A REmark on $\mathbb{Z}_p$-ORBiFOLD CONSTRUCTIONS OF THE MOONSHINE VERTEX OPERATOR ALGEBRA

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Abstract. For $p = 3, 5, 7, 13$, we consider a $\mathbb{Z}_p$-orbifold construction of the Moonshine vertex operator algebra $V^\sharp$. We show that the vertex operator algebra obtained by the $\mathbb{Z}_p$-orbifold construction on the Leech lattice vertex operator algebra $V_\Lambda$ and a lift of a fixed-point-free isometry of order $p$ is isomorphic to the Moonshine vertex operator algebra $V^\sharp$. We also describe the relationship between those $\mathbb{Z}_p$-orbifold constructions and the $\mathbb{Z}_2$-orbifold construction in a uniform manner. In Appendix, we give a characterization of the Moonshine vertex operator algebra $V^\natural$ by two mutually orthogonal Ising vectors.

1. Introduction

Let $V$ be a vertex operator algebra (VOA) and $G \subset \text{Aut} V$ a finite group of automorphisms of $V$. The fixed-point subalgebra $V^G = \{ v \in V | gv = v, g \in G \}$ is called an orbifold subVOA. A simple current extension of an orbifold subVOA provides an effective method to construct a new VOA from a given pair of a VOA and an automorphism of the VOA of finite order.

The first example of such a simple current extension of an orbifold subVOA is the Moonshine VOA $V^\sharp$ constructed in [15], where the VOA $V_\Lambda$ associated with the Leech lattice $\Lambda$ and an automorphism $\theta$ of order 2 lifted from the $-1$ isometry of $\Lambda$ are considered. Using the construction in [15], many maximal 2-local subgroups of the Monster are described as the stabilizers of some subVOAs of $V_\Lambda^{(\theta)}$ in [27]. However, it is not easy to describe $p$-local subgroups based on this construction for $p \neq 2$.

For each $p = 3, 5, 7, 13$, there is a unique, up to conjugacy, fixed-point-free isometry of the Leech lattice $\Lambda$ of order $p$. In the introduction of [15], a similar construction of $V^\sharp$ by using an automorphism $\tau$ of $V_\Lambda$ of order $p$ lifted from a fixed-point-free isometry of $\Lambda$ of order $p$ was conjectured (see also [8]). Such a construction of $V^\sharp$ was obtained in [3] for the case $p = 3$ (see also [23, 28]). Moreover, certain maximal 3-local subgroups of the Monster are described relatively explicitly using the $\mathbb{Z}_3$-orbifold construction.

In this paper, we consider the $\mathbb{Z}_p$-orbifold construction of the Moonshine VOA $V^\sharp$ for the cases $p = 3, 5, 7, 13$. We show that the VOA obtained by the $\mathbb{Z}_p$-orbifold construction on $V_\Lambda$ and a lift of a fixed-point-free isometry of $\Lambda$ of order $p$ is isomorphic to the Moonshine VOA $V^\sharp$. We also describe the relationship between those $\mathbb{Z}_p$-orbifold constructions and the $\mathbb{Z}_2$-orbifold construction [15] in a uniform manner.
Our idea is to use an isometry of the Leech lattice $\Lambda$ of order $2p$ whose $i$th power is fixed-point-free on $\Lambda$ for all $1 \leq i \leq 2p - 1$. Such an isometry is unique up to conjugacy. The isometry can be lifted to an automorphism $\sigma$ of $V_\Lambda$ of the same order $2p$ with $\theta = \sigma^p$ and $\tau = \sigma^{p+1}$. We consider the fixed-point subVOA $V_\Lambda^{(\sigma)} = \{ v \in V_\Lambda | \sigma v = v \}$ by $\sigma$.

It is known that $V_\Lambda^{(\sigma)}$ has exactly $4p^2$ irreducible modules up to equivalence $\mathbb{Z}_2 \times \mathbb{Z}_2$, and all of them are simple current modules $\mathbb{13}, \mathbb{14}, \mathbb{26}$. These irreducible $V_\Lambda^{(\sigma)}$-modules are parametrized by an abelian group $D \cong \mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$ as $W^\alpha$, $\alpha \in D$ so that the fusion products are given by $W^\alpha \boxtimes W^\beta \cong W^{\alpha + \beta}$ $\mathbb{14}, \mathbb{26}$.

It turns out that the VOA obtained by applying a $\mathbb{Z}_{2p}$-orbifold construction to the Leech lattice VOA $V_\Lambda$ and the automorphism $\sigma$ is isomorphic to $V_\Lambda$, also. This isomorphism induces some explicit relations between those $\mathbb{Z}_p$-orbifold constructions and the $\mathbb{Z}_2$-orbifold construction $\mathbb{15}$. In fact, it follows from $\mathbb{14}$ that there are four subgroups $H_r$, $1 \leq r \leq 4$, of $D$ for which a simple current extension $V_r = \bigoplus_{\alpha \in H_r} W^\alpha$ has a structure of a simple, rational, $C_2$-cofinite, holomorphic VOA of CFT-type which extends the VOA structure of $V_\Lambda^{(\sigma)}$. Among these four VOAs $V_r$, $1 \leq r \leq 4$, two are isomorphic to the Leech lattice VOA $V_\Lambda$ and the other two are isomorphic to the Moonshine VOA $V^\flat$. As for the latter two cases, one is a decomposition of the $\mathbb{Z}_p$-orbifold construction of $V^\flat$ by the automorphism $\tau \in \text{Aut} V_\Lambda$ of order $p$ into a direct sum of irreducible $V_\Lambda^{(\sigma)}$-modules, and the other is a decomposition of the $\mathbb{Z}_2$-orbifold construction of $V^\flat$ by the involution $\theta$ obtained in $\mathbb{15}$ into a direct sum of irreducible $V_\Lambda^{(\sigma)}$-modules.

The paper is organized as follows. In Section $\mathbb{2}$ we recall some results on cyclic orbifold constructions, irreducible twisted modules for lattice vertex operator algebras, and certain fixed-point-free isometries of the Leech lattice of order $2p$, $p = 3, 5, 7, 13$. In Section $\mathbb{3}$ we discuss some basic properties of the irreducible $\sigma^i$-twisted $V_\Lambda$-modules. Section $\mathbb{4}$ is devoted to $\mathbb{Z}_{2p}$-orbifold constructions by the automorphism $\sigma$. In Appendix, we give a characterization of the Moonshine VOA $V^\flat$ by two mutually orthogonal Ising vectors.

## 2. Preliminaries

In this section, we recall some results on cyclic orbifold constructions $\mathbb{14}, \mathbb{26}$, irreducible twisted modules for VOAs associated with positive definite even lattices and isometries of finite order $\mathbb{11}, \mathbb{6}, \mathbb{21}$, and some fixed-point-free isometries of the Leech lattice of order $2p$, $p = 3, 5, 7, 13$ $\mathbb{4}$.

### 2.1. Cyclic orbifold constructions

We follow the notation in $\mathbb{14}$. Let $V$ be a simple, rational, $C_2$-cofinite, holomorphic vertex operator algebra of CFT-type and $G = \langle g \rangle$ a cyclic group of automorphisms of $V$ of order $n$. Then there is a unique irreducible $h$-twisted $V$-module $V(h)$ for $h \in G$ by $\mathbb{7}$.

For each $h \in G$, there is a projective representation $\phi_h$ of $G$ on the vector space $V(h)$ such that

$$\phi_h(g)Y_{V(h)}(v, z)\phi_h(g)^{-1} = Y_{V(h)}(gv, z)$$

for $v \in V$. The representation $\phi_h$ is unique up to multiplication by an $n$th root of unity.

If $h = 1$, we have $V(h) = V$ and then assume $\phi_h(g) = g$. We write $\phi_i$ for $\phi_g$.

Let $W^{(i,j)}$ be the eigenspace of $\phi_i(g)$ in $V(g^i)$ with eigenvalue $e^{2\pi i j/n}$, i.e.,

$$W^{(i,j)} = \{ w \in V(g^i) | \phi_i(g)w = e^{2\pi i j/n}w \}.$$
Then \( W^{(i,j)} \) is an irreducible \( V^G \)-module and

\[
V(g^i) = \bigoplus_{j=0}^{n-1} W^{(i,j)}
\]

is an eigenspace decomposition of \( V(g^i) \) for \( \phi_i(g) \). The indices \( i \) and \( j \) are considered to be modulo \( n \). The second index \( j \) depends on the choice of multiplication by an \( n \)th root of unity for the representation \( \phi_i \) if \( i \neq 0 \). Note that \( W^{(0,0)} = V^G \).

The irreducible \( V^G \)-modules \( W^{(i,j)}, i, j \in \mathbb{Z}_n \), form a complete set of representatives of equivalence classes of irreducible \( V^G \)-modules [2, 24, 25], and all of them are simple current modules [13, 14].

The conformal weight of \( V(g^i), i \in \mathbb{Z}_n \) plays an important role in [14]. In the special case where the conformal weight of \( V(g) \) belongs to \( (1/n)\mathbb{Z} \), the fusion algebra of the orbifold subVOA \( V^G \) of \( V \) by \( G \) has particularly nice form. We summarize the results of [14, Section 5] as the following theorem for later use.

**Theorem 2.1.** ([14]) Let \( V \) and \( G = \langle g \rangle \) be as above. If the conformal weight of \( V(g^i) \) belongs to \( (1/n)\mathbb{Z} \), then we can choose multiplication of \( \phi_i \) by an \( n \)th root of unity so that the following conditions hold.

1. \( W^{(i,j)} \otimes W^{(k,l)} \cong W^{(i+k,j+l)} \).
2. The conformal weight of \( W^{(i,j)} \) is \( q_\Delta((i,j)) \equiv ij/n \) (mod \( \mathbb{Z} \)).
3. The fusion algebra of \( V^G \) is the group algebra of \( \mathbb{Z}_n \times \mathbb{Z}_n \) with a quadratic form \( q_\Delta \).
4. Let \( H \) be an isotropic subgroup of \( \mathbb{Z}_n \times \mathbb{Z}_n \) with respect to the quadratic form \( q_\Delta \).

Then

\[
\bigoplus_{(i,j) \in H} W^{(i,j)}
\]

admits a structure of a simple, rational, \( C_2 \)-cofinite, self-contragredient VOA of CFT-type which extends the VOA structure of \( V^G \). Furthermore, if \( H \) is a maximal isotropic subgroup, then it is holomorphic.

The subgroup \( \{(i,0) | i \in \mathbb{Z}_n \} \) is always a maximal isotropic subgroup of \( \mathbb{Z}_n \times \mathbb{Z}_n \) and

\[
\tilde{V}_g = \bigoplus_{i \in \mathbb{Z}_n} W^{(i,0)}
\]

is a simple, rational, \( C_2 \)-cofinite, holomorphic VOA of CFT-type which extends the VOA structure of \( V^G \) [14, page 21]. We say that the VOA \( \tilde{V}_g \) is obtained by a \( \mathbb{Z}_n \)-orbifold construction for \( V \) and \( g \).

2.2. Irreducible twisted modules for \( V_L \). Irreducible twisted modules for a lattice VOA \( V_L \) with respect to a lift of an isometry \( \nu \) of \( L \) of finite order were constructed explicitly in [11, 6, 21]. In this section, we recall some basic properties of those irreducible twisted modules in the special case where \( L \) is unimodular and \( \nu \) is fixed-point-free.

Let \( (L, \langle \cdot, \cdot \rangle) \) be a positive definite even unimodular lattice and \( \nu \) a fixed-point-free isometry of \( L \) of finite order. Let \( m \) be a positive integer such that \( \nu^m = 1 \). Note that \( m \) is not necessarily the order of \( \nu \). We extend the isometry \( \nu \) to \( \mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C} \) linearly. Following [11 (4.17)] and [7, Remark 3.1], let

\[
\mathfrak{h}^{(i,\nu)} = \{ h \in \mathfrak{h} | \nu h = \xi_m^{-i} h \}, \quad \xi_m = e^{2\pi \sqrt{-1}/m}.
\]
Since \( \nu \) is fixed-point-free, we have \( \mathfrak{h}^{(0,\nu)} = 0 \) and \( \mathfrak{h} = \bigoplus_{i=1}^{m-1} \mathfrak{h}^{(i,\nu)} \). The \( \nu \)-twisted affine Lie algebra \( \hat{\mathfrak{h}}[\nu] \) is defined by

\[
\hat{\mathfrak{h}}[\nu] = \left( \bigoplus_{i=1}^{m-1} \mathfrak{h}^{(i,\nu)} \otimes t^{i/m} \mathbb{C}[t, t^{-1}] \right) \oplus \mathbb{C}K
\]

with commutation relations

\[
[x \otimes t^n, y \otimes t^{n'}] = (x, y)n \delta_{n+n',0} K, \quad [K, \hat{\mathfrak{h}}[\nu]] = 0
\]

for \( x \in \mathfrak{h}^{(i,\nu)}, y \in \mathfrak{h}^{(i',\nu)}, n \in i/m + \mathbb{Z}, n' \in i'/m + \mathbb{Z} \). We write \( x(n) \) for \( x \otimes t^n \).

The index \( i \) of \( \mathfrak{h}^{(i,\nu)} \) can be considered to be modulo \( m \). Then \( \hat{\mathfrak{h}}[\nu] \) is also denoted as

\[
\hat{\mathfrak{h}}[\nu] = \left( \bigoplus_{n \in (1/m)\mathbb{Z}} \mathfrak{h}^{(mn,\nu)} \otimes \mathbb{C}t^n \right) \oplus \mathbb{C}K.
\]

Let \( \tilde{\nu} \) be an automorphism of the VOA \( V_L \) which is a lift of \( \nu \). Since \( L \) is unimodular, there is a unique irreducible \( \tilde{\nu} \)-twisted \( V_L \)-module up to equivalence \([7]\). The irreducible \( \tilde{\nu} \)-twisted \( V_L \)-module constructed in \([6, 21]\) is of the form

\[
V_L(\tilde{\nu}) = M(1)\nu \otimes T
\]

as a vector space, where \( M(1)\nu \) is the symmetric algebra \( S(\mathfrak{h}[\nu]^-) \) of an abelian Lie algebra

\[
\hat{\mathfrak{h}}[\nu]^- = \bigoplus_{n \in (1/m)\mathbb{Z}, n < 0} \mathfrak{h}^{(mn,\nu)} \otimes \mathbb{C}t^n
\]

and \( T \) is an irreducible module for a certain finite group. The symmetric algebra \( S(\hat{\mathfrak{h}}[\nu]^-) \) is spanned by the elements of the form

\[
h_r(-n_r) \cdots h_1(-n_1)1
\]

with \( r \in \mathbb{Z}_{\geq 0}, n_j \in (1/m)\mathbb{Z}_{\geq 0} \) and \( h_j \in \mathfrak{h}^{(-mn_j,\nu)}, 1 \leq j \leq r \). The weight of an element \( h_r(-n_r) \cdots h_1(-n_1)1 \otimes u \in V_L(\tilde{\nu}) \) with \( u \in T \) is given by

\[
n_1 + \cdots + n_r + \rho, \quad (2.1)
\]

where

\[
\rho = \rho(V_L(\tilde{\nu})) = \frac{1}{4m^2} \sum_{i=1}^{m-1} i(m - i) \dim \mathfrak{h}^{(i,\nu)} \quad (2.2)
\]

is the conformal weight of \( V_L(\tilde{\nu}) \).

The dimension of \( T \) is determined in \([11, (4.53)]\), \([21, \text{Proposition 6.2}]\) and we have

\[
\dim T = \sqrt{|L/(1-\nu)L|} \quad (2.3)
\]

2.3. Fixed-point-free isometries of \( \Lambda \) of order \( 2p \), \( p = 3, 5, 7, 13 \). The automorphism group \( CO_0 = O(\Lambda) \) of the Leech lattice \( \Lambda \) is a central extension of the largest Conway group \( CO_1 \) by a group of order 2. The central element of \( O(\Lambda) \) of order 2 is the \(-1\) isometry \( \theta : \alpha \mapsto -\alpha \) for \( \alpha \in \Lambda \). The character of the natural representation of \( O(\Lambda) \) on the 24 dimensional space \( \Lambda \otimes \mathbb{C} \) is denoted by \( \chi_{102} \) in \([4, \text{page 186}]\). We see from the values of \( \chi_{102} \) that the following lemma holds.
Lemma 2.2. For $p = 3, 5, 7, 13$, there exists a unique, up to conjugacy, isometry $\tau \in O(\Lambda)$ of order $p$ which acts fixed-point-freely on $\Lambda$. Let $\sigma = \theta \tau \in O(\Lambda)$. Then $\sigma$ is of order $2p$ and $\sigma^i$ acts fixed-point-freely on $\Lambda$ for all $1 \leq i \leq 2p - 1$.

Remark 2.3. For $p = 3, 5, 7, 13$, the isometry $\sigma$ of Lemma 2.2 is the unique, up to conjugacy, isometry of $\Lambda$ of order $2p$ such that $\sigma^i$ acts fixed-point-freely on $\Lambda$ for all $1 \leq i \leq 2p - 1$.

3. Irreducible $\sigma^i$-twisted $V_\Lambda$-modules

For $p = 3, 5, 7, 13$, let $\sigma$ be as in Section 2.2. Thus $\sigma$ is an isometry of the Leech lattice $\Lambda$ of order $m = 2p$ and $\sigma^i$ is fixed-point-free on $\Lambda$ for all $1 \leq i \leq m - 1$. Since $\sigma^{m/2}$ is the $-1$ isometry of $\Lambda$, we have $\langle \alpha, \sigma^{m/2} \alpha \rangle = -\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for $\alpha \in \Lambda$. Moreover, $\Lambda^\sigma = \{ \alpha \in \Lambda | \sigma \alpha = \alpha \} = 0$. Hence there is a lift $\hat{\sigma} \in \text{Aut} V_\Lambda$ of $\sigma$ of order $m$ by [14, Proposition 7.2]. For simplicity of notation, we use the same symbol $\sigma$ to denote $\hat{\sigma}$. That is, $\sigma$ denotes both an isometry of $\Lambda$ of order $m$ whose $i$th power is fixed-point-free on $\Lambda$ for all $1 \leq i \leq m - 1$ and an automorphism of the VOA $V_\Lambda$ of order $m$ which is a lift of the isometry.

Let

$$\theta = \sigma^p, \quad \tau = \sigma^{p+1}.\tag{3.1}$$

Then $\sigma = \theta \tau = \tau \theta$, $|\theta| = 2$, $|\tau| = p$, $\langle \sigma \rangle = \langle \theta, \tau \rangle$ and $\theta$ is a lift of the $-1$ isometry of $\Lambda$.

We follow the notation in Section 2.2 with $L = \Lambda$ and $\nu = \sigma^i$ for the irreducible $\sigma^i$-twisted $V_\Lambda$-modules $V_\Lambda(\sigma^i) = M(1)[\sigma^i] \otimes T$, $1 \leq i \leq m - 1$. Thus $\mathfrak{h} = \Lambda \otimes \mathbb{C}$ and

$$\mathfrak{h}^{(j, \sigma^i)} = \{ h \in \mathfrak{h} | \sigma^i h = \xi_m^j h \}, \quad \xi_m = e^{2\pi \sqrt{\frac{1}{m}}}. \tag{3.2}$$

3.1. Conformal weight of $V_\Lambda(\sigma^i)$. The conformal weight of $V_\Lambda(\sigma^i)$ is

$$\rho(V_\Lambda(\sigma^i)) = \frac{1}{4m^2} \sum_{j=1}^{m-1} \frac{m-1}{j(m-j)} \dim \mathfrak{h}^{(j, \sigma^i)}. \tag{3.2}$$

by Eq. (2.2).

Lemma 3.1. The dimension of $\mathfrak{h}^{(j, \sigma^i)}$, $1 \leq i \leq m - 1$, $j \in \mathbb{Z}_m$ is as follows.

1. If $i$ is odd and $i \neq p$, then

$$\dim \mathfrak{h}^{(j, \sigma^i)} = \begin{cases} 
24/(p-1) & (j \text{ odd}, j \neq p), \\
0 & \text{(otherwise)}.
\end{cases}$$

2. If $i$ is even, then

$$\dim \mathfrak{h}^{(j, \sigma^i)} = \begin{cases} 
24/(p-1) & (j \text{ even}, j \neq 0), \\
0 & \text{(otherwise)}.
\end{cases}$$

3. If $i = p$, then

$$\dim \mathfrak{h}^{(j, \sigma^p)} = \begin{cases} 
24 & (j = p), \\
0 & (j \neq p).
\end{cases}$$
Proof. Let \( h \in h^{(i;\sigma^i)} \). First, assume that \( i \) is odd and \( i \neq p \). If \( j \) is even, then \( jp \equiv 0 \pmod{m} \) and \( (\sigma^i)^p h = \xi_m^{-jp} h = h \). Since \( (\sigma^i)^p = \theta \), this implies that \( h = 0 \). If \( j = p \), then \( \sigma^i h = \xi_m^{-p} h = -h \) and \( \sigma^{-p} h = h \). Since \( \sigma^k \) is fixed-point-free on \( \Lambda \) for \( 0 \neq k \in \mathbb{Z}_m \), it follows that \( h = 0 \). Therefore, we have the eigenspace decomposition

\[
\mathfrak{h} = \bigoplus_{1 \leq k \leq p, k \neq (p+1)/2} h^{(2k-1;\sigma^i)}
\]

of \( \mathfrak{h} \) by \( \sigma^i \). Since \( m \) and \( 2k - 1 \) are coprime for \( 1 \leq k \leq p, k \neq (p+1)/2 \), we have

\[
\dim h^{(2k-1;\sigma^i)} = \dim h^{(2k-1;\sigma^i)^2k-1} = \dim h^{(1;\sigma^i)}.
\]

Thus the assertion (1) holds.

Next, assume that \( i \) is even. Then the order of \( \sigma^i \) is the prime \( p \). The eigenspace decomposition of \( \mathfrak{h} \) by \( \sigma^i \) is

\[
\mathfrak{h} = \bigoplus_{1 \leq k \leq p-1} h^{(2k;\sigma^i)}
\]

and \( \dim h^{(2k;\sigma^i)}, 1 \leq k \leq p-1 \) coincide each other. Hence the assertion (2) holds.

Since \( \sigma^p \) is the \(-1\) isometry of \( \Lambda \), the assertion (3) is clear. \( \square \)

By Lemma 3.1 and Eq. (3.2), we can calculate the conformal weight of \( V_{\Lambda}(\sigma^i) \).

**Lemma 3.2.** The conformal weight of \( V_{\Lambda}(\sigma^i), 1 \leq i \leq m-1 \) is as follows.

\[
\rho(V_{\Lambda}(\sigma^i)) = \begin{cases} 
(2p-1)/2p & (i \text{ is odd, } i \neq p), \\
(p+1)/p & (i \text{ is even}), \\
3/2 & (i = p).
\end{cases}
\]

3.2. Dimension of \( T \). The dimension of \( T \) of the irreducible \( \sigma^i \)-twisted \( V_{\Lambda} \)-module \( V_{\Lambda}(\sigma^i) = M(1)[\sigma^i] \otimes T \) is

\[
\dim T = \sqrt{|\Lambda/(1-\sigma^i)\Lambda|}.
\]

by (2.3).

**Lemma 3.3.** The dimension of \( T \), 1 \leq i \leq m-1 is as follows.

\[
\dim T = \begin{cases} 
1 & (i \text{ is odd, } i \neq p), \\
p^{12/(p-1)} & (i \text{ is even}), \\
2^{12} & (i = p).
\end{cases}
\]

Proof. If \( i \) is odd and \( i \neq p \), then the eigenvalues of \( \sigma^i \) on \( \mathfrak{h} \) are exactly the primitive \( m \)th roots of unity by Lemma 3.1(1), and so the minimal polynomial of \( \sigma^i \) on \( \mathfrak{h} \) is a cyclotomic polynomial

\[
\Phi_m(x) = \sum_{k=0}^{p-1} (-1)^k x^k
\]

\[
= (x-1) \left( \sum_{k=1}^{(p-1)/2} x^{2k-1} \right) + 1.
\]
Hence
\[ \alpha = (1 - \sigma^i) \left( \sum_{k=1}^{(p-1)/2} \sigma^{i(2k-1)} \right) \alpha \in (1 - \sigma^i) \Lambda \]
for \( \alpha \in \Lambda \). Thus \( (1 - \sigma^i) \Lambda = \Lambda \) and \( \dim T = 1 \).

If \( i \) is even, then the order of \( \sigma^i \) is \( p \) and the minimal polynomial of \( \sigma^i \) on \( \mathfrak{h} \) is a cyclotomic polynomial
\[ \Phi_p(x) = x^{p-1} + \cdots + x + 1 \]
by Lemma \[3.1\] (2). Hence \( \Lambda/\sigma^i \Lambda = \mathbb{Z} \) by \[16\], Lemma A.1 and \( \dim T = p^{12/(p-1)} \). If \( i = p \), then \( \Lambda/(1 - \sigma^i) \Lambda = \Lambda/2\Lambda = \mathbb{Z}_2^{24} \) and \( \dim T = 2^{12} \).

4. \( \mathbb{Z}_{2p} \)-ORBITFOLD CONSTRUCTIONS

We keep the notation in Section \[3\]. Since the conformal weight of the irreducible \( \sigma \)-twisted \( V_\Lambda \)-module \( V_\Lambda(\sigma) \) belongs to \((1/m)\mathbb{Z}\) by Lemma \[3.2\], for each \( i \in \mathbb{Z}_m \), we can choose a representation \( \phi_i \) of the group \( \langle \sigma \rangle \) on the irreducible \( \sigma^i \)-twisted \( V_\Lambda \)-module \( V_\Lambda(\sigma^i) \) so that the eigenspace \( W_{(i,j)} \) of \( \phi_i(\sigma) \) in \( V_\Lambda(\sigma^i) \), \( i, j \in \mathbb{Z}_m \) satisfy the four conditions of Theorem \[2.1\] with \( V = V_\Lambda, g = \sigma \) and \( n = m = 2p \). In particular,

1. \( W_{(i,j)} \otimes W_{(k,l)} \cong W_{(i+k,j+l)}. \)

2. The conformal weight of \( W_{(i,j)} \) is \( q_\Delta((i,j)) \equiv ij/m \) (mod \( \mathbb{Z} \)). The eigenspace decomposition of \( V_\Lambda(\sigma^i) \) for \( \phi_i(\sigma) \) is
\[ V_\Lambda(\sigma^i) = \bigoplus_{j \in \mathbb{Z}_m} W_{(i,j)} \quad (4.1) \]

The condition (2) implies that only \( W_{(2k-1,0)} \) is of integral weight among \( W_{(2k-1,j)} \), \( j \in \mathbb{Z}_m \) for \( i = 2k-1, 1 \leq k \leq p, k \neq (p+1)/2; \) only \( W_{(2k,0)} \) and \( W_{(2k,p)} \) are of integral weight among \( W_{(2k,j)} \), \( j \in \mathbb{Z}_m \) for \( i = 2k, 0 \leq k \leq p-1; \) and \( W_{(p,j)} \) is of integral weight if and only if \( j \) is even for \( i = p \).

There are four maximal isotropic subgroups of \( \mathbb{Z}_m \times \mathbb{Z}_m \) with respect to the quadratic form \( q_\Delta \), namely,
\[ H_1 = \{(0,j) | j \in \mathbb{Z}_m \}, \quad H_2 = \{(i,0) | i \in \mathbb{Z}_m \}, \]
\[ H_3 = \{(2k,pk) | k \in \mathbb{Z}_m \}, \quad H_4 = \{(pk,2k) | k \in \mathbb{Z}_m \}. \]

For \( r = 1, 2, 3, 4 \),
\[ \tilde{V}(r) = \bigoplus_{(i,j) \in H_r} W_{(i,j)} \]
admits a structure of a simple, rational, \( C_2 \)-cofinite, holomorphic VOA of CFT-type which extends the VOA structure of \( V_\Lambda(\sigma) \) by Theorem \[2.1\] (4).

In the case \( r = 1 \),
\[ \tilde{V}(1) = \bigoplus_{j=0}^{2p-1} W_{(0,j)} \]
is the eigenspace decomposition of $V_\Lambda$ by the automorphism $\sigma$, that is, $\tilde{V}^{(1)} = V_\Lambda$. Moreover,

$$V_\Lambda^+ = \bigoplus_{k=0}^{p-1} W^{(0,2k)}, \quad V_\Lambda^- = \bigoplus_{k=0}^{p-1} W^{(0,2k+1)},$$

where

$$V_\Lambda^\pm = \{ v \in V_\Lambda | \theta v = \pm v \}.$$  

Next, we consider

$$\tilde{V}^{(2)} = \bigoplus_{i=0}^{2p-1} W^{(i,0)}.$$  

We calculate the weight 1 subspace $\tilde{V}^{(2)}_1$ of $\tilde{V}^{(2)}$. If $i$ is even or $i = p$, then the conformal weight of $V_\Lambda(\sigma^2)$ is greater than 1 by Lemma 3.2. Hence

$$\tilde{V}^{(2)}_1 = \bigoplus_{0 \leq k \leq p-1} W^{(2k+1,0)}_{(2)}.$$

For each $0 \leq k \leq p-1$, $k \neq (p-1)/2$, the conformal weight of $V_\Lambda(\sigma^{2k+1})$ is $(2p-1)/2p$ by Lemma 3.2 and $W^{(2k+1,0)}$ is the only irreducible $V_\Lambda(\sigma)$-module with integral weights among $W^{(2k+1,j)}$, $j \in \mathbb{Z}$ by the condition (2) of Theorem 2.1. Thus by Eq. (2.1) and Lemmas 3.1 and 3.3 we see that

$$\dim(W^{(2k+1,0)})_1 = 24/(p-1).$$

Therefore, $\dim(\tilde{V}^{(2)}_1) = 24$. Furthermore, $a(0)b = 0$ for $a, b \in (\tilde{V}^{(2)})_1$ by the fusion rule $W^{(i,0)} \ltimes W^{(j,0)} \cong W^{(i+j,0)}$ as $i + j$ is even if both $i$ and $j$ are odd. Then $\tilde{V}^{(2)}$ is a holomorphic VOA of central charge 24 whose weight 1 subspace is a 24 dimensional abelian Lie algebra with respect to the bracket $[a,b] = a(0)b$. Thus $\tilde{V}^{(2)} = \bigoplus_{i=0}^{2p-1} W^{(i,0)}$ is isomorphic to $V_\Lambda$ by [11 Theorem 3] and we obtain the following theorem.

**Theorem 4.1.** $V_\Lambda \cong \bigoplus_{i=0}^{2p-1} W^{(i,0)}$.

Define a linear isomorphism $\theta' : \tilde{V}^{(2)} \to \tilde{V}^{(2)}$ by $(-1)^i$ on $W^{(i,0)}$. Then $\theta'$ is an automorphism of the VOA $\tilde{V}^{(2)}$ by the fusion rule $W^{(i,0)} \ltimes W^{(j,0)} \cong W^{(i+j,0)}$. The automorphism $\theta'$ is of order 2 and $-1$ on the weight 1 subspace. Hence it is conjugate to a lift $\tilde{\theta}$ of the $-1$ isometry of $\Lambda$. Indeed, $\theta^{-1}\theta'$ acts as the identity on the weight 1 subspace of $\tilde{V}^{(2)} = V_\Lambda$. Hence $\theta^{-1}\theta'$ is an inner automorphism $e^{h(0)}$ for some $h \in \mathfrak{h}$ by [12 Lemma 2.5] and we have

$$\theta' = \theta e^{h(0)} = e^{-\frac{1}{2} h(0)} \theta e^{\frac{1}{2} h(0)}$$

as required. Then

$$V_\Lambda^+ = V_\Lambda^{(\theta')} \cong (\tilde{V}^{(2)})^{(\theta')} = \bigoplus_{k=0}^{p-1} W^{(2k,0)}.$$  

Therefore, the following theorem holds.

**Theorem 4.2.** $V_\Lambda^+ \cong \bigoplus_{k=0}^{p-1} W^{(2k,0)}$ and $V_\Lambda^- \cong \bigoplus_{k=0}^{p-1} W^{(2k+1,0)}$. 
Next, we consider \( \widetilde{V}^{(3)} = \bigoplus_{k=0}^{2p-1} W^{(2k, pk)} \). Note that \( 2(p + i) \equiv 2i \ (\text{mod } 2p) \). Moreover, \( pi \equiv 0 \ (\text{mod } 2p) \) if \( i \) is even and \( pi \equiv p \ (\text{mod } 2p) \) if \( i \) is odd. Hence

\[ H_3 = \{(2k, 0)|0 \leq k \leq p - 1\} \cup \{(2k, p)|0 \leq k \leq p - 1\}. \]

Thus \( \widetilde{V}^{(3)} \) contains \( \bigoplus_{k=0}^{p-1} W^{(2k, 0)} \cong V_A^+ \). The VOA \( V_A^+ \) has exactly four irreducible modules, namely, \( V_A^\pm \) and \( V_A^{T, \pm} \), where \( V_A^{T, \pm} \) are the eigenspaces with eigenvalues \( \pm1 \) of \( \theta \) in a unique irreducible \( \theta \)-twisted \( V_A \)-module \( V_A(\theta) \). All of those four irreducible \( V_A^+ \)-modules are simple current. We take \( V_A^{T, \pm} \) so that the conformal weight of \( V_A^{T, +} \) is 2 and that of \( V_A^{T, -} \) is \( 3/2 \). The weight 1 subspace of \( \widetilde{V}^{(3)} \) is 0, for the conformal weight of \( V_A(\sigma^{2k}) \), \( 1 \leq k \leq p - 1 \) is greater than 1 by Lemma 3.2. Hence we conclude that \( \widetilde{V}^{(3)} \) is isomorphic to the Moonshine VOA \( V^2 = V_A^+ \oplus V_A^{T, +} \) constructed in [13]. Thus the following theorem holds.

**Theorem 4.3.** \( V^2 \cong \bigoplus_{k=0}^{p-1} W^{(2k, pk)} \) with \( V_A^+ \cong \bigoplus_{k=0}^{p-1} W^{(2k, 0)} \) and \( V_A^{T, +} \cong \bigoplus_{k=0}^{p-1} W^{(2k, p)} \).

Recall that \( \tau = \sigma^{p+1} \) is of order \( p \). We also note that

\[ H_3 = \{((p + 1)k, pk)|0 \leq k \leq 2p - 1\} \]
\[ = \{((p + 1)k, 0)|0 \leq k \leq p - 1\} \cup \{((p + 1)k, p)|0 \leq k \leq p - 1\}. \]  

(4.3)

Let

\[ U^{(k, 0)} = W^{((p + 1)k, 0)} \oplus W^{((p + 1)k, p)}, \quad 0 \leq k \leq p - 1. \]  

(4.4)

For \( v \in V_A \), we have \( \tau v = v \) if and only if \( v \) is a sum of eigenvectors of \( \sigma \) of eigenvalues \( \pm1 \). Hence \( U^{(0, 0)} = W^{(0, 0)} \oplus W^{(0, p)} \) is equal to

\[ V^{(\tau)}_A = \{v \in V_A|\tau v = v\}. \]  

(4.5)

The irreducible \( \tau^k = \sigma^{(p+1)k} \)-twisted \( V_A \)-module \( V_A(\tau^k) = V_A(\sigma^{(p+1)k}) \), \( 1 \leq k \leq p - 1 \) is a direct sum of eigenspaces for \( \tau \). For each \( k \), there is a unique one with integral weights among those eigenspaces for \( \tau \), namely, \( U^{(k, 0)} \). Since the conformal weight of \( V_A(\tau) = V_A(\sigma^{p+1}) \) belongs to \((1/p)\mathbb{Z}\) by Lemma 3.2,

\[ \widetilde{V}_{A, \tau} = \bigoplus_{k=0}^{p-1} U^{(k, 0)} \]  

(4.6)

is a \( \mathbb{Z}_p \)-orbifold construction with respect to \( V_A \) and \( \tau \) by Theorem 2.1.

Both \( V^2 \) and \( \widetilde{V}_{A, \tau} \) are simple current extensions of \( V_A^{(\sigma)} \) with the same simple current components \( W^{(i, j)}, (i, j) \in H_3 \) by Theorem 4.3 Eq. (4.3), (4.4) and (4.6). Since the VOA structure of a simple current extension is unique (see [9] and [10, Proposition 5.3]), it follows that \( V^2 \) and \( \widetilde{V}_{A, \tau} \) are isomorphic as VOAs.

**Theorem 4.4.** \( \widetilde{V}_{A, \tau} \cong V^2 \).

**Remark 4.5.** We have \( V_A^+ = \bigoplus_{k=0}^{p-1} W^{(0, 2k)} \) and \( V_A^{T, +} = \bigoplus_{k=0}^{p-1} W^{(p, 2k)} \), for \( V_A(\theta) = V_A(\sigma^p) \) and \( W^{(p, j)} \) is of integral weight only if \( j \) is even. Since

\[ H_4 = \{(0, 2k)|0 \leq k \leq p - 1\} \cup \{(p, 2k)|0 \leq k \leq p - 1\} \]

\[ \widetilde{V}^{(4)} = \bigoplus_{k=0}^{2p-1} W^{(pk, 2k)} \] is a decomposition of the Moonshine VOA \( V^2 = V_A^+ \oplus V_A^{T, +} \) into a direct sum of irreducible \( V_A^{(\sigma)} \)-modules.
APPENDIX A. A CHARACTERIZATION OF THE MOONSHINE VOA

In this appendix, we give another characterization of the Moonshine VOA $V^2$ using Ising vectors. It also provides an alternative proof that $\tilde{V}_{A,\sigma} \cong V^2$ for $p = 3$ and 5. The main theorem is as follows.

**Theorem A.1.** Let $V$ be a simple, rational, $C_2$-cofinite, holomorphic VOA of CFT-type with central charge 24 such that the weight 1 subspace $V_1 = 0$. If $V$ contains two mutually orthogonal Ising vectors, then $V$ is isomorphic to the Moonshine VOA $V^2$.

The idea is essentially the same as in [19] and is also similar to that in Section 4. We try to obtain the Leech lattice VOA $V_{\Lambda}$ by using some $\mathbb{Z}_2$-orbifold construction on $V$. The extra assumption on Ising vectors is used to define an involution on $V$ such that the corresponding twisted module has conformal weight one.

An element $e \in V_2$ is called an *Ising vector* if the vertex subalgebra $\text{Vir}(e)$ generated by $e$ is isomorphic to the simple Virasoro VOA $L(1/2,0)$ of central charge 1/2. Let $V_e(h)$ be the sum of all irreducible $\text{Vir}(e)$-submodules of $V$ isomorphic to $L(1/2,h)$ for $h = 0, 1/2, 1/16$. Then one has an isotypical decomposition:

$$V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16).$$

Recall from [22] that the linear automorphism $\tau_e$ which acts as 1 on $V_e(0) \oplus V_e(1/2)$ and $-1$ on $V_e(1/16)$ defines an automorphism of the VOA $V$. On the fixed point subVOA $V^{(\tau_e)} = V_e(0) \oplus V_e(1/2)$, the linear automorphism $\sigma_e$ which acts as 1 on $V_e(0)$ and $-1$ on $V_e(1/2)$ also defines an automorphism on $V^{(\tau_e)}$.

Let $e$ and $f$ be two mutually orthogonal Ising vectors in $V$ and let $U$ be the subVOA generated by $e$ and $f$. Then

$$U = \text{Vir}(e) \otimes \text{Vir}(f) \cong L(1/2,0) \otimes L(1/2,0).$$

For any $h_1, h_2 \in \{0, 1/2, 1/16\}$, we define the space of multiplicities of the irreducible $U$-module $L(1/2, h_1) \otimes L(1/2, h_2)$ in $V$ by

$$M(h_1, h_2) = \text{Hom}_U(L(1/2, h_1) \otimes L(1/2, h_2), V).$$

Then we have the isotypical decomposition

$$V = \bigoplus_{h_1, h_2 \in \{0, 1/2, 1/16\}} L(1/2, h_1) \otimes L(1/2, h_2) \otimes M(h_1, h_2).$$

Notice that $M(0, 0) = \text{Com}_V(U) = U^c$ is a subVOA of central charge 23 and $M(h_1, h_2)$, $h_1, h_2 \in \{0, 1/2, 1/16\}$, are $M(0,0)$-modules. Note also that

$$(V^{(\tau_e, \tau_f)} \circ \sigma_e, \sigma_f) = L(1/2,0) \otimes L(1/2,0) \otimes M(0,0).$$

**Proposition A.2.** The subVOA $M(0,0)$ is $C_2$-cofinite and rational. Moreover, $M(h_1, h_2)$, $h_1, h_2 \in \{0, 1/2\}$, are simple current modules of $M(0,0)$. 
Proof. Let \( E = \langle \tau_e, \tau_f \rangle \subset \text{Aut} V \) be the subgroup generated by the Miyamoto involutions \( \tau_e \) and \( \tau_f \). Then \( E \) is elementary abelian of order 4 and the fixed point subVOA is

\[
V^E = \bigoplus_{h_1, h_2 \in \{0, \frac{1}{2}\}} L(\frac{1}{2}, h_1) \otimes L(\frac{1}{2}, h_2) \otimes M(h_1, h_2).
\]

Then \( V^E \) is \( C_2 \)-cofinite and rational by a result of Carnahan and Miyamoto \([2, 24]\).

Let \( S = \langle \sigma_e, \sigma_f \rangle \subset \text{Aut} V^E \). Then \( S \) is also elementary abelian and hence the fixed point subVOA

\[
W = (V^E)^S = L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \otimes M(0, 0)
\]

is \( C_2 \)-cofinite and rational. Therefore, \( M(0, 0) \) is also \( C_2 \)-cofinite and rational.

That \( M(h_1, h_2), h_1, h_2 \in \{0, 1/2\} \), are simple current modules of \( M(0, 0) \) follows from the fact that \( L(\frac{1}{2}, h_1) \otimes L(\frac{1}{2}, h_2) \otimes M(h_1, h_2) \) are common eigenspaces of \( S \) on \( V^E \) and \([5]\) Remark 6.4.

\[\square\]

Notation A.3. For \( i, j \in \{0, 1\} \), let \( V^{(i,j)} = \{ v \in V \mid \tau_e v = (-1)^i v, \tau_f v = (-1)^j v \} \). Then

\[
V^{(0,0)} = V^E = \bigoplus_{h_1, h_2 \in \{0, \frac{1}{2}\}} L(\frac{1}{2}, h_1) \otimes L(\frac{1}{2}, h_2) \otimes M(h_1, h_2),
\]

\[
V^{(1,0)} = L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{1}{2}, 0) \otimes M(\frac{1}{16}, 0) \oplus L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{1}{2}, \frac{1}{2}) \otimes M(\frac{1}{16}, \frac{1}{2}),
\]

\[
V^{(0,1)} = L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, \frac{1}{16}) \otimes M(0, \frac{1}{16}) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{16}) \otimes M(\frac{1}{2}, \frac{1}{16}),
\]

\[
V^{(1,1)} = L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{1}{2}, \frac{1}{16}) \otimes M(\frac{1}{16}, \frac{1}{16}).
\]

Notice that \( V^{(i,j)}, i, j \in \{0, 1\} \), are simple current modules of \( V^E \) \([3]\).

Lemma A.4. Let \( V \) be a simple, rational, \( C_2 \)-cofinite, holomorphic VOA of CFT type with central charge 24 such that \( V_1 = 0 \). Then \( M(h_1, h_2) \neq 0 \) for any \( h_1, h_2 \in \{0, \frac{1}{2}, \frac{1}{16}\} \).

Proof. Recall \( U = \text{Vir}(e) \otimes \text{Vir}(f) \cong L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \). Then the double commutant \((U^c)^c\) is an extension of \( U \). Note that there is only one non-trivial extension of \( U \), which is isomorphic to \( L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2}) \) and the weight one subspace is non-zero. Hence \((U^c)^c = U\), for \( V_1 = 0 \). Therefore, by a result of Krauel and Miyamoto \([17]\), all irreducible modules of \( U \) must appear as a submodule of \( V \) since \( V \) is holomorphic and \( U \) and \( M(0, 0) \) are \( C_2 \)-cofinite and rational. It implies \( M(h_1, h_2) \neq 0 \) for any \( h_1, h_2 \in \{0, \frac{1}{2}, \frac{1}{16}\} \).

The following two results follow immediately from the general arguments on simple current extensions \([20, 29]\).

Lemma A.5. Let \( M = L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2}) \otimes M(0, 0) \) and set

\[
\tilde{M} = \bigoplus_{h_1, h_2 \in \{0, \frac{1}{2}\}} L(\frac{1}{2}, \frac{1}{2} - h_1) \otimes L(\frac{1}{2}, \frac{1}{2} - h_2) \otimes M(h_1, h_2).
\]

Then \( \tilde{M} \) is an irreducible module of \( V^E \).
Theorem A.6. Let \( t = \tau_e \tau_f \) and set
\[
X = \bigoplus_{i,j \in \{0,1\}} V^{(i,j)} \otimes_{\Lambda} \tilde{M}.
\]
Then \( X \) is an irreducible \( t \)-twisted module of \( V \).

Define
\[
\tilde{V} = V^{(t)} \oplus X^{(t)},
\]
where \( X^{(t)} \) is the irreducible \( V^{(t)} \)-submodule of \( X \) which has integral weights. Then \( \tilde{V} \) is a simple, rational, \( C_2 \)-cofinite, holomorphic VOA of CFT-type. Notice that the conformal weight of \( M = L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2}) \otimes M(0,0) \) is 1 and \( M \) is an \( L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \otimes M(0,0) \)-submodule of \( X^{(t)} \). Hence, \( (X^{(t)})_1 \neq 0 \) and \( (\tilde{V})_1 \neq 0 \). Since \( V_1 = 0 \), we have \( \tilde{V}_1 = (X^{(t)})_1 \), and hence the Lie algebra on \( \tilde{V}_1 \) is abelian. Thus we have the following theorem.

Theorem A.7. The VOA \( \tilde{V} \) is isomorphic to the Leech lattice VOA \( V_\Lambda \).

Now we are ready to prove our main theorem.

Theorem A.8. The VOA \( V \) is isomorphic to the Moonshine VOA \( V^2 \).

Proof. By Theorem A.7 we know that the VOA \( \tilde{V} \) is isomorphic to the Leech lattice VOA \( V_\Lambda \). Let \( g \) be the automorphism of \( \tilde{V} \) which acts as 1 on \( V^{(t)} \) and \(-1\) on \( X^{(t)} \). Then \( g \) is conjugate to the lift \( \theta \) of \(-1\) map on \( \Lambda \) since \( g \) acts on \( \tilde{V}_1 \) as \(-1\) (cf. Theorem 4.2). Therefore, we have \( \tilde{V}^{(g)} = V^{(t)} \cong V_\Lambda^+ \). Then by the same argument as in Theorem 4.4 we have
\[
V \cong V_\Lambda^+ \oplus V_\Lambda^{T,+} \cong V^2
\]
as \( V_\Lambda^+ \)-modules. Then by the uniqueness of simple current extensions, we can establish the desired isomorphism between \( V \) and \( V^2 \).

Remark A.9. Recall that the Leech lattice \( \Lambda \) contains a sublattice isometric to \( \sqrt{2}E_8^{\square 3} \). For \( p = 3, 5 \), we can choose a fixed-point-free isometry \( \tau \) of order \( p \) such that each direct summand of \( \sqrt{2}E_8^{\square 3} \) is stabilized; indeed, \( \sqrt{2}E_8^{\square 3} \) contains \( \sqrt{2}A_3^{12} \) and \( \sqrt{2}A_4^{6} \) as sublattices and the fixed-point-free isometry of \( \Lambda \) of order 3 (resp. 5) can be induced by the Coxeter element of \( A_2 \) (resp. \( A_4 \)). Thus, we have \( (V^{(\tau)}_{\sqrt{2}E_8})^{\otimes 3} \subset V_\Lambda^{(\tau)} \). Let \( \theta \in \text{Aut} V_{\sqrt{2}E_8} \) be a lift of the \(-1\)-isometry of \( \sqrt{2}E_8 \). Then \( \theta \) and \( \tau \) commutes. Since \( V_{\sqrt{2}E_8}^{(\theta)} \) has exactly 496 Ising vectors \cite[Proposition 4.3]{18} and 496 is relatively prime to \( p \), there exists an Ising vector in \( V_{\sqrt{2}E_8}^{(\theta)} \) fixed by \( \tau \). Hence \( V_\Lambda^{(\tau)} \) contains two (in fact, three) mutually orthogonal Ising vectors. By Theorem A.8 we have \( \tilde{V}_\Lambda, \tau \cong V^2 \), also.

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