$B \to \rho$ transition form factors within the QCD light-cone sum rules and the $\rho$-meson leading-twist distribution amplitude

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The QCD light-cone sum rules (LCSR) provides an effective way for dealing with the heavy-to-light transition form factors (TFFs), whose non-perturbative dynamics are parameterized into the light-meson’s light-cone distribution amplitudes (LCDAs) with various twist structures. By taking the chiral correlator as the starting point, we calculate the LCSRs for the $B \to \rho$ TFFs up to twist-4 accuracy. As for the TFFs at the large recoil region, we observe that the twist-2 transverse DA $\phi_{2,\rho}^\perp$ provides the dominant contribution, while the contributions from the remaining twist-3 and twist-4 terms are $\delta^2$-suppressed. Thus, our present improved LCSRs provides a good platform for testing the $\phi_{2,\rho}^\perp$ behavior. For the purpose, we suggest a convenient WH-model for the $\rho$-meson leading-twist wavefunction, in which the parameter $B_{2,\rho}^\perp \sim a_2^\perp$ dominantly controls its longitudinal distribution. Typically, its DA $\phi_{2,\rho}^\perp$ is CZ-like as $B_{2,\rho}^\perp \approx -0.20$, which changes to be asymptotic-like as $B_{2,\rho}^\perp \approx 0.00$. By varying $B_{2,\rho}^\perp \in [-0.20, 0.20]$, we present a detailed comparison of the LCSR estimation for the $B \to \rho$ TFFs with those of pQCD and Lattice QCD predictions. Furthermore, by using the extrapolated TFFs, we estimate the CKM-matrix element $|V_{ub}|$ with the help of two $B \to \rho$ semi-leptonic decays. The predicted value for $|V_{ub}|$ increases with the increment of $B_{2,\rho}^\perp$, i.e. we have $|V_{ub}| = (2.91 \pm 0.19) \times 10^{-3}$ for $B_{2,\rho}^\perp = -0.20$ and $|V_{ub}| = (3.11 \pm 0.19) \times 10^{-3}$ for $B_{2,\rho}^\perp = 0.00$. If using the BABAR prediction as a criteria, we observe that $B_{2,\rho}^\perp \in [-0.2, 0.10]$, which indicates that the $\rho$-meson DA prefers a single-peak behavior rather than a double-humped behavior. The $\rho$-meson DA can be further constrained by more data available in the near future, and we hope the $\rho$-meson DA behavior can be determined finally.

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I. INTRODUCTION

The $B \to \rho$ transitions shall manifest themselves in the semileptonic decay $B \to \rho \ell \nu_\ell$ or the rare penguin-induced flavor-changing-neutral-current decays $B \to \rho \gamma$ and $B \to \rho \ell^+ \ell^-$. Since the leptons do not participate in strong interaction, the $B$-meson semileptonic decay shall provide much clearer samples than its hadronic decays and shall be important in studying strong and weak interactions. Moreover, those decays can provide a good platform for determining the $\rho$-meson DAs and the Cabibbo-Kobayashi-Maskawa (CMK) matrix elements.

The QCD light-cone sum rules (LCSR) [1, 2] provides an efficient tool in making predictions for exclusive processes. This method is based on the operator product expansion (OPE) near the light cone $x^2 \to 0$. In different to the conventional SVZ sum rules [3], the LCSR expanses the operator over its twists rather than its dimensions. That is, all non-perturbative dynamics is parameterized into light-cone distribution amplitudes (LCDAs) instead of the quark and gluon condensates as is the case of SVZ sum rules. Since its invention, the LCSR has been widely adopted for studying the $B$-meson decays to light mesons, cf.Refs.[4-9]. It is noted that the LCSRs for the $B \to \rho$ TFFs $A_0(q^2)$, $A_1(q^2)$, $A_2(q^2)$ and $V(q^2)$, being the key components for determining the total width $\Gamma_{B\to \rho}$ for the $B \to \rho$ semi-leptonic decay and the CKM matrix element $|V_{ub}|$, are applicable for low and intermediate $q^2$ region. Here $q$ stands for the momentum transfer between the $B$-meson and the $\rho$-meson. This can be compared with the applicability of the high $q^2$ region for the lattice QCD [10-13] and the low $q^2$ region for pQCD [14, 15]. Those approaches are complementary to each other, and by combining the results derived from those approaches, one may obtain a full understanding of the $B \to \rho$ TFFs in their whole physical region. Particularly, the LCSR provides a link between pQCD and LQCD estimations. Thus, a better LCSR prediction with less uncertainties shall be helpful for a better understanding of those TFFs. Furthermore, by utilizing the LCSR predictions for the $B \to \rho$ TFFs, one may inversely get interesting information on the QCD parameters such as the $\rho$-meson LCDAs via a detailed comparison with the lattice QCD estimation and the experimentally known hadronic parts.

The meson’s LCDA, which relates to the matrix elements of the nonlocal light-ray operators sandwiched between the hadronic state and the vacuum, can exhibit all properties of the meson and provide the underlying links between the hadronic phenomena at the small and large distances. The structures of the LCDAs for the $\rho$-meson are much more complex than the simpler pseudoscalar pion or kaon LCDAs. There are chiral-even or chiral-odd LCDAs for the $\rho$ meson due to the chiral-even and chiral-odd operators; and the $\rho$ meson has two polarization states, longitudinal (||) or transverse (⊥), which may correspond to different twist structures. As sug-
TABLE I. The ρ-meson LCDAs with different twist-structures up to δ3-order [16, 17], where δ ≃ mρ/mB.

| twist-2 | twist-3 | twist-4 |
|---------|---------|---------|
| δ0     | φ_{2,ρ} | /       |
| δ1     | φ_{2,ρ} | ψ_{3,ρ}, Φ_{3,ρ} |
| δ2     | /       | ψ_{3,ρ}, Φ_{3,ρ} |
| δ3     | /       | /       |

suggested in Refs. [16, 17], it is convenient to arrange the ρ-meson LCDAs by a parameter δ, i.e. δ ≃ mρ/mB in an expansion over 1/mB. Following the idea, a classification of twist-2, twist-3 and twist-4 LCDAs up to δ3 are collected in Table I.

To derive the LCSR for the B → ρ TFFs, if taking the standard correlator as the starting point, one observes that all those twist-2, twist-3 and twist-4 LCDAs listed in Table I are there, even though their relative importance follow the suggested δ-counting rule. All those LCDAs, especially those at the order of δ1, shall have sizable contributions, then they should be taken into consideration for a sound estimation [17, 18]. Especially, the accuracy of the LCSR depends heavily on how well we know those δ0 and δ1 LCDAs. At present, all the ρ-meson LCDAs are far from affirmation, then, it would be helpful to find a way to suppress those uncertain sources as much as possible so as to achieve a more reliable estimation. The LCSR derived with the help of a chiral current, the so-called improved LCSR [8, 9, 19–21], can be adopted for such purpose.

By using the improved LCSR, a proper choice of the chiral current correlator shall be taken as the starting point such that the relevant twist-3 LCDAs and the parallel twist-2 LCDA φ_{2,ρ} at the δ-order make no contribution, and the leaving δ2-suppressed twist-3 and twist-4 LCDAs shall lead to small contributions to the LCSR. Thus, the errors due to those uncertain higher-twist LCDAs themselves are largely suppressed. This inversely makes the B → ρ semi-leptonic decays be good places for testing different models of the ρ-meson transverse leading-twist LCDA φ_{2,ρ}^+. For the purpose, we introduce a model for the ρ-meson leading-twist LC wavefunction (LCWF) ψ_{2,ρ}^+ based on the well-known Brodsky-Huang-Lepage (BHL) prescription for constructing the light-meson’s LCWF [22]. As will be shown latter, its longitudinal behavior is dominantly determined by an input parameter B_{2,ρ}^+, i.e. we have B_{2,ρ}^+ ∼ a_2^+ with a_2^+ being the second Gegenbauer moment of φ_{2,ρ}^+. Several LCDA models have been suggested in the literature, either the one in the Gegenbauer expansion [17, 23] or the one based on the AdS/QCD theory [24, 25]. It would be helpful to make a comparison of all those models and show how they affect the LCSR for the B → ρ TFFs.

The LCSR can be extrapolated to any physical region, then we can compare our LCSR prediction for the B → ρ TFFs with the lattice QCD estimations [10–13]. Through such a comparison, one may determine a possible range for the parameter B_{2,ρ}^+ and then a possibly determined behavior for φ_{2,ρ}^+. Moreover, the CKM matrix element |V_{ub}| has been measured by various groups from the B → ρ semi-leptonic decays, cf.Refs. [26–29]. As an application of our present estimations for B → ρ TFFs, we shall predict the CKM matrix element |V_{ub}| and compare it with the experimental predictions.

The remaining parts of the paper are organized as follows. In Sec.II, we present the formulas for the B → ρ semi-leptonic decay and the calculation technology for the B → ρ TFFs under the improved LCSR approach. In Sec.III, we present our numerical results and discussions. Several DA models shall be discussed there. The B → ρ TFFs under various DA models and a comparison of those TFFS with the lattice QCD estimation shall be presented. A comparison of the CKM matrix element |V_{ub}| with the experimental estimation shall also be presented. Sec.IV is reserved for a summary.

II. CALCULATION TECHNOLOGY

The key component of the decay B → ρℓν is the matrix element ⟨ρ(p, λ)|qγμ(1 − γ5)b(B(p + q)), which can be expanded as

\[
\rho(p, λ)|qγ_\mu(1 - γ_5)b(B(p + q)) = -ie^\mu_\sigma(\lambda)(m_B + m_ρ)A_1(q^2) + i(e^\mu_\sigma(\lambda) - q^\sigma)A_2(q^2)(2p + q)_\mu \frac{m_B + m_ρ}{m_B + m_ρ} + iq_\mu(e^\mu_\sigma(\lambda) - q^\sigma)2m_ρ q^2 [A_3(q^2) - A_0(q^2)] + \epsilon_\mu_\rho_\sigma_\beta e^{\lambda}_\rho_\beta \frac{2V(q^2)}{m_B + m_ρ} \frac{2V(q^2)}{m_B + m_ρ},
\]

where e^{(λ)} stands for the ρ-meson polarization vector with λ being its transverse (⊥) or longitudinal (∥) component accordingly. p is the ρ-meson momentum and q = p_B - p_ρ is the four-momentum transfer between those two mesons. A_i with i = (0, · · · , 3) and V are B → ρ TFFs, in which A_{1,2,3} satisfy the following relation

\[
A_3(q^2) = \frac{m_B + m_ρ}{2m_ρ} A_1(q^2) - \frac{m_B - m_ρ}{2m_ρ} A_2(q^2).
\]

At the endpoint, we have A_0(0) = A_3(0).

If the leptonic mass can be ignored (ℓ = e or µ), due to chiral suppression, the total differential decay width for B → ρℓν can be written as

\[
\frac{dΓ}{dq^2} = G|V_{ub}|^2 λ(q^2)^{1/2} q^2[H_0^2(q^2) + H_1^2(q^2) + H_2^2(q^2)],
\]

where G = G_F/(192π^3m_β^3). The G_F = 1.166 × 10^{-5} is the fermi coupling constant and the phase-space factor λ(q^2) = (m_β^2 + m_ρ^2 - q^2)^2 - 4m_β^2m_ρ^2. The transverse and longitudinal helicity amplitudes H_b,±(q^2) are given by
\[
H_\pm(q^2) = (m_B + m_\rho)A_1(q^2) + \frac{\lambda(q^2)^{1/2}}{m_B + m_\rho}V(q^2),
\]
\[
H_0(q^2) = \frac{1}{2m_\rho(q^2)^{1/2}} \left\{ (m_B^2 - m_\rho^2 - q^2)(m_B + m_\rho) \times A_1(q^2) - \frac{\lambda(q^2)}{m_B + m_\rho}A_2(q^2) \right\}.
\]

In the helicity basis, each TFF corresponds to a transition amplitude with definite spin-parity quantum numbers. We can highlight the chiral-odd \(\rho\) meson contributions; while by taking \(B\)-lighted DAs can be tested with a much higher precision. The physical region for the squared four-momentum transfer is \(0 \leq q^2 \leq q_{\text{max}}^2 \equiv (m_B - m_\rho)^2\).

In most of the kinematic region, the \(B \rightarrow \rho\) TFFs are non-perturbative, which can only be treated within some non-perturbative approaches such as LCSR or lattice QCD. In the low and intermediate region, it can be calculated within the framework of LCSR.

The current \(j_B^\mu(x)\) can be conventionally and simply chosen as \(ib(x)\gamma_5q_2(x)\) such that it has the same quantum state as that of the pseudoscalar \(B\)-meson with \(J^P = 0^-\). As mentioned in the Introduction, such a choice of correlator shall result in a complete series of all the possible \(\rho\)-meson twist-structures \([17, 18]\). Large theoretical uncertainties shall be introduced due to unknown/less-known DAs. On the other hand, as suggested by the improved LCSR approach, it is convenient to choose \(j_B^\mu(x)\) as a chiral current, either \(ib(x)(1-\gamma_5)q_2(x)\) or \(ib(x)(1+\gamma_5)q_2(x)\), to do the calculation. The advantage of such a choice lies in that one can highlight the contributions from different twist series of \(\rho\)-meson DAs' to the TFFs by selecting a proper chiral current. Then the properties of those highlighted DAs can be tested with a much higher precision. More explicitly, by taking \(j_B^\mu(x) = ib(x)(1-\gamma_5)q_2(x)\), we can highlight the chiral-even \(\rho\)-meson DAs' contributions; while by taking \(j_B^\mu(x) = ib(x)(1+\gamma_5)q_2(x)\), one can highlight the chiral-odd \(\rho\)-meson DAs' contributions.

In the present paper, we shall adopt \(j_B^\mu(x) = ib(x)(1+\gamma_5)q_2(x)\) to deal with the \(B \rightarrow \rho\) TFFs. Under such choice, it is noted that the resulting hadronic representation of the correlator not only depends on the resonances with \(J^P = 0^-\) state but also depends on those of \(J^P = 0^+\) state. This is the price of introducing a chiral correlator for LCSR. But it is worthwhile, since we can eliminate the large uncertainty from the twist-2 and twist-3 structures at the \(\delta^1\)-order, and we may avoid the pollution from the scalar resonances with \(J^P = 0^+\) by choosing proper continuum threshold \(s_0\). Our final results with slight \(s_0\) dependence also confirm this assumption.

The correlator \((6)\) is an analytic function of \(q^2\) defined at both negative (space-like) and positive (time-like) values of \(q^2\). In the time-like region, the long-distance quark-gluon interactions become important and, eventually, the quarks form hadrons. To deal with the correlator \((6)\) at the time-like region, one can insert a complete series of intermediate hadronic states with the same quantum numbers as the current operator \(\bar{b}i(1+\gamma_5)q_2\) to obtain the hadronic representation. After isolating the pole term of the lowest pseudoscalar \(B\)-meson, we obtain

\[
\Pi^H_\mu(p, q) = \frac{(\rho(p, \lambda)|\bar{q}_1\gamma_\mu(1-\gamma_5)b|B)\langle B|\bar{b}(0)i\gamma_5q_2(0)|0\rangle}{m_B^2 - (p + q)^2} + \sum_H \frac{\langle \rho(p, \lambda)|\bar{q}_1\gamma_\mu(1-\gamma_5)b|B^H\rangle\langle B^H|\bar{b}i(1+\gamma_5)q_2|0\rangle}{m_{B^H}^2 - (p + q)^2},
\]

where \(\langle B|\bar{b}i\gamma_5q_2|0\rangle = m_B^2f_B/m_b\) with \(f_B\) standing for the \(B\)-meson decay constant. Thus, the hadronic expressions

\[
\Pi_1^H[q^2, (p + q)^2] = \frac{m_B^2f_B(m_B + m_\rho)}{m_B[m_B^2 - (p + q)^2]}A_1(q^2) + \int_{s_0}^\infty \frac{\rho_1^H}{s - (p + q)^2} ds,
\]

for the correlator \((6)\) are
\[ \Pi_2^{H}(q^2, (p + q)^2) = \frac{m_B^2 f_B}{m_b (m_B + m_p) [m_B^2 - (p + q)^2] \times A_2(q^2) + \int_{s_0}^{\infty} \frac{\rho_2^H}{s - (p + q)^2} ds, } \]

\[ \Pi_V^{H}(q^2, (p + q)^2) = \frac{2 m_B^2 f_B}{m_b (m_B + m_p) [m_B^2 - (p + q)^2] \times V(q^2) + \int_{s_0}^{\infty} \frac{\rho_V^H}{s - (p + q)^2} ds. } \]

The contributions from higher resonances and the continuum states above \( s_0 \) have been written in terms of dispersion integrations. The spectral densities \( \rho_1^{H}(s) \) can be approximated via the quark-hadron duality ansatz [3], \( \rho_1^{H}(s) = \rho_1^{QCD}(s) \theta(s - s_0) \). As shown by Eqs.(4,5), the decay width for \( B \rightarrow \rho \ell \nu \) with massless leptons involves \( A_1, A_2 \) and \( V \) only, so we present the procedures on how to derive the LCSRs of those TFFs in detail. For completeness, we present the final LCSR for the fourth independent TFF \( A_0 \) in Appendix A.

On the other hand, within the space-like region, we can calculate the correlator via the QCD theory. In large space-like region, which corresponds to small light-cone distance \( x^2 \rightarrow 0 \), we have \( (p + q)^2 - m_B^2 \ll 0 \) with the momentum transfer \( q \sim \mathcal{O}(1 \text{ GeV}) \ll m_b^2 \). In this region, the correlator can be treated by the operator product expansion (OPE) with the coefficients being pQCD calculable. As a basis, we adopt the following \( b \)-quark propagator to do the calculation

\[
\langle 0 | T \{ b(x) \bar{b}(0) \} | 0 \rangle = i \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \frac{k^2 + m_b^2}{m_b^2 - k^2 - i\epsilon} \int d^4 k' \frac{1}{(2\pi)^4} e^{-i k' \cdot x} \int_0^1 dv G^{\mu\nu}(vx) \left[ \frac{k + m_b}{2(m_b^2 - k^2)} \sigma_{\mu\nu} + \frac{v}{m_b^2 - k^2} \sigma_{\mu\nu} \right],
\]

where \( G_{\mu\nu} \) is the gluonic field strength and \( g_s \) denotes the strong coupling constant. Before doing concrete calculation, the following formulas are helpful:

\[
\sigma_{\mu\nu} \gamma_5 = -\frac{i}{2} \epsilon_{\mu\nu\alpha\beta} \sigma^{\alpha\beta},
\]

\[
\gamma_\mu \sigma_{\alpha\beta} = i (g_{\mu\alpha} \gamma_\beta - g_{\mu\beta} \gamma_\alpha) + \epsilon_{\mu\nu\alpha\beta} \gamma^{\nu} \gamma_5,
\]

\[
\gamma_\mu \gamma_\nu = g_{\mu\nu} - i \sigma_{\mu\nu},
\]

\[
\gamma_\mu \gamma_\nu \sigma_{\alpha\beta} = i [g_{\nu\alpha} (g_{\mu\beta} - i \sigma_{\mu\beta}) - g_{\nu\beta} (g_{\mu\alpha} - i \sigma_{\mu\alpha})],
\]

\[
+ \epsilon_{\nu\lambda\alpha\beta} (g_{\mu\lambda} - i \sigma_{\mu\lambda}) \gamma_5. \tag{12}
\]

After applying the OPE to the correlator (6), we obtain

\[
\Pi_\mu^{OPE} = \int \frac{d^4 x d^4 k}{(2\pi)^4} e^{i (q - k) \cdot x} \left\{ \frac{1}{m_b^2 - k^2} \left\{ 2 k^\mu \langle \rho(p, \lambda) \bar{q}_1(x) q_2(0) | 0 \rangle - 2 i k^\nu \langle \rho(p, \lambda) \bar{q}_1(x) \sigma_{\mu\nu} q_2(0) | 0 \rangle \right\} - \int dv \left\{ \frac{k^\nu}{(m_b^2 - k^2)} \left[ -i \langle \rho(p, \lambda) \bar{q}_1(x) g_s G_{\mu\nu}(vx) q_2(0) | 0 \rangle \right] - 2 \langle \rho(p, \lambda) \bar{q}_1(x) g_s G^{\alpha\nu}(vx) q_2(0) | 0 \rangle + 2 i \langle \rho(p, \lambda) \bar{q}_1(x) g_s \tilde{G}_{\mu\nu}(vx) \gamma_5 q_2(0) | 0 \rangle \right\} \right\} \right\}, \tag{13}
\]

where \( \tilde{G}_{\mu\nu}(vx) = \epsilon_{\mu\nu\alpha\beta} G^{\alpha\beta}(vx) / 2 \). Here, we will deal with the meson-to-vacuum matrix element with the \( \gamma \)-structure, \( \Gamma = 1, i \gamma_5 \) and \( \sigma_{\mu\nu} \), for 2- and 3-particles contribution. It is noted that the meson-to-vacuum matrix element with \( \Gamma = i \gamma_5 \) vanishes for two-particle contribution, because it is impossible to construct a pseudoscalar quantity from \( p_\mu, x_\mu \) and \( \epsilon_\mu^B \). Up to twist-4 accuracy, those matrix elements can be expanded as [16]:

\[ e_{\mu}^{B} \]
\[ \langle \rho(p, \lambda) | \bar{q}(x) \sigma_{\mu\nu} q_2(0) | 0 \rangle = -i f^\perp_\rho \int_0^1 \frac{du}{u} e^{iu(p \cdot x)} \left\{ \left( e^{\ast(\lambda)}_{\mu}(p_\nu - e^{\ast(\lambda)}_{\nu} p_\mu) \right) \left[ \phi^\perp_{2\rho}(u) + \frac{m_\rho^2 x^2}{16} \phi^\perp_{4\rho}(u) \right] \right. \]
\[ \left. + \left( p_\mu x_\nu - p_\nu x_\mu \right) \frac{e^{\ast(\lambda)}_{\nu}}{2(p \cdot x)^2} m_\rho \right\} \]
\[ + \frac{1}{2} \left( e^{\ast(\lambda)}_{\mu} x_\nu - e^{\ast(\lambda)}_{\nu} x_\mu \right) m_\rho \int_0^1 \frac{du}{u} e^{iu(p \cdot x)} \left\{ \psi^\perp_{4\rho}(u) - \frac{1}{2} \phi^\perp_{4\rho}(u) \right\} \right\}, \]
\[ (14) \]
\[ \langle \rho(p, \lambda) | \bar{q}(x) q_2(0) | 0 \rangle = - \frac{i}{2} f^\perp_\rho \left( e^{\ast(\lambda)}_\mu \cdot x \right) m_\rho^2 \int_0^1 \frac{du}{u} e^{iu(p \cdot x)} \left[ \psi^\perp_{3\rho}(u) \right], \]
\[ (15) \]
\[ \langle \rho(p, \lambda) | \bar{q}(x) G^{\mu\nu}(vx) q_2(0) | 0 \rangle = \frac{m_\rho^2 f^\perp_\rho}{2(p \cdot x)} \left[ p_\mu \left( p_\alpha g^\perp_{\beta\nu} - p_\beta g^\perp_{\alpha\nu} \right) - p_\nu \left( p_\alpha g^\perp_{\beta\mu} - p_\beta g^\perp_{\alpha\mu} \right) \right] \Phi^\perp_{3\rho}(v, p \cdot x), \]
\[ (16) \]
\[ \langle \rho(p, \lambda) | \bar{q}(x) g G^{\mu\nu}(vx) q_2(0) | 0 \rangle = -i m_\rho^2 f^\perp_\rho \left[ e^{\ast(\lambda)}_{\perp \mu} p_\nu - e^{\ast(\lambda)}_{\perp \nu} p_\mu \right] \Psi^\perp_{4\rho}(v, p \cdot x), \]
\[ (17) \]
\[ \langle \rho(p, \lambda) | \bar{q}(x) g \tilde{G}^{\mu\nu}(vx) \gamma_5 q_2(0) | 0 \rangle = i m_\rho^2 f^\perp_\rho \left[ e^{\ast(\lambda)}_{\perp \mu} p_\nu - e^{\ast(\lambda)}_{\perp \nu} p_\mu \right] \tilde{\Psi}^\perp_{4\rho}(v, p \cdot x), \]
\[ (18) \]

where \( f^\perp_\rho \) represents the \( \rho \)-meson tensor decay constant, \( \langle \rho(p, \lambda) | \bar{q}(0) \sigma_{\mu\nu} q_2(0) | 0 \rangle = i m_\rho^2 \left( e^{\ast(\lambda)}_{\mu} p_\nu - e^{\ast(\lambda)}_{\nu} p_\mu \right), \)

and we have set
\[ g^\perp_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu x_\nu + p_\nu x_\mu}{p \cdot x}, \]
\[ e^{\lambda}_\mu = \frac{e^{\lambda}}{p \cdot x} \left( p_\mu - \frac{1}{2} m_\rho \right) x_\mu + e^{\lambda}_{\perp \mu}, \]

\[ K(v, p \cdot x) = \int D\alpha e^{i(\alpha_1 + \alpha_2)} K(\alpha). \]

Here \( D\alpha = d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) \) and \( K(\alpha) \) stands for the twist-3 or twist-4 DA \( \Phi^\perp_{3\rho}(\alpha), \Psi^\perp_{4\rho}(\alpha) \) or \( \tilde{\Psi}^\perp_{4\rho}(\alpha) \), respectively, in which \( \alpha = \{ \alpha_1, \alpha_2, \alpha_3 \} \) corresponds to the momentum fractions carried by the antiquark, quark and gluon, respectively. As a tricky point, the integration over \( x \) within the above equation can be done by transforming the \( x_\mu \) in the nominator to \( i \partial / \partial (u p)_\mu \), or equivalently to \( -i \partial / \partial q_\mu \), and to transform \( \phi(u)/p \cdot x \) to \( -i \int_0^1 dv \phi(v) \equiv -i \Phi(u) \).

Equaling the correlator within different regions and applying the conventional Borel transformation to suppress the contributions from the unknown continuum states, we obtain the LCSRs for the TFFs:
\[ f_B A_2(q^2)e^{-m_B^2/M^2} = \frac{m_b(m_B + m_\rho)m_\rho^2}{m_B^2} \int_0^1 \frac{du}{u} e^{-(s(u)/M^2)} \left\{ \frac{1}{m_\rho^2} \Theta(c(u, s_0)) \phi_{2,\rho}^\perp(u, \mu) - \frac{1}{M^2} \tilde{\Theta}(c(u, s_0)) \psi^\parallel_{3,\rho}(u) \right\} \]

\[ -\frac{1}{4} \left[ \frac{m_\rho^2}{u^2 M^4} \tilde{\Theta}(c(u, s_0)) + \frac{1}{u M^2} \tilde{\Theta}(c(u, s_0)) \right] \phi_{4,\rho}^\perp(u) + 2 \left[ \frac{C - 2m_\rho^2}{u^2 M^4} \tilde{\Theta}(c(u, s_0)) - \frac{1}{u M^2} \tilde{\Theta}(c(u, s_0)) \right] I_L(u) \]

\[ -\frac{1}{M^2} \tilde{\Theta}(c(u, s_0)) H_3(u) \right\} + \int D\alpha_i \int_0^1 dv e^{-X(u)/M^2} \frac{1}{2 X^2 M^2} \tilde{\Theta}(c(X, s_0)) \left[ (4v - 1) \Psi^+_{4,\rho}(\alpha) - \Psi^+_{4,\rho}(\alpha) \right] \]

\[ + 4v \tilde{\Phi}^+_{3,\rho}(\alpha) \right\} \right) \right) \right) \]

\[ f_B V(q^2)e^{-m_B^2/M^2} = \frac{m_b(m_B + m_\rho)m_\rho^2}{m_B^2} \int_0^1 \frac{du}{u} e^{-(s(u)/M^2)} \left\{ \Theta(c(u, s_0)) \phi_{2,\rho}^\perp(u, \mu) \right. \]

\[ - \frac{1}{m_B^2} \tilde{\Theta}(c(u, s_0)) \psi^\parallel_{3,\rho}(u) \left. \right\} \]

where \( C = m_b^2 + u^2 m_\rho^2 - q^2, \frac{C}{u^2 M^4} = m_b^2 + X^2 m_\rho^2 - q^2 \) and \( s(u) = \frac{m_b^2}{u_0^2 M^4} - u(q^2 - u(m_\rho^2)) \). \( s(X) = \frac{m_b^2}{X M^4} - X(q^2 - X m_\rho^2) \). \( X = a_1 + v_0 s_0, \) \( u = 1 - u \) and \( X = 1 - X \). The simplified functions \( I_L(u) \) and \( H_3(u) \) are defined as

\[ I_L(u) = \int_0^u dv \int_0^v dw \left\{ \phi_{3,\rho}^\perp(w) - \frac{1}{2} \tilde{\phi}_{3,\rho}^\perp(w) \right\} \]

\[ H_3(u) = \int_0^u dv \left\{ \psi_{4,\rho}^\perp(v) - \frac{1}{2} \tilde{\phi}_{4,\rho}^\perp(v) \right\} \]

In deriving the above LCSRs, we need to deal with two typical Borel transformations, i.e. those involving \( E/D^n \) and \( 1/D^n (1 \leq m, n \leq 3) \) with \( D = m_b^2 - (up + q^2)^2 \) and \( E = up^2 + p \cdot q \). We put the needed formulas in the following:

\[ \beta_M^2 \int_0^1 \frac{du}{D} \frac{f(u)}{D} = \int_0^1 \frac{du}{u} e^{-\frac{C}{M^2}} \Theta(c(u, s_0)) f(u), \]

\[ \beta_M^2 \int_0^1 \frac{du}{D^2} \frac{f(u)}{D^2} = \int_0^1 \frac{du}{u^2 M^2} e^{-\frac{C}{M^2}} \Theta(c(u, s_0)) f(u), \]

\[ \beta_M^2 \int_0^1 \frac{du}{D^3} \frac{f(u)}{D^3} = \int_0^1 \frac{du}{2u^3 M^2} e^{-\frac{C}{M^2}} \tilde{\Theta}(c(u, s_0)) f(u), \]

\[ \beta_M^2 \int_0^1 \frac{du}{D^4} \frac{f(u)}{D^4} = \int_0^1 \frac{du}{2u^4 M^2} e^{-\frac{C}{M^2}} \tilde{\Theta}(c(u, s_0)) f(u), \]

\[ \beta_M^2 \int_0^1 \frac{du}{D^5} \frac{f(u)}{D^5} = \int_0^1 \frac{du}{2u^5 M^2} e^{-\frac{C}{M^2}} \tilde{\Theta}(c(u, s_0)) f(u), \]

\[ \beta_M^2 \int_0^1 \frac{du}{D^6} \frac{f(u)}{D^6} = \int_0^1 \frac{du}{2u^6 M^2} e^{-\frac{C}{M^2}} \tilde{\Theta}(c(u, s_0)) f(u), \]

\[ \beta_M^2 \int_0^1 \frac{du}{D^7} \frac{f(u)}{D^7} = \int_0^1 \frac{du}{2u^7 M^2} e^{-\frac{C}{M^2}} \tilde{\Theta}(c(u, s_0)) f(u), \]

where \( \beta_M^2 = \lim_{q^2 \to 0, n \to \infty; (q^2/n) = M^2} \left( \frac{q^2}{n} \right) \) stands for the Borel transformation, \( c(u, s_0) = u s_0 - m_b^2 + u q^2 - u m_\rho^2, \) \( \Theta(c(u, s_0)) \) is the conventional step func-

\[ \text{tion, } \tilde{\Theta}(c(u, s_0)) \) and \( \tilde{\Theta}(c(u, s_0)) \) are defined as

\[ \int_0^1 \frac{du}{u^2 M^2} e^{-\frac{C}{M^2}} \tilde{\Theta}(c(u, s_0)) f(u) = \int_0^1 \frac{du}{u^2 M^2} e^{-\frac{C}{M^2}} \tilde{\Theta}(c(u, s_0)) f(u) + \delta(c(u_0, s_0)), \]

\[ \Delta(c(u, s_0)) = e^{-s_0/M^2} \left[ \frac{1}{2 u_0 M^2} f(u_0) \right] C_0 - \frac{u_0^2}{2 C_0} \left( \frac{f(u_0)}{u C} \right) \bigg|_{u = u_0} \]

\[ C_0 = m_b^2 + u_0^2 m_\rho^2 - q^2 \text{ and } u_0 \text{ is the solution of } c(u_0, s_0) = 0 \text{ with } 0 \leq u_0 \leq 1. \text{ Here we do not present the surface terms for the 3-particle DAs, whose contributions are quite small and can be safely neglected. As a check, by taking only the leading-twist terms in those LCSRs (19, 20, 21), we return to the LCSRs of Ref. [30].} \]

It is noted that the LCSRs (19, 20, 21) only contain the chiral-odd DAs' contributions. Then, we can use a more simpler way to get the LCSRs for those TFFs. That is, we can deal with the correlator (6) within the space-like region by directly treating their Dirac structures. The time-order production of the correlator (6), i.e. \( T\{ q_1(x) \gamma_\mu(1 - \gamma_5)b(x), j_B^\perp(0) \} \), can be rewritten in a simpler form as

\[ \text{Tr } \left\{ q_1(u) q_2(u) \gamma_\mu(1 - \gamma_5)(u \not\! p + g + m_b)(1 + \gamma_5) \right\}, \]
in which the chiral-odd vacuum-to-vector state matrix element can be expanded as

\[
\langle \rho(p, \lambda) | q_1(x) q_2(0) | 0 \rangle^{\text{chiral-odd}} = \frac{1}{4} \int dx^3 p_T x \rho \left\{ -i f_{\rho} \sigma_{\alpha\beta} \left[ e^{i(\lambda + \rho)} \phi^{(\alpha)}_{e,\rho}(u) + \frac{1}{16} m_{\rho}^2 e^{i(\lambda + \rho)} \phi^{(\alpha)}_{e,\rho}(u) \right] - m_{\rho}^2 e^{i(\lambda + \rho)} \phi^{(\alpha)}_{e,\rho}(u) \right\}.
\]

Using this trace form, we can also get the same LCSRs for the TFFs.

III. NUMERICAL ANALYSIS

A. Input parameters

| m_b/GeV | s_0/GeV^2 | M^2/GeV^2 | f_B/GeV |
|---------|-----------|-----------|---------|
| 4.75    | [33.1, 36.9] | [2.09, 2.57] | 0.179(5) |
| 4.80    | [32.8, 35.9] | [1.93, 2.36] | 0.160(5) |
| 4.85    | [32.5, 34.9] | [1.81, 2.17] | 0.141(4) |

Table II. A LCSR estimation on f_{B} for m_b = 4.80 ± 0.05 GeV. The number in the parenthesis shows the uncertainty in the last digit.

In doing the numerical calculation, we take f_{\rho} = 0.165(9) GeV [31] and m_b = 4.80 ± 0.05 GeV for the b-quark pole mass. The \rho-meson and B-meson masses are taken as \rho = 0.775 GeV and m_B = 5.279 GeV [32]. The value of f_B can be consistently determined from a chiral LCSR, following the formulas in Ref.[8], we recalculate it and put the numerical results in Table II. Here, the Borel parameter M^2 and s_0 are determined by the requirements of the continuum state’s contribution to be less than 30% and the six condensates’ contributions to be less than 10% of the total LCSR.

1. Models for the leading-twist LCDA \(\phi^{(\perp)}_{e,\rho}\)

As shown by the LCSRs (19,20,21), the \rho-meson LCDA \(\phi^{(\perp)}_{e,\rho}\), \(\phi^{(\perp)}_{e,\rho}\), \(\psi^{(\perp)}_{e,\rho}\) and \(\Phi^{(\perp)}_{e,\rho}\) are at the \(\delta^1\)-order, provide zero contributions. The dominant contribution comes from the leading-twist LCDA \(\phi^{(\perp)}_{e,\rho}\) while all the remaining twist-3 and twist-4 LCDA contribute totally less than 10% to the LCSRs. Thus, the uncertainties of the LCSR from the uncertainties of those higher-twist LCDA are highly suppressed. For clarity, we take those higher-twist DAs to be the ones suggested by Ref.[33], which are put in Appendix B.

The LCSRs (19,20,21) provide a good platform for determining the leading-twist DA \(\phi^{(\perp)}_{e,\rho}\). For the purpose, we adopt three models for \(\phi^{(\perp)}_{e,\rho}\) to do the calculation, i.e. the Gegenbauer polynomial expansion with specific Gegenbauer moments, the one from the AdS/QCD theory, and the one from the Wu-Huang prescription, respectively.

Conventionally, one can expand the light meson’s LCDA as a Gegenbauer expansion. As for \(\phi^{(\perp)}_{e,\rho}\), it can be expanded as

\[
\phi^{(\perp)}_{e,\rho}(x, \mu_0) = 6 x \bar{x} \left[ 1 + \sum_{n=2,4,\ldots} a_n^{(\perp)}(\mu_0) C_n^{3/2}(2x - 1) \right].
\]

where \(\mu_0\) stands for some hadronic scale \(\sim 1\) GeV. \(a_n^{(\perp)}\) stands for the \(n\)-th Gegenbauer moment and \(C_n^{3/2}\) is the Gegenbauer polynomials. Such a Gegenbauer expansion is convergent and is dominated by its first several moments. Practically, one always takes the first term to do the analysis, i.e.

\[
\phi^{(\perp)}_{e,\rho}(x, \mu_0) = 6 x \bar{x} \left[ 1 + a_2^{(\perp)}(\mu_0) C_2^{3/2}(2x - 1) \right].
\]

Several choices for the second Gegenbauer moment \(a_2^{(\perp)}(\mu_0 = 1\) GeV) from the QCD sum rules have been suggested in the literature, e.g. the one suggested by Chernyak and Zhitnitsky is \(-0.167 \) [23] (we call it the CZ model) and the one by Ball and Brauns (we call it the BB model) is \(0.14 \pm 0.06 \) [33]. It is noted that the CZ model prefers a single-peak behavior, while BB model tends to a double humped behavior. Furthermore, the Gegenbauer moments \(a_2^{(\perp)}\) at any other renormalization scale can be obtain from QCD evolution, e.g. \(a_n^{(\perp)}(\mu) = a_n^{(\perp)}(\mu_0) C_{n/3}^{3/2} \) [31], where \(\beta_0 = 11 - 2n/3, \gamma_0 = a_s(\mu_0)/a_s(\mu_0)\) and one-loop anomalous dimensions \(\gamma_0 = 4C_F [\psi(n + 2) + \gamma_E - 1]\) with \(\psi(n + 1) = \sum_{k=1}^{n} 1/k - \gamma_E\). When the scale tends to infinity, we shall obtain \(a_n^{(\perp)}(\infty) \rightarrow 0\), which corresponds to the asymptotic DA suggested by Ref.[34].

On the other hand, the \(\rho\)-meson DA \(\phi^{(\perp)}_{e,\rho}\) can be derived from its LCWF, since it can be related with the LCWF via the relation

\[
\phi^{(\perp)}_{e,\rho}(x, \mu_0) = \frac{2\sqrt{3}}{f_{\rho}^2} \int_{|k_L|^2 < \mu_0^2} \frac{dk_L}{16\pi^3} \psi^{(\perp)}_{e,\rho}(x, k_L),
\]

where \(f_{\rho}^2 = f_{\rho}/C_{\rho}^{3/2}\) is the improved vector decay constant with \(C_{\rho}^{3/2} = \sqrt{3}\).

One way of constructing the LCWF has been suggested under the AdS/QCD theory [24, 25]. That is, Ref.[35] suggests

\[
\psi^{(\perp)}_{e,\rho}(x, \zeta) = N_1 \sqrt{\frac{x}{x}} \exp \left( -\frac{\kappa^2 x}{2} \right) \exp \left( -\frac{m_{1}^2}{2\kappa^2 x} \right),
\]

which leads to

\[
\phi^{(\perp)}_{e,\rho}(x, \mu_0) = \frac{3m_{1}}{\pi f_{\rho}} \int d\zeta \mu_0 J_1(\mu_0 \zeta) \frac{\psi^{(\perp)}_{e,\rho}(x, \zeta)}{xx},
\]

\[
(29)
\]
We call it the AdS/QCD model. The parameter $m_f = 0.14\text{GeV}$ [35–39] and $N_c = 2.031$, which is fixed by the normalization condition, $\int_0^1 \phi_{2,\rho}^{(A)}(x, \mu_0) = 1$. $\kappa^2 = m_G^2/2 + \zeta = (x \sqrt{1-x})$ with $r$ being the transverse distance between the quark and antiquark at the equal light-front time and $\sqrt{z^2}$ being the variable that maps onto the fifth dimension of the AdS space [40–42].

Another way of constructing the light-meson WF has been suggested by Wu and Huang [43] (we call it the WH model). Following its idea, the radial part $\psi_{2,\rho}$ of $\phi_{2,\rho}(x, \mu_0)$ can be constructed from the BHL-preservation [22] and its spin-space part $\chi_{h1}^{b2}(x, k_\perp)$ can be derived from the Wigner-Melosh rotation [44, 45], that is

$$\psi_{2,\rho}(x, k_\perp) = \sum_{h_1 h_2} \chi_{h1}^{b2}(x, k_\perp) \psi_{2,\rho}^R(x, k_\perp),$$

where $\psi_{2,\rho}^R \propto [1 + B_{2,\rho}^R \xi^{3/2}(\xi)] \exp \left[ -b_{2,\rho}^R (\xi^2 + m_\rho^2) / \xi \right]$ and the spin-space wavefunction $\chi_{h1}^{b2}(x, k_\perp)$ can be found in Ref. [22]. Then, we get

$$\phi_{2,\rho}^R(x, \mu_0) = A_{2,\rho}^{\perp}(\mu_0) \frac{\sqrt{1-x} m_q}{4\pi^3/2} [1 + B_{2,\rho}^R C_n^{3/2} (\xi)] \times \left[ \text{Erf} \left( b_{2,\rho}^R \sqrt{\frac{\mu_0^2 + m_q^2}{x \xi}} \right) - \text{Erf} \left( b_{2,\rho}^R \sqrt{\frac{m_q^2}{x \xi}} \right) \right],$$

where the error function $\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ and the constituent quark mass $m_q \approx 300 \text{ MeV}$. In addition to the normalization condition, we adopt the average value of the transverse momentum square $(k_\perp^2)^{1/2}$ as another constraint, which is defined as

$$(k_\perp^2)^{1/2} = \frac{\int \text{d}x \text{d}^2 k_\perp |k_\perp|^2 |\phi_{2,\rho}^R(x, k_\perp)|^2}{\int \text{d}x \text{d}^2 k_\perp |\psi_{2,\rho}^R(x, k_\perp)|^2}.$$

We take its value to be $0.37 \text{ GeV}$, which is consistent with the choice of Refs [43, 46] for the light-meson. The Gegenbauer moments can be derived from the equation

$$a_n^{b2}(\mu_0) = \frac{\int_0^1 \text{d}x \phi_{2,\rho}^{\perp}(x, \mu_0) C_n^{3/2} (2x - 1)}{\int_0^1 \text{d}x \text{d}^2 x |C_n^{3/2} (2x - 1)|^2}.$$

Using this equation, we can obtain a relation between $B_{2,\rho}^R$ and $a_2^{b2}$.

| $B_{2,\rho}^R$ (GeV$^{-1}$) | $A_{2,\rho}^{\perp}(\mu_0)$ (GeV) | $b_{2,\rho}^{\perp}(\mu_0)$ (MeV) | $a_2^{b2}(1\text{GeV})$ | $a_2^{b2}(2.2\text{GeV})$ |
|-------------------------|-----------------------------|-----------------------------|---------------------|---------------------|
| -0.2                    | 28.56                       | 0.643                       | -0.180              | -0.152              |
| -0.1                    | 27.50                       | 0.628                       | -0.080              | -0.067              |
| 0                       | 25.88                       | 0.604                       | +0.026              | -0.022              |
| +0.1                    | 23.82                       | 0.572                       | +0.140              | +0.118              |
| +0.2                    | 21.61                       | 0.537                       | +0.258              | +0.217              |

TABLE III. The $\rho$-meson leading-twist LCDA parameters $A_{2,\rho}^{\perp}$ and $b_{2,\rho}^{\perp}$ for some typical choices of $B_{2,\rho}^R$. The resultant second Gegenbauer moment $a_2^{b2}$ at $\mu_0 = 1.0\text{GeV}$ and $2.2\text{GeV}$ are also presented.

We put the $\rho$-meson leading-twist DA parameters in Table III, where $B_{2,\rho}^R \in [-0.2, 0.2]$. The resultant second Gegenbauer moment $a_2^{b2}$ at $\mu_0 = 1.0\text{GeV}$ and $2.2\text{GeV}$ are also presented. We observe that $B_{2,\rho}^R \sim a_2^{b2}$, which indicates that $B_{2,\rho}^R$ dominantly determines the longitudinal distribution. We present a comparison of $\phi_{2,\rho}^{\perp}(x, \mu_0)$ under various models in Fig. (1). When $B_{2,\rho}^R$ changes from $-0.20$ to $+0.20$, the $\rho$-meson DA varies from the single-peak behavior to the double-humped behavior. More specifically, we find that $B_{2,\rho}^R = -0.20$ corresponds to CZ-DA, $B_{2,\rho}^R = 0.10$ corresponds to BB-DA and $B_{2,\rho}^R = 0.32$ corresponds to AdS/QCD-DA. Thus the WH-DA provides a convenient form to mimic the behavior of various DA models suggested in the literature. If we have precise measurements for certain processes involving $\rho$-meson, then by comparing the theoretical estimations, we can fix the value of $B_{2,\rho}^R$ and then get the $\rho$-meson DA’s behavior.

As will be shown later that the WH-DA with $B_{2,\rho}^R > 0.10$ leads to a small $A_1(q^2)$ in comparison to the lattice QCD estimation, so we shall take $B_{2,\rho}^R \in [-0.20, +0.10]$ to do our following discussions.

FIG. 1. A comparison of $\phi_{2,\rho}^{\perp}(x, \mu_0 = 1\text{GeV})$ under various models. As for WH-DA model, which are shown by shaded band, we adopt $B_{2,\rho}^R \in [-0.2, 0.2]$. 

...
We have found that both the twist-3 DA $\phi_{3,\rho}$ and twist-4 DA $\psi_{4,\rho}$ shall have large contributions to the LCSR, which does not satisfy the twist-power counting. However, Table V shows that $I_L$ and $H_3$ follow the $\delta$-power counting. Thus, it is better to use the $\delta$-power counting other than the usual twist-powering counting to deal with their contributions. This observation has already been found in Ref.[17]. The 3-particle higher-twist DAs’ contributions such as those of $\Phi_{3,\rho}$ are only about 0.1% of the total LCSR. For example, the non-zero contribution from the 3-particle DA $\Phi_{3,\rho}$ is only about $-0.2\%$ to $A_2(0)$. Then, those 3-particle DAs’ contributions can be safely neglected.

The $\rho$-meson leading-twist DA $\phi_{2,\rho}$ provides the dominant contribution to the $B \to \rho$ TFFs. To show how $\phi_{2,\rho}$ affects the TFFs, we present the $B \to \rho$ TFFs at the large recoil region under the WH-DA, BB-DA, CZ-DA and AdS/QCD-DA in Table VI. In doing the calculation, the scale $\mu$ is set as the typical energy of the process, i.e. $\mu \simeq \sqrt{m_B^2 - m_\rho^2} \sim 2.2\text{GeV}$. As for WH-DA, we calculate the cases with $B_{2,\rho} \sim -0.2, 0.0$ and 0.1, respectively. The results for $B_{2,\rho} = -0.2$ are close to that of CZ-DA and the results for $B_{2,\rho} = 0.1$ are close to that of BB-DA, which are due to their close DA behaviors as shown by Fig.(1). Table VI shows that all the TFFs increases with the increment of $B_{2,\rho}$, i.e. a larger TFFs can be achieved for a larger $B_{2,\rho}$, or equivalently, a larger second Gegenbauer moment $a_2$.

As a further cross check of the present LCSR, if taking the same second Gegenbauer moment $a_2$ as that of Ref.[17, 31], we find that our present TFFs agree with those derived under the usual choice of correlator $[17, 31]$. Moreover, our present results for the TFFs at $q^2 = 0$ agree with the pQCD prediction [47]: $A_1(0) = 0.25 \pm 0.02$, $A_2(0) = 0.21 \pm 0.015$ and $V(0) = 0.318 \pm 0.032$.

The LCSRs for the $B \to \rho$ TFFs are valid when the $\rho$-meson’s energy ($E_\rho$) is large enough, i.e. $E_\rho \gg \Lambda_{\text{QCD}}$, which implies a restriction of not too large $q^2$. Because $q^2 = m_B^2 - 2m_B E_\rho$, we adopt $0 \leq q^2 \leq q^2_{\text{CSR;MAX}} \sim 14\text{GeV}^2$ to evaluate the sum rules. As a comparison, it is noted that the maximum physical allowable value for $q^2$ is $(m_B - m_\rho)^2 \sim 20\text{GeV}^2$. On the other hand, because of the restriction to the $\rho$ energies smaller than the inverse lattice spacing, the lattice QCD calculation becomes more difficult in the large recoil regions. At present, the lattice QCD results of the $B \to \rho$ TFFs are available only for soft regions, i.e., $q^2 > 12\text{GeV}^2$ [10, 11, 13]. Thus, to compare the LCSR estimations with the lattice ones, certain extrapolation has to be done.

For the purpose, we take our present LCSR estimations within the region of $q^2 \in [0, 14]\text{GeV}^2$ as a basis to do the extrapolation. We adopt the following formulae for the extrapolation,

$$F_i(q^2) = \frac{F_i(0)}{1 - a_i q^2/m_B^2 + b_i (q^2/m_B^2)^2},$$

(35)

where $F_i$ stands for the mentioned $B \to \rho$ TFFs, i.e. $A_1$, $A_2$ and $V_1$, accordingly. When $b_i = 0$, we return

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
& WH-DA & BB-like DA & CZ-like DA \\
\hline
$s_0^1$ & 35.5(5) & 33.4(6) & 38.0(5) & 31.0(5) \\
$M_{3A}\rho$ & 7.4(5) & 7.2(5) & 7.2(5) & 7.1(5) \\
$s_0^2$ & 35.9(5) & 32.9(5) & 38.4(5) & 31.0(5) \\
$M_{3A}\perp$ & 7.0(5) & 7.1(5) & 8.1(5) & 6.8(5) \\
$\Lambda_2$ & 36.5(5) & 34.5(5) & 39.0(5) & 30.8(5) \\
$M_2\perp$ & 10.5(5) & 10.0(5) & 9.9(5) & 7.9(5) \\
\hline
\end{tabular}
\caption{The determined continuum threshold $s_0$ and the Borel parameter $M^2$ under the WH-model for $B_{2,\rho} = -0.2, 0.0, +0.10$, respectively. As a comparison, the results for the AdS/QCD-DA are also presented. The central values are for $m_b = 4.80\text{GeV}$ and $f_B = 0.160\text{GeV}$.}
\end{table}
| $B_{2,\rho}^\perp = -0.2$ | $B_{2,\rho}^\perp = 0.0$ | $B_{2,\rho}^\perp = 0.1$ |
|-----------------|-----------------|-----------------|
| $A_1(0)$ | $A_2(0)$ | $V(0)$ | $A_1(0)$ | $A_2(0)$ | $V(0)$ | $A_1(0)$ | $A_2(0)$ | $V(0)$ |
| $\phi_{2,\rho}^\perp$ | 0.214 | 0.267 | 0.301 | 0.244 | 0.305 | 0.340 | 0.245 | 0.306 | 0.343 |
| $\psi_{3,\rho}^\perp$ | 0.007 | -0.036 | / | 0.006 | -0.040 | / | 0.006 | -0.040 | / |
| $\phi_{4,\rho}$ | -0.022 | -0.035 | -0.031 | -0.022 | -0.039 | -0.031 | -0.023 | -0.040 | -0.031 |
| $I_0$ | -0.003 | -0.022 | / | -0.003 | -0.022 | / | -0.003 | -0.022 | / |
| $H_3$ | 0.009 | 0.005 | / | 0.006 | 0.004 | / | 0.005 | 0.003 | / |
| $\Phi_{5,\rho}$ | / | 0.0004 | / | / | 0.0004 | / | / | 0.0003 | / |
| Total | 0.204 | 0.179 | 0.270 | 0.232 | 0.208 | 0.309 | 0.231 | 0.207 | 0.311 |

TABLE V. The $B \rightarrow \rho$ TFFs at the large recoil region, $q^2 = 0$, in which the twist-2, the non-zero twist-3 and twist-4 DAs’ contributions are presented separately. The WH-DA has been adopted in the calculation, in which $B_{2,\rho}^\perp = 0.0, 0.1$ and $-0.2$, respectively. The scale $\mu$ is set as 2.2 GeV.

| $F_1$ | $a_\perp$ | $b_\perp$ | $\Delta$ |
|------|----------|----------|---------|
| $B_{2,\rho}^\perp = -0.2$ | $A_1$ | 1.351 | 0.682 | 0.2 |
| $B_{2,\rho}^\perp = 0.0$ | $A_2$ | 2.159 | 1.430 | 0.4 |
| $B_{2,\rho}^\perp = 0.1$ | $V$ | 2.041 | 1.228 | 0.8 |

TABLE VI. The $B \rightarrow \rho$ TFFs at the large recoil region under the WH-DA, BB-DA, CZ-DA and AdS/QCD-DA, where the errors are squared average of all the mentioned sources. The scale $\mu$ is set as 2.2 GeV.

We put the extrapolated $B \rightarrow \rho$ TFFs $A_1(q^2), A_2(q^2)$ and $V(q^2)$ in Fig. (2), in which the lattice QCD estimations $[10, 13]$ are included as a comparison. The TFFs become smaller with the increment of $a_\perp^2$. This indicates that a larger second Gegenbauer moment $a_\perp^2$ is not allowed by the lattice QCD estimations. If we have a more precise lattice QCD estimation, we can get a more strong constraint on the $\rho$-meson DA behavior.

As mentioned above, the WH-DA provides a convenient $\rho$-meson DA model for mimicking the behaviors of the DA models suggested in the literature, which are shown by Fig. (2). For examples, the TFFs for WH-DA with $B_{2,\rho}^\perp \sim -0.2$ agree with the estimations of the CZ-DA, and the TFFs for WH-DA with $B_{2,\rho}^\perp \sim 0.1$ agree with the estimations of the BB-DA. If taking $B_{2,\rho}^\perp \sim 0.3$ for the WH-DA, we shall get the same prediction of AdS/QCD-model. Among all the $\rho$-meson DA models, the AdS/QCD-model provide the smallest TFFs. It is noted that if taking a larger $m_\perp$ value as 0.35 GeV $[48]$, corresponding to $a_\perp^2 \approx 0.0$, we can obtain satisfiable results consistent with the lattice QCD estimations. In the following, we shall adopt WH-DA model for detailed discussions on the $B \rightarrow \rho$ semi-leptonic decays.

We present the differential decay width $1/[|V_{ub}|^2 \times d\Gamma/dq^2]$ for the WH-DA in Fig. (3), where $B_{2,\rho}^\perp$ is taken as $-0.2, 0.0$ and $0.1$, respectively. The lattice QCD estimations $[10, 13]$ are included for a comparison.

| $B_{2,\rho}^\perp = -0.2$ | $B_{2,\rho}^\perp = 0.0$ | $B_{2,\rho}^\perp = 0.1$ |
|-----------------|-----------------|-----------------|
| $\Gamma/|V_{ub}|^2$ (ps$^{-1}$) | 10.95$^{+1.57}_{-1.39}$ | 9.57$^{+1.34}_{-1.13}$ | 7.97$^{+1.13}_{-0.97}$ |
| $\Gamma^\perp/\Gamma^\parallel$ | 0.79$^{+0.15}_{-0.15}$ | 0.89$^{+0.16}_{-0.15}$ | 0.93$^{+0.17}_{-0.16}$ |

TABLE VII. The fitted parameters $a_\perp$ and $b_\perp$ defined in Eq. (35) for the $B \rightarrow \rho$ TFFs with all the LCSR parameters set to be their central values. $\Delta$ is a measure of the quality of the fit defined in Eq. (36).

to the usual vector meson dominance extrapolation. The parameters $a_\perp$ and $b_\perp$ are fitted by requiring the “quality” of the fit to be within 1%, i.e. $\Delta < 1$. Here, the “quality” of the fit is expressed by a parameter $\Delta$, which is defined as $[5]$

$$\Delta = 100 \sum_i \frac{|F_i(t) - F_i^{\text{fit}}(t)|}{\sum_i |F_i(t)|},$$

where $t \in [0, \frac{1}{2}, \frac{3}{2}, 14]$ GeV$^2$. The fitted parameters are put in Table VII.

$$\Gamma = \Gamma^\parallel + \Gamma^\perp,$$
The lattice QCD estimations \[\text{WH-DA, BB-DA, CZ-DA and AdS/QCD-DA}, \] respectively.

The total decay width \(\Gamma\) and the ratio \(\Gamma^\parallel/\Gamma^\perp\) computed for the WH-DA are presented in Table VIII, where the errors in Table VIII are squared average of the mentioned error sources. It is noted that the total decay width \(\Gamma\) decreases and the ratio \(\Gamma^\parallel/\Gamma^\perp\) increases with the increment of \(B_{2\rho}(a_{2\rho})\).

In the literature, the \(B \to \rho\) semi-leptonic decays has also been adopted for determining the CKM matrix element \([V_{ub}]\). Two types of semi-leptonic decays have been adopted for such purpose. The first type, the so-called “\(B^0\)-type”, is via the process \(B^0 \to \rho^- \ell^+ \nu_\ell\), whose branching ratio and lifetime are [32]

\[
B(B^0 \to \rho^- \ell^+ \nu_\ell) = (2.34 \pm 0.28) \times 10^{-4},
\]

\[
\tau(B^0) = 1.519 \pm 0.007 \text{ps}.
\]  

The second type, the so-called “\(B^+\)-type”, is via the process \(B^+ \to \rho^0 \ell^+ \nu_\ell\), whose branching ratio and lifetime are [32]

\[
B(B^+ \to \rho^0 \ell^+ \nu_\ell) = (1.07 \pm 0.13) \times 10^{-4},
\]

\[
\tau(B^+) = 1.641 \pm 0.008 \text{ps}.
\]

The experimental measurements and the theoretical estimations can be related via the relation

\[
B(B^0/B^+ \to \rho^-/\rho^0 \ell^+ \nu_\ell) = \frac{G|V_{ub}|^2}{\epsilon^2} \int_0^{q^2_{\text{max}}} dq^2 \sqrt{\lambda(q^2)} q^2 \sum_{j=0,\pm} H_j^2(q^2),
\]

where the factor \(\epsilon\) accounts for the \(\rho\)-meson flavor content, and we have

\[
\epsilon = \begin{cases} \sqrt{2} & \rho^0(b \to u) \\ -\sqrt{2} & \rho^0(b \to d) \\ 1 & \rho^0(b \to u,d) \\ \end{cases}
\]

since \(\rho^0 = (\bar{u}u - \bar{d}d)/\sqrt{2}\) and \(\rho^- = \bar{u}d\).
TABLE IX. The values of $|V_{ub}|$ in unit $10^{-3}$ for the WH-DA with $B_{2;\rho}^{\perp} = -0.2$, 0.0 and 0.10, respectively. The central values are obtained by setting all inputs to be their central values. The first (second) error is the squared average of the mentioned theoretical (experimental) uncertainties.

| $B_{2;\rho}^{\perp}$ | $|V_{ub}|$ |
|-------------------|---------|
| $-0.2$            | $2.91 \pm 0.19$ |
| $0.0$             | $3.11 \pm 0.19$ |
| $+0.1$            | $3.41 \pm 0.22$ |

TABLE X. The weighted average of $|V_{ub}|$ in unit $10^{-3}$ from both the $B^{+}$-type and $B^{0}$-type and for the WH-DA with $B_{2;\rho}^{\perp} = -0.2$, 0.0 and 0.10, respectively. The estimations of the BABAR collaboration[26, 27] are also presented as a comparison.

| $B_{2;\rho}^{\perp}$ | $|V_{ub}|$ |
|-------------------|---------|
| $-0.2$            | $2.91 \pm 0.19$ |
| $0.0$             | $3.11 \pm 0.19$ |
| $+0.1$            | $3.41 \pm 0.22$ |

| BABAR [26]        | LCSR [17]   | ISGW [49] |
|-------------------|--------------|
| $2.75 \pm 0.24$   | $2.83 \pm 0.24$ |

| BABAR [27]        | LCSR [17]   | ISGW [49] |
|-------------------|--------------|
| $2.85 \pm 0.40$   | $2.91 \pm 0.40$ |

The WH-prescription provides a convenient way for constructing the LCWF/LCDA of the light mesons such as $\pi$, $K$ and $\rho$ mesons. More specifically, within the WH-prescription, the $\rho$-meson transverse momentum dependence is controlled by the BHL-prescription together with the Wigner-Melosh rotation effects, and its longitudinal tribution is dominantly controlled by a single parameter $B_{2;\rho}^{\perp}$. The WH-model for the $\rho$-meson LCDA is represented by Eq.(31). Varying $B_{2;\rho}^{\perp}$ from $-0.20$ to $+0.20$, the $\rho$-meson DA shall be varied from the single-peak behavior to the double-humped behavior, which covers most of the DA behaviors suggested in the literature. As examples, when taking $B_{2;\rho}^{\perp} \approx -0.20$ and 0.10, we obtain the same shape of CZ-DA and BB-DA.

The improved LCSRs for the $B \to \rho$ TFFs $A_1$, $A_2$ and $V$ have been discussed in detail under various LCDA models. Our present LCSRs agree with previous LCSRs derived via the conventional correlator as done by Ref.[17] but with less uncertainty. At present, the LCDA’s contributions at the $\delta^3$-order have been eliminated, thus we can draw more definite conclusions on the behavior of $\phi_{2;\rho}^{\perp}$. Table V shows the contributions from various DAs at the large recoil region. As required, it shows that the net twist-3 and twist-4 contributions are less than 10% of the LCSR, and leading-twist LCDA $\phi_{2;\rho}^{\perp}$ do provide the dominant contributions. After extrapolation of our present LCSRs to all physical allowable region for the $B \to \rho$ TFFs, we make a comparison to those of lattice QCD calculations. The TFFs become smaller with the increment of $a_{\perp}^2$ and a larger second Gegenbauer moment $a_{\perp}^3$ is not allowed by the lattice QCD estimations. Thus, if we have a more precise lattice QCD estimation, we can get a more strong constraint on the $\rho$-meson DA behavior.

The improved LCSRs of TFFs can be applied to determine the total decay widths or the value of $|V_{ub}|$. We present the total decay widths $\Gamma/|V_{ub}|^2$ and also the $|V_{ub}|$-free ratio $\Gamma/\Gamma^{\perp}$ under different choices of $B_{2;\rho}^{\perp}$ in Table VIII. The total decay width decreases and the ratio $\Gamma/\Gamma^{\perp}$ increases with the increment of $B_{2;\rho}^{\perp}$. We put the values of $|V_{ub}|$ for the WH-DA in Table IX, which shows that $|V_{ub}|$ increases with the increment of $B_{2;\rho}^{\perp}$. To compare with the BABAR prediction on the $|V_{ub}|$, a larger $B_{2;\rho}^{\perp} (a_{\perp}^2)$ is not allowable. For example, using the BABAR prediction based on the LCSR [26] as a criteria, we obtain $B_{2;\rho}^{\perp} \in [-0.2, 0.10]$, which indicates that the $\rho$-meson DA prefers a single-peak behavior rather than a double-humped behavior. The $\rho$-meson DA shall be further constrained/tested by more data available in the near future, and we hope the definite behavior of $\rho$ DA can be concluded finally.

IV. SUMMARY

In the present paper, we have adopted the improved LCSR approach to deal with the $B \to \rho$ TFFs. By using the improved LCSR, it has been noted that the leading-twist LCDA $\phi_{2;\rho}^{\perp}$ provides the dominant contributions to the total LCSRs. This makes the $B \to \rho$ semi-leptonic decays be good places for testing the $\phi_{2;\rho}^{\perp}$ models.

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Appendix A: The LCSR for the TFF $A_0(q^2)$

As a complete analysis of $B \to \rho$ TFFs, we present the results for $A_0$ in the present appendix, even though they are irrelevant to our present analysis of semi-leptonic decays with massless leptons. Following the same procedures as described in Sec.II, we can first get the LCSR for $[A_3(q^2) - A_0(q^2)]$,

\[
\begin{align*}
&f_B[A_3(q^2) - A_0(q^2)] e^{-m_N^2/M^2} = \frac{m_b m_{\rho} f_{\pi}^+ q^2}{2 m_B^2} \left\{ \int_0^1 \frac{du}{u} e^{-s(u)/M^2} \left\{ \frac{1}{m_b^2} \Theta(c(u, s_0)) \phi_{4,\rho}^+(u) - \frac{2 - u}{u M^2} \bar{\Theta}(c(u, s_0)) \psi_{3,\rho}^{||}(u, \mu) \right. \\
&\left. + \frac{1}{4} \left[ \frac{m_b^2}{u^2 M^4} \bar{\Theta}(c(u, s_0)) + \frac{1}{u M^2} \Theta(c(u, s_0)) \right] \phi_{3,\rho}^+(u) + \left[ (4 - 2u) \frac{C}{u^3 M^4} \bar{\Theta}(c(u, s_0)) - \frac{2}{u^2 M^2} \Theta(c(u, s_0)) \right] + 2 \frac{2 m_b^2}{u^2 M^4} \right. \\
&\left. \times \Theta(c(u, s_0)) + \frac{1}{u M^2} \Theta(c(u, s_0)) \right] J_L(u) - \frac{2 - u}{u M^2} \Theta(c(u, s_0)) H_3(u) \right\} - \int \mathcal{D} \alpha_i \int_0^1 d\nu e^{-s(X)/M^2} \frac{1}{2 X^2 M^2} \Theta(c(X, s_0)) \\
&\left[ (4\nu - 1) \Psi_{4,\rho}^{||}(\alpha) - \bar{\Psi}_{4,\rho}^{||}(\alpha) + 4\nu \Phi_{3,\rho}^{||}(\alpha) \right].
\end{align*}
\]

(1)

Then, with the help of Eq.(2), we obtain the LCSR for $A_0(q^2)$. Using the same extrapolation (35) for the $B \to \rho$ TFFs, we obtain

\[
\begin{align*}
&\bar{B}_{4,\rho} = -0.2, \quad a_{A_0} = 2.074, \quad b_{A_0} = 1.193, \quad \Delta_{A_0} = 0.4; \\
&\bar{B}_{3,\rho} = 0.0, \quad a_{A_0} = 1.682, \quad b_{A_0} = 0.622, \quad \Delta_{A_0} = 0.3; \\
&\frac{\bar{B}_{2,\rho}}{\bar{B}_{4,\rho}} = +0.1, \quad a_{A_0} = 1.414, \quad b_{A_0} = 0.209, \quad \Delta_{A_0} = 0.0.
\end{align*}
\]

The extrapolated $A_0(q^2)$ under the WH-DA, BB-DA, CZ-DA and AdS/QCD-DA are presented in Fig.(4).

Appendix B: The $\rho$-meson High-twist LCDAs

As has been shown in the body of the text, the twist-3 and twist-4 LCDAs at the $\delta^2$-order shall provide small contributions to the LCSRs. Thus, we directly adopt the expressions for those higher-twist DAs, i.e. $\phi_{3,\rho}^{||}$, $\psi_{3,\rho}^{||}$, $\phi_{4,\rho}^{||}$, $\psi_{3,\rho}^{||}$, $\phi_{4,\rho}^{||}$, $\psi_{3,\rho}^{||}$, suggested by Refs.[16, 33]:

\[
\begin{align*}
\phi_{3,\rho}^{||}(x) &= 3 \xi^2 + \frac{3}{2} a_2^+ \xi^2 (5 \xi^2 - 3) + \frac{15}{16} \zeta_3 \omega_3^{||} (3 \\
&- 30 \xi^2 + 35 \xi^4), \\
\psi_{3,\rho}^{||}(x) &= 6 x \bar{x} \left[ 1 + \left( \frac{1}{4} a_2^+ + \frac{5}{8} \zeta_3 \omega_3^{||} \right) (5 \xi^2 - 1) \right], \\
\phi_{4,\rho}^{||}(x) &= 30 x^2 \bar{x}^2 \left[ \frac{2}{3} \left( 1 + \frac{2}{7} a_2^+ + \frac{10}{3} \zeta_4^+ - \frac{20}{3} \xi_4 \right) \right. \\
&+ \left. \left( \frac{3}{35} a_2^+ + \frac{1}{40} \zeta_3 \omega_3^{||} \right) c_2^{\delta/2}(\xi) \right] - \frac{18}{11} a_2^+ \\
&- \frac{3}{2} \zeta_3 \omega_3^{||} + \frac{126}{55} c^{(1)} + \frac{70}{11} c^{(3)} \right] \\
&\times \left[ x \bar{x} (2 + 13 x \bar{x}) + 2 x^3 (10 - 15 x + 6 x^2) \\
&\times \ln x + 2 x^3 (10 - 15 \bar{x} + 6 \bar{x}^2) \right],
\end{align*}
\]

(3)
where the non-zero coefficients at two scales 1GeV and 2.2GeV are put in Tables XI and XII.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\(\mu\) & \(\xi_3\) & \(\omega^{3}_3\) & \(\omega^{3}_4\) \\
\hline
1GeV & 0.032 \pm 0.010 & -2.1 \pm 1.0 & 3.8 \pm 1.8 \\
2.2GeV & 0.018 \pm 0.006 & -1.7 \pm 0.9 & 3.6 \pm 1.7 \\
\hline
\end{tabular}
\caption{The parameters for the chiral-even 3-particle DAs.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\(\mu\) & \(\xi_3\) & \(\omega^{3}_3\) & \(\omega^{3}_4\) \\
\hline
1GeV & 7.0 \pm 0.05 & -0.10 \pm 0.05 & -0.15 \pm 0.15 \\
2.2GeV & 7.2 \pm 0.06 & -0.06 \pm 0.03 & -0.07 \pm 0.07 \\
\hline
\end{tabular}
\caption{The parameters for the chiral-odd 3-particle DAs.}
\end{table}
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