ON MATRICES IN PRESCRIBED CONJUGACY CLASSES WITH
NO COMMON INVARIANT SUBSPACE AND SUM ZERO

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Abstract. We determine those $k$-tuples of conjugacy classes of matrices, from which it is possible to choose matrices which have no common invariant subspace and have sum zero. This is an additive version of the Deligne-Simpson problem. We deduce the result from earlier work of ours on preprojective algebras and the moment map for representations of quivers. Our answer depends on the root system for a Kac-Moody Lie algebra.

1. Introduction

A problem considered by Deligne, Simpson [12], and others, concerns the equation

$$A_1 A_2 \ldots A_k = 1$$

for $n \times n$ matrices $A_i$, with entries in an algebraically closed field $K$, say. The additive analogue is the equation

$$A_1 + A_2 + \cdots + A_k = 0.$$  \hspace{1cm} (2)

A solution of the additive equation over the field of complex numbers leads to a solution of the multiplicative equation, as the monodromy of a Fuchsian system of ODEs. See for example [1].

The problem is to describe those $k$-tuples of conjugacy classes $C_1, \ldots, C_k$ for which there exists a solution to (2) or (3) with $A_i \in C_i$, and which is irreducible, meaning that there is no nonzero proper subspace of $K^n$ which is invariant for all the $A_i$. It is stated in this form, and studied, by Kostov [8, 9, 10, 11], who calls it the ‘Deligne-Simpson problem’.

In this paper we solve the problem for equation (2). It is a consequence of our work on preprojective algebras and the moment map for representations of quivers [2, 3, 4].

For each $1 \leq i \leq k$, choose elements $\xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,d_i} \in K$ (not necessarily distinct) with

$$\prod_{j=1}^{d_i} (A - \xi_{i,j} 1) = 0$$  \hspace{1cm} (3)

for the matrices $A \in C_i$. For notational convenience we assume that $d_i \geq 2$ for all $i$. The conjugacy class $C_i$ is uniquely determined by the elements $\xi_{i,j}$ and the ranks $r_{i,j}$ of the partial products $\prod_{j=1}^{d_i} (A - \xi_{i,j} 1)$. Clearly

$$n = r_{i,0} \geq r_{i,1} \geq \cdots \geq r_{i,d_i} = 0$$

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and
\[ r_{i,j-1} - r_{i,j} \geq r_{i,\ell-1} - r_{i,\ell} \quad (4) \]
whenever \(1 \leq j < \ell \leq d_i\) and \(\xi_{i,j} = \xi_{i,\ell}\).

Our answer depends on the root system for the Kac-Moody Lie algebra with symmetric generalized Cartan matrix \(C\) whose diagram is

```
1 2 3 4 5
```

Thus the rows and columns of \(C\) are indexed by the vertex set
\[ I = \{0\} \cup \{(i,j) \mid 1 \leq i \leq k, 1 \leq j \leq d_i - 1\}, \]
the diagonal entries are \(C_{vv} = 2\) for \(v \in I\), and the off-diagonal entries are \(-1\) if there is an edge joining the two vertices, otherwise zero.

The root system can be considered as a subset of the set of column vectors \(Z^I\). It includes the coordinate vectors \(\epsilon_v (v \in I)\) and the fundamental region, which consists of the nonzero elements \(\alpha \in N^I\) which have connected support and all components of \(C\alpha \leq 0\). It is then closed up under change of sign and the action of the Weyl group, which is generated by the reflections \(s_v : Z^I \to Z^I\), defined for \(v \in I\) by \(s_v(\alpha) = \alpha - (\epsilon_v^T C\alpha)\epsilon_v\). The roots coming from a coordinate vector are called real roots; those coming from an element of the fundamental region are called imaginary roots.

If \(\lambda \in K^I\) we denote by \(R_\lambda^+\) the set of positive roots \(\alpha \in N^I\) with the property that \(\lambda \cdot \alpha : \sum_{v \in I} \lambda_v \alpha_v = 0\). For \(\alpha \in Z^I\) we define \(p(\alpha) = 1 - \frac{1}{2} \alpha^TC\alpha \in \mathbb{Z}\). For a root \(\alpha\), one has \(p(\alpha) \geq 0\), with equality if and only if \(\alpha\) is a real root. We denote by \(\Sigma_\lambda\) the set of \(\alpha \in R_\lambda^+\) with the property that
\[ p(\alpha) > p(\beta^{(1)}) + p(\beta^{(2)}) + \ldots \quad (5) \]
for any decomposition \(\alpha = \beta^{(1)} + \beta^{(2)} + \ldots\) as a sum of two or more elements of \(R_\lambda^+\). Other characterizations of this set are given by Theorems 5.6 and 8.1 of [3]. Our result is as follows.

**Theorem 1.** There is an irreducible solution to equation (2) with matrices \(A_i \in C_i\) if and only if \(\alpha \in \Sigma_\lambda\), where \(\alpha\) is defined by \(\alpha_0 = n\) and \(\alpha_{[i,j]} = r_{i,j}\), and \(\lambda\) is defined by \(\lambda_0 = -\sum_{i=1}^{k} \xi_{i,1}\) and \(\lambda_{[i,j]} = \xi_{i,j} - \xi_{i,j+1}\). In this case, if \(\alpha\) is a real root, then any other solution to (2) with matrices in \(C_i\) is conjugate to this solution, while if \(\alpha\) is an imaginary root, then there are infinitely many non-conjugate irreducible solutions.

We prove Theorem 1 in Section 3. The case where there is an irreducible solution which is unique up to conjugacy is called the rigid case, and we discuss it
Another special case worth considering is when the $C_i$ are nilpotent conjugacy classes. We discuss it in Section 5. Finally, at the opposite extreme, we discuss the case when the $C_i$ have generic eigenvalues in Section 6. I would like to thank L. Hille for bringing this problem to my attention, and several referees for their helpful remarks.

2. Deformed preprojective algebras

In this section we recall the deformed preprojective algebras of [4], and a result proved in [3].

Let $Q$ be a quiver with vertex set $I$. Let $C$ be the corresponding generalized Cartan matrix, with rows and columns indexed by $I$, so with diagonal entries $C_{vv} = 2$ and off-diagonal entries $C_{vw} = -(n_{vw} + n_{wv})$ where $n_{vw}$ denotes the number of arrows with head at $v$ and tail at $w$. Let $R_\lambda$, $p(\alpha)$ and $\Sigma_\lambda$ be defined as in the introduction. (For simplicity we are assuming that $Q$ has no loops at vertices, otherwise these definitions need to be modified.)

The double of $Q$ is the quiver $\overline{Q}$ obtained from $Q$ by adjoining a reverse arrow $a^* : w \to v$ for each arrow $a : v \to w$ in $Q$. If $\lambda \in K^I$, the deformed preprojective algebra is

$$\Pi^\lambda = K\overline{Q}/(\sum_{a \in Q}(aa^* - a^*a) - \sum_{v \in I}\lambda_v e_v)$$

where $K\overline{Q}$ is the path algebra of $\overline{Q}$, and $e_v$ denotes the trivial path at vertex $v$.

Recall that a representation $X$ of $\overline{Q}$ is given by a vector space $X_v$ for each vertex and a linear map $a : X_v \to X_w$ for each arrow $a : v \to w$ in $\overline{Q}$. The dimension vector of $X$ is the vector in $\mathbb{N}I$ whose components are the dimensions of the spaces $X_v$.

Representations of $\Pi^\lambda$ are the same as representations of $\overline{Q}$ in which the linear maps satisfy

$$\sum_{a \in Q\ h(a)=v} aa^* - \sum_{a \in Q\ t(a)=v} a^*a = \lambda_v 1,$$

for each $i$. Here $h(a)$ and $t(a)$ denote the head and tail vertices of an arrow $a$. The concepts of ‘isomorphism’ of representations and ‘simple’ representations are straightforward.

In [3, Theorem 1.2] we have proved the following.

**Theorem 2.** There is a simple representation of $\Pi^\lambda$ of dimension vector $\alpha$ if and only if $\alpha \in \Sigma_\lambda$. If $\alpha$ is a real root, the simple representation is unique up to isomorphism, and is the only representation of $\Pi^\lambda$ of dimension vector $\alpha$. If $\alpha$ is an imaginary root there are infinitely many non-isomorphic simple representations.

Some parts are not explicitly stated there, but they follow easily: the number of simple representations is discussed in the remarks following the statement of the theorem; and if there is a non-simple representation $X$ whose dimension vector is a real root $\alpha$, then the decomposition of $\alpha$ as the sum of the dimension vectors of the composition factors of $X$ shows that $\alpha \notin \Sigma_\lambda$. 
3. Proof of Theorem 3

Let \( C \) be the generalized Cartan matrix constructed in the introduction, let \( Q \) be the quiver obtained from its diagram by orienting all arrows towards the vertex 0, and denote by \( a_{i,j} \) the arrow with tail at vertex \([i,j]\) and head at \([i,j-1]\). (For convenience we use the convention that \([i,0]\) denotes the vertex 0 for any \(i\).) Let \( \alpha \) and \( \lambda \) be defined as in the statement of Theorem 1.

Given a solution to equation (2) with matrices \( A_i \in C_i \), we construct a representation \( X \) of \( \Pi^\lambda \) of dimension vector \( \alpha \) in which all the linear maps \( a_{i,[j]} \) are injective and \( a^{*}_{[i,j]} \) are surjective. The vector spaces at each vertex are \( X_0 = K^n \) and \( X_{[i,j]} = \text{Im}(\prod_{\ell=1}^{j}(A_i - \xi_{i,\ell}1)) \). The linear map for arrow \( a_{i,j} \) is the inclusion, and the linear map for \( a^{*}_{i,j} \) is

\[
a^{*}_{i,j} = (A_i - \xi_{i,j}1)|_{X_{[i,j-1]}}.\]

This is well-defined since \((A_i - \xi_{i,j}1)(X_{[i,j-1]}) = X_{[i,j]}\).

We show that \( X \) is a representation of \( \Pi^\lambda \). For \( j < d_i - 1 \) we have

\[
a_{i,j+1}^{*}a_{i,j+1}^{*} - a^{*}_{i,j}a_{i,j} = (A_i - \xi_{i,j+1}1) - (A_i - \xi_{i,j}1) = \lambda_{[i,j]}1
\]

which is the relation at vertex \([i,j]\). Also, since the restriction of \( A_i - \xi_{i,d_i}1 \) to \( X_{[i,d_i-1]} \) is zero by (3), we have

\[
-a^{*}_{i,d_i-1}a_{i,d_i-1} = -(A_i - \xi_{i,d_i-1}1) = \lambda_{[i,d_i-1]}1
\]

which is the relation at \([i,d_i - 1]\). Finally, (2) implies that

\[
\sum_{i=1}^{k} a_{i,1}a^{*}_{i,1} = \sum_{i=1}^{k} (A_i - \xi_{i,1}1) = \lambda_0 1
\]

which is the relation at 0.

Conversely, given a representation \( X \) of \( \Pi^\lambda \) of dimension vector \( \alpha \) in which all the linear maps \( a_{i,[j]} \) are injective and \( a^{*}_{[i,j]} \) are surjective, we construct a solution to equation (2) with matrices \( A_i \in C_i \). Identify \( X_0 \) with \( K^n \). Define \( A_i = a_{i,1}a^{*}_{i,1} + \xi_{i,1}1 \). For \( j < d_i - 1 \) the relation at vertex \([i,j]\) is \( a_{i,j+1}^{*}a_{i,j+1}^{*} - a^{*}_{i,j}a_{i,j} = \lambda_{[i,j]}1 \), which implies that

\[
a_{i,j}^{*}a_{i,j}^{*} + (\xi_{i,j} + c)1 = (a_{i,j+1}^{*}a_{i,j+1}^{*} + (\xi_{i,j+1} + c)1)1
\]

for \( c \in K \). The relation at \([i,d_i - 1]\) is \(-a_{i,d_i-1}^{*}a_{i,d_i-1} = \lambda_{[i,d_i-1]}1 \), which implies that

\[
a_{i,d_i-1}^{*}a_{i,d_i-1}^{*} + (\xi_{i,d_i-1} + c)1 = (\xi_{i,d_i} + c)a_{i,d_i-1}^{*}.
\]

For \( j < d_i - 1 \), equation (3) gives

\[
a_{i,j}^{*}a_{i,j+1}^{*}a_{i,1}^{*}(A_i + c1) = a_{i,j}^{*}a_{i,j+1}^{*}a_{i,1}^{*}(a_{i,1}^{*} + (\xi_{i,1} + c)1)
\]

\[
= a_{i,j}^{*}a_{i,j+1}^{*}a_{i,2}^{*} + (\xi_{i,2} + c)1a_{i,1}^{*}
\]

\[
= \ldots
\]

\[
= a_{i,j}^{*}(a_{i,j}a_{i,j}^{*} + (\xi_{i,j} + c)1)a_{i,j-1}^{*} \ldots a_{i,1}^{*}
\]

\[
= (a_{i,j+1}a_{i,j+1}^{*} + (\xi_{i,j+1} + c)1)a_{i,j}^{*} \ldots a_{i,1}^{*}.
\]

Now equation (5) gives

\[
a_{i,d_i-1}^{*}a_{i,d_i-2}^{*} \ldots a_{i,1}^{*}(A_i + c1) = (\xi_{i,d_i} + c)a_{i,d_i-1}^{*}a_{i,d_i-2}^{*} \ldots a_{i,1}^{*}.
\]
Taking \( c = -\xi_{i,j+1} \) in (8) we have
\[
a_{i,j}^* \cdots a_{i,2}^* a_{i,1}^*(A_i - \xi_{i,j+1}1) = a_{i,j+1} a_{i,j+1}^* a_{i,j}^* \cdots a_{i,1}^*,
\]
and hence by induction
\[
\prod_{\ell=1}^j (A_i - \xi_{i,\ell}1) = a_{i,1} a_{i,2} \cdots a_{i,j} a_{i,j}^* \cdots a_{i,2} a_{i,1}^*
\]
for \( j \leq d_i - 1 \). Now (8) gives
\[
\prod_{\ell=1}^{d_i} (A_i - \xi_{i,\ell}1) = 0.
\]

Using the injectivity of the \( a_{i,j} \) and the surjectivity of the \( a_{i,j}^* \) these two product formulas imply that \( A_i \in C_i \). Finally, the relation at vertex 0 is \( -\sum_{i=1}^k a_{i,1} a_{i,1}^* = \lambda_0 1 \), which implies that the matrices \( (A_i) \) are a solution to equation (8).

Thus we have a correspondence between solutions of equation (8) with matrices \( A_i \in C_i \) and representations of \( \Pi^\lambda \) of dimension vector \( \alpha \) in which all the linear maps \( a_{[i,j]} \) are injective and \( a_{[i,j]}^* \) are surjective. It is clear that this gives a 1-1 correspondence between conjugacy classes of solutions and isomorphism classes of representations. Thus, provided we can show that irreducible solutions correspond to simple representations of \( \Pi^\lambda \), the result follows from Theorem 2.

If the solution to (3) is irreducible, then \( X \) as constructed above is a simple representation of \( \Pi^\lambda \), for if \( Y \) is a subrepresentation, the irreducibility implies that \( Y_0 = 0 \) or \( Y_0 = X_0 \). But if \( Y_0 = 0 \) then \( Y = 0 \) since the linear maps \( a_{i,j} \) are all injective, and if \( Y_0 = X_0 \) then \( Y = X \) since the linear maps \( a_{i,j}^* \) are all surjective.

Conversely, suppose that \( X \) is a simple representation of \( \Pi^\lambda \) of dimension vector \( \alpha \). We show that the linear maps \( a_{i,j} \) are injective. For a contradiction, let \( x \in X_{[i,\ell]} \) be a nonzero element in the kernel of \( a_{i,\ell} \). We define elements \( x_j \in X_{[i,j]} \) for \( j \geq \ell \) by setting \( x_{\ell} = x \) and \( x_{j+1} = a_{i,j+1}^*(x_j) \) for \( j \geq \ell \). An induction, using the relation
\[
a_{i,j+1} a_{i,j+1}^* - a_{i,j}^* a_{i,j} = \lambda_{[i,j]} 1
\]
for \( j < d_i - 1 \), shows that \( a_{i,j+1}(x_{j+1}) \) is a multiple of \( x_j \) for \( j \geq \ell \). It follows that the elements \( x_j \), span a subrepresentation of \( X \). It must be a proper subrepresentation since it is zero at the vertex 0, but \( \alpha_0 = n \geq r_{i,\ell} \neq 0 \). This contradicts the simplicity of \( X \).

A dual argument shows that the linear maps \( a_{i,j}^* \) are surjective.

Now suppose that \( Y_0 \) is an subspace of \( X_0 = K^n \) which is invariant under the \( A_i \). Define subspaces \( Y_{[i,j]} \subseteq X_{[i,j]} \) via \( Y_{[i,j]} = a_{i,j}^*(Y_{[i,j-1]}) \) for \( j \geq 1 \). Thanks to equation (10) this defines a subrepresentation of \( X \). But \( X \) is simple. Thus the solution \( (A_i) \) is irreducible.

4. The rigid case

An irreducible solution is said to be rigid if any other solution (with the matrices in the same conjugacy classes) is conjugate to it. Two additional problems which may be considered as part of the ‘Deligne-Simpson problem’ are to enumerate all cases in which there is a rigid solution, and to construct the corresponding solutions.

For the multiplicative equation (1), Katz [7] has an algorithm which enables one to determine whether or not a given \( k \)-tuple of conjugacy classes has a rigid
solution. Katz does not, however, attempt to enumerate all rigid cases, instead remarking that ‘Even a cursory glance ... leaves one with the impression that there is a fascinating bestiary waiting to be compiled’.

Katz’s algorithm was simplified by Dettweiler and Reiter [5], who also adapted it to the additive equation (2), but again made no attempt to enumerate the cases.

Our results show that for equation (2), there is a rigid solution if and only if

(i) $\alpha$ is a positive real root,
(ii) $\lambda \cdot \alpha = 0$, and
(iii) for any decomposition $\alpha = \beta^{(1)} + \beta^{(2)} + \ldots$ as a sum of two or more positive roots, one has $\lambda \cdot \beta^{(s)} \neq 0$ for some $s$.

Thus we see that the problem of enumerating all rigid cases amounts to enumerating the positive real roots $\alpha$, and for each one, determining the set of $\lambda$ satisfying (ii) and (iii). Note that since a positive real root cannot be proportional to any other positive root, this set contains the general element of $\{\lambda \in K^I \mid \lambda \cdot \alpha = 0\}$.

For example, given 3 conjugacy classes $C_1, C_2, C_3$ of $2 \times 2$ matrices, there is a rigid solution to equation (2) if and only if

(i’) none of the $C_i$ consists of a multiple of the identity matrix,
(ii’) the sum of all six eigenvalues for the three conjugacy classes is zero, and
(iii’) the sum of three eigenvalues, one for each $C_i$, is always nonzero.

Namely, taking $\xi_{i,1}, \xi_{i,2}$ to be the two eigenvalues for $C_i$, we work with the diagram

```
  .
 / \  |
\   \|
  .
```

Since none of the $C_i$ consists of a multiple of the identity matrix, $\alpha$ is the real root

```
  .
 / \  |
\   \|
  .
```

(1)

\[ 2 \rightarrow 1 \]

\[ 1 \]

(where we display the components of $\alpha$ at the appropriate vertices in the diagram).

The sum of all six eigenvalues being zero corresponds to the obvious requirement that $\text{tr}(A_1) + \text{tr}(A_2) + \text{tr}(A_3) = 0$ for $A_i \in C_i$, and also to the condition $\lambda \cdot \alpha = 0$.

Now, for example, the decomposition

```
  .
 / \  |
\   \|
  .
```

\[ 2 \rightarrow 1 = 1 \rightarrow 0 + 1 \rightarrow 1 \]

\[ 1 \]

\[ 1 \]

\[ 1 \]

\[ 0 \]

corresponds to the condition $\xi_{1,1} + \xi_{2,1} + \xi_{3,2} \neq 0$, or equivalently $\xi_{1,2} + \xi_{2,2} + \xi_{3,1} \neq 0$.

As another example, consider 3 conjugacy classes $C_1, C_2, C_3$ of diagonalizable $3 \times 3$ matrices, where $C_1$ and $C_2$ have distinct eigenvalues $\xi_{i,1}, \xi_{i,2}, \xi_{i,3}$ ($i = 1, 2$), and $C_3$ has eigenvalues $\xi_{3,1}, \xi_{3,2}$ of multiplicities 2, 1 respectively. Thus we deal
with the diagram and vector $\alpha$

$$
\begin{array}{c}
2 \\
3 \\
1
\end{array}
\begin{array}{c}
1 \\
2 \\
1
\end{array}
$$

Since $\alpha$ is a real root, there is a rigid solution for general $\lambda$ satisfying $\lambda \cdot \alpha = 0$, or equivalently, for general $\xi_{i,j}$ satisfying

(11) \quad \xi_{1,1} + \xi_{1,2} + \xi_{1,3} + \xi_{2,1} + \xi_{2,2} + \xi_{2,3} + 2\xi_{3,1} + \xi_{3,2} = 0.

There is no rigid solution when

(12) \quad \xi_{1,1} + \xi_{2,1} + \xi_{3,1} = 0,

because of the decomposition

$$
\begin{array}{c}
2 \\
3 \\
1
\end{array}
\begin{array}{c}
1 \\
2 \\
1
\end{array}
\begin{array}{c}
0 \\
0 \\
0
\end{array}
+ 
\begin{array}{c}
2 \\
1 \\
1
\end{array}
\begin{array}{c}
0 \\
2 \\
1
\end{array}
$$

where both terms in the sum are roots. On the other hand, there can be rigid solutions when

(13) \quad \xi_{1,1} + \xi_{2,1} + \xi_{3,2} = 0.

Namely, decompositions of $\alpha$ of the form

$$
\begin{array}{c}
2 \\
3 \\
1
\end{array}
\begin{array}{c}
1 \\
2 \\
1
\end{array}
\begin{array}{c}
0 \\
0 \\
0
\end{array}
+ 
\begin{array}{c}
2 \\
1 \\
1
\end{array}
\begin{array}{c}
2 \\
1 \\
1
\end{array}
$$

could perhaps fail condition (iii) because the dot product of $\lambda$ with the indicated summand is zero, but the difference

$$
\begin{array}{c}
2 \\
1
\end{array}
\begin{array}{c}
2 \\
1 \\
0
\end{array}
$$

is not a root, so this decomposition of $\alpha$ must have at least 3 summands, and the dot product of $\lambda$ with any of the later summands will be nonzero for general $\xi_{i,j}$ satisfying (11) and (13). (I am grateful to a referee of an earlier version of the paper for suggesting this example.)

Using the methods developed in [2, 3, 4], we now give short proof that conditions (i), (ii) and (iii) characterize the existence of a unique simple representation of $\Pi^\lambda$, avoiding Theorem 2, whose proof in [3] is quite complicated. Assuming either that there is a unique simple of dimension vector $\alpha$, or that $\alpha$ satisfies (i)-(iii) we deduce the other.
Since the relations for $\Pi^\lambda$ are given by a moment map, a standard symplectic geometry argument shows that the simple representations of $\Pi^\lambda$, if any, depend on $2p(\alpha)$ parameters. (For example, using the notation of [2, Theorem 10.3], the simple representations form an open subset $S$ of the vector space $\text{Rep}(KQ, \alpha)$ of representations of $Q$ of dimension vector $\alpha$. Representations of $\Pi^\lambda$ correspond to a fibre of $\mu : \text{Rep}(KQ, \alpha) \to \text{End}(\alpha)_{0}$, whose restriction to $S$ is smooth, and isomorphism classes correspond to orbits of the group $\text{GL}(\alpha)$ which acts on $S$ with one-dimensional stabilizers. Thus the number of parameters is $\dim S - \dim \text{End}(\alpha)_{0} - (\dim \text{GL}(\alpha) - 1) = 2p(\alpha)$, as claimed.)

Thus either condition implies that $p(\alpha) = 0$, so $\alpha^T C \alpha > 0$, and hence $\epsilon^T v C \alpha > 0$ for some vertex $v$. Thus the reflection $s_v(\alpha)$ is strictly smaller than $\alpha$.

Now if $\lambda_v \neq 0$ then the reflection functors of [4] show that simple representations of $\Pi^\lambda$ of dimension $\alpha$ correspond to simple representations of $\Pi^{\lambda'}$ of dimension $s_v(\alpha)$ for suitable $\lambda'$. Moreover conditions (i)-(iii) for $\alpha$ correspond to the equivalent conditions for $s_v(\alpha)$ by [3, Lemma 5.2]. Thus the claim follows by an induction.

On the other hand, if $\lambda_v = 0$ then there is a unique simple representation if and only if $\alpha = \epsilon_v$ by [3, Lemma 7.2], and conditions (i)-(iii) hold if and only if $\alpha = \epsilon_v$, for otherwise, either (i) or (ii) fails, or the decomposition $\alpha = s_v(\alpha) + \epsilon_v + \cdots + \epsilon_v$ contradicts (iii).

Finally, we remark that Dettweiler and Reiter’s additive version [5, Appendix] of Katz’s algorithm corresponds to the reflection functor at the vertex 0 for the generalized Cartan matrix introduced in the introduction. (The reflection functors at other vertices have the same effect as re-ordering the elements $\xi_{i,j}$.) Either approach can in principle be used to construct the solutions in the rigid cases. See [6] for an alternative explicit construction of solutions in some special cases.

5. THE NILPOTENT CASE

The case where the $C_i$ are nilpotent conjugacy classes has been considered by Kostov [8, 10]. Here we may take all $\xi_{i,j} = 0$, so that $\lambda = 0$. Now [3, Theorem 8.1] characterizes the elements of $\Sigma_0$ as the coordinate vectors, and the elements of the fundamental region, except those of three special types (I), (II) and (III). For the generalized Cartan matrix constructed in the introduction, the fundamental region consists of the nonzero vectors $\alpha \in N^t$ with connected support, $2\alpha_{[i,j]} \leq \alpha_{[i,j-1]} + \alpha_{[i,j+1]}$ for $1 \leq i \leq k$ and $1 \leq j < d_i - 1$, $2\alpha_{[i,d_i-1]} \leq \alpha_{[i,d_i-2]}$ for $1 \leq i \leq k$, and

$$2\alpha_0 \leq \sum_{i=1}^{k} \alpha_{[i,1]}.$$

For $\alpha$ constructed as in Theorem 1, the first conditions follow automatically from (4), and the last condition becomes

$$2n \leq \sum_{i=1}^{k} r_{i,1}.$$

Deleting any vertices $v$ at which $\alpha_v = 0$, the special types are:

(I) The diagram is an extended Dynkin diagram $\Delta$, and $\alpha$ is a proper multiple of the minimal positive imaginary root $\delta$. 


(II) The diagram is obtained from an extended Dynkin diagram $\Delta$ by adding a new vertex $w$ connected to an extending vertex of $\Delta$. The restriction of $\alpha$ to $\Delta$ is a proper multiple of $\delta$, and $\alpha_w = 1$.

(III) Does not occur.

As an example, suppose that $C_1$, $C_2$ and $C_3$ are all the conjugacy class of nilpotent $12 \times 12$ matrices consisting of four $3 \times 3$ Jordan blocks. In this case the diagram and vector $\alpha$ are

```
       8 ---- 4
      /       \
12 ---- 8 ---- 4
      \
       8 ---- 4
```

This is of type (I), with the diagram being extended Dynkin of type $\tilde{E}_6$, so there is no irreducible solution. Changing $C_3$ to involve Jordan blocks of sizes $4, 3, 3, 2$, we obtain

```
       8 ---- 4
      /       \
12 ---- 8 ---- 4
      \
       8 ---- 4---- 1
```

which is of type (II), so again there is no irreducible solution. Changing $C_3$ again, to involve Jordan blocks of sizes $4, 4, 2, 2$, we obtain

```
       8 ---- 4
      /       \
12 ---- 8 ---- 4
      \
       8 ---- 4---- 2
```

which is in the fundamental region (since $2 \times 12 \leq 8 + 8 + 8$), and is not a special type, so there is an irreducible solution in this case.

Observe that type (I) corresponds to the ‘special’ cases, and type (II) to the ‘almost special’ cases (a1), (b1), (c2) and (d3) of [10, §6]. Thus one implication in our characterization has already been obtained by Kostov [10, Theorem 34].

6. The generic eigenvalues case

Let $\nu_{i,1}, \ldots, \nu_{i,r_i}$ be the distinct eigenvalues of $C_i$, and let $m_{i,\ell}$ be the algebraic multiplicity of $\nu_{i,\ell}$. We assume that

$$
\sum_{i=1}^{k} \sum_{\ell=1}^{r_i} m_{i,\ell} \nu_{i,\ell} = 0,
$$

which is an obvious necessary condition for equation (2) to have a solution. Following Kostov [9, 11], we say that the $C_i$ have generic eigenvalues if, for any integers $0 \leq m'_{i,\ell} \leq m_{i,\ell}$ such that $\sum_{\ell=1}^{r_i} m'_{i,\ell}$ is independent of $i$, an equality of the form

$$
\sum_{i=1}^{k} \sum_{\ell=1}^{r_i} m'_{i,\ell} \nu_{i,\ell} = 0
$$

is satisfied.
implies that \( m'_{i,\ell} = 0 \) for all \( i, \ell \), or \( m'_{i,\ell} = m_{i,\ell} \) for all \( i, \ell \). Clearly, if the \( C_i \) have generic eigenvalues, then the \( m_{i,\ell} \) have no common divisor.

In \([9, 11]\), Kostov determines those \( C_i \) with generic eigenvalues for which equation (2) has an irreducible solution. His answer can be neatly reformulated in terms of roots, and we show that it also follows from Theorem 1. Choose elements \( \xi_{i,j} \) as in the introduction, and let \( \lambda \) and \( \alpha \) be constructed as in Theorem 1. Assuming that the \( C_i \) have generic eigenvalues, we show that \( \alpha \in \Sigma_{\lambda} \) (so that equation (2) has an irreducible solution) if and only if \( \alpha \) is a root.

First note that \( \alpha \) is indivisible, that is, its components have no common divisor, for otherwise the \( m_{i,\ell} \) will have the same common divisor. Also, equation (14) implies that \( \lambda \cdot \alpha = 0 \).

By definition, if \( \alpha \in \Sigma_{\lambda} \), then \( \alpha \) is a root. For the converse, suppose that \( \alpha \) is a root, and that \( \alpha = \beta^{(1)} + \beta^{(2)} + \ldots \) is a nontrivial decomposition of \( \alpha \) as a sum of elements of \( R^+_{\lambda} \). We show that inequality (10) holds.

Let \( \beta^{(s)} \) be a summand occurring in this decomposition. For given \( i \) and \( \ell \), define

\[
m^{(s)}_{i,\ell} = \sum_{j=1}^{d_i} (\beta^{(s)}_{[i,j-1]} - \beta^{(s)}_{[i,j]}),
\]

where, for convenience, we define \( \beta^{(s)}_{[i,0]} = 0 \), and \([i,0]\) denotes the vertex 0. Clearly

\[
\sum_{\ell=1}^{r_i} m^{(s)}_{i,\ell} = \sum_{j=1}^{d_i} (\beta^{(s)}_{[i,j-1]} - \beta^{(s)}_{[i,j]}) = \beta^{(s)}_0,
\]

which is independent of \( i \).

We claim that \( m^{(s)}_{i,\ell} \geq 0 \) for all \( i, \ell \). To prove it, we divide into two cases. If \( \beta^{(s)}_0 \neq 0 \), then the fact that \( \beta^{(s)} \) is a root implies that \( \beta^{(s)}_{[i,j-1]} \geq \beta^{(s)}_{[i,j]} \) for all \( i, j \), which immediately gives the claim. On the other hand, if \( \beta^{(s)}_0 = 0 \), then since any root has connected support, \( \beta^{(s)} \) is supported on one arm of the diagram, say the \( i \)-th. Moreover, using the classification of roots for type \( A_n \), it follows that there are \( 1 \leq p \leq q \leq d_i - 1 \) with

\[
\beta^{(s)}_{[i,p]} = \beta^{(s)}_{[i,p+1]} = \ldots = \beta^{(s)}_{[i,q]} = 1,
\]

and all other components of \( \beta^{(s)} \) are zero. Now

\[
0 = \lambda \cdot \beta^{(s)} = \lambda_{[i,p]} + \lambda_{[i,p+1]} + \ldots + \lambda_{[i,q]} = \xi_{i,p} - \xi_{i,q+1},
\]

so that \( \xi_{i,p} = \xi_{i,q+1} \). From this it follows that \( m^{(s)}_{i,\ell} = 0 \) for all \( \ell \), and clearly also \( m^{(s)}_{i,\ell} = 0 \) for all other arms \( i \).

Now, varying \( s \), we have

\[
m^{(1)}_{i,\ell} + m^{(2)}_{i,\ell} + \cdots = \sum_{j=1}^{d_i} (\alpha_{[i,j-1]} - \alpha_{[i,j]}) = m_{i,\ell}
\]
(where, again, we define \( \alpha[i,d] = 0 \)). Thus \( 0 \leq m_{i,\ell}^{(s)} \leq m_{i,\ell} \). Moreover,
\[
0 = \lambda \cdot \beta^{(s)} = \sum_{i,j} (\xi_{i,j} - \xi_{i,j+1}) \beta_{[i,j]}^{(s)} - (\sum_{i} \xi_{i,1}) \beta_{0}^{(s)} = \sum_{i,j} \xi_{i,j} (\beta_{[i,j]}^{(s)} - \beta_{[i,j-1]}^{(s)}) = - \sum_{i,\ell} \nu_{i,\ell} m_{i,\ell}^{(s)}.
\]
This contradicts the genericity assumption unless \( m_{i,\ell}^{(s)} = m_{i,\ell} \) for all \( i, \ell \), or \( m_{i,\ell}^{(s)} = 0 \) for all \( i, \ell \). Thus, exactly one of the summands \( \beta^{(s)} \) has \( \beta_{0}^{(s)} \neq 0 \). By renumbering we may suppose it is the one with \( s = 1 \).

We show that \( \alpha^T C \beta^{(s)} \leq 0 \) for \( s \neq 1 \), where \( C \) is the generalized Cartan matrix. Namely, using that \( \beta^{(s)} \) is of the form \([3] \), we have
\[
\alpha^T C \beta^{(s)} = (\alpha[i,q] - \alpha[i,q+1]) - (\alpha[i,p-1] - \alpha[i,p]),
\]
so the assertion follows from equation \([3] \).

Now let \( \gamma = \beta^{(2)} + \beta^{(3)} + \cdots = \alpha - \beta^{(1)} \neq 0 \). We have \( \gamma^T C \gamma > 0 \) since the support of \( \gamma \) is contained in the diagram obtained by deleting the vertex 0, which is a union of Dynkin diagrams of type \( A_n \), so has positive definite quadratic form. Since \( C \) is symmetric we have
\[
p(\beta^{(1)}) = 1 - \frac{1}{2} (\beta^{(1)})^T C \beta^{(1)} = 1 - \frac{1}{2} \alpha^T C \alpha - \frac{1}{2} \gamma^T C \gamma + \alpha^T C \gamma
\]
\[
= p(\alpha) - \frac{1}{2} \gamma^T C \gamma + \alpha^T C \beta^{(2)} + \alpha^T C \beta^{(3)} + \cdots < p(\alpha).
\]
Since also \( p(\beta^{(s)}) = 0 \) for \( s \neq 1 \), inequality \([3] \) follows. Thus \( \alpha \in \Sigma_\lambda \).

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