TRANSCENDENTAL BRAUER-MANIN OBSTRUCTION ON A PENCIL OF 
ELLiptIC CURVES

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Abstract. This note gives an explicit example of transcendental Brauer-Manin obstruction to weak 
approximation. It has two features which the only previously known example of such obstruction did 
not have: the class in the Brauer group which is responsible for the obstruction is divisible, and the 
underlying algebraic variety is an elliptic surface.

1. Introduction

Let Br(X) denote the cohomological Brauer group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ of a scheme X. Let k be a number field and $\overline{k}$ be an algebraically closed extension of k. A class in the Brauer group of a projective smooth 
variety X over k is said to be algebraic if it belongs to the kernel of the restriction map Br(X) → Br(X$k$), 
transcendental otherwise; this property does not depend on the choice of $\overline{k}$. For any prime number $\ell$, the $\ell$-primary part of the Brauer group over $\mathbb{C}$ fits into an exact sequence

$$0 \longrightarrow (\mathbb{Q}/\mathbb{Z})^{b_2-\rho} \longrightarrow \text{Br}(X_{\mathbb{C}})[\ell] \longrightarrow H^3(X(\mathbb{C}), \mathbb{Z})[\ell] \longrightarrow 0,$$

where $b_2$ and $\rho$ respectively denote the second Betti number and the Picard number of $X_{\mathbb{C}}$, and $M[\ell]$ denotes the $\ell$-primary part of M. Although this sequence does prove the non-triviality of Br(X$\mathbb{C}$) in many cases, e.g. when X is a K3 surface, transcendental classes are in general difficult to exhibit.

Almost all known instances of Brauer-Manin obstruction are thus explained by algebraic classes, the only exceptions being Harari’s examples [4] with conic bundles over $P^2_\mathbb{Q}$. Besides, in the particular case of pencils of curves of genus 1, results on the Hasse principle have been obtained only under the assumption that the 2-primary part of the Brauer group be “vertical”, and therefore algebraic (see [3], §4.7). The rôle of transcendental elements in the Brauer-Manin obstruction thus seems worthy of investigation. In this note we present an example of transcendental Brauer-Manin obstruction to weak approximation for an elliptic K3 surface over $\mathbb{Q}$, where “elliptic” means that it possesses a fibration in curves of genus 1, with a section, over $P^1_\mathbb{Q}$. It should be noted that the class of order 2 which we will exhibit in Br(X$\mathbb{C}$) enjoys the property of being divisible (because $H^3(X(\mathbb{C}), \mathbb{Z}) = 0$ for a K3 surface), which was not the case in Harari’s examples.

2. Preliminaries: 2-descent and the Brauer group of an elliptic curve

The subscript in $H^1_{\text{ét}}$ will be dropped, as we will only use étale cohomology. If G is an abelian group 
(resp. group scheme), nG will denote the n-torsion subgroup of G. Let k be a perfect field of characteristic 
different from 2. The Hilbert symbol of a pair of elements $f, g \in k^*$ will be denoted $(f, g)$; it is the class of 
a quaternion algebra in $2\text{Br}(k)$. When X is a geometrically integral variety over k and L is an extension 
of k, $L(X)$ will denote the function field of $X_L$. The canonical morphism Br(X) → Br(k(X)) is injective 
if in addition X is regular; this fact will be used without further mention. Let E be an elliptic curve 
over k whose 2-torsion points are rational. Fix an isomorphism of k-group schemes $(\mathbb{Z}/2\mathbb{Z})^2 \cong 2E$. The 
kernel of the evaluation map at the zero section Br(E) → Br(k) will be denoted Br$^0$(E).

Lemma 2.1. The group Br$^0$(E) is canonically isomorphic to $H^1(k, E)$.
\textbf{Proof.} Let us write the Leray spectral sequence for the structure morphism \( f : E \to \text{Spec}(k) \) and the étale sheaf \( \mathbf{G}_m \). Since \( f_! \mathbf{G}_m = \mathbf{G}_m, R^1f_* \mathbf{G}_m = E \otimes \mathbb{Z} \) and \( R^qf_* \mathbf{G}_m = 0 \) for \( q > 1 \) by Tsen’s theorem, we get an exact sequence
\[
\text{Br}(k) \longrightarrow \text{Br}(E) \longrightarrow H^1(k, E) \longrightarrow H^3(k, \mathbf{G}_m) \longrightarrow H^3(E, \mathbf{G}_m).
\]
The zero section induces retractions of \( \text{Br}(k) \to \text{Br}(E) \) and of \( H^3(k, \mathbf{G}_m) \to H^3(E, \mathbf{G}_m) \), hence the lemma.

By symmetry, it is enough to prove that the \textit{lemma}. The zero section induces retractions of \( \text{Br}(k) \to \text{Br}(E) \) and of \( H^3(k, \mathbf{G}_m) \to H^3(E, \mathbf{G}_m) \), hence the lemma. 

The Kummer sequence
\[
0 \longrightarrow 2E \longrightarrow E \xrightarrow{\gamma} 2E \longrightarrow 0,
\]

\[\text{together with the previous lemma and the chosen isomorphism } (\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{\sim} 2E, \text{ yields the exact sequence}
\]
\[
0 \longrightarrow E(k)/2E(k) \xrightarrow{\delta} (k^*/k^{*2})^{2} \xrightarrow{\gamma} 2\text{Br}^0(E) \longrightarrow 0.
\]

We shall need explicit descriptions of the maps \( \delta \) and \( \gamma \). First choose distinct \( p, q \in k^* \) such that the Weierstrass equation
\[
y^2 = x(x-p)(x-q)
\]
defines \( E \) and the points \( P = (p, 0) \) and \( Q = (q, 0) \) are respectively sent to \( (1, 0) \) and \( (0, 1) \) via \( 2E \xrightarrow{\sim} (\mathbb{Z}/2\mathbb{Z})^2 \). It is well-known (see e.g. [9], p. 281) that \( \delta(M) = (x(M) - q, x(M) - p) \) for \( M \in E(k) \) if \( M \notin 2E(k), \) that \( \delta(P) = (p - q, p(p - q)) \) and that \( \delta(Q) = (q(q - p), q - p). \)

\textbf{Proposition 2.2.} Let \( f, g \in k^* \). The classes of the quaternion algebras \( (x-p, f) \) and \( (x-q, g) \in \text{Br}(k(E)) \) actually belong to \( \text{Br}^0(E) \), and \( \gamma(f, g) = (x - p, f) + (x - q, g). \)

\textbf{Proof.} By symmetry, it is enough to prove that \( \gamma(f, 1) = (x - p, f) \) in \( \text{Br}(k(E)). \) Choose a separable closure \( \overline{k} \) of \( k \) and let \( G_k \) be its Galois group over \( k. \) Likewise, choose a separable closure \( k(E) \) of \( k(E) \) and let \( G_{k(E)} \) be its Galois group over \( k(E) \). It follows from the Hochschild-Serre spectral sequence, Tsen’s theorem and Hilbert’s theorem 90 that the inflation map \( H^2(k, \overline{k}(E)^*) \to \text{Br}(k(E)) \) is an isomorphism.

Let \( p : H^1(k, E) \to H^2(k, \overline{k}(E)^*/\mathbb{F}) \) denote the composition of the canonical isomorphism \( H^1(k, E) \xrightarrow{\sim} H^1(k, \text{Pic}(E_{\overline{k}})) \) and the boundary of the exact sequence
\[
0 \longrightarrow \overline{k}(E)^*/\mathbb{F} \longrightarrow \text{Div}(E_{\overline{k}}) \longrightarrow \text{Pic}(E_{\overline{k}}) \longrightarrow 0.
\]

As shown in the annexe of [2], the diagram
\[
\begin{array}{ccc}
\text{Br}(k) & \xrightarrow{\theta} & H^1(k, E) \\
\cap & & \downarrow \rho \\
H^2(k, \overline{k}(E)^*) & \xrightarrow{\iota} & H^2(k, \overline{k}(E)^*/\mathbb{F})
\end{array}
\]

commutes, where \( \theta \) denotes the map which stems from the Leray spectral sequence (see lemma 2.1). This enables us to carry out cocycle calculations for determining the image of \( \gamma(f, 1) \) in \( H^2(k, \overline{k}(E)^*/\mathbb{F}). \) We shall use the standard cochain complexes. Let \( \chi_f : G_k \to \mathbb{Z} \) be the map with image in \( \{0, 1\} \) whose composition with the projection \( \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \) is the quadratic character associated with \( f \in k^*/k^{*2} = H^1(G_k, \mathbb{Z}/2\mathbb{Z}). \) The image of \( (f, 1) \) in \( H^1(k, E) \) is represented by the 1-cocycle \( a : \sigma \mapsto \chi_f(\sigma)p. \) If \( M \in E(k) \), let \( |M| \) denote the corresponding divisor on \( E_{\overline{k}}. \) The 1-cocohain with values in \( \text{Div}(E_{\overline{k}}) \) defined by \( \sigma \mapsto \chi_f(\sigma)(|P| - |0|) \) is a lifting of \( a. \) Its differential \( (\sigma, \tau) \mapsto (\chi_f(\sigma) + \chi_f(\tau) - \chi_f(\sigma\tau))(|P| - |0|) \) is, as expected, a 2-cocycle with values in \( \overline{k}(E)^*/\mathbb{F}, \) which we may rewrite as \( (\sigma, \tau) \mapsto (x - p)^{\chi_f(\sigma)\chi_f(\tau)}; \) it
represents the image of \( \gamma(f, 1) \) in \( H^2(k, k(E)^*/k) \). Since \( x - p \) is invariant under \( G_k \), the same formula defines a 2-cocycle on \( G_k \) with values in \( k(E)^* \). We thus end up with a 2-cocycle 

\[
b: G_k(E) \times G_k(E) \to k(E)^*
\]

which represents the image of \( \gamma(f, 1) \) in \( \text{Br}(k(E)) \), at least modulo \( \text{Br}(k) \), where \( \chi_n \) now denotes the lifting with values in \( [0, 1] \) of the quadratic character on \( k(E) \) associated with \( m \in k(E)^* \). (Note that \( k \) is separably closed in \( k(E) \), so that \( G_k \) identifies with a quotient of \( G_{k(E)} \).) Choose a square root \( s \) of \( x - p \) in \( k(E) \). Dividing \( b \) by the differential of the 1-cochain \( \sigma \to s\chi_1(\sigma) \) gives the 2-cocycle 

\[
(\sigma, \tau) \mapsto (x - p)\chi_1(\sigma)\chi_1(\tau),
\]

which does represent the image of the cup-product \( (x - p)\cup f \) by the composite map \( H^1(k(E), \mathbb{Z}/2\mathbb{Z}) \to H^2(k(E), \mathbb{Z}/2\mathbb{Z}) \to \text{Br}(k(E)) \).

We have now proved that \( \gamma(f, 1) = (x - p, f) \) in \( \text{Br}(k(E))/\text{Br}(k) \), but the equality holds in \( \text{Br}(k(E)) \) since \( (x - p, f) = (y^2/(x - p)^3, f) \) evaluates to 0 at the zero section. \( \square \)

3. An actual example

The reader is referred to [4] for the definitions of weak approximation, Brauer-Manin obstruction, residue maps and unramified Brauer group.

Let \( \Omega \) denote the set of places of \( \mathbb{Q} \). Define the polynomials \( p, q \in \mathbb{Q}[t] \) by \( p(t) = 3(t - 1)^3(t + 3) \) and \( q(t) = p(-t) \). It will be useful to notice that \( p(t) - q(t) = 48t \). Let \( E \) be the elliptic curve over \( \mathbb{Q}(t) \) defined by [2]. Denote by \( \mathcal{E} \) its minimal proper regular model over \( \mathbb{P}^1_{\mathbb{Q}} \) (see [8]); it is a smooth surface over \( \mathbb{Q} \) endowed with a proper flat morphism \( f: \mathcal{E} \to \mathbb{P}^1_{\mathbb{Q}} \) whose generic fibre is isomorphic to \( E \). A geometric fibre of \( f \) is either smooth or is a union of rational curves whose intersection numbers may be computed with Tate’s algorithm [10]. One finds the following reduction types, in Kodaira’s notation [5]: \( I_2 \) above \( t = 0 \), \( t = 3 \) and \( t = -3 \); \( I_6 \) above \( t = 1, t = -1 \) and \( t = \infty \); the other fibres are smooth. Recall that a fibre of type \( I_n \) has \( n \) irreducible components \( (C_i)_{1 \leq i \leq n} \), with \( (C_i, C_{i+1}) = 1 \), \( (C_1, C_n) = 1 \) and \( (C_i, C_j) = 0 \) if \( n - 1 > |j - i| > 1 \). Put

\[
A = \gamma(6(t(t + 1), 6t(t - 1))) = (x - p, 6t(t + 1)) + (x - q, 6t(t - 1)) \in \text{Br}(E).
\]

**Proposition 3.1.** The class \( A \in \text{Br}(E) \) belongs to the subgroup \( \text{Br}(\mathcal{E}) \).

**Proof.** Let \( v \) be a discrete rank 1 valuation on \( \mathbb{Q}(\mathcal{E}) \) whose restriction to \( \mathbb{Q} \) is trivial, and \( \kappa \) be its residue field. We shall prove that \( A \) has trivial residue at \( v \). Let us choose a uniformiser \( \pi \) of \( v \) and put \( \tilde{z} = z\pi^{-v(z)} \) for \( z \in \mathbb{Q}(\mathcal{E})^* \). It will be convenient to denote by \( V: \mathbb{Q}(\mathcal{E})^* \to \mathbb{Z} \times \kappa^* \) the group homomorphism \( z \to (v(z), [\tilde{z}]) \), where \([u]\) denotes the class in \( \kappa \) of \( u \in \mathbb{Q}(\mathcal{E})^* \) if \( v(u) = 0 \). For \( f, g \in \mathbb{Q}(\mathcal{E})^* \), the residue of the quaternion algebra \((f, g)\) at \( v \) is given by the tame symbol formula

\[
\partial_v(f, g) = (-1)^{v(f)\cdot v(g)} \left[ f^{\pi(\tilde{g})} \right] \left[ g^{\pi(\tilde{f})} \right] = (-1)^{v(f)\cdot v(g)} [\tilde{f}]^{v(g)} [\tilde{g}]^{v(f)} \in \kappa^*/\kappa^{\times 2}.
\]

Note that it only depends on \( V(f) \) and \( V(g) \). Furthermore, if \( V(f) \) is a double, i.e. if \( v(f) \) is even and \( \tilde{f} \) is a square modulo \( \pi \), then \( \partial_v(f, g) = 1 \). These remarks will be used implicitly throughout the proof.

**Lemma 3.2.** The class \( (-p, 6t(t + 1)) + (-q, 6t(t - 1)) \in \text{Br}(\mathbb{Q}(t)) \) is unramified over \( \mathbb{P}^1_{\mathbb{Q}} \).

**Proof.** The residue at a closed point of \( \mathbb{P}^1_{\mathbb{Q}} \) other than \( t = \alpha \) for \( \alpha \in \{ -3, -1, 0, 1, 3, \infty \} \) is obviously trivial. It is straightforward to check that the remaining residues are also trivial. \( \square \)

Let us now turn to showing that \( \partial_v(A) = 1 \). As \( A \) is invariant under \( t \mapsto -t \), we may assume \( v(p) \leq v(q) \). If \( v(x) < v(p) \), then \( V(x - p) = V(x - q) = V(x) \), from which we deduce thanks to (2) that \( V(x - p) = V(x - q) = V(\gamma(x)) \), hence the result by lemma 3.2. From now on, we may and will therefore assume \( v(p) \leq v(x) \leq v(q) \). To begin with, suppose \( v(p) < v(q) \). In this case, either \( v(t - 3) > 0 \) or \( v(t + 1) > 0 \). If \( v(x) = v(q) \), then \( V(x - p) = V(\gamma(x)) \), hence \( \partial_v(A) = \partial_v(-q(x - q), 6t(t - 1)) \) by lemma 3.2 but with a look at (2), one finds that both \( v(-q(x - q)) \) and \( v(6t(t - 1)) \) are even. Suppose now \( v(x) < v(q) \). It follows from (2) that
V(x - p) is a double, hence \( \partial_v(A) = \partial_v(x - q, 6t(t - 1)) = \partial_v(x, 6t(t - 1)) \). If \( v(x) \) is even or if \([6t(t - 1)]\) is a square in \( \kappa \), which happens if \( v(t - 3) > 0 \), we get \( \partial_v(A) = 1 \). If on the other hand \( v(t) > 1 \) and \( v(x) \) is odd, then \([6t(t - 1)] = 12\), which (2) shows to be a square in \( \kappa \).

We are now left with the case \( v(p) = v(q) = v(x) \). If \( v(t) = 0 \), then \( v(t - 3) = v(t - 1) = v(t + 1) = v(t + 3) = 0 \), so \( v(6t(t + 1)) = v(6t(t - 1)) = 0 \) and it suffices to prove that \( v(x - p) \) and \( v(x - q) \) are even, which follows from (2) and the equality \( v(p) = v(q) = v(q - p) = 0 \). If \( v(t) < 0 \), then \( V(6t(t + 1)) = V(6(t - 1)) \), so that \( \partial_v(A) = \partial_v(x, 6t(t + 1)) \), which is trivial since both \( v(x) = v(p) = 4v(t) \) and \( v(6t(t + 1)) \) are even. Suppose finally that \( v(t) > 0 \). If \( v(x - p) < v(t) \), then \( V(x - p) = V(x - q) \) since \( v(p - q) = v(t) \), and \( \partial_v(A) = \partial_v(x - p, t(1)(t - 1)) = \partial_v(x - p, -1) \); if \( v(x - p) = 0 \), the residue is obviously trivial, and if \( v(x - p) > 0 \), which means that \( [\tilde{e}] = [\tilde{p}] = -9 \), (2) shows that \( -1 \) is a square in \( \kappa \). We therefore assume \( v(x - p) \geq v(t) \), which still leads to \( [\tilde{e}] = [\tilde{p}] = -9 \). As \( v(p - q) = v(t) \), at least one of \( v(x - p) \) and \( v(x - q) \) is equal to \( v(t) \). In either case, (2) implies that \( v(x - p) + v(t) \) is even, so \((-9)^{v(t)(-1)^{v(x-p)}}\) is a square, hence \( \partial_v(A) = \partial_v(x, 6t(t - 1)) + \partial_v(x - p, t(1)(t - 1)) \) is trivial. \( \square \)

We shall now prove the following.

**Theorem 3.3.** The class \( A \in Br(\mathcal{E}) \) is transcendental and yields a Brauer-Manin obstruction to weak approximation on the projective smooth surface \( \mathcal{E} \) over \( Q \).

**Proof.** Let us first deal with the second part of the assertion. A glance at equation (2) shows that \( \mathcal{E} \) has a \( \mathbb{Q}_2 \)-point \( M_2 \) with coordinates \( x = 1 \) and \( t = 2 \). (Indeed, this equation defines an affine surface over \( Q \), endowed with a morphism to \( P^1_Q \) whose smooth locus identifies with an open subset of \( \mathcal{E} \).) Using the formula given in \( \Pi \), Ch. XIV, §4, one easily checks that \( A(M_2) \) is non-trivial. Now choose \( N \in \mathcal{E}(Q) \) in the image of the zero section and let \( M_n \in \mathcal{E}(Q_n) \) be equal to \( N \) for any \( v \in Q \setminus \{2\} \). This defines an adelic point \( (M_n)_{v \in \Omega} \). The class \( A(N) \in Br(Q) \) is trivial since \( A \in Br^3(E) \); consequently, the evaluation of \( A \) at \( (M_n)_{v \in \Omega} \) is non-trivial, which is an obstruction to weak approximation.

It remains to be shown that \( A \) is transcendental. The exact sequence (1) reduces this to the computation of \( E(C(t))/2E(C(t)) \).

**Lemma 3.4.** The surface \( \mathcal{E} \) is a K3 surface.

**Proof.** The topological Euler-Poincaré characteristic \( e(\mathcal{E}_C) \) of \( \mathcal{E}_C \) can be expressed in terms of that of the fibres and that of the base (\( \Pi \), p. 97, prop. 11.4), which leads to \( e(\mathcal{E}_C) = 24 \). Let \( \chi(\mathcal{E}) \) denote the Euler-Poincaré characteristic of the coherent sheaf \( \mathcal{E} \). The canonical bundle \( \mathcal{K} \) of \( \mathcal{E} \) is simply \( f^*(\mathcal{O}(\chi(\mathcal{E}) - 2)) \) (see \( \Pi \), p. 162, cor. 12.3); in particular it has self-intersection 0, hence \( \chi(\mathcal{E}) = 2 \) by Noether’s formula. We have now proved the triviality of \( \mathcal{K} \). That \( H^1(\mathcal{E}, \mathcal{O}) = 0 \) follows from \( \chi(\mathcal{E}) = 2 \) and Serre duality. \( \square \)

**Lemma 3.5.** The elliptic curve \( E \) has Mordell-Weil rank 0 over \( C(t) \).

**Proof.** Let \( \rho(\mathcal{E}_C) \) be the Picard number of \( \mathcal{E}_C \) and \( R \) be the subgroup of the Néron-Severi group \( NS(\mathcal{E}_C) \) spanned by the zero section and the irreducible components of the fibres. As follows from the output of Tate’s algorithm, \( R \) has rank 20. On the other hand, \( \rho(\mathcal{E}_C) \leq 20 \) since \( \mathcal{E} \) is a K3 surface. The Shioda-Tate formula

\[
\rho(\mathcal{E}_C) = \text{rank}(E(C(t))) + \text{rank}(R)
\]

thus yields the result. \( \square \)

This lemma shows that the \( \mathbb{F}_2 \)-vector space \( E(C(t))/2E(C(t)) \) has dimension 2. Now the classes \( \delta(P) = (t, t(t - 1)(t + 3)) \) and \( \delta(Q) = (t(t + 1)(t - 3), t) \) are independent over \( \mathbb{F}_2 \), hence span the whole kernel of \( \gamma \). On the other hand \( (t(t + 1), t(t - 1)) \) is evidently not a combination of \( \delta(P) \) and \( \delta(Q) \), so that \( A \) has non-zero image in \( Br(C(\mathcal{E})) \) and is therefore transcendental. \( \square \)

**Remark 3.6.** It is actually true that \( A(M) = 0 \) in \( Br(Q) \) for all \( M \in \mathcal{E}(Q) \). This is a consequence of the global reciprocity law and the fact that \( A \) vanishes on \( \mathcal{E}(Q_v) \) for all \( v \in Q \setminus \{2\} \), which can be checked by a tedious computation.
Remark 3.7. It is possible to determine $2\text{Br}(\mathcal{E})$ completely if one is willing to compute explicit equations for $\mathcal{E}$. This involves blowing up the singular surface given by equation (2) a sufficient number of times. Alternatively, one may observe that all fibres have type $I_n$ (in other words, $\mathcal{E} \to \mathbb{P}_Q^1$ is semi-stable), and then use the equations given by Néron in this case in [6], §III. Either way one finds that $2\text{Br}(\mathcal{E})$ is spanned by $\mathcal{A}$ modulo $2\text{Br}(\mathbb{Q})$ after writing out all possible residues of a general class $\gamma(f, g)$. On the other hand, the 2-torsion subgroup of the Brauer group of a complex $K3$ surface with Picard number 20 has rank 2 over $\mathbb{F}_2$, so $2\text{Br}(\mathcal{E}_C)$ is strictly larger than $2\text{Br}(\mathcal{E})/2\text{Br}(\mathbb{Q})$. It turns out that $2\text{Br}(\mathcal{E}_C)$ is spanned by $\mathcal{A}$ and the class of the quaternion algebra $(x, t)$, which unexpectedly belongs to $\text{Br}(\mathbb{Q}(\sqrt{-1}, \sqrt{3}))$.

Remark 3.8. In the semi-stable case, a computer program was written to carry out the calculations alluded to in the previous paragraph, as they often get quite lengthy. Its source code is available on request.

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