Group Decision Making Based on Multi-Granular Distribution Linguistic Assessments and Power Aggregation Operators

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Abstract

In this paper, we focus on multi-attribute group decision making problems with multi-granular distribution linguistic assessments. First, the relationship between a linguistic 2-tuple and a distribution linguistic assessment is analyzed. Second, an approach for unifying multi-granular distribution linguistic assessments is proposed based on the extended linguistic hierarchies model. Third, from the relationship between a linguistic 2-tuple and a distribution linguistic assessment and the DAWA operator, a procedure which can represent the aggregated distribution linguistic assessment using initial linguistic term sets is also presented. Fourth, we develop some power aggregation operators, i.e. distribution linguistic power weighted averaging (DLPWA) operator and distribution linguistic power ordered weighted averaging (DLPOWA) operator to aggregate distribution linguistic assessments, which can relieve the influence of unfair assessments on the final aggregated results. Eventually, two novel approaches for group decision making with multi-granular distribution linguistic assessments are proposed before given a numerical example to illustrate the proposed approaches.

Index Terms

Group decision making, multi-granular distribution linguistic assessment, unification, distribution linguistic power aggregation operator.

I. INTRODUCTION

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Group decision making (GDM) is a common activity occurring in human being’s daily life. For a typical multi-attribute group decision making (MAGDM) problem, a group of decision makers are usually required to express their assessments over alternatives with regard to some predefined criteria. Afterwards, the evaluation information is aggregated to form a group opinion, based on which collective evaluation and a ranking of alternatives can be obtained [1], [2]. For traditional GDM problems, decision makers express their evaluation over alternatives with numerical values. However, in most cases decision makers may have vague or uncertain knowledge about alternatives and cannot express their evaluation with exact numerical values. Therefore a more realistic approach may be to use linguistic assessments instead of numerical values [3], [4]. In the last decades, numerous models and approaches have been developed to deal with MAGDM problems with linguistic information [5]–[11].

When dealing with GDM problems with linguistic information, the techniques for computing with words (CWW) are usually needed [12]. As surveyed by Martínez et al. [13], there are different linguistic computational models: the model based on membership functions [14], the model based on type-2 fuzzy sets [15] and the symbolic model [16]–[18]. The 2-tuple linguistic model proposed by Herrera and Martínez [17] is a useful symbolic model to compute with words without loss of information. Since it was developed, it has been widely used in different areas, such as decision making, recommender systems, information retrieval and sensory evaluation [19]. The 2-tuple linguistic model assumes that the linguistic variables used in a linguistic term set are uniformly and symmetrically distributed. However, there are usually problems that need to assess their variables with linguistic term sets that are not uniformly either symmetrically distributed, which is called unbalanced linguistic term sets [20]. Herrera et al. [20] developed a fuzzy linguistic methodology to deal with unbalanced linguistic term sets based on the 2-tuple linguistic model. Meng and Pei [11] generalized linguistic evaluation values and their weights in group decision making problems based on unbalanced linguistic terms and proposed some weighted aggregation operators for GDM. Another technique to deal with linguistic term sets that are not uniformly either symmetrically distributed is the proportional 2-tuple linguistic model proposed by Wang and Hao [21], [22]. In their model, linguistic information is represented by proportional 2-tuples and symbolic proportions are assigned to two successive linguistic terms in a linguistic term set. Dong et al. [23] defined the concept of numerical scale and showed that the 2-tuple linguistic model and the proportional 2-tuple linguistic model can be obtained by setting different numerical scales. In a recent work, Zhang et al. [24] proposed the concept of the distribution linguistic assessment which can be considered as a generalization of the proportional
2-tuple linguistic model. This linguistic representation model contains not only the linguistic assessments of decision makers, but also the support degree (i.e. symbolic proportion) associated with the linguistic terms.

Due to the difference in culture, cognition, experience and education background, decision makers may express their evaluation using linguistic terms from different linguistic term sets, i.e. multi-granular linguistic term sets, in practical GDM problems. Different approaches have been proposed to deal with GDM problems with multi-granular linguistic information [25]–[32]. However, as far as we know there is no work focuses on aggregating multi-granular distribution linguistic assessments, which is quite important for decision making with large groups/organizations and will be the focus of this paper aiming at presenting interpretable final aggregated results to decision makers according to the CWW scheme [13]. To achieve the interpretability of the aggregated results, some new models that represent the aggregated distribution linguistic assessments using the initial linguistic terms will be developed.

In addition to the concept of the distribution linguistic assessment, Zhang et al. [24] also defined some aggregation operators, but these operators do not take into account the information about the relationship between the values being aggregated. Yager [33] developed a power averaging operator and a power ordered weighted averaging operator to provide aggregation tools which allow exact argument values to support each other in the aggregation process. Subsequently, Xu and Yager [34] proposed some geometric aggregation operators, including the power-geometric (PG) operator, weighted PG operator, and power-ordered weighted geometric (POWG) operator. Based on these operators, some other power aggregation operators have been proposed, such as uncertain power aggregation operators [35], linguistic power aggregation operators [36]–[38], intuitionistic fuzzy power aggregation operators [39], [40] and generalized power aggregation operators [41], [42]. However, there is no aggregation operator that considers the relationship of input arguments for distribution linguistic assessments. Therefore, there is a need to develop some power aggregation operators to aggregate distribution linguistic assessments.

The rest of this paper is organized as follows. Section II reviews some preliminaries related to the 2-tuple linguistic representation model and the distribution linguistic assessment. Section III introduces a definition to transform a linguistic 2-tuple into a distribution linguistic assessment. After that, the unification problems of multi-granular distribution linguistic assessments are investigated in section IV. Section V introduces some power aggregation operators including distribution linguistic power weighted averaging (DLPWA) operator and distribution linguistic power ordered weighted averaging (DLPOWA) operator and discuss
their desirable properties. Based on the proposed aggregation operators, two approaches to multi-attribute group decision making with multi-granular distribution linguistic assessments are proposed in section VI. Section VII presents a numerical example to illustrate the approaches. Finally, we conclude this paper in section VIII.

II. PRELIMINARIES

A. 2-tuple linguistic representation model

Let $S = \{s_0, s_1, \ldots, s_{g-1}\}$ denote a linguistic term set with odd cardinality, the element $s_i$ of which represents the $i$th linguistic term in $S$, and $g$ is the cardinality of the linguistic term set $S$. Moreover, for the linguistic term set $S$, we assume that the midterm representing an assessment of “approximately 0.5”, with the rest of the terms being placed uniformly and symmetrically around it. The linguistic term set should satisfy the following characteristics [13], [17]:

1. The set is ordered: $s_i > s_j$, if $i > j$;
2. There is a negation operator: $\text{Neg}(s_i) = s_j$, such that $j = g - 1 - i$;
3. Maximization operator: $\text{max}(s_i, s_j) = s_i$, if $s_i \geq s_j$;
4. Minimization operator: $\text{min}(s_i, s_j) = s_i$, if $s_i \leq s_j$.

To compute with words without loss of information, Herrera and Martínez [17] proposed the 2-tuple linguistic representation model, which is based on the concept of symbolic translation. The model uses a 2-tuple $(s_k, \alpha)$ to represent linguistic information, where $s_k$ is a linguistic term which belongs to the predefined linguistic term set, $\alpha$ denotes the symbolic translation, and $\alpha \in [-0.5, 0.5)$. Specifically, the 2-tuple linguistic representation model is defined as follows.

**Definition 1.** [17] Let $S = \{s_0, s_1, \ldots, s_{g-1}\}$ be a linguistic term set and $\beta \in [0, g - 1]$ be a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to $\beta$ is obtained with the following function:

$$\Delta : [0, g - 1] \rightarrow S \times [-0.5, 0.5)$$

$$\Delta(\beta) = (s_k, \alpha),$$

with $k = \text{round}(\beta)$, $\alpha = \beta - k$, where “round(·)” is the usual round operation, $s_k$ has the closest index label to $\beta$, and $\alpha$ is the value of symbolic translation.
Definition 2. [17] Let $S = \{s_0, s_1, \ldots, s_{g-1}\}$ be a linguistic term set and $(s_k, \alpha)$ be a 2-tuple, there exists a function $\Delta^{-1}$, which can transform a 2-tuple into its equivalent numerical value $\beta \in [0, g - 1]$. The transformation function is defined as

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [0, g - 1]$$

$$\Delta^{-1}(s_k, \alpha) = k + \alpha = \beta.$$  

Based on the above definitions, a linguistic term can be considered as a linguistic 2-tuple by adding the value 0 to it as a symbolic translation, i.e. $s_k \in S \Rightarrow (s_k, 0)$.

To solve multi-granular linguistic decision making problems, Herrera and Martínez [26] introduced the linguistic hierarchies (LH) model. An LH is the union of all levels $i$: $LH = \bigcup_i l(i, g(i))$, where each level $i$ of an LH corresponds to a linguistic term set with a granularity of $g(i)$ denoted as: $S^{g(i)} = \{s_0^{g(i)}, s_1^{g(i)}, \ldots, s_{g(i)-1}^{g(i)}\}$, and a linguistic term set of level $i + 1$ is obtained from its predecessor as $l(i, g(i)) \rightarrow l(i + 1, 2 \cdot g(i) - 1)$. Based on the LH basic rules, Herrera and Martínez [26] defined a transformation function $TF_i$ between any two linguistic levels $i$ and $i'$ of the LH as follows.

Definition 3. [26] Let $LH = \bigcup_i L(i, g(i))$ be an LH whose linguistic term sets are denoted as $S^{g(i)} = \{s_0^{g(i)}, s_1^{g(i)}, \ldots, s_{g(i)-1}^{g(i)}\}$, and let us consider the 2-tuple linguistic representation. The transformation function from a linguistic label in level $i$ to a label in level $i'$, satisfying the LH basic rules, is defined as

$$TF_i^{s_k} (s_k^{g(i)}, \alpha^{g(i)}) = \Delta \left( \frac{\Delta^{-1}(s_k^{g(i)}, \alpha^{g(i)}) \cdot (g(i') - 1)}{g(i) - 1} \right).$$

B. Distribution linguistic assessments

Definition 4. [24] Let $S = \{s_0, s_1, \ldots, s_{g-1}\}$ denote a linguistic term set and $\beta_k$ be the symbolic proportion of $s_k$, where $s_k \in S$, $\beta_k \geq 0$, $k = 0, 1, \ldots, g - 1$ and $\sum_{k=0}^{g-1} \beta_k = 1$, then an assessment $m = \{ (s_k, \beta_k) | k = 0, 1, \ldots, g - 1 \}$ is called a distribution linguistic assessment of $S$, and the expectation of $m$ is defined by $E(m)$, where $E(m) = \Delta \left( \sum_{k=0}^{g-1} \beta_k \right)^1$. For two distribution linguistic assessments $m_1$ and $m_2$, if $E(m_1) \geq E(m_2)$, then $m_1 \geq m_2$.

However, there may be cases that the expectation values of some distribution linguistic assessments are equal. As a result, the comparison rule mentioned in Definition 4 sometimes cannot distinguish these distribution linguistic assessments. Therefore, we define the inaccuracy function of a distribution linguistic

\footnote{The representation is different from the definition provided in [24], but they have the same meaning.}
assessment based on Shannon’s entropy [43] and give the comparison rules of two distribution linguistic assessments.

**Definition 5.** Let \( m = \{\langle s_k, \beta_k \rangle | k = 0, 1, \ldots, g-1\} \) be a distribution linguistic assessment of a linguistic term set \( S = \{s_0, s_1, \ldots, s_{g-1}\} \), where \( s_k \in S \), \( \beta_k \geq 0 \), \( k = 0, 1, \ldots, g-1 \) and \( \sum_{k=0}^{g-1} \beta_k = 1 \). The inaccuracy function of \( m \) is defined as \( H(m) = -\sum_{k=0}^{g-1} \beta_k \log_2 \beta_k \).

**Definition 6.** Let \( m_1 \) and \( m_2 \) be two distribution linguistic assessments, then the comparison rules are defined as follows: (1) If \( E(m_1) > E(m_2) \), then \( m_1 > m_2 \); (2) If \( E(m_1) = E(m_2) \) and \( H(m_1) < H(m_2) \), then \( m_1 > m_2 \); If \( E(m_1) = E(m_2) \) and \( H(m_1) = H(m_2) \), then \( m_1 = m_2 \).

**Example 1.** Let \( S_{\text{example}} = \{s_0, s_1, \ldots, s_4\} \) be a linguistic term set and there are three distribution linguistic assessments: \( m_1 = \{\langle s_1, 0.3 \rangle, \langle s_2, 0.4 \rangle, \langle s_3, 0.3 \rangle\} \), \( m_2 = \{\langle s_2, 1 \rangle\} \) and \( m_3 = \{\langle s_1, 0.3 \rangle, \langle s_2, 0.7 \rangle\} \).

By Definition 4, we have \( E(m_1) = (s_2, 0) \), \( E(m_2) = (s_2, 0) \), \( E(m_3) = (s_2, -0.3) \). Therefore, \( m_1 = m_2 > m_3 \). However, if we calculate the inaccuracy function values of the three distribution linguistic assessments, we have \( H(m_1) = 1.5710 \), \( H(m_2) = 0 \), \( H(m_3) = 0.8813 \). According to Definition 6, it follows that \( m_2 > m_1 > m_3 \). Obviously, the new comparison rules can distinguish distribution linguistic assessments more effectively.

**Definition 7.** [24] Let \( \{m_1, m_2, \ldots, m_n\} \) be a set of distribution linguistic assessments of \( S \), where \( m_i = \{\langle s_k, \beta_{ik} \rangle | k = 0, 1, \ldots, g-1\} \), \( i = 1, 2, \ldots, n \), and \( w = (w_1, w_2, \ldots, w_n)^T \) be an associated weighting vector that satisfies \( w_i \geq 0 \) and \( \sum_{i=1}^{n} w_i = 1 \), then the weighted averaging operator of \( \{m_1, m_2, \ldots, m_n\} \) is defined as

\[
\text{DAWA}_w(m_1, m_2, \ldots, m_n) = \{\langle s_k, \beta_k \rangle | k = 0, 1, \ldots, g-1\},
\]

where \( \beta_k = \sum_{i=1}^{n} w_i \beta_{ik}, k = 0, 1, \ldots, g-1 \).

**Definition 8.** [24] Let \( \{m_1, m_2, \ldots, m_n\} \) be a set of distribution linguistic assessments of \( S \), where \( m_i = \{\langle s_k, \beta_{ik} \rangle | k = 0, 1, \ldots, g-1\} \), \( i = 1, 2, \ldots, n \), and \( w = (w_1, w_2, \ldots, w_n)^T \) be an associated weighting vector that satisfies \( w_i \geq 0 \) and \( \sum_{i=1}^{n} w_i = 1 \), then the ordered weighted averaging operator of \( \{m_1, m_2, \ldots, m_n\} \) is defined as

\[
\text{DAOWA}_w(m_1, m_2, \ldots, m_n) = \{\langle s_k, \beta_k \rangle | k = 0, 1, \ldots, g-1\},
\]

\(^20 \log_2 0 = 0 \) is defined in this paper.
where $\beta_k = \sum_{i=1}^{n} w_i \beta^{(i)}$ and $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ is a permutation of $\{1, 2, \ldots, n\}$ such that $m_{\sigma(i-1)} \geq m_{\sigma(i)}$ for $i = 2, \ldots, n$.

III. TRANSFORMING A LINGUISTIC 2-TUPLE INTO A DISTRIBUTION LINGUISTIC ASSESSMENT

Due to the fact that our proposal for dealing with multi-granular distribution linguistic assessments and obtaining interpretable results will be based on tools introduced for linguistic 2-tuple values, this section shows how to transform a linguistic 2-tuple into a distribution linguistic assessment. For convenience, let $S = \{s_0, s_1, \ldots, s_{g-1}\}$ be a linguistic term set as defined in section II and $(s_k, \alpha)$ be a linguistic 2-tuple. By the definition of the linguistic 2-tuple, we have the following results.

1. If $\alpha > 0$, $(s_k, \alpha)$ denotes the linguistic information between $s_k$ and $s_{k+1}$.
2. If $\alpha < 0$, $(s_k, \alpha)$ denotes the linguistic information between $s_{k-1}$ and $s_k$.
3. If $\alpha = 0$, $(s_k, \alpha)$ denotes the linguistic information $s_k$.

**Proposition 1.** Let $l$ be the integer part of $\Delta^{-1}(s_k, \alpha)$, then a linguistic 2-tuple $(s_k, \alpha)$ denotes the linguistic information between $s_l$ and $s_{l+1}$ if $\alpha \neq 0$.

**Proof:** We consider two cases.

Case 1: $\alpha > 0$. In this case, $\Delta^{-1}(s_k, \alpha) = k + \alpha > k$. Hence, $l = k$ and $l + 1 = k + 1$.

Case 2: $\alpha < 0$. In this case, $\Delta^{-1}(s_k, \alpha) = k + \alpha < k$. Hence, $l = k - 1$ and $l + 1 = k$.

By the aforementioned results, a linguistic 2-tuple $(s_k, \alpha)$ denotes the linguistic information between $s_l$ and $s_{l+1}$ if $\alpha \neq 0$. This completes the proof of Proposition 1.

Proposition 1 demonstrates that a linguistic 2-tuple $(s_k, \alpha)$, $(\alpha \neq 0)$ can denote the linguistic information between two successive linguistic terms $s_l$ and $s_{l+1}$. From the perspective of distribution linguistic assessments, the linguistic information between $s_l$ and $s_{l+1}$ should be denoted as a distribution linguistic assessment $m = \{(s_l, 1 - \beta), (s_{l+1}, \beta)\}$. The question which needs to be solved is to determine the value of $\beta$.

As the linguistic information between $(s_k, \alpha)$ and $m$ is equivalent, the expectation of $m$ should be equal to $(s_k, \alpha)$. Therefore, we have $\Delta(l \times (1 - \beta) + (l + 1) \times \beta) = (s_k, \alpha)$, i.e.

$$l \times (1 - \beta) + (l + 1) \times \beta = \Delta^{-1}(s_k, \alpha). \quad (6)$$

By solving (6), we have $\beta = \Delta^{-1}(s_k, \alpha) - l$. 
From the above analysis, a linguistic 2-tuple \((s_k, \alpha), (\alpha \neq 0)\) can be denoted as a distribution linguistic assessment \(m = \{(s_l, 1 - \beta), (s_{l+1}, \beta)\}\), where \(l\) is the integer part of \(\Delta^{-1}(s_k, \alpha)\) and \(\beta = \Delta^{-1}(s_k, \alpha) - l\).

It is easy to verify that the above statement also holds for the case \(\alpha = 0\), i.e. a linguistic 2-tuple \((s_k, 0)\) can be denoted as a distribution linguistic assessment \(m = \{(s_l, 1 - \beta), (s_{l+1}, \beta)\}\), where \(l = k\) and \(\beta = 0\).

**Definition 9.** Let \(S = \{s_0, s_1, \ldots, s_{g-1}\}\) be a linguistic term set and \(\Omega\) be the set of all the distribution linguistic assessments of \(S\), and there exists a function \(F\), which can transform a linguistic 2-tuple \((s_k, \alpha)\) into its equivalent distribution linguistic assessment. The transformation function is defined as

\[
F : S \times [-0.5, 0.5) \to \Omega
\]

\[
F(s_k, \alpha) = \{(s_l, 1 - \beta), (s_{l+1}, \beta)\},
\]

where \(l\) is the integer part of \(\Delta^{-1}(s_k, \alpha)\) and \(\beta = \Delta^{-1}(s_k, \alpha) - l\).

For Definition 9, we give the following theorem.

**Theorem 1.** Let \(S = \{s_0, s_1, \ldots, s_{g-1}\}\) be a linguistic term set. The equivalent distribution linguistic assessment of a linguistic 2-tuple \((s_k, \alpha)\) is

\[
F(s_k, \alpha) = \begin{cases} 
\{(s_k, 1 - \alpha), (s_{k+1}, \alpha)\} & \text{if } \alpha \geq 0, \\
\{(s_{k-1}, -\alpha), (s_k, 1 + \alpha)\} & \text{if } \alpha < 0.
\end{cases}
\]

**Proof:** If \(\alpha \geq 0\), \(\Delta^{-1}(s_k, \alpha) \geq k\), then \(l = k\) and \(\beta = \Delta^{-1}(s_k, \alpha) - l = k + \alpha - k = \alpha\). By (7),

\[
F(s_k, \alpha) = \{(s_k, 1 - \alpha), (s_{k+1}, \alpha)\}.
\]

If \(\alpha < 0\), \(\Delta^{-1}(s_k, \alpha) < k\), then \(l = k - 1\) and \(\beta = \Delta^{-1}(s_k, \alpha) - l = k + \alpha - k + 1 = 1 + \alpha\). By (7),

\[
F(s_k, \alpha) = \{(s_{k-1}, -\alpha), (s_k, 1 + \alpha)\}.
\]

This completes the proof of Theorem 1.

Definition 9 and Theorem 1 establish the relationship between a linguistic 2-tuple and a distribution linguistic assessment, which will be helpful in the following section.

**IV. Unifying multi-granular distribution linguistic assessments**

Due to the difference in culture, cognition, experience and education background, decision makers may express their evaluation using linguistic terms from different linguistic terms sets, i.e. multi-granular linguistic term sets. For some complex group decision making problems, decision makers from different
organizations may be involved. For different organizations, different linguistic term sets may be used when providing the preference information. After gathering evaluation values from different organizations, the information in hand will be multi-granular distribution linguistic assessments. Thus a new problem which needs to be solved is how to aggregate such information. One solution is to unify them into a unique granularity.

Espinilla et al. [29] introduced an extended linguistic hierarchies (ELH) model which can fuse linguistic information with any linguistic scale. The essence of the ELH model is to map each linguistic term to a higher linguistic term set without loss of information. In what follows, we will use the ELH model to unify multi-granular distribution linguistic assessments.

**Proposition 2.** [29] Let \( \{S^{g(1)}, S^{g(2)}, \ldots, S^{g(n)}\} \) be a set of linguistic term sets, where the granularity \( g(i) \) is an odd value, \( i = 1, 2, \ldots, n \). A new linguistic term set \( S^{g(i^*)} \) with \( i^* = n + 1 \) that keeps all the formal modal points of the \( n \) linguistic term sets has the minimal granularity:

\[
g(i^*) = LCM(\delta_1, \delta_2, \ldots, \delta_n) + 1, \tag{9}
\]

where \( LCM \) is the least common multiple and \( \delta_i = g(i) - 1, i = 1, 2, \ldots, n \).

Based on Proposition 2, the linguistic terms from linguistic term sets with different granularities can be transformed into the linguistic terms in a linguistic term set \( S^{g(i^*)} = \{s^{g(i^*)}_0, s^{g(i^*)}_1, \ldots, s^{g(i^*)}_{g(i^*)-1}\} \) by Definition 3. For multi-granular distribution linguistic assessments, we give the following definition.

**Definition 10.** Let \( \{S^{g(1)}, S^{g(2)}, \ldots, S^{g(n)}\} \) be a set of linguistic term sets, where the granularity \( g(i) \) with \( i = 1, 2, \ldots, n \) is an odd value, and \( \{m_1, m_2, \ldots, m_n\} \) be a set of distribution linguistic assessments, where \( m_i = \{s_k^{g(i)}, \beta_k^i\}|k = 0, 1, \ldots, g(i) - 1\}, i = 1, 2, \ldots, n \), then each \( m_i \) can be transformed into a distribution linguistic assessment with a granularity of \( g(i^*) \) by

\[
m'_i = \{s_k^{g(i^*)}, \gamma_k^i\}|k = 0, 1, \ldots, g(i^*) - 1\}, i = 1, 2, \ldots, n, \tag{10}
\]

with

\[
\gamma_k^i = \begin{cases} 
\beta_{l(i,k)}^i & \text{if } l(i, k) \in \{0, 1, \ldots, g(i) - 1\}; \\
0 & \text{if } l(i, k) \notin \{0, 1, \ldots, g(i) - 1\}, \tag{11}
\end{cases}
\]

where \( l(i, k) = \frac{k \cdot (g(i) - 1)}{g(i^*) - 1}, k = 0, 1, \ldots, g(i^*) - 1, i = 1, 2, \ldots, n \).
Theorem 2. The transformed \( m'_i \) is a distribution linguistic assessment of \( S^{g(i^*)} \).

Proof: According to [29], the transformation from \( S^{g(i)} \) to \( S^{g(i^*)} \) is one-to-one, i.e. each linguistic term of \( S^{g(i)} \) corresponds to a linguistic term of \( S^{g(i^*)} \). Specifically, we have

\[
\left\{ s_0^{g(i)}, s_1^{g(i)}, \ldots, s_{g(i)-1}^{g(i)} \right\} \leftrightarrow \left\{ s_0^{g(i^*)}, s_1^{g(i^*)} \ldots, s_{g(i^*)-1}^{g(i^*)} \right\}. \tag{12}
\]

If \( k = 0, \frac{g(i^*) - 1}{g(i) - 1}, 2, \frac{g(i^*) - 1}{g(i) - 1}, \ldots, \frac{g(i^*) - 1}{g(i) - 1} \), then \( l(i, k) = 0, 1, 2, \ldots, g(i) - 1 \) and \( \gamma_k = \beta_0, \beta_1, \beta_2, \ldots, \beta_{g(i)-1} \). Thus we have \( \sum_{l(i, k) \in \{0, 1, \ldots, g(i)-1\}} \gamma_k = \sum_{k=0}^{g(i)-1} \beta_k = 1. \)

Since \( \gamma_k = 0, \forall l(i, k) \notin \{0, 1, \ldots, g(i) - 1\}, \) we have \( \sum_{k=0}^{g(i)-1} \gamma_k = 1. \) Therefore, \( m'_i \) is a distribution linguistic assessment of \( S^{g(i^*)} \), which completes the proof of Theorem 2.

We give an example to illustrate Definition 10.

Example 2. Consider three distribution linguistic assessments: \( m_1 = \{ (s_{1}^{5}, 0.3), (s_{2}^{5}, 0.5), (s_{3}^{5}, 0.2) \}, m_2 = \{ (s_{1}^{9}, 0.25), (s_{2}^{9}, 0.3), (s_{3}^{9}, 0.45) \}, m_3 = \{ (s_{2}^{6}, 0.2), (s_{3}^{6}, 0.8) \}. \)

By Proposition 2, we can calculate \( g(i^*) = LCM(4, 6, 6) + 1 = 13. \) For \( m_1 \), since \( 3 \times 4/12 = 1, \) \( 6 \times 4/12 = 2, \) \( 9 \times 4/12 = 3 \), then we have \( \gamma_{3} = \beta_{1} = 0.3, \gamma_{6} = \beta_{2} = 0.5, \gamma_{9} = \beta_{3} = 0.2, \) i.e. \( m'_1 = \{ (s_{3}^{13}, 0.3), (s_{13}^{13}, 0.5), (s_{9}^{13}, 0.2) \}. \) Similarly, we can obtain \( m'_2 = \{ (s_{2}^{13}, 0.25), (s_{4}^{13}, 0.3), (s_{6}^{13}, 0.45) \} \) and \( m'_3 = \{ (s_{2}^{13}, 0.2), (s_{3}^{13}, 0.8) \} \).

Let \( w = (1/3, 1/3, 1/3)^T \) for Example 2, we can then aggregate the three distribution linguistic assessments into a collective one by the DAWA operator, i.e.

\[
m = DAWA_w(m_1, m_2, m_3) = \{ (s_{2}^{12}, 0.083), (s_{3}^{13}, 0.1), (s_{4}^{13}, 0.167), (s_{6}^{13}, 0.583), (s_{9}^{13}, 0.067) \}.
\]

For group decision making problems, we usually need to present the final aggregated results to decision makers. However, the aggregated results are not in the form of the initial linguistic terms, which are difficult to be understood by decision makers. Therefore, we need to consider how to represent the aggregated results using initial linguistic terms. Intuitively we can transform each linguistic term using Definition 3 and attach it with the corresponding symbolic proportion (see Definition 11). However, the transformed result may not be a normative distribution linguistic assessment since the mapping is from a higher linguistic scale to a lower one. Take Example 2 as an example, the distribution linguistic assessment \( m \) can be denoted as \( \{ (s_{1}^{7}, 0.083), (s_{2}^{7}, -0.5), (s_{3}^{7}, 0.167), (s_{9}^{7}, 0.583), (s_{5}^{7}, -0.5), (0.067) \} \). Therefore the transformed distribution linguistic assessment is in the form of linguistic 2-tuples, which is not consistent
with the intuition. To address this issue, we first give the definition of distribution 2-tuple linguistic assessment.

**Definition 11.** Let \( \{S^{g(1)}, S^{g(2)}, \ldots, S^{g(n)}\} \) be a set of linguistic term sets, where the granularity \( g(i) \) with \( i = 1, 2, \ldots, n \) is an odd value, and \( m = \{\langle s_k^{g(i*+1)}, \beta_k \rangle | k = 0, 1, \ldots, g(i*) - 1\} \) be an aggregated distribution linguistic assessment of a linguistic term set \( S^{g(i*)} = \{s_0^{g(i*)}, s_1^{g(i*)}, \ldots, s_{g(i*)-1}^{g(i*)}\} \), where \( g(i*) \) is calculated by (9), then \( m \) can be transformed into a distribution 2-tuple linguistic assessment

\[
m_i = \left\{ \left( T F_i^{g(i*)}(s_k^{g(i*)}, 0), \beta_k \right) | k = 0, 1, \ldots, g(i*) - 1 \right\},
\]

(13)

with \( T F_i^{g(i*)}(s_k^{g(i*)}, 0) = \Delta \left( \frac{k \cdot (g(i) - 1)}{g(i*) - 1} \right), k = 0, 1, \ldots, g(i*) - 1, i = 1, 2, \ldots, n. \)

Following, we attempt to transform a distribution 2-tuple linguistic assessment into a normative distribution linguistic assessment. The basic idea is as follows. First, we transform each linguistic 2-tuple derived by (13) into its equivalent distribution linguistic assessment using Definition 9 and obtain \( g(i*) \) distribution linguistic assessments of \( S^{g(i)} \). Afterwards, we aggregate these distribution linguistic assessments into a new one by the DAWA operator. The final result is a normative distribution linguistic assessment of \( S^{g(i)} \).

The procedures are illustrated as follows.

First, we consider how to transform \( T F_i^{g(i*)}(s_k^{g(i*)}, 0) \) into its equivalent distribution linguistic assessment. By applying \( \Delta^{-1} \) on \( T F_i^{g(i*)}(s_k^{g(i*)}, 0) \), we have

\[
\Delta^{-1} \left( T F_i^{g(i*)}(s_k^{g(i*)}, 0) \right) = \frac{k \cdot (g(i) - 1)}{g(i*) - 1}.
\]

(14)

Let \( l'(i, k) \) denote the integer part of \( \frac{k \cdot (g(i) - 1)}{g(i*) - 1} \) and \( \theta(i, k) = \frac{k \cdot (g(i) - 1)}{g(i*) - 1} - l'(i, k) \). By Definition 9, \( T F_i^{g(i*)}(s_k^{g(i*)}, 0) \) can be denoted as a distribution linguistic assessment

\[
m(i, k) = \{\langle s_{l'(i, k)}^{g(i)}, 1 - \theta(i, k) \rangle, \langle s_{l'(i, k)+1}^{g(i)}, \theta(i, k) \rangle\}.
\]

(15)

Considering the symbolic proportion of each \( T F_i^{g(i*)}(s_k^{g(i*)}, 0) \), the distribution 2-tuple linguistic assessment \( m_i \) can be considered as the weighted average of all the \( m(i, k), k = 0, 1, \ldots, g(i*) - 1 \). Based on the above analysis, we can denote the aggregated distribution linguistic assessment \( m \) with a granularity of \( g(i*) \) using the linguistic terms of the initial linguistic term sets.

**Definition 12.** Let \( \{S^{g(1)}, S^{g(2)}, \ldots, S^{g(n)}\} \) be a set of linguistic term sets, where the granularity \( g(i) \) with \( i = 1, 2, \ldots, n \) is an odd value, and \( m = \{\langle s_k^{g(i*)}, \beta_k \rangle | k = 0, 1, \ldots, g(i*) - 1\} \) be an aggregated distribution
Fig. 1. Flowchart of the transformation

linguistic assessment of a linguistic term set \( S^{g(i^*)} = \{ s_0^{g(i^*)}, s_1^{g(i^*)}, \ldots, s_n^{g(i^*)-1} \} \), where \( g(i^*) \) is calculated by (9), then \( m \) can be denoted by the \( i \)th linguistic term set as

\[
m_i'' = \text{DAWA}_\omega (m(i, 0), m(i, 1), \ldots, m(i, g(i^*) - 1)),
\]

where \( \omega = (\beta_0, \beta_1, \ldots, \beta_{g(i^*)-1})^T \) and \( m(i, k), k = 0, 1, \ldots, g(i^*) - 1 \) is calculated by (15).

The procedures of the transformation are illustrated by Fig. 1.

**Theorem 3.** \( m_i'' \) derived by Definition 12 is a distribution linguistic assessment.

**Proof:** Based on the above analysis, we have that each \( m(i, k), k = 0, 1, \ldots, g(i^*) - 1 \) is a distribution linguistic assessment. Moreover, \( \sum_{k=0}^{g(i^*)-1} \beta_k = 1 \). Since the weighted average of some distribution linguistic assessments is also a distribution linguistic assessment [24], \( m_i'' \) is a distribution linguistic assessment.

In the rest of this section, we use Example 3 to illustrate Definition 12.

**Example 3.** Let us consider the aggregated distribution linguistic assessment \( m = \{ (s^{\frac{13}{2}}, 0.083), (s^{\frac{13}{3}}, 0.1), (s^{\frac{13}{4}}, 0.167), (s^{\frac{13}{5}}, 0.583), (s^{\frac{13}{6}}, 0.067) \} \) of Example 2. As aforementioned, it can be denoted as a distribution 2-tuple linguistic assessment \( m_2 = \{ (s^7_0, 0.083), (s^7_0, -0.5, 0.1), (s^7_2, 0.167), (s^7_3, 0.583), (s^7_5, -0.5, 0.067) \} \).

By Definition 11 and (15), we have \( m(i, 0) = \{ (s^7_0, 0) \}, m(i, 1) = \{ (s^7_0, 0.5), (s^7_1, 0.5) \}, m(i, 2) = \{ (s^7_1, 1) \}, m(i, 3) = \{ (s^7_5, 0.5), (s^7_5, 0.5) \}, m(i, 4) = \{ (s^7_1, 1) \}, m(i, 5) = \{ (s^7_5, 0.5), (s^7_5, 0.5) \}, m(i, 6) = \{ (s^7_1, 1) \}, m(i, 7) = \{ (s^7_5, 0.5), (s^7_5, 0.5) \}, m(i, 8) = \{ (s^7_1, 0) \}, m(i, 9) = \{ (s^7_5, 0.5), (s^7_5, 0.5) \}, m(i, 10) = \{ (s^7_5, 0) \}, m(i, 11) = \{ (s^7_5, 0.5), (s^7_5, 0.5) \}, m(i, 12) = \{ (s^7_1, 0) \} \). Moreover, the weight vector \( \omega = (0, 0, 0.083, 0.1, 0.167, 0, 0.583, 0, 0, 0.067, 0, 0, 0)^T \).
By the DAWA operator, we can denote \( m \) using the linguistic term set \( S^2 = \{ s_0^7, s_1^7, \ldots, s_6^7 \} \) as \( m_i'' = \{ \langle s_1^7, 0.133 \rangle, \langle s_2^7, 0.217 \rangle, \langle s_3^7, 0.583 \rangle, \langle s_4^7, 0.0335 \rangle, \langle s_5^7, 0.0335 \rangle \} \).

\[ \text{V. DISTRIBUTION LINGUISTIC POWER AGGREGATION OPERATORS} \]

In this section, we extend the power aggregation (PWA and POWA) operators proposed in [33] to distribution linguistic assessment environment. We first discuss the distance between two distribution linguistic assessments.

\[ \text{A. Distance between two distribution linguistic assessments} \]

Zhang et al. [24] proposed a formula to calculate the distance between two distribution linguistic assessments as follows.

**Definition 13.** [24] Let \( m_1 = \{ \langle s_k, \beta^1_k \rangle | k = 0, 1, \ldots, g - 1 \} \) and \( m_2 = \{ \langle s_k, \beta^2_k \rangle | k = 0, 1, \ldots, g - 1 \} \) be two distribution assessments of a linguistic term set \( S \), then the distance between \( m_1 \) and \( m_2 \) is defined as

\[
d(m_1, m_2) = \frac{1}{2} \sum_{k=0}^{g-1} |\beta^1_k - \beta^2_k|. \quad (17)
\]

However, we find the definition has some drawbacks, which can be illustrated with Example 4.

**Example 4.** Let \( S_{\text{example}} = \{ s_0, s_1, \ldots, s_4 \} \) be a linguistic term set and there are three distribution linguistic assessments: \( m_1 = \{ \langle s_0, 0 \rangle, \langle s_1, 1 \rangle, \langle s_2, 0 \rangle, \langle s_3, 0 \rangle, \langle s_4, 0 \rangle \} \), \( m_2 = \{ \langle s_0, 1 \rangle, \langle s_1, 0 \rangle, \langle s_2, 0 \rangle, \langle s_3, 0 \rangle, \langle s_4, 0 \rangle \} \) and \( m_3 = \{ \langle s_0, 0 \rangle, \langle s_1, 0 \rangle, \langle s_2, 0 \rangle, \langle s_3, 0 \rangle, \langle s_4, 1 \rangle \} \). By the definition of the distribution linguistic assessment, we know that a linguistic term \( s_i \) of \( S \) is a special case of the distribution linguistic assessment \( m = \{ \langle s_k, \beta_k \rangle | k = 0, 1, \ldots, g - 1 \} \) with \( \beta_i = 1 \) and \( \beta_k = 0 \), for all \( k \neq i \), i.e. \( m_1 = s_1 \), \( m_2 = s_0 \) and \( m_3 = s_4 \). However, by Definition 13, we can obtain

\[
d(m_1, m_2) = \frac{1}{2} (1 + 1 + 0 + 0 + 0) = 1, \quad d(m_1, m_3) = \frac{1}{2} (0 + 1 + 0 + 0 + 1) = 1,
\]

which means that the distance between \( s_1 \) and \( s_0 \) is equal to that between \( s_1 \) and \( s_4 \). Obviously it is unreasonable.

From Definition 13, we can see that (17) just calculates the deviation between symbolic proportions and ignores the importance of linguistic terms. In this paper we call (17) the deviation between two
distribution linguistic assessments and present a new definition of the distance between two distribution linguistic assessments as follows.

**Definition 14.** Let \( m_1 = \{ \langle s_k, \beta^1_k \rangle | k = 0, 1, \ldots, g - 1 \} \) and \( m_2 = \{ \langle s_k, \beta^2_k \rangle | k = 0, 1, \ldots, g - 1 \} \) be two distribution assessments of a linguistic term set \( S \), then the distance between \( m_1 \) and \( m_2 \) is defined as

\[
d(m_1, m_2) = \frac{1}{g - 1} \left| \sum_{k=0}^{g-1} (\beta^1_k - \beta^2_k) \right|.
\] (18)

Reconsider Example 4. By Definition 14, we can calculate \( d(m_1, m_2) = 0.25 \), \( d(m_1, m_3) = 0.75 \), which is more reasonable to the intuition.

**B. Distribution linguistic power weighted averaging (DLPWA) operator**

Yager [33] introduced a nonlinear weighted averaging aggregation operator called power weighted averaging (PWA) operator, which is defined as

\[
PWA(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n} w_i (1 + T(a_i)) a_i}{\sum_{i=1}^{n} w_i (1 + T(a_i))}
\] (19)

where \( T(a_i) = \sum_{j=1}^{n} w_j \text{Sup}(a_i, a_j) \) is the support for \( a_i \) from \( a_j \) and \( a_i(i = 1, 2, \ldots, n) \) is a collection of arguments, \( w_i(i = 1, 2, \ldots, n) \) is the weight of each \( a_i \) which satisfies \( w_i \geq 0, i = 1, 2, \ldots, n \) and \( \sum_{i=1}^{n} w_i = 1 \).

Based on the PWA operator, we define the distribution linguistic power weighted averaging operator as follows.

**Definition 15.** Let \( \{ m_1, m_2, \ldots, m_n \} \) be a set of distribution linguistic assessments of \( S \), where \( m_i = \{ \langle s_k, \beta^i_k \rangle | k = 0, 1, \ldots, g - 1 \} \), \( i = 1, 2, \ldots, n \), and \( w = (w_1, w_2, \ldots, w_n)^T \) be an associated weighting vector that satisfies \( w_i \geq 0, i = 1, 2, \ldots, n \) and \( \sum_{i=1}^{n} w_i = 1 \), then the distribution linguistic power weighted averaging (DLPWA) operator is defined as

\[
\text{DLPWA}_w(m_1, m_2, \ldots, m_n) = \{ \langle s_k, \beta_k \rangle | k = 0, 1, \ldots, g - 1 \},
\] (20)

with \( \beta_k = \frac{\sum_{i=1}^{n} w_i (1 + T(m_i)) \beta^i_k}{\sum_{i=1}^{n} w_i (1 + T(m_i))} \), \( k = 0, 1, \ldots, g - 1 \) and

\[
T(m_i) = \sum_{\substack{j=1 \atop j \neq i}}^{n} w_j \text{Sup}(m_i, m_j).
\] (21)
Moreover, $\text{Sup}(m_i, m_j)$ is the support for distribution linguistic assessment $m_i$ from $m_j$, which satisfies the following conditions:

1. $\text{Sup}(m_i, m_j) \in [0, 1]$;
2. $\text{Sup}(m_i, m_j) = \text{Sup}(m_j, m_i)$;
3. $\text{Sup}(m_i, m_j) \geq \text{Sup}(m_s, m_t)$, if $d(m_i, m_j) < d(m_s, m_t)$, where $d$ is a distance measure for distribution linguistic assessments which can be calculated by (18).

Obviously, the support (i.e. Sup) measure is essentially a similarity index. The more the similarity, the closer the two distribution linguistic assessments are, and the more they support each other.

In what follows, we discuss the properties of the DLPWA operator.

**Theorem 4.** Let $\text{Sup}(m_i, m_j) = b$, for all $i \neq j$, we have $\text{DLPWA}_w(m_1, m_2, \ldots, m_n) = \{(s_k, \beta_k), k = 0, 1, \ldots, g - 1\}$, where

$$
\beta_k = \frac{\sum_{i=1}^{n} w_i (1 + b \sum_{j=1}^{n} w_j) \beta^i_k}{\sum_{i=1}^{n} w_i (1 + b \sum_{j=1}^{n} w_j)}. \quad (22)
$$

**Proof:** Since $\text{Sup}(m_i, m_j) = b$, for all $i \neq j$, we have $T(m_i) = \sum_{j=1}^{n} w_j \text{Sup}(m_i, m_j) = b \sum_{j=1}^{n} w_j$. By Definition 15, (22) holds, which completes the proof of Theorem 4.

**Theorem 5.** (Idempotency). Let $\{m_1, m_2, \ldots, m_n\}$ be a set of distribution linguistic assessments of $S$. If $m_1 = m_2 = \ldots = m_n = m$, then

$$
\text{DLPWA}_w(m_1, m_2, \ldots, m_n) = m. \quad (23)
$$

**Proof:** Since $m_1 = m_2 = \ldots = m_n = m$, let $m = \{(s_k, b_k) | k = 0, 1, \ldots, g - 1\}$, then $\beta^1_k = \beta^2_k = \ldots = \beta^n_k = b_k$, $k = 0, 1, \ldots, g - 1$. It follows that

$$
\beta_k = \frac{\sum_{i=1}^{n} w_i (1 + T(m_i)) \beta^i_k}{\sum_{i=1}^{n} w_i (1 + T(m_i))} = \frac{\sum_{i=1}^{n} w_i (1 + T(m_i)) b_k}{\sum_{i=1}^{n} w_i (1 + T(m_i))} = b_k, \quad k = 0, 1, \ldots, g - 1.
$$

Thus $\text{DLPWA}_w(m_1, m_2, \ldots, m_n) = \{(s_k, b_k) | k = 0, 1, \ldots, g - 1\} = m$. This completes the proof of Theorem 5.
Theorem 6. (Boundary). Let \( \{m_1, m_2, \ldots, m_n\} \) be a set of distribution linguistic assessments of \( S \), then

\[
\min_i \{m_i\} \leq \text{DLPWA}_w(m_1, m_2, \ldots, m_n) \leq \max_i \{m_i\}. \tag{25}
\]

**Proof:** Let \( w_i(1 + T(m_i)) = \lambda_i \). According to the definition of distribution linguistic assessment’s expectation, we have

\[
\Delta^{-1}(E(DLPWA_w(m_1, m_2, \ldots, m_n))) = \frac{1}{\sum_{i=1}^{n} \lambda_i} \sum_{i=1}^{n} \lambda_i \Delta^{-1}(E(m_i)) = \Delta^{-1}(E(DLPWA_w(m_1, m_2, \ldots, m_n)))
\]

By using the \( \Delta \) operation, we have \( E(\min_i \{m_i\}) \leq E(DLPWA_w(m_1, m_2, \ldots, m_n)) \leq E(\max_i \{m_i\}) \).

Thus (25) holds, which completes the proof of Theorem 6. \( \blacksquare \)

Particularly, if \( w = (1/n, 1/n, \ldots, 1/n)^T \), then the DLPWA operator is reduced to a distribution linguistic power averaging (DLPA) operator as

\[
\text{DLPA}(m_1, m_2, \ldots, m_n) = \{ (s_k, \beta_k) | k = 0, 1, \ldots, g - 1 \}, \tag{26}
\]

where \( \beta_k = \sum_{i=1}^{n} \frac{(1 + T'(m_i)) \beta_k^i}{\sum_{i=1}^{n} (1 + T'(m_i))} \), \( k = 0, 1, \ldots, g - 1 \) and \( T'(m_i) = \frac{1}{n} \sum_{j=1, j \neq i}^{n} \text{Sup}(m_i, m_j) \).

For the DLPA operator, we have the following property.

Theorem 7. (Commutativity). Let \( \{m_1, m_2, \ldots, m_n\} \) be a set of distribution linguistic assessments of \( S \) and \( \{\sigma(1), \sigma(2), \ldots, \sigma(n)\} \) is a permutation of \( \{1, 2, \ldots, n\} \), then

\[
\text{DLPA}(m_1, m_2, \ldots, m_n) = \text{DLPA}(m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(n)}). \tag{27}
\]

**Proof:** Let \( \text{DLPA}(m_1, m_2, \ldots, m_n) = \{ (s_k, b_k) | k = 0, 1, \ldots, g - 1 \} \) and \( \text{DLPA}(m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(n)}) = \{ (s_k, c_k) | k = 0, 1, \ldots, g - 1 \} \).
Since \( \{\sigma(1), \sigma(2), \ldots, \sigma(n)\} \) is a permutation of \( \{1, 2, \ldots, n\} \), there exists one and only one \( m_j \) such that \( m_{\sigma(i)} = m_j \), and vice versa. Moreover, we have \( T'(m_{\sigma(i)}) = T'(m_j) \).

By (26), we have
\[
ck = \frac{\sum_{i=1}^{n} (1 + T'(m_{\sigma(i)})) \beta_k^i}{\sum_{i=1}^{n} (1 + T'(m_{\sigma(i)}))} = \frac{\sum_{j=1}^{n} (1 + T'(m_j)) \beta_k^j}{\sum_{j=1}^{n} (1 + T'(m_j))} = b_k, \ k = 0, 1, \ldots, g - 1,
\]
thus (27) holds, which completes the proof of Theorem 7.

It is worth noting that the commutativity property does not hold for the DLPWA operator.

C. Distribution linguistic power ordered weighted averaging (DLPOWA) operator

In this subsection, we extend the POWA operator \[33\] to distribution linguistic assessment environment.

**Definition 16.** Let \( \{m_1, m_2, \ldots, m_n\} \) be a set of distribution linguistic assessments of \( S \), where \( m_i = \{\langle s_k, \beta^i_k \rangle | k = 0, 1, \ldots, g - 1\} \), \( i = 1, 2, \ldots, n \), then the distribution linguistic power ordered weighted averaging (DLPOWA) operator is defined as
\[
\text{DLPOWA}(m_1, m_2, \ldots, m_n) = \{\langle s_k, \beta_k \rangle | k = 0, 1, \ldots, g - 1\},
\]
where \( \beta_k = \sum_{j=1}^{n} u_j \beta^\sigma(j) \) for \( j = 2, \ldots, n \), and
\[
u_j = f \left( \frac{R_j}{TV} \right) - f \left( \frac{R_{j-1}}{TV} \right), \quad R_j = \sum_{i=1}^{j} V_{\sigma(i)},
\]
\[
TV = \sum_{j=1}^{n} V_{\sigma(j)}, \quad V_{\sigma(j)} = 1 + T(m_{\sigma(j)}),
\]
where \( f : [0, 1] \rightarrow [0, 1] \) is a basic unit-interval monotonic(BUM) function with the following properties:
(1) \( f(0) = 0; \) (2) \( f(1) = 1; \) (3) \( f(x) \geq f(y) \), if \( x > y \), and \( T(m_{\sigma(j)}) \) denotes the support of the \( j \)th largest argument from all the other arguments, i.e.
\[
T(m_{\sigma(j)}) = \frac{1}{n} \sum_{j=1}^{n} \text{Sup}(m_{\sigma(j)}, m_{\sigma(i)}),
\]
where \( \text{Sup}(m_{\sigma(j)}, m_{\sigma(i)}) \) is the support of \( m_{\sigma(j)} \) from \( m_{\sigma(i)} \).
For the DLPOA operator, if \( f(x) = x \), then

\[
  u_j = \frac{R_j}{TV} - \frac{R_{j-1}}{TV} = \frac{V_{\sigma(j)}}{TV} = \frac{1 + T(m_{\sigma(j)})}{\sum_{j=1}^{n} (1 + T(m_{\sigma(j)}))}
\]

(31)

In this case, we have

\[
  \beta_k = \frac{\sum_{j=1}^{n} \frac{1 + T(m_{\sigma(j)})}{n} \beta_k^{\sigma(j)}}{\sum_{j=1}^{n} \frac{1}{1 + T(m_{\sigma(j)})}} = \frac{\sum_{i=1}^{n} (1 + T'(m_i)) \beta_k^{i}}{\sum_{i=1}^{n} (1 + T'(m_i))}.
\]

(32)

Obviously, the DLPOA operator is reduced to the DLPA operator.

Like the DLPWA operator, the DLPOA operator also has the following properties.

**Theorem 8.** Let \( \text{Sup}(m_i, m_j) = b \), for all \( i \neq j \) and \( f(x) = x \), then we have \( \text{DLPOA}(m_1, m_2, \ldots, m_n) = \{ (s_k, \beta_k) | k = 0, 1, \ldots, g - 1 \} \), where \( \beta_k = \frac{1}{n} \sum_{i=1}^{n} \beta_k^i \), \( k = 0, 1, \ldots, g - 1 \).

Proof: Since \( \text{Sup}(m_i, m_j) = b \), for all \( i \neq j \), by (30), we have \( T(m_{\sigma(j)}) = \frac{b(n-1)}{n} \), \( j = 1, 2, \ldots, n \).

Thus,

\[
  u_j = \frac{1 + T(m_{\sigma(j)})}{\sum_{j=1}^{n} (1 + T(m_{\sigma(j)}))} = \frac{1 + \frac{b(n-1)}{n}}{\sum_{j=1}^{n} \left( 1 + \frac{b(n-1)}{n} \right)} = \frac{1}{n}.
\]

Hence, we have \( \beta_k = \frac{1}{n} \sum_{i=1}^{n} \beta_k^i \), \( k = 0, 1, \ldots, g - 1 \), which completes the proof of Theorem 8.

Theorem 8 demonstrates that if \( \text{Sup}(m_i, m_j) = b \), for all \( i \neq j \) and \( f(x) = x \), then the DLPOA operator reduces to the simple average of the distribution linguistic assessments.

**Theorem 9.** (Commutativity). Let \( \{m_1, m_2, \ldots, m_n\} \) be a set of distribution linguistic assessments of \( S \) and \( \{\sigma(1), \sigma(2), \ldots, \sigma(n)\} \) is a permutation of \( \{1, 2, \ldots, n\} \), then

\[
  \text{DLPOA}(m_1, m_2, \ldots, m_n) = \text{DLPOA}(m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(n)}).
\]

(33)

**Theorem 10.** (Idempotency). Let \( \{m_1, m_2, \ldots, m_n\} \) be a set of distribution linguistic assessments of \( S \). If \( m_1 = m_2 = \ldots = m_n = m \), then

\[
  \text{DLPOA}(m_1, m_2, \ldots, m_n) = m.
\]

(34)
Theorem 11. (Boundary). Let \( \{m_1, m_2, \ldots, m_n\} \) be a set of distribution linguistic assessments of \( S \), then

\[
\min_i \{m_i\} \leq DLPOWA(m_1, m_2, \ldots, m_n) \leq \max_i \{m_i\}.
\]

VI. MULTI-ATTRIBUTE GROUP DECISION MAKING BASED ON DISTRIBUTION LINGUISTIC POWER AGGREGATION OPERATORS

Consider the following multi-attribute group decision making problem. Let \( G = \{G_1, G_2, \ldots, G_n\} \) be a finite set of alternatives, \( C = \{C_1, C_2, \ldots, C_m\} \) be the set of attributes, \( O = \{O_1, O_2, \ldots, O_q\} \) be the set of decision organizations composed of several experts. The weight vector of the attributes is \( w = (w_1, w_2, \ldots, w_m)^T \), where \( 0 \leq w_j \leq 1, j = 1, 2, \ldots, m, \sum_{j=1}^{m} w_j = 1 \), and the weight vector of the decision organizations is \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)^T \), where \( 0 \leq \lambda_l \leq 1, l = 1, 2, \ldots, q, \sum_{l=1}^{q} \lambda_l = 1 \). In the decision making process, the experts provide their evaluation values using linguistic terms. Assume that experts in different organizations use different linguistic terms. For simplicity, let the linguistic term set used by the \( l \)th organization be \( S^{g(l)} = \{s_0^{g(l)}, s_1^{g(l)}, \ldots, s_{g(l)-1}^{g(l)}\} \). The group decision making problem to be solved is to obtain the collective evaluation of the alternatives and to select the optimal alternative(s) using the information provided by the decision makers.

In the rest of this section, we develop two group decision making methods based on the distribution linguistic power aggregation operators introduced in section V. More specifically, the procedures are summarized as follows.

Method I.

Step 1: Transform the evaluation information of each decision organization into multi-granular distribution linguistic assessments. Let the evaluation value of the \( i \)th alternative with regard to the \( j \)th attribute provided by the \( p \)th expert in the \( l \)th organization be \( x_{ij}^{lp} \in S^{g(l)} \), then the evaluation information of the \( i \)th alternative with regard to the \( j \)th attribute provided by the \( l \)th organization can be denoted as a distribution linguistic assessment \( x_{ij}^{l} = \{s_k^{g(l)}, \beta_{ij,k}^{l}\} | k = 0, 1, \ldots, g(l) - 1 \} \), where

\[
\beta_{ij,k}^{l} = \frac{\#(\{p | x_{ij}^{lp} = s_k^{g(l)}\})}{\#(O_l)}, \quad i \in I = \{1, 2, \ldots, n\}, \quad j \in J = \{1, 2, \ldots, m\}, \quad k = 0, 1, \ldots, g(l) - 1, \quad l \in L = \{1, 2, \ldots, q\}.
\]

Step 2: Calculate the unique granularity \( g(l^*) = LCM(g(1) - 1, g(2) - 1, \ldots, g(q) - 1) + 1 \). According to Definition 10, transform each \( x_{ij}^{l} \) into a distribution linguistic assessment of the linguistic term set \( S^{g(l^*)} = \{s_0^{g(l^*)}, s_1^{g(l^*)}, \ldots, s_{g(l^*)-1}^{g(l^*)}\} \). After the transformation, the evaluation information of
the $i$th alternative with regard to the $j$th attribute provided by the $l$th organization can be denoted as $z_{ij}^l = \{ \langle s_{k}^{g(l)}, \gamma_{ij,k}^l \rangle | k = 0, 1, \ldots, g(l^*) - 1 \}, \ i \in I, \ j \in J, \ l \in L$.

**Step 3:** Calculate the support of $z_{ij}^l$ from $z_{ij}^h$ as

$$\text{Sup}(z_{ij}^l, z_{ij}^h) = 1 - d(z_{ij}^l, z_{ij}^h) = 1 - \left| \sum_{k=0}^{g(l^*)-1} (\gamma_{ij,k}^l - \gamma_{ij,k}^h) \right| / g(l^*) - 1, \ i \in I, \ j \in J, \ l, h \in L, \ l \neq h. \quad (36)$$

Then, the support of $z_{ij}^l$ from other decision organizations is calculated as

$$T(z_{ij}^l) = \sum_{h=1, h \neq l}^{q} \lambda_h \text{Sup}(z_{ij}^l, z_{ij}^h), \ i \in I, \ j \in J, \ l \in L. \quad (37)$$

**Step 4:** Utilize the DLPWA operator to aggregate all the individual evaluation $z_{ij}^l$ into the collective evaluation $z_{ij} = \{ \langle s_{k}^{g(l^*)}, \gamma_{ij,k} \rangle | k = 0, 1, \ldots, g(l^*) - 1 \}$, where

$$\gamma_{ij,k} = \frac{\sum_{l=1}^{q} \lambda_l (1 + T(z_{ij}^l)) \gamma_{ij,k}^l}{\sum_{l=1}^{q} \lambda_l (1 + T(z_{ij}^l))}, \ i \in I, \ j \in J, \ k = 0, 1, \ldots, g(l^*) - 1. \quad (38)$$

**Step 5:** Utilize the DAWA operator to obtain the collective evaluation of each alternative as $z_i = \{ \langle s_{k}^{g(l^*)}, \gamma_{i,k} \rangle | k = 0, 1, \ldots, g(l^*) - 1 \}$, where

$$\gamma_{i,k} = \sum_{j=1}^{m} \gamma_{ij,k} w_j, \ i \in I, \ k = 0, 1, \ldots, g(l^*) - 1. \quad (39)$$

**Step 6:** Calculate the expectation value and inaccuracy function value of each $z_i$ as

$$E(z_i) = \Delta \left( \sum_{k=0}^{g(l^*)-1} k \gamma_{i,k} \right), \ i \in I \quad (40)$$

and

$$H(z_i) = - \sum_{k=0}^{g(l^*)-1} \gamma_{i,k} \log_2 \gamma_{i,k}, \ i \in I. \quad (41)$$

**Step 7:** Based on the values of $E(z_i)$ and $H(z_i)$, rank all the alternatives and select the optimal alternative(s). If the decision organizations want to know the collective evaluation using their linguistic term sets, denote all the $z_i$ using their own linguistic term sets by Definition 12.

Method I can be used to deal with the group decision making problem when the weight vector of the
decision organizations is known. If the weight vector is unknown, we can use the DLPOWA operator to develop another method (Method II) which is summarized as follows.

**Method II.**

**Step 1:** See Step 1 of Method I.

**Step 2:** See Step 2 of Method I.

**Step 3:** Calculate the support of the $l$th largest evaluation value $z_{ij}^{\sigma(l)}$ from the $h$th largest value $z_{ij}^{\sigma(h)}$ as

$$
\text{Sup}(z_{ij}^{\sigma(l)}, z_{ij}^{\sigma(h)}) = 1 - d(z_{ij}^{\sigma(l)}, z_{ij}^{\sigma(h)}) = 1 - \left| \frac{\sum_{k=0}^{g(l^*)-1} (\gamma_{ij,k}^{\sigma(l)} - \gamma_{ij,k}^{\sigma(h)})}{g(l^*) - 1} \right|, \ i \in I, \ j \in J, \ l, h \in L, \ l \neq h. \tag{42}
$$

Then, the support of $z_{ij}^{\sigma(l)}$ from other decision organizations can be calculated as

$$
T(z_{ij}^{\sigma(l)}) = \frac{1}{q} \sum_{h=1, h \neq l}^{q} \text{Sup}(z_{ij}^{\sigma(l)}, z_{ij}^{\sigma(h)}), \ i \in I, \ j \in J, \ l \in L. \tag{43}
$$

In addition, calculate the weight $u_{ij}^{l}$ associated with $z_{ij}^{\sigma(l)}$, where

$$
u_{ij}^{l} = f \left( \frac{R_{lij}}{TV_{ij}} \right) - f \left( \frac{R_{l-1,ij}}{TV_{ij}} \right), \ R_{lij} = \sum_{h=1}^{l} V_{ij}^{\sigma(h)}, \ TV_{ij} = \sum_{l=1}^{q} V_{ij}^{\sigma(l)}, \ V_{ij}^{\sigma(l)} = 1 + T(z_{ij}^{\sigma(l)}), \ i \in I, \ j \in J, \ l \in L, \tag{44}\text{and} \ f \text{ is a BUM function.}
$$

**Step 4:** Utilize the DLPOWA operator to aggregate all the individual evaluation $z_{ij}^{l}$ into a collective evaluation $z_{ij} = \{ (s_{ij}^{\sigma(l)}, \gamma_{ij,k}) | k = 0, 1, \ldots, g(l^*) - 1 \}$, where

$$
\gamma_{ij,k} = \sum_{l=1}^{q} u_{ij}^{l} \gamma_{ij,k}^{\sigma(l)}, \ i \in I, \ j \in J, \ k = 0, 1, \ldots, g(l^*) - 1. \tag{45}
$$

**Step 5:** See Step 5 of Method I.

**Step 6:** See Step 6 of Method I.

**Step 7:** See Step 7 of Method I.

**VII. AN ILLUSTRATIVE EXAMPLE**

Let us consider supplier selection problems in high-tech companies. Suppose that a high-tech company which manufactures electronic products intends to evaluate and select a supplier of USB connectors
(adapted from [44]). There are four suppliers \(G_1, G_2, G_3\) and \(G_4\) to be selected. Three departments of the company including finance department \((O_1)\), engineering department \((O_2)\) and quality control department \((O_3)\) are invited to evaluate the four suppliers. For the decision making problem, four criteria including finance \((C_1)\), performance \((C_2)\), technique \((C_3)\) and organizational culture \((C_4)\) are considered. The weight vector of the four criteria is \(w = (0.25, 0.25, 0.25, 0.25)^T\). The decision makers provide their evaluation on the four alternatives according to the four criteria using linguistic terms. Here decision makers in the same department utilize the same linguistic term set. More specifically, the linguistic term sets used by the three departments are as follows:

\[
O_1 : \quad S^5 = \{s_0^5, s_1^5, s_2^5, s_3^5, s_4^5\}; \\
O_2 : \quad S^7 = \{s_0^7, s_1^7, s_2^7, s_3^7, s_4^7, s_5^7, s_6^7\}; \\
O_3 : \quad S^9 = \{s_0^9, s_1^9, s_2^9, s_3^9, s_4^9, s_5^9, s_6^9, s_7^9, s_8^9\}.
\]

By Step 1 of Method I, we denote the evaluation information of the three departments by using distribution linguistic assessments which are demonstrated in Tables I - III.

| \(O_1\) | \(C_1\) | \(C_2\) | \(C_3\) | \(C_4\) |
|---|---|---|---|---|
| \(G_1\) | \{\(s_3^5, 0.4\), \(s_4^5, 0.6\)\} | \{\(s_2^5, 0.2\), \(s_3^5, 0.8\)\} | \{\(s_5^5, 0.8\), \(s_4^5, 0.2\)\} | \{\(s_3^5, 1\)\} |
| \(G_2\) | \{\(s_2^5, 0.4\), \(s_3^5, 0.2\), \(s_4^5, 0.4\)\} | \{\(s_3^5, 0.8\), \(s_4^5, 0.2\)\} | \{\(s_2^5, 0.2\), \(s_3^5, 0.4\), \(s_4^5, 0.4\)\} | \{\(s_2^5, 0.6\), \(s_3^5, 0.4\)\} |
| \(G_3\) | \{\(s_1^5, 0.2\), \(s_2^5, 0.4\), \(s_3^5, 0.4\)\} | \{\(s_5^5, 0.6\), \(s_3^5, 0.4\)\} | \{\(s_5^5, 1\)\} | \{\(s_1^5, 0.4\), \(s_2^5, 0.4\), \(s_3^5, 0.2\)\} |
| \(G_4\) | \{\(s_4^5, 1\)\} | \{\(s_5^5, 0.6\), \(s_4^5, 0.4\)\} | \{\(s_1^5, 0.4\), \(s_2^5, 0.4\), \(s_3^5, 0.2\)\} | \{\(s_3^5, 0.2\), \(s_4^5, 0.8\)\} |

We will utilize the proposed group decision making methods to select the best supplier.

First of all, we calculate the unique granularity \(g(l^*)\). Since \(g(1) = 5\), \(g(2) = 7\), \(g(3) = 9\), we have \(g(l^*) = LCM(4, 6, 8) + 1 = 25\). By Definition 10, the distribution linguistic assessments in Tables I - III are transformed into Tables IV - VI.

**Case 1:** the weight vector of the three departments is known. Assume that the weight vector is \(\lambda = (0.35, 0.25, 0.4)^T\), then Method I can be utilized to solve the decision making problem.
### Table II
**Evaluation information of department $O_2$**

| $O_2$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|-------|-------|-------|-------|-------|
| $G_1$ | $\{ (s_3^7, 0.25), (s_4^5, 0.5), (s_6^5, 0.25) \}$ | $\{ (s_1^7, 0.25), (s_2^5, 0.25), (s_4^5, 0.5) \}$ | $\{ (s_4^7, 0.5), (s_6^7, 0.5) \}$ | $\{ (s_3^7, 0.25), (s_4^7, 0.75) \}$ |
| $G_2$ | $\{ (s_3^7, 0.5), (s_4^7, 0.5) \}$ | $\{ (s_1^7, 0.25), (s_2^4, 0.25), (s_4^5, 0.5) \}$ | $\{ (s_4^7, 0.25), (s_6^4, 0.5) \}$ | $\{ (s_3^7, 0.5), (s_6^7, 0.5) \}$ |
| $G_3$ | $\{ (s_3^7, 0.5), (s_4^7, 0.5) \}$ | $\{ (s_1^7, 0.25), (s_2^4, 0.25), (s_4^5, 0.5) \}$ | $\{ (s_4^7, 0.25), (s_6^4, 0.5) \}$ | $\{ (s_3^7, 0.5), (s_6^7, 0.5) \}$ |
| $G_4$ | $\{ (s_4^7, 0.5), (s_6^7, 0.5) \}$ | $\{ (s_1^7, 0.25), (s_2^4, 0.25), (s_4^5, 0.5) \}$ | $\{ (s_4^7, 0.25), (s_6^4, 0.5) \}$ | $\{ (s_3^7, 0.5), (s_6^7, 0.5) \}$ |

### Table III
**Evaluation information of department $O_3$**

| $O_3$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|-------|-------|-------|-------|-------|
| $G_1$ | $\{ (s_3^9, 0.333), (s_6^9, 0.167), (s_3^7, 0.5) \}$ | $\{ (s_1^9, 0.167), (s_6^9, 0.833) \}$ | $\{ (s_1^9, 0.5), (s_6^9, 0.167), (s_3^9, 0.333) \}$ | $\{ (s_1^9, 0.5), (s_6^9, 0.167), (s_3^9, 0.333) \}$ |
| $G_2$ | $\{ (s_3^9, 0.333), (s_6^9, 0.167), (s_3^7, 0.5) \}$ | $\{ (s_1^9, 0.5), (s_6^9, 0.5) \}$ | $\{ (s_1^9, 0.333), (s_6^9, 0.167), (s_3^9, 0.333) \}$ | $\{ (s_1^9, 0.5), (s_6^9, 0.167), (s_3^9, 0.333) \}$ |
| $G_3$ | $\{ (s_3^9, 0.333), (s_6^9, 0.5), (s_3^7, 0.167) \}$ | $\{ (s_1^9, 0.333), (s_6^9, 0.667) \}$ | $\{ (s_1^9, 0.5), (s_6^9, 0.5) \}$ | $\{ (s_1^9, 0.333), (s_6^9, 0.333), (s_3^9, 0.333) \}$ |
| $G_4$ | $\{ (s_3^9, 0.333), (s_6^9, 0.667) \}$ | $\{ (s_1^9, 0.333), (s_6^9, 0.667) \}$ | $\{ (s_1^9, 0.333), (s_6^9, 0.5), (s_3^9, 0.167) \}$ | $\{ (s_1^9, 0.5), (s_6^9, 0.167), (s_3^9, 0.333) \}$ |

### Table IV
**Unified evaluation information of department $O_1$**

| $O_1$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|-------|-------|-------|-------|-------|
| $G_1$ | $\{ (s_{18}^{12}, 0.4), (s_{21}^{12}, 0.6) \}$ | $\{ (s_{12}^{18}, 0.2), (s_{18}^{12}, 0.8) \}$ | $\{ (s_{18}^{12}, 0.8), (s_{22}^{12}, 0.2) \}$ | $\{ (s_{18}^{12}, 1) \}$ |
| $G_2$ | $\{ (s_{12}^{12}, 0.4), (s_{18}^{12}, 0.2) \}$ | $\{ (s_{18}^{12}, 0.8), (s_{22}^{12}, 0.2) \}$ | $\{ (s_{18}^{12}, 0.2), (s_{18}^{12}, 0.4), (s_{22}^{12}, 0.4) \}$ | $\{ (s_{12}^{12}, 0.6), (s_{22}^{12}, 0.4) \}$ |
| $G_3$ | $\{ (s_{25}^{25}, 0.2), (s_{12}^{12}, 0.4) \}$ | $\{ (s_{12}^{12}, 0.6), (s_{22}^{12}, 0.4) \}$ | $\{ (s_{22}^{12}, 0.4) \}$ | $\{ (s_{22}^{12}, 0.2) \}$ |
| $G_4$ | $\{ (s_{22}^{22}, 1) \}$ | $\{ (s_{18}^{12}, 0.6), (s_{22}^{12}, 0.4) \}$ | $\{ (s_{25}^{25}, 0.4), (s_{22}^{12}, 0.4), (s_{18}^{12}, 0.2) \}$ | $\{ (s_{18}^{12}, 0.2), (s_{22}^{12}, 0.8) \}$ |
### TABLE V
Unified evaluation information of department $O_2$

| $O_2$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|-------|-------|-------|-------|-------|
| $G_1$ | $\{\langle s_{10}^{25}, 0.25 \rangle, \langle s_{16}^{25}, 0.5 \rangle \}$ | $\{\langle s_{10}^{25}, 0.25 \rangle, \langle s_{16}^{25}, 0.5 \rangle \}$ | $\{\langle s_{10}^{25}, 0.5 \rangle, \langle s_{16}^{25}, 0.5 \rangle \}$ | $\{\langle s_{10}^{25}, 0.25 \rangle, \langle s_{16}^{25}, 0.75 \rangle \}$ |
| $G_2$ | $\{\langle s_{12}^{25}, 0.5 \rangle, \langle s_{16}^{25}, 0.5 \rangle \}$ | $\{\langle s_{12}^{25}, 0.25 \rangle, \langle s_{16}^{25}, 0.5 \rangle \}$ | $\{\langle s_{12}^{25}, 0.25 \rangle, \langle s_{16}^{25}, 0.5 \rangle \}$ | $\{\langle s_{12}^{25}, 0.25 \rangle, \langle s_{16}^{25}, 0.5 \rangle \}$ |
| $G_3$ | $\{\langle s_{12}^{25}, 0.5 \rangle, \langle s_{16}^{25}, 0.5 \rangle \}$ | $\{\langle s_{12}^{25}, 0.25 \rangle, \langle s_{16}^{25}, 0.5 \rangle \}$ | $\{\langle s_{12}^{25}, 0.25 \rangle, \langle s_{16}^{25}, 0.5 \rangle \}$ | $\{\langle s_{12}^{25}, 0.25 \rangle, \langle s_{16}^{25}, 0.5 \rangle \}$ |
| $G_4$ | $\{\langle s_{16}^{25}, 0.5 \rangle, \langle s_{20}^{25}, 0.25 \rangle \}$ | $\{\langle s_{16}^{25}, 0.5 \rangle, \langle s_{20}^{25}, 0.25 \rangle \}$ | $\{\langle s_{16}^{25}, 0.5 \rangle, \langle s_{20}^{25}, 0.25 \rangle \}$ | $\{\langle s_{16}^{25}, 0.5 \rangle, \langle s_{20}^{25}, 0.25 \rangle \}$ |

### TABLE VI
Unified evaluation information of department $O_3$

| $O_3$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|-------|-------|-------|-------|-------|
| $G_1$ | $\{\langle s_{12}^{25}, 0.333 \rangle, \langle s_{12}^{25}, 0.167 \rangle, \langle s_{21}^{25}, 0.5 \rangle \}$ | $\{\langle s_{12}^{25}, 0.167 \rangle, \langle s_{21}^{25}, 0.5 \rangle \}$ | $\{\langle s_{12}^{25}, 0.333 \rangle, \langle s_{21}^{25}, 0.333 \rangle \}$ | $\{\langle s_{12}^{25}, 0.167 \rangle, \langle s_{21}^{25}, 0.333 \rangle \}$ |
| $G_2$ | $\{\langle s_{12}^{25}, 0.333 \rangle, \langle s_{12}^{25}, 0.167 \rangle, \langle s_{21}^{25}, 0.5 \rangle \}$ | $\{\langle s_{12}^{25}, 0.333 \rangle, \langle s_{21}^{25}, 0.5 \rangle \}$ | $\{\langle s_{12}^{25}, 0.333 \rangle, \langle s_{21}^{25}, 0.167 \rangle \}$ | $\{\langle s_{12}^{25}, 0.333 \rangle, \langle s_{21}^{25}, 0.167 \rangle \}$ |
| $G_3$ | $\{\langle s_{12}^{25}, 0.333 \rangle, \langle s_{12}^{25}, 0.5 \rangle, \langle s_{21}^{25}, 0.167 \rangle \}$ | $\{\langle s_{12}^{25}, 0.333 \rangle, \langle s_{21}^{25}, 0.5 \rangle \}$ | $\{\langle s_{12}^{25}, 0.333 \rangle, \langle s_{21}^{25}, 0.5 \rangle \}$ | $\{\langle s_{12}^{25}, 0.333 \rangle, \langle s_{21}^{25}, 0.5 \rangle \}$ |
| $G_4$ | $\{\langle s_{12}^{25}, 0.333 \rangle, \langle s_{21}^{25}, 0.667 \rangle \}$ | $\{\langle s_{21}^{25}, 0.333 \rangle, \langle s_{22}^{25}, 0.667 \rangle \}$ | $\{\langle s_{21}^{25}, 0.333 \rangle, \langle s_{22}^{25}, 0.5 \rangle \}$ | $\{\langle s_{21}^{25}, 0.333 \rangle, \langle s_{22}^{25}, 0.5 \rangle \}$ |

1) Calculate the support of $z_{ij}^l$ from $z_{ij}^h$. For simplicity, we denote the support between the department $O_l$ and $O_h$ as $\text{Sup}_{lh} = (\text{Sup}(z_{ij}^l, z_{ij}^h))_{4 \times 4}$, $l, h = 1, 2, 3$, $l \neq h$, then we have

$$\text{Sup}_{12} = \text{Sup}_{21} = \begin{bmatrix} 0.7667 & 0.7583 & 0.9667 & 0.8750 \\ 0.8333 & 0.9083 & 0.8667 & 0.6833 \\ 0.9667 & 0.9750 & 0.8750 & 0.8 \\ 0.7917 & 0.9333 & 0.9083 & 0.9667 \end{bmatrix},$$

$$\text{Sup}_{13} = \text{Sup}_{31} = \begin{bmatrix} 0.8708 & 0.9042 & 0.9458 & 0.9792 \\ 0.8542 & 0.9875 & 0.8875 & 0.8792 \\ 0.9458 & 0.7667 & 0.9375 & 0.9250 \\ 0.8333 & 0.8917 & 0.9708 & 0.8208 \end{bmatrix}.$$
Sup_{23} = Sup_{32} = \begin{bmatrix}
0.8958 & 0.8542 & 0.9792 & 0.8958 \\
0.9792 & 0.8958 & 0.9792 & 0.5625 \\
0.9792 & 0.7917 & 0.9375 & 0.8750 \\
0.9583 & 0.9583 & 0.9375 & 0.8542 \\
\end{bmatrix}.

2) Calculate the support of \( z_{ij}^l \) from all the other departments. If we denote the support of the department \( O_i \) from other departments as \( T_i = (T_i^l)_{1 \times 4}, \ l = 1, 2, 3 \), then we have

\[
T_1 = \begin{bmatrix}
0.5400 & 0.5513 & 0.6200 & 0.6104 \\
0.5500 & 0.6221 & 0.5717 & 0.5225 \\
0.6200 & 0.5504 & 0.5938 & 0.5700 \\
0.5313 & 0.5900 & 0.6154 & 0.5700 \\
\end{bmatrix},
T_2 = \begin{bmatrix}
0.6267 & 0.6071 & 0.7300 & 0.6646 \\
0.6833 & 0.6763 & 0.6950 & 0.4642 \\
0.7300 & 0.6579 & 0.6812 & 0.6300 \\
0.6604 & 0.7100 & 0.6929 & 0.6800 \\
\end{bmatrix},
T_3 = \begin{bmatrix}
0.5288 & 0.5300 & 0.5758 & 0.5667 \\
0.5437 & 0.5696 & 0.5554 & 0.4483 \\
0.5758 & 0.4662 & 0.5625 & 0.5425 \\
0.5313 & 0.5517 & 0.5742 & 0.5008 \\
\end{bmatrix}.
\]

3) Calculate the collective value of each \( z_{ij}^l \) using the DLPWA operator. The aggregated matrix is shown in Table VII.

|       | \( C_1 \) | \( C_2 \) | \( C_3 \) | \( C_4 \) |
|-------|-------|-------|-------|-------|
| \( G_1 \) | \{ (s_{22}^{25}, 0.065), (s_{12}^{25}, 0.131) \} | \{ (s_{22}^{25}, 0.065), (s_{22}^{25}, 0.065) \} | \{ (s_{22}^{25}, 0.133), (s_{22}^{25}, 0.472) \} | \{ (s_{12}^{25}, 0.065), (s_{12}^{25}, 0.195) \} |
| \( G_2 \) | \{ (s_{25}^{25}, 0.130), (s_{12}^{25}, 0.271) \} | \{ (s_{12}^{25}, 0.065), (s_{22}^{25}, 0.065) \} | \{ (s_{22}^{25}, 0.265), (s_{22}^{25}, 0.133) \} | \{ (s_{12}^{25}, 0.065), (s_{12}^{25}, 0.281) \} |
| \( G_3 \) | \{ (s_{25}^{25}, 0.069), (s_{12}^{25}, 0.401) \} | \{ (s_{12}^{25}, 0.065), (s_{22}^{25}, 0.065) \} | \{ (s_{22}^{25}, 0.267), (s_{22}^{25}, 0.253) \} | \{ (s_{12}^{25}, 0.065), (s_{12}^{25}, 0.270) \} |
| \( G_4 \) | \{ (s_{12}^{25}, 0.133), (s_{12}^{25}, 0.131) \} | \{ (s_{12}^{25}, 0.133), (s_{22}^{25}, 0.133) \} | \{ (s_{25}^{25}, 0.140), (s_{25}^{25}, 0.065) \} | \{ (s_{12}^{25}, 0.134), (s_{12}^{25}, 0.134) \} |
4) By the DAWA operator, calculate the collective evaluation $z_i$ ($i = 1, 2, 3, 4$) of the four suppliers. The results are demonstrated in Table VIII.

**TABLE VIII**
COLLECTIVE EVALUATION OF THE FOUR SUPPLIERS BY METHOD I

| Supplier | Collective evaluation |
|----------|-----------------------|
| $G_1$    | $\{ (s^{25}_{1}, 0.016), (s^{25}_{2}, 0.016), (s^{25}_{12}, 0.066), (s^{25}_{15}, 0.163), (s^{25}_{18}, 0.147), (s^{25}_{18}, 0.343), (s^{25}_{20}, 0.016), (s^{25}_{21}, 0.098), (s^{25}_{21}, 0.135) \}$ |
| $G_2$    | $\{ (s^{25}_{6}, 0.033), (s^{25}_{9}, 0.033), (s^{25}_{12}, 0.220), (s^{25}_{15}, 0.065), (s^{25}_{18}, 0.083), (s^{25}_{20}, 0.080), (s^{25}_{21}, 0.065), (s^{25}_{21}, 0.117) \}$ |
| $G_3$    | $\{ (s^{25}_{6}, 0.016), (s^{25}_{9}, 0.085), (s^{25}_{9}, 0.049), (s^{25}_{12}, 0.033), (s^{25}_{15}, 0.237), (s^{25}_{18}, 0.048), (s^{25}_{18}, 0.100), (s^{25}_{18}, 0.135), (s^{25}_{20}, 0.016), (s^{25}_{21}, 0.112), (s^{25}_{22}, 0.169) \}$ |
| $G_4$    | $\{ (s^{25}_{6}, 0.035), (s^{25}_{9}, 0.016), (s^{25}_{9}, 0.032), (s^{25}_{12}, 0.100), (s^{25}_{15}, 0.064), (s^{25}_{18}, 0.066), (s^{25}_{18}, 0.136), (s^{25}_{20}, 0.083), (s^{25}_{21}, 0.098), (s^{25}_{24}, 0.370) \}$ |

5) Calculate the expectation values of the four suppliers’ collective evaluation. For each supplier, we have $E(z_1) = (s^{25}_{18}, -0.43)$, $E(z_2) = (s^{25}_{17}, -0.31)$, $E(z_3) = (s^{25}_{16}, -0.48)$, $E(z_4) = (s^{25}_{16}, -0.12)$.

6) Rank the four suppliers based on $E(z_i)$, $i = 1, 2, 3, 4$. We obtain $G_4 \succ G_1 \succ G_2 \succ G_3$. Thus the best supplier is $G_4$.

If we want to present the final evaluation to the three departments, we can denote each $z_i$ using the initial linguistic term sets, which are shown in Tables IX - XI. From Tables IX - XI, we can obverse the collective evaluation of the alternatives. For instance, from Table IX we can find that the collective evaluation of supplier $G_1$ is mainly about $s^3_2$ and $s^3_3$, while supplier $G_4$ is about $s^3_3$ and $s^3_4$. Besides, we can also obtain the proportion distribution of the linguistic terms. Therefore, the use of distribution linguistic assessments can provide more information about the evaluation over alternatives.

**TABLE IX**
COLLECTIVE EVALUATION OF THE FOUR SUPPLIERS USING LINGUISTIC TERM SET $S^5$

| Supplier | Collective evaluation |
|----------|-----------------------|
| $G_1$    | $\{ (s^5_0, 0.005), (s^5_1, 0.021), (s^5_2, 0.203), (s^5_3, 0.582), (s^5_4, 0.189) \}$ |
| $G_2$    | $\{ (s^5_1, 0.049), (s^5_2, 0.297), (s^5_3, 0.478), (s^5_4, 0.176) \}$ |
| $G_3$    | $\{ (s^5_0, 0.016), (s^5_1, 0.134), (s^5_2, 0.327), (s^5_3, 0.293), (s^5_4, 0.230) \}$ |
| $G_4$    | $\{ (s^5_1, 0.062), (s^5_2, 0.175), (s^5_3, 0.316), (s^5_4, 0.447) \}$ |

**Case 2:** The weight vector of the three departments is unknown. In this case, we utilize the DLPOWA operator to aggregate the evaluation information of the three departments. Without loss of generality,
let the BUM function be $f(x) = x^{1/2}$. By Step 3 of Method II, we can calculate the values of $u_{ij}^l$, $i, j = 1, 2, 3, 4$, $l = 1, 2, 3$. For simplicity, we denote them as

$$U_1 = (u_{ij}^1)_{4 \times 4} = \begin{bmatrix} 0.5742 & 0.5764 & 0.5772 & 0.5786 \\ 0.5719 & 0.5787 & 0.5733 & 0.5661 \\ 0.5772 & 0.5693 & 0.5761 & 0.5768 \\ 0.5714 & 0.5770 & 0.5755 & 0.5789 \end{bmatrix}.$$  

$$U_2 = (u_{ij}^2)_{4 \times 4} = \begin{bmatrix} 0.2435 & 0.2429 & 0.2399 & 0.2405 \\ 0.2431 & 0.2402 & 0.2422 & 0.2493 \\ 0.2399 & 0.2449 & 0.2413 & 0.2422 \\ 0.2439 & 0.2408 & 0.2410 & 0.2412 \end{bmatrix}.$$  

$$U_3 = (u_{ij}^3)_{4 \times 4} = \begin{bmatrix} 0.1824 & 0.1807 & 0.1829 & 0.1809 \\ 0.1850 & 0.1811 & 0.1845 & 0.1847 \\ 0.1829 & 0.1858 & 0.1826 & 0.1810 \\ 0.1847 & 0.1822 & 0.1836 & 0.1798 \end{bmatrix}.\]
After that, the collective value of each $z_{ij}$ are aggregated using the DLPOWA operator. We summarize the results in Table XII.

### TABLE XII

|       | $C_1$                          | $C_2$                          | $C_3$                          | $C_4$                          |
|-------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| $G_1$ | $(s_{25}^{25}, 0.046), (s_{25}^{25}, 0.081),$ | $(s_{25}^{25}, 0.045), (s_{25}^{25}, 0.045),$ | $(s_{25}^{25}, 0.120), (s_{25}^{25}, 0.435),$ | $(s_{25}^{25}, 0.045), (s_{25}^{25}, 0.120),$ |
|       | $(s_{16}^{25}, 0.091), (s_{15}^{25}, 0.270),$ | $(s_{15}^{25}, 0.156), (s_{15}^{25}, 0.203),$ | $(s_{24}^{25}, 0.096), (s_{24}^{25}, 0.349),$ | $(s_{21}^{25}, 0.136), (s_{18}^{25}, 0.619),$ |
|       | $(s_{24}^{25}, 0.046), (s_{21}^{25}, 0.122),$ | $(s_{18}^{25}, 0.090), (s_{18}^{25}, 0.461),$ |                               | $(s_{21}^{25}, 0.080),$               |
| $G_2$ | $(s_{25}^{25}, 0.081), (s_{12}^{25}, 0.321),$ | $(s_{25}^{25}, 0.045), (s_{12}^{25}, 0.045),$ | $(s_{12}^{25}, 0.242), (s_{12}^{25}, 0.092),$ | $(s_{12}^{25}, 0.062), (s_{12}^{25}, 0.180),$ |
|       | $(s_{12}^{25}, 0.041), (s_{10}^{25}, 0.093),$ | $(s_{12}^{25}, 0.482), (s_{20}^{25}, 0.091),$ | $(s_{12}^{25}, 0.350), (s_{20}^{25}, 0.046),$ | $(s_{12}^{25}, 0.092), (s_{12}^{25}, 0.100),$ |
|       | $(s_{24}^{25}, 0.236), (s_{24}^{25}, 0.229),$ | $(s_{24}^{25}, 0.289), (s_{24}^{25}, 0.048),$ | $(s_{24}^{25}, 0.041), (s_{24}^{25}, 0.229),$ | $(s_{24}^{25}, 0.283), (s_{24}^{25}, 0.283),$ |
| $G_3$ | $(s_{25}^{25}, 0.036), (s_{25}^{25}, 0.386),$ | $(s_{25}^{25}, 0.173), (s_{25}^{25}, 0.184),$ | $(s_{25}^{25}, 0.045), (s_{25}^{25}, 0.045),$ | $(s_{25}^{25}, 0.045), (s_{25}^{25}, 0.312),$ |
|       | $(s_{25}^{25}, 0.289), (s_{25}^{25}, 0.120),$ | $(s_{25}^{25}, 0.264), (s_{21}^{25}, 0.379),$ | $(s_{25}^{25}, 0.121), (s_{21}^{25}, 0.788),$ | $(s_{25}^{25}, 0.135), (s_{25}^{25}, 0.081),$ |
|       | $(s_{21}^{25}, 0.169)$             |                             | $(s_{21}^{25}, 0.169)$             | $(s_{25}^{25}, 0.132), (s_{18}^{25}, 0.115),$ |
| $G_4$ | $(s_{25}^{25}, 0.092), (s_{25}^{25}, 0.081),$ | $(s_{25}^{25}, 0.109), (s_{25}^{25}, 0.121),$ | $(s_{25}^{25}, 0.073), (s_{25}^{25}, 0.144),$ | $(s_{25}^{25}, 0.090), (s_{25}^{25}, 0.146),$ |
|       | $(s_{25}^{25}, 0.046), (s_{21}^{25}, 0.163),$ | $(s_{25}^{25}, 0.192), (s_{24}^{25}, 0.578),$ | $(s_{25}^{25}, 0.080), (s_{12}^{25}, 0.338),$ | $(s_{20}^{25}, 0.120), (s_{24}^{25}, 0.644),$ |
|       | $(s_{24}^{25}, 0.618)$             |                             | $(s_{25}^{25}, 0.040), (s_{18}^{25}, 0.288),$ |                             |
|       |                             |                             | $(s_{25}^{25}, 0.037)$             |                             |

By the DAWA operator, we can calculate the collective evaluation $z_i$ ($i = 1, 2, 3, 4$) of the four suppliers. The results are demonstrated in Table XIII.

### TABLE XIII

| Supplier | Collective evaluation |
|----------|-----------------------|
| $G_1$    | $(s_{25}^{25}, 0.011), (s_{25}^{25}, 0.011), (s_{25}^{25}, 0.062), (s_{25}^{25}, 0.101), (s_{25}^{25}, 0.109), (s_{25}^{25}, 0.446), (s_{25}^{25}, 0.011), (s_{21}^{25}, 0.075), (s_{24}^{25}, 0.174)$ |
| $G_2$    | $(s_{25}^{25}, 0.015), (s_{25}^{25}, 0.020), (s_{25}^{25}, 0.197), (s_{25}^{25}, 0.033), (s_{25}^{25}, 0.058), (s_{25}^{25}, 0.292), (s_{25}^{25}, 0.105), (s_{25}^{25}, 0.083), (s_{25}^{25}, 0.197)$ |
| $G_3$    | $(s_{25}^{25}, 0.011), (s_{25}^{25}, 0.087), (s_{25}^{25}, 0.034), (s_{25}^{25}, 0.202), (s_{25}^{25}, 0.218), (s_{25}^{25}, 0.072), (s_{16}^{25}, 0.088), (s_{18}^{25}, 0.137), (s_{20}^{25}, 0.111), (s_{21}^{25}, 0.125), (s_{25}^{25}, 0.197)$ |
| $G_4$    | $(s_{25}^{25}, 0.018), (s_{25}^{25}, 0.036), (s_{25}^{25}, 0.020), (s_{25}^{25}, 0.084), (s_{25}^{25}, 0.033), (s_{25}^{25}, 0.095), (s_{25}^{25}, 0.093), (s_{25}^{25}, 0.072), (s_{25}^{25}, 0.089), (s_{25}^{25}, 0.460)$ |

The expectation values of the four suppliers’ collective evaluation are calculated as $E(z_1) = (s_{18}^{25}, 0.12)$, $E(z_2) = (s_{25}^{25}, -0.12)$, $E(z_3) = (s_{16}^{25}, 0.12)$, $E(z_4) = (s_{20}^{25}, -0.39)$. Thus the ranking of the four suppliers is $G_4 \succ G_1 \succ G_2 \succ G_3$, which means that the best supplier is $G_4$. 
VIII. Conclusions

The distribution linguistic assessment is an effective tool for decision makers to express their inaccurate linguistic assessments. In this paper, group decision making problems with multi-granular distribution linguistic assessments are investigated. To summarize, the main contributions are as follows.

First, the relationship between a linguistic 2-tuple and a distribution linguistic assessment is analyzed. Some formulae are proposed to transform a linguistic 2-tuple into its equivalent distribution linguistic assessment, which lays a good foundation for the representation of the aggregated results using initial linguistic terms.

Second, an approach to unifying multi-granular distribution linguistic assessments is proposed based on the extended linguistic hierarchies model, which can be used to aggregate distribution linguistic assessments with different granularities. Based on the relationship between a linguistic 2-tuple and a distribution linguistic assessment and the DAWA operator, a procedure which can represent the aggregated distribution linguistic assessments using the initial linguistic term sets is also developed. The proposed approaches enrich the research of group decision making with multi-granular linguistic information and can allow decision makers with different cultures and experiences to express their opinions flexibly.

Third, we redefine the distance between two distribution linguistic assessments. Based on the distance measure, some power aggregation operators including the DLPWA operator and the DLPOWA operator are developed to aggregate distribution linguistic assessments. Some desirable properties of the operators, such as idempotency, boundary and commutativity are discussed. The proposed operators can relieve the influence of unfair arguments on the aggregated results and thus make the aggregated results more reasonable.

Finally, two approaches to multi-attribute group decision making with multi-granular linguistic information are developed. In the proposed approaches, different organizations’ assessments over alternatives are denoted as multi-granular distribution linguistic assessments. The multi-granular distribution linguistic assessments are unified based on the proposed unifying approach and then aggregated using the power aggregation operators to derive the collective assessments of alternatives. The proposed approaches can deal with the situations where the weight vector of decision makers is known and unknown and can represent the collective assessments of alternatives using the initial linguistic term sets.

In terms of future research, we will focus on the aggregation of distribution linguistic assessments with interval symbolic proportions [45] or distribution linguistic assessments with linguistic terms not being
placed uniformly and symmetrically [20], [46]. Moreover, the hesitant fuzzy linguistic term sets defined by Rodríguez et al. [47] have received more and more attention from scholars [48]–[51]. It will be interesting to investigate the relationship between distribution linguistic assessments and the hesitant fuzzy linguistic term sets in the future.

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