GENERALIZED LAWSON TORI AND KLEIN BOTTLES

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ABSTRACT. Using Takahashi theorem we propose an approach to extend known families of minimal tori in spheres. As an example, the well-known two-parametric family of Lawson tau-surfaces including tori and Klein bottles is extended to a three-parametric family of tori and Klein bottles minimally immersed in spheres. Extremal spectral properties of the metrics on these surfaces are investigated. These metrics include i) both metrics extremal for the first non-trivial eigenvalue on the torus, i.e. the metric on the Clifford torus and the metric on the equilateral torus and ii) the metric maximal for the first non-trivial eigenvalue on the Klein bottle.

1. Introduction

1.1. The Lamé equation and the statement of the main Theorem. The well-known Lamé equation is usually written as

$$\frac{d^2\varphi}{dz^2} + (h - n(n + 1)(k \text{sn } z)^2)\varphi = 0, \quad (1)$$

where $k$ is the module of $\text{sn } z$, see e.g. the books [1, 10].

In the case $n = 1$ three wonderful solutions of the Lamé equation (1) given by three Jacobi elliptic functions are known,

$$\text{Ec}_0^1(z) = \text{dn } z, \quad \text{Ec}_1^1(z) = \text{cn } z, \quad \text{Es}_1^1(z) = \text{sn } z,$$

where we use the Ec/Es notation for the solutions used by Ince in the paper [17]. These solutions correspond to

$$h = k^2, \quad h = 1, \quad h = 1 + k^2 \quad (2)$$

respectively.

The change of variable

$$\text{sn } z = \sin y \iff y = \text{am } z, \quad (3)$$

where $\text{am } z$ is Jacobi amplitude function, see e.g. the book [10] Section 13.9], transforms the Lamé equation (1) into a trigonometric form of the Lamé equation

$$\left[1 - (k \sin y)^2\right]\frac{d^2\varphi}{dy^2} - k^2 \sin y \cos y \frac{d\varphi}{dy} + [h - n(n + 1)(k \sin y)^2]\varphi = 0. \quad (4)$$

This trigonometric form of the Lamé equation is used in the book [1]. The change of variable $\text{sn } z = \cos y$ leads to another trigonometric form used in the book [10].

Using standard properties of the Jacobi elliptic functions and the change of variable (3) one obtains three solutions of the Lamé equation in the trigonometric form (4),

$$\text{Ec}_0^0(y) = \sqrt{1 - k^2 \sin^2 y}, \quad \text{Ec}_1^1(y) = \cos y, \quad \text{Es}_1^1(y) = \sin y. \quad (5)$$
Let us consider functions

\[
\begin{align*}
\varphi_1(y) &= \sqrt{\frac{b^2 + c^2 - a^2}{2(c^2 - a^2)}} \sin y, \\
\varphi_2(y) &= \sqrt{\frac{a^2 + c^2 - b^2}{2(c^2 - b^2)}} \cos y, \\
\varphi_3(y) &= \sqrt{\frac{a^2 + b^2 - c^2}{2(b^2 - c^2)}} \sqrt{1 - \frac{b^2 - a^2}{c^2 - a^2}} \sin^2 y. 
\end{align*}
\]

These functions are rescaled three solutions (5) of the Lamé equation in trigonometric form (4) with \( n = 1 \) and \( k = \sqrt{\frac{b^2-a^2}{c^2-a^2}}. \)

Let us denote by \( K(\cdot) \) and \( E(\cdot) \) the complete elliptic integrals of the first and second kind respectively defined as in the book [10] by formulae

\[
K(k) = \int_{0}^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\alpha^2}}, \quad E(k) = \int_{0}^{\pi/2} \frac{\sqrt{1-k^2\alpha^2}}{\sqrt{1-\alpha^2}} \, d\alpha.
\]

The main result of this paper is the following theorem.

**Theorem 1.** Let \( F_{a,b,c} : \mathbb{R}^2 \to S^5 \subset \mathbb{R}^6 \) be a three-parametric doubly-periodic immersion of the plane to the 5-dimensional sphere of radius 1 defined by the formula

\[
F_{a,b,c}(x,y) = (sin\,ax\varphi_1(y), \cos\,ax\varphi_1(y), \\
\sin bx\varphi_2(y), \cos bx\varphi_2(y), \sin cx\varphi_3(y), \cos cx\varphi_3(y)),
\]

where

a) either \( a, b, c \) are integers and \( |c| > \sqrt{a^2 + b^2} \),

b) or \( a, b \) are nonzero integers and \( |c| = \sqrt{a^2 + b^2} \).

Let \( \mathcal{L} = \{(2\pi n, 2\pi m)|n, m \in \mathbb{Z}\} \) and \( \tilde{F}_{a,b,c} : \mathbb{R}^2/\mathcal{L} \to S^5 \subset \mathbb{R}^6 \) be the natural map induced by \( F_{a,b,c}. \)

Let \( S(a,b,c) = \frac{4\pi}{\sqrt{2(a^2+b^2)}} \left( 2(c^2-a^2)E\left(\sqrt{\frac{b^2-a^2}{c^2-a^2}}\right) - (c^2-a^2-b^2)K\left(\sqrt{\frac{b^2-a^2}{c^2-a^2}}\right) \right). \)

Then the following statements hold.

1) The image \( T_{a,b,c} = \tilde{F}_{a,b,c}(\mathbb{R}^2) \) is a minimal compact surface in the 5-dimensional sphere \( (S^5). \)

2) The case b) corresponds to Lawson tau-surfaces \( \tau_{a,b} \cong T_{a,b,\sqrt{a^2+b^2}}. \) Distinct Lawson tau-surfaces correspond to unordered pairs \( a, b \geq 1 \) such that \( (a, b) = 1. \) The surface \( T_{a,b,\sqrt{a^2+b^2}} \) is a Lawson torus \( \tau_{a,b} \) if \( a \) and \( b \) are odd and \( T_{a,b,\sqrt{a^2+b^2}} \) is a Lawson Klein bottle \( \tau_{a,b} \) if either \( a \) or \( b \) is even, where we assume \( (a, b) = 1. \)

3) In the case b) the metric induced on \( \tau_{a,b} \cong T_{a,b,\sqrt{a^2+b^2}} \) is extremal for the functionals \( \Lambda_j(\mathbb{T}^2, g) \) if \( \tau_{a,b} \) is a Lawson torus or \( \Lambda_j(\mathbb{K}^2, g) \) if \( \tau_{a,b} \) is a Lawson Klein bottle, where \( j = 2 \left[ \frac{\sqrt{a^2+b^2}}{a} \right] + a + b - 1 \) and \( [\cdot] \) denotes the integer part. The corresponding value of the functional is

\[
\Lambda_j(\tau_{a,b}) = 8\pi a E\left(\frac{\sqrt{a^2+b^2}}{a}\right).
\]

4) In the case a) for an integer \( k \geq 1 \) one has \( T_{a,b,c} = T_{ka,kb,kc}. \) Moreover, \( T_{a,b,c}, T_{a,-b,c}, T_{a,b,-c} \) and \( T_{a,c} \) are isometric to \( T_{a,b,c}. \) Hence, it is sufficient to consider non-negative integer \( a, b, c \) satisfying conditions a) such that \( (a, b, c) = 1 \) and assume that \( (a, b, c) \) and \( (b, a, c) \) are equivalent.

5) In the case a) depending on the parity of \( a, b \) and \( c \) we have the following three subcases.

1) If \( a \) and \( b \) have different parity and \( c \) is even then the surface \( T_{a,b,c} \) is a Klein bottle and \( \tilde{F}_{a,b,c} : \mathbb{R}^2/\mathcal{L} \to T_{a,b,c} \) is a double covering. The area of \( T_{a,b,c} \) is equal to \( \frac{4}{\pi} S(a,b,c). \)
II) If \(a\) and \(b\) are odd and \(c\) is even then the surface \(T_{a,b,c}\) is a torus and \(\tilde{F}_{a,b,c} : \mathbb{R}^2/\mathbb{L} \to T_{a,b,c}\) is a double covering. The area of \(T_{a,b,c}\) is equal to \(\frac{1}{2}S(a,b,c)\).

III) Otherwise, the surface \(T_{a,b,c}\) is a torus and \(\tilde{F}_{a,b,c} : \mathbb{R}^2/\mathbb{L} \to T_{a,b,c}\) is a one-to-one map. The area of \(T_{a,b,c}\) is equal to \(S(a,b,c)\).

6) In the case a) the metric induced on the torus or the Klein bottle \(T_{a,b,c}\) is extremal for the functional \(\Lambda_S\) found in the paper [33]. Hsiang and Lawson briefly mentioned Otsuki tori but they but a detailed treatment of the Otsuki tori using Hsiang-Lawson approach could be applied in the case of \(S\) minimal tori and the formal proof.

1.2. Minimal tori in spheres. In his 1970 paper [23] Lawson introduced several families of minimal surfaces in \(S^3\) including a family \(\tau_{m,n}\).

**Definition 1.** A Lawson tau-surface \(\tau_{m,n} \ni S^3\) is defined as the image of the doubly-periodic immersion \(\Psi_{m,n} : \mathbb{R}^2 \ni S^3 \subset \mathbb{R}^4\) given by the explicit formula

\[
\Psi_{m,n}(x,y) = (\cos(mx) \cos y, \sin(mx) \cos y, \cos(nx) \sin y, \sin(nx) \sin y).
\]

Here and later \(\ni\) denotes an immersion.

Lawson proved that for each unordered pair of positive integers \((m,n)\) with \((m,n) = 1\) the surface \(\tau_{m,n}\) is a distinct compact minimal surface in \(S^3\). Let us impose the condition \((m,n) = 1\). If both integers \(m\) and \(n\) are odd then \(\tau_{m,n}\) is a torus. We call it a Lawson torus. If one of integers \(m\) and \(n\) is even then \(\tau_{m,n}\) is a Klein bottle. We call it a Lawson Klein bottle. Remark that \(m\) and \(n\) cannot both be even due to the condition \((m,n) = 1\). The torus \(\tau_{1,1}\) is the Clifford torus.

As explained in the statement 2) of Theorem 1 the surfaces \(T_{a,b,c}\) introduced in the Theorem 1 are generalizations of the Lawson tau-surfaces.

Since Lawson paper [25] several methods for constructing or describing minimal tori in spheres were developed. An exhaustive review of all such methods requires writing a book, hence we can mention here only several ones.

Hsiang and Lawson developed in their paper [16] a theory of reduction of a minimal submanifold by a group action. This theory reduces the question about construction of \(S^1\)-invariant minimal tori to the question about construction of closed geodesics, which is much simpler. As the simplest example of application of this approach one can consider a family of Otsuki tori \(O_x\) minimally immersed in \(S^3\). They were introduced by Otsuki in his paper [31] using another approach, but a detailed treatment of the Otsuki tori using Hsiang-Lawson approach could be found in the paper [33]. Hsiang and Lawson briefly mentioned Otsuki tori but they gave also new examples of \(S^1\)-invariant minimal tori in \(S^3\). As another example one can mention the paper [11] by Ferus and Pedit where Hsiang-Lawson approach is applied in the case of \(S^1\)-invariant minimal tori in \(S^4\).
Unfortunately, this approach gives a description of families of minimal tori but does not give explicit formulae for these tori. For example, Otsuki tori \( O_{\frac{p}{q}} \) are in one-to-one correspondence with rational numbers \( \frac{p}{q} \) such that \( \frac{1}{2} < \frac{p}{q} < \frac{\sqrt{2}}{2} \), \( p, q > 0 \), \( (p, q) = 1 \), but we do not know explicit formulae for Otsuki tori since the reconstruction of the torus \( O_{\frac{p}{q}} \) from a fraction \( \frac{p}{q} \) requires solving a transcendental equation and a system of ODEs.

Another approach is based on methods of integrable systems and describes minimal tori in spheres through algebraic geometry. This approach was developed by many authors in different particular cases starting from the paper [15] by Hitchin dealing with the case of \( S^3 \) and finishing by the paper [4] by Burstall dealing with the general case of \( S^n \). In fact, the investigation of minimal tori in spheres was a part of an extended study by many authors of harmonic maps from tori into symmetric spaces. Let us mention here e.g. the paper [12] by Ferus, Pedit, Pinkall and Sterling dealing with the case of \( S^4 \). We refer the reader to the recent paper [6] by Carberry containing a review of the current situation of this approach with an extended list of references.

This method describes all minimal tori in \( S^n \) through data including an algebraic curve of genus \( \gamma \), a divisor \( D = P_1 + \cdots + P_\gamma \) consisting of \( \gamma \) points and some additional data satisfying so called periodicity conditions. The good news is that a minimal torus can be reconstructed from these algebro-geometric spectral data through complicated but in principle explicit formulae involving theta-functions of genus \( \gamma \). The bad news is that there is no constructive description of algebro-geometric spectral data satisfying the periodicity conditions.

In the papers [5, 6] Carberry studied intensively these periodicity conditions but provided only existence results. We would like to cite here one of them interesting for our goals.

**Theorem 2** (Carberry, [5]). For each integer \( n \geq 0 \), there are countably many real \( n \)-dimensional families of minimal immersions from rectangular tori to \( S^3 \). Each family consists of maps from a fixed torus.

There exist also several other approaches generating particular examples of minimal surfaces in spheres. We would like to mention here an approach by Mironov for constructing Hamiltonian-minimal Lagrangian embeddings in \( \mathbb{C}^N \) based on intersections of real quadrics of special type, see the paper [27]. As a by-product of this construction one obtains minimal surfaces in spheres, see the paper [21] by Karpukhin for more details. This approach is interesting for us since it provides a two-parametric family \( M_{m,k} \) of tori minimally immersed in \( S^3 \). It could be easily verified that this family is a subfamily of our family from Theorem 1, \( M_{m,k} \equiv T_{m,k,m+k} \). This family was described in conformal coordinates in the paper [13] by Haskins and in the paper [19] by Joyce, but it seems that this family first appeared in a parametrization similar to \( T_{m,k,m+k} \) in the paper [28] by Mironov. One can find a detailed study of \( M_{m,k} \) in the paper [21] by Karpukhin.

Thus, minimal tori in spheres are described but in implicit way and only several explicitly parametrized examples are known. However, recent progress in study of extremal metrics brings minimal surfaces in spheres back to our attention.

### 1.3. Extremal metrics and minimal surfaces in spheres

Let \( M \) be a closed surface and \( g \) be a Riemannian metric on \( M \). Let us consider the associated Laplace-Beltrami operator \( \Delta : C^\infty(M) \rightarrow C^\infty(M) \),

\[
\Delta f = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j} \right),
\]
and its eigenvalues

\[ 0 = \lambda_0(M, g) < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \lambda_3(M, g) \leq \ldots \]

Since the eigenvalues possess the following rescaling property,

\[ \forall t > 0 \quad \lambda_i(M, tg) = \frac{\lambda_i(M, g)}{t}, \]

it is natural to consider “normalized” eigenvalues

\[ \Lambda_i(M, g) = \frac{\lambda_i(M, g)}{\text{Area}(M, g)} \]

invariant under the rescaling transformation \( g \mapsto tg \).

Let us fix the surface \( M \) and consider \( \Lambda_i(M, g) \) as a functional \( g \mapsto \Lambda_i(M, g) \) on the space of all Riemannian metrics on \( M \).

It turns out that the question about the supremum \( \sup \Lambda_i(M, g) \) of the functional \( \Lambda_i(M, g) \) over the space of Riemannian metrics \( g \) on a fixed surface \( M \) is very difficult and only few results are known.

It is known that this supremum is finite since functionals \( \Lambda_i(M, g) \) are bounded from above. It was proven in the paper [35] by Yang and Yau that for an orientable surface \( M \) of genus \( \gamma \) the following inequality holds,

\[ \Lambda_1(M, g) \leq 8\pi(\gamma + 1). \]

Korevaar proved in the paper [23] that there exists a constant \( C \) such that for any \( i > 0 \) and any compact surface \( M \) of genus \( \gamma \) the functional \( \Lambda_i(M, g) \) is bounded,

\[ \Lambda_i(M, g) \leq C(\gamma + 1)i. \]

**Definition 2.** A metric \( g_0 \) on a fixed surface \( M \) is called maximal for the functional \( \Lambda_i(M, g) \) if

\[ \sup \Lambda_i(M, g) = \Lambda_i(M, g_0), \]

where the supremum is taken over the space of Riemannian metrics \( g \) on the fixed surface \( M \).

Only few maximal metrics are known at this moment. The maximal metric for \( \Lambda_1(S^2, g) \) is the standard metric on the sphere (Hersch, [14]), the maximal metric for \( \Lambda_1(\mathbb{R}P^2, g) \) is the standard metric on the projective plane (Li and Yau, [26]), the maximal metric for \( \Lambda_1(T^2, g) \) is the metric on equilateral torus (Nadirashvili, [29]).

The last known (and quite surprising) maximal metric is the maximal metric for the first eigenvalue \( \Lambda_1(Kl, g) \) on the Klein bottle. As it was proved in El Soufi, Giacomini and Jazar paper [27] using results of Jakobson, Nadirashvili and Polterovich paper [18], this is the metric on the bipolar Lawson surface \( \tilde{\tau}_{3,1} \).

We know also an example where \( \sup \Lambda_1(M, g) \) is known, but however there is no (smooth) maximal metric. It was proved by Nadirashvili in the paper [30] that \( \sup \Lambda_2(S^2, g) = 16\pi \) and the maximum is reached on a singular metric which can be obtained as the metric on the union of two touching spheres of equal radius with canonical metric.

If one would like to find a maximum of a function of several variables, then one usually starts by finding extrema of this function. The same idea is also reasonable for the functionals \( \sup \Lambda_i(M, g) \). However, one should be careful here. The functional \( \Lambda_i(M, g) \) depends continuously on the metric \( g \), but this functional is not differentiable. However, for analytic deformations \( g_t \) the left and right derivatives of the functional \( \Lambda_i(M, g_t) \) with respect to \( t \) exist, see the papers by Berger [3], Bando and Urakawa [2], El Soufi and Ilias [9]. This led to the following definition, see the paper [29] by Nadirashvili and the papers [8, 9] by El Soufi and Ilias.
Definition 3. A Riemannian metric \( g_0 \) on a closed surface \( M \) is called extremal metric for the functional \( \Lambda_1(M,g) \) if for any analytic deformation \( g_t \) the following inequality holds,

\[
\left. \frac{d}{dt} \Lambda_1(M,g_t) \right|_{t=0^+} - \left. \frac{d}{dt} \Lambda_1(M,g_t) \right|_{t=0^-} \leq 0.
\]

Investigation of extremal metrics turned out to be useful. For example, Jakobson, Nadirashvili and Polterovich proved in the paper [18] that the mentioned above metric on the Klein bottle realized as the bipolar Lawson surface \( \tilde{\tau}_{3,1} \) is extremal for \( \Lambda_1(\mathbb{K}^2, g) \) and using this result El Soufi, Giacomini and Jazar proved in the paper [7] the above mentioned result that this metric is the unique extremal metric and the maximal one.

As one can expect, we know more about extremal metrics then about maximal metrics. El Soufi and Ilias proved in the paper [8] that the only extremal metric for \( \Lambda_1(T^2, g) \) different from the maximal one is the metric on the Clifford torus.

Let us remark that the metrics on the surfaces \( T_{a,b,c} \) from Theorem 1 includes both metrics extremal for the first eigenvalue on the torus, i.e. the metric on the Clifford torus \( \tau_{1,1} \) and the metric on the equilateral torus \( M_{1,1} \). Hence, \( T_{a,b,c} \) includes surfaces carrying all extremal metrics for the first eigenvalue on the torus and Klein bottle.

Extremality of several families of metrics on the torus and Klein bottles was investigated recently.

- Lapointe studied metrics on bipolar Lawson surfaces \( \tilde{\tau}_{m,k} \) in his 2008 paper [24].
- The author studied metrics on Lawson surfaces \( \tau_{m,k} \) and metrics on Otsuki tori \( \tilde{O}_{p,q} \) in his 2012 paper [32] and 2013 paper [33] respectively.
- Karpukhin studied metrics on bipolar Otsuki tori \( \tilde{O}_{p,q} \) and on the family of tori \( M_{m,k} \) in his 2013 paper [20] and the paper [21] respectively.
- Karpukhin also proved in the paper [22] that all metrics mentioned in this list are not maximal except metrics on \( M_{1,1} \) (the equivalateral torus) and \( \tilde{\tau}_{3,1} \).

The significant progress in study of extremal metrics in the papers [20, 21, 22, 32, 33] became possible due to El Soufi-Ilias theorem establishing relation between extremal metrics and minimal surfaces in spheres.

Let \( M \) be a two-dimensional minimally immersed submanifold of the standard sphere \( S^n \subset \mathbb{R}^{n+1} \) of radius 1. Let \( \Delta \) be the Laplace-Beltrami operator on \( M \) equipped with the induced metric.

Let us introduce the Weyl eigenvalues counting function

\[
N(\lambda) = \#\{\lambda_i | \lambda_i < \lambda \}.
\]

Remember that we count the eigenvalues starting from \( \lambda_0 = 0 \).

Theorem 3 (El Soufi, Ilias, [9]). The metric induced on \( M \) by the immersion \( M \hookrightarrow S^n \) is an extremal metric for the functional \( \Lambda_{N(2)}(M,g) \).

Thus, investigation of (smooth) extremal metrics on surfaces could be done in the following way:

- find a minimal surface in a sphere,
- find \( N(2) \),
- then the metric induced on the minimal surface is extremal for the functional \( \Lambda_{N(2)} \).
However, it is not easy to follow this approach. As we discussed in Section 1.2, even the descriptions of minimal tori in spheres are quite complicated and implicit. Moreover, it turns out that there is no known general way to find $N(2)$ and in each example one should invent an ad hoc argument.

All mentioned above successful examples of application of this approach in papers [20, 21, 22, 32, 33] share the following features:

- these surfaces were already known to be minimal in spheres,
- their metrics are metrics of revolution,
- either their parametrisation is explicit (Lawson surfaces and Lawson bipolar surfaces) or the structure of zeroes of immersion functions is simple (Otsuki tori and Otsuki bipolar tori).

At this moment there is no hope to investigate all extremal metrics on tori since this requires at least a constructive description of minimal tori in spheres and it seems that the existing implicit description in terms of algebro-geometric data could not be improved. Thus we concentrate now our efforts on investigating particular examples of extremal metrics. This leads us to the problem of finding new explicit examples of minimal tori in spheres.

1.4. Constructing explicit examples of minimal tori via Takahashi theorem. Let us recall the well-known result about description of the minimal surfaces in $\mathbb{R}^n$ in terms of harmonic functions.

**Proposition 1.** A submanifold $M \rightarrow \mathbb{R}^n$ is minimal if and only if the restrictions $x^1|_M, \ldots, x^n|_M$ to $M$ of the coordinate functions in $\mathbb{R}^n$ are harmonic with respect to the Laplace-Beltrami operator $\Delta^M$ on $M$ equipped with the induced metric,

$$\Delta^M x^i|_M = 0.$$ 

One can rewrite this Proposition in terms of isometric immersions.

**Proposition 2.** An isometric immersion $f : M \rightarrow \mathbb{R}^n$ is minimal if and only if the components of the immersion $f = (f^1, \ldots, f^n)$ are harmonic with respect to the Laplace-Beltrami operator $\Delta$ on $M$,

$$\Delta f^i = 0.$$ 

If an isometric immersion by harmonic functions (i.e. eigenfunction of $\Delta$ with eigenvalue 0) is minimal in $\mathbb{R}^n$, what can we say about isometric immersions by eigenfunctions of $\Delta$ with a common eigenvalue $\lambda$? The answer is given by the Takahashi theorem.

**Theorem 4** (Takahashi, [34]). An isometric immersion $f : M \rightarrow \mathbb{R}^{n+1}$, where $f = (f^1, \ldots, f^{n+1})$, defined by eigenfunctions $f^i$ of the Laplace-Beltrami operator $\Delta$ with a common eigenvalue $\lambda$, 

$$\Delta f^i = \lambda f^i,$$

possesses the following properties,

- the image $f(M)$ lies on the sphere $S^m_R$ of radius $R$ with the center at the origin such that

$$\lambda = \frac{\dim M}{R^2},$$

(10)

- the immersion $f : M \rightarrow S^m_R$ is minimal.

If $f : M \rightarrow S^m_R$, where $f = (f^1, \ldots, f^{n+1})$, is a minimal isometric immersion of a manifold $M$ into the sphere $S^m_R$ of radius $R$, then $f^i$ are eigenfunctions of the Laplace-Beltrami operator $\Delta$,

$$\Delta f^i = \lambda f^i,$$

with the same eigenvalue $\lambda$ such that $\lambda = \frac{\dim M}{R^2}$. 


Takahashi theorem describes minimal immersions in terms of eigenfunctions of the Laplace-Beltrami operator. This is a system of PDEs equivalent to the standard system of PDEs describing minimal immersions in terms of mean curvature normal vector. This description is more natural from the point of view of spectral geometry.

Surprisingly, this approach did not generate much interest till very recently. To the best of the author’s knowledge, the only known successful application of this approach to construction of minimal surfaces in spheres was in the above mentioned paper [18]. In this paper using Takahashi theorem and properties of eigenfunctions Jakobson, Nadirashvili and Polterovich constructed a minimal isometric immersion of the Klein bottle to \( S^4 \) such that the corresponding metric is extremal for the first eigenvalue. This metric turned out to be Lawson bipolar surface \( \tilde{\tau}_{3,1} \).

However, solving a PDE system is a difficult task. Is it possible to find at least some new particular examples of minimal surfaces in spheres using the Takahashi theorem? In the present paper we propose an approach based on the Takahashi theorem leading us to an extension of known families of minimal tori in spheres using solving systems of algebraic equations. On this way we obtain Theorem 1.

2. Extension of families of minimal tori in spheres using Takahashi theorem

Let us start with a very naive idea. Let \( x, y \) be coordinates in the plane \( \mathbb{R}^2 \). Let us choose randomly a second order elliptic differential operator \( L \) on the plane invariant with respect to the translations \( (x, y) \mapsto (x + 2\pi, y) \), \( (x, y) \mapsto (x, y + 2\pi) \).

Consider spectral problem for \( L \) with periodic boundary conditions

\[
\begin{cases}
L\psi = \lambda\psi, \\
\psi(x, y) = \psi(x + 2\pi, y) = \psi(x, y + 2\pi).
\end{cases}
\]

Let \( 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \) be the spectrum of this spectral problem. Let us choose an eigenvalue \( \lambda_j \) and several linearly independent eigenfunctions \( f_1, \ldots, f_{n+1} \), corresponding to the eigenvalue \( \lambda_j \). Consider the map

\[ f : \mathbb{R}^2/L \longrightarrow \mathbb{R}^{n+1}, \]

where \( f = (f_1, \ldots, f_{n+1}) \).

Let \( g \) be the pullback \( f^*g_0 \) of the standard euclidean metric \( g_0 \) in \( \mathbb{R}^{n+1} \) to the torus \( \mathbb{R}^2/L \) by the map \( f \).

It follows from Takahashi Theorem 4 that if

(a) \( g \) is a Riemannian metric (i.e. positive definite) and

(b) the Laplace-Beltrami operator for \( g \) coincides with the initial differential operator \( L \),

then \( f \) is a minimal isometric immersion of the torus \( (\mathbb{R}^2/L, g) \) to a sphere \( S^n_R \) of radius \( R \) such that \( \lambda_j = \frac{2}{R^2} \).

This idea is naive since there is practically no chance that starting from a randomly chosen operator \( L \) one can satisfy conditions (a) and (b). Hence one should find a way to start with a smart choice of the initial operator \( L \).

Then it is time to remember Carberry Theorem 2. This theorem is for the case of \( S^3 \) but we can hope that in the general case minimal tori in spheres also like to exist in families. That’s why the key idea is the following: start with the operator \( L \) already known to be the Laplace-Beltrami operator on a minimal torus in a sphere.

Let us now remember that known examples of extremal metrics are metrics on tori of revolution or on quotients of tori on revolution, e.g. Lawson tori [32], Otsuki tori [33], bipolar Lawson tori [24], bipolar Otsuki tori [20] and the family \( M_{m,k} \) considered in [21]. Let us then restrict our attention to tori of revolution.
Since by rescaling we can always restrict our attention to the case of the sphere of radius 1 corresponding by formula (10) to $\lambda = 2$, we are interested in the equation

$$L\psi = 2\psi$$

(11)

and its solutions of the form

$$\psi(x, y) = \varphi(y) \sin mx \quad \text{or} \quad \psi(x, y) = \varphi(y) \cos mx.$$  

After separation of variables in PDE (11) one obtains a linear second order ODE

$$\varphi''(y) + a(m, y)\varphi'(y) + b(m, y)\varphi(y) = 0$$

(12)

with periodic boundary conditions

$$\varphi(y + 2\pi) = \varphi(y).$$

Thus, we propose the following method for constructing minimal tori of revolution in spheres.

- Consider an equation

$$\varphi''(y) + A(\nu, y)\varphi'(y) + B(\nu, \mu, y)\varphi(y) = 0$$

(14)

with a spectral parameter $\mu$ and (possibly) an additional parameter $\nu$ such that this equation is already known to appear after a separation of variables in the spectral problem (11) for the Laplace-Beltrami operator on a minimal torus in the unitary sphere.

- Take solutions $\varphi_1(y), \ldots, \varphi_l(y)$ of equation (14) with periodic boundary conditions (13), corresponding to $\mu = \mu_1, \ldots, \mu = \mu_l$, respectively. These solutions depend on the parameter $\nu$.

- Consider the map $f : \mathbb{R}^2/L \to \mathbb{R}^l$ given by the formula

$$f(x, y) = (c_1\varphi_1(y) \cos m_1 x, c_1\varphi_1(y) \sin m_1 x, \ldots, c_l\varphi_l(y) \cos m_l x, c_l\varphi_l(y) \sin m_l x),$$

where $c_1, \ldots, c_l$ are constants.

- Let $g = f^*g_0$, where $g_0$ is the standard euclidean metric on $\mathbb{R}^l$.

- Find the Laplace-Beltrami operator $L$ for the metric $g$. Remark that $L$ depends on $\nu, c_1, \ldots, c_l, m_1, \ldots, m_l$.

- Separate variables in the equation $L\psi(x, y) = 2\psi(x, y)$ and obtain the equation

$$\varphi''(y) + a_\nu(c, \varphi_1, \ldots, \varphi_l; \nu, \mu, m; y)\varphi'(y) + b_\nu(c, \varphi_1, \ldots, \varphi_l; \nu, \mu, m; y)\varphi(y) = 0.$$  

(15)

- Write down a system of algebraic (i.e. not differential) equations

$$\begin{align*}
g_1(\nu, c_1, \ldots, c_l, m_1, \ldots, m_l) &= 0, \\
&\vdots \\
g_N(\nu, c_1, \ldots, c_l, m_1, \ldots, m_l) &= 0,  \\
\end{align*}$$

(16)

equivalent to the condition of coincidence of initial equation (14) with equation (15), i.e.

$$\begin{align*}
A(\nu, y) &\equiv a_\nu(c, \varphi_1, \ldots, \varphi_l; \nu, \mu_1, m_1; y), \\
B(\nu, \mu_1, y) &\equiv b_\nu(c, \varphi_1, \ldots, \varphi_l; \nu, \mu_1, m_1; m_1; y), \\
&\vdots \\
A(\nu, y) &\equiv a_\nu(c, \varphi_1, \ldots, \varphi_l; \nu, \mu_l, m_l; y), \\
B(\nu, \mu_l, y) &\equiv b_\nu(c, \varphi_1, \ldots, \varphi_l; \nu, \mu_l, m_l; m_l; y),
\end{align*}$$

(17)

and add the condition that $g$ is positive definite. The sign “$\equiv$” in system (17) means “equals identically with respect to $y$.”
If system of equations (16) and the condition that $g$ is positive definite have a solution $(\nu, c_1, \ldots, c_l, m_1, \ldots, m_l)$, then by Takahashi Theorem [11] the map $f$ with these values of parameters is an isometric immersion and the image $f(\mathbb{R}^3)$ is minimal in the unitary sphere $S^{2l-1}$.

Let us consider Lawson tau-surfaces and equation (12) appearing after separation of variables in the spectral problem for the Laplace-Beltrami operator. As we know from the paper [32], this equation is the Lamé equation in trigonometric form (14). Let us then in order to give an example of the proposed method apply the described above algorithm to the Lamé equation in trigonometric form (14). This gives us the family of surfaces from Theorem [11].

In fact, we should remark that in this example we modify a little bit the proposed approach since the image of $f$ could be not only a torus but also a quotient of a torus, e.g. a Klein bottle.

3. Proof of Theorem [11]

Let us apply the algorithm from Section 2 taking the Lamé equation in trigonometric form (14) as equation (14) and its classical solutions (5) as solutions, i.e. $l = 3$ and

$$
\varphi_1(y) = \sin y, \quad \varphi_2(y) = \cos y, \quad \varphi_3(y) = \sqrt{1 - k^2 \sin^2 y},
$$

where $k$ is the module of the Lamé equation. Here $k$ plays the role of the additional parameter $\nu$ in equation (14) that we should also find from system (16). The parameter $h$ in the Lamé equation plays the role of the spectral parameter $\mu$ in equation (14) and as the values $\mu$, we take values of $h$ from formulae (2).

Then we have

$$
f(x, y) = (c_1 \sin y \cos m_1 x, c_1 \sin y \sin m_1 x, c_2 \cos y m_2 x, c_2 \cos y \sin m_2 x, c_3 \sqrt{1 - k^2 \sin^2 y} \cos m_3 x, c_3 \sqrt{1 - k^2 \sin^2 y} \sin m_3 x).
$$

If $c_1, c_2, c_3, m_1, m_2, m_3, k$ satisfy system (16) then by Takahashi theorem [11] the image of $f$ is on the unitary sphere. This condition is equivalent to the equation

$$
-2 + c_1^2 + c_2^2 + 2c_3^2 - c_3^2 k^2 + (-c_1^2 + c_2^2 + c_3^2 k^2) \cos 2y \equiv 0.
$$

It follows that

$$
c_1^2 = 1 - c_3^2 + c_3^2 k^2, \quad c_2^2 = 1 - c_3^2.
$$

A straightforward calculation shows that the metric $g = f^* g_0$ is given by the formula

$$
g = \frac{1}{2} [n_3^2 (1 - c_3^2) + m_3^2 c_3^2 (2 - k^2) + m_1^2 (1 + c_3^2 (k^2 - 1)) + (m_2^2 (1 - c_3^2) + m_3^2 c_3^2 k^2 + m_1^2 (c_3^2 - c_3^2 k^2 - 1)) \cos 2y] dx^2 + k^2 - 2 - 2c_3^2 (k^2 - 1) - k^2 \cos 2y \cos 2y \cos 2y dy^2,
$$

where we already applied formulae (18) to eliminate $c_1$ and $c_2$.

A straightforward calculation shows that the first equation in system (17) implies the equation

$$
((k^2 - 1)(n_2^2 (c_3^2 - 1)^2 - m_2^2 c_3^2 k^2) + m_2^2 (1 + c_3^2 (k^2 - 1)) (2 - k^2 + k^2 \cos 2y) \equiv 0.
$$

It follows that

$$
n_3^2 = m_2^2 (c_3^2 - 1)^2 (2 - k^2 + k^2 \cos 2y) \equiv 0.
$$

We use this formula to eliminate $n_3$ from our system of equations.
In the same way we investigate the second equation in system \((17)\). We do not write down this equation explicitly since it is a huge expression. It turns out that this equation implies that

\[
k^2 = \frac{(m_1^2 - m_2^2)(2c_3^2 - 1)}{2m_1^2c_3^2 + m_2^2(1 - 2c_3^2)}.
\]

(20)

Hence, we have three-parametric family of solutions parametrized by \(m_1, m_2\) and \(c_3\). However, it is more convenient to parametrize by \(m_1, m_2\) and \(m_3\). One can eliminate \(k\) from equation \((19)\) using equation \((20)\) and obtain the equation

\[
m_3^2 = \frac{m_1^2 + m_2^2 - 2m_2^2c_3^2}{1 - 2c_3^2}.
\]

(21)

Then one can find \(c_3\) from equation \((21)\) and obtain the formula

\[
c_3^2 = \frac{m_1^2 + m_2^2 - m_3^2}{2(m_1^2 - m_3^2)}.
\]

(22)

Now we can substitute formula \((22)\) in equations \((20), (18)\) and obtain the formulae

\[
c_1^2 = \frac{-m_1^2 + m_2^2 + m_3^2}{2(-m_1^2 + m_3^2)}, \quad c_2^2 = \frac{m_1^2 - m_2^2 + m_3^2}{2(-m_2^2 + m_3^2)},
\]

\[
c_3^2 = \frac{m_1^2 + m_2^2 - m_3^2}{2(m_2^2 - m_3^2)}, \quad k^2 = \frac{m_1^2 - m_2^2}{m_1^2 - m_3^2}.
\]

Let us now define \(\bar{\varphi}_i = m_i \varphi_i\) and rename \(a = m_1, b = m_2, c = m_3\) for the sake of simplicity. Extracting square roots one obtains formulae \((6), (7)\) and \((8)\).

Now we should satisfy periodicity conditions and also choose such \(a, b, c\) that \(F_{abc}\) is a real map. This is equivalent to conditions

- \(a\) is integer and \(\frac{a^2 + b^2 - c^2}{2(c^2 - a^2)} > 0\) or
- \(a\) is arbitrary and \(\frac{b^2 + c^2 - a^2}{2(c^2 - b^2)} = 0\),

and

- \(b\) is integer and \(\frac{a^2 + b^2 - c^2}{2(c^2 - b^2)} > 0\) or
- \(b\) is arbitrary and \(\frac{a^2 + c^2 - b^2}{2(c^2 - a^2)} = 0\),

and

- \(c\) is integer and \(\frac{a^2 + b^2 - c^2}{2(b^2 - c^2)} > 0\) or
- \(c\) is arbitrary and \(\frac{a^2 + b^2 - c^2}{2(a^2 - c^2)} = 0\).

We should also add the condition that the metric \(g\) is positive definite.

The solution of this system of conditions is exactly written in phrases a) and b) in the statement of Theorem \([1]\). Direct check shows that in fact, the values of \(a, b\) and \(c\) satisfying these conditions give minimal immersion of \(\mathbb{R}^2/L\) to the sphere \(S^5\) and the spectral problem after a separation of variables transforms into the Lamé equation. This finishes the proof of the statement 1) of Theorem \([1]\).

In the case b) our surfaces are Lawson tau-surfaces. One obtains this by direct calculation. If \(c = \sqrt{a^2 + b^2}\) then \(\varphi_1(y) = \sin y, \varphi_2(y) = \cos y\) and \(\varphi_3(y) = 0\). It is sufficient now to remark that i) four first entries of the vector \(F_{a,b,c}\) coincide with \(\Psi_{a,b}\) from \([6]\) and the remaining two entries are zeroes and ii) \(\tau_{a,b,c} \cong \tau_{b,a}\). Then the statements 2) and 3) of Theorem \([1]\) follows from the papers \([25, 32]\).

This finishes the investigation of the case b) and in the following we consider only the case a).

The images \(T_{a,b,c} = F_{a,b,c}(\mathbb{R}^2)\) could be isometric for distinct triples \((a, b, c)\). It is clear that for any integer \(k > 0\) one has \(T_{a,b,c} = T_{ka,kb,kc}\). One can also remark that \(T_{a,b,c}\) and \(T_{-a,b,c}\) are isometric since \(T_{-a,b,c}\) is the image of \(T_{a,b,c}\) under the
reflection of the ambient space $\mathbb{R}^6$ with respect to the hyperplane $x^1 = 0$. Similar statements are true for $T_{a, -b, c}$ and $T_{a, b, -c}$.

Let us denote by $R$ the isometry
\[ R(x^1, x^2, x^3, x^4, x^5, x^6) = (x^3, x^4, -x^1, -x^2, x^5, x^6) \]
of the ambient space $\mathbb{R}^6$. Then we have the identity $R \circ F_{a, b, c}(x, y) = F_{a, b, c}(x, y)$. It follows that $T_{b, a, c}$ is isometric to $T_{a, b, c}$. This implies the statement 4) of Theorem 1.

Next, let us remark that $\tilde{F}_{a, b, c}$ is not necessarily a one-to-one map of $\mathbb{R}^2 / \mathcal{L}$ on $T_{a, b, c}$ because there could exist a non-trivial (i.e. different from shifts by $2\pi$) map $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that
\[ F_{a, b, c} \circ \Phi = F_{a, b, c}. \] (23)

Let $(x_2, y_2) = \Phi(x_1, y_1)$, then equation (23) is equivalent to the system of equations
\[ \sin ax_1 \sin y_1 = \sin ax_2 \sin y_2, \] (24)
\[ \cos ax_1 \sin y_1 = \cos ax_2 \sin y_2, \] (25)
\[ \sin bx_1 \cos y_1 = \sin bx_2 \cos y_2, \] (26)
\[ \cos bx_1 \cos y_1 = \cos bx_2 \cos y_2, \] (27)
\[ \sin cx_1 \sqrt{1 - k^2 \sin^2 y_1} = \sin cx_2 \sqrt{1 - k^2 \sin^2 y_2}, \] (28)
\[ \cos cx_1 \sqrt{1 - k^2 \sin^2 y_1} = \cos cx_2 \sqrt{1 - k^2 \sin^2 y_2}. \] (29)

For generic $x_1, y_1$ one has $\sin y_1 \neq 0$ and $\sin y_2 \neq 0$. Hence we can divide equation (24) by equation (25) and obtain
\[ \tan ax_1 = \tan ax_2 \iff x_1 - x_2 = \frac{\pi}{a} k, \quad k \in \mathbb{Z}. \]

In the same way we obtain
\[ x_1 - x_2 = \frac{\pi}{b} l = \frac{\pi}{c} n, \quad l, n \in \mathbb{Z}. \]

from equations (26)–(29). Hence we are looking for integer $k, l$ and $n$ such that
\[ \frac{k}{a} = \frac{l}{b} = \frac{n}{c}. \]

Let $\frac{k}{a} = \frac{l}{b} = \frac{n}{c}$, where $(p, q) = 1$, $q > 0$. Then $k = \frac{ma}{q}$ and it follows that $q$ divides $a$. In the same way we prove that $q$ divides $b$ and $c$. But we assume $(a, b, c) = 1$, hence $q = 1$ and $k = pa$, $l = pb$, $n = pc$. It follows that
\[ x_1 - x_2 = \frac{\pi}{a} pa = \frac{\pi}{b} pb = \frac{\pi}{c} pc = p\pi, \quad p \in \mathbb{Z}. \]

Since all functions in system (24)–(29) are $2\pi$-periodic, it is sufficient to consider the case $p = 0$ and the case $p = 1$.

In the case $p = 0$ we have $x_2 = x_1$ and system (24)–(26) implies the following system of equations,
\[ \sin y_1 = \sin y_2, \quad \cos y_1 = \cos y_2, \quad \sqrt{1 - k^2 \sin^2 y_1} = \sqrt{1 - k^2 \sin^2 y_2}. \]

This system implies $y_2 = y_1$ and we have only the trivial transformation $\Phi = id$.

In the case $p = 1$ one has $x_2 = x_1 + \pi$ and system (24)–(29) implies the following system of equations,
\[ \sin y_1 = (-1)^a \sin y_2, \] (30)
\[ \cos y_1 = (-1)^b \cos y_2, \] (31)
\[ \sqrt{1 - k^2 \sin^2 y_1} = (-1)^c \sqrt{1 - k^2 \sin^2 y_2}. \] (32)


If $a$ is even then equation (30) implies that either $y_2 = y_1$ or $y_2 = \pi - y_1$. But $y_2 = y_1$ implies from equations (31) and (32) that $b$ and $c$ are even. However, this contradicts our assumption $(a, b, c) = 1$. If $y_2 = \pi - y_1$ then by equations (31) and (32) we obtain that $b$ is odd and $c$ is even. Direct check shows that in fact if $a$ and $c$ is even and $b$ is odd then the transformation $\Phi_1(x, y) = (x + \pi, \pi - y)$ satisfies equation (23).

If $a$ is odd then equation (30) implies that either $y_2 = -y_1$ or $y_2 = y_1 + \pi$. If $y_2 = -y_1$ then equations (31) and (32) imply that $b$ and $c$ are even. Direct check shows that in fact if $a$ is odd and $b$ and $c$ are even then the transformation $\Phi_2(x, y) = (x + \pi, -y)$ satisfies equation (23). If $y_2 = y_1 + \pi$ then equations (31) and (32) imply that $b$ is odd and $c$ is even. Direct check shows that in fact if $a$ and $b$ are odd and $c$ is even then the transformation $\Phi_3(x, y) = (x + \pi, y + \pi)$ satisfies equation (24).

These transformations imply the statement 5) of Theorem 1. The transformations $\Phi_1$ and $\Phi_2$ coincide under isometry $T_{a,b,c} \cong T_{b,a,c}$ and correspond to the case I). Due to the isometry $T_{a,b,c} \cong T_{b,a,c}$ we can consider only the case of odd $a$. Then the points of $T_{a,b,c}$ have unique coordinates $0 \leq x < \pi$ and $-\pi \leq y < \pi$ and functions on $T_{a,b,c}$ could be considered as two-periodic functions on $\mathbb{R}^2$ of period $2\pi$ with additional invariance with respect to the transformation $\Phi_2$.

The transformation $\Phi_3$ corresponds to the case II). The functions on $T_{a,b,c}$ could be considered as two-periodic functions on $\mathbb{R}^2$ of period $2\pi$ with additional invariance with respect to the transformation $\Phi_3$.

The case where there is no transformation $\Phi$ corresponds to the case III). The functions on $T_{a,b,c}$ could be considered as two-periodic functions on $\mathbb{R}^2$ of period $2\pi$.

The induced metric $g$ on $T_{a,b,c}$ is given by the formula

$$
g = P(y) dx^2 + \frac{2P(y)}{Q + 2P(y)} dy^2,
$$

where

$$
P(y) = \frac{1}{2}(c^2 + (b^2 - a^2) \cos 2y), \quad Q = c^2 - a^2 - b^2.
$$

The area of $T_{a,b,c}$ is obtained by a straightforward calculation. Remark that $S(a, b, c) = S(b, a, c)$ since $T_{a,b,c} \cong T_{b,a,c}$.

Next, we shall find $N(2)$ in order to investigate extremal spectral properties of surfaces $T_{a,b,c}$ using El Soufi and Ilias Theorem. We follow the approach proposed in the paper [32] and further developed in the papers [33] and [20, 21]. In order to shorten the text we omit some details and refer the reader to the paper [32].

As it was explained before, the surfaces $T_{a,b,c}$ were constructed in such a way that one has a separation of variables in the spectral problem for the Laplace-Beltrami operator $\Delta \psi = \lambda \psi$. More precisely, since $\Delta$ commutes with $\frac{\partial}{\partial x}$, one can look for an eigenfunction basis consisting of functions of the form

$$
\psi(x, y) = \varphi(y) \sin lx \quad \text{or} \quad \psi(x, y) = \varphi(y) \cos lx,
$$

where we consider $l$ as an integer parameter in $\varphi(y)$. Substituting functions (34) in $\Delta \psi = \lambda \psi$ and separating variables one obtains the following equation,

$$
\left(1 + \frac{Q}{2P(y)}\right) \varphi''(y) + \frac{P(y)}{2P(y)} \varphi'(y) + \left(\lambda - \frac{l^2}{P(y)}\right) \varphi(y) = 0.
$$

The case of Lawson surfaces $\tau_{m,n}$ corresponds to $Q = 0$ and was studied in the paper [32]. We denote occasionally a solution of (35) by $\varphi(y, l)$ when we need to emphasize dependence on the parameter $l$.

In the subcase III) the conditions of $2\pi$-periodicity of $\psi(x, y)$ impose the $2\pi$-periodicity condition on $\varphi(y)$. Thus, we should consider the periodic Sturm-Liouville
problem consisting of equation \(35\) and the periodicity condition
\[
\varphi(y + 2\pi) \equiv \varphi(y).
\] (36)

For each \(l = 0, 1, 2, 3 \ldots\) we obtain the spectrum \(\lambda_0(l) < \lambda_1(l) \leq \lambda_2(l) < \ldots\) of the periodic Sturm-Liouville problem \(35, 36\). Each eigenvalue \(\lambda_l(l)\) corresponds to two eigenvalues \(\lambda_k = \lambda_{k+1}\) of the initial Laplace-Beltrami operator \(\Delta\) on \(T_{a,b,c}\) since one eigenfunction \(\varphi(y)\) of the problem \(35, 36\) corresponds to two eigenfunctions \(34\) of \(\Delta\). The only exception is the case of \(l = 0\). Since \(\sin 0x \equiv 0\), one eigenvalue \(\lambda_0(0)\) corresponds to exactly one eigenvalue \(\lambda_k\) of \(\Delta\). This implies that
\[
N(2) = \#\{\lambda_i < 2\} = \#\{\lambda_i(0) < 2\} + 2\#\{\lambda_i(l) < 2, l \geq 1\}. \tag{37}
\]

According to the Sturm oscillation theorem, one has the inequality
\[
\lambda_0(l) < \lambda_1(l) \leq \lambda_2(l) < \lambda_3(l) \leq \lambda_4(l) < \ldots \tag{38}
\]
On the other hand, one has the inequality
\[
\lambda_0(0) < \lambda_1(1) < \lambda_2(2) < \lambda_3(3) < \lambda_4(4) < \ldots. \tag{39}
\]
A simple proof of this inequality could be found in the paper \(21\). The initial argument in the paper \(32\) uses the particular properties of Lawson tau-surfaces and quite complicated.

Let us now remark that we know three eigenvalues of the problem \(35, 36\) equal to 2. Indeed, Takahashi Theorem \(4\) states that the components of the immersion \(5\) are eigenfunctions of \(\Delta\) with eigenvalue 2. They are exactly of the form \(34\) with \(l = a, l = b\) and \(l = c\). Let us look at \(\sin cx_3(y)\) and \(\cos cx_3(y)\). Since they are eigenfunctions of \(\Delta\), we know that \(\psi_3(y)\) is an eigenfunction with eigenvalue 2 of the problem \(35, 36\) with \(l = c\). Let us remark that \(\psi_3(y)\) has no zeroes.

Then Sturm oscillation theorem implies that this is an eigenfunction corresponding to \(\lambda_0(c)\). Hence, \(\lambda_0(c) = 2\). In a similar way we can establish that if \(a \geq b\) then \(\lambda_1(a) = 2\) and \(\lambda_2(b) = 2\) and if \(a < b\) then \(\lambda_1(b) = 2\) and \(\lambda_2(a) = 2\), see the paper \(32\) for more details.

It follows now from inequality \(39\) that among all eigenvalues \(\lambda_0(l)\) exactly \(\lambda_0(0), \ldots, \lambda_0(c-1)\) are less than 2. The similar statement holds for \(\lambda_1(l)\) and \(\lambda_2(l)\).

Using the theory of the Lamé equation one can prove that \(\lambda_2(0) > 2\), see the papers \(32\) and \(21\) for more details. Then inequality \(39\) implies that for any \(l\) we have \(\lambda_3(l) > 2\). Then inequality \(38\) implies that for any \(i > 3\) and any \(l\) we have \(\lambda_i(l) > 2\). Hence, we can find \(N(2)\) by formula \(47\) and obtain
\[
N(2) = 3 + 2(a - 1 + b - 1 + c - 1) = 2(a + b + c) - 3.
\]
This means that in the subcase III) the metric \(33\) induced on the torus \(T_{a,b,c}\) is extremal for the functional \(I_j(T^2, g)\), where \(j = 2(a + b + c) - 3\). The corresponding value of this functional \(I_j(T_{a,b,c}) = 2\text{Area}(T_{a,b,c}) = 2S(a,b,c)\). This proves the part of the statement 6) of theorem \(11\) concerning the subcase III). The special cases of \(T_{a,0,c}\) and \(T_{0,0,1}\) could be investigated in the same way.

The subcases I) and II) are more complicated since one have to take into account not only the \(2\pi\)-periodicity conditions \(36\). In the subcase I) eigenfunctions \(\psi(x,y)\) have have to satisfy also the condition of invariance with respect to the transformation \(\Phi_2\). Since we look for eigenfunctions of the form \(34\), this condition could be written in the following way: \(\varphi(x,l)\) has to be even function for even \(l\) and odd function for odd \(l\). One can then find \(N(2)\) in the same way as before but taking into the account the parity of solutions. All details could be found in the proof of the Main Theorem in the paper \(32\) and we give here only the answer,
\[
N(2) = a + b + c - 3.
\]
This means that in the subcase I) the metric \( \Lambda_j \) induced on the Klein bottle \( T_{a,b,c} \) is extremal for the functional \( \Lambda_j(K\mathbb{L}, g) \), where \( j = a + b + c - 3 \). The corresponding value of this functional \( \Lambda_j(T_{a,b,c}) = 2 \text{Area}(T_{a,b,c}) = S(a, b, c) \). This proves the part of the statement 6) of theorem \( \text{P} \) concerning the subcase I). The special case of \( T_{a,0,c} \) could be investigated in the same way.

In the subcase II) eigenfunctions \( \psi(x, y) \) have to satisfy also the condition of invariance with respect to the transformation \( \Phi_3 \). This condition means that \( \varphi(y, l) \) has to be \( \pi \)-periodic for even \( l \) and \( \pi \)-antiperiodic for odd \( l \). One can then find \( N(2) \) in the same way as before but taking into the account the \( \pi \)-(anti)periodicty of solutions. All details could be found in the proof of the Main Theorem in the paper \( \text{[32]} \) and we give here only the answer,

\[
N(2) = a + b + c - 3.
\]

This means that in the subcase II) the metric \( \Lambda_j \) induced on the torus \( T_{a,b,c} \) is extremal for the functional \( \Lambda_j(T^2, g) \), where \( j = a + b + c - 3 \). The corresponding value of this functional \( \Lambda_j(T_{a,b,c}) = 2 \text{Area}(T_{a,b,c}) = S(a, b, c) \). This proves the part of the statement 6) of theorem \( \text{I} \) concerning the subcase II).

This finishes the proof. \( \square \)

We should remark that after the author’s talk at the Analysis Seminar at the McGill University I. Polterovich conjectured that one can find tori minimally immersed in spheres using not only three first solutions \( dn, cn \) and \( sn \) of the Lamé equation with \( n = 1 \), but also using next solutions. Is not clear how to prove this conjecture since next solutions are given only by series.

4. The Klein bottle \( T_{1,0,2} \)

It follows from Theorem \( \text{I} \) that \( T_{1,0,2} \) is a Klein bottle and the metric on \( T_{1,0,2} \) is extremal for \( \Lambda_1(K\mathbb{L}, g) \). In the same time, El Soufi, Giacomini and Jazar proved in paper \( \text{[7]} \) that the metric on \( \tau_{3,1} \) is the unique (up to multiplication by a constant) extremal metric for the first eigenvalue on the Klein bottle and hence the maximal one.

It is interesting to compare the values \( \Lambda_1(T_{1,0,2}) \) and \( \Lambda_1(\tau_{3,1}) \). We have

\[
\Lambda_1(T_{1,0,2}) = S(1, 0, 2) = S(0, 1, 2) = 2\pi \left( 8E \left( \frac{1}{2} \right) - 3K \left( \frac{1}{2} \right) \right),
\]

where \( S(a, b, c) = S(b, a, c) \) since \( T_{a,b,c} \cong T_{b,a,c} \), and

\[
\Lambda_1(\tau_{3,1}) = 12\pi E \left( \frac{2\sqrt{2}}{3} \right).
\]

Both values are equal due to the identity (see the book \( \text{[10]} \))

\[
E \left( \frac{2\sqrt{k}}{1 + k} \right) = \frac{2E(k) - (k')^2K(k)}{1 + k},
\]

where \( k' = \sqrt{1 - k^2} \).

Since in both cases \( \lambda_1 = 2 \), the areas are equal. This implies the following Proposition.

**Proposition 3.** The metric on the Klein bottle \( T_{1,0,2} \) is maximal for \( \Lambda_1(K\mathbb{L}, g) \). The Klein bottle \( T_{1,0,2} \) is isometric to the bipolar Lawson Klein bottle \( \tau_{3,1} \).

It would be interesting to find this isometry explicitly. The explicit parametrisation of \( T_{1,0,2} \subset S^4 \) is given by the formula

\[
\left( \frac{1}{\sqrt{2}} \sin x \sin y, \frac{1}{\sqrt{2}} \cos x \sin y, \sqrt{\frac{5}{8}} \cos y, \right)
\]
\[
\sqrt{\frac{3}{8}} \sin 2x \sqrt{1 + \frac{1}{3} \sin^2 y}, \quad \sqrt{\frac{3}{8}} \cos 2x \sqrt{1 + \frac{1}{3} \sin^2 y},
\]
where we omit one of the components equal to zero. It would be also interesting to find whether other bipolar Lawson surfaces \( \tilde{T}_{m,k} \) are among \( T_{a,b,c} \).

**Acknowledgments**

The author thanks D. Jakobson, I. Polterovich and P. Winternitz for fruitful discussions at the Centre de Recherches Mathématiques, Université de Montréal (CRM). The author is very grateful to the CRM for its hospitality.

The author also thanks M. Karpukhin for useful discussions.

This work was partially supported by the Russian Federation Government grant no. 2010-220-01-077, ag. no. 11.G34.31.0005, by the Russian Foundation for Basic Research grant no. 11-01-12067-ofi-m-2011, by the Russian State Programme for the Support of Leading Scientific Schools grant no. 4995.2012.1 and by the Simons-IUM fellowship.

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