On boundaries in social dynamics models with large number of agents

Valeriya Ignatovskaya\textsuperscript{1,2}, Larisa Manita\textsuperscript{1} and Anatoly Manita\textsuperscript{1,2}

\textsuperscript{1} National Research University Higher School of Economics, Moscow Institute of Electronics and Mathematics, Russia
\textsuperscript{2} Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, 119991, GSP-1, Moscow, Russia

E-mail: vaignatovskaya@miem.hse.ru, lmanita@hse.ru, manita@mech.math.msu.su

Abstract. It is known that community boundaries in social systems evolve in a rather complicated manner. We investigate several well-known models of interpersonal interaction as well as their modifications in order to establish some asymptotic results of the long time behaviour of the edge in large (or infinite) configurations of agents. We also show that the introduction of randomness in such models can lead to promising applications of some powerful probabilistic tools.

1. Introduction
The evolution of boundaries is a commonly discussed topic when studying various mathematical and physical models such as interacting particle systems, many-component systems, etc. The notion of a boundary should be understood in a wide sense. Usually, in many models it is easy to observe large domains in multi-dimensional state spaces where trajectories of dynamics behave in a homogeneous manner. Then it is reasonable to think about a “boundary” as a space between neighboring domains of homogeneity. Since shapes of these domains may vary in time it is interesting to follow how the boundaries evolve. Such evolutions can be rather complicated. In the present paper we consider some particular systems in which boundaries change by separating small clusters from large (or infinite) domains of homogeneity. We consider here a number of deterministic and stochastic models of opinion dynamics that provide an interesting clustering behaviour. In Section 2 we recall necessary definitions for the classical HK and DG models. In section 3 we expand HK model to infinite number of agents. In Section 4 we run a special variant of the classical DG dynamics on a set of semi-infinite opinion configurations and introduce a stochastic modification of this model that has a clustering behaviour similar to the one observed for the semi-infinite HK system. In Section 5 we give an example of a simplified model with countable number of agents and obtain some analytical results for it.

2. Classic models
Modern social dynamics models are a very attractive research field for physicists, mathematicians and computer scientists. These models can be regarded as sophisticated multi-component
systems (or particle systems) with special interactions between their components which are usually called participants or agents.

A base framework for models that will be discussed in this paper is the following. We start with the case of a finite number of participants and restrict ourself to discrete time models: $t = 0, 1, 2, \ldots$. At time $t$ each agent $i$ has an opinion expressed by some real number $x_i(t) \in \mathbb{R}$. Agents interact and their opinions change according to some rules. Our interest is to understand the long time behaviour of community boundaries in large configurations of agents.

Here we deal mostly with two popular models called in honor of their creators DeGroot (DG) and Hegselmann-Krause (HK) respectively. The latter of them became extremely popular after the paper [1]. The DeGroot model was proposed a little earlier in [2].

Let us start with a short description of the models. Suppose we have $n$ agents and denote by $x(t)$ the opinion vector of size $n$ at time $t$. Having initial opinion profile $x(0)$, the classical DG dynamics can be written as

$$x(t+1) = Wx(t)$$

(1)

where $W$ is an $n \times n$ stochastic matrix. Thus, the opinion at the next time epoch is a weighted sum of opinions at the previous epoch for each agent. It is important that matrix $W$ does not depend on time $t$ and opinion profile $x(t)$, so we have a linear map.

To introduce the HK model fix some positive parameter $\varepsilon$ (also called “confidence level”) and for each agent $i$ with opinion $x_i(t)$ define a set

$$I_i(t) = \{ j : |x_i(t) - x_j(t)| < \varepsilon \}.$$  

(2)

$I_i(t)$ is interpreted as a set of neighbours of the agent $i$ at the time $t$. Given an initial profile $x(0)$, set the dynamics as the average of neighbours’ opinions

$$x_i(t+1) = \frac{1}{|I_i(t)|} \sum_{j \in I_i(t)} x_j(t)$$

(3)

This is a classical HK model. Similar opinion dynamics are usually called bounded confidence models. The phrase “bounded confidence” means agents to be trusted only by those agents whose opinions are not too far from their own. The set of neighbours includes exactly such agents. As the neighbours change over time, the dynamics (3) cannot be reduced to a linear map as in (1).

Further we explain in details the key difference between these two dynamics. Recall that the weight matrix $W$ in DG model is constant. It follows that if agent $i$ impacts on agent $j$ at the dynamics beginning (i.e. $w_{ji} > 0$), the impact persists through all the dynamics. Moreover, the impact degree remains constant. But this is not true for the HK model. Indeed, each agent has a variable set of neighbours. It means that connections between different agents can arise and disappear from epoch to epoch. And even if the connection remains, the impact degree may change as the number of neighbours may change. We imagine a weighted graph corresponding to connections between agents. Thus, one may say that the DG topology is constant but the HK topology is varying.

These models demonstrate many interesting phenomena. The common problem for the both models is to find conditions implying convergence to consensus [1–3]. The HK dynamics is also useful for modelling a so-called opinion polarization phenomenon [1,3–5]. Other directions of research are multi-dimensional opinions [4] and generalizations of the model the models by adding some randomness (see, for example, [6,7] and references there).

There are many papers containing results of numerical simulations for different modifications of the HK model with uniform initial distributions of agents [1,7,8]. In the present paper we will be interested in equidistant initial configurations or in their small random perturbations.
Figures 1a) and b) shows a typical behaviour of a finite HK model with equidistant initial opinions \(x(0)\). An equidistant or almost equidistant initial configuration (which we imagine as some homogeneous environment) provides us to see sequential separation of small groups of agents. Informally we consider boundaries as separating "strips" between two areas with qualitatively different behaviour. We observe that clusters are created near to the boundaries. The newly created cluster separates from the rest of configuration and they do not communicate anymore. We see that creation of a cluster results in a local perturbation of the remaining homogeneous configuration.

3. The HK model with infinite number of agents

Usually a finite HK model freezes very fast. To analyze the process of cluster formation as long as we want we just expand classical HK model to infinite number of agents as follows. Fix some \(\delta > 0\) and consider equidistant agents lying on a positive halfline \(\mathbb{R}_+\)

\[x_j(0) = j\delta, \quad j = 0, 1, 2, \ldots\]

Set the dynamics according to the rules above (2)-(3). Note that \(\delta\) has to be less than \(\varepsilon\); otherwise, there will be no motion at all. In Fig.1c) you can see a simulation of infinite HK model.

There are a number of papers that deal with HK dynamics with equidistant agents lying on a positive halfline \(\mathbb{R}_+\). A few general words should be said before presenting them to the reader. Two opposite limit situations should be distinguished in models with bounded confidence — high and low density of agents. By density we mean number of agents per unit length. Fixing the confidence level \(\varepsilon\) we can make the density high or low by decreasing or increasing \(\delta\) respectively. If \(\delta \to \varepsilon - 0\) then the lowest agent has only two neighbours at the beginning \((t = 0)\) and other agents only three neighbours. This is the lowest limit situation and it was discussed in [9]. The authors have proved the periodicity of cluster formation for each \(\delta\) lying very close to \(\varepsilon\). More precisely, they have shown that after every 5 time steps, a group of 3 agents become disconnected at either end and collapse to a cluster at the subsequent step. Another limit situation is the high density situation. This case is more common and considered in large
number of articles. For instance, researchers in [5] perform simulations with density 100 on semi-infinite interval. Their interest is focused on the distance between two consecutive clusters. One more paper with high density of agents is [8] where the author compares agent-based models with interactive Markov chains. The main goal of [8] is to analyze the consensus phenomenon, i.e. the situation of establishing agreement between all agents. The author fixes $\delta = 1$ and takes different $\varepsilon = 10, 50, 100, 500$. In each case he observes the formation of one cluster after another roughly in quite regular manner; but no strong proof is given.

During our simulations we see periodic cases of cluster formation. Informally, we call the configuration periodic if the number of agents in cluster and the interval time of cluster formation form periodic sequences. For example, if the size of each cluster is equal to 8 and the sequence of intervals of appearance a new cluster is 6,6,7,6,6,7, etc. then the period of cluster formation is 3. To better understand the regularity of trajectories we have created a metrics which calculates the distance between two clusters' periods. Adhering to our example, we consistently impose trajectories of three first clusters on the trajectories of three second clusters superposing the vertices and compute the distance. Then we do the same with the next triple of clusters. As a result, we have a sequence of distances between each adjacent triples of clusters and show that the sequence limit is 0. We also try to find the rate of cluster formation $v^*$ which we associate with the limit of the average number of agents $C(t)$ that have formed their final opinion by the time $t$. We want to rewrite it in terms of agents' topology. If we start to mark pairs of agents that have been distanced more than $\varepsilon$ at the first time, then we get sequence of so-called splits. With these definitions in hand, we come to the following statement. For $d$-periodic case, the rate of cluster formation $v^*$ is equal to the following limit

$$\lim_{n \to \infty} \frac{m_{k+nd+d} - m_{k+nd}}{s_{k+nd+d} - s_{k+nd}}$$

where $m_{k+id}$ is the opinion of the lowest agent in $k$-th split in $i$-th period and $s_{k+id}$ is the time epoch of the split.

4. Modifications of the DG dynamics. Clustering

Similarly, we generalize the DG model to the case of infinite configurations. To do it we restrict ourself to a specific type of matrices $W$. Fix some natural $k \in \mathbb{N}$ and define the set of neighbours as

$$O_i(t) = \{ j : |i - j| \leq k \}$$

Having equidistant initial opinion profile $x(0)$, we present the DG dynamics

$$x_i(t + 1) = \frac{1}{|O_i(t)|} \sum_{j \in O_i(t)} x_j(t)$$

As the neighbours' set does not change over time, we still have a linear map. Our choice of matrix $W$ and the equidistance of initial opinions provide us to have similar local behaviour of HK and DG dynamics. However, there is no cluster formation yet. This fact motivates us to introduce a DG modification with components of randomness. In finite case suppose up to $r$ agents at the top and bottom can stop the communication and leave the process at each time epoch $t$. One agent from both sides leaves with the probability $p_1$, two with $p_2$, etc. and $p_1 + \ldots + p_r \leq 1$. The leaving process happens independently of previous epochs. Figures 2a) and b) shows the similarity in behaviour with HK model. In the case of infinite number of agents the leaving process happens only at the bottom (see Fig.2c), i.e. one lower agent leaves the process with probability $p_1$, two lower agents with probability $p_2$, etc.
5. Simplified models with infinite number of agents

To illustrate what kind of future analytical results on boundaries we can expect for the above models we consider linear dynamics of equidistant agents. Set the dynamics as

\[
\begin{align*}
x_{j+1}(0) - x_j(0) &= \delta_{j+1} \\
x_{j+1}(t) &= x_{j+1}(0) + vt
\end{align*}
\]

where

\[v\] is "the rate of opinion change", \(\delta_k = \delta + \zeta_k\)

and \(\{\zeta_k\}\) are independent identically distributed random variables such that \(E\zeta_k = 0\) and \(|\zeta_k| < \frac{\delta}{r}\). So at the time \(t = 0\) the agents are spaced at a distance \(\delta\) and slightly disturbed; then they start moving with a constant speed \(v\). Now consider a random variable \(\theta\) with distribution

\[
\theta = \begin{cases} 
0, & \text{with probability } p_0 \\
1, & \text{with probability } p_1 \\
2, & \text{with probability } p_2 \\
\vdots & \text{with probability } p_r
\end{cases}
\]

As earlier, imagine up to \(r\) lower agents leave the dynamics at each time epoch \(t\) according to the \(\theta\) distribution. In this model the boundary \(L(t)\) is just the coordinate of the lowest agent at each time \(t\). Note the following. If no agents disappear, then the coordinate is just increased by \(v\); if \(k\) agents disappear, then the coordinate is additionally increased by \(\sum_{i=1}^{k} \delta_i\).

Thus, \((L(t), t \in \mathbb{Z}_+)\) is a stochastic process. By using a standard probabilistic technique (like the Law of Large Numbers) one can prove that the boundary \(L(t)\) is asymptotically linear in the following sense

\[
\forall \varepsilon > 0 \quad \lim_{t \to \infty} P\left\{ \left| \frac{L(t)}{t} - (v + \delta \sum_{i=1}^{r} ip_i) \right| \geq \varepsilon \right\} = 0
\]
For the sake of brevity, we have intentionally chosen the simplest situation. It seems reasonable to conjecture that the above result should hold under more general assumptions. We plan to develop this topic in subsequent publications.

6. Conclusions and future work
The classic HK dynamical system is one of the famous opinion dynamics models showing nontrivial clustering formation. Despite of the simplicity of the definition a rigorous study of the HK dynamics has proved to be a rather complicated task. We proposed new modifications that are analytically tractable and we hope that this approach will be helpful in future studies of the boundary evolution problem. We plan to extend our research to more general opinion dynamics models with random transitions and randomly perturbed initial configurations.

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