Minimal Non-orthogonal Gate Decomposition for Qubits with Limited Control

Xiao-Ming Zhang,1,2 Jianan Li,3 Xin Wang,2† and Man-Hong Yung1,4,5,

1Shenzhen Key Laboratory of Quantum Science and Engineering, Shenzhen University of Science and Technology, Shenzhen 518055, China
2Department of Physics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong SAR, China, and City University of Hong Kong Shenzhen Research Institute, Shenzhen, Guangdong 518057, China
3Department of Physics, Southern University of Science and Technology, Shenzhen 518055, Guangdong, China
4Shenzhen Key Laboratory of Quantum Science and Engineering, Southern University of Science and Technology, Shenzhen, 518055, China
5Central Research Institute, Huawei Technologies, Shenzhen, 518129, China
(Dated: October 18, 2018)

In quantum control theory, a question of fundamental and practical interest is how an arbitrary unitary transformation can be decomposed into minimum number of elementary rotations for implementation, subject to various physical constraints. Examples include the singlet-triplet (ST) and exchange-only (EO) qubits in quantum-dot systems, and gate construction in the Solovay-Kitaev algorithm. For two important scenarios, we present complete solutions to the problems of optimal decomposition of single-qubit unitary gates with non-orthogonal rotations. For each unitary gate, we give the criteria for determining the minimal number of pieces, the explicit gate construction procedure, as well as a computer code for practical uses. Our results include an analytic explanation to the four-gate decomposition of EO qubits, previously determined numerically by Divincenzo et al [Nature, 408, 339 (2000)]. Furthermore, compared with the approaches of Ramon sequence and its variant [Phys. Rev. Lett., 118, 216802 (2017)], our method can reduce about 50% of gate time for ST qubits. Finally, our approach can be extended to solve the problem of optimal control of topological qubits, where gate construction is achieved through the braiding operations.

A universal gate set for quantum computation can be constructed by any two-qubit entangling gate, together with arbitrary single-qubit gates [1]. In the laboratory, elementary single-qubit gates are normally constructed by switching on and off an external field at certain times (i.e., a square pulse), resulting in a rotation of a Bloch vector along certain axis of the Bloch sphere. The question is, how to optimize the use of these elementary rotations to form arbitrary single-qubit gates? The question becomes crucial for quantum platforms where controls are limited, for example in quantum dot systems [2, 3]. Consequently, a general rotation needs to be decomposed into a sequence of elementary rotations around non-parallel axes. In fact, this “piecewise” decomposition of general operations has inspired the development of composite pulses, which play an important role in quantum control on various types of qubits [4, 10].

Typically, one would like to reduce the complexity of gates: a long sequence of elementary gates implies the need of frequent switching of the applied field. Therefore, a minimal decomposition of single-qubit gates in terms of elementary rotation is of practical and fundamental interest in quantum computing. For the cases where the available elementary controls are rotations around two orthogonal axes, it is well known that arbitrary rotations can be constructed with a three-piece sequence alternating between the two axes [1], for example the z-x-z or x-z-x sequence [11].

However, in many systems, the available elementary rotations are non-orthogonal. For instance, for a singlet-triplet (ST) qubit [2], the x-rotation can be achieved via a magnetic field gradient [12–15], but a pure z-rotation is hardly achievable as the magnetic field gradient has to be completely turned off during execution of a gate, which is impractical. As another example, control of an exchange-only (EO) qubit [16] is only available via two rotation axes 120° apart from each other.

In the literature of quantum dots, much effort has been made to optimize gate sequences involving rotations around a pair of non-orthogonal axes [16, 18, 23]. In particular, if the rotation axes along ẑ and ẑ + ẑ are both available (with an angle 45°), as is typically the case for an ST qubit, a Hadamard gate can convert an x-rotation to a z-rotation, so that an x-z-x sequence can be replaced by a five-piece sequence, namely x-Hadamard-z-Hadamard-x [6, 10]. Moreover, Ramon [19] pointed out that if the acute angle between the two available axes (denoted as ẑ and ẑ) is greater than 45°, the Hadamard gate can be replaced by the rotational gate around ẑ to reduce the gate time [21, 23]. However, the resulting Ramon sequence, namely x-m-x-m-x, still contains five pieces of elementary gates. In the context of controlling quantum-dot qubits, it remains an outstanding problem whether a more efficient decomposition with non-orthogonal elementary rotations is possible.

In an early study of the EO qubit, Divincenzo et al. numerically found that four-piece sequences can be constructed for almost all quantum gates [16], but no analytical explanation was given. On the other hand, in applying the Solovay-Kitaev theorem [1], it was believed that
an arbitrary gate can be decomposed into three pieces \[ U \], but the problem turns out to be far more complicated. Furthermore, in quantum-dot systems including the ST qubit, instead of a pair of fixed rotational axes, it is also possible to access elementary rotational gates for a certain range of rotational angles. The problem of gate sequence optimization depends on the accessible range of rotational angles. Particularly, if the range covers the entire plane, only two pieces of elementary rotations are sufficient for the decomposition of any rotation \[ G \].

Here, we present a complete solution to the problem of minimal decomposition of single-qubit transformation, in terms of non-orthogonal elementary gates. For applications, we focus on two types of quantum-dot qubits, exemplified by the ST and EO qubit respectively; our results can also be extended to other quantum systems with limited control capability.

**Definitions**—A single-qubit rotation, \( R(\hat{n}, \phi) \), around the axis \( \hat{n} = (\sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta) \) for an angle \( \phi \in [0, 4\pi] \), can be generically described by

\[
R(\hat{n}, \phi) = \exp[-i(\sigma \cdot \hat{n})\phi/2],
\]

where \( \sigma = [\sigma_x, \sigma_y, \sigma_z]^T \) contains the Pauli matrices. We are interested in how a unitary gate \( U(\theta, \psi, \phi) \) (up to an overall phase factor) can be minimally decomposed into a sequence of elementary rotations, \( R_i = R(\hat{n}_i, \phi_i) \) in a given set \( \mathcal{G} = \{R(\hat{n}_i, \phi_i)\} \) limited by physical constraints.

For convenience, we define the \( p \)-power of a set \( \mathcal{G} \) to contain all combinations of products of \( p \) elementary rotations, i.e., \( \mathcal{G}^p = \{\prod_{i=1}^p R_i | R_i \in \mathcal{G}\} \). Our task is to solve the following decomposition:

\[
U(\theta, \psi, \phi) = \prod_{i=1}^p R_i \in \mathcal{G}^p,
\]

subject to the condition, \( \hat{n}_i \neq \hat{n}_{i+1} \). Here \( p \) is referred to as "number of pieces". Of course, for each \( U \) the solution of \( p(U) \) satisfying the decomposition is not unique; in fact, there are infinitely many possible solutions.

The goal of this work is to determine the minimum value \( p_{\text{min}}(U) \) for any given unitary transformation \( U(\theta, \psi, \phi) \). Furthermore, the explicit procedure (see Fig. 2) in constructing the minimum decomposition is also provided as a matlab code [25] for practical uses. Our main results are summarized as follows.

**Main results for Type-I qubits**—For Type-I, the rotation axes are allowed to vary in a limited range of a plane enclosed by the boundary rotation axes denoted by \( \hat{z} \) and \( \hat{m} = (\sin \Theta, 0, \cos \Theta) \); the angle between the boundary axes are given by \( \Theta = \arccos \hat{z} \cdot \hat{m} \in [0, \pi] \) [see Fig. 1 (a)]. We define the set containing all possible elementary rotations by \( \mathcal{G}_z \equiv \{R(\hat{z}, \phi) | \hat{z} = (\sin \theta, 0, \cos \theta)\} \).

First of all, we obtain sufficient and necessary conditions [25] for the classes of unitary gates \( U \) decomposable with one \( (p=1) \) and two \( (p=2) \) steps. Furthermore, for those unitary gates \( U \) requiring at least \( p \geq 3 \), we show how the problem of non-orthogonal gate decomposition can be reduced to the case where the only allowed rotations axes are located at the boundaries \( \hat{z} \) and \( \hat{m} \) (without loss of generality), i.e., \( U = R(\hat{z}, \phi)R(\hat{m}, \star)R(\hat{z}, \phi) \cdots \) where the asterisks indicate angles. Therefore, we can focus on the elementary rotations formed by the joint set of rotations:

\[
\mathcal{G}_b \equiv \mathcal{G}_z \cup \mathcal{G}_m ,
\]

where \( \mathcal{G}_z \equiv \{R(\hat{z}, \phi)\} \) and \( \mathcal{G}_m \equiv \{R(\hat{m}, \phi)\} \). This becomes essentially the same problem as the Type-II qubit to be discussed below.

More precisely, for \( p \geq 3 \), we divide our results into two parts, (i) \( \Theta \in (0, \pi/2) \) and (ii) \( \Theta \in [\pi/2, \pi) \), summarized by the following theorem:

**Theorem 1 (Bulk-to-boundary mapping)** (i) For \( 0 < \Theta < \pi/2 \), if a unitary gate \( U \) can be decomposed to \( p \geq 3 \) pieces, \( U \in \mathcal{G}_b^p \), it can always be decomposed into \( p \) pieces with rotation axes at the boundary, i.e.,

\[
\mathcal{G}_b^p = \mathcal{G}_b^p.
\]

(ii) for \( \Theta \geq \pi/2 \), one can always apply the orthogonal \( z-x-z \) decomposition for any single-qubit unitary gate with \( p = 3 \) pieces.

In the existing ST qubits literatures [19, 21, 23], the single-qubit gates are typically decomposed into five or more pieces; our results show that as long as \( \Theta \geq \pi/3 \), all target rotation can be decomposed to four or even less number of pieces [see Eq. (8) and (9) below]. Specifically, when \( J = 30h \) which is a typical experiment value [26, 27], we have found that for the set of 24 Clifford gates, 10 gates can be realized with \( p_{\text{min}}(U) = 2 \), and 13 gates with \( p_{\text{min}}(U) = 3 \). Furthermore, we have compared the performance of our method with previous alternative decompositions [19, 21, 23]. The results indicate that our decomposition offers a significant improvement in reducing both gate time and gate error (see Fig. 1 (d) and Fig. 3).

**Main results for Type-II qubits**—For Type-II, only elementary rotations with two fixed axes are allowed, for example, \( \hat{z} \) and \( \hat{m} \), where the angle between them are given by \( \Theta = \arccos \hat{z} \cdot \hat{m} \in (0, \pi/2) \). The set containing all elementary rotations are given by \( \mathcal{G}_b \) [see Eq. (3) and Fig. 1 (b)]. For any given unitary gate \( U \) and angle \( \Theta \), we have solved the problem of minimal gate decomposition, in terms of a pair of inequalities [see Eq. (5) and (9)].

From the experimental point of view, it is of interest to determine the optimal number of decomposition applicable for all possible unitary transformations, i.e.,

\[
q_{\text{min}} \equiv \max_U p_{\text{min}}(U).
\]

In principle, the values of \( q_{\text{min}} \) for Type-I and Type-II qubits can be different, as they are subject to different physical constraints. However, we prove that the values
II qubits. Rotation axes are fixed to be either \( \hat{z} \) or \( \hat{m} \). (c) The minimum number of pieces for all possible rotations (\( q_{\text{min}} \)) for Type-I qubits. The color scale represents different \( q_{\text{min}} \) when \( \hat{m} \) lies in the corresponding area. (d) Gate time comparison for ST qubits. Red: five-piece "Ramon" sequence [19]; blue: minimal decomposition proposed in this work. Target gates are \( U(\theta, \pi/2, \phi) \), and we set \( J_{\text{max}} = 30h \).

of the \( q_{\text{min}} \) are identical for both Type-I and Type-II qubits.

In particular, for EO qubits, where two available rotation axes are fixed with relative angle \( \Theta = \pi/3 \), our results imply that the minimum number of pieces is given by \( q_{\text{min}} = 4 \), which represents an analytic explanation to the numerical results obtained by Divincenzo et al in 2000 [16].

Singlet-Triplet (ST) qubits—The ST qubit is constructed by double quantum dots in the following computational basis, \( |0\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2} \) and \( |1\rangle = (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2} \), and the Hamiltonian [2] for a ST qubit is given by

\[
H_{ST} = h \sigma_x + J \sigma_z, \tag{6}
\]

where \( h \) is the magnetic field gradient. In the laboratory, the value of \( h \) is usually fixed. The term \( J \) characterizes the exchange interaction that can be varied dynamically. However, the value of \( J \) is bounded within a certain range, \( 0 \leq J \leq J_{\text{max}} \); when \( J > J_{\text{max}} \) exceeds the maximum value \( J_{\text{max}} \), the qubit behaves more like a charge qubit, where decoherence would be significantly increased [2]. In other words, if we let \( \Theta = \arctan(J_{\text{max}}/h) \), the available rotations for the ST qubit is given by \( \mathcal{G}_{ST} = \{ R(\hat{n}, \phi) | \hat{n} = (\sin \theta, 0, \cos \theta), \theta \in [\pi/2 - \Theta, \pi/2] \} \).

Exchange-only (EO) qubits.—On the other hand, the EO qubit is constructed by a coupled triple-quantum-dot system [10]. In the computational basis defined by \( |0\rangle = 1/\sqrt{2}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \) and \( |1\rangle = 1/\sqrt{6}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) - 2/\sqrt{6}|\uparrow\downarrow\rangle \), the Hamiltonian in this subspace can be written as,

\[
H_{EO} = J_{23} \sigma_z - J_{12} \left( \frac{1}{2} \sigma_x - \frac{\sqrt{3}}{2} \sigma_y \right), \tag{7}
\]

where \( J_{12} \geq 0 \) and \( J_{23} \geq 0 \) are coupling constants between the neighboring dots. However, it remains an experimental challenge to simultaneously apply both coupling, which means that either \( J_{23} \) or \( J_{12} \) should be non-zero at each moment of time. In other words, we assume only elementary rotations around \( \hat{z} \) or another axis \( \sqrt{3}\hat{x}/2 - \hat{z}/2 \) can be applied, i.e., \( \mathcal{G}_{EO} = \{ R(\hat{n}, \phi) | \hat{n} = \hat{z} \text{ or } \hat{n} = \sqrt{3}\hat{x}/2 - \hat{z}/2 \} \).

Details of Type-I qubits.—Here, the available elementary rotations are given by \( \mathcal{G}_I \) with \( \Theta \in (0, \pi/2) \) [28]. Below, we will present all the cases where Eq. (2) can be
satisfied for $G = G_{\xi}^z$ and $\Theta \in (0, \pi)$ with a certain value of $p$ (see proofs in supplementary materials [25]). For $p = 1$, Eq. [2] can be satisfied, if and only if one of the following conditions are satisfied: (i) $\theta \in [0, \Theta]$ and $\psi = 0$, (ii) $\phi \in [0, 2\pi]$ (identity rotation) or (iii) $\theta = 0$ (around $\hat{z}$). For $p = 2$, Eq. [2] can be satisfied, if and only if one of the following conditions are satisfied (i) $\phi \in [0, 2\pi]$, (ii) $\theta = 0$, (iii) $\max \{\cot \xi_{\pm}\} \geq \cot \Theta$, with $\cot \xi_{\pm} = \pm (s_{c0}c_{s\phi}/z_{c0})/(s_{c0}z_{s\phi})$, where we defined $c_{\phi} \equiv \cos x$ and $s_{\phi} \equiv \sin x$. (iv) $\Theta = \pi$. In case (iv), the rotation axes can be chosen freely in the entire $x$-$z$ plane; two pieces are sufficient, which is consistent with the result in Ref. [24].

For $p \geq 3$, the results have been summarized in Theorem 1. When (i) $\Theta \in (0, \pi/2)$, it can be reduced to the Type-II with same $\Theta$ apart, so the existence of $p$-piece decomposition is determined by Eq. [8] and [9] below; when (ii) $\Theta \in (\pi/2, \pi)$, decomposition with $p \geq 3$ pieces always exist. The case of (ii) is obvious. We briefly sketch the proof procedure of case (i) here (see [25] for full details):

**Proof** (Sketch) We define the product of two set $G_1, G_2$ as $G_1 G_2 \equiv \{R|R = R_1 R_2, R_1 \in G_1, R_2 \in G_2\}$. (i) We show that $G_{\xi} \subset G_{\xi}^z$, which means that the product of any two rotations in $G_{\xi}$ is equal to the product of a rotation at the boundary $G_b$ and another rotation in $G_{\xi}$. (ii) This result implies that $G_{\xi}^z = G_{\xi}^z \subset G_{\xi}$. (iii) We always show that the product of any two rotations in $G_{\xi}$ can be decomposed in the form of $z$-$m$-$z$ and $m$-$z$-$m$, i.e., $G_{\xi} \subset G_{\xi} G_{\xi} G_{\xi} \cap G_{\xi} G_{\xi} G_{\xi}$.}

**Optimal control of ST qubits**—Here, we apply above results to the ST qubit described by Eq. (6). Since optimal control of ST qubits depends on the charge noise are drawn from $N(0, \sigma^2_{\tilde{J}/F})$. In Fig. 3(a), we set $\sigma_{\tilde{J}/F} = 0.00246$ and $\sigma_{h}/h = 0.575$ which are typical values of the noise in GaAs quantum dots [26]. The results of other noise levels are given in Fig. 3(b)-(d). It shows that the minimal decomposition schemes has substantial improvement in the robustness against nuclear spin noise.

**Details for Type-II qubits**—Different from Type-I, available elementary rotation now are given by $G_{\theta}$ with $\Theta \in (0, \pi/2)$ [23]. We present a set of constraints imposed to the rotation parameters for the decomposition described by Eq. [2] with $G = G_b$ (full proof is given in [25]). The constraint is different when the number of pieces $p$ is an odd or an even. (i) For the odd-piece decomposition, i.e., $p = 2l - 1$, for some $l \in \mathbb{Z^+}$, the decomposition in Eq. [2] can be satisfied for a given rotation if and only if

$$\delta_x < \Theta(l - 1),$$

where the value of $\delta_x \equiv \min \{\delta_1(\theta, \phi), \delta_2(\theta, \psi, \phi, \Theta)\}$ is taken to be the minimum value between $\delta_1(\theta, \phi) \equiv \sin^{-1}(s_{\phi}^2/s_{\phi}/2)$ and $\delta_2(\theta, \psi, \phi, \Theta) \equiv \sin^{-1}(c_{\phi}c_{\psi}/2 + (c_{\phi}c_{\psi} - c_{\phi}c_{\phi})^2) + (c_{\phi}c_{\psi}/2)^2).

Furthermore, the form of $\delta_x$ determines the resulting sequence. For the cases where $\delta_x = \delta(\theta, \phi)$, Eq. [2] can be constructed by the following sequence: $U(\theta, \psi, \phi) = R(\hat{z}, *) R(\hat{\theta}, *) \cdots$; if $\delta_x = \delta' (\theta, \psi, \phi, \Theta)$, Eq. [2] can be constructed in the form of $U(\theta, \psi, \phi) = R(\hat{\theta}, *) R(\hat{\psi}, *) \cdots$.

(ii) For the even-piece decomposition where $p = 2l$, the decomposition in Eq. [2] can be satisfied for a given
rotation, if and only if

\[ \Lambda_s \leq \Theta(l-1), \]

where \( \Lambda_s \equiv \min \{ \Lambda(\theta, \psi, \phi, \Theta), \Lambda(\theta, \psi, -\phi, \Theta) \} \) is taken to be the minimum of \( \Lambda(\theta, \psi, \phi, \Theta) \) defined as follows:

\[ A \equiv (c_\theta c_\phi s_\theta s_{\phi/2} - s_\theta c_\phi s_{\phi/2})^2 + (s_\theta c_\phi s_\phi s_{\phi/2} - s_\theta c_\phi s_{\phi/2})^2, \quad B \equiv (s_\phi s_{\phi/2})^2, \quad C \equiv s_\theta s_\phi s_{\phi/2}(s_\phi s_{\phi/2} c_\theta - c_\phi c_{\phi/2}). \]

**Minimum number of pieces for all possible \( U \)** — It is known that [25], for Type-II qubits, all rotations can be decomposed to \( p \geq 3 \) pieces, if and only if \( \Theta \geq \pi/(p-1) \) (see [25] for alternative proof), which implies

\[ q_{\text{min}} = \left[ \frac{\pi}{\Theta} \right] + 1. \]

From Theorem [1], when \( \Theta \in (0, \pi) \), \( q_{\text{min}} \) is the same for both Type-I and Type-II qubits. Moreover, when \( \Theta = \pi \), criteria (iv) for \( p = 2 \) indicates that \( q_{\text{min}} = 2 \). Therefore, we can conclude that Eq. [10] also holds for Type-I qubit. An illustration of \( q_{\text{min}} \) is given in Fig. [1](c).

To conclude, we have studied the minimal decomposition for two types of qubits: rotation axes are restricted to a range of a plane (Type-I), and rotation axes are fixed at two directions (Type-II). We also present an explicit procedure for minimally applying the elementary gates for an arbitrary single-qubit transformation. Furthermore, we discuss the implications of minimal decomposition for ST qubit, providing numerical evidences showing the effectiveness and robustness of our decomposition. Finally, we provide a code online [25] for experimentalists, who just need to input a target rotation; the code will generate the explicit minimal decomposition. The combination of our work with dynamical decoupling [4, 6, 10] or geometric control [29, 31] may be interesting in the future.

**ACKNOWLEDGEMENTS**

We thank Chengxian Zhang for helpful discussion. This work is supported by the National Natural Science Foundation of China (No. 11604277, 11875160), the Guangdong Innovative and Entrepreneurial Research Team Program (No. 2016ZT06D348), the Research Grants Council of the Hong Kong Special Administrative Region, China (No. CityU 21300116, CityU 11303617), Natural Science Foundation of Guangdong Province (2017B030308003), and the Science, Technology and Innovation Commission of Shenzhen Municipality (JCYJ2017031417390376, JCYJ20170817105046702, ZDSYS201703031659262).

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* x.wang@cityu.edu.hk
† yung@stustc.edu.cn

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## Supplementary material

In this Supplemental Material we provide necessary proofs to claims made in the main text. We will discuss the two types of qubits mentioned in the main text in order: Type I which axes are allowed to vary in a range, and Type II which with two fixed axes. We also provide some instruction about our matlab code for constructing explicit minimal decomposition sequences.

### I. DEFINITION

To facilitate the discussions, for $R(\hat{n}, \phi)$ with $\hat{n} = (\sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta)$, we parametrize it as:

$$R(\hat{n}, \phi) \equiv R(\theta, \psi, \phi) \equiv \begin{bmatrix} \cos \frac{\phi}{2} - \sin \frac{\phi}{2} \cos \theta & -i \sin \frac{\phi}{2} \sin \theta e^{-i\psi} \\ -i \sin \frac{\phi}{2} \sin \theta e^{i\psi} & \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \cos \theta \end{bmatrix},$$

(S-1)

where $\theta \in [0, \pi)$, $\psi \in [0, \pi)$, $\phi \in [0, 4\pi)$ unless otherwise specified. For clarity, we represent all target unitary transformation as $U(\theta, \psi, \phi) \equiv R(\theta, \psi, \phi)$. Inversely, given $R(\theta, \psi, \phi)$, one can calculate angles as follows, which are important for the actual construction of the decomposition:

If $\text{Re}(e_{21}) > 0$,

$$\psi = \text{Arg}(ie_{21}), \quad \phi = 2 \arccos[\text{Re}(e_{11})], \quad \theta = \left\{ \begin{array}{ll} \arccos \left( \frac{\text{Im}(e_{11})}{\sin(\phi/2)} \right), & \phi \neq 0, 2\pi \\ 0, & \phi = 0 \text{ or } 2\pi \end{array} \right. \quad \text{(S-2a, S-2b, S-2c)}$$

If $\text{Re}(e_{21}) < 0$,

$$\psi = \text{Arg}(-ie_{21}), \quad \phi = 4\pi - 2 \arccos[\text{Re}(e_{11})], \quad \theta = \left\{ \begin{array}{ll} \arccos \left( \frac{\text{Im}(e_{11})}{\sin(\phi/2)} \right), & \phi \neq 0, 2\pi \\ 0, & \phi = 0 \text{ or } 2\pi \end{array} \right. \quad \text{(S-3a, S-3b, S-3c)}$$

If $\text{Re}(e_{21}) = 0$,

$$\psi = 0, \quad \theta = \text{Arg}[\text{Im}(e_{11}) - e_{21}], \quad \phi = \left\{ \begin{array}{ll} 2 \text{Arg}[\text{Re}(e_{11}) - i \frac{\text{Im}(e_{21})}{\cos \theta}], & \phi \neq \pi/2 \\ 2 \text{Arg}(e_{11} - e_{21}), & \phi = \pi/2 \end{array} \right. \quad \text{(S-4a, S-4b, S-4c)}$$
Furthermore, we define the set for all possible rotations as:

\[ \mathcal{A} \equiv \{ R(\theta, \psi, \phi) | \theta \in [0, \pi), \psi \in [0, \pi), \phi \in [0, 4\pi) \} \].  

(S-5a)

For both \( \Theta \in (0, \pi] \) for Type I and \( \Theta \in (0, \pi/2] \) for Type II qubits, we define several sets of rotation with \( \phi \in [0, 4\pi) \):

- \( G_p \equiv \{ R(\theta, 0, \phi) | \theta \in [0, \pi), \phi \in (0, 4\pi) \} \), (with axis in \( x-z \) plane)  
  (S-5b)

- \( G_\xi \equiv \{ R(\theta, 0, \phi) | \theta \in [0, \Theta), \phi \in [0, 4\pi) \} \),  
  (all \( \xi \) rotations)  
  (S-5c)

- \( G_z \equiv \{ R(\hat{z}, \phi) | \phi \in [0, 4\pi) \} \), (all \( z \) rotations)  
  (S-5d)

- \( G_m \equiv \{ R(\hat{n}, \phi) | \hat{n} = (\sin \Theta, 0, \cos \Theta), \phi \in [0, 4\pi) \} \), (all \( m \) rotations)  
  (S-5e)

- \( G_\beta \equiv G_z \cup G_m \), (all rotations with axes at the boundary).  
  (S-5f)

and rotation with \( \phi \in (0, 2\pi) \):

- \( S_p \equiv \{ R(\theta, 0, \phi) | \theta \in [0, \pi), \phi \in (0, 2\pi) \} \),  
  (S-5h)

- \( S_p' \equiv \{ R(\theta, 0, \phi) | \theta \in (0, \pi/2), \phi \in (0, 2\pi) \} \),  
  (S-5i)

- \( S_\xi \equiv \{ R(\theta, 0, \phi) | \theta \in [0, \Theta), \phi \in (0, 2\pi) \} \),  
  (S-5j)

- \( S_m \equiv \{ R(\hat{n}, \phi) | \hat{n} = (\sin \Theta, 0, \cos \Theta), \phi \in (0, 2\pi) \} \),  
  (S-5k)

- \( S_z \equiv \{ R(\hat{n}, \phi) | \hat{n} = \hat{z}, \phi \in (0, 2\pi) \} \).  
  (S-5l)

Furthermore, given two set \( \mathcal{G}_1, \mathcal{G}_2 \), we define the product of them as:

\[ \mathcal{G}_1 \mathcal{G}_2 \equiv \{ R = R_1 R_2 | R_1 \in \mathcal{G}_1, R_2 \in \mathcal{G}_2 \} \],  

(S-6)

and for a set \( \mathcal{G} \), we define the \( p \)-power of it as

\[ \mathcal{G}^p \equiv \{ R = \prod_{i=1}^{p} R_i | R_i \in \mathcal{G} \} \].  

(S-7)

II. TYPE I: AXES RESTRICTED IN A RANGE

In this section, we are given axes that are allowed to vary in a range: \( \hat{n}_i = (\sin \theta, 0, \cos \theta) \), where \( \theta \in [0, \Theta] \), with \( \Theta \in (0, \pi] \). We will give the condition for decompositions to exist, and discuss how these decompositions can be constructed or reduced to a Type II qubit case.

A. Lemmas

We first provide several useful lemmas. To begin with, we show that arbitrary rotations can be decomposed into a \( z \)-rotation and another rotation with axis in the \( x-z \) plane.

**Lemma 1** Given any \( U(\theta, \psi, \phi) \in \mathcal{A} \), there exist certain \( R_{1,2}^z = R(\hat{z}, \phi_{1,2}) \in \mathcal{G}_z, R_- = R(\theta_-, 0, \phi_-) \in \mathcal{G}_p, R_+ = R(\theta_+, 0, \phi_+) \in \mathcal{G}_p \), such that

\[ U(\theta, \psi, \phi) = R_1^z R_- \],  

(S-8a)

and

\[ U(\theta, \psi, \phi) = R_+ R_2^z \].  

(S-8b)
Proof

Case I: $\phi \in \{0, 2\pi\}$ or $\theta = 0$
Eq. (S-8) can be satisfied by taking $\phi_{1,2} = \phi$ and $\phi_{\pm} = 0$.

Case II: $\phi \notin \{0, 2\pi\}$ and $\theta \neq 0$
It can be verified that Eq. (S-8) can be uniquely constructed as

\begin{align}
\theta_{\pm} &= \arccot \left( \frac{\pm \sin \psi \cos \frac{\phi}{2} + \cos \psi \sin \frac{\phi}{2} \cos \theta}{\sin \frac{\phi}{2} \sin \theta} \right), \\
\phi_{\pm} &= 2\pi + \left[ 2 \arccos \left( \cos \frac{\phi}{2} \cos \psi \mp \sin \frac{\phi}{2} \sin \psi \cos \theta \right) - 2\pi \right] \text{sgn} \left( \sin \frac{\phi}{2} \right), \\
\phi_1 &= 2\psi, \\
\phi_2 &= -2\psi \mod 4\pi.
\end{align}

In the following, we discuss the decomposition of the product of two rotations in $\mathcal{S}_\xi$.

Lemma 2 given $U_1 = U(\theta_1, 0, \phi_1) \in \mathcal{S}_\xi$, $U_2 = U(\theta_2, 0, \phi_2) \in \mathcal{S}_\xi$ with $\theta_1 < \theta_2$, and $\theta_3 \in [0, \theta_1]$, there exist unique value of $\phi_3$, and unique $R(\theta_4, 0, \phi_4) \in \tilde{\mathcal{S}}_p$, such that

\[ U_1 U_2 = R(\theta_3, 0, \phi_3) R(\theta_4, 0, \phi_4), \]  

and $\theta_4 \neq \theta_3$.

Proof

Existence of $\phi_3$ and $R(\theta_4, 0, \phi_4)$:
Let $\tilde{\theta}_{1,2} = \theta_{1,2} - \theta_3 \in [0, \theta]$, and define

\[ U(\tilde{\theta}_1, 0, \phi_1) U(\tilde{\theta}_2, 0, \phi_2) = U(\tilde{\theta}, \tilde{\psi}, \tilde{\phi}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \]

According to Lemma 1, there exist certain $R(0, 0, \phi_3) \in \mathcal{G}_z$, $R(\tilde{\theta}_4, 0, \phi_4) \in \mathcal{G}_p$, such that

\[ U(\tilde{\theta}, \tilde{\psi}, \tilde{\phi}) = R(0, 0, \phi_3) R(\theta_4, 0, \phi_4). \]

Since

\[ |\text{Re}[a_{12}]| = \left| \sin \frac{\phi_1}{2} \sin \frac{\phi_2}{2} \sin \tilde{\theta}_1 - \tilde{\theta}_2 \right| > 0, \]  

we have $\tilde{\phi} \in (0, 2\pi)$, and $\tilde{\theta} \neq 0$. And combining Eq. (S-11), Eq. (S-12), and Eq. (S-9), after some calculation, one can verify that

\begin{align}
\tilde{\theta}_4 &\in (0, \pi/2), \\
\phi_3 &\in (0, 2\pi), \\
\phi_4 &\in (0, 2\pi).
\end{align}

Then, we apply a transformation on Eq. (S-12) $S \rightarrow R(\hat{y}, \theta_3) R\tilde{S} R(\hat{y}, -\theta_3)$, which then becomes

\[ U(\hat{\theta}_1 + \theta_3, 0, \phi_1) U(\hat{\theta}_2 + \theta_3, 0, \phi_2) = R(\theta_3, 0, \phi_3) R(\theta_4 + \theta_3, 0, \phi_4) \]

\[ U(\theta_1, 0, \phi_1) U(\theta_2, 0, \phi_2) = R(\theta_3, 0, \phi_3) R(\theta_4, 0, \phi_4), \]

where $\theta_1 = \hat{\theta}_1 + \theta_3 \in [0, \pi)$. So obviously, $R(\theta_3, 0, \phi_3) \in \mathcal{S}_{xz}, R(\theta_4, 0, \phi_4) \in \mathcal{S}_p$.

$\theta_4 \neq \theta_3$: 

We denote $R(\check{\theta}_4, 0, \phi_4) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. According to Eq. (S-11) and Eq. (S-12), we have

$$\left| \sin \frac{\phi_4}{2} \sin \check{\theta}_4 \right| = |b_{12}| = |a_{12}| \geq |\text{Re}[a_{12}]| > 0.$$ (S-16)

Therefore, we have $\check{\theta}_4 \neq 0$, which means $\theta_4 \neq \theta_3$.

**Uniqueness:**

Suppose

$$U(\theta_1, 0, \phi_1)U(\theta_2, 0, \phi_2) = R(\theta_3, 0, \phi_3)R(\theta_4, 0, \phi_4) = R(\theta_3, 0, \phi'_3)R(\theta'_4, 0, \phi'_4),$$ (S-17)

for some $\phi_{3,4} \in (0, 2\pi), \phi'_{3,4} \in (0, 2\pi)$, and $\theta_4 \neq \theta_3, \theta'_4 \neq \theta_3$. We can denote

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} = R(\theta_3, 0, \phi_3 - \phi'_3) = R(\theta'_4, 0, \phi'_4)R(\theta_4, 0, -\phi_4).$$ (S-18)

One can find that

$$\text{Re}[e_{12}] = \text{Re} \left[ -i \left( \sin \frac{\phi_3 - \phi'_3}{2} \sin \theta_3 \right) e^{-i0} \right] = 0 = \sin \frac{\phi'_4}{2} \sin \frac{\phi_4}{2} \sin(\theta'_4 - \theta_4).$$ (S-19)

And since $\phi_4, \phi'_4 \neq 0$, we have

$$\theta'_4 = \theta_4.$$ (S-20)

So Eq. (S-18) becomes

$$R(\theta_3, 0, \phi_3 - \phi'_3) = R(\theta_4, 0, \phi'_4 - \phi_4).$$ (S-21)

Since $\theta_4 \neq \theta_3$, and $\phi_{3,4} \in (0, 2\pi), \phi'_{3,4} \in (0, 2\pi)$, we have

$$\phi_3 = \phi'_3,$$ (S-22a)

$$\phi_4 = \phi'_4.$$ (S-22b)

Therefore, the values of $\theta_3, \phi_3, \phi_4$ are unique.

**Lemma 3** Given $U_1 = U(\theta_1, 0, \phi_1) \in \mathcal{S}_\xi$, $U_2 = U(\theta_2, 0, \phi_2) \in \mathcal{S}_\xi$, with $\theta_1 \neq \theta_2$, there exist certain $R^m \in \mathcal{R}_m$, $R^z \in \mathcal{R}_z$ and $R^\xi \in \mathcal{S}_\xi$, such that

(i) if $\theta_1 < \theta_2$

$$U_1U_2 = R^z R^\xi,$$ (S-23)

(ii) if $\theta_1 < \theta_2$

$$U_1U_2 = R^m R^\xi.$$ (S-24)

**Proof**

**Case I $\theta_1 < \theta_2$**:

According to Lemma 2 we can define the following implicit functions $\phi_3(x), \phi_4(x), y(x)$ that satisfy

$$R(x, 0, \phi_3(x))R(y(x), 0, \phi_4(x)) = U_1U_2 = \text{Const},$$ (S-25)

where $x \in [0, \theta_1], \phi_{3,4}(x) \in (0, 2\pi)$ and $y(x) \in (0, \pi)$. From Lemma 2 the above implicit functions have the following properties:
(1) \( y(x), \phi_{3,4}(x) \) are single-value functions (uniqueness);
(2) \( y(x) - x \neq 0; \)
(3) \( y(\theta_1) = \theta_2. \)

To prove case I of Lemma 3, we only need to show that \( y(\Theta) \in [0, \Theta]. \) We first evaluate the continuity and monotonicity of \( y(x). \) For an independent value \( x_0 \in [0, \theta_1], \) we always have

\[
R(x_0, 0, \phi_3(x_0))R(y_0, 0, \phi_4(x_0)) = R(x, 0, \phi_3(x))R(y(x), 0, \phi_4(x)),
\]

which can be rewritten as:

\[
R(x, 0, -\phi_3(x))R(x_0, 0, h(x_0)) = R(y, 0, \phi_4(x))R(y(x_0), 0, -\phi_4(x_0)) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.
\]

We note that

\[
\text{Re}[c_{21}] = \sin \frac{\phi_3(x_0)}{2} \sin \frac{\phi_3(x)}{2} \sin(x_0 - x) = \sin \frac{\phi_4(x_0)}{2} \sin \frac{\phi_4(x)}{2} \sin(y(x_0) - y(x)).
\]

Since \( \phi_{3,4} \in (0, 2\pi), \) when \( x \neq x_0, \) we have

\[
0 < \frac{\sin(y - y_0)}{\sin(x - x_0)} < \infty.
\]

And since \( y \in (0, 2\pi), \) \( y_0 \in (0, 2\pi), \) we have

\[
0 < \lim_{x \to x_0} \frac{y - y_0}{x - x_0} < \infty.
\]

Therefore, \( y(x) \) and \( y(x) - x \) are continuous. Since \( y(x) - x \neq 0 \) [property (2)], and \( y(\theta_1) - \theta_1 < 0 \) [property (3)], according to the intermediate value theorem of continuous function, we have \( y(0) - 0 < 0 \).

Moreover, from Eq. (S-30), we know that \( y(x_1) \leq y(x_2) \) if \( x_1 \leq x_2. \) Since \( y(\theta_1) < \Theta, \) and \( 0 \leq \theta_1, \) we have \( y(0) \leq y(\theta_1) < \Theta. \) Therefore, \( y(\Theta) \in [0, \Theta], \) and (i) of Lemma 3 hold true.

Case II: \( \theta_2 > \theta_1 \)

In this case, we first let \( \tilde{U}_{1,2} = R(\Theta/2, 0, \pi)U_{1,2}R(\Theta/2, 0, -\pi), \) one can verify that \( \tilde{U}_{1,2} = U(\Theta - \theta_{1,2}, 0, \phi_{1,2}) \in S_\xi. \) Since \( \Theta - \theta_1 < \Theta - \theta_2, \) according to case I of Lemma 3, there exist certain \( \tilde{R}_3 = R(\Theta, 0, \phi_3) \in S_m, \tilde{R}_4 = R(\tilde{\theta}_4, 0, \phi_4) \in S_\xi, \) such that

\[
\tilde{U}_1\tilde{U}_2 = \tilde{R}_3\tilde{R}_4,
\]

which is equivalent to

\[
U_1U_2 = \begin{bmatrix} R(\Theta/2, 0, -\pi)\tilde{R}_3R(\Theta/2, 0, \pi) \\ R(\Theta/2, 0, -\pi)\tilde{R}_4R(\Theta/2, 0, \pi) \end{bmatrix} = R(0, 0, \phi_3)R(\Theta - \tilde{\theta}_4, 0, \phi_4).
\]

Obviously, \( R(\Theta, 0, \phi_3) \in S_m, R(\Theta - \tilde{\theta}_4, 0, \phi_4) \in S_\xi. \) So (ii) of Lemma 3 also holds true.

In the following, we generalize the above result to a larger sets of rotations [those with \( \phi \in [0, 4\pi] \)].

**Lemma 4** Given \( U_1^\xi = U(\theta_1, 0, \phi_1) \in G_\xi, U_2^\xi = U(\theta_2, 0, \phi_2) \in G_\xi, \) there exist certain rotations \( R_{1,2}^\xi \in G_z, R_{1,2}^\xi \in G_\xi, \) and \( R_{1,2}^m \in G_m, \) such that either

\[
U_1^\xi U_2^\xi = R_{1,2}^m R_{1,2}^\xi = R_{1,2}^\xi R_{1,2}^m,
\]

or

\[
U_1^\xi U_2^\xi = R_{1,2}^\xi R_{1,2}^m = R_{1,2}^\xi R_{1,2}^m.
\]
Proof
We classify the domain of $\theta_{1,2}$ and $\phi_{1,2}$ into four cases:

**Case I** $\phi_1 \in \{0, 2\pi\}$ or $\phi_2 \in \{0, 2\pi\}$
We can take $R_{1,2}^m = R_{1,2}^m = R(0, 0, 0)$, or $R_{1,2}^\xi = R_{1,2}^\xi = R(0, 0, 2\pi)$.

**Case II** $\phi_{1,2} \in (0, 2\pi) \cup (2\pi, 4\pi)$, and $\theta_1 = \theta_2 = \theta$.
In this case, we have $U_1U_2 = R(\theta, \phi_1 + \phi_2)$, so
\[
U_1U_2 = R(\hat{\theta}, 0)R(\theta, 0, \phi_1 + \phi_2) = R(\theta, 0, \phi_1 + \phi_2)R(\hat{\theta}, 0),
\]
and
\[
U_1U_2 = R(\hat{\theta}, 0)R(\theta, 0, \phi) = R(\theta, 0, \phi)R(\hat{\theta}, 0),
\]
where $\phi = (\phi_1 + \phi_2) \mod 4\pi$. Since $R(\theta, 0, \phi) \in \mathcal{G}_\xi$, $R(\hat{\theta}, 0) \in \mathcal{G}_z$, $R(\hat{\theta}, 0, \phi) \in \mathcal{G}_m$, Lemma 4 hold true in this case.

**Case III** $\phi_{1,2} \in (0, 2\pi) \cup (2\pi, 4\pi)$, and $0 \leq \theta_2 < \theta_1 \leq \Theta$.
Let $\phi_{1',2'} = \phi_{1,2} \mod 2\pi$, and $U_{1',2'} = R(\theta_{1,2}, 0, \phi_{1',2'})$, we have
\[
U_1U_2 = \pm U_{1',2'},
\]
\[
U_1U_2 = \pm \left[(-U_{2'})^\dagger(-U_{1'})^\dagger\right]^\dagger.
\]
(taking the inverse twice)

Obviously, $U_{1',2'} \in \mathcal{S}_\xi$ and $(-U_{1',2'})^\dagger \in \mathcal{S}_\xi$, and since $\theta_2 < \theta_1$, we can apply Lemma 3(i) to the r.h.s. of Eq. (S-37a).

In other words, there exist certain rotations
\[
R_{1'}^m \in \mathcal{G}_m, \quad R_{1'}^\xi \in \mathcal{S}_\xi \subset \mathcal{G}_\xi,
\]
such that
\[
U_{1',2'} = R_{1'}^mR_{1'}^\xi.
\]
Similarly, we can apply Lemma 3(ii) to the r.h.s. of Eq. (S-37b). So there exist certain rotations
\[
R_{2'}^m \in \mathcal{G}_m, \quad R_{2'}^\xi \in \mathcal{S}_\xi \subset \mathcal{G}_\xi,
\]
such that
\[
(-U_{2'})^\dagger(-U_{1'})^\dagger = R_{2'}^\xi R_{1'}^\dagger,
\]
which also leads to
\[
[(U_{2'})^\dagger(U_{1'})^\dagger]^\dagger = (R_{1'}^\dagger)^\dagger(R_{2'}^\xi)^\dagger.
\]
Combining Eq. (S-37), (S-39), (S-42), we have
\[
U_1U_2 = \pm R_{1'}^mR_{1'}^\xi = \pm (R_{1'}^\dagger)^\dagger(R_{2'}^\xi)^\dagger.
\]
Since
\[
\pm R_{1'}^m \in \mathcal{G}_m, \quad \pm R_{1'}^\xi \in \mathcal{G}_\xi,
\]
\[
\pm (R_{1'}^\dagger)^\dagger \in \mathcal{G}_\xi, \quad \pm (R_{2'}^\xi)^\dagger \in \mathcal{G}_z,
\]
Lemma 4 hold true in this case.

**Case IV** $\phi_{1,2} \in (0, 2\pi) \cup (2\pi, 4\pi)$, $\theta_1 \not= \theta_2$ and $0 \leq \theta_1 < \theta_2 \leq \Theta$.
The prove of this case follows the same approach in case III.

Then, we have the following corollary:
Corollary 1

\[ G_\xi G_\xi = G_b G_\xi = G_\xi G_b. \]  \hfill (S-45)

**Proof**

From Lemma 4, we know that \( G_\xi G_\xi \subset (G_z G_\xi \cap G_\xi G_m) \cup (G_m G_\xi \cap G_\xi G_z) \). Since \( G_b \subset G_\xi \) and \( G_b = G_z \cup G_m \), we have \( G_\xi G_\xi = G_b G_\xi = G_\xi G_b \).

\[ \square \]

**B. \( p \geq 3 \) piece decomposition**

In the following, we will provide the proof for (i) of Theorem 1 in the main text.

**Theorem 2 (i)**

For \( p \in \mathbb{N}^* \), \( p \geq 3 \), and \( \Theta \in (0, \frac{\pi}{2}) \)

\[ G_\xi^p = G_b^p. \] \hfill (S-46)

**Proof**

From Corollary 1 we know that

\[ G_\xi G_\xi \subset G_b G_\xi \cap G_\xi G_b. \] \hfill (S-47)

One can verify from Theorem 4 (refer to section III) that when \( \Theta \in (0, \frac{\pi}{2}) \), all rotations in \( G_\xi \) can be decomposed into three pieces both in the form of \( z-m-z \) and \( m-z-m \). In other words, we have

\[ G_\xi \subset G_z G_m G_z \cap G_m G_z G_m, \] \hfill (S-48)

which also gives

\[ G_b G_\xi \subset G_z G_m G_z \cap G_m G_z G_m, \] \hfill (S-49a)

\[ G_\xi G_b \subset G_z G_m G_z \cap G_m G_z G_m. \] \hfill (S-49b)

Combining Eq. (S-47) and Eq. (S-49), we have

\[ G_\xi G_\xi \subset G_z G_m G_z \cap G_m G_z G_m. \] \hfill (S-50)

According to Eq. (S-47) we have

\[ (G_\xi)^p \subset G_b (G_\xi)^{p-1} \subset G_b G_b (G_\xi)^{p-2} \subset \cdots \subset (G_b)^{p-3} G_b G_\xi G_\xi. \] \hfill (S-51)

Combining Eq. (S-50), (S-51), and note that \( G_b = G_z \cup G_m \), we have

\[ G_\xi^p \subset G_b^p, \] \hfill (S-52)

and since

\[ G_b^p \subset G_\xi^p, \] \hfill (S-53)

we finally get

\[ G_\xi^p = G_b^p. \] \hfill (S-54)

\[ \square \]
C. \( p = 2 \) decomposition

Theorem 3

Given \( U(\theta, \psi, \phi) \in \mathcal{A} \),

\[
U(\theta, \psi, \phi) \in \mathcal{G}_z \mathcal{G}_\xi
\]

if and only if one of the following conditions is satisfied:

(i) \( \phi \in \{0, 2\pi\} \),

(ii) \( \theta = 0 \),

(iii) 
\[
\pm \sin \psi \cos \frac{\phi}{2} + \cos \psi \sin \frac{\phi}{2} \cos \theta \geq \cot \Theta
\]

is satisfied for either sign of ‘\( \pm \)’, or

(iv) \( \Theta = \pi \).

Proof

Case I (i) \( \phi \in \{0, 2\pi\} \) or (ii) \( \theta = 0 \):

Eq. (S-55) can always be constructed by taking \( R_1 = R(\hat{z}, \phi) \), \( R_2 = R(\hat{z}, 0) \).

Case II \( \phi \in (0, 2\pi) \cup (2\pi, 4\pi) \), and \( \theta \in (0, \pi) \):  
In such case, we should show that the existence of decomposition as Eq. (S-55) is equivalent to (iii) or (iv).  
According to Lemma \[\] \( U(\theta, \psi, \phi) \) can always be written as

\[
U(\theta, \psi, \phi) = R(\hat{z}, \phi_1)R(\theta, 0, \phi_-),
\]

\[
U(\theta, \psi, \phi) = R(\theta, 0, \phi_+)R(\hat{z}, \phi_2),
\]

for certain values of \( \phi_{1,2} \in [0, 4\pi) \), \( \phi_{\pm} \in [0, 4\pi) \), and

\[
\cot \Theta_{\pm} = \frac{-\sin \psi \cos \frac{\phi_{\pm}}{2} + \cos \psi \sin \frac{\phi_{\pm}}{2} \cos \theta}{\sin \frac{\phi_{\pm}}{2} \sin \theta},
\]

We notice that in case II, the values of \( \Theta_{\pm} \) are unique. We introduce the following statements

a. \( U(\theta, \psi, \phi) \) satisfies (iii) or (iv);

b. \( R(\theta, 0, \phi_-) \in \mathcal{G}_\xi \), or \( R(\theta, 0, \phi_+) \in \mathcal{G}_\xi \);

c. \( U(\theta, \psi, \phi) \in \mathcal{G}_z \mathcal{G}_\xi \cup \mathcal{G}_\xi \mathcal{G}_z \);

d. \( U(\theta, \psi, \phi) \in \mathcal{G}_z \mathcal{G}_\xi \).

From Eq. (S-58), one can verify that \( a \Leftrightarrow b \), and since the value of \( \Theta_{\pm} \) are unique, we have \( b \Leftrightarrow c \). From Lemma \[\] we know that \( \mathcal{G}_z \mathcal{G}_\xi = \mathcal{G}_z \mathcal{G}_\xi \cup \mathcal{G}_\xi \mathcal{G}_z \), so \( c \Leftrightarrow d \). Therefore, \( a \Leftrightarrow d \), and Theorem 3 holds.

III. DECOMPOSITION WITH TWO FIXED AXES

In this section, we are given two fixed axes \( \hat{z} = (0, 0, 0) \) and \( \hat{m} = (\sin \Theta, 0, \cos \Theta) \), and the angle between them is restricted to \( \Theta \in (0, \frac{\pi}{2}] \). We are going to prove the criteria for fixed-axes decomposition [Eq. (8) and Eq. (9) in the main text], and provide methods to construct the decomposition sequences.
A. Odd-piece decomposition

1. Criterion for odd-piece decomposition

For odd-piece decomposition, i.e. \( p = 2l - 1 \) with \( l \in \mathbb{Z}^+ \), Eq. (2) of the main text is equivalent to
\[
U(\theta, \psi, \phi) = R(\hat{z}, \beta_0) R(\hat{m}, \gamma_1) R(\hat{z}, \beta_1) \ldots R(\hat{m}, \gamma_{l-1}) R(\hat{z}, \beta_{l-1}),
\]
(S-59)
or
\[
U(\theta, \psi, \phi) = R(\hat{m}, \beta_0) R(\hat{z}, \gamma_1) R(\hat{m}, \beta_1) \ldots R(\hat{z}, \gamma_{l-1}) R(\hat{m}, \beta_{l-1}),
\]
(S-60)
where \( \beta_i \in [0, 4\pi) \), \( \gamma_i \in [0, 4\pi) \). We define
\[
\delta_1(\theta, \phi) = \arcsin \left( \frac{\sin \theta \sin \phi}{2} \right), \quad (S-61a)
\]
and
\[
\delta_2(\theta, \psi, \phi, \Theta) = \arcsin \left[ \frac{\sin \phi}{2} \sqrt{\left( \cos \Theta \cos \psi \sin \theta - \cos \theta \sin \Theta \right)^2 + (\sin \theta \sin \psi)^2} \right]. \quad (S-61b)
\]
Before giving the proof of theorem, we first provide some useful lemmas.

Lemma 5 (\(z\)-m-\(z\) decomposition)

Given \( U(\theta, \psi, \phi) \in A \), \( \Theta \in (0, \pi/2] \), and \( \Psi \in [0, \pi] \), there exist certain values of \( \beta'_0, 1 \in [0, 4\pi) \), \( \gamma'_1 \in [0, 4\pi) \), such that
\[
U(\theta, \psi, \phi) = R(\hat{z}, \beta'_0) R(\Theta, \Psi, \gamma'_1) R(\hat{z}, \beta'_1),
\]
(S-62)
if and only if \( |\delta_1(\theta, \phi)| \leq \Theta \).

Proof

Necessity of \( |\delta_1(\theta, \phi)| \leq \Theta \):
Let \( U(\theta, \psi, \phi) = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \), according to Eq. (S-1) and Eq. (S-61a), when Eq. (S-62) holds, we have
\[
|e_{12}| = \sin |\delta_1(\theta, \phi)| = \left| \sin \Theta \sin \frac{\gamma'_1}{2} \right|.
\]
Obviously, for \( |\delta_1(\theta, \phi)| > \Theta \), Eq. (S-63) cannot be satisfied for any \( \gamma'_1 \), so the decomposition as Eq. (S-62) does not exist.

Sufficiency of \( |\delta_1(\theta, \phi)| \leq \Theta \):
When \( \delta_1(\theta, \phi) \leq \Theta \) is satisfied, Eq. (S-62) can be constructed as:
\[
\gamma'_1 = \pi \pm \left[ 2 \arcsin \left( \frac{\sin \theta \sin \frac{\phi}{2}}{\sin \Theta} \right) - \pi \right], \quad (both \ signs \ are \ allowed)
\]
(S-64)
and
\[
\beta'_0 = \alpha_3 - \alpha_1, \quad (S-65a)
\]
\[
\beta'_1 = \alpha_4 - \alpha_2, \quad (S-65b)
\]
where \( \alpha_1 = -\psi - \lambda_1, \alpha_2 = \psi + \lambda_1, \alpha_3 = -\Psi - \lambda_2, \alpha_4 = \Psi - \lambda_2 \), and
\[
\lambda_1 = \text{Arg} \left( \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \cos \theta \right), \quad (S-66a)
\]
\[
\lambda_2 = \text{Arg} \left( \cos \frac{\gamma'_1}{2} + i \sin \frac{\gamma'_1}{2} \cos \Theta \right). \quad (S-66b)
\]
Since \( |\delta_1(\theta, \phi)| \leq \pi/2 \), a three-piece decomposition for arbitrary rotations always exists when \( \Theta = \pi/2 \). In particular, we have the following corollary:
Corollary 2 \textit{(z-x-z decomposition)}

Given $U(\theta, \psi, \phi) \in \mathcal{A}$, it can always be decomposed as

$$ U(\theta, \psi, \phi) = R(\hat{z}, \beta_0)R(\hat{x}, \gamma)R(\hat{z}, \beta_1), $$

where

$$ \beta_0 = \text{Arg} \left( \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \cos \theta \right) + \psi, $$

$$ \beta_1 = \text{Arg} \left( \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \cos \theta \right) - \psi, $$

$$ \gamma = 2 \arcsin \left( \sin \theta \sin \frac{\phi}{2} \right). $$

We now generalize Lemma 5 to an arbitrary odd number of pieces.

Lemma 6

Given a rotation $U(\theta, \psi, \phi) \in \mathcal{A}$, there exist certain values of $\beta'_i \in [0, 4\pi)$, $\gamma'_i \in [0, 4\pi)$, and $l \in \mathbb{Z}^+$, such that

$$ U(\theta, \psi, \phi) = R(\hat{z}, \beta'_0)R(\Theta, \Psi, \gamma'_1)R(\hat{z}, \beta'_2) \cdots R(\Theta, \Psi, \gamma'_{l-1})R(\hat{z}, \beta'_{l-1}), $$

if and only if

$$ |\delta_1(\theta, \phi)| \leq (l - 1)\Theta. $$

Proof

Case (i): $l = 1$.

In this case, Eq. (S-69) and Eq. (S-70) become

$$ U(\theta, \psi, \phi) = R(\hat{z}, \beta'_0), $$

$$ |\delta_1(\theta, \phi)| = 0. $$

Obviously, both Eq. (S-71a) and Eq. (S-71b) are equivalent to $\theta = 0$ or $\phi \in \{0, 2\pi\}$.

Case (ii): $l > 1$.

Necessity of $|\delta_1(\theta, \phi)| \leq (l - 1)\Theta$:

Let

$$ \gamma''_i = \gamma'_i \text{ mod } 2\pi, $$

$$ \beta''_i = \beta'_i \text{ mod } 2\pi, $$

Eq. (S-69) is equivalent to

$$ U(\theta, \psi, \phi) = \pm R(\hat{z}, \beta''_0)R(\Theta, \Psi, \gamma''_1)R(\hat{z}, \beta''_2) \cdots R(\Theta, \Psi, \gamma''_{l-1})R(\hat{z}, \beta''_{l-1}). $$

According to corollary 2, one can apply the z-x-z decomposition on each $R(\Theta, \Psi, \gamma''_i)$. So if Eq. (S-69) holds, $U(\theta, \psi, \phi)$ can be further rewritten as

$$ U(\theta, \psi, \phi) = \pm R(\hat{z}, \eta_0)R(\hat{x}, \rho_1)R(\hat{z}, \eta_1) \cdots R(\hat{x}, \rho_{l-1})R(\hat{z}, \eta_{l-1}), $$

for certain values of $\eta_i \in [0, 2\pi)$, and $\rho_i = 2 \arcsin \left( \sin \Theta \sin \frac{\gamma''_i}{2} \right)$. Since $\Theta \in (0, \frac{\pi}{2})$, and $\gamma''_i \in [0, 2\pi)$, we have

$$ 0 \leq \rho_i \leq 2\Theta \leq \pi. $$

We give two statements: (a) $|\delta_1(\theta, \phi)| > (l - 1)\Theta$, and (b) Eq. (S-74) holds.

Since Eq. (S-74) is equivalent to Eq. (S-69), to prove the necessity of Lemma 6, we only need to show that (a) and (b) cannot be satisfied at the same time. In the following, we assume that both (a) and (b) are satisfied.
We define
\[ B_l = \begin{bmatrix} b_{l,11} & b_{l,12} \\ b_{l,21} & b_{l,22} \end{bmatrix} = R(\hat{z}, \eta_l) R(\hat{x}, \rho_l) R(\hat{\zeta}, \eta_l) \ldots R(\hat{x}, \rho_1) R(\hat{\zeta}, \eta_1), \] (S-76)
where \( t \leq l - 1 \). Note that \( B_{l-1} = R(\theta, \psi, \phi) \), and
\[ |b_{l-1,11}| = \cos \delta_l(\theta, \phi). \] (S-77)

Then, the value of \( |b_{l,11}| \) will be bounded by induction as follows:
For \( t = 1 \), Eq. (S-75) implies that \( |b_{1,11}| = \cos \frac{\delta_1}{2} \geq \cos \Theta \).
For \( 1 < t \leq (l - 1) \), we suppose \( |b_{l-1,11}| \geq \cos [(t - 1)\Theta] \) holds.
One can let
\[ b_{l-1,11} = e^{i\varphi_1} \cos \alpha, \] (S-78a)
\[ b_{l-1,12} = e^{i\varphi_2} \sin \alpha, \] (S-78b)
for certain values of \( 0 \leq \alpha \leq (t - 1)\Theta \), and \( 0 \leq \varphi_{1,2} < 2\pi \). Since
\[ \begin{bmatrix} b_{l,11} & b_{l,12} \\ b_{l,21} & b_{l,22} \end{bmatrix} = \begin{bmatrix} b_{l-1,11} & b_{l-1,12} \\ b_{l-1,21} & b_{l-1,22} \end{bmatrix} R(\rho_l) R(\eta_l), \] (S-79)
we have
\[ b_{l,11} = e^{i\varphi_1} \cos \alpha \cos \frac{\rho_l}{2} - ie^{i\varphi_2} \sin \alpha \sin \frac{\rho_l}{2}. \] (S-80)

Then
\[
|b_{l,11}|^2 = \cos^2 \alpha \cos^2 \frac{\rho_l}{2} + \sin^2 \alpha \sin^2 \frac{\rho_l}{2} + 2 \sin (\varphi_1 - \varphi_2) \cos \alpha \cos \frac{\rho_l}{2} \sin \alpha \sin \frac{\rho_l}{2}
\geq \left( \cos \alpha \cos \frac{\rho_l}{2} - \sin \alpha \sin \frac{\rho_l}{2} \right)^2
= \cos^2 \left( \alpha + \frac{\rho_l}{2} \right)
\geq \cos^2 t\Theta. \] (S-81)

The last inequality is due to \( \alpha \leq (t - 1)\Theta \), \( 0 \leq \rho_l \leq 2\Theta \), and \( t\Theta \leq (l - 1)\Theta < \delta_l(\theta, \phi) \leq \frac{\pi}{2} \). Therefore, if both (a) and (b) hold true, we have \( |b_{l,11}| \geq \cos (t\Theta) \) for \( 1 \leq t \leq l \), which also gives
\[ |b_{l-1,11}| \geq \cos (l\Theta). \] (S-82)

Combining Eq. (S-77), Eq. (S-82) and \( \delta_l(\theta, \phi) \in [0, \frac{\pi}{2}] \), we have \( \delta_l(\theta, \phi) \leq (l - 1)\Theta \). However, this is contradicted to (a). Therefore (a) and (b) cannot be satisfied at the same time, which finish the proof of necessity.

**Sufficiency of \( |\delta_l(\theta, \phi)| \leq (l - 1)\Theta \):**
The sufficiency will be proven constructively. According to Corollary 2 \( \hat{U}(\theta, \psi, \phi) \) can first be decomposed as:
\[ U(\theta, \psi, \phi) = R(\hat{z}, \lambda_1 + \psi) R(\hat{x}, 2\delta_l(\theta, \phi)) R(\hat{\zeta}, \lambda_1 - \psi), \] (S-83)
where
\[ \lambda_1 = \text{Arg} \left( \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \cos \theta \right). \] (S-84)

The \( x \) rotation in the middle can be divided into \( l - 1 \) pieces, and we get
\[ U(\theta, \psi, \phi) = R(\hat{z}, \lambda_1 + \psi) \left[ R(\hat{x}, \frac{2\delta_l(\theta, \phi)}{l - 1}) \right]^{l-1} R(\hat{\zeta}, \lambda_1 - \psi). \] (S-85)
We notice that \( R\left( \hat{x}, \frac{2\delta_1(\theta, \phi)}{l-1} \right) = R\left( \frac{\pi}{2}, 0, \frac{2\delta_1(\theta, \phi)}{l-1} \right) \), and \( \delta_1 \left( \frac{\pi}{2}, \frac{2\delta_1(\theta, \phi)}{l-1} \right) = \frac{\delta_1(\theta, \phi)}{l-1} \leq \Theta \). According to Lemma [5],[5] when \( |\delta_1(\theta, \phi)| \leq (l-1)\Theta \), we can have the decomposition \( R(\hat{x}, \frac{2\delta_1(\theta, \phi)}{l-1}) = R(\hat{z}, -\Psi - \chi)R(\Theta, \Psi, \gamma')R(\hat{z}, \Psi - \chi) \), with

\[
\gamma' = \pi \pm \left[ 2 \arcsin \left( \frac{\sin \frac{\delta_1(\theta, \phi)}{l-1}}{\sin \Theta} \right) - \pi \right],
\]

and

\[
\chi = \text{Arg} \left( \cos \frac{\gamma'}{2} + i \sin \frac{\gamma'}{2} \cos \Theta \right).
\]

So Eq. (S-69) can be constructed by taking,

\[
\beta'_i = \begin{cases} 
\lambda_1 + \psi - \Psi - \chi, & i = 0, \\
-2\chi, & 0 < i < l - 1, \\
\lambda_1 - \psi + \Psi - \chi, & i = l - 1,
\end{cases}
\]

and

\[
\gamma'_j = \gamma'
\]

for \( 1 \leq j \leq l - 1 \).

The following theorem corresponds to the odd-piece decomposition.

**Theorem 4** Given \( U(\theta, \psi, \phi) \in \mathcal{A} \), it can be decomposed to \( 2l - 1 \) pieces with \( l \in \mathbb{Z}^+ \)

(i) as Eq. (S-59) with certain values of \( \beta_i \in [0, 4\pi) \), \( \gamma_i \in [0, 4\pi) \), if and only if

\[
|\delta_1(\theta, \phi)| \leq (l - 1)\Theta,
\]

or (ii) as Eq. (S-60) with certain values of \( \beta_i \in [0, 4\pi) \), \( \gamma_i \in [0, 4\pi) \), if and only if

\[
|\delta_2(\theta, \psi, \phi, \Theta)| \leq (l - 1)\Theta.
\]

**Proof**

For case (i), by taking \( \Psi = 0 \) in Lemma [6], one can verify that Theorem 4 holds true.

For case (ii), we first apply the transformation \( U \rightarrow R(\Theta/2, 0, -\pi)UR(\Theta/2, 0, \pi) \) [rotating all axes around \( (\sin \Theta/2, 0, \cos \Theta/2) \) by angle \( \pi \)] on Eq. (S-60), and obtain:

\[
U \left( \hat{\theta}, \hat{\psi}, \hat{\phi} \right) = R(\hat{z}, \beta_0)R(\hat{m}, 0, \gamma_1)R(\hat{z}, 0, \beta_1) \ldots R(\hat{z}, 0, \gamma_{l-1})R(\hat{z}, 0, \beta_{l-1})
\]

where \( U(\hat{\theta}, \hat{\psi}, \hat{\phi}) = R(\Theta/2, 0, -\pi)U(\theta, \psi, \phi)R(\Theta/2, 0, \pi) \). It is straightforward to verify

\[
|\delta_1(\hat{\theta}, \hat{\phi})| = |\delta_2(\theta, \psi, \phi, \Theta)|.
\]

Therefore, case (ii) of Theorem 4 also holds true.

2. Constructing the odd-piece decomposition

When Eq. (S-90) is satisfied, Eq. (S-59) can be constructed as

\[
\beta_i = \begin{cases} 
\lambda_1 - \lambda_2 + \psi, & i = 0, \\
-2\lambda_2, & 0 < i < l - 1, \\
\lambda_1 - \lambda_2 - \psi, & i = l - 1,
\end{cases}
\]

and

\[
\gamma_j = \gamma, \ 1 \leq j \leq l - 1,
\]
where
\[ \gamma = \pi \pm \left[ 2 \arcsin \left( \frac{\sin \delta_1(\theta, \phi)}{\sin \Theta} \right) - \pi \right], \] (S-96a)
\[ \lambda_1 = \text{Arg} \left( \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \cos \theta \right), \] (S-96b)
\[ \lambda_2 = \text{Arg} \left( \cos \frac{\gamma}{2} + i \sin \frac{\gamma}{2} \cos \Theta \right). \] (S-96c)

Similarly, when Eq. (S-91) is satisfied, to construct Eq. (S-60) we first calculate
\[ U(\tilde{\theta}, \tilde{\psi}, \tilde{\phi}) = R(\hat{m}, \beta_1) R(\hat{z}, \gamma_1) \ldots R(\hat{m}, \beta_l) R(\hat{z}, \gamma_l), \] (S-97)
then, we have
\[ \beta_i = \begin{cases} \hat{\lambda}_1 - \hat{\lambda}_2 + \tilde{\psi} & i = 0, \\ -2\lambda_2 & 0 < i < l - 1, \\ \hat{\lambda}_1 - \hat{\lambda}_2 - \tilde{\psi} & i = l - 1, \end{cases} \] (S-98)
and
\[ \gamma_j = \tilde{\gamma}, \quad 1 \leq j \leq l - 1, \] (S-99)
where
\[ \tilde{\gamma} = \pi \pm \left[ 2 \arcsin \left( \frac{\sin \delta_1(\tilde{\theta}, \tilde{\phi})}{\sin \tilde{\Theta}} \right) - \pi \right], \] (S-100a)
\[ \tilde{\lambda}_1 = \text{Arg} \left( \cos \frac{\tilde{\phi}}{2} + i \sin \frac{\tilde{\phi}}{2} \cos \tilde{\theta} \right), \] (S-100b)
\[ \tilde{\lambda}_2 = \text{Arg} \left( \cos \frac{\tilde{\gamma}}{2} + i \sin \frac{\tilde{\gamma}}{2} \cos \tilde{\Theta} \right). \] (S-100c)

It should be notice that this decomposition method is not the unique one.

**B. Even-piece decomposition**

1. **Criterion for odd-piece decomposition**

For even-piece decomposition, i.e. \( p = 2l \) with \( l \in \mathbb{Z}^+ \), Eq. (2) of the main text is equivalent to
\[ U(\theta, \psi, \phi) = R(\hat{m}, \beta_1) R(\hat{z}, \gamma_1) \ldots R(\hat{m}, \beta_l) R(\hat{z}, \gamma_l), \] (S-101)
or
\[ U(\theta, \psi, \phi) = R(\tilde{z}, \beta_1) R(\tilde{m}, \gamma_1) \ldots R(\tilde{z}, \beta_l) R(\tilde{m}, \gamma_l), \] (S-102)
where \( \beta_i = \alpha_{2i-1} \in [0, 4\pi) \) and \( \gamma_i = \alpha_{2i} \in [0, 4\pi) \). we denote
\[ A = \left( \cos \psi \cos \Theta \sin \theta \sin \frac{\phi}{2} - \sin \Theta \sin \theta \sin \frac{\phi}{2} \right)^2 + \left( \sin \psi \cos \Theta \sin \theta \sin \frac{\phi}{2} - \sin \Theta \cos \frac{\phi}{2} \right)^2, \] (S-103a)
\[ B = \left( \sin \theta \sin \frac{\phi}{2} \right)^2, \] (S-103b)
\[ C = \sin \Theta \sin \theta \sin \frac{\phi}{2} \left( \sin \psi \sin \frac{\phi}{2} \cos \theta - \cos \psi \cos \frac{\phi}{2} \right), \] (S-103c)
and
\[ \Lambda(\theta, \psi, \phi, \Theta) = \arcsin \sqrt{\frac{A + B}{2} - \sqrt{C^2 + \frac{(B - A)^2}{4}}}. \] (S-104)
**Theorem 5** Given \( R(\theta, \psi, \phi) \in A \), it can be decomposed to 2\( l \) pieces with \( l \in \mathbb{Z}^+ \)

(i) as Eq. (S-101) with certain values of \( \beta_i \in [0, 4\pi) \), \( \gamma_i \in [0, 4\pi) \), if and only if

\[
\Lambda(\theta, \psi, \phi, \Theta) \leq (l - 1)\Theta,
\]

(S-105)

or (ii) as Eq. (S-102) with certain values of \( \beta_i \in [0, 4\pi) \), \( \gamma_i \in [0, 4\pi) \), if and only if

\[
\Lambda(\theta, \psi, -\phi, \Theta) \leq (l - 1)\Theta.
\]

(S-106)

**Proof**

**Case (i):**
We first define

\[
\begin{bmatrix}
    e_{11}(\beta_1) & e_{12}(\beta_1) \\
    e_{21}(\beta_1) & e_{22}(\beta_1)
\end{bmatrix}
\equiv R(\hat{m}, -\beta_1) U(\theta, \psi, \phi).
\]

(S-107)

According to Eq. (S-61a) and Theorem 4, the existence of Eq. (S-101) is equivalent to the existence of \( \beta_1 \in [0, 4\pi) \), such that

\[
\arcsin |e_{12}(\beta_1)| \leq (l - 1)\Theta.
\]

(S-108)

It can be calculated from Eq. (S-107) that

\[
e_{12}(\beta_1) = e^{-i\psi}
\left(-i \cos \frac{\beta_1}{2} + \cos \Theta \sin \frac{\beta_1}{2}\right) \sin \theta \sin \phi \frac{\phi}{2}
\left(\cos \frac{\phi}{2} + i \cos \theta \sin \frac{\phi}{2}\right).
\]

(S-109)

After some further calculation, one can obtain that

\[
|e_{12}(\beta_1)|^2 = A \sin^2 \frac{\beta_1}{2} + B \cos^2 \frac{\beta_1}{2} + C \sin \beta_1
= \frac{(B - A)}{2} \cos \beta_1 + C \sin \beta_1 + \frac{A + B}{2}.
\]

(S-110)

By varying \( \beta_1 \), the minimum of \( \arcsin |e_{12}(\beta_1)| \) is exactly given by:

\[
\min |\arcsin |e_{12}(\beta_1)|| = \Lambda(\theta, \psi, \phi, \Theta).
\]

(S-111)

Combining Eq. (S-108) and Eq. (S-111), one can conclude that (i) of Theorem 5 holds true.

**Case (ii):**
By taking the inverse operation on both side of Eq. (S-102), it is equivalent to

\[
R(\theta, \psi, -\phi) = R(\hat{m}, -\gamma_1) R(\hat{z}, -\beta_1) \ldots R(\hat{m}, -\gamma_1) R(\hat{z}, -\beta_1),
\]

or

\[
R(\theta, \psi, \tilde{\phi}) = R(\hat{m}, \tilde{\gamma}_1) R(\hat{z}, \tilde{\beta}_1) \ldots R(\hat{m}, \tilde{\gamma}_1) R(\hat{z}, \tilde{\beta}_1),
\]

(S-112)

where

\[
\tilde{\phi} = -\phi \mod 4\pi,
\]

(S-113a)

\[
\tilde{\gamma}_i = -\gamma_i \mod 4\pi,
\]

(S-113b)

\[
\tilde{\beta}_i = -\beta_i \mod 4\pi.
\]

(S-113c)

According to the (i) of Theorem 5 the existence of Eq. (S-112) is equivalent to

\[
\Lambda(\theta, \psi, \tilde{\phi}, \Theta) \leq (l - 1)\Theta.
\]

(S-114)

And since \( \Lambda(\theta, \psi, -\phi, \Theta) = \Lambda(\theta, \psi, \tilde{\phi}, \Theta) \), (ii) of Theorem 5 also holds true.
2. Constructing even-piece decompositions

When Eq. (S-105) is satisfied:
In such case, we should construct Eq. (S-101). Firstly, \( \beta_1 \) can take any values that satisfy Eq. (S-108), or one can simply take
\[
\beta_1 = \pi + \text{Arg} \left( \frac{B - A}{2} + iC \right),
\] (S-115)
which makes the left hand side of Eq. (S-108) reach its minimum. If we denote \( R(\hat{\theta}, \psi, \omega) = R(\hat{m}, -\beta_1)U(\theta, \psi, \phi) \) \[1\], other parameters can be obtained by applying the odd-piece decomposition scheme (cf. Sec. III A 2) on \( R(\theta, \psi, \phi') \).

When Eq. (S-106) is satisfied:
In such case, we should construct Eq. (S-102). To do so, we construct Eq. (S-112) first, then determine the values of \( \tilde{\beta}_i, \tilde{\gamma}_i \) with the same method of (i). After that, one can obtain the values of \( \beta_i, \gamma_i \) in Eq. (S-102) from Eq. (S-113).

C. Minimum number of pieces for arbitrary rotations

We separate the problem into two cases: odd-piece and even-piece decompositions. We recall that \( \Theta \in (0, \pi/2] \) is defined as the angle between two fixed axes.

**Theorem 5.1** (Odd-piece) Arbitrary rotations \( U \in \mathcal{A} \) can be decomposed to \( 2l - 1 \) pieces with \( l \in \mathbb{Z}^+ \), if and only if
\[
\Theta \geq \frac{\pi}{2(l - 1)}. \tag{S-116}
\]

**Proof**

**Sufficiency:**
When Eq. (S-116) is satisfied, we have \( (l - 1)\Theta \geq \frac{\pi}{2} \). Since \( |\delta(\theta, \phi)| \in [0, \frac{\pi}{2}] \) and \( |\delta'(\theta, \psi, \phi, \Theta)| \in [0, \frac{\pi}{2}] \), according to Theorem 4 there exist a decomposition of \( 2l - 1 \) pieces for arbitrary rotations.

**Necessity:**
For \( R \left( \frac{\pi}{2}, \frac{\pi}{2}, \pi \right) \), we have
\[
\delta \left( \frac{\pi}{2}, \pi \right) = \delta' \left( \frac{\pi}{2}, \frac{\pi}{2}, \pi, \Theta \right) = \pi. \tag{S-117}
\]
So when \( \Theta < \frac{\pi}{2l} \), neither Eq. (S-90) nor Eq. (S-91) can be satisfied. According to Theorem 4, the \( 2l - 1 \) pieces decomposition of \( R \left( \frac{\pi}{2}, \frac{\pi}{2}, \pi \right) \) does not exist. \( \blacksquare \)

**Theorem 5.2** (Even-piece) Arbitrary rotations \( U \in \mathcal{A} \) can be decomposed to \( 2l \) pieces, if and only if
\[
\Theta \geq \frac{\pi}{2l - 1}. \tag{S-118}
\]

**Proof**

**Sufficiency:**
According to Lemma 1 (refer to the next section), arbitrary rotations can be written as
\[
U(\theta, \psi, \phi) = R(\hat{\psi}, 2\psi)R(\theta''', 0, \phi'''), \tag{S-119}
\]
for certain values of \( \theta'' \in [0, \pi) \), and \( \phi''' \in [0, 4\pi) \). To show the existence of \( 2l \)-piece decomposition for \( U(\theta, \psi, \phi) \), one only needs to prove that \( R(\theta''', 0, \phi''') \) can always be decomposed into \( 2l - 1 \) pieces when Eq. (S-127) is satisfied.

For \( R(\theta''', 0, \phi''') \) we have
\[
\begin{align*}
\delta(\theta'', \phi''') &= \arcsin \left| \sin \theta'' \sin \frac{\phi'''}{2} \right| \leq |\theta'''|, \tag{S-120a} \\
\delta'(\theta'', 0, \phi''', \Theta) &= \arcsin \left| \sin \frac{\phi'''}{2} \sin (\Theta - \theta'') \right| \leq |\Theta - \theta'''|. \tag{S-120b}
\end{align*}
\]
It is easy to check that when $\theta'' \in [0, (l-1)\Theta] \cup [\pi - (l-1)\Theta, \pi)$, we have
\[
\delta(\theta'', \phi'') \leq (l-1)\Theta,
\] (S-121)
and when $\theta'' \in [\Theta, l\Theta]$, we have
\[
\delta'(\theta'', 0, \phi'') \leq (l-1)\Theta.
\] (S-122)

Therefore, when
\[
\theta'' \in A_\theta = [0, l\Theta] \cup [\pi - (l-1)\Theta, \pi],
\] (S-123)
we have
\[
\Theta \geq \min \{\delta(\theta'', \phi''), \delta'(\theta'', 0, \phi''), \Theta\}. \quad \text{(S-124)}
\]
According to Theorem 4, $R(\theta'', 0, \phi'')$ can be decomposed to $2l-1$ pieces when $\theta'' \in A_\theta$. Furthermore, for $\Theta \geq \frac{\pi}{2l-1}$, we have $[0, \pi) \subset A_\theta$, so $R(\theta'', 0, \phi'')$ can always be decomposed to $2l-1$ pieces.

**Necessity:**
The necessity can be proven by finding specific rotations that fail to be decomposed to $2l$ pieces when $\Theta < \frac{\pi}{2l-1}$.

For $l = 1$, we consider $R(\frac{3\pi}{4}, 0, \pi)$, and notice that for arbitrary values of $\Theta \in (0, \frac{\pi}{2})$, we have $\Lambda(\frac{3\pi}{4}, 0, \pm\pi, \Theta) = \frac{1}{2} [1 + \sin^2(2\Theta)] > 0$. Therefore, $R(\frac{3\pi}{4}, 0, \pi)$ cannot be decomposed in two steps with $\hat{z}$ and $\hat{m}$.

For $l > 1$, we consider the rotation $R(\frac{l}{2l-1}\pi, 0, \pi)$. When $\Theta < \frac{\pi}{2l-1}$, it is easy to check that
\[
\Lambda \left( \frac{l}{2l-1}\pi, 0, \pm\pi, \Theta \right) = \frac{\pi}{2l-1} (l-1) > (l-1)\Theta.
\] (S-125)
So according to Theorem 3, the rotation $R(-\frac{l-1}{2l-1}\pi, 0, \pi)$ cannot be decomposed in $2l$ steps with $\hat{z}$ and $\hat{m}$.

Combining the above results, we have the following:

**Theorem 6** Arbitrary rotations $U \in A$ can be decomposed to $p$ pieces in the form of
\[
U = \prod_{i=1}^{p} R(\hat{n}_i, \phi_i)
\] (S-126)
with $R(\hat{n}_i, \phi_i) \in G_b$ and $\hat{n}_i \neq \hat{n}_{i+1}$, if and only if
\[
\Theta \geq \frac{\pi}{p-1}. \quad \text{(S-127)}
\]

**IV. MATLAB CODE FOR CONSTRUCTING EXACT MINIMAL DECOMPOSITION SEQUENCE**

We have provided matlab code (Type_I.m and Type_II.m) for constructing explicit minimal decomposition of Type I and Type II qubits described in this work.

For a target unitary transformation $U = U(\theta, \psi, \phi)$ and $\Theta$ (we restrict $\Theta \in (0, \pi]$ for Type I or $\Theta \in (0, \pi/2]$ for Type II), the inputs of the function are $\theta, \psi, \phi, \Theta$ respectively. For example, one inputs:

```plaintext
>> Type_I (pi/3, 0, pi, pi/4)
```
the output should be

| 0.7854 | 0       | 0.7854 |
| 4.4411 | 0.7495 | 10.7243 |
The first line corresponds to the polar angles of each elementary rotations in order while the second line corresponds to the rotation angle. One can verify that:

\[ U(\pi/3, 0, \pi) = R(0.7854, 0, 4.4411)R(0, 0, 0.7495)R(0.7854, 0, 10.7243). \]  (S-128)

\[ \text{s.wang@cityu.edu.hk} \]
\[ \text{yung@sustc.edu.cn} \]

[S1] The values of \( \theta', \psi', \phi' \) can be calculated numerically according the Eq. (S-2)-(S-4).