Holonomic protected charge qubits coupled via a superconducting resonator

Chengxian Zhang,1 Guo Xuan Chan,2, 3 Xin Wang,2, 3, ∗ and Zheng-Yuan Xue1, 4, †

1Guangdong Provincial Key Laboratory of Quantum Engineering and Quantum Materials, and School of Physics and Telecommunication Engineering, South China Normal University, Guangzhou 510006, China
2Department of Physics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong SAR, China
3City University of Hong Kong Shenzhen Research Institute, Shenzhen 518057, China
4Guangdong-Hong Kong Joint Laboratory of Quantum Matter, Frontier Research Institute for Physics, South China Normal University, Guangzhou 510006, China

(Dated: December 29, 2020)

A key challenge for semiconductor quantum-dot charge qubits is the realization of long-range qubit coupling and performing high-fidelity gates based on it. Here, we describe a new type of holonomic protected charge (HC) qubit formed by an electron confined in a triple-quantum-dot system, enabling holonomic-protected single and two-qubit gates. We present the form for the dipole coupling between a HC qubit and a transmission-line resonator, the strength of which can exceed 200 MHz experimentally relevant parameters. Based on the hybrid system composed of the HC qubits and the resonator, we present three types of entangling gates: the holonomic entangling gate and the dynamical iSWAP gate. We find that the fidelity for the holonomic entangling gate can reach as high as 98% for the noise level typical in experiments, which is relatively higher compared to a fidelity of less than 85% for the iSWAP gate. Our results suggest that HC qubits in triple quantum dots, coupled via a superconducting resonator, can be a promising platform to implement high-fidelity semiconductor quantum computation.

I. INTRODUCTION

The charge qubit in semiconductor quantum dots [1–7] is a promising candidate to realize universal quantum computing due to its all-electrical control and fast gate operation. However, it suffers heavily from the charge noise [8], resulting in rather short coherent time and thus low gate fidelity [5]. Despite the progress over the past years, the two-qubit quantum gate-fidelity in the experiment remains below 90% [5], which motivates us to search further useful methods to design new types of charge qubits, aiming at mitigating the gate-fidelity.

As the isolated qubits are scaling up, how to implement distant and high-fidelity entangling gate between the neighbouring qubits remains another big challenge. Typically, qubit-qubit interaction can be implemented for two charge qubits via directly (capacitively) coupling between two double quantum dots (DQDs), where the interaction range is only about 100 nm [9]. With this capacitively coupling, the entangling gate can only achieve gate-fidelity lower than 70% [4, 10]. On the other hand, the electron confined in the quantum dots can form relatively large dipole moment when the dots are detuned, due to the delocalized wave-function. Therefore, the charge qubits has great potential to be coupled to the superconducting transmission-line resonator using the dipole moment. Recent experiment has demonstrated that both resonant (real) and nonresonant (virtual) resonator-mediated coherent interactions between two separated DQDs are possible [9]. The range of interaction between two charge qubits there can be substantially increased up to several tens of micrometers [9].

Recently, it is found that one electron confined in a linear triple-quantum dot (TQD) can also be used to encode the so-called charge quadrupole (CQ) qubit [11], which can benefit from the decoherence-free subspace. By using this quadrupole moment of the electron rather than the dipole moment, the long-range coupling between two CQ qubits and the resonator is experimentally realized [12]. Although the CQ qubit can work in the decoherence-free subspace, it still confronts severe leakage induced by the charge noise. To mitigate this leakage, the composite pulses sequences is needed [13], which prolongs the gate time. Worse, in the gate operation if the two tunneling values between the adjacent dots are not equal, i.e. $t_{12} \neq t_{23}$, it will cause extra leakage channel. However, strictly synchronizing these two tunneling during arbitrary gate operation seems difficult in the real experiments.

Inspired by the CQ qubit, we find that (as shown below) one electron confined in a TQD can alternatively form another type of charge qubit, i.e., the holonomic protected charge (HC) qubits. The logical basis states for the HC qubits are different from the ones for the CQ qubits. In this way, the leakage caused by the charge noise is avoided. Further, $t_{12}$ and $t_{23}$ are not required strictly equal during the gate operation. In addition, the HC qubits are using the geometric property of the phase. The geometric phase [14–16], acquired after the cyclic evolution in the parameterized subspace, is believed to be a powerful tool to combat the local noise due to its global property [17, 18]. Therefore, the geometric quantum gate based on the geometric phase has attracted many attentions to achieve high-fidelity quantum computing [19, 20]. Here, we focus on the holonomic geometric gate based on the non-Abelian geometric phase [21–24], which is especially applied to the three-energy level system for the TQD. Recently, the holonomic gate using non-adiabatic evolution have been successfully implemented for various system in experiments, including, for example, the superconducting circuits [23, 25, 26], nitrogen-vacancy centers in diamond [27–32] and nuclear magnetic resonance [33–35]. Several approaches for the non-adiabatic
geometric gates have been proposed for the charge qubits in semiconductor dot [36–40]. However, universal HC qubit manipulation in the semiconductor quantum dot is still lacking in the literature.

Here, we investigate how two spatially separated HC qubits defined in the TQD can be entangled with each other via dipole coupling to a transmission line resonator [41, 42]. The resonator field is coupled to the variation of the dipole detuning of the qubit such that the oscillation in the detuning can be controlled by the resonator voltage. We have derived the specific form of the coupling between the HC qubit and the resonator. We further estimate the coupling strength considering the present experimental parameters for the TQD and the resonator. It is found that the coupling strength is relatively high compared to the present experimental ones implemented between DQDs and the resonator. We have present two approaches to construct the entangling gates. When each HC qubit is in resonance with the resonator, one is able to achieve a holonomic entangling gate. While this hybrid system is working in the dispersive regime, an iSWAP gate is obtained. We numerically simulate the fidelity for these two entangling gates considering with the present experimental decoherence parameters [9]. We surprisingly find that the holonomic entangling gate has a fidelity as high as 98%, which is much higher than that for the iSWAP gate with fidelity less than 85%.

II. SINGLE HC QUBITS IN A TQD

A. Single-qubit implementation

As shown in Fig. 1(a), a single electron confined in a linear TQD can occupy the left, middle, and right dots, which corresponds to position states labeled by $|100\rangle$, $|010\rangle$, and $|001\rangle$, respectively, with $(N_1, N_2, N_3)$ denoting the number of the electron in the corresponding dot. The Hamiltonian within the position-state bases is [11]

$$H^{(0)} = \begin{pmatrix} \epsilon_d & t_{12} & 0 \\ t_{12} & \epsilon_q & t_{23} \\ 0 & t_{23} & -\epsilon_d \end{pmatrix}. \quad (1)$$

Here, $t_{12}$ and $t_{23}$ are the tunneling between the adjacent dots as shown in Fig. 1(b). $\epsilon_d = (U_1 - U_3)/2$ and $\epsilon_q = U_2 - (U_1 + U_3)/2$ are defined as the dipolar and quadrupolar detuning, respectively, where $U_i (i = 1, 2, 3)$ is the site potential. Note that, all the parameters here are real numbers, and we take $\hbar = 1$ for simplicity. Here, we assume each element in $H^{(0)}$ can be controlled independently via the gate voltage [11]. In the “even-odd” bases [11], spanned by $\{|E\rangle = (|100\rangle + |010\rangle)/\sqrt{2}, |C\rangle = |010\rangle, |L\rangle = (|100\rangle - |010\rangle)/\sqrt{2}\}$, $H^{(0)}$ can be transformed to

$$H_{\text{eff}} = \begin{pmatrix} 0 & t_p & 0 \\ t_p & \epsilon_q & t_m \\ 0 & t_m & 0 \end{pmatrix}, \quad (2)$$

where $t_p = (t_{12} + t_{23})/\sqrt{2}, t_m = (t_{12} - t_{23})/\sqrt{2}$. Note that, here we have set $\epsilon_d = 0$. When $t_m = 0$, i.e., $t_{12} = t_{23}$, $H_{\text{eff}}$ can form a CQ qubit in the basis spanned by $\{|E\rangle, |C\rangle\}$ leaving $|L\rangle$ being the leakage state. On the other hand, HC qubit can be alternatively implemented based on the subspace $\{|0\rangle = |E\rangle, |1\rangle = |L\rangle\}$, while the remain state $|C\rangle$ acts as an auxiliary state. To show how a HC qubit can be implemented using the three-level structure of $H_{\text{eff}}$, we parameterize the variables in $H_{\text{eff}}$ as

$$\epsilon_q = 2\Omega \sin \gamma, \quad t_p = \Omega \cos \alpha \cos \gamma, \quad (3)$$
$$t_m = \Omega \sin \alpha \cos \gamma,$$

where $\Omega = \sqrt{t_p^2 + t_m^2 + \epsilon_q^2/4}, \tan \alpha = t_m/t_p$ and $\sin \gamma = \epsilon_q/2\Omega$. Then, $H_{\text{eff}}$ can be rewritten using the dressed-state

$$H_{\text{eff}} = \Omega \sin \gamma (|C\rangle\langle C| + |b_0\rangle\langle b_0|) + \Omega \cos \gamma (|b_0\rangle\langle C| + |C\rangle\langle b_0|) + \sin \gamma (|C\rangle\langle C| - |b_0\rangle\langle b_0|), \quad (4)$$

where the dark state $|d_0\rangle$ and bright state $|b_0\rangle$ are defined as

$$|b_0\rangle = \cos \alpha |0\rangle + \sin \alpha |1\rangle,$$
$$|d_0\rangle = \sin \alpha |0\rangle - \cos \alpha |1\rangle. \quad (5)$$

The dressed states $\{|b_0\rangle, |d_0\rangle \}$ describes the same subspace as that of $\{|0\rangle, |1\rangle\}$. From Eq. 4, it is clear that, the dark
state $|d_0\rangle$ decouples the dynamics and the evolution can be regarded as the oscillation between $|C\rangle$ and $|b_0\rangle$. In this way, we can treat $|C\rangle$ and $|b_0\rangle$ as the two pseudo-spin states, where $|C\rangle\langle C| + |b_0\rangle\langle b_0| \rightarrow \hat{I}$, $|C\rangle\langle b_0| + |b_0\rangle\langle C| \rightarrow \hat{\sigma}_x$, and $|C\rangle\langle C| - |b_0\rangle\langle b_0| \rightarrow \hat{\sigma}_z$. Therefore, in the pseudo-spin representation, we can further simplify $\hat{H}_{\text{eff}}$ as

$$\hat{H}_{\text{eff}} = \Omega \sin \gamma \hat{I} + \Omega \left(\cos \gamma \hat{\sigma}_x + \sin \gamma \hat{\sigma}_z\right). \quad (6)$$

From $\hat{H}_{\text{eff}}$, one can easily calculate the evolution operator

$$\hat{U}_{\text{eff}}(t) = T e^{-i \int_0^t \hat{H}_{\text{eff}} dt},$$

which is determined by

$$\hat{U}_{\text{eff}}(T) = \left( \begin{array}{ccc} e^{-i \phi} & 0 & 0 \\ 0 & e^{-i \phi} & 0 \\ 0 & 0 & 1 \end{array} \right), \quad (8)$$

where $\phi = \pi (\sin \gamma + 1)$. Finally, in the computational bases $\{|0\rangle, |1\rangle\}$, we have

$$U_c(T) = e^{-i \frac{\Omega}{2} t} e^{-i \frac{\bar{\epsilon}}{2} (|b_0\rangle\langle b_0| - |d_0\rangle\langle d_0|)} = e^{-i \frac{\Omega}{2} t} e^{-i \frac{\delta \epsilon}{2} (\sin 2\alpha \sigma_x + \cos 2\alpha \sigma_z)}, \quad (9)$$

where $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$. Here, $U_c(T)$ represents an arbitrary single-qubit rotation around the axis in the $xz$ plane by an angle $\phi$, and the rotation axis is determined by $\alpha$. For example, by choosing $t_p = t_m$ ($t_{12} \neq 0$ and $t_{23} = 0$), we can achieve a rotation around the $x$ axis. While the rotation around the $z$ axis can be obtained by setting $t_p \neq 0$ and $t_m = 0$ ($t_{12} = t_{23}$). Further, it can be easily demonstrated that $U_c(T)$, within the dressed-state subspace $\{|b_0\rangle, |d_0\rangle\}$, actually represents a non-Abelian holonomic gate. At the final time $t = T$, we have $|\Psi(T)\rangle = \sqrt{\frac{1}{2}} (|0\rangle + |1\rangle)$, $|\Psi(0)\rangle = |\Psi(0)\rangle = |\Psi(0)\rangle = |\Psi(0)\rangle$, and $|\Psi[H_{\text{eff}}]\rangle = 0$ is also met. That is to say, both the cyclic evolution and the parallel-transport conditions for the holonomic gate are satisfied [19, 20].

### B. Single-qubit gate fidelity

When manipulating the HC qubit, the decoherence is mainly attributed to the charge noise, which causes fluctuation in both the dipolar and quadrupolar detuning. We model the noise as $\epsilon_d \rightarrow \epsilon_d + \delta \epsilon_d$ and $\epsilon_q \rightarrow \epsilon_q + \delta \epsilon_q$, where $\epsilon_d$ and $\epsilon_q$ are the mean values, while $\delta \epsilon_d$ and $\delta \epsilon_q$ are the dipolar and quadrupolar fluctuations [11]. Note that, here we have assumed $\delta \epsilon_d$ and $\delta \epsilon_q$ as the constant variables. This is always reasonable since the charge noise vary on a time scale about 100 $\mu$s [13, 44], which is much longer than the gate-time (on the scale of ns) for the HC qubit. On the other hand, even though the charge noise can also result in fluctuation for the tunneling, its magnitude is typically orders smaller than that of the detuning fluctuation [12]. In addition, we have $\delta \epsilon_d/\delta \epsilon_q \approx \alpha/R$ [11], where $\alpha$ is the interdot spacing and $R$ is the distance between the qubit and the charge impurity that causes charge fluctuation. Taking $a = 100$ nm and $R = 1$ $\mu$m, the fluctuation in the dipolar detuning can be about 10 times larger than that of the quadrupolar detuning, and thus we also neglect its effect. Therefore, the dipolar detuning fluctuation $\delta \epsilon_d$ is the dominant noise source considered here. It is easy to find that $\delta \epsilon_d$ can cause leakage for the CQ qubit but not for the HC qubit since it only occurs in the logical subspace for the HC qubit.

In Fig. 2, we show the gate fidelity for the $X_\pi$ and $Z_\pi$ rotations as a function of $\delta \epsilon_d/t_p$. The fidelity expression is considered as [13]

$$F = \frac{\text{Tr}(\hat{U}_d^\dagger \hat{U}_d) + \text{Tr}(U_{\text{target}}^\dagger \hat{U})^2}{12}, \quad (10)$$

where $U_{\text{target}}$ is the desired gate operation while $\hat{U}$ the practical one. It is found that the fidelity for both gates are about 98%, considering a typical experimental noise level of $\delta \epsilon_d/t_p \sim 0.1$ [13, 43]. To obtain the thorough understanding of the robustness for the HC qubits, we also performed the randomized benchmarking (RB) [45, 47–49], which is a powerful technique to get the average gate fidelity. Here, instead of focusing on individual gate, we pay attention to the single-qubit Clifford group [40, 44] comprising of 24 special gate operations [45, 49]. For each gate in the Clifford group, we fixed $t_p/2\pi = 10$ GHz [11] and the corresponding $t_m$ and $\bar{\epsilon}_d$ are determine by solving Eq. 3. Then, we can average the gate fidelity over multiple gate sequences, which are randomly drawn from the Clifford group. In our RB simulation we consider the Gaussian distribution noise model [45, 49]. In order to ensure convergence we have performed 1000 times of implementation. In each run of the RB implementation, $\delta \epsilon_d$ is randomly drawn from the normal Gaussian distribution: $N(0, \sigma_{\epsilon_d}^2)$, where $\sigma_{\epsilon_d}$ is related to the deviation. Normally, $\sigma_{\epsilon_d}$ in the experiment is
with the order of μeV and is minimized when \( \epsilon_d = 0 \). According to Ref. [46], \( \sigma_{\epsilon_d} \sim 2 \mu eV (\sigma_{\epsilon_d}/2\pi \sim 0.5 \) GHz). In Fig. 3, we show the benchmarking results, where average gate fidelity is versus Clifford gate number \( n \). We fit the fidelity decaying curve to [45]

\[
F = c + (1 - c)e^{-\mu n}
\]

where \( \mu \) represents the average error per gate. The fidelity curve converges to \( c \) as \( n \) increases. Here we take \( c = 0.42 \) for the simulation. Then, the average fidelity can be obtained via \( \bar{F} = 1 - \mu \). We find that the average gate fidelity is about 98\%, which is similar to the cases for \( X_\pi \) and \( Z_\pi \). Note that the qubit would also couple to the phonon baths [50] making the gate operation non-unitary and resulting in gate errors, which is not considered here and its effect is presented in Appendix A.

III. TWO-QUBIT ENTANGLING GATES MEDIATED VIA MICROWAVE PHOTONS

We now explore how to realize two-qubit gates for the HC qubits mediated by a superconducting transmission-line resonator. In Sec. III A, we begin by deriving the form for the qubit-resonator coupling when a HC qubit (TQD) is coupled to a resonator. Then, in Sec. III B, we show how to entangle two separated HC qubits mediated by the resonator. The two-qubit gates are constructed using two specific approaches, which corresponds to the working point in the resonant and dispersive regimes, respectively.

A. Dipolar coupling of a HC qubit to a superconducting transmission line resonator

We first derive the dipole transition matrix element. Considering the \( ith \) quantum dot centers at \( r_1 = -w\hat{x}, r_2 = 0, \) and \( r_3 = w\hat{x} \). The dipole operator for this triple-dot system is thus \( \mathbf{d} = -e \sum_i r_i n_i \equiv d\hat{x} \) (i=1,2,3), where \( d = \epsilon w (n_1 - n_3) \) and \( n_i = |n_i\rangle \langle n_i| \). In the position bases, the dipole operator reads

\[
d = \epsilon w \langle 100 \rangle \langle 100 | - | 001 \rangle \langle 001 | = \epsilon w \partial_{\epsilon_d} \mathcal{H}^{(0)},
\]

which implies \( n_1 - n_3 = \partial_{\epsilon_d} \mathcal{H}^{(0)} \). Introducing a small variation in the dipolar detuning \( \epsilon_d = \epsilon_0 + F \), where \( \epsilon_0 \) is the chosen operating point and \( F \) is the small variation, then we can (approximately) expand the Hamiltonian \( \mathcal{H}^{(0)} \) near the operating point as

\[
\mathcal{H} \approx \mathcal{H}_{\epsilon_d=\epsilon_0}^{(0)} + \partial_{\epsilon_d} \mathcal{H}^{(0)} \bigg|_{\epsilon_d=\epsilon_0} F,
\]

where the first term of the right hand side of Eq. (13) denotes the component of the Hamiltonian determined by \( \epsilon_0 \), while the second term is proportional to the dipole operator in Eq. (12), which we define as the dipole interaction Hamiltonian. Here, we consider choosing the operating point \( \epsilon_0 = 0 \) for the HC qubits, therefore, \( \mathcal{H}_{\epsilon_0=0}^{(0)} \) is equivalent to the single-qubit Hamiltonian \( \mathcal{H}_{\epsilon_0}^{\text{eff}} \), as described in Eq. (2). Further, defining \( \tau_z = |g\rangle \langle g| - |e\rangle \langle e| \), where \( |g\rangle \) and \( |e\rangle \) are the two lowest eigenstates of \( \mathcal{H}_{\epsilon_0=0}^{(0)} \) (see Appendix B for details), then the Hamiltonian for the small variation of the dipolar detuning can be rewritten as

\[
\mathcal{H}_{\text{HC}} = -\frac{\omega}{2} \tau_z + \mathcal{F} \eta \tau_x,
\]

where \( \omega = \sqrt{4t_p^2 + \epsilon_d^2 - \epsilon_g} \) is the energy difference between the eigenstates \( |g\rangle \) and \( |e\rangle \). Here, we have considered \( t_{12} = t_{23} \), and thus, \( \eta = \cos \theta \) and \( \tan 2\theta = 2t_p/\epsilon_g \) (see Appendix B).

Note that, here we model the TQD as a well-defined two-level system, and neglect the effect due to the second excited eigenstate \( |l\rangle \). This is owing to the fact that \( d_{el} = \langle e|d|l\rangle = 0 \) and \( d_{gl} = \langle g|d|l\rangle = 0 \). This, where \( d_{mn} = \langle m|d|n\rangle \) is defined as the dipole transition matrix element. This means that, there is no transition between \( |l\rangle \) and other lower eigenstates induced by the small variation \( \mathcal{F} \). Further, comparing Eqs. (13) and (14), one finds that

\[
d_{ge} = \langle g|d|e\rangle = \epsilon w \eta = \epsilon w \cos \theta.
\]

Considering the operating regime, \( t_p \ll \epsilon_g \), we have \( \theta \sim 0 \), thus \( d_{ge} \) is maximized. Meanwhile, in this regime, the eigenstates \( |g\rangle, |e\rangle \) are equivalent to the computational bases states \( |0\rangle, |1\rangle \) (see Appendix B). Hereafter we assume \( \theta = 0 \) and the two types of Pauli matrix are therefore identical, i.e. \( \tau = \sigma \).

Next, we determine the effective qubit-resonator coupling strength. We consider a TQD is capacitively coupled to a transmission-line resonator with the lowest-energy mode.
To make the derivation clear, we retake $\hbar$. Here, $a^\dagger (a)$ is the photon creation (annihilation) operator of the resonator, $\omega_r = \pi/LZ_0 C_0$ is the resonator frequency, $L$ denotes the length of the resonator, $C_0$ the capacitance per unit length. The characteristic impedance is $Z_0 = \sqrt{L_0/C_0}$ with $L_0$ being the inductance per unit length. Further, the effective quantized voltage across the resonator is $\hat{V}_{\text{eff}} = C_r \hat{V}/(C_r + C_d) = \chi \hat{V}$, where $C_r$ is the total capacitance between the resonator and the TQD while $C_d$ denotes the capacitance between the TQD and the ground [42]. Therefore, the interaction between the TQD and the resonator due to the voltage is

$$\mathcal{H}_{\text{int}} = -\mathbf{d} \cdot \mathbf{E} = d \hat{V}_{\text{eff}}/s = h\eta_0 (n_1 - n_3) (a + a^\dagger),$$

(17)

where

$$g_0 = \frac{ew \chi}{sLC_0} \sqrt{\frac{\pi}{Z_0 \hbar}} = \frac{ew \chi}{s}\frac{\omega_r}{\sqrt{\pi \hbar}}$$

is the vacuum Rabi coupling strength and $s$ is the effective distance related to $\hat{V}_{\text{eff}}$. On the other hand, comparing the dipole coupling Hamiltonian in Eqs. (13) and (17), the small variation in the dipole detuning is $\mathcal{F} = ew \hat{V}_{\text{eff}}/s = h\eta_0 (a + a^\dagger)$. Now, one is clear that, the resonator controls the oscillation in the dipole detuning for the HC qubit via its voltage and thus induce the transition in the qubit eigenstates. Moreover, substituting $\mathcal{F}$ into Eq. (14), the effective interaction in the computational basis can then be

$$\hat{\mathcal{H}}_{\text{int}} = h g \sigma_x (a + a^\dagger),$$

(19)

where $g = g_0 \eta = g_0 \cos \theta$ is the effective coupling strength. As stated above, we have considered $\eta = \cos \theta = 1$, i.e., $g = g_0$. To estimate the coupling strength, we consider $w = s/2$ and $\chi = 0.28$ according to the data in Refs. [42, 51, 52]. In addition, according to Eq. (18), the coupling strength is proportional to $\sqrt{Z_0}$ and $\omega_r$. From the recent experiments [9, 53, 54], where an array of high-impedance SQUID array is used to design the resonator, $Z_0$ can be as high as $1 \text{k}\Omega$. For a typical value of the resonator frequency $\omega_r/2\pi$ between 1.5 and 6.5 GHz, the coupling strength $g_0/2\pi$ is therefore between 60 and 250 MHz.

### B. Two-qubit entangling gates

We now extend the discussed case in Sec. III A that two separated HC qubits are coupled to the transmission-line resonator. The total Hamiltonian for this hybrid system consisting of two qubits and a resonator reads

$$H_{\text{tot}} = H_{\text{res}} + \sum_{k=1}^{2} \mathcal{H}_0^{(k)} + \sum_{k=1}^{2} \mathcal{H}_{\text{int}}^{(k)},$$

(20)

where $\mathcal{H}_0^{(k)}$ is the Hamiltonian for the $k$th HC qubit (TQD) as described in Eq. (2), $H_{\text{res}} = \omega_r a^\dagger a$ is the Hamiltonian for the resonator, and $\mathcal{H}_{\text{int}}^{(k)}$ represents the dipole interaction Hamiltonian between the $k$th qubit and the resonator. Transforming $H_{\text{tot}}$ into the TQD eigenbasis we have

$$H_{\text{tot}} = \omega_r a^\dagger a + \sum_k \sum_{n,m} E_n^{(k)} \sigma_{nm}^{(k)} + \sum_k \sum_{n,m} g_0^{(k)} d_{nm}^{(k)} (a + a^\dagger) \sigma_{nm}^{(k)}.$$

(21)

where $E_n$ is the eigenvalues of the HC qubits, and $\sigma_{nm} = |n\rangle \langle m|$. Further, as mentioned in Sec. III A, in the operating regime, $t_p \ll \epsilon_g$, the HC qubits can be a well-defined two-level system, and $d_{el} = d_{gl} = 0$. Therefore, $H_{\text{tot}}$ can be reduced to the so-called Tavis-Cummings form as [35]

$$H_{\text{TC}} = \omega_r a^\dagger a - \sum_{k=1}^{2} \left[ \frac{\omega^{(k)}}{2} \sigma_z^{(k)} - g^{(k)} \sigma_x^{(k)} (a + a^\dagger) \right],$$

(22)

where $\omega^{(k)} = E^{(k)}_e - E^{(k)}_g$ and $g^{(k)}$ represent the frequency and coupling strength for the $k$th qubit, respectively. Below, we present two approaches to construct the two-qubit entangling gates.

(a) Both HC qubits and the resonator are resonant, namely, $\Delta^{(k)} = \omega^{(k)} - \omega_r = 0$, where $\Delta^{(k)}$ is defined as the qubit-resonator detuning. In this way we can obtain a holonomic two-qubit gate.

(b) The two HC qubits are in resonant with each other, while they are detuned from the resonator. In the dispersive regime, i.e., $\Delta^{(k)} \gg g^{(k)}$, an iSWAP gate is implemented.

**Holonomic two-qubit gate.** In case (a), the Hamiltonian $H_{\text{TC}}$ is transformed into the rotating frame via the resonator frequency $\omega_r$

$$U_r = e^{-i\omega_r \left( a^\dagger a - \sum_{i=1}^{2} \frac{\pi}{4} \sigma_z^{(i)} \right)t},$$

(23)

which leads to

$$H_r = U_r^\dagger H_{\text{TC}} U_r - iU_r^\dagger \frac{\partial U_r}{\partial t} = \sum_{k=1}^{2} g^{(k)} \left( a^\dagger \sigma_z^{(k)} + \text{H.c.} \right).$$

(24)

Because the total number of excitations is conserved, we can further rewrite $H_r$ in a block-diagonal form. For the subspaces $S_1 = \text{span}\{|000\rangle\}$ and $S_2 = \text{span}\{|111\rangle\}$, we have $H_{r,1} = H_{r,2} = 0$ due to $\Delta^{(k)} = 0$. Here $|mnq\rangle \equiv |m\rangle |n\rangle |q\rangle \quad |g_r\rangle$ denotes the states of the first qubit, the second qubit, and the resonator from left to right. While in the single- and two-excited subspaces, which are
denoted by \( S_3 = \text{span}\{ |100\rangle, |010\rangle, |001\rangle \} \) and \( S_4 = \text{span}\{ |110\rangle, |101\rangle, |111\rangle \} \). \( H_{r,3} \) and \( H_{r,4} \) have the similar forms as

\[
H_{r,3} = g^{(1)} |001\rangle \langle 100| + g^{(2)} |001\rangle \langle 010| + \text{H.c.}, \tag{25}
\]

and

\[
H_{r,4} = g^{(1)} |110\rangle \langle 101| + g^{(2)} |110\rangle \langle 011| + \text{H.c.}. \tag{26}
\]

It is clear that \( H_{r,3} \) can form a three-level \( \Lambda \) structure \([22–24, 40]\) with transitions between \( |001\rangle \leftrightarrow |100\rangle \) and \( |001\rangle \leftrightarrow |010\rangle \). Similarly, \( H_{r,4} \) introduces such transition between \( |110\rangle \leftrightarrow |101\rangle \) and \( |110\rangle \leftrightarrow |011\rangle \).

To make the derivation clear, below we follow Ref. \([40]\) to expand the discussion on how to implement the holonomic operation using \( H_{r,3} \). The case for \( H_{r,4} \) can be understood in the same way since they have the similar Hamiltonian structure. Similar to the single-qubit case as shown in Eq. (5), \( H_{r,3} \) can also be expressed using another dressed-states representation

\[
H_{r,3} = \Omega |001\rangle \langle b'| + \text{H.c.}, \tag{27}
\]

where

\[
|b'| = \sin \frac{\theta'}{2} |100\rangle - \cos \frac{\theta'}{2} |010\rangle,
\]

\[
|d'| = \cos \frac{\theta'}{2} |100\rangle + \sin \frac{\theta'}{2} |010\rangle, \tag{28}
\]

\( \Omega' = \sqrt{(g^{(1)})^2 + (g^{(2)})^2} \) and \( \tan \theta'/2 = -g^{(1)}/g^{(2)} \). Here, the subspace composed of the dressed states \( |001\rangle \) describes the same subspace as \( S_3 \). In this representation, the dark state \( |d'\rangle \) has been dropped out of the dynamics. Therefore, \( H_{r,3} \) can be regarded as the transitions between the dressed state \( |b'\rangle \) and state \( |001\rangle \). Thus, the evolution operator with respect to \( H_{r,3} \) is

\[
U_{r,3}(t) = \exp \left( -i \int_0^t H_{r,3}(t') dt' \right) = \cos \delta(t) (|001\rangle \langle 001| + |b'\rangle \langle b'|) - i \sin \delta(t) (|001\rangle \langle b'| + |b'\rangle \langle 001|) + |d'\rangle \langle d'|, \tag{29}
\]

where \( \delta(t) = \int_0^t \Omega' dt' \). According to Eq. (29), when the cyclic condition is met, i.e., \( \delta(T') = \pi \), the evolution operator in the subspace \( S_3 \) is

\[
U_{r,3}(T') = \begin{pmatrix}
\cos \theta' & \sin \theta' & 0 \\
-\sin \theta' & \cos \theta' & 0 \\
0 & 0 & -1
\end{pmatrix}. \tag{30}
\]

As we can see that after the cyclic evolution, the operator matrix is block-diagonalized. Therefore, the excited state of the resonator \( |001\rangle \), cannot affect the qubit subspace spanned by \( \{|100\rangle, |010\rangle\} \). For an arbitrary state initialized in the subspace spanned by \( S_3 = \text{span}\{ |\psi_1(0)\rangle, |\psi_2(0)\rangle \} \) where

\[
|\psi_1(0)\rangle = \alpha' |100\rangle + \beta' |010\rangle,
\]

\[
|\psi_2(0)\rangle = \beta'^* |100\rangle - \alpha'^* |010\rangle, \tag{31}
\]

the corresponding final states under the action of \( H_{r,3} \) satisfy the parallel-transport condition \([20]\), i.e.,

\[
\langle \psi(0) | U_{r,3}^\dagger(T') | H_{r,3}(t) | U_{r,3}(0) \psi(0) \rangle = 0. \tag{32}
\]

Here, \( \alpha', \beta' \in \mathbb{C} \) and \( |\alpha'|^2 + |\beta'|^2 = 1 \). Therefore, \( U_{r,3}(T') \) represents a holonomic operation in the qubit subspace \( \{|100\rangle, |010\rangle\} \). Then, in the complete computational subspace (also the zero-photon subspace) \( \{|000\rangle, |010\rangle, |100\rangle, |110\rangle\} \), we have the holonomic two-qubit gate as

\[
U_{\text{ent}}(\theta') = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta' & \sin \theta' & 0 \\
0 & -\sin \theta' & \cos \theta' & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}. \tag{33}
\]

Note that the negative sign in the bottom right is owing to the evolution in the single-excited subspace \( H_{r,4} \) \([56]\). As demonstrated in Ref. \([40]\), \( U_{\text{ent}}(\pi/2) \) denotes a SWAP-type two-qubit entangling gate.

**iiSWAP gate.** For approach (b), we expand the discussion of Refs. \([41, 42]\) on the construction of the iiSWAP gate when \( \Delta^{(k)} \gg g^{(k)} \). The effective Hamiltonian for \( H_{TC} \) can be further simplified by using the Schrieffer-Wolff transformation \([41]\), which can eliminate the direct coupling between the HC qubit and the resonator:

\[
H_d = H_{d,0} + \frac{1}{2} [S, V], \tag{34}
\]

where \( H_{d,0} \) denotes the free Hamiltonian for the resonator and the two HC qubits

\[
H_{d,0} = \omega_r a^\dagger a - \sum_{k=1}^2 \frac{g^{(k)}}{\Delta^{(k)}} \sigma_z^{(k)} \tag{35}
\]

\( V \) the dipole interaction Hamiltonian for individual qubit

\[
V = \sum_{k=1}^2 g^{(k)} \sigma_z \ (a + a^\dagger) \tag{36}
\]

and \( S \) the transformation operator

\[
S = \sum_{k=1}^2 \frac{g^{(k)}}{\Delta^{(k)}} (a^\dagger \sigma_-^{(k)} - \sigma_+^{(k)} a). \tag{37}
\]

Combining Eqs. (33) and (36), the resulting approximated Hamiltonian is

\[
\tilde{H}_d = H_{d,0} + \sum_{k=1}^2 \left( \frac{g^{(k)}}{\Delta^{(k)}} \sigma_-^{(k)} \sigma_+^{(k)} - \frac{g^{(k)}}{\Delta^{(k)}} \sigma_+^{(k)} \sigma_-^{(k)} \right) a^\dagger a - \frac{g^{(k)}}{\Delta^{(k)}} \sigma_-^{(k)} \sigma_+^{(k)} - \chi' (\sigma_+^{(1)} \sigma_-^{(2)} + \sigma_-^{(1)} \sigma_+^{(2)}), \tag{38}
\]

where \( \chi' = g^{(1)} g^{(2)} (\Delta^{(1)} + \Delta^{(2)}) / [2 \Delta^{(1)} \Delta^{(2)}] \). If we restrict us to the zero-photon subspace (computational subspace), \( H_d \) can be further reduced to

\[
\tilde{H}_d = \sum_{k=1}^2 \frac{\tilde{g}^{(k)}}{2} \sigma_z^{(k)} - \chi' (\sigma_+^{(1)} \sigma_-^{(2)} + \sigma_-^{(1)} \sigma_+^{(2)}), \tag{39}
\]
where \( \tilde{\omega}^{(k)} = -\omega^{(k)} + (g^{(k)})^2 / \Delta^{(k)} \), lifting the zero-point energy by \( \sum_{k=1}^{2} g^{(k)} / \Delta^{(k)} \). Under the Schrieffer-Wolff transformation, the direct coupling between the qubit and the resonator has been safely eliminated and this approximation is correct to first order in \( g^{(k)} / \Delta^{(k)} \). Further, we transform \( \tilde{H}_d \) into a rotating frame via

\[
U_d = \exp \left[-i \sum_{k=1}^{2} \frac{\tilde{\omega}^{(k)}}{2} \sigma_z^{(k)} t \right],
\]

which leads to the effective Hamiltonian

\[
\tilde{H}_d = U_d \tilde{H}_d U_d - i U_d \frac{\partial U_d}{\partial t} = -\chi' \left( \sigma_+^{(1)} \sigma_-^{(2)} + \sigma_-^{(1)} \sigma_+^{(2)} \right),
\]

where we have considered \( \chi' = \tilde{\omega}^{(2)} \). The evolution operator of the Hamiltonian \( \tilde{H}_d \) is thus

\[
U_d(t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \chi' t & i \sin \chi' t & 0 \\
0 & i \sin \chi' t & \cos \chi' t & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

When \( \chi' t = \pi / 2 \), \( U_d \left( \frac{\pi}{2\chi'} \right) \) is equivalent to an iSWAP gate.

**Two-qubit gate fidelity.** To simulate the gate fidelity for the two-qubit gates, we consider using the master equation as [41]

\[
\dot{\rho} = -i [H_{\text{tot}}, \rho] + \mathcal{L}_1 \rho + \mathcal{L}_\varphi \rho + \mathcal{L}_\alpha \rho,
\]

where

\[
\mathcal{L}_1 \rho = \sum_{k=1,2} \Gamma_{ge}^{(k)} \mathcal{D}[|e\rangle \langle l|] + \Gamma_{gl}^{(k)} \mathcal{D}[|g\rangle \langle l|] + \Gamma_{dI}^{(k)} \mathcal{D}[|l\rangle \langle e|],
\]

\[
\mathcal{L}_\varphi \rho = \sum_{k=1,2} \frac{1}{2} \Gamma_{\varphi}^{(k)} \mathcal{D}[|l\rangle \langle e| - |e\rangle \langle l| - |g\rangle \langle g|],
\]

\[
\mathcal{L}_\alpha \rho = \Gamma_a \mathcal{D}[a],
\]

and

\[
\mathcal{D}[\hat{L}] = (2L\rho L^\dagger - L^\dagger L \rho - \rho L^\dagger L) / 2.
\]

Here, \( \mathcal{L}_1 \rho \) and \( \mathcal{L}_\varphi \rho \) denote the possible relaxation and dephasing baths for each qubit, while \( \mathcal{L}_\alpha \rho \) the internal decay of the resonator. From the recent experiment in Ref. [9], we consider \( g^{(1)} = g^{(2)} = g = 2\pi \times 69 \text{ MHz} \) (corresponding to \( \omega_r / 2\pi \sim 1.77 \text{ GHz} \)), and \( \Gamma_{ge} / 2\pi = \Gamma_{gl} / 2\pi = \Gamma_dL / 2\pi = 11 \text{ MHz} \). While we set the dephasing rate to be zero, i.e., \( \Gamma_{\varphi}^{(1)} = \Gamma_{\varphi}^{(2)} = 0 \), because in the experiment, the relaxation rate and the pure dephasing rate are indistinguishable from the measurement and their effects have been combined together. In addition, according to the recent experiment [12], the internal decay of the resonator can be as low as \( \Gamma_{\alpha} / 2\pi = 4 \text{ MHz} \). We consider the initial state of the coupled system as \( |00\rangle \). Under the action of \( U_{\text{ext}} (\theta = \pi / 2) \), the final state is expected to be \( |100\rangle \) without decoherence effect. As shown in Fig. 4(a), we find the population (fidelity) for the state \( |100\rangle \) is about 96% at the final evolution time \( T = \pi / \Omega' \). We also explore the dependence of the fidelity on the coupling strength \( g \). As shown in Fig. 4(c), by keeping the decoherent parameter unchanged, its fidelity can exceed 98% considering \( g / 2\pi = 100 \text{ MHz} \), which corresponds to the resonator frequency \( \omega_r / 2\pi \sim 2.57 \text{ GHz} \). In Fig. 4(d), we plot the fidelity of the iSWAP gate as function of \( \Delta / g \). We find that, the maximum fidelity appears when \( \Delta / g \sim 3.5 \) with the fidelity less than 85% for comparison, which is significantly small compared to the holonomic gate. We attribute this to the overly long evolution time for the iSWAP gate, whose evolution time is \( \sqrt{2\Delta / g} \sim 4.95 \text{ times longer than the holonomic gate} \). Comparing Fig. 4(c) and (d), we conclude that the holonomic gate can offer substantial fidelity improvement for the two-qubit operation when considering the decoherence level in the realistic environment.

**IV. CONCLUSION**

We have proposed the implementation of coupling two separated HC qubits via the high-impedance resonator. We have derived the specific expression for the qubit-resonator coupling strength. For the typical parameter of the TQD and the resonator we find that the coupling strength can exceed 200 MHz, which is relatively high compared to the present experimental ones. Based on the hybrid system consisting of the HC qubits and the resonator we have implemented high-fidelity entangling gates. In our work, both single- and two-qubit gates can be protected by the holonomic operations resulting gate fidelity around 98%. Our results suggests that the
charge qubit implemented in TQD can benefit from the 
holonomic operation.

**ACKNOWLEDGMENTS**

This work was supported by the Key-Area Research 
and Development Program of GuangDong Province (Grant 
No. 2018B030326001), the National Natural Science 
Foundation of China (Grant Nos. 11905065, 11874156, 11874312), China Postdoctoral Science Foundation (Grant 
No. 2019M652928), the Research Grants Council of 
Hong Kong (No. CityU 11303617), the Guangdong 
Innovative and Entrepreneurial Research Team Program (No. 2016ZT06D348), and Science and Technology Program of 
Guangzhou (No. 2019050001).

**Appendix A: Phonon effect in TQD**

Here, we introduce how the qubit is coupled to the photon 
baths and estimate the magnitude of the relaxation rate. 
The electron-phonon interaction takes the form as [50]

$$
H_{ep} = \Xi_d \text{Tr} \varepsilon + \Xi_u \varepsilon_{zz},
$$

$$
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} \right),
$$

$$
u = \sum_{q,s'} \sqrt{\frac{\rho_v}{2\rho_{Si} V}} e_{q,s'} \left( b_{q,s'} + \frac{\partial \varepsilon_{zz}}{\partial \varepsilon_{zz}} b^\dagger_{q,s'} \right) e^{i \varepsilon_{zz}},
$$

where $b_{q,s'}$ and $b^\dagger_{q,s'}$ are the annihilation and creation operator for 
the phonons with the subscript $q$ denoting the wave 
vector and $s' \in \{ \varepsilon, t_1, t_2 \}$ corresponds to the longitudinal 
and transverse modes, where $e_{q,t_1} = q/s$, $e_{q,t_2} = -e_{q,t_1}$ and 
$e_{q,t_3} = -e_{q,t_2}$. $\Xi_d$ and $\Xi_u$ represent piezoelectric potential 
constants, $\varepsilon$ the displacement operator, $r$ the position vector, 
$\rho_v$ the mass density in bulk silicon, $V$ the considered volume, 
and $v_{s'}$ the speed of sound for different modes. The relaxation 
rate of the transition process between the initial state $|i\rangle$ and 
final state $|f\rangle$ owing to $H_{ep}$ is given by the Fermi’s golden 
rule as

$$
\Gamma_{if} = \frac{2\pi}{h} \sum_{q,s'} \left| \left\langle f | H_{ep}^{q,s'} | i \right\rangle \right|^2 \delta \left( E_i - E_f + i\omega_{q,s'} \right).
$$

Here, $|k\rangle$ $(k = g, e, l)$ and $H_{ep} = \sum_{q,s'} H_{ep}^{q,s'}$ are 
rewritten in the eigenstates basis. A simple model where $\epsilon_q = 0$ and

$t_{12} = t_{23} = t'$ (corresponding to $Z_x$ rotation) suggests that 
the dominant relaxation process are mainly $|l\rangle \rightarrow |g\rangle (\Gamma_{gl})$ 
and $|l\rangle \rightarrow |e\rangle (\Gamma_{el})$ where

$$
\Gamma_{el} \propto |t'|^3 \left| \exp \left( -\frac{i\sqrt{2}t'}{v_{t_1}} \right) - \exp \left( \frac{i\sqrt{2}t'}{v_{t_1}} \right) \right|^2,
$$

$$
\Gamma_{gl} \propto |t'|^3 \left| \exp \left( -\frac{i\sqrt{2}t'}{v_{t_1}} \right) + \exp \left( \frac{i\sqrt{2}t'}{v_{t_1}} \right) \right|^2 - 2.
$$

It is found that, both $\Gamma_{el}$ and $\Gamma_{gl}$ are increasing as $a$ and $t_p$ becomes 
large. Here, we estimate the relaxation rate considering 
typical experimental parameters as $v_{t_1} = 5.4 \times 10^3$ nm/s, 
$\epsilon_d = \epsilon_q = 0$, $t_p/2\pi = 10$ GHz and $a = 90$ nm. It is 
found that $\Gamma_{el}$ and $\Gamma_{gl}$ are with the magnitude of $\sim 10^7$, i.e. 
$\sim 10$ MHz, which is much smaller than the $t_p$ value in the 
single-qubit simulation.

**Appendix B: Eigenstates for TQD**

When performing the two-qubit operation, we consider the 
dipolar detuning $\epsilon_0 = 0$ as the operating point for the HC 
quibs. Then, the eigenvalues for $\mathcal{H}_0^{(0)}$ (or $\mathcal{H}_0^{(0)}$) is

$$
E_g = \left( \epsilon_q - \sqrt{4\epsilon_p^2 + \epsilon_q^2} \right)/2,
$$

$$
E_e = 0,
$$

$$
E_l = \left( \epsilon_q + \sqrt{4\epsilon_p^2 + \epsilon_q^2} \right)/2,
$$

and the energy difference between the two lowest states is

$$
E_e - E_g = \left( \sqrt{4\epsilon_p^2 + \epsilon_q^2} - \epsilon_q \right)/2 = \omega,
$$

which corresponds to the eigenstates as

$$
|g\rangle = \cos \theta |E\rangle - \sin \theta |C\rangle,
$$

$$
|e\rangle = |L\rangle,
$$

$$
|l\rangle = \sin \theta |E\rangle + \cos \theta |C\rangle,
$$

where $|g\rangle$ is the ground state while $|e\rangle$ and $|l\rangle$ are the first and 
second excited state, respectively. Note that, here we have 
considered $t_{12} = t_{23}$ and thus $\tan \theta = 2t_p/\epsilon_q$. In this way we 
have $\langle e | d | l \rangle = \langle g | d | l \rangle = 0$ and $\langle g | d | e \rangle = \cos \theta$. In the 
regime where $t_p \ll \epsilon_q$, sin $\theta \sim 0$ and cos $\theta \sim 1$, therefore we have 
$|g\rangle \sim |E\rangle$ and $|l\rangle \sim |C\rangle$.

[1] G. Shinkai, T. Hayashi, T. Ota, and T. Fujisawa, Phys. Rev. 
Lett. 103, 056802 (2009).
[2] K. D. Petersson, J. R. Petta, H. Lu, and A. C. Gossard, Phys. 
Rev. Lett. 105, 246804 (2010).
[3] G. Cao, H.-O. Li, T. Tu, L. Wang, C. Zhou, M. Xiao, G.-C. Guo, 
H.-W. Jiang, and G.-P. Guo, Nat. Commun. 4, 1401 (2013).
[4] H.-O. Li, G. Cao, G.-D. Yu, M. Xiao, G.-C. Guo, H.-W. Jiang, 
and G.-P. Guo, Nat. Commun. 6, 7681 (2015).
[5] D. Kim, D. Ward, C. Simmons, J. K. Gamble, R. Blume-Kohout, E. Nielsen, D. Savage, M. Lagally, M. Friesen, S. Coppersmith, et al., Nat. Nanotechnol. 10, 243 (2015).
[6] D. R. Ward, D. Kim, D. E. Savage, M. G. Lagally, R. H. Foote,
