Mean Square Error and Limit Theorem for the Modified Leland Hedging Strategy with a Constant Transaction Costs Coefficient

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Abstract We study the Leland model for hedging portfolios in the presence of a constant proportional transaction costs coefficient. The modified Leland’s strategy defined in [2], contrarily to the classical one, ensures the asymptotic replication of a large class of payoff. In this setting, we prove a limit theorem for the deviation between the real portfolio and the payoff. As Pergamenshchikov did in the framework of the usual Leland’s strategy [11], we identify the rate of convergence and the associated limit distribution. This rate turns out to be improved using the modified strategy and non periodic revision dates.

Keywords: Asymptotic hedging – Leland-Lott strategy – Transaction costs – Martingale limit theorem.

1 Introduction

The present paper is concerned with the study of asymptotic hedging in the presence of transaction costs. The asymptotic replication of a given payoff is performed via a modified Leland’s strategy recently introduced in [2].

Let us briefly recall the history and the main known results about Leland’s strategy. In 1985 Leland suggested an approach to price contingent claims under proportional transaction costs. His main idea was to use the classical Black-Scholes formula with a suitably adjusted volatility for a periodically revised portfolio whose terminal value approximates the payoff. The intuition behind this practical method is to compensate for transaction cost by increasing the volatility in the following way:

$$\sigma_t^2 = \sigma^2 + \sigma \sqrt{n} \sqrt{8/\pi} \sqrt{f'(t)},$$ (1.1)

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where $n$ is the number of the portfolio revision dates and $k_n = k_0 n^{-\alpha}$, $\alpha \in [0, \frac{1}{2}]$ is the transaction costs coefficient generally depending of $n$; $f$ is an increasing and smooth function whose inverse $g := f^{-1}$ defines the revision dates $t_i^n := g(\frac{i}{n})$, $1 \leq i \leq n$.

The principal results on convergence for models with transaction costs can be described as follows. First consider the case of approximate hedging of the European call option using the strategy with periodic portfolio revisions (i.e $g(t) = t$). We know the following results with $T = 1$:

(a) For $\alpha = \frac{1}{2}$, Lott gave the first rigorous result on the convergence of the approximating portfolio value $V_1^{n}$ to the payoff $V_1 = (S_1 - K)_+$. The sequence $V_1^{n} - V_1$ tends to zero in probability [10], and a stronger result holds: $n \mathbb{E} (V_1^{n} - V_1)^2$ converges to a constant $A_1 > 0$ [5];

(b) For $\alpha \in (0, \frac{1}{2})$, the sequence $V_1^{n} - V_1$ tends to zero in probability (see [8]), and it is shown in [1] that $n^{p_\alpha} \mathbb{E} (V_1^{n} - V_1)^2 \to 0$ as $n \to \infty$, with $p_\alpha < \alpha$.

(c) For $\alpha = 0$, the terminal values of portfolios do not converge to the European call as shown by Kabanov and Safarian [8]. Namely, there is a negative $\sigma\{S_1\}$-measurable random variable $\xi$ such that $V_1^{n} - V_1 \to \xi$ in probability. Pergamenshchikov [11] then analyzed the rate of convergence and proved a limit theorem: the sequence $n^{\frac{1}{2}} (V_1^{n} - V_1 - \xi)$ converges in law to a mixture of Gaussian distributions [11].

He noticed that one can increase the modified volatility to obtain the asymptotic replication. To do so, he utilizes the explicit form of the systematic hedging error for the European call option. For related results see also [6] and [12].

For models including uniform and non-uniform revision intervals one needs to impose certain conditions on the scale transform $g$. Generalizations of some of the above results to this more technical case as well as extensions to contingent claims of the form $h(S_1)$ can be found in [12], [5], [1]. In particular, $n^{1/2 + \alpha} \mathbb{E} (V_1^{n} - V_1)^2$ converges to a constant in the case $\alpha > 0$. Moreover, for $\alpha = \frac{1}{2}$, the distributions of the process $Y_1^{n} := n^{\frac{1}{2}} (V_1^{n} - V_1)^3$ in the Skorohod space $\mathcal{D}[0,1]$ converges weakly to the distribution of a two-dimensional Markov diffusion process component (see [4]). Notice that the asymptotic replication still does not hold for $\alpha = 0$ in this more general setting. For more details we refer to [1], [3], [4] and references therein.

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Note that $Y_1^{n}$ corresponds to the deviation (up to a multiplicative constant) between the “real world” portfolio and the theoretical Leland’s portfolio $\hat{V}_t = \hat{C}(t,S_t)$ where $\hat{C}$ is the modified heat equation solution suggested by Leland whose terminal value is the payoff function.
We solve the case $\alpha = 0$ for a large class of payoff and with specific revision dates (including uniform dates) by means of the modified strategy introduced in [2]. This one makes the portfolio’s terminal value converge to the contingent claim as $n$ tends to infinity, that is the approximation error vanishes. The analysis we performed here suggests that it might be difficult to obtain a better convergence rate regarding uniform revision dates. In the framework of the non uniform grid we use, concentrating the revision dates near the maturity $T = 1$ accelerates the convergence rate.

The asymptotic behavior of the hedging error is a practical important issue. Since traders obviously prefer gains than losses, measuring the $L^2$-norm of hedging errors is strongly criticized. Of course, the limiting distribution of the hedging error is much more informative. Our present work also aims at tackling this issue: we prove that

$$n^{4 + p} (V_n^0 - h(S_1)) \xrightarrow{n \to \infty} Z,$$

where the law of $Z$ is explicitly identified, $EZ = 0$ and $p > 0$ depends on the choosen grid.

The paper is organized as follows. In Section 2, we introduce the basic notations, models and assumptions of our study; In particular we recall the modified Leland’s strategy defined in [2]. In Section 3, we state our main result: a limit theorem for the renormalized asymptotic hedging error. In Section 4, we establish two lemmas concerning, on one hand, random variables constructed from the geometric Brownian motion, and on the other hand, some change of variables for the revision dates. These auxiliary results will be used repeatedly throughout the paper. In Section 5, we prove the main result. An appendix finally recalls all the known technical results we need for the various proofs.

2 Notations and Models

2.1 Black–Scholes model and hedging strategy

We are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$ on which a standard one-dimensional $(\mathcal{F}_t)$–adapted Brownian motion $W$ is defined. As usual, we denote by $L^2(\Omega)$ the space of square integrable $\mathcal{F}_1$-measurable random variables endowed with its norm $\|X\|_2 := \sqrt{\mathbb{E}X^2}$.

We consider the classical Black–Scholes model composed of two assets without transaction costs, i.e. $k_0 = 0$ and $\sigma = \sigma$. The first one is riskless (bond) with the
interest rate \( r = 0 \) and the second asset is \( S = (S_t), \ t \in [0,1] \), a geometric Brownian motion that is \( S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t}\). It satisfies the SDE \( dS_t = \sigma S_t dW_t \), with positive constants \( S_0, \ \sigma \). It means that the risky asset is seen under the martingale measure.

The well-known Black and Scholes problem without transaction costs is to hedge a payoff \( h(S_1) \), \( h \) being a continuous function of polynomial growth. The pricing function solves the terminal valued Cauchy problem

\[
\begin{cases}
    C_t(t,x) + \frac{\sigma^2}{2}x^2 C_{xx}(t,x) = 0, & t \in [0,1], \ x > 0, \\
    C(1,x) = h(x).
\end{cases}
\]

Its solution can be written as

\[
C(t,x) = \int_{-\infty}^{\infty} h \left( xe^{\rho y - \frac{\rho^2}{2}} \right) \varphi(y)dy
\]  

where \( \rho^2 = (1-t)\sigma^2 \) and \( \varphi \) is the standard Gaussian density.

Without transaction costs (\( \sigma = \tilde{\sigma} \)) the self–financed portfolio process reads

\[
V_t = C(0, S_0) + \int_0^t C_x(u, S_u) dS_u.
\]

In the Itô formula for \( C(t, S_t) \) the integral over \( dt \) vanishes and, therefore, \( V_t = C(t, S_t) \) for all \( t \in [0,1] \). In particular, \( V_1 = h(S_1) \): At maturity the portfolio \( V \) replicates the terminal payoff of the option. Modeling assumptions of the above formulation include frictionless market and continuous trading for instance.

However, an investor revises the portfolio at a finite set of dates

\[ T^n := \{ t_i \in [0,1], i = 0, \cdots, n \} \]

and keeps \( C_x(t_i, S_{t_i}) \) units of the stock until the next revision date. It is well known that this discretized model converges to the Black–Scholes one in the sense that the corresponding portfolio terminal value converges to the payoff as the number of revision dates tends to infinity.

2.2 Reminder about Leland’s strategy

We are now concerned with transaction costs. We directly work in a discrete time setting. Leland suggested to replace \( \sigma \) in the Cauchy problem above by a suitable modified volatility \( \tilde{\sigma} \). In the case where \( \tilde{\sigma} \) does not depend on \( t \), the solution \( \tilde{C} \) satisfies \( \tilde{C}(t,x) = C(t, x, \tilde{\sigma}) \), i.e. practitioners do not need to rectify their algorithms to compute
the strategy. Leland obtained an explicit expression of $\tilde{\sigma}$ by equalizing the transaction costs of the portfolio and the drift term generated by the additional term $\tilde{\sigma} - \sigma^2 > 0$ in the Ito expansion of the payoff $h(S_1) = \tilde{C}(1, S_1)$. In the general case, the pricing function can be written as

$$\tilde{C}(t, x) = \int_{-\infty}^{\infty} h(x e^{\rho t} y - (\rho t)^2 / 2) \varphi(y) dy$$

where

$$\left(\rho t\right)^2 := \int_t^1 \tilde{\sigma}_s^2 ds,$$

$$\tilde{\sigma}_t^2 := \sigma^2 + \sigma \sqrt{\nu k_n} \sqrt{8/\pi} \sqrt{f'(t)},$$

$\varphi$ is the Gaussian density and $g = f^{-1}$ is the revision date function.

2.3 A possible modification of Leland’s strategy

The practically interesting case $\alpha = 0$ (i.e., $k_0$ is constant), where is a systematic error attracted a lot of attention. Limit theorems were obtained by Granditz and Schachinger [6] and Pengamenshchikov [11]. Zhao and Ziemba [13], [14] provides a numerical study of the limiting error for practical values of parameters. Sekine and Yano, [12] suggested some scheme to reduce it. In the paper [11] a modification of the Leland strategy was suggested for the call option eliminating the limiting error. Unfortunately, the approach is based on the explicit formulae and, seemingly, cannot be easily generalized for more general pay-off functions. Our modification of the Leland strategy has the following features:

1) we use the same enlarged volatility;
2) the initial value of the portfolio $V_0^n$ is exactly the same than for the initial Leland strategy (see [11] where the behavior of $V_0^n$ is studied as $n \to \infty$ and a method is suggested to lower the option price);
3) the only difference is at the revision dates $t_i$; We apply not the modified “delta” of the Black-Scholes formula with enlarged volatility, but correct it on the basis of previous revisions, see the formula (2.7). This is a technical modification of Leland’s strategy which is difficult to economically interpret but has the advantage to release the limiting error.
In the model with proportional transaction costs and a finite number of revision
dates the current value of the portfolio process at time $t$ is described as
\[
V^n_t := V^n_0 + \int_0^t D^n_u dS_u - \sum_{t_i < t} k_0 S_{t_i} |D^n_{t+1} - D^n_t| \tag{2.6}
\]
where $D^n$ is a piecewise-constant process with $D^n = D^n_0$ on the interval $(t_{i-1}, t_i]$,
$t_i = t^n_i$, $i \leq n$, are the revision dates, and $D^n_0$ are $\mathcal{F}_{t_{i-1}}$-measurable random variables.

Recall that the transaction costs coefficient is a constant $k_0 > 0$ (that is $\alpha = 0$ in
the Leland model) and the dates $t_i$ are defined by a function $g$, namely $t_i = g(\frac{i}{n})$. Let
us denote by $f$ the inverse of $g$. Set for all $i_0 < n$
\[
J^n_{i_0}(t) = \{i \geq i_0, \ t_i \leq t, \ t_i \in T^n\}
\]
and let us define the dates
\[
t^n_i(t) = t_{(n-1)\land \max J^n_{i_0}(t)}
\]
\[
t^n_1(t) = t_1 + (n-1)\land \max J^n_{i_0}(t).
\]

The “enlarged volatility”, depending on $n$, is given by the formula (2.5). We modify
the usual Leland strategy (see [2]) by considering the process $D^n$ with
\[
D^n_i := \tilde{C}_x(t_i, S_{t_i-1}) - \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \tilde{C}_{xt}(u, S_{t_{j-1}}) du. \tag{2.7}
\]
Moreover, let us define $K^n_t := \sum_{i \in J^n_{i_0}(t)} \Delta K^n_{t_i}$ where $\Delta K^n_{t_0} := 0$ and for $i \geq 1$,
\[
\Delta K^n_{t_i} := - \int_{t_{i-1}}^{t_i} \tilde{C}_{xt}(u, S_u) du. \tag{2.8}
\]
In the same way, we set $L^n_t := \sum_{i \in J^n_{i_0}(t)} \Delta L^n_{t_i}$ where $\Delta L^n_{t_0} := 0$ and for $i \geq 1$,
\[
\Delta L^n_{t_i} := - \int_{t_{i-1}}^{t_i} \tilde{C}_{xt}(u, S_{t_{i-1}}) du. \tag{2.9}
\]

2.4 Assumptions and notational conventions

Throughout the paper, we adopt the following rules:

(i) we will often omit the indexes $n$ and the variable $t$ (especially in the appendix)
  when there is no ambiguity;
(ii) the constants $C$ appearing in the various inequalities is independent of $n$ and may
  change from one line to the next;
(iii) we use the classical Landau notations $O$ and $o$. These quantities will be always deterministic.

As shown in [3], recall that the solution $\tilde{C}$ of the Cauchy problem we consider is infinitely differentiable on $[0,T) \times (0,\infty)$. We use the abbreviations $\delta_t := \tilde{C}_x(t,S_t)$, $\gamma_t := \tilde{C}_{xx}(t,S_t)$. We denote by $(\delta^i_t)$, the process equal to $\delta_{t_i}$ on the interval $[t_i^n, t_{i+1}^n)$ and $(\gamma^i_t)$ is defined similarly. For an arbitrary process $H$, we set $\Delta H_i := H_{t_i} - H_{t_{i-1}}$.

We will work under the following assumptions:

(A1) The function $g$ has the following form:

$$g(t) = 1 - (1 - t)^{\mu}, \quad \mu \in \left[1, \frac{3 + \sqrt{57}}{8}\right).$$

(A2) $h$ is a convex and continuous function on $[0, \infty)$ which is twice differentiable except the points $K_1 < \cdots < K_{p_h}$ where $h'$ and $h''$ admit right and left limits; $|h''(x)| \leq Mx^{-\beta}$ for $x \geq K_{p_h}$ where $\beta \geq 3/2$.

Assumption (A1) is not too restrictive. A trader can in particular choose $\mu = 1$ to balance its portfolio periodically. However, as we will see, it is more preferable to increase $\mu$ to obtain a better rate of convergence.

Note that $f(t) = 1 - (1 - t)^{1/\mu}$, hence the derivative $f'$ for $\mu > 1$ explodes at the maturity date and so does the enlarged volatility. We define the increasing function

$$p := p(\mu) := \frac{\mu - 1}{4(1 + \mu)}.$$

Under Assumption (A1), we have $0 \leq p < 1/16$.

In the sequel, will frequently appear the quantity

$$\mathcal{Q}(\mu) = \frac{\mu^{1/2 - 2p(1 + \mu)^{4p}}}{2^{4p} \left(\sqrt{8/\pi}\right)^{4p + 1}}.$$

3 Main Result

In [2], it is proven that $V_1^n$ converges in probability to $h(S_1)$. We recall this result:

**Theorem 3.1** Let $k_0 > 0$. Suppose that Assumption (A2) hold and $g' > 0$, $g \in C^2[0,1]$. Then

$$P\text{-lim}_n V_1^n = h(S_1).$$

(3.10)
Our main result here provides the rate of convergence for a specific family of revision dates functions including the uniform grid (i.e. \( g(t) = t \)) and identifies the associated limit distribution of the deviation:

**Theorem 3.2** Consider the portfolio \( V^n \) defined by (2.6) and (2.7) under Assumptions (A1) and (A2). The following convergence then holds:

\[
\eta^{\frac{n}{4}} + p(V^n_1 - h(S_1)) \xrightarrow{d} Z, \quad (3.11)
\]

where the law of \( Z \) is a mixed gaussian distribution, i.e. \( Z = \eta N \) where \( N \) is a standard normal independent of \( \eta \) and

\[
\eta^2 := Q(\mu)(k_0\sigma)^{1-4p} S_1^2 \times \\
\int_0^\infty x^{4p} \left\{ \left( \int_x^\infty J(y, S_1) dy \right)^2 + \left( 1 - \frac{2}{\pi} \right) \tilde{J}(x, S_1)^2 \right\} dx,
\]

**Fig. 1** Revision dates with \( f(t) = 1 - (1-t)^\frac{1}{\mu} \), \( \mu = 1.5 \) and \( n = 10 \).
and
\[ J(x, S_1) := \frac{1}{2x} \int_{-\infty}^{\infty} h'(S_1 e^{\sqrt{y^2 + x^2}})(-y^2 - \sqrt{y} + 1) \varphi(y) dy \] (3.12)
\[ \tilde{J}(x, S_1) := \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} h'(S_1 e^{\sqrt{y^2 + x^2}})y \varphi(y) dy. \] (3.13)

Moreover \( n^{1/2} \mathbb{E} (V^n_1 - h(S_1))^2 \xrightarrow{n \to \infty} \mathbb{E} \eta^2. \)

Observe that \( EZ = 0. \) In the proof, \( Z \) will be identified by its characteristic function given by \( \phi_Z(s) = \mathbb{E} e^{-\frac{1}{2} s^2 \eta^2}. \) As we can see, concentrating the revision dates near the horizon date \( (p > 0) \) improves the convergence rate. Actually, we can observe that near \( T = 1, \) the derivative \( f' \) explodes if \( p > 0 \) and so increases the modified volatility, which confirms the main Leland idea; Artificially increase the volatility to compensate for transaction costs. The proof of the theorem above is given in Section 5; To do so, we decompose the difference \( n^{1/2} + 2p (V^n_1 - \tilde{C}(t, S_1)) \) into a martingale which converges in \( L^2 \) and a residual term tending to 0 at \( T = 1. \) We conclude with \( h(S_1) = \tilde{C}(1, S_1). \)

4 Auxiliary results

4.1 Geometric Brownian motion and related quantities

In the sequel, we shall use the decomposition given by Ito formula
\[ \tilde{C}_x(t, S_1) = \tilde{C}_x(0, S_0) + \tilde{M}_t^n + \tilde{A}_t^n \] (4.14)
where
\[ \tilde{M}_t^n := \int_0^t \sigma_u S_u \tilde{C}_{xx}(u, S_u) dW_u, \]
\[ \tilde{A}_t^n := \int_0^t \left[ \tilde{C}_{xt}(u, S_u) + \frac{1}{2} \sigma_u^2 S_u^2 \tilde{C}_{xxx}(u, S_u) \right] du. \]
The process \( \tilde{M}_t^n \) is a square integrable martingale on \([0, 1]\) by virtue of [2].

We set for \( u < v \)
\[ \varepsilon_u^v := \frac{S_v}{S_u} - 1, \]
and
\[ \left[ \varepsilon_u^v \right]_c = \mathbb{E} \left( |\varepsilon_u^v| \right) - |\varepsilon_u^v|. \]
\[ (\varepsilon_u^v)^2 := (\varepsilon_u^v)^2 \text{sgn} \varepsilon_u^v. \]

In the sequel, we will use several times the following basic results.
Lemma 4.1 For all \( i \) the following inequalities and expansions hold:

\[
\begin{align*}
\mathbb{E} \left( \mathcal{L}_u \right)^{2m} & \leq C_m (v-u)^m, \quad u \leq v \\
\mathbb{E} \left( \mathcal{L}_{t_{i-1}}^{2} \right) & = \sigma^2 \Delta t_i (1 + o(1)) \\
\mathbb{E} \left[ \mathcal{L}_{t_{i-1}}^{2} \right] \mathbb{E} \left[ \mathcal{L}_{t_{i-1}}^{2} \right] & = \left( 1 - \frac{2}{\pi} \right) \sigma^2 \Delta t_i (1 + o(1)) \\
\mathbb{E} \left[ \mathcal{L}_{t_{i-1}}^{2} \right] \mathbb{E} \left[ \mathcal{L}_{t_{i-1}}^{2} \right] & = k (\Delta t_i)^{3/2} \left( 1 + o(n^{-1/4}) \right).
\end{align*}
\]

Proof We refer to [1] or [4]. For the sake of completeness we recall the proof of the last one. Let us notice the equality in law

\[
\mathbb{E} \left\{ \mathcal{L}_{t_{i-1}}^{2} \right\} = \exp \left\{ \sigma \sqrt{\Delta t_i} \xi - \sigma^2 \Delta t_i/2 \right\} - 1 \leq \exp \left\{ \sigma \sqrt{\Delta t_i} \xi - \sigma^2 \Delta t_i/2 \right\} - 1,
\]

where \( \xi \) is the standard Gaussian variable. Since \( \xi \) and \( -\xi \) has the same law, this yields

\[
\mathbb{E} \left\{ \mathcal{L}_{t_{i-1}}^{2} \right\} = \mathbb{E} \left\{ \left( e^{u \xi - u^2/2} - 1 \right)^2 - \left( e^{-u \xi - u^2/2} - 1 \right)^2 \right\} 1_{\xi \geq u/2}
\]

where \( u = \sigma \sqrt{\Delta t_i} \). Moreover, we have the inequality

\[
\mathbb{E} \left( e^{-u \xi - u^2/2} - 1 \right)^2 1_{|\xi| \leq u/2} \leq u^4.
\]

From [5], we recall that

\[
\mathbb{E} \left( e^{u \xi - u^2/2} - 1 \right)^2 - \left( e^{-u \xi - u^2/2} - 1 \right)^2 \right\} 1_{\xi \geq u/2} = \frac{2}{\sqrt{2\pi}} u^3 + O(u^4).
\]

We then conclude.

4.2 Basic results concerning the revision dates

The function \( \rho_t \) decreases from \( \rho_0 \) to 0. The following useful bounds are obvious:

\[
\rho_t^2 \geq (\sigma^2 + cn^2)(1-t) \quad (4.16)
\]

\[
\rho_t^2 \leq \sigma^2 (1-t) + \sigma k_0 n^{\frac{1}{2}} \sqrt{8/\pi (1-t)} \left( 1 - f(t) \right)^{\frac{1}{2}}. \quad (4.17)
\]

Moreover, it is straightforward that

\[
\rho_t^2 \geq cn^{\frac{1}{2}} \sqrt{f(t)} (1-t), \quad (4.18)
\]
provided that \( f' \) is no decreasing. Note that there is a constant \( C \) independent of \( n \) such that for all \( i \leq n - 1 \), \( \frac{1 - t_{i-1}}{1 - t_i} \leq C \). From there we deduce

\[
\frac{\Delta t_i}{1 - t_i} \leq C. 
\] (4.19)

We shall often use the inequality

\[
\sum_{i=1}^{n-1} \frac{\Delta t_i}{1 - t_i} \leq C \log(n)
\]

where \( C \) is a constant independent of \( n \).

**Lemma 4.2** Fix \( x > 0 \) and \( t := t(n, x) \in [0, 1) \) such that \( x = \rho^2_t \). Set \( x_{i-1} = \rho^2_{t_{i-1}} \) and \( x_i = \rho^2_{t_i} \) where \( t_{i-1} \), \( t_i \) are such that \( t \in [t_{i-1}, t_i) \). Then, \( x \in (x_i, x_{i-1}] \) with \( |x_{i-1} - x_i| \leq c n^{-1/2} \), \( c \) is a constant. There exists a constant \( C > 0 \) such that

\[
\frac{\Delta t_i n^{1/2+2p}}{x_{i-1} - x_i} \leq C (x + 1). \] (4.20)

Moreover, for a given \( x \), \((1 - t) \leq cn^{-1/2}x \to 0\) as \( n \to \infty \) and

\[
\frac{\Delta t_i n^{1/2+2p}}{x_{i-1} - x_i} \xrightarrow{n \to \infty} \frac{Q(\mu)x^p}{(\sigma k_0)^{2p+1}}. \] (4.21)

**Proof** Let us write

\[
\frac{\Delta t_i n^{1/2+2p}}{x_{i-1} - x_i} = \frac{n^{2p}}{\sigma^2 n^{-1/2} + \sigma k_0 \sqrt{8/\pi} \frac{1}{\Delta t_i} \int_{t_{i-1}}^{t_i} \sqrt{f'(u)}du} \]

\[
\xrightarrow{n \to \infty} \frac{\sigma^2 n^{-1/2} + \int_{t_{i-1}}^{t_i} \sqrt{f'(u)}du}{\sigma k_0 \sqrt{8/\pi} \sqrt{f'(t_i)}}
\]

where \( t_i \in [t_{i-1}, t_i] \). Moreover

\[
x = \rho^2_t = \sigma^2 (1 - t) + \sigma k_0 \sqrt{8/\pi} n^{1/2} \int_0^1 \sqrt{f'(u)}du
\]

\[
= \sigma^2 (1 - t) + \sigma k_0 \sqrt{8/\pi} n^{1/2} \frac{2^1/2}{1 + \mu} (1 - t)^{1+\mu}
\]

and

\[
1 - t = \left( \frac{x - \sigma^2 (1 - t) + \sigma k_0 \sqrt{8/\pi} n^{1/2} \frac{2^1/2}{1 + \mu} (1 - t)^{1+\mu}}{\sigma k_0 \sqrt{8/\pi} n^{1/2} \frac{2^1/2}{1 + \mu}} \right)^{\frac{1}{1+\mu}}
\]

Note that \( x \geq cn^{1/2}(1 - t) \) so that \((1 - t) \leq cn^{-1/2}x \to 0\). In a similar way, we have

\[
x_{i-1} - x_i = \rho^2_{t_{i-1}} - \rho^2_{t_i} = \sigma^2 \Delta t_i + cn^{1/2} \sqrt{f'(t_i)} \Delta t_i
\]
where $t_i \in [t_{i-1}, t_i]$. We deduce that $x_{i-1} - x_i = \sigma^2 \Delta t_i + cn^{1/2} \sqrt{f'(t_i) g'(\theta_i)n^{-1}}$ where $\theta_i \in [(i-1)/n, i/n]$. Moreover,

$$\sqrt{f'(t_i) g'(\theta_i)} = \frac{g'(\theta_i)}{\sqrt{g'(f(t_i))}}$$

is bounded since $f(t_i) \in [(i-1)/n, i/n]$, $i \leq n-1$. Hence there is a constant $c$ satisfying $x_{i-1} - x_i \leq cn^{-1/2}$. Since $\rho$ is decreasing, $x \in [x_i, x_{i-1}]$.

Eventually, $t_i \in [t_{i-1}, t_i]$, is such that $\tilde{x}_i = \rho^2 t_i \in [x_i, x_{i-1}]$ and $\tilde{x}_i \to x$. Similarly we have

$$1 - \tilde{t}_i = \left(\tilde{x}_i - \sigma^2 (1 - \tilde{t}_i) 1 + \mu\right)^{\frac{2p}{1+p}}$$

which yields

$$\sqrt{f'(t_i)} = \mu^{-1/2} \left(\tilde{x}_i - \sigma^2 (1 - \tilde{t}_i) 1 + \mu\right)^{\frac{1-p}{1+p}}$$

and

$$\frac{\Delta t_i n^{1/2+2p}}{x_{i-1} - x_i} \underset{n \to \infty}{\sim} \frac{n^{2p}}{\sigma k_0 \sqrt{S/\pi} \sqrt{f'(t_i)}}$$

$$\quad \sim \frac{n^{2p}}{\sigma k_0 \sqrt{S/\pi}} \mu^{1/2} \left(\frac{\sigma k_0 \sqrt{S/\pi} n^{1/2} 2 \mu^{1/2}}{\tilde{x}_i - \sigma^2 (1 - \tilde{t}_i) 1 + \mu}\right)^{\frac{1-p}{1+p}}$$

$$\quad \sim \frac{n^{2p}}{\sigma k_0 \sqrt{S/\pi}} \mu^{1/2} \left(\frac{\sigma k_0 \sqrt{S/\pi} 2 \mu^{1/2}}{\tilde{x}_i - \sigma^2 (1 - \tilde{t}_i) 1 + \mu}\right)^{\frac{1-p}{1+p}}.$$ 

Since $\tilde{x}_i \to x$ and $\tilde{t}_i \to 0$, we deduce that

$$\frac{\Delta t_i n^{1/2+2p}}{x_{i-1} - x_i} \underset{n \to \infty}{\sim} \frac{1}{\sigma k_0 \sqrt{S/\pi}} \mu^{1/2} \left(\frac{\sigma k_0 \sqrt{S/\pi} 2 \mu^{1/2}}{x} \frac{1}{1 + \mu}\right)^{\frac{1-p}{1+p}}.$$ 

Since $0 \leq (\mu - 1)/(1 + \mu) < 1$, we also find a constant $c$ such that

$$\frac{\Delta t_i n^{1/2+2p}}{x_{i-1} - x_i} \leq c \left|\tilde{x}_i - \sigma^2 (1 - \tilde{t}_i)\right|^{\frac{2p}{1+p}} \leq c (x + 1),$$

which concludes the proof.

We now stress an important remark regarding a slight abuse of notation repeatedly used along the paper.

**Remark 4.3** Throughout the sequel, we shall often use the change of variable $x = \rho^2 t$ with $dx = -\rho^4 dt$. For ease of notation, we will use the abuse of notation $t$ instead of $t(n, x) := (\rho^2)^{-1}(x)$ when applying this change of variable in an integral.
Similarly, a direct computation yields the following lemma.

**Lemma 4.4** Set \( y > 0 \) and \( v := v(n,y) \) such that \( y = \rho^2 \). There exists a constant \( C > 0 \) such that

\[
\frac{(1 - v)n^{1/2 + 2p}}{y} \leq Cy.
\]

Moreover, for a given \( y \), \( (1 - v) \leq cn^{-1/2}y \to 0 \) as \( n \to \infty \) and

\[
\frac{(1 - v)n^{1/2 + 2p}}{y} \overset{n \to \infty}{\to} \mu^{-1/2 - 2p}(1 + \mu)^{4p + 1}y^{4p} \quad 2^{4p} \left( \sigma k_0 \sqrt{S'/\pi} \right)^{4p + 1}.
\]

### 5 Proof of the limit theorem

The proof is divided into three parts. In Step 1 we split the hedging error into a martingale part \( M \) and a residual part \( \epsilon \). In Step 2 we show that the residual terms tend to 0 in \( L^2(\Omega) \) as \( n \) tends to infinity. The convergence rate \( n^{1/4 + p} \) is generated by the revision function \( f \) defining the modified volatility. We identify in Step 3 the asymptotic distribution of the martingale \( n^{1/4 + p}M^n \) and we conclude the proof of the main result.

#### 5.1 Step 1: Splitting of the hedging error

Comparing Expression (2.6) with the Ito expansion of \( h(S_1) = \tilde{C}(1,S_1) \) yields the following decompositions. The hedging error reads

\[
V^n_1 - h(S_1) = M^n_1 + \epsilon^n_1
\]

(5.22)

where for all \( n \in \mathbb{N} \), \( M^n \) is a martingale of terminal value

\[
M^n_1 := k_0 \sum_{i=1}^{n-1} \gamma_{i-1} S_{k_{i-1}}^2 \left[ \xi^i_{k_{i-1}} \right] + \int_0^1 K_u dS_u.
\]

(5.23)

The residual term can be split as

\[
\epsilon^n_1 = R^n_0(t) + R^n_1(t) + R^n_2(t) + R^n_3(t)
\]

(5.24)
where

\[
R_n^0(t) := k_0 \sum_{i \in J_n(t)} \gamma_{t_{i-1}} S^2_{t_{i-1}} \left( \sigma \sqrt{\frac{2}{\pi}} \sqrt{n f'(t_{i-1}) \Delta t_i} - E \left| \frac{\epsilon_{\Delta t_i}}{\Delta t_i} \right| \right) \quad (5.25)
\]

\[
R_n^1(t) := \int_0^t (\delta_n u - \delta u) dS_u,
\]

\[
R_n^2(t) := k_0 \sum_{i \in J_n(t)} (|\Delta \delta_n t_i + \Delta K_n^0| - |\Delta \delta_n t_i + \Delta L_n^0|) S_t,
\]

\[
R_n^3(t) := \int_0^t (L_n u - K_n u) dS_u.
\]

5.2 Step 2: The mean square residue tends to 0 with rate \(n^{1/2+2p}\)

The most technical part of this paper is the following. The deviation of the approximating portfolio from the payoff has been written in an integral form by virtue of the Ito formula. The “real world” portfolio may be interpreted as a discrete-time approximation of the theoretical portfolio \(\tilde{C}(t, S_t)\) yielding the residual terms above. Consequently, the following analysis is mainly based on Taylor approximations involving the successive derivatives of \(\tilde{C}\) and so heavily utilizes estimates of the appendix. Standard tools from stochastic calculus are also frequently used.

**Theorem 5.1** The following convergence holds:

\[
n^{1/2+2p}E \left( \epsilon_n^0 \right)^2 \xrightarrow{n \to \infty} 0.
\]

To prove this theorem, we show the suitable convergence to 0 concerning the \(R_j, \quad 0 \leq j \leq 3\).

**Lemma 5.2** \(n^{1/2+2p}E \left( R_n^0 \right)^2 \xrightarrow{n \to \infty} 0\).

**Proof** We have

\[
E \left| \epsilon_n^{t_i} \right| = 4\Phi \left( \frac{\sigma \sqrt{\Delta t_i}}{2} \right) - 2 = \sigma \sqrt{\frac{2}{\pi}} \sqrt{\Delta t_i} + (\Delta t_i) o(1),
\]

\[
\sigma \sqrt{\frac{2}{\pi}} n^{1/2} \sqrt{f'(t_{i-1}) \Delta t_i} = \sigma \sqrt{\frac{2}{\pi}} \sqrt{\Delta t_i} \epsilon_i
\]

where \(\epsilon_i = n^{1/2} \sqrt{\Delta t_i} \sqrt{f'(t_{i-1})}\) verifies \(|\epsilon_i - 1| \leq \left( \frac{\Delta t_i}{\epsilon_i} \right)^2 \) by virtue of Lemma 6.11. Hence, there is a constant \(C > 0\) such that:

\[
\sup_t |R_n^0(t)| \leq CK_0 \sum_{i=1}^{n-1} \gamma_{t_{i-1}} S^2_{t_{i-1}} \left( \frac{\Delta t_i}{1 - t_i} \right)^2.
\]
Therefore where \( \Delta t \) is small, we write:

\[
\text{where}
\]

the Wiener process, we write:

\[
\lim_{n \to \infty} \frac{1}{n^{3/4}} \sum_{i=1}^{n-1} \frac{(\Delta t_i)^{3/4}}{(1-t_i)^{3/4}} \leq C \frac{n^{1/2+\varepsilon}}{n^{3/4}} \log n \to 0.
\]

A Taylor formula suggests to write the following splitting:

\[
R_{i0}^n = \sigma \left( R_{i0}^{n-1} - R_{i1}^{n-1} - R_{i2}^{n-1} + 2R_{i3}^{n-1} \right),
\]

where

\[
R_{i0}^{n-1}(t) := \sum_{i \leq n} \gamma_{t_{i-1}} S_{t_{i-1}}^2 \int_{t_{i-1}}^{t_{i} \wedge t} \mathcal{C}_x(t_{i-1}, S_{t_{i-1}}) S_u dW_u,
\]

\[
R_{i1}^{n-1}(t) := \frac{1}{2} \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_{i} \wedge t} \mathcal{C}_x(t_{i-1}, S_{t_{i-1}}) \left( \mathcal{C}_x(t_{i-1}, S_{t_{i-1}}) \right)^2 \frac{S_u}{S_{t_{i-1}}} dW_u,
\]

\[
R_{i2}^{n-1}(t) := \frac{1}{2} \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_{i} \wedge t} \mathcal{C}_y(t_{i-1}, S_{t_{i-1}})(u - t_{i-1})^2 \frac{S_u}{S_{t_{i-1}}} dW_u,
\]

\[
R_{i3}^{n-1}(t) := \frac{1}{2} \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_{i} \wedge t} \mathcal{C}_y(t_{i-1}, S_{t_{i-1}})(u - t_{i-1}) S_u dW_u.
\]

Lemma 5.3

\[
n^{1/2+2\varepsilon} \mathbb{E} \left( \sup_{t \in [0,1]} R_{i0}^n(t) \right)^2 \to 0. \quad (5.27)
\]

\[
\textbf{Proof} \quad \text{The Doob inequality yields } n^{1/2+2\varepsilon} \mathbb{E} \sup_t (R_{i0}^n(t))^2 \leq 4n^{1/2+2\varepsilon} \mathbb{E} (R_{i0}^n(1))^2 \text{ where}
\]

the r.h.s tends to 0 as shown below. Indeed, by the independence of the increments of the Wiener process, we write:

\[
\mathbb{E} \left( R_{i0}^n(1) \right)^2 = \sigma^2 \sum_{i=1}^{n} A_i \int_{t_{i-1}}^{t_i} \mathbb{E} \left( \mathcal{C}_x(t_{i-1}) \right)^2 \frac{S_u}{S_{t_{i-1}}} du
\]

where \( A_i := \mathbb{E} \mathcal{C}_x^2(t, S_t) S_t^4 \). It is easy to check the following asymptotic:

\[
\mathbb{E} \left( \mathcal{C}_x(t_{i-1}) \right)^2 \frac{S_u}{S_{t_{i-1}}} = \sigma^2 (u - t_{i-1}) + (u - t_{i-1}) O(n^{-1}).
\]

Therefore

\[
\mathbb{E} \left( R_{i0}^n(1) \right)^2 = \frac{\sigma^4}{2} \sum_{i \leq n} A_{i-1} (\Delta t_i)^2 (1 + O(n^{-1}))
\]

where \( \Delta t_i = g'(\theta_i)/n \) with \( \theta_i \in [(i-1)/n, i/n] \). We then deduce...
Applying the Lebesgue theorem, we deduce that
\[
\Lambda_t u(x) = \frac{\sigma^4(1 + O(n^{-1}))}{2} n^{-1} \sum_{i \leq n} \Delta t_i \frac{\Delta n^{\frac{1}{2} + 2p}}{x_{i-1} - x_i} (x_{i-1} - x_i)
\]
where \(x_i = \rho_{tn}^i\). So, we have:
\[
n^\frac{1}{2} + 2p \mathbf{E} (R_{10}^n(1))^2 = \frac{\sigma^4(1 + O(n^{-1}))}{2} n^{-1} \int_0^1 f_n(x) dx
\]
where
\[
f_n(x) = \sum_{i = 1}^n \frac{\Delta t_i}{n} \frac{\Delta n^{\frac{1}{2} + 2p}}{x_{i-1} - x_i} \delta(x_{i-1} - x_i)
\]
Let us remark the abuse of notations \(\rho_{0n}^i = \rho_{0n}^i\) and \(t_i = t(n, x_i)\) as previously mentioned. First, let us show that \(f_n\) satisfies the dominated convergence bound condition. If \(x \in (x_i, x_{i-1}]\) then from Corollary 6.6, we have
\[
0 \leq \Lambda_{t_i} \leq \frac{C}{\sqrt{x_{i-1}}} e^{-x_{i-1}/4} \leq \frac{C}{\sqrt{x}} e^{-x/4}.
\]
Thus, from (4.20) we obtain \(f_n(x) \leq \frac{C}{\sqrt{n}} e^{-x/4}(1 + x)\).

Regarding the pointwise convergence of \(f_n\), for a given \(x \in (x_i, x_{i-1}]\), there exists \(u \in [t_{i-1}, t_i] \) such that \(x = \rho_{tn}^i \geq cn^{\frac{1}{2}}(1 - u)\). It follows not only \(u \rightarrow 1\) but also \(t_i, t_{i-1} \rightarrow 1\). Recall that \(\Delta t_i = g(\theta_i)n^{-1}\), where \(\theta_i \in [(i - 1)/n, i/n]\). Thus \(g(\theta_i) \rightarrow 1\) and \(\theta_i \rightarrow 1\) since \(f\) is continuous. Therefore \(\Delta t_i n \rightarrow g'(1) \in (0, 1]\). Moreover, note that
\[
\Lambda_{t_{i-1}} = \frac{1}{x_{i-1}} \int_{-\infty}^{\infty} e^{2\sigma \sqrt{1 - z^2 - \sigma^2} \Delta t_{i-1} Y(z)} \varphi(z) dz
\]
where
\[
Y(z) = \left( \int_{-\infty}^{\infty} h' \left( e^{\sigma \sqrt{1 - z^2 - \sigma^2} \Delta t_{i-1} Y(z)} \right) y \varphi(y) dy \right)^2.
\]
Applying the Lebesgue theorem, we deduce that \(\Lambda_{t_{i-1}}\) converges to
\[
A(x) := \frac{1}{2} \int_{-\infty}^{\infty} e^{2\sigma z - \sigma^2} \left( \int_{-\infty}^{\infty} h' \left( e^{\sigma z - \sigma^2 + \sqrt{2} \varphi(y)} dy \right) \varphi(z) dz \right)^2 \varphi(z) dz.
\]
Finally, together with (4.21), \(f_n \xrightarrow{n \to \infty} 0\) a.e. if \(\mu > 1\) and \(f_n \xrightarrow{n \to \infty} f_\infty\) a.e. where \(f_\infty\) is integrable if \(\mu = 1\). We then apply the Lebesgue theorem to conclude the following limit
\[
\frac{\sigma^4(1 + O(n^{-1}))}{2} n^{-1} \int_0^1 f_n(x) dx \xrightarrow{n \to \infty} 0.
\]

**Lemma 5.4** \(n^{\frac{1}{2} + 2p} \mathbf{E} (\sup_t R_{10}^n(t))^2 \xrightarrow{n \to \infty} 0\).
Moreover, using Lemma 6.18 and the Cauchy-Schwarz inequality, we deduce that
\[ n^{\frac{1}{2} + 2p} \mathbb{E} \left( R_{11}^n(1) \right)^2 = n^{\frac{1}{2} + 2p} \sum_{i=1}^{n-1} \mathbb{E} \bar{C}^2_X(t_{i-1}, S_{t_{i-1}}) S^2_{t_{i-1}} \int_{t_{i-1}}^{t_i} (u-t_{i-1})^2 \mathbb{E} \left( \frac{S_u}{S_{t_{i-1}}} \right)^2 du. \]

It follows that
\[ n^{\frac{1}{2} + 2p} \mathbb{E} \left( R_{11}^n(1) \right)^2 \leq cn^{\frac{1}{2} + 2p} \sum_{i=1}^{n-1} \mathbb{E} \bar{C}^2_X(t_{i-1}, S_{t_{i-1}}) S^2_{t_{i-1}} (\Delta t_i)^3 \leq cn^{-\frac{1}{2} + 2p} \log n, \]

since Corollary 6.13 gives
\[ \mathbb{E} \bar{C}^2_X(t_{i-1}, S_{t_{i-1}}) S^2_{t_{i-1}} \leq \frac{n^{\frac{1}{2} + 2p} f(t_{i-1})}{(1-t_{i-1})^2} \]

where \( n f(t_{i-1}) \Delta t_i \) is bounded. We then conclude.

**Lemma 5.5** \( n^{\frac{1}{2} + 2p} \mathbb{E} \left( \sup_t R_{12}^n(t) \right)^2 \rightarrow 0 \) as \( n \rightarrow \infty \).

**Proof** As previously, we have the Doob inequality \( \mathbb{E} \left( \sup_t R_{12}^n(t) \right)^2 \leq 4 \mathbb{E} \left( R_{12}^n(1) \right)^2 \)
and the equality
\[ 4 \mathbb{E} \left( R_{12}^n(1) \right)^2 = \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} \mathbb{E} \left( \bar{C}_{XX}(i_{i-1}, S_{t_{i-1}}) S^2_{t_{i-1}} (1 - \frac{S_t}{S_{t_{i-1}}} \right)^4 \left( \frac{S^2_{t_{i-1}}}{S^2_{t_{i-1}}} \right) dt. \]

From (6.68), there exists a constant \( C \) such that:
\[ \mathbb{E} \bar{C}^4_{XX} (t_{i-1}, S_{t_{i-1}}) \leq \frac{C}{p^4}. \]

Using the Cauchy-Schwarz inequality and (4.15) with \( m = 8 \), we deduce that
\[ n^{\frac{1}{2} + 2p} \mathbb{E} \left( R_{12}^n(1) \right)^2 \leq C n^{\frac{1}{2} + 2p} \sum_{i=1}^{n-1} \frac{(\Delta t_i)^3}{n(1-t_i)^2} \leq C n^{2p} \log n \frac{n}{n^2}, \]

which proves the desired convergence to 0.

**Lemma 5.6** \( n^{\frac{1}{2} + 2p} \mathbb{E} \left( \sup_t R_{13}^n(t) \right)^2 \rightarrow 0 \) as \( n \rightarrow \infty \).

**Proof** We still consider the Doob inequality \( \mathbb{E} \left( \sup_t R_{13}^n(t) \right)^2 \leq 4 \mathbb{E} \left( R_{13}^n(1) \right)^2 \)
and
\[ 4 \mathbb{E} \left( R_{13}^n(1) \right)^2 \leq \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} \mathbb{E} \left( \bar{C}_{XT}(i_{i-1}, S_{t_{i-1}}) (t-t_{i-1})^4 S^2_t \right) dt. \]

Moreover, using Lemma 6.18 and the Cauchy-Schwarz inequality, we deduce that
\[ \mathbb{E} \left( \bar{C}_{XX}(i_{i-1}, S_{t_{i-1}}) S^2_t \right) \leq \frac{c}{(1-t_i)^2}. \]
Then, we obtain
\[ n^{\frac{1}{2}+2p} E \left( R_{14}(1) \right)^2 \leq C n^{\frac{1}{2}+2p} \sum_{i=1}^{n-1} \frac{(\Delta t_i)^5}{(1-t_i)^3} \leq C n^{-\frac{1}{2}+2p} \log n. \]

The conclusion follows.

**Lemma 5.7** \( n^{\frac{1}{2}+2p} E \left( \sup_t R_{14}^n(t) \right)^2 \xrightarrow{n \to \infty} 0. \)

**Proof** We use the Doob inequality \( E \left( \sup_t R_{14}^n(t) \right)^2 \leq 4E \left( R_{14}^n(1) \right)^2 \) and the equality
\[ 4E \left( R_{14}^n(1) \right)^2 = \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} E \left( S_{t_{i-1}}^{i} \tilde{C}_{x,t}^{2}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) \left( 1 - \frac{S_t}{S_{t_{i-1}}} \right)^2 \frac{S^2}{S_{t_{i-1}}^2} \right) \, dt. \]

From (6.69), we deduce that
\[ E \left( S_{t_{i-1}}^{i} \tilde{C}_{x,t}^{2}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) \left( 1 - \frac{S_t}{S_{t_{i-1}}} \right)^2 \frac{S^2}{S_{t_{i-1}}^2} \right) \leq c \frac{t - t_{i-1}}{(1-t_i)^3}. \]

Then,
\[ n^{\frac{1}{2}+2p} E \left( R_{14}^n(1) \right)^2 \leq c n^{\frac{1}{2}+2p} \sum_{i=1}^{n-1} \frac{(t_i - t_{i-1})^4}{(1-t_i)^3} \leq c n^{-\frac{1}{2}+2p} \log n. \]

Thus, we can conclude.

Let us now study the residual term \( R_{14}^n \). Again, a Taylor formula suggests to write the following splitting \( R_{14}^n = R_{20}^n + \cdots + R_{24}^n \) where
\[ R_{20}^n(t) := \sigma_k \sqrt{\frac{2}{\pi}} n^{\frac{1}{2}} \int_{t_n(t)}^{t} S_u^2 \gamma_u \sqrt{f(u)} \, du, \]
\[ R_{21}^n(t) := \sigma k n^{\frac{1}{2}} \sqrt{\frac{2}{\pi}} \sum_{i \in J_n^t(t)} \int_{t_{i-1}}^{t_i} \left( S_u^2 \gamma_u \sqrt{f(u)} - S_{t_{i-1}}^2 \gamma_{t_{i-1}} \sqrt{f(t_{i-1})} \right) \, du, \]
\[ R_{22}^n(t) := \sigma_k \sum_{i \in J_n^t(t)} \gamma_{t_{i-1}} |S_{t_i} - S_{t_{i-1}}| (S_{t_{i-1}} - S_{t_i}), \]
\[ R_{23}^n(t) := \sigma_k \sum_{i \in J_n^t(t)} \Theta_i (S_{t_i} - S_{t_{i-1}}), \]
\[ R_{24}^n(t) := \sigma_k \sum_{i \in J_n^t(t)} \Theta_i S_{t_{i-1}}, \]
\[ \Theta_i := \gamma_{t_{i-1}} |S_{t_i} - S_{t_{i-1}}| - |\tilde{C}_x(t_i, S_{t_i}) - \tilde{C}_x(t_{i-1}, S_{t_{i-1}}) + \Delta K_{1i}^n|. \]  \( \text{(5.28)} \)

**Lemma 5.8** \( n^{\frac{1}{2}+2p} E \left( R_{20}^n(1) \right)^2 \xrightarrow{n \to \infty} 0. \)
Proof We have
\[ n^{\frac{1}{2}+2p} (R_{20}(1))^2 = c \cdot n^{\frac{1}{2}+2p} \int_{\{u_x \leq 1\}} S_t^2 u v \sqrt{f(u)} \sqrt{f(v)} \, dudv. \]

We use the Cauchy–Schwarz inequality, Inequalities (6.6) and (4.18). From the explicit formula of \( f' \), we thus obtain
\[ n^{\frac{1}{2}+2p} (R_{20}(1))^2 \leq c \cdot n^{1+2p} \int_{\{u_x \leq 1\}} 4 \mu \left( 1 - \frac{u}{v} \right)^{\frac{5}{8} - 3/5} \left( 1 - \frac{v}{u} \right)^{\frac{5}{8} - 3/5} \, dudv. \]

Since \( \mu \in [1, 2] \),
\[ \frac{3}{4} + \frac{3}{4\mu} - (1 + 2p) = -3\mu^2 + 5\mu + 3 \]
so that we can conclude.

Lemma 5.9 \( n^{\frac{1}{2}+2p} (R_{21}(1))^2 \to 0 \) as \( n \to \infty \).

Proof Let us consider \( \Psi(t, x) := x^2 \tilde{C}_{xx}(t, x) \sqrt{f(t)} \). The Itô formula yields
\[ \Psi(t, S_t) = \Psi(t, S_{t-1}) + \int_{t-1}^{t} \frac{\partial \Psi}{\partial x} (u, S_u) \sigma S_u dW_u + \int_{t-1}^{t} \frac{\partial \Psi}{\partial t} (u, S_u) du \]
\[ + \frac{1}{2} \int_{t-1}^{t} \frac{\partial^2 \Psi}{\partial x^2} (u, S_u) \sigma^2 S_u^2 du, \]

where
\[ \frac{\partial \Psi}{\partial t} (t, x) = 2x \left[ \tilde{C}_{xt}(t, x) \sqrt{f(t)} + \tilde{C}_{xx}(t, x) \frac{f''(t)}{2 \sqrt{f(t)}} \right], \]
\[ \frac{\partial \Psi}{\partial x} (t, x) = 2x \tilde{C}_{xx}(t, x) \sqrt{f(t)}, \]
\[ \frac{\partial^2 \Psi}{\partial x^2} (t, x) = 2 \tilde{C}_{xx}(t, x) + 4x \tilde{C}_{xxx}(t, x) \sqrt{f(t)}. \]

If we set \( X_t = S_t^2 \tilde{C}_{xx}(t, x) \sqrt{f(t)} \) then \( dX_t = \zeta dt + \beta dW_t \) where
\[ \zeta_t = \frac{\partial \Psi}{\partial t} (t, S_t) + \frac{1}{2} \frac{\partial^2 \Psi}{\partial x^2} (t, S_t) \sigma^2 S_t^2, \quad \beta_t = \frac{\partial \Psi}{\partial x} (t, S_t) \sigma S_t. \]

We write \( n^{\frac{1}{2}+p} R_{21}^n (t) = A^n_t + B^n_t \) with
\[ A^n_t := \sigma_0 n^{\frac{1}{2}+p} \sqrt{\frac{2}{\pi}} \sum_{1 \leq j \leq \frac{1}{4} \sqrt{n}} \int_{t-1}^{t} \left( \int_{t-1}^{t} \beta_u dW_u \right) dt, \]
\[ B^n_t := \sigma_0 n^{\frac{1}{2}+p} \sqrt{\frac{2}{\pi}} \sum_{1 \leq j \leq \frac{1}{4} \sqrt{n}} \int_{t-1}^{t} \zeta_u du \]
From (6.63), there exists a constant $C$ such that:

$$E \beta^2 \leq c \left( E S^2 + E S^2 \right) f'(t) \leq \frac{Cf'(t)^{1/2}}{n^{1/2}(1-t)^{3/4}}.$$

Using Assumption (A1), we claim that there exists a constant $\tilde{c}$ such that

$$\frac{|f''(t)|}{f'(t)} = \frac{\tilde{c}}{(1-t)^{3/4 - 1/(2\mu)}}.$$

Thus, using (6.62–6.67), we obtain some constant $C$ such that the following inequality holds:

$$E Q^2 \leq \frac{c(1-t)^{3/(4\mu)}}{n^{3/4}(1-t)^{3/4}} + \frac{c(1-t)^{3/(4\mu)}}{n^{3/4}(1-t)^{3/4}} + \frac{c}{n^{3/4}(1-t)^{9/4 + 1/(4\mu)}}.$$

By means of the stochastic Fubini theorem, we obtain that

$$A^n = \sigma k \eta + 2p \sqrt{2} \sum_{i \in J^n(t)} \int_{t_i-1}^{t_i} (t_i - u) \beta_u dW_u.$$

Since the Doob inequality $E \left( \sup_t A^n \right)^2 \leq 4E (A^n)^2$ holds, it suffices to estimate $E (A^n)^2$. From the boundedness of $\sqrt{(t_i - u)/(1-u)}$ and $f'(u)(t_i - u)n$ on $u \in [t_{i-1}, t_i)$, we deduce the following estimates:

$$E (A^n)^2 \leq c n^{3/4 + p} \sum_{i = 1}^{n-1} \int_{t_{i-1}}^{t_i} (t_i - u)^2 E \beta_u^2 du \leq c n^{3/4 + p} \sum_{i = 1}^{n-1} \int_{t_{i-1}}^{t_i} (t_i - u)^2 f'(u)^2 du,$$

$$\leq c n^{2p} \sum_{i = 1}^{n-1} \int_{t_{i-1}}^{t_i} (t_i - u)^2 du \leq c n^{2p} \log n \rightarrow 0.$$

Then, we conclude that $E \left( \sup_t A^n \right)^2 \rightarrow 0$.

Secondly, we write:

$$B^n = \sum_{i \in J^n(t)} \int_{t_{i-1}}^{t_i} \zeta_u \int_{t_{i-1}}^{t_i} 1_{t_i \geq u} dt du = \sum_{i \in J^n(t)} \int_{t_{i-1}}^{t_i} (t_i - u) \zeta_u du.$$

Then,

$$\sup_t |B^n| \leq c n^{3/4 + p} \sum_{i = 1}^{n-1} \int_{t_{i-1}}^{t_i} (t_i - u) \zeta_u du.$$

Thus there exists a constant $c$ such that $E \sup_t |B^n|^2 \leq c n^{3/4 + 2p} \gamma^n$.

$$\gamma^n = E \left( \int_0^1 \sum_{i = 1}^{n-1} (t_i - u) |\zeta_u| 1_{[t_{i-1}, t_i]}(u) du \right)^2,$$

$$= E \int_0^1 \int_0^1 \sum_{i,j = 1}^{n-1} (t_i - u)(t_j - v) |\zeta_u| |\zeta_v| 1_{[t_{i-1}, t_i]}(u) 1_{[t_{j-1}, t_j]}(v) du dv.$$
Using the Cauchy–Schwarz inequality and (5.29), we can then bound $T^n$:

$$ T^n \leq \int_0^1 \int_0^1 \sum_{i,j=1}^{n-1} (t_i - u)(t_j - v) E \left( \frac{1}{8^2} \left( E \frac{1}{8} \right)^2 \right) 1_{(t_{i-1} - t_i)}(u) 1_{(t_{j-1} - t_j)}(v) du dv, $$

$$ \leq \left( \int_0^1 \int_0^1 \sum_{i=1}^{n-1} (t_i - u) \left( E \frac{1}{8} \right)^2 1_{(t_{i-1} - t_i)}(u) du \right)^2 \leq c (T^n + T^{2n} + T^{3n}), $$

where

$$ T^{1n} \leq \left( \sum_{i=1}^{n-1} \frac{(\Delta t_i)^2}{(1 - t_i) n^{1/8}(1 - t_i)^{3/8}(1/8)} n^{1/8} \right) \leq c \log n \frac{n}{n^{1+3/(4\mu)}}. \tag{5.30} $$

In the same way, we obtain the following inequalities

$$ T^{2n} \leq \left( \sum_{i=1}^{n-1} \frac{(\Delta t_i)^2}{n^{3/8}(1 - t_i)^{3/8}} \right) \leq \frac{C}{n^3}, \tag{5.31} $$

$$ T^{3n} \leq \left( \sum_{i=1}^{n-1} \frac{(\Delta t_i)^2}{n^{3/8}(1 - t_i)^{3/8}1^{1/(1/8 + 1/(1/8))}} \right) \leq \frac{c \log n}{n^{7/2 + 1/(3\mu)}}. \tag{5.32} $$

Then, from inequalities (5.30), (5.31) and (5.32) we deduce that

$$ E \sup_t |B_t^n|^2 \leq \frac{c n^{3/2 + 3p} \log n}{n^{1+3/(4\mu)}} \leq \frac{c \log n}{n^{3/(4\mu) - 2p}}. $$

where

$$ 3/(4\mu) - 1/2 - 2p = \frac{-4\mu^2 + 3\mu + 3}{4\mu(\mu + 1)}. $$

Assumption (A1) yields $-4\mu^2 + 3\mu + 3 > 0$. Hence the result follows.

**Lemma 5.10** $n^{3/2 + 2p} E (\sup_t R^{22}_t)^2 \to 0$ as $n \to \infty$.

**Proof** We write $-R^{22}_t = k_n \sum_{i \in J_t^n} \gamma_i \gamma_{i-1} S_{i-1}^2 \{ \xi_{i-1}^{\mu} \}_s \leq U^n(t) + V^n(t)$ where $U^n$ is a martingale defined as

$$ U^n(t) := k_n \sum_{i \in J_t^n} \gamma_i \gamma_{i-1} S_{i-1}^2 \{ \xi_{i-1}^{\mu} \}_s - E \{ \xi_{i-1}^{\mu} \}_s, $$

and $V^n(t) := k_n \sum_{i \in J_t^n} \gamma_i \gamma_{i-1} S_{i-1}^2 E \{ \xi_{i-1}^{\mu} \}_s$. Recall that from Lemma 4.1

$$ E \{ \xi_{i-1}^{\mu} \}_s = k(\Delta t_i)^{1/2} \left( 1 + o(n^{-1}) \right). $$

We deduce that for $n$ large enough, $0 \leq E \{ \xi_{i-1}^{\mu} \}_s \leq c(\Delta t_i)^2$. Using the Doob inequality $E (\sup_t U^n(t))^2 \leq 4E (U^n(1))^2$, it suffices to estimate $E (U^n(1))^2$. The independence of the increments of the Brownian motion implies the equality

$$ E (U^n(1))^2 = k^2 \sum_{i=1}^{n-1} E C_{t_i}^2 \{ \xi_{i-1}^{\mu} \}_s S_{i-1}^4 E \{ \xi_{i-1}^{\mu} \}_s - E \{ \xi_{i-1}^{\mu} \}_s^2. $$
Lemma 5.11 We deduce that $C^2_2$ with $n^{2p} E \{ \epsilon_{t_{i-1}} \}^2 \geq 0$. Hence, $0 \leq \sup_t V^n(t) \leq N^n(1)$. In order to prove that $n^{\frac{1}{2}+2p} E \{ \epsilon_{t_{i-1}} \}^2 \rightarrow 0$, we first analyze the following sum

$$n^{\frac{1}{2}+2p} k_2 n^{1} \sum_{i=1}^{n-1} E \{ \dot{C}_{x,t}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^4 \} \leq \frac{c_n^{2p}}{n^{7/4}} \rightarrow 0.$$

Using the Cauchy-Schwarz inequality, we also have

$$n^{\frac{1}{2}+2p} \sum_{t_i < t_j \leq t_{n-1}} E \gamma_{t_i} S_{t_i}^2 \gamma_{t_j} S_{t_j}^2 E \{ \epsilon_{t_i} \}^2 E \{ \epsilon_{t_j} \}^2 \leq \frac{c_n^{2p}}{n} \rightarrow 0.$$

We deduce that $n^{\frac{1}{2}+2p} E \{ \epsilon_{t_{i-1}} \}^2 \rightarrow 0$ and finally $n^{\frac{1}{2}+2p} (\sup_t R_n^{22}(t))^2 \rightarrow 0$

**Lemma 5.11** $n^{\frac{1}{2}+2p} E \{ \sup_t R_n^{23}(t) \}^2 \rightarrow 0$.

**Proof** We write $R_n^{23}(t) = R_n^{231}(t) + R_n^{232}(t)$ where

$$R_n^{231}(t) := k_0 \sum_{i \in J^n_1(t)} \Theta_i \left( S_{t_i} - S_{t_{i-1}} \right)$$

$$R_n^{232}(t) := k_0 \sum_{i \in J^n_1(t)} \left( \Theta_i - \Theta_1 \right) \left( S_{t_i} - S_{t_{i-1}} \right)$$

with $\Theta_1 := \gamma_{t_{i-1}} |S_{t_i} - S_{t_{i-1}}| - |\dot{C}_x(t_i, S_i) - \dot{C}_x(t_{i-1}, S_{t_{i-1}})|$.

We note that $\sup_t |R_n^{231}(t)|$ is bounded by

$$k_0 \sum_{i=1}^{n-1} |\dot{C}_x(t_i, S_i) - \dot{C}_x(t_{i-1}, S_{t_{i-1}}) - \gamma_{t_{i-1}} (S_{t_i} - S_{t_{i-1}})| |S_{t_i} - S_{t_{i-1}}|.$$

Applying Taylor’s formula to the difference $\dot{C}_x(t_i, S_i) - \dot{C}_x(t_{i-1}, S_{t_{i-1}})$ it is sufficient to estimate the following sums (5.33), · · · (5.36). The first one satisfies

$$n^{\frac{1}{2}+p} \left\| k_0 \sum_{i=1}^{n-1} \dot{C}_x(t_{i-1}, S_{t_{i-1}})(\Delta t_i)(S_{t_i} - S_{t_{i-1}}) \right\|_2 \leq \frac{C n^{p}}{n^{7/8}} \rightarrow 0. \quad (5.33)$$

Indeed, from Corollary 6.13, we deduce that:

$$E \dot{C}_x(t_{i-1}, S_{t_{i-1}})(\Delta t_i)^2(S_{t_i} - S_{t_{i-1}})^2 \leq C (\Delta t_i)^3 n^{\frac{3}{2}} f'(t_{i-1}) \frac{t_{i-1}}{1-t_{i-1}^{\frac{3}{2}}}.$$
The second one verifies
\[ n^{1/2+p} \left\| \sum_{i=1}^{n-1} \tilde{C}_{xxt}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}})(S_{t_i} - S_{t_{i-1}})^2 \right\|_2 \leq C \frac{n^p \log n}{n^2} \to 0. \] (5.34)

Thirdly, from (6.69), we deduce that
\[ E \tilde{C}_{xxt}^2(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}})(S_{t_i} - S_{t_{i-1}})^4(\Delta t_i)^2 \leq \frac{C(\Delta t_i)^4}{(1 - t_i)^3} \]
and it follows that
\[ n^{1/2+p} \left\| \sum_{i=1}^{n-1} \tilde{C}_{xxt}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}})(S_{t_i} - S_{t_{i-1}})^2 \Delta t_i \right\|_2 \leq C \frac{n^p \log n}{n^{1/2}} \to 0. \] (5.35)

Finally, from Lemma 6.18, we get that
\[ E \tilde{C}_{xxt}^2(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}})(S_{t_i} - S_{t_{i-1}})^2(\Delta t_i)^4 \leq \frac{C(\Delta t_i)^5}{(1 - t_i)^3} \]
and
\[ n^{1/2+p} \left\| \sum_{i=1}^{n-1} \tilde{C}_{xxt}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}})(S_{t_i} - S_{t_{i-1}})(\Delta t_i)^2 \right\|_2 \leq C \frac{n^p \log n}{n^{1/2}} \to 0. \] (5.36)

From above, we can conclude about that \( n^{1/2+2p} E (\sup_t R_{231}^n(t))^2 \to 0 \).

As for \( R_{232}^n(t) \), we use the inequality \( |\Theta_t - \Theta_i| \leq |\Delta K^n_t| \) and we deduce from Definition (2.8) the bound
\[ |R_{232}^n(t)| \leq c \sup_i |\Delta S_t| \int_0^{t_{n-1}} |\tilde{C}_{xst}(t_u, S_u)| du, \]
with
\[ |\tilde{C}_{xst}(t_u, S_u)| \leq \frac{c}{\sqrt{S_u(1 - u)}} \]
so that \( |R_{232}^n(t)| \leq c \log(n) \sup_t S_t^{-1/2} \sup_i |\Delta S_{t_i}| \).

Using the Cauchy–Schwarz inequality, the boundedness of \( E \sup_t S_t^{-2} \) yields
\[ E \sup_t (R_{232}^n(t))^2 \leq C \log^2(n) \sqrt{E \sup_i (\Delta S_{t_i})^4}. \]

Moreover,
\[ E \sup_i (\Delta S_{t_i})^4 \leq n^{-3/2} + E \sup_i (\Delta S_{t_i})^4 1_{\sup_i (\Delta S_{t_i}) \geq n^{-3/2}} \]
\[ \leq n^{-3/2} + C \sqrt{P \left( \sup_i (\Delta S_{t_i})^4 \geq n^{-3/2} \right)} \].
By virtue of the Bienaymé–Tchebychev inequality \( \mathbb{P}(|X| \geq k) \leq k^{-2} \mathbb{E} X^2 \),
\[
\mathbb{P} \left( \sup_i (\Delta S_t)^4 \geq n^{-\frac{2}{3}} \right) \leq n^{12} \mathbb{E} (\Delta S_t)^{32} \leq C n^{-3}.
\]

We deduce that \( \mathbb{E} \sup_t (\Delta S_t)^4 \leq C n^{-\frac{2}{3}} \) and finally \( \mathbb{E} \sup_t (R_{2k}^p(t))^2 \leq C n^{-3/4} \log^2(n) \) so that we can conclude the lemma.

**Lemma 5.12** We have \( n^{\frac{1}{2} + 2p} \mathbb{E} (\sup_t R_{2k}^p(t))^2 \xrightarrow{n \to \infty} 0 \).

**Proof** Let us notice that \( \sup_t |R_{2k}^p(t)| \) is bounded by the random variable
\[
k_0 \sum_{i=1}^{n-1} \left| C_x(t_i, S_{t_i}) - C_x(t_{i-1}, S_{t_{i-1}}) + \Delta K^p_t - C_{xx}(t_{i-1}, S_{t_{i-1}}) (S_{t_i} - S_{t_{i-1}}) \right| S_{t_{i-1}}.
\]

Using the Ito formula for the increments \( C_x(t_i, S_{t_i}) - C_x(t_{i-1}, S_{t_{i-1}}) \), we obtain
\[
\sup_t |R_{2k}^p(t)| \leq k_0 \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} \sigma S_u \left[ \dot{C}_{xx}(u, S_u) - \dot{C}_{xx}(t_{i-1}, S_{t_{i-1}}) \right] du + \frac{1}{2} \int_{t_{i-1}}^{t_i} \sigma^2 S_u^2 \dot{C}_{xx}(u, S_u) du.
\]

Thus \( n^{\frac{1}{2} + p} \| \sup_t R_{2k}^p(t) \|_2 \leq T_n^1 + T_n^2 \) where
\[
T_n^1 = \sigma k_0 n^{\frac{1}{2} + p} \sum_{i=1}^{n-1} \left( \int_{t_{i-1}}^{t_i} \mathbb{E} S_{t_{i-1}, S_u}^2 (\gamma_u - \gamma_{t_{i-1}})^2 du \right)^{\frac{1}{2}}
\]

and
\[
T_n^2 = k_0 n^{\frac{1}{2} + p} \sigma^4 \sum_{i=1}^{n-1} (\Delta t_i)^{\frac{1}{2}} \left( \int_{t_{i-1}}^{t_i} \mathbb{E} S_{t_{i-1}, S_u}^2 \dot{C}_{xx}(u, S_u) du \right)^{\frac{1}{2}}.
\]

We first prove that \( T_n^1 \xrightarrow{n \to \infty} 0 \). Using the Taylor Formula, we get that
\[
\gamma_u - \gamma_{t_{i-1}} = \gamma_u - \dot{C}_{xx}(u, S_{t_{i-1}}) + \dot{C}_{xx}(u, S_{t_{i-1}}) - \gamma_{t_{i-1}} = \dot{C}_{xx}(u, S_{t_{i-1}})(S_u - S_{t_{i-1}}) + \frac{1}{2} \ddot{C}_{xxx}(u, S_{t_{i-1}})(S_u - S_{t_{i-1}})^2 + \ddot{C}_{xxt}(t_{i-1}, S_{t_{i-1}})(u - t_{i-1}).
\]

Using the suitable estimations from the Appendix, we then obtain
\[
\mathbb{E} S_{t_{i-1}, S_u}^2 (\gamma_u - \gamma_{t_{i-1}})^2 \leq \frac{c \Delta t_i}{n^3 (1 - t_i)^3} + \frac{c(\Delta t_i)^2}{n^4 (1 - t_i)^3 f'(t_i)^2} + \frac{c(\Delta t_i)^2}{n^{5/4} (1 - t_i)^{12}}.
\]

The last estimate follows from Corollary 6.67. Indeed, the proof is the same since \( \rho_{t_{i-1}} \leq \rho_{t_{i-1}} \). We can therefore deduce that \( T_n^1 \xrightarrow{n \to \infty} 0 \).
We then prove that $T_n^2 \xrightarrow{n \to \infty} 0$. We deduce from Appendix the following inequality:

$$\mathbb{E} S_{t_{i-1}}^2 \tilde{S}_u^2 \tilde{C}_{xx}(n, S_u) \leq \frac{c}{n^{7/8}(1-t_i)^{7/4}}.$$ 

It suffices to obtain the convergence

$$n^{\frac{1}{4}+p} \sum_{i=1}^{n-1} \frac{\Delta t_i}{n^{1/16}(1-t_i)^{7/8}} \leq \frac{c n^p}{n^{3/16}} \xrightarrow{n \to \infty} 0$$

to conclude the lemma. This last lemma completes the proof of Theorem 5.1.

5.3 Step 3: Asymptotic distribution

From the previous subsection, it turns out that the deviation between the “real world” terminal portfolio and the payoff $h(S_1)$ is essentially composed of a martingale as $n \to \infty$. To study the asymptotic distribution of $n^{\frac{1}{4}+p} M^n_1$, we consider it as terminal values of the following sequence of martingales $(N^n)_{j=0, \ldots, n}$ with respect to the filtration $\mathcal{F}^n = (\mathcal{F}_{t_i})_{i}:

$$N^n_j := n^{\frac{1}{4}+p} M^n_1 = \sum_{i=1}^{j} (\chi_i + \theta_i),$$

where

$$\chi_i := k_0 n^{1/4+p} \gamma_{t_{i-1}} S_{t_{i-1}}^2 \left[ \epsilon_{t_{i-1}} \right],$$

$$\theta_i := k_0 n^{1/4+p} K^n_{t_{i-1}} (S_{t_i} - S_{t_{i-1}}).$$

Theorem 5.13 The following convergence holds: $N^n_1 \xrightarrow{d} Z$ where the law of $Z$ is given by the characteristic function $\phi_Z(s) = \mathbb{E} e^{-s^2 \eta^2}$ with

$$\eta^2 := Q(\mu)(k_0 \sigma)^{-4p} S_1^2 \times$$

$$\int_0^\infty x^{4p} \left\{ \left( \int_x^\infty J(y, S_1) dy \right)^2 + \left( 1 - \frac{2}{\pi} \right) \tilde{J}(x, S_1) \right\} dx,$$

and

$$J(x, S_1) := \frac{1}{2x} \int_{-\infty}^{\infty} h'(S_u e^{\sqrt{y}+x/2})(-y^2 - \sqrt{xy} + 1) \varphi(y) dy$$

$$\tilde{J}(x, S_1) := \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} h'(S_u e^{\sqrt{y}+x/2}) \varphi(y) dy.$$ (5.38)

Moreover $\mathbb{E} (N^n_1)^2 \xrightarrow{n \to \infty} \mathbb{E} \eta^2$. 

(5.39)
We achieve the proof of this theorem by means of results in [7] recalled by Theorem 6.1 in the Appendix. We thus need to prove the following lemmas.

**Lemma 5.14** The sequence of martingales \((N^i_n)_{i=0, \ldots, n}\) satisfies

\[
\text{for all } \varepsilon > 0, \quad \sum_i \mathbb{E} \left( (\chi_i + \vartheta_i)^2 1_{|\chi_i + \vartheta_i| > \varepsilon} | \mathcal{F}_t_{i-1} \right) \xrightarrow[n \to \infty]{} 0. \tag{5.40}
\]

**Proof** We use the inequality \((\chi_i + \vartheta_i)^2 \leq 2 \chi_i^2 + 2 \vartheta_i^2\) and we deduce the convergence in \(L^1\). First let us show that \(\mathbb{E} \left( \vartheta_i^2 1_{|\chi_i + \vartheta_i| > \varepsilon} \right) \xrightarrow[n \to \infty]{} 0\). By virtue of the Markov inequality, we obtain

\[
\mathbb{E} \left( \vartheta_i^4 1_{|\vartheta_i| > \varepsilon/2} \right) \leq \sqrt{\mathbb{E} \vartheta_i^2 \mathbb{P}(|\vartheta_i| > \varepsilon/2)} \leq C \varepsilon^{-6} \sqrt{\mathbb{E} \vartheta_i^4 \mathbb{E} \vartheta_i^{12}}.
\]

By independance, we have

\[
\begin{align*}
\mathbb{E} \vartheta_i^4 &= k_0^2 n^{1+4p} \mathbb{E} (K^i_{t_{i-1}})^4 S_{t_{i-1}}^i \mathbb{E} (\zeta^i_{t_{i-1}})^4, \\
\mathbb{E} \vartheta_i^{12} &= k_0^2 n^{3+12p} \mathbb{E} (K^i_{t_{i-1}})^{12} S_{t_{i-1}}^{12} \mathbb{E} (\zeta^i_{t_{i-1}})^{12}.
\end{align*}
\]

By virtue of Lemma 6.8 there exists a constant \(C\) such that

\[
|K^i_{t_{i-1}}|^4 \leq C \sup_{0 \leq u \leq T} S_u^{-2} \left( \int_0^{t_{i-1}} \frac{du}{1-u} \right)^4 \leq C \sup_{0 \leq u \leq T} S_u^{-2} \log^4(n).
\]

We deduce that

\[
\begin{align*}
\mathbb{E} \vartheta_i^4 &\leq C \log^4(n) n^{4p-1}, \\
\mathbb{E} \vartheta_i^{12} &\leq C \log^{12}(n) n^{12p-3}, \\
\sum_i \mathbb{E} \vartheta_i^{12} &\leq C \log^{12}(n) n^{12p-2} \to 0. \tag{5.41}
\end{align*}
\]

Since \(p < 1/8\), we deduce

\[
\sum_i \mathbb{E} \left( \vartheta_i^2 1_{|\vartheta_i| > \varepsilon/2} \right) \leq C \varepsilon^{-6} n^{8p-1} \log^8(n) \sum_{i \leq n} n^{-1} \leq C \varepsilon^{-6} n^{8p-1} \log^8(n) \xrightarrow[n \to \infty]{} 0.
\]

Let us study \(\mathbb{E} \left( \vartheta_i^2 1_{|\chi_i| > \varepsilon/2} \right)\). Again,

\[
\mathbb{E} \left( \vartheta_i^2 1_{|\chi_i| > \varepsilon/2} \right) \leq \sqrt{\mathbb{E} \vartheta_i^4 \mathbb{P}(|\chi_i| > \varepsilon/2)} \leq C \varepsilon^{-2} \sqrt{\mathbb{E} \vartheta_i^4 \mathbb{E} \chi_i^4}.
\]
Once again by independence, \( \mathbb{E} \chi^4_i = k_0^4 n^{1+4p} \mathbb{E} \gamma_{t_{i-1}}^4 S_{t_{i-1}}^8 \mathbb{E} \left[ \varepsilon_{t_{i-1}}^4 \right] \). We easily deduce from Lemma 4.1 the inequality \( \mathbb{E} \left[ \varepsilon_{t_{i-1}}^4 \right] \leq C(\Delta t_i)^2 \). Using Inequality (6.62) we obtain

\[
\mathbb{E} \chi^4_i \leq C n^{1+4p} \frac{(\Delta t_i)^2}{(n^{1/4})^{3/4}} \leq C n^{4p-1/4} 
\]

(5.42)

\[
\sum_i \mathbb{E} \chi^4_i \leq C n^{1+4p} \sum_i \frac{(\Delta t_i)^2}{(n^{1/4})^{3/4}} \leq C \log(n) n^{4p-1/4}.
\]

(5.43)

Since \( p < \frac{1}{16} \leq \frac{3}{32} \), then

\[
\sum_i \mathbb{E} \left( \phi_i^2 1_{|\chi_i| > \varepsilon/2} \right) \leq C \varepsilon^{-2} n^{4p} \log^2(n) \sum_i \frac{\Delta t_i}{n^{3/8}} \frac{1}{(1-t_{i-1})^{3/4}}.
\]

\[
\leq C \varepsilon^{-2} n^{4p-3/8} \log^2(n) \sum_i \frac{\Delta t_i}{1-t_{i-1}}
\]

\[
\leq C \varepsilon^{-2} n^{-3/8+4p} \log^3(n) \xrightarrow{n \to \infty} 0.
\]

From the inequality \( 1_{|\chi_i| + \phi_i > \varepsilon} \leq 1_{|\chi_i| > \varepsilon/2} + 1_{|\phi_i| > \varepsilon/2} \) we then deduce that

\[
\sum_i \mathbb{E} \left( \phi_i^2 1_{|\chi_i| + \phi_i > \varepsilon} \right) \xrightarrow{n \to \infty} 0.
\]

Second let us show that \( \mathbb{E} \left( \chi^2_i 1_{|\chi_i| + \phi_i > \varepsilon} \right) \xrightarrow{n \to \infty} 0. \) In the same way, we have

\[
\mathbb{E} \left( \chi^2_i 1_{|\phi_i| > \varepsilon/2} \right) \leq \sqrt{\mathbb{E} \chi^4_i \sqrt{P(|\phi_i| > \varepsilon/2)}} \leq C \varepsilon^{-6} \mathbb{E} \chi^4_i \sqrt{\mathbb{E} \phi_i^2}.
\]

From (5.42) we have \( \mathbb{E} \chi^4_i \leq C n^{4p-1/4} \). Thus, using \( p < \frac{1}{16} \leq \frac{3}{32} \),

\[
\sum_i \mathbb{E} \left( \chi^2_i 1_{|\phi_i| > \varepsilon/2} \right) \leq \frac{C \varepsilon^{-6} n^{8p-5/8} \log^6(n)}{n \to \infty} \xrightarrow{n \to \infty} 0.
\]

(5.44)

Let us now study \( \mathbb{E} \left( \chi^2_i 1_{|\chi_i| > \varepsilon/2} \right) \).

\[
\mathbb{E} \left( \chi^2_i 1_{|\chi_i| > \varepsilon/2} \right) \leq \sqrt{\mathbb{E} \chi^4_i \sqrt{P(|\chi_i| > \varepsilon/2)}} \leq C \varepsilon^{-2} \mathbb{E} \chi^4_i.
\]

Using the bound (5.42), we obtain \( \sum_i \mathbb{E} \left( \chi^2_i 1_{|\chi_i| > \varepsilon/2} \right) \leq C n^{4p-1/4} \xrightarrow{n \to \infty} 0. \)

We finally conclude the lemma. Inspecting the proof above, we deduce the following:

**Corollary 5.15** The sequence of martingales \((N^n_t)_{t=0,\ldots,n}\) satisfies

\[
\max_i \mathbb{E} \left( (\chi_t + \phi_t)^2 |\mathcal{F}_{t_{i-1}} \right) \xrightarrow{n \to \infty} 0.
\]
Proof Indeed, by virtue of Inequalities (5.41) and (5.43), for a given $\varepsilon > 0$

\[
P(\max_i E \left((\xi_i + \vartheta_i)^2 | \mathcal{F}_{t_i-1}\right) > \varepsilon) \leq P(2 \max_i E \left((\xi_i)^2 | \mathcal{F}_{t_i-1}\right) + 2 \max_i E \left(\vartheta_i | \mathcal{F}_{t_i-1}\right) > \varepsilon)
\]

\[
\leq P\left((\max_i E \left((\xi_i)^2 | \mathcal{F}_{t_i-1}\right) > \varepsilon/4\right) + P\left(\max_i E \left(\vartheta_i | \mathcal{F}_{t_i-1}\right) > \varepsilon/4\right)
\]

\[
\leq C \varepsilon^{-2} \sum_i E \xi_i^2 + C \varepsilon^{-6} \sum_i E \vartheta_i^2 \to 0.
\]

Lemma 5.16 The sequence of martingales $(M^n_t)_{t=0,\ldots,n}$ satisfies the following convergence

\[
\sum_i E \left((\xi_i + \vartheta_i)^2 | \mathcal{F}_{t_i-1}\right) \to P_{n \to \infty} \eta^2,
\]

(5.45)

where

\[
\eta := Q(\mu)(k_0 \sigma)^{1-4p} S_1^2 \times \int_0^\infty \int_{-\infty}^\infty \left(\frac{1}{\pi} J(y, S_1)\right)^2 \left(1 - \frac{2}{\pi} \bar{J}(x, S_1)^2\right) dx dy.
\]

Proof First, let us study the term $\xi^n_0 := \sum_i E \left(\vartheta_i^2 | \mathcal{F}_{t_i-1}\right)$. By independence, we obtain $E \left(\vartheta_i^2 | \mathcal{F}_{t_i-1}\right) = k_0^2 n^{1/2+2p} \left(K_{t_i-1}^n\right)^2 S_i^2 E \left(\xi_{t_i-1}^{\vartheta}\right)^2$. Hence, using Lemma 4.1 and the change of variable $y = \rho\delta$ and $x_i = \rho\eta$, we have

\[
E \left(\vartheta_i^2 | \mathcal{F}_{t_i-1}\right) = k_0^2 n^{1/2+2p} K_{t_i-1}^n S_i^2 \sigma^2 \Delta t_i \left(1 + O(n^{-1})\right)
\]

\[
= k_0^2 n^{1/2+2p} S_i^2 \left(1 \right) \Delta t_i \left(1 + O(n^{-1})\right)
\]

\[
= k_0^2 n^{1/2+2p} S_i^2 \left(1 \right) \Delta t_i \left(1 + O(n^{-1})\right)
\]

\[
= k_0^2 n^{1/2+2p} S_i^2 \left(1 \right) \Delta t_i \left(1 + O(n^{-1})\right)
\]

\[
= k_0^2 n^{1/2+2p} S_i^2 \left(1 \right) \Delta t_i \left(1 + O(n^{-1})\right)
\]

We then deduce that

\[
\xi^n_0 = (1 + O(n^{-1})) \int_0^\infty \zeta^n_0(x) dx
\]

(5.46)
where
\[ z_n^\vartheta(x) := S_{t_{i-1}}^2 k_0^2 \sigma^2 \sum_{i} \left( \int_{x_{i-1}}^{x_i} \tilde{C}_{xt}(u, S_u) \tilde{\sigma}_u^{-2} du \right)^2 n^{1/2 + 2p} \Delta t_i 1_{(x, x_{i-1})}(x). \]

Recall that \(|\tilde{C}_{xt}(u, S_u) \tilde{\sigma}_u^{-2}| du \leq c G_1(x, S_u), x = \rho_n^2\) where
\[ G_1(x, y) = \frac{1}{x} e^{-x/8} \left( \sum_{j=1}^p \frac{|\log(y/K_j)|}{\sqrt{x}} \exp \left\{ - \frac{|\log(y/K_j)|}{2x} \right\} + \sqrt{x} + x \right). \]

In particular,
\[ \sqrt{x} G_1(x, y) \leq G(x) \quad (5.47) \]
where \(G(x) = c x^{-\frac{1}{2}} e^{-x/16}, c > 0\) is a constant. Hence, a.s.
\[ \left| \int_{x_{i-1}}^{x_i} \tilde{C}_{xt}(u, S_u) \tilde{\sigma}_u^{-2} du \right| \leq \int_{x}^{\infty} G(x') dx' \leq \int_{0}^{\infty} G(x') dx' < +\infty. \quad (5.48) \]

Therefore, using (4.21),
\[ |z_n^\vartheta(x)| \leq C(1 + x) \left( \int_{x}^{\infty} G(x') dx' \right)^2 \sup_{u \in [0,1]} S_u^2. \quad (5.49) \]

But, due to Hölder’s inequality,
\[ \int_{0}^{\infty} (1 + x) \left( \int_{x}^{\infty} G(x') dx' \right)^2 dx < +\infty. \]

We can thus apply Lebesgue’s theorem using Corollary 6.21 and (4.20):
\[ \xi_n^\vartheta a.s. n \rightarrow +\infty Q(\mu)(k_0\sigma)^{1-4p} S_t^2 \int_{0}^{\infty} x^{4p} \left( \int_{x}^{\infty} J(y, S_t) dy \right)^2 dx. \quad (5.50) \]

Second, let us study the term \(\xi_n^\chi = \sum_i \mathbf{E} \left( \chi_i^2 | F_{t_{i-1}} \right)\). By independence, we obtain
\[ \mathbf{E} \left( \chi_i^2 | F_{t_{i-1}} \right) = k_0^2 \sigma^2 n^{1/2 + 2p} \gamma_{t_{i-1}} s_{t_{i-1}}^4 \mathbf{E} \left[ \xi_{t_{i-1}}^4 \right]. \]

Then \( \mathbf{E} \left( \chi_i^2 | F_{t_{i-1}} \right) = k_0^2 \sigma^2 n^{1/2 + 2p} \gamma_{t_{i-1}} s_{t_{i-1}}^4 \left( 1 - \frac{2}{n} \right) \Delta t_i (1 + o(1)) \). We then deduce that
\[ \sum_i \mathbf{E} \left( \chi_i^2 | F_{t_{i-1}} \right) = (1 + O(n^{-1})) \int_{0}^{\infty} z_n^\chi(x) dx \quad (5.51) \]
where
\[ z_n^\chi(x) := S_{t_{i-1}}^4 k_0^2 \sigma^2 \sum_{i} \gamma_{t_{i-1}}^2 n^{1/2 + 2p} \Delta t_i 1_{(x, x_{i-1})}(x). \]
Let us obtain a suitable bound for $z_n^k(x)$, integrable in $x$. Recall that

$$
\gamma_{t,1} = \hat{C}_{x,x}(t_{i-1}, S_{t,1}) = K \int_{-\infty}^{\infty} h'(S_{t,1} x^\mu + \rho_{t,1}^2) y \varphi(y) dy
$$

Due to Inequality (6.60), we claim that a.s.($\omega$) for $n$ large enough, there is a constant $c_{\omega}$ which does not depend on $n$ such that

$$
|\gamma_{t,1}| \leq C \sup_{u \leq 1} S_u^{-3/2} e^{-x/8} \left( 1_{x \geq 1} + \left( \frac{\sqrt{x}}{\sqrt{2}} + 1 \right) 1_{x \leq 1} \right).
$$

(5.52)

Indeed, this is obvious for $x \geq 1$. Otherwise, $1 \geq x = \rho_n^2 \geq cn^{1/2}(1 - u^n(x))$ implies that $u = u^n(x)$ is close to 1 uniformly in $x \leq 1$ as soon as $n$ is large enough. It then suffices to choose $S_1$ out of the null-set $\{ S_1 = K_1, \cdots, K_{p_h} \}$ to obtain by continuity that $S_n(x)$ is also far enough from the points $K_1, \cdots, K_{p_h}$ if $x \geq 1$.

We conclude that for all $j$, $\log^2 (K_j/S_n(x)) \geq c_{\omega,j}$ for some constants $c_{\omega,j} > 0$.

Therefore,

$$
|S_{t,1}^4| |\gamma_{t,1}|^2 \leq C \xi e^{-x/4} \left( 1_{x \geq 1} + \left( \frac{\sqrt{x}}{\sqrt{2}} + 1 \right) 1_{x \leq 1} \right)
$$

where $\xi := \sup_{0 \leq u \leq 1} S_u^4 \sup_{0 \leq u \leq 1} S_u^{-3}$. Thus, due to (4.20)

$$
|z_n^k(x)| \leq C \xi (1 + x) e^{-x/4} \left( 1_{x \geq 1} + \left( \frac{\sqrt{x}}{\sqrt{2}} + 1 \right) 1_{x \leq 1} \right).
$$

We can then apply the dominated convergence theorem using the limit (4.21). We obtain

$$
\frac{\xi}{n} \xrightarrow{a.s.} \left( 1 - \frac{2}{\pi} \right) Q(\mu)(k_0 \sigma)^{1-4p} S_t^2 \int_0^{\infty} x^{4p} j(x, S_t) dx.
$$

Finally, let us study the term $\sum_j E \left( \chi_i \vartheta_i | F_{t-1} \right)$.

By independence, we have

$$
E \left( \chi_i \vartheta_i | F_{t-1} \right) = k_0^2 n^{1/2+2p} \rho_{t,1}^2 S_t^2 K_{t,1}^{n-1} S_{t,1} \left( E \left[ \epsilon_{t,1}^i \right] \epsilon_{t,1}^i \right).
$$

But

$$
E \left( \left[ \epsilon_{t,1}^i \right] \epsilon_{t,1}^i \right) = E \left( \epsilon_{t,1}^i \right)^2 \text{sgn} \epsilon_{t,1}^i = \left( 1 - \frac{2}{\pi} \right) \sigma^2 (\Delta t)^2 (1 + o(1)).
$$
Due to (4.20), we obtain

\[ (\Delta t_i)^{ \frac{3}{2} n^{1/2 + 2p} } \xrightarrow[\infty]{n \rightarrow 0} 0. \]

From the bounds (4.21), (5.48), (5.52) and by applying again Lebesgue’s theorem, we then deduce the following limit \( \sum E \left( \chi_i \theta_i | \mathcal{F}_{t_i} \right) \xrightarrow[\infty]{n \rightarrow 0} 0. \)

**Lemma 5.17** We have \( E \left( N_i^1 \right)^2 \xrightarrow[\infty]{n \rightarrow 0} E \eta^2. \)

**Proof** Due to the independence of the increments of the Wiener process, we have \( E \left( \chi_i + \theta_i \right) \left( \chi_j + \theta_j \right) = 0 \) whenever \( i \neq j \). We thus obtain

\[ E \left( N_i^1 \right)^2 = \sum E \left( \chi_i + \theta_i \right)^2 = E \sum \left( \chi_i + \theta_i \right)^2 | \mathcal{F}_{t_i} \]

But

\[ \sum E \left( \chi_i + \theta_i \right)^2 | \mathcal{F}_{t_i} \leq 2 \sum E \left( \chi_i^2 + \theta_i^2 | \mathcal{F}_{t_i} \right) = 2(\xi_n^2 + \xi_n^a). \]

Let us show that \( \xi_n := \xi_n^2 + \xi_n^a \) is uniformly integrable. First let us note that \( \xi_n \) is bounded in \( L^1(\Omega) \). Indeed, from Corollary 6.6, Inequalities (5.49) and (4.20), we obtain for all \( n \)

\[ E |\xi_n| \leq C \int_0^\infty (1 + x) \left( E S_n^4 e^{-x} \right)^{e^{-x} G(x') dx'} < \infty. \]

Now, using the Cauchy-Schwarz inequality and then the Markov inequality, we have

\[ E \xi_n^a 1_{\xi_n \geq k} \leq C \int_0^\infty (1 + x) \left( \int_x^\infty G(x') dx' \right)^{2 dx \sqrt{E S_n^2} \sqrt{\mathbb{P}(\xi_n \geq k)}} \leq C \sqrt{\sup_n E |\xi_n|} \xrightarrow[k \rightarrow \infty]{\infty} 0. \]

Recall that

\[ \tilde{\zeta}_n(x) 1_{\xi_n \geq M} := k_0^3 \sigma^2 \sum S_{t_i}^2 \gamma_{t_i-1} \gamma_{t_i-1} 1_{\xi_n \geq M} n^{1/2 + 2p} \Delta t_i \left( x_{i-1} - x_i \right) 1_{(x_i, x_{i-1})}(x). \]

Therefore, applying successively the Cauchy-Schwarz inequality, (4.20), Corollary 6.15, and the Markov inequality, we obtain

\[ E \xi_n^a 1_{\xi_n \geq k} \leq C \int_0^\infty (1 + x) \left( \int_x^\infty G(x') dx' \right)^{4/5} dx \left( \frac{\sup_n E \xi_n}{k} \right)^{1/5} \xrightarrow[k \rightarrow \infty]{\infty} 0. \]

Therefore, \( \xi_n \) is uniformly integrable, and so is \( \sum E \left( \chi_i + \theta_i \right)^2 | \mathcal{F}_{t_i} \), which moreover converges to \( \eta \) a.s. This yields the conclusion of the Lemma.
We easily deduce the following:

**Corollary 5.18** We have \( \sup_{\eta} \mathbb{E} \left( \max_i (\chi_i + \vartheta_i)^2 \right) < \infty \).

These last lemmas and corollaries complete the proof of Theorem 5.13.

5.4 Conclusion

Let us summarize the results of the previous theorems: \( n^{1/4 + 2p} \mathbb{E} \left( \varepsilon_t^n \right)^2 \xrightarrow{n \to \infty} 0 \) and \( n^{1 + p} \mathbb{N}_t \xrightarrow{n \to \infty} Z \). Therefore \( n^{1 + p} (V_t^n - h(S_t)) \xrightarrow{n \to \infty} Z \) and \( n^{1/4 + 2p} \mathbb{E} \left( V_t^n - h(S_t) \right)^2 \xrightarrow{n \to \infty} \mathbb{E} \eta^2 = \mathbb{E} Z^2 \). The proof of the limit theorem is then complete.
6 Appendix

The following limit result combines Theorem 3.4 page 67 and Theorem 3.5 page 71 [7]:

**Theorem 6.1**

Let \( \{M^n_i, \mathcal{F}_t, 0 \leq i \leq n\} \) be a zero-mean square integrable martingale with increments \( \Delta M^n_i = X^n_i \) and let \( \eta^2 \) be an a.s. finite r.v. Suppose that

for all \( \varepsilon > 0 \),

\[
\sum_i \mathbb{E} \left( (X^n_i)^2 | X^n_i | > \varepsilon | \mathcal{F}_{t-1} \right) \xrightarrow{n \to \infty} 0,
\]

(6.53)

\[
V^n_i = \sum_i \mathbb{E} \left( (X^n_i)^2 | \mathcal{F}_{t-1} \right) \xrightarrow{n \to \infty} \eta^2,
\]

(6.54)

\[
\max_i \mathbb{E} \left( (X^n_i)^2 | \mathcal{F}_{t-1} \right) \xrightarrow{n \to \infty} 0,
\]

(6.55)

\[
\sup_n \mathbb{E} \left( \max_i (X^n_i)^2 \right) < \infty.
\]

(6.56)

Then \( M^n \xrightarrow{d} Y \) where the r.v. \( Y \) has the characteristic function \( \mathbb{E} \exp -\frac{1}{2} \eta^2 t^2 \).

**Proof** Under Conditions (6.53), (6.54) and (6.55), we deduce by virtue of Theorem 3.5 page 71 [7] that \( U^n_2 \xrightarrow{L^1} \eta^2 \) where \( U^n_2 := \sum_i (X^n_i)^2 \). Observe that Condition (6.53) implies that \( \max_i |X^n_i| \xrightarrow{P} 0 \). Applying Theorem 3.4 page 67 [7], we then conclude.

### 6.1 Explicit Formulae

We recall from [3] the following expressions for the successive derivatives. They are based on direct computations using the integration by parts formula under suitable assumptions on the payoff function \( h \).

**Lemma 6.2** Let \( \tilde{C}(t,x) \) is given by (2.4). Then

\[
\tilde{C}_x(t,x) = \int_{-\infty}^{\infty} h'(xe^{\rho y + \rho^2/2}) \varphi(y) dy,
\]

\[
\tilde{C}_{xx}(t,x) = \frac{1}{\rho^2} \int_{-\infty}^{\infty} h'(xe^{\rho y + \rho^2/2}) y \varphi(y) dy,
\]

\[
\tilde{C}_{xxx}(t,x) = \frac{1}{\rho^3 x^2} \int_{-\infty}^{\infty} h'(xe^{\rho y + \rho^2/2}) P_2(y) \varphi(y) dy,
\]

\[
\tilde{C}_{xxxx}(t,x) = \frac{1}{\rho^4 x^3} \int_{-\infty}^{\infty} h'(xe^{\rho y + \rho^2/2}) P_3(y) \varphi(y) dy
\]

where

\[
P_2(y) := y^2 - \rho y - 1,
\]

\[
P_3(y) := y^3 - 3\rho y^2 + (2\rho^2 - 3)y + 3\rho.
\]
In particular, $|\hat{C}_x(t, x)| \leq ||h'||_\infty$. Similarly, we obtain the following expressions for the successive derivatives in $t$:

**Lemma 6.3** Let $\hat{C}(t, x)$ is given by (2.4). Then

\[
\hat{C}_t(t, x) = -\frac{\sigma^2 x}{2\rho} \int_{-\infty}^{\infty} h'(xe^{\rho y + \rho^2/2})y\varphi(y)dy, \tag{6.57}
\]

\[
\hat{C}_{tx}(t, x) = \frac{\sigma^2}{2\rho^2} \int_{-\infty}^{\infty} h'(xe^{\rho y + \rho^2/2})Q_2(y)\varphi(y)dy, \tag{6.58}
\]

\[
\hat{C}_{xxt}(t, x) = \frac{\sigma^2}{2\rho^3} \int_{-\infty}^{\infty} h'(xe^{\rho y + \rho^2/2})Q_3(y)\varphi(y)dy, \tag{6.59}
\]

with

\[
Q_2(y) := -y^2 - \rho y + 1,
\]

\[
Q_3(y) := -y^3 - \rho y^2 + 3y + \rho.
\]

We can show the following in a similar way:

**Lemma 6.4** We have:

\[
\hat{C}_{xxt}(t, x) = -\frac{\rho''}{2\rho^4} \int_{-\infty}^{\infty} h'(xe^{\rho y + \rho^2/2})P_1(\rho, y)\varphi(y)dy
\]

\[
+ \frac{\sigma^2}{2\rho^4} \int_{-\infty}^{\infty} h'(xe^{\rho y + \rho^2/2})P_3(\rho, y)\varphi(y)dy,
\]

\[
\hat{C}_{xxx}(t, x) = \frac{\sigma^2}{2\rho^4} \int_{-\infty}^{\infty} h'(xe^{\rho y + \rho^2/2})P_4(\rho, y)\varphi(y)dy
\]

where

\[
P_1(x, y) := -y^3 - xy^2 + 3y + x,
\]

\[
P_2(x, y) := -y^2 - xy + 1,
\]

\[
P_3(x, y) := y^4 - (4 + x^2)y^2 + 2xy + x^2 + 1,
\]

\[
P_4(x, y) := -y^4 + 2xy^3 + (6 - x^2)y^2 - 8xy + x^2 - 3.
\]

### 6.2 Estimates

To study the residual terms generated by the discretization of the theoretical portfolio $\hat{C}(T, S_T)$, we use Taylor approximations. We then need to estimate some bounds of the successive derivatives of $\hat{C}$. 
Lemma 6.5 There is a constant $C > 0$ such that
\[
|\mathcal{C}_{xx}(t,x)| \leq Ce^{-\rho^2/8} x^{3/2} \sum_{j=1}^{p} \frac{1}{\rho^2} \exp \left\{ -\frac{1}{2} \log^2(K_j/x) \right\} + e^{-\rho^2/8} x^{3/2}.
\] (6.60)

Corollary 6.6 There exists a constant $C$ such that for $t \in [0,1]$
\[
\mathbb{E} S_t^2 \mathcal{C}_{xx}(t, S_t) \leq C \frac{e^{-\rho^2/4}}{\rho}.
\]

Similarly, we can deduce the following bounds:

Corollary 6.7 There exists a constant $c$ such that for $t \in [0,1]$
\[
\mathbb{E} S_t^2 \mathcal{C}_{xx}(t, S_t) \leq c \left( \sum_{j=1}^{p} \frac{1}{\rho^2 \sqrt{2u^2 + 1}} \exp \left\{ -\frac{v_j^2}{2u^2 + 1} \right\} + e^{-\rho^2/4} \right)
\]
where $c$ is a constant, $u = \alpha t / \rho$ and
\[
v_j := \frac{\log(S_0/K_j) - \alpha^2 t / 2 \rho}{\rho} + \frac{\rho}{2}.
\]

Lemma 6.8 There exists a constant $c$ such that
\[
|\tilde{\mathcal{C}}_{xx}(t,x)| \leq \frac{ce^{-\rho^2/8}}{\rho^2 x^{3/2}} (L(x, \rho) + \rho),
\]
\[
|\tilde{\mathcal{C}}_{xxx}(t,x)| \leq c e^{-\rho^2/8} x^{-7/2} P_3(\rho^{-1}),
\]
\[
|\tilde{\mathcal{C}}_{tx}(t,x)| \leq c e^{-\rho^2/8} x^{-3/2} (\rho^{-1} + \rho^{-3}),
\]
\[
|\tilde{\mathcal{C}}_{xt}(t,x)| \leq c e^{-\rho^2/8} x^{-3/2} (\rho^{-1} + \rho^{-3}),
\]
where $P_3$ is a polynomial of the third order and
\[
L(x, \rho) := \sum_{j=1}^{p} \frac{|\log(x/K_j)|}{\rho} \exp \left\{ -\frac{\log^2(x/K_j)}{2\rho^2} \right\}.
\]

Lemma 6.9 There exists a constant $c$ and a polynomial $Q$ of third order such that
\[
\mathbb{E} S_t^n \tilde{\mathcal{C}}_{xx}(t, S_t) \leq c \sigma^4 Q(\rho^{-1}) e^{-\rho^2/4}.
\]

Lemma 6.10
\[
|\tilde{\mathcal{C}}_{xxt}(t,x)| \leq \frac{e^{-\rho^2/8}}{x^{3/2}} \sigma^2 \left( \sum_{j=1}^{p} \left( \left( g_j(x) \right)^2 + \frac{\rho^2}{4} + 1 \right) e^{-g_j(x)^2/2} + \rho + \rho_1 \right).
\]
\[
|\tilde{\mathcal{C}}_{xtt}(t,x)| \leq \chi^3(t,x) + \chi^2(t,x)
\]
where
\[ X^1(t, x) := e^{-\frac{\rho^2 t}{8}} \frac{\rho_t^2}{\rho_t} \left( \sum_{j=1}^{p} \varrho_j(x)e^{-\varrho_j(x)^2/2} + \rho_t^2 \right), \]
\[ X^2(t, x) := e^{-\frac{\rho^2 t}{8}} \frac{\rho_t^4}{\rho_t} \left( \sum_{j=1}^{p} \left( \varrho_j(x)^3 + \varrho_j(x) \right) e^{-\varrho_j(x)^2/2} + \sum_{j=1}^{4} \rho_t^4 \right), \]
and \( \varrho_j(x) := \| \log(K_j/x) \| / \rho_t. \)

**Lemma 6.11** Assume that Assumption **A1** holds. Then there exists a constant \( c \) such that \( \varepsilon_i := n^{1/2} \sqrt{\Delta t_i} \sqrt{f'(t_i-1)}, i \leq n-1 \) satisfies the inequality \( |\varepsilon_i - 1| \leq c \Delta t_i/(1-t_i) \) for \( n \) large enough.

**Proof** We have obviously
\[ |\varepsilon_i - 1| \leq |n \Delta t_i f'(t_i-1) - 1|, \]
where \( \Delta t_i = g'(\theta_i)n^{-1} \) and \( \theta_i \in [(i-1)/n, i/n]. \) Then, \( d_i := g(\theta_i) - t_i - 1 \in [0, \Delta t_i]. \)

We deduce that
\[ |\varepsilon_i - 1| \leq \left| \frac{f'(g(\theta_i) - h_i)}{f'(g(\theta_i))} - 1 \right| \leq c \frac{\Delta t_i}{1-t_i}. \]

Indeed, we use a first order Taylor expansion to estimate the difference \( f'(g(\theta_i) - h_i) - f'(g(\theta_i)). \) We conclude by using the explicit expression of \( f, g \) but also the inequality \( (1-t_i-1)/(1-t_i) \leq c \) for \( i \leq n-1. \)

The following lemma is of first importance to get estimations of expectations we need in some of our proofs.

**Lemma 6.12** Suppose that \( t \leq u < 1, m \in \mathbb{R}, q \in 2\mathbb{N} \) and \( K > 0. \) There exists a constant \( c = c(m, q) \) such that
\[ \mathbb{E} \left( S_n^m \log^q \frac{S_n}{K} \exp \left\{ -\frac{\log^2(S_n/K)}{\rho_t^2} \right\} \right) \leq c P_q(\rho_t) \]
where
\[ P_0(\rho_t) := \rho_t, \quad P_2(\rho_t) := \rho_t^3 + \rho_t^5, \]
\[ P_4(\rho_t) := \rho_t^5 + \rho_t^7 + \rho_t^9, \]
\[ P_{2q}(\rho_t) := \rho_t^{2q+1} + \rho_t^{2q+3} + \cdots + \rho_t^{4q+1}. \]
Proof We set \( p = \log \frac{S_0}{K} - \sigma^2 u / 2 \), \( \alpha = \sigma \sqrt{u} \) and

\[
A(q) = E S_u^m \log^q \frac{S_u}{K} \exp \left\{ - \frac{\log^2(S_u/K)}{\rho_t^2} \right\}.
\]

Then,

\[
A(q) = S_u^m \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (p + \alpha y)^q \exp \left\{ \alpha m y^2 - \frac{1}{2} \rho_t^2 (p + \alpha y)^2 - y^2/2 \right\} dy,
\]

\[
A(q) = S_u^m e^{A_1} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (p + \alpha y)^q \exp \left\{ - \frac{1}{2} \left( 1 + \frac{2\alpha^2}{\rho_t^2} \right) y^2 + \alpha \left( m - \frac{2p}{\rho_t^2} \right) y \right\} dy
\]

where

\[
A_1 = - \frac{\alpha^2 m}{2} - \frac{p^2}{\rho_t^2}.
\]

Let \( y = z/A_2 \) with \( A_2 = \sqrt{1 + 2\alpha^2/\rho_t^2} \). Then

\[
A(q) = S_u^m e^{A_4} \frac{1}{\sqrt{2\pi A_2}} \int_{-\infty}^{\infty} (p + \alpha z/A_2)^q \exp \left\{ - \frac{1}{2} \left( 2 - 2(A_3/A_2) z + A_3^2/A_2^2 \right) \right\} dz
\]

where \( A_3 = \alpha \left( m - 2p/\rho_t^2 \right) \) and \( A_4 = A_1 + A_3^2/(2A_2^2) \). After the change of variable \( y = z - A_3/A_2 \), we obtain that

\[
A(2) = S_u^m \rho_t e^{A_4} \sqrt{\rho_t^2 + 2\alpha^2} \left[ \left( p + \frac{\alpha^2 \rho_t^2 A_3}{\rho_t^2 + 2\alpha^2} \right)^2 + \frac{\alpha^2 \rho_t^2 A_4}{\rho_t^2 + 2\alpha^2} \right].
\]

Moreover, if \( u \geq t \), then \( \rho_t^2 \geq \sigma^2(1 - t) \) implies that \( \rho_t^2 + 2\alpha^2 \geq \sigma^2(1 - t) + \sigma^2 u \geq \sigma^2 \).

We have

\[
A_4 = - \frac{m\alpha^2}{2} - \frac{p^2}{\rho_t^2} + \frac{\alpha^2 \rho_t^2 A_3}{2(\rho_t^2 + 2\alpha^2)} \left( m^2 + \frac{4p^2}{\rho_t^2} - \frac{4pm}{\rho_t^2} \right)
\]

where \( p, \alpha \) are bounded. But, the term

\[
\frac{\alpha^2 \rho_t^2}{2(\rho_t^2 + 2\alpha^2)} m^2
\]

is obviously bounded whereas we can establish the following inequality

\[
\frac{\alpha^2 \rho_t^2}{2(\rho_t^2 + 2\alpha^2)} \frac{4p^2}{\rho_t^2} \leq \frac{p^2}{\rho_t^2}.
\]

The following term

\[
\left| \frac{\alpha^2 \rho_t^2}{2(\rho_t^2 + 2\alpha^2)} \frac{4pm}{\rho_t^2} \right|
\]

is also bounded. It follows that \( e^{A_4} \) is bounded and we can conclude easily for \( q = 2 \).

In a similar way, we can conclude for any \( q \in 2\mathbb{N} \) because we use in particular the property

\[
\int_{-\infty}^{\infty} y^k \phi(y) dy = 0, \text{ if } k \in 2\mathbb{N} + 1.
\]
From now on, we can deduce the following results.

**Corollary 6.13** If $m \in \mathbb{R}$ and $u \geq t$, then there exists a constant $c_m > 0$ such that

\[
E \, S_u^m \tilde{C}_{x(t)}(t, u) \leq \frac{c_m \rho_4^{1/2}}{\rho_4^{1/2}} e^{-\rho_4^{1/2}}.
\]

**Proof** Indeed, it suffices to use Lemma 6.8 and apply the previous lemma.

In a similar way, we have:

**Corollary 6.14** If $m \in \mathbb{R}$ and $u \geq t$, then there exists a constant $c_m > 0$ such that

\[
E \, S_u^m \tilde{C}_{xx}(t, u) \leq \frac{c_m \rho_4^{1/2}}{\rho_4^{1/2}} e^{-\rho_4^{1/4}}.
\] (6.61)

**Corollary 6.15** If $m \in \mathbb{R}$ then there exists a constant $c_m > 0$ such that

\[
E \, S_t^m \tilde{C}_{xx}(t, S_t) \leq \frac{c_m \rho_4^{1/2}}{\rho_4^{1/2}} e^{-\rho_4^{1/4}}.
\] (6.62)

**Proof** We write $E \, S_t^m \tilde{C}_{xx}(t, S_t) = E \, S_t^m \tilde{C}_{xx}^{1/2}(t, S_t) \tilde{C}_{xx}(t, S_t)$ and we apply Cauchy-Schwarz' inequality with $p = 4/3$ and $q = 4$ such that $p^{-1} + q^{-1} = 1$. We obtain

\[
E \, S_t^m \tilde{C}_{xx}^{1/2}(t, S_t) \leq \left( E \, S_t^{4m/3} \tilde{C}_{xx}(t, S_t) \right)^{3/4} \left( E \, \tilde{C}_{xx}^4(t, S_t) \right)^{1/4},
\]

\[
\leq \left( c_m E \, \tilde{C}_{xx}(t, S_t) \right)^{3/8} \left( E \, \tilde{C}_{xx}^4(t, S_t) \right)^{1/4},
\]

\[
\leq \left( c_m \rho_4^{1/8} e^{-\rho_4^{1/4}} \right)^{3/8} \left( c_m \rho_4^{1/4} e^{-\rho_4^{1/4}} \right)^{1/4},
\]

where the last inequality is deduced from Corollary 6.62. The conclusion follows.

**Corollary 6.16** If $m \in \mathbb{R}$ and $u \geq t$, then there exists a constant $c_m > 0$ such that

\[
E \, S_u^m \tilde{C}_{xx}(t, u) \leq \frac{c_m \rho_4^{1/2}}{\rho_4^{1/2}} e^{-\rho_4^{1/2}}.
\] (6.63)

\[
E \, S_u^m \tilde{C}_{xx}(t, u) \leq \frac{c_m \rho_4^{1/2}}{\rho_4^{1/2}} e^{-\rho_4^{1/2}}.
\] (6.64)

\[
E \, S_u^m \tilde{C}_{xx}(t, u) \leq \frac{c_m \rho_4^{1/2}}{\rho_4^{1/2}} e^{-\rho_4^{1/2}}.
\] (6.65)

\[
E \, S_u^m \tilde{C}_{xx}(t, u) \leq \frac{c_m \rho_4^{1/2}}{\rho_4^{1/2}} e^{-\rho_4^{1/2}}.
\] (6.66)

\[
E \, S_u^m \tilde{C}_{xx}(t, u) \leq \frac{c_m \rho_4^{1/2}}{\rho_4^{1/2}} e^{-\rho_4^{1/2}}.
\] (6.67)
Let $\tilde{S}_{t_{i-1}} \in [S_{t_{i-1}}, S_{t_i}]$ and $\tilde{t}_{i-1} \in [t_{i-1}, t_i]$ be some random variables. We have the following inequalities:

**Lemma 6.17** There exists a constant $c$ such that

$$E \tilde{C}^4_{x,t}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) \leq \frac{ce^{-\rho^2_t/4}}{(1-t_i)^2}. $$

*Proof* We have $\tilde{S}_{t_{i-1}}^m \leq S_{t_{i-1}}^m + S_{t_i}^m$ and $\rho_{t_{i-1}} \geq \rho_t$. Furthermore, in virtue of Lemma 6.8, recall that we have

$$|\tilde{C}_{x,t}(t,x)| \leq c \sigma^2_x e^{-\rho^2_t/8} x^{1/2} \rho^2_t. $$

Then, the conclusion follows.

In the same way, we can prove the following results:

**Lemma 6.18** There exists a constant $C$ such that

$$E \tilde{C}^4_{x,t}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) \leq \frac{Ce^{-\rho^2_t/4}}{(1-t_i)^5}. $$

*Proof* The arguments are similar to the previous ones but we also use the inequality:

$$\frac{g''(u)}{g'(u)^2} \leq \frac{C}{(1-g(u))^3/2}, \forall u < 1 $$

in order to have $\frac{\rho''_t}{\rho_t} \leq \frac{C}{(1-t_i)^2}$.

**Lemma 6.19** There exists a constant $C$ such that

$$E \tilde{C}^4_{xx,t}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) \leq \frac{Ce^{-\rho^2_t/4}}{\rho^2_t}, \quad (6.68)$$

$$E \tilde{C}^4_{xx,t}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) \leq \frac{Ce^{-\rho^2_t/4}}{n(1-t_i)^6 f'(t_i)}, \quad (6.69)$$

$$E \tilde{C}^4_{xxxx,t}(\tilde{t}_{i-1}, \tilde{S}_{t_{i-1}}) \leq \frac{Ce^{-\rho^2_t/4}}{\rho^2_t f'(t_i)}. \quad (6.70)$$

### 6.3 Technical Lemmas

Recall the two following lemmas (see [2]). These results ensures the convergence of the Leland scheme without any hedging error when using the modified Leland strategy. The change of variable $x = \rho^2_t$ appears to be as essential in the following proofs and points out the significative role of the revision dates near the maturity.
Lemma 6.20 We have the following equality
\[ \int_s^t \tilde{C}_{xt}(u, S_u) du = \int_{\rho_n^2}^{\rho_n^2} \tilde{C}_{xt}(u, S_u) \tilde{\sigma}_u^{-2} dx, \]
where \( u = u(x, n) \) is defined by \( x = \rho_n^2 u \) and verifies \( \lim_{n \to \infty} u(x, n) = 1 \). Moreover,
\[ C_{xt}(u, S_u) \tilde{\sigma}_u^{-2} = \frac{1}{2x} \int_{-\infty}^{\infty} h(S_\psi \exp(\sqrt{x}y + x/2)(-y^2 - \sqrt{xy} + 1) \varphi(y) dy \]
satisfies the following inequality
\[ |\tilde{C}_{xt}(u, S_u) \tilde{\sigma}_u^{-2}| du \leq c G_1(x, S_u) \]
where
\[ G_1(x, S) := \frac{1}{x} e^{-x/8} \left( \sum_{j=1}^n \frac{\log(S/K_j)}{\sqrt{x}} \exp \left( -\frac{\log^2(S/K_j)}{2x} \right) + \sqrt{x} + x \right). \]

Corollary 6.21 Assume that we have two sequences \((t_k)_n \in \mathbb{N}\) and \((s_k)_n \in \mathbb{N}\) in \([0, 1]\) such that \( \rho_{t_k} \) and \( \rho_{s_k} \) respectively converge to \( a \in [0, \infty) \) and \( b \in [0, \infty) \). Then,
\[ \lim_{n \to \infty} \int_{t_k}^{s_k} \tilde{C}_{xt}(u, S_u) du = \int_a^b J(x, S_1) dx < \infty, \text{ a.s.} \]

Proof We apply Lemma 6.20 with the change of variable \( x = \rho_n^2 \). Recall that \( 0 \leq 1 - u \leq c x n^{-1/2} \) so that \( u \to 1 \) as \( n \to \infty \) for a given \( x \geq 0 \). We can apply the Lebesgue theorem by dominating the function \( G_1(x, S_u) \) whether \( x \leq 1 \) or not because \( x \leq 1 \) implies that \( u \) is sufficiently near from 1 independently of \( x \) for \( n \geq n_0 \). Indeed, outside of the null-set \( \bigcup_i \{ S_1 = K_i \} \), we have \( 0 < a \leq |\log(S/K_j)| \leq b \) for some constants \( a, b \) (depending on \( \omega \)) provided that \( u \) is sufficiently near one.

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