On (Co-)morphisms of Lie Pseudoalgebras and Groupoids

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Abstract

We give a unified description of morphisms and comorphisms of Lie pseudoalgebras, showing that the both types of morphisms can be regarded as subalgebras of a Lie pseudoalgebra, called the ψ-sum. We also provide similar descriptions for morphisms and comorphisms of Lie algebroids and groupoids.

1 Introduction

As an algebraic version of Lie algebroids, a Lie pseudoalgebra is a generalized structure for that of a Lie algebra over some field and the structure of the A-module Der(A), all derivations on a commutative algebra A. It consists of a triple (E, A, θ) where E is an A-module and a Lie algebra, θ : E → Der(A) (called the anchor of E), which is an A-module morphism and a Lie algebra morphism satisfying some compatibility conditions.

The language of Lie pseudoalgebras arises from that of Lie algebroids, which were first introduced by Pradines [29] to provide a precise description of the infinitesimal form of Lie groupoids. The reader who wishes to pursue the topic of Lie algebroids and groupoids is referred to Mackenzie’s new book [25] (see also [6, Chp 8, 12], [23, I, III]) for background information. As an abstract algebraic treatment of the category of Lie algebroids performed in [12] and [24], Lie pseudoalgebras are variously called Lie Rinehart algebras [30, 15, 16, 17, 4], Lie d-rings [27], Palais pairs [9], differential Lie algebras [21], or modules with differential [28]. Some other closely related variants are Lie-Cartan spaces [2, 8], Lie modules [26], Lie-Cartan pairs [13](this list is not complete.) It may be seen as an algebraic form of the notion of Lie algebroid in which vector bundles over manifolds are replaced by modules over rings, vector fields by derivations of rings, and so on.

We especially study morphisms and comorphisms, which determine two different categories of Lie pseudoalgebras. The general notion of morphisms of Lie algebroids comes from Higgins and Mackenzie [12]. Comorphisms of Lie pseudoalgebras seem to have first been defined by Huebschmann [15]. There are also corresponding concepts of morphisms for Lie groupoids [13]. Morphisms of Lie pseudoalgebras are well known (for example, see [15]). The analogue concept, i.e., comorphisms of groupoids can also be found in [13]. Our treatment of morphisms and comorphisms of Lie algebroids follows from Higgins and Mackenzie [13] and [25](see also [24]).

One of the main aim of this paper is to show that the two kinds of morphisms of Lie pseudoalgebras can be unified via restriction theory. The conclusion is that both are subalgebras in a Lie pseudoalgebra called the ψ-sum of two Lie pseudoalgebras. We also develop similar theories for Lie algebroids as well as for Lie groupoids. The reader will see that the three versions of our main theorems, Theorems 3.6, 4.8 and 5.5 are actually expressing the same idea from different points of view.

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The paper is organized as follows. In Section 2, we recall the basic definition of Lie pseudoalgebras and we introduce the idea of restricting Lie pseudoalgebras via an ideal of the algebra. Applying this process, we obtain the so-called \( \psi \)-sum of two Lie pseudoalgebras, where \( \psi \) is an algebraic morphism.

Section 3 describes the two kinds of morphisms of Lie pseudoalgebras. The principal objective in this section is the proof of our main result (Theorem 3.6) in this paper. It provides a picture of the relationship of the two different morphisms, whose graphs turn out to be two subalgebras of the \( \psi \)-sum with respect to a given algebra morphism \( \psi \). Moreover, as an application of theorem 3.6, we give a short proof of the fact that with either morphisms or comorphisms, the Lie pseudoalgebras form a category, which is proved originally in [13].

To gain a better understanding of these abstract theories, Section 4 expresses the preceding theory in the language of Lie algebroids. We recall the definition of morphism and comorphism of Lie algebroids and we show how both morphisms and comorphisms are embedded as subalgebras into a common algebroid which we called the \( \phi \)-sum. The \( \phi \)-sum can be regarded as the algebroid version of the preceding \( \psi \)-sum.

Section 5 is an exposition of the theory of the two kinds of morphisms concerning groupoids, analogous to that of Lie algebroids. In this paper, we present an equivalent definition of comorphism between groupoids with generally different bases. It turns out that the action of a groupoid \( \Gamma \) on \( Z \xrightarrow{\theta} M \) is actually a comorphism of groupoids from \( \Gamma \) to the pair groupoid \( Z \times Z \).

As global versions of that of Lie algebroids, we then study the relationship of comorphisms and morphisms of groupoids, with likewise conclusions. Finally, we present various examples.

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2 The \( \psi \)-sum of Lie Pseudoalgebras

Throughout the paper, \( \mathcal{A} \) stands for an associative, commutative algebra over \( K \), where \( K \) is the number field \( \mathbb{R} \) or \( \mathbb{C} \). We shall be concerned exclusively with algebras \( \mathcal{A} \) which are unitary, and hence \( K \) is regarded as a subset of \( \mathcal{A} \). We always consider left \( \mathcal{A} \)-modules, and they are denoted by capital letters \( E, F, \) etc. By “\( \mathcal{A} \)-maps”, or “maps of \( \mathcal{A} \)-modules”, we mean “\( \mathcal{A} \)-module morphisms”. The dual module of \( E \), namely \( \text{Hom}_\mathcal{A}(E, \mathcal{A}) \), will be denoted by \( E^*_\mathcal{A} \).

A derivation of \( \mathcal{A} \) is a \( K \)-linear map \( \delta: \mathcal{A} \to \mathcal{A} \), satisfying the Leibnitz rule \( \delta(ab) = \delta(a)b + a\delta(b) \), \( \forall a, b \in \mathcal{A} \). The \( \mathcal{A} \)-module \( \text{Der}(\mathcal{A}) \) of all derivations of \( \mathcal{A} \) is closed under the bracket, or commutator \( [\delta, \lambda] \triangleq \delta \circ \lambda - \lambda \circ \delta \) and \( \text{Der}(\mathcal{A}) \) is a \( K \)-Lie algebra.

In this section, we recall the basic definition of Lie pseudoalgebras and we mainly introduce the idea of restricting Lie pseudoalgebras via an ideal of the algebra \( \mathcal{A} \). Applying this process, we obtain the so-called \( \psi \)-sum of two Lie pseudoalgebras, where \( \psi \) is an algebraic morphism.

\begin{itemize}
  \item **Lie pseudoalgebras.**
\end{itemize}

In this paper, we adopt the following definition.

**Definition 2.1.** Let \( E \) be an \( \mathcal{A} \)-module and a \( K \)-Lie algebra with the bracket \( [\cdot, \cdot]: E \times E \to E \). If there is a Lie algebra morphism \( \theta: E \to \text{Der}(\mathcal{A}) \) (called the anchor of \( E \)) such that

\[
\begin{align*}
\theta(a_1X_1) &= a_1\theta(X_1), \\
[X_1, a_2X_2] &= a_2[X_1, X_2] + \theta(X_1)(a_2)X_2, \quad \forall X_i \in E, a_i \in \mathcal{A},
\end{align*}
\]

we call \((E, [\cdot, \cdot], \theta)\) a Lie pseudoalgebra over \( \mathcal{A} \). \footnote{Our conclusions also hold if \( K \) is assumed to be a field of characteristic zero.}
Abusing the notation of the brackets, we prefer to write

$$\theta(X)a = [X,a] = -[a,X], \quad X \in E, a \in A.$$  

In fact, one can treat this convention as the Lie bracket in the semidirect sum $E \ltimes A$, where $A$ is considered as an abelian Lie algebra.

In what follows, we write $(E,A)$ to indicate a Lie pseudoalgebra $E$ over a commutative and associative $K$-algebra $A$, with the bracket and anchor $[\cdot,\cdot]_E$. And for simplicity, we will sometimes suppress the subscripts “$E$”.

When $E$ is a finitely generated projective $A$-module, there is an exterior differential operator $d_E : \wedge^2 E^*_A \rightarrow \wedge^{k+1} E^*_A$ $(k \geq 0)$, which is of square zero, defined in the ordinary fashion\footnote{For finitely generated projective $A$-modules $E$, one has $\wedge^2 E^*_A \cong \text{Hom}_A(\wedge^2 E, A)$, and it is also finitely generated projective. And $E \cong \text{Hom}_A(E^*_A, A) = (E^*_A)^\wedge$.} see also [21 16 17]:

\[
\begin{align*}
(d_E a, X) &= \theta(X)a = [X,a], \quad a \in A, X \in E; \\
(d_E \xi, X \wedge_A Y) &= \theta(X)(\xi, Y) - \theta(Y)(\xi, X) - \xi, [X,Y], \quad \xi \in E^*_A, X, Y \in E \\
d_E(\xi_1 \wedge_A \cdots \wedge_A \xi_n) &= \langle d_E \xi_1 \rangle \wedge_A \xi_2 \wedge_A \cdots \wedge_A \xi_n - \xi_1 \wedge_A \cdots \wedge_A (d_E \xi_2) \wedge_A \xi_3 \wedge_A \cdots \wedge_A \xi_n \\
&\quad + \cdots + (-1)^{n-1}\xi_1 \wedge_A \cdots \wedge_A \xi_{n-1} \wedge_A (d_E \xi_n), \quad \xi_i \in E^*_A. \tag{3}
\end{align*}
\]

Conversely, any operator $d_E$ satisfying the derivation rule \ref{differential_operator}, and $d_E^2 = 0$, determines a Lie pseudoalgebra structure on $E$ (over $A$) [21].

- **The restriction of Lie pseudoalgebras.** Fix a Lie pseudoalgebra $(E,A)$ and let $\mathcal{I} \subset A$ be an ideal of $A$. Then, we define

$$E^3 = \{ X \in E | [X, \mathcal{I}] \subset \mathcal{I} \},$$

which is clearly a submodule of $E$, and $(E^3, A)$ is a Lie pseudoalgebra. Let

$$\mathcal{J}E = \left\{ \sum_i a_i X_i | a_i \in \mathcal{J}, X_i \in E \right\}.$$  

Then $\mathcal{J}E \subset E^3$ and the following lemma gives a new Lie pseudoalgebra.

**Lemma 2.2.** The quotient $(A/\mathcal{J})$-module $E^3/(\mathcal{J}E)$ is a Lie pseudoalgebra with structures coming from $(E,A)$.

**Proof.** By definition, $A(\mathcal{J}E) \subset \mathcal{J}E$, $\mathcal{J}E^2 \subset \mathcal{J}E$ and hence $E^3/(\mathcal{J}E)$ is indeed an $(A/\mathcal{J})$-module. We define the induced brackets by the obvious rules

$$[X_1, X_2] = [X_1, X_2]^- = [X_1, X_2]^- \quad \text{and} \quad [X, a] = [X, a].$$

Here by $X$ we denote the element $X + \mathcal{J}E \in E^3/(\mathcal{J}E)$, and similarly by $\mathcal{J}$ we denote $a + \mathcal{J} \in A/\mathcal{J}$. It suffices to prove that they are well defined. In fact, we have

$$[\mathcal{J}E, \mathcal{J}] \subset \mathcal{J}, \quad [E^3, \mathcal{J}] \subset \mathcal{J}, \quad [E^3, \mathcal{J}E] \subset \mathcal{J}E.$$  

The first two relations are obvious. For the last one, notice that if $X_1 \in E^3$, $X' = aX_0 \in \mathcal{J}E$, where $a \in \mathcal{J}$, $X_0 \in E$, we have

$$[X_1, X'] = a[X_1, X_0] - [X_1, a]X_0 \in \mathcal{J}E. \quad \blacksquare$$

Thus we see that $(E^3/(\mathcal{J}E), A/\mathcal{J})$ inherits all the structures of $(E, A)$. As an application, in Example \ref{application} one will find how a Lie algebroid is restricted to a Lie pseudoalgebra through a submanifold. But generally speaking, we can only depict the quotient module $E^3/(\mathcal{J}E) \subset E/(\mathcal{J}E)$ in the following manner.

**Lemma 2.3.**
1) Let $E$ be an $A$-module and $\mathcal{J}$ be an ideal of $A$. Then, $E/(\mathcal{J}E) \cong E \otimes_A (A/\mathcal{J})$ as $(A/\mathcal{J})$-modules, under the isomorphism $\sigma : \overline{X} \mapsto X \otimes \overline{A}$.

2) Let $(E, A)$ be a Lie pseudoalgebra and $\mathcal{J}$ be an ideal of $A$. Then under the isomorphism $\sigma$ defined above, the quotient module $E^2/(\mathcal{J}E) \cong E^2 \otimes_A (A/\mathcal{J})$, and the latter has the induced Lie pseudoalgebra structure (over $A/\mathcal{J}$) defined by

$$[X_1 \otimes_A \overline{a_1}, \overline{a_2}] = \overline{a_1}[X_1, a_2]^-, \quad [X_1 \otimes_A \overline{a_1}, X_2 \otimes_A \overline{a_2}] = [X_1, X_2] \otimes_A \overline{a_1}a_2 + X_2 \otimes_A \overline{a_1}a_2 + X_1 \otimes_A \overline{a_2}[X_2, a_1]^-. \quad (4, 5)$$

for all $X_i \in E^2, a_i \in A$.

**Proof.** For 1), consider the map $\chi : E \times A/\mathcal{J} \to E/(\mathcal{J}E), (X, \overline{a}) \mapsto \overline{aX} \in E, a \in A$. It follows that

$$\chi(a_1 X, \overline{a_2}) = \overline{a_1 a_2 X} = \overline{a_1} \chi(X, \overline{a_2}) = \chi(X, a_1 \overline{a_2}),$$

i.e., $\chi$ is $A$-bilinear and hence it induces a well-defined $A$-map $\tilde{\chi} : E \otimes_A (A/\mathcal{J}) \to E/(\mathcal{J}E), X \otimes_A \overline{a} \mapsto \overline{aX}$. One can check that $\tilde{\chi}$ is exactly the inverse map of $\sigma$.

Now under $\sigma$, the image of $E^2/(\mathcal{J}E)$ is clearly $E^2 \otimes_A (A/\mathcal{J})$, and the bracket in $E^2/(\mathcal{J}E)$ is then transferred to $E^2 \otimes_A (A/\mathcal{J})$, simply given by

$$[X_1 \otimes_A \overline{a_1}, \overline{a_2}] = [X_1, a_2]^-, \quad [X_1 \otimes_A \overline{a_1}, X_2 \otimes_A \overline{a_2}] = [X_1, X_2] \otimes_A \overline{a_1}. \quad (4, 5)$$

Equalities (1) and (2) then follow the bracket laws (1) and (2).

This fact says that $E^2/(\mathcal{J}E)$ can be regarded as an $(A/\mathcal{J})$-submodule of $E \otimes_A (A/\mathcal{J})$. But the latter one does NOT have a Lie pseudoalgebra structure.

**Definition 2.4.** We denote

$$E_2 = E^2 \otimes_A (A/\mathcal{J}) = E^2/(\mathcal{J}E)$$

and call $(E_2, A/\mathcal{J})$ the $\mathcal{J}$-restriction of a Lie pseudoalgebra $(E, A)$ with respect to the ideal $\mathcal{J} \subset A$.

**Remark 2.5.** The terminology “restriction” here is suggested by the referee from the point of view of geometry as follows. Consider the $C^\infty(M)$-module $\mathcal{X}(M)$, the set of smooth vector fields on a smooth manifold $M$. Let $N \subset M$ be a submanifold and consider an ideal $\mathcal{J} \subset C^\infty(M)$ which is the collection of the functions that vanish on $N$. In this case, it is clear that $C^\infty(M)/\mathcal{J} \cong C^\infty(N)$ and $(\mathcal{X}(M))_2 = \mathcal{X}(N)$. So, the $\mathcal{J}$-restriction of $\mathcal{X}(M)$ is the usual restriction of the tangent vector fields on a submanifold. The construction of such restrictions in the Lie algebroid case also corresponds to restricting a Lie algebroid to a submanifold of the base, as Example 2.10 later makes clear.

- **The $\psi$-sum.**

A useful application of the preceding restriction theory is as follows. Consider two Lie pseudoalgebras $(E, A)$ and $(\mathcal{F}, \mathcal{B})$ and define their **direct sum** to be the $A \otimes B$-module $(E \otimes \mathcal{F}) \oplus (A \otimes \mathcal{B})$. Here “$\otimes$” means “$\otimes_K$” and we adopt this convention throughout the paper. We first endow the direct sum with a new Lie pseudoalgebra structure out of the given ones.

**Proposition 2.6.** $(E \otimes \mathcal{B}) \oplus (A \otimes \mathcal{F}), A \otimes \mathcal{B})$ is a Lie pseudoalgebra, the structure being given by the following rules:

$$[X_1 \otimes b_1 + a_1 \otimes Y_1, a_2 \otimes b_2] = [X_1, a_2]_E \otimes b_1 b_2 + a_1 a_2 \otimes [Y_1, b_2]_\mathcal{F};$$

$$[X_1 \otimes b_1 + a_1 \otimes Y_1, X_2 \otimes b_2 + a_2 \otimes Y_2] = [X_1, X_2]_E \otimes b_1 b_2 + a_1 a_2 \otimes [Y_1, Y_2]_\mathcal{F} + [X_1, a_2]_E \otimes b_1 Y_2 + [a_1, X_2]_E \otimes b_2 Y_1 + a_1 X_2 \otimes [Y_1, b_2]_\mathcal{F} + a_2 X_1 \otimes [b_1, Y_2]_\mathcal{F}.$$

Here $a_1, a_2 \in A, b_1, b_2 \in B, X_1, X_2 \in E, Y_1, Y_2 \in \mathcal{F}$. **
We omit the proof of this proposition, since the computations are quite straightforward. For simplicity, in what follows, we will again omit the subscripts \( \mathbf{E}, \mathbf{F} \), etc. One is also referred to Example 2.13 of the direct sum of two Lie algebroids as a guide to understanding \( (\mathbf{E} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{F}), \mathbf{A} \otimes \mathbf{B}) \).

From now on, we fix an algebra morphism \( \psi : \mathbf{A} \rightarrow \mathbf{B} \). And hence \( \mathbf{B} \)-modules can be regarded as \( \mathbf{A} \)-modules via \( \psi \). By \( \tilde{\psi} \) we denote the morphism given canonically by

\[
\mathbf{A} \otimes \mathbf{B} \rightarrow \mathbf{B} : \quad a \otimes b \mapsto \psi(a)b. \tag{6}
\]

Notice that, \( \tilde{\psi} \) is always surjective and \( \mathbf{B} \cong (\mathbf{A} \otimes \mathbf{B})/\text{Ker} \tilde{\psi} \). It is easy to see that

\[
\mathfrak{J} \triangleq \text{Ker} \tilde{\psi} = \text{Span}_K \{a \otimes b - 1 \otimes \psi(a)b | a \in \mathbf{A}, b \in \mathbf{B} \}. \tag{7}
\]

**Definition 2.7.** Let \( (\mathbf{E}, \mathbf{A}) \) and \( (\mathbf{F}, \mathbf{B}) \) be two Lie pseudoalgebras and let \( \psi : \mathbf{A} \rightarrow \mathbf{B} \) be an algebraic morphism. For the direct sum \( \mathbf{D} = (\mathbf{E} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{F}) \) and the ideal \( \mathfrak{J} = \text{Ker} \tilde{\psi} \) defined as above, we denote \( (\mathbf{D} \mathfrak{J}, \mathbf{B}) \), the \( \mathfrak{J} \)-restriction of \( (\mathbf{D}, \mathbf{A} \otimes \mathbf{B}) \) by \( (\mathbf{E} \oplus \psi \mathbf{F}, \mathbf{B}) \) and call this pseudoalgebra the \( \psi \)-sum of \( (\mathbf{E}, \mathbf{A}) \) and \( (\mathbf{F}, \mathbf{B}) \) with respect to the morphism \( \psi \).

The following theorem gives an explicit description of \( \mathbf{E} \oplus \psi \mathbf{F} \).

**Theorem 2.8.** With the above notations,

1) the \( \psi \)-sum \( \mathbf{E} \oplus \psi \mathbf{F} \) is a \( \mathbf{B} \)-submodule of \( (\mathbf{E} \otimes \mathbf{A}) \mathbf{B} \) (here the tensor product \( \mathbf{E} \otimes \mathbf{A} \mathbf{B} \) is both a \( \mathbf{B} \) and \( \mathbf{A} \)-module, considering \( \mathbf{B} \) as an \( \mathbf{A} \)-module through \( \psi \));

2) an element \( \sum_i X_i \otimes a_i b_i + Y \in (\mathbf{E} \otimes \mathbf{A}) \mathbf{B} \oplus \mathbf{F} \) belongs to \( \mathbf{E} \oplus \psi \mathbf{F} \) if and only if

\[
\sum_i \psi([X_i, a])b_i = [Y, \psi(a)], \quad \forall a \in \mathbf{A}. \tag{8}
\]

**Proof.** For 1), by definition, the \( \psi \)-sum

\[
\mathbf{E} \oplus \psi \mathbf{F} = \mathbf{D}^3 \otimes_{\mathbf{A} \otimes \mathbf{B}} \mathbf{B} \subset \mathbf{D} \otimes_{\mathbf{A} \otimes \mathbf{B}} \mathbf{B} \cong (\mathbf{E} \otimes \mathbf{A}) \mathbf{B} \oplus \mathbf{F}.
\]

For 2), since \( \mathbf{D}^3 = \{ x \in \mathbf{D} | \tilde{\psi}(x, \mathfrak{J}) = 0 \} \), and according to (7), an element \( \sum_i (X_i \otimes b_i + a_i \otimes Y_i) \in \mathbf{D}^3 \) if and only if

\[
\tilde{\psi}(\sum_i [X_i \otimes b_i + a_i \otimes Y_i, a \otimes b - 1 \otimes \psi(a)b]) = (\sum_i \psi([X_i, a])b_i - \sum_i \psi(a_i)Y_i, \psi(a)])b = 0,
\]

holds for all \( a \in \mathbf{A}, b \in \mathbf{B} \). This expression shows that

\[
\mathbf{D}^3 = \left\{ X_i \otimes b_i + a_i \otimes Y_i | \sum_i \psi([X_i, a])b_i - \sum_i \psi(a_i)Y_i, \psi(a)] = 0, \forall a \in \mathbf{A} \right\}.
\]

On the other hand,

\[
\sum_i (X_i \otimes b_i + a_i \otimes Y_i) \otimes_{\mathbf{A} \otimes \mathbf{B}} \mathbf{B} = \sum_i X_i \otimes a_i b_i + \sum_i \psi(a_i)Y_i.
\]

Thus, \( \mathbf{E} \oplus \psi \mathbf{F} = \mathbf{D}^3 \otimes_{\mathbf{A} \otimes \mathbf{B}} \mathbf{B} \) is exactly the \( \mathbf{B} \)-submodule described in this theorem. \( \blacksquare \)

By Lemma 2.3, the reader should have no difficulty in obtaining the expressions of brackets of \( \mathbf{E} \oplus \psi \mathbf{F} \), given in the following proposition.
Proposition 2.9. The structure maps of the Lie pseudoalgebra \((E \oplus_f F, B)\) are given as follows.

\[
\left[ \sum_i X_i \otimes_A b_i + Y, b \right] = [Y, b];
\]
\[
\left[ \sum_i X_i \otimes_A b_i + Y, \sum_j X_j' \otimes_A b_j' + Y' \right] = \sum_{i,j} [X_i, X_j'] \otimes_A b_i b_j' + \sum_j X_j' \otimes_A [Y, b_j'] - \sum_i X_i \otimes_A [Y', b_i] + [Y, Y'],
\]

for all \( \sum_i X_i \otimes_A b_i + Y, \sum_j X_j' \otimes_A b_j' + Y' \in E \oplus_f F, b \in B. \)

**Examples.**

We will denote a Lie algebroid \(U\) over a manifold \(M\) by \((U, M, \rho_U)\), with the anchor map \(\rho_U : U \to TM\) and the \((\mathbb{R}\times)\) Lie algebra structure \([\cdot, \cdot]_U\) on \(\Gamma(U)\). We also write \([v, \cdot]_U\) to denote the tangent vector \(\rho_U(v)\), for \(v \in U\). The differential operator on \(\Gamma(\wedge^n U^*)\) will be denoted by \(d_U\). When it introduces no confusion, these subscripts “\(U\)” can be omitted.

**Example 2.10.** The reader should bear in mind this example as a guide to what is going on for the restriction theories. Let \((U, N, \rho)\) be a Lie algebroid, and let \(N_0\) be an embedded submanifold. Let \(i : N_0 \to N\) be the inclusion. Consider \(A = C^\infty(M)\). Of course \(J = \ker^* = \{f \in A|f|_{N_0} = 0\}\) is an ideal of \(A\). Denoting \(E = \Gamma(U)\), we want to find the restriction of the Lie pseudoalgebra \((E, A)\) with respect to \(J\). Clearly,

\[
E^J = \{A \in \Gamma(U)|\rho(A)|_{N_0} \in TN_0\},
\]
\[
JE = \{A \in \Gamma(U)|A|_{N_0} = 0\}.
\]

Thus, the quotient algebra \(E^J\) can be regarded as the space of sections

\[
U_{N_0} = \{v \in U_0|p \in N_0, \rho(v) \in T_p N_0\},
\]

as an \(A/\mathcal{J} \cong C^\infty(N_0)\)-module.

Although \(U_{N_0}\) may not be a vector bundle over \(N_0\) (since the dimension of its fibers may vary), it has formally the anchor map and the Lie algebra structure of the space of sections \(\Gamma(U_{N_0})\). Thus \(U_{N_0}\) can be regarded as a *generalized* Lie algebroid.

**Example 2.11.** Let \(E\) and \(F\) be two \(K\)-Lie algebras. The only \(K\)-morphism from \(K\) to \(K\) is the identity map \(I\), and the \(I\)-sum of \((E, K)\) and \((F, K)\) is just \((E \oplus_f F, K)\), since in this case \(\mathbb{S}\) becomes \(0 = 0\).

**Example 2.12.** Let \((E, A)\) be a Lie pseudoalgebra and \(F\) be a \(K\)-Lie algebra. For an algebra morphism \(\psi : A \to K\), the \(\psi\)-sum of \((E, A)\) and \((F, K)\) is the set

\[
E \oplus_\psi F = \{X \otimes A + Y|X \in E, Y \in F, [X, A]_E \subset \ker \psi\}.
\]

For example, when \(E = \Gamma(U), A = C^\infty(N)\) where \((U, M)\) is a Lie algebroid, and \(\psi(h) = h(p)\), \(\forall h \in C^\infty(N)\), for some fixed point \(p \in N\), we have

\[
E \oplus_\psi F = \{(v, Y) \in U \times F|\rho_U(v) = 0\}.
\]

**Example 2.13.** We recall the direct product of Lie algebroids as in Higgins and Mackenzie [12] and Mackenzie [25]. Let \((U, M, \rho_U)\) and \((V, N, \rho_V)\) be two Lie algebroids. Their direct sum is a bundle \(U \times V\) over \(M \times N\), with elements \((u, v), u \in U_p, v \in V_q\), bundle map \((u, v) \mapsto (p,q)\). We make \(U \times V\) into a Lie algebroid on the base \(M \times N\) as follows. The anchor map is \((u, v) \mapsto (\rho_U(u), \rho_V(v))\). For two sections of \(U \times V\), we define the Lie bracket to be

\[
[(f_1 A_1, g_1 B_1), (f_2 A_2, g_2 B_2)]
\]
\[
= (f_1 f_2 [A_1, A_2]_U + g_1 [B_1, f_2]_V A_2 - g_2 [B_2, f_1]_V A_1,
   g_1 g_2 [B_1, B_2]_V + f_1 [A_1, g_2]_U B_2 - f_2 [A_2, g_1]_U B_1).
\]
Here \( f_i \in C^\infty(N), \ g_i \in C^\infty(M), \ A_i \in \Gamma(U), \ B_i \in \Gamma(V) \). To see that \( (\Gamma(U \times V), C^\infty(M \times N)) \) is exactly the direct sum of \( (\Gamma(U), C^\infty(M)) \) and \( (\Gamma(V), C^\infty(N)) \), one should regard \( C^\infty(M \times N) \approx C^\infty(M) \otimes C^\infty(N) \) and \( \Gamma(U \times V) \) as the \( C^\infty(M \times N) \)-module \( \Gamma(U) \otimes C^\infty(N) \otimes \Gamma(V) \otimes C^\infty(M) \).

Given a smooth map \( \phi : M \to N \), one has its graph
\[
G(\phi) = \{(x, \phi(x)) | x \in M \} \subset M \times N.
\]

Applying the restriction process in Example 2.10 to the direct sum of Lie algebroids, we define the \( \phi \)-sum of Lie algebroids \( U \) and \( V \) to be \( (U \oplus V, M \times N)_{(\phi, \phi)} \) (notice that the result is a subset of \( U \oplus \phi V \)), i.e.,
\[
U \oplus_{\phi} V \equiv \{(u, v) \in U \oplus \phi V | \forall x \in M, u \in U_x, v \in V_{\phi(x)}, \phi_* \circ \rho_U(u) = \rho_V(v) \}.
\]

The space of sections \( \Gamma(U \oplus_{\phi} V) \) is a Lie pseudoalgebra over \( C^\infty(M) \), although \( U \oplus_{\phi} V \) may not be a bundle over \( M \). So it is actually a generalized Lie algebroid, an algebroid version corresponding to the \( \phi^* \)-sum of \( (\Gamma(U), C^\infty(M)) \) and \( (\Gamma(V), C^\infty(N)) \).

3 Morphisms and Comorphisms of Lie Pseudoalgebras

This section describes the two kinds of morphisms of Lie pseudoalgebras. As a main result of this paper, theorem 3.6 provides a picture of the relationship of the two different morphisms, whose graphs turn out to be two subalgebras of the \( \psi \)-sum with respect to a given algebra morphism \( \psi \).

It is proved in [13] that under either morphisms or comorphisms, and together with certain composition laws, the Lie pseudoalgebras are the objects of a category (in sense of morphism or comorphism). However, it is not easy to prove that the comorphisms satisfy the usual axioms of categories. As an application of theorem 3.6, we shall give a short proof of the fact that with either morphisms or comorphisms, the Lie pseudoalgebras form a category.

For an \( \mathcal{A} \)-module \( E \) and a \( \mathcal{B} \)-module \( F \), by \( (E, \mathcal{A}) \overset{(\Psi, \psi)}{\Rightarrow} (F, \mathcal{B}) \) we denote a pair of morphisms \( (\Psi, \psi) \), where \( \psi : \mathcal{A} \to \mathcal{B} \) is a morphism of algebras and \( \Psi : E \to F \) is a map of \( \mathcal{A} \)-modules (considering \( \mathcal{B} \)-modules as \( \mathcal{A} \)-modules through \( \psi \)). We call \( \Psi \) an \( \mathcal{A} \)-map over \( \psi \).

Similarly, by writing \( (F, \mathcal{B}) \overset{(\Psi, \psi)}{\Rightarrow} (E, \mathcal{A}) \), we mean a pair of morphisms \( (\Psi, \psi) \), where \( \psi : \mathcal{A} \to \mathcal{B} \) is a morphism of algebras and \( \Psi : F \to E \otimes_{\mathcal{A}} \mathcal{B} \) is a map of \( \mathcal{B} \)-modules. We also call \( \Psi \) a \( \mathcal{B} \)-map over \( \psi \).

The concept of morphisms of Lie pseudoalgebras is given by [15] [24] as follows, and thus Lie pseudoalgebras with their morphisms form a category.

**Definition 3.1.** Let \( (E, \mathcal{A}), (F, \mathcal{B}) \) be two Lie pseudoalgebras. A morphism of Lie pseudoalgebras from \( E \) to \( F \), is a pair of morphisms \( (E, \mathcal{A}) \overset{(\Psi, \psi)}{\Rightarrow} (F, \mathcal{B}) \) such that

1) \( \psi([X, a]) = [\Psi(X), \psi(a)], \quad \forall X \in E, a \in \mathcal{A}; \)
2) \( \Psi([X_1, X_2]) = [\Psi(X_1), \Psi(X_2)], \quad \forall X_1, X_2 \in E. \)

In particular, if both \( \Psi \) and \( \psi \) are injective, we call \( (E, \mathcal{A}) \) a Lie subpseudoalgebra of \( (F, \mathcal{B}) \).

Meanwhile, there is another kind of morphism defined for Lie pseudoalgebras, also allowing the bases to be changed.

**Definition 3.2.** [24] [13] Let \( (E, \mathcal{A}), (F, \mathcal{B}) \) be two Lie pseudoalgebras. A comorphism of Lie pseudoalgebras from \( F \) to \( E \) over \( \psi \), is a pair of morphisms \( (F, \mathcal{B}) \overset{(\Psi, \psi)}{\Rightarrow} (E, \mathcal{A}) \), such that

1) if \( Y \in F \) and \( \Psi(Y) = \sum_k X_k \otimes \mathcal{A} b_k \), for some \( X_k \in E \) and \( b_k \in \mathcal{B} \), then
\[
[Y, \psi(a)] = \sum_k b_k \psi([X_k, a]), \quad \forall a \in \mathcal{A};
\]
Remark 3.3.

Proposition 3.4. Let \( E, F \) be finitely generated projective \( A \) and \( B \)-modules respectively, and let \( \psi : A \to B \) be an algebra morphism. Then

1) \( E \otimes_A B \) is a finitely generated projective \( B \)-module, and so are

\[
(E \otimes_A B) \otimes_A (E \otimes_A B) = (\wedge^2_A E) \otimes_A B, \quad \cdots, \quad (\wedge^k_A E) \otimes_A B.
\]

2) The map \( I : E \otimes_A B \to \text{Hom}_A(E^*_A, B) \), sending each \( X \otimes_A b \) to

\[
I(X \otimes_A b) : \xi \mapsto \psi((\xi, X)) b, \quad \forall \xi \in E^*_A, b \in B,
\]

is an isomorphism of \( B \)-modules. Similarly, we have

\[
(\wedge^2_A E) \otimes_A B \cong \text{Hom}_A(\wedge^2_A E^*_A, B), \quad \cdots, \quad (\wedge^k_A E) \otimes_A B \cong \text{Hom}_A(\wedge^k_A E^*_A, B).
\]

3) Let \( \Psi : F \to E \otimes_A B \) be a \( B \)-map. There is an induced \( A \)-map \( \Psi^* : E^*_A \to F^*_B \), called the dual map of \( \Psi \), such that

\[
(\Psi^*(\xi), Y) = (I \circ \Psi(Y), \xi), \quad \forall \xi \in E^*_A, Y \in F.
\]

4) Let \( \overline{\Psi} : E^*_A \to F^*_B \) be a \( B \)-map of \( A \)-modules. There is a unique \( B \)-map \( \Psi : F \to E \otimes_A B \) such that

\[
\Psi^* = \overline{\Psi}, \quad (\Psi(Y) = I^{-1} \circ (\overline{\Psi}(\cdot), Y), \text{ for each } Y \in F).
\]

These four conclusions heavily rely on the condition that the modules under consideration are finitely generated and projective. We omit the proofs.

Proposition 3.5. Let \( (E, A) \), \( (F, B) \) be two finitely generated projective Lie pseudoalgebras over the algebras \( A \) and \( B \) respectively. Then the following two statements are equivalent

1) \( (F, B) \xrightarrow{(\Psi, \psi)} (E, A) \) is a comorphism of Lie pseudoalgebras;
2) the dual map $\Psi^* : E^*_A \to F^*_B$ satisfies
\[
d_F \circ \Psi^* = \Psi^* \circ d_E, \text{ as a map } \wedge^k_A E^*_A \to \wedge^{k+1}_B F^*_B \quad (k \geq 0).
\] (13)

Here we regard $\Psi^* = \psi : \wedge^0_A E^*_A = A \to \wedge^0_B F^*_B = B$, and $\Psi^*$ naturally lifts to an $A$-map $\wedge^k_A E^*_A \to \wedge^k_B F^*_B$.

**Proof.** Due to formula (13), the second statement is equivalent to two conditions: for all $a \in A$, $d_F(\psi(a)) = \Psi^*(d_E(a))$ and for all $\xi \in E^*_A$, $d_F(\Psi^*(\xi)) = \Psi^*(d_E(\xi))$. We prove that these two conditions are equivalent to the first statement.

Suppose that $Y \in F$ and $\Psi(Y) = \sum_k X_k \otimes_A b_k$, for some $X_k \in E$ and $b_k \in B$. Then by condition $d_F(\psi(a)) = \Psi^*(d_E(a))$, we get
\[
\[ Y, \psi(a) \] = \langle d_F(\psi(a)), Y \rangle = \langle \Psi^*(d_E(a)), Y \rangle = \langle d_E(a), I \circ \Psi(Y) \rangle = \sum_k \psi(X_k, d_E(a))b_k = \sum_k b_k \psi([X_k, a]) = \theta(\psi(a)).
\] (14)

This proves relation (14). Suppose that $Y_1, Y_2 \in F$ and $\Psi(Y_1) = \sum_k X_{1,k} \otimes_A b_{1,k}$, $\Psi(Y_2) = \sum_k X_{2,t} \otimes_A b_{2,t}$, for some $X_{i,t} \in E$, $b_{i,t} \in B$. Then
\[
\langle \Psi^*(d_E(\xi)), Y_1 \wedge_B Y_2 \rangle = \langle d_E(\xi), I(\Psi(Y_1) \wedge_B \Psi(Y_2)) \rangle = \sum_{k,l} \psi(\langle \xi, X_{1,k} \rangle \wedge_A X_{2,l} \otimes_B b_{1,k} b_{2,l})
\]
\[
= \sum_{k,l} \psi([X_{1,k}, \langle \xi, X_{2,l} \rangle] - [X_{2,t}, \langle \xi, X_{1,k} \rangle] - [\xi, [X_{1,k}, X_{2,l}]] b_{1,k} b_{2,l}
\]
\[
= \sum_l (d_F \circ \psi(\langle \xi, X_{2,l} \rangle)) Y_{1,2} b_{2,l} - \sum_l (d_F \circ \psi(\langle \xi, X_{1,k} \rangle)) Y_{1,2} b_{1,k} - \sum_k \psi(\langle \xi, [X_{1,k}, X_{2,l}] \rangle) b_{1,k} b_{2,l}.
\]

Here we have used Equation (14). On the other hand, we have
\[
\langle \Psi^*(d_E(\xi)), Y_1 \wedge_B Y_2 \rangle = \sum_l \langle d_F(\psi(\langle \xi, X_{2,l} \rangle)), Y_{1,2} b_{2,l} \rangle - \sum_k \langle d_F(\psi(\langle \xi, X_{1,k} \rangle)), Y_{1,2} b_{1,k} \rangle - \langle \xi, I \circ \Psi(Y_1, Y_2) \rangle
\]
\[
= \sum_l \langle d_F(\psi(\langle \xi, X_{2,l} \rangle)), Y_{1,2} b_{2,l} \rangle - \sum_k \langle d_F(\psi(\langle \xi, X_{1,k} \rangle)), Y_{1,2} b_{1,k} \rangle - \langle \xi, I \circ \Psi(Y_1, Y_2) \rangle
\]
\[
= \sum_l \langle d_F(\psi(\langle \xi, X_{2,l} \rangle)), Y_{1,2} b_{2,l} \rangle - \sum_k \langle d_F(\psi(\langle \xi, X_{1,k} \rangle)), Y_{1,2} b_{1,k} \rangle + \sum_l \psi(\langle \xi, X_{2,l} \rangle)) (d_F b_{2,l}, Y_{1,2}) - \langle \xi, I \circ \Psi(Y_1, Y_2) \rangle.
\]

Therefore, by condition $d_F(\Psi^*(\xi)) = \Psi^*(d_E(\xi))$, we obtain
\[
\langle \xi, I \circ \Psi(Y_1, Y_2) \rangle = \sum_{k,l} \psi([X_{1,k}, X_{2,l}]) b_{1,k} b_{2,l} + \sum_l \psi(\langle \xi, X_{2,l} \rangle)) (d_F b_{2,l}, Y_{1,2}) - \sum_k \psi(\langle \xi, X_{1,k} \rangle)) (d_F b_{1,k}, Y_{2}).
\] (15)

Let
\[
Z = \sum_{k,l} [X_{1,k}, X_{2,l}] \otimes_A b_{1,k} b_{2,l} + \sum_l X_{2,l} \otimes_A [Y_1, b_{2,l}] - \sum_k X_{1,k} \otimes_A [Y_2, b_{1,k}].
\]
Then, relation (15) says

\[ \langle \xi, I(\Psi[Y_1, Y_2] - Z) \rangle = 0. \]

By the arbitrariness of \( \xi \in E^*_A \) and \( I \) being an isomorphism, we conclude that \( \Psi[Y_1, Y_2] - Z = 0 \) and this proves relation (12). A similar process shows that 2) implies 1). ■

We call \( \Psi^* \) as a chain map if it enjoys the property described by Equation (13).

The Main Theorem.

We define the graph of a pair of morphisms \((E, A) \mapsto (F, B)\) where \( \psi : A \to B \) is a morphism of algebras and \( \Psi : E \to F \) is a map of \( A \)-modules. Its graph is defined to be the \( B \)-submodule

\[ G_{(\Psi, \psi)} \triangleq \{ x + \tilde{\Psi}(x) | x \in E \otimes_A B \} \subset (E \otimes_A B) \oplus F. \]

Here \( \tilde{\Psi} \) is the \( B \)-map \( E \otimes_A B \to F \) defined by the obvious rule \( X \otimes_A b \mapsto \Psi(X)b, \forall X \in E, b \in B. \)

For a pair of morphisms \((F, B) \mapsto (E, A)\), where \( \psi : A \to B \) is a morphism of algebras and \( \Psi : F \to E \otimes_A B \) is a map of \( B \)-modules, we define its graph to be the \( B \)-submodule

\[ G_{(\Psi, \psi)} \triangleq \{ \Psi(Y) + Y | Y \in F \} \subset (E \otimes_A B) \oplus F. \]

Now, we state the main theorem. First, recall Theorem 2.8 which claims that the \( \psi \)-sum \( E \oplus \psi F \) is a \( B \)-submodule of \((E \otimes_A B) \oplus F\). For the two pairs of morphisms above, the following theorem claims that they are a morphism or comorphism of Lie pseudoalgebras if and only their graphs are contained in the \( \psi \)-sum as Lie subpseudoalgebras.

Theorem 3.6. Let \((E, A), (F, B)\) be two Lie pseudoalgebras and let \( \psi : A \to B \) be a morphism of algebras.

1) Let \((E, A) \mapsto (F, B)\) be a pair of morphisms. It is a morphism of Lie pseudoalgebras if and only if its graph \( G_{(\Psi, \psi)} \) is a Lie subpseudoalgebra (over \( B \)) of the \( \psi \)-sum \( E \oplus \psi F \).

2) Let \((F, B) \mapsto (E, A)\) be a pair of morphisms. It is a comorphism of Lie pseudoalgebras if and only if its graph \( G_{(\Psi, \psi)} \) is a Lie subpseudoalgebra (over \( B \)) of the \( \psi \)-sum \( E \oplus \psi F \).

Before we prove these two statements, we state two facts which deserve attention.

Lemma 3.7.

1) With the same assumptions as in (1) of Theorem 3.6, \( G_{(\Psi, \psi)} \) is contained in \( E \oplus \psi F \) if and only if (1) in Definition 3.1 holds.

2) With the same assumptions as in (2) of Theorem 3.6, \( G_{(\Psi, \psi)} \) is contained in \( E \oplus \psi F \) if and only if (1) in Definition 3.1 holds.

Proof. 1) Consider \( x = X \otimes_A b \). Then \( x + \tilde{\Psi}(x) = X \otimes_A b + b\Psi(X) \in G_{(\Psi, \psi)}. \) By (8) in Theorem 2.8, \( x + \tilde{\Psi}(x) \) belongs to \( E \oplus \psi F \) if and only if

\[ \psi([X, a])b = [b\Psi(X), \psi(a)] = b\Psi(X), \psi(a) \]

holds. The conclusion then comes from the arbitrariness of \( b \).

2) Let \( Y \in F \) and suppose that \( \Psi(Y) = \sum_k X_k \otimes_A b_k \), for some \( X_k \in E \) and \( b_k \in B \). Using the equality again, we know that \( \Psi(Y) + Y \in G_{(\Psi, \psi)} \) belongs to \( E \oplus \psi F \) if and only if relation (11) holds. ■

Proof of Theorem 3.6

1) \( \Psi \) is clearly well defined. The above lemma says that (1) of Definition 3.1 is exactly the condition that \( G_{(\Psi, \psi)} \subset E \oplus \psi F \). And under this condition, it is easily seen that (2) of Definition 3.1 is exactly the condition for \( G_{(\Psi, \psi)} \) to be closed under the bracket given by relation (11) in Proposition 2.8.
2) Let \( Y_i \in F \) and suppose that \( \Psi(Y_i) = \sum_j X_{ij} \otimes A b_{ij} \), \( i = 1, 2 \). We have already shown that the condition \( \Psi(Y_1) + Y_2 \in E \oplus F \) is relation \( (11) \). And by Proposition 2.9 the condition that \( [\Psi(Y_1) + Y_2, \Psi(Y_2) + Y_1] \in E \oplus F \) is exactly relation \( (12) \). ■

It may happen that, given \( \psi \), no morphism or comorphism \( \Psi \) over \( \psi \) exist. But the \( \psi \)- sum always exists. This phenomenon also happens when Lie algebroids or groupoids are concerned (see Remark 1.9 and examples at the end of the paper).

• The Category of Lie Pseudoalgebras.

As an application of Theorem 3.6, we now prove that Lie pseudoalgebras are the objects of a category, with either morphisms or comorphisms, which is proved originally in \[13\]. We will need the following fact which can be drawn directly by the definition.

Proposition 3.8. Let \((E, A), (F, B), (G, C)\) be three Lie pseudoalgebras and \( \psi : A \to B, \theta : B \to C \) be two algebra morphism. Then as a Lie subpseudoalgebra over \( C \),

\[
(E \oplus F) \oplus \Theta G \subset E \oplus \Theta \psi (F \oplus \Theta G).
\]

The concept of category can be found in \[14\]. In the present paper, we define the category \( \mathfrak{P} \) and \( co\mathfrak{P} \) of Lie pseudoalgebras, using the following three pieces of data:

1) a class of objects \((E, A), (F, B), \ldots\), which are respectively pseudoalgebras;

2) to each pair of objects \((E, A), (F, B)\) of \( \mathfrak{P} \) (or \( co\mathfrak{P} \)), a set \( \mathcal{M}((E, A); (F, B)) \), the collection of all morphisms \((E, A) \rightrightarrows (F, B)\) from \((E, A)\) to \((F, B)\) (or, a set \( co\mathcal{M}((E, A); (F, B)) \), the collections of all comorphisms \((E, A) \rightrightarrows (F, B)\) from \((F, B)\) to \((E, A)\);

3) to each triple of objects \((E, A), (F, B), (G, C)\), a law of composition

\[
\mathcal{M}((E, A); (F, B)) \times \mathcal{M}((F, B); (G, C)) \to \mathcal{M}((E, A); (G, C)).
\]

(16)

(17)

Here in \( (16) \), the composition of \((E, A) \rightrightarrows (F, B)\) and \((F, B) \rightrightarrows (G, C)\) is the usual composition of maps

\[
(E, A) \rightrightarrows (G, C).
\]

In \( (17) \), we define the composition of \((E, A) \rightrightarrows (F, B)\) and \((F, B) \rightrightarrows (G, C)\) to be

\[
(E, A) \rightrightarrows (G, C).
\]

Here \( \Psi \Theta : G \to E \otimes A C \) (over \( \theta \circ \psi \)) is in fact \( \overline{\Psi} \circ \Theta \), where

\[
\overline{\Psi} = \Psi \otimes \text{Id} : F \otimes B C \to (E \otimes A B) \otimes B C = E \otimes A C.
\]

They obviously satisfy the usual axioms of associativity and identity. So the only problem is to check that the composition laws are well defined. In what follows we prove that in \( (17) \), the composition of two comorphisms is still a comorphism. And one can do similar procedure for morphisms.

As above, let \((E, A) \rightrightarrows (F, B)\) and \((F, B) \rightrightarrows (G, C)\) be two comorphisms. We prove that their composition \((E, A) \rightrightarrows (G, C)\) is also a comorphism of Lie pseudoalgebras. It suffices to show that the graph

\[
G_{\Psi \Theta} = \{ \Psi \Theta(Z) + Z | Z \in G \} \subset (E \otimes A C) \oplus G.
\]

11
is a subalgebra of \( E \oplus_\Theta \psi G \) (over \( C \)). If we consider the embedding map

\[
i: (E \otimes_A C) \oplus G \to (E \otimes_A C) \oplus (F \otimes_B C) \oplus G,
\]

\[
X \otimes_A c + Z \mapsto X \otimes_A c + \Theta(Z) + Z,
\]

for all \( X \in E, Z \in G, c \in C \), then it suffices to show that \( i(G_{(\psi, \Theta_\psi)}) \) is a subalgebra of

\[
i(E \oplus_\Theta \psi G) = E \oplus_\Theta \psi G.(\Theta_\psi).
\]

In fact, by applying Theorem 3.6 we know that \( E \oplus_\Theta \psi G \) is a subalgebra of \( E \oplus_\Theta \psi (F \oplus_\Theta G) \). And at the same time, \( G_{(\psi, \psi)} \oplus_\psi G \) is a subalgebra of \( (E \oplus_\psi F) \oplus_\psi G \). So it is also a subalgebra of \( E \oplus_\Theta \psi (F \oplus_\Theta G) \) (by Proposition 3.8) and therefore the intersection

\[
(E \oplus_\Theta \psi G)(\Theta_\psi) \cap (G_{(\psi, \psi)} \oplus_\psi G) = i(G_{(\psi, \Theta_\psi)})
\]

is a subalgebra of \( E \oplus_\Theta \psi (F \oplus_\Theta G) \). Of course this implies that it is also a subalgebra of \( i(E \oplus_\Theta \psi G) \) and thus the proof is complete.

There are three crucial points in the above analysis:

1) If \((E_1, A)\) is a subalgebra of \((E_2, A)\), and \((E_2, A)\) is a subalgebra of \((E_3, A)\), then \((E_1, A)\) is a subalgebra of \((E_3, A)\).

2) If \((E_1, A)\) and \((E_2, A)\) are both subalgebras of \((E_3, A)\), then their intersection \((E_1 \cap E_2, A)\) is a subalgebra of \((E_3, A)\).

3) If \((E_1, A)\) is a subalgebra of \((E_3, A)\), and \((E_2, A)\) is a subalgebra of \((E_3, A)\), and \(E_1 \subset E_2\), then \((E_1, A)\) is a subalgebra of \((E_2, A)\).

4 Morphisms and Comorphisms of Lie Algebroids

This part is devoted to expressing the preceding theories in the language of Lie algebroids. We recall the definition of morphism and comorphism of Lie algebroids, which originally appeared in [13] and we recommend Mackenzie’s book [25] for detailed information. However, we adopt a different approach to these two concepts in this paper. Although the original ones are equivalent to the definitions which follow, the latter are quite concise and succinct in language. We finally show how they are embedded into an algebroid which we called the \( \phi \)-sum, as subalgebroids.

A basic fact should be mentioned at the beginning. Consider a smooth map \( \phi : M \to N \). Let \( V \) be a vector bundle over \( N \). We have the pull back bundle \( \phi^*V \) (over \( M \)) and a morphism of algebras \( \psi = \phi^*: C^\infty(N) \to C^\infty(M) \). For another vector bundle \( W \) over \( M \), and a bundle map \( \Psi : \phi^*V \to W \), it determines the dual bundle map

\[
W^* \xrightarrow{\Psi^*} V^*
\]

\[
p \downarrow \quad \downarrow q
\]

\[
M \xrightarrow{\phi} N.
\]

It also naturally induces an additive map \( \tilde{\Psi} : \Gamma(V) \to \Gamma(W) \) satisfying

\[
\tilde{\Psi}(fB) = (\phi^*f)\tilde{\Psi}(B), \quad \forall f \in C^\infty(N), B \in \Gamma(V).
\]

This implies that \( \tilde{\Psi} \) is a map of \( C^\infty(N) \)-modules, in the sense of \( \phi^* \). Conversely, any \( C^\infty(N) \)-module map \( \tilde{\Psi} : \Gamma(V) \to \Gamma(W) \) is uniquely determined by such a bundle map \( \Psi : \phi^*V \to W \), or \( \Psi^* : W^* \to V^* \).

By definition, a Lie algebroid \((\mathcal{A}, M, \rho_\mathcal{A})\) gives rise to a Lie pseudoalgebra \((\Gamma(\mathcal{A}), C^\infty(M))\) and we can therefore extend the two kinds of morphisms described in Section 3 to Lie algebroids, in both cases allowing the bases to be changed [24, 13, 25].
**Definition 4.1.** Let \((\mathcal{U}, M, \rho_\mathcal{U})\) and \((\mathcal{V}, N, \rho_\mathcal{V})\) be Lie algebroids on bases \(M\) and \(N\) respectively. Given a smooth map \(\phi: M \to N\), a **comorphism of Lie algebroids** from \(\mathcal{V}\) to \(\mathcal{U}\) over \(\phi\) is a bundle map \((\Psi, \phi)\)
\[ \Psi: \phi^* \mathcal{V} \to \mathcal{U}, \]
written \((\mathcal{V}, N) \cong (\mathcal{U}, M)\), such that the dual map \(\Psi^*: \mathcal{U}^* \to \mathcal{V}^*\) is a Poisson map. Here \(\mathcal{U}^*\) and \(\mathcal{V}^*\) both carry the Lie-Poisson structures coming from their Lie algebroid structures.

In particular, if \(\phi\) is surjective and \(\Psi\) is injective, then we call \((\mathcal{V}, N)\) a **co-subalgebroid** of \((\mathcal{U}, M)\).

According to the relationship between \(\Psi\) and \(\tilde{\Psi}\), we sometimes directly call \(\tilde{\Psi}: \Gamma(\mathcal{V}) \to \Gamma(\mathcal{U})\) the comorphism of Lie algebroids. It will be convenient at times to consider the comorphisms of Lie algebroids from this alternate point of view, by using the following equivalent description [13].

**Theorem 4.2.** With the assumptions in the above definition, \((\mathcal{V}, N) \cong (\mathcal{U}, M)\) is a comorphism of Lie algebroids if and only if the following two conditions hold:
1) \(\phi_* \circ \rho_\mathcal{U} \circ \Psi = \rho_\mathcal{V}\);
2) the induced map \(\tilde{\Psi}: \Gamma(\mathcal{V}) \to \Gamma(\mathcal{U})\) is a morphism of Lie algebras.

We remark that, the first condition can be restated as: for each \(B \in \Gamma(\mathcal{V})\), the vector field \(\rho_\mathcal{U}(\tilde{\Psi}(B))\) is \(\phi\)-related to \(\rho_\mathcal{V}(B)\). The composition law of morphisms is straightforward.

**Remark 4.3.** Let \(\psi = \phi^*: C^\infty(N) \to C^\infty(M)\). In Theorem 4.2 (1) is equivalent to
\[ \psi([B,g]_\mathcal{V}) = [\tilde{\Psi}(B),\psi(g)]_\mathcal{U}, \quad \forall B \in \Gamma(\mathcal{V}), g \in C^\infty(N). \]
(C.f. relation (1) of Definition 3.1) Hence the **comorphism** in Theorem 4.2 is actually saying that
\[ (\Gamma(\mathcal{V}), C^\infty(N)) \ni (\tilde{\Psi}, \psi) \quad \Rightarrow \quad (\Gamma(\mathcal{U}), C^\infty(M)), \]
is a morphism of Lie pseudoalgebras.

**Example 4.4.** Let \((M, \pi)\) be a Poisson manifold and by \(\Omega(M)\) we denote \(\Gamma(T^*M)\). With the \(\pi\)-bracket defined below
\[ [\xi, f]_\pi \triangleq \pi^\sharp(\xi)(f); \quad \langle [\xi, \eta]_\pi, X \rangle \triangleq \langle [X, \pi], \xi \wedge \eta \rangle + \pi^\sharp(\xi)\langle \eta, X \rangle - \pi^\sharp(\eta)\langle \xi, X \rangle, \]
for any \(\xi, \eta \in \Omega(M), f \in C^\infty(M)\) and \(X \in \mathcal{X}(M)\), it is well known that \((\Omega(M), C^\infty(M))\) is a Lie pseudoalgebra and \(T^*M\) is a Lie algebroid on \(M\). Let \((M, \pi)\) and \((N, \omega)\) be two Poisson manifolds. Assume that \(\phi: M \to N\) is a Poisson map, which induces \(\phi_*: TM \to TN\) and the dual map \(\Phi = \phi^*: \Omega(N) \to \Omega(M)\). Then, \(\Phi\) is a comorphism of Lie algebroids over \(\phi\)
\[ (T^*N, N) \ni (\phi^*, \phi) \quad \Rightarrow \quad (T^*M, M). \]

The definition of morphisms of Lie algebroids appears in many texts such as [12], [21] (see also [13] [25]), stated in the form given in Theorem 4.6. But we prefer to adopt a more concise one as below.

**Definition 4.5.** Let \((\mathcal{U}, M, \rho_\mathcal{U})\) and \((\mathcal{V}, N, \rho_\mathcal{V})\) be Lie algebroids on bases \(M\) and \(N\) respectively. A **morphism of Lie algebroids** from \(\mathcal{U}\) to \(\mathcal{V}\), written \((\mathcal{U}, M) \ni (\mathcal{V}, N)\), is a vector bundle morphism

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\Psi} & \mathcal{V} \\
p & & q \\
M & \xrightarrow{\phi} & N
\end{array}
\]

(20)
such that the induced map \( \tilde{\Psi}^* : \Gamma(\wedge^k V^*) \to \Gamma(\wedge^k U^*) \) is a chain map, i.e.,

\[
d_U \circ \tilde{\Psi}^* = \Psi^* \circ d_V, \quad \text{as a map } \Gamma(\wedge^k V^*) \to \Gamma(\wedge^{k+1} U^*) \quad (k \geq 0).
\]

Here we regard \( \tilde{\Psi}^* = \psi^* : \Gamma(\wedge^0 V^*) = C^\infty(N) \to \Gamma(\wedge^0 U^*) = C^\infty(M) \). In particular, if \( \phi \) and \( \Psi \) are both injective, then we call \((U, M)\) a subalgebroid of \((V, N)\).

**Theorem 4.6.** With the assumptions in the above definition, \((U, M) \x leftarrow (V, N)\) is a morphism of Lie algebroids if and only if

1) \[
\rho_V \circ \Psi = \phi_* \circ \rho_U;
\]

2) if \( A, A' \in \Gamma(U) \) and \( \Psi^1(A) = \sum_i f_i B_i, \ \Psi^1(A') = \sum_i f'_i B'_i \), for some \( f_i, f'_i \in C^\infty(M) \), \( B_i, B'_i \in \Gamma(V) \), where \( \Psi^1 \) is the induced bundle map \( U \rightarrow \phi^1 V \), then

\[
\Psi^1([A, A']) = \sum_{i,j} f_i f'_j [B_i, B'_j] + \sum_j [A, f'_j]_U B'_j - \sum_i [A', f_i]_U B_i.
\]

The reader should bear in mind that the second statement of the theorem is a local condition. The proof is omitted since it is merely a repetition of Proposition 4.3. Although the above two equalities seem quite complicated, by the following remark and according to Proposition 4.5 one is able to understand why \((21)\) together with \((22)\) are equivalent to Definition 4.3.

**Remark 4.7.** Let \( \psi = \phi^* : C^\infty(N) \rightarrow C^\infty(M) \). In Theorem 4.6 relation \((21)\) is equivalent to

\[
[A, \psi(g)]_U = \sum_i f_i \psi([B_i, g]_V), \quad \forall A \in \Gamma(U), g \in C^\infty(N).
\]

(C.f. relation \((11)\).) In addition, one may regard \( \Gamma(\phi^1 V) \approx \Gamma(V) \otimes C^\infty(N) \subset C^\infty(M) \). Therefore, the morphism in Theorem 4.6 is in fact a comorphism of Lie pseudoalgebras

\[
(\Gamma(U), C^\infty(M)) \x leftarrow (\Gamma(V), C^\infty(N))
\]

Note that, by the two equivalent descriptions of morphisms of Lie algebroids in Definition 4.3 and in Theorem 4.6 we recover Theorem 3.1 in [10].

The above two kinds of morphisms are just the algebroid version corresponding to those of Lie pseudoalgebras. Recall the \( \phi \)-sum \( U \oplus_\phi V \) of Lie algebroids \( U \) and \( V \) define in Example 2.13. When passing from the morphisms and comorphisms of Lie pseudoalgebras to those of Lie algebroids, the relationships of the two kinds of morphisms stated in Theorem 3.6 admit straightforward generalizations, formulated as follows.

**Theorem 4.8.** Let \((U, M, \rho_U)\) and \((V, N, \rho_V)\) be two Lie algebroids and let \( \phi : M \rightarrow N \) be a smooth map.

1) For a bundle map \( \Psi : \phi^1 V \rightarrow U \), let its graph be

\[
\mathcal{G}_{(\Psi, \phi)} = \{(\Psi(v), v) \mid x \in M, v \in V_{\phi(x)}\} \subset U \oplus \phi^1 V.
\]

Then, \((V, N) \x leftarrow (U, M)\) is a comorphism of Lie algebroids if and only if \( \mathcal{G}_{(\Psi, \phi)} \) is contained in \( U \oplus_\phi V \) and \( \Gamma(\mathcal{G}_{(\Psi, \phi)}) \) is a Lie subalgebra of \( \Gamma(U \oplus_\phi V) \).
2) For a bundle map $\Psi : U \to V$, let its graph be

$$\mathcal{G}_{(\Psi,\phi)} = \{(u,\Psi(u)) | x \in M, u \in U_x \} \subset U \oplus \phi V.$$ 

Then, $(U,M) \xrightarrow{(\Psi,\phi)} (V,N)$ is a morphism of Lie algebroids if and only if $\mathcal{G}_{(\Psi,\phi)}$ is contained in $U \oplus \phi V$ and $\Gamma(\mathcal{G}_{(\Psi,\phi)})$ is a Lie subalgebra of $\Gamma(U \oplus \phi V)$. 

Remark 4.9. Evidently a comorphism of Lie algebroids $(V,N) \xrightarrow{(\Psi,\phi)} (U,M)$ requires that

$$\phi_*\text{Im}(\rho_{U|x}) \subset \text{Im}(\rho_V|_{\phi(x)}).$$

And a morphism of Lie algebroids $(U,M) \xrightarrow{(\Psi,\phi)} (V,N)$ requires that

$$\phi_*\text{Im}(\rho_{U|x}) \subset \text{Im}(\rho_V|_{\phi(x)}).$$

In the following examples, we show that the traditional representation and action theories can be restated in the language of morphisms and comorphisms.

Example 4.10. Let $V \to M$ be a vector bundle. Then one has the bundle of covariant differential operators, written $\mathcal{CDO}(V) \to M$ (c.f. [23 III], [24], see also [20], where the notation $\mathcal{D}(V)$ is used instead of $\mathcal{CDO}(V)$). A derivative representation of a Lie algebra $\mathfrak{g}$ on $V$, is a morphism of $\mathfrak{g}$ into the Lie algebra $\Gamma(\mathcal{CDO}(V))$ [20]. This is obviously a comorphism of Lie algebroids from $\mathfrak{g}$ to $\mathcal{CDO}(V)$, over the trivial map $M \to pt$.

Example 4.11. [20] Let $(U,M)$ be a Lie algebroid, and let $\varphi : Z \to M$ be a fibred manifold (i.e., $\varphi$ is a surjective submersion onto $M$). An infinitesimal action of $U$ on $Z$ is an $\mathbb{R}$-linear map $\Gamma(U) \to \mathcal{X}(Z)$, $A \mapsto A_Z$, where $A \in \Gamma(U)$, such that

1) $A_Z$ is projective to $\rho_U(A)$ (i.e., they are $\varphi$-related);

2) the map $A \mapsto A_Z$ preserves brackets;

3) the map $A \mapsto A_Z$ is $C^\infty(M)$-linear.

Of course this is equivalently saying that $(\cdot) \to (\cdot)_Z$ over the fibred map $\varphi : Z \to M$, is a comorphism of Lie algebroids from $U$ to $TZ$.

Example 4.12. [20] The derivative representations of a Lie algebra in Example 4.10 can be generalized to a Lie algebroid. Let $(U,M)$ be a Lie algebroid, and let $\varphi : Z \to M$ be a fibred manifold and $V \to Z$ be a vector bundle on $Z$. A derivative representation of $U$ on $V$ associated to a given infinitesimal action of $U$ on $Z$: $A \mapsto A_Z$ (as in Example 4.11), is a morphism, $\gamma$, of Lie algebras from $\Gamma(U)$ to $\Gamma(\mathcal{CDO}(V))$ such that

1) for any $A \in \Gamma(U)$, $\rho_{\mathcal{CDO}(V)} \gamma(A) = [\gamma(A), \cdot ] = A_Z$;

2) $\gamma$ is $C^\infty(M)$-linear in the sense that $\gamma(fA) = (\varphi^* f) \gamma(A)$, $\forall f \in C^\infty(M)$.

We point out that, such a derivative representation is a comorphism of Lie algebroids

$$(U,M) \xrightarrow{(\gamma,\varphi)} (\mathcal{CDO}(V),Z), \quad \text{where we view } \gamma \text{ as a map } : \varphi U \to \mathcal{CDO}(V).$$

Conversely, if a Lie algebroid comorphism is given as above, there are associated

1) an infinitesimal action of $U$ on $Z$, $A \mapsto [\gamma(A), \cdot ]$;

2) a derivative representation of $U$ on $V$, $A \mapsto \gamma(A)$, associated to the action of 1).
In fact, given any comorphism of Lie algebroids which is over a fibred map, we can determine an action of Lie algebroids \[13\] (see also similar conclusions in Theorem 5.15).

**Theorem 4.13.** Let \(\varphi : Z \to M\) be a fibred manifold, and let \((C, Z), (U, M)\) be two Lie algebroids. If 
\[
(U, M) \xrightarrow{\langle \varphi, \varphi \rangle} (C, Z)
\]
is a comorphism of Lie algebroids, then the map 
\[
A \mapsto \rho_C \circ \tilde{\Psi}(A), \quad \forall A \in \Gamma(U),
\]
defines an infinitesimal action of \(U\) on \(Z\). 

**Example 4.14.** [The action algebroid] Let \(g\) be a Lie algebra, \(M\) a smooth manifold and \(\theta : g \to \mathcal{X}(M)\) a morphism of Lie algebras (called an infinitesimal action of \(g\) on \(M\)). Then the vector bundle \(U = M \times g\) admits a Lie algebroid structure by setting 
\[
[fX, gY] = fg[X,Y] + f\theta(X)(g)Y - g\theta(Y)(f)X,
\]
\[
\rho(fX) = f\theta(X),
\]
\(\forall f, g \in C^\infty(M), X, Y \in g\). We call \(U = M \times g\) the action Lie algebroid induced by the action of \(g\) on \(M\). (This is a special case of Examples 4.11 and 4.15.) Evidently, the action algebroid \(U = M \times g\) admits both a trivial morphism to \(g\), and a trivial comorphism from \(g\), both over \(M \to \text{pt}\).

**Example 4.15.** We continue the assumptions in Example 4.11. It is shown in \[12\] that, there is an associated Lie algebroid structure on \(\varphi^!U\) with the base \(Z\), which is called the action Lie algebroid associated to the action \(A \to AZ\). And the diagram
\[
\begin{array}{ccc}
\varphi^!U & \xrightarrow{\text{Id}} & U \\
\downarrow{p} & & \downarrow{q} \\
Z & \xrightarrow{\varphi} & M
\end{array}
\]  
(23)

is a morphism of Lie algebroids. At the same time, there is a comorphism \((U, M) \xrightarrow{\langle \text{Id}, \varphi \rangle} (\varphi^!U, Z)\).

It is also worth noting that the action Lie algebroid \((\varphi^!U, Z)\) can be embedded into the \(\varphi\)-sum of \((TZ, Z)\) and \((U, M)\), by sending \((x, v)\) to \((v_Z, v)\), for each \(x \in Z, v \in U|_{\varphi(x)}\).

## 5 Morphisms and Comorphisms of Lie Groupoids

This section is an exposition of the theory of two kinds of morphisms concerning Lie groupoids, analogous to that of Lie algebroids. We recall here some well-known definitions and certain properties of Lie groupoids and their tangent Lie algebroids, and we refer to \[6, 23, 24, 25\] for more details (note that the composition convention in some of these texts is the opposite of that followed here).

A groupoid \(\Gamma\) on the base \(M\), with respectively source and target maps \(\alpha, \beta\), will be denoted by \((\Gamma \rightrightarrows M; \alpha, \beta)\), or, more briefly, \((\Gamma, M)\). We adopt the convention that, whenever we write a product \(gh\), we are assuming that is defined, i.e., \(\beta(g) = \alpha(h)\). The base \(M \subset \Gamma\) will be regarded as the set of identities. The inversion map \(\iota : \Gamma \to \Gamma\) will be denoted by \(\iota(g) = g^{-1}\).

For \(x \in M\), its **orbit**, denoted by \(O_x\), is the set \(\beta \circ \alpha^{-1}(x) \subset M\).

Let \((\Pi, N)\) be a groupoid and \(\phi : M \to N\) be a map. We define the \(\phi\)-pullback of \(\Pi\) (with respect to the \(\alpha\)-fiber)
\[
M \times_{\phi} \Pi \triangleq \{(x, w) | x \in M, \phi(x) = \alpha(w)\},
\]
which is also denoted by $\phi^*\Pi$, or $\phi^*\Pi$ by many authors. When $\Pi$ is a Lie groupoid on $N$ and $\phi$ is a smooth map between smooth manifolds, $\alpha_{\Pi}$ is a submersion and hence $M \times_{\phi} \Pi$ is a smooth manifold, and $\text{pr}_M$, the projection to $M$ is a submersion.

We are now ready to introduce the two concepts of morphism of groupoids analogous to the morphisms and comorphisms of Lie algebroids. We will prove that they are global formulations in terms of Lie groupoids. In [13], Higgins and Mackenzie had given the definition of comorphisms of groupoids in the language of actions. Here we prefer to adopt a direct description as follows.

**Definition 5.1.** Let $(\Gamma \rightrightarrows M, \alpha_{\Gamma}, \beta_{\Gamma})$ and $(\Pi \rightrightarrows N, \alpha_{\Pi}, \beta_{\Pi})$ be two groupoids on bases $M$ and $N$ respectively. A **comorphism of groupoids** from $\Pi$ to $\Gamma$, over a given map $\phi : M \to N$, is a map $\Phi : M \times_{\phi} \Pi \to \Gamma$, written $(\Pi, N) \rightrightarrows (\Gamma, M)$, such that the diagram

$$
\begin{array}{ccc}
M \times_{\phi} \Pi & \xrightarrow{\Phi} & \Gamma \\
\text{pr}_M \downarrow & & \downarrow \alpha_{\Gamma} \\
M & \xrightarrow{\text{Id}_M} & M
\end{array}
$$

commutes and the following conditions hold

1) for all $x \in M$, $\Phi(x, \phi(x)) = x$;

2) for all $(x, w) \in M \times_{\phi} \Pi$, $\phi \circ \beta_{\Gamma} \circ \Phi(x, w) = \beta_{\Pi}(w)$;

3) for all $(x, w) \in M \times_{\phi} \Pi$, $(\beta_{\Gamma} \circ \Phi(x, w), z) \in M \times_{\phi} \Pi$, there holds

$$
\Phi(x, wz) = \Phi(x, w)\Phi(\beta_{\Gamma} \circ \Phi(x, w), z).
$$

In particular, if $\phi$ is surjective and $\Phi$ is injective, then we call $(\Pi, N)$ a **co-subgroupoid** of $(\Gamma, M)$.

The morphisms of groupoids are already a well-known concept [23], and they are global version of the morphisms of Lie algebroids.

**Definition 5.2.** Let $(\Gamma \rightrightarrows M, \alpha_{\Gamma}, \beta_{\Gamma})$ and $(\Pi \rightrightarrows N, \alpha_{\Pi}, \beta_{\Pi})$ be two groupoids on bases $M$ and $N$ respectively. A **morphism of groupoids** over $\phi : M \to N$ is a map $\Phi : \Gamma \to \Pi$, written $(\Gamma, M) \rightrightarrows (\Pi, N)$, such that

1) $\Phi(M) \subset N$ and $\phi = \Phi|_M$;

2) $\alpha_{\Pi} \circ \Phi = \phi \circ \alpha_{\Gamma}$, $\beta_{\Pi} \circ \Phi = \phi \circ \beta_{\Gamma}$;

3) $\Phi(gh) = \Phi(g)\Phi(h)$, for all composable $g, h \in \Gamma$.

In particular, if $\phi$ and $\Phi$ are both injective, then we call $(\Gamma, M)$ a **subgroupoid** of $(\Pi, N)$.

For the base-preserving case $M \xrightarrow{\phi=\text{Id}} N = M$, these two kinds of morphisms coincide.

A comorphism of Lie groupoids $(\Pi, N) \rightrightarrows (\Gamma, M)$, or a morphism $(\Gamma, M) \rightrightarrows (\Pi, N)$, is defined similarly, with the additional requirement that $\Phi$, as well as $\phi$ in the above definitions to be smooth maps.

For a Lie groupoid $(\Gamma \rightrightarrows M; \alpha, \beta)$, we define the **tangent Lie algebroid** $(\text{Lie}\Gamma, \rho)$ as

$$
\text{Lie}\Gamma \triangleq \bigcup_{x \in M} T_x\alpha^{-1}(x) = \{v \in T_x\Gamma | x \in M, \alpha_*(v) = 0 \}.
$$

The bracket of $\Gamma(\text{Lie}\Gamma)$ is determined by the commutator of left-invariant vector fields and the anchor map is given by $\rho = \beta_{\Gamma*}|_M$.

The infinitesimal counterpart of a comorphism can be determined by differentiation, yielding a co-morphism of Lie algebroids. This fact is illustrated by the following theorem.
Theorem 5.3. Let \((\Pi, N) \rightrightarrows (\Gamma, M)\) be a comorphism of Lie groupoids. Then, the tangent map of \(\Phi\) induces a vector bundle morphism \(\Psi = \Phi_*(\text{Lie}\Pi) \rightarrow \text{Lie}\Gamma\) and \((\Pi, N) \rightrightarrows (\text{Lie}\Pi, M)\) is a comorphism of Lie algebroids.

Proof. By (1) of Definition 5.1 \(\Phi_*\) sends \(T_{x,\phi(x)}\alpha^{-1}_\Pi(\phi(x))\) to \(T_x\alpha^{-1}_\Gamma(x)\) and hence \(\Psi|_x \triangleq \Phi_*|_{(x,\phi(x))}\), for all \(x \in M\), defines a bundle map. According to (2), we have

\[
\phi_* \circ \beta_* \circ \Phi_*(x, v) = \beta_{\Pi_\phi}(v), \quad \forall v \in \text{Lie}\Pi_{\phi(x)},
\]

i.e., \(\Psi\) is subject to the condition \(\phi_* \circ \rho_{\text{Lie}\Pi} \circ \Psi = \rho_{\text{Lie}\Gamma}\). It remains to prove that the induced map \(\tilde{\Psi} : \text{Lie}(\Pi) \rightarrow \text{Lie}(\Gamma)\) is a morphism of Lie algebras.

For each section \(B \in \Gamma\) (\(\Pi\)), we denote the corresponding left-invariant vector field on \(\Pi\) by \(\widehat{B}\) and similarly, \(\Psi(B)\) denotes the left-invariant vector field on \(\Gamma\). Let us prove that \(\widehat{B}\), regarded as a vector field on \(M \times \phi\) \(\Pi\), is \(\Phi\)-related to \(\Psi(B)\). For each \(x \in M\), \(w \in \alpha^{-1}_\Pi(\phi(x))\), if we suppose that \(B_{\beta_{\Pi}(w)} = \frac{d}{dt}c(0)\), where \(c : [0, 1] \rightarrow \alpha^{-1}_\Pi(\beta_{\Pi}(w))\) is a smooth curve, then

\[
\Phi_*(x, w) \widehat{B}(w) = \frac{d}{dt}|_{t=0}\Phi(x, wc(t))
\]

\[
= \frac{d}{dt}|_{t=0}\Phi(x, w)\Phi(\beta_* \circ \Phi(x, w), c(t)) \quad \text{(by (3) in Definition 5.1)}
\]

\[
= L_{\Phi(x, w)}(\Phi(\beta_* \circ \Phi(x, w), c(t)),)
\]

\[
= L_{\Phi(x, w)}(\Psi(\beta_* \circ \Phi(x, w), B_{\beta_{\Pi}(w)})) = \Psi(B)(\Phi(x, w)).
\]

Hence for two \(B_1, B_2 \in \text{Lie}(\Pi), [\widehat{B_1}, \widehat{B_2}] = [\widehat{B_1}, \widehat{B_2}]\) is also \(\Phi\)-related to \([\Psi(B_1), \Psi(B_2)] = [\Psi(B_1), \Psi(B_2)]\).

It follows that

\[
\Psi([B_1, B_2]) = \Phi_*|_M([\widehat{B_1}, \widehat{B_2}]) = [\Psi(B_1), \Psi(B_2)]|_M = [\Psi(B_1), \Psi(B_2)].
\]

Similarly, for a morphism \((\Gamma, M) \rightrightarrows (\Pi, N)\) of Lie groupoids, the tangent map \(\Phi_*\) which evidently sends \(\text{Lie}\Gamma\) to \(\text{Lie}\Pi\), is a morphism of Lie algebroids (over \(\phi\)). The details of the proof can be found in [12].

- The \(\phi\)-product of groupoids.

There is likewise a global version of the restriction theory for groupoids. Let \((\Gamma \rightrightarrows M; \alpha, \beta)\) be a groupoid. \(M_0 \subset M\) a subset. We call

\[
\Gamma_{M_0} \triangleq \{g \in \Gamma| \alpha(g) \in M_0, \beta(g) \in M_0\}
\]

the \(M_0\)-restriction of \(\Gamma\). It is easy to see that \(\Gamma_{M_0}\) is also a groupoid on the base \(M_0\), inheriting the structures of \(\Gamma\) (c.f. Example 2.11).

For two groupoids \((\Gamma \rightrightarrows M, \alpha_\Gamma, \beta_\Gamma)\) and \((\Pi \rightrightarrows N, \alpha_\Pi, \beta_\Pi)\) on bases \(M\) and \(N\) respectively, we can endow their direct product \(\Gamma \times \Pi\) with a groupoid structure on \(M \times N\) by setting

\[
\alpha(g, w) \triangleq (\alpha_\Gamma(g), \alpha_\Pi(w)), \quad \beta(g, w) \triangleq (\beta_\Gamma(g), \beta_\Pi(w));
\]

\[
(g, w)(h, z) = (gh, wz), \quad \text{where} \ g, h \in \Gamma \text{ and } w, z \in \Pi \text{ are respectively composable}.
\]

We will denote this groupoid by \(\Gamma \times \Pi\), and call it the direct product of \(\Gamma\) and \(\Pi\).
Definition 5.4. The $G(\phi)$-restriction of the direct sum groupoid $\Gamma \times \Pi$, denoted by $\Gamma \times_\phi \Pi$, is called the $\phi$-product of $\Gamma$ and $\Pi$.

One may characterize such a groupoid by

$$\Gamma \times_\phi \Pi \triangleq \{(g, w) \in \Gamma \times \Pi | \alpha_\Pi(w) = \phi \circ \alpha_\Gamma(g), \beta_\Pi(w) = \phi \circ \beta_\Gamma(g)\}.$$  

Notice that, $\Gamma \times_\phi \Pi$ is a groupoid on base space $M$. The source and target maps send $(g, w) \in \Gamma \times_\phi \Pi$ to $\alpha_\Gamma(g)$ and $\beta_\Gamma(g)$ respectively (c.f Example 2.13).

On the global level, the analogues of Theorem 3.6 and Theorem 4.8 are as follows. Again we omit the proof.

Theorem 5.5. Let $(\Gamma \Rightarrow M, \alpha_\Gamma, \beta_\Gamma)$ and $(\Pi \Rightarrow N, \alpha_\Pi, \beta_\Pi)$ be two groupoids. Let $\phi : M \rightarrow N$ be a map.

1) A map $\Phi : M \times_\phi \Pi \rightarrow \Gamma$ is a comorphism of groupoids if and only if the graph

$$G(\Phi, \phi) \triangleq \{(\Phi(x, g), g) | x \in M, g \in \alpha_\Pi^{-1}(\phi(x))\}$$

is a subgroupoid of $\Gamma \times_\phi \Pi$, on the same base $M$.

2) A map $\Phi : \Gamma \rightarrow \Pi$ is a morphism of groupoids if and only if the graph

$$G(\Phi, \phi) \triangleq \{(g, \Phi(g)) | g \in \Gamma\}$$

is a subgroupoid of $\Gamma \times_\phi \Pi$, on the same base $M$. $\blacksquare$

Remark 5.6. Evidently a comorphism of groupoids $(\Pi, N) \xrightarrow{\phi} (\Gamma, M)$ requires that the $\phi$-image of each orbit of $\Gamma$ covers some orbit of $\Pi$, i.e., $O_{\phi(x)} \subset \phi(O_x)$. And a morphism of groupoids $(\Gamma, M) \xrightarrow{\phi} (\Pi, N)$ requires that $\phi$ sends each orbit of $\Gamma$ into an orbit of $\Pi$, i.e., $\phi(O_x) \subset O_{\phi(x)}$ (c.f. Remark 4.9).

- Examples.

In the remaining part of the paper, we relate some examples showing various kinds of morphisms and comorphisms of Lie algebroids and groupoids, to be compared with the examples of actions and representations in the preceding section.

Example 5.7. Let $M$ be a smooth manifold. Then $M \times M$ admits a groupoid structure, called the pair groupoid on $M$. The tangent Lie algebroid for $M \times M$ is $TM$. \[6\]

Example 5.8. Let $\phi : M \rightarrow N$ be a smooth map between smooth manifolds. The map $\phi : M \rightarrow N$ naturally induces a morphism of the pair groupoids

$$(M \times M, M) \xrightarrow{(\Phi, \phi)} (N \times N, N),$$

by setting $\Phi(x, y) = (\phi(x), \phi(y)), \forall x, y \in M$. Clearly, $\phi_* : TM \rightarrow TN$ is the corresponding morphism of their tangent Lie algebroids.

Example 5.9. We investigate the comorphisms of the pair groupoids. Suppose that a map $\phi : M \rightarrow N$ is given and there is a comorphism $\Phi : M \times_\phi (N \times N) \rightarrow M \times M$ over $\phi$. Here we see that

$$M \times_\phi (N \times N) = \{(x, \phi(x), y) | x \in M, y \in N\}.$$  

According to Definition 5.4, $\Phi$ can be written in the form

$$\Phi : (x, \phi(x), y) \mapsto (x, \Phi_0(x, y)), \quad \forall x \in M, y \in N,$$

where $\Phi_0 : M \times N \rightarrow M$. And the axioms in Definition 5.1 are now expressed as follows:

\[3\]The $\phi_0$-product of two Lie groupoids is not necessarily a Lie groupoid.
1) \( \Phi_0(x, \phi(x)) = x, \forall x \in M; \)
2) \( \phi \circ \Phi_0(x, y) = y, \forall x \in M, y \in N; \)
3) \( \Phi_0(x, z) = \Phi_0(\Phi_0(x, y), z), \forall x \in M, y, z \in N. \)

Thus \( \phi \) must be a surjection. And for each fixed \( x \in M, \Phi_0(x, \cdot) : N \to M \) is an injection as well as a right inverse of \( \phi \). If \( M \) and \( N \) are both smooth manifolds, these conditions are exactly saying that there is a foliation structure on \( M \) and each leaf is diffeomorphic to \( N \) via \( \phi \). So generally speaking, it is hard to find a comorphism from \( \Pi = N \times N \) to \( \Gamma = M \times M. \)

**Example 5.10.** We now study what a comorphism \( \Psi \), from the algebroid \( TN \) to \( TM \), could be. Suppose that \( \Psi \) is over \( \phi \), then by Remark 4.9 \( Im(\phi) \) must be an open submanifold of \( N \), and \( \phi \) must be a submersion from \( M \) to \( Im(\phi) \). And the map \( \Psi \) must assign every \( Y \in \mathcal{X}(N) \) to a lifted \( \Psi(Y) \in \mathcal{X}(M) \), such that \( \phi_*(\Psi(Y)) = Y \) and the bracket must be preserved

\[
[\Psi(X, Y)] = [\Psi(X), \Psi(Y)].
\]

For each \( x \in M \), let \( F_x = Im(\Psi(\{x\} \times T\phi(x)N)) \). The conditions above imply that each \( F_x \) is isomorphic to \( T\phi(x)N \) via \( \phi_* \). The distribution \( F \) on \( M \) is integrable. And each leaf of \( F \) is a covering space of \( Im(\phi) \).

**Example 5.11.** Let \( M \) be a submanifold of \( N \) and \( T \subset TM \) a regular, integrable distribution. Then the inclusion \( T \subset TN \) together with the embedding \( M \subset N \) is an injective comorphism of Lie algebroids. On the other hand, we have a groupoid with base \( M \),

\[
M \times_T M \equiv \{(x, y) \in M \times M | x \text{ and } y \text{ belong to the same integral submanifold of } M, \text{ via } T\}.
\]

The embedding of \( M \times_T M \) into \( N \times N \) is a morphism of groupoids.

**Example 5.12.** [The action groupoid] Let \( G \) be a group with the unit element \( e \) and let \( G \) act on a set \( M \) (to the right), i.e., we have a map \( \Theta : M \times G \to M, (x, g) \mapsto xg \), satisfying the axioms: \( xe = x \), \( x(g_1g_2) = (xg_1)g_2 \), for all \( x \in M, g_1, g_2 \in G \). Then \( \Gamma = M \times G \) admits a groupoid structure with base space \( M \), as follows

\[
\begin{align*}
\alpha : & \ (x, g) \mapsto x, \\
\beta : & \ (x, g) \mapsto xg, \\
\iota : & \ (x, g) \mapsto (xg, g^{-1}), \\
(x, g_1) & \cdot (xg_1, g_2) = (x, g_1g_2).
\end{align*}
\]

We call \( \Gamma = M \times G \) the action groupoid induced by the action of \( G \) on \( M \). Then \( Lie \Gamma \) is exactly the action algebroid \( M \times LieG \) given in Example 5.14 where \( LieG = T_eG \) has the Lie bracket coming from left-invariant vector fields. This is a particular case of Example 5.14 5.10.

**Example 5.13.** Given a vector bundle \( V \to M \), we denote by \( \mathcal{L}F(V) \) the linear frame groupoid of \( V \) (see [23] III, where it is denoted by \( \Pi(V) \)), which is the collection of all linear isomorphisms from a fiber of \( V \) to some generally different fiber of \( V \). The Lie algebroid of \( \mathcal{L}F(V) \) is canonically isomorphic to \( \mathcal{CDO}(V) \) (this fact was discovered independently by Kumpera [22], Mackenzie [23], Hermann [11], Kosmann-Schwarzbach [19]). The group of bisections \( \mathcal{L}F(V) \) is in fact \( Aut(V) \), the group of vector bundle automorphisms of \( V \). (One can also regard it as the semi-linear isomorphisms of \( \Gamma(V) \), see [19] [20].) In [20] Kosmann-Schwarzbuch and Mackenzie defined the semi-linear representation of a Lie group \( G \) on \( V \) to be a group morphism, say \( R \), from \( G \) to \( Aut(V) \), with the smooth condition that, the map \( (x, g) \mapsto R(g)(\nu)(x), M \times G \to V \), is smooth, for each \( \nu \in \Gamma(V) \) (or equivalently, the map \( (x, g) \mapsto R(g)|_x, M \times G \to \mathcal{L}F(V) \) is smooth).

This is actually a comorphism of Lie groupoids from \( G \) to \( \mathcal{L}F(V) \) (over \( M \to pt \)), and it provides a global formulation for derivative representations of Lie algebras (Example 4.10).
Example 5.14. Let \((\Gamma, M)\) be a groupoid, and let \(\varphi : Z \to M\) be a surjective map. A (right) action of \(\Gamma\) on \(Z\) is a map \(S\) assigning each \(g\) of \(\Gamma\) to another map

\[ S(g) : Z_{\alpha(g)} \to Z_{\beta(g)}, \]

such that

1) \(S(y) = \text{Id} : Z_y \to Z_y\), for any \(y \in M\). (It seems that [20] has missed this condition.)

2) \(S(g_1 g_2) = S(g_2) \circ S(g_1)\), for any composable \(g_1, g_2 \in \Gamma\).

We argue that, an action of \(\Gamma\) is a comorphism of groupoids from \(\Gamma\) to the pair groupoid \(Z \times Z\). In fact, if the action \(S\) is given as above, then by setting

\[ (x, g) \mapsto (x, S(g)(x)), \quad \forall x \in Z, g \in \alpha_{\Gamma}^{-1}(\varphi(x)), \]

we obtain a comorphism \(\Phi\). Conversely, given \(\Phi\), we can determine the action \(S\) by

\[ S(g)(x) \triangleq \text{pr}_2 \circ \Phi(x, g), \quad \text{where pr}_2 \text{ is the projection to the second component.} \]

When considering a Lie groupoid \(\Gamma\) and a fibred manifold \(\varphi : Z \to M\), we require such an action to be a smooth comorphism of Lie groupoids. At the same time, one obtains an infinitesimal action of the corresponding Lie algebroid \(\text{Lie}\Gamma\) on \(Z\). For more details about these groupoids and their infinitesimal forms, see Examples 5.14, 5.15, and 4.15.

Theorem 5.15. Let \(\varphi : Z \to M\) be a fibred manifold, and let \((\Omega, Z), (\Gamma, M)\) be two Lie groupoids. If

\[ (\Gamma, M) \xrightarrow{(\Phi, \varphi)} (\Omega, Z) \]

is a comorphism of Lie groupoids, then the map \(g \mapsto S(g)\), where

\[ S(g) : Z_{\alpha(g)} \to Z_{\beta(g)}, \quad x \mapsto \beta_{\Omega} \circ \Phi(x, g), \quad \forall x \in Z_{\alpha(g)}, \]

defines a Lie groupoid action of \(\Gamma\) on \(Z\).

Example 5.16. Recall Example 5.14. It was shown in [8] (see also [7, 20]) that whenever an action of a groupoid \((\Gamma, M)\) on \(\varphi : Z \to M\) is defined, there is an associated groupoid structure on the pull-back space \(Z \times_{\varphi} \Gamma\), with base \(Z\), known as the action groupoid associated to the given action. The source map is \((x, g) \mapsto x\), while the target map is \((x, g) \mapsto S(g)(x)\). The multiplication is \((x, g)(S(g)(x), h) = (x, gh)\). There is obviously a morphism from \(Z \times_{\varphi} \Gamma\) to \(\Gamma\), \((x, g) \mapsto g\), over \(\varphi\).

Example 5.17. In Example 4.12 we have discussed the derivative representations of Lie algebroids. On the global level, a semi-linear representation of a Lie groupoid associated to an action of \((\Gamma, M)\) on the fibred manifold \(Z \rightarrow M\), defined as in Example 5.14, is a morphism \(\Phi\) from the action groupoid \(Z \times_{\varphi} \Gamma\) to \(\mathcal{ LF}(V)\) (on the same base \(Z\)), where \(V \to Z\) is a vector bundle. This notion was studied in [5]. Here we emphasize that such a semi-linear representation is in fact a comorphism of Lie groupoids,

\[ (\Gamma, M) \xrightarrow{(\Phi, \varphi)} (\mathcal{ LF}(V), Z). \]

And the associated action of \(\Gamma\) on \(Z\) is determined by the relation

\[ S(g)(x) = \beta_{\mathcal{ LF}(V)} \circ \Phi(x, g), \quad \forall x \in Z, g \in \alpha_{\Gamma}^{-1}(\varphi(x)). \]
Example 5.18. Recall the action groupoid $Z \times_\varphi \Gamma$ given in Example 5.16. We point out that it is in fact a subgroupoid of the $\varphi$-product of the pair groupoid $Z \times Z$ and $\Gamma$. In fact, every element $(x, g) \in Z \times_\varphi \Gamma$ can be regarded as $((x, S(g)x); g) \in (Z \times Z) \times_\varphi \Gamma$, where $x \in Z$, $g \in \alpha^{-1}_\Gamma(\varphi(x))$.

Example 5.19. Let $G, H$ be two groups which act on $M$ and $N$ respectively. Suppose that there is a map $\phi : M \to N$ and a group morphism $\tau : G \to H$, and that they are compatible in the following sense

$$\phi(xg) = \phi(x)\tau(g), \quad \forall x \in M, g \in G.$$  \hfill (25)

Then, the map between action groupoids $\Phi : M \times G \to N \times H$ given by

$$\Phi(x, g) = (\phi(x), \tau(g)), \quad \forall x \in M, g \in G,$$

is a morphism of groupoids (over $\phi$).

Example 5.20. Let $G, H$ be two groups which act respectively on $M$ and $N$. Suppose that there is a map $\phi : M \to N$ and a group morphism $\zeta : H \to G$, and that they are compatible in the following sense

$$\phi(x)h = \phi(x\zeta(h)), \quad \forall x \in M, h \in H.$$  \hfill (26)

Then, there is a comorphism of groupoids $((N \times H, N) \xrightarrow{(\Phi, \phi)} (M \times G, M))$, where $\Phi$ is defined by

$$(x, (\phi(x), h)) \xrightarrow{\Phi} (x, \zeta(h)), \quad \forall x \in M, h \in H.$$

Note that the $\phi$-product of the two action groupoids $\Gamma = M \times G$ and $\Pi = N \times H$ in the preceding two examples is

$$\Gamma \times_\phi \Pi = \{(x, g; \phi(x), h)| x \in M, g \in G, h \in H, \phi(xg) = \phi(x)h\}.$$

Example 5.21. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras, and $M, N$ two smooth manifolds. Let $\theta : \mathfrak{g} \to \mathcal{X}(M)$ and $\eta : \mathfrak{h} \to \mathcal{X}(N)$ be two Lie algebra morphisms. Suppose that $\Upsilon : \mathfrak{g} \to \mathfrak{h}$ is also a Lie algebra morphism, $\phi : M \to N$ a smooth map and that they are compatible in the sense that

$$\eta(\Upsilon(X))\big|_{\phi(x)} = \phi_*\left(\theta(X)\big|_x\right), \quad \forall x \in M, X \in \mathfrak{g},$$  \hfill (27)

(i.e., $\theta(X)$ and $\eta(\Upsilon(X))$ are $\phi$-related.) Then, we find a morphism of the action Lie algebroids $\Psi : M \times \mathfrak{g} \to N \times \mathfrak{h}$ (over $\phi$), given by

$$\Psi(x, X) = (\phi(x), \Upsilon(X)), \quad \forall x \in M, X \in \mathfrak{g}.$$  \hfill (28)

Example 5.22. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras, and $M, N$ two smooth manifolds. Let $\theta : \mathfrak{g} \to \mathcal{X}(M)$, $\eta : \mathfrak{h} \to \mathcal{X}(N)$ be two Lie algebra morphisms. Suppose that $\Sigma : \mathfrak{h} \to \mathfrak{g}$ is also a Lie algebra morphism, $\phi : M \to N$ a smooth map and that they are compatible in the sense that

$$\eta(Y)\big|_{\phi(x)} = \phi_*\left(\theta(\Sigma(Y))\big|_x\right), \quad \forall x \in M, Y \in \mathfrak{h},$$  \hfill (29)

(i.e., $\theta(\Sigma(Y))$ and $\eta(Y)$ are $\phi$-related.) Then, there is a comorphism of the action Lie algebroids $\Psi : \Gamma(N \times \mathfrak{h}) \to \Gamma(M \times \mathfrak{g})$ (over $\phi$), given by

$$\Psi(fY) = \phi^*(f)\Sigma(Y), \quad \forall f \in C^\infty(N), Y \in \mathfrak{h}.$$  \hfill (30)

These two morphisms are the infinitesimal versions of Example 5.19 and 5.21.

Example 5.23. [The gauge groupoid] Let $(P \xrightarrow{\rho} M, G)$ be a principal bundle on $P$ with structure group $G$. Then $G$ acts diagonally on $P \times P$. We denote the orbit of $(x_1, x_2)$ by $\langle x_1, x_2 \rangle$ and the orbit manifold by $(P \times P)/G$. Define $\alpha : (P \times P)/G \to M$ by $\langle x_1, x_2 \rangle \mapsto p(x_1)$ and similarly $\beta(\langle x_1, x_2 \rangle) = p(x_2)$. For two elements $\langle x_1, x_2 \rangle$, $\langle y_1, y_2 \rangle \in (P \times P)/G$, if $p(x_2) = p(y_1)$, then there is a unique element $g \in G$ such
that $x_2 = y_1g$ and we define $\langle x_1, x_2 \rangle \langle y_1, y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1g, y_2g \rangle = \langle x_1, y_2g \rangle$. In this way, $(P \times P)/G$ carries a Lie groupoid structure with base $M$, called the gauge groupoid of $(P,M,G)$ [24, 31].

The action of $G$ on $P$ naturally lifts to an action on $TP$. We denote the orbit of $X_x \in T_xP$ by $\langle X_x \rangle$ and the quotient manifold by $\frac{TP}{\langle \cdot \rangle}$. Since this action is free, $\frac{TP}{\langle \cdot \rangle}$ admits a vector bundle structure with base $M$, and bundle projection $q : \langle X_x \rangle \rightarrow p(x)$. Sections of $\frac{TP}{\langle \cdot \rangle}$ can be regarded as vector fields on $P$ which are $G$-invariant. It follows that $\Gamma(\frac{TP}{\langle \cdot \rangle})$ has an induced bracket transferred from $\mathcal{X}(P)$. Besides, the tangent map $p_*$ can also be transferred to $\frac{TP}{\langle \cdot \rangle} \rightarrow TM$. Thus, $(\frac{TP}{\langle \cdot \rangle}, p_*, M)$ is a Lie algebroid, which is in fact the tangent Lie algebroid for the gauge groupoid $(P \times P)/G$ [1] [24].

**Example 5.24.** Continuing the above example, the map $\langle \cdot \rangle$ together with $p$ is obviously a morphism of Lie algebroids from $(TP,Id,P)$ to $(\frac{TP}{\langle \cdot \rangle}, p_*, M)$. Like-wise, we get a typical example of comorphism of Lie algebroids from $(\frac{TP}{\langle \cdot \rangle}, p_*, M)$ to $(TP,Id,P)$, by sending $X \in \Gamma(\frac{TP}{\langle \cdot \rangle})$ identically to $X \in \mathcal{X}(P)$ which is $G$-invariant, over the projection $p : P \rightarrow M$.

**Example 5.25.** Let $(P \xrightarrow{p} M,G)$ and $(Q \xrightarrow{q} N,H)$ be two principal bundles on base spaces $M$, $N$ respectively. Suppose that $\tau : G \rightarrow H$ is a morphism of Lie groups and $\phi : M \rightarrow N$ a smooth map. Given a smooth map $\Phi : P \rightarrow Q$ compatible with $\tau$ and $\phi$ in the sense that
\[ q \circ \Phi(x) = \phi \circ p(x), \quad \Phi(xg) = \Phi(x)\tau(g), \quad \forall x \in M, g \in G, \]
we can then construct a morphism of Lie groupoids $((P \times P)/G, M) \xrightarrow{(\Phi,\phi)} ((Q \times Q)/H,N)$, given by
\[ \Phi((x_1,x_2)) = (\Phi(x_1),\Phi(x_2)), \quad \forall x_1, x_2 \in P. \]
At the same time, we obtain a morphism of Lie algebroids $\Phi_*$ from $(\frac{TP}{\langle \cdot \rangle}, p_*, M)$ to $(\frac{TQ}{\langle \cdot \rangle}, q_*, N)$, over $\phi$.

**Example 5.26.** Let $(P \xrightarrow{p} M,G)$ be a principal bundle. Consider the pair groupoid $P \times P$ and the gauge groupoid $(P \times P)/G$ given in Example 5.24 [24, 31]. There are canonically defined morphism and comorphism of groupoids (both over $p$),
\[ (P \times P, P) \xrightarrow{(\cdot, \cdot),p} (\frac{P \times P}{G}, M), \quad (P \times P, M) \xrightarrow{(\Phi, p)} (P \times P, P). \]
We elaborate on the second one. For any element in $p^1((P \times P)/G)$, say $\langle x, (y, z) \rangle$, where $x, y, z \in P$ and $p(x) = p(y)$, we find $g \in G$ such that $y = xg$ and hence $\langle x, (y, z) \rangle = (\langle x, x \rangle, \langle x, zg^{-1} \rangle)$. Define $\Phi(x, (y, z)) = zg^{-1}$. Clearly $\Phi$ is well defined. One can easily verify that $\Phi$ over $p$ constitutes a comorphism of groupoids, which is the global version of the Lie algebroid comorphism in Example 5.24.

**References**

[1] M.F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* 85(1957), 181-207.

[2] C.M. de Barros, Espaces infinìtésimaux, *Cahiers Topol. Géom. Diff.* (1965), T.7.

[3] C.M. de Barros, Opérateurs infinitésimaux sur l’algèbre des formes différentielles extérieures, *C. R. Acad. Sci. Paris*, T.261, Group 1(1965), 4594-4597.

[4] R. Bкouche, Structures $(K,A)$-linéaires, *C. R. Acad. Sci. Paris*, T.262, série A (1966), 373-376.

[5] R. Brown, Fibrations of groupoids, *J. Algebra*, 15(1970), 103-132.

[6] A. Cannas da Silva and A. Weinstein, *Geometric Models for Noncommutative Algebras*, Berkeley Mathematics Lecture Notes, AMS, Providence, 1999. G/A.
[7] M. Crainic, Differentiable and algebroid cohomology, Van Est isomorphisms, and characteristic classes, *Commentarii Mathematici Helvetici* (78)4(2003), 681-721.

[8] C. Ehresmann, Gattungen von lokalen Strukturen. *Jahresber. d. Deutschen Math. Verein.*, 60:49-77, 1957. Contained in *Euvres complètes et commentées* (A.C. Ehresmann, editor. Seven volumes. Imprimerie Evrard, Amiens, 1984), Partie II-1. French translation, Espèces de structures locales. Séminaire Topologie Géom. Différentielle, 3, 1962. 24 pages.

[9] M. Gerstenhaber and S.D. Schack, *Algebras, bialgebras, quantum groups and algebraic deformations*, in *Deformation theory and quantum groups with applications to mathematical physics*, Gerstenhaber, M. and Stasheff, J. eds., *Contemporary Mathematics* 134:51-92, Amer. Math. Soc., Providence, 1992.

[10] L.-G., He, Z.-J., Liu. and D.S., Zhong, Poisson actions and Lie bialgebroid morphisms, *Quantization, Poisson Brackets and Beyond*, T. Voronov (ed.), *Contemp. Math.*315, Amer.Math. Soc.(2002), 172-181.

[11] R. Hermann, *Vector Bundles in Mathematical Physics*, Vol. 1. Benjamin, New York, 1970.

[12] P.J. Higgins and K. Mackenzie, Algebraic constructions in the category of Lie algebroids, *J.Algebra* 129(1990), 194-230.

[13] P.J. Higgins and K. Mackenzie, Duality for base-changing morphisms of vector bundles, modules, Lie algebroids and Poisson bundles, *Math. Proc. Cambridge Philos. Soc*. 114(1993), 471-488.

[14] P.J. Hilton and U. Stammbach, *A course in homological algebra*. Springer-Verlag 4.

[15] J. Huebschmann, Poisson cohomology and quantization, *J. Reine Angew. Math.* 408(1990), 57-113.

[16] J. Huebschmann, Lie-Rinehart algebras, Gerstenhaber algebras and Batalin-Vilkovisky algebras. *Annales de l’institut Fourier*, 48 no. 2 (1998), 425-440.

[17] J. Huebschmann, Duality for Lie-Rinehart algebras and the modular class, *J. reine angew. Math.* 510 (1999), 103-159.

[18] D. Kastler and R. Stora, Lie-Cartan pairs, *J. Geom. and Physics*, T.2, No.3, 1985.

[19] Y. Kosmann, On Lie transformation groups and the covariance of differential operators, *Differential geometry and relativity*, Mathematical Phys. and Appl. Math. 3(1976), 75-89, Reidel, Dordrecht.

[20] Y. Kosmann-Schwarzbach and K. Mackenzie, Differential operators and actions of Lie algebroids, *Quantization, Poisson Brackets and Beyond*, T. Voronov (ed.), *Contemp. Math.*315, Amer.Math. Soc.(2002), 213-233.

[21] Y. Kosmann-Schwarzbach and F. Magri, Poisson-Nijenhuis structures, *Annales de l’institut Henri Poincaré (A) Physique théorique*, 53 no. 1 (1990), 35-81.

[22] A. KumpeRa, *An introduction to Lie groupoids*, Duplicated notes, Núcleo de Estudos e Pesquisas Científicas, Rio de Janeiro, 1971.

[23] K. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, LMS Lecture Notes Series 124, Cambridge University Press, 1987.

[24] K. Mackenzie, Lie algebroids and Lie pseudoalgebras, *Bull. London Math. Soc.* 27(1995), 97-147.

[25] K. Mackenzie, *General theories of Lie groupoids and Lie algebroids*, Cambridge University Press, 2005.

[26] E. Nelson, *Tensor Analysis*, Princeton University Press, 1967.
[27] R.S. Palais, The Cohomology of Lie rings, *Proc.Symp.Pure Math.*, 3, Amer.Math.Soc.(1961), 130-137.

[28] P. Popescu, Categories of modules with differential, *J. Algebra* 185(1996), 50-73.

[29] J. Pradines, Théorie de Lie pour les groupoïdes différentiables. dans la catégorie des groupoïdes infinitésimaux, *C. R. Acad. Sci. Paris*, 264, série A (1967), 245-248.

[30] G.S. Rinehart, Differential forms on general commutative algebras, *Trans. Amer. Math. Soc.*, 108 (1963), 195-222.

[31] A. Weinstein, Symplectic groupoids and Poisson groupoids, *Bull. Amer. Math. Soc.*, 16(1987), 101-104.