Matrices that are self-congruent only via matrices of determinant one

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Abstract

Docović and Szechtman, [Proc. Amer. Math. Soc. 133 (2005)
2853–2863] considered a vector space $V$ endowed with a bilinear form.
They proved that all isometries of $V$ over a field $\mathbb{F}$ of characteristic not
2 have determinant 1 if and only if $V$ has no orthogonal summands
of odd dimension (the case of characteristic 2 was also consider-
ed). Their proof is based on Riehm’s classification of bilinear forms. Coak-
ley, Dopico, and Johnson [Linear Algebra Appl. 428 (2008) 796–813]
gave another proof of this criterion over $\mathbb{R}$ and $\mathbb{C}$ using Thompson’s
canonical pairs of symmetric and skew-symmetric matrices for con-
gruence. Let $M$ be the matrix of the bilinear form on $V$. We give
another proof of this criterion over $\mathbb{F}$ using our canonical matrices

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for congruence and obtain necessary and sufficient conditions involving canonical forms of $M$ for congruence, of $(M^T, M)$ for equivalence, and of $M^{-T}M$ (if $M$ is nonsingular) for similarity.

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1 Introduction

Fundamental results obtained by Đocović and Szechtman \cite{4} lead to a description of all $n$-by-$n$ matrices $M$ over any field $\mathbb{F}$ such that

$$S \text{ nonsingular and } S^T MS = M \, \implies \, \det S = 1. \quad (1)$$

Over a field of characteristic not 2, we give another proof of their description and obtain necessary and sufficient conditions on $M$ that ensure (1) and involve canonical forms of $M$ for congruence, of $(M^T, M)$ for equivalence, and of $M^{-T}M$ (if $M$ is nonsingular) for similarity. Of course, if $\mathbb{F}$ has characteristic 2 then every nonsingular matrix $M$ satisfies (1).

A vector space $V$ over $\mathbb{F}$ endowed with a bilinear form $B : V \times V \to \mathbb{F}$ is called a bilinear space. A linear bijection $A : V \to V$ is called an isometry if

$$B(Ax, Ay) = B(x, y) \quad \text{for all } x, y \in V.$$ If $B$ is given by a matrix $M$, then the condition (1) ensures that each isometry has determinant 1; that is, the isometry group is contained in the special linear group.

A bilinear space $V$ is called symplectic if $B$ is a nondegenerate skew-symmetric form. It is known that each isometry of a symplectic space has determinant 1 \cite[Theorem 3.25]{1}. If $B$ is given by the matrix

$$Z_{2m} := \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}, \quad (2)$$

then each isometry is given by a symplectic matrix (a matrix $S$ is symplectic if $S^T Z_{2m} S = Z_{2m}$), and so each symplectic matrix has determinant 1.

We denote by $M_n(\mathbb{F})$ the set of $n \times n$ matrices over a field $\mathbb{F}$ and say that $A, B \in M_n(\mathbb{F})$ are congruent if there is a nonsingular $S \in M_n(\mathbb{F})$ such that $S^T AS = B$; they are similar if $S^{-1} AS = B$ for some nonsingular $S \in M_n(\mathbb{F})$. 

The following theorem is a consequence of Đocović and Szechtman’s main theorem [4, Theorem 4.6], which is based on Riehm’s classification of bilinear forms [10].

**Theorem 1.** Let \( M \) be a square matrix over a field \( \mathbb{F} \) of characteristic different from 2. The following conditions are equivalent:

1. \( M \) satisfies (1) (i.e., each isometry on the bilinear space over \( \mathbb{F} \) with scalar product given by \( M \) has determinant 1),
2. \( M \) is not congruent to \( A \oplus B \) with a square \( A \) of odd size.

Đocović and Szechtman [4] also proved that if \( \mathbb{F} \) consists of more than 2 elements and its characteristic is 2 then \( M \in M_n(\mathbb{F}) \) satisfies (1) if and only if \( M \) is not congruent to \( A \oplus B \) in which \( A \) is a singular Jordan block of odd size. (Clearly, each \( M \in M_n(\mathbb{F}) \) satisfies (1) if \( \mathbb{F} \) has only 2 elements.) Coakley, Dopico, and Johnson [3, Corollary 4.10] gave another proof of Theorem 1 for real and complex matrices only: they used Thompson’s canonical pairs of symmetric and skew-symmetric matrices for congruence [14]. We give another proof of Theorem 1 using our canonical matrices for congruence [9, 11]. For the complex field, pairs of canonical forms of 8 different types are required in [3]; our canonical forms are of only three simple types [14]. Our approach to Theorem 1 is via canonical forms of matrices; the approach in [4] is via decompositions of bilinear spaces.

Following [3], we denote by \( \Xi_n(\mathbb{F}) \) the set of all \( M \in M_n(\mathbb{F}) \) that satisfy (1). A computation reveals that \( \Xi_n(\mathbb{F}) \) is closed under congruence, that is,

\[ M \in \Xi_n(\mathbb{F}) \text{ and } M \text{ congruent to } N \text{ imply } N \in \Xi_n(\mathbb{F}). \] (3)

The implication (i) \( \Rightarrow \) (ii) of Theorem 1 is easy to establish: let \( M \) be congruent to \( N = A \oplus B \), in which \( A \in M_r(\mathbb{F}) \) and \( r \) is odd. If \( S := (-I_r) \oplus I_{n-r} \), then \( S^TNS = N \) and \( \det S = (-1)^r = -1 \), and so \( N \notin \Xi_n(\mathbb{F}) \). It follows from (3) that \( M \notin \Xi_n(\mathbb{F}) \).

The implication (ii) \( \Rightarrow \) (i) is not so easy to establish. It is proved in Section 3. In the rest of this section and in Section 2 we discuss some consequences of Theorem 1. The first is

**Corollary 1.** Let \( \mathbb{F} \) be a field of characteristic not 2. If \( n \) is odd then \( \Xi_n(\mathbb{F}) \) is empty. \( M \in \Xi_2(\mathbb{F}) \) if and only if \( M \) is not symmetric.
Indeed, Theorem 1 ensures that $M \not\in \Xi_2(\mathbb{F})$ if and only if $M$ is congruent to $[a] \oplus [b]$ for some $a, b \in \mathbb{F}$, and this happens if and only if $M$ is symmetric.

In all matrix pairs that we consider, both matrices are over $\mathbb{F}$ and have the same size. Two matrix pairs $(A, B)$ and $(C, D)$ are equivalent if there exist nonsingular matrices $R$ and $S$ over $\mathbb{F}$ such that

$$R(A, B)S := (RAS, RBS) = (C, D).$$

A direct sum of pairs $(A, B)$ and $(C, D)$ is the pair

$$(A, B) \oplus (C, D) := (A \oplus C, B \oplus D)$$

The adjoint of $(A, B)$ is the pair $(B^T, A^T)$; thus, $(A, B)$ is selfadjoint if $A$ is square and $A = B^T$. For notational convenience, we write

$$M^{-T} := (M^{-1})^T.$$ 

We say that $(A, B)$ is a direct summand of $(M, N)$ for equivalence if $(M, N)$ is equivalent to $(A, B) \oplus (C, D)$ for some $(C, D)$. A square matrix $A$ is a direct summand of $M$ for congruence (respectively, similarity) if $M$ is congruent (respectively, similar) to $A \oplus B$ for some $B$.

The criterion (ii) in Theorem 1 uses the relation of matrix congruence; one must solve a system of quadratic equations to check that two matrices are congruent. The criteria (iii) and (iv) in the following theorem can be more convenient to use: one must solve only a system of linear equations to check that two matrices are equivalent or similar. In Section 2 we show that Theorem 1 implies

**Theorem 2.** Let $M$ be an $n \times n$ matrix over a field $\mathbb{F}$ of characteristic different from 2. The following conditions are equivalent:

1. $M \not\in \Xi_n(\mathbb{F})$;
2. $M$ has a direct summand for congruence that has odd size;
3. $(M^T, M)$ has a direct summand $(A, B)$ for equivalence, in which $A$ and $B$ are $r \times r$ matrices and $r$ is odd.
4. (in the case of nonsingular $M$) $M^{-T}M$ has a direct summand for similarity that has odd size.
For each positive integer \( r \), define the \((r - 1)\times r\) matrices

\[
F_r := \begin{bmatrix}
1 & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & 1 & 0
\end{bmatrix}, \quad G_r := \begin{bmatrix}
0 & 1 & 0 \\
\vdots & \ddots & \vdots \\
0 & 0 & 1
\end{bmatrix},
\]

and the \( r \times r \) matrices

\[
J_r(\lambda) := \begin{bmatrix}
\lambda & & \\
& \lambda & \\
& & \ddots
\end{bmatrix}, \quad \Gamma_r := \begin{bmatrix}
0 & & & & & \\
& 1 & -1 & & & \\
& -1 & 1 & -1 & & \\
& & & & 1 & -1
\end{bmatrix}.
\]

Note that

\[
\Gamma_r^T \Gamma_r \text{ is similar to } J_r((-1)^{r+1})
\]

since

\[
\Gamma_r^T \Gamma_r = (-1)^{r+1} \begin{bmatrix}
\vdots & \vdots & \vdots & & \vdots \\
-1 & -1 & -1 & \ddots & \\
1 & 1 & 1 & \ddots & \\
-1 & -1 & & \ddots & \\
1 & & & & 0
\end{bmatrix} \cdot \Gamma_r = (-1)^{r+1} \begin{bmatrix}
1 & 2 & * \\
1 & & \ddots \\
0 & & & 2
\end{bmatrix}.
\]

Explicit direct summands in the conditions (ii)–(iv) of Theorem 2 are given in the following theorem.

**Theorem 3.** Let \( M \) be an \( n \times n \) matrix over a field \( \mathbb{F} \) of characteristic different from 2. The following conditions are equivalent:

(i) \( M \notin \Xi_n(\mathbb{F}) \);

(ii) \( M \) has a direct summand for congruence that is either

- a nonsingular matrix \( Q \) such that \( Q^{-T}Q \) is similar to \( J_r(1) \) with odd \( r \) (if \( \mathbb{F} \) is algebraically closed, then we can take \( Q \) to be \( \Gamma_r \), since any such \( Q \) is congruent to \( \Gamma_r \)), or
- \( J_s(0) \) with odd \( s \).
(iii) \((M^T, M)\) has a direct summand for equivalence that is either \((I_r, J_r(1))\) with odd \(r\), or \((F_t, G_t)\) with any \(t\).

(iv) \((in the case of nonsingular \(M\)) \(M^{-T}M\) has a direct summand for similarity that is \(J_r(1)\) with odd \(r\).

In the following section we deduce Theorems 2 and 3 from Theorem 1 and give an algorithm to determine if \(M \in \Xi_n(F)\). In Section 3 we prove Theorem 1.

2 Theorem 1 implies Theorems 2 and 3

Theorem 3 gives three criteria for \(M \notin \Xi_n(F)\) that involve direct summands of \(M\) for congruence, direct summands of \((M^T, M)\) for equivalence, and direct summands of \(M^{-T}M\) for similarity. In this section we deduce these criteria from Theorem 1. For this purpose, we recall the canonical form of square matrices \(M\) for congruence over \(F\) given in [11, Theorem 3], and derive canonical forms of selfadjoint pairs \((M^T, M)\) for equivalence and canonical forms of cosquares \(M^{-T}M\) for similarity. Then we establish conditions on these canonical forms under which \(M \notin \Xi_n(F)\).

2.1 Canonical form of a square matrix for congruence

Every square matrix \(A\) over a field \(F\) of characteristic different from 2 is similar to a direct sum, uniquely determined up to permutation of summands, of Frobenius blocks

\[
\Phi_{p^t} = \begin{bmatrix}
0 & 0 & -c_m \\
1 & \ddots & \ddots \\
& \ddots & 0 & -c_2 \\
0 & 1 & -c_1
\end{bmatrix}, \tag{7}
\]

in which

\[p(x)^t = x^m + c_1x^{m-1} + \cdots + c_m\]

is an integer power of a polynomial

\[p(x) = x^s + a_1x^{s-1} + \cdots + a_s\]

that is irreducible over \(F\). This direct sum is the Frobenius canonical form of \(A\); sometimes it is called the rational canonical form (see [2, Section 6]).
A Frobenius block has no direct summand under similarity other than itself, i.e., it is indecomposable under similarity. Also, the Frobenius block \( \Phi(x-\lambda)^m \) is similar to the Jordan block \( J_m(\lambda) \).

If \( p(0) = a_s \neq 0 \) in (8), we define
\[
p^\vee(x) := a_s^{-1}(1 + a_1 x + \cdots + a_s x^s) = p(0)^{-1}x^s p(x^{-1})
\]  
(9)
and observe that
\[
(p(x)^l)^\vee = p(0)^{-1}x^{sl} p(x^{-1})^l = (p(0)^{-1}x^s p(x^{-1}))^l = (p^\vee(x))^l.
\]  
(10)

The matrix \( A^{-T}A \) is the cosquare of a nonsingular matrix \( A \). If two nonsingular matrices are congruent, then their cosquares are similar because
\[
(S^TAS)^{-T}(S^TAS) = S^{-1}A^{-T}AS.
\]  
(11)
If \( \Phi \) is a cosquare, we choose a matrix \( A \) such that \( A^{-T}A = \Phi \) and write \( \sqrt[\vee]{\Phi} := A \) (a cosquare root of \( \Phi \)).

**Lemma 1.** Let \( p(x) \) be an irreducible polynomial of the form (8) and let \( \Phi_{p^l} \) be an \( m \times m \) Frobenius block (7). Then

(a) \( \Phi_{p^l} \) is a cosquare if and only if
\[
p(x) \neq x, \quad p(x) \neq x + (-1)^{m+1}, \quad \text{and} \quad p(x) = p^\vee(x).
\]  
(12)

(b) If \( \Phi_{p^l} \) is a cosquare and \( m \) is odd, then \( p(x) = x - 1 \).

**Proof.** The conditions in (a) and an explicit form of \( \sqrt[\vee]{\Phi_{p^l}} \) were established in [11, Theorem 7]; see [9, Lemma 2.3] for a more detailed proof.

(b) By (12), \( p(x) = p^\vee(x) \). Therefore, \( a_s = a_s^{-1}, \) so \( a_s = \varepsilon = \pm 1 \) and
\[
p(x) = x^{2k+1} + a_1 x^{2k} + \cdots + a_k x^{k+1} + a_k \varepsilon x^k + \cdots + a_1 \varepsilon x + \varepsilon.
\]
Observe that \( p(-\varepsilon) = 0 \). But \( p(x) \) is irreducible, so \( s = 1 \) and \( p(x) = x + \varepsilon \). By (12) again, \( \varepsilon \neq 1 \). Therefore, \( p(x) = x - 1 \).\( \square \)

Define the *skew sum* of two matrices:
\[
\begin{bmatrix} A \setminus B \end{bmatrix} := \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}.
\]
Theorem 4. Let $M$ be a square matrix over a field $\mathbb{F}$ of characteristic different from 2. Then

(a) $M$ is congruent to a direct sum of matrices of the form

$$[\Phi_p \setminus I_m], \quad Q, \quad J_s(0), \quad (13)$$

in which $\Phi_p$ is an $m \times m$ Frobenius block that is not a cosquare, $Q$ is nonsingular and $Q^{-T}Q$ is similar to a Frobenius block, and $s$ is odd.

(b) $M \notin \Xi_n(\mathbb{F})$ if and only if $M$ has a direct summand for congruence that is either

- a nonsingular matrix $Q$ such that $Q^{-T}Q$ is similar to $J_r(1)$ with odd $r$, or
- $J_s(0)$ with odd $s$.

Proof. (a) This statement is the existence part of Theorem 3 in [11] (also presented in [9, Theorem 2.2]), in which a canonical form of a matrix for congruence over $\mathbb{F}$ is given up to classification of Hermitian forms over finite extensions of $\mathbb{F}$. The canonical block $J_{2m}(0)$ is used in [11] instead of $[J_m(0) \setminus I_m]$, but the proof of Theorem 3 in [11] shows that these two matrices are congruent.

(b) The “if” implication follows directly from Theorem 4. Let us prove the “only if” implication. If $M \notin \Xi_n(\mathbb{F})$, Theorem 4 ensures that $M$ is congruent to $A \oplus B$, in which $A$ is square and has odd size. Part (a) ensures that $A$ is congruent to a direct sum of matrices of the form (13), not all of which have even size. Thus, $A$ (and hence also $M$) has a direct summand for congruence that is either $J_s(0)$ with $s$ odd, or a nonsingular matrix $Q$ of odd size such that $Q^{-T}Q$ is similar to a Frobenius block $\Phi_p$ of odd size. Lemma 1 ensures that $p(x) = x - 1$, so $Q^{-T}Q$ is similar to $\Phi_{(x-1)^r}$, which is similar to $J_r(1)$.

If $\mathbb{F}$ is algebraically closed, then Theorem 4 can be simplified as follows.

Theorem 5. Let $M$ be a square matrix over an algebraically closed field of characteristic different from 2. Then

(a) $M$ is congruent to a direct sum of matrices of the form

$$[J_m(\lambda) \setminus I_m], \quad \Gamma_r, \quad J_s(0), \quad (14)$$

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in which $\lambda \neq (-1)^{m+1}$, each nonzero $\lambda$ is determined up to replacement by $\lambda^{-1}$, $\Gamma_r$ is defined in (5), and $s$ is odd. This direct sum is uniquely determined by $M$, up to permutation of summands.

(b) $M \notin \Xi_n(F)$ if and only if $M$ has a direct summand for congruence of the form $\Gamma_r$ with odd $r$ or $J_s(0)$ with odd $s$.

**Proof.** (a) This canonical form for congruence was obtained in [9, Theorem 2.1(a)]; see also [6, 8].

(b) This statement follows from (a) and Theorem 1. $\square$

The equivalence (i) $\Leftrightarrow$ (ii) in Theorem 3 follows from Theorems 4 and 5. (The equivalence (i) $\Leftrightarrow$ (ii) in Theorem 2 is another form of Theorem 1.)

2.2 Canonical form of a selfadjoint matrix pair for equivalence

Kronecker’s theorem for matrix pencils [5, Chapter 12] ensures that each matrix pair $(A, B)$ over $\mathbb{C}$ is equivalent to a direct sum of pairs of the form

$$(I_m, J_m(\lambda)), \quad (J_r(0), I_r), \quad (F_s, G_s), \quad (F^T, G^T),$$

in which $F_s$ and $G_s$ are defined in (4). This direct sum is uniquely determined by $(A, B)$, up to permutations of summands. Over a field $F$ of characteristic not 2, this canonical form with Frobenius blocks $\Phi_{pl}$ (see (7)) instead of Jordan blocks $J_m(\lambda)$ can be constructed in two steps:

- Use Van Dooren’s regularization algorithm [15] for matrix pencils (which was extended to matrices of cycles of linear mappings in [13] and to matrices of bilinear forms in [7]) to transform $(A, B)$ to an equivalent pair that is a direct sum of the regular part $(I_k, R)$ with nonsingular $R$ and canonical pairs of the form $(J_r(0), I_r), (F_s, G_s)$, and $(F^T, G^T)$.

- Reduce $R$ to a direct sum of Frobenius blocks $\Phi_{pl}$ by a similarity transformation $S^{-1}RS$; the corresponding similarity transformation $S^{-1}(I_k, R)S = (I_k, S^{-1}RS)$ decomposes the regular part into a direct sum of canonical blocks $(I_m, \Phi_{pl})$.

**Theorem 6.** Let $M$ be a square matrix over a field $F$ of characteristic different from 2.
(a) The selfadjoint pair \((M^T, M)\) is equivalent to a direct sum of selfadjoint pairs of the form

\[
\left( [I_m \setminus \Phi^T_{p'}], [\Phi_{p'} \setminus I_m] \right), \quad \left( \sqrt{\Phi_{q'}}^T, \sqrt{\Phi_{q'}} \right), \quad (J_s(0)^T, J_s(0)),
\]

in which \(\Phi_{p'}\) is an \(m \times m\) Frobenius block that is not a cosquare, \(\Phi_{q'}\) is a Frobenius block that is a cosquare, and \(s\) is odd. This direct sum is uniquely determined by \(M\), up to permutations of direct summands and replacement, for each \(\Phi_{p'}\), of any number of summands of the form \((I_m \setminus \Phi^T_{p'}), [\Phi_{p'} \setminus I_m]\) by \((I_m \setminus \Phi^T_{q'}), [\Phi_{q'} \setminus I_m]\), in which \(q(x) := p^r(x)\) is defined in (9).

(b) The following three conditions are equivalent:

(i) \(M \notin \Xi_n(F)\);

(ii) \((M^T, M)\) has a selfadjoint direct summand for equivalence of the form \((\Gamma^T_r, \Gamma_r)\) with odd \(r\), or \((J_s(0)^T, J_s(0))\) with odd \(s\);

(iii) \((M^T, M)\) has a direct summand for equivalence of the form \((I_r, J_r(1))\) with odd \(r\), or \((F_t, G_t)\) with any \(t\).

Proof. Let \(M\) be a square matrix over a field \(F\) of characteristic different from 2.

(a) By Theorem 11(a), \(M\) is congruent to a direct sum \(N\) of matrices of the form (13). Hence, \((M^T, M)\) is equivalent to \((N^T, N)\), a direct sum of pairs of the form (15).

Uniqueness of this direct sum follows from the uniqueness assertion in Kronecker’s theorem and the following four equivalences:

1. \([I_m \setminus \Phi^T_{p(x)}], [\Phi_{p(x)} \setminus I_m]\) is equivalent to \((I_m, \Phi_{p(x)}) \oplus (I_m, \Phi_{p^r(x)})\) for each irreducible polynomial \(p(x) \neq x\).

2. \([I_m \setminus J_m(0)^T], [J_m(0) \setminus I_m]\) is equivalent to \((I_m, J_m(0)) \oplus (J_m(0), I_m)\).

3. \((\sqrt{\Phi_{q^r}}, \sqrt{\Phi_{q^r}})\) is equivalent to \((I, \Phi_{q^r})\).

4. \((J_{2t-1}(0)^T, J_{2t-1}(0))\) is equivalent to \((F_t^T, G_t) \oplus (G_t, F_t)\).

To verify the first equivalence, observe that \((\Phi^T_{p(x)}), I_m)\) is equivalent to \((I_m, \Phi_{p(x)})\) because

\[
\Phi^T_{p(x)} \text{ is similar to } \Phi_{p^r(x)}.
\]

(16)
for each nonsingular $m \times m$ Frobenius block $\Phi := \Phi_p(x)^t$. The similarity follows from the fact that the characteristic polynomials of $\Phi - T$ and $\Phi_p(x)^t$ are equal:

$$\chi_{\Phi - T}(x) = \det(xI - \Phi^{-1}) = \det((-\Phi^{-1})(I - x\Phi))$$

$$= \det(-\Phi^{-1}) \cdot x^m \cdot \det(x^{-1}I - \Phi) = \chi_{\phi}(x) = (p(x)^t)^\vee,$$

which equals $p^\vee(x)^t$ by (10).

The second equivalence is obvious.

To verify the third equivalence, compute

$$T \sqrt{\Phi_q r} - T (T \sqrt{\Phi_q r} T, T \sqrt{\Phi_q r}) = (I, \Phi_q r).$$

The matrix pairs in the fourth equivalence are permutationally equivalent.

(b) “(i) $\Rightarrow$ (ii)” Suppose that $M \notin \Xi_n(\mathbb{F})$. By Theorem 4(b), $M$ has a direct summand $Q$ for congruence such that $Q - T Q$ is similar to $J_r(1)$ with odd $r$, or a direct summand $J_s(0)$ with odd $s$. Then $(Q^T, Q)$ or $(J_s(0)^T, J_s(0))$ is a direct summand of $(M^T, M)$ for equivalence. The pair $(Q^T, Q)$ is equivalent to $(\Gamma_r^T, \Gamma_r)$ since $Q - T Q$ and $\Gamma_r - T \Gamma_r$ are similar (they are similar to $J_r(1)$ by (6)) and because

$$S^{-1}Q - T Q S = \Gamma_r - T \Gamma_r \implies \Gamma_r^T S^{-1}Q - T (Q^T, Q) S = (\Gamma_r^T, \Gamma_r).$$

“(ii) $\Rightarrow$ (iii)” To prove this implication, observe that $(\Gamma_r^T, \Gamma_r)$ with odd $r$ is equivalent to $(I_r, \Gamma_r^{-T} \Gamma_r)$, which is equivalent to $(I_r, J_r(1))$ by (6) and [9, p. 213] ensures that

$$(J_{2t-1}(0)^T, J_{2t-1}(0)) \text{ is equivalent to } (F_t, G_t) \oplus (G_t^T, F_t^T). \quad (17)$$

“(iii) $\Rightarrow$ (i)” Assume the assertion in (iii). By Theorem 4(a), $M$ is congruent to a direct sum $N = \oplus_i N_i$ of matrices of the form (13). Then $(M^T, M)$ is equivalent to $(N^T, N) = \oplus_i (N_i^T, N_i)$. By (iii) and the uniqueness assertion in Kronecker’s theorem, some $(N_i^T, N_i)$ has a direct summand for equivalence of the form $(I_r, J_r(1))$ with odd $r$ or $(F_t, G_t)$ with any $t$.

- Suppose that the direct summand is $(I_r, J_r(1))$ with odd $r$. Since $N_i$ is one of the matrices (13) and $J_r(1)$ with odd $r$ is a cosquare by (12), it follows that $N_i = Q$ and $Q - T Q$ is similar to $J_r(1)$.

- Suppose that the direct summand is $(F_t, G_t)$. Since $N_i$ is one of the matrices (13), (17) ensures that $N_i = J_{2t-1}(0)$. 

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In both the preceding cases, \( N_i \) has odd size, so Theorem 1 ensures that \( M \not\in \Xi_n(F) \). □

The equivalences (i) \( \iff \) (iii) in Theorems 2 and 3 follow from Theorem 6.

2.3 Canonical form of a cosquare for similarity

**Theorem 7.** Let \( M \) be a nonsingular matrix over a field \( F \) of characteristic different from 2.

(a) The cosquare \( M^{-T}M \) is similar to a direct sum of cosquares

\[
\Phi_{p'l} \oplus \Phi_{p'l}^{-T}, \quad \Phi_{q'r},
\]

in which \( \Phi_{p'l} \) is a nonsingular Frobenius block that is not a cosquare and \( \Phi_{q'r} \) is a Frobenius block that is a cosquare. This direct sum is uniquely determined by \( M \), up to permutation of direct summands and replacement, for each \( \Phi_{p'l} \), of any number of summands of the form \( \Phi_{p'l} \oplus \Phi_{p'l}^{-T} \) by \( \Phi_{q'l} \oplus \Phi_{q'l}^{-T} \), in which \( q(x) := p'(x) \) is defined in (9).

(b) \( M \not\in \Xi_n(F) \) if and only if \( M^{-T}M \) has a direct summand for similarity of the form \( J_r(1) \) with odd \( r \).

**Proof.** (a) The existence of this direct sum follows from Theorem 4(a) since \( M \) is congruent to a direct sum of nonsingular matrices \( [\Phi_{p'l} \setminus I_m] \) and \( Q \) (see (13)); the matrices (18) are their cosquares. The uniqueness assertion follows from uniqueness of the Frobenius canonical form and (16).

(b) By Theorem 6(b) and because \( M \) is nonsingular, \( M \not\in \Xi_n(F) \) if and only if \( (M^T, M) \) has a direct summand for equivalence of the form \( (I_r, J_r(1)) \) with odd \( r \). This implies (b) since \( (M^T, M) \) is equivalent to \( (I_n, M^{-T}M) \). □

The equivalences (i) \( \iff \) (iv) in Theorems 2 and 3 follow from Theorem 7.

2.4 An algorithm

The following simple condition is sufficient to ensure that \( M \in \Xi_n(F) \).

**Lemma 2 (3 Theorem 2.3) for \( F = \mathbb{R} \) or \( \mathbb{C} \).** Let \( F \) be a field of characteristic different from 2. If \( M \in M_n(F) \) and if its skew-symmetric part \( M_w = (M - M^T)/2 \) is nonsingular, then \( M \in \Xi_n(F) \).
Proof. Since $M_w$ is skew-symmetric and nonsingular, there exists a nonsingular $C$ such that $M_w = C^T Z_{2m} C$, in which $Z_{2m}$ is defined in (2). If $S^T M S = M$, then

$$S^T M_w S = M_w, \quad (C S C^{-1})^T Z_{2m} (C S C^{-1}) = Z_{2m},$$

and so $C S C^{-1}$ is symplectic. By [1, Theorem 3.25], $\det C S C^{-1} = 1$, which implies that $\det S = 1$. □

Independent of any condition on $M_w$, one can use the regularization algorithm described in [7] to reduce $M$ by a sequence of congruences (simple row and column operations) to the form

$$B \oplus J_{n_1}(0) \oplus \cdots \oplus J_{n_p}(0), \quad B \text{ nonsingular and } 1 \leq n_1 \leq \cdots \leq n_p. \quad (19)$$

Of course, the singular blocks are absent and $B = M$ if $M$ is nonsingular.

According to Theorem (7)(b), the only information needed about $B$ in (19) is whether it has any Jordan blocks $J_r(1)$ with odd $r$. Let $r_k = \text{rank}(B^{-T} B - I)^k$ and set $r_0 = n$. For each $k = 1, \ldots, n$, $B^{-T} B$ has $r_{k-1} - r_k$ blocks $J_j(1)$ of all sizes $j \geq k$ and exactly $(r_{2k} - r_{2k+1}) - (r_{2k+1} - r_{2k+2}) = r_{2k} - 2r_{2k+1} + r_{2k+2}$ blocks of the form $J_{2k+1}(1)$ for each $k = 0, 1, \ldots, \left[\frac{n-1}{2}\right]$.

The preceding observations lead to the following algorithm to determine whether a given $M \in M_n(\mathbb{F})$ is in $\Xi_n(\mathbb{F})$:

1. If $M - M^T$ is nonsingular, then stop: $M \in \Xi_n(\mathbb{F})$.

2. If $M$ is singular, use the regularization algorithm [7] to determine a direct sum of the form (19) to which $M$ is congruent, and examine the singular block sizes $n_j$. If any $n_j$ is odd, then stop: $M \notin \Xi_n(\mathbb{F})$.

3. If $M$ is nonsingular or if all $n_j$ are even, then $M \in \Xi_n(\mathbb{F})$ if and only if $r_{2k} - 2r_{2k+1} + r_{2k+2} = 0$ for all $k = 0, 1, \ldots, \left[\frac{n-1}{2}\right]$.

Notice that if $M - M^T$ is nonsingular, then (a) no $n_j$ is odd since $J_r(0) - J_r(0)^T$ is singular for every odd $r$, (b) $B - B^T$ is nonsingular, and (c) $\text{rank}(B^{-T} B - I) = \text{rank}(B^{-T} (B - B^T)) = n$, so $r_k = n$ for all $k = 1, 2, \ldots$ and $r_{2k} - 2r_{2k+1} + r_{2k+2} = 0$ for all $k = 0, 1, \ldots$. 

13
3 Proof of Theorem 1

The implication (i) ⇒ (ii) of Theorem 1 was established in Section 1. In this section we prove the remaining implication (ii) ⇒ (i): we take any $M \in M_n(F)$ that has no direct summands for congruence of odd size, and show that $M \in \Xi_n(F)$. We continue to assume, as in Theorem 1, that $F$ is a field of characteristic different from 2.

By (3) and Theorem 4(a), we can suppose that $M$ is a direct sum of matrices of even sizes of the form $[\Phi_p \setminus I_m]$ and $Q$; see (13). Rearranging summands, we represent $M$ in the form

$$M = M' \oplus M'', \quad M' \text{ is } n' \times n', \quad M'' \text{ is } n'' \times n'', \quad (20)$$

in which

(α) $M'$ is the direct sum of all summands of the form $[\Phi_{(x-1)m} \setminus I_m]$ ($m$ is even by Lemma 1(a)), and

(β) $M''$ is the direct sum of the other summands; they have the form $[\Phi_{p' \setminus I_m}]$ with $p(x) \neq x - 1$ and $Q$ of even size, in which $\Phi_{p'}$ is an $m \times m$ Frobenius block that is not a cosquare and $Q^{-T}Q$ is similar to a Frobenius block.

Step 1: Show that for each nonsingular $S$,

$$S^TMS = M \quad \implies \quad S = S' \oplus S'', \quad S' \text{ is } n' \times n', \quad S'' \text{ is } n'' \times n''. \quad (21)$$

If $S^TMS = M$, then $S^T(M^T,M)S = (M^T,M)$, and so with $R := S^{-T}$ we have

$$(M^T,M)S = R(M^T,M). \quad (22)$$

To prove (21), we prove a more general assertion: (22) implies that

$$S = S' \oplus S'', \quad R = R' \oplus R'', \quad S', R' \text{ are } n' \times n', \quad S'', R'' \text{ are } n'' \times n''. \quad (23)$$

Using Theorem 5(a), we reduce $M'$ and $M''$ in (20) by congruence transformations over the algebraic closure $\overline{F}$ of $F$ to direct sums of matrices of the form $[J_m(1) \setminus I_m]$ and, respectively, of the form $[J_m(\lambda) \setminus I_m]$ with $\lambda \neq 1$ and $\Gamma_r$ with even $r$. Then
\( (M'\top, M') \) is equivalent over \( \overline{F} \) to a direct sum of pairs of the form 
\((I_m, J_m(1)) \oplus (J_m(1), I_m), \) and

\( (M''\top, M'') \) is equivalent over \( \overline{F} \) to a direct sum of pairs of the form 
\((I_m, J_m(\lambda)) \oplus (J_m(\lambda), I_m) \) with \( 1 \neq \lambda \in \overline{F} \) and \( (\Gamma_T^r, \Gamma_r) \) with even \( r. \)

The pair \((J_m(1), I_m)\) is equivalent to \((I_m, J_m(1)). \) The pair \((\Gamma_T^r, \Gamma_r)\) is equivalent to \((I_r, \Gamma_r^{-T} \Gamma_r), \) which is equivalent to \((I_r, J_r(-1))\) by \((6) \) since \( r \) is even.

Thus, \((\alpha') \) \( (M'\top, M') \) is equivalent to a direct sum of pairs of that are of the form 
\((I_m, J_m(1)), \) and

\((\beta') \) \( (M''\top, M'') \) is equivalent to a direct sum of pairs that are either of the form 
\((I_m, J_m(\lambda)) \) with \( \lambda \neq 1 \) or of the form \((J_m(0), I_m)). \)

We choose \( \gamma \in \overline{F}, \gamma \neq -1, \) such that \( M''\top + \gamma M'' \) is nonsingular (if \( M'' \) is nonsingular, then we may take \( \gamma = 0; \) if \( M'' \) is singular, then we may choose any \( \gamma \neq 0, -1 \) such that \( (M'\top, M) \) has no direct summands of the form \( (I_m, J_m(-\gamma^{-1})). \)

Then \((22) \) implies that
\[
(M'\top + \gamma M, M)S = R(M'\top + \gamma M, M).
\]

The pair \( (M'\top + \gamma M, M) \) is equivalent to \((I_n, (M'\top + \gamma M)^{-1} M), \) whose Kronecker canonical pair has the form
\[
(I_n, N) := (I_{n'} , N'') \oplus (I_{n''} , N''),
\]
in which \( (\alpha') \) and \( (\beta') \) ensure that

\( (\alpha'') \) \( N' \) (of size \( n' \times n' \)) is a direct sum of Jordan blocks with eigenvalue \((1 + \gamma)^{-1}, \) and

\( (\beta'') \) \( N'' \) (of size \( n'' \times n'' \)) is a direct sum of Jordan blocks with eigenvalues distinct from \((1 + \gamma)^{-1}. \)

If \((I_n, N)\tilde{S} = \tilde{R}(I_n, N), \) then \( \tilde{S} = \tilde{R}, N\tilde{S} = \tilde{S}N, \) and \( (\alpha'') \) and \( (\beta'') \) ensure that \( \tilde{S} = \tilde{S}' \oplus \tilde{S}'' \) in which \( \tilde{S}' \) is \( n' \times n' \) and \( \tilde{S}'' \) is \( n'' \times n''. \) Since \((I_n, N)\) is obtained from \((M'\top, M)\) by transformations within \((M'\top, M')\) and within \((M''\top, M''), \) \((22) \) implies \((23) \). This proves \((21). \)
Since \( \det S = \det S' \det S'' \), it remains to prove that
\[
M' \in \Xi_n'(F), \quad M'' \in \Xi_{n''}(F).
\]

**Step 2: Show that \( M'' \in \Xi_{n''}(F) \).**

By Lemma 2 it suffices to show that \( 2M''_w = M'' - M''^T \) is nonsingular. This assertion is correct since \((\beta)\) ensures that the matrix \( M'' \) is a direct sum of matrices of the form \([\Phi_{p'l} \setminus I_m]\) with \( p(x) \neq x - 1 \) and \( Q \) of even size, and

- for each summand of the form \([\Phi_{p'l} \setminus I_m]\),
  \[
  [\Phi_{p'l} \setminus I_m]_w = \begin{bmatrix}
  0 & I_m - \Phi_{p'l}^T \\
  \Phi_{p'l} - I_m & 0
  \end{bmatrix}
  \]
  is nonsingular since 1 is not an eigenvalue of \( \Phi_{p'l} \);  

- for each summand of the form \( Q, Q - Q^T = Q^T(Q^{-T}Q - I_r) \) is nonsingular since \( Q^{-T}Q \) is similar to a Frobenius block \( \Phi_{p'l} \) of even size, in which \((12)\) ensures that \( p(x) \neq x - 1 \), and so 1 is not an eigenvalue of \( Q^{-T}Q \).

**Step 3: Show that \( M' \in \Xi_{n'}(F) \).**

By \((\alpha)\), \( M' \) is a direct sum of matrices of the form
\[
[\Phi_{(x - 1)m} \setminus I_m], \quad m \text{ is even}, \quad (24)
\]
in which \( \Phi_{(x - 1)m} \) is a Frobenius block that is not a cosquare; \((12)\) ensures that \( m \) is even.

Since \( C^{-1}\Phi_{(x - 1)m}C = J_m(1) \) for some nonsingular \( C \), each summand \([\Phi_{(x - 1)m} \setminus I_m]\) is congruent to
\[
\begin{bmatrix}
  0 & I_m \\
  J_m(1) & 0
\end{bmatrix} = \begin{bmatrix}
  C^T & 0 \\
  0 & C^{-1}
\end{bmatrix} \begin{bmatrix}
  0 & I_m \\
  \Phi_{(x - 1)m} & 0
\end{bmatrix} \begin{bmatrix}
  C & 0 \\
  0 & C^{-T}
\end{bmatrix},
\]
which is congruent to
\[
\begin{bmatrix}
  0 & \tilde{I}_m \\
  \tilde{J}_m(1) & 0
\end{bmatrix} = \begin{bmatrix}
  \tilde{I}_m & 0 \\
  0 & \tilde{J}_m(1)
\end{bmatrix} \begin{bmatrix}
  0 & I_m \\
  J_m(1) & 0
\end{bmatrix} \begin{bmatrix}
  \tilde{I}_m & 0 \\
  0 & I_m
\end{bmatrix},
\]
in which
\[
\tilde{I}_m := \begin{bmatrix}
  0 & 1 \\
  \vdots & \ddots \\
  1 & 0
\end{bmatrix}, \quad \tilde{J}_m(1) := \begin{bmatrix}
  0 & 1 \\
  \vdots & \ddots \\
  1 & 1 & 0
\end{bmatrix} \quad (m\text{-by-}m).
\]
The matrix $[J_m(1) \setminus \tilde{I}_m]$ is congruent via a permutation matrix to
\[
\begin{bmatrix}
0 & K_2 \\
K_2 & L_2 \\
& \\
& \\
& \\
& \\
K_2 & L_2 & 0
\end{bmatrix}, \text{ in which } K_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad L_2 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
\]

We have proved that $[\Phi_{(x-1)^m} \setminus I_m]$ is congruent to (25). Respectively,

$[\Phi_{(x-1)^m} \setminus I_m] \oplus \cdots \oplus [\Phi_{(x-1)^m} \setminus I_m]$ (r summands)

is congruent to

\[
A_{m,r} := \begin{bmatrix}
0 & K_r \\
& K_r \\
& L_r \\
K_r & L_r & 0
\end{bmatrix} \quad (m^2 \text{ blocks}),
\]

in which

\[
K_r := \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix}, \quad L_r := \begin{bmatrix} 0 & 0_r \\ 0_r & 0 \end{bmatrix}.
\]

Therefore, $M'$ is congruent to some matrix

$N = A_{m_1,r_1} \oplus A_{m_2,r_2} \oplus \cdots \oplus A_{m_t,r_t}, \quad m_1 > m_2 > \cdots > m_t,$

in which $r_i$ is the number of summands $[\Phi_{(x-1)^{m_i}} \setminus I_{m_i}]$ of size $2m_i$ in the direct sum $M'$. In view of (3), it suffices to prove that $N \in \Xi_n'(\mathbb{F})$.

If $S^TNS = N$, then (11) implies that

$N^{-T}NS = SN^{-T}N,$

in which

\[
N^{-T}N = \begin{bmatrix}
A^{-T}_{m_1,r_1}A_{m_1,r_1} & 0 \\
& \ddots & \vdots \\
& 0 & A^{-T}_{m_t,r_t}A_{m_t,r_t}
\end{bmatrix}.
\]

Since

\[
A^{-1}_{m_i,r_i} = \begin{bmatrix}
* & \cdots & * & -L^T_{r_i} & K_{r_i} \\
: & \ddots & \ddots & \ddots & \ddots \\
: & & \ddots & \ddots & \ddots \\
-\quad L^T_{r_i} & \cdots & K_{r_i} & \ddots & \ddots \\
K_{r_i} & \ddots & \ddots & \ddots & 0
\end{bmatrix},
\]

then (11) implies that

$N^{-T}NS = SN^{-T}N,$

in which

\[
A^{-T}_{m_1,r_1}A_{m_1,r_1} = \begin{bmatrix}
* & \cdots & * & -L^T_{r_1} & K_{r_1} \\
: & \ddots & \ddots & \ddots & \ddots \\
: & & \ddots & \ddots & \ddots \\
-\quad L^T_{r_1} & \cdots & K_{r_1} & \ddots & \ddots \\
K_{r_1} & \ddots & \ddots & \ddots & 0
\end{bmatrix}.
\]
we have

\[
A_{m_i,r_i}^{-T} A_{m_i,r_i} = \begin{bmatrix}
I_{2r_i} & H_{r_i} & \ast & \ldots & \ast \\
I_{2r_i} & H_{r_i} & \ddots & \vdots \\
I_{2r_i} & \ddots & \ddots & \ast \\
0 & \cdots & H_{r_i} & I_{2r_i}
\end{bmatrix}, \quad H_{r_i} := \begin{bmatrix}
I_{r_i} & 0 \\
0 & -I_{r_i}
\end{bmatrix}; \quad (30)
\]

the stars denote unspecified blocks.

Partition \( S \) in (28) into \( t^2 \) blocks

\[
S = \begin{bmatrix}
S_{11} & \ldots & S_{1t} \\
\vdots & \ddots & \vdots \\
S_{t1} & \ldots & S_{tt}
\end{bmatrix}, \quad S_{ij} \text{ is } 2m_i r_i \times 2m_j r_j,
\]

conformally to the partition (29), then partition each block \( S_{ij} \) into subblocks of size \( 2r_i \times 2r_j \) conformally to the partition (30) of the diagonal blocks of (29). Equating the corresponding blocks in the matrix equation (28) (much as in Gantmacher’s description of all matrices commuting with a Jordan matrix, \[5, \text{ Chapter VIII, } \S 2\]), we find that

- all diagonal blocks of \( S \) have the form

\[
S_{ii} = \begin{bmatrix}
C_i & \ast \\
C_i^H & \ddots \\
0 & \ddots & C_i \\
0 & \ddots & C_i^H
\end{bmatrix}, \quad C_i^H := H_{r_i} C_i H_{r_i},
\]

(the number of diagonal blocks is even by (24)), and

- all off-diagonal blocks \( S_{ij} \) have the form

\[
\begin{bmatrix}
\ast & \ldots & \ast \\
\vdots & \ddots & \vdots \\
0 & \ast & \ast
\end{bmatrix} \text{ if } i < j, \quad \begin{bmatrix}
\ast & \ldots & \ast \\
\vdots & \ddots & \vdots \\
0 & 0 & \ast
\end{bmatrix} \text{ if } i > j.
\]
in which the stars denote unspecified subblocks.¹

For example, if

\[ N = A_{6,r_1} \oplus A_{4,r_2} \oplus A_{2,r_3} \]

\[
\begin{bmatrix}
0 & K_{r_1} & L_{r_1} \\
K_{r_1} & L_{r_1} & 0 \\
K_{r_1} & L_{r_1} & 0
\end{bmatrix}
\oplus
\begin{bmatrix}
0 & K_{r_2} & L_{r_2} \\
K_{r_2} & L_{r_2} & 0 \\
K_{r_2} & L_{r_2} & 0
\end{bmatrix}
\oplus
\begin{bmatrix}
0 & K_{r_3} & L_{r_3}
\end{bmatrix},
\]

then

\[
S =
\begin{bmatrix}
C_1 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
C_1^H & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
C_1 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
C_1^H & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
C_2 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
C_2^H & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
C_3 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
C_3^H & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{bmatrix},
\]

in which

\[ C_1^H := H_{r_1}C_1H_{r_1}, \quad C_2^H := H_{r_2}C_2H_{r_2}, \quad C_3^H := H_{r_3}C_3H_{r_3}. \]

Now focus on equation (27). The subblock at the upper right of the \( i \)th diagonal block \( A_{m_i,r_i} \) of \( N \) is \( K_{r_i} \); see (26). Let us prove that the corresponding subblock of \( S^TNS \) is \( C_i^TK_iC_i^H \); that is,

\[ C_i^TK_iC_i^H = K_{r_i}. \quad (31) \]

¹Each Jordan matrix \( J \) is permutation similar to a Weyr matrix \( W_J \) and all matrices commuting with \( W_J \) are block triangular; see [12, Section 1.3]. If we reduce the matrix (29) by simultaneous permutations of rows and columns to its Weyr form, then the same permutations reduce \( S \) to block triangular form.
Multiplying the first horizontal substrip of the $i$th strip of $S^T$ by $N$, we obtain
\[
(0 \ldots 0 \ast | \ldots | 0 \ldots 0 \ast | 0 \ldots 0 C_i^T K_r | 0 \ldots 0 | \ldots | 0 \ldots 0);
\]
multiplying it by the last vertical substrip of the $i$th vertical strip of $S$, we obtain $C_i^T K_r C_i^H$, which proves (31). Thus, $\det C_i \det C_i^H = 1$. But

\[
\det S = \det C_1 \det C_1^H \ldots \det C_1 \det C_1^H \det C_2 \det C_2^H \ldots
\]

Therefore, $\det S = 1$, which completes the proof of Theorem 1.

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