Universal bounds and semiclassical estimates for eigenvalues of abstract Schrödinger operators

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1 Introduction

Universal spectral bounds for Laplace and Schrödinger operators, i.e., bounds that control eigenvalues with expressions that do not depend on the specific geometry of the domain or on details of the potential (cf. [11]), can be derived from fundamental identities involving traces of operators and their commutators [12]. This insight has proved useful both for unifying numerous previously known inequalities of this kind...
and for discovering new ones \[12,10,11,16\]. Related methods have also been used to obtain control on the spectrum of Laplace and Schrödinger operators in terms of curvature \[6,7,9,8\]. Many of the universal inequalities related to trace identities are sharp in the sense that they are saturated for particular examples: For the Schrödinger operators treated in \[12\] the upper bounds on eigenvalue gaps \(\lambda_{n+1} - \lambda_n\) become identities for all \(n\) in the case of the harmonic oscillator, while for Laplacians on embedded manifolds discussed in \[9,8\] all of the gap bounds become identities for embedded spheres. More recently, in some circumstances (e.g., \[13,14,15,11\]) universal bounds on moments of eigenvalues have been connected to “semiclassical” theorems about the spectrum such as asymptotic behavior as the index \(k \to \infty\) and nonasymptotic bounds in the spirit of the Berezin-Li-Yau inequality \[3,18\].

One of the motivations of this work is to sharpen the understanding of moments of eigenvalues and of Riesz means of the spectrum, which plays something like the role of a dual version of moments. Among the applications of our analysis will be a family of differential inequalities for functions determined by the spectrum, extending the analysis of \[12,10\]. By Legendre duality as in \[10\] these imply bounds on ratios of averages of eigenvalues. We also introduce a novel type of inequality relating arithmetic and geometric means of eigenvalues.

A second motivation is to better unify the subject of universal bounds with formally analogous semiclassical spectral theorems.

In the next section we present some more abstract versions of the essential trace identity of \[12\] for a class of self-adjoint operators \(H\) enjoying algebraic properties modeled on those of Schrödinger operators. We also identify a special family of functions for which \(tr(H)\) can be sharply controlled, \(viz.:\)

**Definition 1.1** Let \(H\) be a self-adjoint operator and let \(J \subset \sigma(H)\) be a distinguished subset of the spectrum. We let \(\hat{J}\) denote the smallest closed interval containing \(J\). A \(C^1\) function \(f : \hat{J} \to \mathbb{R}\) belongs to the set \(\mathcal{S}_J\) of trace-controllable functions provided that on \(\hat{J}\),

\[
\begin{align*}
H1. & \quad f(\lambda) \geq 0; \\
H2. & \quad f'(\lambda) \leq 0; \\
H3. & \quad f'(\lambda) \text{ is concave;} \\
H4. & \quad \text{If } \sup(J) < \infty, \text{ there exists } a > \sup(J) \text{ such that} \\
& \quad g_f(x) := (a - \lambda)^3 \frac{d}{d\lambda} \left( \frac{f(\lambda)}{(a - \lambda)^2} \right) = 2f(\lambda) + f'(\lambda)(a - \lambda) \\
& \quad \text{is nondecreasing in } \lambda; \\
H5. & \quad tr(P_J(f(H))) < \infty, \text{ where } P \text{ denotes the spectral projector for the set } J.
\end{align*}
\]

For reasons of parsimony we shall sometimes assume only a subset of the hypotheses in the statements of some theorems.

The model situation is that at least the lower part of the spectrum consists of eigenvalues \(\lambda_1 < \lambda_2 \leq \cdots\) and \(J = \{\lambda_1, \ldots, \lambda_n\}\). Among familiar functions in \(\mathcal{S}_J\) we mention \(\exp(-t\lambda)\) and \((z - \lambda)^p\) with \(p \geq 2\). Note that conditions H1–H4 are preserved by multiplication \(f(\lambda), g(\lambda) \to f(\lambda)g(\lambda)\) and that conditions H1–H3 are preserved by compositions in the form \(f(\lambda), g(\lambda) \to f(-g(\lambda))\).
The only condition in Definition 1.1 that may not be familiar is H4, so we observe some sufficient conditions for its validity, depending on some elementary facts about concavity, in particular.

**Proposition 1.2** If the function \( h(x) \) is concave for \( 0 < x < x_0 \), then

\[
xh(x) - 2 \int_0^x h(s) ds
\]

is concave on the same interval.

We observe that this is immediate when \( h \in C^2 \) by a calculation of the second derivative. We give a proof without this hypothesis.

**Proof** Recall that a function \( f \) is concave on an interval \( I \) iff its right and left derivatives exist at all interior points of \( I \) and \( f'(x) \) is nonincreasing, in the extended sense that if the right and left derivatives differ at \( x \), then \( f'_r(x) < f'_l(x) \). (For this and other basic facts about concave functions see chapter 5 of [21].) Therefore we compare the derivative \( \frac{d}{dx} \left( xh(x) - 2 \int_0^x h(s) ds \right) = xh'(x) - h(x) \) at \( a \) (right derivative) and \( a + \delta \) (left derivative), for \( \delta > 0 \). (We shall not complicate the notation by distinguishing right and left derivatives in the following calculation.)

\[
(x + \delta)h'(x + \delta) - h(x + \delta) - (xh'(x) - h(x))
\]

\[
= \delta h'(x + \delta) + x(h'(x + \delta) - h'(x)) - (h(x + \delta) - h(x))
\]

\[
= x(h'(x + \delta) - h'(x)) + \delta h'(x + \delta) - ((h(x + \delta) - h(x)) - ((h(x + \delta) - h(x)))
\]

\[
\leq \delta h'(x + \delta) - (h(x + \delta) - h(x)).
\]

By the mean value theorem of convex functions, for some \( y \in (x, x + \delta) \), \( (h'_r(y) \cdot \delta \leq h(x + \delta) - h(x)) \leq (h'_l(y) \cdot \delta \), and thus the final term is \( \leq 0 \). \( \Box \)

**Corollary 1.3** Suppose that \( f \) satisfies hypotheses H2 and H3 of Definition 1.1 and that

H4' There exists a \( a > \operatorname{sup}(J) \) such that

\[
g'_f(\operatorname{sup}(J)) = f'(\operatorname{sup}(J)) + f''(\operatorname{sup}(J))(a - \operatorname{sup}(J)) \geq 0.
\]

Then \( f \) satisfies hypothesis H4.

**Proof** We study the function \( g_f \) occurring in Hypothesis H4. With the change of variable \( a - \lambda \rightarrow x, \ h(x) := f'(\lambda) \) satisfies the conditions of Proposition 1.2 so \( xh(x) - 2 \int_0^x h(s) ds \) is concave for positive \( x \). But this expression evaluates to \( g_f(\lambda) - g_f(a) \), establishing that \( g_f \) is concave for \( \lambda \leq a \). Therefore \( g'_f \) is nonincreasing. At the same time we know by H4' that \( g'_f(\operatorname{sup}(J)) \geq 0 \), so it follows that \( g'_f(\lambda) \geq 0 \) on \( J \). \( \Box \)
2 Abstract trace inequalities

We consider a self-adjoint operator $H$ with domain $\mathcal{D}_H$ on a Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$. We suppose that $H$ has nonempty point spectrum, and that $\mathcal{J}$ is a finite-dimensional subspace of $\mathcal{H}$ spanned by an orthonormal set $\{\phi_j\}$ of eigenfunctions of $H$. We further let $P_A$ denote the spectral projector associated with $H$ and a Borel set $A$, and $J := \{\lambda_j : H\phi_j = \lambda_j\phi_j\}$. We refer to [19] for terminology, notation, and details about the spectral theorem.

**Theorem 2.1** Let $H$ and $G$ be self-adjoint operators with domains $\mathcal{D}_H$ and $\mathcal{D}_G$ such that $G(\mathcal{J}) \subseteq \mathcal{D}_H \subseteq \mathcal{D}_G$. Then, for any real-valued $C^1$-function $f$ the derivative $f'$ of which is a concave function (i.e., Hypothesis H3 of Definition [19]),

$$\frac{1}{2} \sum_{\lambda_j \in J} f'(\lambda_j) \langle [H,G]\phi_j, [H,G]\phi_j \rangle + f(\lambda_j) \langle [G,[H,G]], \phi_j, \phi_j \rangle \leq \sum_{\lambda_j \in J} \int (f(\lambda_j) + \frac{1}{2} f'(\lambda_j)(\kappa - \lambda_j))(\kappa - \lambda_j) \langle G\phi_j, dP_{\kappa}\mathcal{J}G\phi_j \rangle^2$$

(2.1)

In case the spectrum of $H$ is purely discrete, we may write the inequality as

$$\frac{1}{2} \sum_{\lambda_j \in J} f'(\lambda_j) \langle [H,G]\phi_j, [H,G]\phi_j \rangle + f(\lambda_j) \langle [G,[H,G]], \phi_j, \phi_j \rangle \leq \sum_{\lambda_j \in J} \sum_{\lambda_k \in \mathcal{J}} (f(\lambda_j) + \frac{1}{2} f'(\lambda_j)(\lambda_k - \lambda_j))(\lambda_k - \lambda_j) \langle G\phi_j, \phi_k \rangle^2.$$  

(2.2)

The proof will use the following lemma:

**Lemma 2.2** Let $f \in C^1(\mathbb{R})$ such that $f'$ is a concave function. Then for all $x,y \in \mathbb{R}$

$$\frac{f(y) - f(x)}{y - x} \geq \frac{1}{2} f'(y) + \frac{1}{2} f'(x).$$

(2.3)

**Proof** By the fundamental theorem of calculus and the concavity of $f'$ we have

$$\frac{f(y) - f(x)}{y - x} = \int_0^1 f'((1-t)x + ty) \, dt$$

$$\geq \int_0^1 (1-t)f'(x) + tf'(y) \, dt = \frac{1}{2} f'(y) + \frac{1}{2} f'(x).$$  

Q.E.D.

**Proof** We begin with an observation that is an abstract version of what is known in quantum theory as the oscillator-strength sum rule of Thomas, Reiche, and Kuhn [4]: By a straightforward calculation, the self-adjoint operators $(H,G)$ satisfy

$$\langle [G,[H,G]]\phi_j, \phi_j \rangle = 2 \langle (H - \lambda_j)G\phi_j, G\phi_j \rangle,$$  

(2.4)

which, with the spectral resolution, equals $2 \int (\kappa - \lambda_j) \langle dP_{\kappa}G\phi_j, G\phi_j \rangle$. Thus

$$\frac{1}{2} \langle [G,[H,G]]\phi_j, \phi_j \rangle = \int (\kappa - \lambda_j) dG^2_{j\kappa},$$

(2.5)
where $dG_{jk}^2 := |\langle G\phi_j, dP_k G\phi_j \rangle|$. When $\kappa = \lambda_k$ for $\phi_k \in \mathcal{F}$, we also write the discrete matrix elements as $G_{jk} := \langle G\phi_j, \phi_k \rangle$.

Multiplying by $f(\lambda_j)$ and summing over $\lambda_j$ in $J$, we get

\[
\frac{1}{2} \sum_{\lambda_j \in J} f(\lambda_j) \langle [G, [H, G]] \phi_j, \phi_j \rangle = \sum_{\lambda_j \in J} \int f(\lambda_j)(\kappa - \lambda_j) dG_{jk}^2
\]

\[
= \sum_{\lambda_j \in J} \sum_{\lambda_k \in J} f(\lambda_j)(\lambda_k - \lambda_j)G_{jk}^2 + \sum_{\lambda_j \in J} \int f(\lambda_j)(\kappa - \lambda_j) dG_{jk}^2.
\]

Using the symmetry of the matrix elements $G_{jk}$ we rewrite the first double sum as follows:

\[
\sum_{\lambda_j \in J} \sum_{\lambda_k \in J} f(\lambda_j)(\lambda_k - \lambda_j)G_{jk}^2 = \frac{1}{2} \sum_{\lambda_j \in J} \sum_{\lambda_k \in J} (f(\lambda_j) - f(\lambda_k))(\lambda_k - \lambda_j)G_{jk}^2
\]

\[
= -\frac{1}{2} \sum_{\lambda_j \in J} \sum_{\lambda_k \in J} \frac{f(\lambda_k) - f(\lambda_j)}{\lambda_k - \lambda_j}(\lambda_k - \lambda_j)^2 G_{jk}^2.
\]

Applying Lemma 2.2 and once again using the symmetry of the $G_{jk}$ we get

\[
\sum_{\lambda_j \in J} \sum_{\lambda_k \in J} f(\lambda_j)(\lambda_k - \lambda_j)G_{jk}^2 \leq -\frac{1}{2} \sum_{\lambda_j \in J} \sum_{\lambda_k \in J} f'(\lambda_j)(\lambda_k - \lambda_j)^2 G_{jk}^2.
\]

At the same time, the pair $(H, G)$ satisfies the trace formula

\[
\int (\kappa - \lambda_j)^2 dG_{jk}^2 = \langle [H, G] \phi_j, [H, G] \phi_j \rangle.
\]

Multiplying by $-\frac{1}{2} f'(\lambda_j)$ and summing over $\lambda_j \in J$ we get

\[
-\frac{1}{2} \sum_{\lambda_j \in J} f'(\lambda_j) \langle [H, G] \phi_j, [H, G] \phi_j \rangle
\]

\[
= -\frac{1}{2} \sum_{\lambda_j \in J} \int f'(\lambda_j)(\kappa - \lambda_j)^2 dG_{jk}^2
\]

\[
= -\frac{1}{2} \sum_{\lambda_j \in J} \sum_{\lambda_k \in J} f'(\lambda_j)(\lambda_k - \lambda_j)^2 G_{jk}^2
\]

\[
-\frac{1}{2} \sum_{\lambda_j \in J} \int_{\kappa \in \mathcal{F}} f'(\lambda_j)(\kappa - \lambda_j)^2 dG_{jk}^2.
\]

Combining (2.6), (2.7) and (2.8) yields the statement of the theorem. \qed

**Corollary 2.3** If $f(\lambda) = a\lambda^2 + b\lambda + c$, then (2.2) holds with equality. In particular, for any $z$ we have

\[
\sum_{\lambda_j \in J} (z - \lambda_j)^2 \langle [G, [H, G]] \phi_j, \phi_j \rangle - 2(z - \lambda_j) \langle [H, G] \phi_j, [H, G] \phi_j \rangle
\]

\[
= 2 \sum_{\lambda_j \in J} \int_{\kappa \in \mathcal{F}} (z - \lambda_j)(z - \kappa)(\kappa - \lambda_j) dG_{jk}^2.
\]
Consider the right side of (2.2). Since we also may assume without loss of generality that the values of eigenvalues. Indeed, if \( \tilde{\lambda} \) of \( \tilde{a} \), a universal relationship between the lower part of the spectrum for \( f = 1, 2, \ldots, n \) and the values of \( \tilde{\lambda}_n \) and \( \lambda_{n+1} \).

**Theorem 2.5** Let \( H \) and \( G \) be self-adjoint operators with domains \( \mathcal{D}_H \) and \( \mathcal{D}_G \) such that \( G(\mathcal{F}) \subseteq \mathcal{D}_H \subseteq \mathcal{D}_G \). Let the subset \( J = \{ \lambda_1, \ldots, \lambda_n \} \) lie below the rest of the spectrum of \( H \). Then for any \( f \in \mathcal{S}_J \), with \( f'(\lambda_n) + f''(\lambda_n)(\lambda_{n+1} - \lambda_n) \geq 0 \),

\[
\frac{1}{2} \sum_{j=1}^{n} \left( f'(\lambda_j) \langle [H, G]\phi_j, [H, G]\phi_j \rangle + f(\lambda_j) \langle [G, [H, G]]\phi_j, \phi_j \rangle \right) \geq \frac{1}{2} f(\lambda_n) + \frac{1}{2} f'(\lambda_n)(\lambda_{n+1} - \lambda_n) \sum_{j=1}^{n} \langle [G, [H, G]]\phi_j, \phi_j \rangle. \tag{2.10}
\]

**Proof** Consider the right side of (2.2). Since \( \lambda_k \geq \lambda_{n+1} \geq \lambda_j \) and \( f' \leq 0 \), we have

\[
f(\lambda_k) + \frac{1}{2} f'(\lambda_k)(\lambda_k - \lambda_k) \leq f(\lambda_k) + \frac{1}{2} f'(\lambda_k)(\lambda_{n+1} - \lambda_k) = \frac{1}{2} g_f(\lambda_k).
\]

To prove (2.10) it suffices to show that \( g \) is nondecreasing so \( g_f(\lambda_k) \) can be replaced with \( g_f(\lambda_n) \). As a concave function, \( g' \) is nondecreasing on \( \hat{J} \), so this is true because of the assumption that \( g'(\lambda_n) \geq 0 \). \( \square \)

Under slightly weakened assumptions on \( f \) we get the following (weaker) inequality:

**Corollary 2.6** Let \( H \) and \( G \) be self-adjoint operators with domains \( \mathcal{D}_H \) and \( \mathcal{D}_G \) such that \( G(\mathcal{F}) \subseteq \mathcal{D}_H \subseteq \mathcal{D}_G \). Let the subset \( J = \{ \lambda_1, \ldots, \lambda_n \} \) lie below the rest of the spectrum of \( H \). Then, for any real \( C^1 \)-function \( f \) satisfying Hypothesis H2 with \( f'(\lambda_{n+1}) = 0 \),

\[
\frac{1}{2} \sum_{j=1}^{n} \left( f'(\lambda_j) \langle [H, G]\phi_j, [H, G]\phi_j \rangle + f(\lambda_j) \langle [G, [H, G]]\phi_j, \phi_j \rangle \right) \leq \frac{1}{2} f(\lambda_n) \sum_{j=1}^{n} \langle [G, [H, G]]\phi_j, \phi_j \rangle. \tag{2.11}
\]

In applications to Laplace, Schrödinger, and similar differential operators, the commutators typically simplify as follows: There are constants \( \alpha, \beta, \gamma \), with \( \beta, \gamma > 0 \), such that

\[
\gamma = \frac{1}{2} [H, G], \quad \beta H + \alpha \geq [H, G]^2. \tag{2.12}
\]

(Recall that \( -[H, G]^2 = [H, G][H, G]^\dagger \geq 0 \).) For instance, see \([1, 2, 5, 12, 15, 17, 22]\). Therefore we also may assume without loss of generality that \( H \) has only nonnegative eigenvalues. Indeed, if \( \tilde{H} = H + \eta \) for some real constant \( \eta \), then

\[
\tilde{\lambda} = \lambda + \eta, \quad \tilde{\gamma} = \gamma, \quad \tilde{\beta} = \beta, \quad \tilde{\alpha} = \alpha - \beta \eta.
\]
For the model case of the Dirichlet Laplacian \( H = -\Delta_D \) on a domain \( \Omega \) in \( \mathbb{R}^d \) there is a choice of Cartesian system coordinate system for such that with \( G = x_1 \), \( \alpha = 0, \beta = \frac{4}{d}, \) and \( \gamma = 1 \). In the literature these same effective constants are often obtained by averaging over all the coordinates, but the coordinate system can always be chosen to make this unnecessary.

Values of \( \alpha \neq 0 \) arise for several reasons. In the case of a Schrödinger operator \( H = -\Delta + V(x) \), the potential energy disappears from all commutators, and the term \(-[H, G]^2\) is typically dominated by the kinetic energy term, i.e., \(-[H, G]^2 \leq \beta (-\Delta)\) rather than \( \beta H\). In an elementary way, the addition of \( \alpha \) can compensate for the absence of \( V \) if, say, the negative part of \( V \) is bounded. Even if the negative part of the potential \( V \) is unbounded, if it lies in certain function classes, there are constants \( a < 1 \) and \( b < \infty \) such that for all functions \( \phi \) in the quadratic-form domain of \( H \),

\[
|\langle \phi, V_-(x)\phi \rangle| \leq a\|\nabla \phi\|^2 + b\|\phi\|^2,
\]

(2.13)
in which case \(-\Delta + V_- \geq (1-a)(-\Delta) - b\), and consequently

\[-\Delta \leq \frac{1}{1-a}(H + b).\]

Examples of function classes guaranteeing the estimate (2.13) are that \( V_- \) is a Rollnik potential in three dimensions or that \( V_- \in L^p(\mathbb{R}^d) \) with \( p > \frac{d}{2} \) when \( d \geq 4 \). For discussion of these conditions refer to \([20, \text{section X.2.}]\).

Another instance where \( \alpha \neq 0 \) is of interest is in the case of Laplace or Schrödinger operators on hypersurfaces \( M \) in \( (\mathbb{R}^d) \). By letting \( G \) be the Cartesian coordinate \( x_1 \) in the ambient space, and choosing the orientation of the coordinate system appropriately (or averaging over all coordinates),

\[-[H, G]^2 = -4\Delta + h^2(x),\]

where \( h(x) \) is the sum of the principal curvatures at the point \( x \in M \) and \( \Delta \) now denotes the Laplace-Beltrami operator on \( M \) [9]. Our estimates therefore apply to Laplace-Beltrami operators on \( M \) with a \( \alpha = \|h\|_{\infty} \). Schrödinger operators on \( M \) will require \( \alpha \) to be the sum of this curvature effect and any contribution owing to the negative part of \( V \). The situation is analogous for Laplace or Schrödinger operators on manifolds immersed in other symmetric spaces [8].

Under these conditions Theorem 2.5 is simplified:

**Corollary 2.7** If in addition to the assumptions of Theorem 2.5 the relations (2.12) hold, then

\[
\frac{1}{n} \sum_{j=1}^{n} \left( f(\lambda_j) + \frac{\beta \lambda_j + \alpha}{2\gamma} f'(\lambda_j) \right) \leq f(\lambda_n) + \frac{1}{2} f'(\lambda_n)(\lambda_{n+1} - \lambda_n).
\]

(2.14)

If, in addition, the spectrum of \( H \) is purely discrete and all sums over the full spectrum \( \sigma(H) \) are finite, then

\[
\sum_{\lambda_j \in \sigma(H)} \left( f(\lambda_j) + \frac{\beta \lambda_j + \alpha}{2\gamma} f'(\lambda_j) \right) \leq 0.
\]

(2.15)

In the following sections we apply Corollary 2.7 for appropriate functions \( f(\lambda) \).
3 Inequalities for moments of eigenvalues

In this section we prove various inequalities for eigenvalues under the assumptions of Corollary 2.7 for appropriate functions \( f(\lambda) \), and we restrict ourselves to operators \( H \) with purely discrete spectrum. As a first result we generalize the result of [12] on the partition function \( \text{tr}(e^{-tH}) \).

**Proposition 3.1** Let \( f \) and \( H \) satisfy the assumptions of Corollary 2.7 and suppose that
\[
\text{tr} \left( f \left( t \left( H + \frac{\alpha}{\beta} \right) \right) \right) = \sum_{j=1}^{\infty} f \left( t \left( \lambda_j + \frac{\alpha}{\beta} \right) \right)
\]
and
\[
\frac{d}{dt} \text{tr} \left( f \left( t \left( H + \frac{\alpha}{\beta} \right) \right) \right) = \sum_{j=1}^{\infty} \left( \lambda_j + \frac{\alpha}{\beta} \right) f \left( t \left( \lambda_j + \frac{\alpha}{\beta} \right) \right)
\]
are finite for all \( t > 0 \), then
\[
t \mapsto t^{\frac{2}{\beta}} \text{tr} \left( f \left( t \left( H + \frac{\alpha}{\beta} \right) \right) \right) \tag{3.1}
\]
is nonincreasing.

**Proof** If \( f(\lambda) \) satisfies the assumptions of Theorem 2.1, then \( f(t\lambda + \frac{\alpha}{\beta}) \) satisfies the same assumptions for any \( t > 0 \). Then inequality (2.15) of Corollary 2.7 reads as follows:
\[
\text{tr} \left( f \left( t \left( H + \frac{\alpha}{\beta} \right) \right) \right) + \frac{\beta}{2\gamma} t \frac{d}{dt} \text{tr} \left( f \left( t \left( H + \frac{\alpha}{\beta} \right) \right) \right) \leq 0,
\]
which proves the proposition. \( \square \)

The proposition applies to \( f(\lambda) = \lambda^{-p} e^{-\lambda} \), for any \( p \geq 0 \) and therefore

**Corollary 3.2** If
\[
Z_p(t) := \text{tr} \left( (H + \frac{\alpha}{\beta})^{-p} e^{-tH} \right) = \sum_{j=1}^{\infty} \left( \lambda_j + \frac{\alpha}{\beta} \right)^{-p} e^{-t\lambda_j}
\]
is finite for all \( t > 0 \) and \( H \) satisfies the assumptions of Corollary 2.7 then
\[
t \mapsto Z_p(t) t^{\frac{2}{\beta} - p} e^{-\frac{\alpha}{\beta}t} \tag{3.2}
\]
is nonincreasing.

**Remark 3.3** In particular, Corollary 3.2 shows that
\[
\lim_{t \to 0^+} Z_p(t) = +\infty
\]
for all \( p < \frac{2\gamma}{\beta} \).
As a second application, we shall show that certain moments of eigenvalues are dominated by their geometric mean. Let \( z > 0 \) be a parameter to be chosen later and \( p > q > 0 \) such that \( q \leq \min(1,p) \) and \( p + q \leq 3 \). For \( \lambda \in [0,z] \) the function \( f_z(\lambda) \) defined by

\[
f_z(\lambda) := q\lambda^p - p\lambda^q z^{p-q} + (p-q)z^p \tag{3.3}
\]

satisfies the assumptions of Theorem 2.5 and Corollary 2.7 for all \( z \in [\lambda_n, \lambda_{n+1}] \).

Indeed, with

\[
\begin{aligned}
f_z'(\lambda) & = pq(\lambda^{p-1} - \lambda^{q-1}z^{p-q}) \\
f_z''(\lambda) & = pq\lambda^{q-2}((p-1)\lambda^{p-q} - (q-1)z^{p-q}) \\
\end{aligned}
\]

we see that \( f_z''(\lambda) \geq 0 \) if \( q \leq 1 \), using the estimate \( (1-q)z^{p-q} \geq (1-q)\lambda^{p-q} \). Furthermore, \( f_z''(\lambda) \leq 0 \) since \( (p-1)(p-2)\lambda^{p-q} - (q-1)(q-2)z^{p-q} \leq (p-q)(p+q-3)\lambda^{p-q} \). As in the proof of Theorem 2.5 we show that \( g'(z) \geq 0 \) if \( z \leq \lambda_{n+1} \), and hence

\[
g(\lambda_k) \leq g(\lambda_n) \leq g(z) = 0,
\]

provided that \( z \in [\lambda_n, \lambda_{n+1}] \). We define

\[
F_n(z) := \sum_{j=1}^n f_z(\lambda_j) + \frac{\beta \lambda_j + \alpha}{2\gamma} f_z'(\lambda_j).
\]

Applying Corollary 2.7 with 0 as an upper bound, we have \( F_n(z) \leq 0 \) for all \( z \in [\lambda_n, \lambda_{n+1}] \). Since

\[
F_n(z) = n(p-q)z^p - npS_n(q)z^{p-q} + nqS_n(p),
\]

where

\[
S_n(r) := \left(1 + \frac{\beta r}{2\gamma}\right) - \frac{1}{n} \sum_{j=1}^n \lambda_j^r + \frac{\alpha r}{2\gamma n} \sum_{j=1}^n \lambda_j^{-1},
\]

we see that \( F_n(z) \) attains a global nonnegative minimum at \( z = S_n(q)\frac{1}{p} \). Therefore we have the following result:

**Theorem 3.4** Let \( p > q > 0 \) such that \( q \leq \min(1,p) \) and \( p + q \leq 3 \). Then for all \( n \) we have

\[
S_n(p)\frac{1}{p} \leq S_n(q)\frac{1}{q}. \tag{3.4}
\]

In particular, for \( 0 < p \leq 1 \) the function \( p \mapsto S_n(p)^\frac{1}{p} \) is nonincreasing, and for all \( 0 < p \leq 3 \) we have

\[
S_n(p)^\frac{1}{p} \leq \exp \left( \frac{\alpha}{2\gamma n} \sum_{j=1}^n \lambda_j^{-1} \right), \tag{3.5}
\]

where

\[
G_n := \left( \prod_{j=1}^n \lambda_j \right)^{\frac{1}{n}}
\]

denotes the geometric mean of the first \( n \) eigenvalues.
Inequality (3.5) is obtained from (3.4) by taking the limit \( q \to 0 \).

**Remark 3.5** We note that \( F_{n+1}(\lambda_{n+1}) = F_n(\lambda_{n+1}) \) for all \( n \).

### 4 Inequalities on generalized Riesz means

Recently, it was shown in [10] that Riesz means of eigenvalues of the Laplacian on bounded domains satisfy differential inequalities, which in turn imply universal eigenvalue bounds that are sharp in the sense of having the correct behavior as expected from the Weyl law for \( \lambda_n \) as \( n \to \infty \). We note here how some of the results of [10] can be extended as a consequence of Theorem 2.1.

Let \( f \) be a function in the class \( \mathcal{S}_J \) such that \( f(1) = f'(1) = 0 \). We define a generalized Riesz mean by

\[
R_f(t) := \sum_j f(t\lambda_j)\theta(1-t\lambda_j),
\]

where \( \theta(x) = 1 \) if \( x > 0 \) and zero otherwise and \( f \) satisfies the hypotheses of Theorem 2.1.

For simplicity we consider the case \( \alpha = 0 \), which can always be arranged by shifting \( \lambda_j \to \lambda_j + \alpha/\beta \).

**Corollary 4.1** If in addition to the assumptions of Theorem 2.5 the relations (2.12) hold, then

\[
\frac{d}{dt} t^{2\frac{\gamma}{\beta}} R_f(t) \leq 0.
\]

In [10] the Riesz means

\[
R_\rho(z) := \sum_j (z-\lambda_j)^\rho
\]

for \( \rho > 1 \) have been studied for the Dirichlet problem. We obtain these means from Corollary 4.1 choosing \( f(\lambda) = (1-\lambda)^\rho \) and putting \( z = 1/t \). We therefore have

**Corollary 4.2** Suppose that the pair \((H,G)\) satisfy the relations (2.12). Then

\[
\frac{R_\rho(z)}{z^{\rho+2\gamma/\beta}}
\]

is a nondecreasing function for \( 0 < z < \inf \sigma_{ess}(H) \).

Formula (4.5) is identical in form to an inequality in [10]; the constant \( 2\gamma/\beta \) has simply replaced \( d/2 \) in the earlier article. As a consequence we obtain a Weyl-sharp bound,

\[
\frac{\lambda_n}{\lambda_k} \leq \left( \frac{2(\beta + \gamma)^{1+\frac{\beta}{\gamma}}}{\beta(\beta + 2\gamma)^{\frac{\beta}{\gamma}}} \right) \left( \frac{n}{k} \right)^{\frac{\beta}{\gamma}},
\]

provided that \( n \geq \left( 1 + \frac{2\gamma}{\beta} \right) k \). (In case the constant \( \alpha \) has not been set to 0, the averages \( \lambda_{n,k} \) on the left side are both replaced by \( \lambda_{n,k} + \alpha/\beta \).)
5 Application to the Dirichlet Laplacian

For the Dirichlet Laplacian \( H = -\Delta_D \) on a domain \( \Omega \) in \( \mathbb{R}^d \) such that \( H \) has only eigenvalues, we put \( G = x_i \), the multiplication operator by a suitable Cartesian coordinate. As shown for example in [12] we then have \( \alpha = 0 \), \( \beta = \frac{d}{2} \) and \( \gamma = 1 \). For the Dirichlet Laplacian the inequality (2.10) of our main Theorem 2.5 reads

\[
\sum_{j=1}^{n} \left( f(\lambda_j) + \frac{2}{d} \lambda_j f'(\lambda_j) \right) \leq n \left( f(\lambda_n) + \frac{1}{2} f'(\lambda_n)(\lambda_{n+1} - \lambda_n) \right).
\]

We claim that all estimates of the form (5.1) are sharp in the semiclassical limit. Indeed, recall that according to the Weyl law, on any bounded domain the semiclassical limit of the eigenvalue \( \lambda_n \) is given by

\[
\lambda_n \sim C_d \left( \frac{n}{V} \right)^{\frac{2}{d}}
\]

as \( n \to \infty \), where \( V \) denotes the volume of \( \Omega \). In terms of the counting function \( N(\lambda) \) this is equivalent to

\[
N(\lambda) \sim C_d^{-d/2} V^{d/2}. \tag{5.2}
\]

Now, for any function \( f \), we have

\[
\sum_{j=1}^{N(\lambda)} f(\lambda_j) = N(\lambda) f(\lambda) - \int_{0}^{\lambda} f'(t) N(t) \, dt
\]

\[
\sim C_d^{-d/2} V \left( f(\lambda) - \int_{0}^{\lambda} f'(t) t^{d/2} \, dt \right),
\]

from which it easily follows that

\[
\sum_{j=1}^{N(\lambda)} \left( f(\lambda_j) + \frac{2}{d} \lambda_j f'(\lambda_j) - f(\lambda_n) \right)
\]

\[
\sim C_d^{-d/2} V \left( 2 \frac{d}{d}^{d/2+1} f'(\lambda) - \int_{0}^{\lambda} \left( 1 + 2 \frac{d}{d} f'(t)  t^{d/2} + 2 \frac{d}{d} f''(t)  t^{d/2+1} \right) \, dt \right)
\]

\[
= 2 \frac{d}{d} C_d^{-d/2} V \left( \lambda^{d/2+1} f'(\lambda) - \int_{0}^{\lambda} (f'(t) t^{d/2+1})' \, dt \right)
\]

\[
\sim o(N(\lambda)f(\lambda)).
\]

Consequently, Theorem 3.4 for the moments

\[
S_n(r) = \left( 1 + \frac{2r}{d} \right) \frac{1}{n} \sum_{j=1}^{n} \lambda_j^r
\]

is sharp. Since the semiclassical limit of \( S_n(r)^{1/2} \) does not depend on \( r \) and is given by

\[
S_n(r)^{1/2} \sim C_d \left( \frac{n}{V} \right)^{\frac{1}{2}}
\]
as \( n \to \infty \), phase-space bounds on \( S_n(r) \) for \( r \leq 1 \) follow from the Berezin-Li-Yau bound \([3][14][18]\) for \( r = 1 \) by the monotonicity property \((3.4)\).

We can further refine Theorem \((3.4)\) for the Dirichlet Laplacian by exploiting the right side of inequality \((2.10)\) of Theorem \((2.5)\) or respectively Corollary \((2.7)\) in order to relate the arithmetic and geometric means of eigenvalues to the sizes of the eigenvalue gaps. We begin by choosing

\[
f_z(\lambda) = \lambda^p - pz^p \ln \lambda + pz^p \ln z - z^p
\]

for \( 0 < p \leq 3 \), which corresponds to the choice \( q = 0 \) in \((3.3)\). Therefore \( f_z(\lambda) \) in \((5.5)\) satisfies the assumptions of Theorem \((2.5)\) and Corollary \((2.7)\) for all \( z \) in an interval of the form \([\lambda_n, \Lambda_n(p)]\) for some \( \Lambda_n(p) \) which is determined by the condition \( f_z'(\lambda_n) + f_z''(\lambda_n)(\lambda_{n+1} - \lambda_n) \geq 0 \). Defining

\[
\gamma_n := \frac{\lambda_{n+1} - \lambda_n}{\lambda_n}
\]

we find

\[
\Lambda_n(p)^p = \begin{cases} 
\frac{\lambda_n^p}{\lambda_{n-1}^p} & \text{if } \gamma_n < 1 \\
\Lambda_n(p) & \text{otherwise.}
\end{cases}
\]

Defining

\[
F_n(z) := npz^p \ln z - nz^p - npz^p \ln(e^2 G_n) + nS_n(p)
\]

and

\[
\tilde{F}_n(z) := nf_z(\lambda_n) + \frac{1}{2} f_z'(\lambda_n)(\lambda_{n+1} - \lambda_n),
\]

we have

\[
F_n(z) \leq \tilde{F}_n(z)
\]

for all \( z \in [\lambda_n, \Lambda_n(p)] \). We see that \( \tilde{F}_n(z) \) has a global minimum at \( \tilde{z} = \lambda_n e^{\gamma_n/2} \leq \Lambda_n(p) \). As the global minimum of \( F_n(z) \) is below the global minimum of \( \tilde{F}_n(z) \), we obtain the inequality

\[
S_n(p) - (e^2 G_n)^p \leq \lambda_n^p \left(1 + \frac{p\gamma_n}{2} - e^{\gamma_n/2}\right).
\]

We note that the left side of \((5.6)\) can be bounded above by \(-\frac{1}{3}(\frac{p}{2})^2\) which yields an explicit upper bound on the gap \( \lambda_{n+1} - \lambda_n \). However, we find it more convenient to optimize the inequality

\[
z^{-p}F_n(z) \leq z^{-p} \tilde{F}_n(z)
\]

with respect to \( z \). The right side has then a global minimum at \( \tilde{z} = \lambda_n(1 + \frac{p\gamma_n}{2})^{\frac{1}{p}} \leq \Lambda_n(p) \), while the left side has its global minimum at \( z = S_n(p)^{\frac{1}{p}} \). After taking the exponential on both sides we therefore obtain the inequality

\[
\frac{\lambda_{n+1}^{\frac{1}{p}}}{(e^2 G_n)^{\frac{1}{p}}} \leq \left(1 + \frac{p\gamma_n}{2} - e^{\gamma_n/2}\right). \tag{5.7}
\]

Extending the above discussion to all pairs \((p, q)\) of Theorem \((3.4)\) we obtain the following refinement for the Dirichlet Laplacian:
Theorem 5.1 Let $p > q > 0$ such that $q \leq \min(1, p)$ and $p + q \leq 3$. Then for all $n$ we have
\[
S_n(p)^{\frac{1}{2p}} \left( 1 + \frac{p}{2} \gamma_n \right)^{\frac{1}{2}} \leq S_n(q)^{\frac{1}{2q}} \left( 1 + \frac{q}{2} \gamma_n \right)^{\frac{1}{2}}.
\] (5.8)
In particular, for $0 < p \leq 1$ the function
\[
p \mapsto S_n(p)^{\frac{1}{2p}} \left( 1 + \frac{p}{2} \gamma_n \right)^{\frac{1}{2}}
\]
is nonincreasing.

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