An integrable noncommutative version of the sine–Gordon system

Marcus T. Grisaru

Physics Department, McGill University
Montreal, QC Canada H3A 2T8

Silvia Penati

Dipartimento di Fisica, Università degli studi di Milano-Bicocca
and INFN, Sezione di Milano, piazza della Scienze 3, I-20126 Milano, Italy

ABSTRACT

Using the bicomplex approach we discuss a noncommutative system in two–dimensional Euclidean space. It is described by an equation of motion which reduces to the ordinary sine–Gordon equation when the noncommutation parameter is removed, plus a constraint equation which is nontrivial only in the noncommutative case. We show that the system has an infinite number of conserved currents and we give the general recursive relation for constructing them. For the particular cases of lower spin nontrivial currents we work out the explicit expressions and perform a direct check of their conservation. These currents reduce to the usual sine-Gordon currents in the commutative limit. We find classical “localized” solutions to first order in the noncommutativity parameter and describe the Backlund transformations for our system. Finally, we comment on the relation of our noncommutative system to the commutative sine-Gordon system.

PACS: 03.50.-z, 11.10.-z, 11.30.-j
Keywords: Noncommutative geometry, Integrable systems, sine–Gordon.
1 Introduction

Field theories defined on noncommutative (NC) manifolds \[1\] have been receiving considerable attention in the last few years, primarily because of the appearance of noncommutative geometries in strings \[2\] and matrix theory \[3\]. The presence of spacetime noncommutativity has dramatic consequences on the dynamics of the fields and the quantum properties of the related theories (see \[4\] for a review of the subject and a quite complete list of references). In this context it is interesting to investigate two-dimensional systems defined on a manifold where the two coordinates do not commute.

Implementing noncommutativity on a two-dimensional Minkowski spacetime would necessarily involve the time coordinate in the nonzero commutation relations. However, it has been proven \[5\] that in general a noncommutation of the time variable affects the causality of the theory and its unitarity. Therefore, a well defined problem would be to consider systems defined on a two-dimensional NC euclidean space. It is well known that a selected class of two-dimensional euclidean theories, i.e. conformal and integrable theories, give a continuum description of two-dimensional statistical models at the critical point \[6\] or perturbed away from the critical point along integrable directions \[7\]. In particular, one may be interested in the formulation of two-dimensional integrable theories in NC geometry and their possible connections with statistical mechanics.

Some examples of NC equations which admit an infinite number of conserved currents have been constructed in \[8, 9\], by using a gauged bi-differential calculus. In this approach, which works in the ordinary commuting case \[10\] and can be extended in NC geometries, the equations of motion of integrable systems are obtained as nontrivial consistency conditions for the existence of two flat covariant derivatives. As a consequence, by solving an associated linear equation, one can establish the existence of an infinite chain of conservation laws.

In this paper we use this procedure to construct a NC integrable system whose equations of motion reduce to the ordinary sine–Gordon equation in the commutative limit. Precisely, what emerges in the NC case is a system of two coupled equations; one of them contains a sine interaction term and can be thought as a natural noncommutative analogue of ordinary sine–Gordon equation, whereas the other one has the structure of a conservation equation and can be seen as imposing an extra constraint on the system. Only in the limit of commuting geometry the constraint becomes trivial and the second one reduces to the standard well–known equation. At first sight, the appearance of two equations seems quite unexpected and restrictive. However, as will be clear later on, this can be traced back to the fact that the $SU(2)$ group which is the natural symmetry group of ordinary sine–Gordon, in the noncommutative case is not closed under $\ast$–product and any noncommutative extension of the system must naturally rely on $U(2)$. This implies that the group valued fields which enter the bicomplex construction take values in $U(2)$. Since $U(2)$ contains a noncommutative $U(1)$ subgroup, they develop a nontrivial trace part which is responsible for the appearance of an extra constraint equation \[\text{[1]}\].

\[\text{[1]}\]This is similar to what happens in the $U(1)$ WZNW model which, in the NC case, becomes nontrivial \[11, 12\] and does not have an immediate relation with its commutative counterpart (free scalar theory).
We then give a general prescription to generate conserved currents, thus proving the classical integrability of the system. These currents reduce to those of the sine-Gordon system in the commutative limit. To complete the classical analysis we also study “localized” (pseudo–solitonic) solutions. For the one–soliton solution we determine its distortion away from the ordinary one due to the noncommutation of the coordinates. This analysis is carried out perturbatively in the deformation parameter.

In this work we have not succeeded in constructing an action from which the equations of motion emerge. Although this is an important issue, we feel that integrability is an equally important feature of the sine-Gordon system. The alternative possibility, of starting with the usual sine-Gordon action and making it noncommutative is not a viable option in this respect.

The paper is organized as follows. In the next Section we summarize the general procedure based on the definition of a bicomplex. In the third Section we construct the NC equations of motion of our system. In Section 4 the iterative formula for constructing an infinite number of conserved currents is given and the first two nontrivial currents are written explicitly. For the first, the NC stress tensor, an explicit check of its conservation is performed up to second order in the noncommutation parameter θ. For the spin 3 current which in the ordinary case is a total derivative, triviality is checked up to the first order in θ. Section 5 is devoted to the study of one–“soliton” solutions perturbatively in θ, and a generalization of Backlund transformations is given to generate n–“solitons” solutions. Finally, Section 6 contains our conclusions and an outlook on possible developments. Two Appendices follow: The first one presents the detailed derivation of the conserved currents, whereas in the second one we have collected the trigonometric ∗–calculus and all the identities required to perform perturbative θ–expansions.

### 2 Generalities on NC bi–differential calculus

In this Section we summarize the general procedure to obtain integrable equations in a NC geometry as given in [8]. The basic idea is to write these equations as the nontrivial flatness conditions for two covariant derivatives suitably defined.

Given a noncommutative two–dimensional space with euclidean signature and complex coordinates

\[
z = \frac{x^0 + i x^1}{\sqrt{2}} ; \quad \bar{z} = \frac{x^0 - i x^1}{\sqrt{2}} \tag{2.1}
\]

noncommutativity is encoded in the relation

\[
[z, \bar{z}] = \theta \tag{2.2}
\]

where θ is a real parameter. The algebra \( \mathcal{F} \) of smooth functions on NC \( \mathbb{R}^2 \) is endowed with the product

\[
(f * g)(z, \bar{z}) = e^{\frac{\theta}{2}(\partial_z \bar{\xi} - \partial_{\bar{z}} \xi)} f(z, \bar{z})g(\xi, \bar{\xi})|_{\xi = z, \bar{\xi} = \bar{z}} = e^{\frac{\theta}{2} P} fg \tag{2.3}
\]
where
\[ Pfg \equiv (\partial f \bar{\partial} g - \bar{\partial} f \partial g) \] (2.4)

As basic ingredients we consider:
1) The \( N_0 \)-graded linear space \( \mathcal{M} = \bigoplus_{r \geq 0} \mathcal{M}^r \), where \( \mathcal{M}^0 = \mathcal{F} \) or more generally a noncommutative algebra of functions in \( \mathcal{F} \);
2) Two linear maps \( d, \delta : \mathcal{M}^r \rightarrow \mathcal{M}^{r+1} \) satisfying \( d^2 = \delta^2 = \{d, \delta\} = 0 \).

The triple \( (\mathcal{M}, d, \delta) \) is called a \textit{bicomplex}. Given the two differential maps, we consider the associated linear equation
\[ \delta \chi = ld\chi \] (2.5)
where \( \chi \in \mathcal{M}^s \) for a given “spin” \( s \) and \( l \) is a real parameter. If a nontrivial solution exists, we can write
\[ \chi = \sum_{l=0}^{\infty} l^l \chi^{(l)} \] (2.6)
with \( \chi^{(l)} \in \mathcal{M}^s \) satisfying
\[ \delta \chi^{(0)} = 0 \quad ; \quad \delta \chi^{(l)} = d\chi^{(l+1)} \quad , \quad l > 0 \] (2.7)

Therefore we can construct a chain of \( \delta \)-closed and \( \delta \)-exact forms in \( \mathcal{M}^{s+1} \)
\[ \chi^{(l+1)} \equiv d\chi^{(l)} = \delta \chi^{(l+1)} \quad , \quad l \geq 0 \] (2.8)

We note that in order to obtain an actual chain of \( \delta \)-closed and \( \delta \)-exact forms it is necessary to require \( \chi^{(0)} \) not to be \( \delta \)-exact, i.e. the cohomology \( H^{(s)}_\delta \) must be nontrivial.

In general, if the cohomology \( H^{(s+1)}_\delta \) is trivial we are guaranteed that an infinite chain of \( \delta \)-closed and \( \delta \)-exact forms exists. If the cohomology is not trivial, it is the solvability of the linear problem which assures the possible existence of an infinite chain of \( \delta \)-closed and \( \delta \)-exact forms.

When the two differential maps are defined in terms of the ordinary derivatives with respect to the two coordinates in \( \mathcal{R}^2 \) (see for instance eq. (3.1) below), the integrability conditions for the linear equation are trivially satisfied \( (\delta^2 = d^2 = \{d, \delta\} = 0 \) by definition). In this case the eqs. (2.7) have the appearance of an infinite number of conservation laws. However they are not the ones we are interested in since they are not associated to any second order integrable equation.

Nontrivial integrable equations can be obtained by considering a \textit{gauged} bicomplex.

We introduce two connections \( A \) and \( B \) and define
\[ D_d = d + A * \quad D_\delta = \delta + B * \] (2.9)

The flatness conditions \( D^2_d = D^2_\delta = \{D_d, D_\delta\} = 0 \) imply
\[ \mathcal{F}(A) \equiv dA + A * A = 0 \]
\[ \mathcal{F}(B) \equiv \delta B + B * B = 0 \]
\[ \mathcal{G}(A, B) \equiv dB + \delta A + A * B + B * A = 0 \] (2.10)
which, for a suitable choice of the bicomplex, give rise to non-trivial equations of motion. In the commutative \([10]\) and noncommutative cases \([8, 9]\), many known examples of integrable equations can be obtained from \((2.10)\).

Again, one may consider the linear problem associated to \((2.9)\)

$$D\chi \equiv (D\delta - lDd)\chi = 0 \quad (2.11)$$

The equations \((2.10)\) can then be seen as integrability conditions for the linear equation, since

$$0 = D^2\chi = \left[\mathcal{F}(B) + l^2\mathcal{F}(A) - l\mathcal{G}(A, B)\right]\chi \quad (2.12)$$

If this equation has solutions \(\chi \in \mathcal{M}^s\) of the form \(\chi = \sum_l l^l\chi^{(l)}\), we obtain an infinite chain of identities

$$D\delta\chi^{(0)} = 0 \quad ; \quad D\delta\chi^{(l)} = D_d\chi^{(l+1)} \quad , \quad l > 0 \quad (2.13)$$

which can be used to construct \(D\delta\)-closed and \(D\delta\)-exact forms \(\chi^{(l)}\), if \(\chi^{(0)}\) is not cohomologically trivial.

As above, when the differential maps are defined in terms of ordinary derivatives, these equations can be interpreted sometime as an infinite set of (nontrivial) conservation equations. However, in general, the \(\chi^{(l)}\) are nonlocal functions of the coordinates (in the sense that they are defined in terms of integrals) with no obvious physical interpretation. As we will see in Section 4, conserved local objects can be constructed out of the functions \(\chi^{(l)}\) \([10]\).

### 3 A NC sine–Gordon

We apply the procedure described in the previous Section to construct a noncommutative version of the sine–Gordon equation.

We consider the linear space \(\mathcal{M} = \mathcal{M}^0 \otimes L\), where \(\mathcal{M}^0\) is the space of \(2 \times 2\) matrices with entries in \(\mathcal{F}\), and \(L = \otimes \tau_{r=0}^2 L^r\) is a two-dimensional graded vector space with the \(L^1\) basis \((\tau, \sigma)\) satisfying \(\tau^2 = \sigma^2 = \tau\sigma + \sigma\tau = 0\).

For any matrix function \(f \in \mathcal{M}^0\) we define two linear maps

$$\delta f = \bar{\partial}f\tau - Rf\sigma \quad ; \quad df = -Sf\tau + \partial f\sigma \quad (3.1)$$

where \(R, S\) are constant matrices with \([R, S] = 0\). It is easy to check that, as a consequence, the bicomplex conditions \(\delta^2 = d^2 = (d\delta + \delta d) = 0\) are trivially satisfied.

To get nontrivial conditions, we introduce a gauged bicomplex by dressing the \(d\) operator as

$$Df \equiv G^{-1} * d(G * f) = -L * f\tau + (\partial + M*)f\sigma \quad (3.2)$$

where \(G\) is a generic invertible \((G * G^{-1} = G^{-1} * G = I)\) matrix in \(\mathcal{M}^0\) and

$$L = G^{-1} * SG \quad ; \quad M = G^{-1} * \partial G \quad (3.3)$$
Now we require \((\mathcal{M}, \delta, D)\) to be a bicomplex. The condition \(D^2 = 0\) implies \(\partial L = [L, M]_*\) which one can check to be identically satisfied. The last condition \(\{D, \delta\} = 0\) gives instead the nontrivial equation
\[
\bar{\partial} M = [R, L]_* \quad (3.4)
\]

In order to obtain a noncommutative version of the sine–Gordon equations we choose the \(U(2)\) group valued fields
\[
R = S = \sqrt{\gamma} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]
\[
G = e^{i\sigma_2 \phi} = \begin{pmatrix} \cos_* \frac{\phi}{2} & \sin_* \frac{\phi}{2} \\ -\sin_* \frac{\phi}{2} & \cos_* \frac{\phi}{2} \end{pmatrix} \quad (3.5)
\]
where \(*\)-functions are defined through their \(*\)-power series (see Appendix B). As a consequence we have
\[
M = \frac{1}{2} \begin{pmatrix} e^{i\frac{\phi}{2}} \partial e^{-i\frac{\phi}{2}} + e^{i\frac{\phi}{2}} \partial e^{i\frac{\phi}{2}} & -i(e^{-i\frac{\phi}{2}} \partial e^{i\frac{\phi}{2}} - e^{i\frac{\phi}{2}} \partial e^{-i\frac{\phi}{2}}) \\ i(e^{-i\frac{\phi}{2}} \partial e^{i\frac{\phi}{2}} - e^{i\frac{\phi}{2}} \partial e^{-i\frac{\phi}{2}}) & e^{i\frac{\phi}{2}} \partial e^{-i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}} \partial e^{i\frac{\phi}{2}} \end{pmatrix}
\]
\[
L = \sqrt{\gamma} \begin{pmatrix} \sin^2 \frac{\phi}{2} & -\sin_* \frac{\phi}{2} \cos_* \frac{\phi}{2} \\ -\cos_* \frac{\phi}{2} & \sin_* \frac{\phi}{2} \cos_* \frac{\phi}{2} \end{pmatrix} \quad (3.6)
\]
Computing \([R, L]\) we obtain
\[
[R, L] = \gamma \begin{pmatrix} 0 & \sin_* \frac{\phi}{2} \cos_* \frac{\phi}{2} \\ -\cos_* \frac{\phi}{2} \sin_* \frac{\phi}{2} & 0 \end{pmatrix} \quad (3.7)
\]
The equation \((3.4)\) is a matrix equation in \(U(2)\). In particular, the matrix \(M\) has a nontrivial trace part, as a consequence of the noncommutative nature of the \(U(1)\) subgroup. Therefore, writing eq. \((3.4)\) in components we obtain the two nontrivial equations for the field \(\phi\)
\[
\bar{\partial} \left( e^{i\frac{\phi}{2}} \partial e^{-i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}} \partial e^{i\frac{\phi}{2}} \right) = 0
\]
\[
\bar{\partial} \left( e^{-i\frac{\phi}{2}} \partial e^{i\frac{\phi}{2}} - e^{i\frac{\phi}{2}} \partial e^{-i\frac{\phi}{2}} \right) = i\gamma \sin_* \phi \quad (3.8)
\]
where the identity \((B.5)\) has been used. We note that in the limit \(\theta \to 0\) the first equation becomes trivial, whereas the second one reduces to the ordinary sine–Gordon equation
\[
\partial \bar{\partial} \phi = \gamma \sin \phi \quad (3.9)
\]
In the noncommutative case, since \(\partial e_*^* \neq e_*^* \partial \phi\) (see eq.\((B.4)\)), both equations are meaningful and describe the dynamics of the field \(\phi(z, \bar{z}, \theta)\).

5
4 Conserved currents

In the ordinary, commutative case, the derivation of the sine–Gordon equations from a bicomplex automatically guarantees \[10\] the existence of an infinite chain of currents satisfying the conservation equations
\[
\bar{\partial} J^{(l)} = \partial \tilde{J}^{(l)} , \quad l \geq 0
\] (4.1)

In this Section we extend those arguments to the noncommutative case in order to prove the classical integrability of the system whose dynamics is given by (3.8). We will find a recursive procedure to determine an infinite set of \(*\)-functions satisfying (4.1).

Quite generally, given a \(*\)-invertible function \( \tilde{\chi} \in \mathcal{M}^0 \) we define functions
\[
J = \text{Tr}((\partial \tilde{\chi}) \ast \tilde{\chi}^{-1}) , \quad \tilde{J} = \text{Tr}((\bar{\partial} \tilde{\chi}) \ast \tilde{\chi}^{-1})
\] (4.2)

which satisfy the following identity
\[
\bar{\partial} J = \partial \tilde{J} + [\tilde{J}, J]_\ast
\] (4.3)

This is almost a conservation law, up to the commutator. To get rid of it, we first observe that it can be written as \[10\]
\[
[\tilde{J}, J]_\ast = \theta(\partial \tilde{J} \circ \bar{\partial} J - \bar{\partial} \tilde{J} \circ \partial J)
\] (4.4)

where we have introduced the new product
\[
f \circ g \equiv \frac{\sinh (\frac{\theta}{2} P)}{\frac{\theta}{2} P} fg
\] (4.5)

with the operator \( P \) given in (2.4). Therefore, if we introduce
\[
J \equiv J - \theta J \circ \partial \tilde{J} + \partial \mathcal{T} , \quad \tilde{J} \equiv \tilde{J} - \theta J \circ \bar{\partial} \tilde{J} + \bar{\partial} \mathcal{T}
\] (4.6)

they satisfy the conservation equation
\[
\bar{\partial} J = \partial \tilde{J}
\] (4.7)

where \( \mathcal{T} \) represents possible trivial terms. In particular, for an invertible solution of
\[
\delta \chi = lD\chi
\] (4.8)

defined as a power series in \( l \), we can write for the functions \( J \) and \( \tilde{J} \)
\[
J = \sum_{l=0}^{\infty} l^l J^{(l)} , \quad \tilde{J} = \sum_{l=0}^{\infty} l^l \tilde{J}^{(l)}
\] (4.9)

and from (4.7) we obtain an infinite set of conserved currents associated to the equations of motion (2.10). We stress that it is the solvability of the linear equation (4.8) which
guarantees that the number of currents is infinite, i.e. that the system is classically integrable.

We now turn to the construction of the quantities $\mathcal{J}^l$ and $\tilde{\mathcal{J}}^l$. To simplify the notation we introduce the following functions

\[ a \equiv \frac{1}{2} \left( e^{-\frac{i}{2} \phi} \partial e^{\frac{i}{2} \phi} + e^{\frac{i}{2} \phi} \partial e^{-\frac{i}{2} \phi} \right) \]

\[ b \equiv \frac{i}{2} \left( e^{\frac{i}{2} \phi} \partial e^{-\frac{i}{2} \phi} - e^{-\frac{i}{2} \phi} \partial e^{\frac{i}{2} \phi} \right) \]

subject to the conditions

\[ \bar{\partial} a = 0 \quad ; \quad \bar{\partial} b = \frac{\gamma}{2} \sin_* \phi \]

as follows from the equations of motion. We define

\[ M = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \equiv U + aI \]

where, in this formulation, the off-diagonal matrix $U$ appears in the ordinary sine–Gordon system while $a$ represents the noncommutative contribution.

Following closely the notation of [10] we introduce the chiral matrices

\[ e^\pm = \frac{1}{2} (I \pm \sigma_3) \]

so that the matrix $R$ takes the form $R = \sqrt{\gamma} e^-$. The linear equation

\[ \delta \chi = l D \chi \]

with $\chi$ an element of $\mathcal{M}^0$ (a $2 \times 2$ matrix), can be decomposed into the two equations

\[ \begin{aligned} (i) & \quad \bar{\partial} \chi = -l L \ast \chi \\
(ii) & \quad \sqrt{\gamma} e^- \chi = -l (\partial \chi + M \ast \chi) = -l [\partial \chi + (U + aI) \ast \chi] \end{aligned} \]

We look for solutions defined as power series in $l$. As explained in Appendix A, $\chi$ is not invertible and therefore we will define the currents in terms of $\tilde{\chi} = \chi + e_-$ (see eq. (A.6)). We introduce the matrices

\[ j \equiv \partial \tilde{\chi} \ast \tilde{\chi}^{-1} \quad ; \quad \tilde{j} \equiv \bar{\partial} \tilde{\chi} \ast \tilde{\chi}^{-1} \]

Following the details given in Appendix A, from the system (4.15) we obtain the following equations which have to be satisfied by $j$ and $\tilde{j}$

\[ \sqrt{\gamma} j = -l \partial j - lj \ast j + l [-a - e_+ U \ast a \ast U^{-1} + e_+ \partial U \ast U^{-1} - e_- U + e_-(\partial a) \ast U^{-1}] \ast j \]

\[ + l [e_+ U \ast U - e_+ \partial a - e_+ U \ast a \ast U^{-1} \ast a + e_+ \partial U \ast U^{-1} \ast a \]

\[ - e_- \partial U + e_-(\partial a) \ast U^{-1} \ast a \] 

\[ - \sqrt{\gamma} e_+ a \]
and
\[ \tilde{j} = -lL \ast (e_+ - e_- U^{-1} \ast a) + lL \ast e_- U^{-1} \ast \tilde{j} \quad (4.18) \]

We expand \( j = \sum_{l \geq 0} l^l j^{(l)} \) and \( \tilde{j} = \sum_{l \geq 0} l^l \tilde{j}^{(l)} \). Substituting in the previous equations, up to the second order in \( l \) we find
\[
\begin{align*}
j^{(0)} &= -e_+ a \\
j^{(1)} &= \frac{1}{\sqrt{\gamma}} e_+ U \ast U + \frac{1}{\sqrt{\gamma}} e_- (U \ast a - \partial U) \\
j^{(2)} &= \frac{1}{\gamma} e_+ (-U \ast \partial U + U \ast [U, a]_*) \\
&\quad + \frac{1}{\gamma} e_- (-2\partial U \ast a - U \ast \partial a + [U, a]_* \ast a - U \ast U \ast U + \partial(a \ast U) + \partial^2 U)
\end{align*}
\]
and
\[
\begin{align*}
\tilde{j}^{(0)} &= 0 \\
\tilde{j}^{(1)} &= -L e_+ + L \ast e_- U^{-1} \ast a \\
\tilde{j}^{(2)} &= -L \ast e_- U^{-1} \ast Le_+ + L \ast e_- U^{-1} L \ast e_- U^{-1} \ast a
\end{align*}
\]

If we now introduce \( J^{(l)} \equiv \text{Tr}_j j^{(l)} \) and \( \tilde{J}^{(l)} \equiv \text{Tr}_j \tilde{j}^{(l)} \) as the functions which enter the definitions (4.6) of the conserved currents, up to second order we find
\[
\begin{align*}
J^{(0)} &= -a \\
J^{(1)} &= -\frac{1}{\sqrt{\gamma}} b \ast b \\
J^{(2)} &= \frac{1}{\gamma} b \ast (\partial b - [b, a]_*)
\end{align*}
\]
and
\[
\begin{align*}
\tilde{J}^{(0)} &= 0 \\
\tilde{J}^{(1)} &= \sqrt{\gamma} (-\sin^2 \phi \frac{\partial}{2} - \frac{1}{2} \sin \phi \ast b^{-1} \ast a) \\
\tilde{J}^{(2)} &= \gamma \left( \frac{1}{2} \sin \phi \ast b^{-1} \sin^2 \phi \frac{\partial}{2} + \frac{1}{4} \sin \phi \ast b^{-1} \ast \sin \phi \ast b^{-1} \ast a \right)
\end{align*}
\]

A quite lengthy but straightforward calculation, along the same steps explained above, gives
\[
\begin{align*}
J^{(3)} &= \frac{1}{\gamma^{3/2}} \left( -b \ast b \ast b \ast b + \partial b \ast \partial b \\
&\quad - \partial a \ast \partial a + \partial a \ast a \ast a + \partial a \ast b \ast b + \partial a \ast b^{-1} \ast \partial^2 b \\
&\quad -b \ast a \ast \partial b - \partial b \ast b \ast a + \partial b \ast a \ast b + b \ast \partial b \ast a \\
&\quad + 2b \ast a \ast b \ast a - b \ast b \ast a \ast a - b \ast a \ast a \ast b \\
&\quad + \partial a \ast b^{-1} \ast \partial (a \ast b) - 2\partial a \ast b^{-1} \ast \partial b \ast a - \partial a \ast b^{-1} \ast a \ast b \ast a \right)
\end{align*}
\]

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\[ \mathcal{J}^{(3)} = \gamma^{3/2} \left( -\frac{1}{4} \sin_* \phi \ast b^{-1} \ast \sin_* \phi \ast b^{-1} \ast \sin_2^* \phi \right) \\
\quad \quad - \frac{1}{4} \cos^2 \frac{\phi}{2} \ast b^{-1} \ast \sin_* \phi \ast b^{-1} \ast \sin_* \phi \\
\quad \quad - \frac{1}{8} \sin_* \phi \ast b^{-1} \ast \sin_* \phi \ast b^{-1} \ast \sin_* \phi \ast b^{-1} \ast a \right) \quad (4.24) \]

We have now all the ingredients to write the first non trivial conservation laws associated to the equations (3.8). Using the general recipe (4.6), we find:

1) Order zero in \( l \)

\[ \mathcal{J}^{(0)} = J^{(0)} = -a \quad ; \quad \tilde{\mathcal{J}}^{(0)} = 0 \quad (4.25) \]

This current is trivially conserved \((\bar{\partial} \mathcal{J}^{(0)} = 0)\) once the equations of motion (4.11) are taken into account.

2) Order one in \( l \)

\[ \mathcal{J}^{(1)} = J^{(1)} - \theta J^{(0)} \diamond \bar{\partial} \tilde{J}^{(1)} \quad ; \quad \tilde{\mathcal{J}}^{(1)} = \tilde{J}^{(1)} - \theta J^{(0)} \diamond \bar{\partial} \tilde{J}^{(1)} \quad (4.26) \]

with \( J^{(0)}, J^{(1)}, \tilde{J}^{(1)} \) given in (4.21, 4.22). Note that, as a consequence of \( \bar{\partial} J^{(0)} = 0 \), the second term in \( \tilde{\mathcal{J}}^{(1)} \) is trivial (it is a \( \bar{\partial} \)-derivative).

3) Second order in \( l \)

\[ \mathcal{J}^{(2)} = J^{(2)} - \theta (J^{(0)} \diamond \bar{\partial} \tilde{J}^{(2)} + J^{(1)} \diamond \bar{\partial} \tilde{J}^{(1)}) \]
\[ \tilde{\mathcal{J}}^{(2)} = \tilde{J}^{(2)} - \theta (J^{(0)} \diamond \bar{\partial} \tilde{J}^{(2)} + J^{(1)} \diamond \bar{\partial} \tilde{J}^{(1)}) \quad (4.27) \]

4) Third order in \( l \)

\[ \mathcal{J}^{(3)} = J^{(3)} - \theta (J^{(0)} \diamond \bar{\partial} \tilde{J}^{(3)} + J^{(1)} \diamond \bar{\partial} \tilde{J}^{(2)} + J^{(2)} \diamond \bar{\partial} \tilde{J}^{(1)}) \]
\[ \tilde{\mathcal{J}}^{(3)} = \tilde{J}^{(3)} - \theta (J^{(0)} \diamond \bar{\partial} \tilde{J}^{(3)} + J^{(1)} \diamond \bar{\partial} \tilde{J}^{(2)} + J^{(2)} \diamond \bar{\partial} \tilde{J}^{(1)}) \quad (4.28) \]

The general argument presented at the beginning of this Section automatically guarantees that these currents are conserved. From our procedure, they are obviously constructed out of our sine–Gordon field \( \phi \). Furthermore, our construction is based on the existence of a solution \( \chi \) of the linear system and the existence of this solution, i.e. the bicomplex integrability condition, is guaranteed only when the field \( \phi \) satisfies the system of equations (3.8).

More importantly, the currents \( \mathcal{J}, \tilde{\mathcal{J}} \) are local functions of the field \( \phi \). (We use the term “local” in its standard meaning: the currents depend on the field \( \phi \) and its derivatives, but not on integrals of \( \phi \). Of course, the intrinsic nonlocality of a noncommutative theory – \( \phi \)-derivatives to infinite order, multiplying the parameter \( \theta \) – is present.) We note that the \( \chi^{(l)} \) themselves are not local in the above sense. Indeed, it is not difficult to ascertain, by examining the solution of the system (4.15), that it will depend nonlocally on the field \( \phi \). For example, even in the commutative case, the trace of the first equation integrates to

\[ \text{Tr} \chi(z, \bar{z}) = \text{Tr}c \exp \left[ -l \int \phi(z, \bar{z}) \right] \quad (4.29) \]
as can be verified by direct differentiation, and this exhibits directly the nonlocality we are discussing, after expanding in powers of \( l \), of (at least some of) the matrix elements of \( \chi^{(l)} \). On the other hand, the quantities \( j, \tilde{j} \) in [4.17, 4.18] satisfy local algebraic equations and lead to local conserved currents. The nonlocality present in the solutions \( \chi \) disappears from \( \text{Tr} \bar{\partial} \chi / \chi = \bar{\partial} \text{Tr} \ln \chi \) and what, superficially, appears to be a trivial total derivative turns into a local function of the field \( \phi \). (There is a subtlety here: these equations, and their solutions, involve the *-inverse \( U^{-1} \) and it is conceivable that this quantity involves an integral; however, we must again consider this kind of nonlocality as acceptable since it is intrinsic to the noncommutativity of the theory).

## 5 Perturbative expansion in \( \theta \)

The currents constructed in the previous section depend explicitly on the noncommutation parameter \( \theta \).

To better understand their dependence on \( \theta \), their connection with the ordinary currents and the role of \( b^{-1} \) in their expressions, we evaluate them perturbatively in \( \theta \) and check explicitly their conservation up to second order. In doing this, we will make repeated use of the *-identities contained in Appendix B. To begin with we work out the explicit \( \theta \) dependence of the equations of motion.

We can evaluate the expressions \( a \) and \( b \) by using the identity (B.9). We obtain

\[
\begin{align*}
a &= \frac{1}{2} \left( \frac{i}{2} \right)^2 k \[\partial \phi, \phi \] + \text{even powers} \\
b &= \frac{1}{2} \partial \phi + \frac{i}{3!} \left( \frac{i}{2} \right)^3 k [k \[\partial \phi, \phi \] + \phi] + \text{odd powers}
\end{align*}
\]

The perturbative expansion of the commutator is given in (B.10). Making use of that result we can write

\[
\begin{align*}
a &= -\frac{1}{8} \theta (\partial^2 \phi \bar{\partial} \phi - \bar{\partial} \partial \phi \partial \phi) + O(\theta^3) \\
b &= \frac{1}{2} \partial \phi \\
&\quad + \frac{1}{48} \theta^2 \left( 2 \partial \partial^2 \phi \partial \bar{\partial} \phi - \partial^3 \phi (\bar{\partial} \phi)^2 - \partial \bar{\partial} \partial \phi (\partial \phi)^2 + \partial^2 \phi \partial \bar{\partial} \partial \phi - (\partial \bar{\partial} \phi)^2 \partial \phi \right) \\
&\quad + O(\theta^4)
\end{align*}
\]

These are formal expansions where only the dependence on \( \theta \) from the *--product has been considered. To complete the expansion we have to take into account also the \( \theta \)--dependence of the dynamical field \( \phi(z, \bar{z}, \theta) \). Writing

\[
\phi(z, \bar{z}, \theta) = \sum_{n \geq 0} \theta^n \phi_n
\]
and inserting in the expressions (5.2), we finally have

\[ a \equiv \sum_{n \geq 0} \theta^n a_n = -\frac{1}{8}[\partial^2 \phi_0 \bar{\partial} \phi_0 - \bar{\partial} \partial \phi_0 \bar{\partial} \phi_0] + \frac{1}{8} \theta^2 (\partial^2 \phi_0 \bar{\partial} \phi_1 + \partial^2 \phi_1 \bar{\partial} \phi_0 - \bar{\partial} \partial \phi_0 \partial \phi_1 - \bar{\partial} \bar{\partial} \phi_1 \partial \phi_0) + O(\theta^3) \]

\[ b \equiv \sum_{n \geq 0} \theta^n b_n = \frac{1}{2} \partial \phi_0 + \theta \frac{1}{2} \partial \phi_1 \]

\[ + \theta^2 \left[ \frac{1}{2} \partial \phi_2 + \frac{1}{48} (2 \bar{\partial} \partial^2 \phi_0 \partial \phi_0 - \partial^3 \phi_0) - \bar{\partial} \bar{\partial}^2 \phi_0 (\partial \phi_0)^2 \bar{\partial} \partial \phi_0 \partial \phi_0 - \bar{\partial} \bar{\partial} \phi_0 \partial \phi_0 - (\bar{\partial} \bar{\partial} \phi_0 \partial \phi_0) \right] + O(\theta^3) \]

(5.4)

From the equations of motion (3.8) we can read the equations of motion for the coefficient functions \( a_n, b_n \). The \( a_n \) equations are simply \( \bar{\partial} a_n = 0 \), whereas for \( b_n \) we need to use the \( \theta \)-expansion (3.10) for \( \sin^* \) on the r.h.s. of (3.8). Taking into account the explicit expressions for \( a_n \) and \( b_n \) as read from (5.2) we finally have that, up to second order, the equations of motion satisfied by the various components of \( \phi \) in the \( \theta \)-expansion are:

1) Order zero in \( \theta \)

\[ a_0 = 0 , \quad b_0 = \frac{1}{2} \partial \phi_0 \]

(5.5)

and the equations are

\[ \partial \bar{\partial} \phi_0 = \gamma \sin \phi_0 \]

(5.6)

2) Order one

\[ a_1 = -\frac{1}{8} (\partial^2 \phi_0 \bar{\partial} \phi_0 - \bar{\partial} \partial \phi_0 \partial \phi_0) \]

\[ b_1 = \frac{1}{2} \partial \phi_1 \]

\[ \sin_\theta \phi |_{\theta^2} = \phi_1 \cos \phi_0 \]

(5.7)

The equations of motion then read

\[ \partial (\partial^2 \phi_0 \bar{\partial} \phi_0 - \bar{\partial} \partial \phi_0 \partial \phi_0) = 0 \]

\[ \partial \bar{\partial} \phi_1 = \gamma \phi_1 \cos \phi_0 \]

(5.8)

We notice that, using the equation of motion at order zero, which also implies \( \partial^2 \bar{\partial} \phi_0 \bar{\partial} \phi_0 = \bar{\partial} \bar{\partial}^2 \phi_0 \partial \phi_0 \), the first equation in (5.8) can be written as

\[ \partial^2 \phi_0 \bar{\partial}^2 \phi_0 - (\bar{\partial} \bar{\partial} \phi_0)^2 = 0 \]

(5.9)

From this identity it also follows \( [\partial \phi_0, \bar{\partial} \phi_0] |_{\theta^2} = 0 \) and \( (\partial \phi_0 \ast \phi_0) |_{\theta^2} = 0 \).
3) Order two: Using the previous identities for $\phi_0$ we have

$$a_2 = -\frac{1}{8}(\partial^2\phi_0 \partial\phi_1 + \partial^2\phi_1 \partial\phi_0 - \bar{\partial}\partial\phi_0 \partial\phi_1 - \bar{\partial}\partial\phi_1 \partial\phi_0)$$

$$b_2 = \frac{1}{2} \partial\phi_2 + \frac{1}{48} (\partial^2 \bar{\partial}\phi_0 \partial\phi_0 - \partial^2 \phi_0 (\bar{\partial}\phi_0)^2)$$

$$= \frac{1}{2} \partial\phi_2 + \frac{1}{6} \partial\phi_0 \partial a_1$$

(5.10)

and the equations of motion are

$$\bar{\partial}(\partial^2\phi_0 \partial\phi_1 + \partial^2\phi_1 \partial\phi_0 - \bar{\partial}\partial\phi_0 \partial\phi_1 - \bar{\partial}\partial\phi_1 \partial\phi_0) = 0$$

$$\partial\bar{\partial}\phi_2 + \frac{1}{3} \bar{\partial}(\partial\phi_0 \partial a_1) = \gamma(\sin\phi)|_{\sigma^2}$$

(5.11)

The conserved currents $\mathcal{J}^{(l)}$ are given in terms of the field $\phi$ and its derivatives. Therefore, using the expansion (5.3) we can write

$$\mathcal{J}^{(l)} = \sum_{n=0}^{\infty} \theta^n \mathcal{J}_n^{(l)}$$

(5.12)

where, according to the general conservation law (4.1) each coefficient has to satisfy

$$\bar{\partial}\mathcal{J}_n^{(l)} = \partial\tilde{\mathcal{J}}_n^{(l)}$$

(5.13)

Setting $\theta = 0$ in (5.12) we should recover the ordinary conserved currents for the commutative sine–Gordon system. Indeed, we find $\mathcal{J}_0^{(0)} = 0$, while

$$\mathcal{J}_0^{(1)} = J_0^{(1)} = -\frac{b_0^2}{\sqrt{\gamma}} = -\frac{1}{4\sqrt{\gamma}} (\partial\phi_0)^2$$

(5.14)

is the spin 2 stress tensor of the ordinary sine–Gordon system. Its conservation law reads

$$\bar{\partial} \left( -\frac{1}{4\sqrt{\gamma}} (\partial\phi_0)^2 \right) = \partial \left( -\sqrt{\gamma} \sin^2\frac{\phi_0}{2} \right)$$

(5.15)

where the r.h.s. coincides with $\partial\tilde{\mathcal{J}}_0^{(1)}$ as given in (4.26) for $\theta = 0$.

For the next two currents, we find $\mathcal{J}_0^{(2)}$ to be a total derivative, whereas

$$\mathcal{J}_0^{(3)} = \frac{1}{4\gamma^{3/2}} \left[ -\frac{1}{4} (\partial\phi_0)^4 + (\partial^2\phi_0)^2 \right]$$

(5.16)

It coincides with the spin 4 nontrivial current of the ordinary sine–Gordon. The on–shell conservation reads

$$\bar{\partial} \left[ -\frac{1}{4} (\partial\phi_0)^4 + (\partial^2\phi_0)^2 \right] = \partial \left[ \frac{\gamma}{4} (\partial\phi_0)^2 \cos\phi_0 \right]$$

(5.17)
5.1 Perturbative evaluation of $\mathcal{J}^{(1)}$ current

We now concentrate on the perturbative evaluation of the stress tensor $\mathcal{J}^{(1)}$ as given in (4.26). We are interested in computing the explicit expressions of the coefficients $\mathcal{J}_n^{(1)}$, $n > 0$ and check their conservation

$$\bar{\partial}\mathcal{J}_n^{(1)} = \partial(\text{something}) \quad (5.18)$$

We will push the calculation up to second order in $\theta$.

The first non trivial deformation of the ordinary stress tensor due to the noncommutativity is given by

$$\mathcal{J}^{(1)}_1 = -\frac{1}{\sqrt{\gamma}} (b \ast b) |_{\theta} = -\frac{2}{\sqrt{\gamma}} b_0 b_1 = -\frac{1}{2\sqrt{\gamma}} \partial\phi_0 \partial\phi_1 \quad (5.19)$$

Its conservation reads

$$\bar{\partial}\mathcal{J}^{(1)}_1 = -\frac{1}{2\sqrt{\gamma}} \bar{\partial}(\partial\phi_0 \partial\phi_1) = -\frac{1}{2\sqrt{\gamma}} (\partial\bar{\partial}\phi_0 \partial\phi_1 + \partial\phi_0 \partial\bar{\partial}\phi_1)$$

$$= -\frac{\sqrt{\gamma}}{2} (\sin\phi_0 \partial\phi_1 + \partial\phi_0 (\cos\phi_0) \phi_1)$$

$$= -\frac{\sqrt{\gamma}}{2} \partial((\sin\phi_0) \phi_1) \quad (5.20)$$

The last line coincides with $\bar{\partial}\mathcal{J}^{(1)}_1$, up to total derivative terms.

At second order in $\theta$ the current is given by

$$\mathcal{J}^{(1)}_2 = J^{(1)}_1 - J^{(0)}_1 \bar{\partial} \mathcal{J}^{(1)}_0 = -\frac{1}{\sqrt{\gamma}} (b \ast b) |_{\theta^2} - a_1 \partial(\sqrt{\gamma} \sin^2 \frac{\phi_0}{2}) \quad (5.21)$$

We evaluate $(b \ast b) |_{\theta^2}$ by observing that contributions at order $\theta^2$ come both from the expansion of $b$ and of the $\ast$–product (see identity (B.7)). Collecting all the terms we have

$$b \ast b = b_1^2 + 2b_0 b_2 + \frac{1}{4}(\partial^2 b_0 \bar{\partial}^2 b_0 - (\partial\bar{\partial} b_0)^2) \quad (5.22)$$

and the final expression for the stress tensor at second order is

$$\mathcal{J}^{(1)}_2 = -\frac{1}{\sqrt{\gamma}} \left[ b_1^2 + 2b_0 b_2 + \frac{1}{4}(\partial^2 b_0 \bar{\partial}^2 b_0 - (\partial\bar{\partial} b_0)^2) - \frac{\gamma}{2} a_1 \partial \cos\phi_0 \right] \quad (5.23)$$

with $a_1$, $b_0$, $b_1$ and $b_2$ given in (5.3, 5.4) and (5.10) respectively.

We now apply the $\bar{\partial}$–derivative and prove that on–shell it can be written as a $\partial$–derivative of some quantity. Already at this order the check is quite complicated but it is worth pursuing it to understand how noncommutativity works. First of all, as proven in Appendix B, the following identity holds

$$\bar{\partial} \left[ \frac{1}{4}(\partial^2 b_0 \bar{\partial}^2 b_0 - (\partial\bar{\partial} b_0)^2) - \frac{\gamma}{2} a_1 \partial \cos\phi_0 \right] = \partial \left[ -\frac{\gamma}{2} \bar{\partial} (a_1 \cos\phi_0) \right]$$

$$= \frac{\sqrt{\gamma}}{2} \bar{\partial}(a_1 \cos\phi_0) \quad (5.24)$$
Therefore, the second and the third terms in (5.22) satisfy a conservation equation. For the first two terms, after inserting the explicit expressions for $b_0$, $b_1$ and $b_2$, we have

$$
\delta(b_1^2 + 2b_0b_2) \equiv \bar{\delta} \left( \frac{1}{4}(\partial \phi_1)^2 + (\partial \phi_0) b_2 \right)
$$

$$
= \frac{1}{2} \partial \phi_1 \delta \phi_1 + (\partial \phi_0)b_2 + (\partial \phi_0)\bar{\delta} b_2
$$

$$
= \frac{\gamma}{2}(\partial \phi_1) \phi_1 \cos \phi_0 + \frac{\gamma}{2} \partial \phi_2 \sin \phi_0 + \frac{\gamma}{6}(\partial \phi_0) \partial a_1 \sin \phi_0
$$

$$
+ \frac{\gamma}{2}(\partial \phi_0)(\sin \phi)|_{\theta^2}
$$

(5.25)

where the equations of motion for $b_2$ have been used. We concentrate on the last term $(\partial \phi_0)(\sin \phi)|_{\theta^2}$. Since we expect it to appear in $(\partial \phi \ast \sin \phi)|_{\theta^2}$ we first evaluate this expression up to second order

$$
(\partial \phi \ast \sin \phi)|_{\theta^2} =
$$

$$
(\partial \phi_0 + \theta \partial \phi_1 + \theta^2 \partial \phi_2) \ast (\sin \phi_0 + \theta \phi_1 \cos \phi_0 + \theta^2(\sin \phi)|_{\theta^2})
$$

(5.26)

We perform the $\ast$-product and keep only quadratic terms in $\theta$. Using the identity (B.13) and

$$
\partial^3 \phi_0 \partial \phi_0 - \partial^2 \partial \phi_0 \partial \phi_0 = -8\partial a_1
$$

(5.27)

which are consequences of the zero order equations of motion, we end up with

$$
(\partial \phi \ast \sin \phi)|_{\theta^2} =
$$

$$
\bar{\partial} \phi_0(\sin \phi)|_{\theta^2} - 4a_2 \cos \phi_0 + 4\phi_1 a_1 \sin \phi_0
$$

$$
+(\partial a_1)(\bar{\partial} \phi_0) \sin \phi_0 + \phi_1(\partial \phi_1) \cos \phi_0 + (\partial \phi_2) \sin \phi_0
$$

(5.28)

On the other hand, from the identity (B.20) proven in Appendix B we read

$$
(\partial \phi \ast \sin \phi)|_{\theta^2} =
$$

$$
-\partial(\cos \phi)|_{\theta^2} + 4\phi_1 a_1 \sin \phi_0 - \frac{2}{3} \bar{\partial}(\partial a_1 \cos \phi_0) - 4a_2 \cos \phi_0
$$

$$
= -\partial(\cos \phi)|_{\theta^2} + 4\phi_1 a_1 \sin \phi_0 + \frac{2}{3} \partial a_1(\partial \phi_0) \sin \phi_0 - 4a_2 \cos \phi_0
$$

(5.29)

Comparing the equations (5.28) and (5.29) we finally obtain

$$
\partial \phi_0(\sin \phi)|_{\theta^2} =
$$

$$
-\partial(\cos \phi)|_{\theta^2} - \frac{1}{3}(\partial a_1)(\partial \phi_0) \sin \phi_0 - \phi_1(\partial \phi_1) \cos \phi_0
$$

$$
-(\partial \phi_2) \sin \phi_0
$$

(5.30)

It is now easy to see that if we insert this result in (5.25) a lot of cancellations occur and we are left with

$$
\delta(b_1^2 + 2b_0b_2) = -\frac{\gamma}{2} \partial(\cos \phi)|_{\theta^2}
$$

(5.31)
Therefore, the conservation law at second order reads
\[ \bar{\partial} J^{(1)}_2 = \partial \left( \frac{\sqrt{7}}{2} (\cos \phi) |_{\theta^2} + \frac{\sqrt{7}}{2} \bar{\partial} (a_1 \cos \phi) \right) \] (5.32)

The r.h.s. is related to \( \tilde{J}^{(2)}_1 \) up to total \( \bar{\partial} \)-derivatives.

To summarize, the conserved stress tensor up to second order in \( \theta \) is (rescaling by a constant factor)
\[
J^{(1)} = \frac{1}{2} \partial \phi_0 \partial \phi_0 + \theta \partial \phi_0 \partial \phi_1 \\
+ \theta^2 \left( \partial \phi_0 \partial \phi_2 + \frac{1}{2} (\partial \phi_1)^2 + \frac{1}{3} \partial \phi_0 \partial \phi_0 \partial a_1 \\
+ \frac{1}{8} \partial^3 \phi_0 \partial^2 \phi_0 - \frac{1}{8} (\partial^2 \phi_0)^2 - \gamma a_1 \partial \cos \phi_0 \right) 
\] (5.33)
with \( a_1 \) given in (5.7).

### 5.2 Perturbative evaluation of \( J^{(2)} \) current

As we have already remarked, the current \( J_0^{(2)} \) is a total derivative, according to the well known fact that a spin 3 current does not appear in the spectrum of the conserved quantities for the ordinary sine–Gordon. The natural question which arises is whether \( J^{(2)} \) remains trivial when the noncommutation parameter is turned on. To answer this question we compute the first \( \theta \)-correction to \( J^{(2)} \).

According to the general relation (4.27) we have
\[ J^{(2)}_1 = J^{(2)}_1 - J^{(1)}_0 \partial \bar{J}^{(1)}_0 \] (5.34)
where, from (4.21) using the fact that \( a \) is already order one,
\[ J^{(2)}_1 = \frac{1}{\gamma} b \ast \partial b |_{\theta} = \partial \left( \frac{1}{4 \gamma} \partial \phi_0 \partial \phi_1 \right) + \frac{1}{8 \gamma} \left( \partial^2 \phi_0 \partial^2 \phi_0 \partial \phi_0 \partial^3 \phi_0 - \partial^3 \phi_0 \partial^2 \phi_0 \right) \] (5.35)

Now, extracting \( J^{(1)}_0 \) and \( \bar{J}^{(1)}_0 \) from eq. (5.15) we finally obtain
\[ J^{(2)}_1 = \partial \left( \frac{1}{4 \gamma} \partial \phi_0 \partial \phi_1 + \frac{1}{8} (\partial \phi_0)^2 \cos \phi_0 + \frac{1}{8} \partial^2 \phi_0 \sin \phi_0 \right) \] (5.36)

Therefore, at first order the \( J^{(2)} \) is still trivial.

To summarize the results of this section, we notice that the perturbative analysis of the conserved currents has revealed the following features:
a) The spin of the conserved currents are the same as the ordinary ones. Therefore, the spin spectrum of the corresponding integrals of motion
\[ Q^{(s)} = \int dz J^{(s)} + \int d\bar{z} \bar{J}^{(s)} \] (5.37)
still coincides with the exponents of the SU(2) algebra, modulo the Coxeter number. This means that noncommutativity does not affect the algebraic structure which underlies the model.

b) Despite the appearance of $b^{-1}$ in the general expression for $\tilde{F}^{(1)}$, the conservation law at spin 2, order by order in $\theta$, only involves the $\ast$–product of $\phi$ and its derivatives, but not their inverses.

6 Localized solutions

Since the system of equations (3.8) describes a constrained field, the class of solutions for $\phi$ will be in general smaller than the one corresponding to the unconstrained case. In order to show that, in spite of the constraint, nontrivial solutions exist, we look for localized solutions of the equations of motion up to first order in $\theta$, $\phi = \phi_0 + \theta \phi_1$,

$$\frac{\partial \bar{\phi}_0}{\partial \bar{\phi}_1} = \gamma \sin \phi_0$$
$$\frac{\partial \bar{\phi}_1}{\partial \bar{\phi}_0} = \gamma \phi_1 \cos \phi_0$$
$$\partial^2 \bar{\phi}_0 \partial^2 \bar{\phi}_0 - (\partial \bar{\phi}_0)^2 = 0$$

(6.1)

Although the bicomplex approach has given us suitable equations of motion which guarantee integrability, we have not been able yet to find the corresponding action from which they can be derived (but see below). Consequently, we use the term “localized” as follows: we construct the standard euclidean solitons at order zero in $\theta$. Since the solution at order zero determines the solutions at succeeding orders, we call “localized” the all-orders solution constructed in this fashion.

The last equation in (6.1) is automatically satisfied by any function of $(x^1 - \text{i}v x^0)$. In particular, it is automatically satisfied by any function of $x^1$ only. To solve the other equations we first reduce the problem to a one–dimensional problem by looking for solutions of equations of motion which do not depend on $x^0$. We then need to solve

$$\phi_0'' = 2\gamma \sin \phi_0$$
$$\phi_1'' = 2\gamma \phi_1 \cos \phi_0$$

(6.2)

The first one is the ordinary equation for static euclidean solitons of the sine–Gordon system. Thus, we apply the standard procedure to integrate it. A first integration gives

$$\phi_0' = \pm 2\sqrt{2\gamma} \sin \frac{\phi_0}{2}$$

(6.3)

A second integration (taking the plus sign in the previous equation) gives the one–(anti)soliton solutions

$$\phi_0^{\text{sol}}(x^1) = 4 \arctg e^{\sqrt{2\gamma}(x^1 - \bar{x}^1)}$$
$$\phi_0^{\text{antisol}}(x^1) = -4 \arctg e^{\sqrt{2\gamma}(x^1 - \bar{x}^1)}$$

(6.4)

§It is easy to show that also at second order in $\theta$ the equation of motion which does not contain the potential (first equation in (5.13)) is automatically satisfied by any function of $(x^1 - \text{i}v x^0)$. We conjecture that this pattern repeats at every order.
We now look for solutions to the second equation in (6.1). Taking the product

\[ \phi'_1 \phi''_0 + \phi'^0 \phi''_1 = (\phi'_0 \phi'_1)' \]  

(6.5)

and inserting the equations of motion on the l.h.s. we can perform a first integration to obtain

\[ \phi'_0 \phi'_1 = 2\gamma \phi_1 \sin \phi_0 \]  

(6.6)

Inserting eq. (6.3) on the l.h.s. and dividing by \( \phi_1 \), we have

\[ \frac{\phi'_1}{\phi_1} = \sqrt{2\gamma} \cos \frac{\phi_0}{2} \]

\[ = \sqrt{2\gamma} \frac{1 - e^{2\sqrt{2\gamma}(x^1 - \bar{x}^1)}}{1 + e^{2\sqrt{2\gamma}(x^1 - \bar{x}^1)}} \]  

(6.7)

This equation can be easily integrated and gives

\[ \phi_1(x^1) = \frac{1}{\cosh\sqrt{2\gamma}(x^1 - \bar{x}^1)} \]  

(6.8)

A plot of this function for \( 2\gamma = 1 \), \( \bar{x}^1 = 0 \) is given in Fig. 1.

![Figure 1: First order correction to the one–(anti)soliton solution](image)

It represents the first order correction generated by noncommutativity to the euclidean “one–soliton” solutions of the ordinary sine–Gordon equation. The first order correction to the antisoliton solution is again (6.8).

In Fig. 2 we have plotted the “one–soliton” solution (again \( 2\gamma = 1 \) and \( \bar{x}^1 = 0 \)) at first order in \( \theta \) for different values of the deformation parameter. As it can be easily seen, the
perturbation due to the noncommutativity mainly affects the “soliton” around \( x^1 = 0 \), while leaving its asymptotic behavior at large \(|x^1|\) unmodified.

This fact has immediate consequences for the topological charge. Even for noncommutative solitons we can define the topological charge as

\[
\mathcal{T} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx^1 \frac{d\phi}{dx^1} = \sum_n \theta^n \mathcal{T}^{(n)}
\]

where \( \phi \) is the localized solution at all orders in \( \theta \). For \( \theta = 0 \) the topological charge is \( \mathcal{T}^{(0)} = 1 \) for the soliton and \( \mathcal{T}^{(0)} = -1 \) for the antisoliton. Computing it for the first order correction (6.8) we find \( \mathcal{T}^{(1)} = 0 \). At this order noncommutativity does not affect the topological properties of the solution.

It is interesting to note that the first two equations of motion in (6.1) can be derived from the following action

\[
S = \int d^2x \left\{ \frac{1}{2} \partial \phi_0 \bar{\partial} \phi_0 + \gamma (1 - \cos \phi_0) + \theta [\partial \phi_0 \bar{\partial} \phi_1 + \gamma \phi_1 \sin \phi_0] \right\}
\]

(6.10)

For static solutions, using (6.3, 6.6) we can write

\[
S = 2\gamma \int dx^0 dx^1 \left[ (1 - \cos \phi_0) + \theta \phi_1 \sin \phi_0 \right]
\]

(6.11)

![Figure 2: One-soliton solution including the first order correction for various values of \( \theta \)](image)

If we compactify the system on a cylinder \( 0 < x^0 < 2\pi, -\infty < x^1 < \infty \), for the soliton
we obtain
\[ S = \sqrt{2\gamma} \int_0^{2\pi} dx^0 \left[ -4 \frac{1}{1 + e^{2\sqrt{2}\gamma(x^1 - \bar{x}^1)}} + \theta \frac{1}{\text{ch}^2 \sqrt{2\gamma}(x^1 - \bar{x}^1)} \right] \bigg|_{-\infty}^{+\infty} = 8\pi \sqrt{2\gamma} \quad (6.12) \]

which is the value of the action for the ordinary sine–Gordon euclidean soliton. We then conclude that at this order noncommutativity does not change the value of the classical action. It would be interesting to investigate whether this is a peculiarity of the first order or it is a general feature.

Starting from the solutions (6.4, 6.8) which do not depend on the euclidean time $x^0$, we can generate “nonstatic” solutions. As already noticed, the third eq. in (6.1) constrains $\phi_0$ to depend on $(x^1 - ivx^0)$, so that
\[ \phi_0^{\text{sol}}(x^0, x^1) = 4 \text{arctg} e^{\frac{2\gamma(x^1 - \bar{x}^1 - ivx^0)}{\sqrt{1 - v^2}}} \]
\[ \phi_0^{\text{antisol}}(x^0, x^1) = -4 \text{arctg} e^{\frac{2\gamma(x^1 - \bar{x}^1 - ivx^0)}{\sqrt{1 - v^2}}} \quad (6.13) \]
is easily verified to be a solution of the first eq. in (6.1). For the $x^0$ dependence of $\phi_1$, at this order we do not have any constraint. However, since $\phi_0$ enters its equation of motion, we expect $\phi_1$ to have the same dependence on $x^0$. Indeed, by direct inspection, one realizes that a “nonstatic” solution is
\[ \phi_1(x^0, x^1) = \frac{1}{\text{ch} \left(\frac{2\gamma(x^1 - \bar{x}^1 - ivx^0)}{\sqrt{1 - v^2}}\right)} \quad (6.14) \]

Finally, we look for a generalization of Backlund transformations to the noncommutative case. In the ordinary case, these are first order equations which generate multi–“soliton” solutions starting from a given solution with fewer “solitons”.

Consider first the case $\theta = 0$. Localized solutions $\tilde{\phi}_0$ of the equations of motion (6.1) can be generated by solving the first order equations
\[ \frac{1}{2} \bar{\partial}(\tilde{\phi}_0 - \phi_0) = \alpha \sqrt{\gamma} \sin \left( \frac{\tilde{\phi}_0 + \phi_0}{2} \right) \]
\[ \frac{1}{2} \bar{\partial}(\tilde{\phi}_0 + \phi_0) = \frac{1}{\alpha} \sqrt{\gamma} \sin \left( \frac{\tilde{\phi}_0 - \phi_0}{2} \right) \quad (6.15) \]
where $\phi_0$ is a known solution and $\alpha$ an arbitrary real parameter. Indeed, by applying $\bar{\partial}$ to the first equation it is easy to check that $\tilde{\phi}_0$ satisfies the equations of motion, once $\phi_0$ does.

At the first order in $\theta$, given a solution $\phi_1$, we look for a function $\tilde{\phi}_1$ which satisfies
\[ \frac{1}{2} \bar{\partial}(\tilde{\phi}_1 - \phi_1) = \alpha \sqrt{\gamma} \frac{\tilde{\phi}_1 + \phi_1}{2} \cos \left( \frac{\tilde{\phi}_0 + \phi_0}{2} \right) \]
\[ \frac{1}{2} \bar{\partial}(\tilde{\phi}_1 + \phi_1) = \frac{1}{\alpha} \sqrt{\gamma} \frac{\tilde{\phi}_1 - \phi_1}{2} \cos \left( \frac{\tilde{\phi}_0 - \phi_0}{2} \right) \quad (6.16) \]
where $\tilde{\phi}_0$ satisfies (6.13). Again, by application of $\bar{\partial}$ to the first equation, it is easy to verify that $\tilde{\phi}_1$ is a solution of the equation of motion at first order. This system can be used to generate first order corrections to multi–“soliton” solutions.
7 Conclusions and outlook

We have discussed in this paper an integrable noncommutative two-dimensional field theory whose equations of motion reduce to the ordinary sine-Gordon equation in the commutative limit. In considering generalizations of the ordinary sine-Gordon system to the NC case one might be tempted to start from the usual action \( \int d^2x \left[ \frac{1}{2} \partial \phi \bar{\partial} \phi + \gamma (1 - \cos \phi) \right] \) and replace ordinary products by \(*\)-products [12]. However, since the currents obtained as a natural extension of the ordinary ones are not conserved, the corresponding system is not guaranteed to be integrable. Instead our approach, which expresses the field equations as integrability conditions of a bicomplex system, constructs directly classically conserved currents which reduce to the standard currents of the commutative theory in the limit of \( \theta \to 0 \).

Constructing a NC extension of sine–Gordon directly at the level of equations of motion, necessarily generates a NC system of two equations, one of them being a natural extension of the ordinary one, whereas the other, which has the structure of a constraint equation, has only a NC origin. The appearance of the second equation is quite unavoidable and seems to be necessary in order to guarantee integrability. It is a consequence of the fact that, in the NC case, the \( SU(2) \) symmetry group of ordinary sine–Gordon is enlarged to \( U(2) \) which contains a noncommutative \( U(1) \) factor. Therefore, the group valued fields involved in eq. (3.4) have a nontrivial trace part which is responsible for the appearance of the constraint equation.

We have recursively constructed an infinite set of conservation laws. In the present approach, writing down a conservation equation for suitably defined objects is not a difficult task. What is essential however is that these objects be local, i.e. not involve integrals of the field \( \phi \). Our currents, defined in terms of nonlocal solutions of a certain bidifferential equation, satisfy this requirement. We have performed some calculations to verify the conservation of the currents to low order in the parameter \( \theta \). In particular, from our results it appears that noncommutativity does not affect the spin spectrum of the conserved currents.

We have presented “localized” solutions, to first order in the NC parameter, as well as the corresponding Backlund transformations. For the former, a better understanding of their significance and of the corresponding topological charges would require a knowledge of the classical action. We were able to make some progress by working to first order in \( \theta \).

Since the system we are describing is a constrained system, the class of solutions for the field \( \phi \) will be in general smaller than the one of the corresponding unconstrained system (the system which would satisfy only the second equation in (3.8)). This is already clear at the perturbative level where for instance, the constraint at first order (see last equation in (3.1)) selects a particular subclass among all the solutions \( \phi_0 \) of the ordinary sine–Gordon equation (even if we have shown that it does not restrict the spectrum of localized solutions). We are faced with the question whether our NC system can be considered a natural NC extension of the sine–Gordon system. To answer this question, we should define what we mean in general with “NC extension” of a field theory. The
The most natural definition is that the NC system should reproduce the ordinary field theory when the limit $\theta \to 0$ is done appropriately. However, our example shows that there can be situations where the limit is not smooth. In fact, if we take the limit at the level of equations of motion, we obtain the ordinary equations and, consequently, the whole spectrum of ordinary solutions. If we perform the limit directly on the solutions we seem to lose part of the ordinary spectrum. A similar pattern can be observed in the NC extension of the $U(1)$ WZNW model \cite{11,12}, since the NC equations of motion contain a nontrivial constraint which restricts the set of solutions. Also in this case, performing the $\theta \to 0$ limit at the level of solutions one does not recover the whole dynamics of the ordinary model.

The main motivation of our work was to construct a NC system which is integrable and is related to the commutative sine-Gordon system. Therefore we concentrated on the currents and the equations of motion which guarantee their conservation. As we mentioned above, a direct NC generalization of the sine-Gordon action appears not to lead to a set of conserved currents. At the same time, it is not trivial to find an action from which our equations of motion follow. Although we were able to do this to first order in the $\theta$ parameter, this issue remains an open problem. Furthermore, even if such an action were found, or guessed at (and we do have some possible candidates), because manipulations involving NC exponentials are so cumbersome, checking that it would lead to the correct equations of motion might be an equally difficult task.

A final comment: It is evident, from what we have presented in this paper, that explicit calculations in NC theories are rather complicated. It is to be hoped that better techniques – a $\ast$-calculus – can be developed to facilitate manipulations which are trivial in the commutative case but extremely difficult here. This seems essential if one has any hope to move on to a quantum formulation of the theory.

**Acknowledgements**

We would like to thank the referee, whose objections and comments were quite helpful in focusing our thoughts on some of the issues.

This work has been supported in part by INFN, MURST and the European Commission RTN program HPRN–CT–2000–00131, in which S.P. is associated to the University of Padova. M.T.G acknowledges support by the NSF under grant PHY-00-70475.
A Derivation of conserved currents

In this Appendix we give the detailed derivation of the equations (4.17, 4.18) which have been used in the text to obtain the conserved currents.

We concentrate on the system (4.15) which, for convenience, we write again

\[(i) \quad \bar{\partial} \chi = -lL \ast \chi \]
\[(ii) \quad \sqrt{\gamma} e_\pm = -l(\partial \chi + M \ast \chi) = -l[\partial \chi + (U + aI \ast \chi)] \quad (A.1)\]

Setting \(l = 0\), from eq. \((ii)\) we see that at zero order \(\chi\) satisfies \(e_\pm \chi(0) = 0\). A solution is then \(\chi(0) = e_+\). For \(l \neq 0\) instead, we apply \(e_+\) to eq. \((ii)\) to obtain (in the derivation we make often use of the identity \(e_+ U = U e_+\))

\[e_+ \partial \chi = -e_+(U + aI) \ast \chi = -U \ast (e_\pm \chi) - e_+ a \ast \chi \quad (A.2)\]

Substituting \(e_\pm \chi\) as given by eq. \((ii)\) we can write

\[\sqrt{\gamma} e_\pm \partial \chi = l[U \ast \partial \chi + U \ast U \ast \chi + U \ast a \ast \chi] = \sqrt{\gamma} a \ast e_+ \chi \quad (A.3)\]

This can be added to \(\sqrt{\gamma} e_\pm \partial \chi\) obtained by differentiating eq. \((ii)\)

\[\sqrt{\gamma} e_\pm \partial \chi = -l[\partial^2 \chi + \partial U \ast \chi + U \ast \partial \chi + \partial a \ast \chi + a \ast \partial \chi] \quad (A.4)\]

to write

\[\sqrt{\gamma} \partial \chi = l[-\partial^2 \chi - a \ast \partial \chi + U \ast U \ast \chi + U \ast a \ast \chi + \partial U \ast \chi - \partial a \ast \chi] = \sqrt{\gamma} a \ast e_+ \chi \quad (A.5)\]

Since \(\chi^{(0)} = e_+\) is not invertible, it follows that \(\chi\) will not be invertible for \(l \to 0\). Therefore, we consider instead the shifted function

\[\tilde{\chi} = \chi + e_- \quad (A.6)\]

Computing \(e_\pm \chi\) from eq. \((A.2)\) we have

\[\chi = e_+ \chi + e_- \chi = e_+ \tilde{\chi} - e_- U^{-1} \ast \partial \tilde{\chi} - e_- U^{-1} \ast a \ast \tilde{\chi} \quad (A.7)\]

Substituting in \((A.3)\) we find a differential equation for \(\tilde{\chi}\). If we multiply that equation by \(\tilde{\chi}^{-1}\) we obtain an equation for \(j \equiv \partial \tilde{\chi} \ast \tilde{\chi}^{-1}\). Making use of the identity \(\partial^2 \tilde{\chi} \ast \tilde{\chi}^{-1} = \partial j + j \ast j\) we finally have

\[\sqrt{\gamma} j = -l \partial j - lj \ast j \]
\[+ l[-a - e_+ U \ast a \ast U^{-1} + e_+ \partial U \ast U^{-1} - e_+ U + e_- (\partial a) \ast U^{-1}] \ast j \]
\[+ l[e_+ U \ast U - e_+ \partial a - e_+ U \ast a \ast U^{-1} \ast a + e_+ \partial U \ast U^{-1} \ast a \]
\[\quad - e_- \partial U + e_- (\partial a) \ast U^{-1} \ast a] - \sqrt{\gamma} e_+ a \quad (A.8)\]
which is eq. (4.17) in the text.

We now go back to the system (A.1) and consider eq. (i). Using (A.7) we find

\[
\bar{\partial} \tilde{\chi} = -lL^* (e_+ \tilde{\chi} - e_- U^{-1} * \partial \tilde{\chi} - e_- U^{-1} * a * \tilde{\chi})
\]

(A.9)

and for \( \tilde{j} \equiv \bar{\partial} \tilde{\chi} * \tilde{\chi}^{-1} \)

\[
\tilde{j} = -lL^* (e_+ - e_- U^{-1} * a) + lL * e_- U^{-1} * \tilde{j}
\]

(A.10)

which is eq. (4.18) used in the text.

B  \( *-\)calculus

In this Appendix we collect and prove identities which are useful when dealing with the NC equations and their \( \theta \)-expansion.

We start by recalling that \( *-\)functions are defined in terms of their \( *-\)series expansion. In particular, we have

\[
e^{a\phi}_* = \sum_{n=0}^{\infty} \frac{a^n}{n!} \phi * \phi * \cdots \phi \equiv \sum_{n=0}^{\infty} \frac{a^n}{n!} \phi^n_*
\]

\[
\cos_* \phi = \frac{e^{i_\phi}_* + e^{-i_\phi}_*}{2}
\]

\[
\sin_* \phi = \frac{e^{i_\phi}_* - e^{-i_\phi}_*}{2i}
\]

(B.1)

As a consequence of the general identity

\[
e^{a_\phi}_* * e^{b_\phi}_* = e^{(a+b)_\phi}_*
\]

(B.2)

which follows from the definition of the \( *-\)exponential, it is easy to prove that the main trigonometric identities are still valid. Among them we list

\[
\cos^2_* \phi + \sin^2_* \phi = 1
\]

(B.3)

\[
\cos^2_* \phi \frac{2}{2} = \frac{1 + \cos_* \phi}{2}
\]

\[
\sin^2_* \phi \frac{2}{2} = \frac{1 - \cos_* \phi}{2}
\]

(B.4)

\[
\sin_* \frac{\phi}{2} * \cos_* \frac{\phi}{2} = \cos_* \frac{\phi}{2} * \sin_* \frac{\phi}{2} = \frac{1}{2} \sin_* \phi
\]

(B.5)

On the other hand, due to the lack of commutativity, the derivatives of exponentials and trigonometric functions do not satisfy the nice properties they have in the commutative
case. This is due to the fact that the derivative of the exponential is not proportional to the exponential itself; instead

$$\partial e^{a\phi} = a \int_0^1 dt e_t^{a\phi} \star \partial \phi \star e_t^{(1-t)a\phi}$$  \hspace{1cm} (B.6)

It follows that $\partial \cos \star \phi \neq -\sin \star \phi$ and the check of the conservation laws at any order in $\theta$ is a hard problem.

We now list the main identities which can be useful to write the explicit expressions of the currents perturbatively in $\theta$. Since in the text we perform perturbative calculations up to second order, we will stop our identities at that order. We give formal expansions in terms of the field $\phi$ and its derivatives, keeping in mind that $\phi$ itself may depend on $\theta$ and eventually must be expanded in power series in the noncommutation parameter. We have

$$\phi \star \phi = \phi^2 + \frac{\theta^2}{4} (\partial^2 \phi \bar{\partial}^2 \phi - (\bar{\partial} \partial \phi)^2) + O(\theta^3)$$  \hspace{1cm} (B.7)

where the $\star$-product has been defined in (4.5). Other useful identities are

$$e^{\frac{\phi}{2}} \star \partial e^{-\frac{\phi}{2}} = -\frac{i}{2} \partial \phi + \frac{1}{2!} \left(-\frac{i}{2}\right)^2 [\bar{\partial} \partial \phi]_* + \frac{1}{3!} \left(-\frac{i}{2}\right)^3 [[[\bar{\partial} \partial \phi], \phi], \phi]_* + \cdots$$  \hspace{1cm} (B.9)

$$\sin_\star \phi = \sin \phi + O(\theta^2)$$

$$= \sin (\phi_0 + \theta \phi_1 + \cdots) + O(\theta^2)$$

$$= \sin \phi_0 + \theta \phi_1 \cos \phi_0 + O(\theta^2)$$  \hspace{1cm} (B.10)

$$\sin^2 \star \frac{\phi}{2} = \frac{1 - \cos_\star \phi}{2}$$

$$= \frac{1 - \cos \phi_0 - \theta \phi_1 \sin \phi_0}{2} + O(\theta^2)$$  \hspace{1cm} (B.11)

Less trivial identities which have been used in checking the conservation of $\mathcal{J}_1$ at second order are (5.24) and (5.29). Here we give a proof of these two identities.

A way to check (5.24) is to compute $(\partial \phi_0 \star \sin \phi_0)_{\theta^2}$, where $\sin$ is the ordinary sine, in two different ways:

1) using the definition of star product at that order

$$(\partial \phi_0 \star \sin \phi_0)_{\theta^2}$$

$$= \frac{1}{8} (\partial^2 \phi_0 \bar{\partial}^2 \sin \phi_0 + \bar{\partial}^2 \partial \phi_0 \partial^2 \sin \phi_0 - 2 \partial^2 \bar{\partial} \phi_0 \bar{\partial} \sin \phi_0)$$

$$= -\frac{1}{8} (\partial^3 \phi_0 \bar{\partial} \phi_0 - \bar{\partial} \partial^2 \phi_0 \partial \phi_0) \bar{\partial} \phi_0 \sin \phi_0 = \partial a_1 \bar{\partial} \phi_0 \sin \phi_0$$

$$= -\bar{\partial} (\partial a_1 \cos \phi_0)$$  \hspace{1cm} (B.12)
Note that the equations of motion at order zero and one have been used and also the following identity
\[ \partial^2 \phi_0 \bar{\partial}^2 \phi_0 + \bar{\partial}^2 \partial \phi_0 \partial^2 \phi_0 - 2\partial^2 \bar{\partial} \partial \phi_0 = \partial (\partial^2 \phi_0 \bar{\partial}^2 \phi_0 - (\bar{\partial} \partial \phi_0)^2) = 0 \] (B.13)
which follows from \((5.9)\).

2) using the equations of motion from the very beginning
\[ (\partial \phi_0 \ast \sin \phi_0)|_{\theta^2} \]
\[ = \frac{1}{\gamma} (\partial \phi_0 \ast \partial \bar{\partial} \phi_0)|_{\theta^2} = \frac{1}{2\gamma} \bar{\partial} (\partial \phi_0 \ast \partial \phi_0)|_{\theta^2} \]
\[ = \frac{1}{8\gamma} \bar{\partial} (\partial^2 \phi_0 \bar{\partial}^2 \phi_0 - (\bar{\partial} \partial \phi_0)^2) \] (B.14)

Comparing the results from the two procedures we obtain
\[ \bar{\partial} (\partial^2 \phi_0 \bar{\partial}^2 \phi_0 - (\bar{\partial} \partial \phi_0)^2) = -8\gamma \bar{\partial} (\partial a_1 \cos \phi_0) \] (B.15)
which is the identity \([5,24]\).

To check the identity \((5.29)\) we evaluate \(\partial (\cos \phi)|_{\theta^2} = (\partial \cos \phi)|_{\theta^2}\). By expanding the cosine in power series, we are left with the evaluation of
\[ \partial (\phi \ast \cdots \ast \phi)|_{\theta^2} = \{(\partial \phi) \ast \cdots \ast \phi + \phi \ast (\partial \phi) \ast \cdots \ast \phi + \cdots \ast \phi \ast \cdots \ast (\partial \phi)\}_{\theta^2} \] (B.16)
for the \(\ast\)-product of \(2n\) fields. It can be written as
\[ \partial (\phi \ast \cdots \ast \phi)|_{\theta^2} = \left\{ 2n(\partial \phi) \ast \phi_{\ast}^{2n-1} + \sum_{j=1}^{2n-1} [\phi_{\ast}^j, \partial \phi]_{\ast} \ast \phi_{\ast}^{2n-1-j} \right\}_{\theta^2} \] (B.17)

At this order we can replace any \(\ast\)-power \(\phi_{\ast}^m\) in the sum with the ordinary power \(\phi^m\) (note that the commutator is already order one in \(\theta\) and \(\phi_{\ast}^m = \phi^m + O(\theta^2)\) according to \((B.7))\). Now consider the following identity
\[ [\phi^j, \partial \phi]_{\ast} \text{ up to } \theta^2 = \theta j (\phi_0 + \theta \phi_1)^{j-1} \{(\partial \phi_0 + \theta \partial \phi_1)(\partial \bar{\partial} \partial \phi_0 + \theta \bar{\partial} \partial \phi_1)
- (\partial \bar{\partial} \partial \phi_0 + \theta \bar{\partial} \partial \phi_1)(\partial \partial \phi_0 + \theta \partial \partial \phi_1)\} \text{ up to } \theta^2 \]
\[ = \theta 8j \phi_0^{-1} a_1 + \theta^2 (8j(j - 1) \phi_0^{-2} \phi_1 a_1 + 8j \phi_0^{-1} a_2) \] (B.18)

We substitute in \((B.17)\), perform the \(\ast\)-product and keep only \(\theta^2\)-order terms
\[ \partial (\phi \ast \cdots \ast \phi)|_{\theta^2} = 2n(\partial \phi) \ast \phi_{\ast}^{2n-1} \]
\[ + \sum_{j=1}^{2n-1} [8j(j - 1) \phi_0^{-2} \phi_1 a_1 + 8j(2n - 1 - j) \phi_0^{-2} \phi_1 a_1 + 8j \phi_0^{-2} a_2] \]
\[ + \sum_{j=1}^{2n-2} \frac{1}{2} \left[ \partial (\phi_0^{-1} a_1) \bar{\partial} \phi_0^{2n-1-j} - \bar{\partial} (\phi_0^{-1} a_1) \partial \phi_0^{2n-1-j} \right] \] (B.19)
In the last term, using the equations of motion $\bar{\partial}a_1 = 0$, we are left with $(2n-1-j)\phi_0^{2n-3}\partial a_1\bar{\partial}\phi_0$. Now perform the sums over $j$ and substitute in the original equation to find

\[
(\partial \cos_* \phi)|_{\theta^2} = \\
\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{(2n)!} 2n(\partial\phi) * \phi_*^{2n-1}\right)|_{\theta^2} + 8 \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{(2n)!} n(2n-1)(2n-2)\phi_0^{2n-3}\phi_1 a_1\right) \\
+ \frac{4}{3} \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{(2n)!} n(2n-1)(\bar{\partial}\phi_0^{2n-2})\partial a_1\right) + 8 \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{(2n)!} n(2n-1)\phi_0^{2n-2} a_2\right) \\
= -(\partial\phi * \sin_* \phi)|_{\theta^2} + 4\phi_1 a_1 \sin \phi_0 - \frac{2}{3} \bar{\partial}(\partial a_1 \cos \phi_0) - 4a_2 \cos \phi_0 \\
\text{as claimed.}
\]
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