A NOTE ON THE NUMBER OF REGIONS IN A LINE ARRANGEMENT

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Abstract. For an arrangement of $n$ lines in the real projective plane, we denote by $f$ the number of regions into which the real projective plane is divided by the lines. Using Bojanowski’s inequality, we establish a new lower bound for $f$. In particular, we show that if no more than $\frac{2}{3}n$ lines intersect at any point, then $f \geq \frac{1}{6}n^2$.

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1. Introduction

Let $\mathcal{L}$ be an arrangement of $n \geq 2$ lines in the real projective plane $\mathbb{RP}^2$ and let $m$ denote the maximum number of lines from $\mathcal{L}$ intersecting at one point. The lines from $\mathcal{L}$ divide $\mathbb{RP}^2$ into polygonal regions which are the connected components of the complement of the union of the lines. Denote the number of regions by $f$. The question we are interested in is: how many regions can be obtained (under all possible arrangements $\mathcal{L}$ of $n$ lines)?

Below we collect some known lower bounds for $f$ in terms of $n$ and $m$.

- $f \geq 2n - 2$, if $m < n$, Grünbaum [5];
- $f \geq 3n - 6$, if $m \leq n - 2$, Grünbaum [5];
- $f \geq m(n + 1 - m)$, Arnol’d [1];
- $f \geq \frac{n(n-1)}{2m+1}$, if $m > 2$, Arnol’d [1];
- $f \geq (m+1)(n-m)$, Arnol’d [1] and Purdy [9];
- $f \geq 2\left(\frac{n^2-n+2m}{m+1}\right)$, Shnurnikov [10];
- $f \geq \frac{(3m-10)n^2+(m^2-6m+12)n}{m^2+4m-18} + 1$, if $5 \leq m < n - 2$, Shnurnikov [11].

In this paper we use Bojanowski’s inequality [2] to establish a new lower bound for $f$. The main result states that:

**Theorem 1.** Let $\mathcal{L}$ be an arrangement of $n$ lines in the real projective plane such that $m \leq \frac{2}{3}n$. Then

$$f \geq \frac{(m+2)n^2 + (3m-6)n}{6m} + 1 \geq \frac{1}{6}n^2.$$
2. Bounds for Number of Regions

For an arrangement of lines \( \mathcal{L} \) in the projective plane we denote by \( t_k, 2 \leq k \leq n - 1 \), the number of intersection points where exactly \( k \) lines of the arrangement are incident. The following are some known relations for values of \( t_k \).

- \( t_2 \geq 3 + \sum_{k \geq 4} (k - 3) t_k \), Melchior [7];
- \( t_2 \geq \frac{6}{7} n \) for \( n \geq 8 \), Csima and Sawyer [3];
- \( t_2 + \frac{4}{3} t_3 \geq \sum_{k \geq 5} (2k - 9) t_k \), if \( t_{n-1} = t_{n-2} = 0 \), Hirzebruch [6];
- \( t_2 + \frac{4}{3} t_3 \geq \sum_{k \geq 5} (\frac{1}{3} k^2 - k) t_k \), if \( t_k = 0 \) for \( k > \frac{2}{3} n \), Bojanowski [2];
- \( t_2 \geq \frac{1}{2} n \) and \( t_2 \geq 3\lfloor \frac{1}{4} n \rfloor \) for sufficiently large, even and odd \( n \), respectively, Green and Tao [4].

Maybe it is worth to mention here that both Bojanowski [2] and Hirzebruch [6] inequalities hold for arrangements of complex lines in the complex projective plane and consequently, they also hold for arrangements of lines in the real projective plane. To the best of our knowledge, Bojanowski’s inequality [2] is the strongest known inequality for line arrangements with \( m \leq \frac{2}{4} n \) [8].

**Proof of Theorem 1.** Let \( \mathcal{L} \) be an arrangement of \( n \) lines. If we add lines one by one, then the number of new regions created by each line is equal to the number of intersection points with previously added lines. In this process, a point with \( k \) lines passing through it is intersected \( k - 1 \) times.

Thus, the number of regions, including 1 for the first line, is

\[
f = 1 + \sum_{k=2}^{m} (k-1) t_k.
\]

Note that (1) can be obtained by using the fact that the Euler characteristic of the real projective plane is 1. The number of pairs of lines in \( \mathcal{L} \) is equal to \( \frac{n(n-1)}{2} \). In a projective plane, every pair of lines intersects at exactly one point, and if \( k \) lines meet at a point, we get \( k(k-1)/2 \) of such pairs which cross at that point. Since \( t_k = 0 \) for \( k > m \), we obtain

\[
n(n-1) = \sum_{k=2}^{m} k(k-1) t_k.
\]

Suppose we are given an inequality

\[
\sum_{k=2}^{m} \alpha_k t_k \geq \alpha_0
\]

where \( \alpha_0, \alpha_2, \alpha_3, \cdots, \alpha_m \) are some real numbers, and suppose that for some \( c_1, c_2 > 0 \) the inequality

\[
c_1 k(k-1) + c_2 \alpha_k \leq k - 1
\]

is satisfied for all \( 2 \leq k \leq m \). Multiply both sides of (4) by \( t_k \) and sum up for \( k = 2, 3, \cdots, m \) to obtain

\[
c_1 \sum_{k=2}^{m} k(k-1) t_k + c_2 \sum_{k=2}^{m} \alpha_k t_k \leq \sum_{k=2}^{m} (k - 1) t_k
\]

since \( t_k \geq 0 \). This is equivalent to

\[
c_1 n(n-1) + c_2 \sum_{k=2}^{m} \alpha_k t_k \leq f - 1.
\]

Using (3) and (5) and the fact that \( c_2 > 0 \), we obtain

\[
f \geq c_1 n(n-1) + c_2 \alpha_0 + 1
\]
for positive $c_1, c_2$, satisfying (1). For $m \leq 2/3n$ we use Bojanowski’s inequality (2)
\[ t_2 + \frac{3}{4}t_3 + \sum_{k \geq 5} \left( k - \frac{1}{4} k^2 \right) t_k \geq n \]
in the form (3) to obtain the following
\[ \alpha_0 = n, \quad \alpha_2 = 1, \quad \alpha_3 = \frac{3}{4}, \quad \alpha_4 = 0, \quad \alpha_k = \left( k - \frac{1}{4} k^2 \right) \text{ for } k \geq 5. \]

From (4) we get
\[ 1 \geq 2c_1 + c_2, \quad 2 \geq 6c_1 + \frac{3}{4}c_2, \quad 3 \geq 12c_1, \text{ for } k = 2, 3, \text{ and } 4, \text{ respectively,} \]
\[ 0 \geq c_1 k(k - 1) + c_2 \left( k - \frac{1}{4} k^2 \right) - (k - 1) \text{ for } 5 \leq k \leq m. \]

For $m \geq 2$, let us take the positive numbers
\[ c_1 = \frac{m + 2}{6m}, \quad c_2 = \frac{2(m - 1)}{3m}. \]

Now we need to check these inequalities for $2 \leq k \leq m$ and for the given $c_1, c_2$. The first three are easy to check, so we verify the last one for $5 \leq k \leq m$. Thus,
\[ c_1 k(k - 1) + c_2 \left( k - \frac{1}{4} k^2 \right) - (k - 1) = \frac{0.5(k - 2)(k - m)}{m} \leq 0, \]
because $k \leq m$ and $k \geq 5$. So, we obtain (6) for the given $c_1, c_2$, and hence, the inequality of the theorem. \[ \square \]

Note that lower bounds on $f$ in the form (6) were obtained by Shnurnikov in (11). In (11) he applied Hirzebruch’s inequality (6) to obtain the result mentioned in Section 1.

It is natural to ask under which assumptions the inequality of Theorem 1 is stronger than previously known inequalities. The inequality in Theorem 1 is quadratic in $n$. So, it suffices to compare it to those inequalities mentioned in Section 1 that are quadratic in $n$ for some function $m(n)$. In particular, the results of Arnol’d (1) and Purdy (9) become quadratic in $n$ if $m(n) = \frac{n}{p}$ where $p$ is a real number greater than 1. A simple calculation shows that Theorem 1 is weaker than those inequalities when $p \in (3 - \sqrt{3}, 3 + \sqrt{3})$ and it is also weaker than Shnurnikov (10) when $m \leq 5$. On the other hand, Theorem 1 is stronger than all the inequalities mentioned in Section 1 whenever $7 \leq m \leq \frac{n}{5}$ and for $m = 6$ we have equality with Shnurnikov (11).

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