Quantization of the Schwarzschild black hole: a Noether symmetry approach

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Abstract

We study the canonical formalism of a spherically symmetric space-time. In the context of the $3 + 1$ decomposition with respect to the radial coordinate $r$, we set up an effective Lagrangian in which a couple of metric functions play the role of independent variables. We show that the resulting $r$-Hamiltonian yields the correct classical solutions which can be identified with the space-time of a Schwarzschild black hole. The Noether symmetry of the model is then investigated by utilizing the behavior of the corresponding Lagrangian under the infinitesimal generators of the desired symmetry. According to the Noether symmetry approach, we also quantize the model and show that the existence of a Noether symmetry yields a general solution to the Wheeler-DeWitt equation which exhibits a good correlation with the classical regime. We use the resulting wave function in order to (qualitatively) investigate the possibility of the avoidance of classical singularities.

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1 Introduction

Black hole physics has played a central role in conceptual discussion of general relativity at the classical and quantum levels. For example regarding event horizons, space-time singularities and also studying the aspects of quantum field theory in curved space-time. In classical point of view, the horizon of a black hole which is a one way membrane, and the space-time singularities are some interesting features of the black hole solutions in general relativity [1]. In spirit of the Ehrenfest principle, any classical adiabatic invariant corresponds to a quantum entity with discrete spectrum, Bekenstein conjectured that the horizon area of a non extremal quantum black hole should have a discrete eigenvalue spectrum [2]. Also, the black hole thermodynamics is based on applying quantum field theory to the curved space-time of a black hole [3]. According to this formalism, the Hawking radiation of a black hole is due to random processes in the quantum fields near the horizon. The mechanism of this thermal radiation can be explained in terms of pair creation in the gravitational potential well of the black hole [4]. The conclusions of the above works are that the temperature of a black hole is proportional to the surface gravity and that the area of its event horizon plays the role of its entropy. In this scenario, the black hole is akin to a thermodynamical system obeying the usual thermodynamic laws, often called the laws of black hole mechanics, first formulated by Hawking [3]. In more recent times, this issue has been at the center of concerted efforts to describe and make clear various aspects of the problem that still remain unclear, for a review see [5]. With the birth of string theory [6], as a candidate for quantum gravity and loop quantum gravity [7], a new window was opened to the problem of black hole radiation. This was because the nature of black hole radiation is such that quantum gravity effects cannot be neglected [8]. According to all of the

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above remarkable works, it is believed that a black hole is a quantum mechanical system and thus like any other quantum system its physical states can be described by a wave function. Indeed, due to its fundamental conceptual role in quantum general relativity, we may use it as a starting point for testing different constructions of quantum gravity [9, 10].

In this paper we deal with the Hamiltonian formalism of a static spherically symmetric space-time. We show that the classical solution of such a system can be identified with the space-time of a Schwarzschild black hole. In this model a couple of metric functions play the role of independent variables which with their conjugate momenta construct the corresponding phase space. We then study the existence of Noether symmetry in this phase space by utilizing the behavior of the corresponding Lagrangian under the infinitesimal generators of the desired symmetry. By the Noether symmetry of a given phase space we mean that there exists a vector field $X$ as the infinitesimal generator of the symmetry on the tangent space of the configuration space such that the Lie derivative of the Lagrangian with respect to this vector field vanishes. Since the existence of a symmetry results in a constant of motion (Noether charge), we show that the mass of the black hole plays the role of the Noether charge in black hole system. Finally, we consider a canonical quantum theory by replacing the classical phase space variables by their Hermitian operators. We study the various aspects of the resulting quantum model and corresponding Wheeler-DeWitt equation and present closed form expressions for the wave function of the black hole.

2 The model

We start with the Einstein-Hilbert action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + S_{YGH},$$  \hspace{1cm} (1)

where $R$ is the Ricci scalar, $g$ is the determinant of the metric tensor and $S_{YGH}$ is the York-Gibbons-Hawking boundary term. Following [11] we assume that the geometry of space-time is described by a general static spherically symmetric line element as [12]

$$ds^2 = -a(r)dt^2 + N(r)dr^2 + 2B(r)dt dr + b^2(r)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2),$$  \hspace{1cm} (2)

where $a(r)$, $B(r)$, $N(r)$ and $b(r)$ are undetermined functions of $r$. If we introduce a new radial coordinate by the transformation $b(r) \rightarrow r'$ and define a new time coordinate by $I(r)[a(r)dt - B(r)dr] \rightarrow dt'$, it easy to show that the line element (2) (after dropping the primes) takes the standard form

$$ds^2 = -A(r)dt^2 + C(r)dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2),$$  \hspace{1cm} (3)

in which there are two unknown functions $A(r)$ and $C(r)$ to be determined by the Einstein field equations. Although, the line element (3) is the standard spherically symmetric metric line element in four dimension, as in [11] our starting point to construct the Hamiltonian formalism of the model will be the line element (2). Indeed, we shall see that a combination of the metric functions in (2) forms a Lagrange multiplier as

$$n(r) = a(r)N(r) + B^2(r).$$  \hspace{1cm} (4)

On the other hand, in the metric (2), $N(r)$ plays the role of a lapse function with respect to the $r$-slicing in the ADM terminology. Therefore, since the functions $N(r)$ and $B(r)$ are related to the Lagrange multiplier $n(r)$ by (4), they can be arbitrarily chosen. These two functions represent the freedom in the definition of the coordinates $r$ and $t$ in the metric (2). Hence, we are left with two functions $a(r)$ and $b(r)$ which can be determined by the Einstein field equations.

Now, let us deal with the canonical structure of the model at hand. Putting equation (2) in the expression (1), we find that the action for the gravitational field can be written as [11]

$$S = \int dt \int dr L(a, b, n),$$  \hspace{1cm} (5)
where
\[ \mathcal{L} = 2\sqrt{n} \left( \frac{a'b'b}{n} + \frac{ab'^2}{n} + 1 \right), \]
is an effective Lagrangian in which primes denotes differentiation with respect to \( r \) and \( n \) is given by (4). In order to pass to the Hamiltonian formalism, conjugate momenta must be calculated. They are given by
\[ P_a = \frac{\partial \mathcal{L}}{\partial a'} = \frac{2bb'}{\sqrt{n}}, \quad P_b = \frac{\partial \mathcal{L}}{\partial b'} = 2\left( \frac{2ab' + a'b}{\sqrt{n}} \right). \]
Also, the primary constraint is given by
\[ P_n = \frac{\partial \mathcal{L}}{\partial \eta} = 0, \]
which is a consequence of the gauge invariance of general relativity. In terms of these conjugate momenta the canonical Hamiltonian is given by
\[ H = a'P_a + b'P_b - \mathcal{L}, \]
leading to
\[ H = \sqrt{n} \left( \frac{P_aP_b}{2b} - \frac{a}{2b^2}P_a^2 - 2 \right) = \sqrt{n}\mathcal{H}. \]
Because of the existence of constraint (8), the Lagrangian of the system is singular and the total Hamiltonian can be constructed by adding to \( H \) the primary constraints multiplied by arbitrary functions \( \lambda(r) \)
\[ H_T = \sqrt{n} \left( \frac{P_aP_b}{2b} - \frac{a}{2b^2}P_a^2 - 2 \right) + \lambda P_n. \]
The requirement that the primary constraint should hold during the \( r \)-evolution of the system means that \( (\approx \text{is the Dirac weak equality}) \)
\[ P'_n = \{P_n, H_T\} \approx 0, \]
where \( \{,\} \) denotes the Poisson bracket in the minisuperspace \((a, b, n)\) and for any phase space function can be straightforward obtained by its standard definition
\[ \{f(q, p), g(q, p)\} = \{\eta^A, \eta^B\} \frac{\partial f}{\partial \eta^A} \frac{\partial g}{\partial \eta^B}; \]
where \( \eta = (q, p) = (a, b, n; P_a, P_b, P_n) \). This leads to the secondary constraint
\[ -\frac{1}{2\sqrt{n}}\mathcal{H} = -\frac{2}{2\sqrt{n}} \left( \frac{P_aP_b}{2b} - \frac{a}{2b^2}P_a^2 - 2 \right) \approx 0, \]
which represents the invariance of the theory under \( r \)-reparametrization. We see that this expression has the form \( f(q, p)g(q, p) \approx 0 \). As is argued in [13] and [14] from such relations one cannot conclude \( f(q, p) \approx 0 \) or \( g(q, p) \approx 0 \) since this is in contrast with the statement that all secondary first class constraints must generate symmetries. However, if \( g(q, p) \approx 0 \) then for any function \( f(q, p) \) we have \( f(q, p)g(q, p) \approx 0 \). As a result, in spite of the usual case where we obtain \( \mathcal{H} \approx 0 \) from the secondary constraint, the constraint equation (14) in this case does not lead us to \( \mathcal{H} \approx 0 \). The Poisson bracket of the secondary constraint with the total Hamiltonian reads
\[ \{-\frac{1}{2\sqrt{n}}\mathcal{H}, \sqrt{n}\mathcal{H} + \lambda P_n\} = -\frac{\lambda}{2\sqrt{n}} \left( \frac{1}{\sqrt{n}} P_n \right) \mathcal{H} \approx -\frac{\lambda}{2n} \left( -\frac{1}{2\sqrt{n}}\mathcal{H} \right) \approx 0, \]
which shows that the secondary constraint is preserved with varying of \( r \). Now, following [13], we define the extended Hamiltonian as
\[ H_E = (\sqrt{n} - \frac{\eta}{2\sqrt{n}})\mathcal{H} + \lambda P_n, \]
where \( \eta \) is a Lagrange multiplier. The \( r \)-evolution of a function is thus given by \( f' \approx \{f, H_E\} \). The preliminary setup for describing the model is now complete. In what follows, we will study the classical and quantum solutions of the minisuperspace model described by Hamiltonian (16).
3 Classical solutions

The classical field equations are governed by the Hamiltonian equations, that is

\[
\begin{align*}
    n' &\approx \{n,H_E\} = \lambda, \\
    P_n' &\approx \{P_n,H_E\} = -\frac{1}{2\sqrt{n}} \mathcal{H} + \frac{n}{2n}(-\frac{1}{2\sqrt{n}} \mathcal{H}) = 0, \\
    a' &\approx \{a,H_E\} = \frac{1}{\sqrt{n}}(n - \frac{\eta}{2}) \left( \frac{1}{2a} P_b - \frac{\alpha}{\sqrt{a}} P_a \right), \\
    P_a' &\approx \{P_a,H_E\} = \frac{1}{\sqrt{n}}(n - \frac{\eta}{2}) \frac{1}{2a^2} P_a^2, \\
    b' &\approx \{b,H_E\} = \frac{1}{\sqrt{n}}(n - \frac{\eta}{2}) \sqrt{P_a}, \\
    P_b' &\approx \{P_b,H_E\} = \frac{1}{\sqrt{n}}(n - \frac{\eta}{2}) \left( \frac{1}{2a^2} P_a P_b - \frac{\alpha}{\sqrt{a}} P_a^2 \right).
\end{align*}
\]

We see from the first equation of the above system that the derivative of \( n \) is equal to a Lagrange multiplier. This shows that \( n \) is completely arbitrary and therefore from (4) we have the freedom of choosing \( B(r) \) or \( N(r) \). Now, since \( n \) can be arbitrarily chosen the rest equations of the system (17) can be cast in the form

\[
\begin{align*}
    a' &\approx \{a,H_E\} = \sqrt{n} \left( \frac{1}{2a} P_b - \frac{\alpha}{\sqrt{a}} P_a \right), \\
    P_a' &\approx \{P_a,H_E\} = \sqrt{n} \frac{1}{2a^2} P_a^2, \\
    b' &\approx \{b,H_E\} = \sqrt{n} \frac{1}{2a} P_a, \\
    P_b' &\approx \{P_b,H_E\} = \sqrt{n} \left( \frac{1}{2a^2} P_a P_b - \frac{\alpha}{\sqrt{a}} P_a^2 \right).
\end{align*}
\]

Up to this point the model, in view of the concerning issue of gauges, has been rather general and of course under-determined. Before trying to solve these equations we must decide on a choice of gauge in the theory. This is important because in order to measure the physical quantities one should employ gauge conditions. The under-determinacy problem at the classical level may be removed by using the gauge freedom via fixing the gauge. From now on we restrict ourselves to a certain class of gauges, namely \( n = \text{const.} \), which is equivalent to the choice \( \lambda = 0 \) in the first equation of (17). With a constant \( n \) we assume \( n = 1 \) without losing general character of the solutions. Although, it seems that this is the most trivial choice for the function \( \lambda(r) \) in equation (11), but this is compatible with the spirit of gauge invariance, that is, there is no physical difference between multiple gauge fixing. The measured value of the system parameters could be different for different gauges but they have the same behavior. The second and the third equations of (17) can easily be integrated to yield

\[
b(r) = r, \quad P_a = 2r, \tag{19}
\]

where with the help of them the rest equations can be put into the form

\[
\begin{align*}
    a' &= \frac{1}{2a} P_b - \frac{2a}{r}, \\
    P_b' &= \frac{P_b}{r} - \frac{4a}{r}.
\end{align*}
\]

Upon integration, the solution of this system can be found as

\[
a(r) = 1 - \frac{2M}{r}, \quad P_b = 4 - \frac{4M}{r}, \tag{21}
\]
where $M$ is an integration constant. Therefore, the above Hamiltonian formalism lead us to the following static spherically symmetric metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + N(r)dr^2 \pm 2\left[1 - \left(1 - \frac{2M}{r}\right)N(r)\right]^{1/2}dtdr + r^2d\Omega^2,$$

(22)

in which we have used equation (4) to eliminate $B(r)$. This is exactly the solution obtained in [11] using the Lagrangian formalism. Now, it is clear that with the choice of $N(r) = (1 - \frac{2M}{r})^{-1}$, we can recover the standard Schwarzschild black hole solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2,$$

(23)

with $M$ being now the mass parameter of the black hole. Other choices of $N(r)$ are correspond to the other forms of the Schwarzschild solution, see [11]. It is clear from the condition $a(r) > 0$ that this solution is only valid for $r > 2M$. The above metric has an apparent singularity at $r = 2M$. This singularity is the coordinate singularity associated with horizon in the Schwarzschild space-time, and as is well known, there are other coordinate system for which this type of singularity is removed [1]. Another singularity associated with the metric (23) is its essential singularity at $r = 0$. As we know, in general relativity, to investigate the types of singularities one has to study the invariants characteristics of space-time and to find where these invariants become infinite so that the classical description of space-time breaks down. In a 4- dimensional Riemannian space-time there are 14 independent invariants, but to detect the singularities it is sufficient to study only three of them, the Ricci scalar $R$, $R_{\mu\nu}R^{\mu\nu}$ and the so-called Kretschmann scalar $R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta}$. For the metric (23) the Kretschmann scalar reads

$$K = R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta} \sim \frac{1}{r^6}.$$

(24)

Now, it is clear that the space-time described by the metric (23) has an essential singularity at $r = 0$, which can not be removed by a coordinate transformation.

4 Noether symmetry

As is well known, Noether symmetry approach is a powerful tool in finding the solution to a given Lagrangian, including the one presented above. In this approach, one is concerned with finding the cyclic variables related to conserved quantities and consequently reducing the dynamics of the system to a manageable one. The investigation of Noether symmetry in the model presented above is therefore the goal we shall pursue in this section. Following [15], we define the Noether symmetry induced on the model by a vector field $X$ on the tangent space $TQ = (a, b, a', b')$ of the configuration space $Q = (a, b)$ of Lagrangian (6) through

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b} + \frac{d\alpha}{dr} \frac{\partial}{\partial a'} + \frac{d\beta}{dr} \frac{\partial}{\partial b'},$$

(25)

such that the Lie derivative of the Lagrangian with respect to this vector field vanishes

$$L_X L = 0.$$

(26)

In (25), $\alpha$ and $\beta$ are functions of $a$ and $b$ and $\frac{d}{dr}$ represents the Lie derivative along the dynamical vector field, that is,

$$\frac{d}{dr} = a' \frac{\partial}{\partial a} + b' \frac{\partial}{\partial b}.$$

(27)

It is easy to find the constants of motion corresponding to such a symmetry. Indeed, equation (26) can be rewritten as

$$L_X L = \left(\alpha \frac{\partial L}{\partial a} + \frac{d\alpha}{dr} \frac{\partial L}{\partial a'}\right) + \left(\beta \frac{\partial L}{\partial b} + \frac{d\beta}{dr} \frac{\partial L}{\partial b'}\right) = 0.$$

(28)
Noting that \( \frac{\partial L}{\partial q} = \frac{dP}{dr} \), we have
\[
\left( \alpha \frac{dP_a}{dr} + \frac{d\alpha}{dr} P_a \right) + \left( \beta \frac{dP_b}{dr} + \frac{d\beta}{dr} P_b \right) = 0,
\] (29)
which yields
\[
\frac{d}{dr} (\alpha P_a + \beta P_b) = 0.
\] (30)
Therefore, the quantity
\[
Q = \alpha P_a + \beta P_b,
\] (31)
is constant with respect to \( r \). In terms of the Hamiltonian formalism equation (28) can be written as
\[
\alpha \{ P_a, H \} + \beta \{ P_b, H \} + P_a \left[ \frac{\partial \alpha}{\partial a} \{ a, H \} + \frac{\partial \alpha}{\partial b} \{ b, H \} \right] \\
+ P_b \left[ \frac{\partial \beta}{\partial a} \{ a, H \} + \frac{\partial \beta}{\partial b} \{ b, H \} \right] = 0.
\] (32)
In order to obtain the functions \( \alpha \) and \( \beta \) we use equation (32) (or (28)). In general these equations give a quadratic polynomial in terms of \( P_a \) and \( P_b \) (or \( a' \) and \( b' \)) with coefficients being partial derivatives of \( \alpha \) and \( \beta \) with respect to the configuration variables \( a \) and \( b \). Thus, the resulting expression is identically equal to zero if and only if these coefficients are zero. This leads to a system of partial differential equations for \( \alpha \) and \( \beta \). For Lagrangian (6), condition (32) results in
\[
\alpha \left( \frac{P_a^2}{2b^2} \right) + \beta \left( \frac{P_a P_b}{2b^2} - \frac{a}{b^3} P_a^2 \right) + P_a \left[ \frac{\partial \alpha}{\partial a} \left( \frac{P_b}{2b} - \frac{a}{b^2} P_a \right) + \frac{\partial \alpha}{\partial b} P_a \right] \\
+ P_b \left[ \frac{\partial \beta}{\partial a} \left( \frac{P_b}{2b} - \frac{a}{b^2} P_a \right) + \frac{\partial \beta}{\partial b} P_a \right] = 0,
\] (33)
which leads to the following system of equations
\[
\begin{align*}
\frac{1}{2b^2} \alpha - \frac{a}{b^3} \beta - \frac{a}{2b} \frac{\partial \alpha}{\partial a} + \frac{1}{2b} \frac{\partial \alpha}{\partial b} &= 0, \\
\frac{1}{2b} \frac{\partial \beta}{\partial a} &= 0, \\
\frac{1}{2b} \beta + \frac{1}{2b} \frac{\partial \alpha}{\partial a} - \frac{a}{b^2} \frac{\partial \beta}{\partial a} + \frac{1}{2b} \frac{\partial \beta}{\partial b} &= 0.
\end{align*}
\] (34)
From the second equation of this system we obtain \( \beta = \beta(b) \), that is, the function \( \beta \) is independent of \( a \). Then, the third equation gives us
\[
\frac{1}{2b^2} \beta + \frac{1}{2b} \frac{\partial \alpha}{\partial a} + \frac{1}{2b} \frac{d\beta}{db} = 0,
\] (35)
where upon integration with respect to \( a \) we obtain
\[
\alpha(a, b) = -a \left( \frac{1}{b} \beta + \frac{d\beta}{db} \right).
\] (36)
Substituting this result into the first equation of the system (34) we get \( \frac{d^2 \beta}{db^2} = 0 \) and therefore
\[
\beta = k_1 b + k_2,
\] (37)
where \( k_1 \) and \( k_2 \) are integration constants. Now, it is easy to find \( \alpha \) from (36) as
\[
\alpha = -a \left( 2k_1 + \frac{k_2}{b} \right).
\] (38)
The corresponding constant to the Noether symmetry can be computed from \( Q = \alpha P_a + \beta P_b \). If we use the classical solutions (19) and (21) in this relation we obtain \( Q = 4k_1M + 2k_2 \). On the other hand we know that from the point of an observer being outside the black hole horizon view, its mass is constant and if one moves along the radial coordinate \( r \) this is the only constant corresponding to the black hole system. Therefore, the integration constants in (37) can be chosen as \( k_1 = \frac{1}{4} \) and \( k_2 = 0 \) yielding \( \alpha = -\frac{1}{2}a \) and \( \beta = \frac{1}{4}b \). Hence, the Noether symmetry is generated by the following vector field

\[
X = -\frac{1}{2}a \frac{\partial}{\partial a} + \frac{1}{4}b \frac{\partial}{\partial b} - \frac{1}{2}a' \frac{\partial}{\partial a'} + \frac{1}{4}b' \frac{\partial}{\partial b'},
\]

(39)

while, as a direct consequence of (31), the corresponding Noether charge is given by

\[
Q = -\frac{1}{2}aP_a + \frac{1}{4}bP_b,
\]

(40)

where its numerical value is equal to the mass of the black hole \( Q = M \).

5 Quantization of the model

We now focus attention on the study of the quantization of the model described above. In the canonical quantization procedure, the extended Hamiltonian (16) contains two first class constraints (8) and (14). Therefore, if the wave function \( \Psi(a, b, n) \) describes the quantum version of the theory, according to the Dirac prescription we demand that it is annihilated by the operator version of these constraints, that is

\[
\hat{P}_n \Psi(a, b, n) = -i \frac{\partial}{\partial n} \Psi(a, b, n) = 0,
\]

(41)

which implies that the wave function is independent of \( n \). And

\[
(\sqrt{n} - \frac{\eta}{2\sqrt{n}}) \hat{\mathcal{H}} \Psi(a, b) = 0 \Rightarrow \hat{\mathcal{H}} \Psi(a, b) = 0,
\]

(42)

which is known as the Wheeler-DeWitt equation. Before going any further, some remarks are in order. Although, the classical equations of motion resulting from the Lagrangian (6) or Hamiltonian (10) give the corresponding classical metric (22) or (23), the operator version of the Hamiltonian constraint (42) does not present a complete description of the quantum version of the model. This is because that the Lagrangian (6) does not exhibit the existence of a cyclic variable corresponding to the Noether symmetry. To be more precise, we seek a point transformation \( (a, b) \rightarrow (u, v) \) on the vector field (25) such that in terms of the new variables \( (u, v) \), the Lagrangian includes one cyclic variable. A general discussion of this issue can be found in [16]. Under such point transformation it is easy to show that the vector field (25) takes the form

\[
\tilde{X} = (Xu) \frac{\partial}{\partial u} + (Xv) \frac{\partial}{\partial v} + \frac{d}{dr}(Xu) \frac{\partial}{\partial u'} + \frac{d}{dr}(Xv) \frac{\partial}{\partial v'}.
\]

(43)

One can show that if \( X \) is a Noether symmetry of the Lagrangian, \( \tilde{X} \) has also this property, that is

\[
X \mathcal{L} = 0 \Rightarrow \tilde{X} \mathcal{L} = 0.
\]

(44)

Thus, if we demand

\[
Xu = 1, \quad Xv = 0,
\]

(45)

we get\(^1\)

\[
\tilde{X} = \frac{\partial}{\partial u} \Rightarrow \tilde{X} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial u} = 0.
\]

(46)

\(^1\)The existence of such a variable which satisfies the condition \( Xu = 1 \) is indeed a consequence of the Noether theorem [17]. However, the change of variables which gives this kind of variable may be not unique. On the other hand, it is possible that the system (34) admit more than one set of solutions. In this case we have more symmetries and therefore more cyclic variables exist. For instance, if \( X_1 \) and \( X_2 \) are two independent Noether symmetries ([\( X_1, X_2 \) = 0]), we may obtain two cyclic variables \( u^1 \) and \( u^2 \) by solving \( X_1 u^1 = 1, \quad X_1 u^{\prime 1} = 0 \) and \( X_2 u^2 = 1, \quad X_2 u^{\prime 2} = 0 \).
This means that \( u \) is a cyclic variable and the dynamics can be reduced. On the other hand, one can show that \([18]\), the constant of motion \( Q \) which corresponds to the Noether symmetry is nothing but the momentum conjugated to the cyclic variable, that is, \( Q = P_u \). To find the explicit form of the above mentioned point transformation we should solve the equations (45), which give

\[
-\frac{1}{2} a \frac{\partial u}{\partial a} + \frac{1}{4} b \frac{\partial u}{\partial b} = 1,
\]

(47)

\[
-\frac{1}{2} a \frac{\partial v}{\partial a} + \frac{1}{4} b \frac{\partial v}{\partial b} = 0.
\]

(48)

These differential equations admit the following general solutions

\[
u(a, b) = \ln \frac{b^2}{a} + f_1(ab^2), \quad v(a, b) = f_2(ab^2),
\]

(49)

where \( f_1 \) and \( f_2 \) are two arbitrary functions of \( ab^2 \). As is indicated in \([16]\), "the change of coordinates is not unique and a clever choice is always important". With a glance at the Lagrangian (6), we choose the functions \( f_1 \) and \( f_2 \) as

\[
f_1(ab^2) = f_2(ab^2) = \ln(ab^2).
\]

(50)

With this choice, the Lagrangian (6) takes the form

\[
\mathcal{L} = e^v \left( -\frac{1}{8} u'^2 + \frac{1}{2} u' v' \right) + 2.
\]

(51)

It is clear from this Lagrangian that \( u \) is cyclic and the Noether symmetry is given by \( P_u = Q = \text{const.} \). Also, the momenta conjugate to \( u \) and \( v \) are

\[
P_u = \frac{\partial \mathcal{L}}{\partial u'} = e^v \left( -\frac{1}{4} u' + \frac{1}{2} v' \right), \quad P_v = \frac{\partial \mathcal{L}}{\partial v'} = \frac{1}{2} u' e^v,
\]

(52)

which give rise to the following Hamiltonian for our model

\[
\mathcal{H} = e^{-v} \left( 2P_u P_v + \frac{1}{2} P_v^2 \right) - 2.
\]

(53)

Since this Hamiltonian and \( P_u \) commute with each other \([\mathcal{H}, P_u] = 0\), they have simultaneous eigenfunctions. Therefore, the quantum description of our Noether symmetric black hole model can be viewed by the following equations

\[
\mathcal{H} \Psi(u, v) = \left[ e^{-v} \left( 2\hat{P}_u \hat{P}_v + \frac{1}{2} \hat{P}_v^2 \right) - 2 \right] \Psi(u, v) = 0,
\]

(54)

\[
\hat{P}_u \Psi(u, v) = Q \Psi(u, v).
\]

(55)

Choice of the ordering \( e^{-v} \hat{P}_v \to \frac{1}{2} (e^{-v} \hat{P}_v + \hat{P}_v e^{-v}) \) and \( e^{-v} \hat{P}_v^2 \to \hat{P}_v e^{-v} \hat{P}_v \) to make the Hamiltonian Hermitian and use of \( \hat{P}_u \to -i \partial_u \) and similarly for \( \hat{P}_v \), the above equations read

\[
\left( -\frac{1}{2} \frac{\partial^2}{\partial u^2} + \frac{1}{2} \frac{\partial}{\partial v} - 2 \frac{\partial^2}{\partial u \partial v} + \frac{\partial}{\partial u} - 2e^v \right) \Psi(u, v) = 0,
\]

(56)

\[
-i \frac{\partial}{\partial u} \Psi(u, v) = Q \Psi(u, v).
\]

(57)

The solutions of the above differential equations are separable and may be written in the form \( \Psi(u, v) = U(u) V(v) \). Equation (57) can be immediately integrated leading to a oscillatory behavior for the wave function in \( u \) direction, i.e. in the direction of symmetry, that is

\[
\Psi(u, v) = e^{iQu} V(v).
\]

(58)
Therefore, in the semiclassical approximation region we obtain the phase function 

\[ V(v) = 0, \]  

where its solutions can be written in terms of the Hankel (or Bessel) functions as

\[ V(v) = e^{(\frac{1}{2} - 2iQ)v} \left[ c_1 H^{(1)}_{i\sqrt{16Q^2 - 1}} \left( 4e^{v/2} \right) + c_2 H^{(2)}_{i\sqrt{16Q^2 - 1}} \left( 4e^{v/2} \right) \right]. \]

Thus, the eigenfunctions of the Wheeler-DeWitt equation can be written as

\[ \Psi_Q(u, v) = e^{v/2} e^{iQ(u - 2v)} \left[ c_1 H^{(1)}_{i\sqrt{16Q^2 - 1}} \left( 4e^{v/2} \right) + c_2 H^{(2)}_{i\sqrt{16Q^2 - 1}} \left( 4e^{v/2} \right) \right]. \]

We may write the general solutions to the Wheeler-DeWitt equation as a suitable superposition of these eigenfunctions. In the classical limit, i.e. for large values of \( r \) we have \( b(r) \sim r \) and \( a(r) \sim 1 \) and from equations (49) and (50) we get the behavior \( u(r) \sim \ln r^4 \) and \( v(r) \sim \ln r^2 \) for \( u \) and \( v \) in this limit. On the other hand, in view of the asymptotically behavior of the Hankel functions \( H^{(1,2)}_{i} \sim z^{-1/2} e^\pm \sqrt{-(2\nu + 1)\pi/4} \) we obtain the following behavior for the Wheeler-DeWitt eigenfunction for large values of \( r \)

\[ \Psi_Q(u, v) \sim e^{v/4} e^{iQ(u - 2Qv + 4e^{v/2}),} \]

Therefore, in the semiclassical approximation region we obtain the phase function \( S(u, v) \) as

\[ S(u, v) = \pm \left( Qu - 2Qv + 4e^{v/2} \right). \]

where the positive sign corresponds to an increasing \( r \) model. In the WKB method, the correlation between classical and quantum solutions is given by the relation \( P_q = \frac{\partial S}{\partial q} \). Thus, using the definition of \( P_u \) and \( P_v \) in (52), the equation for the classical trajectories becomes

\[
\begin{align*}
&v'\left( -\frac{1}{4}u' + \frac{1}{2}v' \right) = Q, \\
&\frac{1}{2}u'e^v = -2Q + 2e^{v/2}.
\end{align*}
\]

Eliminating \( u' \) from this system results \( v' e^{v/2} = 2 \), which can easily be integrated to yield \( v(r) = \ln r^2 \). With the help of this expression for \( v \), the equation for \( u' \) can be put into the form \( u' = -\frac{4Q}{r^2} + \frac{4}{r} \), which admits the solution \( u(r) \sim \ln r^4 \) for large values of \( r \). Therefore, our analysis shows that in the large-\( r \) limit the behavior of the classical solution is exactly recovered. The meaning of this result is that for large values of \( r \) the effective action is very large and the system can be described classically. On the other hand, near the black hole singularities we cannot neglect the quantum effects and the classical description breaks down. Since the WKB approximation is no longer valid in this regime, one should go beyond the semiclassical approximation. In the cases where \( r \) approaches the classical singularities \( r \rightarrow 0 \) and \( r \rightarrow 2M \) the variables \( u \) and \( v \) behave as \( u, v \rightarrow -\infty \) and \( u \rightarrow \ln(2M)^4 \), \( v \rightarrow -\infty \) respectively. We see from (61) that the Hankel functions have small argument in these limits with a oscillatory behavior. Therefore, since the eigenfunctions tend to zero, the quantum solutions are regular for \( r \rightarrow 0, 2M \). These quantum solutions may be interpreted as being responsible of the avoidance of classical singularity. We are not here representing a complete discussion about this subject since it depends on the various aspects of the Wheeler-DeWitt equation such as the choice of ordering, representation for the momenta and choice of suitable superposition of its eigenfunctions to construct the wave packets.
6 Conclusions

In this paper we have studied a static spherically symmetric space-time in a canonical point of view. To construct the Hamiltonian formalism we have followed a method proposed in [11] in which a 3 + 1 decomposition is performed with respect to the radial coordinate $r$. The resulting $r$-Hamiltonian is shown to satisfy the Hamiltonian constraint which represents that the underlying theory is invariant under $r$-reparametrization. We have shown that the classical Hamiltonian equations admit solutions which can be identified with the Schwarzschild black hole. The existence of Noether symmetry implies that the Lie derivative of the Lagrangian with respect to the infinitesimal generator of the desired symmetry vanishes. By applying this condition to the Lagrangian of the model, we showed that the mass parameter of the black hole plays the role of the Noether charge, i.e. the mass is a constant of $r$-motion. We have then quantized the model and shown that the corresponding quantum black hole model and the ensuing Wheeler-DeWitt equation are amenable to exact solutions in terms of the Hankel functions. In classical regime we have seen that these solutions are expressed in terms of a superposition of states of the form $e^{iS}$ due to the existence of Noether symmetry. In semiclassical approximation for quantum gravity, this type of state represents the correlations between classical trajectories and the quantum wave function. Using this interpretation we have shown that the corresponding classical metric can be recovered by the quantum solutions counterparts. Finally, we have presented a short discussion to clear the regular behavior of the wave function near the classical singularities and the possibility of the avoidance of these singularities due to quantum effects.

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