Boyle’s Conjecture and perfect localizations

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Abstract

In this article we study the behaviour of left QI-rings under perfect localizations. We show that a perfect localization of a left QI-ring is a left QI-ring. We prove that Boyle’s conjecture is true for left QI-rings with finite Gabriel dimension such that the hereditary torsion theory generated by semisimple modules is perfect. As corollary we get that Boyle’s conjecture is true for left QI-rings which satisfy the restricted left socle condition, this result was proved first by C. Faith in [3].

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1 Introduction

Through all this paper $R$ will denote an associative ring with unit element. We will work with unitary left $R$-modules and the category of modules will be denoted by $R$-Mod. For general background on module and ring theory we refer the reader to [1], [10], [11] and [12].

Remember that an $R$-module $M$ is quasi-injective if every morphism $f : N \to M$ with $N \leq M$ can be extended to an endomorphism of $M$. Equivalently $M$ is quasi-injective if and only if $M$ is fully invariant in its injective hull. A ring $R$ is called left QI-ring if every quasi-injective left $R$-module is injective, these rings were introduced by A. Boyle in [2]. Also in [2] is shown that a left QI-ring is left noetherian and left $V$-ring; recall that a left $V$-ring is defined as a ring where every simple left module is injective.

In [4] the author introduces two examples of non semisimples left QI-rings. These examples are left hereditary rings, that is, every left ideal is projective.

In [2, Theorem 5] A. Boyle characterizes two-sided hereditary, right noetherian, left QI-rings and she conjectured that every left QI-ring is left hereditary. In [6] C. Faith gave an approach to this conjecture. In [6, Corrolary 3] is shown that every left QI-ring is a finite product of simple left QI-rings, so it is enough formulate Boyle’s conjecture for simple left QI-rings.

C. Faith, in [6, Theorem 18], answers in affirmative Boyle’s conjecture for simple left QI-rings which satisfy the restricted left socle condition (RLS):

If $I \neq R$ is an essential left ideal, then $R/I$ has non zero socle.

The Theorem 18 in [6] was extended by T. Rizvi and D. Van Huynh in [9]. But the conjecture is still open.

This paper is organized in three sections, the first one is this introduction and the second one concerns to present the necessary preliminaries.

Section 3 is where we develop our work and it contains the main results, we prove that if $R$ is a left QI-ring and $\tau$ is a perfect torsion theory then the ring of quotients $\tau R$ is left QI (Proposition 3.10). Also, we prove that Boyle’s conjecture is true for all those left QI-rings with finite Gabriel dimension such that the hereditary torsion theory generated by the semisimple modules is perfect (Theorem 3.12).
2 Preliminaries

One useful tool to characterize rings, is the hereditary torsion theories. Let us recall the definition of hereditary torsion theory:

**Definition 2.1.** A pair of nonempty classes of modules $\tau = (\mathcal{T}, \mathcal{F})$ is a torsion theory if

1. $Hom_R(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
2. If $M$ is such that $Hom_R(M, F) = 0$ for all $F \in \mathcal{F}$ then $M \in \mathcal{T}$.
3. If $N$ is such that $Hom_R(T, N) = 0$ for all $T \in \mathcal{T}$ then $N \in \mathcal{F}$.

It is said $\tau$ is hereditary if $\mathcal{T}$ is closed under submodules. The set of hereditary torsion theories in $R$-Mod is denoted $R$-tors. The class $\mathcal{T}$ is called the torsion class and $\mathcal{F}$ the torsion free class.

It can be proved $R$-tors is a frame with order the inclusion of torsion classes, the least and greatest elements of $R$-tors are denoted by $\xi$ and $\chi$ respectively. Given a class of modules $C$ there exists the least hereditary torsion theory $\xi(C)$ such that $C$ is contained in the torsion class and there exist the greatest hereditary torsion theory $\chi(C)$ such that $C$ is contained in the torsion free class. If $\tau = (\mathcal{T}, \mathcal{F})$, the modules in $\mathcal{T}$ are called $\tau$-torsion modules and the modules in $\mathcal{F}$ are called $\tau$-torsion free. For more details see [7].

There exist a bijective correspondence between $R$-tors and $R$-Gab, where $R$-Gab denotes the set of Gabriel topologies in $R$. see [12] VI, Theorem 5.1.

An $R$-module $N$ is called $\tau$-cocritical with $\tau \in R$-tors if $N$ is $\tau$-torsion free but every proper factor module is $\tau$-torsion. With this modules can be defined the Gabriel filtration in $R$-tors as:

\[ \tau_0 = \xi \]

If $i$ is a non limit ordinal:

\[ \tau_i = \tau_{i-1} \vee \bigvee \{\xi(N) \mid N \text{ is } \tau_{i-1}\text{-cocritical}\} \]

If $i$ is a limit ordinal:

\[ \tau_i = \bigvee_{j<i} \tau_j \]
Since $R$-tors is a set it must exist a least ordinal $\alpha$ such that $\tau_\alpha = \tau_{\alpha + \beta}$ for all ordinals $\beta$. If $\tau_\alpha = \chi$ then we say $R$ has Gabriel dimension equal to $\alpha$ and we denote it as $Gdim(R) = \alpha$.

The concept of QI-ring can be generalized to modules, one paper in this sense is [5]. In that paper it is shown ([5, Theorem 3.9]) that (in a more general context) a ring $R$ is left QI if and only if $R$ has Gabriel dimension and every hereditary pretorsion class is a hereditary torsion class. A hereditary pretorsion class in $R$-Mod is a class of modules closed under submodules, direct sums and quotients.

Let us recall the concept of perfect localization.

**Definition 2.2.** If $\varphi : R \to S$ is an epimorphism in the category of rings which makes $S$ into a flat right $R$-module, then we will call $S$ a left perfect localization of $R$.

**Remark 2.3.** Given a hereditary torsion theory $\tau \in R$-tors we will say that $\tau$ is perfect if the ring of quotients $\tau R$ is a perfect left localization of $R$.

**Remark 2.4.** Given $\tau \in R$-tors, let us denote the localization functor as $q_\tau : R - Mod \to \tau R - Mod$. If $\tau$ is perfect then $q_\tau$ is exact. [12, XI, Proposition 3.4]. We will write $\tau N = q_\tau(N)$.

### 3 Left QI-rings and perfect torsion theories

**Remark 3.1.** Let $\tau \leq \sigma \in R$-tors. Note that if $N$ is $\sigma$-torsion free then $q_\sigma(N)$ is $\sigma$-torsion free. In fact, since $\tau \leq \sigma$ then $N$ is $\tau$-torsion free, so we have an essential monomorphism $\psi_N : N \to q_\tau(N)$.

**Lemma 3.2.** Let $\tau \leq \sigma \neq \chi \in R$-tors perfect torsion theories with $R$ $\sigma$-torsion free. Let $q_\tau : R - Mod \to \tau R - Mod$ denote the localization functor.

1. If $\sigma = (\mathfrak{I}, \mathfrak{F})$ then $\hat{\sigma} := (q_\tau(\mathfrak{I}), q_\tau(\mathfrak{F})) \in \tau$-tors.

2. If $\mathcal{F}_\sigma$ and $\mathcal{F}_{\hat{\sigma}}$ denote the Gabriel topologies in $R$ and $\tau R$ associated to $\sigma$ and $\hat{\sigma}$ respectively, then:

   \[ J \in \mathcal{F}_{\hat{\sigma}} \iff J = \tau I \text{ with } I \in \mathcal{F}_\sigma \]

3. $\sigma R$ is $\hat{\sigma}$-closed (as $\tau R$-module).
4. There is a ring isomorphism $\sigma R \cong (\tau R)$

5. $\hat{\sigma} \in \tau R$-tors is perfect.

Proof. 1. Let $\tau \leq \sigma \in R$-tors, with $\sigma = (\mathcal{T}, \mathcal{F})$. Then $\hat{\sigma} = (q_{\tau}(\mathcal{T}), q_{\tau}(\mathcal{F}))$ where

$$q_{\tau}(\mathcal{T}) = \{ q_{\tau}(M) | M \in \mathcal{T} \}$$

$$q_{\tau}(\mathcal{F}) = \{ q_{\tau}(N) | N \in \mathcal{F} \}$$

Let $q_{\tau}(M) \in q_{\tau}(\mathcal{T})$ and $q_{\tau}(N) \in q_{\tau}(\mathcal{F})$. Since $\tau$ is perfect, $q_{\tau}(M) = \tau R \otimes R M$. Then

$$\text{Hom}_{\tau R}(q_{\tau}(M), \tau N) = \text{Hom}_{\tau R}(\tau R \otimes R M, \tau N) \cong \text{Hom}_{\tau R}(M, \text{Hom}_{\tau R}(\tau R, \tau N))$$

$$\cong \text{Hom}_{\tau R}(M, \tau N) = 0$$

by remark 3.1.

Now, let $q_{\tau}(N) = \tau N$ an $\tau$-module such that $\text{Hom}_{\tau R}(\tau R \otimes R M, \tau N) = 0$ for all $M \in \mathcal{T}$. Following the above isomorphisms, we get that $\text{Hom}_{\tau R}(M, \tau (N)) = 0$, i.e., $\tau N \in \mathcal{F}$. Since $\tau \leq \sigma$, we have a monomorphism $\psi_N : N \rightarrow \tau N$ so $N \in \mathcal{F}$. Thus $\tau N \in q_{\tau}(\mathcal{F})$.

On the other hand, suppose $\tau M$ is an $\tau$-module such that $\text{Hom}_{\tau R}(\tau M, \tau N) = 0$ for all $\tau N \in \mathcal{F}$. Then, we have that $\text{Hom}_{\tau R}(M, \tau N) = 0$. Let $N \in \mathcal{F}$ and suppose $\text{Hom}_{\tau R}(M, N) \neq 0$. Hence there exists $0 \neq f : M \rightarrow N$, this implies that $0 \neq \psi_N f : M \rightarrow \tau N$. Contradiction. Thus $\text{Hom}_{\tau R}(M, N) = 0$ for all $N \in \mathcal{F}$, hence $M \in \mathcal{T}$.

Thus, we have that $\hat{\sigma}$ is a torsion theory. Let us see it is hereditary.

Let $\tau M \in q_{\tau}(\mathcal{T})$ and $K \leq \tau M$ an $\tau$-submodule. There is a monomorphism $\psi : M/\tau(M) \rightarrow \tau M$. Consider $K \cap (M/\tau(M))$ which is a $\sigma$-torsion $R$-module. By [12, XI, Proposition 3.7] $q_{\tau}(K \cap (M/\tau(M))) = K$. Thus $K \in q_{\tau}(\mathcal{T})$.

2$\Rightarrow$. Let $I \in \mathcal{F}_\sigma$. We have the following commutative diagram with exact rows:

$$
\begin{array}{c}
0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0 \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
0 \rightarrow \tau I \rightarrow \tau R \rightarrow \tau(R/I) \rightarrow 0
\end{array}
$$

Since $R/I$ is $\sigma$-torsion then $\tau(R/I) \in q_{\tau}(\mathcal{T})$ and $\tau R / \tau I \cong \tau(R/I)$. Thus $\tau I \in \mathcal{F}_\hat{\sigma}$. 

5
\( \Rightarrow \). Let \( E \) be an injective \( R \)-module such that \( \chi(E) = \sigma \). Since \( E \) is \( \sigma \)-torsion free then it is \( \tau \)-torsion free. Hence \( E \) is \( \tau \)-closed, so \( E = \tau E \). Let \( J \in \mathcal{J}_\theta \), since \( R \) is \( \tau \)-torsion free then \( J = \tau (J \cap R) \). Therefore:

\[
\text{Hom}_R(R/J \cap R, E) = \text{Hom}_R(R/J \cap R, \tau E) \cong \text{Hom}_R(R/J \cap R, \text{Hom}_R(\tau R, E)) \\
\cong \text{Hom}_\tau(\tau R \otimes_R R/J \cap R, E) \cong \text{Hom}_\tau(\tau (R/J \cap R), E) = \text{Hom}_\tau(\tau R/J, E)
\]

Since \( E \) is \( \sigma \)-torsion free then \( E \) is \( \hat{\sigma} \)-torsion free but \( \tau R/J \) is \( \hat{\sigma} \)-torsion. Thus \( \text{Hom}_\tau(\tau R/J, E) = 0 \). This implies \( \text{Hom}_R(R/J \cap R, E) = 0 \) and hence \( J \cap R \in \mathcal{F}_\sigma \).

3. Let us see first that \( \sigma R \) is \( \tau \)-closed (as \( R \)-module). By [12, IX, Proposition 1.8] \( \sigma R \) is \( \sigma \)-closed, that is, \( \sigma R \) is \( \sigma \)-torsion free and \( \mathcal{F}_\sigma \)-injective. Thus \( \sigma R \) is \( \tau \)-torsion free and since \( \mathcal{F}_\tau \subseteq \mathcal{F}_\sigma \) then \( \sigma R \) is \( \mathcal{F}_\tau \)-injective.

Since \( R \) is \( \sigma \)-torsion free then \( \tau R \) is \( \hat{\sigma} \)-torsion free and we have following inclusions

\[
R \hookrightarrow \tau R \hookrightarrow \sigma R
\]

Also, we have to note that these inclusions are essential, so \( \sigma R \) is \( \hat{\sigma} \)-torsion free. Now, let \( J \in \mathcal{J}_\theta \) and \( g : J \to \sigma R \) an \( \tau R \)-morphism. Then \( J = \tau (J \cap R) \) and by (2) \( R/J \cap R \) is \( \tau \)-torsion. If \( \psi : R/J \cap R \to \tau (R/J \cap R) = \tau R/J \) is the canonical homomorphism then \( \text{Ker}(\psi) \) and \( \text{Coker}(\psi) \) are \( \tau \)-torsion, therefore they are \( \sigma \)-torsion. So we have the following short exact sequence:

\[
0 \to \frac{R/J \cap R}{\text{Ker}(\psi)} \to \tau R/J \to \text{Coker}(\psi) \to 0
\]

Thus, \( \tau R/J \) is \( \sigma \)-torsion.

Since \( \sigma R \) is \( \mathcal{F}_\sigma \)-injective there exists an \( R \)-morphism \( \bar{g} : \tau R \to \sigma R \) such that \( \bar{g}|_J = g \). We have that \( \sigma R \) is \( \tau \)-closed, so \( \bar{g} \) is an \( R \)-morphism between \( \tau \)-closed \( R \)-module, hence \( \bar{g} \) is an \( \tau R \)-morphism. Thus \( \sigma R \) is \( \mathcal{F}_\tau \)-injective.

4. Since \( \sigma R \) is an \( \tau R \)-module which is \( \hat{\sigma} \)-closed and \( \tau R \leq \sigma R \) then \( \sigma R \cong \hat{\sigma}(\tau R) \).

5. We have \( \sigma R \cong \hat{\sigma}(\tau R) \). Let \( N \) be an \( \sigma R \)-module. Since \( \sigma \) is perfect then \( N = \sigma N \) with \( R N \) \( \sigma \)-torsion free. Then \( \tau N \in \mathcal{q}_\tau(\mathfrak{F}) \). Thus \( \sigma N \) is \( \hat{\sigma} \)-torsion free. By [12, XI, Ex. 6] \( \hat{\sigma} \in \tau R \)-tors is perfect.

\[ \square \]

**Lemma 3.3.** Let \( \tau \leq \sigma \) be perfect torsion theories in \( R \)-Mod and let \( q_\tau : R - \text{Mod} \to \tau R - \text{Mod} \) the localization functor.
1. If $M$ is $\sigma$-cocritical then $\tau M$ is $q_r(\sigma)$-cocritical.

2. If an $\tau R$-module $K$ is $q_r(\sigma)$-cocritical then $K$ as $R$-module is $\sigma$-cocritical.

Proof. 1. Since $M$ is $\sigma$-torsion free then $\tau M$ is $q_r(\sigma)$-torsion free. Let $N \leq \tau M$ be an $\tau R$-submodule. Since $M$ is $\sigma$-torsion free then it is $\tau$-torsion free, so the canonical morphism $\psi_M : M \to \tau M$ is a monomorphism. By [12, IX, Proposition 4.3] $N = \tau(N \cap M)$. Since $\tau$ is perfect, then $q_r$ is exact, hence $M_N \cong q_r(\frac{M}{N \cap M})$. By hypothesis $\frac{M}{M \cap N}$ is $\sigma$-torsion thus $\frac{M}{N \cap N}$ is $q_r(\sigma)$-torsion.

2. Let $K$ be a $q_r(\sigma)$-cocritical. Since $K$ is $q_r(\sigma)$-torsion free then $K = \tau M$ for some $M$ $\sigma$-torsion free. So, as $R$-module $K$ is $\sigma$-torsion free. Now, let $L < K$ be an $R$-submodule such that $K/L$ is $\sigma$-torsion free. Since $q_r$ is exact $\frac{K}{L} \cong q_r(\frac{K}{L})$ but $q_r(\frac{K}{L})$ is $q_r(\sigma)$-torsion and $\frac{K}{L}$ $q_r(\sigma)$-torsion, this is a contradiction. This implies that $N/L = t_r(K/L) \neq 0$ and $\frac{K}{N} \cong \frac{K/L}{N/L}$ is $\sigma$-torsion free, hence $N = K$. Thus $K/L$ is $\sigma$-torsion.

Lemma 3.4. Let $R$ be a ring with finite Gabriel dimension, $Gdim(R) = n$. Let $\{\tau_i\}_{i=0}^n$ be the Gabriel filtration in $R$-tors with every $\tau_i$ perfect. If $q_{\tau_i} : R - Mod \to \tau_i R - Mod$ is the localization functor and $\{\omega_j\}$ is the Gabriel filtration in $\tau_i R$-tors, then $q_{\tau_i}(\tau_{i+1}) = \omega_i$ for all $0 \leq i$. 

Proof. By induction over $i$. If $i = 0$ then $\tau_1 = \xi \in \tau_1 R$-tors and $\omega_0 = \xi$. Now suppose the result is valid for each natural less than $i$.

By hypothesis of induction $q_{\tau_i}(\tau_i) = \omega_{i-1}$, so

$$\omega_i = \omega_{i-1} \vee \bigvee \{\xi(K)|K \text{ is } \omega_{i-1}\text{-cocritical}\}$$

$$= q_{\tau_i}(\tau_i) \vee \bigvee \{\xi(K)|K \text{ is } q_{\tau_i}(\tau_i)\text{-cocritical}\}$$

Let $K$ be a $q_{\tau_i}(\tau_i)$-cocritical, then by Lemma 3.3, 2 $K$ as $R$-module is $\tau_i$-cocritical, hence $K$ is $\tau_{i+1}$-torsion. Therefore $\omega_i \leq q_{\tau_i}(\tau_{i+1})$. By Lemma 3.3, 1 if $N$ is $\tau_i$-cocritical then $\tau_i N$ is $q_{\tau_i}(\tau_i) = \omega_{i-1}$-cocritical, so $\tau_i N$ is $\omega_{i-1}$-torsion.

Let $L$ be a $\omega_{i-1}$-torsion free. Then $L$ is $\omega_{i-1} = q_{\tau_i}(\tau_i)$-torsion free and hence $RL$ is $\tau_i$-torsion free. If $L$ is not $\tau_{i+1}$-torsion free there exists an $R$-morphism $0 \neq f : N \to L$ with $N$ $\tau_i$-cocritical. Then we have a non zero $\tau_i$ $R$-morphism $\tau_i f : \tau_i N \to L$ with $\tau_i N$ $\omega_{i-1}$-cocritical. Contradiction. Thus $L$ is $\tau_{i+1}$-torsion free. This implies that $L$ is $q_{\tau_i}(\tau_{i+1})$-torsion free. Thus $q_{\tau_i}(\tau_{i+1}) \leq \omega_i$. 

\[\Box\]
Corollary 3.5. Let $R$ be a ring with finite Gabriel dimension, $\text{Gdim}(R) = n$. Let $\{\tau_i\}_{i=0}^n$ be the Gabriel filtration in $R\text{-Mod}$. Suppose that every $\tau_i$ is perfect, then $\text{Gdim}(\tau_1 R) < n$.

Proof. If $\{\omega_j\}$ is the Gabriel filtration in $\tau_1 R$ then, by Lemma 3.4 $q_{\tau_1}(\tau_{i+1}) = \omega_i$ for all $0 \leq i$. Since $\text{Gdim}(R) = n$ then $\tau_n = \chi$ so $\omega_{n-1} = q_{\tau_1}(\tau_n) = q_{\tau_1}(\chi) = \chi \in \tau_1 R\text{-tors}$. This implies that $\text{Gdim}(\tau_1 R) \leq n - 1$. □

Lemma 3.6. Let $\tau \in R\text{-tors}$ and $\{\sigma_i\}_{i \in I} \subseteq R\text{-tors}$ be a family of perfect torsion theories such that $\tau \leq \sigma_i$ for all $i \in I$. If $q_{\tau} : R - \text{Mod} \rightarrow \tau R - \text{Mod}$ is the localization functor then

$$q_{\tau}(\bigvee_{i \in I} \sigma_i) = \bigvee_{i \in I} q_{\tau}(\sigma_i)$$

Proof. Write $\bigvee_{i \in I} \sigma_i = (\mathcal{F}_{\bigvee \sigma_i}, \mathcal{F}_{\bigvee \sigma_i})$. Then

$$q_{\tau}(\bigvee_{i \in I} \sigma_i) = (q_{\tau}(\mathcal{F}_{\bigvee \sigma_i}), q_{\tau}(\mathcal{F}_{\bigvee \sigma_i}))$$

The torsion free class of $\bigvee_{i \in I} \sigma_i$ is described as $\mathcal{F}_{\bigvee \sigma_i} = \bigcap_{i \in I} \mathcal{F}_{\sigma_i}$. So, if $q_{\tau}(N) \in q_{\tau}(\mathcal{F}_{\bigvee \sigma_i})$ then $N \in \bigcap_{i \in I} \mathcal{F}_{\sigma_i}$, hence $q_{\tau}(N) \in \bigcap_{i \in I} q_{\tau}(\mathcal{F}_{\sigma_i})$. Thus

$$\bigvee_{i \in I} q_{\tau}(\sigma_i) \leq q_{\tau}(\bigvee_{i \in I} \sigma_i)$$

Now, suppose $\bigvee_{i \in I} q_{\tau}(\sigma_i) < q_{\tau}(\bigvee_{i \in I} \sigma_i)$, that is, $\mathcal{F}_{\bigvee q_{\tau}(\sigma_i)} < q_{\tau}(\mathcal{F}_{\bigvee \sigma_i})$ then there exists $0 \neq q_{\tau}(N) \in q_{\tau}(\mathcal{F}_{\bigvee \sigma_i})$ such that $q_{\tau}(N) \in q_{\tau}(\mathcal{F}_{\bigvee q_{\tau}(\sigma_i)}) = \bigcap_{i \in I} q_{\tau}(\mathcal{F}_{q_{\tau}(\sigma_i)})$.

Since $\tau(N/\tau(N)) = 0$ then $q_{\tau}(N) = q_{\tau}(N/\tau(N))$, we have that $N \in \mathcal{F}_{\bigvee \sigma_i}$ then $N/\tau(N) \in \mathcal{F}_{\bigvee \sigma_i}$. Therefore, we can assume $N$ is $\tau$-torsion free. By the choice of $N$ there exists $j \in I$ such that $N \notin \mathcal{F}_{\sigma_j}$, so $\sigma_j(N) = N' \neq 0$. On the other hand, $q_{\tau}(N) \in \bigcap_{i \in I} \mathcal{F}_{q_{\tau}(\sigma_i)}$ then there exists $N_j \in \mathcal{F}_{\sigma_j}$ such that $q_{\tau}(N) = q_{\tau}(N_j)$. Since $\tau < \sigma_j$, $N_j$ is $\tau$-torsion free, thus $N_j \leq e \mathfrak{M}_j = q_{\tau}(N)$. This implies that $0 \neq N' \cap N_j$ but $N'$ is $\sigma_j$-torsion and $N_j$ is $\sigma_j$-torsion free, this is a contradiction. Thus

$$\bigvee_{i \in I} q_{\tau}(\sigma_i) = q_{\tau}(\bigvee_{i \in I} \sigma_i)$$

□
**Remark 3.7.** In general Lemma 3.4 is not true for infinite ordinals. Let $R$ be a ring with Gabriel dimension, $Gdim(R) = \alpha$, $\omega < \alpha$. Let $\{\tau_i\}_{i=0}^\alpha$ the Gabriel filtration in $R$-tors and suppose that every $\tau_i$ is perfect. Then, by the proof of Lemma 3.4 if $\{w_i\}$ is the Gabriel filtration in $\tau_1 R$-tors we have that $q_{\tau_1}(\tau_{i+1}) = w_i$ for all $i \in \mathbb{N}$. If $\omega$ is the first infinite ordinal, by Lemma 3.6
\[
q_{\tau_1}(\tau_\omega) = q_{\tau_1}(\bigvee_{i \in \mathbb{N}} \tau_i) = \bigvee_{i \in \mathbb{N}} q_{\tau_1}(\tau_i) = \bigvee_{i \in \mathbb{N}} w_{i-1} = w_\omega
\]

**Definition 3.8.** An injective left $R$-module $E$ is called completely injective if every factor module of $E$ is injective.

The following result is well known, see [1, 18, Ex. 10]

**Proposition 3.9.** Let $R$ be a ring. $R$ is left hereditary if and only if every injective module is completely injective.

**Proposition 3.10.** Let $R$ be a left QI-ring and $\tau \in R$-tors a perfect torsion theory. Then the ring of quotients $\tau R$ is a left QI-ring.

**Proof.** Let $\tau A$ be a quasi-injective $\tau R$-module. Then we can consider $A$ as a $\tau$-torsion free $R$-module and by [12, IX, Proposition 2.5] $E(A)$ is an injective envelope of $\tau A$ in $\tau R$-Mod. Hence $\tau A \leq E(A)$ is a fully invariant $\tau R$-submodule.

Let $f \in End_R(E(A))$, since $E(A)$ is $\tau$-closed then $f$ is an $\tau R$-morphism. Then $f(\tau A) \leq \tau A$, i.e., $\tau A$ is a quasi-injective $R$-module. Since $R$ is left QI, $\tau A$ is an injective $R$-module. Thus by [12, IX, Proposition 2.7] $\tau A$ is an injective $\tau R$-module. \qed

**Remark 3.11.** Let $R$ be a left QI-ring. Consider the class of all semisimple left $R$-modules, it is known that this class is a hereditary pretorsion class but since $R$ is left QI then, semisimple modules form a hereditary torsion class [5, Theorem 3.9]. Let us denote the hereditary torsion theory associated to the semisimple torsion class by $\tau_{ss}$. The radical associated to $\tau_{ss}$ is $\text{Soc}$.

In the same way, if we consider the pretorsion class of all singular modules it is a hereditary torsion class. We will denote the hereditary torsion class by $\tau_g$ and the radical associated by $\mathcal{Z}$. Notice that if $R$ is a simple ring then $\tau_g$ is the unique coatom in $R$-tors.

**Theorem 3.12.** Let $R$ be a (simple) left QI-ring. Suppose that $Gdim(R) = n$ and let $\{\tau_i\}_{i=1}^n$ be the Gabriel filtration in $R$-tors. Suppose $\tau_1$ is perfect. If $\tau_1 R$ is left hereditary then $R$ is left hereditary.
Proof. Since $R$ is a left QI-ring then $\tau_1 = \tau_{ss}$ and by hypothesis $\tau_1 R$ is left hereditary.

Now, let $E$ be an indecomposable non singular injective left $R$-module. Let $E \to F$ be an epimorphism. Since $R$ is a left noetherian and left $V$-ring $F = \text{Soc}(F) \oplus F'$ where $F'$ is $\tau_1$-torsion free and $\text{Soc}(F)$ is injective. So, to prove $R$ is left hereditary is enough to prove that every factor module $F$ of $E$ with $\text{Soc}(F) = 0$ is injective.

Let $\rho : E \to F$ be an epimorphism such that $\text{Soc}(F) = 0$. This implies that $\text{Ker}(\rho) \in \text{Sat}_{\tau_1}(E)$. By $[12]$ Proposition 4.2 $\text{Sat}_{\tau_1}(E)$ consist of the $\tau_1$-closed submodules of $E$. Consider the following diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ker}(\rho) & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Ker}(\rho) & \longrightarrow & E & \longrightarrow & \tau_1 F & \longrightarrow & 0
\end{array}
$$

Since $\tau_1$ is perfect the localization functor is exact and $\text{Ker}(\rho)$ and $E$ are $\tau_1$-closed, so the second row is exact. This implies $F$ is $\tau_1$-closed. Thus $\rho$ is an $\tau_1 R$-morphism. Since $\tau_1 R$ is left hereditary and $E$ is an injective $\tau_1 R$-module then $F$ is an injective $\tau_1 R$-module. Thus $F$ is an injective $R$-module.

By $[6]$ Proposition 14A] every injective $R$-module is completely injective, thus $R$ is left hereditary.

Theorem 3.13. Let $R$ be a (simple) left QI-ring. Suppose that $Gdim(R) = n$ and let $\{\tau_i\}_{i=1}^n$ be the Gabriel filtration in $R$-tors. Then the following conditions are equivalent:

1. Every $\tau_i$ is perfect
2. $R$ is left hereditary.

Proof. $\Rightarrow$ By induction over $n$.

If $n = 1$ then $R$ is semisimple, and thus $R$ is hereditary.

Suppose the result is true for all left QI-rings $R$ with $Gdim(R) < n$ such that every element in the Gabriel filtration is perfect. Let $R$ be a left QI-ring with $Gdim(R) = n$. By hypothesis $\tau_1 = \tau_{ss}$ is perfect.

By Proposition 3.10 $\tau_1 R$ is a left QI-ring and by Lemma 3.5 $Gdim(\tau_1 R) < n$. Since the Gabriel filtration in $\tau_1 R$-tors is $\{q(\tau_i)\}_{1 \leq i < n}$ then, by
Lemma 3.2 we can apply the induction hypothesis. Hence $\tau_1 R$ is left hereditary. Thus by Theorem 3.12 $R$ is left hereditary.

$\Leftarrow$. If $R$ is left hereditary and left QI-ring then every hereditary torsion theory is perfect [12 XI, Corollary 3.6].

**Definition 3.14.** An $R$-module $M$ satisfies the restricted left socle condition (RLS) if for any essential submodule $N \neq M$, the factor module $M/N$ has non zero socle.

**Proposition 3.15.** Let $R$ be a simple left QI-ring which is non semisimple. The following are equivalent:

1. $R$ satisfies RLS.
2. $\tau_{ss} = \tau_g$
3. $Gdim(R) = 2$.
4. There exists a non singular indecomposable completely injective module $E$ which is $\tau_{ss}$-cocritical.

**Proof.** 1 $\Rightarrow$ 2. Since $R$ is non semisimple, then by Remark 3.11 $\tau_{ss} \leq \tau_g$. Now let $M$ be a singular module and $0 \neq m \in M$. Then $(0 : m)$ is an essential left ideal of $R$. Thus $0 \neq Soc(R/(0 : m)) = Soc(Rm)$. This implies that $Soc(M) \leq M$ but $Soc(M)$ is a direct summand, so $M = Soc(M)$.

2 $\Rightarrow$ 3. If $\{\tau_i | i \geq 0\}$ is the Gabriel filtration in $R$-tors then $\tau_1 = \tau_{ss} = \tau_g$. Since $R$ is simple $\tau_g$ is a coatom in $R$-tors, thus $\tau_2 = \chi$.

3 $\Rightarrow$ 4. If $Gdim(R) = 2$, then the Gabriel filtration in $R$-tors is $\{\xi, \tau_{ss}, \chi\}$. Therefore

$\chi = \tau_{ss} \lor \bigvee \{\xi(N) | N \text{ $\tau_{ss}$-cocritical}\}$

Assume that every $\tau_{ss}$-cocritical module is singular, then $\chi \leq \tau_g$, this is a contradiction. Hence, there exists a non singular $\tau_{ss}$-cocritical module $N$. Since $N$ is cocritical, it is uniform and so $E(N)$ is a non singular indecomposable injective module. By [3] proposition 2.1 $E(N)$ is also $\tau_{ss}$-cocritical, thus we are done.

4 $\Rightarrow$ 1. Since $R$ is a simple ring then $R$ is a prime ring. By [3 Corollary 2.15], $E$ is up to isomorphism the only non singular indecomposable injective module. Now, by [3 Theorem 2.20] $E(R) \cong E^k$ for some $k > 0$; since $E$ is completely injective then $E(R)$ so does by [6 Proposition 14A]. Let $I \leq_e R$, then $I \leq_e E(R)$ and so $E(R)/I$ is semisimple. Thus $R/I$ is semisimple. 

\[ \square \]
Remark 3.16. In [6, Theorem 17] it is constructed an indecomposable injective module $E$ such that $E$ is non semisimple and satisfies \( RLS \). Then \( \text{Soc}(E) = 0 \), that is, $E$ is $\tau_{ss}$-torsion free. Since $E$ is uniform and satisfies \( RLS \) then $E$ is $\tau_{ss}$-cocritical. Thus if $E$ is nonsingular then $E$ satisfies condition 4 of Proposition 3.15.

As Corollary we have the next result due to C. Faith [6, Theorem 18]

**Corollary 3.17.** Any left QI-ring $R$ with restricted left socle condition is left hereditary.

**Proof.** By Proposition 3.15 $R$ has $Gdim(R) = 2$. Hence the Gabriel filtration in $R$-tors is \( \{\xi, \tau_g, \chi\} \) where $\xi$ and $\chi$ are the least and greatest elements of $R$-tors respectively. The element $\tau_g \in R$-tors is the Goldie’s torsion theory and it is perfect because $R$ is left noetherian [12, Proposition 2.12 and Proposition 3.4]. Thus, by Theorem 3.13 $R$ is left hereditary. \( \square \)

**Lemma 3.18.** Let $R = R_1 \times \cdots \times R_n$ be a finite product of rings. Then $R$ satisfies \( RLS \) if and only if $R_i$ satisfies \( RLS \) for all $1 \leq i \leq n$.

**Proof.** \( \Rightarrow \). Let $1 \leq i \leq n$ and $I_i$ an essential left ideal of $R_i$. Then $I = R_1 \times \cdots \times I_i \times \cdots R_n$ is an essential left ideal of $R$. By hypothesis, $R/I$ contains a simple $R$-module $S$, and we have that $R/I \cong R_i/I_i$. Thus $S$ is a simple $R_i$-module and it can be embedded in $R_i/I_i$.

\( \Leftarrow \). Let $I$ be an essential left ideal of $R$, then $I = I_1 \times \cdots \times I_n$ with $I_i$ an essential left ideal of $R_i$. By hypothesis $R_i/I_i$ contains a simple $R_i$-module and we have that $R/I \cong (R_1/I_1) \oplus \cdots \oplus (R_n/I_n)$. Thus $R/I$ contains a simple $R$-module. \( \square \)

**Remark 3.19.** Let $R = R_1 \times \cdots \times$ be a product of rings. Notice that $E$ is a non singular indecomposable injective $R$-module then $E$ is a non singular indecomposable injective $R_i$-module for some $1 \leq i \leq n$. On the other hand, if $E_i$ is a non singular indecomposable injective $R_i$-module then $E_i$ is a non singular indecomposable injective $R$-module.

**Theorem 3.20.** Let $R = R_1 \times \cdots \times R_n$ be a left QI-ring such that each $R_i$ is a simple left QI-ring and non semisimple. The following conditions are equivalent:

1. $R$ satisfies \( RLS \).
2. For each \( 1 \leq i \leq n \) there exists a non singular indecomposable injective \( R_i \)-module \( E_i \) which are \( \tau_{ss} \)-cocritical as \( R \)-modules.

3. \( Gdim(R) = 2 \).

4. \( \tau_{ss} = \tau_g \) in \( R \)-tors.

Proof. \( 1 \Rightarrow 2 \). Since \( R \) satisfies \( RLS \) then by Lemma 3.18 each \( R_i \) satisfies \( RLS \). Hence by Proposition 3.15 there exist a non singular indecomposable injective \( R_i \)-module \( E_i \) which satisfies \( RLS \).

\( 2 \Rightarrow 3 \). Let \( \{ \tau_j \} \) be the Gabriel filtration in \( R \)-tors. By Remark 3.19 each \( E_i \) is a non singular indecomposable injective \( R_i \)-module, so each \( E_i \) is \( \tau_{ss} \)-torsion free. Since each \( E_i \) is \( \tau_{ss} \)-cocritical then

\[
\tau_{ss} \lor \bigvee \xi(E_i) \leq \tau_2
\]

Now, if \( E \) is a non singular indecomposable injective \( R \)-module then \( E \) is a non singular indecomposable injective \( R_i \)-module for some \( 1 \leq i \leq n \). But since \( R_i \) is simple and hence a prime ring, by [3, Corollary 2.20] \( E \cong E_i \). Thus all non singular indecomposable injective \( R \)-modules, up to isomorphism, are \( E_1, ..., E_n \). Again by [3, Corollary 2.20] \( \hat{R} \cong E_1^{k_1} \oplus \cdots \oplus E_n^{k_n} \) where \( \hat{R} \) denotes the injective hull of \( R \) for some natural numbers \( k_1, ..., k_n \). Thus \( \tau_{ss} \lor \bigvee \xi(E_i) = \chi \). So, \( Gdim(R) = 2 \).

\( 3 \Rightarrow 4 \). If \( Gdim(R) = 2 \) then the Gabriel filtration in \( R \)-tors is \( \{ \xi, \tau_{ss}, \chi \} \). Since every \( R_i \) is a simple left QI-ring non semisimple, then all simple \( R \)-modules are singular. Thus \( \tau_{ss} \leq \tau_g \).

Suppose \( \tau_{ss} < \tau_g \) then there exists \( C \) such that \( C \) is \( \tau_{ss} \)-cocritical and \( \tau_g \)-torsion. If \( c \in C \), \( \text{ann}(c) \leq_e R_i \), and \( \text{ann}(c) = I_1 \times \cdots \times I_n \) with \( I_i \leq_e R_i \). Hence \( R_i/I_i \) is singular and we have a monomorphism

\[
R/I \cong (R_1/I_1) \oplus \cdots \oplus (R_n/I_n) \to C
\]

Since every \( R_i \) is a simple left QI-ring and \( R_i/I_i \) is singular then by Proposition 3.15 \( R_i/I_i \) is a semisimple \( R_i \) module, hence it is semisimple as \( R \)-module. Thus \( C \) is semisimple and \( \tau_{ss} = \tau_g \).

\( 4 \Rightarrow 1 \). Let \( I \leq_e R \), then \( R/I \) is \( \tau_g \)-torsion then it is \( \tau_{ss} \)-torsion. This implies that \( R/I \) contains a simple \( R \)-module. Thus \( R \) satisfies \( RLS \). \( \square \)
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