A Gap Theorem for Self-shrinkers of the Mean Curvature Flow in Arbitrary Codimension*

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Abstract

In this paper, we prove a classification theorem for self-shrinkers of the mean curvature flow with $|A|^2 \leq 1$ in arbitrary codimension. In particular, this implies a gap theorem for self-shrinkers in arbitrary codimension.

1 Introduction

Let $x : M^n \to \mathbb{R}^{n+p}$ be an $n$-dimensional submanifold in the $(n+p)$-dimensional Euclidean space. If we let the position vector $x$ evolve in the direction of the mean curvature $H$, then it gives rise to a solution to the mean curvature flow:

$$x : M \times [0, T) \to \mathbb{R}^{n+p}, \quad \frac{\partial x}{\partial t} = H$$

(1.1)

We call the immersed manifold $M$ a self-shrinker if it satisfies the quasilinear elliptic system:

$$H = -x^\perp$$

(1.2)

where $\perp$ denotes the projection onto the normal bundle of $M$.

Self-shrinkers are an important class of solutions to the mean curvature flow (1.1). Not only they are shrinking homothetically under mean curvature flow (see, e.g., [5]), but also they describe possible blow ups at a given singularity of the mean curvature flow.

In the curve case, U. Abresch and J. Langer [1] gave a complete classification of all solutions to (1.2). These curves are so-called Abresch-Langer curves.

In the hypersurface case (i.e. codimension 1), K. Ecker and G. Huisken [4] proved that if an entire graph with polynomial volume growth is a self-shrinker, then it is necessarily a hyperplane. Recently L. Wang [10] removed the condition of polynomial volume growth in Ecker-Huisken’s Theorem. Let $|A|^2$ denote the norm square of the second fundamental form of $M$. In [9] and [10], G. Huisken proved a classification theorem that $n$-dimensional self-shrinkers satisfying (1.2) in $\mathbb{R}^{n+1}$ with non-negative mean curvature, bounded $|A|$, and polynomial volume growth are $\Gamma \times \mathbb{R}^{n-1}$, or $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ ($0 \leq m \leq n$). Here, $\Gamma$ is a Abresch-Langer curve and $\mathbb{S}^m(\sqrt{m})$ is a $m$-dimensional sphere of radius $\sqrt{m}$. Recently, T.H. Colding and W.P. Minicozzi [5] showed that G. Huisken’s classification theorem still holds without the assumption that $|A|$ is bounded. Moreover, they showed that the only embedded entropy stable self-shrinkers with polynomial volume growth in $\mathbb{R}^{n+1}$ are hyperplanes, n-spheres, and cylinders.

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In arbitrary codimensional case, K. Smoczyk [15] proved the following two results: (i) For any $n$-dimensional compact self-shrinker $M^n$ in $\mathbb{R}^{n+p}$ satisfying (1.2), if $H \neq 0$ and unit mean curvature vector field $\nu = H/|H|$ is parallel in the normal bundle, then $M^n = S^n(\sqrt{n})$ in $\mathbb{R}^{n+1}$; (ii) For any $n$-dimensional compact self-shrinker $M^n$ in $\mathbb{R}^{n+p}$ satisfying (1.2), if $M^n$ is a complete self-shrinker with $H \neq 0$ and unit mean curvature vector field $\nu = H/|H|$ is parallel in the normal bundle, and having uniformly bounded geometry, then $M^n$ is either $\Gamma \times \mathbb{R}^{n-1}$, or $N^m \times \mathbb{R}^{n-m}$. Here $\Gamma$ is an Abresch-Langer curve and $N^m$ is an $m$-dimensional minimal submanifold in $S^{m+p-1}(\sqrt{m})$. On the other hand, Q. Ding and Z. Wang [7] recently have extended the result of L. Wang [16] to higher codimensional case under the condition of flat normal bundle.

Very recently, based on an identity of Colding and Minicozzi (see (9.42) in [3]), N. Q. Le and N. Sesum [11] proved a gap theorem (cf. Theorem 1.7 in [11]) for self-shrinkers of codimension 1: if a hypersurface $M^n \subset \mathbb{R}^{n+1}$ is a smooth complete embedded self-shrinker without boundary and with polynomial volume growth, and satisfies $|A|^2 < 1$, then $M^n$ is a hyperplane. Motivated by this result of Le and Sesum, we prove in this paper the following classification theorem for self-shrinkers in arbitrary codimensions:

**Theorem 1.1** If $M^n \rightarrow \mathbb{R}^{n+p}$ ($p \geq 1$) is an $n$-dimensional complete self-shrinker without boundary and with polynomial volume growth, and satisfies

$$|A|^2 \leq 1,$$

then $M$ is one of the followings:

(i) a round sphere $S^n(\sqrt{n})$ in $\mathbb{R}^{n+1}$,

(ii) a cylinder $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, $1 \leq m \leq n-1$, in $\mathbb{R}^{n+1}$,

(iii) a hyperplane in $\mathbb{R}^{n+1}$.

Here $|A|^2$ is the norm square of the second fundamental form of $M$.

As an immediate consequence, we have the following gap theorem valid for arbitrary codimensions:

**Corollary 1.1** If $M^n \rightarrow \mathbb{R}^{n+p}$ ($p \geq 1$) is a smooth complete embedded self-shrinker without boundary and with polynomial volume growth, and satisfies

$$|A|^2 < 1,$$

then $M$ is a hyperplane in $\mathbb{R}^{n+1}$.

**Remark 1.1** We expect that the condition on volume growth in Theorem 1.1 and Corollary 1.1 can be removed. In fact, it was conjectured by the first author that a complete self-shrinker automatically has polynomial volume growth. Note that D. Zhou and the first author [3] proved that a complete Ricci shrinker necessarily has at most Euclidean volume growth.

**Remark 1.2** Shortly after our work was finished, Q. Ding and Y. L. Xin [8] proved that any complete non-compact properly immersed self-shrinker $M^n$ in $\mathbb{R}^{n+p}$ has at most Euclidean volume growth.

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2 Preliminaries

In this section, we recall some formulas and notations for submanifolds in Euclidean space by using the method of moving frames.

Let \( x : M^n \to \mathbb{R}^{n+p} \) be an \( n \)-dimensional submanifold of the \((n+p)\)-dimensional Euclidean space \( \mathbb{R}^{n+p} \). Let \( \{e_1, \ldots, e_n\} \) be a local orthonormal basis of \( M \) with respect to the induced metric, and \( \{\theta_1, \ldots, \theta_n\} \) be their dual 1-forms. Let \( e_{n+1}, \ldots, e_{n+p} \) be the local unit orthonormal normal vector fields.

In this paper we make the following convention on the range of indices:

\[ 1 \leq i, j, k \leq n; \quad n + 1 \leq \alpha, \beta, \gamma \leq n + p. \]

Then we have the following structure equations,

\[ dx = \sum_i \theta_i e_i, \quad (2.1) \]

\[ de_i = \sum_j \theta_{ij} e_j + \sum_{\alpha,j} h_{ij}^\alpha \theta_j e_\alpha, \quad (2.2) \]

\[ de_\alpha = -\sum_{i,j} h_{ij}^\alpha \theta_j e_i + \sum_\beta \theta_{\alpha\beta} e_\beta, \quad (2.3) \]

where \( h_{ij}^\alpha \) denote the components of the second fundamental form of \( M \) and \( \theta_{ij}, \theta_{\alpha\beta} \) denote the connections of the tangent bundle and normal bundle of \( M \), respectively.

The Gauss equations are given by

\[ R_{ijkl} = \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha) \quad (2.4) \]

\[ R_{ik} = \sum_\alpha H^\alpha h_{ik}^\alpha - \sum_{\alpha,j} h_{ij}^\alpha h_{jk}^\alpha \quad (2.5) \]

\[ R = H^2 - |A|^2 \quad (2.6) \]

where \( R \) is the scalar curvature of \( M \), \( |A|^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2 \) is the norm square of the second fundamental form, \( H = \sum_\alpha H^\alpha e_\alpha = \sum_\alpha (\sum_i h_{ii}^\alpha) e_\alpha \) is the mean curvature vector field, and \( H = |H| \) is the mean curvature of \( M \).

The Codazzi equations are given by (see, e.g., [12])

\[ h_{ij}^\alpha = h_{ik}^\alpha, \quad (2.7) \]

where the covariant derivative of \( h_{ij}^\alpha \) is defined by

\[ \sum_k h_{ijk}^\alpha \theta_k = dh_{ij}^\alpha + \sum_k h_{kj}^\alpha \theta_{ki} + \sum_k h_{ik}^\alpha \theta_{kj} + \sum_\beta h_{ij}^\beta \theta_{\alpha\beta}. \quad (2.8) \]

If we denote by \( R_{\alpha\beta ij} \) the curvature tensor of the normal connection \( \theta_{\alpha\beta} \) in the normal bundle of \( x : M \to \mathbb{R}^{n+p} \), then the Ricci equations are

\[ R_{\alpha\beta ij} = \sum_k (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta). \quad (2.9) \]
By exterior differentiation of (2.8), we have the following Ricci identities (see, e.g., [12])

\[ h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum_m h^{\alpha}_{mj} R^{mikl} + \sum_m h^{\alpha}_{km} R^{mjkl} + \sum_{\beta} h^{\beta}_{ij} R^{\beta\alpha kl}. \]  

(2.10)

We define the first and second covariant derivatives, and Laplacian of the mean curvature vector field \( H = \sum_\alpha H^\alpha e_\alpha \) in the normal bundle \( N(M) \) as follows (cf. [4], [12])

\[ \sum_i H^{\alpha}_{,i} \theta_i = dH^{\alpha} + \sum_\beta H^{\beta}_{,i} \theta_\beta, \]  

(2.11)

\[ \sum_j H^{\alpha}_{,j} \theta_j = dH^{\alpha}_{,i} + \sum_j H^{\alpha}_{j} \theta_{ji} + \sum_\beta H^{\beta}_{i} \theta_\beta, \]  

(2.12)

\[ \Delta_H^{\perp} H^{\alpha} = \sum_i H^{\alpha}_{,ii}, \quad H^{\alpha} = \sum_k h^{\alpha}_{kk}. \]  

(2.13)

Let \( f \) be a smooth function on \( M \), we define the covariant derivatives \( f_i, f_{ij}, \) and the Laplacian of \( f \) as follows

\[ df = \sum_i f_i \theta_i, \quad \sum_j f_{ij} \theta_j = df_i + \sum_j f_j \theta_{ji}, \quad \Delta f = \sum_i f_{ii}. \]  

(2.14)

3 A Key Lemma

As we mentioned in the introduction, the proof of Le-Sesum’s gap theorem relies on an important identity of Colding and Minicozzi [5] for hypersurfaces. The identity, see (9.42) in [5] or (4.1) in [11], is obtained in terms of certain second order linear operator for hypersurfaces (which is part of the Jacobi operator for the second variation). In this section, we derive a similar inequality for arbitrary codimensions.

Let \( a \) be any fixed vector in \( \mathbb{R}^{n+p} \), we define the following height functions in the \( a \) direction on \( M \),

\[ f = \langle x, a \rangle, \]  

(3.1)

and

\[ g_\alpha = \langle e_\alpha, a \rangle \]  

(3.2)

for a fixed normal vector \( e_\alpha \).

From (2.13) for \( f_i \) and the structure equation (2.1), we have

\[ f_i = \langle e_i, a \rangle. \]  

(3.3)

Similarly, from (2.14) for \( f_{ij} \) and the structure equation (2.2), we have

\[ f_{ij} = \sum_\alpha h^{\alpha}_{ij} \langle e_\alpha, a \rangle. \]  

(3.4)

Since \( a \) can be arbitrary in (3.3) and (3.4), we obtain (see [4])

\[ x_i = e_i, \quad x_{ij} = \sum_\alpha h^{\alpha}_{ij} e_\alpha. \]  

(3.5)
Define the first derivative $g_{\alpha,i}$ of $g_{\alpha}$ by
\begin{equation}
\sum_{i} g_{\alpha,i} \theta_{i} = d g_{\alpha} + \sum_{\beta} g_{\beta} \theta_{\beta \alpha}.
\end{equation}
(3.6)
We have, by use of (2.3),
\begin{equation}
g_{\alpha,i} = - \sum_{k} h_{ik}^{\alpha} \langle e_{k}, a \rangle.
\end{equation}
(3.7)
Taking covariant derivatives on both sides of (3.7) in the $e_{j}$ direction and using (3.5), we have
\begin{equation}
g_{\alpha,ij} = - \sum_{k} h_{ikj}^{\alpha} \langle e_{k}, a \rangle - \sum_{k,\beta} h_{ik}^{\alpha} h_{kj}^{\beta} \langle e_{\beta}, a \rangle,
\end{equation}
(3.8)
where the second derivative $g_{\alpha,ij}$ of $g_{\alpha}$ is defined by
\begin{equation}
\sum_{j} g_{\alpha,ij} \theta_{j} = d g_{\alpha,i} + \sum_{j} g_{\alpha,j} \theta_{ji} + \sum_{\beta} g_{\beta,i} \theta_{\beta \alpha}.
\end{equation}
(3.9)
Again, since $a$ is arbitrary in (3.7) and (3.8), we obtain (see 4)
\begin{equation}
e_{\alpha,i} = - \sum_{j} h_{ij}^{\alpha} e_{j}, \quad e_{\alpha,ij} = - \sum_{k} h_{ikj}^{\alpha} e_{k} - \sum_{k,\beta} h_{ik}^{\alpha} h_{kj}^{\beta} e_{\beta},
\end{equation}
(3.10)
where the covariant derivative $h_{ijk}^{\alpha}$ of the second fundamental form $h_{ij}^{\alpha}$ is defined by (2.8).

Now the self-shrinker equation (1.2) is equivalent to
\begin{equation}
-H^{\alpha} = \langle x, e_{\alpha} \rangle, \quad n + 1 \leq \alpha \leq n + p.
\end{equation}
(3.11)
Taking covariant derivative of (3.11) with respect to $e_{i}$ by use of (3.5) and (3.10), we have
\begin{equation}
-H^{\alpha}_{,i} = - \sum_{j} h_{ij}^{\alpha} < x, e_{j} >, \quad 1 \leq i \leq n, \quad n + 1 \leq \alpha \leq n + p.
\end{equation}
(3.12)
Taking covariant derivative of (3.12) with respect to $e_{k}$ by use of (3.5) and (3.11), we have
\begin{equation}
-H^{\alpha}_{,ik} = - \sum_{j} h_{ij}^{\alpha} < x, e_{j} > - h_{ik}^{\alpha} - \sum_{\beta,j} h_{ij}^{\beta} h_{\beta jk}^{\alpha} < x, e_{\beta} >
\end{equation}
\begin{equation}
= - \sum_{j} h_{ij}^{\alpha} < x, e_{j} > - h_{ik}^{\alpha} + \sum_{\beta,j} H_{\beta j}^{\alpha} h_{ij}^{\beta}.
\end{equation}
(3.13)
Writing
\begin{equation}
\sigma_{\alpha \beta} = \sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta},
\end{equation}
(3.14)
we have
\begin{equation}
\sum_{\alpha,\beta} \sigma_{\alpha \beta} H^{\alpha} H^{\beta} \leq |A|^{2} |H|^{2}.
\end{equation}
(3.15)
We are now ready to prove the following key lemma:

**Lemma 3.1** Let $M^{n}$ be an $n$-dimensional complete self-shrinker in $\mathbb{R}^{n+p}$ without boundary and with polynomial volume growth, if $|A|^{2}$ is bounded on $M^{n}$, then
\begin{align*}
\int_{M} |\nabla H|^{2} e^{-\frac{|x|^{2}}{4}} dv &= \int_{M} \sum_{\alpha,\beta} \sigma_{\alpha \beta} H^{\alpha} H^{\beta} - |H|^{2} e^{-\frac{|x|^{2}}{4}} dv \\
&\leq \int_{M} [ |A|^{2} - 1 ] |H|^{2} e^{-\frac{|x|^{2}}{4}} dv.
\end{align*}
Proof. Letting $i = k$ in (3.13) and summing over $i$, we get

$$\Delta^\perp H^\alpha = \sum_j H^\alpha_{j < x, e_j} > +H^\alpha - \sum_\beta \sigma_{\alpha\beta} H^\beta. \tag{3.16}$$

Since $M^n$ has polynomial volume growth and $|A|^2$ is bounded on $M^n$, (3.11), (3.12), (3.14) and (3.16) imply that

$$\int_M |\nabla^\perp H|^2 e^{-|x|^2/2} \, dv < +\infty,$$

and

$$\int_M \sum_{\alpha, i} H^\alpha_{i < x, e_i} e^{-|x|^2/2} \, dv < +\infty.$$

Let $\varphi_r(x)$ be a smooth cut-off function with compact support in $B_{x_0}(r+1) \subset M$,

$$\varphi_r(x) = \begin{cases} 1, & \text{in } B_{x_0}(r) \\ 0 & \text{in } M \setminus B_{x_0}(r+1) \end{cases} \quad 0 \leq \varphi_r(x) \leq 1, \quad |\nabla \varphi_r| \leq 1.$$

Then, by integration by parts, we get

$$\int_M \sum_{\alpha} \Delta^\perp H^\alpha (\varphi_r H^\alpha) e^{-|x|^2/2} \, dv = \int_M \varphi_r H^\alpha H^\alpha_{i < x, e_i} e^{-|x|^2/2} \, dv - \int_M \sum_{\alpha, i} H^\alpha (\varphi_r H^\alpha)_{i < x, e_i} e^{-|x|^2/2} \, dv$$

$$= \int_M \varphi_r \left( \sum_{\alpha, i} H^\alpha H^\alpha_{i < x, e_i} > -|\nabla^\perp H|^2 \right) e^{-|x|^2/2} \, dv$$

$$- \int_M \sum_{\alpha, i} H^\alpha H^\alpha_{i < x, e_i} (\varphi_r)_{i < x, e_i} e^{-|x|^2/2} \, dv.$$

Letting $r \to +\infty$, the dominated convergence theorem implies that

$$\int_M \sum_{\alpha} \Delta^\perp H^\alpha H^\alpha e^{-|x|^2/2} \, dv = \int_M \left( \sum_{\alpha, i} H^\alpha H^\alpha_{i < x, e_i} > -|\nabla^\perp H|^2 \right) e^{-|x|^2/2} \, dv. \tag{3.17}$$

Putting (3.16) into (3.17), we obtain:

$$\int_M |\nabla^\perp H|^2 e^{-|x|^2/2} \, dv = \int_M \left( \sum_{\alpha, \beta} \sigma_{\alpha\beta} H^\alpha H^\beta - |H|^2 \right) e^{-|x|^2/2} \, dv$$

$$\leq \int_M (|A|^2 - 1) |H|^2 e^{-|x|^2/2} \, dv.$$ 

\[\square\]

**Remark 3.1** From the proof of Lemma 3.1, one can see that the conclusion of Lemma 3.1 is valid even if $|A|^2$ has certain growth in $|x|^2$. 

4 Proof of Theorem 1.1

Now we present the proof of Theorem 1.1.

Proof of Theorem 1.1. Under the assumptions of Theorem 1.1, from Lemma 3.1, we know that either $H \equiv 0$, or $H \neq 0$ but with $\nabla^\perp H \equiv 0$ and $|A|^2 \equiv 1$.

If $H \equiv 0$, we have $\langle x, e_\alpha \rangle \geq 0$, $n + 1 \leq \alpha \leq n + p$, from which we easily conclude from (3.12) that $M$ is totally geodesic, that is, a hyperplane in $\mathbb{R}^{n+1}$.

Next, suppose that $H \neq 0$, $\nabla^\perp H \equiv 0$, and $|A|^2 \equiv 1$. In this case, (3.13) becomes

$$\sum_{\beta,j} H^\beta h^\alpha_{ij} h^\beta_{jk} = h^\alpha_{ik} + \sum_{j} h^\alpha_{ijk} < x, e_j >, \quad 1 \leq i, k \leq n; n + 1 \leq \alpha \leq n + p. \quad (4.1)$$

Multiplying both sides of (4.1) by $h^\alpha_{ik}$ and summing over $\alpha, i, k$, we get

$$\sum_{\alpha,\beta,i,j,k} H^\beta h^\alpha_{ij} h^\beta_{jk} h^\alpha_{ik} = |A|^2 + \frac{1}{2} (|A|^2)_j < x, e_j > = |A|^2 = 1. \quad (4.2)$$

Next we choose a local orthonormal frame $\{e_\alpha\}$ for the normal bundle of $x : M \to \mathbb{R}^{n+p}$, such that $e_{n+p}$ is parallel to the mean curvature vector $H$; i.e.,

$$e_{n+p} = \frac{H}{|H|}, \quad H^{n+p} = H, \quad H^\alpha = 0, \quad \alpha \neq n + p. \quad (4.3)$$

Because now the equality holds in (3.15), we have

$$h^\alpha_{ij} = 0, \quad \alpha \neq n + p, \quad |A|^2 = \sum_{i,j} h^{n+p}_{ij} h^{n+p}_{ij} = 1. \quad (4.4)$$

Since $\nabla^\perp H \equiv 0$ and $|A|^2 \equiv 1$, by the definition of $\Delta$ and using (2.7), (2.10), (2.4), (2.5) and (2.9), we have (c.f. [14],[13],[12],[17])

$$0 = \frac{1}{2} \Delta |A|^2$$

$$= \sum_{\alpha,i,j,k} (h^\alpha_{ij})^2 + \sum_{\alpha,i,j,k} h^\alpha_{ij} h^\alpha_{jik}$$

$$= \sum_{\alpha,i,j,k} (h^\alpha_{ij})^2 + \sum_{\alpha,i,j,k,m} h^\alpha_{ij} h^\alpha_{mk} R_{mijk} + \sum_{\alpha,i,j,m} h^\alpha_{ij} h^\alpha_{im} R_{mj} + \sum_{\alpha,i,j,k} h^\alpha_{ij} h^\beta_{ik} R_{\beta ajk}$$

$$= \sum_{\alpha,i,j,k} (h^\alpha_{ij})^2 + \sum_{\alpha,i,j,k,m} H^\beta h^\alpha_{mj} h^\beta_{ij} h^\alpha_{im} - \sum_{\alpha,i,j,k,m} h^\alpha_{ij} h^\beta_{ijk} h^\alpha_{mk} h^\beta_{mk} + 2 \sum_{\alpha,i,j,k} h^\alpha_{ij} h^\beta_{ik} R_{\beta ajk}.$$ 

Plugging (4.2), (4.3) and (4.4) into the above identity, we conclude that

$$h^\alpha_{ij} = 0, \quad n + 1 \leq \alpha \leq n + p. \quad (4.5)$$

Because $e_{n+1} \wedge e_{n+2} \wedge \cdots \wedge e_{n+p-1}$ is parallel in the normal bundle of $M$ and $h^\alpha_{ij} \equiv 0, \quad \alpha \neq n + p$, by Theorem 1 of Yau [15], we see that $M$ is a hypersurface in $\mathbb{R}^{n+1}$. So (4.5) implies that $M$ is an isoparametric hypersurface, thus from $|A|^2 = 1$ we conclude that $M$ is either a round sphere $S^n(\sqrt{n})$, or a cylinder $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, $1 \leq m \leq n - 1$ in $\mathbb{R}^{n+1}$. This completes the proof of Theorem 1.1. $\square$
5 Further Remarks

In this section, we make several simple observations:

**Proposition 5.1** If a submanifold $M^n \to \mathbb{R}^{n+p}$ is an $n$-dimensional complete self-shrinker without boundary and with polynomial volume growth, such that

$$|H|^2 \geq n, \quad (5.1)$$

then $|H|^2 \equiv n$ and $M$ is a minimal submanifold in the sphere $S^{n+p-1}(\sqrt{n})$.

**Proof of Proposition 5.1.** From (3.5) and (3.11), we have

$$\frac{1}{2}\Delta|\mathbf{x}|^2 = n < \mathbf{x}, \Delta \mathbf{x}> = n + \sum_{\alpha} H^\alpha < \mathbf{x}, e_\alpha> = n - |H|^2 \quad (5.2)$$

Under the polynomial volume growth assumption, (1.2) and (5.2) guarantee that

$$\int_M (\Delta|\mathbf{x}|^2)e^{-\frac{|\mathbf{x}|^2}{2}}d\mathbf{v} < +\infty \quad \text{and} \quad \int_M |\nabla|\mathbf{x}|^2|e^{-\frac{|\mathbf{x}|^2}{2}}d\mathbf{v} < +\infty.$$  

Then, by integrating by parts and the dominated convergence theorem, it follows that (similar to the proof of Lemma 3.1)

$$\frac{1}{4}\int_M |\nabla|\mathbf{x}|^2|e^{-\frac{|\mathbf{x}|^2}{2}}d\mathbf{v} = \frac{1}{2}\int_M (\Delta|\mathbf{x}|^2)e^{-\frac{|\mathbf{x}|^2}{2}}d\mathbf{v} = \int_M (n - |H|^2)e^{-\frac{|\mathbf{x}|^2}{2}}d\mathbf{v}. \quad (5.3)$$

From (5.1) and (5.3), we get $|H|^2 = n$ and $< \mathbf{x}, \mathbf{x}> = r^2$. Thus by (1.2) we conclude that $r = \sqrt{n}$ and $M$ is a minimal submanifold in the sphere $S^{n+p-1}(\sqrt{n})$. $\square$

**Proposition 5.2** If a submanifold $M \to \mathbb{R}^{n+p}$ is an $n$-dimensional compact self-shrinker without boundary and satisfies either $|H|^2 = \text{constant}$, or

$$|H|^2 \leq n, \quad (5.4)$$

then $|H|^2 \equiv n$ and $M$ is a minimal submanifold in the sphere $S^{n+p-1}(\sqrt{n})$.

**Proof of Proposition 5.2.** Integrating (5.2) over $M$ and using the Stokes theorem, we have

$$\int_M (n - |H|^2)d\mathbf{v} = 0. \quad (5.5)$$

Hence Proposition 5.2 follows from (5.5), (5.4), and (1.2). $\square$

**Remark 5.1** Let $x : M \to \mathbb{R}^{n+p}$ be an $n$-dimensional submanifold. If $x$ satisfies

$$\lambda H^\alpha = < \mathbf{x}, e_\alpha>, \quad n + 1 \leq \alpha \leq n + p \quad (5.6)$$

for some positive constant $\lambda$, then we call $M$ a self-expander of the mean curvature flow. Observe that for a self-expander, we have

$$\frac{1}{2}\Delta|\mathbf{x}|^2 = n+ < \mathbf{x}, \Delta \mathbf{x}> = n + n \sum_{\alpha} H^\alpha < \mathbf{x}, e_\alpha> = n + n\lambda |H|^2. \quad (5.7)$$
From (5.7), we immediately get

**Proposition 5.3** There exists no n-dimensional compact self-expander without boundary in \( \mathbb{R}^{n+p} \).

Finally, we list some simple examples of self-shrinkers of higher codimensions.

**Example 5.1** For any positive integers \( m_1, \ldots, m_p \) such that \( m_1 + \cdots + m_p = n \), the submanifold

\[
M^n = S^{m_1}(\sqrt{m_1}) \times \cdots \times S^{m_p}(\sqrt{m_p}) \subset \mathbb{R}^{n+p}
\]

is an n-dimensional compact self-shrinker in \( \mathbb{R}^{n+p} \) with

\[
H = -X, \quad |H|^2 = n, \quad |A|^2 = p.
\]

Here

\[
S^{m_i}(r_i) = \{ X_i \in \mathbb{R}^{m_i+1} : |X_i|^2 = r_i^2 \}, \quad i = 1, \ldots, p
\]

is an \( m_i \)-dimensional round sphere with radius \( r_i \).

**Example 5.2** For positive integers \( m_1, \ldots, m_p, q \geq 1 \), with \( m_1 + \cdots + m_p + q = n \), the submanifold

\[
M^n = S^{m_1}(\sqrt{m_1}) \times \cdots \times S^{m_p}(\sqrt{m_p}) \times \mathbb{R}^q \subset \mathbb{R}^{n+p}
\]

is an n-dimensional complete non-compact self-shrinker in \( \mathbb{R}^{n+p} \) with polynomial volume growth which satisfies

\[
H = -X^\perp, \quad |H|^2 = \sum_{i=1}^{p} m_i, \quad |A|^2 = p.
\]

**Remark 5.2** In Example 5.1 and Example 5.2, if we let \( p \geq 2 \), then we have an n-dimensional self-shrinker of codimension \( p \) with \( |A|^2 = p \geq 2 \), thus not one of the three cases in Theorem 1.1.

**Remark 5.3** From Example 5.2, we can see that the condition "\( |H|^2 \geq n \)" in Proposition 5.1 is necessary.

**Example 5.3** (cf. [2]) Let

\[
X : S^2(\sqrt{m(m+1)}) \hookrightarrow S^{2m}(\sqrt{2}) \subset \mathbb{R}^{2m+1}, \quad m \geq 2
\]

be a minimal surface in \( S^{2m}(\sqrt{2}) \). Consider it as a surface in \( \mathbb{R}^{2m+1} \), then it is a self-shrinker with

\[
H = -X, \quad |H|^2 = 2, \quad |A|^2 = 2 - \frac{2}{m(m+1)} < 2.
\]

**Remark 5.4** By choosing local orthogonal frame \( \{ e_{\alpha} \} \) for the normal bundle of \( x : M^n \to \mathbb{R}^{n+p} \), such that \( e_{n+p} \) is parallel to the mean curvature vector \( H \), by Lemma 3.1, if \( |A|^2 \) is bounded, and

\[
\sum_{i,j} h_{ij}^{n+p} h_{ij}^{n+p} \leq 1,
\]

we have \( \nabla^\perp H = 0 \), that is, \( |H|^2 = \text{constant} \) and unit mean curvature vector field \( \nu = H/|H| \) is parallel in the normal bundle. From Proposition 5.2 and Theorem 1.3 of Smoczyk [15], we have

**Proposition 5.4** Let \( M^n \) be an n-dimensional complete self-shrinker in \( \mathbb{R}^{n+p} \) without boundary and with polynomial volume growth. If \( |A|^2 \) is bounded on \( M^n \) and (5.15) holds, then

\[
M^n = N^m \times \mathbb{R}^{n-m}, \quad 0 \leq m \leq n,
\]

where \( N^m \) is a m-dimensional minimal submanifold in \( S^{m+p-1}(\sqrt{m}) \).
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