Non-Markovian qubit dynamics in the presence of 1/f noise

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Within the lowest-order Born approximation, we calculate the exact dynamics of a qubit in the presence of 1/f noise, without Markov approximation. We show that the non-Markovian qubit time-evolution exhibits asymmetries and beatings that can be observed experimentally and cannot be explained within a Markovian theory. The present theory for 1/f noise is relevant for both spin- and superconducting qubit realizations in solid-state devices, where 1/f noise is ubiquitous.

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I. INTRODUCTION

Random telegraph noise has been encountered in a wide range of situations in many different areas of physics. A typical example in condensed matter physics is that of a resistor coupled to an ensemble of randomly switching impurities, producing voltage fluctuations with a spectral density that scales inversely proportional with the frequency, hence the name “1/f noise”. The quest to build and coherently control quantum two-level systems functioning as qubits in various solid state systems has once more highlighted the importance of understanding 1/f noise, being a limitation to the quantum coherence of such devices.

The description of low-frequency noise (such as 1/f noise) is complicated by the presence of long-time correlations in the fluctuating environment which prohibit the use of the Markov approximation. Only in few cases, non-Markovian effects have been taken into account exactly, e.g., for the relaxation of an atom to thermal equilibrium. Here, we are interested in the decoherence and relaxation of a qubit, i.e., a single two-level system (spin 1/2). For the spin-boson model, i.e., a qubit coupled to a bath of harmonic oscillators, the dynamics has been calculated within a rigorous Born approximation without making a Markov approximation. Here, we carry out a similar analysis for 1/f noise and find even stronger effects than in the spin-boson case (see Fig. 1).

Charge and to some extent (via the spin-orbit interaction) spin qubits in quantum dots5 formed in semiconductor or carbon structures are subject to 1/f noise. In superconducting (SC) Josephson junctions, SC interference devices (SQUIDs), and SC qubits, 1/f noise has been extensively studied experimentally2,9,10,11,12,13,14,15 and theoretically16,17.

Even where the origin of 1/f noise is known, the induced decoherence is not fully understood. Most theoretical work is either restricted to longitudinal fluctuations or employs a Markov approximation. Here, we present a calculation of the qubit dynamics in the presence of 1/f noise which is exact within the lowest-order Born approximation. In particular, we make no use of a Markov approximation. In contrast to earlier calculations18,19,21,22, we allow for arbitrary qubit Hamiltonians and include transverse as well as longitudinal (phase) 1/f noise. Non-Gaussian 1/f noise originating from few fluctuators was studied in23,24,25, while numerical studies using an adiabatic approximation were carried out in26. The coupling to a single fluctuator was also studied27.

II. MODEL

We model the qubit (spin 1/2) coupled to a bath of two-level fluctuators with the Hamiltonian

\[ H = H_S + H_B + H_{SB} \]  

with

\[ H_S = \Delta \sigma_x + \epsilon \sigma_z, \]  

\[ H_{SB} = \sigma_z X, \]

where \( \sigma_x \) and \( \sigma_z \) are Pauli matrices describing the qubit and \( X = \sum_{i=1}^{N} v_i \sigma_i^z \) where \( \sigma_i^z \) operates on the \( i \)-th fluctuator. In a SC qubit, \( \Delta \) and \( \epsilon \) denote the tunneling
and energy bias between the two qubit states. In a spin qubit, $\epsilon$ is the Zeeman splitting and $\Delta$ a transverse field. The bath Hamiltonian $H_B$ need not be provided explicitly; it is sufficient to know the auto-correlator $C(t) = \langle X(0)X(t) \rangle$ of the bath operator $X(t)$, where $\langle \ldots \rangle = \text{Tr}_B(\ldots \rho_B)$ denotes a trace over the bath degrees of freedom with the bath density matrix $\rho_B$. We can further assume that the fluctuators are unbiased, $\langle X(t) \rangle = 0$. For independent two-level fluctuators with switching rates $\gamma_i$, one obtains

$$C(t) = \sum_i v_i^2 \langle \sigma_i(t)\sigma_i(0) \rangle = \sum_i v_i^2 e^{-\gamma_i |t|}. \quad (4)$$

The noise spectral density is the Fourier transform

$$S(\omega) = \int_{-\infty}^{\infty} dt C(t)e^{-i\omega t} = \sum_i (2v_i^2\gamma_i)/(\gamma_i^2 + \omega^2). \quad (5)$$

While this correlator describes essentially classical bath dynamics (as is commonly assumed for 1/f noise), it should be emphasized that our model is not classical, because the $[H_{SB}, H_S] \neq 0$. In the case of a large number of fluctuators, the sum in $C(t)$ can be converted into an integral. For 1/f noise, one typically assumes a distribution of fluctuators of the form $P(v, \gamma) \propto 1/\gamma v^2$, where both $v$ and $\gamma$ are limited by upper and lower cut-offs. The spectral density of the ensemble of fluctuators then becomes

$$S(\omega) \propto \int_{v_{\text{min}}}^{v_{\text{max}}} dv \int_{\gamma_0}^{\gamma_c} d\gamma P(v, \gamma) \frac{2v^2 \gamma}{\gamma^2 + \omega^2}. \quad (6)$$

For $\gamma_0 = 0$ this yields 1/f noise of the form $S(\omega) \propto 1/|\omega|$. The divergence at low frequencies is cut off by the finite duration of a qubit measurement, if not by other effects at even shorter times. A low-frequency cut-off $\gamma_0 > 0$ yields

$$S(\omega) = 2\pi A \frac{\text{arctan}(\omega/\gamma_0)}{\pi} \frac{1}{\omega}. \quad (7)$$

where $A$ depends on the cut-offs and the exponent $\beta$. For $\gamma_0 \to 0$, we recover $S(\omega) \to 2\pi A/|\omega|$. Inverting the above Fourier transform, we obtain

$$C(t) = -A Ei(-\gamma_0 |t|), \quad (8)$$

where $Ei$ denotes the exponential integral function.

### III. QUBIT DYNAMICS

The density matrix $\rho$ of the total system, consisting of the qubit and the bath, obeys the Liouville equation,

$$\dot{\rho}(t) = -i[H, \rho(t)].$$

The time evolution of the reduced density matrix of the qubit alone $\rho_S(t) = \text{Tr}_B[\ldots \rho(t)]$ is determined by the generalized master equation (GME)

$$\dot{\rho}_S(t) = -i[H_S, \rho_S(t)] - i \int_0^t \Sigma(t-t')\rho_S(t')dt', \quad (9)$$

where the self-energy superoperator $\Sigma(t)$ gives rise to memory effects, i.e., the time evolution of $\rho_S(t)$ depends on the state $\rho_S(t')$ at all earlier times $t' \leq t$. Therefore, the qubit dynamics is inherently non-Markovian. Expanding the right-hand side of the GME in orders of $H_{SB}$ and only keeping the lowest (second) order, one obtains $\Sigma$ in (lowest-order) Born approximation $\Sigma(t)\rho_S = -i\text{Tr}_B[H_{SB}, e^{-iH_0}t\rho_S e^{iH_0}]$, where $H_0 = H_S + H_B$.

Introducing the Bloch vector $\langle \sigma(t) \rangle = \text{Tr}_S \rho_S \sigma(t)$, where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli operators, we write the GME as a generalized Bloch equation

$$\langle \dot{\sigma} \rangle = R \star \langle \sigma \rangle + k, \quad (10)$$

where the star denotes convolution and $\frac{3,4}{\Sigma(t)}$.

$$R(t) = \begin{pmatrix}
-\frac{\epsilon}{2} & \frac{\epsilon}{2} & \Gamma_1(t)
-rac{\epsilon}{2} & -\frac{\epsilon}{2} & 0
\Gamma_1(t) & 0 & -\Delta(t)
\end{pmatrix}. \quad (11)$$

with $E = \sqrt{\Delta^2 + \epsilon^2}$ and $\frac{3,4}{\Sigma(t)}$.

$\Gamma_1(t) = (2\Delta/E)^2 \cos(\Theta)C'(t)$, $\Gamma_2(t) = (2\Delta/E)^2(1 + \epsilon/\Delta) \cos(\Theta)C'(t)$, and $K_s(t) = (4\epsilon \Delta/E^2) \sin(\Theta)C'(t)$, where $C'(t)$ and $C''(t)$ denote the real and imaginary parts of $C(t)$. Since for 1/f noise, $C''(t) = 0$, we find $k(t) = \frac{3,4}{\Sigma(t)}$. As shown in $\frac{3,4}{\Sigma(t)}$, Eq. (10) can be solved by means of the Laplace transform (LT) $f(s) = \int_0^\infty f(t)e^{-ts} dt$, where

$$\langle \sigma(s) \rangle = (s - R(s))^{-1}((\sigma(t = 0)) - k(s)). \quad (12)$$

The LT $R(s)$ of $R(t)$, has entries according to Eq. (11), with $\delta(t)$ replaced by 1, and, for 1/f noise

$$\Gamma_1(s) = (2A/E^2)\Delta^2(C(s+iE) + C(s-iE)),$$

$$\Gamma_2(s) = (2A/E^2)(2\Delta^2 C(s) + \epsilon^2(C(s+iE) + C(s-iE))),$$

$$K_s(s) = i(2A/E^2)\Delta \epsilon (C(s+iE) - C(s-iE)). \quad (14)$$

where the LT of the correlator $C(t)$ in Eq. (4) is

$$C(s) = \frac{A}{s} \log(1 + s/\gamma_0). \quad (16)$$

We recover $\langle \sigma(t) \rangle$ from $\langle \sigma(s) \rangle$ by way of an inverse LT as carried out below, first for the special case of an unbiased qubit ($\epsilon = 0$) and then for the general case.

### IV. UNBIASED QUBIT

We first assume that the qubit is prepared at time $t = 0$ in one of the eigenstates $|0\rangle = |\uparrow\rangle$ of $\sigma_z$, i.e., $\langle \sigma \rangle = (0, 0, 1)$, and that the qubit is unbiased, $\epsilon = 0$. If the fluctuators were absent the qubit would undergo a precession about the $x$ axis, $\langle \sigma_x(t) \rangle = \cos(\Delta t)$. Due
to the presence of the fluctuators, we find (see also Appendix A)

$$\langle \sigma_z(s) \rangle = \frac{s^2 + 4A \log(1 + s/\gamma_0)}{s(s^2 + \Delta^2 + 4A \log(1 + s/\gamma_0))}. \quad (17)$$

We expand $\langle \sigma_z(s) \rangle$ in leading order of $A$,

$$\langle \sigma_z(s) \rangle = \frac{s}{s^2 + \Delta^2} + 4A\Delta^2 \log(1 + s/\gamma_0) + O(A^2). \quad (18)$$

The coherent spin oscillations in the time domain are obtained from the inverse LT, the so-called Bromwich integral (see Fig. 2). $\langle \sigma_z(t) \rangle = \frac{1}{2\pi i} \int_{-\infty+i\eta}^{\infty+i\eta} \langle \sigma_z(s) \rangle e^{st} ds$. The integral contour can be closed in the left complex half-plane $\text{Re}(s) < 0$ (Fig. 2). The behavior of $\langle \sigma_z(t) \rangle$ is therefore given by the analytic structure of $\langle \sigma_z(s) \rangle$ in the left half-plane, see Fig. 2. In the absence of the fluctuating environment ($A = 0$), $\langle \sigma_z(s) \rangle$ has two poles at $s = \pm i\Delta$ which yield $\langle \sigma_z(t) \rangle = \cos(\Delta t)$, as expected. The coupling to the environment has two effects: (i) a shift of the poles, and (ii) the appearance of a branch point (bp) due to the logarithm in Eq. (18) and the associated branch cut (bc) that we choose to lie on the real axis between $-\gamma_0$ and $-\infty$. Here, it should be noted that in the case of an unbiased qubit, the presence of 1/f noise does not lead to the appearance of a pole on the real axis, and thus there is only pure dephasing and no $T_1$ type decay (spin relaxation), in contrast to other types of environment. The exact shift of the poles has been calculated numerically from Eq. (17). To lowest order in $A$, we find $\Delta_r = \Delta_r' + i\Delta_r'' \approx \Delta + \frac{\Delta}{\gamma_0} \log \left(1 + \frac{\Delta}{\gamma_0}\right) \pm 2i\frac{\Delta}{2\gamma_0} \arctan \frac{\Delta}{\gamma_0}$, where the real part $\Delta_r'$ is the renormalized frequency of the coherent oscillations, while the imaginary part $\Delta_r''$ describes an exponential decay of those oscillations. If a Markovian approximation were made by setting $s = 0$ in $\Gamma_1(s)$, $\Gamma_y(s)$, and $K_y(s)$, then the bc would be missed completely and only an exponential decay with a rate $2A/\gamma_0$ would be obtained. The Markov approximation is only justified if $\gamma_0 \gg \Delta$, i.e., if the bath dynamics is much faster than the system dynamics. Here, we entirely avoid making a Markov approximation.

The Bromwich integral can then be divided into two parts, $\langle \sigma_z(t) \rangle = \langle \sigma_z(t) \rangle_{\text{poles}} + \langle \sigma_z(t) \rangle_{\text{bc}}$. The integration in the first term along the contour $C$, not including the line integrals along the bc (Fig. 2) yields the sums of the residues from the poles $\langle \sigma_z(t) \rangle_{\text{poles}} = \frac{1}{2\pi i} \int_{-\infty+i\eta}^{\infty+i\eta} \langle \sigma_z(s) \rangle e^{st} ds = r' \cos(\Delta t) e^{-\Delta t} - r'' \sin(\Delta t) e^{-\Delta t}$, where $r' = (2A/\Delta^2) \log(1 + \Delta^2/\gamma_0^2) + O(A^2)$ and $r'' = (4A/\Delta^2) \arctan(\Delta/\gamma_0) + O(A^2)$. For $A = 0$, this reduces to $\cos(\Delta t)$.

The branch-cut contribution to lowest order in $A$ is

$$\langle \sigma_z(t) \rangle_{\text{bc}} = \frac{4A}{\Delta^2} I_1(\gamma_0/\Delta, \Delta t) \quad (19)$$

with the integral $I_1(a, b) = \int_a^\infty dy e^{-\eta y} e^{-b y}$. We have used Eq. (18) and introduced dimensionless variables and where $a > 0$ and $b \geq 0$. For $n = 1$, we find (Fig. 3)

$$I_1(a, b) = \frac{1}{2} \text{Re} \left[(ib + 2)e^{-ib}(-i\pi + \text{Ei}(ib - ab))\right] \equiv \frac{1}{2} \frac{1}{1 + ab} e^{-ab} - \text{Ei}(-ab). \quad (20)$$

For $a = \gamma_0/\Delta > 1$ and $b > 0$ ($t > 0$), the effect of the environment from the bc integral is exponentially suppressed: $I_1(a, b) < e^{-ab}/b$ and thus $\langle \langle \sigma_z(t) \rangle_{\text{bc}} \rangle < (4A/\Delta^2) e^{-\gamma_0 t}$. The physically more interesting regime is $a = \gamma_0/\Delta \ll 1$. Within this regime, we can distinguish two temporal regimes: short times $ab \ll 1$ ($t \ll \gamma_0^{-1}$) and long times $ab \gg 1$ ($t \gg \gamma_0^{-1}$). In the short-time case, the integral is cut off from above by a combination of the $y^{1/3}$ and the exponential factor. The effect of the latter can be approximated by cutting off the integral at 1/b, with the result $I_1(a, 0) \approx -I_1(1/b, 0) + I_1(a, 0)$, where $I_1(a, 0) = -\frac{1}{4}(1 + a^2)^{-1} + \frac{1}{4} \log(1 + a^{-2})$ is the bc integral for $t = 0$ ($b = 0$). Note that $I_1(a, 0) > 0$ due to the logarithmic term. In the long-time case, the integral is cut off by the exponential whereas the $(y^2 + 1)^2$ factor in the denominator becomes irrelevant, $I_1(a, b) \approx -\text{Ei}(-ab)$.

At this point, the parameter that controls the strength of the non-Markovian effects due to 1/f noise can be identified as $\xi = (A/\Delta^2) \log(1 + \Delta^2/\gamma_0^2)$. The regime of validity of the Born approximation (the only approximation required in this paper) is confined by the condition $\xi \ll 1$. The resulting damped qubit oscillation is plotted in Fig. 1 for $A/\Delta^2 = 0.05$ and $\gamma_0/\Delta = 0.05$ where $\xi \approx 0.1$. If the infrared cutoff is lowered, the non-Markovian effects due to 1/f noise become more pronounced. However, since the dependence on the infrared cutoff $\gamma_0$ is only logarithmic, the result does not change drastically even if $\gamma_0$ is much smaller than in our example, as long as $A$ is chosen sufficiently small to ensure the validity of the Born approximation. E.g., for $\Delta \approx 10$ GHz and $\gamma_0 \approx 1$ Hz (cf. Ref. [10]) then $\gamma_0/\Delta = 10^{-10}$. With $A/\Delta^2 = 0.005$, one finds a long-lived asymmetry as shown in the inset of

![Figure 2: (Color online) Analytic structure of $\langle \sigma_z(s) \rangle$ in the complex $s$ plane, for (a) the unbiased case, $\epsilon = 0$ and (b) the biased case, $\epsilon \neq 0$. Red dots denote poles, blue lines branch cuts.](image-url)
The intermediate asymptotics of this contribution is \( \langle \sigma_z \rangle_{bc} \approx \xi \approx 0.1 \), while for longer times this contribution also decays logarithmically to zero. A similar long-time behavior has been found also for longitudinal coupling\(^\text{10}\).}

**V. THE BIASED CASE**

We again assume that the qubit prepared at time \( t = 0 \) in one of the eigenstates \( |0\rangle = |\uparrow\rangle \) of \( \sigma_z \), i.e., \( \langle \sigma \rangle = (0,0,1) \), but now the qubit is biased, \( \epsilon \neq 0 \). In the absence of the fluctuators \( (A = 0) \), the qubit would now undergo a precession about an axis in the \( xz \) plane with frequency \( E/2\pi \), where \( E = \sqrt{\Delta^2 + \epsilon^2} \). In this unperturbed situation, \( \langle \sigma_z(s) \rangle \) has three poles at \( s = \pm iE \) and \( s = 0 \), the former two giving rise to undamped oscillations of \( \langle \sigma_z(t) \rangle \) with frequency \( E/2\pi \) and amplitude \( \Delta^2/E^2 \), while the latter allows for a non-vanishing stationary value \( \epsilon^2/E^2 \) of \( \langle \sigma_z(t) \rangle \) in the long-time limit.

Including \( 1/i \) noise we find in leading order in \( A \) (see Appendix \( A \)),

\[
\langle \sigma_z(s) \rangle = \frac{s^2 + \epsilon^2}{s(s^2 + E^2)} + 4A \frac{\Delta^2}{E^2} \text{Re} \left[ \frac{\Delta^2}{(E^2 + s^2)^2} C(s) \right] + \frac{\epsilon^2}{s(s + iE)} C(s + iE) + O(A^2). \tag{21}
\]

Analogously to the unbiased case, the poles are shifted in the presence of the fluctuators. In leading order in \( A \), we find three poles at \( -E_{\epsilon}^0 = -(4A\Delta^2/E^3) \arctan(E/\gamma_0), \) and \( \pm iE_{\epsilon} = \pm iE \pm (iA\Delta^2/E^3) \log(1 + E^2/\gamma_0^2) - (2A\Delta^2/E^3) \arctan(E/\gamma_0) \). From the shift of these poles (Fig. \( 2b \)), we obtain \( \langle \sigma_z(t) \rangle_{\text{poles}} = \frac{\Delta^2}{E^2} \cos(E_{\epsilon}^0 t) e^{-E_{\epsilon}^0 t} + \epsilon^2 e^{-2\epsilon E_{\epsilon}^0 t}. \) However, while in the unbiased case a Markovian treatment at least qualitatively describes the pole contribution correctly, in the biased case, there is another effect that is elusive in a Markovian analysis. As shown in Fig. \( 2b \), there are three bp’s in the biased case, lying at \(-\gamma_0 \) and \( \gamma_0 \pm \pm \). We find that as the two poles near \( \pm iE \) approach the bp’s at \( \gamma_0 \pm iE \) as \( A \) is increased, these poles split into two poles. This behavior is illustrated in Fig. \( 3 \). The significance of this splitting is that it leads to beating patterns already in the pole part of \( \langle \sigma_z(t) \rangle \), as shown in Fig. \( 3 \). It should be noted that, again, the precise value of \( \gamma_0 \) is not critical for the possibility to observe the effect, since \( \gamma_0 \) only enters in the argument of a logarithm; even a much smaller value of \( \gamma_0 \) can thus be compensated by only a slight increase of the system-environment coupling constant \( A \).

The three bc’s give rise to a contribution to \( \langle \sigma_z(t) \rangle \),

\[
\langle \sigma_z(t) \rangle_{bc} = -\frac{4A\Delta^2}{E^4} \left( \frac{\Delta^2 + \epsilon^2 \cos(Et)}{E^2} I_1 + \epsilon^2 e^{-2\epsilon E_{\epsilon}^0 t} \sin(Et) I_2 - \cos(Et) I_3 \right), \tag{22}
\]

where the functions \( I_n \) are as defined above and are evaluated at the arguments \( a = \gamma_0/E \) and \( b = Et \). For the unbiased case \( \epsilon = 0 \) and \( E = \Delta \), one retrieves the previous result. The integrals \( I_2 \) and \( I_3 \) can be calculated in closed form, but not will not be given here. The damped oscillations \( \langle \sigma_z(t) \rangle \), consisting of both pole and bc contributions, are plotted in Fig. \( 3 \).
FIG. 5: (Color online) Oscillation \( \langle \sigma_z(t) \rangle \) of the biased qubit for \( \epsilon/\Delta = 0.3 \), \( A/\Delta^2 = 0.05 \) and \( \gamma_0/\Delta = 0.05 \). The beating due to the splitting of the poles at \( \pm i \epsilon \) can be observed in \( \langle \sigma_z(t) \rangle_{\text{poles}} \).

VI. COMPARISON WITH AN EXACTLY SOLVABLE CASE

The circumstance that in the case \( \Delta = 0 \) the coupling Hamiltonian between the system and the environment \( H_{SE} \) commutes with the system Hamiltonian \( H_S \) makes this special case exactly solvable. A state prepared transverse to the common direction of the fixed precession axis and the fluctuating field, e.g., as \( \langle \sigma(t = 0) \rangle = (1, 0, 0) \), for low-frequency noise essentially leads to a Gaussian decay behavior \( \langle \sigma_z(t) \rangle = \cos(\epsilon t) \exp(-\epsilon^2 t^2) \).

The Born approximation which we have employed here can only be expected to yield this result in lowest-order of the coupling constant, i.e.,

\[
\langle \sigma_z(t) \rangle \simeq \cos(\epsilon t) \left( 1 - \epsilon t^2 + O(\epsilon^2 t^4) \right).
\] (23)

Here, we show that our result indeed has this form in the special case \( \Delta = 0 \).

To this end, we take the limit \( \Delta \to 0 \) in the propagator, Eq. (12), as shown in the Appendix A. We then find

\[
\langle \sigma_z(s) \rangle = P_{zz}(s) = \frac{s + \Gamma_y(s)}{(s + \Gamma_y(s))^2 + \left( \epsilon - K^+_y(s) \right)^2}.
\] (24)

From Eq. (19) and omitting logarithmic corrections, we can use \( C(s) \simeq A/s \) and thus \( \Gamma_y(s) \simeq 4A/s(s^2 + \epsilon^2) \) and \( K^+_y(s) \simeq 4A \epsilon/(s^2 + \epsilon^2) \). Substituting this into Eq. (24) and expanding to lowest order in \( A \), we find

\[
\langle \sigma_z(s) \rangle \simeq \frac{s}{s^2 + \epsilon^2} + A \frac{s(3\epsilon^2 - s^2)}{(s^2 + \epsilon^2)^2},
\] (25)

which equals the LT of Eq. (23) to lowest order, with the identification \( c = A/2 \). Therefore, our result is consistent with the known exact result for \( \Delta = 0 \), but, within the Born approximation, goes far beyond it, in that it includes arbitrary values of \( \epsilon \) and \( \Delta \).

VII. DISCUSSION

We find the following essentially non-Markovian features in the decay of the \( z \)-component of the spin: (i) The spin decay is non-exponential and asymmetric. For relatively large infrared cutoff \( \gamma_0 \), there is an “initial loss” of coherence on a typical time scale \( 1/\gamma_0 \), as seen in Figs. 11 and 12. More importantly, for the typical case of small \( \gamma_0 \), there is a long-time asymmetry favouring the qubit near its initial state. (ii) In the biased case, \( 1/f \) noise can lead to a two-frequency oscillation, exhibiting a characteristic beating pattern. Here, we have concentrated on the longitudinal component \( \langle \sigma_x(t) \rangle \) of the qubit under the influence of both longitudinal and transverse \( 1/f \) noise. The transverse component \( \langle \sigma_z(t) \rangle \) shows similar behavior. The predicted non-Markovian effects are observable in free induction decay (Ramsey fringe) experiments. Indeed, such asymmetries are clearly visible in superconducting flux qubits. Measurements on a superconducting flux qubit have shown deviations from the exponential decay and beatings. The question whether these effects are due to the mechanisms described here or not require further investigation.

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APPENDIX A: FORM OF THE PROPOGATOR

The propagator (resolvent) for solving the generalized Bloch equation in Laplace space is defined in Eq. (12) as

\[
P(s) = (s - R(s))^{-1}.
\] (A1)

Using the form of the relaxation matrix \( R(s) \), we obtain the following expressions for the matrix elements of \( P(s) \),

\[
P_{xx}(s) = \frac{1}{D(s)} \left( s + \Gamma_y(s) + \frac{\Delta^2}{s} \right),
\] (A2)

\[
P_{yy}(s) = \frac{1}{D(s)} \left( s + \frac{E^2}{\Delta^2} \Gamma_1(s) \right),
\] (A3)

\[
P_{zz}(s) = \frac{1 - \Delta^2}{s^2} P_{yy}(s),
\] (A4)

\[
P_{xy}(s) = -P_{yx}(s) = -\frac{1}{D(s)} \left( \epsilon - E \frac{\Delta K_y^+(s)}{\Delta^2} \right),
\] (A5)

\[
P_{xz}(s) = -P_{zx}(s) = -\frac{\Delta}{s} P_{xx}(s),
\] (A6)

\[
P_{yz}(s) = -P_{zy}(s) = -\frac{\Delta}{s} P_{yy}(s),
\] (A7)

with the definition

\[
D(s) = \left( s + \Gamma_y(s) + \frac{\Delta^2}{s} \right) \left( s + \frac{E^2}{\Delta^2} \Gamma_1(s) \right) + \left( \epsilon - E \frac{\Delta K_y^+(s)}{\Delta^2} \right)^2.
\] (A8)
The solution in Laplace space is now obtained according to Eq. (12), with $k = 0$,

$$\langle \sigma_i(s) \rangle = \sum_{j=x,y,z} P_{ij}(s) \langle \sigma_j(t = 0) \rangle. \quad (A9)$$

E.g., for $\langle \sigma(t = 0) \rangle = (0, 0, 1)$, we find $\langle \sigma_i(s) \rangle = P_{zz}(s)$. Using Eqs. (A2), (A3), and (A5), we recover the known results from Ref. 4 in the special case $k = 0$. The remaining matrix elements, Eqs. (A2), (A3), and (A5), allow us the use different initial conditions.

1. The case $\epsilon = 0$

For an unbiased qubit, $\epsilon = 0$ and thus $E = \Delta$, so that the quantities discussed above are reduced to the form

$$D(s) = \left( s + \Gamma_y(s) + \frac{\Delta^2}{s} \right) (s + \Gamma_1(s)), \quad (A10)$$

$$P_{xx}(s) = (s + \Gamma_1(s))^{-1}, \quad (A11)$$

$$P_{yy}(s) = \frac{s + \Gamma_1(s)}{D(s)} = \left( s + \Gamma_y(s) + \frac{\Delta^2}{s} \right)^{-1}, \quad (A12)$$

$$P_{yz}(s) = (s + \Gamma_y(s))P_{yy}(s)/s, \quad (A13)$$

$$P_{yx}(s) = P_{xy}(s) = P_{xx}(s) = P_{zz}(s) = 0. \quad (A15)$$

2. The case $\Delta = 0$

In the case of a diagonal system Hamiltonian $H_S$, we set $\Delta = 0$ and thus $E = \epsilon$, and

$$\Gamma_y(s) = \frac{E^2}{\Delta^2} \Gamma_1(s) = 2A(C(s + i\epsilon) + C(s - i\epsilon)), \quad (A16)$$

$$\hat{K}_y^+(s) = \frac{E}{\Delta} K_y^+(s) = 2iA(C(s + i\epsilon) - C(s - i\epsilon)), \quad (A17)$$

$$D(s) = (s + \Gamma_y(s))^2 + \left( \epsilon - \hat{K}_y^+ \right)^2. \quad (A18)$$

With Eqs. (A2–A7), we obtain

$$P_{xx}(s) = P_{yy}(s) = \frac{s + \Gamma_y(s)}{D(s)}, \quad (A19)$$

$$P_{zz}(s) = \frac{1}{s}, \quad (A20)$$

$$P_{xy}(s) = -P_{yx}(s) = \frac{s - \epsilon - \hat{K}_y^+}{D(s)}, \quad (A21)$$

$$P_{zz}(s) = P_{xx}(s) = P_{yy}(s) = P_{zz}(s) = 0. \quad (A22)$$

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For a SC flux qubit, $A \approx (E_J/\Delta)^2 A_{n,\Phi}$ with the 1/f flux noise $S_{n,\Phi} = A_{n,\Phi}/\omega$ with noise power (in units of flux quanta) $A_{n,\Phi} \approx 5 \cdot 10^{-6}$ (see11) and $E_J/\Delta \approx 30$.

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We assume $\beta > 3$ to ensure that a large number of fluctuators over the entire range of $v$ couples to the qubit and the assumption of Gaussian noise is justified.