UNCERTAINTY PRINCIPLES FOR THE CLIFFORD-FOURIER
TRANSFORM

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Abstract. In this paper, we establish an analogue of Hardy’s theorem and Miyachi’s theorem for the Clifford-Fourier transform.

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1. Introduction

Initially, uncertainty principles were given by two theorems: Heisenberg’s inequality [19] and Hardy’s theorem which ensured that is impossible for a nonzero function and its Fourier transform to be simultaneously small. Afterwards, many theorems were devoted to describe this smallness for example Miyachi [15], Cowling and Price [5], Donoho and Stark [9], Benedicks-Amrein-Berthier [1] and Beurling [2].

Hardy’s theorem [11] for the classical Fourier transform states: If we suppose $p$ and $q$ be positive constants and $f$ be a measurable function on the real line satisfying

$$|f(x)| \leq Ce^{-px^2}$$

and

$$|\mathcal{F}(f)(\lambda)| \leq Ce^{-q\lambda^2}$$

for some positive constant $C$, then (i) $f = 0$ if $pq > \frac{1}{4}$; (ii) $f = Ae^{-px^2}$ for some constant $A$ if $pq = \frac{1}{4}$; (iii) there are many $f$ if $pq < \frac{1}{4}$.

Miyachi’s theorem [15] asserts that if $f$ is a measurable function on $\mathbb{R}$ satisfying :

$$e^{ax^2}f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$$

and

$$\int_{\mathbb{R}} \log^+ \left( \frac{e^{ax^2}f(y)}{\lambda} \right) dy < \infty,$$

for some positive constants $a$ and $\lambda$, then $f$ is a constant multiple of $e^{-ax^2}$.

In Dunkl analysis, analogues of Heisenberg’s inequality [16] [17], Hardy’s [17], Miyachi’s [4], Cowling and Price’s [14], Donoho and Stark’s [14] and Beurling’s [14] theorems were established.

The last decades, there has been an increasing interest in the study of the Clifford-Fourier transform on $\mathbb{R}^m$ introduced in [3] and studied in [7] [8].

In 2010, H. De Bie and Y. Xu in [8] wrote the Clifford-Fourier transform as an integral transform

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\[ \mathcal{F}_\pm(f)(y) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(x, y) f(x) dx, \]

where the kernel function \( K_\pm(x, y) \) was given by an explicit expression. Recently, H. De Bie and Y. Xu in [7] show that the Clifford-Fourier transform is a continuous operator on Schwartz class functions. Besides, they define a translation operator related to the Clifford-Fourier transform and introduce a convolution structure based on translation operator. In the even dimension case, they give an inversion formula for the Clifford-Fourier transform.

Heisenberg’s inequality [6, 10], Donoho and Stark’s theorem and Benedicks’s theorem [13] are obtained for the Clifford-Fourier transform.

The purpose of this paper is to generalize Hardy’s theorem and Miyachi’s theorem for the Clifford-Fourier transform on \( \mathbb{R}^m \).

Our paper is organized as follows. In section 2, we present basic notions and notations related to the Clifford algebra. In section 3, we recall some results for the Clifford-Fourier transform that will be useful in the sequel. Also, we establish some new properties associated to the kernel of the Clifford-Fourier transform as well as for the Clifford-Fourier transform. In section 4, Clifford-heat kernel is introduced and studied. In section 5, we provide Hardy’s theorem for the Clifford-Fourier transform on \( \mathbb{R}^m \) when \( m \) even. Section 6 is devoted to Miyachi’s theorem for the Clifford-Fourier transform when \( m \) even.

Throughout this paper, the letter C indicates a positive constant that is not necessarily the same in each occurrence.

2. Notations and preliminaries

The Clifford algebra \( Cl_{0,m} \) over \( \mathbb{R}^m \) is a non commutative algebra generated by the basis \( \{e_1, \ldots, e_m\} \) satisfying the rules:

\[
\begin{align*}
e_i e_j &= -e_j e_i, & \text{if } i \neq j; \\
e_i^2 &= -1, & \forall 1 \leq i \leq m.
\end{align*}
\]

This algebra can be decomposed as

\[(2.2) \quad Cl_{0,m} = \bigoplus_{k=0}^{m} Cl_{0,m}^k,
\]

with \( Cl_{0,m}^k \) the space of vectors defined by

\[(2.3) \quad Cl_{0,m}^k = \text{span}\{e_{i_1} \ldots e_{i_k}, i_1 < \ldots < i_k\}.
\]

Hence, \( \{1, e_1, e_2, \ldots, e_{12}, \ldots, e_{12m}\} \) forms a basis of \( Cl_{0,m} \).

A Clifford number \( x \) in \( Cl_{0,m} \) is written as follows:

\[
(2.4) \quad x = \sum_{A \in J} e_A x_A,
\]
where \( J := \{0, 1, ..., m, 12, ..., 12..m\} \), \( x_A \) is a real number and \( e_A \) belongs to the basis of \( Cl_{0,m} \) defined above.

The norm of such element \( x \) is given by :

\[
||x||_c = \left( \sum_{A \in J} x_A^2 \right)^{\frac{1}{2}}.
\]

In particular, if \( x \) is a vector in \( Cl_{0,m} \), then

\[
||x||_c^2 = -x^2.
\]

The Clifford-Dirac operator and Clifford-Laplace operator are defined respectively by :

\[
\partial_x = \sum_{i=1}^{m} e_i \partial_{x_i},
\]

and

\[
\Delta_x = \sum_{i=1}^{m} \partial_{x_i}^2.
\]

We have the following relation :

\[
\Delta_x = -\partial_x^2.
\]

We introduce respectively the Clifford-Gamma operator associated to a vector \( x \), the inner product and the wedge product of two vectors \( x \) and \( y \) :

\[
\Gamma_x := -\sum_{j<k} e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j});
\]

\[
\langle x, y \rangle := \sum_{j=1}^{m} x_j y_j = -\frac{1}{2} (xy + yx);
\]

\[
 x \wedge y := \sum_{j<k} e_j e_k (x_j y_k - x_k y_j) = \frac{1}{2} (xy - yx).
\]

In the sequel, we consider functions defined on \( \mathbb{R}^m \) and taking values in \( Cl_{0,m} \). Such functions can be decomposed as :

\[
f(x) = f_0(x) + \sum_{i=1}^{m} e_i f_i(x) + \sum_{i<j} e_i e_j f_{ij}(x) + ... + e_1..e_m f_{1..m}(x),
\]

with \( f_0, f_i, ..., f_{1..m} \) all real-valued functions.

Let us recall some functional spaces :

- \( S(\mathbb{R}^m) \otimes Cl_{0,m} \) the Schwartz space of infinitely differentiable functions on \( \mathbb{R}^m \) and taking values in \( Cl_{0,m} \) which are rapidly decreasing as their derivatives,

- \( \mathcal{P}_k \) the space of homogenous polynomials of degree \( k \) taking values in \( Cl_{0,m} \),

- \( \mathcal{M}_k := \ker \partial_x \cap \mathcal{P}_k \) the space of spherical monogenic of degree \( k \),
\( \bullet \mathbb{L}^p(\mathbb{R}^m) \otimes Cl_{0,m} \) the space of integrable functions taking values in \( Cl_{0,m} \) equipped with the norm
\[
\| f \|_{p,c} = \left( \int_{\mathbb{R}^m} \| f(x) \|_c^p \, dx \right)^{1/p} = \left( \int_{\mathbb{R}^m} \left( \sum_{A \in J} (f_A(x))^2 \right)^{\frac{m}{2}} \, dx \right)^{\frac{1}{p}},
\]
where \( J = \{0, 1, \ldots, m, 12, 13, 23, \ldots, 12m\} \).

\( \bullet \mathcal{B}(\mathbb{R}^m) \otimes Cl_{0,m} \) a class of integrable functions taking values in \( Cl_{0,m} \) and satisfying
\[
\| f \|_{B,c} := \int_{\mathbb{R}^m} (1 + \| y \|_c)^{m-2} \| f(y) \|_c \, dy < \infty.
\]

\( \bullet \mathbb{L}^\infty(\mathbb{R}^m) \otimes Cl_{0,m} \) the space of essentially bounded functions on \( \mathbb{R}^m \) taking values in \( Cl_{0,m} \) endowed with the norm
\[
\| f \|_{\infty,c} := \inf \{ C \geq 0 : \| f(x) \|_c \leq C \text{ for almost every } x \in \mathbb{R}^m \}.
\]

3. Clifford-Fourier Transform

**Definition 3.1.** Let \( f \in \mathcal{B}(\mathbb{R}^m) \otimes Cl_{0,m} \). The Clifford-Fourier transform is given by (see [8]):
\[
\mathcal{F}_\pm(f)(y) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(x,y) f(x) \, dx,
\]
where
\[
K_\pm(x,y) = e^{\mp i \frac{\pi}{2} \Gamma_y e^{-i <x,y>}}.
\]

**Lemma 3.1.** Let \( m \) be even. Then
\[
\| K_\pm(x,y) \|_c \leq C e^{\| x \|_c \| y \|_c}, \quad \forall x, y \in \mathbb{R}^m,
\]
with \( C \) a positive constant.

**Proof.** By proposition 3.4 of [8], it is enough to prove the lemma for \( K_-(x,y) \).
For \( m = 2 \), it is shown in [8, p.15], that for \( x = x_1 e_1 + x_2 e_2 \) and \( y = y_1 e_1 + y_2 e_2 \), we have
\[
K_-(x,y) = \cos(x_1 y_2 - x_2 y_1) + e_{12} \sin(x_1 y_2 - x_2 y_1).
\]
Thus
\[
\| K_-(x,y) \|_c = 1.
\]
For \( m > 2 \). Recall that the kernel \( K_-(x,y) \), for \( x, y \in \mathbb{R}^m \), can be decomposed as the following
\[
K_-(x,y) = K_0^-(x,y) + \sum_{i<j} e_{ij} K_{ij}^-(x,y),
\]
with \( K_0^-(x,y) \) and \( K_{ij}^-(x,y) \) scalar functions,
By lemma 5.2 in [8], for \( x, y \in \mathbb{R}^m \),
\[
| K_0^-(x,y) | \leq c (1 + \| x \|_c \| y \|_c)^{\frac{m-2}{2}},
\]
\[
| K_{ij}^-(x,y) | \leq c (1 + \| x \|_c \| y \|_c)^{\frac{m-2}{2}}.
\]
Thus
\[ |K_-(x,y)| \leq C (1 + ||x||c||y||c) \frac{m-2}{2}. \]

Since for \( u \geq 0 \)
\[ (1 + u)^n e^{-u} \leq \frac{n^n}{e^{n-1}}, \forall n \in \mathbb{N}, \]
we conclude.

\[ \blacksquare \]

**Lemma 3.2.** (see [6]) Let \( c > 0 \) and \( f \in B(\mathbb{R}^m) \otimes Cl_{0,m} \), then
\[ K_\pm(x, cy) = K_\pm(cx, y), \quad \forall x, y \in \mathbb{R}^m. \]
Assume that \( f_c(x) := f(cx), x \in \mathbb{R}^m \). Then
\[ (3.4) \quad \mathcal{F}_\pm(f_c)(\lambda) = c^n \mathcal{F}_\pm(f)(c^{-1} \lambda). \]

**Theorem 3.3.** (see [7])

i) The Clifford-Fourier transform defines a continuous operator mapping \( S(\mathbb{R}^m) \otimes Cl_{0,m} \) to \( S(\mathbb{R}^m) \otimes Cl_{0,m} \) (see [7, Theorem 6.3]).

In particular, when \( m \) is even, we have
\[ \mathcal{F}_\pm \mathcal{F}_\pm = id_{S(\mathbb{R}^m) \otimes Cl_{0,m}}. \]

ii) The Clifford-Fourier transform extends from \( S(\mathbb{R}^m) \otimes Cl_{0,m} \) to a continuous map on \( L^2(\mathbb{R}^m) \otimes Cl_{0,m} \).

In particular, when \( m \) is even, we have
\[ ||\mathcal{F}_\pm(f)||_{2,c} = ||f||_{2,c}, \]
for all \( f \in L^2(\mathbb{R}^m) \otimes Cl_{0,m} \) (see [7, Theorem 6.4]).

**Theorem 3.4.** Let \( \delta > 0 \) and \( P \in \mathcal{P}_k(\mathbb{R}^m) \). Then, there exists \( Q \in \mathcal{P}_k(\mathbb{R}^m) \) which satisfies :
\[ (3.5) \quad \mathcal{F}_\pm(P(.e^{-\delta||.||^2})(x) = Q(x)e^{-\frac{||x||^2}{4\delta}}. \]

Proof. Considering the basis of \( L^2(\mathbb{R}^m) \otimes Cl_{0,m} \)
\[ \phi_{k,l,j}(x) = e^{-\frac{||x||^2}{4\delta}} H_{k,m,l}(\sqrt{2}x) P_i^{(j)}(\sqrt{2}x), \]
with \( k, l \in \mathbb{N} \cup \{0\} \) and \( j = 1, \ldots, dim(\mathcal{M}_{n}^+(l)) \).

\( H_{k,m,l} \) are the generalized Clifford-Hermite polynomials and
\[ \{ P_i^{(j)}, j = 1, 2, ..., dim(\mathcal{M}_{n}^+(l)) \} \]
denotes an orthogonal basis of \( \mathcal{M}_{n}^+(l) \) where \( \mathcal{M}_{n}^+(l) \) the set of all left inner spherical monogenics of degree \( l \). This basis constitutes eigenfunction of the Clifford-Fourier transform (see [3]). Note \( H_{k,m,l} \) is a polynomial of deree \( s \) in the variable \( x \) with the real coefficients depending on \( k \). Thus, by unicity of eigenfunction of the Clifford-Fourier transform, for every \( P \in \mathcal{P}_k(\mathbb{R}^m) \) there exists a unique polynomial \( Q \) of degree \( k \) such that
\[ \mathcal{F}_\pm(P(.e^{-\frac{||.||^2}{2\delta}}))(x) = Q(x)e^{-\frac{||x||^2}{4\delta}}. \]
with degree \(Q\) is \(k\).

The proof is completed by lemma 3.2.

\[\square\]

4. Clifford-Heat kernel

In this section, we introduce the Heat kernel in Clifford analysis. Then, we establish some properties of the Clifford-Heat kernel.

**Definition 4.1.** We define the Clifford-heat kernel by

\[
N_c(x, s) := \frac{1}{(2\pi)\frac{m}{2}(2s)^\frac{m}{2}} e^{-\frac{||x||^2}{4s}}, \quad \forall x \in \mathbb{R}^m, \; s > 0,
\]

associated with the Clifford-Laplace operator \(\Delta_x\).

**Theorem 4.1.** Let \(x \in \mathbb{R}^m\) and \(s > 0\). Then \(N_c(x, s)\) satisfies:

\[
\frac{\partial N_c(x, s)}{\partial s} - \Delta_x N_c(x, s) = 0.
\]

**Proof.** On one hand, applying the Clifford-Dirac operator to the Clifford-Heat kernel, we obtain

\[
\partial_x N_c(x, s) = \frac{1}{(2\pi)\frac{m}{2}(2s)^\frac{m}{2}} \partial_x e^{-\frac{||x||^2}{4s}}
\]

\[
= -\frac{1}{(2\pi)\frac{m}{2}(2s)^\frac{m}{2}+1} \sum_{i=1}^{m} e_i x_i e^{-\frac{||x||^2}{4s}}
\]

\[
= -\frac{1}{(2\pi)\frac{m}{2}(2s)^\frac{m}{2}+1} x e^{-\frac{||x||^2}{4s}}.
\]

Thus

\[
\partial_x^2 N_c(x, s) = -\frac{1}{(2\pi)\frac{m}{2}(2s)^\frac{m}{2}+1} \partial_x (x e^{-\frac{||x||^2}{4s}})
\]

\[
= -\frac{1}{(2\pi)\frac{m}{2}(2s)^\frac{m}{2}+1} (-m - \frac{x^2}{2s}) e^{-\frac{||x||^2}{4s}}.
\]

By (2.6) and (2.9), we get

\[
\Delta_x N_c(x, s) = \frac{1}{(2\pi)\frac{m}{2}(2s)^{\frac{m}{2}+1}} (-m + \frac{||x||^2}{2s}) e^{-\frac{||x||^2}{4s}}.
\]

On the other hand, we have

\[
\partial_s N_c(x, s) = \frac{1}{(2\pi)\frac{m}{2}} \partial_s \left( \frac{1}{(2s)^\frac{m}{2}} e^{-\frac{||x||^2}{4s}} \right)
\]

\[
= \frac{1}{(2\pi)\frac{m}{2}(2s)^{\frac{m}{2}+1}} (-m + \frac{||x||^2}{2s}) e^{-\frac{||x||^2}{4s}}
\]

\[
= \Delta_c N_c(x, s).
\]

\[\square\]
**Theorem 4.2.** The Clifford-Heat kernel satisfies the following properties:

i) For all $x \in \mathbb{R}^m$ and $s > 0$,

\[
\mathcal{F}_\pm(N_c(.,s))(x) = \frac{1}{(2\pi)^\frac{m}{2}} e^{-s||x||^2}.
\]

ii) Let $m$ be even. For $x \in \mathbb{R}^m$ and $s > 0$,

\[
N_c(x, s) = \frac{1}{(2\pi)^\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(y, x)e^{-s||y||^2}dy.
\]

iii) For all $\lambda > 0$, $x \in \mathbb{R}^m$ and $s > 0$,

\[
N_c(\lambda^{\frac{1}{2}}x, \lambda s) = \lambda^{-\frac{m}{2}} N_c(x, s).
\]

iv) For $s > 0$,

\[
||N_c(., s)||_{1,c} = 1.
\]

v) For all $s, t > 0$ and $x \in \mathbb{R}^m$,

\[
N_c(., t) *_{Cl} N_c(., s)(x) = (2\pi)^{-\frac{m}{2}} N_c(x, s + t),
\]

where $*_{Cl}$ denotes the Clifford-Fourier convolution (see [8]).

**Proof.**

i) One has

\[
\mathcal{F}_\pm(N_c(., s))(x) = \frac{1}{(2\pi)^\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(y, x)N_c(y, s)dy
\]

\[
= \frac{1}{(2\pi)^\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(y, x) e^{-\frac{||y||^2}{4s}} dy.
\]

By a change of variable, we get

\[
\mathcal{F}_\pm(N_c(., s))(x) = \frac{1}{(2\pi)^{\frac{m}{2}}(2s)^{\frac{m}{2}}} \int_{\mathbb{R}^m} K_\pm(\sqrt{2sz}, x)e^{-\frac{||y||^2}{4s}} dz.
\]

Lemma 3.2 implies that

\[
\mathcal{F}_\pm(N_c(., s))(x) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} K_\pm(z, \sqrt{2sx})e^{-\frac{||z||^2}{4s}} dz
\]

\[
= \mathcal{F}_\pm(\phi)(\sqrt{2sx}),
\]

with $\phi(x) = e^{-\frac{||x||^2}{2}}$. Since $\mathcal{F}_\pm(\phi)(x) = \phi(x)$ (see [8]), we deduce the result.

ii) Using Theorem 3.3 and (4.3), we obtain

\[
N_c(x, s) = \mathcal{F}_\pm \circ \mathcal{F}_\pm(N_c(x, s)) = \frac{1}{(2\pi)^\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(y, x)\mathcal{F}_\pm(N_c(y, s))dy
\]

\[
= \frac{1}{(2\pi)^\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(y, x) e^{-s||y||^2} dy.
\]

iii) Since

\[
N_c(\lambda^{\frac{1}{2}}x, \lambda s) = \frac{1}{(2\pi)^{\frac{m}{2}}(2\lambda s)^{\frac{m}{2}}} e^{-\frac{||\lambda^{\frac{1}{2}}x||^2}{4\lambda s}},
\]

The Clifford-Heat kernel satisfies the following properties:

i) For all $x \in \mathbb{R}^m$ and $s > 0$, \( \mathcal{F}_\pm(N_c(., s))(x) = \frac{1}{(2\pi)^\frac{m}{2}} e^{-s||x||^2}. \)

ii) Let $m$ be even. For $x \in \mathbb{R}^m$ and $s > 0$, \( N_c(x, s) = \frac{1}{(2\pi)^\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(y, x)e^{-s||y||^2}dy. \)

iii) For all $\lambda > 0$, $x \in \mathbb{R}^m$ and $s > 0$, \( N_c(\lambda^{\frac{1}{2}}x, \lambda s) = \lambda^{-\frac{m}{2}} N_c(x, s). \)

iv) For $s > 0$, \( ||N_c(., s)||_{1,c} = 1. \)

v) For all $s, t > 0$ and $x \in \mathbb{R}^m$, \( N_c(., t) *_{Cl} N_c(., s)(x) = (2\pi)^{-\frac{m}{2}} N_c(x, s + t), \) where $*_{Cl}$ denotes the Clifford-Fourier convolution (see [8]).

**Proof.**

i) One has

\[
\mathcal{F}_\pm(N_c(., s))(x) = \frac{1}{(2\pi)^\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(y, x)N_c(y, s)dy
\]

\[
= \frac{1}{(2\pi)^\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(y, x) e^{-\frac{||y||^2}{4s}} dy.
\]

By a change of variable, we get

\[
\mathcal{F}_\pm(N_c(., s))(x) = \frac{1}{(2\pi)^\frac{m}{2}(2s)^\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(\sqrt{2sz}, x)e^{-\frac{||y||^2}{4s}} dz.
\]

Lemma 3.2 implies that

\[
\mathcal{F}_\pm(N_c(., s))(x) = \frac{1}{(2\pi)^\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(z, \sqrt{2sx})e^{-\frac{||z||^2}{4s}} dz
\]

\[
= \mathcal{F}_\pm(\phi)(\sqrt{2sx}),
\]

with $\phi(x) = e^{-\frac{||x||^2}{2}}$. Since $\mathcal{F}_\pm(\phi)(x) = \phi(x)$ (see [8]), we deduce the result.

ii) Using Theorem 3.3 and (4.3), we obtain

\[
N_c(x, s) = \mathcal{F}_\pm \circ \mathcal{F}_\pm(N_c(x, s)) = \frac{1}{(2\pi)^\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(y, x)\mathcal{F}_\pm(N_c(y, s))dy
\]

\[
= \frac{1}{(2\pi)^\frac{m}{2}} \int_{\mathbb{R}^m} K_\pm(y, x) e^{-s||y||^2} dy.
\]

iii) Since

\[
N_c(\lambda^{\frac{1}{2}}x, \lambda s) = \frac{1}{(2\pi)^\frac{m}{2}(2\lambda s)^\frac{m}{2}} e^{-\frac{||\lambda^{\frac{1}{2}}x||^2}{4\lambda s}},
\]
the result is obvious.

iv) By (2.14)

\[ ||N_c(\cdot, s)||_{1,c} = \int_{\mathbb{R}^m} ||N_c(x, s)||_{1} dx \]

\[ = \frac{1}{(2\pi)^{m/4} (2s)^{m/4}} \int_{\mathbb{R}^m} e^{-\frac{||x||^2}{4s}} dx. \]

A change of variable yields the desired result.

v) Since \( N_c(\cdot, s) \) is a radial function, then the Clifford-convolution coincides with the classical convolution (see [8]). Thus

\[ N_c(\cdot, t) *_{Cl} N_c(\cdot, s)(x) = (2\pi)^{-m/2} 2^{-m} (st)^{-m/2} \int_{\mathbb{R}^m} e^{-\frac{||x-y||^2}{4(\sqrt{s} + t)}} e^{-\frac{||y||^2}{4\sqrt{s}}} dy. \]

Note that

\[ s||x-y||^2_c = s||x||^2_c + s||y||^2_c - 2s < x, y >. \]

\[ N_c(\cdot, t) *_{Cl} N_c(\cdot, s)(x) = (2\pi)^{-m/2} 2^{-m} (st)^{-m/2} \int_{\mathbb{R}^m} e^{-\frac{||y\sqrt{s+t} - x\sqrt{s+t}||^2_c}{4s+t}} dy. \]

It follows from a change of variable and (4.6) that

\[ N_c(\cdot, t) *_{Cl} N_c(\cdot, s)(x) = (2\pi)^{-m/2} 2^{-m} (st)^{-m/2} e^{-\frac{||z||^2_c}{4(\sqrt{s} + t)}} \int_{\mathbb{R}^m} e^{-\frac{||y||^2}{4s+t} - \frac{||z||^2}{4s+t}} dy. \]

\[ = (2\pi)^{-m} e^{-\frac{||z||^2}{4(\sqrt{s} + t)}} (2(s + t))^{-m/2} \int_{\mathbb{R}^m} N_c(z, st) dz \]

\[ = (2\pi)^{-m} N_c(x, s + t). \]

5. **Hardy’s theorem**

In this section, we give a generalization of Hardy’s theorem for the Clifford-Fourier transform.

**Theorem 5.1.** Assume that \( m \) is even. Let \( p \) and \( q \) be positive constants. Suppose \( f \) is a measurable function on \( \mathbb{R}^m \) such that :

\[ ||f(x)||_{c} \leq C e^{-p||x||^2}, \quad x \in \mathbb{R}^m \]

and

\[ ||\mathcal{F}(f)(\lambda)||_{c} \leq C e^{-q||\lambda||^2}, \quad \lambda \in \mathbb{R}^m, \]

for some positive constant \( C \). Then, three cases can occur :

i) If \( pq > \frac{1}{4} \), then \( f = 0 \).

ii) If \( pq = \frac{1}{4} \), then \( f(x) = A e^{-p||x||^2} \), where \( A \) is a constant.

iii) If \( pq < \frac{1}{4} \), then there are many functions satisfying the assumptions.
Proof. The basis \( \{ \psi_{j,k,l} \} \) defined in [8] by:

\[
\begin{align*}
\psi_{2j,k,l}(x) &:= L_{j}^{m+k-1}(|x|c)M_{k}^{(l)}e^{-\frac{||x||^2}{2}}, \\
\psi_{2j+1,k,l}(x) &:= L_{j}^{m+k}(|x|c)M_{k}^{(l)}e^{-\frac{||x||^2}{2}},
\end{align*}
\]

where \( j, k \in \mathbb{N}, M_{k}^{(l)} \in \mathcal{M}_k; \ l = 1, \ldots, \text{dim} \mathcal{M}_k \) gives an infinite number of examples for iii).

It is well known that by scaling (lemme 3.2), we may assume \( p = q \) without loss of
generality. The proof of i) is a simple deduction of ii).

Assume \( p = q = \frac{1}{2} \). Since \( f \) satisfying (5.1), then \( f \in B(\mathbb{R}^m) \otimes Cl_{0,m} \).
Moreover, we have for \( \lambda \in \mathbb{R}^m \otimes \mathbb{C} \)

\[
||F_{\pm}(f)(\lambda)||_c \leq (2\pi)^{-m} \int_{\mathbb{R}^m} ||K_{\pm}(x, \lambda)||_c ||f(x)||_c dx 
\leq C(2\pi)^{-m} \int_{\mathbb{R}^m} ||K_{\pm}(x, \lambda)||_c e^{-\frac{||x||^2}{2}} dx.
\]

Applying Lemma 3.1, we get:

\[
||F_{\pm}(f)(\lambda)||_c \leq C(2\pi)^{-m} \int_{\mathbb{R}^m} e^{\frac{||x||^2}{2}} ||K_{\pm}(x, \lambda)||_c dx 
\leq C e^{\frac{||\lambda||^2}{2}} (2\pi)^{-m} \int_{\mathbb{R}^m} e^{-\frac{(|x|c+||\lambda||c)^2}{2}} dx.
\]

Thus

\[(5.3) \quad ||F_{\pm}(f)(\lambda)||_c \leq Ce^{\frac{||\lambda||^2}{2}},\]

where \( C \) is a positive constant.
As \( F_{\pm}(f) \) is an entire function verifying (5.2) and (5.3), lemma 2.1 in [18] allows to
express \( F_{\pm}(f) \) as follows:

\[
F_{\pm}(f)(x) = Ae^{-\frac{||x||^2}{2}},
\]

with \( A \) is a constant.
Since \( f \) satisfies (5.1), then \( f \in L^2(\mathbb{R}^m) \otimes Cl_{0,m} \). By the inversion formula (Theorem
3.3), the proof is completed.\[\blacksquare\]

6. MIYACHI’S THEOREM FOR THE CLIFFORD-FOURIER TRANSFORM

In this section, we provide an analogue of Miyachi’s theorem for the Clifford-Fourier transform.

Lemma 6.1. Let \( m \) be even. Suppose \( f \) is a measurable function on \( \mathbb{R}^m \) such that

\[(6.1) \quad e^{a||x||^2} f \in L^p(\mathbb{R}^m) \otimes Cl_{0,m} + L^q(\mathbb{R}^m) \otimes Cl_{0,m},\]

for some \( a > 0 \) and \( 1 \leq p, q \leq +\infty \).
Then $F_\pm(f)$ is well defined. Moreover, there exists $C > 0$ such that,

\begin{equation}
||F_\pm(f)(z)||_c \leq C e^{||z||^2/4a}, \quad \forall z \in \mathbb{R}^m \otimes \mathbb{C}.
\end{equation}

**Proof.** By (6.1) there exists $u \in L^p(\mathbb{R}^m) \otimes Cl_{0,m}$ and $v \in L^q(\mathbb{R}^m) \otimes Cl_{0,m}$ such that

\[ f(x) = e^{-a||x||^2} (u(x) + v(x)). \]

Using Hölder’s inequality, one has

\[ ||f||_{B,c} \leq C(||u||_{p,c} + ||v||_{q,c}). \]

Thus, $f \in B(\mathbb{R}^m) \otimes Cl_{0,m}$. Lemma 3.1 yields for $z \in \mathbb{R}^m \otimes \mathbb{C}$

\[ ||K_\pm(x,z)||_c \leq e^{||x||_c||z||_c} \]

and

\[ ||F_\pm(f)(z)||_c \leq C(2\pi)^{-m} \int_{\mathbb{R}^m} e^{||x||_c||z||_c} ||f(x)||_c dx \]

\[ \leq C(2\pi)^{-m} e^{-a||x||_c||z||_c} \int_{\mathbb{R}^m} e^{-a(||x||_c||z||_c ||f(x)||_c)^2} e^{a||x||_c^2} ||f(x)||_c dx. \]

Applying Hölder’s inequality, we deduce that

\[ ||F_\pm(f)(z)||_c \leq C(2\pi)^{-m} e^{-a||x||_c^2} (||U||_{p,c} + ||V||_{q,c}). \]

Thus, we are done. \hfill \blacksquare

**Lemma 6.2.** (see [4]) Let $h$ be an entire function on $\mathbb{C}^m$ such that

\[ ||h(z)|| \leq A e^{B||Rez||^2} \]

and

\[ \int_{\mathbb{R}^m} \log^+ ||h(y)|| dy < \infty, \]

for some positive constants $A$ and $B$ where $\log^+ x = \log x$ if $x > 1$ and $\log^+ x = 0$ if $x \leq 1$. Then $h$ is a constant.

**Theorem 6.3.** Let $m$ be even. Let $a, b$ and $\lambda$ be positive constants and $p, q \in [1, +\infty]$. Suppose that $f$ is a measurable function such that

\begin{equation}
e^{a||x||^2} f \in L^p(\mathbb{R}^m) \otimes Cl_{0,m} + L^q(\mathbb{R}^m) \otimes Cl_{0,m}
\end{equation}

and

\begin{equation}
\int_{\mathbb{R}^m} \log^+ \frac{||e^{b||y||^2} F_\pm(f)(y)||_c}{\lambda} dy < \infty.
\end{equation}

Then, we have the following results:

- If $ab > \frac{1}{4}$, then $f = 0$.
- If $ab = \frac{1}{4}$, then $f = CN_c(.,b)$ with $|C| \leq \lambda$. 

• If \( ab < \frac{1}{4} \), then all functions \( f \) written as \( f = P(x)N_c(x, \delta) \) with \( P \in \mathcal{P}_k \) and \( \delta \in [b, \frac{1}{4a}] \) satisfy the assumptions of the theorem.

**Proof.** Put

\[
h(z) = e^{\frac{-z^2}{4a}} \mathcal{F}_\pm(f)(z), \quad \forall z \in \mathbb{R}^m \otimes \mathbb{C}.
\]

On one hand, we have for \( z = \epsilon + i\eta \in \mathbb{R}^m \otimes \mathbb{C} \)

\[
z^2 = \epsilon^2 + i\epsilon\eta + i\eta\epsilon - \eta^2 = -||\epsilon||_c^2 - 2i < \epsilon, \eta > + ||\eta||_c^2,
\]

and

\[
||z||_c^2 = ||\epsilon||_c^2 + ||\eta||_c^2.
\]

Lemma 6.1 yields

(6.5)

\[
||h(\epsilon + i\eta)||_c \leq Ce^{\frac{||\eta||_c^2}{4a}}.
\]

On the other hand, since \( \log^+(\text{cd}) \leq \log^+(c) + d \) for all \( c, d > 0 \), for \( ab > \frac{1}{4} \) it follows that

\[
\int_{\mathbb{R}^m} \log^+ ||h(y)||_c dy = \int_{\mathbb{R}^m} \log^+ \left( \frac{||e^{\text{bd}}||_{||y||_c^2} \mathcal{F}_\pm(f)(y)||_c}{\lambda} \right) dy
\]

\[
\leq \int_{\mathbb{R}^m} \log^+ \left( \frac{||e^{\text{bd}}||_{||y||_c^2} \mathcal{F}_\pm(f)(y)||_c}{\lambda} \right) dy + \int_{\mathbb{R}^m} \lambda e^{||y||_c^2(\frac{1}{4a} - b)} dy < +\infty.
\]

As \( h \) is an entire function, applying Lemma 6.2, we get \( h \) is a constant. Thus for \( ab > \frac{1}{4} \), we have

\[
\mathcal{F}_\pm(f)(y) = Ce^{\frac{y^2}{4a}} = Ce^{-\frac{||y||_c^2}{4a}}, \forall y \in \mathbb{R}^m.
\]

Subsequently, if \( ab > \frac{1}{4} \) refering to (6.4), \( C \) must be zero.

It is clear that if \( f \) satisfies (6.3), then \( f \in L^2(\mathbb{R}^m) \otimes \text{Cl}_{0,m} \). Thus, Theorem 3.3 implies that

\[
f = 0.
\]

If \( ab = \frac{1}{4} \), then as in the previous case we have

\[
\int_{\mathbb{R}^m} \log^+ \frac{||h(y)||_c}{\lambda} dy = \int_{\mathbb{R}^m} \log^+ \frac{||e^{\text{bd}}||_{||y||_c^2} \mathcal{F}_\pm(f)(y)||_c}{\lambda} dy < +\infty.
\]

Thus, we deduce from (6.5) and Lemma 6.2 that

\[
\mathcal{F}_\pm(f)(y) = Ce^{\frac{-y^2}{4a}}.
\]

Under condition (6.4), we should have \( |c| \leq \lambda \).

Hence, by Theorem 3.3 and (4.3), it follows that

\[
f = CN(., b), \text{ with } |c| \leq \lambda.
\]
For the last case \((ab < \frac{1}{4})\), suppose that \(f = P(x)N_c(x, \delta)\) with \(\delta \in [b, \frac{1}{4a}]\) and \(P \in \mathcal{P}_k\). Since \(\delta < \frac{1}{4a}\), we get
\[
e^{a|x|^2}f = e^{a|x|^2}\psi(x)e^{-\frac{|x|^2}{4a}} = \psi(x)e^{\frac{|x|^2}{4a}(a - \frac{1}{4a})} \in L^p(\mathbb{R}^m) \otimes Cl_{0,m} + L^q(\mathbb{R}^m) \otimes Cl_{0,m},
\]
where \(\psi(x) = \frac{1}{(2\pi)^{\frac{m}{2}}(2\delta)^{\frac{m}{2}}}P(x)\).

Using Theorem 3.4, we obtain
\[
\mathcal{F}_\pm(f)(x) = Q(x)e^{-\delta|x|^2},
\]
where \(Q\) is a polynomial with the same degree of \(P\). Since \(b < \delta\) we find that
\[
\int_{\mathbb{R}^m} \log^{\frac{1}{2}} \left| e^{b|y|^2} \mathcal{F}_\pm(f)(y) \right| dy = \int_{\mathbb{R}^m} \log^{\frac{1}{2}} \left| e^{b|y|^2}Q(y)e^{-\delta|y|^2} \right| dy < +\infty.
\]

**Corollary 6.4.** Let \(f\) be a measurable function on \(\mathbb{R}^m\) such that
\[
e^{a|x|^2}f \in L^p(\mathbb{R}^m) \otimes Cl_{0,m} + L^q(\mathbb{R}^m) \otimes Cl_{0,m}
\]
(6.6)

\[
\int_{\mathbb{R}^m} \left| \mathcal{F}_\pm(f)(y) \right|^{1+} e^{r|y|^2} dy < \infty,
\]
(6.7)

for some constants \(a, b > 0\), \(1 \leq p, q \leq +\infty\) and \(0 < r \leq \infty\).

(i) If \(ab \geq \frac{1}{4}\), then \(f = 0\).

(ii) If \(ab < \frac{1}{4}\) then all functions \(f\) of the form \(f(x) = P(x)N_c(x, \delta)\) with \(P \in \mathcal{P}_k\) and \(\delta \in [b, \frac{1}{4a}]\) satisfy (6.6) and (6.7).

**References**

[1] M. Benedicks, *On Fourier transforms of function supported on sets of finite Lebesgue measure*, J. Math. Anal. Appl., 106 , 180-183.
[2] A. Beurling, *The collect works of Arne Beurling*, Birkhauser. Boston (1989), 1-2.
[3] F. Brackx, N. de Schepper and F.Sommen, *The Clifford-Fourier transform*, J. Fourier Analysis and Applications (1989), Volume 11, Issue 6.
[4] F. Chouchene, R. Daher, T. Kawazoe and H. Mejjaoli, *Miyachi’s theorem for the Dunkl transform*, Integral Transforms and Special Functions Vol. 22, No. 3 (2011), 167-173.
[5] M.G. Cowling and J.F. Price, *Generalizations of Heisenberg inequality*, Lecture Notes in Math., 992. Springer Berlin (1983), 443-449.
[6] H. De Bie, R. Oste and J. Van der Jeugt, *Generalized Fourier arising from the enveloping algebras of sl(2) and osp(1|2)* Oxford University Press(2015).
[7] H. De Bie, N. De Schepper, F. Sommen, *The Class of Clifford-Fourier Transforms*. J. Fourier Anal Appl,17 (2011):1198-1231 .
[8] H. De Bie and Y. Xu, *On the Clifford-Fourier transform*. J. Fourier Anal. Appl. 11(2005), 669-681.
[9] D.L. Donoho and P.B. Stark, *Uncertainty principles and signal recovery*, SIAM J. Appl. Math., 49 (1989), 906-931.
[10] J. Elkamel R. Jday *The uncertainty principle in Clifford analysis* arXiv:submit/1281037 [math.CA] (16 Jun 2015).
[11] G.H. Hardy, *A theorem concerning Fourier transform*, J. London Math. Soc. 8 (1933), 227-231.
[12] L. Gallardo and K. Trimèche, *An Lp version of Hardys theorem for the Dunkl transform*. 
[13] S. Ghobber, Ph. Jaming, *Uncertainty principles for integral operators* [arXiv:1206.1195v1 [math.CA]] (6 Jun 2012).

[14] T. Kawazoe and H. Mejjaoli, *Uncertainty principles for the Dunkl transform* Hiroshima Mathematical Journal (2010).

[15] A. Miyachi, *A generalization of theorem of Hardy*, Harmonic Analysis Seminar held at Izumagaoka, Shizuoka-Ken, Japon (1997), 44-51.

[16] M. Rosler, *An uncertainty principle for the Dunkl transform*, Bull. Austral. Math. Soc. 59 (1999), 353-360.

[17] N. Shimeno, *A Note on the Uncertainty Principle for the Dunkl Transform* J. Math. Univ. Tokyo (2001), 33-42.

[18] A. Sitaram and M. Sundari, *An analogue of Hardy’s theorem for very rapidly decreasing functions on semi-simple Lie groups*, Pacific J. Math. 177 (1997), 187-200.

[19] Weyl, *Gruppentheorie and Quantenmechanik*, S. Hirzel, Leipzig, 1928. Revised English edition: The Theory of Groups and Quantum Mechanics, Methuen, London (1931); reprinted by Dover, New York, 1950.

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