RECURRENCE OF THE TWISTED RANDOM WALK IN THE PLANE

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Abstract. Suppose that \((X_k)_{k \geq 1}\) is a stationary process taking values in the complex plane. For any choice of \(\beta\) from the interval \([0, 2\pi)\) we consider the random walk recursively defined by the equations \(S_0^{(\beta)} = 0\) and \(S_n^{(\beta)} = e^{i\beta} S_{n-1}^{(\beta)} + X_{n-1}\) for \(n \geq 1\), and prove recurrence under diverse additional assumptions on the increment process \((X_k)_{k \in \mathbb{Z}}\). For example if the increment process is \(\alpha\)-mixing and \(E(|X_k|^2)\) is finite, then \(S_n^{(\beta)}\) is recurrent for every fixed choice of the angle \(\beta\) out of a set of full Lebesgue measure, no matter how slowly the mixing coefficients decay.

1. The main results and their proofs

Assume that \((X_k)_{k \in \mathbb{Z}}\) is a stationary process taking values in the complex plane \(\mathbb{C}\). For any fixed angle \(\beta\) in \([0, 2\pi)\) we define the process \((S_n^{(\beta)})_{n \geq 0}\) by setting

\[
S_0^{(\beta)} = 0, \\
S_n^{(\beta)} = e^{i\beta} S_{n-1}^{(\beta)} + X_{n-1}, \quad n \geq 1,
\]

(1)

These random sums can be considered as (the first coordinate of a) random walk in the locally compact group \(G = \mathbb{C} \rtimes S^1 = \{(z, e^{i\beta}) : z \in \mathbb{C}, \beta \in [0, 2\pi)\}\) with the usual product topology and group operation defined by

\[
(z_2, e^{i\beta_2}) \cdot (z_1, e^{i\beta_1}) = (z_2 + e^{i\beta_2} z_1, e^{i(\beta_2 + \beta_1)}).
\]

Indeed, if we put \(Z_k(\omega) = (X_k(\omega), e^{i\beta})\) for all \(k \in \mathbb{Z}\) then it follows immediately that the sums \(Y_n^{(\beta)} = Z_{n-1} \cdot Z_{n-2} \cdots Z_0\) satisfy that

\[
Y_n^{(\beta)} = (e^{i(n-1)\beta} \sum_{k=0}^{n-1} X_k \cdot e^{-i\beta k}, e^{in\beta}) = (S_n^{(\beta)}, e^{in\beta}),
\]

(2)

for every \(n \geq 1\). This random walk has been considered in [Pe] (although in topological setting) over sofic shift spaces, i.e. finite-to-one factors of shifts of finite type, but it is also connected with the construction of invariant measures of non-hyperbolic toral automorphisms (cf. [LS1], [LS2]).

Throughout this paper we restrict our considerations to the case that

\[
\beta \in [0, 2\pi) \setminus 2\pi\mathbb{Q},
\]

the ‘rational’ case will be discussed in Remark 1.5.
For any integer \( n \in \mathbb{Z} \) we denote by \( \mathcal{F}_n \) the sigma algebra generated by all random variables \( \{X_k\}_{k \leq n} \) and by \( \mathcal{F}_n \) the sigma algebra generated by the random variables \( \{X_k\}_{k > n} \). The process \( (X_k)_{k \in \mathbb{Z}} \) is \( \alpha \)-mixing if
\[
\alpha(n) = \sup_{A \in \mathcal{F}_n, B \in \mathcal{F}_n} |P(A \cap B) - P(A)P(B)| \to 0, \tag{3}
\]
as \( n \) tends to infinity. For \( \alpha \)-mixing processes we found the following criterion for recurrence of the twisted random walk.

**Theorem 1.1.** Suppose \( (X_k)_{k \in \mathbb{Z}} \) is a stationary, \( \alpha \)-mixing, complex valued process with \( X_k \) being square integrable. Then for every choice of \( \beta \) from a set of full Lebesgue measure in \( I = [0, 2\pi) \setminus 2\pi\mathbb{Q} \) the random walk \( Y_n^{(\beta)} \) defined by (2) is recurrent. Furthermore, under the additional assumption that \( \sum_{k=0}^{\infty} |E(X_k \cdot X_0)| < \infty \) we have recurrence for every \( \beta \) belonging to \( I \).

Note that if we do not impose restrictions on the decay of correlations one cannot expect recurrence for every \( \beta \) belonging to \( [0, 2\pi) \setminus 2\pi\mathbb{Q} \). At the end of the present section we provide an example of a mixing Gaussian process whose twisted random walk is transient for an arbitrary fixed angle \( \beta \).

Theorem 1.1 applies in particular to the case of a topologically mixing sofic subshift of \( \{-1, 1\}^\mathbb{Z} \) equipped with its measure of maximal entropy: With respect to that measure the process \( X_k = \pi_k \), where \( \pi_k \) is the projection onto the \( k \)-th coordinate, is \( \alpha \)-mixing with exponential decay of the correlations (this follows from [CP]).

It is also worth to mention that under the assumption of very strong dependencies of the increment process (e.g. being generated by a minimal rotation on a compact group) an analogous result holds.

**Theorem 1.2.** Suppose \( (X_k)_{k \in \mathbb{Z}} \) is a stationary, ergodic, complex valued process with \( X_k \) being square integrable. If \( (X_k)_{k \in \mathbb{Z}} \) has singular spectrum, i.e. the spectral measure is singular with respect to the Lebesgue measure, then \( Y_n^{(\beta)} \) is recurrent for almost every \( \beta \in [0, 2\pi) \).

Both theorems are simple applications of the following recurrence criterion for stationary random walks in the locally compact group \( G = \mathbb{C} \times S^1 \).

**Theorem 1.3** (cf. Theorem 2.1 in Section 2). Suppose that \( (Z_k)_{k \in \mathbb{Z}} \) is a stationary and ergodic process with values in \( \mathbb{C} \times S^1 \), and set \( S_n = Z_{n-1} \cdots Z_1 \cdot Z_0 \) for \( n \geq 1 \). If there exist a constant \( c > 0 \) such that
\[
\liminf_{n \to \infty} P \left[ n^{-1/2} |\pi_\mathbb{C}(S_n)| \leq \eta \right] \geq c\eta^2
\]
for all \( \eta \in (0, 1) \), then the random walk \( S_n \) is recurrent. Here \( \pi_\mathbb{C} \) denotes the projection of \( \mathbb{C} \times S^1 \) onto \( \mathbb{C} \).

Its proof, which we postpone to Section 2, makes use of abstract ergodic theory: it relies on a theorem on the growth of transient \( \mathbb{C} \times S^1 \)-valued cocycles over the action of an ergodic probability-preserving transformation, the proof of which is based on the same arguments as the results in [S2] (treating the case \( G = \mathbb{R}^d \), cf. also the survey [S3]) and [Gr] (in which the ideas from [S2] were generalised to the group of unipotent \( d \times d \)-matrices).

Let us show how Theorem 1.1 and Theorem 1.2 can be deduced from Theorem 1.3. Assume that \( (X_k)_{k \in \mathbb{Z}} \) is stationary with finite second moments, and let \( m =
$h d \lambda_{S^1} + m_\perp$ be the decomposition of its spectral measure into absolutely continuous and singular part (with respect to the Lebesgue measure $\lambda_{S^1}$ on the torus $S^1$). Note that for almost every angle $\beta$ in $[0, 2\pi]$ the $L^2$-norms of the sums

$$S_n^{(\beta)} = \sum_{k=0}^{n-1} e^{i(n-1-k)\beta} X_k,$$

grow at most with rate $n^{1/2}$. In fact, a direct computation shows that

$$\frac{1}{n} \cdot E(|S_n^{(\beta)}|^2) = \sum_{k=-(n-1)}^{n-1} (1 - \frac{k}{n}) E(X_0 \tilde{X}_k) \cdot e^{i2\pi \beta k} = m \ast K_{n-1}(e^{i\beta}),$$

with $K_{n-1}(e^{i\alpha}) = \frac{1}{n} \cdot \frac{\sin^2(n\alpha/2)}{\sin^2(\alpha/2)}$ being the Fejér kernel on $S^1$. By a well-known property of the Fejér kernel (see [Ka], e.g.),

$$\lim_{n \to \infty} m \ast K_{n-1}(e^{i\beta}) = h(e^{i\beta}),$$

for Lebesgue-almost every $\beta$.

There are different ways to establish a limit behaviour as needed for the recurrence criterion. In many situations (under additional assumptions on the moments and mixing properties, cf. the classical results in [E]) one could prove a central limit theorem for the sums $S_n^{(\beta)}$. However, we do not aim proving such a limit theorem and follow an alternative approach: using the 'structure' of the set of limiting distributions of $\{n^{-1/2}S_n^{(\beta)}\}_{n \geq 1}$ we directly show the following lemma.

**Lemma 1.4.** Suppose that $(X_k)_{k \in \mathbb{Z}}$ is a stationary, $\alpha$-mixing, complex valued process, and $\beta \in [0, 2\pi) \setminus 2\pi\mathbb{Q}$. If the distributions of the normed sums $n^{-1/2}S_n^{(\beta)}$, $n \geq 0$, are uniformly tight then there exists a constant $c > 0$ such that

$$\liminf_{n \to \infty} P[n^{-1/2}|S_n^{(\beta)}| \leq \eta] \geq c \cdot \eta^2,$$

for all $\eta \in (0, 1)$.

The proof of the lemma uses the well-known property of the mixing coefficient $\alpha(n)$ from (3) that

$$|E(fg) - E(f)E(g)| \leq 4\alpha(n) \cdot \|f - E(f)\|_{\infty} \|g - E(f)\|_{\infty}$$

for every two bounded complex functions $f$ and $g$ measurable with respect to the sigma algebras $\mathfrak{F}_0$ and $\mathfrak{G}_n$ as defined in the introduction, respectively (cf. [E], for example).

**Proof of Lemma 1.4.** Let $\sigma_n$ be the distribution of the normed sum $n^{-1/2}S_n^{(\beta)}$ and $\Sigma$ the set of all weak limits of the sequence $\{\sigma_n\}_{n \geq 0}$.

We first show rotation-invariance of every measure $\sigma$ belonging to $\Sigma$. Assume that $\lim_{j \to \infty} \sigma_{n_j} = \sigma$. Then for every $m \geq 0$,

$$n_j^{-1/2}S_{n_j+m}^{(\beta)} = n_j^{-1/2}S_m^{(\beta)} \circ T^{n_j} + e^{i\beta m} n_j^{-1/2}S_n^{(\beta)},$$

where we introduce the notation $S_m^{(\beta)} \circ T^{n_j} = \sum_{k=0}^{m-1} e^{i(m-1-k)} X_{n_j+k}$. When $j \to \infty$ the distributions of the left side converge to $\sigma$ whereas the distributions of the right side converge to rotated measure $\sigma(e^{-i\beta m} \cdot )$. Hence $\sigma(e^{-i\beta m} \cdot ) = H(\cdot)$ for every $m \geq 1$. Since $\beta \notin 2\pi\mathbb{Q}$ the sequence $\{e^{-i\beta m}\}_{m \geq 0}$ is dense in the unit circle and $\sigma$ is rotation-invariant.
In the next step we show that to every measure $\sigma$ from $\Sigma$ we can find another measure $\rho$ belonging to $\Sigma$ such that

$$\sigma(\cdot) = \rho \ast \rho(2^{1/2} \cdot \cdot \cdot).$$

This is shown by standard arguments using the $\alpha$-mixing condition. As above we assume that $\lim_{j \to \infty} \sigma_{n_j} = \sigma$ and choose sequences of positive integers $\{m_j\}_{j \geq 1}$ and $\{d_j\}_{j \geq 1}$ satisfying $n_j = 2m_j + d_j, d_j/n_j \to 0$, and $\alpha(d_j) \to 0$. Then

$$n_j^{-1/2} S_{n_j}^{(\beta)} = A_j + B_j + C_j,$$

with

$$A_j = n_j^{-1/2}(S_{m_j}^{(\beta)} \circ T^{m_j + d_j}), \quad B_j = e^{im_j \beta} n_j^{-1/2}(S_{d_j}^{(\beta)} \circ T^{m_j}),$$

and

$$C_j = e^{im_j + d_j} n_j^{-1/2} S_{m_j}^{(\beta)},$$

the ‘shifted’ sums defined as above. Passing to a subsequence we assume that also $\lim_{j \to \infty} S_{m_j}^{(\beta)} \circ T^{m_j}$ converge to a limit $\rho$ in $\Sigma$. Since $\rho$ is rotation-invariant and $\lim_{j \to \infty} m_j/n_j = 2/1$, the distributions both of $A_j$ and $C_j$ converge to the measure $\rho(2^{1/2} \cdot \cdot \cdot)$. Applying (5),

$$\left| E(e^{i(tA_j + C_j)}) - E(e^{i(tA_j)}) \cdot E(e^{i(tC_j)}) \right| \leq 16 \beta(d_j) \to 0,$$

and as $B_j \to 0$ in probability we conclude that for every $t$ in $\mathbb{R}^2$,

$$\hat{\sigma}(t) = \lim_{j \to \infty} E(e^{i(A_j + B_j + C_j)}) = \hat{\rho}(2^{-1/2}t)^2,$$

where $\hat{\sigma}$ and $\hat{\rho}$ denote the Fourier transform of the respective measures.

To prove the assertion of the lemma we use the same argument as in the proof of Theorem 14 in [S3]. For $r \in (0, 1)$ let $h_r : \mathbb{C} \to \mathbb{R}$ be the normed indicator function

$$h_r = \frac{1}{r^2} \cdot 1_{[-r/r, r/r] \cdot},$$

and set $g_r = h_r \ast h_r$. Then $\int g_r d\lambda = 1$, where $\lambda$ denotes the 2-dimensional Lebesgue measure, and $0 \leq g_r \leq 1/r^2 \cdot 1_{[-r/r, r/r]}$. For every measure $\sigma$ belonging to $\Sigma$ define the function

$$\phi_r(z) = g_r \ast (\sigma \ast \sigma)(z) = (h_r \ast \sigma) \ast (h_r \ast \sigma)(z),$$

and choose $K > 0$ so that $\sigma([-K/2, K/2]^2) > 1/2$ for all $\sigma \in \Sigma$. Then

$$\int_{[-K-1,K+1]^2} \phi_r(u) \ du \geq (\sigma \ast \sigma)([-K, K]^2) > 1/4$$

for every $\sigma \in \Sigma$. Hence $\lambda(\{u \in \mathbb{R}^2 : \phi_r(u) > 1/4 \cdot 1/(2K + 2)^2\}) > 0$. By rotation-invariance $\sigma$ is symmetric and therefore the function $\phi_r(z)$ attains its maximum at $z = 0$. This implies that

$$\phi_r(0) > 1/4 \cdot 1/(2K + 2)^2$$

for every $\sigma \in \Sigma$ and $r \in (0, 1)$. Using (6) we finally obtain that

$$\inf_{\sigma \in \Sigma} \frac{1}{r^2} \cdot \sigma([-r/2, r/2]^2) \geq \inf_{\sigma \in \Sigma} \frac{1}{r^2} \cdot \sigma(\sqrt{2} \cdot [-r/2, r/2]^2)$$

$$\geq \frac{1}{2} \cdot \frac{1}{\int g_r \sqrt{\sigma} d(\sigma \ast \sigma)} > \frac{1}{8} \cdot \frac{1}{(2K + 2)^2}$$

for every $r \in (0, 1)$, which immediately implies formula (4).
Proof of Theorem 1.1 and Theorem 1.2 using Theorem 1.3. Suppose that the process \((X_k)_{k \in \mathbb{Z}}\) is \(\alpha\)-mixing. It follows from the preceding discussion that for almost every \(\beta\) the \(L^2\)-norms of the normed sums \(n^{-1/2} S_n^{(\beta)}\) are bounded and therefore their distributions are uniformly tight. Under the additional assumption that \(\sum_k |E(X_0X_k)| < \infty\), the spectral measure \(m\) is absolutely continuous with continuous density \(h\) and we have uniformly tightness even for every \(\beta\). For those \(\beta\) which are not contained in the null set \(2\pi \mathbb{Q}\), Lemma 1.4 together with Theorem 1.3 implies recurrence of the random walk \(Y_n^{(\beta)}\). This shows Theorem 1.1.

In the case of an ergodic process \((X_k)_{k \in \mathbb{Z}}\) with discrete spectrum, the density function \(h\) is zero almost everywhere and we conclude that \(n^{-1/2} S_n^{(\beta)} \to 0\) in probability, for almost every angle \(\beta\). For these \(\beta\), Theorem 1.3 again yields recurrence of the random walk \(Y_n^{(\beta)}\) and Theorem 1.2 is proved.

Remark 1.5. Let us shortly discuss the case when \(\beta \in 2\pi \mathbb{Q}\). Suppose that \(\beta = 2\pi p/q\) with relatively prime integers \(p\) and \(q\). If \(q\) is even, then the proof of Lemma 1.4 still works out as we only used symmetry of the limit measures \(\sigma\) belonging to \(\Sigma\) in order to prove (4). However, it is not clear to the author how to show (4) in the case when \(q\) is odd. This case would be clear if one could show the following conjecture: If \(\mu\) is any probability measure in the complex plane then the convolution of the rotated measures

\[
\rho = \mu \ast (\mu \circ R_\beta) \ast \cdots \ast (\mu \circ R_{(p-1)\beta})
\]

satisfies the following maximal inequality:

\[
\rho(B(0, r)) \geq \sup_{z \in \mathbb{C}} \rho(B(z, r)),
\]

for arbitrary \(r > 0\). Note that this inequality is obviously valid for even \(q\) since \(\rho\) is the symmetrisation of the measure \(\rho' = \mu \ast (\mu \circ R_\beta) \ast \cdots \ast (\mu \circ R_{(p/2-1)\beta})\).

Anyway, recurrence of \(Y_n^{(\alpha)}\) in the case \(\beta = 2\pi p/q\) can be treated as follows: The process \((Y_{nq}^{(\beta)})_{n \geq 0}\) is an ordinary random walk in the complex plane with stationary increment process

\[
X_k' = e^{i(q-1)\beta} \sum_{m=0}^{q-1} e^{-im\beta} X_{kq+m}, \quad k \in \mathbb{Z},
\]

and this random walk is recurrent if its increment process \((X_k')_{k \in \mathbb{Z}}\) is ergodic and the central limit theorem holds, which applies to a large variety of examples.

Let us give an example of a process satisfying the assumptions of Theorem 1.1 and for which the random walk \(Y_n^{(\beta)}\) is transient at an arbitrary point \(\beta \in [0, 2\pi) \setminus 2\pi \mathbb{Q}\). Set

\[
m = f d\lambda_{\Sigma}, \quad \text{with} \quad f(e^{ix}) = 1/|e^{ix} - e^{i\beta}|^{1/2}
\]

and let \((X_k)_{k \in \mathbb{Z}}\) be the uniquely determined real Gaussian process with zero mean and \(E(X_0X_k) = \int_0^{2\pi} e^{-ikx} dm(x)\). As \(m\) is absolutely continuous the so constructed process is mixing\(^1\) and it is easily verified that

\[
\sigma_n^2 = n \cdot (m \ast K_{n-1}(e^{i\beta})) \geq c \cdot n^{3/2}
\]

for some constant \(c > 0\).

\(^1\)and therefore \(\alpha\)-mixing, as it is Gaussian
We claim that the normed sums $\sigma_n^{-1}S_n^{(\beta)}$, which are Gaussian distributed with zero mean and covariance matrix $\beta_n = (a_{i,j}^{(n)})_{i,j=1}^2$, converge in distribution to the Gaussian law with zero mean and covariance matrix $A = 1/2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In fact, the set $\Sigma$ of all limit distributions can only consist of Gaussian laws which, as shown in the proof of Lemma 1.4, are invariant under rotations of the complex plane. But since the covariance matrices $\beta_n = (a_{i,j}^{(n)})_{i,j=1}^2$ satisfy that $a_{1,1}^{(n)} + a_{2,2}^{(n)} + 2a_{1,2}^{(n)} = 1$, the same relation holds for the covariance matrices of the measures belonging to $\Sigma$. Thus $\Sigma$ can only consist of a single distribution, namely the Gaussian law with zero mean and covariance matrix $A$.

Hence for any $\eta > 0$,
$$P[|S_n^{(\beta)}| \leq \eta] = P[|\sigma_n^{-1}S_n^{(\beta)}| \leq \sigma_n^{-1}\eta] =$$
$$= \frac{1}{(2\pi \det A_n)^{1/2}} \int_{|\mathbf{x}| < \sigma_n^{-1}\eta} e^{-\frac{1}{2}(\mathbf{x},A_n^{-1}\mathbf{x})} d\lambda_C(\mathbf{x})$$
$$\sim \sqrt{2}\eta^2 \pi \frac{1}{\sigma_n^2},$$
as $A_n \to A$, which proves that $\sum_{n \geq 1} P[|\sigma_n^{(\beta)}| \leq \eta] < \infty$ and therefore
$$P[|S_n^{(\beta)}| \leq \eta \text{ for infinitely many } n \geq 1] = 0.$$
function \( f : X \to G \). The cocycle \( f(n, \cdot) \) generated by the function \( f = X_0 \) then satisfies that

\[
f(n, \cdot) = X_{n-1} \cdots X_1 \cdot X_0,
\]

for every \( n \geq 1 \). Both notions of recurrence, the probabilistic and the one for cocycles, coincide (this is shown in [S1] for real valued cocycles but its proof is valid in any locally compact group).

The group \( G = \mathbb{C} \times S^1 \) provides the following situation which differs only slightly from the one in [Gr]: there exist a one-parameter group of scaling automorphisms

\[
\alpha_\eta : \mathbb{C} \rtimes S^1 \to \mathbb{C} \rtimes S^1, \quad (x, e^{i\beta}) \mapsto (\eta x, e^{i\beta}),
\]

where \( \eta > 0 \) (one-parameter group in the sense that \( \alpha_{\eta_1} \circ \alpha_{\eta_2} = \alpha_{\eta_1 \eta_2} \) for every \( \eta_1, \eta_2 > 0 \)) and these automorphisms contract \( \mathbb{C} \rtimes S^1 \) to its compact subgroup \( \{0\} \times S^1 \), i.e. for every compact subset \( C \) and \( \varepsilon > 0 \) we have that

\[
\alpha_\eta(C) \subseteq \{ z \in \mathbb{C} : |z| < \varepsilon \} \times S^1
\]

for \( \eta \) small enough.

With help of these scaling automorphisms we define for any cocycle \( f(n, \cdot) \) the probability measures \( \sigma_n \) and \( \tau_n \) by setting

\[
\sigma_n(B) = \mu\left( \{ x \in X : \alpha_{n-1/2}f(n, x) \in B \} \right),
\]

and

\[
\tau_n(B) = \frac{1}{n} \sum_{k=1}^{n} \sigma_k(B),
\]

for every Borel set \( B \subseteq G \) and \( n \geq 1 \), the scaling rate \( n^{-1/2} \) chosen in connection with the fact that the right Haar measure \( \lambda \), which in our case is the product measure \( \lambda_{\mathbb{C}} \times \lambda_{S^1} \) of the two-dimensional Lebesgue measure with the normed Haar measure of \( S^1 \), is transformed by the scaling automorphisms according to the equation

\[
\lambda(\alpha_\eta(B)) = \eta^2 \lambda(B),
\]

for every Borel set \( B \subseteq G \) and \( \eta > 0 \).

The aim of this section is to prove - as counterpart to [S2] and [Gr] - the following theorem on the ‘weak’ growth of transient cocycles.

**Theorem 2.1.** Suppose that \( T \) is a ergodic and measure preserving automorphism of a standard probability space \((X, \mathcal{B}, \mu)\), and \( f(n, \cdot) \) is a transient cocyle taking values in the semi-direct product \( G = \mathbb{C} \rtimes S^1 \). Then, in analogy to [S2] and [Gr],

\[
\sup_{\eta > 0} \lim_{n \to \infty} \sup \tau_n(B(0, \eta) \times S^1)/\eta^2 < \infty
\]

and

\[
\liminf_{\eta \to 0^+} \liminf_{n \to \infty} \tau_n(B(0, \eta) \times S^1)/\eta^2 = 0,
\]

with \( B(0, \eta) = \{ z \in \mathbb{C} : |z| < \eta \} \).

Note that the recurrence criterion Theorem 1.3 is an immediate corollary. As already mentioned in the introduction, the proof of Theorem 2.1 is based on the same arguments used in [S2]. Let us start with its preparations.

If \( f(n, \cdot) \) is transient then we can find a Borel set \( B \subseteq X \) of positive measure and a relatively compact open neighborhood \( U \) of the identity \( 1_G \) such that

\[
\mu(T^nB \cap B \cap \{ x \in C : f(n, x) \in U \}) = 0
\]
for every $n \in \mathbb{Z}$. Decreasing the set $B$ if necessary we may even assume that
\[ \mu(B) = 1/L \]
for some integer $L \geq 1$. As in \[S2\] and \[Gr\] we find a measurable function $b : X \rightarrow G$ with $b(x) = 1_G$ on $B$ so that the cocycle defined by setting
\[ f'(n, x) = b(T^n x) \cdot f(n, x) \cdot b(x)^{-1} \]
satisfies the following properties at $\mu$-almost every point $x$ in $X$:
\[ g \cdot h^{-1} \notin U \text{ for every two distinct } g, h \in V_x = \{f'(n, x) : n \in \mathbb{Z}\}, \quad (16) \]
and
\[ |\{n \in \mathbb{Z} : f'(n, x) = g\}| = L \quad (17) \]
for every $g \in V_x$.

With help of these properties one proves the following lemma.

**Lemma 2.2.** Let $U$ and $B$ be as above and $W$ be an open neighbourhood of $1_G$ such that $W^{-1} \cdot W \subseteq U$. Then for every $\eta > 0$ and integer $N \geq 1$,
\[ \limsup_{n \to \infty} \tau_n(B_\eta) \leq L \lambda(W)^{-1} \lambda(B_\eta), \quad (18) \]
and
\[ \limsup_{n \to \infty} \sum_{k=0}^N 2^k \tau_{2^n+k}(B_{2^{-k}/2\eta}) \leq \frac{L}{\log 2} \lambda(W)^{-1} \lambda(B_\eta), \quad (19) \]
with $B_\eta = \{z \in \mathbb{C} : |z| < \eta\} \times S^1$, the measures $\tau_n$ being defined by the equations (10) and (11).

**Proof of Lemma 2.2.** First of all note that both relations (18) and (19) are cohomology invariant in the sense that they remain true when replacing $\tau_n$ by the analogously defined measures $\tau'_n$ corresponding to the cocycle $f'(n, x)$ and vice-versa. Indeed, whenever $\alpha_{n-1/2} f(n, x)$ belongs to $B_\eta$ and both $\alpha_{n-1/2} b(T^n x)$ and $\alpha_{n-1/2} b(x)^{-1}$ are in $B_{\eta'/2}$ then
\[ \alpha_{n-1/2} f'(n, x) = \alpha_{n-1/2} b(T^n x) \cdot \alpha_{n-1/2} f(n, x) \cdot \alpha_{n-1/2} b(x)^{-1} \]
is contained in $B_{\eta + \eta'} \times S^1$, for arbitrary $\eta, \eta' > 0$. Together with the contraction property (9), we see immediately that $\limsup_{n \to \infty} \sigma_n(B_\eta) - \sigma'_n(B_{\eta+\eta'}) \leq 0$ and hence
\[ \limsup_{n \to \infty} \tau_n(B_\eta) - \tau'_n(B_{\eta+\eta'}) \leq 0, \]
for every $\eta, \eta' > 0$. By symmetry the same is true when interchanging $\tau_n$ and $\tau'_n$ and our claim is proved.

As immediate consequence of property (16) we conclude the following estimate for almost every $x \in X$ and $n \geq 1$:
\[ |\{1 \leq k \leq n : f'(k, x) \in \alpha_{k/2}(B_\eta)\}| \leq \leq \begin{array}{l}
|\{1 \leq k \leq n : f'(k, x) \in \alpha_{n/2}(B_\eta)\}| \\
\leq L \lambda(W \cdot \alpha_{n/2}(B_\eta)) / \lambda(W) \\
= L \lambda(\alpha_{n/2}(W) \cdot B_\eta) / \lambda(W),
\end{array} \]
since $\mathcal{W}g \cap \mathcal{W}h = \emptyset$ for every two distinct $g, h$ from $V_z$. Integrating this ‘strong’ estimate with respect to the measure $\mu$ we conclude the following ‘weak’ estimate for the growth of $f'(n, \cdot)$:

$$
\tau_n'(B_\eta) = \frac{1}{n} \sum_{k=1}^{n} \sigma_k'(B_\eta) \leq L\lambda(\alpha_{n-1/2}(W) \cap B_\eta) / \lambda(W),
$$

and therefore $\limsup_{n \to \infty} \tau_n'(B_\eta) \leq L\lambda(B_\eta) / \lambda(W)$. By cohomology the same relation holds for $\tau_n$.

We turn to the proof of (19). For any group element $g = (z, e^{\gamma})$ we set $\|g\| = |z|$. Let $\varepsilon > 0$ and choose $r \geq 0$ large enough so that $\alpha_{\varepsilon r}(W) \cdot B_1 \subseteq B_{1 + \varepsilon}$ for every $\eta \geq r$. For arbitrary integer $m \geq 1$ and $\eta > 0$,

$$
\sum_{n=1}^{\infty} \left| \left\{ 1 \leq k \leq 2^n : f'(k, x) \in \alpha_{k/2}(B_{2-n/2\eta}) \setminus B_r \right\} \right|
= \sum_{g \in V_z \setminus B_r} \left| \left\{ n \geq 1 : g = f'(k, x) \text{ for some } k \in (0, 2^n] \text{ with } \|g\| \leq k^{1/2} 2^{-n/2\eta} \right\} \right|
= \sum_{g \in V_z \setminus B_r} \left| \left\{ n \geq 1 : g = f'(k, x) \text{ for some } k \text{ with } k \leq 2^n \leq k\eta^2 / \|g\|^2 \right\} \right|
\leq 2L / \log 2 \sum_{g \in V_z \setminus B_r} \log (1 \vee \eta \|g\|^{-1}),
$$

since $\left| \left\{ n \geq 0 : k \leq 2^n \leq k\eta^2 \|g\|^2 \right\} \right| \leq (2/ \log 2) \cdot \log(1 \vee \eta \|g\|^{-1})$. By our assumption on $r$,

$$
\sup_{h \in W \cdot g} \|h\| / \|g\| \leq \sup_{h \in \alpha_{1\|g\|^2-1}(W) \cdot B_1} \|h\| \leq 1 + \varepsilon
$$

for every $g$ outside $B_r$ and therefore

$$
\sum_{g \in V_z \setminus B_r} \log (1 \vee \eta \|g\|^{-1}) \leq \sum_{g \in V_z \setminus B_r} \inf_{h \in W \cdot g} \log (1 \vee \eta (1 + \varepsilon) \|h\|^{-1})
\leq \lambda(W)^{-1} \int_{G \setminus \{0\} \times S^1} \log (1 \vee \eta (1 + \varepsilon) \|h\|^{-1}) \, d\lambda(h)
= \eta^2 (1 + \varepsilon)^2 \lambda(W)^{-1} \int_{G \setminus \{0\} \times S^1} \log (1 \vee \|h\|^{-1}) \, d\lambda(h),
$$

where $\int_{G \setminus \{0\} \times S^1} \log (1 \vee \|h\|^{-1}) \, d\lambda(h) = \int_{0 < |z| < 1} \log |z|^{-1} \, d\lambda_C(z) = \pi/2$. We thus may conclude that for every $N \geq 1$

$$
\sum_{n=0}^{N} \left| \left\{ 1 \leq k \leq 2^n : f'(k, x) \in \alpha_{k/2}(B_{2-n/2\eta}) \right\} \right|
\leq \sum_{n=0}^{N} \left| \left\{ 1 \leq k \leq 2^n : f'(k, x) \in B_r \right\} \right| +
+ \sum_{n=0}^{N} \left| \left\{ 1 \leq k \leq 2^n : f'(k, x) \in \alpha_{k/2}(B_{2-n/2}) \setminus B_r \right\} \right|
\leq \frac{N L\lambda(W \cdot B_r)}{\lambda(W)} + \frac{L (1 + \varepsilon)^2}{\lambda(W) \log 2} \pi \eta^2
$$
from which we again by integration follow that
\[ \sum_{n=0}^{N} 2^n \tau_{2n}^r (B) \leq \frac{N L \lambda(W \cdot B)}{\lambda(W)} + \frac{L(1 + \varepsilon)^2}{\lambda(W) \log 2} \pi \eta^2. \]
Substituting \( \eta \) by \( 2^{n/2} \eta \), omitting the first \( m \) terms in the series and dividing by \( 2^m \) we arrive at
\[ \sum_{n=0}^{N} 2^n \tau_{2n+m}^r (B) \leq \frac{N L \lambda(W \cdot B)}{2^m \lambda(W)} + \frac{L(1 + \varepsilon)^2}{\lambda(W) \log 2} \pi \eta^2, \]
for every \( m \geq 1 \). This shows that for arbitrary \( N \geq 1 \) and \( \eta > 0 \)
\[ \limsup_{N \to \infty} \sum_{k=0}^{N} 2^k \tau_{2m+k}^r (B) \leq \frac{L(1 + \varepsilon)^2}{\lambda(W) \log 2} \lambda(B). \]
Since \( \varepsilon > 0 \) was arbitrary we conclude by cohomology that \( (19) \) holds.

Proof of Theorem 2.1. The first assertion of the theorem is already contained in Lemma 2.2. Furthermore from
\[ \limsup_{n \to \infty} \sum_{k=0}^{N} 2^{k/2} \tau_{2m+k}^r (B) \leq \frac{4L}{\log 2} \lambda(W)^{-1} \lambda(B), \]
we conclude that
\[ \liminf_{n \to \infty} \tau_{2m+k}^r (B) / \lambda(B) \leq \frac{4L}{N \log 2} \lambda(W)^{-1}, \]
for some \( 0 \leq k \leq N \). As \( N \) was arbitrary equation \( (14) \) holds and the theorem is proved.

Remark 2.3. Of course Theorem 2.1 can be proved in more general setting: Suppose that \( G \) is a locally compact second countable group for which there exists a compact subgroup \( K \leq G \) and a one-parameter group of scaling automorphisms \( \{ \alpha_n \}_{n>0} \) such that
(i) the map \((g, \eta) \mapsto \alpha_\eta(g)\) is jointly continuous,
(ii) \( \alpha_\eta(K) = K \) for every \( \eta > 0 \), thus \( \{ \alpha_\eta \}_{n>0} \) acts on the homogeneous space \( G/K = \{ g \cdot K g \in G \} \) by setting \( \alpha_\eta(g \cdot K) = K \cdot \alpha_\eta(g) \),
(iii) the sets \( \bigcup_{\eta' \leq \eta} \alpha_\eta'(U \cdot K) \) increase to \( G \) as \( \eta \to \infty \), for every open neighbourhood \( U \) of the identity \( 1_G \). In other words the group \( \{ \alpha_\eta \}_{n>0} \) contracts \( G/K \) to its origin as \( \eta \to 0^+ \).

It is an immediate consequence of (i) that the right-invariant Haar measure \( \lambda \) is transformed according to the equation
\[ \lambda \circ \alpha_\eta^{-1} (B) = \eta^d \lambda(B), \]
for every Borel set \( B \subseteq G \), for some fixed constant \( d > 0 \). For the definition of the measures \( \sigma_n \) and \( \tau_n \) the appropriate scaling of the cocycle is then \( n^{-1/d} \) instead of \( n^{-1/2} \). If we fix a relatively compact neighbourhood \( U_0 \) of \( 1_G \) the open sets
\[ B_\eta = \bigcup_{n' \leq \eta} \alpha_{\eta'}(K \cdot U_0 \cdot K), \]
with \( \eta > 0 \), form a basis of \( K \)-invariant neighbourhoods of the origin in \( G/K \). The proof of \( (18) \) in Lemma 2.2 is verbatim and setting \( ||g|| = \inf \{ \eta : g \in B_\eta \} \) in the
second part of the proof of Lemma 2.2 yields the estimate (19) with the constant $L/\log 2$ replaced by $2L/(d \log 2)$, since

$$
\int_{G \setminus \mathcal{K}} \log(1 \vee \|h\|^{-1})d\lambda(h) = \lambda(B_1) \int_0^1 dr^{d-1} \log r^{-1} dr = \lambda(B_1)/d.
$$

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