Considering corrections to all orders in the Planck length on the density of quantum states from a generalized uncertainty principle (GUP), we calculate the statistical entropy of a scalar field on the background of the Schwarzschild black hole in massive gravity. As a result, we newly obtain the generalized Bekenstein-Hawking entropy depending on a gravitational mass without any artificial cutoff.

Keywords: Schwarzschild black hole in massive gravity; generalized uncertainty principle; brick wall model

I. INTRODUCTION

Einstein’s theory of general relativity (GR) is a theory of a massless spin-2 graviton, which has been successfully tested to date as the description of the force of gravity. However, quantum gravity phenomenology [1] that focuses on modifications of the existing theory at extreme limits has pushed forward to search for alternatives to GR. One of them is to introduce a massive graviton to GR. In 1930s, by extending GR with a quadratic mass term, Fierz and Pauli developed a massive spin-2 theory [2]. It was later known to suffer from the Boulware-Deser ghost problem [3] and the van Dam, Veltman and Zakharov (vDVZ) discontinuity [4,5] in the massless graviton limit. The vDVZ discontinuity was cured by the Vainshtein mechanism [6] due to certain low scale strongly coupled interactions. Moreover, de Rham, Gabadadze and Trolley (dRGT) [7,8] successfully obtained a ghost free massive gravity, which has nonlinearly interacting mass terms constructed from the metric coupled with a symmetric reference metric tensor. This was confirmed by a Hamiltonian analysis of the untruncated theory [9,10] and other works [11–16]. Furthermore, Vegh [17] introduced a nonlinear massive gravity with a special singular reference metric which keeps the diffeomorphism symmetry for coordinates \((t,r)\) intact but breaks it in angular directions so that gravitons acquire the mass because of a broken momentum conservation [18–20]. Since then, this kind of holographic massive gravity has been extensively exploited to investigate many black hole models [21–31]. Very recently, we have investigated the tidal effects in the Schwarzschild black hole in the holographic massive gravity, showing that massive gravitons effectively affect the angular component of the tidal force, while the radial component remains the same as in massless gravity [32].

On the other hand, quantum gravity phenomenology predicts the possible existence of a minimal length on the smallest scale [33]. This implies the modification of the Heisenberg uncertainty principle (HUP) in quantum mechanics to a generalized uncertainty principle (GUP) [34–38], which paves the way for deeper understanding on black hole thermodynamics [39–46] including the final stage of the Hawking radiation. As is well known, studies on black holes in terms of thermodynamics started with the discovery of the Bekenstein-Hawking entropy [47–51] proportional to the surface area of a black hole at the event horizon. In order to provide a microphysical explanation to the Bekenstein-Hawking entropy, ’t Hooft developed the statistical method of finding the black hole’s entropy by introducing a scalar field propagating just outside the event horizon [52]. One of main characteristics in his so-called brick wall method is to introduce an ad hoc cutoff which removes ultraviolet divergences due to the infinite blue shift at the event horizon. Since then, a lot of works have been devoted to study on the statistical properties of various types of black holes [53–72]. Later, it was shown that the unnatural cutoff in the brick wall method can be removed in terms of a minimal length to the order of the Planck length which is derived from a GUP [73]. Making use of these ideas, the authors in Refs. [74–76] calculated the statistical entropy of black holes to leading order in the Planck length. The ultraviolet divergences of the just vicinity near the horizon in the usual brick wall method is drastically solved by the newly modified equation of the density states motivated by GUPs [35–38]. Nouicer has further developed the GUP effect to all orders in the Planck length [77] by arguing that the GUP up to leading order correction in the Planck length is not enough because the wave vector \(k\) does not satisfy the asymptotic property in the modified dispersion relation [78–81].
After his work, according to this approach, we have obtained the desired Bekenstein-Hawking entropy to all orders in the Planck length units without any artificial cutoff and little mass approximation for the case of the Schwarzschild black hole in massless gravity \[82\]. However, since these works are mostly concentrated on studying entropies of the Schwarzschild black hole in massless gravity, it would be interesting to extend them to massive gravity.

In this paper, we calculate the statistical entropy of a scalar field on the Schwarzschild black hole in massive gravity to all orders in the Planck length by carefully considering modified density of quantum states and corresponding entropy integrals near the event horizon. As a result, we newly obtain the generalized Bekenstein-Hawking entropy with correction terms. In Sec. II, we briefly introduce all order corrections of the GUP. In Sec. III, we recapitulate the solution of the Schwarzschild black hole in massive gravity for self-consistency. In Sec. IV, we consider a scalar field propagating on the Schwarzschild black hole in massive gravity, which satisfies the Klein-Gordon equation. In Sec. V, we calculate the generalized entropy to all orders in the Planck length by counting modified density states due to the existence of massive gravitons with the GUP. Finally, conclusions are drawn in Sec. VI.

II. ALL ORDER CORRECTIONS OF GUP

Quantum gravity phenomenology has been tackled with effective models which incorporate a minimal length as a natural ultraviolet cutoff \[78, 79\]. Such a minimal length leads to deformed Heisenberg algebras \[33–36, 39\] which show a GUP. For a particle with the momentum \(p\) and the wave vector \(k\) having a nonlinear relation \(p = f(k)\), the commutator between two operators \(\hat{x}\) and \(\hat{p}\) can be generalized to

\[
[\hat{x}, \hat{p}] = i \frac{\partial p}{\partial k} \Leftrightarrow \Delta x \Delta p \geq \frac{1}{2} \left| \frac{\partial p}{\partial k} \right| \tag{2.1}
\]

at the quantum mechanical level \[78, 79\]. Without loss of generality, in the following, let us restrict ourselves to the isotropic case in one space-like dimension. Kempf et al. \[33–36, 39\] have considered the following relation

\[
\frac{\partial p}{\partial k} = 1 + \lambda p^2, \tag{2.2}
\]

which came in the context of perturbative string theory. The GUP parameter \(\lambda\) is of order of the Planck length \(l_p^{-2}\). This exhibits the features of UV/IR correspondence such that as \(\Delta p\) is large, \(\Delta x\) is proportional to \(\Delta p\). More importantly, it provides us the existence of the minimal length as \((\Delta x)_{\text{min}} = \sqrt{\lambda}\) beyond which distances cannot be probed.

On the other hand, it was extended to all orders in the Planck length by Nouicer et al. \[83, 85\] as

\[
\frac{\partial p}{\partial k} = e^{\lambda p^2}. \tag{2.3}
\]

For this case, the GUP in Eq. (2.1) for mirror symmetric states as \(\langle \hat{p} \rangle = 0\) can be solved by the multi-valued Lambert function \[86\]

\[
W(\xi)e^{W(\xi)} = \xi \tag{2.4}
\]

by putting it to \(W(\xi) = -2\lambda(\Delta p)^2\) and \(\xi = \frac{\lambda}{\pi(\Delta x)^2}\). Since in order to have a real solution for \(\Delta p\) it is required to satisfy \(\xi \geq -1/e\), this gives naturally the position uncertainty of

\[
\Delta x \geq \sqrt{\frac{e\lambda}{2}}. \tag{2.5}
\]

Therefore, one can obtain the minimal length \((\Delta x)_{\text{min}} = \sqrt{e\lambda/2}\) for the GUP to the all orders in the Planck length.

Note that this includes the leading order correction in the relation (2.2). However, since this correction only of the GUP does not satisfy the property that the wave vector \(k\) asymptotically reaches the cutoff in large energy region as in Ref. \[78, 79\], we will consider the all order corrections in the Planck length in the followings.

III. SCHWARZSCHILD BLACK HOLE IN MASSIVE GRAVITY

The (3+1)-dimensional Schwarzschild black hole in massive gravity \[17, 32\] is described by the action

\[
S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R + \tilde{m}^2 \sum_{i=1}^{4} c_i U_i(g_{\mu\nu}, f_{\mu\nu}) \right], \tag{3.1}
\]
where $\mathcal{R}$ is the scalar curvature of the metric $g_{\mu\nu}$, $\tilde{m}$ is a graviton mass\(^1\), $c_i$ are constants, and $\mathcal{U}_i$ are symmetric polynomial potentials of the eigenvalue of the matrix $\mathcal{K}^\mu_\nu = \sqrt{g^{\mu\alpha}f_{\alpha\nu}}$ as

\[
\begin{align*}
\mathcal{U}_1 &= |\mathcal{K}|, \\
\mathcal{U}_2 &= |\mathcal{K}|^2 - |\mathcal{K}|^2, \\
\mathcal{U}_3 &= |\mathcal{K}|^3 - 3|\mathcal{K}||\mathcal{K}^2| + 2|\mathcal{K}^3|, \\
\mathcal{U}_4 &= |\mathcal{K}|^4 - 6|\mathcal{K}^2||\mathcal{K}|^2 + 8|\mathcal{K}^3||\mathcal{K}| + 3|\mathcal{K}^2|^2 - 6|\mathcal{K}^4|.
\end{align*}
\]

(3.2)

Here, the square root in $\mathcal{K}$ means $\sqrt{\mathcal{A}} = \sqrt{\mathcal{A}}^\mu_\nu = A^\mu_\nu$ and square brackets denote the trace $|\mathcal{K}| = K^\mu_\mu$. Indices are raised and lowered with the dynamical metric $g_{\mu\nu}$, while the reference metric $f_{\mu\nu}$ is a non-dynamical, fixed symmetric tensor which is introduced to construct nontrivial interaction terms in massive gravity.

Variation of the action (3.1) with respect to the metric $g_{\mu\nu}$ leads to the equations of motion given by

\[
\mathcal{R}_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} + \tilde{m}^2 \chi_{\mu\nu} = 0.
\]

(3.3)

Here,

\[
\chi_{\mu\nu} = -\frac{c_1}{2}(U_1 g_{\mu\nu} - K_{\mu\nu}) - \frac{c_2}{2}(U_2 g_{\mu\nu} - 2U_1 K_{\mu\nu} + 2K^2_{\mu\nu}) - \frac{c_3}{2}(U_3 g_{\mu\nu} - 3U_2 K_{\mu\nu} + 6U_1 K^2_{\mu\nu} - 6K^3_{\mu\nu})

- \frac{c_4}{2}(U_4 g_{\mu\nu} - 4U_3 K_{\mu\nu} + 12U_2 K^2_{\mu\nu} - 24U_1 K^3_{\mu\nu} + 24K^4_{\mu\nu}).
\]

(3.4)

When one considers the spherically symmetric black hole solution ansatz as

\[
ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)
\]

(3.5)

with the following degenerate reference metric $f_{\mu\nu} = \text{diag}(0, 0, c_0^2, c_0^2 \sin^2\theta)$, one can find

\[
K^0_\phi = K^\phi_0 = \frac{c_0}{r}.
\]

(3.7)

Note that the choice of the reference metric in Eq. (3.6) preserves general covariance in $(t, r)$ but not in the angular directions. This gives the symmetric potentials as

\[
U_1 = \frac{2c_0}{r}, \quad U_2 = \frac{2c_0^2}{r^2}, \quad U_3 = U_4 = 0.
\]

(3.8)

It should be pointed out that there are no contributions from $c_3$ and $c_4$ terms which appear in $(4+1)$ and $(5+1)$-dimensional spacetimes, respectively $[21, 23, 26, 29]$. Then, one can have the solution

\[
f(r) = 1 - \frac{2m}{r} + 2Rr + C
\]

(3.9)

with $R = c_0 c_1 \tilde{m}^2 / 4$ and $C = c_0^2 c_2 \tilde{m}^2 [32]$, where $m$ is an integration constant related to the mass of the black hole and $c_0$ is a positive constant $[17, 31]$.

Now, by solving $f(r) = 0$, one can find the event horizons of the Schwarzschild black hole in massive gravity as

\[
r_{\pm} = \frac{-(1 + C) \pm \sqrt{16mR + (1 + C)^2}}{4R}.
\]

(3.10)

The allowed event horizon can be classified according to the relative signs of $R$ and $C$ as shown in Table I. We note that the event horizon $r_+$ is reduced to $r_H = 2m$ of the Schwarzschild black hole in massless gravity as $R$ and $C \rightarrow 0$. It is also appropriate to comment that in Table I $r_-$ is discarded since it is either negative or imaginary in each

\[1\] In this paper, we shall call it massless when $\tilde{m}$ is zero.
TABLE I: Event horizons of the Schwarzschild black hole in massive gravity.

| $C > -1$ | $C = -1$ | $C < -1$ |
|----------|----------|----------|
| $R > 0$  | $r_+ = \frac{1 + \mathcal{C}}{4R} \left(1 + \sqrt{1 + \frac{8R_H}{(1+C)^2}}\right)$ | $r_+ = \sqrt{\frac{2R}{1+C}}$ | $r_+ = -\frac{1 + \mathcal{C}}{4R} \left(1 + \sqrt{1 + \frac{8R_H}{(1+C)^2}}\right)$ |
| $R = 0$  | (IV) $r_+ = \frac{1 + \mathcal{C}}{4R}$ | NA | NA |
| $-\frac{(1+C)^2}{8R_H} < R < 0$ | (V) $r_+ = \frac{1 + \mathcal{C}}{4R} \left(1 + \sqrt{1 + \frac{8R_H}{(1+C)^2}}\right)$ | NA | NA |
| $R = -\frac{(1+C)^2}{8R_H}$ | (VI) $r_+ = -\frac{1 + \mathcal{C}}{4R}$ | NA | NA |
| $R < -\frac{(1+C)^2}{8R_H}$ | NA | NA | NA |

ranges. Only in the special case of $C > -1$ and $R = -\frac{(1+C)^2}{16m}$, the physical solution of $r_-$ exists and coincides exactly with $r_+$. In Table II the abbreviation NA denotes that there is no available event horizon in the specified range. Moreover, from the solution (3.9), one can find the surface gravity [87]

$$\kappa = \frac{1 + \mathcal{C}}{2r_+} + 2R,$$

(3.11)

and the Hawking temperature $T_H$ for the Schwarzschild black hole in massive gravity [32] as

$$T_H = \frac{1 + \mathcal{C}}{4\pi r_+} + \frac{R}{\pi}.$$

(3.12)

In Table II we have summarized the Hawking temperatures in massive gravity which behave differently according to the relative signs of $R$ and $C$. First of all, in the case of (I) and (IV), the Hawking temperatures are depicted as the one in massless gravity, proportional to $\frac{1}{4\pi r_+}$ but asymptotically approach to a constant of $\frac{R}{2\pi}$. In the case of (II) with $C = -1$, the Hawking temperature is given by a constant, and in the case of (III) with $C < -1$, the curve is flipped due to the negative sign of $C$ in front of $\frac{1}{4\pi r_+}$ in Eq. (3.12) and it becomes $\frac{R}{2\pi}$ as $R_H \to 0$. In the case of (V), the Hawking temperature is the same with (I), however, the range of $r_H$ is limited to $r_H < -\frac{(1+C)^2}{8R_H}$ (note that $R < 0$). Finally, in the case of (VI), the Hawking temperature vanishes.

TABLE II: Hawking temperatures of the Schwarzschild black hole in massive gravity.

| $C > -1$ | $C = -1$ | $C < -1$ |
|----------|----------|----------|
| $R > 0$  | $T_H = \frac{R}{\pi} \left[1 + \frac{(1+C)^2}{8R_H} \left(1 + \sqrt{1 + \frac{8R_H}{(1+C)^2}}\right)\right]$ | $T_H = \frac{R}{\pi} \left[1 + \frac{(1+C)^2}{8R_H} \left(1 - \sqrt{1 + \frac{8R_H}{(1+C)^2}}\right)\right]$ | $T_H = \frac{R}{\pi} \left[1 + \frac{(1+C)^2}{8R_H} \left(1 - \sqrt{1 + \frac{8R_H}{(1+C)^2}}\right)\right]$ |
| $R = 0$  | (IV) $T_H = \frac{(1+C)^2}{8R_H}$ | NA | NA |
| $-\frac{(1+C)^2}{8R_H} < R < 0$ | (V) $T_H = \frac{R}{\pi} \left[1 + \frac{(1+C)^2}{8R_H} \left(1 + \sqrt{1 + \frac{8R_H}{(1+C)^2}}\right)\right]$ | NA | NA |
| $R = -\frac{(1+C)^2}{8R_H}$ | (VI) $T_H = 0$ | NA | NA |
| $R < -\frac{(1+C)^2}{8R_H}$ | NA | NA | NA |

These Hawking temperatures are drawn in Fig. 1. As shown in Fig. 1(a), the Hawking temperatures in massive gravity with $C > -1$ and $R \geq 0$ are similar to the one in massless gravity except approaching $\frac{R}{2\pi}$ as $R_H \to \infty$ for the case of (I). On the other hand, Fig. 1(b) shows rather unexpected aspects of massive gravitons in the Hawking temperatures. The case (II) gives us a constant Hawking temperature due to the absence of the first term in Eq. (3.12), and the case (III) a reversed Hawking temperature near $R_H = 0$ due to the flip of sign in front of $\frac{1}{4\pi r_+}$ while approaching $\frac{R}{2\pi}$ as $R_H \to \infty$. Finally, in the case of (V), the Hawking temperature decreases and eventually vanishes at $r_H = -\frac{(1+C)^2}{8R_H}$. In the followings, without any loss of generality, we will concentrate on the cases of (I) and (IV) unless otherwise mentioned.
IV. SCALAR FIELD ON THE SCHWARZSCHILD BLACK HOLE IN MASSIVE GRAVITY

We begin by considering the Schwarzschild black hole solution in massive gravity found in the previous section as

$$ds^2 = -\left(1 - \frac{2m}{r} + 2Rr + C\right)dt^2 + \left(1 - \frac{2m}{r} + 2Rr + C\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

(4.1)

Then, let us consider a free scalar field with a mass $\mu$ in the background described by the solution (4.1), which satisfies the Klein-Gordon equation given by

$$(\Box - \mu^2)\Phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) - \mu^2\Phi = 0.$$ 

(4.2)

Substituting the ansatz of wave function $\Phi(t, r, \theta, \phi) = e^{-i\omega t}\psi(r, \theta, \phi)$ into Eq. (4.2), we find that the Klein-Gordon equation in the spherical coordinates becomes

$$\partial_r^2 \psi + \left(\frac{f'}{f} + \frac{2}{r}\right)\partial_r \psi + \frac{1}{f} \left[\frac{1}{r^2} \left(\partial_\theta^2 + \cot\theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2\right) + \frac{\omega^2}{f} - \mu^2\right] \psi = 0,$$

(4.3)

where $f(r) = 1 - \frac{2m}{r} + 2Rr + C$ and the prime denotes the derivative with respect to $r$. By using the Wenzel-Kramers-Brillouin approximation [52] with $\psi \sim e^{iS(r, \theta, \phi)}$ and keeping the real parts, we have the following modified dispersion relation

$$p_\mu p^\mu = p_0p^0 + \mathbf{p}^2 = \frac{-\omega^2}{f} + fp_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2 \theta} = -\mu^2,$$

(4.4)

where

$$p_r = \frac{\partial S}{\partial r}, \quad p_\theta = \frac{\partial S}{\partial \theta}, \quad p_\phi = \frac{\partial S}{\partial \phi}. \quad (4.5)$$

Furthermore, we also obtain the square module of momentum as follows

$$p^2 = p_\mu p^\mu = g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2 + g^{\phi\phi}p_\phi^2 = \frac{\omega^2}{f} - \mu^2.$$ 

(4.6)

Then, the volume in the momentum phase space is given by

$$V_p(r, \theta) = \int dp_r dp_\theta dp_\phi = \frac{4\pi}{3}r^2\sin\theta \left(\frac{\omega^2}{f} - \mu^2\right)^\frac{3}{2},$$ 

(4.7)

with the condition $\omega \geq \mu\sqrt{f}$. 
V. ENTROPY TO ALL ORDERS IN THE PLANCK LENGTH

Now, let us calculate the statistical entropy of the scalar field on the Schwarzschild black hole background to all orders in the Planck length units. When the gravity is turned on, the number of quantum states in a volume element in phase cell space based on the GUP in (3+1)-dimensions is given by

\[ dn_A = \frac{d^3xd^3p}{(2\pi)^3} e^{-\lambda p^2}, \]  

(5.1)

where \( p^2 = p_i^2 \) (\( i = r, \theta, \phi \)) and one quantum state corresponding to a cell of volume is changed from \((2\pi)^3\) into \((2\pi)^3 e^{\lambda p^2}\) in the phase space [73–76]. Here, the subscript \( A \) denotes the quantity for all orders in the Planck length. Note that in the limit of \( \lambda \to 0 \), we have the number of quantum states with HUP [52].

From Eqs. (4.0) and (5.1), the number of quantum states related to the radial mode with energy less than \( \omega \) is given by

\[ n_A(\omega) = \frac{1}{(2\pi)^3} \int dr\phi V_p(r, \theta)e^{-\lambda p^2} \]
\[ = \frac{2}{3\pi} r^2 \int_{r_+}^\infty dr \frac{\omega^2}{\sqrt{f}} \left( \frac{\omega^2}{f} - \mu^2 \right)^{\frac{3}{2}} e^{-\lambda(\frac{\omega^2}{f} - \mu^2)}. \]

(5.2)

It is interesting to note that \( n_A(\omega) \) is convergent at the horizon without any artificial cutoff because of the existence of the suppressing exponential \( \lambda \)-term induced from the GUP.

For the bosonic case, the free energy of a thermal ensemble of scalar fields at inverse temperature \( \beta \) is given by

\[ e^{-\beta F} = \prod_K (1 - e^{-\beta \omega K})^{-1}, \]

(5.3)

where \( K \) represents the set of quantum numbers. Then, we deduce the free energy by using Eq. (5.2) as

\[ F_A = \frac{1}{\beta} \sum_K \ln (1 - e^{-\beta \omega K}) \]
\[ = \frac{1}{\beta} \int_{\mu<\beta} d\omega \frac{dn_A(\omega)}{d\omega} \ln (1 - e^{-\beta \omega}) \]
\[ = \int_{\mu<\beta} d\omega \frac{n_A(\omega)}{e^{\beta \omega} - 1} \]
\[ = -\frac{2}{3\pi} \int_{r_+} \frac{r^2}{\sqrt{f}} \int_{\mu<\beta} d\omega \frac{\omega^2}{e^{\beta \omega} - 1} e^{-\lambda(\frac{\omega^2}{f} - \mu^2)}. \]

(5.4)

Here, we have taken the continuum limit in the second line and integrated it by parts in the third line. In the last line, we have used the number of quantum states (5.2). Since \( f \to 0 \) near the event horizon, \( i.e., \) in the range of \( (r_+, r_+ + \epsilon) \), \( \frac{\omega^2}{f} - \mu^2 \) becomes \( \frac{\omega^2}{2} \) so that we do not need to require the little mass approximation. Then, the free energy can be rewritten as

\[ F_A = -\frac{2}{3\pi} \int_{r_+}^{r_++\epsilon} dr \frac{r^2}{\sqrt{f(r)}} \int_0^{\infty} d\omega \frac{\omega^3}{e^{\beta \omega} - 1} e^{-\lambda(\frac{\omega^2}{2} - \mu^2)}. \]

(5.5)

On the other hand, we are also interested in the contribution from just the vicinity near the horizon in the range of \( (r_+, r_+ + \epsilon) \), where \( \epsilon \) is related to a proper distance of order of the minimal length as follows

\[ (\Delta x)_{min} = \sqrt{\frac{\epsilon \lambda}{2}} = \int_{r_+}^{r_++\epsilon} \frac{dr}{\sqrt{f(r)}} \approx \int_{r_+}^{r_++\epsilon} \frac{dr}{\sqrt{2\kappa_+(r-r_+)}}, \]

(5.6)

where the metric function \( f(r) \) is expanded near the event horizon as

\[ f(r) \approx f(r_+) + 2\kappa+(r-r_+) + O((r-r_+)^2), \]

(5.7)
and the surface gravity \( \kappa_+ \) replaces \( \frac{1}{\sqrt{2\pi}} \) in the second term. For massive gravity, \( \kappa_+ \) is given by Eq. (3.11).

Then, from \( F_A \) in Eq. (5.5), the entropy can be obtained as

\[
S_A = \beta^2 \frac{\partial F_A}{\partial \beta} \bigg|_{\beta=\beta_+} = \frac{\beta_+^2}{6\pi} \int_{r_+}^{r_+ + \epsilon} dr \frac{r^2}{f^2} \int_0^\infty d\omega \frac{\omega^4}{\sinh^2(\frac{\omega}{2}))} e^{-\lambda\omega^2}.
\]

By defining \( x \equiv \sqrt{\lambda} \omega \), we have

\[
S_A = \frac{\beta_+^2}{6\pi \lambda^2 \sqrt{\lambda}} \int_0^\infty dx \frac{x^4}{\sinh^2(\frac{\sqrt{x}}{2\sqrt{\lambda}})} \Lambda(x, \epsilon),
\]

where

\[
\Lambda(x, \epsilon) \equiv \int_{r_+}^{r_+ + \epsilon} dr \frac{r^2}{f^2} e^{-\frac{r^2}{\lambda}}.
\]

Near the horizon, using the expansion of Eq. (5.6), we have

\[
\Lambda(x, \epsilon) \approx \int_{r_+}^{r_+ + \epsilon} dr \frac{r^2}{[2\kappa_+(r - r_+)]^2} e^{-\frac{r^2}{\lambda\kappa_+(r - r_+)}}.
\]

Again, by redefining \( t = x^2/2\kappa_+(r - r_+) \), we have

\[
\Lambda(x, \epsilon) = \frac{1}{2\kappa_+ x^2} \int_{2\kappa_+ x^2}^{\infty} dt \left( \frac{r_+^2 + r_+ x^2}{\kappa_+ t^2} + \frac{x^4}{4\kappa_+^2 t^2} \right) e^{-t} \\
= \frac{r_+^2}{2\kappa_+ x^2} \Gamma(1, \frac{x^2}{2\kappa_+ \epsilon}) + \frac{r_+}{2\kappa_+^2} \Gamma(0, \frac{x^2}{2\kappa_+ \epsilon}) + \frac{x^2}{8\kappa_+^2} \Gamma(-1, \frac{x^2}{2\kappa_+ \epsilon}),
\]

where the incomplete Gamma function is given by

\[
\Gamma(a, z) = \int_z^\infty dt t^{a-1} e^{-t}.
\]

Then, the exact form of all order GUP corrected entropy is written as

\[
S_A = \frac{\beta_+^2 r_+^2}{12\pi \lambda^2 \sqrt{\lambda \kappa_+}} \int_0^\infty dx \frac{x^2 \Gamma(1, \frac{x^2}{2\kappa_+ \epsilon})}{\sinh^2(\frac{\sqrt{x}}{2\sqrt{\lambda}})} + \frac{\beta_+^2 r_+}{12\pi \lambda^2 \sqrt{\lambda \kappa_+}} \int_0^\infty dx \frac{x^4 \Gamma(0, \frac{x^2}{2\kappa_+ \epsilon})}{\sinh^2(\frac{\sqrt{x}}{2\sqrt{\lambda}})} + \frac{\beta_+^2}{48\pi^2 \sqrt{\lambda \kappa_+}} \int_0^\infty dx \frac{x^6 \Gamma(-1, \frac{x^2}{2\kappa_+ \epsilon})}{\sinh^2(\frac{\sqrt{x}}{2\sqrt{\lambda}})}.
\]

By redefining \( y \equiv \frac{\beta_+ x}{2\sqrt{\lambda}} \) and making use of Eq. (5.6) and \( \beta_+ \kappa_+ = 2\pi \), we have

\[
S_A = \frac{r_+^2}{3\pi^2 \lambda} \int_0^\infty dy \frac{y^2 \Gamma(1, \frac{2y^2}{x^2})}{\sinh^2(y)} + \frac{r_+}{3\pi^4} \int_0^\infty dy \frac{y^4 \Gamma(0, \frac{2y^2}{x^2})}{\sinh^2(y)} + \frac{\lambda \kappa_+}{12\pi^6} \int_0^\infty dy \frac{y^6 \Gamma(-1, \frac{2y^2}{x^2})}{\sinh^2(y)}.
\]

The integrals can be numerically integrated as

\[
\delta_1 = \int_0^\infty dy \frac{y^2 \Gamma(1, \frac{2y^2}{x^2})}{\sinh^2(y)} \approx 1.4509,
\]

\[
\delta_2 = \int_0^\infty dy \frac{y^4 \Gamma(0, \frac{2y^2}{x^2})}{\sinh^2(y)} \approx 3.0709,
\]

\[
\delta_3 = \int_0^\infty dy \frac{y^6 \Gamma(-1, \frac{2y^2}{x^2})}{\sinh^2(y)} \approx 18.4609.
\]
so that the desired entropy can be finally rewritten as

$$S_A = \frac{\delta_1}{3\pi^3} \lambda^{-1} \left( \frac{A}{4} \right) + \frac{\delta_2}{3\pi^3} r_+ \kappa_+ + \frac{\delta_3}{12\pi^6} \lambda \kappa_+^2,$$

(5.17)

where $A = 4\pi r_+^2$ is the surface area at the event horizon of the Schwarzschild black hole in massive gravity. It seems appropriate to comment that the massive graviton effect is included in the event horizon $r_+$ through $R$ and $C$. Moreover, the second term in $S_A$ is independent of the GUP parameter $\lambda$ and the last term depends linearly on it.

When we take the GUP parameter $\lambda$ as

$$\lambda = \frac{\delta_1}{3\pi^4} \approx 0.0156,$$

(5.18)

we can find the entropy satisfying the area law with correction terms depending on the surface gravity

$$S_A = \frac{A}{4} + \alpha r_+ \kappa_+ + \beta \lambda \kappa_+^2,$$

(5.19)

where

$$\alpha = \frac{\delta_2}{3\pi^4} \approx 0.0105, \quad \beta \lambda = \frac{\delta_1 \delta_4}{36\pi^9} \approx 2.4959 \times 10^{-5}.$$  

(5.20)

Making use of the surface gravity (3.11), the modified entropy in massive gravity can be rewritten as

$$S_A = S_{\text{HUP},0} + S_{\text{GUP},0} + S_{\text{GUP},\tilde{\mu}},$$

(5.21)

where

$$S_{\text{HUP},0} = \frac{\pi r_+^2}{4},$$

(5.22)

$$S_{\text{GUP},0} = \frac{\alpha}{2} + \frac{\beta \lambda}{4r_+^2},$$

(5.23)

$$S_{\text{GUP},\tilde{\mu}} = C \left[ \frac{\alpha}{2} + \frac{\beta \lambda}{4r_+^2} (2 + C) \right] + 2R \left[ 2\beta \lambda R + \alpha r_+ + \frac{\beta \lambda (1 + C)}{r_+} \right].$$

(5.24)

Here, $S_{\text{HUP},0}$ shows the area’s law of entropy with HUP in massive gravity. Also, $S_{\text{GUP},0}$ is the extension of all order GUP corrected entropy in massless gravity [82] to massive gravity. Finally, $S_{\text{GUP},\tilde{\mu}}$ has new contribution to entropy of massive gravity. In the massless limit of $R \to 0$ and $C \to 0$, the all order GUP corrected entropy is reduced to

$$S_A = S_{\text{HUP},0} + S_{\text{GUP},0},$$

(5.25)

where

$$S_{\text{HUP},0} = \frac{\pi r_H^2}{4},$$

(5.26)

$$S_{\text{GUP},0} = \frac{\alpha}{2} + \frac{\beta \lambda}{4r_H^2}. $$

(5.27)

Note that $S_{\text{GUP},\tilde{\mu}} = 0$ in the massless limit. On the other hand, by turning off the all order GUP correction to entropy in Eq. (5.21), we have

$$S_A = \pi r_+^2,$$

(5.28)

which is the area’s law of the Schwarzschild black hole in massive gravity without the GUP.

In Fig. 2, we have drawn the all order GUP corrected entropies in massless/massive gravity by comparing with the ones in massless/massive gravity without the GUP. In the figure, the dotted line is for the entropy of the Schwarzschild black hole in massless gravity increasing with $\pi r_H^2$. The dashed line is for the all order GUP corrected entropy of the Schwarzschild black hole in massless gravity. As shown in the figure, the all order GUP correction makes the entropies divergent near $r_H = 0$, which was predicted in loop quantum gravity [88, 89]. On the other hand, when we introduce massive gravitons, the area’s law increasing with $\pi r_+^2$ is changed to $\pi r_+^2$ since $r_+$ is the event horizon in massive gravity. This is represented by the dot-dashed line in Fig. 2. Accordingly, the all order GUP corrected entropy of the Schwarzschild black hole in massive gravity is drawn by solid line in Fig. 2 which corresponds to the all order GUP corrected entropy in massless gravity drawn by the dashed line.

In Fig. 3, we have drawn the all order GUP corrected entropy of the Schwarzschild black hole in massive gravity by varying $R$ and $C$. Fig. 3(a) is drawn by varying $C$ with a fixed $R$, while in Fig. 3(b), by varying $R$ with a fixed $C$, respectively. These figures all show results that near $r_H = 0$ all order GUP corrections give dominant effects on the generalized Bekenstein-Hawking entropy.
In summary, we have newly studied the statistical entropy of a scalar field on the Schwarzschild black hole in massive gravity to all orders in the Planck length units through the rigorous integrations, and compared the result with the ones in massless gravity without/with all order GUP corrections. By carefully counting the number of quantum states in the just vicinity near the horizon based on the GUP, we have obtained the generalized Bekenstein-Hawking entropy to all orders in the Planck length units without any artificial cutoff. Furthermore, we have also found that all order GUP corrections to the Bekenstein-Hawking entropy both in massless and massive gravities give dominant effects near $r_H = 0$.

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