AN EINSTEIN-HILBERT ACTION

FOR

AXI-DILATON GRAVITY IN 4-DIMENSIONS

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ABSTRACT

We examine the axi-dilatonic sector of low energy string theory and demonstrate how the gravitational interactions involving the axion and dilaton fields may be derived from a geometrical action principle involving the curvature scalar associated with a non-Riemannian connection. In this geometry the antisymmetric tensor 3-form field determines the torsion of the connection on the frame bundle while the gradient of the metric is determined by the dilaton field. By expressing the theory in terms of the Levi-Civita connection associated with the metric in the “Einstein frame” we confirm that the field equations derived from the non-Riemannian Einstein-Hilbert action coincide with the axi-dilaton sector of the low energy effective action derived from string theory.

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1. Introduction

In the absence of matter Einstein’s theory of gravity can be derived from a geometrical action principle. The action density may be expressed elegantly in terms of the curvature scalar associated with the curvature of the Levi-Civita connection. Such a connection $\nabla$ is torsion-free and metric compatible. Thus for all vector fields $X, Y$ on the spacetime manifold:

$$T(X, Y) \equiv \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

and $Q \equiv \nabla g = 0$ where $g$ denotes the metric tensor, $T$ the torsion tensor of $\nabla$ and $Q$ the gradient tensor of $g$ with respect to $\nabla$. The Levi-Civita connection provides a useful reference connection since it depends entirely on the metric structure of the manifold. Such a metric structure can be used to construct additional terms in the action describing the interaction of matter with gravity although one needs a guiding principle in order to remove the adhoc nature of this construction. Such interactions can sometimes be derived from more general connections. Torsional connections have been shown to accommodate gravitational interactions between spinors [1] as well as scalar fields [2]. In these approaches the theory can be rewritten in terms of the Levi-Civita connection so that the torsion and non-metricity tensors can then be interpreted as matter induced couplings for Einsteinian gravity, [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]. An alternative approach is to induce matter couplings from string theory. Low energy string theory predicts specific couplings between fields for massless excitations in space-time. Besides the graviton, gauge fields and hypothetical gravitinos one must consider the antisymmetric tensor field and dilaton. Depending on the topology of the compactification of internal dimensions there appear further scalar excitations with well defined coupling schemes. The pattern of all these couplings gives rise to new kinds of “duality” symmetries that may be used to discover new kinds of dilatonic black holes in 4-dimensions. These symmetries may be expressed either in terms of the geometry of the metric of the string graviton or in terms of a Weyl scaled metric (the so called “Einstein frame”).

In this note we concentrate on the axi-dilatonic sector of low energy string theory and demonstrate how the gravitational interactions involving the axion and dilaton may be derived from a geometrical action principle involving the curvature scalar associated with a non-Riemannian connection.

2. The Curvature Scalar

The traditional metric variation of the Einstein-Hilbert action assumes that the connection is torsion free and metric compatible [13]. A simpler variational approach relaxes the torsion free condition but maintains the constraint of metric compatibility [14]. In the absence of matter varying such a connection and metric (by varying a set of orthonormal coframe fields) yields the same result. Alternatively varying a metric and a connection that is constrained to be torsion free but not metric compatible yields the same (pseudo-) Riemannian geometry [15]. The results of such traditional constrained variations can be unified using an action principle in which the structure equations for the torsion and non-metricity appear as constraints [16]. This approach has the virtue that it can accommodate consistent variations of the metric, connection, torsion and non-metricity forms.

In the following it will prove convenient to choose a $g$-orthonormal coframe in which $g$ takes the form $g = \eta_{ab} e^a \otimes e^b$ with $\eta_{ab} = \pm 1$. In this coframe we define the connection 1-forms $\Lambda^a_b$ by

$$\nabla_X e^c = -\Lambda^c_b (X_a) e^b$$

(1)
It follows that the torsion and curvature forms of $\nabla$ are given by
\[
T^a = De^a \equiv de^a + \Lambda^a_b \wedge e^b \quad (2)
\]
\[
R^a_b(A) = d\Lambda^a_b + \Lambda^a_c \wedge \Lambda^c_b \quad (3)
\]
We shall often use the exterior covariant derivative $D$ whose general definition can be found in [17]. We induce a variation of the curvature scalar $R$ by varying consistently the set of $g$-orthonormal coframe fields $\{e^a\}$ and the set of connection 1-forms $\Lambda^{ab}$ representing $\nabla$ in such a coframe. The curvature scalar $R$ is defined as the coefficient of the canonical volume form $\ast 1$ in the 4-form $R^a_b(A) \wedge \ast(e^a \wedge e^b)$ where $\ast$ denotes the Hodge map. The torsion tensor can be written in terms of a set of 2-forms $\{T^a\}$ with respect to the coframe $\{e^a\}$:
\[
T^a(X, Y) = \frac{1}{2} e^a(T(X, Y)) \quad (4)
\]
The symmetric non-metricity forms $Q_{ab}$ are tensorial and
\[
D\eta_{ab} = -2Q_{ab}
\]
We shall work in the so called “Einstein frame” and denote the dilaton field by $\phi$, (with $q = e^{-\phi}$) and the axion field by $A$. $A$ is related to the antisymmetric tensor 3-form field $H$ by the relation:
\[
q^2 \ast H = dA
\]
In terms of these fields we assume that the connection $\nabla$ has torsion
\[
T_a = G(q)i_aH \quad (5)
\]
and non-metricity
\[
Q^a_b = \eta^a_b d(f(q)) \quad (6)
\]
in terms of real functions $f$ and $G$. We may decompose the connection forms
\[
\Lambda^a_b = \Omega^a_b + q^a_b + Q^a_b
\]
where
\[
q^a_b = -(i_aQ_{bc})e^c + (i_bQ_{ac})e^c
\]
\[
\Omega^a_b = \omega^a_b + K^a_b
\]
The Levi-Civita connection forms $\omega^a_b$ are defined by $de^a + \omega^a_b \wedge e^b = 0$ and $K_{ab} = \frac{G}{2}i_a i_b H$ are the contorsion forms defined by $T^a = K^a_b \wedge e^b$. Corresponding to the decomposition (7) of the connection one finds
\[
R^a_b(A) = R^a_b(\Omega) + D(\Omega)(q^a_b + Q^a_b) + q^a_c \wedge \Omega^c_b + Q^a_c \wedge Q^c_b + q^a_c \wedge Q^c_b + Q^a_c \wedge q^c_b
\]
Evaluating the exterior covariant derivative and using (5) and (6) one finds after discarding exact forms:
\[
R^a_b(A) \wedge \ast(e^a \wedge e^b) = R^a_b(\omega) \wedge \ast(e^a \wedge e^b) - \frac{3}{2} G^2(q)H \wedge \ast H - 6df(q) \wedge \ast df(q)
\]
With the choice $G(q) = q$, $f(q) = \sqrt{(1/6)ln(q)}$ one recognises precisely the action for axidilaton gravity in the “Einstein frame”. This observation suggests that this sector of the theory can be obtained from a variational principle in which the Lagrangian density is the
curvature scalar of the above connection. Such a variational procedure must yield the correct field equations for the axion and dilaton fields when the theory is re-expressed in terms of the standard (pseudo-)Riemannian geometry.

3. The Einstein-Hilbert Variational Problem for a non-Riemannian Geometry

To consistently vary the curvature scalar when the torsion and non-metricity are constrained by (5) and (6) introduce multiplier 2-forms $\lambda^a$ and 3-forms $\rho_{ab}$. We may use these to impose the torsion and non-metricity constraints in the variational procedure. Since we are disregarding Maxwell and Chern-Simons terms in the effective action we shall also assume that $H$ is a closed form. This is accommodated by introducing a further 0-form multiplier $\mu$ and writing the action as:

$$\mathcal{L}[\Lambda, e, \lambda, \rho, \mu, q, H] = \mathcal{R} + \lambda^a \wedge (de^a + \Lambda^a_b \wedge e^b - \mathcal{G}(q)i^a H) + \rho^a_b \wedge (Q^b_a - \eta^b_a df(q)) + \mu dH$$ (9)

Varying $q$, $H$ and $\mu$ immediately gives

$$f' dp + \mathcal{G}' H \wedge (i_a \lambda^a) = 0$$ (10)

$$(i^a \lambda_a) \mathcal{G} = d\mu$$ (11)

$$dH = 0$$ (12)

where $\rho = \rho^a_a$. In order to evaluate (10) and (11) further we must solve for the forms $\rho$ and $i_a \lambda^a$. This is achieved by analysing the equation obtained from the connection $\Lambda^a_b$ variations

$$D \ast (e_a \wedge e^a) + \lambda_a \wedge e^b - \rho^b_a = 0$$ (13)

The first term may be expressed in terms of the torsion and non-metricity forms:

$$D \ast (e_a \wedge e^a) = D(\eta_{ac} \ast (e^c \wedge e^a)) =$$

$$= -2Q_{ac} \wedge \ast (e^c \wedge e^b) + \eta_{ac} \ast (e^c \wedge e^b \wedge e^d) \wedge T_d$$

and using (5) and (6) one finds

$$D \ast (e_a \wedge e^a) = -2df \wedge \ast (e_a \wedge e^b) + \mathcal{G} \ast (e_a \wedge e^b \wedge e^c) \wedge i_c H$$

The symmetric and antisymmetric parts of (13) now give

$$\lambda_a \wedge e_b - \lambda_b \wedge e_a = 4df \wedge \ast (e_a \wedge e_b) - 2\mathcal{G} \ast (e_a \wedge e_b \wedge e_c) \wedge i_c H$$

$$\lambda_a \wedge e_b + \lambda_b \wedge e_a = 2\rho_{ba}$$

Tracing the symmetric part yields

$$\rho = \lambda_a \wedge e^a$$

while the anti-symmetric part yields after some calculation

$$i_a \lambda^a = 3\mathcal{G} \ast H$$

Hence

$$\rho = -12 \ast df$$ (14)

and

$$\lambda_a = \mathcal{G} e_a \wedge \ast H - 4i_a \ast df$$ (15)
Thus with the choice \( \mathcal{G} = kq \), \( f = \frac{1}{2} \ln(q) \) \( \frac{k^2}{6} > 0 \) the equations (11) and (10) become

\[
\frac{d \ast dq}{q^2} - \frac{dq \wedge \ast dq}{q^3} - qH \wedge \ast H = 0
\] (16)

\[
d \ast (q^2H) = 0
\] (17)

In terms of the axion field, \( dH = 0 \) implies

\[
d(\ast dA \frac{q^2}{2}) = 0
\] (18)

and (16) becomes

\[
d \ast dq - \frac{dq \wedge \ast dq}{q} + \frac{dA \wedge \ast dA}{q} = 0
\] (19)

These are the correct field equations for the axion and dilaton predicted by low energy effective string actions.

Finally we examine the metric variations induced by varying the coframes. With the aid of the identity

\[
\lambda^a \wedge \delta e(i_aH) = -\delta^a \wedge i_b(\lambda^b \wedge i_aH)
\]

the Einstein equation follows from the action (8) as

\[
R_{bc}(\Lambda) \wedge \ast (e_b \wedge \ast e_c \wedge e_a) + D(\lambda)(\lambda) + kqi_b(\lambda^b \wedge i_aH) = 0
\] (20)

It is a somewhat tedious calculation to decompose this into Levi-Civita components. The first term becomes

\[
R_{bc}(\omega) \wedge \ast (e_b \wedge \ast e_c \wedge e_a) + kqi_b(\lambda^b \wedge i_aH) = \frac{2k}{q^2} dq \wedge i_aqa + \frac{2kD(\omega)i_a \ast dq}{q}
\]

\[
+ \frac{k}{q^2} (2q^2 dq \wedge \ast d(q \wedge \ast H) - \frac{2k}{q^2} dq \wedge i_a \ast dq - \frac{2k}{q} D(\omega)(i_a \ast dq)
\]

The second and third terms are respectively

\[
D(\lambda)(\lambda) = -ke_a \wedge d(q \wedge \ast H) - \frac{2k}{q^2} dq \wedge i_a \ast dq - \frac{2k}{q} D(\omega)(i_a \ast dq)
\]

\[
+ k^2 q^2 i_aH \wedge \ast H + k^2 e_a \wedge dq \wedge \ast H + k^2 i_b i_aH \wedge i_b \wedge dq + \frac{3k^2}{q^2} i_a dq \wedge i_a dq + \frac{k^2}{q^2} dq \wedge i_a \ast dq
\]

\[
kqi_b(\lambda^b \wedge i_aH) = k^2 q^2 i_aH \wedge \ast H - 2k^2 i_b \wedge dq \wedge i_b iq_a H
\] (22)

Thus (20) may be written

\[
R_{bc}(\omega) \wedge \ast (e_b \wedge \ast e_c \wedge e_a) + \frac{\eta}{4} q^2 \tau_a[H] + \frac{\eta}{4q^2} \tau_a[q] = 0
\] (24)

where the stress forms are

\[
\tau_a[H] \equiv H \wedge i_a \ast H + i_aH \wedge \ast H
\]

\[
\tau_a[q] \equiv dq \wedge i_a \ast dq + i_a dq \wedge \ast dq
\]
String theory couplings correspond to $\eta = 1$. This confirms that the field equations derived from the non-Riemannian Einstein-Hilbert action above coincide with the axi-dilaton sector of the low energy effective action derived from string theory.

4. Discussion

We have expressed the axi-dilatonic sector of low energy string theory in terms of a geometry with torsion and a metric gradient. This formulation emphasises the geometrical nature of the axion and dilaton fields and raises questions about the most appropriate geometry for the discussion of physical phenomena involving these fields. In particular if the quanta of these fields can give rise to classical particles then their gravitational interactions may correspond to autoparallels of this non-Riemannian connection rather than the geodesics corresponding to the Levi-Civita connection.

5. Acknowledgment

The authors are grateful to EPSRC for providing facilities at the 15-th UK Institute for Theoretical High Energy Physicists at the University of Southampton where this work was begun and to the Human Capital and Mobility Programme of the European Union for partial support. TD is grateful to the School of Physics and Materials, University of Lancaster, UK for its hospitality.

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