Pair Correlation of the zeros of the Riemann zeta function in longer ranges

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Abstract

In this paper, we study a more general pair correlation function, \( F_h(x, T) \), of the zeros of the Riemann zeta function. It provides information on the distribution of larger differences between the zeros.

1 Introduction

First of all, we assume the Riemann Hypothesis on the Riemann zeta function \( \zeta(s) \) throughout this paper; \( \rho = \frac{1}{2} + i\gamma \) denotes a non-trivial zero of the Riemann zeta function.

In the early 1970s, Hugh Montgomery considered the pair correlation function

\[
F(x, T) = \sum_{0<\gamma, \gamma' \leq T} x^{i(\gamma - \gamma')} w(\gamma - \gamma') \text{ with } w(u) = \frac{4}{4 + u^2}.
\]

Here the sum is a double sum over the imaginary parts of the non-trivial zeros of \( \zeta(s) \). He proved in [9] that, as \( T \to \infty \),

\[
F(x, T) \sim \frac{T}{2\pi} \log x + \frac{T}{2\pi x^2} \log^2 T
\]

for \( 1 \leq x \leq T \) (actually he only proved for \( 1 \leq x \leq o(T) \) and the full range was done by Goldston [5]). He conjectured that

\[
F(x, T) \sim \frac{T}{2\pi} \log T
\]

for \( T \leq x \) which is known as the Strong Pair Correlation Conjecture. From this, one has the (Weak) Pair Correlation Conjecture:

\[
\sum_{0<\gamma, \gamma' \leq T, 0<\gamma - \gamma' \leq 2\pi \alpha/\log T} \frac{1}{2\pi} \log T \int_0^\alpha 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 du.
\]

which draws connections with random matrix theory.
The author studied these further in his thesis [1] (see also [2] and [3]) and derived more precise asymptotic formulas for $F(x, T)$ when $x$ is in various ranges under the Twin Prime Conjecture TPC (see section 4). In the present paper, we generalize $F(x, T)$ further to

$$F_h(x, T) = \sum_{0 < \gamma, \gamma' \leq T} \cos ( (\gamma - \gamma' - h) \log x ) w(\gamma - \gamma' - h).$$

Note that $F_h(x, T) = F_{-h}(x, T)$ and $F_0(x, T) = F(x, T)$. This leads to a better understanding of the distribution of larger differences between the zeros. Our main results are the following theorems: Here and throughout the paper, $\tilde{h} = |h| + 1$.

**Theorem 1.1.** For $1 \leq x \leq \frac{T}{\log T}$,

$$F_h(x, T) = \frac{T}{2\pi} \left[ \frac{4 \cos (h \log x)}{4 + h^2} \log x - \frac{8h \sin (h \log x)}{(4 + h^2)^2} \right]$$

$$+ \frac{T}{2\pi x^2} \left[ \left( \log \frac{T}{2\pi} \right)^2 - 2 \log \frac{T}{2\pi} \right] + O(x \log x) + O\left( \frac{\tilde{h}T}{x^{1/2-\epsilon}} \right).$$

**Theorem 1.2.** Assume TPC. For $M \geq 3$ and $\frac{T}{\log M} \leq x$,

$$F_h(x, T) = \frac{T}{\pi} \left[ \frac{2 \cos (h \log x)}{4 + h^2} \log x - \frac{4h \sin (h \log x)}{(4 + h^2)^2} \right]$$

$$+ \frac{T}{\pi} \int_1^\infty \left[ - \frac{2 \cos (h \log x)}{4 + h^2} \frac{1}{y} - \frac{4f(y)}{y^2} \cos (h \log x) + G_1(y) + G_2(y) \right] \frac{\sin \frac{Ty}{x}}{\frac{Ty}{x}} dy$$

$$- \frac{x}{\pi} \int_0^{T/x} \frac{\sin u}{u} du \left[ \frac{3 \cos (h \log x)}{9 + h^2} + \frac{h \sin (h \log x)}{9 + h^2} \right]$$

$$- \frac{x}{\pi} \int_0^{T/x} \frac{\sin u}{u} du \left[ \frac{\cos (h \log x)}{1 + h^2} - \frac{h \sin (h \log x)}{1 + h^2} \right]$$

$$+ \frac{T}{\pi} \sum_{k=1}^{\infty} \frac{g(k)}{k^2} \int_0^1 y \cos (h \log \frac{kx}{y}) \frac{\sin \frac{Ty}{x}}{\frac{Ty}{x}} dy$$

$$+ O\left( \frac{\tilde{h}x^{1+6\epsilon}}{T} \right) + O(\tilde{h}x^{1/2+7\epsilon}) + O\left( \frac{\tilde{h}x^{2}}{T^{2-2\epsilon}} \right) + O\left( \frac{\tilde{h}T}{\log M-2} \right).$$

where $G_1(y)$ and $G_2(y)$ are defined in Lemma 4.2.

**Theorem 1.3.** Assume TPC. For $M \geq 3$ and $\frac{T}{\log M} \leq x \leq T$,

$$F_h(x, T) = \frac{T}{\pi} \left[ \frac{2 \cos (h \log x)}{4 + h^2} \log x - \frac{4h \sin (h \log x)}{(4 + h^2)^2} \right] + O(\tilde{h}x) + O\left( \frac{\tilde{h}T}{\log M-2} \right).$$
Theorem 1.4. Assume TPC. For $M \geq 3$ and $T \leq x \leq T^{2-2\epsilon}$,

$$F_h(x, T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} \left[ \frac{4 \cos (h \log x)}{4 + h^2} \right] + O \left( \frac{\hat{T}T}{\pi \log M-2T} \right).$$

For real $\alpha$, let $F_h(\alpha) := \left( \frac{T}{2\pi \log T} \right)^{-1} F_h(T, T)$. Then $F_h(\alpha) = F_h(-\alpha)$.

Based on the above theorems, one may make the following Conjecture 1.1.

For any arbitrary large $A$, as $T \to \infty$,

$$F_h(\alpha) = \begin{cases} (1 + o(1))T^{-2\alpha \log T} + o(1), & \text{if } 0 \leq \alpha \leq 1, \\ \frac{4 \cos (h \log T \alpha)}{4 + h^2} + o(1), & \text{if } 1 \leq \alpha \leq A. \end{cases}$$

By convolving $F_h(\alpha)$ with an appropriate kernel $\hat{r}(\alpha)$,

$$\left( \frac{T}{2\pi \log T} \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} r(\gamma - \gamma' - h) \log T \frac{T}{2\pi} w(\gamma - \gamma' - h) = \int_{-\infty}^{+\infty} F_h(\alpha) \hat{r}(\alpha) d\alpha$$

leads to

Conjecture 1.2. For fixed $\alpha > 0$,

$$\left( \frac{T}{2\pi \log T} \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} 1 \sim \int_{-\alpha + h \log T / (2\pi)}^{\alpha + h \log T / (2\pi)} 1 - \frac{4}{4 + h^2} \left( \frac{\sin \pi u}{\pi u} \right)^2 du.$$

Conjecture 1.3. For $0 < \alpha < \beta < \log T$,

$$\left( \frac{T}{2\pi \log T} \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} 1 \sim \int_{\alpha}^{\beta} \frac{1}{1 + (\pi u / \log T)^2} \left( \frac{\sin \pi u}{\pi u} \right)^2 du.$$

2 Some Lemmas

Lemma 2.1.

$$2 \sum_{\gamma} \frac{x^{i(\gamma-t)}}{1 + (t-\gamma)^2} = - \frac{1}{x} \sum_{n \leq x} \frac{\Lambda(n)}{n^{-1/2+it}} - \frac{x}{\pi} \sum_{n > x} \frac{\Lambda(n)}{n^{3/2+it}} + \frac{x^{1/2-it}}{2} + \frac{x^{1/2-it}}{2 - it}$$

$$+ \frac{\log \tau}{x} + \frac{1}{x} \left( \frac{\zeta'}{\zeta} \left( \frac{3}{2} - it \right) - \log 2\pi \right) + O \left( \frac{1}{x \tau} \right)$$

(2)

where the sum is over all the imaginary parts of the zeros of the Riemann zeta function, and $\tau = |t| + 2$. $\Lambda(n)$ is von Mangoldt’s lambda function.
Proof: This is Lemma 2.2 in [2].

Write (2) as \( \text{Left}(x, t) = \text{Right}(x, t) \). Let
\[
\begin{align*}
P(x, T) &= \frac{1}{x} \sum_{n \leq x} \frac{\Lambda(n)}{n^{1/2 + it}} + x \sum_{n > x} \frac{\Lambda(n)}{n^{3/2 + it}} - \frac{x^{1/2 - it}}{2 + it} - \frac{x^{1/2 - it}}{2 - it}, \\
Q(x, T) &= \frac{\log \tau}{x}, \\
R(x, T) &= \frac{1}{x} \left[ \frac{\zeta'}{\zeta} \left( \frac{3}{2} - it \right) - \log 2 \pi \right], \\
S(x, T) &= O\left( \frac{1}{x \tau} \right).
\end{align*}
\]

Lemma 2.2.
\[
\int_0^T |\text{Left}(x, t) + \text{Left}(x, t - h)|^2 dt = 2\pi F(x, T) + 2\pi F(x, T - h) + 4\pi F_h(x, T) + O(\log^3 T) + O(h \log^2 h).
\]

Proof: This follows from page 188 of Montgomery [9] and the fact that \( F(x, T) \ll T \log 2 \).

Lemma 2.3. For \( x \geq 1 \),
\[
\int_0^T |\text{Right}(x, t) + \text{Right}(x, t - h)|^2 dt = \int_0^T |P(x, t) + P(x, t - h)|^2 dt + \frac{4T}{x^2} \left[ \left( \log \frac{T}{2\pi} \right)^2 - 2 \log \frac{T}{2\pi} \right]
+ \left( \frac{1}{2} \sum_{n=1}^\infty \frac{\Lambda^2(n)(1 + \cos (h \log n))}{n^3} + 2 \right)
+ O(h \log^2 T).
\]

Proof: This is similar to the proof of Theorem 3.1 in [2].

Lemma 2.4. For \( x \geq 1 \),
\[
4\pi F_h(x, T) = \int_0^T |P(x, t) + P(x, t - h)|^2 dt - \int_0^T |P(x, t)|^2 dt - \int_0^T |P(x, t - h)|^2 dt
+ \frac{2T}{x^2} \left[ \left( \log \frac{T}{2\pi} \right)^2 - 2 \log \frac{T}{2\pi} + \left( \sum_{n=1}^\infty \frac{\Lambda^2(n) \cos (h \log n)}{n^3} + 2 \right) \right]
+ O(h \log^3 T).
\]

Proof: It follows from Lemma 2.2 and Lemma 2.3 as well as their special cases when \( h = 0 \).

Lemma 2.5. For any sequence of complex numbers \( \{a_n\}_{n=1}^\infty \) with \( \sum_{n=1}^\infty n|a_n|^2 < \infty \),
\[
\int_0^T \left| \sum_{n=1}^\infty a_n e^{-itn} \right|^2 dt = \sum_{n=1}^\infty |a_n|^2 \left( T + O(n) \right).
\]
Proof: This is Parseval’s identity for Dirichlet series. See \[10\].

**Lemma 2.6.**

\[
\sum_{n \leq x} \Lambda^2(n) n = \frac{1}{2} x^2 \log x - \frac{1}{4} x^2 + O(x^{1/2 + \varepsilon}).
\]

\[
\sum_{n > x} \frac{\Lambda^2(n)}{n^3} = \frac{1}{2} \frac{\log x}{x^2} + \frac{1}{4 x^2} + O\left(\frac{1}{x^{5/2 - \varepsilon}}\right).
\]

Proof: Use partial summation and the prime number theorem.

**Lemma 2.7.** For any real \(a\) and \(b\) not both zero,

\[
\int e^{ax} \sin bx \, dx = \frac{a}{a^2 + b^2} e^{ax} \sin bx - \frac{b}{a^2 + b^2} e^{ax} \cos bx.
\]

\[
\int e^{ax} \cos bx \, dx = \frac{a}{a^2 + b^2} e^{ax} \cos bx + \frac{b}{a^2 + b^2} e^{ax} \sin bx.
\]

\[
\int xe^{ax} \sin bx \, dx = \left[ \frac{ax}{a^2 + b^2} - \frac{a^2 - b^2}{(a^2 + b^2)^2} \right] e^{ax} \sin bx - \left[ \frac{bx}{a^2 + b^2} - \frac{2ab}{(a^2 + b^2)^2} \right] e^{ax} \cos bx.
\]

\[
\int xe^{ax} \cos bx \, dx = \left[ \frac{ax}{a^2 + b^2} - \frac{a^2 - b^2}{(a^2 + b^2)^2} \right] e^{ax} \cos bx + \left[ \frac{bx}{a^2 + b^2} - \frac{2ab}{(a^2 + b^2)^2} \right] e^{ax} \sin bx.
\]

Proof: One can use \(\int e^{(a+ib)x} \, dx\), \(\int e^{(a-ib)x} \, dx\), \(\int xe^{(a+ib)x} \, dx\) and \(\int xe^{(a-ib)x} \, dx\) which are simple to compute.

**Lemma 2.8.**

\[
\frac{1}{x^2} \sum_{n \leq x} \Lambda^2(n) \cos (h \log n) = \frac{2 \cos (h \log x)}{4 + h^2} \log x + \frac{h^2 - 4}{(4 + h^2)^2} \cos (h \log x)
\]

\[
+ \frac{h \sin (h \log x)}{4 + h^2} \log x - \frac{4h}{(4 + h^2)^2} \sin (h \log x)
\]

\[
+ O\left(\frac{h}{x^{1/2 - \varepsilon}}\right).
\]

\[
x^2 \sum_{n > x} \frac{\Lambda^2(n)}{n} \cos (h \log n) = \frac{2 \cos (h \log x)}{4 + h^2} \log x - \frac{h^2 - 4}{(4 + h^2)^2} \cos (h \log x)
\]

\[
- \frac{h \sin (h \log x)}{4 + h^2} \log x - \frac{4h}{(4 + h^2)^2} \sin (h \log x)
\]

\[
+ O\left(\frac{h}{x^{1/2 - \varepsilon}}\right).
\]
Proof: We shall prove the first one. The other one is very similar. Let 
\[ A(x) = \frac{1}{x^2} \sum_{n \leq x} \Lambda^2(n)n. \] 
By integration by parts and Lemma 2.6,
\[
\frac{1}{x^2} \sum_{n \leq x} \Lambda^2(n)n \cos (h \log n) = A(x) \cos (h \log x) + \frac{1}{x^2} \int_1^x A(u) \frac{\sin (h \log u)}{u} \, du 
\]
which gives the desired result after applying Lemma 2.7 with \( a = 2 \) and \( b = h \), and some algebra.

3 Proof of Theorem 1.1

First, note that 
\[
P(x, t) = \frac{1}{x^{1/2}} \left[ \sum_{n \leq x} \Lambda(n) \left( \frac{x}{n} \right)^{-1/2 + it} + \sum_{n > x} \Lambda(n) \left( \frac{x}{n} \right)^{3/2 + it} \right] + O \left( \frac{x^{1/2}}{\tau} \right).
\]
Thus,
\[
P(x, t) + P(x, t - h) = \frac{1}{x^{1/2}} \left[ \sum_{n \leq x} \Lambda(n) (1 + n^{ih}) \left( \frac{x}{n} \right)^{-1/2 + it} 
+ \sum_{n > x} \Lambda(n) (1 + n^{ih}) \left( \frac{x}{n} \right)^{3/2 + it} \right] + O \left( \frac{x^{1/2}}{\tau} \right)
\]
So, the first integral in Lemma 2.4

\[
= \frac{1}{x} \int_0^T \left[ \sum_{n \leq x} \Lambda(n)(1 + n^h) \left( -\frac{x}{n} \right)^{-1/2 + it} + \sum_{n > x} \Lambda(n)(1 + n^h) \left( -\frac{x}{n} \right)^{3/2 + it} \right]^2 \, dt
\]

\[
+ O\left( \int_0^T 1 \, dt \right) + O\left( \int_0^T \frac{x}{t^4} \, dt \right)
\]

\[
= \frac{1}{x} \sum_{n \leq x} \Lambda^2(n) |1 + n^h|^2 \left( -\frac{x}{n} \right)^{-1} (T + O(n))
\]

\[
+ \frac{1}{x} \sum_{n > x} \Lambda^2(n) |1 + n^h|^2 \left( -\frac{x}{n} \right)^3 (T + O(n)) + O(x)
\]

\[
= \frac{2T}{x^2} \sum_{n \leq x} \Lambda^2(n)n(1 + \cos (h \log n)) + 2T x^2 \sum_{n > x} \frac{\Lambda^2(n)}{n^3} (1 + \cos (h \log n))
\]

\[
+ O(x \log x)
\]

Similarly (or by setting \( h = 0 \)), each of the second and third integral in Lemma 2.4

\[
= \frac{T}{x^2} \sum_{n \leq x} \Lambda^2(n)n + T x^2 \sum_{n > x} \frac{\Lambda^2(n)}{n^3} + O(x \log x)
\]

Therefore,

\[
4\pi F_h(x, T) = 2T \left[ \frac{1}{x^2} \sum_{n \leq x} \Lambda^2(n)n \cos (h \log n) + x^2 \sum_{n > x} \frac{\Lambda^2(n)}{n^3} \cos (h \log n) \right]
\]

\[
+ \frac{2T}{x^2} \left[ \left( \log \frac{T}{2\pi} \right)^2 - 2 \log \frac{T}{2\pi} \right]
\]

\[
+ O\left( \frac{T}{x^2} \right) + O(h \log^3 T) + O(x \log x)
\]

\[
= 2T \left[ \frac{4 \cos (h \log x)}{4 + h^2} \log x - \frac{8h \sin (h \log x)}{(4 + h^2)^2} \right]
\]

\[
+ \frac{2T}{x^2} \left[ \left( \log \frac{T}{2\pi} \right)^2 - 2 \log \frac{T}{2\pi} \right] + O(x \log x) + O\left( \frac{T}{x^{1/2 - \epsilon}} \right)
\]

by Lemma 2.8. The theorem follows after dividing through by \( 4\pi \).

4 Twin Prime Conjecture and smooth weight

We shall use a quantitative form of the Twin Prime Conjecture TPC as follow:

For any \( \epsilon > 0 \),

\[
\sum_{n=1}^N \Lambda(n) \Lambda(n + d) = \Theta(d) N + O(N^{1/2 + \epsilon}) \text{ uniformly in } |d| \leq N.
\]
\[ S(d) = 2 \prod_{p > 2} (1 - \frac{1}{p^{1/2}}) \prod_{p|d, p > 2} \frac{p-1}{2} \text{ if } d \text{ is even, and } S(d) = 0 \text{ if } d \text{ is odd.} \]

Let \( K \) and \( M \) be some large positive integers (\( K \) may depend on \( \epsilon \)). Set \( U = \log M \) and \( \Delta = 1/(2^K U) \). We recall the smooth weight \( \Psi_U(t) \) in [3] with:

1. support in \([-1/U, 1+1/U]\),
2. \( 0 \leq \Psi_U(t) \leq 1 \),
3. \( \Psi_U(t) = 1 \) for \( 1/U \leq t \leq 1 - 1/U \),
4. \( \Psi^{(j)}_U(t) \ll U^j \) for \( j = 1, 2, \ldots, K \).

This weight function satisfies the requirements in Goldston and Gonek [4]. One more thing to note is that

\[ \text{Re} \hat{\Psi}_U(y) = \frac{\sin 2\pi y}{2\pi y} \left( \frac{\sin 2\pi \Delta y}{2\pi \Delta y} \right)^{K+1} \]

where \( \hat{f}(y) = \int_{-\infty}^{\infty} f(t) e^{yt} dt \).

We also need to study

\[ S^h_\alpha(y) := \sum_{k \leq y} \mathcal{S}(k) k^\alpha \cos (h \log \frac{kx}{y}) - \int_0^y u^\alpha \cos (h \log \frac{ux}{y}) du \text{ for } \alpha \geq 0, \]

and

\[ T^h_\alpha(y) := \sum_{k > y} \frac{\mathcal{S}(k)}{k^\alpha} \cos (h \log \frac{kx}{y}) - \int_y^\infty \frac{1}{u^\alpha} \cos (h \log \frac{ux}{y}) du \text{ for } \alpha > 1. \]

Then from [4],

\[ S_0(y) := S^0_0(y) = -\frac{1}{2} \log y + O((\log y)^{2/3}) = -\frac{1}{2} \log y + \epsilon(y). \quad (3) \]

By partial summation and Lemma 2.7,

\[ S^h_\alpha(y) = \epsilon(y)y^\alpha \cos (h \log x) - \frac{\alpha \cos (h \log x)}{2(\alpha^2 + h^2)} y^\alpha - \frac{h \sin (h \log x)}{2(\alpha^2 + h^2)} y^\alpha \]

\[ - \int_0^y \epsilon(u) u^{\alpha-1} \left[ \alpha \cos \left( h \log \frac{ux}{y} \right) - h \sin \left( h \log \frac{ux}{y} \right) \right] du, \quad (4) \]

and

\[ T^h_\alpha(y) = -\frac{\epsilon(y)}{y^\alpha} \cos (h \log x) - \frac{\alpha \cos (h \log x)}{2(\alpha^2 + h^2)} \frac{1}{y^\alpha} + \frac{h \sin (h \log x)}{2(\alpha^2 + h^2)} \frac{1}{y^\alpha} \]

\[ + \int_y^\infty \frac{\epsilon(u)}{u^{\alpha+1}} \left[ \alpha \cos \left( h \log \frac{ux}{y} \right) + h \sin \left( h \log \frac{ux}{y} \right) \right] du. \quad (5) \]
Let
\[ f(y) := \int_0^y \epsilon(u) - \frac{B}{2} \, du \]
where \( B = -C_0 - \log 2\pi \) and \( C_0 \) is Euler’s constant. Note that
\[ f(y) \ll y^{1/2+\epsilon} \]  
(see Lemma 2.2 of [3]). From (4) and (5),
\begin{align*}
  S_h^b(y) \frac{1}{y^3} + T_h^b(y) y \\
  & = - \frac{2 \cos (h \log x)}{4 + h^2} \frac{1}{y^3} \int_0^y u \epsilon(u) \left[ 2 \cos \left( h \log \frac{ux}{y} \right) - h \sin \left( h \log \frac{ux}{y} \right) \right] du \\
  & \quad + y \int_y^{\infty} \frac{\epsilon(u)}{u^3} \left[ 2 \cos \left( h \log \frac{ux}{y} \right) + h \sin \left( h \log \frac{ux}{y} \right) \right] du. 
\end{align*}

Lemma 4.1.

\[ I + J = - \frac{1}{y^3} \int_0^y u \epsilon(u) \left[ 2 \cos \left( h \log \frac{ux}{y} \right) - h \sin \left( h \log \frac{ux}{y} \right) \right] du \\
\quad + y \int_y^{\infty} \frac{\epsilon(u)}{u^3} \left[ 2 \cos \left( h \log \frac{ux}{y} \right) + h \sin \left( h \log \frac{ux}{y} \right) \right] du \\
\quad = - \frac{4f(y)}{y} \cos (h \log x) \\
\quad \quad + \frac{1}{y^3} \int_0^y f(u) \left[ (2 - h^2) \cos \left( h \log \frac{ux}{y} \right) - 3h \sin \left( h \log \frac{ux}{y} \right) \right] du \\
\quad \quad + y \int_y^{\infty} f(u) \left[ (6 - h^2) \cos \left( h \log \frac{ux}{y} \right) + 5h \sin \left( h \log \frac{ux}{y} \right) \right] du. 
\]

Proof: \( I \) can be rewritten as
\begin{align*}
  - \frac{1}{y^3} \int_0^y u \left( \epsilon(u) - \frac{B}{2} \right) \left[ 2 \cos \left( h \log \frac{ux}{y} \right) - h \sin \left( h \log \frac{ux}{y} \right) \right] du \\
  - \frac{B}{2} \frac{1}{y^3} \int_0^y u \left[ 2 \cos \left( h \log \frac{ux}{y} \right) - h \sin \left( h \log \frac{ux}{y} \right) \right] du = -I_1 - I_2.
\end{align*}

By a substitution \( v = \log \frac{ux}{y} \) and Lemma 2.7
\[ I_2 = \frac{B}{2} \frac{1}{y} \cos (h \log x). \]  
(8)

By integration by parts and (6),
\begin{align*}
  I_1 &= \frac{f(y)}{y^2} \left[ 2 \cos (h \log x) - h \sin (h \log x) \right] \\
  \quad \quad - \frac{1}{y^3} \int_0^y f(u) \left[ (2 - h^2) \cos \left( h \log \frac{ux}{y} \right) - 3h \sin \left( h \log \frac{ux}{y} \right) \right] du. 
\end{align*}
(9)
Similarly, $J$ can be rewritten as
\[
y \int_1^\infty \frac{e(u)}{u^3} \left[ \frac{B}{2} \right] 2 \cos \left( h \log \frac{ux}{y} \right) + h \sin \left( h \log \frac{ux}{y} \right) du
\]
\[
+ \frac{B}{2} y \int_1^\infty \frac{1}{u^3} \left[ 2 \cos \left( h \log \frac{ux}{y} \right) + h \sin \left( h \log \frac{ux}{y} \right) \right] du = J_1 + J_2
\]

By a substitution $v = \log \frac{ux}{y}$ and Lemma 2.7,
\[
J_2 = \frac{B}{2} \left( \frac{1}{y} \cos \left( h \log x \right) \right).
\]

By integration by parts and (6),
\[
J_1 = -\frac{f(y)}{y^2} \left[ 2 \cos \left( h \log x \right) + \frac{1}{4 + h^2} \right] \left[ 2 \cos \left( h \log \frac{ux}{y} \right) - 3h \sin \left( h \log \frac{ux}{y} \right) \right]
\]
\[
+ \frac{4f(y)}{y^2} \cos \left( h \log x \right) + G_1(y) + G_2(y)
\]

where
\[
G_1(y) = \frac{1}{y^n} \int_0^y f(u) \left[ (2 - h^2) \cos \left( h \log \frac{ux}{y} \right) - 3h \sin \left( h \log \frac{ux}{y} \right) \right] du,
\]

and
\[
G_2(y) = y \int_0^\infty \frac{f(u)}{u^4} \left[ (6 - h^2) \cos \left( h \log \frac{ux}{y} \right) + 5h \sin \left( h \log \frac{ux}{y} \right) \right] du.
\]

Proof: Combine (7) and Lemma 4.1.

Lemma 4.2. $S_2(y) + T_2(y) = -\frac{2 \cos \left( h \log x \right)}{4 + h^2} \left[ 2 \cos \left( h \log \frac{ux}{y} \right) - 3h \sin \left( h \log \frac{ux}{y} \right) \right] \left( 6 - h^2 \right) \cos \left( h \log \frac{ux}{y} \right) + 5h \sin \left( h \log \frac{ux}{y} \right) + G_1(y) + G_2(y)$

Proof: Combine (9), (10), and (11) together gives the lemma.

Lemma 4.3. For any integer $n \geq 1$,
\[
\int_1^\infty \frac{1}{y^n} \Re \hat{\Psi}_U \left( \frac{Ty}{2\pi x} \right) dy = \int_1^\infty \frac{1}{y^n} \sin \frac{Ty}{2y} \frac{dy}{x} + O \left( \Delta \log \frac{1}{\Delta} \right).
\]

When $n \neq 2$, the error term can be replaced by $O(\Delta)$.

Proof: This is Lemma 3.3 in [3].

Lemma 4.4. If $F(y) \ll y^{-3/2+c}$ for $y \geq 1$, then
\[
\int_1^\infty F(y) \Re \hat{\Psi}_U \left( \frac{Ty}{2\pi x} \right) dy = \int_1^\infty F(y) \sin \frac{Ty}{2y} \frac{dy}{x} + O(\Delta).
\]

Proof: This is Lemma 3.4 in [3].
5 Proof of Theorem 1.2

Throughout this section, we assume $\tau = T^{1-\epsilon} \leq T/\log^M T \leq x \leq T^{2-2\epsilon}$, $U = \log^M T$ for $M > 2$, $H^* = \tau^{-2}x^{2/(1-\epsilon)}$, and $\Psi_U(t)$ is defined as in the previous section. The implicit constants in the error terms may depend on $\epsilon$, $K$, and $M$.

Our method is that of Goldston and Gonek $[6]$ and it is very similar to $[3]$. Let $s = \sigma + it$,

$$A_h(s) := \sum_{n \leq x} \frac{\Lambda(n)(1 + n^h)}{n^s}$$

and $A^*_h(s) := \sum_{n > x} \frac{\Lambda(n)(1 + n^h)}{n^s}$.

By Lemma 2.4 with slight modifications, one has

$$4\pi F_h(x, T) = \int_0^T \left| \frac{1}{x} (A_h(-\frac{1}{2} + it) - \int_1^x (1 + u^h)u^{1/2-it}du) \right|^2 dt$$

$$+ x \left| A^*_h(-\frac{3}{2} + it) - \int_1^\infty (1 + u^h)u^{-3/2-it}du \right|^2$$

$$-2 \int_0^T \left| \frac{1}{x} (A(-\frac{1}{2} + it) - \int_1^x u^{1/2-it}du) \right|^2 dt$$

$$+ x \left| A^*(\frac{3}{2} + it) - \int_1^\infty u^{-3/2-it}du \right|^2 dt + O(\tilde{h} \log^3 T).$$

Inserting $\Psi_U(t/T)$ into the integral and extending the range of integration to the whole real line, we have

$$4\pi F(x, T) = \frac{1}{x^2} I_1(x, T) + x^2 I_2(x, T) - \frac{2}{x^2} I_3(x, T) - 2x^2 I_4(x, T)$$

$$+ O\left(\frac{T(\log T)^2}{U}\right) + O\left(\frac{x^{1+6\epsilon}}{T}\right),$$

(12)

where

$$I_1(x, T) = \int_{-\infty}^\infty |A_h(-\frac{1}{2} + it) - \int_1^x (1 + u^h)u^{1/2-it}du|^2 \Psi_U(t/T)dt,$$

$$I_2(x, T) = \int_{-\infty}^\infty |A^*_h(\frac{3}{2} + it) - \int_1^\infty (1 + u^h)u^{-3/2-it}du|^2 \Psi_U(t/T)dt,$$

$$I_3(x, T) = \int_{-\infty}^\infty |A(-\frac{1}{2} + it) - \int_1^x u^{1/2-it}du|^2 \Psi_U(t/T)dt,$$

$$I_4(x, T) = \int_{-\infty}^\infty |A^*(\frac{3}{2} + it) - \int_1^\infty u^{-3/2-it}du|^2 \Psi_U(t/T)dt$$

by Lemma 1 of $[3]$ with modification $V = -T/U$ and $T - T/U$, and $W = 2T/U$. The contribution from the cross terms are estimated via Theorem 3 of $[3]$. Note
that by partial summation with the Riemann Hypothesis and TPC,
\[
\sum_{n \leq x} \Lambda(n)(1 + n^h) = \int_1^x (1 + u^h) \, du + O(\tilde{h}x^{3/2+\epsilon}),
\]
\[
\sum_{n \leq x} \Lambda(n)\Lambda(n + k)(1 + n^h)(1 + (n + k)^{-ih})
\]
\[= \mathcal{G}(k) \int_1^x (1 + u^h)(1 + (u + k)^{-ih}) \, du + O(\tilde{h}x^{3/2+\epsilon}).
\]

By Corollary 1 of [8] (see also the calculations at the end of [7] and [9]),
\[
I_1(x, T) = \hat{\Psi}_U(0)T \sum_{n \leq x} \Lambda^2(n)n[1 + n^h]^2
\]
\[+ 4\pi \left( \frac{T}{2\pi} \right)^3 \int_{T/2\pi}^{\infty} \int_{T/2\pi}^{\infty} \left( \sum_{k \leq 2\pi xv/T} \mathcal{G}(k)k^2\left(1 + \left( \frac{kT}{2\pi v} \right)^{ih}\right) \right.
\]
\[\times \left(1 + \left( \frac{kT}{2\pi v} + k\right)^{-ih}\right) Re\hat{\Psi}_U(v) \frac{dv}{v^3}
\]
\[\left. - 4\pi \left( \frac{T}{2\pi} \right)^3 \int_{T/2\pi}^{\infty} \int_{0}^{2\pi xv/T} u^2\left[1 + \left( \frac{uT}{2\pi v} \right)^{ih}\right]^2 du\right]\frac{dv}{v^3}
\]
\[+ O\left( \frac{\tilde{h}x^{3+6\epsilon}}{T} \right) + O(\tilde{h}x^{5/2+\epsilon}).
\]

Note that
\[
\left(1 + \left( \frac{kT}{2\pi v} \right)^{ih}\right)\left(1 + \left( \frac{kT}{2\pi v} + k\right)^{-ih}\right)
\]
\[= \left|1 + \left( \frac{kT}{2\pi v} \right)^{ih}\right|^2 + \left(1 + \left( \frac{kT}{2\pi v} \right)^{ih}\right)\left( \frac{kT}{2\pi v} + k\right)^{-ih} - \left( \frac{kT}{2\pi v} \right)^{-ih}
\]
\[= \left|1 + \left( \frac{kT}{2\pi v} \right)^{ih}\right|^2 + \left(1 + \left( \frac{kT}{2\pi v} \right)^{ih}\right)\left( \frac{kT}{2\pi v} \right)^{-ih} \left(1 + \frac{2\pi v}{T}\right)^{-ih} - 1
\]
\[= \left|1 + \left( \frac{kT}{2\pi v} \right)^{ih}\right|^2 + O\left( \min\left( \frac{hv}{T}, 1 \right) \right).
\]

Thus,
\[
\int_{T/2\pi}^{\infty} \left[ \sum_{k \leq 2\pi xv/T} \mathcal{G}(k)k^2\left(1 + \left( \frac{kT}{2\pi v} \right)^{ih}\right)\left(1 + \frac{kT}{2\pi v} + k\right)^{-ih}\right] Re\hat{\Psi}_U(v) \frac{dv}{v^3}
\]
\[= \int_{T/2\pi}^{\infty} \left[ \sum_{k \leq 2\pi xv/T} \mathcal{G}(k)k^2\left|1 + \left( \frac{kT}{2\pi v} \right)^{ih}\right|^2 \right] Re\hat{\Psi}_U(v) \frac{dv}{v^3}
\]
\[+ O\left( \int_{T/2\pi}^{T+\epsilon/x} \frac{xv}{T} \frac{dv}{T} \frac{1}{v^3} \right) + \int_{T/2\pi}^{\infty} \left( \frac{xv}{T} \right)^3 \frac{1}{\Delta v^2} \frac{dv}{v^8}
\]
\[= \int_{T/2\pi}^{\infty} \left[ \sum_{k \leq 2\pi xv/T} \mathcal{G}(k)k^2\left|1 + \left( \frac{kT}{2\pi v} \right)^{ih}\right|^2 \right] Re\hat{\Psi}_U(v) \frac{dv}{v^3} + O\left( \frac{\tilde{h}x^2}{T^{2+\epsilon}} + \frac{x^4}{\Delta T^{5-\epsilon}} \right).
\]
as \( \sum_{k \leq x} \mathcal{S}(k) \sim x \) and \( \text{Re} \hat{\Psi}_U(v) \ll \min \left( \frac{1}{v}, \frac{1}{\Delta v^2} \right) \). Therefore,

\[
I_1(x, T) = T \sum_{n \leq x} \Lambda^2(n) n (2 + 2 \cos (h \log n)) \\
+ 4\pi \left( \frac{T}{2\pi} \right)^3 \int_{T/2\pi}^\infty \left[ \sum_{k \leq 2\pi xv/T} \mathcal{S}(k) k^2 \left( 2 + 2 \cos \left( h \log \frac{kT}{2\pi v} \right) \right) \\
- \int_0^{2\pi xv/T} u^2 \left( 2 + 2 \cos \left( h \log \frac{uT}{2\pi v} \right) \right) du \text{Re} \hat{\Psi}_U(v) \frac{dv}{v^3} \right] \\
- 4\pi \left( \frac{T}{2\pi} \right)^3 \int_{T/2\pi}^{T} \int_{2\pi xv/T}^{2\pi xv/T} u^2 \left( 2 + 2 \cos \left( h \log \frac{uT}{2\pi v} \right) \right) du \text{Re} \hat{\Psi}_U(v) \frac{dv}{v^3} \\
\quad + O \left( \frac{\hat{h}x^{3+6\epsilon}}{T} \right) + O(\hat{h}x^{5/2+7\epsilon}) + O(\hat{h}x^2 T^{-1-\epsilon}) + O \left( \frac{x^4}{\Delta T^{2-\epsilon}} \right) .
\]

Similarly, by Corollary 2 of [6],

\[
I_2(x, T) = T \sum_{n \leq x} \frac{\Lambda^2(n)}{n^3} (2 + 2 \cos (h \log n)) \\
+ 8\pi^2 T \int_0^{T H^*/2\pi} \left[ \sum_{2\pi xv/T \leq k \leq H^*} \frac{\mathcal{S}(k)}{k^2} \left( 2 + 2 \cos \left( h \log \frac{kT}{2\pi v} \right) \right) \\
- \int_{2\pi xv/T}^{H^*} \frac{1}{u^2} \left( 2 + 2 \cos \left( h \log \frac{uT}{2\pi v} \right) \right) du \text{Re} \hat{\Psi}_U(v) \frac{dv}{v^3} \right] \\
+ O \left( \frac{\hat{h}T^{-1} x^{-1+6\epsilon}}{\Delta} \right) + O(\hat{h}x^{-3/2+7\epsilon}) + O(\hat{h}T^{-1/2} x^{-2}) + O \left( \frac{\hat{H}^*}{\Delta x^2} \right)
\]

where the last error term comes from the error term in (13). \( I_3(x, T) \) and \( I_4(x, T) \) are computed in [3] or one can simply set \( h = 0 \) in \( I_1(x, T) \) and \( I_2(x, T) \), and divide by 4. Putting these into (12) with a substitution \( \gamma = \frac{2\pi xv}{T} \) and using

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Lemma 2.8

\[4\pi F_h(x, T)\]

\[= 2T \left[ \frac{4 \cos (h \log x)}{4 + h^2} \log x - \frac{8h \sin (h \log x)}{(4 + h^2)^2} \right] + 4T \int_1^\infty \left[ \sum_{k \leq y} \mathcal{S}(k) k^2 \cos \left( k \frac{h}{y} \right) - \int_0^y u^2 \cos \left( \frac{ux}{y} \right) du \right] \text{Re} \hat{\Psi}_U \left( \frac{T y}{2\pi x} \right) \frac{dy}{y^3} - 4T \int_0^1 \int_0^y u^2 \cos \left( \frac{ux}{y} \right) du \text{Re} \hat{\Psi}_U \left( \frac{T y}{2\pi x} \right) \frac{dy}{y^3} + 4T \int_1^H \left[ \sum_{y \leq k \leq H} \mathcal{S}(k) \cos \left( \frac{h k}{y} \right) \frac{k}{y^2} - \int_y^H \cos \left( \frac{h}{y} \right) dy \right] \text{Re} \hat{\Psi}_U \left( \frac{T y}{2\pi x} \right) \frac{dy}{y^3} + 4T \int_1^H \left[ \sum_{y \leq k \leq H} \mathcal{S}(k) \cos \left( \frac{h k}{y} \right) \frac{k}{y^2} - \int_y^H \cos \left( \frac{h}{y} \right) dy \right] \text{Re} \hat{\Psi}_U \left( \frac{T y}{2\pi x} \right) \frac{dy}{y^3} + O \left( \frac{h x^{1+6\epsilon}}{T} \right) + O \left( h x^{1/2+7\epsilon} \right) + O \left( \frac{h^2}{\log M - 2T} \right) \right] + O \left( \frac{\tilde{h} x^2 T^2}{x^3} \right) + O \left( \frac{\tilde{h} T \log M}{x} \right).

From Lemma 2.7

\[\int e^{ax} \cos bx = \frac{a}{a^2 + b^2} e^{ax} \cos bx + \frac{b}{a^2 + b^2} e^{ax} \sin bx. \quad (14)\]

Also,

\[\int_0^1 \text{Re} \hat{\Psi}_U \left( \frac{T y}{2\pi x} \right) dy = \frac{x}{T} \int_0^{T/x} \sin u \frac{dy}{u} (1 + O(\Delta^2 u^2)) du = \frac{x}{T} \int_0^{T/x} \sin u \frac{dy}{u} du + O \left( \frac{\Delta^2 T}{x} \right). \quad (15)\]

By appropriate change of variables, (14) and (15),

\[\int_0^1 \int_0^y u^2 \cos \left( \frac{ux}{y} \right) du \text{Re} \hat{\Psi}_U \left( \frac{T y}{2\pi x} \right) \frac{dy}{y^3} = \frac{x}{T} \int_0^{T/x} \sin u \frac{dy}{u} \left[ \frac{3}{9 + h^2} \cos \left( \frac{h x}{y} \right) + \frac{h}{9 + h^2} \sin \left( \frac{h x}{y} \right) \right] + O \left( \frac{\Delta^2 T}{x} \right). \]

\[\int_0^1 \left[ \sum_{k \leq H} \mathcal{S}(k) \cos \left( \frac{h k}{y} \right) \frac{k}{y^2} - \int_y^H \cos \left( \frac{h}{y} \right) dy \right] \text{Re} \hat{\Psi}_U \left( \frac{T y}{2\pi x} \right) \frac{dy}{y^3} = \sum_{k=1}^\infty \mathcal{S}(k) \frac{k}{y^2} \int_0^y \cos \left( \frac{h k}{y} \right) \frac{dy}{y^2} + O \left( \frac{1}{H^2} \right) + O \left( \frac{\Delta^2 T}{x} \right) \right] - \frac{x}{T} \int_0^{T/x} \sin u \frac{dy}{u} \left[ \frac{1}{1 + h^2} \cos \left( \frac{h x}{y} \right) - \frac{h}{1 + h^2} \sin \left( \frac{h x}{y} \right) \right] + O \left( \frac{\Delta^2 T}{x} \right) + \frac{\tilde{h} x^2 T^2}{x^3} + \frac{\tilde{h} T \log M}{x}. \]
Therefore, with the notation $S^h_\alpha(y)$ and $T^h_\alpha(y)$,

\[
4\pi F_h(x, T) = 2T \left[ \frac{4\cos(h \log x)}{4 + h^2} \log x - \frac{8h \sin(h \log x)}{(4 + h^2)^2} \right] + 4T \int_1^\infty S^h_2(y) \Re \Psi_U \left( \frac{Ty}{2\pi x} \right) \frac{dy}{y^3} + 4T \int_1^{H^*} (T^h_2(y) - T^h_2(H^*)) \Re \Psi_U \left( \frac{Ty}{2\pi x} \right) dy - 4x \int_0^{T/x} \frac{\sin u}{u} du \left[ \frac{3 \cos(h \log x)}{9 + h^2} + \frac{h \sin(h \log x)}{9 + h^2} \right] - 4x \int_0^{T/x} \frac{\sin u}{u} du \left[ \frac{\cos(h \log x)}{1 + h^2} - \frac{h \sin(h \log x)}{1 + h^2} \right] - 4x \int_0^{T/x} \frac{\sin u}{u} du \left[ \log(1 + ud) - \log(1 + ud) \right] + 4T \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{1} y \cos(h \log kx/y) \frac{\sin \left( \frac{T_y}{T_x} \right)}{x} dy + O \left( \frac{h x^{1/2+\epsilon}}{T} \right) + \frac{\hbar x^2}{T^{2-2\epsilon}} + O \left( \frac{h T}{\log x} \right).
\]

By (3) and (5), $T^h_2(H^*) \ll \frac{h(\log H^*)^{2/3}}{(H^*)^2}$. It follows that the contribution from $T^h_2(H^*)$ in the second integral is $O(hT^{-\epsilon})$. Also, one can extend the upper limit of the second integral to $\infty$ with an error $O(hT^{-\epsilon})$ by (3) and (5) again. Finally, we obtain the theorem by applying Lemma 1.2, Lemma 1.3, Lemma 1.4, (6) and dividing by $4\pi$.

6 Proof of Theorem 1.3 and 1.4

Proof of Theorem 1.3. It follows directly from Theorem 1.2 by observing that all the other main terms besides the first one are $O(x)$ because of (5).

Before proving Theorem 1.4, we need the following lemmas.

Lemma 6.1.

\[
\int_1^\infty \frac{\sin ax}{x^{2n}} dx = \frac{a^{2n-1}}{(2n-1)!} \sum_{k=1}^{2n-1} \frac{(2n-k-1)!}{a^{2n-k}} \left[ \sin \left( a + (k-1) \frac{\pi}{2} \right) + (-1)^{k+1}ci(a) \right]
\]

where $ci(x) = -\int_x^\infty \frac{\cos t}{t} dt = C_0 + \log x + \int_0^x \frac{\cos t - 1}{t} dt$ and $C_0$ is Euler’s constant.

Proof: This is formula 3.761(3) on P.430 of [8] which can be proved by integration by parts repeatedly.
Lemma 6.2. If $F(y) \ll y^{-3/2+\epsilon}$ for $y \geq 1$, then for $T \leq x$,
\[
\int_{1}^{\infty} F(y) \sin \frac{T y}{T x} \, dy = \int_{1}^{\infty} F(y) \, dy + O \left( \left( \frac{T x}{x} \right)^{1/2-\epsilon} \right).
\]

Proof: This is Lemma 5.2 in [3].

Lemma 6.3.

\[
I = \int_{1}^{\infty} \int_{1}^{\infty} f(u) \left[ (2 - h^2) \cos (h \log \frac{ux}{y}) - 3h \sin (h \log \frac{ux}{y}) \right] \, du \, dy
\]
\[
= \int_{1}^{1} f(u) \cos (h \log ux) - h \sin (h \log ux) \, du
\]
\[
+ \int_{1}^{\infty} \frac{f(u)}{u^2} \, du \left[ \cos (h \log x) - h \sin (h \log x) \right].
\]

Proof: Because of [6], we can change the order of integration.

\[
I = \int_{1}^{1} f(u) \int_{1}^{\infty} \frac{1}{y^2} \left[ (2 - h^2) \cos (h \log \frac{ux}{y}) - 3h \sin (h \log \frac{ux}{y}) \right] \, dy \, du
\]
\[
+ \int_{1}^{\infty} f(u) \int_{u}^{\infty} \frac{1}{y^2} \left[ (2 - h^2) \cos (h \log \frac{ux}{y}) - 3h \sin (h \log \frac{ux}{y}) \right] \, dy \, du
\]
\[
= \int_{1}^{1} f(u) \left\{ \left(2 - h^2\right) \left[ \frac{2}{4 + h^2} \cos (h \log ux) + \frac{h}{4 + h^2} \sin (h \log ux) \right]
\right.
\]
\[
- 3h \left[ \frac{2}{4 + h^2} \sin (h \log ux) - \frac{h}{4 + h^2} \cos (h \log ux) \right] \right\} \, du
\]
\[
+ \int_{1}^{\infty} \frac{f(u)}{u^2} \left\{ \left(2 - h^2\right) \left[ \frac{2}{4 + h^2} \cos (h \log x) + \frac{h}{4 + h^2} \sin (h \log x) \right]
\right.
\]
\[
- 3h \left[ \frac{2}{4 + h^2} \sin (h \log x) - \frac{h}{4 + h^2} \cos (h \log x) \right] \right\} \, du.
\]

by substituting $v = \log \frac{ux}{y}$ and applying Lemma 2.4. The lemma follows after some simple algebra.

Lemma 6.4.

\[
J = \int_{1}^{\infty} \int_{1}^{\infty} f(u) \left[ (6 - h^2) \cos (h \log \frac{ux}{y}) + 5h \sin (h \log \frac{ux}{y}) \right] \, du \, dy
\]
\[
= - \int_{1}^{\infty} \frac{f(u)}{u^4} \left[ 3 \cos (h \log ux) + h \sin (h \log ux) \right] \, du
\]
\[
+ \int_{1}^{\infty} \frac{f(u)}{u^2} \left[ 3 \cos (h \log x) + h \sin (h \log x) \right].
\]
Proof: Again, because of (6), we can change the order of integration.

\[ J = \int_1^\infty \int_1^u \left[ (6 - h^2) \cos \left( h \log \frac{ux}{y} \right) + 5h \sin \left( h \log \frac{ux}{y} \right) \right] dy \, du \]

\[ = \int_1^\infty \frac{f(u)}{u^2} \left\{ (6 - h^2) \left[ \frac{-2}{4 + h^2} \cos \left( h \log \frac{ux}{y} \right) + \frac{h}{4 + h^2} \sin \left( h \log \frac{ux}{y} \right) \right] 
+ 5h \left[ \frac{-2}{4 + h^2} \sin \left( h \log \frac{ux}{y} \right) - \frac{h}{4 + h^2} \cos \left( h \log \frac{ux}{y} \right) \right] 
- (6 - h^2) \left[ \frac{-2}{4 + h^2} \cos \left( h \log \frac{ux}{y} \right) + \frac{h}{4 + h^2} \sin \left( h \log \frac{ux}{y} \right) \right] 
- 5h \left[ \frac{-2}{4 + h^2} \sin \left( h \log \frac{ux}{y} \right) - \frac{h}{4 + h^2} \cos \left( h \log \frac{ux}{y} \right) \right] \right\} du. \]

By substituting \( v = \log \frac{ux}{y} \) and applying Lemma 2.7, the lemma follows after some simple algebra.

**Lemma 6.5.**

\[ S = \sum_{k=1}^{\infty} \frac{\mathcal{S}(k)}{k^2} \int_0^1 y \cos \left( h \log \frac{kx}{y} \right) dy \]

\[ = \left[ \frac{1}{1 + h^2} \cos \left( h \log \frac{kx}{y} \right) - \frac{h}{1 + h^2} \sin \left( h \log \frac{kx}{y} \right) \right] \]

\[ \quad - \left[ \frac{4 - h^2}{2(4 + h^2)^2} \cos \left( h \log \frac{kx}{y} \right) - \frac{2h}{(4 + h^2)^2} \sin \left( h \log \frac{kx}{y} \right) \right] \]

\[ + \frac{B}{2} \left[ \frac{2}{4 + h^2} \cos \left( h \log \frac{kx}{y} \right) - \frac{h}{4 + h^2} \sin \left( h \log \frac{kx}{y} \right) \right] + \left( 1 + \frac{B}{2} \right) \cos \left( h \log \frac{kx}{y} \right) 
+ \int_1^\infty \frac{f(u)}{u^4} \left[ 3 \cos \left( h \log \frac{ux}{y} \right) + h \sin \left( h \log \frac{ux}{y} \right) \right] du. \]

Proof: By substituting \( v = \log \frac{kx}{y} \) and Lemma 2.7,

\[ S = \frac{2}{4 + h^2} \sum_{k=1}^{\infty} \frac{\mathcal{S}(k)}{k^2} \cos \left( h \log kx \right) - \frac{h}{4 + h^2} \sum_{k=1}^{\infty} \frac{\mathcal{S}(k)}{k^2} \sin \left( h \log kx \right). \]
Recall the definition of $S_0(u)$ from (3) and use partial summation,

\[ S = \frac{2}{4 + h^2} \int_1^\infty S_0(u) + u \left[ 2 \cos (h \log ux) + h \sin (h \log ux) \right] du \]

\[ - \frac{h}{4 + h^2} \int_1^\infty S_0(u) + u \left[-h \cos (h \log ux) + 2 \sin (h \log ux) \right] du \]

\[ = \int_1^\infty \frac{S_0(u) + u}{u^3} \cos (h \log ux) du \]

\[ - \frac{h}{u^3} \int_1^\infty \log u + \epsilon(u) \cos (h \log ux) du \]

\[ = \int_1^\infty \frac{1}{u^3} \cos (h \log ux) du - \frac{h}{2} \int_1^\infty \frac{\log u}{u^3} \cos (h \log ux) du \]

\[ + \frac{B}{2} \int_1^\infty \frac{1}{u^3} \cos (h \log ux) du + \int_1^\infty \frac{\epsilon(u) - B}{u^3} \cos (h \log ux) du \]

\[ = I_1 - \frac{1}{2} I_2 + \frac{B}{2} I_3 + I_4. \]

By appropriate substitution and Lemma 2.4,

\[ I_1 = \frac{1}{1 + h^2} \cos (h \log x) - \frac{h}{1 + h^2} \sin (h \log x), \]

\[ I_2 = \frac{4 - h^2}{(4 + h^2)^2} \cos (h \log x) - \frac{2h}{(4 + h^2)^2} \sin (h \log x), \]

\[ I_3 = \frac{2}{4 + h^2} \cos (h \log x) - \frac{h}{4 + h^2} \sin (h \log x). \]

Finally, by integration by parts,

\[ I_4 = \int_1^\infty \frac{\cos (h \log ux)}{u^3} df(u) \]

\[ = \left(1 + \frac{B}{2}\right) \cos (h \log x) + \int_1^\infty \frac{f(u)}{u^4} [3 \cos (h \log ux) + h \sin (h \log ux)] du \]

because $f(1) = -1 - \frac{B}{2}$. Combining the results for $I_1$, $I_2$, $I_3$ and $I_4$, we have the lemma.

**Lemma 6.6.**

\[ \int_0^1 f(u) [\cos (h \log ux) - h \sin (h \log ux)] du \]

\[ = - \frac{1}{2} \left[ \frac{4 + 3h^2}{(4 + h^2)^2} \cos (h \log x) + \frac{h^3}{(4 + h^2)^2} \sin (h \log x) \right] \]

\[ - \left( \frac{1}{2} + \frac{B}{2} \right) \left[ \frac{2 + h^2}{4 + h^2} \cos (h \log x) - \frac{h}{4 + h^2} \sin (h \log x) \right] \]

\[ - \frac{1}{2} \left[ \frac{3 + h^2}{9 + h^2} \cos (h \log x) - \frac{2h}{9 + h^2} \sin (h \log x) \right]. \]
Proof: The key is $\epsilon(u) = \frac{1}{2} \log u - u$ when $0 \leq u \leq 1$ (see (3)). So,

$$f(u) = \int_0^u \epsilon(v) - \frac{B}{2} \, dv = \frac{1}{2} u \log u - \left( \frac{1}{2} + \frac{B}{2} \right) u - \frac{1}{2} u^2.$$  

Putting this into the integral and evaluating the integral piece by piece with suitable substitution and Lemma 2.7, one gets the lemma.

Proof of Theorem 1.4: First observe that when $T \leq x \leq T^2 - 29e$, the error terms in Theorem 1.2 is $O\left( \frac{hT}{\log M - 2} \right)$. Rewrite Theorem 1.2 as

$$F_h(x, T) = T_1 + T_2 + T_3 + T_4 + T_5 + O\left( \frac{\tilde{h}T}{\log M - 2} \right).$$

Since $\frac{\sin u}{u} = 1 + O(u^2),$

$$T_3 = -\frac{T}{\pi} \left[ \frac{3 \cos (h \log x)}{9 + h^2} + \frac{h \sin (h \log x)}{9 + h^2} \right] + O\left( T \left( \frac{T}{x} \right)^2 \right),$$

$$T_4 = -\frac{T}{\pi} \left[ \frac{\cos (h \log x)}{1 + h^2} - \frac{h \sin (h \log x)}{1 + h^2} \right] + O\left( T \left( \frac{T}{x} \right)^2 \right).$$

By Lemma 3.5

$$T_5 = \frac{T}{\pi} \sum_{k=1}^{\infty} \frac{\mathcal{S}(k)}{k^2} \int_0^1 y \cos \left( \frac{h \log kx}{y} \right) \, dy + O\left( T \left( \frac{T}{x} \right)^2 \right)$$

$$= \frac{T}{\pi} \left[ \frac{\cos (h \log x)}{1 + h^2} - \frac{h \sin (h \log x)}{1 + h^2} \right]$$

$$- \frac{T}{\pi} \left[ \frac{4 - h^2}{2(4 + h^2)^2} \cos (h \log x) - \frac{2h}{(4 + h^2)^2} \sin (h \log x) \right]$$

$$+ \frac{T}{\pi} \left[ \frac{2 \cos (h \log x)}{4 + h^2} - \frac{h \sin (h \log x)}{4 + h^2} \right] + \frac{T}{\pi} \left( 1 + \frac{B}{2} \right) \cos (h \log x)$$

$$+ \frac{T}{\pi} \int_1^{\infty} \frac{f(u)}{u^2} \left[ 3 \cos (h \log ux) + h \sin (h \log ux) \right] du + O\left( T \left( \frac{T}{x} \right)^2 \right).$$

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By Lemma 6.1 and Lemma 6.2, Lemma 6.3, Lemma 6.4, and Lemma 6.6

\[ T_2 = - \frac{2x \cos (h \log x)}{\pi (4 + h^2)} \int_1^\infty \frac{Ty}{y^2} dy - \frac{4T}{\pi} \cos (h \log x) \int_1^\infty \frac{f(y)}{y^2} dy + \frac{T}{\pi} \int_1^\infty (G_1(y) + G_2(y)) dy + O\left( \frac{T}{x} \right)^{1/2-\epsilon} + O\left( \frac{hT}{\log^M T} \right) \]

\[ = - \frac{2x \cos (h \log x)}{\pi (4 + h^2)} \int_1^\infty \frac{Ty}{y^2} dy + O\left( \frac{hT}{\log^M T} \right) \]

\[ + \frac{T}{\pi} \int_0^1 \frac{f(u)}{u^4} \left[ 3 \cos (h \log ux) + h \sin (h \log ux) \right] du + O\left( \frac{T}{x} \right)^{1/2-\epsilon} \]

\[ = - \frac{2x \cos (h \log x)}{\pi (4 + h^2)} \int_1^\infty \frac{Ty}{y^2} dy + O\left( \frac{hT}{\log^M T} \right) \]

\[ - \frac{T}{\pi} \left( \frac{1}{2} + \frac{B}{2} \right) \left[ \frac{2 + h^2}{4 + h^2} \cos (h \log x) - \frac{h}{4 + h^2} \sin (h \log x) \right] \]

\[ - \frac{T}{\pi} \left( \frac{1}{2} + \frac{B}{2} \right) \left[ \frac{3 + h^2}{9 + h^2} \cos (h \log x) - \frac{2h}{9 + h^2} \sin (h \log x) \right] \]

\[ - \frac{T}{\pi} \int_1^\infty \frac{f(u)}{u^4} \left[ 3 \cos (h \log ux) + h \sin (h \log ux) \right] du + O\left( \frac{T}{x} \right)^{1/2-\epsilon}. \]

Therefore, with miraculous cancellations,

\[ T_2 + T_3 + T_4 + T_5 = - \frac{2x \cos (h \log x)}{\pi (4 + h^2)} \int_1^\infty \frac{Ty}{y^2} dy + \frac{T}{\pi} \frac{2B \cos (h \log x)}{4 + h^2} \]

\[ + \frac{T}{\pi} \frac{4h \sin (h \log x)}{(4 + h^2)^2} + O\left( \frac{T}{x} \right)^{1/2-\epsilon} + O\left( \frac{hT}{\log^M T} \right). \]

By Lemma 6.1 and \( B = -C_0 - \log 2\pi, \)

\[ F_h(x, T) = \frac{T}{\pi} \left[ \frac{2 \cos (h \log x)}{4 + h^2} \log x \right] - \frac{2T \cos (h \log x)}{\pi (4 + h^2)} \left[ \sin \left( \frac{T}{x} \right) \frac{T}{x} \right] - \frac{4T \cos (h \log x)}{\pi (4 + h^2)} \left[ \cos \left( \frac{T}{x} \right) \frac{T}{x} \right] \]

\[ + \frac{T}{\pi} \frac{2B \cos (h \log x)}{4 + h^2} + O\left( \frac{T}{x} \right)^{1/2-\epsilon} + O\left( \frac{hT}{\log^M T} \right) \]

\[ = \frac{T}{\pi} \left[ \frac{2 \cos (h \log x)}{4 + h^2} \log x \right] - \frac{2T \cos (h \log x)}{\pi (4 + h^2)} \left[ 1 - C_0 - \log \frac{T}{x} \right] \]

\[ + C_0 + \log 2\pi \right] + O\left( \frac{T}{x} \right)^{1/2-\epsilon} + O\left( \frac{hT}{\log^M T} \right) \]

\[ = \frac{T}{2\pi} \log \frac{T}{2\pi e} \left[ \frac{4 \cos (h \log x)}{4 + h^2} \right] + O\left( \frac{T}{x} \right)^{1/2-\epsilon} + O\left( \frac{hT}{\log^M T} \right). \]
7 Sketch for Conjecture 1.2

Fix $\alpha > 0$. Let $r(u)$ be an even function which is almost the characteristic function of the interval $[-\alpha, \alpha]$ with $\hat{r}(\alpha) \ll 1$ (see page 87 of [1] for detail construction). We use Conjecture 1.1 to compute the right hand side of (1).

$$I = \int_{-\infty}^{\infty} F_h(\alpha) \hat{r}(\alpha) d\alpha = 2 \int_{0}^{1} F_h(\alpha) \hat{r}(\alpha) d\alpha$$

$$= 2(1 + o(1)) \log T \int_{0}^{1} T^{-2} \hat{r}(\alpha) d\alpha + \frac{4}{4 + h^2} \int_{0}^{1} \alpha \cos (h \log T \alpha) \hat{r}(\alpha) d\alpha$$

$$+ 2 \frac{4}{4 + h^2} \int_{1}^{\infty} \cos (h \log T \alpha) \hat{r}(\alpha) d\alpha + O \left( \frac{1}{A} \right) + o(1)$$

$$= \frac{4}{4 + h^2} \int_{-\infty}^{\infty} \hat{r}(\alpha) d\alpha - \frac{4}{4 + h^2} \int_{-1}^{1} (1 - |\alpha|) \hat{r}(\alpha) d\alpha$$

$$+ (1 + o(1)) \log T \int_{-\infty}^{\infty} T^{-2} |\hat{r}(\alpha)| d\alpha + O \left( \frac{1}{A} \right) + o(1)$$

$$= \frac{4}{4 + h^2} \int_{-\infty}^{\infty} \hat{r}(\alpha) d\alpha - \frac{4}{4 + h^2} I_2 + (1 + o(1)) I_3 + O \left( \frac{1}{A} \right) + o(1)$$

where $r_1(u) = r(u + \frac{h \log T}{2\pi})$. As $\int_{-\infty}^{\infty} \hat{r}(\alpha) d\alpha = r_1(0)$,

$$I_1 = r_1(0) = r \left( \frac{h \log T}{2\pi} \right).$$

By $f \hat{g} = \int f \hat{g}$, the transform pair and the definition of $r(u),$

$$f(t) = \max (1 - |t|, 0), \quad \hat{f}(u) = \left( \frac{\sin \pi u}{\pi u} \right)^2,$$

$$I_2 = \int_{-\infty}^{\infty} r_1(u) \left( \frac{\sin \pi u}{\pi u} \right)^2 du = \int_{-\alpha + h \log T/(2\pi)}^{\alpha + h \log T/(2\pi)} \left( \frac{\sin \pi u}{\pi u} \right)^2 du + o(1).$$

Similarly, by the transform pair

$$f(t) = e^{-2|t|}, \quad \hat{f}(u) = \frac{4a}{4a^2 + (2\pi u)^2},$$

$$I_3 = \int_{-\alpha}^{\alpha} \frac{4 \log^2 T}{4 \log^2 T + (2\pi u)^2} du + o(1) = \int_{-\alpha + h \log T/(2\pi)}^{\alpha + h \log T/(2\pi)} 1 du + o(1).$$

Therefore,

$$I = \frac{4}{4 + h^2} r \left( \frac{h \log T}{2\pi} \right) + \int_{-\alpha + h \log T/(2\pi)}^{\alpha + h \log T/(2\pi)} 1 - \frac{4}{4 + h^2} \left( \frac{\sin \pi u}{\pi u} \right)^2 du + O \left( \frac{1}{A} \right) + o(1).$$

(16)
Now, the left hand side of (1) is
\[
\frac{4}{4 + h^2 r} \left( \frac{h \log T}{2\pi} \right) + \left( \frac{T}{2\pi \log T} \right)^{-1} \sum_{0 < \gamma \neq \gamma' \leq T \atop |\gamma - \gamma' - h| \leq 2\pi \alpha / \log T} (1 + o(1)). \tag{17}
\]
Combining (16) and (17), we have Conjecture 1.2 by making \( A \) arbitrarily large. The only shaky point in the above argument is the error analysis. All of these become rigorous following page 87 – 90 of [1].

References

[1] T.H. Chan, *Pair Correlation and Distribution of Prime Numbers*, Thesis, U. of Mich., Ann Arbor, 2002.

[2] T.H. Chan, *On a conjecture of Liu and Ye*, to appear.

[3] T.H. Chan, *More precise Pair Correlation Conjecture on the zeros of the Riemann zeta function*, submitted.

[4] J.B. Friedlander and D.A. Goldston, *Some singular series averages and the distribution of Goldbach numbers in short intervals*, Illinois J. Math. 39 (1995), 158-180.

[5] D.A. Goldston, *Large Differences between Consecutive Prime Numbers*, Thesis, U. of Calif., Berkeley, 1981.

[6] D.A. Goldston and S.M. Gonek, *Mean value theorems for long Dirichlet polynomials and tails of Dirichlet series*, Acta Arith. (2) 84 (1998), 155-192.

[7] D.A. Goldston, S.M. Gonek, A.E. Oziuk and C. Synder, *On the pair correlation of zeros of the Riemann zeta-function*, Proc. London Math. Soc. (3) 80 (2000), 31-49.

[8] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, sixth edition, 2000.

[9] H.L. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic Number Theory (St. Louis Univ., 1972), Proc. Sympos. Pure Math. 24, Amer. Math. Soc., Providence, 1973, pp. 181-193.

[10] H.L. Montgomery and R.C. Vaughan, *Hilbert’s inequality*, J. London Math. Soc. (2) 8 (1974), 73-82.

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