Critical phenomena in gravitational collapse

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This article reviews classical and quantum aspects of critical phenomena in gravitational collapse. We pay special attention to the origin of the scaling law for black hole mass, and to phase transitions in which black hole formation turns on at finite mass. We present some new results for perfect fluids with pressure proportional to density.

1 Introduction

The discovery, by Choptuik, of critical point behavior in gravitational collapse is one of the most significant achievements of numerical relativity to date. Efforts to understand the Choptuik phenomena have opened an exciting area of research in general relativity, one which has benefited from the interplay between mathematical and numerical investigations.

The problem addressed by Choptuik was suggested by the work of Christodoulou on the gravitational collapse of spherically symmetric configurations of massless, minimally coupled scalar fields. Christodoulou proved the global existence and uniqueness of regular solutions for initial scalar field configurations in a neighborhood of trivial initial data (i.e., flat, empty space). When the initial data is strong, in a precise sense, he also established that the black hole mass has a positive lower bound. Choptuik examined the intermediate regime numerically. By constructing interpolating families of solutions depending on a single parameter $\eta$, Choptuik demonstrated that there generally exists a critical value $\eta_*$ such that $S[\eta < \eta_*]$ are solutions in which the field disperses to infinity, and $S[\eta > \eta_*]$ are solutions in which the field collapses to form a black hole; he called these *subcritical* and *supercritical* solutions, respectively. He also observed two important properties of near critical evolutions $S[\eta \approx \eta_*]$:

1. For each family of initial data, the masses of black holes formed in marginally supercritical collapse obey a scaling relation

$$M_{\text{BH}} = K|\eta - \eta_*|^\beta$$  \hspace{1cm} (1)
where $\beta \approx 0.37$ is universal, i.e., independent of the initial data, although the constant $K$ is a family dependent.

2. The evolution of the scalar field is characterized by scale-invariant echoing with a period, in logarithmic time, given by $\Delta \approx 3.4$.

These two observations led Choptuik to speculate that precisely critical evolutions approach a unique solution in which scale-invariant echoes accumulate at a massless, central singularity.

It is now known that critical point behavior is a general feature of gravitational collapse. Consider the evolution of a physical system consisting of matter coupled to gravity, or simply of the gravitational field itself. One can think of the equations which govern this system as a map $\mathcal{E}$ from the space of initial data into the space of solutions. Suppose the strength of the initial data $I[\eta]$ is characterized by the single tunable parameter $\eta$ (taken to be in the range $0 \leq \eta \leq \infty$ for concreteness), then one constructs a family of solutions $S[\eta]$ by

$$\mathcal{E} : I[\eta] \rightarrow S[\eta] .$$

Provided weak cosmic censorship holds, critical point behavior may be expected whenever these solutions interpolate between black hole formation and stable, regular solutions. Some of the questions which then arise are:

1. Is cosmic censorship upheld in all families of interpolating solutions?
2. What is the origin of the scaling relation for the black hole mass?
3. Does black hole formation always turn on at infinitesimal mass?
4. Is the critical point always associated with scale invariant solutions?
5. What are the semi-classical corrections to the critical point behavior?

In this contribution, we provide only a flavor of the work that has been done to answer these, and other questions. Our presentation is not intended to be complete, rather it focuses on issues that were discussed in the parallel session at the 8th Marcel Grossmann meeting (MG8). A more complete review, including technical details of the many interesting results which have been obtained, has recently been prepared by Gundlach.[8]

Nevertheless, we would be remiss if we did not mention some of the notable contributions which were not represented at MG8. All of the work to date has been restricted to spherical symmetry except for the study of axisymmetric collapse of pure gravitational waves by Abrahams and Evans.[9] They demonstrated the existence of critical point behavior and derived a scaling law for the black-hole mass similar to that in Eq. (1) with $\beta \approx 0.37$. Despite the extreme complexity of the analysis, they also presented tentative evidence for scale invariant echoing of the

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[8] By weak cosmic censorship we mean that, in the evolution of generic initial conditions, singularities are always hidden behind an event horizon.

[9] Minkowski spacetime is a special case where the material disperses completely.
metric on the symmetry axis of the spacetime. The black-hole mass scaling exponent is intriguingly similar to that observed in scalar field collapse, although the echoing period, $\Delta \approx 0.6$, of the solution is quite different.

The initial explanation of the scaling relation \( (1) \) for black hole mass was made possible by the work of Evans and Coleman \[10\]. They showed that the critical solution, in gravitational collapse of radiation fluid, is continuously self-similar by examining one parameter families of interpolating solutions, and direct construction of the intermediate self-similar attractor. They indicated that the critical exponent might be derived by considering linearized perturbations about the critical solution. This suggestion was followed up by Koike \[2\] et al. who argued that the critical exponent, determined to be $\beta \approx 0.36$ by Evans and Coleman, is directly related to the largest Lyapunov exponent of perturbations about the critical solution. (This idea was also explored by Eardley and Hirschmann \[11\] in the context of complex scalar field collapse.) In this way Koike \[2\] et al. provided the first direct computation of the critical exponent as $\beta = 0.355$ for radiation fluid. Maison \[12\] then extended these results to more general equations of state for the perfect fluid.

It is worth summarizing the argument which relates the Lyapunov exponent to the critical exponent; the discussion follows Maison \[12\]. The general spherically symmetric line element can be written as

$$ds^2 = -\alpha^2 dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) .$$ \hspace{1cm} (3)

The self-similar critical solution has $\alpha = \alpha_*(\xi)$, and $f = f_*(\xi)$, where $\xi = r/t$ and the time coordinate is normalized so that $t = 0$ at the singularity in the critical solution. An important feature of this solution is the existence of a sonic point $\xi_{sp}$ where the fluid moves at the speed of sound relative to $\xi = \xi_{sp}$. Now, consider the linear stability of these solutions against perturbations with the general form

$$\eta - \eta_*(f_1(\xi)t^{-\lambda}) ,$$ \hspace{1cm} (4)

where $\lambda$ is a positive constant. These perturbations arise as small deviations from precisely critical initial data — consequently, they are proportional to $(\eta - \eta_*)$. There is a unique value of $\lambda$ such that the perturbations are regular both at $r = 0$ and at $\xi_{sp}$. An apparent horizon in the perturbed spacetime is given by $(\xi_h, t_h)$ such that

$$f_*(\xi_h) - (\eta - \eta_*)f_1(\xi_h)t_h^{-\lambda} = 0 .$$ \hspace{1cm} (5)

The mass $M_{BH}$ inside the apparent horizon is related to its radius $r_h$ by $M_{BH} = r_h/2$. Substituting $t_h = r_h/\xi_h$ into Eq. (5), we can solve for $M_{BH}$ as

$$M_{BH} = r_h/2 \propto |\eta - \eta_*|^{1/\lambda} .$$ \hspace{1cm} (6)

The scaling exponent is then read off as $\beta = 1/\lambda$. In addition to these works, Gundlach has made significant contributions to the analytic understanding of critical phenomena when the critical solutions have self-similar echoes. He has termed this symmetry discrete self-similarity (DSS), and has constructed solutions with DSS by writing the metric as a Fourier series, and recasting Einstein’s equations as an eigenvalue problem for the echoing period. As
in the case of perfect fluids, where Maison was able to predict the critical exponent for situations which had not been numerically explored, the power of Gundlach’s analysis was demonstrated by his prediction that there should be periodic fine structure on the mass scaling relations such that

\[ M_{\text{BH}} \propto |\eta - \eta_*|^{\beta} e^{\Psi(\ln |\eta - \eta_*|)} , \]  

(7)

where \( \Psi \) has period \( \Delta/(2\beta) \approx 4.61 \ln |\eta - \eta_*| \). Similar arguments were presented independently by Hod and Piran who confirmed the existence of this effect numerically.

In collaboration with Martin-Garcia, Gundlach considered charged scalar field collapse and again correctly predicted the scaling relation that has been observed for the final black hole charge. Recently he has extended his analysis to include small deviations from spherical symmetry, and has derived a scaling relation for the angular momentum parameter of spinning black holes near the critical point.

The remainder of this paper is organized as follows. In subsection 2.1, we present a simple argument for the periodic fine structure discovered by Gundlach, and independently by Hod and Piran. Interesting new phenomenology is observed when scale invariance of the underlying mathematical equations is broken. This was first demonstrated in the work of Choptuik et al. by considering the gravitational collapse of a Yang-Mills field where the SU(2) charge introduces a fundamental scale into the problem. In subsection 2.2 we discuss similar results for massive scalar field collapse; in particular, we focus on a simple criterion to determine when the mass term is important in critical evolutions. We then outline the results which have been obtained by Choptuik et al. who have shown that the approximate black hole solution discovered by Van Putten is unstable. This result also has implications for massless scalar field collapse since van Putten’s solution may lie at the threshold of black hole formation. Some preliminary results from a study of fluid collapse with \( p = k\rho \) are presented in subsection 2.4; in particular, the critical exponents are tabulated for several values of \( k \) between zero and unity, and graphical evidence for self-similarity in near critical collapse with \( k = 0.95 \) is presented in Fig. 3. Concluding the section on classical results, we discuss the implications of critical point behavior for cosmic censorship in subsection 2.5. To fully understand the physical significance of critical point behavior in gravitational collapse requires the inclusion of quantum effects into the picture. Unfortunately, we do not have a complete theory of quantum gravity, so the best we can do is address some model problems. Several interesting results have been obtained in this context. Peleg et al. have investigated a 2-dimensional dilaton theory of gravity in which they demonstrate the existence of critical behavior and scaling at the classical level. When one loop quantum effects are included into the model, a mass gap is observed at the threshold of black hole formation. These results are reinforced in similar work by Ayal and Piran who have considered a semi-classical model for spherically symmetric scalar field collapse which is outlined in subsection 3.1. General arguments can be brought to bear on the four-dimensional problem when the critical solution is continuously self-similar. We present a brief summary of a recent analysis by Brady and Ottewill which suggests the existence of a quantum mass-gap at the threshold of black hole formation for sufficiently stiff
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perfect fluids. Finally, we conclude with a brief discussion of future directions.

2 Classical results

2.1 Massless scalar fields

Since the initial discovery of critical phenomena, spherically symmetric scalar field collapse has been one of the most studied systems. In contrast to perfect fluid collapse the critical solution is not self-similar in the familiar sense, rather it has an infinite train of self-similar echoes which accumulate at the singularity. In precise terms, scale invariant quantities, such as \( f \) in Eq. (3), satisfy

\[
f(\xi, \ln |t|) = f(\xi, \ln |t| - n\Delta/2)
\]

where \( n \) is an integer and \( \Delta \approx 3.4 \) is the period computed by Choptuik for the scalar field. In words, scale invariant quantities are periodic in \( \ln |t| \) on surfaces of constant \( \xi \). The discrete self-similarity of the critical solution has implications for the mass scaling law in Eq. (1) — there is periodic fine structure superimposed on the power-law. This was first noticed by Gundlach \[13\] and subsequently verified numerically by Hod and Piran \[14\].

A simple argument demonstrates this result. The mass scaling law arises because a single unstable mode governs the way that marginally supercritical solutions run away from criticality. This mode can be written as

\[
\eta - \eta_\ast t^{-\lambda} f_1(\xi, \ln |t|)
\]

where \( f_1(\xi, \ln |t|) \) inherits the symmetry of the background, namely that it is periodic in \( \ln |t| \). Let \( t_h \) satisfy

\[
f_\ast - (\eta - \eta_\ast) t_h^{-\lambda} f_1 = 0
\]

for some fixed \( \xi = \xi_h \). Substitute \( t_h = r_h/\xi_h \) into Eq (10) and solve for \( r_h \) to get

\[
\ln r_h = \frac{1}{\lambda} \ln |\eta - \eta_\ast| + \Psi(\xi_\ast, \ln r_h),
\]

where \( \Psi(\xi_h, \ln r_h) \) is periodic in \( \ln r_h \) with period \( \Delta/2 \). If \( \Psi \equiv 0 \) we have a precise power-law relation, however \( \Psi \) is generally non-zero so an approximate solution for \( r_h \) is

\[
\ln r_h = \frac{1}{\lambda} \ln |\eta - \eta_\ast| + \Psi(\xi_\ast, \lambda^{-1} \ln |\eta - \eta_\ast|).
\]

Since \( \Psi(\xi_h, \ln r_h) \) is periodic in \( \ln r_h \), Eq. (12) implies that \( r_h \), and equivalently the black hole mass, has periodic fine structure with period \( \lambda\Delta/2 \) in \( \ln |\eta - \eta_\ast| \) superimposed on the familiar power-law.

A natural extension of Choptuik’s results which was considered by Gundlach and Martin-Garcia \[14\] and also by Hod and Piran \[14\] is scalar electro-dynamics coupled to gravity. The different scaling of the mass and charge in the self-similar echoing regime suggests that the significance and influence of the charge decreases during a near-critical evolution. As \( \eta - \eta_\ast \to 0 \) the black-hole charge tends to zero
faster than its mass, i.e., $Q_{BH}/M_{BH} \to 0$ as $\eta \to \eta_*$ where $Q_{BH}$ is the black hole charge. Gundlach and Martin-Garcia computed the critical solution and predicted that the charge should scale as

$$Q_{BH} \propto |\eta - \eta_*|^{0.883\pm0.007}.$$  

(13)

This scaling relation was confirmed by Hod and Piran in their numerical simulations.

2.2 Type I phase transitions

By analogy with statistical mechanics, phase transitions in which black-hole formation turns on at infinitesimal mass have been termed Type II transitions. Choptuik, Chmaj and Bizon observed that black hole formation sometimes turn on at a finite mass in the collapse of Yang-Mills field coupled to gravity. They called this a Type I transition since the order parameter, the black hole mass, is not continuous at the critical point. The critical solution in this sector of the theory is the Bartnik-McKinnon soliton, an unstable static solution of the Einstein-Yang-Mills (EYM) equations. The mass gap at the threshold of black hole formation is approximately equal to the mass of the Bartnik-McKinnon solution. A fundamental difference between the EYM system and the massless scalar field is the presence of a length scale, the Yang-Mills charge.

It has been argued that Type II transitions in scalar field collapse should be stable to the introduction of a mass. Gundlach has also presented a mathematical argument to this effect in his recent review. Nevertheless, it was demonstrated by Brady, Chambers and Gonçalves that Type I transitions can occur in massive scalar field collapse. In contrast to EYM collapse the mass gap at the threshold of black hole formation is not universal; marginally supercritical black holes have masses ranging from $\sim 0.3\mu^{-1}$ to $\sim 0.6\mu^{-1}$, where $\mu$ is the scalar field mass. A plausible explanation for this is provided by studying the critical solutions in Type I transitions; they are oscillating soliton stars. Seidel and Suen have discussed these solutions in some detail. In particular, they demonstrated that there is a family of solutions parameterized by the mass and the effective radius of the star. A schematic representation of the mass-radius curve is shown in Fig. note that the maximum mass of such a star is $\sim 0.6\mu^{-1}$, and solutions to the left of the maximum are unstable. The numerical evidence suggests that all of the unstable solutions can act as critical point solutions, perhaps explaining the range of values for the mass gap.

Why do some initial configurations lead to Type I transitions while others lead to Type II? A complete answer to this question is unavailable at this time, however there is a simple criterion which provides some guidance. Let $\lambda$ denote the radial extent of the initial shell of scalar field, i.e., its thickness. Type I transitions occur when the radial extent $\lambda$ of the initial pulse is bigger than the Compton wavelength of the scalar field, that is

$$\lambda\mu \gg 1.$$  

(14)

This observation, and the evidence in support of it that is presented in Chambers’ article validates local arguments about the relevance of the mass term in Type II transitions.
Unstable branch

0.6

Mass

Effective Radius

Figure 1: A schematic representation of the mass-radius curve for oscillating soliton stars. (See Seidel and Suen for a quantitatively correct version.) The maximum mass that one of these configurations can have is $\sim 0.6\mu^{-1}$. Solutions on the unstable branch of the mass radius curve may act as critical solutions.

It is likely that unstable, confined solutions play a role in critical point behavior of other matter models. Indeed, this is expected in the astrophysical context where stars which exceed the Chandrasekhar limit must either shed some material or collapse to form a black hole. The possible end-points of gravitational collapse in this context are: (i) black hole formation, (ii) complete disruption of the star, or (iii) a dead star with a mass less than $\sim 1.4M_\odot$ (where $M_\odot$ is the mass of the sun). In the most general context, phase transitions between any pairwise combination of these types can be expected, although all of the possibilities need not occur. Recent work by Chmaj and Bizon lends support to this viewpoint. They have shown that stable skyrmion solutions are a possible endpoint of gravitational collapse in Type I transitions, i.e., phase transitions are observed between stable stars and black holes.

### 2.3 Brans-Dicke Theory

Critical point behavior in Brans-Dicke theory has been studied by Liebling and Choptuik. The critical solution exhibits continuous self-similarity or discrete self-similarity depending on the strength of the coupling of the Brans-Dicke field to the matter (taken to be massless scalar field by Liebling and Choptuik). In a broader context, van Putten has presented a one-parameter family of static solutions to vacuum Brans-Dicke theory which he suggests might be thought of as approximate black holes. These solutions can be arbitrarily close to the exterior Schwarzschild spacetime, but are globally regular. An important application of solutions like this could be to provide approximate boundary conditions at the surface of a black hole.
Choptuik, Hirschmann and Liebling have argued that, in addition to the static solution behaving like a black hole, the solution should also be stable. Unfortunately, a linear stability analysis indicates that van Putten’s solutions have an unstable mode. Using the code developed previously, Choptuik et al. also studied the non-linear stability of these solutions. They evolved initial data corresponding to the static solution and small additive perturbations. Generically, the solution either collapses to a black hole, or disperses depending on the sign of the perturbation. An interesting point was raised by van Putten during the session on critical phenomena at MG8. He noticed, in animations of the perturbed solutions, that the instability is evident only after the perturbation reaches the center of symmetry and reflects back out toward larger radii. One may therefore wonder if the solutions might be stable when absorbing boundary conditions would be applied at the origin; if so, the solutions would become interesting as approximate black holes once again.

Choptuik et al. have shown that van Putten’s solutions have a single unstable mode, and they either disperse or collapse when perturbed. This suggests that the solutions may be black-hole threshold critical solutions. It is intriguing to consider this possibility, since it would imply that there is a basin of attraction in the space of initial data, for massless scalar field collapse, in which van Putten’s solutions represent the critical solution. Once again this begs the question: what makes one solution a critical attractor, and another not? Clearly, there is a gap in our understanding of critical phenomena.

2.4 Perfect fluid collapse

Despite the early study of critical phenomena in the collapse of radiation fluid by Evans and Coleman, no results have been available on the evolution of perfect fluids with the general equation of state $p = k\rho$, where $0 < k \leq 1$ is a constant, $\rho$ is the energy density of the fluid, and $p$ is the pressure. Solutions expected to be at the threshold of black hole formation have been computed by Maison and Koike et al. when $0 < k < \sim 0.899$. In each case a single unstable mode has been found, and a scaling exponent for the black hole mass in slightly super-critical collapse has been predicted.

Brady and Cai have developed spherically symmetric code to study this problem. The numerical scheme uses polar slicing and a flux conservative version of the fluid equations of motion which are differenced using a two-step Lax-Wendroff scheme. The differencing scheme is second order accurate in both space and time; this has been verified by performing a sequence of evolutions at various levels of discretization. The code was used to confirm the results obtained by Evans and Coleman when $k = 1/3$; at the threshold of black hole formation self-similarity is observed over two orders of magnitude in scale, and the critical exponent is computed to be $\beta \approx 0.352$.

For equations of state with $0 < k \lesssim 0.899$, the critical exponents which have been computed from sequences of numerical evolutions agree well with those obtained using perturbation methods. Several values are tabulated in Table 1. In near critical evolutions, continuous self-similarity is observed over approximately two orders of magnitude in scale before the matter either disperses or collapses into
Table 1: The scaling exponent for black hole mass for several values of $k$. The values predicted using perturbation theory are in the second column labelled $\beta_{\text{predicted}}$. The numerically observed values and estimates of their errors are also presented.

| $k$   | $\beta_{\text{predicted}}$ | $\beta_{\text{observed}} \pm \epsilon$ | $k$   | $\beta_{\text{predicted}}$ | $\beta_{\text{observed}} \pm \epsilon$ |
|-------|--------------------------|---------------------------------|-------|--------------------------|---------------------------------|
| 0.1   | 0.1875                   | 0.196 ± 0.009                   | 0.6   | 0.5556                   | 0.565 ± 0.011                   |
| 0.2   | 0.2614                   | 0.265 ± 0.009                   | 0.7   | 0.6392                   | 0.653 ± 0.005                   |
| 0.3   | 0.3322                   | 0.337 ± 0.002                   | 0.8   | 0.7294                   | 0.740 ± 0.003                   |
| 0.4   | 0.4035                   | 0.408 ± 0.005                   | 0.9   | —                        | 0.814 ± 0.001                   |
| 0.5   | 0.4774                   | 0.486 ± 0.004                   | 0.95  | —                        | 0.850 ± 0.005                   |

Near criticality, the solution develops a highly evacuated region outside the
collapsing matter, however there is evidence for self-similarity over two orders of magnitude in scale in slightly sub-critical solutions. Several profiles of $f$ [see Eq. (3)] are shown in Fig. 3. Some of the profiles are overlayed when scaled according to the self-similar ansatz; continuous self-similarity is apparent. Since stiff fluid solutions can be recast as scalar field solutions, one might expect that DSS should turn on as $k \rightarrow 1$. No evidence to this effect has been found in the numerical simulations. The reason appears to be connected with a lack of analyticity at the outer boundary of the collapsing material in perfect fluid collapse.

2.5 Cosmic censorship and critical collapse

The scaling law for black hole mass in Type II phase transitions suggests that black holes of arbitrarily small mass can be formed in gravitational collapse. At the critical point, the central region of collapsing material is self-similar; an infinite train of scale invariant echoes accumulates within finite proper time in the echoing solutions. These two observations suggest that a singularity must form at the center of critical solutions. Is it naked? Numerical simulations by Hamade and Stewart provide direct evidence that it is. Using an evolution scheme based on double null coordinates, they find a regular Cauchy horizon with almost vanishing flux of scalar field across it. Gundlach has analytically continued the critical solution, constructed by his pseudo-spectral method, to the Cauchy horizon which is regular. Furthermore, self-similar, perfect fluid solutions are known to have naked singularities at the origin. Thus, suitably chosen, regular initial data can form a naked singularity in gravitational collapse; however, the data are not generic in the sense that they
belong to a sub-space of codimension one in the space of all initial data.

Weak cosmic censorship states that, in the evolution of generic initial conditions, singularities are always hidden behind event horizons. Since critical solutions violate the generic condition, they are not counter-examples to cosmic censorship. Two further assumptions are implicit in the above statement: First, gravitational collapse is governed by the classical Einstein equations. Second, matter is described by fundamental fields on spacetime. Thus the formation of naked singularities in the collapse of dust is not usually considered a violation of cosmic censorship since dust only provides an effective description of matter, and it can become singular in flat spacetime. Nevertheless, it is remarkable that naked singularities are so easy to form in such matter. Jhingan et al. have examined dust collapse using the Tolman-Bondi-Lemaître metric, relating the formation of black holes and naked, shell-focusing singularities in such a collapse to the generic form of regular initial data. Such data are characterized by the density and velocity profiles of the matter on some initial time slice. Given a generic initial density profile, they have shown that there exists a corresponding velocity field which gives rise to a strong curvature, naked singularity in the evolution. This establishes that strong naked singularities arise for generic density profiles in the spherically symmetric collapse of dust.

3 Quantum effects in critical spacetimes

It is well known, since the work of Hawking, that a black hole decreases in size by emitting particles via quantum processes. The radiated particles have a thermal spectrum: for a static black hole the temperature is inversely proportional to the mass of the black hole. Thus, the smaller the black hole, the more it radiates. A black hole of mass $M$ radiates all its mass in approximately $10^{-27}(M/1g)^3$ seconds. Black holes formed in marginally super-critical collapse will therefore evaporate almost instantaneously; quantum effects are important in a full description of Type II phase transitions.

In the absence of a complete theory of quantum gravity, semi-classical calculations can provide some information about the underlying quantum evolution. Since renormalization breaks conformal invariance, scale invariant critical solutions surely are modified when curvatures approach Planck scales. Furthermore, the introduction of a fundamental length scale, the Planck length, into the picture suggests that a mass-gap might occur at the threshold of black hole formation in a semi-classical treatment of critical point behavior. These speculations have been confirmed in several model problems.

3.1 Two-dimensional dilaton models

Peleg et al. have studied one-loop quantum effects on the collapse of a massless scalar field in two-dimensional (2D) dilaton gravity. Reflecting boundary conditions are imposed at some finite value of the dilaton $\phi = \phi_c$ in order to avoid the strong coupling regime of the theory which has a timelike singularity where $\exp(2\phi) \to \infty$. The classical solutions exhibit critical point behavior: in supercritical evolutions the black hole mass $M_{bh}$ scales as $M_{bh} \propto |\eta - \eta_c|^2$ near to criticality. In this 2D
model, the critical exponent is $\beta \simeq 0.53$.

There is some freedom in constructing the effective action which describes the semi-classical theory since one is free to add local counter-terms to the Polyakov-Liouville term derived from the trace anomaly in two dimensions. Peleg et al. add a term which makes the theory exactly soluble at the semi-classical level. The quantum coupling constant $\kappa = Nh/12$ depends on the number of fields $N$. As expected, quantum effects are not relevant in the formation of sufficiently large black holes. The classical scaling relation holds over four orders of magnitude when $\kappa = 0.001$, and over two orders of magnitude when $\kappa = 0.01$. The threshold of black hole formation is characterized by a mass-gap in semi-classical evolutions, i.e. $M_{bh}$ approaches a non-zero lower limit as $\eta$ tends to its critical value from above. The mass-gap depends both on $\kappa$, and on the initial data. This is not surprising since the semi-classical action is non-local. Peleg has constructed analytic arguments supporting these results (see his contribution to this volume).

### 3.2 Semi-classical models of spherical collapse

The benefit of 2D models is that the quantum stress energy tensor can be calculated exactly with very little effort. Ayal and Piran have considered the four-dimensional, semi-classical Einstein equations

$$G = 8\pi (T + \langle T \rangle)$$

where $G$ is the Einstein tensor, $T$ is the stress energy of classical matter, and $\langle T \rangle$ is the renormalized stress-energy tensor of quantum matter. Since direct computation of $\langle T \rangle$ is extremely difficult, if not beyond current techniques, they use a model stress-energy tensor constructed from a two-dimensional one computed on the radial two sections of the four dimensional spacetime. Tangential stresses are ignored, and (in a slight abuse of notation) $4\pi r^2 \langle T \rangle = [1 + (\alpha/r^2)^2]^{-1} \langle T^{(2)} \rangle$ where $\sqrt{\alpha}$ is a length scale of order the Planck scale. The prefactor is required so that the stress-energy tensor is not singular at the origin in four dimensions. Even though this term leads to small violations of covariant conservation, it is a useful model of semi-classical effects including Hawking evaporation.

Solving the semi-classical equations numerically for scalar field collapse, Ayal and Piran show that quantum effects reduce the mass of a black hole compared to its classical value. When the length scale $\sqrt{\alpha}$ is increased the black hole no longer forms. As one might expect, the results are qualitatively similar to those of Peleg et al.

### 3.3 Four-dimensional model calculation

An alternative approach exploits the symmetry of spacetimes with continuous self-similarity to compute the general form of the renormalized stress-energy tensor (RSET) for conformally coupled fields. Spherically, symmetric self-similar spacetimes are conformally static, therefore one can apply the transformation rule for

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*By radial two section we mean the 2D spacetime obtained by ignoring the angular dimensions in the spherical line element.*
the RSET, as derived by Page, to show that

$$\langle T \rangle = \hbar t^{-4} \left[ \text{functions of } \xi = r/t \text{ only} \right].$$

(16)

Here it is assumed that $t$ is the coordinate in Eq. (3) normalized so that $t = 0$ at the singularity in the critical solution.

Figure 4: The horizon location is determined by the roots of the function $F(r_h, \xi_h)$ in Eq. (17). We show here a schematic representation for several values of $\eta$ which determine deviations from classically critical initial data, and $\lambda > 2$. For sufficiently large $\eta$ classical collapse takes hold and a black hole forms at $R_c$. The function has a minimum, however, and another root $R_q$ exists. As $\eta$ is tuned to a critical value $\eta_q$ the two roots coincide. When $\eta < \eta_q$ no black hole forms.

The critical solution in perfect fluid collapse is $f = f_*(\xi)$. The classical argument which relates the Lyapunov exponent to the scaling relation for black hole mass can now be repeated including perturbations which originate from the quantum stress-energy tensor in Eq. (16). The semi-classical equations imply that the perturbations to the self-similar solution can be written as

$$F(\xi, t) = f_*(\xi) - (\eta - \eta_*) f_1(\xi) t^{-\lambda} + \hbar f_q(\xi) t^{-2},$$

(17)

where the last term arises from quantum effects. Clearly, when $\lambda < 2$ the quantum perturbations will dominate as $t \to 0$. Thus, we restrict attention to equations of state with $k \gtrsim 0.53$ for which $\lambda < 2$. (See Brady and Ottewill for a discussion of $k < 0.53$.) Now, substitute $t = r/\xi$ into Eq. (17). For fixed $\xi = \xi_h$ the solutions of

$$F(\xi_h, r_h) = f_*(\xi_h) - (\eta - \eta_*) f_1(\xi_h) (r_h/\xi_h)^{-\lambda} + \hbar f_q(\xi_h) (r_h/\xi_h)^{-2} = 0$$

(18)

determine the radii of apparent horizons in spacetime as a function of $\eta$. Assuming that quantum effects compete with classical perturbations to reduce the mass of a
black hole that forms in gravitational collapse, \( F(\xi_h, r_h) \) is plotted schematically in Fig. 4. For \( \eta \) sufficiently greater than \( \eta^* \), gravitational collapse and black hole formation is dominated by the classical perturbations; the apparent horizon is at \( R_c \). Note that the function has a minimum and a second root at \( R_q < R_c \). As \( \eta \) decreases \( R_c \rightarrow R_q \) until the roots coincide at some \( \eta_q \). When \( \eta < \eta_q \) no black hole forms; therefore, we infer that quantum effects induce a mass gap at the threshold of black hole formation in critical collapse of perfect fluids. It is also apparent that the classical scaling law for black-hole mass remains valid for black holes with apparent horizons significantly above the Planck length.

4 Concluding remarks

Critical point behavior in gravitational collapse is an exciting area of research in general relativity, one which continues to bring new insights into gravitational collapse and black hole formation. Significant progress has been made to understand the phenomenology observed in near critical evolutions. Nevertheless, important open questions remain. For example, is cosmic censorship upheld in all families of interpolating solutions for realistic matter fields? Or, when both Type I and Type II phase transitions occur, which properties of the initial data determine the critical point behavior? Numerical studies of gravitational collapse can provide a great deal of insight into the answers to these and other questions, however real progress will be made at the interface between numerical and mathematical approaches.

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