We present a computational method to directly calculate and visualize the directional components of the Coulomb, radiation, and total electromagnetic fields, as well as the scalar and vector potentials, generated from moving point charges in arbitrary motion with varying speeds. We explicitly calculate the retarded time of the point charge along a discretized grid which is then used to determine the fields and potentials. Our computational approach, implemented in Python, provides an intuitive understanding of the electromagnetic waves generated from moving point charges and can be used in conjunction with grid-based numerical modeling methods to solve real-world computational electromagnetics problems. The method can also be used to help students visualize problems related to moving potentials, which are often only treated analytically for very simple problems, and can be used to compute electromagnetic sources for non-trivial electron beams with other approaches in computational electromagnetics.

I. INTRODUCTION

The electric and magnetic fields generated from moving point charges are often complex and unexpected compared to their static counterparts. In particular, for moving point charges, there is a “correction” to the retarded Coulomb field which is proportional to the rate of change of the retarded Coulomb field multiplied by the retardation delay, defined as the time it would take to travel from the charge to the field point at speed \( c \). If the charge is accelerating, there is an additional term in the electromagnetic (EM) field expressions, known as the radiation field, which is proportional to the second derivative of the vector field directed from the charge at the retarded time towards the field point [1]. In homogeneous environments, the radiation field decreases as \( 1/r \) from the point charge and is responsible for the EM radiation produced by moving charges that propagates to infinity.

Retarded potentials and their solutions represent a special place in electrodynamics, generalizing Poisson’s equations to account for time dependence and retardation effects. Unlike Maxwell’s equations, the potentials have a freedom of gauge where the potential need not satisfy causality, though the “physical fields” derived from these potential fields must. Traditionally, such potentials are introduced in advanced or retarded forms, and involve complicated integrations with respect to advanced or retarded time [2]. The potentials also appear naturally in quantum field theories, and some may argue they are more fundamental than the physical fields, explaining such peculiarities as the Aharonov–Bohm effect [3]. While elegant and insightful, there are only a few known cases where an analytical solution for the potentials exists. These often involve fairly complex vector calculus and can leave many students struggling to visualize or appreciate the solution. It is therefore unfortunate that few computational approaches have been developed to model such moving potential problems [4]. Unlike many acclaimed simulation tools with Maxwell’s equations, such as the finite-difference time-domain (FDTD) method, there are no well known approaches to computationally solving problems with moving charges directly in space and time.

In this paper, we introduce a direct numerical modelling technique to solve retarded time problems of moving point charges. The simulation is implemented in PYTHON 3.8 for full three-dimensional problems and is optimized for computational efficiency. Our program allows the user to directly visualize how the different EM fields and potentials are formed using numerical methods to determine the retarded time of the particle at different points in space. Solutions of retarded potential problems are not just an interesting academic study, but have direct relevance on emerging experiments with high-energy electron sources, such as electron-energy loss spectroscopy (EELS) [5, 6] and vortex EELS [7, 8].

The rest of our paper is organized as follows: in Sec. II, we present the main theoretical background of moving point charges in electrodynamics, including the celebrated Liénard–Wiechert potentials [9] and expressions for the electric and magnetic fields. In Sec. III, we show a selection of results and examples that can be studied with our computational code, including the EM fields generated from a fast oscillating charge and an oscillating dipole. We also explore the phenomena of synchrotron radiation, Bremsstrahlung, and the fields formed from a simple moving charge with constant velocity, as well as with constant acceleration. We present our conclusions in Sec. IV. The simulation code is freely available for download and use at Ref. 10.

II. THEORY AND COMPUTATIONAL IMPLEMENTATION

The charge and current densities of a point charge \( q \) at the position \( \mathbf{r}_p(t) \) with velocity \( c\beta(t) \) are, respectively,

\[
\rho(r, t) = q\delta[r - \mathbf{r}_p(t)], \quad (1)
\]
\[ \mathbf{J}(\mathbf{r}, t) = q c \beta(t) \delta[\mathbf{r} - \mathbf{r}_p(t)]. \]  

The vector and scalar potentials of a moving point charge in the Lorentz gauge, known as the Liénard–Wiechert potentials [9], are derived from Maxwell’s equations as

\[ \Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{\kappa R} \right]_{\text{ret}}, \quad (3) \]

\[ \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi} \left[ \mathbf{\beta} \right]_{\text{ret}}, \quad (4) \]

where \( R = |\mathbf{r} - \mathbf{r}_p(t')|, \kappa = 1 - \mathbf{n}(t') \cdot \mathbf{\beta}(t') \) such that \( \mathbf{n} = (\mathbf{r} - \mathbf{r}_p(t'))/R \) is a unit vector from the position of the charge to the field point, and the quantity in brackets is to be evaluated at the retarded time \( t' \) given by

\[ t' = t - \frac{R(t')}{c}. \quad (5) \]

The physical (gauge-invariant) electric and magnetic fields generated from a moving point charge can be obtained using various approaches, including deriving them directly from their scalar and vector potentials [2, 11]:

\[ \mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \left[ \frac{\mathbf{n} - \mathbf{\beta}}{\kappa^3 R^2} \right]_{\text{ret}}, \quad (6) \]

\[ \mathbf{B} = \frac{1}{c} [\mathbf{n} \times \mathbf{E}]_{\text{ret}}, \quad (7) \]

where \( \mathbf{\beta} \) is the derivative of \( \mathbf{\beta} \) with respect to \( t' \). The first term in Eq. (6) is known as the “Coulomb field” and is independent of acceleration, while the second term is known as the “radiation field” and is linearly dependent on \( \mathbf{\beta} \):

\[ \mathbf{E}_{\text{Coul}} = \frac{q}{4\pi\epsilon_0} \left[ \frac{\mathbf{n} - \mathbf{\beta}}{\kappa^3 R^2} \right]_{\text{ret}}, \quad (8) \]

\[ \mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c} \left[ \frac{\mathbf{n}}{\kappa^3 R} \times \left[ \frac{\mathbf{n} - \mathbf{\beta}}{\kappa^3 R} \right]_{\text{ret}}. \right. \quad (9) \]

Notably, the Coulomb field falls off as \( 1/R^2 \), similar to the static field, while the radiation field decreases as \( 1/R \). The Coulomb and radiation field terms from the total magnetic field can be determined by substituting Eqs. (8) and (9) into Eq. (7).

To computationally solve the above equations, our simulation determines the retarded time \( t' \) of a moving point charge in arbitrary motion at each time step using Newton’s method to calculate the approximate solution of Eq. (5) for \( t' \). For the simulations, the trajectories of the moving point charges, including the velocities and accelerations at each time step, are known \textit{a priori}. We simulate an arbitrary number of point charges by exploiting the superposition principle for EM fields and potentials. In addition to specifying the particles’ trajectories beforehand, their motion could be determined numerically at each time step using the Lorentz force: \( \mathbf{F} = q \mathbf{E} + q \mathbf{v} \times \mathbf{B} \), where \( \mathbf{v} \) is the instantaneous velocity of the point charge and the \( \mathbf{E} \) and \( \mathbf{B} \) fields are generated from other charges in the simulation.

The scalar and vector potentials are calculated at each time step from Eqs. (3) and (4) using the previously determined retarded times at each grid point. The total, Coulomb, and radiation fields are computed using Eqs. (6), (8), and (9) for the respective electric fields; the corresponding magnetic fields are calculated from Eq. (7).

The simulation was written in Python 3.8 and is available at Ref. 10, which includes the code used to produce the figures in this paper and the corresponding figure animations. The program allows the user to specify the charge and trajectory of each point charge in the simulation. The code is optimized using vectorized operations from the NumPy package to compute the retarded time of the particles, as well as the potentials and fields, along the grid. The program runs \( \mathcal{O}(n) \) with respect to both the number of particles in the simulation and the number of grid points. Using an Intel Core i5-8250U 1.60 GHz CPU, the electric and magnetic fields generated by a single moving point charge can be calculated at each point in a three dimensional grid of size \( 100 \times 100 \times 100 \) in approximately 2.1 seconds.

The figures shown in Sec. III are the steady-state solutions of the EM fields generated from various moving point charge configurations. These snapshots were captured after a sufficient time duration had elapsed since the point charges began moving from their initial stationary state. The transient behaviour of the fields can be seen in the accompanying animations.

III. RESULTS

A. Oscillating Single Charge

First, we simulate a single positive charge oscillating sinusoidally along the \( x \) axis, with an amplitude of 2 nm and reaching a maximum speed of 0.5c. At this speed, the particle produces radiation with a wavelength \( \lambda \approx 25.133 \) nm. The total, Coulomb, and radiation fields produced from the accelerating particle are shown in the \( yz \) and \( xz \) planes in Figs. 1 and 2, respectively. The total field comprises both the Coulomb and radiation field, and only the EM field components with non-zero values are shown. The plots use a symmetric logarithmic scale that is linear around zero to allow both positive and negative values. The scalar and vector potentials, as well as the Poynting vector for the radiation field \( \mathbf{S}_{\text{rad}} = \frac{1}{\mu_0} (\mathbf{E}_{\text{rad}} \times \mathbf{B}_{\text{rad}}) \), are shown in the \( xz \) plane in Fig. 3.

B. Oscillating Electric Dipole

Electric dipole radiation is usually responsible for the radiation emitted in the electronic transitions of atoms or molecules, and can describe the scattered light by a nanoparticle in a laser beam to the first approximation [12].
Here, we simulate a physical dipole oscillating sinusoidally along the x axis with an amplitude of 2 nm and a maximum particle speed of 0.5c. The EM fields generated from the charges are shown in the yz and xz plane in Figs. 4 and 5, respectively. The potentials and Poynting vector for the radiation field are shown in the xz plane in Fig. 6. As expected from the analytical solutions for a moving dipole, there is no radiation emitted along the axis of the dipole.

C. Synchrotron Radiation

Synchrotron radiation is produced when charged particles are accelerated radially. This can be achieved in synchrotrons, which are high-energy particle machines that inject charged particles into roughly circular orbits. Synchrotrons behave like huge excited antennas radiating EM energy with a broad spectrum that contains the frequency of revolution and the corresponding harmonics [13].

We directly simulate a radially accelerating positive point charge that follows a circular trajectory with a radius of 2 nm and a speed of 0.5c along the xy plane. The EM fields in the xz and xy planes are shown in Figs. 7 and 8, respectively, and the potentials and Poynting vector for the radiation field in the xy plane are shown in Fig. 9.
Figure 4. EM fields from a sinusoidal oscillating electric dipole along the x axis with an amplitude of 2 nm and a maximum point charge speed of 0.5c, yielding an angular frequency \( \omega \approx 7.495 \times 10^{16} \) rad/s. Snapshots in the yz plane where the positive charge is at position \( x = -2 \) nm and the negative charge is at \( x = 2 \) nm. The total, Coulomb, and radiation fields are plotted from left to right: (a)–(c) \( E_x \); (d)–(f) \( B_y \); (g)–(i) \( B_z \).

D. Linear Acceleration

The total power radiated by a point charge as it accelerates has a well-known solution given by the Larmor formula for non-relativistic point charges \([14]\), while the Liénard’s generalization of the Larmor formula is used to evaluate particles moving at relativistic speeds. Figure 10 plots the total electric and magnetic fields of a linearly accelerating point charge at different times. At \( t = 0 \) s, the charge is stationary and therefore only the Coulomb field is present. As the particle begins accelerating the EM information travels outwards radially at the speed \( c \). The EM waves in the radiation field are emitted in a toroidal shape about the direction of acceleration that is stretched forward, with respect to the velocity, by the factor \( 1/(1 - \beta \cos \theta)^2 \), where \( \theta \) is the angle between the acceleration vector and the unit vector \( \mathbf{n} \) \([2]\).

E. Bremsstrahlung

Bremsstrahlung, also known as “braking radiation”, is produced when a charged particle decelerates. Bremsstrahlung can occur when a charged particle is deflected by other charged particles as it travels through matter. Here, we plot the total electric and magnetic fields of a linearly decelerating point charge at different points in time, as shown in Fig. 11. Similar to the linearly accelerating point charge in Fig. 10, the charge is moving at a constant velocity at \( t = 0 \) s, so only the Coulomb field is present.

Figure 5. EM fields in the xz plane from a sinusoidal oscillating electric dipole as shown in Fig. 4. The total, Coulomb, and radiation fields are plotted from left to right: (a)–(c) \( E_x \); (d)–(f) \( E_z \); (g)–(i) \( B_y \).

Figure 6. Potentials and Poynting vector in the xz plane of a sinusoidal oscillating electric dipole as shown in Fig. 4. (a) x component of the vector potential; (b) scalar potential; (c) Poynting vector for the radiation field in radial direction \( \mathbf{F} \).

F. Constant Velocity

Finally, we study the case of a moving point charge with constant velocity, which is known to not generate a radiation field. Interestingly, the direction of the electric field points along the line from the present position of the point charge.
charge, and the field flattens in the direction perpendicular to motion as the charge’s speed increases. The electric field is reduced in the forward and backwards direction by a factor \((1 - \beta^2)\) relative to the field at rest, and enhanced in the perpendicular direction by a factor \(1/\sqrt{1 - \beta^2}\) [2]. These relativistic effects can be seen directly in Figs. 12 and 13, which plot the fields generated from point charges moving at different constant velocities along the \(x\) axis in the \(xz\) and \(yz\) planes, respectively.

**IV. CONCLUSIONS**

We have introduced a computational method to directly simulate and visualize the EM fields and potentials generated from moving point charges following several different trajectories of motion, including sinusoidal oscillations, linear and radial acceleration, linear deceleration, and constant velocity. Our program, written in PYTHON 3.8, allows the user to specify the number of charges and their trajectories in three dimensions. The simulation calculates the Coulomb, radiation, and total fields, as well as the scalar and vector potentials, at specified grid points in time by

Figure 7. EM fields from a radially accelerating positive point charge orbiting in the \(xy\) plane with a radius of 2 nm and a speed of 0.5c, yielding an angular frequency \(\omega \approx 7.495 \times 10^{10} \text{ s}^{-1}\). Snapshots in the \(xz\) plane where the charge is at position \(x = 2\ \text{nm}\) and \(y = 0\ \text{nm}\). The total, Coulomb, and radiation fields are plotted from left to right: (a)–(c) \(E_x\); (d)–(f) \(E_y\); (g)–(i) \(E_z\); (j)–(l) \(B_z\); (m)–(o) \(B_y\); (p)–(r) \(B_x\).

Figure 8. EM fields in the \(xy\) plane from a radially accelerating positive point charge as shown in Fig. 7. The total, Coulomb, and radiation fields are plotted from left to right: (a)–(c) \(E_x\); (d)–(f) \(E_y\); (g)–(i) \(B_z\).

Figure 9. Potentials and Poynting vector in the \(xy\) plane of a radially accelerating positive point charge as shown in Fig. 7. (a) \(x\) component of the vector potential; (b) \(y\) component of the vector potential; (c) scalar potential; (d) Poynting vector for the radiation field in radial direction \(F\).
Figure 10. EM fields from a linearly accelerating positive point charge along the $x$ axis. The charge accelerates from $x = 0$ nm while stationary and reaches a speed of $0.99c$ at $x = 30$ nm ($\tau \approx 2.022 \times 10^{-16}$ s), accelerating at approximately $1.468 \times 10^{24}$ m/s$^2$. Snapshots in the $xz$ plane starting at $t = 0$ s and increasing with equal increments of $\tau/5$ in time from left to right: (a)–(f) total $E_x$; (g)–(l) total $E_z$; (m)–(r) total $B_y$.

Figure 11. EM fields from a linearly decelerating positive point charge along the $x$ axis. The charge begins decelerating from an initial speed of $0.99c$ at $x = 0$ nm and reaches a speed of zero at $x = 30$ nm ($\tau = 2.022 \times 10^{-16}$ s), decelerating at approximately $1.468 \times 10^{24}$ m/s$^2$. Snapshots in the $xz$ plane starting at $t = 0$ s and increasing with equal increments of $\tau/5$ in time from left to right: (a)–(f) total $E_x$; (g)–(l) total $E_z$; (m)–(r) Total $B_y$. 
Figure 12. EM fields from a positive point charge moving at a constant velocity. Snapshots in the $xz$ plane of point charges with different velocities at position $x = 0$ nm. The point charge speeds from left to right are zero, $0.2c$, $0.5c$, $0.8c$, $0.9c$, $0.99c$: (a)–(f) total $E_x$; (g)–(l) total $E_z$; (m)–(r) total $B_y$.

Figure 13. EM fields from a positive point charge moving at a constant velocity. Snapshots in the $yz$ plane of point charges with different velocities at position $x = 0$ nm. The point charge speeds from left to right are zero, $0.2c$, $0.5c$, $0.8c$, $0.9c$, $0.99c$: (a)–(f) total $E_y$; (g)–(l) total $E_z$; (m)–(r) total $B_y$; (s)–(x) total $B_z$. 
first determining the retarded time at each point. These simulations provide useful insights to gain an intuitive understanding of point charge radiation, and can be used as teaching tools for advanced undergraduate and graduate-level EM theory courses. Future work includes implementing numerical modeling methods, such as FDTD, to investigate the interactions of these generated fields with physical nanostructures.

ACKNOWLEDGEMENTS

This work was supported by the Natural Sciences and Engineering Research Council of Canada, Queen’s University, and the Canadian Foundation for Innovation.

[1] R. P. Feynman, R. Leighton, and M. Sands, The Feynman Lectures on Physics, Vol. 2 (Addison-Wesley, 1964).
[2] D. J. Griffiths, Introduction to Electrodynamics, 4th ed. (Cambridge University Press, 2017).
[3] Y. Aharonov and D. Bohm, “Significance of electromagnetic potentials in the quantum theory,” Phys. Rev. 115, 485–491 (1959).
[4] D. Ruhlandt, S. Mühle, and J. Enderlein, “Electric field lines of relativistically moving point charges,” Am. J. Phys. 88, 5–10 (2020).
[5] F. J. García de Abajo and A. Howie, “Relativistic electron energy loss and electron-induced photon emission in inhomogeneous dielectrics,” Phys. Rev. Lett. 80, 5180–5183 (1998).
[6] A. W. Blackstock, R. H. Ritchie, and R. D. Birkhoff, “Mean free path for discrete electron energy losses in metallic foils,” Phys. Rev. 100, 1078–1083 (1955).
[7] J. Verbeeck, H. Tian, and P. Schattschneider, “Production and application of electron vortex beams,” Nature 467, 301–304 (2010).
[8] B. J. McMorrnan et al., “Electron vortex beams with high quanta of orbital angular momentum,” Science 331, 192–195 (2011).
[9] E. Wiechert, “Elektrodynamische elementargesetze,” Ann. Phys. (Berlin) 309, 667–689 (1901).
[10] M. Filipovich, “moving-point-charges,” (2020), note: Python repository includes the code used to produce the figures and animations in this paper.
[11] J. D. Jackson, Classical Electrodynamics, 3rd ed. (Wiley, 1999).
[12] X. Li and H. F. Arnoldus, “Electric dipole radiation near a mirror,” Phys. Rev. A 81, 053844 (2010).
[13] Christof Kunz, “Synchrotron radiation,” (CERN, 1974) pp. 155–166.
[14] J. Larmor, “On the theory of the magnetic influence on spectra; and on the radiation from moving ions,” Philos. Mag. 44, 503–512 (1897).