theta.
\]

This Proposition is a simplified version of Proposition 5.24 below, since the martingale there is more general. To avoid repetition, we refer to that Proposition for the proof.

**Proposition 5.11** For all \( T > 0 \),

\[
\sup_{N \geq 1} \mathbb{E} \left( \sup_{0 \leq t \leq T} \| Z^N_t \|_{H^{-s}}^2 \right) < \infty.
\]

**Proof** Recall that the semigroup \( \gamma(t) \) generated by \( \gamma \Delta \) satisfies \( |\gamma(t)|_{\mathcal{L}(H^{-s})} \leq 1 \), where \( |.|_{\mathcal{L}(H^{-s})} \) denotes the operator norm on \( H^{-s} \).

From Eq. (5.1), we have

\[
Z^N_t = \gamma(t) Z^N_0 + \int_0^t \gamma(t - r) d\tilde{H}^N_r.
\]

Thus

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \| Z^N_t \|_{H^{-s}}^2 \right) \leq 2 \mathbb{E} (\| Z^N_0 \|_{H^{-s}}^2) + 2 \mathbb{E} \left( \sup_{0 \leq t \leq T} \left\| \int_0^t \gamma(t - r) d\tilde{H}^N_r \right\|_{H^{-s}}^2 \right).
\]

Furthermore from Lemma 2.9 and Corollary 5.4, we have

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left\| \int_0^t \gamma(t - r) d\tilde{H}^N_r \right\|_{H^{-s}}^2 \right) \leq \mathbb{E} (\| \mathcal{H}^N_T \|_{H^{-s}}^2) \leq CT.
\]

Combined with Proposition 3.2, this shows that \( \sup_{N \geq 1} \sup_{0 \leq t \leq T} \mathbb{E} (\| Z^N_t \|_{H^{-s}}^2) < \infty. \)

**5.3.2 Proof of Theorem 5.7**

We first prove that \( Z^N \) is tight in \( C(\mathbb{R}_+, H^{-s}) \) then we show that all converging subsequences have the same limit.

**Proposition 5.12** The sequence \( Z^N \) is tight in \( C(\mathbb{R}_+, H^{-s}) \).

**Proof** We prove that \( Z^N \) satisfies the conditions of Proposition 5.8.

1. Proof of (T1). It suffices to show that

\[
\forall t \geq 0, \forall \varepsilon > 0 \text{ there exists a compact subset } \mathcal{K} \text{ of } H^{-s} \text{ such that } \mathbb{P} (Z^N_t \notin \mathcal{K}) < \varepsilon.
\]

This follows from Proposition 5.11 by the exact same argument as used in the proof of Proposition 3.4, see also Remark 3.5.
Proof of (T2). Let $T > 0$, $\varepsilon_1, \varepsilon_2 > 0$, $(\tau^N)_N$ a family of stopping times such that $\tau^N \leq T$.

By noticing that $\forall 0 \leq u \leq t$, $Z^N_t = \Upsilon(t - u)Z^N_u + \int_u^t \Upsilon(t - r)d\tilde{H}^N_r$, we have

$$Z^N_{\tau^N + \theta} - Z^N_{\tau^N} = (\Upsilon(\theta) - I_d)Z^N_{\tau^N} + \int_{\tau^N}^{\tau^N + \theta} \Upsilon(\tau^N + \theta - r)d\tilde{H}^N_r.$$  

We want to find $\delta > 0$ and $N_0 \geq 1$ such that

$$\sup_{N \geq N_0} \sup_{\delta \geq \theta} \mathbb{P}(\| (\Upsilon(\theta) - I_d)Z^N_{\tau^N} \|_{H^{-s}} \geq \varepsilon_1) \leq \varepsilon_2,$$  

(5.4)

$$\sup_{N \geq N_0} \sup_{\delta \geq \theta} \mathbb{P}\left( \left\| \int_{\tau^N}^{\tau^N + \theta} \Upsilon(\tau^N + \theta - r)d\tilde{H}^N_r \right\|_{H^{-s}} \geq \varepsilon_1 \right) \leq \varepsilon_2.$$  

(5.5)

Proof of (5.4) Recall that $(\lambda_{n_1,n_2})_{n_1,n_2}$ denotes the family of eigenvalues of the operator $-\gamma \Delta$. Let $n_1, n_2 \in \mathbb{N}^*$, such that

$$\left( 32 \sup_{N \geq 1} \mathbb{E}( \sup_{0 \leq t \leq T} \| Z^N_t \|^2_{H^{-s+\sigma}} ) \right)^{\frac{1}{\sigma}} < 1 + \lambda_{n_1,n_2}, \text{ for some } 0 < \sigma < s - 2.$$  

(5.6)

Note that we can choose $n_1$ and $n_2$ such that (5.6) is satisfied since, from Proposition 5.11,

$$\sup_{N \geq 1} \mathbb{E}( \sup_{0 \leq t \leq T} \| Z^N_t \|^2_{H^{-s+\sigma}} ) < \infty \text{ for } 0 < \sigma < s - 2,$$

and $(\lambda_{n_1,n_2})_{n_1,n_2}$ is a non-decreasing sequence which converges to $+\infty$ as either $n_1 \to \infty$ or $n_2 \to \infty$ (see Proposition 2.3).

Let $F_{n_1,n_2}$ denotes the sub-space of $H^s$ generated by

$$\{ \rho^0_0, (\rho^i_{k_1,0}, i \in \{5, 6\}), (\rho^i_{k_2}, i \in \{7, 8\}), (\rho^i_{k_1,k_2}, i \in [1, 4]), \kappa_1, \kappa_2 \text{ even and } \kappa_1 \leq n_1, \kappa_2 \leq n_2 \}.$$

Let $Z^N_t|_{F_{n_1,n_2}}$ be the orthogonal projection of $Z^N_t$ on the dual space of $F_{n_1,n_2}$.

We have

$$\mathbb{P}(\| (\Upsilon(\theta) - I_d)Z^N_{\tau^N} \|_{H^{-s}} \geq \varepsilon_1) \leq \mathbb{P}(\| (\Upsilon(\theta) - I_d)Z^N_{\tau^N} \|_{F_{n_1,n_2}} \|_{H^{-s}} \geq \frac{\varepsilon_1}{2})$$

$$+ \mathbb{P}(\| (\Upsilon(\theta) - I_d)(Z^N_{\tau^N} - Z^N_{\tau^N}|_{F_{n_1,n_2}}) \|_{H^{-s}} \geq \frac{\varepsilon_1}{2}).$$
Let us bound each of the two terms of the above right hand side.

\[-P(\| (\Upsilon(\theta) - I_d)Z_N^{tN} | F_{n_1,n_2} \|_{H^{-s}} \geq \frac{\varepsilon_1}{2}) \leq \frac{4}{\varepsilon_1^2} \mathbb{E}( \sup_{0 \leq t \leq T} (\| (\Upsilon(\theta) - I_d)Z_t | F_{n_1,n_2} \|_{H^{-s}})^2 ).\]

Furthermore from (2.1) in Proposition 2.7 \( \Upsilon(t)f_{i_1,k_2} = e^{-\lambda_{k_1,k_2}t}f_{i_1,k_2} \), thus (we refer again the reader to Remark 2.5 for the meaning of the notation \( \sum_{i,k_1,k_2} \))

\[
\| (\Upsilon(\theta) - I_d)Z_t^N | F_{n_1,n_2} \|_{H^{-s}}^2
\]

\[
= \sum_{i,k_1,k_2} (1 + \gamma \pi^2 (k_1 + k_2))^{-s}((\Upsilon(\theta) - I_d)Z_t^N, f_{i_1,k_2})^2
\]

\[
= \sum_{i,k_1,k_2} (1 + \gamma \pi^2 (k_1 + k_2))^{-s}(Z_t^N, (\Upsilon(\theta) - I_d)f_{i_1,k_2})^2
\]

\[
= \sum_{i,k_1,k_2} (e^{-\theta \lambda_{k_1,k_2}} - 1)^2(1 + \gamma \pi^2 (k_1 + k_2))^{-s}(Z_t^N, f_{i_1,k_2})^2
\]

\[
\leq (e^{-\delta \lambda_{n_1,n_2}} - 1)^2 \| Z_t^N \|_{H^{-s}}^2,
\]

hence

\[
\mathbb{P}(\| (\Upsilon(\theta) - I_d)Z_t^N | F_{n_1,n_2} \|_{H^{-s}} \geq \frac{\varepsilon_1}{2}) \leq \frac{4(e^{-\delta \lambda_{n_1,n_2}} - 1)^2}{\varepsilon_1^2} \mathbb{E}\left( \sup_{N \geq 1} \| Z_t^N \|_{H^{-s}}^2 \right). \tag{5.7}
\]

Since \( \| \Upsilon(t)Z_t^N \|_{H^{-s}} \leq \| Z_t^N \|_{H^{-s}} \),

\[
\mathbb{P}(\| (\Upsilon(\theta) - I_d)(Z_t^N - Z_t^{tN}) | F_{n_1,n_2} \|_{H^{-s}} \geq \frac{\varepsilon_1}{2}) \leq \mathbb{P}(\| Z_t^N - Z_t^{tN} | F_{n_1,n_2} \|_{H^{-s}} \geq \frac{\varepsilon_1}{4}) \leq \frac{16}{\varepsilon_1^2} \sup_{0 \leq t \leq T} \mathbb{E}\left( \sup_{0 \leq t \leq T} \| Z_t^N - Z_t^{tN} | F_{n_1,n_2} \|_{H^{-s}}^2 \right).
\]

On the other hand, since \( (\lambda_{k_1,k_2})_{k_1,k_2} \) is a non-decreasing sequence, for any \( 0 < \sigma < s - 2 \),

\[
\| Z_t^N - Z_t^{tN} | F_{n_1,n_2} \|_{H^{-s}}^2 = (I_d - \gamma \Delta) \frac{\varepsilon}{2} (I_d - \gamma \Delta) \frac{\varepsilon}{2} (Z_t^N - Z_t^{tN} | F_{n_1,n_2})^2\]

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\[ \sum_{i,\kappa_1,\kappa_2} (1 + \lambda_{\kappa_1,\kappa_2})^{-\sigma} ((I_d - \gamma \Delta)^{-\frac{s}{2}} (I_d - \gamma \Delta)^{\frac{s}{2}} (Z_t^N - Z_t^N |_{F_{n_1,n_2}}), \]

\[ \sum_{i,\kappa_1,\kappa_2} (1 + \lambda_{\kappa_1,\kappa_2})^{-\sigma} (Z_t^N - Z_t^N |_{F_{n_1,n_2}}, (I_d - \gamma \Delta)^{\frac{s}{2}} (I_d - \gamma \Delta)^{-\frac{s}{2}}) \]

\[ \sum_{i,\kappa_1,\kappa_2} (1 + \lambda_{\kappa_1,\kappa_2})^{-s+\sigma} (1 + \lambda_{\kappa_1,\kappa_2})^{-\sigma} (Z_t^N - Z_t^N |_{F_{n_1,n_2}}, f_{i,\kappa_1,\kappa_2})^2 \]

\[ \leq (1 + \lambda_{n_1,n_2})^{-\sigma} \sum_{i,\kappa_1=1}^{\infty,\kappa_2=2} (1 + \lambda_{\kappa_1,\kappa_2})^{-s+\sigma} (Z_t^N, f_{i,\kappa_1,\kappa_2})^2 \]

\[ \leq (1 + \lambda_{n_1,n_2})^{-\sigma} \|Z_t^N\|_{H^{-s+\sigma}}^2. \]

Thus

\[ \mathbb{P}(\|\mathcal{Y}(\theta) - I_d)(Z_t^N - Z_t^N |_{F_{n_1,n_2}})\|_{H^{-s}} \geq \frac{\varepsilon_1}{2}) \]

\[ \leq \frac{16(1 + \lambda_{n_1,n_2})^{-\sigma}}{\varepsilon_1^2} \mathbb{E} \left( \sup_{N \geq 1} \mathbb{E} \left( \sup_{0 \leq t \leq T} \|Z_t^N\|_{H^{-s+\sigma}}^2 \right) \right). \] (5.8)

So from (5.6), (5.7) and (5.8) we deduce (5.4).

**Proof of (5.5)** From Proposition 5.10, we have

\[ \mathbb{P} \left( \left\| \int_{\tau^N}^{\tau^N + \theta} \mathcal{Y}(\tau^N + \theta - r) d\widetilde{N}_r \right\|_{H^{-s}} \geq \varepsilon_1 \right) \]

\[ \leq \frac{1}{\varepsilon_1^2} \mathbb{E} \left( \left\| \int_{\tau^N}^{\tau^N + \theta} \mathcal{Y}(\tau^N + \theta - r) d\widetilde{N}_r \right\|_{H^{-s}}^2 \right) \]

\[ \leq \frac{C}{\varepsilon_1^2} \theta \leq \frac{C}{\varepsilon_1^2} \delta. \]

(5.5) follows. (T1), (T2) are proved, hence \((Z_t^N)\) is tight in \(C(\mathbb{R}_+, H^{-s})\).

We end the proof of Theorem 5.7 by showing the next Proposition.

**Proposition 5.13** Every limit point \(Z\) of \((Z_t^N)\) is a solution of

\[ Z_t = \mathcal{Y}(t) Z_0 + \int_0^t \mathcal{Y}(t - r) d\widetilde{N}_r. \]

**Proof** We have \(Z_t^N = \mathcal{Y}(t) Z_0^N + \int_0^t \mathcal{Y}(t - r) d\widetilde{N}_r^N\). Furthermore
- according to Proposition 5.9, for each fixed $t > 0$,
\[
\int_0^t \Upsilon(t-r) d\widehat{\mathcal{H}}_r^N \overset{L}{\to} \int_0^t \Upsilon(t-r) d\widehat{\mathcal{H}}_r;
\]
- according to Theorem 3.3 $\Upsilon(t)Z_0^N \overset{L}{\to} \Upsilon(t)Z_0$.

Hence the result. \hfill \Box

5.4 Convergence of $(U^N, V^N)_{N \geq 1}$

We recall that $(U^N, V^N)$ satisfies the equations (5.2) and (5.3), and that we assume that $2 < s < 3$. The Next Lemma is proved by the same argument as used in the proof of Lemma 5.5.

Lemma 5.14 For all $N \geq 1$, for all $T > 0$, there exists $C > 0$ such that:
\[
\mathbb{E}(\sup_{0 \leq t \leq T} \|U^N_t\|_{H^{-s}}) < CN, \quad \mathbb{E}(\sup_{0 \leq t \leq T} \|V^N_t\|^2_{H^{-s}}) < CN.
\]

Lemma 5.15 For all $N \geq 1$, the processes $U^N$ and $V^N$ belong to $D(\mathbb{R}_+, H^{-s})$

Proof Since $U^N_t = \sqrt{N}(\mu_{t,N} - \mu_t)$ and $V^N_t = \sqrt{N}(\mu_{t,N} - \mu_t)$, with $\mu_{t,N} = \frac{1}{N} \sum_{i=1}^N 1_{\{E_i = S\}} \delta X^i_t$ and $\mu_{t,N} = \frac{1}{N} \sum_{i=1}^N 1_{\{E_i = I\}} \delta X^i_t$, the result in the statement follows easily from the fact that $E^i \in D(\mathbb{R}_+, \mathbb{R})$ and $\mu_{t,S}, \mu_{t,I} \in C(\mathbb{R}_+, L^2(\mathbb{T}^2))$ and $\delta X^i \in C(\mathbb{R}_+, H^{-s})$ (with $s > 1$) (see the proof of Lemma 5.6). \hfill \Box

We stress the fact that, contrary to $Z^N$, $U^N$ and $V^N$ are discontinuous, due to infections and healings. We now state the main result of this section.

Theorem 5.16 Under (H2), the sequence of processes $(U^N, V^N)_{N \geq 1}$ converges in law in $(D(\mathbb{R}_+, H^{-s}))^2$ to the pair of processes $(U, V)$ which satisfies
\[
\begin{align*}
U_t &= U_0 + \gamma \int_0^t \Delta U_r dr + \beta \int_0^t (G_r^{S,I})^* Z_r dr - \beta \int_0^t (G_r^{I})^* U_r dr \\
&\quad - \beta \int_0^t (G_r^{S})^* V_r dr + W^1_t, \\
V_t &= V_0 + \gamma \int_0^t \Delta V_r dr - \beta \int_0^t (G_r^{S,I})^* Z_r dr + \beta \int_0^t (G_r^{I})^* U_r dr \\
&\quad + \int_0^t (\beta(G_r^{S})^* - \alpha I_d) V_r dr + W^2_t,
\end{align*}
\]

where for any $r > 0$, $G_r^I$ and $G_r^{S,I}$ are defined as in Sect. 5.1.2 (replacing $\mu_{r,N}$ and $\mu_{r,N}$ by $\mu_{r}$ and $\mu_{r}$ respectively) and $\forall \varphi, \psi \in H^s$, $((W^1, \varphi), (W^2, \psi))$ is a centered continuous Gaussian martingale satisfying
\[
< (W^1, \varphi) >_t = \beta \int_0^t \left( \mu_r^S, \varphi^2 \left( \mu_r^I, \frac{K}{(\mu_r, K)} \right) \right) dr \\
+ 2\gamma \int_0^t (\mu_r^S, (\nabla \varphi)^2) dr,
\]
\[
< (W^2, \psi) >_t = \beta \int_0^t \left( \mu_r^S, \psi^2 \left( \mu_r^I, \frac{K}{(\mu_r, K)} \right) \right) dr \\
+ 2\gamma \int_0^t (\mu_r^I, (\nabla \psi)^2) dr + \alpha \int_0^t (\mu_r^I, \psi^2) dr,
\]
\[
< (W^1, \varphi), (W^2, \psi) >_t = -\beta \int_0^t \left( \mu_r^S, \varphi \psi \left( \mu_r^I, \frac{K}{(\mu_r, K)} \right) \right) dr.
\]

The proof of this Theorem is the content of a subsection below, however let us first prove a few preliminary results.

### 5.4.1 Preliminary results

**Proposition 5.17** Both sequences \((\tilde{M}^N)_N \geq 1\) and \((\tilde{L}^N)_N \geq 1\) are tight in \(D(\mathbb{R}_+, H^{-s})\).

**Proof**
- Tightness of \((\tilde{M}^N)_N \geq 1\).
  We prove that \((\tilde{M}^N)_N \geq 1\) satisfies the conditions (T1) and (T2) of Proposition 5.8.
  - Based on Corollary 5.4, we deduce (T1) by the same argument as used in the proof of (T1) in Proposition 5.12.
  - Proof of (T2). First note that
    \[
    < \tilde{M}^N, \varphi >_t = \int_0^t \gamma_t^N (\varphi) dr,
    \]
    where
    \[
    \gamma_t^N (\varphi) = \beta \left( \mu_r^S, \varphi^2 \left( \mu_r^I, \frac{K}{(\mu_r, K)} \right) \right) + 2\gamma (\mu_r^S, (\nabla \varphi)^2).
    \]

According to Theorem 2.3.2 of [10] or Corollary 1.5 and Theorem 1.6 of [16], it is enough to prove that \(\forall T > 0\: \forall \varepsilon_1, \varepsilon_2 > 0\: \exists \delta > 0, N_0 \geq 1\) such that for any stopping times \(\tau^N \leq T\),
\[
\sup_{N \geq N_0} \sup_{\theta \leq \delta} \mathbb{P}( | < \tilde{M}^N >_{(\tau^N + \theta)} - < \tilde{M}^N >_{\tau^N} | > \varepsilon_1 ) < \varepsilon_2.
\]

where \(< \tilde{M}^N >\) is the increasing, continuous process such that \(\| \tilde{M}^N_t \|_{H^{-s}} < \tilde{M}^N >_t\) is a martingale.

Let \(T > 0, \varepsilon_1, \varepsilon_2 > 0\), from Lemma 7.1 below, we have (again the meaning of the notation \(\sum_{i,n_1,n_2}\) is explained in Remark 2.5)
\[
| < \tilde{M}^N >_{(\tau^N + \theta)} - < \tilde{M}^N >_{\tau^N} |
\]
\[
= \left| \sum_{i,n_1,n_2} < \tilde{M}^N, \rho_{i,n_1,n_2}^{l,s} >_{(\tau^N + \theta)} - < \tilde{M}^N, \rho_{i,n_1,n_2}^{l,s} >_{\tau^N} \right|
\]
Since Proposition 3.26 page 315 in [9] concerning the continuity of the limit in law of \( R \)

Lemma 5.19

Every limit point \( (T_2) \) follows.

We prove that the processes \( W \)

Proof

Let \( T \)

Thus \( P \)

The same argument yields the fact that \( \tilde{L}^N \) is an element of \( C(\mathbb{R}_+, H^{-s}) \times C(\mathbb{R}_+, H^{-s}) \).

Proposition 5.18

Every limit point \( (W^1, W^2) \) of the sequence \( (\tilde{M}^N, \tilde{L}^N)_{N \geq 1} \) is such that for any \( \varphi, \psi \in H^s \), \((W^1, \varphi), (W^2, \psi))\) is a martingale.

\( \square \)

Lemma 5.19

Every limit point \( (W^1, W^2) \) of the sequence \( (\tilde{M}^N, \tilde{L}^N)_{N \geq 1} \) is such that for any \( \varphi, \psi \in H^s \), \((W^1, \varphi), (W^2, \psi))\) is a martingale.

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Proof — Martingale property of $(W^1, \varphi)$

A sufficient condition for $(W^1, \varphi)$ to be a martingale, is that, for all $k \in \mathbb{N}^*$, $\Phi_k \in C_b(\mathbb{R}^k)$, $\varphi_1, \varphi_2, \ldots, \varphi_k \in H^s$, and $0 \leq s_0 < s_1 < s_2 < s_3 < \ldots < s_k \leq s < t$,

$$\mathbb{E}(\phi((W^1, \varphi))) = 0,$$

where $\phi((W^1, \varphi)) = \Phi_k((W^1_{s_1}, \varphi_1), (W^1_{s_2}, \varphi_2), \ldots, (W^1_{s_k}, \varphi_k))(\varphi((W^1_s, \varphi)) - (W^1_t, \varphi))$.

However given that $\tilde{M}^{N, \varphi}$ is a martingale, $\mathbb{E}(\phi(\tilde{M}^{N, \varphi})) = 0$, moreover $\phi$ is continuous thus,

$$\phi(\tilde{M}^{N, \varphi}) \xrightarrow{L} \phi((W^1, \varphi)), \text{ as } N \to \infty.$$

On the other hand $\phi(\tilde{M}^{N, \varphi})$ is uniformly integrable since $\mathbb{E}[(\phi(\tilde{M}^{N, \varphi}))^2] \leq C$, hence $\mathbb{E}[\phi((W^1, \varphi))] = \lim_{N \to \infty} \mathbb{E}[\phi(\tilde{M}^{N, \varphi})] = 0$.

So we conclude that $(W^1, \varphi)$ is a martingale. A similar argument shows that $(W^2, \psi)$ is a martingale. $\square$

Proposition 5.20 The sequence $(\tilde{M}^N, \tilde{L}^N)_{N \geq 1}$ converges in law in $(D(\mathbb{R}^+, H^{-s}))^2$ towards the processes $(W^1, W^2) \in C(\mathbb{R}^+, H^{-s}) \times C(\mathbb{R}^+, H^{-s})$ where $\forall \varphi, \psi \in H^s$, $(W^1, \varphi), (W^2, \psi))$ is a centered Gaussian martingale having the same law as:

$$
(W^1_t, \varphi) = -\int_0^t \int_{\mathbb{T}^2} \beta f_S(r, x) \frac{K(x, y)}{f_T K(x', y)} \int_{\mathbb{T}^2} \frac{K(x, y)}{r} f_I(r, y) d\varphi(x) W_1(dr, dx) + 
\int_0^t \int_{\mathbb{T}^2} \frac{\partial \varphi}{\partial x_1}(x) \sqrt{2} \gamma f_S(r, x) W_2(dr, dx) + 
\int_0^t \int_{\mathbb{T}^2} \frac{\partial \varphi}{\partial x_2}(x) \sqrt{2} \gamma f_S(r, x) W_2(dr, dx),
$$

(5.11)

$$
(W^2_t, \psi) = \int_0^t \int_{\mathbb{T}^2} \beta f_S(r, x) \frac{K(x, y)}{f_T K(x', y)} \int_{\mathbb{T}^2} \frac{K(x, y)}{r} f_I(r, y) d\psi(x) W_1(dr, dx) + 
\int_0^t \int_{\mathbb{T}^2} \frac{\partial \psi}{\partial x_1}(x) \sqrt{2} \gamma f_I(r, x) W_4(dr, dx) + 
\int_0^t \int_{\mathbb{T}^2} \frac{\partial \psi}{\partial x_2}(x) \sqrt{2} \gamma f_I(r, x) W_5(dr, dx) + 
\int_0^t \int_{\mathbb{T}^2} \psi(x) \sqrt{\alpha f_I(r, x)} W_6(dr, dx),
$$

(5.12)

where $W_1, W_2, W_3, W_4, W_5, W_6$ are independent spatio-temporal white noises.

Proof From Proposition 5.17 $(\tilde{M}^N, \tilde{L}^N)_{N \geq 1}$ is tight in $D(\mathbb{R}^+, H^{-s}) \times D(\mathbb{R}^+, H^{-s})$, hence according to Prokhorov’s theorem there exists a subsequence still denoted $(\tilde{M}^N, \tilde{L}^N)_{N \geq 1}$ which converges in law in $(D(\mathbb{R}^+, H^{-s}))^2$ towards $(W^1, W^2)$. By Proposition 5.18 and Lemma 5.19, $\forall \varphi, \psi \in H^s$, $(W^1, \varphi), (W^2, \psi))$ is a continuous martingale, thus we end the proof of Proposition 5.20 by showing that the centered, continuous martingale $((W^1, \varphi), (W^2, \psi))$ is Gaussian and satisfies (5.11) and (5.12).
We have

\[
\tilde{M}_t^{N, \varphi} = -\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_0^t \int_0^\infty 1\{E_{i,-}^r = S\} \varphi (X_{i,r}^j) 1\{u \leq \frac{\beta}{N} \sum_{j=1}^{N} \frac{K (X_{i,r}^j, X_{i,r}^j)}{\nu_r^N (K, X_{i,r}^j)} 1\{E_{i,-}^r = l\}) \overline{M}_t^i (dr, du)
\]

\[
+ \sqrt{2 \gamma} \sum_{i=1}^{N} \int_0^t 1\{E_{i,-}^r = S\} \nabla \varphi (X_{i,r}^j) dB_r^i
\]

\[
= -M_t^{1, N, \varphi} + M_t^{2, N, \varphi},
\]

\[
\tilde{L}_t^{N, \psi} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_0^t \int_0^\infty 1\{E_{i,-}^r = S\} \psi (X_{i,r}^j) 1\{u \leq \frac{\beta}{N} \sum_{j=1}^{N} \frac{K (X_{i,r}^j, X_{i,r}^j)}{\nu_r^N (K, X_{i,r}^j)} 1\{E_{i,-}^r = l\}) \overline{M}_t^i (dr, du)
\]

\[
+ \sqrt{2 \gamma} \sum_{i=1}^{N} \int_0^t 1\{E_{i,-}^r = S\} \nabla \psi (X_{i,r}^j) dB_r^i
\]

\[
- \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_0^t \int_0^\alpha 1\{E_{i,-}^r = S\} \psi (X_{i,r}^j) \overline{Q}_t^i (dr, du)
\]

\[
= M_t^{1, N, \psi} + M_t^{3, N, \psi} + M_t^{4, N, \psi}.
\]

Consider for \( \varphi, \psi \in C^2 (\mathbb{T}^2) \) and \( N \geq 1 \) the following martingale

\[
\tilde{M}_t^{N, \varphi} + \tilde{L}_t^{N, \psi} = -M_t^{1, N, \varphi} + M_t^{2, N, \varphi} + M_t^{1, N, \psi} + M_t^{3, N, \psi} + M_t^{4, N, \psi}.
\]

The martingales \( M_t^{1, N, \varphi}, M_t^{2, N, \varphi}, M_t^{3, N, \psi}, M_t^{4, N, \psi} \) being two by two orthogonal,

\[
< \tilde{M}_t^{N, \varphi} + \tilde{L}_t^{N, \psi} >_t = < M_t^{1, N, \varphi} >_t + < M_t^{2, N, \varphi} >_t
\]

\[
+ < M_t^{1, N, \psi} >_t + < M_t^{3, N, \psi} >_t + < M_t^{4, N, \psi} >_t
\]

\[
- 2 < M_t^{1, N, \varphi}, M_t^{1, N, \psi} >_t.
\]

In addition we have the following convergences in probability

\[
< M_t^{1, N, \varphi} >_t \overset{p}{\to} \beta \int_0^t \left( \mu_r^S, \varphi^2 (\mu_r^1, \frac{K}{(\mu_r, K)}) \right) dr,
\]

\[
< M_t^{2, N, \varphi} >_t \overset{p}{\to} 2 \gamma \int_0^t (\mu_r^S, (\nabla \varphi)^2) dr
\]

\[
< M_t^{3, N, \psi} >_t \overset{p}{\to} 2 \gamma \int_0^t (\mu_r^1, (\nabla \psi)^2) dr,
\]

\[
< M_t^{4, N, \psi} >_t \overset{p}{\to} \alpha \int_0^t (\mu_r^1, \psi^2) dr.
\]

On the other hand:
\( \tilde{M}^N, \tilde{L}^N \) along a subsequence since
\( (\tilde{M}^N, \tilde{L}^N) \overset{L}{\rightarrow} ((W^1, \varphi), (W^2, \psi)) \).

• \((W^1, \varphi) + (W^2, \psi)\) is a continuous martingale since \((W^1, \varphi)\) and \((W^2, \psi)\) have this property.

Thus \((W^1, \varphi) + (W^2, \psi)\) is a time changed Brownian motion.

The quadratic variation
\[
< (W^1, \varphi) + (W^2, \psi) >_t = \int_0^t \beta \left( \mu^S_r, \varphi^2(\mu^l_r, K) \right) + \left( \mu^S_r, \psi^2(\mu^l_r, K) \right) dr
- 2 \left( \mu^S_r, \varphi \psi(\mu^l_r, K) \right) \vert_0^t dW_1(\mu^S_r, \varphi)
+ \int_0^t \{ 2\gamma (\mu^S_r, (\nabla \varphi)^2) + 2\gamma (\mu^l_r, (\nabla \psi)^2) + \alpha (\mu^l_r, \psi^2) \} dr
\]

of \((W^1, \varphi) + (W^2, \psi)\) being deterministic then we conclude that \((W^1, \varphi) + (W^2, \psi)\)
is a Gaussian martingale having the same law as

\[
- \int_0^t \int_{\mathbb{T}^2} \sqrt{\beta f_S(r, x) \int_{\mathbb{T}^2} K(x, y) \frac{f_I(r, y) dy}{f_S(r, x)}} \frac{K(x, y) dx}{f_S(r, x)} W_1(dr, dx)
- \psi(x)) W_1(dr, dx)
+ \int_0^t \int_{\mathbb{T}^2} \frac{\partial \varphi}{\partial x_1}(x) \sqrt{2\gamma f_S(r, x)} W_2(dr, dx)
+ \int_0^t \int_{\mathbb{T}^2} \frac{\partial \varphi}{\partial x_2}(x) \sqrt{2\gamma f_S(r, x)} W_3(dr, dx)
+ \int_0^t \int_{\mathbb{T}^2} \frac{\partial \psi}{\partial x_1}(x) \sqrt{2\gamma f_I(r, x)} W_4(dr, dx)
+ \int_0^t \int_{\mathbb{T}^2} \frac{\partial \psi}{\partial x_2}(x) \sqrt{2\gamma f_I(r, x)} W_5(dr, dx)
+ \int_0^t \int_{\mathbb{T}^2} \psi(x) \sqrt{\alpha f_I(r, x)} W_6(dr, dx)
\]

where \( W_1, W_2, W_3, W_4, W_5, W_6 \) are independent spatio-temporal white noises defined in Proposition 5.9.

So taking \( \psi \equiv 0, \varphi \equiv 0 \) respectively in the above equation we see that \((W^1, \varphi)\) and \((W^2, \psi)\) satisfy (6.12) and (6.13).

5.4.2 Proof of Theorem 5.16

We first prove that \( U^N \) and \( V^N \) are tight in \( D(\mathbb{R}_+, H^{-s}) \) then we show that all converging subsequences of \((U^N, V^N)_{N \geq 1}\) have the same limit which we shall identify.
Let us recall the following embeddings which follow from Proposition 2.6, Lemma 2.8 above, and from Theorem 1.69 page 47 of [4].

- If \( s > 1 \) then \( H^s(\mathbb{T}^2) \subset C(\mathbb{T}^2) \) and for all \( \varphi \in H^s \), \( \Upsilon(t) \varphi \in H^s \) and
  \[ \| \Upsilon(t) \varphi \|_{H^s} \leq \| \varphi \|_{H^s}. \]
  Furthermore if \( \varphi, \psi \in H^s \) there exists \( C > 0 \) such that \( \| \varphi \psi \|_{H^s} \leq C \| \varphi \|_{H^s} \| \psi \|_{H^s}. \)
- If \( s > 2 \) then, \( H^{s+1}(\mathbb{T}^2) \subset C^2(\mathbb{T}^2) \).
- \( \forall s_1, s_2 \in \mathbb{R} \) such that \( s_1 > s_2 \) the embedding \( H^{s_1}(\mathbb{T}^2) \hookrightarrow H^{s_2}(\mathbb{T}^2) \) is compact.

**Proposition 5.21** There exists a constant \( C > 0 \) such that for any \( s > 0 \), \( \varphi \in H^s \), we have

\[
\| G^{S,I,N}_r \varphi \|_{H^s} \leq C \| \varphi \|_{H^s} \sup_y \| K(., y) \|_{H^s}, \quad (5.13)
\]

\[
\| G^{I,N}_r \varphi \|_{H^s} \leq C \| \varphi \|_{H^s} \sup_y \| K(., y) \|_{H^s}, \quad (5.14)
\]

\[
\| G^{S}_r \varphi \|_{H^s} \leq C \| \varphi \|_{H^s} \sup_x \left\| \frac{K(x, .)}{\int_{\mathbb{T}^2} K(x', .) \mu_r(dx')} \right\|_{H^s}^2, \quad (5.15)
\]

**Proof of (5.13)** We first recall that \( \int_{\mathbb{T}^2} K(x, y) f(t, x) dx \) is lower bounded by a positive constant \( C \) independently of \( y \in \mathbb{T}^2 \) and

\[
\left| \frac{\int_{\mathbb{T}^2} K(x, y) \varphi(x) \mu^{S,N}_r(d x)}{\int_{\mathbb{T}^2} K(u, y) \mu^{I,N}_r(d u)} \right| \leq \| \varphi \|_{\infty}.
\]

Now we have

\[
\| G^{S,I,N}_r \varphi \|_{H^s}^2 = \left\| \left( \mu^{I,N}_r, \frac{\mu^{S,N}_r, K \varphi}{(\mu^{N}_r, K)(\mu_r, K)} \right) \right\|_{H^s}^2
\]

\[
= \sum_{i,n_1,n_2} (1 + y \pi^2 (n_1^2 + n_2^2))^s \left( \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} K(z, y)
\frac{\int_{\mathbb{T}^2} K(x, y) \varphi(x) \mu^{S,N}_r(d x)}{\int_{\mathbb{T}^2} K(u, y) \mu^{I,N}_r(d u)} \int_{\mathbb{T}^2} K(u, y) \mu_r(d u)
\times f^{i}_{n_1,n_2}(z) \mu^{I,N}_r(dy) dz \right)^2
\]

\[
\leq \sum_{i,n_1,n_2} (1 + y \pi^2 (n_1^2 + n_2^2))^s
\int_{\mathbb{T}^2} \left( \frac{\int_{\mathbb{T}^2} K(x, y) \varphi(x) \mu^{S,N}_r(d x)}{\int_{\mathbb{T}^2} K(u, y) \mu^{I,N}_r(d u)} \int_{\mathbb{T}^2} K(u, y) \mu_r(d u) \right)^2 \mu^{I,N}_r(dy)
\times \int_{\mathbb{T}^2} \left( \int_{\mathbb{T}^2} K(z, y) f^{i}_{n_1,n_2}(z) dz \right)^2 \mu^{I,N}_r(dy)
\leq C \| \varphi \|^2_{\infty} \int_{\mathbb{T}^2} \sum_{i,n_1,n_2} (1 + y \pi^2 (n_1^2 + n_2^2))^s \left( \int_{\mathbb{T}^2} K(z, y) f^{i}_{n_1,n_2}(z) dz \right)^2 \mu^{I,N}_r(dy)
\leq C \| \varphi \|^2_{H^s} \sup_y \| K(., y) \|^2_{H^s}.
\]
Proof of (5.14)

\[ \| G_r^{I,N} \varphi \|_{H^s} = \| \varphi (\mu_r^{I,N}, \frac{K}{(\mu_r, K)}) \|_{H^s} \leq C \| \varphi \|_{H^s} \left( \frac{K}{(\mu_r, K)} \right) \| \varphi (\mu_r^{I,N}, \frac{K}{(\mu_r, K)}) \|_{H^s} \]

\[ \| (\mu_r^{I,N}, K) \|_{H^s}^2 = \sum_{i,n_1,n_2} (1 + \gamma \pi^2 (n_1^2 + n_2^2))^s \left( \frac{K}{(\mu_r, K)} \right) f_{n_1,n_2}^{I,N} \|_{L^2} \]

\[ = \sum_{i,n_1,n_2} (1 + \gamma \pi^2 (n_1^2 + n_2^2))^s \left( \frac{K}{(\mu_r, K)} \right) f_{n_1,n_2}^{I,N} \|_{L^2} \]

\[ \leq \sum_{i,n_1,n_2} (1 + \gamma \pi^2 (n_1^2 + n_2^2))^s \left( \frac{K}{(\mu_r, K)} \right) f_{n_1,n_2}^{I,N} \|_{L^2} \]

Proof of (5.15) Since \( \int_{\mathbb{T}^2} \varphi^2(x) \mu_r^S(dx) \leq \int_{\mathbb{T}^2} \varphi^2(x)dx \), we have

\[ \| G_r^S \varphi \|_{H^s} = \left\| \frac{K}{(\mu_r, K)} \right\|_{H^s}^2 = \sum_{i,n_1,n_2} (1 + \gamma \pi^2 (n_1^2 + n_2^2))^s \left( \frac{K}{(\mu_r, K)} \right) f_{n_1,n_2}^{I,N} \|_{L^2} \]

\[ = \sum_{i,n_1,n_2} (1 + \gamma \pi^2 (n_1^2 + n_2^2))^s \left( \frac{K}{(\mu_r, K)} \right) f_{n_1,n_2}^{I,N} \|_{L^2} \]

\[ \leq \sum_{i,n_1,n_2} (1 + \gamma \pi^2 (n_1^2 + n_2^2))^s \left( \frac{K}{(\mu_r, K)} \right) f_{n_1,n_2}^{I,N} \|_{L^2} \]

\[ \times \left( \sum_{i,n_1,n_2} (1 + \gamma \pi^2 (n_1^2 + n_2^2))^s \left( \frac{K}{(\mu_r, K)} \right) f_{n_1,n_2}^{I,N} \|_{L^2} \right) \]

\[ \leq C \| \varphi \|_{L^2}^2 \int_{\mathbb{T}^2} \sum_{i,n_1,n_2} (1 + \gamma \pi^2 (n_1^2 + n_2^2))^s \right) \left( \frac{K}{(\mu_r, K)} \right) f_{n_1,n_2}^{I,N} \|_{L^2} \]

\[ \leq C \| \varphi \|_{H^s} \sup_{x} \left( \frac{K}{(\mu_r, K)} \right) \|_{H^s}^2 \]

We have the following immediate consequence of Proposition 5.21.

\[ \square \]
Corollary 5.22 There exists a constant $C > 0$ such that for any $s > 0$, $\mathcal{V} \in H^{-s}$, we have

$$
\|(G_r^{S,1,N})^s \mathcal{V}\|_{H^{-s}} \leq C \sup_y \| K(., y) \|_{H^s} \| \mathcal{V} \|_{H^{-s}},
$$

$$
\|(G_r^{I,N})^s \mathcal{V}\|_{H^{-s}} \leq C \sup_y \| K(., y) \|_{H^s} \| \mathcal{V} \|_{H^{-s}},
$$

$$
\|(G_r^S)^s \mathcal{V}\|_{H^{-s}} \leq C \sup_x \left\| \frac{K(x,.)}{\int_{\mathbb{T}^2} \mu_r(dx')} \right\|_{H^s} \| \mathcal{V} \|_{H^{-s}}.
$$

Let us prove the following results which will be useful to prove the tightness of $(U^N_{N\geq 1})$ and $(V^N_{N\geq 1})$ in $D(\mathbb{R}_+, H^{-s})$.

Lemma 5.23 The pair of processes $(U^N, V^N)$ satisfies $\forall 0 \leq u < t$,

$$
U_t^N = \Upsilon(t-u)U_0^N + \beta \int_u^t \Upsilon(t-r)(G_r^{S,1,N})^s Z_r^N dr - \beta \int_u^t \Upsilon(t-r)(G_r^{I,N})^s U_r^N dr
$$

- $\beta \int_u^t \Upsilon(t-r) (G_r^S)^s V_r^N dr + \int_u^t \Upsilon(t-r) d\tilde{M}_r^N,$

$$
V_t^N = \Upsilon(t-u)V_0^N - \beta \int_u^t \Upsilon(t-r)(G_r^{S,1,N})^s Z_r^N dr + \beta \int_u^t \Upsilon(t-r)(G_r^{I,N})^s U_r^N dr
$$

- $\int_u^t \Upsilon(t-r)[\beta (G_r^S)^s - \alpha] V_r^N dr + \int_u^t \Upsilon(t-r)d\tilde{L}_r^N.$

(5.16) (5.17)

Proof Let us consider a function $\phi$ belonging to $C^{1,2}(\mathbb{R}_+ \times \mathbb{T}^2)$. By the Itô formula applied to $\phi(t, X^N_t)$ and using a computation similar to those in Sects. 4.1 and 5.1, we obtain that for $0 \leq u < t$,

$$(U_t^N, \phi_r) = (U_0^N, \phi_u) + \gamma \int_u^t (U_r^N, \Delta \phi_r) dr + \int_u^t (U_r^N, \frac{\partial \phi_r}{\partial r}) dr$$

$$+ \beta \int_u^t \left( Z_r^N, \left( \mu_r^{I,N}, K \left( \frac{\mu_r, N, \phi_r K}{(\mu_r, K)} \right) \right) \right) dr$$

$$- \beta \int_u^t \left( U_r^N, \phi_r \left( \frac{K}{(\mu_r, K)} \right) \right) dr$$

$$- \beta \int_u^t \left( V_r^N, \left( \frac{\mu_r, N, \phi_r K}{(\mu_r, K)} \right) \right) dr + \int_u^t (\phi_r, d\tilde{M}_r^N).$$

Let $\varphi \in H^{s+1}$ and $0 \leq u < t$, consider for $r \in [u, t]$ the mapping $\psi_r(x) = \Upsilon(t-r)\varphi(x)$.

We have that $\psi_r(\cdot) \in C^{1,2}([u, t] \times \mathbb{T}^2)$. Indeed:

- For any $r \in [u, t]$, $\psi_r(\cdot) \in H^{s+1}(\mathbb{T}^2) \subset C^2(\mathbb{T}^2)$.
- $\forall x \in \mathbb{T}^2$, the map $r \in [u, t] \mapsto \psi_r'(x) = -\gamma \Delta (\Upsilon(t-r)\varphi(x))$ is continuous since $\Upsilon(t)$ is a strongly continuous semi-group and $-\gamma \Delta (\Upsilon(t-r)\varphi(x)) = \Upsilon(t-
Thus replacing \( \phi \) by \( \psi \) in the above equation, we obtain

\[
(U^N_t, \varphi) = (U^N_u, \Upsilon(t-u)\varphi)
\]

\[
+ \beta \int_u^t \left( Z^N_r, \left( \mu_r^{I,N}, K \frac{\varphi_r^S}{(\mu_r^N, K)} \right) \Upsilon(t-r)\varphi K \right) \, dr
\]

\[
- \beta \int_u^t \left( U^N_r, \Upsilon(t-r)\varphi \frac{\mu_r^{I,N}}{(\mu_r^N, K)} K \right) \, dr
\]

\[
- \beta \int_u^t \left( V^N_r, \frac{\varphi_r^S}{(\mu_r^N, K)} \Upsilon(t-r)\varphi K \right) \, dr
\]

\[
+ \int_u^t (\Upsilon(t-r)\varphi, d\tilde{M}^N_r).
\]

This proves (5.16). We obtain (5.17) by a similar argument. \( \square \)

**Proposition 5.24** There exists \( C > 0 \) such that for any stopping times \( \tau < \infty \) a.s. and \( \theta > 0 \),

\[
\mathbb{E} \left( \left\| \int_\tau^{\tau+\theta} \Upsilon(\tau+r) \, d\tilde{M}^N_r \right\|^2_{H^{-s}} \right) \leq C\theta, \tag{5.18}
\]

\[
\mathbb{E} \left( \left\| \int_\tau^{\tau+\theta} \Upsilon(\tau+r) \, d\tilde{L}^N_r \right\|^2_{H^{-s}} \right) \leq C\theta. \tag{5.19}
\]

**Proof of (5.18)** We first recall that \( (f_{i,n_1,n_2}^i)_{i,n_1,n_2} \) (as defined is Proposition 2.3) is a family of eigenfunctions of the operator \( \gamma \Delta \) associated to the family of eigenvalues \( (-\lambda_{n_1,n_2})_{n_1,n_2} \) and \( (\rho_{n_1,n_2})_{i,n_1,n_2} \) is an orthonormal basis of \( H^s \).

We also recall that

\[
\Upsilon^N_r(\varphi) = \beta \left( \mu_r^S, \varphi^2 \frac{\mu_r^{I,N}}{(\mu_r^N, K)} \right) + 2\gamma (\mu_r^S, (\nabla \varphi)^2),
\]

and

\[
\int_\tau^{\tau+\theta} (\Upsilon(\tau+r) \varphi, d\tilde{M}^N_r)
\]

\[
= \sqrt{\frac{2\gamma}{N}} \sum_{i=1}^N \int_\tau^{\tau+\theta} 1_{[E^i_r=S]} \nabla \Upsilon(\tau+r) \varphi(X^i_r) \, dB^i_r
\]

\[
- \sqrt{\frac{1}{N}} \sum_{i=1}^N \int_\tau^{\tau+\theta} \varphi(X^i_r) \, dB^i_r.
\]
\[
\int_0^\infty 1_{\{E_{i,j}^r = S\}} \Upsilon(\tau^N + \theta - r) \varphi(X_i^r) \left\{ u \leq \beta \sum_{j=1}^N \frac{k(x_i^r, x_j^r)}{\sum_{i=1}^N k(x_i^r, x_j^r)} 1_{\{E_{j}^r = I\}} \right\} \overline{M}^{i}(dr, du).
\]

Now since from (2.1) in Proposition 2.7, we have \(\nabla \Upsilon(t) \rho^i_{n_1, n_2} = e^{-\lambda_{n_1, n_2}} \nabla \rho^i_{n_1, n_2}\), from Lemma 7.1 below, we have (again the meaning of the notation \(\sum_{i,n_1,n_2}\) is explained in Remark 2.5)

\[
\mathbb{E} \left( \left\| \int_{\mathcal{T}}^{T+\theta} \Upsilon(\tau + \theta - r) d\tilde{M}^N_r \right\|^2_{H^{-s}} \right) 
\leq \sum_{i,n_1,n_2} \mathbb{E} \left( \left( \int_{\mathcal{T}}^{T+\theta} \Upsilon(\tau + \theta - r) \rho^i_{n_1, n_2} d\tilde{M}^N_r \right)^2 \right) 
= \sum_{i,n_1,n_2} \mathbb{E} \left( \int_0^{T+\theta} \Upsilon(\theta - r) \rho^i_{n_1, n_2} dr \right) 
= \sum_{i,n_1,n_2} \mathbb{E} \left( \int_0^\theta \Upsilon(\theta - r) e^{-(\theta - r)\lambda_{n_1, n_2}} \rho^i_{n_1, n_2} dr \right) 
\leq \mathbb{E} \left( \int_0^\theta \Upsilon(\theta - r) \sum_{i,n_1,n_2} \rho^i_{n_1, n_2} dr \right) \leq \theta C.
\]

A similar argument yields (5.19). \(\square\)

**Proposition 5.25** For all \(T > 0\),

\[
\sup_{N \geq 1} \mathbb{E} (\sup_{0 \leq t \leq T} \|U^N_t\|^2_{H^{-s}}) < \infty,
\]

\[
\sup_{N \geq 1} \mathbb{E} (\sup_{0 \leq t \leq T} \|V^N_t\|^2_{H^{-s}}) < \infty.
\]

**Proof** Choosing \(u = 0\) in Eqs. (5.16) and (5.17), we get the estimates

\[
\|U^N_t\|^2_{H^{-s}} \leq 5\|\Upsilon(t)U^N_0\|^2_{H^{-s}} + 5\beta^2 \int_0^t \|\Upsilon(t - r)(G^{S,I,N}_r)^* Z^N_r\|^2_{H^{-s}} dr 
+ 5\beta^2 \int_0^t \|\Upsilon(t - r)(G^{S,I,N}_r)^* U^N_r\|^2_{H^{-s}} dr 
+ 6\|\Upsilon(t) d\tilde{M}_t\|^2_{H^{-s}},
\]

\[
\|V^N_t\|^2_{H^{-s}} \leq 6\|\Upsilon(t)V^N_0\|^2_{H^{-s}} + 6\beta^2 \int_0^t \|\Upsilon(t - r)(G^{S,I,N}_r)^* Z^N_r\|^2_{H^{-s}} dr 
+ 6\beta^2 \int_0^t \|\Upsilon(t - r)(G^{S,I,N}_r)^* U^N_r\|^2_{H^{-s}} dr 
+ 6\beta^2 \int_0^t \|\Upsilon(t - r)(G^{S,I,N}_r)^* V^N_r\|^2_{H^{-s}} dr
\]

\(\square\) Springer
+6t\alpha^2 \int_0^t \|V_r^N\|_{H^{-s},r}^2 \, dr + 6 \left\| \psi(t-r) d\tilde{L}_r^N \right\|^2_{H^{-s}}.

From Corollary 5.22, we have

\[
\|U_t^N\|_{H^{-s}}^2 \leq 5\|U_0^N\|_{H^{-s}}^2 + 5t\beta^2 C \sup_x \|K(x, .)\|_{H^s}^2 \int_0^t \|U_r^N\|_{H^{-s}}^2 \, dr
+ 5t\beta^2 \sup_x \left\| \frac{K(x, .)}{\int_{\mathbb{T}^2} K(x', .) \mu_r(dx')} \right\|^2_{H^s} \int_0^t \|V_r^N\|_{H^{-s}}^2 \, dr
+ 5t\beta^2 C \sup_x \|K(x, .)\|_{H^s}^2 \int_0^t \|Z_r^N\|_{H^{-s}}^2 \, dr
+ \left\| \psi(t-r) d\tilde{M}_r^N \right\|^2_{H^{-s}},
\]

\[
\|V_t^N\|_{H^{-s}}^2 \leq 6\|V_0^N\|_{H^{-s}}^2 + 6t\beta^2 C \sup_x \|K(x, .)\|_{H^s}^2 \int_0^t \|U_r^N\|_{H^{-s}}^2 \, dr
+ 6t \left( \beta^2 C \sup_x \left\| \frac{K(x, .)}{\int_{\mathbb{T}^2} K(x', .) \mu_r(dx')} \right\|^2_{H^s} + \alpha^2 \right) \int_0^t \|V_r^N\|_{H^{-s}}^2 \, dr
+ 6t\beta^2 C \sup_x \|K(x, .)\|_{H^s}^2 \int_0^t \|Z_r^N\|_{H^{-s}}^2 \, dr + \left\| \psi(t-r) d\tilde{L}_r^N \right\|^2_{H^{-s}}.
\]

Thus from Lemmas 5.2 and 7.2 in the “Appendix” below, and Lemma 2.9 above, we have

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|U_t^N\|_{H^{-s}}^2 \right) \leq 5\mathbb{E}(\|U_0^N\|_{H^{-s}}^2) + 5T \beta^2 C
\int_0^T \left\{ \mathbb{E}(\sup_{0 \leq r \leq t} \|U_r^N\|_{H^{-s}}^2) + \mathbb{E}(\sup_{0 \leq r \leq t} \|V_r^N\|_{H^{-s}}^2) \right\} dt
+ 5T^2 \beta^2 C \mathbb{E}(\sup_{0 \leq t \leq T} \|Z_t^N\|_{H^{-s}}) + C \mathbb{E}(\|\tilde{M}_T^N\|_{H^{-s}}^2),
\]

(5.20)

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|V_t^N\|_{H^{-s}}^2 \right) \leq 6\mathbb{E}(\|V_0^N\|_{H^{-s}}^2) + 6T (\beta^2 C + \alpha^2)
\int_0^T \left\{ \mathbb{E}(\sup_{0 \leq r \leq t} \|U_r^N\|_{H^{-s}}^2) + \mathbb{E}(\sup_{0 \leq r \leq t} \|V_r^N\|_{H^{-s}}^2) \right\} dt
+ 6T^2 \beta^2 C \mathbb{E}(\sup_{0 \leq t \leq T} \|Z_t^N\|_{H^{-s}}^2) + C \mathbb{E}(\|\tilde{L}_T^N\|_{H^{-s}}^2).
\]

(5.21)

Summing up (5.20) and (5.21), and applying Gronwall’s lemma, we deduce the result from Proposition 3.2 in Sect. 3.2, Proposition 5.11 and Corollary 5.4. □

Now we can establish the tightness of the sequences \((U^N)_N\) and \((V^N)_N\).

**Proposition 5.26** Both sequences of processes \(U^N\) and \(V^N\) are tight in \(D(\mathbb{R}_+, H^{-s})\).
Proof We only establish the tightness of $U^N$ by showing that the conditions of the Proposition 5.8 are satisfied.

- (T1) is obtained by using the Proposition 5.25 and applying an argument similar to that of the proof of (T1) in Theorem 5.7.
- Proof of (T2). Let $T > 0$, $\epsilon_1, \epsilon_2 > 0$, $(\tau^N)_N$ a family of stopping times with $\tau^N \leq T$. From Eq. (5.16), we have

$$U^N_{\tau^N + \theta} - U^N_{\tau^N} = (\Upsilon(\theta) - I_d)U^N_{\tau^N} + \beta \int_{\tau^N}^{\tau^N + \theta} \Upsilon(\tau^N + r)(G^S_r, I^N_r)^*Z^N_r dr$$

$$- \beta \int_{\tau^N}^{\tau^N + \theta} \Upsilon(\tau^N + r)(G^I_r, N^N_r)^*U^N_r dr$$

$$- \beta \int_{\tau^N}^{\tau^N + \theta} \Upsilon(\tau^N + r)(G^S_r)^*V^N_r dr$$

$$+ \int_{\tau^N}^{\tau^N + \theta} \Upsilon(\tau^N + r)d\tilde{M}^N_r$$

where

$$J^{S, I, N}_r(Z^N, U^N, V^N) = (G^S_r, I^N_r)^*Z^N_r - (G^I_r, N^N_r)^*U^N_r - (G^S_r)^*V^N_r.$$ 

We will next show that there exist $\delta > 0$ and $N_0 \geq 1$ such that

$$\sup_{N \geq N_0} \sup_{\delta \geq \theta} \mathbb{P}\left( \left\| (\Upsilon(\theta) - I_d)U^N_{\tau^N} \right\|_{H^{-s}} \geq \epsilon_1 \right) \leq \epsilon_2,$$  

$$\sup_{N \geq N_0} \sup_{\delta \geq \theta} \mathbb{P}\left( \left\| \beta \int_{\tau^N}^{\tau^N + \theta} \Upsilon(\tau^N + r)(G^S_r, I^N_r)^*Z^N_r - (G^I_r, N^N_r)^*U^N_r - (G^S_r)^*V^N_r \right\|_{H^{-s}} \geq \epsilon_1 \right) \leq \epsilon_2,$$  

$$\sup_{N \geq N_0} \sup_{\delta \geq \theta} \mathbb{P}\left( \left\| \int_{\tau^N}^{\tau^N + \theta} \Upsilon(\tau^N + r)d\tilde{M}^N_r \right\|_{H^{-s}} \geq \epsilon_1 \right) \leq \epsilon_2.$$  

- (5.22) is proved by a reasoning similar to that in the proof of (5.4) in Proposition 5.12.
- Proof of (5.23). Let $l \in \mathbb{R}_+ \setminus [0, 1]$, we find $\delta > 0$ such that $\tau^N + \delta \leq lT$ and such that (5.23) is satisfied. Since $\forall \varphi \in H^s, \left\| \Upsilon(\tau^N + \theta)\varphi \right\|_{H^s} \leq \left\| \varphi \right\|_{H^s}$, from
Corollary 5.22, Lemmas 5.2 and 7.2 below, and Propositions 5.11 and 5.25, we have

\[
\mathbb{P} \left( \left\| \int_{\tau}^{\tau+\theta} \gamma(\tau + \theta - r) J_r^{S.1.1}(Z^N, U^N, V^N) dr \right\|_{H_s^{H-s}} \geq \varepsilon_1 \right) 
\]

\[
\leq \frac{\beta^2}{\varepsilon_1^2} \mathbb{E} \left( \left\| \int_{\tau}^{\tau+\theta} \gamma(\tau + \theta - r) J_r^{S.1.1}(Z^N, U^N, V^N) dr \right\|_{H_s^{H-s}}^2 \right) 
\]

\[
\leq \frac{\beta^2 \theta}{\varepsilon_1^2} \mathbb{E} \left( \left\| \gamma(\tau + \theta - r) J_r^{S.1.1}(Z^N, U^N, V^N) \right\|_{H_s^{H-s}} \right)
\]

\[
\leq \frac{\beta^2 \theta C}{\varepsilon_1^2} \sup_{\gamma(\cdot, y)} \left\| K(\cdot, \cdot) \right\|_{H_s^{H-s}} \mathbb{E} \left( \left\| \int_{\tau}^{\tau+\theta} \left\{ \left\| Z_r^N \right\|_{H_s^{H-s}}^2 + \left\| U_r^N \right\|_{H_s^{H-s}}^2 \right\} dr \right) 
\]

\[
\leq \frac{\beta^2 \theta C}{\varepsilon_1^2} \sup_{\gamma(\cdot, y)} \left\| K(\cdot, \cdot) \right\|_{H_s^{H-s}} \mathbb{E} \left( \int_{\tau}^{\tau+\theta} \left\| V_r^N \right\|_{H_s^{H-s}}^2 dr \right)
\]

So (5.23) is established.

Proof of (5.24). From (5.18) in Proposition 5.24 we have,

\[
\mathbb{P} \left( \left\| \int_{\tau}^{\tau+\theta} \gamma(\tau + \theta - r) \tilde{M}_r^N \right\|_{H_s^{H-s}} \geq \varepsilon_1 \right) 
\]

\[
\leq \frac{1}{\varepsilon_1^2} \mathbb{E} \left( \left\| \int_{\tau}^{\tau+\theta} \gamma(\tau + \theta - r) \tilde{M}_r^N \right\|_{H_s^{H-s}}^2 \right) 
\]

\[
\leq \frac{1}{\varepsilon_1^2} \delta C,
\]

hence (5.24) is proved.

\[\square\]

To show that all converging subsequences of \((U^N, V^N)_{N \geq 1}\) have the same limit we will need the next two Lemmas.

**Lemma 5.27** For any \(t \geq 0, \varphi \in H^3(T^2)\), as \(N \rightarrow \infty\),

\[
\int_0^t \mathbb{E} \left( \| [G_r^{I,N} - G_r^I] \gamma(t - r) \varphi \|_{H_s}^2 \right) dt \rightarrow 0.
\]
Proof Since \( s < 3 \), \( H^3 \hookrightarrow H^s \), thus

\[
\int_0^t \mathbb{E}\left( \| [G_r^{I,N} - G_r^I] \gamma (t - r) \varphi \|_{H^3}^2 \right) dt \leq C \int_0^t \mathbb{E}\left( \| [G_r^{I,N} - G_r^I] \gamma (t - r) \varphi \|_{H^3}^2 \right) dt.
\]

Furthermore as \( H^3 \) is a Banach algebra (see Proposition 2.6) and \( \| \gamma (t) \varphi \|_{H^3} \leq C \| \varphi \|_{H^3} \),

\[
\| [G_r^{I,N} - G_r^I] \gamma (t - r) \varphi \|_{H^3} = \left\| \gamma (t - r) \varphi \left( \mu_r^{I,N} - \mu_r^I, \frac{K}{\mu_r, K} \right) \right\|_{H^3} \leq C \| \varphi \|_{H^3} \left\| \mu_r^{I,N} - \mu_r^I, \frac{K}{\mu_r, K} \right\|_{H^3}.
\]

On the other hand

\[
\left\| \left( \mu_r^{I,N} - \mu_r^I, \frac{K}{\mu_r, K} \right) \right\|_{H^3}^2 = \sum_{|\eta| \leq 3} \int_{T^2} \left| \int_{T^2} \frac{D^{\eta} K(x, y)}{K(x', y) \mu_r(x') \mu_r(y)} (\mu_r^{I,N} - \mu_r^I)(dy) \right|^2 dx,
\]

furthermore from Lemma 5.1 the map \( y \mapsto D^{\eta} K(x, y) \) is continuous and bounded by \( C \max_{0 \leq |\eta| \leq 3} \| k^{(|\eta|)} \|_{\infty} \). Thus as the map \( y \mapsto \int_{T^2} K(x', y) \mu_r(dx') \) is also continuous and lower bounded by a positive constant, the map \( y \mapsto \int_{T^2} \frac{D^{\eta} K(x, y)}{K(x', y) \mu_r(dx')} (\mu_r^{I,N} - \mu_r^I)(dy) \) is continuous and bounded by \( C \max_{0 \leq |\eta| \leq 3} \| k^{(|\eta|)} \|_{\infty} \). So we deduce from Theorem 4.7 that

\[
\int_{T^2} \int_{T^2} \frac{D^{\eta} K(x, y)}{K(x', y) \mu_r(dx')} (\mu_r^{I,N} - \mu_r^I)(dy) \left\| \left( \mu_r^{I,N} - \mu_r^I, \frac{K}{\mu_r, K} \right) \right\|_{H^3}^2 \underset{N \to \infty}{\to} 0.
\]

According to Lebesgue’s dominated convergence theorem,

\[
\mathbb{E}\left( \left\| \left( \mu_r^{I,N} - \mu_r^I, \frac{K}{\mu_r, K} \right) \right\|_{H^3}^2 \right) \underset{N \to \infty}{\to} 0.
\]

Thus

\[
\int_0^t \mathbb{E}\left( \| [G_r^{I,N} - G_r^I] \gamma (t - r) \varphi \|_{H^3}^2 \right) dt \leq C \| \varphi \|_{H^3}^2 \int_0^t \mathbb{E}\left( \left\| \left( \mu_r^{I,N} - \mu_r^I, \frac{K}{\mu_r, K} \right) \right\|_{H^3}^2 \right) dr \to 0,
\]

as \( N \to \infty \).

Hence the result. \( \square \)
Lemma 5.28 For any $t \geq 0, \varphi \in H^s(\mathbb{T}^2)$, as $N \to \infty$,

$$
\int_0^t \mathbb{E} \left( \| [G_r^{S,I,N} - G_r^{S,I}] \Upsilon (t - r) \varphi \|_{\mathcal{H}^s}^2 \right) dt \to 0.
$$

Proof We have

$$G_r^{S,I,N} (\Upsilon (t - r) \varphi) (x) - G_r^{S,I} (\Upsilon (t - r) \varphi) (x)$$

$$= \left( \mu_{r,I,N}, K (x, y) \left( \mu_{r,S,N}^{I}, K \Upsilon (t - r) \varphi \right) / (\mu_{r,N}, K) (\mu_r, K) \right)$$

$$- \left( \mu_{r,I,N}, K (x, y) \left( \mu_{r,S}^{I}, K \Upsilon (t - r) \varphi \right) / (\mu_r, K) (\mu_r, K)^2 \right)$$

$$= \left( \mu_{r,I,N}^{I} - \mu_{r,I,N}, K (x, y) \left( \mu_{r,S,N}^{I}, K \Upsilon (t - r) \varphi \right) / (\mu_{r,N}, K) (\mu_r, K)^2 \right)$$

Furthermore:

(a) Since:

$$\left| \frac{\int_{\mathbb{T}^2} K (x', y) \Upsilon (t - r) \varphi (x') \mu_r^{S} (dx')} {\int_{\mathbb{T}^2} K (x'', y) \mu_r (dx'')} \right| \leq \| \varphi \|_{\infty}$$

- The map $y \in \mathbb{T}^2 \mapsto \int_{\mathbb{T}^2} K (x', y) \mu_r (dx')$ is continuous and lower bounded by a positive constant.

- Under (H2), The map $y \in \mathbb{T}^2 \mapsto \int_{\mathbb{T}^2} K (x', y) \Upsilon (t - r) \varphi (x') \mu_r^{S} (dx')$ is continuous.

- From Lemma 5.1, for any $x \in \mathbb{T}^2, |\eta| \leq 3$, the map $y \in \mathbb{T}^2 \mapsto D^\eta K (x, y)$ is continuous and bounded by $C \max_{0 \leq |\eta| \leq 3} \| k^{(\eta)} \|_{\infty}$, the map $y \in \mathbb{T}^2 \mapsto D^\eta K (x, y) \times \frac{\int_{\mathbb{T}^2} K (x', y) \Upsilon (t - r) \varphi (x') \mu_r^{S} (dx')}{\left( \int_{\mathbb{T}^2} K (x'', y) \mu_r (dx'') \right)^2}$ is continuous and bounded by $C \| \varphi \|_{\infty} \max_{0 \leq |\eta| \leq 3} \| k^{(\eta)} \|_{\infty}$, hence

$$\left| \int_{\mathbb{T}^2} D^\eta K (x, y) \times \frac{\int_{\mathbb{T}^2} K (x', y) \Upsilon (t - r) \varphi (x') \mu_r^{S} (dx')}{\left( \int_{\mathbb{T}^2} K (x'', y) \mu_r (dx'') \right)^2} (\mu_{r,I,N}^{I} - \mu_{r,I}^{I}) (dy) \right|^2 \to 0.$$
Furthermore, according to Lebesgue’s dominated convergence theorem, we have

\[
\int_0^t \mathbb{E}\left(\left\| \left( \mu_{r,N}^{1,N} - \mu_r^I, K \frac{\left( \mu_r^S, K \Upsilon(t-r) \varphi \right)}{\left(\mu_r, K\right)^2} \right) \right\|^2_{H^3}\right) dr \\
\leq C \int_0^t \mathbb{E}\left(\left\| \left( \mu_{r,N}^{1,N} - \mu_r^I, K \frac{\left( \mu_r^S, K \Upsilon(t-r) \varphi \right)}{\left(\mu_r, K\right)^2} \right) \right\|^2_{H^3}\right) dr \\
\leq C \int_0^t \sum_{|\eta| \leq 3} \int_{\mathbb{T}^2} \mathbb{E}\left( \int_{\mathbb{T}^2} D^\eta K(x, y) \times \left( \int_{\mathbb{T}^2} K(x'', y) \mu_r(dx'') \right)^2 \right) dx dr \\
\rightarrow 0, \quad \text{as } N \rightarrow \infty.
\]

(b) Let us prove that \( \lim_{N \rightarrow \infty} \int_0^t \mathbb{E}\left(\left\| \left( \mu_{r,N}^{1,N}, K \frac{\left( \mu_r^S,N - \mu_r^S, K \Upsilon(t-r) \varphi \right)}{\left(\mu_r, K\right)^2} \right) \right\|^2_{H^3}\right) dt = 0. \)

Under (H2):

- for any \( y \in \mathbb{T}^2, 0 \leq r \leq t, \) the map \( x \in \mathbb{T}^2 \mapsto K(x, y) \Upsilon(t-r) \varphi(x) \) is Lipschitz and bounded and its \( \| \cdot \|_\infty \) and \( \| \cdot \|_2 \) norms are bounded by \( C \| k \|_\infty \| \varphi \|_\infty \) and \( C \| k \|_\infty + 2\sqrt{2}C_k \| \varphi \|_\infty \) (where \( 2\sqrt{2}C_k \) is the Lipschitz constant of \( K \) (see Lemma 4.8)) respectively.

- For any \( x \in \mathbb{T}^2, |\eta| \leq 3, \) the map \( y \in \mathbb{T}^2 \mapsto D^\eta K(x, y) \) is continuous and bounded by \( C \max_{0 \leq |\eta| \leq 3} \| k^{(\|\eta\|)} \|_\infty. \)

- \( \forall y \in \mathbb{T}^2, \frac{1}{(\mu_r, K(\cdot, y))} \) is bounded by a positive constant independent of \( y. \)

Thus

\[
\left\| \left( \mu_{r,N}^{1,N}, K \frac{\left( \mu_r^S,N - \mu_r^S, K \Upsilon(t-r) \varphi \right)}{\left(\mu_r, K\right)^2} \right) \right\|^2_{H^3} \\
\leq C \left\| \left( \mu_{r,N}^{1,N}, K \frac{\left( \mu_r^S,N - \mu_r^S, K \Upsilon(t-r) \varphi \right)}{\left(\mu_r, K\right)^2} \right) \right\|^2_{H^3} \\
= C \sum_{|\eta| \leq 3} \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} D^\eta K(x, y) \\
\times \left( \int_{\mathbb{T}^2} K(x'', y) \mu_r(dx'') \right)^2 \mu_r^{1,N} (dy) \right) \right\|^2 dx \\
\leq C \sum_{|\eta| \leq 3} \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} D^\eta K(x, y)
\]
Thus

\[
\mathbf{E}\left( \left\| \mu_r^{1, N} \right\|_{H^3}^2 \right) \leq C \sum_{|\eta| \leq 3} \int_{T^2} \int_{T^2} |D^{\eta} K(x, y)| \times \left| \int_{T^2} Y(t - r) \varphi(x') K(x', y) \mu_r^{S,N} (dx') \right| \left| \int_{T^2} K(u, y) \mu_r^{N} (du) \left( \int_{T^2} K(u, y) \mu_r (du) \right)^{-1} \right|^2 dx dy
\]

as \( N \to \infty \).

(c) Finally we show that

\[
\lim_{N \to \infty} \int_0^t \mathbf{E}\left( \left\| \mu_r^{1, N} \right\|_{H^3}^2 \right) dt = 0.
\]

Since for all \( x \in T^2 \), \( K(x, \cdot) \) is Lipschitz and bounded by \( \|k\|_{\infty} \) and for any \( x \in T^2 \), \( |\eta| \leq 3 \), the map \( y \in T^2 \mapsto D^{\eta} K(x, y) \) is bounded by \( C \max_{0 \leq |\eta| \leq 3} \|k(\eta)\|_{\infty} \). Therefore, we have

\[
\mathbf{E}\left( \left\| \mu_r^{1, N} \right\|_{H^3}^2 \right) \leq C \sum_{|\eta| \leq 3} \int_{T^2} \int_{T^2} |D^{\eta} K(x, y)| \times \left| \int_{T^2} Y(t - r) \varphi(x') K(x', y) \mu_r^{S,N} (dx') \right| \left| \int_{T^2} K(u, y) \mu_r^{N} (du) \left( \int_{T^2} K(u, y) \mu_r (du) \right)^{-1} \right|^2 dx dy
\]

Thus

\[
\int_0^t \mathbf{E}\left( \left\| \mu_r^{1, N} \right\|_{H^3}^2 \right) dt \leq C \sum_{|\eta| \leq 3} \int_{T^2} \int_{T^2} |D^{\eta} K(x, y)| \times \left| \int_{T^2} K(u, y) \mu_r^{N} (du) \left( \int_{T^2} K(u, y) \mu_r (du) \right)^{-1} \right|^2 dx dy
\]

as \( N \to \infty \).
as $N \to \infty$. \hfill \Box

The next Proposition establishes the evolution equations of all limit points $(U, V)$ of the sequence $(U^N, V^N)$.

**Proposition 5.29** Any limit point $(U, V)$ of the sequence $(U^N, V^N)$ satisfies

\[ U_t = \Upsilon(t)U_0 + \beta \int_0^t \Upsilon(t-r)(G_r^{S,I})^*Z_r dr - \beta \int_0^t \Upsilon(t-r)(G_r^I)^*U_r dr \]

\[ - \beta \int_0^t \Upsilon(t-r)(G_r^S)^*V_r dr + \int_0^t \Upsilon(t-r)V_r^1, \] (5.25)

\[ V_t = \Upsilon(t)V_0 - \beta \int_0^t \Upsilon(t-r)(G_r^{S,I})^*Z_r dr \]

\[ + \beta \int_0^t \Upsilon(t-r)(G_r^I)^*U_r dr + \beta \int_0^t \Upsilon(t-r)(G_r^S)^*V_r dr \]

\[ - \alpha \int_0^t \Upsilon(t-r)V_r dr + \int_0^t \Upsilon(t-r)dW_r^2. \] (5.26)

**Proof** We prove this Proposition by taking the weak limit in the Eqs. (5.16) and (5.17). Note first that from Propositions 5.12 and 5.26 there exists a subsequence along which the sequences $(U^N, V^N, Z^N)_N$ converges in law towards $(U, V, Z)$. For any $\varphi \in H^s$, one has

\[ (\Upsilon(t)U_0^N, \varphi) + \int_0^t (\Upsilon(t-r)\varphi, d\tilde{M}_r^N) = (U_t^N, \varphi) \]

\[ - \beta \int_0^t (\Upsilon(t-r)(G_r^{S,I})^*Z_r^N, \varphi)dr + \beta \int_0^t (\Upsilon(t-r)(G_r^I)^*U_r^N, \varphi)dr \]

\[ + \beta \int_0^t (\Upsilon(t-r)(G_r^S)^*V_r^N, \varphi)dr + \beta \int_0^t (\Upsilon(t-r)(G_r^{I,N})^* - (G_r^I)^*U_r^N, \varphi)dr \]

\[ - \beta \int_0^t (\Upsilon(t-r)(G_r^{S,I,N})^* - (G_r^{S,I})^*Z_r^N, \varphi)dr, \]

\[ (\Upsilon(t)V_0^N, \varphi) + \int_0^t (\Upsilon(t-r)\varphi, d\tilde{L}_r^N) = (V_t^N, \varphi) \]

\[ + \beta \int_0^t (\Upsilon(t-r)(G_r^{S,I})^*Z_r^N, \varphi)dr - \beta \int_0^t (\Upsilon(t-r)(G_r^I)^*U_r^N, \varphi)dr \]
Thus

\[
(\Upsilon(t)U_0^N, \varphi) + \int_0^t \left( \Upsilon(t-r)\varphi, d\tilde{M}_r^N \right) = \Psi_{1,t,\varphi}(U^N, V^N, Z^N)
\]

\[
+ \beta \int_0^t ([\Upsilon(t-r)(G_r^{1,N})^{*} - (G_r^{1})^{*}]U_r^N, \varphi) dr
\]

\[
- \beta \int_0^t (\Upsilon(t-r)[(G_r^{S,1,N})^{*} - (G_r^{S,1})^{*}]V_r^N, \varphi) dr,
\]

\[
(\Upsilon(t)V_0^N, \varphi) + \int_0^t \left( \Upsilon(t-r)\varphi, d\tilde{L}_r^N \right) = \Psi_{2,t,\varphi}(U^N, V^N, Z^N)
\]

\[
- \beta \int_0^t ([\Upsilon(t-r)(G_r^{I,1,N})^{*} - (G_r^{I})^{*}]U_r^N, \varphi) dr
\]

\[
+ \beta \int_0^t (\Upsilon(t-r)[(G_r^{S,1,N})^{*} - (G_r^{S,1})^{*}]Z_r^N, \varphi) dr.
\]

With

\[
\Psi_{1,t,\varphi}(U^N, V^N, Z^N) = (U^N_t, \varphi) - \beta \int_0^t (\Upsilon(t-r)(G_r^{S,1})^{*}Z_r^N, \varphi) dr
\]

\[
+ \beta \int_0^t (\Upsilon(t-r)(G_r^{I})^{*}U_r^N, \varphi) dr
\]

\[
+ \beta \int_0^t (\Upsilon(t-r)(G_r^{S})^{*}V_r^N, \varphi) dr,
\]

\[
\Psi_{2,t,\varphi}(U^N, V^N, Z^N) = (V^N_t, \varphi) + \beta \int_0^t (\Upsilon(t-r)(G_r^{S,1})^{*}Z_r^N, \varphi) dr
\]

\[
- \beta \int_0^t (\Upsilon(t-r)(G_r^{I})^{*}U_r^N, \varphi) dr
\]

\[
- \int_0^t (\Upsilon(t-r)[\beta(G_r^{S})^{*} - \alpha]V_r^N, \varphi) dr.
\]

Furthermore.

1- From Lemma 5.27 and Proposition 5.25 \( \int_0^t U_r^N, [G_r^{I,N} - G_r^{I}]\Upsilon(t-r)\varphi \, dr \rightarrow 0 \) in \( L^1(\mathbb{P}) \), locally uniformly in \( t \).
Indeed,
\[ \mathbb{E}\left( \sup_{0 \leq t \leq T} \int_0^t \left( U_r^N, [G_r^{I,N} - G_r^{I}] \mathcal{Y}(t - r) \varphi \right) dr \right) \leq \sqrt{T} \sup_{N \geq 1} \mathbb{E}\left( \sup_{0 \leq t \leq T} \left\| U_r^N \right\|_{H^{-s}}^2 \right)^{\frac{1}{2}} \left[ \int_0^T \mathbb{E}\left( \left\| [G_r^{I,N} - G_r^{I}] \mathcal{Y}(t - r) \varphi \right\|_{H^s} \right) dr \right]^{\frac{1}{2}}. \]

2- The fact that \( \int_0^T (\mathcal{Y}(t - r) [(G_r^{S,N})^* - (G_r^{S,I})^*] Z_r^N, \varphi) dr \to 0 \) in \( L^1(\mathbb{P}) \), locally uniformly in \( t \), follows from the same argument as the one in 1-, using this time Lemma 5.28 and Proposition 5.11.

3- Using Proposition 5.21, it is easy to see that the maps \((\Psi_1, \varphi, \Psi_2, \varphi)\) is continuous from \( D(\mathbb{R}^+, H^{-s})^3 \) into \( C(\mathbb{R}^+, \mathbb{R}^2) \). Thus as \((U^N, V^N, Z^N)\) converges in law in \((D(\mathbb{R}^+, H^{-s}))^3 \) towards \((U, V, Z)\), then \((\Psi_1, \varphi(U^N, V^N, Z^N), \Psi_2, \varphi(U^N, V^N, Z^N))\) converges in law towards \((\Psi_1, \varphi(U, V, Z), \Psi_2, \varphi(U, V, Z))\).

4- \((\mathcal{Y}(\cdot) U_0^N, \varphi) + \int_0^T (\mathcal{Y}(\cdot - r) \varphi, d\tilde{M}_r^N), (\mathcal{Y}(\cdot) V_0^N, \varphi) + \int_0^T (\mathcal{Y}(\cdot - r) \varphi, dW^2_r)\) converges in law towards \((\mathcal{Y}(\cdot) U_0, \varphi) + \int_0^T (\mathcal{Y}(\cdot - r) \varphi, dW^1_r), (\mathcal{Y}(\cdot) V_0, \varphi) + \int_0^T (\mathcal{Y}(\cdot - r) \varphi, dW^2_r)\) in \((D(\mathbb{R}^+, H^{-s})^2)\) since \((\mathcal{Y}(\cdot) U_0^N, \varphi), (\mathcal{Y}(\cdot) V_0^N, \varphi), \int_0^T (\mathcal{Y}(\cdot - r) \varphi, dW^1_r), \int_0^T (\mathcal{Y}(\cdot - r) \varphi, dW^2_r)\) converges in law towards \((\mathcal{Y}(\cdot) U_0, \varphi), (\mathcal{Y}(\cdot) V_0, \varphi), \int_0^T (\mathcal{Y}(\cdot - r) \varphi, dW^1_r), \int_0^T (\mathcal{Y}(\cdot - r) \varphi, dW^2_r)\) in \((C(\mathbb{R}^+, H^{-s}))^2 \) times \((D(\mathbb{R}^+, H^{-s})^2)\), which in turn follows from the fact that \((\mathcal{Y}(\cdot) U_0^N, \varphi), (\mathcal{Y}(\cdot) V_0^N, \varphi)\) converges in law towards \((\mathcal{Y}(\cdot) U_0, \varphi), (\mathcal{Y}(\cdot) V_0, \varphi)\) in \((C(\mathbb{R}^+, H^{-s}))^2\) (see Theorem 3.3) and \((\int_0^T (\mathcal{Y}(\cdot - r) \varphi, d\tilde{M}_r^N), \int_0^T (\mathcal{Y}(\cdot - r) \varphi, d\tilde{L}_r^N)\) converges in law towards \((\int_0^T (\mathcal{Y}(\cdot - r) \varphi, dM_r^N), \int_0^T (\mathcal{Y}(\cdot - r) \varphi, dW^1_r)\) in \((D(\mathbb{R}^+, H^{-s}))^2\) (which follows from Proposition 5.20) and \((\mathcal{Y}(\cdot) U_0^N, \varphi), (\mathcal{Y}(\cdot) V_0^N, \varphi)\) are globally independant of \((\int_0^T (\mathcal{Y}(\cdot - r) \varphi, d\tilde{M}_r^N), \int_0^T (\mathcal{Y}(\cdot - r) \varphi, d\tilde{L}_r^N)\).

Thus from 1-, 2-, 3- and 4-, we obtain the result of the statement. \(\square\)

From Proposition 5.18 we deduce that all limit points \((U, V)\) of \((U^N, V^N)_{N \geq 1}\) are elements of \((C(\mathbb{R}^+, H^{-s}))^2\), thus since it is so easy to see that Eqs. (5.25) and (5.26) are the same as Eqs. (5.9) and (5.10) respectively, we end the proof of Theorem 5.16 by showing that the system formed by the Eqs. (5.25) and (5.26) admits a unique solution \((U, V) \in (C(\mathbb{R}^+, H^{-s}))^2\).
Proposition 5.30 Suppose that \((U^1, V^1)\) and \((U^2, V^2)\) both belong to \((C([\mathbb{R}^+, H^{-s}])^2\) and are solutions to equations (6.26) and (6.27) with \((U^1_0, V^1_0) = (U^2_0, V^2_0)\) then \((U^1, V^1) = (U^2, V^2)\)

**Proof** We have

\[
U^1_t - U^2_t = -\beta \int_0^t \Upsilon(t - r)(G^1_r)^*(U^1_r - U^2_r)dr - \beta \int_0^t \Upsilon(t - r)(G^2_r)^*(V^1_r - V^2_r)dr,
\]

thus

\[
\|U^1_t - U^2_t\|_{H^{-s}} \leq \beta \int_0^t \| \Upsilon(t - r)(G^1_r)^*(U^1_r - U^2_r)\|_{H^{-s}}dr + \beta \int_0^t \| \Upsilon(t - r)(G^2_r)^*(V^1_r - V^2_r)\|_{H^{-s}}dr.
\]

Moreover from Corollary 5.22, we deduce that

\[
\|U^1_t - U^2_t\|_{H^{-s}} \leq \beta C \sup_y \|K(., y)\|_{H^s} \int_0^t \|U^1_r - U^2_r\|_{H^{-s}}dr + \beta C \sup_x \left\| \frac{K(x, .)}{\int_{\mathbb{T}^2} K(x', .) \mu(x') dx'} \right\|_{H^s} \int_0^t \|V^1_r - V^2_r\|_{H^{-s}}dr,
\]

hence using Lemma 7.2 below and 5.2, we obtain

\[
\|U^1_t - U^2_t\|_{H^{-s}} \leq \beta C \int_0^t \left\{ \|U^1_r - U^2_r\|_{H^{-s}} + \|V^1_r - V^2_r\|_{H^{-s}} \right\}dr. \tag{5.27}
\]

On the other hand

\[
V^1_t - V^2_t = -(U^1_t - U^2_t) - \alpha \int_0^t \Upsilon(t - r)(V^1_r - V^2_r)dr,
\]

hence

\[
\|V^1_t - V^2_t\|_{H^{-s}} \leq \|U^1_t - U^2_t\|_{H^{-s}} + \alpha \int_0^t \|V^1_r - V^2_r\|_{H^{-s}}dr. \tag{5.28}
\]

Summing (5.27) and (5.28) and applying Gronwall’s lemma we obtain \((U^1, V^1) = (U^2, V^2)\). \(\square\)

### 5.4.3 Additional regularity properties of \(U\) and \(V\)

Again in this subsection we assume that \(s > 2\) Recall the abstract SPDE existence and uniqueness result from reference [18] or [19], which has been stated above in Proposition 2.10. It states an existence and uniqueness result of a solution with trajectories in the space \(L^2((0, T); F) \cap C([0, T]; H)\). It turns out that for the present
purpose, $H = H^{-s}$ and $F = H^{-s+1}$, for some arbitrarily chosen $s > 2$. If we have by some other argument existence and uniqueness of a solution with trajectories in $C([0, T]; H)$, then Proposition 2.10 provides us some additional regularity. We cannot prove existence of a solution in any space $C([0, T]; H^s(\mathbb{T}^2))$ with $s \leq 2$, but for any $s > 1$, the $H^s(\mathbb{T}^2)$ norm of the solution is square integrable as a function of $t$ (but it need not be continuous!). Since this result is a direct consequence of well-known results, we present it in this subsection.

However, there are two aspects which make the following presentation a bit tricky. First, we do not have one single equation, but a system of two coupled equations. This is not a real problem. The advantage of the abstract statement in Proposition 2.10 is that it can be directly adapted to a system, as we will see below. The other small trouble is that the Hilbert space $H$, into which the driving martingale takes its values, is not the usual space $L^2(\mathbb{T}^2)$, but a space of distributions $H^{-s}$. In order to be back in the standard situation, we take the image of the solutions by the operator $(I - \gamma \Delta)^{-s/2}$, which maps $H^{-s}(\mathbb{T}^2)$ into $L^2(\mathbb{T}^2)$.

The aim of this subsection is to establish the following result.

**Proposition 5.31** The pair of processes $(U, V)$ belongs to $L^2_{\text{loc}}(\mathbb{R}_+, (H^{-s+1})^2)$.

**Proof** Let us set

$$
U^s_t = (I_d - \gamma \Delta)^{-\frac{s}{2}} U_t, \quad V^s_t = (I_d - \gamma \Delta)^{-\frac{s}{2}} V_t, \quad Z^s_t = (I_d - \gamma \Delta)^{-\frac{s}{2}} Z_t,
$$

$$
W^{1,s}_t = (I_d - \gamma \Delta)^{-\frac{s}{2}} W^1_t, \quad W^{2,s}_t = (I_d - \gamma \Delta)^{-\frac{s}{2}} W^2_t.
$$

Given that $U_t \in H^{-s}$, $U^s_t \in L^2(\mathbb{T}^2)$. The same conclusion remains true for $V^s_t$, $Z^s_t$, $W^{1,s}_t$ and $W^{2,s}_t$. We now notice that we have the commutation $\Delta (I_d - \gamma \Delta)^{-\frac{s}{2}} = (I_d - \gamma \Delta)^{-\frac{s}{2}} \Delta$. Hence we deduce from the evolution equations (5.9) and (5.10) of $(U, V)$ given in Theorem 5.16 that

$$
U^s_t = U^s_0 + \gamma \int_0^t \Delta U^s_r dr + \beta \int_0^t (I_d - \Delta)^{-\frac{s}{2}} (G^{S,f}_r)^* (I_d - \gamma \Delta)^{\frac{s}{2}} Z^s_r dr - \beta \int_0^t (I_d - \gamma \Delta)^{-\frac{s}{2}} (G^f_r)^* (I_d - \gamma \Delta)^{\frac{s}{2}} U^s_r dr
$$

$$
- \beta \int_0^t (I_d - \gamma \Delta)^{-\frac{s}{2}} (G^S_r)^* (I_d - \gamma \Delta)^{\frac{s}{2}} V^s_r dr + W^{1,s}_t,
$$

$$
V^s_t = V^s_0 + \gamma \int_0^t \Delta V^s_r dr - \beta \int_0^t (I_d - \gamma \Delta)^{-\frac{s}{2}} (G^S_r)^* (I_d - \gamma \Delta)^{\frac{s}{2}} Z^s_r dr + \beta \int_0^t (I_d - \gamma \Delta)^{-\frac{s}{2}} (G^f_r)^* (I_d - \gamma \Delta)^{\frac{s}{2}} U^s_r dr
$$

$$
+ \beta \int_0^t (I_d - \gamma \Delta)^{-\frac{s}{2}} (G^S_r)^* (I_d - \gamma \Delta)^{\frac{s}{2}} V^s_r dr - \alpha \int_0^t V^s_r dr + W^{2,s}_t.
$$
Moreover the above system can be rewritten as one equation as follows

\[(dU_t^s, dV_t^s)' + A(t, (U_t^s, V_t^s))dt = J(t)dt + (dW_t^{1,s}, dW_t^{2,s})',\]

where

\[A(t, (U_t^s, V_t^s)) = \begin{pmatrix}
-\gamma \Delta + \beta(I_d - \gamma \Delta) \overline{z} (G_1^s)^*(I_d - \gamma \Delta) \overline{z} & \beta(I_d - \gamma \Delta) \overline{z} (G_2^s)^*(I_d - \gamma \Delta) \overline{z} \\
-\beta(I_d - \gamma \Delta) \overline{z} (G_1^s)^*(I_d - \gamma \Delta) \overline{z} & -\gamma \Delta - \beta(I_d - \gamma \Delta) \overline{z} (G_2^s)^*(I_d - \gamma \Delta) \overline{z} + \alpha
\end{pmatrix} (U_t^s, V_t^s)\]

\[J(t) = ((I_d - \gamma \Delta) \overline{z} (G_1^s)^*(I_d - \gamma \Delta) \overline{z} Z_t^s, -(I_d - \gamma \Delta) \overline{z} (G_2^s)^*(I_d - \gamma \Delta) \overline{z} Z_t^s)'.\]

and \((u, v)'\) denotes the column vector which is the transpose of the row vector \((u, v)\).

On the other hand given that the dual spaces of \((L^2(\mathbb{T}^2))^2\) and \((H^1)^2\) are isomorphic to \((L^2(\mathbb{T}^2))' \times (L^2(\mathbb{T}^2))'\) and \(H^{-1} \times H^{-1}\) respectively, by identifying \(L^2(\mathbb{T}^2)\) to its dual space \((L^2(\mathbb{T}^2))')', we have \((H^1)^2 \subset (L^2(\mathbb{T}^2))^2 \subset (H^{-1})^2\).

Let us now prove that the family of operators \(A(t, \cdot)\) satisfies the assumptions stated in Proposition 2.10 in Sect. 2, with \(H = (L^2(\mathbb{T}^2))^2, F = (H^1)^2\) and \(F' = (H^{-1})^2\).

We first recall that the family of eigenfunctions \((f_{1,n_1,n_2}^{i,s})_{i,n_1,n_2}\) (as defined in Proposition 2.3) associated to the family of eigenvalues \((\lambda_{n_1,n_2})_{n_1,n_2}\) of the operator \(-\gamma \Delta\) is an orthonormal basis of \(L^2(\mathbb{T}^2)\).

Now we start by showing that \(\forall (U, V) \in H^1 \times H^1, A(t, (U, V)) \in H^{-1} \times H^{-1}\).

We have

\[A(t, (U, V)) = \begin{pmatrix}
-\gamma \Delta U + \beta(I_d - \gamma \Delta) \overline{z} (G_1^s)^*(I_d - \gamma \Delta) \overline{z} U + \beta(I_d - \gamma \Delta) \overline{z} (G_2^s)^*(I_d - \gamma \Delta) \overline{z} V \\
-\beta(I_d - \gamma \Delta) \overline{z} (G_1^s)^*(I_d - \gamma \Delta) \overline{z} U - \gamma \Delta V - \beta(I_d - \gamma \Delta) \overline{z} (G_2^s)^*(I_d - \gamma \Delta) \overline{z} V + \alpha V
\end{pmatrix}.\]

Moreover:

(i) \[
\|(I_d - \gamma \Delta) \overline{z} (G_1^s)^*(I_d - \gamma \Delta) \overline{z} U\|_{H^{-1}} \\
\leq C \|(I_d - \gamma \Delta) \overline{z} (G_1^s)^*(I_d - \gamma \Delta) \overline{z} U\|_{L^2} \\
= C \|(G_1^s)^*(I_d - \gamma \Delta) \overline{z} U\|_{H^{-s}} \\
\leq C \|(I_d - \gamma \Delta) \overline{z} U\|_{H^{-s}} \\
= C \|U\|_{L^2},
\]

where the second inequality follows from Corollary 5.22.

(ii) Similarly, from Corollary 5.22, we deduce

\[
\|\beta(I_d - \gamma \Delta) \overline{z} (G_1^s)^*(I_d - \gamma \Delta) \overline{z} V\|_{H^{-1}} \\
\leq C \|(I_d - \gamma \Delta) \overline{z} (G_1^s)^*(I_d - \gamma \Delta) \overline{z} V\|_{L^2} \\
= C \|(G_1^s)^*(I_d - \gamma \Delta) \overline{z} V\|_{H^{-s}}
\]
\leq C \| (I_d - \gamma \Delta)^{\frac{1}{2}} \mathcal{V} \|_{H^{-s}}
= C \| \mathcal{V} \|_{L^2}. 

(ii)\n\| -\gamma \Delta \mathcal{U} \|_{H^{-1}}^2 = \sum_{i, n_1, n_2} (1 + \gamma \pi^2(n_1^2 + n_2^2))^{-1} (-\gamma \Delta \mathcal{U}, f_{n_1, n_2}^i)^2 \leq \sum_{i, n_1, n_2} (1 + \gamma \pi^2(n_1^2 + n_2^2))(\mathcal{U}, f_{n_1, n_2}^i)^2_L = \| \mathcal{U} \|_{H^1}^2.

So from (i), (ii), (iii) we conclude that $A(t, (\mathcal{U}, \mathcal{V})) \in H^{-1} \times H^{-1}$. Let us now prove that the four assumptions in Proposition 2.10 are satisfied. In the rest of this proof we will note by $< .. >$ the duality product between $H^{-1} \times H^{-1}$ and $H^1 \times H^1$.

(1). Given that $A(t, ..)$ is linear, to prove the first point, it is enough to show that $\mathcal{V}(\mathcal{U}, \mathcal{V})$, $(\mathcal{U}_1, \mathcal{V}_1) \in H^1 \times H^1$ the map $\theta \mapsto < A(t, \theta(\mathcal{U}, \mathcal{V})), (\mathcal{U}_1, \mathcal{V}_1) >$ is continuous on $\mathbb{R}$, which in turn is not hard to verify.

(2). The fact that there exists $\delta > 0$ such that $\| A(t, (\mathcal{U}, \mathcal{V})) \|_{H^{-1} \times H^{-1}} \leq \delta \| (\mathcal{U}, \mathcal{V}) \|_{H^1 \times H^1}$ follows from (i), (ii), and (iii).

(3). Let us prove that there exists $\sigma_1 > 0, \sigma_2 \in \mathbb{R}$, such that

$$< A(t, (\mathcal{U}, \mathcal{V})), (\mathcal{U}, \mathcal{V}) > + \sigma_2 \| (\mathcal{U}, \mathcal{V}) \|_{L^2 \times L^2}^2 \geq \sigma_1 \| (\mathcal{U}, \mathcal{V}) \|_{H^1 \times H^1}^2.$$ 

We have

$$< A(t, (\mathcal{U}, \mathcal{V})), (\mathcal{U}, \mathcal{V}) >$$

$$= < -\gamma \Delta \mathcal{U} + \beta(I_d - \gamma \Delta)^{\frac{1}{2}}(G_i^1)^*(I_d - \gamma \Delta)^{\frac{1}{2}} \mathcal{U} + \beta(I_d - \gamma \Delta)^{\frac{1}{2}}(G_i^1)^*(I_d - \gamma \Delta)^{\frac{1}{2}} \mathcal{V}, \mathcal{U} >$$

$$= < -\beta(I_d - \gamma \Delta)^{\frac{1}{2}}(G_i^1)^*(I_d - \gamma \Delta)^{\frac{1}{2}} \mathcal{U} - \gamma \Delta \mathcal{V} - \beta(I_d - \gamma \Delta)^{\frac{1}{2}}(G_i^1)^*(I_d - \gamma \Delta)^{\frac{1}{2}} \mathcal{V} + \alpha \mathcal{V}, \mathcal{V} > .$$

Furthermore:

(3.1). $< -\gamma \Delta \mathcal{U}, \mathcal{U} > = \gamma < \nabla \mathcal{U}, \nabla \mathcal{U} > = \gamma \| \nabla \mathcal{U} \|_{L^2}^2.$

(3.2). From (i) and (ii), we note that

$$\beta(I_d - \gamma \Delta)^{\frac{1}{2}}(G_i^1)^*(I_d - \gamma \Delta)^{\frac{1}{2}} \mathcal{U} + \beta(I_d - \gamma \Delta)^{\frac{1}{2}}(G_i^1)^*(I_d - \gamma \Delta)^{\frac{1}{2}} \mathcal{V} \in L^2(\mathbb{T}^2),$$

thus as $H^1 \subset L^2(\mathbb{T}^2)$, from (i) and (ii) and from Young’s inequality we have

$$| < \beta(I_d - \gamma \Delta)^{\frac{1}{2}}(G_i^1)^*(I_d - \gamma \Delta)^{\frac{1}{2}} \mathcal{U} + \beta(I_d - \gamma \Delta)^{\frac{1}{2}}(G_i^1)^*(I_d - \gamma \Delta)^{\frac{1}{2}} \mathcal{V}, \mathcal{U} |$$
Now in addition to the fact that the family of operators
\(A\) satisfies the four assumptions given in Proposition 2.10 in Sect. 2, with
\(H = (L^2(\mathbb{T}^2))^2, F = (H^1)^2\) and \(F' = (H^{-1})^2\), we note that:
\[
\begin{align*}
-J(t) &= ((I_d - \gamma \Delta)^{\frac{\gamma}{2}}(G^I_t)^* (I_d - \gamma \Delta)^{\frac{\gamma}{2}} Z_t^i, - (I_d - \gamma \Delta)^{\frac{\gamma}{2}} (G^S_t)^* (I_d - \\
&\quad \gamma \Delta)^{\frac{\gamma}{2}} Z_t^i)' \quad \text{belongs to } L^2_{\text{loc}}(\mathbb{R}_+, (L^2(\mathbb{T}^2))^2). \quad \text{Indeed, } ||(I_d - \gamma \Delta)^{\frac{\gamma}{2}} (G^S_t)^* (I_d - \gamma \Delta)^{\frac{\gamma}{2}} Z_t^i||_{L^2_{\text{loc}}} = ||(G^S_t)^* Z_t||_{H^{-s}} \leq C ||Z_t||_{H^{-s}} \quad \text{and } Z \in C(\mathbb{R}_+, H^{-s}).
\end{align*}
\]
\(\left(U^s_0, V^s_0\right) \in (L^2(\mathbb{T}^2))^2\).

Thus we deduce from Proposition 2.10 that for \((A(t, \cdot))_{t \in [0, T]}\) being the above family of operators, the stochastic differential equation
\[
(dU(t), dV(t))' + A(t, (U(t), V(t))dt = J(t)dt + (dW^{1,s}_t, dW^{2,s}_t)',
\]
\((U(0), V(0)) = (U^s_0, V^s_0)\)

admits a unique solution \((U, V) \in L^2_{\text{loc}}(\mathbb{R}_+, (H^1)^2) \cap C(\mathbb{R}_+, (L^2(\mathbb{T}^2))^2)\) a.s. On the other hand, it has been shown in Proposition 5.30, that the pair \((U^s, V^s)\) is the unique solution of the Eq. (5.29) in \(C(\mathbb{R}_+, (L^2(\mathbb{T}^2))^2)\) (since
Eq. (5.29) is equivalent to the system formed by Eqs. (5.9) and (5.10), which in turn is the same as the system formed by Eqs. (5.25) and (5.26). Thus we conclude that the pair of processes \((U^S, V^S) = (U, V)\) belongs to 
\[L^2_{\text{loc}}(\mathbb{R}^+,(H^1(\mathbb{T}^2))^2) \cap C(\mathbb{R}^+,(L^2(\mathbb{T}^2))^2),\]
consequently the pair of processes \((U, V)\) belongs to 
\[L^2_{\text{loc}}(\mathbb{R}^+, (H^{-s+1})^2).\]  
\(\Box\)

**Remark 5.32** Given that 
\[dZ^N_t = \gamma \Delta Z^N_t dt + d\tilde{N}^N_t\] (see Eq. (5.1)), by proceeding as in the proof of Proposition 5.31, it is easy to see that \(\forall N \in \mathbb{N}^*, Z^N \in L^2_{\text{loc}}(\mathbb{R}^+, H^{-s+1}).\) Furthermore by using the Itô formula established in [18] page 62 we can see that the sequence \((Z^N)_N\) is bounded in 
\[L^2_{\text{loc}}(\mathbb{R}^+, H^{-s+1})\] (although we do not establish this result in this paper). Consequently since in a Hilbert space the bounded sets are relatively compact for the weak topology, the convergence in law of \((Z^N)_N\) in 
\[L^2_{\text{loc}}(\mathbb{R}^+, H^{-s+1})\] equipped with its weak topology can be easily deduced.

The convergence in law of the sequence \((U^N, V^N)_N\) in 
\[(L^2_{\text{loc}}(\mathbb{R}^+, H^{-s+1}))^2\]
where \[L^2_{\text{loc}}(\mathbb{R}^+, H^{-s+1})\] is equipped with its weak topology follows also by the fact it is bounded in \[(L^2_{\text{loc}}(\mathbb{R}^+, H^{-s+1}))^2,\] which in turn is not difficult to prove by using the Proposition 5.31. However we do not establish that result in this paper.

### 6 Law of Large Numbers, Central Limit Theorem of the sequence \((\mu^{S,N}, \mu^{I,N})_{N \geq 1}\): the case \(\gamma = 0\)

In this section we consider the second model presented in the Introduction ie when the diffusion coefficient \(\gamma = 0.\) More precisely we consider a compartmental SIR stochastic epidemic model for a population distributed on the two dimentional torus such that:

- The position of an individual \(i\) is independent of time and is represented by \(X^i\) defined as in the Introduction.
- A susceptible \(i\) become infected at time \(t\) at the rate \(\beta I\{E^i_1 = S\} \sum_{j=1}^{N} \frac{K(X^i, X^j)}{\sum_{i=1}^{N} K(X^i, X^j)} 1\{E^i_1 = I\},\)

where \(\alpha, \beta, E^i_1\) and the function \(K\) are defined as in the Introduction.

The temporal evolution of \(S(\cdot), I(\cdot)\) and \(R(\cdot)\) are defined as in the Introduction. The assumptions made at time \(t = 0,\) are presented in the Introduction, in other words, the sequence of empirical measures \((\mu^{S,N}_0, \mu^{I,N}_0, \mu^{R,N}_0)_N\) has already been defined in the introduction.

Thus the renormalized processes \(\mu^{S,N}, \mu^{I,N}, \mu^{R,N}\) and \(\mu^N\) are defined as follows. \(\forall t > 0,\)

\[
\mu^{S,N}_t = \frac{1}{N} \sum_{i=1}^{N} 1\{E^i_1 = S\} \delta_{X^i}
\]

\[
\mu^{I,N}_t = \frac{1}{N} \sum_{i=1}^{N} 1\{E^i_1 = I\} \delta_{X^i}
\]
\[ \mu_t^{R,N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i}^{E_i = R} \]

\[ \mu_t^N = \mu_t^{S,N} + \mu_t^{I,N} + \mu_t^{R,N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i}. \]

We suppose again that Assumption (H0) from Sect. 3 holds. Now since we have already shown in Sects. 3 and 3.2 respectively that \((\mu_0^{S,N}, \mu_0^{I,N}, \mu_0^N) \xrightarrow{a.s.} (\mu_S^0, \mu_I^0, \mu)\) and that \((U_0^N = \sqrt{N}(\mu_0^{S,N} - \mu_S^0), V_0^N = \sqrt{N}(\mu_0^{I,N} - \mu_I^0), Z_N^N = \sqrt{N}(\mu_0^N - \mu))\) converges in law in \((H^{-s})^3\) (for any \(s > 1\)) towards the Gaussian vector \((U_0, V_0, Z)\), then the aim in this section is to study the law of large numbers and the central limit theorem of the sequence \(((\mu_t^{S,N}, \mu_t^{I,N}), t \geq 0, N \geq 1)\).

### 6.1 Law of large numbers

The following is assumed to hold throughout Sect. 6.1.

**Assumption (H3)** \(k\) is Lipschitz.

#### 6.1.1 System of evolution equations of the pair \((\mu_t^{S,N}, \mu_t^{I,N})\)

For \(\varphi \in C(T^2)\), since \(\gamma = 0\), the evolution equations of \(\mu_t^{S,N}\) and \(\mu_t^{I,N}\) simplify as follows.

\[
(\mu_t^{S,N}, \varphi) = (\mu_0^{S,N}, \varphi) - \beta \int_0^t \left( \mu_r^{S,N}, \varphi(\mu_r^{I,N}, \frac{K}{(\mu_r^N, K)}) \right) dr + M_t^{I,N,\varphi},
\]

\[
(\mu_t^{I,N}, \varphi) = (\mu_0^{I,N}, \varphi) + \int_0^t \left( \mu_r^{S,N}, \varphi(\mu_r^{I,N}, \frac{K}{(\mu_r^N, K)}) \right) dr - \alpha \int_0^t (\mu_r^{I,N}, \varphi) dr + L_t^{I,N,\varphi},
\]

where

\[
M_t^{I,N,\varphi} = -\frac{1}{N} \sum_{i=1}^{N} \int_0^t \int_0^\infty 1_{E_i = S} \varphi(X_i^t) 1_{u \leq \beta \sum_{j=1}^{N} \frac{K(X_i^t, X_j)}{\sum_{l=1}^{N} K(X_l^t, X_j)}} \overline{M}^i(dR, du),
\]

\[
L_t^{I,N,\varphi} = \frac{1}{N} \sum_{i=1}^{N} \int_0^t \int_0^\infty 1_{E_i = S} \varphi(X_i^t) 1_{u \leq \beta \sum_{j=1}^{N} \frac{K(X_i^t, X_j)}{\sum_{l=1}^{N} K(X_l^t, X_j)}} \overline{M}^i(dR, du) - \frac{1}{N} \sum_{i=1}^{N} \int_0^t \int_0^\alpha 1_{E_i = I} \varphi(X_i^t) \overline{Q}^i (dR, du).
\]
Let us state the main result of this subsection.

**Theorem 6.1** The sequence \((\mu_{S,N}, \mu_{I,N})_{N \geq 1}\) converges in probability in \((D(\mathbb{R}_{+}, M(\mathbb{T}^2)))^2\) towards \((\mu^S, \mu^I)\) in \((C(\mathbb{R}_{+}, M(\mathbb{T}^2)))^2\) where \((\mu^S_t, \mu^I_t), t \geq 0\) satisfies, \(\forall \varphi \in C(\mathbb{T}^2),\)

\[
\begin{align*}
(\mu^S_t, \varphi) &= (\mu^S_0, \varphi) - \beta \int_0^t \left( \mu^S_r, \varphi(\mu^I_r, \frac{K}{(\mu, K)}) \right) dr, \\
(\mu^I_t, \varphi) &= (\mu^I_0, \varphi) + \beta \int_0^t \left( \mu^S_r, \varphi(\mu^I_r, \frac{K}{(\mu, K)}) \right) dr \\
&\quad - \alpha \int_0^t (\mu^I_r, \varphi) dr.
\end{align*}
\]

**Proof** We obtain the tightness of the sequence \((\mu_{S,N}, \mu_{I,N})_{N \geq 1}\) by an adaptation of the proof of Proposition 4.5, thus by Prokhorov’s theorem we deduce the existence of a subsequence which converges in law towards \((\mu^S_t, \mu^I_t), t \geq 0\). Furthermore, adapting the proof of Theorem 4.7 we prove that \((\mu^S_t, \mu^I_t), t \geq 0\) is continuous and verifies the system formed by the Eqs. (6.1) and (6.2), and \(\forall t \geq 0, \mu^S_t\) and \(\mu^I_t\) are absolutely continuous with respect to the Lebesgue measure with densities \(f_S(t, .)\) and \(f_I(t, .)\) bounded by \(\delta_2(\delta_2\text{ is defined in Sect. 3})\). On the other hand \((f_S(t, .), f_I(t, .))\) satisfies the following system.

\[
\begin{align*}
f_S(t) &= f_S(0) - \beta \int_0^t f_S(r) \int_{\mathbb{T}^2} \frac{K(., y)}{K(x', y)g(x')} f_I(r, y) dy dr, \\
f_I(t) &= f_I(0) + \beta \int_0^t f_S(r) \int_{\mathbb{T}^2} \frac{K(., y)}{K(x', y)g(x')} f_I(r, y) dy dr \\
&\quad - \alpha \int_0^t f_I(r) dr,
\end{align*}
\]

where \(g\) is the density of \(\mu\). Moreover it is easy to prove that the above system admits a unique solution in the set \(A = \{(f_1, f_2)/0 \leq f_i \leq \delta_2, i \in \{1, 2\}\}\). Thus we conclude that the sequence \((\mu_{S,N}, \mu_{I,N})_{N \geq 1}\) weakly converges in \((D(\mathbb{R}_{+}, M(\mathbb{T}^2)))^2\) towards \((\mu^S, \mu^I)\). However as the initial measures \(\mu^S_0\) and \(\mu^I_0\) are deterministic (see Theorem 3.1 in Sect. 3), the measures \(\mu^S_t\) and \(\mu^I_t\) also have this property, consequently we have the convergence in probability.

\[\square\]

### 6.2 Central limit theorem

The aim here is to study the convergence of the sequence \((U^N = \sqrt{N}(\mu_{S,N} - \mu^S), V^N = \sqrt{N}(\mu_{I,N} - \mu^I))\), as \(N \rightarrow \infty\) in \(D(\mathbb{R}_{+}, H^{-s}) \times D(\mathbb{R}_{+}, H^{-s})\) with \(s>1\) (the choice of such a ’s’ is justified as in Sect. 5.2). To this end we make the following assumptions.

**Assumption (H4)** \(k \in C^2(\mathbb{R}_{+}).\)
Following the same argument as that used in the proof of Lemma 5.2, we note that:

**Remark 6.2** Under the assumption (H4), provided \( s \leq 2, \sup_{x} \| K(x, \cdot) \|_{H^s} < \infty. \)

### 6.2.1 Equations verified by the pair \((U^N, V^N)\)

Let \( \varphi \in C(T^2) \), by a similar reasoning as in Sect. 5.1, we see that

\[
(U^N_t, \varphi) = (U^N_0, \varphi) + \beta \int_0^t (Z^N_r, G_r^{S,1,N} \varphi) dr - \beta \int_0^t (U^N_r, G_r^{I,N} \varphi) dr - \beta \int_0^t (V^N_r, G_r^S \varphi) dr + M^{'N,\varphi}_t,
\]

\[
(V^N_t, \varphi) = (V^N_0, \varphi) - \beta \int_0^t (Z^N_r, G_r^{S,1,N} \varphi) dr + \beta \int_0^t (U^N_r, G_r^{I,N} \varphi) dr + \beta \int_0^t (V^N_r, G_r^S \varphi) dr - \alpha \int_0^t (V^N_r, \varphi) dr + \tilde{L}^{'N,\varphi}_t,
\]  

where

- \( \tilde{M}'_t, \tilde{L}'_t = \sqrt{N} M'_t, \sqrt{N} L'_t \) and \( \tilde{L}'_t = \sqrt{N} L'_t \),
- \( \forall x, y, x' \in T^2 \),

\[
G_r^{S,1,N} \varphi(x') = \left( \mu_r^{L,N}, K(x', \cdot) \right) \left( \mu_r^{S,N}, \varphi K \right) = \int_{T^2} K(x', y) \frac{\int_{T^2} \varphi(x) K(x, y) \mu_r^{S,N}(dx)}{\int_{T^2} K(y', y) \mu_r^{L,N}(dy)} \mu_r^{L,N}(dy),
\]

\[
G_r^{I,N} \varphi(x) = \varphi(x) \left( \mu_r^{L,N}, K(x, \cdot) \right) = \varphi(x) \int_{T^2} \frac{K(x, y)}{\mu_r^{L,N}(dy)} \mu_r^{I,N}(dy),
\]

\[
G_r^S \varphi(y) = \left( \mu_r^{S,N}, \varphi K(\cdot, y) \right) = \int_{T^2} \frac{\varphi(x) K(x, y) \mu_r^S(dx)}{\int_{T^2} K(y', y) \mu_r^{S,N}(dy)},
\]

**In the rest of this section we arbitrarily choose** \( 1 < s < 2 \), and we equipped \( D(\mathbb{R}^+, H^{-s}) \) with the Skorhokod topology.

**Theorem 6.3** Under (H4), the sequence \((U^N, V^N)_{N \geq 1}\) converges in law in \((D(\mathbb{R}^+, H^{-s}))^2\) to the pair of process \((U, V)\) which belongs to \((C(\mathbb{R}^+, H^{-s}))^2\) and satisfies:

\[
U_t = U_0 + \beta \int_0^t (G_r^{S,1})^* Z_r dr - \beta \int_0^t (G_r^I)^* U_r dr - \beta \int_0^t (G_r^S)^* V_r dr + W^1_t,
\]  

\[
(V_t, \varphi) = (V_0, \varphi) - \beta \int_0^t (Z^N_r, G_r^{S,1,N} \varphi) dr + \beta \int_0^t (U^N_r, G_r^{I,N} \varphi) dr + \beta \int_0^t (V^N_r, G_r^S \varphi) dr - \alpha \int_0^t (V^N_r, \varphi) dr + \tilde{L}^{'N,\varphi}_t.
\]
Proposition 6.4 The sequence is an easy adaptation of the proof of Proposition 5.20. The proof of the next Proposition, which will be useful for the proof of Theorem 6.3 where \( g \) is the density of \( \mu \).

\[
\mathbb{E}(\sup_{N \geq 1} \| U_t^N \|^2_{H^{-s}}) < \infty,
\]

\[
\mathbb{E}(\sup_{N \geq 1} \| V_t^N \|^2_{H^{-s}}) < \infty.
\]

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\[V_t = V_0 - \beta \int_0^t (S_t^I)^* Z_r dr + \beta \int_0^t (I_t^I)^* U_r dr + \beta \int_0^t (S_t^I)^* V_r dr - \alpha \int_0^t V_r dr + W_t^2, \] (6.6)

where \( \forall \varphi, \psi \in H^s, ((W^1, \varphi), (W^2, \psi)) \) is a centered Gaussian martingale verifying

\[
< (W^1, \varphi) >_t = \beta \int_0^t \left( \mu^S_r, \varphi^2(\mu^I_r, \frac{K}{(\mu, K)}) \right) dr,
\]

\[
< (W^2, \psi) >_t = \beta \int_0^t \left( \mu^S_r, \psi^2(\mu^I_r, \frac{K}{(\mu, K)}) \right) dr + \alpha \int_0^t (\mu^I_r, \psi^2) dr,
\]

\[
< (W^1, \varphi), (W^2, \psi) >_t = -\beta \int_0^t \left( \mu^S_r, \varphi \psi(\mu^I_r, \frac{K}{(\mu, K)}) \right) dr.
\]

The proof of the next Proposition, which will be useful for the proof of Theorem 6.3 is an easy adaptation of the proof of Proposition 5.20.

**Proposition 6.4** The sequence \( \{(\tilde{M}^N_t, \tilde{L}^N_t), t \geq 0\}_{N \geq 1} \) converges in law in \( (D(\mathbb{R}_+, H^{-s}))^2 \) towards the process \( (W^1, W^2) \in (C(\mathbb{R}_+, H^{-s}))^2 \) and \( \forall \varphi, \psi \in H^s, ((W^1, \varphi), (W^2, \psi)) \) is a centered Gaussian martingale of the form

\[
(W^1, \varphi) = -\int_0^t \int_{\mathbb{T}^2} \sqrt{\beta f_S(r, x) \int_{\mathbb{T}^2} K(x, y) g(x') dx'} f_I(r, y) d\varphi(x) \mathbb{W}_1(dr, dx)
\]

\[
(W^2, \psi) = \int_0^t \int_{\mathbb{T}^2} \sqrt{\beta f_S(r, x) \int_{\mathbb{T}^2} K(x, y) g(x') dx'} f_I(r, y) d\psi(x) \mathbb{W}_1(dr, dx)
\]

\[\quad - \int_0^t \int_{\mathbb{T}^2} \sqrt{\psi(x)} f_I(r, x) \mathbb{W}_2(dr, dx), \]

where \( g \) is the density of \( \mu \) and \( \mathbb{W}_1, \mathbb{W}_2 \) are independent spatio-temporal white noises.

**6.2.2 Proof of Theorem 6.3**

We establish the tightness of the sequence \( (U^N, V^N) \) first, then we show that all converging subsequences of \( (U^N, V^N)_{N \geq 1} \) have the same limit which we shall identify. The next proposition is useful to prove the tightness of the sequence \( (U^N, V^N)_{N \geq 1} \).

**Proposition 6.5** \( \forall T > 0, \)

\[
\sup_{N \geq 1} \mathbb{E}(\sup_{0 \leq t \leq T} \| U_t^N \|^2_{H^{-s}}) < \infty,
\]

\[
\sup_{N \geq 1} \mathbb{E}(\sup_{0 \leq t \leq T} \| V_t^N \|^2_{H^{-s}}) < \infty.
\]
Proof From Eqs. (6.3) and (6.4), we have

\[
\|U^N_t\|_{H^{-s}}^2\leq 5\|U_0^N\|_{H^{-s}}^2+5\beta^2 t \int_0^t \|(G_r^{S,I,N})^* Z^N\|_{H^{-s}}^2 dr + 5\beta^2 t \\
\int_0^t \|(G_r^{I,N})^* U^N_r\|_{H^{-s}}^2 dr \\
+5\beta^2 t \int_0^t \|(G_r^S)^* V_r^N\|_{H^{-s}}^2 dr \\
+5\beta^2 t \int_0^t \|(G_r^I)^* V^N_r\|_{H^{-s}}^2 dr + 5\|\tilde{M}_{t}^{N,\varphi}\|_{H^{-s}}^2,
\]

\[
\|V^N_t\|_{H^{-s}}^2\leq 6\|V_0^N\|_{H^{-s}}^2+6\beta^2 t \\
\int_0^t \|(G_r^S)^* Z^N\|_{H^{-s}}^2 dr \\
+6\beta^2 t \int_0^t \|(G_r^I)^* U^N_r\|_{H^{-s}}^2 dr \\
+6\beta^2 t \int_0^t \|(G_r^S)^* V^N_r\|_{H^{-s}}^2 dr + 6\alpha^2 t \\
\int_0^t \|V^N_r\|_{H^{-s}}^2 dr + 6\|\tilde{L}^{N,\varphi}_t\|_{H^{-s}}^2.
\]

So since Proposition 5.21 and Corollary 5.22 remain true for \(\mu_t = \mu\), we have

\[
\|U^N_t\|_{H^{-s}}^2\leq 5\|U_0^N\|_{H^{-s}}^2+5\beta^2 t^2 C \sup_y \|K(., y)\|_{H^s} \|Z^N\|_{H^{-s}}^2 \\
+5\beta^2 t C \sup_y \|K(., y)\|_{H^s} \int_0^t \|U^N_r\|_{H^{-s}}^2 dr \\
+5\beta^2 t C \sup_x \left| \frac{K(x, \cdot)}{\int_{T^2} K(x', \cdot) \mu(dx')} \right|_{H^s} \int_0^t \|V^N_r\|_{H^{-s}}^2 dr \\
+5\|\tilde{M}_{t}^{N,\varphi}\|_{H^{-s}}^2,
\]

\[
\|V^N_t\|_{H^{-s}}^2\leq 6\|V_0^N\|_{H^{-s}}^2+6\beta^2 t^2 C \sup_y \|K(., y)\|_{H^s} \|Z^N\|_{H^{-s}}^2 \\
+6\beta^2 t C \sup_y \|K(., y)\|_{H^s} \int_0^t \|U^N_r\|_{H^{-s}}^2 dr \\
+6t \left( C \beta^2 \sup_x \left| \frac{K(x, \cdot)}{\int_{T^2} K(x', \cdot) \mu(dx')} \right|_{H^s}^2 + \alpha^2 \right) \int_0^t \|V^N_r\|_{H^{-s}}^2 dr \\
+6\|\tilde{L}^{N,\varphi}_t\|_{H^{-s}}^2.
\]

Thus from Remark 6.2 and Lemma 7.2 below, one has

\[
\mathbb{E}( \sup_{0\leq t \leq T} \|U^N_t\|_{H^{-s}}^2 ) \leq 5 \sup_{N \geq 1} \mathbb{E}( \|U_0^N\|_{H^{-s}}^2 ) + 5t \beta^2 C \\
\int_0^t \left\{ \mathbb{E}( \sup_{0\leq u \leq T} \|U^N_u\|_{H^{-s}}^2 ) + \mathbb{E}( \|V^N_u\|_{H^{-s}}^2 ) \right\} dr \\
+ 5t^2 \beta^2 C \sup_{N \geq 1} \mathbb{E}( \|Z^N\|_{H^{-s}} ) + \sup_{N \geq 1} \mathbb{E}( \|\tilde{M}_{t}^{N,\varphi}\|_{H^{-s}}^2 ),
\]

(6.7)
\[ \mathbb{E} \left( \sup_{0 \leq t \leq T} \| V_t^N \|^2_{H^{-s}} \right) \leq 6 \sup_{N \geq 1} \mathbb{E} \left( \| V_0^N \|^2_{H^{-s}} \right) + 6t (\beta^2 C + \alpha^2) \]
\[ \int_0^T \left\{ \mathbb{E} \left( \sup_{0 \leq u \leq r} \| U_u^N \|^2_{H^{-s}} \right) + \mathbb{E} \left( \sup_{0 \leq u \leq r} \| V_u^N \|^2_{H^{-s}} \right) \right\} dr \]
\[ + 6t^2 \beta^2 C \sup_{N \geq 1} \mathbb{E} \left( \| Z_N \|^2_{H^{-s}} \right) + \sup_{N \geq 1} \mathbb{E} \left( \sup_{0 \leq t \leq T} \| \tilde{L}_t^N \|^2_{H^{-s}} \right). \quad (6.8) \]

Hence summing (6.7) and (6.8) and applying Gronwall’s lemma we deduce the result from Corollary 5.4 and Proposition 3.2 in section 4.

Now we prove the tightness of the sequence \((U^N, V^N)_{N \geq 1}\) in \((D(\mathbb{R}^+, H^{-s}))^2\).

**Proposition 6.6** Both sequences \((U^N)_{N \geq 1}\) and \((V^N)_{N \geq 1}\) are tight in \(D(\mathbb{R}^+, H^{-s}))^2\).

**Proof** We only establish the tightness of \((U^N)_{N \geq 1}\) by showing that the conditions (T1) and (T2) of Proposition 5.8 are satisfied.

The condition (T1) is obtained by using the Proposition 6.5 and applying an argument similar to that of the proof of (T1) in Theorem 5.7.

– Proof of (T2). we have

\[ U_t^N = U_0^N + \beta \int_0^t (G_r^S, I, N)^* Z_N dr - \beta \int_0^t (G_r^I, N)^* U_r^N dr \]
\[ - \beta \int_0^t (G_r^S)^{\tilde{N}} V_r^N dr + \tilde{M}_t^{\tilde{N}, \psi} \]
\[ = U_0^N + \beta \int_0^t \Gamma_r^N dr + \tilde{M}_t^{\tilde{N}, \psi} \]

with \(\Gamma_r^N = (G_r^S, I, N)^* Z_N - (G_r^I, N)^* U_r^N - (G_r^S)^{\tilde{N}} V_r^N\).

We want to prove that \(\forall T > 0, \forall \varepsilon_1, \varepsilon_2 > 0, \exists \delta > 0, N_0 \geq 1\) such that for any family of stopping times \((\tau^N)_N\) with \(\tau^N \leq T\),

\[ \sup_{N \geq N_0} \mathbb{P} \left( \int_{\tau^N}^{\tau^N + \delta} \Gamma_r^N dr \right)_{H^{-s}} > \varepsilon_1 < \varepsilon_2, \quad (6.9) \]
\[ \sup_{N \geq N_0} \mathbb{P} \left( \| \tilde{M}_{\tau^N}^{\tilde{N}} - \tilde{M}_{\tau^N}^{\tilde{N}} \|_{H^{-s}} > \varepsilon_1 < \varepsilon_2. \quad (6.10) \right) \]

– (6.10) is proved by using an argument similar to that of the proof of (T2) in Proposition 5.17

– Proof of (6.9). Let \(T > 0\). Given \(l \in \mathbb{R}^+ \backslash [0, 1], \varepsilon_1, \varepsilon_2 > 0\), we find \(\delta > 0, N_0 \geq 1\) such that \(\delta + \tau^N \leq lT\) and

\[ \sup_{N \geq N_0} \mathbb{P} \left( \int_{\tau^N}^{\tau^N + \delta} \Gamma_r^N dr \right)_{H^{-s}} > \varepsilon_1 < \varepsilon_2. \]
We have \[ \left\| \int_{\tau(N)}^{\tau(N+\theta)} \Gamma_r \, dr \right\|_{H^{-s}} \leq \int_{\tau(N)}^{\tau(N+\theta)} \left\| \Gamma_r \right\|_{H^{-s}} \, dr \leq \theta \sup_{0 \leq r \leq t} \left\| \Gamma_r \right\|_{H^{-s}}. \]

Thus (6.9) will follow from \( \sup_{N \geq 1} E(\sup_{0 \leq r \leq l_T} \left\| \Gamma_r \right\|_{H^{-s}}) < C \) in view of the last inequality, which in turn follows readily from Lemma 7.2 below, Remark 6.2 and Propositions 3.2 and 6.5, combined with the fact that Corollary 5.22 remains true for \( \mu_t = \mu \). So (T2) is established.

An easy adaptation of the proof of Lemmas 5.27 and 5.28 yields respectively the next two Lemmas.

**Lemma 6.7** For any \( t \geq 0, \varphi \in H^2(\mathbb{T}^2) \), as \( N \to \infty \),
\[
\int_0^t E\left( \left\| (G_r^1,N - G_r^1)\varphi \right\|_{H^1}^2 \right) dt \to 0.
\]

**Lemma 6.8** For any \( t \geq 0, \varphi \in H^s(\mathbb{T}^2) \), as \( N \to \infty \),
\[
\int_0^t E\left( \left\| (G_r^S,I,N - G_r^S,I)\varphi \right\|_{H^s}^2 \right) dt \to 0.
\]

From Proposition 6.6, we deduce that the sequence \( (U^N, V^N)_N \) is tight in \( (D(\mathbb{R}_+, H^{-s}))^2 \), so there exists a subsequence still denoted \( (U^N, V^N)_N \), which converges in law in \( (D(\mathbb{R}_+, H^{-s}))^2 \) towards \( (U, V) \). Moreover from Proposition 6.4, we deduce that \( (U, V) \in (C(\mathbb{R}_+, H^{-s}))^2 \), thus we end the proof of Theorem 6.3 as follows.

**Proposition 6.9** The pair \( (U, V) \) is the unique solution in \( (C(\mathbb{R}_+, H^{-s}))^2 \) of the system formed by Eqs. (6.5) and (6.6).

**Proof** By adapting the proof of Proposition 5.29 we see that the pair \( (U, V) \) satisfies the Eqs. (6.5) and (6.6). By adapting the proof of Proposition 5.30 we also see that the system formed by Eqs. (6.5) and (6.6) admits a unique solution in \( (C(\mathbb{R}_+, H^{-s}))^2 \). \( \square \)

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### 7 Appendix

We first recall that for any \( s > 0 \), the family \((\rho_{n_1,n_2}^{i,s})_{i,n_1,n_2}\) (as defined in Proposition 2.3) is an orthonormal basis of \( H^s(\mathbb{T}^2) \).

In this appendix we prove the next two Lemmas.
Lemma 7.1 We have,

\[
\sup_{x \in \mathbb{T}^2} \sum_{i,n_1,n_2} (\rho_{n_1,n_2}^{i,s}(x))^2 < \infty \text{ iff } s > 1,
\]

\[
\sup_{x \in \mathbb{T}^2} \sum_{i,n_1,n_2} (\nabla \rho_{n_1,n_2}^{i,s}(x))^2 < \infty \text{ iff } s > 2.
\]

**Proof** As for any \( x \in \mathbb{T}^2, i \in [1, 8], 0 \leq (f_{n_1,n_2}^{i})^2 \leq 4, \)

\[
\sum_{i,n_1,n_2} (\rho_{n_1,n_2}^{i,s}(x))^2 = \sum_{i,n_1,n_2} \frac{(f_{n_1,n_2}^{i})^2(x)}{(1 + \gamma \pi^2(n_1^2 + n_2^2))^s}, \text{ and}
\]

\[
\sum_{i,n_1,n_2} (\nabla \rho_{n_1,n_2}^{i,s}(x))^2 = \pi^2 \sum_{i=1}^{4} \sum_{n_1 > 0, n_2 > 0, \text{even}} \frac{n_1^2 + n_2^2}{(1 + \gamma \pi^2(n_1^2 + n_2^2))^s} (f_{n_1,n_2}^{i})^2(x)
\]

\[
+ \sum_{n_1 > 0, \text{even}} \frac{\pi^2 n_1^2 [(f_{n_1,0}^{i,6})^2(x) + (f_{n_1,0}^{i,5})^2(x)]}{(1 + \gamma \pi^2 n_1^2)^s}
\]

\[
+ \sum_{n_2 > 0, \text{even}} \frac{\pi^2 n_2^2 [(f_{n_1,0}^{i,8})^2(x) + (f_{n_1,0}^{i,7})^2(x)]}{(1 + \gamma \pi^2 n_2^2)^s}
\]

So

\[
\sum_{i,n_1,n_2} ((\rho_{n_1,n_2}^{i,s}(x))^2 \leq 1 + 16 \sum_{n_1 > 0, n_2 > 0, \text{even}} \frac{1}{(1 + \gamma \pi^2(n_1^2 + n_2^2))^s}
\]

\[
+ 8 \sum_{i=1}^{2} \sum_{n_1 > 0, \text{even}} \frac{1}{(1 + \gamma \pi^2 n_1^2)^s}, \text{ and}
\]

\[
\sum_{i,n_1,n_2} (\nabla \rho_{n_1,n_2}^{i,s}(x))^2 \leq 16 \pi^2 \sum_{n_1 > 0, n_2 > 0, \text{even}} \frac{n_1^2 + n_2^2}{(1 + \gamma \pi^2(n_1^2 + n_2^2))^s}
\]

\[
+ 8 \pi^2 \sum_{i=1}^{2} \sum_{n_1 > 0, \text{even}} \frac{n_i^2}{(1 + \gamma \pi^2 n_i^2)^s}
\]

Hence we see that:

\[- \sum_{i,n_1,n_2} (\rho_{n_1,n_2}^{i,s}(x))^2 < \infty \text{ provided the series } \sum_{n_1 > 0, n_2 > 0} \frac{1}{(1 + \gamma \pi^2(n_1^2 + n_2^2))^s};
\]

\[\sum_{n_1 > 0} \frac{1}{(1 + \gamma n_1^2)^s} \text{ and } \sum_{n_2 > 0} \frac{1}{(1 + \gamma n_2^2)^s} \text{ converge.}\]
- Convergence of

$$\sum_{i,n_1,n_2} \left( \nabla \rho_{n_1,n_2}^i (x) \right)^2 < \infty$$

provided the series

$$\sum_{n_1>0,n_2>0} \frac{n_1^2+n_2^2}{(1+\gamma \pi^2 (n_1^2+n_2^2))^s};$$

$$\sum_{n_1>0} \frac{n_1^2}{(1+\gamma \pi^2 n_1^2)^s}$$

and

$$\sum_{n_2>0} \frac{n_2^2}{(1+\gamma \pi^2 n_2^2)^s}$$

converge.

- Convergence of the series

1. Convergence of

$$\sum_{n_1>0,n_2>0} \frac{n_1^2+n_2^2}{(1+\gamma \pi^2 (n_1^2+n_2^2))^s}$$

It is so easy to see that

$$\sum_{n_1 \geq 1, n_2 \geq 1} \frac{n_1^2+n_2^2}{(1+\gamma \pi^2 (n_1^2+n_2^2))^s}$$

and

$$\int_1^{+\infty} \int_1^{+\infty} \frac{x^2+y^2}{(1+\gamma \pi^2 (x^2+y^2))^s} dx dy$$

are of the same type (either convergent or divergent), and the latter is of the same type as

$$\int_1^{+\infty} \frac{r^3}{(1+\gamma \pi^2 r^2)^s} dr \leq \frac{1}{\gamma s \pi^2 s} \int_1^{+\infty} r^{3-2s} dr$$

and

$$\int_1^{+\infty} r^{3-2s} dr < \infty \text{ iff } s > 2.$$ 

Thus

$$\sum_{n_1 \geq 1, n_2 \geq 1} \frac{n_1^2+n_2^2}{(1+\gamma \pi^2 (n_1^2+n_2^2))^s}$$

converges iff

$$s > 2.$$

2. By the same argument as previously

$$\sum_{n_1>0,n_2>0} \frac{1}{(1+\gamma \pi^2 (n_1^2+n_2^2))^s}$$

converges for

$$s > 1.$$

3. By the comparison criterion the series

$$\sum_{n_1>0} \frac{1}{(1+\gamma \pi^2 n_1^2)^s}$$

and

$$\sum_{n_2>0} \frac{1}{(1+\gamma \pi^2 n_2^2)^s}$$

converge for

$$s > \frac{1}{2}.$$

4. By the comparison criterion the series

$$\sum_{n_1>0} \frac{n_1^2}{(1+\gamma \pi^2 n_1^2)^s}$$

and

$$\sum_{n_2>0} \frac{n_2^2}{(1+\gamma \pi^2 n_2^2)^s}$$

converge for

$$s > \frac{3}{2}.$$

Lemma 7.2 Under the assumption (H2), for any

$$t \geq 0,$$

we have

$$\sup_x \left\| \frac{K(x,\cdot)}{\int_{\mathbb{T}^2} K(x',\cdot) \mu_t(dx')} \right\|_{H^3}^2 < \infty$$

Proof We have

$$\left\| \frac{K(x,\cdot)}{\int_{\mathbb{T}^2} K(x',\cdot) \mu_t(dx')} \right\|_{H^3}^2 = \sum_{|\eta| \leq 3} \left| \int_{\mathbb{T}^2} D^\eta \frac{K(x,y)}{\int_{\mathbb{T}^2} K(x',y) \mu_t(dx')} \right|^2 dy,$$

Now if we let

$$w_t(x,y) = \frac{K(x,y)}{\int_{\mathbb{T}^2} K(x',y) \mu_t(dx')}$$

for any

$$y \in \mathbb{T}^2,$$

one has

$$\frac{\partial w_t}{\partial y_1}(x,y) = \frac{\partial K(x,y)}{\partial y_1} \frac{\partial K(x,y)}{\partial y_1} \frac{K(x,y) f_{\mathbb{T}^2} \frac{\partial K(u,y)}{\partial y_1} \mu_t(du)}{f_{\mathbb{T}^2} K(u,y) \mu_t(du)^2}.$$
\[ \frac{\partial^2 w}{\partial y_2^2}(x, y) = \frac{\partial K}{\partial y_2 y_1}(x, y) \int_{\mathbb{T}^2} K(u, y) \mu_t(du) - \frac{\partial K}{\partial y_2}(x, y) \int_{\mathbb{T}^2} \frac{\partial^2 K}{\partial y_2^2}(u, y) \mu_t(du) \frac{(\int_{\mathbb{T}^2} K(u, y) \mu_t(du))^2}{(\int_{\mathbb{T}^2} K(u, y) \mu_t(du))^2} \]

\[ \frac{\partial^3 w}{\partial y_1 y_2 (y_1)}(x, y) = \frac{\partial^3 K}{\partial y_2^2 y_1}(x, y) \int_{\mathbb{T}^2} K(u, y) \mu_t(du) - 2 \frac{\partial^3 K}{\partial y_2^2}(x, y) \int_{\mathbb{T}^2} \frac{\partial^2 K}{\partial y_2^2 y_1}(u, y) \mu_t(du) \frac{(\int_{\mathbb{T}^2} K(u, y) \mu_t(du))^2}{(\int_{\mathbb{T}^2} K(u, y) \mu_t(du))^2} \]

Furthermore from Lemma 5.1, \( \forall |\eta| \leq 3, \ x \in \mathbb{T}^2, \ D^\eta K(x, y) \) is bounded by a constant independent of \( x \). Thus since \( \forall y \in \mathbb{T}^2, \int_{\mathbb{T}^2} K(u, y) \mu_t^N(du) = \int_{\mathbb{T}^2} K(u, y) f(t, u)(du) \) is lower bounded by a constant independent of \( y \) and \( f(t, .) \leq \delta_2 \), then we deduce from the above calculations that

\[ \sum_{|\eta| \leq 3} \int_{\mathbb{T}^2} \left| D^\eta \frac{K(x, y)}{\int_{\mathbb{T}^2} K(x, y) \mu_t(du)} \right|^2 dy \]

is bounded by a constant independent of \( x \). Hence the result.

\[ \square \]

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