Black and White holes in four-dimensional Chern-Simons gravity

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Abstract

We discuss a four-dimensional gravitational action which was obtained replacing a Randall-Sundrum type metric in the so called five-dimensional Einstein-Chern-Simons gravity action. We studied black hole solutions of the corresponding 4-dimensional gravitational field equations. It is found that for a spherically symmetric metric such equations lead to a spacetime with a cosmological constant inversely proportional to the square of the compactification radius and to one solution dependent on an arbitrary constant $C$. If this constant is negative, we find a Schwarzschild-de Sitter black hole. If $C$ is positive, the solution can be understood as a white hole solution which is obtained applying to the solution with $C < 0$ the discrete coordinate transformation $PT$ accompanied by the transformation $C \rightarrow -\tilde{C}$, with $\tilde{C} > 0$, corresponding to a transformation known as mass reversal.

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I. INTRODUCTION

In [1] was shown that the standard 5-dimensional General Relativity (GR), without cosmological constant, can be obtained from Chern-Simons gravity theory for a certain Lie algebra \(\mathcal{B}\) \([2], [3]\). The Chern-Simons Lagrangian for the \(\mathcal{B}\) algebra is given by

\[
L_{\text{ChS}}^{(5)} = \alpha_1 l^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e + \alpha_3 \varepsilon_{abcde} \left( \frac{2}{3} R^{ab} e^c e^d e^e + 2 l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right),
\]

where \(\alpha_1\) and \(\alpha_3\) are arbitrary constants and \(l\) can be interpreted as a coupling constant that characterizes different regimes within the theory.

The field content induced by the \(\mathcal{B}\) algebra includes the vielbein \(e^a\), the spin connection \(\omega^{ab}\) and two extra bosonic fields \(h^a\) and \(k^{ab}\). From (1) we can see that it is possible to recover the 5-dimensional Einstein gravity theory from a Chern-Simons gravity theory in the limit where the coupling constant \(l\) equals to zero while keeping the effective Newton’s constant fixed.

In the presence of matter described by the langragian \(L_M = L_M(e^a, h^a, \omega^{ab})\), in Ref. [4] was found that the field equations admit black hole-type solutions for a spherically symmetric metric.

On the other hand, in Ref. [5], [6] was found that if one replaces a Randall-Sundrum type metric \([7] [8]\) in the Einstein-Chern-Simons (EChS) gravity Lagrangian (1), the following action can be obtained

\[
\tilde{S}[\tilde{e}, \tilde{h}] = \int_{\Sigma_T} \tilde{\varepsilon}_{mnpq} \left( -\frac{1}{2} \tilde{R}^{mn} \tilde{e}^p \tilde{e}^q + K \tilde{R}^{mn} \tilde{e}^p \tilde{h}^q - \frac{K}{4 r_c^2} \tilde{e}^m \tilde{e}^n \tilde{e}^p \tilde{h}^q \right),
\]

which is a gravity action with a cosmological constant for a 4-dimensional brane embedded in the 5-dimensional spacetime of the so-called EChS gravity. \(\tilde{\varepsilon}_{mnpq}\), \(\tilde{e}^m\), \(\tilde{R}^{mn}\) and \(\tilde{h}^m\) represent, respectively, the 4-dimensional versions of the Levi-Civita symbol, the vielbein, the curvature form, a matter field and \(r_c\) is the so-called compactification radius. \(K\) is a constant related to \(\alpha_3\), \(l\) and \(r_c\). It is of interest to note that the field \(h^a\) gives rise to a form of positive cosmological constant which appears as a consequence of modifications of
the Poincaré symmetries, carried out through the expansion procedure. The corresponding version of Eq. (2) in tensor language Ref. [5], [6] is given by

\[ \tilde{S}[\tilde{g}, \tilde{h}] = \int d^4 \tilde{x} \sqrt{-\tilde{g}} \left[ \tilde{R} + 2K \left( \tilde{R} \tilde{h} - 2 \tilde{R}^\mu_\nu \tilde{h}^\nu_\mu \right) - \frac{3K}{2r_c^2} \tilde{h} \right], \tag{3} \]

where \( K = \frac{4\pi \alpha_g l^2}{r_c}, \) so that when \( l \rightarrow 0 \) then \( K \rightarrow 0 \) and the actions (2) and (3) becomes the 4-dimensional Einstein-Hilbert action.

An interesting observation that if we consider a maximally symmetric spacetime, then the equation 13.4.6 of Ref. [9] allows us to write the field \( \tilde{h}_{\mu\nu} \) as

\[ \tilde{h}_{\mu\nu} = \frac{1}{4} \tilde{F}(\tilde{\phi}) \tilde{g}_{\mu\nu}, \tag{4} \]

where \( \tilde{F} \) is an arbitrary function of an 4-scalar field \( \tilde{\phi} = \tilde{\phi}(\tilde{x}) \). This means

\[ \tilde{R}^\mu_\nu \tilde{h}^\nu_\mu = \frac{1}{4} \tilde{F}(\tilde{\phi}) \tilde{R}, \quad \tilde{h} = \tilde{h}_{\mu\nu} \tilde{g}^{\mu\nu} = \tilde{F}(\tilde{\phi}), \tag{5} \]

so that the action (3) takes the form

\[ \tilde{S}[\tilde{g}, \tilde{\phi}] = \int d^4 \tilde{x} \sqrt{-\tilde{g}} \left[ \tilde{R} + K \tilde{R} \tilde{F}(\tilde{\phi}) - \frac{3K}{2r_c^2} \tilde{F}(\tilde{\phi}) \right], \tag{6} \]

which corresponds to an action for the 4-dimensional gravity coupled non-minimally to the scalar field. Note that this action has the form \( \tilde{S} = \tilde{S}_g + \tilde{S}_{g\phi} + \tilde{S}_\phi \), where, \( \tilde{S}_g \) is a pure gravitational action term, \( \tilde{S}_{g\phi} \) is a non-minimal interaction term between gravity and the scalar field, and \( \tilde{S}_\phi \) represents the action for a kind of scalar field potential. In order to write down the action in the usual way, we define the constant \( \varepsilon \) and the potential \( V(\tilde{\phi}) \) as (removing the symbols \( \sim \) in (6))

\[ \varepsilon = \frac{4\kappa r_c^2}{3}, \quad V(\phi) = \frac{3K}{4\kappa r_c^2} \tilde{F}(\phi), \tag{7} \]

where \( \kappa \) is a constant. This permits to rewrite the action for a 4-dimensional brane non-minimally coupled to a scalar field, immersed in a 5-dimensional space-time as

\[ S[g, \varphi] = \int d^4 x \sqrt{-g} [R + \varepsilon RV(\phi) - 2\kappa V(\phi)]. \tag{8} \]

Note that this action was obtained from the Lagrangian (1) that does not consider the presence of a cosmological constant.
The corresponding field equations describing the behavior of the 4-dimensional brane in the presence of the scalar field $\phi$ are given by \cite{5}, \cite{6}

$$ G_{\mu\nu} \left( 1 + \varepsilon V \right) + \varepsilon H_{\mu\nu} = -\kappa g_{\mu\nu} V, \quad (9) $$

$$ \frac{\partial V}{\partial \phi} \left( 1 - \frac{\varepsilon R}{2\kappa} \right) = 0, \quad (10) $$

where

$$ H_{\mu\nu} = g_{\mu\nu} \nabla^\lambda \nabla_\lambda V - \nabla_\mu \nabla_\nu V. \quad (11) $$

In this article we study black hole solutions of the 4-dimensional gravitational field equations (9),(10),(11). In Section II we find that field equations (9),(10) and (11), for a spherically symmetric metric, lead to a spacetime with a cosmological constant inversely proportional to the square of the compactification radius and to a solution dependent on an arbitrary constant $C$. In Section III we discuss the case when this constant is negative, in particular, when such negative constant is equal to the Schwarzschild radius. This solution is reminiscent of a Schwarzschild-de Sitter (SdS) black hole. A procedure used in the context of general relativity \cite{10} is generalized in Section IV in order to apply it to Chern-Simons gravity. This method is based on the symmetries of parity $r \rightarrow -\tilde{r}$, time reversal $t \rightarrow -\tilde{t}$ and mass reversal $M \rightarrow -\tilde{M}$ called $PTM$ symmetry. This symmetry: (i) leaves the metric invariant in form, (ii) relates the different regions of the extended solution of the Einstein field equations written in terms of the Kruskal-Szekeres coordinates \cite{11}, \cite{12} and (iii) reverse the signs of the Kruskal-Szekeres coordinates. In Ref. \cite{10} the Schwarzschild case was studied and in Refs. \cite{13}, \cite{14} the Reissner Nordström’s solution was considered. In this article, the solutions of the field equations obtained from a four-dimensional gravitational action, that was obtained by compactification of the so-called EChS gravity in five dimensions, will be analyzed. Finally, in Section V we present our conclusions. We will use $G = c = 1$ units.

II. FIELD EQUATIONS FOR A SPHERICALLY SYMMETRIC METRIC

In 4-dimensions a static and spherically symmetric metric can be written in the form

$$ ds^2 = -e^{\alpha(r)} dt^2 + e^{\beta(r)} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \quad (12) $$

where $r, \theta$ and $\varphi$ are the usual spherical polar coordinates. So, according to (9) and (10), the following field equations are obtained (see Appendix)
\[
\left[ e^\alpha \left( \frac{\alpha'}{r} + \frac{1}{r^2} \right) \right] (1 + \varepsilon V) + \varepsilon H_{00} = \kappa e^\alpha V, \quad (13)
\]

\[
\left[ \frac{\alpha'}{r} + \frac{1}{r^2} - e^{-\alpha} \right] (1 + \varepsilon V) + \varepsilon H_{11} = -\kappa e^{-\alpha} V, \quad (14)
\]

\[
\left[ 1 - \frac{r^2}{2} R - \frac{e^{-\alpha}}{2} \left( (\alpha' - \beta') r + 2 \right) \right] (1 + \varepsilon V) + \varepsilon H_{22} = -\kappa r^2 V, \quad (15)
\]

\[
\left[ 1 - \frac{r^2}{2} R - \frac{e^{-\beta}}{2} \right] (1 + \varepsilon V) \sin^2 \theta + \varepsilon H_{33} = -\kappa r^2 V \sin^2 \theta, \quad (16)
\]

\[
\frac{\partial V}{\partial \phi} \left( 1 - \frac{\varepsilon R}{2\kappa} \right) = 0, \quad (17)
\]

\[
g_{\mu\nu} \nabla^\lambda \nabla_\lambda V - \nabla_\mu \nabla_\nu V = H_{\mu\nu}, \quad (18)
\]

where the "prime" symbol denotes derivative respect to \( r \). We have seen that when \( l \to 0 \), i.e., \( V \to 0 \), the equations \( (9),(10),(11) \) lead to Einstein’s equations. This means that the equations \( (13-18) \) must lead, in this limit, to the corresponding Einstein field equations for a spherically symmetric metric. For this limit to be fulfilled, it must be satisfied that \( \alpha(r) = -\beta(r) \). On the other hand, the spherical symmetry condition implies that \( V = V(r) \) and \( F = F(r) \), so that \( \partial V/\partial \phi \to \partial V/\partial r \).

By imposing these conditions on the equations \( (13-18) \) we obtain

\[
\left[ e^\alpha \left( \frac{\alpha'}{r} + \frac{1}{r^2} \right) \right] (1 + \varepsilon V) + \varepsilon H_{00} = \kappa e^\alpha V, \quad (19)
\]

\[
\left[ \frac{\alpha'}{r} + \frac{1}{r^2} - e^{-\alpha} \right] (1 + \varepsilon V) + \varepsilon H_{11} = -\kappa e^{-\alpha} V, \quad (20)
\]

\[
\left[ 1 - \frac{r^2}{2} R - e^\alpha (r\alpha' + 1) \right] (1 + \varepsilon V) + \varepsilon H_{22} = -\kappa r^2 V, \quad (21)
\]

\[
\left[ 1 - \frac{r^2}{2} R - e^\alpha (r\alpha' + 1) \right] (1 + \varepsilon V) \sin^2 \theta + \varepsilon H_{33} = -\kappa r^2 V \sin^2 \theta, \quad (22)
\]

\[
\frac{\partial V}{\partial r} \left( 1 - \frac{\varepsilon R}{2\kappa} \right) = 0, \quad (23)
\]
\[ H_{00} = -e^{2\alpha} \left[ \frac{V'\alpha'}{2} + V'' + \frac{2}{r} V' \right], \quad (24) \]

\[ H_{11} = \frac{V'\alpha'}{2} + \frac{2}{r} V' = -e^{-2\alpha} \left( H_{00} - V'' \right), \quad (25) \]

\[ H_{22} = r^2 e^\alpha \left[ V'' + \left( \frac{\alpha'}{r} + \frac{1}{r} \right) V' \right], \quad (26) \]

\[ H_{33} = H_{22} \sin^2 \theta. \quad (27) \]

From (25), (19) and (20) it direct to see that \( V'' = 0 \), which implies that \( V(r) \sim r \). From (23) it is direct to see that when \( \partial V/\partial r \neq 0 \) we have

\[ R = \frac{3}{2r_c^2}, \quad (28) \]

which indicates the presence of a cosmological constant given by

\[ \Lambda = \frac{3}{8r_c^2}. \quad (29) \]

This result allow us to conjecture that the 4-dimensional cosmological constant could have its origin in the extra dimension. In other words, the cosmological constant could have its origin in the projection of the extra dimension on the brane.

The equations (9),(10) can be written in the form

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}; \quad (30) \]

\[ T_{\mu\nu} = -\frac{1}{1 + \epsilon V} \left( \frac{2}{R} H_{\mu\nu} + g_{\mu\nu} V \right), \quad (31) \]

such that, in absence of sources \( G_{\mu\nu} = 0 \) and then \( R = 0 \). If we include cosmological constant we obtain \( R = 4\Lambda \). In the present case \( R = 3/2r_c^2 \), i.e., \( \Lambda = 3/8r_c^2 \) and we have a source given by \( T_{\mu\nu} (V \neq 0) \).

Introducing these results into (19) and (21) and carrying out the change of variables \( \alpha(r) = \ln F(r) \), we can write

\[ e^{\alpha(r)} = F(r) = 1 + \frac{C}{r} - \frac{\Lambda}{3} r^2, \quad (32) \]

where \( C \) is a constant of integration. So, the line element

\[ ds^2 = - \left( 1 + \frac{C}{r} - \frac{\Lambda}{3} r^2 \right) dt^2 + \left( 1 + \frac{C}{r} - \frac{\Lambda}{3} r^2 \right)^{-1} dr^2 + r^2 d\Omega^2, \quad (33) \]
is a $\text{sgn}(C)$-dependent solution. This means that the $F(r)$ function is subject to the discussion of $C$, that is, its sign and size. As we will see in the next Section, if $C = 0$ we obtain the usual solution that leads to a black hole whose even horizon is a cosmological one. For $C < 0$ we obtain a SdS black hole if we consider $C = -r_s$, where $r_s$ is the Schwarzschild radius.

The case $C > 0$, discussed in Section IV, leads to a white hole solution which is obtained applying to the solution with $C < 0$, the discrete coordinate transformation $PT$ accompanied by the transformation $C \rightarrow \tilde{C}$, which can be interpreted as the mass reversal.

III. $C < 0$ CASE. SDS BLACK HOLE SOLUTION

From (33), we write

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2d\Omega^2,$$  \hspace{1cm} (34)

where $F(r)$ is given by (32). If $C = 0$, then the function $F(r)$ takes the form

$$F(r) = 1 - \frac{\Lambda}{3}r^2,$$  \hspace{1cm} (35)

and we have a divergence at $r_0 = \sqrt{3/\Lambda}$, case that can be seen as the limit at large $r$ ($dS$ space limit) of the SdS black hole.

Let us consider now the conditions under which the field equations admit black hole-type solutions. The solution (34) with $F(r)$ given by (32) shows a singular behaviour at $F(r_0) = 0$, i.e.,

$$r_0^3 + (-8r_c^2) r_0 + (-8r_c^2 C) = 0,$$  \hspace{1cm} (36)

i.e., $C = \frac{1}{8r_c^2} \left( r_0 - 2\sqrt{2}r_c \right) \left( r_0 + 2\sqrt{2}r_c \right) r_0.$  \hspace{1cm} (37)

From (36) we see that the discriminant is

$$\Delta = -64r_c^4 \left( -32r_c^2 + 27C^2 \right) > 0,$$  \hspace{1cm} (38)

so that $-32r_c^2 + 27C^2 < 0$. This means that $C < 0$ as long as $r_0 < 2\sqrt{2}r_c$ with

$$C = -\Sigma^2 < -\sqrt{\frac{2}{3}} \left( \frac{4}{3} r_c \right).$$  \hspace{1cm} (39)

Hence (33) can be written in the form

$$ds^2 = -\left( 1 - \frac{\Sigma^2}{r} - \frac{\Lambda}{3} r^2 \right) dt^2 + \left( 1 - \frac{\Sigma^2}{r} - \frac{\Lambda}{3} r^2 \right)^{-1} dr^2 + r^2d\Omega^2.$$  \hspace{1cm} (40)
On another hand, from (36) and (38) we have three real roots

\[ r_{0k}(C) = -\frac{4\sqrt{2}}{\sqrt{3}} r_c \cos \left[ \frac{1}{3} \arccos \left( \frac{3\sqrt{3} \Sigma^2}{4\sqrt{2} r_c} \right) + \frac{2}{3} (k - 1) \pi \right], \]  

where \( k = 0, 1, 2 \). The solution for \( k = 1 \) can be withdrawn as non-physical. Then, the equation \( F(r_0) = 0 \) tells us that the physical horizons are \( r = r_{02} \) and \( r = r_{00} \), where \( r_{02} \leq r_{00} \). So, according to (39) and with \( \Sigma^2 = r_s \)

\[ r_s > \sqrt{\frac{2}{3}} \left( \frac{4}{3} r_c \right), \]

we identify \( r_{02} \) as the black hole event horizon and \( r_{00} \) as the cosmological horizon.

The solution (40) coincides with the SdS black hole solution when \( C = -\Sigma^2 = -r_s \), where \( r_s = 2M \) is the Schwarzschild radius.

Defining \( x = 4\sqrt{2} r_c / \sqrt{3} r_s \) and then taking the limit when \( r_c \to \infty \), we have

\[
\lim_{r_c \to \infty} r_{00} = r_s \lim_{x \to \infty} \left( x \cos \frac{1}{3} \left[ \arccos \left( \frac{3}{x} \right) - \frac{2\pi}{3} \right] \right) = r_s = \lim_{r_c \to \infty} r_{02}.
\]

This shows that the solution (40) with \( \Sigma^2 = r_s \) is the SdS black hole solution, where \( r_s = 2M \), where \( M \) is the black hole mass.

Now, from (32) and (40) we can see that \( F(r_0) = 0 \) leads in this case to

\[ \Lambda r_0^3 = 3 (r_0 - r_s), \]

and from here it is straightforward to see that \( r_c \to \infty \leftrightarrow \Lambda \to 0 \), then \( r_0 \to r_s \). Similarly, when \( r_s << r_0 \) we have to \( r_0 \to \sqrt{3/\Lambda} \), i.e., ”we live” in a \( r_0 \)-range given by

\[ r_s < r_0 < 2\sqrt{2} r_c, \]

i.e., we live in an inflationary (\( \Lambda \)) universe whose ”center” is the singularity of the black hole, singularity that we do not see given the event horizon. As we know, in GR can have black holes with event horizon and also with cosmological horizon being the SdS black hole the simplest solution which has both [15].

In order to have a rough idea of the present scales, it is good to remember that according to the current observational data \( \Lambda \sim 10^{-52} [m^{-2}] \) and \( a_0 \sim 10^{26} [m] \) (\( a_0 \) is the causal ”size” of the observable universe). In our case, from the current value for \( \Lambda = 3/8r_c^2 \) we obtain
$r_c \sim 10^{25} \ [m]$, i.e., $r_c \sim a_0/10$. Thus, we could say that for $z \gtrsim 9$ our universe remained inside $r_c$. What implications could it have this fact if we are thinking as feasible the existence of one extra dimension? It is an open question [16].

We finish this Section highlighting the relation between $\Lambda$ and $r_c$. This situation is interesting in itself. As we already said, the cosmological constant could have its origin in the projection of the extra dimension over our 4-spacetime. Seen in reverse, the observational value of $\Lambda$ could give us information on $r_c$ accepting that our 4-reality is a sort of "shadow" of a 5-reality.

IV. $C > 0$ CASE. WHITE HOLE SOLUTION

This case leads to a solution that can be understood as a white hole. To prove this we will generalize to the case of a Chern-Simons theory, a procedure introduced in the context of GR [10] to construct a white hole solution from a black hole solution.

From (34) we see that exist a singular behaviour at $F(r_0) = 0$, i.e., at $r_0^3 = 8 (r_0 + C) r_c^2$, such that

$$C = \frac{1}{8r_c^2} \left( r_0 - 2\sqrt{2}r_c \right) \left( r_0 + 2\sqrt{2}r_c \right) r_0,$$

and this means that $C > 0$ as long as $r_0 > 2\sqrt{2}r_c$. The equation $F(r_0) = 0$ has only a real and positive root and its discriminant tells us that

$$C > \frac{4}{3} \sqrt{\frac{2}{3}r_c} = \left( \frac{2}{3\sqrt{3}} \right) 2\sqrt{2}r_c,$$

and the solution for $r_0$ is given by

$$r_0 = \frac{1}{3} \left[ \left( 108C \pm 12\sqrt{3C} \sqrt{27 - 32r_c^2/C^2} \right) r_c^2 \right]^{1/3} + \frac{8r_c^2}{\left[ \left( 108C \pm 12\sqrt{3C} \sqrt{27 - 32r_c^2/C^2} \right) r_c^2 \right]^{1/3}}.$$

The solution (34) appears to have a singularity at $r = 0$ and $r = r_0$. The singularity at $r = r_0$ divides the coordinates in two disconnected patches. The exterior solution with $r > r_0$ is the one that is related to the gravitational fields of massive objects. The interior solution with $0 \leq r \leq r_0$, which contains the singularity at $r = 0$, is completely separated from the outer patch by the singularity at $r = r_0$. 

9
The singularity at \( r = r_0 \) is a coordinate singularity. This singularity arises from a bad choice of coordinates or coordinate conditions. An easiest way to see that at \( r = r_0 \) is to consider the metric in the Eddington-Finkelstein (EF) coordinates. For the ingoing EF coordinates \((v, r, t)\), which describe the line element in the \( I-II \) regions (the form of this regions as well as the regions \( III \) and \( IV \) are similar to that of the case of GR which are shown in Ref.[13]), we have

\[
ds^2 = -F(r) dt^2 + 2dvdr + r^2 d\Omega^2, \tag{47}
\]

which is regular at future horizon and the past horizon is at \( v = t + r^* = \infty \).

Correspondingly, the outgoing EF coordinates \((u, r, t)\) which describe the metric in the \( III-IV \) regions has the following form

\[
ds^2 = -F(r) dt^2 - 2dudr + r^2 d\Omega^2, \tag{48}
\]

which is regular at past horizon and the future horizon is at \( u \equiv t - r^* \), which can be interpreted as a new time coordinate where \( r^* \) is given by

\[
r^* = \beta \ln \left( \frac{r - r_0}{2r_0 \sqrt{(r - 2r_c)^2 + r_0 r/4r^2 + 2C/2}} \right) - \frac{8r_c (3C + r_0)}{(2r_0 + 3C) \sqrt{8(3C - r_0)/r_0}} \arctan \left( \frac{2r + r_0}{r_0 \sqrt{8(3C - r_0)/r_0}} \right), \tag{49}
\]

and

\[
\beta = -\frac{r_0^2}{2r_0 + 3C}. \tag{50}
\]

The coordinates used in (47) and (48) have the advantage that they describe the neighbourhood of the surface \( r = r_0 \) in a satisfactory way. However, the metrics (47) and (48) has still a deficiency, analogue to that appears in the Schwarzschild solution of GR. This deficiency is avoided in the Kruskal coordinates, which describe a geodesically complete space. Defining

\[
U = -\exp \left( -\frac{u}{2\beta} \right), \quad V = \exp \left( \frac{v}{2\beta} \right), \tag{51}
\]

we have that the Kruskal coordinates are given by

\[
T = \frac{1}{2}(U + V), \quad X = \frac{1}{2}(U - V). \tag{52}
\]
with which the line element takes the form

\[ ds^2 = -F_C(r) \left( dT^2 - dX^2 \right) + r^2 d\Omega^2, \quad (53) \]

where

\[ F_C > 0(r) = \frac{32 r_0 r_c^2}{3 r_0^2 - 8 r_c^2} F(r) \exp \left( \frac{3 r_0^2 - 8 r_c^2 r_0^*}{8 r_c^2} \right), \quad r > 0, \quad (54) \]

Following the usual procedure we construct a spacetime diagram in the \((T, X)\) plane (with \(\theta, \varphi\) suppressed) known as the Kruskal diagram, which represents the extended spacetime corresponding to the metric \((53)\), whose form is similar to that of the case of GR (see Ref. [13] where the aforementioned regions \(I, II, III, IV\) are shown).

Following the reference [10], the next step is to continue the metric in the region of the additive inverse of \(C\) carrying out the substitution \(C \rightarrow -\tilde{C}\), with \(\tilde{C} > 0\), which must be complemented with the transformation \(t \rightarrow -\tilde{t}, \ r \rightarrow -\tilde{r}, \ r_c \rightarrow -\tilde{r}_c\), because it is necessary to preserve the dynamics. Under this transformation we see that \((46)\) transforms as

\[ r_0 \rightarrow -\tilde{r}_0 = \frac{1}{3} \left[ - \left( 108 \tilde{C} \pm 12 \sqrt{3} \tilde{C} \sqrt{27 - 32 \tilde{r}_c^2 / \tilde{C}^2} \right) \tilde{r}_c^2 \right]^{1/3} + \frac{8 \tilde{r}_c^2}{\left[ - \left( 108 \tilde{C} \pm 12 \sqrt{3} \tilde{C} \sqrt{27 - 32 \tilde{r}_c^2 / \tilde{C}^2} \right) \tilde{r}_c^2 \right]^{1/3}}. \quad (55) \]

This means that \(C \rightarrow -\tilde{C}\) implies \(r_0 \rightarrow -\tilde{r}_0\). On the other hand under this transformation it is straightforward to see that \(\tilde{\beta} = -\beta\) and that the EF coordinate \(v = t + r^*\) takes the form

\[
\begin{align*}
\tilde{\nu} & = -\tilde{t} - \left\{ \tilde{\beta} \ln \left( \frac{(\tilde{r} - \tilde{r}_0)}{2 \tilde{r}_c \sqrt{\tilde{r} / 2 \tilde{r}_c} + \tilde{r}_0 \tilde{r} / 4 \tilde{r}_c^2 + 2 \tilde{C} / \tilde{r}_0} \right) \\
& \quad - \frac{8 \tilde{r}_c \left( 3 \tilde{C} + \tilde{r}_0 \right)}{\left( 2 \tilde{r}_0 + 3 \tilde{C} \right) \sqrt{8 \left( 3 \tilde{C} - \tilde{r}_0 \right) / \tilde{r}_0}} \arctan \left( \frac{2 \tilde{r} + \tilde{r}_0}{\tilde{r}_c \sqrt{8 \left( 3 \tilde{C} - \tilde{r}_0 \right) / \tilde{r}_0}} \right) \right\} - 2i \pi \tilde{\beta}, \\
\tilde{\nu} & = -\tilde{t} - \tilde{r}^* - 2i \pi \tilde{\beta} = -\nu - 2i \pi \tilde{\beta}, \quad (56)
\end{align*}
\]

where \(\tilde{r}^*\) is given by the expression in parentheses brace.
In the same way it is found that the Eddington-Finkelstein coordinate \( u = -t + r^* \) transform as

\[
u = -\tilde{t} + \tilde{r}^* + 2i\pi\tilde{\beta} = -u + 2i\pi\tilde{\beta}.
\] (57)

Equations (56) and (57) allow finding the continuation of equations (51) in the region of the additive inverse of \( C \), which can be interpreted as the negative mass region. In fact, under the transformation \( C \rightarrow -\tilde{C}, \ t \rightarrow -\tilde{t}, \ r \rightarrow -\tilde{r}, \ r_c \rightarrow -\tilde{r}_c \), the equations (51) transform as

\[
U \rightarrow \tilde{U} = \exp\left(-\frac{\tilde{u}}{2\tilde{\beta}}\right) = -U
\]

\[
V \rightarrow \tilde{V} = -\exp\left(\frac{\tilde{v}}{2\tilde{\beta}}\right) = -V.
\]

It is straightforward to see that this transformation implies that the Kruskal coordinates transform as

\[
T \rightarrow \tilde{T} = \frac{1}{2}(\tilde{U} + \tilde{V}) = -T,
\]

\[
\tilde{X} = \frac{1}{2}(\tilde{V} - \tilde{U}) = -X,
\]

that is, the inversion of the Kruskal coordinates. As in the case of general relativity, studied in [10], the spacetime with the variables \((\tilde{U}, \tilde{V})\), found using the transformations \( C \rightarrow -\tilde{C}, \ t \rightarrow -\tilde{t}, \ r \rightarrow -\tilde{r}, \ r_c \rightarrow -\tilde{r}_c \), can be understood as a space corresponding to a spacetime of the a white hole. This means that region \( I \) with coordinates \((U, V)\) corresponds to region \( III \) when we use coordinates \((U, V)\).

In the same way, region \( IV \), which in the usual coordinates corresponds to a white hole, is transformed into region \( II \) which corresponds, in the new coordinates, to a black hole. Obviously, region \( II \), which in the usual coordinates corresponds to a black hole, will be transformed into region \( IV \) which, in the new coordinates, corresponds to a white hole.

The interesting thing about this result is that the solutions for \( C < 0 \) and \( C > 0 \) are related through the transformation \( C \rightarrow -\tilde{C} \), with \( \tilde{C} > 0 \), \( t \rightarrow -\tilde{t}, \ r \rightarrow -\tilde{r}, \ r_c \rightarrow -\tilde{r}_c \).
V. CONCLUDING REMARKS

We have found black hole solutions from the 4-dimensional gravitational field equations corresponding to a four-dimensional gravity action, which was obtained by compactifying the so-called five-dimensional EChS gravity action using a mechanism developed in [7] and [8]. We have shown that such field equations lead to a spacetime with a cosmological constant inversely proportional to the square of the compactification radius and to a solution dependent on an arbitrary constant $C$ which gives rise to black white holes solutions: (i) when $C$ is negative, we find a SdS black hole, (ii) when $C = 0$, we have a black hole whose event horizon is a cosmological one and (iii) it is shown that the application of the $PTM$ transformation to the solution for $C < 0$ leads to a solution for $C > 0$ equivalent to an SdS white hole solution. The $PTM$ transformation means changing $r \rightarrow -\tilde{r}$, $t \rightarrow -\tilde{t}$, $r_c \rightarrow \tilde{r}_c$ in addition to changing $M \rightarrow -\tilde{M}$. The change of sign in the mass can be understood as a consequence of the requirement of the invariance of the metric under the $PTM$ symmetry.

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VI. APPENDIX. FIELD EQUATIONS

In 4-dimensions the static and spherically symmetric solution is given by

\[ ds^2 = -e^\alpha(r)dt^2 + e^\beta(r)dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \]  

(A1)

so that the non-zero Christoffel symbols turn out to be

\[
\begin{align*}
\Gamma^1_{00} &= \frac{\alpha'}{2} e^{\alpha-\beta}, \\
\Gamma^1_{11} &= \frac{\beta'}{2}, \\
\Gamma^1_{22} &= -re^{-\beta}, \\
\Gamma^2_{12} &= r^{-1} = \Gamma^2_{21}, \\
\Gamma^2_{33} &= -\sin \theta \cos \theta, \\
\Gamma^3_{13} &= r^{-1} = \Gamma^3_{31}, \\
\Gamma^3_{33} &= -re^{-\beta} \sin^2 \theta,
\end{align*}
\]  

(A2)

where a ”prime” denotes derivative with respect to \( r \). The next step is to find the nonzero components of the Ricci tensor

\[ R_{\alpha\beta} = \partial_\rho \Gamma^\rho_{\alpha\beta} - \partial_\beta \Gamma^\rho_{\alpha\rho} + \Gamma^\sigma_{\alpha\beta} \Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\alpha\rho} \Gamma^\rho_{\sigma\beta}, \]  

(A3)

and it is straightforward to obtain
\[
R_{00} = \frac{e^{\alpha - \beta}}{2} \left( \frac{\alpha''}{2} + \frac{\alpha'^2}{2} - \frac{\alpha' \beta'}{2} + 2 \frac{\alpha'}{r} \right), \\
R_{11} = -\frac{1}{2} \left( \frac{\alpha''}{2} - \frac{\alpha' \beta'}{2} - \frac{2 \beta'}{r} + \frac{\alpha'^2}{2} \right), \\
R_{22} = 1 - e^{-\beta} \left( 1 + \frac{r}{2} (\alpha' - \beta') \right), \\
R_{33} = R_{22} \sin^2 \theta, \\
R = g^{\mu \nu} R_{\mu \nu} = \frac{2}{r^2} - e^{-\beta} \left( \frac{\alpha''}{2} + \frac{\alpha'^2}{2} - \frac{\alpha' \beta'}{2} + \frac{2}{r} (\alpha' - \beta') + \frac{2}{r^2} \right), 
\]

(A4)

and then we obtain the non-zero components of \( G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R \)

\[
G_{00} = \frac{e^{\alpha}}{r^2} + e^{\alpha - \beta} \left( \frac{\beta'}{r} - \frac{1}{r^2} \right), \\
G_{11} = \frac{\alpha'}{r} + \frac{1}{r^2} - e^{\beta}, \\
G_{22} = -\frac{e^{-\beta}}{2} \left( \frac{\alpha' \beta'}{2} r^2 - \frac{\alpha'^2}{2} r^2 - \frac{\alpha'' r^2}{2} - r (\alpha' - \beta') \right), \\
G_{33} = G_{22} \sin^2 \theta. 
\]

(A5)