Abstract—Permutation and multipermutation codes in the Ulam metric have been suggested for use in non-volatile memory storage systems such as flash memory devices. In this paper we introduce a new method to calculate permutation ball sizes in the Ulam metric using Young Tableaux and prove the non-existence of non-trivial perfect permutation codes in the Ulam metric. We then extend the study to multipermutations, providing upper and lower bounds on multipermutation Ulam ball sizes and resulting upper and lower bounds on the maximal size of multipermutation codes in the Ulam metric.

Index Terms—Error correction codes, channel coding, upper bound, nonvolatile memory, flash memory.

I. INTRODUCTION

The history of permutation codes dates as far back as the 1960’s and 70’s, with Slepian, Berger, Blake, and others [2], [3], [19]. However, the application of permutation codes and multipermutation codes for use in non-volatile memory storage systems such as flash memory has received attention in the coding theory literature in recent years [1], [12], [13], [17], [21], [22]. One of the main distance metrics in the literature has been the Kendall-τ metric, which is suitable for correction of the type of error expected to occur in flash memory devices [5], [11]–[13], [23]. Errors occur in these devices when the electric cell charges storing information leak over time or there is an overshoot of charge level in the rewriting process. For relatively small leak or overshoot errors the Kendall-τ metric is appropriate. However, it may not be well-suited for large errors within a single cell.

In 2013, Farnoud et al. proposed permutation codes using the Ulam metric [6]. They showed that the use of the Ulam metric would allow a large leakage or overshoot error within a single cell to be viewed as a single error. Subsequent papers expounded on the use of Ulam metric in multipermutation codes and bounds on the size of permutation codes in the Ulam metric [7], [10]. Meanwhile, Buzaglo and Etzion discovered the existence of a perfect permutation code under the cyclic Kendall-τ metric, and proved the non-existence of perfect permutation codes under the Kendall-τ metric for certain parameters [4]. However, the possibility of perfect permutation codes in the Ulam metric had not previously been considered. Exploring this possibility requires first understanding the sizes of Ulam permutation balls, of which only limited research exists. Even less is known about the size of multipermutation Ulam balls.

In this paper we consider four main questions. Their answers are the main contributions of this paper. First: How can permutation Ulam ball sizes be calculated? One answer to this question is to use Young Tableaux and the RSK (Robinson-Schensted-Knuth) Correspondence (Theorem III.1).

Second: Do perfect Ulam permutation codes exist? The answer to this question is that nontrivial perfect Ulam permutation codes do not exist (Theorem IV.1). These two questions are closely related to each other since perfect Ulam permutation code sizes are characterized by Ulam ball sizes. These results are summarized in Tables I and II on the following page. Notation appearing in the tables is defined in subsequent sections.

The discussion is then extended to multipermutation codes, where we consider the third question: How can multipermutation Ulam ball sizes be calculated? Theorem III.1 and Theorem V.1 show how to calculate ball sizes for certain parameters. Finally, the final question: What is the maximum possible Ulam multipermutation code size? Lemmas VI.2, VI.4, and VI.5 (as well as Lemmas VI.15 and VI.16 for the special binary case) provide new upper and lower bounds on the maximal code size. These results are summarized in Tables III and IV. Notation appearing in the tables is defined in subsequent sections.

The organization is as follows: Section II defines notation and basic concepts used throughout the paper. Sections III and IV focus primarily on permutations, although many results apply to multipermutations as well. Section III

TABLE I

| Permutation Ulam Ball Size Formulas | Reference |
|-------------------------------------|-----------|
| \(|B(\sigma, t)| = \sum_{\lambda \in \Lambda} (\lambda^T)^2 \) |
| \(|B(\sigma, 1)| = 1 + (n-1)^2 \) | Proposition III.2 |

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introduces a method of calculating Ulam ball sizes using Young tableaux and the RSK-Correspondence. Section IV focuses on proving the non-existence of nontrivial perfect permutation codes. The remaining sections focus on multipermutations. Section V discusses how to calculate \( r \)-regular multipermutation Ulam ball sizes. Section VI discusses minimum and maximum ball sizes and provides new upper and lower bounds on maximal multipermutation code size. Included in the last two subsections of Section VI is an explanation of how to determine the maximum ball size in the binary case, which presents unique challenges. Finally Section VII gives concluding remarks.

II. Preliminaries and Notation

In this paper we utilize the following notation and definitions, generally following conventions established in [6] and [7]. Throughout the paper we will assume that \( n \) and \( r \) are positive integers, with \( r \) dividing \( n \). The symbol \([n]\) denotes the set of integers \( \{1, 2, \ldots, n\} \) and for \( a, b \in \mathbb{Z} \) with \( a < b \), the notation \([a, b] := \{a, a+1, \ldots, b\} \). The symbol \( S_n \) stands for the set of permutations (automorphisms) on \([n]\), i.e., the symmetric group of order \( n \). For a permutation \( \sigma \in S_n \), we use the notation \( \sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in [n]^n \). Where for all \( i \in [n] \), \( \sigma(i) \) is the image of \( i \) under \( \sigma \). With some abuse of notation, we may also use \( \sigma \) to refer to the sequence \( (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in [n]^n \). In other words, we define multiplication of permutations by composition. e.g., \([2, 1, 5, 4, 3][5, 1, 4, 2, 3] = [3, 2, 4, 1, 5]\). The identity permutation \([1, 2, \ldots, n] \in S_n \) is denoted by \( e \).

An \( r \)-regular multiset is a multiset such that each of its elements appear exactly \( r \) times (i.e., each element is

### TABLE II
NEW RESULTS 2) THEORETICAL LIMIT ON MAXIMUM ULM PERMUTATION CODE SIZE

| Theorem on perfect Ulam permutation codes | Reference |
|------------------------------------------|-----------|
| Nontrivial perfect \( t \)-error correcting permutation codes do not exist | Theorem IV.1 |

### TABLE III
NEW RESULTS 3) MULTIPERMUTATION ULM BALL SIZES

| Multipermutation Ulam ball Size Formulas and Bounds | Reference |
|-----------------------------------------------------|-----------|
| \(|B(m_r, t)| = \sum(L_{\lambda}(K_r^\Lambda))_\Lambda\in\Lambda| Theorem III.1 |
| \(|B(m_r, 1)| = 1 + (n - 1)^2 - |SD(m_r)| - |AD(m_r)|| Theorem V.1 |
| \(1 + (n - 1)(n/r - 1) = \frac{|B(m_r, 1)|}{|B(m_r, 1)|} \leq \frac{|B(m_r, 1)|}{|B(m_r, 1)|} \) | Lemma VI.1 and Theorem III.1 |
| Non-Binary Case: \(|B(m_r, 1)| \leq |B(m_r, 1)| = 1 + (n - 1)^2 - (r - 1)n| Lemma VI.3 |
| Binary Case: \(|B(m_r, 1)| < U(r)| Corollary VI.14 |

### TABLE IV
NEW RESULTS 4) THEORETICAL LIMITS ON MAXIMUM ULM MULTIPERMUTATION CODE SIZE

| Ulam Multipermutation Code Max Size Bounds | Value | Reference |
|-------------------------------------------|-------|-----------|
| 1-error correcting code upper bound | \(|C| \leq \frac{n!}{(r!)^{n/r} (1 + (n-1)(n/r-1))^{n/r}}| Lemma VI.2 |
| Non-Binary Case: Perfect 1-error correcting code lower bound | \(|C| \leq \frac{n!}{(r!)^{n/r} (1 + (n-1)^2 - (r-1)n)^{n/r}}| Lemma VI.4 |
| Binary Case: Perfect 1-error correcting code lower bound | \(|C| \leq \frac{n!}{(r!)^{n/r} (1 + (n-1)^2 - (r-1)n)^{n/r}}| Lemma VI.15 |
| Non-Binary Case: MPC_\( r \)(n, r, d) lower bound | \(|C| \leq \frac{n!}{(r!)^{n/r} (1 + (n-1)^2 - (r-1)n)^{n/r}}| Lemma VI.5 |
| Binary Case: MPC_\( r \)(n, r, d) lower bound | \(|C| \leq \frac{n!}{(r!)^{n/r} (1 + (n-1)^2 - (r-1)n)^{n/r}}| Lemma VI.16 |
Definition (m^r_r). Given σ ∈ S_n, we define a corresponding r-regular multipermutation m^r_σ as follows: for all i ∈ [n] and j ∈ [n/r],

\[ m^r_σ(i) := j \text{ if and only if } (j - 1)r + 1 \leq σ(i) \leq jr, \]

and \( m^r_σ := (m^r_σ(1), m^r_σ(2), \ldots, m^r_σ(n)) \in [n/r]^n \).

As an example of \( m^r_σ \), let \( n = 6, r = 2, \) and \( σ = [1, 5, 2, 4, 3, 6] \). Then \( m^r_σ = (1, 3, 1, 2, 2, 3) \). Note that this definition differs slightly from the correspondence defined in [7], which was defined in terms of the inverse permutation. This is so that certain properties (Lemmas II.1 and II.2) of the Ulam metric for permutations (the case when \( r = 1 \)) will also hold for general multipermutations. Notice that \( m^r_σ = (σ(1), \ldots, σ(n)) \in [n/r]^n \), so based on our abuse of notation described in the first paragraph of this section, we may denote \( m^r_σ \) simply with \( σ \). In other words, whenever \( r = 1, \) r-regular multipermutations reduce to permutations, or more accurately their associated sequences.

With the correspondence above, we may define an equivalence relation between elements of \( S_n \). For permutations \( σ, π ∈ S_n \), we say that \( σ \equiv_r π \) if and only if \( m^r_σ = m^r_π \). The equivalence class \( R_r(σ) \) of \( σ ∈ S_n \) is defined by \( R_r(σ) := \{ π ∈ S_n : π \equiv_r σ \} \). Note that if \( r = 1 \), then \( R_r(σ) \) is simply the singleton \( \{ σ \} \). For a subset \( S ⊆ S_n \), define \( M_r(S) := \{ m^r_σ : σ ∈ S \} \), i.e. the set of r-regular multipermutations corresponding to elements of \( S \). When \( r = 1 \), we may identify \( M_r(S) \) simply by \( S \).

We next define the r-regular Ulam distance. For the definition, it is first necessary to define \( ℓ(x, y) \). Given sequences \( x, y ∈ Z^n \), then \( ℓ(x, y) \) denotes the length of the longest common subsequence of \( x \) and \( y \) (not to be confused with the longest common substring). More precisely, \( ℓ(x, y) \) is the largest integer \( k ∈ Z_{>0} \) such that there exists a sequence \( (a_1, a_2, \ldots, a_k) \) where for all \( l ∈ [k] \), we have \( a_l = x(i_l) = y(j_l) \) with \( 1 ≤ i_1 < i_2 < \cdots < i_k ≤ n \) and \( 1 ≤ j_1 < j_2 < \cdots < j_k ≤ n \). For example, \( ℓ((3, 1, 2, 1, 2, 3), (1, 1, 2, 2, 3, 3)) = 4 \), since \((1, 1, 2, 3)\) is a common subsequence of both \((3, 1, 2, 1, 2, 3)\) and \((1, 1, 2, 2, 3, 3)\) and its length is 4. If \( σ ∈ S_n \), then \( ℓ(σ, e) \) is the length of the longest increasing subsequence of \( σ \), which we denote simply by \( ℓ(σ) \). Similarly, for an r-regular multipermutation \( m^r_σ \), we denote the length of the longest non-decreasing subsequence \( ℓ(m^r_σ, m^r_π) \) of \( m^r_σ \) simply by \( ℓ(m^r_π) \).

Definition (d_0(m^r_σ, m^r_π), r-regular Ulam distance). Let \( m^r_σ, m^r_π ∈ M_r(S_n) \). Define

\[ d_0(m^r_σ, m^r_π) := \min_{σ' ∈ R_r(σ), π' ∈ R_r(π)} d_0(σ', π'), \]

where \( d_0(σ, π) := n - ℓ(σ, π) \). We call \( d_0(m^r_σ, m^r_π) \) the r-regular Ulam distance between \( m^r_σ \) and \( m^r_π \). In the case when \( r = 1 \), we may simply refer to the Ulam distance between \( σ \) and \( π \) and use the notation \( d_0(σ, π) \).

The definition of r-regular Ulam distance above follows the convention of [7], defining the distance in terms of equivalence classes comprised of permutations, although our notation differs. However, it is convenient to think of the distance instead in terms of the multipermutations themselves. A simple argument shows that the r-regular Ulam distance between multipermutations \( m^r_σ \) and \( m^r_π \) is equal to \( n \) minus the length of their longest common subsequence. The details of the argument can be found in Appendix A.

Lemma II.1. Let \( m^r_σ, m^r_π ∈ M_r(S_n) \). Then

\[ d_0(m^r_σ, m^r_π) = n - ℓ(m^r_σ, m^r_π). \]

Viewed this way, it is easily verified that the r-regular Ulam distance \( d_0(m^r_σ, m^r_π) \) is a proper metric between the multipermutations \( m^r_σ \) and \( m^r_π \). Additionally, it is known that in the permutation case, the case when \( r = 1 \), that the Ulam distance can be characterized in terms of permutations of a specific type, known as translocations [6, 10]. We can show a similar relationship for multipermutations. We define translocations below and then give the relationship between the Ulam distance and translocations.

Definition (φ(i, j), translocation). Given distinct \( i, j ∈ [n] \), define \( φ(i, j) ∈ S_n \) as follows:

\[ φ(i, j) := \begin{cases} [1, 2, \ldots, i - 1, i + 1, i + 2, \ldots, j, i, i + 1, \ldots, n] & \text{if } i < j \\ [1, 2, \ldots, j - 1, j, i, j + 1, \ldots, i - 1, i + 1, \ldots, n] & \text{if } i > j \end{cases} \]

If \( i = j \), then define \( φ(i, j) := e \). We refer to \( φ(i, j) \) as a translocation, and if we do not specify the indexes \( i \) and \( j \) we may denote a translocation simply by \( φ \). The notation \( φ \), as well as \( φ \) with lower indices (such as \( φ_i \)), will henceforth be reserved for translocations.

Intuitively, a translocation is the permutation that results in a delete/insertion operation. More specifically, given \( σ ∈ S_n \) and the translocation \( φ(i, j) ∈ S_n \), the product \( σφ(i, j) \) is the result of deleting \( σ(i) \) from the \( i \)th position of \( σ \), then shifting all positions between the \( i \)th and \( j \)th position by one (left if \( i < j \) and right if \( i > j \)), and finally reinserting \( σ(i) \) into the new \( j \)th position. The top half of Figure 1 illustrates the permutation \( σ = [6, 2, 8, 5, 4, 1, 3, 9, 7] \) (or its related 3-regular multipermutation \( m^3_σ = (2, 1, 3, 2, 1, 1, 3, 3) \)) represented physically by relative cell charge levels and the effect of multiplying \( σ \) (or \( m^3_σ \)) on the right by the translocation \( φ(1, 9) \). The bottom half of Figure 1 illustrates the same \( σ \) (or \( m^3_σ \)) and the effect of \( φ(7, 4) \). Notice that multiplying by \( φ(1, 9) \) corresponds to the error that occurs when the highest (1st) ranked cell suffers charge leakage that results in it being the lowest (9th) ranked cell. Multiplying by \( φ(7, 4) \) corresponds
to the error that occurs when the 7th highest cell is overfilled so that it is the 4th highest cell.

It is well-known that \( d_{\sigma}(\sigma, \pi) \) equals the minimum number of translocations needed to transform \( \sigma \) into \( \pi \) [6, 10]. That is, \( d_{\sigma}(\sigma, \pi) = \min\{k \in \mathbb{Z}_{\geq 0} : \text{there exists } \phi_1, \phi_2, \ldots, \phi_k \text{ such that } \sigma \phi_1 \phi_2 \cdots \phi_k = \pi \} \).

By applying Lemma II.1, it is also a simple matter to prove that an analogous relationship holds for the \( r \)-regular Ulam distance. First, it is necessary to define multiplication between multipermutations and permutations.

We define the product \( m^r_{\sigma} \cdot \pi \) as \( m^r_{\sigma} \cdot \pi := m^r_{\sigma \cdot \pi} \). Technically speaking, this can be seen as a right group action of the set \( S_n \) of permutations on the set \( M_r(S_n) \). Since it is possible for different permutations to correspond to the same multipermutation, we should clarify that \( m^r_{\sigma} = m^r_{\tau} \) implies \( m^r_{\sigma \cdot \pi} = m^r_{\tau \cdot \pi} \), i.e. we should show that the product is well-defined. Indeed this is true because if \( m^r_{\sigma} = m^r_{\tau} \), then for all \( i \in [n] \) we have \( m^r_{\sigma}(i) = m^r_{\tau}(i) \), which implies for \( j := m^r_{\sigma}(i) \) that \( (j - 1)r + 1 \leq \sigma(i) \leq jr \) and \( (j - 1)r + 1 \leq \tau(i) \leq jr \). This in turn implies that \( (j - 1)r + 1 \leq \sigma \pi(\pi^{-1}(i)) \leq jr \) and \( (j - 1)r + 1 \leq \tau \pi(\pi^{-1}(i)) \leq jr \), which means \( m^r_{\sigma \pi}(\pi^{-1}(i)) = m^r_{\tau \pi}(\pi^{-1}(i)) \). Since \( \pi \) is a permutation and \( i \in [n] \) was chosen arbitrarily, we conclude that \( m^r_{\sigma \pi} = m^r_{\tau \pi} \).

Intuitively speaking, the same corresponding elements of the sequences \( \sigma \) and \( \tau \) still correspond (with a different index) after being multiplied on the right by \( \pi \). Hence \( m^r_{\sigma \pi} = m^r_{\tau \pi} \), or by our notation \( m^r_{\sigma \cdot \pi} = m^r_{\tau \cdot \pi} \).

If two multipermutations \( m^r_{\sigma} \) and \( m^r_{\tau} \) have a longest common subsequence of length \( k \), then \( m^r_{\sigma} \) can be transformed into \( m^r_{\tau} \) with \( n - k \) (but no fewer) delete/insert operations.

As with permutations, delete/insert operations correspond to applying (multiplying on the right by) a translocation. Hence by Lemma II.1 we can state the following lemma about the \( r \)-regular Ulam ball.

**Lemma II.2.** Let \( m^r_{\sigma} \in M_r(S_n) \). Then

\[
d_{\sigma}(m^r_{\sigma}, m^r_{\tau}) = \min\{k \in \mathbb{Z}_{\geq 0} : \text{there exists a sequence of translocations } \phi_1, \phi_2, \ldots, \phi_k \text{ such that } m^r_{\sigma} \cdot \phi_1 \phi_2 \cdots \phi_k = m^r_{\tau} \}.
\]

We now define the notions of a multipermutation code and an \( r \)-regular Ulam ball.

**Definition (r-regular multipermutation code, MPC\((n, r, d)\)).** Recall that \( n, r \in \mathbb{Z}_{\geq 0} \) with \( r \mid n \). An \( r \)-regular multipermutation code (or simply a multipermutation code) is a subset \( C \subseteq M_r(S_n) \). Such a code is denoted by \( \text{MPC}(n, r, d) \).

**Definition (r-regular multipermutation Ulam ball).** Let \( t \in \mathbb{Z}_{\geq 0} \), and \( m^r_{\sigma} \in M_r(S_n) \). Define

\[
B(m^r_{\sigma}, t) := \{m^r_{\tau} \in M_r(S_n) : d_{\sigma}(m^r_{\sigma}, m^r_{\tau}) \leq t \}
\]

We call \( B(m^r_{\sigma}, t) \) the \( r \)-regular multipermutation Ulam ball, (or simply the multipermutation Ulam ball) centered at \( m^r_{\sigma} \) of radius \( t \). We refer to any 1-regular multipermutation Ulam ball as a permutation Ulam ball and use the simplified notation \( B(\sigma, t) \) instead of \( B(m^r_{\sigma}, t) \).

By Lemma II.1, \( B(m^r_{\sigma}, t) = \{m^r_{\tau} \in M_r(S_n) : n - \ell(m^r_{\sigma}, m^r_{\tau}) \leq t \} \). The \( r \)-regular Ulam ball definition can also be viewed in terms of translocations. Lemma II.2 implies that \( B(m^r_{\sigma}, t) \) is equivalent to \( \{m^r_{\tau} \in M_r(S_n) : \text{there exists } k \in \{0, 1, \ldots, t\} \text{ and } (\phi_1, \phi_2, \ldots, \phi_k) \text{ such that } m^r_{\sigma} \cdot \phi_1 \phi_2 \cdots \phi_k = m^r_{\tau} \} \). This is the set of all multipermutations reachable by applying up to \( t \) translocations to the center multipermutation \( m^r_{\sigma} \).

It is well-known that an \( \text{MPC}(n, r, d) \) code is \( t \)-error correcting if and only if \( d \geq 2t + 1 \) [7]. This is because if the distance between two codewords is greater than or equal to \( 2t + 1 \), then after \( t \) or fewer errors (multiplication by \( t \) or fewer translocations), the resulting multipermutation remains closer to the original multipermutation than any other element in a code was a subset of the code. Next, we define \( r \)-regular Ulam balls.

**Definition (B(m^r_{\sigma}, t), r-regular multipermutation Ulam ball).** Let \( t \in \mathbb{Z}_{\geq 0} \), and \( m^r_{\sigma} \in M_r(S_n) \). Define

\[
B(m^r_{\sigma}, t) := \{m^r_{\tau} \in M_r(S_n) : d_{\sigma}(m^r_{\sigma}, m^r_{\tau}) \leq t \}
\]
multipermutation. We finish this section by defining perfect t-error correcting codes.

**Definition** (perfect code). Let \( C \subseteq \mathcal{M}_t(S_n) \) be an MPC\((n, r)\). Then \( C \) is a perfect t-error correcting code if and only if for all \( m^r_\sigma \in \mathcal{M}_t(S_n) \), there exists a unique \( m^c_\sigma \in C \) such that \( m^r_\sigma \in B(m^c_\sigma, t) \). We call such \( C \) a perfect t-error correcting MPC\((n, r)\), or simply a perfect code if the context is clear. A permutation code that is perfect is called a perfect permutation code.

A perfect MPC\((n, r)\) partitions \( \mathcal{M}_t(S_n) \). This means the balls centered at codewords fill the space without overlapping. A perfect code \( C \subseteq \mathcal{M}_t(S_n) \) is said to be trivial if either (1) \( C = \mathcal{M}_t(S_n) \) (occurring when \( t = 0 \)); or (2) \( |C| = 1 \) (occurring when \( t = n - r \)).

### III. Permutation Ulam Ball Size

This section focuses on answering the question: how can we calculate the sizes of permutation Ulam balls? Theorem III.1 is one answer to this question. The theorem is actually stated in terms of multipermutations, making it also a partial answer to the question of how to calculate multipermutation Ulam ball sizes. However, unlike permutations, in the case of multipermutations ball sizes may depend upon the choice of center, limiting the applicability of the theorem for multipermutation Ulam balls. The proof of the theorem is provided after a necessary lemma is recalled and notation used in the theorem is clarified.

**Theorem III.1.** Let \( t \in \{0, 1, 2, \ldots, n - r\} \), and \( \Lambda := \{\lambda \vdash n : \lambda_1 \geq n - t\} \). Then

\[
|S(m^r_\sigma, t)| = \sum_{\lambda \in \Lambda} (f^{\lambda}) (K^r_\lambda).
\]

(1)

Although this section is primarily concerned with permutation Ulam ball sizes, many of the results hold for multipermutation Ulam balls as well, and lemmas and propositions in this section are stated with as much generality as possible. In the case of permutation codes, perfect codes and ball sizes are related as follows: a perfect \( t \)-error correcting permutation code \( C \subseteq \mathcal{S}_n \), if it exists, will have cardinality \( |C| = n! / B(c, t) \), where \( c \in C \). Hence one of the first questions that may be considered in exploring the possibility of a perfect code is the feasibility of a code of such size.

As noted in [6], for any \( \sigma \in \mathcal{S}_n \), we have \( |B(\sigma, t)| = |B(e, t)| \). Hence calculation of permutation Ulam ball sizes can be reduced to the case when the center of the ball is the identity permutation \( e \).

One way to calculate permutation Ulam ball sizes centered at \( e \) is to use Young tableaux and the RSK (Robinson-Schensted-Knuth) Correspondence [9]. It is first necessary to introduce some basic notation and definitions regarding Young diagrams and Young tableaux. Additional information on the subject can be found in [18] and [20].

A **Young diagram** is a left-justified collection of cells with a (weakly) decreasing number of cells in each row below (see Figure 2). Listing the number of cells in each row gives a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) of the integer \( n \), where \( n \) is the total number of cells in the Young diagram, that is, the notation \( \lambda \vdash n \) is used to mean \( \lambda \) is a partition of the integer \( n \). Because the partition \( \lambda \vdash n \) defines a unique Young diagram and vice versa, a Young diagram may be referred to by its associated partition \( \lambda \vdash n \). For example, the partition \( \lambda := (4, 3, 3, 2) \vdash 12 \) has the corresponding Young diagram pictured on the left side of Figure 2.

A **Young tableau**, abbreviated as SYT, is a filling of a Young diagram \( \lambda \vdash n \) with the following three qualities: (1) cell values are strictly increasing across each row; and (2) cell values are strictly increasing down each column. One possible Young tableau is pictured in the center of Figure 2. A **standard Young tableau**, abbreviated as SYT, is a filling of a Young diagram \( \lambda \vdash n \) with the following three qualities: (1) cell values are strictly increasing across each row; (2) cell values are strictly increasing down each column; and (3) each of the integers 1 through \( n \) appears exactly once. One possible SYT on \( \lambda := (4, 3, 3, 2) \) is pictured on the right side of Figure 2.

Among other things, the famous RSK-correspondence ([9], [20]) establishes a bijection between the set of all \( r \)-regular multipermutations \( m^r_\sigma \) and the set of all ordered pairs \( (P, Q) \) on the same Young diagram \( \lambda \vdash n \), where \( P \) is a Young tableau whose members come from \( m^r_\sigma \) and \( Q \) is a SYT. The bijection is the result of running what is known as the Schensted algorithm. In this paper we are not concerned with the particulars of the bijection, but the details of the correspondence can be found on pages 30–57 of [9], and especially in the discussion of the “R-S-K Theorem” on page 40. The next lemma, a stronger form of which appears on page 32 of [9], is an application of the RSK-correspondence.

**Lemma III.1.** Let \( m^r_\sigma \in \mathcal{M}_r(S_n) \) and let \( P \) and \( Q \), both on \( \lambda \vdash n \), be the pair of Young tableaux associated with \( m^r_\sigma \) by the RSK-correspondence. Then

\[
\lambda_1 = \ell(m^r_\sigma).
\]

In words, the above lemma says that \( \lambda_1 \), the number of columns in the \( P \) (or equivalently \( Q \)) associated with \( m^r_\sigma \) by the RSK-correspondence, is equal to \( \ell(m^r_\sigma) \), the length of the longest non-decreasing subsequence of \( m^r_\sigma \). The lemma implies that for all \( k \in [n] \), the size of the set \( \{m^r_\sigma \in \mathcal{M}_r(S_n) \mid \ell(m^r_\sigma) = k\} \) is equal to the sum of the number of ordered pairs \( (P, Q) \) on each Young diagram \( \lambda \vdash n \) of exactly \( k \) columns. Following conventional notation ([9], [20]), \( f^k_\lambda \) denotes the number of SYT on \( \lambda \vdash n \). We denote by \( K^r_\lambda \) the number of Young tableaux on \( \lambda \vdash n \) such that each \( i \in [n/r] \) appears exactly \( r \) times. We are now able to...
prove Theorem III.1, which states the relationship between $|B(m^n, t)|$, $f^t$, and $K^f_r$.

**Proof of Theorem III.1:**
Assume $t \in \{0, 1, \ldots, n-1\}$, and let $\Lambda := \{ \lambda \vdash n : \lambda_1 \geq n - t \}$. Furthermore, let $A^{(l)} := \{ \lambda \vdash n : \lambda_1 = l \}$, the set of all partitions of $n$ having exactly $l$ columns. By the RSK-Correspondence and Lemma III.1, there is a bijection between the set $\{ m^n_{\pi} : \ell(m^n_{\pi}) = l \}$ and the set of ordered pairs $(P, Q)$ where both $P$ and $Q$ have exactly $l$ columns. This implies that $\#m^n_{\pi} : \ell(m^n_{\pi}) = l = \sum_{\lambda \in A^{(l)}} (f^\lambda(K^f_r))$ (here $\#A$ is an alternate notation for the cardinality of a set that we prefer for conditionally defined sets). By Lemma II.1, $|S(m^n_{\pi}, t)| = \#m^n_{\pi} : d_t(m^n_{\pi}, m^n_{\pi}) \leq t = \#m^n_{\pi} : \ell(m^n_{\pi}) \geq n-t$. Hence it follows that $|S(m^n_{\pi}, t)| = \sum_{\lambda \in A} (f^\lambda(K^f_r))$. □

Because $K^f_r$ is equivalent to $f^\lambda$ by definition, in the case of permutation Ulam balls, equation (1) simplifies to

$$|S(e, t)| = \sum_{\lambda \in \Lambda} (f^\lambda)^2. \quad (2)$$

In both equation (1) and (2), the famous hook length formula, due to Frame et al. [8], [9], provides a way to calculate $f^\lambda$. Within the hook length formula, the notation $(i, j) \in \lambda$ is used to refer to the cell in the $i$th row and $j$th column of a Young diagram $\lambda \vdash n$. The notation $h(i, j)$ denotes the hook length of $(i, j) \in \lambda$, i.e., the number of boxes below or to the right of $(i, j)$, including the box $(i, j)$ itself. More formally, $h(i, j) := \#\{(t, s) \in \lambda : s \geq j \cup \{i^*, j \in \lambda : i^* \geq i\}$.

The hook-length formula is as follows:

$$f^\lambda = \frac{n!}{\prod_{(i, j) \in \lambda} h(i, j)}.$$

Applying the hook length formula to Theorem III.1, we may explicitly calculate Ulam permutation ball sizes, as demonstrated in the following propositions. These propositions will be useful later to show the nonexistence of nontrivial $t$-error correcting perfect permutation codes for $t \in \{1, 2, 3\}$. Proposition III.2 is stated in terms of general multipermutation Ulam balls.

**Proposition III.2.** $|S(m^{n}_{\lambda}, 1)| = 1+(n-1)(n/r-1)$.

**Proof.** First note that $|S(m^{n}_{\lambda}, 0)| = |m^{n}_{\lambda}| = 1$. There is only one possible partition $\lambda \vdash n$ such that $\lambda_1 = n-1$, namely $\lambda := (n-1, 1, 1)$, with its Young diagram pictured below.

$$\begin{array}{cccc}
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
n-1 & n-2 & n-3 & n-4 \\
\hline
\end{array}$$

Therefore by Theorem III.1, $|S(m^{n}_{\lambda}, 1)| = 1 + f^\lambda(K^f_r)$. Applying the hook length formula, we obtain $f^\lambda = n-1$. The value $K^f_r$ is characterized by possible fillings of row 2 with the stipulation that each $i \in [n/r]$ must appear exactly $r$ times in the diagram. In this case, since there is only a single box in row 2, the possible fillings are $i \in \{2, 3, \ldots n/r\}$, each of which yields a unique Young tableau of the desired type. Hence $K^f_r = n/r - 1$, which implies that $|S(m^{n}_{\lambda}, 1)| = 1+(n-1)(n/r-1)$.

Setting $r = 1$, Proposition III.2 implies that $|B(e, 1)| = 1+(n-1)^2$. The next two propositions continue the same vein of reasoning, but focus on permutation Ulam balls. Such individual cases could be considered indefinitely. In fact, a recurrence equation providing an alternative method of calculating permutation Ulam ball sizes for reasonably small radii is also known [16]. However, the following two propositions are the last instances of significance in this paper as their results will be necessary to prove the main result of this paper.

**Proposition III.3.** Let $n > 3$ and $\sigma \in \mathbb{S}_n$. Then

$$|B(\sigma, 2)| = 1+\left(\frac{n(n-3)}{2}\right)^2 + \left(\frac{(n-1)(n-2)}{2}\right)^2.$$

**Proof.** Assume $n > 3$ and $\sigma \in \mathbb{S}_n$. Note first that $|B(\sigma, 2)| = |B(\sigma, 1)| + |\pi \in \mathbb{S}_n : \ell(\pi) = n-2\}$. The only partitions $\lambda \vdash n$ such that $\lambda_1 = n-2$ are $\lambda^{(1)} := (n-2, 1, 1)$ and $\lambda^{(2)} := (n-2, 2, 2)$, with their respective Young diagrams pictured below.

Using the hook length formula, $f^{\lambda^{(1)}}$ and $f^{\lambda^{(2)}}$ may be calculated to yield: $f^{\lambda^{(1)}} = ((n-1)(n-2))/2$ and $f^{\lambda^{(2)}} = ((n)(n-3))/2$. Following the same reasoning as in Proposition III.2 yields the desired result. □

**Lemma III.4.** Let $n > 5$ and $\sigma \in \mathbb{S}_n$ then

$$|B(\sigma, 3)| = 1+(n-1)^2 + \left(\frac{n(n-3)}{2}\right)^2 + \left(\frac{(n-1)(n-2)}{2}\right)^2$$

$$+ \left(\frac{(n)(n-1)(n-5)}{6}\right)^2 + \left(\frac{(n)(n-2)(n-4)}{6}\right)^2.$$  

**Proof.** The proof is essentially the same as the proof for Proposition III.3. In this case $|\pi \in \mathbb{S}_n : \ell(\pi) = n-3\}$ can be calculated by considering the partitions $\lambda^{(1)} := (n-3, 3, 1)$, $\lambda^{(2)} := (n-3, 2, 1, 2)$, and $\lambda^{(3)} := (n-3, 1, 1, 1, 1)$, the only Young diagrams having $n-3$ columns. These Young diagrams are pictured below.

Applying the hook length formula to $\lambda^{(1)}$, $\lambda^{(2)}$, and $\lambda^{(3)}$ and adding the value from Proposition III.3 yields the result. □
IV. NONEXISTENCE OF NONTRIVIAL PERFECT ULAM PERMUTATION CODES

The previous section demonstrated how to calculate permutation Ulam ball sizes. In this section, again focusing on permutation codes, we utilize ball size calculations to prove the following theorem, establishing a theoretical limit on the maximum size of Ulam permutation codes. The proof of the theorem can be found at the end of the current section.

Theorem IV.1. There do not exist any nontrivial perfect permutation codes in the Ulam metric.

In 2013, Farnoud et al. [6] proved the following upper bound on the size of an Ulam permutation code $C \subseteq S_n$ with minimum Ulam distance $d$ (i.e. $C$ is an MPC$_d(n,1,d)$).

$$|C| \leq (n-d+1)! \quad (3)$$

Hence one strategy to prove the non-existence of perfect permutation codes is to show that the size of a perfect code must necessarily be larger than the upper-bound given above. Note that equation (3) makes sense whenever $d \leq n-1$. We can assume this, since the maximum always true since the maximum Ulam distance between any two permutations in $S_n$ is $n-1$, achieved when permutations are in reverse order of each other (e.g., $d_5(e, [n,n-1,...,1]) = n-1$).

Lemma IV.1. There do not exist any (nontrivial) single-error correcting perfect permutation codes.

Proof. Assume that $C \subseteq S_n$ is a perfect single-error correcting permutation code. Recall that $C$ is a trivial code if either $C = S_n$ or $|C| = 1$. If $n \leq 2$, then for all $\pi \in S_n$, we have $\pi \in B(\sigma, 1)$, which implies that $C$ is a trivial code. Thus we may assume that $n > 2$.

We proceed by contradiction. Since $C$ is a perfect single-error correcting permutation code, $C$ is an MPC$_d(n,1,d)$ with $3 \leq d \leq n-1$ and $|C| = n! / |B(\sigma, 1)| = n! / (1 + (n-1)^2)$ by Proposition III.2. However, inequality (3) implies that the code size $|C| \leq (n-2)!$. Hence, it suffices to show that $|C| = n! / (1 + (n-1)^2) > (n-2)!$, which is true whenever $n > 2$.

Similar arguments may also be applied to show that no nontrivial perfect $t$-error correcting codes exist for $t \in [2,3]$. This is the subject of the next two lemmas. The remaining cases, when $t > 3$, are treated toward the end of this section.

Lemma IV.2. There do not exist any (nontrivial) perfect 2-error correcting permutation codes.

Proof. Assume that $C$ is a perfect 2-error correcting permutation code. Similarly to the proof of Lemma IV.1, if $n \leq 3$, then $C$ is a trivial code consisting of a single element, so we may assume $n > 3$. Again we proceed by contradiction.

Since $C \subseteq S_n$ is a perfect 2-error correcting code, then $C$ is an MPC$_d(n,1,d)$ code with $5 \leq d \leq n-1$ and Proposition III.3 implies

$$|C| = \frac{n!}{|B(\sigma, 2)|} = \frac{n!}{1 + (n-1)^2 + \left(\frac{(n-3)}{2}\right)^2 + \left(\frac{(n-1)(n-2)}{2}\right)^2}.$$

By Inequality (3), $|C| \leq (n-4)!$, so it suffices to prove that

$$\frac{n!}{1 + (n-1)^2 + \left(\frac{(n-3)}{2}\right)^2 + \left(\frac{(n-1)(n-2)}{2}\right)^2} - (n-4)! > 0,$$

which is easily shown by elementary methods to be true for $n > 3$.

For small values of $t$, explicit ball calculations work well for showing the non-existence of nontrivial perfect $t$-error correcting codes. However, for each radius $t$, the size of the ball $B(e,t)$ is equal to $|B(e,t)| + \#\{\pi \in S_n : \ell(\pi) = n-t\}$. This means each ball size calculation of radius $t$ requires calculation of ball sizes for radii from 0 through $t-1$. Hence such explicit calculations are impractical for large values of $t$. For values of $t > 3$, another method can be used to show that nontrivial perfect codes do not exist. We begin by showing that if $n \leq 2t+1$, then it is impossible for a nontrivial perfect $t$-error correcting permutation code to exist.

Lemma IV.3. If $t \in \mathbb{Z}_{> 0}$ such that $n \leq 2t+1$, then it is impossible for a nontrivial perfect $t$-error correcting permutation code to exist.

Proof. Consider two permutations within $S_n$ of maximal Ulam distance apart. The most obvious example of which would be the identity element $e$ and the only-decreasing permutation $\omega_* := [n,n-1,...,1]$. Notice that $B(e,t) = \{\pi \in S_n : \ell(\pi) \geq n-t\}$, which means that every permutation whose longest increasing subsequence is at least $n-t$ is in the ball centered at $e$. Meanwhile, there is at least one permutation $\sigma \in S_n$ such that $\ell(\sigma) = 1 + t$ and $\sigma \in B(\omega_*, t)$, since we may apply successive translocations to $\omega_*$ in such a way that the longest increasing subsequence is increased with each translocation. As long as $n \leq 2t+1$, then $n-t \leq t+1 = 1 + t$, implying that $\ell(\sigma) = 1 + t \geq n-t$, which implies that $\sigma \in B(e,t) \cap B(\omega_*, t)$. Therefore the only perfect code possible when $n \leq 2t+1$ is a single element code, i.e. a trivial code.

The next lemma provides a sufficient condition to conclude that perfect codes do not exist. We will use this condition to complete the proof of Theorem IV.1 in the proof of the lemma below, the notation $\binom{n}{i}$ denotes the usual binomial coefficient.

Lemma IV.4. Let $t$ be a fixed non-negative integer. If the following inequality holds, then no nontrivial perfect $t$-error correcting permutation codes exist in $S_n$:

$$F(n,t) := \frac{(n-t)!^2 t!}{n!} > 1. \quad (4)$$

We call the above inequality the overlapping condition.
Proof. By Lemma IV.3, we may assume that $t$ is a nonnegative integer such that $n \geq 2t + 2$. We proceed by contrapositive. Suppose $C \subseteq S_n$ is a nontrivial perfect $t$-error correcting permutation code. We want to show that $F(n, t) \leq 1$. Since $C$ is a perfect code, we know it is also an MPC$_c(n, 1, d)$ code with $2t + 1 \leq d$ and by inequality (3), $|C| \leq (n - 2t)!$. At the same time, for any $\sigma \in S_n$, we have $|B(\sigma, t)| = |B(e, t)|$, which is less than or equal to $\left(\frac{n}{t}\right)!/(n-t)!$, since any permutation $\pi \in B(e, t)$ can be obtained by first choosing $n-t$ elements of $[n]$ to be in increasing order in $\pi$, and then arranging the remaining $t$ elements into $\pi$. Of course this method will generally result in double counting some permutations in $B(e, t)$, hence the inequality. Now

$$|B(\sigma, t)| \leq \left(\frac{n}{t}\right)!/(n-t)!$$

implies that

$$\frac{(n-t)!}{\left(\frac{n}{t}\right)!} \leq \frac{n!}{|B(\sigma, t)|} = |C| \leq (n-2t)!.$$ 

Moreover, $(n-t)!/(\left(\frac{n}{t}\right)!)$ is less than or equal to $(n-2t)!$ if and only if $F(n, t) \leq 1$.

Notice that the overlapping condition is never satisfied for $t = 1$. However, the following proposition will imply that as long as $t > 1$, then the overlapping condition may be satisfied for sufficiently large $n$.

**Proposition IV.5.** Let $t$ be a nonnegative integer such that $n \geq 2t + 2$. Then $\lim_{n \to \infty} F(n, t) = t!$.

**Proof.** Assume $t$ is a nonnegative integer such that $n \geq 2t + 2$. Then

$$\lim_{n \to \infty} F(n, t) = \lim_{n \to \infty} \frac{(n-t)(n-t-1)\cdots(n-2t+1)(n-2t)!}{(n)(n-1)\cdots(n-t+1)(n-t)!}$$

$$= \lim_{n \to \infty} \frac{(n-t)(n-t-1)\cdots(n-2t+1)}{n^t} = t!$$

The proposition above means that for any nonnegative integer $t$ less than or equal to $n/2 - 1$, there is some value $k$ such that for all values of $n$ larger than $k$, there does not exist a perfect $t$-error correcting code. The question remains: how large must the value of $k$ be before it is guaranteed that perfect $t$-error correcting codes do not exist?

Table V compares positive integer values $t$ versus $\min(n \in \mathbb{Z}_{>0}: F(n, t) > 1)$. Values were determined via numerical computer calculation. The table suggests that for $t > 2$, the minimum value of $n$ satisfying the overlapping condition is $n = 2t + 2$. If what the table suggests is true, then in view of Proposition IV.5, we may rule out perfect $t$-correcting codes for any $t > 2$. The next lemma formalizes what is implied in the table by providing parameters for which the overlapping condition is always satisfied. In combination with Lemma IV.4, the implication is that nontrivial perfect permutation codes do not exist for these parameters.

**Lemma IV.6.** Let $t$ be an integer greater than 2. Then $n \geq 2t + 2$ implies that the overlapping condition is satisfied.

**Proof.** Assume $t$ is an integer greater than 2 and that $n \geq 2t + 2$. We begin the proof of the lemma by showing that if $n = 2t + 2$, then the desired inequality holds. We assume that $n = 2t + 2$ and proceed by induction on $t$.

For the base case, let $t = 3$. Then $n = 8$, and $F(n, t) = ((5!)^23!)/((8!)^2) \approx 1.07 > 1$. As the induction hypothesis, suppose it is true that $F(2t + 2, t) = ((t + 2)!^2t!)/((2t + 2)! > 1$. We wish to show that the following inequality holds:

$$F(2t + 4, t + 1) = \frac{(t + 3)!^2(t + 1)!}{(2t + 4)!} > 1.$$ 

Here

$$\frac{(t + 3)!^2(t + 1)!}{(2t + 4)!} = \frac{(t + 2)!^2t!}{2(2t + 2)!} \cdot \frac{(t + 3)^2(t + 1)}{(2t + 3)(2t + 4)}.$$ 

By our induction hypothesis, the first term of the right hand side, $((t + 2)!^2t!)/(2(2t + 2)!)$, is greater than 1, so it suffices to show that $((t + 3)^2(t + 1))/(2(t + 3)(2t + 4)) \geq 1$. Note here that

$$\frac{(t + 3)^2(t + 1)}{(2t + 3)(2t + 4)} \geq \frac{(t + 3)^2(t + 1)}{(2t + 4)(2t + 4)} \geq \frac{(t + 3)^2(t + 1)}{4(t + 2)^2} > \frac{t + 1}{4},$$

which is greater than or equal to 1 whenever $t > 2$. Of course $t > 2$ by assumption, so the desired conclusion follows.

Thus far we have technically only proven that $F(n, t) > 1$ whenever $n = 2t + 2$. However, it is a simple matter to show that the same is true whenever $n > 2t + 2$ as well. We begin by supposing that $F(n, t) > 1$. Then

$$F(n + 1, t) = \frac{(n + 1 - t)!^2 t!}{(n + 1)(n + 1 - 2t)!}$$

$$= F(n, t) \frac{(n + 1 - t)^2}{(n + 1)(n + 1 - 2t)}$$

$$= \frac{(n + 1 - t)^2}{(n + 1 - t + 1)^2} > 1$$

whenever $n \geq 2t + 2$.

Consolidating all previous results, we are now able to prove Theorem IV.1.

**Proof of Theorem IV.1:** First, by Lemmas IV.1 and IV.2, there do not exist any nontrivial perfect $t$-error correcting
permutation codes for $t \in \{1, 2\}$. By Lemmas IV.4 and IV.6, there are no nontrivial perfect $t$-error correcting permutation codes for $t > 2$.

V. MULTIPERMUTATION ULAM BALL SIZE AND DUPLICATION SETS

Thus far we have focused primarily on permutations, but we wish to extend the discussion to multipermutations. With both permutations and multipermutations, the number of possible messages is limited by the number of distinguishable relative rankings in the physical scheme. However, multipermutations may significantly increase the total possible messages compared to ordinary permutations, as observed in [7]. For example, if only $k$ different charge levels are utilized at a given time, then permutations of length $k$ can be stored. Hence, in $r$ blocks of length $k$, one may store $(k!)^r$ potential messages.

On the other hand, if one uses $r$-regular multipermutations in the same set of blocks, then $(kr)!/(r!)^k$ potential messages are possible.

The $r$-regular multipermutation Ulam ball sizes play an important role in understanding the potential code size for $\text{MPC}_c(n, r, d')$'s. For example, the well-known sphere-packings bounds and Gilbert-Varshamov type bounds rely on calculating, or at least bounding ball sizes. In this section we analyze how to calculate $r$-regular multipermutation Ulam ball sizes. Recall that a partial answer to this question was given in Theorem III.1, but the theorem was applicable to the special case when $m_r'$ was chosen as the center. The next theorem provides a way to calculate radius 1 balls for any center using the concept of duplication sets. Notation used in the theorem is defined subsequently and the proof is given toward the end of the section.

**Theorem V.1.** Recall that $n, r \in \mathbb{Z}_{\geq 0}$ and $r|n$. Let $m_r' \in \mathcal{M}_r(S_n)$. Then

$$|B(m_r', 1)| = 1 + (n-1)^2 - |SD(m_r')| - |AD(m_r')|.$$ 

In the permutation case, the Ulam metric is known to be left-invariant, i.e. given $\sigma, \pi, \tau \in S_n$, we have $d_\ell(\sigma, \pi) = d_\ell(\tau \sigma, \tau \pi)$ [6]. Left-invariance implies that permutation ball sizes do not depend on the choice of center. Unfortunately, it is easily confirmed by counterexample that left invariance does not generally hold for the $r$-regular Ulam metric. Moreover, it is also easily confirmed that in the multipermutation Ulam ball case, the choice of center has an impact on the size of the ball, even when the radius remains unchanged (e.g. compare Proposition III.2 to Proposition VI.3 in the next section). Hence we wish to consider balls with various center multipermutations.

To aid with calculating such ball sizes, we first find it convenient to introduce the following subset of the set of translocations.

**Definition** $(T_n, \text{minimal set of translocations})$. Define $T_n := \{\phi(i, j) \in S_n : i-j \neq 1\}$.

We call $T_n$ the **minimal set of translocations**.

In words, $T_n$ is the set of all translocations, except translocations of the form $\phi(i, i-1)$. We exclude translocations of this form because they can be modeled by translocations of the form $\phi(i, 1)$, and are therefore redundant. Note that the identity permutation, $\phi(1, 1) = \phi(2, 2) = \cdots = \phi(n, n) = e$, is included in $T_n$. We claim that the set $T_n$ is precisely the set of translocations needed to obtain all unique permutations within the Ulam ball of radius 1 via multiplication (right action). Moreover, there is no redundancy in the set, meaning no smaller set of translocations yields the entire Ulam ball of radius 1 when multiplied with a given center permutation. These facts are stated in the next lemma.

**Lemma V.1.** Let $\sigma \in S_n$. Then $B(\sigma, 1) = \{\phi \in S_n : \phi \in T_n\}$, and $|T_n| = |B(\sigma, 1)|$.

**Proof.** Let $\sigma \in S_n$. We will first show that $B(\sigma, 1) = \{\phi \in S_n : \phi \in T_n\}$. Note that

$$B(\sigma, 1) = \{\pi \in S_n : d_\ell(\sigma, \pi) \leq 1\}$$

$$= \{\sigma \phi(i, j) \in S_n : i, j \in [n]\}.$$ 

It is trivial that

$$T_n = \{\phi(i, j) \in S_n : i-j \neq 1\} \subseteq \{\phi(i, j) \in S_n : i, j \in [n]\}.$$ 

Therefore $\{\sigma \phi \in S_n : \phi \in T_n\} \subseteq B(\sigma, 1)$.

To see why $B(\sigma, 1) \subseteq \{\phi \in S_n : \phi \in T_n\}$, consider any $\sigma \phi(i, j) \in \{\phi \in S_n : i, j \in [n]\} = B(\sigma, 1)$. If $i-j \neq 1$, then $\phi(i, j) \in T_n$, and thus $\sigma \phi(i, j) \in \{\phi \in S_n : \phi \in T_n\}$. Otherwise, if $i-j = 1$, then $\sigma \phi(i, j) = \sigma \phi(j, i)$, and $i-j = 1$ implies $j-i = -1 \neq 1$, so $\phi(j, i) \in T_n$. Hence $\sigma \phi(i, j) = \sigma \phi(j, i) \in \{\phi \in S_n : \phi \in T_n\}$.

Next we show that $|T_n| = |B(\sigma, 1)|$. By Proposition III.2, $|B(\sigma, 1)| = 1+(n-1)^2$. On the other hand, $|T_n| = \#\{\phi(i, j) \in S_n : i-j \neq 1\}$. If $i = 1$, then there are $n$ values $j \in [n]$ such that $i-j \neq 1$. Otherwise, if $i \in [n]$ but $i \neq 1$, then there are $n-1$ values $j \in [n]$ such that $i-j \neq 1$. However, for all $i, j \in [n]$, $\phi(i, i) = \phi(j, j) = e$ so that there are $n-1$ redundancies. Therefore $|T_n| = n + (n-1)(n-1) - (n-1) = 1 + (n-1)^2$.

Although the Ulam ball centered at $\sigma \in S_n$ of radius 1 can be characterized by all permutations obtainable by applying (multiplying on the right) a translocation to $\sigma$, the previous lemma shows that some translocations are redundant. That is, there are translocations $\phi_1 \neq \phi_2$ such that $\sigma \phi_1 = \sigma \phi_2$. In the case of permutations, the set $T_n$ has no such redundancies. If $\phi_1, \phi_2 \in T_n$, then $\sigma \phi_1 = \sigma \phi_2$ implies $\phi_1 = \phi_2$. However, in the case of multipermutations, the set $T_n$ can generally be shrunken further to exclude redundancies.

Given $m_r' \in \mathcal{M}_r(S_n)$, the ball $B(m_r', 1) = \{m_r' : \phi \in \mathcal{M}_r(S_n) :$ there exists a translocation $\phi$ such that $m_r' \cdot \phi = m_r'\} = \{m_r' : \phi \in \mathcal{M}_r(S_n) : \phi \in T_n\}$. However, it is possible that there exist $\phi_1, \phi_2 \in T_n$ such that $\phi_1 \neq \phi_2$, but $m_r' \cdot \phi_1 = m_r' \cdot \phi_2$. In such an instance we may refer to either $\phi_1$ or $\phi_2$ as a **duplicate translocation** for $m_r'$. If we remove all duplicate translocations for $m_r'$ from $T_n$, then the resulting set will have the same cardinality as the $r$-regular Ulam ball of radius 1 centered at $m_r'$. The next definition is a standard set of duplicate translocations. It is called standard because as long as $r \neq 1$ it always exists and is of predictable size.
Definition \((SD(m), \text{standard duplication set})\). Given a tuple \(m \in \mathbb{Z}^n\), define
\[SD(m) := \{\phi(i, j) \in T_n \setminus \{e\} : m(i) = m(j) \text{ or } m(i) = m(i - 1)\}\]
We call \(SD(m)\) the \textbf{standard duplication set} for \(m\).

The intuition for the definition is that it is sufficient to delete the leftmost number in a substring consisting of equal numbers, and it is sufficient to insert a number next to a substring consisting of numbers equal to the inserted number without moving the substring, i.e., on the closer side next to the substring. Translocations other than those will belong to the standard duplication set. If we take an \(r\)-regular multipermutation \(m^r\), then removing the general set of duplications from \(T_n\) equates to removing a set of duplicate translocations. These duplications come in two varieties. The first variety corresponds to the first condition of the \(SD(m)\) definition, when \(m(i) = m(j)\). For example, if \(m^2_\sigma(1, 3, 2, 2, 3, 1)\), then we have \(m^2_\sigma \cdot \phi(1, 5) = (3, 2, 2, 3, 1, 1) = m^2_\sigma \cdot \phi(1, 6)\), since \(m^2_\sigma(2) = 3 = m^2_\sigma(4)\). This is because moving the first 1 to the left or to the right of the last 1 results in the same tuple. The second variety corresponds to the second condition of the \(SD(m)\) definition above, when \(m(i) = m(i - 1)\). For example, if \(m^2_\sigma = (1, 3, 2, 2, 3, 1)\) as before, then for all \(j \in [6]\), we have \(m^2_\sigma \cdot \phi(3, 3, j) = m^2_\sigma \cdot \phi(4, j)\). This is because any translocation that deletes and inserts the second of the two adjacent 2’s does not result in a different tuple when compared to deleting and inserting the first of the adjacent 2’s.

**Lemma V.2.** Let \(m^r_\sigma \in M_r(S_n)\). Then \(B(m^r_\sigma, 1) = \{m^r_\sigma \cdot \phi \in M_r(S_n) \mid \phi \in T_n \setminus SD(m^r_\sigma)\}\).

**Proof.** The idea of the proof is to show that for every translocation that is in the standard duplication set, there is a translocation outside of the standard duplication set but in the minimal set of translocations that when applied has the same effect. It is helpful to keep in mind the example in the paragraph preceding the statement of the lemma for intuition.

Assume \(m^r_\sigma \in M_r(S_n)\). First note that \(B(m^r_\sigma, 1) = \{m^r_\sigma \cdot \phi \in M_r(S_n) : \phi \in T_n\}\). Hence it suffices to show that for all \(\phi(i, j) \in SD(m^r_\sigma)\), there exists some \(i', j' \in [n]\) such that \(\phi(i', j') \in T_n \setminus SD(m^r_\sigma)\) and \(m^r_\sigma \cdot \phi(i, j) = m^r_\sigma \cdot \phi(i', j')\). We proceed by dividing the proof into three main cases. Case I is when \(m^r_\sigma(i) = m^r_\sigma(i + 1) = \cdots = m^r_\sigma(j)\). Case II is when \(m^r_\sigma(i) \neq m^r_\sigma(i - 1)\) or \(i = 1\). Case III is when \(m^r_\sigma(i) = m^r_\sigma(i - 1)\).

Case I can be immediately solved since in this case \(m^r_\sigma \cdot \phi(i, j) = m^r_\sigma \cdot e\), and \(e \in T_n\) but \(e \not\in SD(m^r_\sigma)\).

Case II (when \(m^r_\sigma(i) \neq m^r_\sigma(i - 1)\) or \(i = 1\) can be split into two subcases:

- **Case IIA:** \(i < j\)
- **Case IIB:** \(i > j\)

We can ignore the instance when \(i = j\). This is because \(\phi(i, j) \in SD(m^r_\sigma)\) implies \(i \neq j\) since \(e \not\in SD(m^r_\sigma)\). For case IIA, if for all \(p \in [i, j]\) (recall \([i, j] = [i, i + 1, \ldots, j]\)) we have \(m^r_\sigma(i) = m^r_\sigma(p)\), then \(m^r_\sigma \cdot \phi(i, j) = m^r_\sigma \cdot e\). Thus setting \(i' = j' = 1\) yields the desired result. Otherwise, if there exists \(p \in [i, j]\) such that \(m^r_\sigma(i) \neq m^r_\sigma(p)\), then let \(j^* := j - \min\{k \in \mathbb{Z}_+ : m^r_\sigma(i) \neq m^r_\sigma(j - k)\}\). Then \(\phi(i, j^*) \in T_n \setminus SD(m^r_\sigma)\) and \(m^r_\sigma \cdot \phi(i, j) = m^r_\sigma \cdot \phi(i, j^*)\). Thus setting \(i' = i\) and \(j' = j^*\) yields the desired result. Case IIB is similar to Case IIA.

Case III (when \(m^r_\sigma(i) = m^r_\sigma(i - 1)\), can also be divided into two subcases.

- **Case IIIA:** \(i < j\)
- **Case IIIB:** \(i > j\)

As in Case II, we can ignore the instance when \(i = j\). For Case IIIA, we can essentially reduce it to Case II A. This is because there is some \(i^* < i\) for which \(i^* \neq i\), but \(i^* + 1 = i^* + 2 = \cdots = i\). Then \(\phi(i^*, j)\) has the same effect on \(m^r_\sigma\) as \(\phi(i, j)\), that is, \(m^r_\sigma \cdot \phi(i^*, j) = m^r_\sigma \cdot \phi(i, j)\). If \(\phi(i^*, j) \in T_n \setminus SD(m^r_\sigma)\) then we are finished. Otherwise, we are in Case II A above. Case IIIB is similar to Case IIIA.

While Lemma V.2 shows that \(SD(m^r_\sigma)\) is a set of duplicate translocations for \(m^r_\sigma\), we have not shown that \(T_n \setminus SD(m^r_\sigma)\) is the set of minimal size having the quality that \(B(m^r_\sigma, 1) = \{m^r_\sigma \cdot \phi \in M_r(S_n) : \phi \in T_n \setminus SD(m^r_\sigma)\}\). In fact it is not minimal in general. In some instances it is possible to remove further duplicate translocations to reduce the set size. We will define another set of duplicate translocations, but a few preliminary definitions are first necessary.

We say that \(m \in \mathbb{Z}^n\) is alternating if for all odd integers \(1 \leq i \leq n\), \(m(i) = m(1)\) and for all even integers \(2 \leq i' \leq n\), \(m(i') = m(2)\) but \(m(1) \neq m(2)\). In other words, any alternating tuple is of the form \((a, b, a, b, \ldots, a, b)\) or \((a, b, a, b, \ldots, a)\) where \(a, b \in \mathbb{Z}\) and \(a \neq b\). Any singleton is also said to be alternating. Now for integers \(1 \leq i \leq n\) and \(0 \leq k \leq n - i\), the substring \([m[i, i + k]] := [m(i), m(i + 1), \ldots, m(i + k)]\). Given a substring \([m[i, j]]\) of \(m\), the length of \([m[i, j]]\), denoted by \([m[i, j]]\), is defined as \([m[i, j]] := j - i + 1\). As an example, if \(m' := (1, 2, 2, 4, 2, 4, 3, 1, 3)\), then \(m'[3, 6] = (2, 4, 2, 4)\) is an alternating substring of \(m'\) of length 4.

**Definition (AD(m), alternating duplication set).** Given \(m \in \mathbb{Z}^n\), define
\[AD(m) := \{\phi(i, j) \in T_n \setminus SD(m) : i \leq j - 2 \text{ and there exists } k \in [i, j - 2] \text{ such that } (\phi(j, k) \in T_n \setminus SD(m)) \}\]
and \((m \cdot \phi(i, j) = m \cdot \phi(j, k))\).

We call \(AD(m)\) the alternating duplication set for \(m\) because it is only nonempty when \(m\) contains an alternating substring of length at least 4. For each \(i \in [n]\), also define
\[AD_i(m) := \{\phi(i, j) \in AD(m) : j \in [n]\}\]. Notice that
\[AD(m) = \bigcup_{i=1}^{n} AD_i(m)\].

In the example of \(m' := (1, 2, 2, 4, 2, 4, 3, 1, 3)\) above, \(m' \cdot \phi(2, 6) = m' \cdot \phi(2, 6)\) and \(\phi(2, 6), \phi(2, 6) \in T_n \setminus SD(m')\), implying that \(\phi(2, 6) \in AD(m')\). In fact, it can easily be shown that \(AD(m') = \{\phi(2, 6)\}\). In order to simplify the discussion of the alternating duplication set, we find the following lemma useful.
Lemma V.3. Let $m \in \mathbb{Z}^n$ and $i, j \in [n]$. Then $\phi(i, j) \in AD_1(m)$ if and only if

1) $m(i) \neq m(i - 1)$
2) There exists $k \in [i, j - 2]$ such that
   i) For all $p \in [i, k - 1]$, $m(p) = m(p + 1)$
   ii) $m[k, j]$ is alternating
   iii) $|m[k, j]|$ is at least 4 and even.

Proof. Let $m \in \mathbb{Z}^n$ and $i, j \in [n]$. We will first assume 1) and 2) in the lemma statement and show that $\phi(i, j) \in AD_1(m)$. Suppose $m(i) \neq m(i - 1)$, and that there exists $k \in [i, j - 2]$ such that for all $p \in [i, k - 1]$, we have $m(p) = m(p + 1)$. Suppose also that $m[k, j]$ is alternating with $|m[k, j]|$ greater than or equal to 4 and even.

For ease of notation, let $a := m(k) = m(k + 2) + 2$ and $b := m(k + 1) = m(k + 3)$ so that $m[k, j] = (a, b, a, b, \ldots, a, b)$. Then

$$\begin{align*}
(m \cdot \phi(i, j))[k, j] &= (m \cdot \phi(k, j))[k, j] \\
&= (a, b, a, b, \ldots, a, b) \\
&= (m \cdot \phi(j, k))[k, j].
\end{align*}$$

Moreover, for all $p \notin [k, j]$, we have $(m \cdot \phi(i, j))(p) = m(p) = (m \cdot \phi(k, j))(p)$. Therefore $m \cdot \phi(i, j) = m \cdot \phi(j, k)$. Also notice that $m(i) \neq m(i - 1)$ implies that $m \cdot \phi(i, j) \notin SD(m)$. Hence $\phi(i, j) \notin AD_1(m)$.

We now prove the second half of the lemma. That is, we assume that $\phi(i, j) \in AD_1(m)$ and then show that 1) and 2) necessarily hold. Let $\phi(i, j) \in AD_1(m)$. Then $m(i) \neq m(i - 1)$, since otherwise $\phi(i, j)$ would be in $SD(m)$, contradicting the fact that $\phi(i, j) \notin AD_1(m)$.

Let $k \in [i, j - 2]$ such that $\phi(k, j) \in T_n \setminus SD(m)$ and $m \cdot \phi(i, j) = m \cdot \phi(j, k)$. Existence of such $k$ and $\phi(i, j)$ is guaranteed by the definition of $AD_1(m)$ and the fact that $\phi(i, j) \in AD_1(m)$. Then for all $p \in [i, k - 1]$, we have $m(p) = m(p + 1)$ and for all $p \in [k, j - 2]$, we have $m(p) = m(p + 2)$. Hence either $m[k, j]$ is alternating, or else for all $p, q \in [k, j]$, we have $m(p) = m(q)$. However, the latter case is impossible, since it would imply that for all $p, q \in [i, j]$, $m(p) = m(q)$, which would mean $\phi(k, j) \notin T_n \setminus SD(m)$, a contradiction. Therefore $m[k, j]$ is alternating.

It remains only to show that $|m[k, j]| \geq 4$ and even. Since $k \in [i, j - 2]$, it must be the case that $|m[k, j]| \geq 4$. However, if $|m[k, j]|$ is odd then $(m \cdot \phi(i, j))(j) = m(i) = m(k) \neq m(k + 1) = (m \cdot \phi(j, k))(j)$, which implies that $m \cdot \phi(i, j) \neq m \cdot \phi(j, k)$, a contradiction. Hence $|m[k, j]|$ is greater than or equal to 4 and even.

One implication of Lemma V.3 is that there are only two possible forms for $m[i, j]$ where $\phi(i, j) \in AD_1(m)$. The first possibility is that $m[i, j]$ is an alternating substring of the form $(a, a, a, b, \ldots, a, b)$ (here $a, b \in \mathbb{Z}$), so that $m[i, j] \cdot \phi(i, j)$ is of the form $(b, a, b, a, \ldots, b, a)$. In this case, as long as $|m[i, j]| \geq 4$, then setting $k = i$ implies that $k \in [i, j - 2]$, that $\phi(j, k) \in T_n \setminus SD(m)$, and that $m[i, j] \cdot \phi(i, j) = m[i, j] \cdot \phi(j, k)$.

The other possibility is that $m[i, j]$ is of the form $(a, a, a, \ldots, a, b, a, b, \ldots, a, b)$ (again $a, b \in \mathbb{Z}$), so that $m[i, j] \cdot \phi(i, j)$ is of the form $(a, a, a, \ldots, a, b, a, b, \ldots, a, b)$. Again in this case, as long as $|m[i, j]| \geq 4$, then $k \in [i, j - 2]$ with $\phi(j, k) \in T_n \setminus SD(m)$ and $m[i, j] \cdot \phi(i, j) = m[i, j] \cdot \phi(j, k)$. To simplify the calculation of $|AD_1(m)|$, we wish to define a set of equal size that is easier to count.

Definition ($AD^*(m)$). Given $m \in \mathbb{Z}^n$, define

$$AD^*(m) := \{ (i, j) \in [n] \times [n], i < j : (m[i, j] \text{ is alternating}), (|m[i, j]| \geq 4), \text{ and } (m[i, j]) \text{ is even} \}. $$

For each $i \in [n]$, also define $AD^*_i(m) := \{ (i, j) \in AD^*(m) : j \in [n] \}$. Notice that $AD^*(m) = \bigcup_{i=1}^n AD^*_i(m)$.

Lemma V.4. Let $m \in \mathbb{Z}^n$. Then $|AD_1(m)| = |AD^*(m)|$.

Proof. Let $m \in \mathbb{Z}^n$. The idea of the proof is simple. Each element $\phi(i, j) \in AD_1(m)$ involves exactly one alternating sequence of length greater than or equal to 4, so the set sizes must be equal. We formalize the argument by showing that $|AD_1(m)| \leq |AD^*(m)|$ and then that $|AD^*(m)| \leq |AD_1(m)|$.

To see why $|AD_1(m)| \leq |AD^*(m)|$, we define a mapping $\map : [n] \rightarrow [n]$, which maps index values either to the beginning of the nearest alternating substring to the right, or else to $n$. Note that the map essentially maps the first position of a non-alternating substring to its last position. For all $i \in [n]$, let

$$\map(i) := \begin{cases} 
\map(i) := i + \min\{p \in \mathbb{Z}_{\geq 0} : (m(i) \neq m(i + p + 1)) \text{ or } (i + p = n) 
& \text{ if } m(i) \neq m(i - 1) \text{ or } i = 1 \\
& \text{ (otherwise)} 
\end{cases} $$

Notice by definition of $\map$, if $i, i' \in [n]$ such that $i \neq i'$, and if $m(i) \neq m(i - 1)$ or $i = 1$ and at the same time $m(i') \neq m(i' - 1)$ or $i' = 1$, then $\map(i) \neq \map(i')$.

Now for each $i \in [n]$, if $m(i) \neq m(i - 1)$ or $i = 1$, then $|AD_1(m)| = |AD^*_i(m)|$ by Lemma V.3. Otherwise, if $m(i) = m(i - 1)$, then $|AD_1(m)| = |AD^*_i(m)| = 0$. Therefore $|AD_1(m)| \leq |AD^*_i(m)|$. This is true for all $i \in [n]$, so $|AD_1(m)| \leq |AD^*_i(m)|$.

The argument to show that $|AD^*(m)| \leq |AD_1(m)|$ is similar, except it uses the following function $\map^* : [n] \rightarrow [n]$ instead of $\map$. Note that $\map^*$ is essentially the inverse of $\map$. For all $i \in [n]$, let

$$\map^*(i) := \begin{cases} 
\map^*(i) := i - \min\{p \in \mathbb{Z}_{\geq 0} : (m(i) \neq m(i - p - 1)) \text{ or } (i - p = 1) 
& \text{ if } m(i) \neq m(i - 1) \text{ or } i = n 
\end{cases} $$

By definition, calculating $|AD^*(m)|$ means counting the number of alternating substrings $m[i, j]$ of $m$ such that the
length of the substring is both even and at least 4. We can simplify the calculation of $|AD(m)|$ further by establishing a relation to the following quantity.

**Definition** ($\psi(n)$, $\psi(x)$). Define

$$\psi(n) := \left\lfloor \frac{(n-2)^2}{4} \right\rfloor,$$

and for $x \in \mathbb{Z}^+$, define

$$\psi(x) := \sum_{i=1}^{\lfloor x \rfloor} \psi(x(i)),$$

where $|x|$ denotes the length of the tuple $x$.

While we define both $\psi(n)$ and $\psi(x)$ here, we will not make use of $\psi(x)$ until the following section. The next lemma relates $\psi(n)$ to the calculation of $|AD(m)|$.

**Lemma V.5.** Let $m$ be an alternating string. Then

$$|AD(m)| = \psi(|m|).$$

**Proof.** Assume $m$ is an alternating string and let $|m| = n$.

By Lemma V.4, $|AD(m)| = |AD^*(m)| = |\bigcup_{i=1}^{n} AD^*_i(m)|$.

Since $m$ was assumed to be alternating,

$$|\bigcup_{i=1}^{n} AD^*_i(m)| = \#[(i, j) \in [n] \times [n], j-i \geq 3 : m[i, j] \in A \text{ and } |m[i, j]| \text{ is even}] = \#[(i, j) \in [n] \times [n] : j-i+1 \in A],$$

where $A$ is the set of even integers between 4 and $n$, i.e. $A := \{a \in [4, n] : a \text{ is even}\}$. For each $a \in A$, we have

$$\#[(i, j) \in [n] \times [n] : j-i+1 = a] = \#[i \in [n] : i \in [1, n-a+1]] = n-a+1.$$

Therefore $|AD(m)| = \sum_{a \in A} (n-a+1)$. In the case that $n$ is even, then

$$\sum_{a \in A} (n-a+1) = \sum_{i=2}^{n/2} (n-2i+1) = \left( \frac{n-2}{2} \right)^2 = \psi(n).$$

In the case that $n$ is odd, then

$$\sum_{a \in A} (n-a+1) = \sum_{i=2}^{(n-1)/2} (n-2i+1) = \left( \frac{n-3}{2} \right) \left( \frac{n-1}{2} \right) = \psi(n).$$

Notice that by Lemma V.5, it suffices to calculate $|AD(m)|$ for locally maximal length alternating substrings of $m$. An alternating substring $m[i, j]$ is of **locally maximal length** if and only if 1) $m[i-1, j]$ is not alternating or $i = 1$; and 2) $m[i, j+1]$ is not alternating or $j = n$.

Finally, we define the general set of duplications, $D(m)$. The lemma that follows the definition also shows that removing the set $D(m'_\sigma)$ from $T_n$ removes all duplicate translocations associated with $m'_\sigma$.

**Definition** ($D(m)$, duplication set). Given $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}^n$, define

$$D(m) := SD(m) \cup AD(m).$$

We call $D(m)$ the **duplication set** for $m$. For each $i \in [n]$, we also define $D_i(m) := \{\phi(i, j) \in D(m) : j \in [n]\}$.

**Lemma V.6.** Let $m_\sigma \in \mathcal{M}_\sigma(S_n)$ and $\phi_1, \phi_2 \in T_n \setminus D(m'_\sigma)$. Then $\phi_1 = \phi_2$ if and only if $m'_\sigma \cdot \phi_1 = m'_\sigma \cdot \phi_2$.

**Proof.** Assume $m_\sigma \in \mathcal{M}_\sigma(S_n)$ and $\phi_1, \phi_2 \in T_n \setminus D(m'_\sigma)$. If $\phi_1 = \phi_2$ then $m'_\sigma \cdot \phi_1 = m'_\sigma \cdot \phi_2$ trivially. It remains to prove that $m'_\sigma \cdot \phi_1 = m'_\sigma \cdot \phi_2$ implies $\phi_1 = \phi_2$. We proceed by contrapositive. Suppose that $\phi_1 \neq \phi_2$. We want to show that $m'_\sigma \cdot \phi_1 \neq m'_\sigma \cdot \phi_2$. Let $\phi_1 := \phi(i_1, j_1)$ and $\phi_2 := \phi(i_2, j_2)$.

We begin by solving the case when $\phi_1$ or $\phi_2$ is the identity permutation, $e$. Without loss of generality, let $\phi_1 = e$. Now if $m'_\sigma \cdot \phi_2 = m'_\sigma$, then it must be the case that for all $k$ between $i_2$ and $j_2$ that $m'_\sigma(k) = m'_\sigma(i_2) = m'_\sigma(j_2)$. But this means that $\phi_2 \in SD(m'_\sigma)$, a contradiction. Thus we may conclude that $m'_\sigma \cdot \phi_1 \neq m'_\sigma \cdot \phi_2$. Hence for the rest of the proof we assume that neither $\phi_1$ nor $\phi_2$ is equal to $e$.

The remainder of the proof can be split into two main cases: Case I is if $i_1 = i_2$ and Case II is if $i_1 \neq i_2$.

Case I (when $i_1 = i_2$), can be further divided into two subcases:

- **Case IA:** $m'_\sigma(i_1) = m'_\sigma(i_1 - 1)$
- **Case IB:** $m'_\sigma(i_1) \neq m'_\sigma(i_1 - 1)$

Case IA is easy to prove. We have $D_i(m'_\sigma) = D_i(e) = \{\phi(i_1, j) \in T_n \setminus \{e\} : j \in [n]\}$, so $\phi_1 = e = \phi_2$, a contradiction. For Case IB, we can first assume without loss of generality that $j_1 < j_2$ and then split into the following smaller subcases:

- i) $i_1 < j_1 < j_2$
- ii) $j_1 < i_1 < j_2$
- iii) $j_1 < j_2 < i_1$.

Each subcase is proven by noting that there is some element in the multipermutation $m'_\sigma \cdot \phi_1$ that is necessarily different from $m'_\sigma \cdot \phi_2$. For example, in subcase i), we have $m'_\sigma \cdot \phi_1(j_2) = m'_\sigma(i_1) \neq m'_\sigma(i_1) = m'_\sigma(i_2) = m'_\sigma \cdot \phi_2(j_2)$. Subcases ii) and iii) are solved similarly.

Case II (when $i_1 \neq i_2$) can be divided into three subcases:

- **Case IIA:** $m'_\sigma(i_1) = m'_\sigma(i_1 - 1)$ and $m'_\sigma(i_2) = m'_\sigma(i_2 - 1)$
- **Case IIB:** either $m'_\sigma(i_1) = m'_\sigma(i_1 - 1)$ and $m'_\sigma(i_2) \neq m'_\sigma(i_2 - 1)$ or $m'_\sigma(i_1) \neq m'_\sigma(i_1 - 1)$ and $m'_\sigma(i_2) = m'_\sigma(i_2 - 1)$
- **Case IIC:** $m'_\sigma(i_1) \neq m'_\sigma(i_1 - 1)$ and $m'_\sigma(i_2) \neq m'_\sigma(i_2 - 1)$

Case IIA is easily solved by mimicking the proof of Case IA. Case IIB is also easily solved as follows. First, without loss of generality, we assume that $m'_\sigma(i_1) = m'_\sigma(i_1 - 1)$
and \( m'_\sigma(i_2) \neq m'_\sigma(i_2 - 1) \). Then \( D_{\phi}(m'_\sigma) = \{ \phi(i_1, j) \in T_n \setminus \{ e \} : j \in [n] \} \), so \( \phi_1 = e \), a contradiction.

Finally, for Case IIC, without loss of generality we may assume that \( i_1 < i_2 \) and then split into the following subcases:

- i) \( i_2 \leq j_1, j_2 \)
- ii) \( j_1 \leq i_2 < j_1 \)
- iii) \( j_2 < i_2 \leq j_1 \)
- iv) \( j_1, j_2 \leq i_2 \)

As in Case IB, subcases i), ii), and iv) are proven by noting some element in \( m'_\sigma \setminus \phi_1 \) that differs from \( m'_\sigma \setminus \phi_2 \). For instance, in subcase i), notice that \( m'_\sigma \setminus \phi_1(i_2 - 1) = m'_\sigma(i_2) \neq m'_\sigma(i_2 - 1) = m'_\sigma \setminus \phi_2(i_2 - 1) \). Subcases ii) and iv) are solved in a similar manner.

For subcase iii), if \( i_2 < j_1 \), then \( m'_\sigma \setminus \phi_1(j_1) = m'_\sigma(i_1) \neq m'_\sigma(j_1) \neq m'_\sigma \setminus \phi_2(j_2) \). Otherwise, if \( i_2 = j_1 \), then \( \phi_1 = \phi(i_1, i_2) \) and \( \phi_1 = \phi(i_1, j_2) \). Thus if \( m'_\sigma \setminus \phi_1 = m'_\sigma \setminus \phi_2 \) then \( \phi_1 \in AD_{\phi}(m'_\sigma) \), which implies that \( \phi_1 \notin T_n \setminus D(m'_\sigma) \), a contradiction.

Lemma V.6 implies that we can calculate \( r \)-regular Ulam ball sizes of radius 1 whenever we can calculate the appropriate duplication set. This calculation can be simplified by noting that for a sequence \( m \in \mathbb{Z}^n \) that \( SD(m) \cap AD(m) = \emptyset \) (by the definition of \( AD(m) \)) and then decomposing the duplication set into these components. This idea is stated in Theorem V.1 at the beginning of this section, which like Theorem III.1, is a partial answer to the question:

We now have the machinery to prove Theorem V.1.

**Proof of Theorem V.1**

Let \( m'_\sigma \in \mathcal{M}_r(S_n) \). By the definition of \( D(m'_\sigma) \) and lemma V.2,

\[
\{ m'_\sigma \setminus \phi : \phi \in \mathcal{M}_r(S_n) : \phi \in T_n \setminus D(m'_\sigma) \} = \{ m'_\sigma \setminus \phi : \phi \in \mathcal{M}_r(S_n) : \phi \in T_n \setminus D(m'_\sigma) \} = \{ m'_\sigma \}.
\]

This implies \( |T_n \setminus D(m'_\sigma)| \geq |B(m'_\sigma, 1)| \). By lemma V.6, for \( \phi_1, \phi_2 \in T_n \setminus D(m'_\sigma) \), if \( \phi_1 \neq \phi_2 \), then \( m'_\sigma \setminus \phi_1 \neq m'_\sigma \setminus \phi_2 \). Hence we have \( |T_n \setminus D(m'_\sigma)| \leq |B(m'_\sigma, 1)| \), which implies that \( |T_n \setminus D(m'_\sigma)| = |B(m'_\sigma, 1)| \). It remains to show that \( |T_n \setminus D(m'_\sigma)| = 1 + (n-1)^2 - |SD(m'_\sigma)| - |AD(m'_\sigma)| \). This is an immediate consequence of the fact that \( |T_n| = 1 + (n-1)^2 \) and \( SD(m'_\sigma) \cap AD(m'_\sigma) = \emptyset \).

Theorem V.1 reduces the calculation of \( |B(m'_\sigma, 1)| \) to calculating \( |SD(m'_\sigma)| \) and \( |AD(m'_\sigma)| \). The next lemma states how to calculate \( |SD(m'_\sigma)| \).

**Lemma V.7.** Let \( m \in \mathcal{M}_r(S_n) \). Then

\[
|SD(m'_\sigma)| = (n-2)\#i \in [n] : m'_\sigma(i) = m'_\sigma(i-1) + (r-1)\#i \in [n] : m'_\sigma(i) \neq m'_\sigma(i-1) \text{ or } i=1.
\]

**Proof.** The lemma follows from two facts. First, if \( m'_\sigma(i) = m'_\sigma \), then \( \phi(i, j) \in SD(m'_\sigma) \) for every \( j \in [n] \) as long as \( j \neq i \) and \( i - j \neq 1 \). Second, if \( m'_\sigma(i) \neq m'_\sigma(i-1) \) or \( i = 1 \), then \( \phi(i, j) \in SD(m'_\sigma) \) for every \( j \in [n] \) such that \( m'_\sigma(i) = m'_\sigma(j) \) and \( i \neq j \), of which there are \( r-1 \) instances.

We showed how to calculate \( |AD(m'_\sigma)| \) previously. The next example is an application of Theorem V.1.

**Example V.8.** Suppose \( m'_\sigma = (1, 1, 1, 2, 3, 2, 3, 2, 4, 4, 3, 4) \). There are 3 values of \( i \in [12] \) such that \( m'_\sigma(i) = m'_\sigma(i-1) \), which implies by Lemma V.7 that \( |SD(m'_\sigma)| = (3)(12-2) + (12-3)(3-1) = 48 \). Meanwhile, by Lemmas V.4 and V.5, \( |AD(m'_\sigma)| = ((3-3)/(3-2)) = 2 \). By Theorem V.1, \( |B(m'_\sigma)| = 1 + (12-1)^2 - 48 - 2 = 72 \).

**VI. MIN/MAX BALLS AND CODE SIZE BOUNDS**

In this section we show choices of center achieving minimum and maximum \( r \)-regular Ulam ball sizes for the radius \( t = 1 \) case. As an application, we also state new upper and lower bounds on maximal code size in Lemmas VI.2, VI.4, and VI.5 (Lemmas VI.15 and VI.16 may also be included in this list, which are bounds in the special case when \( n/r = 2 \)). These bounds represent the final main contribution of this paper.

The binary case, when \( n/r = 2 \), presents unique challenges because of the nature of its alternating duplication sets. In particular, the choice of center multipermutation yielding the maximal ball size in the non-binary cases does not yield the maximal size in the binary case. Thus we divide this section into two parts -- the first subsection treating the non-binary case, and the remaining two subsections treating the binary case.

**A. Non-Binary Case**

We begin by discussing the non-binary case in this subsection. The non-binary case is the general case where \( n/r \neq 2 \). Tight minimum and maximum values of ball sizes are explicitly given. We then discuss resulting bounds on code size. First let us consider the \( r \)-regular Ulam ball of minimal size. The first two lemmas presented in this section apply to all cases, both non-binary and binary, while the remaining results only apply when \( n/r \neq 2 \).

**Lemma VI.1.** Recall that \( n, r \in \mathbb{Z}_{>0} \) and \( r \mid n \). Let \( m'_\sigma \in \mathcal{M}_r(S_n) \). Then

\[
|B(m'_\sigma, 1)| \leq |B(m'_\sigma, 1)|.
\]

**Proof.** First note that in the special case when \( r = 1 \), then \( |B(m'_\sigma, 1)| = |B(e, 1)| = |B(\sigma, 1)| = |B(m'_\sigma, 1)| \), so we may assume that \( r \geq 2 \). Let \( m'_\sigma \in \mathcal{M}_r(S_n) \). In the case that \( n/r = 1 \), then \( m'_\sigma = (1, 1, \ldots, 1) = m'_\sigma \), so that \( |B(m'_\sigma, 1)| = |B(m'_\sigma, 1)| \). Therefore we may assume that \( n/r \geq 2 \). By Theorem VI.1, \( \min_{\sigma \in S_n} |B(m'_\sigma, 1)| = 1 + (n-1)^2 - \max_{\sigma \in S_n} (|SD(m'_\sigma)| + |AD(m'_\sigma)|) \). Since \( n/r \geq 2 \) and \( r \geq 2 \), we know that \( n-2 > r-1 \), which implies for all \( \sigma \in S_n \), that \( |SD(m'_\sigma)| \) is maximized by maximizing the number of integers \( i \in [n] \) such that \( m'_\sigma(i) = m'_\sigma(i-1) \). This is accomplished by choosing \( \sigma = e \), and hence for all \( \sigma \in S_n \), we have \( |SD(m'_\sigma)| \geq |SD(m'_\sigma)| \).

We next will show that for any increase in the size of \( |AD(m'_\sigma)| \) compared to \( |AD(m'_\sigma)| \), that \( |SD(m'_\sigma)| \) is decreased by a larger value compared to \( |SD(m'_\sigma)| \), so that
Let $\omega \in \Sigma_n$ be defined as follows:

$$w(i) := ((i - 1) \mod (n/r)) r + [ir/n] \quad (5)$$

and $\omega := [\omega(1), \omega(2), \ldots, \omega(n)]$. With this definition, for all $i \in [n]$, we have $\omega(i) = i \mod (n/r)$. For example, if $r = 3$ and $n = 12$, then $\omega = [1, 4, 7, 10, 2, 5, 8, 11, 3, 6, 9, 12]$ and $m_\omega = (1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4)$. We can use Theorem V.1 to calculate $|B(m_\omega, 1)|$, and then show that this is the maximal $r$-regular Ulam ball size (except for the case when $n/r = 2$).

**Lemma VI.3.** Suppose $n/r \neq 1$. Then

$$|B(m_\omega, 1)| \leq |B(m_\omega', 1)| = 1 + (n - 1)^2 - (r - 1)n.$$  

**Proof.** Assume $n/r \neq 1$. First notice that if $n/r = 1$ then for any $\sigma \in \Sigma_n$, the ball $B(m_\sigma, 1)$ contains exactly one element (the tuple of the form $(1, 1, \ldots, 1)$). Hence the lemma holds trivially in this instance. Next, assume that $n/r > 2$. We will first prove that $|B(m_\omega, 1)| = 1 + (n - 1)^2 - (r - 1)n$.

Since $n/r > 2$, it is clear that $m_\omega$ contains no alternating subsequences of length greater than 2. Thus by Lemma V.3, $|B(m_\omega, 1)| = 1 + (n - 1)^2 - |SD(m_\omega)|$. Since there does not exist $i \in [n]$ such that $m_\omega(i) = m_\omega'(i - 1)$, we have $|SD(m_\omega)| = (r - 1)n$, completing the proof of the first statement in the lemma.

We now prove that $|B(m_\omega, 1)| \leq |B(m_\omega', 1)|$. Recall that $|SD(m_\omega)|$ is equal to $(n - 2)$ times the number of $i \in [n]$ such that $m_\omega(i) = m_\omega'(i - 1)$. This is because $r - 1 < n - 2$, which implies $|SD(m_\omega)| = (r - 1)n$. Therefore

$$|B(m_\omega, 1)| \leq 1 + (n - 1)^2 - \min_{\omega \in \Sigma_n} |SD(m_\omega, 1)| - \min_{\omega \in \Sigma_n} |AD(m_\omega, 1)|
\leq 1 + (n - 1)^2 - \min_{\omega \in \Sigma_n} |SD(m_\omega', 1)|
= 1 + (n - 1)^2 - (r - 1)n
= |B(m_\omega', 1)|.$$

The upper bound of lemma VI.3 implies a lower bound on a perfect single-error correcting $MPC(n, r)$.

**Lemma VI.4.** Suppose $n/r \neq 2$. If $C$ is a perfect single-error correcting $MPC(n, r)$, then

$$\frac{n!}{(r^n)!/(1 + (n - 1)!^2 - (r - 1)n|\leq |C|.$$  

**Proof.** Assume $n/r \neq 2$, and that $C$ is a perfect single-error correcting $MPC(n, r)$. Then $\min_{\omega \in \Sigma_n} |B(m_\omega, 1)| = (n!/((r^n)!/(1 + (n - 1)!^2 - (r - 1)n|)$. This means

$$\frac{n!}{(r^n)!/(1 + (n - 1)!^2 - (r - 1)n| \leq (|C| \cdot \max_{\omega \in \Sigma_n} |B(m_\omega, 1)|),$$

which by Lemma VI.3 implies the desired result.

A more general lower bound is easily obtained by applying Lemma VI.3 with a standard Gilbert-Varshamov bound argument. While the lower bound of Lemma VI.4 applies only to perfect codes that are $MPC(n, r, d)$ with $d \geq 3$, the next lemma applies to any $MPC(n, r, d)$, which may or may not be perfect.

**Lemma VI.5.** Suppose $n/r \neq 2$, and let $C \subseteq M_r(\Sigma_n)$ be an $MPC(n, r, d)$ code of maximal cardinality. Then

$$\frac{n!}{(r^n)!/(1 + (n - 1)!^2 - (r - 1)n|d - 1 \leq |C|.$$  

**Proof.** Assume $n/r \neq 2$, and that $C$ is an $MPC(n, r, d)$ code of maximal cardinality. For all $m_\omega \in M_r(\Sigma_n)$, there exists $c \in C$ such that $d_c(m_\omega, c) \leq d - 1$. Otherwise, we could
add \( \mathbf{m}_r \notin C \) to \( C \) while maintaining a minimum distance of \( d \), contradicting the assumption that \(|C|\) is maximal.

Therefore \( \bigcup_{\mathbf{m}_r \in C} B(\mathbf{m}_r, d-1) = \mathcal{M}_r(S_n) \). This in turn implies that

\[
\frac{n!}{(r!)^{n/r}} \leq \sum_{\mathbf{m}_r \in C}|B(\mathbf{m}_r, d-1)|. 
\]

Of course, the right hand side of the above inequality is less than or equal to \((|C|) \cdot \left( \max_{\mathbf{m}_r \in C} |B(\mathbf{m}_r, d-1)| \right) \). Finally Lemma VI.3 implies that

\[
\max_{\mathbf{m}_r \in C} |B(\mathbf{m}_r, d-1)| \leq (1 + (n - 1)^2 - (r - 1)n)^{d-1}
\]

so the conclusion holds.

B. Binary Case – Cut Location Maximizing Ball Size

In the previous subsection we were able to find center multipermutations whose ball sizes were both minimal (Lemma VI.1) and maximal (Lemma VI.3). These were used to provide bounds on the maximum code size (Lemas VI.2, VI.4, VI.5). However, a complication arises from applying the binary case, the case when \( n/r = 2 \). We say that \( \mathbf{m}_r \in \mathcal{M}_r(S_n) \) is a binary multipermutation if and only if \( n/r = 2 \). The next two subsections focus on determining the maximum ball size for binary multipermutations. The current subsection addresses the question of cut location. The notion of cuts is defined in the following paragraphs. For the remainder of the paper we assume that \( n \) is an even integer and that \( n/r = 2 \) (equivalently \( r = n/2 \)).

Since we are assuming that \( n/r = 2 \), by definition \( \mathbf{m}_o \) (see Equation 5 to recall the definition of \( \omega \)) is an alternating string of length \( n \), which results in the size of the alternating duplication set \( AD(\mathbf{m}_o) \) increasing rapidly as \( n \) increases. This is often in turn results in \( |B(\mathbf{m}_o, d)\) increasing rapidly (in the sense of Lemma VI.3). For example, if \( n = 12 \), then we have \( \mathbf{m}_o = (1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1) \), which would imply that \( |\mathcal{M}(\mathbf{m}_o)| = \psi(12) = 25 \).

To compensate for this problem, it is best to “cut” the original \( \mathbf{m}_o \) into some number \( c \) of locally maximal alternating substrings. Whenever \( \mathbf{m} \) is a tuple in two symbols, for example when \( \mathbf{m} \in \{1, 2\}^n \), we use the term \( \text{cut} \) to refer to any locally maximal alternating substring of \( \mathbf{m} \). This language applies to binary multipermutations. Considering the example above when \( n = 12 \), we could instead take the binary multipermutation \( (1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1) \), which has two cuts of length 6, namely \((1, 2, 1, 2, 1, 2)\) and \((2, 1, 2, 1, 2, 1)\) as opposed to a single length 12 cut in the original \( \mathbf{m}_o \). Notice that the standard duplication set increases by 5 but the new alternating duplication set size is now \( \psi(6) + \psi(6) = 8 \), a decrease by 17.

Intuitively, these cuts should be chosen so that each is as similar in length as possible in order to minimize the total size of the alternating duplication set. For example, \((1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1)\), which has an alternating duplication set of size 8 is preferable to \((1, 2, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1)\), which has an alternating duplication set of size 16. This idea is proven subsequently. Another question concerns the optimal number of such cuts, since each time a cut is introduced the standard duplication set size necessarily increases. This question is addressed in the next subsection, and it turns out that having approximately \( \sqrt{n} \) cuts minimizes total duplications and thus results in the maximum ball size.

To start this subsection, we will show that given a multipermutation with a fixed number \( c \) of cuts, the alternating duplication set is minimized when these cut lengths are as similar in length as possible. In order to simplify the argument, the following two lemmas reduce the discussion to the lengths of these alternating substrings.

Lemma VI.6. Let \( \mathbf{m} \in \{1, 2\}^n \). Then there exists a binary multipermutation \( \mathbf{m}_o \in \mathcal{M}_r(S_n) \) such that \( \mathbf{m}_o = \mathbf{m} \) and only if \#\( i \in [n] : \mathbf{m}(i) = 1 \) = \#\( i \in [n] : \mathbf{m}(i) = 2 \), i.e. the number of 1’s and 2’s of \( \mathbf{m} \) are equal.

Proof. Assume \( \mathbf{m} \in \{1, 2\}^n \). First suppose that there exists a binary multipermutation \( \mathbf{m}_o \in \mathcal{S}_n \) such that \( \mathbf{m}_o = \mathbf{m} \). Then by the definition of binary multipermutations, \#\( i \in [n] : \mathbf{m}(i) = 1 \) = \#\( i \in [n] : \mathbf{m}(i) = 2 \), completing the first direction.

For the second direction of the proof, suppose that \#\( i \in [n] : \mathbf{m}(i) = 1 \) = \#\( i \in [n] : \mathbf{m}(i) = 2 \) = \( n/2 \). Then we can construct a binary multipermutation \( \mathbf{m}_o \) with the property that \( \mathbf{m}_o = \mathbf{m} \) as follows: Define \( \{i_1, i_2, \ldots, i_q\} := \{i \in [n] : \mathbf{m}(i) = 1\} \) and \( \{i_{q+1}, i_{q+2}, \ldots, i_n\} := \{i \in [n] : \mathbf{m}(i) = 2\} \). For all \( j \in [n] \), set \( \sigma(i_j) := j \) and define \( \sigma := (\sigma(1), \sigma(2), \ldots, \sigma(n)) \). Then \( \mathbf{m}_o = \mathbf{m} \). \( \square \)

Lemma VI.7. Let \( c \in [n-1] \), and \( (q(1), q(2), \ldots, q(c)) \in \mathbb{Z}_{\geq 0}^c \) such that \( \sum_{i=1}^c q(i) = n \). Then there exists \( i \in [c] \) such that \( q(i) \) is even if and only if there exists a binary multipermutation \( \mathbf{m}_o \in \mathcal{M}_r(S_n) \) such that \( \mathbf{m}_o = (\mathbf{m}_o'[a_1, b_1], \mathbf{m}_o' [a_2, b_2], \ldots, \mathbf{m}_o' [a_c, b_c]) \),

where for all \( i \in [c] \), \( a_i, b_i \in [n] \), and \( a_i \leq b_i \) such that \( \mathbf{m}_o'[a_i, b_i] \) is a cut (locally maximal alternating substring).

The proof of Lemma VI.7 can be found in the Appendix C. In words, the lemma states that given any tuple of positive integers \( (q(1), q(2), \ldots, q(c)) \) whose entries sum to \( n \), as long as there is at least one even integer in the tuple, then the entries can be made to correspond to the lengths of the cuts of some binary multipermutation \( \mathbf{m}_o \). Notice that in the formulation resulting from Lemma VI.7, the number of cuts \( c \) in a binary multipermutation \( \mathbf{m}_o \) is one more than the number of repeated adjacent digits. In other words, \( c = \#i \in [2, n] : \mathbf{m}_o(i) = \mathbf{m}_o(i+1) = 1 \). Hence for a fixed number of cuts \( c \), the standard duplication set size \( |SD(\mathbf{m}_o)| \) does not depend on the lengths of individual cuts.

On the other hand, the size of the alternating duplication set \( |AD(\mathbf{m}_o)| \) does depend on the lengths of the cuts. This means that if the number of cuts is fixed at \( c \) then by
Lemma V.5 and Lemma VI.7, finding the maximum ball size equates to minimizing \( \psi(q(1), q(2), \ldots, q(c)) \), where \((q(1), q(2), \ldots, q(c)) \in \mathbb{Z}_{\geq 0}^c \) has at least one even entry and whose entries sum to \( n \). We claim that the tuple defined next minimizes the sum in question.

**Definition** \((q_c, rem_c, q_c)\). Let \( c \in [n-1] \). Denote by \( q_c \in \mathbb{Z}_{\geq 0} \) and \( rem_c \in \mathbb{Z}_{\geq 0} \) the unique quotient and remainder when \( n \) is divided by \( c \), i.e. \( c \cdot q_c + rem_c = n \) where \( rem_c < c \). Define

\[
q_c := \begin{cases} 
(q_c + 1, q_c, \ldots, q_c, -1) & \text{if } q_c \text{ is odd and } rem_c = 0 \\
(q_c + 1, \ldots, q_c + 1, q_c, \ldots, q_c) & \text{otherwise}
\end{cases}
\]

We also use the notation \( q_c = (q_c(1), \ldots, q_c(c)) \in \mathbb{Z}_{\geq 0}^c \).

The above definition guarantees that two important conditions are satisfied: (1) the entries of \( q_c \) sum to \( n \); and (2) there exists some \( i \in [c] \) such that \( q_c(i) \) is even. These two conditions correspond with the conditions and statement of Lemma VI.7. Additionally, by definition, \( q_c \) is a weakly decreasing (non-increasing) sequence with all entries being positive integers, and thus it is a partition of \( n \).

Standard calculation (see Lemma D.3 in Appendix D) indicates that if two cuts of a binary multipermutation differ by 2 or more, then the size of the alternating duplication set associated with that multipermutation can be reduced by bringing the length of those two cuts closer together. Generalizing over all the cuts in the multipermutation, we may minimize the alternating duplication set and hence maximize ball size by choosing all cuts to be as similar in length as possible. Another way of saying that the cut sizes are as similar in length as possible is to say that the cut sizes are precisely the values of \( q_c \). The fact that cut sizes equaling the values of \( q_c \) minimizes the associated alternating duplication set size is stated in the next theorem.

**Theorem VI.1.** Let \( c \in [n-1] \). Then

\[ \min_{m_c' \in \mathcal{M}_c'(\mathbb{S}_n)} |AD(m_c')| = \psi(q_c), \]

where \( \mathcal{M}_c'(\mathbb{S}_n) := \{m_c' \in \mathcal{M}_c(\mathbb{S}_n) : \#(m_c'_{r}(i) = m_c'_{r}(i-1) + 1 = c)\} \), i.e. \( \mathcal{M}_c'(\mathbb{S}_n) \) is the set of binary multipermutations with exactly \( c \) cuts.

**Proof.** Assume \( c \in [n-1] \). Note first that by Lemma VI.7, there exists a binary multipermutation with exactly \( c \) cuts, whose cut lengths correspond to \( q_c \). Now let \( (a(1), a(2), \ldots, a(c)) \in \mathbb{Z}^c_{\geq 0} \) such that \( \sum_{i=1}^c a(i) = n \) and there exists \( i \in [c] \) such that \( a(i) \) is even. Again by Lemma VI.7, \( (a(1), a(2), \ldots, a(c)) \) corresponds to the cut lengths of an arbitrary binary multipermutation with exactly \( c \) cuts. Hence by Lemma V.5 it suffices to show that \( \psi(q_c) \leq \psi((a(1), a(2), \ldots, a(c)) \). We divide the remainder of the proof into two halves corresponding to the split definition of \( q_c \).

First, suppose \( q_c \) is odd and \( rem_c = 0 \) so that \( q_c = (q_c + 1, q_c, \ldots, q_c, -1) \). Then since there exists \( i \in [c] \) such that \( a(i) \) is even, there must be distinct \( i' \) and \( j' \) in \( [c] \) such that \( a_i = q_c + h_{i'} \) and \( a_j = q_c - h_{j'} \) where \( h_{i'}, h_{j'} \in \mathbb{Z}_{\geq 0}^c \). Hence, by Lemma D.3 (see appendix D),

\[
\psi((q_c, q_c, \ldots, q_c)) + 1 \leq \psi((a(1), a(2), \ldots, a(c)),
\]

but also by Lemma D.3 (applied to the first and last entry of \( q_c \)),

\[
\psi(q_c) = \psi(q_c + 1) + \psi((q_c, q_c, \ldots, q_c)) + \psi(q_c - 1) = \psi((q_c, q_c, \ldots, q_c)) + 1.
\]

For the second half, suppose that \( q_c \) is even or that \( rem_c \neq 0 \). Then \( q_c = (q_c + 1, \ldots, q_c + 1, q_c, \ldots, q_c) \). This means that for all \( i, j \in [c] \), that \( |q_c(i) - q_c(j)| \leq 1 \). Hence, by Lemma D.3,

\[
\psi(q_c) \leq \psi((a(1), a(2), \ldots, a(c)) \).
\]

We have shown that choosing cuts to be as evenly distributed as possible results in minimizing the alternating duplication set. However, as mentioned before, while increasing cuts generally decreases the size of the alternating duplication set, it also increases the size of the standard duplication set. The question of the optimal number of cuts in a multipermutation \( m_c' \) minimizing \( |SD(m_c')| + |AD(m_c')| \) remains.

**C. Binary Case — Number of Cuts Maximizing Ball Size**

The previous subsection demonstrated the nature of cuts maximizing ball size in the binary case once the number of cuts \( c \) is fixed. This subsection focuses on determining the number of cuts maximizing binary multipermutation ball size. Computer analysis for values of \( r \) up to 10, 000 suggests that \( c \approx \sqrt{r} \) cuts minimizes the sum of \( |SD(m_c')| \) and \( |AD(m_c')| \) (and therefore maximizes the ball size). The next lemmas prove that this is indeed the case. We therefore call \( \sqrt{r} \) the **ideal cut value** and use the notation

\[
\hat{c} := \sqrt{r}.
\]

In practice the actual optimal number of cuts is only approximately equal to \( \hat{c} \) since \( \hat{c} \) is not generally an integer. As in the previous subsection, recall that we assume \( n \) is a positive even integer and that \( n/r = 2 \) for the remainder of this paper.

**Lemma VI.8.** Let \( m_c' \in \mathcal{M}_c(\mathbb{S}_n) \) be a binary multipermutation. Then

\[
|SD(m_c')| = (c(1) - 1)(n-2) + (n - (c-1))(r-1) = c(r-1) + (n-1)(r-1),
\]

where \( c := \#(i \in [2, n] : m_c'_{i}(i) = m_c'_{i}(i-1) + 1) \).

Note that the lemma could technically be simplified by rewriting \( n \) as \( 2r \), but here and elsewhere \( n \) is kept in favor of \( 2r \) to retain intuition behind the meaning and for ease of comparison with previous results in the non-binary case. Although the lemma is obvious, its significance is that the
only component that depends upon \(c\) is \(c(r - 1)\). This means that each time the number of cuts is increased by 1, the size of the standard duplication set is increased by \(r - 1\).

Therefore to show that \(\hat{c}\) cuts minimizes duplications, it is enough to show the following two facts: (1) if the number of cuts is greater or equal to \(\hat{c}\), then increasing the number of cuts by one causes a decrease in the alternating duplication set by at most \(r - 1\); and (2) if the number of cuts is less than or equal to \(\hat{c}\), then a further decrease in cuts by one will enlarge the alternating duplication set by at least \(r - 1\). These two facts are expressed in the next two lemmas.

**Lemma VI.9.** Let \(c < n - 1\) and \(\hat{c} \leq c\). Then

\[
\psi(q_c) - \psi(q_{c+1}) \leq r - 1. 
\]

\(7\)

The proof for Lemma VI.9 is in Appendix E. The next example demonstrates how to construct \(q_{c+1}\) from \(q_c\) when \(\hat{c} \leq c < n - 1\). This corresponds to increasing the number of cuts from \(c\) to \(c + 1\). Notice that in the example \(q_c = c\) and that each cut is decreased by at most 2, with some cuts decreased by only 1. This corresponds to the second case in the proof of Lemma VI.9.

**Example VI.10.** Let \(n = 30\) and \(c = 4\). Notice that \(\hat{c} = \sqrt{15} \approx 3.873\) so that \(\hat{c} < c < n - 1\). We also have \(q_4 = 8\) and rem\_4 = 5 while \(d = 6\) and rem\_5 = 0. Therefore \(q_4 = (8, 8, 7, 7)\) and \(q_5 = (6, 6, 6, 6)\).

We may visualize \(q_4\) and \(q_5\) respectively as the left and right diagrams in Figure 3, with the \(i\)th row of the diagram corresponding to the \(i\)th cut, \(q_i(i)\) or \(q_{i+1}(i)\). The numbers in the blocks in the left diagram of Figure 3 represent the order in which each row would be shortened to construct the last cut of \(q_4\).

If \(m_r\) is a multipermutation with four cuts whose lengths correspond to \(q_4\), then applying Lemmas VI.8 and V.5, \(|SD(m_r)| + |AD(m_r)| = 492\). By Theorem VI.1, this means \(|B(m_r, 1)| = 238\). On the other hand, if \(m_r\) is a multipermutation with five cuts whose lengths correspond to \(q_5\), then similar methods show \(|SD(m_r)| + |AD(m_r)| = 496\), which implies \(|B(m_r, 1)| = 234\), a smaller value.

**Theorem VI.2.**

\[
\max_{m_r \in \mathcal{M}_r(G_n)} |B(m_r, 1)| = 1 + (n - 1)^2 - \min_{c \in [\hat{c}, c]} \left( 1 - \frac{c(r - 1) + (n - 1)(r - 1) + \psi(q_c)}{c} \right).
\]

\(9\)

Proof. The proof is an immediate consequence of Theorem V.1, Theorem VI.1, and Lemmas VI.8, VI.9 and VI.11.

The proof for Lemma VI.11 is in the Appendix F. The next example demonstrates how to construct \(q_4\) from \(q_{c-1}\) when \(c \leq \hat{c}\). Notice that each cut length is decreased by at least 2.

**Example VI.12.** Let \(n = 34\) and \(c = 4\). Notice that \(\hat{c} = \sqrt{17} \approx 4.123\) so that \(c < \hat{c}\). We also have \(q_3 = 11\) and rem\_3 = 1 while \(q_4 = 8\) and rem\_4 = 2. Therefore \(q_3 = (12, 11, 11)\) and \(q_4 = (9, 9, 8, 8)\). We can visualize \(q_3\) and \(q_4\) respectively as the left and right diagrams in Figure 4, with the \(i\)th row of each diagram corresponding to the \(i\)th cut, \(q_i(i)\) or \(q_{i+1}(i)\). The numbers in the blocks in the left diagram of Figure 4 represent the order in which each row would be shortened to construct the last cut of \(q_4\).

If \(m_r\) is a multipermutation with three cuts whose lengths correspond to \(q_3\) above, then applying Lemmas VI.8 and V.5, \(|SD(m_r)| + |AD(m_r)| = 641\). By Theorem VI.1, this means \(|B(m_r, 1)| = 449\). On the other hand, if \(m_r\) is a multipermutation with four cuts whose lengths correspond to \(q_4\) above, then similar methods show \(|SD(m_r)| + |AD(m_r)| = 634\), which implies \(|B(m_r, 1)| = 456\), a larger value.

Lemmas VI.9 and VI.11 imply that the number of cuts \(c\) minimizing the sum \(|SD(m_r)| + |AD(m_r)|\) (and thus maximizing ball size) observes the inequalities \(\hat{c} < c < \hat{c} + 1\). This answers the question of the optimal number of cuts. For a particular value \(c\), it is a relatively simple matter to calculate the exact size of the maximal Ulam multipermutation ball. One simply has to determine whether \(c = \lfloor \hat{c}\rfloor\) or \(c = \lceil \hat{c}\rceil\) yields a smaller \(|SD(m_r)| + |AD(m_r)|\) (here \(\lfloor x\rfloor\) denotes the ceiling function on \(x \in \mathbb{R}\), i.e. the least integer greater than or equal to \(x\)). Once the best choice for \(c\) is ascertained, an application of Theorem VI.1 will yield the maximum size for that particular \(r\). The above statements are summarized in the next theorem.

Once again we retain \(n\) instead of \(2r\) in the theorem statement for intuition purposes. Theorem VI.2 indicates that the number of cuts \(c\) maximizing ball size should be either \(\lfloor \hat{c}\rfloor\) or \(\lceil \hat{c}\rceil\). Stated another way, the optimal number of cuts is within 1 of the true square root of \(r\). We may apply this fact to obtain an upper bound on maximum ball size. This is shown in the next lemma and corollary.
Lemma VI.13. Let $c \in [n-1]$. Then
\[ \psi(q_c) \geq c \left( \frac{r}{c} - 1 \right)^2 \frac{1}{4} \cdot \frac{1}{4}. \]

Proof. Suppose $c \in [n-1]$ and let $a := (a(1), a(2), \ldots, a(c)) \in \mathbb{R}^c_{>0}$ such that $\sum_{i=1}^c a(i) = n$. Note first that
\[ \sum_{i=1}^c \left( \frac{a(i) - 2}{2} \right)^2 = \sum_{i=1}^c \left( \frac{a(i)^2}{4} - a(i) + 1 \right) = \frac{1}{4} \sum_{i=1}^c a(i)^2 - n + c. \] (10)

Note also that by applying the Cauchy-Schwarz inequality to $a$ and $1 \in \mathbb{R}^c$, the all-1 vector of length $c$, we obtain
\[ \sum_{i=1}^c a(i)^2 \geq \frac{\left( \sum_{i=1}^c a(i) \right)^2}{c} = \frac{n^2}{c} = \sum_{i=1}^c \left( \frac{n}{c} \right)^2. \] (11)

Equation (10) and inequality (11) imply that choosing $a = (n/c, n/c, \ldots, n/c)$ minimizes the sum on the far left of Equation (10). The minimum of the left side of Equation (10) is less than or equal to $\sum_{i=1}^c ((q_c(i) - 2)/2)^2$, with equality only holding when $n/c \in \mathbb{Z}$. Thus an application of Lemma D.1 completes the proof. □

Corollary VI.14. Let $m'_p$ be a binary multipermutation. Also define
\[ U(r) := 1 + (n-1)^2 - \left( \left( \hat{c} - 1 \right)(r-1) \right. \]
\[ + (n-1)(r-1) + \left( \hat{c} - 1 \right) \left( \frac{r}{\hat{c}} - 1 \right) \left( \frac{1}{4} \right) \right). \]

Then
\[ |B(m'_p, 1)| < U(r). \]

Proof. The proof follows from Theorem VI.2, Lemma VI.13, and the fact that $\hat{c} - 1 < |\hat{c}| < \hat{c} + 1$. □

The following table compares values from Corollary VI.14 versus the size of the actual largest multipermutation ball for given values of $r$. The actual values of largest ball sizes were calculated using Theorem VI.2.

| $r$ | Max ball size (9) | Inequality (10) | ratio |
|-----|------------------|-----------------|-------|
| 10  | 148              | $\sim 168$      | 8819  |
| 100 | 18,101           | $\sim 18,423$  | 9825  |
| 1000| 1,937,753        | $\sim 1,941,489$ | 9981  |

TABLE VI
MAXIMUM BALL SIZE VERSUS BOUNDED VALUE

between the true maximum ball size, $\max_{m'_p \in \mathcal{M}_r(S_a)} |B(m'_p, 1)|$, and the upper bound value, $U(r)$ from Corollary VI.14, approaches 1 as $r$ approaches infinity. This can be confirmed by observing that if
\[ L(r) := 1 + (n-1)^2 - \left( \left( \hat{c} + 1 \right)(r-1) \right. \]
\[ + (n-1)(r-1) + \left( \hat{c} + 1 \right) \left( \frac{r}{\hat{c} + 1} - 1 \right) \left( \frac{1}{4} \right) \right). \]

then $L(r) < \max_{m'_p \in \mathcal{M}_r(S_a)} |B(m'_p, 1)|$. After making this observation, the Squeeze Theorem can then be applied with $(\max_{m'_p \in \mathcal{M}_r(S_a)} |B(m'_p, 1)|)/U(r)$ being squeezed between $L(r)/U(r)$ and $U(r)/L(r)$. As before, in the definitions of both $U(r)$ and $L(r)$, we keep $n$ in $\mathbb{N}$.

Finally, Corollary VI.14 can be applied to establish a new lower bound on perfect single-error correcting MPC$(n, r)$’s in the binary case. It can also be applied to establish a new Gilbert Varshamov type lower bound. These two bounds are stated as the last two lemmas. Note that Lemma VI.16 is similar to a Gilbert Varshamov type lower bound stated as Lemma 7 in [7], but our estimate of the multipermutation Ulam ball size is tighter, leading to a larger lower bound on code size.

Lemma VI.15. Let $C$ be a perfect single-error correcting MPC$(n, r)$. Also let $U(r)$ be defined as in Corollary VI.14. Then
\[ \frac{n!}{(r!)^2 U(r)} \leq |C|. \]

Proof outline. The proof follows from Corollary VI.14. □

Lemma VI.16. Let $C$ be an MPC$(n, r, d)$. Also let $U(r)$ be defined as in Corollary VI.14. Then
\[ \frac{n!}{(r!)^2 U(r)^{d-1}} \leq |C|. \]

Proof outline. The proof follows from Corollary VI.14 and a standard Gilbert-Varshamov argument (see the proof of Lemma VI.5 for such an argument). □

VII. Conclusion

This paper first considered and answered two questions. The first question concerned Ulam ball sizes and the second concerned the possibility of perfect codes. It was shown that Ulam ball sizes can be calculated explicitly for reasonably small radii using an application of the RSK-correspondence. It was then shown, partially using the aforementioned ball-calculation method, that nontrivial perfect Ulam permutation codes do not exist. These new results are summarized in Tables I and II, found in the introduction.

Following the discussion of permutation codes, the multipermutation code case was considered next, and two more questions were addressed. The third question of calculating $r$-regular Ulam balls was addressed for the cases when the
center is \( m'_r \) or when the radius \( t = 1 \). This lead to new upper and lower bounds on maximal code size. These new results are summarized in the Tables III and IV, found in the introduction.

Many remaining problems remain. One problem is to find a method for calculating \( r \)-regular Ulam balls for more general parameters. Our current work began to show how to calculate sizes for any radius when the center is \( m'_r \) (using Young tableaux) or for any center when the radius is 1, but not for general parameters. While we proved the nonexistence of nontrivial perfect Ulam permutation codes, it is unknown whether or not perfect multipermutation codes exist. A general formula for any center or radii, or at least bounds on general ball sizes, would help in understanding bounds on the size of multipermutation Ulam codes and the possibility of perfect multipermutation codes.

**APPENDIX A**

**Proof of Lemma II.1:**

Let \( m'_r, m'_s \in M_r(S_n) \). We will first show that \( d_{\sigma'}(m'_r, m'_s) \geq n - \ell(m'_r, m'_s) \). By definition of \( d_{\sigma'}(m'_r, m'_s) \), there exist \( \sigma' \in R_r(\sigma) \) and \( \pi' \in R_r(\pi) \) such that \( d_{\sigma'}(m'_r, m'_s) = d_{\sigma'}(\sigma, \pi) = n - \ell(\sigma', \pi') \). Hence for all \( \sigma' \in R_r(\sigma) \) and \( \pi' \in R_r(\pi) \) we have \( \ell(\sigma', \pi') \leq \ell(m'_r, m'_s) \), then \( d_{\sigma'}(m'_r, m'_s) \geq n - \ell(m'_r, m'_s) \). Therefore it suffices to show that for all \( \sigma' \in R_r(\sigma) \) and \( \pi' \in R_r(\pi) \), that \( \ell(\sigma', \pi') \leq \ell(m'_r, m'_s) \). This is simple to prove because if two permutations have a common subsequence, then their corresponding \( r \)-regular multipermutations will have a related common subsequence. Let \( \sigma' \in R_r(\sigma) \), \( \pi' \in R_r(\pi) \), and \( \ell(\sigma', \pi') = k \). Then there exist indexes \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \) and \( 1 \leq j_1 < j_2 < \cdots < j_k \leq n \) such that for all \( p \in [k] \), \( \sigma'(i_p) = \pi'(j_p) \).

Next, we will show that \( d_{\sigma'}(m'_r, m'_s) \leq n - \ell(m'_r, m'_s) \).

Note that

\[
d_{\sigma'}(m'_r, m'_s) = \min_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} \ell(\sigma', \pi')
= \min_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} (n - \ell(\sigma', \pi'))
= n - \max_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} \ell(\sigma', \pi').
\]

Here if \( \max_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} \ell(\sigma', \pi') \geq \ell(m'_r, m'_s) \), then \( d_{\sigma'}(m'_r, m'_s) \leq n - \ell(m'_r, m'_s) \). It is enough to show that there exist \( \sigma' \in R_r(\sigma) \) and \( \pi' \in R_r(\pi) \) such that \( \ell(\sigma', \pi') \geq \ell(m'_r, m'_s) \). To prove this fact, we take a longest common subsequence of \( m'_r \) and \( m'_s \) and then carefully choose \( \sigma' \in R_r(\sigma) \) and \( \pi' \in R_r(\pi) \) to have an equally long common subsequence. The next paragraph describes how this can be done.

Let \( \ell(m'_r, m'_s) = k \) and let \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \) and \( 1 \leq j_1 < j_2 < \cdots < j_k \leq n \) be integer sequences such that for all \( p \in [k] \), \( m'_r(i_p) = m'_s(j_p) \). The existence of such sequences is guaranteed by the definition of \( \ell(m'_r, m'_s) \). Now for all \( p \in [k] \), define \( \sigma'(i_p) \) to be the smallest integer \( l \in [n] \) such that \( m'_r(l) = m'_s(l) \) and if \( q \in [k] \) with \( q < p \), then \( m'_r(i_q) = m'_s(i_q) \) implies \( \sigma'(i_q) < \sigma'(i_p) = l \). For all \( p \in [k] \), define \( \pi'(j_p) \) similarly. Then for all \( p \in [k] \), \( \sigma'(i_p) = \pi'(j_p) \). The remaining terms of \( \sigma' \) and \( \pi' \) may easily be chosen in such a manner that \( \sigma' \in R_r(\sigma) \) and \( \pi' \in R_r(\pi) \). Thus there exist \( \sigma' \in R_r(\sigma) \) and \( \pi' \in R_r(\pi) \) such that \( \ell(\sigma', \pi') \geq \ell(m'_r, m'_s) \).

**APPENDIX B**

**Proof of Lemma II.2:**

Suppose \( m'_r, m'_s \in M_r(S_n) \). If \( \ell(m'_r, m'_s) < n \) then there exists a translation \( \phi \in S_n \) such that \( \ell(m'_r \cdot \phi, m'_s \cdot \phi) \geq \ell(m'_r, m'_s) + 1 \). Since it is always possible to arrange one element with a single translation. This then implies that \( \min(k \in \mathbb{Z} : \phi \in S_n \) such that \( m'_r \cdot \phi_1 \cdots \phi_k = m'_s \) \) \( n - \ell(m'_r, m'_s) = d_{\sigma'}(m'_r, m'_s) \). At the same time, given \( \ell(m'_r, m'_s) \leq n - 1 \), then for all translations \( \phi \in S_n \), we have that \( \ell(m'_r \cdot \phi, m'_s \cdot \phi) \leq \ell(m'_r, m'_s) + 1 \), since a single translation can only arrange one element at a time. This can be seen, as upon deletion of the rearranged element from the sequence \( m'_r \) and any element from the sequence \( m'_s \), any new longest common subsequence has length at most \( \ell(m'_r, m'_s) \), as it is part of an old common subsequence. This follows by monotonicity of \( \ell : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{N} \) upon the insertion or deletion of elements. Therefore by Lemma II.1, \( \min(k \in \mathbb{Z} : \phi_1 \cdots \phi_k = m'_s \) \) \( n - \ell(m'_r, m'_s) = d_{\sigma'}(m'_r, m'_s) \).

**APPENDIX C**

**Proof of Lemma VI.7:**

Recall that \( n \) is an even integer. Assume \( c \in [n - 1] \) and \( q := (q_1, q_2, \ldots, q(c)) \in \mathbb{Z}_{-c} \) such that \( \sum_{i=1}^{c} q(i) = n \). For the first direction, suppose there exists some \( i \in [c] \) such that \( q(i) \) is even. Since \( n \) is even, the number of odd values in \( q \) is even, i.e. \( \#[i \in [c] : q(i) \text{ is odd}] = 2k \) for some nonnegative integer \( k \). We will now construct an \( m \in [1, 2]^n \) with an equal number of 1’s and 2’s, whose cuts correspond to \( q \). We begin by defining two sets: first \( \{a_1^*(1), a_2^*(2), \ldots, a_k^*(2k)\} := \{q(i) : q(i) \text{ is odd}\} \) and then \( \{a_1^*(2k + 1), a_2^*(2k + 2), \ldots, a_k^*(4c)\} := \{q(i) : q(i) \text{ is even}\} \).

Then define \( m \) as follows:

\[
m := (m[a_1^*, b_1^*], m[a_2^*, b_2^*], \ldots, m[a_k^*, b_k^*], m[a_{k+1}^*, b_{k+1}^*],
= (m[a_{k+2}^*, b_{k+2}^*], m[a_{k+3}^*, b_{k+3}^*], \ldots, m[a_{2k+1}^*, b_{2k+1}^*],
= (m[a_{2k+2}^*, b_{2k+2}^*], \ldots, m[a_c^*, b_c^*]),
= g^*(1) \quad g^*(2) \quad g^*(k) \quad g^*(2k-1)
= g^*(k+1) \quad g^*(2k) \quad g^*(2k+1)
\]

where \( m(1) = 1 \) and for all \( j \in [c] \), \( m[a_j^*, b_j^*] \) is a cut.

The idea here is simple. By the definition of \( m \), each of the first \( k \) cuts begin and end with 1 since they are all odd length cuts, and thus each will have one more 1 than 2. The \( (k + 1) \)th cut, \( m[a_{k+1}^*, b_{k+1}^*] \), is taken to be of even length, which reverses the order of the subsequent \( k \) cuts. Hence the \( k \) cuts from \( m[a_{k+2}^*, b_{k+2}^*] \) through \( m[a_{2k+1}^*, b_{2k+1}^*] \) each begin and end with 2, so each will have one more 2 than 1. The remaining cuts from \( m[a_{2k+2}^*, b_{2k+2}^*] \) through \( m[a_c^*, b_c^*] \) are even, which implies that the number of 1’s and 2’s in each of
these cuts is equal. Hence, we may write the following:

\[
\# \{ i \in [n] : m(i) = 1 \} = \# \{ i \in [a_1, b_k] : m(i) = 1 \} + \# \{ i \in [a_{k+2}, b_{k+1}] : m(i) = 1 \} + \# \{ i \in [a_{k+1}, b_{k+1}] \cup [a_{2k+2}, b_c] : m(i) = 1 \}
\]

Applying Lemma D.2 to the four cases when \( a \) is either even or odd and \( b \) is either even or odd, a routine calculation shows the following: If both \( a \) and \( b \) are even, then the left side of inequality (14) equals \((a - b)/2 > 0\). If \( a \) is even and \( b \) is odd or if \( a \) is odd and \( b \) is even, then the left side of inequality (14) equals \((a - b - 1)/2 > 0\). Finally, if both \( a \) and \( b \) are odd, then the left side of inequality (14) equals \((a - b - 2)/2 > 0\). □

APPENDIX E

Proof of Lemma VI.9:
Assume \( c \in [n - 2] \) and \( c \leq c \). We will split the proof into two cases, when \( q_{c+1} \leq c \) and when \( q_{c+1} > c \). Let us first suppose \( q_{c+1} \leq c \). This corresponds roughly to the case where \( c \leq c < \sqrt{n} \). In this instance, for all \( i \in [2, c] \), we have \( q_c(i) - q_{c+1}(i) \leq 1 \) and \( q_c(1) - q_{c+1}(1) \leq 2 \). This is because \( q_{c+1} \) can be constructed from \( q_c \) by shortening each cut \( q_c \) in order to create the \((c + 1)\)st cut. Since \( q_{c+1} \leq c \), each cut will decrease by at most one, except for one exceptional case when \( q_c \) is an odd integer and \( q_{c+1} = c \), in which case \( q_c(1) - q_{c+1}(1) = 2 \).

Since for all \( i \in [2, c] \), \( q_c(i) - q_{c+1}(i) \leq 1 \) and \( q_c(1) - q_{c+1}(1) \leq 2 \), the left hand side of inequality (7) is less than or equal to

\[
\psi(q_c) - \psi(q_c(1) - 2) - \psi(q_c(2) - 1, q_c(3) - 1, \ldots, q_c(c) - 1) - \psi(q_{c+1}(c + 1)).
\]

(15)

By adding and subtracting \( \psi(q_c(1) - 1) \) from expression (15) and also disregarding the last term, \( -\psi(q_{c+1}(c + 1)) \), after some rearrangement expression (15) is less than or equal to

\[
\psi(q_c) - \psi(q_c - 1) + \psi(q_c(1) - 1) - \psi(q_c(2) - 2),
\]

which by Lemma D.2 is less than or equal to

\[
\sum_{i=1}^{c} \left( \frac{q_c(i)}{2} - 1 \right) + \frac{(q_c(c) - 1)}{2} - 1
\]

\[
\leq \left( \frac{n}{2} - c \right) + \frac{(q_c + 1 - 1)}{2} - 1
\]

\[
= c - c + \frac{q_c}{2} - 1,
\]

where this last expression is less than or equal to \( r - 1 \) since \( q_{c+1} \leq c \) implies that \( q_c \leq q_{c+1} + 1 \leq 2c \). This concludes the case where \( q_{c+1} \leq c \).

Next assume that \( q_{c+1} > c \). This corresponds roughly to the case where \( c < c < \sqrt{n} \). Note that

\[
q_{c+1} = \frac{n - \text{rem}_{c+1}}{c + 1} < \frac{n}{c} \leq \frac{2r}{c} = 2c \leq 2c.
\]

Since \( q_{c+1} \) is strictly less than \( 2c \), for all \( i \in [c] \) we have \( q_c(i) - q_{c+1}(i) \leq 2 \). Moreover, because \( c < q_{c+1} < 2c \), the number of \( i \in [c] \) such that \( q_c(i) - q_{c+1}(i) = 2 \) is \( q_{c+1} - c \) or \( q_{c+1} - c - 1 \). This means then that the number of \( i \in [c] \) such that \( q_c(i) - q_{c+1}(i) = 1 \) is equal to \( c - (q_{c+1} - c) \) or \( c - (q_{c+1} - c - 1) \). As before, the reasoning for these set sizes comes from constructing \( q_{c+1} \) from \( q_c \) and considering how
much each cut length is decreased to construct the final cut, 
$q_{k+1}(c+1)$. Example (VI.10) helps here to aid comprehension.

It should be noted that there is one exceptional case, when 
$q_{k+1}$ is an odd integer and $rem_{k+1} = 0$. In this instance,
$q_{k+1}(c+1) = q_{k+1} - 1$, which means the number of $i \in [c]$
such that $q_c(i) - q_{k+1}(i) = 2$ is decreased by one, the number
of $i \in [c]$ such that $q_c(i) - q_{k+1}(i) = 1$ is increased by one.
By Equation (6), the final effect on the size of the left hand
side of Inequality (7) is a decrease by 1, so it is enough to
prove the inequality in the typical case, when 
$\#i \in [c] : q_c(i) - q_{k+1}(i) = 2 = q_{k+1} - c$ and 
$\#i \in [c] : q_c(i) - q_{k+1}(i) = 1 = c - q_{k+1} - c$. These set sizes imply that the
left hand side of Inequality (7) is less than or equal to

$$\psi(q_c) - \psi(q_c(1) - 2), q_c(2) - 2, \ldots, q_c(q_{c+1} - c) - 2$$

$$- \psi(q_c(q_{c+1} - c + 1) - 1, q_c(q_{c+1} - c + 2) - 1, \ldots, q_c(1) - 1).$$

By adding and subtracting $\sum_{i=1}^{q_{k+1} - c} ((q_c(i) - 3)/2)^2$, after
some rearrangement the expression above can be rewritten as

$$\psi(q_c) - \psi(q_c(1) - 1, q_c(2) - 1, \ldots, q_c(q_{c+1} - c) - 1)$$

$$- \psi(q_c(q_{c+1} + 1)) - \psi(q_c - 1)$$

$$- \psi(q_c(1) - 2, q_c(2) - 2, \ldots, q_c(q_{c+1} - c) - 2)$$

which by Lemma D.2 is less than or equal to

$$\sum_{i=1}^{c} \left( \frac{q_c(i)}{2} - 1 \right) + \sum_{i=1}^{q_{c+1} - c} \left( \frac{q_c(i) - 1}{2} - 1 \right) - \psi(q_{c+1}(c + 1))$$

$$\leq \left( \frac{n - c}{2} \right) + \left( \frac{q_{c+1} - c}{2} \right) \left( \frac{q_{c+1} - c}{2} - 1 \right) - \left( \frac{q_{c+1} - 2}{2} \right)^2 - 1/4 \right)$$

$$\leq (r - c) + q_c(c - c) \left( \frac{q_c - 3}{2} - 1 \right) - \left( \frac{q_c - 3}{2} - 1 \right)$$

which reduces to

$$r + q_c^2/4 - (c q_c)/2 - 1. \quad (16)$$

Notice here that $q_c \leq 2c$, since otherwise if $q_c > 2c$, then $q_c(c) > 2c(c) \geq 2c^2 = n$, contradicting the fact that $q_c(c) \leq q_c(c) + rem_c = n$. Hence the final expression above is
guaranteed to be less than or equal to $r - 1$, completing the
proof.

---

**APPENDIX F**

**Proof of Lemma VI.11:**
Assume $c \in [n - 1]$ and $c \leq \hat{c}$. The left hand side of
inequality (8) depends on the difference between $q_{c-1}(i)$ and
$q_c(i)$ for each $i \in [c]$. We wish to show that these
differences are sufficiently large to cause inequality (8) to be
satisfied. Note that

$$q_c = \frac{n - rem_c}{c} > \frac{n - c}{c} = \frac{n}{c - 1} > \frac{n}{c^2} (c - 1).$$

Since $c \leq \hat{c}$, we have $n/c^2 = 2c/c^2 \geq 2$, which implies
that $q_c > 2(c - 1)$. Therefore, for all $i \in [c - 1]$, we have
$q_{c-1}(i) - q_c(i) \geq 2$. This is because $q_c$ can be constructed
from $q_{c-1}$ by shortening each cut of $q_{c-1}$ in order to create the

**c**th cut of $q_c$, whose length is at least $q_c - 1$. Example VI.12
helps comprehension here.

Next, let $k := \#i \in [c - 1] : q_{c-1}(i) - q_c(i) > 2$. In other
words, $k$ is the number of cuts in $q_{c-1}$ that are decreased by
more than 2 in the construction of $q_c$ from $q_{c-1}$. Notice that
if $q_{c-1}(i) - q_c(i) = 2$ then by Lemma D.1,

$$\psi(q_{c-1}) - \psi(q_c) = \left( \frac{q_{c-1}(i) - 2}{2} \right)^2 - \left( \frac{q_c(i) - 2}{2} \right)^2.$$

Moreover, Lemma D.1 also implies that in all other instances,

$$\psi(q_{c-1} - \psi(q_c) \geq \left( \frac{q_{c-1}(i) - 2}{2} \right)^2 - \left( \frac{q_c(i) - 2}{2} \right)^2 - \frac{1}{4}$$

Hence the left hand side of inequality (8) is greater than or
equal to

$$\sum_{i=1}^{c-1} \left( \frac{q_{c-1}(i) - 2}{2} \right)^2 - \sum_{i=1}^{c-1} \left( \frac{q_c(i) - 2}{2} \right)^2 - \frac{k}{4} \left( \frac{q_c(c) - 2}{2} \right)^2.$$

(17)

At this point, we split the remainder of the proof into
two possibilities, the general case where $q_c(c) = q_c$ and
the exceptional case where $q_c(c) = q_c - 1$, which only occurs
when $n/c$ is an odd integer. We will treat the general case first
and then end with some comments about the exceptional case.
Since we are first assuming that $q_c(c) = q_c$, expression (17)
is equal to

$$\sum_{i=1}^{c-1} \left( \frac{q_{c-1}(i) - 2}{2} \right)^2 - \sum_{i=1}^{c-1} \left( \frac{q_c(i) - 2}{2} \right)^2 - \frac{k}{4} \left( \frac{q_c(c) - 2}{2} \right)^2$$

$$= \sum_{i=1}^{c-1} \left( \frac{q_{c-1}(i) - 2}{2} \right)^2 - \sum_{i=1}^{c-1} \left( \frac{q_c(i) - 2}{2} \right)^2 - \frac{k}{4} \left( \frac{q_c(c) - 2}{2} \right)^2$$

$$= \sum_{i=1}^{c-1} \left( \frac{q_{c-1}(i) - 2}{2} \right)^2 - \sum_{i=1}^{c-1} \left( \frac{q_c(i) - 2}{2} \right)^2 - \frac{k}{4} \left( \frac{q_c(c) - 2}{2} \right)^2$$

$$= \sum_{i=1}^{c-1} \left( \frac{q_{c-1}(i) - 2}{2} \right)^2 - \sum_{i=1}^{c-1} \left( \frac{q_c(i) - 2}{2} \right)^2 - \frac{k}{4} \left( \frac{q_c(c) - 2}{2} \right)^2$$

(18)

From here, we focus on the summation $\sum_{i=1}^{c-1} \left( \frac{q_{c-1}(i) - q_c(i)}{2} \right)$ to prove that the overall expression is sufficiently large.

The summation can be viewed as the sum of all shaded areas in
Figure 5. In the figure, squares of area $q_{c-1}(i) - q_c(i)$ (with $i \in [c - 1]$), are placed along the diagonal of an $n$-by-$n$ square.

Within the bottom left corner of each of these squares is placed
another square of area $q_c^2(i)$ (again $i \in [c - 1]$). By carefully
examining the total area of all shaded regions in the figure,
we can lower bound $\sum_{i=1}^{c-1} \left( \frac{q_{c-1}(i) - q_c^2(i)}{2} \right)$ to satisfy the
desired inequality.

Figure 6 depicts the difference $q_{c-1}(i) - q_c(i)$ for an
individual $i \in [c - 1]$. This is a closer view of one of the
individual squares along the main diagonal in Figure 5.
From Figure 6, we can observe that the value \(q_c(i)\) in Figure 6. The dimensions of this square are \(n\) in Figure 5 is at least 2. Hence, the sum of the area of all the lightly shaded rectangles is at least \(2q_c\). Moreover, each time the difference between \(q_{c-1}(i)\) and \(q_c(i)\) is greater than 2, this means an overall increase of \((q_{c-1}(i) - q_c(i))^2\) by at least 3. Hence \(\sum_{i=1}^{c-1}(q_{c-1}(i) - q_c(i))^2 \geq 2q_c + 3k\), which implies that expression (18) is greater than or equal to

\[
\frac{q_c^2}{2} + \frac{q_c}{2} + \frac{3k}{4} - \frac{q_c^2}{4} - \frac{k}{4} - 1 = \frac{q_c^2}{4} + \frac{q_c}{2} + \frac{k}{2} - 1.
\]

(19)

To complete the proof in the general case, recall that \(q_c > n/c - 1\). Thus, by replacing \(q_c\) with \(n/c - 1\), we have that the right hand side of equation (19), after some basic reduction, is greater than

\[
\frac{r^2}{c^2} - \frac{1}{4} + \frac{k}{2} - 1.
\]

(20)

In this final expression, since \(c \leq \hat{c}\), we have \(r^2/c^2 \geq r\). Also, since \(q_c\) was strictly greater than \(2(c-1)\), we know that \(k \geq 1\), completing the proof in the general case.

For the exceptional case, when \(q_c(c) = q_c - 1\), we can follow the same argument as in the general case with a slight modification. In this instance the last term in expression (18) is reduced since \(q_c(c) = q_c - 1\) rather than \(q_c\), resulting in a larger overall value. Using this fact, we can then show that whenever \(q_c(c) = q_c - 1\), expression (18) is greater or equal to

\[
\frac{1}{4} \sum_{i=1}^{c-1} (q_{c-1}(i) - q_c(i)) - \frac{k}{4} + \frac{q_c^2}{4} + \frac{q_c}{2} - \frac{5}{4}.
\]

(21)

As in the general case, we now focus on lower bounding the summation \(\sum_{i=1}^{c-1}(q_{c-1}(i) - q_c(i))\), keeping Figures 5 and 6 in mind. Since \(n/c\) is an odd integer, by the definition of \(q_c\), we have \(q_c(1) = q_c + 1\). For all other \(i \in [c-1]\), by definition of \(q_c\) we have \(q_c(i) = q_c\) exactly. Hence for \(i = 1\), the area of each lightly shaded rectangle in Figure 6 is equal to \((q_{c-1}(1) - q_c(1))(q_c + 1)\) and for \(i \in [2, c-1]\), the area is \((q_{c-1}(i) - q_c(i))q_c\). Since \(\sum_{i=1}^{c-1}(q_{c-1}(i) - q_c(i))q_c\), the total area of all of the lightly shaded rectangles yields

\[
2 \left( q_c + 1 \right) (q_{c-1}(1) - q_c(1)) + q_c\sum_{i=2}^{c-1} (q_{c-1}(i) - q_c(i))
\]

\[
= 2 \left( q_c \sum_{i=1}^{c-1} (q_{c-1}(i) - q_c(i)) + q_c(1) - q_c(1) \right)
\]

\[
\geq 2(q_c(q_c - 1) + 2)
\]

\[
= 2q_c^2 - 2q_c + 4.
\]

(22)

Since \(q_{c-1}(i) - q_c(1)\), and \(\sum_{i=1}^{c-1}(q_{c-1}(i) - q_c(i)) = q_c\), by the same argument as in the general case, the sum of the areas of the darkly shaded rectangles is

\[
\sum_{i=1}^{c-1}(q_{c-1}(i) - q_c(i))^2 \geq 2(q_c - 1) + 3k = 2q_c - 2 + 3k.
\]

(23)
Combining Expression (22) and the right hand side of Equation (23), we have that
\[ \sum_{i=1}^{c-1} \left( q_{c-1}^2(i) - q_c^2(i) \right) \geq 2q_c^2 + 3k + 2. \]
Applying this inequality, we have that Expression (21) is greater than or equal to
\[ \frac{1}{4} \left( 2q_c^2 + 3k + 2 \right) - \frac{k}{4} - \frac{q_c^2}{2} + \frac{q_c}{2} - \frac{3}{4} \]
\[ = \frac{q_c^2}{4} + \frac{k}{2} + \frac{q_c}{2} - \frac{3}{4}. \]
(24)
As in the general case, we now apply the fact that \( g_c > n/c - 1 \).
Replacing \( g_c \) with \( n/c - 1 \), after some basic reduction we have that Expression (24) is strictly greater than
\[ \frac{r^2}{c^2} + \frac{k}{2} - 1 > r - 1. \]

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References
[1] A. Barg and A. Mazumdar, “Codes in permutations and error correction for rank modulation,” IEEE Trans. Inf. Theory, vol. 56, no. 7, pp. 3158–3165, Jul. 2010.
[2] T. Berger, F. Jelinek, and J. K. Wolf, “Permutation codes for sources,” IEEE Trans. Inf. Theory, vol. IT-18, no. 1, pp. 160–169, Jan. 1972.
[3] I. F. Blake, G. Cohen, and M. Deza, “Coding with permutations,” Inf. Control, vol. 43, pp. 1–19, Sep. 1979.
[4] S. Buzaglo and T. Etzion, “Perfect permutation codes with the Kendall’s \( \tau \) metric,” in Proc. IEEE Int. Symp. Inf. Theory, Honolulu, HI, USA, Jun. 2014, pp. 2391–2395.
[5] E. E. Gad, E. Yaakobi, A. A. Jiang, and J. Bruck, “Rank-modulation rewrite coding for flash memories,” IEEE Trans. Inf. Theory, vol. 61, no. 8, pp. 4209–4226, Aug. 2015.
[6] F. Farnoud, V. Skachek, and O. Milenkovic, “Error-correction in flash memories via codes in the Ulam metric,” IEEE Trans. Inf. Theory, vol. 59, no. 5, pp. 3003–3020, May 2013.
[7] F. F. Hassanzadeh and O. Milenkovic, “Multipermutation codes in the Ulam metric for nonvolatile memories,” IEEE J. Sel. Areas Commun., vol. 32, no. 5, pp. 919–932, May 2014.
[8] J. Frame, G. de Robinson, and R. M. Thrall, “The hook graphs of the symmetric group,” Can. J. Math., vol. 6, pp. 316–324, Feb. 1954.
[9] W. Fulton, “Young Tableaux,” London Mathematical Society Student Texts, Cambridge, U.K.: Cambridge Univ. Press, 1996.
[10] F. Gölo˘glu, J. Lember, A.-E. Riet, and V. Skachek, “New bounds for permutation codes in Ulam metric,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Jun. 2015, pp. 1726–1730.
[11] M. Hagiwara and J. Kong, “Consolidation for compact constraints and Kendall tau LP decodable permutation codes,” Des., Codes Cryptogr., vol. 85, no. 3, pp. 483–521, Dec. 2017.
[12] A. Jiang, R. Mateescu, M. Schwartz, and J. Bruck, “Rank modulation for flash memories,” IEEE Trans. Inf. Theory, vol. 55, no. 6, pp. 2659–2673, Jun. 2009.
[13] A. Jiang, M. Schwartz, and J. Bruck, “Correcting charge-constrained errors in the rank-modulation scheme,” IEEE Trans. Inf. Theory, vol. 56, no. 5, pp. 2112–2120, May 2010.
[14] J. Kong and M. Hagiwara, “Nonexistence of perfect permutation codes in the Ulam metric,” in Proc. ISITA, Oct. 2016, pp. 691–695.
[15] J. Kong and M. Hagiwara, “Multipermutation Ulam sphere analysis toward characterizing maximal code size,” in Proc. IEEE ISIT, Jun. 2017, pp. 1628–1632.
[16] K. Kobayashi, H. Morita, and M. Hoshi, “Enumeration of permutations classified by length of longest monotone subsequences,” in Proc. IEEE ISIT, Jun. 1994, p. 318.
[17] F. Lim and M. Hagiwara, “Linear programming upper bounds on permutation code sizes from coherent configurations related to the Kendall-tau distance metric,” in Proc. IEEE Int. Symp. Inf. Theory, Jul. 2012, pp. 2998–3002.
[18] C. Schensted, “Longest increasing and decreasing subsequences,” Can. J. Math., vol. 13, pp. 179–191, Feb. 1961.
[19] D. Slepian, “Permutation modulation,” Proc. IEEE, vol. 53, no. 3, pp. 228–236, Mar. 1965.
[20] R. P. Stanley, “Algebraic Combinatorics: Walks, Trees, Tableaux, and More,” New York, NY, USA: Springer, 2013.
[21] A. J. H. Vinck, “Coded modulation for power line communications,” AEU Int. J. Electron. Commun., vol. 54, no. 1, pp. 45–49, Jan. 2000. [Online]. Available: http://arxiv.org/abs/11041528
[22] T. Wadayama and M. Hagiwara, “LP-decodable permutation codes based on linearly constrained permutation matrices,” IEEE Trans. Inf. Theory, vol. 58, no. 8, pp. 5454–5470, Aug. 2012.
[23] X. Wang and F.-W. Fu, “On the snake-in-the-box codes for rank modulation under Kendall’s \( r \)-metric,” Des., Codes Cryptogr., vol. 83, no. 2, pp. 455–465, May 2017.

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