On the nullification of threshold amplitudes

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Abstract

The nullification of threshold amplitudes is considered within the conventional framework of quantum field theory. The relevant Ward identities for the reduced theory are derived both on path-integral and diagrammatic levels. They are then used to prove the vanishing of tree-graph threshold amplitudes.

I Introduction

Threshold amplitudes have attracted much attention in the last decade (for a review see [1]). In particular, the tree-graph approximation has been analysed in great detail [2]. One of the most interesting phenomena discovered here is the so-called nullification of amplitudes; it appeared that, in some theories, a number of threshold amplitudes vanish. For example, in unbroken $\Phi^4$-theory all threshold amplitudes $2 \to n$ vanish except $n = 2$ and $n = 4$ [3]; in the case of spontaneously broken $\Phi \to -\Phi$ symmetry the only nonvanishing amplitude is the $2 \to 2$ one [4]. Other examples concern theories with more bosonic fields where the nullification phenomenon occurs provided certain relations between the parameters of the theory hold [5].

Particularly interesting nullification phenomena occur in bosonic theory with softly broken $O(2)$ symmetry with two real fields in basic representation of $O(2)$. The relevant lagrangian reads

\[ L = \frac{1}{2}((\partial_{\mu}\Phi_1)^2 + (\partial_{\mu}\Phi_2)^2 - m_1^2\Phi_1^2 - m_2^2\Phi_2^2) - \frac{\lambda}{4!}(\Phi_1^2 + \Phi_2^2)^2, \]  

(1)

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with $m_1 \neq m_2$. It has been found that the tree amplitudes $n_1 \Phi_1 \to n_2 \Phi_2$, with all particles (initial as well as final) on the threshold, vanish \[6\]. This result, obtained in Ref.\[6\] by explicit calculations, admits more general and interesting explanation \[7\]. The generating functional for tree amplitudes is obtained by solving the classical field equations. For threshold amplitudes they reduce to one-dimensional equations for dynamical system of two degrees of freedom. One can show that the nonvanishing amplitudes can occur only if the resonances appear when the equations of motion are solved perturbatively. Now, the reduced system posses a special symmetry which excludes such possibility. This relation between symmetries and threshold nullification phenomena is more general and can be described within the framework of modern theory of integrable systems \[8\].

On the other hand, the nullification is ultimately a result of subtle cancellations between contributions coming from different tree graphs. One can suspect that these cancellations result from Ward identities related to the symmetry of reduced system. This more traditional point of view is the subject of the present paper. We derive the Ward identities for the reduced system, both by path-integral and diagrammatical methods. Then the coincidence of threshold tree amplitudes with those of reduced theory is used to show that the tree-level contributions do cancel.

II Ward identities

We start with the Hamiltonian of the reduced system corresponding to translational-invariant version of eq.(1)

$$H = \frac{1}{2}(\Pi_1^2 + m_1^2 \phi_1^2) + \frac{1}{2}(\Pi_2^2 + m_2^2 \phi_2^2) + \frac{\lambda}{4!}(\phi_1^2 + \phi_2^2)^2,$$

(2)

where $\Pi_i \equiv \dot{\phi}_i, i = 1, 2$. The system is integrable \[9\] \[10\], the two independent commuting integrals being

$$F_i = \frac{\lambda J^2}{4! \Delta m_i^2} + \frac{1}{2}(\Pi_i^2 + m_i^2 \phi_i^2) + \frac{\lambda}{4!} \phi_i^2(\phi^2);$$

(3)

here $\phi^2 \equiv \phi_1^2 + \phi_2^2$, $\Delta m_i^2 \equiv m_j^2 - m_i^2, i \neq j$ and

$$J \equiv \sum_{i,j} \varepsilon_{ij} \phi_i \Pi_j = \phi_1 \Pi_2 - \phi_2 \Pi_1$$
is two-dimensional angular momentum. The existence of two integrals quadratic in momenta is implied by the separability of the potential (in elliptic coordinates \([\mathbb{Q}]\)).

For any pair \(\alpha_1, \alpha_2 \in \mathbb{R}\) one can define the generator of symmetry transformations

\[ F_{(\alpha)} \equiv \alpha_1 F_1 + \alpha_2 F_2; \quad (4) \]

in particular \(F_{(1,1)} = H\). Let us define for further use

\[ \alpha \equiv \alpha_1 + \alpha_2 \]

\[ \beta \equiv \sum_{i=1}^{2} \frac{\alpha_i}{\Delta m_i^2} = \frac{\alpha_1 - \alpha_2}{m_2^2 - m_1^2} \quad (5) \]

The relevant symmetry transformations read (in infinitesimal form)

\[ \varphi_i \rightarrow \varphi'_i = \varphi_i + \varepsilon\{\varphi_i, F_{(\alpha)}\} = \varphi_i + \varepsilon(\alpha_i \Pi_i - \frac{2\lambda\beta}{4!} J \sum_{j=1}^{2} \varepsilon_{ij} \varphi_j) \equiv \]

\[ \equiv \varphi_i + \varepsilon Q^{(a)}_i \quad (6) \]

\[ \Pi_i \rightarrow \Pi'_i = \Pi_i + \varepsilon\{\Pi_i, F_{(\alpha)}\} = \Pi_i + \varepsilon(-\alpha_i m_i^2 \varphi_i - \frac{2\lambda}{4!} \alpha_i \phi_i \phi^2 + \]

\[ -\frac{2\lambda}{4!} \varphi_i(\sum_{j=1}^{2} \alpha_j \phi_j^2) - \frac{2\lambda\beta}{4!} J \sum_{j=1}^{2} \varepsilon_{ij} \Pi_j) \equiv \]

\[ \Pi_i + \varepsilon P^{(a)}_i \quad (7) \]

Let us remind the Noether theorem in the Hamiltonian framework. The canonical transformation \(\varphi_i \rightarrow \varphi'_i, \Pi_i \rightarrow \Pi'_i\) is a symmetry transformation if the following condition holds

\[ \sum_i \Pi'_i \varphi'_i - H(\varphi', \Pi') = \sum_i \Pi_i \varphi_i - H(\varphi, \Pi) + \tilde{\Psi}(\varphi, \Pi), \quad (8) \]

where \(\tilde{\Psi}(\varphi, \Pi)\) is some function (in general it may depend also on time).

Taking an infinitesimal form of symmetry transformations

\[ \varphi'_i = \varphi_i + \varepsilon Q_i, \]

\[ \Pi'_i = \Pi_i + \varepsilon P_i \]

\[ \tilde{\Psi}(\varphi, \Pi) = \varepsilon\Psi(\varphi, \Pi) \quad (9) \]
one obtains from eq. (8)
\[ \sum_i (\dot{\varphi}_i - \frac{\partial H}{\partial \Pi_i})P_i - \sum_i (\dot{\Pi}_i + \frac{\partial H}{\partial \varphi_i})Q_i + \frac{d}{dt}(\sum_i \Pi_i Q_i - \Psi) = 0. \] (10)

This is the Hamiltonian form of Noether theorem. It is easy to check that the conserved quantity generates the symmetry transformations. Indeed, eq. (8) implies
\[ \sum_i \Pi_i' d\varphi'_i - \sum_i \Pi_i d\varphi_i = d\Psi(\varphi, \Pi) \] (11)
or, infinitesimally,
\[ \sum_i (P_i d\varphi_i + \Pi_i dQ_i) = d\Psi \] (12)
so that
\[ P_i + \sum_j \Pi_j \frac{\partial Q_j}{\partial \varphi_i} = \frac{\partial \Psi}{\partial \varphi_i} \] (13)
\[ \sum_j \Pi_j \frac{\partial Q_j}{\partial \Pi_i} = \frac{\partial \Psi}{\partial \Pi_i}. \]

Eqs. (13) can be written as
\[ \{ \varphi_i, \sum_j \Pi_j Q_j - \Psi \} = Q_i \]
\[ \{ \Pi_i, \sum_j \Pi_j Q_j - \Psi \} = P_i \]
which imply that \( F \equiv \sum_j \Pi_j Q_j - \Psi \) is a conserved generator of symmetry transformations.

Now, we can prove the relevant Ward identities. We start with path-integral representation of the generating functional
\[ Z[\vec{J}, \vec{K}] = \int D\varphi' D\Pi' e^{i \int (\sum_i \Pi_i' \dot{\varphi}_i' - H(\varphi', \Pi')) dt + i \int \sum_i (J_i \varphi_i' + K_i \Pi_i') dt} \] (14)

Let us make a canonical transformation
\[ \varphi'_i = \varphi_i + \varepsilon Q_i \]
\[ \Pi'_i = \Pi_i + \varepsilon P_i \] (15)
with time-dependent parameter $\varepsilon = \varepsilon(t)$. Canonicity implies formal measure invariance, $D\varphi' D\Pi' = D\varphi D\Pi$. Moreover, assuming that for a constant $\varepsilon$ is a symmetry transformation, one gets

$$\sum_i \Pi_i \dot{\varphi}_i' - H(\varphi', \Pi') = \sum_i \Pi_i \dot{\varphi}_i - H(\varphi, \Pi) + \dot{\varepsilon} \sum_i \Pi_i Q_i + \varepsilon \dot{\Psi} \quad (16)$$

Therefore

$$Z[\vec{J}, \vec{K}] = \int D\varphi D\Pi e^{i \int (\sum_i \Pi_i \dot{\varphi}_i - H) dt + i \int \sum_i (J_i \dot{\varphi}_i + K_i \Pi_i) dt + \varepsilon \dot{\Psi}) dt} \quad (17)$$

or, to the first order,

$$\int D\varphi D\Pi \left( \int (\sum_i (\varepsilon \Pi_i Q_i + \varepsilon J_i Q_i + \varepsilon K_i P_i) + \varepsilon \dot{\Psi}) dt \right) \cdot e^{i \int (\sum_i \Pi_i \dot{\varphi}_i - H) dt + i \int \sum_i (J_i \dot{\varphi}_i + K_i \Pi_i) dt + \varepsilon \dot{\Psi}) dt} = 0 \quad (18)$$

Assuming that $\varepsilon(t)$ vanishes outside a finite interval and integrating by parts we arrive finally at

$$\int D\varphi D\Pi \left( \sum_i (J_i Q_i + K_i P_i)(t) + \tilde{F}(t) e^{i \int (\sum_i \Pi_i \dot{\varphi}_i - H) dt + i \int \sum_i (J_i \dot{\varphi}_i + K_i \Pi_i) dt + \varepsilon \dot{\Psi}) dt} = 0 \quad (19)$$

Differentiating $n$ times with respect to $J_i$ and putting $\vec{J} = 0, \vec{K} = 0$ we get

$$\frac{d}{dt} < T(F(t))\varphi_{i_1}(t_1)\ldots\varphi_{i_n}(t_n) > = \frac{1}{i} \sum_{k=1}^n \delta(t - t_k) \cdot < T(Q_{i_k}(t)\varphi_{i_1}(t_1)\ldots\varphi_{i_k}(t_k)\ldots\varphi_{i_n}(t_n)) > \quad (20)$$

These are Ward identities following from the symmetry. More general identities can be obtained by differentiating with respect to $\vec{J}$ and $\vec{K}$ but we shall not need them here. The Ward identities (20) can be also obtained from canonical commutation rules and Heisenberg equations of motion.
III Diagrammatical proof

The derivation given above is slightly formal. It is desirable to give a diagrammatical proof which allows for a better insight into the cancellation mechanism which makes the amplitudes vanishing.

Let us consider the generator

$$F(\alpha) = \frac{\lambda \beta}{4!} J^2 + \sum_{i=1}^{2} \alpha_i \left( \frac{\Pi_i^2}{2} + \frac{m_i^2}{2} \varphi_i^2 + \frac{\lambda}{4!} \varphi_i^2 \vec{\varphi}^2 \right).$$  \hspace{1cm} (21)

We decompose $F(\alpha)$,

$$F(\alpha) \equiv F^{(0)}(\alpha) + F^{(1)}(\alpha)$$  \hspace{1cm} (22)

into the $\lambda$-independent and $\lambda$-linear parts:

$$F^{(0)}(\alpha) = \sum_{i=1}^{2} \alpha_i \left( \frac{\Pi_i^2}{2} + \frac{m_i^2}{2} \varphi_i^2 \right)$$  \hspace{1cm} (23)

$$F^{(1)}(\alpha) = \frac{\lambda}{4!} (\beta J^2 + \left( \sum_{i=1}^{2} \alpha_i \varphi_i^2 \right) \vec{\varphi}^2) \equiv \frac{\lambda}{4!} (\beta (\vec{\varphi}^2 \Pi^2 - (\vec{\varphi} \Pi)^2) + \left( \sum_{i=1}^{2} \alpha_i \varphi_i^2 \right) \vec{\varphi}^2).$$

The same applies to $Q(\alpha) \equiv \partial F(\alpha) / \partial \Pi_i$:

$$Q(\alpha)i = Q^{(0)}(\alpha)i + Q^{(1)}(\alpha)i$$  \hspace{1cm} (24)

with

$$Q^{(0)}(\alpha)i = \alpha_i \Pi_i$$  \hspace{1cm} (25)

$$Q^{(1)}(\alpha)i = -\frac{2\lambda \beta}{4!} J \sum_{j=1}^{2} \varepsilon_{ij} \varphi_j = -\frac{2\lambda \beta}{4!} ((\Pi \cdot \varphi) \varphi_i - \varphi^2 \Pi_i).$$

The momentum-space counterpart of eq.\( (20) \) reads, in obvious notation,

$$(-q) F_{i_1 \ldots i_n}(q; p_1, \ldots, p_n) = \sum_{k=1}^{n} Q_{i_k:i_1 \ldots \hat{i}_k \ldots i_n}(p_k + q; p_1, \ldots, \hat{p}_k, \ldots, p_n)$$  \hspace{1cm} (26)
\[ \frac{i\delta_{ij}}{p^2 - m^2_i + i\epsilon} \]

\[ -\frac{i\lambda}{3}(2\pi)^4\delta^{(4)}(\sum p)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \]

Figure 1: Feynman rules for \( \phi^4 \) theory

We shall give the diagrammatical proof of eq. (26). The standard Feynman rules for \( \phi^4 \) theory are shown on fig. 1. Further one needs the additional vertices related to the symmetry transformations. They are shown on fig. 2-5 where the notation introduced in eqs. (13) and (24) has been used. All momenta are directed inwards.

Eq. (26) is represented graphically as follows:

\[ \sum_k = \sum_k \]

the blobs stand here for the sums of all relevant Feynman graphs.

We shall give a diagrammatical proof of Ward identities (27) for arbitrary number of loops using the Feynman rules described above (with obvious modifications: \( (2\pi)^4\delta^{(4)}(p) \rightarrow (2\pi)\delta(p), \int \frac{d^4p}{(2\pi)^4} \rightarrow \int \frac{dp}{(2\pi)}, \) etc.)

We start from lowest order identities which follow from the invariance of the hamiltonian to the orders 0, 1 and 2 in \( \lambda \). First, free field theory is invariant under the canonical transformations generated by \( F^{(0)}_{(\alpha)} \). For two-
\[(2\pi)^4 \delta^{(4)}(q + p + p')(\sum_k \alpha_k (m_k^2 - p_0p'_0)\delta_{ik}\delta_{jk})\]

Figure 2: Rule for $F_{(\alpha)}^{(0)}$

\[(2\pi)^4 \delta^{(4)}(q + \sum p)(-\frac{2\beta\lambda}{4!}(\delta_{i_1i_2}\delta_{i_3i_4}((p_{10} - p_{30})(p_{20} - p_{40}) + (p_{20} - p_{30})(p_{10} - p_{40})) + \frac{\lambda}{3!} \sum_k \alpha_k (\delta_{k_1i_1}\delta_{k_2i_2}\delta_{i_3i_4} + \delta_{k_1i_1}\delta_{k_3i_3}\delta_{i_1i_2}) + + \text{perm.})\]

Figure 3: Rule for $F_{(\alpha)}^{(1)}$
\[ (2\pi)^{4} \delta^{(4)}(q + p)(-ip_{0})\alpha_{i}\delta_{ij} \]

Figure 4: Rule for \(Q^{(0)}_{(\alpha)_{i}}\)

\[ (2\pi)^{4} \delta^{(4)}(q + \sum p)(\frac{2i\beta \lambda}{4!}(p_{20} + p_{30})\delta_{ii_{1}}\delta_{ij_{3}} - \frac{i\lambda \beta}{3!}p_{10}\delta_{ii_{1}}\delta_{ij_{3}} + \text{perm}) \]

Figure 5: Rule for \(Q^{(1)}_{(\alpha)_{i}}\)
point Green function we obtain:

\[=+\] (28)

This is the only nontrivial identity to this order because all other Green functions are disconnected.

To the first order in \(\lambda\) we must consider four–point functions. We modify the four–vertex by replacing one leg by \(Q^{(0)}_{(\alpha)}\):

\[\rightarrow\] (29)

Then one checks easily that

\[+\sum=\sum\] (30)

indeed, eq. (30) is simply a statement that all terms of the first order in \(\lambda\) cancel against each other when eqs. (3) and (4) are inserted into the hamiltonian (2).

Eqs. (28) and (30) imply the lowest order counterpart of eq. (27) for
Finally there is an identity due to the cancellation of $\lambda^2$-terms. To write it out we modify the four vertex by replacing one leg by $Q_{(\alpha j)}^{(1)}$:

\begin{equation}
\sum = 0 \quad (33)
\end{equation}

where the summation goes over all permutations of external lines leading to new graphs.
Now one can give the proof of general identities (26). Let

\[ (34) \]

stand for the sum of all graphs of a given order contributing to \( n \)-point Green function. The relevant graphs contributing to the left hand side of eq. (26) to the same order are obtained either by inserting \( F^{(0)}(\alpha) \) into arbitrary line of any graph contained in (34) or by replacing any vertex by \( F^{(1)}(\alpha) \). Therefore one can write

\[ = + \]

\[ (35) \]

Let us consider the first term on the right hand side. Using eq. (28) one
obtains

\[ \sum_k (36) \]

The last term on the right hand side of eq. (36) contributes to the right hand side of eq. (26). On the other hand the first two terms together with
the last term of eq. (35) can be written as

\[
\sum + \sum + \sum = \sum
\]

By the identity (30) the last sum can be rewritten as

\[
\sum_k + \sum = \sum \quad (37)
\]

By the identity (30) the last sum can be rewritten as

\[
\sum_k + \sum = \sum \quad (38)
\]

The last term vanishes due to the identity (33). Collecting together eqs. (34)–(40) one obtains the basic identity (26).
IV Nullification

The threshold tree-graph Green functions for four-dimensional theory coincide (up to some irrelevant factors) with those of our reduced theory. Therefore the Ward identities (27) hold for those Green functions. Consider the connected tree amplitude with $n_1$ initial $\varphi_1$- and $n_2$ final $\varphi_2$-lines; here $n_1$ and $n_2$ are coprime numbers up to one common divisor 2 such that $n_1m_1 = n_2m_2$. Consider the Ward identity (27) with $n_1$ $\varphi_1$-lines and $n_2$ $\varphi_2$-lines, $n = n_1 + n_2$. It is easy to see that, due to the definition of $n_1$ and $n_2$, the following properties hold:

(i) one can put $q = 0$ making the left-hand side of eq.(27) vanishing;
(ii) amputating external propagators and passing to mass-shell limit makes the $Q_{(\alpha)1}^{(1)}$ contributions vanishing.

Taking all that into account we obtain from eq.(27)

$$(\alpha_1n_1m_1 - \alpha_2n_2m_2)A(n_1\varphi_1 \rightarrow n_2\varphi_2) = 0 \quad (39)$$

the parameters $\alpha_1$, $\alpha_2$ being arbitrary. Therefore

$$A(n_1\varphi_1 \rightarrow n_2\varphi_2) = 0 \quad (40)$$

The above proof allows for some insight into the cancellation mechanism. It is also obvious that such a cancellation works in a wider context: it is sufficient that the reduced system posses a symmetry containing a linear part in canonical variables which survives the amputation of external lines propagators. This conclusion has been obtained in the framework of Ref.(8).

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