TWO-TIME PHYSICS

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ABSTRACT

We give an overview of the correspondance between one-time-physics and two-
time-physics. This is characterized by the presence of an SO(d, 2) symmetry and
an Sp(2) duality among diverse one-time-physics systems all of which can be lifted
to the same more symmetric two-time-physics system by the addition of gauge
degrees of freedom. We provide several explicit examples of physical systems that
support this correspondance. The example of a particle moving in $AdS_D \times S^n$,
with SO($D + n, 2$) symmetry which is larger than the popularly known symmetry
SO($D - 1, 2$) $\times$ SO($n + 1$) for this case, should be of special current interest in
view of the proposed AdS-CFT duality.

1 Introduction and summary

One of the conceptual advances around and after Strings-95 is to understand
how to go beyond perturbative String Theory as part of an as yet poorly
understood non-perturbative theory called M-theory. Duality symmetries in
M-theory connect different limits of M-theory including strings in various
backgrounds, p-branes, supergravity and Super Yang-Mills theories. The
supersymmetry structure of the theory has provided global hints of higher
dimensions, including two timelike dimensions \[.\] Hints of two-times have
been noted from various points of view \[.\]

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quium for Group Theoretical Methods in Physics, Hob-
bart, Tasmania, Australia, July 1998.
If two-time physics is real physics it should be possible to formulate ordinary physical systems in the language of two time physics without any ghosts. In this lecture we will describe some important steps in that direction. Some technical details and specific examples, including spinning systems, have appeared in several recent papers [13]-[15]. Here we will provide an overview as well as a few new examples that connect one-time-physics to two-time-physics.

The formalism is a simple Sp(2, R) gauge theory for particles $X^M(\tau)$ (zero-branes) which arises from a basic idea as follows. Sp(2, R) is the automorphism symmetry of the quantum relations $[x, p] = i$ and treats $(x, p)$ as a doublet. The idea is to turn this global symmetry of Quantum Mechanics into a local symmetry of a theory. The 3-parameter local symmetry Sp(2, R) includes $\tau$-reparametrizations as one of its local transformations, and therefore it can be regarded as a generalization of gravity on the worldline. The Sp(2, R) gauge theory is non-trivial and physically consistent only if the zero brane has two timelike coordinates $X^0(\tau), X^0'(\tau)$ in target space, and has a global symmetry SO($d, 2$), which is the Lorentz group with two times.

Various gauge choices produce an infinite number of sectors of one-time-physics, including free relativistic or non-relativistic particles with or without mass, hydrogen atom, harmonic oscillator, particles in various curved spacetimes (such as anti de Sitter space and others), and even particles in arbitrary potentials $V(r)$. This is possible because in the two-time theory there are an infinite number of ways to make a gauge choice that defines physical “time” as known in one-time-physics. All these sectors are connected to each other by Sp(2) gauge transformations (duality). The quantum Hilbert space for each of these systems provide realizations of the SO($d, 2$) symmetry, which is recognized as the conformal symmetry in massless systems, and has other interpretations in other systems, from the point of view of one-time-physics.

The SO($d, 2$) group theoretical aspects are especially interesting in the quantum theory. For diverse one-time physical sectors SO($d, 2$) is realized in the same unique representation, with the same eigenvalues of the Casimir operators. Each one-time physical system provides a different basis within the same representation of SO($d, 2$), while duality transformations are unitary transformations from one basis to another.

The two-time system has been generalized to include spin $\frac{n}{2}$ by considering the gauge supergroup OSp($n/2$) [15]. This generalizes supergravity with $n$ supercharges on the worldline to a theory with a larger local symmetry. Multi-particles are treated by using Sp($2N$) (instead of Sp(2)) and
its supersymmetric generalizations [17] and this makes a connection with the multiparticle formalism in [11].

Open problems include generalizing to target spacetime supersymmetric version, investigating the two-time system in the presence of interactions with background gravitational and gauge fields [13], formulating the second quantized version of the two-time theory, and generalizing the scheme to strings or p-branes.

The main message is that two-time-physics is not only possible, but is a basis for unifying many features of one-time-physics in a geometrical manner.

2 Sp(2, R) gauge theory

In the 1970’s we gradually learned that what used to be considered global symmetries became part of local gauge symmetries that unified all interactions. There is a global symmetry that was not included in the unification scheme, namely the Sp(2, R) global symmetry of all Quantum Mechanics. Sp(2, R) treats generalized position and momentum \((x, p)\) as a doublet in phase space. The basic idea put forward by BDA [13] is to turn this global symmetry into the gauge symmetry of a theory².

So far this idea has been used to construct the simplest model involving particles (zero-branes), but the basic idea is more general and one may look forward to applying it to more general cases including p-branes.

It is worth noting that a common factor in all duality transformations is a transformation which mixes canonical conjugates and that belongs to Sp(2, R). This is already the case in the oldest example of Maxwell duality that acts between the canonical pairs of the electric and magnetic fields \((E, B)\) or electric-magnetic charges \((e, g)\). Similarly there is an Sp(2, R) in the Seiberg-Witten duality in a supersymmetric Yang-Mills theory. Furthermore, T-duality in string theory transforms Kaluza-Klein momenta with winding numbers in position space. Finally, the more general U-duality transforms canonically conjugate electric-magnetic quantum numbers of p-branes. In all

²Historically, the Sp(2, R) gauge theory formalism gradually developed from formalism that was used to construct ghost free two-time models [9]-[11]. In turn these were motivated by general supersymmetric structures that emerged in trying to understand duality in M-theory and Super Yang-Mills theory, which provided hints for two-times and higher dimensions from various points of view. However, the idea can be stated as a principle which can be pursued quite independently than the historical steps that led to it.
these cases duality is a gauged discrete group which should be contrasted with our continuously gauged Sp(2, R). Although these dualities are discrete gauge transformations, it is not excluded that they may arise from a more general continuous gauge theory after some gauge fixing. In fact, examples of the discrete duality after gauge fixing exist in our model with zero-branes: some of the one-time-physics models listed in the introduction are related to each other by discrete Sp(2, R) transformations at fixed time in a Hamiltonian formalism. These discrete transformations are part of the continuous Sp(2, R).

A consequence of the Sp(2, R) gauge theory is that duality and two-times are inextricably connected to each other. In fact, local Sp(2, R) symmetry requires one extra timelike coordinate plus one extra spacelike coordinate to lift a system from one-time physics to its most symmetric SO(d, 2) covariant form in two-time-physics. The requirement of the extra dimensions to exhibit a higher symmetry is consistent with similar observations involving duality in M-theory and its extensions. In particular it is worth noting that (i) Type-IIA $\leftrightarrow$ 11D supergravity duality lifts 10D string theory to M-theory in 11 dimensions, (ii) the 11D superalgebra (2-brane + 5-brane) of M-theory is automatically a 12D superalgebra with signature (10, 2), (iii) Type IIA$\leftrightarrow$ Type IIB dualities lead to F-theory in 12 dimensions, and (iv) requiring a sufficiently large structure to unify TypeIIA + TypeIIB superalgebras leads to S-theory in 14 dimensions. By now we have become more accustomed to the idea that the fundamental theory may take its most symmetric form when formulated in higher dimensions. In fact the construction of the elusive fundamental theory may first require a deeper understanding of the relationship between one-time physics and the formulation of physics in higher dimensions with more timelike coordinates.

In the remainder of this section we review the construction for zero-branes given by BDA [13]. The theory is based on turning the global Sp(2, R) automorphism symmetry of the commutation relations in Quantum Mechanics into a local symmetry of an action. Sp(2, R) treats position and momentum $(x, p)$ as a doublet in phase space. Consider the particle (zero-brane) described by $X^M (\tau)$ and its canonical conjugate $P^M (\tau)$. The signature of target spacetime $\eta_{MN}$ and its relation to ordinary spacetime will be determined below. To remove the distinction between position and momentum we rename them $X^M_1 \equiv X^M$ and $X^M_2 \equiv P^M$ and define the doublet
\[ X_i^M = (X_1^M, X_2^M). \] The local \( \text{Sp}(2, R) \) acts as follows

\[ \delta_\omega X_i^M (\tau) = \varepsilon_{ik} \omega^{kl} (\tau) X_l^M (\tau). \]  

(1)

Here \( \omega^{ij} (\tau) = \omega^{ji} (\tau) \) is a symmetric matrix containing three local parameters of \( \text{Sp}(2, R) \), and \( \varepsilon_{ij} \) is the Levi-Civita symbol that is invariant under \( \text{Sp}(2, R) \) and serves to raise or lower indices. The \( \text{Sp}(2, R) \) gauge field \( A^{ij} (\tau) \) is symmetric in \( (ij) \) and transforms in the standard way

\[ \delta_\omega A^{ij} = \partial_\tau \omega^{ij} + \omega^{ik} \varepsilon_{kl} A^{lj} + \omega^{jk} \varepsilon_{kl} A^{il}. \]

The covariant derivative is

\[ D_\tau X_i^M = \partial_\tau X_i^M - \varepsilon_{ik} A^{kl} X_l^M. \]

An action that is invariant under \( \text{Sp}(2, R) \) gauge symmetry is

\[ S_0 = \frac{1}{2} \int_0^T d\tau \left( D_\tau X_i^M \right) \varepsilon^{ij} X_j^N \eta_{MN} \]

(2)

\[ = \int_0^T d\tau \left( \partial_\tau X_1^M X_2^N - \frac{1}{2} A^{ij} X_i^M X_j^N \right) \eta_{MN}. \]

where we have dropped a total derivative \( \partial_\tau \left( \frac{1}{2} X_1 \cdot X_2 \right) \) from the Lagrangian.

The canonical conjugates are \( X_1^M = X^M \) and \( \partial S/\partial \dot{X}_1^M = X_2^M = P^M. \) They are consistent with the idea that \( (X_1^M, X_2^M) \) is the doublet \( (X^M, P^M) \). The equations of motion for \( A^{ij} \) that follows from the Lagrangian (2) give the first class constraints

\[ X \cdot X = X \cdot P = P \cdot P = 0. \]  

(3)

Their Lie algebra is \( \text{Sp}(2, R) \). If the signature of \( \eta_{MN} \) corresponds to a single time-like coordinate the only classical solution of the constraints is that \( X^M, P^M \) are parallel and lightlike. This is trivial in the sense that there is no angular momentum. Non-trivial solutions are possible provided the signature of \( \eta_{MN} \) corresponds to two time-like coordinates. More timelike coordinates are not allowed because the gauge symmetry is insufficient beyond two time-like dimensions to remove ghosts.

The action has a global symmetry under global Lorentz transformations \( \text{SO}(d, 2) \) which leave the metric \( \eta_{MN} \) invariant. The generators of this symmetry are

\[ L^{MN} = \varepsilon^{ij} X_i^M X_j^N = X^M P^N - X^N P^M. \]  

(4)

These \( L^{MN} \) are gauge invariant under \( \text{Sp}(2, R) \) for each \( M, N \). Other gauge invariants include \( \varepsilon^{ij} X_i^M D_\tau X_j^N, \varepsilon^{ij} D_\tau X_i^M D_\tau X_j^N, \) etc., but these vanish on shell as a result of the equations of motion \( D_\tau X_i^M = 0. \)
The full \textit{physical information of the theory is contained in the gauge invariant} $L^{MN}$. Using the constraints (3) it is straightforward to show that all the Casimir operators of SO($d,2$) vanish at the classical level

$$\text{Classical : } C_n (SO (d,2)) = \frac{1}{n!} Tr (iL)^n = 0.$$ \hfill (5)

In the first quantized theory the quantum states are labelled by both Sp($2, R$) and SO($d,2$) Casimir eigenvalues in the form $|C_2 (Sp (2, R)) ; C_n (SO (d,2)) >$ since the generators of these groups commute, and we need to find their eigenvalues for physical states. Generally the possible $Sp (2, R)$ quantum numbers are $|jm >$. In contrast to the classical theory, the quantized $C_n (SO (d,2))$ are not zero after taking quantum ordering into account. The following relations are proven by writing out all the Casimir operators in terms of $X, P$ while respecting their order. First, all Casimir eigenvalues $C_n (SO (d,2))$ are rewritten in terms of $C_2 (SO (d,2))$ and $d$. For example $C_3 (SO (d,2)) = \frac{d}{3!} C_2 (SO (d,2))$, etc.. Second, the quadratic Casimir of Sp($2, R$) is related to the quadratic Casimir of SO($d,2$) by $C_2 (SO (d,2)) = 4 C_2 (Sp (2, R)) + 1 - \frac{d^2}{4}$. Third, since physical states are gauge invariant, the quadratic Casimir of Sp($2, R$) must vanish in the physical sector (i.e. $j = 0$ and $m = 0$). The last condition fixes all the Casimir eigenvalues for SO($d,2$) to unique non-zero values in terms of $d$. Therefore the quantum system can exist only in a unique unitary representation of $SO (d,2)$ characterized by [13]

$$\text{Quantum : } C_2 (Sp (2)) = 0,
\begin{cases}
C_2 (SO (d,2)) = 1 - \frac{d^2}{4}, \\
C_3 (SO (d,2)) = \frac{d}{3!} \left(1 - \frac{d^2}{4}\right)
\end{cases} \quad \ldots$$ \hfill (6)

This information completely specifies the physical sector of the Hilbert space in a unique $SO (d,2)$ representation. It should be possible to obtain a similar result by using the methods of BRST quantization [18] but this remains to be done.

From this gauge invariant result it follows that the diverse one-time-physics models that emerge by gauge fixing must have precisely zero SO($d,2$) Casimir eigenvalues at the classical level, and also the same non-trivial Casimir eigenvalues (6) that label the unique physical Hilbert space in their first quantized versions. This is, of course, a natural result of the formalism, however the prediction it makes for diverse one-time-physics systems was not recognized to be true before, and seems to be amazing. For example it suggests
that the free relativistic massless particle in \((d - 1)\) space dimensions should have a Hilbert space dual to that of the particle moving in the \(1/r\) potential in the same number of dimensions, and (except for the choice of basis) should be described by the same unique \(\text{SO}(d, 2)\) representation, etc.. This prediction of two-time-physics has been explicitly verified\[14\] -\[15\] to be correct not only for this example, but for many other cases as well, including spinning particles (for which the Casimir eigenvalues change according to the value of the spin). The fact that this test succeeded is encouraging evidence for two-time-physics.

3 Gauge choices, dual physics

First we describe in general terms how diverse one-time physical systems emerge from the same two-time theory by taking various gauge choices that embed physical time in different ways in the extra dimensions. It is evident that, by the very procedure in which they are derived, these diverse physical systems are \(\text{Sp}(2, R)\) duals of each other.

We have the freedom to fix up to 3 functions since \(\text{Sp}(2)\) has 3 gauge parameters. The procedure is as follows. (1) Make \(n\) gauge choices (using \(n=2\) or 3) for some of the \(2d+4\) functions \(X^M(\tau), P^M(\tau)\). (2) Solve \(n\) constraints which determines \(n\) additional functions, thus obtaining a gauge fixed configuration \(X^M_0(\tau), P^M_0(\tau)\) parametrized in terms of \(2(d + 2 - n)\) independent functions \(x(\tau), p(\tau)\). (3) The dynamics for the remaining degrees of freedom \(x, p\) is determined by inserting \(X^M_0(\tau), P^M_0(\tau)\) in the original action \[2\], thus constructing a new one-time-physics action \(^3\)

\[
S(x, p) = S_0\left( X^M_0, P^M_0 \right) = \int d\tau \ L\left( x(\tau), p(\tau), A(\tau) \right).
\]

(7)

Here \(A(\tau)\) is a remaining gauge potential if \(n = 2\), but \(A(\tau)\) is absent if \(n = 3\) (the \(A\)'s in eq.\[2\] drop out when the corresponding constraint is solved explicitly). The one-time physical system that emerges is recognized by studying the form of the Lagrangian \(L\).

In this approach it is no surprise to find that \(S(x, p)\) inherits the \(\text{SO}(d, 2)\) symmetry, which however is now realized non-linearly. The presence of this

\(^3\)Thus, for \(d = 4\), if we choose all the gauges in a particular way and solve all the constraints the remaining three positions and three momenta correspond to ordinary 3D physical phase space \(\mathbf{r}, \mathbf{p}\).
hidden symmetry was not suspected for most of the diverse physical systems constructed by this procedure, although it was known before for a couple of examples (free relativistic particle, and hydrogen atom). Since the original generators of the symmetry $L^{MN}$ in (4) are gauge invariant, they must be the generators of the symmetry of the new action. Indeed the symmetry generators for the new action can be constructed easily at any $\tau$ by inserting $X_0^M(\tau), P_0^M(\tau)$ in the $L^{MN}$ of eq.(4)

$$L^{MN}(x(\tau), p(\tau), \tau) = X_0^M(\tau) P_0^N(\tau) - X_0^N(\tau) P_0^M(\tau).$$

(8)

It can be checked explicitly that these $L^{MN}$ form the algebra of $SO(d, 2)$ under Poisson brackets by using the fundamental Poisson brackets for $\{x, p\} = \eta,

$$\{L^{MN}, L^{RS}\} = \eta^{MR} L^{NS} + \eta^{NS} L^{MR} - \eta^{NR} L^{MS} - \eta^{MS} L^{NR}.$$  

(9)

Explicit $\tau$ dependence appears in the new $L^{MN}$ when $n = 3$ (but not when $n = 2$). If $\tau$ appears explicitly it is treated as a parameter in the Poisson brackets as opposed to a canonical degree of freedom $x(\tau), p(\tau)$. Then the $SO(d, 2)$ algebra holds for all $\tau$. The symmetry transformations of the canonical coordinates $\delta x(\tau), \delta p(\tau)$ are obtained by evaluating the Poisson brackets

$$\delta x(\tau) = \frac{1}{2} \varepsilon_{MN} \{L^{MN}(\tau), x(\tau)\}, \quad \delta p(\tau) = \frac{1}{2} \varepsilon_{MN} \{L^{MN}(\tau), p(\tau)\},$$  

(10)

while continuing to treat $\tau$ as a parameter. The explicit $\tau$ dependence in these transformation laws is vital for demonstrating the symmetry of the action when $n = 3$. Indeed one can verify explicitly that the one-time-physics action $S(x, p)$ is invariant under $SO(d, 2)$ because the Lagrangian transforms like a total derivative at any $\tau$

$$\delta L = \partial_\tau \Lambda(\tau, \varepsilon_{MN}).$$  

(11)
It must be emphasized that we obtain invariance of the action under $SO(d, 2)$, which must be distinguished from invariance of the Hamiltonian. This distinction arises when all three gauge choices are made and a Hamiltonian is defined. Recall that in that case the generators of the symmetry depend explicitly on $\tau$ as emphasized before, which means they are conserved in the sense that the total time derivative, including derivative with respect to the explicit $\tau$, is zero. On the other hand, the generators that commute with the Hamiltonian are only those that do not depend on $\tau$ explicitly. In this sense $SO(d, 2)$ should be understood as the dynamical symmetry of the system. For example, for the H-atom, the symmetry of the Hamiltonian is $SO(d)$ while the dynamical symmetry is $SO(d, 2)$. We will see below more examples of physical systems all of which have $SO(d, 2)$ dynamical symmetry (equivalently symmetry of the action) as a trivial consequence of our formulation, but which was not known and was unexpected before for those systems.

The one-time physical system described by $L(x, p)$ can be first quantized in the usual way. One may then construct the quantum generators of $SO(d, 2)$ from the classical ones. In a Hamiltonian formalism we take $\tau = 0$, and order the operators to insure hermitian $L^{MN}(x, p)$. In this procedure we find that we also need corrections of some $L^{MN}$ by including some anomaly terms to insure closure of the $SO(d, 2)$ algebra at the quantum level (i.e. orders of operators respected). Once this is achieved in some fixed gauge we know that the physical space for the corresponding physical system is described by a representation of the $SO(d, 2)$ algebra. The remaining question is whether the representation is the same one as the one specified in covariant quantization in eq.(6). Indeed we find complete agreement in every case in which this procedure has been carried out. This includes the free relativistic massless particle, the hydrogen atom, the harmonic oscillator, the particle moving on $AdS_{d-n} \times S^n$ and a few of these cases including spinning particles (for which the original action and the Casimirs are generalized to include the effects of spin).

\[\text{This is a familiar phenomenon. For example for the standard relativistic particle the action is invariant under rotations as well as boosts, but the Hamiltonian defined after gauge fixing } x^0(\tau) = \tau \text{ is invariant only under rotations.}\]
4 Examples

We will give here some examples to illustrate the ideas and procedures described above. More cases including spinning generalizations are available in the literature.

4.1 Free massless particle

We use the basis $X^M = (X^+, X^-, x^\mu)$ with the metric $\eta^{MN}$ taking the values $\eta^{+ -} = -1$ and $\eta^{\mu \nu} =$ Minkowski. We choose 2 gauges $X^+ = 1$, $P^+ = 0$, and solve 2 constraints $X^2 = X \cdot P = 0$

$$M = (+', -', \mu)$$

$$X_0^M (\tau) = \left( 1, \frac{x^2(\tau)}{2}, x^\mu(\tau) \right),$$

$$P_0^M (\tau) = (0, p(\tau) \cdot x(\tau), p^\mu(\tau)).$$

Inserting these in the action (2) we find

$$S(x, p) = \int_0^T d\tau \left( -\dot{X}^+ P^- - \dot{X}^- P^+ + \dot{x}^\mu p_\mu - \frac{A^{22}}{2} p^2 - 0 - 0 \right)$$

$$= \int_0^T d\tau \left( \dot{x}^\mu p_\mu - \frac{1}{2} A^{22} p^2 \right) \Rightarrow \frac{1}{2} \int_0^T d\tau \frac{\dot{x}^2}{A^{22}}$$

(12)

which is obviously interpreted as the action for the massless relativistic particle.

A third gauge choice can be made by taking $x^+(\tau) = \tau$ and then solving the constraint $P^2 = 0$ which gives $p^- = \vec{p}_2/2p^+$ namely the Hamiltonian in the lightcone gauge. Inserting these either in the original action (2) or in the intermediate action (12) produces the action for the remaining independent degrees of freedom $(x^-, p^+)$ and $(\vec{x}, \vec{p})$

$$S(x, p) = \int_0^T d\tau \left( -\partial_\tau x^- p^+ - \partial_\tau \vec{x} \cdot \vec{p} - \vec{p}_2/2p^+ \right).$$

(13)

The gauge invariant observables are the $L^{MN}$. They can be constructed explicitly as described above [13]-[14] either for the action (12) or the action (13). One can easily compute $\delta x(\tau)$ and $\delta p(\tau)$ and verify explicitly that all forms of the action are symmetric under all SO($d, 2$) transformations.
Note that the transformations generated by $L^{MN}$ for the action (12) are independent of $\tau$ ($n = 2$ case) but those for the action (13) depend on $\tau$ ($n = 3$ case).

In a Hamiltonian formalism at $\tau = 0$ the first quantized generators for either action (12) or (13) have anomalous terms due to quantum ordering, which is necessary for hermiticity and closure of the algebra. Using the corrected quantum generators it has been verified\cite{13}-\cite{14} that the quadratic Casimir is precisely $C_2 = (1 - d^2/4)$, in agreement with the prediction in eq.(6). This result is also obtained in the field theory version of the free massless particle\cite{13}.

4.2 Particle in a potential $V(r)$

In this section we follow the ideas of \cite{14}-\cite{15}. We use the basis $X^M = (X^0', X^0, X^I)$, with metric $\eta^{0'0} = \eta^{00} = -1$ and $\eta^{IJ} = \delta^{IJ}$. The Sp(2) gauge symmetry permits us to make three gauge choices and then solve the three constraints $X^2 = P^2 = X \cdot P = 0$. This eliminates six functions from phase space or expresses them in terms of independent degrees of freedom. In \cite{14}-\cite{15} the following choices of gauge and solution of constraints was given

$$M = [0', 0', 1', i]$$

$$X^M = \left[ \begin{array}{c}
\cos u, -\sin u, -\hat{R} \cdot P \frac{\sqrt{-2H}}{V}, \quad (\hat{R}^i + \frac{1}{V} \hat{R} \cdot PP^i) \gamma \end{array} \right] \frac{VR}{\gamma}, \quad (14)$$

$$P^M = \left[ \begin{array}{c}
\sin u, \cos u, \quad (1 + \frac{P^2}{V}), \quad \sqrt{-2H} \frac{1}{V} P^i \gamma \end{array} \right] \frac{\gamma}{\sqrt{-2H}}. \quad (15)$$

The independent degrees of freedom are $R, P$. Here $V(R, P)$ is any function of the canonical variables while $H$ is the Hamiltonian

$$H = \frac{P^2}{2} + V < 0. \quad (16)$$

To determine which class of potentials $V(R)$ is possible we consider the generators of SO($d, 2$) $L^{MN}$ which must be conserved in any gauge since they correspond to the global symmetries of the action. Since both the action and the $L^{MN}$ are gauge invariant, the $L^{MN}$ expressed in any gauge must be conserved when we use the equations of motion that follow from the gauge fixed action. Hence consider $L^{00}$ which becomes in this gauge

$$L^{0'0} = X^0 P^0 - X^0 P^{0'} = \frac{RV}{\sqrt{-2H}} \quad (17)$$
Since $H$ is guaranteed to be a constant of motion, the remaining possibility is that $RV$ must be a constant number, or more generally a constant of motion. Let us first consider the case $V = -\alpha/R$ where $\alpha$ is a positive constant. In that case we further make the additional gauge choice

$$u(\tau) = (r \cdot p - 2\tau H) \sqrt{-\frac{2H}{\alpha}}. \tag{18}$$

We also take $\gamma$ a constant, although $\gamma$ plays no role in the classical theory because it drops out in all gauge invariant expressions, but it can play a role in the quantum theory due to quantum ordering as discussed in [15]. In this form we have expressed the original $(d+2)$ degrees of freedom $(X^M, P^M)$ in terms of $(d-1)$ independent canonical degrees of freedom $(R^i, P^i)$.

Inserting these expressions in the original gauge invariant action gives the dynamics for the independent degrees of freedom $(R, P)$. Since the constraints are already solved we get (see [14]-[15])

$$S = \int_0^T d\tau \left( \partial_\tau X_1 X_2 \eta_{MN} - 0 - 0 - 0 \right) \tag{19}$$

$$= \int_0^T d\tau \int_0^T d\tau \left( P^i \partial_\tau R^i - H \right),$$

where $H$ is identified as the Hamiltonian given above. In addition, the last line shows that the $(R, P)$ used above are indeed canonical conjugates.

### 4.2.1 Positive energies

A similar gauge exists for a positive Hamiltonian. It is obtained by exchanging the roles of $X^0$ and $X^1$ as follows, which is equivalent to an analytic continuation from $H < 0$ to $H > 0$ in all expressions. Thus, consider the gauge choice and solutions of constraints given by

$$M = \begin{bmatrix} 0' & 1' & 0 & i \end{bmatrix}$$

$$\tilde{X}^M = \begin{bmatrix} \cosh w, \ 
\sinh w, \ 
\hat{r} \cdot \hat{p} \sqrt{\frac{2H}{V}}, \ 
(\hat{r}^i + \frac{1}{V} \hat{r} \cdot pp^i) \end{bmatrix} VR \gamma \tag{20}$$

$$\tilde{P}^M = \begin{bmatrix} \sinh w, \ 
\cosh w, \ 
(1 + \frac{p^2}{V}), \ 
\sqrt{\frac{2H}{V}} p^i \end{bmatrix} \frac{\gamma}{\sqrt{2H}} \tag{21}$$

and

$$w = (r \cdot p + 2\tau H) \frac{\sqrt{2H}}{\alpha}, \tag{22}$$
with
\[ H = \frac{p^2}{2} + V(r, p) > 0. \] (23)

These expressions are related to the previous ones by the analytic continuation \( \sqrt{-2H} \to i\sqrt{2H} \), \( u \to iw \), and then switching \( X^0 \) and \( X^1 \) to eliminate the complex number \( i \) (the factor of \( i \) converts a spacelike coordinate to a timelike coordinate and vice-versa). Inserting these gauge choices in the action we get again
\[
S = \int_0^T d\tau \left( \partial_\tau X_1^M X_2^N \eta_{MN} - 0 - 0 - 0 \right)
\]
\[
= \int_0^T d\tau \left( p^i \partial_\tau r^i - H \right)
\]

Thus, in switching from negative to positive energies we need to make an analytic continuation which is equivalent to an \( \text{Sp}(2) \) gauge transformation \( (X^M, P^M) \to (\tilde{X}^M, \tilde{P}^M) \). Hence the canonical conjugates \( (r, p) \) used for positive energies must be related to \( (R, P) \) used for negative energies by a local \( \text{Sp}(2) \) gauge transformation.

### 4.2.2 \( \text{SO}(d) \) or \( \text{SO}(d-1,1) \) symmetric Hamiltonians

Next we discuss the symmetries of \( H \) (as opposed to the symmetries of the action \( S \)) for a special class of Hamiltonians. For an arbitrary potential \( V \) the evident symmetry is rotation symmetry \( \text{SO}(d-1) \). Next consider a potential \( V \) of the form \( V = -\alpha \frac{R}{2} \). If \( \alpha \) is a constant this Hamiltonian describes the \( H \)-atom or the Kepler problem. For this case it is well known that \( H \) has a hidden \( \text{SO}(d) \) symmetry. However, one may go beyond a constant \( \alpha \) and still have \( \text{SO}(d) \) symmetry. In particular if we choose \( \alpha \) to be a function of \( H \) then the \( \text{SO}(d) \) symmetry is preserved. That is, if the Hamiltonian \( H \) is solved from the equation
\[
H = \frac{p^2}{2} - \frac{\alpha(H)}{R}
\]
for any function \( \alpha(H) \), the resulting \( H(R, P) \) will be shown to have \( \text{SO}(d) \) symmetry. This is seen by constructing the \( L^{MN} \) as described above. The \( \text{SO}(d,2) \) generators are (classical, and \( \tau = 0 \))
\[
L^{0'i} = \frac{\alpha(H)}{\sqrt{-2H}} m^i, \quad L^{0'i} = -\frac{\alpha(H)}{\sqrt{-2H}} n^i
\]
\[
L^0 = \frac{\alpha(H)}{\sqrt{-2H}}, \quad L^{ij} = \frac{\alpha(H)}{\sqrt{-2H}} \left(n_i m_j - n_j m_i\right).
\]

where \(n^i, m^i\) are unit vectors that are orthogonal (as seen from (14)). The SO\((d)\) generators \(L^{ij}\) include rotations \(L^{ij}\) and the \(L^{1'i}\) - Runge-Lenz vector \(L^{1'i}\) in \(d - 1\) dimensions

\[
L^{ij} = R^i P^j - R^j P^i, \quad L^{1'i} = \frac{\alpha(H)}{\sqrt{-2H}} \left(\frac{1}{2} L^{1'j} P_j + \frac{1}{2} P_j L^{1'i} - \frac{\alpha(H)}{R} R^i\right).
\]

We see that both the SO\((d)\) quadratic Casimir and the SO\((d)\) singlet generator \(L^0\) are functions of only \(H\) at the classical level

\[
C_2(SO(d)) = \left(L^{1'i}\right)^2 + \frac{1}{2} \left(L^{1'i}\right)^2 = \frac{\alpha^2(H)}{-2H},
\]

\[
\left(L^0\right)^2 = \frac{\alpha^2(H)}{-2H}.
\]

Of course the relation between \(H\) and the Casimir \(C_2(SO(d))\) receives quantum corrections but the relation between \(L^0\) and \(H\) remains the same despite quantization. Furthermore \(H\) is obviously invariant under SO\((d)\) since \(L^0\) commutes with all the SO\((d)\) generators \(L^{ij}\).

To do the quantum mechanics it is sufficient to use that \(H\) is a function of the generator \(L^0\). Since the three generators \(L^0, L^{0'i}, L^{1'i}\) form an SO\((1, 2)\) algebra we can immediately determine group theoretically the spectrum of \(H\) from the spectrum of \(L^0\). This procedure was followed in [14] to discuss the spectrum of the H-atom, and now it can be generalized to the more general case \(\alpha(H)\) in a straightforward manner. Using the permitted eigenvalues of \(L^0\) as determined in [14], \(L^0 = \left(\frac{1}{2}(d - 4) + n\right)\), with \(n = 0, 1, 2, \ldots\), the spectrum is obtained from the relation between \(L^0\) and \(H\)

\[
E_n = -\frac{\alpha^2(E_n)}{2\left(\frac{1}{2}(d - 4) + n\right)^2}. \quad (26)
\]

Thus, this method provides a new class of exactly solvable Hamiltonians \(H(R, P)\) that have an SO\((d)\) symmetry, just like the H-atom does. Since the excitation spectrum is now a different function of \(n\) (determined by the choice of \(\alpha(H)\)) such a Hamiltonian may have interesting applications.

In summary, If \(RV(R, P) = \alpha(H)\) is a function of \(H(R, P)\) we have argued that the Hamiltonian is symmetric under SO\((d)\) symmetry for \(H < 0\),
and similarly symmetric under an SO\((d-1,1)\) symmetry when \(H > 0\). There may be a bigger class of SO\((d)\) symmetric Hamiltonians that remain to be found by using a more general gauge. The action \(S(R,P)\) is invariant under a dynamical symmetry SO\((d,2)\) as described in the general discussion. When the Hamiltonian can be written as a simple expression of the generators of SO\((d,2)\) the dynamical symmetry can be used to provide a group theoretical solution of the eigenstates, eigenenergies and other physical properties of the system. A class of such simple physical systems is obtained when \(\alpha(H)\) is an arbitrary function of \(H\).

### 4.3 Particle in curved space - AdS\(_{d-n} \times S^n\) gauge

The example of a particle moving in an AdS\(_{d-n} \times S^n\) background should be of special current interest in view of the proposed AdS – CFT duality \[16\]. We begin with AdS\(_d\). Using the basis \(X^M = (X^0', X^1', X^0, X^i)\) we choose 2 gauges \(X^1' = 1, P^1' = 0\), and solve the 2 constraints \(X^2 = X \cdot P = 0\). The solution is parametrized as follows

\[
M = (0', 1', 0', i)
\]

\[
X_0^M = (A(r) \cos t, 1, A(r) \sin t, B(r)\hat{r}), A^2 - B^2 = 1,
\]

\[
P^M_0 = \left( -\frac{\rho^0}{A^2} \sin t, 0, -\frac{\rho^0}{A^2} \cos t, \frac{\rho^0}{A^2} \nabla_B \xi + \frac{\rho^0}{A^2} \hat{r} \right)
\]

Inserting this gauge in the action we find

\[
S_0 = \int_0^T d\tau \left( \partial_\tau X_1^M X_2^N \eta_{MN} - \frac{1}{2} A^{22} X_2 \cdot X_2 - 0 - 0 \right)
\]

\[
= \int_0^T d\tau \left( \dot{x}^\mu \cdot p_\mu - \frac{1}{2} A^{22} G^{\mu\nu} (x) p_\mu p_\nu \right) \rightarrow \int_0^T d\tau \frac{G^{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{2A^{22}}
\]

where the last expression is obtained by using the equation of motion for \(p\) or by integrating out \(p\) in the path integral. This describes a particle moving in a curved background. The metric \(G_{\mu\nu}\) (lower indices) is given by

\[
d s^2 = dX \cdot dX = G_{\mu\nu} dx^\mu dx^\nu = -A^2 dt^2 + \frac{(\partial_\tau B)^2}{A^2} dr^2 + B^2 d\Omega^2.
\]

For example, \(A = \sqrt{1 + r^2}\), and \(B = r\) gives

\[
(ds^2)_{AdS_d} = - \left( 1 + r^2 \right) dt^2 + \frac{1}{1 + r^2} dr^2 + r^2 d\Omega^2,
\]

15
which is recognized as a particular parametrization of anti de Sitter space $AdS_d$ in $(d - 1, 1)$ dimensions. Another example with $A = \frac{1 + r^2}{1 - r^2}$, and $B = \frac{2r}{1 - r^2}$ gives

$$(ds^2)_{AdS_d} = -\left(1 + \frac{r^2}{1 - r^2}\right)^2 dt^2 + \left(\frac{2r}{1 - r^2}\right)^2 dr^2.$$ 

which is a different form of the $AdS_d$ metric.

To construct the particle in $AdS_{d-n} \times S^n$ we divide the $d + 2$ components of $X^M$ into two sets, $X^M = (x_{d-n+1}^m, y_{n+1}^i)$. The first set $x_{d-n+1}^m$ contains $d-n+1$ components that include the two timelike dimensions and the second set $y_{n+1}^i$ contains $n+1$ components that are purely spacelike. Similarly with $P^M$. Then we make the two gauge choices

$$y \cdot y = 1, \quad y^i p_i = 0. \quad (27)$$

That is, $y$ is a unit vector while the corresponding radial component of $p^i$ vanishes. This is to be compared to the $n = 0$ case treated above. Solving the constraints $X^2 = X \cdot P = 0$, and inserting the result in the action we derive the action for the particle moving in a curved background with the metric computed from $ds^2 = dX \cdot dX = dx \cdot dx + dy \cdot dy$. We find the $AdS_{d-n} \times S^n$ metric

$$(ds^2) = ds^2_{AdS_{d-n}} + (d\Omega_n)^2 \quad (28)$$

where $\Omega_n$ describes the $n$-sphere defined by the unit vector $y_{n+1}^i$.

Just like all previous cases the action with this metric has an $SO(d,2)$ symmetry. This contains hidden symmetries not noticed before since $SO(d,2)$ is larger than the popularly known symmetry in the $AdS_{d-n} \times S^n$ background. Namely the $AdS_{d-n}$ piece has an $SO(d-n-1,2)$ symmetry and the $S^n$ piece has an $SO(n+1)$ symmetry, while our approach shows that the overall system has a larger symmetry $SO(d,2)$

$$SO(d,2) \supset SO(d-n-1,2) \times SO(n+1). \quad (29)$$

For example, the action for a particle moving in $AdS_3$ has an $SO(3,2)$ symmetry, which is larger than the popularly known $SO(2,2)$. The action for the particle moving on $AdS_5 \times S^5$ has an $SO(10,2)$ symmetry which is larger than the expected symmetry $SO(4,2) \times SO(6)$. Again, the presence of the larger symmetry is the evidence for the presence of two-time-physics. The symmetry transformations and the quantum generators for this case are discussed elsewhere in more detail [19].
4.4 Conformal factors

Consider a gauge that has a solution $X^M_0(x,p), P^M_0(x,p)$ which satisfies the constraints $X^2_0 = X_0 \cdot P_0 = 0$, and which gives $\dot{X}^M_0 \cdot P_M = \dot{x}^\mu \cdot p_\mu$ and $P^\mu_0 = G^\mu\nu_0(x) p_\mu p_\nu$. Then the action is

$$S = \int_0^T d\tau \left( \dot{x}^\mu \cdot p_\mu - \frac{1}{2} A^{22} G^\mu\nu_0(x) p_\mu p_\nu \right) \rightarrow \int_0^T d\tau \frac{G^\mu\nu_0 \dot{x}^\mu \dot{x}^\nu}{2 A^{22}}$$

as we have seen above in some examples. From this solution we can construct a new solution $X^M = F(x) X^M_0, P^M = P^M_0 F(x)$, which automatically satisfies the constraints and gives and $\dot{X}^M \cdot P_M = \dot{x}^\mu \cdot p_\mu$. But it also produces a new metric that differs from the previous one by a conformal factor

$$S = \int_0^T d\tau \frac{G^\mu\nu \dot{x}^\mu \dot{x}^\nu}{2 A^{22}}$$

For example the free particle solution in flat Minkowski space $G^0_{\mu\nu} = \eta_{\mu\nu}$, gives the conformal particle solution with $G^\mu\nu = F^2(x) \eta^\mu_\nu$, any $F(x)$.

As seen from eq.(8) the classical gauge invariant $L^{MN}$ are the same in the two models related to each other by a conformal factor. However, this is not so in the quantum theory in which operators must be ordered. In [15] we argued that there is an ordering of the quantum generators in the new theory that is simply related to the quantum generators in the old theory, and that the new theory automatically has the same Casimir eigenvalues as the old theory, in agreement with the general prediction obtained in the SO $(d,2)$ covariant quantization.

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