CONTACT GEOMETRY OF ONE DIMENSIONAL HOLOMORPHIC FOLIATIONS

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ABSTRACT. Let $V$ be a real hypersurface of class $C^k$, $k \geq 3$, in a complex manifold $M$ of complex dimension $n+1$. $HT(V)$ the holomorphic tangent bundle to $V$ giving the induced CR structure on $V$. Let $\theta$ be a contact form for $(V, HT(V))$, $\xi_0$ the Reeb vector field determined by $\theta$ and assume that $\xi_0$ is of class $C^k$. In this paper we prove the following theorem (cf. Theorem 4.1): if the integral curves of $\xi_0$ are real analytic then there exist an open neighbourhood $M_0 \subset M$ of $V$ and a solution $u \in C^k(M_0)$ of the complex Monge-Ampère equation $(dd^c u)^{n+1} = 0$ on $M_0$ which is a defining equation for $V$. Moreover, the Monge-Ampère foliation associated to $u$ induces on $V$ that one associated to the Reeb vector field. The converse is also true. The result is obtained solving a Cauchy problem for infinitesimal symmetries of CR distributions of codimension one which is of independent interest (cf. Theorem 5.1 below).

1. INTRODUCTION

We follow [7], [8] for standard notations in differential geometry and complex manifolds.

Let $M$ be a complex manifold of complex dimension $n+1$ with complex structure $J$ on the tangent bundle $T(M)$.

A function $u \in C^2(M)$ is a solution of the complex Monge-Ampère equation if and only if

$$MA(u) = (dd^c u)^{n+1} = (2i\partial \bar{\partial} u)^{n+1} = 0$$

where $d^c = i(\bar{\partial} - \partial)$; in local holomorphic coordinates $z_1, \ldots, z_{n+1}$

$$\det \left( \frac{\partial^2 u}{\partial \bar{z}_\alpha \bar{z}_\beta} \right) = 0.$$ 

If $\omega$ is a 1–differential form, then we denote by $\omega^c$ the 1–differential form which satisfies $\omega^c(X) = -\omega(JX)$, for each vector field $X$, so that if $f \in C^1(M)$ then $(df)^c = d^c f$.

If $V \subset M$ is a real hypersurface of class $C^k$, $k > 1$, set

$$HT(V) = \{ X \in T(V) \mid JX \in T(V) \}.$$
Then $HT(V)$ is a $J$-invariant distribution of real dimension $2n$, called the \textit{holomorphic tangent bundle} to $V$. The distribution $HT(V)$ gives $V$ a CR structure, that induced by the complex structure of $M$; $V$ endowed with this CR structure is denoted by $(V, HT(V))$.

We recall that the Levi form of $V$ is non degenerate if, and only if, locally there exists a real differential form $\theta$ of degree 1 on $V$ such that the restriction to $HT(V)$ of the skew 2-form $d\theta$ is non degenerate. In this case $(V, HT(V))$ is a contact manifold with a contact form $\theta$, also said a \textit{contact CR hypersurface}.

We refer to \cite{4} and \cite{9} for basic facts on contact geometry.

A vector field $\xi$ on the contact manifold $(V, HT(V))$ is an \textit{infinitesimal symmetry} if $[\xi, HT(V)] \subset HT(V)$; an infinitesimal symmetry such that $\xi(p) \in HT_p(V)$ for each point $p \in V$ is a \textit{characteristic vector field} for the distribution $HT(V)$ (see e. g. Section 1.2 of \cite{9}).

If $\theta$ is a contact form for the contact manifold $(V, HT(V))$ then there exists a unique vector field $\xi_0$ on $V$, the \textit{Reeb vector field}, which satisfies $\theta(\xi_0) = 1$ and $d\theta(\xi_0, X) = 0$ for each vector field $X$ on $V$. It is easy to show that $\xi_0$ is an infinitesimal symmetry of the distribution $HT(V)$.

Let $V \subset M$ be a hypersurface of class $C^k$. A real function $u \in C^k(M)$ is called an \textit{equation} of $V \subset M$ if $V = \{u = 0\}$ and $du \neq 0$ near $V$. Then the restriction of the form $d^c u$ to $T(V)$ is a real 1-form which defines the CR structure $HT(V)$. Observe that $V$ is a contact CR hypersurface if, and only if, the $2(n+1)$ form $du \wedge d^c u \wedge (dd^c u)^n$ does not vanish on (a neighbourhood of) $V$.

In this paper we are interested in studying the existence of equations $u \in C^k(M)$ of a given hypersurface $V \subset M$ which are solution of the complex Monge-Ampère equation $(dd^c u)^{n+1} = 0$ (where $n+1$ is the complex dimension of $M$). We prove the following theorem (cf. Theorem \cite{4,1}): if the integral curves of $\xi_0$ are real analytic then there exist an open neighbourhood $M_0 \subset M$ of $V$ and a solution $u \in C^k(M_0)$ of the complex Monge-Ampère equation $(dd^c u)^{n+1} = 0$ on $M_0$ which is a defining equation for $V$. Moreover, the Monge-Ampère foliation associated to $u$ induces on $V$ that one associated to the Reeb vector field. The converse is also true.

The result is obtained solving a Cauchy problem for infinitesimal symmetries (cf. Theorem \cite{3,1} below) of CR distributions of codimension one which is of independent interest.

As for the contents of the paper, in Section \cite{2} we define the notion of \textit{calibrated foliation} which is nothing but than a pair $(\xi, u)$ where $\xi$ is a vector field on the complex manifold $M$ such that $[\xi, J\xi] = 0$ and $u$ is a function on $M$ which satisfy $d^c u(\xi) = 0$ and $du(\xi) = 1$. If $(\xi, u)$ is calibrated foliation then the vector field $\xi$ induces on $M$ a holomorphic foliation whose leaves are Riemann surfaces. The main result of the section is, roughly speaking, that the set $Z$ of the points of the complex manifold $M$ where the vector field $\xi$ is an infinitesimal symmetry for the distribution $\text{Ker} du \cap \text{Ker} d^c u$ intersects each leaf $S$ of the holomorphic foliation along an analytic subset of
S; hence either \( S \subset Z \) or \( S \cap Z \) is a discrete subset of \( S \) (cf. Theorem 2.1). Here the basic tools are provided by the theory of the generalized analytic functions, developed extensively in [6], dealing with functions which satisfy a first-order complex linear differential system of equations having the Cauchy-Riemann operator as principal part symbol (cf. Theorem 2.2).

In Section 3 we prove that if \( V \subset M \) is a real hypersurface of the complex manifold \( M \) and \( \xi_0 \) is a vector field on \( V \) having real analytic integral curves then there exists locally a unique calibrated foliation \((\xi, u)\) in a neighborhood of \( V \) in \( M \) such that \( u \) vanishes on \( V \) and \( \xi \) extends \( \xi_0 \). The results of the previous section are used in order to show that the vector field \( \xi \) is an infinitesimal symmetry of the distribution \( \text{Ker} \, du \cap \text{Ker} \, d^c u \) if, and only, if \( \xi_0 \) is an infinitesimal symmetry of the distribution \( H^T(V) \) (cf. Theorem 3.1).

If \((\xi, u)\) is a calibrated foliation and \( \xi \) is an infinitesimal symmetry of the distribution \( \text{Ker} \, du \cap \text{Ker} \, d^c u \) then \( u \) is a solution of the complex Monge-Ampère equation on \( M \). This elementary observation yields to the main result of Section 4 on the existence, for a Levi non degenerate CR real hypersurface \( V \) with assigned contact form \( \theta \), of an equation which is solution of the complex Monge-Ampère. The result is that such equation exists if, and only if, the Reeb vector field \( \xi_0 \) associated to the contact manifold \((V, \theta)\) has real analytic integral curves (cf. Theorem 4.1). Indeed in this case by the results of the previous section there exists a unique calibrated foliation \((\xi, u)\) in a neighborhood of \( V \) such that \( \xi \) extends the Reeb vector field \( \xi_0 \); then, since the Reeb vector field is an infinitesimal symmetry of the corresponding contact structure, \( u \) is a solution of the complex Monge-Ampère equation.

Finally, let us observe that, by a result of Andreotti and Fredricks (cf. [1], Theorem 1.12) any real analytic codimension one CR manifold with integrable real analytic CR distribution embeds in some complex manifold as a CR hypersurface. Thus, our theory applies to these abstract CR manifolds too.

2. CALIBRATED ONE DIMENSIONAL FOLIATIONS

Let \( M \) be a complex manifold of dimension \( n + 1 \) with (integrable) complex structure \( J \).

Let \( \xi \) be a vector field on \( M \) and let \( u \) be a function on \( M \). We say that the pair \((\xi, u)\) is a calibrated foliation of dimension one (or simply a calibrated foliation) of class \( C^k \), \( k > 0 \), if \( \xi \) and \( u \) are of class \( C^k \) and satisfies the conditions

\[
(1) \quad [\xi, J\xi] = 0, \\
(2) \quad d^c u(\xi) = 0, \quad d^c u(\xi) = 1.
\]

From \( d^c u(\xi) = 1 \) it follows that \( d^c u \) never vanishes identically at any point \( p \in M \) and hence for each constant \( c \in \mathbb{R} \) the subset of points \( p \in M \) which satisfy \( u(p) = c \) is either the empty set or is a real hypersurface of \( M \) of class \( C^k \).
Clearly the vector fields $\xi$ and $J_\xi$ generate a $J$-invariant bidimensional distribution on $TM$ whose maximal (connected) integral submanifolds are Riemann surfaces which fill the manifold $M$. We call such Riemann surfaces the leaves of the calibrated foliation $(\xi, u)$.

Let $S$ be a leaf of the calibrated foliation $(\xi, u)$. An adapted holomorphic coordinate on $S$ is a holomorphic map $z : A \to \mathbb{C}$ where $A \subset S$ is open in $S$ and $\text{Im} z = u|_A$. It is straightforward to prove that for each $p \in S$ there exists an adapted holomorphic coordinate on $S$ defined in a neighbourhood of $p$ in $S$. This shows in particular that the restriction of the function $u$ to each leaf $S$ is a harmonic function on $S$.

If $(\xi, u)$ is a calibrated foliation then $\text{Ker} du \subset T(M)$ is an integrable distribution having as maximal integrable submanifold the connected components of the hypersurfaces defined by $u = c$, $c$ real constant. Let us observe that if $V$ is a (maximal) integral submanifold of $\text{Ker} du$ and if $S$ is a leaf of $(\xi, u)$ then $V \cap S$ is a (maximal) integral curve of the vector field $\xi$, and each (maximal) integral curve of the vector field $\xi$ is obtained in this way.

Let $(\xi, u)$ be a calibrated foliation. We define the contact locus of the calibrated foliation $(\xi, u)$ as the set $Z$ of the points $p \in M$ where the differential form $L_\xi (d^c u)$ vanishes. Here $L_\xi$ stands for the Lie derivative with respect to the vector field $\xi$. We also say that $(\xi, u)$ is a contact calibrated foliation if $Z = M$. It is easy to show that $\xi$ is a characteristic vector field of the distribution $\text{Ker} du$ and that $(\xi, u)$ is a contact calibrated foliation if, and only if the vector field $\xi$ is an infinitesimal symmetry for the distribution $\text{Ker} du \cap \text{Ker} d^c u$ (cf. e. g. Theorem 1.2.1 of [9]).

The main result of this section is the following:

**Theorem 2.1.** Let $(\xi, u)$ be a calibrated foliation of class $C^k$, $k \geq 2$, on the complex manifold $M$ with contact locus $Z$.

If $S$ is a leaf of the calibrated foliation $(\xi, u)$ then $S \cap Z$ is an analytic subset of $S$, that is $S \cap Z \subset S$ is locally defined by the common zeroes of holomorphic function on $S$ and hence either $S \subset Z$ or $S \cap Z$ is a discrete subset of $S$.

Before getting involved in the proof let us observe an immediate consequence of Theorem 3.12 of [6]:

**Theorem 2.2.** Let $D \subset \mathbb{C}$ be an open domain and let $w_i \in C^1(D), i = 1, \ldots, p$ and $A_{ij}, B_{ij} \in C^0(D), i, j = 1, \ldots, p$ be complex functions. Assume that

$$\frac{\partial w_i}{\partial \bar{z}} = \sum_{j=1}^{p} A_{ij} w_j + B_{ij} \bar{w}_j \quad i = 1, \ldots, p$$

holds on $D$. Then the common zeroes of the functions $w_i$ is an analytic subset of $D$ and hence either the functions $w_i$ vanishes identically on $D$ or the common zeroes of the functions $w_i$ is a discrete set of $D$. 

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Solutions of (4) are called generalized analytic functions. We also summarize by the following lemma some elementary fact that will be used in the sequel.

**Lemma 2.1.** Let $M$ a complex manifold and let $X, Y$ be vector fields on $M$.

If $f \in C^2(M)$ then

(4) \[ \dd^c f (JX, JY) = \dd^c f (X, Y). \]

If $\theta$ is a differential form on $M$ of degree one and $\theta(X) = c_1$, $\theta(Y) = c_2$ with $c_1$ and $c_2$ constant then

(5) \[ d\theta(X, Y) = (L_X \theta)(Y) = -\theta([X, Y]). \]

If $f \in C^2(M)$ and $d f(X) = c_1$, $d f(Y) = c_2$ with $c_1$ and $c_2$ constant then

(6) \[ d f ([X, Y]) = 0. \]

**Proof of theorem 2.1.** We will prove the theorem showing that for each leaf $S$ the set $S \cap Z$ is locally the set of common zeroes of a set of functions $w_1, \ldots, w_n$ which satisfy a system of equations of the form (3). Let $n + 1$ be the dimension of $M$. Let $S$ be a leaf. Let $\rho \in S$. For a neighbourhood $U$ of $\rho$ in $M$ small enough there exist an adapted holomorphic coordinate $z : S \cap U \to \mathbb{C}$ and $C^2$ vector fields $X_1, \ldots, X_n$ on $U$ such that $\xi, X_1, JX_1, \ldots, X_n, JX_n$ generate the distribution $\text{Ker} \, du$.

Of course $X_1, JX_1, \ldots, X_n, JX_n$ generate the distribution $\text{Ker} \, du \cap \text{Ker} \, d^c u$ and $\xi, J_\xi, X_1, JX_1, \ldots, X_n, JX_n$ generate the whole $T(M)$.

Setting $\omega = L_\xi d^c (u)$ and for $i = 1, \ldots, n$ let define $u_i = \omega(X_i)$, $v_i = \omega^c (X_i)$.

Since $\omega(\xi) = \omega(J_\xi) = 0$ it follows that $Z \cap U$ is exactly the common zero set of the functions $u_i, v_i$, $i = 1, \ldots, n$.

We now first prove that for $i = 1, \ldots, n$,

(7) \[ d^c u([X_i, \xi]) = u_i, \]

(8) \[ d^c u([JX_i, \xi]) = v_i, \]

(9) \[ d^c u([X_i, J\xi]) = -v_i, \]

(10) \[ d^c u([JX_i, J\xi]) = u_i. \]

Indeed, since $d^c u(X_i) = d^c u(JX_i) = 0$,

\[ u_i = L_\xi d^c u(X_i) = \xi \left( d^c u(X_i) \right) = d^c u([\xi, X_i]) = d^c u([X_i, \xi]). \]

and

\[ v_i = L_\xi d^c u(JX_i) = \xi \left( d^c u(JX_i) \right) = d^c u([\xi, JX_i]) = d^c u([JX_i, \xi]), \]

which proves (7) and (8).

Using (4), (5), and the identity $J^2(X) = -X$ we obtain

\[ d^c u([X_i, J\xi]) = -d^c u(JX_i) = d^c u(JX_i, \xi) = -v_i. \]
and
\[ d^c u([JX_i, J\xi]) = -dd^c u(JX_i, J\xi) = -dd^c u(X_i, \xi) = d^c u([X_i, \xi]) = u_i. \]

which proves (9) and (10).

Then we prove that for \( i = 1, \ldots, n \)

(11) \[ [X_i, \xi] = u_i \xi + \sum_{j=1}^{n} (a_{ij}X_j + b_{ij}JX_j), \]

(12) \[ [JX_i, \xi] = v_i \xi + \sum_{j=1}^{n} (c_{ij}X_j + d_{ij}JX_j), \]

(13) \[ [X_i, J\xi] = -v_i \xi + \sum_{j=1}^{n} (e_{ij}X_j + f_{ij}JX_j), \]

(14) \[ [JX_i, J\xi] = u_i \xi + \sum_{j=1}^{n} (g_{ij}X_j + h_{ij}JX_j), \]

where \( a_{ij}, \ldots, h_{ij} \) are \( C^1 \) functions on \( U \).

Indeed consider first \([X_i, \xi]\). Since \( d^c u(X_i) = 0, d^c u(\xi) = 0 \) by (6) it follows that \([X_i, \xi] \in \text{Ker} \, d^c u \) and hence

(15) \[ [X_i, \xi] = \lambda_i \xi + \sum_{j=1}^{n} (a_{ij}X_j + b_{ij}JX_j), \]

where \( \lambda_i, a_{ij} \) and \( b_{ij} \) are some \( C^1 \) functions on \( U \).

Since \( d^c u(\xi) = 1 \) and \( d^c u(X_i) = d^c u(JX_i) = 0 \) it follows that

(16) \[ u_i = d^c u([X_i, \xi]) = \lambda_i, \]

and this proves (11).

The proofs of (12), (13) and (14) are similar.

Finally we claim that

(17) \[ d^c u([X_i, \xi], J\xi)) = -J\xi(u_i) + \sum_{j=1}^{n} (b_{ij}u_j - a_{ij}v_j), \]

(18) \[ d^c u([JX_i, \xi], J\xi)) = -J\xi(v_i) + \sum_{j=1}^{n} (d_{ij}u_j - c_{ij}v_j), \]

(19) \[ d^c u([X_i, J\xi], \xi)) = J\xi(v_i) + \sum_{j=1}^{n} (e_{ij}u_j + f_{ij}v_j), \]

(20) \[ d^c u([JX_i, J\xi], \xi)) = -\xi(u_i) + \sum_{j=1}^{n} (g_{ij}u_j + h_{ij}v_j). \]
Indeed from (11) we obtain
\[
[[X_i, \xi], J\xi] = [u_i \xi + \sum_{j=1}^{n} (a_{ij}X_j + b_{ij}JX_j), J\xi]
\]
\[
= -J\xi (u_i) \xi + \sum_{j=1}^{n} (a_{ij}[X_j, J\xi] + b_{ij}[JX_j, J\xi])
\]
\[
- \sum_{j=1}^{n} (J\xi (a_{ij})X_j + J\xi (b_{ij})JX_j).
\]
Applying \( \text{d}^c u \), using (9) and (10) we obtain
\[
d^c u (\mathbb{[} \mathbb{[}X_i, \xi], J\xi \mathbb{]} \mathbb{])} = -J\xi (u_i)
\]
\[
+ \sum_{j=1}^{n} \left( a_{ij} \text{d}^c u (\mathbb{[}X_j, J\xi \mathbb{]} \mathbb{])} + b_{ij} \text{d}^c u (\mathbb{[}JX_j, J\xi \mathbb{]} \mathbb{])} \right)
\]
\[
= -J\xi (u_i) + \sum_{j=1}^{n} \left( b_{ij}u_j - a_{ij}v_j \right),
\]
and this proves (17).

The proofs of (18), (19) and (20) are similar.

The relation \([\xi, J\xi] = 0\) and the Jacobi identity for the Poisson bracket yield
\[
[[JX_i, \xi], J\xi] = [[JX_i, J\xi], \xi],
\]
\[
[[X_i, \xi], J\xi] = [[X_i, J\xi], \xi].
\]
Applying \( \text{d}^c u \) and using (17), . . . , (20), after some rearrangement we obtain that for \( i = 1, \ldots, n \)

(21) \[ \xi (u_i) - J\xi (v_i) = \sum_{j=1}^{n} \left[ (g_{ij} - d_{ij})u_j + (h_{ij} + c_{ij})v_j \right], \]

(22) \[ J\xi (u_i) + \xi (v_i) = \sum_{j=1}^{n} \left[ (b_{ij} - e_{ij})u_j - (a_{ij} + f_{ij})v_j \right]. \]

Considering the holomorphic coordinate \( z : S \cap U \), if \( z = x + y \) with \( x \) and \( y \) real functions we have
\[
\xi = \frac{\partial}{\partial x},
\]
\[
J\xi = \frac{\partial}{\partial y},
\]
and hence in such coordinate system the equations (21) and (22) reduce to

(23) \[ \frac{\partial u_i}{\partial x} - \frac{\partial v_i}{\partial y} = \sum_{j=1}^{n} \left[ (g_{ij} - d_{ij})u_j + (h_{ij} + c_{ij})v_j \right], \]

(24) \[ \frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x} = \sum_{j=1}^{n} \left[ (b_{ij} - e_{ij})u_j - (a_{ij} + f_{ij})v_j \right]. \]
If we set \( w_i = u_i + \sqrt{-1}v_i \) we easily obtain that the functions \( w_1, \ldots, w_n \) satisfy
\[
\frac{\partial w_i}{\partial z} = \sum_{j=1}^{n} \left[ A_{ij} w_j + B_{ij} w_j \right], \quad i = 1, \ldots, n,
\]
where \( A_{ij} \) and \( B_{ij} \) are complex functions of class \( C^1 \). By Theorem 2.2 it follows that the common zeroes of the functions \( w_1, \ldots, w_n \), that is \( S \cap Z \cap U \) either coincides with \( S \cap U \) or is a discrete set in \( S \cap U \).

The following theorem is an easy consequence of Theorem 2.1.

**Theorem 2.3.** Let \( (\xi, u) \) be a calibrated foliation of class \( C^2 \) on the complex manifold \( M \) with contact locus \( Z \).

Let \( c \in \mathbb{R} \) and let \( V \subset M \) be the set of points \( p \in M \) which satisfies \( u(p) = c \).

Let \( D \subset M \) be the open set which is the union of all the leaf \( S \) of \( (\xi, u) \) such that \( S \cap V \neq \emptyset \).

If \( V \subset Z \) then \( D \subset Z \).

**Proof.** Indeed if \( S \) is a leaf of \( (\xi, u) \) such that \( S \cap V \neq \emptyset \) then by hypothesis \( S \cap V \subset Z \). Since \( S \cap V \) is an integral curve of the vector field \( \xi \) it follows that \( S \cap V \) is not a discrete subset of \( S \) and hence Theorem 2.1 implies that \( S \subset Z \).

\[ \Box \]

### 3. A Cauchy Problem

Let \( (\xi, u) \) be a calibrated foliation of class \( C^k \) on the complex manifold \( M \). Then it follows from the definitions that if \( V = \{ u = c \} \) is not empty then \( V \) is a real hypersurface of class \( C^k \), and for each \( p \in V \) we have \( \xi(p) \in T_p(M) \) and \( J\xi(p) \notin T_p(M) \). Moreover we have:

**Proposition 3.1.** Let \( (\xi, u) \) be a calibrated foliation of class \( C^k \), \( k \geq 1 \) on the complex manifold \( M \). Then the integral curves of \( \xi \) are real analytic maps.

**Proof.** Let \( g_t : M \rightarrow M \) the one parameter group of local transformations associated to the vector field \( \xi \). Let \( p \in M \). Since \( g_{t+s}(p) = g_t(g_s(p)) \) it suffices to prove that for each \( p \in M \) the map \( t \mapsto g_t(p) \) is real analytic in a neighbourhood of \( t = 0 \).

Let \( S \) be the leaf passing through \( p \) and let \( z \) be an adapted holomorphic coordinate in a neighbourhood of \( p \) in \( S \). Then, by construction, \( z(g_t(p)) = t \), and hence the map \( t \mapsto g_t(p) \in S \) is real analytic. Since the inclusion \( S \subset M \) is holomorphic the assertion easily follows. \( \Box \)

Conversely we have:

**Theorem 3.1.** Let \( M \) be a complex manifold of complex dimension \( n \). Let \( V \subset M \) be a closed real hypersurfaces of class \( C^k \), \( k \geq 2 \) with holomorphic tangent bundle \( HT(V) \subset T(V) \).
Let $\xi_0$ be a $C^k$ vector field on $V$ and let $g^0_t : V \to V$ be the one parameter group of local transformations associated to the vector field $\xi_0$. Assume that for each $p \in V$ we have $J\xi(p) \notin T_p(M)$ and the map $t \mapsto j(g^0_t(p))$ is real analytic.

Then there exists a neighbourhood $M_0$ of $V$ in $M$ and a calibrated foliation $(\xi, u)$ of class $C^k$ on $M_0$ such that $u|_V = 0$, $\xi|_V = \xi_0$ and for each leaf $S$ of $(\xi, u)$ we have $S \cap V \neq \emptyset$.

Such a calibrated foliation is locally unique, that is if $(\xi_1, u_1)$ is calibrated foliation on a neighbourhood $M_1$ of $V$ in $M$ such that $u_1|_V = 0$ and $\xi_1|_V = \xi_0$ then there exists a neighbourhood $M_2$ of $V$ in $M$ contained in $M_0 \cap M_1$ such that $\xi = \xi_1$ and $u = u_1$ in $M_2$.

Moreover $(\xi, u)$ is a contact calibrated foliation on $M_0$, that is $L_\xi d^c u$ vanishes identically on $M_0$ if, and only if, $\xi_0$ is an infinitesimal symmetry of the distribution $HT(V)$, that is $[\xi_0, HT(V)] \subset HT(V)$.

**Proof.** Let denote by $j : V \to M$ the inclusion map.

By the hypotheses there exists a map $W \ni (p, z) \mapsto g_z(p) \in M$, where $W \subset V \times \mathbb{C}$ is an open subset of $V \times \mathbb{C}$ containing $V \times \{0\}$ such that for each $p \in V$ the set $W_p = \{z \in \mathbb{C} \mid (p, z) \in W\}$ is an open connected neighbourhood of 0 in $\mathbb{C}$, the function $W_p : \mathbb{C} \ni z \mapsto g_z(p) \in M$ is holomorphic and if $(p, z) \in W$ with $z = t \in \mathbb{R}$ then $g_t(p) = j(g^0_t(p))$. Of course when $t, s \in \mathbb{R}$, denoting by $j_s : T(V) \to T(M)$ the differential of the inclusion map $j : V \to M$, we have

$$\frac{d}{dt} g_t(p) \bigg|_{t=0} = j_*(\xi_0(p))$$

$$\frac{d}{ds} g_{ts}(p) \bigg|_{s=0} = Jj_* \xi_0(p)).$$

Now set $W_0 = W \cap V \times \mathbb{R}$ and define $\varphi : W_0 M$ putting $\varphi(p, s) = g_is(p)$.

Being $\varphi(p, 0) = j(p)$ for each $p \in V$ and $d\varphi(p, s) \left( \frac{\partial}{\partial t} \right) = J\xi_j(p))$ it follow that after shrinking $W$ if necessary the map $\varphi : W_0 \to M$ is a diffeomorphism between $W_0$ and an open subset $\varphi(W_0) = M_0$ of $M$.

Denoting $\pi : W_0 \to \mathbb{R}$, $\pi(p, s) = s$ the canonical projection we set $u = -\pi \circ \varphi^{-1} : M_0 \to \mathbb{R}$. Then $u$ is by construction a function of class $C^k$ with non vanishing differential anywhere on $M_0$.

Moreover the formula

$$G_t(g_{is}(p)) = g_{is}(g_t(p))$$

defines an one parameter group of local diffeomorphisms of $M_0$.

Let $\xi$ be the infinitesimal generator of $G_t$. We shall prove that $(\xi, u)$ is a calibrated foliation with the required properties.

Observe that $u$ is characterized by

$$u(g_{is}(p)) = -s$$
for each $p \in V$ and for each $s$ small enough. Setting $s = 0$ we see that $u_{|V} = 0$.

We also have

$$G_t(j(p)) = G_t(g_0(p)) = g_0(g_t(p)) = g_t(p) = j(g_t^0(p)),$$

and hence, for each $p \in V$,

$$\xi(j(p)) = \frac{d}{dt} G_t(j(p)) \bigg|_{t=0} = \frac{d}{dt} j(g_t^0(p)) \bigg|_{t=0} = j_*(p)(\xi_0(p)).$$

From

$$u\left(G_t(g_{is}(p))\right) = u\left(g_{is}(g_t(p))\right) = -s = u(g_{is}(p))$$

we see that the hypersurfaces $\{u = c\}$ are $G_t$-invariant and hence $\xi(u) = du(\xi) = 0$.

We now prove that $d^c u(\xi) = 1$.

We first show that given $p \in \xi$, for $z \in \mathbb{C}$, $t \in \mathbb{R}$, with $|t|, |z|$ small enough

$$g_z(g_t(p)) = g_{z+t}(p).$$

Indeed, both sides of (26) for $p$ and $t$ fixed are holomorphic functions of $z$. Since they coincide when $z \in \mathbb{R}$ then they coincide by analytic continuation.

It follows then that

$$u(g_z(p)) = -\text{Im} z.$$

Indeed, if $z = t + is$ then $s = \text{Im} z$ and

$$u(g_z(p)) = u\left(g_{is}(g_t(p))\right) = -s = -\text{Im} z.$$

The formula $H_t(g_{is}(p)) = g_{i(s-t)}(p)$ defines an one parameter group of local diffeomorphisms of $M_0$. We now show that the infinitesimal generator of $H_t$ is $-J\xi$. Indeed we have

$$\frac{d}{dt} H_t(g_{is}(p)) \bigg|_{t=0} = \frac{d}{dt} g_{is}(g_t(p)) \bigg|_{t=0} = -J \frac{d}{dt} g_{is}(g_t(p)) \bigg|_{t=0}$$

$$= -J \frac{d}{dt} g_{is}(g_t(p)) \bigg|_{t=0} = -J \frac{d}{dt} G_t(g_{is}(p)) \bigg|_{t=0}$$

$$= -J\xi(p).$$

Thus we obtain

$$u\left(H_t(g_{is}(p))\right) = u(g_{i(s-t)}(p)) = t - s,$$

and hence

$$d^c u(\xi)(g_{is}(p)) = -du(\xi)(g_{is}(p))$$

$$= \frac{d}{dt} u\left(H_t(g_{is}(p))\right) \bigg|_{t=0} = \frac{d(t-s)}{dt} \bigg|_{t=0} = 1.$$

We end the proof that $(\xi, u)$ is a calibrated foliation showing that $[\xi, J\xi] = 0$.  

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We have to prove that \( H_{t_1} \circ G_{t_2} (g_{is}(p)) = G_{t_2} \circ H_{t_1} (g_{is}(p)) = g_{i(s-t_1)} (g_{t_2}(p)), \)
and the proof of the existence of a calibrated foliation is completed.

We now prove the uniqueness of \((\xi, u)\). Let \((\xi_1, u_1)\) another calibrated foliation on a neighbourhood \(M_1\) of \(V\) in \(M\) such that \(u_1 | V = 0\) and \(\xi_1 | V = \xi_0\).

The leaves of the foliation \((\xi, u)\) and the ones of \((\xi_1, u_1)\) which intersect \(V\) both intersect \(V\) along the integral curves of \(\xi_0\) and therefore are the same.

Let hence \(M_2\) be the union of all the leaves of the foliation \((\xi, u)\) which intersect \(V\). Then the restriction of the function \(w = u - u_1\) to each leaf \(S\) is a harmonic function on \(S\) which vanishes \(S \cap V\). Since \(d^c w(p)(\xi_0) = d^c u_1(p)(\xi_0) - d^c u_1(p)(\xi_0) = 1 - 1 = 0\) for each \(p \in V\) it follows that \(d^c w|_{S \cap V} = 0\). By lemma 3.1 below it follows that \(w|_S = 0\), that is \(u|_S = u_1|_S\). Since the leaf \(S \subset M_2\) is arbitrary then \(u|_{M_2} = u_1|_{M_2}\) and also \(\xi|_{M_2} = \xi_1|_{M_2}\) easily follows.

It remains to prove the last assertion of the Theorem.

Let \(\omega = L_\xi d^c u\). We have to prove that \(\omega\) vanishes identically on \(M_0\) if, and only if, \([\xi_0, HT(V)] \subset HT(V)\).

By Theorem 2.3 it suffices to prove that \([\xi_0, HT(V)] \subset HT(V)\) if and only if \(\omega\) vanishes identically on \(V\).

Let \(p_0 \in V\). Let \(U\) be a neighbourhood of \(p_0\) in \(M\) and vector fields \(X_1, \ldots, X_n\) such that \(\xi, J\xi, X_1, JX_1, \ldots, X_n, JX_n\) is a frame for \(T(M)\) on \(U\) such that \(\xi, X_1, JX_1, \ldots, X_n, JX_n\) is a frame for \(\ker du\) and \(X_1, JX_1, \ldots, X_n, JX_n\) is a frame for \(\ker du \cap \ker d^c u\).

As in the proof of Theorem 2.1 we see that that \(\omega\) vanishes identically on \(V\) if, and only if for \(i = 1, \ldots, n\) the functions \(d^c u([\xi, X_i])\) \(\mathbb{R}\) vanish identically on \(V\), that is, since \(T(V) = \ker du|_V\) and \(HT(V) = \ker du|_V \cap \ker d^c u|_V\), if, and only if for \(i = 1, \ldots, n\) we have \([\xi, X_i]|_V \in HT(V)\) and \([\xi, JX_i]|_V \in HT(V)\).

But we have
\[
[\xi, X_i]|_V = [\xi|_V, X_i]|_V = [\xi_0, X_i]|_V
\]
and
\[
[\xi, JX_i]|_V = [\xi|_V, JX_i]|_V = [\xi_0, JX_i]|_V
\]

Being \(X_1|_V, JX_1|_V, \ldots, X_n|_V, JX_n|_V\) be a frame for \(HT(V)\) it follows that \(\omega\) vanishes identically on \(V\) if, and only if, \([\xi_0, HT(V)] \subset HT(V)\).

The proof of the Theorem is therefore completed.

The idea to construct the function \(u\) as the imaginary part of the function obtained by complex analytic continuation of the integral curves of a vector field is taken from [5].

**Lemma 3.1.** Let \(D \subset \mathbb{C}\) be a domain such that \(D \cap \mathbb{R} \neq \emptyset\) and let \(w : D \rightarrow \mathbb{R}\) be a harmonic function. If \(w, \partial w/\partial x\) and \(\partial w/\partial y\) vanish on \(D \cap \mathbb{R}\) then \(w\) vanishes identically on \(D\).
Proof. Let $x_0 \in D \cap \mathbb{R}$. Then there exists a convex neighbourhood $U \subset D$ of $x_0$ and an holomorphic function $f : U \to \mathbb{C}$ such that $w = \text{Re} f$. By the Cauchy-Riemann equations it follows that $f'(x) = 0$ on the interval $U \cap \mathbb{R}$. By the analytic continuation principle for the holomorphic functions it follows that $f'(z) = 0$ on $U$ and hence $f(z)$ is constant on $U$. But then $w = \text{Re} f$ also is constant on $U$. Since $w$ vanishes on $U \cap \mathbb{R}$ then it vanishes also on $U$. Since $w$ is real analytic then it must vanish identically on $D$. //

4. THE COMPLEX MONGE-AMPERE EQUATIONS

Let $M$ be a connected complex manifold of complex dimension $n + 1$. Let $u \in C^k(M)$, $k \geq 2$ be a function without critical point. For each constant $c$ let denote by $V_c$ the set of points of $M$ where $u$ assume the value $c$.

We denote by $H_u \subset T(M)$ the distribution $\text{Ker} du \cap \text{Ker} d^c u$. Assume that $du \wedge d^c u \wedge (d^c u)^n$ do not vanish on $M$. Then each hypersurface $V_c$ is a CR contact manifold with $HT(V) = H_u|V$ and contact form $d^c u|_{T(V)}$. We then denote by $\xi_u$ the unique vector field on $M$ (of class at least $C^{k-2}$) which satisfies $du(\xi_u) = 0$, $d^c u(\xi_u) = 1$ and $dd^c u(\xi_u, X) = 0$ for each vector field $X$ which satisfies $du(X) = 0$.

In other word $\xi_u$ is the vector field on $M$ which is tangent to each hypersurface $V_c$ and coincides on $V_c$ with the Reeb vector field associated to the contact form $d^c u|_{T(V)}$.

Observe that $\xi_u$ can be characterize by the conditions $du(\xi_u) = 0$, $d^c u(\xi_u) = 1$ and $[\xi_u, H_u] \subset H_u$.

Lemma 4.1. Let $(\xi, u)$ be a calibrated foliation of class $C^2$ on the complex manifold $M$ of complex dimension $n + 1$ with contact locus $Z$.

Then we have the identity

$$\xi \mathcal{L}_{d^c u} = L_{\xi} d^c u$$

and the form $(dd^c u)^{n+1}$ vanishes on $Z$.

Proof. Let $X$ be a vector field on $M$. Since $X(d^c u(\xi)) = X(1) = 0$ then

$$dd^c u(\xi, X) = \xi(d^c u(X)) - X(d^c u(\xi)) - d^c u([\xi, X]) = L_{\xi} d^c u(X) - X(d^c u(\xi)) = L_{\xi} d^c u(X).$$

It follows that if $p \in Z$ then $\xi(p) \neq 0$ belongs to the radical of the bilinear form $(X, Y) \mapsto dd^c u(X, JY)$, and hence its rank is strictly less than $n + 1$ and this implies that $(dd^c u)^{n+1}$ vanishes at $p$.

//

Theorem 4.1. Let $M$ be a connected complex manifold of complex dimension $n + 1$ Let $V \subset M$ be a closed contact CR hypersurface of class $C^k$, $k \geq 3$ and let $\theta$ be a contact form for $(V, HT(V))$ of class $C^{k-1}$ with associated Reeb vector field $\xi_0$. Let denote by $j : V \to M$ the inclusion map.

Assume that the Reeb vector field $\xi_0$ is of class $C^k$.

Then the following conditions are equivalent:

1. $(\xi_0, \theta)$ be a calibrated foliation of class $C^2$ on $V$.
2. The distribution $\text{Ker} \theta \cap \text{Ker} d^c \theta$ do not vanish on $V$.
3. There exists a convex neighbourhood $U \subset V$ such that $\theta = \text{Re} f$ on $U \cap \mathbb{R}$ and $f(z)$ is constant on $U$. But then $\theta = \text{Re} f$ also is constant on $U$. Since $\theta$ vanishes on $U \cap \mathbb{R}$ then it vanishes also on $U$. Since $\theta$ is real analytic then it must vanish identically on $V$. //
(1) for each integral curve $\gamma(t)$ of $\xi_0$ the map $t \mapsto j(\gamma(t))$ is real analytic;

(2) there exists a open neighbourhood $M_0 \subset M$ of $V$ and a function $u \in C^k(M_0)$ which satisfies

$$
\begin{align*}
(dd^c u)^{n+1} &= 0 \text{ on } M_0, \\
du \wedge d^c u \wedge (dd^c u)^n &\neq 0 \text{ on } M_0, \\
u|_V &= 0, \\
d^c u|_{T(V)} &= \theta.
\end{align*}
$$

(27)

**Proof.** Assume (1). By theorem [3,1] there exists a neighbourhood $M_0 \subset M$ of $V$ and a contact calibrated foliation $(\xi, u)$ on $M_0$ such that $\xi|_M = \xi_0$. We claim that the function $u$ satisfies (27). By lemma [4,1] the function $u$ satisfies the Monge-Ampère equation.

By construction $u|_V = 0$ and for each $p \in V$ we have $\text{Ker} \, d^c u|_{T_p(V)} = \text{Ker} \, \theta(p)$. Since $d^c u|_{T(V)}(\xi_0) = d^c u(\xi)_V = 1 = \theta(\xi_0)$, it follows that $d^c u|_{T(V)} = \theta$.

Finally, since $V$ and a contact calibrated foliation then $\theta \wedge (d\theta)^n$ does not vanish on $V$ and it is then easy to show that shrinking the neighbourhood $M_0$ if necessary, the function $u$ satisfies $du \wedge d^c u \wedge (dd^c u)^n \neq 0$ in a neighbourhood of $V$ in $M$.

Conversely assume that $u$ is a solution of class $C^k$, $k \geq 3$ of (27) in a neighbourhood $M_0$ of $V$ in $M$.

Set $\xi = \xi_u$. Then $\xi$ is a $C^{k-2}$ vector field on $M$ which at each point $p \in V$ is tangent to $V$ and on coincides with the Reeb vector field $\xi_0$.

It suffices then to prove that the integral curves of the vector field $\xi$ are analytic curves in $M$.

Let $p$ be an arbitrary point of $M$ Let $U$ be a neighbourhood of $p$ in $M$ and let $X_1, \ldots, X_n$ be $C^k$ vector fields such that $X_1, JX_1, \ldots, X_n, JX_n$ is a local frame on $U$ for the distribution $H_{\text{Eq}}$.

Then $\xi, J\xi, X_1, JX_1, \ldots, X_n, JX_n$ is a local frame on $U$ for the tangent bundle $T(M)$.

By construction $\xi$ is a $C^{k-2}$ vector field which satisfies $du(\xi) = 0, d^c u(\xi) = 1$ and $dd^c u(\xi, X_i) = dd^c u(\xi, JX_i) = 0$ for $i = 1, \ldots, n$.

Since $(dd^c u)^{n+1} = 0$ then

$$
0 = \det \left( \begin{array}{cccc}
dd^c u(\xi, J\xi) & dd^c u(\xi, X_1) & \cdots & dd^c u(\xi, X_n) \\
\vdots & dd^c u(\xi, J\xi) & \cdots & dd^c u(\xi, X_n) \\
\vdots & \vdots & dd^c u(\xi, J\xi) & dd^c u(\xi, X_1) \end{array} \right)
$$
The last determinant does not vanish so
\[ \dd^c u(\xi, J\xi) = 0 \]
and since clearly \( \dd^c u(\xi, J\xi) = 0 \) we obtain \( \xi \perp \dd^c u = 0 \) on \( U \).

Since the point \( p \in M \) is arbitrary it follows that \( \xi \perp \dd^c u = 0 \) on \( M \).

For each vector field \( X \) we also have
\[ \dd^c u(J\xi, X) = -\dd^c u(\xi, JX) = 0 \]
that is \( J\xi \perp \dd^c u = 0 \).

Let us prove that \( [\xi, J\xi] = 0 \). By Theorem 2.4, pag. 549 of \([3]\) it follows in particular that for each \( p \in M \) the the vector subspace of \( T_p(M) \) of the vectors \( Z \in T_p(M) \) which satisfy \( Z \perp \dd^c u = 0 \) is a \( J \)-invariant subspace of real dimension 2.

Hence we have \( [\xi, J\xi] = a\xi + bJ\xi \) for some functions \( a \) and \( b \).

We prove that the functions \( a \) and \( b \) vanish on \( M \). By (6) of Lemma 2.1 it follows that \( du([\xi, J\xi]) = 0 \) and hence
\[ 0 = du([\xi, J\xi]) = a du(\xi) + b du(\xi) = \text{a}. \]

By (5) of Lemma 2.1 it follows that \( d^c u([\xi, J\xi]) = -\dd^c u(\xi, J\xi) \) and hence
\[ 0 = d^c u([\xi, J\xi]) = a d^c u(\xi) + b d^c u(\xi) = \text{a}. \]

Thus we have proved that \( (\xi, u) \) is a calibrated foliation of class at least \( C^1 \). The analytcity of the integral curves of \( \xi \) follows then from Proposition 3.1. //

**Remark 4.1.** When \( V \) and \( \theta \) are real analytic the previous theorem is an immediate consequence of Proposition 1.5 of \([2]\). Proposition 1.1 of \([2]\) gives the uniqueness of a \( C^3 \) solution of the problem (27).

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