Collective Field Formulation of the Multispecies Calogero Model and its Duality Symmetries

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Abstract

We study the collective field formulation of a restricted form of the multispecies Calogero model, in which the three-body interactions are set to zero. We show that the resulting collective field theory is invariant under certain duality transformations, which interchange, among other things, particles and antiparticles, and thus generalize the well-known strong-weak coupling duality symmetry of the ordinary Calogero model. We identify all these dualities, which form an Abelian group, and study their consequences. We also study the ground state and small fluctuations around it in detail, starting with the two-species model, and then generalizing to an arbitrary number of species.

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# 1 Introduction

The Calogero Model (CM) [1] - [6] is a well-known exactly solvable many-body system, both at the classical and quantum levels. The CM and its various descendants continue to draw considerable interest due to their diverse physical applications in systems such as random matrices [7], fractional statistics [8]-[11], gravity and black hole physics [12, 13], spin chains [14, 15], solitons [16]-[20], 2D Yang-Mills theory [21, 22], lowest Landau level (LLL) anyon models [23, 24], Chern-Simons matrix model [25]-[27], Laughlin-Hall states [28]-[30], and unoriented superstrings in two dimensions [31].

Calogero’s original model describes $N$ indistinguishable particles on the line which interact through an inverse-square two-body interaction. It is well-known, however, that the CM may alternatively be interpreted in terms of $N$ free particles obeying generalized exclusion statistics [9, 10, 32, 33].

Haldane’s formulation of statistics may be extended to systems made of different species of particles, in which the interspecies statistical coupling depends on the species being coupled. This may be implemented in a multi-species generalization of the CM in which particles have different masses and different couplings to each other [34]-[38].

Quite a few such generalized multi-species Calogero models exist, but contrary to the original CM, knowledge about their exact solvability was rather tenuous. The recent breakthrough in this front derives from the papers [39]-[42]. The authors of [39] introduced deformed Calogero models, related to root systems of superalgebras, and gave effectively a proof of their integrability. In [40] they presented a more
conceptual proof by using shifted super-Jack polynomials. In related developments, the authors of [41, 42] introduced a supersymmetric generalizations of the CM which was based on Jacobians for the radial coordinates on certain superspaces. Both aforementioned models are closely related to the multi-family generalization of the CM introduced in [43, 44].

Motivated by these developments, in the present paper we investigate the latter model in the limit in which each family contains a large number of particles. In this limit, the high-density limit, the system is amenable to large-$N$ collective-field formulation. As is well-known, the collective theory offers a continuum field-theoretic framework for studying interesting aspects of many-particle systems, somewhat analogous to the continuum hydrodynamic description of fluids. It is appropriate to mention at this point the recent review on the collective-field and other continuum approaches to the spin-Calogero-Sutherland model [45]. The collective formulation has several virtues. In the large-$N$ limit dynamics is governed by saddle points of the effective collective action, which contains the leading quantum effects. Thus, by extremizing this action, one is able to compute the uniform ground-state collective-field configuration, as well as topological and non-topological soliton configurations, and their corresponding energy eigenvalues. By expanding around these extrema it is possible to go beyond the large-$N$ leading order and obtain the spectrum and wave-functionals of the quadratic fluctuations around these semiclassical configurations, and also to compute the corresponding density-density correlation functions.

Beside these obvious advantages, the collective-field theory provides a natural framework for analyzing symmetries of the system which cannot be seen directly in the original (finite) $N$-particle quantum system. An important example in this
respect is the strong-weak coupling duality symmetry of the one-family Calogero model discussed in [46]. In this paper we generalize this approach to the multi-family Calogero model. We show that the resulting collective field theory is invariant under certain duality transformations, which interchange, among other things, particles and antiparticles, and thus generalize the duality symmetry [46] of the ordinary Calogero model. We identify all these dualities, which form an Abelian group, and study their consequences. In particular, the investigations carried in this paper will enable us to find the conditions under which collective quasi-particles describing density fluctuations in the the $F$-family Calogero model can be identified with those of an effective one-family Calogero model. As a by-product, this may help to better understand the exact solvability of some of the recently proposed two-family Calogero models [39]-[42]. We stress that the duality relations derived and discussed in this paper are exact symmetries of the collective-field hamiltonian, as opposed to the approxiamte duality symmetries discussed in [60, 61].

This paper is organized as follows: In Section 2 we review some known facts about the two-family Calogero model [47]. We then derive the quantum collective-field Hamiltonian, including divergent terms which are inherent to the collective-field formulation. In Section 3 we show that this collective field theory is invariant under certain duality transformations, which interchange, among other things, particles and antiparticles, and thus generalize the well-known strong-weak coupling duality symmetry of the ordinary Calogero model. We make a complete list of these dualities, show that they form an Abelian group, and study their consequences. In Section 4 we concentrate on the leading large-$N$ behavior of the collective Hamiltonian of the two-family model and identify the effective potential which dominates dynamics in this limit. The large-$N$ uniform ground state configuration is then
found by minimizing the effective potential, and the fluctuations around it are then investigated. The Hamiltonian which governs these small quadratic density fluctuations contains divergent terms. By carefully choosing the creation and annihilation operators describing these excitations, these divergent terms can be renormalized away. Section 5 is devoted to diagonalizing the latter Hamiltonian, resulting in two \textit{decoupled} quasi-particle dispersions. One of these dispersions, at small momenta, has the structure of a single-species Calogero model with some effective particle density, mass and statistical parameter, which we calculate. The other dispersion is that of free massive particles. In Section 6 we consider the ground-state wave-functional of small fluctuations, and calculate the corresponding static density-density correlation functions of the two-family model. Not surprisingly, the long-distance behavior of these correlation functions coincides with that of the effective single-species Calogero model alluded to above. In Section 7, we generalize our study of the two-family case to the $F$-family Calogero model (with no three-body interactions). We find all the exact duality transformations which leave the collective Hamiltonian of the $F$-family Calogero model invariant, and identify the Abelian group they form. We also obtain the ground-state and study the small fluctuations around it. In particular, we compute the low-momentum behavior of the dispersion relations of the quasi-particles corresponding to these fluctuations. As in the two-family case, one of these dispersions has the structure of a single-species Calogero model with some effective particle density, mass and statistical parameter. The remaining $F - 1$ dispersions are those of free massive particles. We then compare our dispersion laws with those found before in the generalized Thomas-Fermi approach [36]. Finally, in Section 8 we argue how our collective-field theory approach may explain the exact solvability of the two-family Calogero models proposed in recent papers.
Two-family collective-field Hamiltonian

In order to get oriented, and in preparation for studying the multi-species Calogero model, let us start by recalling the essential features of the two-family Calogero system. The Hamiltonian of this model reads [44]

\[
H = \frac{1}{2m_1} \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2} + \frac{\lambda_1(\lambda_1 - 1)}{2m_1} \sum_{i \neq j}^{N_1} \frac{1}{(x_i - x_j)^2} - \frac{1}{2m_2} \sum_{\alpha=1}^{N_2} \frac{\partial^2}{\partial x_\alpha^2} + \frac{\lambda_2(\lambda_2 - 1)}{2m_2} \sum_{\alpha \neq \beta}^{N_2} \frac{1}{(x_\alpha - x_\beta)^2} + \frac{1}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \lambda_{12}(\lambda_{12} - 1) \sum_{i=1}^{N_1} \sum_{\alpha=1}^{N_2} \frac{1}{(x_i - x_\alpha)^2}.
\]

Here, the first family contains \( N_1 \) particles of mass \( m_1 \) at positions \( x_i, \ i = 1, 2, ..., N_1 \), and the second one contains \( N_2 \) particles of mass \( m_2 \) at positions \( x_\alpha, \ \alpha = 1, 2, ..., N_2 \). All particles interact via two-body inverse-square potentials. The interaction strengths within each family are parametrized by the coupling constants \( \lambda_1 \) and \( \lambda_2 \), respectively. The interaction strength between particles of the first and the second family is parametrized by \( \lambda_{12} \). For reasons that should become clear in later sections, we shall assume in this paper that \( \lambda_1 \) and \( \lambda_2 \) are always positive, whereas the sign of \( \lambda_{12} \) is unrestricted. In any case, these couplings are always assumed to be set such that (1) has a well-defined ground state.

Note that we did not include in (1) a confining potential. This is not really a problem, as we can always add a very shallow confining potential to regulate the problem (in the case of purely repulsive interactions), or else, consider the particles confined to a very large circle (i.e., consider (1) as the large radius limit of the Calogero-Sutherland model [2]). We shall henceforth tacitly assume that the system is thus properly regularized at large distances.

In (1) we imposed the restriction that there be no three-body interactions, which
requires [43, 44, 48, 49, 50]

\[
\frac{\lambda_1}{m_1^2} = \frac{\lambda_2}{m_2^2} = \frac{\lambda_{12}}{m_1 m_2}. \tag{2}
\]

This particular relation was also displayed in [36], as a consequence of the requirement that the asymptotic Bethe Ansatz should be applicable to the ground state of the multi-species model. It follows from (2) that

\[
\lambda_{12}^2 = \lambda_1 \lambda_2. \tag{3}
\]

We assume that (3) holds throughout this paper. The Hamiltonian (1) describes the simplest multi-species Calogero model for particles on the line, interacting only with two-body potentials. The singularities of the Hamiltonian (1) at points where particles coincide implies that the many-body eigenfunctions contain a product of Jastrow-type prefactors \( \Pi_1 \Pi_2 \Pi_{12} \), where

\[
\Pi_1 = \prod_{i<j}^{N_1} (x_i - x_j)^{\lambda_1},
\]

\[
\Pi_2 = \prod_{\alpha<\beta}^{N_2} (x_{\alpha} - x_{\beta})^{\lambda_2},
\]

\[
\Pi_{12} = \prod_{i,\alpha}^{N_1,N_2} (x_i - x_{\alpha})^{\lambda_{12}}. \tag{4}
\]

These Jastrow factors vanish (for positive \( \lambda \)'s) at particle coincidence points, and multiply that part of the wave-function which is totally symmetric under any permutation of identical particles. It is precisely these symmetric wave-functions on which the collective field operators act, as explained below.

Let us make a couple of elementary comments at this point: First, we note from (4) that for \( \lambda_i = 0 \) and \( \lambda_i = 1 \), the model describes two families of interacting bosons and fermions, respectively. Second, superficial glance at (1) may suggest
that the model breaks into two independent Calogero systems at $\lambda_{12} = 1$. This is wrong, however. The two families remain correlated at $\lambda_{12} = 1$, and display mutual fermionic behavior. Namely, the relevant prefactor in (4) picks up a factor $(-1)^{\lambda_{12}}$ under the exchange of particle indices $(i \leftrightarrow \alpha)$. We will see, after making the transformation to collective variables, that this gives a non-trivial interacting bosonic theory of the two collective fields describing different families of particles.

Let us recall at this point some of the basic ideas of the collective-field method (adapted for our two-family Calogero model [47]): Instead of solving the Schrödinger equation associated with (1) for the many-body eigenfunctions, subjected to the appropriate particle statistics (Bosonic, Fermionic of fractional), we restrict ourselves to functions which are totally symmetric under any permutation of identical particles. This we achieve by stripping off the Jastrow factors (4) from the eigenfunctions, which means performing on (1) the similarity transformation

$$H \to \tilde{H} = \Pi_{12}^{-1} \Pi_2^{-1} \Pi_1^{-1} H \Pi_1 \Pi_2 \Pi_{12},$$

where the Hamiltonian

$$\tilde{H} = -\frac{1}{2m_1} \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2} - \frac{1}{m_1} \left( \lambda_1 \sum_{i \neq j}^{N_1} \frac{1}{x_i - x_j} + \lambda_{12} \sum_{i,\alpha}^{N_1} \frac{1}{x_i - x_{\alpha}} \right) \frac{\partial}{\partial x_i}$$

$$-\frac{1}{2m_2} \sum_{\alpha=1}^{N_2} \frac{\partial^2}{\partial x_{\alpha}^2} - \frac{1}{m_2} \left( \lambda_2 \sum_{\alpha \neq \beta}^{N_2} \frac{1}{x_\alpha - x_\beta} + \lambda_{12} \sum_{i,\alpha}^{N_2} \frac{1}{x_\alpha - x_i} \right) \frac{\partial}{\partial x_\alpha}.$$ (6)

Note that $\tilde{H}$ does not contain the singular two-body interactions. By construction, this Hamiltonian is hermitian with respect to the measure

$$d\mu(x_i, x_\alpha) = (\Pi_1 \Pi_2 \Pi_{12})^2 d^{N_1} x_i d^{N_2} x_\alpha,$$

(as opposed to the original Hamiltonian $H$ in (1), which is hermitian with respect to the flat Cartesian measure).
We can think of the symmetric many-body wave-functions acted upon by $\tilde{H}$ as functions depending on all possible symmetric combinations of particle coordinates (symmetric separately in the coordinates of each family of particles). These combinations form an overcomplete set of variables. However, as explained below, in the continuum limit, redundancy of these symmetric variables has a negligible effect.

The set of these symmetric variables can be generated, for example, by products of moments of the collective - or density - fields

$$
\rho_1(x) = \sum_{i=1}^{N_1} \delta(x - x_i), \quad \rho_2(x) = \sum_{\alpha=1}^{N_2} \delta(x - x_\alpha).
$$

(7)

The collective-field theory for the two-family Calogero model is obtained by changing variables from the particle coordinates $x_i$ and $x_\alpha$ to the density fields $\rho_1(x)$ and $\rho_2(x)$. This transformation replaces the finitely many variables $x_i$ and $x_\alpha$ by two continuous fields, which is just another manifestation of overcompleteness of the collective variables. Clearly, description of the particle systems in terms of continuous fields becomes an effectively good description in the high density limit. In this limit the mean interparticle distance is much smaller than than any relevant physical length-scale, and the $\delta$-spikes in (7) can be smoothed out into well-behaved continuum fields. All this is in direct analogy to the hydrodynamical effective description of fluids, which replaces the microscopic atomistic formulation. Of course, the large density limit means that we have taken the large-$N_1, N_2$ limit. (It is understood throughout this paper that $N_1$ and $N_2$ tend to infinity at comparable rates.)

Changing variables from particle coordinates $x_i, x_\alpha$ to the collective fields (7) implies that we should express all partial derivatives in the Hamiltonian $\tilde{H}$ in (6) as

$$
\frac{\partial}{\partial x_i} = \int dx \frac{\partial \rho_1(x)}{\partial x_i} \frac{\delta}{\delta \rho_1(x)}, \quad \frac{\partial}{\partial x_\alpha} = \int dx \frac{\partial \rho_2(x)}{\partial x_\alpha} \frac{\delta}{\delta \rho_2(x)},
$$

(8)
where we applied the differentiation chain rule.

In the large \(-N_1, N_2\) limit, the Hamiltonian \(\tilde{H}\) can be expressed entirely in terms of the collective fields \(\rho_1, \rho_2\) and their canonical conjugate momenta

\[
\pi_1(x) = -i \frac{\delta}{\delta \rho_1(x)}, \quad \pi_2(x) = -i \frac{\delta}{\delta \rho_2(x)},
\]

as we show below. It follows from (8) and (9) that the particle momentum operators (acting on symmetric wave-functions) may be expressed in terms of the collective-field momenta at particular points on the line as

\[
p_i = -\pi_1'(x_i), \quad p_\alpha = -\pi_2'(x_\alpha),
\]

where \(\pi'_a(x) = \partial_x \pi_a(x)\). Finally, note from (7) that the collective fields obey the normalization conditions

\[
\int dx \rho_1(x) = N_1, \quad \int dx \rho_2(x) = N_2.
\]

The density fields \(\rho_1, \rho_2\), and their conjugate momenta \(\pi_1, \pi_2\), satisfy the equal-time canonical commutation relations\(^1\)

\[
[\rho_a(x), \pi_b(y)] = i \delta_{ab} \delta(x - y), \quad a, b = 1, 2,
\]

\[
[\rho_a(x), \rho_b(y)] = [\pi_a(x), \pi_b(y)] = 0.
\]

By substituting (7)-(10) in (6), we obtain the continuum-limit expression for \(\tilde{H}\) as

\[
\tilde{H} = \frac{1}{2m_1} \int dx \rho_1(x)(\partial_x \pi_1(x))^2
\]

\(^1\)According to (11), the zero-momentum modes of the density fields are constrained, i.e., non-dynamical. This affects the first set of commutation relations in (12), whose precise form is \([\rho_a(x), \pi_b(y)] = i \delta_{ab} \delta(x - y) - (1/l)\), where \(l\) is the size of the large one-dimensional box in which the system is quantized, which is much larger than the macroscopic size \(L\) of the particle condensate in the system. In what follows, we can safely ignore this \(1/l\) correction in the commutation relations.
\[-\frac{i}{m_1} \int dx \rho_1(x) \left( \frac{\lambda_1 - 1}{2} \frac{\partial_x \rho_1}{\rho_1} + \lambda_1 \int \frac{dy \rho_1(y)}{x-y} + \lambda_12 \int \frac{dy \rho_2(y)}{x-y} \right) \partial_x \pi_1(x) + \frac{1}{2m_2} \int dx \rho_2(x) (\partial_x \pi_2(x))^2 \]

\[-\frac{i}{m_2} \int dx \rho_2(x) \left( \frac{\lambda_2 - 1}{2} \frac{\partial_x \rho_2}{\rho_2} + \lambda_2 \int \frac{dy \rho_2(y)}{x-y} + \lambda_12 \int \frac{dy \rho_1(y)}{x-y} \right) \partial_x \pi_2(x), \quad (13)\]

where \( \int \) denotes Cauchy’s principal value.

It can be shown [51] that (13) is hermitian with respect to the functional measure\(^2\)

\[\mathcal{D} \mu[\rho_1, \rho_2] = J[\rho_1, \rho_2] \prod_x d\rho_1(x) d\rho_2(x), \quad (14)\]

where \(J[\rho_1, \rho_2]\) is the Jacobian of the transformation from \( \{x_i, x_a\} \) to the collective fields \( \{\rho_1(x), \rho_2(x)\} \). In the large - \(N_1, N_2\) limit it is given by [47]

\[\ln J = (1 - \lambda_1) \int dx \rho_1(x) \ln \rho_1(x) + (1 - \lambda_2) \int dx \rho_2(x) \ln \rho_2(x) - \lambda_1 \int dxdy \rho_1(x) \ln |x-y| \rho_1(y) - \lambda_2 \int dxdy \rho_2(x) \ln |x-y| \rho_2(y) - 2\lambda_12 \int dxdy \rho_1(x) \ln |x-y| \rho_2(y). \quad (15)\]

It is more convenient to work with an Hamiltonian, which unlike (13), is hermitian with respect to the flat functional Cartesian measure \( \prod_x d\rho_1(x) d\rho_2(x) \). This we achieve by means of the similarity transformation \( \psi \to J^{\frac{1}{2}} \psi, \tilde{H} \to H_{\text{coll}} = J^{\frac{1}{2}} H J^{-\frac{1}{2}} \), where the continuum collective Hamiltonian is

\[H_{\text{coll}}\]

\[= \frac{1}{2m_1} \int dx \pi_1'(x) \rho_1(x) \pi_1'(x) + \frac{1}{2m_1} \int dx \rho_1(x) \left( \frac{\lambda_1 - 1}{2} \frac{\partial_x \rho_1}{\rho_1} + \lambda_1 \int \frac{dy \rho_1(y)}{x-y} + \lambda_12 \int \frac{dy \rho_2(y)}{x-y} \right)^2 + \frac{1}{2m_2} \int dx \pi_2'(x) \rho_2(x) \pi_2'(x) + \frac{1}{2m_2} \int dx \rho_2(x) \left( \frac{\lambda_2 - 1}{2} \frac{\partial_x \rho_2}{\rho_2} + \lambda_2 \int \frac{dy \rho_2(y)}{x-y} + \lambda_12 \int \frac{dy \rho_1(y)}{x-y} \right)^2\]

\(^2\)By definition (recall (7)), this measure is defined only over positive values of \( \rho_1, \rho_2 \).
where \( H_{\text{sing}} \) denotes a singular boundary contribution\(^3\):
\[
H_{\text{sing}} = -\int dx \left( \frac{\lambda_1}{2m_1} \rho_1(x) + \frac{\lambda_2}{2m_2} \rho_2(x) \right) \partial_x \frac{P}{x - y} \bigg|_{y=x} - \int dx \left( \frac{\lambda_1 - 1}{4m_1} + \frac{\lambda_2 - 1}{4m_2} \right) \partial_x^2 \delta(x - y) \bigg|_{y=x},
\]
and \( P \) is the principal part symbol.

It is well-known [52] that to leading order in the \( \frac{1}{N_{1,2}} \) expansion, collective dynamics of our system (properly represented by well-behaved observables) is determined by the classical equations of motion resulting from (16).

As a concluding remark, note that the \( \frac{\partial_x \rho_{1,2}}{\rho_{1,2}} \) terms in (16) are independent of the normalization (11) of \( \rho_1 \) and \( \rho_2 \). Thus, they are clearly of next-to-leading order in the \( \frac{1}{N_{1,2}} \) expansion relative to the other terms. Nevertheless, these terms play an important role in dynamics of the model, e.g., in governing small fluctuations around the ground state (see Sections 4 and 5), in the soliton sector of the model [47], and in establishing the duality symmetries of (16), which we do in the next section.

### 3 Duality transformations and symmetries

In this section we introduce and study the duality transformation and the corresponding symmetries of the collective-field Hamiltonian (16). It is straightforward...\(^3\)Note that the singular coefficients of \( \rho_1 \) and \( \rho_2 \) in (17) are in fact independent of \( x \), as evidently, \( \partial_x \frac{P}{x - y} \bigg|_{y=x} = \partial_x \frac{P}{x} \bigg|_{x=0} = \epsilon^{-2} \), where in the last step we used the standard representation \( \frac{P}{x} = \frac{x}{x^2 + \epsilon^2} \).
ward to check that the Hamiltonian (16) is invariant under the following set of transformations\(^4\) of the parameters:

\[
\tilde{\lambda}_1 = \frac{1}{\lambda_1}, \quad \tilde{\lambda}_2 = \frac{1}{\lambda_2}, \quad \tilde{m}_1 = -\frac{m_1}{\lambda_1}, \quad \tilde{m}_2 = -\frac{m_2}{\lambda_2}, \quad \tilde{\lambda}_{12} = \frac{1}{\lambda_{12}} \tag{18}
\]

and of the operators:

\[
\tilde{\rho}_1 = -\lambda_1 \rho_1, \quad \tilde{\rho}_2 = -\lambda_2 \rho_2, \quad \tilde{\pi}_1 = -\frac{\pi_1}{\lambda_1}, \quad \tilde{\pi}_2 = -\frac{\pi_2}{\lambda_2}. \tag{19}
\]

Derivation of these transformations, as well as of the other sets of transformations discussed below in this section is straightforward: Consider two two-family Calogero systems with collective Hamiltonians of the form (16). Assume that the tilded quantities, corresponding say, to the second system, are simple homogeneous functions of the untilded ones (allowing sign flips). Assume further that the tilded and untilded quantities appearing in (2) have the same common value. The transformations (18) and (19) then follow as the unique solution of equating the two Hamiltonians, which is neither an identity, nor a trivial permutation. (The \((\lambda_i - 1)\partial_x \rho_i/\rho_i\) terms in (16), are crucial in obtaining these transformations uniquely.)

Let us denote the set of transformations (18) and (19) by \(T_{12}\). These transformations are canonical, as they preserve the commutation relations (12). For obvious reasons, we refer to the transformations \(T_{12}\) as the strong-weak coupling duality transformation. Thus, we see that our Hamiltonian, expressed in terms of the new tilded parameters and operators, is identical in form to the original one, but with \(\lambda_1\) and \(\lambda_2\) and the inter-family coupling \(\lambda_{12}\) turned into their reciprocal values; with \(N_1\) and \(N_2\) turned, respectively, into \(\tilde{N}_1 = -\lambda_1 N_1\) and \(\tilde{N}_2 = -\lambda_2 N_2\) and, finally, with masses \(m_1\) and \(m_2\) turned into \(-\frac{m_1}{\lambda_1}\) and \(-\frac{m_2}{\lambda_2}\). The minus

\(^4\)Note that (18) and (19) do not constitute a symmetry of the original Hamiltonian (1).
signs which occur in these identifications are all important: By drawing analogy to a similar situation in the one-family case [46, 53], we interpret all negative values of the parameters and densities as those pertaining to holes, or anti-particles. Now, strictly speaking, since $N_i$ and $\tilde{N}_i$ are integers, this interpretation is consistent only for rational couplings, as was discussed in [46, 53].

We further note that (16) is invariant also under two more sets of canonical duality transformations. The first one, which we denote by $T_1$, is comprised of the set of transformations of parameters

$$\begin{align*}
\tilde{\lambda}_1 &= \lambda_1, \quad \tilde{\lambda}_2 = \frac{1}{\lambda_2}, \quad \tilde{m}_1 = m_1, \quad \tilde{m}_2 = -\frac{m_2}{\lambda_2}, \quad \tilde{\lambda}_{12} = -\frac{\lambda_{12}}{\lambda_2} \quad (20)
\end{align*}$$

and of the operators

$$\begin{align*}
\tilde{\rho}_1 &= \rho_1, \quad \tilde{\rho}_2 = -\lambda_2 \rho_2, \quad \tilde{\pi}_1 = \pi_1, \quad \tilde{\pi}_2 = -\frac{\pi_2}{\lambda_2}. \quad (21)
\end{align*}$$

Negative values of masses, densities and momenta, as in the previous case, refer to holes. These transformations map the two-family Calogero model of particles (positive $m_1, m_2, \rho_1$ and $\rho_2$) with inter-family interaction strength $\lambda_{12}$ into the dual two-family Calogero model of particles ($m_1, \rho_1$) and holes ($\tilde{m}_2, \tilde{\rho}_2$) with the inter-family interaction strength $-\frac{\lambda_{12}}{\lambda_2}$. The second (and last) set of duality symmetries of (16), which we denote by $T_2$, is obtained from (20) and (21) simply by permuting the family indices $1 \leftrightarrow 2$.

It is easy to check that the duality transformations $T_1$, $T_2$, $T_{12}$, together with the identity transformation $I$, form an Abelian group under composition, in which each element squares to $I$, and where $T_1 T_2 = T_{12}$, $T_1 T_{12} = T_2$ and $T_2 T_{12} = T_1$. This is readily identified as Klein’s four-group. The latter is isomorphic to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, where the two $\mathbb{Z}_2$ factors are $\{I, T_1\}$ and $\{I, T_2\}$. As we shall see in Section 7, this
group-theoretic pattern of dualities persists also in the generic case.

### 3.1 Resemblence of special two-family Calogero models to single-family Calogero models

As an interesting application of the duality symmetry group, consider the special case of the two-family Calogero model (1) in which\(^5\) \(\lambda_2 = \frac{1}{\lambda_1}\) and \(m_2 = -\frac{m_1}{\lambda_1}\). From (2) we then find that \(\lambda_{12} = \frac{m_2}{m_1} \lambda_1 = -1\). Note that the two particle families in this system are generically manifestly distinct. Nevertheless, this distinction is, in some sense, an illusion. To see this, note that by the duality transformations (20) and (21), namely, the element \(T_1\) of the duality group, this system is equivalent to a two-family system with parameters \(\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_{12} = \lambda_1\), \(\tilde{m}_1 = \tilde{m}_2 = m_1\) and densities \(\tilde{\rho}_1 = \rho_1, \tilde{\rho}_2 = -\frac{\rho_2}{\lambda_1}\). In the latter dual system, the two families are identical!

For this reason, we may refer to the two dimensional locus

\[
\lambda_2 = \frac{1}{\lambda_1}, \quad m_2 = -\frac{m_1}{\lambda_1}, \quad \lambda_{12} = -1 \tag{22}
\]

in parameter space as the “surface of hidden identity”, or SOHI. Thus, the special two-family Calogero model we started with resembles the single-family Calogero model specified by

\[
\lambda = \lambda_1, \quad m = m_1, \quad \rho = \tilde{\rho}_1 + \tilde{\rho}_2 = \rho_1 - \frac{1}{\lambda_1} \rho_2. \tag{23}
\]

Similarly, by inverting the roles of family indices \(1 \leftrightarrow 2\) in the previous discussion, which leaves us on the SOHI (22), and then applying the duality transformation \(T_2\), we shall conclude that the special two-family Calogero model we started with

---

\(^5\)If we ignore inter-family coupling, we can think of this system as made of two single-family models, related by the one-family version of the strong-weak coupling duality, save for the relation between densities.
resembles the single-family Calogero model specified by

\[ \lambda = \lambda_2, \quad m = m_2, \quad \rho = \rho_2 - \frac{1}{\lambda_2} \rho_1. \]  

(24)

In both cases, the effective single-species collective field \( \rho \) actually shares the statistics \( \lambda \) and the mass \( m \) with the first or the second family, respectively. Note that these two cases can be mapped one each other by the duality transformation \( T_{12} \). Thus, the SOHI (22) is left invariant under \( T_{12} \). However, the latter does not act on it freely, as \( \lambda_1 = \lambda_2 = -\lambda_{12} = 1 \) and \( m_1 = -m_2 \) is a fixed line. Models lying on this line are comprised of particles and their antiparticles, and only particles and antiparticles interact (repulsively).

Note that we described the relation between the original special two-family models and the corresponding single-family models merely as “resemblance”. They are certainly not identical! The density operator \( \rho \) appearing on the LHS of (23) and (24), which corresponds to the single-family Calogero model, is defined in a Hilbert space made of many-body wave functions which are completely symmetric in the coordinates of all particles. \( \rho_1 \) and \( \rho_2 \), on the other hand, are symmetric only in the coordinates of particles of each family separately. The best one could do is perhaps to consider the two-family system with identical families (the one dual to the special two-family systems we started with) as a one-family system divided into two parts, differing by some internal quantum number, in which one symmetrizes in each sector separately. However, this means one should also contrapt an actual physical context to justify such separate symmetrization.
4 Ground state and small fluctuations

In this section we shall concentrate on the ground-state of the two-family Calogero system and the small excitations above it. To this end, it is sufficient to consider only static density fields, for which, of course, \( \pi_1 = \pi_2 = 0 \). Evaluating the Hamiltonian (16) on such fields and imposing the normalization constraints (11), yields the effective potential, given (to leading order in the \( 1/N_1 \) and \( 1/N_2 \) expansion) by

\[
V = \frac{1}{2m_1} \int dx \rho_1(x) \left( \frac{\lambda_1 - 1}{2} \frac{\partial_x \rho_1}{\rho_1} + \lambda_1 \int \frac{dy \rho_1(y)}{x-y} + \lambda_{12} \int \frac{dy \rho_2(y)}{x-y} \right)^2 \\
+ \frac{1}{2m_2} \int dx \rho_2(x) \left( \frac{\lambda_2 - 1}{2} \frac{\partial_x \rho_2}{\rho_2} + \lambda_2 \int \frac{dy \rho_2(y)}{x-y} + \lambda_{12} \int \frac{dy \rho_1(y)}{x-y} \right)^2 \\
+ \mu_1 \left( N_1 - \int dx \rho_1(x) \right) + \mu_2 \left( N_2 - \int dx \rho_2(x) \right). 
\]

Here \( \mu_1 \) and \( \mu_2 \) are the chemical potentials which impose (11).\(^6\)

The expression in (25) contains bilocal as well as trilocal terms in the densities. We can avoid the menacing trilocal terms by applying standard tricks as follows: As was mentioned at the beginning of the previous section, we tacitly assume that the system is properly regularized at large distances, meaning in particular, that \( \rho_1(x) \) and \( \rho_2(x) \) are always of compact support. The principal value distribution, acting on such functions, satisfies the identity

\[
P \frac{P}{x-y} \frac{P}{x-z} + \frac{P}{y-z} \frac{P}{y-x} + \frac{P}{z-x} \frac{P}{z-y} = \pi^2 \delta(x-y)\delta(x-z).
\]

Making use of (26) in (25), we obtain

\[
V = \frac{\pi^2}{6\lambda_1 m_1} \int dx (\lambda_1 \rho_1 + \lambda_{12} \rho_2)^3 +
\]

\(^6\)In (25) we have absorbed the constant singular coefficients of \( \rho_1 \) and \( \rho_2 \) which appear in \( H_{\text{sing}} \) in (17) as infinite shifts in \( \mu_1 \) and \( \mu_2 \), and also omitted the infinite, density independent constant in \( H_{\text{sing}} \).
This expression for $V$ is evidently devoid of trilocal terms. In Appendix A we present an alternative derivation of (27).

As was mentioned at the end of the previous section, in the large-$N$ limit, dynamics of the system is semiclassical, governed by the saddle points of (16). In particular, the ground-state energy and the corresponding density distributions can be found, to leading order in $\frac{1}{N_{1,2}}$, by varying the effective potential (27) with respect to $\rho_1$ and $\rho_2$, which yields the two coupled integro-differential, non-linear equations

\[
\frac{\pi^2}{2m_1}(\lambda_1 \rho_1 + \lambda_{12} \rho_2)^2 - \frac{(\lambda_1 - 1)^2}{8m_1} \left( \frac{\partial_x \rho_1}{\rho_1} \right)^2 - \frac{(\lambda_1 - 1)^2}{4m_1} \partial_x \left( \frac{\partial_x \rho_1}{\rho_1} \right) 
\]
\[
- \frac{\lambda_1(\lambda_1 - 1)}{m_1} \int dy \frac{\partial_y \rho_1(y)}{x-y} - \frac{\lambda_{12}}{2} \left( \frac{\lambda_1 - 1}{m_1} + \frac{\lambda_2 - 1}{m_2} \right) \int dy \frac{\partial_y \rho_2(y)}{x-y} = \mu_1, \tag{28}
\]
\[
\frac{\pi^2}{2m_2}(\lambda_2 \rho_2 + \lambda_{12} \rho_1)^2 - \frac{(\lambda_2 - 1)^2}{8m_2} \left( \frac{\partial_x \rho_2}{\rho_2} \right)^2 - \frac{(\lambda_2 - 1)^2}{4m_2} \partial_x \left( \frac{\partial_x \rho_2}{\rho_2} \right) 
\]
\[
- \frac{\lambda_2(\lambda_2 - 1)}{m_2} \int dy \frac{\partial_y \rho_2(y)}{x-y} - \frac{\lambda_{12}}{2} \left( \frac{\lambda_1 - 1}{m_1} + \frac{\lambda_2 - 1}{m_2} \right) \int dy \frac{\partial_y \rho_1(y)}{x-y} = \mu_2. \tag{29}
\]

In the ground state, the two particle types evidently condense into uniform density configurations

\[
\rho_{1,0} = \frac{N_1}{L}, \quad \rho_{2,0} = \frac{N_2}{L}, \tag{30}
\]
inside the spatial box of length $L$, which confines the particles. These uniform densities are clearly solutions of the variational equations (28) and (29), since for
uniform densities the latter equations reduce simply to relations which determine
the chemical potentials. To conclude our discussion of the semiclassical limit, note
that to leading order in $\frac{1}{N_{1,2}}$, the ground-state energy corresponding to (30) follows
from (27):

$$E_0 = \frac{\pi^2}{6\lambda_1 m_1} \int dx (\lambda_1 \rho_{1,0} + \lambda_{12} \rho_{2,0})^3 = \frac{\pi^2}{6\lambda_1 m_1 L^2} (\lambda_1 N_1 + \lambda_{12} N_2)^3. \quad (31)$$

This is, in fact, the exact result for finite $N_1$ and $N_2$ [36]. It is worth mentioning
here that (28) and (29) possess also nonuniform static soliton solutions, which were
studied extensively in [47, 54].

The collective-field formalism provides a systematic framework for the $\frac{1}{N_{1,2}}$
expansion [55, 56]. In particular, by expanding around the classical uniform configu-
trations (30) (with their corresponding null momenta), we can go beyond the leading
order and obtain the spectrum of low-lying excitations above the ground state. Here
we shall concentrate on the next-to-leading terms. To this end we write

$$\rho_1(x) = \rho_{1,0} + \eta_1(x), \quad \rho_2(x) = \rho_{2,0} + \eta_2(x), \quad (32)$$

where $\eta_1(x)$ and $\eta_2(x)$ are small density fluctuations, with $\eta_a$ being typically of order
$1/N_a$. Substituting (32) in the Hamiltonian (16), and expanding it to quadratic order
in $\pi_1, \pi_2, \eta_1$, and $\eta_2$, we obtain

$$H_{\text{coll}} = \frac{\rho_{1,0}}{2m_1} \int dx (\partial_x \pi_1(x))^2 + \frac{\rho_{1,0}}{2m_1} \int dx \left( \frac{\lambda_1 - 1}{2} \frac{\partial_x \eta_1}{\rho_{1,0}} + \lambda_1 \int \frac{dy \eta_1(y)}{x-y} + \lambda_{12} \int \frac{dy \eta_2(y)}{x-y} \right)^2$$

$$+ \frac{\rho_{2,0}}{2m_2} \int dx (\partial_x \pi_2(x))^2 + \frac{\rho_{2,0}}{2m_2} \int dx \left( \frac{\lambda_2 - 1}{2} \frac{\partial_x \eta_2}{\rho_{2,0}} + \lambda_2 \int \frac{dy \eta_2(y)}{x-y} + \lambda_{12} \int \frac{dy \eta_1(y)}{x-y} \right)^2$$

$$+ H_{\text{sing}} + E_0. \quad (33)$$

The quadratic approximation (33) to $H_{\text{coll}}$, as the original collective Hamiltonian
(16), is invariant under the duality transformations $T_1, T_2$ and $T_{12}$ in Section 3. Eq.
(33) governs the leading-order fluctuations of the system around the ground state. Its quadratic structure clearly calls for introducing the two operators

\[ A_1(x) = \partial_x \pi_1(x) + i \left( \frac{\lambda_1 - 1}{2} \frac{\partial_x \eta_1}{\rho_{1,0}} + \lambda_1 \int \frac{d\eta_1(y)}{x-y} + \lambda_{12} \int \frac{d\eta_2(y)}{x-y} \right), \]

\[ A_2(x) = \partial_x \pi_2(x) + i \left( \frac{\lambda_2 - 1}{2} \frac{\partial_x \eta_2}{\rho_{2,0}} + \lambda_2 \int \frac{d\eta_2(y)}{x-y} + \lambda_{12} \int \frac{d\eta_1(y)}{x-y} \right), \]

along with their hermitian adjoints. Evidently, these operators are the key to computing the collective quasiparticle spectrum of small fluctuations around the ground state. Higher order terms in the expansion of \( H_{\text{coll}} \) around the uniform densities yield the interactions among these quasiparticles.

It can be checked that the duality transformations of Section 3 act on \( A_1(x) \) and \( A_2(x) \) as follows: \( A_1 \) is invariant under \( T_1 \) while \( T_1 A_2 \rightarrow -\frac{1}{\lambda_2} A_2 \); \( A_2 \) is invariant under \( T_2 \) while \( T_2 A_1 \rightarrow -\frac{1}{\lambda_1} A_1 \); finally, \( T_{12} A_{1,2} \rightarrow -\frac{1}{\lambda_{1,2}} A_{1,2} \) (the latter transformation is consistent with the group multiplication \( T_1 T_2 = T_{12} \)).

The operators (34), (35) and their adjoints satisfy the commutation relations

\[ [A_1(x), A_1^\dagger(y)] = \frac{1 - \lambda_1}{\rho_{1,0}} \partial_x \partial_y \delta(x-y) + 2 \lambda_1 \partial_x \frac{P}{x-y}, \]

\[ [A_2(x), A_2^\dagger(y)] = \frac{1 - \lambda_2}{\rho_{2,0}} \partial_x \partial_y \delta(x-y) + 2 \lambda_2 \partial_x \frac{P}{x-y}, \]

\[ [A_2(x), A_1^\dagger(y)] = [A_1(x), A_2^\dagger(y)] = 2 \lambda_{12} \partial_x \frac{P}{x-y}, \]

with all other commutators vanishing. In terms of these operators the quadratic Hamiltonian (33) takes the form

\[ H_{\text{coll}} = E_0 + \frac{\rho_{1,0}}{2m_1} \int dx A_1^\dagger(x) A_1(x) + \frac{\rho_{2,0}}{2m_2} \int dx A_2^\dagger(x) A_2(x) \]

\[ + \frac{\rho_{1,0}}{4m_1} \int dx [A_1(x), A_1^\dagger(x)] + \frac{\rho_{2,0}}{4m_2} \int dx [A_2(x), A_2^\dagger(x)] + H_{\text{sing}}. \]
As can be seen from (36), the commutator terms in the last equation, which are 
essentially the sum over zero-point energies originating from the quadratic pieces in (33), cancel the the singular term $H_{\text{sing}}$ defined in (17). This is a typical behavior in collective field theory. Thus, we obtain the finite result

$$H_{\text{coll}} = E_0 + \frac{\rho_{1,0}}{2m_1} \int dx A_1^\dagger(x) A_1(x) + \frac{\rho_{2,0}}{2m_2} \int dx A_2^\dagger(x) A_2(x). \quad (37)$$

As our next step towards diagonalization of (37), we move to momentum space and express the operators $A_1$ and $A_2$ in terms of of their Fourier transforms

$$A_1(x) = \sqrt{\frac{m_1}{\pi \rho_{1,0}}} \int_{-\infty}^{\infty} dk e^{ikx} a_1(k), \quad A_2(x) = \sqrt{\frac{m_2}{\pi \rho_{2,0}}} \int_{-\infty}^{\infty} dk e^{ikx} a_2(k). \quad (38)$$

By definition of the duality transformations in Section 3, and from their action on $A_1$ and $A_2$ as described following (35), we obtain that their action on $a_1$ and $a_2$ reduces to either flipping a sign or not: $a_1$ is invariant under $T_1$ while $T_1 a_2 \rightarrow -a_2$; $a_2$ is invariant under $T_2$ while $T_2 a_1 \rightarrow -a_1$; finally, $T_{12} a_{1,2} \rightarrow -a_{1,2}$.

From (34) and (35), and from the identity

$$\int_{-\infty}^{\infty} dx \frac{e^{ikx}}{x-y} = i\pi e^{iky} \text{sign } k \quad (39)$$

it is easy to verify that

$$a_1(k) = \sqrt{\frac{\pi \rho_{1,0}}{m_1}} \left[ ik \tilde{\eta}_1(k) + \frac{1 - \lambda_1}{2 \rho_{1,0}} k \tilde{\eta}_1(k) + \pi \left( \lambda_1 \tilde{\eta}_1(k) + \lambda_{12} \tilde{\eta}_2(k) \right) \text{sign } k \right]$$

$$a_2(k) = \sqrt{\frac{\pi \rho_{2,0}}{m_2}} \left[ ik \tilde{\eta}_2(k) + \frac{1 - \lambda_2}{2 \rho_{2,0}} k \tilde{\eta}_2(k) + \pi \left( \lambda_2 \tilde{\eta}_2(k) + \lambda_{12} \tilde{\eta}_1(k) \right) \text{sign } k \right]. \quad (40)$$

The Fourier modes

$$\tilde{\eta}_a(k) = \int_{-\infty}^{\infty} dx \frac{e^{-ikx}}{2\pi} \eta_a(x), \quad \tilde{\pi}_a(k) = \int_{-\infty}^{\infty} dx \frac{e^{-ikx}}{2\pi} \pi_a(x) \quad (41)$$
obey the commutation relations\textsuperscript{7,8}

\begin{equation}
\left[\tilde{\eta}_a(k), \tilde{\pi}_b(q)\right] = \frac{i}{2\pi} \delta(k + q),
\end{equation}

with all other commutators being null. For convenience, we also write down the hermitian adjoints of (40):

\begin{align*}
a_1(k) &= \sqrt{\frac{\pi \rho_{1,0}}{m_1}} \left[ -ik\tilde{\pi}_1(-k) + \frac{1 - \lambda_1}{2\rho_{1,0}} k\tilde{\eta}_1(-k) + \pi (\lambda_1 \tilde{\eta}_1(-k) + \lambda_{12} \tilde{\eta}_2(-k)) \text{sign } k \right] \quad (42) \\
a_2(k) &= \sqrt{\frac{\pi \rho_{2,0}}{m_2}} \left[ -ik\tilde{\pi}_2(-k) + \frac{1 - \lambda_2}{2\rho_{2,0}} k\tilde{\eta}_2(-k) + \pi (\lambda_2 \tilde{\eta}_2(-k) + \lambda_{12} \tilde{\eta}_1(-k)) \text{sign } k \right]. \quad (43)
\end{align*}

Note that by combining the equations for \(a_b(k)\) and \(a_b^\dagger(-k)\) we can invert (40) and (43), and express \(\eta_b(k)\) and \(\pi_b(k)\) as linear combinations of \(a_b(k) \pm a_b^\dagger(-k)\).

Finally, it follows from (40), (42) and (43) that\textsuperscript{9}

\begin{align*}
[a_1(k), a_1^\dagger(k')] &= \left( \frac{1 - \lambda_1}{2m_1} k^2 + \frac{\lambda_1 \pi \rho_{1,0}}{m_1} |k| \right) \delta(k - k') \equiv \omega_1(k) \delta(k - k'), \quad (44) \\
[a_2(k), a_2^\dagger(k')] &= \left( \frac{1 - \lambda_2}{2m_2} k^2 + \frac{\lambda_2 \pi \rho_{2,0}}{m_2} |k| \right) \delta(k - k') \equiv \omega_2(k) \delta(k - k'), \quad (45) \\
[a_1(k), a_2^\dagger(k')] &= [a_2(k), a_1^\dagger(k')] = \lambda_{12} \pi \sqrt{\frac{\rho_{1,0}\rho_{2,0}}{m_1 m_2}} |k| \delta(k - k') \equiv \omega_{12}(k) \delta(k - k'), \quad (46)
\end{align*}

and all other commutators vanish. For later use, it will be convenient to lump (44) - (46) into a real symmetric matrix \([a_a(k), a_b^\dagger(k')] = \omega_{ab}(k) \delta(k - k')\), with

\begin{equation}
\omega \equiv \begin{pmatrix}
\omega_1(k) & \omega_{12}(k) \\
\omega_{12}(k) & \omega_2(k)
\end{pmatrix}. \quad (47)
\end{equation}

\textsuperscript{7}Due to the fact that \(\int_{-\infty}^{\infty} dx \eta_a(x) = 0\), the zero-momentum modes of the fluctuation fields are nondynamical commuting numbers. See the footnote just above (12) for a similar matter concerning the full density fields. As was commented upon there, in the limit of very large spatial box, we can safely use (42).

\textsuperscript{8}Note that \(\tilde{\eta}_a(k) = \tilde{\eta}_a(-k)\) and \(\tilde{\pi}_a(k) = \tilde{\pi}_a(-k)\), since \(\eta_a(x)\) and \(\pi_a(x)\) are hermitian field operators.

\textsuperscript{9}Alternatively, Eqs. (44) - (46) can be verified by Fourier-transforming (36) directly.

22
It is easy to verify that the duality transformations of Section 3 act on $\omega$ as follows: $T_{12}$ leaves it invariant, while both $T_1$ and $T_2$ transform it to $\sigma_3 \omega \sigma_3$ (i.e., they flip the sign of the off-diagonal terms $\omega_{12}$). All this is of course consistent with the action of these duality transformations on $a_1$ and $a_2$, as described following (38).

Note in passing that we can infer a special case of the exact duality equivalence of systems with parameters on the SOHI (22) and two-family models made of identical families, as discussed in Section 3.1. This we achieve simply by demanding that $\omega_{11}(k) = \omega_{22}(k)$, which immediately leads to the pair of equations $\frac{1-\lambda_1}{m_1} = \frac{1-\lambda_2}{m_2}$ and $\frac{\lambda_1 \rho_{1,0}}{m_1} = \frac{\lambda_2 \rho_{2,0}}{m_2}$. Discarding the trivial solution (i.e., that the two families are identical to begin with), we obtain that the conditions defining the SOHI (22) comprise a solution of the first equation, which upon substitution into the second equation leads to the further relation $\rho_{2,0} = -\lambda_1 \rho_{1,0}$ (which did not arise in Section 3.1).

In terms of the operators (40) and (43) the Hamiltonian (37) becomes

$$H_{coll} = E_0 + \int dk a_1^\dagger(k) a_1(k) + \int dk a_2^\dagger(k) a_2(k).$$

(48)

In Section 5 we will show that the ground state of the fluctuation fields is annihilated by $a_1(k)$ and $a_2(k)$. Thus, the ground-state energy $E_0$ is not affected by the next-to-leading-order corrections stemming from the quantum collective-field fluctuations $\eta_1$ and $\eta_2$.

5 Diagonalization and dispersion laws

Expression (48) is not really the diagonal form of the Hamiltonian, since the operators $a_1$ and $a_2$ do not commute - see (46). As usual, complete decomposition of (48) into independent normal modes can be achieved by properly rotating $a_1(k)$ and
$a_2(k)$ into two new *commuting* bosonic operators $b_1(k)$ and $b_2(k)$. Thus, we write

$$b_a(k) = \sum_{b=1}^{2} u_{ab}(k) a_b(k),$$

(49)

where $u_{ab}(k)$ are $c$-number coefficients. They are chosen such that the matrix $u$ diagonalizes the commutator matrix (47). Since $\omega$ is real and symmetric, $u$ is real orthogonal. Thus,

$$u \omega u^T = \Omega,$$

(50)

where

$$\Omega \equiv \begin{pmatrix} \Omega_1(k) & 0 \\ 0 & \Omega_2(k) \end{pmatrix}.$$  

(51)

Recall that the duality transformations $T_1, T_2$ and $T_{12}$ transform $\omega$ either to itself or to $\sigma_3 \omega \sigma_3$, which is similar to $\omega$. Thus, the eigenvalues $\Omega_a(k)$ of (47) are *invariant* under these duality transformations. These eigenvalues are given by the standard formula

$$\Omega_{1,2}(k) = \frac{1}{2} \left( \omega_1(k) + \omega_2(k) \pm \sqrt{(\omega_1(k) - \omega_2(k))^2 + 4 \omega_{12}^2(k)} \right).$$

(52)

By construction, therefore, the new operators satisfy the decoupled commutation algebra

$$[b_1(k), b_1^\dagger(k')] = \Omega_1(k) \delta(k - k'),$$

$$[b_2(k), b_2^\dagger(k')] = \Omega_2(k) \delta(k - k'),$$

$$[b_1(k), b_2^\dagger(k')] = 0.$$  

(53)

The coefficients $u_{ab}(k)$ of the orthogonal transformation can be expressed in terms of the rotation angle $\phi(k)$ as

$$u_{11} = u_{22} = \cos \phi(k) = \frac{2 \omega_{12}}{\sqrt{4 \omega_{12}^2 + [\omega_2 - \omega_1 + \sqrt{4 \omega_{12}^2 + (\omega_2 - \omega_1)^2}]^2}}.$$

24
\[ u_{12} = -u_{21} = \sin \phi(k) = \frac{\omega_2 - \omega_1 + \sqrt{4\omega_{12}^2 + (\omega_2 - \omega_1)^2}}{\sqrt{4\omega_{12}^2 + [\omega_2 - \omega_1 + \sqrt{4\omega_{12}^2 + (\omega_2 - \omega_1)^2}]^2}}. \] (54)

Note from (44) - (46) that the \( u_{ab}(k) \) are all even functions of \( k \), in addition to being real. A useful and simple identity which we mention in passing is

\[ \cot 2\phi = \frac{\omega_1 - \omega_2}{2\omega_{12}}. \] (55)

By rescaling

\[ b_1(k) \to \sqrt{\Omega_1(k)} \ b_1(k), \quad b_2(k) \to \sqrt{\Omega_2(k)} \ b_2(k), \] (56)

we finally obtain the diagonal Hamiltonian

\[ H_{\text{coll}} = E_0 + \int dk \Omega_1(k)b_1^\dagger(k)b_1(k) + \int dk \Omega_2(k)b_2^\dagger(k)b_2(k). \] (57)

\( \Omega_1(k) \) and \( \Omega_2(k) \) are therefore the energy spectra of the physical fluctuations, i.e. the two decoupled collective oscillator modes, or quasiparticles of the system.

As can be readily seen from the commutation algebra (44)-(46), the dispersions \( \omega_1(k) \) and \( \omega_2(k) \), evaluated at \( \lambda_{12} = 0 \), actually represent the low-energy dispersions for the two non-interacting one-family Calogero systems. Stability of these latter two systems requires that the coefficients of the terms in \( \omega_1(k) \) and \( \omega_2(k) \) which are proportional to \( |k| \), the leading term in the low-momentum behavior of these systems, be positive. Thus, we shall henceforth make the assumption that these terms are positive. Note further that these terms are invariant under the duality transformations of Section 3, and so are their signs.

Since we know from [57] that the low-energy one-family dispersions \( \omega_1(k) \) and \( \omega_2(k) \) are correctly given only up to \( k^2 \) (near \( k = 0 \)), the same must be also true for
\( \Omega_1(k) \) and \( \Omega_2(k) \). By expanding the square root in (52) up to order \( k^2 \), we obtain\(^{10} \)

\[
\Omega_1(k) = \frac{(1 - \lambda_1)\lambda_1 \rho_{1,0}/m_1^2 + (1 - \lambda_2)\lambda_2 \rho_{2,0}/m_2^2 k^2} {\lambda_1 \rho_{1,0}/m_1 + \lambda_2 \rho_{2,0}/m_2} k^2 + \pi \left( \frac{\lambda_1 \rho_{1,0}} {m_1} + \frac{\lambda_2 \rho_{2,0}} {m_2} \right) |k|,
\]

\[
\Omega_2(k) = \frac{(1 - \lambda_2)\lambda_1 \rho_{1,0}/(m_1 m_2) + (1 - \lambda_1)\lambda_2 \rho_{2,0}/(m_1 m_2) k^2} {\lambda_1 \rho_{1,0}/m_1 + \lambda_2 \rho_{2,0}/m_2} k^2.
\]

(58)

The first dispersion \( \Omega_1(k) \) has the same structure as that of a single-species Calogero model [56] with some effective particle density \( \rho \), mass \( m \) and interaction parameter \( \lambda \) satisfying the following two relations:

\[
\frac{(1 - \lambda_1)\lambda_1 \rho_{1,0}/m_1^2 + (1 - \lambda_2)\lambda_2 \rho_{2,0}/m_2^2} {\lambda_1 \rho_{1,0}/m_1 + \lambda_2 \rho_{2,0}/m_2} = \frac{1 - \lambda} {m},
\]

(59)

\[
\frac{\lambda_1 \rho_{1,0}} {m_1} + \frac{\lambda_2 \rho_{2,0}} {m_2} = \frac{\lambda \rho} {m}.
\]

(60)

The second dispersion \( \Omega_2(k) \) describes a free-particle. The effective mass \( m^* \) of this quasi-particle is given by

\[
\frac{(1 - \lambda_2)\lambda_1 \rho_{1,0}/(m_1 m_2) + (1 - \lambda_1)\lambda_2 \rho_{2,0}/(m_1 m_2)} {\lambda_1 \rho_{1,0}/m_1 + \lambda_2 \rho_{2,0}/m_2} = \frac{1} {m^*}.
\]

(61)

As was mentioned earlier, the exact expressions for \( \Omega_{1,2}(k) \) are invariant under the duality transformations of Section 3. It can be further checked that the approximate expressions in (58) retain this invariance. Thus, in particular, the expressions on the LHS’s of (59) - (61) are all invariant under the duality transformations of Section 3.

It is interesting to observe that for the special values of uniform densities configurations \( \rho_{2,0} = \lambda_1 \rho_{1,0} \), parameters \( \lambda_1 \lambda_2 = \lambda_{12} = 1 \), and masses \( m_2 = m_1/\lambda_1 \), the pieces of both dispersions \( \Omega_1(k) \) and \( \Omega_2(k) \) which are proportional to \( k^2 \) vanish, as should be evident from (58) - (61). Thus, to order \( k^2 \), \( \Omega_2(k) \) vanishes identically, \(^{10} \)As was stated above, here we assume that the coefficient of \( |k| \) in the first equation in (58) is positive. In case it is negative, the term linear in \( |k| \) would appear in \( \Omega_2(k) \) instead of \( \Omega_1(k) \).
while $\Omega_1(k)$ attains the value of the dispersion corresponding to the single-family Calogero model \(^{11}\) at its fermionic point $\lambda = 1$. In other words, at this particular point in parameter space, our two-family Calogero model appears to be similar (but not entirely equivalent) to a system of free fermions. The vanishing of $\Omega_2(k)$ at $k \neq 0$ is an indication of an instability at that point in parameter space. Evidently, one has to go beyond the quadratic approximation to resolve this problem.

To summarize our discussion thus far, we have found that the low-momentum behavior of the two-family Calogero model is effectively that of an ordinary single-species Calogero model and a decoupled system of free massive bosons. This conclusion will manifest itself again when we compute density-density correlation functions in the next section.

Let us discuss now how to determine the effective one-family parameters $m, \lambda$ and $\rho$. Eqs. (59) and (60) comprise two relations among the three effective parameters $\rho, m$ and $\lambda$. The third relation, needed to determine them unambiguously, arises from identifying the ground-state energy (31) with that of the single-species collective field \(^{12}\), namely,

$$\frac{\pi^2}{6\lambda_1 m_1} (\lambda_1 \rho_{1,0} + \lambda_{12} \rho_{2,0})^3 = \frac{\pi^2 \lambda^2}{6m} \rho^3.$$  \hspace{1cm} (62)

Note that although, strictly speaking, the first equality in the expression (31) for the ground state energy is based on the leading large-$N$ behavior of the effective potential (27), the result quoted there is, in fact, the exact value of the ground state energy, valid also for finite $N_{1,2}$. Indeed, the integrand appearing in that equation (namely, the LHS of (62)) is invariant under the duality transformations of Section

\(^{11}\)That is, $\Omega(k) = \frac{1-\lambda}{2m} k^2 + \frac{\lambda m}{m} |k|$.  
\(^{12}\)Clearly, the free quasi-particles described by $\Omega_2(k)$ cannot contribute to the ground state energy.
3, which hold also beyond leading order. Thus, we can treat (62) on the same footing as (59) and (60), as far as large-$N$ accuracy is concerned.

Recall that the couplings and masses are related according to (2). Thus, eliminating, for example, $\lambda_2$ and $\lambda_{12}$ in terms of $\lambda_1$ and the ratio of masses, and then substituting these into (59), (60) and (62), it is possible to cast these three equations into forms from which elimination of $\lambda$ and $\rho$ in terms of $m$ leads to the simple relations

$$\frac{\lambda}{m^2} = \frac{\lambda_1}{m_1^2}, \quad m\rho = m_1\rho_{1,0} + m_2\rho_{2,0}, \quad (63)$$

while $m$ itself satisfies the quadratic equation

$$\left(\rho_{1,0} + \frac{m_2}{m_1}\rho_{2,0}\right) \left[\lambda_1\left(\frac{m}{m_1}\right)^2 - 1\right] + \left((1 - \lambda_1)\rho_{1,0} + (1 - \lambda_2)\rho_{2,0}\right)\frac{m}{m_1} = 0. \quad (64)$$

(Note that Eqs.(63) and (64) are invariant under the duality transformations of Section 3, having been derived from duality-invariant equations.)

After solving (64) for $m$ we can obtain $\lambda$ from (63). It is natural to conjecture that the quasi-particles of the effective one-family Calogero model have fractional statistics with statistical parameter $\lambda$. It will be interesting to prove this conjecture.

As an example for the discussion above, consider the case in which all couplings and masses are positive. Under these conditions, (64) has two real solutions of opposite signs (which either remain invariant or are flipped by the duality transformations). Let us choose the positive root as desired mass $m$. Substituting the latter into (63) we obtain $\lambda$ and $\rho$, which are also positive. (We do not present the corresponding expressions here.)

Finally, note that $\lambda = \lambda_1$, $m = m_1$, subjected to the constraints (22) defining the SOHI, namely, the situation corresponding to (23), is also a solution of (63) and
6 Ground-state wave-functional and correlation functions

Having completed diagonalization of $H_{\text{col}}$ in (57), we next compute the corresponding collective-field ground-state wave functional. The ground state $|0\rangle$ is defined by $b_1(k)|0\rangle = b_2(k)|0\rangle = 0$, for every $k$. Due to the orthogonal transformation (49), the latter conditions are equivalent to $a_1(k)|0\rangle = a_2(k)|0\rangle = 0$, or in real space, to

$$A_1(x)|0\rangle = A_2(x)|0\rangle = 0. \quad (65)$$

We shall now concentrate on the latter representation. Following the definitions (34) and (35) we write (65) as

$$
\begin{align*}
&\left(-\partial_x \frac{\delta}{\delta \eta_1(x)} + \lambda_1 - \frac{1}{2\rho_{1,0}} \partial_x \eta_1(x) + \lambda_1 \int dy \frac{\eta_1(y)}{x-y} + \lambda_{12} \int dy \frac{\eta_2(y)}{x-y}\right) \Psi_0[\eta_1, \eta_2] = 0 \\
&\left(-\partial_x \frac{\delta}{\delta \eta_2(x)} + \lambda_2 - \frac{1}{2\rho_{2,0}} \partial_x \eta_2(x) + \lambda_2 \int dy \frac{\eta_2(y)}{x-y} + \lambda_{12} \int dy \frac{\eta_1(y)}{x-y}\right) \Psi_0[\eta_1, \eta_2] = 0,
\end{align*}
\quad (66)
$$

where $\Psi_0[\eta_1, \eta_2] = \langle\{\eta_1, \eta_2\}|0\rangle$ is the wave functional we sought for. The solution of these linear and homogeneous coupled functional differential equations is evidently a Gaussian in the fluctuating fields $\eta_1$ and $\eta_2$, which we readily find as

$$
\Psi_0[\eta_1, \eta_2] = \frac{1}{\sqrt{Z_0}} \exp \int dx dy \left\{ \eta_1(x) \left( \frac{\lambda_1 - 1}{4\rho_{1,0}} \delta(x-y) + \frac{\lambda_1}{2} \ln |x-y| \right) \eta_1(y) \\
+ \eta_2(x) \left( \frac{\lambda_2 - 1}{4\rho_{2,0}} \delta(x-y) + \frac{\lambda_2}{2} \ln |x-y| \right) \eta_2(y) + \lambda_{12} \eta_1(x) \ln |x-y| \eta_2(y) \right\}, \quad (67)
$$

(64).
where the normalization constant $Z_0$ is fixed by the requirement that $\int \mathcal{D}\eta_1 \mathcal{D}\eta_2 |\Psi_0[\eta_1,\eta_2]|^2 = 1$. We may write (67) more compactly as

$$\Psi_0[\eta_1,\eta_2] = \frac{1}{\sqrt{Z_0}} \exp\left(-\frac{1}{4} \int \int dx dy \sum_{a,b=1}^2 \eta_a(x)K_{ab}(x - y)\eta_b(y)\right),$$

where

$$K_{11}(x) = \frac{1 - \lambda_1}{\rho_{1,0}} \delta(x) - 2\lambda_1 \ln |x|,$$

$$K_{22}(x) = \frac{1 - \lambda_2}{\rho_{2,0}} \delta(x) - 2\lambda_2 \ln |x|,$$

$$K_{12}(x) = K_{21}(x) = -2\lambda_{12} \ln |x|.$$  (69)

By comparing (36) and (69) we obtain the simple relation

$$[A_a(x), A_b^\dagger(y)] = \partial_x \partial_y K_{ab}(x - y).$$  (70)

Now that we have the collective-field vacuum wave-functional, we can calculate the density-density correlation functions

$$\langle 0|\eta_a(x)\eta_b(y)|0 \rangle = \frac{1}{Z_0} \int \mathcal{D}\eta_1 \mathcal{D}\eta_2 \eta_a(x)\eta_b(y) \exp\left(-\frac{1}{2} \int \int dx dy \sum_{c,d=1}^2 \eta_c(x)K_{cd}(x - y)\eta_d(y)\right)$$

by standard quantum field theoretic methods. We thus obtain

$$\langle 0|\eta_a(x)\eta_b(y)|0 \rangle = K_{ab}^{-1}(x - y),$$  (72)

where $K_{ab}^{-1}(x - y)$ denotes the inverse kernel

$$\int dz \sum_{c=1}^2 K_{ac}(x - z)K_{cb}^{-1}(z - y) = \delta_{ab}\delta(x - z).$$  (73)

Throughout the discussion from (67) to (72) we have assumed, of course, that $K_{ab}(x - y)$ is positive definite. Since the kernels $K_{ab}(x - y)$ are translation-invariant, it is more appropriate to calculate the inverse kernels in momentum space. Thus,
by Fourier transforming both sides of (70), we find, using (38), (40) and (43) - (47) that

\[ K_{ab}(x) = \int_{-\infty}^{\infty} dk \ e^{ikx} \tilde{K}_{ab}(k), \]  

(74)

with

\[ \tilde{K}_{ab}(k) = \sqrt{\frac{m_a}{\pi \rho_{a0}}} \sqrt{\frac{m_b}{\pi \rho_{b0}}} \frac{\omega_{ab}(k)}{k^2}. \]  

(75)

Note that \( \tilde{K}_{ab}(k) \) diverges like \( 1/|k| \) as \( k \to 0 \). Eq. (75) can be also verified by Fourier transforming all equations in (69), by employing the Fourier integral

\[ \ln |x| = -\frac{1}{2} \int_{-\infty}^{\infty} dk \ e^{ikx} \]  

(76)

(and the standard Fourier representation of \( \delta(x) \)). The desired inverse kernel is then found to be

\[ K_{ab}^{-1}(x-y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk \ e^{ik(x-y)} \tilde{K}_{ab}^{-1}(k), \]  

(77)

where \( \tilde{K}^{-1}(k) \) is the matrix inverse to (75). From (75) and (44) - (47) we obtain the latter as

\[ \tilde{K}^{-1}(k) = \frac{2\pi \rho_{1,0} \rho_{2,0}}{2\pi \lambda_2} \kappa \begin{pmatrix} \frac{1-\lambda_2}{2\pi \rho_{2,0}} |k| + \lambda_2 & -\lambda_{12} \\ -\lambda_{12} & \frac{1-\lambda_1}{2\pi \rho_{1,0}} |k| + \lambda_1 \end{pmatrix}, \]  

(78)

where

\[ \kappa = (1-\lambda_1)\lambda_2 \rho_{2,0} + (1-\lambda_2)\lambda_1 \rho_{1,0}. \]  

(79)

In obtaining (78) we used

\[ \det \tilde{K}(k) = \frac{1}{2\pi \rho_{1,0} \rho_{2,0} |k|} \left[ \kappa + \frac{(1-\lambda_1)(1-\lambda_2)}{2\pi} |k| \right], \]  

(80)

where (2) was invoked. By substituting (78) in (77) we obtain the integral representation

\[ \langle 0 | \eta_a(x) \eta_b(0) | 0 \rangle = K_{ab}^{-1}(x) = \frac{1}{2\pi} \begin{pmatrix} \rho_{1,0} & 0 \\ 0 & \rho_{2,0} \end{pmatrix} \delta(x) + \]
\[
-2 \left( \begin{array}{cc}
\frac{\lambda_1 \rho_1}{(1-\lambda_1)^2} & \frac{\lambda_1 \rho_2}{(1-\lambda_1)(1-\lambda_2)} \\
\frac{\lambda_2 \rho_1 \rho_2}{(1-\lambda_1)(1-\lambda_2)} & \frac{\lambda_2 \rho_2}{(1-\lambda_2)^2}
\end{array} \right) \int_0^\infty \frac{dk}{k} \cos(kx) \frac{\cos(2\pi\kappa)}{k + \frac{2\pi\kappa}{(1-\lambda_1)(1-\lambda_2)}}
\]
(81)

for the desired correlators (72). The latter integral is well defined for

\[
q = \frac{2\pi\kappa}{(1-\lambda_1)(1-\lambda_2)} = 2\pi \left( \frac{\lambda_1 \rho_1}{1-\lambda_1} + \frac{\lambda_2 \rho_2}{1-\lambda_2} \right) > 0,
\]
(82)
in which case it can be expressed as a combination of the Sine-integral \( \text{si}(z) = \int_{z}^{\infty} dt \frac{\sin(t)}{t} \) and the Cosine-integral \( \text{ci}(z) = \int_{z}^{\infty} dt \frac{\cos(t)}{t} \) [58], leading to the final expression (for \( x \neq 0 \))

\[
\langle 0 | \eta_a(x) \eta_b(0) | 0 \rangle = 2 \left( \begin{array}{cc}
\frac{\lambda_1 \rho_1}{(1-\lambda_1)^2} & \frac{\lambda_1 \rho_1 \rho_2}{(1-\lambda_1)(1-\lambda_2)} \\
\frac{\lambda_2 \rho_1 \rho_2}{(1-\lambda_1)(1-\lambda_2)} & \frac{\lambda_2 \rho_2}{(1-\lambda_2)^2}
\end{array} \right) F(q|x),
\]
(83)

where

\[
F(q|x) = \sin(q|x|) \text{si}(q|x|) + \cos(q|x|) \text{ci}(q|x|).
\]
(84)

Eq. (83) generalizes a similar result known for the one-family Calogero model [56]. The function \( F(q|x|) \) has oscillatory behavior for small to moderate values of its argument (as one would typically expect of correlation functions), and falls like \( 1/(qx)^2 \) at large values. In fact, \( F(q|x|) \) has a rather simple asymptotic expansion [58] \(^{13}\) in inverse powers of \((qx)^2\):

\[
F(q|x|) \sim \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!}{(qx)^{2n}}, \quad |q|x| >> 1.
\]
(85)

In the case \( q < 0 \) the integral representation (81) is ill-defined, due to the pole on the integration contour, however, the asymptotic series (85) still makes sense.

\(^{13}\)It is straightforward to obtain this asymptotic expansion by repeatedly integrating-by-parts the integral in (81).
as the large-\(|x|\) behavior of the correlation function comes from momentum values \(|k| \ll |q|\). In either case, we can trust the asymptotic series

\[
\langle 0| \eta_a(x) \eta_b(0)|0 \rangle = K_{ab}^{-1}(x)
\]

\[
\sim \left( \begin{array}{cc}
\frac{2\lambda_1 \rho_{1,0}^2}{(1-\lambda_1)^2} & \frac{2\lambda_2 \rho_{1,0} \rho_{2,0}}{(1-\lambda_1)(1-\lambda_2)} \\
\frac{2\lambda_2 \rho_{1,0} \rho_{2,0}}{(1-\lambda_1)(1-\lambda_2)} & \frac{2\lambda_2 \rho_{2,0}^2}{(1-\lambda_2)^2}
\end{array} \right) \sum_{n=1}^{\infty} (-1)^n \left( \frac{(1-\lambda_1)(1-\lambda_2)}{2\pi \kappa x} \right)^{2n} (2n-1)!, \quad (86)
\]

valid for \(\frac{2\pi \kappa |x|}{(1-\lambda_1)(1-\lambda_2)} > 1\), to capture the leading long-distance behavior of the correlators.

Close inspection of (83) reveals that the coefficient matrix there is of rank-1. Indeed, due to (2) it can be written as \(w w^T\) with \(w_a = \sqrt{\lambda_a \rho_a} \left( 1 - \lambda_a \right) \). Thus, \(\det \langle 0| \eta_a(x) \eta_b(0)|0 \rangle = 0\) for all \(x \neq 0\). More importantly, this means that only one combination of the density fluctuation fields \(\eta_a(x)\) can have long-range correlations. In order to reveal the reason for this behavior, it is instructive to rewrite (68) in momentum space. Starting from the identity

\[
S = \int dx dy \sum_{a,b} \eta_a(x) K_{ab}(x - y) \eta_b(y) = (2\pi)^2 \int dk \sum_{a,b} \tilde{\eta}_a(-k) \tilde{K}_{ab}(k) \tilde{\eta}_b(k)
\]

we obtain, using (75), (50) and \(u_{ab}(-k) = u_{ab}(k)\) (see (54)) that

\[
S = (2\pi)^2 \int dk \sum_a \tilde{\zeta}_a(-k) \frac{\Omega_{aa}(k)}{k^2} \tilde{\zeta}_a(k), \quad (87)
\]

where

\[
\tilde{\zeta}_a(k) = \sum_b u_{ab}(k) \sqrt{\frac{m_b}{\pi \rho_{b,0}}} \tilde{\eta}_b(k). \quad (88)
\]

Thus, (68) factors into two uncorrelated Gaussian pieces, describing the vacuum fluctuations of the two fields \(\zeta_a(x) = \int dk e^{ikx} \tilde{\zeta}_a(k)\). From the low-momentum

\[14\]Related to this is the fact that the matrix \(\tilde{\mathbf{K}}^{-1}(0)\) exists, but due to (2) it is not invertible.
behavior (58) of $\Omega_1(k)$ and $\Omega_2(k)$, and from (59) - (61), we conclude immediately that at long-distances, the vacuum fluctuations of the combination $\zeta_1(x)$ coincide with those of a single-species Calogero model with parameters $\lambda, m$ and $\rho$, whereas $\zeta_2(x)$ appears like a white-noise random field with ultra-local correlations proportional to $\delta(x - y)$. All this, of course, is in full accordance with our earlier conclusion that the two-family Calogero model is equivalent, in the large-$N$ limit and at long-distances, to two decoupled systems, one - an effective one-species Calogero model, and the other, a system of free massive particles.

As a simple application of this observation, consider the special two-family systems living on the SOHI (22) in parameter space. Let us use our result (83) to evaluate the correlation function of the effective one-family density (23). We shall denote the fluctuating part of $\rho(x)$ in (23) by $\eta(x) = \eta_1(x) - \frac{1}{\lambda} \eta_2(x)$. Substituting the parameters corresponding to (22) and (23) in the appropriate places, we obtain

$$
\langle 0|\eta(x)\eta(0)|0 \rangle = \langle 0| \left( \eta_1(x) - \frac{1}{\lambda} \eta_2(x) \right) \left( \eta_1(0) - \frac{1}{\lambda} \eta_2(0) \right) |0 \rangle
$$

$$
= \langle 0|\eta_1(x)\eta_1(0)|0 \rangle - \frac{2}{\lambda} \langle 0|\eta_1(x)\eta_2(0)|0 \rangle + \frac{1}{\lambda^2} \langle 0|\eta_2(x)\eta_2(0)|0 \rangle
$$

$$
= \frac{2\lambda}{(1 - \lambda)^2} \left( \rho_{1,0} - \frac{1}{\lambda} \rho_{2,0} \right)^2 F(q|x|),
$$

with

$$
q = \frac{2\pi \lambda}{\lambda - 1} \left( \rho_{1,0} - \frac{1}{\lambda} \rho_{2,0} \right).
$$

From (85) we obtain the leading long-distance asymptotic behavior of this correlator simply as

$$
\langle 0|\eta(x)\eta(0)|0 \rangle = -\frac{1}{2\pi^2 \lambda} \frac{1}{x^2},
$$

with

$$
q = \frac{2\pi \lambda}{\lambda - 1} \left( \rho_{1,0} - \frac{1}{\lambda} \rho_{2,0} \right).
$$

From (85) we obtain the leading long-distance asymptotic behavior of this correlator simply as

$$
\langle 0|\eta(x)\eta(0)|0 \rangle = -\frac{1}{2\pi^2 \lambda} \frac{1}{x^2},
$$

with
which does not depend on the particle density at all, in accordance with the one-family correlation function of [57, 59].

6.1 The low-lying collective excitations

We close this section by analyzing of the low-lying collective excitations above the vacuum. These excitations are the pseudo-particles created by $b_{1,2}^\dagger(k)$ acting on the vacuum. The corresponding states are

$$b_b^\dagger(k)|0\rangle = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, e^{ikx} \sum_b u_{ab} \sqrt{\frac{\rho_{b0}}{m_b}} A_b^\dagger(x)|0\rangle ,$$

(92)

where we used (49), (38), (34), (35) and the fact that the $u_{ab}$ are real. Next, we observe from (34), (35) and (67) that

$$A_a^\dagger(x)|0\rangle = -2i \left( \frac{\lambda_a}{2} - \frac{1}{\rho_{a0}} \int \frac{dy}{x-y} \eta_a(y) + \lambda_{12} \int \frac{dy}{x-y} \bar{\eta}_a(y) \right) |0\rangle$$

(93)

where $\bar{a}$ is the index obtained from $a$ by permutation, i.e., $\bar{1} = 2, \bar{2} = 1$. By substituting (93) in (92), and using (39), we then obtain, after some algebra,

$$b_a^\dagger(k)|0\rangle = \frac{1}{\sqrt{\pi k}} \int_{-\infty}^{\infty} dx \, e^{ikx} \sum_b u_{ab} \left( \sqrt{\frac{m_b}{\rho_{b0}}} \omega_{bb} \eta_b(x) + \sqrt{\frac{m_\bar{b}}{\rho_{\bar{b}0}}} \omega_{\bar{b}\bar{b}} \bar{\eta}_b(x) \right) |0\rangle ,$$

(94)

where we also used (44) - (46) on the way. Performing summation over the index $b$ and using (50), which is equivalent to the relations $u_{aa} \omega_{aa} + u_{a\bar{a}} \omega_{\bar{a}a} = \Omega_a u_{aa}$ and $u_{a\bar{a}} \omega_{a\bar{a}} + u_{a\bar{a}} \omega_{\bar{a}a} = \Omega_a u_{a\bar{a}}$, we may simplify (94) further into

$$b_a^\dagger(k)|0\rangle = \frac{\Omega_a(k)}{k} \int_{-\infty}^{\infty} dx \, e^{ikx} \left( \sqrt{\frac{m_a}{\pi \rho_{a0}}} u_{aa} \eta_a(x) + \sqrt{\frac{m_\bar{a}}{\pi \rho_{\bar{a}0}}} u_{a\bar{a}} \bar{\eta}_a(x) \right) |0\rangle .$$

(95)

Using the definitions of $u_{ab}$ given above (55) we thus obtain

$$b_1^\dagger(k)|0\rangle \propto \sqrt{\frac{m_1\rho_{2,0}}{m_2\rho_{1,0}}} \cot \phi \int_{-\infty}^{\infty} dx \, e^{ikx} \eta_1(x)|0\rangle + \int_{-\infty}^{\infty} dx \, e^{ikx} \eta_2(x)|0\rangle$$

$$b_2^\dagger(k)|0\rangle \propto -\sqrt{\frac{m_1\rho_{2,0}}{m_2\rho_{1,0}}} \tan \phi \int_{-\infty}^{\infty} dx \, e^{ikx} \eta_1(x)|0\rangle + \int_{-\infty}^{\infty} dx \, e^{ikx} \eta_2(x)|0\rangle .$$

(96)
Let us now compute cot $\phi$. Substituting (55) in the identity cot $\phi = \cot 2\phi + \sqrt{1 + \cot^2 2\phi}$ and then using (44) - (46) we obtain

$$ \cot \phi = \frac{\omega_1 - \omega_2 + \Omega_1 - \Omega_2}{2\omega_12}, \quad (97) $$

which upon expansion to leading order in $|k|$ yields

$$ \cot \phi \simeq \sqrt{\frac{m_2\rho_{1,0}}{m_1\rho_{2,0}} \frac{\lambda_1}{\lambda_{12}}} \left( 1 + \frac{1}{2\pi} \frac{1 - \lambda_1}{\lambda_{12}} \frac{m_2}{m_1} \right |k| \right) \quad (98) $$

Finally, returning to the original configuration space coordinates

$$ \int dx e^{ikx} \eta_1(x) \rightarrow \sum_{i=1}^{N_1} e^{ikx_i}, \quad \int dx e^{ikx} \eta_2(x) \rightarrow \sum_{\alpha=1}^{N_2} e^{ikx_\alpha}, \quad (99) $$

and substituting it and (98) in (96) we find

$$ b_1^\dagger(k)|0\rangle \propto \frac{m_1}{m_2} \left( 1 + \frac{1}{2\pi} \frac{1 - \lambda_1}{\lambda_{12}} \frac{m_2}{m_1} \right |k| \right) \sum_{i=1}^{N_1} e^{ikx_i}|0\rangle + \sum_{\alpha=1}^{N_2} e^{ikx_\alpha}|0\rangle \quad (100) $$

and

$$ b_2^\dagger(k)|0\rangle \propto \frac{\rho_{2,0}}{\rho_{1,0}} \left( 1 - \frac{1}{2\pi} \frac{1 - \lambda_1}{\lambda_{12}} \frac{m_2}{m_1} \right |k| \right) \sum_{i=1}^{N_1} e^{ikx_i}|0\rangle + \sum_{\alpha=1}^{N_2} e^{ikx_\alpha}|0\rangle $$

where we used $\lambda_1/\lambda_{12} = m_1/m_2$ from (2). Our low-energy excitations are in fact long-wavelength density oscillations (phonons) in which the velocities of all particles point in the same direction, either positive or negative, depending on the sign of $k$.

Comparing the excitation (100) with that obtained in [36], we see that our relative weight approaches the relative weight $m_1/m_2$ from [36], but only in the small $k$ limit. The second excitation (101), which describes the free-particle motion, does not have any counterpart in Sen’s approach [36].
7 The multi-species generalization

Let us now apply the methods we used in our analysis of the two-family Calogero model to investigate its $F$-family generalization. Thus, consider a collection of $F$ species of particles. The $a$–th family contains $N_a$ particles of mass $m_a$ ($a = 1, 2, \ldots F$), which interact among themselves with coupling constant $\lambda_a \equiv \lambda_{aa}$. The inter-family mutual interaction strengths are $\lambda_{ab} = \lambda_{ba}$ ($a \neq b$). All couplings satisfy the constraint

$$\frac{\lambda_{ab}}{m_a m_a} = c,$$

(102)

where the same constant $c$ holds for all pairs of indices. This condition generalizes (2) and guarantees that there be no three-body interactions [43, 44]. It follows from (102) that

$$\lambda_a \lambda_b = \lambda_{ab}^2,$$

(103)

similarly to (3). We take the limit in which all particle numbers $N_a$ tend to infinity at the same rate. The collective-field Hamiltonian of the $F$–family Calogero model is given by

$$H_{\text{coll}} = \sum_{a=1}^{F} \frac{1}{2m_a} \int dx \left( \partial_x \pi_a(x) \rho_a(x) \partial_x \pi_a(x) + \rho_a(x) \left( \frac{\lambda_a - 1}{2} \frac{\partial_x \rho_a}{\rho_a} + \sum_{b=1}^{F} \lambda_{ab} \int \frac{dy \rho_b(y)}{x-y} \right)^2 \right),$$

(104)

up to singular boundary-contributions (compare to (16)). The collective density fields and their conjugate momenta obey canonical equal-time commutation relations analogous to (12). The $a$–th density field is subjected, of course, to the normalization condition $\int dx \rho_a(x) = N_a$.

The ground-state energy and density distributions can be found, as in the 2-family case, by varying the effective potential corresponding to (104) with respect
to all \( \rho_a \). We thus obtain the uniform ground-state densities

\[
\rho_{a,0} = \frac{N_a}{L},
\]

(105)

where \( L \) is the length of the large confining box, and the ground-state energy

\[
E_0 = \frac{\pi^2 e^2}{6L^2} \left( \sum_{a=1}^{F} m_a N_a \right)^3 = \frac{\pi^2}{6\lambda_1 m_1 L^2} \left( \sum_{a=1}^{F} \lambda_{1a} N_a \right)^3.
\]

(106)

In the last equation, which is a direct generalization of (31), we used (102).

Similarly to our conventions in the \( F = 2 \) case, we shall take all \( \lambda_a > 0 \), and anticipating the possibility of negative masses and densities, we shall also set \( \text{sign} \ m_a = \text{sign} \ \rho_{a,0} \) for all families.

We now expand around the uniform configurations (105)

\[
\rho_a(x) = \rho_{a,0} + \eta_a(x),
\]

(107)

and introduce the \( F \) operators \( A_c(x) \) analogous to (34) and (35). The Fourier transforms \( a_a(k) \) of these operators, and their hermitian conjugates, defined as in (40) and (43), obey the commutation relations \( [a_a(k), a_b^\dagger(k')] = \omega_{ab}(k) \delta(k - k') \), where the \( F \times F \) real symmetric matrix

\[
\omega_{ab}(k) = \frac{1 - \lambda_a}{2m_a} k^2 \delta_{ab} + \lambda_{ab} \pi \sqrt{\frac{\rho_{a,0}\rho_{b,0}}{m_a m_b}} |k|
\]

(108)

is an immediate generalization of (47).

Let us now diagonalize \( \omega_{ab} \) and obtain its eigenvalues \( \Omega_a(k) \). We shall contend ourselves with the low-momentum behavior of these dispersions, and compute them to order \( k^2 \). Details of this computation are given in Appendix B. We find that one of these eigenvalues is given approximately by

\[
\Omega_1(k) = \frac{\sum_{a=1}^{F} \frac{(1-\lambda_a)\lambda_a \rho_{a,0}}{m_a^2} k^2}{\sum_{a=1}^{F} \frac{\lambda_a \rho_{a,0}}{m_a}} \frac{k^2}{2} + \pi \sum_{a=1}^{F} \frac{\lambda_a \rho_{a,0}}{m_a} |k|.
\]

(109)
Since in our conventions \( \lambda_a > 0 \) and \( \text{sign} \, m_a = \text{sign} \, \rho_a,0 \), the coefficient of \( |k| \) in the last term is manifestly positive. The remaining \( F-1 \) eigenvalues \( \Omega_2(k) \sim \Omega_3(k) \sim \ldots \sim \Omega_F(k) \sim k^2 \). Thus, as in the \( F = 2 \) case, the first dispersion \( \Omega_1(k) \) looks effectively like that of a one-family Calogero model. The remaining \( F-1 \) dispersions correspond evidently to free massive bosonic quasi-particles, decoupled from each other and from the Calogero-like mode.

Let us now compare our Calogero-like dispersion law (109) with that given by Eq.(58) of [36], which was derived to leading order in \( \frac{1}{L} \). The terms linear in momentum \( k \) agree, as can be easily seen by virtue of the relation

\[
v_F = \frac{\pi}{2} \sum_a \frac{\lambda_a \rho_a,0}{m_a},
\]

where \( v_F \) denotes the ”Fermi” velocity in Sen’s approach. We observe that the \( k^2 \) correction in our dispersion (109) arises from both the linear and the quadratic differential operators in the Hamiltonian (104). Taken together, they give a contribution of the typical weight \( \frac{\lambda_{a-1}}{2} \). These terms have been completely ignored in Sen’s corresponding dispersion relation mentioned above, because they did not seem to be analytically computable in that approach. In this respect, our method seems to do better than the generalized Thomas-Fermi method of [36].

7.1 Duality transformations and symmetries of the \( F \)-family model

The duality symmetries of the two-family model, discussed in Section 3, have a natural generalization in the \( F \)-family model. The main idea in searching for these duality symmetries in the generic case is to find all classes of physically equivalent \( F \)-family models. Here, the equivalence of two systems in a given class means that
despite the fact that these systems are defined in terms of different sets of collective fields, momenta and parameters, their collective Hamiltonians (104) coincide.

As our first step in classifying all these duality symmetries, it is easy to check that the Hamiltonian (104) is invariant under the following set of elementary duality transformations, associated with a prescribed family index \( a \):

\[
\tilde{\lambda}_a = \frac{1}{\lambda_a}; \quad \tilde{\lambda}_{ab} = -\frac{\lambda_{ab}}{\lambda_a}, \quad \text{for } b \neq a; \quad \tilde{\lambda}_{cd} = \lambda_{cd}, \quad \text{for } c, d \neq a; \\
\tilde{m}_a = -\frac{m_a}{\lambda_a}; \quad \tilde{m}_b = m_b, \quad \text{for } b \neq a; \\
\tilde{\rho}_a = -\lambda_a \rho_a; \quad \tilde{\rho}_b = \rho_b, \quad \text{for } b \neq a; \\
\tilde{\pi}_a = -\frac{\pi_a}{\lambda_a}; \quad \tilde{\pi}_b = \pi_b, \quad \text{for } b \neq a.
\]

Let us denote all these transformations collectively as \( T_a, \ a = 1, \ldots, F \). As can be readily seen, \( T_a \) acts non-trivially only on the \( a \)-th family. Clearly, \( T_a^2 = I \), where \( I \) is the identity transformation. Therefore, \( T_a \) and \( I \) form a \( \mathbb{Z}_2 \) group of of symmetries of (104). It is possible to compose any pair of such elementary transformations, corresponding to different family indices. Moreover, this composition is obviously commutative. In this way we can generate the more complicated duality symmetries \( T_a T_b = T_b T_a \) of (104). Continuing this process, we can generate all transformations involving triplets of different family indices \( T_a T_b T_c \) all the way up to the maximally possible composition \( T_1 T_2 \cdots T_F \). In this way, the elementary transformations \( T_a \), together with the identity \( I \) generate an Abelian group of order \( 2^F \), which exhaust all possible duality symmetries of (104). Evidently, each element in this group is of order 2. We can readily identify this group is as \( \otimes^F \mathbb{Z}_2 \).

Within this general picture, we can interpret the two-family duality transformations (20), (21) as an elementary transformation on the second family, while the
transformations (18),(19) may be understood as the simultaneous action of the two elementary transformations on the first and the second family.

All physical properties of the $F$-family system should remain invariant under the duality symmetries of (104). In particular, the $F$ dispersions $\Omega_a(k)$ should be duality-invariants. This can be proved in a manner similar to the $F=2$ case, namely by showing that the effect of a duality transformation on the matrix $\omega$ in (108) is simply a conjugation by an orthogonal matrix.\textsuperscript{15} It is enough to prove this conjugation for an elementary transformation (111): It can be easily checked that the elementary duality transformation $T_a$ leaves $\omega_{aa}, \omega_{bb}$ and $\omega_{cd}$ ($b, c, d, \neq a$) invariant, while $\omega_{ab} = \omega_{ba}$ flips sign. All this can be summarized by saying that under $T_a$, $\omega \rightarrow \tau_a \omega \tau_a$, where the diagonal orthogonal matrix $\tau_a$ is obtained from the unit matrix by flipping the sign of its $a$-th diagonal entry. Thus, all the dispersions $\Omega_a(k)$ are invariant under the duality transformations of (104). In particular, all duality transformations preserve positivity of the coefficient of $|k|$ in the second term in (109).

7.1.1 Special $F$-family Calogero models which resemble single-family Calogero models

In Section 3.1 we applied the duality group to identify a special subset of two-family models, with apparently different families, which nevertheless resemble, in almost all respects, a Calogero system composed of identical particles. Those are the two-family systems living on the so-called “surface of hidden identity” (SOHI) (22) in parameter space. Not surprisingly, this construction can be generalized to the $F$-family case, as we now show.

\textsuperscript{15}See the discussions following (47) and (51).
In order to find special multi-family Calogero models, which resemble a single-species Calogero model with some effective particle density $\rho$, mass $m$ and interaction strength $\lambda$, let us split the latter effective one-family system into $F$ different sub-families whose particles share the same mass $m$, the same coupling constant $\lambda$ and the same mutual interaction strength $\lambda_{ab} = \lambda$. This splitting means of course that we no longer consider totally symmetric many-body wave functions, and replace them by wave functions symmetric separately in the coordinates of the particles of each family. Let $\rho_a$ denote the density of $a$-th family particles. Therefore, $\rho = \sum_a \rho_a$. We shall refer to the $F$-family system thus obtained as maximally degenerate (MD).

Let us consider our MD $F$-family model as the image under one of the elementary duality transformations $T_k$ in (111) for some $1 \leq k \leq F$. Since $T_k^2 = I$, its origin must have been the less-degenerate $F$-family Calogero system with parameters

$$\lambda_k = \frac{1}{\lambda}, \quad \lambda_{ak} = \lambda_{ka} = -1, \quad \lambda_a = \lambda_{ab} = \lambda \quad \text{for} \quad a, b \neq k;$$
$$m_k = -\frac{m}{\lambda}, \quad m_a = m \quad \text{for} \quad a \neq k;$$

(112)

partial densities

$$-\frac{\rho_k}{\lambda}, \quad \rho_a \quad \text{for} \quad a \neq k;$$

(113)

and total density

$$\rho(x) = \sum_{a \neq k} \rho_a(x) - \frac{1}{\lambda} \rho_k(x).$$

(114)

To reverse the argument, the $k$-th family in the less degenerate $F$-family system defined by (112) - (113) is singled out, yet the latter is dual to the MD $F$-family system, and therefore resembles a single family Calogero model. We can now apply another element $T_l$ ($l \neq k$) of (111) on the less degenerate $F$-family system, and obtain an
even lesser-degenerate $F$-family system, now with two families singled out. Yet, the latter system is still dual to the MD system. We can clearly continue this process thus generating all images of the MD $F$-family system under all $2^F - 1$ non-unit elements of the duality group. Each of these images comprises a special $F$-family system, whose families are not all identical, which nevertheless resembles a single family Calogero model. Just to give a nontrivial example, consider the image of the MD $F$-family system under the combined duality transformation $T_{k+1} T_{k+2} \cdots T_F$. This image is the $F$-family Calogero system with masses $m_a$, coupling constants $\lambda_a$ and mutual interaction strengths $\lambda_{ab}$ given by

$$\lambda_1 = \lambda_2 = \ldots = \lambda_k = \lambda;$$
$$m_1 = m_2 = \ldots = m_k = m;$$
$$\lambda_{ab} = \lambda, \text{ for } a, b \leq k; \quad \lambda_{k+1} = \lambda_{k+2} = \ldots = \lambda_F = \frac{1}{\lambda};$$
$$m_{k+1} = m_{k+2} = \ldots = m_F = -\frac{m}{\lambda};$$
$$\lambda_{ab} = \frac{1}{\lambda}, \text{ for } a, b > k; \quad \lambda_{ab} = -1, \text{ for } a \leq k \text{ and } b > k. \quad (115)$$

The effective one-family collective-field is obviously given by

$$\rho(x) = \sum_{a=1}^{k} \rho_a(x) - \frac{1}{\lambda} \sum_{a=k+1}^{F} \rho_a(x). \quad (116)$$

For $F = 2$ this result clearly coincides with (23).

\section{Concluding remarks}

We conclude this paper with a few comments on some recent papers and also on another set of duality transformations which were discussed in the literature.
Sergeev and Veselov [39] constructed supersymmetric extensions of the Calogero-Sutherland model which actually correspond to our two-family Calogero model with $\lambda_1 \lambda_2 = 1$, $\lambda_{12} = -1$ and $m_1 m_2 < 0!$ They gave solutions in terms of deformed Jack polynomials. In a recent paper, Kohler and Guhr [41] introduced a supersymmetric generalization of the Calogero-Sutherland model. Their construction is based on Jacobians for the radial coordinates on certain superspaces. This approach allowed them to explicitly construct the solutions in terms of recursion formulae for a non-trivial ($\lambda_1 \lambda_2 = 1$) one-parameter subspace in the $(\lambda_1, \lambda_2)$ plane. The underlying model involves two kinds of interacting particles, one with the positive and the other one with the negative mass. Needless to say, this again corresponds to our two-family Calogero model with $\lambda_{12} = -1$. It is interesting to observe that the authors of Refs.[39]-[42] were probably unaware of the constraints (2). Namely, in their approaches, these constraints remain hidden, but still present, as can be easily checked by direct substitutions. Consequently, the two types of models discussed in [39]-[42], share the very same parametric structure, which enables one to transform them to the one-family Calogero model. This connection then guarantees their exact integrability. Although our collective-field approach is applicable only to the multi-species Calogero system with an infinitely large number of particles within each family, we believe that our findings shed some light on the problem of their exact integrability in general.

Let us finally remark on an additional approximate duality invariance of the multi-species model (104), known in the literature [60, 61]. For simplicity, we illustrate it on the two-family Hamiltonian (16), in which case this approximate duality symmetry involves special two-family systems, in which $\lambda_2 = \frac{1}{\lambda_1}$ but $\lambda_{12} = +1$, as opposed to (22). This duality symmetry is destroyed by subleading terms in (16) as
we now explain.

Estimating the $\frac{1}{N}$ dependence of the terms in the effective potential (25), we see that the $\partial_x \ln \rho_1$ and $\partial_x \ln \rho_2$ terms are down by $\frac{1}{N_1}$ and by $\frac{1}{N_2}$, respectively, when compared with the Hilbert-transform terms. The effective potential to this order is then

$$V = \frac{1}{2m_1} \int dx \rho_1(x) \left( \lambda_1 \int \frac{dy \rho_1(y)}{x-y} + \lambda_{12} \int \frac{dy \rho_2(y)}{x-y} \right)^2 + \frac{1}{2m_2} \int dx \rho_2(x) \left( \lambda_2 \int \frac{dy \rho_2(y)}{x-y} + \lambda_{12} \int \frac{dy \rho_1(y)}{x-y} \right)^2. \quad (117)$$

One can verify that the above potential is invariant under the following set of transformations of the parameters:

$$\tilde{\lambda}_1 = \lambda_1; \quad \tilde{\lambda}_2 = \frac{1}{\lambda_2}; \quad \tilde{m}_1 = m_1; \quad \tilde{m}_2 = \pm \frac{m_2}{\lambda_2}; \quad \tilde{\lambda}_{12} = \pm \frac{\lambda_{12}}{\lambda_2}$$

and of the densities

$$\tilde{\rho}_1 = \rho_2; \quad \tilde{\rho}_2 = \pm \lambda_2 \rho_2. \quad (118)$$

Arguing along the same lines as before, we can show that the two-family Calogero model is equivalent to the one-family Calogero model if the following conditions are satisfied:

$$\lambda_1 = \lambda; \quad \lambda_2 = \frac{1}{\lambda}; \quad m_1 = m; \quad m_2 = \pm \frac{m}{\lambda}; \quad \lambda_{12} = \pm 1,$$

where $\lambda$ denotes the coupling constant and $m$ the mass of the effective one-family model. The effective one-family collective density is given by

$$\rho(x) = \rho_1(x) \pm \frac{1}{\lambda} \rho_2(x). \quad (119)$$

We stress that the above picture is valid in the leading approximation only. Consequently, this description is inevitably destroyed by the next-to-leading-order terms.
stemming, for instance, from the quantum collective-field fluctuations. In contrast, the various dualities $T_1, T_2, T_{12}$ (and, of course, the identity $I$), discussed in Section 3, are exact symmetries of the collective Hamiltonian (16). Consequently, the dual equivalence of any two-family system on the SOHI (22) to a two-family system of identical families given by (23) or (24) is also an exact property of (16).
Appendix A: An alternative derivation of (27)

For completeness, we present here an alternative derivation of the expression (27) for the collective effective potential. To this end, it is useful to introduce the weighted density

\[ \rho(x) = \sqrt{\lambda_1} \rho_1(x) + \sqrt{\lambda_2} \rho_2(x), \quad (A.1) \]

and consider the resolvent

\[ G(z) = \int \frac{dy \rho(y)}{z - y}, \quad (A.2) \]

associated with it, in which \( z \) is a complex variable. As in the single family case, the particles are expected to condense in the ground state in a large but finite segment of some length \( L \) - the support of \( \rho(x) \). The resolvent \( G(z) \) is therefore analytic in the complex plane, save for a cut along that segment, which lies on the real axis. Consequently, as usual, we have

\[ G(x \mp i0) = F(x) \pm i \pi \rho(x) \quad (A.3) \]

where

\[ F(x) = \int \frac{dy \rho(y)}{x - y}. \quad (A.4) \]

The combinations of principal part integrals in the first two lines in (25) are evidently proportional to \( F(x) \). This fact, together with (3) can be used to rewrite (25) as

\[
V = \frac{(\lambda_1 - 1)^2}{8m_1} \int dx \left( \frac{\partial_x \rho_1}{\rho_1} \right)^2 + \frac{(\lambda_2 - 1)^2}{8m_2} \int dx \left( \frac{\partial_x \rho_2}{\rho_2} \right)^2 \\
+ \frac{\sqrt{\lambda_1}}{m_1} \int dx \ F(x) \partial_x \left( \frac{\lambda_1 - 1}{2} \rho_1 + \frac{\lambda_2 - 1}{2} \rho_2 \right) \\
+ \frac{\sqrt{\lambda_1}}{2m_1} \int dx \ \rho(x) \ F^2(x) \\
+ \mu_1 \left( N_1 - \int dx \rho_1(x) \right) + \mu_2 \left( N_2 - \int dx \rho_2(x) \right). \quad (A.5)
\]
The term $\rho(x) F^2(x)$ in the third line in (A.5) appears to be a trilocal cubic functional of the densities. In fact, it can be brought to local form by a standard trick [51], based on the analytic structure of $G(z)$ and its asymptotic behavior

$$G(z) \sim \frac{N_1\sqrt{\lambda_1} + N_2\sqrt{\lambda_2}}{z}$$

as $z \to \infty$. Note that the fact that $\rho_1(x)$ and $\rho_2(x)$ have compact support is necessary for (A.6) to hold. It follows from (A.6) that $\oint_{C_\infty} dz G^3(z) = 0$, where $C_\infty$ is a circle around the point at infinity. Then, shrink $C_\infty$ to wrap around the cut to obtain $2i \int dx \Im G^3(x - i0) = 0$, namely,

$$\int dx \rho(x) F^2(x) = \frac{\pi^2}{3} \int dx \rho^3(x).$$

(A.7)

Substituting (A.7) in (A.5) we obtain the effective potential in its final form as

$$V = \frac{(\lambda_1 - 1)^2}{8m_1} \int dx \frac{(\partial_x \rho_1)^2}{\rho_1} + \frac{(\lambda_2 - 1)^2}{8m_2} \int dx \frac{(\partial_x \rho_2)^2}{\rho_2} + \frac{\sqrt{\lambda_1}}{m_1} \int dx F(x) \partial_x \left( \frac{\lambda_1 - 1}{2} \rho_1 + \frac{\lambda_2 - 1}{2} \rho_2 \right) + \frac{\pi^2 \sqrt{\lambda_1}}{6m_1} \int dx \rho^3(x) + \mu_1 \left( N_1 - \int dx \rho_1(x) \right) + \mu_2 \left( N_2 - \int dx \rho_2(x) \right),$$

(A.8)

which can be shown easily to coincide with (27).
Appendix B: Diagonalization of $\omega_{ab}(k)$ in (108)

The matrix $\omega_{ab}(k)$ in (108) is of the general form

$$\omega = Dk^2 + vv^T|k|,$$  \hspace{1cm} (B.1)

where $D$ is a diagonal matrix with diagonal entries $d_a = \frac{1-\lambda_a}{2m_a}$, and $v$ is an $F$-dimensional vector with components $v_a = cm_a\sqrt{p_{\mu a}/m_a} = \text{sign}(m_a)\sqrt{p_{\mu a}\lambda_a/m_a}$, where we used (102) twice. After a standard computation we obtain the characteristic polynomial of (B.1) as

$$P_F(z) \equiv \det(z - Dk^2 + vv^T|k|) = \det(z - Dk^2) \left(1 - v^T \frac{|k|}{z - Dk^2} v\right).$$  \hspace{1cm} (B.2)

In the generic case, where all the diagonal elements $d_a$ of $D$ are different from each other (and we shall henceforth assume this case), none of the eigenvalues of $\omega$ is an eigenvalue of $Dk^2$.

\hspace{1cm} 16 Therefore, all roots of $P_F(z)$ are obtained by finding the zeros of the second factor in (B.2), namely, from

$$v^T \frac{|k|}{z - Dk^2} v = \sum_{a=1}^F v_a^2|k| \frac{1}{z - d_a k^2} = 1.$$  \hspace{1cm} (B.3)

It is clear from (B.1) that all eigenvalues of $\omega$ vanish at least as fast as $|k|$ when the latter tends to zero. Alternatively, this can be seen by taking the limit $|k| \to 0$ in (B.3). Since all $v_a^2 > 0$, the roots must vanish at least like $|k|$. Thus, let us set $z = \zeta|k|$ in (B.3), and solve

$$\sum_{a=1}^F \frac{v_a^2}{\zeta - d_a|k|} = 1.$$  \hspace{1cm} (B.4)

\hspace{1cm} 16This should be well-known. The reason for this is simple: the second factor in (B.2) has simple poles located at the eigenvalues of $Dk^2$, which cancel against zeros at the same points in the first factor in (B.2). If $D$ is not degenerate, all those zeros are simple, and we end up with a nonvanishing result.
for \( \zeta \). In fact, we need to find \( \zeta \) only to first order in \( |k| \), as we are interested in the eigenvalues of (B.1) only to order \( k^2 \).

Let us first look for roots of (B.4) which do not vanish as \( |k| \to 0 \). We find only one such root, which at \( |k| = 0 \) is given simply by \( \zeta^{(0)} = \sum_a v_a^2 > 0 \). To find the leading correction to this root we expand (B.4) in inverse powers of \( \zeta \) to order \( |k| \):

\[
1 - \frac{1}{\zeta} \sum_{a=1}^{F} v_a^2 - \frac{|k|}{\zeta^2} \sum_{a=1}^{F} v_a^2 d_a + O(k^2) = 0 .
\]

(B.5)

Only one root of this approximate quadratic equation converges to \( \zeta^{(0)} \), and to order \( |k| \) it is given by

\[
\zeta^{(1)} = \sum_{a=1}^{F} v_a^2 + \frac{\sum_{a=1}^{F} v_a^2 d_a}{\sum_{a=1}^{F} v_a^2} |k| .
\]

(B.6)

Upon substituting the appropriate values of \( v_a \) and \( d_a \), (B.6) coincides with the expression (109) quoted in the text. In our conventions \( \lambda_a > 0 \) and \( \text{sign} \, m_a = \text{sign} \, \rho_{a,0} \). Therefore all \( v_a \) are real, and consequently the coefficient \( \sum_{a=1}^{F} v_a^2 \) of \( |k| \) in (109) is positive.

All other \( F - 1 \) roots of (B.4) vanish as \( |k| \) when the latter tends to zero. To see this, restore the original roots \( z = \zeta |k| \) and consider \( \det(-\omega) = P_F(0) \sim (k^2)^{F-1} \left( k^2 + |k| v^T D^{-1} v \right) \). As \( |k| \to 0 \) this behaves like \( (k^2)^{F-1} |k| \), proving our claim.

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