Failed zero forcing and critical sets on directed graphs

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Abstract

Let $D$ be a simple digraph (directed graph) with vertex set $V(D)$ and arc set $A(D)$ where $n = |V(D)|$, and each arc is an ordered pair of distinct vertices. If $(v, u) \in A(D)$, then $u$ is considered an out-neighbor of $v$ in $D$. Initially, we designate each vertex to be either filled or empty. Then, the following color change rule (CCR) is applied: if a filled vertex $v$ has exactly one empty out-neighbor $u$, then $u$ will be filled. The process continues until either all of the vertices in $V(D)$ are filled, or the CCR restricts the remaining empty vertices from being filled. If all vertices in $V(D)$ are eventually filled, then the initial set is called a zero forcing set (ZFS); if not, it is a failed zero forcing set (FZFS). We introduce the failed zero forcing number $F(D)$ on a digraph, which is the maximum cardinality of any FZFS. The zero forcing number, $Z(D)$, is the minimum cardinality of any ZFS. We characterize oriented graphs that have $F(D) < Z(D)$, present a list of digraphs with the same property, and determine $F(D)$ for several classes of digraphs including directed acyclic graphs, weak paths and cycles, and weakly connected line digraphs such as de Bruijn and Kautz graphs. We also characterize digraphs with $F(D) = n - 1$, $F(D) = n - 2$, and $F(D) = 0$, which also gives us a characterization of digraphs in which any vertex is a ZFS. Finally, we show that for any integer $n \geq 3$ and any non-negative integer $k$ with $k < n$, there exists a weak cycle $D$ with $F(D) = k$.

1 Introduction

In this paper, we study failed zero forcing on simple digraphs (directed graphs). Zero forcing problems, including failed zero forcing, are based on a color change rule (CCR) applied to an initial coloring on the vertex set, where there are only two colors: filled or empty. The CCR is: if a filled vertex has exactly one empty out-neighbor, then the out-neighbor will change from empty to filled. In [4], the authors relate this to rumor spreading: if Astrid knows a secret, and all of Astrid’s friends except Zoe know the secret, then Astrid will share the secret with Zoe. The zero forcing number is the smallest number of vertices that initially must be filled in order for all vertices in the digraph to eventually be filled.

There has been a great deal of work on determination of the zero forcing number [3, 4, 5, 7, 12]. A related question has also been studied for finite simple graphs: what is the largest number of vertices that initially could be filled, yet never lead to the entire graph being filled? In the context of the rumor example, how many people could initially know the secret, yet the secret never spread to all the people in the network? This is called the failed zero forcing number of a graph, and has been studied for finite simple graphs [1, 11]. The problem of computing the failed zero forcing number has been shown to be NP-hard [19]. Zero forcing was studied for digraphs in [3, 5]. In this paper, we expand the study of failed zero forcing to digraphs, including oriented graphs.

1.1 Definitions and notation

We denote by $D = (V, A)$ a finite simple digraph with vertex set $V$ and arc set $A$, or $V(D)$ and $A(D)$ respectively in the case the digraph in question is ambiguous. We primarily use digraph
Figure 1: From left to right: a ZFS, a stalled FZFS, a critical set, and a FZFS that is not stalled.

notation based on [2]. Simple indicates that the digraph has no loops (that is, no arcs of the form \((u, u)\) or more than one copy of any arc (no multiple or parallel arcs), where an arc \((u, v)\) is an ordered pair of vertices with tail \(u\) and head \(v\). Note that \((u, v), (v, u) \in A\) is permitted, since the head and tail of each is swapped. Some digraphs with loops are investigated in Section 4.3. An oriented graph \(D\) is a digraph with no cycle of length 2. That is, if \(D\) is an oriented graph with \((u, v) \in A(D)\), then \((v, u) \notin A(D)\). We use \(uv\) in place of \((u, v)\) throughout the paper. For any \(S \subseteq V\), we refer to \(|S|\) as the order of \(S\) and to \(|V|\) as the order of the digraph.

For a vertex \(u \in V(D)\), the open in-neighborhood of \(u\) in \(D\), denoted \(N_D^+(u)\), is \(N_D^+(u) = \{v \in V : vu \in A\}\). The closed in-neighborhood of \(u\), denoted \(N_D[u]\), is the set \(N_D^+(u) \cup \{u\}\). The open out-neighborhood of \(u\) is the set \(N_D^-(u) = \{v \in V(D) : uv \in A\}\). The closed out-neighborhood of \(u\) is the set \(N_D^-[u] = N_D^-(u) \cup \{u\}\). The in-degree and out-degree of \(u \in V(D)\) are given by \(\deg_D^-(u) = |N_D^-(u)|\) and \(\deg_D^+(u) = |N_D^+(u)|\) respectively. For \(S \subseteq V(D)\), we use \(N_D^+(S), N_D^+[S], N_D^-(S), N_D^-[S]\) to denote the respective neighborhoods. If the digraph \(D\) is understood, at times we omit mention of \(D\), using \(\deg^-\) instead of \(\deg_D^-(u)\) for example if the digraph \(D\) is understood. A source is a vertex \(v \in V(D)\) such that \(\deg^-_D(v) = 0\).

We describe zero forcing formally as follows on a simple digraph \(D\). Let \(S \subseteq V\), and let \(i \in \{0, 1, 2, 3, \ldots\}\). Then

- \(B^i(S) := S\)
- \(B^{i+1}(S) := B^i(S) \cup \{w : \{w\} = N^+(v) \setminus B^i(S)\text{ for some }v \in B^i(S)\}\)

Note that for any \(i \geq 0\), \(B^i(S) \subseteq B^{i+1}(S)\), and there exists some \(j \geq 0\) such that \(B^k(S) = B^j(S)\) for any \(k \geq j\). This formal definition is equivalent to the definition described in terms of the CCR. If \(u \in B^i(S)\) for some \(i\), and \(\{v\} = N^+[u] \setminus B^i(S)\) (that is, if the CCR dictates that \(v\) will be filled in the next iteration), we refer to this as a color change.

**Definition 1.1.** We say that \(S\) is a

- zero forcing set (ZFS) if \(B^t(S) = V\) for some \(t \geq 0\)
- failed zero forcing set (FZFS) otherwise.

The zero forcing number \(Z(D)\) is the smallest order of any ZFS of \(D\). The failed zero forcing number \(F(D)\) is the largest order of any FZFS of \(D\). If \(S \subseteq V\) is a set with \(|S| = F(D)\), then we say that \(S\) is a maximum ZFS. If \(B^1(S) = B^0(S)\), we say that \(S\) is stalled. Note that any maximum FZFS is stalled. The concept of a stalled zero forcing set was introduced in the context of failed skew zero forcing in [1]. In [10], the authors introduced the idea of a critical set.

**Definition 1.2.** A nonempty set \(W \subseteq V(D)\) is called (weakly) critical if for every \(v \in V(D) \setminus W\), \(|N_D^+(v) \cap W| \neq 1\), and strongly critical if for every \(v \in V(D)\), \(|N_D^+(v) \cap W| \neq 1\).

Note that \(W\) is a critical set in \(D\) if and only if \(V(D) \setminus S\) is stalled, and that every strongly critical set is a critical set. Figure 1 shows examples of a ZFS, a FZFS that is stalled, a critical set, and a FZFS that is not stalled. Note that for the digraph \(D\) shown, neither FZFS is a maximum FZFS, since we can see that \(F(D) = n - 1\) by letting \(S = V \setminus \{v_3\}\), for example. Throughout the paper, we use the relationship between stalled sets and critical sets to establish results about FZFS as well as about critical sets.
For any digraph $D$, we say that $G$ is the underlying graph of $D$, denoted $G = UG(D)$, if $G$ is the unique simple, finite undirected graph obtained by replacing every arc $uv \in A(D)$ with an undirected edge $\{u, v\}$. The digraph $D$ is weakly connected if $UG(D)$ is connected. We present several results related to paths and cycles in this paper. Digraphs whose underlying graphs are paths or cycles have special terminology associated with them.

**Definition 1.3.** A weak path (resp. weak cycle) is a digraph whose underlying graph is a path (resp. cycle). Given a digraph $D$ with an alternating sequence $P = v_1a_1v_2a_2v_3a_3\ldots v_{k−1}a_{k−1}v_k$ of vertices $v_i \in V(D)$ and distinct arcs $a_j \in A(D)$ such that the tail of $a_i$ is $v_i$ and the head of $a_i$ is $v_{i+1}$, if all vertices are distinct, then $P$ is a (directed) path. If vertices $v_1$ through $v_{k−1}$ are distinct, and $v_k = v_1$, then $P$ is a (directed) cycle.

For both weak paths and weak cycles, we include the possibilities that $D = K_1$ (a single vertex) and that $UG(D) = K_2$. An example of a directed cycle is shown in Figure 3.

This remainder of this paper is organized as follows. We now describe motivation for the study of failed zero forcing. In Section 2, we characterize digraphs with high and low values for a vector $v$ forcing to matrices here.

Given a square matrix $M$ with $n$ rows, we use $\ker(M)$ to denote the kernel of $M$. That is, for a vector $v$ of length $n$, $v \in \ker(M)$ if and only if $MV = 0$. The support of a vector $x = [x_i]$, denoted supp($x$), is given by $\{i : x_i \neq 0\}$. Given a digraph $D$ with $|V| = n$, let $S(D)$ denote the set of $n \times n$ matrices such that the entry in Row $i$, Column $j$ is nonzero if and only if $ij \in A$ for $i \neq j$, with diagonal entries unrestricted.

We note a proposition, similar to [12] Proposition 2.3 but extended to digraphs. The proof is similar to that of [12] Proposition 2.3 but included here for completeness.

**Proposition 1.4.** Let $Z$ be a ZFS of digraph $D$ with $M \in S(D)$. If $x \in \ker(M)$ and supp($x$) $\cap Z = \emptyset$, then $x = 0$.

**Proof.** If $Z = V$, then supp($x$) is empty, giving us $x = 0$, so suppose $Z \subsetneq V$. Then $B^0(Z) \subseteq B^1(Z)$, and there exist $u, v \in V$ with $\{v\} = N_D^+[u]\setminus B^0(Z)$. By assumption, $x_u = 0$ and $(Mx)_u = 0$. The only nonzero entries in Row $u$ of $M$ are those corresponding to $N_D^+(u)$. Since $(N_D^+(u)\setminus\{v\}) \subseteq Z$, the corresponding entries of $x$ are 0, other than $x_v$. We have the equation $M_{uv}x_v = 0$, giving $x_v = 0$. This is true for each color change. Thus, $x = 0$.

Since $F(D)$ is the maximum order of any FZFS, any set of order $F(D) + 1$ or bigger is a ZFS. Combining this fact with the above proposition gives us the following.

**Proposition 1.5.** Let $S \subseteq V(D)$ with $|S| \geq F(D) + 1$. Let $M \in S(D)$. If $x \in \ker(M)$ and all entries of $x$ corresponding to $S$ are 0, then $x = 0$.

Thus, if $M \in S(D)$ for a digraph $D$, and $x \in \ker(M)$ with $x \neq 0$, then $x$ has at least $n − F(D)$ nonzero entries.
2 Extreme values

In this section, we establish the relationship between failed zero forcing and critical sets to characterize digraphs with high and low values of $F(D)$.

**Observation 2.1.** For any critical set $W$ in a digraph $D$, $V(D) \setminus W$ is a failed zero forcing set.

**Proposition 2.2.** Let $1 \leq k \leq n$. Then $F(D) = n - k$ if and only if the smallest cardinality of any critical set in $D$ is $k$.

**Proof.** If $D$ contains a critical set $W$ of cardinality $k$, then $V \setminus W$ is a FZFS. Thus, $F(D) \geq n - k$ where $k$ is the smallest cardinality of any critical set. For the reverse direction, suppose $S$ is a largest FZFS in $D$. Then for any $v \in S$, $v$ has either no out-neighbors or two out-neighbors in $V \setminus S$, since otherwise, either $S$ is a ZFS or there is a larger FZFS than $S$. That is, $V \setminus S$ is a critical set. Hence $F(D) \leq n - k$.

For high values, we can describe these digraphs as follows.

**Corollary 2.3.** $F(D) = n - 1$ if and only if $D$ has a source, and $F(D) = n - 2$ if and only if there exist $u, v \in V$ with $N^{-}_D(u) \setminus \{v\} = N^{-}_D(v) \setminus \{u\}$ and $\deg^{-}(w) > 0$ for all $w \in V$. $F(D) = n - 3$ if and only if all of the following conditions are satisfied.

1. $\deg^{-}(v) > 0$ for all $v \in V$
2. For any distinct vertices $u, v \in V$, $N^{-}_D(u) \setminus \{v\} \neq N^{-}_D(v) \setminus \{u\}$
3. There exist vertices $u, v, w \in V$ such that
   - $N^{-}_D(u) \setminus \{v, w\} \subseteq N^{-}_D(v) \cup N^{-}_D(w)$,
   - $N^{-}_D(v) \setminus \{u, w\} \subseteq N^{-}_D(u) \cup N^{-}_D(w)$, and
   - $N^{-}_D(w) \setminus \{u, v\} \subseteq N^{-}_D(u) \cup N^{-}_D(v)$

![Figure 2: Examples of digraphs with $F(D) = n - 2$.](image)

We now characterize digraphs that have $F(D) = 0$. Note that this is of particular interest because a digraph $D$ with $F(D) = 0$ has the property that $\{v\}$ is a ZFS for any $v \in V(D)$.

![Figure 3: A digraph with $F(D) = 0$. Every vertex is a ZFS.](image)

**Theorem 2.4.** $F(D) = 0$ if and only if $D$ is a directed cycle.

**Proof.** Suppose $D$ is a directed cycle. For any $W \subseteq V(D)$, there exists at least one $w \in W$ with an in-neighbor $v \in V \setminus W$. Since $N^+(w) = \{v\}$, $W$ is not a critical set. Thus, the smallest critical set is $W = V$, giving us $F(D) = 0$ by Proposition 2.2.

For the other direction, suppose $F(D) = 0$. Then $S = \{v\}$ is a ZFS for any $v \in V$. If $|V| = 1$, then we’re done. Otherwise, $\deg^+(v) = 1$ for any $v \in V$ to allow $B^0(S) \subseteq B^1(S)$, and by Corollary 2.3, $\deg^{-}(v) \geq 1$. Since $\sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v) = n$, $\deg^-(v) = 1$ for all
\( v \in V \). Noting that \( D \) must be connected (else, all vertices in the largest connected component form a FZFS), we have that \( D \) is a directed cycle.

An example of a digraph with \( F(D) = 0 \) is shown in Figure 3. The following corollaries are immediate from Theorem 2.4.

**Corollary 2.5.** For every \( v \in V \), \( \{v\} \) is a ZFS of \( D \) if and only if \( D \) is a directed cycle.

**Corollary 2.6.** The digraph \( D \) has no critical sets of cardinality less than \( n \) if and only if \( D \) is a directed cycle on \( n \) vertices.

3 Comparing \( F(D) \) with \( Z(D) \)

In this section, we compare \( F(D) \) with \( Z(D) \). Specifically, we identify digraph families for which \( F(D) < Z(D) \). For oriented graphs, we provide a characterization, and produce a list for digraphs in general.

**Observation 3.1.** The following are equivalent.

1. \( F(D) < Z(D) \)
2. \( F(D) = Z(D) - 1 \).
3. \( S \subseteq V(D) \) is a ZFS if and only if \( |S| \geq Z(D) \).

**Lemma 3.2.** Suppose \( D \) is a digraph with \( F(D) < Z(D) \). Then for each \( v \in V \), \( \deg^+(v) = 0 \) or \( \deg^+(v) \geq Z(D) \).

**Proof.** Let \( v \in V \). Suppose \( 0 < \deg^+(v) < Z(D) \). Then \( \deg^+(v) \leq Z(D) - 1 = F(D) \). Let \( u \in N_D^+(v) \), and let \( S = (N_D^+(v) \setminus \{u\}) \cup S' \), where \( S' \) is any \( F(D) - \deg^+(v) \) vertices in \( V \setminus N_D^+[v] \). Then \( |S| = F(D) \), so \( S \) is a maximum FZFS. But since \( \{u\} = N_D^+[v] \setminus S \), it follows that \( S \) is not stalled, a contradiction. Hence \( \deg^+(v) \geq Z(D) \) or \( \deg^+(v) = 0 \). □

A tournament \( \overrightarrow{K_n} \) is an oriented graph obtained by assigning an orientation to each edge in a complete graph, \( K_n \). A tournament is regular if \( \deg^+(v) = \deg^-(v) \) for every \( v \in V \), and transitive if for any \( \{u, v, w\} \subseteq V \), \( uv, vu \in A \) implies that \( uw \in A \).

**Lemma 3.3.** Suppose \( \overrightarrow{G} \) is an oriented graph with \( 1 \leq F(\overrightarrow{G}) < Z(\overrightarrow{G}) \). If \( u, v \in V \) with \( \deg^+(u), \deg^+(v) > 0 \), then \( uv \in A \) or \( vu \in A \). In particular, if \( \deg^+(v) > 0 \) for all \( v \in V \), then \( \overrightarrow{G} \) is a tournament.

**Proof.** Let \( \overrightarrow{G} \) be an oriented graph with \( 1 \leq F(\overrightarrow{G}) < Z(\overrightarrow{G}) \). Suppose \( u, v \in V(\overrightarrow{G}) \) with \( \deg^+(u) > 0 \) and \( \deg^+(v) > 0 \), and \( uv, vu \notin A \). By Lemma 3.2, \( \deg^+(u), \deg^+(v) \geq Z(\overrightarrow{G}) \). Let \( S' \) consist of \( Z(\overrightarrow{G}) - 2 \) out-neighbors of \( u \), and let \( S = S' \cup \{u\} \cup \{v\} \). Then \( |S| = Z(\overrightarrow{G}) \), so \( S \) is a ZFS. Since \( u \) and \( v \) each have at least 2 out-neighbors in \( V \setminus S \), there exists \( w \in S' \) such that \( |N^+(w) \setminus S'| = 1 \). If \( Z(\overrightarrow{G}) = 2 \), then we have a contradiction and are done. Otherwise, since \( |S' \setminus \{u\}| = Z(\overrightarrow{G}) - 1 = F(\overrightarrow{G}) \), \( S' \setminus \{u\} \) is a FZFS and stalled, indicating that \( uv \in A \). But since \( \overrightarrow{G} \) is an oriented graph and \( uv \in A \) by assumption, this is a contradiction. Hence, \( uv \in A \) or \( vu \in A \).

**Lemma 3.4.** Suppose \( \overrightarrow{G} \) is an oriented graph with \( F(\overrightarrow{G}) < Z(\overrightarrow{G}) \). Then \( \deg^+(v) = Z(\overrightarrow{G}) \) for any vertex \( v \) with \( \deg^+(v) > 0 \).

**Proof.** Let \( \overrightarrow{G} \) be an oriented graph with \( F(\overrightarrow{G}) < Z(\overrightarrow{G}) \), and \( v \in V \) with \( \deg^+(v) > 0 \). By Lemma 3.2, we know that \( \deg^+(v) \geq Z(\overrightarrow{G}) \). If \( \deg^+(v) > Z(\overrightarrow{G}) \), then let \( S' \) be any \( Z(\overrightarrow{G}) - 1 \) vertices in \( N^+(v) \), and let \( S = S' \cup \{v\} \). Then \( |S| = Z(\overrightarrow{G}) \), so \( S \) is a ZFS. Since \( |N^+(v) \setminus S| \geq 2 \), there must exist \( u \in S' \) with \( |N^+(u) \setminus S| = 1 \). Then \( |S \setminus \{v\}| = F(\overrightarrow{G}) \), so \( S' = S \setminus \{v\} \) is stalled. However, \( |N^+(u) \setminus S| = |N^+(u) \setminus S'| = 1 \) because \( v \notin N^+(u) \), a contradiction. Hence \( \deg^+(v) = Z(\overrightarrow{G}) \). □
Lemma 3.5. Let $\vec{G}$ be an oriented graph with $F(\vec{G}) < Z(\vec{G})$, and $A(\vec{G}) \neq \emptyset$. Let $\vec{H}$ be the subgraph of $\vec{G}$ induced by $\{v \in V(\vec{G}) : \deg^+(v) > 0\}$. Then $\vec{H}$ is a regular, nontransitive tournament.

Proof. By Lemmas 3.3, 3.4 $\vec{H}$ is a regular tournament. Every transitive tournament has a source [13]. Suppose $\vec{H}$ is transitive, and let $v$ be a source. Then $v$ is also a source in $\vec{G}$ since for any $v \in V(\vec{G}) \setminus V(\vec{H})$, $\deg^+(v) = 0$, giving us $F(\vec{G}) = n - 1$ by Corollary 2.3. Hence $\vec{H}$ is a nontransitive tournament. □

Lemma 3.6. Let $\vec{G}$ be an oriented graph with $F(\vec{G}) < Z(\vec{G})$. Then for any $u, v \in V$ with $\deg^+(u) > 0$ and $\deg^+(v) = 0$, $uv \notin A$. If $\vec{G} \neq \overrightarrow{K}_n$, then there is at most one vertex $v$ with $\deg^+(v) = 0$.

Proof. Let $\vec{G}$ be an oriented graph with $F(\vec{G}) < Z(\vec{G})$, and a vertex $v$ with $\deg^+(v) = 0$. If $\deg^+(w) = 0$ for all $w \in V$, then we are done, so suppose $\deg^+(w) > 0$ with $uv \notin A$. By Lemma 3.3 $\deg^+(u) = Z(\vec{G})$. Let $S = \{u, v\} \cup S'$ where $S'$ consists of any $Z(\vec{G}) - 2$ neighbors of $u$. Then $S$ is a ZFS since $|S| = Z(\vec{G})$. Since $|N^+(u) \setminus S| = 2$ and $|N^+(v) \setminus S| = 0$, it follows that there exists $w \in S$ such that $|N^+(w) \setminus S| = 1$. However, $|S \setminus \{u\}| = Z(\vec{G}) - 1 = F(\vec{G})$, so $S \setminus \{u\}$ is a stalled FZFS, implying $wu \in A$. Since $uw \notin A$, however, and $\vec{G}$ is an oriented graph, this is impossible. Hence, $uv \in A$.

Assume $\vec{G} \neq \overrightarrow{K}_n$. We next show that there is at most one vertex with $\deg^+(v) = 0$. Let $W = \{w \in V(\vec{G}) : \deg^+(w) > 0\}$. Note $W \neq \emptyset$. Suppose $\deg^+(u) = \deg^+(v) = 0$. We have just shown that $wu, uv \notin A$ for every $w \in W$. Hence, $N^-(u) = N^-(v)$, giving us $F(\vec{G}) \geq n - 2$ by Corollary 2.3. The assumption that $F(\vec{G}) < Z(\vec{G})$ gives us that $Z(\vec{G}) = n - 1$ and $F(\vec{G}) = n - 2$. But by Lemma 3.3, then $\deg^+(w) = n - 1$ for every $w \in W$, giving us that $w$ is a source for all $w \in W$ since $\vec{G}$ is an oriented graph, implying that $F(\vec{G}) = n - 1$, a contradiction. Hence there is at most one vertex with out-degree 0. □

![Figure 4: The outjoin from a two-cycle to a directed cycle on 3 vertices.](image)

Let $D, H$ be digraphs. We define the outjoin from $D$ to $H$ denoted $D \overrightarrow{\vee} H$ to be the digraph with vertex set $V(D \overrightarrow{\vee} H) = V(D) \cup V(H)$ and arc set $A(D \overrightarrow{\vee} H) = A(D) \cup A(H) \cup \{v_D, v_H \mid v_D \in V(D), v_H \in V(H)\}$. An example is shown in Figure 4.

Theorem 3.7. An oriented graph $\vec{G}$ has $F(\vec{G}) < Z(\vec{G})$ if and only if $\vec{G}$ is one of the following.

1. $\overrightarrow{K}_n$,
2. a directed cycle,
3. a directed 3-cycle with all 3 vertices outjoined to one additional vertex,
4. a regular, non-transitive tournament on 5 vertices.

Proof. For the forward direction, note that by Theorem 2.3 $F(\vec{G}) = 0$ if and only if $\vec{G}$ is a directed cycle. We assume $F(\vec{G}) > 0$. By Lemmas 3.5, 3.6 if $F(\vec{G}) < Z(\vec{G})$, then $\vec{G} = \overrightarrow{K}_n$ or $\vec{G}$ is a regular, non-transitive tournament, possibly outjoined with one additional vertex. We first show that if $\vec{G} = \overrightarrow{K}_n$, then $n = 3$ or 5, and if $\vec{G} = \overrightarrow{K}_n \overrightarrow{\vee} \{x\}$, then $n = 3$. 


Since $\overrightarrow{K}_n$ is not transitive, it contains a directed 3-cycle $uvw$. Note $Z(\overrightarrow{K}_3) = 3$ by Lemma 3.4, so $\{u, v, w\}$ must be a ZFS, but each has out-degree 2 outside of $\{u, v, w\}$, meaning that $\{u, v, w\}$ is a FZFS, a contradiction. So we assume $n > 7$. Since $\deg^+(v) = Z(\overrightarrow{K}_n)$ for every $v \in V$ by Lemma 3.3, $|N^+[u]\{u, v, w\}| = Z(\overrightarrow{K}_n) - 1$. Let $S = \{u, v, w\} \cup S'$ where $S'$ consists of any $Z(\overrightarrow{K}_n) - 3$ out-neighbors of $u$. Since $|S| = Z(\overrightarrow{K}_n)$, $S$ is a ZFS. However, since $|N^+(y) \cap S| \geq 1$ for any $y \in S$, $|N^+(y) \cap S| = Z(\overrightarrow{K}_n) - 2 = \deg^+(y) - 2$, so $|N^+(y) \cap S| \geq 2$. Hence, $S$ is a FZFS, a contradiction. The same arguments apply to $\overrightarrow{K}_n \setminus \{x\}$ by adding $x$ to $S$. Thus, noting that $n$ is odd for any regular non-transitive tournament, $n = 3$ or 5.

We now eliminate the case $\overrightarrow{G} = \overrightarrow{K}_5 \setminus \{x\}$. Let $S = \{u, v, w\}$ where $uvw$ forms a directed 3-cycle. $S$ is a FZFS since each vertex is adjacent to $x$ and one additional out-neighbor. By Lemma 3.4, $Z(\overrightarrow{G}) = \deg^+(v) = 3$, a contradiction.

We have shown that $F(\overrightarrow{G}) < Z(\overrightarrow{G})$, then $\overrightarrow{G}$ is either $\overrightarrow{K}_n$, a directed cycle, a regular non-transitive tournament on 5 vertices, or a directed 3 cycle with all 3 vertices outjoined to one additional vertex.

For the reverse direction, [1] was established in [11]. For [2] note that from Theorem 2.3 we know that a directed cycle has $F(\overrightarrow{G}) = 0$, and it follows that $Z(\overrightarrow{G}) = 1$.

For [3] note that $\deg^+(v) = 0$ or 2 for every $v \in V$. Hence, $S = \{v\}$ is a FZFS for any $v \in V$. For any $u, v, w \in V$, $uw \in A$ or $uw \in A$. Without loss of generality, assume $uw \in A$. Since $\deg^+(u) = 2$, there exists $w \in V$ with $\{w\} = N^+[u] \setminus \{u, v\}$. Thus $B^1(S) = V \setminus \{z\}$ for some $z \in V$. Since $\deg^+(z) > 0$, $B^2(S) = V$, and $\{u, v\}$ is a ZFS.

For [4] since $\deg^+(v) = 2$ for every $v \in V$, every vertex is a FZFS. Let $S = \{u, v\}$ for any $u, v \in V$. Either $uw \in A$ or $v\in A$. Assume without loss of generality $uw \in A$. Then $N^+[u] \setminus S = \{w\}$ for some $w \in V$, so $B^1(S) = \{u, v, w\}$. Either $uw \in A$ or $vv \in A$, so there exists $x \in V$ with $\{x\} = N^+[w] \setminus B^1(S)$. Finally, $B^2(S) = V \setminus \{y\}$ for some $y \in V$, and $\deg^+(y) = 2$, so $B^3(S) = V$, and $S$ is a ZFS.

![Figure 5: All oriented graphs with $F(\overrightarrow{G}) < Z(\overrightarrow{G})$. For the first two digraphs, $|V| \geq 1.$](image)

**Theorem 3.7** allows us to make the following statement about critical sets in oriented graphs.

**Corollary 3.8.** The following list contains the only oriented graphs with the property that there exists $k$ such that $W \subseteq V(\overrightarrow{G})$ is a critical set if and only if $|W| \geq k$.

1. $\overrightarrow{K}_n$, $(k = 1)$,
2. a directed cycle, $(k = n)$,
3. a directed 3-cycle with all 3 vertices outjoined to one additional vertex, $(k = 3)$,
4. a regular, non-transitive tournament on 5 vertices, $(k = 4)$.

![Figure 6: $F(K_5 \setminus K_2) = 5$ and $Z(K_5 \setminus K_2) = 6.$](image)

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We now present a list of digraphs that have \( F(D) < Z(D) \). In addition to the oriented graphs from Theorem 3.7 and undirected graphs from [11], there is one additional digraph on this list, an example of which is shown in Figure 6. We use \( K_n \) to denote a digraph \( D \) such that \( uv, vu \in A(D) \) for every \( u, v \in V(D) \).

**Theorem 3.9.** The following digraphs have \( F(D) < Z(D) \).

1. \( K_n \),
2. \( \overrightarrow{K}_n \),
3. a directed cycle,
4. a directed 3-cycle with all 3 vertices outjoined to one additional vertex,
5. a regular, non-transitive tournament on 5 vertices,
6. \( K_j \overrightarrow{\cup} \overrightarrow{K}_\ell \) where \( j > 1 \) and \( \ell > 0 \).

**Proof.** \( \square \) and \( \square \) were shown in [11], \( \overrightarrow{K}_n \) in Theorem 5.7. Hence, we need only show \( \square \).

Let \( S = V \setminus \{v\} \) for any \( v \in V \). Since \( \deg^-(v) \geq 1 \), \( B^1(S) = V \), and \( S \) is a ZFS with \( |S| = n - 1 \). Now let \( S' = V \setminus \{u, v\} \) for any \( u, v \in V \). Then \( N^-(u) \cap S' = N^-(v) \cap S' \); hence \( B^1(S') = S' \), and \( S' \) is a FZFS with \( |S'| = n - 2 \). Hence, \( F(K_j \overrightarrow{\cup} \overrightarrow{K}_\ell) < Z(K_j \overrightarrow{\cup} \overrightarrow{K}_\ell) \).

**Corollary 3.10.** The digraphs \( K_n \) and \( K_j \overrightarrow{\cup} \overrightarrow{K}_\ell \) where \( j > 1 \) and \( \ell > 0 \) have the property that any set \( W \subseteq V \) is a critical set if and only if \( |W| \geq 2 \).

### 4 Select digraphs

For a digraph consisting of two or more components, we can determine the failed zero forcing number in terms of the failed zero forcing numbers and orders of the components. The result is similar to the result for undirected graphs in [11].

**Theorem 4.1.** Let \( D \) be a digraph that consists of \( k \) components, where \( k \geq 1 \), and let \( D_i \) denote the \( i^{\text{th}} \) component of \( D \), \( 1 \leq i \leq k \). Then

\[
F(D) = \max_{1 \leq j \leq k} \left( F(D_j) + \sum_{i=1,i \neq j}^{k} |V(D_i)| \right)
\]

**Proof.** If \( k = 1 \), the result is trivial. Otherwise, for any FZFS \( S_i \) of any component \( D_i \), \( S = (V(D) \setminus V(D_i)) \cup S_i \) is a FZFS.

If \( S' \subseteq V \) with \( |S'| > \max_{1 \leq j \leq k} \left( F(D_j) + \sum_{i=1,i \neq j}^{k} |V(D_i)| \right) \), then for any \( \ell \), \( |S' \cap V(D_\ell)| > F(D_\ell) \), so \( S' \cap V(D_\ell) \) is a ZFS of \( D_\ell \), and consequently \( S' \) is a ZFS of \( D \).

A directed acyclic graph is a digraph that contains no directed cycles. The following proposition follows directly from Corollary 2.3, since any directed acyclic graph has a source.

**Proposition 4.2.** For any directed acyclic graph \( D \), \( F(D) = n - 1 \).

We turn our attention to special cases of directed trees, starting with oriented trees. For any vertex \( v \) in a directed tree, if \( |N^+(v) \cup N^-(v)| = 1 \), then we say that \( v \) is a leaf. The following corollary follows immediately from Proposition 4.2, since every oriented tree is a directed acyclic graph.

**Corollary 4.3.** For any oriented tree \( \overrightarrow{T} \), \( F(\overrightarrow{T}) = n - 1 \).

The (undirected) graph \( K_{1,t} \) has a single vertex adjacent to \( t = n - 1 \) other vertices, and no other edges.
Theorem 4.4. If $UG(D) = K_{1,t}$ for any $t \geq 1$, then

$$F(D) = \begin{cases} t, & \text{if } D \text{ is oriented, or if at least one leaf has in-degree 0.} \\ t - 1, & \text{otherwise.} \end{cases}$$

Proof. If $D$ is oriented, then $F(D) = t$ by Corollary 4.3. If $\deg^-(v) = 0$ for a leaf $v$, then $v$ is a source, and by Corollary 2.3 $F(D) = t$.

Otherwise, there exist $u, w \in V(D)$ such that $uw, wu \in A(D)$, and $\deg^-(v) = 1$ for every leaf $v \in V(D)$. Thus, $D$ has no source, giving us that $F(D) \leq t - 1$. If $UG(D) = K_2$, $D$ is a 2-cycle, and $F(D) = 0$, so we are done. Let $u$ be the non-leaf vertex in $V(D)$, and let $v, w$ be any two leaves. Then $\{v, w\}$ is a critical set because $u$ is the unique in-neighbor of both. By Proposition 2.2, $F(D) = t - 1$.

4.1 Weak paths

To establish $F(D)$ if $UG(D) = P_n$, we assume that the vertices of $D$ are labeled in order from one end-vertex to the other: $v_1, v_2, \ldots, v_n$.

![Figure 7: A weak path with maximum FZFS shown in blue.](image)

Theorem 4.5. Suppose $D$ is a weak path. Let $V_- = \{v_1\} \cup \{v_k : v_kv_{k-1} \in A \text{ and } v_{k-1}v_k \notin A\}$, and $V_+ = \{v_n\} \cup \{v_k : v_kv_{k+1} \in A \text{ and } v_{k+1}v_k \notin A\}$. Let $\ell = \min\{i - j : v_j \in V_-, v_i \in V_+, i - j \geq 0\}$. Then

$$F(D) = n - 1 - \left\lfloor \frac{\ell}{2} \right\rfloor.$$  

Proof. Let $S$ be defined as follows, where $i = i^*$ and $j = j^*$ are the indices achieving $\min\{i - j : v_j \in V_-, v_i \in V_+, i - j \geq 0\}$.

$$S = \begin{cases} V \setminus \{v_j, v_{j+2}, v_{j+4}, \ldots, v_{j+2k}\} & \text{if } \ell = 2k \\ V \setminus \{v_j, v_{j+2}, v_{j+4}, \ldots, v_{j+2k}\} \cup \{v_i\} & \text{if } \ell = 2k + 1. \end{cases}$$

An example with $n = 17, i^* = 9$ and $j^* = 5$ is shown in Figure 7.

To show that $S$ is a FZFS, let $v_s \in S$. If $s > i$ or $s < j$, then $N^+(v_s) \subseteq S$. If $j < s < i$, then $N^+(v_s) = \{v_{s-1}, v_{s+1}\}$ (otherwise the minimum assumption is violated). By construction of $S$, $v_{s-1}, v_{s+1} \notin S$. Thus $B^1(S) = S$ and $S$ is a FZFS.

We show that $S$ is a maximum FZFS. If $|S| = n - 1$, then we are done. Note that this includes any case with a source, so we can assume that $\deg^-(v) \geq 1$ for all $v \in V$. Suppose there exists $S' \subseteq V$ with $|S| < |S'| < n$, and $S'$ is stalled. For any closest pair of $u \in V_+$ and $w \in V_-$, if a pair of adjacent vertices between them is in $S'$, then all vertices from $u$ to $w$ are in $S'$, since $S'$ is stalled. If this is true for all such pairs $u$ and $w$ then $S' = V$, so there must exist a pair $v_s \in V_+$ and $v_t \in V_-$ for which this is not the case. Note since $|S'| > |S|$, there is at most one such pair and that $v_s, v_t \in S'$. Also since $|S'| > |S|$, if $s > 1$ then $v_{s-1} \in S'$ and if $t < n$, then $v_{t+1} \in S'$. That is, $N^+[v_s] \setminus S' = \{v_{s+1}\}$, which is a contradiction: either $S'$ is not stalled, or two adjacent vertices between $v_s$ and $v_t$ are in $S'$. Thus, $S$ is a maximum FZFS.

In many cases, the formula from Theorem 4.5 can be simplified. If a weak path $D$ contains a source including if $D$ is an oriented path, for example, then there exists at least one vertex in $V_+ \cap V_-$, giving us $i = j$, and consequently $F(D) = n - 1$. By setting $i = n$ and $j = 1$ in Theorem 4.5 we have the following corollary, established for undirected graphs in [11].
Corollary 4.6. For any undirected path \( P_n \) (or a weak path with \( \{v_i, v_{i+1}, v_{i+1}v_i\} \subseteq A \) for \( 1 \leq i \leq n-1 \)), \( F(P_n) = \left\lfloor \frac{n-2}{2} \right\rfloor \).

4.2 Weak cycles

We now turn to weak cycles, starting with oriented cycles.

Observation 4.7. Any oriented cycle \( \overrightarrow{C}_n \) that is not a directed cycle has a source.

As we know from Corollary 2.3 if \( D \) has a source then \( F(D) = n - 1 \). Combining with Theorem 2.4 completes the proof of the following theorem.

Theorem 4.8. An oriented cycle \( \overrightarrow{C}_n \) has

\[
F(\overrightarrow{C}_n) = \begin{cases} 
0, & \text{if it is a directed cycle} \\
(n-1), & \text{otherwise}
\end{cases}
\]

Finally, we turn to weak cycles in general. We present the failed zero forcing numbers of weak cycles depending on the orientations of the edges. Given a weak cycle \( D \), pick any vertex and label the vertices in order around the cycle, so \( V = \{v_0, v_1, \ldots, v_{n-1}\} \). Let

\[
V_- = \{ i : v_{i+1}v_i \in A \text{ and } v_i v_{i+1} \notin A \},
\]

\[
V_+ = \{ j : v_j v_{j+1} \in A \text{ and } v_{j+1} v_j \notin A \},
\]

\[
V_0 = \{ k : v_k v_{k+1} \in A \text{ and } v_k v_{k+1} \notin A \}
\]

where we assume addition is modulo \( n \), so for example if \( v_i = v_{n-1} \), then \( v_{i+1} = v_0 \).

We define a run on \( k \) vertices to be a consecutive sequence of vertices along the cycle all from the same set: \( V_- \), \( V_+ \), or \( V_0 \). We say that the run is maximal if no vertex can be added to the run without violating the definition.

Theorem 4.9. Suppose \( V_- \) or \( V_+ = \emptyset \) in a weak cycle \( D \). Let \( \ell \) be the number of maximal runs of vertices in \( V_0 \). Let \( n_i \) denote the number of vertices in the \( i \)-th maximal run of vertices from \( V_0 \), in order around the cycle. Then

\[
F(D) = \sum_{i=1}^{\ell} \left\lfloor \frac{n_i}{2} \right\rfloor
\]

Proof. Suppose \( V_- = \emptyset \) (without loss of generality). If \( V_0 = \emptyset \) then \( D \) is a directed cycle, so \( F(D) = 0 \) by Theorem 4.8. Assume that \( V_0 \neq \emptyset \).

We define \( S \subseteq V \) as follows. The first run has \( n_1 \) vertices. We can assume that the vertex labels begin with the first run, so the first run vertices are \( v_0, v_1, \ldots, v_{n_1-1} \). Add \( v_1, v_2, \ldots \) up to \( v_{n_1-1} \) or \( v_{n_1} \) (whichever is odd) to \( S \). This gives us \( \left\lceil \frac{v_1}{2} \right\rceil \) vertices. We do this for each maximal run of vertices from \( V_0 \), giving us \( |S| = \sum_{i=1}^{\ell} \left\lfloor \frac{n_i}{2} \right\rfloor \).

We show that \( S \) is a maximum FZFS. Since \( \deg^+(v) = 2 \) for every \( v \in S \) by construction, and since \( N^+(S) \subseteq V \setminus S \), \( S \) is stalled and therefore a FZFS. Now, suppose \( S' \subseteq V \) with \( |S'| > |S| \), and \( S' \) is stalled. Then either there must be some \( v \in V \) that has \( \deg^+(v) = 1 \) with \( v \in S' \), or there exists an \( i \)-th run with more than \( \left\lceil \frac{n_i}{2} \right\rceil \) vertices in \( S' \). In the first case, let \( \{u\} = N^+(v) \). Since \( S' \) is stalled, \( u \in S' \). However, recalling that \( V_- = \emptyset \), since \( S' \) is stalled \( w \in S' \), where \( \{w\} = N^+(w) \). We can continue the same argument for each vertex along the cycle, giving us \( S' = V \). In the second case, suppose the \( i \)-th run has more than \( \left\lceil \frac{n_i}{2} \right\rceil \) vertices in \( S' \). Then either the first vertex in the run is in \( S' \), or there are two adjacent vertices in the run that are not in \( S' \). If there are two or more adjacent vertices in the run that are in \( S' \), let \( x \) be the last such vertex. Then \( |N^+(x) \setminus S'| = 1 \), a contradiction since \( S' \) is stalled. Otherwise, let \( x \in S' \) be the first vertex in the run. We assumed that no adjacent vertices are in \( S' \), so the next vertex in the run, \( y \), is not in \( S' \). But \( \{y\} = N^+(x) \), contradicting our assumption that \( S' \) is stalled. Hence, \( S \) is a maximum FZFS, and \( F(D) = \sum_{i=1}^{\ell} \left\lfloor \frac{n_i}{2} \right\rfloor \). \( \square \)
An Example of Theorem 4.9 with \( V_- = \emptyset \) is shown in Figure 8. Theorem 4.10 establishes \( F(D) \) in the case that \( D \) has \( V_0, V_+, \) and \( V_- \) nonempty. Figure 9 shows an example of Theorem 4.10.

**Theorem 4.10.** Let \( D \) be a weak cycle such that \( V_0, V_- \), and \( V_+ \) are nonempty. Let \( d(i, j) = j - i \mod n \) and set \( \ell = \min\{d(i, j) : i \in V^-, j \in V^+\} \). Then

\[
F(D) = n - 1 - \left\lfloor \frac{\ell}{2} \right\rfloor.
\]

**Proof.** Let \((\hat{i}, \hat{j})\) be the indices that achieve \( \ell = \min\{d(i, j) : i \in V^-, j \in V^+\} \). Define

\[
S = \begin{cases} 
V \setminus \left\{v_{i+1}, v_{i+3}, v_{i+5}, \ldots, v_{i+\ell} = v_j\right\} & \text{if } \ell \text{ is odd} \\
V \setminus \left\{v_{i+1}, v_{i+3}, v_{i+5}, \ldots, v_{i+\ell-1}\right\} \cup \{v_j = v_{i+\ell}\} & \text{if } \ell \text{ is even}
\end{cases}
\]

where all indices are taken modulo \( n \). We show that \( V \setminus S \) is a critical set. If \( v \in V \setminus S \), then \( v = v_{i+2m} \) for some nonnegative \( m \), so if \( u \in N^-(v) \) then \( u = v_{i+2m+1} \) or \( u = v_{i+2m-1} \). If \( u = v_{i+2m+1} \), then \( N^+(u) = \{v, v_{i+2m+2}\} \). If \( u = v_{i+2m-1} \), then \( N^+(u) = \{v, v_{i+2m-2}\} \). Thus, \( V \setminus S \) is a critical set, and by Observation 2.1 \( S \) is a FZFS.

Let \( W \) be a critical set in \( D \). We show that \( |W| \geq \left\lceil \ell/2 \right\rceil + 1 \). Choose any \( i \in V_- \) and \( j \in V_+ \) such that \( d(i, j) \) is minimal. That is, if there exists \( j^* \) with \( d(i, j^*) < d(i, j) \) or \( i^* \) such that \( d(i^*, j) < d(i, j) \), then replace \( j \) with \( j^* \) or \( i \) with \( i^* \) as appropriate (or if both cases are true, pick one). Do this until there no such \( j^* \) or \( i^* \).

Let \( P \) denote the weak path \( v_x, v_{i+1}, \ldots, v_j, v_{j+1} \). Note that for any \( v_x \in V_0 \), if \( s \notin \{i, j, j+1\} \), then \( s \in V_0 \). Let \( P_1 \) denote the weak path starting from \( v_i \) and descending modulo \( n \) (i.e., the weak path that is edge-disjoint from \( P \)) until the first vertex \( v_x \) such that \( x-1 \in V_0 \). Note that \( v_x \) exists, because if no other vertex before satisfies the property, then \( v_{j+1} \) is such a vertex. Similarly, let \( P_2 \) be the weak path starting from \( v_{j+1} \) and ascending modulo \( n \) (i.e., the weak path that is edge-disjoint from \( P \)) until the first vertex \( v_y \) with \( y \in V_+ \).

Note that if there exist adjacent vertices in \( V(P) \cap V \setminus W \), then \( (V(P) \cup V(P_1) \cup V(P_2)) \setminus W = \emptyset \), because otherwise there exists a vertex \( v \in V \setminus W \) with \( |N^+(v) \cap W| = 1 \). So, either \( (V(P) \cup V(P_1) \cup V(P_2)) \setminus W = \emptyset \), or \( V(P) \cap W \geq \left\lceil \frac{d(i, j)}{2} \right\rceil \).

If \( V(P) \cup V(P_1) \cup V(P_2) = V(D) \), then we have shown that that \( V(P) \cap W \geq \left\lceil \frac{d(i, j)}{2} \right\rceil \), since otherwise, \( W = \emptyset \), violating the definition of a critical set. If \( V(D) \setminus (V(P) \cup V(P_1) \cup V(P_2)) \neq \emptyset \), let \( P' \) be the weak path from \( v_x \) to \( v_y \) whose internal vertices are exactly those vertices in \( V(D) \setminus (V(P) \cup V(P_1) \cup V(P_2)) \). Then we can choose \( i' \in V_- \cap V(P') \) and \( j' \in V_+ \cap V' \) such that \( d(i, j) \) is minimal (note that \( i' = y, j' = x \) satisfies \( i' \in V_- \cap V(P') \) and \( j' \in V_+ \cap V' \), so there exists such a minimal \( i' \) and \( j' \)). We can repeat the same argument as above for this set of vertices, giving us that either \( V(P'') \cap W = \emptyset \), or \( V(P') \cap W \geq \left\lceil \frac{d(i', j')}{2} \right\rceil \) where \( P'' \) is a nonempty weak path containing \( P' \) as a sub-weak-path. We can do this repeatedly until there are no remaining vertices in \( V(D) \), giving us that \( W = \emptyset \) or \( W \geq \left\lceil \frac{d(i, j)}{2} \right\rceil \) for some \( i \in V_- \) and \( j \in V_+ \). Since the former violates the definition of critical set and \( \ell = d(i, j) \) minimizes the latter, it follows that \( |W| \geq \left\lceil \ell/2 \right\rceil + 1 \) for any critical set \( W \).

Thus, by Observation 2.1, \( F(D) \geq n - 1 - \left\lceil \ell/2 \right\rceil \). \( \square \)

We establish that a weak cycle \( D \) on \( n \) vertices can achieve any value of \( F(D) \) between 0 and \( n - 1 \) by choosing appropriate arc orientations. There are two constructions, depending on whether \( F(D) > \frac{n}{2} \) or \( F(D) \leq \frac{n}{2} \). Examples are shown in Figures 8-9.

**Theorem 4.11.** For any \( n \) and \( k \) with \( 0 \leq k \leq n - 1 \), there exists a weak cycle \( D \) on \( n \) vertices such that \( F(D) = k \).

**Proof.** Case 1: \( k \leq \frac{n}{2} \). In this case, we use the construction from Theorem 4.9. Let \( V_0 = \{0, 2, \ldots, 2k-2\} \). Let \( V_+ \) consist of all remaining indices between 0 and \( n - 1 \). An example with
Figure 8: Weak cycle with $n = 10$, $F(D) = 3$. Note $V_\emptyset$.

Figure 9: Weak cycle with $n = 10$, $F(D) = 6$. Note $V_0, V_-, V_+$ are nonempty.

$n = 10$ and $k = 3$ is shown in Figure 8. By Theorem 4.10 $F(D) = \sum_{i=1}^{k \left\lfloor \frac{n}{2} \right\rfloor}$. Note that using $k = 0$ runs gives us a directed cycle, the special case $F(D) = 0$.

**Case 2:** $\frac{n}{2} < k \leq n - 1$. In this case, we use the construction from Theorem 4.10. Let $V_+ = \{0\}$, let $V_- = \{2k-n+1\}$, and let $V_0$ consist of all remaining indices between 0 and $n - 1$. Then by Theorem 4.10

$$F(D) = n - 1 - \left\lfloor \frac{k}{2} \right\rfloor = n - 1 - \left\lfloor \frac{n - (2k - n + 1)}{2} \right\rfloor = k.$$

Note that this includes the case $F(D) = n - 1$, where $v_0$ is a source.

We can restate Theorem 4.11 in terms of critical sets by applying Proposition 2.2.

**Corollary 4.12.** For any $n$ and $k$ with $1 \leq k \leq n$, there exists a weak cycle $D$ on $n$ vertices whose smallest critical set is of cardinality $k$.

### 4.3 Line graphs including de Bruijn and Kautz graphs

We now look at $F(D)$ in the case that $D$ is an iterated line graph. Since some of the digraphs we consider here have loops, we introduce the following modified CCR that applies only to digraphs with loops. We continue to use the CCR introduced earlier if $D$ has no loops.

- $B^0(S) := S$
- $B^{i+1}(S) := B^i(S) \cup \{w : \{w\} = N^+(v) \setminus B^i(S) \text{ for some } v \in V\}$

We could also state the CCR as follows: if any vertex $v \in V$ has exactly one empty out-neighbor $u$, then $u$ will be filled. In other words, the CCR that applies to digraphs with loops is identical to the CCR that applies to digraphs without loops, except that a vertex $u$ may become filled if $u$ is the unique empty out-neighbor of any vertex $v$, whether or not $v$ is filled. We make the following observation, similar to Observation 2.1 but for digraphs with loops.

**Observation 4.13.** In a digraph with loops, $W$ is a strongly critical set if and only if $V \setminus W$ is a stalled zero forcing set.

Note that although $F(D)$ is defined for any digraph without loops, there exists $D$ with loops that have $Z(D) = 0$ and therefore $F(D)$ is undefined. Also note that if a loop $vu \in A(D)$, then $u \in N^+(v)$ and $u \in N^-(u)$. A digraph consisting of $V = \{v\}$ and $A = \{(v, v)\}$ then has $Z(D) = 0$ and $F(D)$ undefined. We do not characterize $D$ with $F(D)$ undefined here, but note the following observation and lemma.

**Observation 4.14.** If $D$ is a digraph with loops, then $F(D)$ is undefined if and only if $Z(D) = 0$.

**Lemma 4.15.** If $N^-(v) = 0$ for some $v \in V$, or $N^+(v) \geq 2$ for all $v \in V$ and $V \neq \emptyset$, then $F(D)$ is defined.

**Proof.** If $N^-(v) = 0$ for some $v \in V$, then $v \in S$ for any ZFS $S$. Thus, $Z(D) \geq 1$, and $F(D)$ is defined. If $N^+(v) \geq 2$ for all $v \in V$, and $|V| \geq 1$, then each $u \in V$ has at least two out-neighbors, so the empty set is not a ZFS. That is, $Z(D) \geq 1$, and $F(D)$ is defined. \qed
Definition 4.16. For a digraph $D = (V, A)$, the line digraph of $D$ is the digraph $L(D)$ where

- $V(L(D)) = A(D)$, and
- $A(L(G)) = \{ab : a, b \in V(L(D)), \text{ and the head of } a \text{ is the tail of } b \text{ in } D\}$.

We use the following result from [10] Lemma 3.5.

Lemma 4.17. Let $D$ be a digraph and let $uv$ be a vertex of $L(D)$. If $\deg_{L(D)}^+(uv) \geq 2$, then every subset $T \subseteq N_{L(D)}^+(uv)$ with $|T| \geq 2$ is a strongly critical set in $L(D)$.

Observation 4.18. For any weakly connected digraph $D$, $L(D)$ has a source vertex if and only if $D$ has a source vertex.

We note the following proposition, analogous to Proposition 2.2 but for graphs with loops.

Proposition 4.19. In a digraph $D$ with loops, $F(D) = n - k$ if and only if the minimum cardinality of any strongly critical set in $D$ is $k$.

Lemma 4.20. Suppose a digraph $D$ with $|V(D)| \geq 2$ and no source that has $\deg^+(v) \leq 1$ for all $v \in V(D)$. Then $D$ is a set of disjoint directed cycles.

Proof. We have $|V(D)| \leq \sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v) \leq |V(D)|$. Hence, $\deg^-(v) = \deg^+(v) = 1$ for every $v \in V(D)$, giving us that $D$ is a set of disjoint directed cycles. In particular, if $D$ is weakly connected, then $D$ is a directed cycle. 

The following theorem establishes $F(L(D))$, with the added assumption that $F(L(D))$ is defined in the case that $D$ has loops.

Theorem 4.21. For any weakly connected digraph $D$ with $|V(D)| \geq 2$, set $m = |A(D)|$. Then if $D$ does not have loops, or if $D$ has loops and $Z(L(D)) > 0$, then

$$F(L(D)) = \begin{cases} 0 & \text{if } D \text{ is a directed cycle} \\ m - 1 & \text{if } D \text{ has a source,} \\ m - 2 & \text{otherwise.} \end{cases}$$

Proof. If $D$ is a directed cycle, then $L(D)$ is as well, and we know that $F(L(D)) = 0$ from Theorem 4.2. Suppose $D$ has a source $u$. Then since $D$ is weakly connected, $N^+(u) \geq 0$. Thus there exists $uv \in V(L(D))$ for some $v \in V(D)$, and also note that $N_{L(D)}^-(uv) = 0$, so $uv$ is a source in $L(D)$. Then $\{uv\}$ forms a critical set in $L(D)$, giving us $F(L(D)) = |V(L(D))| - 1 = m - 1$.

Finally, assume that $D$ is not a directed cycle and does not have a source. Note then that $L(D)$ does not have a source, and is not a directed cycle. Since $D$ and therefore $L(D)$ is weakly connected, by Lemma 4.20 there exists a vertex $uv \in V(L(D))$ such that $\deg^+_{L(D)}(uv) \geq 2$. By Lemma 4.17 $S = \{xy, uv\}$ is a strongly critical set for any $xy, uv \in N_{L(D)}^+(uv)$. Then, recalling that $L(D)$ does not have a source, by Propositions 4.19 or 2.2 depending on whether $D$ has loops, $F(L(D)) = |V(L(D))| - 2 = m - 2$.

Two graph families that can each be defined iteratively by using line graphs and are used in multiple applications are the de Bruijn and the Kautz digraphs. See [13] [16] for examples of the de Bruijn graph and [15] [18] for examples of the Kautz graph in applications. We will use the following definitions. For integers $d \geq 2$ and $M \geq 1$, the de Bruijn graph $B(d, M)$ is defined to be the digraph with $V(B(d, M)) = \{x_0x_1\ldots x_{M-1} : x_i \in \mathbb{Z}_d\}$, and $A(B(d, M)) = \{(x_0x_1\ldots x_{M-1}, x_1x_2\ldots x_M)\}$. The Kautz graph $K(d, M)$ is defined to be the digraph with $V(K(d, M)) = \{x_0x_1\ldots x_{M-1} : x_i \in \mathbb{Z}_{d+1} \text{ and } i \neq i + 1\}$, and $A(K(d, M)) = \{(x_0x_1\ldots x_{M-1}, x_1x_2\ldots x_M)\}$. Also, each Kautz graph and each de Bruijn graph has vertices of out-degree at least 2, leading to the following corollary.

Corollary 4.22. If $D$ is a de Bruijn or a Kautz graph, then $F(D) = |V(D)| - 2$. 

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Open problems

Since computing $F(G)$ for undirected graphs was found to be NP-hard in [19], it follows that the same is true for digraphs. However, it remains open whether or not this remains true if we restrict to oriented graphs. Indeed, at the time this paper was written, this result had not been established for the zero forcing number $Z(G)$ where $G$ is an oriented graph.

While we were able to characterize oriented graphs with $F(G) < Z(G)$ and produce a list of digraphs in general with this property, a characterization of all digraphs with $F(D) < Z(D)$ is open. Also, while we considered line digraphs with loops in Section 4.3, more general investigation of $F(D)$ in the case $D$ has loops would be interesting. In particular, a characterization of digraphs with loops that have $F(D)$ undefined (and therefore $Z(D) = 0$) is a possible starting point.

We can also consider the following generalizations of this problem. A ZFS $S$ is a minimal ZFS if deleting any vertex from $S$ results in the new set being a FZFS. Similarly, a FZFS $S$ is a maximal FZFS if adding any vertex to $S$ results in the new set being a ZFS. Certainly, any minimal ZFS is also minimal, and any maximal FZFS is also maximal. However, for some digraphs there exist examples of minimal ZFS and maximal FZFS that are not minimum and not maximum respectively, as in Figure 11.

Let $Z_m(D)$ denote the set of minimal ZFS of $D$, and let $F_M(D)$ denote the set of maximal FZFS of $D$. For digraphs with $F(D) = Z(D) - 1$, then $F_M(D)$ is precisely the set of maximum FZFS, and $Z_m(D)$ is precisely the set of minimum ZFS. However, it would be interesting to study these parameters for digraphs with $F(D) \geq Z(D)$. For example, if a digraph $D$, we can ask which integers $k$ with $0 < k < n$ have the property that there exists a ZFS $S \in Z_m(D)$ with $|S| = k$. We can ask the analogous question for maximal FZFS as well.

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References

[1] Thomas Ansill, Bonnie Jacob, Jaime Penzella, and Daniel Saavedra. Failed skew zero forcing on a graph. Linear Algebra and its Applications, 509:40–63, 2016.

[2] Jørgen Bang-Jensen and Gregory Z. Gutin. Digraphs: Theory, Algorithms and Applications. Springer Science & Business Media, 2008.
Francesco Barioli, Wayne Barrett, Shaun M. Fallat, H. Tracy Hall, Leslie Hogben, Bryan Shader, P. van den Driessche, and Hein van der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*, 433(2):401–411, 2010.

Adam Berliner, Chassidy Bozeman, Steve Butler, Minerva Catral, Leslie Hogben, Brenda Kroschel, Jephian C-H Lin, Nathan Warnberg, and Michael Young. Zero forcing propagation time on oriented graphs. *Discrete Applied Mathematics*, 224:45–59, 2017.

Adam Berliner, Cora Brown, Joshua Carlson, Nathanael Cox, Leslie Hogben, Jason Hu, Katrina Jacobs, Kathryn Manternach, Travis Peters, Nathan Warnberg, and Michael Young. Path cover number, maximum nullity, and zero forcing number of oriented graphs and other simple digraphs. *Involve. A Journal of Mathematics*, 8(1):147–167, 2015.

Daniel Burgarth, Sougato Bose, Christoph Bruder, and Vittorio Giovannetti. Local controllability of quantum networks. *Physical Review A*, 79(6):060305, 2009.

Daniel Burgarth, Domenico D’Alessandro, Leslie Hogben, Simone Severini, and Michael Young. Zero forcing, linear and quantum controllability for systems evolving on networks. *IEEE Transactions on Automatic Control*, 58(9):2349–2354, 2013.

Daniel Burgarth and Vittorio Giovannetti. Full control by locally induced relaxation. *Physical Review Letters*, 99(10):100501, 2007.

Daniel Burgarth and Koji Maruyama. Indirect Hamiltonian identification through a small gateway. *New Journal of Physics*, 11(10):103019, 2009.

Daniela Ferrero, Thomas Kalinowski, and Sudeep Stephen. Zero forcing in iterated line digraphs. *Discrete Applied Mathematics*, 255:198–208, 2019.

Katherine Fetcie, Bonnie Jacob, and Daniel Saavedra. The failed zero forcing number of a graph. *Involve. A Journal of Mathematics*, 8(1):99–117, 2014.

AIM Minimum Rank-Special Graphs Work Group. Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428(7):1628–1648, 2008.

Frank Harary and Leo Moser. The theory of round robin tournaments. *The American Mathematical Monthly*, 73(3):231–246, 1966.

M. Frans Kaashoek and David R. Karger. Koorde: A simple degree-optimal distributed hash table. In *International Workshop on Peer-to-Peer Systems*, pages 98–107. Springer, 2003.

Dongsheng Li, Xicheng Lu, and Jinshu Su. Graph-theoretic analysis of Kautz topology and DHT schemes. In *IFIP International Conference on Network and Parallel Computing*, pages 308–315. Springer, 2004.

Pavel A. Pevzner, Haixu Tang, and Michael S. Waterman. An Eulerian path approach to DNA fragment assembly. *Proceedings of the National Academy of Sciences*, 98(17):9748–9753, 2001.

Simone Severini. Nondiscriminatory propagation on trees. *Journal of Physics A: Mathematical and Theoretical*, 41(48):482002, 2008.

Haiying Shen and Ze Li. A Kautz-based wireless sensor and actuator network for real-time, fault-tolerant and energy-efficient transmission. *IEEE Transactions on Mobile Computing*, 15(1):1–16, 2015.

Yaroslav Shitov. On the complexity of failed zero forcing. *Theoretical Computer Science*, 660:102–104, 2017.