Edge-local equivalence of graphs

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Abstract

The local complement $G * i$ of a simple graph $G$ at one of its vertices $i$ is obtained by complementing the subgraph induced by the neighborhood of $i$ and leaving the rest of the graph unchanged. If $e = \{i, j\}$ is an edge of $G$ then $G * e = ((G * i) * j) * i$ is called the edge-local complement of $G$ along the edge $e$. We call two graphs edge-locally equivalent if they are related by a sequence of edge-local complementations. The main result of this paper is an algebraic description of edge-local equivalence of graphs in terms of linear fractional transformations of adjacency matrices. Applications of this result include (i) a polynomial algorithm to recognize whether two graphs are edge-locally equivalent, (ii) a formula to count the number of graphs in a class of edge-local equivalence, and (iii) a result concerning the coefficients of the interlace polynomial, where we show that these coefficients are all even for a class of graphs; this class contains, as a subset, all strongly regular graphs with parameters $(n, k, a, c)$, where $k$ is odd and $a$ and $c$ are even.

1 Introduction

Let $G$ be a simple graph with vertex set $V$ and edge set $E$. The local complement of $G$ at a vertex $i \in V$, denoted by $G * i$, is obtained by replacing the subgraph induced by the neighborhood of $i$ by its complement, and leaving the rest of the graph unchanged. Two graphs are called locally equivalent if they are related by a sequence of local complementations. If $e = \{i, j\} \in E$, the edge-local complement $G * e$ of $G$ along the edge $e$ is defined by $G * e = ((G * i) * j) * i$. We call two graphs edge-locally equivalent if they are related by a sequence of edge-local complementations.

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Local equivalence of graphs was studied in detail in the 1980s, primarily by Bouchet [1, 2, 3, 4, 5], where, among other results, this work lead to a polynomial algorithm to recognize circle graphs. Bouchet studied local equivalence of graphs in the framework of certain algebraic and combinatorial structures called isotropic systems, which can be regarded as subgroups of $K \times \ldots \times K$ that are self-dual with respect to some bilinear form, where $K$ is the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Recently, local equivalence of graphs has found a renewed interest in a number of distinct fields of research. First, local equivalence of graphs turns out to be a relevant equivalence relation in the context of quantum information theory\footnote{We note that quantum information theory is the field of research of the present authors.}, where graphs correspond to an important class of (pure) states of distributed quantum systems, called graph states [9]; in this context local equivalence of graphs corresponds to so-called local Clifford equivalence, or LC equivalence, of such graph states, an equivalence relation that is of importance in the study of multi-partite entanglement in graph states [10, 11, 12], measurement-based quantum computation [13] and the theory of quantum error-correction [14, 15]. Second, there is an intimate connection between (edge-)local complementation of graphs and the interlace polynomial [16, 17, 18, 19, 20], which is a recently introduced graph polynomial motivated by questions arising from DNA sequencing by hybridization [21]; the interlace polynomial can be defined recursively through identities involving edge-local complementation. Third, LC equivalence of graph states, and thus local equivalence of graph, has recently been studied in classical cryptography in the theory of (quadratic) generalized bent functions [22, 23].

Whereas local complementation of graphs is understood reasonably well, a general theory for edge-local equivalence seems to be missing. It is the aim of this paper to take the first significant steps towards this end. In the following we perform an algebraic analysis of edge-local complementation of graphs, relating this notion to certain transformations over $\mathbb{F}_2$ of adjacency matrices, namely linear fractional transformations (LFTs) having the form

$$\Gamma \mapsto ((A + I)\Gamma + A)(A\Gamma + A + I)^{-1},$$

where $\Gamma$ is the adjacency matrix of a graph on $n$ vertices and $A$ is an $n \times n$ diagonal matrix. Our main result is a proof that two graphs are edge-locally equivalent if and only if their adjacency matrices are related by an LFT of the above type. We will then use this result in a number of applications: first, a polynomial time algorithm to recognize edge-local equivalence of two arbitrary graphs is obtained; the complexity of this algorithm is $O(n^4)$, where $n$ is the number of vertices of the considered graphs. Second, the description of edge-local equivalence in terms of LFTs allows to obtain a formula to count the number of graphs in a class of edge-local equivalence. Third, we consider a number of invariants of edge-local equivalence. Fourth, we prove a result concerning the coefficients of the interlace polynomial of a graph, where we show that these coefficients are all even for a class of graphs; this class contains, among others, all strongly regular graphs with parameters $(n, k, a, c)$, where $k$ is odd.
and a and c are even. Fifth, we show that a nonsingular adjacency matrix can be inverted by performing a sequence of edge-local complementations on the corresponding graph. Finally, we consider the connection between edge-local equivalence of graphs and the action of local Hadamard matrices on the corresponding graph states, yielding a restricted version of LC equivalence of graph states. We believe that this last result is of interest in the theory of generalized bent functions.

This paper is organized as follows. In section 2 we introduce the basic definitions regarding local complementation and edge-local complementation. In section 3 we present the basic theory involving linear fractional transformations of adjacency matrices and we state our main result, which is subsequently proven in section 4. Then in section 5 a number of applications of this result are considered.

**Notations**

We note that in this paper all arithmetic is considered over the field \( \mathbb{F}_2 = \text{GF}(2) \), except in section 5.5, where also complex numbers are involved.

The following notations will be used. The identity matrix is denoted by \( I \), and \( E_i \) is a square matrix where \((E_i)_{ii} = 1\) and all other entries are zero (the dimensions of \( I \) and \( E_i \) follow from the context). Let \( n \in \mathbb{N}_0 \). If \( \omega \subseteq \{1, \ldots, n\} \), then \( \bar{\omega} \) denotes the complement of \( \omega \) in \( \{1, \ldots, n\} \). Let \( X \) be an \( n \times n \) matrix. For every \( \omega \subseteq \{1, \ldots, n\} \), define the \(|\omega| \times |\omega|\) principal submatrix \( X[\omega] \) of \( X \) by

\[
X[\omega] = (X_{ij})_{i,j \in \omega},
\]

and define the \(|\omega| \times (n - |\omega|)\) off-diagonal submatrix \( X\langle\omega\rangle \) by

\[
X\langle\omega\rangle = (X_{ij})_{i \in \omega, j \not\in \omega}.
\]

Also, for every \( n \times n \) diagonal matrix \( A \), we denote \( \hat{A} := (A_{11}, \ldots, A_{nn}) \in \mathbb{F}_2^n \).

For every \( v \in \mathbb{F}_2^n \), \( \hat{v} \) is the \( n \times n \) diagonal matrix with the entries of \( v \) on the diagonal. Finally, we denote the support of the vector \( v \) and its associated diagonal matrix by

\[
\text{supp}(v) = \text{supp}(\hat{v}) = \{i \in \{1, \ldots, n\} \mid v_i = 1\}.
\]

In this paper all graphs are finite, undirected and simple (no loops, no multiple edges). A graph \( G \) with vertex set \( V \) and edge set \( E \) is denoted by \( G = (V, E) \) and, for simplicity and without loss generality, we will always take \( V = \{1, \ldots, n\} \) for some \( n \in \mathbb{N}_0 \). The adjacency matrix \( \Gamma(G) = \Gamma \) of the graph \( G \) is the unique \( n \times n \) 0–1 matrix such that \( \Gamma_{ij} = 1 \) if and only if \( \{i, j\} \in E \). Adjacency matrices are symmetric and have zeros on their diagonals.

The neighborhood \( N(i) \subseteq V \) of a vertex \( i \in V \) is the set of all vertices that are adjacent to \( i \). For a subset \( \omega \subseteq V \) of vertices, the induced subgraph \( G[\omega] \subseteq G \) is the graph with vertex set \( \omega \) and edge set

\[
\{\{i, j\} \in E \mid i, j \in \omega\}.
\]
Figure 1: (i) Graph on 5 vertices, and (ii) its local complement at vertex 1. The neighborhood of vertex 1 in (i) consists of the vertices 2, 3 and 4. Hence, the graph in (ii) is obtained by complementing the induced subgraph on these vertices, and leaving the rest of the graph unchanged.

Note that the adjacency matrix of $G[\omega]$ is $\Gamma[\omega]$. The complement $\tilde{G}$ of $G = (V, E)$ is the graph on the same vertex set $V$ with the property that $\{i, j\}$ is an edge of $\tilde{G}$ if and only if it is not an edge of $G$.

## 2 Edge-local equivalence

Let $G = (V, E)$ be a graph with adjacency matrix $\Gamma$ and let $i \in V$ be an arbitrary vertex of $G$. The local complement of $G$ at vertex $i$, denoted by $G * i$, is the graph on the same vertex set $V$ obtained by replacing the subgraph $G[N(i)]$ by its complement and leaving the rest of the graph unchanged. Note that $(G * i) * i = G$. One can easily verify that the adjacency matrix $\Gamma * i$ of $G * i$ is equal to

$$\Gamma * i = \Gamma + \Gamma_i \Gamma_i^T + \hat{\Gamma}_i,$$

where $\Gamma_i$ is the $i$th column of $\Gamma$. An example of local complementation is given in Fig. 1.

The local complementation rule gives rise to an equivalence relation on the set of graphs, called local equivalence. Two graphs $G$ and $G'$ on the same vertex set $V$ are called locally equivalent if there exist $i_1, \ldots, i_N \in V$ such that

$$G' = (((G * i_1) * i_2) \ldots) * i_N.$$

Local complementation is also used as an elementary building block for a second graph transformation, which we call edge-local complementation. Letting $G = (V, E)$ be a graph and $e = \{i, j\} \in E$, one defines the edge-local complement of $G$ along the edge $e$, denoted by $G * e$, by

$$G * e = (((G * i) * j) * i = ((G * j) * i) * j.$$
Figure 2: Edge-local complement of graph (i) in Figure 1 along the edge \{1, 3\}, which can be obtained by first complementing at vertex 1, then at vertex 3, and again at vertex 1.

More explicitly, the graph \(G*e\) is the unique graph with the following properties.

- Let \(k, l \in V \setminus e\). Then \(k\) is adjacent to \(i\) in \(G*e\) if and only if it is adjacent to \(j\) in \(G\).
- Let \(k, l \in V \setminus e\) and suppose that one of the following situations occurs:
  1. \(k \in N(i) \setminus N(j)\) and \(l \in N(j) \setminus N(i)\);
  2. \(k \in N(j) \setminus N(i)\) and \(l \in N(i) \setminus N(j)\);
  3. \(k \in N(i) \setminus N(j)\) and \(l \in N(i) \cup N(j)\);
  4. \(k \in N(j) \setminus N(i)\) and \(l \in N(i) \cup N(j)\);
  5. \(k \in N(i) \cup N(j)\) and \(l \in N(i) \setminus N(j)\);
  6. \(k \in N(i) \cup N(j)\) and \(l \in N(j) \setminus N(i)\);

then \(\{k, l\}\) is an edge of \(G*e\) if and only if \(\{k, l\}\) is not an edge of \(G\). If none of the cases (i)-(vi) occur, then \(\{k, l\}\) is an edge of \(G*e\) if and only if it is an edge of \(G\).

Note that \((G*e)*e = G\). An example of local complementation along an edge is given in Fig. 2. Note that local complementation of \(G\) along an edge \(e\) can also be formulated elegantly in terms of adjacency matrices. Letting \(\Gamma\) and \(\Gamma*e\) be the adjacency matrices of \(G\) and \(G*e\), respectively, the matrix \(\Gamma*e\) is obtained as follows:

\[
(\Gamma*e)[e] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
(\Gamma*e)\langle e\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Gamma\langle e\rangle,
(\Gamma*e)[\bar{e}] = \Gamma[\bar{e}] + \Gamma\langle e\rangle^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Gamma\langle e\rangle.
\]

We will need this representation in section 4.
Analogous to the case where local complementation is used to define local equivalence of graphs, edge-local complementation gives rise to an equivalence relation as well, which we call edge-local equivalence. Letting \([V]^2\) be the set of all 2-element subsets of \(V\), two graphs \(G\) and \(G'\) are called edge-locally equivalent if there exist \(e_1, \ldots, e_N \in [V]^2\) such that

(i) \(e_1\) is an edge of \(G\) and \(e_\alpha\) is an edge of \((G \ast e_1) \ast \ldots \ast e_{\alpha-1}\), for every \(\alpha = 2, \ldots, N\), and

(ii) \((G \ast e_1) \ast \ldots \ast e_N = G'\).

3 Linear fractional transformations

In this section we consider matrix transformations of the form

\[
\Gamma \mapsto (A\Gamma + B)(C\Gamma + D)^{-1},
\]

where are \(\Gamma\), \(A\), \(B\), \(C\) and \(D\) are \(n \times n\) matrices. These transformations are called linear fractional transformations (LFTs). In particular, we will consider the case where \(\Gamma\) is the adjacency matrix of a graph and \(A\), \(B\), \(C\) and \(D\) are diagonal matrices satisfying \(AD + BC = I\); this last constraint is equivalent to stating that the \(2n \times 2n\) matrix

\[
Q := \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

is nonsingular. The set of all such matrices \(Q\) is a group isomorphic to \(GL(2, F_2)^n\) and will be denoted by \(C_n\). We will also write \(Q = [A, B, C, D]\) instead of (11). If \(\Gamma\) is an \(n \times n\) adjacency matrix, \(Q = [A, B, C, D] \in C_n\) and \(C\Gamma + D\) is nonsingular, we will use the notation

\[
Q(\Gamma) = (A\Gamma + B)(C\Gamma + D)^{-1}.
\]

Note that \(Q(\Gamma)\) is symmetric whenever \(\Gamma\) is symmetric. However, it is possible that \(Q(\Gamma)\) does not have zero diagonal when \(\Gamma\) does. Therefore, we associate to every \(Q = [A, B, C, D] \in C_n\) a domain of definition \(\Delta(Q)\), consisting of all \(n \times n\) adjacency matrices \(\Gamma\) such that

(i) \(C\Gamma + D\) is nonsingular, and

(ii) \(Q(\Gamma)\) has zero diagonal.

There is an intimate connection between local complementation of graphs and the above LFTs of adjacency matrices, as we showed in Ref. [10], where we studied local complementation in the context of quantum information theory. The main result in [10] is the following:

**Theorem 1** Let \(G\), \(G'\) be graphs with \(n \times n\) adjacency matrices \(\Gamma\) and \(\Gamma'\), respectively. Then \(G\) and \(G'\) are locally equivalent if and only if there exists a matrix \(Q \in C_n\) such that \(\Gamma \in \Delta(Q)\) and \(Q(\Gamma) = \Gamma'\).
For example, letting $\Gamma$ be an $n \times n$ adjacency matrix and $i \in \{1, \ldots, n\}$, and defining $Q^\Gamma_i = [I, \Gamma_i, E_i, I]$, where $\Gamma_i$ is the $i$th column of $\Gamma$, one has
\[ \Gamma \ast i = Q^\Gamma_i(\Gamma). \] (13)

We note that the result in theorem 1 is also implicit in the work of Bouchet regarding isotropic systems and local complementation [1]. We elaborate on this matter in appendix A. Glynn also proved an equivalent of theorem 1 independently of our own work [15].

It is the aim of this paper to derive a result analogous to theorem 1 for edge-local complementation of graphs. To do so, we will consider LFTs having the following specific structure. Let $\mathcal{H}_n$ be the subset of $\mathcal{C}_n$ consisting of all elements of the form
\[ H = [A + I, A, A, A + I] =: H^A \] (14)
for some diagonal $n \times n$ matrix $A$. Note that $\mathcal{H}_n$ is a subgroup of $\mathcal{C}_n$. Also, $H^A H^B = H^{A+B}$ for all diagonal matrices $A$ and $B$, such that $\mathcal{H}_n$ is isomorphic to the additive group of $\mathbb{F}_2^n$. The main result of this paper can now be formulated as follows:

**Theorem 2** Let $G$ and $G'$ be graphs with adjacency matrices $\Gamma$ and $\Gamma'$, respectively. Then $G$ and $G'$ are edge-locally equivalent if and only if there exists an operator $H \in \mathcal{H}_n$ such that $\Gamma \in \Delta(H)$ and $H(\Gamma) = \Gamma'$.

The proof of this result will be given in section 4. For the remainder of this section, we analyze some basic properties of LFTs that we will need below. A first basic result is the following:

**Proposition 1** Let $\Gamma$, $\Gamma'$ be $n \times n$ adjacency matrices and let $Q = [A, B, C, D] \in \mathcal{C}_n$. Then the following statements are equivalent:

(i) $\Gamma \in \Delta(Q)$ and $Q(\Gamma) = \Gamma'$;

(ii) $\Gamma' CT + A \Gamma + \Gamma'D + B = 0$.

**Proof:** The implication from (i) to (ii) is trivial. We prove the reverse implication. First, define the $n$-dimensional linear space
\[ V_\Gamma = \{(\Gamma u, u) \mid u \in \mathbb{F}_2^n\} \subseteq \mathbb{F}_2^{2n} \] (15)
and the space $V_{\Gamma'}$ analogously. Note that the columns of the matrix $[\Gamma' I^T \ (I' \ I)^T]$ are a basis of $V_\Gamma$ ($V_{\Gamma'}$). Second, consider the following (symplectic) inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{F}_2^{2n}$:
\[ \langle (u, v), (u', v') \rangle = u^T v' + v^T u', \] (16)
where $u, v, u', v' \in \mathbb{F}_2^n$. It is easy to verify that $\langle x, x \rangle = 0$ for every $x \in V_\Gamma$, such that this space is self-dual with respect to the above inner product, and the same holds for $V_{\Gamma'}$. Furthermore, note that the identity (ii) holds if and only if
\[ \begin{bmatrix} I & \Gamma' \\ \Gamma & I \end{bmatrix} Q \begin{bmatrix} \Gamma \\ I \end{bmatrix} = 0, \] (17)
which is in turn equivalent to stating that $\langle x, y \rangle = 0$ for every $x \in V_Y$ and $y \in QV_T$. Since the space $V_T$ is self-dual, it follows that $QV_T = V_T^*$. Therefore, the columns of the matrix $Q[\Gamma I]^T$ and those of the matrix $[\Gamma' I]^T$ form bases of the same linear space, such that there must exist a nonsingular matrix $R$ satisfying $Q[\Gamma I]^T = [\Gamma' I]^T R$. More explicitly, this last equation reads

$$
\begin{bmatrix}
A\Gamma + B \\
CT + D
\end{bmatrix} = 
\begin{bmatrix}
\Gamma' R \\
R
\end{bmatrix},
$$

(18)

showing that $\Gamma \in \Delta(Q)$ and $Q(\Gamma) = \Gamma'$. This completes the proof. □

The above result will be used in section 5.1 to obtain an efficient algorithm to recognize edge-local equivalence of two given graphs.

Next we state a criterion to verify whether an adjacency matrix belongs to the domain of a given element of $C_n$.

**Proposition 2** Let $Q = [A, B, C, D] \in C_n$ with $\omega := \text{supp}(C)$ and let $\Gamma$ be an $n \times n$ adjacency matrix. Then $\Gamma \in \Delta(Q)$ if and only if $\Gamma[\omega] + D[\omega]$ is nonsingular and $\Gamma AC = BD$.

**Proof:** First we show that $CT + D$ is invertible if and only if $\Gamma[\omega] + D[\omega]$ is invertible. It follows from $Q \in C_n$ that $D[\bar{\omega}] = I$. Therefore, $CT + D$ has the form

$$
\begin{bmatrix}
I & 0 \\
\Gamma[\omega] + D[\omega]
\end{bmatrix}
$$

(19)

up to a simultaneous permutation of its rows and columns. It is now easy to see that $CT + D$ is nonsingular if and only $\Gamma[\omega] + D[\omega]$ is, which proves the first part of the proposition.

Second, we show that the matrix $(A\Gamma + B)(CT + D)^{-1}$ has zeros on its diagonal if and only if $\Gamma AC = BD$. To see this, note that for every $n \times n$ adjacency matrix $X$ and $R \in GL(n, F_2)$ the matrix $R^T XR$ is symmetric and has zero diagonal. This follows from the observation that $(R^T XR)_{ij} = R_i^T X R_j = 0$, where $R_i$ is the $i$th column of $R$ (we have $R_i^T X R_i = 0$ since $X$ is symmetric and has zero diagonal). It follows from the above that $(A\Gamma + B)(CT + D)^{-1}$ has zero diagonal if and only if $(\Gamma C + D)(A\Gamma + B)$ has, which is equivalent to

$$
(\Gamma AC + AD\Gamma + \Gamma BC + BD)_{ii} = 0
$$

(20)

for every $i = 1, \ldots, n$. Since $\Gamma$ has zero diagonal one has $(AD\Gamma)_{ii} = (\Gamma BC)_{ii} = 0$. Moreover,

$$
(\Gamma AC)_{ii} = \sum_{k=1}^n \Gamma_{ik} A_k C_k \Gamma_{ki}
$$

$$
= \sum_{k=1}^n \sum_{k=1}^n \Gamma_{ik} A_k C_k
$$

$$
= \sum_{k=1}^n \Gamma_{ik} A_k C_k = \Gamma_i^T AC,
$$

(21)
where $\Gamma_i$ is the $i$th column of $\Gamma$. This shows that (20) is equivalent to $\Gamma \hat{A}C = BD$. This proves the result. □

**Proposition 3** Let $\Gamma$ be an $n \times n$ adjacency matrix and let $Q = [A, B, C, D], Q' = [A', B', C', D'] \in C_n$ such that $\Gamma \in \Delta(Q)$ and $Q(\Gamma) \in \Delta(Q')$. Then $\Gamma \in \Delta(Q'Q)$ and $Q'(Q(\Gamma)) = (Q'Q)(\Gamma)$.

**Proof:** denoting $\Gamma' := Q'(Q(\Gamma))$, it follows from proposition 1 that $\Gamma \in \Delta(Q'Q)$ and $(Q'Q)(\Gamma) = \Gamma'$ if and only if

$$\Gamma'V\Gamma + TT + \Gamma'W + U = 0,$$

(22)

where

$$\begin{pmatrix} T & U \\ V & W \end{pmatrix} = Q'Q = \begin{pmatrix} A'A + B'C & A'B + B'D \\ C'A + D'C & C'B + D'D \end{pmatrix}. \tag{23}$$

Substituting

$$\Gamma' = (A'(A\Gamma + B)(CT + D)^{-1} + B')(C'(A\Gamma + B)(CT + D)^{-1} + D')^{-1} \tag{24}$$

yields the result after a straightforward calculation. □

**Proposition 4** Let $\Gamma$ be an $n \times n$ adjacency matrix and let $H^A \in H_n$ with $\omega = \text{supp}(A)$. Then the following statements are equivalent.

(i) $\Gamma \in \Delta(H^A)$;

(ii) $A\Gamma + A + I$ is nonsingular;

(iii) $\Gamma[\omega]$ is nonsingular.

**Proof:** the implication from (i) to (ii) is trivial. The implication from (i) to (iii) follows from proposition 2. The reverse implication is proven by using proposition 2 and noting that $A(A + I) = 0$. The equivalence of (ii) and (iii) is proven by an argument analogous to the (the first paragraph of the) proof of proposition 2. This completes the proof. □

**Proposition 5** Let $\Gamma$ be an $n \times n$ adjacency matrix and $H^A \in H_n$ with $\Gamma \in \Delta(H^A)$. Let $\omega = \text{supp}(A)$. Then $H^A(\Gamma)$ has the following structure:

$$H^A(\Gamma)[\omega] = \Gamma[\omega]^{-1} \tag{25}$$

$$H^A(\Gamma)\langle \omega \rangle = \Gamma[\omega]^{-1}\Gamma(\omega) \tag{26}$$

$$H^A(\Gamma)[\bar{\omega}] = \Gamma[\bar{\omega}] + \Gamma(\omega)^T\Gamma[\omega]^{-1}\Gamma(\omega). \tag{27}$$

**Proof:** the proof is obtained by a straightforward calculation. □

**Proposition 6** Let $\Gamma$ be an $n \times n$ adjacency matrix and let $A$ and $B$ be $n \times n$ matrices such that $\Gamma \in \Delta(H^A)$ and $H^A(\Gamma) \in \Delta(H^B)$. Then $\Gamma \in \Delta(H^{A+B})$ and $H^{B}(H^{A}(\Gamma)) = H^{A+B}(\Gamma)$.

**Proof:** This is an immediate corollary of proposition 3. □
4 Edge-local equivalence of graphs and LFTs

In this section it is our aim to prove theorem 2. Letting $G$ be a graph with $n \times n$ adjacency matrix $\Gamma$ and letting $e = \{i, j\}$ be an edge of $G$, it immediately follows from (27) and proposition 5 that

$$\Gamma \ast e = H^{E_i + E_j}(\Gamma).$$

Moreover, according to proposition 6 successively complementing $G$ along edges can also be realized as an LFT. Indeed, let $e_1, \ldots, e_N \in [V]^2$ such that $e_1$ is an edge of $G$ and $e_\alpha$ is an edge of $((G \ast e_1) \ast \ldots) \ast e_{\alpha-1}$, for every $\alpha = 2, \ldots, N$. Denoting $e_\alpha = \{i_\alpha, j_\alpha\}$ and

$$A = \sum_{\alpha=1}^N E_{i_\alpha} + E_{j_\alpha},$$

proposition 6 shows that $\Gamma \in \Delta(H^A)$ and

$$((\Gamma \ast e_1) \ast \ldots) \ast e_N = H^A(\Gamma).$$

This proves the forward implication of theorem 2:

**Proposition 7** Let $G$ and $G'$ be edge-locally equivalent graphs with $n \times n$ adjacency matrices $\Gamma$, $\Gamma'$, respectively. Then there exists a matrix $H \in \mathcal{H}_n$ such that $\Gamma \in \Delta(H)$ and $\Gamma' = H(\Gamma)$.

We now prove the reverse implication of theorem 2. To do so, define the set $\mathcal{T}_n'$ by

$$\mathcal{T}_n' = \{A\Gamma + A + I \mid A \text{ is } n \times n \text{ diagonal, } \Gamma \text{ is an } n \times n \text{ adjacency matrix}\},$$

and let $\mathcal{T}_n := \mathcal{T}_n' \cap \text{GL}(n, \mathbb{F}_2)$. For every $i = 1, \ldots, n$ consider the transformation $f_i : \mathbb{F}_2^{n \times n} \to \mathbb{F}_2^{n \times n}$ defined by

$$f_i(X) = X(E_iX + X_iE_i + I),$$

for every $n \times n$ matrix $X$. We also denote $f_{ij} := f_if_j$. We are now ready to state the following lemma.

**Lemma 1** Let $R = A\Gamma + A + I \in \mathcal{T}_n$. Then there exist $i_\alpha, j_\alpha \in \text{supp}(A)$, where $\alpha = 1, \ldots, N$, such that $f_{i_{N-1}j_{N-1}} \ldots f_{i_1j_1}(R) = I$ and

$$1 = R_{i_1j_1} = (f_{i_1j_1}(R))_{i_2j_2} = \ldots = (f_{i_{N-1}j_{N-1}}(\ldots f_{i_1j_1}(R)))_{i_{N}j_{N}} .$$

**Proof:** For every $i \in \{1, \ldots, n\}$, let $e_i$ be the $i$th canonical basis vector of $\mathbb{F}_2^n$. Fix $i_1 \in \text{supp}(A)$ and apply $f_{i_1}$ to $R$. It can easily be verified that the diagonal of $f_{i_1}(R)$ is equal to $\text{diag}(R) + R_{i_1}$, where $R_{i_1}$ is the $i_1$th column of $R$ and $\text{diag}(R)$ is the diagonal of $R$. Since $R$ is nonsingular, $R_{i_1}$ has some nonzero element, say $R_{j_1i_1} = 1$ (and one moreover has $j_1 \in \text{supp}(A)$). Therefore, the $j_1j_1$-entry of
$f_{i_1}(R)$ is equal to 1. Now we apply $f_{j_1}$, and one can verify that the $j_1$th row of $f_{j_1}f_{i_1}(R)$ is equal to $e^{T}_{j_1}$. Furthermore, the $i_1j_1$-entry of $f_{j_1}f_{i_1}(R)$ is $R_{i_1j_1}$, where, from the symmetry of $R$, one has $R_{i_1j_1} = R_{j_1i_1} = 1$. By again applying $f_{i_1}$, one finds that the $i_1$th row of $f_{i_1j_1}(R)$ is equal to $e^{T}_{i_1}$ and the $j_1$th row remains equal to $e^{T}_{j_1}$. Moreover, one can also verify that $f_{i_1}f_{j_1}(R) = A\Gamma' + A' + I \in T_n$, (33) for some $n \times n$ adjacency matrix $\Gamma'$ and some diagonal matrix $A'$ with $\text{supp}(A') \subseteq \text{supp}(A) \setminus \{i_1, j_1\}$. The rest of the proof follows by induction on $|\text{supp}(A)|$. This proves the result. □

The above result will now be used to complete the proof of theorem 2.

Proposition 8 Let $\Gamma$ be an $n \times n$ adjacency matrix and let $A$ be a diagonal matrix such that $\Gamma \in \Delta(A)$. Let $i_\alpha, j_\alpha \in \text{supp}(A)$, for every $\alpha = 1, \ldots, N$, be the sequence of vertices produced by applying lemma 1 to $A \Gamma + A + I$. Then, denoting $e_\alpha = \{i_\alpha, j_\alpha\}$, one has

(i) $e_1$ is an edge of $G$ and $e_\alpha$ is an edge of $((G * e_1) \ast \ldots) * e_{\alpha - 1}$, for every $\alpha = 2, \ldots, N$, and

(ii) $((\Gamma * e_1) \ast \ldots) * e_N = H^A(\Gamma)$,

implying that $\Gamma$ and $H^A(\Gamma)$ are edge-locally equivalent.

Proof: The result is proven by induction on $N$. We start with the basis of the induction, i.e., $N = 1$, where we consider a diagonal matrix $A$ such that $\Gamma \in \Delta(A)$ and

$$f_{i_j}(A \Gamma + A + I) = I.$$  

(34)

for some $i, j \in \text{supp}(A)$. Denoting $e = \{i, j\}$, the identity (33) implies that $R = A \Gamma + A + I$ satisfies

$$R[e] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad R(\bar{e}) = 0, \quad R[\bar{e}] = I,$$

(35)

showing that $A = E_i + E_j$. Recalling that $H^{E_i + E_j}(\Gamma) = \Gamma * e$ proves the basis of the induction.

In the induction step, we assume that the result is true for sequences $f_{j_1k_1}, \ldots, f_{j_Nk_N}$ of length $N$ and prove that this implies the result is true for sequences of length $N + 1$. Thus, we consider a diagonal matrix $A$ such that $\Gamma \in \Delta(A)$ and, denoting $R = A \Gamma + A + I$, we assume that

$$f_{i_Nj_N} \ldots f_{i_1j_1}f_{ij}(R) = I$$

(36)

and

$$1 = R_{ij} = (f_{ij}(R))_{i_1j_1} = \ldots = (f_{i_{N-1}j_{N-1}}(\ldots f_{ij}(R)))_{i_Nj_N},$$

(37)
for some $i,j,i_\alpha,j_\alpha \in \text{supp}(A)$ for every $\alpha = 1,\ldots,N$. Denoting $e = \{i,j\}$, we then define

$$
R' = f_{ij}(R),
G' = G \ast e,
\Gamma' = \Gamma \ast e,
A' = A + E_i + E_j
$$

(38)

Note that $R_{ij} = 1$ implies that $e$ is an edge of $G$, such that $G \ast e$ is well-defined. It is now straightforward to show that $R' = A'T' + A' + I$, which is the crucial observation of the proof. We then have the following situation:

- $\Gamma' \in \Delta(H^{A'})$: this follows from the invertibility of $A'T' + A' + I = R'$ and proposition 4, and
- $f_{i_1j_1} \cdots f_{i_Nj_N}(R') = I$

The induction hypothesis then implies that

(i) $e_1$ is an edge of $G'$ and $e_\alpha$ is an edge of $((G' \ast e_1) \ast \ldots) \ast e_{\alpha-1}$, for every $\alpha = 2,\ldots,N$, and

(ii) $((\Gamma' \ast e_1) \ast \ldots) \ast e_N = H^{A'}(\Gamma')$.

Moreover, we have $\Gamma' = \Gamma \ast e = H^{E_i+E_j}(\Gamma)$, such that

$$
H^{A'}(\Gamma') = H^{A'}H^{E_i+E_j}(\Gamma) = H^{A}(\Gamma).
$$

(39)

This completes the proof of the proposition. □

The proof of theorem 2 is now obtained by combining propositions 8 and 9.

5 Applications

In this section we present some applications of theorem 2. First, we present an efficient algorithm to recognize whether two given graphs are edge-locally equivalent; second, we derive a formula to count the number of graphs in a class of edge-local equivalence; third, we consider some graph invariants under edge-local equivalence; fourth, we use theorem 2 to obtain some results regarding the coefficients of the interlace polynomial of a graph; fifth, we show that a nonsingular adjacency matrix can be inverted by performing a sequence of edge-local complementations on the corresponding graph; finally, we state an equivalent version of theorem 2 in terms of graph states and local Hadamard transformations.

5.1 Recognizing edge-local equivalence efficiently

Theorem 2 allows to efficiently recognize whether two given graphs are edge-locally equivalent. To see this, let $G$ and $G'$ be two graphs with $n \times n$ adjacency
matrices $\Gamma$ and $\Gamma'$, respectively. Then $G$ and $G'$ are edge-locally equivalent if and only if there exists an $n \times n$ diagonal matrix $A$ such that $H^A(\Gamma) = \Gamma'$. Following proposition 1, this happens if and only if

$$\Gamma' A \Gamma + (A + I) \Gamma + \Gamma'(A + I) + A = 0,$$

or, equivalently,

$$\Gamma' + (A + I) \Gamma + \Gamma'(A + I) + A = 0,$$

Regarding the adjacency matrices $\Gamma$ and $\Gamma$ as given and the entries of the matrix $A$ as unknowns, this is a system of $n^2$ affine equations in $n$ unknowns, which can be solved in $O(n^4)$ time. The graphs $G$ and $G'$ are edge-locally equivalent if and only if (40) has a solution; moreover, if a solution $A$ is found, the following algorithm produces a sequence of edge-local complementations transforming $G$ into $G'$: first, using the method presented in the proof of lemma 1, one obtains $i_\alpha, j_\alpha \in \text{supp}(A)$, where $\alpha = 1, \ldots, N$, such that

$$f_{iN,jN} \ldots f_{i1,j1}(A \Gamma + A + I) = I$$

Denoting $e_\alpha = \{i_\alpha, j_\alpha\}$, it follows from proposition 9 that

$$((\Gamma * e_1) * \ldots) * e_N = H^A(\Gamma) = \Gamma',$$

yielding the desired sequence of edge-local complementation transforming $G$ into $G'$.

Note that the system of linear equations (40) can be written more explicitly as follows. Letting $A = \hat{a}$, where $a = (a_1, \ldots, a_n) \in \mathbb{F}_2^n$, and denoting by $\tilde{\Gamma}_i$ ($\tilde{\Gamma}'_i$) the $i$th column of $\Gamma + I$ ($\Gamma' + I$), for every $i = 1, \ldots, n$, the identity (40) is equivalent to

$$\sum_{i=1}^n a_i \tilde{\Gamma}'_i \otimes \tilde{\Gamma}_i = \Gamma + \Gamma'.$$

Now we let $\gamma$ ($\gamma'$) be the $n^2$-dimensional vector which is obtained by assembling the entries of $\Gamma$ ($\Gamma'$) row per row in a vector. The reader can now verify that (40) is equivalent to

$$\sum_{i=1}^n a_i \tilde{\Gamma}'_i \otimes \tilde{\Gamma}_i = \gamma + \gamma'.$$

Here $\otimes$ denotes the tensor product, or Kronecker product, of vectors. Note that (46) is an equality of $n^2$-dimensional vectors, whereas (40) was an equality of $n \times n$ matrices. Thus, we have shown that $G$ and $G'$ are edge-locally equivalent if and only if the system

$$\begin{bmatrix} \tilde{\Gamma}'_1 \otimes \tilde{\Gamma}_1 & \ldots & \tilde{\Gamma}'_n \otimes \tilde{\Gamma}_n \end{bmatrix} a = \gamma + \gamma'$$

has a solution.
5.2 Number of graphs in a class of edge-local equivalence

A second application of theorem 2 is a formula to count the number of graphs edge-locally equivalent to a given one. Let $G$ be a graph with $n \times n$ adjacency matrix $\Gamma$ and let $L_e(G)$ be the set of all graphs edge-locally equivalent to $G$. Define the following two sets

$$
\Delta_e(G) := \{ x \in \mathbb{F}_2^n \mid \Gamma \in \Delta(H^x) \} \\
\Sigma_e(G) := \{ x \in \mathbb{F}_2^n \mid H^x(\Gamma) = \Gamma \}.
$$

(47)

The set $\Delta_e(G)$ collects all operations in $H_n$ that have $\Gamma$ in their domain. The set $\Sigma_e(G)$, which is a subset of $\Delta_e(G)$, collects all operations in $H_n$ that have $\Gamma$ as a fixed point. Using theorem 2, it is clear that

$$
\left| L_e(G) \right| = \left| \Delta_e(G) \right| \left| \Sigma_e(G) \right|.
$$

(48)

First, it follows from proposition 4 that

$$
\left| \Delta_e(G) \right| = \left| \{ \omega \subseteq \{1, \ldots, n\} \mid \Gamma[\omega] \text{ is nonsingular} \} \right|.
$$

(49)

Thus, $\left| \Delta_e(G) \right|$ simply counts the number of nonsingular principal submatrices of $\Gamma$. Second, using proposition 1 we find that $\Sigma_e(G)$ consists of all $x \in \mathbb{F}_2^n$ satisfying

$$
0 = \Gamma \hat{x} \Gamma + \hat{x} \Gamma + \Gamma \hat{x} + \hat{x} = (\Gamma + I) \hat{x} (\Gamma + I).
$$

(50)

Note that $\Sigma_e(G)$ is a linear vector space of dimension at most $n$, such that $\left| \Sigma_e(G) \right| = 2^l$, where $l$ be calculated efficiently. The discussion at the end of section 5.1 shows that the space $\Sigma_e(G)$ is the null space of the $n^2 \times n$ matrix

$$
\begin{bmatrix}
\hat{\Gamma}_1 \otimes \hat{\Gamma}_1 & \ldots & \hat{\Gamma}_n \otimes \hat{\Gamma}_n
\end{bmatrix}.
$$

(51)

This implies that the dimension of $\Sigma_e(G)$ is equal to the corank of the $n^2 \times n$ matrix $\left[ \hat{\Gamma}_1 \otimes \hat{\Gamma}_1 \ldots \hat{\Gamma}_n \otimes \hat{\Gamma}_n \right]$. Further, we recall the definition of the bineighborhood space of $G$, as defined in [2]: using the notation $\nu_{ij} := (\Gamma_i \Gamma_j, \ldots, \Gamma_i \Gamma_n) \in \mathbb{F}_2^n$, for every $i, j \in \{1, \ldots, n\}$, the bineighborhood space $\nu(G)$ of $G$ is the subspace of $\mathbb{F}_2^n$ spanned by the sets of vectors

$$
\{ \sum_{(i,j) \in C} \nu_{ij} \mid C \text{ is a cycle of } G \} \quad \text{and} \quad \{ \nu_{ij} \mid (i, j) \notin E \}.
$$

(52)

We can then formulate the following properties.

**Proposition 9** Let $G$ be a graph. Then

$$
\Sigma_e(G) \subseteq \ker(\Gamma + I) \cap \nu(G)\perp.
$$

(53)
Proof: Let \( x \in \Sigma_e(G) \). Then by definition

\[
(\Gamma_i + e_i)x = ((\Gamma + I)\hat{x}(\Gamma + I))i = 0
\]

(54)

\[
\sum_{k=1}^{n} \Gamma_{ik}\Gamma_{jk}x_k + (x_i + x_j)\Gamma_{ij} = 0
\]

(55)

for every \( i,j = 1, \ldots, n \), \( i \neq j \), where \( e_i \) is the \( i \)th canonical basis vector of \( F^2_n \).

Equation (54) shows that \((\Gamma + I)x = 0\). Second, (55) implies that \( \nu^T_{ij}x = 0 \) for every \( \{i,j\} \notin E \) and that \( \nu^T_{ij}x + x_i + x_j = 0 \) for every \( \{i,j\} \in E \). It immediately follows from this last equation that \( \sum_{(i,j) \in C} \nu^T_{ij}x = 0 \) for every cycle \( C \), showing that \( x \in \nu(G)^\perp \). This ends the proof. \( \square \)

**Corollary 1** Let \( G \) be a graph with \( n \times n \) adjacency matrix \( \Gamma \). Then

\[ \log_2 |\Sigma_e(G)| \leq n - \text{rank}_2(\Gamma + I). \]  

(56)

**Corollary 2** Let \( G \) be a graph with adjacency matrix \( \Gamma \). If the girth of \( G \) is \( \geq 5 \) then \( |\Sigma_e(G)| = 1 \).

Proof: It was proven in [2] that \( \nu(G)^\perp \) is trivial for every graph with girth \( \geq 5 \). \( \square \)

5.3 **Invariants of edge-local complementation**

In this section we present some graph invariants under edge-local equivalence, where we will systematically use theorem 2 to prove that a given function is an invariant. Of course, there exist many invariants under edge-local complementation (e.g. all invariants under local complementation), and we will restrict ourselves to a handful of interesting ones.

Let \( G = (V,E) \) be a graph with \( n \times n \) adjacency matrix \( \Gamma \). First, by definition \( |L_e(G)| \) is invariant under edge-local complementation. Second, we show that the space \( \Sigma_e(G) \) is invariant. To see this, let \( x \in \Sigma_e(G) \). Further, let \( G' \) be an arbitrary graph edge-locally equivalent to \( G \), and let \( \Gamma' \) be the adjacency matrix of \( G' \). Then there exists an \( n \times n \) diagonal matrix \( A \) such that \( H^A(\Gamma) = \Gamma' \). We then have

\[ H^\hat{x}(\Gamma') = H^\hat{x}(H^A(\Gamma)) = H^{A+\hat{x}}(\Gamma) = H^A(H^\hat{x}(\Gamma)) = H^A(\Gamma) = \Gamma', \]

(57)

proving that \( x \in \Sigma_e(\Gamma') \). We have proven:

**Proposition 10** Let \( G \) be a graph with \( n \times n \) adjacency matrix \( \Gamma \). Then the space \( \Sigma_e(G) \) is invariant under edge-local complementation.

**Corollary 3** \( |\Sigma_e(G)| \) is invariant under edge-local complementation.

**Corollary 4** \( |\Delta_e(G)| \) is invariant under edge-local complementation.
Proof: this follows from the fact that $|L_e(G)|$ and $|\Sigma_e(G)|$ are invariants. □

**Corollary 5** If $G$ is a graph with adjacency matrix $\Gamma$ satisfying $\Gamma^2 = I$, i.e., $\Gamma$ is an orthogonal matrix, then every graph edge-locally equivalent to $G$ also has an orthogonal adjacency matrix.

Proof: the result follows by noting that $\Gamma^2 = I$ if and only if $(\Gamma + I)^2 = 0$, which is equivalent to stating that the all-ones vector belongs to $\Sigma_e(G)$. The result then follows from proposition 10. □

Next we show that the kernel of $\Gamma + I$ is invariant under edge-local complementation. To see this, let $G'$ be a graph which is edge-locally equivalent to $G$ and let $\Gamma'$ be its adjacency matrix, where $\Gamma' = H^{A}(\Gamma)$ for some diagonal matrix $A$. Let $x \in \mathbb{F}_2^n$ belong to the kernel of $\Gamma + I$, which is equivalent to stating that $(x, x) \in V_{\Gamma}$, where the space $V_{\Gamma}$ is defined as in (15). It follows that

$$ (x, x) = H^{A}(x, x) \in H^{A}V_{\Gamma} = V_{\Gamma'}, $$

(58)

showing that $x$ belongs to the kernel of $\Gamma' + I$. We have proven:

**Proposition 11** Let $G$ be a graph with $n \times n$ adjacency matrix $\Gamma$. Then the kernel of the matrix $\Gamma + I$ is invariant under edge-local complementation.

**Corollary 6** The rank of $\Gamma + I$ is invariant under edge-local complementation.

Two vertices $i, j \in V$ of a graph $G = (V, E)$ are called twins if $\{i, j\} \in E$ and $N(i) \setminus \{i\} = N(j) \setminus \{j\}$, i.e., if these vertices have common neighborhoods; we can now state the following result.

**Corollary 7** Edge-locally equivalent graphs have the same twins.

Proof: let $G, G'$ be graphs with adjacency matrices $\Gamma, \Gamma'$ and let $i, j \in V$ be a pair of twins of $G$; this is equivalent to stating that the $i$th and $j$th column of $\Gamma + I$ are equal, showing that the vector $x = e_i + e_j$ belongs to the kernel of this matrix, where $e_i$ ($e_j$) is the $i$th ($j$th) canonical basis vector of $\mathbb{F}_2^n$; it follows from proposition 11 that $x$ also belongs to the kernel of $\Gamma' + I$, showing that $i$ and $j$ are also twins of $G'$. □

### 5.4 Interlace polynomial

For every $k \times k$ matrix $X$ over $\mathbb{F}_2$, we let $s(X)$ denote the corank of $X$, i.e., the dimension of its kernel. Letting $G$ be a graph with adjacency matrix $\Gamma$, the interlace polynomial $q$ of $G$ has the following expansion: [19] [20]

$$ q(G, x) = \sum_{\omega \subseteq \{1, \ldots, n\}} (x - 1)^{s(\Gamma[\omega])}, $$

(59)

where by definition $s(\Gamma[\emptyset]) = 0$. It is well known [19] that edge-local complementation leaves $q(G, x)$ invariant, such that edge-locally equivalent graphs have the
same interlace polynomial (in fact, \(q\) can be defined through a recursive relation involving edge-local complementation). We will use the results presented so far in the present work to gain some insight in the coefficients of \(q(G,x)\). First, it is immediately clear from (48) and (49) that

\[
q(G,1) = |L_e(G)||\Sigma_e(G)|, \tag{60}
\]

linking this evaluation of \(q\) with the number of graphs that are edge-locally equivalent to \(G\). In particular, this result shows that \(|\Sigma_e(G)|\) divides \(q(G,1)\).

**Proposition 12** Let \(G\) be a graph with adjacency matrix \(\Gamma\) and let \(q(G,x) = \sum_{i=0}^{n} a_i x^i\). Then \(|\Sigma_e(G)|\) divides every coefficient \(a_i\). In particular, if \(\Sigma_e(G) \neq \{0\}\) then all coefficients \(a_i\) are even.

**Proof:** Let \(S \subseteq \mathbb{F}_2^n\) be an arbitrary \(n\)-dimensional linear space and let a basis of \(S\) be given by the columns of the matrix, \([Z^T X^T]^T\) where \(Z\) and \(X\) are \(n \times n\) matrices. We will say that the rank of \(S\) is equal to \(k\) if the rank of \(X\) is \(k\) (note that this definition is independent of the chosen basis). Defining \(V_{\Gamma} = \{(\Gamma u, u) \mid u \in \mathbb{F}_2^n\}\) as before, we denote

\[
L_e^{(k)}(G) = \{HV_{\Gamma} \mid H \in \mathcal{H}_n, \ \text{rank}(HV_{\Gamma}) = k\}. \tag{61}
\]

Defining

\[
\Delta_e^{(k)}(G) = \{H \in \mathcal{H}_n \mid \text{rank}(HV_{\Gamma}) = k\}, \tag{62}
\]

one has

\[
|L_e^{(k)}(G)| = \frac{|\Delta_e^{(k)}(G)|}{|\Sigma_e(G)|}. \tag{63}
\]

Moreover, it is easy to verify that \(H^A \in \Delta_e^{(k)}(G)\) if and only if \(s(\Gamma[\omega]) = n - k\), where \(A\) is an \(n \times n\) diagonal matrix with \(\omega = \text{supp}(A)\). Letting

\[
q(G,x) = \sum_{k=0}^{n} b_k (x - 1)^k \tag{64}
\]

we therefore have

\[
|L_e^{(k)}(G)||\Sigma_e(G)| = |\Delta_e^{(k)}(G)|
\]

\[
= |\{\omega \subseteq \{1, \ldots, n\} \mid s(\Gamma[\omega]) = n - k\}|
\]

\[
= b_{n-k}, \tag{65}
\]

proving that \(|\Sigma_e(G)|\) divides all coefficients \(b_k\). Since

\[
a_i = \sum_{k=1}^{n} (-1)^{i+k} \binom{k}{i} b_k, \tag{66}
\]

17
it follows that \(|\Sigma_e(G)|\) also divides the coefficients \(a_i\). □

Note that it can efficiently be tested if \(\Sigma_e(G) \neq \{0\}\) for a given graph \(G\), since one simply has to verify whether the matrix (51) is rank deficient. This condition is quite strong, such that a typical graph will not satisfy \(\Sigma_e(G) \neq \{0\}\). However, there are interesting classes of graphs which do satisfy this property. In the next proposition some sufficient conditions for \(\Sigma_e(G) \neq \{0\}\) to hold are given; this result will be used below to show that the interlace polynomials of a subclass of strongly regular graphs have even coefficients.

**Proposition 13** Let \(G = (V,E)\) be a graph such that one of the following situations (i)-(ii) occurs. Then \(\Sigma_e(G) \neq \{0\}\).

(i) \(G\) has a pair of twins;

(ii) every vertex of \(G\) has odd degree and \(|N(i) \cap N(j)|\) is even for every two different \(i, j \in V\).

**Proof:** let \(i, j\) be a pair of twins of \(G\). Letting \(\hat{\Gamma}_i\) and \(\hat{\Gamma}_j\) denote the \(i\)th and \(j\)th column of \(\Gamma + I\) as before, property (i) implies that \(\hat{\Gamma}_i = \hat{\Gamma}_j\), such that a fortiori \(\hat{\Gamma}_i \otimes \hat{\Gamma}_i = \hat{\Gamma}_j \otimes \hat{\Gamma}_j\); this shows that the matrix (46) is rank deficient, such that \(\Sigma_e(G)\) is nontrivial. If property (ii) holds, then one has \(\Gamma^2 = I\) or, equivalently, \((\Gamma + I)^2 = 0\), proving that the all-ones vector belongs to \(\Sigma_e(G)\). This yields the result. □

**Corollary 8** Let \(G\) be a strongly regular graph with parameters \((n,k,a,c)\) such that \(k\) is odd and \(a\) and \(c\) are even. Then the coefficients of \(q(G,x)\) are all even.

**Proof:** by definition, the degree of each vertex is \(k\), \(|N(i) \cap N(j)| = a\) for every \(\{i, j\} \in E\), and \(|N(i) \cap N(j)| = c\) for every \(\{i, j\} \notin E\). Using proposition 13(ii) then yields the result. □

**Example.** The Clebsch graph \(G_C\) is the unique strongly regular graph with parameters \((16, 5, 0, 2)\). Hence, the interlace polynomials of graphs \(G_C\) has even coefficients (A computer calculation showed that \(|\Sigma_e(G_C)| = 2\)).

### 5.5 Inverting an adjacency matrix

Let \(G\) be a graph with adjacency matrix \(\Gamma \in GL(n,F_2)\), i.e., the matrix \(\Gamma\) is nonsingular over \(F_2\). This is equivalent to stating that the all-ones vector \(d\) belongs to \(\Delta(G)\). Moreover, one has

\[
H^d(\Gamma) = H^I(\Gamma) = (0 \cdot \Gamma + I)(I \cdot \Gamma + 0)^{-1} = \Gamma^{-1}.
\] (67)

Thus, inverting a nonsingular adjacency matrix can be realized as an LFT. Consequently, theorem 2 shows that \(\Gamma\) and \(\Gamma^{-1}\) are edge-locally equivalent. Finally, given \(\Gamma\) one can calculate \(\Gamma^{-1}\) by applying the following sequence of
edge-local complementations: first, using the method presented in the proof of lemma 1, one obtains \(f_{i_N\ldots i_1} = I\) (68)

Denoting \(e_\alpha = \{i_\alpha, j_\alpha\}\), it follows from proposition 9 that

\[((\Gamma \ast e_1) \ast \ldots) \ast e_N = H^I(\Gamma) = \Gamma^{-1},\]  (69)

yielding the desired sequence of edge-local complementation transforming \(\Gamma\) into \(\Gamma^{-1}\).

### 5.6 Graph states and local Hadamard transformations

Let \(G\) be a graph with \(n \times n\) adjacency matrix \(\Gamma\) and let \(k_\Gamma\) be its associated quadratic form over \(\mathbb{F}_2\), i.e., \(k_\Gamma(x) = \sum_{i<j} \Gamma_{ij} x_i x_j\), for every \(x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n\). Further, define the canonical basis vectors \(u_0 = (1, 0)\) and \(u_1 = (0, 1)\) in the real vector space \(\mathbb{R}^2\). Then one defines the following \(2^n\)-dimensional real vector to be the graph state \(\psi_G\) associate by \(G\):

\[\psi_G = \frac{1}{2^{n/2}} \sum_{x \in \mathbb{F}_2^n} (-1)^{k_\Gamma(x)} u_{x_1} \otimes \ldots \otimes u_{x_n}.\]  (70)

In this section we will see that transforming a graph \(G\) by successively applying edge-local complementation or, equivalently, an LFT \(H \in \mathcal{H}_n\), has a direct translation in terms of a linear action on the graph state \(\psi_G\). To state this result, we need some additional definitions. Let \(\omega \subseteq \{1, \ldots, n\}\) and define the \(2 \times 2\) real matrices \(H_\omega^i\) by

\[H_\omega^i = \begin{cases} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \text{if } i \in \omega \\ I & \text{otherwise}, \end{cases}\]  (71)

where \(I\) is here the \(2 \times 2\) real identity matrix; the matrix \(H\) is a \(2 \times 2\) Hadamard matrix. Furthermore, the \(2^n \times 2^n\) real matrix \(H^\omega\) is defined by

\[H^\omega = H_1^\omega \otimes \ldots \otimes H_n^\omega.\]  (72)

We call the matrix \(H^\omega\) a local Hadamard matrix\(^2\). One can now formulate the following result:

**Theorem 3** Let \(G, G'\) be graphs with \(n \times n\) adjacency matrices \(\Gamma, \Gamma'\), respectively, and let \(A\) be a diagonal matrix over \(\mathbb{F}_2\). Then \(H^A(\Gamma) = \Gamma'\) if and only if

\[H^{\text{supp}(A)} \psi_G \sim \psi_{G'},\]  (73)

where \(\sim\) denotes equality up to a multiplicative constant.

\(^2\)The term *local* refers to the fact that \(H^\omega\) is an operator acting on \(\mathbb{R}^2 \otimes \ldots \otimes \mathbb{R}^2\) that can be written as a tensor product of \(n\) \(2 \times 2\) matrices.
This theorem is proven by using the so-called quantum stabilizer formalism \cite{14}. The interested reader is referred to \cite{24} for the necessary material to prove the above theorem. An immediate corollary is the following.

**Corollary 9** Two graphs $G$ and $G'$ are edge-locally equivalent if and only if there exists a set $\omega \subseteq \{1, \ldots, n\}$ such that $H^\omega \psi_G \sim \psi_{G'}$.

Thus, this result connects edge-local equivalence of graphs with the action of local Hadamard transformations on graph states.

\section{Conclusion}

In this paper we have studied edge-local equivalence of graphs. Our main result was an algebraic description of edge-local equivalence of graphs in terms of linear fractional transformations (LFTs) over GF(2) of the corresponding adjacency matrices. Using this equivalent description, we obtained, first, a polynomial time algorithm to recognize edge-local equivalence of two arbitrary graphs; the complexity of this algorithm is $O(n^4)$, where $n$ is the number of vertices of the considered graphs. Second, the description of edge-local equivalence in terms of LFTs allows to obtain a formula to count the number of graphs in a class of edge-local equivalence. Third, we considered a number of invariants of edge-local equivalence. Fourth, we proved a result concerning the coefficients of the interlace polynomial of a graph, where we showed that these coefficients are all even for a class of graphs; this class contains, among others, all graphs having an orthogonal adjacency matrix (over GF(2)), all strongly regular graphs with parameters $(n,k,a,c)$, where $k$ is odd and $a$ and $c$ are even, and all graphs having a pair of twins. Fifth, we showed that a nonsingular adjacency matrix can be inverted by performing a sequence of edge-local complementations on the corresponding graph; finally, we considered the connection between edge-local equivalence of graphs and the action of local Hadamard matrices on the corresponding graph states.

\section{Appendix: Isotropic systems and LFTs}

In this appendix it is shown that the result in theorem 1 is in fact also present in the work of Bouchet regarding local complementation and isotropic systems \cite{9,7}. In fact, the following analysis will show that the descriptions of local equivalence of graphs in terms of isotropic systems and in terms of LFTs are completely equivalent (while they constitute very different approaches to study the present subject).

We start by defining isotropic systems \cite{6}. Let $K = \{0, x, y, z\}$ be a two-dimensional vector space over $\mathbb{F}_2$. There exists a unique inner product $\langle \cdot, \cdot \rangle$ on $K$ satisfying

$$\langle a, b \rangle = \begin{cases} 1 & \text{if } 0 \neq a \neq b \neq 0 \\ 0 & \text{otherwise} \end{cases}$$
for every $a, b \in K$. Further, let $V$ be a finite set with $n := |V|$ and consider the $2^n$-dimensional vector space $K^V$ over $\mathbb{F}_2$ consisting of all $4^n$ functions
\[ v : i \in V \mapsto v(i) \in K. \tag{74} \]

One equips the space $K^V$ with the inner product $\langle \cdot, \cdot \rangle_V$, defined by
\[ \langle v, w \rangle_V = \sum_{i \in V} \langle v(i), w(i) \rangle, \tag{75} \]
for every $v, w \in K^V$. A subspace $L \subseteq K^V$ is called \textit{totally isotropic} if $\langle v, w \rangle = 0$ for every $v, w \in L$. An \textit{isotropic system} is then a pair $S = (L, V)$, where $V$ is a finite set of cardinality $n$ and $L$ is an $n$-dimensional totally isotropic subspace of $K^V$.

Let $G = (V, E)$ be a graph on the vertex set $V$. Second, let $a, b \in K^V$ be two \textit{supplementary vectors}, i.e., $0 \neq a(i) \neq b(i) \neq 0$ for every $i \in V$. Third, for every $v \in V$ and $\omega \subseteq V$, define $v[\omega] \in K^V$ by
\[ v[\omega](i) = \begin{cases} v(i) & \text{if } i \in \omega \\ 0 & \text{otherwise}. \end{cases} \tag{76} \]
It was showed in [7] that the subspace $L$ of $K^V$ spanned by the vectors
\[ \{ a[N(i)] + b[i] \mid i \in V \} \tag{77} \]
is totally isotropic and has dimension $n$, and therefore $S = (L, V)$ is an isotropic system. The ordered triple $(G, a, b)$ is called a graphic presentation of the isotropic system $S$, and $G$ is called a fundamental graph of the system $S$. Moreover, the following results hold [7]:

\textbf{Theorem 4} Every isotropic system has a graphic presentation.

\textbf{Theorem 5} Two graphs are fundamental graphs of the same isotropic system if and only if they are locally equivalent.

We will now make the connection of the above results with the theory of LFTs.

First, for every $i \in V$, define the vectors $z^i, x^i \in K^V$ by $z^i(i) = z, x^i(i) = x$ and $z^i(j) = x^i(j) = 0$ for every $j \in V \setminus \{i\}$. Note that the set $\{z^i, x^i\}_{i \in V}$ is a basis of $K^V$, i.e., every vector $v \in K^V$ can be written as a linear combination
\[ v = \sum_{i \in V} (v_z)_i z^i + (v_x)_i x^i, \tag{78} \]
where the coefficients $(v_z)_i, (v_x)_i \in \mathbb{F}_2$ are uniquely defined, for every $i \in V$. Defining $v_z, v_x \in \mathbb{F}_2^V$ by $v_z(i) = (v_z)_i$ and $v_x(i) = (v_x)_i$, for every $i \in V$, it follows that the mapping
\[ \phi : v \in K^V \mapsto \phi(v) = (v_z, v_x) \in \mathbb{F}_2^V \times \mathbb{F}_2^V \tag{79} \]
is an isomorphism between the vector spaces $K^V$ and $\mathbb{F}_2^V \times \mathbb{F}_2^V$. We will also identify the space $\mathbb{F}_2^V \times \mathbb{F}_2^V$ with the isomorphic space $\mathbb{F}_2^{2n}$. It is now straightforward to verify that $\langle v, w \rangle_V = \langle \phi(v), \phi(w) \rangle$ for every $v, w \in K^V$, where $\langle \cdot, \cdot \rangle$ is the symplectic inner product on $\mathbb{F}_2^n$ defined in the proof of proposition 1. This leads to the following result.

**Theorem 6** Let $V$ be a finite set with cardinality $n$ and let $\phi$ be the isomorphism defined in (79). Then $S = (L, V)$ is an isotropic system if and only if $\phi(L)$ is an $n$-dimensional linear subspace of $\mathbb{F}_2^{2n}$ which is self-dual w.r.t. the symplectic inner product.

Further, let $(G, a, b)$ be a graphic presentation of an isotropic system $S = (L, V)$. Denote $\phi(a) = (a_z, a_x), \phi(b) = (b_z, b_x)$ as in (79) and let $\Gamma$ be the $n \times n$ adjacency matrix of $G$. Finally, let $\Gamma_k$ denote the $k$th column of $\Gamma$ and let $e_k$ denote the $k$th canonical basis vector of $\mathbb{F}_2^n$, for every $k \in \{1, \ldots, n\}$. With these notations, it is readily verified that the image of the set (77) under the mapping $\phi$ is equal to the set

$$\{ \left[ \begin{array}{c} \hat{a}_z \\ \hat{a}_x \end{array} \right] \Gamma_k + \left[ \begin{array}{c} \hat{b}_z \\ \hat{b}_x \end{array} \right] e_k \mid k \in \{1, \ldots, n\} \}.$$  

(80)

Straightforward manipulations show that the elements of the set (80) are the columns of the matrix

$$\left[ \begin{array}{c} \hat{a}_z \\ \hat{a}_x \end{array} \right] \Gamma_k + \left[ \begin{array}{c} \hat{b}_z \\ \hat{b}_x \end{array} \right] e_k$$

(81)

Recall that the space $L \subseteq K^V$ is spanned by the set (77), as $(G, a, b)$ is a graphic presentation of $S$. It follows that the space $\phi(L) \subseteq \mathbb{F}_2^{2n}$ is spanned by the columns of the matrix (81), i.e.,

$$\phi(L) = \left\{ \left[ \begin{array}{c} \hat{a}_z \\ \hat{a}_x \end{array} \right] \Gamma u \mid u \in \mathbb{F}_2^n \right\}.$$  

(82)

This shows, in particular, that $\phi(L)$ is the image of the linear space

$$V_\Gamma := \{(\Gamma u, u) \mid u \in \mathbb{F}_2^n\} \subseteq \mathbb{F}_2^{2n}$$

(83)

under the mapping

$$Q^{a,b} := \left[ \begin{array}{c} \hat{a}_z \\ \hat{a}_x \end{array} \right] \Gamma_k + \left[ \begin{array}{c} \hat{b}_z \\ \hat{b}_x \end{array} \right] e_k.$$  

(84)

Furthermore, we have the following lemma:

**Lemma 2** Let $a, b \in K^V$ and let the corresponding matrix $Q^{a,b}$ be defined as in (84). Then $a$ and $b$ are supplementary vectors if and only if $Q^{a,b} \in \mathbb{C}_n$.
Proof: It is sufficient to prove the lemma for the case where \(|V| = 1\); the general case follows immediately. If \(|V| = 1\) then there exist exactly 6 pairs of supplementary vectors, namely

\[(z, x), (z, y), (x, y), (x, z), (y, x), (y, z).\] (85)

Also, the isomorphism \(\phi\) in this case maps \(z \mapsto (1, 0), x \mapsto (0, 1), y \mapsto (1, 1)\) and one can verify that the lemma is correct by explicitly constructing the matrices \(Q_{a,b}^n\) for all 6 pairs of supplementary vectors in \((85)\). □

We now arrive at the following result:

**Proposition 14** Let \(S = (L, V)\) be an isotropic system and denote \(n := |V|\). Let \(G = (V, E)\) be a graph with adjacency matrix \(\Gamma\). Then \((G, a, b)\) is a graphic presentation of \(S\) if and only if \(\phi(L) = Q_{a,b}^nV\Gamma\), where \(Q_{a,b}^n \in \mathbb{C}_n\).

This leads to the sought-after connection between isotropic systems and linear fractional transformations.

**Corollary 10** Let \(G\) and \(G'\) be two graphs on the same vertex set \(V\) and let \(\Gamma\) and \(\Gamma'\) be the \(n \times n\) adjacency matrices of these graphs. Then \(G\) and \(G'\) are fundamental graphs of the same isotropic system if and only if there exist a matrix \(Q \in \mathbb{C}_n\) such that \(\Gamma \in \Delta(Q)\) and \(Q(\Gamma) = \Gamma'\).

Proof: Let \(S = (L, V)\) be the isotropic system such that \(G\) and \(G'\) are fundamental graphs of \(S\). Proposition 14 yields two matrices \(Q_1, Q_2 \in \mathbb{C}_n\) such that \(Q_1V\Gamma = \phi(L) = Q_2V\Gamma'\) and therefore \(Q_2^{-1}Q_1V\Gamma = V\Gamma'\). Note that, since \(\mathbb{C}_n\) is a group, we have \(Q := Q_2^{-1}Q_1 \in \mathbb{C}_n\). Furthermore, the identity \(QV\Gamma = V\Gamma'\) is equivalent to stating that \(\Gamma \in \Delta(Q)\) and \(Q(\Gamma) = \Gamma'\) (see the proof of proposition 1). This completes the proof. □

As a final corollary, we see that theorem 1 now immediately follows from corollary 10 and theorem 5.

The above analysis shows that the descriptions of local equivalence of graphs in terms of (graphic presentations of) isotropic systems and in terms of LFTs are equivalent.

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