Invited Comment

Multipartite entanglement dynamics in a cavity

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Abstract

We study the dynamics of two kinds of entanglement and their interplay. On one hand, we consider the intrinsic entanglement within a central system composed of three two-level atoms and measured by multipartite concurrence; on the other, we consider the entanglement between the central system and a cavity, acting as an environment and measured with purity. Using dipole-dipole and Ising interactions between atoms, we propose two Hamiltonians: one homogeneous and one quasi-homogeneous. We find an upper bound for concurrence as a function of purity, associated with the evolution of the W state. A lower bound is also observed for the homogeneous case. In both situations, we show the existence of critical values of the interaction, for which the dynamics of entanglement seem complex.

Keywords: decoherence, entanglement, multipartite

(Some figures may appear in colour only in the online journal)

1. Introduction

Entanglement was appreciated by Schrödinger not as ‘one but rather the characteristic trait of quantum mechanics’ [1]. Its study has opened its own field [2, 3], which remains active because the understanding of entanglement has proven valuable to comprehending the transition from the quantum to the classical world. Also, from a pragmatic point of view, to build ever more complex quantum technologies, one needs to tame entanglement’s effects. Entanglement has been identified as a key resource in both, quantum simulation and quantum computing [4].

Several theoretical developments have been devoted to understanding the evolution of the entanglement of bipartite systems, both from theoretical and experimental points of view (see [5–7] and many articles citing these references). Nowadays, two-party entanglement is routinely produced, controlled, studied, and exploited in the laboratory. Moreover, it is well understood. Many aspects of multipartite entanglement, despite a huge effort, still remain as open questions (see [8] and [9]). In this paper, we will study the simplest case of multipartite entanglement—namely three qubits—and its interplay with bipartite entanglement, which is often associated with decoherence. For this, we use the simplest possible ‘reservoir’ with an infinite spectrum: a harmonic oscillator.

Building upon [10], we generalize their model to allow for more qubit while retaining its simplicity. We consider three two-level atoms, coupled to each other via dipole and Ising interaction. Additionally, we consider a cavity that will be regarded as an environment, to which the atoms are coupled to a single mode. Both the qubits and the cavity have free dynamics. Except for very special cases, this model is not analytically solvable. However, number conservation will allow us to work on a finite Hilbert space and perform numerics. We will focus on the evolution of the entanglement within the three qubits, and of those, with the cavity as a measure of decoherence. Moreover, we will also study the relation between these two very different quantities, via a generalized concurrence-purity (CP) map [6, 10, 11].

Our paper is organized as follows. In section 2, we introduce the model and discuss some of its symmetries. We use section 3, to recall some aspects of entanglement that will be discussed here. Special attention will be given to multipartite mixed entanglement, as quantified by the concurrence.
In section 4, we present our findings, and we finish with some conclusions in section 5.

2. The model

We will work in the Hilbert space associated with a harmonic oscillator and three qubits—that is, \( \mathcal{H} = H_{\text{ho}} \bigotimes H_{q,1} \bigotimes H_{q,2} \bigotimes H_{q,3} \), with \( H_{\text{ho}} \) being the Hilbert space of the oscillator, and the others being the Hilbert spaces of each of the three qubits (\( \dim H_{q,i} = 2 \)). The Hamiltonian that will determine the evolution of the system is

\[
\hat{H} = \frac{3}{2} \sum_{j=1}^{n} \Delta \hat{a}^{(j)} + \sum_{j=1}^{n} g_{j} \left( \hat{a} \hat{a}^{(j)} + \text{h.c.} \right) \\
+ 2 \sum_{j \neq k = 1}^{3} \kappa_{jk} \left( \hat{a}^{(j)} \hat{a}^{(k)} + \text{h.c.} \right) \\
+ \sum_{j \neq k = 1}^{3} J_{jk} \hat{a}^{(j)} \hat{a}^{(k)},
\]

in which we use Pauli matrices, \( \hat{\sigma}^{(j),x} \), acting in the \( j \)th spin 1/2 particle, the lowering and rising operators of a harmonic oscillator (\( \hat{a}, \hat{a}^\dagger \)), and the rising and lowering operators of spin 1/2 particle, \( j, \hat{\sigma}^{(j),z} = \hat{\sigma}^{(j),x} \pm i \hat{\sigma}^{(j),y} \). We also introduce several parameters, namely the energy splitting in each qubit (\( \Delta \)), the intensity of their interaction with the harmonic oscillator, \( g_{j} \), the pairwise dipole-dipole interaction, \( \kappa_{jk} \), and the Ising interaction, \( J_{jk} \). Intrinsic dynamics in the oscillator can be safely ignored using the appropriate interaction picture.

Note that despite the simplicity of the model, it will admit several very distinct dynamical configurations, ranging from all qubits interacting with each other; a ‘line’ configuration, in which there is no interaction between qubits 1 and 3; a spectator configuration, in which one qubit does not interact with the other qubits; and finally, a configuration in which all qubits are decoupled from each other. This can be controlled by setting several of the coupling parameters, \( \kappa_{jk} \) and \( J_{jk} \), to zero.

A very important feature that allows us to treat the model is that this Hamiltonian preserves the number of excitations, characterized by the operator

\[
\hat{N} = \frac{1}{2} \sum_{j=1}^{3} \hat{\sigma}^{(j),z} + \hat{a}^\dagger \hat{a} + \frac{3}{2} \mathbf{1}.
\]

One can thus write the Hamiltonian in block-diagonal form on a suitable basis. We take this basis to be

\[
|\phi^{(n)}_{0}\rangle = |n\rangle |000\rangle, \quad |\phi^{(n)}_{1}\rangle = |n-2\rangle |110\rangle, \\
|\phi^{(n)}_{2}\rangle = |n-1\rangle |001\rangle, \quad |\phi^{(n)}_{3}\rangle = |n-2\rangle |101\rangle, \\
|\phi^{(n)}_{4}\rangle = |n-1\rangle |010\rangle, \quad |\phi^{(n)}_{5}\rangle = |n-2\rangle |011\rangle, \\
|\phi^{(n)}_{6}\rangle = |n-1\rangle |100\rangle, \quad |\phi^{(n)}_{7}\rangle = |n-3\rangle |111\rangle.
\]

so that \( \hat{N}|\phi^{(n)}_{i}\rangle = n|\phi^{(n)}_{i}\rangle \). We use the convention in which \( |1\rangle \) means an excitation and \( |0\rangle \) means no excitation, so \( \sigma_{z}|0\rangle = |1\rangle \). Note that when using matrix representation, even though there is a one-to-one correspondence with the state of the qubits, there is additional information regarding the state of the harmonic oscillator via the superindex, \( (n) \). The order chosen for the basis will allow us to write the partial trace in a particularly nice fashion. Given the matrix representations of an arbitrary mixed state, \( \rho \), acting in the subspace of \( n \) excitations, we can write

\[
\rho_{\text{osc}} = \begin{pmatrix}
\rho_{00} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{11} & \rho_{12} & \rho_{13} & 0 & 0 & 0 & 0 \\
0 & \rho_{21} & \rho_{22} & \rho_{23} & 0 & 0 & 0 & 0 \\
0 & \rho_{31} & \rho_{32} & \rho_{33} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \rho_{44} & \rho_{45} & \rho_{46} & 0 \\
0 & 0 & 0 & 0 & \rho_{54} & \rho_{55} & \rho_{56} & 0 \\
0 & 0 & 0 & 0 & \rho_{64} & \rho_{65} & \rho_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{77}
\end{pmatrix}
\]

A generic block of \( n \geq 3 \) excitations is given by

\[
\rho^{(n)}_{ij} = \begin{pmatrix}
H^{(n)}_{AA} & H^{(n)}_{AB} \\
(H^{(n)}_{AB})^\dagger & H^{(n)}_{BB}
\end{pmatrix} - \sum_{j \neq k = 1}^{3} J_{jk} \mathbf{1},
\]

with the first block being

\[
H_{AA} = \begin{pmatrix}
2J & g_{1}^{+} & g_{2}^{+} & g_{3}^{+} \\
g_{1}^{-} & \delta_{12}^{-} & \delta_{12}^{+} & \delta_{12}^{+} \\
g_{2}^{-} & \delta_{12}^{+} & \delta_{23}^{-} & \delta_{23}^{+} \\
g_{3}^{-} & \delta_{12}^{+} & \delta_{23}^{+} & \delta_{23}^{-}
\end{pmatrix} - \sum_{j=1}^{3} \Delta_{j} \mathbf{1},
\]

if \( g_{j}^{\pm} = g_{j} \sqrt{n \pm \frac{1}{2}} \), \( \delta_{i,j}^{-} = 2J_{ji} \pm \Delta_{j} \), and \( T_{a,b} = \sum_{i} J_{ai} - \Delta_{i} \). Note that we use the subindex, \( \neq i \), to refer to a pair of subindices that are different from \( i \) and from each other. The off-diagonal matrix is

\[
H_{AB} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g_{1}^{-} & g_{2}^{+} & 0 & 0 & 0 & 0 & 0 \\
0 & g_{3}^{-} & g_{1}^{+} & 0 & 0 & 0 & 0 & 0 \\
0 & g_{2}^{-} & g_{1}^{-} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

with \( g_{j}^{0} = g_{j} \sqrt{n} \). Finally, the second diagonal block is given by

\[
H_{BB} = \begin{pmatrix}
\delta_{12}^{-} & \kappa_{12} & \kappa_{13} & g_{1}^{+} \\
\kappa_{12} & \delta_{12}^{-} & \delta_{23}^{-} & g_{2}^{+} \\
\kappa_{13} & \delta_{23}^{-} & \delta_{23}^{+} & g_{3}^{+} \\
g_{1}^{-} & g_{2}^{-} & g_{3}^{-} & 2J
\end{pmatrix} + \sum_{j=1}^{3} \Delta_{j} \mathbf{1}.
\]

For smaller values of \( n \), the matrix will be the first block of 1, 4, and 7 states corresponding to (5), for \( n = 0, 1, 2 \), respectively.

This model has a large number of parameters. A simplification would be to consider a homogeneous situation, such as all \( \kappa_{ij} = \kappa \), \( J_{ij} = J \), \( g_{j} = g \), and \( \Delta_{i} = \Delta \). Moreover, we will set the detuning to be \( \Delta = 0 \) and the coupling to the oscillator to be \( g = 1 \) in our numerical calculations, so that we
have the *homogeneous* Hamiltonian

\[
\hat{H}_{h,j} = \sum_{j=1}^{3} \left( \hat{a}_x^{(j)} + \text{h.c.} \right) + 2\kappa \sum_{j,k=1}^{3} \left( \hat{a}_x^{(j)} \hat{a}_x^{(k)} + \text{h.c.} \right) + \frac{J}{r} \sum_{j,k=1}^{3} \hat{a}_x^{(j)} \hat{a}_x^{(k)}.
\]

To appreciate the effect of homogeneity in the Hamiltonian, we also consider a slightly inhomogeneous case. We change one dipole–dipole interaction, to obtain the *quasi-homogeneous* Hamiltonian

\[
\hat{H}_{q,h,j} = \hat{H}_{h,j} + \kappa \left( \hat{a}_x^{(1)} \hat{a}_x^{(2)} + \text{h.c.} \right).
\]

Several other simple ways of introducing the inhomogeneity are also available. However, this one displays very clearly the effects we want to underline in this work.

Note that one further consideration, in the homogeneous case, could be considered: the Hamiltonian is invariant under rotations. In this case, the rotation operator, \( \hat{R} \), that acts on the computational basis as \( \hat{R} |n \rangle |i_1 i_2 i_3 \rangle = |n \rangle |i_3 i_1 i_2 \rangle \), with \( i_j = \pm \), has three eigenvalues, namely \( \exp(2\pi ij/3) \), with \( j = 0, 1, 2 \). Let us define the vectors

\[
\begin{align*}
|\varphi_0^{(n,k)} \rangle &= \alpha^k |\varphi_0^{(n)} \rangle + \alpha^{2k} |\varphi_2^{(n)} \rangle + \alpha^{3k} |\varphi_6^{(n)} \rangle, \\
|\varphi_1^{(n,k)} \rangle &= \alpha^k |\varphi_1^{(n)} \rangle + \alpha^{2k} |\varphi_4^{(n)} \rangle + \alpha^{3k} |\varphi_5^{(n)} \rangle, \\
|\varphi_2^{(n,k)} \rangle &= \alpha^k |\varphi_2^{(n)} \rangle + \alpha^{2k} |\varphi_5^{(n)} \rangle + \alpha^{3k} |\varphi_6^{(n)} \rangle,
\end{align*}
\]

(\( k = 0, 1, 2 \)) with \( \alpha = \exp(2\pi i/3) \) and the additional \( |\varphi_2^{(n,0)} \rangle = |\varphi_0^{(n)} \rangle \) and \( |\varphi_6^{(n,0)} \rangle = |\varphi_6^{(n)} \rangle \). These are eigenvectors of \( \hat{R} \) with eigenvalues \( \alpha^k \); that is, \( \hat{R} |\varphi_1^{(n,k)} \rangle = \alpha^k |\varphi_1^{(n,k)} \rangle \). This leads to a splitting of the subspace in spaces of dimension 4, 2, and 2. Very lengthy expressions, however, make it very difficult to extract the general behavior, and one would be forced to fall back on numerics. On the other hand, if one wishes to restrict to the symmetric subspace, the machinery of the Dicke states could be used, where some analytical results regarding entanglement are available [12, 13].

### 3. Multipartite entanglement

The notion of entanglement is defined using separability. Separable pure states, \( |\psi\rangle \), are those which, in a multipartite Hilbert space, \( \mathcal{H} = \otimes \mathcal{H}_i \), can be written as a tensor product. Thus, \( |\psi\rangle \in \mathcal{H} \) is separable if \( |\psi\rangle = \otimes_i |\psi_i\rangle \), with \( |\psi_i\rangle \in \mathcal{H}_i \). Entangled states are those that are not a mixture of pure separable states. The problem of determining if a mixed state is or is not entangled is difficult.

However, for the special case of pure bipartite states, all the information regarding entanglement is encoded in their Schmidt coefficients. One can choose any convex function of the Schmidt coefficients the von Neumann entropy and purity being the most common choices. We will use purity, defined for mixed states as

\[
P(\rho) = \text{tr} \rho^2,
\]

due to its algebraic simplicity. Purity can be regarded as an entanglement measure, provided that the state over which purity is calculated is given by

\[
\rho = \text{tr}_{\text{env}} |\psi\rangle \langle \psi|.
\]

If no *a priori* information is given about \( \rho \), purity is simply a measure of mixedness. The value of purity ranges between 1/\( N \leq P \leq 1 \), where \( N \) is the dimension of the Hilbert space in which \( \rho \) acts, with the minimum value corresponding to the maximally mixed state, and the maximum to a pure state.

Characterizing multipartite entanglement, on the other hand, proves more challenging, as even a unique maximally entangled state does not exist for more than two parties [14]. Most measures provide either well-founded physical grounds or numerically simple recipes. A convenient compromise is given by the multipartite concurrence. This measure, which is a generalization of the two-party concurrence [15], is inspired by the symmetry properties of pure states—that is, in expected values of projections over antisymmetric subspaces. The detailed construction is out of reach within this presentation, but it reduces to the very simple form [16] for pure states:

\[
C(|\psi\rangle) = \frac{1}{2^{N/2-1}} \sqrt{(2^N - 2) - \sum_i \text{tr} \rho_i^2}
\]

where the index, \( i \), runs over all proper subsets of particles, except for the empty set. For example, in the three-particle case, we must consider all three partitions in which the particles can be divided: particle A against BC; B with AC; and C with AB. Thus, \( C(|\psi\rangle)^2 \propto \langle 1 - P(\rho) \rangle \), where here, the average \( \langle \cdot \rangle \) is taken again over all nontrivial subset particles, thus relating the measure with the entanglement over all possible partitions.

This measure, as presented above, will not suit our purposes, as it is defined only for pure states. We will use the *convex roof* construction, in which the measure is averaged over a particular realization of our mixed state, such as our state \( \rho = \sum p_i |\psi_i\rangle \langle \psi_i| \), with normalized \( |\psi_i\rangle \) and positive \( p_i \) for all \( i \). Then, we associate with this particular realization of \( \rho \) the measure \( \sum p_i C(|\psi_i\rangle) \). The measure is obtained by finding the realization that minimizes this expression. This would mean, if one thinks about entanglement as a resource, the cheapest way of realizing this ensemble. Summarizing, we define

\[
C(\rho) = \inf_{\rho = \sum p_i |\psi_i\rangle \langle \psi_i|} \sum p_i C(|\psi_i\rangle),
\]

with \( \rho = \sum p_i |\psi_i\rangle \langle \psi_i| \), \( p_i > 0 \), and \( \langle \psi_i | \psi_i \rangle = 1 \). Even though this is a very meaningful definition, the process of exact evaluation is normally difficult, as the landscape can be very complex. Upper bounds can be easily obtained using gradient methods to numerically optimize (14), and simple but useful lower algebraic bounds are also available. In particular, we will use the method of *quasi-pure approximation* [17]. It is derived by approximating a multimatrix representation of a mixed state.
with a single matrix that captures the first few order terms in expressions involving the Schmidt coefficients of $\rho$. We underline that this is not only an approximation, but also a lower bound, reasonably tight even for some states close to the maximally mixed state $[16, 18]$. To obtain this bound, no optimization procedure is involved, and only the diagonalization of a matrix of the same size as $\rho$ is needed. A detailed description requires some technicalities that we do not wish to introduce. Instead, we refer the reader to $[16]$. Some properties of the concurrence should be highlighted: it is invariant under unitary local operations, it vanishes only for completely separable states, and it has the nice scaling property $C(|\psi\rangle \langle \psi| \otimes |\phi\rangle \langle \phi|) = C(|\psi\rangle \langle \psi|)$ if $|\psi\rangle$ belongs to the Hilbert space of just one of the particles. Finally, the bound coincides with the concurrence for pure states.

The quasi-pure approximation is well suited to our needs, where repeated evaluation of such a quantity is required for many parameters and times. Moreover, as we will explore, qualitative properties, the small errors inherent in this approximation can be ignored.

4. Results

In this section we calculate the entanglement and purity as a function of time, and we explore how they depend on each other. We consider two families of initial product states, with the condition that any member of the family must belong to the eigenspace characterized by a fixed eigenvalue of the operator $\hat{N}$.

The first family corresponds to the normalized product state, with $n - 1$ photons in the cavity and the superposition of the states $|001\rangle$ and $|010\rangle$.

$$|\Phi^{(n)}(\alpha)\rangle = |n - 1\rangle \otimes (\sin \alpha |001\rangle + \cos \alpha |010\rangle). \quad (15)$$

The states have no genuine tripartite entanglement, as one of the parties has an uncorrelated state, and the others share bipartite entanglement, parametrized by $\alpha$. Its concurrence is $C(|\Phi^{(n)}(\alpha)\rangle) = \sin (2\alpha)$. The second family corresponds to the superposition of the states $|001\rangle$, $|010\rangle$, and $|100\rangle$.

$$|\Psi^{(n)}(\alpha)\rangle = |n - 1\rangle \otimes \begin{pmatrix} \sin \alpha \sqrt{2} \\ \cos \alpha \sqrt{2} \end{pmatrix} |001\rangle + \begin{pmatrix} \sin \alpha \sqrt{2} \\ \cos \alpha \sqrt{2} \end{pmatrix} |010\rangle + \begin{pmatrix} \sin \alpha \sqrt{2} \\ \cos \alpha \sqrt{2} \end{pmatrix} |100\rangle. \quad (16)$$

The states do have genuine tripartite entanglement, as one can see by setting $\alpha = \alpha_0 = \arctan \sqrt{2}$, for which we retrieve the $|W\rangle$ state. Moreover, its concurrence is

$$C(|\Psi^{(n)}(\alpha)\rangle) = \frac{\sin \alpha}{\sqrt{2}} \sqrt{5 + 3 \cos (2\alpha)}, \quad (17)$$

with a maximum given by $C(|\Psi^{(n)}(\alpha_0)\rangle) = 2/\sqrt{3} \approx 1.15$. However, in both cases, we can range from a totally separable state for $\alpha = 0$ to a maximally entangled state for some critical $\alpha$. These families of states also have the important characteristic that they will remain pure when the oscillator is traced out. The system evolves with the unitary evolution generated by Hamiltonian (1), and the state of the three qubits is given at time, $t$, by

$$\rho(t) = \text{tr}_\mu e^{-iHt} |\psi(0)\rangle \langle \psi(0)| e^{iHt}, \quad (18)$$

where $|\psi(0)\rangle$ is the initial state of the system, which will be taken as either $|\Phi^{(0)}(\alpha)\rangle$ or $|\Psi^{(0)}(\alpha)\rangle$. One can then study the evolution of concurrence and purity by studying $C(t) = C(\rho(t))$ and $P(t) = P(\rho(t))$, respectively. Moreover, we shall study a homogeneous case, in which all $J_i = J$ and all $\kappa_i = \kappa$.

In figures 1 and 2, we present concurrence against purity with time as a parameter—i.e., the so-called CP plane. We studied the sector $n = 1$ in these figures, but the conclusions drawn here can be extended to higher excitation numbers, unless we explicitly say otherwise. We also show the evolution of concurrence in the insets of the figures. We fix the Hamiltonians to the homogeneous and quasi-homogeneous with parameters $J = 0.5$ and $\kappa = 1$; using the notation of (9) and (10), we use $H_{2,0.5}^{\text{h}}$ and $H_{1,0.5}^{\text{qh}}$. In all cases, an important benchmark will be considered: a maximally mixed state, $|\rho^{\text{mix}}(\alpha_0)\rangle \propto |001\rangle + |010\rangle + |100\rangle$, evolved with an interaction-free model, $H_{2,0.5}^{\text{h}}$. This is plotted as a thick red curve.

In figure 1, we initially study nonmaximally entangled states, both with and without interaction. In both parts of figure 1, one can note that the red curve serves as an upper bound for the evolution of the states without maximal initial entanglement, similar to the two-atom case $[10]$. Remarkably, for three atoms, the dynamics of the noninteracting case is a lower bound in the homogeneous case (a) (as in the two qubit case), whereas that is no longer the case when the system is not homogeneous (b). We stress here that these observations hold in all cases studied (and not shown), namely other parameters and excitation numbers. The behavior of concurrence and purity with time (displayed in the insets of the figures) will be discussed later, when we have a global picture with respect to the parameters and time. However, note that the nonhomogeneous case displays a richer behavior.

In figure 2, we study the effect of interactions on both maximally entangled initial states, $|\Psi^{(n)}(\alpha_0)\rangle$, and partially entangled states, $|\Psi^{(n)}(\alpha/10)\rangle$. In this case, the behavior of states that initially have maximum entanglement, with already-large interaction, closely resembles the case with no interaction. In particular, for the homogeneous Hamiltonian, the two curves coincide, while for the quasi-pure Hamiltonian, the noninteracting case acts as an imperfect, but very good, guide. This is not the case if we start with a state with smaller entanglement. The red curve acts only as an upper bound for the interacting case. For larger values of the Ising interaction, the dynamics display an oscillatory behavior with almost the same periodicity. These observations are robust with respect to varying the systems and the total number of excitations.

To give a general view of the dynamics of purity and concurrence, we proposed a density diagram, which is displayed in figures 3 and 4. There, concurrence and purity are color-coded for several values of the Ising interaction, for a given time span, under the influence of the homogeneous Hamiltonian. The initial state chosen was $|\Psi^{(2)}(\alpha = \pi/4)\rangle$. 

4
In Figure 3, we compare the dynamics of purity (figures 3(a) and (c)) and concurrence (figure 3(b)). Indeed, concurrence mimics the behavior of purity, with some additional nodes, which may be caused by the internal dynamics of the three qubits.

Purity displays a regular behaviour, with some some oscillations that seem to be independent of $J$ for large or small values of $J$. However, a critical area (around $J = 0.5$ for $\kappa = 1$, as seen in figure 3(a)) is clearly present. Varying the dipole-dipole coupling shifts this critical value of $J$ to the right.
(figure 3(b)). For higher excitation numbers, this region shows higher complexity, whereas the regular areas remain largely unchanged (see figure 4). As concurrence mimics the behavior of purity, these observations are also valid for the internal entanglement.

A remarkable fact is also seen: the existence of critical points seems to agree for all cases in the density diagrams (figures 3 and 4).

5. Conclusions and outlook

In this paper, we studied the entanglement dynamics of three interacting two-level atoms inside a cavity with dipole-dipole and Ising interactions. Entanglement within the atoms was measured by concurrence, and the atoms with the cavity were measured by purity. Despite measuring totally different properties of the system, concurrence and purity are quite related. In particular, in a concurrence-purity plane for a given state, concurrence is bounded by above, by the curve described by a noninteractive system, and initially in a W state. We believe that this upper bound is closely related to the with monogamy of entanglement, where entanglement cannot be freely shared among multiple parties [19]. Recall that our system is a four-partite state (three qubits and a bath) in which entanglement must be considered in this setting, not as a tripartite problem. In this respect, our analysis contributes to the existence of a hierarchy of strong monogamy inequalities (as proposed by Regula et al [20]), or alternatively by viewing multipartite entanglement from the point of view of frustration [21]. In the homogeneous case, a lower bound, drawn by the evolution of the same initial state but with a noninteracting Hamiltonian, is also apparent. The inhomogeneous case, however, does not have a simple lower bound. In fact, for this lower bound, calculations could be pursued using the Dicke states and the symmetric subspace, where the block-diagonal Hamiltonian and the basis resemble the two-qubit case. In the present work, this bound is very sensitive to the presence of a small perturbation, which suggests a strong connection with the symmetric properties of the Dicke basis. A deeper study of these bounds might prove useful in the context of multipartite entanglement. Finally, we presented a global view for the dynamics of the concurrence and purity as function of time and Ising interaction, and we showed that there is a translation of the intervals where the dynamics exhibits complexity when the dipole-dipole interaction increases.

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Figure 4. In these set of figures, we study the behavior of purity with the Ising interaction (in the x-axis) and time (in the y-axis), for a varying number of total excitations, $|\Phi^{12.5,20}(\pi/4)|$ in (a), (b) and (c) respectively, evolved with $\hat{H}_{20,J}$. The region with complex behavior increases with the number of excitations.
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