Differential–difference equations associated with the fractional Lax operators

V E Adler\textsuperscript{1} and V V Postnikov\textsuperscript{2}

\textsuperscript{1} LD Landau Institute for Theoretical Physics, 1A Ak. Semenov, Chernogolovka 142432, Russia
\textsuperscript{2} Sochi Branch of Peoples’ Friendship University of Russia, 32 Kuibyshev str., 354000 Sochi, Russia

E-mail: adler@itp.ac.ru and postnikovvv@mail.ru

Received 12 July 2011, in final form 29 August 2011
Published 22 September 2011
Online at stacks.iop.org/JPhysA/44/415203

Abstract

We study integrable hierarchies associated with spectral problems of the form $P\psi = \lambda Q\psi$, where $P$ and $Q$ are difference operators. The corresponding nonlinear differential–difference equations can be viewed as inhomogeneous generalizations of the Bogoyavlensky-type lattices. While the latter turn into the Korteweg–de Vries equation under the continuous limit, the lattices under consideration provide discrete analogs of the Sawada–Kotera and Kaup–Kupershmidt equations. The $r$-matrix formulation and several of the simplest explicit solutions are presented.

PACS numbers: 02.30.Ik, 02.30.Jr
Mathematics Subject Classification: 35Q53, 37K10

1. Introduction

The simplest example studied in this paper is the lattice equation

$$u, t = u^2(u_2u_1 - u_{-1}u_{-2}) - u(u_1 - u_{-1}),$$

(1)

where we use the shorthand notations

$$u = u(n, t), \quad u, t = \partial_t(u), \quad u_j = u(n + j, t).$$

This equation was derived first by Tsujimoto and Hirota [1, equation (4.12)] as the continuous limit of the reduced discrete BKP hierarchy. Recall that both equations

$$u, t = u(u_1 - u_{-1}) \quad \text{and} \quad u, t = u^2(u_2u_1 - u_{-1}u_{-2})$$

(2)

are very well known integrable models: the Volterra lattice [2, 3] and the modified Narita–Itoh–Bogoyavlensky lattice of the second order [4–6], respectively. One can easily verify that the flows $\partial_t$ and $\partial_r$ do not commute, that is, these equations belong to the different hierarchies. Hence, one should not expect \textit{a priori} that their linear combination remains
integrable. Nevertheless, this is the case: we will show that equation (1) admits the Lax representation

$$L_t = [A, L]$$

with the operator $L$ equal to a ratio of two difference operators, namely $L = (T^2 + u)^{-1} (uT^2 + 1)T$, where $T$ denotes the shift operator $u_k \rightarrow u_{k+1}$.

Equation (1) can be cast into Hirota’s bilinear form which admits a family of generalizations depending on a pair of integer parameters $(l, m)$. These generalizations were discovered by Hu, Clarkson and Bullough [7, equation (4)] who searched for bilinear equations admitting $N$-soliton solutions. One of the goals of our paper is to demonstrate that this family of equations is associated with the fractional Lax operators of the form

$$L = (T^m + u)^{-1} (uT^m + 1)T^l. \quad (3)$$

As usual, any such $L$ is associated with a whole commutative hierarchy of equations corresponding to the sequence of difference operators $A$ of increasing order. We denote this hierarchy by $dSK^{(l,m)}$, since it can be viewed as a discretization of the hierarchy containing the Sawada–Kotera (SK) equation [8, 9]

$$U_{\tau} = U_5 + 5UU_3 + U_1U_2 + 5U^2U_1, \quad (SK)$$

where we denote

$$U = U(x, \tau), \quad U_{\tau} = \partial_{\tau}(U), \quad U_j = \partial^j_x(U).$$

For instance, equation (1) belongs to $dSK^{(1,2)}$. The concrete formula of the continuous limit in this example is the following, at $\varepsilon \rightarrow 0$:}

$$u(n, t) = \frac{1}{3} + \frac{\varepsilon^2}{9} U \left( x - \frac{4}{9} \varepsilon t + \frac{2\varepsilon^5}{135} \right), \quad x = \varepsilon n \quad (4)$$

and an analogous formula exists for any $(l, m)$. It should be noted that each of equations (2) apart defines a discretization of the Korteweg–de Vries (KdV) equation $U_{\tau} = U_3 + 6UU_1$, rather than the SK one. Moreover, it is well known that actually all Bogoyavlensky-type lattices serve as discretizations of the KdV equation or its higher symmetries, so that an infinite family of discrete hierarchies correspond to just one continuous. Quite analogously, the whole family of $dSK^{(l,m)}$ hierarchies serve as discrete analogs of the SK hierarchy. We hope that this observation makes clear the place of these equations in the big picture of integrable systems.

On the other hand, the differential and difference cases are not quite parallel. First, the Lax operator for the SK equation

$$L = D^3 + UD = (D - f)(D + f)D$$

is not fractional. Lax operators given by the ratio of differential operators were studied by Krichever [10]; however, it seems that these examples and (3) are unrelated.

Second, let us consider the problem of discretization for another important example, the Kaup–Kupershmidt (KK) equation [11–13]

$$U_{\tau} = U_5 + 5UU_3 + \frac{25}{2}U_1U_2 + 5U^2U_1. \quad (KK)$$

Recall that it is associated with the operator

$$L = D^3 + UD + \frac{1}{2}U_{\tau} = (D + f)D(D - f)$$

and both SK and KK equations are connected through the Miura substitutions obtained by factorization of Lax operators [14, 15]:

$$U_{SK} = f_{,x} - f^2, \quad U_{KK} = -2f_{,x} - f^2.$$
Despite this close relation, it was noted that some properties of the SK and KK equations are rather different, see e.g. [16]. It seems that distinctions between the lattice analogs of these equations are even more deep. A discretization of the KK equation is presented in section 4; however, we were able to find just one operator \( L \) in this case compared to infinite family (3) in the SK case, and no discrete analog of Miura-type substitution between dSK and dKK is known.

The contents of this paper are as follows. Section 2 contains some necessary information on the lattices of Bogoyavlensky type, see also [17, 18]. Section 3 is devoted to discretization of the SK equation and contains the main results of the paper. A general construction of the Lax pairs with operator (3) is given in section 3.1. In section 3.3, the \( r \)-matrix approach in the difference setting [19, 20, 18] is used to obtain explicit formulas for the operator \( A \) and to prove the commutativity of the dSK\((l,m)\) hierarchy. The continuous limit to SK hierarchy, the bilinear representation, the simplest breather-type solutions are presented in sections 3.4 and 3.5. Section 4 is devoted to discretization of the KK equation and section 5 contains several examples of coupled lattice equations associated with more general fractional Lax operators.

### 2. Preliminaries

#### 2.1. Definitions and notations

We consider differential–difference (lattice) equations of the evolutionary form

\[
 u_{j,t} = f(u_{m}, \ldots, u_{-m}), \quad u = u(n, t), \quad u_{j} = \partial_{j}(u), \quad u_{j} = u(n+j, t). \tag{5}
\]

Such equations can be viewed as discrete analogs of continuous evolutionary equations such as KdV or SK

\[
 U_{,t} = F(U_{k}, \ldots, U), \quad U = U(x, \tau), \quad U_{,t} = \partial_{t}(U), \quad U_{j} = \partial_{j}(U)
\]

(the orders \( m \) and \( k \) may not coincide under the continuous limit). The shift operator \( T : u_{j} \mapsto u_{j+1} \) plays the same role for equations (5) as the total \( x \)-derivative \( D : U_{j} \mapsto U_{j+1} \) plays in the continuous case. Differential operators are polynomials with respect to \( D \), with the multiplication defined by the Leibniz rule \( DA = D(A) + AD \) and the conjugation defined by the rule \( D^{\dagger} = -D \). In contrast, difference operators are in general Laurent polynomials, that is they contain powers of both \( T \) and \( T^{-1} \), and the rules for the multiplication and the conjugation are \( TA = T(A)T \) and \( T^{\dagger} = T^{-1} \). For short, we will use subscripts also for denoting the action of \( T \) on operators, \( A_{j} = T^{j}(A) \).

A lattice equation

\[
 u_{,t} = g(u_{k}, \ldots, u_{-k})
\]

is called symmetry of (5) if the compatibility condition \( D_{,t}(g) = D_{,t}(f) \) is fulfilled, that is

\[
 [f, g]_{s} := \sum_{j=-m}^{m} \partial_{j}(f)T^{j}(g) - \sum_{j=-k}^{k} \partial_{j}(g)T^{j}(f) = 0. \tag{6}
\]

The lattice is called integrable if it admits an infinite sequence of symmetries with the order \( k \) greater than any fixed number. The linear space of all symmetries is called hierarchy. A conservation law is a relation of the form

\[
 D_{,t}(\rho(u_{k}, \ldots, u)) = (T-1)(\sigma(u_{k+m-1}, \ldots, u_{-m}))
\]

which holds true in virtue of equation (5). The discussion of these notions and applications to the problem of classification of integrable lattice equations can be found in the review article by Yamilov [21].

---

J. Phys. A: Math. Theor. 44 (2011) 415203

V E Adler and V V Postnikov
2.2 Bogoyavlensky lattices

Understanding the structure of $dSK^{(l,m)}$ hierarchy is not possible without understanding the homogeneous hierarchies of Bogoyavlensky type. A general pattern of (local) equations from $dSK^{(l,m)}$ is given by the formula

$$u_{l,i} = F^{(L+KM)} + \cdots + F^{(L+M)} + F^{(L)}_l,$$

where $F^{(s)}$ denotes a homogeneous polynomial of degree $s$ with respect to the variables $u_j$ and $K, L, M$ are related somehow to the parameters $l, m$ and the order $k$ of the flow. Moreover, the first and the last terms in the sum always correspond to some (modified) lattices of Bogoyavlensky type belonging to the different hierarchies.

This structure is explained by the following arguments, starting from the Lax representation with the operator $L$ (3). Let us consider the scaling $u \to \delta^{-m} u$, $T \to \delta T$; then it is easy to see that the limit $\delta \to \infty$ sends $L$ to the operator $L' = u_{-m} T^l + T^{l-m}$ and the limit $\delta \to 0$ leads to $L'' = T^{m+l} + u^{-1} T^l$. Each of these operators corresponds to its own hierarchy of homogeneous lattice equations. The total inhomogeneous equation contains both of them together with the intermediate terms which are necessary for preserving commutativity of the flows.

Let us consider the concrete example. One can check that the lattice

$$u_{l,i} = u_1 (w_3 + w_2 + w_1 + w) - w_{-1} (w + w_{-1} + w_{-2} + w_{-3}) - u_1 (w_3 + w_{-1}) + u_{-1} (w_1 + w_{-3}), \quad w := u (1 - u_1 u_{-1})$$

is a higher symmetry of equation (1). Collecting the homogeneous terms yields

$$u_{l,i} = F^{(4)} + F^{(2)}, \quad u_{l,r} = G^{(7)} + G^{(5)} + G^{(3)}$$

and the consistency condition of the flows splits into the relations

$$[F^{(4)}, G^{(7)}]_s = 0, \quad [F^{(4)}, G^{(5)}]_s + [F^{(2)}, G^{(7)}]_s = 0,$$

$$[F^{(4)}, G^{(3)}]_s + [F^{(2)}, G^{(5)}]_s = 0, \quad [F^{(2)}, G^{(5)}]_s = 0,$$

where commutator $[,]_s$ is defined by equation (6). As was already said in the introduction, polynomials $F^{(4)}$ and $F^{(2)}$ correspond to the modified Bogoyavlensky and Volterra lattices. Polynomials $G^{(7)}$ and $G^{(3)}$ correspond to their symmetries and the intermediate polynomial $G^{(5)}$ compensates inconsistency of the hierarchies.

The Bogoyavlensky hierarchy $B^{(m)}$ is associated with the operator $L = T + u T^{-m}$ and we recall here several basic formulas regarding this case. A detailed theory can be found in [6, 18]. More general operators of the form $L = T^l + u T^{-m}$ were considered recently in the paper [22].

The simplest equation from the $B^{(m)}$ hierarchy reads

$$u_m = u (u_m + \cdots + u_1 - u_{-1} - \cdots - u_{-m}).$$

This equation and its higher symmetries are associated with the difference spectral problem

$$\psi_1 + u \psi_{-m} = \lambda \psi$$

and admit the Lax representations

$$L_{l,i} = [A^{(k)}, L], \quad L = T + u T^{-m}, \quad A^{(k)} = \pi_+ (L^{m+1})^k,$$

where $\pi_+$ denotes the projection of any formal series $A = \sum_{j<\infty} a^{(j)} T^j$ onto the linear space of polynomials with respect to $T$:

$$\pi_+ (A) = \sum_{0 \leq j < \infty} a^{(j)} T^j, \quad \pi_- (A) = \sum_{j > 0} a^{(j)} T^j.$$
In particular,

\[ A^{(1)} = T^{m+1} + v, \quad v := u_m + \cdots + u \]

and equation (9) at \( k = 1 \) is equivalent to lattice (8). The check is easy:

\[
L_t = [A^{(1)}, L] = u_j T^{-m} - [T^{m+1} + v, T + u T^{-m}]
\]

\[
= u_j T^{-m} - (u_{m+1} - u + v - v_1) T - u(v - v_{-m}) T^{-m},
\]

the terms with \( T \) cancel and the rest yields the equation.

In order to prove that equation (9) correctly defines the lattice for any \( k \), we have to check that all powers of \( T \) except for \( T^{-m} \) vanish in the commutator \([A^{(k)}, L] \). Since \( L^{m+1} \) is a Laurent polynomial with respect to \( T^{m+1} \), \( A^{(k)} \) is a polynomial with respect to \( T^{m+1} \). Therefore, the commutator contains only powers of the form \( T^{(m+1)j+1}, j \geq -1 \). On the other hand,

\[ [A^{(k)}, L] = -[\pi - (L^{m+1})^k, L], \]

so that the commutator does not contain positive powers of \( T \) and only one possible power \( T^{-m} \) remains.

It can be proven that equations (9) define a special reduction in the Lax pair with a generic operator \( L = T + u(0) + u(1) T^{-1} + \cdots + u(m) T^{-m} \). In this case, one can choose operators \( A \) in the form \( A = \pi_x (L^k) \) with arbitrary \( k \). For instance, the Toda lattice hierarchy appears at \( m = 1 \). This type of multi-field systems was studied, for instance, [20, 22].

3. Discretizations of the SK equation

3.1. Lax representation

Let us consider the difference spectral problem

\[ u \psi_{m+l} + \psi_l = \lambda (\psi_m + u \psi) \]

(11)

where \( m, l \) are integers. We assume that \( m, l \) are positive and coprime, without loss of generality, since the general case can be obtained by refinement of the mesh and/or change of its directions.

It is less obvious that the numbers \( m \) and \( l \) can be exchanged: spectral problem (11) is equivalent to

\[ u \phi_{m+l} + \phi_m = \mu (\phi_l + u \phi) \]

under the change

\[ \psi(n) = \kappa^n \phi(n), \quad \lambda = -\kappa, \quad \mu = -\kappa^{-m}. \]

(12)

In the operator form, equation (11) reads

\[ P \psi = \lambda Q \psi, \quad P = (u T^m + 1) T^l, \quad Q = T^m + u. \]

(13)

The isospectral deformations are defined by the equation \( \psi_t = A \psi \) with some difference operator \( A \). The corresponding Lax equation

\[ L_t = [A, L], \quad L = Q^{-1} P \]

(14)

can be rewritten as the system

\[ P \psi_t - B P - PA, \quad Q \phi_t = B Q - QA, \]

(15)

where one of the equations can be considered just as a definition of \( B \). Let \( P, Q \) be as in (13); then this system is equivalent to the equations

\[ u \phi_t = B(T^m + u) - (T^m + u) A, \]

\[ B(T^{2m} - 1) = A_{10} T^{2m} - A_1 + u A T^m - u A_{m+1} T^m. \]

(16)
In order to resolve the latter, we make the assumption that the operator $A$ is of the form
\begin{equation}
A = F(T^m - T^{-m});
\end{equation}
then $B$ is found as the difference operator
\begin{equation}
B = F_m T^m - F_1 T^{-m} + u(F - F_{m+1})
\end{equation}
while first equation (16) turns into
\begin{equation}
u_j = T^m F u + u F T^{-m} - uT^m F_1 - F_1 T^{-m} u + F_m - F_1 + u(F - F_{m+1}) u.
\end{equation}
It is clear that the same evolution of the variable $u$ is defined by the conjugated operator $F^\dagger$ and, moreover, all terms $T^j, j \neq m$, can be thrown away. This means that we can find $F$ as a self-adjoint operator $F = F^\dagger$ which is a Laurent polynomial with respect to the powers $T^m$:
\begin{equation}
F = f^{(k)} T^m m + \cdots + f^{(1)} T^m + f^{(0)} + T^{-m} f^{(1)} + \cdots + T^{-km} f^{(k)}, \quad k \geq 0.
\end{equation}
Certainly, the coefficients depend on $k$, $l$, $m$, so that it would be more rigorous to write $f^{(j,k,l,m)}$ instead of $f^{(j)}$, but we will consider these numbers fixed at the moment.

Collecting the coefficients at $T^j m, j > 0$, yields the relations
\begin{equation}
u_{mj} f_m^{(j-1)} - u f_{m+1}^{(j-1)} = f_j^{(j)} - f_m^{(j)} + uu_{jm} (f_{m+1}^{(j)} - f_j^{(j)})
+ uu_{jm} f_{j-1}^{(j-1)} - u f_{j+1}^{(j-1)}, \quad j = 1, \ldots, k + 1,
\end{equation}
where it is assumed for convenience that $f_j^{(j)} = 0$ at $j > k$. The coefficient at $T^0$ gives an evolutionary equation for $u$:
\begin{equation}v_j = 2u (f^{(1)} - f_j^{(1)}) + uu^2 (f^{(0)} - f_{m+1}^{(0)}) + f_m^{(0)} - f_j^{(0)}.
\end{equation}
System of equations (21) and (22) defines the $k$th flow in the hierarchy $\text{dSK}^{(k,m)}$.

If we are interested in the local evolution only, then we require that all $f^{(j)}$ can be recurrently found as functions of a finite set of variables $u_i$. In this case, a certain restriction on the values of $k$ appears and a part of the flows is rejected. Indeed, consider equation (21) at $j = k + 1$,
\begin{equation}
u_{(k+1)m} f_m^{(k)} = u f_m^{(k)},
\end{equation}
or
\begin{equation}(T^l - 1)(\log f_m^{(k)}) = (T^{(k+1)m} - 1)(\log u).
\end{equation}
It can be proven that it is solvable with respect to $f_m^{(k)}$ if and only if $(k+1)m$ is divisible by $l$ and the solution is, up to a constant factor,
\begin{equation}f_m^{(k)} = u - u_{m-1} \cdots u_{(k-1)m-3} - u_{(k-1)m-1} - u_{(k-1)m}.
\end{equation}
Since $l$ and $m$ are coprimes, the local flows may appear only if $k = pl - 1$ and $s = mp$. The fact that the rest equations (21) for such $k$ are solvable indeed will be verified later in section 3.3. The case $l = 1$ is the only one when there are no restrictions on $k$ and the simplest choice $k = 0$ brings in this case to the following family of lattices.

**Theorem 1.** For any $m > 0$, the simplest equation in the hierarchy $\text{dSK}^{(1,m)}$
\begin{equation}v_j = u (u_{m} \cdots u_{1} - u_{-1} \cdots u_{-m}) - u (u_{m-1} \cdots u_{1} - u_{-1} \cdots u_{-1})
\end{equation}
possesses the Lax representation (14) with the operators
\begin{equation}P = u T^{m+1} + T, \quad Q = T^m + u,
\end{equation}
\begin{equation}A = f (T^{-m} - T^m), \quad B = f T^{-m} - f T^m + u (f_{m+1} - f),
\end{equation}
where $f = u_{-1} \cdots u_{-m}$.
Proof. A direct computation (cf (10)) proves that both equations (15) with given \( P, Q, A, B \) are equivalent to the relations
\[
\begin{align*}
\psi_{m,f_l} &= \psi_{m+1,f_l}, \quad \psi_{l} = u^2(f_{m+1} - f) - f_m + f_l.
\end{align*}
\]
The former defines the variable \( f \) (up to a constant factor) and the latter is equivalent to lattice (25).

In particular, equation (25) at \( m = 2 \) coincides with (1) and at \( m = 1 \) it is just the modified Volterra lattice
\[
\psi_{l} = u^2(u_1 - u_{-1}).
\]

It should be remarked that gauge equivalence (12) between the spectral problems can be extended on the level of nonlinear equations and the same flow (25) appears also as a member of the \( \text{dSK}^{(m,1)} \) hierarchy. However, operator (20) is much more complicated in this case: it contains all powers \( T^{m-1}, T^{m-2}, \ldots, T^{1-m} \) compared with just \( F = f^{(0)} \) in the \( \text{dSK}^{(1,m)} \) case.

Computing higher symmetries quickly becomes involved because finding \( F \) requires (discrete) integration of rather bulky expressions. For instance, the second flow in the hierarchy \( \text{dSK}^{(1,m)} \) is, according to (22), of the form
\[
\psi_{l} = 2u(f^{(1)} - f^{(1)}) + u^2(f^{(0)} - f^{(0)}) + f^{(0)} - f^{(0)},
\]
where the functions \( f^{(1)}, f^{(0)} \) are defined by the relations
\[
\begin{align*}
\psi_{m,f_l} &= \psi_{m+1,f_l}, \quad \psi_{m,0} = \psi_{m+1,0} = f^{(0)} - f^{(0)}, \quad \psi_{m,0} = \psi_{m+1,0} = f^{(1)} - f^{(1)} - \psi_{m,1}.
\end{align*}
\]
This yields, up to integration constants,
\[
\begin{align*}
\psi^{(1)} &= u_{m-1} \cdots u_{-m}, \quad \psi^{(0)} = (w + \cdots + w_{2m+1})u_{-1} \cdots u_{-m}, \quad w := (1 - \psi_{m,1})u_{m-2} \cdots u_{-1},
\end{align*}
\]
(at \( m = 2 \) equation (7) appears). One can check straightforwardly that the obtained flow commutes with (25) indeed. A general proof and a way to bypass the integration are given below in section 3.3.

Adopting nonlocal variables leads to some extension of the hierarchy. In this case, we consider equation (23) as a constraint which defines the variable \( f^{(k)} \) for any \( k \). Then, we arrive at the following system which generalizes (25) for any \( l \), making the picture more uniform. We will return to this system in section 3.5.

Theorem 2. For any coprime \( m, l \), the simplest system in the extended \( \text{dSK}^{(l,m)} \) hierarchy
\[
\begin{align*}
\psi_{m,f_l} &= \psi_{m+1,f_l}, \quad \psi_{l} = u^2(f - f_{m+1}) + f_m - f_l
\end{align*}
\]
possesses the Lax representation (14) with operators
\[
\begin{align*}
P &= uT^{m+1} + T^l, \quad Q = T^m + u, \quad A = f(T^{-m} - T^m), \quad B = f(T^{-m} - f_m T^m + u(f_{m+1} - f)).
\end{align*}
\]

3.2. Modified lattices

The equations under consideration can be rewritten in several ways by the use of difference substitutions. The simplest kind of substitution is introducing a potential. Let \( A \) be a constant operator; then the substitution \( u = A(v) \) maps solutions of the equation \( \psi_{l} = f[A(v)] \) into solutions of the equation \( \psi_{l} = A(f[u]) \). Table 1 contains several instances of such kind, up to the change \( u \rightarrow e^u, v \rightarrow e^v \).
Another kind of substitution is Miura-type transformations. Let \( \psi \) be a particular solution of the spectral problem (11) corresponding to a value \( \lambda = \alpha \) of the spectral parameter. Then, one readily finds that the ratio \( h = \psi_1/\psi \) is related to the potential \( u \) by the formula

\[
M^- : \quad u = \frac{ah_{m-1} \cdots h_{l-1} \cdots h_{1} \cdots h}{h_{m+l-1} \cdots h - \alpha}.
\]

This defines a difference substitution, according to the following statement.

**Theorem 3.** Let \( u \) satisfy equation (22) from dSK\((l,m)\); then \( h = \psi_1/\psi \) also satisfies a lattice equation which can be written as a conservation law

\[(\log h)_l = (T - 1)S[h].\] (27)

**Proof.** Since \( \psi \) is governed by the equation \( \psi_\tau = A\psi = F(\psi_m - \psi_{-m}) \),

\[(\log h)_l = (T - 1)(\log \psi)_l = (T - 1) \left( \frac{1}{\psi} F(\psi_m - \psi_{-m}) \right).
\]

Coefficients of the operator \( F \) are functions of the variables \( h_j \), being functions of \( u_j \)'s. The ratios of the form \( \psi_k/\psi \) can be expressed through \( h_j \) as well and therefore an equation of the form (27) holds. \( \Box \)

It is worth noting that an infinite sequence of conservation laws for the original lattice (22) can be obtained from (27) by use of the classical trick with the inversion of the Miura map \( u = M^- (h, \alpha) \) as a formal power series with respect to \( \alpha \) [23].

The second Miura map is obtained by replacing \( h \rightarrow 1/h, \alpha \rightarrow 1/\alpha \), which results in the mapping

\[
M^+ : \quad u = \frac{ah_{m+1} \cdots h_{l} - h - h_{m+l-1} \cdots h_{1}}{h_{m+l-1} \cdots h - \alpha}.
\]

This substitution relates the same equations as \( M^- \), due to invariance of the spectral problem with respect to the change \( n \rightarrow -n, \lambda \rightarrow 1/\lambda \). Therefore, the composition \( M^- \circ M^+ \) defines a Bäcklund transformation which relates two copies of the dSK\((l,m)\) hierarchy. Recall that the Bäcklund transformation for the continuous SK equation was derived in [24].

A particular example at \( l = 2, m = 1 \) is given by the substitutions

\[M^- : u = \frac{(\alpha - h_1)h}{h_2 h_1 h - \alpha}, \quad M^+ : u = \frac{h_2 (\alpha - h_1)}{h_2 h_1 h - \alpha}\]

which map solutions of the modified equation

\[h_j = \frac{h(\alpha - h)}{h_1 h_{j-1} - \alpha} \left( \frac{h_1 h_{j-1} (\alpha - h_{j-1})}{h_2 h_1 h - \alpha} - h_1 - h_{j-1} \right)\]

into solutions of (1).
3.3. r-matrix formulation

In this section, we prove that

(i) if the constraint (23) is resolved by formula (24), then the further recurrent relations (21)
are solved in the local form as well, so that the (local) hierarchy dSK\((l,m)\) is correctly
defined;

(ii) the flows corresponding to the different \(k\) commute.

In achieving this goal the \(r\)-matrix approach is an indispensable tool, see e.g. [19, 20, 18].

Let us consider the Lie algebra of the formal Laurent series with respect to the powers \(T^m\) of
the shift operator:

\[
g^{(m)} = \left\{ \sum_{j<\infty} g^{(j)} T^j \right\}
\]

with the commutator \([A, B] = AB - BA\). It is easy to see that any element

\[
G = g^{(k+1)} T^{(k+1)m} + g^{(k)} T^{km} + g^{(k-1)} T^{(k-1)m} + \ldots
\]

of this Lie algebra admits a unique decomposition of the form

\[
G = F (T^m - T^{-m}) + H,
\]

(28)

where \(F = F^\dagger\) is a self-conjugated difference operator and \(H\) is a formal series which contains
only nonpositive powers of \(T^m\). Each of the linear spaces

\[
g_+^{(m)} = \{ F (T^m - T^{-m}) | F = F^\dagger \}, \quad g_-^{(m)} = \left\{ \sum_{j\leq 0} h^{(j)} T^j \right\}
\]

constitutes a Lie algebra: for \(g_+^{(m)}\) this is obvious and for \(g_-^{(m)}\) we have

\[
[F (T^m - T^{-m}), F'(T^m - T^{-m})] = (P + P^\dagger) (T^m - T^{-m}),
\]

where \(P = F (T^m - T^{-m}) F^\dagger\).

Thus, formula (28) is the decomposition (in the vector space sense)

\[
g^{(m)} = g_+^{(m)} \oplus g_-^{(m)}
\]

of the Lie algebra into the direct sum of two Lie subalgebras. This decomposition defines
the projections \(\pi_{\pm}\) on the \(g_{\pm}^{(m)}\) component and the \(r\)-matrix \(r = \frac{1}{2} (\pi_+ - \pi_-)\). Now, we can formulate the following theorem about the Lax equations (13), (14) with the fractional \(L\)
operator.

**Theorem 4.** Let \(l, m\) be coprime, \(P = (uT^m + 1)T^l\), \(Q = T^m + u\) and let \(L = Q^{-1} P\) be expanded as a formal Laurent series. Then, the flows

\[
L_{tp} = [\pi_+(L^m), L]
\]

(29)

are correctly defined for all \(p = 1, 2, \ldots\), coincide with the dSK\((l,m)\) flows introduced by equations (21), (22) and commute with each other.

**Proof.** After expanding, \(L\) takes the form

\[
L = (1 - u_{-m} T^{-m} + (u_{-m} T^{-m})^2 - \ldots) (u_{-m} + T^{-m}) T^l
\]

\[
= u_{-m} T^l + (1 - u_{-m} u_{-2m}) T^{l-m} + \ldots
\]

Differentiating this series turns (29) into an infinite system of equations for a single variable \(u\),
and the correctness means that all these equations must coincide. To prove this, we compare
representation (29) with the Lax equation (14) in the fractional form.
Note that $L$ itself does not belong to the Lie algebra $\mathfrak{g}^{(m)}$, but its power $G = L^m$ does, so that the projection $A = \pi_+(G) = F(T^m - T^{-m})$ makes sense. We denote the order of the operator $F$ as $k = pl - 1$, in agreement with (20) and (24). The coefficients of $F$ are uniquely computed from the coefficients of $G$ according to the recurrent relations

$$f^{(k+2)} = f^{(k+1)} = 0, \quad f^{(j)} = g^{(j+1)} + f^{(j+2)}, \quad j = k, k - 1, \ldots, 0,$$

so that all coefficients are local functions of $u_j$ (in particular, $f^{(k)}$ is given by (24)). Moreover, the order of the right-hand side of (29) is equal to $l$, because $[\pi_+(G), L] = -[\pi_-(G), L]$. This proves that $F$ provides a solution of the recurrent relations (21) as well (which is unique up to integration constants). Indeed, these relations were derived from the condition that terms $T^{(k+1)m}, \ldots, T^m$ in equation (19) cancel which is equivalent to cancellation of the powers $T^{(k+1)m+l}, \ldots, T^{m+l}$ in the original Lax equation (14). Thus, flow (29) coincides with a flow from the dSK$^{(l,m)}$ which is, therefore, local. On the other hand, this proves the correctness of (29), since the whole infinite set of equations turns out to be equivalent to the single equation (22).

The proof of the commutativity is standard. Let $G' = L^p m$ and $A' = \pi_+(G')$; then

$$(L_{t_{L'}})_{t_{L'}} - (L_{t_{A'}})_{t_{A'}} = [A_{L'}, - A'_{L'} + [A, A'], L],$$

so it is sufficient to prove that

$$A_{L'} - A'_{L'} + [A, A'] = 0.$$  

Since $A_{L'} = \pi_+(\{A', G\})$ and $[G, G'] = 0$, this is equivalent to

$$\pi_+(\{A', G\} - [A, G'] + [A, A']) = \pi_+(\{G - \pi_-(G'), G\} - [G - \pi_-(G), G'] + [G - \pi_-(G), G' - \pi_-(G')])$$

$$= \pi_+(\{\pi_-(G), \pi_-(G')\}) = 0$$

as required. $\Box$

### 3.4. Continuous limit

Here, we compute the continuous limit for the basic flow of the extended hierarchy dSK$^{(l,m)}$ defined by equation (26). There is certain technical difficulty in the prolongation of the continuous limit on the variable $f$ which is not local at $l \neq 1$. In order to solve the constraint, this variable should be considered as a series with respect to the small parameter. Up to this complication the continuous limit is very similar to example (4) from the introduction. We postulate that, at $\varepsilon \to 0$, the variables $u, f$ are of the form

$$u(n, t) = a + ab\varepsilon^2 U(x + c\varepsilon t, \tau + d\varepsilon^2 t),$$

$$f(n, t) = 1 + \sum_{s=2}^{\infty} \varepsilon^s Y_s(x + c\varepsilon t, \tau + d\varepsilon^2 t), \quad x = \varepsilon n$$

with the undetermined coefficients $a, b, c, d$. The functions $Y_s$ are expressed through the function $U$ and its partial derivatives with respect to $x$ after substituting into first equation (26) and taking the Taylor expansion about $\varepsilon = 0$ (clearly, one can neglect the dependence on $t$ here). We find, omitting the unessential integration constants,

$$Y_2 = \frac{mb}{l} U,$$

$$Y_3 = -\frac{m(m + l)b}{2l} U_1,$$
we consider a linear combination with the next dSK appearance is an artifact of the continuous limit. A straightforward computation shows that if hierarchy dSK 

(29) which show that in the discrete case there are no gaps multiple of 3. It turns out that their combinations of the dSK 

vanish while the coefficients at 

\[ \alpha \approx 7 \]

and we restrict ourselves by the following concrete example corresponding to the local 

This is enough, since we need only terms up to \( \varepsilon \) when substituting into the second equation (26). The coefficients \( a, c \) are found from the requirement that the low-order terms vanish and the coefficients \( b, d \) are responsible for the scaling of \( U \) and \( t \) and can be chosen arbitrarily. Finally, we come to the following statement.

**Theorem 5.** Continuous limit (30) with the values of parameters

\[ a = \frac{m - l}{m + l}, \quad b = \frac{ml}{6}, \quad c = 2m, \quad d = \frac{m^3(l^2 - m^2)}{180} \]

sends systems (26) into the SK equation

\[ U, t = U_5 + 5UU_3 + 5U_1U_2 + 5U^3U_1. \]

The higher flows of the SK hierarchy can be derived analogously from suitable linear combinations of the dSK\( ^{(l,m)} \) flows. However, the general formulas become rather complicated and we restrict ourselves by the following concrete example corresponding to the local hierarchy dSK\(^{(1,2)} \). Let \( u_\tau = 88u_{1,1} + 27u_{1,2} \) where the flows \( \partial_h \) and \( \partial_t \) are defined by equations (1) and (7), respectively; then the formula

\[ u(n, \varepsilon t) = \frac{1}{3} \varepsilon^2 \left( x - \frac{200}{9} \varepsilon t, \tau + \frac{16\varepsilon^7}{189} \right), \quad x = \varepsilon n \]

defines the continuous limit to the seventh-order symmetry of the SK equation

\[ U, \varepsilon t = U_7 + 7UU_5 + 14U_1U_4 + 21U_3U_2 + 14U_2^2U_3 + 42UU_1U_2 + 7U_1^3 + \frac{28}{3}U_3^2U_1. \]

It is well known that there are gaps in the sequence of orders \( k \) of equations from the SK hierarchy; namely, the restrictions \( k \notdivides 2, 3 \) are fulfilled, so that the next higher symmetry is of 11th order. The natural question appears as to how this agrees with relations (20)–(22) or (29) which show that in the discrete case there are no gaps multiple of 3. It turns out that their appearance is an artifact of the continuous limit. A straightforward computation shows that if we consider a linear combination with the next dSK\(^{(1,2)} \) flow \( u, \varepsilon t = u_{1,1} + au_{1,2} + \beta u_{1,3} \) and set

\[ u(n, \varepsilon t) = a + be^2U(x + c\varepsilon t, \tau + d\varepsilon^9 t), \quad x = \varepsilon n, \]

then all parameters are uniquely determined by the condition of vanishing the terms up to \( \varepsilon^{10} \); however, then the coefficients at \( \varepsilon^{11} \) cancel automatically and only the trivial flow \( U, \varepsilon t = 0 \) appears.

### 3.5. Bilinear equations

The constraint (23) can be solved by introducing additional variables and this leads to a convenient representation of the basic system (26) of the extended dSK\(^{(l,m)} \) hierarchy. Let

\[ u = \frac{v_l}{v}, \quad f = \frac{v}{v_{l-m}}; \]

This is enough, since we need only terms up to \( \varepsilon \) when substituting into the second equation (26). The coefficients \( a, c \) are found from the requirement that the low-order terms vanish and the coefficients \( b, d \) are responsible for the scaling of \( U \) and \( t \) and can be chosen arbitrarily. Finally, we come to the following statement.

**Theorem 5.** Continuous limit (30) with the values of parameters

\[ a = \frac{m - l}{m + l}, \quad b = \frac{ml}{6}, \quad c = 2m, \quad d = \frac{m^3(l^2 - m^2)}{180} \]

sends systems (26) into the SK equation

\[ U, t = U_5 + 5UU_3 + 5U_1U_2 + 5U^3U_1. \]

The higher flows of the SK hierarchy can be derived analogously from suitable linear combinations of the dSK\(^{(l,m)} \) flows. However, the general formulas become rather complicated and we restrict ourselves by the following concrete example corresponding to the local hierarchy dSK\(^{(1,2)} \). Let \( u_\tau = 88u_{1,1} + 27u_{1,2} \) where the flows \( \partial_h \) and \( \partial_t \) are defined by equations (1) and (7), respectively; then the formula

\[ u(n, \varepsilon t) = \frac{1}{3} \varepsilon^2 \left( x - \frac{200}{9} \varepsilon t, \tau + \frac{16\varepsilon^7}{189} \right), \quad x = \varepsilon n \]

defines the continuous limit to the seventh-order symmetry of the SK equation

\[ U, \varepsilon t = U_7 + 7UU_5 + 14U_1U_4 + 21U_3U_2 + 14U_2^2U_3 + 42UU_1U_2 + 7U_1^3 + \frac{28}{3}U_3^2U_1. \]

It is well known that there are gaps in the sequence of orders \( k \) of equations from the SK hierarchy; namely, the restrictions \( k \notdivides 2, 3 \) are fulfilled, so that the next higher symmetry is of 11th order. The natural question appears as to how this agrees with relations (20)–(22) or (29) which show that in the discrete case there are no gaps multiple of 3. It turns out that their appearance is an artifact of the continuous limit. A straightforward computation shows that if we consider a linear combination with the next dSK\(^{(1,2)} \) flow \( u, \varepsilon t = u_{1,1} + au_{1,2} + \beta u_{1,3} \) and set

\[ u(n, \varepsilon t) = a + be^2U(x + c\varepsilon t, \tau + d\varepsilon^9 t), \quad x = \varepsilon n, \]

then all parameters are uniquely determined by the condition of vanishing the terms up to \( \varepsilon^{10} \); however, then the coefficients at \( \varepsilon^{11} \) cancel automatically and only the trivial flow \( U, \varepsilon t = 0 \) appears.

### 3.5. Bilinear equations

The constraint (23) can be solved by introducing additional variables and this leads to a convenient representation of the basic system (26) of the extended dSK\(^{(l,m)} \) hierarchy. Let

\[ u = \frac{v_l}{v}, \quad f = \frac{v}{v_{l-m}}; \]
then first equation (26) is satisfied identically and the second one is equivalent to
\[(T^l - 1) \frac{v_t}{v} = (T^m - 1) \left( \frac{v}{v_{l-m}} - \frac{v_l}{v_{-m}} \right).\]
Further substitutions
\[v = \frac{w_m}{w} \Rightarrow u = \frac{w_{m+l}w}{u_m w_l}, \quad f = \frac{w_m w_{-m}}{w^m}\]
bring to the bilinear equation
\[w_{1,t} w - w_{1} w_{-1, m} = w_{m} w_{1-m} - w_{-m} w_{1+m}. \tag{31}\]
It appeared first in [7], in a slightly more general form
\[w_{1,t} w - w_{1} w_{-1, m} = w_{m} w_{1-m} - \alpha w_{-m} w_{1+m} + \beta w w_{l}\]
which is reduced to (31) by the point change \(w(n, t) = e^{\alpha^2 n^2} w(n, t)\). In particular, it was proven in [7] that this equation admits N-soliton solutions. Here, we consider in more detail a specification of the 2-soliton formula which leads to the breather solution.

The substitution of the 2-soliton Ansatz
\[w(n, t) = 1 + e_1 + e_2 + A_{12} e_1 e_2, \quad e_i = q_i^m \exp(-\omega_i t + \delta_i) \tag{32}\]
into (31) gives us the dispersion relation and the phase shift:
\[\omega_i = q_i^m - q_i^{-m}, \quad A_{ij} = \frac{(q_i^m - q_j^m)(q_i^{-m} - q_j^{-m})}{(1 - q_i^m q_j^m)(1 - q_i^{-m} q_j^{-m})}. \tag{33}\]
The direct check proves that then the 3-soliton Ansatz
\[w = 1 + e_1 + e_2 + e_3 + A_{12} e_1 e_2 + A_{13} e_1 e_3 + A_{23} e_2 e_3 + A_{123} e_1 e_2 e_3\]
satisfies (31) automatically. It is interesting to compare these formulas with their counterparts for the continuous SK equation [8, 9, 25, 26]
\[e_i = \exp(\xi_i x - \omega_i t + \delta_i), \quad \omega_i = \xi_i^2, \quad A_{ij} = \frac{(\xi_i - \xi_j)^2(\xi_i^2 - \xi_i \xi_j + \xi_j^2)}{(\xi_i + \xi_j)^2(\xi_i^2 + \xi_i \xi_j + \xi_j^2)}. \tag{33}\]
Formula (32) allows us to obtain the breather-type solutions as well, if we choose
\[q_1 = \rho e^{i\psi}, \quad q_2 = \rho e^{-i\psi}, \quad \delta_1 = \alpha + i\beta, \quad \delta_2 = \alpha - i\beta.\]
The regularity of the potential \(u(n, t)\) is achieved under certain restrictions on the value of \(q\). In order to show this, rewrite relations (33) as follows:
\[\omega = \mu + iv, \quad \mu = (\rho^m - \rho^{-m}) \cos m\varphi, \quad v = (\rho^m + \rho^{-m}) \sin m\varphi, \quad A_{12} = \frac{4\rho^{m+l} \sin l\varphi \sin m\varphi}{(1 - \rho^2)(1 - \rho^{2m})}\]
then a simple algebra brings (32) to the form
\[w = 1 + 2z \cos y + A_{12} z^2, \quad y = \varphi n - vt + \beta, \quad z = \rho^n e^{\alpha^2 n^2}. \tag{33}\]
In particular, if \(\varphi = \frac{3\pi}{2m}, \) then \(\mu = 0\) and solution \(w\) is periodic in \(t\). The necessary and sufficient condition for \(u\) to be regular is that the function \(w\) does not vanish at any \(n, t\). In the generic case, the variables \(y, z\) are independent and then this is equivalent to the condition that the trinomial \(1 + 2z + A_{12} z^2\) does not vanish at real \(z\), that is
\[\left(\rho^l - \rho^{-l}\right)(\rho^m - \rho^{-m}) + 4 \sin l\varphi \sin m\varphi < 0.\]
Thus, we see that already two-phase solutions in these models exhibit a nontrivial zone structure of the spectrum. The corresponding domains in the plane \(q = \rho e^{i\varphi}\) are shown in figure 1, and the examples of solutions \(u(n, t)\) are shown in figure 2.
Figure 1. The values of $q = \rho e^{i\phi}$ inside the bounded domains in $\mathbb{C}$ correspond to the regular potentials $u(n, t)$. The values along the dashed lines correspond to the potentials periodic in $t$.

Figure 2. A moving and a stable breather. The values of parameters: $\rho = 1.2, \phi = 2\pi/3$ (left); $\rho = 1.6, \phi = 3\pi/4$ (right); in both cases $l = 1, m = 2, \alpha = \beta = 0$. 
4. A discrete analog of the KK equation

The KK equation

\[ U, \tau = U_5 + 5U_3 + \frac{2\epsilon}{9}U_2 + 5U^2U_1 \]

is associated with the spectral problem \( L\psi = \lambda \psi \), where \( L \) is the skew-symmetric ordinary differential operator of third order

\[ L = D^3 + UD + \frac{1}{2}U_2 = (D - f)D(D + f), \quad U = 2f_3 - f^2. \]

When we find a discrete analog, a difficulty is that a symmetric or skew-symmetric difference operator can be of even order only. A way to overcome this is to consider a sixth-order difference problem, but on the odd nodes of the lattice only, so that effectively it is of third order with respect to the double shift \( T^2 \) (however, the coefficients may depend on the variables associated with the even nodes as well). Let us consider the spectral problem

\[ u_{-3}\psi_{-3} + \psi_{-1} = \lambda (\psi_1 + u\psi_3) \tag{34} \]

or, in the operator form, denoting \( K = uT^3 + T \):

\[ K^4\psi = \lambda K\psi. \]

The Lax equation for the operator \( L = K^{-1}K^4 \) can be written in the form of system (15). It admits the reduction \( B = -A^\dagger \) which yields the equation

\[ K, + A^\dagger K + KA = 0. \tag{35} \]

The operator \( A \) is found as a Laurent polynomial with respect to the even powers of \( T \),

\[ A = a^{(k)}T^{2k} + \cdots + a^{(-k)}T^{-2k}, \]

and a direct analysis of equation (35) at \( k = 1, 2 \) proves the following statement.

**Theorem 6.** Equation (35) with \( K = uT^3 + T \) is equivalent to the nonlocal lattice equation

\[ u_{n,t} = u(f_2u_2 - f_1u_1 + f_{-1}u_{-1} - f_{-2}u_{-2}) + f_1 - f_{-1}, \quad f_3u = f_{-1}u_2 \]

under the choice

\[ A = -fT^2 + f_{-2}u_{-2} - f_{-1}u_{-1} + f_{-3}T^{-2}, \]

and it is equivalent to the local lattice equation

\[ u_{t_2} = u(v_3 - v_2 + v_1 - v_{-1} + v_{-3} - u_2 + u_{-2}), \quad v := u_1u_{-1} \tag{36} \]

under the choice

\[ A = u_1T^4 - u_{-4}T^{-4} + (1 - u_{-1}u_{-2})(T^2 - T^{-2}) + u_{-1} - u_{-2} - v + v_{-1} - v_{-2} + v_{-3}. \]

It is worth noting that, alternatively, one can use the following pair of operators (cf the gauge equivalence (12)):

\[ \tilde{K} = uT^3 + T^{-1}, \quad \tilde{A} = -u_1u_{-1}T^4 + u_{-3}u_{-4}T^{-4} - v + v_{-1} - v_{-2} + v_{-3}. \tag{37} \]

The continuous limit to the KK equation is of the same general form as before, namely, for the flow (36) it reads

\[ u(n, t_2) = \frac{1}{3} + \frac{4}{9} \epsilon^2 U \left( x - \frac{8}{9} \epsilon t_2, \tau + \frac{64\epsilon^5}{135} t_2 \right), \quad x = \epsilon n. \]
5. Examples related to generic operators

Recall that, according to [20], the Bogoyavlensky-type lattices can be viewed as reductions of more general multi-field models associated with the spectral problems \(L \psi = \lambda \psi\) for generic difference operators \(L = u^{(m)} T^m + u^{(m-1)} T^{m-1} + \ldots + u^{(1)} T + u^{(0)}\). Here, \(m, l\) are any positive integers, and one can adopt the normalization \(u^{(m)} = 1\) or \(u^{(l)} = 1\) without loss of generality. A part of the flows from the corresponding hierarchy is consistent with the constraints \(u^{(m-1)} = \ldots = u^{(1)} = 0\) and this reduction brings to the Bogoyavlensky lattices. A detailed study of some other reductions can be found in [22].

The lattices introduced in the previous sections are related to the spectral problems \(P \psi = \lambda Q \psi\), where the operators \(P, Q\) are binomial. It is natural to expect that these lattices also define reductions for some multi-field equations related to more general operators \(P, Q\). The study of such models is beyond the scope of this paper and we restrict ourselves to three typical examples.

**Example 1.** First, let us consider the Lax equations \(P_t = BP - PA, Q_t = BQ - QA\) for the binomial operators \(P, Q\) with different potentials:

\[P = uT^3 + T, \quad Q = T^2 + v.\]

If \(v = u\), then the operators \(A, B\) are given by formulas (18), (20) with a self-adjoint operator \(F\) which contains only even powers of \(T^2\). In the general case two sets of operators \(A, B\) appear, containing positive or negative powers of \(T^2\). The simplest operators and corresponding flows are as follows:

\[A^- = v_2 v_1 T^{-2} + f_{-3} + f_{-2}, \quad B^- = v_{-1} v T^{-2} + f_{-1} + f,\]

\[u_{-1}^r = u(f_{-1} - f_1),\]

\[v_{-1}^r = v(f + f_{-1} - f_{-2} - f_{-3} - v_1 + v_{-1}), \quad f := uv_1 v_2;\]

\[A^+ = u_{-2} u_{-1} T^2 + g_{-1} + g, \quad B^+ = u u_{-1} T^2 + g + g_1,\]

\[u_{-1}^+ = u(g + g_1 - g_{-2} - g_{-3} - u_{-1} + u_1),\]

\[v_{-1}^+ = v(g_{-1} - g_{-g_1}), \quad g := u_{-2} u_{-1} v.\]

The flows \(\partial_+\) and \(\partial_-\) commute, and the flow \(\partial_+ = \partial_- - \partial_+\) admits the reduction \(v = u\) which brings to the dSK equation (1). It should be noted that the same flows can be obtained starting from the gauge equivalent operators \(P = u T^3 + T, Q = T + v\).

**Example 2.** Now let us consider the trinomial operators

\[P = u T^3 + p T^2 + T, \quad Q = T^2 + q T + v.\]

In this case, the operators \(A, B\) contain the odd powers of \(T\) as well. The simplest operators and the corresponding flows are of the form

\[A^- = v_{-1} T^{-1} + v_{-1} p_{-2}, \quad B^- = v T^{-1} + v_{1} p,\]

\[u_{-1}^r = u(u_{-1} q - u_{2} q_2 - p + p_1),\]

\[p_{-1}^r = p(u_{-1} q - u q_1) + u - u_{-1},\]

\[v_{-1}^r = v(u_{-1} q - u_{2} q_1),\]

\[q_{-1}^r = uv_1 - u_{-2} v;\]

\[A^+ = v_{-2} v_{1} T^{-2} + f_{-3} + f_{-2}, \quad B^+ = v_{-1} v T^{-2} + f_{-1} + f,\]

\[u_{-1}^+ = u(f_{-1} - f_1),\]

\[v_{-1}^+ = v(f + f_{-1} - f_{-2} - f_{-3} - v_1 + v_{-1}), \quad f := uv_1 v_2;\]

\[A^+ = u_{-2} u_{-1} T^2 + g_{-1} + g, \quad B^+ = u u_{-1} T^2 + g + g_1,\]

\[u_{-1}^+ = u(g + g_1 - g_{-2} - g_{-3} - u_{-1} + u_1),\]

\[v_{-1}^+ = v(g_{-1} - g_{-g_1}), \quad g := u_{-2} u_{-1} v.\]
Example 3. Let us consider the following generalization of the spectral problem (37):

\[ K^+ \psi = \lambda K \psi, \quad K = uT^3 + v^{-1}T + T^{-1}. \]

The isospectral deformations are defined by the operators

\[ A = u_{-2}T^2 + u_{-2}q_{-1}, \quad B = uT + u_{-1}q, \]

\[ u_{i+} = u_{i}(v_{1}p - v_{1}p_{1}), \quad p_{i+} = u_{i}v - uv_{2}, \]

\[ v_{i+} = v_{i}(v_{1}p - v_{1}p_{i-2} + q_{i-1} - q), \quad q_{i+} = q_{i}(v_{1}p - vp_{i-1}) + v_{1}. \]

The higher symmetry corresponding to \( k = 2 \) is too bulky and we do not write it down; however, one can check that it admits the reduction \( v = 0 \) to the dKK equation (36). In contrast, the flow \( \partial_{t} \) itself does not admit this reduction.

6. Conclusion

In this paper, we introduced a family of integrable lattice hierarchies associated with fractional Lax operators. In particular, these hierarchies contain equations found earlier in [1, 7] by use of the Hirota bilinear formalism. We proved that these equations serve as semi-discrete analogs of the SK and KK equations. An important question which remains open is about the Hamiltonian structure of the presented equations. As usual, the existence of the Lax representation allows us to obtain a set of conserved quantities which presumably are Hamiltonians, and moreover, the applicability of the \( r \)-matrix approach suggests that some more or less standard Poisson brackets should exist. However, no explicit answer is found yet. Another intriguing question is about possible relations with the models introduced in [27, 28] within the theory of the lattice \( W \) algebras. The fractional-type Lax operators may become important in the completely discrete setup as well, cf [29].

Acknowledgments

We are grateful to Ya P Pugay, Yu B Suris and A K Svinin for many stimulating discussions. The research of VA was supported by grant NSh–6501.2010.2.

References

[1] Tsujimoto S and Hirota R 1996 Pfaffian representation of solutions to the discrete BKP hierarchy in bilinear form J. Phys. Soc. Japan 65 2797–806
[2] Zakharov V E, Musher S L and Rubenchik A M 1974 Nonlinear stage of parametric wave excitation in a plasma JETP Lett. 19 151–2
[3] Manakov S V 1975 Complete integrability and stochastization of discrete dynamical systems Sov. Phys.—JETP 40 269–74
[4] Narita K 1982 Soliton solution to extended Volterra equation J. Phys. Soc. Japan 51 1682–5
[5] Itoh Y 1987 Integrals of a Lotka–Volterra system of odd number of variables *Prog. Theor. Phys.* 78 507–10
[6] Bogoyavlensky O I 1991 Algebraic constructions of integrable dynamical systems—extensions of the Volterra system *Rus. Math. Surv.* 46 1–64
[7] Hu X B, Clarkson P A and Bullough R 1997 New integrable differential–difference systems *J. Phys. A: Math. Gen.* 30 L669–76
[8] Sawada K and Kotera T 1974 A method for finding *n*-soliton solutions of the KdV equation and KdV-like equations *Prog. Theor. Phys.* 51 1555–67
[9] Caudrey P J, Dodd R K and Gibbon J D 1976 A new hierarchy of Korteweg–de Vries equations *Proc. R. Soc. A* 351 407–22
[10] Krichever I 1995 Linear operators with self-consistent coefficients and rational reductions of KP hierarchy *Physica D* 87 14–9
[11] Kaup D J 1980 On the inverse scattering problem for cubic eigenvalue problems of the class \( \psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi \) *Stud. Appl. Math.* 62 189–216
[12] Drinfel’d V G and Sokolov V V 1985 Lie algebras and equations of Korteweg–de Vries type *J. Math. Sci.* 30 1975–2036
[13] Sokolov V V and Shabat A B 1980 \((L, A)\)-pairs and a Riccati type substitution *Funct. Anal. Appl.* 14 148–50
[14] Musette M and Verhoeven C 2000 Nonlinear superposition formula for the Kaup–Kupershmidt partial differential equation *Physica D* 144 211–20
[15] Bogoyavlensky O I 1991 *Breaking Solitons. Nonlinear Integrable Equations* (Moscow: Nauka)
[16] Suris Y B 2003 *The Problem of Integrable Discretization: Hamiltonian Approach* (Basel: Birkhäuser)
[17] Reyman A G and Semenov-Tian-Shansky M A 2003 *Integrable Systems. Group-Theoretical Approach* (Izhevsk: Institute of Computer Science) (in Russian)
[18] Fordy A P and Gibbons J D 1980 Some remarkable nonlinear transformations *Phys. Lett. A* 75 325
[19] Miyazaki M and Marciniai K 1994 \( R \)-matrix approach to lattice integrable systems *J. Math. Phys.* 35 4661–82
[20] Yamilov R I 2006 Symmetries as integrability criteria for differential difference equations *J. Phys. A: Math. Gen.* 39 R541–62
[21] Svinin A K 2011 On some class of homogeneous polynomials and explicit form of integrable hierarchies of differential–difference equations J. Phys. A: Math. Theor. 44 165206
[22] Miura R M, Gardner C S and Kruskal M D 1968 Korteweg–de Vries equation and generalizations: II. Existence of conservation laws and constants of motion *J. Math. Phys.* 9 1204–9
[23] Satsuma J and Kaup D J 1977 A Bäcklund transformation for a higher order Korteweg–de Vries equation *J. Phys. Soc. Japan* 43 692–7
[24] Date E, Jimbo M, Kashiwara M and Miwa T 1981 KP hierarchies of orthogonal and symplectic type. Transformation groups for soliton equations: VI *J. Phys. Soc. Japan* 50 3813–8
[25] Parker A 2001 A reformulation of the ‘dressing method’ for the Sawada–Kotera equation *Inverse Problems* 17 885–95
[26] Pugay Y P 1994 Lattice \( W \) algebras and quantum groups *Theor. Math. Phys.* 100 900–11
[27] Hikami K 1999 Generalized lattice KdV type equation reduction of the lattice \( W_3 \) algebra *J. Phys. Soc. Japan* 68 46–50
[28] Kakei S, Nimmo J and Willox R 2010 Yang–Baxter maps from the discrete BKP equation *SIGMA* 6 028