WHITHAM HIERARCHIES, INSTANTON CORRECTIONS
AND SOFT SUPERSYMMETRY BREAKING
IN $\mathcal{N} = 2$ $SU(N)$ SUPER YANG-MILLS THEORY

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Abstract

We study $\mathcal{N} = 2$ super Yang-Mills theory with gauge group $SU(N)$ from the point of view of the Whitham hierarchy. We develop a new recursive method to compute the whole instanton expansion of the prepotential using the theta function associated to the root lattice of the group. Explicit results for the one and two-instanton corrections in $SU(N)$ are presented. Interpreting the slow times of the hierarchy as additional couplings, we show how they can be promoted to spurion superfields that softly break $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 0$. This provides a family of nonsupersymmetric deformations of the theory, associated to higher Casimir operators of the gauge group. The $SU(3)$ theory is analyzed in some detail.

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1 Introduction and Conclusion

Among the large number of toy-models that have been proposed in the literature, in the aim to capture the essentials of non-perturbative QCD, the one solved exactly by Seiberg and Witten [1] stands out as a breakthrough. Apart from its unquestionable beauty, this work is remarkable in that it condenses and gives shape to the most beautiful ideas and conjectures about Yang-Mills theories that have been suggested over the last 30 years like, for instance, duality or quark confinement by monopole condensation.

Certainly, as a toy model, the Seiberg-Witten solution only represents an approximation to the real world: in order to get an exact answer, the price to pay is the need for $\mathcal{N} = 2$ supersymmetry. The generalization of the initial breakthrough from $SU(2)$ to $SU(N)$ was soon unraveled [2]. Another step towards the real world was to avoid supersymmetry. Since in the Seiberg-Witten ansatz supersymmetry was an essential ingredient from the very beginning, the strategy was to break it softly, trying to preserve the analytic properties of the solution. The spurion formalism [3] proved to be instrumental, and results were reported for the first time in [4], in the context of $SU(2)$ with and without additional matter. These results were generalized in [5] to $SU(N)$ and refined in [6, 7].

Seemingly, a totally unrelated topic is that of integrable hierarchies. In its origin this is a subject related to non-linear differential equations that appear in classical mechanics of systems with either finite or infinite number of degrees of freedom. Although this is a vast subject in itself, it is only recently that some unifying language has emerged. A close relationship between integrable models and supersymmetric quantum field theory was observed sometime ago in the context of two-dimensional topological conformal field theories, obtained by twisting $\mathcal{N} = 2$ superconformal models [8].

It is by now a well established fact that the Seiberg-Witten solution for the effective theory of $\mathcal{N} = 2$ super Yang-Mills can be embedded into the Whitham hierarchy associated to the periodic Toda lattice [9, 10]. The link between both constructions is summarized in the statement that the prepotential of the $\mathcal{N} = 2$ Yang-Mills theory corresponds to the logarithm of the quasiclassical tau function. The interplay between Whitham hierarchies and two-dimensional superconformal models suggests understanding the times of the Whitham hierarchy as coupling constants of composite operators also in the four-dimensional context, and, thereafter, the prepotential as the generating function of correlation functions for them. Recently, this interpretation has proven to be very useful in understanding some aspects of the twisted version of $\mathcal{N} = 2$ Yang-Mills theory [11, 12, 13, 14, 15, 16]. In particular, it has been shown in [17] that the slow times of the Whitham hierarchy are the appropriate variables to understand the structure of contact
terms in the twisted theory.

In this paper the structure of the effective action of $\mathcal{N} = 2$ theories from the point of view of the underlying integrable hierarchy will be explored, and we will show that the results in [17] have interesting applications for the dynamics of the $\mathcal{N} = 2$, $SU(N)$ theory. We will see that the integrable structure constraints the semiclassical expansion of the prepotential in such a way that the knowledge of the one-loop contribution essentially determines the instanton corrections. This provides in fact a new method to compute the prepotential to any given instanton number. The inputs for this computation are the following: first of all, the RG equation [18, 19], which relates the first derivative of the prepotential with respect to the quantum scale, $\partial \mathcal{F} / \partial \Lambda$, to the quadratic Casimir. The second ingredient is the equation derived in [15, 17], which relates $\partial^2 \mathcal{F} / \partial \Lambda^2$ to the theta function associated to the root lattice of the gauge group. These relations allow for a recursive computation, starting from the one-loop contribution to the prepotential. Explicit results for the one and two instanton corrections that agree with those previously obtained in the literature will be presented. A general formula for the three instanton correction is also written down.

We will also show that the slow times of the Whitham hierarchy can be understood as spurion superfields that softly break supersymmetry down to $\mathcal{N} = 0$, in the spirit of [4, 5, 6]. In the original approach to the soft breaking of $\mathcal{N} = 2$ supersymmetry, the quantum scale $\Lambda$ is promoted to a spurion superfield, and this generates a series of terms that explicitly break supersymmetry. However, these terms are associated to the quadratic Casimir of the gauge group, as this Casimir is in essence the dual variable to the quantum scale. In this way, the soft breaking to $\mathcal{N} = 0$ using $\Lambda$ as a spurion is the analog of the soft breaking to $\mathcal{N} = 1$ using the operator $\text{Tr} \Phi^2$. But one can not implement, in this restricted approach, the $\mathcal{N} = 0$ analog of the $\mathcal{N} = 1$ supersymmetry breaking operator associated to a higher Casimir operator, like $\text{Tr} \Phi^3$ for $SU(3)$. A natural extension of this formalism is provided by the Whitham hierarchy. In principle, the variables that appear in the prepotential in the framework of the Whitham hierarchy are different from the original variables of the Seiberg-Witten ansatz, but can be related in a precise way. In fact, the first slow time of the hierarchy, $T_1$, can be identified with the quantum scale $\Lambda$, and the times $T_n$, with $n = 2, \ldots, N−1$, are dual to particular homogeneous combinations of the higher Casimir operators. In this way, the Whitham hierarchy can be interpreted as a family of supersymmetry breaking deformations of the original theory associated to the higher Casimir operators of the gauge group.

The results presented in this paper can be extended in many ways. We have restricted ourselves to the theory without matter hypermultiplets and to the gauge group $SU(N)$. 
One could generalize this approach to other gauge groups and/or matter content, and this would provide a powerful computational tool to obtain instanton expansions, along the lines explained in this paper. Another avenue for future research is the connection with string theory and D-branes. In [20] it has been shown that some nonsupersymmetric configurations of branes can be interpreted as softly broken $\mathcal{N} = 2$ theories, and on the other hand the approach via integrable systems has also been casted in the context of D-brane configurations [21]. It would be very interesting to explore the connection between these two problems, and obtain in this way a new family of nonsupersymmetric deformations of MQCD.

The organization of this paper is the following: in section 2, the relation between Whitham hierarchies and the Seiberg-Witten solution is reviewed, following [17], and clarify the relation between the slow time variables and the variables of the Seiberg-Witten prepotential. In the remaining sections we present two independent applications of the equations so far. In section 3, we study the instanton expansion of the prepotential in the semiclassical region and explain the new method to compute this expansion. Explicit computations for the one and two-instanton corrections are presented for comparison. In section 4, we promote the slow times to spurion superfields and we analyze the resulting theory once supersymmetry is broken down to $\mathcal{N} = 0$. Finally, the $SU(3)$ theory is discussed in some detail in section 5.

## 2 Whitham Hierarchies and Seiberg-Witten Ansatz

The low-energy dynamics of $N = 2$ super Yang-Mills theory with gauge group $SU(N)$ is described by the hyperelliptic curve [2]

$$y^2 = P^2(\lambda, u_k) - 4\Lambda^{2N},$$

where $P(\lambda, u_k) = \lambda^N - \sum_{k=2}^{N} u_k \lambda^{N-k}$ is the characteristic polynomial of $SU(N)$ and $u_k, k = 2, \ldots, N$ are the Casimirs of the gauge group. They provide Weyl invariant coordinates on $\mathcal{M}_\Lambda$, the quantum moduli space of vacua of the theory. This curve has genus $g = r$, where $r = N - 1$ is the rank of the group. As explained in [3], the curve (2.1) can be identified with the spectral curve of the $N$ site periodic Toda lattice and, moreover [1, 10], the prepotential of the effective theory is essentially the logarithm of the quasiclassical tau function and hence depends on the slow times of the corresponding Whitham hierarchy. Here the integrable approach along the lines of [17] will be followed, and a subfamily of slow times $T_n$ with $1 \leq n \leq N - 1$ will be considered. As we mentioned in the introduction, when the Seiberg-Witten ansatz is embedded into the
Whitham hierarchy, one has to clarify the relation between the variables and parameters in both approaches. Although there is an indication of how this goes in [17] (see also [22]) we shall pause here to discuss this point, focusing on $\Lambda$ and $T_1$.

In principle, $\Lambda$ and $T_1$ are different variables. $\Lambda$ appears, in the Seiberg-Witten solution, in the hyperelliptic curve (2.1) describing the moduli space of vacua $\mathcal{M}_\Lambda$. Let $A_i$ and $B_i$ denote a symplectic basis of homology cycles for this curve, $i = 1, \ldots, r$. The $a^i$ variables of the prepotential, for the duality frame associated to the cycles $A_i$, are given by the integrals over these cycles of a certain meromorphic one-form:

$$a^i(u_k, \Lambda) = \frac{1}{2\pi i} \oint_{A_i} \frac{\lambda P'(u_k)}{\sqrt{P^2(u_k) - 4\Lambda^2}} d\lambda,$$

where $P' = dP/d\lambda$, and the same expression holds for the dual variables $a_{D,i}$ with $B_i$ instead of $A_i$. A dependence of $u_k$ upon $\Lambda$ is induced by solving $a^i(u_k, \Lambda)$ = constant, as $u_k = f_k(a^j, \Lambda)$. That is to say, by using $(a^i, \Lambda)$ as coordinates for the moduli space.

On the other hand, in the context of the Whitham hierarchy, the slow times $T_n$ appear associated to meromorphic differentials of second kind $d\hat{\Omega}_n$ (in the notation of [17])

$$\alpha^i(u_k, T_1, T_2, \ldots) = \sum_{n \geq 1} \frac{T_n}{2\pi i} \oint_{A_i} d\hat{\Omega}_n = \sum_{n \geq 1} \frac{T_n}{2\pi i} \oint_{A_i} \frac{P(u_k)^{n/N} P'(u_k)}{\sqrt{P^2(u_k) - 4}} d\lambda = T_1 a^i(u_k, 1) + O(T_{n>1}),$$

(2.3)

where $\left(\sum_{k=-\infty}^{\infty} c_k \lambda^k\right)_+ = \sum_{k=0}^{\infty} c_k \lambda^k$. Also here the $\alpha_{D,i}$ are defined by the same expression with $B_i$ replacing $A_i$. Similarly, we may choose to solve for $u_k(\alpha^i, T_1, T_2, \ldots)$ by demanding that $\alpha^i$ in (2.3) be independent of all $T_n$. The induced dependence $u_k = g_k(a^j, T_n)$ solves the Whitham equations.

The recovery of the Seiberg-Witten solution goes as follows. First we define the rescaled times $\hat{T}_n$ and “vevs” $\hat{u}_k$:

$$\hat{T}_n = T_n/T_1^n, \quad \hat{u}_k = T_1^k u_k,$$

(2.4)

with $\hat{T}_1 = 1$. It is easy to see that (2.3) can be written as

$$\alpha^i(u_k, T_1, \hat{T}_2, \hat{T}_3, \ldots) = \sum_{n \geq 1} \frac{\hat{T}_n}{2\pi i} \oint_{A_i} \frac{P^{n/N}(\hat{u}_k) P'(\hat{u}_k)}{\sqrt{P^2(\hat{u}_k) - 4T_1^{2N}} d\lambda.}$$

(2.5)

In particular, after setting $\hat{T}_2 = \hat{T}_3 = \ldots = 0$ we find that

$$\alpha^i(u_k, T_1, \hat{T}_{n>1} = 0) = T_1 a^i(u_k, 1) = a^i(\hat{u}_k, \Lambda = T_1).$$

(2.6)
In conclusion, we may identify $T_1$ with $\Lambda$ in the submanifold $\dot{T}_2 = \dot{T}_3 = ... = 0$, provided the moduli space is parametrized with $\dot{u}_k$ instead of $u_k$.

With this correspondence in mind, let us now focus on the derivatives of the prepotential $F(\alpha, T)$. The computation of such derivatives from the Whitham hierarchy was the main result of the paper [17]. We just list them here for completeness:

$$
\frac{\partial F}{\partial T_n} = \frac{\beta}{2\pi i} \sum_m m T_m H_{m+1,n+1},
$$

$$
\frac{\partial^2 F}{\partial \alpha^i \partial T_n} = \frac{\beta}{2\pi i} \frac{\partial H_{n+1}}{\partial a^i}, \quad (2.6)
$$

$$
\frac{\partial^2 F}{\partial T_m \partial T_n} = -\frac{\beta}{2\pi i} \left( H_{m+1,n+1} + \frac{\beta}{m n} \frac{\partial H_{m+1}}{\partial a^i} \frac{\partial H_{n+1}}{\partial a^j} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta(E(0|\tau)) \right).
$$

In these equations, $\Theta(E(0|\tau))$ denotes Riemann’s theta function with a certain characteristic $E$, evaluated at the origin $4$; $\beta = 2N$, $m, n = 1, ..., r = N - 1$, and derivatives with respect to $T_n$ are taken at constant $\alpha^i$. The functions $H_{m,n}$ are certain homogeneous combinations of the Casimirs $u_k$, given by

$$
H_{m+1,n+1} = \frac{N}{mn} \text{res}_{P} \left( P^{m/N}(\lambda) dP^{n/N}(\lambda) \right) = H_{n+1,m+1},
$$

and

$$
H_{m+1} \equiv H_{m+1,2} = \frac{N}{m} \text{res}_{P} P^{m/N}(\lambda) d\lambda = u_{m+1} + O(u_m).
$$

Here res$_P$ stands for the usual Cauchy residue at the point $P$. We have for instance $H_{2,2} = H_2 = u_2$, $H_{3,2} = H_3 = u_3$ and $H_{3,3} = u_4 + \frac{N-2}{2N} u_2^2$.

As they stand, the expressions given in (2.6) are not suitable for application to the Seiberg-Witten solution. Therefore, and in view of the previous considerations, we define the following change of variables

$$
\log \Lambda = \log T_1, \quad \dot{T}_n = T_1^{-n} T_n, \quad (n \geq 2),
$$

and, consequently,

$$
\frac{\partial}{\partial \log \Lambda} = \sum_{m \geq 1} m T_m \frac{\partial}{\partial T_m}, \quad \frac{\partial}{\partial \dot{T}_n} = T_1^n \frac{\partial}{\partial T_n}, \quad (n \geq 2) \quad (2.9)
$$

4 We follow the convention of [23] in the definition of the theta function $\Theta(\alpha, \beta)(\xi|\tau)$ associated to the genus $r$ hyperelliptic curve, and with characteristics $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r), \quad \vec{\beta} = (\beta_1, \ldots, \beta_r)$:

$$
\Theta(\vec{\alpha}, \vec{\beta})(\xi|\tau) = \sum_{n \in \mathbb{Z}} \exp \left[ i\pi \tau_{ij}(n_i + \alpha_i)(n_j + \alpha_j) + 2\pi i(n_i + \alpha_i)(\xi + \beta_i) \right]. \quad (2.7)
$$

Therefore, in this normalization $\partial_{\tau_{ij}} = \frac{1}{4\pi i} \partial^2_{ij}$. See also [2].
With the help of these expressions, it is now straightforward to reexpress all the formulae in (2.6) as derivatives of $F$ with respect to $a^i, T_n$ and $\Lambda$. Most of the factors $T_1$ can be used to promote $u_k$ to $\hat{u}_k$ or, rather, to the homogeneous combinations thereof:

$$\hat{H}_{m+1,n+1} = T_1^{m+n} H_{m+1,n+1} \Rightarrow \hat{H}_{m+1} = T_1^{m+1} H_{m+1}$$

(2.10)

(since $H_{m+1} = H_{m+1,2}$). The remaining $T_1$’s are absorbed in making up $\hat{a}^i \equiv T_1 a^i(u_k,1) = a^i(\hat{u}_k, T_1)$. Altogether we find

$$\frac{\partial F}{\partial \log \Lambda} = \beta \frac{\beta}{2\pi i} \sum_{m,n \geq 1} m \hat{T}_m \hat{H}_n H_{m+1,n+1}, \quad \frac{\partial F}{\partial T_n} = \beta \frac{\beta}{2\pi i} \sum_{m \geq 1} m \hat{T}_m \hat{H}_{m+1,n+1},$$

$$\frac{\partial^2 F}{\partial \alpha^i \partial \log \Lambda} = -\beta^2 \frac{\beta}{2\pi i} \sum_{m \geq 1} \hat{T}_m \hat{H}_n \frac{\partial \hat{H}_{m+1}}{\partial \alpha^i} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau),$$

$$\frac{\partial^2 F}{\partial \log \Lambda \partial T_n} = \beta \frac{\beta}{2\pi i} \hat{H}_n \frac{\partial \hat{H}_{m+1,n+1}}{\partial \alpha^i} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau),$$

$$\frac{\partial^2 F}{\partial T_m \partial T_n} = -\beta^2 \frac{\beta}{2\pi i} \frac{\partial \hat{H}_2}{\partial \alpha^i} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau),$$

$$\frac{\partial^2 F}{\partial T_m \partial T_n} = -\beta \frac{\beta}{2\pi i} \left( \hat{H}_{m+1,n+1} + \beta \frac{\partial \hat{H}_{m+1,n+1}}{\partial \alpha^i} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau) \right),$$

with $m, n \geq 2$. In these expressions $\hat{T}_1 = 1$. The restriction to the submanifold $\hat{T}_2 = \hat{T}_3 = ... = 0$ yields formulae which are suited for the Seiberg-Witten analysis. Notice that in this subspace $a^i(u_k, T_1, T_{n>1} = 0) = \hat{a}^i$, hence

$$\frac{\partial F}{\partial \log \Lambda} = \beta \frac{\beta}{2\pi i} \hat{H}_2, \quad \frac{\partial F}{\partial T_n} = \beta \frac{\beta}{2\pi i} \hat{H}_{n+1},$$

$$\frac{\partial^2 F}{\partial \alpha^i \partial \log \Lambda} = \beta \frac{\beta}{2\pi i} \hat{H}_2 \frac{\partial \hat{H}_2}{\partial \alpha^i} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau),$$

$$\frac{\partial^2 F}{\partial \log \Lambda \partial T_n} = \beta \frac{\beta}{2\pi i} \hat{H}_2 \frac{\partial \hat{H}_{n+1}}{\partial \alpha^i} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau),$$

$$\frac{\partial^2 F}{\partial T_m \partial T_n} = -\beta^2 \frac{\beta}{2\pi i} \frac{\partial \hat{H}_2}{\partial \alpha^i} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau),$$

$$\frac{\partial^2 F}{\partial T_m \partial T_n} = -\beta \frac{\beta}{2\pi i} \left( \hat{H}_{m+1,n+1} + \beta \frac{\partial \hat{H}_{m+1,n+1}}{\partial \alpha^i} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau) \right),$$

with $m, n \geq 2$. As a check, notice that the formula (1.11) in [17] follows directly from combining the second and the sixth formulae of the list, i.e.

$$\frac{\partial \hat{H}_m}{\partial \log \Lambda} = -\beta \frac{\partial \hat{H}_2}{\partial \alpha^i} \frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau).$$

(2.13)
and the first equation in (2.12) is precisely the RG equation derived in [18, 19] (see below eq.(3.1)). Hereafter, we will always work with the coordinates (2.8) and, therefore, hats will be omitted everywhere.

According to [17], the characteristic \( E \) appearing in (2.11)–(2.12) is an even, half-integer characteristic associated to a particular splitting of the roots of the discriminant. The explicit form of \( E \) can be obtained using the connection with the twisted version of the \( \mathcal{N} = 2 \) theory investigated in [11, 12, 13, 15, 16]. The theta function involved in (2.11)–(2.12) is the same one that appears in the blow-up formula derived in [13] and further discussed in [16] from the point of view of the Toda–Whitham hierarchy. When no non-abelian magnetic fluxes are turned on, the blow-up factor of the lattice sum is [13]:

\[
\Theta[\vec{\alpha}, \vec{\beta}](t\vec{V}|\tau) = \sum_{n_i} e^{i\pi\tau_i n_in_j + itV_i n_i - i\pi \sum_i n_i},
\]

where \( V_i = \frac{\partial u}{\partial a_i} \). From here, we read off

\[
\vec{\alpha} = (0, \ldots, 0) \quad \text{and} \quad \vec{\beta} = (1/2, \ldots, 1/2).
\]

This is the characteristic \( E \) of the theta function in (2.11)–(2.12) when we express it in terms of electric variables. Notice that it is even, half-integer, and for \( SU(2) \) the associated theta function is the Jacobi \( \vartheta_4(z|\tau) \), in agreement with the result in Appendix B of [17].

We will see in the next section that the above identification of Seiberg–Witten and Whitham variables, together with this choice of the characteristic, allows us to find the appropriate instanton expansion in the semiclassical region.

## 3 Instanton Corrections

Instanton calculus provides one of the few non-perturbative links between the Seiberg-Witten ansatz and the microscopic non-abelian field theory that it is supposed to describe effectively at low energies. For this reason, since the very advent of the work of Seiberg and Witten, there has been a lot of work on this particular topic with the aim of relating the explicit non-perturbative computations with the predictions of the exact solutions. On the one hand, different techniques have been developed to extract instanton expansions from the hyperelliptic curves [18, 25, 26, 27, 28, 29]. On the other hand, instanton corrections have also been explicitly computed in the microscopic theory [30, 31] (see [32] for a review and a list of references). In [18] it was realized that the non perturbative relation

\[
\mathcal{F} - \frac{1}{2} \partial_{D,i} a^i = \frac{\beta}{4\pi i} u
\]

7
is very useful to derive a recursion relation for the instanton contributions. In order to compute the instanton corrections, however, (3.1) is not sufficient and one needs additional input. This is usually provided by the Picard-Fuchs equations for the periods. The Picard-Fuchs equations are difficult to derive and solve when the rank of the gauge group is larger than one, although techniques from topological Landau-Ginzburg theories can make them more instrumental in order to obtain the one-instanton correction to the prepotential for the ADE series [28]. The procedure we will use here is rather different. As we will see, it turns out that the equation for $\partial^2 \mathcal{F} / \partial (\log \Lambda)^2$ in (2.12), together with (3.1), provides enough information to obtain the instanton expansion of the prepotential in the semiclassical region to any order, and we don’t have to make use of the Picard-Fuchs equations. We then see that the connection of $SU(N)$, $\mathcal{N} = 2$ super Yang–Mills theory with Toda–Whitham hierarchies embodies in a natural way a recursive procedure to compute all instanton corrections\(^5\). The essential ingredient that makes this possible is the relation of the derivatives of the prepotential with the theta function associated to the root lattice of the gauge group.

### 3.1 Recursive Procedure from the Prepotential Theory

To begin with, let us fix our conventions for the expansion of the prepotential in the semiclassical region. We choose the basis $H_k = E_{k,k} - E_{k+1,k+1}$ for the Cartan subalgebra and $E_{k,j}, k \neq j$ for the raising and lowering operators. Let $\{\alpha_i\}_{i=1,...,r}$ stand for the simple roots of $SU(N)$ and $(\alpha, \beta)$ denote the usual inner product constructed with the Cartan-Killing form. The dot product $\alpha \cdot \beta \equiv 2(\alpha, \beta)/(\beta, \beta) = (\alpha, \beta^\vee)$. We have that $\alpha_i \cdot \alpha_j = C_{ij}$, with $C_{ij}$ the Cartan matrix, while $\lambda^i \cdot \alpha_j = \delta^i_j$ define the fundamental weights. In particular this means that $\alpha_i = \sum_j C_{ij} \lambda^j$. The simple roots generate the root lattice $\Delta = \{\alpha = n^i \alpha_i | n^i \in \mathbb{Z}\}$, and the fundamental weights its dual, the weight lattice $\Delta^\vee$.

The instanton expansion of the prepotential is:

$$
\mathcal{F} = \frac{1}{2N} \tau_0 \sum_{\alpha_+} Z_{\alpha_+}^2 + \frac{i}{4\pi} \sum_{\alpha_+} Z_{\alpha_+}^2 \log \frac{Z_{\alpha_+}^2}{\Lambda^2} + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \mathcal{F}_k(Z) \Lambda^{2Nk}. \quad (3.2)
$$

In this expression $a = a^i \alpha_i$ and $Z_{\alpha} = \alpha \cdot a$. Also, $\alpha_+$ denotes a positive root and $\sum_{\alpha_+}$ a sum over positive roots. The expansion is in powers of $\Lambda^{\beta}$, where $\beta = 2N$ for $SU(N)$,

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\(^5\)When this work was finished, we noticed that recursive relations for the prepotential were recently derived from modular anomaly equations for mass deformed $\mathcal{N} = 4$ super Yang-Mills theories in powers of the mass of the adjoint hypermultiplet [33]. In a particular limit, this theory reproduces the results corresponding to pure $\mathcal{N} = 2$ super Yang-Mills theory.
and \( k \) is the instanton number. We then have
\[
\frac{\partial^2 F}{\partial (\log \Lambda)^2} = \frac{1}{2\pi i} \sum_{k=1}^{\infty} (2Nk)^2 F_k(Z) \Lambda^{2Nk},
\]
which, after (2.12), should be equated to
\[
\frac{\partial^2 F}{\partial (\log \Lambda)^2} = \frac{\beta^2}{2\pi i} \frac{\partial H}{\partial a^i} \frac{1}{i\pi} \partial_{\gamma\gamma} \log \Theta_E(0|\tau).
\]

The derivative of the quadratic Casimir also has an expansion that can be obtained from the RG equation and (3.2):
\[
\frac{\partial H}{\partial a^i} = \frac{2\pi i}{\beta} \frac{\partial^2 F}{\partial a^i \partial \log \Lambda} = C_{ij} a^j + \sum_{k=1}^{\infty} kF_{k,i} \Lambda^{2Nk}
\equiv \sum_{k=0}^{\infty} H_i^{(k)} \Lambda^{2Nk}
\]
where \( F_{k,i} = \partial F_k/\partial a^i \), and use has been made of the fact that \( \frac{1}{2N} \sum_{\alpha} Z_{\alpha}^2 = \frac{1}{2} a^i C_{ij} a^j \).

The couplings in the semiclassical region are obtained again from the expansion (3.2):
\[
\tau_{ij} = \frac{\partial^2 F}{\partial a^i \partial a^j} = \frac{i}{2\pi} \sum_{\alpha} \frac{\partial Z_{\alpha}}{\partial a^i} \frac{\partial Z_{\alpha}}{\partial a^j} \log \left( \frac{Z_{\alpha}^2}{\Lambda^2} \right) + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \mathcal{F}_{k,ij} \Lambda^{2Nk}.
\]
with \( \mathcal{F}_{k,ij} = \frac{\partial^2 F_k}{\partial a^i \partial a^j} \). For convenience, in (3.6) a term \( \frac{i}{2\pi} \sum_{\alpha} \frac{\partial Z_{\alpha}}{\partial a^i} \frac{\partial Z_{\alpha}}{\partial a^j} (2\pi i \tau_0 - 3) \) has been set to zero by a suitable adjustment of the bare coupling \( 2\pi i \tau_0 = 3N \). Of course, we may shift \( \tau_0 \) to any value by appropriately rescaling \( \Lambda \), and this will be reflected in our choice for the normalization of the \( \mathcal{F}_k \). One has to be careful with this normalization when comparing our final expressions with similar computations in the literature.

The term involving the couplings that appear in the theta function \( \Theta_E \) is now
\[
i\pi n^i \tau_{ij} n^j = \sum_{\alpha} \log \left( \frac{Z_{\alpha}}{\Lambda} \right)^{-(\alpha - \alpha_+)^2} + \frac{1}{2} \sum_{k=1}^{\infty} (\alpha \mathcal{F}''_{k,\alpha}) \Lambda^{2Nk},
\]
where \( \alpha = n^i \alpha_i \) and
\[
\alpha \mathcal{F}''_{k,\alpha} \equiv \sum_{i,j} n^i \mathcal{F}_{k,ij} n^j
= \sum_{\beta, \gamma \in \Delta} (\alpha \cdot \beta) \frac{\partial^2 F_k}{\partial Z_\beta \partial Z_\gamma} (\gamma \cdot \alpha).
\]
The appropriate characteristic $E$ for the theta function $\Theta_E$ in the semiclassical region has been given in (2.15). Inserting (3.7) in the theta function, we obtain

$$
\Theta_E(0|\tau) = \sum \exp \left[ i\pi n^i \tau n^j + i\pi \sum n_k \right] \\
= \sum (1)^{\rho \cdot \alpha} \prod_{\alpha^+} \left( \frac{Z_{\alpha^+}}{\Lambda} \right)^{-(\alpha \cdot \alpha)^2} \prod_{k=1}^{\infty} \exp \left( \frac{1}{2} (\alpha \cdot F_{k/\alpha} \Lambda^{2N_k}) \right) \\
= \sum \sum_{r=0}^{\infty} (1)^{\rho \cdot \alpha} \prod_{\alpha^+} Z_{\alpha^+}^{-(\alpha \cdot \alpha)^2} \prod_{k=1}^{\infty} \left( \sum_{m=0}^{\infty} \frac{1}{2^m m!} (\alpha \cdot F_{k/\alpha} \Lambda^{2N_k}) \right) \Lambda^{2N_r} \\
= \sum_{r=0}^{\infty} \Theta^{(r)} \Lambda^{2N_r}.
$$

(3.9)

In the previous expression, $\rho = \sum_{i=1}^{N-1} \lambda^i$. $\Delta_r \subset \Delta$ is a subset of the root lattice composed of those lattice vectors $\alpha$ that fulfill the constraint $\sum_{\alpha^+} (\alpha \cdot \alpha)^2 = 2N_r$. In particular $\Delta_1$ is the root system, i.e. the simple roots together with their Weyl reflections. In other words, as the root system itself, $\Delta_1$, forms an orbit of the Weyl group, the one- and two-instanton contributions will come from a sum over the Weyl orbit of, say, $\alpha_1$. On the other hand $\Delta_r$, for $r > 1$, will be in general a union of Weyl orbits, since Weyl reflections are easily seen to be automorphisms of $\Delta_r$. Therefore $\Theta^{(r)}$ is Weyl invariant by construction. The first few terms in the expansion (3.9) are given by

$$
\Theta^{(0)} = 1, \quad \Theta^{(1)} = \sum_{\alpha \in \Delta_1} (1)^{\rho \cdot \alpha} \prod_{\alpha^+} Z_{\alpha^+}^{-(\alpha \cdot \alpha)^2}, \\
\Theta^{(2)} = \sum_{\alpha \in \Delta_1} (1)^{\rho \cdot \alpha} \frac{1}{2} (\alpha \cdot F_{1/\alpha} \Lambda^{2N_1}) \prod_{\alpha^+} Z_{\alpha^+}^{-(\alpha \cdot \alpha)^2} + \sum_{\beta \in \Delta_2} (1)^{\rho \cdot \beta} \prod_{\alpha^+} Z_{\alpha^+}^{-(\beta \cdot \alpha)^2}.
$$

In the logarithmic derivative, the theta function appears in the denominator, and we have the expansion

$$
\Theta(0|\tau)^{-1} = \sum_{l=0}^{\infty} \Xi(l)(\Theta) \Lambda^{2N_l}.
$$

(3.10)

Here $\Xi(0)(\Theta) = 1$ and for $\Xi(l)(\Theta)$ we can write in general

$$
\Xi(l)(\Theta) = \sum_{(p_1, ..., p_k) \in \mathbb{N}^k} \xi(p_1, ..., p_k) \prod_{i=1}^{l} (\Theta^{(i)})^{p_i},
$$

(3.11)

where the coefficients $\xi$ are parametrized by the partition elements $(p_1, ..., p_k)$. The first few values for these parameters are, for example,

$$
\xi(1) = -1, \quad \xi(2,0) = 1, \quad \xi(0,1) = -1, \quad \xi(3,0,0) = -1, \quad \xi(1,1,0) = 2, \quad \xi(0,0,1) = -1.
$$
and using these values we can immediately obtain the lower $\Xi^{(l)}(\Theta)$. Similarly, the derivative of the theta function with respect to the period matrix is given by

$$\frac{1}{i\pi} \partial_{\tau_{ij}} \Theta_E(0, \tau) = \sum_n n^i n^j \exp \left[ i\pi n^k \tau_{kl} + i\pi \sum_k n_k \right]$$

$$= \sum_{r=1}^{\infty} \sum_{\alpha \in \Delta_r} (-1)^p \alpha \cdot \lambda^i \alpha \cdot \lambda^j \prod_{\alpha_+} Z_{\alpha_+}^\alpha (\alpha \cdot \alpha_+)^2 \prod_{k=1}^{\infty} \exp \left( \frac{1}{2} (\alpha \cdot F_k^\alpha) \Lambda^{2Nk} \right) \Lambda^{2Nr}$$

$$\equiv \sum_{p=1}^{\infty} \Theta^{(p)}_{ij} \Lambda^{2Np} . \quad (3.12)$$

Now, collecting all the pieces and inserting them back into (3.4), we find for $F_k(Z)$ the following expression:

$$F_k(Z) = -k^{-2} \sum_{p,q,l=0}^{p+q+l=k-1} \sum_{ij} H_i^{(p)} H_j^{(q)} \Theta^{(k-p-q-l)}_{ij} \Xi^{(l)} , \quad (3.13)$$

in terms of the previously defined coefficients.

If we look at the coefficients in (3.13), it is easy to see that the expressions they involve depend on $F_1, F_2, \ldots$ up to $F_{k-1}$. In fact, although both $H^{(p)}$ and $\Theta^{(p)}$ depend on $F_1, \ldots, F_p$, the indices within parenthesis in (3.13) reach at most the value $k - 1$ as $\Theta^{(0)}_{ij} = 0$. Moreover $\Theta^{(k)}_{ij}$ depends on $F_1, \ldots, F_{k-1}$ since the vector $\alpha = 0$ is missing from the lattice sum. This fact implies the possibility to build up a recursive procedure to compute all the instanton coefficients by starting just from the perturbative contribution to $F(a)$ in (3.2).

### 3.2 Lower Instanton Corrections

As we have seen, (3.13) gives the instanton expansion of the prepotential in the semiclassical region, for $N = 2$ super Yang–Mills theory with gauge group $SU(N)$. We emphasize the fact that the essential ingredients are summarized in (2.12). We believe that a direct comparison of (3.13) with other computations of $F_k$ presented in the literature provides an independent test of the proposal made in [17]. Also, our results show that the embedding of the Seiberg–Witten theory inside an integrable hierarchy, aside from its theoretical interest, provides an alternative device for some computations in $SU(N)$ $\mathcal{N} = 2$ super Yang–Mills theory. Here we shall give a fairly manageable general expression for the one-, two-, and three-instanton contributions in $SU(N)$, and compare it with the answers that can be found in the literature. We would like to point out that, although we have focused on $SU(N)$, the form of the instanton corrections that we have presented should be generalizable to other cases. In this respect we point out that relations such as (2.13) hold for all the simply laced algebras [13].
The one-instanton contribution is given by (3.13) with $k = 1$. In this case the expression is rather simple and reads

$$F_1 = -\sum_{ij} H_i^{(0)} H_j^{(0)} \Theta_{ij}^{(1)}$$

$$= -\sum_{\alpha \in \Delta_1} Z_\alpha^2 (-1)^{\rho + \alpha} \prod_{\alpha^+} Z_{\alpha^+}^{-(\alpha + \cdot))^2}, \quad (3.14)$$

As pointed out in [25], in general there is not a unique form for the $F_k$ when written in terms of the $Z_\alpha$'s, since these are not independent variables. An unambiguous expression should come out for $F_k$ when written in terms of the $a_i$'s or, for example, in terms of symmetric polynomials thereof, such as the classical values of the Casimirs.

The two-instanton contribution can also be easily worked out from (3.13), and turns out to be

$$F_2 = -\frac{1}{4} \left( \Theta_{ij}^{(2)} H_i^{(0)} H_j^{(0)} + \Theta_{ij}^{(1)} (2 H_i^{(1)} H_j^{(0)} - H_i^{(0)} H_j^{(0)} \Theta_{ij}^{(1)}) \right)$$

$$= -\frac{1}{4} \left( \sum_{\alpha \in \Delta_1} (-1)^{\rho + \alpha} \prod_{\alpha^+} Z_{\alpha^+}^{-(\alpha + \cdot))^2} \left[ F_1 + 2(\alpha \cdot F_1') Z_\alpha + \frac{1}{2} (\alpha \cdot F_1'' \cdot \alpha) Z_\alpha^2 \right]$$

$$+ \sum_{\beta \in \Delta_2} Z_\beta^2 (-1)^{\rho + \beta} \prod_{\alpha^+} Z_{\alpha^+}^{-(\beta + \cdot))^2} \right), \quad (3.15)$$

where $\alpha \cdot F_k' = n_i F_{k,i}$, $= \sum_{\beta \in \Delta}(\alpha \cdot \beta) \partial F_k/\partial Z_\beta$. Furthermore, our proposal for the three-instanton correction gives

$$F_3 = -\frac{1}{9} \left( \sum_{\alpha \in \Delta_1} (-1)^{\rho + \alpha} \prod_{\alpha^+} Z_{\alpha^+}^{-(\alpha + \cdot))^2} \left[ 4F_2 + 4(\alpha \cdot F_2') Z_\alpha + (\alpha \cdot F_1')^2 \right.$$\

$$+ \frac{1}{2} (\alpha \cdot F_1'' \cdot \alpha) (F_1 + 2(\alpha \cdot F_1') Z_\alpha) + \frac{1}{8} (\alpha \cdot F_1'' \cdot \alpha)^2 Z_\alpha^2 + \frac{1}{2} (\alpha \cdot F_2'' \cdot \alpha) Z_\alpha^2 \right]$$

$$+ \sum_{\beta \in \Delta_2} (-1)^{\rho + \beta} \prod_{\alpha^+} Z_{\alpha^+}^{-(\beta + \cdot))^2} \left[ F_1 + 2(\beta \cdot F_1') Z_\beta + \frac{1}{2} (\beta \cdot F_1'' \cdot \beta) Z_\beta^2 \right]$$

$$+ \sum_{\gamma \in \Delta_3} (-1)^{\rho + \gamma} \prod_{\alpha^+} Z_{\alpha^+}^{-(\gamma + \cdot))^2} Z_\gamma^2 \right),$$

etc. The above expressions make patent the recursive character of the procedure.

It is quite cumbersome to show that our formulae match the results in the literature for a generic value of $N$. Nevertheless, we have checked several particular cases using symbolic computation. For the one-instanton contribution we found the following results

$$F^{SU(2)}_1 = 2^{-3} \Delta_0^{-1}, \quad F^{SU(3)}_1 = \frac{3}{2} u_0 \Delta_0^{-1}, \quad F^{SU(4)}_1 = [8u_0^3 - 36u_0^2 + 32u_0 w_0] \Delta_0^{-1}$$
where the zero subindex signals the classical expression for the Casimirs. These results fully agree with those obtained by other authors\cite{25,27}.

Concerning the following corrections, $F_2$ and $F_3$, the previous formulae give

$$
\begin{align*}
F_2^{SU(2)} &= \frac{5}{2^8 \Delta_0^3}, \\
F_2^{SU(3)} &= \frac{3^2 u_0}{2^4 \Delta_0^3} (17 u_0^3 + 189 v_0^2), \\
F_3^{SU(2)} &= \frac{3}{2^8 \Delta_0^3}, \\
F_3^{SU(3)} &= \frac{u_0}{2^4 \Delta_0^3} (3080 u_0^6 + 119529 u_0^3 v_0^2 + 248589 v_0^4),
\end{align*}
$$

which also agree with those in Refs.\cite{25,27,33}. Despite these coincidences, we believe it would be useful to carry out an exhaustive comparison with different methods, as well as a generalization to other Lie algebras.

\section{Soft Supersymmetry Breaking with Higher Casimir Operators}

The prepotential of the Seiberg-Witten solution in the Toda-Whitham framework depends on infinitely many slow times $T_n$, and for $n = 1, \ldots, N - 1$ we can find explicit expressions for its first and second derivatives. In this section, we will interpret these slow times as parameters of a nonsupersymmetric family of theories, by promoting them to spurion superfields in the spirit of \cite{4,5,6}. The higher order slow times, as (2.12) shows, are dual to the $H_m$, which are homogeneous combinations of the Casimir operators of the group. This means that we will be able to parametrize soft supersymmetry breaking terms induced by all the Casimirs of the group, and not just the quadratic one (the only case considered in \cite{5}). This also generalizes to the $\mathcal{N} = 0$ case the family of $\mathcal{N} = 1$ supergravities.

\footnote{A word of caution is in order concerning the normalization of the $F_k$. As mentioned before, it is related to the classical value of the coupling $\tau_0$ through $\Lambda$. For example in the case of $SU(2)$, if we want to have $\tau_0 = (i/\pi)(2 \log 2 - 3)$, in agreement with \cite{25}, we have to multiply every $F_k$ in (3.13) with a scale factor $1/16^k$.}

\footnote{Note that the result for $F_3^{SU(3)}$ in Ref.\cite{25} has a wrong global factor equal to 27.}
supersymmetry breaking terms considered for instance in [34]. As the dependence on the slow times as spurion superfields is encoded in the prepotential, we will be able to obtain the exact effective potential of the theory, generalizing in this way the results of [3].

4.1 Properties of the prepotential under duality transformations

To understand the softly broken model it is useful to derive first the properties of the prepotential and its derivatives under a duality transformation in the effective theory, following the strategy of [4, 5]. In the case of $SU(r+1)$ Yang-Mills theory, the duality group is the symplectic group $Sp(2r, \mathbb{Z})$. An element of this group has the structure

$$\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$ (4.1)

where the $r \times r$ matrices $A$, $B$, $C$, $D$ satisfy:

$$A^t D - C^t B = 1, \quad A^t C = C^t A, \quad B^t D = D^t B.$$ (4.2)

In what follows it will be convenient to define the spurion variables $s_n$ as

$$s_1 = -i \log \Lambda, \quad s_n = -i \hat{T}_n, \quad n = 2, \ldots, r.$$ (4.3)

We take as our independent coordinates in the prepotential $\alpha^i, s_n$. The dual spurions are defined by

$$s_{D,n} = \frac{\partial F}{\partial s_n},$$ (4.4)

and we introduce a generalized $(2r) \times (2r)$ matrix of couplings as follows:

$$\tau_{ij} = \frac{\partial^2 F}{\partial \alpha^i \partial \alpha^j}, \quad \tau^n_i = \frac{\partial^2 F}{\partial \alpha^i \partial s_n}, \quad \tau^{mn} = \frac{\partial^2 F}{\partial s_m \partial s_n},$$ (4.5)

where $i, j, m, n = 1, \ldots, r$.

The symplectic group acts on the $\alpha^i, \alpha_{D,i}$ variables as $v \rightarrow \Gamma v$, where $v^i = (\alpha_{D,i}, \alpha^i)$,

$$\begin{pmatrix} \alpha^i_{D,i} \\ \alpha^i \end{pmatrix} = \begin{pmatrix} A_i^k & B_{ik} \\ C_{ik} & D_i^k \end{pmatrix} \begin{pmatrix} \alpha_{D,i}^k \\ \alpha^i \end{pmatrix}.$$ (4.6)

The spurion variables, coming from the slow times of the Toda-Whitham hierarchy, parametrize deformations of the Seiberg-Witten differential, as we have seen in (2.3), therefore they are invariant under the duality transformations (which are symplectic
transformations of the homology cycles of the curve): \( s_m^\Gamma = s_m \). The jacobian matrix of the change of coordinates is then given by

\[
\begin{pmatrix}
\frac{\partial \alpha^\Gamma_i}{\partial \alpha^\Gamma j} & \frac{\partial \alpha^\Gamma_i}{\partial \alpha^\Gamma m} \\
\frac{\partial s^\Gamma}{\partial \alpha^\Gamma j} & \frac{\partial s^\Gamma}{\partial \alpha^\Gamma m}
\end{pmatrix} = \begin{pmatrix}
C^{ik} \tau_{kj} + D^i_j & C^{ik} \tau_{kn} \\
0 & \delta_m^n
\end{pmatrix},
\]

(4.7)

and we obtain the transformation laws for the operators

\[
\frac{\partial}{\partial \alpha^\Gamma i} = \frac{\partial \alpha^j}{\partial \alpha^\Gamma i} \frac{\partial}{\partial \alpha^j} + \frac{\partial s_m}{\partial \alpha^\Gamma i} \frac{\partial}{\partial s_m} = (C \tau + D)^{-1} j_i \frac{\partial}{\partial \alpha^j},
\]

\[
\frac{\partial}{\partial s^\Gamma m} = \frac{\partial \alpha^i}{\partial s^\Gamma m} \frac{\partial}{\partial \alpha^i} + \frac{\partial s_n}{\partial s^\Gamma m} \frac{\partial}{\partial s_n} = \frac{\partial}{\partial s_m} - (C \tau + D)^{-1} j_i C^{ik} \tau_{km} \frac{\partial}{\partial \alpha^i}.
\]

(4.8)

The transformation law for the prepotential has been found in \([18, 19]\) and reads

\[
\mathcal{F}^\Gamma = \mathcal{F} + \frac{1}{2} \alpha^i (D^T B)_{ij} \alpha^j + \frac{1}{2} \alpha^j (C^T A)^{ij} \alpha_{D,j} + \alpha^i (B^T C)^{ij} \alpha_{D,j}.
\]

(4.9)

Using (4.8), (4.9) and (4.2) one easily finds that the dual times are invariant under duality transformations:

\[
s_D^\Gamma = \frac{\partial \mathcal{F}^\Gamma}{\partial s^\Gamma m} = \frac{\partial \mathcal{F}}{\partial s_m} = s_D^m
\]

(4.10)

and that the second derivatives of the prepotential transform as follows

\[
\tau^\Gamma_{ij} = (A \tau + B)(C \tau + D)^{-1} j_i \tau_{ij},
\]

\[
\tau^\Gamma_{i m} = \left[(C \tau + D)^{-1} j_i \tau_{j m} \right],
\]

\[
\tau^{\Gamma mn} = \tau_{mn} - \tau_{mi} \left[(C \tau + D)^{-1} C \right]^{ij} \tau_{j n}.
\]

(4.11)

Notice that the matrix \((C \tau + D)^{-1} C\) is symmetric (\([24]\), p. 91).

Using (4.11) one can find explicit expressions for the dual spurions and couplings in terms of the gauge-invariant functions \(H_{n,m}\) and quantities associated to the hyperelliptic curve:

\[
s_D^1 = \frac{\beta}{2 \pi} \left[H_2 + i \sum_{m \geq 2} m s_m H_{m+1} - \sum_{m,n \geq 2} m s_m s_n H_{m+1,n+1} \right],
\]

\[
s_D^n = \frac{\beta}{2 \pi} \left[H_{n+1} + i \sum_{m \geq 2} m s_m H_{m+1,n+1} \right],
\]

15
\[ \tau^1_i = \frac{\beta}{2\pi} \left[ \frac{\partial H_2}{\partial a^i} + \sum_{n \geq 2} s_n \frac{\partial H_{n+1}}{\partial a^i} \right], \]

\[ \tau^n_i = \frac{\beta}{2\pi n} \frac{\partial H_{n+1}}{\partial a^i}, \quad (4.12) \]

\[ \tau^{11} = -2\tau^1_i \tau^1_j \partial_{\tau_{ij}} \log \Theta_E(0|\tau), \]

\[ \tau^{1n} = -2\tau^1_i \tau^n_j \partial_{\tau_{ij}} \log \Theta_E(0|\tau), \]

\[ \tau^{nm} = \frac{\beta}{2\pi} H_{m+1,n+1} - 2\tau^n_i \tau^m_j \partial_{\tau_{ij}} \log \Theta_E(0|\tau). \]

with \( n, m \geq 2 \). Notice that, when the spurions \( s_m \) are zero, we recover for the variable \( s_1 \) the results of [5]. One can check that the explicit expressions for the couplings and the dual spurions obtained in (4.12) satisfy the transformation properties given in (4.11). The invariance of the dual times \( s_{D,n} \), as expressed in (4.10), is consistent with the fact that they depend on \( s_m \) and \( H_{m+1,n+1} \), which are duality invariant. To verify the transformation properties of \( \tau^n_i \), we can now appeal to the transformation properties of the derivatives of the Casimirs,

\[ \frac{\partial H_{n+1}}{\partial \alpha^i} \to [(C\tau + D)^{-1}]^t \frac{\partial H_{n+1}}{\partial \alpha^j}, \quad (4.13) \]

which is again a consequence of the duality invariance of the \( H_{n+1} \). Finally, to obtain the transformation law of \( \tau^{mn} \), we need the behaviour of the theta function under the symplectic transformation (4.1). The arguments \( \xi, \tau \) of the theta function change as follows:

\[ \tau \to \tau^\Gamma = (A\tau + B)(C\tau + D)^{-1}, \]

\[ \xi \to \xi^\Gamma = [(C\tau + D)^{-1}]^t \xi, \quad (4.14) \]

and the characteristics (understood as row vectors) transform as

\[ \alpha \to \alpha^\Gamma = D\alpha - C\beta + \frac{1}{2} \text{diag}(CD^t), \quad (4.15) \]

\[ \beta \to \beta^\Gamma = -B\alpha + A\beta + \frac{1}{2} \text{diag}(AB^t). \quad (4.16) \]

The transformation law for the theta function is then given by [24]

\[ \Theta[\alpha^\Gamma, \beta^\Gamma](\tau^\Gamma|\xi^\Gamma) = K \exp[\pi i \xi^t (C\tau + D)^{-1} C\xi] \Theta[\alpha, \beta](\tau|\xi), \quad (4.17) \]

where \( K \) is given by

\[ K = e^{i\phi} (\det(C\tau + D))^{1/2} \quad (4.18) \]
and $\phi$ is a $\xi$-independent phase that will cancel in the logarithmic derivative. Using (4.17) we see that, under a symplectic transformation, $\partial_{\tau i j} \log \Theta(0|\tau)$ gets shifted by a term of the form

$$\frac{1}{2} (C\tau + D)^i_k (C\tau + D)^j_l [(C\tau + D)^{-1} C]^{kl},$$

(4.19)

and in this way we recover (4.11) directly from (4.12).

### 4.2 The Microscopic Lagrangian

As anticipated above, to break $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 0$, we promote the variables $s_n$ to $\mathcal{N} = 2$ vector superfields $S_n$, and then freeze the scalar and auxiliary components to constant vacuum expectation values. Indeed, we are really interested in non-supersymmetric deformations of pure $SU(r+1)$ Yang–Mills theory that preserve the nice holomorphic properties of the Seiberg–Witten solution. Thus, for all $S_n, n \geq 2$, we will set the corresponding scalar components $s_n$ to zero, while keeping the top components $D_n$ and $F_n$ as supersymmetry breaking parameters. A similar scenario was considered in [6], where the bare masses of the hypermultiplets were also promoted to $\mathcal{N} = 2$ superfields, and non-supersymmetric deformations of the massless theory were studied by turning on the auxiliary components of the mass spurion superfield while setting the scalar components (the bare masses of the original model) equal to zero.

The couplings of the spurions $S_n$ are encoded in the holomorphic dependence on the slow times. In terms of $\mathcal{N} = 1$ superfields we have,

$$S \equiv S_1 = s_1 + \theta^2 F_1 , \quad V_s \equiv V_1 = \frac{1}{2} D_1 \theta^2 \bar{\theta}^2 ,$$

(4.20)

$$S_n = \theta^2 F_n , \quad V_n = \frac{1}{2} D_n \theta^2 \bar{\theta}^2 , \quad n \geq 2$$

(4.21)

where $s_1$ is related to the dynamical scale of the theory, $\Lambda = e^{is_1}$. Although the prepotential of the Whitham hierarchy is defined for the low-energy theory, it is important to know what is the microscopic, non-supersymmetric theory whose effective action is encoded in the prepotential $F(\alpha_i, S_n)$ (where the $S_n$ are given in (4.20)–(4.21)). When we only consider the first slow time, $S_1$, we can identify it with the classical gauge coupling in the bare Lagrangian. In fact, the RG equation for the $SU(r+1)$ theory gives a relation between $\Lambda$ and the coupling $\tau = \frac{\theta}{2\pi} + \frac{4n_i}{\theta^2}$, $\Lambda^{2(r+1)} \sim e^{2i\pi \tau}$. We then see that the scalar component of $S_1$ can be written as $s_1 = \pi \tau/(r+1)$, and the spurion superfield appears in the classical prepotential as follows [6, 7]

$$F = r + \frac{1}{2\pi} S_1 \text{tr} A^2 = \frac{r+1}{\pi} S_1 \mathcal{H}_2 .$$

(4.22)
The nonsupersymmetric microscopic Lagrangian is then obtained from (4.22) by turning on the scalar and auxiliary components of $S_1$.

Let us now analyze what happens when the rest of the slow times are turned on. We can expand the prepotential around $S_2 = \cdots = S_r = 0$, but then, in the supersymmetric case, we should take into account the higher derivatives of the prepotential to obtain the terms of order $O(S^3)$ in this expansion. If we consider however the case of softly broken supersymmetry, where the $S_{n \geq 2}$ have the form (4.20)–(4.21), then the terms with more that two powers of the $S_{n \geq 2}$ will not give any term in the Lagrangian, because they involve too many $\theta$’s and the integral in superspace will vanish. Notice that this is a general fact that holds for both the macroscopic and the microscopic Lagrangian: as we only know the first and second derivatives of the prepotential, we will be able to write the complete Lagrangian only when we restrict ourselves to the spurion configurations (4.20)–(4.21). Also notice that, as the derivatives of the prepotential are evaluated at $S_2 = \cdots = S_r = 0$, they are obtained by evaluating the right hand side of (2.12) in the original Seiberg-Witten solution.

To write the microscopic Lagrangian, we have to consider the classical limit of the expressions (4.12) evaluated at $s_2 = \cdots = s_r = 0$. From the expressions we have obtained in section 3, it is easy to see that:

$$\partial H_{r+1} \partial H_{r+1} \frac{1}{i \pi} \partial_{\tau_{ij}} \log \Theta_E(0|\tau) \sim \Lambda^{2r} O\left(\frac{\Lambda^2}{Z_{\alpha_+}}\right),$$

(4.23)

then the terms involving the theta function vanish in the semiclassical limit ($\Lambda/Z_{\alpha_+} \to 0$) (for all the positive roots $\alpha_+$). This is also the expected behaviour of these terms in the twisted theory: they are contact terms that should vanish at tree level \([11]\). The microscopic Lagrangian we are considering is then given by the following deformed, classical prepotential

$$\mathcal{F} = \frac{r+1}{\pi} \sum_{n=1}^{r} \frac{1}{n} S_n H_{n+1} + \frac{r+1}{2 \pi i} \sum_{n,m \geq 2} S_n S_m H_{n+1,m+1},$$

(4.24)

where the spurions $S_n$, $n \geq 1$ are of the form (4.20)–(4.21). We then see that we are studying nonsupersymmetric deformations of the $\mathcal{N} = 2$ theory induced by the higher Casimirs. We will be mainly interested in the “reduced” prepotential,

$$\mathcal{F}^{\text{red}} = \mathcal{F} - \frac{r+1}{2 \pi i} \sum_{n,m \geq 2} S_n S_m H_{n+1,m+1},$$

(4.25)

as in this case the perturbations are linear in the higher order slow times (at the classical level) and only involve the Casimirs $H_{n+1}$. Notice that the second derivative of the
reduced, quantum prepotential (when the higher order times are set to zero) is then given by

$$\frac{\partial^2 \mathcal{F}_{\text{red}}}{\partial S_m \partial S_n} \bigg|_{S_2=\ldots=S_n=0} = \frac{\beta^2}{2\pi i mn} \frac{\partial \mathcal{H}_{m+1}}{\partial a^i} \frac{1}{\partial_{\bar{a}^j}} \log \Theta_E(0|\tau), \quad (m, n \geq 2) \quad (4.26)$$

an expression which is invariant under the semiclassical monodromy and vanishes semiclassically. This reduced prepotential is in fact the relevant one for Donaldson-Witten theory, and the second derivatives (4.26) are essentially the contact terms for higher Casimirs found in [15]. From now on we will consider the reduced prepotential and omit the superscript.

If we expand (4.25) in superspace, we find the microscopic Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}}, \quad (4.27)$$

where,

$$\mathcal{L}_{\text{kin}} = \frac{1}{4\pi} \text{Im} \left[ (\nabla_\mu \phi)^\dagger_\alpha (\nabla^\mu \mathcal{F})^\alpha + i (\nabla_\mu \psi)^\dagger_\alpha \bar{\sigma}^\mu \psi^b \mathcal{F}^a_b - i \mathcal{F}_{ab} \lambda^a \sigma^\mu (\nabla_\mu \bar{\lambda})^b - \frac{1}{4} \mathcal{F}_{ab} \left( F^\alpha_{\mu\nu} F^{\mu\nu} + i F^a_{\mu\nu} \bar{F}^{b\mu\nu} \right) \right], \quad (4.28)$$

$$\mathcal{L}_{\text{int}} = \frac{1}{4\pi} \text{Im} \left[ \mathcal{F}_{AB} F^A F_{\text{class}}^B - \frac{1}{2} \mathcal{F}_{abC} \left( (\psi^a \psi^b)^\dagger F^{*C} + (\lambda^a \lambda^b) F^C + i \sqrt{2}(\psi^a \lambda^b) D^C \right) \right] + \frac{1}{2} \mathcal{F}_{AB} D^A D^B + ig \left[ (\phi^a \psi^b)^\dagger D_b \mathcal{F}^c + \sqrt{2} \left[ (\phi^a \lambda^b) \mathcal{F}^a \psi^b - (\bar{\psi} \bar{\lambda}) a \mathcal{F}^a \right] \right]. \quad (4.29)$$

In (4.28)–(4.29), $\lambda$, $\psi$ are the gluinos and $\phi$ is the scalar component of the $\mathcal{N} = 2$ vector superfield. The $f^a_{bc}$ are the structure constants of the Lie algebra. The indices $a, b, c, \ldots$ belong to the adjoint representation of $SU(N)$, and are raised and lowered with the invariant metric. $A, B, \ldots$ run over both the indices in the adjoint, $a, b, \ldots$, and over those of the slow times, $n, m, \ldots$. Since all spurions corresponding to higher Casimirs are purely auxiliary superfields, (4.21), the Lagrangian can be easily presented in a more transparent form. To this end we first integrate the auxiliary fields out

$$D^a = -b^{-1}_{\text{class}}^{ac} \left( b^{(\text{class})}_c^m D_m + \text{Re}(g a^*_b f^b_{ca} \mathcal{F}^a) \right), \quad (4.30)$$

$$F^a = -b^{-1}_{\text{class}}^{ac} b^{(\text{class})}_c^m F_m. \quad (4.31)$$

In these equations, we have introduced the classical matrix of couplings $b_{AB}$.

$$b^{(\text{class})} = \frac{1}{4\pi} \text{Im} \tau^{(\text{class})}, \quad (4.32)$$
which are given by the derivatives of $\mathcal{F}^{\text{red}}$ with respect to the lower components of the vector superfields. We obtain the following expressions for them,

\begin{equation}
\tau_{ab}^{(\text{class})} = \tau \delta_{ab} ,
\end{equation}

\begin{equation}
\tau_{a}^{(\text{class}) m} = \frac{N}{\pi \text{i} m} \frac{\partial \mathcal{H}_{m+1}^{(\text{class})}}{\partial \phi^a} = \frac{N}{\pi \text{i} m} \text{tr} (\phi^m T_a) + \ldots ,
\end{equation}

\begin{equation}
\tau_{mn}^{(\text{class})} = 0 ,
\end{equation}

where the dots (in eq. (4.34)) denote the derivative with respect to $\phi^a$ of lower order Casimir operators. Inserting back (4.30)–(4.31) in (4.27), we find

\begin{equation}
\mathcal{L} = \mathcal{L}_{N=2} - B_{mn}^{(\text{class})} \left( F_m F_m^* + \frac{1}{2} D_m D_n \right) + f_{bc}^{e} b_{a}^{(\text{class})} m b_{b}^{(\text{class})} - 1 \alpha \epsilon D_m \phi^b \bar{\phi}^c
+ \frac{1}{8\pi} \text{Im} \frac{\partial \tau^{(\text{class})}}{\partial \phi^a} \left[ (\psi^a \psi^b) F_m^* + (\lambda^a \lambda^b) F_m + i \sqrt{2} (\lambda^a \psi^b) D_m \right] ,
\end{equation}

where $B_{mn}^{(\text{class})}$ is the classical value of the duality invariant quantity

\begin{equation}
B_{mn}^{(\text{class})} = b_a^{m} b_a^{-1} b_b^{n} - b_{mn} .
\end{equation}

The dilaton spurion gives mass to the gauginos of the $\mathcal{N} = 2$ vector multiplet and to the imaginary part of the Higgs field $\phi$. The spurions corresponding to higher Casimirs, on the other hand, give couplings between the Higgs field and the gauginos. Finally, note that the bare Lagrangian (4.36) is not CP invariant, since $\tau$ and $F_m$ are arbitrary complex parameters. Thus, the corresponding low energy effective Lagrangian on which we will focus from now on, is not in general CP invariant. Notice that the spurion superfields $S_n$ have dimensions $1 - n$, therefore the supersymmetry breaking parameters $F_n$, $D_n$ have dimension $2 - n$. For $n > 2$, they will give nonrenormalizable (i.e. irrelevant) interactions in the microscopic Lagrangian. This does not mean that the resulting perturbations do not change the low-energy structure of the theory: the operators we are considering can be dangerously irrelevant operators, as in the related theory analyzed in [35], and in this case they will affect the infrared physics.

### 4.3 The Effective Potential

As the prepotential has an analytic dependence on the spurion superfields, the effective Lagrangian up to two derivatives and four fermion terms for the $\mathcal{N} = 0$ theory described by (4.27) is given by the exact Seiberg-Witten solution once the spurion superfields are taken into account. This gives the exact effective potential at leading order and the
vacuum structure can be determined. The computation of the effective potential and the condensates is very similar to the one in [5]. If we are near a submanifold of the moduli space of vacua where $n_H$ hypermultiplets become massless, the full Lagrangian contains the vector multiplet contribution and the hypermultiplet contribution involving (in $\mathcal{N} = 1$ language) two chiral superfields $H_a, \tilde{H}_a, a = 1, \ldots, n_H$. We choose an appropriate duality frame whose variables will be generically denoted by $a_i$. We will denote the charge of the $a$th hypermultiplet with respect to the $i$th $U(1)$ factor by $n_{a_i}$, where $a = 1, \ldots, n_H$. In this section, the indices $m,n = 1, \ldots, r$ label the spurion variables, $s_m$, and the indices $i,j = 1, \ldots, r$ the period variables, $a^i$. The matrix of couplings appearing in the effective potential is given by

$$b = \frac{1}{4\pi} \text{Im} \tau . \tag{4.38}$$

If we now define the quantities

$$(n^a, n^b) = n_i^a b^{-1ij} n_j^b,$$

$$(n^a, b^m) = n_i^a b^{-1ij} b_j^m, \tag{4.39}$$

the effective potential can be written as

$$V = B^{mn} \left( F_m F_n^* + \frac{1}{2} D_m D_n \right) + (n^a, b^m) D^m \left( |h_a|^2 - |\tilde{h_a}|^2 \right) + \sqrt{2} (n^a, b^m) \left( F_m \tilde{h}_a h_a + \tilde{F}_m h_a \tilde{h}_a \right) + 2 (n^a, n^b) (h_a \tilde{h}_a h_b \tilde{h}_b) + \frac{1}{2} (n^a, n^b) (|h_a|^2 - |\tilde{h}_a|^2) (|h_b|^2 - |\tilde{h}_b|^2) + 2 |n^a| \cdot a^i (|h_a|^2 + |\tilde{h}_a|^2) , \tag{4.40}$$

where $n^a \cdot a = \sum_i n_i^a a^i$, and $h_a (\tilde{h}_a)$ is the scalar component of $H_a (\tilde{H}_a)$. This expression is identical to the one derived in [5], with the only difference that we have now $r$ spurion superfields associated to the different Casimirs of the group. We can then adapt the results derived there to this context, where we also set $D_m = 0$, these being no real restriction since $F_n$ and $D_n$ transform as doublets under $SU(2)_R$. To obtain the values of the condensates, we minimize $V$ with respect to $h_a, \tilde{h}_a$. One finds that $|h_a| = |\tilde{h}_a|$. It is convenient to fix the gauge in the $U(1)^r$ factors in such a way that

$$h_a = \rho_a , \quad \tilde{h}_a = \rho_a e^{i\beta_a} \tag{4.41}$$

If the charge vectors $n^a$ are linearly independent, the non trivial condensates are given by

$$|n^a \cdot a|^2 + \sum_b (n^a, n^b) \rho_b^2 e^{i(\beta_b - \beta_a)} + \frac{1}{\sqrt{2}} (n^a, b^m) F_m e^{-i\beta_a} = 0, \tag{4.42}$$

21
The effective potential is then given by
\[ V = B^{mn} F_m F_n^* - 2 \sum_{ab} (n^a, n^b) \rho_a^2 \rho_b^2 \cos(\beta_a - \beta_b) \] (4.43)

To make use of the previous equations one needs explicit knowledge of the values of the couplings as functions over the moduli space \( \tau(a) \). This is achieved by means of eqs.(4.12) where the terms on the right hand side can be computed from the original Seiberg-Witten solution. We shall see two examples in the following section.

5 Analysis of SU(3)

\( N = 2 \) supersymmetric Yang-Mills theory with gauge group SU(3) has been analyzed in detail in \[25, 34\]. As it is well-known, there are two sets of distinguished points in the moduli space of the theory. The three \( \mathbb{Z}_2 \) vacua are located at \( u^3 = (3 \Lambda^2)^3, v = 0 \), and give rise to the \( \mathcal{N} = 1 \) vacua when the theory is perturbed with a mass term of the form \( \text{Tr} \Phi^2 \) (we denote \( u_2 = u, u_3 = v \)). Then we have the two \( \mathbb{Z}_3 \) vacua, located at \( u = 0 \) and \( v = \pm 2 \Lambda^2 \). They are also known as the Argyres-Douglas (AD) points, and there are three mutually nonlocal BPS states becoming massless at this point. The low-energy theory there is an \( \mathcal{N} = 2 \) superconformal theory and the two \( U(1) \) factors are decoupled.

In this section we will briefly examine the softly broken theory near these vacua. This will also illustrate the structure of the formalism we have been using in the strong coupling regime, in particular the use of theta functions. We will set \( \Lambda = 1 \), as in \[34, 36, 17\].

5.1 The \( \mathbb{Z}_2 \) vacua

In this subsection we study the soft breaking of the theory near the \( \mathcal{N} = 1 \) points, which have been studied in detail in \[36\]. To evaluate the second derivatives of the prepotential, we need the values of the periods of the hyperelliptic curve and the structure of the gauge couplings. We will focus on the \( \mathcal{N} = 1 \) point where \( N - 1 \) magnetic monopoles become massless (corresponding in \( SU(3) \) to the point \( u = 3 \Lambda^2, v = 0 \)). The values of the quantities at the other points can be obtained using the \( \mathbb{Z}_N \) symmetry in the moduli space. The eigenvalues for the field \( \phi \) are given by
\[ \phi_n = 2 \cos \frac{\pi (n - \frac{1}{2})}{N}, \quad n = 1, \cdots, N. \] (5.1)

The derivatives of the dual variables satisfy
\[ \sum_{j=1}^r \frac{\partial a_{D,j}}{\partial \phi_n} \sin \frac{\pi j l}{N} = i \cos \frac{\pi l (n - \frac{1}{2})}{N}, \] (5.2)
and from this relation one can easily derive the general result

$$\frac{1}{n} \frac{\partial \text{Tr} \phi^n}{\partial a_{D,j}} = -2i \sum_{l=0}^{[n/2]-1} \binom{n-1}{l} \sin \frac{\pi j(n-2l-1)}{N}$$

(5.3)

for $SU(N)$. In $SU(3)$, $u = \frac{1}{2} \text{Tr} \phi^2$ and $v = \frac{1}{3} \text{Tr} \phi^3$, so that

$$\frac{\partial u}{\partial a_{D,j}} = -2i \sin \frac{\pi j}{N}, \quad \frac{\partial v}{\partial a_{D,j}} = -2i \sin \frac{2\pi j}{N}.$$  

(5.4)

The gauge couplings near the $\mathcal{N} = 1$ point have the structure

$$\tau_{ij}^D = \frac{1}{2\pi i} \log \left( \frac{a_{D,i}}{\Lambda_i} \right) \delta_{ij} + (1 - \delta_{ij}) \tau_{ij}^{\text{off}} (0) + \mathcal{O}(a_{Di}),$$

(5.5)

where $\Lambda_i/\Lambda \sim \sin (\pi i/N)$, and $\tau_{ij}^{\text{off}} (0)$, $i \neq j$ are the values of the off-diagonal entries of the coupling constant at the $\mathcal{N} = 1$ point $a_{Di} = 0$. For $SU(3)$, $\tau_{12} = \frac{\pi}{4} \log 2$ [27]. For $SU(N)$, the $\tau_{ij}^{\text{off}} (0)$ can be obtained from the results on the scaling trajectory in section 5 of [36].

To compute the theta function in magnetic variables, we have to take into account the change of the “electric” characteristics under the symplectic transformation

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(5.6)

to the magnetic variables $a_{Di}$. Using (4.16) we find

$$\vec{\alpha} = (1/2, \ldots, 1/2), \quad \vec{\beta} = (0, \ldots, 0).$$

(5.7)

One can then obtain, at the $\mathcal{N} = 1$ point of $SU(3)$,

$$\frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta_D (0|\tau^D) = \frac{1}{4} \delta_{ij} - \frac{1}{12} (1 - \delta_{ij}).$$

(5.8)

Using (5.4) and (5.8), it is easy to check the relation (2.13) for the $\Lambda$ derivatives of $u$ and $v$ at this $\mathcal{N} = 1$ point. For $SU(N)$, the diagonal part of the matrix (5.8) is still of the form $(1/4) \delta_{ij}$, but the off-diagonal part is more involved and one needs the values of the couplings $\tau_{ij}^{\text{off}} (0)$.

With this information we can already discuss the structure of the condensates at the $\mathcal{N} = 1$ points, following the discussion in section 4 of [3]. At the point where $N - 1$ monopoles become massless, there is a symplectic basis for the hyperelliptic curve, such that the magnetic charge vectors are given by $n^a_j = \delta^a_j$, and the equation (4.42) becomes

$$\rho_1^2 = -\sum_j e^{i(\beta_j - \beta_i)} b_{ij} |a_j|^2 - \frac{e^{-i\beta_i}}{\sqrt{2}} \sum_{n=1,2} F_n b^{n,i}$$

(5.9)
At the $\mathcal{N} = 1$ point, $a_{D,i} = 0$ and the condensates are essentially given by the absolute value of the order parameters $b^i = (N/4\pi^2)\text{Im}(\partial H_{n+1}/\partial a^i)$, which in the case of $SU(3)$ can be obtained from (5.4)

$$b^1_i = -\frac{3}{2\pi^2} \sin \frac{\pi i}{3}, \quad b^2_i = -\frac{3}{4\pi^2} \sin \frac{2\pi i}{3}.$$  

(5.10)

Hence

$$\rho^2_1 = \sqrt{\frac{3}{2}} \frac{3}{4\pi^2} |F_1 + \frac{1}{2}F_2|, \quad \rho^2_2 = \sqrt{\frac{3}{2}} \frac{3}{4\pi^2} |F_1 - \frac{1}{2}F_2|.$$  

(5.11)

and we see that the soft breaking induced by the quadratic and cubic Casimirs gives rise to monopole condensation in both $U(1)$ factors, although the condensates (and therefore the string tension) are bigger for the soft breaking coming from $u$ (for equal values of the supersymmetry breaking parameters $F_1, F_2$). In the same way, the vacuum energy associated to these condensates is

$$V_{\text{eff}} = -b^{mn} F_m F^*_n = -\frac{9}{4\pi^2} \left(|F_1|^2 + \frac{1}{2}|F_2|^2\right).$$  

(5.12)

As expected, the soft breaking associated to $u$ gives lower energy to the vacuum.

### 5.2 The $Z_3$ vacua

Next we explore the behaviour near the Argyres-Douglas point at $v = 2\Lambda^3, u = 0$. It is convenient to use the parameters $\rho$ and $\epsilon$ introduced in [34] and defined by

$$u = 3\epsilon^2 \rho, \quad v - 2\Lambda^3 = 2\epsilon^3.$$  

(5.13)

The three submanifolds $\rho^3 = 1$ correspond to three massless BPS states which after an appropriate symplectic transformation can be seen to be charged with respect to only one of the $U(1)$ factors, with variables denoted by $a^1, a_{D,1}$. Using the symplectic transformation of [43], the charges of these states with respect to the $a^1, a_{D,1}$ are $(n_e, n_m) = (-1, 0), (1, -1)$ and $(0, 1)$, i.e. we have one electron, one dyon, and one monopole. These submanifolds come together at the AD point, where we have a nontrivial superconformal field theory. The two $U(1)$ ’s are weakly coupled near the AD point, and the hyperelliptic curve splits into a small torus (corresponding to two mutually nonlocal periods $a^1, a_{D,1}$ which go to zero) and a big torus with periods $a^2, a_{D,2} \sim \Lambda$. The small torus is given by the elliptic curve

$$w^2 = z^3 - 3\rho z - 2,$$  

(5.14)
and the meromorphic Seiberg-Witten differential degenerates on (5.14) to

$$\lambda_{SW} = \frac{1}{2\pi \Lambda^{3/2}} w dz. \quad (5.15)$$

The matrix $\partial a^i / \partial u_j$ near the AD point reads, at leading order [13]:

$$\begin{pmatrix}
\frac{\partial a_1}{\partial u_1} & \frac{\partial a^1}{\partial v} \\
\frac{\partial a_2}{\partial u_1} & \frac{\partial a^2}{\partial v}
\end{pmatrix} = \begin{pmatrix}
\frac{2}{\pi \Lambda^{3/2}} \eta & -\frac{e^{1/2}}{4\pi \Lambda^{1/2} \omega_p} \\
\frac{c}{\pi} & \frac{d}{\pi}
\end{pmatrix}, \quad (5.16)
$$

where $\omega_p$ is the period of the elliptic curve (5.14) corresponding to $a_1$ (with $\text{Im}(\omega_p / \omega) > 0$), $\eta = \zeta(\omega_p / 2)$ is the value of the Weierstrass zeta function at the half-period, and $c, d$ are nonzero constants (which can be obtained from the explicit computations in [37, 38]). For the dual variables we have similar expressions with $\omega_{p,D}, \eta_D, c_D$ and $d_D$. Using these expressions one can obtain the matrix of couplings near the AD point [13, 34, 38]

$$\tau_{11} = \tau(\rho) + O(\epsilon),$$
$$\tau_{12} = -\frac{2i}{c \Lambda^{1/2} \omega_p} + O(\epsilon^{3/2}),$$
$$\tau_{22} = \omega + O(\epsilon), \quad (5.17)$$

where $\omega = e^{\pi i / 3}$.

To analyze the theta function in these variables, we need the symplectic transformation from the electric variables to the variables appropriate for the large and the small torus. We first compute the transformation of the characteristics under this symplectic transformation. Using (4.16) and the results in [13] we find

$$\vec{\alpha} = \vec{\beta} = (1/2, 1/2). \quad (5.18)$$

We can already obtain the behaviour of the theta function as an expansion in $\epsilon$:

$$\Theta(0|\tau) = -\frac{1}{\pi c \Lambda^{1/2}} \frac{\epsilon^{1/2}}{\omega_p} \vartheta_1(0|\tau(\rho)) \vartheta_1'(0|\omega) + O(\epsilon^{3/2}), \quad (5.19)$$

where $\vartheta_1(\xi|\tau)$ is the Jacobi theta function with characteristic $[1/2, 1/2]$. Using that

$$\frac{\vartheta'''_1(0|\tau)}{\vartheta''_1(0|\tau)} = -\pi^2 E_2(\tau), \quad (5.20)$$

we find

$$\frac{1}{i\pi} \partial_{\tau_{ij}} \log \Theta = \begin{pmatrix}
\frac{1}{4} E_2(\tau(\rho)) & \frac{c \Lambda^{1/2}}{4\pi} e^{-1/2} \omega_p \\
\frac{c \Lambda^{1/2}}{4\pi} e^{-1/2} \omega_p & \frac{1}{4} E_2(\omega)
\end{pmatrix}. \quad (5.21)$$
Again, using (5.16) and (5.21), one can check the relation (2.13) for \( v \) (for \( u \) one needs the explicit values of the constants appearing in the above expressions).

The analysis of the condensates near the AD point is difficult because one has to take into account mutually nonlocal degrees of freedom, and there is not a Lagrangian description of this theory. In fact, one expects that, in the softly broken theory, a cusp singularity will appear in the effective potential near the AD point, as it happens in \( \mathcal{N} = 2 \) QCD with gauge group \( SU(2) \) and one massive flavour [6]. But we can analyze the monopole condensates along the divisors \( \rho^3 = 1 \) and their evolution as we approach the AD point. Near each of the submanifolds \( \rho^3 = 1 \) there is a massless BPS state, and we expect it to condense after breaking supersymmetry down to \( \mathcal{N} = 0 \). These condensates correspond to mutually nonlocal states, but we can assume, as in [5, 6], that these states do not interact and that the condensates are given by the equation

\[
\rho_k^2 = -\frac{1}{(b-1)_{11}}|a_k|^2 - \frac{e^{-i\beta_k}}{\sqrt{2(b-1)_{11}}} \sum_{n=1,2} F_n(b^{-1})_{1j} b^n_{j},
\]

(5.22)

where \( k = 1, 2, 3 \) and \( a_k \) are the appropriate local coordinates for each of the massless states (i.e. \( a_k = a^1_k, a^2_{D,1}, a^3 - a^2_{D,1} \)). The equation (5.22) can be obtained from (4.42) taking into account that the states are only charged with respect to the first \( U(1) \) factor. The quantities \( (b^{-1})_{ij}, b^n_j \) should be also computed in the duality frame dictated by the \( a_k \). For example, for \( a^1 \) we use the “electric” period of the \( \rho \) curve, \( \omega_{\rho} \), and for \( a^2_{D,1} \) we use the “magnetic” period \( \omega_{\rho,D} \). The approximation where the mutually nonlocal states do not interact should be good far enough from the AD point. These condensates give only a magnetic Higgs mechanism in one of the \( U(1) \) factors, and correspond to the half-Higgsed vacua of [34]. Notice that one should perform a careful numerical study of the equations for the condensates and for the effective potential to know if these partial condensates give the true vacua of the \( \mathcal{N} = 0 \) theory. As we approach the AD point, \( \epsilon \rightarrow 0 \), we see that the parameters for condensation go to zero for both the quadratic and the cubic Casimir:

\[
\frac{\partial u}{\partial a^1}, \quad \frac{\partial v}{\partial a^1} \sim \mathcal{O}(\epsilon^{1/2}),
\]

(5.23)

and the mass gap associated to the condensates vanishes at the AD point, like in the \( \mathcal{N} = 1 \) breaking considered in [34].

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