Capillary soft singularities and ejection: application to the physics of bubble bursting

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Abstract

Bubble bursting at the surface of a liquid open to a gas environment may exhibit the appearance of a spatiotemporal singularity for a specific choice of flow parameters. The flow can be described as a non-simple two-phase dipole centered at the surface of the liquid whose mechanical energy comes from the free surface energy of the initial bubble. Its topology shows both a rapid ejection of a microjet into the gas environment and the thrust of a relatively large ellipsoidal liquid blob into the liquid to ensure that the total momentum in the axial direction is always preserved. We propose a general physical model where the radius $R$ of the ejected droplet sizes and their initial speeds $V$ scale as $R/R_o \sim \left( (a^2 + \psi)^{1/2} + a \right) ^{-1} \delta$, and $V/V_o \sim \left( (a^2 + \psi)^{1/2} + a \right) \delta^{-1/2}$, where $\psi = \left( (b\varphi)^{1/4} - (b\varphi)^{-1/4} \right)^{-2}$. $a$ and $b$ are universal constants. Here, $\delta = \| \text{Oh}^* - \text{Oh} \|$ and $\varphi = \text{Oh}^{-2}\delta$, where Oh= $\mu/(\rho\sigma R_o)^{1/2}$ is the Ohnesorge number ($\rho$, $\sigma$ and $\mu$ are the liquid density, surface tension and viscosity). $R_o$ and $V_o = (\sigma/(\rho R_o))^{1/2}$ are the initial bubble radius and the reference capillary speed, respectively. The experimental and numerical results yield $\text{Oh}^* \simeq 0.038$, $a \simeq 0.15$, and $b \simeq 1.7$. These laws are modified with the Bond number as here proposed. Previously proposed models can be derived from our model by choosing different values of the parameters $\{\text{Oh}^*, a, b\}$. As an intriguing result, this model anticipates the appearance of two relatively close critical values of Oh, which would explain the striking discrepancy among previous proposed critical values of this parameter.
I. INTRODUCTION

Everyday experience teaches that radially convergent flows close to a liquid surface produce vigorous transient liquid ejections in the form of a jet perpendicular to the surface, as those seen after bubble bursting, droplet impact on a liquid pool, or cavity collapse. Under the light of recent research on those phenomena [1, 2], performing exhaustive simulations [3], and by dimensional and similarity analyses [4–6], their physics has been progressively unveiled. The conditions under which the flow produces a radial implosion and subsequent vigorous ejection hinge on an elusive temporal singularity that can be termed a “soft singularity”, somehow akin to the singularity assumed in cosmology for pristine or future universes. To this end, the necessary existence of self-similar flow regimes up to times close to the point where the surface curvature diverges at the point of collapse has been demonstrated [6–8]. The self-similar structure of the flow shows a rich topology currently under intense scrutiny.

The appearance of spatiotemporal singularities in liquid flows bound by free surfaces is a complex and ubiquitous phenomenon in nature requiring a critical condition: either before or after the singularity, or in both cases, local surface curvature should diverge as the time left before the singularity vanishes. This condition is approximately fulfilled by radially collapsing flows, producing locally axisymmetric flow geometries. Although some of the conclusions of this work may be extended to other configurations (e.g. planar flows), this is the flow geometry here considered.

We propose the distinction between two main types of axisymmetric singularities. To begin with, what could be termed as hard singularities: This type of singularities comprises the inescapable breakup and pinch-off of any liquid thread, drop, or bubble [9]. It is characterized by the obvious existence of an instant where a hard discontinuity or split of the fluid domain occurs at the point of the spatiotemporal singularity. With sufficient generality, since the local radius vanishes at the singularity, the condition of axial symmetry leads to a slender geometry that allows a drastic reduction of the problem, where the only outstanding variables are the axial coordinate and the time to the singularity, and the unknowns are the axial velocity and the local shape (or the radius of the interface). Eggers [7] masterly demonstrated that this type of singularity eventually develops self-similar flow solutions sufficiently close to the spatiotemporal singularity. Vicinity to the singularity is scaled by the natural time and length scales of the flow $t_\mu = \mu^3/(\rho\sigma^2)$ and $l_\mu = \mu^2/(\rho\sigma)$, respectively,
where \( \rho \), \( \sigma \) and \( \mu \) are the density, surface tension and viscosity of the liquid, respectively. While the flow exhibits self-similarity where inertia and surface tension dominate for times \( t \) to the singularity larger than \( t_\mu \), other self-similar regimes appear for times shorter than \( t_\mu \). When the pinch-off geometry is slender, the consistency of those self-similar solutions and the equations show that viscosity, surface tension and inertia forces should necessarily be balanced \([7, 10]\). Experiments and numerical simulations in general support these findings, although other possibilities have been proposed and tested which consider memory effects from initial conditions \([11]\).

Other more subtle types of singularities are what one may term *soft singularities*, where the liquid domain is not actually split into separate domains. Hence, strictly speaking, the flow does not develop a genuinely noticeable discontinuity anywhere in the liquid domain, except at the point of collapse where local curvature may diverge. Many transient flows that exhibit ejections are not necessarily associated with a singularity. However, a sudden burst or change in the flow pattern (i.e. the appearance of a rapid ejection) signals a behavior akin to a singularity, so it is worth investigating whether there is a true singularity parametrically close to those flow conditions.

In bubble bursting, the value of the *critical time* \( t_0 \) can be fixed as the instant when the interface develops an apparent *curvature reversal* where the axis of symmetry of the radially collapsing flow meets the interface \([4, 12–14]\). As a consequence of this reversal, the flow experiences a transient vigorous *focusing*. In general, the mechanical energy excess of the convergent flow leads to a transient liquid ejection in the form of an unsteady capillary column or jet after the critical time. In turn, the same energy excess will determine whether the issued jet may break up and eject a droplet or not. Many flow configurations with convergent velocity patterns at the vicinity of a free surface are known to develop this type of ejection. A few examples are given in figure \([1]\). A soft singularity may occur for a specific set of flow parameters such that the size of the ejection vanishes and its velocity diverges, as previously described \([13, 15]\).

A. Soft singularities: suggested meaning

In contrast with hard singularities, in general, soft singularities do not induce a discontinuity in the liquid domain; they occur under very specific values of the flow parameters
FIG. 1. Rapid liquid ejections following the release of a potential energy associated to the presence of an axisymmetric liquid free surface, with different origins: The collapse of a cavity produced (a) after a droplet impact on a deep liquid pool [16], (b) after bubble bursting on a free surface [17], or (c) outflow through a hole on a plate [18]. (d) Bubble collapse by surrounding liquid volume oscillations close to a solid surface [19]. (e) Conical collapse of a suddenly electrified liquid droplet [20]. (d) Overdriven Faraday waves [21], or the sudden vertical acceleration of a semi-enclosed cylindrical liquid volume [22, 23].

leading to a lower-dimensional manifold of the entire flow parameter domain. For example, if the flow is characterized by the Ohnesorge Oh and Bond Bo numbers, the soft singularities may appear at a specific line in the \{Oh, Bo\} space [15]. For parameter values belonging to the manifold (e.g. a line in space), the size of the issued jet goes to zero while its velocity becomes enormous. This is a unique feature of these singularities.

Convergent flows do not always exhibit self-similar solutions sufficiently close to the critical time. Yet, several studies [6, 13, 14, 24] have determined that self-similar solutions indeed develop before and after the critical time, while self-similarity is generally lost in a small spatiotemporal region around it. In particular, Zeff et al. [13] showed that for times \( t \) larger than \( t_\mu \), the 2D flow develops self-similarity where inertia and surface tension forces are balanced while viscous forces are subdominant, and the magnitudes of lengths \( l \) (including coordinates and interface shapes) and speeds \( v \) scale as \( l \sim t^{2/3} \) and \( v \sim t^{-1/3} \), respectively.

Nothing prevents the occurrence of self-similarity in the theoretical parametrical manifold
of a singularity when $t < t \mu$, though. Indeed, in the case of hard singularities, self-similarity has been extensively confirmed for $t < t \mu \ [7, 25]$. However, for soft singularities no existing studies (either experimental, numerical or theoretical) have yet reported any example where, even admitting the actual impossibility of being in the manifold of singularity, the flow could continue exhibiting self-similarity for $t < t \mu$. One reason could be the extreme narrowness of a domain whose time and length scales are smaller than $t \mu$ and $l \mu$, not easily accessible by experimental means.

First, an extreme care and accuracy is required to assess the flow parameters and initial conditions. For example, Walls et al. \[15\] and Ghabache et al. \[17\] report exquisite experimental explorations of the parameter space for surface bubble bursting, while failing to show examples of these extreme situations simply because experimental error and measuring instruments do not allow it: $t \mu$ is extremely small for the usual low viscosity liquids like water and liquid metals (about 0.2 nanoseconds for water and about 1 femtosecond for mercury at ambient temperatures), and on the other hand, the flow should develop length scales comparable or smaller than $l \mu$ (about 14 and 0.3 nm for water and mercury, respectively) when $t$ becomes smaller than $t \mu$. In many cases, those scales openly prevent direct observation with currently available instruments. Thus, a blunt fundamental question may be posed: is self-similarity really possible for $t < t \mu$, as in the case of the more mundane hard singularities? Its importance comes, as we will see, from the link between the existence of self-similarity and that of a strict singularity. The first is the only way to demonstrate the second for these flow configurations where the liquid domain is not divided.

In summary, soft singularities could be thought of as situations where the flow circumvents a strict spatiotemporal singularity somewhere in the liquid domain, resulting in a local ejection of a vanishing amount of matter at a divergent velocity but not leading to the breakup of the liquid domain. This happens when the parameter controlling the flow belong to a zero-volume (singular) manifold of the parametrical space. The natural questions are: (i) do these strict singularities really exist for precise combinations of parameters and initial conditions?; and (ii) do radially convergent axisymmetric flows in the presence of a free surface always circumvent a strict singularity? In other, simpler words, can we really talk about soft singularities at all? These questions are akin to the ones in astronomy on the possibility that soft singularities would have taken place or will occur at pristine or future universes or the existence of a black hole at the center of every galaxy. The enormous interest
of the answers to these questions lies in the fact that if a given geometry or flow configuration can develop a soft singularity, it can develop extremely small scales and ejecta (e.g. [18]). Thus, developing extremely accurate liquid deposition technologies taking benefit from this knowledge is a sheer matter of precision, skills and ingenuity.

Hence, due to their immense scientific and technologic interest and potential (for example, many everyday inkjet printers use this kind of flows), this work is partially devoted to the analysis of soft singularities, their existence, and the scaling laws of ejections produced around them. The study is here particularized for bubble bursting since this has been a widely studied fundamental problem, and a lot of experimental and numerical data are available for testing proposals.

B. Previous approaches

The nearness to a soft singularity is signaled by the flow parameters [3, 15]: for given initial conditions and geometries, there are certain values of the parameters for which, theoretically, one may approach the singularity as much as desired, and experiments seem to support this assumption [4, 17, 24, 26]. This would suggest that the parameter space should exhibit manifolds or subspaces where the values of the parameters would lead to the occurrence of strict spatiotemporal singularities. A clear example of this can be seen in Walls et al. [15], where the soft singularity would be achieved for parameter values in the line \( \psi(Oh,Bo) = 0 \) (or manifold) of the parameter space \{Oh,Bo\} (a plane) for bubble bursting, where \( Oh= \mu/(\rho\sigma R_o) \) and \( Bo= \rho g R_o^2/\sigma \) are the Ohnesorge and Bond numbers, and \( R_o \) is the radius of the sphere whose volume equals the initial bubble volume. Although the exact points in the manifold might not be physically attainable, the possible nature and topology of the flow in its vicinity suggested by experiments, and the scaling laws governing the characteristics of ejecta have been the subject of recent studies [4–6, 24, 27], with some degree of deviation. However, they seem to agree in the existence of flow self-similarity and its temporal power laws close to the critical time [6, 13, 27]. Self-similarity is lost sufficiently close to the critical time, except -presumably- under a parametrical combination where the soft singularity occurs.

In summary, while hard singularities exhibit self-similarity as close to the critical time as desired regardless of the position within the parameter space, because the spatiotemporal
singularity is always reached, the nature and structure of flows displaying soft singularities, and even the mere existence of these singularities are still a matter of intense debate. In bubble bursting, the authors have not agreed yet in the values of the critical parameters \[3–6, 15, 24, 27\].

C. Overview from dimensional analysis

Consider a perfectly axisymmetric convergent flow of a Newtonian liquid with a free surface in the presence of a dynamically inactive environment. Now, by hypothesis, consider that the flow parameters and initial conditions lead to a strict singularity. Then, at certain location the surface should exhibit a maximum curvature that increases in time without limit, around which the flow develops the most relevant values of the (dimensional) velocity field \(v\). Let us call \(h_{\text{min}}\) the inverse of that maximum curvature. At this point, we will not yet consider the particular case of a quasi-cylindrical collapsing flow [28]. Due to the 2D nature of the radially convergent flow, that minimum length \(h_{\text{min}}\) cannot scale differently from either the radial \(r\) or axial \(z\) coordinates centered around the point of maximum curvature since all those lengths should be comparable in that region. Thus, one should have two functions \(f\) and \(f_v\) such that

\[
f(h_{\text{min}}, \Delta t, \rho, \sigma, \mu) = 0, \quad f_v(v, x, \Delta t, \rho, \sigma, \mu) = 0
\] (1)

where \(x\) is the coordinate vector, and \(\Delta t\) is the time to the singularity. From those functions, one could generically choose the natural length and time scales \(l_\mu\) and \(t_\mu\) to write

\[
h_{\text{min}}/l_\mu = \chi_{\text{min}} [\Delta t/t_\mu], \quad v/v_\mu = \nu [x/l_\mu, \Delta t/t_\mu]
\] (2)

where \(v_\mu = \sigma \mu^{-1}\) is the natural scale of velocity, and \(\chi_{\text{min}}\) and \(\nu\) represent non dimensional scalar and vectorial functions of the indicated variables, respectively. In the search for time self-similarity, one should have

\[
\chi_{\text{min}} = (\Delta t/t_\mu)^\beta \xi_{\text{min}}, \quad \nu = (\Delta t/t_\mu)^\alpha u [\xi],\n\] (3)

where \(\xi = (\Delta t/t_\mu)^{-\beta} (x/l_\mu)\) is the self-similar coordinate vector, and \(\xi_{\text{min}}\) and \(u\) would be a constant and a function of \(\xi = x/(\Delta t/t_\mu)^\beta\), respectively, depending on the particular flow configuration. Self similarity demands that neither \(u\) nor \(\xi\) should depend on time. One
can analyze the three possible choices of \( \{\alpha, \beta\} \) for which either \( \mu, \sigma \) or \( \rho \) can be absent in expressions (3).

The first choice, with \( \mu \) absent, corresponds to \( \beta = 2/3 \) and \( \alpha = -1/3 \). It would indicate a subdominant role of viscosity, leading to the scales of length and velocity:

\[
l_1 = l_\mu (\Delta t/t_\mu)^{\beta} = \left(\frac{\sigma \Delta t^2}{\rho}\right)^{1/3} \quad \text{and} \quad v_1 = v_\mu (\Delta t/t_\mu)^{\alpha} = \left(\frac{\sigma}{\rho \Delta t}\right)^{1/3},
\]

supporting the self-similarity laws:

\[
h_{\text{min},1} = \left(\frac{\sigma \Delta t^2}{\rho}\right)^{1/3} \xi_{\text{min},1}, \quad v_1 = \left(\frac{\sigma}{\rho \Delta t}\right)^{1/3} u_1.
\] (4)

The second choice, with \( \sigma \) (\( \beta = 1/2 \) and \( \alpha = -1/2 \)) absent and surface tension force subdominant, yields:

\[
l_2 = \left(\frac{\mu \Delta t}{\rho}\right)^{1/2} \quad \text{and} \quad v_2 = \left(\frac{\mu}{\rho \Delta t}\right)^{1/2},
\]

with the corresponding self-similar laws:

\[
h_{\text{min},2} = \left(\frac{\mu \Delta t}{\rho}\right)^{1/2} \xi_{\text{min},2}, \quad v_2 = \left(\frac{\mu}{\rho \Delta t}\right)^{1/2} u_2.
\] (5)

The third choice, with \( \rho \) absent (\( \beta = 1 \) and \( \alpha = 0 \)), is physically irrelevant since inertia would be subdominant. Here, one would have:

\[
h_{\text{min},3} = \left(\frac{\sigma \Delta t}{\mu}\right)^{1/3} \xi_{\text{min},3} \quad \text{and} \quad v_3 = v_\mu u_3,
\]

which in general would become negligible once the self-similar solutions \( v_1 \) or \( v_2 \) take over as \( \Delta t \) decreases.

Considering the relative values in (4) and (5) as \( \Delta t \) is large (or small) compared to \( t_\mu \), one should conclude that the natural scales \( t_\mu \) and \( l_\mu \) provide a threshold above (or below) which the solutions \( h_{\text{min},1} \) (or \( h_{\text{min},2} \)) and \( v_1 \) (or \( v_2 \)) would prevail due to their larger \((2/3 > 1/2)\) and less negative \((-1/3 > -1/2)\) powers of \( \Delta t \), respectively.

Thus, while in hard singularities the occurrence of a local quasi-cylindrical flow is the norm, soft singularities may either exhibit (i) two subsequent self-similar regimes close to the critical time, or (ii) the eventual development of a local, asymptotically quasi-cylindrical flow as in hard singularities. For the latter, see the excellent analysis of the quasi-cylindrical collapse of a gas cavity by Eggers et al. In the two cases, a very specific flow configuration should develop close to the critical time. However, we anticipate that the second possibility is precisely what happens when a tiny bubble is engulfed in the bubble bursting process close to the critical time (see figure in section where Oh=0.032). Indeed, we will show that the radial implosion of momentum driven by a steep collapsing capillary wave leads to the surface overturning and the engulfment of a bubble at the axis (a similar problem with gravity waves was considered in ). When its wavelength is very small, the radial collapse of the walls of this tiny capillary wave opens the possibility
of a locally quasi one-dimensional radial collapsing flow where inertia, viscous and surface tension forces are present up to the time when curvature diverges \[10\]. This would sustain the existence of true soft singularities that cannot develop if one neglects viscous forces, as observations demonstrated in bubble bursting when one approaches a critical Ohnesorge number \[15\].

In reality, viscosity is effectively the bridge that allows the flow to approach a self-similar regime towards the singularity. This does not mean that jetting cannot occur neglecting viscosity along the entire process, naturally. However, in this latter case self-similar solutions degenerate close to the critical time \(t_0\). This would lead to a singularity wash out or smoothing. In the following, we show the critical importance of viscosity in the appearance of soft singularities through the analysis of bubble bursting and the physics and scaling laws of the ejection, namely the size and speed of ejected droplets.

### II. BUBBLE BURSTING. SCALING LAWS AND THE APPEARANCE OF SOFT SINGULARITIES

Consider a gas bubble initially tangent to the free surface of a liquid with density, surface tension and viscosity \(\rho\), \(\sigma\) and \(\mu\), respectively, in static equilibrium. At a certain instant, the infinitesimal film at the point of tangency breaks and the process of bubble bursting starts. The liquid rim that is initially formed pilots a wave front that advances along the bubble surface towards its bottom. When this wave front approaches the bottom, it becomes steep. The wave front steepness is such that the flow becomes predominantly radial in a region with characteristic length \(L\). One can distinguish three phases of the flow development around the critical time \(t_0\) (see figure 2) where the surface collapses (i.e., it develops a point of infinite curvature): (a) When \(t < t_0\), the surface collapses due to the predominantly radial speed \(U\). (b) Right after the surface collapse, the kinetic energy excess is diverted in the axial direction producing a liquid spout with speed \(V\) and characteristic radial length \(R\), while the main flow keeps running radially with speed \(U\). (c) The advancing front of the resulting capillary jet eventually pinches off in the form of a droplet or droplet train scaling as \(R\).

In [4] and [5] we formulated a set of relations among the radial and axial characteristic lengths and velocities using dimensional analysis based on the arguments that all forces per
FIG. 2. General overview of the three phases of the local flow development around the critical time \( t_0 \) and the point of collapse, for \( \text{Oh} = 0.032 \): (a) surface collapse \((t < t_0)\): the flow is predominantly radial, with speed \( U \), (b) and (c): finger ejection and droplet formation \((t > t_0)\), with the development of a vigorous axial velocity \( V \). Distances are scaled with the original bubble radius \( R_o \). Note the smallness of the region depicted (around 10% of the bubble size). Thin discontinuous lines represent the interface shape at time intervals \( \delta t = 5 \times 10^{-5} \) (scaled with the capillary time \( t_c = \left( \frac{\rho R_o^3}{\sigma} \right)^{1/2} \)). At the critical time, a nearly conical shape develops. The entrapment of a tiny bubble can be observed. \( L \) is the characteristic length scale of the region where the collapse develops (large speeds), and \( R \) is the characteristic size of the ejection. \( U \) and \( V \) denote the scales of the radial and axial velocities. Unless otherwise specified, in all numerical simulations in the present work, the gas-liquid ratios of density and viscosity are 0.001 and 0.02, respectively.

A unit volume in the liquid domain should be comparable very close to the instant of collapse of the free surface. While the previously proposed relations were fundamentally consistent, a more rigorous derivation of those relations is here offered.

**A. An integral relations-based dimensional analysis**

The momentum equation of the liquid can be written as:

\[
\rho \mathbf{v}_t + \nabla \left( p - P_a + \rho \mathbf{v}^2/2 + \rho g z \right) = \mathbf{v} \wedge \nabla \wedge \mathbf{v} + \mu \nabla^2 \mathbf{v}.
\]  

where \( \mathbf{v} \) is the velocity vector, subindex \( t \) denotes partial derivative with time, \( p \) is the liquid pressure, \( z \) the axial coordinate, \( \mathbf{n} \) the unit normal on the liquid surface, and \( P_a \) the gas
pressure. Although numerical simulations using Basilisk to obtain figures 2 and 3 take into account the gas motion, the gas has a density and viscosity are much smaller than those of the liquid (see figure 2). Thus, the dynamical effects are assumed negligible for the purpose of the following analysis.

Equation (6) can be multiplied by the unit vector $l$ tangent to any instantaneous streamline, in particular the streamline flowing through a point $A$ where it meets the free surface to a point $B$ at the vicinity of the point of collapse (see figure 3(a)), and integrated with respect to the streamline coordinate $s$ from $A$ to $B$, yielding:

$$
\rho \int_A^B l \cdot v_t ds + \sigma \nabla \cdot n|_B + \rho v^2/2|_B + \rho g \Delta z|_A^B = \mu \int_A^B l \cdot \nabla^2 v ds
$$

(7)

since the velocity is negligible at $A$, and pressure is $P_a$. $\Delta z$ is the depth of point $B$ respect to $A$. As a general consideration, the liquid velocities are very small everywhere compared to the velocity at distances $L$ to the collapsing region, which may exhibit a self-similar flow structure [3, 6, 13]. The length scale $L$ also characterizes the inverse of the mean local curvature of the liquid surface around the region of collapse, for any given time $t$. Thus, $L$ obviously changes with time around the instant of collapse. Let us consider two situations, one for $t < t_0$ and the other for $t > t_0$ such that their characteristic length scales $L$ are the same (see figure 2). Then, one may estimate the characteristic values of each term of equation (7) in these two cases:

$t < t_0$: Considering the flow structure shown in figure 3(b), both the left integral in (7) and the kinetic energy term at $B$ should scale as $\rho W^2$. The surface tension term at $B$ should be proportional to $\sigma/L$, and the gravity term to $\rho g R_o$. Finally, an inspection of the configuration of streamlines at the surface, where the velocity is predominantly in the normal direction, suggests that the viscous stresses should be predominantly extensional, and the integral should scale as $\mu W/L$. Thus, one has an overall balance among the different terms that can be formulated as:

$$
\rho \left( W^2 + \beta_1 g R_o \right) \sim \sigma L^{-1} + \alpha_1 \mu W L^{-1},
$$

(8)

where $\alpha_1$ and $\beta_1$ should be universal constants assuming that a global self-similarity can be found for this flow configuration. Using $R_o$ and $V_o = (\sigma/\rho R_o)^{1/2}$ nondimensionalize length and speed, one arrives to

$$
w^2 + \beta_1 Bo \sim (1 + \alpha_1 Oh \, w) x^{-1},
$$

(9)
where $x = L/R_o$, and $w = W/V_o$. This equation can also be written in terms of variables normalized with the natural length and velocity, $l_\mu = \mu^2/(\rho \sigma)$ and $v_\mu = \sigma/\mu$ respectively, such that $\zeta = L/l_\mu = x \text{Oh}^{-2}$ and $\omega = W/v_\mu = w\text{Oh}$, as:

$$\omega^2 + \epsilon_1 \sim (1 + \alpha_1 \omega)^{-1},$$

where $\epsilon_1 = \beta_1 \text{Oh}^2 \text{Bo} \ll 1$.

FIG. 3. (a) Geometry of the flow streamlines, similar to dipole contours at a liquid-gas surface, for $\text{Oh} = 0.032$ and $\text{Bo} = 0$. At the top, the streamlines for $t_0 - t = 3.54 \times 10^{-3} t_c$; in the center, those for $t_0 - t = 8.4 \times 10^{-6} t_c$, and at the bottom, those for $t - t_0 = 4 \times 10^{-4} t_c$. The figure shows that the streamline function does not change appreciably as $t_0$ is approached. (b) Close-up view of the flow streamlines for $t_0 - t = 3.54 \times 10^{-3} t_c$: the flow is roughly spherical. The characteristic length scale $L$, and the approximate position of point B where the integration streamline in (7) is located, are indicated.

$t > t_0$: Once the critical time is passed, either after the entrapment of a bubble and the radial pinch-off of a gas neck, or after the bottom surface curvature reverses (e.g. [3, 12, 14, 15, 30, 32]), a vigorous ejection in the axial direction with characteristic velocity $V$ takes place. While the scale of the radial velocity $W$ remains the same
through the collapse and beyond, a large velocity in the axial direction develops over
the point where the surface collapses at $t = t_0$ (see central panel in figure 4 for $Oh =
0.032$, $Bo = 0$), leading to the ejection of a liquid jet into the gas environment, with
characteristic axial and transversal length scales $L$ and $R$, respectively. Strikingly,
even before collapse, the flow develops a stagnant region in the liquid just below the
collapsing point which leads to an ellipsoidal liquid blob growing at the same pace
as $L^3$. While the upper point of this liquid ellipsoid remains stagnant, coinciding in
space with the point where the surface collapsed at $t = t_0$ (observe the horizontal
line connecting the three panels of figure 4), its center of mass moves downstream
against the axial flow stream coming from below. This fact embraces the fundamental
physics of the phenomenon: the total momentum in the axial direction should globally
remain the same (or, equivalently, the total mechanical energy should remain constant)
through the collapse and ejection process. It means that the momentum of the vigorous
ejection in the form of a microjet should be mirrored by that of the slow moving liquid
blob in the opposite direction. Writing this in terms of length and velocity scales, one
should have:

\[ \rho V^2 R^2 L \sim \rho W^2 L^3 \implies V R \sim W L \implies u r \sim w x \quad (11) \]

where $u = V/V_o$ and $r = R/R_o$. This is the sheer Newton’s third law of motion, which
takes place locally at the point of collapse, with the proper geometrical adjustments
to account for the global liquid motion: if a mass rate is ejected in one direction
with certain total mechanical energy, an equivalent injection in the opposite direction
should take place. Yet, it is also exactly what is demanded by mass continuity, as
equation (11) shows with stunning consistency. Due to the disparate values of density
in both liquid and gas domains in this problem, the flow does not develop a symmetric
jet into the liquid domain, but a very slow and large blob whose mechanical energy is
equivalent to that of the jet. As a secondary consequence, the nearly conical surface
raises due to the incoming nearly conical flow, but this raise is much slower than that
of the jet: this kinematics is exactly what happens in the explosion of a conically
shaped charge, which creates a strongly perforating jet of fire whose nature is purely
kinetic. Another important observation is that the jet is fed primarily by an axial
stream coming from below, which surrounds the ellipsoidal liquid blow (see figure 4
right panel): if one observes the flow pattern below the jet, it is almost strictly axial
in the upper direction from the stagnation point at the upper surface of the blob. Thus, the flow idealization discussed by Gordillo and Rodríguez-Rodríguez [27], who assumed that the flow was fundamentally cylindrical (radial) below the jet, proposing a simplified kinematics (a line of sinks below the jet) to eventually explain its raise, is starkly inconsistent with present results. In addition, one can observe in figure 4 that the size of the trapped microbubble is much smaller than that of the liquid blob, and consequently its global dynamical effect should be negligible once the ejection is initiated. With this physics in mind, one can now easily estimate the scaling of the different terms of (7) for a streamline ending at a point B at the surface of the jet (see figure 4) and write the following balance:

$$\rho \left( V^2 + \beta_2 g R_o \right) \sim \alpha_2 \sigma R^{-1} + \mu W L^{-1},$$

(12)

which using non-dimensional variables reads:

$$u^2 + \beta_2 Bo \sim \alpha_2 r^{-1} + Oh w x^{-1},$$

(13)

where $\alpha_2$ and $\beta_2$ are expected to be, again, universal constants. Again, in terms of variables normalized with the natural scales, one obtains:

$$v^2 + \epsilon_2 \sim \alpha_2 \chi^{-1} + \omega \zeta^{-1},$$

(14)

where $v = V / v_\mu = u \text{Oh}$, $\chi = R / l_\mu = r \text{Oh}^{-2}$, and $\epsilon_2 = \beta_2 \text{Oh}^2 \text{Bo} \ll 1$. Here, the choice of the terms affected by $\alpha_{1,2}$ is not whimsical since those terms are expected to be of secondary importance in relevant parametrical ranges: indeed, making $\alpha_{1,2} = 0$ and resolving $\chi, \zeta$ and $\omega$ as functions of $v$ with $\epsilon_{1,2} = 0$, one obtains $\chi \sim v^{-5/3}, \zeta \sim v^{-4/3}$, and $\omega \sim v^{2/3}$, exactly as predicted in [4], where a good agreement with experimental results was found.

Unfortunately, like it was observed in the simpler argument presented in [4], while the three equations (9)-(13) or their equivalent forms using natural scales provide interesting relationships among $\{U, V, R, L\}$, they do not close the problem, neither any of these equations can be tested against experimental measurements. But still, one can establish a useful relationship integrating the total energy equation in a sufficiently ample fluid volume $\Omega(t)$ around the initial bubble from the instant of bubble bursting $t = 0$ up to the point $t_e$ when
FIG. 4. Oh = 0.032 and Bo = 0: the geometry of the surface and topology of the streamlines for $t_0 - t = 8.4 \times 10^{-6} t_c$ (left panel), $t_0 - t = 10^{-6} t_c$ (center), and $t - t_0 = 4 \times 10^{-4} t_c$. The characteristic radial and axial lengths $R$ and $L$, and velocities $W$ and $V$ are indicated.

the jet issues a first droplet. To do so, the fluid volume $\Omega(t)$ should be initially as large as for example a hemisphere with a radius about twice or three times larger than $R_o$: 

$$
\int_{\Omega(t)} \rho \left( e + \frac{v^2}{2} + gz \right) d\Omega \bigg|_{t=t_e}^{t=t_0} = - \int_{t=0}^{t=t_e} \int_{S(t)} \mathbf{v} \cdot (\mathbf{\tau}' - p \mathbf{I}) \cdot \mathbf{n} \, dA \, dt \quad \text{(15)}
$$

where $\mathbf{\tau}'$ is the viscous stress and $\mathbf{I}$ the identity matrix. An estimation of the overall value of the left term for the whole volume proportional to $R_o^3$ would necessarily entail averaging the liquid speeds in that volume. Let us call $V_{R_o}$ the scale of those velocities. Then, the left term would be proportional to $\rho V_{R_o}^2 R_o^3$. On the other hand, considering the values of the Ohnesorge number in this problem for all cases where droplets are ejected, published literature clearly establishes that Oh should necessarily be smaller than 0.05 (Oh = 0 for the inviscid case). Consequently, neglecting viscous effects and assuming that the liquid speeds vanish as $\sim 1/r^2$ at distances $r$ from the bottom of the bubble larger than $R_o$, the right hand side should be proportional to $\frac{\pi}{R_o} R_o^2 V_{R_o} t_e$, where $t_e$ should necessarily be proportional
to $R_o/V_{R_o}$. Thus, one would have:

$$\rho V_{R_o}^2 R_o^3 \sim \sigma R_o^2 \implies V_{R_o} \sim \left(\frac{\sigma}{\rho R_o}\right)^{1/2} \equiv V_o$$  \hspace{1cm} \text{(16)}$$

This result is naturally consistent with the fact that the main velocities induced by the bursting should be proportional to the capillary ones corresponding to a length comparable to $R_o$. However, it does not produce any additional useful information to close the problem.

Now, considering terms of the order $\text{Oh} \ll 1$ in equation (16), the second order term of the left hand side should involve the axial velocity $V$, the largest one in the liquid domain, which appear in a very small portion of that domain with volume $R^2 R_o \ll R_o^3$. Therefore, retaining the small gravitational total energy as well, of the order $\rho g R_o^4$, the small parts of the left term of (16) previously neglected should be comparable to $\rho (V^2 R^2 + \beta_3 g R_o^3) R_o$. Furthermore, expecting that the flow is nearly radial at the internal side of the liquid surface closing $\Omega(t)$ in the liquid and that the very small, likely extensional viscous stresses should be proportional to $\frac{\mu V_o}{R_o}$, the viscous term in the right side of equation (15) would be proportional to $\frac{\mu V_o}{R_o} R_o V_{R_o} t_e \sim \mu V_o R_o^2$. On the other hand, the mean surface stresses at the liquid free surface should be comparable to a small fraction of the total surface energy $\sigma R_o^2$. That small fraction should be a small universal constant which we may call $\text{Oh}^*$ for convenience (an obvious choice for the informed reader). Hence, one finally has:

$$\rho \left( V^2 R^2 + \beta_3 g R_o^3 \right) R_o \sim \text{Oh}^* \sigma R_o^2 - \mu V_o R_o^2$$  \hspace{1cm} \text{(17)}$$

The negative sign affecting the last term is consistent with the expectation that the extensional viscous stresses nearly everywhere at the inner surface of the fluid domain should point in the same direction as the velocities (consistently with velocities increasing as $1/r^2$ for decreasing distances $r$), while the unit normal points in the opposite direction. Dividing by $\sigma R_o^2$, one obtains

$$\left( V^2 R^2 + \beta_3 g R_o^3 \right) V_o^{-2} R_o^{-2} \sim \text{Oh}^* - \text{Oh}$$  \hspace{1cm} \text{(18)}$$

This equation is equivalent to the energy equation closing the problem in [4]. Yet, the right hand side of (17) can be negative, while ejection is still observed experimentally up to a certain limit value of Oh. It is thus plausible that the global contributions of the corresponding surface tension and viscous terms reverse as the parameters go beyond $\text{Oh}^*$, and therefore one could define $\delta = \|\text{Oh}^* - \text{Oh}\|$ and write:

$$u^2 r^2 + \beta_3 \text{Bo} \sim \delta$$  \hspace{1cm} \text{(19)}$$
Summarizing equations (9), (11), (13) and (19) using variables \( \{u, r, w, x\} \), one reaches to the following system of algebraic equations:

\[
\begin{align*}
ur &\sim wx \\
w^2 + \beta_1 \text{Bo} &\sim (1 + \alpha_1 \text{Oh} w)x^{-1} \\
u^2 + \beta_2 \text{Bo} &\sim \alpha_2 r^{-1} + \text{Oh} w x^{-1} \\
u^2 r^2 &\sim \text{Oh}^* - \text{Oh} - \beta_3 \text{Bo},
\end{align*}
\]

The six universal constants \( \{\text{Oh}^*, \alpha_{i=1,2}, \beta_{i=1,2,3}\} \) will be easily obtained from experiments. Indeed, defining and assuming that \( \delta \) should always be a positive number, the fourth equation in system (20) can be verified against published measurements from experiments and simulations as shown in figure 5, since both \( R \) and \( V \) can be experimentally determined. Even though a large experimental and numerical errors can be expected as the Oh number approaches the critical value suggested by the model proposed, an excellent fit is obtained with \( \text{Oh}^* = 0.038 \ll 1 \) (the same value as in [5]) and \( \beta_3 \simeq 5 \).

FIG. 5. Experimental measurements of the product of ejected droplet radii and their velocities from [2, 3] and numerical simulations here reported. The prediction \( u^2 r^2 + \beta_3 \text{Bo} \sim \delta \equiv ||\text{Oh}^* - \text{Oh}|| \) is plotted as a black diashed line \( (u^2 r^2 + \beta_3 \text{Bo} = k_\delta \delta, \text{where the best fit is for } k_\delta = 5/\text{Oh}^* \text{ with } \text{Oh} = 0.038) \).
The alternative form of system (20) using variables with natural scales is:

\[ u \chi \sim \omega \zeta \]
\[ \omega^2 + \epsilon_1 \sim (1 + \alpha_1 \omega) \zeta^{-1} \]
\[ v^2 + \epsilon_2 \sim \alpha_2 \chi^{-1} + \omega \zeta^{-1} \]
\[ v^2 \chi^2 \sim (\delta - \beta_3 \text{Bo}) \text{Oh}^{-2}, \quad (21) \]

Under the initial ansatz that \( \beta_{i=1,2,3} \) are small numbers, the resolution of system (21) yields the following meaningful linearized solution for \( \chi \) and \( v \) around \( \beta_{i=1,2,3} = 0 \) among the four possible solutions (three of which yield complex numbers and thus are physically meaningless):

\[ \chi = k_r \varphi \left( (a^2 + \psi)^{1/2} + a \right)^{-1} (1 + \epsilon_r \text{Bo}), \quad (22) \]
\[ v = k_v \varphi^{-1/2} \left( (a^2 + \psi)^{1/2} + a \right) (1 + \epsilon_v \text{Bo}), \quad (23) \]
\[ \zeta = k_z \psi^{1/2}, \quad (24) \]
\[ \omega = k_w \psi^{1/2}, \quad (25) \]

where \( \psi = \left( (b \varphi)^{1/4} - (b \varphi)^{-1/4} \right)^{-2} \) and \( \varphi = \text{Oh}^{-2} \delta \). The algebraic relations among the small fitting constants \( \epsilon_1 \) and \( \epsilon_2 \) with \( \epsilon_r \) and \( \epsilon_v \) in (25) are not relevant here since the latter (which both scale with Bo) are the ones experimentally fitted.

Note that \( \varphi \) is exactly the same variable as that used in [4], except for the generalization taking the absolute value of \( \|\text{Oh}^* - \text{Oh}\| \). Also note the beautiful symmetry of the variable \( \psi \) (always positive, too). Here again, the constants \( \{a, b, k_r, k_v, \epsilon_r, \epsilon_v\} \) are algebraically related to \( \{\alpha_{i=1,2}, \beta_{i=1,2,3}\} \), but the former are the relevant ones since those are the ones fitted to experiments. Moreover, when \( \alpha_{i=1,2} = 0 \), one obtains \( \chi = k_r \varphi^{5/4} \) and \( v = k_v \varphi^{-3/4} \), exactly as the solutions proposed in [4]: a proof of consistency which is added to the study in [6] verifying the validity and robustness of those originally proposed solutions.

In summary, solutions (25) support the existence of singularities at least when \( \psi \to 0 \), which in the case of collapsing flows as bubble bursting would at least theoretically support an affirmative answer to the question initially formulated in the introduction on the existence of local spatiotemporal singularities which (i) do not imply a discontinuity of the liquid domain, and (ii) exhibit a local divergence of velocities with vanishing local scales.
B. Experimental verification

To verify solutions (25), six hundred measurements of first ejected droplets and about one hundred of their corresponding initial velocities (a laudable experimental task, also achievable by numerical simulation) are plotted in figure 6(a) and (b).

To enhance potential deviations that may result imperceptible when the same parameters appear in both axes with powers 1 or larger, which yields an apparently better visual correlation (for example, plotting $R/l_\mu$ as a function of Oh or its combinations), we plot $r = R/R_o$ and $u = V/V_o$ instead of $\chi = R/l_\mu = r\text{Oh}^{-2}$ and $\omega = V/v_\mu = u\text{Oh}$. Those measurements are plotted against $\varphi_r = \varphi \left( (a^2 + \psi)^{1/2} + a \right)^{-1} (1 + \epsilon_r \text{Bo})$ and $\varphi_v = \varphi^{-1/2} \left( (a^2 + \psi)^{1/2} + a \right) (1 + \epsilon_v \text{Bo})$. Dashed lines are the theoretical predictions assuming constant liquid properties $\{\rho, \sigma, \mu\}$. Here, the fitting parameters are $\text{Oh}^* = 0.038$, $a = 0.15$, $b = 1.7$, with $\epsilon_r = 0.25$ and $\epsilon_v = 3.5$. Although this latter value is not small in the case of ejected velocities, the excellent fitting found provides assurance on the proposed linearized model expressions (25), in particular for the droplet diameters.

C. Multiplicity of critical Ohnesorge numbers and soft singularities

The striking lack of agreement among the different authors concerning the specification of the critical Oh has already been noticed in recent publications [4, 5, 14, 15, 24, 27]. Even the mere existence of any singularity has been questioned [24, 27]: a minimum finite value of the emitted droplet size based on the role of viscous forces on the development of the liquid jet is proposed in [24]. The same is proposed for the velocity of ejections in [27]. However, figure 4 of [24] supports a theoretical vanishing of droplet sizes, while figure 12 of [3] and even figure 8(b) of [27] clearly suggest the possibility of divergence of velocities, in the range of Oh between 0.03 and 0.04.

A key reason for this discrepancy could be explained by our proposed model. Interestingly, observing solutions (25) for $\chi$ and $\upsilon$, one may have two different values of the Ohnesorge number for which $R$ vanishes while $V$ diverges. One may have this concurrence in the following situations:

1. When $\psi \to \infty$ (i.e., when $b\varphi \to 1$). This implies $\text{Oh} = \text{Oh}^*$. 

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FIG. 6. Experimental measurements of (a) first ejected droplet size, and (b) initial velocity of ejected droplets. Dashed lines represent the theoretically predicted scaling laws.

2. When $\phi \to 0$. This yields $\text{Oh} = \frac{b}{2} \left( (1 + \frac{4\text{Oh}^*}{b})^{1/2} - 1 \right)$. 

Given the smallness of $\text{Oh}^*$, the second expression can be approximated as

$$\frac{b}{2} \left( (1 + \frac{4\text{Oh}^*}{b})^{1/2} - 1 \right) \simeq \text{Oh}^* \left( 1 - \frac{\text{Oh}^*}{b} \right). \quad (26)$$

This striking result suggests that, even in our linearized approximation assuming a small role of gravitational effects, a duality of critical points can be obtained in the parametrical domain of the flow. A multiplicity of critical values, and therefore a multiplicity of soft singularities can be expected when gravitational effects without linearizing the solution to system (21) are included and augmenting the complexity of the phenomenon including surface viscosity [33], Marangoni and non-Newtonian effects. In addition, the possible effect of the outer gas environment on the critical values should also be considered. This is a subject of subsequent studies.
Appendix A: Other proposals and criticisms

While other authors [6, 14, 24] partially or completely supported and extended previous results [4, 5], Gordillo and Rodríguez-Rodríguez departed significantly from that trend, proposing a very different model that cannot be (at least easily) derived from solution (25):

\[ r \sim 1 - \left( \frac{Oh}{Oh_c} \right)^{1/2} \]  
(A1)

\[ u \sim \begin{cases} 
\left( 1 - \left( \frac{Oh}{Oh_c} \right)^{1/2} \right)^{-1/2} & \text{for } Oh \approx Oh_c \\
Oh^{1/2} \left( 1 - \left( \frac{Oh}{Oh_c} \right)^{1/2} \right)^{-3/2} & \text{for } Oh \ll Oh_c
\end{cases} \]  
(A2)

This model is tested against experiments in figures 7, where Oh\(_c\) = 0.02 in that work (it has been recently corrected by those authors to Oh\(_c\) = 0.03). Here, \( \varphi_{GRR} = 1 - \left( \frac{Oh}{Oh_c} \right)^{1/2} \), \( \varphi_{GRV1} = \left( 1 - \left( \frac{Oh}{Oh_c} \right)^{1/2} \right)^{-1/2} \), and \( \varphi_{GRV2} = Oh^{1/2} \left( 1 - \left( \frac{Oh}{Oh_c} \right)^{1/2} \right)^{-3/2} \).

While the model proposed in [27] reasonably predicts droplet sizes for Oh \( \ll Oh_c \), following a similar trend as our model (25), it fails for Oh \( \approx Oh_c \) or larger (indeed, it cannot predict anything above Oh\(_c\)), and is completely inconsistent predicting ejected droplet velocities.

1. Other inconsistencies of criticisms

Our previous results [4, 5] were questioned in [27] arguing that those were inconsistent with numerical results. According to [27], the inconsistency lies on the assumption that the high-speed jet emerges as a consequence of viscous shear stress. This statement cannot be sustained by the explicit indications given in both [4, 5], which were further discussed in extenso in [5] (see in particular expression (9) from that work), and supported in [6]. The authors of [27] went on stating that those previous results were inconsistent with boundary layer theory (should that be applicable to this problem), and that the critical Oh numbers reported were not in accord with published results. The unquestionable consistency with experiments shown in both [4] and [5] for both droplet sizes and speeds, and the subsequent
FIG. 7. Experimental measurements of (a) first ejected droplet size, and (b) initial velocity of ejected droplets compared to the theoretically predicted scaling laws (shown as dashed lines) in [27].

Support from other works [6, 14] show Gordillo’s statements to be inaccurate. Moreover, the consistency of our previous models and results is reinforced under the light of the more rigorous present derivation and the enhanced consistency of present extended model with physical principles and experiments (either numerical or experimental).

One of the most unquestionable tests that the proposed model of Gordillo and Rodríguez-Rodríguez fails to fulfil (see figure 5) is the comparison with the product \( v^2 r^2 \), which our model predicts satisfactorily. Another test is to compare the models with the product of the experimentally measured values of the product \( \chi^{3/5} v \), which is predicted in [4] based on arguments criticized in [27]. This is shown in figure 8. Gañán-Calvo [4] predicted \( \chi^{3/5} v = \text{const.} \) for \( \text{Bo} \ll 1 \), whose remarkable agreement with the experiments is slightly improved by present model. The model of Gordillo and Rodríguez-Rodríguez, for both \( \text{Oh} \sim \text{Oh}_{c2} \) and \( \text{Oh} \ll 1 \) aimed to criticize and supposedly improve predictions in [4, 5], is also shown.
FIG. 8. Experimental measurements of the product $\chi^{3/5}v$, corrected for non small Bo numbers as suggested in [14]. The theoretically predicted scaling laws in [4, 14, 27], and present model are shown for comparison. Predictions in [5] are indistinguishable from present model in the range of the plot.

In any case though, the observation made in [27] about the discrepancy in the critical values of Oh leaves an interesting open question that is addressed in the main text.

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