Abstract

In this article we give in analytical closed form the solutions of the
Dirichlet problems for the Laplace equations with inverse square and
singular Pöschl-Teller potentials.

Keywords: Poisson kernel, Inverse square potential, Dirichlet problem, sin-
gular Pöschl-Teller potentials, Legendre function, Hypergeometric function.
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1 Introduction

The Laplace equation is used in many contexts. For example, in potential
theory, in electrostatics and in complex analysis as well as in Riemannian
geometry.

This study shows in the unified way the explicit formulas for the following
three equations of Laplace type:

a) The Laplace equation associated to the Schrödinger operator with the
inverse square potential on \( R^+ \)

\[
\begin{align*}
\left\{ 
& L_\nu u(Y, X) + \frac{\partial^2}{\partial X^2} u(Y, X) = 0, \quad (Y, X) \in \mathbb{R}_+ \times \mathbb{R}_+ \\
& u(0, X) = u_0(X) \in C^\infty_0(\mathbb{R}_+^*). 
\end{align*}
\]

where

\[
L_\nu = \frac{\partial^2}{\partial X^2} + \frac{1/4 - \nu^2}{X^2},
\]

is the Schrödinger operators with inverse square potential.
b) The Laplace type equations associated to the Schrödinger operators with the singular trigonometric Pöschl-Teller potentials

\[
\begin{aligned}
& \left\{ L^T_{\nu} v(y, \theta) + \partial^2_y v(y, \theta) = 0, \quad (y, \theta) \in \mathbb{R}_+ \times [0, \pi] \\
& v(0, \theta) = v_0(\theta) \in C_0^\infty([0, \pi]).
\end{aligned}
\]  

(1.3)

where

\[
L^T_{\nu} = \frac{\partial^2}{\partial \theta^2} + \frac{1/4 - \nu^2}{16 \sin^2 \theta},
\]  

(1.4)

is the Schrödinger operators with singulars trigonometric Pöschl-Teller potential.

c) The Laplace type equations associated to the Schrödinger operators with the singular hyperbolic Pöschl-Teller potentials

\[
\begin{aligned}
& \left\{ L^H_{\nu} w(y, x) + \partial^2_y w(y, x) = 0, \quad (y, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \\
& w(0, x) = w_0(x) \in C_0^\infty(\mathbb{R}_+).
\end{aligned}
\]  

(1.5)

where

\[
L^H_{\nu} = \frac{\partial^2}{\partial x^2} + \frac{1/4 - \nu^2}{16 \sinh^2 x},
\]  

(1.6)

is the Schrödinger operators with singulars hyperbolic Pöschl-Teller potential. That is the researcher computes explicitly the Schwartz integral kernels of the following Poisson semi-groups \( \exp (-Y \sqrt{-L_{\nu}}) \), \( \exp (-\theta \sqrt{-L^T_{\nu}}) \), and \( \exp (-y \sqrt{-L^H_{\nu}}) \).

The Schrödinger operator with inverse square potential \( L_{\nu} \) arises in the contexts of the Schrödinger equation in non relativistic quantum mechanics [15], the molecular physics and the quantum cosmology as well as the linearization of combustion models [1] and [2].

For example, the Hamiltonian for a spinzero particle in Coulomb field gives rise to a Schrödinger operator involving the inverse square potential [5].

The Schrödinger equation with the Pöschl-Teller potentials was studied a long time ago. First in 1882 by Darboux [8], then independently by Pöschl and Teller in 1933 [13].

The importance of the Pöschl-Teller potentials in mathematics, physics and chemistry is well known. These potentials represent one of the most studied anharmonic systems. On the one hand, the Schrödinger equation with this potential plays an important role in many body integrable systems [6] and [11] in soluton mathematics, from which the multi-soluton solutions of the nonlinear Korteweg-de Vries (KdV) equation can be explicitly constructed [9] and [16], and in the Hartree mean field equation of many body...
systems interacting through a delta force [7] and [17]. On the other hand the Schrödinger equation with singular trigonometric and hyperbolic Pöschl-Teller potentials can be also regarded as Schrödinger equations with inverse square potential on the one dimensional spherical and the hyperbolic spaces respectively.

The rest of the paper is organised as follows, this section is ended by recall about the Hankel transform, in section 2, the solution of Laplace equation with inverse square potential (1.1) is given in a closed form in terms of Legendre functions of the second kind. In sections 3 and 4 the solutions of Laplace equations with singular trigonometric and hyperbolic Pöschl-Teller potentials are given in a closed form.

Now some facts about the Hankel transform are given (see [4], [12]). When \( \nu > -1 \), the Hankel transform of order \( \nu \) is defined as

\[
(\mathcal{H}_\nu f)(\Omega) = \int_0^\infty (X\Omega)^{1/2} J_\nu(X\Omega) f(X) dX
\]

(1.7)

where \( f \in C_0^\infty(\mathbb{R}^*_+) \) and \( J_\nu \) is the Bessel function of the first kind and order \( \nu \).

**Proposition 1.1.** (see [4] and [12]) The Hankel transform has the following properties:

i) \( \mathcal{H}_\nu^2 = \text{Id} \).

ii) \( \mathcal{H}_\nu \) is self-adjoint.

iii) \( \mathcal{H}_\nu \) is an \( L^2 \)-isometry

iv) \( \mathcal{H}_\nu L^E = -\Omega^2 \mathcal{H}_\nu \).

2 Laplace equation with inverse square potential on \( \mathbb{R}^+ \)

**Theorem 2.1.** The Dirichlet problem (1.1) for the Laplace equation with inverse square potential on the Euclidian line has the unique solution given by

\[
u(Y, X) = \int_0^\infty \mathcal{P}_\nu(Y, X, X') u_0(X') dX'
\]

(2.1)
where

\[ P_\nu(Y, X, X') = \frac{-4Y}{\sqrt{Y^2 + (X + X')^2} \sqrt{Y^2 + (X - X')^2}} Q_{\nu - 1/2}^1 \left( \frac{Y^2 + X^2 + X'^2}{2XX'} \right) \tag{2.2} \]

where \( Q_{\nu - 1/2}^\mu \) is the associated Legendre function of the second kind given in terms of the Gauss hypergeometric as

\[ Q_{\nu - 1/2}^\mu(z) = \frac{\sqrt{\pi e^{i\mu\pi}}}{2^{\nu} \Gamma(\nu + 1)} \Gamma(\mu + \nu + 1/2) \Gamma(2\nu + 1/2) \times \]

\[ F \left( \frac{\mu + \nu + 1/2}{2}, \frac{\mu + \nu + 3/2}{2}; \nu + 1; \frac{1}{z^2} \right) \tag{2.3} \]

The Gauss hypergeometric function is defined by:

\[ F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1, \tag{2.4} \]

where as usual \((a)_n\) is the Pochhammer symbol defined by

\[ (a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} \tag{2.5} \]

and \(\Gamma\) is the classical Euler function.

**Proof** By the Hankel transform I can transform the Dirichlet problem (1.1) into the following

\[
\begin{align*}
-\omega^2 (Hu)(Y, \omega) + \partial_Y^2 (Hu)(Y, \omega) &= 0, & (Y, X) \in \mathbb{R}_+ \times \mathbb{R}_+ \\
(Hu)(0, \omega) &= (Hu_0)(\omega) \in C^\infty_0 (\mathbb{R}_+^*). 
\end{align*}
\tag{2.6}
\]

The solution of (2.6) is given by

\[ (Hu)(Y, \omega) = \exp (-Y\omega) (Hu_0)(\omega) \tag{2.7} \]

Using the inverse Hankel transform and in view of the following asymptotic formulas for the Bessel functions ([10] p.134):

\[ J_\nu(x) \approx \frac{x^\nu}{2^\nu \Gamma(1 + \nu)}; x \to 0 \tag{2.8} \]

\[ J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x - (\nu \pi/2) - (\pi/4)); x \to \infty \tag{2.9} \]
we can use the Fubini theorem to obtain

\[ u(Y,X) = \int_0^\infty u_0(X') \int_0^\infty (XX')^{1/2} J_\nu(X\Omega)J_\nu(X'\Omega)e^{-Y\Omega} \Omega d\Omega dX' \quad (2.10) \]

Now using the formula in ([14], p.286 – 287)

\[ \int_0^\infty e^{-px} J_\nu(ax)J_\nu(bx)xdx = -\frac{pk^2(ab)^{-3/2}}{\sqrt{1-k^2}} Q^1_{\nu-1/2} \left( \frac{2-k^2}{k^2} \right) \quad (2.11) \]

\( \Re(\nu) > -1/2, \Re p > |\Im a| + |\Im b| \), and \( k = 2\sqrt{ab \left( p^2 + (a + b)^2 \right)^{-1/2}} \), we obtain the result of the theorem.

We end this section by the following lemma

**Lemma 2.2.** Set \( X = \varphi(\xi) + \psi(\eta), Y = \varphi(\xi) - \psi(\eta) \) then we have:

i) For \( F \in C^2 \) and \( \varphi, \psi \in C^1 \) the following formula holds

\[
\begin{bmatrix}
\frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} + V(X)
\end{bmatrix}
F(X,Y) = \left( \varphi'(\xi)\psi'(\eta) \right)^{-1} \left[ 4\frac{\partial^2}{\partial \xi \partial \eta} + \varphi'(\xi)\psi'(\eta)V(\varphi(\xi) + \psi(\eta)) \right] G(\xi,\eta) 
\]

ii) For \( \varphi(\xi) = \tan \xi \) and \( \psi(\eta) = \tan \eta \) with \( \xi = \frac{\theta + iy}{2} \) and \( \eta = \frac{\theta - iy}{2} \), the following formula hold:

\[ X_1 =: \varphi(\xi) + \psi(\eta) = \tan \xi + \tan \eta = \frac{2\sin \theta}{\cos \theta + \cosh y} \]

\[ Y_1 =: \varphi(\xi) - \psi(\eta) = \tan \xi - \tan \eta = \frac{2i \sinh y}{\cos \theta + \cosh y} \]

iii) For \( \varphi(\xi) = \tanh \xi \) and \( \psi(\eta) = \tanh \eta \) with \( \xi = \frac{x + iy}{2} \) and \( \eta = \frac{x - iy}{2} \) the following formulas hold:

\[ X_2 =: \varphi(\xi) + \psi(\eta) = \tanh \xi + \tanh \eta = \frac{2 \sinh x}{\cosh x + \cosh y} \]

\[ Y_2 =: \varphi(\xi) - \psi(\eta) = \tanh \xi - \tanh \eta = \frac{2 \sinh y}{\cosh x + \cosh y} \]

The proof of this lemma is simple and hence is left to the reader.
3 Laplace equation with singular trigonometric Pöschl-Teller potential

Theorem 3.1. The Dirichlet problem (1.3) for Laplace equation with singular trigonometric Pöschl-Teller on the Spherical line has the unique solution given by

\[ v(y, \theta) = \int_0^\pi P^T_\nu (y, \theta, \theta') v_0(\theta') d\theta' \]  

(3.1)

where

\[ P^T_\nu (y, \theta, \theta') = P_\nu(Y_1, X_1, \theta') \]  

(3.2)

where \( P_\nu \) is the Schwartz integral kernel of the Poisson semi-group \( \exp (-Y\sqrt{-L_\nu}) \) with inverse square potential given in (2.2) with \( Y_1 \) and \( X_1 \) as in Lemma 2.2.

Proof We take \( V(X_1, Y_1) = \frac{1}{4 - \nu^2 X_1^2} \) and set

\[ X_1 =: \varphi(\xi) + \psi(\eta) = \tan \xi + \tan \eta; \ Y_1 =: \varphi(\xi) - \psi(\eta) = \tan \xi - \tan \eta \]  

(3.3)

with \( \xi = \frac{\theta + iy}{2} \) and \( \eta = \frac{\theta - iy}{2} \) in Lemma 2.2 we obtain

\[ \left[ \frac{\partial^2 F}{\partial X_1^2} + \frac{\partial^2 F}{\partial Y_1^2} + \frac{1/4 - \nu^2}{X_1^2} \right] F = (1 + \tan^2 \xi)(1 + \tan^2 \eta)^{-1} \left[ 4 \frac{\partial^2}{\partial \xi \partial \eta} + \frac{1/4 - \nu^2}{\sin^2(\xi + \eta)} \right] F \]  

(3.4)

we obtain

\[ \left[ \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial Y_1^2} + \frac{1/4 - \nu^2}{X_1^2} \right] F = 16 \cos^2((\theta + iy)/2) \cos^2((\theta - iy)/2) \left[ \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial y^2} + \frac{1/4 - \nu^2}{16 \sin^2 \theta} \right] F \]  

(3.5)

To see the limit condition we use the formulas (3.1), (3.2) and the corresponding limit condition in the inverse square potential case (1.1).

4 Laplace equation with singular hyperbolic Pöschl-Teller potential

Theorem 4.1. The Dirichlet problem (1.5) for the Laplace equation with singular hyperbolic Pöschl-Teller potential has the unique solution given by

\[ w(y, x) = \int_0^\infty P^H_\nu (y, x, x') w_0(x') dx' \]  

(4.1)

where

\[ P^H_\nu (y, x, x') = P_\nu(Y_2, X_2, x') \]  

(4.2)
with $P_\nu$ is the Schwartz integral kernel of the Poisson semi-group $\exp \left( -Y \sqrt{-L_\nu} \right)$ with inverse square potential given in (2.2) with $Y_2$ and $X_2$ as in Lemma 2.2.

**Proof** We take $V(X_2, Y_2) = \frac{1}{4 - \nu^2} X_2^2$ and set

$$X_2 =: \varphi(\xi) + \psi(\eta) = \tan \xi + \tan \eta; \quad Y_2 =: \varphi(\xi) - \psi(\eta) = \tan \xi - \tan \eta \quad (4.3)$$

with $\xi = \frac{x+y}{2}$ and $\eta = \frac{x-y}{2}$ in Lemma 2.2 we obtain

$$\left[ \frac{\partial^2 F}{\partial X_2^2} + \frac{\partial^2 F}{\partial Y_2^2} + \frac{1}{4 - \nu^2} \right] F = \left( (1 + \tanh^2 \xi)(1 + \tanh^2 \eta) \right)^{-1} \left[ \frac{4}{\partial \xi \partial \eta} + \frac{1}{\sin^2(\xi + \eta)} \right] F \quad (4.4)$$

we obtain

$$\left[ \frac{\partial^2}{\partial X_2^2} + \frac{\partial^2}{\partial Y_2^2} + \frac{1}{4 - \nu^2} \right] F = 16 \cos^2((x+y)/2) \cos^2((x+y)/2) \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{16 \sin^2 x} \right] F \quad (4.5)$$

To see the limit condition we use the formulas (4.1), (4.2) and the corresponding limit condition in the inverse square potential case (1.1).

## 5 Applications

In this section we give as an application of our results the following heat Schwartz integral kernels with inverse square and singular trigonometric and hyperbolic Pöschl-Teller potentials. That is by using the transmutation formulas of Bragg and Dettman [3] we give explicit solutions to the following heat equation with inverse square and singular trigonometric and hyperbolic Pöschl-Teller potentials:

1) The heat equation associated to the Schrödinger operator with the inverse square potential on $\mathbb{R}^+$

$$\left\{ \begin{array}{l}
L_\nu U(t, X) = \partial_t U(t, X) = 0, \quad (t, X) \in \mathbb{R}_+ \times \mathbb{R}_+ \\
U(0, X) = U_0(X) \in C_0^\infty(\mathbb{R}_+^+) \end{array} \right. \quad (5.1)$$

2) The heat type equations associated to the Schrödinger operators with the singular trigonometric Pöschl-Teller potentials

$$\left\{ \begin{array}{l}
L_\nu^TV(t, \theta) = \partial_t V(t, \theta), \quad (t, \theta) \in \mathbb{R}_+ \times [0, \pi] \\
V(0, \theta) = V_0(\theta) \in C_0^\infty([0, \pi]) \end{array} \right. \quad (5.2)$$

3) The heat type equations associated to the Schrödinger operators with the singular hyperbolic Pöschl-Teller potentials
\[
\begin{align*}
L^H_\nu W(t, x) &= \partial_t W(t, x) \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \\
W(0, x) &= W_0(x) \in C^\infty_0(\mathbb{R}_+).
\end{align*}
\]

(5.3)

where \( L_\nu, L^H_\nu \) and \( L^T_\nu \) are given in (1.2), (1.4) and (1.6).

**Corollary 5.1** The Cauchy problems for the heat equations (5.1), (5.2) and (5.3) has at least formally the following Schwartz integral kernels

\[
H_\nu(t, X, X') = \frac{1}{4i\sqrt{\pi t}} \int_{\gamma-i\infty}^{\gamma+i\infty} s^{-1/2} e^{st} P_\nu(s^{1/2}, X, X') ds
\]

(5.4)

\[
H^S_\nu(t, \theta, \theta') = \frac{1}{4i\sqrt{\pi t}} \int_{\gamma-i\infty}^{\gamma+i\infty} s^{-1/2} e^{st} P^S_\nu(s^{1/2}, \theta, \theta') ds
\]

(5.5)

\[
H^H_\nu(t, X', X') = \frac{1}{4i\sqrt{\pi t}} \int_{\gamma-i\infty}^{\gamma+i\infty} s^{-1/2} e^{st} P^H_\nu(s^{1/2}, x, x') ds
\]

(5.6)

where \( P_\nu(Y, X, X'); P^S_\nu(y, \theta, \theta') \) and \( P^H_\nu(y, x, x') \) are the Poisson Schwartz integral kernels given in (2.2), (3.2) and (4.2) respectively.

**Proof** We use essentially the formula (1.2) of [3] and the theorems (2.1), (3.1) and (4.1).

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