Vertical Liouville foliations on the big-tangent manifold of a Finsler space

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Abstract

The present paper unifies some aspects concerning the vertical Liouville distributions on the tangent (cotangent) bundle of a Finsler (Cartan) space in the context of generalized geometry. More exactly, we consider the big-tangent manifold $\mathcal{T}M$ associated to a Finsler space $(M, F)$ and of its $L$-dual which is a Cartan space $(M, K)$ and we define three Liouville distributions on $\mathcal{T}M$ which are integrable. We also find geometric properties of both leaves of Liouville distribution and the vertical distribution in our context.

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1 Introduction and preliminary notions

1.1 Introduction

The vertical Liouville distribution on the tangent bundle of a (pseudo) Finsler space was defined for the first time in [3] where some aspects of the geometry of the vertical bundle are derived via vertical Liouville distribution. A similar study on the cotangent bundle of a Cartan space can be found in [9]. Also, other significant studies concerning the interrelations between natural foliations defined by Liouville fields on the tangent bundle of a Finsler space and the geometry of the Finsler space itself, as well as similar problems on Cartan spaces are intensively studied in [5] and [11], respectively. See also [9, 11, 16, 17].

As it is well known, in the generalized geometry initiated in [8], the tangent bundle $\mathcal{T}M$ of a smooth $n$-dimensional manifold $M$ is replaced by the big-tangent bundle (or Pontryagin bundle) $\mathcal{T}M \oplus T^*M$. On its total space the velocities and momenta are considered as independent variables. This idea was proposed and developed in [18, 19] and later was used in the study of Hamiltonian-Jacoby theory for singular Lagrangian systems [10]. The geometry of the total space of the big-tangent bundle, called big-tangent manifold, is intensively studied in [22] and some its applications to mechanical systems can be found in [7].

Using the framework of the geometry on the big-tangent manifold, our aim in this paper is to extend some results concerning the vertical Liouville foliation in the context of generalized geometry. In this sense, we consider the big-tangent manifold $\mathcal{T}M$ associated to a Finsler space $(M, F)$ and of its $L$-dual which is a Cartan space $(M, K)$. As usual, we reconsider the vertical Liouville distributions $V_{E_1}$ and $V_{E_2}$ from the case of vertical tangent (cotangent) bundle of a Finsler (Cartan) space, see [3, 9], for the case of vertical subbundles $V_1$ and $V_2$, respectively, with respect to Liouville vector fields $E_1$ and $E_2$. Next we define the Liouville distribution $V_{E}$ with respect to
the Liouville vector field $E = E_1 + E_2$, we prove that it is integrable (Theorem 2.1) and we study some of its properties (Theorems 2.2 and 2.3). Also, some links between the vertical Liouville foliations $V_{E_1}$, $V_{E_2}$ and $V_E$, respectively, are established.

1.2 Preliminaries and notations

Let $M$ be a $n$-dimensional smooth manifold, and we consider $\pi : TM \to M$ its tangent bundle, $\pi^* : T^*M \to M$ its cotangent bundle and $\tau \equiv \pi \oplus \pi^* : TM \oplus T^*M \to M$ its big-tangent bundle defined as Whitney sum of the tangent and cotangent bundles of $M$. The total space of the big-tangent bundle, called big-tangent manifold, is a $3n$-dimensional smooth manifold denoted here by $T M$. Let us briefly recall some elementary notions about the big-tangent manifold $T M$. For a detailed discussion about its geometry we refer [22].

Let $(U, (x^i))$ be a local chart on $M$. If $\{dx^i|_x\}, x \in U$ is a local frame of sections in the tangent bundle over $U$ and $\{dx^i|_x\}, x \in U$ is a local frame of sections in the cotangent bundle over $U$, then by definition of the Whitney sum, $\{\frac{\partial}{\partial y^i}, dx^i|_x\}, x \in U$ is a local frame of sections in the big-tangent bundle $TM \oplus T^*M$ over $U$. Every section $(y, p) \in \tau$ over $U$ takes the form $(y, p) = y^i \frac{\partial}{\partial y^i} + p_i dx^i$ and the local coordinates on $\tau^{-1}(U)$ will be defined as the triples $(x^i, y^i, p_i)$, where $i = 1, \ldots, n = \dim M$, $(x^i)$ are local coordinates on $M$, $(y^i)$ are vector coordinates and $(p_i)$ are covector coordinates.

The change rules of these coordinates are:

$$\tilde{x}^i = \tilde{x}^i(x^i), \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j, \tilde{p}_i = \frac{\partial \tilde{x}^i}{\partial y^j} p_j$$  \hspace{1cm} (1.1)

and the local expressions of a vector field $X$ and of a 1-form $\varphi$ on $T M$ are

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^j \frac{\partial}{\partial y^j} + \zeta^j \frac{\partial}{\partial p_j} \quad \text{and} \quad \varphi = \alpha_i dx^i + \beta_j dy^j + \gamma^i dp_i.$$  \hspace{1cm} (1.2)

For the big-tangent manifold $T M$ we have the following projections

$$\tau : TM \to M, \tau_1 : TM \to TM, \tau_2 : TM \to T^*M$$

on $M$ and on the total spaces of tangent and cotangent bundle, respectively.

As usual, we denote by $V = V(TM)$ the vertical bundle on the big-tangent manifold $T M$ and it has the decomposition

$$V = V_1 \oplus V_2,$$  \hspace{1cm} (1.3)

where $V_1 = \tau_1^{-1}(V(TM))$, $V_2 = \tau_2^{-1}(V(T^*M))$ and have the local frames $\{\frac{\partial}{\partial y^i}\}$, $\{\frac{\partial}{\partial p_i}\}$, respectively. The subbundles $V_1$, $V_2$ are the vertical foliations of $T M$ by fibers of $\tau_1$, $\tau_2$, respectively, and $T M$ has a multi-foliate structure [20]. The Liouville vector fields (or Euler vector fields) are given by

$$E_1 = y^i \frac{\partial}{\partial y^i} \in \Gamma(V_1), E_2 = p_i \frac{\partial}{\partial p_i} \in \Gamma(V_2), E = E_1 + E_2 \in \Gamma(V).$$  \hspace{1cm} (1.4)

In the following we consider that manifold $M$ is endowed with a Finsler structure $F$, and we present a metric structure on $V$ induced by $F$. According to [2] [4] [14], a function $F : TM \to [0, \infty)$ which satisfies the following conditions:

i) $F$ is $C^\infty$ on $TM^0 = TM \setminus \{\text{zero section}\}$;
ii) $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda \in \mathbb{R}_+$;

iii) the $n \times n$ matrix $(g_{ij})$, where $g_{ij} = \frac{1}{2} \frac{\partial^2 K^2}{\partial p_i \partial p_j}$, is positive definite at all points of $TM^0$, is called a Finsler structure on $M$ and the pair $(M, F)$ is called a Finsler space. We notice that in fact $F(x, y) > 0$, whenever $y \neq 0$.

There are some useful facts which follow from the above homogeneity condition ii) of the fundamental function of the Finsler space $(M, F)$. By the Euler theorem on positively homogeneous functions we have, see [2, 4, 14]:

$$y_i = g_{ij}y^j, \quad y^i = g^{ij}y_j, \quad F^2 = g_{ij}y^i y^j = y_i y^i, \quad C_{ijk}y^k = C_{ikj}y^k = C_{kij}y^k = 0, \quad (1.5)$$

where $(g^{ij})$ is the inverse matrix of $(g_{ij})$ and we have put $y_i = \frac{1}{2} \frac{\partial^2 K^2}{\partial p_i}$. \(C_{ijk}p_k = C_{ikj}p_k = C_{kij}p_k = 0\).

Also, for a Finsler structure $F$ on $TM^0$ there is Cartan structure $K = F^*$ on $T^*M^0 := T^*M – \{\text{zero section}\}$ obtained by Legendre transformation of $F$ (the $\mathcal{L}$-duality process, see [12, 13, 15]), that is a function $K : T^*M \rightarrow [0, \infty)$ which has the following properties:

i) $K$ is $C^\infty$ on $T^*M^0$;

ii) $K(x, \lambda p) = \lambda K(x, p)$ for all $\lambda > 0$;

iii) the $n \times n$ matrix $(g^{*ij})$, where $g^{*ij} = \frac{1}{2} \frac{\partial^2 K^2}{\partial p_i \partial p_j}$, is positive definite at all points of $T^*M_0$.

Also $K(x, p) > 0$, whenever $p \neq 0$. The properties of $K$ imply that

$$p^i = g^{*ij}p_j, \quad p_i = g^{*ij}p_j, \quad K^2 = g^{*ij}p_ip_j = p_ip^i, \quad C^{iij}p_k = C^{ikj}p_k = C^{kij}p_k = 0, \quad (1.6)$$

where $(g^{*ij})$ is the inverse matrix of $(g^{*ij})$ and we have put $p^i = \frac{1}{2} \frac{\partial K^2}{\partial p_i}$, $C^{iij}p_k = -\frac{1}{2} \frac{\partial^2 K^2}{\partial p_i \partial p_j \partial p_k}$.

It is well-known that $g_{ij}$ determines a metric structure on $V(TM)$ and $g^{*ij}$ determines a metric structure on $V(T^*M)$. Similarly, every Finsler structure $F$ on $M$ determines a metric structure $G$ on $V$ by setting

$$G(X, Y) = g_{ij}(x, y)X^i_t(x, y, p)Y^j_t(x, y, p) + g^{*ij}(x, p)X^2_t(x, y, p)Y^2_t(x, y, p), \quad (1.7)$$

for every $X = X^i_t(x, y, p) \frac{\partial}{\partial p_t} + X^2_t(x, y, p) \frac{\partial}{\partial p_t}$, $Y = Y^i_t(x, y, p) \frac{\partial}{\partial p_t} + Y^2_t(x, y, p) \frac{\partial}{\partial p_t} \in \Gamma(V)$.

## 2 Vertical Liouville foliations on $TM$

In this section we reconsider the vertical Liouville distributions $V_{E_1}$ and $V_{E_2}$ from the case of vertical tangent (cotangent) bundle of a Finsler (Cartan) space, see [3, 9], for the case of vertical subbundles $V_1$ and $V_2$, respectively, with respect to Liouville vector fields $E_1$ and $E_2$. Next we define the Liouville distribution $V_E$ with respect to the Liouville vector field $E = E_1 + E_2$, we prove that it is integrable and we study some of its properties. Also, some links between the vertical Liouville foliations $V_{E_1}$, $V_{E_2}$ and $V_E$, respectively, are established.
2.1 Vertical Liouville distributions $V_{\mathcal{E}_1}$ and $V_{\mathcal{E}_2}$

Following [3], [9] we define two vertical Liouville distributions on $\mathcal{T}M$ as the complementary orthogonal distributions in $V_1$ and $V_2$ to the line distributions spanned by the Liouville vector fields $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively.

By (1.3) and (1.5) we have

$$G(\mathcal{E}_1, \mathcal{E}_1) = F^2.$$  \hspace{1cm} (2.1)

Using $G$ and $\mathcal{E}_1$, we define the $V_1$-vertical one form $\zeta_1$ by

$$\zeta_1(X_1) = \frac{1}{F}G(X_1, \mathcal{E}_1), \quad \forall X_1 = X^i_1(x, y, p)\frac{\partial}{\partial y^i} \in \Gamma(V_1).$$  \hspace{1cm} (2.2)

Let us denote by $\{\mathcal{E}_1\}$ the line vector bundle over $\mathcal{T}M$ spanned by $\mathcal{E}_1$ and we define the first vertical Liouville distribution as the complementary orthogonal distribution $V_{\mathcal{E}_1}$ to $\{\mathcal{E}_1\}$ in $V_1$ with respect to $G$. Thus, $V_{\mathcal{E}_1}$ is defined by $\zeta_1$, that is

$$\Gamma(V_{\mathcal{E}_1}) = \{X_1 \in \Gamma(V_1) : \zeta_1(X_1) = 0\}.$$  \hspace{1cm} (2.3)

We get that every $V_1$-vertical vector field $X_1 = X^i_1(x, y, p)\frac{\partial}{\partial y^i}$ can be expressed in the form:

$$X_1 = P_1X_1 + \frac{1}{F}\zeta_1(X_1)\mathcal{E}_1,$$  \hspace{1cm} (2.4)

where $P_1$ is the projection morphism of $V_1$ on $V_{\mathcal{E}_1}$.

Also, by direct calculus, we get

$$G(X_1, P_1Y_1) = G(P_1X_1, P_1Y_1) = G(X_1, Y_1) - \zeta_1(X_1)\zeta_1(Y_1), \quad \forall X_1, Y_1 \in \Gamma(V_1).$$  \hspace{1cm} (2.5)

Let us consider $\{\theta^i\}$ the dual basis of $\{\frac{\partial}{\partial y^i}\}$. Then, with respect to the basis $\{\theta^i\}$ and $\{\frac{\theta^i \otimes \frac{\partial}{\partial y^j}}{F^2}\}$, respectively, $\zeta_1$ and $P_1$ are locally given by

$$\zeta_1 = \zeta^1 \theta^i, \quad P_1 = P^i_j \theta^i \otimes \frac{\partial}{\partial y^j}, \quad \zeta_1 = \frac{y_i}{F}, \quad P^i_j = \delta^i_j - \frac{y_j y^i}{F^2},$$  \hspace{1cm} (2.6)

where $\delta^i_j$ are the components of the Kronecker delta.

As usual for tangent bundle of a Finsler space (see Theorem 3.1 from [3]), the first vertical Liouville distribution $V_{\mathcal{E}_1}$ is integrable and it defines a foliation on $\mathcal{T}M$, called the first vertical Liouville foliation on the big-tangent manifold $\mathcal{T}M$. Also, some geometric properties of the leaves of vertical foliation $V_1$ can be derived via the first vertical Liouville foliation $V_{\mathcal{E}_1}$.

Similarly, by (1.4) and (1.6) we have

$$G(\mathcal{E}_2, \mathcal{E}_2) = K^2,$$  \hspace{1cm} (2.7)

and using $G$ and $\mathcal{E}_2$, we define the $V_2$-vertical one form $\zeta_2$ by

$$\zeta_2(X_2) = \frac{1}{K}G(X_2, \mathcal{E}_2), \quad \forall X_2 = X^i_2(x, y, p)\frac{\partial}{\partial p_i} \in \Gamma(V_2).$$  \hspace{1cm} (2.8)

Let us denote by $\{\mathcal{E}_2\}$ the line vector bundle over $\mathcal{T}M$ spanned by $\mathcal{E}_2$ and we define the second vertical Liouville distribution as the complementary orthogonal distribution $V_{\mathcal{E}_2}$ to $\{\mathcal{E}_2\}$ in $V_2$ with respect to $G$. Thus, $V_{\mathcal{E}_2}$ is defined by $\zeta_2$, that is

$$\Gamma(V_{\mathcal{E}_2}) = \{X_2 \in \Gamma(V_2) : \zeta_2(X_2) = 0\}.$$  \hspace{1cm} (2.9)
We get that every $V_2$-vertical vector field $X_2 = X^2_1(x, y, p)\frac{\partial}{\partial p}$ can be expressed in the form:

$$X_2 = P_2 X_2 + \frac{1}{K} \zeta_2(X_2) E_2,$$  \hspace{1cm} (2.10)

where $P_2$ is the projection morphism of $V_2$ on $V_{E_2}$.

Similarly, by direct calculus, we get

$$G(X_2, P_2 Y_2) = G(P_2 X_2, P_2 Y_2) = G(X_2, Y_2) - \zeta_2(X_2) \zeta_2(Y_2), \, \forall X_2, Y_2 \in \Gamma(V_2). \hspace{1cm} (2.11)$$

Let us consider $\{k_i\}$ the dual basis of $\left\{\frac{\partial}{\partial p^i}\right\}$. Then, with respect to the basis $\{k_i\}$ and $\left\{k_j \otimes \frac{\partial}{\partial p^i}\right\}$, respectively, $\zeta_2$ and $P_2$ are locally given by

$$\zeta_2 = 2 \zeta^i k_i, \, P_2 = P^j_k k_j \otimes \frac{\partial}{\partial p^i}, \, \zeta^i = \frac{p^i}{K}, \, P^j_k = \delta^j_k - \frac{p^j p^i}{K^2}. \hspace{1cm} (2.12)$$

As usual for cotangent bundle of a Cartan space (see Theorem 2.1 from [9]), the second vertical Liouville distribution $V_{E_2}$ is integrable and it defines a foliation on $TM$, called the second vertical Liouville foliation on the big-tangent manifold $TM$. Also, some geometric properties of the leaves of vertical foliation $V_2$ can be derived via the second vertical Liouville foliation $V_{E_2}$.

2.2 Vertical Liouville distribution $V_{E}$

In this subsection we unify the concepts presented in the previous subsection and we define a vertical Liouville distribution on $TM$ as the complementary orthogonal distribution in $V$ to the line distribution spanned by the Liouville vector field $E = E_1 + E_2$. We prove that this distribution is an integrable one, and also, we find some geometric properties of both leaves of Liouville distribution and the vertical distribution on the big-tangent manifold $TM$. Finally, some links between the vertical Liouville foliations $V_{E_1}$, $V_{E_2}$ and $V_{E}$, respectively, are established.

By (1.4), (1.5) and (1.6) we have

$$G(E, E) = F^2 + K^2. \hspace{1cm} (2.13)$$

Now, by means of $G$ and $E$, we define the vertical one form $\zeta$ by

$$\zeta(X) = \frac{1}{\sqrt{F^2 + K^2}} G(X, E), \forall X = X_1^1(x, y, p) \frac{\partial}{\partial y} + X_2^2(x, y, p) \frac{\partial}{\partial p} \in \Gamma(V). \hspace{1cm} (2.14)$$

Let us denote by $\{E\}$ the line vector bundle over $TM$ spanned by $E$ and we define the vertical Liouville distribution as the complementary orthogonal distribution $V_{E}$ to $\{E\}$ in $V$ with respect to $G$. Thus, $V_{E}$ is defined by $\zeta$, that is

$$\Gamma(V_{E}) = \{X \in \Gamma(V) : \zeta(X) = 0\}. \hspace{1cm} (2.15)$$

We get that every vertical vector field $X = X_1^1(x, y, p) \frac{\partial}{\partial y} + X_2^2(x, y, p) \frac{\partial}{\partial p}$ can be expressed in the form:

$$X = P X + \frac{1}{\sqrt{F^2 + K^2}} \zeta(X) E, \hspace{1cm} (2.16)$$

where $P$ is the projection morphism of $V$ on $V_{E}$. 

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Similarly, differentiate (2.23) with respect to $\tau$ using an argument similar to that used in [3]. Let $\zeta$ and $P$ be locally given by

$$\zeta = \zeta^i \theta^i + \zeta^j k_j, \quad P = P^i_j \theta^i \otimes \frac{\partial}{\partial y^i} + P^j_i k_j \otimes \frac{\partial}{\partial p_i} + P_{ij} \theta^i \otimes \frac{\partial}{\partial p_i} + P^{ij} k_j \otimes \frac{\partial}{\partial y^i},$$

where their local components are expressed by

$$\zeta_i = \frac{y_i}{\sqrt{F^2 + K^2}}, \quad \zeta^i = \frac{p^i}{\sqrt{F^2 + K^2}}.$$

(2.19)

Also, by direct calculus, we get

$$G(X, P Y) = G(P X, P Y) = G(X, Y) - \zeta(X) \zeta(Y), \quad \forall X, Y \in \Gamma(V).$$

(2.17)

With respect to the basis $\{\theta^i, k_i\}$ and $\{\theta^i \otimes \frac{\partial}{\partial y^i}, \theta^j \otimes \frac{\partial}{\partial p_i}, k_j \otimes \frac{\partial}{\partial y^i}, k_j \otimes \frac{\partial}{\partial p_i}\}$, respectively, $\zeta$ and $P$ are locally given by

$$\zeta = \zeta^i \theta^i + \zeta^j k_j, \quad P = P^i_j \theta^i \otimes \frac{\partial}{\partial y^i} + P^j_i k_j \otimes \frac{\partial}{\partial p_i} + P_{ij} \theta^i \otimes \frac{\partial}{\partial p_i} + P^{ij} k_j \otimes \frac{\partial}{\partial y^i},$$

(2.18)

Remark 2.1. We have the following relations between $\zeta$, $P$, $\zeta_1$, $\zeta_2$, $P_1$ and $P_2$:

$$\zeta(X) = \frac{F}{\sqrt{F^2 + K^2}} \zeta_1(X_1) + \frac{K}{\sqrt{F^2 + K^2}} \zeta_2(X_2),$$

(2.21)

$$P(X) = P_1(X_1) + P_2(X_2) + \frac{1}{F^2 + K^2} \left( \zeta_1(X_1) - \zeta_2(X_2) \right) (K^2 \xi_1 - F^2 \xi_2),$$

(2.22)

for every vertical vector field $X = X_1 + X_2 = X_1(x, y, p) \frac{\partial}{\partial y} + X_2(x, y, p) \frac{\partial}{\partial p}$.

Theorem 2.1. The vertical Liouville distribution $\mathcal{V}$ is integrable and it defines a foliation on $\mathcal{T}M$, called vertical Liouville foliation on the big-tangent manifold $\mathcal{T}M$.

Proof. Follows using an argument similar to that used in [3]. Let $X, Y \in \Gamma(\mathcal{V})$. As $V$ is an integrable distribution on $\mathcal{T}M$, it is sufficient to prove that $[X, Y]$ has no component with respect to $\mathcal{E}$.

It is easy to see that a vertical vector field $X = X_1(x, y, p) \frac{\partial}{\partial y} + X_2(x, y, p) \frac{\partial}{\partial p}$ is in $\Gamma(\mathcal{V})$ if and only if

$$g_{ij}(x, y) X_1^i y^j + g^{*ij}(x, p) X_2^j p_j = 0.$$  

(2.23)

Differentiate (2.23) with respect to $y^k$ we get

$$\frac{\partial g_{ij}}{\partial y^k} X_1^i y^j + g_{ik} X_1^i + g_{jk} \frac{\partial X_1^i}{\partial y^j} y^j + g^{*ij} p_j \frac{\partial X_2^j}{\partial y^k} = 0, \quad \forall k = 1, \ldots, n$$

(2.24)

and taking into account the relation $\frac{\partial g_{*ij}}{\partial y^k} y^j = 0$ (see (1.5)), one gets

$$g_{ik} X_1^i + g_{jk} \frac{\partial X_1^i}{\partial y^j} + g^{*ij} p_j \frac{\partial X_2^j}{\partial y^k} = 0, \quad \forall k = 1, \ldots, n.$$  

(2.25)

Similarly, differentiate (2.23) with respect to $p_k$ we get

$$g_{ij} y^i \frac{\partial X_1^i}{\partial p_k} + g^{*ik} X_1^i + g_{ik} \frac{\partial X_1^i}{\partial p_k} X_2^j p_j + g^{*ij} p_j \frac{\partial X_2^j}{\partial p_k} = 0, \quad \forall k = 1, \ldots, n.$$  

(2.26)
and taking into account the relation \( \frac{\partial g^{ij}}{\partial p_k} p_j = 0 \) (see (1.6)), one gets
\[
g^{ik} X_i^2 + g_{ij} y \frac{\partial X_i}{\partial p_k} + g^{ij} p_j \frac{\partial X_i^2}{\partial p_k} = 0, \quad \forall k = 1, \ldots, n. \tag{2.27}
\]
Let \( X = X_i(x, y, p) \frac{\partial}{\partial y^i} + X_i^2(x, y, p) \frac{\partial}{\partial p_k} \), \( Y = Y_i(x, y, p) \frac{\partial}{\partial y^i} + Y_i^2(x, y, p) \frac{\partial}{\partial p_k} \in \Gamma(V) \). Then, by direct calculations using (2.25) and (2.27), we have
\[
G([X, Y], E) = g_{ijk} X^i_1 \frac{\partial Y^j}{\partial y^k} - Y^i_1 \frac{\partial X^j}{\partial y^k} + g^{ijk} p_k \frac{\partial X^i_1}{\partial p_j} - g^{ijk} Y^i_1 \frac{\partial X^j}{\partial p_k} \frac{\partial Y^i}{\partial p_j} = 0,
\]
which completes the proof. \( \square \)

**Remark 2.2.** The proof of Theorem 2.1 can be also obtained using an argument similar to [6].

More exactly, if we consider \( P(\frac{\partial}{\partial y^i}) = P_2^i \frac{\partial}{\partial y^i} + P_3^i \frac{\partial}{\partial p_k} \) and \( P(\frac{\partial}{\partial p_i}) = P_4^i \frac{\partial}{\partial y^i} + P_5^i \frac{\partial}{\partial p_k} \), by direct calculus we obtain
\[
P(\frac{\partial}{\partial y^i}) (\sqrt{F^2 + K^2}) = P(\frac{\partial}{\partial p_i}) (\sqrt{F^2 + K^2}) = 0. \tag{2.28}
\]
Now, since \( V = V_\mathcal{E} \oplus \{ E \} \) is integrable, the Lie brackets of vector fields from \( V_\mathcal{E} \) are given by
\[
\begin{align*}
[P(\frac{\partial}{\partial y^i}), P(\frac{\partial}{\partial y^j})] &= A^k_{ij} P(\frac{\partial}{\partial y^k}) + B_{ijk} P(\frac{\partial}{\partial p_k}) + C_{ij} E, \tag{2.29} \\
[P(\frac{\partial}{\partial y^i}), P(\frac{\partial}{\partial p_j})] &= D^k_{ij} P(\frac{\partial}{\partial y^k}) + E_{ijk} P(\frac{\partial}{\partial p_k}) + F^i_{ij} E, \tag{2.30} \\
[P(\frac{\partial}{\partial p_i}), P(\frac{\partial}{\partial p_j})] &= G^{ijk} P(\frac{\partial}{\partial y^k}) + H^i_{ijk} P(\frac{\partial}{\partial p_k}) + L^i_{ij} E, \tag{2.31}
\end{align*}
\]
for some locally defined functions \( A^k_{ij}, B_{ijk}, C_{ij}, D^k_{ij}, E_{ijk}, F^i_{ij}, G^{ijk}, H^i_{ijk} \) and \( L^i_{ij} \), respectively.

We notice that by the homogeneity condition of \( F \) and \( K \) we have \( E(\sqrt{F^2 + K^2}) = \sqrt{F^2 + K^2} \).

Now, if we apply the vector fields in both sides of formulas (2.29), (2.30) and (2.31) to the function \( \sqrt{F^2 + K^2} \) and using (2.28), we obtain \( C_{ij} \sqrt{F^2 + K^2} = F^i_{ij} \sqrt{F^2 + K^2} = L^i_{ij} \sqrt{F^2 + K^2} = 0 \). This implies that \( C_{ij} = F^i_{ij} = L^i_{ij} = 0 \), and then the vertical Liouville distribution \( V_\mathcal{E} \) is integrable.

As usual, the Theorem 2.1 we may say that the geometry of the leaves of vertical foliation \( V \) should be derived from the geometry of the leaves of vertical Liouville foliation \( V_\mathcal{E} \) and of integral curves of \( \mathcal{E} \). In order to obtain this interplay, we consider a leaf \( L_V \) of \( V \) given locally by \( x^i = a^i, \ i = 1, \ldots, n \), where the \( a^i \)'s are constants. Then, \( g_{ij}(a, y) \) and \( g^{ij}(a, p) \) are the components of a Riemannian metric \( G_{L_V} = G|_{L_V} \) on \( L_V \). If we denote by \( \nabla \) the Levi-Civita connection on \( L_V \) with respect to \( G_{L_V} \), then its local expression is
\[
\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = C^i_{ij}(a, y) \frac{\partial}{\partial y^k}, \quad \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial p_j} = 0, \quad \nabla_{\frac{\partial}{\partial p_i}} \frac{\partial}{\partial y^j} = 0, \quad \nabla_{\frac{\partial}{\partial p_i}} \frac{\partial}{\partial p_j} = C^i_{k}(a, p) \frac{\partial}{\partial p_k}, \tag{2.32}
\]
where \( C^k_{ij}(a,y) = \frac{1}{2} g^k_{ij}(a,y) \frac{\partial g_{ij}(a,y)}{\partial y} \) and \( C^k_{ij}(a,p) = -\frac{1}{2} g^k_{ij}(a,p) \frac{\partial g_{ij}(a,p)}{\partial p} \).

Contracting \( C^k_{ij}(a,y) \) by \( y^i \) and \( C^k_{ij}(a,p) \) by \( p_j \), respectively, we deduce

\[
C^k_{ij}(a,y)y^i = 0, \quad C^k_{ij}(a,p)p_j = 0.
\]

In the following lemma we obtain the covariant derivatives with respect to \( \nabla \) of \( E, \zeta \) and \( P \), respectively.

**Lemma 2.1.** On any leaf \( L_V \) of \( V \), we have

\[
\nabla X \left( \frac{E}{\sqrt{F^2 + K^2}} \right) = \frac{PX}{\sqrt{F^2 + K^2}},
\]

(2.34)

\[
(\nabla X \zeta) Y = \frac{1}{\sqrt{F^2 + K^2}} G_{L_V}(PX, PY),
\]

(2.35)

and

\[
(\nabla X P) Y = -\frac{1}{F^2 + K^2} \left[ G_{L_V}(PX, PY) E + \sqrt{F^2 + K^2} \zeta(Y) PX \right]
\]

(2.36)

for any \( X, Y \in \Gamma(TL_V) \).

Proof. We take \( X = X^1_1(a,y,p) \frac{\partial}{\partial y_1} + X^2_1(a,y,p) \frac{\partial}{\partial y_2} \), \( Y = Y^1_1(a,y,p) \frac{\partial}{\partial y_1} + Y^2_1(a,y,p) \frac{\partial}{\partial y_2} \in \Gamma(TL_V) \) and the relation (2.34) follows by:

\[
\nabla X \left( \frac{E}{\sqrt{F^2 + K^2}} \right) = \frac{X^1_1}{\sqrt{F^2 + K^2}} \left( \delta_i^j - \frac{y^i y_j}{F^2 + K^2} \frac{\partial}{\partial y_j} - \frac{p_j y_i}{F^2 + K^2} \frac{\partial}{\partial p_j} \right) + \frac{X^2_1}{\sqrt{F^2 + K^2}} \left( \delta_i^j - \frac{p_i p_j}{F^2 + K^2} \frac{\partial}{\partial p_j} - \frac{y^i y_j}{F^2 + K^2} \frac{\partial}{\partial y_j} \right)
\]

\[
= \frac{1}{\sqrt{F^2 + K^2}} \left( X^1_1 P^1_1 \frac{\partial}{\partial y_1} + X^1_1 P^3_1 \frac{\partial}{\partial p_j} + X^2_1 P^2_1 \frac{\partial}{\partial p_j} \right)
\]

\[
= \frac{PX}{\sqrt{F^2 + K^2}}
\]

For the relation (2.35) we have

\[
(\nabla X \zeta) Y = X(\zeta(Y)) - \zeta(\nabla X Y)
\]

\[
= X^1_1 Y^1_1 \frac{\partial \zeta_j}{\partial y^1} + X^1_1 Y^2_1 \frac{\partial \zeta^j}{\partial y^2} + X^2_1 Y^1_1 \frac{\partial \zeta_j}{\partial p_i} + X^2_1 Y^2_1 \frac{\partial \zeta^j}{\partial p_i}
\]

\[
= \frac{X^1_1 Y^1_1}{\sqrt{F^2 + K^2}} \left( g_{ij} - \frac{y_i y_j}{F^2 + K^2} \right) - \frac{X^1_1 Y^2_1 p_i y_i}{(F^2 + K^2) \sqrt{F^2 + K^2}}
\]

\[
- \frac{X^2_1 Y^2_1 p_j y_j}{(F^2 + K^2) \sqrt{F^2 + K^2}} + \frac{X^2_1 Y^2_1}{\sqrt{F^2 + K^2}} \left( g^{*ij} - \frac{p^{*i} p^j}{F^2 + K^2} \right).
\]

On the other hand we have

\[
G_{L_V}(PX, PY) = G_{L_V}(X, Y) - \zeta(X) \zeta(Y)
\]

\[
= X^1_1 Y^1_1 g_{ij} + X^2_1 Y^2_1 g^{*ij} - \frac{(X^1_1 y_i + X^2_1 p^j)(Y^1_1 y_j + Y^2_1 p^i)}{F^2 + K^2}.
\]
and the relation (2.35) follows easy.

The relation (2.36) follows using (2.16), (2.34) and (2.35).

**Theorem 2.2.** Let \((M, F)\) be a \(n\)-dimensional Finsler space and \(L_V, L_{V\xi}\) and \(\gamma\) be a leaf of \(V\), a leaf of \(V_{\xi}\) that lies in \(L_V\), and an integral curve of \(\frac{\mathbf{E}}{\sqrt{F^2 + K^2}}\), respectively. Then the following assertions are valid:

i) \(\gamma\) is a geodesic of \(L_V\) with respect to \(\nabla\).

ii) \(L_{V\xi}\) is totally umbilical immersed in \(L_V\).

iii) \(L_{V\xi}\) lies in the generalized indicatrix \(I_a = \{(y, p) \in T_a M^0 \oplus T^*_a M^0 : F^2(a, y) + K^2(a, p) = 1\}\) and has constant mean curvature equal to \(-1\).

**Proof.** Replace \(X\) by \(\frac{\mathbf{E}}{\sqrt{F^2 + K^2}}\) in (2.34) and we obtain i). Taking into account that \(\frac{\mathbf{E}}{\sqrt{F^2 + K^2}}\) is the unit normal vector field of \(L_{V\xi}\), the second fundamental form \(B\) of \(L_{V\xi}\) as a hypersurface of \(L_V\) is given by

\[
B(X, Y) = \frac{1}{\sqrt{F^2 + K^2}} G_{L_V}(\nabla_X Y, \mathbf{E}), \forall X, Y \in \Gamma (TL_{V\xi}).
\]

(2.37)

On the other hand, by using (2.34) and taking into account that \(G_{L_V}\) is parallel with respect to \(\nabla\), we deduce that

\[
G_{L_V}(\nabla_X Y, \mathbf{E}) = -G_{L_V}(X, Y), \forall X, Y \in \Gamma (TL_{V\xi}).
\]

(2.38)

Hence,

\[
B(X, Y) = -\frac{1}{\sqrt{F^2 + K^2}} G_{L_V}(X, Y), \forall X, Y \in \Gamma (TL_{V\xi}),
\]

(2.39)

that is, \(L_{V\xi}\) is totally umbilical immersed in \(L_V\). Now, we have

\[
\frac{g_{ij}y^i}{\sqrt{F^2 + K^2}} + \frac{g^{ij}p_i}{\sqrt{F^2 + K^2}} = \frac{\partial \sqrt{F^2 + K^2}}{\partial y^i} + \frac{\partial \sqrt{F^2 + K^2}}{\partial p_j}
\]

(2.40)

which says that \(\frac{\mathbf{E}}{\sqrt{F^2 + K^2}}\) is a unit normal vector field for both \(L_{V\xi}\) and the component \(I_a\). Thus, \(L_{V\xi}\) lies in \(I_a\) and \(F^2(a, y) + K^2(a, p) = 1\) at any point \((y, p) \in L_{V\xi}\). Then (2.39) becomes

\[
B(X, Y) = -G_{L_V}(X, Y), \forall X, Y \in \Gamma (TL_{V\xi})
\]

(2.41)

which implies that

\[
\frac{1}{2n-1} \sum_{i=1}^{2n-1} \varepsilon_i B(E_i, E_i) = -1,
\]

(2.42)

where \(\{E_i\}\) is an orthonormal frame field on \(L_{V\xi}\) of signature \(\{\varepsilon_i\}\). Hence, the mean curvature of \(L_{V\xi}\) is \(-1\) which completes the proof.

**Theorem 2.3.** Let \((M, F)\) be a \(n\)-dimensional Finsler space and \(L_V\) be a leaf of the vertical foliation \(V\). Then the sectional curvature of any nondegenerate plane section on \(L_V\) which contain the vertical Liouville vector field \(\mathbf{E}\) is equal to zero.
Proof. Denote by $R_{LV}$ the curvature tensor field of $\nabla$ on $L_V$. Then, by using (2.34) and (2.36), we obtain

$$R_{LV} (X, E)E = - \left(1 - \frac{E(\sqrt{F^2 + K^2})}{\sqrt{F^2 + K^2}}\right) PX$$

(2.43)

for every vector field $X$ on $L_V$. Now, taking into account $E(\sqrt{F^2 + K^2}) = \sqrt{F^2 + K^2}$, the sectional curvature of a plane section $\{X, E\}$ vanishes on $L_V$. \hfill $\square$

Corollary 2.1. Let $(M, F)$ be a $n$-dimensional Finsler space. Then there exist no leaves of $V$ which are positively or negatively curved.

Finally, let us study certain relations between the vertical Liouville foliations $V_{E_1}, V_{E_2}$ and $V_E$, respectively.

We notice that we have the following decompositions of the vertical distribution:

$$V = V_{E_1} \oplus V_{E_2} \oplus \{E_1\} \oplus \{E_2\} \quad \text{and} \quad V = V_E \oplus \{E\}. \quad (2.44)$$

Taking into account that $[P^1_j \frac{\partial}{\partial y^j}, P^k_l \frac{\partial}{\partial y^k}] = 0$ and $[E_1, E_2] = 0$ we get that both distributions $V_{E_1} \oplus V_{E_2}$ and $\{E_1\} \oplus \{E_2\}$ are integrable. Evidently, $\{E\} \subset \{E_1\} \oplus \{E_2\}$ and by (2.21) we have also $V_{E_1} \oplus V_{E_2} \subset V_E$. Thus, we have the following vertical subfoliations on $TM$:

$$\{E\} \subset \{E_1\} \oplus \{E_2\} \subset V, \ V_{E_1} \oplus V_{E_2} \subset V_E \subset V. \quad (2.45)$$

The relations (2.44) says that $\{E\}$ and $V_{E_1} \oplus V_{E_2}$ have the same orthogonal complement in $\{E_1\} \oplus \{E_2\}$ and in $V_E$, respectively. It is a line distribution $\{E'\}$, where $E' = K^2E_1 - F^2E_2$, see (2.22) (or by direct calculation in $G(\alpha_1E_1 + \alpha_2E_2, E) = 0$ it results $\alpha_1 = K^2$ and $\alpha_2 = -F^2$). Thus

$$\{E_1\} \oplus \{E_2\} = \{E\} \oplus \{E'\}, \ V_E = V_{E_1} \oplus V_{E_2} \oplus \{E'\}. \quad (2.46)$$

Proposition 2.1. The leaves of the foliation $\{E_1\} \oplus \{E_2\}$ are totally geodesic submanifolds of the leaves of vertical foliation $V$.

Proof. Follows easily taking into account that $\nabla E_1, E_1 = E_1, \ \nabla E_1, E_2 = \nabla E_2, E_1 = 0, \ \nabla E_2, E_2 = E_2$. \hfill $\square$

Also by direct calculus we obtain $\nabla E, E' = -K^2F^2E + (K^2 - F^2)E' \notin \Gamma(\{E'\})$, which leads to

Proposition 2.2. If $\gamma$ is an integral curve of $E'$ then it is not a geodesic of a leaf of vertical foliation $V$.

A natural question is if between the foliations $V_{E_1} \oplus V_{E_2}$ and $V_E$ exists certain relations. Although, the leaves of $V_{E_1}$ are totally umbilical submanifolds of the leaves of $V$, the leaves of $V_{E_2}$ are totally umbilical submanifolds of the leaves of $V_2$ and the leaves of $V_E$ are totally umbilical submanifolds of the leaves of $V$, we have

Theorem 2.4. The leaves of $V_{E_1} \oplus V_{E_2}$ are not totally umbilical submanifolds of the leaves of $V_E$.

Proof. Taking into account that $\frac{E'}{FK\sqrt{F^2 + K^2}}$ is the unit normal vector field of $L_{V_{E_1} \oplus V_{E_2}}$, the second fundamental form $B'$ of $L_{V_{E_1} \oplus V_{E_2}}$ as hypersurface of $L_{V_E}$ is given by

$$B'(X', Y') = \frac{1}{FK\sqrt{F^2 + K^2}}G_{LV} (\nabla X', Y', E') , \forall X', Y' \in \Gamma (TL_{V_{E_1} \oplus V_{E_2}}). \quad (2.47)$$
Taking into account that $G_{L_V}$ is parallel with respect to $\nabla$, we deduce that
\[
G_{L_V}(\nabla X', Y', E') = -G_{L_V}(Y', \nabla X', E'), \quad \forall \ X', Y' \in \Gamma (TL_{V_1} \oplus V_2).
\] (2.48)

Now, let us take $X' = P_1(X_1) + P_2(X_2)$ and $Y' = P_1(Y_1) + P_2(Y_2)$ for every $X_1, Y_1 \in \Gamma(V_1)$ and $X_2, Y_2 \in \Gamma(V_2)$. Then by direct calculus we get
\[
\nabla X' E' = K^2 P_1(X_1) - F^2 P_2(X_2).
\] (2.49)

Thus the relation (2.47) becomes
\[
B'(X', Y') = \frac{-1}{FK^2 - F^2} G_{L_V}(K^2 P_1(X_1) - F^2 P_2(X_2), Y') \neq \lambda G_{L_V}(X', Y'),
\] (2.50)

that is, $L_{V_1} \oplus V_2$ is not totally umbilical immersed in $L_{V_2}$.

\[\square\]

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