Measure of invertible channels under mixing of non-invertible channels

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We study the uniform mixing of the \((d + 1)\) generalized Pauli channels in a Hilbert space of dimension \(d\), where each channel is characterized by the decoherence function \((1 - e^{-ct})/n\), with the decoherence parameter \(n\) and decay factor \(c\). The channels are invertible if and only if \(n > \frac{d}{c} - 1\). We show that if the input Pauli channels are invertible, then so is the resultant mixture. If they are non-invertible, then for sufficiently small \(n\), the resultant mixture is also non-invertible. In the intermediate range of \(n\), where the input channels are non-invertible, the mixture can be invertible or not, depending on the mixing coefficients. We evaluate the fraction of invertible channels obtained upon mixing, which is found to increase super-exponentially with dimension \(d\). We briefly discuss a convex resource theory of non-invertibility of quantum channels.

I. INTRODUCTION

For a system undergoing open evolution [1], the finite time evolution \(\rho(0) \rightarrow \rho(t) = \Phi[\rho]\) of the system of interest in a state \(\rho(0)\) is characterized by a completely positive, trace-preserving dynamical map [2]. \(\Phi(\cdot)\), also known as the quantum channel [3]. One could also have a differential form of the map [4], \(\dot{\Phi} = L\Phi\).

\[
L[\rho] = -\frac{i}{\hbar}[H, \rho] + \sum_i \gamma_i \left( G_i^\dagger \rho G_i - \frac{1}{2} \{ G_i^\dagger G_i, \rho \} \right),
\]

where \(H\) is the effective Hamiltonian, \(G_i\)'s are the noise operators, and \(\gamma_i \geq 0\) the decoherence rates, describing the continuous-in-time description of the dynamics of the system of interest. Eq. (1), could be generalized to the time-dependent case with the time-local generator \(L(t)\). In general, the decoherence rates \(\gamma_i(t)\) need not be positive. The dynamical map is divisible if

\[
\Phi(t_f, t_i) = \Phi(t_f, t)\Phi(t, t_i), \quad \forall t_f \geq t \geq t_i.
\]

The evolution is CP-divisible if, \(\forall t_f, t, \Phi(t_f, t)\) is CP and satisfies the above divisibility condition. Otherwise, the map is CP- indivisible.

Recently, convex combinations of quantum channels has been studied in detail. We studied the [5] convex combinations of the set of Pauli channels, each assumed to be Quantum Dynamical Semigroups (QDS), and characterized the geometry of the resultant set along with providing a measure. In [6], we considered the the mixing of Pauli channels that are non-semigroups. These results showed that the set of CP-divisible Pauli channels is non-convex which was quantified via a non-zero measure of the resulting CP-divisible set. The non-convexity of the set of generalized Pauli Markovian channels was studied in [7].

In Eq. 2, if for a particular instant of time \(t = t_s\) such that the composition law is not satisfied, the map \(\Phi(t_s, t_i)\) is non-invertible at the point \(t_s\), where the map \(\Phi(t_f, t_i) = \Phi(t_f, t_s)\Phi(t_s, t_i)\) is undefined. Such points \(t_s\) are referred to as singular points [8] of the corresponding channel. The decay rates go to infinity at the singular points of the map. Even in the presence of singularities in a map, the dynamics can be perfectly regular [9, 10]. Convex combinations of channels with singularities have also caught attention recently [11, 12].

In Refs. [5, 6], the present authors considered the question of measure of Markovian channels obtained when Pauli qubit invertible channels are mixed. Continuing the line of investigation, the conditions under which a semigroup is obtained upon mixing the set of (possibly non-invertible) generalized Pauli channels was studied in [13]. Building on these works, here we address the problem of mixing generalized Pauli channels that are non-invertible. We find that the resulting mixture of such non-invertible channels can be invertible. We then evaluate the measure of the set of invertible regions that are obtained.

A resource theory (RT) classifies the set under consideration into resource states and free states. Associated with the free states is a set of free operations such that the set of free states is closed under the action of free operations. RT therefore addresses the question of tasks which are (im)possible using free operations [14]. In a convex RT, the free states form a convex set. For the set of invertible maps, let the free operation considered be the composition of maps, i.e., a sequential application. Invertible maps being bijective, the composition of bijective maps is also a bijection. Since bijective maps have a unique inverse, the composed map therefore is also invertible. Thus the set of invertible channels do not form a resource under composition.

The paper is organized as follows. We discuss the case of mixing non-invertible Pauli channels in the following Section. In Sec. III, we extend our analysis to the case of generalized Pauli channels. Finally, we discuss the results and conclude in Sec. IV.
II. MIXING NON-INVERTIBLE PAULI CHANNELS: QUBIT CASE

Consider the case of three qubit Pauli channels
\[ \Phi_t(p) = (1 - p(t))\rho + p(t)\sigma_i \rho \sigma_i, \]  
with a suitable \( p \). Consider the three Pauli channels Eq. (3), each mixed in proportions of \( x_i, i = 1, 2, 3 \) such that \( \sum x_i = 1 \). We consider the case where all the maps are each characterized by the decoherence function as in Eq. (14) The case \( n \geq 2 \) corresponds to invertible maps, with \( n = 2 \) being the semigroup. The case \( n < 2 \) corresponds to non-invertible maps. Upon mixing the three Pauli channels, the eigenvalues of the output map turn out to be
\[ \lambda_i = 1 - 2(1 - x_i) \left( \frac{1 - e^{-\alpha}}{n} \right) \]  
The eigenvalues \( \lambda_i \) becomes singular at
\[ t^* = \frac{1}{c} \log \left[ \frac{2(1 - x_i)}{2(1 - x_i) - n} \right]. \]  
We then have the following result (which is a direct consequence of Theorem 1).

When the input map is invertible \( (n \geq 2) \), the resultant map is necessarily invertible. For non-invertible inputs, if \( n \leq \frac{4}{3} \), the resultant map is necessarily non-invertible. For input maps that are of the intermediate non-invertible range \( (2 > n \geq \frac{4}{3}) \), the resultant can be invertible or not, depending on the mixing coefficients.

We have with Eq. (5) that the requirement for invertibility is
\[ x_i \geq 1 - \frac{n}{2} \]  
For invertible inputs, \( n \geq 2 \), the r.h.s above is negative and thus any mixing coefficient \( x_i \) satisfies the lower bound in Eq. (6). This bound monotonically increases as \( n \) decreases in Eq. (6). If \( n < \frac{4}{3} \), then \( x_i < \frac{1}{2} \), a constraint that cannot be satisfied by all three channels (since in that case we would have \( \sum x_i < 1 \)).

In the intermediate range of non-invertible inputs \( n \in \left( \frac{4}{3}, 2 \right) \), a fraction of the output maps will be invertible. For the case \( n = 4/3 \), Eq. (17) implies that \( x_i \geq \frac{1}{3} \). Therefore, the map is non-invertible for all choices of mixing except at the point of equal mixing where all \( x_i \)'s are \( 1/3 \) where we get a semigroup. For the intermediate case, i.e., \( n \in \left( 4/3, 2 \right) \), the measure \( \Delta_{\text{invert}} \) of invertible maps over all possible mixtures is given by \( \frac{(4 - 3n)^2}{8} \).

To show this, we note that \( x_1 \) can take values in the range \( [1 - \frac{2}{3}, 1 - 2 \times (1 - \frac{2}{3})] = [1 - \frac{2}{3}, n - 1] \). The variable \( x_2 \) then is allowed to take values in the range \( 1 - \frac{n}{2} \) and \( \frac{n}{2} - x_1 \). We have
\[ \Delta_{\text{invert}}(2, n) = \frac{1}{\mu} \int_{1 - \frac{2}{3}}^{\frac{n}{2} - x_1} dx_2 \int_{1 - \frac{2}{3}}^{n - 1} dx_1 = \frac{(4 - 3n)^2}{8}, \]  
where the normalization factor \( \mu = \int_{0}^{1-x_1} dx_2 \int_{0}^{1} dx_1 = \frac{1}{4} \). Eq. (7) shows that the measure of invertible channels falls monotonically through the range \([1, 0]\) as \( n \) is varied in the intermediate range \( (\frac{2}{3}, 2) \). The above arguments can be readily extended to the case of qudits, as done in the following Section.

III. MIXING NON-INVERTIBLE PAULI CHANNELS: GENERALIZED CASE

Based on the concept of Mutually Unbiased bases (MUBs), Nathanson and Ruskai introduced [15] a generalization of the Pauli channels to the case of channels on qudits. A set of \( d + 1 \) orthonormal bases \( \{ |\xi_i^{(a)}\rangle, i = 0, \ldots, d - 1 \} \) in \( \mathbb{C}^d \) such that \( \langle \xi_i^{(a)} | \xi_j^{(b)} \rangle^2 = 1/d \) whenever \( a \neq b \), is a MUB. Defining the rank-1 projectors \( P_i^{(a)} = |\xi_i^{(a)}\rangle\langle \xi_i^{(a)}| \), one introduces the \( d + 1 \) unitary operators
\[ U_\alpha = \sum_{i=0}^{d-1} \omega^i P_i^{(a)}, \quad \omega = e^{2\pi i/d}. \]  
The time-dependent generalized Pauli channels are defined as [15]
\[ \Phi_G = p_0(t)I + \frac{1}{d-1} \sum_{i=1}^{d+1} p_\alpha(t)U_\alpha, \]  
where \( p_\alpha(t) \) is a probability distribution and the \( d + 1 \) CP maps \( U_\alpha \) are
\[ U_\alpha[\rho] = \sum_{k=1}^{d-1} U_\alpha^k \rho U_\alpha^{k\dagger}. \]  
For \( d = 2 \), Eq. 9 leads to the familiar Pauli channel. In \( d \) dimensions, one has \( d + 1 \) generalized Pauli channels. We consider the convex combination of \( d + 1 \) generalized Pauli channels \( (i = 1, \ldots, d + 1) \),
\[ \Phi^p(t) = (1 - p(t))\rho + p(t) \sum_{k=0}^{d-1} U_i^k \rho U_i^{k\dagger}, \]  
with the same \( p(t) \), each mixed in proportions of \( x_i \) such that \( \sum_{i=1}^{d+1} x_i = 1 \). The map \( \Phi_i \) satisfies the eigenvalue relation
\[ \Phi_i[U_i^k] = \lambda_i U_i^k, k = 1, \ldots, d - 1. \]  
The eigenvalues of the resultant map under convex combination can be evaluated to be
\[ \lambda_i = 1 - \frac{d}{d-1} (1 - x_i)p(t). \]  
We consider the case where all the maps are characterized by the decoherence function
\[ p(t) = \frac{1 - e^{-ct}}{n}. \]
For this choice, the eigenvalues becomes singular at

\[ t^* = \frac{1}{c} \log \left[ \frac{d(1 - x_i)}{d(1 - x_i) - n(d - 1)} \right]. \]  

(15)

It can be seen that for the generalized Pauli channel to be a semigroup, the decoherence function should be

\[ p(t) = \frac{d - 1}{d} (1 - e^{-ct}). \]  

(16)

We have the following result.

Theorem 1. When the input map is invertible \((n \geq \frac{d}{d-1})\), the resultant map is necessarily invertible. For non-invertible inputs, if \(n < \frac{d^2}{d^2 - 1}\), the resultant map is necessarily non-invertible. For input maps that are of the intermediate non-invertible range \(\left(\frac{d}{d-1} > n \geq \frac{d^2}{d^2 - 1}\right)\), the fraction of invertible maps is given by \(\Delta_{\text{invert}}(d, n)\) in Eq. (22).

Proof. From Eq. (15) it follows that the requirement for invertibility is

\[ x_i \geq 1 - \frac{n(d - 1)}{d} \equiv g(d, n) \]  

(17)

The output map is invertible for \(n \geq \frac{d}{d-1}\), and is a semigroup for \(n = \frac{d}{d-1}\). For invertible inputs, \(n \geq \frac{d}{d-1}\), the r.h.s above is negative and thus any mixing coefficient \(x_i\) satisfies the lower bound in Eq. (17). This bound monotonically increases as \(n\) decreases in Eq. (17). If \(n < \frac{d^2}{d^2 - 1}\), then \(x_i < \frac{1}{d^2 - 1}\), a constraint that cannot be satisfied by all three channels (since in that case we would have \(\sum x_i < 1\)).

In the intermediate range of non-invertible inputs \(n \in \left(\frac{d^2}{d^2 - 1}, \frac{d}{d-1}\right)\), a fraction of the output maps will be invertible. To evaluate this, first we note that

\[ x_i \in [g(d, n), 1 - dq(d, n)] = [g(d, n), (d-1)(n-1)]. \]  

(18)

Similarly,

\[ x_2 \in [g(d, n), 1 - (d - 1)g(d, n) - x_1]. \]  

(19)

For a general mixing variable \(x_j\), we then have

\[ x_{j+1} \in [g(d, n), f(j) - X_j], \]  

(20)

where \(f(j)\) and \(X_j\) are as defined by

\[ f(j) \equiv 1 - (d - 1)g(d, n), \]  

\[ X_j \equiv x_j + x_{j-1} + \cdots + x_1. \]  

(21)

Given dimension \(d\) and parameter \(n\), the measure of the invertible region in the space of resultant channels is evaluated to be

\[ \Delta_{\text{invert}}(d, n) = \mu \int_{g(d, n)}^{f(d-1)(n-1)} dx_1 \cdots \int_{g(d, n)}^{f(j) - X_j} dx_j \cdots \int_{g(d, n)}^{f(d-1) - X_{d-1}} dx_{d-1} \]  

(22)

\[ = \left[ \frac{(d^2(n - 1) - n)}{d} \right]^{d}. \]

Here,

\[ \mu = \int_0^{1-x_j} dx_d \int_0^{1-x_{j-1}} dx_{d-1} \cdots \int_0^1 dx_1 = \frac{1}{d!} \]  

is the normalization constant.

(23)

Eq. (22) implies that for a fixed dimension \(d\), the invertible fraction \(\Delta_{\text{invert}}\) falls monotonically with \(n\) in the intermediate range. For fixed \(n\), and sufficiently large \(d\), we find that \(\Delta_{\text{invert}}\) approaches unity with a super-exponential rate \(O(d^2)\). However, to plot this, we remark that the intermediate region \(\left[\frac{d^2}{d^2 - 1}, \frac{d}{d-1}\right]\) quadratically shrinks into a narrow interval that converges towards unity. Therefore to depict the dependence of the invertible fraction on \(d\), we must choose a range of \(d\) values such that a given \(n\) value falls within their corresponding interval. An example is given in Figure 1.

IV. DISCUSSIONS AND CONCLUSIONS

Our results imply that in the context of qudit Pauli channels, non-invertibility is a convex resource. Mixing invertible channels can never result in non-invertibility. On the other hand, a mixture of non-invertible channels satisfying the condition \(n \in \left[\frac{d}{d-1}, \frac{d^2}{d^2 - 1}\right]\) produces a set of invertible channels with a finite non-zero measure. The convexity of the invertible set may be contrasted with non-convexity of the set of quantum Markovian (CP-indissolved) channels [6]. On that basis, it was argued that CP-indissolvibility doesn’t yield a convex resource theory [5].

In this paper, we have studied the properties of channels that result under mixing of generalized Pauli channels that are non-invertible. In any given dimension \(d\), (non)-invertibility of the input channels is parametrized through a parameter \(n\), which ensures invertibility when sufficiently large. We show that when the mixed channels are invertible, then the resultant channel is always invertible. Analogously, for non-invertible channels such that
\(n\) is sufficiently small, the resultant channel is necessarily non-invertible. We quantify the fraction of invertible channels in the intermediate range of \(n\). Specifically, we have shown that the set of such invertible channels is of non-zero measure.

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