PAIRS TRADING WITH ILLIQUIDITY AND POSITION LIMITS

MENGLU FENG
Societe Generale Hong Kong, Three Pacific Place
1 Queen’s Road East, Hong Kong

MEI CHOI CHIU
Department of Mathematics and Information Technology
Education University of Hong Kong, Hong Kong SAR, China

HOI YING WONG*
Department of Statistics
The Chinese University of Hong Kong, Hong Kong SAR, China

(Communicated by Ken Siu)

Abstract. We investigate the optimal investment among the money market account, a liquid risky asset (e.g. stock index) and an illiquid risky asset (e.g. individual stock), where the two risky assets are cointegrated. The illiquid risky asset is subject to a proportional transaction cost and the portfolio of the three assets faces certain position limits. We develop the optimal investment strategy to maximize the gain function, which is realized through an expected sum of discounted utilities given transaction costs and position limits. The problem formulation uses a singular control framework with cointegration that determines optimal trading boundaries among holding, selling and no-trading regions. We conduct comprehensive numerical analysis on the optimal investment strategy and features of the optimal trading boundaries given various levels of position limits.

1. Introduction. The concept of cointegration proposed by Engle and Granger [15] describes an equilibrium relation among non-stationary economic series such that a linear combination of the series is stationary. Cointegration provides an alternative to correlation coefficient for describing relations between series and better captures long-term equilibrium. Granger won the Nobel Prize in Economics for the contribution of cointegration. Cointegration techniques are widely used in financial studies for various asset classes such as stock markets [5] and foreign currency exchange markets [3].

Not only is cointegration useful for investigating common equilibrium among economic series, it is also useful in pairs-trading. Chiu and Wong ([8] and [9]) solve mean-variance portfolio and asset-liability management problems with cointegrated assets. Liu and Timmermann [20] consider optimal strategy for convergence trades with cointegrated assets and illustrate the result with Chinese bank shares. Tourin and Yan [22] investigate optimal pairs-trading in the sense of exponential utility

2020 Mathematics Subject Classification. 91G10, 91G80.
Key words and phrases. Cointegration, liquidity, pairs trading, position limits, singular control problem.

*Corresponding author.
maximization. Chiu and Wong [10] extend the result to the optimal investment problem with cointegrated risky assets for insurers with CRRA utility. Chiu and Wong [11] prove mathematically that cointegration leads to statistical arbitrage when there is no market friction.

However, market frictions are practically and theoretically relevant for investment strategies. Transaction cost is an important concept for reflecting the cost of market frictions. Wermers [24] finds that transaction costs and expenses decrease the return of mutual funds by 1.6%. Karceski et al. [18] suggest that around 46% of small cap funds suffer from transaction costs greater than the annual fees their investors pay. Gatev et al. [17] show that trading strategies that ignore transaction costs require too much trading and lead to poor decisions when maturity nears. Davis and Norman [13] study optimal consumption-portfolio problems with proportional transaction costs and reveal that the presence of transaction costs can have a huge effect on optimal trading strategy. Dai et al. [12] study optimal trading boundaries for mutual funds investing in both liquid and illiquid assets with transaction costs and position limits. However, these studies do not take the cointegration effect into account.

In this paper, we investigate pairs-trading with cointegration subject to transaction costs on an illiquid asset with position limits. This amounts to a generalization of the study by Dai et al. [12] to the cointegration financial market. Our work is also related to the study of Lei and Xu [19] that considers the liquidity effect and a fixed portfolio of two cointegrated assets subject to a proportional transaction cost on the portfolio, without position limits. Lei and Xu [19] document that the transaction cost makes the cointegration statistical arbitrage costly. Position limits are practical constraints that regulate the holding of assets to maintain market stability and fairness. Liquidities of risky assets are generally different. For example, the transaction cost of index futures can be much lower than that of an individual stock. When two risky assets are cointegrated but one is more liquid than the other, the consideration of simultaneous transactions on the two risky assets may result in a suboptimal decision and a hardly executable strategy in practice. Unlike the impulse control framework in [19], we use a singular control framework in our problem formulation. In other words, in our framework, the fund manager can adjust the holding of the liquid asset at any time, realized as a classical control variable, and the holding of the illiquid asset at some optimal time points to reduced transaction costs, realized as an impulse control variable.

When the fund manager maximizes a gain function of an expected sum of discounted utilities over a finite investment horizon, similarly to the discounted cash flow value proposed by [19], we derive Hamilton-Jacobi-Bellman (HJB) equations of singular control type for cases both with and without position limits. We numerically solve the HJB equations for both cases by a penalty method and then analyze the characteristics of the investment regions. With a traded liquid asset, the investment strategy exhibits strong incentive to gain arbitrage profits from the cointegration effect. Adding position limits to the problem changes the strategy at the very beginning of the investment period, and the investor trades more actively during that period.

The remainder of this paper is structured as follows. Section 2 introduces basic concepts of cointegration, the financial market and the optimal investment problem with or without position limits. Section 3 presents the HJB framework for solving the optimization problem. Section 4 provides numerical analysis and comparative
2. Problem formulation.

2.1. Cointegration. As suggested by Engle and Granger in [15], cointegrated time series can be presented as an error-correction model. Cointegration implies even if the time series are non-stationary, they can form long-run equilibrium with certain cointegrating vectors, and deviations from equilibrium are stationary. The concept of cointegration provides an alternative to correlation coefficient for describing relations among time series. Cointegrating factors describe the relations between series even if their correlation coefficients equal zero.

As presented in [8], in an error-correction model, the k cointegrating factors \((z_{1t}, \cdots, z_{kt})\) form a stationary time series with bounded variance at each time point for each \(z_{jt}\). In a continuous time economy, the diffusion limit of an error-correction model for \(n\) asset price processes with \(k\) cointegrating factors, where \(1 \leq k < n\), is derived as follows:

\[
d\ln S_{it} = \left( \beta_i - \sum_{j=1}^{k} \delta_{ij} z_{jt} \right) dt + \sigma_i dW_{it}, \text{ for } i = 1, \cdots, n,\]

\[
z_{jt} = a_j + b_j t + \sum_{i=1}^{k} c_{ij} \ln S_{it}, \text{ for } j = 1, \cdots, k,\]

where \((c_{1j}, \cdots, c_{nj})\) is a matrix with linearly independent column vectors for \(j = 1, \cdots, k\) and \(W_{it}\) are correlated Brownian motions for \(i = 1, \cdots, n\).

When \(k = 1\), \(z_t\) follows the Ornstein-Uhlenbeck process [14]:

\[
dz_t = (\beta_z - \lambda z_t) dt + \sigma_z dW_{zt},\]

where \(\beta_z\), \(\alpha\) and \(\sigma_z\) are constants and \(\alpha\) and \(\sigma_z\) are positive. In this study, we focus on the situation where two cointegrated risky assets \((S_1, S_2)\) have one cointegrating factor \(z\). In other words, the pair of risky assets have a common long-term equilibrium. The detail market setting is given as follows.

2.2. The market. Consider a portfolio of three assets: a money market asset (e.g. a bond) that grows at a constant rate \(r\), a liquid risky asset \(S_1\) (e.g. a stock index) and an illiquid risky asset \(S_2\) (e.g. individual stocks). Returns of the two risky assets are cointegrated and the price processes evolve as follows:

\[
d\ln S_{it} = (\beta_i - \delta_i z_t) dt + \sigma_i dW_{it}, \quad i = 1, 2,\]

(1)

where \(\delta_i\) and \(\sigma_i\) are constants, \(\beta_i - \delta_i z_t\) denotes the instantaneous expected rate of return for risky asset \(i\) and \(\sigma_i\) denotes volatility, \(i = 1, 2\). The constant \(\delta_i\) is the level of dependency on the cointegrated factor, \(z_t\), for the risky asset \(i\). \(W_{1t}\) and \(W_{2t}\) are correlated Brownian motions defined on a filtered probability space \((\Omega, \mathcal{F}_t, \mathbb{P})\) with a correlation coefficient \(\rho \in [-1, 1]\).

The cointegration factor \(z_t\) is defined as \(z_t = \ln S_{1t} - \lambda \ln S_{2t}\) so that

\[
dz_t = [\beta_1 - \lambda \beta_2 - (\delta_1 - \lambda \delta_2) z_t] dt + \sigma_1 dW_{1t} - \lambda \sigma_2 dW_{2t},\]

(2)

with \(\delta_1 - \lambda \delta_2 > 0\). This implies that \(z_t\) is stationary and \(z_t\) captures relative price between the two risky assets. The mean reversion nature of \(z_t\) captures the temporary relative mispricing between the two risky assets, resulting in a statistical arbitrage opportunity in the absence of market friction.
We introduce market friction in the manner of [12]. Specifically, both the money market asset and liquid risky asset are traded without transaction cost. The illiquid asset bears transaction costs while trading so that the asset is bought at the ask price $S^a_2 > S_2$ and sold at the bid price $S^b_2 < S_2$. To take advantage of the cointegration effect between the risky assets, we define an illiquid instrument whose price is $e^{z_t}$.

Such an illiquid instrument can be approximated by holding the portfolio associated with the cointegration factor $z_t = \ln S_{1t} - \lambda \ln S_{2t}$. This consideration aligns with those in [19]. Trading with the illiquid instrument incurs proportional transaction costs. This instrument is then purchased at the ask price $(1 + \theta) e^{z_t}$ and sold at the bid price $(1 - \alpha) e^{z_t}$, where $\theta \geq 0$ and $0 \leq \alpha < 1$ are constants denoting the proportional rates of transaction cost.

The introduction of the illiquid instrument $e^{z_t} = S_{1t}/S_{2t}^\lambda$ aims to simplify the problem and the numerical computation. Such an artificial instrument can be constructed by a practical approximation. For instance, the holding of the instrument can be realized as purchasing $1/S_{2t}^\lambda$ units of $S_{1t}$ at time $t$. When time passes, the value of $S_1$ changes but the number of units so that the fund manager can track the value of $e^{z_t}$ by dynamically adjusting the position in the liquid asset $S_{2t}$ using, for instance, the mean-variance cointegration tracking in [7] in which the illiquid asset is treated as a non-tradable asset. As the tracking is imperfect, the tracking error can be subsumed into a part of transaction cost. The fund manager can then take a larger proportional transaction cost as a proxy to cover both costs from the tacking error and the actual transaction of $S_1$.

We also use a liquid instrument consisting of the money market asset and the liquid risky asset. Let $x_t$ be the dollar amount invested in the liquid instrument, where $\epsilon_t$ is the dollar amount of the liquid risky asset. Let $S_1$ be the price of the liquid risky asset. Thus, we have the following:

$$dx_t = rx_t dt + \epsilon_t \left( \beta_1 - \delta_1 z_t + \frac{1}{2} \sigma_1^2 - r \right) dt + \epsilon_t \sigma_1 dW_{1t}. \quad (3)$$

Our setting of an asset market is similar to that in [12], except that they treat the optimal trading problem as a portfolio choice problem in which the dependency between two risky assets is reflected solely by the conventional correlation coefficient. We consider cointegration between the two risky assets, and form part of our optimal investment problem focusing on optimal timing for trading the cointegrated assets and taking advantage of cointegration effects. Similar to [12] and [19], we consider proportional transaction costs for trading illiquid risky assets.

2.3. The optimal investment problem. In a finite time horizon $T$, a fund manager (or pairs-trader) can either long or short one unit of the illiquid instrument (i.e., the instrument defined with price $e^{z_t}$) or not invest in the illiquid asset at all. However, she adjusts the holding of the liquid instrument continuously in time. The fund manager aims to maximize the utility gained throughout the investment horizon by determining the set of proper times of entry or exit from a position in the illiquid instrument while trading with the liquid instrument continuously. Such an investment policy is designed to capture the price divergence caused by the cointegration effect and thus takes advantage of possible statistical arbitrage opportunities.

Let $\tau_1 \in [0, T)$ be a time point at which a position is created in the illiquid instrument and $\eta_1 \in [0, T]$ a time point for closing the position. As the manager only trades with a single unit of the illiquid instrument, we have $\tau_1 < \eta_1 < \tau_2 < \eta_2$. 
\( \eta_2 < \cdots \leq T \). On creating a position, the manager can either long or short the illiquid instrument. At any time point \( t < T \), three states are possible:

1. a short position with the illiquid asset,
2. no position with the illiquid asset,
3. a long position with the illiquid asset.

The states are denoted as \( j = -1, 0, 1 \), respectively.

We consider a discounted utility gain function, \( U_{jt} \), throughout the investment horizon, where \( j = -1, 0, 1 \) indicates the three possible states at time \( t \). This setting generalizes the considerations in [19] from a purely cumulative cash flow to a general utility function. If the fund manager has no position in the illiquid instrument at time \( t < T \), then the discounted utility gained reads as follows:

\[
U_{0t} = u(x_T) e^{-\nu(T-t)} + \sum_{i=1}^{\infty} \left[ e^{-\nu(\eta_i \wedge T-t)} u((1 - \alpha) e^{\bar{z}_{\eta_i} \wedge T}) - e^{-\nu(\tau_i - t)} u((1 + \theta) e^{\bar{z}_{\tau_i}}) \right] I_{A_i} I_{(\tau_i < T)} + \sum_{i=1}^{\infty} \left[ e^{-\nu(\tau_i - t)} u((1 - \alpha) e^{\bar{z}_{\tau_i}}) - e^{-\nu(\eta_i \wedge T-t)} u((1 + \theta) e^{\bar{z}_{\eta_i} \wedge T}) \right] I_{B_i} I_{(\tau_i < T)},
\]

where \( A_i \) denotes the event that the manager takes a long position in the illiquid asset at time \( \tau_i \) and \( B_i \) a short position. \( \mathbb{I}_K \) is an indicator function for event \( K \):

\[
\mathbb{I}_K(x) = \begin{cases} 
1 & \text{if } x \in K \\
0 & \text{if } x \notin K 
\end{cases}
\]

The concave function \( u(y) \) is the manager’s utility and the discount rate \( \nu \) shows the time preference of the manager.

If the manager longs the illiquid instrument at time \( t < T \), the discounted utility gain function is:

\[
U_{1t} = u(x_T) e^{-\nu(T-t)} + e^{-\nu(\eta_i - t)} u((1 - \alpha) e^{\bar{z}_{\eta_i}})
+ \sum_{i=2}^{\infty} \left[ e^{-\nu(\eta_i - t)} u((1 - \alpha) e^{\bar{z}_{\eta_i}}) - e^{-\nu(\tau_i - t)} u((1 + \theta) e^{\bar{z}_{\tau_i}}) \right] I_{A_i} I_{(\tau_i < T)}
+ \sum_{i=2}^{\infty} \left[ e^{-\nu(\tau_i - t)} u((1 - \alpha) e^{\bar{z}_{\tau_i}}) - e^{-\nu(\eta_i - t)} u((1 + \theta) e^{\bar{z}_{\eta_i}}) \right] I_{B_i} I_{(\tau_i < T)}.
\]

Alternatively, if she shorts the illiquid instrument at \( t < T \), the discounted utility gain function is:

\[
U_{-1t} = u(x_T) e^{-\nu(T-t)} - e^{-\nu(\eta_i - t)} u((1 + \theta) e^{\bar{z}_{\eta_i}})
+ \sum_{i=2}^{\infty} \left[ e^{-\nu(\eta_i - t)} u((1 - \alpha) e^{\bar{z}_{\eta_i}}) - e^{-\nu(\tau_i - t)} u((1 + \theta) e^{\bar{z}_{\tau_i}}) \right] I_{A_i} I_{(\tau_i < T)}
+ \sum_{i=2}^{\infty} \left[ e^{-\nu(\tau_i - t)} u((1 - \alpha) e^{\bar{z}_{\tau_i}}) - e^{-\nu(\eta_i - t)} u((1 + \theta) e^{\bar{z}_{\eta_i}}) \right] I_{B_i} I_{(\tau_i < T)}.
\]

When the fund manager aims to maximize the expected utility gained, the problem is formulated as a singular control problem:

\[
V^j(x, z, t) = \sup_{\epsilon, \tau, \eta} E_t [U_{jt}],
\]

(5)
where \( j = -1, 0, 1 \) indicates the illiquid asset position state at time \( t \). This is the situation where the position limits are inactive.

2.3.1. *Position limits*. We further introduce position limits into the framework. Position limits force the fund manager to create a long position or close a short position in the illiquid instrument when \( z \) reaches a lower bound \( l \). In addition, she creates a short position or close a long position with the illiquid instrument when \( z \) reaches an upper bound \( \bar{l} \).

The position limits regarding the cointegrating factor \( z \) suggest certain constraints on the sequence of stopping times \( \tau_i \) and \( \eta_i \). With position limits, the stopping times are defined on the set:

\[
(\tau_i, \eta_i) \in \mathcal{T} = \{ \tau : 0 \leq \tau_i < T, 0 \leq \eta_i < T : \tau_1 < \eta_1 < \tau_2 < \eta_2 < \cdots \leq T, \eta_i - t < l \text{ and } \eta_i = t \}.
\]

The optimal investment problem in (3.1) becomes

\[
V^j(x, z, t) = \sup_{\epsilon, \tau, \eta} E_t[U_{jt}]
\]

where \((\tau_i, \eta_i) \in \mathcal{T} \) and \( j = -1, 0, 1 \) indicates the position state of the illiquid instrument at time \( t \).

This setting is similar to that in [12] so that the illiquid instrument has been traded throughout the investment horizon. According to [1], investors are very likely to set similar constraints for fund managers because of the asymmetric interests between them.

3. HJB framework.

3.1. *Unconstrained case*. For the time being, we abstract the position limits and consider the problem (3.1). At any stopping time \( \tau \), the manager creates a position in the illiquid instrument so that the state changes from \( V^0 \) to either \( V^1 \) or \( V^{-1} \). At time \( \tau \),

\[
V^0(x, z, \tau) = [V^1(x, z, \tau) - u((1 + \theta)e^{z\tau})]I_A + [V^{-1}(x, z, \tau) + u((1 - \alpha)e^{z\tau})]I_B.
\]

Similarly, at any stopping time \( \eta \), the manager closes the position so that the objective function changes immediately from either \( V^1 \) or \( V^{-1} \) to \( V^0 \): \( V^1(x, z, \eta) = V^0(x, z, \eta) + u((1 - \alpha)e^{z\eta}) \) and \( V^{-1}(x, z, \eta) = V^0(x, z, \eta) - u((1 + \theta)e^{z\eta}) \).

For a small positive value of \( t_0 - t \) such that \( t_0 < T \), there is at most one stopping time \( \tau \) in \([t, t_0]\). The following dynamic programming principle holds for \( V^0 \):

\[
V^0(x, z, t) = \sup_{\epsilon, \tau, \eta} E_t\{[V^1(x, z, \tau) - u((1 + \theta)e^{z\tau})]I_{\{\tau < t_0\}}I_A + [V^{-1}(x, z, \tau) + u((1 - \alpha)e^{z\tau})]I_{\{\tau < t_0\}}I_B + V^0(x, t_0, t_0)I_{\{\tau \geq t_0\}}\}. \tag{7}
\]

Similarly, there exists at most one stopping time \( \eta \) in \([t, t_1]\) and \([t, t_2]\) for small values of \( t_1 - t \) and \( t_2 - t \). The dynamic programming principles for \( V^1 \) and \( V^{-1} \) imply

\[
V^1(x, z, t) = \sup_{\epsilon, \tau, \eta} E_t\{[V^0(x, \eta, \eta) + u((1 - \alpha)e^{z\eta})]I_{\{\eta < t_1\}} + V^1(x, z_1, t_1)I_{\{\eta \geq t_1\}}\} \tag{8}
\]
Putting all these together, we have the following theorem.

**Theorem 3.1.** Suppose $V^j \in C^{1,2}([0, T] \times \mathcal{S})$, where $\mathcal{S}$ is the solvency region defined as $\mathcal{S} = \{(x, z) \in \mathbb{R}^2 : x > 0\}$. Then $V^j$ satisfies the HJB equations:

$$\begin{align*}
\min(-V^0_t - \mathcal{L}V^0, V^0 - V^1 + u((1 + \theta)e^{z_t}), V^0 - V^{-1} - u((1 - \alpha)e^{z_t}) &= 0 \\
\min(-V^1_t - \mathcal{L}V^1, V^1 - V^0 - u((1 - \alpha)e^{z_t}) &= 0 \\
\min(-V_t^{-1} - \mathcal{L}V^{-1}, V^{-1} - V^0 + u((1 + \theta)e^{z_t}) &= 0
\end{align*}$$

(10)

with terminal conditions

$$\begin{align*}
V^0(x, z, T) &= u(x), \\
V^1(x, z, T) &= u(x) + u((1 - \alpha)e^{z_T}), \\
V^{-1}(x, z, T) &= u(x) - u((1 + \theta)e^{z_T}).
\end{align*}$$

The differential operator $\mathcal{L}$ is defined as:

$$\mathcal{L}V^j = rxV^j_x + ((\beta_1 - \lambda\beta_2 - (\delta_1 - \lambda\delta_2)z)V^j_x + \frac{1}{2}(\sigma_1^2 + \lambda^2\sigma_2^2 - 2\rho\lambda\sigma_1\sigma_2)V^j_{zz} - \nu V^j + \max(\frac{1}{2}\sigma_1^2e^{z_T} + e(\beta_1 - \delta_1)z + \frac{1}{2}\sigma_1^2 - r)V^j_x + e\sigma_1(\sigma_1 - \theta\rho\sigma_2)V^j_{xz}).$$

(11)

If $V^j_{xx} < 0$, then

$$\mathcal{L}V^j = rxV^j_x + ((\beta_1 - \lambda\beta_2 - (\delta_1 - \lambda\delta_2)z)V^j_x + \frac{1}{2}(\sigma_1^2 + \lambda^2\sigma_2^2 - 2\rho\lambda\sigma_1\sigma_2)V^j_{zz} - \nu V^j + \frac{[(\beta_1 - \delta_1)z + \frac{1}{2}\sigma_1^2 - r)V^j_x + \sigma_1(\sigma_1 - \lambda\rho\sigma_2)V^j_{xz}]^2}{2\sigma_1^2V^j_{xx}}. $$

(12)

because, for each $j \in \{-1, 0, 1\}$, the optimal $\epsilon^*_j$ is given by

$$\epsilon^*_j = -\frac{(\beta_1 - \delta_1)z + \frac{1}{2}\sigma_1^2 - r)V^j_x + \sigma_1(\sigma_1 - \lambda\rho\sigma_2)V^j_{xz}}{\sigma_1^2V^j_{xx}}.$$  

**Proof.** We arrive directly at the following by simply comparing the objective function value $V^j$ to the stopping rule given by $\tau = t$ or $\eta = t$:

$$\begin{align*}
V^0 - V^1 + u((1 + \theta)e^{z_t}) &\geq 0 \\
V^0 - V^{-1} - u((1 - \alpha)e^{z_t}) &\geq 0 \\
V^1 - V^0 - u((1 - \alpha)e^{z_t}) &\geq 0 \\
V^{-1} - V^0 + u((1 + \theta)e^{z_t}) &\geq 0
\end{align*}$$

For any $t < T$, consider the interval $(t, t + \Delta t)$ that contains no stopping times for some sufficiently small $\Delta t$. For any $j = -1, 0, 1$, we have:

$$V^j(x_t, z_t, t) \geq E_t[e^{-\nu\Delta t}V^j(x_{t+\Delta t}, z_{t+\Delta t}, t + \Delta t)].$$

By the smooth assumption of the value function, Itô’s lemma shows,

$$-V^j_t - \mathcal{L}V^j \geq 0.$$

If $V^j_{xx} < 0$, then the optimal $\epsilon^*$ can be directly obtained from the quadratic equation.
Therefore, we have:
\[
\begin{align*}
\min(-V_t^0 - LV^0, V^0 - V^1 + u((1 + \theta)e^{z_t}), V^0 - V^{-1} - u((1 - \alpha)e^{z_t})) & \geq 0, \\
\min(-V_t^1 - LV^1, V^1 - V^0 - u((1 - \alpha)e^{z_t})) & \geq 0, \\
\min(-V_t^{-1} - LV^{-1}, V^{-1} - V^0 + u((1 + \theta)e^{z_t})) & \geq 0.
\end{align*}
\]

Then we prove the equal sign by contradiction, with reference to [23]. We only prove for the first equation with regard to $V^0$. The proofs of the other two equations, $V^1$ and $V^{-1}$, are similar. Assume that

\[-V_t^0 - LV^0 > 0, V^0 - V^1 + u((1 + \theta)e^{z_t}) > 0 \text{ and } V^0 - V^{-1} - u((1 - \alpha)e^{z_t}) > 0\]

at some $(x_{t_0}, z_{t_0}, t_0)$.

Consider,
\[
\phi^0(x, z, t) = V^0(x, z, t) + \xi(\|x - x_{t_0}\| + |z - z_{t_0}| + |t - t_0|^2),
\]

where $\xi > 0$. For a sufficiently small $\xi$, we can find $\omega_1 > 0, \omega_2 > 0$ and $h > 0$ such that

\[V^0 - V^1 + u((1 + \theta)e^{z_t}) - \omega_1 \geq 0, V^0 - V^{-1} - u((1 - \alpha)e^{z_t}) - \omega_2 \geq 0 \text{ and } -\phi^0_t - L\phi^0 \geq 0,\]

on $\mathcal{N}_h = [t_0, t_0 + h] \times hB$, where $B$ is the unit ball of $\mathcal{S}$ centered at $(x_{t_0}, z_{t_0})$. Let
\[
c = \max_{\partial \mathcal{N}_h}(V^0 - \phi^0) < 0
\]

and define
\[s = \inf\{t > t_0 : (x_t, z_t, t) \notin \mathcal{N}_h\}.\]

For an arbitrary control law $(\epsilon, \tau, \eta)$, an application of Itô’s lemma shows,
\[
\begin{align*}
E_{t_0}[V^0(x_{\tau \land s}, z_{\tau \land s}, \tau \land s)] &= E_{t_0}[V^0(x_{t_0}, z_{t_0}, t_0)] \\
&= E_{t_0}[(V^0 - \phi^0)(x_{t_0}, z_{t_0}, \tau \land s)] \\
&+ E_{t_0}[\phi^0(x_{t_0}, z_{t_0}, \tau \land s) - \phi^0(x_{t_0}, z_{t_0}, t_0)] \\
&= E_{t_0}[(V^0 - \phi^0)(x_{t_0}, z_{t_0}, \tau \land s)] \\
&+ E_{t_0}[\int_{t_0}^{\tau \land s} \phi^0_t(x_t, z_t, t) + L\phi^0_t(x_t, z_t, t)dt].
\end{align*}
\]

As $-\phi^0_t - L\phi^0 \geq 0$ on $\mathcal{N}_h$, we have
\[
E_{t_0}[V^0(x_{\tau \land s}, z_{\tau \land s}, \tau \land s)] - V^0(x_{t_0}, z_{t_0}, t_0)] \leq E_{t_0}[(V^0 - \phi^0)(x_{t_0}, z_{t_0}, \tau \land s)] \\
\leq cP(\tau \geq s).
\]

As $V^0 - V^1 + u((1 + \theta)e^{z_t}) - \omega_1 \geq 0$ and $V^0 - V^{-1} - u((1 - \alpha)e^{z_t}) - \omega_2 \geq 0$ on $\mathcal{N}_h$, we obtain
\[
\begin{align*}
V^0(x_{t_0}, z_{t_0}, t_0) &\geq -cP(\tau \geq s) + E_{t_0}[[V^1(x_{\tau}, z_{\tau}, \tau) - u((1 + \theta)e^{z_{\tau}}) + \omega_1]_{\{\tau < s}\}I_A \\
&+ [V^{-1}(x_{\tau}, z_{\tau}, \tau) + u((1 - \alpha)e^{z_{\tau}}) + \omega_2]_{\{\tau < s\}}I_B + V^0(x, z, s)I_{\{\tau \geq s\}}] \\
&\geq \min(-c, \omega_1, \omega_2) + E_{t_0}[[V^1(x_{\tau}, z_{\tau}, \tau) - u((1 + \theta)e^{z_{\tau}})]_{\{\tau < s\}}I_A \\
&+ [V^{-1}(x_{\tau}, z_{\tau}, \tau) + u((1 - \alpha)e^{z_{\tau}})]_{\{\tau < s\}}I_B + V^0(x, z, s)I_{\{\tau \geq s\}}].
\end{align*}
\]
As the control law \((\epsilon, \tau, \eta)\) is arbitrary, the above relationship also holds for the optimal control law \((\epsilon^*, \tau^*, \eta^*)\). By
\[
V^0(x_{t_0}, z_{t_0}, t_0) = \sup_{\epsilon, \tau, \eta} E_{t_0} \{ [V^1(x, \tau, z) - u((1 + \theta)e^{z_t})] I_{\{\tau < s\}} I_{\{A\}} + [V^{-1}(x, \tau, z) + u((1 - \alpha)e^{z_t})] I_{\{\tau < s\}} I_{\{B\}} + V^0(x, z, s) I_{\{\tau \geq s\}} \}.
\]
This implies \(\min(-c, \omega_1, \omega_2) \leq 0\), which results in a contradiction. Following the same argument, we can prove the equal sign for the other two equations, for \(V^1\) and \(V^{-1}\), and get the HJB equations (10).

Our HJB equations have a similar format to those in [19], except that the equations in the problem setting of [19] and also of [12] can be reduced into a two-dimensional system. Ours cannot reduce the dimensionality. In particular, we cannot avoid the cross derivative term \(V^0_{xz}\) corresponding to the correlation coefficient \(\rho\) and thus the HJB equations become much more difficult to solve. Theorem 3.1 provides a base to numerically solve for the problem. The terminal conditions satisfy the assumption of \(V^0_{xz} < 0\) because all of them take the form as the concave function \(u(x)\) plus a function of \(z\). As the value functions are continuous, the assumption holds true near the terminal investment time, making the theorem useful at least for a short horizon problem. In fact, all of our numerical examples show stable results and satisfy that assumption numerically. A rigorous proof for the negativity of \(V^0_{xz}\) for all \(t \in [0, T]\) and \(T > 0\) without an explicit solution is a very difficult task and we leave it for a future research.

The HJB equations (10) imply the following four investment regions on the solvency region \(S = \{(x, z) \in \mathbb{R}^2 : x > 0\}:
\[
\begin{align*}
ENL &= \{(x, z, t) \in S \times [0, T) : V^0 - V^1 + u((1 + \theta)e^{z_t}) = 0\} \\
ENS &= \{(x, z, t) \in S \times [0, T) : V^0 - V^{-1} - u((1 - \alpha)e^{z_t}) = 0\} \\
EXL &= \{(x, z, t) \in S \times [0, T) : V^1 - V^0 - u((1 - \alpha)e^{z_t}) = 0\} \\
EXS &= \{(x, z, t) \in S \times [0, T) : V^{-1} - V^0 + u((1 + \theta)e^{z_t}) = 0\},
\end{align*}
\]
where ENL refers to the region in which the manager creates a long position with the illiquid asset and ENS, a short position with the illiquid asset. EXL and EXS describe the regions in which the manager closes the long or short position with the illiquid asset, respectively. The first two regions determine when to enter the illiquid market, whereas the other two determine when to exit the illiquid market.

3.2. Position limits. The position limits on the cointegrating factor \(z\) impose additional constraints on the sequence stopping times \(\tau_i\) and \(\eta_i\), and the optimal investment problem changes to (6). To solve this problem, we derive the HJB equations in a similar manner:
\[
\begin{cases}
\min(-V^0 - \mathcal{L}V^0, V^0 - V^1 + u((1 + \theta)e^{z_t}), V^0 - V^{-1} + u((1 - \alpha)e^{z_t})) = 0 \\
\min(-V^0 - \mathcal{L}V^1, V^1 - V^0 - u((1 - \alpha)e^{z_t})) = 0 \\
\min(-V^0 - \mathcal{L}V^{-1}, V^{-1} - V^0 + u((1 + \theta)e^{z_t})) = 0,
\end{cases}
\]for \((x, z) \in S',\)
\[
\text{where } S' = \{(x, z) \in \mathbb{R}^2 : x > 0 \text{ and } \underline{z} \leq z \leq \bar{z}\} \text{ with boundary conditions}
\begin{align*}
V^0 - V^1 + u((1 + \theta)e^{z_t}) &= 0 \text{ on } z = \underline{z}, \\
V^{-1} - V^0 + u((1 + \theta)e^{z_t}) &= 0 \text{ on } z = \bar{z}.
\end{align*}
\]
Table 1. Summary of Default Parameters for Numerical Analysis

| Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|
| $\beta_1$ | 0.1   | $\rho$    | 0.8   |
| $\beta_2$ | 0.15  | $\alpha$  | 0.01  |
| $\delta_1$| 1     | $\theta$  | 0.01  |
| $\delta_2$| 0.4   | $\gamma$  | 0.5   |
| $\sigma_1$| 0.2   | $r$       | 0.01  |
| $\sigma_2$| 0.25  | $\nu$     | 0.02  |
| $\lambda$ | 1     |           |       |

\[ V^0 - V^{-1} - u((1 - \alpha)e^{zt}) = 0 \quad \text{on} \quad z = \bar{l}, \quad (18) \]
\[ V^1 - V^0 - u((1 - \alpha)e^{zt}) = 0 \quad \text{on} \quad z = \bar{l}, \quad (19) \]

and terminal conditions
\[
\begin{cases}
  V^0(x, z, T) = u(x), \\
  V^1(x, z, T) = u(x) + u((1 - \alpha)e^{zt}), \\
  V^{-1}(x, z, T) = u(x) - u((1 + \theta)e^{zt}).
\end{cases}
\]

The differential operator $\mathcal{L}$ is the same as in the above section. The boundary conditions (16) – (19) reflect the position limits. In practice, a pairs-trading fund should not hold no position for a long period of time. The boundary conditions (16) and (17) reveal that a long position is enforced once the cointegrating factor $z$ drops to a certain downside threshold $\bar{l}$. The left-handed side of (16) shows the cost of changing from no position to the long position while (16) and (17) together imply the cost of jumping from the short position to the long position once the down side threshold is hit. An analogy can be drawn for boundary conditions (18) and (19). As the domain of $z$ is $\mathbb{R}$, we assume that the fund maintains a long position for $z \leq \bar{l}$ and a short position for $z \geq \bar{l}$.

4. Numerical analysis. As obtaining an analytic solution is difficult, we conduct a numerical analysis of the optimal investment strategy with or without position limits to distinguish the investment regions in this section. We numerically solve the HJB equations (10) and (15) using a penalty method, then apply a finite difference scheme to discretize the HJB equations and use a nonsmooth version of Newton’s method at each time iteration. The work of Forsyth and Vetzal [16] gives evidence of convergence for such a numerical scheme.

Our numerical analysis uses the default parameters as shown in Table 1. We assume that the proportional transaction costs for purchasing and selling the illiquid asset are the same. We also assume the investor’s utility follows a power utility function, which is given by
\[ u(x) = \frac{x^\gamma}{\gamma}, \quad (20) \]
where $\gamma \in \Gamma = \{ \gamma \in \mathbb{R} : \gamma < 1 \text{ and } \gamma \neq 0 \}$. Moreover, the setting of the parameters also implies that the long-term expected return and the volatility of the liquid risky asset $S_1$ are higher than those of the illiquid risky asset $S_2$. 
4.1. Optimal investment strategies.

4.1.1. Unconstrained case. Fig. 1 specifies the four investment regions for the illiquid asset. The left graph determines the boundaries of when the investor should create a position with the illiquid asset, $e^z$. When the cointegrating factor $z$ falls below the lower surface in the left graph, the investor should create a long position with the illiquid asset, and this region corresponds to the first investment region ENL. When the cointegrating factor $z$ lies above the upper surface in the left graph, it is in the second investment region ENS, and the investor should create a short position with the illiquid asset. Similarly, the right graph corresponds to the regions in which the investor should close the positions with the illiquid asset $e^{-z}$. When the cointegrating factor $z$ lies above the upper surface in the right graph, the investor is in the third investment region EXL and should close the long position, whereas if the cointegrating factor $z$ falls below the lower surface in the right graph, the investor should close the short position with the illiquid asset.

The optimal investment strategy, as Fig. 1 suggests, exhibits a clear incentive of arbitrage. The investor creates a position with the illiquid asset when the prices diverge. According to the divergence direction, the investor would create long or short positions respectively. When the prices converge to a certain level, the investor would close the position to make arbitrage profits, which could cover the transaction costs involved. Moreover, the investment boundaries of creating positions with the illiquid asset approach negative and positive infinity respectively, suggesting that near maturity $T$, the investor should not enter the illiquid asset market to create positions with $e^z$ no matter the cointegrating effect.

Lei and Xu [19] suggest a certain stop-loss feature for pairs trading with transaction costs when there is no liquidity discrepancy between the two cointegrated assets. With a liquid asset, such as an index future, the optimal investment strategy does not present such a feature.

4.1.2. Position limits. The optimal investment boundaries that account for position limits are presented in Fig. 2. We set the lower and upper bounds of the cointegrating factor $z$ to $\bar{l} = -0.5$ and $\bar{l} = 0.7$, respectively. We define the four investment
regions in the same way as in the unconstrained case. In the left graph, the investment boundary for creating a long position on the lower surface reaches the lower bound near maturity instead of going to negative infinity as in the unconstrained case. Similarly, the investment boundary for creating a short position on the upper surface meets the upper bound near maturity $T$.

![Figure 2. Optimal investment boundaries for the illiquid asset with position limits. Parameter values: $\beta_1 = 0.1, \beta_2 = 0.15, \sigma_1 = 0.2, \sigma_2 = 0.25, \delta_1 = 1, \delta_2 = 0.4, \lambda = 1, \rho = 0.8, \alpha = 0.01, \theta = 0.01, \gamma = 0.5, r = 0.01, \nu = 0.02, \bar{l} = -0.5, \bar{\ell} = 0.7$](image)

4.1.3. **Comparison.** To visualize the difference between constrained and unconstrained cases clearly, we look at the image projected onto the 2D-plane of time and $Z$ as shown in Fig. 3. All four investment regions change at the beginning of the investment horizon for a given position limit, among which the boundaries for creating positions present more obvious patterns. In the left graph, the boundary for creating a long position with position limit lies above that in the unconstrained case throughout the investment horizon. The boundary for creating a short position with a limit lies slightly below the unconstrained case at the beginning and then move across later. Intuitively, the investment strategy with position limits may have the same boundaries as in the unconstrained case and only change once on reaching the limits. However, the numerical analysis implies that the investment strategy should be adjusted at the very beginning stage for a given set of position limits.

For the unconstrained case, the numerical results suggest that the investor should not enter the illiquid asset market to create positions near the end of the investment horizon. However, the existence of position limits forces the investor to take actions bounded by the limits near maturity. Consequently, instead of engaging in possible compulsory transactions near maturity with transaction costs that can hardly be covered by profits, the investor changes the investment behavior at the beginning. With position limits, the investor tends to create positions when the prices are not diverged as heavily as in the unconstrained case, to wait for convergence as the cointegration effect suggests. Investors would reasonably enter the illiquid market earlier instead of reaching the position limits near maturity and thereby incurring transaction costs that cannot be covered by profits. Moreover, with position limits, the investor tends to close the position when prices approach a level below that in the unconstrained case, i.e., with slightly less arbitrage profits. Overall, adding
PAIRS TRADING WITH ILLIQUIDITY AND POSITION LIMITS

4.2. Comparative statics. In addition to position limits, we also examine the optimal strategy with different parameter values. We first explore how the optimal strategy changes in terms of transaction costs, which implies the change in liquidity demands, and in correlation coefficient $\rho$. We then investigate optimal investment behavior with different position limits. We also check how cointegration effects, i.e., convergence speed of the two risky asset price processes, affect optimal investment boundaries for both constrained and unconstrained cases. Selecting 2D-plane sections along the time axis of the boundary surfaces for comparison makes a better illustration.

4.2.1. Change in transaction costs. Fig. 4 plots the four optimal investment regions against time at the section $x = 15$ for different transaction cost rates $\alpha$. Without loss of generality, we assume the same transaction cost rates for purchasing and selling the illiquid asset $e^z$, i.e., $\alpha = \theta$. When the transaction cost rate $\alpha$ decreases, the optimal strategy tends to create positions once the two risky asset prices are not far apart from each other. In other words, the first two investment regions for entering the illiquid market widen and the no-transaction region shrinks. Similarly, the two investment regions corresponding to closing positions also widen as $\alpha$ decreases. In such a situation, the investor closes the positions with less arbitrage profits to cover the costs. Generally, market liquidity provides wider investment regions to the optimal pairs-trading strategy.

4.2.2. Change in correlation coefficients. Fig. 5 plots the four optimal investment boundaries against time at the section $x = 15$ for various correlation coefficients $\rho$. As the correlation coefficient $\rho$ decreases, the investor tends to create a position with the illiquid asset when the prices diverge more and to close the position when the prices converge to the level that induces more arbitrage profits at the beginning of the investment horizon. In other words, the investment regions widen as $\rho$ increases. The respective investment regions in which to create a short position and close a long position, i.e., when the investor needs to sell the illiquid asset, keep the position limits to the investment policy widens the investment regions and forces investors to trade more actively.
FIGURE 4. Optimal investment boundaries for the illiquid asset at \( x = 15 \) with various transaction cost rates \( \alpha \). Parameter values: \( \beta_1 = 0.1, \beta_2 = 0.15, \sigma_1 = 0.2, \sigma_2 = 0.25, \delta_1 = 1, \delta_2 = 0.4, \lambda = 1, \rho = 0.8, \gamma = 0.5, r = 0.01, \nu = 0.02 \)

same pattern throughout the investment horizon. When the investment horizon shortens, the investment boundaries with lower \( \rho \) get closer to those with higher \( \rho \). Interestingly, for the investment regions to create a long position and to close a short position, i.e. when the trader need to trade the illiquid asset, the investment boundaries with lower \( \rho \) run across those with higher \( \rho \) later in the investment horizon.

FIGURE 5. Optimal investment boundaries for the illiquid asset at \( x = 15 \) with various correlation coefficients \( \rho \). Parameter values: \( \beta_1 = 0.1, \beta_2 = 0.15, \sigma_1 = 0.2, \sigma_2 = 0.25, \delta_1 = 1, \delta_2 = 0.4, \lambda = 1, \alpha = 0.01, \theta = 0.01, \gamma = 0.5, r = 0.01, \nu = 0.02 \)

Intuitively, the lower the correlation coefficient \( \rho \) the greater the fluctuation in the illiquid asset \( e^z \) so that the incentive of trading is reduced for a low correlation at the beginning of the investment horizon. However, the effect of cointegration accelerates the convergence process and the trading of the liquid assets compensates for such a fluctuation through the diversification effect. Thus, the investment boundaries with lower \( \rho \) get closer and even run across those with higher \( \rho \) at a later time.

The optimal investment boundaries act in a similar manner at the beginning of the investment horizon, as [19] suggests. However, their trading behaviors are solely explained by the incentive of arbitrage, while [12] explains trading boundaries
regarding \( \rho \) only through the diversification effect. Interestingly, our optimal investment boundaries give a mixture of incentives so that the boundaries run across each other.

![Figure 6](image)

**Figure 6.** Optimal investment boundaries for creating positions with the illiquid asset at \( x = 15 \) with various lower bound \( \ell \). Parameter values: \( \beta_1 = 0.1, \beta_2 = 0.15, \sigma_1 = 0.2, \sigma_2 = 0.25, \delta_1 = 1, \delta_2 = 0.4, \lambda = 1, \alpha = 0.01, \theta = 0.01, \gamma = 0.5, r = 0.01, \nu = 0.02 \)

4.2.3. **Change in position limits.** As the position limits most affect the investment regions in which to create positions with the illiquid asset, we focus on comparing first two investment regions for entering the illiquid asset market. Fig. 6 plots the investment boundaries for creating a position with the illiquid asset against time with various lower bounds \( \ell \) and upper bounds \( \bar{\ell} \) for \( z \). Evidently, as \( \ell \) decreases, the later part of the investment boundaries approach those in the unconstrained case. However, at the beginning of the investment horizon, there exists a gap between the constrained and unconstrained cases. Cointegration effects imply that with the existence of position limits, no matter their value, the trader tends to enter the illiquid market earlier to wait for price convergence rather than hitting the position limits near maturity, consistent with section 3. At the same time, as \( \bar{\ell} \) increases, and the later part of the investment boundaries approaches those in the unconstrained case. Similarly, there exists a gap between the constrained and unconstrained cases at the beginning of the investment horizon, although with much smaller size. We can explain it in the same way namely investors’ incentive to enter the illiquid market earlier due to the cointegration effect.

[12] also concludes that with position limits, the optimal trading boundary changes at the beginning of the investment horizon. Without considering cointegration, [12] points out that with position limits, the manager tends to hold a lower proportion in the illiquid asset. However, in our problem, wherein we consider cointegration, the investment regions widen with the position limits. The gap between investment strategies with or without position limits is mainly explicable by cointegration effects.

4.2.4. **Change in cointegration effects.** To examine the changes in investment boundaries with regard to cointegration effects, we introduce a parameter \( \kappa \) to be a multiplier for \( \delta_1 \) and \( \delta_2 \) to control the speed of convergence for the two price processes:

\[
d\ln S_{it} = (\beta_i - \kappa \delta_i z_t)dt + \sigma_i dW_{1t}, \ i = 1, 2.
\]

(21)
Fig. 7 plots the four investment regions with various cointegration effects $\kappa$. We observe that as $\kappa$ decreases, which implies the weakening of the cointegration effect, all four investment regions move downward. This pattern implies that with a weaker cointegration effect, it is optimal to trade the illiquid asset at a lower price, and thus the proportion of the illiquid asset in the whole portfolio decreases. When the cointegration effect is weak, the arbitrage profit from cointegration decreases, and thus the manager tends to invest more in the liquid asset instead of the high-risk illiquid asset. Besides, the boundaries for creating a short position with the illiquid asset have a steeper pattern for weaker cointegration effects to run across those boundaries with larger $\kappa$. Smaller $\kappa$ discourages trading of the illiquid asset, and when approaching maturity, it is optimal to avoid entering the illiquid asset market in the presence of transaction costs. Thus, the investment region for creating a short position becomes much narrower for smaller $\kappa$ compared to that for larger $\kappa$. Similar patterns can also be observed in the first trading region for creating a long position with the illiquid asset. The investment region is much narrower for smaller $\kappa$, especially near maturity.

Our method of examining the impact by different cointegration effects $\kappa$ is similar to that of [19] in that we both introduce the parameter $\kappa$. [19] concludes the trading regions widen as $\kappa$ increases and explains that solely through arbitrage incentives. However, when we incorporate liquid asset trading into our problem, we find that the optimal investment boundaries exhibit more complicated motivations such that the boundaries run across each other.

5. Conclusion. We formulate the optimal investment problem for a portfolio of cointegrated risky assets with transaction costs and position limits, and study the behavior of the optimal investment strategy. It shows strong incentives to make arbitrage profits from the temporary relative mis-pricing advised by the cointegration factor. Assuming certain position limits, the optimal investment strategy becomes more active and the investor adjust it from the beginning of the investment horizon, rather than only changing after reaching the limits. When there are positive limits, cointegration effects provide an incentive to create a position with the illiquid asset earlier and wait for price convergence. When one considers both cointegration
and trading with liquid assets, one finds that the optimal investment strategy exhibits more complicated incentives. Investigating analytical features of the optimal investment problem could be a future research direction.

**Acknowledgment.** We thank the Editor, the Associate Editor and two anonymous referees for their constructive comments and suggestions which greatly improve the quality and readability of the paper. M.C. Chiu acknowledges the support of Research Grants Council of Hong Kong, GRF project: EdUHK18200114.

**Appendix A. Penalty method.** With reference to [16] and [19], the penalty method for solving the HJB equations (10) and (15) is given by

\[
\begin{aligned}
V_t^0 + LV^0 + K \max(-V^0 + V^1 - u((1 + \theta)e^{z^i}), 0) \\
+ K \max(-V^0 - V^{-1} + u((1 - \alpha)e^{z^i}), 0) = 0 \\
V_t^1 + LV^1 + K \max(-V^1 + V^0 + u((1 - \alpha)e^{z^i}), 0) = 0 \\
V_t^{-1} + LV^{-1} + K \max(-V^{-1} + V^0 - u((1 + \theta)e^{z^i}), 0) = 0,
\end{aligned}
\]  

(22)

where \( K > 0 \) is the penalty parameter. As \( K \to \infty \), the solution to (B) satisfies the original HJB equations (10) and (15).

**Appendix B. Numerical scheme.** The system of HJB equations are discretized with a standard finite difference scheme. As for \( V_t^j + LV^j \), where \( j = 0, 1, -1 \), in each equation, the discretization is essentially the same. We here only show a general case of the value function \( V \) with \( x \in [\bar{x}, \bar{x}] \) and \( z \in [\bar{z}, \bar{z}] \). Let \( V_{i,m}^n \) denote \( V(x_i, z_m, t_n) \) at time \( t_n \). The discretization of \( V_t + LV \) is given by:

\[
(V_t)_i^j + LV_{i,j}^n = \frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta t} + \frac{V_{i+1,j}^n - V_{i-1,j}^n}{2\Delta x} + \frac{(\beta_1 - \lambda \beta_2 - (\delta_1 - \lambda \delta_2) z_j)V_{i,j}^{n+1} - V_{i,j}^{n-1}}{2\Delta z} \\
+ \frac{1}{2} (\sigma_1^2 + \lambda^2 \sigma_2^2 - 2\rho \lambda \sigma_1 \sigma_2) \left[ \frac{V_{i,j}^{n+1}}{2\Delta x} + \frac{V_{i,j}^{n-1}}{2\Delta z} - \nu V_{i,j}^n \right] \\
- \left[ (\beta_1 - \delta_1 z + \frac{1}{2} \sigma_1^2 - r) \frac{V_{i+1,j}^n - V_{i-1,j}^n}{2\Delta x} + \sigma_1 (\sigma_1 - \lambda \rho \sigma_2) \right] \\
\frac{V_{i+1,j+1}^n - V_{i+1,j-1}^n - V_{i-1,j+1}^n + V_{i-1,j-1}^n}{4\Delta x \Delta z} / \left( 2\sigma_1^2 V_{i+1,j}^{n+1} + V_{i,j}^{n+1} + V_{i-1,j}^n \right). \quad (23)
\]

According to [2], on the boundaries of \( x \) and \( z \), the following adjustments are added to the discretization:

\[
V_{i,j}^{n-1} = V_{i,j}^n \quad \text{on} \quad x_i = x; \quad V_{i,j}^{n-1} = V_{i,j}^n \quad \text{on} \quad z_i = z; \\
V_{i+1,j}^n = V_{i,j}^n \quad \text{on} \quad x_i = \bar{x}; \quad V_{i,j+1}^n = V_{i,j}^n \quad \text{on} \quad z_i = \bar{z}.
\]

We then apply an implicit scheme and a non-smooth version of Newton’s method at each time iteration to solve the non-smooth equation. First, the discretized HJB
equations are given by:

\[
\begin{aligned}
(V_t^n)_{i,j} + \mathcal{L}(V^0)_{i,j} + K\max(-(V^0)_{i,j} + (V^1)_{i,j} - u((1 + \theta)e^{z_j}),0) \\
+ K\max(-(V^0)_{i,j} + (V^1)_{i,j} + u((1 - \alpha)e^{z_j}),0) = 0 \\
(V_t^n)_{i,j} + \mathcal{L}(V^1)_{i,j} + K\max(-(V^1)_{i,j} + (V^0)_{i,j} + u((1 - \alpha)e^{z_j}),0) = 0 \\
(V_t^{-n})_{i,j} + \mathcal{L}(V^{-1})_{i,j} + K\max(-(V^{-1})_{i,j} + (V^0)_{i,j} - u((1 + \theta)e^{z_j}),0) = 0,
\end{aligned}
\]

corresponding to a full implicit discretization. Then to solve , we apply Newton’s method at each time step to find the solution. In particular, we need to define the derivatives of the penalty terms. Let the penalty terms generalized be \(K_{\text{max}}(f(x),0)\). With reference to \([16]\), we define the derivative of the penalty terms as follows:

\[
\frac{\partial K_{\text{max}}(f(x),0)}{\partial x} = \begin{cases} 
K \frac{\partial f(x)}{\partial x} & \text{if } f(x) > 0 \\
0 & \text{otherwise.}
\end{cases}
\]

REFERENCES

[1] A. Almazan, K. C. Brown, M. Carlson and D. Chapman, Why constrain your mutual fund manager?, *J. of Financial Econ.*, 73 (2004), 289–321.

[2] M. Akian, J. L. Meadaldi, and A. Sulem, On an investment-consumption model with transaction costs, *SIAM J. on Control and Optimization*, 34 (1996), 329–364.

[3] R. Baillie and T. Bollerslev, Common stochastic trends in a system of exchange rates, *J. of Financial Econ.*, 33 (1993), 329–364.

[4] S. Basak, A. Pavlova and A. Shapiro, Optimal asset allocation and risk shifting in money funds, *European J. of Oper. Research*, 52 (2013), 52–64.

[5] M. C. Chiu and H. Y. Wong, Mean-variance portfolio selection of cointegrated assets, *J. of Computational and Applied Math.*, 290 (2015), 516–534.

[6] K. Chen, M. C. Chiu and H. Y. Wong, Time-consistent mean-variance pairs-trading under regime-switching cointegration, *SIAM J. on Control and Optimization*, 146 (2011), 1598–1630.

[7] M. Davis and A. Norman, Portfolio selection with transaction costs, *Math. of Ops. Research*, 15 (1990), 676–713.

[8] J. Duan and S. R. Pliska, Option valuation with co-integrated asset prices, *J. of Econ. Dynamics and Control*, 28 (2004), 727–754.

[9] R. Engle and C. Granger, Co-integration and error correction: Representation, estimation, and testing, *Econometrica*, 55 (1987), 251–276.

[10] P. A. Forsyth and K. R. Vetzal, Quadratic convergence for valuing American options using a penalty method, *SIAM J. on Scientific Computing*, 23 (2002), 2095–2122.

[11] E. Gatev, W. N. Goetzmann and K. G. Rouwenhorst, Pairs trading: Performance of a relative-value arbitrage rule, *The Review of Finan. Studies*, 19 (2006), 797–827.

[12] J. Kacperski, M. Livingston and E. S. O’Neal, Portfolio transactions costs at US equity mutual funds, Working Paper, (2004).

[13] Y. Lei and J. Xu, Costly arbitrage through pairs trading, *J. of Econ. Dynamics and Control*, 56 (2015), 1–19.
[20] J. Liu and A. Timmermann, Optimal convergence trade strategies, The Review of Finan. Studies, 26 (2013), 1048–1086.

[21] R. Merton, Optimum consumption and portfolio rules in a continuous-time model, Journal of Econ. Theory, 3 (1971), 373–413.

[22] A. Tourin and R. Yan, Dynamic pairs trading using the stochastic control approach, J. of Econ. Dynamics and Control, 37 (2013), 1972–1981.

[23] N. Touzi, Optimal Stochastic Control, Stochastic Target Problems, and Backward SDE, Fields Institute Monographs, 29, Springer, New York, 2013.

[24] R. Wermers, Mutual fund performance: An empirical decomposition into stock-picking talent, style, transactions costs, and expenses, The J. of Finance, 55 (2000), 1655–1695.

Received August 2018; revised February 2019.

E-mail address: lucy.feng@sgcib.com
E-mail address: mcchiu@eduhk.hk
E-mail address: hywong@cuhk.edu.hk