MULTIVARIATE INTERACTIONS MODELING THROUGH THEIR MANIFESTATIONS: LOW DIMENSIONAL MODEL BUILDING VIA THE CUMULANT GENERATING FUNCTION

JHAN RODRÍGUEZ AND ANDRÁS BÁRDÖSSY

Abstract. Growing dimensionality of data calls for beyond-pairwise interactions quantification. Measures of multidimensional interactions quantification are hindered, among others, by two issues: 1. Interpretation difficulties, 2. the curse of dimensionality. We propose to deal with multidimensional interactions by identifying subject-matter specific interaction manifestations and then building a low-dimensional model that reproduces as close as possible such manifestations. We argue that an adequate model building approach is to build the model in the form of a cumulant generating function, i.e. to use joint cumulants as building blocks. The whole approach resembles that of probability inversion in the area of expert knowledge based risk assessment, where a discrimination is made between “elicitation” variables, familiar to the experts, and “target” (or model) variables, consisting of the more abstract parameters of a mathematical model. A synthetic example is provided to illustrate these ideas.

Introduction

Technological innovation has lead to a world full of data of an increasingly growing dimension. These data in turn contain information, the extraction of which is a basic task of Statistics (c.f. Lindsay et al. (2004)). An important type of information is the kind of interdependence among variables being represented by data. This calls for statistical means of extracting, quantifying and, if possible, modeling such interdependence. At the very least, coefficients that somehow summarize the type and intensity of multivariate interdependence are very desirable in applied science.

The introduction of the correlation concept by Francis Galton (1822-1911) had a tremendous impact on many sciences due to its straightforward interpretation as a measure of “partial causation” or “average association”, as summarized in a single parameter. However elementary this concept now may seem, it was welcomed as an important scientific contribution at the end of the XIX century. As Pearson (2011) writes, it did “open to quantitative analysis wide fields of medical, psychological and social research [...] [it] was to replace not only in the minds of many of us the old category of causation, but deeply to influence our outlook on the universe”.

Karl Pearson would develop the original correlation coefficient of Galton into the widely used product-moment correlation coefficient. Given the first paradigmatic step, new implementations of the concept would appear, in the form of other association coefficients more adequate for specific applications in psychology and the social sciences: Spearman’s $\rho$, Kendall’s $\tau$, Ginni’s $\gamma$, Blonqvist’s $q$, etc. (The reader is referred to Joe (1989) for more coefficients). These are coefficients intended to represent the degree of association between two random variables.
Later on, Rényi (1959) would attempt to give some mathematical rigor to the concept of dependence, providing “seven rather natural postulates which should be fulfilled by a suitable measure of dependence”. Rényi’s work would be revised by Schweizer and Wolff (1981), who made some “reasonable modifications” to the postulates, since they were found to be too restrictive. Additionally, Schweizer and Wolff (1981) used the concept of copulas to introduce a number of measures of pair-wise dependence which fulfilled their new postulates. With these conceptual tools (i.e. a set of reasonable postulates and the unifying concept of copulas), Wolff (1980) extends the measures of dependence between two variables given by Schweizer and Wolff (1981), and proposes an extension to more than two variables of Spearman’s $\rho$. This course of action has been further followed and developed by Schmid et al. (2010). Indeed, Schmid et al. (2010) introduce a series of measures that can be considered as extension to more that two variables of some of the well-established, pair-wise measures of dependence mentioned above.

Another course of action, traceable back to Linfoot (1957), is to use entropy or mutual information as association coefficient. Joe (1989) proposes a number of measures of this type that apply to more than two variables; Peña and Linde (2007) introduce a measure which adjusts itself to dimension, so as to compare the intensity of association of two vectors of different dimensions. Micheas and Zografos (2006) deal with the general case of $\phi$-dependence, of which mutual information is one particular case. The intensity of association is measured by Micheas and Zografos (2006) in terms of the deviance of the joint distribution from the distribution given by the product of the marginal distributions (the independence case). The specific definition of deviance depends on the specific selection of function $\phi : [0, +\infty) \to \mathbb{R}$, which is continuous and convex, satisfying some basic conditions.

Apart from their theoretical interest, measures of dependence for more than two variables are required and sought in applied research. In the area of neuronal science, an influential theory of behavior introduced by Hebb (1949) suggests that “fundamental insight into the nature of neuronal computation requires the understanding of the cooperative dynamics of populations of neurons” (Grün and Rotter (2010), chapter 12), and further evidence in the course of the years has lead brain theorists to build models that “rely on groups of neurons, rather than single nerve cells, as the functional building blocks for representation and processing of information” (Grün and Rotter (2010), preface); this has lead to the development of techniques to quantify beyond pair-wise association in that research area. Concerning applied atmospheric research, Bárdossy and Pegram (2009) in the context of daily precipitation modeling, and Bárdossy and Pegram (2012) in the context of downscaling, have found evidence that explicit quantification of interactions among more than two variables, and their proper incorporation into modeling and forecasting, may be of an importance hitherto unexplored: predictions based on statistical models can be otherwise severely biased, particularly for very high (or extreme) values of the multivariate process modeled. More recently, Rodríguez and Bárdossy (2013) investigate the consequences for inference of ignoring multivariate interdependence in the context of Spatial Statistics, and propose a model that can deal with this type of interdependence explicitly. The present paper comprises the theoretical basis for the work of Rodríguez and Bárdossy (2013). In the area of finance, “herding” behavior, the degree to which several economic actors behave as a herd (Dhaene et al. (2012)), doing basically the same thing, is important for estimation
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of loss risks: If a single underlying factor or small number of factors are inducing a high degree of herd behavior, financial assets practically independent or very loosely correlated with each other can interact en bloc, rendering portfolio diversification ineffective. As Dhaene et al. (2012) indicate, pair-wise correlations, or a measure based on these, may be misleading in this case. Dhaene et al. (2013) present a related interdependence measure for aggregating risks.

In a recent paper, Reimherr and Nicolae (2013) note that most of the theory on measures of association has left out the important issue of interpretability of the measures for the research at hand. These authors argue (correctly, in our opinion) that the lack of interpretability limits their use as summary tools. This problem is greatly exacerbated if what we intend to quantify or represent is the association among several variables. In a more general manner, the interpretability of parameters and coefficients of a statistical model has been considered an important characteristic of the model by Cox et al. (1995).

This paper deals with some of the issues inherent in formulating interaction coefficients that pertain to more than two variables. We propose an approach for dealing with these issues. Section 1 introduces some issues that one encounters when dealing with measures of interaction for more than two variables. Section 2 states the approach we suggest for dealing with these issues: to discriminate between interaction “parameters” and interaction “manifestations”. We illustrate what we mean by the names interaction parameter and interaction manifestation. Section 3 introduces joint cumulants and Lancaster Interactions. The relation between the two is exhibited, and a justification of joint cumulants as legitimate extensions to covariance coefficients indicated. Section 4 exhibits the relation between joint cumulants and some illustrative interaction manifestations, as defined in this paper. Section 5 illustrates the ideas presented, in that a specific model is introduced, and the ideas of this paper applied to simulated data. In section 6 a discussion of the results is provided.

1. DIFFICULTIES OF DEFINING A MEASURE OF MULTIVARIATE INTERACTION

1.1. Interpretability. For a two dimensional dataset, interpretability of a dependence coefficient is aided by the possibility of plotting the data. One looks at several datasets and computes the respective coefficient of dependence. After many such data sets, one has an idea of what, say, a correlation coefficient with a value of $-0.8$ stands for. This visual aid is still possible for three dimensional datasets, but is not available for higher dimensions. Assuming we have a coefficient of interdependence, $\lambda$, applicable to multivariate vectors; how is one supposed to interpret a value of $\lambda(X_1, X_2, X_3, X_4) = -0.8$? Can one visualize a dataset producing such a coefficient, so as to relate it to the phenomenon one is investigating?

It has been claimed that major advances in the science of statistics usually occur as a result of the theory-practice interaction (Box (1976)), and that the parameters of a model should have clear subject-matter interpretations (Cox et al. (1995)). These statements suggest that interaction parameters as mere abstract constructions will not find much application in statistical modeling, unless one can “paraphrase” their meaning and relate them to the problem at hand.

Our approach to interaction quantification and modeling consists in discriminating between interaction manifestations and interaction parameters. So, we can focus on quantification and modeling of what really interests us about dependence
in data (i.e. its subject-matter relevant manifestation), while trying to reproduce such manifestations with as few parameters as possible.

Relation to the “probability inversion” technique in Probabilistic Risk Assessment. An analogous approach has found successful application in the area of probabilistic risk assessment (Bedford and Cooke (2001)). With the aid of mathematical models, it is often possible to predict (approximately) what the consequences of a given event may be. This mathematical model has parameters that ideally should be calibrated on the basis of past data. However, absence of data for certain events (e.g. a nuclear accident in a given region) makes the use of expert knowledge necessary, whereby the model parameters are to be estimated on the basis of the experience of a group of experts. Experts usually cannot give an adequate direct evaluation of the joint probability distribution of the model parameters, or target variables. Hence each expert is asked to express his uncertainty judgments in terms of elicitation variables, i.e. observable quantities within the area of his/her expertise. A target variable-set for the model is then recovered, such that the elicitation variables produced by the mathematical model look as similar as possible like the elicited variables provided by the experts. This is an inverse problem, labeled “probabilistic inversion”. The interested reader is referred for more details to (Bedford and Cooke (2001); Du et al. (2006)) and the references therein.

We suggest in section 2 a course of action that is analogous to “probabilistic inversion” for the problem of interactions quantification and modeling.

1.2. High parametric dimensionality. A second issue when defining an interaction coefficient, is the issue of high dimensionality. As dimension of the random vector under analysis increases, a naive use of interaction coefficients becomes prohibiting. For example, the correlation matrix of a 10-dimensional random vector is an array having 45 correlation coefficients. Assume symmetry on the variables with respect to the association coefficient (i.e. the order of the variables plays no role on the coefficient’s value): If, for the same 10-dimensional vector, one intends to consider 3-wise, 4-wise and 5-wise "correlation coefficients", the corresponding arrays would have 450, 4500, and 45000 coefficients, respectively.

Hence, it is necessary to be able to select judiciously the interaction parameters with which to work, and impose reasonable constraints on them.

Another aspect that can be considered a sort of “curse” of dimensionality, is the coefficient of interdependence to use: there are too many features that multivariate datasets can exhibit.

In the one-dimensional case, parameters such as mean, standard deviation, skewness and kurtosis (basically, the first four cumulants) give a lot of information about the distribution of data, provided these data come from an unimodal distribution. Those parameters (mean, skewness coefficient, etc.) describe data to some extent, since they can be readily connected to specific questions about data: the location of data, how informative this location about data is, how symmetric the distribution is, to what extend can one expect values very far away from the mean. As a reference one may have in mind these characteristics for the normal distribution.

But as dimension grows, one must focus on that feature of data interaction which is most connected with the research questions at hand, rather than on an abstract dependence coefficient.
1.2.1. How high dimensionality is dealt with in the realm of Spatial Statistics. We present now an example of how the issue of high dimensionality has been addressed in the context of Spatial Statistics. This will give us a basis method from which to generalize.

In the area of spatial statistics (see, for example, Cressie (1991); Cressie and Wikle (2011); Diggle and Ribeiro (2007)), the studied random vector \( X \in \mathbb{R}^J \) spans hundreds or thousands of components, each of which component represents the value of an environmental process at a given location \( j = 1, \ldots, J \). The way high dimensionality is addressed in spatial statistics is an apt introduction for the method we advocate in this paper. We give here a very basic form of a spatial statistical model, since it suffices for our introducing purposes.

One focuses on the correlation between every two components of \( X \). The covariance among every two components, \((X_i, X_j)\), of \( X \), is expressed as a function \( \text{Cov}(d) \) of the distance between the locations represented by these two components, \( d \geq 0 \). The covariance function \( \text{Cov}(d) \) must be such that the resulting covariance matrix is positive definite. To this end there are a number of covariance functions often used in practice, for example, one popular covariance function is the powered exponential one,

\[
\text{Cov}(d) = \sigma_0^2 I(d = 0) + \sigma_1^2 \exp \left(-\frac{d}{\theta_1}\right)^\theta_2
\]

where \( \theta_1 > 0, 0 < \theta_2 \leq 2, \sigma_0^2 \geq 0, \sigma_1^2 \geq 0 \) are the covariance function parameters.

Note that:

1. Function (1.1) allows to have the covariance between every two components of \( X \) as a function of the distance between the locations these components represent, and only 4 additional parameters. In this way, the whole dependence structure of \( X \in \mathbb{R}^J \) (with \( J >> 2 \)) is low dimensionally obtained, built on the basis of 2-dimensional dependence coefficients.

2. The interesting dependence manifestation to recover is covariance between every two components of \( X \), whereas the (interaction) parameters to estimate are the function parameters, \( \theta_1, \theta_2, \sigma_0^2 \) and \( \sigma_1^2 \). This is entirely analogous to the probability inversion technique mentioned in section 1.1; covariance takes the place of the elicitation variables, whereas the covariance function parameters are the target variables.

3. There is a functional relation between \( \theta_1, \theta_2, \sigma_0^2, \sigma_1^2 \) and the dependence manifestation. Covariance can be written in terms of the (interaction) parameters, \( \theta_1, \theta_2, \sigma_0^2 \) and \( \sigma_1^2 \).

Items 1 through 3 summarize a technique to tackle the problem of high dimensionality in an ingenious low-dimensional way. The issue of interpretability goes relatively unnoticed, since in this case parameters have a relatively straightforward interpretation: \( \sigma_0^2 \) represents a micro scale variability of the environmental process; \( \sigma_0^2 + \sigma_1^2 \) represents the variance of the marginal distribution of each component \( X_j \) of \( X \); \( \theta_1 \) (often called “range”) represent the distance at which correlation between data from two locations is relatively insignificant. The parameter \( \theta_2 \) might even receive a suitable interpretation, depending on the context.

In the next section, this approach is extended to deal with the interdependence among more than two variables at a time, keeping basically the same ideas.
2. Interaction parameters versus interaction manifestations

The approach we advocate in this paper can be summarized as follows: first select an interaction “manifestation” relevant for the research in question. Then fit (low-dimensional) interactions “parameters” that make the fitted distribution reproduce, as close as possible, the observed interaction manifestation. In this way, we circumvent the issues of interpretability and high dimensionality mentioned above.

By interaction manifestation, we mean any function of more than one component of the random vector analyzed, $\mathbf{X} \in \mathbb{R}^J$, which can be interpreted as relevant for the research objectives at hand. For the sake of illustration:

1. The distribution of the sum of subsets of components of a random vector. In the context of financial analysis, this sum is readily interpreted as “risk” (see also section 3 below).
2. The joint distribution of subsets of components, or the probability of trespassing simultaneously a threshold defined for each component. This is useful in many applications. For example, in the context of series systems reliability, such trespassing probability is the probability of “failure”.
3. Differential entropy, any information-based dependence measure, or any of the copula-based generalizations to correlation measures studied by Schmid et al. (2010), of subsets of components. Depending on the specific research carried out, these may have subject-matter interpretations, or can readily provide the versed researcher of a specific area with a summary picture of the dependence in the data.

Interaction manifestations are interesting for the problem at hand, we would like our model to reproduce them properly. But they are not very helpful for building a model that integrates them, let alone a low-dimensional model. If we had interaction parameters or coefficients which:

1. Provide us with an idea of the number of variables interacting within the random vector analyzed, $\mathbf{X} \in \mathbb{R}^J$.
2. Can be somehow (functionally) connected with the interaction manifestations that are interesting for the research carried out.
3. Can be built into a parametric or semi-parametric model. This would immediately open up the possibility of a low-dimensional model, via a judicious selection of assumptions and/or constraints on the interaction parameters.

Then we could proceed, in the manner of an inverse problem, as follows:

1. We find data-based estimates or approximations to the interesting interaction manifestations
2. We fit the interactions parameters so as to match best the observed interaction manifestations

In the next section, we introduce a reasonable interaction measure, and through it, a reasonable type of interaction parameter with which one can work along the lines above; namely the joint cumulant. We claim that using joint cumulants as building blocks of a multivariate statistical model allows for an adequate consideration of dependence, both of pairs of variables, and of groups of more variables.

It might be argued that moments (and hence cumulants) of sufficiently high orders might not exist for the “true” probability distribution of the process under analysis. We would answer that such distributions can always be sufficiently (i.e.
for practical purposes) approximated by a distribution with existing moments of all orders. See, for example Gallant and Nychka (1987), where the authors introduce a semi-parametric model, similar to an Edgeworth expansion. This model possesses moments of all orders. Yet, under minimal conditions it can approximate any continuous distribution on $\mathbb{R}^J$, provided sufficiently many factors are added to the sum defining the model. Additionally, Del Brio et al. (2009); Mauleon and Perote (2000); Perote (2004) present variants of the model of Gallant and Nychka (1987), and show how they can be effectively applied to modeling heavy tailed data, both univariate and multivariate.

3. The Lancaster Interaction Measure and Joint Cumulants

In this section, the connection between the Lancaster Interaction measure of a random variable and its joint cumulants is established. To our knowledge, this connection has not been pointed out before as a justification of joint cumulants as reasonable interdependence parameters.

3.1. A review of Lancaster Interactions. We review now the function called “additive interaction measure” or “Lancaster interaction measure”, introduced by Lancaster (1969) and later modified by Streitberg (1990). This function can be built for every random vector $X \in \mathbb{R}^J$, and has the property of being identically zero if any sub-vector of $X$ is independent of the other. An additive interaction measure $\Delta F(X)$ is a signed measure determined by a given distribution $F(X)$ on $\mathbb{R}^J$. Its defining characteristic is that it is equal to zero for all $X \in \mathbb{R}^J$, if $F(X)$ can be written as the non-trivial product of two or more of its (multivariate) marginal distributions (Streitberg 1990). For example, if $J = 4$ and $F$ can be written as $F_{124}F_3$, being $F_{124}$ and $F_3$ the marginal distributions of $(X_1, X_2, X_4)$ and $X_3$, respectively, then $\Delta F(X) \equiv 0$, for all $X \in \mathbb{R}^J$.

An alternative explanation is that $\Delta F \equiv 0$, if one subset of $X$’s components is independent of another subset of components. If $\Delta F \equiv 0$, then $F$ is said to be "decomposable".

Lancaster Interaction measure is defined by
\begin{equation}
\Delta F(X) = \sum_{\pi} \left\{ \left( (-1)^{|\pi|-1} (|\pi| - 1)! \right) F_\pi(X) \right\}
\end{equation}

where the sum is over all partitions, $\pi$, of index set $C = \{1, \ldots, J\}$.

An example will help clarify the notation: for index set $C = \{1, 2, 3, 4\}$ there are 15 partitions, three of which are: $\pi_1 = \{1\}, \{2\}, \{3, 4\}$, $\pi_2 = \{1, 4\}, \{2, 3\}$, $\pi_3 = \{1, 2, 3, 4\}$. Their cardinalities are $|\pi_1| = 3$, $|\pi_2| = 2$ and $|\pi_3| = 1$, respectively. In general, a set of $J$ elements has a total of $B_J$ possible partitions where $B_0 = B_1 = 1$ and any subsequent $B_{k>1}$ can be found (see e.g. Rota (1964)) by the recurrence relation $B_{k+1} = \sum_{r=0}^{k} \binom{k}{r} B_r$. The reader is referred to the textbook of Aigner (2006) for more on partitions and their enumeration.

The symbol $F_\pi$ is further to be interpreted as
\begin{equation}
F_\pi(X) = F_1(X_1) F_2(X_2) F_{34}(X_3, X_4)
\end{equation}

that is, the product of the (multivariate) marginal distributions defined by partition $\pi_1$. The same explanation holds at (3.1) for any of the $B_J$ partitions, $\pi$, of index set $C = \{1, \ldots, J\}$.

\footnote{The number $B_J$ is often called Bell’s number.}
It will be convenient to define partition operator $J_{\pi}$, to be applied to $F$ for a given partition $\pi$, by

\[(3.3) \quad J_{\pi}F \rightarrow F_{\pi}\]

where $F_{\pi}$ is as in the example at equation (3.2).

Streitberg (1990, 1999) shows an important result concerning $\Delta F$: given a probability distribution function $F$, function $\Delta F$ as in (3.1) is the only function built as a linear combination of products of (multivariate) marginal distributions of $F$, such that $\Delta F (X) := 0$, whenever one subset of $X$'s components is independent of another components subset.

Since the interaction measure is defined in terms of a given distribution $F$, we can define the interaction operator:

\[(3.4) \quad \Delta = \sum_{\pi} \left\{ \left( -1 \right)^{|\pi| - 1} \left( |\pi| - 1 \right)! \right\} J_{\pi}\]

which, upon application to the distribution in question, returns the additive interaction measure.

3.2. A review of Joint Cumulants. Moments and cumulants can be defined as constants summarizing important information about a probability distribution and sometimes, even determining it completely (cf. Kendall and Stuart (1969)). In this section we deal with random variables having a probability density function. The development is also valid for discreet distributions, under simple modifications.

The reader is referred to Kendall and Stuart (1969); Muirhead (1982); Billingsley (1986); McCullagh (1987) for more details on moments and cumulants.

The Cumulant Generating Function (c.g.f.), $K_X(t)$, of a random vector, $X \in \mathbb{R}^J$, is defined as the logarithm of the moment generating function (m.g.f.),

\[(3.5) \quad K_X(t) = \log (M_X(t)) = E \left( \exp \left( \sum_{j=1}^{J} t_j X_j \right) \right)\]

where $t \in \mathbb{R}^J$, assuming these functions exist.

Joint cumulants are then defined to be the coefficients of the Taylor expansion for $K_X(t)$,

\[(3.6) \quad K_X(t) \sim \sum_{r_1=0}^{\infty} \cdots \sum_{r_J=0}^{\infty} \frac{\kappa_{r_1, \ldots, r_J} t_1^{r_1} \cdots t_J^{r_J}}{r_1! \cdots r_J!}\]

and hence can be found by differentiating $K_X(t)$ and evaluating at $t = 0$,

\[(3.7) \quad \kappa_{r_1, \ldots, r_J} = \frac{\partial^{r_1 + \cdots + r_J}}{\partial t_1^{r_1} \cdots \partial t_J^{r_J}} K_X(t) \big|_{t=0}\]

where $r_j \geq 0$ is a non-negative integer. An important particular case is the covariance coefficient, or second order joint cumulant,

\[\frac{\partial^2}{\partial t_i \partial t_j} K_X(t_i, t_j) \big|_{(t_i, t_j)=(0,0)} = \text{cov} (X_i, X_j)\]

The c.g.f. of a sub-vector $Y = (X_{j_1}, \ldots, X_{j_k})$, with indexes in an index set, $j_i \in I$, can be readily found in terms of that of $X$, by setting the indexes not corresponding to $Y$ to zero:
\[ K_Y(s) = \left( E \left( \exp \left( \sum_{i=1}^{k} s_i X_{j_i} \right) \right) \right) = \log \left( \left( E \left( \exp \left( \sum_{j=1}^{j} g_j(s) X_j \right) \right) \right) \right) = K_X(g(s)) \]

where \( g : \mathbb{R}^k \to \mathbb{R}^J \), and
\[ g_j(s) = \begin{cases} 1, & j \in I \\ 0, & j \notin I \end{cases} \]

An alternative definition for joint cumulants uses product moments as departing point (see, for example, Brillinger (1974)). Let \( X \in \mathbb{R}^J \) be a random vector. For a set \( (X_{j_1}, \ldots, X_{j_d}) \) of \( X \)'s components, where some sub-indexes \( j_r \) may be repeated, consider joint moments
\[ E(X_{j_1} \ldots X_{j_d}) \]

Consider partition operator \( J_\pi^* \), analogous to (3.3), related to each partition \( \pi \) of \( (j_1, \ldots, j_d) \). This operator converts \( E(X_{j_1} \ldots X_{j_d}) \) into the product of the factors determined by partition \( \pi \).

For example, for \( d = 4 \), \( (j_1, j_2, j_3, j_4) \) and \( \pi = \{\{1\}, \{2, 3\}, \{4\}\} \), one has partition components \( v_1 = \{1\} \), \( v_2 = \{2, 3\} \) and \( v_3 = \{4\} \). Upon application of \( J_\pi^* \), we have,
\[ J_\pi^* E(X_{j_1} \ldots X_{j_4}) = E(X_{j_1}) E(X_{j_2} X_{j_3}) E(X_{j_3}) \]

In the general case
\[ J_\pi^* E(X_{j_1} \ldots X_{j_d}) = \prod_{v \in \pi} E \left( \prod_{j_r \in v} X_{j_r} \right) \]

The alternative definition of joint cumulants can now be given.

For random variables \( (X_{j_1}, \ldots, X_{j_d}) \), their joint cumulant of order \( d \) is given by,
\[ \text{cum}(X_{j_1}, \ldots, X_{j_d}) := \sum_{\pi} \left\{ (-1)^{|\pi|-1} (|\pi|-1)! \right\} J_\pi^* E(X_{j_1} \ldots X_{j_d}) \]

Two examples are:
\[ \text{cum}(X_1, X_2) = E(X_1 X_2) - E(X_1) E(X_2) \]

and
\[ \text{cum}(X_1, X_2, X_3) = E(X_1 X_2 X_3) - E(X_1 X_2) E(X_3) - E(X_1 X_3) E(X_2) - E(X_2 X_3) E(X_1) + 2E(X_1) E(X_2) E(X_3) \]

Hence joint cumulants can be seen, from a merely formalistic point of view, to form a kind of higher order covariance coefficient. The second order joint cumulant is just the typical covariance coefficient.

3.3. Relationship between Lancaster Interactions and Joint Cumulants. The similarity between (3.1) and (3.5) is evident. Indeed, if we concentrate for now
on the case $X \in \mathbb{R}^2$, then [Lehmann (1966)] reports that:

\begin{equation}
(3.9) \quad \text{Cov} (X_1, X_2) = \text{cum} (X_1, X_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [F_{12} (x_1, x_2) - F_1 (x_1) F_2 (x_2)] \, dx_1 \, dx_2
\end{equation}

under the condition that $E \left( \left| X_{1k}^k, X_{2k}^k \right| \right) < +\infty$, for $k_j = 0, 1$.

This equation is often called "Hoeffding’s formula" since it was first discovered by [Hoeffding (1940)]. Of course, the above equation can be written in terms of the Lancaster interaction measure (3.1), as

\begin{equation}
(3.10) \quad \text{cum} (X_1, X_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Delta F (x_1, x_2) \, dx_1 \, dx_2
\end{equation}

It turns out that this equation can be extended to higher dimensions. Let $X \in \mathbb{R}^J$ be a random vector. As shown by [Block and Fang (1988)], we have that (page 1808):

\begin{equation}
(3.11) \quad \text{cum} (X) = (-1)^J \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \sum_{\pi} \left\{ \left( (-1)^{|\pi| - 1} (|\pi| - 1)! \right) F_{\pi} \right\} \, dX
\end{equation}

under the condition that $E \left( \left| X^j_j \right| \right) < +\infty$, for $j = 1, \ldots, J$. Again, this is the same as saying that

\begin{equation}
(3.12) \quad \text{cum} (X) = (-1)^J \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \Delta F (X) \, dX
\end{equation}

Thus, joint cumulants are equal (up to a known constant) to the integral of Lancaster Interaction measure; they are “summary” or “integral” measures of additive interaction. To our knowledge, this connection had not been pointed out elsewhere.

It goes without much explanation that the joint cumulants of a random vector $X$ vanish whenever a subset of the vector is independent of another, since then the integrating function is identically zero. This property is well-known and oftentimes the reason why joint cumulants are used in practice (e.g. in [Brillinger (1974); Mendel (1991)]). The inverse is true only if the distribution of $X$ is determined by its moments, which may or may not be a reasonable assumption, depending on the application. Again, based on the work of [Gallant and Nychka (1987); Perote (2004); Maulen and Perote (2000); Del Brio et al. (2009), we argue that this is not an extreme limitation to our approach, since all we are seeking is a good approximation to the distribution under analysis.

In particular, whenever we have $\text{cum} (X_{j_1}, \ldots, X_{j_d}) \neq 0$, where no index $j_k$ is repeated, this means that one cannot decompose the distribution of $(X_{j_1}, \ldots, X_{j_d})$: At least $d$ variables within $X$ are interacting simultaneously with each other.

Our theoretical contribution here is that joint cumulants are seen as the integral of the Lancaster interaction measure. As shown by [Streitberg (1990)], $\Delta F$ is the only additive measure, built very elementarily with the marginal distributions of the random vector, which vanishes whenever one subset of $X$’s components is independent of another subset of components.
We have provided a theoretical basis for declaring joint cumulants “interaction parameters”, and the cumulant generating function a “dependence structure”. The functional character of the c.g.f. opens up the possibility of parametric modeling, with its respective low-dimensionality advantage. It is just another way of defining a model, alternative to the density specification.

We shall see below, how the parameters of a model expressed as a c.g.f. can be connected with some interesting interaction manifestations.

4. Interaction manifestations in terms of interaction parameters

The connection between interaction parameters (i.e. joint cumulants) and interaction manifestations relies on the concepts of the Edgeworth expansion and the saddlepoint approximation to the density of a random vector. A brief review of these topics is provided at the appendix.

4.1. Connection of dependence structure with interaction manifestations.

We shall show explicitly the connection of joint cumulants and the c.g.f. with three of the interaction manifestations listed at section 2, which manifestations refer to subsets of components, \((X_{j_1}, \ldots, X_{j_k})\), \(1 \leq k \leq J\), of the random vector \(X \in \mathbb{R}^J\). Namely: the distribution of the sum of components; parameters related to the joint probability of the components; and the differential entropy of the components.

A relevant point here is that, except for the distribution of the sum of components, even with a lot of data at hand, estimation of the interaction manifestations mentioned can be done only for (multivariate) marginals of relatively low dimension, such as \(k\) equal to 3, 4 or 5. But armed with a sensible c.g.f., we can consistently integrate these manifestations into the whole distribution (in much the same way as thousand of covariance coefficients are integrated into a Spatial Statistics model that spans thousands of variables). This we can attain with the aid of the overarching dependence structure, that is, the c.g.f.

Assume for the moment you have a reasonable type of c.g.f., that is, one that seems reasonable for the problem at hand (for an illustration see section 5).

4.1.1. Connection of dependence structure with Sums of components. Given a random vector \(X \in \mathbb{R}^J\) representing the variables under analysis, we are interested in the distribution of variable \(S_X = \sum_{i=1}^{k} X_{j_i}\), where \((X_{j_1}, \ldots, X_{j_k})\), \(1 \leq k \leq J\), is a sub-vector of the random vector \(X \in \mathbb{R}^J\). The distribution of \(S_X\) is the interaction manifestation we in which we are interested. We want to fit the distribution of the whole vector, \(X \in \mathbb{R}^J\), in such a way the we fit this interaction manifestation properly.

One course of action is to find the cumulants of \(S_X\) in terms of the joint cumulants of \(X\), and then approximate the density of \(S_X\), by using the Edgeworth Expansion. Since \(S_X\) is a one-dimensional random variable, one can alternatively find research-relevant quantiles of its distribution by inverting the Edgeworth Expansion, i.e. by using the Cornish-Fisher inversion.

To find the cumulants of \(S_X\), note that two of the properties of joint cumulants are [Brillinger 1974]: symmetry and multi-linearity. Symmetry means that \(\text{cum}(X_{j_1}, \ldots, X_{j_k}) = \text{cum}(P(X_{j_1}, \ldots, X_{j_k}))\) for any permutation \(P(j_1, \ldots, j_k)\) of the indexes \((j_1, \ldots, j_k)\). Concerning multi-linearity, for any random variable \(Z \in \mathbb{R}\), one has

\[
\text{cum}(Z + X_{j_1}, \ldots, X_{j_k}) = \text{cum}(Z, \ldots, X_{j_k}) + \text{cum}(X_{j_1}, \ldots, X_{j_k})
\]
Combining these two properties, it can be shown that

\[(4.1) \quad \kappa_r (S_X) = \text{cum} \left( \sum_{i=1}^{k} \sum_{i_2=1}^{k} \ldots \sum_{i_r=1}^{k} \text{cum} \left( X_{j_{i_1}}, \ldots, X_{j_{i_r}} \right) \right) \]

where \( \kappa_r (S_X) \) denotes the \( r \)-th cumulant of random variable \( S_X = \sum_{i=1}^{k} X_{j_i} \). Then the interesting quantiles of \( S_X \) can be (approximately) written in terms of the \( \kappa_r \) via the Cornish-Fisher inversion.

As the dimension \( k \) of the sub-vector increases, this approach becomes impractical, since the sum at (4.1) comprises too many elements. Fortunately, knowing the c.g.f. of \( X \) tells much about the c.g.f. of sums of its components.

A second course of action uses all the information provided by the c.g.f. and is now given.

In a somewhat more general context as before, consider a random vector \( X = (X_1, \ldots, X_J) \). One wishes to study the joint distribution of aggregated variables of the form:

\[(4.2) \quad \xi_1 = \sum_{j_1 \in I_1} X_{j_1} \]
\[(4.3) \quad \xi_2 = \sum_{j_2 \in I_2} X_{j_2} \]
\[\vdots \quad \vdots \]
\[(4.4) \quad \xi_l = \sum_{j_l \in I_l} X_{j_l} \]

where \( I_k \), for \( k = 1, \ldots, l \) represent non-overlapping index sets such that

\[I_1 \cup \ldots \cup I_l = \{1, \ldots, J\} \]

(Note that \( S_X \) above is the specific case in which \( I_1 = \{1, \ldots, J\} \).

The cumulant generating function of the \( l \)-dimensional vector so obtained is given by

\[(4.4) \quad K_\xi (t) = \log \left( E \left( \exp \left( t \cdot \xi \right) \right) \right) = \log \left( E \left( \exp \left( t_1 \xi_1 + \ldots + t_l \xi_l \right) \right) \right) = \log \left( E \left( \exp \left( t_1 \sum_{i=1}^{I_1} X_{j_i} + \ldots + t_l \sum_{i=1}^{I_l} X_{j_i} \right) \right) \right) = \log \left( E \left( \exp \left( g_1 (t) X_1 + \ldots + g_J (t) X_J \right) \right) \right) = \log \left( E \left( \exp \left( g(t) \cdot X \right) \right) \right) = K_X (g(t)) \]

Function \( g : \mathbb{R}^l \to \mathbb{R}^J \) is a vector function defined by

\[(4.5) \quad g(t) = (g_1(t), \ldots, g_J(t)) \]
\[(4.5) \quad g_j(t) = t \cdot (1 (j \in I_1), \ldots, 1 (j \in I_l)) \]
where
\[ 1\left(j \in I_k\right) = \begin{cases} 1, & j \in I_k \\ 0, & j \notin I_k \end{cases} \]

It is hence possible to find the cumulant generating function of random vector \( \xi \in \mathbb{R}^l \) in terms of that of the original vector \( X \in \mathbb{R}^J \). If we know the c.g.f. of the original random vector \( X \), then the cumulants, the cumulant generating function, and hence the approximate joint density of the aggregated variables, via Saddlepoint approximation at (A.5) of \( \xi \in \mathbb{R}^l \) are also determined (see section 5). We can use this fact in order to fit the model for \( X \) in such a way that the interesting interaction manifestation (the sums of components) are explicitly considered in the estimation.

4.1.2. Joint probabilities of (multivariate) marginals. Joint marginal distributions are usually important interaction manifestations. Given a sub-vector \( Y := (X_{j_1}, \ldots, X_{j_k}) \) of \( X \), in order to find probabilities of the form
\[ \Pr (X_{j_1} \geq x_{j_1}, \ldots, X_{j_k} \geq x_{j_k}) \]
one should in principle integrate expression (A.5), for the c.f.g. of \( Y \).

In the univariate case, it is a well-established practice Huzurbazar (1999) to employ instead an accurate approximation to that integral, which is due to Lugannani and Rice (1980). Namely, in the univariate case, we have:

\[
F_X(x_0) \approx \exp \left( \frac{1}{2} \left[ \frac{d^2 K_X(\lambda)}{d\lambda^2} \right]_{\lambda = \hat{\lambda}(x)} \right)^{1/2} dx
\approx \Phi (r) + \varphi (r) \left\{ \frac{1}{r} - \frac{1}{q} \right\}
\]

Where \( \hat{\tau} \) is such that \( K'_X(\hat{\tau}) = x_0 \), and:
\[
\begin{align*}
    r & = \text{sign} (\hat{\tau}) \{ 2 [\hat{\tau} x_0 - K_X(\hat{\tau})] \}^{1/2} \\
    q & = \hat{\tau} \left\{ \frac{d^2 K_X(\lambda)}{d\lambda^2} \right\}^{1/2}_{\lambda = \hat{\tau}}
\end{align*}
\]

Thus, one must not perform the numerical integration at all.

For the multivariate case, Kolassa and Li (2010) have provided a generalization of the Lugannani-Rice formula, which produces an approximation to probability \( \Pr (Y \geq y) \) of order \( O(n^{-1}) \), for \( X \in \mathbb{R}^J \). This formula is extremely complicated and writing it here will most likely obscure rather than clarify anything. Only the probability distribution function of a multivariate Normal distribution with covariance matrix given by
\[
\Gamma_{ij} = \frac{\partial^2}{\partial t_i \partial t_j} K_X(t) \big|_{t=0}
\]
must be computed. For this task there are accurate methods available for up to 20 dimensions Genz (1993).

If one intends to deal with vectors of dimension at most 5, corresponding to multidimensional marginals of the random field modeled, we consider more convenient to use numerical integration of (A.5). For higher dimensions it would be better to
use the result of Kolassa and Li (2010) in order to avoid difficult and inaccurate integrations.

4.1.3. **Differential entropy.** This also an important interaction manifestation, often encountered in statistical research. Using the shorthand notation of A.1, define $Z(x) := \frac{1}{3!}\kappa^{j_1,j_2,j_3}h^{j_1,j_2,j_3}(x;\Gamma)$. Hulle (2005) studies an approximation to the differential entropy of $X$, which utilizes only the first correction term in A.2:

\[
\hat{f}_X(x) \log(\hat{f}_X(x))dx = H(\phi_{\Gamma}) - H(\phi_{\Gamma}) - \int \phi_{\Gamma}(x)(1 + Z(x)) \log(1 + Z(x)) dx
\]

\[
\approx H(\phi_{\Gamma}) - \int \phi_{\Gamma}(x)
\left[
Z(x) + \frac{1}{2}Z(x)^2
\right]dx
= H(\phi_{\Gamma}) - \frac{1}{12}\left\{\sum_{j=1}^{J}(\kappa^{j,j,j})^2
+ 3\sum_{i,j=1,i\neq j}(\kappa^{i,i,j})^2
+ \frac{1}{6}\sum_{i,j,k=1,i,j<k}(\kappa^{i,j,k})^2\right\}
\]

The value of $H(\phi_{\Gamma})$ can be found in closed form, $H(\phi_{\Gamma}) = \frac{1}{2}\log(\det(\Gamma)) + \frac{1}{2}\log(2\pi) + \frac{1}{2}$. The approximation (4.7) is accurate to order $O(n^{-2})$.

4.2. **Summarizing.** As we have seen in this section, joint cumulants provide us not only with a lower bound for the number of variables interacting within a vector; joint cumulants can also be connected with relevant interaction manifestations, that may have a specific subject-matter interpretation. The fitting of these interaction manifestations can be integrated into model parameter estimation explicitly.

Please note that the idea of the approach here presented is not to estimate the joint cumulants of a random vector by means of sample joint cumulants. The goal is to fit as well as possible the relevant interaction manifestation. The joint cumulants of the vector are fitted, in that the set of joint cumulants that best recovers the interesting interaction manifestation is kept, even if they are very different from the sample ones.

5. **Illustration: Extending the Gaussian model**

In this section we illustrate the ideas put forward in this paper, taking as interesting interaction manifestation the distribution of the sums of sub-vectors of a random vector, mentioned in section 2.

Another, more extended application of these ideas in the context of spatial statistics can be found at Rodriguez and Bárdossy (2013). See also section 5.5.

The multivariate Normal model is a widely applied model in multivariate analysis. A random vector $X \in \mathbb{R}^J$ having mean vector $\mathbf{m}$ and covariance matrix $\Gamma$, has c.g.f. given by,

\[
K_X(s) = s.m^T + \frac{1}{2}s\Gamma s^T
\]

A similar c.g.f. was studied by Steyn (1993),

\[
K_X(s) = s.m^T + \frac{c_1}{1!}\left(\frac{1}{2}s\Gamma s^T\right) + \frac{c_2}{2!}\left(\frac{1}{2}s\Gamma s^T\right)^2 + \frac{c_3}{3!}\left(\frac{1}{2}s\Gamma s^T\right)^3 + \ldots
\]
Indeed, this c.g.f. reduces to that of the Gaussian model by setting \( c_1 = 1 \) and \( c_{r>1} = 0 \). In order to avoid identifiability problems of the covariance matrix, we set \( c_1 = 1 \) and declare \( \Gamma \) to be a true covariance matrix. This model is treated in detail at Rodríguez and Bárdossy (2013), in the context of spatial statistics; it is shown at Rodríguez and Bárdossy (2013) that it covers a span of tail dependence going from zero (i.e. Normal) to that of the Student-t.

5.1. Some data. In figure 5.1 an 8-dimensional dataset is presented, with a size of \( n = 10950 \) realization. This dataset may represent the daily (log) return of 8 stocks, or they could represent some daily measured environmental variable at 8 locations, possibly after transformation. In either case this dataset would amount to a 30 year record. A plot of the data appears in figure 5.1. We are interested in fitting a model that recovers properly the distribution of the sum of the components of the 8-dimensional random vector, \( S_X = \sum_{i=1}^{8} X_i \).

We shall employ the model given by c.g.f. (5.2), due to the shape of data, and to the flexibility of the mentioned model to represent tail dependence. Specifically, we are interested in fitting a model that captures the correlation among the 8 components properly, but additionally provides a good estimation to the distribution of interaction manifestation

\[
S_X = \sum_{j=1}^{8} X_j
\]
We assume for simplicity a mean vector $\mathbf{m} = (0, \ldots, 0)$ of zeros (otherwise, data could be standardized to have zero means, first). As in section 4.1.1 we have that the c.g.f. of $S_X$ is given by

$$K_{S_X}(t) = K_X(g(t)) = c_1 \left( \frac{1}{2} g(t) \Gamma g(t)^T \right)^2 + c_2 \left( \frac{1}{2} g(t) \Gamma g(t)^T \right)^3 + \ldots$$

where

$$g(t) = \left( t, \ldots, t \right)^T$$

5.2. Parameter estimation. Our estimating strategy consists of:

Step 1: Estimate Covariance matrix $\Gamma$. In this way capture much of the 8-dimensional dependence structure. Since our model is a member of the elliptical family, we can use the estimator for the correlation matrix which uses Kendall’s $\tau$ correlation coefficient (see Lindskog et al. (2003)),

$$\hat{c}_{\tau r}(X_i, X_j) = \sin \left( \frac{\pi}{2} \tau(X_i, X_j) \right)$$

whereby a complete correlation matrix, $\hat{R}$, is obtained.

Then the covariance matrix estimate can be found by

$$\hat{\Gamma} = \Sigma^{1/2} \hat{R} \Sigma^{1/2}$$

with

$$\Sigma = \begin{pmatrix}
S^2(X_1) & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & S^2(X_8)
\end{pmatrix}$$

and $S^2(X_j)$ stands for the sample variance of $X_j$. This procedure was followed, resulting in the covariance matrix given in table 2 at the appendix.

Alternatively, if data represents an environmental variable sampled at several locations, standard geostatistical tools can be used to estimate $\Gamma$ (see Rodríguez and Bárdossy (2013)). The covariance matrix will be in the following considered as known.

Step 2: Interaction manifestation fitting. We do this in a “method-of-moments” fashion (method of cumulants, should we say). The $r$-th order cumulant of $S_X$, $\kappa_r(S_X)$, can be found by differentiating (5.4) $r$ times with respect to $t$, and then setting $t = 0$. Performing the necessary computations, one has for the mean and the variance:

$$\kappa_1(S_X) = 0$$
$$\kappa_2(S_X) = \frac{c_1}{11} 2 \sum_{i,j=1}^{8} \Gamma_{ij}$$

and in general, odd-ordered cumulants will be zero, while even-ordered cumulants are given by

$$\kappa_{2r}(S_X) = \frac{c_r}{r!} \frac{(2r)!}{2^r} \left( \sum_{i_1,\ldots,i_r=1}^{8} \sum_{j_1,\ldots,j_r=1}^{8} \Gamma_{i_1 j_1} \cdots \Gamma_{i_r j_r} \right)$$
We compute the sample cumulants, \( \hat{\kappa}_r \) (for \( r = 1, 2, 3 \)), of \( S_X \). These are found to be 37.426, 463.509 and 105098.112, respectively. Substituting these sample cumulants for the theoretical cumulants in (5.10), and using the already available covariance matrix, \( \Gamma \), we can estimate \( c_1, c_2 \) and \( c_3 \). These estimates are given by \( \hat{c}_1 = 0.999, \hat{c}_2 = 0.1101 \) and \( \hat{c}_3 = 0.1332 \). Note that by considering cumulants of \( S_X \) of order \( \geq 4 \), we can capture important tail characteristics of its distribution.

### 5.3. Evaluation of the fit

We use the Monte Carlo approach to evaluate the fit carried out in the previous sub-section. One can sample from a random vector, \( Y \in \mathbb{R}^8 \), having c.g.f. as in (5.2), by sampling two independent random variables:

1. a non-negative random variable \( V > 0 \), with cumulants \( c_1, \ldots, c_r \) (in our case, \( r = 3 \));
2. a normally distributed random vector \( X \sim N(0, \Gamma) \).

Then one sets:

\[
Y = m + \sqrt{V} \times X
\]

For more details, the reader is referred to Rodríguez and Bárdossy (2013). We fitted \( V \) as a mixture of 5 gamma random variables, in such a way that the cumulants of this mixture are \( \hat{c}_1 = 0.999, \hat{c}_2 = 0.1101 \) and \( \hat{c}_3 = 0.1332 \), up to a small error. Then we were able to simulate 1000 samples of \( Y \), each of size \( n = 10950 \), using the fitted parameters. One of the realizations is shown in figure 5.2. Note that the covariance structure is mostly recovered, though there are some outliers of a magnitude somewhat larger than those displayed in figure 5.1. This is because, once we fitted covariance matrix \( \Gamma \), we focus on recovering the distribution of the sum of the components of the vector \( X \), i.e. \( S_X \). The outliers there presented are part of the mechanism that helps recover the distribution of the components sum.

To see how well the fitted parameters reproduce \( S_X \), we present several sample quantiles of it, together with confidence bands built out of the 1000 Monte Carlo simulations. See table [1]. We see an excellent cover of the given quantiles, particularly at the tails of the distribution of \( S_X \).

Additionally, the distribution of the 365-block maxima of the components sums is also acceptably recovered. In figure 5.3 we show the empirical distribution function of the 30 sample 365-block maxima (i.e. yearly maxima). The Monte Carlo based 95% confidence bands for the 365-block maxima of \( S_X \) are also presented in figure 5.3.

### 5.4. More complicated questions

The techniques presented in this section can also be used to investigate more complex situations. For example, one would like to model jointly the random variables

\[
Z_1 := X_1 + \ldots + X_4
\]
\[
Z_2 := X_5 + \ldots + X_8
\]

This may be the case if each group of components, \( X_1, \ldots, X_4 \) and \( X_5, \ldots, X_8 \), refers each to a geographical area (in environmental modeling); or if there is some economical reason to group them (stock price modeling). We may then wish to model the distributions of \( Z_1 \) and \( Z_2 \), but also model properly at least the correlation between them.

Applying a similar computation as before, we find that

\[
cov(Z_1, Z_2) = \frac{c_1}{2} \sum_{i=1}^{4} \sum_{j=5}^{8} \Gamma_{ij}
\]
for the covariance. Regarding each $Z_j$, all odd-ordered cumulants are zero, whereas all even-ordered cumulants are given by

\[(5.15)\quad \kappa_{2r}(Z_j) = (2r - 1)! \times c_r \times \left(\frac{R_j}{2}\right)^r\]

for $j = 1, 2$, where

\[
R_1 = 2 \sum_{i=1}^{4} \Gamma_{ii} + 4 \sum_{1<i<j<4} \Gamma_{ij}
\]

\[
R_2 = 2 \sum_{i=5}^{8} \Gamma_{ii} + 4 \sum_{5<i<j<8} \Gamma_{ij}
\]

Using equations (5.14) and (5.15), and the sample estimates for these quantities, we can fit parameters $c_1, \ldots, c_r$ of (5.2), as in section 5.2. In this new case, we shall have parameters that reproduce well the correlation among the aggregation vectors, and produce a good match of the cumulants of each marginal distribution, thereby modeling each marginal adequately.

5.5. **A model for Spatial Statistics.** The model given by (5.2) can be used to incorporate multivariate interdependence into a spatial model, while keeping the spatial consistency requirement that any subvector of the spatial field must have the same distribution of the vector containing it (Rodríguez and Bárdossy (2013)).
covariance matrix, \( \Gamma \), is thereby estimated using the standard technique of fitting a covariance function to the spatially labeled data. The additional parameters, \( c_2, c_3, \ldots \), can be used to obtain a better fit of any subject-matter relevant interaction manifestation.

Using data from the Saalach river catchment, in southeast Germany, Rodriguez and Bárdossy (2013) fitted a spatio-temporal model to precipitation data of nine gauging stations lying in the catchment area. A model very similar to the one investigated by Sansó and Guenni (1999) was fitted, because it can easily accommodate missing data as well as the truncated nature of daily precipitation. The model relies on a latent Gaussian field for spatial dependence modeling; that is a latent model with cumulant generating function as (5.2), with \( 0 = c_2 = c_3 = \ldots \).

Rodríguez and Bárdossy (2013) then study the implications of selecting \((c_1, \ldots, c_5) = (0.999, 0.079, 0.152, 0.521, 1.971)\), instead of \(c_{r>1} = 0\) as in the original model by Sansó and Guenni (1999). As shown by Rodriguez and Bárdossy (2013), a random field with \((c_1, \ldots, c_5)\) as above is practically indistinguishable in its one and two dimensional marginal distributions from a Gaussian field with the same covariance function and mean. However, implications for the interaction manifestation “average of fields components”, where each component represents daily precipitation over a 500 m × 500 m squared area on the Saalach river catchment, are significant.

| Quantile (%) | 2.75%  | 97.5%  | Observed |
|--------------|--------|--------|----------|
| 0 (min)      | -41.921| -22.644| -29.191  |
| 0.1          | -21.549| -18.876| -20.72   |
| 0.5          | -16.956| -15.72  | -17.032  |
| 1            | -15.033| -14.124 | -14.771  |
| 5            | -10.248| -9.748  | -10.049  |
| 10           | -7.897 | -7.518  | -7.774   |
| 20           | -5.159 | -4.838  | -5.194   |
| 25           | -4.138 | -3.838  | -4.212   |
| 50           | -0.136 | 0.137   | -0.18    |
| 75           | 3.836  | 4.145   | 4.013    |
| 80           | 4.84   | 5.155   | 4.987    |
| 90           | 7.507  | 7.903   | 7.573    |
| 95           | 9.765  | 10.273  | 9.911    |
| 99           | 14.114 | 15.058  | 14.293   |
| 99.5         | 15.718 | 17.034  | 16.01    |
| 99.9         | 18.93  | 21.625  | 20.159   |
| 99.99        | 21.908 | 29.07   | 28.542   |
| 100 (max)    | 22.897 | 43.735  | 28.983   |

Table 1. Representative quantiles of \( S_X \) and confidence bands of 1000 Monte Carlo simulations of 10950 sized samples each. The parameters fitted in section 5.2 have been used for the simulation. Simulations reproduce quantiles very similar to those observed.
Figure 5.3. Empirical Cumulative Distribution Function of the 365-block maxima, made out of the 10950 sized sample presented at figure 5.1. Monte Carlo simulation based 95% confidence bands have been added from data simulated using the parameters fitted in this section.

The authors obtained 3000 conditional simulations, given the rainfall data available, of the rainfall field over the Saalach river catchment for June 1st 2013, a day of intense rainfall during the 2013 central European floods. In figure 5.4, two of the obtained conditional fields are presented, using the Gaussian and the almost-Gaussian latent structure. In figure 5.5, we show the distribution of the conditional values of mean precipitation over the catchment, for both latent structures. Note that the multivariate interactions, hardly noticeable on the one and two dimensional marginal distributions, increase dramatically the probability of a very high mean precipitation over the studied catchment. The consequence is that substantial under-estimation of flood return periods may me incurred, if one does not account for interaction among more than tow components, in one’s spatio-temporal precipitation models.

6. Discussion

An approach for considering interactions that go beyond correlations has been presented. We have seen that the discrimination between interactions “parameters” and interactions “manifestations” can help to circumvent two major problems one is confronted with when attempting to quantify and model higher order interactions:
the problem of interpretability, by working with subject-matter relevant manifestations of interdependence; and the problem of high dimensionality, by recouring to joint cumulants as building blocks of a dependence model. By using the cumulant generating function, we are recouring to a well-studied object: the characteristic function of a distribution.

As dimension of vector $X$ increases, interactions of high order may be more and more difficult to assess. For example, a random vector having c.f.g. \( \phi(t) \), with $c_1 = 1$, $c_r \approx 0$ for $2 \leq r \leq 3$ but then $c_{r>3} \neq 0$, would have one and two dimensional marginals practically equal to those of a Guassian distribution. But the interaction coefficients of groups of 14 components or more will be very different, producing very different interaction manifestations. The difference in the overall dependence structures may grow tremendously as the dimension of the random vector $X$ grow (i.e. $J >> 2$), even though these fact may go totally unnoticed in the one and two dimensional marginal analysis of data.

In Rodriguez and Bárdossy (2013), these issues are dealt with and illustrated in the context of Spatial Statistics, where the issue of low dimensionality is essential, and where interaction manifestations can differ drastically between two models having very similar 1 and 2 dimensional marginals, due to the big dimension of the field.

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Figure 5.5. Boxplots of the average of the conditionally simulated random fields for June 1st 2013, in millimeters, for the Saalach river catchment. The field with high order interacting latent structure shows much more variability. In particular, average precipitation over the catchment above 120 mm are quite probable under this model.

Appendix A. Review of Edgeworth Expansion and the Saddlepoint Approximation

We recall well-known results about density approximation. Details for all topics of this appendix can be found in Barndorff-Nielsen and Cox (1990); Kolassa (2006); we present here just the approximations, in the context of a distribution having a probability density function.

The Edgeworth Expansion is a series expansion of the probability density and of the probability distribution in terms of the joint cumulants (performing as coefficients) and of the multivariate normal distribution (performing as basis function).

We employ below the shorthand notation for summations used in Barndorff-Nielsen and Cox (1990), in order to avoid an overflow of symbols in these pages. Arrays are represented by symbols with superscripts and under-scripts. For example a matrix is represented by $a^{i,j}$ or by $b_{ij}$. An array with three dimensions would be $c^{i,j,k}$ or $d_{ijk}$, and so on. The product of these symbols indicates summation along all dimensions for which the index is repeated. For example the term...
Approximations as if we were dealing with a variable being the average of
are often referred to as "correction terms".

\begin{equation}
\frac{1}{6 \sqrt{n}} K_{j_1,j_2,j_3}^j h_{j_1,j_2,j_3}, \text{ to be used below, should be interpreted as}
\end{equation}

\begin{equation}
\frac{1}{6 \sqrt{n}} K_{j_1,j_2,j_3}^j h_{j_1,j_2,j_3} = \frac{1}{6 \sqrt{n}} \sum_{j_1=1}^{J_1} \sum_{j_2=1}^{J_2} \sum_{j_3=1}^{J_3} K_{j_1,j_2,j_3}^j h_{j_1,j_2,j_3}
\end{equation}

where, for example,

\begin{align*}
K_{j_1,j_2,j_3}^j &= \text{cum} (X_{j_1}, X_{j_2}, X_{j_3}) \\
K_{j_1,j_2,j_3}^{j_4} &= \text{cum} (X_{j_1}, X_{j_2}, X_{j_3}, X_{j_4})
\end{align*}

Let \( \mathbf{Z} \in \mathbb{R}^J \) be a random vector with probability density function \( f \). Assume also, without loss of generality, that \( \mathbf{Z} \) has mean a vector of zeros, a \( J \times J \) covariance matrix \( \kappa^{ij} = \Gamma \), and joint cumulants \( \{ \kappa^{j_1,j_2,j_3} \}, \{ \kappa^{j_1,j_2,j_3,j_4} \}, \ldots \). If we have a random sample of \( n \) i.i.d. random vectors with the same distribution as \( \mathbf{Z} \), namely \( \mathbf{Z}_1, \ldots, \mathbf{Z}_n \), then we can form the average random vector \( \mathbf{X} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{Z}_i \). This latter random vector has a density function \( f_\mathbf{X} \) which can be formally written as the following series expansion, in terms of the summation shorthand notation:

\begin{equation}
f_\mathbf{X} (\mathbf{x}) = \phi_\Gamma (\mathbf{x}) \left\{ 1 + \frac{1}{6 \sqrt{n}} K_{j_1,j_2,j_3}^j h_{j_1,j_2,j_3} (\mathbf{x}; \Gamma) + \frac{1}{24n} K_{j_1,j_2,j_3}^{j_4} h_{j_1,j_2,j_3,j_4} + \frac{1}{72n} K_{j_1,j_2,j_3}^{j_4,j_5,j_6} h_{j_1,j_2,j_3,j_4,j_5,j_6} (\mathbf{x}; \Gamma) \right\} + O \left( n^{-\frac{3}{2}} \right)
\end{equation}

Where \( \phi_\Gamma \) is the multivariate Normal density function with zero mean and covariance matrix \( \Gamma \), and \( h_{j_1,...,j_k} (\mathbf{x}; \Gamma) \) represents the evaluation at \( \mathbf{x} \) of the \( k \)-order Hermite polynomial determined by the identity

\begin{equation}
\phi_\Gamma (\mathbf{x}) h_{j_1,...,j_k} (\mathbf{x}; \Gamma) = (-1)^k \frac{\partial^k \phi_\Gamma (\mathbf{x})}{\partial x_{j_1} \cdots \partial x_{j_k}}
\end{equation}

Actually, \( \phi_\Gamma (\mathbf{x}) \) is a Normal approximation to \( f_\mathbf{X} \), and the factors within brackets are often referred to as "correction terms".

It could be protested that we have considered only the case of an average \( \mathbf{X} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{Z}_i \), of random vectors. However, if the distribution of \( \mathbf{Z} \) is unimodal and not wildly skewed or leptokurtic, then the Edgeworth Approximation given in \( A.2 \) is often a good approximation in practice even with \( n = 1 \), as we shall use it. After all, a random variable does not have to be the result of averaging \( n \) variables in order to have cumulants as such an average variable. This is the case of the chi-squared distribution with \( n \) degrees of freedom, for example, which can be interpreted as the sum of \( n \) standard Normal variables after raising each to the second power.

The usefulness of retaining the dependence on \( n \) is that we are reminded of when the Edgeworth Expansion is useful in practice: When the cumulants of \( \mathbf{X} \), of which the density must be approximated, do not explode as their order increases, i.e. they behave as if \( \mathbf{X} \) were approximately an average.

The Edgeworth expansion is practically accurate near the expected value of the distribution, but degenerates as one moves towards the tails of the distribution.

The Saddlepoint Approximation, also called "tilted" Edgeworth Approximation, is a more accurate approximation to the density of \( \mathbf{X} \) at the tails, which we can apply if we know its cumulant generating function \( K_{\mathbf{X}} (t) \). In the context of considering \( \mathbf{X} \) as the mean of \( n \) copies of \( \mathbf{Z} \), the relation between the cumulant generating functions is \( K_{\mathbf{X}} (t) = n K_{\mathbf{Z}} \left( \frac{t}{\sqrt{n}} \right) \). As mentioned above, we shall be using this approximations as if we were dealing with a variable being the average of \( n = 1 \).
random variables. Thus we remove in the following the dependence on such an underlying $n$ and work directly with $K_X(t)$.

In order to introduce the Saddlepoint Approximation, assume for a moment we are trying to find the Edgeworth Expansion not of $f_X(x)$, but of a related family of density functions, defined in terms of an auxiliary vector $\lambda \in \mathbb{R}^J$,

(A.4) $f_X(x; \lambda) = \exp (x^T \lambda - K_X(\lambda)) f_X(x)$

The idea is, for each $x \in \mathbb{R}^J$ to choose the most advantageous value $\hat{\lambda}$ of $\lambda \in \mathbb{R}^J$ in order to make the Edgeworth approximation $\hat{f}_X(x; \hat{\lambda})$ to $f_X(x; \lambda)$ as accurate as possible. Of course, this will provide automatically an approximation for $f_X(x)$,

$$
\hat{f}_X(x) = \exp \left( K_X(\hat{\lambda}) - x^T \hat{\lambda} \right) \hat{f}_X(x; \hat{\lambda})
$$

which is in fact what we want.

The optimum value $\hat{\lambda}$ can be proved to be the one fulfilling $x = \nabla K_X(\hat{\lambda})$, for the particular $x \in \mathbb{R}^J$ in question, because then density $f_X(x; \hat{\lambda})$ corresponds to a random vector having its mean at $x$, where the Edgeworth Approximation is most accurate. Now, under suitable regularity conditions, the leading term of the Edgeworth expansion of $f_X(x; \hat{\lambda})$ is a multivariate Normal density with covariance matrix with entries

$$
\left( \Sigma_{i,j} \right) = \left. \frac{\partial^2 K_X(\lambda)}{\partial \lambda_i \partial \lambda_j} \right|_{\lambda = \hat{\lambda}}
$$

evaluated at its mean; that is,

$$
f_X(x; \hat{\lambda}) \approx \frac{e^{0}}{(2\pi)^{J/2} \det(\Sigma)^{1/2}}
$$

Thus, the looked for approximation is given by

(A.5) $f_X(x) = \exp \left( K_X(\hat{\lambda}) - x^T \hat{\lambda} \right) f_X(x; \hat{\lambda}) \approx \frac{\exp \left( K_X(\hat{\lambda}) - x^T \hat{\lambda} \right)}{(2\pi)^{J/2} \det(\hat{\Sigma})^{1/2}}$

The error of this approximation is of order $O(n^{-1})$ for all $x \in \mathbb{R}^J$, if the joint cumulants of random vector $X$ behave like an average of $n$ iid random vectors. Suitable normalization can bring this order down to $O(n^{-2})$.

In spite of the apparent disadvantage of having to re-compute the density estimation for each $x$, the computational cost becomes considerably smaller than that of the Edgeworth Approximation as dimension increases, since the number of multivariate Hermite polynomials at A.2 to evaluate increases exponentially with the dimension of $x$. 


### APPENDIX B. ESTIMATED COVARIANCE FOR THE ILLUSTRATION

|   | 1    | 2   | 3    | 4    | 5    | 6    | 7    | 8   |
|---|------|-----|------|------|------|------|------|-----|
| 1 | 1.00 | 0.72 | 0.62 | 0.77 | 0.62 | 0.71 | 0.77 | 0.77 |
| 2 | 0.72 | 0.99 | 0.32 | 0.51 | 0.49 | 0.53 | 0.63 | 0.64 |
| 3 | 0.62 | 0.32 | 1.00 | 0.29 | 0.47 | 0.37 | 0.49 | 0.49 |
| 4 | 0.77 | 0.49 | 0.47 | 0.50 | 0.99 | 0.48 | 0.55 | 0.59 |
| 5 | 0.62 | 0.53 | 0.37 | 0.41 | 0.48 | 0.99 | 0.55 | 0.52 |
| 6 | 0.72 | 0.48 | 0.80 | 0.49 | 0.28 | 1.02 | 0.52 | 0.53 |
| 7 | 0.77 | 0.63 | 0.49 | 0.50 | 0.49 | 0.59 | 0.86 | 1.00 |

Table 2. Estimated covariance for the illustrative dataset of section 5.

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**References**

Aigner, M. (2006). *Diskrete Mathematik*. Vieweg+Teubner Verlag, 6. auflage edition.

Bárdossy, A. and Pegram, G. (2009). Copula based multisite model for daily precipitation simulation. *Hydrology and Earth System Sciences Discussions*, 6(3):4485–4534.

Bárdossy, A. and Pegram, G. (2012). Multiscale spatial recorrelation of RCM precipitation to produce unbiased climate change scenarios over large areas and small. *Water Resources Research*, 48(9).

Barndorff-Nielsen, O. E. and Cox, D. R. (1990). *Asymptotic techniques for use in statistics*. Chapman and Hall, London [u.a.].

Bedford, T. and Cooke, R. (2001). *Probabilistic risk analysis: foundations and methods*. Cambridge University Press.

Billingsley, P. (1986). *Probability and measure*. Wiley series in probability and mathematical statistics. Wiley, New York, 2nd ed edition.

Block, H. W. and Fang, Z. (1988). A multivariate extension of hoeffding’s lemma. *The Annals of Probability*, 16(4):1803–1820.

Box, G. E. P. (1976). Science and statistics. *Journal of the American Statistical Association*, 71(356):pp. 791–799.

Brillinger, D. R. (1974). *Time series: data analysis and theory*. Holt, Rinehart, and Winston, New York.

Cox, D. R., Bayarri, M., Cuadras, C., Bernadro, J. M., Girón, F., Moreno, E., Keiding, N., Lindley, D., Pericchi, L., Piccinato, L., et al. (1995). The relation between theory and application in statistics. *Test*, 4(2):207–261.

Cressie, N. and Wikle, C. K. (2011). *Statistics for spatio-temporal data*. Wiley.

Cressie, N. A. C. (1991). *Statistics for spatial data*. Wiley series in probability and mathematical statistics. Wiley, New York.

Del Brio, E. B., Ñiguez, T.-M., and Perote, J. (2009). Gram-Charlier densities: a multivariate approach. *Quantitative Finance*, 9(7):855–868.

Dhaene, J., Linders, D., Schoutens, W., and Vyncke, D. (2012). The herd behavior index: A new measure for the implied degree of co-movement in stock markets. *Insurance: Mathematics and Economics*, 50(3):357 – 370.
Dhaene, J., Linders, D., Schoutens, W., and Vyncke, D. (2013). A multivariate dependence measure for aggregating risks. Open access publications from katholieke universiteit leuven, Katholieke Universiteit Leuven.

Diggle, P. J. and Ribeiro, P. J. (2007). Model-based Geostatistics. Springer.

Du, C., Kurowicka, D., and Cooke, R. (2006). Techniques for generic probabilistic inversion. *Computational Statistics and Data Analysis*, 50(5):1164 – 1187.

Gallant, A. R. and Nychka, D. W. (1987). Semi-nonparametric maximum likelihood estimation. *Econometrica*, 55(2):pp. 363–390.

Genz, A. (1993). Comparison of methods for the computation of multivariate normal probabilities. *Computing Sciences and Statistics*, 25:400–405.

Grünn, S. and Rotter, S. (2010). Analysis of parallel spike trains. Springer, New York.

Hebb, D. O. (1949). *The organisation of behavior: a neuropsychological theory*. Wiley and Sons, New York.

Hoeffding, W. (1940). Masstabinvariante korrelations-theorie. *Schriften Math. Inst. Univ. Berlin*, 5:181–233.

Hulle, M. M. V. (2005). Edgeworth approximation of multivariate differential entropy. *Neural Computation*, 17(9):1903–1910.

Huzurbazar, S. (1999). Practical saddlepoint approximations. *The American Statistician*, 53(3):pp. 225–232.

Joe, H. (1989). Relative entropy measures of multivariate dependence. *Journal of the American Statistical Association*, 84(405):pp. 157–164.

Kendall, M. G. and Stuart, A. (1969). *The advanced theory of statistics Vol. 1, Distribution theory*. Griffin, London.

Kolassa, J. (2006). *Series approximation methods in statistics*, volume 88. Springer.

Kolassa, J. and Li, J. (2010). Multivariate saddlepoint approximations in tail probability and conditional inference. *Bernoulli*, 16(4):1191–1207.

Lancaster, H. O. (1969). *The chi-squared distribution*. Wiley, New York.

Lehmann, E. L. (1966). Some concepts of dependence. *The Annals of Mathematical Statistics*, 37(5):1137–1153.

Lindsay, B. G., Kettenring, J., and Siegmund, D. O. (2004). A report on the future of statistics. *Statistical Science*, 19(3):pp. 387–407.

Lindskog, F., McNeil, A., and Schmock, U. (2003). Kendall’s tau for elliptical distributions. In Müller, W. A., Bihn, M., Bol, G., Nakhcizadeh, G., Rachev, S. T., Ridder, T., and Vollmer, K.-H., editors, *Credit Risk*, pages 149–156. Physica-Verlag HD, Heidelberg.

Linfoot, E. (1957). An informational measure of correlation. *Information and control*, 1(1):85–89.

Lugannani, R. and Rice, S. (1980). Saddle point approximation for the distribution of the sum of independent random variables. *Advances in Applied Probability*, 12(2):pp. 475–490.

Mauleon, I. and Perote, J. (2000). Testing densities with financial data: an empirical comparison of the Edgeworth–Sargan density to the student t. *The European Journal of Finance*, 6(2):225–239.

McCullagh, P. (1987). *Tensor methods in statistics*. Chapman and Hall, London; New York.

Mendel, J. (1991). Tutorial on higher-order statistics (spectra) in signal processing and system theory: theoretical results and some applications. *Proceedings of the*
IEEE, 79(3):278–305.
Micheas, A. C. and Zografos, K. (2006). Measuring stochastic dependence using $\varphi$-divergence. *Journal of Multivariate Analysis*, 97(3):765 – 784.
Muirhead, R. (1982). *Aspects of multivariate statistical theory*. John Wiley & Sons, New York.
Peña, D. and Linde, A. v. d. (2007). Dimensionless measures of variability and dependence for multivariate continuous distributions. *Communications in Statistics - Theory and Methods*, 36(10):1845–1854.
Pearson, K. (2011). *The life, letters and labours of Francis Galton*. Cambridge University Press, Cambridge.
Perote, J. (2004). The multivariate Edgeworth–Sargan density. *Spanish Economic Review*, 6(1):77–96.
Reimherr, M. and Nicolae, D. L. (2013). On quantifying dependence: A framework for developing interpretable measures. *Statistical Science*, 28(1):116–130.
Rényi, A. (1959). On measures of dependence. *Acta Mathematica Hungarica*, 10(3):441–451.
Rodriguez, J. and Bárdossy, A. (2013). Beyond correlation in spatial statistics modeling. Technical report, Institute for Modelling Hydraulic and Environmental Systems, Universität Stuttgart.
Rota, G.-C. (1964). The number of partitions of a set. *The American Mathematical Monthly*, 71(5):498.
Sansó, B. and Guenni, L. (1999). Venezuelan rainfall data analysed by using a bayesian space-time model. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 48(3):345–362.
Schmid, F., Schmidt, R., Blumentritt, T., Gaßner, S., and Ruppert, M. (2010). Copula-based measures of multivariate association. In Jaworski, P., Durante, F., Härdle, W. K., and Rychlik, T., editors, *Copula Theory and Its Applications*, volume 198, pages 209–236. Springer Berlin Heidelberg, Berlin, Heidelberg.
Schweizer, B. and Wolff, E. F. (1981). On nonparametric measures of dependence for random variables. *The Annals of Statistics*, 9(4):879–885.
Steyn, H. (1993). On the problem of more than one kurtosis parameter in multivariate analysis. *Journal of Multivariate Analysis*, 44(1):1 – 22.
Streitberg, B. (1990). Lancaster interactions revisited. *The Annals of Statistics*, 18(4):1878–1885.
Streitberg, B. (1999). Exploring interactions in high-dimensional tables: a bootstrap alternative to log-linear models. *The Annals of Statistics*, 27(1):405–413.
Wolff, E. F. (1980). N-dimensional measures of dependence. *Stochastica: revista de matemática pura y aplicada*, 4(3):175–188.