Non-causal correlations certify the lack of a definite causal order among localized space-time regions. In stark contrast to scenarios where a single region influences its own causal past, some processes that distribute non-causal correlations satisfy a series of natural desiderata: logical consistency, linear and reversible dynamics, and computational tameness. Here, we present such processes among arbitrary many regions where each region influences every other but itself, and show that the above desiderata are altogether insufficient to limit the amount of “acausality” of non-causal correlations. This leaves open the identification of a principle that forbids non-causal correlations. Our results exhibit qualitative and quantitative parallels with the non-local correlations due to Ardehali and Svetlichny.

The succession of events is usually assumed to follow a fixed causal order: The causal structure is directed acyclic [1]. General relativity describes dynamic causal order, and according to quantum theory physical quantities are indefinite. Therefore, it is reasonable to expect that a satisfactory theory of quantum gravity exhibits both features [2]. This is exemplarily demonstrated by the quantum switch [3]: A quantum system coherently controls the causal order between two events in its future [4, 5]. It is known [6–9] that the quantum switch and generalizations thereof do not violate causal inequalities. Causal inequalities [10, 11], similar to Bell inequalities [12], are theory independent and confine the observable correlations among a set of agents under the assumption of a definite causal order. Although this assumption is natural, there exist motivations to study the world beyond. In a world beyond, for instance, one can ask: How can we derive causal order without presupposing causal order, and what is the logical origin of causal order? This question is of foundational interest and relevant to general relativity and quantum gravity [14]. In general relativity, for instance, no causal order is enforced, and Einstein’s suspicion [15] that closed time-like curves are consistent with that theory proved true [16, 17]. Thus, this question asks for a principle with which such exotic space-time structures are excluded.

The recent process-matrix framework [10] describes such a “non-causal” world, and relates to quantum theory in the same way as general relativity relates to special relativity: While special relativity holds in sufficiently small space-time regions of general relativity, the process-matrix framework postulates that physics in a discrete number of local regions is described by quantum theory—and no causal order among the regions is enforced (see Figure 1). This framework is known to describe indefinite causal order—e.g., the quantum switch [6]—, and moreover, violates causal inequalities [10, 18, 19, 11, 20, 21, 6, 22–24]. In fact, this latter quality is independent of quantum theory. Causal inequalities are also violated in the classical-probabilistic [21] and classical-deterministic limit [25]. The process-matrix framework and its classical limits describe linear dynamics and comply with various desiderata: The restriction to reversible (unitary) dynamics does not reestablish causal order [25–27], and the computational capabilities seem highly restricted [28–30]. In stark contrast, alternative models of violations of causal order [31–36] lead to non-linear dynamics and bare unnatural features, e.g., quantum-

---

**Figure 1**: In each region an experiment on a system provided by the environment is performed (the knobs illustrate the settings and the meters the results). After the experiment, systems are released back to the environment. Process matrices [10] describe the most general dynamics (functions from quantum instruments $\mu_a^x, \nu_b^y, \tau_z^\alpha$), which describe the experiments, to behaviors $P_{A,B,C|X,Y,Z}$ such that locally no deviation from quantum theory is observed. Some process matrices violate causal order.
state cloning [37, 38], and extravagant computational power [39–42].

Here, we further investigate the process-matrix framework, and ask whether violations of causal inequalities vanish by increasing the number of regions—as suggested by previous studies [19, 22, 28]. The analogous question had been asked [43] for violations of Bell inequalities, with the result that quantum non-local correlations are unlimited: They are non-vanishing for any number of bodies [44, 45]. We show that this is also the case here: For any number \( n \geq 3 \) of regions, robust violations of causal inequalities are theoretically possible, and, in contrast to the previous results, the degree of the violation increases with the number of regions. More concretely, we design a bi-causal game \( G_n \) for \( n \) parties that is asymptotically the hardest: As \( n \) becomes increasingly large, the maximal winning probability of the game \( G_n \) approaches 1/2. Then, we show that this game is won deterministically in the classical-deterministic limit of the process-matrix framework. Finally, we prove that every classical-deterministic process is a process matrix: The game \( G_n \) is won deterministically with unitarily extendible [26] process matrices. These main findings are compactly illustrated in Figure 2. Moreover, we provide evidence that “acausality” in these frameworks and Bell non-locality are intimately connected. The derived game resembles the Ardehali-Svetlichny game [43, 46], and shares qualitative and quantitative features. We show that this analogy also holds in another setting [47, 11]. Our results therefore motivate to further investigate the connection between “non-causality” and Bell non-locality, to envisage field-theoretic frameworks with no causal order, and to search for principles with which “acausality” is banned in these worlds, in general relativity, and in theories of quantum gravity.

We present our findings in the three subsequent sections. In the first, we state our results, and in the second, we give a general discussion including the connections to the Ardehali-Svetlichny game and to causal structures. In the last section, finally, we prove the theorems.

1 Results

Before we present our results, we briefly comment on the notation used. Usually we use calligraphic letters for sets, and uppercase letters for random variables. The set \( Z_n \) is defined as \( \{0, 1, \ldots, n - 1\} \). If a symbol appears with and without subscripts from some \( Z_n \), then the bare symbol denotes the collection under the natural composition, e.g., \( A = (A_0, \ldots, A_{n-1}) \).

We extend this notation further: If \( S \subseteq Z_n \), then \( A_S = (A_k)_{k \in S} \) and \( A_{S'} = (A_k)_{k \in \mathbb{Z}_n \setminus S'} \), and if \( k \in Z_n \), then \( A_{k} = A_{\{k\}} \). We use \( \oplus \) for the addition modulo two, and \( \equiv_2 \) for the equivalence relation modulo two. Finally, if \( H \) is a Hilbert space, then \( \mathcal{L}(H) \) denotes the set of linear operators on \( H \), and \( I_H \) is the identity operator on \( H \).

1.1 Bi-causal inequalities for arbitrary many parties

Consider \( n \) parties (regions) where each party \( k \in Z_n \) is given a random variable \( X_k \) and outputs a random variable \( A_k \). Such a setup is described by a behavior (a conditional probability distribution) \( P_{A|X} \). The possible behaviors \( P_{A|X} \) depend on how the parties interact. Under the assumption of causal order, i.e., every party can influence her or his future only, one obtains restrictions that are mirrored by causal inequalities [20]. If some \( P_{A|X} \) violates such an inequality, then \( P_{A|X} \) is not compatible with a causal ordering of the parties and is called non-causal. The set of causal correlations among \( n \) parties is \( \mathcal{C}_n^{\text{causal}} \).

In a multi-party setting, however, a violation of a causal inequality could also arise because only some parties violate causal order but not all. In analogy to multi-party non-local correlations [43–45, 48], Abbott et al. [22] show that correlations among \( n \) parties are genuinely multi-party non-causal (they are non-causal among all parties) if and only if they are not bi-causal: The correlations cannot be simulated by partitioning the \( n \) parties in two subsets such that the subsets are causally ordered.

\textbf{Definition 1 (Bi-causal correlations [22]).} An \( n \)-party behavior \( P_{A|X} \), for \( n \geq 2 \), is bi-causal if and only if

\begin{equation}
P_{A|X} = \sum_{\theta \subseteq K \subseteq Z_n} P_K(K)P_{A|X|X_{\theta},K=K}P_{A|X|A_{\theta},X,K=K},
\end{equation}

(1)
where $K$ is a random variable with sample space $\{K \mid \emptyset \subseteq K \subseteq \mathbb{Z}_n\}$. The set of bi-causal behaviors among $n$ parties is $C^n_{\text{bi-causal}}$. Behaviors among $n$ parties that lie outside this set are called genuinely multi-party non-causal.

Again, the restrictions imposed by bi-causality are mirrored in bi-causal inequalities. A behavior $P_{A|X}$ that violates a bi-causal inequality cannot be decomposed as above, and therefore is genuinely multi-party non-causal.

We describe a game that is played among $n$ parties, for arbitrary $n$, and derive bi-causal inequalities by upper bounding the winning probability for any bi-causal strategy (see Figure 2).

**Game ($G_n$).** Every party $k \in \mathbb{Z}_n$ receives a uniformly distributed binary random variable $X_k$, and must deterministically produce a random variable $A_k$ that equals $\omega^n_k(X)$, where

$$\omega^n_k : \mathbb{Z}_2^n \to \mathbb{Z}_2 \quad x \mapsto \bigoplus_{i,j \in \mathbb{Z}_n \setminus \{k\}} x_i x_j \oplus \bigoplus_{i \in \mathbb{Z}_n} \gamma_{k,i} x_i$$

with

$$\gamma_{k,i} := \begin{cases} 1 & (i < k \land i \neq 2k) \lor (k < i \land i \equiv 2k) \\ 0 & \text{otherwise} \end{cases}$$

In the three-party case, $G_3$ is won whenever

$$A_0 = (\neg X_1 \land X_2) \quad A_1 = (\neg X_2 \land X_0) \quad A_2 = (\neg X_0 \land X_1),$$

which is the three-party game by Araújo and Feix [49], and first published in Ref. [21]. Thus, the game $G_n$ is a generalization of that three-party game to any number of parties. Another generalization of that game is known [28], however, with the drawback that the winning probability with causal strategies approaches one.

**Theorem 1 (Bi-causal inequalities).** For $n \geq 2$, the probability of winning the game $G_n$ with bi-causal correlations $P_{A|X} \in C^n_{\text{bi-causal}}$ is bounded as follows:

$$\Pr[A = \omega^n(X)] \leq \frac{1}{2} + \frac{1}{2^{n/2}}.$$  

We prove this theorem in Section 3.1. From Definition 1, moreover, it is evident that $G_n$ is asymptotically the hardest bi-causal game where at least one party guesses a binary variable.

**Lemma 1 (Least bi-causal winning probability).** If $\mathbb{Z}_n$ is a set of parties, $d_{\text{min}} := \min_{k \in \mathbb{Z}_n} |\mathbb{A}_k|$, and $\sigma$ a function $\bigtimes_{k \in \mathbb{Z}_n} \mathbb{A}_k \to \bigtimes_{k \in \mathbb{Z}_n} \mathbb{A}_k$, then

$$\max_{P_{A|X} \in C^n_{\text{bi-causal}}} \Pr[A = \sigma(X)] \geq \frac{1}{d_{\text{min}}}. $$

**Proof.** This lower bound is bi-causally achieved by placing a single party $k$ with $d_{\text{min}} = |\mathbb{A}_k|$ in the first subset, i.e., $P_k(\{k\}) = 1$, and by letting that party make a uniformly random guess, i.e., $P_{A_k|X_k}(a, x) = |\mathbb{A}_k|^{-1}$ for all $(a, x) \in \mathbb{A}_k \times \mathbb{X}_k$. Every other party $\ell \in \mathbb{Z}_n \setminus \{k\}$ has access to all random variables $X_0, \ldots, X_{n-1}$ and deterministically generates $\sigma_{\ell}(X)$.

We also establish that the above bound on the probability of winning $G_n$ with bi-causal behaviors is tight (see Figure 2).

**Theorem 2 (Faces).** The inequalities of Theorem 1 represent the faces of the bi-causal polytopes: The value of the game $G_n$ with bi-causal behaviors is

$$\max_{P_{A|X} \in C^n_{\text{bi-causal}}} \Pr[A = \omega^n(X)] = \frac{1}{2} + \frac{1}{2^{n/2}}$$

for at least $n/2$ bi-causal extremal points if $n$ is even, and at least $n$ bi-causal extremal points if $n$ is odd.

We prove this theorem in Section 3.2.

### 1.2 Classical violations

In the classical-deterministic limit [25] of the process-matrix framework [10], each party obtains a system from the environment, locally applies an arbitrary function (intervention), and outputs a system to the environment (cf. Figure 1). Let $I_k$ be the set of possible states party $k \in \mathbb{Z}_n$ can receive from the environment, and $O_k$ the set of possible states party $k$ can release to the environment. It is known that the most general dynamics in this setup (without assuming causal order) is described with process functions.

**Definition 2 (Process function [25, 50]).** An $n$-party process function is a function $\omega : \mathcal{O} \to \mathcal{I}$ for some sets $\mathcal{O} = \bigtimes_{k \in \mathbb{Z}_n} O_k, \mathcal{I} = \bigtimes_{k \in \mathbb{Z}_n} I_k$ such that

$$\forall f \exists i : i = \omega(f(i)),$$

where $f = (f_k : I_k \to O_k)_{k \in \mathbb{Z}_n}$ is a collection of functions.

In words, a process function accounts for the interaction among the parties and has a unique fixed point for each intervention $f$ of the parties. Thus, given a choice of interventions $f$, the process function uniquely determines the states the parties receive from the environment (see Figure 3). This is intuitive: No fixed point corresponds to a logical contradiction, and multiple fixed points to an ambiguity [25].

For three parties or more, there exist process functions that do not reflect a causal order

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among the parties (Equation (4) interpreted as a function \((X_0, X_1, X_2) \mapsto (A_0, A_1, A_2)\) is an example).

The parties in this classical-deterministic world have access to a process function to generate some behavior \(P_{A|X}\). Each party \(k\) receives an element from the set \(\mathcal{I}_k\) and some \(X_k\), and applies a function (intervention) \(\mu_k : X_k \times \mathcal{I}_k \to \mathcal{A}_k \times \mathcal{O}_k\) to generate the output \(\mathcal{A}_k\) and the system that is released to the environment. The process function illustrated in Figure 3, for instance, leads to behaviors \(P_{A|X}\) where party 0 is in the causal past of party 1.

**Definition 3** (Process-function behavior). An \(n\)-party behavior \(P_{A|X}\) is a deterministic process-function behavior if and only if there exists some \(n\)-party process function \(\omega\) and interventions \(\mu = (\mu_0, \ldots, \mu_{n-1})\) such that

\[
P_{A|X}(a, x) = \begin{cases} 1 & \exists i : \alpha(i, x) = a \land \omega(x, i) = i \\ 0 & \text{otherwise,} \end{cases}
\]

where \(\alpha_k : X_k \times \mathcal{I}_k \to \mathcal{A}_k\) and \(\beta_k : X_k \times \mathcal{I}_k \to \mathcal{O}_k\) are the components of \(\mu_k = (\alpha_k, \beta_k)\). The set of \(n\)-party process-function behaviors is the convex hull of all \(n\)-party deterministic process-functions behaviours and is denoted by \(\mathcal{C}_n^{\text{process}}\).

**Theorem 3.** The function \(\omega^n\) of the game \(G_n\) is an \(n\)-party process function.

This theorem—proven in Section 3.3—immediately implies that the game \(G_n\) is won deterministically in this framework (see Figure 2).

**Corollary 1** (Non-causal value of \(G_n\)). The value of the game \(G_n\) with classical-process behaviors is

\[
\max_{P_{A|X} \in \mathcal{C}_n^{\text{process}}} \Pr[A = \omega^n(X)] = 1.
\]

**Proof.** Let the \(n\) parties have access to the process function \(\omega^n\) of the game \(G_n\). Each party \(k \in \mathbb{Z}_n\) relays the input \(x_k\) to the environment \(o_k = x_k\), and uses the input \(i_k\) from the environment as guess \(a_k\). Formally, party \(k\) implements the function \(\mu_k : (x_k, i_k) \mapsto (i_k, x_k)\), by which the behavior \(P_{A|X}\) that equals \(\omega^n\) is obtained. \(\square\)

This corollary shows that for any number of parties, the classical-deterministic limit of the process-matrix framework leads to genuinely multi-party non-causal correlations (\(\forall n \geq 3 : \mathcal{C}_n^{\text{process}} \nsubseteq \mathcal{C}_n^{\text{non-causal}}\)), and that the violation is bounded by a constant (\(\forall n \geq 3 : \text{the gap is at least } 1/4\)). While previous games [22] share the former feature, their non-causal correlations are limited—the gap vanishes for increasing number of parties.

### 1.3 Quantum violations

In the process-matrix framework [10], each party \(k\) receives a quantum state on the Hilbert space \(\mathcal{I}_k\), applies a quantum instrument \(\mu_k = (\mu_k^{a,x})_{(a,x) \in \mathcal{A}_k \times \mathcal{X}_k}\), and releases a quantum state on the Hilbert space \(\mathcal{O}_k\). A quantum instrument \(\mu_k\) is a family of completely positive trace-non-increasing maps from \(\mathcal{L}(\mathcal{I}_k)\) to \(\mathcal{L}(\mathcal{O}_k)\) such that for any \(x \in \mathcal{X}_k\), the map \(\sum_{a \in \mathcal{X}_k} \mu_k^{a,x}\) is trace preserving.

**Definition 4** (Process matrix and process-matrix behaviors [10]). An \(n\)-party process matrix is a positive-semi-definite matrix \(W \in \mathcal{L}(\mathcal{O} \otimes \mathcal{I})\) for some Hilbert spaces \(\mathcal{O} = \bigotimes_{k \in \mathbb{Z}_n} \mathcal{O}_k, \mathcal{I} = \bigotimes_{k \in \mathbb{Z}_n} \mathcal{I}_k\) such that

\[
\forall \mu : \text{Tr} \left[ \bigotimes_{k \in \mathbb{Z}_n} M_k W \right] = 1,
\]

where \(\mu = (\mu_k : \mathcal{L}(\mathcal{I}_k) \to \mathcal{L}(\mathcal{O}_k))_{k \in \mathbb{Z}_n}\) is a family of completely positive trace-preserving maps, and where \(M_k\) is the Choi operator [52, 53] of \(\mu_k\). An \(n\)-party behavior \(P_{A|X}\) is a process-matrix behavior if and only if there exists some \(n\)-party process matrix \(W\) and quantum instruments \(\mu = (\mu_k)_{k \in \mathbb{Z}_n}\) such that

\[
P(a | x) = \text{Tr} \left[ \bigotimes_{k \in \mathbb{Z}_n} M_k^{a,x} W \right].
\]

The set of \(n\)-party process-matrix behaviors is \(\mathcal{C}_n^{\text{process}}\).

Just as in the classical case, the process matrix accounts for the interaction among the parties. A process matrix yields a conditional probability distribution under any choice of quantum instruments (interventions) of the parties: No matter what experiment the parties perform, and even if they share entangled states, the probabilities of their observations

\[3\text{Note that in the process-matrix framework, the Choi operator of a map } \mathcal{E} \text{ is defined as } [\mathbb{1} \otimes \mathcal{E}(\mathbb{1})](\mathbb{1})^T, \text{ whereas some define the Choi operator with a } \text{partial transpose only, or without transpose. For our results, this distinction is irrelevant.}

\[4\text{Allowing the parties to share entangled states forces } W \text{ to be positive semi-definite. Oreshkov, Costa, and Brukner [10] show that if the Hilbert space } \mathcal{O} \text{ is trivial and the parties share arbitrary entangled states, then every process matrix is a quantum state (see Figure 4). In contrast, Barnum et al. [54] and Acín et al. [55] show that if } \mathcal{O} \text{ is trivial but the parties do not share quantum states, then } W \text{ is positive on pure tensors as opposed to positive semi-definite; for three parties or more, this yields correlations beyond the quantum set.}
are well-defined. The example illustrated in Figure 3 is obtained by the process matrix $W = |0⟩⟨0| ⊗ |1⟩⟨1|$ where $|1⟩$ is the non-normalized maximally entangled state $\sum_0 |o⟩⟨o|$, and the sum is taken over a basis of $O_0$.

**Theorem 4.** If $\omega : O \to I$ is an $n$-party process function, then

$$W := \sum_{o \in O} |o⟩⟨o| \otimes |\omega(o)⟩⟨\omega(o)|_I$$

is an $n$-party process matrix.

This theorem—shown in Section 3.4—implies that every process-function behavior is a process-matrix behavior (see Figure 2), and therefore

$$\max_{P_{A|X} \in C_{\text{process}}} \Pr[A = \omega^O(X)] = 1.$$ 

Moreover, from Ref. [25] it is known that every process function and every mixture of process functions is embeddable into a reversible process function with two additional parties: A party in the global past with a trivial input, and a party in the global future with a trivial output. By the above theorem, the same behaviours are unitarily extensible [26]—the corresponding process matrices can be extended to unitary dynamics. This does not hold for all process matrices. Moreover, Barrett, Lorenz, and Oreshkov [56] show that every unitarily extensible two-party process matrix is causal.

Note that the above theorem is not trivial: It is not proven by referring to the classical-deterministic limit of process matrices. The reason for this is the following. Call an operator $W$ a deterministic-diagonal process matrix if and only if $W$ has zero-one entries only and satisfies Equation (11) restricted to interventions $\otimes_{j \in Z} M_{\lambda}^{i,j}$ diagonal in the same basis as $W$. The limit theorem [21, 27] states that there exists a bijection (through Equation (13)) between process functions and deterministic-diagonal process matrices. A priori, however, it could be that some deterministic-diagonal process matrices are not process matrices; it could be that there exist quantum instruments $\mu$ such that the probabilities are not well-defined for some deterministic-diagonal process matrices. This potentiality arises because the enlargement of the set of possible interventions from functions to quantum instruments restricts the set of objects $W$ that satisfy Equation (11). While a few process functions and stochastic processes (probabilistic generalizations of process functions) were translated case-by-case to process matrices [19, 27, 26], the above theorem was unknown.

## 2 Discussion

Violations of causal order arise naturally—not in the sense that such violations are known to arise in our physical world, but in the sense that they arise and are advocated by our best physical theories. This leaves open a binary alternative: Either causal order does not hold in our physical world, or these violations are mathematical artifacts of our theories. To show the latter, one must describe a reasonable and physical principle from which causal order is reestablished. All attempts so far, however, fail to do so, and leave us with the “chronology protection conjecture” [14]. While this conjecture is well motivated, it also leaves room to speculate that “acausal” dynamics arise in yet unprobed physical regimes, e.g., in the minuscule of quantum foam or in black holes [57–60]. Overall, our results show that causal order cannot be derived in the classical limit [19], with the requirement of determinism and reversibility [25], with the NP-hardness assumption [40, 29], and also not—as we show here—in the many-body limit. Note that the process functions and process matrices described here can be made reversible [21, 26]. In the context of general relativity [50, 61], our findings present non-trivial and reversible closed time-like curves that traverse any number of local space-time regions.

### 2.1 Relation to non-locality

We not only rule out that “acausal” processes must be restricted to few regions, but also establish a link between non-causal and non-local correlations. To see this, we briefly describe the Ardehali-Svetlichny non-local game.\(^5\) This game is played among $n$ parties that are space-like separated—they cannot communicate (see Figure 4).

**Definition 5** (Ardehali-Svetlichny function). An $n$-ary Ardehali-Svetlichny function is a function $S^n_\lambda$ where

$$S^n_\lambda : \mathbb{Z}_2^n \to \mathbb{Z}_2, \quad z \mapsto \bigoplus_{i,j \in \mathbb{Z}_m} z_i z_j \oplus \bigoplus_{i \in \mathbb{Z}_m} \lambda_i z_i,$$

for some $\lambda \in \mathbb{Z}_2^n$.

\(^5\)The three-body Bell inequality was introduced by Svetlichny [43] and generalized by Ardehali [46] based on insights by Mermin [62]. A reformulation of these inequalities as a game is found in the work by Ambainis et al. [63, 64].
The winning probability
non-causal
local
same

G
While in
Svetlichny bi-local
Ardehali-Svetlichny Game. Every party \( k \in \mathbb{Z}_n \) receives a uniformly distributed binary random variable \( X_k \) and must produce a random variable \( A_k \) such that

\[
\bigoplus_{i \in \mathbb{Z}_n} A_i = S^{\lambda}_n(X).
\] (16)

Lemma 2 (Ardehali-Svetlichny inequalities [44–46]). The winning probability

\[
\Pr \left[ \bigoplus_{i \in \mathbb{Z}_n} A_i = S^{\lambda}_n(X) \right]
\] (17)
of the Ardehali-Svetlichny game for \( n \geq 2 \) is

- upper bounded by \( 1/2 + 1/2^{n/2} + 1 \) for local behaviors \( P_{A|X} \).
- upper bounded by \( 3/4 \) for Svetlichny bi-local behaviors \( P_{A|X} \).
- and reaches the Tsirelson bound \([65] (2 + \sqrt{2})/4 \) for quantum behaviors \( P_{A|X} \).

A first connection between the game \( G_n \) and the Ardehali-Svetlichny game is that \( \omega_n^\lambda(x) = S^{\lambda-1}_n(x|k) \) for \( \lambda \) being the alternating string \( (0,1,0,1,\ldots) \) when \( k \) is even, and \( (1,0,1,0,\ldots) \) when \( k \) is odd. While in \( G_n \), each party must guess the value of an Ardehali-Svetlichny function, in the Ardehali-Svetlichny game, the parties must jointly guess an Ardehali-Svetlichny function. A second connection is that \( G_n \) can be used to certify genuinely multi-party non-causal correlations, and the Ardehali-Svetlichny game can be used to certify genuinely multi-party non-local correlations.6 Third, the bi-causal (and therefore also the causal) bound of \( G_n \) is essentially the local bound of the Ardehali-Svetlichny game. And finally, with the frameworks employed, the parties win \( G_n \) with constant probability, as it is the case for the Ardehali-Svetlichny game in quantum theory. This last feature implies that non-local and non-causal correlations are unlimited and yield an exponential non-local versus local, and respectively non-causal versus causal advantage for an increasing number of parties.

Note that we cannot simply connect the results on (non-)locality and (a)causality by considering the same game, but must aim for a translation. A reason for this is that the infamous game where every party must guess its neighbours input (GYNI) does not allow for a quantum-over-classical advantage with shared resources [67], however, it allows for a non-causal-over-causal advantage when played with process matrices [11]. The similarities displayed above, however, lead us to ask whether any parity local game is translatable to a causal game. A parity local game is a game where the parity of the parties’ outputs must equal a function of their inputs. A proposal for general translations to causal games is to ask each party separately to guess the function value. We briefly apply this recipe to the CHSH game [47]. In that two-party parity game, each party is given a binary random variable \( X \) and \( Y \), and produces \( A \) and \( B \), respectively, satisfying \( A \oplus B = XY \). By following the above recipe, we ask the parties to produce \( A \) and \( B \) such that \( A = XY \) and \( B = XY \). This resulting game is the lazy guess-your-neighbour’s-input game [11]: \( X (A \oplus Y) = Y (B \oplus X) = 0 \). The causal bound for this game is \( 3/4 \) and coincides with the local bound of the CHSH game we started with. Two parties using the process-matrix framework violate that causal bound and win this derived game with probability 0.819 (the maximal winning probability is unknown) [11], the CHSH game is quantum mechanically won with probability at most \((2 + \sqrt{2})/4 \approx 0.854 \) [65].

2.2 Geometric and causal structure

By Theorem 2, the bi-causal inequalities describe faces of the bi-causal polytopes. However, these inequalities do not describe facets. This can be seen for \( n = 3 \). The hyperplane specified by that inequality is 7-dimensional:

\[
\begin{align*}
p_{000}^{000} + p_{001}^{001} + p_{010}^{010} + p_{011}^{011} + p_{100}^{100} + p_{101}^{101} + p_{110}^{110} + p_{111}^{111} &= 6, \\
\end{align*}
\] (18)

where \( p_{xyz}^{abc} := P_{A,X}(a,b,c,x,y,z) \). However, there are exactly three bi-causal extremal points that lie on that hyperplane. For \( n \) parties, the hyperplane specified by the bi-causal inequality is \((2^n - 1)\)-dimensional. We have identified \( n/2 \) for \( n \) even, and \( n \) for \( n \) odd, bi-causal extremal points on that hyperplane (see Theorem 2 and the proof in Section 3.2 for the strategies), and leave open whether more optimal strategies exist for \( n \geq 4 \).

Following the work by Barrett, Lorenz, and Orešíkov [56], we are in position to describe the causal structure corresponding to the process function \( \omega^0 \).

In their article, the authors show that the causal structure of \( \omega^3 \) is the fully connected directed graph as shown in Figure 5a, where each node denotes a party, and where an edge \( i \rightarrow j \) indicates that party \( i \) influences party \( j \). For \( n \) parties, the causal structure again is such a graph. If every undirected edge represents a bi-directional edge, then the causal structure of \( \omega^k \) is given by the complete graph \( K_n \) (see Figure 5b). Every party \( i \in \mathbb{Z}_n \) influences every other party \( j \in \mathbb{Z}_n \setminus \{i\} \), and, no matter what intervention \( f = (f_0,\ldots,f_n) \) the parties perform, no party \( k \) influences her or himself.
2.3 Open questions

Our findings bring forward a series of open questions. While the main question “how to derive causal order without presupposing causal order” stated in the introduction persists, we now ask whether a general translation between Bell non-local correlations and non-causal correlations is possible. Such a translation enriches the understanding of the “non-causal” world—just as the translation between Bell non-locality and contextuality [68–73] (see Budroni et al. [74] for a recent preprint on that topic)—and speculatively brings forward experimental setups to violate causal order (cf. Oreshkov [75], Purves and Short [8], Wechs et al. [9], and Wechs, Branciard, and Oreshkov [76]).

Because the process functions presented here can be embedded in reversible functions [25], they are physically implementable e.g., by means of billiard-ball collisions [77]. This yields a physical approach to produce the presented correlations: Construct closed time-like curves, e.g., with wormholes [78–80], which connect the output of the process function \( \omega^n \) with the past space-time boundaries of the \( n \) regions. The general-relativistic properties of these space-time geometries, however, are unknown. More details on this approach are found in the articles by Baumeler et al. [50], and by Tobar and Costa [61]. This latter article moreover provides process functions inequivalent to those presented here for \( n = 4 \) regions.

To bridge a gap still present between the process-matrix framework and relativity, and towards a theory of quantum gravity, one can ask whether a field-theoretic version of the process-matrix framework allows for violations of causal order. In such a framework, every localized space-time region can be regarded as a party, and thus, parties may be overlapping. The derived non-causal correlations for any number of parties support such a possibility. In connection to that, Giacomini, Castro-Ruiz, and Brukner [81] describe process matrices with continuous-variable systems.

Moreover, while we show that the causal structure of the process function \( \omega^n \) is the complete graph \( K_n \), we leave open the question of singling out the causal structures that are attainable with process functions: Does there exist a graph-theoretic criterion to reject a function as a process function? The absence of self-loops is a necessary condition (see Lemma 5 below) but insufficient.

Finally, it is intriguing to further compare the process-matrix framework with process functions. The reason is that the only known deterministic violations of causal inequalities are present in the quantum and in the classical framework (this is also the case here). Moreover, all unitarily extendible two-party process matrices are causal [56], just as it is in the classical case. Thus, we ask: Is \( C_{\text{causal}}^n \setminus C_{\text{non-causal}}^n \cap D = C_{\text{causal}}^n \setminus C_{\text{causal}}^n \cap D \), where \( D \) is the set of deterministic behaviors? An affirmative answer would imply that in the “acausal” regime, quantum and classical theories are equivalent (cf.Aaronson and Watrous [41]).

3 Proofs

3.1 Derivation of bi-causal inequalities

We derive the bi-causal inequalities (Theorem 1) in two main steps. Note that in any bilateral partition \((\mathcal{K}, \mathbb{Z}_n \setminus \mathcal{K})\) of \( n \) parties, where the parties in \( \mathcal{K} \) causally precede the remaining parties, every party \( j \) not in \( \mathcal{K} \) can deterministically guess the random variable \( \omega_j^n(\mathcal{X}) \); only the parties in \( \mathcal{K} \) have to make non-trivial guesses. Since the parties in \( \mathcal{K} \) can communicate as they wish, it might be the case that a single party only has to make a non-trivial guess, and that all other parties can base their guesses on that party. First, we show that this almost never happens. In the second step, we invoke the bound on the winning probability of the Ardehali-Svetlichny game.

3.1.1 Guesses cannot be recycled

We define the event \( \mathcal{E}^\mathcal{K} \): There is some party \( k_0 \in \mathcal{K} \) such that every party \( \ell \in \mathcal{K} \) can deterministically compute \( \omega_{\ell}^n(\mathcal{X}) \) from \( \omega_{k_0}^n(\mathcal{X}) \) and \( X_{\mathcal{K}} \). In other words, there exists some \( k_0 \in \mathcal{K} \) such that the term \( \omega_{\ell}^n(\mathcal{X}) \oplus \omega_{k_0}^n(\mathcal{X}) \) for any \( \ell \in \mathcal{K} \) is independent of the variables \( x_{\mathcal{K}} \). Note that if such a \( k_0 \) exists, then clearly \( \forall k, \ell \in \mathcal{K} \) the expression \( \omega_{k_0}^n(\mathcal{X}) \oplus \omega_{k_0}^n(\mathcal{X}) \) is independent of \( x_{\mathcal{K}} \).

Definition 6. For a non-empty \( \mathcal{K} \subseteq \mathbb{Z}_n \) and \( n \geq 2 \), the event \( \mathcal{E}^\mathcal{K} \) is

\[
\mathcal{E}^\mathcal{K} := \left\{ x_{\mathcal{K}} \in \mathbb{Z}_2^{n-|\mathcal{K}|} \mid \forall k, \ell \in \mathcal{K}, \exists c \in \mathbb{Z}_2 : \forall x_{\mathcal{K}} \in \mathbb{Z}_2^{n-|\mathcal{K}|} : \omega_{k_0}^n(\mathcal{X}) \oplus \omega_{k_0}^n(\mathcal{X}) = c \right\}.
\]

We give an upper bound on the probability for this event to occur.

Lemma 3. Let \( \mathcal{K} \subseteq \mathbb{Z}_n \) be non-empty, \( n \geq 2 \) and let \( X_{\mathcal{K}} \) be a uniformly distributed random variable. The
probability of the event $\mathcal{E}^K$ over $\mathcal{X}_K$ is upper bounded as follows:

$$\Pr_{\mathcal{X}_K}[\mathcal{E}^K] \leq 2^{-|K|+1}. \quad (20)$$

Proof. We express $\omega_k^n(x)$ in a form where the terms involving $x_K$ are separated from the rest:

$$\omega_k^n(x) = \bigoplus_{i,j \notin K} x_i x_j \bigoplus \bigoplus_{i \in K} \alpha_{k,i} x_i \bigoplus \beta_k, \quad (21)$$

with

$$\alpha_{k,i} := \gamma_{k,i} \bigoplus \bigoplus_{j \notin K \setminus \{i\}} x_j, \quad (22)$$

$$\beta_k := \bigoplus_{i \in K \setminus \{j\}} x_i x_j \bigoplus \bigoplus_{i \notin K} \gamma_{k,i} x_i, \quad (23)$$

where $\gamma_{k,i}$ is defined as in Equation (3). This allows us to express $\omega_k^n(x) \oplus \omega_k^n(x)$, for $k, \ell \in K$, compactly as

$$\bigoplus_{i \in K} x_i (\gamma_{k,i} \oplus \gamma_{\ell,i} \oplus x_k \oplus x_\ell) \bigoplus \beta_k \bigoplus \beta_\ell. \quad (24)$$

Therefore, if $x_K \in \mathcal{E}^K$, then

$$\forall k, \ell \in K, i \notin K : \gamma_{k,i} \oplus \gamma_{\ell,i} = x_k \oplus x_\ell, \quad (25)$$

and moreover

$$\forall k, \ell \in K, i, j \notin K : \gamma_{k,i} \oplus \gamma_{\ell,j} = \gamma_{k,j} \oplus \gamma_{\ell,j}. \quad (26)$$

Thus, for some $x_K \in \mathcal{E}^K$ we can define $c_{k,\ell} := \gamma_{k,i_0} \oplus \gamma_{\ell,i_0}$ for an arbitrary $i_0 \notin K$. Since $c_{k,\ell}$ is independent of $x_K$, we have

$$\forall x_K : x_K \in \mathcal{E}^K \implies x_k \oplus x_\ell = c_{k,\ell}; \quad (27)$$

the bits of every sequence $x_K \in \mathcal{E}^K$ are related by the same constants $c_{k,\ell}$. Now, we pick some $k_0 \in K$ and find that $x_\ell$ for every $\ell \in K$ is uniquely determined by $x_{k_0}$; there are at most two distinct sequences $x_K$ in $\mathcal{E}^K$. Finally, since $X_K$ is uniformly distributed, we obtain

$$\Pr_{X_K}[\mathcal{E}^K] = \frac{|\mathcal{E}^K|}{2^{|K|}} \leq 2^{-|K|+1}. \quad (28)$$

\[ \square \]

3.1.2 Proof of Theorem 1 via an application of the Ardehali-Svetlichny inequality

We prove the bi-causal inequalities by conditioning the winning probabilities of the event $\mathcal{E}^K$. In the event $\mathcal{E}^K$, only one party must guess an Ardehali-Svetlichny function, and otherwise, at least two parties must guess independent Ardehali-Svetlichny functions. It turns out that these independent functions are parity relative.

\[ \square \]

Lemma 4 (Guessing two parity-relative functions). \textit{If $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ and $g : \mathbb{Z}_2^n \to \mathbb{Z}_2$ are two functions that satisfy}

$$g(z) = f(z) \oplus c \bigoplus_{i \in S} z_i, \quad (29)$$

for some constant $c \in \mathbb{Z}_2$ and for some non-empty set $S \subseteq \mathbb{Z}_m$, and if the random variable $Z$ is uniformly distributed, and $P_{A,B,Z} = P_{A,B,Z}$, then

$$\forall P_{A,B} : \Pr[A = f(Z) \land B = g(Z)] \leq \frac{1}{2}. \quad (30)$$

Proof. Since $|S| > 1$, the expression $\bigoplus_{i \in S} z_i$ is a parity function. Furthermore, since $Z$ is uniformly distributed, we have

$$\Pr[f(Z) = g(Z)] = \Pr[f(Z) \neq g(Z)] = \frac{1}{2}. \quad (31)$$

Denote by $p$ the probability that $A = B$, and note that this probability is independent of the random variable $Z$. We thus have

$$\Pr[A = f(Z) \land B = g(Z)] = p \Pr[A = f(Z) = g(Z) | A = B] + (1 - p) \Pr[A = f(Z) \neq g(Z) | A \neq B] \quad (32)$$

$$\leq \frac{1}{2}. \quad (33)$$

\[ \square \]

Now, we acquired the tools to prove our first theorem.

\textit{Proof of Theorem 1.} We start with the analysis on the upper bound of the winning probability of the game $G_n$, where the bi-causal correlations are restricted: The set $K$ is fixed. The parties in the set $\mathbb{Z}_n \setminus K$ have access to $X$, \textit{i.e.}, to all inputs. This means that for every $k \in \mathbb{Z}_n \setminus K$, party $k$ can guess $\omega_k^n(X)$ deterministically. Thus, the maximum probability of winning $G_n$ in this restricted setup equals the maximum probability that the parties in $K$ produce the correct guess:

$$\max_{P_{A_{1}X} = P_{A_{K}X|K} P_{A|K_{-K}}} \Pr[A = \omega^n(X)] \quad (34)$$

For every $P_{A_{K}|X_K}$, we decompose the winning probability as

$$\Pr[A = \omega_k^n(X)] = \Pr[A = \omega_k^n(X) | \mathcal{E}^K] \Pr[\mathcal{E}^K] + \Pr[A = \omega_k^n(X) | \mathcal{E}_K^{\complement}] \Pr[\mathcal{E}_K^{\complement}]. \quad (35)$$

First, let us consider the winning probability conditioned on the event $\mathcal{E}^K$, and let $k_0 \in K$ be the only
party that has to make a non-trivial guess. By Equation (21) and Definition 5, we observe that there exists some $\lambda$ and some function $f : Z_2^{[K]} \to Z_2$ such that

$$\omega_{k_0}^n(x) = S_{\lambda}^{n-|K|}(x) \oplus f(x_K),$$

where $S_{\lambda}^{n-|K|}$ is an $(n-|K|)$-ary Ardehali-Svetlichny function. Thus, for party $k_0$ it is sufficient to guess $S_{\lambda}^{n-|K|}(X_K)$ only—the variable $f(X_K)$ is known to party $k$. In combination with Lemma 2, we obtain

$$\Pr [A_{k_0} = \omega_{k_0}^n(X)] \leq \frac{1}{2} + 2^{(-\frac{n-|K|}{2}) - 1},$$

from which the bound

$$\Pr [A_K = \omega_K^n(X) \mid E^K] \leq \frac{1}{2} + 2^{(-\frac{n-|K|}{2}) - 1}$$

(38)

follows. Note that we can use Lemma 2 in this setting because a single party $k_0$ cannot guess an Ardehali-Svetlichny function better than multiple parties together.

In the converse case, at least two parties $k_0, k_1 \in K$ have to jointly guess the random variables $\omega_{k_0}^n(X)$ and $\omega_{k_1}^n(X)$. These two functions, however, are parity relative:

$$\omega_{k_0}^n(x) \oplus \omega_{k_1}^n(x) = \beta_{k_0} \oplus \beta_{k_1} \oplus \bigoplus_{i \in K} x_i (\alpha_{k_0, i} \oplus \alpha_{k_1, i}),$$

where for some $i \not\in K$ the value of $\alpha_{k_0, i} \oplus \alpha_{k_1, i}$ equals to one. Thus, by Lemma 4, we get

$$\Pr [A_K = \omega_K^n(X) \mid Z_2^{[K]} \setminus E^K] \leq \frac{1}{2}.$$  

(40)

By Lemma 3, Equations (38) and (40), the winning probability for bi-causal correlations with a fixed $K$ is

$$\Pr [A_K = \omega_K^n(X)] \leq \left(\frac{1}{2} + 2^{(-\frac{n-|K|}{2}) - 1}\right) 2^{-|K|+1} + \frac{1 - 2^{-|K|+1}}{2}$$

(41)

$$= \frac{1}{2} + 2^{(-\frac{n+|K|}{2})}. $$

(42)

Since this holds for every $\emptyset \subseteq K \subseteq Z_n$, and since bi-causal correlations are arbitrary convex combinations over $K$, we obtain

$$\Pr [A = \omega^n(X)] \leq \frac{1}{2} + 2^{-\frac{n}{2}}$$

(43)

for all bi-causal correlations.

3.2 Saturation of bi-causal inequalities

We inductively show Theorem 2 by specifying a bi-causal strategy that reaches the upper bound on the winning probability of Theorem 1.

Proof of Theorem 2. Place party 0 in the first subset, i.e., $K = \{0\}$, and let party 0 deterministically guess the value 0. In this setting, the winning probability $\Pr [A = \omega^n(X)]$ reduces to the probability $\Pr [0 = \omega_0^n(X)]$. As observed above, we can express $\omega_0^n(x)$ with an Ardehali-Svetlichny function:

$$\omega_0^n(x) = S_{\lambda}^{n-1}(x_{\emptyset}),$$

(44)

where $\lambda$ is the alternating sequence $(0, 1, 0, \ldots)$. Party 0 guesses correctly if and only if the random variable $X$ takes a value in the set

$$Z^n := \{ x \in Z_2^n \mid S_{\lambda}^{n-1}(x_{\emptyset}) = 0 \}.$$  

(45)

Base case. It is easily verified that $|Z^2| = 4$, and $|Z^3| = 6$, from which $\Pr[Z^2] = 1$, and $\Pr[Z^3] = 3/4$ follow.

Induction step. We take steps of two. First, observe that for every $x \in Z_2^{n+2}$ we have

$$S_{\lambda}^{n+2}(x) = S_{\lambda}^{n}(x_{\{0, n, n+1\}} \oplus x_n x_{n+1} \oplus x_{n+1},) \oplus \bigoplus_{i \in Z_n \setminus \{0\}} x_i (x_n \oplus x_{n+1})$$

(46)

where the single term $x_{n+1}$ is $x_n$ if $n$ is even, and $x_{n+1}$ otherwise. Thus, if $x \in Z^n$, then $(x, 0, 0), (x, 1, 1)$, and exactly one of $(x, 0, 1)$ and $(x, 1, 0)$ are in $Z^{n+2}$. In the alternative case, if $x \not\in Z^n$, then exactly one of $(x, 0, 1)$ and $(x, 1, 0)$ is in $Z^{n+2}$. Therefore, the cardinalities of these sets are related by

$$|Z^{n+2}| = 2 |Z^n| + 2^n.$$  

(47)

By the induction hypothesis $|Z^n| = 2^n(1/2 + 2^{-[n/2]})$, we therefore obtain

$$|Z^{n+2}| = 2 \left(2^n \left(\frac{1}{2} + 2^{-[n/2]}\right) + 2^n\right)$$

(48)

$$= 2^{n+1} + 2^{n+1-\frac{n}{2}}$$

(49)

$$= 2^{n+2} \left(\frac{1}{2} + 2^{-\left([n+2]/2\right)}\right).$$

(50)

from which $\Pr[Z^{n+2}] = 1/2 + 2^{\left([-n+2]/2\right)}$ follows. Therefore, for all $n \geq 2$ there exists at least one bi-causal strategy with which the bi-causal bound is saturated.

Now, observe that $\omega_0^n(x)$ is invariant under any relabelling of the parties $0 \leftrightarrow k$ where $k \in Z_n$ is even. Moreover, if we consider an odd number of parties $n$, then $\omega_0^n(x)$ is additionally invariant under any relabelling $0 \leftrightarrow k$ when $k$ is odd. Thus, if $n$ is even, then there exist at least $n/2$ bi-causal strategies that saturate the bound, and if $n$ is odd, then there exist at least $n$ such strategies.

3.3 Deterministic classical violation

We exploit various properties of process functions in order to show Theorem 3.
3.3.1 Properties of process functions

Before we present the properties, we introduce element-wise constant functions and their respective reduced functions.

Definition 7 (Element-wise constant and reduced function). An \( n \)-ary function \( \omega: \times_{k \in \mathbb{Z}_n} \mathcal{O}_k \rightarrow \times_{k \in \mathbb{Z}_n} \mathcal{T}_k \) is element-wise constant if and only if

\[
\forall k \in \mathbb{Z}_n, o \in \mathcal{O}_k, \bar{o}_k \in \mathcal{O}_k : \omega_k(o) = \omega_k(o_k, \bar{o}_k). \tag{51}
\]

Let \( \omega \) be such a function, and let \( f_\ell : \mathcal{T}_\ell \rightarrow \mathcal{O}_\ell \) be some function for \( \ell \in \mathbb{Z}_n \). The reduced function \( \omega^{f_\ell} : \mathcal{O}_\ell \rightarrow \mathcal{I}_\ell \) is \( (\omega_0^{f_\ell}, \ldots, \omega_{n-1}^{f_\ell}) \) with

\[
\omega_k^{f_\ell} : o_\ell \mapsto \omega_k(o_\ell, \bar{o}_\ell), \tag{52}
\]

where, for some arbitrary \( \bar{o}_\ell \in \mathcal{O}_\ell \),

\[
\bar{o}_\ell = f_\ell(\omega_\ell(o_\ell, \bar{o}_\ell)). \tag{53}
\]

Lemma 5 (Constant \([50, 51]\)). If \( \omega \) is an \( n \)-party process function, then \( \omega \) is element-wise constant.

We now relate process functions with their reduced functions.

Lemma 6 (Transitivity \([50]\)). Let \( \omega : \mathcal{O} \rightarrow \mathcal{I} \) be an \( n \)-ary element-wise constant function. If there exists some \( k \in \mathbb{Z}_n \) such that for all \( f_k : \mathcal{I}_k \rightarrow \mathcal{O}_k \) the reduced function \( \omega^{f_k} \) is a process function, then \( \omega \) is a process function.

The above lemma, proven in Ref. \([50]\), can be made stronger: It is sufficient that only for some functions \( f_k \) the reduced function \( \omega^{f_k} \) is a process function. We show this in the special case where for all \( k \), all sets \( \mathcal{O}_k \) and \( \mathcal{I}_k \) are binary.

Lemma 7. Let \( \omega : \mathbb{Z}_n^2 \rightarrow \mathbb{Z}_n^2 \) be element-wise constant. If the reduced functions \( \omega^{f_k} \) for \( f_k \) being the constant-zero, constant-one, and identity function, are process functions, then for all \( f_k \), \( \omega^{f_k} \) is a process function.

Proof. Without loss of generality, and for better presentation, we set \( k \) to 0. If the reduced functions of \( \omega \) for \( f_0 \) constant zero, constant one, and identity, are process functions, we have that for all \( f_0 \) there exist \( \alpha \) unique fixed points for the functions

\[
x \mapsto \omega_0(0, f_0(x)) \tag{54}, \quad x \mapsto \omega_0(1, f_0(x)) \tag{55}, \quad x \mapsto \omega_0(\alpha, f_0(x), f_0(x)) \tag{56}.
\]

Let \( \alpha \) and \( \beta \) be the fixed points of the first two functions, i.e., we have the identities

\[
\alpha = \omega_0(0, f_0(\alpha)) \tag{57}, \quad \beta = \omega_0(1, f_0(\beta)) \tag{58}.
\]

We define the bits the first party receives upon applying the function \( \omega \) to these fixed points:

\[
\bar{\alpha} := \omega_0(0, f_0(\alpha)), \quad \bar{\beta} := \omega_0(0, f_0(\beta)). \tag{58}
\]

Note that these bits are independent of the first argument; the function \( \omega \) is element-wise constant. We now show that this implies that the reduced function \( \omega^{f_0} \), where \( f_k \) is the bit-flip function, i.e., the function

\[
x \mapsto \omega_0(1, f_0(\omega(x)), f_0(x)) \tag{59},
\]

also has a fixed point. In the case where \( \bar{\alpha} = 1, \) \( \alpha \) is a fixed point of Equation (59):

\[
\omega_0(1, f_0(0, f_0(\alpha)), f_0(\alpha)) = \omega_0(0, f_0(\alpha)) \tag{60}.
\]

\[
\alpha. \tag{61}
\]

In the case where \( \bar{\beta} = 0, \) \( \beta \) is a fixed point of Equation (59):

\[
\omega_0(1, f_0(0, f_0(\beta)), f_0(\beta)) = \omega_0(1, f_0(\beta)) \tag{62}.
\]

\[
\beta. \tag{63}
\]

The last case, i.e., \( \bar{\alpha} = 0 \) and \( \bar{\beta} = 1 \) cannot arise. Assume towards a contradiction that \( \bar{\alpha} = 0 \) and \( \bar{\beta} = 1 \). This implies that \( \alpha \) and \( \beta \) are fixed points of Equation (56):

\[
\omega_0(0, f_0(\alpha), f_0(\alpha)) = \omega_0(0, f_0(\alpha)) = \alpha, \tag{64}
\]

\[
\omega_0(0, f_0(\beta), f_0(\beta)) = \omega_0(1, f_0(\beta)) = \beta. \tag{65}
\]

Since Equation (56) has a unique fixed point, we conclude \( \alpha = \beta \) which contradicts \( \bar{\alpha} \neq \bar{\beta} \). By noting that there is a total of four functions from a bit to a bit (i.e., constant zero, constant one, identity, and flip-bit), the proof is concluded.

\[
\square
\]

3.3.2 Proof of Theorem 3

The above properties allow us to prove our statement: The functions \( \omega^n \) of the game \( G_n \) are process functions. A schematic representation of the proof is given in Figure 6.

Proof of Theorem 3. We show this theorem by induction. Assume towards a contradiction that the function \( \omega^n \) is not a process function. Clearly, \( \omega^n \) is element-wise constant. Thus, by Lemma 6 there exists some function \( f_{n-1} : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \) such that the reduced function \( \omega^{f_{n-1}} \), i.e., the function \( \omega^n \) where party \( n-1 \) implements \( f_{n-1} \), is not a process function. Furthermore, by Lemma 7 we know that this will be the case where \( f_{n-1} \) is the constant-zero, constant-one, or identity function.
The function \(\omega\) is the constant-zero function, the reduced function, in this case, is \(\Gamma_{0,1}\). Appendix B shows that the reduced function, in this case, is \(\Gamma_{0,1}\). Appendix C shows that \(\Gamma_{0,1}\) is a process function.

Since \(\omega^n\) (see Equation (4)) and \(\Gamma_{0,1}\) are process functions, we reach a contradiction: \(\omega^n\) is a process function. \(\square\)

### 3.4 All process functions are process matrices

We prove Theorem 4 constructively with the limit theorem [21, 27].

**Proof of Theorem 4.** Let \(\omega: \times_{k \in \mathbb{Z}_n} O_k \rightarrow \times_{k \in \mathbb{Z}_n} I_k\) be a process function and define the operator

\[
W^\omega := \sum_{o \in O} |o\rangle\langle o| \otimes |\omega(o)\rangle\langle \omega(o)|_I
\]

with appropriate Hilbert spaces, and where \(|\{o\}\rangle\langle o|\}_{o \in O}\) and \(|\{i\}\rangle\langle i|\) is a fixed basis for these Hilbert spaces. The limit theorem states that for every collection of completely positive trace-preserving maps \((\mu_k: \mathcal{L}(I_k) \rightarrow \mathcal{L}(O_k))_{k \in \mathbb{Z}_n}\), for \(M_k\) are the corresponding Choi operators and \(M = \bigotimes_{k \in \mathbb{Z}_n} M_k\) is diagonal in the same basis as \(W^\omega\): 

\[
\text{Tr}[MW^\omega] = 1.
\]

Now let \(M' = \bigotimes_{k \in \mathbb{Z}_n} M'_{k}\) be the Choi operator of the completely positive trace-preserving maps \((\mu_k': \mathcal{L}(I_k) \rightarrow \mathcal{L}(O_k))_{k \in \mathbb{Z}_n}\). Then

\[
\text{Tr}[M'W^\omega] = \text{Tr}[M'_{\text{diag}}W^\omega]
\]

with

\[
M'_{\text{diag}} := \sum_{(o, i) \in O \times I} |o, i\rangle\langle o, i| M'[o, i] |o, i\rangle\langle o, i|
\]

Since \(M'_{\text{diag}}\) is the Choi operator of a completely positive trace-preserving map \((M_{\text{diag}} \geq 0, \text{Tr}_{O} M_{\text{diag}} = 1)\) and diagonal in the same basis as \(W^\omega\), we have

\[
\text{Tr}[M'W^\omega] = \text{Tr}[M'_{\text{diag}}W^\omega] = 1.
\]

We conclude the proof by noting that \(W^\omega\) is positive semi-definite and that it implements the same dynamics as \(\omega\). \(\square\)

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References

[1] Judea Pearl. “Causality”. Cambridge University Press. Cambridge (2009).

[2] Lucien Hardy. “Probability Theories with Dynamic Causal Structure: A New Framework for Quantum Gravity” (2005). arXiv:gr-qc/0509120.

[3] Giulio Chiribella, Giacomo Mauro D’Ariano, Paolo Perinotti, and Benoît Valiron. “Quantum computations without definite causal structure”. Physical Review A 88, 022318 (2013).

[4] Timoteo Colnaghi, Giacomo Mauro D’Ariano, Stefano Facchini, and Paolo Perinotti. “Quantum computation with programmable connections between gates”. Physics Letters A 376, 2940–2943 (2012).

[5] Magdalena Zych, Fabio Costa, Igor Pikovski, and Časlav Brukner. “Bell’s theorem for temporal order”. Nature Communications 10, 3772 (2019).

[6] Ognyan Oreshkov and Christina Giarmatzi. “Causal and causally separable processes”. New Journal of Physics 18, 093020 (2016).

[7] Mateus Araújo, Cyril Branciard, Fabio Costa, Adrien Feix, Christina Giarmatzi, and Časlav Brukner. “Witnessing causal nonseparability”. New Journal of Physics 17, 102001 (2015).

[8] Tom Purves and Anthony J Short. “Quantum Theory Cannot Violate a Causal Inequality”. Physical Review Letters 127, 110402 (2021).

[9] Julian Wechs, Hippolyte Dourdent, Alastair A Abbott, and Cyril Branciard. “Quantum Circuits with Classical Versus Quantum Control of Causal Order”. PRX Quantum 2, 030335 (2021).

[10] Ognyan Oreshkov, Fabio Costa, and Časlav Brukner. “Quantum correlations with no causal order”. Nature Communications 3, 1092 (2012).

[11] Cyril Branciard, Mateus Araújo, Adrien Feix, Fabio Costa, and Časlav Brukner. “The simplest causal inequalities and their violation”. New Journal of Physics 18, 013008 (2015).

[12] John S Bell. “On the Einstein Podolsky Rosen paradox”. Physics Physique Fizika 1, 195–200 (1964).

[13] John Archibald Wheeler. “World as system self-synthesized by quantum networking”. IBM Journal of Research and Development 32, 4–15 (1988).

[14] Stephen W Hawking. “Chronology protection conjecture”. Physical Review D 46, 603–611 (1992).

[15] Albert Einstein. “Die formale Grundlage der allgemeinen Relativitätstheorie”. In Georg Reimer, editor, Sitzungsberichte der Königliche Preussischen Akademie der Wissenschaften. Volume Zweiter Halbband, pages 1030–1085. Verlag der Königlichen Akademie der Wissenschaften, Berlin (1914).

[16] Kornel Lanczos. “Über eine stationäre Kosmologie im Sinne der Einsteinschen Gravitationstheorie”. Zeitschrift für Physik 21, 73–110 (1924).

[17] Kurt Gödel. “An Example of a New Type of Cosmological Solutions of Einstein’s Field Equations of Gravitation”. Reviews of Modern Physics 21, 447–450 (1949).

[18] Āmin Baumeler and Stefan Wolf. “Perfect signaling among three parties violating predefined causal order”. In 2014 IEEE International Symposium on Information Theory. Pages 526–530. Piscataway (2014). IEEE.

[19] Āmin Baumeler, Adrien Feix, and Stefan Wolf. “Maximal incompatibility of locally classical behavior and global causal order in multiparty scenarios”. Physical Review A 90, 042106 (2014).

[20] Alastair A Abbott, Christina Giarmatzi, Fabio Costa, and Cyril Branciard. “Multiparticle causal correlations: Polytopes and inequalities”. Physical Review A 94, 032131 (2016).

[21] Āmin Baumeler and Stefan Wolf. “The space of logically consistent classical processes without causal order”. New Journal of Physics 18, 013036 (2016).

[22] Alastair A Abbott, Julian Wechs, Fabio Costa, and Cyril Branciard. “Genuinely multipartite noncausality”. Quantum 1, 39 (2017).

[23] Christina Giarmatzi. “Rethinking causality in quantum mechanics”. Springer Theses. Springer, Cham (2019).

[24] Juan Gu, Longsuo Li, and Zhi Yin. “Two Multi-Setting Causal Inequalities and Their Violations”. International Journal of Theoretical Physics 59, 97–107 (2020).

[25] Āmin Baumeler and Stefan Wolf. “Device-independent test of causal order and relations to fixed-points”. New Journal of Physics 18, 035014 (2016).

[26] Mateus Araújo, Adrien Feix, Miguel Navascués, and Časlav Brukner. “A purification postulate for quantum mechanics with indefinite causal order”. Quantum 1, 10 (2017).

[27] Āmin Baumeler. “Causal Loops: Logically Consistent Correlations, Time Travel, and Computation”. PhD thesis. Università della Svizzera italiana. (2017). url: cqi.inf.usi.ch/publications/these_amin.pdf.

[28] Mateus Araújo, Philippe Allard Guérin, and Āmin Baumeler. “Quantum computation with indefinite causal structures”. Physical Review A 96, 052315 (2017).

[29] Āmin Baumeler and Stefan Wolf. “Computational tameness of classical non-causal models”. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 474, 20170698 (2018).

[30] Martin J Renner and Časlav Brukner. “Reassessing the computational advantage of quantum-
controlled ordering of gates”. Physical Review Research 3, 043012 (2021).
[31] David Deutsch. “Quantum mechanics near closed timelike lines”. Physical Review D 44, 3197–3217 (1991).
[32] James B Hartle. “Unitarity and causality in generalized quantum mechanics for nonchronal spacetimes”. Physical Review D 49, 6543–6555 (1994).
[33] George Svetlichny. “Effective Quantum Time Travel” (2009). arXiv:0902.4898.
[34] George Svetlichny. “Time Travel: Deutsch vs. Teleportation”. International Journal of Theoretical Physics 50, 3903–3914 (2011).
[35] Seth Lloyd, Lorenzo Maccone, Raul Garcia-Patron, Vittorio Giovannetti, and Yutaka Shikano. “Quantum mechanics of time travel through post-selected teleportation”. Physical Review D 84, 025007 (2011).
[36] John-Mark A Allen. “Treating time travel quantum mechanically”. Physical Review A 90, 042107 (2014).
[37] D Ahn, C R Myers, Timothy C Ralph, and R B Mann. “Quantum-state cloning in the presence of a closed timelike curve”. Physical Review A 88, 022332 (2013).
[38] Todd A Brun, Mark M Wilde, and Andreas Winter. “Quantum State Cloning Using Deutschian Closed Timelike Curves”. Physical Review Letters 111, 190401 (2013).
[39] Scott Aaronson. “Quantum computing, postselection, and probabilistic polynomial-time”. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 461, 3473–3482 (2005).
[40] Scott Aaronson. “Guest Column: NP-complete problems and physical reality”. SIGACT News 36, 30 (2005).
[41] Scott Aaronson and John Watrous. “Closed timelike curves make quantum and classical computing equivalent”. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 465, 631–647 (2009).
[42] Scott Aaronson, Mohammad Bavarian, and Giulio Gueltrini. “Computability Theory of Closed Timelike Curves” (2016). arXiv:1609.05507.
[43] George Svetlichny. “Distinguishing three-body from two-body nonseparability by a Bell-type inequality”. Physical Review D 35, 3066–3069 (1987).
[44] Daniel Collins, Nicolas Gisin, Sandu Popescu, David Roberts, and Valerio Scarani. “Bell-type inequalities to detect true n-body nonseparability”. Physical Review Letters 88, 170405 (2002).
[45] Michael Seevinck and George Svetlichny. “Bell-type inequalities for partial separability in N-particle systems and quantum mechanical violations”. Physical Review Letters 89, 060401 (2002).
[46] Mohammad Ardehali. “Bell inequalities with a magnitude of violation that grows exponentially with the number of particles”. Physical Review A 46, 5375–5378 (1992).
[47] John F Clauser, Michael A Horne, Abner Shimony, and Richard A Holt. “Proposed Experiment to Test Local Hidden-Variable Theories”. Physical Review Letters 23, 880–884 (1969).
[48] Rodrigo Gallego, Lars Erik Würflinger, Antonio Acín, and Miguel Navascués. “Operational Framework for Nonlocality”. Physical Review Letters 109, 070401 (2012).
[49] Mateus Araújo and Adrien Feix. private communication (2014). The process was communicated to Baumeler before it was found by inspecting the extremal points of the non-causal polytope characterized in Baumeler and Wolf [21] (see also Ref. [56] in the latter article).
[50] Amin Baunmeler, Fabio Costa, Timothy C Ralph, Stefan Wolf, and Magdalena Zych. “Reversible time travel with freedom of choice”. Classical and Quantum Gravity 36, 224002 (2019).
[51] Amin Baunmeler and Eleftherios Tselentis. “Equivalence of grandfather and information antimony under intervention”. In Benoît Valiron, Shane Mansfield, Pablo Arrighi, and Prakash Panangaden, editors, Proceedings 17th International Conference on Quantum Physics and Logic. Volume 340, pages 1–12. Electronic Proceedings in Theoretical Computer Science (2021).
[52] Man-Duen Choi. “Completely positive linear maps on complex matrices”. Linear Algebra and its Applications 10, 285–290 (1975).
[53] Andrzej Jamiołkowski. “Linear transformations which preserve trace and positive semidefiniteness of operators”. Reports on Mathematical Physics 3, 275–278 (1972).
[54] Howard Barnum, Salman Beigi, Sergio Boixo, Matthew B Elliott, and Stephanie Wehner. “Local Quantum Measurement and No-Signaling Imply Quantum Correlations”. Physical Review Letters 104, 140401 (2010).
[55] Antonio Acín, Remigiusz Augusiak, Daniel Cavalcanti, Christopher Hadley, Jaroslaw K Korbicz, Maciej Lewenstein, Lluis Masanes, and Marco Piani. “Unified Framework for Correlations in Terms of Local Quantum Observables”. Physical Review Letters 104, 140404 (2010).
[56] Jonathan Barrett, Robin Lorenz, and Ognyan Oreshkov. “Cyclic quantum causal models”. Nature Communications 12, 885 (2021).
[57] Kip S Thorne. “Do the Laws of Physics Permit Closed Timelike Curves?”. Annals of the New York Academy of Sciences 631, 182–193 (1991).
[58] Kip S Thorne. “Black Holes & Time Warps: Einstein’s Outrageous Legacy”. W.W. Norton
[59] George Svetlichny. “Nonlinear Quantum Mechanics at the Planck Scale”. International Journal of Theoretical Physics 44, 2051–2058 (2005).

[60] John Archibald Wheeler and Kenneth Ford. “Geons, Black Holes, and Quantum Foam: A Life in Physics”. W.W. Norton & Company. New York (1998).

[61] Germain Tobar and Fabio Costa. “Reversible dynamics with closed time-like curves and freedom of choice”. Classical and Quantum Gravity 37, 205011 (2020).

[62] N David Mermin. “Extreme quantum entanglement in a superposition of macroscopically distinct states”. Physical Review Letters 65, 1838–1840 (1990).

[63] Andris Ambainis, Dmitry Kravchenko, Nikolajs Nahimovs, and Alexander Rivosh. “Nonlocal Quantum XOR Games for Large Number of Players”. In Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics). Volume 6108 LNCS, pages 72–83. Springer, Berlin, Heidelberg (2010).

[64] Andris Ambainis, Dmitry Kravchenko, Nikolay Nahimov, Alexander Rivosh, and Madars Virza. “On symmetric nonlocal games”. Theoretical Computer Science 494, 36–48 (2013).

[65] Boris S Cirel’son. “Quantum generalizations of Bell’s inequality”. Letters in Mathematical Physics 4, 93–100 (1980).

[66] Jean-Daniel Bancal, Jonathan Barrett, Nicolas Gisin, and Stefano Pironio. “Definitions of multipartite nonlocality”. Physical Review A 88, 014102 (2013).

[67] N David Mermin. “Nonlinear Quantum Mechanics at the Planck Scale”. International Journal of Theoretical Physics 44, 2051–2058 (2005).

[68] Allen Stairs. “Quantum Logic, Realism, and Value Definiteness”. Philosophy of Science 50, 578–602 (1983).

[69] Peter Heywood and Michael L G Redhead. “Nonlocality and the Kochen-Specker paradox”. Foundations of Physics 13, 481–499 (1983).

[70] Renato Renner and Stefan Wolf. “Quantum pseudo-telepathy and the Kochen-Specker theorem”. In International Symposium on Information Theory, 2004. ISIT 2004. Proceedings. Pages 322–322. IEEE (2004).

[71] Renato Renner and Stefan Wolf. “Ernst Specker and the Hidden Variables”. Elemente der Mathematik 67, 122–133 (2012).

[72] Gilles Brassard, Anne Broadbent, and Alain Tapp. “Quantum Pseudo-Telepathy”. Foundations of Physics 35, 1877–1907 (2005).

[73] Adán Cabello. “Reversible dynamics with closed time-like curves and freedom of choice”. Classical and Quantum Gravity 37, 205011 (2020).

[74] Costantino Budroni, Adán Cabello, Otfried Gühne, Matthias Kleinmann, and Jan-Åke Larsson. “Quantum Contextuality” (2021). arXiv:2102.13036.

[75] Ognyan Oreshkov. “Time-delocalized quantum subsystems and operations: on the existence of processes with indefinite causal structure in quantum mechanics”. Quantum 3, 206 (2019).

[76] Julian Wechs, Cyril Branciard, and Ognyan Oreshkov. “Existence of processes violating causal inequalities on time-delocalised subsystems” (2022). arXiv:2201.11832.

[77] Edward Fredkin and Tommaso Toffoli. “Conservative logic”. International Journal of Theoretical Physics 21, 219–253 (1982).

[78] Michael S Morris, Kip S Thorne, and Ulvi Yurtsever. “Wormholes, Time Machines, and the Weak Energy Condition”. Physical Review Letters 61, 1446–1449 (1988).

[79] Igor Dmitriyevich Novikov. “An analysis of the operation of a time machine”. Journal of Experimental and Theoretical Physics 68, 439 (1989). url: http://www.jetp.ras.ru/cgi-bin/e/index/e/68/3/p439?a=list.

[80] Valery P Frolov and Igor D Novikov. “Physical effects in wormholes and time machines”. Physical Review D 42, 1057–1065 (1990).

[81] Flaminia Giacomini, Esteban Castro-Ruiz, and Časlav Brukner. “Indefinite causal structures for continuous-variable systems”. New Journal of Physics 18, 113026 (2016).
We define the function $\bar{\omega}^n : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^n$ as $\bar{\omega}^n(x) = (\tilde{\omega}^n_0(x), \tilde{\omega}^n_1(x), \ldots, \tilde{\omega}^n_{n-1}(x))$, where for all $k \in \mathbb{Z}_n$ we have

$$\bar{\omega}^n_k : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$$

$$x \mapsto \bigoplus_{i,j \in \mathbb{Z}_n \setminus \{k\} \quad i < j} x_i x_j \bigoplus_{i \in \mathbb{Z}_n} \tilde{\gamma}_{k,i} x_i \, , \quad (76)$$

with

$$\tilde{\gamma}_{k,i} := \begin{cases} 1 & (i < k \land i \equiv_2 k) \lor (k < i \land i \not\equiv_2 k) \\ 0 & \text{otherwise.} \end{cases} \quad (77)$$

In comparison to $\omega^n$, this function uses the alternative single terms (cf. Equation (3)).

**Lemma 8.** The function $\bar{\omega}^n$ is equivalent to $\omega^n$ under a relabeling of the parties.

**Proof.** Reversing the order of parties in $\omega^n$ gives $\bar{\omega}^n$, i.e., if party $k$ in the function $\omega^n$ becomes party $n - k - 1$, then we obtain $\bar{\omega}^n$. We denote with $\omega^{n,(c)}_k$ the function of the $k$-th party where the order of parties in $\omega^n_k$ is reversed. More precisely, the input to and the output of the $k$-th party is $x_k^c = x_{n-k-1}$ and $\omega^{n,(c)}_{n-k-1}(x_k^c)$ where $x_k^c$ represents the reversed input string. This equivalence is shown in the following calculation, where we use $i' := n - i - 1$ and $j' := n - j - 1$:

$$\omega^{n,(c)}_k(x) = \omega^{n-1}_{n-k-1}(x_k^c)$$

$$= \bigoplus_{i,j \in \mathbb{Z}_n \setminus \{n-k-1\} \quad i < j} x_{n-i-1} x_{n-i-1} \bigoplus_{i \in \mathbb{Z}_n} \tilde{\gamma}_{n-k-1,i} x_{n-i-1}$$

$$= \bigoplus_{i',j' \in \mathbb{Z}_n \setminus \{k\}} x_{i'} x_{j'} \bigoplus_{i' \in \mathbb{Z}_n} \tilde{\gamma}_{k,i'} x_{i'} \quad (80)$$

$$= \bar{\omega}^n_k(x) \, . \quad (81)$$

\[ \square \]

**B Reduced function of $\omega^n$ where party $n - 1$ implements the identity**

Let the $f_{n-1}$ be the identity function. The reduced function in this case is

$$\omega^{n,f_{n-1}}_k(x) = \omega^n_k(x_{\setminus n-1}, \omega^n_{n-1}(x_{\setminus n-1}, 0)) \, . \quad (82)$$

The input to party $n - 1$, i.e., $\omega^{n}_{n-1}(x_{\setminus n-1}, 0)$, is

$$\bigoplus_{i \in \mathbb{Z}_n - \{k\}} x_i \, , \quad (83)$$

This expression is now plugged into Equation (82), i.e., the variable $x_{n-1}$ in Equation (66) takes the above value. We evaluate the two expressions in Equation (66) involving $x_{n-1}$ separately. The first term is

$$\bigoplus_{i \in \mathbb{Z}_n - \{k\}} x_i \omega^n_{n-1}(x_{\setminus n-1}, 0) = \bigoplus_{i,j \in \mathbb{Z}_n - \{k\} \quad i < j} x_i x_j x_k \bigoplus_{i \in \mathbb{Z}_n - \{k\}} x_i x_k \bigoplus_{i,j \in \mathbb{Z}_n - \{k\} \quad i < j} x_i x_j \bigoplus_{i \in \mathbb{Z}_n - \{k\}} x_i \bigoplus_{i \in \mathbb{Z}_n - \{k\} \quad i \equiv_2 n} x_i \bigoplus_{i \in \mathbb{Z}_n - \{k\}} x_i x_k \, , \quad (84)$$

and the second is

$$[k \not\equiv_2 n] \omega^{n}_{n-1}(x_{\setminus n-1}, 0) = [k \not\equiv_2 n] \left( \bigoplus_{i,j \in \mathbb{Z}_n - \{k\} \quad i < j} x_i x_j \bigoplus_{i \in \mathbb{Z}_n - \{k\}} x_i x_k \bigoplus_{i \in \mathbb{Z}_n - \{k\} \quad i \equiv_2 n} x_i \right) \, . \quad (85)$$
Thus, for all $k \in \mathbb{Z}_{n-1}$, Equation (82) is the parity of $\omega_{k}^{n-1}(x_{\cap n-1})$ and these last two expressions:

$$
\omega_{k}^{n,f_{n-1}}(x) = \bigoplus_{i,j \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i}x_{j}x_{\ell} \oplus \bigoplus_{i,j \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i}x_{j} \oplus [k \equiv 2 \ n] \oplus x_{i}x_{j} \\
\bigoplus_{i \in \mathbb{Z}_{n-1}} \gamma_{k,i}x_{i} \oplus [k \equiv 2 \ n] \oplus x_{i}.
$$

Therefore, the function $\omega_{k}^{n,f_{n-1}}$, where $f_{n-1}$ is the identity function for party $n-1$, equals the function $\Gamma_{0,0,k}^{n-1}$ defined in Appendix C. That same appendix also shows that $\Gamma_{0,0}^{n-1}$ is a process function.

C Family of process functions

For all $\alpha, \beta \in \{0, 1\}$, we define the function $\Gamma_{\alpha, \beta}^{n} : \mathbb{Z}_{2}^{n} \rightarrow \mathbb{Z}_{2}$ as $\Gamma_{\alpha, \beta}^{n}(x) = \Gamma_{\alpha, \beta, 0}^{n}(x), \Gamma_{\alpha, \beta, 1}^{n}(x), \ldots, \Gamma_{\alpha, \beta, n-1}^{n}(x)$, where for all $k \in \mathbb{Z}_{n}$

$$
\Gamma_{\alpha, \beta, k}^{n} : \mathbb{Z}_{2}^{n} \rightarrow \mathbb{Z}_{2} \\
\begin{array}{c}
\centering
\begin{array}{c}
\Gamma_{\alpha, \beta}^{n}(x) = \Gamma_{\alpha, \beta, k}^{n}(x) \oplus \bigoplus_{i,j \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i}x_{j}x_{\ell} \oplus \bigoplus_{i,j \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i}x_{j} \oplus [\alpha \neq 2 \ k + n] \oplus x_{i}x_{j} \\
\bigoplus_{i \in \mathbb{Z}_{n-1}} x_{i}x_{j} \oplus \bigoplus_{i \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i} \oplus x_{1}.
\end{array}
\end{array}
\end{array}
$$

Theorem 5. The function $\Gamma_{\alpha, \beta}^{n}$ for all $\alpha, \beta \in \{0, 1\}$ and for all $n \geq 1$ is a process function.

Proof. We explicitly prove this statement for $\alpha = \beta = 0$; the other cases are analogous. For better presentation, define $\Gamma_{k}^{n} := \Gamma_{0,0,k}^{n}$ and $\Gamma^{n} := \Gamma_{0,0}^{n}$. The proof idea is the same as for Theorem 3. First, we assume that $\Gamma^{n}$ is not a process function. Since this function is element-wise constant, it follows that at least one of the reduced functions $\Gamma_{f_{n-1}}^{n}$, where $f_{n-1}$ is the constant-zero, constant-one, or identity function, is not a process function (see Lemma 6 and Lemma 7). First, we express $\Gamma_{k}^{n}$ in terms of $\Gamma_{k}^{n-1}$:

$$
\Gamma_{k}^{n}(x) = \Gamma_{k}^{n-1}(x_{\cap n-1}) \oplus \bigoplus_{i,j \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i}x_{j}x_{\ell} \oplus \bigoplus_{i,j \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i}x_{j} \oplus [\alpha \neq 2 \ k + n] \oplus x_{i}x_{j} \\
\bigoplus_{i,j \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i}x_{j} \oplus \bigoplus_{i \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i} \oplus x_{1}.
$$

In the case where $f_{n-1}$ is the constant-zero function, we obtain for the reduced function

$$
\Gamma_{k}^{n,f_{n-1}}(x) = \Gamma_{k}^{n}(x, 0) = \Gamma_{k}^{n-1}(x) \oplus \bigoplus_{i,j \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i}x_{j} \oplus \bigoplus_{i \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i} = \Gamma_{\alpha, \beta, 0}^{n-1}(x).
$$

In the case where $f_{n-1}$ is the constant-one function, we obtain for the reduced function

$$
\Gamma_{k}^{n,f_{n-1}}(x) = \Gamma_{k}^{n}(x, 1) = \Gamma_{k}^{n-1}(x) \oplus \bigoplus_{i \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i} \oplus \bigoplus_{i \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i} \oplus [\alpha \neq 2 \ n] \oplus x_{i} = \Gamma_{k}^{n-1}(x).
$$

Finally, let $f_{n-1}$ be the identity function. In this case $\Gamma_{n-1}^{n,f_{n-1}}$ is

$$
\Gamma_{n-1}^{n,f_{n-1}}(x_{\cap n-1}) = \Gamma_{n-1}^{n}(x_{\cap n-1}, \Gamma_{n-1}^{n}(x)).
$$

We first express $\Gamma_{n-1}^{n}(x)$:

$$
\Gamma_{n-1}^{n}(x) = \bigoplus_{i \in \mathbb{Z}_{n-1}} x_{i}x_{j}x_{\ell} \oplus \bigoplus_{i \in \mathbb{Z}_{n-1} \setminus \{k\}} x_{i}x_{j} \oplus \bigoplus_{i \in \mathbb{Z}_{n-1}} x_{i}.
$$
Now, we evaluate the terms in \( \Gamma^n(x) \) that involve \( x_{n-1} \) (see Equation (88))—the term \( x_{n-1} \) is replaced by \( \Gamma^n_{n-1}(x) \). The first term involving \( x_{n-1} \) becomes

\[
\bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j \Gamma^n_{n-1}(x).
\] (93)

Let us perform this calculation step by step. By taking the product with the first term in \( \Gamma^n_{n-1}(x) \), we obtain

\[
\bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j \left( \bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j x_{\ell} \right) = \bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j x_{\ell} \bigoplus_k \left( \bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j \right).
\] (94)

Then again, the product with the second term in \( \Gamma^n_{n-1}(x) \), gives

\[
\bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j \left( \bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j x_{\ell} \right) \bigoplus_{i,j,m \leq Z_{n-1}\backslash \{k\}, i j < \ell < m, i + j + \ell = m} x_{i,j,\ell} x_{m} \bigoplus_k \left( \bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j \right).
\] (95)

Finally, the product with the third term in \( \Gamma^n_{n-1}(x) \) is

\[
\bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j \left( \bigoplus_{i \in Z_{n-1}} x_i \right) = \bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j x_{\ell}.
\] (96)

By taking the sum modulo two of the above three expressions, we obtain that Equation (93) equals

\[
\bigoplus_{i,j,\ell,m \leq Z_{n-1}\backslash \{k\}, i < j < \ell < m, i + j + \ell + m = m} x_{i,j,\ell} x_{m} \bigoplus_k \left( \bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j \right).
\] (97)

By similar calculations, we replace the variable \( x_{n-1} \) appearing in the second and third term of Equation (88) with \( \Gamma^n_{n-1}(x) \), and obtain the expressions

\[
\bigoplus_{i \in Z_{n-1}\backslash \{k\}, i = 2n} x_i \Gamma^n_{n-1}(x) = \bigoplus_{i,j,\ell,m \leq Z_{n-1}\backslash \{k\}, i < j < \ell < m, i + j + \ell + m = m} x_{i,j,\ell} x_{m} \bigoplus_k \left( \bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j \right) \bigoplus_{i \in Z_{n-1}\backslash \{k\}, i = 2n} x_i,
\] (98)

and

\[
\bigoplus_{i \in Z_{n-1}\backslash \{k\}, i = 2n} x_i \Gamma^n_{n-1}(x) = \bigoplus_{i,j,\ell,m \leq Z_{n-1}\backslash \{k\}, i < j < \ell < m, i + j + \ell + m = m} x_{i,j,\ell} x_{m} \bigoplus_k \left( \bigoplus_{i,j \leq Z_{n-1}\backslash \{k\}, i < j} x_i x_j \right) \bigoplus_{i \in Z_{n-1}\backslash \{k\}, i = 2n} x_i,
\] (99)
Table 1: The reduced function of $\Gamma^n_{\alpha,\beta}$ for all functions $f_{n-1}$ of party $n-1$.

| $\Gamma^n_{0,0,k}$ | $\Gamma^{n-1}_{1,0,k}$ | $\Gamma^{n-1}_{0,0,k}$ | $\Gamma^{n-1}_{0,1,k}$ | $\Gamma^{n-1}_{1,0,k}$ |
|---------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| $\Gamma^n_{0,1,k}$  | $\Gamma^{n-1}_{1,1,k}$ | $\Gamma^{n-1}_{0,1,k} \oplus 1$ | $\Gamma^{n-1}_{1,1,k}$ | $\Gamma^{n-1}_{0,1,k} \oplus 1$ |
| $\Gamma^n_{1,0,k}$  | $\Gamma^{n-1}_{0,0,k}$ | $\Gamma^{n-1}_{1,0,k} \oplus [k \neq n]$ | $\Gamma^{n-1}_{0,0,k}$ | $\Gamma^{n-1}_{1,0,k} \oplus [k \neq n]$ |
| $\Gamma^n_{1,1,k}$  | $\Gamma^{n-1}_{0,1,k}$ | $\Gamma^{n-1}_{1,0,k} \oplus [k = n]$ | $\Gamma^{n-1}_{0,1,k}$ | $\Gamma^{n-1}_{1,0,k} \oplus [k = n]$ |

Now, we have everything at hand to evaluate Equation (91): The resulting function is $\Gamma^n_{k}$.

Hence, if $f_{n-1}$ is the constant-zero function, we retrieve $\Gamma^n_{1,0,k}$, if $f_{n-1}$ is the constant-one or the identity function, we retrieve $\Gamma^n_{k}$. For the other values of $\alpha$ and $\beta$ the proof is similar. The reduced functions are summarized in Table 1, we also include—simply for completeness—the case where $f_{n-1}$ is the bit-flip function.

What remains to show is the base case. For $n = 3$, we get that $\Gamma^3_{1,\beta}$ is the process function described in Equation (4), and thus is yet another generalization of that three-party function. Then again, $\Gamma^3_{0,\beta}$ is causal and therefore a process function as well.