Variational Multi-Time Green’s Functions for Nonequilibrium Quantum Fields

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Abstract

The time-dependent variational principle proposed by Balian and Vénéroni is used to provide the best approximation to the generating functional for multi-time Green’s functions of a set of (bosonic) observables $Q_\mu$. By suitably restricting the trial spaces, the computation of the two-time Green’s function, obtained by a second order expansion in the sources, is considerably simplified. This leads to a tractable formalism suited to quantum fields out of equilibrium. We propose an illustration on the finite temperature $\Phi^4$-theory in curved space and coupled to gravity.

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1 Introduction

In the past few years, a time-dependent variational principle was proposed [1] in order to deal with the question of how to predict correctly, within an approximation scheme, the expectation value of a measured observable, when the system is in a mixing of states and is therefore described by a density operator. The variational principle has been applied to a variety of quantum processes including heavy ion reactions [2], quantum fields out of equilibrium [3], attempts to go beyond the gaussian approximation for fermion [4] and boson systems as well as studies of tunneling through a gaussian barrier [5].

More recently, a more elaborate version of the variational principle was developed [6] in order to evaluate the generating functional for multi-time correlation functions in equilibrium and non-equilibrium statistical mechanics.

The purpose of the present paper is to extend the formalism of [6] to quantum field theory and to make it more transparent in this area. Among the net advantages of the approach proposed here and in [6], is the ability to handle in a consistent and non-perturbative way out-of-equilibrium phenomena such as the quantum non-equilibrium evolution of the inflaton field(s) in the inflationary scenario [7] or that of the quark-gluon plasma [8, 9].

We will concentrate in this paper on boson fields because of their direct connections with the previously mentioned problems. Extension to fermion fields is quite trivial.

The paper is organized as follows: in section 2, we recall some results concerning the exact determination of multi-time correlation functions of a set of observables. In section 3, we present the variational action-like which leads to the desired generating functional as its stationary value. A peculiarity of the variational principle is the appearance of two variational objects akin to a density operator and to an observable. This is however analogous to the bra and ket in ordinary quantum mechanics or to the position and its conjugate momentum in classical dynamics. In section 4, we select for the variational objects gaussian operators and derive the corresponding evolution equations. Being coupled, these equations are quite complicated because of the mixed boundary conditions inherent in the variational principle. Fortunately, the computation of the one and two-time correlation functions does not require the solution of the whole set of equations. Indeed, we show in section 5 that it is sufficient to expand these equations up to second order in a set of sources, and this transforms the problem into two initial value problems.

In order to illustrate the formalism, we choose in section 6 to focus on a $O(1)$ quantum field theory in Robertson-Walker metric. Furthermore, we restrict the set of relevant observables (that is, those we wish to calculate the correlation functions,) to the boson field operator $\Phi(x)$ and the composite operator $\Phi(x)\Phi(y)$. This allows the
determination of the one-, two-, three- and four-point connected Green’s functions of the theory. These are manifestly finite since the underlying dynamical equations are already renormalized.

Finally, we present some perspectives and discuss possible improvements of the approximations involved in this work.

2 Summary of Some Exact Results

We wish to determine in this section the exact multi-time correlation functions of a set of operators, which may be composite, denoted in the Schrödinger picture by $Q_\mu$ ($\mu = 1, 2 \ldots$). Following ref. [6], we introduce the generating functional

$$W\{\xi\} \equiv -i \log \text{Tr} A(t_0)D(t_0),$$

(2.1)

where $D(t_0)$ is the exact density operator of the system at some initial time $t_0$ (possibly equal to $-\infty$) and $A(t_0)$ is given by

$$A(t_0) = T \exp \left\{ i \int_{t_0}^{\infty} dt' \sum_\mu \xi_\mu(t') \bar{Q}_\mu(t') \right\}.$$  

(2.2)

In expression (2.2), $\xi_\mu(t')$ are the sources and $Q_\mu(t')$ are the Heisenberg representations of the observables $Q_\mu$.

$$Q_\mu^H(t') = U^+(t', t_0) Q_\mu U(t', t_0),$$

(2.3)

$U$ being the evolution operator. The index $\mu$ may contain both discrete indices and space variables that the observables $Q$ may possess. By expanding (2.2) in powers of the sources, we obtain the following formal development for the generating functional

$$iW\{\xi\} = \log \text{Tr} D(t_0) + i \int_{t_0}^{\infty} dt' \sum_\mu \xi_\mu(t') \langle Q_\mu^H(t') \rangle$$

$$+ \frac{i^2}{2} \int_{t_0}^{\infty} dt' dt'' \sum_{\mu, \nu} \xi_\mu(t') \xi_\nu(t'') \langle T \left( \bar{Q}_\mu^H(t') \bar{Q}_\nu^H(t'') \right) \rangle + \ldots,$$

(2.4)

where $\langle O \rangle = \text{Tr} OD(t_0)/\text{Tr} D(t_0)$ and $\bar{O} = O - \langle O \rangle$ for any operator $O$. It is now clear that $W\{\xi\}$ generates the connected Green’s functions of the observables $Q_\mu$. Indeed, by multiple differentiation with respect to the sources, one has for instance

$$\langle Q_\mu^H(t') \rangle = \left. \frac{\delta W\{\xi\}}{\delta \xi_\mu(t')} \right|_{\xi=0},$$

(2.5)

$$\langle T \left( \bar{Q}_\mu^H(t') \bar{Q}_\nu^H(t'') \right) \rangle \equiv G^{(2)}_{\mu\nu}(t', t'') = -i \left. \frac{\delta^2 W\{\xi\}}{\delta \xi_\mu(t') \delta \xi_\nu(t'')} \right|_{\xi=0}.$$  

(2.6)
The next step is to define the generating functional for the "1–PI" Green’s functions, that is the effective action \( S_{\text{eff}} \) as a (multiple) Legendre transform of \( W\{\xi\} \). Upon setting
\[
(Q_{\mu}(t')) = \frac{\delta W\{\xi\}}{\delta \xi_{\mu}(t')} = (Q_{\mu}^H(t')) + \sum_{n=2}^{\infty} i^{n-1} \int_0^\infty dt_1 \cdots dt_n \\
\sum_{\nu_1 \cdots \nu_{n-1}} \xi_{\nu_1}(t_1) \cdots \xi_{\nu_{n-1}}(t_{n-1}) G_{\mu_1 \cdots \nu_{n-1}}^{(n)}(t', t_1, \ldots, t_{n-1}),
\]
(2.7)
where
\[
G_{\nu_1 \cdots \nu_n}^{(n)}(t_1, \ldots, t_n) = \langle T \left( \hat{Q}_{\nu_1}^H(t_1) \cdots \hat{Q}_{\nu_n}^H(t_n) \right) \rangle,
\]
(2.8)
we can define the effective action as
\[
S_{\text{eff}}\{\langle Q \rangle\} = W\{\xi\} - \int_{t_0}^\infty dt' \sum_{\mu} \xi_{\mu}(t')\langle Q_{\mu}(t') \rangle.
\]
(2.9)
Now since
\[
\frac{\delta S_{\text{eff}}\{\langle Q \rangle\}}{\delta \langle Q_{\mu}(t') \rangle} = -\xi_{\mu}(t'),
\]
(2.10)
the physical theory will be defined by the vanishing of the sources or equivalently by the condition
\[
\langle Q_{\mu}(t') \rangle\{\xi = 0\} = \langle Q_{\mu}^H(t') \rangle.
\]
(2.11)
The computation of the 1–PI Green’s functions proceeds by multiple differentiation of \( S_{\text{eff}} \) with respect to \( \langle Q_{\mu}(t') \rangle \). But this is a rather difficult task in general since it requires the inversion of the functional relation (2.7). However, considerable simplifications occur when one intends to calculate the two-point 1–PI function \( \Gamma^{(2)} \). Indeed, up to first order in the sources, the expansion (2.7) yields
\[
\xi_{\mu}(t) \simeq -i \int_{t_0}^\infty dt' \sum_{\nu} \left( [G^{(2)}]^{-1} \right)_{\mu\nu} (t, t') \left( \langle Q_{\nu}(t') \rangle - \langle Q_{\nu}^H(t') \rangle \right),
\]
(2.12)
where one notes that the condition (2.11) is preserved. Omitting trivial constant terms, the expressions for \( W\{\xi\} \) and \( S_{\text{eff}}\{\langle Q \rangle\} \) now read
\[
W\{\xi\} \simeq \int_{t_0}^\infty dt' \sum_{\mu} \xi_{\mu}(t')\langle Q_{\mu}^H(t') \rangle + \frac{i}{2} \int_{t_0}^\infty dt' dt'' \sum_{\mu\nu} \xi_{\mu}(t') G^{(2)}_{\mu\nu}(t', t'') \xi_{\nu}(t''),
\]
(2.13)
\[
S_{\text{eff}}\{\langle Q \rangle\} \simeq \frac{i}{2} \int_{t_0}^\infty dt' \sum_{\mu\nu} \langle \langle Q_{\mu} \rangle - \langle Q_{\mu}^H \rangle \rangle(t') \Gamma^{(2)}_{\mu\nu}(t', t'') \langle Q_{\nu} \rangle - \langle Q_{\nu}^H \rangle \rangle(t''),
\]
(2.14)
where
\[
\Gamma^{(2)}_{\mu\nu}(t', t'') \equiv -i \frac{\delta^2 S_{\text{eff}}\{\langle Q \rangle\}}{\delta \langle Q_{\mu}(t') \rangle \delta \langle Q_{\nu}(t') \rangle}\big|_{\langle Q \rangle = \langle Q^H \rangle} = \left( [G^{(2)}]^{-1} \right)_{\mu\nu} (t', t '').
\]
(2.15)
One should note that, although (2.13) and (2.14) are truncated, the expression (2.13) is exact since \( \Gamma^{(2)} \) is defined at the "stationary point" where the sources vanish.
The determination of higher order equal time Green’s functions proceeds simply by enlarging the set of operators $Q_{\mu}$. For instance, one may take $Q_1 = Q$ (where $Q$ is some simple observable) and $Q_2, Q_3, \ldots$ as successive powers of $Q$. We will see an example in section 6.

The expressions that we have derived for the correlation functions require the exact computation of $W\{\xi\}$ and $S_{\text{eff}}\{\langle Q \rangle\}$. But this in turn needs the evaluation of the exact evolution operator which is a quite formidable task in general. In the next section, we present a method first developed in ref. [6] and devised to provide the best approximation to $W\{\xi\}$ within a variational framework.

3 Variational Formulation

We recall in this section the major steps in a variational derivation of the dynamics for the generating functional $W\{\xi\}$. For a more detailed analysis, we refer the reader to sect. 3.

The authors of [6] introduce the functional (with its two variants)

$$ I\{A(t); D(t)\} = \text{Tr} A(t_0) D(t_0) + \int_{t_0}^{\infty} \text{Tr} D(t) \left\{ \frac{dA(t)}{dt} - i[A(t), H] + iA(t) \sum_\mu \xi_\mu(t) Q_\mu \right\}, \quad (3.1) $$

$$ I\{A(t); D(t)\} = \text{Tr} A(\infty) D(\infty) - \int_{t_0}^{\infty} \text{Tr} A(t) \left\{ \frac{dD(t)}{dt} + i[H, D(t)] - i \sum_\mu \xi_\mu(t) Q_\mu D(t) \right\}. \quad (3.2) $$

In (3.1) and (3.2), $D(t)$ and $A(t)$ are two time-dependent trial operators, the latter being subject to the constraint

$$ A(\infty) = A(\infty) = 1, \quad (3.3) $$

where $A(t)$ is given by (2.2) with $t_0$ replaced by $t$. To account for the dynamics, the functional $I$ should be made stationary with respect to variations of $D(t)$ and $A(t)$. This leads to the following equations

$$ \text{Tr} \delta D(t) \left[ \frac{dA(t)}{dt} - i[A(t), H] + iA(t) \sum_\mu \xi_\mu(t) Q_\mu \right] = 0, \quad (3.4) $$

$$ \text{Tr} \delta A(t) \left[ \frac{dD(t)}{dt} + i[H, D(t)] - i \sum_\mu \xi_\mu(t) Q_\mu D(t) \right] = 0. \quad (3.5) $$

One should in principle also enforce the optimization of the initial state by imposing

$$ \text{Tr} \delta A(t_0) [D(t_0) - D(t_0)] = 0. \quad (3.6) $$
However, since we are not directly interested in this paper in the initial state problem, we will suppose that $D(t)$ and $D(t_0)$ belong to the same class so that one can impose $D(t_0) = D(t_0)$ as initial condition. For a careful study of this problem, see [6] sect. 6.

When the variations $\delta A(t)$ and $\delta D(t)$ are unrestricted, one obtains from (3.4) and (3.5)

$$\frac{dA(t)}{dt} - i[A(t), H] + iA(t) \sum_\mu \xi_\mu(t)Q_\mu = 0, \quad (3.7)$$

$$\frac{dD(t)}{dt} + i[H, D(t)] - i \sum_\mu \xi_\mu(t)Q_\mu D(t) = 0. \quad (3.8)$$

One recognizes in (3.7) and (3.8) the backward Heisenberg equation for $A(t)$ and the forward Liouville-von Neumann equation for $D(t)$ with source terms. The stationary value of the action-like (3.1) reduces in this case to

$$I_{st} = Tr A(t_0)D(t_0) = Tr A(t_0)D(t_0). \quad (3.9)$$

Therefore, the generating functional (2.1) writes

$$W\{\xi\} = -i \log I_{st}. \quad (3.10)$$

Hence, one has succeeded in constructing a variational principle which has the desired quantity as its stationary value.

Now, one can build approximation schemes by choosing for $D(t)$ and $A(t)$ some trial spaces. However, the important relation (3.11) will remain true only when $\delta D \propto D$ is an allowed variation [1, 3] because the integrand in (3.1) vanishes at the stationary point. This excludes for instance normalized operators. In addition, when $\delta A \propto A$ is also allowed, the quantity $Tr A(t)D(t)$ does not depend on the time $t$, as can be seen by adding (3.4) and (3.3). In this case ($\delta D \propto D, \delta A \propto A$), the approximate stationary value writes

$$I_{st} = \exp iW\{\xi\} = Tr A(t)D(t) \quad t \geq t_0, \quad (3.11)$$

as in the exact situation. Nevertheless, the dynamics will be of course different since it is induced by (3.4,3.3) which lead in general to non-linear, coupled equations of motion for $A(t)$ and $D(t)$ [3, 4]. Moreover, differentiation of (3.11) with respect to $\xi$ yields the interesting result

$$\frac{\delta W\{\xi\}}{\delta \xi_\mu(t)} = \frac{Tr D(t)A(t)Q_\mu}{Tr D(t)A(t)}. \quad (3.12)$$

In the following, we shall select for both $D(t)$ and $A(t)$ trial spaces that meet with the above restrictions: the boson gaussian operators.
4 Approximate Evolution Equations for Gaussian State and Observable

Before proceeding further, let us introduce our notation. In what follows, $x$ denotes a $D$-dimensional vector and $f_x = f d^D x$. In addition, $x$ is the $(D + 1)$-vector $(x, t)$. Although the formalism presented here applies to quantum field theory, we will use a many-body notation, hiding as much as possible space integrals and discrete sums.

Let us first introduce the $2N$-component operator $\alpha(x)$ in the Schrödinger picture

$$
\alpha_j(x) = \frac{1}{\sqrt{2}} (\Phi_j(x) + i \Pi_j(x)) \quad j = 1, 2 \ldots N,
$$

$$
\alpha_j(x) = \frac{1}{\sqrt{2}} (\Phi_{j-N}(x) - i \Pi_{j-N}(x)) \quad j = N + 1, N + 2 \ldots 2N,
$$

(4.1)

where $\Phi(x)$ is the ($N$-component) boson field operator and $\Pi(x)$ its conjugate momentum. The usual boson commutation relations write in term of the operator $\alpha$ as

$$
[\alpha_i(x), \alpha_j(y)] = \tau_{ij}(x, y),
$$

(4.2)

where the $2N \times 2N$ matrix $\tau$ has the block form

$$
\tau(x, y) = \begin{pmatrix}
0 & \delta^D(x - y) \\
-\delta^D(x - y) & 0
\end{pmatrix}.
$$

(4.3)

We now choose for both $D(t)$ and $A(t)$ the class of gaussian operators in $\alpha$:

$$
\begin{cases}
D(t) = N_d \exp(\lambda_d \tau \alpha) \exp(\frac{1}{2} \alpha \tau S_d \alpha), \\
A(t) = N_a \exp(\lambda_a \tau \alpha) \exp(\frac{1}{2} \alpha \tau S_a \alpha).
\end{cases}
$$

(4.4)

In (4.4), $N_d$ and $N_a$ are c-numbers, $\lambda_d(x, t)$ and $\lambda_a(x, t)$ are $2N$-component vectors and $S_d(x, y, t)$ and $S_a(x, y, t)$ are $2N \times 2N$ matrices which can be chosen so that $\tau S_d$ and $\tau S_a$ are symmetric.

The quantities $N_q$, $\lambda_q$ and $S_q$ (with $q = d$ or $a$) uniquely define the operator to which they refer ($D$ or $A$). It is however more convenient to characterize these operators by $Z_q(t)$, $\langle \alpha \rangle_q(x, t)$ and $R_q(x, y, t)$ defined as

$$
\begin{cases}
Z_q(t) = \text{Tr} \mathcal{T}_q(t), \\
\langle \alpha_i \rangle_q(x, t) = \frac{\text{Tr} \alpha_i(x) \mathcal{T}_q(t)}{Z_q(t)}, \\
(R_q)_{ij}(x, y, t) = \frac{\text{Tr} (\tau \tilde{\alpha}_j(y) \tilde{\alpha}_i(x)) \mathcal{T}_q(t)}{Z_q(t)}.
\end{cases}
$$

(4.5)
with $T_d = D$ and $T_a = A$. We have introduced in (1.3) the shifted operators $\tilde{\alpha} = \alpha - \langle \alpha \rangle$. The relations between the two sets ($N_q$, $\lambda_q$, $S_q$) and ($Z_q$, $\langle \alpha \rangle_q$, $R_q$) are given in the appendix. From now on, we consider ($Z_d$, $\langle \alpha \rangle_d$, $R_d$) and ($Z_a$, $\langle \alpha \rangle_a$, $R_a$) as variational parameters. To proceed further, we introduce the mixed operators $T_{ad} = AD$ and $T_{da} = DA$ which are also gaussian operators and can therefore be characterized by ($Z_{ad}$, $\langle \alpha \rangle_{ad}$, $R_{ad}$) and ($Z_{da}$, $\langle \alpha \rangle_{da}$, $R_{da}$) given by (1.5) with $q = ad$ or $da$. We give in the appendix the relations between ($Z_{ad,da}$, $\langle \alpha \rangle_{ad,da}$, $R_{ad,da}$) and ($Z_{a,d}$, $\langle \alpha \rangle_{a,d}$, $R_{a,d}$). Furthermore, for any operator $O$, we will denote by $\langle O \rangle_q$ (with $q = a$, $d$, $ad$ or $da$) its expectation value with respect to $A$, $D$, $T_{ad}$ or $T_{da}$. Our previous notation anticipated this convention. Because of the cyclic invariance of the trace, we have $Z_{ad} = Z_{da}$. Their common value will be denoted by $Z$. According to equation (3.11), it is precisely the quantity of interest since we have

$$I_{st} = \exp iW\{\xi\} = Z(t).$$

The reduced functional (3.13,2) now takes the form

$$I = Z(t_0) + \int_{t_0}^{\infty} Z(t) \left\{ \frac{d\log Z}{dt} \Big|_D + i(\mathcal{E}_{ad} - \mathcal{E}_{da}) + i \sum_{\mu} \xi_{\mu}(Q_{\mu})_{da} \right\},$$

or

$$I = Z(\infty) - \int_{t_0}^{\infty} Z(t) \left\{ \frac{d\log Z}{dt} \Big|_A - i(\mathcal{E}_{ad} - \mathcal{E}_{da}) - i \sum_{\mu} \xi_{\mu}(Q_{\mu})_{da} \right\},$$

where $\mathcal{E}_{ad}$ and $\mathcal{E}_{da}$ are the expectation values of $H$ with respect to $T_{ad}$ and $T_{da}$

$$\mathcal{E}_q \equiv \langle H \rangle_q = \frac{\text{Tr} HT_q}{\text{Tr} I_q},$$

with $q = ad$ or $da$ and $\frac{d\log Z}{dt}|_D$ (resp. $\frac{d\log Z}{dt}|_A$) denotes a time-differentiation in which the parameters of $D$ (resp. $A$) are kept fixed.

The equations of motion for $R_a$ and $\langle \alpha \rangle_a$ are obtained from the stationarity conditions of (1.7) with respect to $R_d$ and $\langle \alpha \rangle_d$. A straightforward calculation yields the following equations

$$\left\{ \begin{array}{l}
i \frac{dR_a}{dt} = R_aH_{da}(1 + R_a) - (1 + R_a)H_{ad}R_a, \\
i \frac{d\langle \alpha \rangle_a}{dt} = -R_aV_{da} + (1 + R_a)V_{ad} + i\frac{dR_a}{dt}(\langle \alpha \rangle_{ad} - \langle \alpha \rangle_{da}). \end{array} \right.$$  

The stationarity conditions of (4.8) with respect to $R_d$ and $\langle \alpha \rangle_d$ yield the equations of motion for $R_a$ and $\langle \alpha \rangle_a$:

$$\left\{ \begin{array}{l}
i \frac{dR_d}{dt} = R_dH_{ad}(1 + R_d) - (1 + R_d)H_{da}R_d, \\
i \frac{d\langle \alpha \rangle_d}{dt} = -R_dV_{ad} + (1 + R_d)V_{da} - i\frac{dR_d}{dt}(\langle \alpha \rangle_{ad} - \langle \alpha \rangle_{da}). \end{array} \right.$$  

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In (4.10, 4.11), we have introduced the vector $V_{ad}$ and the matrix $H_{ad}$ given by
\begin{equation}
\begin{cases}
(V_{ad})(x, t) = \left( \tau \frac{\partial E_{ad}}{\partial (\alpha)_{ad}} \right)_i (x, t) = \sum_{j=1}^{2N} \int \tau_{ij}(x, y) \frac{\partial E_{ad}}{\partial ((\alpha)_{ad})_j(y, t)}, \\
(H_{ad})_{ij}(x, y, t) = -2 \left( \frac{\partial E_{ad}}{\partial (R_{ad})_{ij}} \right) (x, y, t) = -2 \frac{\partial E_{ad}}{\partial (R_{ad})_{ji}(y, x, t)}.
\end{cases}
\end{equation}

The primed quantities are obtained from (4.12) by replacing the index $ad$ by $da$ and the "energy" $E_{da}$ by
\begin{equation}
E'_{da} \equiv E_{da} - \sum_\mu \xi_\mu \langle Q_\mu \rangle_{da}.
\end{equation}

This amounts to replace the Hamiltonian $H$ by the source-dependent one $H' \equiv H - \sum_\mu \xi_\mu Q_\mu$ in the $da$–indexed quantities as can be seen from (4.17, 4.18). Furthermore, since $Z_a$ and $Z_d$ present no particular interest, we have preferred to omit their equations.

The system (4.10, 4.11), which to our knowledge, has never been derived elsewhere, requires several comments. First of all, the coupling between $D$ and $A$ occurs via the $ad$ and $da$–indexed variables. This coupling renders the resolution of the problem quite complicated because of the mixed boundary conditions which are given both at the initial time $t_0$ and at the final time $t_1 = \infty$. Supposing that the initial density operator is a gaussian operator characterized by $R_0$ and $\langle \alpha \rangle_0$, we have
\begin{equation}
R_a(x, y, t_0) = R_0(x, y), \quad \langle \alpha \rangle_a(x, t_0) = \langle \alpha \rangle_0(x).
\end{equation}

On the other hand, the condition (3.3) implies that $N_a(\infty) = 1$, $\lambda_a(x, \infty) = 0$ and $S_a(x, y, \infty) = 0$. Since these two last conditions cannot be expressed conveniently in terms of $\langle \alpha \rangle_a$ and $R_a$ (see (4.6)), it is preferable to consider the equations of motion of $\lambda_a$ and $T_a \equiv \exp (-S_a)$ (with $T_a(x, y, \infty) = \delta^D(x - y)$)
\begin{equation}
\begin{cases}
i \frac{d}{dt} T_a = \dot{T} a - H_{ad} T_a, \\
i \frac{d}{dt} \lambda_a = V_{ad} - T_a \langle \alpha \rangle_{da} - i \frac{d}{dt} [\langle \alpha \rangle_{da}] .
\end{cases}
\end{equation}

An interesting feature of (4.10, 4.11 or 4.13) is that the "contractions" $(R_{ad}, \langle \alpha \rangle_{ad})$ and $(R_{da}, \langle \alpha \rangle_{da})$ associated with $T_{ad}$ and $T_{da}$ satisfy the (formally) simple equations
\begin{equation}
\begin{cases}
i \frac{d}{dt} R_{ad} = [R_{ad}, H_{ad}] \\
i \frac{d}{dt} \langle \alpha \rangle_{ad} = V_{ad}
\end{cases}, \quad \begin{cases}
i \frac{d}{dt} R_{da} = [R_{da}, H_{da}'] \\
i \frac{d}{dt} \langle \alpha \rangle_{da} = V'_{da}
\end{cases}.
\end{equation}

It is important to note here that these equations are not sufficient because of the mixed boundary conditions which cannot be implemented conveniently for $\langle \alpha \rangle_{ad, da}$ and $R_{ad, da}$. Notice further that only $\langle \alpha \rangle_{da}$ and $R_{da}$ depend directly on the sources. This dissymmetry between $T_{ad}$ and $T_{da}$ is due to the way we have introduced the source term in (3.1, 3.2).

Finally, upon using (3.12), one obtains
\begin{equation}
\frac{\delta W\{\xi\}}{\delta \xi_\mu(t)} = \langle Q_\mu \rangle_{da}(t).
\end{equation}
This result shows that the connected Green’s functions are embodied in $\langle Q_\mu \rangle_{da}$ and not in $\langle Q_\mu \rangle_d$ as one would have expected from a first view. Following the exact case, one may define the effective action as in section 2, with $\langle Q_\mu(t) \rangle$ (given by (2.7)) replaced by $\langle Q_\mu \rangle_{da}(t)$. The procedure now is to expand the whole set of equations in powers of the sources in order to determine approximate expressions for (2.3), (2.6) and more generally for the successive terms of (2.4). This is the purpose of the next section.

5 Expansion in Powers of the Sources

When there are no sources, we have at any time $A(t) = 1$. Therefore, $R_{ad}^{(0)} = R_{da}^{(0)} = R^{(0)}$ and $\langle \alpha \rangle_{ad}^{(0)} = \langle \alpha \rangle_{da}^{(0)} = \langle \alpha \rangle^{(0)}$. The equations (4.10, 4.11, 4.16) become identical and lead to the following equations

\[
\begin{align*}
\frac{i}{\hbar} \frac{dR^{(0)}}{dt} &= [R^{(0)}, H^{(0)}] = -2 \left[ R^{(0)}, \frac{\partial \mathcal{S}^{(0)}}{\partial R^{(0)}} \right], \\
\frac{i}{\hbar} \frac{d\langle \alpha \rangle^{(0)}}{dt} &= V^{(0)} = \tau \frac{\partial \mathcal{S}^{(0)}}{\partial \langle \alpha \rangle^{(0)}},
\end{align*}
\]

(5.1)

which are the Time-Dependent Hartree-Fock-Bogoliubov (TDHFB) equations[9, 11] submitted to the initial conditions $R^{(0)}(x, y, t_0) = R_0(x, y)$ and $\langle \alpha \rangle^{(0)}(x, y, t_0) = \langle \alpha \rangle_0(x, y)$. These equations are the bosonic counterpart of the well-known TDHF equations and belong to the class of the time-dependent mean-field approximation. An interesting property, which will be used later, is that the ”von-Neumann entropy” $-\text{Tr} \mathcal{D}(t) \log \mathcal{D}(t)$ is conserved. This leads to the fact that an initial pure state $|\psi_0\rangle$ remains pure during the evolution.

From expressions (2.5) and (4.17), one sees that the best variational approximation to $\langle Q_\mu^H(t) \rangle$ within our trial spaces is given by

\[
\langle Q_\mu \rangle_{da}^{(0)}(t) = \langle Q_\mu \rangle_d^{(0)}(t) \equiv \langle Q_\mu \rangle^{(0)}(t).
\]

(5.2)

Hence, to evaluate the expectation value of $Q_\mu$ at a time $t$, one runs the TDHFB equations (5.1) from $t_0$ to $t$ and then computes the quantity (5.2) by means of Wick’s theorem if necessary.

The next order in the sources allows the determination of the two-point connected Green’s function $G^{(2)}_{\mu\nu}$. Indeed, upon using (2.6) and (4.17), one finds

\[
G^{(2)}_{\mu\nu}(t', t'') = -i \frac{\delta \langle Q_\mu \rangle_{da}(t')}{\delta \xi_{\nu}(t'')} \bigg|_{\xi = 0}.
\]

(5.3)

Since $\langle Q_\mu \rangle_{da}$ depends solely on $\langle \alpha \rangle_{da}$ and $R_{da}$, we expand these last variables up to first order in the sources

\[
\langle \alpha \rangle_{da} = \langle \alpha \rangle^{(0)} + \delta \langle \alpha \rangle_{da}, \quad R_{da} = R^{(0)} + \delta R_{da},
\]

(5.4)
with
\[
\delta \langle \alpha \rangle_{da}(x, t) = -i \int_{t_0}^{\infty} dt' \sum_{\mu} \gamma_{\mu}(x, t, t') \xi_{\mu}(t'),
\]
\[
\delta \mathcal{R}_{da}(x, y, t) = -i \int_{t_0}^{\infty} dt' \sum_{\mu} \mathcal{R}_{\mu}(x, y, t, t') \xi_{\mu}(t').
\]  \tag{5.5}

The connected Green's function \( G_{\mu\nu}^{(2)} \) now reads
\[
G_{\mu\nu}^{(2)}(t', t'') = -\frac{\partial(Q_{\mu}^{(0)}(t'))}{\partial \langle \alpha \rangle^{(0)}(t')} \gamma_{\nu}(t', t'') - \frac{\partial(Q_{\mu}^{(0)}(t'))}{\partial \mathcal{R}^{(0)}(t')} \mathcal{R}_{\nu}(t', t'').
\]  \tag{5.6}

The equations of motion for \( \delta \langle \alpha \rangle_{da} \) and \( \delta \mathcal{R}_{da} \) are obtained by developing \( \langle \alpha \rangle^{(0)}, \mathcal{R}^{(0)} \) around \( \langle \alpha \rangle^{(0)}, \mathcal{R}^{(0)} \). In terms of the corrections \( \gamma_{\mu} \) and \( \mathcal{R}_{\mu} \), they take the form
\[
\left\{ \begin{array}{l}
i \frac{d\gamma_{\mu}}{dt}(t, t') = \tau \frac{\partial F_{\mu}(t, t')}{\partial \langle \alpha \rangle^{(0)}(t)}, \\
i \frac{d\mathcal{R}_{\mu}}{dt}(t, t') = [\mathcal{R}_{\mu}(t, t'), \mathcal{H}^{(0)}(t)] - 2 \left[ \mathcal{R}^{(0)}(t), \frac{\partial F_{\mu}(t, t')}{\partial \mathcal{R}^{(0)}(t)} \right],
\end{array} \right.
\]  \tag{5.7}

where
\[
F_{\mu}(t, t') = V^{(0)}(t) \gamma_{\mu}(t, t') - \frac{1}{2} \text{tr} H^{(0)}(t) \mathcal{R}_{\mu}(t, t') - i(Q_{\mu}^{(0)}(t)) \delta(t - t').
\]  \tag{5.8}

The symbol \( \text{tr} \) in \( \langle 5.8 \rangle \), not to be confused with \( \text{Tr} \), denotes a trace on both the discrete indices and the space variables. One has for instance
\[
\text{tr} H^{(0)}(t) \mathcal{R}_{\mu}(t, t') = \sum_{i,j=1}^{2N} \int_{x,y} \left( H^{(0)} \right)_{ij}(x, y, t)(\mathcal{R}_{\mu})_{ji}(y, x, t, t').
\]  \tag{5.9}

The equations \( \langle 5.4 \rangle \) are not sufficient since \( \langle \alpha \rangle_{da} \) and \( \mathcal{R}_{da} \) (and hence \( \gamma_{\mu} \) and \( \mathcal{R}_{\mu} \)) are neither known at \( t_0 \) nor at \( \infty \). Let us instead linearize the equations \( \langle 5.13 \rangle \) by setting \( T_{a}^{(1)} = T_{a}^{(0)} + \delta T_{a} \equiv 1 - S_{a}^{(1)} \) and \( \lambda_{a}^{(1)} = \lambda_{a}^{(0)} + \delta \lambda_{a} \equiv \alpha_{a} \), where \( S_{a}^{(1)} \) and \( \delta \lambda_{a} \) are of first order in the sources. By using a parametrization similar to \( \langle 5.3 \rangle \), namely
\[
L_{a}^{(1)}(x, t) \equiv \delta \lambda_{a}(x, t) - \int_{y} S_{a}^{(1)}(x, y, t) \langle \alpha \rangle^{(0)}(y, t)
= -i \int_{t_0}^{\infty} dt' \sum_{\mu} \xi_{\mu}(t') L_{\mu}(x, t'),
\]  \tag{5.10}

we obtain the following equations for \( L_{\mu} \) and \( S_{\mu} \)
\[
\left\{ \begin{array}{l}
i \frac{dL_{\mu}}{dt}(t', t) = \tau \frac{\partial K_{\mu}(t', t)}{\partial \langle \alpha \rangle^{(0)}(t)}, \\
i \frac{dS_{\mu}}{dt}(t', t) = -2 \frac{\partial K_{\mu}(t', t)}{\partial \mathcal{R}^{(0)}(t)},
\end{array} \right.
\]  \tag{5.11}

where
\[
K_{\mu}(t', t) = V^{(0)}(t) \tau L_{\mu}(t', t) + \frac{1}{2} \text{tr} [\mathcal{R}^{(0)}(t), H^{(0)}(t)] S_{\mu}(t', t) + i(Q_{\mu}^{(0)}(t)) \delta(t - t').
\]  \tag{5.12}
According to the general philosophy, these equations are to be solved backward in time since the final conditions read

\[ L_\mu(x, t', \infty) = 0, \quad S_\mu(x, y, t', \infty) = 0. \] (5.13)

Therefore, for \( t' < t \), we have \( L_\mu(t', t) = 0 \) and \( S_\mu(t', t) = 0 \). For \( t' > t \), \( L_\mu(t', t) \) and \( S_\mu(t', t) \) are solutions of (5.11) without the \( \delta(t - t') \) term in \( K_\mu \). The jumps at \( t = t' - 0 \) read

\[
\left\{ \begin{array}{l}
L_\mu(x, t', t' - 0) = -\tau \frac{\partial \langle Q_\mu \rangle^{(0)}(t')}{\partial \langle \alpha \rangle^{(0)}(x, t')}, \\
S_\mu(x, y, t', t' - 0) = -2 \frac{\partial \langle Q_\mu \rangle^{(0)}(t')}{\partial \mathcal{R}^{(0)}(y, x, t')}. 
\end{array} \right. \tag{5.14}
\]

We now want to derive an alternative and tractable expression for \( G_{\mu\nu}^{(2)} \) in terms of \( L_\mu \) and \( S_\mu \). One can show first that the quantity

\[ g_{\mu\nu} \equiv L_\mu(t', t) \tau \gamma_{\nu}(t, t'') - \frac{1}{2} \text{tr} S_\mu(t', t) \mathcal{R}_\nu(t, t'') \]

does not depend on the time \( t \) except for jumps at \( t = t' \) and \( t = t'' \). Indeed, upon using (5.7, 5.11), one finds the expression

\[
\frac{dg_{\mu\nu}}{dt} = \left\{ \frac{\partial \langle Q_\mu \rangle^{(0)}(t)}{\partial \langle \alpha \rangle^{(0)}(t)} \gamma_{\nu}(t, t'') + \frac{\partial \langle Q_\mu \rangle^{(0)}(t)}{\partial \mathcal{R}^{(0)}(t)} \mathcal{R}_\nu(t, t'') \right\} \delta(t - t')
+ \left\{ L_\mu(t', t) \frac{\partial \langle Q_\mu \rangle^{(0)}(t)}{\partial \langle \alpha \rangle^{(0)}(t)} - \text{tr} S_\mu(t', t) \left[ \mathcal{R}^{(0)}(t), \frac{\partial \langle Q_\mu \rangle^{(0)}(t)}{\partial \mathcal{R}^{(0)}(t)} \right] \right\} \delta(t - t''),
\]

where, by choosing \( t'' > t' \), it is clear that just the first term survives the integration over \( t \). One recognizes in this remaining term the expression (5.4) for \( G_{\mu\nu}^{(2)} \). Hence, \( g_{\mu\nu} \) can be evaluated at \( t = t_0 \) which yields

\[
G_{\mu\nu}^{(2)}(t', t'') = L_\mu(t', t_0) \tau \gamma_{\nu}(t_0, t'') - \frac{1}{2} \text{tr} S_\mu(t', t_0) \mathcal{R}_\nu(t_0, t'')
= L_\mu(t', t_0) \tau \mathcal{R}_0 L_\nu(t'', t_0) + \frac{1}{2} \text{tr} S_\mu(t', t_0) \mathcal{R}_0 \mathcal{S}_\nu(t'', t_0)(1 + \mathcal{R}_0). \tag{5.15}
\]

The second line in expression (5.15) follows from the linearization of the relations between \((\langle \alpha \rangle_{da}, \mathcal{R}_{da})\) and \((\langle \alpha \rangle_{a}, \langle \alpha \rangle_{d}, \mathcal{R}_a, \mathcal{R}_d)\) (8.4). When evaluated at time \( t_0 \), the result writes

\[
\left\{ \begin{array}{l}
\delta \mathcal{R}_{da}(t_0) = -\mathcal{R}_0 S_{\alpha}^{(1)}(t_0)(1 + \mathcal{R}_0), \\
\delta \langle \alpha \rangle_{da}(t_0) = \mathcal{R}_0 L_{\alpha}^{(1)}(t_0), 
\end{array} \right. \tag{5.16}
\]

or equivalently

\[
\left\{ \begin{array}{l}
\mathcal{R}_\mu(t_0, t'') = -\mathcal{R}_0 S_\mu(t'', t_0)(1 + \mathcal{R}_0), \\
\gamma_\mu(t_0, t') = \mathcal{R}_0 L_\mu(t'', t_0). 
\end{array} \right. \tag{5.17}
\]

The anti-causal Green’s function is obtained from (5.13) by exchanging \((\mu, \nu)\) and \((t', t'')\).
The computation of the two-point Green’s function now proceeds as follows: one first solves the TDHFB equations \((5.1)\) from \(t_0\) to \(t\) in order to determine \(\langle \alpha \rangle^{(0)}(t)\) and \(R^{(0)}(t)\). Then, one runs backward \((5.11)\) from \(t = t'\) down to \(t = t_0\) to obtain \(L_\mu(t', t_0)\) and \(S_\mu(t', t_0)\). We note in particular that one is spared solving \((5.7)\).

As discussed in [6] sect. 4, this procedure reveals several interesting features that are missing in the naive application of Wick’s theorem. Moreover, since \(\langle Q_\mu \rangle_{da}(t)\) and \(\langle Q_\mu \rangle^{(0)}(t)\) as given by \((4.17)\) and \((5.2)\) are the best second order approximations to \(\langle Q_\mu \rangle(t)\) and \(\langle Q_\mu(t)\rangle\) (respectively given by \((2.7)\) and \((2.11)\)), we can use \((2.13)\) and \((2.14)\) to derive directly the expressions

\[
W\{\xi\} \simeq \int_{t_0}^{\infty} dt' \sum_\mu \xi_\mu(t')\langle Q_\mu \rangle^{(0)}(t') + \frac{i}{2} \int_{t_0}^{\infty} dt' dt'' \sum_\mu \xi_\mu(t')G^{(2)}_{\mu \nu}(t', t'')\xi_\nu(t''),
\]

\[
S_{eff}\{\langle Q \rangle_{da}\} \simeq \frac{i}{2} \int_{t_0}^{\infty} dt' dt'' \sum_{\mu \nu} \langle Q_\mu \rangle_{da} - \langle Q_\mu \rangle^{(0)} \rangle(t')\Gamma^{(2)}_{\mu \nu}(t', t'')\langle Q_\nu \rangle_{da} - \langle Q_\nu \rangle^{(0)}\rangle(t''),
\]

with the relation

\[
\langle Q_\mu \rangle_{da}(t) = \langle Q_\mu \rangle^{(0)}(t) + i \int_{t_0}^{\infty} dt' \sum_\nu \xi_\nu(t')G^{(2)}_{\mu \nu}(t, t').
\]

### 6 Illustration

Since the TDHFB equations \((5.1)\) remain the main ingredient whatever the set \(\{Q_\mu\}\), they deserve a deeper analysis. Let us focus on a \(O(1)\) quantum field theory in \((D + 1)\)–dimensional Robertson-Walker spacetime described by the metric \(ds^2 = dt^2 - a^2(t)dx^2\). The Hamiltonian density is taken to be

\[
H = \frac{1}{2} a^{-D}(t)\Pi^2(x) + a^D(t)V(\Phi),
\]

where

\[
V(\Phi) = \frac{1}{2} a^{-2}(\nabla \Phi(x))^2 + \frac{1}{2}(m_0^2 + g_0R)\Phi^2(x) + \frac{\lambda_0}{6}\Phi^4(x),
\]

with the scalar curvature given by \((3.12)\) \(R = 2D\dot{a}/a + D(D - 1)(\dot{a}/a)^2\).

Upon defining the expectation values

\[
\begin{align*}
\phi(x, t) &= \langle \Phi(x) \rangle, \\
\pi(x, t) &= \langle \Pi(x) \rangle, \\
G(x, y, t) &= \langle \Phi(x)\Phi(y) \rangle, \\
F(x, y, t) &= \langle \Pi(x)\Pi(y) \rangle, \\
C(x, y, t) &= \langle \Phi(x)\Pi(y) + \Pi(y)\Phi(x) \rangle.
\end{align*}
\]

\[ \]
which are related to $\langle \alpha \rangle$ and $R$ by means of (A.11–A.12), the energy density writes
\[
E = \frac{1}{2}a^{-D}(t)\left(\pi^2(x, t) + F(x, x, t)\right) + a^D(t)\langle V\rangle(\phi, G).
\tag{6.4}
\]
For a gaussian density operator, $\langle V\rangle(\phi, G)$ can be computed by means of Wick’s theorem which yields
\[
\langle V\rangle = \frac{1}{2}\left(-a^{-2}\Delta x + m_0^2 + g_0R + 2\lambda_0 G(x, x, t) + \frac{\lambda_0}{3}\phi^2(x, t)\right)\phi^2(x, t)
+ \frac{1}{2}\left(-a^{-2}\Delta x + m_0^2 + g_0R + \lambda_0 G(x, x, t)\right)G(x, x, t).
\tag{6.5}
\]
In terms of the Fourier transforms of the new variables
\[
\begin{align*}
\hat{\phi} &= a^{D/2}\phi \\
\hat{\pi} &= a^{-D/2}\pi + \frac{DH}{2}a^{D/2}\phi
\end{align*}
\]
the TDHFB equations take the form
\[
\begin{align*}
\dot{\hat{\phi}} &= \hat{\pi}, \\
\dot{\hat{\pi}} &= -(\mu^2 - \frac{4\lambda_0}{3}a^{-D}\hat{\phi}^2)\hat{\phi},
\end{align*}
\tag{6.7}
\]
\[
\begin{align*}
\dot{\hat{G}} &= a^D G \\
\dot{\hat{F}} &= a^{-D}F + \left(\frac{DH}{2}\right)^2a^D G + \frac{DH}{4}(C + \hat{C}) \\
\dot{\hat{C}} &= C + Dh a^D G
\end{align*}
\tag{6.8}
\]
where we have introduced the time-dependent self-consistent mass $\mu(t)$:
\[
\mu^2(t) = m_0^2 + \frac{DH^2}{4} + (g_0 - \frac{1}{4})R + 2\lambda_0 a^{-D}\langle \phi^2 \rangle + \int_k \hat{G}(k, t),
\tag{6.9}
\]
$H = \dot{a}/a$ being the Hubble parameter and $f_k = \int d^D k/(2\pi)^D$. In deriving (6.7–6.8), we have assumed that the field $\hat{\phi}$ is homogeneous, which is compatible with translation invariance.

The divergences in the $k$–integral of (6.9) are easily removed by using the adiabatic expansion method\[12, 13\]. In the limit $D = 3$, this leads to the following prescription
\[
\begin{align*}
\frac{m_0^2}{\lambda_R} &= \frac{m_0^2}{\lambda_0}, \\
\frac{1}{\lambda_R} &= \frac{1}{\lambda_0} + \frac{4}{(4\pi)^{2/3}}\frac{1}{3 - D}, \\
g\frac{D - 1}{\lambda_R} &= g_0 - \frac{D - 1}{\lambda_0}.
\end{align*}
\tag{6.10}
\]
Having now established the renormalizability of the TDHFB equations, we will concentrate on the case where $Q_{\mu=1}$ is the $O(1)$ boson field operator $\Phi(x)$ and $Q_{\mu=2}$ is
the composite operator $\Phi(x)\Phi(y)$. We have therefore to consider for the set $\{\xi_\mu\}$ a local source $J(x,t)$ as well as a bilocal one $K(x,y,t)$.

According to (5.10), $L_\mu$ are $2-$dimensional vectors and $S_\mu$ are $2 \times 2$ matrices, which, for convenience, we will decompose in the form

$$L_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} l_\mu + i e_\mu \\ l_\mu - i e_\mu \end{pmatrix}, \quad S_\mu = \frac{1}{4} \begin{pmatrix} (u_\mu + v_\mu) & -(u_\mu - v_\mu) + 2w_\mu \\ (u_\mu - v_\mu) + 2w_\mu & -(u_\mu + v_\mu) \end{pmatrix}. \quad (6.11)$$

The evolution equations for these five parameters are obtained from (5.11-5.12). They read

$$\begin{cases} \frac{dl_\mu}{dt} = a^{-D} e_\mu \\ \frac{de_\mu}{dt} = a^D \left( -(-a^2 \Delta + M^2)l_\mu + 2\lambda_R \hat{\phi}(\frac{1}{2} \hat{C}u_\mu + g_f w_\mu) \right), \quad (6.12) \\ \frac{du_\mu}{dt} = 2ia^{-D} w_\mu \\ \frac{dv_\mu}{dt} = 2ia^D \left( 4\lambda_R \hat{\phi}l_\mu - (-a^2 \Delta + M^2 + 2\lambda_R g_f)w_\mu - i\lambda_R \hat{C}u_\mu \right), \quad (6.13) \\ \frac{dw_\mu}{dt} = 2ia^D (-a^2 \Delta + M^2)u_\mu - 2ia^{-D} v_\mu \end{cases}$$

with $M^2 = m_R^2 + g_R R + 2\lambda_R(\hat{\phi}^2 + g_f)$ and $g_f$ being the finite part of $\int_k G(k,t)$. Their boundary conditions (3.14) write for $\mu = 1$

$$\begin{cases} l_1(x,y,t',t') = 0 \\ e_1(x,y,t',t') = i\delta^D(x - y) \\ u_1(x,y,z,t',t') = 0 \\ v_1(x,y,z,t',t') = 0 \end{cases}. \quad (6.14)$$

and for $\mu = 2$

$$\begin{cases} l_2(x,y,z,t',t') = 0 \\ e_2(x,y,z,t',t') = i\delta^D(x - y)\hat{\phi}(z,t') + i\delta^D(x - z)\hat{\phi}(y,t') \\ u_2(x,y,z,r,t',t') = 0 \\ v_2(x,y,z,r,t',t') = -2(\delta^D(x - r)\delta^D(y - z) + \delta^D(x - z)\delta^D(y - r)) \\ w_2(x,y,z,r,t',t') = 0 \end{cases}. \quad (6.15)$$

Hence, as mentioned earlier, the optimization of the $n-$point connected Green’s functions (here $n = 2, 3$ or $4$) proceeds by solving first the TDHFB equations (6.7-6.8) forward and then, (6.12-6.13) backward in time. For instance, one may solve (6.12-6.13) with $\mu = 1$ and $\mu = 2$. Then, one computes $G^{(2)}_{11}(t',t'') \equiv G_{\Phi-ph}(x,y,t',t''), G^{(2)}_{12}(t',t'') \equiv G_{\Phi-ph}(x,y,z,t',t'')$ and $G^{(2)}_{22}(t',t'') \equiv G_{\Phi-\Phi-ph}(x,y,z,r,t',t'')$ by means of (5.13).

At equilibrium, that is for a static solution of the TDHFB equations, one can notice that the equations (6.12-6.13) can be solved exactly for a wide class of metrics. One
can therefore check in particular that the Green’s functions $G_{\mu\nu}^{(2)}$ do only depend on the time-difference $t' - t''$ as expected.

We can also notice that the two sets (6.12) and (6.13) completely decouple in the symmetric case $\hat{\phi} = 0$. Moreover, the boundary conditions (6.14) imply that $u_1, v_1$ and $w_1$ vanish for all times $t$. Therefore, the two-point causal function, given by (see eq.5.15)

$$G_{\Phi\Phi}(x, y, t', t'') = -\int_{z, r} l_1(x, z, t', t_0)F_0(z, r)l_1(r, y, t'', t_0) - \int_{z, r} e_1(x, z, t', t_0)G_0(z, r)e_1(r, y, t'', t_0) + \frac{1}{2}\int_{z, r} l_1(x, z, t', t_0)C_0(z, r)l_1(r, y, t'', t_0),$$

(6.16)

where $F_0, G_0$ and $C_0$ refer to the initial density matrix $R_0$, is obtained by solving just one equation, namely

$$\left(\frac{d^2}{dt^2} + DH\frac{d}{dt} + (-a^{-2}\Delta_x + M^2)\right)l_1(t', t) = 0.$$  

(6.17)

Moreover, for $\mu = 2$, one has only to solve (6.13) since $l_2$ and $e_2$ identically vanish.

### 7 Conclusions and Perspectives

We have presented in this work a variational approach suited to the optimization of multi-time Green’s functions of a set of observables. By choosing gaussian boson operators for the variational objects, we obtained two sets of coupled non-linear evolution equations with two-time boundary conditions. The resolution of this type of equations is quite complicated. Fortunately, in order to compute the one and two-time Green’s functions, one must expand these equations up to second order in a set of sources, and this in turn simplifies drastically the problem, since it reduces to two initial value problems. The first one, forward in time, turns out to be the time-dependent mean-field equations and the second, backward in time, is linear in the variational parameters which precisely determine the desired Green’s functions.

We have illustrated this formalism on a $O(1)$ quantum field theory in a RW metric. For the set of observables, we have chosen the boson field operator $\Phi(x)$ and the composite operator $\Phi(x)\Phi(y)$. This allowed us to optimize the one-, two-, three- and four-point connected Green’s functions of the observable $\Phi$.

The extension of the present formalism to fermion fields is quite simple since it only depends on the way we parametrize the contraction matrix $R$ (remember that in this case $\langle \alpha \rangle = 0$.) However, the generalization to gauge field theories still remains an open question. In particular, it is not clear whether one should introduce the gauge conditions.
as constraints in the variational principle (which seems very natural) or one should better enforce these conditions in the choice of the trial spaces. The latter procedure has in fact been attempted\cite{14} but has several drawbacks.

So far we have only discussed gaussian trial operators. The question that naturally arises is how one can go beyond the gaussian approximation while keeping the tractability of the formalism and the consistency of the approximations. Some promising methods do indeed exist in the literature. Among them, we can quote in particular the ”post-gaussian” approximation\cite{4,5} which consists of an expansion in cumulants around a gaussian density operator, or the so-called non-gaussian calculation\cite{15}. More recently, a new method, based on the background field method, has been developed\cite{16}.

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In the final elaboration of this work, we received a paper\cite{17} dealing with the same approach. We thank C. Martin for sending us the manuscript.
Appendix

For the sake of completeness, we recall in this appendix some useful properties of bosonic gaussian operators of the form:

\[ \mathcal{T} = \mathcal{N} \exp (\lambda \tau \alpha) \exp \left( \frac{1}{2} \alpha \tau S \alpha \right). \]  

(A.1)

In (A.1), \( \mathcal{N} \) is a c-number, \( \lambda(x, t) \) a \( 2N \)-component vector and \( S(x, y, t) \) is a \( 2N \times 2N \) symplectic matrix. The \( (2N \times 2N) \) matrix \( \tau \) is defined as

\[ \tau(x, y) = \begin{pmatrix} 0 & \delta^D(x - y) \\ -\delta^D(x - y) & 0 \end{pmatrix}, \]

(A.2)

and \( \alpha(x) \) is the \( 2N \)-component boson operator in the Schrödinger picture

\[ \alpha_j(x) = \frac{1}{\sqrt{2}} \begin{cases} \Phi_j(x) + i \Pi_j(x) & j = 1, 2, \ldots, N \\ \Phi_{j-N}(x) - i \Pi_{j-N}(x) & j = N + 1, \ldots, 2N \end{cases} \]

(A.3)

obeying the usual commutation relations

\[ [\alpha_i(x), \alpha_j(y)] = \tau_{ij}(x, y). \]  

(A.4)

We are adopting in (A.1) compact notations for discrete sums and space integrals. For instance

\[ \lambda \tau \alpha = \sum_{i,j=1}^{2N} \int_{x,y} \lambda_i(x, t) \tau_{ij}(x, y) \alpha_j(y), \]

\[ \alpha \tau S \alpha = \sum_{i,j,k=1}^{2N} \int_{x,y,z} \alpha_i(x) \tau_{ij}(x, y) S_{jk}(y, z, t) \alpha_k(z). \]  

(A.5)

As mentioned in the text, it is more convenient to work with the set of three parameters consisting of the "partition function" \( Z \), the vector \( \langle \alpha \rangle \) and the contraction matrix \( \mathcal{R} \). These are given by

\[ \begin{cases} Z \equiv \text{Tr} \mathcal{T} = \mathcal{N} \exp \left( \frac{1}{2} \lambda \tau \mathcal{R} \lambda \right) \sqrt{\det \sigma \mathcal{R}} \\ \langle \alpha \rangle \equiv \text{Tr} \mathcal{T} \alpha / Z = \frac{1}{1 - \exp(-S)} \lambda = \frac{1}{1 - T} \lambda \\ \mathcal{R} \equiv \text{Tr} (\mathcal{T} \tau \tilde{\alpha} \tilde{\alpha}) / Z = \frac{1}{1 - \exp(S) - 1} = \frac{T}{1 - T} \end{cases} \]

(A.6)

where \( T = e^{-S} \), \( \tilde{\alpha} = \alpha - \langle \alpha \rangle \) and

\[ \sigma(x, y) = \begin{pmatrix} 0 & \delta^D(x - y) \\ -\delta^D(x - y) & 0 \end{pmatrix}. \]  

(A.7)

The product of two gaussian operators \( \mathcal{T} = \mathcal{T}_1 \mathcal{T}_2 \) of the form (A.1) with parameters \( \mathcal{Z}_1, \langle \alpha \rangle_1, \mathcal{R}_1 \) and \( \mathcal{Z}_2, \langle \alpha \rangle_2, \mathcal{R}_2 \), is also a gaussian operator of the form (A.1) with parameters \( \mathcal{Z}, \langle \alpha \rangle \) and \( \mathcal{R} \). These can be written

\[ \begin{cases} \mathcal{Z} = \mathcal{Z}_1 \mathcal{Z}_2 \exp \left\{ \frac{1}{2} \langle \alpha \rangle_1 - \langle \alpha \rangle_2 \rangle \tau \mathcal{P} \langle \langle \alpha \rangle_1 - \langle \alpha \rangle_2 \rangle \right\} \sqrt{\det \sigma \mathcal{P}} \\ \langle \alpha \rangle = \mathcal{R}_1 \mathcal{P} \langle \alpha \rangle_2 + (\mathcal{R}_2 + 1) \mathcal{P} \langle \alpha \rangle_1 \\ \mathcal{R} = \mathcal{R}_1 \mathcal{P} \mathcal{R}_2 \end{cases} \]

(A.8)
with $P = (1 + R_1 + R_2)^{-1}$. As an immediate consequence, a projector on coherent states (that is a pure state density matrix): $T^2 = Z T$ satisfies the property

$$R^2 + R = 0. \quad (A.9)$$

The expressions of $\langle \alpha \rangle$ and $R$ become more intuitive in terms of the usual averages defined as $(i, j = 1, \ldots N)$

$$\begin{align*}
\left\{ \begin{array}{l}
\phi_i(x, t) = \langle \Phi_i(x) \rangle \\
\pi_i(x, t) = \langle \Pi_i(x) \rangle
\end{array} \right., \quad \left\{ \begin{array}{l}
G_{ij}(x, y, t) = \langle \Phi_i(x) \Phi_j(y) \rangle \\
F_{ij}(x, y, t) = \langle \Pi_i(x) \Pi_j(y) \rangle \\
C_{ij}(x, y, t) = \langle \Phi_i(x) \Pi_j(y) + \Pi_j(y) \Phi_i(x) \rangle
\end{array} \right., \quad (A.10)
\end{align*}$$

with $\bar{Q} = Q - \langle Q \rangle$. Indeed, with the help of the canonical transformations (A.3), one gets easily

$$\langle \alpha \rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
\phi + i\pi \\
\phi - i\pi
\end{array} \right), \quad (A.11)$$

and

$$R = \frac{1}{2} \left( \begin{array}{cc}
G + F - \frac{i}{2}(C - \bar{C}) & -(G - F) - \frac{i}{2}(C + \bar{C}) \\
(G - F) - \frac{i}{2}(C + \bar{C}) & -G + F - \frac{i}{2}(C - \bar{C})
\end{array} \right). \quad (A.12)$$
References

[1] R. Balian and M. Vénéroni, Ann. of Phys. (N.Y.) 164 (1985), 334.

[2] R. Balian, P. Bonche, H. Flocard and M. Vénéroni, Nucl. Phys. A428 (1984), 79; P. Bonche and H. Flocard, Nucl. Phys. A437 (1985), 189; J. B. Marston and S. E. Koonin, Phys. Rev. Lett. 54 11 (1985), 1139.

[3] O. Eboli, R. Jackiw and S-Y Pi, Phys. Rev. D37, 12 (1988), 3557.

[4] H. Flocard, Ann. of Phys. (N.Y.) 191 (1989), 382.

[5] M. Benarous, Thesis. IPN-Orsay-France, October 1991.

[6] R. Balian and M. Vénéroni, Nucl. Phys. B408 (1993), 445 and references therein.

[7] R. H. Brandenberger, Rev. of Mod. Phys. 57 (1985), 1; R. H. Brandenberger and H. A. Feldman, Physica A 158 (1989), 343; S. Y. Pi, Physica A (158) (1989), 366; R. H. Bandenberger "Lectures on Modern Cosmology and Structure Formation", Proc. of the 7th Swieca Summer School in Particles and Fields, Campos do Jordao, Brazil, Jan. 1993. Eds. O. Eboli and V. Richelles, World Scientific, Singapore 1993; M. S. Turner "Inflation after COBE, Lectures on Inflationary Cosmology", Proc. of the Cargese Summer School on Quantitative Particle Physics, Cargese, Corsica, July 1992.

[8] F. Cooper, Y. Kluger, E. Mottola and J. P. Paz, Phys. Rev. D51, 5 (1995), 2377.

[9] D. Boyanovsky, H. J. de Vega and R. Holman, Phys. Rev. D51 2 (1995), 734.

[10] R. Balian and E. Brezin, Nuovo Cimento, B64 (1969), 37.

[11] M. Benarous "On the Poisson Structure of the Time-Dependent Mean-Field Equations for Systems of Bosons out of Equilibrium". To be published in Ann. of Phys.

[12] N. D. Birrel and P. C. W. Davies, ”Quantum Fields in Curved Space”, Cambridge University Press, Cambridge, England, 1982.

[13] O. Eboli, So-Young Pi and M. Samiullah, Ann. of Phys. (N.Y.) 193 (1989), 102.

[14] A. Kerman and D. Vautherin, Ann. of Phys. (N.Y.) 192, (1989), 408.

[15] L. Polley and U. Ritschel, Phys. Lett B221, 44 (1989).

[16] J. H. Yee ”Variational Approach to Quantum Field Theory: Gaussian Approximation and the Perturbative Expansion around It”. Report hep-th/9707234.

[17] C. Martin, Phys. Rev. D52, 12 (1995), 7121.