The development of inner product spaces and its generalization: a survey

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Abstract. An inner product space is a vector space with an additional structure called the inner product. This additional structure associates each vector pair in space with a scalar quantity known as the product. This paper will discuss a survey related to the development of the inner product space and its generalization. These generalization include semi-inner product space, sesquilinear space, indefinite inner product space, and bilinear space.

1. Introduction

One of the topics in algebra that is often used as research material or is involved in research, especially in the field of algebra, is inner product space. Some of the research carried out is still in the scope of pure algebra, some are in the applied scope either in mathematics or algebra itself or in other areas of research. The research carried out related to inner product space is also very diverse and has experienced so many developments. Even the inner product space itself has undergone much expansion. This paper discusses a survey regarding developments in the inner product space and its generalizations.

An inner product allows the introduction of geometric ideas such as the length of a vector or the angle between two vectors. The concept of inner product space also provides a way of defining the orthogonality between vectors. Let $P$ be a vector space over field $L$ (real or complex). A function $\langle -, - \rangle$ from $P \times P$ to $L$ is an inner product on $P$ if the function satisfy these conditions: (i) for any $p_1$ in $P$, $\langle p_1, p_1 \rangle \geq 0$ and also the condition $\langle p_1, p_1 \rangle = 0$ applies if and only if $p_1 = 0$; (ii) (a) for the field of complex numbers, for all $p_1, p_2 \in P$, $\langle p_1, p_2 \rangle = \bar{\langle p_2, p_1 \rangle}$, (b) for the field of real numbers, for all $p_1, p_2 \in P$, $\langle p_1, p_2 \rangle = \langle p_2, p_1 \rangle$; (iii) for all $p_1, p_2, p_3 \in P$ and scalar $\alpha, \beta$, this condition apply $\langle \alpha p_1 + \beta p_2, p_3 \rangle = \alpha \langle p_1, p_3 \rangle + \beta \langle p_2, p_3 \rangle$ [1]. A real (or complex) vector space $P$, together with an inner product, is called a real (or complex) inner product space. The norm or length of an element $p_1$ in $P$ is defined by $\|p_1\| = \sqrt{\langle p_1, p_1 \rangle}$ [1]. There is a very close relationship between the inner product concept and the length concept of a vector or what is often called a norm. One of them can be seen from a very popular lidentities, that is polarization identities. For a real vector space $P$, the polarization identitites is defined by $\langle p_1, p_2 \rangle = \frac{1}{4} \left( \|p_1 + p_2\|^2 - \|p_1 - p_2\|^2 \right)$, for any $p_1, p_2$ in $P$. On the other hand, For a complex vector space $P$, the polarization identitites is defined by $\langle p_1, p_2 \rangle = \frac{1}{4} \left( \|p_1 + p_2\| - \|p_1 - p_2\|^2 \right) + \frac{1}{4} i \left( \|p_1 + ip_2\|^2 - \|p_1 - ip_2\|^2 \right)$ for any $p_1, p_2$ in $P$[1].

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An inner product allows us to define the concept of the perpendicularity between two vectors, or commonly known as the concept of orthogonality. Suppose $P$ be an inner product space. Any two vectors in $P$ are refer as mutually orthogonal if the inner product between the two vectors is 0, or it can be written as $p_1 \perp p_2$ ($p_1$ orthogonal to $p_2$) if $\langle p_1, p_2 \rangle = 0$, for all $p_1, p_2 \in P$.

One of the things that are currently being researched, especially concerning the inner product space, is the linear preserver problem. Informally, the linear preserver problem is a problem that focuses on linear mappings that preserve certain properties [2]. One research that is quite active in its development is research related to mapping that preserves orthogonality. One of the results of research that is quite popular and has become the basis for the development of research related to the problem of preserving orthogonality is a study conducted by Chmielinski in 2005. Regarding the issue of orthogonality, two mappings are said to preserve orthogonality or orthogonality preserving if for any two vectors that mutually orthogonal in the domain then the image of that two vectors also mutually orthogonal di codomain.

Furthermore, two mappings are said to be strongly orthogonality preserving if the condition applies in both directions. Formally, the definition of the two things is explained as follows. Let $P$ and $Q$ be two inner product spaces. A mapping $\Gamma$ from $P$ to $Q$ is said orthogonality preserving, if for all $p_1, p_2$ in $P$ that satisfy $p_1 \perp p_2$ then $\Gamma(p_1) \perp \Gamma(p_2)$, and called strongly orthogonality preserving if for all $p_1, p_2$ two elements in $P$, $p_1 \perp p_2$ if and only if $\Gamma(p_1) \perp \Gamma(p_2)$ [3]. In his paper, Chmielinsky [5] suggested the linkages between orthogonality preserving mapping with an isometry. Let $(P, \langle -,- \rangle_P)$ and $(Q, \langle -,- \rangle_Q)$ be two inner product spaces, and let $\Gamma : P \rightarrow Q$ be a linear mapping. $\Gamma$ is an isometry if $\langle \Gamma(p_1), \Gamma(p_2) \rangle_Q = \langle p_1, p_2 \rangle_P$ for all $p_1, p_2 \in P$ [4]. Chmielinsky proofed that a linear map that has the property of orthogonality preserving indeed is a scalar multiple with an isometry. He characterizes a mapping that has the nature of preserving the orthogonality, especially inner product spaces. Some of his characterizations are illustrated in the theorem below.

**Theorem 1.1.** [5] Let $P$ and $Q$ be two inner product spaces over the same field $L$. For a non zero mapping $\Gamma : P \rightarrow Q$ the following conditions are equivalent:

(i) A mapping $\Gamma$ is linear and there’s exist $\gamma > 0$ for all $p \in P : \|\Gamma(p)\| = \gamma \|p\|;$

(ii) There’s $\gamma > 0$ for all $p_1, p_2$ in $P$ where $\langle \Gamma(p_1), \Gamma(p_2) \rangle = \langle p_1, p_2 \rangle;$

(iii) A mapping $\Gamma$ is linear and strongly orthogonality preserving;

(iv) A mapping $\Gamma$ is linear and orthogonality preserving.

Chmielinski also studied operators that preserved orthogonality, both those that preserved orthogonality in general, and which preserved strong orthogonality. As mentioned above, an operator is said to maintain strong orthogonality if every two members in the mutual domain are orthogonal if and only if the map results of the two members are also orthogonal. Chmielinski argues that the operators that preserve orthogonality need not be linear. More specifically, the main study of his paper is the linear operator that preserves $\varepsilon$-orthogonality or it is also called the approximate orthogonality. An operator is said to preserve the approximate orthogonality if each member that is orthogonal in the domain, the mapping results are orthogonal approximations for a certain value $\varepsilon$. He also showed that the operator preserving approximate orthogonality does not need to meet the usual orthogonality requirements. Chmielinski notes that the class of operator preserving the orthogonality (or approximate orthogonality) can be generalized to a space larger than the inner product space. The paper also describes several open problems or other research that can be carried out by examining several forms of orthogonality in normed space (semi-inner product space). The research conducted by Chmielinski uses operators in inner product spaces (real and complex). Although the main focus in his research is the linear operator that preserves the approximate orthogonality, there are characterization results of the
linear operator that preserve the ordinary orthogonality. Many researchers have developed this characterization.

2. Method
In general, the methodology used is an exploration method of existing results through literature studies. In particular, what was used as the basis for the survey was the existence of research topics that were actively developed in the field of algebra related to the linear preserver problem. More specifically, there are the results of the Chmielinsky [5] which are still being developed in various directions today. However, this is not a limitation in the survey regarding the development of inner product space and its generalizations. Chmiliensky also departed from the long-standing and quite popular results presented by Kohler and Rosenthal in 1970. They obtained results whose aim was to simplify the isometric characterization of normed spaces in the form of semi-inner products. The definition of the semi inner product will be explained in the next section. The results are described below.

Theorem 2.1. [6] Let \( M \) be a normed linear space (over real or complex field) also let \( R \) be a mapping from \( M \) to itself. Then \( R \) is an isometry if and only if there exists a semi inner product \([-,-]\) such that \([Rm_1,Rm_2]=[m_1,m_2]\) for any \( m_1 \) and \( m_2 \) in \( M \).

3. Result and Discussion
3.1. Sesquilinear Forms
In this section, it will explained the sesquilinear space. Wojcik [3] defines a sesquilinear form of a normed space as follows. Let \( P \) be a normed space, a function \( \tau : P \times P \rightarrow L \) is a sesquilinear form if for any \( p_1,p_2,p_3 \in P \) and \( \gamma,\delta \in L \), satisfy the following conditions: (i) \( \tau(\gamma p_1 + \delta p_2, p_3) = \gamma \cdot \tau(p_1,p_3) + \delta \cdot \tau(p_2,p_3) \), (ii) \( \tau(p_1, \gamma p_2 + \delta p_3) = \overline{\gamma} \cdot \tau(p_1,p_2) + \overline{\delta} \cdot \tau(p_1,p_3) \).

The concept of boundedness in the sesquilinear form is also defined similarly to the standard boundedness concept, that is, a sesquilinear form is said to be bounded if it has an upper and lower bound. Formally, it is written as follows, if there exist a constant \( K \) such that \( |\tau(p_1,p_2)| \leq K \|p_1\| \cdot \|p_2\| \) for all \( p_1, p_2 \in P \), then the sesquilinear form is bounded. A bound for \( \tau \) is the constant \( K \). Furthermore, if \( \tau(p_1,p_1) \neq 0 \) if \( p_1 \in P \) then the sesquilinear form is nontrivial.

Some basic properties such as the law of parallelograms and polarization identities that apply in the inner product space also apply to this sesquilinear space. Wojcik [3] has made developments related to the results of the research that has been conducted by Chmielinsky [5]. He developed part of the result of [5] for the sesquilinear form. The results are as follows.

Theorem 3.1. [3] Let \( P \) and \( Q \) be two normed spaces over the field \( L \). Let \( \tau_a : P \times P \rightarrow L, \tau_b : Q \times Q \rightarrow L \), be two nontrivial bounded sesquilinear forms. For a non zero linear map \( E \) from \( P \) to \( Q \), the following conditions are equivalent:

(i) for all \( p_1,p_2 \) in \( P \), if \( \tau_a(p_1,p_2) = 0 \) then \( \tau_b(Ep_1,Ep_2) = 0 \);

(ii) there exists \( \delta \) that is non zero for all \( p_1,p_2 \) in \( P \),such that \( \tau_b(Ep_1,Ep_2) = \delta \cdot \tau_a(p_1,p_2) \).

Furthermore, Wojcik views that the theorem above can be strengthened. Based on the research results, \( A \) does not need to be assumed to be continuous and \( \tau_a \) and \( \tau_b \) also need not be assumed to be bounded. Thus the theorem can be generalized as follows.

Theorem 3.2. [3] Let \( P \) and \( Q \) be two normed spaces over the field \( L \). Let \( \tau_a : P \times P \rightarrow L, \tau_b : Q \times Q \rightarrow L \), be two sesquilinear forms. For a nonzero linear mapping \( E \) from \( P \) to \( Q \), the following conditions are equivalent:

(i) for all \( p_1,p_2 \) in \( P \), if \( \tau_a(p_1,p_2) = 0 \) then \( \tau_b(Ep_1,Ep_2) = 0 \);

(ii) there exists \( \delta \) that is non zero for all \( p_1,p_2 \) in \( P \),such that \( \tau_b(Ep_1,Ep_2) = \delta \cdot \tau_a(p_1,p_2) \).
3.2. Indefinite Inner Product Spaces

This part will discuss the further development of the inner product space, namely the indefinite inner product. Broadly speaking, the indefinite inner product is the development of the inner product by removing the positive definite property. Formally, Gohberg (2000) defines indefinite inner product as follow. Let $S^n$ be the $n$-dimensional complex Hilbert space include of all column vectors $r = (r_1, r_2, ..., r_n)$. A function $[-, -]: S^n \times S^n \Rightarrow S$, where $S$ is the set of all complex numbers, is indefinite inner product in $S^n$ if the following conditions apply: (i) $[\gamma r_1 + \delta r_2, r_3] = \gamma [r_1, r_3] + \delta [r_2, r_3]$, for all $r_1, r_2, r_3 \in S^n$ and $\gamma, \delta \in S$; (ii) $[r_1, r_2] = [r_2, r_1]$, for all $r_1, r_2 \in S^n$; (iii) Let $r_1 \in S^n$, if $[r_1, r_2] = 0$ for all $r_2 \in S^n$, then $r_1 = 0$ [7]. A vector spaces equipped with indefinite inner product is called indefinite inner product space. Indefinite inner product space has some of the same properties as it is in an inner product space. One of them is orthogonality, that is, two vectors is mutually orthogonal in an indefinite product space if its indefinite inner product is 0.

**Theorem 3.3.** [7] Let $S^n$ be an indefinite inner product space. For all $r_1, r_2 \in S^n$ the following property apply

$$[r_1, r_2] = \frac{1}{4} \{ [r_1 + r_2, r_1 + r_2] + i [r_1 + ir_2, r_1 + ir_2] - [r_1 - r_2, r_1 - r_2] - i [r_1 - ir_2, r_1 - ir_2] \} .$$

Sararei et al (2019) conducted one of the most recent studies related to indeterminate product space. Sararei et al. Investigated the orthogonality in the Krein space. Krein space is an example of an indefinite inner product space. An indefinite inner product space $(Q, [-, -])$ is a krein space if $Q$ can be written as the orthogonal direct sum of $Q = Q^+ \oplus Q^-$, where $(Q^+, [-, -])$ and $(Q^-, [-, -])$ are Hilbert Space [8]. Sararei also carried out a research development carried out by Chmielinsky [5] for the Krein space. But what Sararei did was only the development of some of the results obtained by Chmielinsky, not all of them. These results include the following.

**Theorem 3.4.** [9] Let $(Q_1, [\cdot, \cdot]_1)$, and $(Q_2, [\cdot, \cdot]_2)$ be two real Krein spaces, and let $\Gamma: Q_1 \rightarrow Q_2$ be a nonzero linear mapping such that $\Gamma(Q_1^+) = Q_2^+$ dan $\Gamma(Q_1^-) = Q_2^-$. Then $\Gamma$ preserves orthogonality, if and only if there exists $\delta > 0$ such that $\|\Gamma q_1\| = \delta \|q_1\|$, for each $q_1 \in Q_1$.

Apart from that, sararei also introduces four types of orthogonality from the perspective of the Krein space and shows that the usual orthogonality is equivalent to the four new types of orthogonality. The four orthogonality is as follows. Let $(Q, [\cdot, \cdot])$ be a Krein space, let $J$ be the canonical symmetry operator associated with $Q$ and let $q_1, q_2 \in Q$. We say that

(i) $q_1 \perp J q_2$ if $[q_1, q_2] = 0$;
(ii) $q_1 \perp_{PK} q_2$ if $\|q_1^+ + q_2^+\|^2 - \|q_1^- + q_2^-\|^2 = \|q_1^+\|^2 - \|q_1^-\|^2 + \|q_2^+\|^2 - \|q_2^-\|^2$;
(iii) $q_1 \perp_{BK} q_2$ if $\|q_1 + \lambda J q_2\| \geq \|q_1\|$ for all $\lambda \in \mathbb{C}$;
(iv) $q_1 \perp_{IK} q_2$ if $\|q_1 + J q_2\| = \|q_1 - J q_2\|$;
(v) $q_1 \perp_{WK} q_2$ if $\|q_1^+\| \|q_2^+\| + \|q_1^-\| \|q_2^-\| = \|q_1^+\|^2 \|q_2^-\|^2 - \|q_1^-\|^2 \|q_2^+\|^2$;

and the four types of orthogonality is equivalent to standard orthogonality.

**Theorem 3.5.** [9] Let $(Q, [\cdot, \cdot])$ be a real Krein space, and let $J$ be the canonical symmetry operator associated with $Q$. The following statements are equivalent.

(i) $q_1 \perp J q_2$
(ii) $q_1 \perp_{PK} q_2$
(iii) $q_1 \perp_{BK} q_2$
(iv) $q_1 \perp_{IK} q_2$
(v) $q_1 \perp_{WK} q_2$
Other most recent studies conducted is research conducted by Saltenberger (2020). In his paper, he examined the indefinite inner product associated with a structured matrix [10]. He departed from many uses of structured matrices in various fields, including engineering, physics, and statistics [11], optimal control problems, and analysis of mechanical and electrical vibrations [12], as well as computational analysis and the theory of matrix functions itself [13] or the matrix equation [7].

3.3. Semi-Iinner Product Spaces
This section will discuss regarding the concept of a semi-inner product space introduced by G. Lumer in 1961 [14] and its main properties introduced by J.R. Giles [15]. Let $M$ together with $\|\cdot\|$ be a normed space over field $L$ (real or complex). The mapping $[-,-]$ from $M \times M$ to $L$ called semi-inner product if the following conditions apply: (i) $[m_1 + m_2, m_3] = [m_1, m_3] + [m_2, m_3]$, for all $m_1, m_2, m_3 \in M$; (ii) $[\delta m_1, m_2] = \delta [m_1, m_2]$, for all $m_1, m_2 \in M$, and a scalar $\delta \in L$; (iii)$[m_1, \delta m_2] = \delta [m_1, m_2]$, for all $m_1, m_2 \in M$, and a scalar $\delta \in L$; (iv) $[m_1, m_2] \leq \|m_1\|\|m_2\|$, for all $m_1, m_2 \in M$; (v) $[m_1, m_1] = \|m_1\|^2$, for all $m_1 \in M [16]$. Mappings that satisfy the above properties are called semi inner product (s.i.p) in $M$ (generate $\|\cdot\|$). The space $M$ that is associated with an inner semi product is called the inner semi product space.

An element $m_1 \in M$ is said to be Giles-orthogonal over the element $m_2 \in M$ relative to L-G-s.i.p $[-,-]$ or G-orthogonal, for short, if the condition $[m_2, m_1] = 0$ holds, we denote this by $m_1 \perp m_2 (G)$ [16]. Related to this G-orthogonal, Nur and Gunawan (2018) have made developments related to this. They developed the orthogonality in the semi-inner product space to become $gg$-orthogonal. They also propose the definition of $gg$-angle which can exists among two vectors in a normed space. The $gg$-orthogonal is defined as follows. Consider $g$ as a semi-inner product on $M$, the mapping $[-,-]_{gg}$ that define on $M \times M$ by $[m_1, m_2]_{gg} = \sqrt{g(m_1, m_2)g(m_2, m_1)}$ [17]. Any two vectors in $M$ is said perpendicular to $gg$-angle or $gg$-orthogonal if its $gg$-semi inner product is equal to 0. In other words, for any $m_1$ and $m_2$ two elements in $M$ we say that $m_1$ is $gg$-orthogonal to $m_2$, or $m_1 \perp_{gg} m_2$ if and only if $[m_1, m_2]_{gg} = 0$.

Another research result is that conducted by Wojcik [3] who also carried out the development of the chmielisnky [5] results for the semi inner product space. The results are as follows.

**Theorem 3.6.** [3] Let $M$ and $N$ be two normed spaces. Assume that $M$ is smooth. Let $E$ and $H$ be two linear mapping from $M$ to $N$. Assume that $\|E\| \cdot \|H\| \leq 1$. Then the following conditions are equivalent:

(i) For all $m_1 \in M$ where $[Em_1, Hm_1]_N = \|m_1\|^2$;

(ii) For all $m_1, m_2 \in M$, where $[Em_2, Hm_1]_N = [m_2, m_1]_M$.

The above result is also an extension of the results obtained by Koehler and Rosenthal [6] which show that a linear operator of a normed space is itself isometric if and only if it preserves a semi-inner product. Wojcik extends the result for a smooth, normed space. The results are then generalized to:

**Theorem 3.7.** [3] Let $M$ and $N$ be two normed spaces. Assume that $M$ is smooth. Let $E$, and $H$ be two linear mapping from $M$ to $N$. Assume that $\|E\| \cdot \|H\| \leq \varrho$. Then the following conditions are equivalent:

(i) For all $m_1 \in M$ where $[Em_1, Hm_1]_N = \varrho \cdot \|m_1\|^2$;

(ii) For all $m_1, m_2 \in M$, where $[Em_2, Hm_1]_N = \varrho \cdot [m_2, m_1]_M$.
3.4. Bilinear Spaces
A bilinear space is a vector space over the field $L$ together with a bilinear form. The bilinear form is closely related to the inner product. In general, an inner product over a field of a real number is indeed a symmetric bilinear form. According to these properties, a bilinear form can be viewed as an extension of an inner product. Therefore, in the bilinear space, there are also concepts developed from the inner product space, including the orthogonality concept of two vectors, the orthogonal subspace concept of a subset. These concepts are similarly defined and can be viewed as the expression of the concepts in the inner product space.

First, it will be explained the basic concepts related to the bilinear, especially in a finite dimensional vector space. Let $P$ be a finite dimensional vector space over the field $L$ and $[-,-]$ be a mapping $[-, -] : P \times P \rightarrow L$ that satisfy: (i) $[\delta_1 p_1 + \delta_2 p_2, p_3] = \delta_1 [p_1, p_3] + \delta_2 [p_2, p_3]$ and $[p_1, \delta_1 p_2 + \delta_2 p_3] = \delta_1 [p_1, p_2] + \delta_2 [p_1, p_3]$, for all $p_1, p_2, p_3$ in vector space $P$ and for all $\delta_1, \delta_2$ in field $L$. A pair $(P, [-, -])$ over a field $L$ is a bilinear space [18].

In general, bilinear spaces are not always symmetrical. A bilinear space $(P, [-, -])$ is said to be symmetric if $[p_1, p_2] = [p_2, p_1]$ for every $p_1$ and $p_2$ element in $P$. In bilinear space there is also a definition where two vectors are perpendicular to each other. The definition is similar to the concept of orthogonality in the inner product space. A vector $p_1$ are said to be perpendicular or orthogonal with vector $p_2$, written as $p_1 \perp p_2$, if $[p_1, p_2] = 0$. Like the bilinear space itself, the $\perp$ relation in general is not always symmetrical. In other words, it is possible that we have $p_1$ which is orthogonal with $p_2$ but $p_2$ is not orthogonal with $p_1$.

In recent research, the topic of the bilinear form is widely used in research which is associated with applicable problems. The bilinear form is widely used in solving problems such as control systems, cryptography, and many more. One of them is research conducted by Mameri and Aissani (2018). They use a bilinear form of symmetry in making algorithms One of them is research conducted by Mameri and Aissani (2018). They use a bilinear form of symmetry in making algorithms that generate orthogonal matrices over a finite field and also a proactive variant of Bloom’s threshold secrete sharing scheme that is used in Cryptography [19]. Another study was conducted by Gao, et al. In 2020. They built a system by using the bilinear form through the binary Bell polynomial. The results they get are different from previous results and do not depend on certain constraints [20].

4. Conclusion
Let $(M, \langle - , - \rangle)$ be an inner product space over field $L$ (complex or real). The development of the concept of inner product has been carried out in various directions, including the following directions

(i) Sesquilinear form $\langle - , - \rangle$ with properties for each $m \in M$, $x$ nonzero and $\langle m, m \rangle \neq 0$ [21].
(ii) Indefinite inner product space $\langle - , - \rangle$ which is a nondegenerate sesquilinear form [8].
(iii) The norm of a vector space equivalent to the existence of a semi inner product.
(iv) Bilinear form which is a generalization of indefinite inner product space over $\mathbb{R}$ by expanding the field to any field (it doesn’t have to be real).

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