GEOMETRIC REALIZATIONS AND DUALITY
FOR DAHMEN-MICCHELLI MODULES
AND DE CONCINI-PROCESI-VERGNE MODULES

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Abstract. We give an algebraic description of several modules and
algebras related to the vector partition function, and we prove that
they can be realized as the equivariant K-theory of some manifolds
that have a nice combinatorial description. We also propose a more natural
and general notion of duality between these modules, which corresponds
to a Poincaré duality-type correspondence for equivariant K-theory.

1. Introduction

In recent years, the multivariate spline and the vector partition function
have been studied by several authors. While the former is an essential tool
in Approximation Theory, the latter has been studied in Combinatorics at
least since Euler. Although they may seem quite different in nature, they
can be viewed as the volume, and the number of integer points respectively,
of a variable polytope; thus the partition function is the discretization of
the spline. In their book [15], De Concini and Procesi brought these func-
tions to the attention of geometers, showing their relation with hyperplane
arrangements and toric arrangements.

These functions are piecewise polynomial/quasi-polynomial respectively,
meaning that their support can be divided in regions called big cells, such
that on every big cell Ω, the spline agrees with a polynomial pΩ, and the
partition function agrees with a quasi-polynomial qΩ.

The polynomials pΩ, together with their derivatives, form a vector space
D(X). In a more algebraic language, D(X) is generated by the elements
pΩ as module over the ring of polynomials, acting as derivations. On the
other hand, the elements qΩ generate DM(X), as a module over the Laurent
polynomials, acting as difference operators. The Dahmen-Micchelli modules
(DM modules for short) D(X) and DM(X) are defined by a system of
differential/difference equations respectively, having a simple combinatorial
description in terms of the cocircuits of the associated matroid ([12, 13, 14]).

For every big cell Ω, it is also natural to consider DΩ(X), the cyclic
submodule of D(X) generated by the single polynomial pΩ, and DMΩ(X),

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the cyclic submodule of $DM(X)$ generated by the quasi-polynomial $q_\Omega$. We call them the local DM modules.

Furthermore, studying the partition function and the spline led to the definition of two flags of modules $\tilde{F}_i(X)$ and $\tilde{G}_i(X)$, of which $DM(X)$ and $D(X)$ are the smallest elements ([16] [17]). We call the modules $\tilde{F}_i(X)$ and $\tilde{G}_i(X)$ the De Concini-Procesi-Vergne modules (or DPV modules for short).

The DM modules and the local DM modules naturally come together with their dual modules $D^*(X)$, $D^*_\Omega(X)$, $DM^*(X)$, $DM^*_\Omega(X)$, all described as quotients of the ring of polynomials by suitable ideals, and hence endowed by a structure of algebras. As we show in Theorem 3.7 this duality can be viewed in a more natural way via an $Ext$ functor. This approach also allows us to define the dual DPV modules $F^*_i(X)$, $G^*_i(X)$ (see Definitions 3.9 and 3.11).

Surprisingly, all the modules and algebras above appear as invariants of geometric objects. In particular, the modules $D(X)$ and $D^*_\Omega(X)$ can be “geometrically realized” as the equivariant cohomology of two differentiable manifolds, while their discrete counterparts $DM(X)$ and $DM^*_\Omega(X)$ can be “geometrically realized” as the equivariant K-theory of the same manifolds.

The construction goes as follows. Let $X$ be the list of vectors in $\mathbb{Z}^d$ that defines the spline and the partition function. Each element of $X$ defines a 1-dimensional representation of the torus $G = (S^1)^d$, hence there is a representation $M_X$ which is the direct sum of all them. Let $M^\text{fin}_X$ be the open subset of $M_X$ of points with finite stabilizer; this is the complement of a linear subspace arrangement. We have:

\begin{align*}
\text{a)} & \quad H^*_{c,G}(M^\text{fin}_X) \cong D(X); \\
\text{b)} & \quad H^*_G(M^\text{fin}_X) \cong D^*(X).
\end{align*}

Here the duality between $D(X)$ and $D^*(X)$ is realized by the Poincaré duality between the ordinary cohomology $H$ and compact support cohomology $H_c$ of the toric orbifold $M^\text{fin}_X/G_C$ (see Section 2.7).

The first aim of this paper is to provide an analogue of this result for discrete DM modules; this is done by considering equivariant K-theory. However, this is intrinsically a compact support cohomology theory, thus lacking of a natural non-compact-support counterpart. Instead of looking for a different definition of equivariant K-theory non involving compact support, our approach will be based on compactifying the manifold $M^\text{fin}_X$. In fact we intersect it with the equivariant unit sphere and then remove open tubular neighborhoods of every resulting hypersurface, thus obtaining a compact manifold with corners $S^\text{fin}_X$. Then we have:

\begin{align*}
\text{c)} & \quad K^*_G(M^\text{fin}_X) \cong DM(X); \\
\text{d)} & \quad K^*_G(S^\text{fin}_X) \cong DM^*(X).
\end{align*}

Facts a), b) and c) are consequences of statements proved by De Concini, Procesi and Vergne for DPV modules, that we will recall in Theorems 5.1, 5.2, 5.3. In particular the proof of a), given in [17], is based on index theory
of transversally elliptic operators, and answers to a question that Atiyah raised about four decades ago ([4]). The proof of c), given in [18], required the development of a further tool, the infinitesimal index. In this paper we prove Theorem 5.4 for dual DPV modules, that implies fact d).

The second aim of this paper is to provide an analogue of statement c) for the local DM modules: this is Theorem 6.4. Here $M_{X}^{\text{fin}}$ is replaced by an open submanifold $M_{X}^{\Omega} \subset M_{X}^{\text{fin}}$ which has a nice combinatorial description; the algebraic property that the $R(G)$–modules $\text{DM}_{\Omega}(X)$ generate $\text{DM}(X)$ is reflected in the geometrical fact that the pushforward $K_{G}(M_{X}^{\Omega}) \to K_{G}(M_{X}^{\text{fin}})$ realizes generators of $K_{G}(M_{X}^{\text{fin}})$. We actually prove a more general statement (Theorem 6.2), by using Atiyah’s index.

As we stated in Conjecture 6.3, we believe that the same path may be followed for the differentiable counterpart, by using the infinitesimal index; this will provide geometric realizations for the local modules $D_{\Omega}(X)$.

We think that also other objects, such as the modules arising from the power ideals studied by Ardila and Postnikov [3], the semi-internal zonotopal modules introduced by Holtz, Ron and Xu in [25], and their generalizations proposed by Lenz [27], may admit a similar geometric realization, that we hope to study in future papers.

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2. Recalls on Representation Theory and Combinatorics

2.1. First notations. Let $G$ be an abelian compact Lie group, and let $\Gamma \cong \text{Hom}(G, S^1)$ be its character group. We will assume for simplicity that $G$ is connected, i.e., it is a torus. Let $\mathfrak{g}$ be the Lie algebra of $G$: this can be identified to the tangent space at 1 of $G$, so that the differential $d\lambda$ of every character $\lambda$ is an element of the dual space $\mathfrak{g}^*$; by a slight abuse of notation we will sometimes identify $d\lambda$ with $\lambda$. We denote by $S[\mathfrak{g}^*]$ the symmetric algebra of polynomial functions on $\mathfrak{g}$, and by $R(G)$ the character ring of $G$, that is, the group algebra of $\Gamma$.

For the sake of concreteness, let us say that $\mathfrak{g}^*$ (as well as $\mathfrak{g}$) is a real vector space $V$, and if we denote its dimension by $d$, $G$ is isomorphic to $(S^1)^d$, while $\Gamma$ is isomorphic to $\mathbb{Z}^d$. $S[\mathfrak{g}^*]$ is isomorphic to the ring of polynomials $\mathbb{R}[x_1, \ldots, x_d]$ while $R(G)$ is isomorphic to the ring of Laurent of polynomials $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$. Then $\Gamma$ embeds in $S[\mathfrak{g}^*]$ and in $R(G)$ as

$$(m_1, \ldots, m_d) \mapsto m_1 x_1 + \cdots + m_d x_d$$

and

$$(m_1, \ldots, m_d) \mapsto x_1^{m_1} \cdots x_d^{m_d}$$

respectively.

All the results in this paper may be extended to the case of $G$ not being connected. In this case $G$ is isomorphic to the product of a compact torus $(S^1)^d$ and a finite group $G_f$, and $\Gamma$ is isomorphic to $\mathbb{Z}^d \times G_f$. Then we have a projection $\Gamma \to V$ which forgets the torsion part of $\Gamma$.

Let $\mathcal{C}[\Gamma]$ be the space of $\mathbb{Z}$-valued functions on $\Gamma$. On this space every element $a \in \Gamma$ acts as the translation $\tau_a$; this extends to an action of $R(G)$. We define the difference operator $\nabla_a = 1 - \tau_a$, i.e.:

$$\nabla_a f(x) \equiv f(x) - f(x - a).$$

Let $X = [a_1, \ldots, a_n]$ be a finite list of elements of $\Gamma$. We will always assume that $rk(X) = rk(\Gamma)$, and that none of the elements of $X$ is zero in $V$ (otherwise, it is simple to reduce to this case).

Following [16], we say that a linear subspace $\mathfrak{s} \subseteq V$ is rational if it is spanned by elements of $X$; we denote by $\mathcal{R}_X$ the set of all rational subspaces.
With a little abuse of notation, we will indicate by $s$ both the linear subspace of $V$ and the sublattice of $\Gamma$ generated by the same elements.

2.2. DM modules. We recall that $A \subset X$ is a cocircuit if $A = X \setminus s$ for some rational hyperplane $s$ (i.e. for some $s \in \mathcal{R}_X$ such that $\text{codim}(s) = 1$).

Let us define the set $L(X) = \{ A \subseteq X | C \subseteq A \text{ for some cocircuit } C \}$.

For every $A \subseteq X$, we consider the difference operator $\nabla_A = \prod_{a \in A} \nabla_a$ acting on $C[\Gamma]$. The discrete DM module is defined as $DM(X) = \{ f \in C[\Gamma] | \nabla_A f = 0 \text{ for every } A \in L(X) \}$.

This can be seen as the “discretization” of the differentiable DM module $D(X) = \{ f : V \to \mathbb{R} | \partial_A f = 0 \text{ for every } A \in L(X) \}$ where the differential operator $\partial_A = \prod_{a \in A} \partial_a$ is just the product of directional derivatives. Of course, as in many of the definitions that will follow, it is enough to check the equations above for the minimal elements of $L(X)$, that is, the cocircuits.

The space $D(X)$ is naturally a module over $S[\mathfrak{g}^*] \simeq \mathbb{R}[x_1, \ldots, x_d]$, the action being given by derivation: $x_i \cdot f = \partial_{x_i} f$. On the other hand, $DM(X)$ is naturally a module over $R(G) \simeq \mathbb{Z}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$, the action being given by difference operators: $x_i \cdot f = \nabla_{x_i} f$.

The vector space underlying to $D(X)$ is the space $D(X)$ studied in [24]. We say that the list $X$ is unimodular (or totally unimodular) if every $k \times k$ minor of $X$ is equal to 1, $-1$ or 0, for every $1 \leq k \leq d$; in other words, if every bases taken out from $X$ generates the whole lattice $\Gamma$. In this case it is easy to see that $D(X) = DM(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Remark 2.1. One can see from the definition that $D(X)$ essentially depends on the linear algebra of the vectors in $X$, while $DM(X)$ also depends on the arithmetic of the vectors. The need to encode this arithmetic information in a combinatorial object led to the introduction of arithmetic matroids ([10, 8]) and matroids over $\mathbb{Z}$ ([21]).

As proved in [11] and [12] respectively, the dimension of $D(X)$ (as a real vector space) is equal to the number of bases which can be extracted from $X$, while the rank of $DM(X)$ (as a free $\mathbb{Z}$-module) is equal to the volume of the zonotope

$$Z(X) = \left\{ \sum_{i=1}^{d} t_i a_i, \quad 0 \leq t_i \leq 1 \right\}.$$ 

These spaces were introduced in order to study two important functions, that we are going to describe in the next subsection.
2.3. Vector partition function and multivariate spline. For every \( \lambda \in \Gamma \), we define \( P_X(\lambda) \) as the number of ways we can write
\[
\lambda = \sum_{i=1}^{n} x_i a_i, \quad x_i \in \mathbb{N}.
\]
Since we want this number to be finite, we assume that all the elements \( a_i \) of the list \( X \) lie on the same side of a hyperplane in \( V \). We can always do that, eventually replacing some vectors by their opposites. We call \( P_X(\lambda) \) the \textit{vector partition function}, or simply the \textit{partition function}.

The equation above is indeed a system of diophantine equations, one for every coordinate of \( \lambda \). We can rewrite this system as \( \overline{X}x = \lambda \), where \( \overline{X} \) is the matrix whose columns are the vectors \( a_i \in X \), and \( x \) is the vector whose entries are the variables \( x_i \).

This defines a subspace of \( \mathbb{R}^n \). The intersection of this subspace with the positive orthant is a \textit{variable polytope}
\[
P_X(\lambda) = \{ x \in (\mathbb{R}_{\geq 0})^n \mid \overline{X}x = \lambda \}
\]
and the partition function is the number of its integer points:
\[
P_X(\lambda) = |P_X(\lambda) \cap \mathbb{Z}^n|.
\]

Then \( P_X(\lambda) \) is related with another function: \( M_X(\lambda) \doteq \text{vol}(P_X(\lambda)) \). Indeed the number of integer points of a polytope the “discrete analogue” of its volume. The function \( M_X(\lambda) \), which is well defined for every \( \lambda \in \mathbb{R}^n \), is known as the \textit{multivariate spline} (or simply the \textit{spline}). Splines are used in Numerical Analysis to approximate functions. The word “spline” means that \( M_X \) is piecewise polynomial and “as smooth as possible”; we will now make more precise this statement.

First of all, notice that both the functions \( M_X, P_X \) are supported on the cone
\[
C(X) = \left\{ \sum_{i=1}^{n} t_i a_i, \quad t_i \geq 0 \right\}
\]
For every cocircuit \( A \), we consider the cone \( C(X \setminus A) \) spanned by \( X \setminus A \). We define a \textit{big cell} as a connected component of
\[
C(X) \setminus \bigcup_{A \in \mathcal{L}(X)} C(X \setminus A).
\]
Then we have:

\textbf{Theorem 2.2} (de Boor-Hollig). \textit{For every big cell } \( \Omega \), \textit{there is a polynomial } \( p_{\Omega} \in D(X) \) \textit{such that } \( M_X \) \textit{and } \( p_{\Omega} \) \textit{coincide on } \( \Omega \). \textit{Moreover, all the polynomials } \( p_{\Omega} \) \textit{have degree } \( n-d \) \textit{and } \( M_X \) \textit{is continuous of class } \( C^{n-d-1}(V) \).
Since \( P_X \) is the discretization of \( M_X \), it is natural to wish a discrete analogue of the theorem above. Such an analogue exists, but with two important differences.

The first can be understood by looking at the 1-dimensional example given by the list \([2, 1] \in \mathbb{Z}\). Here \( P_X(\lambda) = \lambda/2 + 1 \) when \( \lambda \) is even, and \( P_X(\lambda) = \lambda/2 + 1/2 \) when \( \lambda \) is odd. Then, in general, we recall the following definition: a function \( q : \Gamma \to \mathbb{Z} \) is a quasi-polynomial if there exist a finite index subgroup of \( \Gamma \) such that on every coset, \( q \) coincides with a polynomial. As we will see, the partition function is piecewise quasi-polynomial.

The second issue is what a “discrete analogue” of smoothness should be. The natural idea is to require that the regions of quasi-polynomiality overlap a bit, that is, given two neighboring big cells, there is a stripe on which the corresponding quasi-polynomials agree.

More precisely we have:

**Theorem 2.3** (Dahmen-Micchelli). For every big cell, there is a quasi-polynomial \( q_\Omega \in DM(X) \) such that \( P_X \) and \( q_\Omega \) coincide on \( \Omega \). Moreover, they coincide on a larger region, the Minkowski sum of \( \Omega \) and \( -Z(X) \).

The two theorems above motivate the interest for the modules \( D(X), DM(X) \). In fact these modules contain the “local pieces” \( p_\Omega, q_\Omega \) respectively; more precisely, \( D(X) \) is the \( S[\mathfrak{g}^*] \)-module generated by the polynomials \( p_\Omega \), and \( DM(X) \) is the \( R(G) \)-module generated by the quasi-polynomials \( q_\Omega \), where \( \Omega \) ranges over all the big cells.

**2.4. Local DM modules and their generalizations.** For every big cell \( \Omega \), it is also natural to consider \( D_\Omega(X) \), the cyclic submodule of \( D(X) \) generated by the polynomial \( p_\Omega \), and \( DM_\Omega(X) \), the cyclic submodule of \( DM(X) \) generated by the quasi-polynomial \( q_\Omega \). These modules admit a simple combinatorial description. Let us define

\[
L_\Omega(X) = \{ A \subseteq X \mid C(X \setminus A) \notin \Omega \}.
\]

Notice that \( L_\Omega(X) \) contains all the cocircuits and is closed under taking supsets. We have:

\[
D_\Omega(X) = \{ f : V \to \mathbb{R} \mid \partial_A f = 0 \forall A \in L_\Omega(X) \}
\]

\[
DM_\Omega(X) = \{ f \in C[\Gamma] \mid \nabla_A f = 0 \forall A \in L_\Omega(X) \}
\]

More generally, given any subset \( T \subseteq 2^X \) closed under taking supsets (that is, if \( A \in T \) and \( A \subset B \in 2^X \), then \( B \in T \)), we can consider the \( S[\mathfrak{g}^*] \)-module

\[
D_T(X) = \{ f : V \to \mathbb{R} \mid \partial_A f = 0 \forall A \in T \}
\]

and the \( R(G) \)-module

\[
DM_T(X) = \{ f \in C[\Gamma] \mid \nabla_A f = 0 \forall A \in T \}
\]
Lemma 2.4. The module $DM_T(X)$ has finite rank over $\mathbb{Z}$ if and only if $T$ contains all the cocircuits. The module $D_T(X)$ has finite dimension over $\mathbb{R}$ if and only if $T$ contains all the cocircuits.

Proof. If $T$ contains all the cocircuits, then $DM_T(X) \subseteq DM(X)$, because it is defined by the same difference equations, plus further ones.

Now let us assume that $T$ does not contain a cocircuit $A$, and let $s$ be a rational hyperplane such that $A = X \setminus s$. Then we will show that any function that is constant on $s$ and on all its translates belongs to $DM_T(X)$. These functions are annihilated by all the operators $\nabla_a$ with $a \in s$, and then by all the $\nabla_B$ for all $B$ such that $B \not\subseteq A$; now, notice that all elements of $T$ satisfy this condition, so that all these functions belong to $DM_T(X)$. Now, being $s$ a proper subspace, its cosets in $\Gamma$ are infinitely many, so such functions are a subset of $DM_T(X)$ of infinite rank over $\mathbb{Z}$ (in fact, a basis of it consists of uncountably many elements).

The same proof holds for $D_T(X)$. □

2.5. DPV modules. Although $DM(X)$ contains all the local pieces $q_\Omega$ of the partition function $P_X$, it does not contain $P_X$ itself. In fact all the elements of $DM(X)$ are genuine quasi-polynomials, while the partition function is piecewise quasi-polynomial. It is then desirable to have a space that contains both $DM(X)$ and $P_X$. This is the $\mathbb{Z}$-module $F(X) = \{ f \in C[\Gamma] | \nabla_{X \setminus s} f$ is supported on $s \forall s \in R_X \}$. By “supported on $s$” we mean that the support of $f$ (i.e., the subset of the domain on which $f$ takes nonzero values) is contained in $s$. This space comes with a natural filtration

$$DM(X) = F_d(X) \subset F_{d-1}(X) \subset \ldots \subset F_0(X) = F(X)$$

where

$$F_i(X) = \{ f \in C[\Gamma] | \begin{cases} \nabla_{X \setminus s} f = 0 & \text{if } \dim(s) < i \\ \nabla_{X \setminus s} f \text{ is supported on } s & \text{otherwise} \end{cases} \}.$$ 

Clearly, these $\mathbb{Z}$-modules are not invariant for the action by translations of $R(G)$ (because after a translation $\nabla_{X \setminus s} f$ is going to be supported on a translate of $s$). In fact they generate the discrete De Concini-Procesi-Vergne modules (discrete DPV modules for short)

$$\tilde{F}(X) = \{ f \in C[\Gamma] | \nabla_{X \setminus s} f \text{ is supported on a finite number of translates of } s \forall s \in R_X \}$$

$$\tilde{F}_i(X) = \{ f \in C[\Gamma] | \begin{cases} \nabla_{X \setminus s} f = 0 & \text{for every rational proper subspace } s \text{ of dimension } < i \\ \nabla_{X \setminus s} f \text{ is supported on a finite number of translates of } s & \text{otherwise} \end{cases} \}.$$ 

The space $F(X)$ is a free $\mathbb{Z}$-module of rank equal to the number of integer points of the zonotope $Z(X)$. On the other hand, $\tilde{F}(X)$ and $\tilde{F}_i(X)$ have
clearly infinite rank over \( \mathbb{Z} \). All these spaces have been introduced and studied in \([16, 17]\).

In the same way, we can define a real vector space \( \mathcal{G}(X) \) containing both the multivariate spline \( \mathcal{M}_X \) and the space \( \mathcal{D}(X) \), with a filtration \( \mathcal{G}_i(X) \). The definitions are exactly the same, except for \( \nabla \) that is replaced by \( \partial \). Again, these vector spaces generate the differentiable DPV \( S[\mathfrak{g}^*] \)-modules \( \tilde{\mathcal{G}}(X), \tilde{\mathcal{G}}_i(X) \) respectively (see \([19]\)). These modules have a natural grading, which is given by the degree of the homogeneous polynomials (or piecewise polynomials).

**Remark 2.5.** Interestingly, the external zonotopal space \( \mathcal{D}_+(X) \) studied in \([24]\) has the same dimension as \( \mathcal{G}(X) \) over \( \mathbb{R} \). When the list \( X \) is unimodular, this dimension is equal to the number of integer points of the zonotope \( Z(X) \). However, \( \mathcal{D}_+(X) \) and \( \mathcal{G}(X) \) are different spaces: the former contains polynomials, the latter distributions. It is not surprising, then, that only the first space is closed under derivations, i.e., is an \( S[\mathfrak{g}^*] \)-module. It would be interesting, nevertheless, to establish some canonical correspondence between these two spaces.

The same considerations hold for the semi-external zonotopal spaces \( \mathcal{D}_+(X, I_i) \) studied in \([25]\), when \( I_i \) is the family of linearly independent sublists of \( X \) of rank at least \( i \) \((0 \leq i \leq d)\). In fact, these spaces have the same dimension as the spaces \( \mathcal{G}_i(X) \).

### 2.6. Some representations.

For every rational subspace \( \mathfrak{s} \in \mathcal{R}_X \), let \( G_\mathfrak{s} \) be the subgroup of \( G \) of the elements that are annihilated by all the characters in \( \Gamma \cap \mathfrak{s} \). Notice that \( \dim(G_\mathfrak{s}) = \text{codim}(\mathfrak{s}) \).

To every element \( a \) of \( \Gamma \) correspond a 1-dimensional representation \( M_a \) of \( G \), on which every \( g \in G \) acts as the multiplication by the scalar \( a(g) \). Then for every list \( X \) of elements of \( G \) we consider the representation

\[
M_X = \bigoplus_{a \in X} M_a.
\]

which has dimension \( n \) over \( \mathbb{C} \).

Given a \( G \)-invariant Hermitian product on \( M_X \), let \( S_X \) be the unit sphere. This is a \( G \)-manifold of dimension \( 2n - 1 \) over \( \mathbb{R} \).

For every \( A \subset X \) we can consider the coordinate subspace \( M_A \subset M_X \) given by

\[
M_A = \{(z_a)_{a \in X} \in M_X \mid z_a = 0 \text{ for every } a \notin A\}.
\]

Notice that the stabilizer of \( M_\mathfrak{s} = M_X \cap \mathfrak{s} \) is \( G_\mathfrak{s} \).

Given any subset \( T \subseteq 2^X \) closed under taking supsets, we can consider

\[
M_X^T = M_X \setminus \bigcup_{A \in T} M_X \setminus A.
\]

In particular when \( T = \mathcal{L}(X) \), we get
\[ M^\mathcal{L}(X) = M_X \setminus \bigcup_{A \in \mathcal{L}(X)} M_{X \setminus A} = M_X \setminus \bigcup_{\mathcal{L} \in \mathcal{R}_X} M_\mathcal{L} = M^{\text{fin}}_X. \]

Notice that in the formula above it is sufficient to take the first union over the minimal elements (i.e. cocircuits) and the second union over the maximal elements (i.e. rational hyperplanes).

In the same way, we can stratify \( M_X \) by dimension of orbits, and for every \( i \leq d \) call \( M^i_X \) the subset of points having \( i \)-dimensional orbit. Then the subset \( M^{\geq i}_X \) of points whose orbit is \( i \)-dimensional or more is

\[ M^{\geq i}_X = M_X \setminus \bigcup_{\mathcal{L} \in \mathcal{R}_X, \dim(\mathcal{L}) < i} M_\mathcal{L}. \]  

Then we have:

\[ M^{\text{fin}}_X = M^d_X \subseteq M^{d-1}_X \subseteq \cdots \subseteq M^0_X = M_X \]

Finally, when \( \mathcal{T} = L_\Omega(X) \), we get

\[ M^\Omega_X = \{(z_a)_{a \in X} \in M_X \mid \exists A \subset X \mid C(A) \supseteq \Omega \text{ and } z_a \neq 0 \forall a \in A\}. \]

### 2.7. Toric varieties and orbifolds.

For every big cell \( \Omega \), we have defined a subset \( M^\Omega_X \) of \( M^{\text{fin}}_X \). We now look at its quotient by the action of \( G_\mathbb{C} \simeq (\mathbb{C}^*)^d \), the algebraic torus which is the complexification of the compact torus \( G \).

\textbf{Proposition 2.6.} \( M^\Omega_X/G_\mathbb{C} \) is a projective toric variety. Furthermore, \( M^\Omega_X/G_\mathbb{C} \) is an orbifold, and it is smooth whenever \( X \) is unimodular.

(see for example [17, Section 8] or [19, Remark 3.8]).

We recall that an orbifold is a space that is locally homeomorphic to the quotient of a vector space by a linear action of a finite group, with the usual compatibility relations. In particular, this means that an orbifold has just “mild” singularities.

\textbf{Remark 2.7.} When \( \Omega \) ranges over the big cells, the sets \( M^\Omega_X \) cover \( M^{\text{fin}}_X \). So

\[ M^{\text{fin}}_X/G_\mathbb{C} = \bigcup_{\Omega \text{ big cell}} M^\Omega_X/G_\mathbb{C} \]

is still a smooth (or orbifold) toric variety, but in general it is not separated, and hence not projective, as the example in Section 2.8 shows.

These toric varieties provide a new insight on our geometric realizations; in fact, instead of looking at equivariant cohomology, we can take the cohomology of the quotient:

\[ H^*_G(M^{\text{fin}}_X) = H^*(M^{\text{fin}}_X/G_\mathbb{C}), \quad \text{and} \quad H^*_G(M^\Omega_X) = H^*(M^\Omega_X/G_\mathbb{C}) \]  

(2)
and the same for cohomology with compact support:

\[ H^*_c(G\mathbb{C})(M^\text{fin}_X) = H^*_c(M^\text{fin}_X/G\mathbb{C}), \quad \text{and} \quad H^*_c(G\mathbb{C})(M^\Omega_X) = H^*_c(M^\Omega_X/G\mathbb{C}). \]

Thus, as we will see in Section 5, the algebraic duality between \( D(X) \) and \( D^*(X) \) is realized by the Poincaré duality between ordinary cohomology and compact support cohomology (see for example [7, Chapter I.5]).

**Remark 2.8.** Unfortunately, the analogous relations of [2] do not hold for K-theory and equivariant K-theory, unless the list \( X \) is unimodular. However, this problem can be addressed by replacing equivariant K-theory by the orbifold K-theory studied in [2]. See in particular [1, Proposition 6.5].

**2.8. An example.** Let us take the list \( X = [(1,0), (0,1), (k,k)] \) in \( \Gamma = \mathbb{Z}^2 \), where \( k \) is a positive integer. Then we have two big cells; let us call \( \Omega \) the one whose extremal rays are spanned by the vectors \((1,0)\) and \((k,k)\) and \( \Omega' \) the other one.

The torus \( G\mathbb{C} = (\mathbb{C}^*)^2 \) acts on \( M_X = \mathbb{C}^3 \) by \((t,s),(z_1,z_2,z_3) = (tz_1,sz_2,t^k s^k z_3)\). Since the cocircuits are the three couples of vectors, we have that

\[ M^\text{fin}_X = \{(z_1,z_2,z_3) \in M_X \mid z_1 z_2 \neq 0 \text{ or } z_1 z_3 \neq 0 \text{ or } z_2 z_3 \neq 0\}. \]

If \( k = 1 \), \( D(X) \) and \( DM(X) \) have a basis given by the three functions, over \( \mathbb{R} \) and \( \mathbb{Z} \) respectively. A basis of the local modules corresponding to \( \Omega \) is given by \( y \) and \( 1 \), while a basis of the local modules corresponding to \( \Omega' \) is given by \( x \) and \( 1 \).

The subset \( M^\Omega_X \) is given by the condition \( z_1 z_2 \neq 0 \) or \( z_1 z_3 \neq 0 \), that is

\[ M^\Omega_X = \{(z_1,z_2,z_3) \in M_X \mid z_1 \neq 0 \text{ and } (z_2,z_3) \neq (0,0)\}. \]

Therefore \( M^\Omega_X / G\mathbb{C} \simeq \mathbb{P}_1(\mathbb{C}) \), and similarly for \( \Omega' \). These projective lines intersect in an affine line, hence their union \( M^\text{fin}_X / G \), which can be seen as a “projective line with double point at infinity”, is not separated.

If \( k > 1 \), \( D(X) \) is unchanged while \( DM(X) \) is the free \( \mathbb{Z} \)-module of rank \( 2k + 1 = \text{vol} \ Z(X) \) that is spanned by \( x \), \( y \) and by all the functions that are constant in one of the two variables and \( k \)-periodic in the other. Furthermore \( M^\Omega_X / G\mathbb{C} \) and \( M^\Omega_{X'} / G\mathbb{C} \) are isomorphic to the weighted projective space \( \mathbb{P}_{1,k}(\mathbb{C}) \). The module \( DM_{\Omega}(X) \simeq K^*_G(M^\Omega_X) \) has rank \( k + 1 \) over \( \mathbb{Z} \).

**3. Duality of modules**

In this section we describe some duality relations between DM and DPV modules, that will be reflected in duality statements in equivariant K-theory.

**3.1. Duality for DM modules.** Let \( T \subseteq 2^X \) contain all the cocircuits, as in Lemma 2.4. Let us consider the two embeddings of \( \Gamma \) in \( S[g^*] \) and in \( R(G) \) described in Section 2.1. Then we can define an ideal of \( S[g^*] \) as

\[ J_T^0 = (d_A)_{A \in T}, \text{ where } d_A = \prod_{a \in A} a. \]
In the same way we can define an ideal of $R(G)$ as
\[ J_T^\nabla = (\nabla_A)_{A \in T}, \text{ where } \nabla_A = \prod_{a \in A} (1 - a). \]

Notice that these ideals are annihilators of $D_T(X)$ as $S[\mathfrak{g}^*]$-module and of $DM_T(X)$ as $R(G)$-module respectively. Then we define:
\[ D_T^\ast(X) \doteq S[\mathfrak{g}^*]/J_T^\nabla \]
and
\[ DM_T^\ast(X) \doteq R(G)/J_T^\nabla. \]

By definition these are a $S[\mathfrak{g}^*]$-module and a $R(G)$-module respectively; but, unlike $D_T(X)$ and $DM_T(X)$, they also have a multiplicative structure, i.e. they are algebras.

In particular when $T = \mathcal{L}(X)$ we get two algebras that we will denote by $D^\ast(X)$, $DM^\ast(X)$, and call the dual DM modules. While when $T = \mathcal{L}_\Omega(X)$ for a big cell $\Omega$, we will denote the two corresponding algebras by $D^\ast_\Omega(X)$, $DM^\ast_\Omega(X)$.

**Remark 3.1.** The ideal $J_T^\partial(\mathcal{L}(X))$ has been studied in [24], where is denoted by $J(\mathcal{P}(X))$. The vector space underlying to $D^\ast(X)$ is the space therein denoted by $P(X)$, which is defined using a power ideal associated to $X$.

**Lemma 3.2.** We have the isomorphism of $S[\mathfrak{g}^*]$-modules
\[ D_T(X) \cong \text{Hom}_\mathbb{R}(D_T^\ast(X), \mathbb{R}). \]

**Proof.** Clear by definition: the homomorphism $D_T^\ast(X) \rightarrow \mathbb{R}$ are the homomorphisms $S[\mathfrak{g}^*] \rightarrow \mathbb{R}$ that are zero on $J_T^\partial$, which correspond precisely to the polynomial functions on $V = \mathfrak{g}^*$ that satisfy the defining conditions for $D_T(X)$.

**Lemma 3.3.** We have the isomorphism of $R(G)$-modules
\[ DM_T(X) \cong \text{Hom}_\mathbb{Z}(DM_T^\ast(X), \mathbb{Z}). \]

**Proof.** Since $\Gamma$ is a set of generators of $R(G)$ as $\mathbb{Z}$-module, we have an isomorphism of $R(G)$-modules
\[ C[\Gamma] \cong \text{Hom}_\mathbb{Z}(R(G), \mathbb{Z}). \]

Then, we can see by definition $DM_T(X)$ inside $C[\Gamma]$ as the vanishing locus of the endomorphisms given by $\nabla_A$ for $A \in T$; and we can see naturally as well $\text{Hom}_\mathbb{Z}(DM_T^\ast(X), \mathbb{Z})$ sitting inside $\text{Hom}_\mathbb{Z}(R(G), \mathbb{Z})$ as vanishing locus of the same elements of $R(G)$, because $DM_T^\ast(X)$ is the quotient of $R(G)$ by these elements. \qed
3.2. Ext functor and duality. From an algebraic point of view, the duality described above is not completely satisfactory. In fact, since we are dealing with $S[\mathfrak{g}^*]$-modules and $R(G)$-modules, it would be more natural to have a duality involving these rings. Furthermore, in the case of DPV modules the attempt to build duals via the functors $\text{Hom}_{R}(\cdot, R)$ and $\text{Hom}_{Z}(\cdot, Z)$ does not give good results, because these modules have infinite rank (over $R$ and over $Z$ respectively).

On the other hand, defining a duality via the functors $\text{Hom}_{S[\mathfrak{g}^*]}(\cdot, S[\mathfrak{g}^*])$ and $\text{Hom}_{R(G)}(\cdot, R(G))$ would not yield desirable results either: in fact, we have

$$\text{Hom}_{S[\mathfrak{g}^*]}(D^*_T(X), S[\mathfrak{g}^*]) = 0$$

and

$$\text{Hom}_{R(G)}(DM^*_T(X), R(G)) = 0,$$

as one can easily see by the fact that $S[\mathfrak{g}^*]$ and $R(G)$ are domains.

Hence we propose to take a more abstract perspective, realizing the duality via the functors $\text{Ext}^*_S(\cdot, S[\mathfrak{g}^*])$ and $\text{Ext}^*_R(\cdot, R(G))$.

The advantages of this choice will be multiple: it gives rise to a genuine duality in the category of finitely generated $S[\mathfrak{g}^*]$-modules (or $R(G)$-modules respectively), which corresponds to the Verdier duality in the derived categories. Furthermore, this algebraic duality is reflected in (and inspired by) the geometric duality we will describe in the next Sections. Finally, this notion allows to build duals also for the DPV modules.

We recall that $\text{Ext}^*$ is the collection of the $\text{Ext}^i$, which are the derived functors of $\text{Hom}$. In particular, they can be nonzero only for $0 \leq i \leq d$ ($0 \leq i \leq d + 1$ in the discrete case) and $\text{Ext}^0$ is $\text{Hom}$ itself.

First of all, we check that we are actually extending the duality defined in the previous Subsection. In fact we have:

**Proposition 3.4.** Let $V$ be an $S[\mathfrak{g}^*]$-module, that is finite dimensional as a vector space. Then we have the isomorphisms of $S[\mathfrak{g}^*]$-modules:

$$\text{Ext}^i_S(\cdot, S[\mathfrak{g}^*]) = 0,$$

for $0 \leq i < d$

$$\text{Ext}^d_S(\cdot, S[\mathfrak{g}^*]) \simeq \text{Hom}_R(V, R).$$

**Proposition 3.5.** Let $M$ be an $R(G)$-module, that is free over $Z$ and has finite rank over it. Then we have the isomorphisms of $R(G)$-modules:

$$\text{Ext}^i_R(\cdot, R(G)) = 0,$$

for $0 \leq i < d$ and $i = d + 1$

$$\text{Ext}^d_R(\cdot, R(G)) \simeq \text{Hom}_Z(M, Z).$$

We have not found these statements in literature. Indeed, there are many analogues involving different categories, for instance [6, Section 5]; a general version implying Proposition 3.4 can be found in [9, Theorem 7.0.5].
Proof. We are going to prove the two statements together. We will indicate by $R$ the ring $S[q^s]$ when dealing with the first statement and the ring $R(G)$ when dealing with the second, denoting by $x_1, \ldots, x_d$ the variables in both rings. In the same way, $\mathbb{F}$ will stand for $\mathbb{R}$ in the first case and for $\mathbb{Z}$ in the second. Let $\{e_1, \ldots, e_s\}$ be a basis of $M$ as a free $\mathbb{F}$-module, and let us denote by $M_R$ the module $R^s$, with a basis denoted by $\{p_1, \ldots, p_s\}$, and the surjective $R$–linear map

$$M_R \xrightarrow{\delta_0} M$$

 sending $p_i$ to $e_i$. Note that the action $\Psi$ of $R$ on $M$ is determined by specifying the $d$ matrices $\Psi(x_i)$; these have to be $s \times s$ matrices with coefficients in $\mathbb{F}$, and commuting one with each other. In the second (i.e. discrete) case, we also need them to be invertible in $GL_s(\mathbb{Z})$.

These matrices also act on $M_R$; in particular, in $M_R$ the operators $x_i$ and $\Psi(x_i)$ are different, one acting as a constant, one as linear transformation. Furthermore, in the module $\text{Hom}_{\mathbb{F}}(M, \mathbb{F})$, denoting by $\{e_1^*, \ldots, e_s^*\}$ the $\mathbb{F}$-dual basis, the action of $R$ is given by the transpose matrices $^t\Psi(x_i)$. Starting from $\delta_0$, we will create a free resolution of $M$ as $R$-module

$$0 \to M^{(d)}_R \xrightarrow{\delta_d} M^{(d-1)}_R \to \cdots \to M^{i}_R \xrightarrow{\delta_i} M^{i-1}_R \to M^{0}_R \xrightarrow{\delta_0} M \to 0,$$

in which we need to describe the maps $\delta_i : M^{(i)}_R \to M^{(i-1)}_R$; we are going to describe these maps block by block, using the decomposition

$$M^{(i)}_R = \bigoplus_{|I|=i} (M_R)_I.$$

Given $I = \{b_1, \ldots, b_i\}$, the image of a vector $v$ of $(M_R)_I$ by $\delta_i$ is going to be zero on the sets $J \not\subseteq I$, while on $J = I \setminus \{b_j\}$ the coordinate is going to be

$$(-1)^{j+1}(x_{b_j}v - \Psi(x_{b_j})v).$$

The reason of this choice is, at first, to have the image of $\delta_1$ in $M_R$ generated by all the vectors of the form $x_{b_j}v - \Psi(x_{b_j})v$, that indeed is the kernel of $\delta_0$. As any other kind of de Rham complex, proving exactness is only a matter of symbol chasing; crucial point to prove it is that matrices $\Psi(x_i)$ commute each other.

With this free resolution, we can explicitly evaluate the $\text{Ext}^i$ functors, as cohomology of the following complex

$$0 \to \text{Hom}_R(M, R) \xrightarrow{^t\delta_0} \text{Hom}_R(M_R, R) \xrightarrow{^t\delta_1} \text{Hom}_R(M^{d-1}_R, R) \xrightarrow{^t\delta_2} \cdots$$

$$\cdots \to \text{Hom}_R(M^{i-1}_R, R) \xrightarrow{^t\delta_i} \text{Hom}_R(M^{(i)}_R, R) \to 0,$$

where the maps are called $^t\delta_i$ because the matrices giving them are exactly the transposes of the one given in the previous complex. Now, as we already noticed, $\text{Hom}_R(M, R)$ is 0; moreover, we have that

$$\text{Hom}_R(M^{(i)}_R, R) \simeq M^{(i)}_R \simeq M^{(d-i)}_R.$$
and up to sign changes the complex is precisely the same as before, except for using $\Psi(x_i)$ in the maps $\delta_i$. Therefore, in the end, we get that the sequence is exact, besides at the last spot, in which the cokernel (indeed, $Ext^d_R(M, R)$) is a module $\hat{M}$ in which $R$ acts in a transpose way (by transpose matrices), that as we have seen before is exactly $Hom_F(M, F)$.

**Remark 3.6.** As a complex in the dual category of $S[g^*]$-modules (resp. $R(G)$-modules), the (derived) dual of any finite dimensional $\mathbb{R}$-vector space (resp. free $\mathbb{Z}$-module) has cohomology only supported in degree $d$; for the same reason, in some categories the definition of the dual includes also a shift in the derived category. This happens for instance for the category of coherent algebraic $D$-modules over an algebraic variety $\mathbb{Z}$. It is interesting to remark that in this setting we have an analogue of Propositions 3.4 and 3.5: if $M$ is an integrable connections (that is, a coherent $D$-module that is also a coherent sheaf), than its dual as $D$-module (obtained as $Ext^{dim(Z)}(M, D_Z)$ because of the shift) is isomorphic to its dual as coherent sheaf, that is, $Hom(M, O_Z)$; for all details, see [26, Chapter 2.6].

We can now focus on the modules $DM_T(X)$ and $D_T(X)$, which are free and (under the assumption that $T$ contains all the cocircuits) have finite rank over $\mathbb{Z}$ and $\mathbb{R}$ respectively. We have:

**Theorem 3.7.**

\[
\begin{align*}
\text{i)} & \quad Ext^d_{S[g^*]}(D_T^*(X), S[g^*]) \cong D_T(X) \\
\text{ii)} & \quad Ext^d_{S[g^*]}(D_T^*(X), S[g^*]) = 0 \text{ when } i \neq d \\
\text{iii)} & \quad Ext^d_{R(G)}(DM_T^*(X), R(G)) \cong DM_T(X) \\
\text{iv)} & \quad Ext^d_{R(G)}(DM_T^*(X), R(G)) = 0 \text{ when } i \neq d
\end{align*}
\]

**Proof.** By Lemma 2.4 $D_T(X)$ has finite dimension over $\mathbb{R}$. Then by Lemma 3.2 also $D_T^*(X)$ has finite dimension over $\mathbb{R}$. Therefore statements i) and ii) follow from Proposition 3.4, since for every finite dimensional vector space $V$ we have that $Hom_{\mathbb{R}}(Hom_{\mathbb{R}}(V, \mathbb{R}), \mathbb{R}) \cong V$.

In the same way, by Lemmas 2.4 and 3.3, $DM_T(X)$ and $DM_T^*(X)$ have finite rank over $\mathbb{Z}$. Then claims iii) and iv) follow from Proposition 3.5, since $Hom_{\mathbb{Z}}(Hom_{\mathbb{Z}}(M, \mathbb{Z}), \mathbb{Z}) \cong M$ for every free module of finite rank over $\mathbb{Z}$.  

### 3.3. Duality for DPV modules.

Since DPV modules have not finite rank, we cannot invoke Lemmas 3.4 and 3.5. In order to study duality of these objects, we need first to better examine their structure.
We will treat only the discrete case, since everything is the same in the differentiable setting.

From [16], we get that a canonical isomorphism

\[ \mathcal{F}_i(X)/\mathcal{F}_{i+1}(X) \cong \bigoplus_{\text{dim}(\mathfrak{s})=i} DM(X \cap \mathfrak{s}). \]

Here \( DM(X \cap \mathfrak{s}) \) is a submodule of \( \mathcal{C}[\Gamma \cap \mathfrak{s}] \), the module of \( \mathbb{Z} \)-valued functions on the lattice \( \Gamma \cap \mathfrak{s} \), which can be identified to functions in \( \mathcal{C}[\Gamma] \) that are supported in \( \Gamma \cap \mathfrak{s} \).

By definition (see Section 2.6) the corresponding group is \( G/G_\mathfrak{s} \), so that the dual space \( DM^*(X \cap \mathfrak{s}) \) is a quotient of \( R(G/G_\mathfrak{s}) \).

Furthermore, the quotient actually splits in a non-canonical way, meaning that we have isomorphisms (depending on a choice of some bases)

\[ \mathcal{F}_i(X) \cong \bigoplus_{\text{dim}(\mathfrak{s}) \geq i} DM(X \cap \mathfrak{s}). \]

By taking on both sides the \( R(G) \)-modules generated, we get the isomorphisms

\[ \tilde{\mathcal{F}}_i(X) \cong \bigoplus_{\text{dim}(\mathfrak{s}) \geq i} R(G) DM(X \cap \mathfrak{s}). \] (3)

Given this decomposition, we prove the following Lemma.

Lemma 3.8. We have the following isomorphisms:

\[ \text{Ext}^k_{R(G)}(\tilde{\mathcal{F}}_i(X), R(G)) \cong \bigoplus_{\text{codim}(\mathfrak{s})} R(G) \otimes_{R(G/G_\mathfrak{s})} DM^*(X \cap \mathfrak{s}) \text{ when } i \leq k \leq d \]

\[ \text{Ext}^k_{R(G)}(\tilde{\mathcal{F}}_i(X), R(G)) = 0 \text{ otherwise.} \]

Proof. Given the splitting in Formula (3), we can work on each component separately; in this way, we just need to check that

\[ \text{Ext}^k_{R(G)}(R(G) DM(X \cap \mathfrak{s}), R(G)) = R(G) \otimes_{R(G/G_\mathfrak{s})} DM^*(X \cap \mathfrak{s}) \]

when \( k = \text{dim}(\mathfrak{s}) \), and 0 otherwise.

By applying Theorem 3.7, everything follows from the sequence of isomorphisms

\[ \text{Ext}^k_{R(G)}(R(G) DM(X \cap \mathfrak{s}), R(G)) \cong \text{Ext}^k_{R(G/G_\mathfrak{s})}(DM(X \cap \mathfrak{s}), R(G)) \cong \]

\[ \cong R(G) \otimes_{R(G/G_\mathfrak{s})} \text{Ext}^k_{R(G/G_\mathfrak{s})}(DM(X \cap \mathfrak{s}), R(G/G_\mathfrak{s})) \]

coming from the fact that \( R(G) \) is a flat \( R(G/G_\mathfrak{s}) \)-module. \( \square \)

Thus for DPV modules several \( \text{Ext}^i \)’s can be nonzero and must be taken into account. We give the following definition.

Definition 3.9. The (discrete) dual DPV modules are the \( R(G) \)-modules

\[ \tilde{\mathcal{F}}^*_i(X) \cong R(G)/J^\nabla_i \]

where \( J^\nabla_i = (\nabla_X \setminus \mathfrak{s})_{\text{dim}(\mathfrak{s}) < i} \).
We believe that this is a good definition for many reasons. First, the ideal \( \mathcal{J}_i^\nabla \) is the annihilator of the \( R(G) \) module \( \tilde{F}_i(X) \); this can be shown using the decomposition \([3]\) above. Therefore \( \tilde{F}_i^*(X) \) is defined is the same spirit as the dual DM module \( DM^*(X) \).

Second, this algebraic notion of duality corresponds to a geometric duality: that is, the dual DPV modules appear as the equivariant K-theory of some spaces related with \( \tilde{F}_i(X) \), as we will see in the following sections.

Third, these modules actually keep track of all information contained in the whole \( Ext^* \). In fact, they admit filtration in which successive quotients are all the components of \( Ext^*(\tilde{F}_i(X), R(G)) \). In particular, we have the following proposition.

**Proposition 3.10.** The module \( \tilde{F}_i^*(X) \) admits a filtration

\[
\tilde{F}_{i,d}^*(X) \subset \tilde{F}_{i,d-1}^*(X) \subset \ldots \subset \tilde{F}_i^*(X) = \tilde{F}_i^*(X)
\]
such that

\[
\tilde{F}_{i,j}^*(X)/\tilde{F}_{i,j+1}^*(X) = Ext_{R(G)}^{d-j+i}(\tilde{F}_i(X), R(G))
\]

**Proof.** From the sequence of inclusions \( \mathcal{J}_i^\nabla \subset \mathcal{J}_{i+1}^\nabla \subset \ldots \subset \mathcal{J}_d^\nabla \) we get the sequence of projections

\[
\tilde{F}_i^*(X) \twoheadrightarrow \tilde{F}_{i+1}^*(X) \twoheadrightarrow \ldots \twoheadrightarrow \tilde{F}_d^*(X) \twoheadrightarrow 0
\]
and we take \( \{\tilde{F}_{i,j}^*(X)\} \) as sequence of successive kernels. In this way we get isomorphisms

\[
\tilde{F}_{i,j}^*(X)/\tilde{F}_{i,j+1}^*(X) \cong \frac{F_{d-j+i}(X)}{R(G) \otimes_{R(G/\mathcal{J}_{i})} DM^* (X \cap \mathcal{J}_{i})} \cong Ext_{R(G)}^{d-j+i}(\tilde{F}_i(X), R(G))
\]

where the last isomorphism comes from Lemma \([3,8]\) \( \Box \)

In the same way, in the differentiable case we can give the following definition. The same considerations and statements apply.

**Definition 3.11.** The (differentiable) dual DPV modules are the \( S[g^*]- \) modules

\[
\tilde{G}_{i}^*(X) \cong S[g^*]/\mathcal{J}_{i}^0
\]
where \( \mathcal{J}_i^0 = (d_{X_{\nabla}})_{\text{dim}(\nabla) < i} \).

4. Recalls on equivariant K-theory

We will briefly recall some notions about equivariant K-theory for a compact group \( G \). The reader is suggested to refer to \([5]\) and \([28]\) for details and proofs.
4.1. Definition in the compact case. Given a compact topological space $M$ with a continuous action of a compact Lie group $G$, one can consider *equivariant complex vector bundles* on $M$, that is, complex vector bundles $E \to M$ with a $G$-action on the total space $E$, respecting the action on $M$ and acting linearly on fibers. The *equivariant $K$-theory* $K^*_G(M)$ of $M$ is the group of integer linear combination of isomorphism classes of such objects, with sum operation given by direct sum of vector bundles.

This group is naturally a ring endowed with tensor product, having zero element given by the 0-dimensional vector bundle and identity given by the bundle $\mathbb{C} \times M$ with the action of $G$ only on the second coordinate. Furthermore, we have a class of trivial bundles $V \times M$ where $V$ is a representation of $G$, so that we get an homomorphism $R(G) \to K_G(M)$ and hence giving $K_G(M)$ the structure of a $R(G)$-algebra with identity.

4.2. Definition in the locally compact case. To define equivariant $K$-theory in the locally compact case, we will need a different definition. Following [28], in a compactly supported fashion, equivariant $K$-theory is defined in the following way.

**Definition 4.1.** The equivariant $K$-theory of a locally compact $G$-space $M$ classifies objects of the kind

\[ \{ E \xrightarrow{\phi} F \} \]

where $E$ and $F$ are equivariant vector bundles on $M$, and $\phi$ is an isomorphism outside a compact subspace of $M$.

We are not going to explicitly describe the meaning of the word “classifies”; the only things to keep in mind are that there are (quite obvious) notions of isomorphism and homotopy between such objects, and that objects like $\{ E \xrightarrow{id} E \}$ are set to be equivalent to 0.

This definition agrees with the one given for compact spaces; in particular, the correspondence is obtained sending the object $\{ E \xrightarrow{\phi} F \}$ to the formal difference $[E] - [F]$; note that in the compact case the map $\phi$ may be nowhere an isomorphism, and part of the proof of the equivalence is showing that we can move $\phi$ in an homotopic way to make it the zero function.

We have again a ring structure, in which the sum is given by the direct sum piece by piece, and the product is given by the formula

\[
\{ E \xrightarrow{\phi} F \} \circ \{ E' \xrightarrow{\phi'} F' \} = \{(E\otimes E')\oplus(F\otimes F') \xrightarrow{[1_E \otimes \phi' \otimes 1_{E'} + 1_F \otimes (\phi')^*]} (E\otimes F')\oplus(F\otimes E')\}
\]

where $\phi^*$ is the dual map, defined by Hermitian metrics on $E$, $E'$, $F$ and $F'$.

We have again an action of $R(G)$ by the tensoring both $E$ and $F$ with the trivial elements $V \times M$. So $K_G(M)$ is again an $R(G)$-algebra, but not necessarily with identity: in fact, we do not have anymore a ring homomorphism $R(G) \to K_G(M)$. 

Finally, we define $K^{-i}_G(M) = K_G(M \times \mathbb{R}^i)$, where $\mathbb{R}^i$ is given a trivial action of $G$, that again carries an $R(G)$-algebra structure (for $i = 0$ we get the same object we just defined). As we will see in Section 4.4 all the information is actually contained in $K^0_G(M)$ and $K^{-1}_G(M)$.

4.3. **Examples.**

1. If $M$ is a point, then $K_G(M) = R(G)$, and $K^{-1}_G(M) = 0$.
2. If $M$ is given a trivial $G$-action, then $K_G(M) \cong R(G) \otimes K(M)$, where $K(M)$ is the usual $K$-theory of $M$.
3. If $M$ is given a free $G$-action, then we have an isomorphism of rings $K_G(M) \cong K(M/G)$.
4. If $G$ is abelian, and a subgroup $H$ stabilizes all points in $M$, then we have $K_G(M) \cong K_{G/H}(M) \otimes_{R(G/H)} R(G)$.

Proof of (4) is a slight modification of the classic proof of (2).

4.4. **Thom isomorphism.** If $E \xrightarrow{\pi} M$ is a $G$-vector bundle, then we have an isomorphism as $R(G)$-modules (but not as rings)

$$K_G(M) \xrightarrow{Th} K_G(E).$$

If $M$ is compact, we can give an explicit description of the image of the identity element of $K_G(M)$, so that we can see it as a generator of $K_G(E)$ as $K_G(M)$-module.

We can consider $E$ as an equivariant vector bundle on $E$ itself (pulling back from the projection $E \rightarrow M$), and then consider all its exterior powers $\wedge^i E$ still as equivariant vector bundles on $E$. We will call $\wedge^{odd} E$ and $\wedge^{even} E$ the direct sum of all odd (resp. even) exterior powers of $E$; between this two vector bundles on $E$ we have a special map $c(E)$ (coming from the wedge product), called the *Clifford map* (see [17], pag. 795). So now, the data of

$$\bigwedge^{odd} E \xrightarrow{c(E)} \bigwedge^{even} E$$

is an element of $K_G(E)$, because $c(E)$ is an isomorphism everywhere except possibly on the zero section of $E$ (namely, $M$), which is indeed compact; this is the generator we were talking about.

In particular, using as $E$ the space $\mathbb{C} \times M$ with trivial action on $\mathbb{C}$, we get an isomorphism of $R(G)$-modules $K^{-2}_G(M) \cong K^0_G(M)$, and more in general we have $K^{n-2}_G(M) \cong K^{n}_G(M)$ (the *Bott periodicity*). On one hand, this allows us to define $K^i_G(M)$ also for positive $i$, and on the other hand this let us focus just on $K_G = K^0_G$ and $K^1_G$, that contain all the information.

4.5. **Functoriality and Mayer-Vietoris sequence.** Equivariant K-theory is a contravariant functor of $R(G)$-algebras for proper maps (by pullback), and a covariant functor of $R(G)$-modules for open embeddings (by extension).
Given a closed equivariant embedding \( Z \overset{i}{\to} M \) with an equivariant complex tubular neighbourhood (that is, an open neighbourhood \( U \) of \( Z \) in \( M \) that can be seen as a complex \( G \)-vector bundle over \( Z \)), then we have a shriek pushforward \( i_! : K_G(Z) \to K_G(M) \), that is the composition of the Thom isomorphism from \( K_G(Z) \) to \( K_G(U) \) and of the pushforward given by the open embedding \( U \hookrightarrow M \).

Given \( Z \overset{i}{\to} M \) closed embedding, calling \( U = M \setminus Z \overset{j}{\to} M \) the open embedding, we have an exact sequence

\[
K^0_G(U) \xrightarrow{j_*} K^0_G(M) \xrightarrow{i^*} K^0_G(Z) \xrightarrow{\delta} K^1_G(Z) \xrightarrow{i_*} K^1_G(M) \xrightarrow{j_*} K^1_G(U).
\]

5. Geometric realization of dual DM and DPV modules

While the theorems in this Section can be stated without any Index Theory, their proof will make much use of it. The interested reader will find some basic notions on this topic in the Appendix.

Let \( M^i_X \) \((0 \leq i \leq d)\) be the spaces introduced in Section 2.6. We denote by \( TM_X \) and \( T^*_X \) the tangent and cotangent bundle of \( M_X \) respectively. Let \( T^*_G M_X \) be the subspace of the covectors in \( T^*_X \) that are normal to the vectors in \( T_M X \) that are tangent to orbits of the \( G \)-action.

Notice that orbits in \( M^i_X = M^{i+d}_X = M^d_X \) are all of the same dimension, so \( T^*_G M^i_X \) is actually a vector bundle over \( M^i_X \) (of rank \( n - d \)), while for \( i < d \) this is not true.

In [17], by using the index theorem, the following result is proved:

**Theorem 5.1** (De Concini-Procesi-Vergne). There is an isomorphisms of \( R(G) \)-modules

\[
K^0_G(T^*_G M^{2i}_X) \cong \widetilde{F}_i(X)
\]

and

\[
K^1_G(T^*_G M^{2i}_X) = 0
\]

The extremal cases are particularly interesting: the fact that

\[
K^0_G(T^*_G M_X) \cong \widetilde{F}(X)
\]

is the answer to a question that Atiyah raised in [4]. On the other hand, when \( i = d \), by the Thom isomorphism we get

\[
K^2_G(M^{2n-d}_X) \cong K^0_G(T^*_G M^{2n}_X) \cong DM(X)
\]

(and \( K^1_G(M^{2n}_X) = K^1_G(T^*_G M^{2n}_X) = 0 \)).

In [19] there are analogues of these facts for equivariant cohomology with compact support:
Theorem 5.2 (De Concini-Procesi-Vergne). There is an isomorphism of graded $S[\mathfrak{g}^*]$-modules
\[ H^*_{c,G}(T^*_G M^\geq_i X) = \tilde{\mathcal{G}}_i(X) \]
In particular for $i = d$ we have
\[ H^*_{c,G}(M^\text{fin}_X) = H^*_{c,G}(T^*_G M^\text{fin}_X) = D(X) \]
where, by retracting the total space $T^*_G M^\text{fin}_X$ on the base space $M^\text{fin}_X$, the grading on the compact support cohomology is shifted by the (real) rank of the vector bundle, that is $2n - d$.

The theorem above is proved in [18] by introducing an analogue of the index of transversally elliptic operators, the infinitesimal index; the correspondence is naturally with cohomology with compact support because of the “compact support nature” of equivariant K-theory.

In the same paper, there is also a calculation of the standard (meaning not with compact support) cohomology of the spaces $M^\geq_i X$, which does not use any index theory and turn out to yield the dual modules described in Section 3:

Theorem 5.3 (De Concini-Procesi-Vergne). There is an isomorphism of graded $S[\mathfrak{g}^*]$-algebras
\[ H^*_G(T^*_G M^\geq_i X) = \tilde{\mathcal{G}}^*_i(X) \]
In particular for $i = d$ we have
\[ H^*_G(M^\text{fin}_X) = H^*_G(T^*_G M^\text{fin}_X) = D^*(X) \]
since of course cohomology (with its natural grading) is preserved by the deformation retract.

We will now provide an analogue of Theorem 5.3 for equivariant K-theory. Instead of looking for a different definition of equivariant K-theory not involving compact support, we will compactify the spaces $M^\geq_i X$ (actually, a deformation retract of them) to find spaces to perform the same inductive process on.

The geometric idea behind this kind of compactification is the following: instead of removing a closed subspace from a compact space (then, losing compactness), one removes a tubular neighborhood. In this way, the resulting space is still compact, but another property is lost: smoothness. More precisely, to perform differential geometry one has to enlarge the class of spaces to manifold with corners (for instance, see [22] and [23]); anyway, this is not an issue, because we are not requiring any smoothness or boundarylessness condition (we are not either going to use the tangent space).

We now describe in more detail our construction. First, let us restrict to the unitary sphere $S_X \subset M_X$, to have a compact space to start with. By Formula (1) $S_X \cap M^\text{fin}_X$, and more generally $S_X \cap M^\geq_i X$, are obtained by the compact space by removing some closed subsets $S_X \cap M^\geq_i$. Now, for any such...
subset of $S_X$, we consider a small $G$-invariant tubular neighbourhood

$$U_\varepsilon = \{(z_a)_{a \in X} \in S_X \text{ such that } |z_a| < \varepsilon \text{ if } a \notin s\}$$

where $\varepsilon$ is a sufficiently small number (for the calculations we are going to do, we can assume $\varepsilon$ being any number smaller than 1) and finally we are ready to define

$$S^{\geq i}_X = S_X \setminus \bigcup_{\substack{s \in R_X \dim(s) < i}} U_s$$

(in fact, we can take the union only on the rational subspaces of dimension $i-1$) and to state one of the main results of this paper.

**Theorem 5.4.** We have

$$K^0_G(S^{\geq i}_X) = \tilde{F}^*_i(X) \text{ and } K^1_G(S^{\geq i}_X) = 0 \text{ for } 1 \leq i \leq d$$

In particular,

$$K^0_G(S^{fin}_X) = DM^*(X) \text{ and } K^1_G(S^{fin}_X) = 0$$

Our proof will be based on a multiple induction. This requires to generalize it to a broader family of spaces; in order to do that, we define an admissible set of nonzero rational subspaces $Q$ as a set closed under inclusions (if $s \subset t$ and $t \in Q$, then also $s \in Q$), and let

$$S^Q_X = S_X \setminus \bigcup_{s \in Q} U_s.$$}

Of course, the set of all proper rational subspaces, and more in general the sets $\{\dim(s) < i\}$ are admissible.

Then Theorem 5.4 is a corollary of the following:

**Theorem 5.5.** If $Q$ is an admissible set of rational subspaces, then

$$K^0_G(S^Q_X) = R(G)/\nabla_{X \setminus s} \in Q \text{ and } K^1_G(S^Q_X) = 0.$$}

The beginning step of our induction is given by the following lemma.

**Lemma 5.6.** $K^0_G(S_X) \cong \tilde{F}^*_1(X) \cong R(G)/\nabla_X$, and $K^1_G(S_X) = 0$;

**Proof.** Let us consider the Mayer-Vietoris exact sequence

$$
\begin{array}{ccc}
K^0_G(M_X \setminus \{\emptyset\}) & \longrightarrow & K^0_G(M_X) \longrightarrow K^0_G(\{\emptyset\}) \\
\uparrow & & \uparrow \\
K^1_G(\{\emptyset\}) & \longrightarrow & K^1_G(M_X) \longrightarrow K^1_G(M_X \setminus \{\emptyset\})
\end{array}
$$

The bottom left and center elements of this sequence are 0 (because of Thom isomorphism), so we have to look at kernel and cokernel of $K^0_G(M_X) \rightarrow$
$K^0_G(\emptyset)$; now, both this two modules are isomorphic to $R(G)$ (the latter being isomorphic to $R(G)$ as ring too), and the map is the multiplication by the Clifford class $[\bigwedge^{odd} M_X] - [\bigwedge^{even} M_X]$ that by a straightforward calculation appears to be exactly $\nabla_X$; so, $K^0_G(M_X \setminus \{f\}) = 0$ and $K^1_G(M_X \setminus \{f\}) = R(G)/\nabla_X$. But now, since $M_X \setminus \{f\} = \mathbb{R} \times S_X$, we are done. \hfill \square

5.1. Proof of Theorem 5.5. Let us proceed by a double induction on the cardinality of $Q$ and on the dimension of the group $G$; we start with $Q = \emptyset$ and we will add one rational subspace at the time; this step will be based on the statement of the theorem for a lower dimensional group. When $Q = \emptyset$, lemma 5.6 gives us the statement of the theorem.

Now, suppose we want to add a rational subspace $s$ to $Q$, such that all rational subspaces $t$ of $s$ already belong to $Q$. On the geometric side, we have to remove from $S_X^Q$ the open set $S_X^Q \cap U_s$ to get $S_X^{Q \cup \{s\}}$. Now, the open set $S_X^Q \cap U_s$ retracts onto the compact space $S_X^Q \cap S_s$; more precisely, $S_X^Q \cap U_s$ is equivariantly homeomorphic to the normal bundle of $S_X^Q \cap S_s$ into $S_X^Q$, that is, $M_X \setminus s \times S_X^Q \cap S_s$. We are going to investigate the maps

$$K^0_G(S_X^Q \cap S_s) \xrightarrow{\cong} K^0_G(S_X^Q \cap U_s) \rightarrow K^0_G(S_X^Q)$$

the second one of which we hope to fill in the exact sequence of the inclusions $S_X^Q \cap U_s \hookrightarrow S_X^Q \hookrightarrow S_X^{Q \cup \{s\}}$, to get the inductive step.

At first, let us focus on $S_X^Q \cap S_s$; by the fact that $Q$ contains all rational subspaces of $s$, we have

$$S_X^Q \cap S_s = S_s \setminus \bigcup_{t \subseteq s} U_t.$$ 

Considering now $H = \ker(s)$ the kernel of all characters belonging to $s$ we have an action of $G/H$ on this space, because this space is contained in $M_s$, so $H$ will stabilize the space all; considering this action, this space is exactly $S_s^{fin}$. So, by inductive hypothesis, we get that $K^1_{G/H}(S_X^Q \cap S_s) = 0$ and

$$K^0_{G/H}(S_X^Q \cap S_s) = R(G/H)/\langle \nabla_X \rangle_{L \subseteq s}.$$ 

By the fourth Example in Section 4.3 we get that $K^1_G(S_X^Q \cap S_s) = 0$ and $K^0_G(S_X^Q \cap S_s) = R(G)/\langle \nabla_X \rangle_{L \subseteq s}$.

Coming back to Formula (1), it is easy to see that the element 1 of $K^0_G(S_X^Q \cap S_s)$ is sent to the element $\nabla_X \setminus s$ in

$$K^0_G(S_X^Q) = R(G)/\langle \nabla_X \rangle_{L \subseteq Q}.$$ 

Note that the fact that $\nabla_X \setminus s = 0$ in $K^0_G(S_X^Q \cap S_s)$ is reflected into the fact that

$$\nabla_X \setminus \nabla_X \setminus s = \nabla_X \setminus s = 0$$

in $K^0_G(S_X^Q)$, because $t \in Q$ for all proper subspaces of $s$. 


So, we are ready to consider the exact sequence for the inclusions $S^Q_X \cap U_\Delta \hookrightarrow S^Q_X \leftarrow S^Q_X \cup \{s\}$

\[
\begin{array}{cccc}
K^0_G(S^Q_X \cap U_\Delta) & \rightarrow & K^0_G(S^Q_X) & \rightarrow & K^0_G(S^Q_X \cup \{s\}) \\
\downarrow & & \downarrow & & \downarrow \\
K^1_G(S^Q_X \cup \{s\}) & \leftarrow & K^1_G(S^Q_X) & \leftarrow & K^1_G(S^Q_X \cap U_\Delta)
\end{array}
\]

where we already know that the lower middle and right modules are 0, and the upper left maps is the inclusion

\[R(G)/(\nabla_{x\cup\{s\}}) \xrightarrow{\nabla_{x\cup\{s\}}} R(G)/(\nabla_{x\cup\{s\}})_{x \in \Omega} \]

In this way, considering kernel and cokernel, we immediately get $K^0_G(S^Q_X \cup \{s\}) = 0$ and

\[K^0_G(S^Q_X \cup \{s\}) = R(G)/(\nabla_{x\cup\{s\}})_{x \in \Omega, \nabla_{x\cup\{s\}}} = R(G)/(\nabla_{x\cup\{s\}})_{x \in \Omega, \nabla_{x\cup\{s\}}} = K^1_G(S^Q_X \cap U_\Delta),
\]

that is exactly what we needed to prove.

**Remark 5.7.** In this way we found a geometric realization for $\tilde{\mathcal{F}}^*_i(X)$ only for $i \geq 1$, because these are the only ones that can be got from admissible set; for $\tilde{\mathcal{F}}^*(X)$, the extremal case, we haven’t either defined a compact space to match with it; this is because we should consider also points that are fixed for the $G$ action, so the whole $M_X$, that doesn’t retract onto the unit sphere $S_X$. But if we retract $M_X$ onto the origin, we immediately get

\[K^0_G(\{pt\}) \cong \tilde{\mathcal{F}}^*(X) \cong R(G) \text{ and } K^1_G(\{pt\}) = 0\]

How this theorem may lead to a more general statement about duality in equivariant K-theory is still a subject of investigation.

6. Geometric realization of local DM modules

As stated in Theorems 5.2 and 5.4, the modules $D(X)$ and $DM(X)$ have a geometric realization in terms of the manifold $M_X^{fin}$.

We also know that these modules have interesting submodules, the local modules $D_\Omega(X)$ and $DM_\Omega(X)$; it is then natural to wonder if these submodules have geometric realizations in terms of submanifolds of $M_X^{fin}$.

More generally, following definitions given in Section 2.3, we may ask whether we have $K^*_G(T^*_G M_T^\xi) = DM_T(X)$ and $H^*_G(T^*_G M_T^\xi) = D_T(X)$. Of course, this is not true in general: for instance, for $T = \emptyset$ we have $K^0_G(T^*_G M_T^\xi) = \mathcal{F}(X)$ on one side, and on the other $DM_\emptyset(X) = C[\Gamma]$. We can restate the question as finding for which subsets $T$ we have such isomorphisms.
**Definition 6.1.** We say that a set $T \subseteq 2^X$ is semilocal if it is closed under taking supsets and there exists a big cell $\Omega$ such that

$$L(X) \subseteq T \subseteq L_\Omega(X).$$

Notice that the definition of semilocal can be restated in the following way: a set $T$ is semilocal if and only if $DM_T(X)$ contains one of the quasi polynomials $q_\Omega$, as in Theorem 2.3, that means one of the quasi polynomials agreeing with the partition function $P_X$ in one of the big cells of $C(X)$. We will prove the following theorem.

**Theorem 6.2.** Let $T$ be semilocal. Then there is an isomorphism of $R(G)$-modules

$$K^0_T(T_G^*M^T_X) \cong DM_T(X)$$

that is given by the Atiyah’s index; furthermore, $K^1_G(T_G^*M^T_X) = 0$.

This Theorem will be proved in the next Subsection. We believe that, with completely analogous methods, the following statement can be proved:

**Conjecture 6.3.** Let $T$ be semilocal. Then there is an isomorphism of graded $S[\mathfrak{g}^*]$-modules

$$H^*_T(T_G^*M^T_X) \cong D_T(X)$$

that is given by the infinitesimal index.

This will provide, as special cases, geometric realizations of $D_\Omega(X)$.

Also many of the semi-internal spaces $D_\Omega(X, I)$ introduced in [25] seem to be semilocal, and then would be geometrically realized by Conjecture 6.3.

As a Corollary of Theorem 6.2 we get the second fundamental result of this paper:

**Theorem 6.4.** There is an isomorphism of $R(G)$-modules

$$K^0_T(T_G^*M^\Omega_X) \cong DM_\Omega(X)$$

that is given by the Atiyah’s index; furthermore, $K^1_G(T_G^*M^\Omega_X) = 0$.

A nice feature of this theorem is that we already know from the theory of splines that the various spaces $DM_\Omega(X)$ are cyclic $R(G)$-modules generated by the quasi-polynomial $q_\Omega$ coming from the partition function; now, natural injections $DM_\Omega(X) \hookrightarrow DM(X)$ give exactly a minimal set of generators of $DM(X)$ as $R(G)$-module. So, this theorem provides the geometric part of this, giving commutative diagrams

$$
\begin{array}{ccc}
DM_\Omega(X) & \longrightarrow & DM(X) \\
\downarrow & & \downarrow \\
K^*_T(T_G^*M^\Omega_X) & \longrightarrow & K^*_T(T_G^*M^{fin}_X)
\end{array}
$$
where the last row comes from the open embedding $T^*_G M^\Omega_X \hookrightarrow T^*_G M^\text{fin}_X$, that means, a geometric realization of the generators of $K^*_G(T^*_G M^\Omega_X)$ as $R(G)$-module, as cycles supported in the subvarieties $T^*_G M^\Omega_X$.

**Remark 6.5.** Since $M^T_X \subseteq M^\text{fin}_X$, all the points in $M^T_X$ have finite stabilizer, so $T^*_G M^T_X$ is a vector bundle over $M^T_X$ of (real) rank $2n - d$. By Thom isomorphism, we then get, as a consequence, $K^{2n-d}_G(M^\Omega_X) \cong DM_\Omega(X)$ and $K^{2n-d+1}_G(M^\Omega_X) = 0$.

6.1. **Proof of Theorem 6.2.** The proof will be based on an induction, as $T$ increases from the set of all cocircuits to one of the sets $L_\Omega(X)$. Throughout the proof, we will need to use also the extremal situation where we consider $S = 2^X \setminus \{\emptyset\}$ (that in general is not semilocal). Note that if $S = 2^X \setminus \{\emptyset\}$, then $DM_S(X)$ are just the function on $\Gamma$ constant modulo the subgroup $\mathbb{Z}X$. In the Appendix, we are going then to prove the following.

**Lemma 6.6.** If we denote $2^X \setminus \{\emptyset\}$ by $S$, then, if $\text{rk}(X) = \text{rk}(\Gamma)$, the index gives an isomorphism

$$K^0_G(T^*_G M^S_X) \cong DM_S(X)$$

and furthermore we have $K^1_G(T^*_G M^S_X) = 0$.

Let’s now proceed with the inductive step. Let us consider two semilocal sets $T$ and $T' = T \cup \{A\}$ that differ only by one element, and suppose we know the theorem for $T$; note that $T$ contains all subsets of $X$ of the kind $A \cup \{a\}$ for $a \in X \setminus A$, because $A$ has to be necessarily a minimal element for $T'$.

On the geometric side, we have that $M^T_X$ contains $M^{T'}_X$, and their difference is

$$M^S_{X \setminus A} = \{(v_a)_{a \in X} | v_a = 0 \ \forall a \in A, \ v_a \neq 0 \ \forall a \notin A\}.$$ 

Passing to cotangent spaces, we still have that $T^*_G M^T_X$ is an open set of $T^*_G M^T_X$ (let us call $j$ the open embedding), whose complement is $T^*_G M^S_{X \setminus A} \times M_A$ (let us call $i$ the closed embedding), for which we have a Thom isomorphism

$$C : K^0_G(T^*_G M^S_{X \setminus A}) \rightarrow K^0_G(T^*_G M^S_{X \setminus A} \times M_A).$$

Now, we have that $K^1_G(T^*_G M^S_{X \setminus A}) = 0$ (from the lemma above; here we are using the fact that $T$ contains all cocircuits, because $X \setminus A$ must have maximal span) and that $K^1_G(T^*_G M^{T'}_X) = 0$ (by inductive hypothesis); so we get the following short exact sequence

$$0 \rightarrow K^0_G(T^*_G M^{T'}_X) \xrightarrow{j_*} K^0_G(T^*_G M^{T}_X) \xrightarrow{C^{-1}_*} K^0_G(T^*_G M^S_{X \setminus A}) \rightarrow 0.$$ 

On the combinatoric side, we have the following Lemma:

**Lemma 6.7.** The following is a short exact sequence
0 \longrightarrow DM_T(X) \longrightarrow DM_T(X) \xrightarrow{\nabla_A} DM_S(X \setminus A) \longrightarrow 0,

where the map $DM_T(X) \rightarrow DM_T(X)$ is just the inclusion.

Proof. Well-definedness of $\nabla_A$ comes right from the definitions, because $A \cup \{a\} \in \mathcal{T}$ for every $a \notin A$. Let us prove exactness. In $DM_T(X)$ this is clear; furthermore, $DM_T(X)$ is defined in $DM_T(X)$ as the vanishing locus of $\nabla_A$, so we have exactness also in $DM_T(X)$.

To prove surjectivity of $\nabla_A$, let’s recall the fact that $DM_T(X)$ contains a quasi-polynomial $q_{\Omega}$ agreeing with the partition function $P_X$ on the big cell $\Omega$; this means, by definition of $L_{\Omega}(X)$, that the cone $C(X \setminus A)$ contains $\Omega$ as well; let now $q'_{\Omega}$ be the quasi polynomial agreeing in $\Omega$ with the partition function $P_{X \setminus A}$; note that by properties of the partition function, we have $\nabla_A q_{\Omega} = q'_{\Omega}$. Let now $X \setminus A = B \cup C$, where $B$ is a $\mathbb{Q}$-basis of $\Gamma$ such that $C(B) \supseteq \Omega$; we have now $\nabla_C P_{X \setminus A} = P_B$, that on $\Omega$ agrees with a quasi polynomial with value 1 on the lattice points in $\mathbb{Z}B$ and 0 otherwise, that is $\nabla_C q'_{\Omega}$. Now, let’s take an element $f$ of $DM_S(X \setminus A)$; this function is constant modulo the sublattice $\mathbb{Z}(X \setminus A)$ (that contains the lattice $\mathbb{Z}B$); this means that $f$ can be obtained by $\nabla_C q'_{\Omega}$ by an element $\tau$ of $R(G)$. We have then

$$f = \tau \nabla_C q'_{\Omega} = \tau \nabla_C \nabla_A q_{\Omega} = \nabla_A (\tau \nabla_C q_{\Omega}),$$

thus $f$ is in the image of $\nabla_A$, because $\tau \nabla_C q_{\Omega}$ is an element of the $R(G)$-module $DM_T(X)$.

Now that we have two exact sequences, we are going to glue them in a commutative diagram, with vertical arrows given by the index map, and all vertical arrows but one being an isomorphism; by the five lemma, this will lead us to the conclusion.

First, let’s verify that such a diagram exist; the only check to do is proving the following lemma, that will be done in the Appendix.

**Lemma 6.8.** The index map sends $K_G^0(T_G^* M^T_X)$ into $DM_T(X)$.

Note that the two other vertical arrows are well defined because of lemma 6.6 and by the inductive step on $\mathcal{T}$. We know have the diagram

$$
\begin{array}{ccc}
0 & \xrightarrow{\text{ind}} & K_G^0(T_G^* M^T_X) \\
\downarrow \text{ind} & & \downarrow \text{ind} \\
0 & \xrightarrow{\subseteq} & DM_T(X)
\end{array}
\begin{array}{ccc}
\text{ind} & & \text{ind} \\
\downarrow \nabla_A & & \downarrow \text{ind} \\
0 & \xrightarrow{\text{ind}} & DM_S(X \setminus A)
\end{array}
\xrightarrow{\text{ind}}
\begin{array}{ccc}
0 & \xrightarrow{\text{ind}} & 0
\end{array}
\begin{array}{ccc}
0 & \xrightarrow{\text{ind}} & 0
\end{array}

The only thing to be checked now is the commutativity of the two center squares: this will be done in the Appendix. Now, by the five lemma, we have that the index map $K_G^0(T_G^* M^T_X) \rightarrow DM_T(X)$ is an isomorphism, and
completing the Mayer-Vietoris sequence we get $K^0_G(T^*_G M^T_X)$. This proves the theorem.

**APPENDIX: SOME INDEX THEORY**

We shall recall some notion and properties of the index of transversally elliptic operators, and in particular the applications we need in equivariant $K$-theory. References with details and proofs are [4] and [17].

Let $M$ be a manifold that has a smooth action of a compact abelian Lie group $G$ and is embedded equivariantly in a compact $G$-manifold. We consider the subspace $T^*_G M$ of the cotangent bundle $T^* M$, given by conormal vector that are perpendicular to tangent spaces of orbits, that still carries a smooth action of $G$. Let then $\mathcal{C}^{-\infty}(G)$ be the space of distributions on $G$, that we can see by Fourier transform inside the space

$$\mathcal{C} \Gamma_G = \{ f : \Gamma_G \to \mathbb{Z} \}$$

where $\Gamma_G$ is the lattice of irreducible representations of $G$ (whenever unnecessary, we will omit the subscript $G$ in $\Gamma_G$).

The *index map* is an $R(G)$-module homomorphism $K^0_G(T^*_G M) \xrightarrow{\text{ind}_M} \mathcal{C}[\Gamma_G]$ coming from the theory of elliptic operators; for the definitions and preparatory work needed, we suggest [4]. Some of its properties we are going to need are the following, that can be found in [17]:

1. If $U \xrightarrow{j} M$ is an open embedding, then the following diagram is commutative:

$$
\begin{array}{c}
K^0_G(T^*_G U) \xrightarrow{j_*} K^0_G(T^*_G M) \\
\downarrow \text{ind}_U \quad \downarrow \text{ind}_M
\end{array}
\Rightarrow
\begin{array}{c}
\mathcal{C}[\Gamma_G] \\
\end{array}
$$

2. Let us consider $M \times \mathbb{R}$ where $\mathbb{R}$ is given a trivial $G$-action; we have $T^*_G(M \times \mathbb{R}) = T^*_G M \times T^*_G \mathbb{R} = T^*_G M \times \mathbb{C}$ so that we have a Thom isomorphism $\phi_*$, and the following diagram commutes:

$$
\begin{array}{c}
K^0_G(T^*_G M) \xrightarrow{\phi_*} K^0_G(T^*_G(M \times \mathbb{R})) \\
\downarrow \text{ind}_M \quad \downarrow \text{ind}_{M \times \mathbb{R}}
\end{array}
\Rightarrow
\begin{array}{c}
\mathcal{C}[\Gamma_G] \\
\end{array}
$$

3. Let $H$ be a subgroup of $G$, and assume $N$ is a manifold on which $H$ acts. Then, if $M = G \times_H N$, we have an isomorphism

$$
K^*_H(T^*_H N) \xrightarrow{\iota^*_G} K^*_G(T^*_G M),
$$
and an induction map $\text{Ind}_G^H : \mathcal{C}[\Gamma_H] \to \mathcal{C}[\Gamma_G]$. The following diagram is commutative:

$$
\begin{array}{ccc}
K^0_H(T^*_HN) & \xrightarrow{\sim} & K^0_G(T^*_GM) \\
\downarrow \text{ind}_N & & \downarrow \text{ind}_M \\
\mathcal{C}[\Gamma_H] & \xrightarrow{\text{Ind}_H^G} & \mathcal{C}[\Gamma_G].
\end{array}
$$

(4) ([17], Lemma 6.11) Let $M_X$ be as in the previous sections, and let $Y$ be a sublist of $X$, such that we have a decomposition

$$M_X = M_{X \setminus Y} \oplus M_Y.$$ 

Let also $U$ be an open $G$-invariant set contained in $M_{X \setminus Y} \times (M_Y \setminus 0)$, and $\sigma$ an element of $K^0(T^*_G U)$; then we have

$$\nabla_Y \cdot \text{ind}(\sigma) = 0.$$

(5) ([17], Corollary 6.15) Let $C^{-1}i^* : K^0_G(T^*_G M_X) \to K^0_G(T^*_G M_{X \setminus A})$ as defined in the previous section; let $\sigma$ be an element of $K^0_G(T^*_G M_X)$, and $\sigma_0$ its image in $K^0_G(T^*_G M_{X \setminus A})$. Then we have

$$\nabla_A \cdot \sigma = \sigma_0.$$

Let’s prove now the lemmas we need for the proof of Theorem 6.4.

**Proof of Lemma 6.6.** In this case, the manifold $M^S_X$ is just

$$(\mathbb{C}^*)^n = \mathbb{R}^n \times (S^1)^n.$$ 

now, calling $H$ the finite stabilizer of this space, we get $(S^1)^n = G \times_H (S^1)^{n-d}$, where now $H$ acts on $(S^1)^{n-d}$ trivially; by applications of properties (2) and (3), we get the following commutative diagram

$$
\begin{array}{ccc}
K^0_H(T^*_H(S^1)^{n-d}) & \sim & K^0_G(T^*_G M^S_X) \\
\downarrow \text{ind}_N & & \downarrow \text{ind}_M \\
\mathcal{C}[\Gamma_H] & \xrightarrow{\text{Ind}_H^G} & \mathcal{C}[\Gamma_G].
\end{array}
$$

Now, we know that

$$\text{Im}(\text{Ind}_H^G) = \{ f \in \mathcal{C}[\Gamma_G] | \nabla_a f = 0 \ \forall a \in \Gamma_{G/H} \} =$$

$$= \{ f \in \mathcal{C}[\Gamma_G] | \nabla_a f = 0 \ \forall a \in X \} = DM_S(X)$$

so the bottom row is an isomorphism into $DM_S(X)$; if we prove now that the left arrow is an isomorphism, we get by commutativity that the index map gives an isomorphism $K^0_G(T^*_G M^S_X) \to DM_S(X)$. Now, we have

$$K^0_H(T^*_H(S^1)^{n-d}) = K^0_H((S^1)^{n-d}) = R(H) \otimes \mathbb{Z} K^0((S^1)^{n-d}) = R(H)$$
(and by the way, in this way we also get \( K_G^0(T_G^*M_X^S) = 0 \)), and on the other hand \( C[\Gamma_H] \cong R(H) \) because \( H \) is finite. To conclude we just need to show that a generator of \( K_H^0(T_H^*S^{n-d}) \cong K_H^0(T_H(C^*)^{n-d}) \) is sent in \( C[\Gamma_H] \); to do it, we can first reduce to the case \( n - d = 1 \) and \( H = \{0\} \); the result will follow taking the tensor product. Then we are now examining the index map \( K^0(T^*C^* \to Z) \); embedding \( C^* \) into \( \mathbb{P}^1(C) \) (using property (1)) we need to check the value of the index on the class \([O(1)] - [O]\) (that is, the image in \( K^0(T^*\mathbb{P}^1) \) of the generator of \( K^0(T^*C^*) \)); but on compact manifold, index coincides with Euler characteristic, that in this case is 2-1=1, that is, a generator of \( \mathbb{Z} \).

Proof of Lemma 6.8 This is a direct consequence of property (4); \( M^T_X \) is obtained by \( M_X \) removing \( M_Y \) such that \( X \setminus Y \in \mathcal{T} \); so, the index of every element of \( K^0_G(T_G^*M^T_X) \) is annihilated by all \( \nabla_Y \) for \( Y \in \mathcal{T} \), that means, lies in \( DM_T(X) \).

Proof of the squares commuting. The left square commutes because of property (1). About the right square, property (5) says that the following square is commutative:

\[
\begin{array}{ccc}
K^0_G(T^*_G M_X) & \xrightarrow{C^{-1}i^*} & K^0_G(T^*_G M_{X\setminus A}) \\
\downarrow \text{ind} & & \downarrow \text{ind} \\
C[\Gamma] & \xrightarrow{\nabla_A} & C[\Gamma].
\end{array}
\]

But now, given the open inclusions \( M^T_X \hookrightarrow M_X \) and \( M^S_{X\setminus A} \hookrightarrow M_{X\setminus A} \), we can apply property (1) to conclude.

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