Abstract

The structure of a previously developed representation of the Leech lattice, $\Lambda_{24}$, is exposed to further light with this unified and very simple construction.
1. Introduction.

In [1] I presented a representation of $\Lambda_{24}$ (see [2]) over $O^{3}$ (octonionic 3-space). The octonion multiplication I am accustomed to using (see [3-7]) is one of four for which index cycling and doubling are automorphisms; and the representation of $\Lambda_{24}$ developed in [1] is invariant under both kinds of maps.

The purpose of the present paper is to give this representation an elegant characterization that highlights the invariances and the relationship to the octonion algebra.

As usual, I’m going to ask the interested reader to look at [4-7] for background material on the relationship of the octonions to $E_{8}$ and $\Lambda_{16}$. The notation I employ here, and in those previous papers, is the same I employed in [3].

2. Foundation.

Let $X$ be an arbitrary octonion of unit norm. Let $A, B \in O$. Then in general,

$$A \circ_{X} B \equiv (AX)(X^{\dagger}B) = (A(BX))X^{\dagger} = X((X^{\dagger}A)B).$$  

(1)

This defines the octonion X-product $[3,4,8].$

The octonion multiplication I employ is characterized by the following rule cyclic in the indices:

$$e_{a}e_{a+1} = e_{a+5}, \quad a = 1, \ldots, 7$$  

(2)

(indices in (2) taken from 1 to 7, modulo 7) (see [3,4].) The index doubling invariance of the resulting multiplication leads to the following equivalent rules:

$$e_{a}e_{a+2} = e_{a+3},$$

and

$$e_{a}e_{a+4} = e_{a+6}.$$

Define:

$$\Xi_{0} = \{ \pm e_{a} \},$$

$$\Xi_{1} = \{ (\pm e_{a} \pm e_{b})/\sqrt{2} : a, b \text{ distinct} \},$$

$$\Xi_{2} = \{ (\pm e_{a} \pm e_{b} \pm e_{c} \pm e_{d})/2 : a, b, c, d \text{ distinct},$$

$$e_{a}(e_{b}e_{c}e_{d}) = \pm 1 \},$$

$$\Xi_{3} = \{ (\sum_{a=0}^{7} \pm e_{a})/\sqrt{8} : \text{odd number of} \ '+ \text{'s}, a, b, c, d \in \{0, \ldots, 7\}. \}.$$
Let \( X \in \Xi_m \), for some \( m = 0, 1, 2, 3 \). Given the multiplication (2), for all \( a, b \in \{0, ..., 7\} \), there is some \( c \in \{0, ..., 7\} \) such that
\[
e_a \circ_X e_b = \pm e_c
\] (see [4]). Therefore, by (1),
\[
e_a(e_bX) = \pm e_cX,
\]
\[
(X^\dagger e_a)e_b = \pm X^\dagger e_c.
\]
By induction this implies that for all \( a, b, \ldots, c \in \{0, ..., 7\} \), there exists some \( d \in \{0, ..., 7\} \), such that if \( X \in \Xi_m \), for some \( m = 0, 1, 2, 3 \), then
\[
e_a(e_b(\ldots(e_cX))\ldots)) = \pm e_dX,
\]
\[
(\ldots((X^\dagger e_a)e_b)e_c\ldots)e_d = \pm X^\dagger e_d. \tag{5}
\]
Define
\[
\mathcal{E}_{E_8}^{even} = \Xi_0 \cup \Xi_2,
\]
\[
\mathcal{E}_{E_8}^{odd} = \Xi_1 \cup \Xi_3. \tag{6}
\]
These are the inner shells (normalized to unity) of \( E_8 \) lattices [5].

3. \( \Lambda_{24} \).

Let \( P \in \mathcal{E}_{E_8}^{even} \), and define
\[
\ell_0 = \frac{1}{2}(1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7). \tag{7}
\]
Define the subset of \( O^3 \),
\[
\Lambda_{24}^3 = \langle \frac{1}{2}P, \pm \frac{1}{2}e_aP, \pm \frac{1}{2}(e_a\ell_0e_a)(e_bP) : a, b \in \{0, ..., 7\} \rangle \cup \{\text{all permutations of such elements}\}, \tag{8}
\]
where the two \( \pm \)'s are independent.

Let \( Q = \pm e_aP \in \mathcal{E}_{E_8}^{even} \Rightarrow P = \mp e_aQ \).

Also, by (5) there exists some index \( d \in \{0, ..., 7\} \) such that
\[
e_bP = \mp e_b(e_aQ) = \pm e_dQ.
\]
Therefore permuting the first two terms of the octonion triple in (8) leaves the form of that element intact. As a consequence there are only three inequivalent permutations on that element, each characterized by the position of the component with the term $\ell_0$.

Given that there are three inequivalent permutations on the triple in (8), 16 values for $\pm e_a$, 16 for $\pm e_b$, and 240 possible values $P$, the order of $\Lambda_{24}^3$ is therefore

$$3 \times 16 \times 16 \times 240 = 184320.$$ 

The term $(e_a \ell_0 e_a)(e_b P)$ can be written in other ways. Using (5) and the Moufang identities it is not hard to prove that there are indices $c, d \in \{0, \ldots, 7\}$ such that

$$
(e_a \ell_0 e_a)(e_b P) = \pm e_a(\ell_0(e_c P)) \\
= \pm'(e_a \circ_P (\ell_0 \circ_P e_d))P.
$$

Therefore the triple in (8) may be written

$$\frac{1}{2} < 1, \pm e_a, \pm e_a \circ_P (\ell_0 \circ_P e_d) > P.$$ 

Note that the element $P$ is not completely factored out, since it is part of the $\circ_P$ product in the third component.

Finally define

$$\Lambda_{24}^1 = \{ < A, 0, 0 >, < 0, A, 0 >, < 0, 0, A > : A \in \mathcal{E}_{8}^{\text{even}} \},$$

of order $3 \times 240$; and

$$\Lambda_{24}^2 = \{ < A, B, 0 >, < 0, A, B >, < B, 0, A > : A, B \in \frac{1}{\sqrt{2}}\mathcal{E}_{8}^{\text{odd}} \},$$

$$AB^\dagger = \pm \frac{1}{2} e_a, \ a \in \{0, \ldots, 7\},$$

of order $3 \times 16 \times 240$. Together with $\Lambda_{24}^3$ these three sets form the inner shell of a representation of $\Lambda_{24}$ normalized to unity (also see [1]). There are a total of

$$3 \times 240 + 3 \times 16 \times 240 + 3 \times 16^2 \times 240 = 196560$$

elements in this inner shell.
4. Invariance.

Both $\Lambda_{24}^1$ and $\Lambda_{24}^2$ are easily seen to be invariant under the index doubling and cycling maps. That $\Lambda_{24}^3$ is also invariant rests on the invariance of

$$\ell_0 = \frac{1}{2}(1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7).$$

In fact, it is easy to see that any element of $O$ of the form

$$u + v(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7),$$

$u, v$ real, will be index doubling and cycling invariant given that the underlying $O$ multiplication is index doubling and cycling invariant.

I’d like to acknowledge several electronic conversations with Tony Smith, who maintains a fascinating Web site at Georgia Tech: www.gatech.edu/tsmith/home.html

References

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