Dimension via Waiting time and Recurrence

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Abstract

Quantitative recurrence indicators are defined by measuring the first entrance time of the orbit of a point $x$ in a decreasing sequence of neighborhoods of another point $y$. It is proved that these recurrence indicators are a.e. greater or equal to the local dimension at $y$, then these recurrence indicators can be used to have a numerical upper bound on the local dimension of an invariant measure.

1 Introduction

The first well known results about recurrence in a dynamical system $(X, T)$ state that under suitable assumptions a typical trajectory of the system comes back infinitely many times in any neighborhood of its starting point. These quantitative results does not give a quantitative estimation about the speed of this coming back to the starting point.

A more precise analysis of recurrence was done by defining quantitative recurrence indicators. In the literature such indicators have been defined in several ways by measuring the first return time of an orbit in a decreasing sequence of neighborhoods of the starting point. These sequences of neighborhoods have been defined by the metric of the space $X$, considering a decreasing sequence of balls ([BS], [BS]) or with respect to the symbolic dynamics induced by a partition, considering a decreasing sequence of cylinders $c_k$ on the associated symbolic space ([OW]). Other definitions consider the forward images of the whole cylinder $c_k$ and consider as a first return for the cylinder the minimum $n$ such that $T^n(c_k) \cap c_k \neq \emptyset$ ([HSV], [ACS2], [STV], [BGI], [CD]). In the above cited papers many relations have been then proved between
these indicators and other important features of dynamics (for example dimension, entropy, orbit complexity, Lyapunov exponents, mixing properties).

Barreira and Saussol in [BS] prove some relations between quantitative recurrence and the local dimension of an invariant measure. In an applicative, framework, this relation can be used to estimate efficiently this local dimension. If some technical assumptions are satisfied their recurrence indicator is indeed a.e. equal to the dimension of the invariant measure. Their assumptions are satisfied for example in hyperbolic systems, with an equilibrium measure supported on a locally maximal hyperbolic set.

In computer simulations or experimental situations, the quantitative recurrence indicators can be easily estimated by looking to the behavior of a "typical, random" orbit and to its first entrance time in a sequence of balls centered in the starting point. By the above results this can give a numerical estimation of the pointwise dimension of the underlying invariant measure, which is not easy to be known in general. In general systems (where the additional assumptions are not satisfied) however the Barreira and Saussol recurrence indicator gives only a lower bound on the dimension.

A natural generalization of the quantitative recurrence indicators can be defined by measuring how fast the orbit of a point \( x \) approaches near to another point \( y \). In the literature about finite alphabet stochastic processes and symbolic systems indicators of this type were called waiting times. Relations between waiting time and entropy similar to the Oerstein Weiss theorem were proved ([Sh]). Such relations holds for Markov chains and in a weaker version in weak Bernoulli processes. In the case of finite type shifts with a Gibbs measure other results were given by [Ch]. Another recent work ([KS]) calculates the waiting time for irrational translations on the circle.

In this work we consider general systems acting on a metric space and we consider a recurrence-waiting time indicator that generalizes the former ones. Then we prove some result relating this indicator to the local dimension (of a measure defined on the space were the dynamics acts).

In [BCI] a quantitative recurrence argument of this kind was used to estimate the initial condition sensitivity and the orbit complexity of interval exchange transformations \(^1\). In the present work this argument is extended. The results we present give an upper bound to the local dimension of the

\(^1\)In these maps the main source of initial conditions sensitivity is given by the fact that nearby starting orbits can be separated by the discontinuities of the map. For this reason the initial condition sensitivity is estimated when we estimate how near we go to the discontinuity points.
measure at the point \( y \) in terms of the generalized recurrence indicator. Moreover, these generalized recurrence indicators are also easy to be estimated numerically by looking to the behavior of a ”typical” orbit, measuring its first entrance time in a decreasing sequence of balls centered in \( y \). These results and the ones given in [BS] can be combined to have upper and lower bounds on the (upper and lower) local dimension of general systems. One last remark is that the following results also hold in systems with an infinite invariant measure.

## 2 Recurrence and dimension

In the following we will consider a discrete time dynamical system \((X, T)\) were \( X \) is a separable metric space equipped with a Borel locally finite measure \( \mu \) and \( T : X \to X \) is a measurable map (we remark that we do not assume \( \mu(X) = 1 \)).

Let us consider the first entrance time of the orbit of \( x \) in the ball \( B(y, r) \) with center \( y \) and radius \( r \)

\[
\tau_r(x, y) = \min(\{n \in \mathbb{N}, n > 0, T^n(x) \in B(y, r)\}).
\]

By this let us define the quantitative recurrence indicators

\[
\overline{R}(x, y) = \limsup_{r \to 0} \frac{\log(\tau_r(x, y))}{-\log(r)} = \limsup_{n \to \infty} \frac{\log(\tau_{2^{-n}}(x, y))}{n}
\]

\[
\underline{R}(x, y) = \liminf_{r \to 0} \frac{\log(\tau_r(x, y))}{-\log(r)} = \limsup_{n \to \infty} \frac{\log(\tau_{2^{-n}}(x, y))}{n}.
\]

If for some \( r \) \( \tau_r(x, y) \) is infinite then \( \overline{R}(x, y) \) and \( \underline{R}(x, y) \) are set to be equal to infinity. The indicators \( \overline{R}(x) \) and \( \underline{R}(x) \) of quantitative recurrence defined in [BS] are obtained as a special case, \( \overline{R}(x) = \overline{R}(x, x), \underline{R}(x) = \underline{R}(x, x) \).

We state some first properties of \( R(x, y) \). The proof follows directly from the definitions.

**Proposition 1** \( R(x, y) \) satisfies the following properties

- \( \overline{R}(x, y) = \overline{R}(T(x), y), \underline{R}(x, y) = \underline{R}(T(x), y) \).
- If \( T \) is Lipschitz, then \( \overline{R}(x, y) \geq \overline{R}(x, T(y)), \underline{R}(x, y) \geq \underline{R}(x, T(y)) \).
• If $T$ is $\alpha$–Hoelder, then $\overline{R}(x,y) \geq \alpha \overline{R}(x,T(y))$, $\underline{R}(x,y) \geq \alpha \underline{R}(x,T(y))$.

Now we are interested to prove relations with dimension. If $X$ is a metric space and $\mu$ is a measure on $X$ the upper local dimension at $x \in X$ is defined as

$$d_{\mu}(x) = \limsup_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log(r)} = \limsup_{k \to \infty, k \in \mathbb{N}} \frac{-\log(\mu(B(x,2^{-k})))}{k}$$

the lower local dimension $\underline{d}_{\mu}(x)$ is defined in an analogous way by replacing $\limsup$ with $\liminf$.

In general (even in examples that are interesting in dynamical system theory) $d_{\mu}(x)$ and $\underline{d}_{\mu}(x)$ can differ on a positive measure set. If $\overline{d}_{\mu}(x) = \underline{d}_{\mu}(x) = d$ almost everywhere the system is called exact dimensional. In this case all notions of dimension of a measure (Hausdorff, box counting, information dimension) will coincide (See for example the book [P]) and then we have a precise description of the fractal structure of the system. For these and other reasons is important to have estimations for both $d_{\mu}(x)$ and $\underline{d}_{\mu}(x)$.

With the above notations, Theorem 1 of [BS] can be rewritten as follows

**Theorem 2** If $X$ is a closed subset of $\mathbb{R}^n$ then for almost each $x \in X$

$$\overline{R}(x,x) \leq \overline{d}_{\mu}(x) \ , \underline{R}(x,x) \leq \underline{d}_{\mu}(x).$$

In uniformly hyperbolic systems [BS] also proved that recurrence and dimension are a.e. equal. The equality also holds in some nonuniformly hyperbolic example, however it is not difficult to see ([BS] example 3) that there are uniquely ergodic irrational rotations $(S^1, x \to x + \alpha)$ such that $\overline{R}(x,x) < \overline{d}_{\mu}(x)$ for each $x \in S^1$. In such systems and in general systems $\overline{R}(x,x)$ then gives only a lower bound for the dimension. We will see how it is possible to obtain a general upper bound for the dimension in term of $R(x,y)$. In [BGI] is indeed proved

**Lemma 3** Let $(X,T)$ be as above, $\mu$ is an invariant measure, and $y \in X$. If $\alpha > \overline{d}_{\mu}(y)^{-1}$ then for $\mu$-almost all $x \in X$ it holds

$$\liminf_{n \to \infty} n^\alpha \min_i d(y, T^n x) = \infty.$$
Here we reformulate and extend this fact in the following way.

**Theorem 4** If $\mu$ is invariant, for each fixed $y$

$$\underline{R}(x, y) \geq d_\mu(y), \quad \overline{R}(x, y) \geq \overline{d}_\mu(y)$$  \hspace{1cm} (1)

holds for $\mu$ almost each $x$.

**Proof.** First we prove $\underline{R}(x, y) \geq d_\mu(y)$. We remark that if $n^{-\alpha} \leq r \leq (n + 1)^{-\alpha}$, since $\tau_r(x, y)$ is decreasing in $r$ then

$$\frac{\log(\tau_{n^{-\alpha}}(x, y))}{-\log((n + 1)^{-\alpha})} \leq \frac{\log(\tau_r(x, y))}{-\log(r)} \leq \frac{\log(\tau_{(n + 1)^{-\alpha}}(x, y))}{-\log(n^{-\alpha})}$$

by this we can see that $\liminf_{r \to 0} \frac{\log(\tau_r(x, y))}{-\log(r)} = \liminf_{n \in \mathbb{N}, n \to \infty} \frac{\log(\tau_{n^{-\alpha}}(x, y))}{-\log(n^{-\alpha})}$. Now Lemma 3 implies that if $n$ is big enough $\tau_{n^{-\alpha}}(x, y) \geq n$ for each $\alpha > \frac{1}{\underline{d}_\mu(y)}$. Then $\liminf_{n \in \mathbb{N}, n \to \infty} \frac{\log(\tau_{n^{-\alpha}}(x, y))}{-\log(n^{-\alpha})} \geq \frac{1}{\alpha}$. Since $\alpha$ can be chosen as near as we want to $\frac{1}{\underline{d}_\mu(y)}$ we have the statement.

Now we prove $\overline{R}(x, y) \geq \overline{d}_\mu(y)$. Suppose $d' < \overline{d}_\mu(y)$, let us consider the set

$$A(d', y) = \{ x \in X | \overline{R}(x, y) < d' \}.$$

By the assumption on the dimension, if $0 < d' < d < \overline{d}_\mu(y)$ then there is a sequence $n_k$ such that

$$\mu(B(y, 2^{-n_k})) < 2^{-dn_k} \text{ for each } k.$$  \hspace{1cm} (2)

On the other side for each $x \in A(d', y)$ the relation $\tau_{2^{-n}}(x, y) < 2^{d'n}$ must hold eventually. Let us consider $C(m) = \{ x \in A(d', y) | \forall n \geq m, \tau_{2^{-n}}(x, y) < 2^{d'n} \}$. This is an increasing sequence of sets. If we prove that $\liminf_{m \to \infty} \mu(C(m)) = 0$ the statement is proved. By the definition of $C(m)$ we see that

$$C(n_k) \subset \bigcup_{i \leq 2^{d'n_k}} T^{-i}(B(y, 2^{-n_k}))$$

the latter is made of $2^{d'n_k}$ sets, whose measure can be estimated by Eq. 2 because $T$ is measure preserving. Then $\mu(C(n_k)) \leq 2^{d'n_k} \ast 2^{-dn_k}$ and $\mu(C(n_k))$ goes to 0 as $k \to \infty$. □

5
Remark 5 If the measure $\mu$ is not invariant inequality $\mathcal{I}$ can fail at some point. For example let us consider a system $(X,T)$, where the map $T$ sends all the space $X$ in a point $y$ ($\forall x, T(x) = y$) with $d_\mu(y) > 0$. Here $\mathcal{R}(x,y) < d_\mu(y)$. We remark that in this example the inequality fails only at one point $(y)$. Next results shows that even when the measure is not preserved the inequality can fail only on a zero measure set.

In the previous result the invariance of the measure was an important ingredient. The following results (where $x$ is fixed and $y$ varies) are more general, they do not require the invariance of $\mu$.

Theorem 6 For each $x \in X$
\[ \mathcal{R}(x,y) \geq d_\mu(y), \quad \mathcal{R}(x,y) \geq d_\mu(y) \]
for $\mu$ almost each $y$.

Theorem 7 For each $x \in X$ the set $Y_h \subset X$ such that
\[ Y_h = \{ y \in X, \mathcal{R}(x,y) \leq h \} \]
has Hausdorff dimension $\leq h$.

We remark that since obviously $\mathcal{R}(x,y) \geq \mathcal{R}(x,y)$ then the above result holds also with $\mathcal{R}(x,y)$ instead of $\mathcal{R}(x,y)$.

Remark 8 [KS] Before to prove these results we remark that one cannot expect in general stronger results like $\mathcal{R}(x,y) = d_\mu(y)$. This can be realized by thinking to the following trivial example: let us consider a periodic rotation $(S^1, x \to x + \alpha, \lambda)$ with $\alpha \in \mathbb{Q}$ and $\lambda$ is the Lesbegue measure, here $\mathcal{R}(x,y) = \mathcal{R}(x,y) = \infty$ for each $y$ that in not contained in the orbit of $x$ (that is a finite set) while $\lambda$ has dimension 1. A less trivial example is in a certain sense a small perturbation of this latter one.

An irrational $\alpha$ is said to be of type $\nu$ if
\[ \nu = \sup\beta | \liminf_{n \to \infty} j^\beta \min_{n \in \mathbb{N}} | j\alpha - n | = 0 \].

Lesbegue almost each irrational is of type 1, but there are irrationals with type $> 1$. From the main result of [KS] it can be deduced (with some technical work) that an irrational rotation with angle of type $\nu > 1$ satisfies $\mathcal{R}(x,y) = \nu > 1$ almost everywhere (while $\mathcal{R}(x,y) = 1$ a.e.).
3 Proof of Theorems 6 and 7

Theorems 6 and 7 come from the following more general results. Let us consider a sequence $x_i : \mathbb{N} \to X$, we define recurrence indicators indicating how the sequence comes near some given points. For this let us consider $y \in X$, and the first entrance time of $x_i$ in a ball with center $y$

$$
\tau(x_i, y, r) = \min\{n \in \mathbb{N}, x_n \in B(y, r)\}.
$$

Let us define the quantitative recurrence indicators

$$
\overline{R}(x_i, y) = \limsup_{r \to 0} \frac{\log(\tau(x_i, y, r))}{-\log(r)} = \limsup_{n \to \infty} \frac{\log(\tau(x_i, y, 2^{-n}))}{n},
$$

$$
\underline{R}(x_i, y) = \liminf_{r \to 0} \frac{\log(\tau(x_i, y, r))}{-\log(r)} = \liminf_{n \to \infty} \frac{\log(\tau(x_i, y, 2^{-n}))}{n}.
$$

Theorem 7 comes from the following proposition

**Proposition 9** For each sequence $x_i$ the set $Y_h \subset X$ such that $Y_h = \{y \in X, \overline{R}(x_i, y) \leq h\}$ has Hausdorff dimension $\leq h$.

**Proof.** We have that $\forall y \in Y_h \minlim_{k \to \infty} \frac{\log(\tau(x_i, y, 2^{-k}))}{k} \leq h$. This means that $\forall \epsilon > 0, \forall y \in Y_h$ and $\forall k_0 \in \mathbb{N}$ there is $k > k_0$ and an index $j$ with $j \leq 2^{(h+\epsilon)k}$ with $y \in B(x_j, 2^{-k})$.

Let us call $S_{\epsilon, k}$ the union of all the balls $B(x_j, 2^{-k})$ for all index $j$ such that $j \leq 2^{(h+\epsilon)k}$, $S_{\epsilon, k} = \bigcup_{j \leq 2^{(h+\epsilon)k}} B(x_j, 2^{-k})$. Then for each $k_0$

$$
Y_h \subset \bigcup_{k \geq k_0} S_{\epsilon, k}
$$

by this $Y_h$ (and each $S_{\epsilon, k}, k > k_0$) is covered by a family of balls of diameter less than $2^{-k_0}$ and we can estimate the $d$–dimensional Hausdorff measure of $S_{\epsilon, k}$

$$
\mathcal{H}^{d}_{2^{-k+1}} (S_{\epsilon, k}) \leq 2^{(h+\epsilon)k+1} 2^{(-k+1)d} = 2^{k(h+\epsilon-d)+1+d}
$$

and

$$
\mathcal{H}^{d}_{2^{-k_0}} (Y_h) \leq \sum_{k \leq k_0} 2^{1+d} 2^{k(h+\epsilon-d)}
$$
if \( d > h + \epsilon \) we can set \( k_0 \) so big that \( \mathcal{H}^d_{2-k_0}(Y_h) \leq \delta \) for each fixed \( \delta \) and \( Y_h \) is covered by balls of arbitrary small size. This proves \( \mathcal{H}^d(Y_h) = \lim_{k_0 \to \infty} \mathcal{H}^d_{2-k_0}(X_h) = 0 \) for each \( d > h + \epsilon \). Since \( \epsilon \) is arbitrary the statement follows. \( \blacksquare \)

**Remark 10** By [BS] (example 3) we have that if \( \alpha \) is “well approximable” by rationals then \( R(x, x) < 1 \). By theorem 7 the set of other points \( y \) such that \( R(x, y) < d < 1 \) is very small, indeed it must have dimension less or equal than \( d \).

Theorem 8 comes from

**Proposition 11** For \( \mu \) almost each \( y \in X \)

\[
\overline{R}(x_i, y) \geq d_\mu(y), \quad R(x_i, y) \geq d_\mu(y).
\]

The proof of proposition 11 is based on the following lemmas

**Lemma 12** Let \( A = \{ y \in X, \overline{d}_\mu(y) > d \} \). If \( h < d \) and

\[
Y_h = \{ y \in A, \text{s.t.} \overline{R}(x_i, y) < h \}
\]

then \( \mu(Y_h) = 0 \).

**Proof.** Let us consider \( 0 < \epsilon < h - d \) and

\[
Y^n_h = \{ y \in A, \text{s.t.} \forall m > n \log(\tau(x_i, y, 2^{-m})) < (h + \epsilon) m \}
\]

we have that \( Y_h \subset \bigcup_{n \geq n_0} Y^n_h \subset Y^{n+1}_h \). If we prove that \( \mu(Y^n_h) = 0 \) eventually with respect to \( n \) the assertion is proved.

If \( y \in Y^n_h \) then \( \forall m > n \exists x_i, s.t. y \in B(x_i, 2^{-m}) \) where \( i < 2^{m(h+\epsilon)} \) in other words if we consider the set of all ball of radius \( 2^{-m} \) with centers \( x_i \) with \( i < 2^{m(h+\epsilon)} \)

\[
\mathcal{B}^m = \{ B(x_i, 2^{-m}) \text{ s.t. } i < 2^{m(h+\epsilon)} \}
\]

we have that \( \forall m > n, Y^n_h \subset \bigcup_{\beta \in \mathcal{B}^m} \beta \).

For each \( y \in A \) we have that \( \overline{d}(y) = \lim_{n \to \infty} \sup_{n} \frac{-\log \mu(B(y, 2^{-n}))}{n} > d \), this implies that \( \forall y \in A \) there exist an infinite sequence \( B(y, 2^{-n_k}) \) of balls centered in \( y \)
with radius $2^{-nk_d}$ such that $\mu(B(y, 2^{-nk})) \leq 2^{-nk_d}$. Let us call this family of balls $y$-estimated balls.

Now let us consider the balls in $B^n$ for which we have an estimation about their measure: we say that a ball in $\beta \in B^n$ is “nice” if there exist an $y$ such that $\beta$ is contained in some $y$-estimated ball of radius $2^{-n+1}$ found above (we recall that all the balls in $B^n$ have radius $2^{-n}$), thus if $\beta$ is nice then $\mu(\beta) \leq 2^{-(n-1)d}$. Every $y \in Y^n_\delta$ has a sequence of $y$-estimated balls, let us consider one of these balls $\beta(y, 2^{-k+1})$: $y$ is also contained in a ball $\beta' \in B^k$, then $\beta' \subset \beta(y, 2^{-k+1})$. This implies that $\forall j \geq n$ each point of $Y^n_\delta$ is contained in some “nice” ball with radius not greater than $2^{-j}$ that is: $Y^n_\delta \subseteq \bigcup_{m \geq n} \bigcup_{\beta \in \{\text{nice balls in } B^m\}} \beta$. Now we are ready to estimate the total measure of the nice balls: we remark that the number of nice balls with radius $2^{-m}$ is not greater than $2^{m(h+\epsilon)+1}$ and the measure of a nice ball is not greater than the measure of the corresponding $y$-estimated ball. This implies that $\forall j > n$

$$\mu(Y^n_\delta) \leq \sum_{m \geq j} 2^{m(h+\epsilon)+1}2^{-(m-1)d}$$

if $n$ is big enough this sum can be set as small as we want, then $\mu(X^n_\delta) = 0$. □

**Lemma 13** Let $d, c, \delta > 0$, let $B = \{y \in X, d_\mu(y) > d + c\}$. If $A = \{y \in B \text{ s.t. } R(x, y) \leq d - \delta\}$, then $\mu(A) = 0$.

**Proof.** Conversely let us suppose that $\mu(A) > 0$. Since $\forall x \in A$, $d_\mu(x) \geq d + c$ then $x \in A$, implies that if $m$ is big enough (depending on $x$)

$$\mu(B(x, 2^{-m})) < 2^{-m(d + \frac{\epsilon}{2})}$$

(3)

then there is an $m > 0$ and a set $A' \subset A$ with $\mu(A') > 0$ such that if $x \in A'$

$$\forall m > m_0, \mu(B(x, 2^{-m})) < 2^{-m(d + \frac{\epsilon}{2})} \text{ uniformly on all } A'.$$

By the definition of $A'$ for each $k$, each $x \in A'$ is contained in some ball $B(x_j, 2^{-j})$ with $j > k$ and $j < 2^{d_i}$, that is $\forall k \in N \ A' \subset \bigcup_{\{j|k \leq j\}} B(x_j, 2^{-\log_{2}(d) \frac{\log_{2}(d)}{d}})$. Now the measure of these balls can be estimated as before by Eq. 3 and then the total measure of $A'$ can be estimated as in the previous proof, concluding that $\mu(A) = 0$. □

**Proof of proposition 11** If conversely $\overline{R}(x, y) < \overline{d}(x)$ on a set $A'$ with $\mu(A') > 0$ we can find a constant $c$ and a set $A''$, $\mu(A'') > 0$ such that $\overline{R}(x, y) < c < \overline{d}(x)$ on $A''$, by lemma 16 we obtain $\mu(A'') = 0$. Similarly the other inequality can be obtained □
References

[AC2] AFRAIMOVICH V, CHAZOTTES J R, SAUSSOL B, Pointwise dimensions for Poincaré recurrences associated with maps and special flows, Disc. Cont. Dyn. Syst. - A 9 (2003), 263-280

[BPS] BARREIRA L, PESIN, Y SCHMELING J Dimension and product structure of hyperbolic measures Ann. of Math. (2) 149 (1999), 755-783.

[BS] BARREIRA L, SAUSSOL B, Hausdorff dimension of measures via Poincaré recurrence, Commun. Math. Phys., 219 (2001), 443-463.

[BGI] BONANNO C, ISOLA S, GALATOLO S Recurrence and algorithmic information preprint.

[Bo] BOSHERNITZAN M D, Quantitative recurrence results, Invent. Math. 113 (1993), 617-631

[CD] CHAZOTTES J R, DURAND F Local rates of Poincaré recurrence for rotations and weak mixing, to appear in Disc. Cont. Dyn. Sys. A (2003).

[Ch] CHAZOTTES J R Dimensions and waiting times for Gibbs measures. J. Statist. Phys. 98 (2000), no. 1-2, 305–320.

[KS] KIM D H, SEO B K The waiting time for irrational rotations, Nonlinearity V. 16, N. 5, Sept. 2003

[HSV] HIRATA M, SAUSSOL B, VAIENTI S, Statistics of return times: a general framework and new applications, Commun. Math. Phys. 206 (1999), 33-55

[OW] ORNSTEIN D S, WEISS H, Entropy and data compression schemes, IEEE Trans. Inf. Th. 39 (1993), 78-83

[P] PESIN Y Dimension theory in dynamical systems Chicago lectures in Mathematics (1997).

[STV] SAUSSOL B, TROUBETZKOY S, VAIENTI S, Recurrence, dimensions and Lyapunov exponents, J. Stat. Phys. 106 (2002), 623-634
[Sh] SHIELDS P C Waiting times: positive and negative results on the Wyner-Ziv problem. J. Theoret. Probab. 6 (1993), no. 3, 499–519.