GKM THEORY FOR TORUS ACTIONS WITH NON-ISOLATED FIXED POINTS

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ABSTRACT. Let $M^{2d}$ be a compact symplectic manifold and $T$ a compact $n$-dimensional torus. A Hamiltonian action, $\tau$, of $T$ on $M$ is a GKM action if, for every $p \in M^T$, the isotropy representation of $T$ on $T_p M$ has pair-wise linearly independent weights. For such an action the projection of the set of zero and one-dimensional orbits onto $M/T$ is a regular $d$-valent graph; and Goresky, Kottwitz and MacPherson have proved that the equivariant cohomology of $M$ can be computed from the combinatorics of this graph. (See [GKM].) In this paper we define a “GKM action with non-isolated fixed points” to be an action, $\tau$, of $T$ on $M$ with the property that for every connected component, $F$ of $M^T$ and $p \in F$ the isotropy representation of $T$ on the normal space to $F$ at $p$ has pair-wise linearly independent weights. For such an action, we show that all components of $M^T$ are diffeomorphic and prove an analogue of the theorem above.

1. INTRODUCTION

Let $M^{2d}$ be a compact symplectic manifold, $T$ an $n$-dimensional torus and $\tau$ a Hamiltonian action of $T$ on $M$. We will denote by $M^T$ the fixed point set of $\tau$ and by $M_1$ the set

$$\{x \in M, \dim T \cdot x = 1\};$$

and we will say that $\tau$ is a GKM action if

(1.1) \hspace{1cm} \#M^T < \infty

and

(1.2) \hspace{1cm} \dim M_1 = 2.

Let

$$V = \{p_1, \ldots, p_e\}$$

be the elements of $M^T$ and

$$E = \{e_1, \ldots, e_N\}$$

the connected components of $M_1$. For each $e_i$ let $\bar{e}_i$ be its closure in $M$. From the assumptions (1.1) and (1.2) one easily deduces:

(1) Each $\bar{e}_i$ is an embedded copy of $\mathbb{C}P^1$

(2) The set $\bar{e}_i - e_i$ is a two element subset of $V$.

(3) For $i \neq j$, $\bar{e}_i \cap \bar{e}_j$ is empty or is a one element subset of $V$. 

For $p \in V$, the set $\{e_i, p \in \tilde{e}_i\}$ is a $d$-element subset of $E$.

In other words, $V$ and $E$ are the vertices and edges of a regular $d$-valent graph, $\Gamma$.

Moreover, by noting how $T$ acts on each of the $\tilde{e}_i$'s one gets a labeling of the oriented edges of $\Gamma$ by elements of the weight lattice of $T$. Explicitly, if $e$ is an oriented edge of $\Gamma$ joining $p$ to $q$ one can assign to $e$ the weight of the isotropy representation of $T$ on the tangent space to $\tilde{e}$ at $p$. Denoting this weight by $\alpha_e$, the assignment, $e \mapsto \alpha_e$, defines a labeling of the type above. We will call this labeling the axial function on the graph $\Gamma$.

We will show in §2 how to construct from the data $(\Gamma, \alpha)$ a commutative ring, $H(\Gamma, \alpha)$, and sketch a proof of the main result of [GKM] which asserts that

\[(1.3) \quad H(\Gamma, \alpha) \simeq H_T(M; \mathbb{C}) = H^*_T(M).\]

Our main result is a generalization of the GKM theorem. To describe it, we recall that the hypotheses (1.1) and (1.2) can be reformulated somewhat differently.

**Proposition 1.1.** The conditions (1.1) and (1.2) are satisfied if and only if, for every $p \in M^T$, the weights, $\alpha_{i,p}$, $i = 1, \ldots, d$, of the isotropy representation of $T$ on $T_p M$ are pair-wise linearly independent, i.e., for $i \neq j$ $\alpha_{i,p}$ is not a multiple of $\alpha_{j,p}$.

This suggests imposing a slightly weaker “GKM hypothesis” on the action $\tau$:

**Definition 1.2.** The action, $\tau$, is a GKM action with non-isolated fixed points if, for every connected component, $F$ of $M^T$ and $p \in F$ the isotropy representation of $T$ on $T_p M$ has pair-wise linearly independent weights.

This relatively innocuous assumption has some surprising implications. Let

\[(1.4) \quad \{F_i; i = 1, \ldots, \ell\}\]

be the connected components of $M^T$, and let

\[(1.5) \quad \{W^0_i; i = 1, \ldots, N\}\]

be the connected components of $M_1$.

**Theorem 1.3.** The sets above have the following properties:

(a) The $F_i$'s are all diffeomorphic and, in particular, are all of the same dimension, $2m$.
(b) The closure, $W_i$, of $W^0_i$ is a symplectic submanifold of $M$ of dimension $2m + 2$.
(c) $W_i^T$ is the union of two $X_j$'s.
(d) $W_i \cap W_j$ is either empty or is a single $X_k$.
(e) For $X = X_j$

\[\#\{i, X \subseteq W_i\} = d - m.\]
Thus, as above, the sets (1.4) and (1.5) are the vertices, \( V \) and the edges, \( E \) of a regular \((d - m)\)-valent graph, \( \Gamma \). Moreover, as above, this is a labeled graph. If \( W \) is one of the \( W_i \)'s and \( X \) one of the two connected components of \( W^T \), the isotropy representation of \( T \) on the normal space to \( X \) in \( W \) at \( p \in X \) does not depend on \( p \). The weight of this representation gives one a labeling, \( e \rightarrow \alpha_e \), of the oriented edges of \( \Gamma \) by elements of the weight lattice of \( T \). We will prove in §4 the following generalization of (1.3).

**Theorem 1.4.** For \( F = F_i \),
\begin{align*}
H_T(M) & \simeq H(\Gamma, \alpha) \otimes H(F).
\end{align*}

The \( W_i \)'s, unlike the \( X_i \)'s, are not all diffeomorphic. However, if \( W \) is one of the \( W_i \)'s and \( F \) one of the two connected components of \( W^T \), we will prove:

**Theorem 1.5.** The normal bundle, \( \mathbb{L} \), of \( X \) in \( W \) is a complex line bundle and
\begin{align*}
W & \simeq \mathbb{P}(\mathbb{L} \oplus \mathbb{C}).
\end{align*}

We will conclude this introduction with a brief summary of the contents of this article. In §2 we will discuss in more detail the graph theoretic aspects of GKM theory. In particular we will describe what we mean by an “action” of a torus on a graph, define the “equivariant cohomology ring” of a graph, and sketch a proof of (1.3).

In §3 we will prove theorems 1.3 and 1.5 and in §4 we will prove theorem 1.4. In §5 we will discuss a few examples of “GKM actions with non-isolated fixed points”. All these examples are fiber bundles

\[ X \hookrightarrow M \xrightarrow{\pi} F \]

with the property that \( T \) acts fiberwise and that the action on the fiber, \( X \), is a GKM action in the usual sense. In §6 we discuss the symplectic structure of the \( X_i \)'s and show that the question of when two \( X_i \)'s are symplectomorphic is closely related to the question of when two \( W_i \)'s are diffeomorphic. Finally, in §7 we discuss some holonomy invariants of \( M \) whose vanishing may imply that \( M \) is a fiber product of the type above.

2. **T ACTIONS ON GRAPHS**

Let \( \Gamma \) be a regular \( d \)-valent graph and let \( V \) and \( E \) be the vertices and the oriented edges of \( \Gamma \). For every \( e \in E \) we will denote by \( \bar{e} \), the edge \( e \), with its orientation reversed and we will denote by \( i(e) \) and \( t(e) \) the initial and terminal vertices of \( e \). Thus \( t(\bar{e}) = i(e) \) and \( i(\bar{e}) = t(e) \). Let \( \mathcal{R}_k(T) \) be the set of (equivalence classes of) \( k \)-dimensional representations of \( T \).

**Definition 2.1.** We define an action of \( T \) on \( \Gamma \) to be a pair of maps
\[ \tau : V \rightarrow \mathcal{R}_d(T) \]
and
\[ \gamma : E \rightarrow \mathcal{R}_1(T) \]
satisfying

(A1) \( \tau_p = \bigoplus_{i(e) = p} \gamma_e \),

(A2) \( \gamma_e = \gamma_e^* \), and

(A3) \( \tau_p(g) = \tau_q(g) \) for \( p = i(e), q = t(e) \) and \( g \in \text{Ker}(\gamma_e) \).

**Remark 2.2.** By (A1), the mapping \( \tau \) is determined by the mapping \( \gamma \). Moreover, if for every oriented edge, \( e \), we let \( \alpha_e \) be the weight of the representation \( \gamma_e \), \( \gamma_e \) is determined by \( \alpha_e \). Hence \( \gamma \) and \( \tau \) are determined by the axial function, \( \alpha \), which assigns to each oriented edge, \( e \), the element, \( \alpha_e \), of the weight lattice of \( T \).

**Remark 2.3.** The axioms (A1)–(A3) translate into axioms on \( \alpha \). For instance axiom (A2) is equivalent to: \( \alpha_e = -\alpha_e \).

**Remark 2.4.** Let \( \tau \) be a GKM action of \( T \) on \( M \), and for each \( p \in M^T \) let \( \tau_p \) be the isotropy representation of \( T \) on the tangent space to \( M \) at \( p \). If \( e \) is a connected component of \( M^1 \) we can orient \( e \) by specifying that one of the two points in \( \bar{e} - e \) is the initial vertex \( i(e) = p \), of \( e \) and the other the terminal vertex, \( t(e) = q \). If we let \( \gamma_e \) be the isotropy representation of \( T \) on the tangent space to \( e \) at \( p \), the mappings, \( p \to \tau_p \) ad \( e \to \gamma_e \) define an action of \( T \) on \( \Gamma \).

**Remark 2.5.** Let \( \tau \) be a GKM action of \( T \) on \( M \) “with non-isolated fixed points”. Let \( X_i \), \( i = 1, \ldots, \ell \), and \( W^0_i \), \( i = 1, \ldots, N \), be the connected components of \( M^T \) and \( M^1 \); and for each \( W^0_i \), let \( W_i \) be its closure in \( M \). Then, by Theorem 1.3 the sets

\[ V = \{ X_i, \quad i = 1, \ldots, \ell \} \]

and

\[ E = \{ W_i, \quad i = 1, \ldots, N \} \]

are the vertices and edges of a regular \( (d - m) \)-valent graph. For each \( X_i \), let \( \tau_i \) be the isotropy representation at \( p \in X_i \) of \( T \) on the normal space to \( X_i \) at \( p \). (Since \( X_i \) is connected this representation doesn’t depend on \( p \).) Similarly, for every \( W_i \) et \( X_j \) and \( X_k \) be the connected components of \( W_i - W^0_i \) and let \( \gamma_{j,k} \) be the isotropy representation, at \( p \in X_j \), of \( T \) on the normal space at \( p \) to \( X_j \) in \( W_i \). Then the mappings, \( i \to \tau_i \) and \( (j, k) \to \gamma_{j,k} \) define an action of \( T \) on \( \Gamma \).

Let \( t = \text{Lie}(T) \) and let \( S(t^*) \) be the ring of polynomial functions on \( t \). Given a graph \( T \) and an action of \( T \) on \( \Gamma \) we will define the equivariant cohomology ring, \( H(\Gamma, \alpha) \), of \( \Gamma \) to be the set of maps

\[ f : V \to S(t^*) \]

which, for all \( e \in E \), satisfy the compatibility condition

\[ f_p - f_q \in \alpha_e \cdot S(t^*) \]

for \( p = i(e) \) and \( q = t(e) \). The GKM theorem asserts

**Theorem 2.6.** If \( M \) is a GKM manifold

\[ H(\Gamma, \alpha) \cong H_T(M) \]

We will give a brief sketch of how to prove this since we will prove Theorem 1.4 by mimicking this proof.
Step 1. (The Kirwan formality theorem.) This asserts that as an $S(t^*)$ module,
\begin{equation}
H_T(M) = H(M) \otimes S(t^*).
\end{equation}
By a theorem of Borel, the restriction map
\begin{equation}
r : H_T(M) \to H_T(M^T)
\end{equation}
has, as kernel, the torsion elements in $H_T(M)$; however, Kirwan’s theorem implies that $H_T(M)$ is a free $S(t^*)$ module hence (2.3) is injective. Moreover, since $T$ acts trivially on $M^T$ and $M^T$ is finite, $H_T(M^T)$ is a sum of copies of $S(t^*)$
\begin{equation}
\bigoplus_{p \in M^T} S(t^*)_p
\end{equation}
or alternatively is the set of maps,
\begin{equation}
f : V \to S(t^*).
\end{equation}
We claim
\begin{lemma}
The image of $r$ is contained in the subring, $H(\Gamma, \alpha)$ of the ring of maps of $V$ into $S(t^*)$.
\end{lemma}
Proof. Let $e$ be a connected component of $M_1$, let $e$ be its closure and let $p$ and $q$ be the elements of $e - e^0$. The kernel $T_e$ of $\gamma_e : T \to S^1$ acts trivially on $e$; therefore, denoting by $t_e$ the Lie algebra of $T_e$
\begin{equation}
H_T(e) = H(e) \otimes S(t_e^*).
\end{equation}
In particular, letting $i_p$ and $i_q$ be the inclusions of $p$ and $q$ into $e$ the induced maps
\begin{equation}
i_p^* : H_T(e) \to S(t^*_e)
\end{equation}
and
\begin{equation}
i_q^* : H_T(e) \to S(t^*_e)
\end{equation}
are identical. In particular for every $f$ in the image of the restriction map $f_p$ and $f_q$ have to satisfy the compatibility condition (2.1).

Step 2. Betti numbers. Fix a vector $\xi \in t$ such that for all $e \in E \alpha_e(\xi) \neq 0$ and for every $p \in V$ let
\begin{equation}
\sigma_p = \# \{ \alpha_e ; 2(e) = p \text{ and } \alpha_e(\xi) < 0 \}.
\end{equation}
We define the $i$th Betti number $\beta_i(\Gamma)$ of the graph $\Gamma$ to be:
\begin{equation}
\# \{ p \in V, \sigma_p = i \}.
\end{equation}
The numbers (2.5) depend on the choice of $\xi$, however one can show that the numbers (2.6) don’t. Moreover, one can prove by elementary Morse theory that $\beta_{2i+1}(M) = 0$ and
\begin{equation}
\beta_i(\Gamma) = \beta_{2i}(M).
\end{equation}
Thus by the Kirwan formality theorem
\begin{equation}
\dim H_T^{2k}(M) = \sum \beta_i(\Gamma) \dim S(t^*)^{k-i}.
\end{equation}
Note that since the odd Betti numbers of $M$ are zero, (2.2) implies that the odd equivariant cohomology groups of $M$ are zero.

**Step 3. Graph theoretic Morse inequalities.** These assert that
\begin{equation}
\dim H^k(\Gamma, \alpha) \leq \sum \beta_i(\Gamma) \dim S(t^*)^{k-\ell}.
\end{equation}
For the relatively elementary proof of these inequalities, see [GZ]. Combining steps 1, 2 and 3 we conclude that the map (2.3) maps $H_T(M)$ bijectively onto $H(\Gamma, \alpha).$ \hfill $\Box$

### 3. The fixed points

Let $M$ be a compact connected symplectic manifold of dimension $2d$, and let $T$ be an $n$-torus and let $\tau$ be a Hamiltonian action of $T$ on $M$ with moment map $\Phi : M \to t^*$. We make the following GKM assumption: For every connected component $F$ of $M^T$, the weights
\begin{equation}
\alpha_{i,F} \quad i = 1, \ldots, r
\end{equation}
of the isotropy representation of $T$ on the normal bundle to $F$ are pairwise linearly independent. As above, we call such a manifold a non-isolated GKM manifold. We begin by examining the fixed point components of $M^T$, in the case when $T = S^1$ is a circle.

**Lemma 3.1.** Let $S^1$ act on a compact, connected symplectic manifold $M$ in Hamiltonian fashion, with moment map $\phi : M \to \mathbb{R}$. Every connected component of $M^{S^1}$ is of codimension 2 in $M$.

**Proof.** Let $E$ be such a component and let $N \to E$ be the normal bundle to $E$ in $M$. The weights of the isotropy representation of $T$ on $N^H$ all have to be multiples of $\alpha$, since $t^*$ is one-dimensional. So, because the weights must be two-independent by the GKM assumption above, there can be only one weight, and so $\dim_{\mathbb{R}}(N^H) = 2$. \hfill $\Box$

At any point $p \in F$, the Darboux theorem for $\phi$ says that there exists a Darboux coordinate system centered at $p$: coordinates $x_1, y_1, \ldots, x_d, y_d$ such that locally near $p$, $F$ is defined by $x_1 = y_1 = 0$ and
\[ \phi = \phi(p) + \alpha(\xi)(x_1^2 + y_1^2). \]
Thus, the component $F$ is either a maximum or a minimum of $\phi$ depending on whether $\alpha(\xi) < 0$ or $\alpha(\xi) > 0$. The same is true for every other component of $M^{S^1}$. However, by the Atiyah convexity theorem, $\phi$ has at most one connected level set where it takes its maximum value and one connected level set where it takes its minimum value. Thus, $M^{S^1}$ has exactly two connected components, $F$ and $E$. 

Lemma 3.2. The components $F$ and $E$ are diffeomorphic.

Proof. We can assume $\phi = 0$ on $F$ and $\phi = 1$ on $E$. Let $\mathbb{L}$ be the normal bundle to $F$ in $X$. We can regard $\mathbb{L}$ as a complex line bundle. By the equivariant tubular neighborhood theorem, the action of $T$ on $X$ is identical, near $F$ with the linear action of $T$ on $\mathbb{L}$ and $\phi$ is just the length-squared function for a Hermitian metric on $\mathbb{L}$. Thus, for $c$ close to 0,

$$\phi^{-1}(c)/S^1 = F.$$ 

But all level sets, $\phi^{-1}(c)$ for $0 < c < 1$ are equivariantly diffeomorphic, since there are no critical values between 0 and 1. So all the reduced spaces

$$\phi^{-1}(c)/S^1 \quad 0 < c < 1$$

are diffeomorphic. But for $c$ close to 1,

$$\phi^{-1}(c)/S^1 = E.$$ 

by the same argument as above. Hence, $F$ and $E$ are diffeomorphic. □

The result can be sharpened. From the action of $T$ on $\mathbb{L}$ one gets an action of $T$ on the bundle

$$P(\mathbb{L} \oplus \mathbb{C}).$$

Theorem 3.3. The component $M$ and $P(\mathbb{L} \oplus \mathbb{C})$ are isomorphic as $S^1$-manifolds.

Proof. Equip $L$ with an inner product, and let $\psi : L \rightarrow [0, 1)$ be the function defined by

$$\psi(x, v) = \frac{|v|^2}{|v|^2 + 1}.$$ 

This extends to a Morse-Bott function

$$\psi : P(\mathbb{L} \oplus \mathbb{C}) \rightarrow [0, 1]$$

whose critical sets, $\psi^{-1}(1)$ and $\psi^{-1}(0)$, are copies of $F$. The set $\phi^{-1}([\varepsilon, 1 - \varepsilon])$ can be identified with $F \times [\varepsilon, 1 - \varepsilon])$. So

$$M_\varepsilon = \phi^{-1}([\varepsilon, 1 - \varepsilon]) \rightarrow \mathbb{C} = F \times [\varepsilon, 1 - \varepsilon])$$

is a circle bundle with Chern class $c(\mathbb{L})$. Similarly,

$$P^\varepsilon(\mathbb{L} \oplus \mathbb{C}) = \psi^{-1}([\varepsilon, 1 - \varepsilon])$$

is a circle bundle over this set with Chern class $c(\mathbb{L})$. Hence, these bundles are isomorphic as manifolds with boundary. One obtains $P(\mathbb{L} \oplus \mathbb{C})$ and $M$ from these manifolds by collapsing the circle orbits on the boundary to points; however, the boundaries are isomorphic as circle bundles. Thus, the spaces obtained by this collapsing are isomorphic. □

Now let $T$ be an $n$-torus, for $n > 1$, and $\tau$ a GKM action on $M$ with non-isolated fixed points.

Theorem 3.4. All connected components of $M^T$ are diffeomorphic.
Proof. Let $F$ be a connected component of $M^T$. If $\alpha$ is one of the isotropy weights on the normal bundle to $F$, let
\[ \mathfrak{h} = \{ \xi \in t \mid \alpha(\xi) = 0 \}, \]
and let $H$ be the subtorus of $T$ with $\text{Lie}(H) = \mathfrak{h}$. Let $X$ be the connected component of $M^H$ containing $F$.

Now let $\xi \in t \setminus \mathfrak{h}$ and let $\phi^\xi$ be the restriction of $\Phi^\xi$ to $X$. We can regard $\phi^\xi$ as the moment map associated with the action of the circle $T/H$ on $X$. By Lemma 3.2, $X$ contains precisely two components of $M^T$, and they are diffeomorphic. Let $\Gamma$ be the GKM graph associated with the $T$-action on $M$.

To show that all connected components of $M$ are diffeomorphic, we must show that $\Gamma$ is connected. Let $\phi$ be a Bott-Morse component of $\Phi$ with $\text{Crit}(\phi) = M^T$.

That is, $\phi$ is a generic component of $\Phi$. From $\phi$, $\Gamma$ inherits a poset structure, and for each connected component $\Gamma_0$ of $\Gamma$, the vertex of $\Gamma_0$ at which $\phi$ takes its minimum corresponds to a component of $M^T$ at which $\phi$ takes on a minimum value. But the Atiyah convexity theorem says that there is a unique component of $M^T$ on which this can occur. Thus, $\Gamma$ is connected, completing the proof of the theorem. \[ \square \]

Remark 3.5. The generic component $\phi$ of $\Phi$ in the proof above is a perfect Bott-Morse function, with diffeomorphic critical sets. We will use this fact below to analyze the equivariant topology of $M$.

Remark 3.6. The connected components of $M^T$ are not symplectomorphic. We will further discuss these symplectic structures in Section 6.

4. GKM Theory

Suppose $M$ is a compact, connected symplectic manifold, and $\tau : M \times T \to M$ a GKM action with non-isolated fixed points. Let $\Gamma$ be the GKM graph associated to $M$ and $\tau$. Each edge $e$ of $\Gamma = (V, E)$ is labeled by a one-dimensional representation $\gamma_e$ of $T$. Let $T_e$ be the kernel of $\gamma_e$, and $t_e$ its Lie algebra. To each vertex of the graph, $\Gamma$, we attach the ring $R = H^*_T(F) = H^*(F) \otimes S(t^*)$, and to each edge, $e$, the ring $R_e = H^*_T_e(F) = H^*(F) \otimes S(t_e^*)$. The map
\[ S(t^*) \to S(t_e^*) \]
induces a map
\[ \pi_e : R \to R_e. \]

Definition 4.1. A map $f : V \to R$ is $\Gamma$-compatible if, for every edge $e = (p, q)$,
\[ \pi_e(f(p)) = \pi_e(f(q)). \]

Let $H^*(\Gamma, F)$ be the ring of all of these maps.
We recall that Theorem 1.4 asserts
\[ H^*_T(M) = H^*(\Gamma, F). \]
To prove this, we note that the Kirwan map
\[ H^*_T(M) \hookrightarrow H^*_T(M^T) \]
is an injection. Hence, if \( F_\ell \) are the connected components of \( M^T \), for \( \ell = 1, \ldots, N \), then \( H^*_T(M) \) sits inside the ring
\[ H^*_T(M^T) = \bigoplus \ell H^*_T(F_\ell) \]
and since \( F_i \cong F \), \( H^*_T(F_i) = H^*_T(F) = R \). Thus, \( H^*_T(M^T) \) consists of \( N \) copies of \( R \), each labeled by a vertex of \( \Gamma \). In other words,
\[ H^*_T(M^T) = \text{Maps}(V, R), \]
and in particular, \( H^*(\Gamma, F) \) is a subring of \( H^*_T(M^T) \). We now prove the main theorem. We break the proof of the theorem into three steps.

**Proof of Theorem 1.4**

**Step One:** We will show that the restriction map
\[ (4.2) \quad H^*_T(M) \hookrightarrow H^*_T(M^T) \]
maps \( H^*_T(M) \) into \( H^*(\Gamma, F) \).

Let \( L \) be a line bundle over \( F \), and let \( X = P(L \oplus \C) \). Let \( \mathbf{L} \) be the tautology bundle over \( X \), and let \( x \) be its Chern class. By the Leray-Hirsch theorem,
\[ H^*(X) = H^*(F) \oplus x \cdot H^*(F). \]
Let \( \iota_+: F \to X \) be the embedding of \( F \) onto \( P(\{0\} \oplus \C) \). Then \( \iota_+^* \mathbf{L} = 1 \), so \( \iota_+^* x = 0 \). Similarly, identifying \( P(L \oplus \C) \) with \( P(C \oplus L^{-1}) \), and letting \( \iota_-: F \to X \) be the embedding onto \( P(C \oplus \{0\}) \), one has \( \iota_-^* x = 0 \). Thus, by (4.3),
\[ (4.4) \quad \iota_+^* c = \iota_-^* c \]
for every cohomology class \( c \in H^*(X) \).

Now let \( F = F_i \), let \( T_e \) be a codimension one subgroup of \( T \) and let \( X \) be a connected component of \( M^T \) containing \( F \). Let \( \iota_\pm: F \to X \) be the embeddings onto the two components of \( X^T \). Then by (4.2), the maps
\[ H^*_T(X) \rightarrow H^*_T(X) = H^*(X) \otimes S(t^*_e) \rightarrow R_e \]
are identical. Thus, if \( f \) is in the image of (4.2), it has to satisfy compatibility conditions (4.1) for every edge \( e \) of the graph \( \Gamma \).
Step Two: We first notice that $H^*(\Gamma, F) = H^*(\Gamma) \otimes H^*(F)$ and $R = H^*(F) \otimes S(t^*)$. Hence, tensoring (2.9) with $H^*(F)$, we get the inequality
\begin{equation}
\dim H^k(\Gamma, F) \leq \sum_i \dim R^{k-i} \beta_i,
\end{equation}
where the $\beta_i$ are the Betti numbers of $\Gamma$.

Step Three: Recall that $F_\ell$ is a connected component of the critical set of the Bott-Morse function $\phi = \Phi^\xi$. Using Bott-Morse theory, Atiyah proves
\begin{equation}
\dim H^k_T(M) = \sum_i \dim H^{k-d_i}_T(F_\ell),
\end{equation}
where $d_\ell$ is the index of $F_\ell$ (see [A]). Moreover, by elementary Morse theory, it is easy to show that
\begin{equation}
\frac{d_\ell}{2} = \sigma_p
\end{equation}
where $p$ is the vertex of $\Gamma$ corresponding to $F_\ell$. Thus, from (4.6) and (4.7), one deduces
\begin{equation}
\dim H^*_T(M) = \sum_i \dim R^{k-i} \beta_i.
\end{equation}
Finally, the identity (4.8) and the inequality (4.5) imply that the Kirwan injection is a bijection of $H^*_T(M)$ onto $H^*(\Gamma, F)$. \qed

5. Examples

We now describe several examples of GKM actions with non-isolated fixed points, all of which start with the same basic set-up. Let $F$ be a symplectic manifold, and let $\mathbb{L}_i$, for $i = 1, \ldots, n$ be complex line bundles over $F$. Let $E = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n$.

The group $T = T^n$ acts on $E$ by acting fiberwise on the fiber $E_p = (\mathbb{L}_1)_p \oplus \cdots \oplus (\mathbb{L}_n)_p$ by the action
\[\tau(e^{i\theta})(v_1, \ldots, v_n) = (e^{i\theta_1} \cdot v_1, \ldots, e^{i\theta_n} \cdot v_n)\]
We use minimal coupling to produce a symplectic form $\omega$ on $E$, and $T$ acts in a Hamiltonian fashion with respect to $\omega$.

5.1. Projective bundles. Let
\begin{equation}
P(E) \rightarrow F
\end{equation}
be the projectivization of $E$. From the action of $T$ on $E$, one gets a fiberwise action of $T$ on $P(E)$ whose fixed points are copies of $F$, and which satisfies our non-isolated GKM axiom. The graph associated to this space in the $n$-simplex, the same graph associated to complex projective space in the ordinary GKM setting.
5.2. Grassmannian bundles. In a similar vein, let

\[ G_r^k(E) \to F \]

be the fiber bundle over \( F \) whose fiber at \( p \) is the Grassmannian \( G_r^k(E_p) \). The \( T \) action on \( E \) defines a fiberwise action of \( T \) on the fiber of (5.2) which also satisfies our non-isolated GKM axiom. The graph associated to this space in the Johnson graph \( J(n, k) \), whose vertices consist of \( k \)-element subsets of an \( n \)-element set. This is the same graph associated to the complex Grassmannian \( G_r(n, k) \) in the ordinary GKM setting.

5.3. Partial flag bundles. One can continue in the vein of (5.1) and (5.2) and take fiber bundles with fiber some partial flag variety of \( E_p \).

5.4. Toric bundles. In the last example, we apply symplectic reduction to the above examples. The torus \( T^n \) acts fiberwise on the bundle \( E \). We may make a symplectic reduction by \( T^k \) to obtain

\[ \mathbb{C}^n // T^k \to E // T^k \]

a fiber bundle over \( F \) whose fiber is a complex toric variety. In this case, the moment image of \( E \) is a simple convex polytope.

These examples all exemplify the situation where there is a fiber bundle \( M \to F \) with fiber \( X \), and a fiberwise action of \( T \). Modulo assumptions on \( F \) and \( X \), the Leray-Hirsch theorem asserts

\[ H^*_T(M) \cong H^*(F) \otimes H^*_T(X). \]

Hence, if \( X \) is a GKM manifold, one gets from (5.3) the same result as Theorem 1.4.

6. The symplectic forms on the fixed point sets

Let \( F \) be a connected component of \( M^T \), \( H \) a codimension one subgroup of \( T \), and \( X \) a connected component of \( M^H \) containing \( F \). Let \( F' \) be the other connected component of \( M^T \) in \( X \). Without loss of generality, we may assume that \( T/H \) acts faithfully on \( X \). Let \( \xi \in t \setminus h \) be the generator of this group, normalized so that \( \exp(2\pi \xi) = 1 \), and let \( \phi^\xi \) be the \( \xi \)-component of the moment map \( \Phi \). Replacing \( \xi \) by \(-\xi \) if necessary, we may assume that the restriction of \( \phi^\xi \) takes its minimum value on \( F \).

Now let \( e \) denote the oriented edge of \( \Gamma \) joining the vertices corresponding to \( F \) and \( F' \). We will call the difference

\[ a_e = \phi^\xi(F') - \phi^\xi(F) \]
the length of $e$. Let $\mathbb{L}_e$ be the normal bundle to $F$ in $X$. If $\pi$ is the edge $e$ with its orientation reversed, then $\mathbb{L}_\pi$ is the normal bundle to $F'$ in $X$. So in view of the isomorphisms

$$X \cong \mathbb{C}P(\mathbb{L} \oplus 1) \cong \mathbb{C}P(1 \oplus \mathbb{L}^{-1}),$$

we have

$$\mathbb{L}_e \cong \mathbb{L}_\pi,$$

and hence if $c_e$ and $c_\pi$ are the Chern classes of $\mathbb{L}_e$ and $\mathbb{L}_\pi$ respectively, then

$$c_\pi = -c_e.$$

From the Duistermaat-Heckman theorem, one deduces the following theorem.

**Theorem 6.1.** Let $\omega_F$ and $\omega_{F'}$ be the symplectic forms on $F$ and $F'$. Then

$$[\omega_{F'}] = [\omega_F] + a_e c_e.$$

We will discuss a few other properties of the assignment $e \mapsto c_e$. We recall that a GKM action with non-isolated fixed points has the following property: for every connected component $F$ of $M^T$, and for every $p \in F$, the weights of the isotropy representation of $T$ on the normal space to $F$ at $p$ are pairwise linearly independent. We will call a representation with this property 2-independent. More generally, we will call a representation of $T$ with weights $\alpha_i$, for $i = 1, \ldots, N$ k-independent if every subset of $k$ weights is linearly independent. Now let $\Gamma$ be a regular $d$-valent graph, as in Section 2, and $(\tau, \gamma)$ an action of $T$ on $\Gamma$. For $p \in V$, let

$$E_p = \{ e \in E \mid i(e) = p \}.$$

A connection on $\Gamma$ is a function, $\nabla$, which assigns to every edge $e \in E$ with $p = i(e)$ and $q = t(e)$ a bijective map

$$\nabla_e : E_p \rightarrow E_q$$

with the property

$$\nabla_\pi = \nabla_e^{-1}.$$

We say the connection is compatible with the action of $T$ if, for every edge $e \in E$,

$$\nabla_e e' = e'' \implies a_{e'} \equiv a_{e''} \mod a_e.$$

We leave the following strengthening of Axiom (A3) of Definition 2.1 as an easy exercise.

**Lemma 6.2.** Suppose that for all $p \in V$, $\tau_p$ is 3-independent. Then there exists a unique $T$-compatible connection $\nabla$ on $\Gamma$.

In particular, let $\Gamma$ be the graph associated to $M$. Then by definition, $M$ is GKM if for every $p \in V$, $\tau_p$ is 2-independent. We make the stronger assumption that for every $p \in V$, $\tau_p$ is 3-independent. Then the connection $\nabla$ is not only compatible with the action of $T$, but is also compatible with
the assignment $e \mapsto c_e$. More explicitly, let $e_1, \ldots, e_k$ be the oriented edges of $\Gamma$ with initial vertex $F$, and let $e'_1, \ldots, e'_k$, with $e'_1 = \overline{e_1}$, be the oriented edges of $\Gamma$ with initial vertex $F'$. We will order these edges so that $\nabla_{e_1}$ maps $e_i$ to $e'_i$. We claim that

\begin{equation}
(6.6) c_{e'_i} = c_{e_i} \text{ for } i = 2, \ldots, k,
\end{equation}

and

\begin{equation}
(6.7) c_{e_1} = -c_{e'_1}.
\end{equation}

**Proof.** The second identity (6.7) follows immediately from (6.3). To prove (6.6), let $a_i = a_{e'_i}$, and let $K_i = \exp(\xi_i)$, where

$$
\xi_i = \{ \xi \in t \mid a_i(\xi) = a_1(\xi) = 0 \}.
$$

Let $Y_i$ be the connected component of $M^{K_i}$ containing $X$. Since the action of $T$ is $3$-independent, the codimension of $X$ in $Y_i$ is $2$. Let $L_i$ be the normal bundle of $X$ in $Y_i$. Then the restriction of $L_i$ to $F$ is $L_{e_i}$, and the restriction of $L_i$ to $F'$ is $L_{e'_i}$. Hence if $c_i$ is the Chern class of $L_i$ in $H^2(X; \mathbb{Z})$, its restriction to $F$ is $c_{e_i}$, and its restriction to $F'$ is $c_{e'_i}$. Thus, by (4.4), $c_{e_i} = c_{e'_i}$. \(\square\)

### 7. Holonomy and Diffeomorphisms of the Fixed Point Sets

The GKM manifolds $M$ discussed in Section 5 are all twisted products of $F$ with GKM manifolds with isolated fixed points. We now describe a holonomy invariant of the graph of $M$ which measures the failure of $M$ to be such a twisted product. Let $E$ and $F$ be components of $M^T$. Then, as we explain in Section 3, one can construct a diffeomorphism of $E$ onto $F$ using Morse theory. This diffeomorphism is not unique, but it is easy to see that it is unique up to isotopy.

In particular, suppose that $\gamma$ is a closed path in $\Gamma$ whose initial and terminal vertices are the vertex corresponding to $F$. Then by composing the diffeomorphisms above, one can associate to $\gamma$ a diffeomorphism of $F$ onto itself which is unique up to isotopy. Thus, letting $G = \pi_0(Dif(M))$ and letting $\pi_1(\Gamma, F)$ be the fundamental group of $\Gamma$ with base point $F \in V$, one gets a homomorphism

\begin{equation}
(7.1) \Theta : \pi_1(\Gamma, F) \to G,
\end{equation}

which is the holonomy invariant alluded to above.

If $M$ is Kähler, and the action of $T$ preserves the Kähler structure, one has a slightly more refined version of this invariant. Namely, in this case, the action of $T^n$ on $M$ extends to a holomorphic action of the complex torus $T_C^n = (\mathbb{C}^*)^n$. In particular, if $n = 1$, one has a $\mathbb{C}^*$ action on $M$ and the diffeomorphism between $E$ and $F$ in Lemma 3.2 is given explicitly by the map

$$
e \in F \mapsto y \in F$$
if and only if, for some \( m \in M \),

\[
x = \lim_{z \to 0} \tau_z(m)
\]

and

\[
y = \lim_{z \to \infty} \tau_z(m).
\]

In this case, the diffeomorphism is canonically defined and is a biholomorphism. Hence, in the Kähler case, the holonomy invariant becomes a homomorphism

(7.2) \[ \Theta : \pi_1(\Gamma, F) \to \text{Bihol}(F). \]

It is clear that if \( M \) is a fiber product of the type discussed in \( \S 5 \), then these invariants vanish. We do not know if the converse is true.

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