Kinetic structures of shear Alfvén and acoustic wave spectra in burning plasmas

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Abstract. We present a general theoretical framework for discussing the physics of low frequency fluctuation spectra of shear Alfvén and acoustic waves in toroidal plasmas of fusion interest. This framework helps identifying the relevant dynamics and, thus, interpreting experimental observations. We also discuss the roles of such general theoretical framework for verification and validation of numerical simulation codes vs. analytic predictions and experimental results.

1. Introduction

The low frequency fluctuation spectra of shear Alfvén and acoustic waves are important to burning plasmas of fusion interest, for they can be excited by both thermal as well as supra-thermal particles in various parameter regimes. There is an increasing consensus that accurate treatments of key physics processes of these spectra do require a kinetic approach [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]; although whether a fully kinetic approach is really necessary is still being debated [11].

Here, we briefly summarize recently derived analytic theories, which demonstrate the crucial importance of employing kinetic analyses for fundamental processes in collisionless plasmas of fusion interest in the low frequency regime, i.e., where geometry effects, magnetically trapped particles [9, 10], wave particle interactions and finite (kinetic) plasma compressibility play major roles. Analytic expressions show that the acoustic branch is generally more strongly damped than the shear Alfvén branch, which is the one of practical interest for interpreting experimental observations. Meanwhile, analytic calculations can be used to benchmark results of numerical codes in the relevant limits [10, 12]; thus helping their verification processes. In this work, we also discuss possible further extensions of present analytic theories, which are already underway.

2. The general fishbone-like dispersion relation

Beta induced Alfvén Eigenmodes (BAE), observed originally in DIII-D [13], are particularly important in the study of low frequency fluctuations of the shear Alfvén wave (SAW) spectrum,
since they can be excited by both fast ions (at long wavelengths) as well by thermal ions (at short wavelengths), as in the case of Alfvénic Ion Temperature Gradient driven modes (AITG) [14]. This was predicted theoretically [4] and then observed experimentally [15].

The BAE dispersion relation is yet another special case of the general fishbone-like dispersion relation (GFLDR) [1, 2, 3, 4, 5, 6], which can be written in the form

\[-i\Lambda + \delta W_f + \delta W_k = 0,\]  

(1)

where \(\delta W_f\) and \(\delta W_k\) play the role of fluid (core plasma, frequency independent) and kinetic (e.g. fast ion) contribution to the potential energy, respectively, while \(\Lambda\) represents a generalized inertia term. In ideal MHD, for frequencies much lower than the ion sound transit frequency, \(\Lambda^2 = (\omega^2/\omega_A^2)(1+2q^2)\), where \(\omega_A = v_A/(qR_0)\), \(v_A\) is the Alfvén speed and \(R_0\) is the torus major radius, while \(q\) is the safety factor. In this case, Eq. (1) reduces to the well known fishbone dispersion relation [1]. More generally, \(\Lambda\) reflects finite thermal plasma diamagnetic responses [16, 17] as well as kinetic wave-particle interactions with the thermal plasma [4, 18].

Note that Eq. (1) is applicable both for low mode-number MHD fluctuations as well as for higher mode numbers [2, 3, 4]. In fact, the GFLDR can be generally derived by asymptotic matching the regular (ideal MHD) mode structure with the general (known) form of the SAW field in the singular (inertial layer) region. Following the idea that ballooning formalism can be generally used for computing the layer response of resistive MHD modes [19], the generic expression of \(\Lambda\) [4] can be used for application to low mode-number fluctuations as well [8, 10]. Actually, a more general Mode Structure Decomposition (MSD) approach can be used [20], which is based on the Poisson summation formula and reduces to the well-known ballooning formalism in the limit where spatial scale separation applies [20]. The MSD approach naturally yields [20, 21] the twisted field line coordinates discussed in [22], which have been successfully implemented as coordinate representation in numerical simulations, e.g. [23, 24]. It also suggests a straightforward way to handle both finite as well as vanishing magnetic shear \(s = r(d\ln q/dr)\) [25]. Here, for brevity, we have written the GFLDR for \(s \neq 0\) only.

The GFLDR generally demonstrates the existence of two types of modes [5]: a discrete gap mode, or Alfvén Eigenmode (AE), for \(\Re \Lambda^2 < 0\); and an Energetic Particle continuum Mode (EPM [3]) for \(\Re \Lambda^2 > 0\). Meanwhile, the SAW continuous spectrum is given by

\[\Lambda^2 = kqR_0^2 = (nq - m)^2,\]  

(2)

for a fluctuation with given parallel wave vector \(k_{||}\) and poloidal/toroidal mode numbers \(m/n\). Equations (1) and (2) demonstrate that, for both AE and EPM, the SAW accumulation point, \(\Lambda = 0\), is the natural gateway through which modes are born at marginal stability [6]. Meanwhile, the GFLDR consistently treats SAW continuum, for which waves with local support \(\Delta r\) decay by phase mixing on a time scale \(\propto (\Delta r\partial \tilde{\omega}_A(r)/\partial r)^{-1}\) [26], where \(\tilde{\omega}_A(r)\) is generally to be intended as the solution of Eq. (2). As a consequence of the wide range of scale lengths on which SAW are excited [4, 14, 15], AE form a dense population of eigenmodes (lighthouses) with unique (equilibrium-dependent) frequencies and locations [6], nearby positions where \(\partial \tilde{\omega}_A(r)/\partial r = 0\), and with characteristic properties depending on the peculiar equilibrium that is being considered via \(\delta W_f\) and \(\delta W_k\). Note that \(\delta W_f\) and \(\delta W_k\) can be computed numerically, using different types of codes and models and, thus, even realistic mode structures. So, even if the GFLDR is given in the form of an energy functional, its validity is very general and, thus, it can be used as a generic theoretical framework, which helps identifying the underlying dynamics and nature of the mode, as well as guidance for interpreting experimental observations. In conclusion, the GFLDR provides the unified theoretical framework for discussing kinetic structures of SAW and acoustic wave spectra in burning plasmas [5, 6], and can be used for a variety of MHD modes as well [19]. Here, for brevity, we will discuss the general fluctuation features described by Eqs. (1) and (2), which can be deduced from the properties of the generalized inertia term \(\Lambda\).
3. BAE dispersion relation

In order to calculate the generalized inertia term $\Lambda$ using kinetic theory, we follow [4] and consider finite mode numbers $[8, 10, 21]$, in order to be able to use Eqs. (1) and (2) for all mode numbers $(m, n)$. We also consider mode structures that may have a kinetic singular (inertial) layer located away from a mode rational surface, so that $k || q R_0 = (n q - m)$ is generally non vanishing. This allows us to derive $\Lambda$ expressions that apply also for $s = 0$, although Eq. (1) in such a case needs to be modified [25, 27], while Eq. (2) remains unaltered.

For BAE one typically has $\omega \approx \omega_{ti} \approx \omega_{sp} \approx k || v_A$, where $\omega_{ti} = (2T_i/m_i)^{1/2}/q R_0$ is the thermal ion transit frequency and $\omega_{sp} = \omega_{st} + \omega_{Ti}$ is the thermal ion diamagnetic frequency, separated as density and temperature gradient components, $\omega_{st} = (T_i/c)(k \times b) \cdot \nabla (n_i)/n_i$ and $\omega_{Ti} = (T_i/c)(k \times b) \cdot \nabla (T_i)/T_i$, with $b = B/B$. Thus, $k || q R_0 \approx \beta^{1/2}$, with $\beta$ the ratio of kinetic to magnetic pressures. In the case analyzed here, we have $k || = k ||_0 + \Delta k ||$, with $k ||_0 = (n q - m)/(q R_0)$ computed at the singular layer and $\Delta k || q R_0 \approx \beta^{1/2}$.

As in [4], the problem of coupled SAW and acoustic fluctuations is defined by the coupled quasi-neutrality and vorticity equations, i.e.,

\[
(1 + \tau^{-1}) (\delta \phi - \delta \psi) = (T_i/ne) \langle \delta K_i \rangle ,
\]

\[
Bb \cdot \nabla \left[ \frac{k^2}{B k^2_B} \nabla \delta \psi \right] + \frac{\omega^2}{v_A} \left( 1 - \frac{\omega_{sp}}{\omega} \right) \frac{k^2}{k^2_0} \delta \phi + \frac{\alpha}{q^2 R^2} g(\theta) \delta \psi = \left\langle \frac{4n e}{k^2_0 c^2} \omega_{d0} \delta K_i \right\rangle .
\]

Here, $\tau = T_e/T_i$, angular brackets denote velocity space integration and the scalar fields $\delta \phi$ and $\delta \psi$ are related to the vector potential fluctuation $\delta A ||$ and parallel electric field $\delta E ||$ by $\delta A || \equiv -ic/\omega \cdot \nabla \delta \psi$ and $\delta E || \equiv -b \cdot \nabla (\delta \phi - \delta \psi)$. The fluctuating parallel magnetic field $\delta B ||$ is given by the condition of perpendicular pressure balance [5]. Meanwhile, for the sake of simplicity, we have assumed massless electrons and one thermal ion species with unit electric charge $e$. The drift-kinetic thermal ion distribution function response is decomposed as

\[
\delta f_i = \left( \frac{e}{m_i} \right) \left[ \frac{\partial F_{0i}}{\partial \phi} \delta \phi - \frac{Q F_{0i}}{\omega} \delta \psi \right] + \delta K_i ,
\]

where $F_{0i}$ is the equilibrium distribution function, $E = v^2/2$ the energy per unit mass, $Q F_{0i} = \omega \delta \phi + \omega_{st} F_{0i}$, $\omega_{st} F_{0i} = m_i c \delta \phi/G \nabla F_{0i}$. We also adopted straight field line toroidal flux coordinates $(r, \theta, \zeta)$ [20, 23, 24], as in the MSD approach, assuming a simple $(s, \alpha = -R_0 m c/\beta \delta R)$ model equilibrium with shifted circular magnetic surfaces, for which the perpendicular wave vector is $k^2 = k^2_0 [1 + (s \theta - \alpha \sin \theta)^2]$ and the magnetic drift frequency $\omega_{d0} = g(\theta) k_0 m_i c (v^2_R/2 + v^2_t)/eBR$, $g(\theta) = \cos \theta + |s \theta - \alpha \sin \theta| \sin \theta$. Finally, the drift-kinetic thermal ion compressibility response is obtained from

\[
\left[ \omega_{tr} \partial_\theta - i (\omega - \omega_{d0}) \right] \delta K_i = i \left( e/\omega \right) Q F_{0i} [ (\delta \phi - \delta \psi) + (\omega_d/\omega) \delta \psi ] ,
\]

where $\omega_{tr} = \nu_///q R$ is the transit frequency. The kinetic expression of the generalized inertia term $\Lambda$ is obtained from the formal manipulation of the layer Eqs. (3), (4) and (6) via asymptotic expansion in $\beta^{1/2}$ [4, 14, 30]. From the lowest order solution of Eq. (6), substituted back into Eq. (3), we have the ideal MHD response $\delta \psi^{(0)} = \delta \phi^{(0)}$ for the mode numbers $m/n$.

At the next order, $\delta \phi^{(1)} = e^{i \Gamma} \delta \phi^{(1+)} + e^{-i \Gamma} \delta \phi^{(1-)}$ with a corresponding $\delta K_i^{(1)} = e^{i \Gamma} \delta K_i^{(1+)} + e^{-i \Gamma} \delta K_i^{(1-)}$. The $\delta \psi^{(1)}$ component is negligible, as it is readily verified from Eq. (4). Let $\omega_{tr}^{(\pm)} = [1 \pm (n q - m)] (\nu_///q R_0)$. Then

\[
i \left[ \pm \omega_{tr}^{(\pm)} - \omega \right] \delta K_i^{(1 \pm)} = i \left( e/\omega \right) Q F_{0i}^{(\mp)} \delta \phi^{(1 \pm)} + i \left( e/\omega \right) Q F_{0i}^{(t)} \frac{v^2_t/2 + v^2_\perp}{R_0 \omega c_i} \frac{k \omega_{d0}}{2} .
\]

Equation (1) needs also to be modified for $k ||$ values satisfying the Bragg reflection condition $[3, 28, 29]$. 

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Note that the dominant effect comes from the geodesic curvature (large $k_r \propto \theta$), causing radial magnetic drifts. Sideband generation is evident in the wave-particle interaction as well as in the diamagnetic effect in $QF_{\text{bi}}^{(\pm)}$, which is computed at $m \mp 1$ [8, 10]. Substituting back into the quasi-neutrality condition and letting $\omega_{ti}^{(\pm)} = (2T_i/m_i)^{1/2} [1 \pm (nq - m)]/qR_0$, one obtains

$$\delta \phi^{(\pm)} = \mp i \frac{eT_i}{eB_0} \frac{N_m(\omega/\omega_{ti}^{(\pm)})}{D_{m \mp 1}(\omega/\omega_{ti}^{(\pm)})} \frac{k_r}{\omega R_0} \delta \phi^{(0)}. \tag{8}$$

Here, $Z(x) = \pi^{-1/2} \int_0^\infty e^{-y^2}/(y - x)dy$ and subscripts $m$ and $\mp 1$ indicate poloidal mode numbers to compute the functions $N_m$ and $D_m$ [4]

$$N_m(x) = \left( 1 - \frac{\omega_{ni}}{\omega} \right) \left[ x + (1/2 + x^2) Z(x) \right] - \frac{\omega_{ni}}{\omega} \left[ x (1/2 + x^2) + (1/4 + x^4) Z(x) \right], \tag{9}$$

$$D_m(x) = \left( 1 + \frac{1}{\pi} \right) \left[ 1 - \frac{\omega_{ni}}{\omega} \right] (x + (1 - \omega_{ni}/\omega) Z(x) - \frac{\omega_{ni}}{\omega} \left[ x + (2x^2 - 1/2) Z(x) \right]. \tag{10}$$

Corresponding solutions of the drift-kinetic equation are

$$\delta K_1^{(\pm)} = \mp i \frac{eT_i}{eB_0} \frac{m_i}{\omega} \left[ \frac{m_i}{2T_i} \left( \frac{v_i^2}{2} + u_i^2 \right) \right] QF_{\text{bi}} - \frac{N_m(\omega/\omega_{ti}^{(\pm)})}{D_{m \mp 1}(\omega/\omega_{ti}^{(\pm)})} QF_{\text{bi}}^{(\pm)} \frac{k_r}{\omega R_0} \delta \phi^{(0)}. \tag{11}$$

When substituted back into the second order vorticity equation for the singular layer (large $|\theta|$), reducing to the canonical form yielding the GFLDR [4, 14, 30],

$$(\partial_{\theta}^2 + \Lambda^2) \delta \psi^{(0)} = 0, \tag{12}$$

these results give the following expression of $\Lambda^2$ [8, 10], which accounts for charge separation effects due to geodesic curvature, but neglects higher order terms due to finite Larmor radius (FLR) and finite magnetic drift orbit widths (FOW) [30], which will be considered elsewhere (see final section). Accounting for FLR and FOW introduces finer scales in the problem and modifies the structure of the vorticity equation in the singular layer [30], similar to considering small but finite electron inertia [31]; the GFLDR is readily recovered from such treatments in the appropriate (long) wavelength limit [7, 30, 31].

$$\Lambda^2 = \frac{\omega_0^2}{\omega_A^2} - \frac{\omega_{ni}}{\omega} \left( 1 - \frac{\omega_{ni}}{\omega} \right) \left( \frac{\omega}{\omega_A} (\omega/\omega_{ti}^{(\mp)}) F(\omega/\omega_{ti}^{(\mp)}) + (\omega/\omega_{ti}^{(\pm)}) F(\omega/\omega_{ti}^{(\pm)}) \right)$$

$$- \frac{\omega_{ni}}{\omega} \left( \frac{\omega}{\omega_A} (\omega/\omega_{ti}^{(\mp)}) G(\omega/\omega_{ti}^{(\mp)}) + (\omega/\omega_{ti}^{(\pm)}) G(\omega/\omega_{ti}^{(\pm)}) \right)$$

$$- \left( \frac{\omega}{\omega_A} (\omega/\omega_{ti}^{(\mp)}) N_m(\omega/\omega_{ti}^{(\mp)}) \frac{N_{m-1}(\omega/\omega_{ti}^{(\pm)})}{D_{m-1}(\omega/\omega_{ti}^{(\pm)})} + (\omega/\omega_{ti}^{(\pm)}) N_m(\omega/\omega_{ti}^{(\pm)}) \frac{N_{m+1}(\omega/\omega_{ti}^{(\mp)})}{D_{m+1}(\omega/\omega_{ti}^{(\mp)})} \right). \tag{13}$$

Here, the functions $F(x)$ and $G(x)$ are defined as [4]

$$F(x) = x (x^2 + 3/2) + (x^4 + x^2 + 1/2) Z(x),$$

$$G(x) = x (x^4 + x^2 + 2) + (x^5 + x^4/2 + x^2 + 3/4) Z(x). \tag{14}$$

Details of diamagnetic sideband generation are unnecessary for drift Alfvén waves (DAW) at short wavelength; they are needed for moderate wavelength MHD-type modes. The importance of these effects, which are missing in Ref. [11], has been lately emphasized in Refs. [8, 10].

The structures of the low-frequency SAW continuum can be studied with the given $\Lambda^2$ expression and solving Eq. (2). Such analyses, however, must be limited to exploring the frequency range above the thermal ion bounce frequency, since the expression of $\Lambda^2$, given above, does not describe trapped particle dynamics (cf. later discussions). The importance (dominance at low frequency) of trapped particle dynamics has been recently discussed both analytically [9] and numerically [10].
4. Acoustic wave couplings

The notion of Beta induced Alfvén Acoustic Eigenmode (BAAE) was originally formulated [32, 33] on the basis of a fluid approach, which may not be valid in collisionless plasmas of fusion interest. This approach was extended later on to kinetic analyses accounting for circulating thermal particle effects [11]. A simple pictorial view of the BAAE gap formation is given in Fig. 1 of Ref. [11]. The BAAE gap is formed essentially because of the coupling of two (sideband, \(m \pm 1/n\)) acoustic waves both among themselves as well as with the reference (flute-like, \(m/n\)) SAW. That figure, however, neglects crucially important kinetic effects, which completely alter that picture, since Landau damping essentially rules out the possibility of a significant plasma response to the acoustic and Alfvén acoustic polarizations.

In order to show this, we give a simple derivation of the kinetic BAAE response, following the analogous procedure used for BAE in the previous section. For Alfvénic polarization, the dominant component is flute-like and satisfies \(\delta \psi^{(0)} = \delta \phi^{(0)}\) (ideal Ohm’s law), as shown above. For acoustic wave coupling to become important, as in the BAAE picture, we must as well consider an \(O(1)\) sideband \(\delta \phi_s = e^{i\theta} \delta \phi^{(+)} + e^{-i\theta} \delta \phi^{(-)}\); i.e., the sideband does not enter as first order modulation as for BAE/KBM (kinetic ballooning mode [17]). Note that the \(\delta \psi_s\) sideband component is negligible at low-\(\beta\), due to the vorticity equation constraints.

At the lowest order of the \(\beta^{1/2}\) asymptotic expansion, solution of the quasi-neutrality condition implies \(\delta \psi^{(0)} = \delta \phi^{(0)}\) (as for the BAE/KBM problem) and

\[
D_{m \pm 1}(\omega/\omega_{ti}^{(\pm)})\delta \phi^{(\pm)} = 0 .
\]

So, no connection is provided at the lowest order between \(\delta \phi^{(\pm)}\) and \(\delta \phi^{(0)}\). This result shows that, at the lowest order, the “true” acoustic mode (no SAW coupling) is merely the usual (sideband) electrostatic drift wave (e.s. DW), given by \(D_{m \pm 1}(\omega/\omega_{ti}^{(\pm)}) = 0\), which is known to be generally affected by strong Landau damping unless, e.g., \(\tau \gg 1\).

At the next \(O(\beta^{1/2})\) order, the solution of the quasi-neutrality condition gives back Eq. (8) with \(\delta \phi^{(1\pm)}\) replaced by \(\delta \phi^{(\pm)}\), i.e. the same link between sidebands and flute-like component that was written for the BAE/KBM case. The only difference is that, in the present case, we may generally have \(\delta \phi^{(\pm)} \approx \delta \phi^{(0)}\) for \(D_{m \pm 1}(\omega/\omega_{ti}^{(\pm)}) \approx \beta^{1/2}\), or even \(|\delta \phi^{(\pm)}| \gg |\delta \phi^{(0)}|\) for a purely electrostatic polarization [4]. The final solubility condition is obtained from the vorticity equation at \(O(\beta)\), which yields the same expression for \(\Lambda^2\), given above in Eq. (13).

Thus, the kinetic BAAE dispersion relation reduces trivially to the GFLDR in the limit where trapped particle dynamics are neglected, which are crucially important at low-frequencies [9, 10] (see next section on trapped particle effects). This result is not unexpected, since the first derivation of BAE/KBM kinetic responses solved the coupled vorticity and quasi-neutrality equations [4], thus completely accounting for SAW coupling to acoustic waves.

It was also noted [4] that the low-frequency kinetic SAW and acoustic fluctuation response is analytically expressed in terms of transcendental functions, which in general have an infinite number of (stable) roots. Only the least stable must be considered when computing the plasma behavior. In fact, when solving the initial value problem, the contribution of the more stable branch points and cuts becomes rapidly negligible.

The solution of Eq. (2), with \(\Lambda^2\) given by Eq. (13), is shown in Figs. 1 and 2 for both DAW (BAE) as well as e.s. DW (BAAE) branches. Due to the intrinsic difficulties in comparing experimental observations with theoretical predictions of mode frequencies (see Sec. III.D of [11]), we have chosen \(\beta_i = 0.01, \omega_{sni}/\omega_{ti} = 0.1, \omega_{ei}/\omega_{ti} = 0.2\) and \(\tau = 2\) as fixed parameters, reasonably close to experimental values, and considered fluctuations with \(m/n = 6/3\) localized near \(q = 2\). The close resemblance of the present Fig. 1 with Fig. 1 of Ref. [11] is evident for the two higher frequency branches. Lower frequency modes of [11] are not considered here, since their correct treatment requires the inclusion of trapped particle dynamics (see below).
The acoustic polarization has stronger Landau damping due to the typically lower fluctuation frequency and the stronger a.c. electric field component. Thus, most unstable modes tend to have Alfvénic polarization and a general flute-like structure. Meanwhile, the various kinetic interactions with the thermal plasma are properly described by the generalized inertia term $\Lambda^2$ and account for the different fluctuation types in the low frequency Kinetic Thermal Ion (KTI) gap [6], which are connected with different physics: e.g., diamagnetic drift for KBM [17], thermal ion compressibility for BAE [13], kinetic thermal ion compressibility and wave-particle resonances for AITG [14, 30]. Note that fluctuation frequencies below the SAW accumulation points are not indicative of the existence of BAAE; they simply show that AE/EPM can be shifted down in frequency with respect to the accumulation point, as prescribed by the GFLDR (see section 2). In addition, SAW accumulation points, computed with $\Lambda^2$ from Eq. (13), tend to overestimate the actual frequency, due to the absence of trapped particle responses [9, 10]. Plasma shaping, such as elongation, may also lower the SAW accumulation point frequency [10].

5. Trapped particle effects
Detailed descriptions of trapped particle effects [9, 10], including finite $k_\parallel q R_0$ and diamagnetic sideband generation, are beyond the scope of the present work. Here, we briefly remark two important elements. First, the low frequency ($|\omega| < \omega_{bi}$, with $\omega_{bi}$ the thermal ion bounce frequency) renormalized plasma inertia is dominated by trapped particle dynamics. Second, trapped particle compressions generally lower the BAE accumulation point frequency and are needed for a correct description of experimental observations [10]. At low frequency, the SAW dynamics in toroidal system reflects the compression effects of trapped thermal ions [34]:

$$\Lambda^2 = \left( \frac{\omega^2}{\omega_A^2} \right) \left( 1 - \frac{\omega_{spit}}{\omega} \right) \left( 1 + 1.6q^2 \epsilon^{-1/2} + 0.5q^2 \right).$$

(16)

It is possible to show that the renormalized inertia expression in Eq. (16) must be equal [27] to the zonal flow (ZF) polarizability [35]. This is a consequence of the same physics analogy argument that is used to demonstrate BAE accumulation point degeneracy with the Geodesic acoustic mode (GAM) for long wavelengths [6, 36] and has implications on cross-scale couplings mediated by zonal structures and long time-scale behaviors in burning plasmas [37, 38]. Note the difference of this expression with that for the modified Alfvén branch, “shielded” by acoustic sidebands, given in [11] when ignoring trapped particles. Besides not including trapped particle responses, the expression given in [11] fails, for it assumes isotropic thermal plasma compressibility, whereas at low-frequency the plasma behavior is strongly anisotropic. The low

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**Figure 1.** $\text{Re}(\omega/\omega_{ti})$ is plotted vs. $(nq - m)$ for DAW (BAE) and e.s. DW (BAAE).

**Figure 2.** $-\text{Im}(\omega/\omega_{ti})$ is plotted vs. $(nq - m)$ for DAW (BAE) and e.s. DW (BAAE).
frequency limit of Eq. (13), which accounts for anisotropic compressibility but not for trapped particles, correctly reproduces the $\propto (1 + 0.5q^2)$ real part of Eq. (16) [27]; however, it must be corrected for the imaginary part, in order to account for the modified circulating ion response due to finite trapped particle fraction [9]. A general expression of $\Lambda^2$, smoothly connecting that of Eq. (16) with that of Eq. (13), has been derived for the case of deeply trapped particles [9].

6. Interpretation of experimental data, conclusions and discussions

One of the key experimental evidences [15] is the observation of low frequency SAW in a wide range of mode numbers, driven by both energetic (long wavelength) as well as thermal (short wavelength) ions, confirming theoretical predictions [4, 14]. New interest was attracted by observations of BAE [32, 33] at frequencies below the BAE accumulation point of the SAW continuum, by the more recent evidence of “Sierpes modes” in ASDEX Upgrade [39], interpreted as BAE excited by energetic ions generated by Ion Cyclotron Resonance Heating (ICRH) [40], and also by measurements of ICRH driven BAE in Tore Supra [41]. For interpretation of these observations, as shown above, it is necessary to use kinetic theories for the proper treatment of wave-particle interactions with circulating as well as trapped thermal plasma particles [9, 10]. The features of the kinetic dispersion relations and the intrinsic non-linear (non additive) interplay of trapped and circulating particle dynamics are complex [9, 10] and make simplified extensions and/or renormalizations of fluid theories inappropriate [11]. Meanwhile, for sufficiently strong energetic particle drive (ions and electrons), experimental observations show evidence of a continuous transition between various SAW and MHD fluctuation branches [8, 42, 43, 44, 45], consistent with theoretical predictions [1, 2, 3, 28] based on the GFLDR.

The present theoretical framework suggests that low-frequency SAW and acoustic oscillations should be seen as either EPM or AE, localized within the Kinetic Ion Thermal (KTI) gap [6]. The notion of BAAE, which was lately emphasized [11, 32, 33], seems instead to be based on a paradigmatic approach dictated by improper analogies with fluid theories. In fact, the kinetic BAAE dispersion relation reduces trivially to the GFLDR in the limit where trapped particle dynamics are neglected, which are crucially important at low-frequencies [9, 10]. The intrinsic limits of MHD/fluid analyses at low frequency in collisionless tokamak plasmas of fusion interest suggest developing new numerical investigation tools, with aim at bridging simulations of meso- and micro-scale phenomena [6, 36, 37, 38, 46]. The linear gyrokinetic code LIGKA is capable to address these issues [10, 46]. Meanwhile, an eXtended version of the HMGC code (XHMGC [47], nonlinear hybrid MHD-gyrokinetic code), has been developed to handle kinetic thermal ion compressibility effects [48]. Successful comparisons of XHMGC numerical studies of kinetic BAE excited by energetic particles vs. analytic predictions have been already carried out [12]. Global non-perturbative (full-f) electromagnetic gyrokinetic simulations are also becoming available, such as for the GTC [49] and GYRO [50] codes. Work is ongoing to complete the present theoretical-analytical framework; e.g., extending the $\Lambda^2$ expression including (deeply) trapped particle dynamics [9] with finite $k_{\parallel}qR_0$ and diamagnetic sideband generation, as well as finite Larmor radius and finite magnetic-orbit width effects [14, 30] in the whole frequency range. With these results, the further generalization of the GFLDR at short scales [14, 30] will complete the present theoretical framework and allow us to verify local analytic theories vs. local and global numerical simulations, consolidating the basis for a complete toolbox for numerical computation and application of the GFLDR in situations of practical interest.

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