Bosonization, Pairing, and Superconductivity of the Fermionic Tonks-Girardeau Gas

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We determine some exact static and time-dependent properties of the fermionic Tonks-Girardeau (FTG) gas, a spin-aligned one-dimensional Fermi gas with infinitely strongly attractive zero-range odd-wave interactions. We show that the two-particle reduced density matrix exhibits maximal superconducting off-diagonal long-range order, and on a ring an FTG gas with an even number of atoms has a highly degenerate ground state with quantization of Coriolis rotational flux and high sensitivity to rotation and to external fields and accelerations. For a gas initially under harmonic confinement we show that during an expansion the momentum distribution undergoes a “dynamical bosonization”, approaching that of an ideal Bose gas without violating the Pauli exclusion principle.

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If an ultracold atomic vapor is confined in a de Broglie wave guide with transverse trapping so tight and temperature so low that the transverse vibrational excitation quantum $\hbar \omega$ is larger than available longitudinal zero point and thermal energies, the effective dynamics becomes one-dimensional (1D) [1, 2], a regime currently under intense experimental investigation [3, 4]. Confinement-induced 1D Feshbach resonances (CIRs) are reachable by tuning the 1D coupling constant via 3D Feshbach scattering resonances occur for both Bose gases [5] and spin-aligned Fermi gases [6]. Near a CIR the 1D interaction is very strong, leading to strong short-range correlations, breakdown of effective-field theories, and emergence of highly-correlated $N$-body ground states. In the bosonic case with very strong repulsion (1D hard-core Bose gas with coupling constant $g_{1D}^B \to +\infty$, the Tonks-Girardeau (TG) gas), the exact $N$-body ground state was determined some 45 years ago by a Fermi-Bose (FB) mapping to an ideal Bose gas [7], leading to “fermionization” of many properties of this Bose system, as recently confirmed experimentally [8]. The “fermionic TG” (FTG) gas [9, 10], a spin-aligned Fermi gas with very strong attractive 1D odd-wave interactions, can be realized by 3D Feshbach resonance mediated tuning to the attractive side of the CIR with 1D coupling constant $g_{1D}^D \to -\infty$. It has been pointed out [9, 10] that the generalized FB mapping [11] can be exploited in the opposite direction to map this system to the trapped ideal Bose gas, leading to determination of the exact $N$-body ground state and “bosonization” of many properties of this Fermi system. We recently examined the equilibrium one-body density matrix and exact dynamics following sudden confinement of the FTG gas [11] by detuning from the CIR [12, 13]. Here we determine some other exact properties of the untrapped, ring-trapped, and harmonically trapped fermionic TG gas, the most striking of which are pairing, superconductive off-diagonal long-range order (ODLRO) of the two-body density matrix, a highly degenerate ground state of an even number of atoms on a ring with quantization of Coriolis rotational flux and high sensitivity to rotation and to external fields and accelerations, and a “dynamical bosonization” of the momentum distribution following sudden relaxation of the trap frequency.

Untrapped FTG gas: The Hamiltonian is $\hat{H} = \sum_{j=1}^{N} \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_j^2} + \sum_{1 \leq i < j \leq N} v_{ij} \phi_i \phi_j \right]$ where $v_{ij}$ is the two-body interaction. Since the spatial wave function is antisymmetric due to spin polarization, there is no s-wave interaction, but it has been shown [14] that a strong, attractive, short-range odd-wave interaction (1D analog of 3D p-wave interactions) occurs near the CIR. This can be modeled by a narrow and deep square well of depth $V_0$ and width $2x_0$. The contact condition at the edges of the well is $\psi_F(x_{j\ell} = x_0) = -\psi_F(x_{j\ell} = -x_0) = -a_{1D}^F \psi_F(x_{j\ell} = \pm x_0)$ where $a_{1D}^F$ is the 1D scattering length and the prime denotes differentiation. Consider first the relative wave function $\psi_F(x)$ in the case $N = 2$. The FTG limit is $a_{1D}^F \to -\infty$, a zero-energy scattering resonance.

\[ \psi_F(x_1, \cdots, x_N) = A(x_1, \cdots, x_N) \prod_{j=1}^{N} \phi_0(x_j) \]  

with $A(x_1, \cdots, x_N) = \prod_{1 \leq i < j \leq N} \text{sgn}(x_i - x_j)$ the “unit antisymmetric function” employed in the original discovery of fermionization [14] and $\phi_0 = 1/\sqrt{L}$ the ideal Bose gas ground orbital, $L$ being the periodicity length. Its
energy is zero and it satisfies periodic boundary conditions for odd $N$ and antiperiodic boundary conditions for even $N$.

The exact single-particle density matrix $\rho_1(x, x') = N \int \psi^*_j(x, x_1, x_2, \ldots, x_N) \psi_j(x', x_2, \ldots, x_N) dx_2 \cdots dx_N$ is [8, 12]

$$\rho_1(x, x') = N \phi_0(x) \phi_0(x') |F(x, x')|^2 = 1 - 2|x - x'|/L.$$ In the thermodynamic limit $N \to \infty$, $L \to \infty$, $N/L = n$ this gives an exponential decay [12]: $\rho_1(x, x') = n e^{-2n|x - x'|}$. Its Fourier transform $n_k$, normalized to $\sum k n_k = N$ (allowed momenta $n2\pi/L$ with $n = 0, \pm 1, \pm 2, \ldots$), is the momentum distribution function $n_k = [1 + (k/2n)^2]^{-1}$. It satisfies the exclusion principle limitation $n_k \leq 1$, but not necessarily for $n \to 0$. The continuous momentum density $n(k) = (L/2\pi)n_k$ reduces to $N$ times a representation of the Dirac delta function, simulating the ideal Bose gas distribution: $n(k) \to N\delta(k)$ [12].

The two-particle density matrix $\rho_2(x_1, x_2; x_1', x_2') = N(N-1) \int \psi^*_j(x_1, x_2, x_3, \ldots, x_N) \psi_j(x_1', x_2', x_3, \ldots, x_N) dx_3 \cdots dx_N$ also has a simple closed form:

$$\rho_2(x_1, x_2; x_1', x_2') = N(N-1) \rho_1(x_1, x_2) \rho_1(x_1', x_2') \times \text{sgn}(x_1' - x_2') \phi_0(x_1) \phi_0(x_2) \phi_0(x_1') \phi_0(x_2')$$

$$\times \text{sgn}(x_1' - x_2') e^{-2n|x_1' - x_2'|}[G(x_1, x_2; x_1', x_2')]^{-1}$$

where $|G(x_1, x_2; x_1', x_2')| = [J_{L/2}^2 \rho_1(x_1 - x) \rho_1(x_2 - x) \rho_1(x_1' - x) \rho_1(x_2' - x)]^{1/2}$ in the thermodynamic limit and $y_1 \leq y_2 \leq y_3 \leq y_4$ are the arguments $(x_1, x_2; x_1', x_2')$ in ascending order. $\rho_2$ is of order $n^2$ in the following cases: (a) $|x_1 - x_1'| \leq O(1/n)$, $|x_2 - x_2'| \leq O(1/n)$; (b) $|x_1 - x_1'| \leq O(1/n)$, $|x_2 - x_2'| \leq O(1/n)$; (c) $|x_1 - x_2| \leq O(1/n)$, $|x_1' - x_2'| \leq O(1/n)$. These are just Yang’s criteria [14] for superconducting ODLRO of $\rho_2$ in the absence of ODLRO of $\rho_1$. In case (c) $\rho_2$ remains of order $n^2$ for arbitrarily large separation of the centers of mass $X = (x_1 + x_2)/2$ and $X' = (x_1' + x_2')/2$, the hallmark of ODLRO. On the other hand, in cases (a) and (b) $\rho_2$ decays exponentially with $|X - X'|$. In the thermodynamic limit only configurations (c) contribute to the largest eigenvalue of $\rho_2$, and $\rho_2$ separates apart from negligible contributions (a) and (b) [13].

$$\rho_2(x_1, x_2; x_1', x_2') = n^2 \text{sgn}(x_1 - x_2) e^{-2n|x_1 - x_2|} \times \text{sgn}(x_1' - x_2') e^{-2n|x_1' - x_2'|} + \text{terms negligible for } \lambda_1.$$  

By Yang’s argument [13] the largest eigenvalue is $\lambda_1 = N$, and this is confirmed by comparison with the $\lambda_1$ contribution to the spectral representation of $\rho_2$, implying that the corresponding eigenfunction is $\psi_1(x_1, x_2) = C \text{sgn}(x_1 - x_2) e^{-2n|x_1 - x_2|}$ with $C = \sqrt{n/2L}$ confirming the value $\lambda_1 = n^2/C^2 = N$.

The range $1/2n$ of $\psi_1$ is of order of onset of a BEC-BCS crossover between tightly bound bosons and loosely bound Cooper pairs. There is an upper bound on $\lambda_1 \leq N$ on the largest eigenvalue, so the untrapped FTG gas is maximally superconductive in the sense of Yang’s ODLRO criterion.

**FTG gas on a ring:** If the FTG gas is trapped on a circular loop of radius $R$, with particle coordinates $x_j$ measured around the circumference $L = 2\pi R$, the FTG gas must satisfy periodic boundary conditions for both odd and even $N$ because of single-valuedness of its wave function. Since the mapping function $A(x_1, \ldots, x_N) = \prod_{l=1}^{n_{\phi_0}} \text{sgn}(x_l - x_j)$ is periodic (antiperiodic) for odd (even) $N$ as a result of its definition, it follows that the mapped ideal Bose gas used to solve the FTG problem must satisfy periodic (antiperiodic) boundary conditions for odd (even) $N$. The ground state of a FTG gas on a ring is then dependent on the particle number parity. For odd $N$ the FTG ground state in Eq. (1) is built from the zero-momentum orbital $\phi_0 = 1/\sqrt{L}$ and corresponds to mapping the FTG gas onto the ideal Bose gas ground state, the usual complete Bose-Einstein condensate (BEC), and is nondegenerate. On the other hand, for even $N$, which we henceforth assume, antiperiodicity requires that the only plane-wave orbitals allowed are $e^{ikx_j}/\sqrt{L}$ with $k = \pm \pi/L$, $\pm 3\pi/L$, \ldots. The ground state of this fictitious ideal Bose gas, and hence that of the mapped FTG gas, is then $(N + 1)$-fold degenerate, with energy eigenvalue $N(h^2/2m)\pi/L^2$. These degenerate ground states are fragmented BECs with $wN$ atoms in the orbital $e^{i\pi x_j}/L$ and $(1 - w)N$ in $e^{-i\pi x_j}/L$ with $0 \leq w \leq 1$, and are conveniently labelled by a quantum number $\ell_z = (w - \frac{1}{2})N = 0, \pm 1, \pm 2, \ldots, \pm N/2$ related to the eigenvalue $P$ of circumferential linear momentum and that $L_z$ of angular momentum $z$-component by $P = \ell_z \hbar/R$ and $L_z = \ell_z \hbar$. The angular momentum per particle is half-integral due to antiperiodicity of the orbitals, and the degenerate ground states are in one-one correspondence with the eigenstates of spin angular momentum $z$-component of $N$ spin-1/2 fermions.

The ground state degeneracy makes the FTG gas on a ring a good candidate for detecting small external fields and linear accelerations. Suppose that there is a potential gradient parallel to a diameter of the ring, or an acceleration leading to a gradient in the inertial potential arising from Einstein’s principle of equivalence, with the circumferential minimum of this potential occurring at a point $x_0$. Then the degeneracy is lifted and to lowest order in degenerate perturbation theory all $N$ atoms occupy the orbital $\phi_0(x) = \sqrt{2/\piL} \cos[\pi(x - x_0)/L]$, leading to an observable asymmetric density profile $n(x) = 2n \cos^2[\pi(x - x_0)/L]$. 

Due to its quantum coherence the FTG gas is also a good candidate for a sensitive rotation detector. Suppose that the ring trap is rotating with angular velocity $\omega$ perpendicular to the plane of the ring. In the rotating coordinate system each atom sees an effective Coriolis force $F_{Cor} = 2m\bar{v} \times \bar{\omega}$. Comparing this with the usual magnetic force $F_{mag} = (e/c)\bar{v} \times \bar{B}$, one sees that the kinetic energy operators in the Hamiltonian in the rotating system are
FIG. 1: Dependence of energies $E$ on rotational flux $\Phi$. Heavy line: Ground state energy $E_0(\Phi)$. Lighter lines: Lowest energy for each value of total angular momentum.

$[\hat{p}_j - \frac{\hbar}{2} \frac{\Phi}{\Phi_0}]^2/2m$ where $\hat{p}_j = (\hbar/i)\partial/\partial x_j$, $\Phi = \pi R^2 \omega$ is the Coriolis flux through the loop, and $\Phi_0 = \hbar/2m$ is the Coriolis flux quantum. The energy of each state $|\ell_z\rangle$ then becomes $E = E_0(\Phi = 0) + \frac{N N^2}{2m R^2} (\frac{\Phi}{\Phi_0})^2 - 2\ell_z \Phi_0$, which is minimized when $\ell_z = \frac{N}{2}$ if $\Phi > 0$ and $\ell_z = -\frac{N}{2}$ if $\Phi < 0$, i.e., even a very small angular velocity leads to a nondegenerate ground state with all $N$ atoms at either $k = \pi/L$ or $k = -\pi/L$. Generalizing to states differing from the $\Phi = 0$ ground states by displacement in $k$-space by integral multiples of $2\pi/L$ one obtains the $\Phi$-dependent ground state energy $E_0(\Phi)$ shown by the heavy line in Fig. 1 in which the lighter lines show the lowest energies for $\ell_z = \pm \frac{N}{2}, \pm \frac{3N}{2}, \cdots$. The ground state energy is a periodic function of $\Phi$ with period $\Phi_0$ in accord with a general theorem \[14\], but unlike the usual situation for a superconductor, (a) there is no smaller period $\Phi_0/2$, and (b) for even $N$, $\Phi_0 = 0$ is a relative maximum of $E_0$ rather than a minimum (as is the case of odd $N$), the first minima occurring at $\Phi = \pm \Phi_0/2$.

The barrier heights of the energy landscape in Fig 1 diminish like $1/N$ for $N \to \infty$, so flux quantization will not be observable for a macroscopic ring. However, it may be observable for mesoscopic rings using BEC-on-a-chip technology. For example, assuming a ring radius $R = 5 \mu m$, one finds that for $^6$Li, $\Delta E > k_B T$ for $T < 50 \text{ nK}$.

Flow properties on a nonrotating ring: According to the FB mapping the excitation spectrum of the FTG gas is the same as that of an ideal Bose gas, and hence it is sufficient to analyze the latter. Since the excitation energy of the ideal Bose gas is quadratic in the excitation momentum $\hbar q$, the FTG gas does not satisfy the Landau-Bogoliubov criterion for superfluidity. We investigate here the possibility of flow metastability associated with barriers in the excitation energy landscape as a function of the transferred momentum. It was shown by F. Bloch \[10\] that for the usual ideal Bose gas, which corresponds to the case of odd $N$ in our treatment, no such barriers exist. In the case of even $N$ both the ground state and the excitation branches are $(N + 1)$-fold degenerate, but it is sufficient here to consider the $\ell_z = 0$ ground state and the excitations arising from it by promoting atoms to higher $k$-values. Generalizing Bloch’s analysis, we note that for $0 < \nu \leq N/2$ the lowest branch corresponds to excitation of $\nu$ atoms from $k = -\pi/L$ to $k = 3\pi/L$, yielding a state with angular momentum $z$-component $\ell_z \hbar$ with $\ell_z = 2\nu$, and with excitation energy $\epsilon(\ell_z) = \ell_z \hbar^2/2m R^2$. At $\nu = N/2$ one has reached a state differing from the ground state by translation of all atoms by an amount $2\pi/L$ in $k$-space, and one can repeat this process, promoting atoms from $k = \pi/L$ to $5\pi/L$, yielding another straight-line segment connecting the points $\ell_z = N$ and $\ell_z = 2N$ on a parabolic curve $(\ell_z \hbar^2/2m R^2$, etc. Together with symmetry $\epsilon(\ell_z) = \epsilon(-\ell_z)$ this yields an excitation energy curve composed of straight-line segments as in the dashed curve of Bloch’s Fig. 2 \[16\] with the notation $P = \ell_z \hbar / R$. Hence for both odd and even $N$ there are no energy barriers, and the FTG gas on a nonrotating ring does not exhibit flow metastability.

Expansion from a longitudinal harmonic trap: We focus finally on a 1D expansion, as could be achieved by keeping on the transverse confinement. If the 1D interactions are suddenly turned off before the gas is let free to expand from a longitudinal harmonic trap, the density profile at long times reflects the initial momentum distribution \[8\]. If instead the interactions are kept on during the expansion we find that the density profile expands self-similarly, while the momentum distribution evolves from an initial overall Lorentzian shape \[12\] to that of an ideal Bose gas. These properties can be demonstrated with the aid of an exact scaling transformation as we outline below. Since the FB mapping holds also for time-dependent phenomena induced by one-body external fields \[17\], the exact many-body wavefunction $\psi_F(x_1, \cdots, x_N; t) = A(x_1, \cdots, x_N) \prod_{j=1}^N \phi_j(x_j; t)$ during the dynamics is fully determined by the solution of the single-particle Schrödinger equation for the orbital $\phi_0(x_j; t)$. For the case of an external potential $V_{ext}(x, t) = m \omega^2(x)^2 x^2/2$ with $\omega(0) = \omega_0$ the solution is known \[15\] to be $\phi_0(x; t) = \phi_0(x/b(t); 0) e^{i m \omega_0 b(t) \hbar} E_0(\tau(t))/\hbar$ where $b(t)$ is the solution of the differential equation $\dot{b} + \omega^2(t)b = \omega_0^2/b^3$ with $b(0) = 1$ and $b(0) = 0$, $\tau(t) = \int_0^t dt' / b^2$ and $E_0 = \hbar \omega_0/2$. Since the unit antisymmetric wavefunction $A$ under the scaling transformation, we immediately obtain the expression for the many-body wavefunction, $\psi_F(x_1, \cdots, x_N; t) = b^{-N/2} \psi_F(x_1/b, \cdots, x_N/b; 0) e^{i(b/h\omega_0) \sum_{j=1}^N x_j^2 / 2 x_0^2 e^{-i N E_0(\tau(t))/\hbar}}$, and for the one-body density matrix, $\rho_1(x, x'; t) = \frac{1}{2} \rho_1(\frac{\mathbf{x}}{b(t)}, \frac{\mathbf{x}'}{b(t)}) \exp \left[-i \frac{\mathbf{x}^2 + \mathbf{x}'^2}{2 b^2} \right]$. This yields the momentum distribution as a function of time. While the intermediate-time dynamics has to be determined numerically, the stationary-phase method determines
the long-time evolution of the momentum distribution in the same way as for the bosonic TG gas. For the case of a 1D expansion the scaling parameter is $b(t) = \sqrt{1 + \omega_0^2 t^2}$ and the momentum distribution tends to that of an ideal Bose gas under harmonic confinement,

$$n(k, t \to \infty) \simeq \frac{\omega_0}{b} n_B(k \omega_0/b),$$  

where $n_B(k) = \frac{2\pi N}{\omega_0^2} |\tilde{\phi}_0(k)|^2$, with $\tilde{\phi}_0(k) = \pi^{-1/4} k_{\text{osc}}^{-1/2} e^{-k^2/2k_{\text{osc}}^2}$ and $k_{\text{osc}} = 1/x_{\text{osc}}$. This behavior is illustrated in Fig. 2. Quite noticeably, the “bosonization” time appears to be much longer than the “fermionization” time of the momentum distribution of the bosonic TG gas. Note that the “dynamical bosonization” described above does not violate the Pauli exclusion principle: by using the above scaling solution for the one-body density matrix and fixing unit normalization of the natural orbitals at all times it follows that the eigenvalues $\alpha_j$ of $\rho_1(x, x'; t)$ are invariant during the expansion and hence always satisfy the condition $\alpha_j \leq 1$.

In conclusion, we have found that (a) the untrapped system exhibits superconductive ODLRO of the two-body density matrix $\rho_2$ associated with its maximal eigenvalue $N$ and pair eigenfunction $C \text{sgn}(x_1 - x_2)e^{-2\omega_1 |x_1 - x_2|}$; (b) on a ring it has a highly degenerate ground state for even atom number, and it exhibits quantization of rotational Coriolis flux and high sensitivity to rotation and to accelerations, making it a good candidate for high-sensitivity detectors; (c) the harmonically trapped system undergoes a “dynamical bosonization” of its momentum distribution during a 1D expansion.

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