SOLUTIONS OF $D_\alpha = 0$
FROM HOMOGENEOUS INVARIANT FUNCTIONS

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Abstract

We prove that the existence of a homogeneous invariant of degree $n$ for a representation of a semi-simple Lie group guarantees the existence of non-trivial solutions of $D_\alpha = 0$: these correspond to the maximum value of the square of the invariant divided by the norm of the representation to the $n^{th}$ power.

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The search of solutions of the equation:

\[ D_\alpha = 0 \]  

is a necessary tool for the classification of non-trivial supersymmetric vacua \([1]\). Several years ago a sufficient condition was proposed for the validity of (1), namely the existence of an invariant \( F(z) \), such that:

\[ \left( \frac{\delta F}{\delta z_a} \right) z_a = k z_a^* \]  

with \( k \neq 0 \) \([2]\). On the basis of absence of known counterexamples the condition has been conjectured to be also necessary \([3]\) and the proof has been given by Procesi and Schwarz \([4]\). Here we want to show that a sufficient condition for the existence of non-trivial solutions of (1) is the existence of a homogeneous invariant \( F^n(z) \) of degree \( n \). The proof goes as follows.

Be \( F(z_a) \) a homogeneous invariant of degree \( n \); let us consider the function:

\[ G(z_a, z_a^*) = \frac{F(z_a)F(z_a^*)}{N(z_a)^n} \]  

where:

\[ N(z_a) = \Sigma_a z_a z_a^* . \]  

Let us look for the maxima of \( G \), which is real, positive and homogeneous of degree 0 in the domain:

\[ \frac{1}{2} \leq N(z_a) \leq 1 . \]  

A maximum certainly exists, since \( G \) is a continuous function of the real variables \( x_a = \frac{z_a + z_a^*}{2} \) and \( y_a = \frac{z_a - z_a^*}{2i} \) defined on a closed and limited set (Weierstrass theorem). Since \( G \) is a homogeneous function of degree 0, it is constant along the intersection of the radii, which start from the origin, with the set defined by (3). Therefore the maximum will occur also on an internal point \( z_a^0 \) of the domain, where:

\[ \frac{\delta G}{\delta z_a z_a = z_a^0} = \left( \frac{\delta F}{\delta z_a} F(z_a) N(z_a) - n z_a^* F(z_a) F(z_a^*)}{N(z_a)^{n+1}} \right) \]  

should vanish. At that point, \( F(z_a) \) and \( F(z_a^*) \) are different from zero, since the maximum of a positive function, which is not identically zero, is positive. \( N(z) \) is also positive, so from (3) we get:

\[ \frac{\delta F}{\delta z_a z_a = z_a^0} = k z_a^{0*} \]  

with \( k = \frac{n F(z_a)}{N(z_a)} \neq 0 \), which guarantees \([4]\) the vanishing of the \( D \) term for \( z_a^{(*)} = z_a^{0(*)} \). To give some application of the theorem just shown, let us consider an irreducible representation \( \phi \) of a semi-simple Lie group \( G \). We may decompose the symmetric product of two \( \phi \)'s along irreducible representations of \( G \):

\[ (\phi \times \phi)_S = \Sigma_l \phi_l \]
The number of independent quartic invariants, bilinear in $\phi$ and its complex conjugate $\phi^*$, is given by the number of terms in the r.h.s. of (8). More precisely they are:

$$I_l = N(\phi \times \phi)_{l}.$$  

(9)

Also the invariant:

$$I_\alpha = N(\phi^* \times \phi)_{adjoint},$$

(10)

which is proportional to $D_{\alpha}D_{\alpha^*}$, is a combination of the $I_l$'s. If the sum in (8) has only one term, there is only one independent quartic invariant, bilinear in $\phi$ and $\phi^*$, $I_\alpha$ is proportional to $N(\phi)^2$ and there is no non-trivial solution to (1). In that case we can conclude that we cannot write an invariant, which depends only on $\phi$. Let us now consider three cases, where there is a cubic invariant built in terms of an irreducible complex representation of a simple group: according to the theorem just shown, there will be solutions of (1). We shall consider the 6 of $SU(3)$, the 15 of $SU(6)$ and the 27 of $E(6)$ and their symmetric products:

$$(6 \times 6)_S = 15 + \bar{6}$$

$$(15 \times 15)_S = 105 + 15$$

$$(27 \times 27)_S = 351 + 27$$

(11)

So there are two independent quartic invariants bilinear in the 6 (or the 15 or the 27) and in the $\bar{6}$ (or the $\bar{15}$ or the $\bar{27}$), and one has:

$$18 \ N(6 \times 6)_6 + 15 \ N(6 \times \bar{6})_8 = 8 \ N(6)^2$$

$$9 \ N(15 \times 15)_{15} + 12 \ N(15 \times \bar{15})_{35} = 8 \ N(15)^2$$

$$15 \ N(27 \times 27)_{27} + 9 \ N(27 \times \bar{27})_{78} = 2 \ N(27)^2$$

(12)

where the second terms in the l.h.s.'s of (12) are just proportional to $D_{\alpha}D_{\alpha^*}$. The two terms in the l.h.s.'s of (12) have the intriguing property that one vanishes when the other takes its maximum. The vanishing of the second term when the second takes its maximum is a consequence of the theorem we have just shown. In fact, when $\phi_a$ is on a critical orbit of an irreducible representation $\phi$, the invariants:

$$\frac{N(6 \times 6)_6}{N(6)^2} \quad \text{and} \quad \frac{N(6 \times 6 \times 6)_1}{N(6)^3}$$

are both proportional to $(C\phi_a \bar{\phi}_a \bar{\phi}_a \phi_a)^2$, which implies that, if $\phi_a$ is a maximum for the first one, it is a maximum for the second one as well. (The implication in the opposite direction does not hold for a non-critical orbit, since in that case $N(\phi \times \phi)_{\phi^*}$ receives contributions also from $\phi^* \neq \phi_a^*$. A similar property holds for the 15 of $SU(6)$ and the 27 of $E(6)$. The $D$ term vanishes in the $SO(3)$, $Sp(6)$ or $F(4)$ invariant direction respectively. From the other side the first terms in (12) vanish in the $SU(2)$, $SU(4) \times SU(2)$ or $SO(10)$ invariant direction, respectively. This is not surprising, because, when $\varphi_a$ correspond to the maximal weight of the representation, as in the cases just mentioned, $(\varphi_a \times \varphi_a)$ has components only along the representation with the highest maximal weight. The necessity of the existence of an invariant for the existence of solutions of (1), already proved in [3], implies that no invariant can be constructed with a
representation unable to supply non-trivial solutions to \([\Xi]\). It applies to the case with only one term in the r.h.s. of \((8)\) and \(D_\alpha D^\alpha\) proportional to \(N(\phi)^2\), but also to cases with more than one term present in \((8)\). As an example, let us consider the 16 spinorial representation of \(SO(10)\), for which:

\[
(16 \times 16)_S = 126 + 10 \\
32 \ N(16 \times 16)_{10} + 16 \ N(16 \times \bar{16})_{45} = 5 \ N(16)^2 .
\]  (13)

While the maximum of the second term in the l.h.s. of \((13)\) in the \(SU(5)\) invariant direction corresponds to the vanishing of the first term \([\Xi]\), the maximum of the first term in the \(SO(7)\) invariant direction corresponds to a minimum, but not to a vanishing value for \(D_\alpha D^\alpha\). In fact no \(SO(10)\) invariant can be built only with a 16.

For semi-simple groups we shall consider the bifundamental \((N, \bar{M})\) representation of \(SU(N) \times SU(M)\). For \(M = N\), the existence of the invariant:

\[
\varepsilon^{\gamma_1 \ldots \gamma_N} \phi_{\alpha_1}^\gamma \ldots \phi_{\gamma_N}^\alpha
\]  (14)

implies the existence of a solution of \((\Xi)\), namely the singlet under the sum of the two \(SU(N)\). If \(N > M\) the contribution \(\phi_M^2\) to \(D_\alpha D^\alpha\) from \(SU(N)\) cannot vanish, since

\[
\frac{(D_\alpha D^\alpha)_{SU(N)}}{g_N^2} > \frac{(D_\alpha D^\alpha)_{SU(M)}}{g_M^2}.
\]  (15)

and the r.h.s. of \((15)\) is non-negative: in fact no invariant can be built in that case. For \(N = 3, M = 2\) one has

\[
(\phi \times \phi)_s = (6,3) + (3,1)
\]  (16)

and \(N(\phi \times \phi)_{(3,1)}\) takes its maximum in the direction where the contribution to \(D_\alpha D^\alpha\) from \(SU(2)\) (but not from \(SU(3)\)) vanishes.

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