A New Generalized Kumaraswamy Distribution

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Abstract
A new five-parameter continuous distribution which generalizes the Kumaraswamy and
the beta distributions as well as some other well-known distributions is proposed and studied.
The model has as special cases new four- and three-parameter distributions on the standard
unit interval. Moments, mean deviations, Rényi’s entropy and the moments of order statistics
are obtained for the new generalized Kumaraswamy distribution. The score function is given
and estimation is performed by maximum likelihood. Hypothesis testing is also discussed.
A data set is used to illustrate an application of the proposed distribution.

Keywords: Beta distribution; Continuous proportions; Generalized Kumaraswamy distribution;
Kumaraswamy distribution; Maximum likelihood; McDonald Distribution; Moments.

1 Introduction
We introduce a new five-parameter distribution, so-called generalized Kumaraswamy (GKw)
distribution, which contains some well-known distributions as special sub-models as, for example,
the Kumaraswamy (Kw) and beta (B) distributions. The GKw distribution allows us to define
new three- and four-parameter generalizations of such distributions. The new model can be
used in a variety of problems for modeling continuous proportions data due to its flexibility in
accommodating different forms of density functions.

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The **GKw** distribution comes from the following idea. Wahed (2006) and Ferreira and Steel (2006) demonstrated that any parametric family of distributions can be incorporated into larger families through an application of the probability integral transform. Specifically, let \( G_1(\cdot;\omega) \) be a cumulative distribution function (cdf) with corresponding probability density function (pdf) \( g_1(\cdot;\omega) \), and \( g_2(\cdot;\tau) \) be a pdf having support on the standard unit interval. Here, \( \omega \) and \( \tau \) represent scalar or vector parameters. Now let

\[
F(x;\omega,\tau) = \int_0^{G_1(x;\omega)} g_2(t;\tau)dt. \tag{1}
\]

Note that \( F(\cdot;\omega,\tau) \) is a cdf and that \( F(\cdot;\omega,\tau) \) and \( G_1(x;\omega) \) have the same support. The pdf corresponding to (1) is

\[
f(x;\omega,\tau) = g_2(G_1(x;\omega);\tau)g_1(x;\omega). \tag{2}
\]

This mechanism for defining generalized distributions from a parametric cdf \( G_1(\cdot;\omega) \) is particularly attractive when \( G_1(\cdot;\omega) \) has a closed-form expression.

The beta density is often used in place of \( g_2(\cdot;\tau) \). However, different choices for \( G_1(\cdot;\omega) \) have been considered in the literature. Eugene et al. (2002) defined the beta normal distribution by taking \( G_1(\cdot;\omega) \) to be the cdf of the standard normal distribution and derived some of its first moments. More general expressions for these moments were obtained by Gupta and Nadarajah (2004a). Nadarajah and Kotz (2004) introduced the beta Gumbel (BG) distribution by taking \( G_1(\cdot;\omega) \) to be the cdf of the Gumbel distribution and provided closed form expressions for the moments, the asymptotic distribution of the extreme order statistics and discussed maximum likelihood estimation. Also, Nadarajah and Kotz (2006) dealt with the beta exponential (BE) distribution and obtained its moment generating function, its first four cumulants, the asymptotic distribution of its extreme order statistics and discussed maximum likelihood estimation.

The starting point of our proposal is the Kumaraswamy (Kw) distribution (Kumaraswamy, 1980; see also Jones, 2009). It is very similar to the beta distribution but has a closed-form cdf given by

\[
G_1(x;\omega) = 1 - (1 - x^\alpha)^\beta, \quad 0 < x < 1, \tag{3}
\]

where \( \omega = (\alpha,\beta)^T \), \( \alpha > 0 \) and \( \beta > 0 \). Its pdf becomes

\[
g_1(x;\omega) = \alpha\beta x^{\alpha-1}(1 - x^\alpha)^{\beta-1}, \quad 0 < x < 1. \tag{4}
\]

If \( X \) is a random variable with pdf (4), we write \( X \sim \text{Kw}(\alpha,\beta) \). The **Kw** distribution was originally conceived to model hydrological phenomena and has been used for this and also for
other purposes. See, for example, Sundar and Subbiah (1989), Fletcher and Ponnambalam (1996), Seifi et al. (2000), Ganji et al. (2006), Sanchez et al. (2007) and Courard-Hauri (2007).

In the present paper, we propose a generalization of the $Kw$ distribution by taking $G_1(\cdot; \omega)$ as cdf (3) and $g_2(\cdot; \tau)$ as the standard generalized beta density of first kind (McDonald, 1984), with pdf given by

$$g_2(x; \tau) = \frac{\lambda x^{\gamma-1}(1 - x^\lambda)^{\eta-1}}{B(\gamma, \eta)}, \quad 0 < x < 1,$$

(5)

where $\tau = (\gamma, \eta, \lambda)^\top$, $\gamma > 0, \eta > 0$ and $\lambda > 0$, $B(\gamma, \eta) = \Gamma(\gamma)\Gamma(\eta)/\Gamma(\gamma + \eta)$ is the beta function and $\Gamma(\cdot)$ is the gamma function. If $X$ is a random variable with density function (5), we write $X \sim GB1(\gamma, \eta, \lambda)$. Note that if $X \sim GB1(\gamma, \eta, 1)$ then $X \sim B(\gamma, \eta)$, i.e., $X$ has a beta distribution with parameters $\gamma$ and $\eta$.

The article is organized as follows. In Section 2, we define the $GKw$ distribution, plot its density function for selected parameter values and provide some of its mathematical properties. In Section 3, we present some special sub-models. In Section 4, we obtain expansions for the distribution and density functions. We demonstrate that the $GKw$ density can be expressed as a mixture of $Kw$ and power densities. In Section 5, we give general formulae for the moments and the moment generating function. Section 6 provides an expansion for the quantile function. Section 7 is devoted to mean deviations about the mean and the median and Bonferroni and Lorenz curves. In Section 8, we derive the density function of the order statistics and their moments. The Rényi entropy is calculated in Section 9. In Section 10, we discuss maximum likelihood estimation and determine the elements of the observed information matrix. Section 11 provides an application to a real data set. Section 12 ends the paper with some conclusions.

2 The New Distribution

We obtain an appropriate generalization of the $Kw$ distribution by taking $G_1(\cdot; \omega)$ as the two-parameter $Kw$ cdf (3) and associated pdf (4). For $g_2(\cdot; \tau)$, we consider a three-parameter generalized beta density of first kind given by (3). To avoid non-identifiability problems, we allow $\eta$ to vary on $[1, \infty)$ only. We then write $\delta = \eta - 1$ which varies on $(0, \infty)$. Using (1), the cdf of the $GKw$ distribution, with five positive parameters $\alpha, \beta, \gamma, \delta$ and $\lambda$, is defined by

$$F(x; \theta) = \frac{\lambda}{B(\gamma, \delta + 1)} \int_0^{1-(1-x^\alpha)^\beta} y^{\gamma-1}(1 - y^\lambda)^\delta dy,$$

(6)

where $\theta = (\alpha, \beta, \gamma, \delta, \lambda)^\top$ is the parameter vector.

The pdf corresponding to (6) is straightforwardly obtained from (2) as

$$f(x; \theta) = \frac{\lambda \alpha \beta x^{\alpha-1}}{B(\gamma, \delta + 1)} (1 - x^\alpha)^{\beta-1}[1 - (1 - x^\alpha)^\beta]^{\gamma-1} \{1 - (1 - x^\alpha)^\beta \}^{\delta}, \quad 0 < x < 1.$$

(7)
Based on the above construction, the new distribution can also be referred to as the McDonald Kumaraswamy (McKw) distribution. If $X$ is a random variable with density function (7), we write $X \sim \text{GKw}(\alpha, \beta, \gamma, \delta, \lambda)$.

An alternative, but related, motivation for (6) comes through the beta construction (Eugene et al., 2002). We can easily show that

$$F(x; \theta) = I_{[1-(1-x^\alpha)^\beta]}^{\lambda}(\gamma, \delta + 1),$$

where $I_x(a, b) = B(a, b)^{-1} \int_0^x \omega^{a-1}(1-\omega)^{b-1}d\omega$ denotes the incomplete beta function ratio. Thus, the \textit{GKw} distribution can arise by taking the beta construction applied to a new distribution, namely the exponentiated Kumaraswamy (EKw) distribution, to yield (7), which can also be called the beta exponentiated Kumaraswamy (BEKw) distribution, i.e., a beta type distribution defined by the baseline cumulative function $G(x) = [1-(1-x^\alpha)^\beta]^{\lambda}$.

Immediately, inverting the transformation motivation (8), we can generate $X$ following the \textit{GKw} distribution by

$$X = [1-(1-V^{1/\lambda})^{1/\beta}]^{1/\alpha},$$

where $V$ is a beta random variable with parameters $\gamma$ and $\delta + 1$. This scheme is useful because of the existence of fast generators for beta random variables. Figure 1 plots some of the possible shapes of the density function (7). The \textit{GKw} density function can take various forms, bathtub, \textit{J}, inverted \textit{J}, monotonically increasing or decreasing and upside-down bathtub, depending on the parameter values.

We now provide two properties of the \textit{GKw} distribution.

**Proposition 1.** If $X \sim \text{GKw}(1, \beta, \gamma, \delta, \lambda)$, then $Y = X^{1/\alpha} \sim \text{GKw}(\alpha, \beta, \gamma, \delta, \lambda)$ for $\alpha > 0$.

**Proposition 2.** Let $X \sim \text{GKw}(\alpha, \beta, \gamma, \delta, \lambda)$ and $Y = -\log(X)$. Then, the pdf of $Y$ is given by

$$f(y; \theta) = \frac{\lambda \alpha \beta}{B(\gamma, \delta + 1)} e^{-\alpha y}(1 - e^{-\alpha y})^{\beta-1}[1 - (1 - e^{-\alpha y})^\beta]^{\gamma \lambda - 1}\{1 - [1 - (1 - e^{-\alpha y})^\beta]^{\lambda}\}^\delta, \quad y > 0.$$ (9)

We call (9) the log-generalized Kumaraswamy (LGKw) distribution.

### 3 Special Sub-Models

The \textit{GKw} distribution is very flexible and has the following distributions as special sub-models.

**The Kumaraswamy distribution (Kw)**

If $\lambda = \gamma = 1$ and $\delta = 0$, the \textit{GKw} distribution reduces to the \textit{Kw} distribution with parameters $\alpha$ and $\beta$, and cdf and pdf given by (3) and (4), respectively.
Figure 1: $GKW$ density curves. (a) $\theta = (\alpha, 3.5, 1.5, 2.5, 0.5)^T$, (b) $\theta = (3.5, \beta, 1.5, 2.5, 0.5)^T$, (c) $\theta = (1.0, 1.5, \gamma, 2.5, 0.5)^T$, (d) $\theta = (1.0, 1.5, 2.5, \delta, 0.5)^T$, (e) $\theta = (0.5, 0.7, 0.1, 3.0, \lambda)^T$, (f) $\theta = (\alpha, \beta, 2.5, 0.1, 0.5)^T$, (g) $\theta = (\alpha, 1.5, 2.5, \delta, 0.5)^T$, (h) $\theta = (0.5, 0.7, 0.15, \delta, \lambda)^T$, (i) $\theta = (0.5, 1.0, \gamma, 0.3, \lambda)^T$. 
The McDonald distribution (Mc)

For $\alpha = \beta = 1$, we obtain the Mc distribution with parameters $\gamma$, $\delta + 1$ and $\lambda$.

The beta distribution

If $\alpha = \beta = \lambda = 1$, the GKw distribution reduces to the beta distribution with parameters $\gamma$ and $\delta + 1$.

The beta Kumaraswamy distribution (BKw)

If $\lambda = 1$, (7) yields

$$f(x; \alpha, \beta, \gamma, \delta, 1) = \frac{\alpha \beta x^{\alpha - 1}(1 - x^{\alpha})^{\beta - 1}[1 - (1 - x^{\alpha})^{\beta}]^{\gamma - 1}}{B(\gamma, \delta + 1)}$$

This distribution can be viewed as a four-parameter generalization of the Kw distribution. We refer to it as the BKw distribution since its pdf can be obtained from (2) by setting $G_1(x; \omega)$ to be the Kw($\alpha, \beta$) cdf and $g_2(x; \tau)$ to the $B(\gamma, \delta + 1)$ density function.

The Kumaraswamy-Kumaraswamy distribution (KwKw)

For $\gamma = 1$, (7) reduces to (for $0 < x < 1$)

$$f(x; \alpha, \beta, 1, \delta, \lambda) = \lambda \alpha \beta (\delta + 1)x^{\alpha - 1}(1 - x^{\alpha})^{\beta - 1}[1 - (1 - x^{\alpha})^{\beta}]^{\lambda - 1}\{1 - [1 - (1 - x^{\alpha})^{\beta}]^{\lambda}\}^{\delta}.$$ 

Again, this distribution is a four-parameter generalization of the Kw distribution. It can be obtained from (2) by replacing $G_1(x; \omega)$ by the cdf of the Kw($\alpha, \beta$) distribution and $g_2(x; \tau)$ by the pdf of the Kw($\gamma, \delta + 1$) distribution. Its cdf has a closed form given by

$$F(x; \alpha, \beta, 1, \delta, \lambda) = 1 - \{1 - [1 - (1 - x^{\alpha})^{\beta}]^{\lambda}\}^{\delta + 1}.$$ 

The EKw distribution

If $\delta = 0$ and $\gamma = 1$, (7) gives

$$f(x; \alpha, \beta, 1, 0, \lambda) = \lambda \alpha \beta x^{\alpha - 1}(1 - x^{\alpha})^{\beta - 1}[1 - (1 - x^{\alpha})^{\beta}]^{\lambda - 1}, \quad 0 < x < 1.$$ 

It can be easily seen that the associated cdf can be written as

$$F(x; \alpha, \beta, 1, 0, \lambda) = G_1(x; \alpha, \beta)^{\lambda},$$

where $G_1(x; \alpha, \beta)$ is the cdf of the Kw($\alpha, \beta$) distribution. This distribution was defined before as the EKw distribution which is a new three-parameter generalization of the Kw distribution.
The beta power distribution (BP)

For $\alpha = 1$ and $\beta = 1$, (9) reduces to

$$f(x; 1, 1, \gamma, \delta, \lambda) = \frac{\lambda}{B(\gamma, \delta + 1)} x^{\gamma \lambda - 1} (1 - x^{\lambda})^{\delta}, \quad 0 < x < 1.$$  

This density function can be obtained from (2) if $G_1(x) = x^{\lambda}$ and $g_2(x)$ is taken as the beta density with parameters $\gamma$ and $\delta + 1$. We call this distribution as the BP distribution.

The LGKw distribution (9) contains as special sub-models the following distributions.

The beta generalized exponential distribution (BGE)

For $\lambda = 1$, (9) gives

$$f(y; \alpha, 1, \gamma, \delta, \lambda) = \frac{\alpha \beta}{B(\gamma, \delta + 1)} e^{-\alpha y} (1 - e^{-\alpha y})^{\delta (\delta + 1) - 1} [1 - (1 - e^{-\alpha y})^\delta]^\gamma - 1, \quad y > 0, \quad (10)$$

which is the density function of the BGE distribution introduced by Barreto-Souza et al. (2010). If $\gamma = 1$ and $\delta = 0$ in addition to $\lambda = 1$, the LGKw distribution becomes the generalized exponential distribution (Gupta and Kundu, 1999). If $\lambda = \beta = \gamma = 1$ and $\delta = 0$, (10) coincides with the exponential distribution with mean $1/\alpha$.

The beta exponential distribution (BE)

For $\beta = 1$ and $\lambda = 1$, (9) reduces to

$$f(y; \alpha, 1, \gamma, \delta, 1) = \frac{\alpha}{B(\gamma, \delta + 1)} e^{-\alpha y} (1 - e^{-\alpha y})^{\delta}, \quad y > 0,$$

which is the density of the BE distribution introduced by Nadarajah and Kotz (2006).

4 Expansions for the Distribution and Density Functions

We now give simple expansions for the cdf of the GKw distribution. If $|z| < 1$ and $\delta > 0$ is a non-integer real number, we have

$$(1 - z)^{\delta} = \sum_{j=0}^{\infty} (-1)^j (\delta)_j z^j,$$  

(11)
where \( (\delta)_j = \delta(\delta - 1) \ldots (\delta - j + 1) \) (for \( j = 0, 1, \ldots \)) is the descending factorial. Clearly, if \( \delta \) is a positive integer, the series stops at \( j = \delta \). Using the series expansion (11) and the representation for the \( \text{GKw} \) cdf (6), we obtain

\[
F(x; \theta) = \int_0^{G_1(x; \alpha, \beta)} \frac{\lambda}{B(\gamma, \delta + 1)} y^{\gamma - 1} \sum_{j=0}^{\infty} (\delta)_j (-1)^j y^{\gamma j} dy
\]

if \( \delta \) is a non-integer real number. By simple integration, we have

\[
F(x; \theta) = \sum_{j=0}^{\infty} \omega_j [G_1(x; \alpha, \beta)]^{\lambda(\gamma + j)},
\]

(12)

where

\[
\omega_j = \frac{(-1)^j (\delta)_j}{(\gamma + j)B(\gamma, \delta + 1)},
\]

(13)

and \( G_1(x; \alpha, \beta) \) is given by (3). If \( \delta \) is a positive integer, the sum stops at \( j = \delta \).

The moments of the \( \text{GKw} \) distribution do not have closed form. In order to obtain expansions for these moments, it is convenient to develop expansions for its density function. From (12), we can write

\[
f(x; \theta) = \sum_{j=0}^{\infty} \omega_j \lambda(\gamma + j) g_1(x; \alpha, \beta)[G_1(x; \alpha, \beta)]^{\lambda(\gamma + j) - 1}.
\]

If we replace \( G_1(x; \alpha, \beta) \) by (3) and use (4), we obtain

\[
f(x; \theta) = \sum_{k=0}^{\infty} p_k g_1(x; \alpha, (k + 1)\beta),
\]

(14)

where \( p_k = \sum_{j=0}^{\infty} \omega_j t_{j,k}, \) with \( t_{j,k} = (\phi)_k \lambda(\gamma + j)(-1)^k/(k + 1). \) Here, \( \phi = (\gamma + j)\lambda - 1 \) and \( g_1(x; \alpha, (k + 1)\beta) \) denotes the Kw\((\alpha, (k + 1)\beta) \) density function with parameters \( \alpha \) and \( (k + 1)\beta \). Further, we can express (14) as a mixture of power densities, since the Kw density (4) can also be written as a mixture of power densities. After some algebra, we obtain

\[
f(x; \theta) = \sum_{i=0}^{\infty} v_i x^{(i+1)\alpha - 1},
\]

(15)

where

\[
v_i = (-1)^i \alpha \beta \sum_{k=0}^{\infty} (k + 1)((k + 1)\beta - 1)_i p_k.
\]

Equations (14) and (15) are the main results of this section. They can provide some mathematical properties of the \( \text{GKw} \) distribution from the properties of the Kw and power distributions, respectively.
5 Moments and Moment Generating Function

Let $X$ be a random variable having the $GKw$ distribution (17). First, we obtain an infinite sum representation for the $r$th ordinary moment of $X$, say $\mu'_r = E(X^r)$. From (14), we can write

$$\mu'_r = \sum_{k=0}^{\infty} p_k \tau_r(k),$$

(16)

where $\tau_r(k) = \int_0^1 x^r g_1(x; \alpha, (k+1)\beta) dx$ is the $r$th moment of the $Kw(\alpha, (k+1)\beta)$ distribution which exists for all $r > -\alpha$. Using a result due to Jones (2009, Section 3), we have

$$\tau_r(k) = (k+1)\beta B\left(1 + \frac{r}{\alpha}, (k+1)\beta\right).$$

(17)

Hence, the moments of the $GKw$ distribution follow directly from (16) and (17). The central moments ($\mu_s$) and cumulants ($\kappa_s$) of $X$ are easily obtained from the ordinary moments by

$$\mu_s = \sum_{k=0}^{s} \binom{s}{k} (-1)^k \mu'_k \mu_{s-k},$$

and

$$\kappa_s = \mu'_s - \mu_1^s + 2\mu_2^s - \mu_3^s + 3\mu_4^s,$$

where $s = 5, 6, 7, \ldots$. Using a result due to Jones (2009, Section 3), we have

$$\tau_r(k) = (k+1)\beta B\left(1 + \frac{r}{\alpha}, (k+1)\beta\right).$$

(17)

Hence, the moments of the $GKw$ distribution follow directly from (16) and (17). The central moments ($\mu_s$) and cumulants ($\kappa_s$) of $X$ are easily obtained from the ordinary moments by

$$\mu'_r = E[X^r] = E[X(X-1) \times \cdots \times (X-r+1)] = \sum_{m=0}^{r} s(r, m) \mu'_m,$$

where $s(r, m)$ is the Stirling number of the first kind given by $s(r, m) = (m!)^{-1} d^m x(r) / dx^m |_{x=0}$. It counts the number of ways to permute a list of $r$ items into $m$ cycles. Thus, the factorial moments of $X$ are given by

$$\mu'_r = \sum_{k=0}^{\infty} p_k \sum_{m=0}^{r} s(r, m) \tau_m(k).$$

The moment generating function of the $GKw$ distribution, say $M(t)$, is obtained from (15) as

$$M(t) = \sum_{i=0}^{\infty} v_i \int_0^1 x^{(i+1)\alpha-1} \exp(tx) dx.$$

By changing variable, we have

$$M(t) = \sum_{i=0}^{\infty} v_i t^{-(i+1)\alpha} \int_0^t u^{(i+1)\alpha-1} \exp(-u) du$$

and then $M(t)$ reduces to the linear combination

$$M(t) = \sum_{i=0}^{\infty} v_i \frac{\gamma((i+1)\alpha, t)}{t^{(i+1)\alpha}},$$

where $\gamma(a, x) = \int_0^x u^{a-1} \exp(-u) du$ denotes the incomplete gamma function.
6 Quantile Function

We can write (8) as $F(x; \theta) = I_z(\gamma, \delta + 1) = u$, where $z = [1 - (1 - x^\alpha)^\beta]^{\lambda}$. From Wolfram’s website\footnote{http://functions.wolfram.com/06.23.06.0004.01} we can obtain some expansions for the inverse of the incomplete beta function, say $z = Q_B(u)$, one of which is

$$z = Q_B(u) = a_1 v + a_2 v^2 + a_3 v^3 + a_4 v^4 + O(v^5/\gamma),$$

where $v = \gamma u B(\gamma, \delta + 1)^{1/\gamma}$ for $\gamma > 0$ and $a_0 = 0, a_1 = 1, a_2 = \delta/(\gamma + 1), a_3 = \delta [\gamma^2 + 3(\delta + 1)\gamma - \gamma + 5\delta + 1][2(\gamma + 1)^2(\gamma + 2)],

$$a_4 = \delta \{\gamma^4 + (6\delta + 5)\gamma^3 + (\delta + 3)(8\delta + 3)\gamma^2 + [33(\delta + 1)^2 - 30\delta + 26]\gamma + (\delta + 1)(31\delta - 16) + 18\}/[3(\gamma + 1)^3(\gamma + 2)(\gamma + 3)],\ldots$$

The coefficients $a_i$'s for $i \geq 2$ can be derived from the cubic recursion (Steinbrecher and Shaw, 2007)

$$a_i = \frac{1}{i^2 + (\gamma - 2)i + (1 - \gamma)} \left\{ (1 - \rho_{i,2}) \sum_{r=2}^{i-1} a_r a_{i+1-r} [r(1-\gamma)(i-r) - r(r-1)] + \sum_{r=1}^{i-1} \sum_{s=1}^{i-r} a_r a_s a_{i+1-r-s} [r(r-\gamma) + s(\gamma + \beta - 2) \times (i + 1 - r - s)] \right\},$$

where $\rho_{i,2} = 1$ if $i = 2$ and $\rho_{i,2} = 0$ if $i \neq 2$. In the last equation, we note that the quadratic term only contributes for $i \geq 3$. Hence, the quantile function $Q_{GKW}(u)$ of the $GKW$ distribution can be written as $Q_{GKW}(u) = \{1 - [1 - Q_B(u)^{1/\gamma}]^{1/\beta}\}^{1/\alpha}$.

7 Mean Deviations

If $X$ has the $GKW$ distribution, we can derive the mean deviations about the mean $\mu'_1 = E(X)$ and about the median $M$ from

$$\delta_1 = \int_0^1 |x - \mu'_1| f(x; \theta)dx \quad \text{and} \quad \delta_2 = \int_0^1 |x - M| f(x; \theta)dx,$$
respectively. From (8), the median \( M \) is the solution of the nonlinear equation

\[
I_{1-(1-M^\alpha)^\beta}(\gamma, \delta + 1) = 1/2.
\]

These measures can be calculated using the relationships

\[
\delta_1 = 2[\mu'_1 F(\mu'_1; \theta) - J(\mu'_1; \theta)] \quad \text{and} \quad \delta_2 = \mu'_1 - 2J(M; \theta).
\]

Here, the integral \( J(a; \theta) = \int_0^a x f(x; \theta) \, dx \) is easily calculated from the density expansion (15) as

\[
J(a; \theta) = \sum_{i=0}^{\infty} \frac{v_i a^{(i+1)\alpha+1}}{(i+1)\alpha+1}.
\]

We can use this result to obtain the Bonferroni and Lorenz curves. These curves have applications not only in economics to study income and poverty, but also in other fields, such as reliability, demography, insurance and medicine. They are defined by

\[
B(p; \theta) = \frac{J(q; \theta)}{p \mu'_1} \quad \text{and} \quad L(p; \theta) = \frac{J(q; \theta)}{\mu'_1},
\]

respectively, where \( q = F^{-1}(p; \theta) \).

8 Moments of Order Statistics

The density function of the \( i \)th order statistic \( X_{i:n} \), say \( f_{i:n}(x; \theta) \), in a random sample of size \( n \) from the \( GKw \) distribution, is given by (for \( i = 1, \ldots, n \))

\[
f_{i:n}(x; \theta) = \frac{1}{B(i, n - i + 1)} f(x; \theta) F(x; \theta)^{i-1} \{1 - F(x; \theta)\}^{n-i}, \quad 0 < x < 1.
\]  

(18)

The binomial expansion yields

\[
f_{i:n}(x; \theta) = \frac{1}{B(i, n - i + 1)} f(x; \theta) \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j F(x; \theta)^{i+j-1},
\]

and using and integrating (15) we arrive at

\[
f_{i:n}(x; \theta) = \frac{1}{B(i, n - i + 1)} \left( \sum_{t=0}^{\infty} v_t x^{(t+1)\alpha-1} \right) \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \left( \sum_{s=0}^{\infty} v^*_s x^{(s+1)\alpha} \right)^{i+j-1},
\]

where \( v^*_s = v_s [(s+1)\alpha]^{-1} \).
We use the following expansion for a power series raised to a integer power (Gradshteyn and Ryzhik, 2000, Section 0.314)

\[
\left( \sum_{j=0}^{\infty} a_j x^j \right)^p = \sum_{j=0}^{\infty} c_{j,p} x^j, \tag{19}
\]

where \( p \) is any positive integer number, \( c_{0,p} = a_0^p \) and \( c_{s,p} = (sa_0)^{-1} \sum_{j=1}^{s} (jp-s+j)a_je_{s-j,p} \) for all \( s \geq 1 \). We can write

\[
f_{i:n}(x; \theta) = \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \sum_{s,t=0}^{\infty} v_t e_{s,i+j-1} x^{(s+t+i+j)\alpha-1},
\]

where \( e_{0,i+j-1} = \nu_0^{*(i+j-1)} \) and (for \( s \geq 1 \))

\[
e_{s,i+j-1} = (sv_{s+i-j-1})^{-1} \sum_{m=1}^{s} [m(i+j-1) - s + m]v_{s-m,i+j-1}. \]

The \( r \)th moment of the \( i \)th order statistic becomes

\[
E(X_{i:n}^r) = \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \sum_{s,t=0}^{\infty} \frac{v_t e_{s,i+j-1}}{(r+s+t+i+j)\alpha}. \tag{20}
\]

We now obtain another closed form expression for the moments of the \( GKw \) order statistics using a general result due to Barakat and Abdelkader (2004) applied to the independent and identically distributed case. For a distribution with pdf \( f(x; \theta) \) and cdf \( F(x; \theta) \), we can write

\[
E(X_{i:n}^p) = r \sum_{m=n-i+1}^{n} (-1)^{m-n+i-1} \binom{m-1}{n-i} \binom{n}{m} I_m(r),
\]

where

\[
I_m(r) = \int_0^1 x^{r-1} (1-F(x; \theta))^m \, dx.
\]

For a positive integer \( m \), we have

\[
I_m(r) = \int_0^1 x^{r-1} \sum_{p=0}^{m} \binom{m}{p} (-1)^p [F(x; \theta)]^p \, dx.
\]

By replacing (12) in the above equation we have

\[
I_m(r) = \sum_{p=0}^{m} (-1)^p \binom{m}{p} \int_0^1 x^{r-1} \left( \sum_{j=0}^{\infty} \omega_j [G\alpha(x; \alpha, \beta)]^{\lambda(j+1)} \right)^p \, dx. \tag{21}
\]
Equations (19) and (21) yield

\[ I_m(r) = \sum_{p=0}^{m} \left( \begin{array}{c} m \\ p \end{array} \right) (-1)^p \int_0^1 x^{r-1} \sum_{j=0}^{\infty} c_{j,p}(G_1(x; \alpha, \beta))^j \lambda^{j} \, dx. \]

By replacing \( G_1(x; \alpha, \beta) \) by (3) and using (11) we obtain

\[ I_m(r) = \sum_{p=0}^{m} \left( \begin{array}{c} m \\ p \end{array} \right) (-1)^p \sum_{j,w=0}^{\infty} c_{j,p}(\psi)^w \int_0^1 x^{r-1}(1-x^\alpha)^{w\beta} \, dx, \]

where \( \psi = \lambda(\gamma + j) \). Since \( B(a/b,c) = b \int_0^1 w^{a-1}(1-w^b)^{c-1} \, dw \) for \( a, b, c > 0 \) (Gupta and Nadarajah, 2004b), we have

\[ I_m(r) = \sum_{p=0}^{m} \sum_{j,w=0}^{\infty} s_{p,j,w} B\left(\frac{r}{\alpha}, \beta w + 1\right), \]

where

\[ s_{p,j,w} = \frac{(-1)^{p+w}m!}{\alpha(m-p)!p! c_{j,p}(\psi)^w}. \]

Finally, \( E(X_{i:n}^r) \) reduces to

\[ E(X_{i:n}^r) = r \sum_{m=n-i+1}^{n} \left( \begin{array}{c} m-n-i+1 \\ n-i \end{array} \right) \left( \begin{array}{c} m-1 \\ n-i \end{array} \right) \sum_{p=0}^{m} \sum_{j,w=0}^{\infty} s_{p,j,w} B\left(\frac{r}{\alpha}, \beta w + 1\right). \]  (22)

Equations (20) and (22) are the main results of this section. The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They are linear functions of expected order statistics defined by (Hoskings, 1990)

\[ \lambda_{r+1} = (r+1)^{-1} \sum_{k=0}^{r} (-1)^k \binom{r}{k} E(X_{r+1-k:n}^r), \quad r = 0, 1, \ldots \]

The first four L-moments are \( \lambda_1 = E(X_{1:1}) \), \( \lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2}) \), \( \lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3}) \) and \( \lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}) \). These moments have several advantages over the ordinary moments. For example, they exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. From (22) applied for the means \( (r = 1) \), we can obtain expansions for the L-moments of the GKw distribution.
9 Rényi Entropy

The entropy of a random variable $X$ with density function $f(x)$ is a measure of variation of the uncertainty. One of the popular entropy measures is the Rényi entropy given by

$$J_R(\rho) = \frac{1}{1-\rho} \log \left[ \int f^\rho(x) dx \right], \quad \rho > 0, \quad \rho \neq 1.$$ \hspace{1cm} (23)

From (15), we have

$$f(x; \theta)^\rho = \left( \sum_{i=0}^{\infty} v_i x^{(i+1)\alpha-1} \right)^\rho.

In order to obtain an expansion for the above power series for $\rho > 0$, we can write

$$f(x; \theta)^\rho = \sum_{j=0}^{\infty} \sum_{r=0}^{j} (-1)^{j+r} \binom{\rho}{j} \binom{j}{r} d_{i,r} x^{(i+r)\alpha-r},

where $d_{0,r} = v_0^r$ and $d_{s,r} = (sv_0)^{-1} \sum_{m=1}^{s} (mr-s+m)v_m d_{s-m,r}$ for all $s \geq 1$. Hence,

$$J_R(\rho) = \frac{1}{1-\rho} \log \left[ \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} (-1)^{j+r} \binom{\rho}{j} \binom{j}{r} d_{i,r} \right].$$

10 Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample from the GKw($\lambda, \alpha, \beta, \gamma, \delta$) distribution. From (17), the log-likelihood function is easy to derive. It is given by

$$\ell(\theta) = n \log(\lambda) + n \log(\alpha) + n \log(\beta) - n \log[B(\gamma, \delta + 1)] + (\alpha - 1) \sum_{i=1}^{n} \log(x_i)

+ (\beta - 1) \sum_{i=1}^{n} \log(1 - x_i^\alpha) + (\gamma \lambda - 1) \sum_{i=1}^{n} \log[1 - (1 - x_i^\alpha)^\beta] + \delta \sum_{i=1}^{n} \log[1 - \{1 - (1 - x_i^\alpha)^\beta \}^\lambda].$$
By taking the partial derivatives of the log-likelihood function with respect to $\lambda, \alpha, \beta, \gamma$ and $\delta$, we obtain the components of the score vector, $U(\theta) = (U_\alpha, U_\beta, U_\gamma, U_\delta, U_\lambda)$. They are given by

$$U_\alpha(\theta) = \frac{n}{\alpha} + \sum_{i=1}^{n} \left[1 - (\beta - 1)z_i\right] \log(x_i) + (\gamma\lambda - 1) \sum_{i=1}^{n} \frac{\hat{y}_i(\alpha)}{y_i} - \delta \lambda \sum_{i=1}^{n} v_i \hat{y}_i(\alpha),$$

$$U_\beta(\theta) = \frac{n}{\beta} + \sum_{i=1}^{n} \log(1 - x_i^{\alpha}) + (\gamma\lambda - 1) \sum_{i=1}^{n} \frac{\hat{y}_i(\beta)}{y_i} - \lambda \delta \sum_{i=1}^{n} v_i \hat{y}_i(\beta),$$

$$U_\gamma(\theta) = -n[\psi(\gamma) - \psi(\gamma + \delta + 1)] + \lambda \sum_{i=1}^{n} \log(y_i),$$

$$U_\delta(\theta) = -n[\psi(\delta + 1) - \psi(\gamma + \delta + 1)] + \sum_{i=1}^{n} \log(1 - y_i^{\lambda}),$$

$$U_\lambda(\theta) = \frac{n}{\lambda} + \sum_{i=1}^{n} \left[\gamma - \delta y_i v_i\right] \log(y_i),$$

where $\psi(\cdot)$ is the digamma function, $y_i = 1 - (1 - x_i^{\alpha})^{\beta}$, $v_i = y_i^{\lambda - 1}(1 - y_i^{-\lambda})^{-1}$, $z_i = x_i^{\alpha}(1 - x_i^{\alpha})^{-1}$, $\hat{y}_i(\alpha) = \partial y_i / \partial \alpha = -\beta x_i^{\alpha}(1 - x_i^{\alpha})^{\beta - 1} \log(x_i)$ and $\hat{y}_i(\beta) = \partial y_i / \partial \beta = -(1 - x_i^{\alpha})^{\beta} \log(1 - x_i^{\alpha})$. For interval estimation and hypothesis tests on the model parameters, the observed information matrix is required. The observed information matrix $J = J(\theta)$ is given in the Appendix.

Under conditions that are fulfilled for parameters in the interior of the parameter space, the approximate distribution of $\sqrt{n}(\theta - \hat{\theta})$ is multivariate normal $N_5(0, I^{-1}(\theta))$, where $\hat{\theta}$ is the maximum likelihood estimator (MLE) of $\theta$ and $I(\theta)$ is the expected information matrix. This approximation is also valid if $I(\theta)$ is replaced by $J(\theta)$.

The multivariate normal $N_5(0, J^{-1}(\hat{\theta}))$ distribution can be used to construct approximate confidence regions. The well-known likelihood ratio (LR) statistic can be used for testing hypotheses on the model parameters in the usual way. In particular, this statistic is useful to check if the fit using the $GKw$ distribution is statistically superior to a fit using the $BKw$, $EKw$ and $Kw$ distributions for a given data set. For example, the test of $H_0 : \lambda = 1$ versus $H_1 : \lambda \neq 1$ is equivalent to compare the $BKw$ distribution with the $GKw$ distribution and the LR statistic reduces to $w = 2[\ell(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, \hat{\lambda}) - \ell(\hat{\alpha}, \beta, \gamma, \delta, 1)]$, where $\theta$ and $\hat{\theta}$ are the unrestricted and restricted MLEs of $\theta$, respectively. Under the null hypothesis, $w$ is asymptotically distributed as $\chi^2_1$. For a given level $\zeta$, the LR test rejects $H_0$ if $w$ exceeds the $(1 - \zeta)$-quantile of the $\chi^2_1$ distribution.

11 Application

This section contains an application of the $GKw$ distribution to real data. The data are the observed percentage of children living in households with per capita income less than R$ 75.50 in 1991 in 5509 Brazilian municipal districts. The data were extracted from the Atlas of
Brazil Human Development database available at [http://www.pnud.org.br/](http://www.pnud.org.br/). The histogram of the data is shown in Figure 2 along with the estimated densities of the $GKw$ distribution and some special sub-models. Apparently, the $GKw$ distribution gives the best fit.

The $GKw$ model includes some sub-models described in Section 3 as special cases and thus allows their evaluation relative to each other and to a more general model. As mentioned before, we can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain the LR statistics for testing some sub-models of the $GKw$ distribution. We test $H_0: (\alpha, \beta, \lambda) = (1, 1, 1)$ versus $H_1: H_0$ is not true, i.e. we compare the $GKw$ model with the beta model. In this case, $w = 2\{1510.7 - 1271.6\} = 239.1$ (p-value< 0.001) indicates that the $GKw$ model gives a better representation of the data than the beta distribution. Further, the LR statistic for testing $H_0: \lambda = 1$ versus $H_1: \lambda \neq 1$, i.e. to compare the $GKw$ model with the $BKw$ model, is $w = 2(1510.7 - 1383.6) = 254.2$ (p-value < 0.001). It also yields favorable indication for the $GKw$ model. Table 1 lists the MLEs of the model parameters (standard errors in parentheses) for different models. The computations were carried out using the subroutine MAXBFGS implemented in the Ox matrix programming language (Doornik, 2007).

Table 1: MLEs of the model parameters.

| Distribution | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\lambda$ | $\ell(\theta)$ |
|--------------|----------|---------|----------|----------|-----------|----------------|
| $GKw$        | 18.1161  | 1.8132  | 0.7303   | 0.0609   | 15.7803   | 1510.6670     |
|              | (0.1829) | (0.0219)| (0.0057) | (0.0008) | (0.8908)  |                |
| $BKw$        | 0.0247   | 0.1849  | 26.0933  | 17.3768  |           | 1383.5690     |
|              | (0.0003) | (0.0005)| (0.1054) | (0.0739) |           |                |
| $KKw$        | 2.7191   | 0.4654  | 0.0968   | 79.9999  |           | 1405.0650     |
|              | (0.0086) | (0.0060)| (0.0005) | (1.0915) |           |                |
| $PKw$        | 17.9676  | 0.1647  | 1.1533   |           |           | 1237.5800     |
|              | (0.2421) | (0.0019)|           |           |           |                |
| $BP$         |          |         | 0.1590   | 16.7313  | 0.2941    | 1269.9760     |
|              |          |         | (0.0018) | (0.1998) | (0.0129)  |                |
| $Kw$         | 2.4877   | 1.3369  |          |          | 2.4877    | 1278.7860     |
|              | (0.0295) | (0.0180)|          |          | (0.0147)  |                |
| $Beta$       |          |         | 2.5678   | 0.3010   |           | 1271.5610     |
|              |          |         | (0.0317) | (0.0147) |           |                |

12 Conclusions

We introduce a new five-parameter continuous distribution on the standard unit interval which generalizes the beta, Kumaraswamy (Kumaraswamy, 1980) and McDonald (McDonald,
Figure 2: Histogram and estimated pdf’s for the percentage of children living in households with per capita income less than R$ 75.50 (1991) in 5509 Brazilian municipal districts.
1984) distributions and includes as special sub-models other distributions discussed in the literature. We refer to the new model as the generalized Kumaraswamy distribution and study some of its mathematical properties. We demonstrate that the generalized Kumaraswamy density function can be expressed as a mixture of Kumaraswamy and power densities. We provide the moments and a closed form expression for the moment generating function. Explicit expressions are derived for the mean deviations, Bonferroni and Lorenz curves and Rényi’s entropy. The density of the order statistics can also be expressed in terms of an infinite mixture of power densities. We obtain two explicit expressions for their moments. Parameter estimation is approached by maximum likelihood. The usefulness of the new distribution is illustrated in an analysis of real data. We hope that the proposed extended model may attract wider applications in the analysis of proportions data.

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Appendix

The elements of the observed information matrix $J(\theta)$ for $(\alpha, \beta, \gamma, \delta, \lambda)$ are

\[
J_{\alpha\alpha} = -\frac{n}{\alpha^2} - (\beta - 1) \sum_{i=1}^{n} \frac{\hat{z}_i}{x_i} \log(x_i) + (\gamma \lambda - 1) \sum_{i=1}^{n} \left\{ \frac{\hat{y}_i}{y_i} - \left( \frac{\hat{y}_i}{y_i} \right)^2 \right\} - \\
\delta \lambda \sum_{i=1}^{n} (\hat{v}_i \hat{y}_i + v_i \hat{y}_i),
\]

\[
J_{\alpha\beta} = -\sum_{i=1}^{n} \frac{\hat{z}_i}{x_i} \log(x_i) + (\gamma \lambda - 1) \sum_{i=1}^{n} \left\{ \frac{\hat{y}_i}{y_i} - \left( \frac{\hat{y}_i}{y_i} \right)^2 \right\} - \delta \lambda \sum_{i=1}^{n} (\hat{v}_i \hat{y}_i + v_i \hat{y}_i),
\]

\[
J_{\alpha\gamma} = \lambda \sum_{i=1}^{n} \frac{\hat{y}_i}{y_i}, \quad J_{\alpha\delta} = -\lambda \sum_{i=1}^{n} v_i \hat{y}_i, \quad J_{\alpha\lambda} = \sum_{i=1}^{n} \left\{ \frac{\gamma}{y_i} - \delta \hat{v}_i \right\} \hat{y}_i,
\]

18
\[ J_{\beta\beta} = -\frac{n}{\beta^2} + (\gamma\lambda - 1) \sum_{i=1}^{n} \left\{ \frac{\ddot{y}_i(\beta)}{y_i} - \left( \frac{\ddot{y}_i(\beta)}{y_i} \right)^2 \right\} - \delta \lambda \sum_{i=1}^{n} \left( \dot{v}_i(\beta) \ddot{y}_i(\beta) + v_i \dddot{y}_i(\beta) \right), \]

\[ J_{\beta\gamma} = \lambda \sum_{i=1}^{n} \frac{\ddot{y}_i(\beta)}{y_i}, \quad J_{\beta\delta} = -\lambda \sum_{i=1}^{n} v_i \dot{y}_i(\beta), \quad J_{\beta\lambda} = \gamma \sum_{i=1}^{n} \frac{\ddot{y}_i(\beta)}{y_i} - \delta \sum_{i=1}^{n} v_i \dddot{y}_i(\beta), \]

\[ J_{\gamma\gamma} = -n \left\{ \psi'(\gamma) - \psi'(\gamma + \delta + 1) \right\}, \quad J_{\gamma\delta} = n \psi'(\gamma + \delta + 1), \quad J_{\gamma\lambda} = \sum_{i=1}^{n} \log(y_i), \]

\[ J_{\delta\delta} = -n \left\{ \psi'(\delta + 1) - \psi'(\gamma + \delta + 1) \right\}, \quad J_{\delta\lambda} = -\sum_{i=1}^{n} y_i v_i \log(y_i), \]

\[ J_{\lambda\lambda} = -\frac{n}{\lambda^2} - \delta \sum_{i=1}^{n} y_i \dot{v}_i(\lambda) \log(y_i), \]

where \( \dot{z}_i(\alpha) = \partial z_i / \partial \alpha = (1 + z_i) z_i \log(x_i) \), \( \ddot{y}_i(\alpha) = \partial^2 y_i / \partial \alpha^2 = \{1 - (\beta - 1) z_i\} \ddot{y}_i(\alpha) \log(x_i) \),
\( \ddot{y}_i(\beta) = \partial^2 y_i / \partial \beta^2 = \dot{y}_i(\beta) \log(1 - x_i^\alpha) \), \( \ddot{y}_i(\alpha, \beta) = \partial^2 y_i / \partial \alpha \partial \beta = \{1/\beta + \log(1 + x_i^\alpha)\} \ddot{y}_i(\alpha) \), \( \dot{v}_i(\alpha) = \partial v_i / \partial \alpha = \{(\lambda - 1)/y_i + \lambda v_i\} v_i \dot{y}_i(\alpha) \), \( \dot{v}_i(\beta) = \partial v_i / \partial \beta = \{(\lambda - 1)/y_i + \lambda v_i\} v_i \dot{y}_i(\beta) \), \( \dot{v}_i(\lambda) = \partial v_i / \partial \lambda = (1 + y_i v_i) v_i \log(y_i) \) and \( \psi'(\cdot) \) is the first derivative of the digamma function.

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