GLOBAL EXISTENCE FOR A TWO-SPECIES CHEMOTAXIS-NAVIER-STOKES SYSTEM WITH $p$-LAPLACIAN

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Abstract. We consider a two-species chemotaxis-Navier-Stokes system with $p$-Laplacian in three-dimensional smooth bounded domains. It is proved that for any $p \geq 2$, the problem admits a global weak solution.

1. Introduction. In this paper, we are concerned with the following two-species chemotaxis-Navier-Stokes system:

\[
\begin{aligned}
(n_1)_t + u \cdot \nabla n_1 &= \nabla \cdot (|\nabla n_1|^{p-2} \nabla n_1) - \chi_1 \nabla \cdot (n_1 \nabla c) \\
&\quad + \mu_1 n_1 (1 - n_1 - a_1 n_2), \\
(n_2)_t + u \cdot \nabla n_2 &= \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) + \mu_2 n_2 (1 - a_2 n_1 - n_2), \\
c_t + u \cdot \nabla c &= \Delta c - (\alpha n_1 + \beta n_2) c, \\
u_t + (u \cdot \nabla)u &= \Delta u + \nabla P + (n_1 + n_2) \nabla \Phi, \\
\nabla \cdot u &= 0,
\end{aligned}
\]

in $Q = \Omega \times [0, \infty)$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $\mu_1, \mu_2, \alpha, \beta, \chi_1, \chi_2$, are positive constants, and $a_1, a_2 \geq 0$ are constants. This system describes the evolution of two kinds of aerobic bacteria, that compete according to Iotka-Volterra competitive kinetics in a liquid surrounding environment. Here $n_1$ and $n_2$ represent the population densities of two species respectively, $c$ stands for the concentration of oxygen, $u$ shows the fluid velocity field, and $P$ represents the pressure of the fluid. The given function $\Phi$ represents the gravitational potential.

The problem (1.1) is a generalized system to the chemotaxis-fluid system, which is proposed by Tuval et al. in [21]. The chemotaxis-Navier-Stokes system models have been widely studied by many researchers ([22, 23, 25]).

What’s more, the investigation of the problems involving chemotaxis-Navier-Stokes system models with p-Laplacian has been addressed by several authors. Tao
Liu [10] investigated the following problem:

\[ p > \frac{27}{15}, \]

bounded weak solutions of the chemotaxis-Stokes system exist whenever solution in three dimensional spaces. Furthermore, Tao and Li [20] proved that global and Li [19] discussed the following chemotaxis-Navier-Stokes system:

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totic stability of solution. Cao, Kurima and Mizukami [3] considered the following studied in [11, 12].

They got that if \( p > \frac{27}{15} \), under appropriate assumptions on \( f \) and \( \chi \), for all sufficiently smooth initial data \((n_0, c_0, u_0)\), the system owns at least one global weak solution in three dimensional spaces. Furthermore, Tao and Li [20] proved that global bounded weak solutions of the chemotaxis-Stokes system exist whenever \( p > \frac{27}{11} \). Liu [10] investigated the following problem:

\[
\begin{aligned}
\rho_t + u \cdot \nabla \rho &= \nabla \cdot (|\nabla \rho|^{p-2} \nabla \rho) - \nabla \cdot (\rho \nabla c) - \rho m, \\
c_t - u \cdot \nabla c &= \Delta c - c + m, \\
m_t + u \cdot \nabla m &= \Delta m - \rho m, \\
u_t + \kappa (u \cdot \nabla) u &= \Delta u + \nabla P + (\rho + m) \nabla \phi, \\
\nabla \cdot u &= 0,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary. It is proved that if either \( p > 2 \) for \( \kappa \in \mathbb{R} \), \( N = 2 \) or \( p > \frac{27}{15} \) for \( \kappa = 0 \), \( N = 3 \) is satisfied, then for each properly chosen initial data and associated initial-boundary problem admits a global weak solution which is bounded. The relevant equations have also been studied in [11, 12].

On the other hand, two-species competitive chemotaxis systems have been studied by many authors [1, 15] recently, mainly about the global existence and asymptotic stability of solution. Cao, Kurima and Mizukami [3] considered the following two-species chemotaxis-Stokes system:

\[
\begin{aligned}
(n_1)_t + u \cdot \nabla n_1 &= \Delta n_1 - \chi_1 \nabla \cdot (n_1 \nabla c) \\
&+ \mu_1 n_1 (1 - n_1 - a_1 n_2), \\
(n_2)_t + u \cdot \nabla n_2 &= \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) \\
&+ \mu_2 n_2 (1 - a_2 n_1 - n_2), \\
c_t + u \cdot \nabla c &= \Delta c - c(\alpha n_1 + \beta n_2), \\
u_t + \kappa (u \cdot \nabla) u &= \Delta u + \nabla P + (\gamma n_1 + \delta n_2) \nabla \phi, \\
\nabla \cdot u &= 0,
\end{aligned}
\]

They proved the global existence, boundedness and stabilization of solutions to the above system in the 3-dimensional case. Hirata et al [6] gave more complete stabilization of solutions for (1.2) in the 2-dimensional case. Moreover Liu and Li [13] proved that the system (1.2) admits a time periodic solution under some conditions. The relevant equations have also been studied in [8].

As mentioned above, two-species chemotaxis-Stokes system and one species chemotaxis-Stokes system with p-Laplacian were studied by many authors. However the combination of these two kinds of problems has not been studied. Thus, we are inspired to investigate the case that the two species have different diffusion law, namely one according to the p-Laplacian diffusion and the other according to standard Laplacian diffusion. From a physical point of view, in the same liquid surrounding environment, one species is influenced by ions and molecules and thus
its mobility is described by a nonlinear function of the cells, but the other species is not affected by ions or molecules thus diffuse by linear Laplacian diffusion.

Obviously, Cao, Kurima and Mizukami solved the problem (1.1) when \( p = 2 \). If \( p > 2 \), from a mathematical point of view, \( p \)-Laplacian diffusion term \( \nabla \cdot (|\nabla n_1|^{p-2}\nabla n_1) \) lead to a lot of difficulties in our proof of main result because of its nonlinear character. To be specific, a main difficulty arises in the estimate of \( n_1 \) in contrast to the case of Laplacian, and the property of the Neumann heat semigroup becomes useless and so on. In order to overcome the difficulties bring by the \( p \)-Laplacian diffusion term, we consider a regularized problems of (1.1) and establish an energy-type inequality in Section 3 as a starting point to discuss the problem.

In this paper, we shall consider (1.1) along with the boundary conditions

\[
|\nabla n_1|^{p-2}\frac{\partial n_1}{\partial n} - \frac{\partial n_2}{\partial n} = \frac{\partial c}{\partial n} = 0 \quad \text{and} \quad u = 0, \quad \text{on} \ Q_\Gamma, \tag{1.3}
\]

where \( Q_\Gamma = \partial \Omega \times [0, \infty) \), and the initial conditions

\[
n_i(x, 0) = n_{i, 0}(x), \quad c(x, 0) = c_0(x), \quad u(x, 0) = u_0(x), \quad x \in \Omega, \quad i = 1, 2. \tag{1.4}
\]

Assume that \( n_{1, 0}, n_{2, 0}, c_0, u_0 \) and \( \Phi \) are given functions satisfying

\[
\Phi \in W^{1, \infty}(\Omega), \tag{1.5}
\]

and

\[
\begin{cases}
n_{1, 0} \in L^2(\Omega), \quad \text{and} \quad n_{1, 0} > 0, \\
n_{2, 0} \in L \log L(\Omega), \quad \text{and} \quad n_{2, 0} > 0, \\
c_0 \in L^\infty(\Omega), \quad c_0 > 0, \quad \text{and} \quad \sqrt{c_0} \in W^{1,2}(\Omega), \\
u_0 \in L^2(\Omega),
\end{cases} \tag{1.6}
\]

where \( L \log L(\Omega) \) is the standard Orlicz space associated with the Young function \((0, \infty) \ni z \mapsto z \ln(1 + z)\) and \( L^2(\Omega) := \{ \varphi \in (L^2(\Omega))^3 | \nabla \cdot \varphi = 0 \} \).

Let us first give the definition of weak solution.

**Definition 1.1.** We call \((n_1, n_2, c, u)\) a global weak solution of (1.1), (1.3) and (1.4) if

\[
n_1 \in L^1_{\text{loc}}([0, \infty); L^1(\Omega)), \quad n_2 \in L^1_{\text{loc}}([0, \infty); W^{1,1}(\Omega)),
\]

\[
c \in L^1_{\text{loc}}([0, \infty); W^{1,1}(\Omega)), \quad u \in (L^1_{\text{loc}}([0, \infty); W^{1,1}_0(\Omega)))^3,
\]

such that \( n_1 \geq 0, n_2 \geq 0 \) and \( c \geq 0 \) a.e in \( Q \) and that

\[
\mu_1 n_1(1 - n_1 - a_1 n_2), \quad \mu_2 n_2(1 - a_2 n_1 - n_2), \quad (\alpha n_1 + \beta n_2)c \in L^1_{\text{loc}}([0, \infty); L^1(\Omega)),
\]

\[
|\nabla n_1|^{p-2}\nabla n_1, \quad n_1 \nabla c, \quad n_2 \nabla c, \quad n_1 u, \quad n_2 u, \quad cu \in (L^1_{\text{loc}}([0, \infty); L^1(\Omega)))^3,
\]

\[
u_0 \in (L^1_{\text{loc}}([0, \infty); L^1(\Omega)))^3 \times 3,
\]

and that
\[
\int_0^\infty \int_\Omega (n_1) \phi_1 - \int_0^\infty \int_\Omega n_1 u \cdot \nabla \phi_1 = - \int_0^\infty \int_\Omega |\nabla n_1|^{p-2} \nabla n_1 \cdot \nabla \phi_1 \\
+ \int_0^\infty \int_\Omega \chi_1 n_1 \nabla c \cdot \nabla \phi_1 + \int_0^\infty \int_\Omega \mu_1 n_1 (1 - n_1 - a_1 n_2) \phi_1,
\]
\[
\int_0^\infty \int_\Omega (n_2) \phi_2 - \int_0^\infty \int_\Omega n_2 u \cdot \nabla \phi_2 = - \int_0^\infty \int_\Omega \nabla n_2 \cdot \nabla \phi_2 \\
+ \int_0^\infty \int_\Omega \chi_2 n_2 \nabla c \cdot \nabla \phi_2 + \int_0^\infty \int_\Omega \mu_2 n_2 (1 - a_2 n_1 - n_2) \phi_2,
\]
\[
\int_0^\infty \int_\Omega c \phi_3 - \int_0^\infty \int_\Omega cu \cdot \nabla \phi_3 = - \int_0^\infty \int_\Omega \nabla c \cdot \nabla \phi_3 - \int_0^\infty \int_\Omega (\alpha n_1 + \beta n_2) c \phi_3,
\]
as well as
\[
\int_0^\infty \int_\Omega u_4 \phi_4 - \int_0^\infty \int_\Omega u \nabla \phi_4 = - \int_0^\infty \int_\Omega \nabla u \cdot \nabla \phi_4 + \int_0^\infty \int_\Omega (n_1 + n_2) \nabla \Phi \cdot \phi_4,
\]
hold for all \( \phi_1, \phi_2, \phi_3 \in C_c^\infty(\bar{\Omega} \times [0, \infty)) \) and \( \phi_4 \in (C_c^\infty(\bar{\Omega} \times [0, \infty]))^3 \) satisfying \( \nabla \phi_4 = 0 \).

The plan of this paper is as follows. In Section 2, we list some lemmas, which will be used throughout this paper. In Section 3, we consider a family of regularized problems and show the global existence of the regularized problems, by establishing an energy-type inequality and using the Moser-Alikakos iteration procedure. Finally, in Section 4, we show that the problem (1.1), (1.3) and (1.4) admits a global-in-time weak solution.

2. Preliminaries. In this section, we recall some lemmas, which will be used throughout the paper. Before going further, we first list the Gagliardo-Nirenberg interpolation inequality [16] for the convenience of application.

**Lemma 2.1.** For functions \( u : \Omega \to \mathbb{R} \) defined on a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^N \), we have
\[
\|D^j u\|_{L^p} \leq C\|D^m u\|_{L^q}^b \|u\|_{L^r}^{1-b} + C\|u\|_{L^r},
\]
where \( 1 \leq q, r \leq \infty, \frac{j}{m} \leq b \leq 1, \)
\[
\frac{1}{p} = \frac{j}{N} + \left(\frac{1}{r} - \frac{m}{N}\right)b + \frac{1-b}{q},
\]
and \( s > 0 \) is arbitrary.

Next, we list the following Lemma 2.2 [18].

**Lemma 2.2.** Let \( T > 0, \tau \in (0, T), a > 0, b > 0 \), and suppose that \( y : [0, T) \to [0, \infty) \) is absolutely continuous such that
\[
y'(t) + ay(t) \leq h(t), \quad \text{for } t \in [0, T),
\]
where \( h \geq 0, h(t) \in L^1_{loc}([0, T)) \) and
\[
\int_{t-\tau}^t h(s) ds \leq b, \quad \text{for all } t \in [\tau, T).
\]
Then
\[
y(t) \leq \max\{y(0) + b, \frac{b}{a\tau} + 2b\} \quad \text{for all } t \in [0, T).
\]
Finally, we also give a generalized lemma of Lemma 2.2 [7].

Lemma 2.3. Let \( T > 0, \tau \in (0,T), \sigma \geq 0, a > 0, b \geq 0, \) and suppose that \( f : [0,T] \rightarrow [0, \infty) \) is absolutely continuous and satisfies

\[
  f'(t) + af^{1+\sigma}(t) \leq h(t), \quad t \in \mathbb{R},
\]

where \( h \geq 0, h(t) \in L^1_{loc}([0,T]) \) and

\[
  \int_{t-\tau}^{t} h(s)ds \leq b, \quad \text{for all } t \in [\tau, T).
\]

Then

\[
  \sup_{t \in (0,T)} f(t) + a \sup_{t \in (\tau,T)} \int_{t-\tau}^{t} f^{1+\sigma}(s)ds \leq b + 2 \max\{f(0) + b + \alpha t, \frac{b}{\alpha t} + 1 + 2b + 2\alpha\}.
\]

3. Regularized problem. Inspired by the idea from [22], in order to construct a global weak solution of (1.1), (1.3) and (1.4), we first consider the following appropriately regularized problem:

\[
\begin{aligned}
\partial_t n_{1\varepsilon} + u_{\varepsilon} \cdot \nabla n_{1\varepsilon} &= \nabla \cdot \left( (|\nabla n_{1\varepsilon}|^2 + \varepsilon) \frac{\nabla^2}{2} \nabla n_{1\varepsilon} \right) \\
- \chi_1 \nabla \cdot (n_{1\varepsilon} F_1'(n_{1\varepsilon}) \nabla c_{\varepsilon}) &+ \mu_1 n_{1\varepsilon} (1 - n_{1\varepsilon} - a_1 n_{2\varepsilon}), (x,t) \in Q, \\
\partial_t n_{2\varepsilon} + u_{\varepsilon} \cdot \nabla n_{2\varepsilon} &= \Delta n_{2\varepsilon} - \chi_2 \nabla \cdot (n_{2\varepsilon} F_2'(n_{2\varepsilon}) \nabla c_{\varepsilon}) \\
+ \mu_2 n_{2\varepsilon} (1 - a_2 n_{1\varepsilon} - n_{2\varepsilon}), \quad (x,t) \in Q, \\
\partial_t c_{\varepsilon} + u_{\varepsilon} \cdot \nabla c_{\varepsilon} &= \Delta c_{\varepsilon} - (\alpha F_1(n_{1\varepsilon}) + \beta F_2(n_{2\varepsilon})) c_{\varepsilon}, \quad (x,t) \in Q, \\
\partial_t u_{\varepsilon} + (Y \cdot u_{\varepsilon} \cdot \nabla) u_{\varepsilon} &= \Delta u_{\varepsilon} + \nabla p_{\varepsilon} + (n_{1\varepsilon} + n_{2\varepsilon}) \nabla \Phi, \quad (x,t) \in Q, \\
\nabla \cdot u_{\varepsilon} &= 0, \quad (x,t) \in Q, \\
\frac{\partial n_{1\varepsilon}}{\partial \nu} = \frac{\partial n_{2\varepsilon}}{\partial \nu} = \frac{\partial c_{\varepsilon}}{\partial \nu} = 0, & \quad \text{on } \partial Q, \\
(\alpha, \beta, \sigma, i, j) = 1, 2, & \quad \text{for all } \varepsilon \in (0,1).
\end{aligned}
\]

for \( \varepsilon \in (0,1) \). We take \( F_\varepsilon(s) := \frac{1}{2} \ln(1 + \varepsilon s), \) for \( s \geq 0 \) and utilize the standard Yosida approximation \( Y_\varepsilon \) [14, 17], which was defined by

\[
Y_\varepsilon v := (1 + \varepsilon A)^{-1} v \quad \text{for all } v \in L^2_0(\Omega).
\]

By \( \mathbf{P} \) we mean the Helmholtz projection in \( L^2(\Omega) \), then we represent \( A \) as the realization of Stokes operator \( -\Delta u \) in \( L^2_0(\Omega) \), with domain \( D(A) = W^{2,2}(\Omega) \cap W^{1,2}_{0,\sigma}(\Omega) \), where \( W^{1,2}(\Omega) = W^{1,2}_{0,\sigma}(\Omega) \cap L^2(\Omega) = C_0^{\infty}(\Omega)^{||1||_{W^{1,2}(\Omega)}} \), with \( C_0^{\infty}(\Omega) = C_0^{\infty}(\Omega) \cap L^2(\Omega) \). Thereafter, it is obvious to see that our choice of \( F_\varepsilon \) ensures that

\[
0 \leq F_\varepsilon(s) \leq s, \quad \text{for all } s \geq 0,
\]

\[
0 \leq F_\varepsilon(s) = \frac{1}{1 + \varepsilon s} \leq 1 \quad \text{and} \quad 0 \leq \varepsilon F_\varepsilon'(s) = \frac{s}{1 + \varepsilon s} \leq \frac{1}{\varepsilon}, \quad \text{for all } s \geq 0,
\]

and

\[
F_\varepsilon(s) \nearrow s \quad \text{and} \quad F_\varepsilon'(s) \nearrow 1, \quad \text{as } \varepsilon \searrow 0, \quad \text{for all } s \geq 0.
\]
The families of approximate initial date \( n_{01\varepsilon}(x) > 0, n_{02\varepsilon}(x) > 0, c_{0\varepsilon}(x) > 0 \) and \( u_{0\varepsilon} \) satisfy that
\[
\begin{align*}
\{ & n_{01\varepsilon} \in C_0^\infty(\Omega), \int_\Omega n_{01\varepsilon} = \int_\Omega n_{1,0} \text{ for all } \varepsilon \in (0, 1), \\
& n_{01\varepsilon} \to n_{1,0} \text{ in } L^2(\Omega) \text{ as } \varepsilon \searrow 0, \\
& n_{02\varepsilon} \in C_0^\infty(\Omega), \int_\Omega n_{02\varepsilon} = \int_\Omega n_{2,0} \text{ for all } \varepsilon \in (0, 1), \\
& n_{02\varepsilon} \to n_{2,0} \text{ in } L^2(\Omega) \text{ as } \varepsilon \searrow 0,
\end{align*}
\] (3.5)

\[
\begin{align*}
\{ & \sqrt{c_{0\varepsilon}} \in C_0^\infty(\Omega), \|c_{0\varepsilon}\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} \text{ for all } \varepsilon \in (0, 1), \\
& \sqrt{c_{0\varepsilon}} \to \sqrt{c_0} \text{ a.e. in } \Omega \text{ and } W^{1,2}(\Omega) \text{ as } \varepsilon \searrow 0,
\end{align*}
\] (3.6)

and
\[
u_{0\varepsilon} \in C_{0,\varepsilon}^\infty(\Omega), \text{ with } \|\nu_{0\varepsilon}\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)} \text{ for all } \varepsilon \in (0, 1).
\] (3.7)

Firstly, we give the local smooth solutions existence result of the above approximate problem as follows.

**Lemma 3.1.** Taking \( p \geq 2 \), then for each \( \varepsilon \in (0, 1) \), there exist \( T_{\text{max},\varepsilon} \in (0, \infty) \) and uniquely determined functions
\[
\begin{align*}
n_{1\varepsilon} & \in C^0(\Omega \times [0, T_{\text{max},\varepsilon}) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max},\varepsilon})), \\
n_{2\varepsilon} & \in C^0(\Omega \times [0, T_{\text{max},\varepsilon}) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max},\varepsilon})), \\
c_{\varepsilon} & \in C^0(\Omega \times [0, T_{\text{max},\varepsilon}) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max},\varepsilon})) \text{ and } \\
u_{\varepsilon} & \in C^0(\Omega \times [0, T_{\text{max},\varepsilon}); \mathbb{R}^3) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max},\varepsilon}); \mathbb{R}^3),
\end{align*}
\]

such that \((n_{1\varepsilon}, n_{2\varepsilon}, c_{\varepsilon}, u_{\varepsilon})\) is a classical solution of (3.1) in \( \Omega \times (0, T_{\text{max},\varepsilon}) \) with some \( P_{\varepsilon} \in C^{1,0}(\Omega \times (0, T_{\text{max},\varepsilon})) \). Moreover, we have \( n_{1\varepsilon} > 0, n_{2\varepsilon} > 0, c_{\varepsilon} > 0 \) in \( \Omega \times (0, T_{\text{max},\varepsilon}) \) and if \( T_{\text{max},\varepsilon} < \infty \), then
\[
\|n_{1\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} + \|n_{2\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} + \|c_{\varepsilon}(\cdot, t)\|_{W^{1,2}(\Omega)} + \|A^* u_{\varepsilon}(\cdot, t)\|_{L^q(\Omega)} \to \infty,
\]
as \( t \uparrow T_{\text{max},\varepsilon} \), for all \( q > 3 \) and \( 3 < \frac{3}{p} \).

**Proof.** Combination of arguments Lemma 2.1 in [23] and Lemma 2.1 in [19], which is based on a standard Schauder fixed point argument and a parabolic regularity theory, entails the existence of classical solution. Since \((n_{1\varepsilon}, n_{2\varepsilon}, c_{\varepsilon}, u_{\varepsilon})\) is a smooth solution of (3.1), the nonnegativity of \( n_{1\varepsilon}, n_{2\varepsilon} \) and \( c_{\varepsilon} \) follows from the maximum principle [23, 5, 2].

We are now ready to construct some basic estimates of \((n_{1\varepsilon}, n_{2\varepsilon}, c_{\varepsilon}, u_{\varepsilon})\). In what follows, without special explanation, we take \( \tau = \min\left\{1, \frac{T_{\text{max},\varepsilon}}{2}\right\} \). What’s more, all constants \( C \) denote some different constants from line to line, which are only depend on the given coefficients, initial date and \( \Omega \). All these estimates of this form \( \int_{t-\tau}^t \cdots ds \) in this paper can be replaced by \( \int_{0}^{T_{\text{max},\varepsilon}} \cdots ds \) if \( \tau < 1 \), so these estimates do not dependent on \( \tau \). The following estimates of \( n_{1\varepsilon}, n_{2\varepsilon} \) and \( c_{\varepsilon} \) are obvious but important in the proof of our result.

**Lemma 3.2.** For each \( \varepsilon \in (0, 1) \), the solution of (3.1) satisfies
\[
\sup_{t \in (0, T_{\text{max},\varepsilon})} \|n_{1\varepsilon}(\cdot, t)\|_{L^1(\Omega)} + \sup_{t \in (\tau, T_{\text{max},\varepsilon})} \int_{t-\tau}^t \int_{\Omega} n_{1\varepsilon}^2 dx ds \leq C,
\] (3.9)

\[
\sup_{t \in (0, T_{\text{max},\varepsilon})} \|n_{2\varepsilon}(\cdot, t)\|_{L^1(\Omega)} + \sup_{t \in (\tau, T_{\text{max},\varepsilon})} \int_{t-\tau}^t \int_{\Omega} n_{2\varepsilon}^2 dx ds \leq C,
\] (3.10)
and
\[ \|c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} = s_0 \quad \text{for all} \quad t \in (0, T_{\max, \varepsilon}). \tag{3.11} \]

**Proof.** Integrating the first equation in (3.1) to see

\[ \frac{d}{dt} \int_\Omega n_1 \varepsilon + \mu_1 \int_\Omega (n_1^2 + a_1 n_1 n_2) = \mu_1 \int_\Omega n_1, \]

in both cases \( a_1 = 0 \) and \( a_1 > 0 \), we have

\[ \frac{d}{dt} \int_\Omega n_1 + \mu_1 \int_\Omega n_1^2 \leq \mu_1 \int_\Omega n_1 \leq \frac{\mu_1}{2} \int_\Omega n_1^2 + C(\Omega), \]

with some \( C(\Omega) > 0 \). We derive from the Hölder inequality that

\[ \left( \int_\Omega n_1 \right)^2 \leq \int_\Omega n_1^2, \]

hence we further have

\[ \frac{d}{dt} \int_\Omega n_1 + \frac{\mu_1}{2} \int_\Omega n_1^2 \leq C(\Omega). \]

By Lemma 2.3, we can obtain (3.9). Then completely similar to the proof of (3.10). Finally, an application of the maximum principle to the third equation in (3.1) gives (3.11). \( \square \)

Now we are in the position to derive an energy-type inequality which will be used in the reduction of further estimates. To this end, we first list an inequality which is crucial in the proof of the energy-type inequality. More details of the proof please refer to [4].

**Lemma 3.3.** Suppose that \( h \in C^2(\mathbb{R}) \). Then for all \( \varphi \in C^2(\bar{\Omega}) \) fulfilling \( \partial \varphi / \partial \nu = 0 \) on \( \partial \Omega \) we have

\[ \int_\Omega h'(\varphi) |\nabla \varphi|^2 \Delta \varphi = -\frac{2}{3} \int_\Omega h(\varphi) |\Delta \varphi|^2 + \frac{2}{3} \int_\Omega h(\varphi) |D^2 \varphi|^2 - \frac{1}{3} \int_\Omega h''(\varphi) |\nabla \varphi|^4 - \frac{1}{3} \int_{\partial \Omega} h(\varphi) \frac{\partial}{\partial \nu} |\nabla \varphi|^2. \]

Then we derive the decisive energy-type inequality from the first three equations in (3.1). The main idea of the proof is similar to the strategy introduced in [22] (see also [19]).

**Lemma 3.4.** Assume that \( p \geq 2 \). There exist constants \( C > 0, \ K > 0 \) such that for any \( \varepsilon \in (0, 1) \), the solution of (3.1) satisfies

\[
\frac{d}{dt} \left( \bar{A} \int_\Omega \ln n_1 - n_1 \right) + \bar{B} \int_\Omega \ln n_2 - n_2 + \frac{1}{2} \int_\Omega |\nabla c_\varepsilon|^2 \frac{c_\varepsilon}{c_\varepsilon} + \frac{1}{k} \left( \int_\Omega |\nabla c_\varepsilon|^4 \frac{c_\varepsilon}{c_\varepsilon} + \int_\Omega (|\nabla n_1|^2 + \varepsilon) \frac{\varepsilon}{n_1} + \int_\Omega |\nabla n_2|^2 \frac{n_2}{n_2} \right) \right)
\leq K \int_\Omega |\nabla u|^2 + C, \tag{3.12}
\]

where \( \bar{A} = \frac{a}{x_1}, \ \bar{B} = \frac{b}{x_2} \).
Proof. Testing the first equation of (3.1) by \(\ln n_{1\varepsilon}\) and integrating by parts to compute

\[
\frac{d}{dt} \int_{\Omega} (n_{1\varepsilon} \ln n_{1\varepsilon} - n_{1\varepsilon}) + \int_{\Omega} \left( (|\nabla n_{1\varepsilon}|^2 + \varepsilon)^{p-2} \frac{|\nabla n_{1\varepsilon}|^2}{n_{1\varepsilon}} \right)
\]

\[
= \chi_1 \int_{\Omega} \nabla F_{\varepsilon}(n_{1\varepsilon}) \cdot \nabla c_{\varepsilon} + \mu_1 \int_{\Omega} n_{1\varepsilon}(1 - n_{1\varepsilon} - a_1 n_{2\varepsilon}) \ln n_{1\varepsilon}
\]

\[
= \chi_1 \int_{\Omega} \nabla F_{\varepsilon}(n_{1\varepsilon}) \cdot \nabla c_{\varepsilon} + \mu_1 \int_{\Omega} n_{1\varepsilon}(1 - n_{1\varepsilon}) \ln n_{1\varepsilon} - \mu_1 a_1 \int_{\Omega} n_{1\varepsilon} n_{2\varepsilon} \ln n_{1\varepsilon},
\]

for all \(t \in (0, T_{\text{max},\varepsilon})\). Here we use that \(s \ln s \geq -\frac{1}{e}s\), for all \(s \geq 0\) to see that

\[-\mu_1 a_1 \int_{\Omega} n_{1\varepsilon} n_{2\varepsilon} \ln n_{1\varepsilon} \leq \frac{\mu_1 a_1}{e} \int_{\Omega} n_{2\varepsilon},\]

for all \(t \in (0, T_{\text{max},\varepsilon})\). Moreover, due to the fact that \((1 - n_{1\varepsilon}) \ln n_{1\varepsilon} \leq 0\) for all \((x, t) \in \Omega \times (0, T_{\text{max},\varepsilon})\), we have

\[
\mu_1 \int_{\Omega} n_{1\varepsilon}(1 - n_{1\varepsilon}) \ln n_{1\varepsilon} \leq 0, \quad \text{for all } t \in (0, T_{\text{max},\varepsilon}).
\]

Furthermore, we see that

\[
\frac{d}{dt} \int_{\Omega} (n_{1\varepsilon} \ln n_{1\varepsilon} - n_{1\varepsilon}) + \int_{\Omega} \left( (|\nabla n_{1\varepsilon}|^2 + \varepsilon)^{p-2} \frac{|\nabla n_{1\varepsilon}|^2}{n_{1\varepsilon}} \right)
\]

\[
\leq \chi_1 \int_{\Omega} \nabla F_{\varepsilon}(n_{1\varepsilon}) \cdot \nabla c_{\varepsilon} + \frac{\mu_1 a_1}{e} \int_{\Omega} n_{2\varepsilon}.
\]

(3.13)

Proceeding similarly, we obtain

\[
\frac{d}{dt} \int_{\Omega} (n_{2\varepsilon} \ln n_{2\varepsilon} - n_{2\varepsilon}) + \int_{\Omega} \frac{|\nabla n_{2\varepsilon}|^2}{n_{2\varepsilon}}
\]

\[
= \chi_2 \int_{\Omega} \nabla F_{\varepsilon}(n_{2\varepsilon}) \cdot \nabla c_{\varepsilon} + \mu_2 \int_{\Omega} n_{2\varepsilon}(1 - a_2 n_{1\varepsilon} - n_{2\varepsilon}) \ln n_{2\varepsilon}
\]

\[
\leq \chi_2 \int_{\Omega} \nabla F_{\varepsilon}(n_{2\varepsilon}) \cdot \nabla c_{\varepsilon} + \frac{\mu_2 a_2}{e} \int_{\Omega} n_{1\varepsilon}.
\]

(3.14)

Finally, we have the following inequality

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2}{c_{\varepsilon}} + \frac{1}{2(2 + \sqrt{3})} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^4}{c_{\varepsilon}^2}
\]

\[
\leq - \int_{\Omega} \nabla c_{\varepsilon} \cdot \nabla (\alpha F_{\varepsilon}(n_{1\varepsilon}) + \beta F_{\varepsilon}(n_{2\varepsilon})) + \frac{2 + \sqrt{3}}{2} s_0 \int_{\Omega} |\nabla u_{\varepsilon}|^2,
\]

(3.15)

for all \(t \in (0, T_{\text{max},\varepsilon})\). Indeed, by a straightforward calculation and integration by parts we see

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2}{c_{\varepsilon}} = \frac{1}{2} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2}{c_{\varepsilon}^2} c_{\varepsilon} - \int_{\Omega} \frac{\Delta c_{\varepsilon}}{c_{\varepsilon}} c_{\varepsilon},
\]

for all \(t \in (0, T_{\text{max},\varepsilon})\). In view of the third equation of (3.1) and integration by parts, we further have

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2}{c_{\varepsilon}}
\]

\[
= \frac{1}{2} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2}{c_{\varepsilon}^2} \Delta c_{\varepsilon} - \int_{\Omega} \frac{|\Delta c_{\varepsilon}|^2}{c_{\varepsilon}} - \frac{1}{2} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2}{c_{\varepsilon}} (\alpha F_{\varepsilon}(n_{1\varepsilon}) + \beta F_{\varepsilon}(n_{2\varepsilon}))
\]

(3.16)
and Combining the above three equations, we further have
\[\int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon} \cdot (\alpha F_\varepsilon(n_1\varepsilon) + \beta F_\varepsilon(n_2\varepsilon)) - \frac{1}{2} \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon^2} u_\varepsilon \cdot \nabla c_\varepsilon + \int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon} (u_\varepsilon \cdot \nabla c_\varepsilon)
= \frac{1}{2} \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon^2} \Delta c_\varepsilon - \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon^2} \cdot \nabla (\alpha F_\varepsilon(n_1\varepsilon) + \beta F_\varepsilon(n_2\varepsilon))
- \int_\Omega \nabla c_\varepsilon \cdot \nabla (\alpha F_\varepsilon(n_1\varepsilon) + \beta F_\varepsilon(n_2\varepsilon)) - \frac{1}{2} \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon^2} (u_\varepsilon \cdot \nabla c_\varepsilon) + \int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon} (u_\varepsilon \cdot \nabla c_\varepsilon),
\]
for all \(t \in (0, T_{\text{max,}\varepsilon})\). On the other hand, considering the Lemma 3.3 with \(\varphi = c_\varepsilon\) and \(h(\varphi) = -\frac{1}{\varphi}\) to observe that
\[- \int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon} = - \int_\Omega \frac{D^2 c_\varepsilon}{c_\varepsilon} - \frac{3}{2} \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon^2} \Delta c_\varepsilon + \int_\Omega \frac{\nabla c_\varepsilon^4}{c_\varepsilon^3} + \int_\partial \frac{1}{c_\varepsilon} \frac{\partial}{\partial \nu} |\nabla c_\varepsilon|^2.
\]
What’s more, we observe that
\[\int_\Omega c_\varepsilon |D^2 \ln c_\varepsilon|^2 = \int_\Omega \frac{1}{c_\varepsilon} |D^2 c_\varepsilon|^2 - 2 \int_\Omega \frac{1}{c_\varepsilon^2} (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) \cdot \nabla c_\varepsilon + \int_\Omega \frac{\nabla c_\varepsilon^4}{c_\varepsilon^3} - \int_\Omega \nabla c_\varepsilon^2 \cdot \nabla c_\varepsilon^2 + \int_\Omega \frac{\nabla c_\varepsilon^4}{c_\varepsilon^3}.
\]
Combining the above three equations, we further have
\[\frac{1}{2} \frac{d}{dt} \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon^2} = - \int_\Omega c_\varepsilon |D^2 \ln c_\varepsilon|^2 + \frac{1}{2} \int_\partial \frac{1}{c_\varepsilon} \frac{\partial}{\partial \nu} |\nabla c_\varepsilon|^2
- \frac{1}{2} \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon^2} (\alpha F_\varepsilon(n_1\varepsilon) + \beta F_\varepsilon(n_2\varepsilon)) + \int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon} (u_\varepsilon \cdot \nabla c_\varepsilon)
- \int_\Omega \nabla c_\varepsilon \cdot \nabla (\alpha F_\varepsilon(n_1\varepsilon) + \beta F_\varepsilon(n_2\varepsilon)) - \frac{1}{2} \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon^2} (u_\varepsilon \cdot \nabla c_\varepsilon).
\]
Then applying Lemma 3.3 in [23], the inequality
\[\int_\Omega \frac{\nabla c_\varepsilon^4}{c_\varepsilon^3} \leq (2 + \sqrt{3}) \int_\Omega c_\varepsilon |D^2 \ln c_\varepsilon|^2
\]
holds for all \(t \in (0, T_{\text{max,}\varepsilon})\). Since the convexity of \(\Omega\) and \(\frac{\partial c_\varepsilon}{\partial \nu} = 0\) on \(\partial \Omega\), it follows that \(\frac{\partial}{\partial \nu} |\nabla c_\varepsilon|^2 \leq 0\) on \(\partial \Omega([4])\). Moreover because of \(\nabla \cdot u_\varepsilon = 0\), integration by parts and the Young’s inequality we find
\[\int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon} (u_\varepsilon \cdot \nabla c_\varepsilon)
= \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon^2} (u_\varepsilon \cdot \nabla c_\varepsilon) - \int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon} \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) - \int_\Omega \frac{u_\varepsilon}{c_\varepsilon} \cdot (D^2 c_\varepsilon \cdot \nabla c_\varepsilon)
= \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon^2} (u_\varepsilon \cdot \nabla c_\varepsilon) - \int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon} \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) - \frac{1}{2} \int_\Omega \frac{u_\varepsilon}{c_\varepsilon} \cdot \nabla |\nabla c_\varepsilon|^2
= \frac{1}{2} \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon^2} (u_\varepsilon \cdot \nabla c_\varepsilon) - \int_\Omega \frac{\nabla c_\varepsilon}{c_\varepsilon} \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon)
\leq \frac{1}{2} \int_\Omega \frac{\nabla c_\varepsilon^2}{c_\varepsilon^2} (u_\varepsilon \cdot \nabla c_\varepsilon) + \frac{1}{2(2 + \sqrt{3})} \int_\Omega \frac{\nabla c_\varepsilon^4}{c_\varepsilon^2} + \frac{2 + \sqrt{3}}{2} \int_\Omega c_\varepsilon |\nabla u_\varepsilon|^2.
\]
show that application of the Gagliardo-Nirenberg inequality shows that for all 

\[ t \]

Proof. Combined the above discussion with (3.2), we arrive at (3.15).

Finally, considering the consequence of Lemma 3.2 and combining (3.13)-(3.15) with \( C = \frac{4}{3}\int_\Omega n_{2\varepsilon} + B \ell_{\varepsilon} \int_\Omega n_{1\varepsilon}, \) and \( K = \max \left\{ \frac{2\varepsilon}{\sqrt{3}} s_0, 2(2 + \sqrt{3}), \frac{1}{\varepsilon}, \frac{1}{B} \right\}, \) we arrive at (3.12). \( \square \)

**Lemma 3.5.** Assume that \( p \geq 2. \) \( \tilde{A}, \tilde{B} \) and \( K \) be given in Lemma 3.4. Then for any \( \varepsilon \in (0,1) \) there exists \( K_0 > 0 \) large enough such that

\[
\frac{d}{dt} \left( \tilde{A} \int_\Omega (n_{1\varepsilon} \ln n_{1\varepsilon} - n_{1\varepsilon}) + \tilde{B} \int_\Omega (n_{2\varepsilon} \ln n_{2\varepsilon} - n_{2\varepsilon}) + \frac{1}{2} \int_\Omega \frac{|\nabla c_e|^2}{c_e} + K \int_\Omega |u_e|^2 \right) \\
+ \frac{1}{K_0} \left\{ \int_\Omega \frac{|\nabla c_e|^4}{c_e^2} + \int_\Omega \frac{\nabla n_{1\varepsilon}}{n_{1\varepsilon}} \right\} \int_\Omega \frac{|\nabla n_{2\varepsilon}|^2}{n_{2\varepsilon}} + \int_\Omega |\nabla u_e|^2 \right\} \right) \\
\leq K_0,
\]

for all \( t \in (0, T_{\text{max}, \varepsilon}). \) Here we have used that

\[
\int_\Omega (Y_e u_e \cdot \nabla u_e) u_e = - \int_\Omega (\nabla \cdot Y_e u_e)|u_e|^2 - \frac{1}{2} \int_\Omega Y_e u_e \cdot \nabla |u_e|^2 \\
= - \frac{1}{2} \int_\Omega (\nabla \cdot Y_e u_e)|u_e|^2 = 0,
\]

since \( \nabla \cdot u_e = 0 \) and \( \nabla \cdot (1 + \varepsilon A)^{-1} u_e = 0. \)

Let \( \theta := \frac{p-1}{4(2p-3)} \in (0,1), \) then \( \theta \) satisfies \( \frac{p-1}{6p} = \theta(\frac{1}{p} - \frac{1}{3}) + (1 - \theta) \frac{p-1}{p}. \) An application of the Gagliardo-Nirenberg inequality shows that

\[
\|n_{1\varepsilon}\|^2_{L^p(\Omega)} = \| \nabla n_{1\varepsilon} \|^2_{L^\frac{2p}{p-1}(\Omega)} \\
\leq C_1 \| \nabla n_{1\varepsilon} \|^2_{L^p(\Omega)} \| \nabla n_{1\varepsilon} \|^2_{L^\frac{2p}{p-1}(\Omega)} + C_1 \| n_{1\varepsilon} \|^2_{L^\frac{2p}{p-1}(\Omega)} \\
= C_1 \| \nabla n_{1\varepsilon} \|^2_{L^p(\Omega)} \| n_{1\varepsilon} \|^2_{L^\frac{2p}{p-1}(\Omega)} + C_1 \| n_{1\varepsilon} \|^2_{L^\frac{2p}{p-1}(\Omega)},
\]
for all $t \in (0, T_{\text{max}, \varepsilon})$, with some $C_1 > 0$. Due to our assumption $p \geq 2 > \frac{7}{4}$, we have $rac{2p}{p-1} \theta < p$. Using the Young’s inequality together with (3.9) to estimate
\[
\|n_{1\varepsilon}\|_{2\varepsilon}^2 \leq \delta \left\| \nabla n_{1\varepsilon}^{\frac{p-1}{2}} \right\|_p + C_2 C_\delta,
\]
for all $t \in (0, T_{\text{max}, \varepsilon})$, with some $C_2 > 0$.

Let $\eta := \frac{\varepsilon}{2} \in (0, 1)$, then $\eta$ satisfies $\frac{\varepsilon}{2} = \eta \left( \frac{1}{2} - \frac{1}{2} \right) + \frac{1}{2} (1 - \eta)$. Once again we use the Gagliardo-Nirenberg inequality to see that
\[
\|n_{2\varepsilon}\|_{L^{2\varepsilon}(\Omega)}^2 = \left\| n_{2\varepsilon} \right\|_{L^{4\varepsilon}(\Omega)}^4 \\
\leq C_3 \left\| \nabla n_{2\varepsilon} \right\|_{L^2(\Omega)}^{4\eta} \left\| n_{2\varepsilon} \right\|_{L^1(\Omega)}^{4(1-\eta)} + C_3 \left\| n_{2\varepsilon} \right\|_{L^1(\Omega)}^4 \\
= C_3 \left\| n_{2\varepsilon} \right\|_{L^2(\Omega)}^{4\eta} \left\| n_{2\varepsilon} \right\|_{L^1(\Omega)}^{2(1-\eta)} + C_3 \|n_{2\varepsilon}\|_{L^1(\Omega)},
\]
for all $t \in (0, T_{\text{max}, \varepsilon})$, with some $C_3 > 0$. Since $4\eta < 2$, we use Young’s inequality together with (3.10) to estimate
\[
\|n_{2\varepsilon}\|_{L^{2\varepsilon}(\Omega)}^2 \leq \eta \left\| \nabla n_{2\varepsilon} \right\|_{L^2(\Omega)}^2 + C_4 C_\delta,
\]
for all $t \in (0, T_{\text{max}, \varepsilon})$, with some $C_4 > 0$.

Noticing that
\[
\int_{\Omega} (|\nabla n_{1\varepsilon}|^2 + \varepsilon) \frac{c_{\varepsilon}}{n_{1\varepsilon}} \frac{\|\nabla n_{1\varepsilon}\|_n^2}{n_{1\varepsilon}} \geq \int_{\Omega} |\nabla n_{1\varepsilon}|^p.
\]
Then considering the consequence of Lemma 3.4 and combining (3.17)-(3.19) with $\delta = \frac{p}{4K\|\nabla \Phi\|_{\infty}^{p-1})$ and $\theta = \frac{1}{K\|L\Phi\|_{\infty}}$, we arrive at (3.16).

We can thereby establish the following consequences.

**Lemma 3.6.** Assume that $p \geq 2$. $A$, $B$ and $K$ be given in Lemma 3.4. Then for any $\varepsilon \in (0, 1)$, we have
\[
\bar{A} \int_{\Omega} (n_{1\varepsilon} \ln n_{1\varepsilon} - n_{1\varepsilon}) + \bar{B} \int_{\Omega} (n_{2\varepsilon} \ln n_{2\varepsilon} - n_{2\varepsilon}) \\
+ \frac{1}{2} \int_{\Omega} \left\| \nabla c_{\varepsilon} \right\|_{c_{\varepsilon}} + K \int_{\Omega} |u_{\varepsilon}|^2 \leq C,
\]
for all $t \in (0, T_{\text{max}, \varepsilon})$, and
\[
\int_0^T \int_{\Omega} \left| \nabla c_{\varepsilon} \right|_{c_{\varepsilon}}^4 \cdot + \int_0^T \int_{\Omega} \left| \nabla n_{1\varepsilon} \right|_{n_{1\varepsilon}}^{\frac{p-1}{2}} + \int_0^T \int_{\Omega} \left| n_{1\varepsilon} \right|_{n_{1\varepsilon}}^2 + \int_0^T \int_{\Omega} \left| \nabla n_{2\varepsilon} \right|_{n_{2\varepsilon}}^2 \\
\leq C(T + 1), \text{ for any fixed } T \in (0, T_{\text{max}, \varepsilon}).
\]

**Proof.** Set
\[
y_{\varepsilon}(t) := \int_{\Omega} \left\{ \bar{A}(n_{1\varepsilon} \ln n_{1\varepsilon} - n_{1\varepsilon}) + \bar{B}(n_{2\varepsilon} \ln n_{2\varepsilon} - n_{2\varepsilon}) + \frac{1}{2} \left| \nabla c_{\varepsilon} \right|_{c_{\varepsilon}}^2 + K |u_{\varepsilon}|^2 \right\} (\cdot, t),
\]
for all $t \in (0, T_{\text{max}, \varepsilon})$, and
\[
h_{\varepsilon}(t) := \int_{\Omega} \left\{ \left| \nabla c_{\varepsilon} \right|_{c_{\varepsilon}}^4 + \left| \nabla n_{1\varepsilon} \right|_{n_{1\varepsilon}}^{\frac{p-1}{2}} + \left| n_{1\varepsilon} \right|_{n_{1\varepsilon}}^2 + \left| \nabla n_{2\varepsilon} \right|_{n_{2\varepsilon}}^2 \right\} (\cdot, t),
\]
for all $t \in (0, T_{\max, \varepsilon})$. Obviously, Lemma 3.5 shows that

$$y_\varepsilon'(t) + \frac{1}{K_0} h_\varepsilon(t) \leq K_0 \quad \text{for all } t \in (0, T_{\max, \varepsilon}).$$

Now, we are going to show that $y_\varepsilon(t)$ is dominated by $h_\varepsilon(t)$. Firstly, employing the standard Poincaré inequality, there exists $C_1 > 0$ such that

$$K \int_\Omega |u_\varepsilon|^2 \leq C_1 \int_\Omega |\nabla u_\varepsilon|^2 \quad \text{for all } t \in (0, T_{\max, \varepsilon}). \tag{3.22}$$

Obviously, using that $s \ln s \leq 2s^{\frac{p}{p-1}}$ for all $s > 0$ and combining the Young’s inequality together with (3.18), we see that

$$\int_\Omega A(n_{1\varepsilon} \ln n_{1\varepsilon} - n_{1\varepsilon}) \leq \tilde{A} \int_\Omega n_{1\varepsilon} \ln n_{1\varepsilon} \leq 2 \tilde{A} \int_\Omega n_{1\varepsilon}^\varepsilon \leq C_2 \left\| \nabla n_{1\varepsilon} \right\|_{L^p(\Omega)}^p + C_2, \tag{3.23}$$

with some $C_2 > 0$. Similar to the proof of (3.23), due to the Young’s inequality and (3.19), we can also obtain

$$B \int_\Omega (n_{2\varepsilon} \ln n_{2\varepsilon} - n_{2\varepsilon}) \leq C_3 \left\| \nabla n_{2\varepsilon} \right\|_{L^2(\Omega)}^2 + C_3, \tag{3.24}$$

with some $C_3 > 0$. Finally, according to the Young’s inequality and (3.11), there exists $C_4$ such that

$$\frac{1}{2} \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \leq \frac{1}{\Omega} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^4} + \frac{1}{16} \int_\Omega c_\varepsilon \leq \frac{1}{\Omega} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^4} + C_4. \tag{3.25}$$

(3.22)-(3.25) ensure that $y_\varepsilon \leq C_5 h_\varepsilon(t) + C_9$, with some $C_9 > 0$. And then

$$y_\varepsilon'(t) + \frac{1}{2C_5 \varepsilon K_0} y_\varepsilon + \frac{1}{2K_0} h_\varepsilon(t) \leq K_0 + \frac{1}{2K_0}, \quad \text{for all } t \in (0, T_{\max, \varepsilon}).$$

By Lemma 2.2, it is easy to see that (3.20) and (3.21) hold. The proof is complete.

\[ \square \]

**Lemma 3.7.** Assume that $p \geq 2$, $\xi \in (1, 1 - \frac{12}{p}]$, $\zeta \in (1, 5]$. For any fixed $T \in (0, T_{\max, \varepsilon})$, there exists $C > 0$ such that the solution of (3.1) satisfies

$$\int_\Omega n_{1\varepsilon}^\zeta + \int_0^T \int_\Omega n_{1\varepsilon}^{\xi+1} + \int_0^T \int_\Omega n_{1\varepsilon}^{\xi-2} |\nabla n_{1\varepsilon}|^p \leq C(T+1), \tag{3.26}$$

and

$$\int_\Omega n_{2\varepsilon}^\zeta + \int_0^T \int_\Omega n_{2\varepsilon}^{\xi+1} + \int_0^T \int_\Omega n_{2\varepsilon}^{\xi-2} |\nabla n_{2\varepsilon}|^2 \leq C(T+1). \tag{3.27}$$

**Proof.** By Lemma 3.6, we have

$$\int_0^T \int_\Omega |\nabla c_\varepsilon|^4 = \int_0^T \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^4} \cdot c_\varepsilon \leq s_0^3 C(T+1). \tag{3.28}$$

When multiplying the first equation of (3.1) by $\xi n_{1\varepsilon}^{\xi-1}$, and integrating by parts, due to Young’s inequality and (3.3) we can estimate

$$\frac{d}{dt} \int_\Omega n_{1\varepsilon}^\xi + \xi(\xi - 1) \int_\Omega n_{1\varepsilon}^{\xi-2} |\nabla n_{1\varepsilon}|^p + \xi \mu_1 \int_\Omega n_{1\varepsilon}^{\xi+1}$$

$$\leq \frac{d}{dt} \int_\Omega n_{1\varepsilon}^\xi + \xi(\xi - 1) \int_\Omega n_{1\varepsilon}^{\xi-2} (|\nabla n_{1\varepsilon}|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla n_{1\varepsilon}|^2 + \xi \mu_1 \int_\Omega n_{1\varepsilon}^{\xi+1}$$
\[
= \xi (\xi - 1) \chi \int_{\Omega} n_{1e} F'(n_{1e}) n_{1e}^{\xi - 2} \nabla n_{1e} \cdot \nabla c_e + \xi \mu_1 \int_{\Omega} n_{1e}^\xi - \xi \mu_1 a_1 \int_{\Omega} n_{1e}^\xi n_{2e}
\]
\[
\leq \frac{\xi (\xi - 1) \chi}{\varepsilon} \int_{\Omega} n_{1e}^{\xi - 2} \nabla n_{1e} \cdot \nabla c_e + \xi \mu_1 \int_{\Omega} n_{1e}^\xi
\]
\[
\leq \frac{1}{2} \xi (\xi - 1) \int_{\Omega} n_{1e}^{\xi - 2} \left| \nabla n_{1e} \right|^p + C_2 \int_{\Omega} (n_{1e}^{\xi - 2}) \left| \nabla c_e \right|^4 + \xi \mu_1 \int_{\Omega} n_{1e}^\xi,
\]
with \( C_2 = 2^{\frac{1}{4-p}} \left( \frac{\varepsilon}{\xi} \right)^{\frac{p}{2-p}} \xi (\xi - 1) \). Using the Young’s inequality again, we further have
\[
\frac{d}{dt} \int_{\Omega} n_{1e}^\xi + \frac{1}{2} \xi (\xi - 1) \int_{\Omega} n_{1e}^{\xi - 2} \left| \nabla n_{1e} \right|^p + \xi \mu_1 \int_{\Omega} n_{1e}^{\xi + 1}
\]
\[
\leq \frac{\xi \mu_1}{2} \int_{\Omega} n_{1e}^{(\xi - 2) \frac{4(p - 1)}{3p}} + C_3 \int_{\Omega} \left| \nabla c_e \right|^4 + \frac{\xi \mu_1}{4} \int_{\Omega} n_{1e}^{\xi + 1} + C_4 |\Omega|,
\]
where \( C_3 = \frac{(2 \xi^2 - \varepsilon)}{\xi^2} \), \( C_4 = 4 \xi \mu_1 \). If \( \xi \in (1, 11 - \frac{12}{p}] \), we have \((\xi - 2) \frac{4(p - 1)}{3p} \leq \xi + 1\), this implies that
\[
\frac{d}{dt} \int_{\Omega} n_{1e}^\xi + \frac{1}{2} \xi (\xi - 1) \int_{\Omega} n_{1e}^{\xi - 2} \left| \nabla n_{1e} \right|^p + \frac{1}{4} \xi \mu_1 \int_{\Omega} n_{1e}^{\xi + 1} \leq C_3 \int_{\Omega} \left| \nabla c_e \right|^4 + C_5 |\Omega|,
\]
where \( C_5 = \frac{\xi \mu_1}{2} + C_4 \). Considering Lemma 2.3 and (3.28), then there exists \( C_6 > 0 \) such that \( \int_{\Omega} n_{1e}^\xi \leq C_6 \), for all \( t \in (0, T_{\text{max}, e}) \) and \( \xi \in (1, 11 - \frac{12}{p}] \). Since \( p \geq 2 \), ensuring that \( 11 - \frac{12}{p} \geq 5 \), we obtain \( \int_{\Omega} n_{1e}^\xi \leq C_6 \), for all \( t \in (0, T_{\text{max}, e}) \).

Next we test the second equation from (3.1) by \( \zeta n_{2e}^{\xi - 1} \) with \( \zeta \in (1, 5] \), to see by neglecting several non positive contributions and employing (3.3) as well as Young’s inequality that
\[
\frac{d}{dt} \int_{\Omega} n_{2e}^\xi + \zeta (\xi - 1) \int_{\Omega} n_{2e}^{\xi - 2} \left| \nabla n_{2e} \right|^2 + \zeta \mu_2 \int_{\Omega} n_{2e}^{\xi + 1}
\]
\[
\leq \zeta (\xi - 1) \chi \int_{\Omega} n_{2e} F'(n_{2e}) n_{2e}^{\xi - 2} \nabla n_{2e} \cdot \nabla c_e + \zeta \mu_2 \int_{\Omega} n_{2e}^\xi
\]
\[
\leq \chi \xi \varepsilon \zeta (\xi - 1) \int_{\Omega} n_{2e}^{\xi - 2} \nabla n_{2e} \cdot \nabla c_e + \zeta \mu_2 \int_{\Omega} n_{2e}^\xi
\]
\[
\leq \frac{1}{2} \zeta (\xi - 1) \int_{\Omega} n_{2e}^{\xi - 2} \left| \nabla n_{2e} \right|^2 + \frac{\chi^2}{2} \zeta (\xi - 1) \int_{\Omega} n_{2e}^{\xi - 2} \left| \nabla c_e \right|^2 + \zeta \mu_2 \int_{\Omega} n_{2e}^\xi.
\]
Considering the fact that \( 2(\xi - 2) \leq \zeta + 1 \), since \( \zeta \in (1, 5] \), and combining the Young’s inequality, we see that
\[
\frac{\chi^2}{2} \zeta (\xi - 1) \int_{\Omega} n_{2e}^{\xi - 2} \left| \nabla c_e \right|^2
\]
\[
\leq \frac{1}{2} \zeta \mu_2 \int_{\Omega} n_{2e}^{(\xi - 2)} + C_7 \int_{\Omega} \left| \nabla c_e \right|^4
\]
\[
\leq \frac{1}{2} \zeta \mu_2 \int_{\Omega} n_{2e}^{\xi + 1} + C_7 \int_{\Omega} \left| \nabla c_e \right|^4 + \zeta \mu_2 |\Omega|,
\]
and that
\[
\zeta \mu_2 \int_{\Omega} n_{2e}^\xi \leq \frac{1}{4} \zeta \mu_2 \int_{\Omega} n_{2e}^{\xi + 1} + 4 \zeta \mu_2 |\Omega|,
\]
where \( C_7 = \frac{8\zeta(\zeta-1)^2\chi^2}{\mu_2^2}. \) Letting \( C_8 = \frac{\zeta\mu_2}{2} + 4\zeta\mu_2, \) then we have
\[
\frac{d}{dt} \int_{\Omega} n_2^\zeta + \frac{1}{2}\zeta(\zeta-1) \int_{\Omega} n_2^\zeta |\nabla n_2|^2 + \frac{1}{4}\zeta\mu_2 \int_{\Omega} n_2^{\zeta+1} \leq C_7 \int_{\Omega} |\nabla c_\varepsilon|^4 + C_8 |\Omega|.
\]
Finally, by Lemma 2.3 and (3.28), we also have that there exists \( C_{10} > 0 \) such that \( \int_{\Omega} n_2^{\beta} \leq C_{10}, \) for all \( t \in (0, T_{\text{max},\varepsilon}). \)

**Lemma 3.8.** Assume that \( p \geq 2. \) For all \( \varepsilon \in (0,1), \) the solution of (3.1) is global in time; that is, we have \( T_{\text{max},\varepsilon} = \infty. \)

**Proof.** Assume that \( T_{\text{max},\varepsilon} < \infty, \) for some \( \varepsilon \in (0,1). \) By Lemma 3.6, it is easy to see that the following inequality holds
\[
\int_{0}^{T_{\text{max},\varepsilon}} \int_{\Omega} |\nabla c_\varepsilon(\cdot,t)|^4 \leq C_1 \quad \text{and} \quad \int_{\Omega} |u_\varepsilon(\cdot,t)|^2 \leq C_2,
\]
for all \( t \in (0, T_{\text{max},\varepsilon}), \) with some \( C_1, C_2 > 0. \) Combining (3.26), (3.27) with (3.29), we obtain
\[
\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C_3 \quad \text{for all} \quad t \in (0, T_{\text{max},\varepsilon}),
\]
and
\[
\|\nabla c_\varepsilon\|_{L^1(\Omega)} \leq C_4 \quad \text{for all} \quad t \in (0, T_{\text{max},\varepsilon}),
\]
with some \( C_3 > 0, \) \( C_4 > 0. \) Because the proof of (3.30) and (3.31) is similar to [22] Lemma 3.9, we refer the reader to it for more details. Noticing the third equation of (3.1) and using the variation-of-constant formula, we can obtain
\[
c_\varepsilon(t) = e^{-t}e^{t\Delta}c_0\varepsilon + \int_{0}^{t} e^{-(t-s)}e^{(t-s)\Delta}(c_\varepsilon - \alpha F_\varepsilon(n_1\varepsilon)c_\varepsilon - \beta F_\varepsilon(n_2\varepsilon)c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon)(s)ds,
\]
for all \( t \in (0, T_{\text{max},\varepsilon}), \) where the \( \{e^{t\Delta}\}_{t \geq 0} \) is the Neumann heat semigroup in \( \Omega. \) For more details of Neumann heat semigroup, please refer to [24]. In conjunction with (3.2), (3.7), (3.26), (3.27), (3.30) and (3.31), we further have
\[
\|\nabla c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq e^{-t}\|e^{t\Delta}c_0\varepsilon\|_{L^\infty(\Omega)} + \int_{0}^{t} e^{-(t-s)}\|e^{(t-s)\Delta}(c_\varepsilon - \alpha F_\varepsilon(n_1\varepsilon)c_\varepsilon - \beta F_\varepsilon(n_2\varepsilon)c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon)(s)ds\|_{L^\infty(\Omega)} \leq C_5 t^{-\frac{1}{2}} e^{-t}\|c_0\varepsilon\|_{L^\infty(\Omega)} + C_5 \int_{0}^{t} e^{-(t-s)}(t-s)^{-\frac{1}{2} - \frac{1}{4}} \|c_\varepsilon - \alpha F_\varepsilon(n_1\varepsilon)c_\varepsilon - \beta F_\varepsilon(n_2\varepsilon)c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon\|_{L^1(\Omega)}ds \leq C_6 t^{-\frac{1}{2}} e^{-t}\|c_0\varepsilon\|_{L^\infty(\Omega)} + C_6 \int_{0}^{t} e^{-(t-s)}(t-s)^{-\frac{1}{2} - \frac{1}{4}} (1 + \|n_1\varepsilon\|_{L^4(\Omega)} + \|n_2\varepsilon\|_{L^4(\Omega)} + \|\nabla c_\varepsilon\|_{L^4(\Omega)})ds \leq C_7,
\]
for all \( t \in (\tau, T_{\text{max},\varepsilon}) \) and any fixed \( \tau \in (0, T_{\text{max},\varepsilon}) \), with some \( C_5, C_6, C_7 > 0 \) since \( 1 - \frac{1}{2} - \frac{3}{4} > 0. \)
In what follows, we are in the position to estimate \( \|n_{1c}\|_{L^\infty(\Omega)} \) and \( \|n_{2c}\|_{L^\infty(\Omega)} \). We first combine the estimates for Neumann heat semigroup and Young’s inequality to obtain that

\[
\begin{align*}
\|n_{2c}(\cdot,t)\|_{L^\infty(\Omega)} &= \left\| e^{(t-\tau)\Delta} n_{2c}(\cdot,\tau) - \int_\tau^t \nabla e^{(t-s)\Delta} \left( \chi_{2n_{2c}} F_1'(n_{2c}) \nabla c_\varepsilon + n_{2c} u_\varepsilon \right) ds \right\|_{L^\infty(\Omega)} \\
&+ \int_\tau^t e^{(t-s)\Delta} (1-a_2 n_{1c} - n_{2c}) ds \right\|_{L^\infty(\Omega)} \\
&\leq C_8 (t-\tau)^{-\frac{3}{2}} \|n_{2c}(\cdot,\tau)\|_{L^1(\Omega)} + C_8 \int_\tau^t (t-s)^{-\frac{1}{2} - \frac{3}{4}} (1 + \|n_{2c}\|_{L^1(\Omega)}) ds \\
&+ C_9 \int_\tau^t (t-s)^{-\frac{1}{2} - \frac{3}{4}} \left( \|n_{2c}\|_{L^1(\Omega)} + \|n_{1c}\|^2_{L^1(\Omega)} + \|n_{2c}\|^2_{L^1(\Omega)} \right) ds \\
&\leq C_{10},
\end{align*}
\]

for all \( t \in (2\tau, T_{\text{max},c}) \) and any fixed \( \tau \in (0, T_{\text{max},c}) \), with some \( C_8, C_9, C_{10} > 0 \), since (3.10), (3.26), (3.27), (3.30) and (3.32) hold.

Once again, we multiply the first equation in (3.1) by \( \xi n_{1c}^{-1} \), \( \xi > 1 \) to see upon integrating by parts that again by Young’s inequality, (3.3) and (3.32)

\[
\begin{align*}
\frac{d}{dt} \int_\Omega n_{1c} \xi + \xi (\xi - 1) \int_\Omega n_{1c}^{-2} |\nabla n_{1c}|^p + \xi \mu_1 \int_\Omega n_{1c}^{\xi+1} \\
&\leq \frac{d}{dt} \int_\Omega n_{1c} \xi + \xi (\xi - 1) \int_\Omega n_{1c}^{-2} (|\nabla n_{1c}|^2 + \varepsilon) \frac{\varepsilon^{\frac{p-1}{2}}}{\xi^{\frac{p-1}{2}}} |\nabla n_{1c}|^2 + \xi \mu_1 \int_\Omega n_{1c}^{\xi+1} \\
&= \xi (\xi - 1) \chi_1 \int_\Omega n_{1c} F_1'(n_{1c}) n_{1c}^{-2} \nabla c_\varepsilon \cdot \nabla c_\varepsilon + \xi \mu_1 \int_\Omega n_{1c}^{-2} - \xi \mu_1 a_1 \int_\Omega n_{1c} n_{2c} \\
&\leq C_{11} \int_\Omega n_{1c}^{-2} |\nabla n_{1c}| + \xi \mu_1 \int_\Omega n_{1c}^{\xi} \\
&\leq \int_\Omega n_{1c}^{\xi-2} (\xi (\xi - 1) |\nabla n_{1c}|^p + C_{12}) + \frac{1}{2} \xi \mu_1 \int_\Omega n_{1c}^{\xi+1} + C_{13} \\
&\leq \xi (\xi - 1) \int_\Omega n_{1c}^{\xi-2} |\nabla n_{1c}|^p + \frac{3}{4} \xi \mu_1 \int_\Omega n_{1c}^{\xi+1} + C_{14},
\end{align*}
\]

for all \( t \in (\tau, T_{\text{max},c}) \). Then by Lemma 2.3, we have \( \int_\Omega n_{1c}^{\xi} \leq C_{15} \), for all \( t \in (\tau, T_{\text{max},c}) \), with any \( \xi > 1 \) and some \( C_{15} > 0 \).

Finally, based on a Moser-type iteration method, we achieve \( L^\infty \) bounds for \( n_{1c} \).

Taking \( r_k = 2r_{k-1} + 2 - p, \ k = \{1, 2, 3, \ldots\} \) and \( r_0 > p \) is a positive constant.

It is not hard to see that \( \{r_k\}_{k \in \mathbb{N}} \) is a nonnegative strictly increasing sequence, \( r_k > p, \) for all \( k \) and \( r_k \not\rightarrow \infty, \) as \( k \to \infty. \) What’s more, there exist \( c_1, c_2 > 0 \) which are independent of \( k \) such that

\[
c_1 2^k \leq r_k \leq c_2 2^k, \text{ for all } k \in \mathbb{N}. \tag{3.33}
\]
Letting

\[ M_k := \sup_{t \in (\tau, T_{\max})} \int_{\Omega} \overline{n}_1 \overline{r}_k(x, t) dx, \quad k \in \mathbb{N} , \text{ where } \overline{n}_1 := \max\{n_1(x, t), 1\}, \]

for all \( x \in \overline{\Omega} \) and \( t \in (\tau, T_{\max}) \).

We multiply the first equation of (3.1) by \( r_k n_1^{r_k-1} \) and invoke the Young’s inequality along with (3.3) and (3.32) to see that

\[
\frac{d}{dt} \int_{\Omega} n_1^{r_k} + r_k(r_k - 1) \int_{\Omega} n_1^{r_k-2} \left( |\nabla n_1| + \varepsilon \right)^{\frac{p}{2}} |\nabla n_1|^2 \\
+ r_k \mu \int_{\Omega} n_1^{r_k+1} + \int_{\Omega} n_1^{r_k} \\
\leq r_k(r_k - 1) \chi \int_{\Omega} n_1^{r_k-1} P'(n_1) \nabla c \cdot \nabla n_1 + \left( r_k \mu_1 + 1 \right) \int_{\Omega} n_1^{r_k} \\
\leq C_1 \int_{\Omega} \left( \frac{1}{2C_1} n_1^{r_k-2} |\nabla n_1|^p + \frac{p - 1}{p} \left( \frac{p}{2C_1} \right) \cdot n_1^{2(r_k - 1)} \right) + \left( r_k \mu_1 + 1 \right) \int_{\Omega} n_1^{r_k} \\
\leq \frac{1}{2} r_k(r_k - 1) \int_{\Omega} n_1^{r_k-2} |\nabla n_1|^p + C_1 \int_{\Omega} n_1^{r_k+1} \\
+ \left( r_k \mu_1 + 1 \right) \int_{\Omega} n_1^{r_k},
\]

for all \( t \in (\tau, T_{\max}) \). Taking \( \theta_k := 2 \cdot \frac{r_k + p - 2}{r_k + p - 2} \geq 2 \) and considering the Hölder inequality, there must exist \( C_{18} > 0 \) fulfilling

\[
\frac{d}{dt} \int_{\Omega} n_1^{r_k} + \frac{1}{2} r_k(r_k - 1) \left( \frac{p}{r_k - 2 + p} \right)^p \int_{\Omega} |\nabla n_1^{r_k-2}|^p + \int_{\Omega} n_1^{r_k} \\
= \frac{d}{dt} \int_{\Omega} n_1^{r_k} + \frac{1}{2} r_k(r_k - 1) \int_{\Omega} n_1^{r_k-2} |\nabla n_1|^p + \int_{\Omega} n_1^{r_k} \\
\leq C_{18} r_k(r_k - 1) \left( \int_{\Omega} n_1^{(r_k+1)p-2} \theta_k \right)^{\frac{1}{p}} + C_{18} r_k(r_k - 1) \left( \int_{\Omega} n_1^{r_k+2} \lambda_k \right)^{\frac{1}{p}}, \quad (3.34)
\]

for all \( t \in (\tau, T_{\max}) \), since \( r_k - 1 > 1 \), for all \( k \). Using the Gagliardo-Nirenberg inequality and the Young’s inequality, there exists \( C_{19} > 0 \) such that

\[
C_{18} \left( \int_{\Omega} n_1^{(r_k+2)-p} \theta_k \right)^{\frac{1}{p}} \\
= C_{18} \left( \frac{r_k + p - 2}{p} \right)^{\frac{2p}{p}} \| n_1^{r_k} \|_{L^{2p}(\Omega)} \\
\leq C_{19} \| \nabla n_1^{p} \|_{L^p(\Omega)}^{\frac{2p}{p}} \cdot \| n_1^{r_k+2} \theta_k \|_{L^p(\Omega)}^{\frac{2p(1-\alpha)}{p}} + C_{19} \| n_1^{r_k+2} \|_{L^p(\Omega)}^{\frac{2p}{p}} \\
= C_{19} \| \nabla n_1^{p} \|_{L^p(\Omega)}^{\frac{2p}{p}} \cdot \| n_1^{r_k-1} \|_{L^1(\Omega)}^{\frac{4(1-\alpha)}{p}} + C_{19} \| n_1^{r_k-1} \|_{L^1(\Omega)}^{\frac{4}{p}}.
\]
for all $t \in (\tau, T_{\text{max}, \varepsilon})$, with $a = \frac{9}{2p+6} \in (0, 1)$ satisfying $\frac{1}{2p} = (\frac{1}{p} - \frac{1}{4})a + \frac{2}{p}(1-a)$. Similarly, we also have

$$C_{18} \left( \int_{\Omega} n_{1e}^{r_k+\lambda_k} \right)^{\frac{1}{r_k}}$$

$$= C_{18} \left( \int_{\Omega} n_{1e}^{r_k+\lambda_k} \right)^{\frac{2p}{2p+2}}$$

$$\leq C_{20} \left( \int_{\Omega} n_{1e}^{r_k+\lambda_k} \right)^{\frac{2p}{2p+2}} \cdot \left( \int_{\Omega} n_{1e}^{\frac{4(1-a)}{3}} \right)^{\frac{2p}{2p+2}} + C_{20} \left( \int_{\Omega} n_{1e}^{r_k+\lambda_k} \right)^{\frac{2p}{2p+2}}$$

$$\leq C_{20} \left( \int_{\Omega} n_{1e}^{r_k+\lambda_k} \right)^{\frac{2p}{2p+2}} + C_{20} \left( \int_{\Omega} n_{1e}^{r_k+\lambda_k} \right)^{\frac{2p}{2p+2}}$$

$$\leq C_{20} \left( \int_{\Omega} n_{1e}^{r_k+\lambda_k} \right)^{\frac{2p}{2p+2}} + C_{20} \left( \int_{\Omega} n_{1e}^{r_k+\lambda_k} \right)^{\frac{2p}{2p+2}}$$

for all $t \in (\tau, T_{\text{max}, \varepsilon})$, with some $C_{27} > 0$. Taking $\eta = \frac{1}{4c_{19}} \left( \frac{p}{r_k+\lambda_k} \right)^{\frac{2p}{2p+2}}$, $\delta = \frac{1}{4c_{20}} \left( \frac{p}{r_k+\lambda_k} \right)^{\frac{2p}{2p+2}}$ and substituting (3.35) and (3.36) into (3.34), we obtain

$$\frac{d}{dt} \int_{\Omega} n_{1e}^{r_k} + \int_{\Omega} n_{1e}^{r_k} \leq C_{19} C_{\eta} r_k (r_k - 1) M_{k-1}^{\frac{4(1-a)}{3}} + C_{19} r_k (r_k - 1) M_{k-1}^{\frac{4(1-a)}{3}}$$

$$+ C_{20} C_{\delta} r_k (r_k - 1) M_{k-1}^{\frac{4(1-a)}{3}} + C_{20} r_k (r_k - 1) M_{k-1}^{\frac{4(1-a)}{3}}$$

where

$$C_{\eta} = \left( \frac{\theta_k}{\eta^{2a}} \right)^{\frac{1}{2a-1}} + \frac{\theta_k}{\eta^{2a}} - 1 = \frac{\theta_k - 2a}{\theta_k} \left( \frac{\theta_k}{2a} \right)^{\frac{2a}{2a-2}}$$

Letting

$$\bar{b} = 2^{\frac{ap}{2a-2}} > 1,$$

then by (3.33) we have

$$C_{\eta} \leq \eta^{-\frac{2a}{2a-2}}$$

$$= (4C_{19})^{\frac{2a}{2a-2}} \left( \frac{r_k + p - 2}{p} \right)^{\frac{2a}{2a-2}p}$$

$$\leq (4C_{19})^{\frac{2a}{2a-2}} \left( r_k^{\frac{2a}{2a-2}p} + 1 \right)$$

$$\leq (4C_{19})^{\frac{2a}{2a-2}} 2 r_k^{\frac{2a}{2a-2}p}$$

$$\leq C_{21} \bar{b}.$$
A quite similar computation gives
\[ C_\delta \leq C_{22} \bar{b}^k, \]
with some \( C_{22} > 0 \). Noticing that \( \frac{4(1-p)}{p-2a} \frac{1}{\sqrt{\lambda_k - 2a}} \) and \( \frac{1}{\delta_k} < 2 \), then we have
\[
\frac{d}{dt} \int_\Omega n_{1\varepsilon}^k + \int_\Omega n_{01\varepsilon}^k \leq C_{19} C_{21} r_k(r_k - 1) \bar{b}^k M_{k-1}^k + C_{19} r_k(r_k - 1) M_{k-1}^k + C_{20} C_{22} r_k(r_k - 1) \bar{b}^k M_{k-1}^k + C_{20} r_k(r_k - 1) M_{k-1}^k \leq 2C_{19} C_{21} r_k(r_k - 1) \bar{b}^k M_{k-1}^2 + 2C_{20} C_{22} r_k(r_k - 1) \bar{b}^k M_{k-1}^2.
\]
An integration of this ODI shows that
\[ M_k \leq \max \left\{ \int_\Omega n_{01\varepsilon}^k, (2C_{19} C_{21} + 2C_{20} C_{22}) r_k(r_k - 1) \bar{b}^k M_{k-1}^2 \right\}.
\]
On the one hand, if
\[ (2C_{19} C_{21} + 2C_{20} C_{22}) r_k(r_k - 1) \bar{b}^k M_{k-1}^2 < \int_\Omega n_{01\varepsilon}^k, \]
holding for infinitely many \( k \geq 1 \), we see
\[
sup_{t \in (0, \infty)} \left( \int_\Omega n_{1\varepsilon}^{k-1} \right)^{\frac{1}{k-1}} \leq \left( \int_\Omega n_{01\varepsilon}^k \right)^{\frac{1}{k}},
\]
for all such \( k \), and hence conclude that \( \| n_{1\varepsilon}(t) \|_{L^\infty(\Omega)} \leq \| n_{01\varepsilon} \|_{L^\infty(\Omega)} \), for all \( t > 0 \).

On the other hand, in the opposite case, upon enlarging \( C_{19} \) and \( C_{20} \) if necessary we have that \( M_k \leq (2C_{19} C_{21} + 2C_{20} C_{22}) r_k(r_k - 1) \bar{b}^k M_{k-1}^2 \), for all \( k \geq 1 \). Let \( C_{23} := (2C_{19} C_{21} + 2C_{20} C_{22}) c_2^2 \), then from the definition of \( \bar{b} \) and (3.33), we see
\[
M_k \leq (2C_{19} C_{21} + 2C_{20} C_{22}) r_k^2 \bar{b}^k M_{k-1}^2 \leq (2C_{19} C_{21} + 2C_{20} C_{22}) (c_2 2^k)^2 \bar{b}^k M_{k-1}^2 \leq C_{23} (4\bar{b})^k M_{k-1}^2,
\]
for all \( k \geq 1 \). Furthermore, we have
\[
M_k \leq C_{23} \sum_{j=0}^{k-1} 2^j (4\bar{b})^j \sum_{j=0}^{k-1} (k-j) 2^j M_0^2,
\]
for all \( k \geq 1 \). A straightforward calculation shows that \( \sum_{j=0}^{k-1} 2^j = 2^k - 1 \leq 2^k \) and \( \sum_{j=0}^{k-1} (k-j) 2^j = 2^{k+1} - 2 - k \leq 2^{k+1} \), then we have
\[
M_k \leq C_{23}^2 (4\bar{b})^{2k+1} M_0^2.
\]
Finally, combining the definition of \( M_k \), we concludes that
\[
\| n_{1\varepsilon}(t) \|_{L^\infty(\Omega)} \leq C_{24} \cdot \| n_{01\varepsilon} \|_{L^\infty(\Omega)}, \quad \text{for all } t \in (\tau, T_{\text{max}, \varepsilon}),
\]
with \( C_{24} = C_{23} (4\bar{b})^2 \), which confirms \( \| n_{1\varepsilon} \|_{L^\infty(\Omega)} \) is bounded. By Lemma 3.1, \( T_{\text{max}, \varepsilon} = \infty \).

\begin{lemma}
There exists \( C > 0 \) such that for any \( \varepsilon \in (0, 1) \), the solution of (3.1) satisfies
\[
\int_0^T \int_\Omega |u_{\varepsilon}|^\frac{m}{2} \leq C(T + 1),
\]
\end{lemma}
for all $T > 0$. And
\[ \int_0^T \int_\Omega n_{1\varepsilon}^r \leq C(T + 1), \quad (3.38) \]
for all $T > 0$, where $r \in [1, \frac{4p}{3} + 3]$. And
\[ \int_0^T \int_\Omega n_{2\varepsilon}^m \leq C(T + 1), \quad (3.39) \]
for all $T > 0$, where $m \in [1, \frac{17}{4}]$.

**Proof.** Fixing $C_1 > 0$ and $C_2 > 0$ such that in accordance with Lemma 3.6, we have
\[ \int_\Omega |u_{\varepsilon}|^2 \leq C_1(T + 1), \quad (3.40) \]
and that
\[ \int_0^T \int_\Omega |\nabla u_{\varepsilon}|^2 \leq C_2(T + 1). \quad (3.41) \]

Then we combine the Gagliardo-Nirenberg inequality with Poincaré inequality to fix $C_3, C_4 > 0$ such that
\[ \|u_{\varepsilon}\|_{L^{\frac{10}{3}}((0, T) \times \Omega)} = \int_0^T \|u_{\varepsilon}\|_{L^{\frac{10}{3}}(\Omega)} \]
\[ \leq C_3 \int_0^T \|\nabla u_{\varepsilon}\|_{L^2(\Omega)} \|u_{\varepsilon}\|_{L^2(\Omega)} \]
\[ \leq C_4 \int_0^T \|\nabla u_{\varepsilon}\|_{L^2(\Omega)} \leq C_2 C_4 (T + 1). \]

Recalling Lemma 3.7, particularly, when $\xi = 5$, we have
\[ \int_0^T \int_\Omega |\nabla n_{1\varepsilon}|^p = \left( \frac{3 + p}{p} \right) \int_0^T \int_\Omega n_{1\varepsilon} |\nabla n_{1\varepsilon}|^p \leq C_5(T + 1), \]
with some $C_5 > 0$. Taking $\tilde{a} = \frac{p+3-(p+3)r}{2 - \frac{3}{p+3}} = \frac{3(p+3)(r-1)}{(4p+6)r}$, when $r \leq \frac{4p}{3} + 3$, we see that $\tilde{a} \in [0, 1)$ and $\frac{p}{p+3} \cdot \tilde{a} \leq p$. Thus, invoking the Gagliardo-Nirenberg inequality along with (3.9), we obtain $C_6 > 0$ and $C_7 > 0$ such that
\[ \int_0^T \|n_{1\varepsilon}\|_{L^{r}(\Omega)} = \int_0^T \left\| n_{1\varepsilon} \right\|_{L^{\frac{p}{p+3}}(\Omega)} \]
\[ \leq C_6 \left( \int_0^T \left\| \nabla n_{1\varepsilon} \right\|_{L^{p}(\Omega)} \right) ^{\frac{p+3}{p}} \left( \int_0^T \left\| n_{1\varepsilon} \right\|_{L^{\frac{p}{p+3}}(\Omega)} \right) ^{\frac{p}{p+3} \cdot \tilde{a}} \]
\[ + C_6 \int_0^T \left( \left\| \nabla n_{1\varepsilon} \right\|_{L^{p}(\Omega)} \right) ^{\frac{p+3}{p}} \left( \int_0^T \left\| n_{1\varepsilon} \right\|_{L^{\frac{p}{p+3}}(\Omega)} \right) ^{\frac{p}{p+3} \cdot (1-\tilde{a})} \]
\[ \leq C_7 (T + 1), \]
with $\sigma = \frac{p+3}{p+3-r\tilde{a}}$. 

Again by Lemma 3.7 we also have
\[
\int_0^T \int_\Omega \left| \nabla n_{2\varepsilon}^3 \right|^2 = \left( \frac{5}{2} \right)^2 \int_0^T n_{2\varepsilon}^3 |\nabla n_{2\varepsilon}|^2 \leq C_8 (T + 1),
\]
with some \( C_8 > 0 \). Upon another application of the Gagliardo-Nirenberg inequality in precisely the same way we see
\[
\int_0^T \left\| n_{2\varepsilon} \right\|_{L^m(\Omega)}^m = \int_0^T \left\| \frac{3}{2} n_{2\varepsilon} \right\|_{L^{\frac{2m}{3}}(\Omega)}^{\frac{2m}{3}} \leq C_9 \int_0^T \left\| \nabla n_{2\varepsilon}^3 \right\|_{L^2(\Omega)} \left\| n_{2\varepsilon}^\frac{2m}{3} (1 - \hat{a}) \right\|_{L^{\frac{2m}{3}}(\Omega)} + C_9 \int_0^T \left\| n_{2\varepsilon}^\frac{2m}{3} \right\|_{L^{\frac{2m}{3}}(\Omega)}^{\frac{2m}{3}},
\]
where \( \hat{a} = \frac{\frac{3}{2} - \frac{5}{2}}{\frac{5}{3} - \frac{2}{3}} = \frac{15}{14} (1 - \frac{1}{m}) \). Since \( m \leq \frac{14}{9} \), we have \( 2m \cdot \hat{a} \leq 2 \). Then employing (3.10) and Young’s inequality, we have
\[
\int_0^T \left\| n_{2\varepsilon} \right\|_{L^m(\Omega)}^m \leq C_{10} (T + 1).
\]

**Lemma 3.10.** There exists \( C > 0 \) such that for any \( \varepsilon \in (0, 1) \) and any \( T > 0 \), the solution of (3.1) satisfies
\[
\int_0^T \left\| \partial_t n_{1\varepsilon} \right\|_{W^{1, p}(\Omega), \varepsilon}^p \leq C (T + 1),
\]
(3.42)
\[
\int_0^T \left\| \partial_t n_{2\varepsilon} (\cdot, t) \right\|_{W^{1, \frac{2}{3}}(\Omega), \varepsilon}^\frac{5}{2} \leq C (T + 1),
\]
(3.43)
\[
\int_0^T \left\| \partial_t c_{\varepsilon} (\cdot, t) \right\|_{W^{1, \frac{4}{3}}(\Omega), \varepsilon}^{\frac{10}{9}} \leq C (T + 1),
\]
(3.44)
\[
\int_0^T \left\| \partial_t u_{\varepsilon} (\cdot, t) \right\|_{W^{1, \frac{4}{3}}(\Omega), \varepsilon}^{\frac{5}{2}} \leq C (T + 1).
\]
(3.45)

**Proof.** Testing the first equation in (3.1) by any \( \varphi \in C^\infty (\bar{\Omega}) \), and integrating by parts, due to Hölder’s inequality, and (3.3) we can estimate
\[
\left| \int_\Omega \partial_t n_{1\varepsilon} (\cdot, t) \varphi \right| = - \int_\Omega \left( |\nabla n_{1\varepsilon}|^2 + \varepsilon \right)^{\frac{p-2}{2}} \nabla n_{1\varepsilon} \cdot \nabla \varphi + \int_\Omega \chi_{1n_{1\varepsilon}} F'(n_{1\varepsilon}) \nabla c_{\varepsilon} \cdot \nabla \varphi + \int_\Omega n_{1\varepsilon} u_{\varepsilon} \cdot \nabla \varphi
\]
\[
+ \int_\Omega \mu n_{1\varepsilon} (1 - n_{1\varepsilon} - a_1 n_{2\varepsilon})
\]
\[
\leq \int_\Omega \left( |\nabla n_{1\varepsilon}|^2 + \varepsilon \right)^{\frac{p-2}{2}} |\nabla n_{1\varepsilon}| \|
\varphi\| + C_1 |n_{1\varepsilon} \nabla c_{\varepsilon} \|_{L^{p'}(\Omega)} \|
\varphi\|_{L^{p}(\Omega)}
\]
\[
+ |n_{1\varepsilon} u_{\varepsilon} \|_{L^{p'}(\Omega)} \|
\varphi\|_{L^{p}(\Omega)} + \|\mu n_{1\varepsilon} (1 - n_{1\varepsilon} - a_1 n_{2\varepsilon}) \|_{L^{p'}(\Omega)} \|
\varphi\|_{L^{p}(\Omega)}
\]
\[
\leq C_1 \left( \left( \left| |\nabla n_{1\varepsilon}|^2 + \varepsilon \right)^{\frac{p-1}{2}} \right) \|
\varphi\|_{L^{p}(\Omega)} + |n_{1\varepsilon} \nabla c_{\varepsilon} \|_{L^{p'}(\Omega)} + |n_{1\varepsilon} u_{\varepsilon} \|_{L^{p'}(\Omega)}
\right)
Upon the Hölder’s inequality, we furthermore see that

\[
\int_0^T \| \partial_t n_{1\varepsilon}(\cdot, t) \|_{W^{1,p}(\Omega)}^{p'} \leq C_2 \left( \int_0^T \int_\Omega (|\nabla n_{1\varepsilon}|^2 + 1)^{\frac{p-1}{2}} + \int_0^T \int_\Omega |n_{1\varepsilon} \nabla c_\varepsilon|^p + \int_0^T \int_\Omega |n_{1\varepsilon} u_{\varepsilon}|^{p'} \right)
\]

with some \( C_2 > 0 \). Then we use Young’s inequality (3.26), (3.27), (3.28), (3.37) and the fact that \( \frac{1}{5} + \frac{1}{4} < \frac{1}{5} + \frac{3}{10} < \frac{3}{7} \) to see

\[
\int_0^T \| \partial_t n_{1\varepsilon}(\cdot, t) \|_{W^{1,p}(\Omega)}^{p'} \leq C_2 \left( \int_0^T \int_\Omega (|\nabla n_{1\varepsilon}|^2 + 1)^{\frac{p-1}{2}} \right) + C_3 \left( \int_0^T \int_\Omega |n_{1\varepsilon}|^5 + \int_0^T \int_\Omega |\nabla c_\varepsilon|^4 \right)
\]

By Lemma 3.6, we see there exists \( C_4 > 0 \) such that

\[
\int_0^T \int_\Omega \frac{\| \nabla n_{2\varepsilon} \|^2}{n_{2\varepsilon}} \leq C_4(T + 1).
\]

Upon the Hölder’s inequality, we furthermore see that

\[
\int_0^T \int_\Omega |\nabla n_{2\varepsilon}|^\frac{\hat{\beta}}{2} = \int_0^T \int_\Omega \left( \frac{|\nabla n_{2\varepsilon}|^\frac{2}{2}}{n_{2\varepsilon}} \right) \cdot \frac{n_{2\varepsilon}}{2} \leq \left( \int_0^T \int_\Omega \left( \frac{|\nabla n_{2\varepsilon}|^\frac{2}{2}}{n_{2\varepsilon}} \right) \right)^\frac{\hat{\beta}}{2} \left( \int_0^T \int_\Omega \frac{n_{2\varepsilon}}{2} \right)^{\frac{\hat{\beta}}{2}} \leq C_5(T + 1),
\]

with some \( C_5 > 0 \). We next test the second equation in (3.1) by any \( \varphi \in C^\infty(\bar{\Omega}) \), and integrate by parts, due to Hölder’s inequality and (3.3) we can obtain

\[
\left| \int_\Omega \partial_t n_{2\varepsilon}(\cdot, t) \varphi \right| = \left| - \int_\Omega \nabla n_{2\varepsilon} \cdot \nabla \varphi + \int_\Omega \chi_2 n_{2\varepsilon} F'_2(n_{2\varepsilon}) \nabla c_\varepsilon \cdot \nabla \varphi + \int_\Omega n_{2\varepsilon} u_{\varepsilon} \cdot \nabla \varphi \right|
\]

\[
+ \mu_2 \int_\Omega n_{2\varepsilon} (1 - a_2 n_{1\varepsilon} - n_{2\varepsilon}) \varphi
\]

\[
\leq C_6 \left( \| \nabla n_{2\varepsilon} \|_{L^\frac{4}{3}(\Omega)} + \| n_{2\varepsilon} \nabla c_\varepsilon \|_{L^\frac{4}{3}(\Omega)} \right) \| \varphi \|_{W^{1,\frac{4}{3}}(\Omega)} + \| n_{2\varepsilon} u_{\varepsilon} \|_{L^\frac{4}{3}(\Omega)} + \| n_{2\varepsilon}(1 - a_2 n_{1\varepsilon} - n_{2\varepsilon}) \|_{L^\frac{4}{3}(\Omega)} \| \varphi \|_{W^{1,\frac{4}{3}}(\Omega)}.
\]

We notice that \( \frac{1}{5} + \frac{1}{4} < \frac{1}{5} + \frac{3}{10} < \frac{3}{7} \). By the Young’s inequality, we can find \( C_7 > 0 \) such that

\[
\int_0^T \| \partial_t n_{2\varepsilon}(\cdot, t) \|_{W^{1,\frac{4}{3}}(\Omega)}^{\frac{3}{2}} \leq C_7(T + 1).
\]
Combining (3.26), (3.27), (3.28) and (3.37), we obtain (3.43).

Recalling (3.26), (3.27), (3.28) with (3.37), we obtain (3.43).

4. The main result and its proof. In this section, we are going to proof the existence of weak solution for the problem (1.1), (1.3) and (1.4). With the above compactness properties at hand, by means of a standard extraction procedure we can now derive the following theorem which actually is our main result.
According to the Aubin-Lions lemma ([9]), there exists at least one global weak solution (in the sense of Definition 1.1) of (1.1), (1.3) and (1.4).

Proof. If \((n_{1\varepsilon}, n_{2\varepsilon}, c_{\varepsilon}, u_{\varepsilon})\) is the global solution of (3.1), considering Lemma 3.6-Lemma 3.10 to see that

\[
\|n_{1\varepsilon}\|_{L^p_{\text{loc}}([0,\infty);W^{1,p}(\Omega))} \leq C(T + 1),
\]  

(4.1)

\[
\|(n_{1\varepsilon})_t\|_{L^{p'}_{\text{loc}}([0,\infty);(W^{1,p}(\Omega))')} \leq C(T + 1),
\]  

(4.2)

\[
\|n_{2\varepsilon}\|_{L^\frac{5}{2}_{\text{loc}}([0,\infty);W^{1,\frac{5}{2}}(\Omega))} \leq C(T + 1),
\]  

(4.3)

\[
\|(n_{2\varepsilon})_t\|_{L^\frac{5}{2}_{\text{loc}}([0,\infty);(W^{1,\frac{5}{2}}(\Omega))')} \leq C(T + 1),
\]  

(4.4)

and that

\[
\|c_{\varepsilon}\|_{L^4_{\text{loc}}([0,\infty);W^{1,4}(\Omega))} \leq C(T + 1),
\]  

(4.5)

\[
\|(c_{\varepsilon})_t\|_{L^\frac{10}{9}_{\text{loc}}([0,T);(W^{1,\frac{10}{9}}(\Omega))')} \leq C(T + 1),
\]  

(4.6)

\[
\|u_{\varepsilon}\|_{L^2_{\text{loc}}([0,\infty);W^{1,2}(\Omega))} \leq C(T + 1),
\]  

(4.7)

\[
\|u_{\varepsilon}\|_{L^\frac{5}{2}_{\text{loc}}([0,\infty);(W^{1,\frac{5}{2}}(\Omega))')} \leq C(T + 1).
\]  

(4.8)

According to the Aubin-Lions lemma ([9]), there exists \((\varepsilon_j)_{j\in\mathbb{N}} \in (0, 1)\) such that \(\varepsilon_j \searrow 0\) as \(j \to \infty\), and such that as \(\varepsilon = \varepsilon_j \searrow 0\), we have

\[
n_{1\varepsilon} \rightharpoonup n_1, \quad \text{in } L^p_{\text{loc}}(\Omega \times [0, \infty)) \text{ and a.e. in } \Omega \times [0, \infty),
\]  

(4.9)

\[
n_{2\varepsilon} \rightharpoonup n_2, \quad \text{in } L^\frac{5}{2}_{\text{loc}}(\Omega \times [0, \infty)) \text{ and a.e. in } \Omega \times [0, \infty),
\]  

(4.10)

\[
c_{\varepsilon} \rightharpoonup c, \quad \text{in } L^4_{\text{loc}}(\Omega \times [0, \infty)) \text{ and a.e. in } \Omega \times [0, \infty),
\]  

(4.11)

\[
u_{\varepsilon} \rightharpoonup u, \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty)) \text{ and a.e. in } \Omega \times [0, \infty).
\]  

(4.12)

Considering Lemma 3.7-Lemma 3.9, we also have

\[
n_{1\varepsilon} \rightharpoonup n_1, \quad \text{in } L^r_{\text{loc}}(\Omega \times [0, \infty)), \quad \text{for } r \in [1, \frac{4p}{3} + 3],
\]  

(4.13)

\[
\nabla n_{1\varepsilon}^{\frac{\xi - 2 + \varepsilon}{r}} \rightharpoonup \nabla n_1^{\frac{\xi - 2 + \varepsilon}{r}}, \quad \text{in } L^p_{\text{loc}}(\Omega \times [0, \infty)), \quad \text{for } \xi \in (1, 11 - \frac{12}{p}]
\]  

(4.14)

\[
n_{2\varepsilon} \rightharpoonup n_2, \quad \text{in } L^\frac{5}{2}_{\text{loc}}(\Omega \times [0, \infty)),
\]  

(4.15)

\[
\nabla n_{2\varepsilon}^\frac{\zeta}{2} \rightharpoonup \nabla n_2^\frac{\zeta}{2}, \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty)), \quad \text{for } \zeta \in (1, 5],
\]  

(4.16)

as well as

\[
c_{\varepsilon} \rightharpoonup c, \quad \text{in } L^\infty(\Omega \times [0, \infty)),
\]  

(4.17)

\[
\nabla c_{\varepsilon} \rightharpoonup \nabla c, \quad \text{in } L^4_{\text{loc}}(\Omega \times [0, \infty)),
\]  

(4.18)
\[ u_\varepsilon \to u, \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.19) \]
\[ \nabla u_\varepsilon \to \nabla u, \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.20) \]
as \( \varepsilon \searrow 0 \). Combining the Lemma 6.1 and Lemma 6.2 in [19], we further see
\[ n_{1\varepsilon} \to n_1, \quad \text{in } L^5_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.21) \]
\[ n_{2\varepsilon} \to n_2, \quad \text{in } L^5_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.22) \]
\[ c_\varepsilon u_\varepsilon \to c u, \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.23) \]
\[ n_{1\varepsilon} u_\varepsilon \to n_1 u, \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.24) \]
and
\[ n_{2\varepsilon} u_\varepsilon \to n_2 u, \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.25) \]
\[ Y_\varepsilon u_\varepsilon \otimes u_\varepsilon \to u \otimes u, \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.26) \]
\[ n_{1\varepsilon} F_\varepsilon'(n_{1\varepsilon}) \nabla c_\varepsilon \to n_1 \nabla c, \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.27) \]
\[ n_{2\varepsilon} F_\varepsilon'(n_{2\varepsilon}) \nabla c_\varepsilon \to n_2 \nabla c, \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.28) \]
as well as
\[ (\alpha F_\varepsilon(n_{1\varepsilon}) + \beta F_\varepsilon(n_{2\varepsilon})) c_\varepsilon \to (\alpha n_1 + \beta n_2) c, \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.29) \]
\[ (|\nabla n_{1\varepsilon}|^2 + \varepsilon) \frac{\varepsilon}{n_{1\varepsilon}} \nabla n_{1\varepsilon} \to |\nabla n_1|^p - 2 \nabla n_1, \quad \text{in } L^{\frac{2}{p-2}}_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.30) \]
as \( \varepsilon = \varepsilon_j \searrow 0 \). According to these convergence properties, by using the standard arguments and letting \( \varepsilon = \varepsilon_j \searrow 0 \) in each term of the natural weak formulation of (3.1) separately. Then we can verify that \((n_1, n_2, c, u)\) is a weak solution of (1.1), (1.3) and (1.4). The proof is complete. \( \Box \)

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