High-Order Oracle Complexity
of Smooth and Strongly Convex Optimization

Guy Kornowski  Ohad Shamir
Weizmann Institute of Science

October 15, 2020

Abstract

In this note, we consider the complexity of optimizing a highly smooth (Lipschitz $k$-th order derivative) and strongly convex function, via calls to a $k$-th order oracle which returns the value and first $k$ derivatives of the function at a given point, and where the dimension is unrestricted. Extending the techniques introduced in Arjevani et al. [2019], we prove that the worst-case oracle complexity for any fixed $k$ to optimize the function up to accuracy $\epsilon$ is on the order of $(\mu_k D^{k-1})^\frac{2}{k+1} + \log \log \left(\frac{1}{\epsilon}\right)$ (up to log factors independent of $\epsilon$), where $\mu_k$ is the Lipschitz constant of the $k$-th derivative, $D$ is the initial distance to the optimum, and $\lambda$ is the strong convexity parameter.

1 Introduction

The complexity of optimizing functions of a given class using iterative methods is a fundamental question in the theory of optimization. A standard way to approach this is via oracle-based complexity (cf. Nemirovsky and Yudin [1983]): Given local access to the function’s values and derivatives at various points, how many such points do we need to sequentially query in order to optimize any function of a given class to some target accuracy $\epsilon$? This forms a good model for generic, unstructured optimization problems, where most practical methods are iterative and rely on such local information.

Classical results on oracle complexity focus mostly on zero-order and first-order oracles, which provide information about the function’s values or gradients, and thus capture algorithms which rely on such local information, such as gradient descent and accelerated gradient descent [Nemirovsky and Yudin, 1983, Nemirovski, 2005, Nesterov, 2018]. However, there has been much progress in recent years in understanding methods which rely on Hessians and higher-order derivatives, both in terms of new methods and complexity upper bounds (e.g., Nesterov and Polyak [2006], Nesterov [2008], Baes [2009], Bubeck et al. [2019], Jiang et al. [2019], Gasnikov et al. [2019], Kamzolov and Gasnikov [2020], Nesterov [2020a,b]), as well as complexity lower bounds [Arjevani et al., 2019, Agarwal and Hazan, 2018].

In this note, we focus on a particularly well-behaved class of functions on $\mathbb{R}^d$: Those which are both highly-smooth (with a globally $\mu_k$-Lipschitz $k$-th order derivative, for some fixed $k$, $\mu_k$), and strongly convex with parameter $\lambda$\footnote{In our case of twice differentiable functions, this can be defined as $\nabla^2 f(x) \succeq \lambda I$ for all $x \in \mathbb{R}^d$.} . Generalizing and extending the techniques introduced in [Arjevani et al., 2019], we...
formally prove an oracle complexity lower bound of

\[ \Omega \left( \left( \frac{\mu_k D^{k-1}}{\lambda} \right)^{\frac{2}{3k+1}} + \log \log \left( \frac{1}{\epsilon} \right) \right) \]

for any sufficiently small \( \epsilon \), where \( D \) is the initial distance from the global optimum. This lower bound holds for any deterministic algorithm\(^2\). Moreover, under a mild assumption on the size of the derivatives at the optimum, we show that the lower bound above can be attained (up to logarithmic factors independent of \( \epsilon \)), using a combination of the accelerated Taylor descent algorithm of Bubeck et al. [2019] and the cubic regularized Newton algorithm from Nesterov [2008].

The results generalize those of Arjevani et al. [2019] (which in the strongly convex case considered only second-order oracles), and provide qualitatively similar conclusions. For example, the bound implies that even though the convergence order is eventually quadratic (as captured by the \( \log \log(1/\epsilon) \) term, growing extremely slowly in \( \epsilon \)), the optimization complexity is largely influenced by geometry-dependent factors such as the initial distance \( D \) and the Lipschitz/strong convexity parameters \( \mu_k \) and \( \lambda \) (although the latter two are attenuated as \( k \) increases, due to the \( \frac{2}{3k+1} \) power term). Moreover, the bound implies that higher-order methods (with any \( k > 1 \) and assuming only Lipschitzness of the \( k \)-th order derivatives) must have a polynomial dependence on \( D \), in sharp contrast to first-order methods (\( k = 1 \)) which can actually attain logarithmic dependence on \( D \), assuming the gradient is Lipschitz.

### 2 Main results

Consider a high-order oracle, which given a point \( x \in \mathbb{R}^d \) returns some function’s value and all of its derivatives up to order \( k \) evaluated at the point: \( f(x), \nabla f(x), \nabla^2 f(x), \ldots, \nabla^k f(x) \). Given access to such an oracle, an algorithm produces a sequence of points \( x_1, x_2, \ldots, x_T \), with each \( x_t \) being some deterministic function of the oracle’s responses at \( x_1, x_2, \ldots, x_{t-1} \). The algorithm’s goal is to approximate the function’s global minimum \( x^* \). That is, after some \( T \) queries, produce \( x_T \) such that \( f(x_T) - f(x^*) \leq \epsilon \). We will only consider objective functions \( f \) which come from the relatively well behaved class of \( \lambda \)-strongly convex, \( k \)-times differentiable functions with a \( \mu_k \)-Lipschitz \( k \)-th order derivative (with respect to the tensor operator norm).

To state our lower bound, we will make two weak assumptions, which essentially ensure that all terms in the bound are at least some positive constant, and that we take logarithms of positive quantities:

**Assumption 1.**

1. (We do not start off too lucky)

\[ \|x^1 - x^*\| > \max \left\{ \sqrt{3} \left( \frac{k! 2^{k+3}}{\mu_k \lambda} x^{\frac{1}{k-1}} \right)^{\frac{1}{2}} , \sqrt{3} \frac{12}{2^{(k+1)}} \left( \frac{k! 2^{k+3}}{\mu_k \lambda} \right)^{\frac{3}{2}} \right\} \]

2. (Our standards are not too low) \( \epsilon < \min \left\{ \left( \frac{(4-k)^2 \lambda^{k+1}}{\mu_k} \right)^{\frac{1}{k-1}}, \frac{1}{k} \left( \left( \frac{k! 2^{k+3}}{\mu_k \lambda} \right)^{\frac{3}{k-1}} \right) \right\} \)

**Theorem 1.** For any \( k \in \mathbb{N} \) and positive \( \lambda, \mu_k, D, \epsilon \) that satisfy assumption 1, and an algorithm based on a \( k \)-th order oracle as described above, there exists a function \( f: \mathbb{R}^d \to \mathbb{R} \) such that:

\(^2\)This captures standard algorithms for this setting, and is mostly for simplicity. Indeed, the proof can be extended to any randomized algorithm using, for example, the techniques of Woodworth and Srebro [2017], which performed a similar extension for first-order oracles, at the cost of a considerably more involved proof.
• \( f \) is \( \lambda \)-strongly convex, \( k \) times differentiable with \( \mu_k \)-Lipschitz \( k \)-th order derivative and has a global minimum \( x^* \) satisfying \( \| x^1 - x^* \| \leq D \).

• The index \( T \) required to ensure \( f(x^T) - f(x^*) \leq \epsilon \) is at least

\[
c \cdot \left( \frac{\mu_k D^{k-1}}{3 + \frac{1}{2} + \frac{k}{k!} \lambda} \right)^{\frac{2}{3k+1}} + \log \log \left( \frac{(4 \cdot k!)^2 \lambda^{k+1}}{\mu_k^2} + 1 \right) \frac{1}{\epsilon}
\]

for some absolute constant \( c > 0 \). Consequently,

\[
T \geq c_k \left( \frac{\mu_k D^{k-1}}{\lambda} \right)^{\frac{2}{3k+1}} + \log \left( \frac{\lambda^{k+1}}{\mu_k} + 1 \right) \frac{1}{\epsilon}
\]

for some constant \( c_k > 0 \) which depends only on \( k \).

In the same setting as above, denote \( M := \max \{ \| \nabla f(x^*) \|, \ldots, \| \nabla^k f(x^*) \| \} \) (under the operator norm). Once again, for technical reasons we will make two weak assumptions:

**Assumption 2.**

1. (We do not start off too lucky) \( D > 2 \)

2. (Our standards are not too low) \( \epsilon < \frac{\lambda^3}{2M^2} \)

The following theorem states that the lower bound is essentially tight up to a logarithmic factor.

**Theorem 2.** Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a function which satisfies the same assumptions as in Theorem 1. Assume the parameters satisfy assumption 2 (instead of 1). Then there exists a deterministic algorithm that utilizes a \( k \)-th order oracle of \( f \) and produces a sequence \( x^1, x^2, \ldots \) such that \( f(x^T) - f(x^*) < \epsilon \) where

\[
T = C_k \cdot \tilde{O} \left( \left( \frac{\mu_k D^{k-1}}{\lambda} \right)^{\frac{2}{3k+1}} + \log \left( \frac{\lambda^3}{M^2 \epsilon} \right) \right)
\]

for some constant \( C_k > 0 \) which depends only on \( k \).

**Remark.** More specifically, the \( \tilde{O}(\cdot) \) hides the fact that the first summand is multiplied by

\[
\log \left( \frac{64 M^2 D^2 \lambda}{\lambda} \cdot \left( \frac{\mu_k}{M} \right)^{\frac{2}{3k+2}} \right)
\]

Interestingly, note that \( M \) appears in the bound only inside logarithmic factors.

The proofs of the theorems appear in the next sections, and are a generalization of the techniques used in Arjevani et al. [2019] for the convex and 2nd-order oracle cases. In a nutshell, the lower bound is based on a rotation of a strongly convex function of roughly the form

\[
f(x_1, x_2, \ldots) = \sum_{j=1}^{T-1} |x_i - x_{i+1}|^{k+1} - x_1 + \frac{\lambda}{2} \| x \|^2 .
\]

The structure of the function is inspired by Nesterov’s quadratic “worst function in the world” for first-order oracles Nesterov [2018]. Here, higher-order powers and algorithm-dependent rotations are used to ensure
that with a \( k \)-th order oracle, the algorithm is forced to query only on a certain subspace, whose points are relatively far from the global optimum. The main technical difficulty of the proof is in understanding the structure of the global optimum, and proving a lower bound on its distance from the relevant subspace. As to the upper bound in Theorem 2, the proof is algorithmic in nature. We start by using a recent high order optimization algorithm called “Accelerated Taylor Descent” (ATD) [Bubeck et al., 2019] which was designed and analyzed for highly smooth convex functions (not strongly convex ones). We utilize the facts that in the strongly convex case, decreasing the error necessarily decreasing the distance to the optimum, and that the error bounds for ATD decay with the initial distance. Thus, by repeatedly applying and restarting ATD, we get a linear convergence towards the optimal point (a similar idea was also utilized in Arjevani et al. [2019] for the second-order oracle case). Once this gets us close enough to the optimum, we switch to performing cubic regularized Newton (CRN) steps [Nesterov, 2008]. Naively we would like to gain the quadratic convergence rate that CRN is proved to achieve for strongly convex functions when initialized close enough to the optimum. The difficulty here lies in the fact that CRN achieves this rate only under the additional assumption that the Hessians are globally Lipschitz, whereas we here we assume that the \( k \)-th order derivatives (for possible \( k > 2 \)) are Lipschitz. To circumvent this, we make the additional mild assumption that the derivatives are bounded at the optimum by \( M \), and combine this with the \( k \)-th order Lipschitz assumption to guarantee sufficiently Lipschitz Hessians in a neighborhood of the optimum.

3 Proof of Theorem 1

The proof is constructed of several parts. In subsection 3.1 we define a parameterized family of functions and prove their respective minima satisfy certain qualities. Then, in subsection 3.2 we introduce yet another, richer, parameterized family of functions and relate it to the earlier class. Afterwards, in subsection 3.3 we provide an oracle complexity lower bound for the class we constructed. Finally, in subsection 3.4 we choose the remaining parameters such that they fix a function \( f \) which proves the theorem.

3.1 Simplified function

Fix \( k, d, \tilde{T} \in \mathbb{N}, \gamma, \tilde{\lambda} > 0 \). Assume \( d > \tilde{T} \geq \frac{4\gamma}{\tilde{\lambda}^{k-1}}, \gamma \geq \max\{\tilde{\lambda}^{k-1}, 12k+1\tilde{\lambda}^{2k-1}\} \) and consider the function

\[
\tilde{f}_\gamma(x_1, \ldots, x_d) = \frac{1}{k+1} \sum_{i=1}^{\tilde{T}-1} |x_i - x_{i+1}|^{k+1} - \gamma x_1 + \frac{\tilde{\lambda}}{2} \|x\|^2
\]

For the sake of notational simplicity, we will assume \( \gamma \) is fixed and denote the function by \( \tilde{f} \). Note that \( \tilde{f} \) is a \( \tilde{\lambda} \)-strongly convex function as it is the sum of \( \tilde{T} \) convex functions and a \( \tilde{\lambda} \)-strongly convex function. Thus \( \bar{x}^* := \arg\min_{x \in \mathbb{R}^d} \tilde{f}(x) \) exists and is unique. We will now prove some of it’s properties.

Lemma 3. 

1. \( \forall t \in [\tilde{T}] : \bar{x}^*_t \geq 0 \)
2. \( \bar{x}^*_1 \geq \bar{x}^*_2 \geq \ldots \geq \bar{x}^*_\tilde{T} \)
3. \( \forall t \in [\tilde{T} - 1] : \bar{x}^*_{t+1} = \bar{x}^*_t - (\gamma - \tilde{\lambda}\sum_{j=1}^{t}\bar{x}^*_j)^{1/k} \)
4. \( \sum_{t=1}^{\tilde{T}} \bar{x}^*_t = \frac{\tilde{\lambda}}{\gamma} \)
Lemma 4. \( \forall t \in [\bar{T}] : \bar{x}^*_t \geq max \left\{ 0, \frac{k+1}{2} \gamma^{1/k} + (1 - t) \gamma^{1/k} \right\} \)
Proof. From Lemma 1.3, using Lemma 1.1 we get that $\forall t \in [T - 1]: x_{t+1}^* \geq x_t^* - \gamma^{1/k}$. Inductively:

$$\forall t \in [T]: x_{t+1}^* \geq x_t^* - (t - 1)\gamma^{1/k} \quad (2)$$

Using Lemma 1.4 and rolling up our sleeves:

$$\frac{\gamma}{\lambda} = \sum_{t=1}^T x_t^* \geq \sum_{t=1}^T \max\{0, x_1^* - (t - 1)\frac{\gamma^{1/k}}{t}\} = \sum_{t=1}^T \left(x_1^* - (t - 1)\frac{\gamma^{1/k}}{t}\right)$$

$$= \left\lfloor \frac{x_1^*}{\gamma^{1/k}} \right\rfloor \frac{1}{\gamma^{1/k}} \geq \frac{1}{\gamma^{1/k}} \left(\sum_{j=1}^{t-1} x_j^* \right)$$

By rearranging we obtain: $$(x_1^*)^2 - \gamma^{1/k} \cdot x_1^* \leq \frac{2^{k-1} - 1}{2^k} \gamma$$

On the other hand, using Lemma 1.3 again, we have that if $t \in [T - 1]$ satisfies $\sum_{j=1}^t x_j^* \leq \frac{2^k - 1}{2^k} \gamma$, then: $x_{t+1}^* \leq x_t^* - (\gamma - \frac{2^k - 1}{2^k} \gamma) = \frac{2^k - 1}{2^k} \gamma$. Inductively:

$$\forall t \in [T - 1]: \sum_{j=1}^t x_j^* \leq \frac{2^k - 1}{2^k} \gamma \implies x_{t+1}^* \leq x_t^* - \frac{\gamma^{1/k}}{2} \quad (4)$$

Let $t_0$ be the minimal index such that $\sum_{j=1}^{t_0} x_j^* > \frac{2^k - 1}{2^k} \gamma$ (such an index exists from Lemma 1.4). Combining Lemma 1.2 with (3) reveals:

$$\frac{2^k - 1}{2^k} \gamma < \sum_{j=1}^{t_0} x_j^* \leq t_0 x_1^* \leq t_0 \left(\frac{\gamma^{1/k}}{2} + \sqrt{\frac{2(\gamma^{1/k})}{\lambda}}\right)$$

$$\implies t_0 \geq \frac{2^k - 1}{2^k} \cdot \frac{\gamma}{\lambda \gamma^{1/k} + \sqrt{2\lambda (\gamma^{1/k})}}$$

From the minimality of $t_0$, we know that (4) applies to $t_0 - 1$, thus: $0 \leq x_{t_0}^* \leq x_1^* - \frac{(t_0 - 1)}{2^k} (\gamma)^{1/k}$. We obtain:

$$x_1^* \geq \frac{t_0 \gamma^{1/k}}{2} - \frac{\gamma^{1/k}}{2} \geq \frac{2^k - 1}{2^{k+1}} \cdot \frac{\gamma^{1+1/k}}{\lambda \gamma^{1/k} + \sqrt{2\lambda (\gamma^{1+1/k})}}$$

Finally, plugging the last inequality into (2) and rearranging gives

$$x_t^* \geq \max\left\{0, \frac{2^k - 1}{2^{k+1}} \left(\frac{\gamma}{\lambda + \sqrt{2\lambda (\gamma^{1-1/k})}}\right) + \left(\frac{1}{2} - t\right) \gamma^{1/k}\right\}$$

The lemma follows easily from our assumption that $\gamma \geq \lambda^{k-1}$. \hfill \Box
Lemma 5. There exists an index $t_0 \leq \frac{T}{2}$ such that:
\[\overline{x}_{t_0+j} \geq \lambda^{\frac{1}{k-1}} (6)^{-k^{j+1}}, \forall j \in \{0, 1, ..., \overline{T} - t_0\}\]

Proof. By Lemma 3.4, $\forall t \in [\overline{T} - 1]$

\[\overline{x}_t^* = \overline{x}_{t+1}^* + \left( \gamma - \lambda \sum_{j=1}^{t} \overline{x}_j^* \right)^{1/k} = \overline{x}_{t+1}^* + \left( \gamma - \lambda \overline{x}_j^* \right)^{1/k} = \overline{x}_{t+1}^* + \left( \lambda \sum_{j=t+1}^{\overline{T}} \overline{x}_j^* \right)^{1/k}\]

In particular, due to 3.1:
\[\overline{x}_t^* \geq \left( \overline{x}_{t+1}^* \right)^{1/k} \geq \lambda^{\overline{x}_{t+1}^*} \iff \forall t \in [\overline{T} - 1] : \overline{x}_{t+1}^* \leq \frac{1}{\lambda} (\overline{x}_t^*)^k\]

Which by induction takes the form:
\[\overline{x}_{t+j}^* \leq \frac{(\overline{x}_t^*)^k}{(\lambda)^{j+1}}, \forall t, j \geq 0 : j + t \leq \overline{T}\]

(6)

Now let’s fix an index $t \in [\overline{T} - 1]$ for which $\overline{x}_{t+1}^* \leq \frac{\lambda^{1/2}}{2}$. Notice that such an index exists due to our assumption on $\overline{T}$. For such an index, we get from (6):

\[\sum_{j=t+1}^{\overline{T}} \overline{x}_j^* = \sum_{j=0}^{\overline{T}-t-1} \overline{x}_{t+1+j}^* \leq \sum_{j=0}^{\overline{T}-t-1} \frac{(\overline{x}_{t+1}^*)^{k^j}}{(\lambda)^{j+1}} = \sum_{j=0}^{\overline{T}-t-1} \overline{x}_{t+1}^* \left( \frac{1}{2} \right)^{k^j-1} \leq 2 \overline{x}_{t+1}^*\]

Combining the last inequality, (5), and Lemma 3.2 we get that for all $t \in [\overline{T} - 1]$ such that $\overline{x}_{t+1}^* \leq \frac{\lambda^{1/2}}{2}$:

\[\overline{x}_t^* = \overline{x}_{t+1}^* + \left( \overline{x}_{t+1}^* \right)^{1/k} \leq \overline{x}_{t+1}^* + \left( 2 \lambda \overline{x}_{t+1}^* \right)^{1/k} = \overline{x}_{t+1}^* \left( \frac{1}{2} \right)^{1/k} \leq 2 \overline{x}_{t+1}^*\]

\[\overline{x}_{t+1}^* \leq \lambda^{\frac{1}{k}} \left( \frac{1}{2} \right)^{1/k} \leq 3 \lambda x_{t+1}^* \]

\[\sum_{j=1}^{\overline{T}} \overline{x}_j^* \geq \overline{T} \overline{x}_{t+1}^* > \frac{\overline{x}_{t+1}^*}{\lambda^{1/k}} = \overline{x}_{t+1}^* \iff \overline{x}_{t+1}^* \leq \frac{\lambda^{1/k}}{2}\]
Where the last inequality can be easily verified for \( k \in \mathbb{N} \). Overall we have:

\[
\forall t \in [\tilde{T} - 1] : \bar{x}_{t+1}^* \leq \frac{\bar{x}_{t+1}^*}{2} \implies \bar{x}_{t+1}^* \geq \frac{1}{\lambda} \left( \frac{\bar{x}_{t+1}^*}{3} \right)^k
\]  

(7)

Now let’s fix \( t_0 \) as the unique index for which \( \bar{x}_{t_0}^* > \frac{\bar{x}_{t_0}^*}{2} \) and \( \bar{x}_{t_0+1}^* \leq \frac{\bar{x}_{t_0}^*}{2} \). Note that \( t_0 \leq \tilde{T}/2 \) because of our extra assumptions. Indeed, if we denote by \( t_1 \) the maximal index such that \( \bar{x}_{t_1}^* > \frac{\bar{x}_{t_0}^*}{2} \) then

\[
\frac{\gamma}{\lambda} \geq \sum_{t=1}^{t_1} \bar{x}_t^* > t_1 \cdot \frac{\bar{x}_{t_0}^*}{2} \implies t_1 < \frac{2\gamma}{\lambda} \leq \tilde{T}/2
\]

Where the last inequality follows from our assumption on \( \tilde{T} \). On the other hand, it can be verified using Lemma 4 that \( \bar{x}_t^* \geq \frac{\bar{x}_{t_0}^*}{2} \) under our assumption \( \gamma \geq (12) \frac{2k}{\lambda} \). Now, for all \( j \leq \tilde{T} - t_0 \) we can use (7) inductively:

\[
\bar{x}_{t_0+j}^* \geq \frac{1}{\lambda} \left( \frac{\bar{x}_{t_0+j-1}^*}{3} \right)^k \geq \frac{(\bar{x}_{t_0+j-2}^*)^k}{3k^{k^2+k}k^{k+1}} \geq \ldots \geq \frac{(\bar{x}_{t_0}^*)^{k^j}}{3^{k+k^2+\ldots+k^j}k^{1+k^2+k^3+\ldots+k^{j-1}}} \geq \frac{\bar{x}_{t_0}^*}{3^{k^j}2^{k+1}k^{1+k^2+k^3+\ldots+k^{j+1}}} = \frac{\bar{x}_{t_0}^*}{\lambda^{\frac{k^j}{2}}}
\]

Which finishes the proof. \( \square \)

**Lemma 6.** \( \sum_{i=1}^{\tilde{T}} (\bar{x}_i^*)^2 \leq \frac{3\gamma \bar{x}_{t_0}^*}{\lambda^2} \)

**Proof.** First we use the simple inequality:

\[
\sum_{i=1}^{\tilde{T}} \bar{x}_i^2 \leq (\max_{i \in [\tilde{T}]} |\bar{x}_i^*|) \sum_{i=1}^{\tilde{T}} |\bar{x}_i^*| = \bar{x}_1^* \frac{\gamma}{\lambda}
\]

Where the last equality follows from Lemma 3. Furthermore, we can upper bound the right hand side using (3):

\[
\bar{x}_1^* \frac{\gamma}{\lambda} \leq \left( \gamma^{1/k} + \sqrt{2} \frac{2\gamma \frac{k+1}{k}}{\lambda} \right) \frac{\gamma}{\lambda} = \left( 1 + \sqrt{2} \frac{2\gamma \frac{k+1}{k}}{\lambda} \right) ^{\frac{k+1}{k}} \frac{\lambda}{\lambda}
\]

Due to our assumption \( \gamma \geq \lambda^{\frac{k+1}{k}} \implies 1 \leq \sqrt{\frac{2\gamma \frac{k+1}{k}}{\lambda}} \) so we can further upper bound the last term by

\[
\left( \sqrt{\frac{2\gamma \frac{k+1}{k}}{\lambda}} + \sqrt{2(\gamma) \frac{k+1}{k}} \right) \frac{\gamma}{\lambda} \leq (1 + \sqrt{2} \gamma \frac{2k}{\lambda}) \frac{\lambda}{\lambda^2}
\]

Which finishes the proof since \( 1 + \sqrt{2} < 3 \). \( \square \)
3.2 Rotated and rescaled function

We will now shift our focus to a “rotated and rescaled” version of \( \tilde{f} \). Formally, let \( v_1, ..., v_T \in \mathbb{R}^d \) be orthogonal unit vectors, and define:

\[
f_{\gamma,v_1,...,v_T}(x_1, ..., x_d) = \frac{\mu_k}{k!2^{k+1}} \left( \sum_{i=1}^{T-1} g(\langle v_i - v_{i+1}, x \rangle) - \gamma \langle v_1, x \rangle \right) + \frac{\lambda}{2} \|x\|^2,
\]

\[
g(x) = \frac{1}{k+1} |x|^{k+1}
\]

For the sake of notational simplicity, we will assume \( \gamma, v_1, ..., v_T \) are somehow fixed (we will describe later on how to choose them) and denote the function by \( f \) (abbreviating the parameters’ subscript). Furthermore, we will assume the relation \( \lambda = \frac{\mu_k \lambda}{k!2^{k+1}} \) (where \( \lambda \) is needed in order to define \( \tilde{f} \) from the previous section).

Denote \( x^* = \arg \min_{x \in \mathbb{R}^d} f(x) \). In order to derive properties of \( x^* \) (analogous to subsection 3.1) we will connect between \( x^* \) and \( \tilde{x}^* \) (from subsection 3.1) through the following lemmas.

**Lemma 7.** \( \forall i \in [T] : \langle v_i, x^* \rangle = \tilde{x}_i^* \)

**Proof.** Since the global minimizer \( \tilde{x}^* \) is invariant under multiplication of the function \( \tilde{f} \) by a positive constant, we deduce that \( \tilde{x}^* \) also minimizes:

\[
\frac{\mu_k}{k!2^{k+1}} \tilde{f}(x) = \frac{\mu_k}{k!2^{k+1}} \left( \sum_{i=1}^{T-1} g(x_i - x_{i+1}) - \gamma x_1 + \frac{\lambda}{2} \|x\|^2 \right)
\]

\[
= \frac{\mu_k}{k!2^{k+1}} \left( \sum_{i=1}^{T-1} g(x_i - x_{i+1}) - \gamma x_1 \right) + \frac{\lambda}{2} \|x\|^2
\]

Now recall that \( f(x) = \frac{\mu_k}{k!2^{k+1}} \left( \sum_{i=1}^{T-1} g(\langle v_i - v_{i+1}, x \rangle) - \gamma \langle v_1, x \rangle \right) + \frac{\lambda}{2} \|x\|^2 \) and that \( v_1, ..., v_T \) are orthogonal unit vectors. We deduce that \( f(x) = \tilde{C} f(V x) \) for some positive constant \( \tilde{C} \) and for any orthogonal \( V \) such that it’s first \( T \) columns are \( v_1, ..., v_T \). Therefore \( x^* \) satisfies \( V x^* = (\langle v_1, x^* \rangle, \langle v_2, x^* \rangle, ..., \langle v_T, x^* \rangle) = \tilde{x}^* \). \( \square \)

**Lemma 8.** \( \|x^*\|^2 = \sum_{i=1}^{T} \langle v_i, x^* \rangle^2 \)

**Proof.** The proof follows immediately by combining the representer theorem and the Pythagorean theorem.

Combining the previous lemmas with subsection 3.1 results in an immediate consequence:

**Proposition 1.** Suppose that

\[
\gamma > \max \left\{ \left( \frac{k!2^{k+1}}{\mu_k} \right)^{\frac{1}{k+1}}, 12^{\frac{2k}{2k-1}} \left( \frac{k!2^{k+1}}{\mu_k} \right)^{\frac{2k}{2k-1}} \right\}, \quad \tilde{T} \geq \frac{4\gamma}{\left( \frac{k!2^{k+1}}{\mu_k} \right)^{\frac{1}{k+1}}}
\]

Then \( f \) has a unique minimizer \( x^* \) which satisfies:
1. \( \forall t \in [\tilde{T}] : \langle v_t, x^* \rangle \geq \max \left\{ 0, \frac{\gamma_{k+1}}{12\sqrt{\frac{k+3}{\mu_k}}}, \left(\frac{1}{2} - t\right)\gamma^{1/k} \right\} \)

2. There exists an index \( t_0 \leq \tilde{T} \) such that for all \( j \in [\tilde{T} - t_0] : \)
\[
\langle v_{t_0+j}, x^* \rangle \geq \left( \frac{k!2^{\frac{k+3}{2}}}{\mu_k} \lambda \right)^{1/4} \left( 6 \right)^{-k^{j+1}}
\]

3. \[
\|x^*\|^2 \leq \frac{3\gamma \frac{2k+1}{2k}}{\left( \frac{k!2^{\frac{k+3}{2}}}{\mu_k} \lambda \right)^{1/4}} \tag{9}
\]

### 3.3 Oracle complexity

We will now derive an oracle complexity lower bound using the function we introduced in the previous section, with a particular choice of \( v_1, \ldots, v_{\tilde{T}} \) which will depend on the optimization algorithm. Both our function and our lower bound will still depend on the choice of \( \gamma \), which we will pick in the next section. We start with a simple yet crucial lemma:

**Lemma 9.** If \( x \in \mathbb{R}^d \) is orthogonal to \( v_t, v_{t+1}, \ldots, v_{\tilde{T}} \), then \( f(x), \nabla f(x), \ldots, \nabla^k f(x) \) do not depend on \( v_{t+1}, v_{t+2}, \ldots, v_{\tilde{T}} \).

**Proof.** The proof follows from looking at \( f(x) - \frac{\lambda}{2}\|x\|^2 \), and applying the chain rule while noticing that \( \forall i \in [k] : g^{(i)}(0) = 0 \).

This observation allows us to provide the following oracle complexity lower bound:

**Proposition 2.** Assume that
\[
\epsilon < \min \left\{ \frac{2^{\frac{k+1}{2}}(k!)^{\frac{1}{2-k}}(\lambda)^{\frac{k+1}{k-1}}}{(\mu_k)^{\frac{1}{k-1}}} \frac{\lambda \gamma^{1/k}}{4} \right\} \tag{10}
\]

Assuming (8), it is possible to choose the vectors \( v_1, \ldots, v_{\tilde{T}} \) (in the function \( f \)) such that the number of iterations \( T \) required to have \( f(x^T) - f(x^*) \leq \epsilon \) is at least
\[
\max \left\{ \log_k \log_0 \left( \frac{2^{\frac{k+1}{2}}(k!)^{\frac{1}{2-k}}(\lambda)^{\frac{k+1}{k-1}}}{(\mu_k)^{\frac{1}{k-1}}\sqrt{\epsilon}} \right) - 1, \frac{\gamma \frac{2k+1}{2k}}{12\left( \frac{k!2^{\frac{k+3}{2}}}{\mu_k} \lambda \right)^{1/4}} \right\} \tag{11}
\]

**Proof.** Fix the number of iterations \( T \leq \tilde{T} \). Given some algorithm, we will describe a method of picking \( v_1, \ldots, v_{\tilde{T}} \) in such a way that will result in a the desired lower bound.

- First, compute \( x^1 \) (which is the deterministic algorithm’s first output, thus chosen without any oracle calls are even made).
• Pick \( v_1 \) to be some unit vector orthogonal to \( x^1 \). Assuming \( v_2, ..., v_T \) will also be orthogonal to \( x^1 \) (which we will soon ensure by construction), we have by Lemma 9 that \( f(x^1), ..., \nabla^k f(x^1) \) do not depend on \( v_2, ..., v_T \), thus depend only on \( v_1 \) which is already fixed. Since the algorithm is deterministic, this fixes the next point \( x^2 \).

• Repeat the same process for \( t = 2, 3, ..., T - 1 \): Compute \( x^t \), and pick \( v_t \) to be some unit vector orthogonal to \( x^1, ..., x^t \) and also to \( v_1, ..., v_{t-1} \) (recall the dimension is assumed to be large enough). Similar to the previous step, Lemma 9 insures the next point \( x^{t+1} \) is fixed.

• At the end of the process, pick \( v_T, ..., v_T \) to be some unit vectors which are orthogonal to all previously chosen \( v \)’s as well as \( x^1, ..., x^T \) (which is possible due to the large dimension).

We get by this construction (where \( t_0 \) is defined in proposition 1):

\[
\|x^T - x^*\|^2 \geq \sum_{i=1}^{\tilde{t}} \langle v_i, x_T - x^* \rangle^2 \geq \langle v_{t_0+T}, x_T - x^* \rangle^2 = \langle v_{t_0+T}, x^* \rangle^2
\]

Which can be lower bounded using proposition 1 by \( \left( \frac{k!2k+1}{\mu_k} \right)^{2/\gamma} \) \( (6)^{-2kT+1} \). Using the strong convexity of \( f \), we get:

\[
f(x^T) - f(x^*) \geq \frac{\lambda}{2} \|x^T - x^*\|^2 \geq \frac{\lambda}{2} \left( \frac{k!2k+1}{\mu_k} \right)^{2/\gamma} \|x_T - x^*\|^2 = \left( \frac{k!2k+1}{\mu_k} \right)^{2/\gamma} \|x_T - x^*\|^2
\]

In order to make the last expression smaller than \( \epsilon \), \( T \) must satisfy:

\[
\frac{\lambda}{2} \left( \frac{k!2k+1}{\mu_k} \right)^{2/\gamma} \geq \epsilon \iff k^{T+1} \geq \log_6 \left( \frac{2^{k-1}(k!)^{k-1}}{(\mu_k)^{k-1} \sqrt{\epsilon}} \right)
\]

Since we assume \( \epsilon < \frac{2^{k-1}(k!)^{k-1}}{(\mu_k)^{k-1}} \) the expression inside the logarithm is \( > 1 \), so we get

\[
T \geq \log_k \log_6 \left( \frac{2^{k-1}(k!)^{k-1}}{(\mu_k)^{k-1} \sqrt{\epsilon}} \right) - 1
\]

which is the first part of the lower bound. For the second part of the lower bound, notice that:

\[
\|x^T - x^*\|^2 \geq \sum_{i=1}^{T} \langle v_i, x_T - x^* \rangle^2 \geq \langle v_T, x_T - x^* \rangle^2 = \langle v_T, x^* \rangle^2
\]

\[
\iff f(x^T) - f(x^*) \geq \frac{\lambda}{2} \|x^T - x^*\|^2 \geq \frac{\lambda}{2} \langle v_T, x^* \rangle^2
\]

\[
\implies \frac{\text{prop.} 1}{2} \geq \frac{\lambda}{2} \left( \frac{\gamma^{2k}}{12 \sqrt{\frac{k!2k+1}{\mu_k}}} + \frac{1}{2} - T \right) (\gamma)^{1/k}
\]
In order to make the right hand side smaller than \( \epsilon \), \( T \) must satisfy:

\[
\lambda \left( \frac{\gamma^{k+1}}{2e^{\lambda^2/k}} + \frac{(\frac{1}{2} - T)(\gamma)^{1/k}}{\mu_k} \right) \leq \epsilon \iff T \geq \frac{\gamma^{k+1}}{12 \left( \frac{k! \cdot (\gamma^2 + 1)}{\mu k} \right)^{\frac{1}{2}}} - \frac{2\epsilon}{\lambda \gamma^{1/k}} + \frac{1}{2}
\]

And since we assumed \( \epsilon < \frac{\lambda \gamma^{k+1}}{4} \), we have that \( -\frac{2\epsilon}{\lambda \gamma^{1/k}} + \frac{1}{2} \geq 0 \), finishing the proof of the proposition.

### 3.4 Setting \( \gamma \) and wrapping up

We can assume without loss of generality that the algorithm initializes at \( x^1 = 0 \) (otherwise, simply replace \( f(x) \) by \( f(x - x^1) \)). Thus, the theorem requires that our function’s minimizer \( x^* \) will satisfy \( \|x^*\| \leq D \).

We will soon establish this using previous sections, but first we’ll show that \( f \) indeed has the structural properties we desire regardless of the choice of \( \gamma \).

**Lemma 10.** \( f \) is \( \lambda \)-strongly convex, \( k \) times differentiable, with \( \mu_k \)-Lipschitz \( k \)-th derivative.

**Proof.** \( f \) is \( \lambda \)-strongly convex as a sum of convex functions and the \( \lambda \)-strongly convex \( \lambda \frac{\gamma^{k+1}}{2e^{\lambda^2/k}} \), and is easily verified to be \( k \) times smooth. It remains to show the desired Lipschitz property. Since Lipschitzness is invariant under orthogonal transformations, we may assume without loss of generality that

\[
v_i = e_i \implies f(x) = \frac{\mu_k}{k!2^{k+1}} \left( \sum_{i=1}^{k-1} g(\langle e_i - e_{i+1}, x \rangle) - \gamma x_1 \right) + \frac{\lambda}{2} \|x\|^2
\]

Denote \( r_i = e_i - e_{i+1} \), and assume for simplicity that \( k > 2 \). The chain rule implies

\[
\nabla^{(k)} f(x) = \frac{\mu_k}{k!2^{k+1} \cdot \lambda} \left( \sum_{i=1}^{k-1} g^{(k)}(\langle r_i, x \rangle) r_i^{\otimes k} \right)
\]

where \( v^{\otimes k} = v \otimes v \otimes \ldots \otimes v \). Since \( g^{(k)}(x) = k! |x| \) we have

\[
\|\nabla^{(k)} f(x) - \nabla^{(k)} f(\bar{x})\| \leq \frac{\mu_k k!}{2^{k+1} \cdot \lambda} \left( \sum_{i=1}^{k-1} (|\langle r_i, x \rangle| - |\langle r_i, \bar{x} \rangle|) r_i^{\otimes k} \right)
\]

where the last inequality is due to the reverse triangle inequality, and the linearity of the dot product. Recall that the operator norm of a tensor is defined as \( \|T\| = \max_{\|z\|=1} \left| \sum_{i_1, \ldots, i_k} T_{i_1, \ldots, i_k} z_{i_1} \cdots z_{i_k} \right| \), so
So overall we got
\[
\|\nabla^{(k)} f(x) - \nabla^{(k)} f(\bar{x})\| \leq \frac{\mu_k}{2^{k+3}} 2^{k+3} \|x - \bar{x}\| = \mu_k \|x - \bar{x}\|
\]
\[
\]

Now all that remains is to pick \( \gamma \) in order to finish the proof of the Theorem. Our goal is to maximize the oracle complexity lower bound (11) under the constraints (8), (10), \( \|x^*\| \leq D \). Note that in order to satisfy the latter, it is enough to bound the right hand side of (9) by \( D^2 \). An elementary computation shows one should pick \( \gamma = \left( \frac{2^{k+3} 2^{k+3} D^4}{9 \mu_k^4} \right)^{k \over 3k+7} \), which results in the main theorem.

\[
\]

4 Proof of Theorem 2

Our proof is constructive - we will describe such an algorithm. Our algorithm is based on a certain combination of accelerated Taylor descent (ATD) Bubeck et al. [2019] with cubic regularized Newton (CRN) Nesterov [2008], while analyzing them under the highly smooth strongly convex regime. More specifically, we suggest an algorithm which is composed of several different phases. We build the theoretical background during subsections 4.1, 4.2, and combine them into 4.3.

4.1 Highly smooth strongly convex ATD

Assume \( f \) is as stated in the Theorem. We start by proving a basic property of high order smoothness.

**Lemma 11.** For all \( x \in \mathbb{R}^d \) such that \( \|x - x^*\| \leq D \):

\[
f(x) - f(x^*) \leq \frac{M}{4} \cdot D^{k-1} + \frac{\mu_k}{2} \cdot D^{k+1}
\]
Proof. The proof is based on recursively applying Taylor’s theorem.

\[ f(x) - f(x^*) = \nabla f(x^*) \cdot (x - x^*) + \frac{1}{2} \nabla^2 f(x^{(2)}) \cdot (x - x^*)^2 \]

\[ \leq \frac{1}{2} \|\nabla^2 f(x^{(2)})\| \cdot \|x - x^*\|^2 \]

where \( x^{(2)} \) lies on the segment between \( x^* \), \( x \). Furthermore

\[ \|\nabla^2 f(x^{(2)}) - \nabla^2 f(x^*)\| \leq \left\| \nabla^3 f(x^{(3)}) \cdot (x^{(2)} - x^*) \right\| \]

\[ \leq \|\nabla^3 f(x^{(3)})\| \cdot \|x^{(2)} - x^*\| \leq \|\nabla^3 f(x^{(3)})\| \cdot \|x - x^*\| \]

And similarly for all \( 2 \leq j < k \):

\[ \|\nabla^j f(x^{(j)}) - \nabla^j f(x^*)\| \leq \left\| \nabla^{j+1} f(x^{(j+1)}) \cdot (x^{(j)} - x^*) \right\| \]

\[ \leq \|\nabla^{j+1} f(x^{(j+1)})\| \cdot \|x - x^*\| \]

with each \( x^{(j+1)} \) lying on the line segment between \( x^{(j)} \), \( x^* \). Eventually,

\[ \|\nabla^k f(x^{(k)}) - \nabla^k f(x^*)\| \leq \mu_k \|x^{(k)} - x^*\| \leq \mu_k \|x - x^*\| \]

So recursively combining the triangle inequality with the previous inequalities gives

\[ \|\nabla^2 f(x^{(2)})\| \leq \|\nabla^2 f(x^*)\| + \|\nabla^2 f(x^{(2)}) - \nabla^2 f(x^*)\| \]

\[ \leq \|\nabla^2 f(x^*)\| + \|\nabla^3 f(x^{(3)})\| \cdot \|x - x^*\| \]

\[ \leq \|\nabla^2 f(x^*)\| + \left( \|\nabla^3 f(x^*)\| + \|\nabla^3 f(x^{(3)}) - \nabla^3 f(x^*)\| \right) \cdot \|x - x^*\| \]

\[ \ldots \leq \sum_{j=2}^{k-1} \|\nabla^j f(x^*)\| \cdot \|x - x^*\|^{j-2} + \|\nabla^k f(x^{(k)}) - \nabla^k f(x^*)\| \cdot \|x - x^*\|^{k-2} \]

\[ \leq \sum_{j=2}^{k-1} \|\nabla^j f(x^*)\| \cdot \|x - x^*\|^{j-2} + \mu_k \cdot \|x - x^*\|^{k-1} \]

\[ \leq M \cdot \sum_{j=2}^{k-1} D^{j-2} + \mu_k \cdot D^{k-1} \]

So overall

\[ f(x) - f(x^*) \leq \frac{1}{2} \left( M \cdot \sum_{j=2}^{k-1} D^{j-2} + \mu_k \cdot D^{k-1} \right) \cdot D^2 \]

\[ = \frac{1}{2} \left( M \cdot \sum_{j=2}^{k-1} D^j + \mu_k \cdot D^{k+1} \right) \leq \frac{M}{2} \cdot \sum_{j=2}^{k-1} D^j + \frac{\mu_k}{2} \cdot D^{k+1} \]

\[ = \frac{M}{2} \cdot \frac{D^k - D^2}{D - 1} + \frac{\mu_k}{2} \cdot D^{k+1} \leq \frac{M}{2} \cdot \frac{D^{k+1} - D^2}{D/2} + \frac{\mu_k}{2} \cdot D^{k+1} \]

\[ = \frac{M}{4} \cdot D^{k-1} + \frac{\mu_k}{2} \cdot D^{k+1} \]

\[ \square \]
We are now ready to describe how to utilize ATD in the presence of high smoothness and strong convexity. ATD satisfies
\[
f(x^t) - f(x^*) \leq \frac{c_k \mu_k \|x^1 - x^*\|^{k+1}}{t^{\frac{3}{k+1}}}
\]
Strong convexity implies
\[
\forall x : \frac{\lambda}{2} \|x - x^*\|^2 \leq f(x) - f(x^*)
\]
And by assumption \(\|x^1 - x^*\| \leq D\) so we can get
\[
f(x^t) - f(x^*) \leq \frac{2c_k \mu_k D^{k-1}}{\lambda t^{\frac{3}{k+1}}} (f(x^1) - f(x^*))
\]
So if we set \(\tau\) such that
\[
\frac{2c_k \mu_k D^{k-1}}{\lambda \tau^{\frac{3}{k+1}}} = \frac{1}{2} \iff \tau = \left(\frac{4c_k \mu_k D^{k-1}}{\lambda}\right)^{\frac{2}{3k+1}}
\]
we get after \(\tau\) iterations
\[
f(x^\tau) - f(x^*) \leq \frac{1}{2} (f(x^1) - f(x^*))
\]
Now notice that although accelerated schemes are not monotone in general, we have not gotten further away than our original starting point. In order to see this notice that our choice of \(\tau\) satisfies
\[
\tau \geq \left(\frac{2c_k \mu_k D^{k-1}}{\lambda}\right)^{\frac{2}{3k+1}} \implies \frac{c_k \mu_k D^{k+1}}{\tau^{\frac{3}{k+1}}} \leq \frac{\lambda D^2}{2}
\]
Thus we can utilize strong convexity and ATD’s sub-optimality guarantee and get
\[
\|x^\tau - x^*\| \leq \sqrt{\frac{2}{\lambda} (f(x^\tau) - f(x^*))} \leq \sqrt{\frac{2}{\lambda} \left(\frac{c_k \mu_k \|x^1 - x^*\|^{k+1}}{\tau^{\frac{3}{k+1}}}\right)} \leq \sqrt{\frac{2}{\lambda} \cdot \frac{\lambda D^2}{2} = D}
\]
Using this crucial observation, we can now initialize ATD with \(x^1 = x^\tau\). Iterating this process eventually gives
\[
f(x^T) - f(x^*) \leq \frac{f(x^1) - f(x^*)}{2T/\tau}
\]
In order to make the right hand side smaller than some \(\epsilon_0\), a simple rearrangement reveals it is enough to set
\[
T \geq \left(\frac{4c_k \mu_k D^{k-1}}{\lambda}\right)^{\frac{2}{3k+1}} \cdot \log_2 \left(\frac{f(x^1) - f(x^*)}{\epsilon}\right)
\]
Using Lemma 11, we conclude that it is enough to set
\[
T = \left(\frac{4c_k \mu_k D^{k-1}}{\lambda}\right)^{\frac{2}{3k+1}} \cdot \log_2 \left(\frac{MD^{k-1} + 2\mu_k \cdot D^{k+1}}{4\epsilon}\right)
\]
4.2 Cubic regularized Newton

If we were to assume that $\nabla^2 f$ is $\mu_2$-Lipschitz, then provided that we start inside the region

$$\left\{ f(x) - f(x^*) \leq \frac{\lambda^3}{2\mu_2^2} \right\}$$

CRN guarantees $\epsilon$ sub-optimality after $O\left( \log \log \left( \frac{\lambda^3}{\mu_2^2 \epsilon} \right) \right)$ steps (under our assumption on $\epsilon$). Naively, we would like to substitute $\epsilon = \frac{\lambda^3}{2\mu_2^2}$ into the iteration bound from the previous subsection and introduce a second phase in which the algorithm follows CRN. The obstacle is that we do not want to assume that $\mu_2$ exists globally. Thus we define

$$\tilde{\mu}_2(r) := \sup\{ L > 0 \mid \forall x, y \in B(x^*, r) : \|\nabla^2 f(x) - \nabla^2 f(y)\| \leq L\|x - y\| \}$$

that is, a $\mu_2$ proxy in a ball of radius $r$ around the optimum. Notice that

$$\tilde{\mu}_2(r) \leq \sup \left\{ \|\nabla^2 f(x)\| \mid x \in B(x^*, r) \right\}$$

The right hand side can be bounded exactly the same way as in the proof of Lemma 11, resulting in

$$M \cdot \sum_{j=3}^{k-1} r^{j-3} + \mu_k \cdot r^{k-2}$$

which if $r \neq 1$ equals

$$M \cdot \frac{r^{k-3} - 1}{r - 1} + \mu_k \cdot r^{k-2}$$

Overall, for all $r \neq 1$

$$\tilde{\mu}_2(r) \leq M \cdot \frac{r^{k-3} - 1}{r - 1} + \mu_k \cdot r^{k-2} \tag{13}$$

In particular, we get the following lemma:

**Lemma 12.** If $r < \min \left\{ 1, \left( \frac{M}{\mu_k} \right)^{\frac{1}{k-2}} \right\}$ then $\tilde{\mu}_2(r) \leq 2M$.

**Proof.** For such $r$ it holds that $\frac{r^{k-3} - 1}{r - 1} \leq 1$ and also $\mu_k \cdot r^{k-2} < M$. Thus

$$M \cdot \frac{r^{k-3} - 1}{r - 1} + \mu_k \cdot r^{k-2} \leq 2M$$

which by (13) finishes the proof. \hfill \square

4.3 Combining the ingredients

Our suggested algorithm starts by performing ATD for $t_1$ steps, where $t_1$ is the minimal index for which $\|x^{t_1} - x^*\| < \min \left\{ 1, \left( \frac{M}{\mu_k} \right)^{\frac{1}{k-2}} \right\}$. We will now calculate $t_1$. Using strong convexity, we know that it is
enough to ensure
\[
\sqrt{\frac{2}{\lambda}} (f(x^{t_1}) - f(x^*)) \leq \min \left\{ 1, \left( \frac{M}{\mu_k} \right)^{\frac{1}{k-2}} \right\}
\]
\[\iff f(x^{t_1}) - f(x^*) < \min \left\{ \frac{\lambda}{2}, \frac{\lambda}{2} \left( \frac{M}{\mu_k} \right)^{\frac{2}{k-2}} \right\} \]

Equation (12) allows us to convert the latter into a precise statement about \( t_1 \). A simple calculation turns out to give
\[
t_1 = \left( \frac{4c_k \mu_k D^{k-1}}{\lambda} \right)^{\frac{2}{k+1}} \cdot \log_2 \left( \frac{\left( MD^{k-1} + 2\mu_k \cdot D^{k+1} \right)^2}{8\lambda} \cdot \left( \frac{\mu_k}{M} \right)^{\frac{2}{k-2}} \right)
\]

(14)

Afterwards, we perform \( t_2 \) more steps of ATD, where \( t_2 \) is the minimal index for which
\[
f(x^{t_2}) - f(x^*) \leq \frac{\lambda^3}{32M^2}
\]

Once again we use equation (12), in order to deduce
\[
t_2 = \left( \frac{4c_k \mu_k D^{k-1}}{\lambda} \right)^{\frac{2}{k+1}} \cdot \log_2 \left( \frac{8M^3 D^{k-1} + 16M^2 \mu_k \cdot D^{k+1}}{\lambda^3} \right)
\]

(15)

At this point, the algorithm starts performing cubic regularized Newton steps. Notice that the whole point of waiting the first \( t_1 \) steps was to ensure Lemma (12) is applicable. Furthermore, the additional \( t_2 \) steps ensure we are in the region of quadratic convergence for CRN, with \( \mu_2 = \Theta(M) \). Thus, from this point onward the number of steps required to ensure \( \epsilon \) sub-optimality is
\[
\mathcal{O} \left( \log \log \left( \frac{\lambda^3}{M^2 \epsilon} \right) \right)
\]

(16)

Summing (14), (15), (16) and slightly rearranging finishes the proof.

**References**

Naman Agarwal and Elad Hazan. Lower bounds for higher-order convex optimization. In *Conference On Learning Theory*, pages 774–792. PMLR, 2018.

Yossi Arjevani, Ohad Shamir, and Ron Shiff. Oracle complexity of second-order methods for smooth convex optimization. *Mathematical Programming*, 178(1-2):327–360, 2019.

Michel Baes. Estimate sequence methods: extensions and approximations. *Institute for Operations Research, ETH, Zürich, Switzerland*, 2009.

Sébastien Bubeck, Qijia Jiang, Yin Tat Lee, Yuanzhi Li, and Aaron Sidford. Near-optimal method for highly smooth convex optimization. In *Conference on Learning Theory*, pages 492–507. PMLR, 2019.

Alexander Gasnikov, Pavel Dvurechensky, Eduard Gorbunov, Evgeniya Vorontsova, Daniil Selikhanovych, César A Uribe, Bo Jiang, Haoyue Wang, Shuzhong Zhang, Sébastien Bubeck, et al. Near optimal methods for minimizing convex functions with lipschitz \( p \)-th derivatives. In *Conference on Learning Theory*, pages 1392–1393, 2019.
Bo Jiang, Haoyue Wang, and Shuzhong Zhang. An optimal high-order tensor method for convex optimization. In Conference on Learning Theory, pages 1799–1801, 2019.

Dmitry Kamzolov and Alexander Gasnikov. Near-optimal hyperfast second-order method for convex optimization and its sliding. arXiv preprint arXiv:2002.09050, 2020.

Arkadi Nemirovski. Efficient methods in convex programming – lecture notes, 2005.

Arkadi Nemirovsky and David Yudin. Problem Complexity and Method Efficiency in Optimization. Wiley-Interscience, 1983.

Yurii Nesterov. Accelerating the cubic regularization of newton’s method on convex problems. Mathematical Programming, 112(1):159–181, 2008.

Yurii Nesterov. Lectures on convex optimization, volume 137. Springer, 2018.

Yurii Nesterov. Inexact accelerated high-order proximal-point methods. Technical report, Technical report, Technical Report CORE Discussion paper 2020, Université . . ., 2020a.

Yurii Nesterov. Superfast second-order methods for unconstrained convex optimization. CORE DP, 7:2020, 2020b.

Yurii Nesterov and Boris T Polyak. Cubic regularization of newton method and its global performance. Mathematical Programming, 108(1):177–205, 2006.

Blake Woodworth and Nathan Srebro. Lower bound for randomized first order convex optimization. arXiv preprint arXiv:1709.03594, 2017.