The Ultimate Energy Density of Observable Cold Matter

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We demonstrate that the largest measured mass of a neutron star establishes an upper bound to the energy density of observable cold matter. An equation of state-independent expression satisfied by both normal neutron stars and self-bound quark matter stars is derived for the largest energy density inside stars as a function their masses. The largest observed mass sets the lowest upper limit to the density. Implications from existing and future neutron star mass measurements are discussed.

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The number of neutron stars with measured masses has grown in recent years \[1, 2\]. The most accurately measured masses are from timing observations of radio binary pulsars \[3\] and, until recently, were consistent with neutron star masses in the range 1.26 to 1.45 M\(_\odot\) \[1\]. Recent data on binaries containing pulsars and white dwarfs, however, indicate a larger range of masses \[2\]. An example is the binary containing PSR J0751-1807, with 2.2 ± 0.2 M\(_\odot\) with 1σ errors \[4\]. Data from x-ray binaries \[1\] also suggest a wide range in masses, but are subject to greater theoretical and observational uncertainties. As neutron stars are expected to contain the densest cold matter outside black holes, the maximum neutron mass mass and the corresponding maximum energy density are of great interest.

We demonstrate here that a precisely measured neutron star mass sets an upper limit to the mass density, or, equivalently, the energy density, inside the star. The larger the measured mass, the smaller the density limit. A sufficiently large mass could delimit classes of possible equations of state (EOS’s). A limit for this maximum density is preferred utilizing an analytic solution of Einstein’s equations. This limit is checked by comparing numerical results for a variety of EOS’s of both normal and self-bound stars. Recent neutron star mass measurements are summarized and inferences drawn.

From the general relativistic structure equations \[5, 6\], the maximum compactness of a star is set by the limit \(R > (9/4)GM/c^2\) \[7\], where \(R\) and \(M\) are the stellar radius and mass, respectively. With the additional requirements that (i) nowhere in the star is the speed of sound \(c_s\) greater than the speed of light \(c\), (ii) \(c_s\) is everywhere real, and (iii) the EOS matches smoothly to calculable low density EOS’s near the nuclear saturation density \(\rho_s \approx 2.6 \times 10^{14} \text{ g cm}^{-3}\), Ref. \[5\] showed that the compactness limit is increased to

\[R \gtrsim 2.94GM/c^2.\] (1)

This result improves the limit \(R \gtrsim 3.05GM/c^2\) established \[8\] using the prescription \(c_s = c\) above a fiducial energy density \(\rho_f\) \[10\]. The maximum mass inferred from this prescription is proportional to \(\rho_f^{-1/2}\), but the compactness limit is independent of \(\rho_f\) for \(\rho_f << \rho_c\), where \(\rho_c\) is the central density of the star \[9\].

The central mass density of a star must be greater than the average density \(\rho_\text{av} = 3M/(4\pi R^3)\), the value for a uniform density star with the same mass and radius. Combining the compactness limit, Eq. (1), with the constant density relation \(\rho_s = \rho_c\), yields

\[\rho_{c,s} \approx 5.80 \times 10^{15} \left(\text{M}_\odot/M\right)^2 \text{ g cm}^{-3}.\] (2)

This is a plausible approximate lower limit to the central density \(\rho_c\) for a star of a given mass, but it is not an absolute lower limit. (This lower limit cannot be made firm as causality has been imposed on a uniform density fluid in which transmission of signals is instantaneous.) A firm lower limit can be established, however, if an upper limit to \(R\) exists. One observational limitation originates from the most rapidly spinning pulsar, PSR B1937+21 \[11\], which has a frequency \(\nu = 641\ \text{Hz}\). This leads to a lower limit to \(M/R^3\) \[12\] and a lower limit

\[\rho_{c,rat} \approx 1.79 \times 10^{14} (\nu/641\ \text{Hz})^2 \text{ g cm}^{-3},\] (3)

which is, however, not very restrictive. A far more stringent limit could be achieved from a redshift observed from a neutron star. The largest observed redshift \(z_{\text{obs}}\) sets a lower limit to \(M/R\), implying

\[\rho_{c,z} > \frac{3}{4\pi M^2} \left(\frac{c^2 z_{\text{obs}} (2 + z_{\text{obs}})}{2G(1 + z_{\text{obs}})^2}\right)^3.\] (4)

Recently, \(z_{\text{obs}} = 0.35\) was reported \[13\] for the x-ray bursting source XTE J1814-338. With this value,

\[\rho_{c,z} > 1.69 \times 10^{15} \left(\text{M}_\odot/M\right)^2 \text{ g cm}^{-3}.\] (5)

The central question is, how much greater can \(\rho_c\) be compared to any of the above expressions for physically motivated EOS’s? If this question can be answered, an upper limit to the density inside a star of a given mass can found. An important consequence of the existence of an upper limit is that the largest measured neutron star mass would set an upper limit to the density of cold matter. (In a dynamical environment, such as the gravitational collapse of a stellar core to a black hole or a high
energy heavy ion collision, matter becomes hot and may achieve higher densities.) An additional consequence is that one could infer whether or not non-nucleonic degrees of freedom, such as hyperons, Bose condensates or quarks, which generally reduce the maximum mass, can exist in the cores of neutron stars.

Some insights can be gained by comparing analytical solutions to Einstein’s equations with numerical solutions employing model EOS’s. The known analytic solutions fall into two classes: (i) the class that describes “normal” neutron stars for which \( \rho_c \) vanishes at the surface where the pressure vanishes, and (ii) the class that describes “self-bound” stars for which \( \rho_c \) is finite at the surface. In the first class, there are only three known analytic solutions: the Tolman VII solution [3], Buchdahl’s solution [4], and the Nariai IV solution [15]. In the second class, an infinite number of analytic solutions exist, but the useful ones are variants of the Tolman IV and VII solutions [3,4,14], as well as the uniform density case [17].

All known analytic solutions are scale-free; they depend parametrically on the compactness ratio \( \beta = GM/Rc^2 \). However, by coupling these solutions with Eq. (1), i.e.,

\[ \frac{\rho - \rho_c}{\rho_c} = \alpha(1 - \frac{\beta}{\sqrt{\beta_c}}) \]

where the parameter \( \alpha = \frac{12 + \beta_c}{2 + \beta_c + 2\sqrt{1 - 2\beta_c}} \) [10], the central density for a given mass decreases from \( \rho_c \) throughout most of the star, similar to the behavior of strange quark matter. For this case,

\[ \rho_c,TIV = \frac{3}{4\pi} \left( \frac{2 - 5\beta}{2 - 5\beta_c} \right)^{2/3} \left( \frac{c^2\beta_c}{G} \right) \frac{1}{M^2} \approx 1.56 \times 10^{16} (M_\odot/M)^2 \text{ g cm}^{-3}. \]

This solution is valid for \( \beta < 2/5 \). The ratio of the surface to the central densities for the case \( N = 2 \) is

\[ w = \frac{6 - 10\beta}{3} \left( \frac{2 - 5\beta}{2 - 2\beta} \right)^{2/3} \]

which is approximately 0.32 for \( \beta = \beta_c \). With increasing \( N \), the central density for a given mass decreases from that of Eq. (12), and in the limit \( N >> 1 \), \( \rho_{c,TIV}(M) \) approaches the uniform density result.

To investigate the relevance of analytic relations between the central density and mass, we carried out numerical integrations of the TOV structure equations for a multitude of EOS’s, including potential and field-theoretical models, and models that contain strong softening due to the occurrence of hyperons, Bose condensates or quark matter, including the case of self-bound strange...
FIG. 1: The central energy density and mass of maximum mass configurations. Symbols reflect the nature of the EOS’s selected from Refs. \[18\]. NR are non-relativistic potential models, R are field-theoretical models, and Exotica refers to NR or R models in which strong softening occurs, due to the occurrence of hyperons, a Bose condensate, or quark matter. The Exotica points include self-bound strange quark matter stars. For comparison, the central density–maximum mass relations for the redshift, Tolman VII (\(w = 0\)), and Tolman IV (\(N = 2\)) bounds from Eqs. (5), (7), and (12) are shown. The dashed lines for 1.44 and 2.2 \(M_\odot\) serve to guide the eye.

The above results are given in terms of the central mass or energy density \(\rho\). However, most models of dense matter are formulated in terms of the baryon number density \(n\). Therefore, it is of interest to examine relationships between the central baryon density and mass for the maximum mass configurations. A good rule of thumb for converting \(\rho\) to \(n\), using \(n_s \approx \frac{0.16}{r_m^{3/4}}\), is

\[
\frac{\rho}{\rho_s} \approx 0.9(n/n_s)[1 + 0.11(n/n_s)^{3/4}].
\]

The number density so obtained is indicated on the top scale of Fig. 1. We emphasize that the plotted points are positioned using \(\rho_c\), not \(n_c\), in this figure.

The most accurately measured neutron star masses are from timing observations of radio binary pulsars. These include pulsars orbiting another neutron star, a white dwarf, or a main-sequence star. Measured masses are summarized in Fig. 2 and are plotted with 1\(\sigma\) uncertainties. Ordinarily, observations of pulsars in binaries yield orbital sizes and periods from Doppler phenomenon, from which the total mass of the binary can be deduced. But the compact nature of several binary pulsars permits detection of relativistic effects, such as Shapiro delay or orbit shrinkage due to gravitational radiation reaction, which constrains the inclination angle and permits measurement of each mass.
in the binary. The largest accurately measured mass originates from the binary pulsar system PSR 1913+16, whose masses are $1.3867 \pm 0.0002$ and $1.4414 \pm 0.0002 M_\odot$, respectively [34].

A significant development concerns mass determinations in binaries with white dwarf companions, which show a broader mass range than binary pulsars having neutron star companions. Ref. [12] suggests that a narrow set of evolutionary circumstances conspire to form double neutron star binaries, leading to a restricted range of neutron star masses. The implication of this restriction for other binaries remains to be explored. A few cases of white dwarf binaries that contain neutron stars considerably larger than the canonical 1.4 M_\odot value have been reported. A striking case is PSR J0751+1807 [4] in which the estimated mass with 1\sigma error bars is $2.2 \pm 0.2 M_\odot$. For this neutron star, a mass of 1.4 M_\odot is 4\sigma away. If this mass determination holds up after further observations, the central density constraints become intriguingly close to the estimated density for the quark-hadron phase transition. Raising the limit for the neutron star maximum mass could also mark the boundaries of other families of EOS’s in which substantial softening begins around 2 to 3\sigma. This is significant, since exotica generally reduce the maximum mass appreciably.

The simple mean of the measured neutron star masses in white dwarf-neutron star binaries exceeds that of the double neutron star binaries by about 0.22 M_\odot (Fig. 2). Nevertheless, caution is in order since the 2\sigma errors of most of these systems extend below 1.4 M_\odot. Continued observations are required to reduce these errors.

Masses can also be estimated for binaries which contain an accreting neutron star emitting x-rays. Some of these stars are characterized by relatively large masses but also large estimated errors (Fig. 2). The system Vela X-1 is noteworthy, because its lower mass limit (1.6 to 1.7 M_\odot) is constrained, albeit mildly, by geometry [22]. The source 4U 1700-37 might be a black hole, due to lack of oscillations in its x-ray spectrum [20]. Another object, 2S 0921-630 [22], could either be a high-mass neutron star or a low-mass black hole. Although not yet demonstrated, it is widely believed that black holes formed in gravitational collapse have masses that exceed the neutron star maximum mass. These latter two objects could play a significant role in determining the neutron star maximum and the black hole minimum masses.

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