Thermal entanglement in the nanotubular system Na$_2$V$_3$O$_7$

T. Vértesi$^1$ and E. Bene$^2$

1 Institute of Nuclear Research of the Hungarian Academy of Sciences, H-4001 Debrecen, P.O. Box 51, Hungary
2 Institute of Chemistry, Chemical Research Center, Hungarian Academy of Sciences, Budapest, Hungary

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Macroscopic entanglement witnesses have been put forward recently to reveal nonlocal quantum correlations between individual constituents of the solid at nonzero temperatures. Here we apply a recently proposed universal entanglement witness, the magnetic susceptibility [New J. Phys. 7, 258 (2005)] for the estimation of the critical temperature $T_c$ in the nanotubular system Na$_2$V$_3$O$_7$ below which thermal entanglement is present. As a result of an analysis based on the experimental data for dc-magnetic susceptibility, we show that $T_c \approx 365$ K, which is approximately three times higher than the critical temperature corresponding to the bipartite entanglement.

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I. Introduction

Entanglement lies in the heart of quantum mechanics, revealing the existence of nonlocal correlations that cannot be described by classical physics alone \[1\]. Because of such strong correlations over and above the classical one, entangled states have been recognized as a crucial resource for quantum information processing \[2\]. Thus far, experimental demonstrations of entanglement have been confined yet mostly to atomic scales. This is due to the subtle nature of quantum entanglement, since extending the size of the physical object increases the number of constituents, and the number of degrees of freedom that can interact with the environment, which is responsible for reducing and eventually destroying entanglement in the system.

However, it was demonstrated recently that entanglement can affect the macroscopic properties (such as heat capacity or magnetic susceptibility) of an insulating magnetic compound LiHo$_x$Y$_{1-x}$F$_4$ \[3\], but at a very low (below 1 K) temperature. The basic reason for it is that magnetic susceptibility is proportional to the two-site correlations between the spins in the magnetic solid, which can be higher than the one allowed by classical physics alone, when entanglement is present in the material as well. In the light of the result of Ghosh et al \[3\] that entanglement can have a significant effect in the macroscopic world, the question arises whether we can experience entanglement at higher temperatures as well.

As a next step, Brukner et al \[4\] showed that for uncoupled dimers at zero magnetic field the magnetic susceptibility serves as an entanglement witness (EW), i.e., an observable which can distinguish between entangled and separable states in a quantum system \[4\]. It has been demonstrated \[4\] that the experimental data on magnetic susceptibility in the spin-1/2 alternating bond antiferromagnet CN (Cu(NO$_3$)$_2$·5D$_2$O) \[5\] implies the presence of entanglement below the temperature 5 Kelvin.

On the other hand, Bose and Tribedi \[5\] showed, based on energy considerations \[8\], that the magnetic susceptibility as an EW can be applied to compounds which can be modeled as a collection of independent clusters of $N$ spins, when the Hamiltonian is isotropic in spin space. As a special case, Bose and Tribedi considered the $S = 1/2$ polyoxovanadate AFM compound V12, which can interact treated as a collection of independent $S = 1/2$ tetramers and estimated that the critical temperature below which entanglement exists in the system is $T_c \approx 25.4$ K.

Other thermodynamical observables, such as internal energy, magnetization, or heat capacity \[4, 8, 9, 10, 11, 12\], have also been proposed for EW but either they rely on a model Hamiltonian of the system (and usually not a directly measurable quantity) or are applicable only for relatively low temperatures \[12\].

However, recently Wieśniak et al \[12\] constructed an EW based on the sum of magnetic susceptibilities measured along the three orthogonal spatial directions. This entanglement witness is applicable to systems without full knowledge of the specific model of the solid. The separability criterion of Wieśniak et al \[12\] for $N$ spin-$l$ particles reads as follows:

\[
\bar{\chi} = \chi_x + \chi_y + \chi_z \geq C \frac{1}{T^6}, \tag{1}
\]

with $C = (g \mu_B)^2 N/k$, where $g$ is the $g$-factor, $\mu_B$ is the Bohr magneton, $k$ is the Boltzmann constant, and $N$ denotes the number of particles in the volume. Further in the right-hand side of (1), $\bar{\chi}$ denotes the sum of magnetic susceptibilities per volume measured along the orthogonal $x, y,$ and $z$ directions. In the isotropic case for spin-1/2 particles inequality \[1\] simplifies to the inequality

\[
\chi_x \geq \frac{1}{T^6}. \tag{2}
\]

Therefore, if $\chi_x < \frac{1}{T^6}$, the isotropic solid state system contains entanglement between the individual spin-1/2 systems. Note, that $\chi_x$ entanglement witness in \[2\] is the same as the one in Ref. \[1\], although that has been introduced only for uncoupled dimers using a simpler argumentation. Since the entanglement witness $\bar{\chi}$ of Wieśniak...
et al.\cite{14} is completely general, the entanglement criterion (2) can be applied for a large class of quantum spin systems, as it neither rely on the existence of small spin clusters or on the full knowledge of the system’s Hamiltonian.

The low-dimensional quantum magnets have recently been in the focus of both experimental and theoretical research. A large class of these materials are the spin-$1/2$ quantum magnets with chain, ladder, or planar geometries. In this respect new possibilities emerged, when Millet et al.\cite{15} recognized that the variation of some two-dimensional compounds results in a nanotubular structure, and as a result the first transition-metal oxide-based nanotubular system Na$_2$V$_3$O$_7$ was synthesized. In this system the VO$_2$ pyramids share edges and corners to form a nanotubular structure with Na atoms located inside each individual tube. The ab initio analysis in Ref.\cite{16} showed that Na$_2$V$_3$O$_7$ can be described as formed by nanotubes consisting of weakly coupled nine-site rings.

The main purpose of the present paper is to demonstrate that the entanglement witness $\chi_2$ in criterion (2) implies entangled states in the system Na$_2$V$_3$O$_7$ at the room temperature range, and we estimate the critical temperature $T_{c\text{exp}} \approx 365$ K. The validity of this result is also supported by additional calculations. In particular we find that neglecting the inter-ring couplings as a first approximation, the inequality $|\langle S_i S_j \rangle| \leq 1/4$, where $i,j$ denotes spin sites within the ring, may indicate pairwise entanglement. Based on the parameters of the model Hamiltonian of the system Na$_2$V$_3$O$_7$ which are derived from ab initio molecular structure calculations\cite{16}, we calculate the critical temperature corresponding to $|\langle S_i S_j \rangle| = 1/4$, and obtain $T_{c\text{pair}} \approx 123$ K, below which pairwise entanglement is present in the system. The three times smaller temperature bound for the bipartite entanglement indicates that a large portion of entanglement is stored in correlations between more than two spins within the nine-site ring. Therefore we conclude that the entanglement above the temperature $T_{c\text{pair}}$ is multipartite in nature. To account for this effect we analyze the symmetric trimer (described by $S = 1/2$ Heisenberg antiferromagnetic (AFM) isotropic Hamiltonian), since by small temperatures the nine-spin cluster model of the Na$_2$V$_3$O$_7$ system corresponds to a three-site ring. Although the symmetric trimer is known not to have pairwise entanglement both at $T = 0$ and at finite temperatures\cite{17}, we find multipartite entanglement witnessed by criterion (2) in the system. We argue that the "missing entanglement" in the case of the symmetric trimer, not explored by the $|\langle S_i S_j \rangle| \leq 1/4$ condition, appears in the nanotubular system Na$_2$V$_3$O$_7$ as well, and the existence of this multipartite entanglement causes the large difference in the values of critical temperatures $T_{c\text{exp}}$ and $T_{c\text{pair}}$. Further, we claim that due to the weakly coupled nine-site rings with respect to the intraring couplings, the entangled state itself in the nanotubular system is shared with a good approximation among the nine spins within a ring. Therefore the entangled state is confined to the rings of the nanotube, and does not extend to macroscopic sizes. This fact may partially explain why the entanglement in this system persists up to such high temperatures.

II. Entanglement witnesses based on two-site spin correlation functions

When a system is in its thermal equilibrium under a certain temperature $T$, it can be described by the density operator $\rho(T) = Z^{-1} \exp(-\beta H)$, where $H$ is the Hamiltonian, $Z$ is the partition function, and $\beta = 1/kT$. A thermal state remains entangled up to a critical temperature $T_c$ above which the state becomes separable, i.e., the amount of entanglement vanishes.

In order to measure the thermal entanglement based on the thermal density matrix we need to know both the eigenvalues and eigenvectors of the Hamiltonian. On the other hand, bulk properties of solids such as such as internal energy $U = -Z^{-1} \partial Z/\partial \beta$, magnetization $\langle M_z \rangle = -(Z\beta)^{-1} \partial Z/\partial \beta$, or magnetic susceptibility $\chi_z = \partial \langle M_z \rangle/\partial \beta$, can be derived from the partition function alone; thus these quantities are defined merely by the eigenvalues of the system $H$.

The latter relationship between magnetization and magnetic susceptibility has given rise to the notion of thermodynamical complementarity\cite{18}. On the other hand, if $[M,H] = 0$ the equality

$$\chi_z = \frac{C_z}{N} \Delta^2 M_z = \frac{C_z}{N T} \left( \langle M^2_z \rangle - \langle M_z \rangle^2 \right)$$  \hspace{1cm} (3)

holds linking the fluctuation of magnetization to the macroscopic observable $\chi_z$. Here $\Delta^2 M_z$ denotes the variance of the magnetization, while the constant $C_z$ has the same meaning as in inequality (1) and $N$ is the number of particles in the volume. On the other hand, noticing that $M_z = \sum_{i=1}^N S_z^i$, formula (3) also connects the magnetic susceptibility to its microscopic origin in the form of spin-spin correlation functions.

Now we represent two entanglement witnesses from the literature, based on spin-spin correlation functions, which are crucial to construct macroscopic entanglement witness from the magnetic susceptibility through equation (3). This promise to reveal thermal entanglement in macroscopic systems even when the eigenvectors of the system’s Hamiltonian are not known. For any separable states of $N$ spins of length $l$, that is, for any convex sum of $N$ product states, $\rho = \sum_k \omega_k \rho_k^1 \otimes \rho_k^2 \otimes \ldots \otimes \rho_k^N$, one has\cite{11}

$$\sum_{i=1}^N |\langle S_i S_{i+1} \rangle| \leq Nl^2,$$

which especially for two qubits (at sites $i$ and $j$) yields
the inequality

\[ |\langle S_i S_j \rangle| \leq \frac{1}{4}. \tag{5} \]

The other inequality for separable states of \(N\) spins of length \(l\) is based on entanglement detection applying the sum uncertainty relations \cite{14, 10}.

\[
\Delta^2 M_x + \Delta^2 M_y + \Delta^2 M_z \geq N l. \tag{6}
\]

Now we restrict ourselves to Heisenberg spin-1/2 lattices at zero field with isotropic, but in general inhomogeneous coupling constants \(J_{ij}\),

\[
H = \sum_{i<j} J_{ij} S_i S_j. \tag{7}
\]

Due to the isotropy of the Hamiltonian in spin space, the magnetization vanishes in all three orthogonal directions, and thus \(\Delta^2 M_\alpha = \langle M_\alpha^2 \rangle\) for \(\alpha = x, y, z\). Using the relation \(M_\alpha = \sum_{i=1}^{N} S_i^\alpha\) and assuming spin-1/2 lattices Eq. \(6\) simplifies to

\[
\sum_{i,j=1}^{N} \langle S_i S_j \rangle \geq \frac{N}{2}, \tag{8}
\]

From now on for brevity we denote the entanglement witness corresponding to criterion \(3\) by EW1, and by EW2 of the criterion \(5\). Note that combining inequality \(6\) with \(3\) implies the criterion \(1\) (see Wiecki et al \cite{14}). On the other hand for the isotropic Heisenberg Hamiltonian \(H\) magnetization \(M\) commutes with the Hamiltonian \(H\), \([M, H] = 0\), thus the witness \(\chi_z\) in criterion \(2\) is valid. As a consequence the power of EW2 in \(5\) to signal entanglement is equivalent with the witness \(\chi_z\) in criterion \(2\).

Wang and Zanardi \cite{20} showed that for the Heisenberg Hamiltonian \(7\) the concurrence \(21\), which gives a measure of pairwise entanglement between qubits at site \(i\) and site \(j\), becomes \(C_{ij} = 1/2 \max \{0, 2 \langle S_i^x S_j^x \rangle - \langle S_i^y S_j^y \rangle - 1/4\}\). The concurrence \(C_{ij} = 0\) corresponds to an unentangled state. Regarding to the isometry of the model the \(\alpha = x, y, z\) components of the spin-spin correlation functions are all equal, and assuming that \(\langle S_i S_j \rangle\) is negative, we arrive at \(C_{ij} = 1/2 \max \{0, -\langle S_i S_j \rangle - 1/4\}\). Thus, whenever \(\langle S_i S_j \rangle \leq 0\) holds, pairwise entanglement between qubits \(i\) and \(j\) vanishes for \(\langle S_i S_j \rangle \leq 1/4\). Since the condition above is just the separability criterion \(6\), this implies that EW1 detects pairwise entanglement between any two qubits of a system with Hamiltonian \(7\) when the corresponding spin-spin correlation has negative value.

Now we compare the power of EW1 and EW2 detecting entanglement in a three-qubit ring interacting via the AFM isotropic Heisenberg Hamiltonian and show that EW2 can be more effective. For the AFM symmetric trimer the Hamiltonian is given by \(H = J (S_i S_{i+2} + S_{i+1} S_{i+3})\), with \(J > 0\) thus criterion \(5\) gives \(\sum_{i,j=1}^{3} \langle S_i S_j \rangle = 3 \langle S_i^2 \rangle + 2 \langle S_i S_{i+1} \rangle \geq \frac{3}{4}\) and consequently \(\langle S_i S_{i+1} \rangle \geq -1/8\), for which the AFM interaction implies the relation \(\langle S_i S_{i+1} \rangle \leq 1/8\) for separable states. Thus in this case the condition \(\langle S_i S_{i+1} \rangle \leq 1/8\) of EW2 obviously implies a lower bound on the separability than the condition \(\langle S_i S_{i+1} \rangle \leq 1/4\) of EW1. Since direct calculation shows that for the ground state of the symmetric trimer \(\langle S_i S_{i+1} \rangle = -1/4\), EW2 indicates entangled ground state, whereas EW1 could not reveal entanglement. The result that there exists entanglement in the symmetric trimer over the two-site entanglement is supported by the definition of 1-tangle \(22\), which quantifies the entanglement of one site with the rest of the chain at \(T = 0\), and is given as \(\gamma_1 = 4 \det (\rho_1)\), where \(\rho_1 = \left( \langle I + \sum_{\alpha} \langle S_i^\alpha \rangle \sigma_{\alpha} \rangle \right)^2\) is the one-site reduced density matrix, and \(\sigma_{\alpha}\) are the Pauli matrices with \(\alpha = x, y, z\). In terms of spin expectation values \(\langle S_i^\alpha \rangle\) \(\gamma_1\) takes the form \(\gamma_1 = 1 - 4 \sum_{\alpha} \langle S_i^\alpha \rangle^2\). The vanishing of \(\gamma_1\) implies that there is no entanglement in the ground state, i.e. the state is factorized. The isotropy of the symmetric trimer \(\langle S_i^\alpha \rangle = \langle S_i^\beta \rangle = \langle S_i^\gamma \rangle = 0\) entails that \(\gamma_1 = 1\), which means that indeed the ground state of the symmetric trimer is entangled. Although in the above analysis we treated merely the three qubit isotropic Heisenberg ring, we conjecture that the results obtained above can be generalized to systems with general Heisenberg Hamiltonian \(7\) for cluster size \(N > 3\) and for different but positive \(J_{ij}\) values as well. We intend to support this conjecture in the next sections with the analysis of the quantum magnet \(Na_2V_3O_7\), which can be considered as a nine-leg nanotubular system, and is well described by the Hamiltonian \(7\) with \(J_{ij} > 0\) exchange constants.

### III. Magnetic susceptibility of the quantum magnet \(Na_2V_3O_7\)

\(Na_2V_3O_7\) is a material whose structure may be considered as composed by nanotubes oriented along the \(c\) axis of the crystal lattice. A detailed description of the structure of this compound can be found in Ref. \cite{15}. Following the discussion and ab initio results given in Ref. \cite{16} we construct the model Hamiltonian of the system \(Na_2V_3O_7\), which is given by

\[
H = \sum_{i=1}^{9} J_1 S_i S_{i+1} + J_2 S_i S_{i+2}, \tag{9}
\]

where periodic boundary conditions were imposed \((S_{i+9} = S_i)\), and the weak inter-ring couplings as a first approximation were neglected in the model. Thus in this model we treat the system as the collection of independent nine-site rings. The optimal parameters, neglecting also the differences in between the nearest neighbor (NN) and in between the next nearest neighbor (NNN) intraring interactions, are given by \(J_1 = 200\) K and \(J_2 = 140\) K \cite{16, 10}. Due to the structure of the material \cite{15, 10} the ring
is partially frustrated, where $J^i_j = 0$ for $i = 1, 4, 7$ and $J^i_j = J_2$ for the indices $i = 2, 3, 5, 6, 8, 9$. The two kinds of couplings in the frustrated model are depicted in Fig. 1.

The ground state of the model is fourfold degenerate, composed of two doublets, corresponding to the right and left chirality [22]. Since this model is invariant under translations $i \rightarrow i + 3$ it corresponds by low energies to a three-site ring. In Fig. 2 we display the experimentally observed inverse magnetic susceptibility of the Na$_2$V$_3$O$_7$ at temperatures between 1.9 K and 315 K (data is taken from Ref. [24]). As a comparison we also reproduced and plotted in Fig. 2 the calculated inverse magnetic susceptibility for the partially frustrated model from Ref. [16] with couplings $J_1 = 200$ K and $J_2 = 140$ K in the temperature range $T = 0 - 600$ K. The temperature dependence of the magnetic susceptibility was obtained in Ref. [16] from the thermal state $\rho(T)$ by exact diagonalization of the Hamiltonian [4]. The nice agreement between the experimental and calculated values supports the validity of the ab initio derived parameters $J_1$ and $J_2$ in the model.

IV. Calculation of thermal entanglement in the system Na$_2$V$_3$O$_7$

In this section we give a lower bound on the critical temperature below which the thermal state of the system Na$_2$V$_3$O$_7$ is entangled. The two estimations for this temperature limit will be calculated based on entanglement witnesses EW1 and EW2 (in criterions [4] and [5], respectively). EW1 is able to reveal pairwise entanglement between individual spins, while EW2 may reveal residual entanglement which is not detected by EW1, as it was demonstrated in the example of symmetric trimer.

The critical temperature related to the pairwise entanglement and detected by EW1 will be called $T^{\text{pair}}$, while the possibly higher critical temperature indicated by EW2 is denoted by $T^{\text{exp}}$. The relation [3] between magnetic susceptibility and two-site spin correlation functions links EW2 to the entanglement witness $\chi_z$ expressed by condition [2]. Therefore the critical temperature revealed by the witness $\chi_z$ also applies to EW2, and in the following it is jointly denoted by $T^{\text{exp}}$.

Since the magnetic susceptibility of the system Na$_2$V$_3$O$_7$ is available from Sec. III, we next calculate the critical temperature $T^{\text{exp}}$ signaled by $\chi_z$. Applica-

![FIG. 1: Illustration of exchange couplings $J_1$, $J_2$ in the spin-$1/2$ nine-leg ring model of the nanotube Na$_2$V$_3$O$_7$.](image1)

![FIG. 2: Temperature dependence of the inverse magnetic susceptibility in units of mol/emu of the system Na$_2$V$_3$O$_7$. The filled dots correspond to the experimental data from Ref. [24]. The solid curve is the calculated curve by reproducing the results of Ref. [16], and the dotted line represents the macroscopic entanglement witness $\chi_z$.](image2)

tion of criterion [2] implies that entanglement is present in the system, when the inverse magnetic susceptibility per volume satisfies the inequality

$$\chi_z^{-1} > \frac{T}{C},$$

(10)

where the concrete value of $C = (g \mu_B)^2 N/k$ for the material Na$_2$V$_3$O$_7$ is given by $C \approx 5.58$ emu K/mol. The equality in (10) is represented by the dotted line in Fig. 2. The intersection point of the solid curve (calculated inverse susceptibility versus $T$) and the dotted line provides the estimate of the critical temperature $T^{\text{exp}} \approx 365$ K. One may note that experimental data was measured up to 315 K and therefore the estimation was obtained by the calculated curve, which is not so reliable. However, data of inverse susceptibility is larger than the right-hand side of (10) up to $T = 315$ K, thus $T^{\text{exp}}$ is definitely higher than 315 K, and therefore the experimental data in this temperature range cannot be described fully without taking into account entanglement.

Now relying on the assumption that the partially frustrated model [4] describes well the real physical system, we calculate a lower bound on the presence of entanglement, witnessed by EW1 of criterion [4]. If for any two sites $(i, j)$ of the ring of nine qubits the inequality $|\langle S_i S_j \rangle| > 1/4$ holds, then the state of the pair $(i, j)$, described by the reduced density matrix of those two, is entangled. Furthermore, regarding the result of Ref. [20],
neighboring of the model system, there are two inequivalent neighbors (j = 2, 9) and second neighboring (j = 8, 3) sites. The solid curves correspond to nearest neighboring sites, while the chain curves represent spin-spin correlations between second neighbors. The horizontal dotted line represents the value of 1/4 for the EW1. The intersection point of this line with the solid curve corresponding to j = 2 gives the critical temperature \( T_{\text{pair}} \approx 123 \, \text{K} \). The critical temperature is designated by a vertical solid line which defines the temperature range (left from the solid line) with pairwise entanglement in the system Na\(_2\)V\(_3\)O\(_7\).

We have also calculated the spin-spin correlation for the \( T = 0 \) ensemble, where the ground state density matrix was assumed to be an equal mixture of all eigenstates, corresponding to the fourfold degenerate ground state. For the next nearest neighboring (NNN) spin pairs the values \( \langle S_i S_k \rangle = -0.011 \) and \( \langle S_i S_3 \rangle = +0.1807 \) were obtained. As it can be observed in Fig. 3, these are just the limiting values of these NNN spin-spin correlation functions by \( T = 0 \), which indicates smooth spin-spin correlation functions near \( T = 0 \). The large difference in the values of \( \langle S_i S_k \rangle \) and \( \langle S_i S_3 \rangle \) implies strong partial frustration in the system due to the large coupling \( J_2 \), which diminishes the spin-spin correlation between sites 1 and 8 at \( T = 0 \). Although this effect may reduce the critical temperature pertaining to the pairwise entanglement, qualitatively the result obtained in this section for the Na\(_2\)V\(_3\)O\(_7\) is consistent with the result for the symmetric trimer we arrived at in Sec. II. In the latter case the bound revealed by EW2 (\( \langle \langle S_i S_j \rangle \rangle = 1/8 \)) was half of the bound corresponding to EW1 (\( \langle \langle S_i S_j \rangle \rangle = 1/4 \)). On the other hand, we can calculate the power of EW2 with respect to EW1 in the system Na\(_2\)V\(_3\)O\(_7\) as well. By eye inspection of the spin-spin correlation functions in Fig. 3, the value \( \langle \langle S_i S_j \rangle \rangle \) by the critical temperature \( T_{\text{c}}^{\text{exp}} = 365 \, \text{K} \) approximately yields \( \langle \langle S_i S_j \rangle \rangle = 1/10 \). This demonstrates the fact that despite of the frustration, there exists entanglement in the system Na\(_2\)V\(_3\)O\(_7\) beyond the pairwise one. The residual thermal entanglement, stored in three-spin, four-spin, and higher-spin correlations, could not be detected by the \( \langle \langle S_i S_j \rangle \rangle = 1/4 \) bound of EW1, but was revealed by the macroscopic witness \( \chi_z \) (based on EW2).

V. Conclusion

In conclusion we have shown that the nanotubular system Na\(_2\)V\(_3\)O\(_7\) is an ideal candidate to demonstrate thermal entanglement at the range of room temperature. The magnetic susceptibility as a macroscopic entanglement witness, proposed in Ref. [14] (and based on EW2 in criterion (3)) gives the \( T_{\text{c}}^{\text{exp}} \approx 365 \, \text{K} \) lower bound on the critical temperature beyond which entanglement disappears, while the calculation of EW1 (of criterion (3)) for the model Hamiltonian of the Na\(_2\)V\(_3\)O\(_7\) system revealed pairwise entanglement up to the temperature \( T_{\text{pair}} \approx 123 \, \text{K} \). This result indicates that pairwise entanglement is not the sole entanglement in the system, and the magnetic susceptibility \( \chi_z \) is able to exhaust a probably large portion of residual part of the total entanglement. However we have to mention that the total entanglement in this particular material does not extend to the whole system, but rather is confined to the rings of the tube. Although macroscopic observables, such as magnetic susceptibility, can be explained only on the ground of entangled states, the entangled state itself is not macroscopic but shared among nine spins within a cluster. We emphasize that while the calculation of the temperature bound \( T_{\text{pair}} \) for the pairwise entanglement depends on the validity of the model of the real physical system, the critical temperature \( T_{\text{c}}^{\text{exp}} \) revealed by the entanglement witness \( \chi_z \) does not require any assump-
tion about the explicit values of the exchange-coupling parameters of the system. Moreover, it does not rely on the existence of independent or weakly coupled spin clusters in the material, thus the result for $T_{c_{\exp}}$ is reliable even if our model assumptions do not hold exactly. The existence of large exchange coupling parameters are of great importance in order to demonstrate quantum entanglement at high temperatures in the solid. Since this requirement is fulfilled for a large class of materials, such as some low-dimensional 25 or finite 26 quantum spin systems, we believe that entanglement as a crucial resource for quantum information processing in various types of solids might be harnessed at room temperature range as well.

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