Having the same wild ramification is preserved
by the direct image

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Abstract

Let $S$ be the spectrum of an excellent henselian discrete valuation ring of residue characteristic $p$ and $X$ a separated scheme over $S$ of finite type. Let $\Lambda$ and $\Lambda'$ be finite fields of characteristics $\ell \neq p$ and $\ell' \neq p$ respectively. For elements $F \in K_c(X, \Lambda)$ and $F' \in K_c(X, \Lambda')$ of the Grothendieck groups of constructible sheaves of $\Lambda$-modules and $\Lambda'$-modules on $X$ respectively, we introduce the notion that $F$ and $F'$ have the same wild ramification and prove that this condition is preserved by four of Grothendieck’s six operations except the derived tensor product and $R\text{Hom}$.

Introduction

Let $X$ be a separated scheme over a field $k$ of characteristic $p$ of finite type and $\bar{X}$ a proper normal scheme over $k$ containing $X$ as a dense open subscheme. Let $\mathcal{F}$ and $\mathcal{F}'$ be constructible complexes of $\Lambda$-modules on $X$, where $\Lambda$ is a finite field of characteristic $\ell \neq p$. Deligne-Illusie [I] have given a sufficient condition for $\mathcal{F}$ and $\mathcal{F}'$ to have the same Euler-Poincaré characteristic in terms of wild ramification of $\mathcal{F}$ and $\mathcal{F}'$.

Let $S$ be the spectrum of an excellent henselian discrete valuation ring of residue characteristic $p > 0$ or the spectrum of a field of characteristic $p > 0$. Let $K_c(X, \Lambda)$ be the Grothendieck group of constructible sheaves of $\Lambda$-modules on $X$. Vidal [V1] has extended Deligne-Illusie’s result to the case where $X$ is a separated scheme over $S$ of finite type and where $\mathcal{F}$ and $\mathcal{F}'$ are elements of $K_c(X, \Lambda)$. More precisely, Vidal has defined a subgroup $K_c(X, \Lambda)_I^0 \subset K_c(X, \Lambda)$ called the Grothendieck group of constructible sheaves of virtual wild ramification 0 and proved that $Rf_! : K_c(X, \Lambda) \to K_c(S, \Lambda)$ and $Rf_* : K_c(X, \Lambda) \to K_c(S, \Lambda)_I$ induce $Rf_! : K_c(X, \Lambda)_I^0 \to K_c(S, \Lambda)_I^0$ and $Rf_* : K_c(X, \Lambda)_I^0 \to K_c(S, \Lambda)_I^0$ respectively for the structure morphism $f : X \to S$. This result gives a sufficient condition for $\mathcal{F}$ and $\mathcal{F}'$ to have the same Swan conductor ([V1]). Vidal [V2] has further extended this result to the case where $S$ is the spectrum of an excellent henselian discrete valuation ring of residue characteristic $p$ (possibly $p = 0$) and where $f$ is an $S$-morphism of separated schemes over $S$ of finite type.

In this paper, we give a definition of the Grothendieck group $K_c(X, \Lambda)_I^0 \subset K_c(X, \Lambda)$ of constructible sheaves of $\Lambda$-modules on $X$ of wild ramification 0 in Definition 2.7 along the notion that two constructible complexes have the same wild ramification introduced in [SY] and we prove an analogue of Vidal’s result in [V2] for this group $K_c(X, \Lambda)_I^0$ and an $S$-morphism $f$ of separated schemes over $S$ of finite type. More precisely, the group
of $K_c(X, \Lambda)_0$ is defined to be the subgroup of $K_c(X, \Lambda)$ consisting of the elements which have the same wild ramification with 0. The same wild ramification condition is expressed in terms of the wild ramification of two complexes. The difference from Deligne-Illusie’s condition is that our condition is given in terms of the dimensions of the fixed parts by elements of inertia groups of $p$-power orders instead of the Brauer traces of these elements. The subgroup $K_c(X, \Lambda)_0$ contains Vidal’s subgroup $K_c(X, \Lambda)_0'$ of $K_c(X, \Lambda)$.

Let $\Lambda'$ be another finite field of characteristic $\ell' \neq p$ and let $\Delta_c(\Lambda, \Lambda')$ be the subgroup of $K_c(X, \Lambda) \times K_c(X, \Lambda')$, defined in Definition 2.4 consisting of the elements $(a, b) \in K_c(X, \Lambda) \times K_c(X, \Lambda')$ such that $a$ and $b$ have the same wild ramification. The main theorem of this article is the following:

**Theorem 0.1.** Let $S$ be the spectrum of an excellent strict local henselian discrete valuation ring of residue characteristic $p$ (possibly $p = 0$). Let $f : X \to Y$ be an $S$-morphism of separated schemes over $S$ of finite type. Let $f_1 \times f_1 : K_c(X, \Lambda) \times K_c(X, \Lambda') \to K_c(Y, \Lambda) \times K_c(Y, \Lambda')$ be the morphism induced by $Rf_1 \times Rf_1$. Then $f_1 \times f_1$ induces $f_1 \times f_1 : \Delta_c(X, \Lambda) \to \Delta_c(Y, \Lambda')$.

The analogue of Vidal’s result in [V2] for $K_c(X, \Lambda)_0$ is proved in Corollary 4.2 (iii) as a corollary of this theorem. Theorem 0.1 also leads in Corollary 4.2 to the compatibility of $K_c(X, \Lambda)_0$ with four of Grothendieck’s six operations except the derived tensor product and $R\text{Hom}$ as well as Vidal’s result in [V2]. A partial result of Theorem 0.1, which is under the assumption that $\dim Y \leq 2$, and a similar result for two complexes having the same Artin conductor when restricted to a curve have been obtained by Kato [K].

We describe the construction of this paper. In Section II we recall the Brauer trace. We define the same wild ramification condition in Definition 2.4 and give the definition of $K_c(X, \Lambda)_0$ in Definition 2.7. The proof of Theorem 0.1 is given in Section III In Section IV we give two corollaries of Theorem 0.1 on the compatibility with Grothendieck’s six operations.

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1 Brauer Trace

We briefly recall the definition of the Brauer trace. Let $G$ be a profinite group and $\Lambda$ a finite field of characteristic $\ell$. Let $W(\Lambda)$ be the Witt ring of $\Lambda$. The subgroup of $G$ consisting of $\ell$-regular elements is denoted by $G_{\text{reg}}$. Let $K(\Lambda[G])$ be the Grothendieck group of finite dimensional $\Lambda$-vector spaces with continuous $G$-actions. Let $M$ be an element of $K(\Lambda[G])$. The Brauer trace $\text{Tr}_M^\Lambda : G_{\text{reg}} \to W(\Lambda)$ is a central function of $G_{\text{reg}}$, and if $M$ is the class of a finite dimensional $\Lambda$-vector space with continuous $G$-action, then it is given by $\text{Tr}_M^\Lambda(g) = \sum [\lambda]$ for $g \in G_{\text{reg}}$. Here $\lambda$ runs through every eigenvalue of the action of $g$ on $M$ and $[\lambda]$ denotes the unique lift of $\lambda$ such that $[\lambda]$ is a root of unity of order prime to $\ell$. We note that the Brauer trace is an additive function, and is multiplicative with respect to the elements of $K(\Lambda[G])$.

**Lemma 1.1** (cf. [SY] Lemma 4.1]). Let $M$ be an element of $K(\Lambda[G])$ and $g$ an element of $G$ of $p$-power order for a prime number $p$ different from $\ell$ as an automorphism of $M$. For
every subfield $E$ of the fractional field of $W(\Lambda)$ of finite degree over $\mathbb{Q}$ containing $\text{Tr}^{\text{Br}}_{M}(g)$, we have

$$
\frac{1}{[E : \mathbb{Q}]} \text{Tr}_{E/\mathbb{Q}} \text{Tr}^{\text{Br}}_{M}(g) = \frac{1}{p - 1}(p \cdot \dim M^{g} - \dim M^{g^{p}}).
$$

**Proof.** If $M$ is the class of a finite dimensional $\Lambda$-vector space with continuous $G$-action, the assertion follows from [SY] Lemma 4.1.

Suppose that $M$ is the linear combination over $\mathbb{Z}$ of finitely many classes $\{M_i\}$, of finite dimensional $\Lambda$-vector spaces with continuous $G$-actions. If $E$ contains $\text{Tr}^{\text{Br}}_{M_i}(g)$ for every $i$, then the assertion follows since $\text{Tr}_{E/\mathbb{Q}}$ is an additive function on $E$ and the Brauer trace is additive with respect to the elements of $K(\Lambda[G])$. If $E$ does not contain $\text{Tr}^{\text{Br}}_{M_i}(g)$ for some $i$, take a finite extension $E'$ of $E$ containing $\text{Tr}^{\text{Br}}_{M_i}(g)$ for every $i$. Then we have

$$
\frac{1}{[E' : \mathbb{Q}]} \text{Tr}_{E'/\mathbb{Q}} \text{Tr}^{\text{Br}}_{M}(g) = \frac{1}{p - 1}(p \cdot \dim M^{g} - \dim M^{g^{p}})
$$

by the case where $E$ contains $\text{Tr}^{\text{Br}}_{M_i}(g)$ for every $i$. Since $E$ contains $\text{Tr}^{\text{Br}}_{M}(g)$, the left hand side of (1.2) is equal to that of (1.1). Hence the assertion follows. \hfill \Box

**Lemma 1.2.** Let $M$ and $N$ be elements of $K(\Lambda[G])$ and $K(\Lambda'[G])$ respectively. Let $p$ be a prime number different from $\ell$ and $\ell'$. Let $g$ be an element of $G$ of $p$-power order. Then the following are equivalent:

(i) $p \cdot \dim M^{g^{n}} - \dim M^{g^{n+1}} = p \cdot \dim N^{g^{n}} - \dim N^{g^{n+1}}$ for every $n \in \mathbb{Z}_{\geq 0}$.

(ii) $\dim M^{g^{n}} = \dim N^{g^{n}}$ for every $n \in \mathbb{Z}_{\geq 0}$.

**Proof.** The condition (ii) obviously implies the condition (i).

Suppose that the condition (i) holds. Then we have

$$
p \cdot (\dim M^{g^{n}} - \dim N^{g^{n}}) = \dim M^{g^{n+1}} - \dim N^{g^{n+1}}
$$

for every $n \in \mathbb{Z}_{\geq 0}$. Let $s$ be an integer $\geq 0$ such that $g^{s} = \text{id}_{M}$ and $g^{s} = \text{id}_{N}$. If $n \geq s$, we have $\dim M^{g^{n}} = \dim M^{g^{n+1}} = \dim M$ and $\dim N^{g^{n}} = \dim N^{g^{n+1}} = \dim N$. Hence, by (i), we have $(p - 1) \cdot \dim M = (p - 1) \cdot \dim N$. Since $p - 1 \neq 0$ in $\mathbb{Q}$, we have $\dim M = \dim N$ and hence $\dim M^{g^{n}} = \dim N^{g^{n}}$ for every integer $n \geq s$.

If $0 \leq n < s$, then we have $p^{s-n} \cdot (\dim M^{g^{n}} - \dim N^{g^{n}}) = \dim M - \dim N$ by (1.3). Since $\dim M = \dim N$ and $p^{s-n} \neq 0$ in $\mathbb{Q}$, the assertion follows. \hfill \Box

## 2 Same wild ramification

Let $S$ be an excellent trait of residue characteristic $p$, namely $S$ is the spectrum of an excellent henselian discrete valuation ring of residue characteristic $p$. Note that $p = 0$ is admitted. For example, a complete discrete valuation ring is an excellent henselian discrete valuation ring. The generic point of $S$ is denoted by $\eta$ and the closed point of $S$ is denoted by $s$. 

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Definition 2.1 ([V1], Subsection 2.1, [V2] Section 1). Let $Z$ be a normal connected scheme over $S$ of finite type and $\bar{v}_0$ a geometric generic point of $Z$.

(i) Let $\bar{Z}$ be a normal compactification of $Z \to S$ containing $Z$ as a dense open subscheme, namely $\bar{Z}$ is a proper normal scheme over $S$ containing $Z$ as a dense open subscheme. Let $\bar{z}$ be a geometric point of $\bar{Z}$ and $\bar{v}$ a geometric point of $\bar{Z}(\bar{z}) \times \bar{Z} Z$ lying above $\bar{v}_0$. We define a subset $E_{\bar{z}, \bar{v}}$ of $\pi_1(Z, \bar{v}_0)$ to be the union of the image of $p$-Sylow subgroups of $\pi_1(\bar{Z}(\bar{z}) \times \bar{Z} Z, \bar{v})$, where $\bar{Z}(\bar{z})$ denotes the strict localization of $Z$ at $\bar{z}$.

(ii) We define a subset $E_{\bar{Z}/S, \bar{z}}$ of $\pi_1(Z, \bar{v}_0)$ to be the union of conjugates of $E_{\bar{z}, \bar{v}}$.

(iii) We define a subset $E_{\bar{Z}/S, Z}$ of $\pi_1(Z, \bar{v}_0)$ to be the union of the image of $E_{\bar{Z}/S, \bar{z}}$ for every geometric point $\bar{z}$ of $\bar{Z}$.

(iv) We define a subset $E_{\bar{Z}/S}$ of $\pi_1(Z, \bar{v}_0)$ to be the intersection of $E_{\bar{Z}/S, Z}$ for every normal compactification $\bar{Z}$ of $Z \to S$ containing $Z$ as a dense open subscheme.

(v) Let $\tau$ be the generic point of $Z$. We define a subset $E_{\tau/S}(Z)$ of $E_{\bar{Z}/S}$ to be the intersection of the image of $E_{U/S} \to E_{\bar{Z}/S}$ for every dense open subscheme $U$ of $Z$.

The definition of $E_{\bar{Z}/S, \bar{z}}$ is independent of the choice of the geometric point $\bar{v}$ of $\bar{Z}(\bar{z}) \times \bar{Z} Z$ lying above $\bar{v}_0$. In fact, for two geometric points $\bar{v}_1$ and $\bar{v}_2$ of $\bar{Z}(\bar{z}) \times \bar{Z} Z$ lying above $\bar{v}_0$, the subsets $E_{\bar{z}, \bar{v}_1}$ and $E_{\bar{z}, \bar{v}_2}$ of $\pi_1(Z, \bar{v}_0)$ are conjugate. We note that $E_{\bar{Z}/S, \bar{z}}$, $E_{\bar{Z}/S, Z}$, and $E_{\bar{Z}/S}$ are stable under conjugate.

Let $Q$ be a finite quotient of $\pi_1(Z, \bar{v}_0)$. Let $E_{\bar{Z}/S, \bar{z}}(Q)$, $E_{\bar{Z}/S, Z}(Q)$, $E_{\bar{Z}/S}(Q)$, and $E_{\tau/S}(Q)$ be the images of $E_{\bar{Z}/S, \bar{z}}$, $E_{\bar{Z}/S, Z}$, $E_{\bar{Z}/S}$, and $E_{\tau/S}(Z)$ in $Q$ respectively. Since the category of normal compactifications of $Z \to S$ containing $Z$ as a dense open subscheme is cofiltered and $Q$ is a finite group, there exists a normal compactification $\bar{Z}$ of $Z \to S$ containing $Z$ as a dense open subscheme such that $E_{\bar{Z}/S}(Q) = E_{\bar{Z}/S, Z}(Q)$. Since the category of dense open subschemes of $Z$ is cofiltered and $Q$ is a finite group, there exists a dense open subscheme $U$ of $Z$ such that $E_{\tau/S}(Q)$ is the image of $E_{U/S}$ in $Q$.

Let $X$ be a separated scheme over $S$ of finite type. Let $\Lambda$ and $\Lambda'$ be finite fields of characteristic $\ell \neq p$ and $\ell' \neq p$ respectively. Let $K_c(X, \Lambda)$ be the Grothendieck group of constructible sheaves of $\Lambda$-modules on $X$ and $K_{coh}(X, \Lambda)$ the subgroup of $K_c(X, \Lambda)$ generated by the classes of locally constant constructible sheaves of $\Lambda$-modules on $X$. If $X$ is normal connected and if $\bar{t}_0$ is a geometric generic point of $X$, then the Grothendieck group $K_{coh}(X, \Lambda)$ is equal to the Grothendieck group $K_c(\Lambda[\pi_1(X, \bar{t}_0)])$ of finite dimensional $\Lambda$-vector spaces with continuous $\pi_1(X, \bar{t}_0)$-actions.

Lemma 2.2 ([V1], Lemme 2.1.1)]. Let $f : X \to Y$ be an $S$-morphism of normal connected schemes over $S$ of finite type. Let $\bar{t}_0$ be a geometric generic point of $X$ and let $\bar{u}_0$ be a geometric generic point of $Y$. Let $\phi_f : \pi_1(X, \bar{t}_0) \to \pi_1(Y, f(\bar{t}_0)) \to \pi_1(Y, \bar{u}_0)$ be the composition of the morphism $\pi_1(X, \bar{t}_0) \to \pi_1(Y, f(\bar{t}_0))$ induced by $f$ and an isomorphism $\pi_1(Y, f(\bar{t}_0)) \to \pi_1(Y, \bar{u}_0)$ of fundamental groups. Let $E_{X/S}$ and $E_{Y/S}$ be the subsets of $\pi_1(X, \bar{t}_0)$ and $\pi_1(Y, \bar{u}_0)$ respectively defined in Definition 2.1 (iv).
Lemma 2.5. Let $a, b$ be elements of $K_c(X, \Lambda)$ and $K_c(X, \Lambda')$ respectively. We say that $a$ and $b$ have the same wild ramification if $(a, b) \in \Delta_c(X, \Lambda, \Lambda')$.

Let $(a, b)$ be an element of $K_c(X, \Lambda) \times K_c(X, \Lambda')$. By the induction on the dimension of $X$, there always exists a finite decomposition $X = \coprod_i X_i$ of $X$ into normal connected locally closed subschemes $\{X_i\}$ of $X$ such that $(a|_{X_i}, b|_{X_i}) \in \Delta_c(X_i, \Lambda, \Lambda')$ for every $i$. If $\Lambda = \Lambda'$, then we have $(a, b) \in \Delta_c(X, \Lambda, \Lambda)$ if and only if $(b, a) \in \Delta_c(X, \Lambda, \Lambda)$, and we have $(a, b) \in \Delta_c(X, \Lambda, \Lambda)$ if and only if $(b, a) \in \Delta_c(X, \Lambda, \Lambda)$.

Definition 2.3. Assume that $X$ is normal connected. Let $\bar{t}_0$ be a geometric generic point of $X$ and let $E_{X/S}$ be the subset of $\pi_1(X, \bar{t}_0)$ defined in Definition 2.2 (iv).

(i) We put $A_{X/S} = \prod_{g \in E_{X/S}} \mathbb{Z}$. We define the morphism $\varphi_\Lambda: K_{coh}(X, \Lambda) \to A_{X/S}$ of modules by $\varphi_\Lambda(a) = (\dim a^g)_g$.

(ii) We define a subgroup $\Delta_{coh}(X, \Lambda, \Lambda')$ of $K_{coh}(X, \Lambda) \times K_{coh}(X, \Lambda')$ to be the kernel of the morphism

$$
\varphi_\Lambda - \varphi_{\Lambda'}: K_{coh}(X, \Lambda) \times K_{coh}(X, \Lambda') \to A_{X/S}; (a, b) \mapsto \varphi_\Lambda(a) - \varphi_{\Lambda'}(b).
$$

Let $\bar{t}_0$ and $\bar{t}'_0$ be geometric generic points of $X$. Since the fundamental groups $\pi_1(X, \bar{t}_0)$ and $\pi_1(X, \bar{t}'_0)$ are isomorphic and since $E_{X/S} \subset \pi_1(X, \bar{t}_0)$ and $E_{X/S} \subset \pi_1(X, \bar{t}'_0)$ are isomorphic by the isomorphism of $\pi_1(X, \bar{t}_0)$ and $\pi_1(X, \bar{t}'_0)$, the definition of $\Delta_{coh}(X, \Lambda, \Lambda')$ is independent of the choice of the geometric generic point $\bar{t}_0$ of $X$. By this reason, we sometimes omit the base point $\bar{t}_0$ of $\pi_1(X, \bar{t}_0)$ when we consider $\Delta_{coh}(X, \Lambda, \Lambda')$.

Definition 2.4. Let the notation be as before Lemma 2.2.

(i) We define a subgroup $\Delta_c(X, \Lambda, \Lambda')$ of $K_c(X, \Lambda) \times K_c(X, \Lambda')$ to be the subgroup of $K_c(X, \Lambda) \times K_c(X, \Lambda')$ consisting of the pairs $(a, b)$ such that there exists a finite decomposition $X = \coprod_i X_i$ of $X$ into normal connected locally closed subschemes $\{X_i\}_i$ of $X$ such that $(a|_{X_i}, b|_{X_i}) \in \Delta_{coh}(X_i, \Lambda, \Lambda')$ for every $i$.

(ii) Let $a$ and $b$ be elements of $K_c(X, \Lambda)$ and $K_c(X, \Lambda')$ respectively. We say that $a$ and $b$ have the same wild ramification if $(a, b) \in \Delta_c(X, \Lambda, \Lambda')$.

Lemma 2.5. Let $a$ and $b$ be elements of $K_c(X, \Lambda)$ and $K_c(X, \Lambda')$ respectively.

(i) Assume that $a \in K_{coh}(X, \Lambda)$ and $b \in K_{coh}(X, \Lambda')$. Then the following are equivalent:

(a) $(a, b) \in \Delta_{coh}(X, \Lambda, \Lambda')$.

(b) $a$ and $b$ satisfy the following condition for a normal compactification $\bar{X}$ of $X \to S$ containing $X$ as a dense open subscheme and for every geometric point $\bar{x}$ of $\bar{X}$:
(W) Let $M$ and $N$ be the elements of $K(\Lambda[G])$ and $K(\Lambda'[G])$ corresponding to the pull-backs of $a$ and $b$ to $\tilde{X}(\bar{x}) \times_{\bar{X}} X$ respectively, where $G$ is a finite quotient of the inertia group $I_x = \pi_1(\tilde{X}(\bar{x}) \times_{\bar{X}} X, \bar{t})$ for a geometric point $\bar{t}$ of $\tilde{X}(\bar{x}) \times_{\bar{X}} X$. For every element $g \in G$ of $p$-power order, we have $\dim M^g = \dim N^g$.

(ii) The following are equivalent:

(a) $(a, b) \in \Delta_c(X, \Lambda, \Lambda')$, namely $a$ and $b$ have the same wild ramification.

(b) There exists a finite decomposition $X = \coprod_i X_i$ of $X$ into normal connected locally closed subschemes $\{X_i\}$ of $X$ and normal compactifications $\{\tilde{X}_i\}$ of $\{X_i \to S\}$ containing $\{X_i\}$ as dense open subschemes respectively such that $(a|_{X_i}, b|_{X_i}) \in K_{coh}(X_i, \Lambda) \times K_{coh}(X_i, \Lambda')$ for every $i$ and that $a|_{X_i}$ and $b|_{X_i}$ satisfy the condition (W) in (i) (b) for every $i$ and every geometric point $\bar{x}$ of $X_i$.

Proof. The assertion (ii) follows from (i). We prove (i). Let $Q$ be a finite quotient of $\pi_1(X)$ through which $\pi_1(X)$ acts $a$ and $b$. Take a normal compactification $\tilde{X}$ of $X \to S$ containing $X$ as a dense open subscheme such that $E_{X/S}(Q) = E_{X/S,X}(Q)$. Then (a) implies (b) for the compactification $\tilde{X}$. Suppose that the condition (b) holds. Let $\bar{x}$ be a geometric point of $\tilde{X}$ and $\bar{t}_0$ a geometric generic point of $X$. Suppose that $\bar{t}$ is lying above a geometric point $\bar{t}_0$ of $X$. Then $E_{X/S,\bar{x}}$ is equal to the union of conjugates of the union of images of $p$-Sylow subgroups of $I_\bar{x}$ by an isomorphism $\pi_1(X, \bar{t}_0) \to \pi_1(X, \bar{t}_0)$. Hence (b) implies (a).

For an $S$-morphism $f : X \to Y$ of separated schemes over $S$ of finite type, let $f_1 : K_c(X, \Lambda) \to K_c(Y, \Lambda)'$ and $f^* : K_c(Y, \Lambda) \to K_c(X, \Lambda)$ denote the morphisms induced by the functors $Rf_1$ and $f^*$ respectively.

**Proposition 2.6.** Let $f : X \to Y$ be an $S$-morphism of separated schemes over $S$ of finite type.

(i) $f^* \times f^* : K_c(Y, \Lambda) \times K_c(Y, \Lambda') \to K_c(X, \Lambda) \times K_c(X, \Lambda')$ induces $f^* \times f^* : \Delta_c(Y, \Lambda, \Lambda') \to \Delta_c(X, \Lambda, \Lambda')$.

(ii) If $f$ is quasi-finite, then $f_1 \times f_1 : K_c(X, \Lambda) \times K_c(X, \Lambda') \to K_c(Y, \Lambda) \times K_c(Y, \Lambda')$ induces $f_1 \times f_1 : \Delta_c(X, \Lambda, \Lambda') \to \Delta_c(Y, \Lambda, \Lambda')$.

Proof. The proof goes similarly as the proof of [V1] Proposition 2.3.3 (i), (ii).

(i) Let $(a, b)$ be an element of $\Delta_c(Y, \Lambda, \Lambda')$. Take a finite decomposition $Y = \coprod_i Y_i$ of $Y$ into normal connected locally closed subschemes $\{Y_i\}$ of $Y$ such that $(a|_{Y_i}, b|_{Y_i}) \in \Delta_{coh}(Y_i, \Lambda, \Lambda')$ for every $i$. We put $X_i = f^{-1}(Y_i)$ and let $X_i = \coprod_j X_{ij}$ be a finite decomposition of $X_i$ into normal connected locally closed subschemes $\{X_{ij}\}$ of $X_i$ for $i$. Let $f_{ij} : X_{ij} \to Y_i$ be the morphism induced by $f$ for $i$ and $j$. Then, by Lemma 2.2 (i), we have $(f_{ij}^*(a|_{Y_i}), f_{ij}^*(b|_{Y_i})) \in \Delta_{coh}(X_{ij}, \Lambda, \Lambda')$ for every $i$ and $j$. Hence the assertion follows.

(ii) Let $(a, b)$ be an element of $\Delta_c(X, \Lambda, \Lambda')$. By decomposing $Y$ into the disjoint union of connected components of $Y$, we may assume that $Y$ is connected. By Zariski’s main theorem, it is sufficient to prove the assertion in the case where $f$ is an open immersion or a finite morphism. Suppose that $f$ is an open immersion. Take the decomposition
\[ Y = X \amalg (Y \setminus X). \] Since \((f_\text{a})|_X = a\) and \((f_b)|_X = b\), it follows that \((f_\text{a})|_X\) and \((f_b)|_X\) have the same wild ramification. Since \((f_\text{a})|_{Y \setminus X} = 0\) and \((f_b)|_{Y \setminus X} = 0\), by taking a finite decomposition of \(Y \setminus X\) into normal connected locally closed subschemes of \(Y \setminus X\), we see that \((f_\text{a})|_{Y \setminus X}\) and \((f_b)|_{Y \setminus X}\) have the same wild ramification. Hence the assertion follows if \(f\) is an open immersion.

Assume that \(f\) is a finite morphism. Then we have \(f_\text{a} = f_\text{a}\) and \(f_b = f_b\). By Lemma 2.5 (ii) and the induction on the dimension of \(Y\), it is sufficient to prove that there exists a normal dense open subscheme \(U\) of \(Y\) such that \(((f_\text{a})_U, (f_b)_U) \in \Delta_\text{coh}(U, \Lambda, \Lambda')\). By shrinking \(Y\) to a normal affine dense open subscheme if necessary, we may assume that \(Y\) is normal affine and thus that \(X\) is affine. Since \((f_\text{a}, f_b)\) are constructible, we may assume that \((f_\text{a}, f_b) \in K_\text{coh}(Y, \Lambda) \times K_\text{coh}(Y, \Lambda')\) by shrinking \(Y\) to a dense open subscheme. Let \(X = \coprod X_i\) be a finite decomposition of \(X\) into normal connected locally closed subschemes \(X_i\) of \(X\) such that \((a|_{X_i}, b|_{X_i}) \in \Delta_\text{coh}(X_i, \Lambda, \Lambda')\) for every \(i\). Let \(i_{X_i}: X_i \to X\) be the immersion for \(i\). Since \(X = \text{normal}, \text{a normal dense open subscheme if necessary, we may assume that } X \text{ is normal and thus that } X \text{ is affine. Since } f_\text{a} \text{ and } f_b \text{ are constructible, we may assume that } (f_\text{a}, f_b) \in \Delta_\text{coh}(X, \Lambda, \Lambda')\).

Further, we have \((f_i_{X_i}) = X\) be the immersion for \(i\). Since \(a = \sum_i i_{X_i}(a|_{X_i})\) and \(b = \sum_i i_{X_i}(b|_{X_i})\), we may replace \(f, a,\) and \(b\) by \(f \circ i_{X_i}, a|_{X_i},\) and \(b|_{X_i}\), respectively. Hence we may assume that \(X\) is normal connected and that \((a, b) \in \Delta_\text{coh}(X, \Lambda, \Lambda')\).

Since \(f\) is factorized to the composition of a finite surjective morphism and a closed immersion, we may assume that \(f\) is a finite surjective morphism or a closed immersion. Suppose that \(f\) is a closed immersion. Then we have \((f_\text{a})|_X = a\) and \((f_b)|_X = b\). Further we have \((f_\text{a})|_{Y \setminus X} = 0\) and \((f_b)|_{Y \setminus X} = 0\). Hence the assertion follows similarly as the case where \(f\) is an open immersion. Therefore we may assume that \(f\) is a finite surjective morphism. Let \(F_X\) and \(F_Y\) be the function fields of \(X\) and \(Y\) respectively. Let \(F_x\) be the separable closure of \(F_Y\) in \(F_X\). By replacing \(X\) by the normalization of \(Y\) in \(F_x\) if necessary, we may assume that \(f\) is generically étale. By shrinking \(Y\) to a dense open subscheme if necessary, we may assume that \(f\) is finite étale. Then \(f_\text{a}\) and \(f_b\) are \(a \otimes_{\Lambda[\pi_1(X)]} \Lambda[\pi_1(Y)]\) and \(b \otimes_{\Lambda[\pi_1(X)]} \Lambda'[\pi_1(Y)]\) respectively. Further, by Lemma 2.2 (ii), we have \(E_{X/S} = E_{Y/S} \cap \pi_1(X)\).

Let \(g\) be an element of \(E_{Y/S}\) and let \(R\) be a representative system of \(\pi_1(Y)/\pi_1(X)\). By [5 Exercise 18.2], we have \(\text{Tr}_{f_\text{a}}^\text{Br}(g) = \sum_{r \in R, rgr^{-1} \in \pi_1(X)} \text{Tr}_{a}^\text{Br}(rgr^{-1})\) and \(\text{Tr}_{f_b}^\text{Br}(g) = \sum_{r \in R, rgr^{-1} \in \pi_1(X)} \text{Tr}_{b}^\text{Br}(rgr^{-1})\). Let \(r\) be an element of \(R\) such that \(rgr^{-1} \in \pi_1(X)\). Since \(E_{X/S} = E_{Y/S} \cap \pi_1(X)\), we have \(rgr^{-1} \in E_{X/S}\). Let \(E\) be a subfield of the fractional field of \(W(\Lambda)\) of finite degree over \(Q\) containing \(\text{Tr}_{a}^\text{Br}(rgr^{-1})\) for every \(r \in R\) such that \(rgr^{-1} \in \pi_1(X)\) and let \(E'\) be a subfield of the fractional field of \(W(\Lambda')\) of finite degree over \(Q\) containing \(\text{Tr}_{b}^\text{Br}(rgr^{-1})\) for every \(r \in R\) such that \(rgr^{-1} \in \pi_1(X)\). Since \(\dim a^{\overline{g}} = \dim b^{\overline{g}}\) for every \(g' \in E_{X/S}\), we have

\[
\frac{1}{[E : Q]} \text{Tr}_{E/Q} \text{Tr}_{a}^\text{Br}(rgr^{-1}) = \frac{1}{[E' : Q]} \text{Tr}_{E'/Q} \text{Tr}_{b}^\text{Br}(rgr^{-1})
\]

for every \(r \in R\) such that \(rgr^{-1} \in \pi_1(X)\) by Lemma 2.1. Since \([E : Q]\) and \([E' : Q]\) are positive integers and since \(\text{Tr}_{E/Q}\) and \(\text{Tr}_{E'/Q}\) are additive functions, we have

\[
\frac{1}{[E : Q]} \text{Tr}_{E/Q} \text{Tr}_{f_\text{a}}^\text{Br}(g) = \frac{1}{[E' : Q]} \text{Tr}_{E'/Q} \text{Tr}_{f_b}^\text{Br}(g).
\]

Hence the assertion follows by Lemma 2.1 and Lemma 2.2.

\[\square\]
Definition 2.7 (cf. [V1, Définition 2.3.1]). We define a subgroup $K_c(X, \Lambda)_0$ of $K_c(X, \Lambda)$ to be the subgroup of $K_c(X, \Lambda)$ consisting the elements $a \in K_c(X, \Lambda)$ such that $a$ and $0$ have the same wild ramification with $\Lambda' = \Lambda$. We call the subgroup $K_c(X, \Lambda)_0$ the Grothendieck group of constructible sheaves of $\Lambda$-modules on $X$ of wild ramification 0.

Corollary 2.8 (cf. [V1 Proposition 2.3.3]). Let $f : X \to Y$ be an $S$-morphism of separated schemes over $S$ of finite type.

(i) The functor $f^* : K_c(Y, \Lambda) \to K_c(X, \Lambda)$ induces $f^* : K_c(Y, \Lambda)_0 \to K_c(X, \Lambda)_0$.

(ii) If $f$ is quasi-finite, then the functor $f^* : K_c(X, \Lambda) \to K_c(Y, \Lambda)$ induces $f^* : K_c(X, \Lambda)_0 \to K_c(Y, \Lambda)_0$.

(iii) Let $X = \bigsqcup_i X_i$ be a finite decomposition into locally closed subschemes $\{X_i\}_i$ of $X$. Then we have $K_c(X, \Lambda)_0 = \bigoplus_i K_c(X_i, \Lambda)_0$.

Proof. Since $\Delta_c(X, \Lambda, \Lambda) \cap (K_c(X, \Lambda) \times \{0\})$ and $\Delta_c(Y, \Lambda, \Lambda) \cap (K_c(Y, \Lambda) \times \{0\})$ are isomorphic to $K_c(X, \Lambda)_0$ and $K_c(Y, \Lambda)_0$ respectively by the first projections, the assertions (i) and (ii) follow from Proposition 2.6 (i) and (ii) respectively. We prove (iii). Let $a$ be an element of $K_c(X, \Lambda)$ and let $i_{X_i} : X_i \to X$ be the immersion for $i$. Since $a = \sum_i i_{X_i!}(a|_{X_i})$ and $i_{X_i}$ is quasi-finite for every $i$, the assertion follows by (i) and (ii).

3 Proof of Theorem 0.1

Let the notation be as in Section 2. We devote this section to the proof of Theorem 0.1. The proof goes similarly as Vidal’s proof of [V2 Théorème 0.1].

Let $(a, b)$ be an element of $\Delta_c(X, \Lambda, \Lambda')$ and assume that $S$ is strict local. Let $f : X \to Y$ be an $S$-morphism of separated schemes over $S$ of finite type. Take the decompositions $X = X_\eta \amalg X_s$ and $Y = Y_\eta \amalg Y_s$, and decompose $f$ into $f_\eta \oplus f_s$, where $f_\eta : X_\eta \to Y_\eta$ and $f_s : X_s \to Y_s$ are induced by $f$. Let $i_{Y_\eta} : Y_\eta \to Y$ and $i_{Y_s} : Y_s \to Y$ be immersions. Since $f_\eta|a = i_{Y_\eta!}f_\eta|a(X_\eta) + i_{Y_s!}f_s|a(X_s)$ and $f_s|b = i_{Y_\eta!}f_\eta|b(X_\eta) + i_{Y_s!}f_s|b(X_s)$, we may replace $f$ by $f_\eta$ or $f_s$ by Proposition 2.6.

Let $z$ be the generic point $\eta$ of $S$ or the closed point $s$ of $S$. Since the assertion is local, we may assume that $Y = \text{Spec} B$ is affine. Further, by taking a finite decomposition of $X$ into locally closed affine subschemes of $X$ if necessary and applying Proposition 2.6 (i), we may assume that $X = \text{Spec} A$ is affine. We identify $A$ with $B[t_1, \ldots, t_d]/I$ for some $d \in \mathbb{Z}_{\geq 0}$ and an ideal $I$ of $B[t_1, \ldots, t_d]$. Since the assertion follows if $f$ is a closed immersion by Proposition 2.6 (ii), by taking the factorization $B \to B[t_1] \to \cdots \to B[t_1, \ldots, t_d] \to B[t_1, \ldots, t_d]/I = A$ of $f^* : B \to A$, we may assume that $f$ is (smooth) of relative dimension 1. Since $f_\eta a$ and $f_s b$ are constructible, as in the proof of Proposition 2.6 (ii), we may assume that $Y$ is normal connected and that $(f_\eta a, f_s b) \in K_{\text{coh}}(Y, \Lambda) \times K_{\text{coh}}(Y, \Lambda')$. Further, as in the proof of Proposition 2.6 (ii), we may assume that $X$ is normal connected and that $(a, b) \in \Delta_{\text{coh}}(X, \Lambda, \Lambda')$.

Let $p : V \to X$ be a galois étale covering trivializing $a$ and $b$ with galois group $H$. By shrinking $Y$ to a dense open subscheme if necessary, we may assume that $(f \circ p)|\Lambda \in K_{\text{coh}}(Y, \Lambda)$ and $(f \circ p)_!\Lambda' \in K_{\text{coh}}(Y, \Lambda')$. Let $\tau$ be the generic point of $Y$. We apply the following proposition proved by Vidal in [V2 Section 3]:


Proposition 3.1 ([V2 Corollary 3.0.5 and the proof of Théorème 0.1]). Let \( f: X \to Y \) be a \( z \)-morphism of relative dimension \( \leq 1 \) of normal affine connected schemes over \( z \) of finite type. Let \( a \) be an element of \( K_{coh}(X, \Lambda) \) and let \( p: V \to X \) be a galois étale covering trivializing \( a \) with galois group \( H \). Let \( E_\ell \) be a finite extension of \( \mathbb{Q}_\ell \) whose integer ring has the residue field \( \Lambda \). Assume that \( f_*a \in K_{coh}(Y, \Lambda) \) and \( (f \circ p)_!\Lambda \in K_{coh}(Y, \Lambda) \). Let \( g \) be an element of \( E_\ell/S(Y) \) and let \( P \) be the subgroup of \( \pi_1(Y) \) generated by \( g \). Let \( P' \) be a pro-p-subgroup of \( \text{Gal}(\overline{\tau}/\tau) \) whose image by the canonical morphism \( \text{Gal}(\overline{\tau}/\tau) \to \pi_1(Y) \) is \( P \). We put \( H' = P' \times H \). Then we have

\[
\text{Tr}^\text{Br}_{f_*a}(g) = \frac{1}{|H|} \sum_{h' \in H' \atop \text{Tr}_{\text{RT}_{c}}(V_{\bar{\tau}, E_\lambda})(h') \times \text{Tr}^\text{Br}_{a}(p_{H}(h'))},
\]

where \( p_{P'}: H' \to P' \) and \( p_{H}: H' \to H \) are projections and \( g' \in P' \) is a lift of \( g \).

Let \( Q \) be a finite quotient of \( \pi_1(Y) \) through which \( \pi_1(Y) \) acts on \( f_*a, f_*b, (f \circ p)_!\Lambda \), and \( (f \circ p)_!N' \). By shrinking \( Y \) to a dense open subscheme if necessary, we may assume that \( E_{Y/S}(Q) = E_{\tau/S}(Q) \). Hence we may replace \( E_{Y/S} \) by \( E_{\tau/S} \). Let \( g \) be an element of \( E_{Y/S} \). With the notation in Proposition 3.1, we have

\[
\text{Tr}^\text{Br}_{f_*a}(g) = \frac{1}{|H|} \sum_{h' \in H' \atop \text{Tr}_{\text{RT}_{c}}(V_{\bar{\tau}, E_\lambda})(h') \times \text{Tr}^\text{Br}_{a}(p_{H}(h'))},
\]

Let \( E'_\lambda \) be a finite extension of \( \mathbb{Q}_\ell \) whose integer ring has the residue field \( \Lambda' \). With the notation in Proposition 3.1, we have

\[
\text{Tr}^\text{Br}_{f_*b}(g) = \frac{1}{|H|} \sum_{h' \in H' \atop \text{Tr}_{\text{RT}_{c}}(V_{\bar{\tau}, E'_\lambda})(h') \times \text{Tr}^\text{Br}_{b}(p_{H}(h'))}.
\]

Further we apply the following proposition proved by Vidal in [V2 Section 3]:

Proposition 3.2 ([V2 Corollaire 3.0.7 and the proof of Théorème 0.1]). Let the notation and the assumption be as in Proposition 3.1. Let \( h' \) be an element of \( H' \) such that \( \text{Tr}_{\text{RT}_{c}}(V_{\bar{\tau}, E_\lambda})(h') \neq 0 \). Then \( p_{H}(h') \) is the image of an element of \( E_{X/S} \) by the surjection \( \pi_1(X) \to H \).

Let \( E \) be a subfield of the fractional field of \( W(\Lambda) \) of finite degree over \( \mathbb{Q} \) containing \( \text{Tr}^\text{Br}_{a}(p_{H}(h')) \) for every \( h' \in H' \) such that \( p_{P'}(h') = g' \) and \( \text{Tr}_{\text{RT}_{c}}(V_{\bar{\tau}, E_\lambda})(h') \neq 0 \). Further, let \( E' \) be a subfield of the fractional field of \( W(\Lambda') \) of finite degree over \( \mathbb{Q} \) containing \( \text{Tr}^\text{Br}_{b}(p_{H}(h')) \) for every \( h' \in H' \) such that \( p_{P'}(h') = g' \) and \( \text{Tr}_{\text{RT}_{c}}(V_{\bar{\tau}, E'_\lambda})(h') \neq 0 \). By [V2 Proposition 2.2.1] and [V2 Proposition 2.3.1], the trace \( \text{Tr}_{\text{RT}_{c}}(V_{\bar{\tau}, E_\lambda})(h') \) is an integer independent of \( \ell \neq p \) and the extension \( E_\lambda \) of \( \mathbb{Q}_\ell \) for every \( h' \in H' \). By Lemma 3.1 and Proposition 3.2, we have

\[
\frac{1}{[E: \mathbb{Q}] \text{Tr}_{E/\mathbb{Q}} \text{Tr}^\text{Br}_{a}(p_{K}(h'))} = \frac{1}{[E': \mathbb{Q}] \text{Tr}_{E'/\mathbb{Q}} \text{Tr}^\text{Br}_{b}(p_{K}(h'))}.
\]
for every $h' \in K'$ such that $p_{P'}(h') = q'$ and that $\text{Tr}_{R_{\mathbf{E}}}(\gamma_{s_{\mathbf{E}}}) (h') \neq 0$. Since $[E : Q]$, $[E' : Q]$, and $|H|$ are positive integers, we have

$$\frac{1}{[E : Q]} \text{Tr}_{E/Q} \text{Tr}_{f_{\mathbf{E}}} (g) = \frac{1}{[E' : Q]} \text{Tr}_{E'/Q} \text{Tr}_{f_{\mathbf{E}}} (g).$$

Hence the assertion follows by Lemma 1.1 and Lemma 1.2.

### 4 Compatibility with functors

Let $S$ be an excellent trait of residue characteristic $p$. Let $\Lambda$ and $\Lambda'$ be finite fields of characteristic $\ell \neq p$ and $\ell' \neq p$ respectively. We prove two corollaries of Theorem 0.1.

For an $S$-morphism $f : X \to Y$ of separated schemes over $S$ of finite type, let $f_* : K_c(X, \Lambda) \to K_c(Y, \Lambda)$ and $f^! : K_c(Y, \Lambda) \to K_c(X, \Lambda)$ denote the morphisms induced by the functors $Rf_*$ and $Rf^!$ respectively. Let $D_X : K_c(X, \Lambda) \to K_c(X, \Lambda)$ be the morphism induced by the dualizing functor $\text{[SGA4.2 4.2]}$.

**Corollary 4.1.** Let $f : X \to Y$ be an $S$-morphism of separated schemes over $S$ of finite type.

(i) $f^* \times f^* : K_c(Y, \Lambda) \times K_c(Y, \Lambda') \to K_c(X, \Lambda) \times K_c(X, \Lambda')$ induces $f^* \times f^* : \Delta_c(Y, \Lambda, \Lambda') \to \Delta_c(X, \Lambda, \Lambda')$.

(ii) $f_* \times f_* : K_c(X, \Lambda) \times K_c(X, \Lambda') \to K_c(Y, \Lambda) \times K_c(Y, \Lambda')$ induces $f_* \times f_* : \Delta_c(X, \Lambda, \Lambda') \to \Delta_c(Y, \Lambda, \Lambda')$.

(iii) $f_1 \times f_1 : K_c(X, \Lambda) \times K_c(X, \Lambda') \to K_c(Y, \Lambda) \times K_c(Y, \Lambda')$ induces $f_1 \times f_1 : \Delta_c(X, \Lambda, \Lambda') \to \Delta_c(Y, \Lambda, \Lambda')$.

(iv) $f^! \times f^! : K_c(Y, \Lambda) \times K_c(Y, \Lambda') \to K_c(X, \Lambda) \times K_c(X, \Lambda')$ induces $f^! \times f^! : \Delta_c(Y, \Lambda, \Lambda') \to \Delta_c(X, \Lambda, \Lambda')$.

(v) $D_X \times D_X : K_c(X, \Lambda) \times K_c(X, \Lambda') \to K_c(X, \Lambda) \times K_c(X, \Lambda')$ induces $D_X \times D_X : \Delta_c(X, \Lambda, \Lambda') \to \Delta_c(X, \Lambda, \Lambda')$.

**Proof.** We have already proved (i) in Proposition 2.6 (i).

We prove (ii) and (iii). By [1 Théorème 1.1], if $S$ is strict local then we have $f_1 = f_*$ as a morphism of Grothendieck groups. Let $(a, b)$ be an element of $\Delta_c(X, \Lambda, \Lambda')$ and let $\pi : S^{sh} \to S$ be the strict localization. We use the same notation $\pi$ for the base change $X_{S^{sh}} \to X$ of $X$ by $\pi : S^{sh} \to S$. Since $(\pi^* a, \pi^* b) \in \Delta_c(X_{S^{sh}}, \Lambda, \Lambda')$ and $\pi^* \circ f_1 = f_1 \circ \pi^*$, we have $(\pi^* f_1 a, \pi^* f_1 b) \in \Delta_c(Y_{S^{sh}}, \Lambda, \Lambda')$ by Theorem 0.1. Since $\pi^* \circ f_* = f_* \circ \pi^* = f_1 \circ \pi^*$, we have $(\pi^* f_* a, \pi^* f_* b) \in \Delta_c(Y_{S^{sh}}, \Lambda, \Lambda')$. Since the assertion is étale local, the assertions (ii) and (iii) follow.

We prove (iv). Since the assertion is local, we may assume that $X = \text{Spec} \ A$ and $Y = \text{Spec} \ B$ are affine. We identify $A$ with $B[t_1, \ldots, t_n]/I$ for some $d \in \mathbb{Z}_{\geq 0}$ and an ideal $I$ of $B[t_1, \ldots, t_n]$. Then the morphism $h$ is factorized to the composition of a closed immersion $X \to A^d_Y$ and the projection $A^d_Y \to Y$. Hence we may assume that $f$ is a closed immersion.
or a smooth morphism of relative dimension $d$. Suppose that $f$ is a closed immersion and let $j : Y \setminus X \to Y$ denote the open immersion. Since we have $f^! = f^* - f^* \circ j_* \circ j^*$, the assertion follows by (i) and (ii). Suppose that $f$ is a smooth morphism of relative dimension $d$. Since we have an isomorphism $t_f : f^*(d)[2d] \to Rf^!$ ([D, 3.2]), the assertion follows by (i).

We prove (v). Let $h : X \to S$ be the structure morphism. Let $(a,b)$ be an element of $\Delta_c(X, \Lambda, \Lambda')$. By devissage, we may assume that $(a,b) \in \Delta_{coh}(X, \Lambda, \Lambda')$ and that $(h'^1\Lambda, h'^1\Lambda') \in K_{coh}(S, \Lambda) \times K_{coh}(S, \Lambda')$. Since $(\Lambda, \Lambda') \in \Delta_{coh}(X, \Lambda, \Lambda')$, we have $(h'^1\Lambda, h'^1\Lambda') \in \Delta_{coh}(X, \Lambda, \Lambda')$ by (iv). Since $Tr^0_{D_X(a)}(g)$ for $g \in E_{X/S}$ is the product of $Tr^{Br}_{h'^1\Lambda}(g)$ and the conjugate of $Tr^{Br}_{a}(g)$ and similarly for $Tr^{Br}_{D_X(b)}(g)$ for $g \in E_{X/S}$, it is sufficient to prove that $Tr^{Br}_{h'^1\Lambda}(g)$ and $Tr^{Br}_{h'^1\Lambda}(g)$ are integers for every $g \in E_{X/S}$.

As in the proof of (iv), we may assume that $h$ is a closed immersion or a smooth morphism of relative dimension $d$ for some $d \in \mathbb{Z}_{\geq 0}$. Suppose that $h$ is a closed immersion. Let $j : S \setminus X \to S$ be the open immersion. Since $h'^1 = h^* - h^* \circ j_* \circ j^*$, the assertion follows by applying [S, Exercise 18.2] to $j_*\Lambda$ and $j_*\Lambda'$. Suppose that $h$ is a smooth morphism of relative dimension $d$. Since we have the isomorphism $t_h : h^*(d)[2d] \to Rh^!$, the assertion follows.

**Corollary 4.2** (cf. [V2, Corollaire 0.2]). Let $f : X \to Y$ be an $S$-morphism of separated schemes over $S$ of finite type.

(i) $f^* : K_c(Y, \Lambda) \to K_c(X, \Lambda)$ induces $f^* : K_c(Y, \Lambda)_0 \to K_c(X, \Lambda)_0$.

(ii) $f_* : K_c(X, \Lambda) \to K_c(Y, \Lambda)$ induces $f_* : K_c(X, \Lambda)_0 \to K_c(Y, \Lambda)_0$.

(iii) $f_! : K_c(X, \Lambda) \to K_c(Y, \Lambda)$ induces $f_! : K_c(X, \Lambda)_0 \to K_c(Y, \Lambda)_0$.

(iv) $f^! : K_c(Y, \Lambda) \to K_c(X, \Lambda)$ induces $f^! : K_c(Y, \Lambda)_0 \to K_c(X, \Lambda)_0$.

(v) $D_X : K_c(X, \Lambda) \to K_c(X, \Lambda)$ induces $D_X : K_c(X, \Lambda)_0 \to K_c(X, \Lambda)_0$.

**Proof.** Since $\Delta_c(X, \Lambda, \Lambda) \cap (K_c(X, \Lambda) \times \{0\})$ and $\Delta_c(Y, \Lambda, \Lambda) \cap (K_c(Y, \Lambda) \times \{0\})$ are isomorphic to $K_c(X, \Lambda)_0$ and $K_c(Y, \Lambda)_0$ respectively by the first projections, the assertions (i)–(v) follow from Corollary 4.1 (i)–(v) respectively.

Vidal’s subgroup $K^0_c(X, \Lambda)_t$ of $K_c(X, \Lambda)$ consists of the elements $a \in K_c(X, \Lambda)$ such that there exists a finite decomposition $X = \bigsqcup_i X_i$ of $X$ into normal connected locally closed subschemes $\{X_i\}_i$ of $X$ such that $a|_{X_i} \in K_{coh}(X_i, \Lambda)$ for every $i$ and that $Tr^0_{a|X_i}(g) = 0$ for every $i$ and $g \in E_{X_i/S}$ ([V1, Définition 2.3.1]). For Vidal’s subgroup $K^0_c(X, \Lambda)_t$, the same assertions in Corollary 4.2 hold and further the compatibility with the derived tensor product and $R\mathcal{H}om$ hold by [V2, Corollaire 0.2]. Namely, $K^0_c(X, \Lambda)_t$ is an ideal of $K_c(X, \Lambda)$ with respect to the multiplication induced by the derived tensor product and $R\mathcal{H}om : K^0_c(X, \Lambda) \times K_c(X, \Lambda) \to K_c(X, \Lambda)$ induces $R\mathcal{H}om : K^0_c(X, \Lambda)_t \times K_c(X, \Lambda) \to K_c(X, \Lambda)_t$ and $R\mathcal{H}om : K_c(X, \Lambda) \times K^0_c(X, \Lambda)_t \to K_c(X, \Lambda)_t$. However, the compatibility with the derived tensor product or $R\mathcal{H}om$ does not hold for $K_c(X, \Lambda)_0$ in general.
Example 4.3. Let $G$ be a finite group $\mathbb{F}_p$. Let $M$ and $N$ be 1-dimensional representations of $G$ over $\Lambda$. Let $m$ and $n$ be bases of $M$ and $N$ respectively. Assume that $\Lambda$ has a $p$-th root of unity $\zeta_p$ not equal to 1 and that $\zeta_p^{-1} \neq \zeta_p$. Further assume that the action of $1 \in \mathbb{F}_p$ on $M$ is given by $1 \cdot m = \zeta_p \cdot m$ and the action of $1 \in \mathbb{F}_p$ on $N$ is given by $1 \cdot n = \zeta_p^{-1} \cdot n$. Then we have $\dim M^g = \dim N^g$ for every $g \in G$. However, we have $\dim(M \otimes_{\Lambda} M)^1 = 0$ and $\dim(N \otimes_{\Lambda} M)^1 = 1$.

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