Existence of multiple closed CMC hypersurfaces with small mean curvature

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Abstract

Let $(M^{n+1}, g)$ be a closed Riemannian manifold, $n + 1 \geq 3$. We will prove that for all $m \in \mathbb{N}$, there exists $c^*(m) > 0$, which depends on $g$, such that if $0 < c < c^*(m)$, $(M, g)$ contains at least $m$ many closed $c$-CMC hypersurfaces with optimal regularity. More quantitatively, there exists a constant $\gamma_0$, depending on $g$, such that for all $c > 0$, there exist at least $\gamma_0 c^{-n+1}$ many closed $c$-CMC hypersurfaces (with optimal regularity) in $(M, g)$. This extends the theorem of Zhou and Zhu [ZZ19], where they proved the existence of at least one closed $c$-CMC hypersurface in $(M, g)$.

1 Introduction

Constant mean curvature (CMC) hypersurfaces are the critical points of the area functional with respect to the variations which preserve the enclosed volume. The hypersurfaces with constant mean curvature zero are called minimal hypersurfaces; they are the critical points of the area functional.

There is a well-developed existence theory for the closed minimal hypersurfaces in closed Riemannian manifolds. By the combined work of Almgren [Alm65], Pitts [Pit81] and Schoen-Simon [SS81], every closed Riemannian manifold $(M^{n+1}, g)$, $n + 1 \geq 3$, contains a closed minimal hypersurface, which is smooth and embedded outside a singular set of Hausdorff dimension $\leq n - 7$.

Since the cohomology ring of $Z_n(M^{n+1}, \mathbb{Z}_2)$ (with coefficients in $\mathbb{Z}_2$) is isomorphic to the polynomial ring $\mathbb{Z}_2[\xi]$, from the finite dimensional Morse theory one expects that $(M, g)$ contains infinitely many closed minimal hypersurfaces. This was conjectured by Yau [Yau82]. Yau’s conjecture has been completely resolved when the ambient dimension $3 \leq n + 1 \leq 7$. By the works of Marques-Neves [MN17] and Song [Son18], in every closed Riemannian manifold $(M^{n+1}, g)$, $3 \leq n + 1 \leq 7$, there exist infinitely many closed minimal hypersurfaces. In higher dimensions, Li [Li19] has proved that every closed manifold $M^{n+1}$, $n + 1 \geq 3$, equipped with a generic metric, contains infinitely many closed minimal hypersurfaces with optimal regularity.

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When $3 \leq n + 1 \leq 7$, stronger results have been obtained for generic set of metrics. In \cite{IMN18}, Irie, Marques and Neves proved that for a generic metric, the union of all closed minimal hypersurfaces is dense in $M$. This theorem was later quantified in \cite{MNS19} by Marques, Neves and Song, where they proved the existence of an equidistributed sequence of closed minimal hypersurfaces in $(M, g)$ (for a generic metric $g$). The density and the equidistribution theorems were proved in the Allen-Cahn setting by Gaspar and Guaraco \cite{GG19}. Using the multiplicity one theorem, proved by Zhou \cite{Zho19}, Marques and Neves \cite{MN18} proved that for a generic (bumpy) metric $g$, there exists a sequence of closed, two sided, minimal hypersurfaces $\{\Sigma_k\}_{k=1}^{\infty}$ in $(M, g)$ such that $\text{Ind}(\Sigma_k) = k$ and $H^{n}(\Sigma_k) \sim k^{\frac{1}{n+1}}$. This theorem was previously proved for $n + 1 = 3$, in the Allen-Cahn setting, by Chodosh and Mantoulidis \cite{CM20}.

In \cite{ZZ19}, Zhou and Zhu developed a min-max theory to construct closed CMC hypersurface in an arbitrary closed Riemannian manifold. More precisely, they proved that if $(M^{n+1}, g)$ is a closed Riemannian manifold, $3 \leq n + 1 \leq 7$, given any $c \in \mathbb{R}^+$, there exists a closed, almost embedded hypersurface in $(M, g)$ with constant mean curvature $c$. Here almost embedded means that all the self-intersections are one sided, tangential intersections. (For a precise definition, see Definition 3.1.) Later they generalized this theorem in \cite{ZZ18} and proved that for a generic function $h \in C^\infty(M)$, there exists a closed, almost embedded hypersurface in $(M, g)$ with prescribed mean curvature (PMC) $h$. The min-max construction of the PMC hypersurfaces was used by Zhou \cite{Zho19} to prove the multiplicity one conjecture of Marques and Neves \cite{MN16}.

Given the above mentioned results regarding the abundance of closed minimal hypersurfaces in closed Riemannian manifolds, it is natural to ask, for a given $c > 0$ and a closed Riemannian manifold $(M, g)$, whether there exist multiple closed $c$-CMC hypersurfaces in $(M, g)$. We will show that for any closed Riemannian manifold $(M, g)$, the number of closed $c$-CMC hypersurfaces in $(M, g)$ tends to infinity as $c \to 0^+$. This will be a consequence of the following theorem. (We refer to Section 2 and 3 for the relevant notations and definitions.)

**Theorem 1.1.** Let $(M^{n+1}, g)$ be a closed Riemannian manifold, $n + 1 \geq 3$. Suppose $k \in \mathbb{N}$ such that $\omega_k < \omega_{k+1}$, $c \in \mathbb{R}^+$ such that $c\text{Vol}(g) < \omega_{k+1} - \omega_k$ and $\eta \in \mathbb{R}^+$ is arbitrary. Then there exists $\Omega \in \mathcal{C}(M)$ such that $\partial\Omega$ is a closed $c$-CMC (with respect to the inward unit normal) hypersurface with optimal regularity and $\omega_k < A^c(\Omega) < \omega_k + W_0 + \eta$.

As a corollary of Theorem 1.1, we obtain the following theorem.

**Theorem 1.2.** Let $(M^{n+1}, g)$ be a closed Riemannian manifold, $n + 1 \geq 3$. For all $m \in \mathbb{N}$, there exists a constant $c^*(m) > 0$, which depends on the metric $g$, such that if $0 < c < c^*(m)$, $(M, g)$ contains at least $m$ many closed $c$-CMC hypersurfaces with optimal regularity. More quantitatively, there exists a constant $\gamma_0$, depending on $g$, such that for all $c > 0$, there exist at least $\gamma_0 c^{-\frac{1}{n+1}}$ many closed $c$-CMC hypersurfaces (with optimal regularity) in $(M, g)$.
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At the end of the proof of Theorem 1.2, we will obtain an explicit lower bound for \( \gamma_0 \).

On the other hand, by the work of Pacard and Xu [PX09], if \( \Lambda_M \) is the Lusternik-Shnirelman category of \( M \) (i.e. the minimum number of critical points of a smooth function \( f : M \to \mathbb{R} \)), for \( c \) sufficiently large, there exist at least \( \Lambda_M \) many closed, embedded \( c \)-CMC hypersurfaces in \( (M, g) \).

The proof of Theorem 1.1 relies on the works by Zhou-Zhu [ZZ19] and Zhou [Zho19]. We will also use the regularity theory of stable CMC hypersurfaces, developed by Bellettini-Wickramasekera [BW18,BW19] and Bellettini-Chodosh-Wickramasekera [BCW19], to extend the theorem of Zhou and Zhu [ZZ19] in higher dimensions. Finally, Theorem 1.2 is deduced from Theorem 1.1 using the growth estimate of the volume spectrum, proved by Gromov [Gro03], Guth [Gut09], Marques-Neves [MN17] and Liokumovich-Marques-Neves [LMN18].

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2 Notation and Preliminaries

2.1 Notation

Here we summarize the notation which will be frequently used later.

- \( \mathcal{C}(M) \), the space of Caccioppoli sets in \( M \).
- \( \mathcal{Z}_n(M; F; \mathbb{Z}_2) \), \( \mathcal{Z}_n(M; F; \mathbb{Z}_2) \), the space of mod 2 flat hypercycles in \( M \), which are boundaries, equipped with the \( F \) norm and the \( F \)-metric.
- \( \omega_p = \omega_p(M, g) \), the \( p \)-width of \( (M, g) \).
- \( W_0 = W_0(M, g) \), the one parameter Almgren-Pitts width of \( (M, g) \).
- \( \mathcal{H}^s \), the Hausdorff measure of dimension \( s \).
- \( B(p, r) \), the open ball of radius \( r \), centered at \( p \).
- \( A(p, r, R) \), the open annulus, centered at \( p \), with radii \( r < R \).
- \( |\Sigma| \), the \( n \)-varifold associated to an \( n \)-rectifiable set \( \Sigma \).
- \( \|V\| \), the Radon measure associated to a varifold \( V \).
- \( CA \), the cone over a topological space \( A \).
- \( SA \), the suspension of a topological space \( A \).
- \( \Delta^k, [v_0, \ldots, v_k] \), the standard \( k \)-simplex with vertices \( v_0, \ldots, v_k \).
- \( C_i(A, \mathbb{Z}_2), C^i(A, \mathbb{Z}_2) \), the abelian group of \( i \) dimensional singular chains and cochains in \( A \) with \( \mathbb{Z}_2 \) coefficient.
- \( H_i(A, \mathbb{Z}_2), H^i(A, \mathbb{Z}_2) \), the \( i \)-th singular homology and cohomology group of \( A \) with \( \mathbb{Z}_2 \) coefficient.
2.2 Caccioppoli sets and the $A^c$ functional

In this subsection, we will briefly recall the notion of the Caccioppoli set; further details can be found in Simon’s book [Sim83]. $E \subset (M, g)$ is called a Caccioppoli set if $E$ is $\mathcal{H}^{n+1}$-measurable and $D\chi_E$, the distributional derivative of the characteristic function $\chi_E$, is a Radon measure. This is equivalent to

$$\sup \left\{ \int_E \text{div} \omega \, d\mathcal{H}^{n+1} : \omega \in \mathfrak{X}^1(M) \text{ and } \|\omega\|_\infty \leq 1 \right\} < \infty,$$

where $\mathfrak{X}^1(M)$ is the space of $C^1$ vector-fields on $M$. The space of Caccioppoli sets in $M$ will be denoted by $\mathcal{C}(M)$. If $E \in \mathcal{C}(M)$, there exists an $n$-rectifiable set, denoted by $\partial E$, such that the total variation measure $|D\chi_E| = \mathcal{H}^n \mathbf{1}_{\partial E}$. Further, there exists a $|D\chi_E|$-measurable vector-field $\nu_{\partial E}$ such that $\|\nu_{\partial E}\| = 1$ $|D\chi_E|$-a.e. and

$$\int_E \text{div} \omega \, d\mathcal{H}^{n+1} = \int_{\partial E} g(\omega, \nu_{\partial E}),$$

for all $\omega \in \mathfrak{X}^1(M)$. $\nu_{\partial E}$ is called the outward unit normal to $\partial E$. For $c > 0$, the $A^c$ functional on $\mathcal{C}(M)$ is defined by

$$A^c(\Omega) = \mathcal{H}^n(\partial\Omega) - c\mathcal{H}^{n+1}(\Omega).$$

Let $X$ be a vector-field on $M$ and $\varphi_t$ be the flow of $X$. The first variation of $A^c$ along $X$ is given by

$$\delta A^c|_{\Omega}^t(X) = \left. \frac{d}{dt} \right|_{t=0} A^c(\varphi_t(\Omega)) = \int_{\partial\Omega} \text{div}_{\partial\Omega} X \, d\mathcal{H}^n - c \int_{\partial\Omega} g(X, \nu_{\partial\Omega}) \, d\mathcal{H}^n.$$

Suppose $\Omega$ is an open set such that $\partial\Omega$ is a smooth hypersurface with mean curvature vector $H$. Then $\text{div}_{\partial\Omega} X = -g(X, H)$. Therefore, $\Omega$ is a critical point of the $A^c$ functional if and only if $H = -c\nu_{\partial\Omega}$, i.e. $\partial\Omega$ has constant mean curvature $c$ with respect to the inward unit normal $-\nu_{\partial\Omega}$. In this case, the second variation of $A^c$ along $X$ (which is assumed to be normal along $\partial\Omega$) is given by

$$\delta^2 A^c|_{\Omega}^t(X, X) = \left. \frac{d^2}{dt^2} \right|_{t=0} A^c(\varphi_t(\Omega))$$

$$= \int_{\partial\Omega} \left( |\nabla^\perp X|^2 - \text{Ric}(X, X) - |A_{\partial\Omega}|^2 |X|^2 \right) d\mathcal{H}^n.$$

Here $A_{\partial\Omega}$ stands for the second fundamental form of $\partial\Omega$.

2.3 The space of hypercycles and the volume spectrum

Let us introduce the following notation. $I_l(M^{n+1}; \mathbb{Z}_2)$ is the space of $l$-dimensional mod 2 flat chains in $M$; we only need to consider $l = n, n + 1$. $Z_n(M; \mathbb{Z}_2)$ denotes the space of flat chains $T \in I_n(M; \mathbb{Z}_2)$ such that $T = \partial U$ for some $U \in I_{n+1}(M; \mathbb{Z}_2)$. For $T \in I_n(M; \mathbb{Z}_2)$, $|T|$
stands for the varifold associated to $T$ and $\|T\|$ is the Radon measure associated to $|T|$. $\mathcal{F}$ and $\mathbf{M}$ denote the flat norm and the mass norm on $\mathbf{I}_l(M;\mathbb{Z}_2)$. When $l = n + 1$, these two norms coincide. The $\mathbf{F}$-metric on the space of currents is defined as follows.

$$
\mathbf{F}(U_1, U_2) = \mathcal{F}(U_1 - U_2) + \mathbf{F}(|\partial U_1|, |\partial U_2|) \text{ if } U_1, U_2 \in \mathbf{I}_{n+1}(M;\mathbb{Z}_2);
$$
$$
\mathbf{F}(T_1, T_2) = \mathcal{F}(T_1 - T_2) + \mathbf{F}(|T_1|, |T_2|) \text{ if } T_1, T_2 \in \mathcal{Z}_n(M;\mathbb{Z}_2).
$$

$\mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)$ and $\mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)$ will stand for the space $\mathcal{Z}_n(M;\mathbb{Z}_2)$ equipped with the $\mathcal{F}$ norm and the $\mathbf{F}$-metric respectively. We will identify $\mathcal{C}(M)$ with $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$, i.e. $E \in \mathcal{C}(M)$ will be identified with $[E]$, the current associated with $E$. Similarly, $\partial E$ (with $E \in \mathcal{C}(M)$) will be identified with $[\partial E] = \partial[E]$.

By the constancy theorem, if $\Omega_1, \Omega_2 \in \mathcal{C}(M)$ such that $\partial \Omega_1 = \partial \Omega_2$, then either $\Omega_1 = \Omega_2$ or $\Omega_1 = M - \Omega_2$. In [MN18], Marques and Neves proved that the space $(\mathcal{C}(M), \mathcal{F})$ is contractible and the boundary map

$$
\partial : (\mathcal{C}(M), \mathcal{F}) \to \mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)
$$

is a 2-sheeted covering map. By the definition of the $\mathbf{F}$-metric on $\mathcal{C}(M)$ and $\mathcal{Z}_n(M;\mathbb{Z}_2)$, this implies

$$
\partial : (\mathcal{C}(M), \mathbf{F}) \to \mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)
$$

is also a 2-sheeted covering map. Furthermore, $\pi_1(\mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)) = \mathbb{Z}_2$ and for $i \geq 2$, $\pi_i(\mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)) = 0$. It was also proved in [MN18] that $\mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)$ is weakly homotopy equivalent to $\mathbb{R}P^{\infty}$. Let $\overline{\lambda}$ denote the unique nonzero element of $H^1(\mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2), \mathbb{Z}_2)$ (i.e. $\mathbb{Z}_2$); then the cohomology ring $H^*(\mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2[\overline{\lambda}]$.

$X$ is called a cubical complex if $X$ is a subcomplex of $[0, 1]^l$ for some $l \in \mathbb{N}$. The cells of $[0, 1]^l$ are of the form $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_l$, where each $\alpha_i \in \{[0], [1], [0, 1]\}$. By [BP02, Chapter 4], every simplicial complex is homeomorphic to a cubical complex and vice-versa.

Let $X$ be a cubical complex. A continuous map $\Phi : X \to \mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)$ is called a $k$-sweepout if

$$
(\varphi^*\overline{\lambda})^k \not\in H^k(X, \mathbb{Z}_2).
$$

$\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is said to have the property no concentration of mass if

$$
\lim_{r \to 0} \sup \{ \|\Phi(x)\| (B(p,r)) : x \in X, p \in M \} = 0.
$$

A continuous map $\Phi : X \to \mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)$ has no concentration of mass [MN16, Proof of Theorem 3.8]. The set of all the $k$-sweepouts with no concentration of mass is denoted by $\mathcal{P}_k$. The $k$-width of $(M, g)$ is defined by

$$
\omega_k(M, g) = \inf_{\Phi \in \mathcal{P}_k} \{ \mathbf{M}(\Phi(x)) : x \in \mathrm{dmn}(\Phi) \},
$$

5
where $\text{dnn}(\Phi)$ is the domain of $\Phi$. The volume spectrum $\{\omega_k\}_{k=1}^{\infty}$ satisfies the following inequality, proved by Gromov [Gro03], Guth [Gut09], Marques-Neves [MN17] and Liokumovich-Marques-Neves [LMN18].

$$\omega_k \geq K_0 k^{\frac{1}{n+1}} \quad \forall \ k \in \mathbb{N},$$

where $K_0 > 0$ is a constant, which depends on the metric $g$.

Let $\mathcal{S}$ be the set of all continuous maps $\Lambda : [0, 1] \to (\mathcal{C}(M), \mathcal{F})$ such that $\Lambda(0) = M$, $\Lambda(1) = \emptyset$ and $\partial \circ \Lambda : [0, 1] \to \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ has no concentration of mass. The one parameter Almgren-Pitts width of $(M, g)$, denoted by $W_0(M, g)$, is defined by

$$W_0(M, g) = \inf_{\Lambda \in \mathcal{S}} \sup \{M(\partial \Lambda(t)) : t \in [0, 1]\}.$$

### 2.4 Some notions from topology

Given a topological space $A$, let $CA$ denote the cone over $A$, which is defined as follows.

$$CA = \frac{A \times [0, 1]}{\sim},$$

where the equivalence relation ‘$\sim$’ collapses $A \times \{1\}$ to a point. The cone construction is functorial, i.e. if $f : A \to B$ is a continuous map between the topological spaces $A$ and $B$, there exists a continuous map $Cf : CA \to CB$ defined by $Cf(a, t) = (f(a), t)$. Here, by the abuse of notation, we are denoting an element of $CA$ by a pair $(a, t)$ with $a \in A$ and $t \in [0, 1]$. We note that if $p_0 \in CA$ is the collapsed image of $A \times \{1\}$ and $q_0 \in CB$ is the collapsed image of $B \times \{1\}$, then $Cf(p_0) = q_0$.

The suspension of $A$, denoted by $SA$, is defined as follows.

$$SA = \frac{A \times [-1, 1]}{\sim},$$

where the equivalence relation ‘$\sim$’ collapses $A \times \{1\}$ to a point and $A \times \{-1\}$ to another point. $SA = C_+ A \cup C_- A$, where

$$C_+ A = \frac{A \times [0, 1]}{\sim}, \quad C_- A = \frac{A \times [-1, 0]}{\sim}.$$ 

$C_+ A$ and $C_- A$ are cones over $A$. Since the cone over a topological space is contractible, using van Kampen’s Theorem, one can show that if $A$ is path-connected, $SA$ is simply-connected. We note that the cone over a $k$-simplex is a $(k + 1)$-simplex. Therefore, if $A$ is a cubical complex, $CA$ and $SA$ are also cubical complexes.
3 Regularity of the min-max CMC hypersurfaces in higher dimensions

In this section, we will modify the argument of Zhou and Zhu [ZZ19, Zho19] to extend their min-max theory (Theorem 3.5) in higher dimensions. We will use the compactness theory of stable CMC hypersurfaces (Theorem 3.3), developed by Bellettini-Wickramasekera [BW18, BW19] and Bellettini-Chodosh-Wickramasekera [BCW19]. When the ambient dimension $3 \leq n + 1 \leq 7$, the compactness theorem for stable CMC hypersurfaces was also proved by Zhou and Zhu [ZZ19].

Definition 3.1. $\Sigma \subset (U^{n+1}, g)$ is called a $c$-CMC hypersurface with optimal regularity ($c \geq 0$) if $\Sigma$ is a closed $n$-rectifiable subset of $(U, g)$ and there exists a closed subset of $\Sigma$, denoted by $\text{sing}(\Sigma)$, such that the following conditions are satisfied.

(i) $H^s(\text{sing}(\Sigma)) = 0$ for all $s > n - 7$.

(ii) $\text{reg}(\Sigma) = \Sigma \setminus \text{sing}(\Sigma)$ is almost embedded, i.e. if $p \in \text{reg}(\Sigma)$ such that $\Sigma$ is not embedded near $p$, there exists an open set $B \subset U$, containing $p$, such that

- $B \cap \Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1$ and $\Sigma_2$ are smoothly embedded hypersurfaces in $B$ (since $\Sigma$ is not embedded at $p$, we necessarily have $p \in \Sigma_1 \cap \Sigma_2$);
- $B \setminus \Sigma_1 = B_1 \cup B_2$, where $B_1$ and $B_2$ are connected and $\Sigma_2 \subset \overline{B_1}$ (hence $\Sigma_1$ and $\Sigma_2$ intersect each other tangentially).

(iii) $|\Sigma|$ has bounded first variation and $\text{reg}(\Sigma)$ has constant mean curvature $c$ with respect to a globally defined unit normal $\nu$. (If $c = 0$, we do not require that $\text{reg}(\Sigma)$ is two sided.)

Given such a $\Sigma$, following [ZZ19], let us define

$$\mathcal{R}(\Sigma) = \{ p \in \text{reg}(\Sigma) : \Sigma \text{ is embedded near } p \} ; \quad \mathcal{S}(\Sigma) = \text{reg}(\Sigma) \setminus \mathcal{R}(\Sigma).$$

If $\Sigma$ is a minimal hypersurface (i.e. $c = 0$) with optimal regularity, then, by the maximum principle, $\mathcal{S}(\Sigma) = \emptyset$. If $c > 0$, by [ZZ19, Proposition 2.9] and [BW18, Remark 2.6], $\mathcal{S}(\Sigma)$ is $(n - 1)$-rectifiable.

Definition 3.2. Let $\Sigma \subset U$ be a $c$-CMC hypersurface with optimal regularity ($c \geq 0$). $\Sigma$ is called stable if for all compactly supported, normal vector-field $\mathcal{X}$ on $\text{reg}(\Sigma)$,

$$\int_{\text{reg}(\Sigma)} \left( |\nabla^\perp \mathcal{X}|^2 - \text{Ric}(\mathcal{X}, \mathcal{X}) - |A_\Sigma|^2 |\mathcal{X}|^2 \right) d\mathcal{H}^n \geq 0.$$ 

Here $A_\Sigma$ is the second fundamental form of $\Sigma$, which is defined on $\text{reg}(\Sigma)$.

Theorem 3.3 ([BW18, BW19, BCW19], [ZZ19] when $3 \leq n + 1 \leq 7$). Suppose $\Sigma_k \subset U$ is a sequence of stable, $c_k$-CMC hypersurfaces with optimal regularity, $\sup_k \mathcal{H}^n(\Sigma_k) < \infty$.
and \( c_k \to c_\infty \). Then, possibly after passing to a subsequence, \(|\Sigma_k|\) converges to an integral \( n \)-varifold \( W \) such that \( \text{spt} \| W \| = \Sigma_\infty \) is a closed \( c_\infty \)-CMC hypersurface with optimal regularity. The convergence is smooth on compact subsets of \( \text{reg}(\Sigma_\infty) \). Moreover, if each \( \Sigma_k = \partial \Omega_k \) for some \( \Omega_k \in \mathcal{C}(U) \) and \( c_\infty > 0 \), then \( \Sigma_\infty = \partial \Omega_\infty \) for some \( \Omega_\infty \in \mathcal{C}(U) \) and the density of \( W \) is 1 on \( \mathcal{R}(\Sigma_\infty) \) and 2 on \( \mathcal{S}(\Sigma_\infty) \).

The following definition is taken from [Zho19]. We remark that in [Zho19, Definition 1.1], the homotopies are continuous in the flat norm. However, if we examine the proof of Theorem 1.7 in [Zho19], we see that the homotopies can be taken to be continuous in the \( \mathbf{F} \)-metric. (In [Zho19, Proposition 1.14 and 1.15] it is proved that the Almgren extensions are homotopic to each other and to the initial map in the \( \mathbf{F} \)-metric; see also [MN18, Section 3].)

**Definition 3.4.** Suppose \( X \) is a cubical complex, \( Z \subset X \) is a subcomplex, \( F_0 : X \to (\mathcal{C}(M), \mathbf{F}) \) is a continuous map. Let \( \Pi \) denote the set of all sequence of continuous maps \( \{F_i : X \to (\mathcal{C}(M), \mathbf{F})\}_{i=1}^\infty \) such that for every \( i \in \mathbb{N} \), there exists homotopy \( G_i : X \times [0,1] \to (\mathcal{C}(M), \mathbf{F}) \) with the properties \( G_i(-,0) = F_0 \), \( G_i(-,1) = F_i \) and

\[
\limsup_{i \to \infty} \sup_{x \in X} \{\mathbf{F}(G_i(x,s), F_0(x)) : x \in Z, s \in [0,1]\} = 0.
\]

\( \Pi \) is called the \((X,Z)\)-homotopy class of \( F_0 \). For \( c > 0 \), we define

\[
\mathbf{L}^c(\Pi) = \inf_{\{F_i\} \in \Pi} \limsup_{i \to \infty} \sup_{x \in X} \{\mathcal{A}^c(F_i(x))\}.
\]

To prove the following theorem, we only need to slightly modify the argument given in the paper [ZZ19]. We will only mention the necessary modifications; further details can be found in [ZZ19, Section 5 and 6].

**Theorem 3.5 ([ZZ19,Zho19]).** Let \((M^{n+1},g)\) be a closed Riemannian manifold, \( n + 1 \geq 3 \), and \( c > 0 \). Suppose \( F_0 : X \to (\mathcal{C}(M), \mathbf{F}) \) is a continuous map, \( Z \subset X \) and \( \Pi \) is the \((X,Z)\)-homotopy class of \( F_0 \) such that

\[
\mathbf{L}^c(\Pi) > \sup_{x \in Z} \mathcal{A}^c(F_0(x)).
\]

Then there exists \( \Omega \in \mathcal{C}(M) \) such that \( \mathcal{A}^c(\Omega) = \mathbf{L}^c(\Pi) \) and \( \partial \Omega \) is a closed \( c \)-CMC (with respect to the inward unit normal) hypersurface with optimal regularity.

**Proof.** Throughout the proof, we will use the notation used in [ZZ19]. It suffices to show that if \( V \in \mathcal{V}_n(M) \) has \( c \)-bounded first variation in \( M \) and is \( c \)-almost minimizing in small annuli, then \( V = |\Sigma| \), where \( \Sigma \) is a \( c \)-CMC hypersurface with optimal regularity. We will assume that \( M \) is isometrically embedded inside some Euclidean space \( \mathbb{R}^L \). In the rest of this section, \( B(p,r) \) and \( A(p,r,R) \) will respectively denote the ball and the annulus in \( \mathbb{R}^L \) with respect to the Euclidean metric.
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For \( p \in \mathbb{R}^L \) and \( r > 0 \), \( \eta_{p,r} : \mathbb{R}^L \to \mathbb{R}^L \) is defined by \( \eta_{p,r}(x) = \frac{x-p}{r} \). We start with the following proposition.

**Proposition 3.6.** Let \( V \) be \( c \)-almost minimizing in \( U \). Suppose \( \{p_i\} \) is a sequence in \( U \) converging to \( p \in U \) and \( r_i > 0 \) is a sequence converging to \( 0 \). If \( V \) is the varifold limit of \( \{(\eta_{p_i,r_i})_*V\} \), \( \eta \) is induced by a minimal hypersurface with optimal regularity (possibly with integer multiplicities). In particular, if \( V \in \mathcal{V}_n(M) \) has \( c \)-bounded first variation in \( M \) and is \( c \)-almost minimizing in small annuli, then \( V \) is induced by a minimal hypersurface with optimal regularity (possibly with integer multiplicities).

The proof of this proposition is same as the proof of Lemma 5.10 and Proposition 5.11 in [ZZ19]. One can show that \( \eta \) satisfies the good replacement property. This implies the proposition by the results of Schoen-Simon [SS81] and De Lellis-Tasnady [DT13].

**Lemma 3.7.** Let \( H = \{ x \in \mathbb{R}^{n+1} : x_{n+1} = 0 \} \) and \( S \neq H \) be another hyperplane in \( \mathbb{R}^{n+1} \), \( S \) is not parallel to \( H \). Suppose \( W \) is a stationary, integral varifold in \( \mathbb{R}^{n+1} \) such that \( \text{spt} \|W\| \) is a minimal hypersurface with optimal regularity and \( W \mathbb{L} \{ x \in \mathbb{R}^{n+1} : x_{n+1} > 0 \} = m |S| \mathbb{L} \{ x \in \mathbb{R}^{n+1} : x_{n+1} > 0 \} \). Then \( W = m |S| \).

**Proof.** The proof is an adaptation of the argument of Hoffman and Meeks [HM90, Proof of Theorem 3] in the singular setting. Let \( \Xi = \text{spt} \|W\| \). By our assumption,

\[
\mathcal{H}^s(\text{sing}(\Xi)) = 0 \quad \text{for all } s > n - 7. \tag{3.1}
\]

Therefore, there exists a collection \( \{\Xi_j\}_{j=0}^J \) of disjoint, connected minimal hypersurfaces with optimal regularity and positive integers \( \{m_j\}_{j=0}^J \) such that

\[
W = \sum_{j=0}^J m_j |\Xi_j|.
\]

Since \( \Xi \cap \{x_{n+1} > 0\} = S \cap \{x_{n+1} > 0\} \), by (3.1), we have \( S \subset \text{reg}(\Xi) \) (see [SS81, (7.23)]). By the theorem of Ilmanen [Ilm96, Theorem A(ii)] and Wickramasekera [Wic14], (3.1) implies that the connected components of \( \text{reg}(\Xi) \) are precisely \( \{\text{reg}(\Xi_j)\}_{j=0}^J \). Thus \( S = \text{reg}(\Xi_0) \), for some \( l \in \{0, \ldots, J\} \). Let us assume that \( S = \text{reg}(\Xi_0) \); (3.1) gives \( S = \Xi_0 \) and hence \( m_0 = m \). If \( W \neq m |S| \), let us consider \( \Xi_1 \). Since \( W \mathbb{L} \{x_{n+1} > 0\} = m |S| \mathbb{L} \{x_{n+1} > 0\} \), \( \Xi_1 \subset \{x_{n+1} < 0\} \). We claim that \( \Xi_1 \subset \{x_{n+1} < 0\} \). Indeed, by the maximum principle proved by White [Whi10, Theorem 4] and Solomon-White [SW89], if \( \Xi_1 \cap H \neq \emptyset \), then \( H \subset \Xi_1 \), which contradicts the fact that \( \Xi_1 \) is disjoint from \( \Xi_0 = S \). As \( \Xi_1 \) is connected, \( \Xi_1 \) lies in a region \( R \) which is the intersection of the open half-space \( \{x_{n+1} < 0\} \) and another open half-space defined by \( S \). Using an orthogonal transformation, we can obtain a new coordinate system \( \{y_1, \ldots, y_{n+1}\} \) such that \( R \subset \{y_{n+1} > 0\} \), \( \partial R \) is a graph over the hyperplane \( \{y_{n+1} = 0\} \) and the \( y_1 \)-axis is contained in \( H \cap S \). As \( \Xi_1 \) is a closed set and is disjoint from \( H \cup S \), there exists \( r > 0 \) such that

\[
B_r = \{ y \in \mathbb{R}^{n+1} : (y_1 - r)^2 + y_2^2 + \cdots + y_{n+1}^2 \leq r^2 \}.
\]
is disjoint from $\Xi_1$. We note that $\partial B_\tau \cap \partial \mathcal{R}$ is a graph over the boundary of a convex domain in $\{y_{n+1} = 0\}$. Therefore, by [GT01, Theorem 16.8], there exists a smooth minimal hypersurface $N$ with boundary $\partial N = \partial B_\tau \cap \partial \mathcal{R}$. By the convex hull property, $N \subset B_\tau$; hence $N \cap \Xi_1 = \emptyset$. For $t \in [1, \infty)$, let us consider

$$N_t = \{tp : p \in N\}.$$

We claim that for all $t \in [1, \infty)$, $N_t \cap \Xi_1 = \emptyset$. Otherwise, there exists a smallest $\tau > 1$ such that $N_{\tau} \cap \Xi_1 \neq \emptyset$. This implies, by [SW89, Step 1, p. 687], $N_{\tau} \subset \Xi_1$, which contradicts the fact that $\Xi_1 \cap \partial \mathcal{R} = \emptyset$. However,

$$\mathcal{R} \cap \{y_1 > 0\} \subset B_\tau \cup (\bigcup_{t \in [1, \infty)} N_t).$$

Therefore, $\Xi_1 \subset \mathcal{R} \cap \{y_1 \leq 0\}$. An analogous argument gives that for any $T \in \mathbb{R}$, $\Xi_1 \subset \mathcal{R} \cap \{y_1 \leq T\}$, which forces $\Xi_1 = \emptyset$. This finishes the proof of the lemma. \hfill \Box

We proceed as in [ZZ19]. Let us fix $p \in \text{spt} \|V\|$. We choose $0 < r_0 < r_{am}(p)$ such that

- for all $0 < r \leq r_0$, $\partial B(p, r) \cap M$ is a smooth hypersurface in $M$ with mean curvature greater than $c$;
- $x \mapsto d_{RL}(p, x)$ is a smooth function on $(B(p, r_0) \cap M) \setminus \{p\}$.

For $r \leq r_0$ and $W \in \mathcal{V}_n(M)$, if $W \neq 0$ in $B(p, r)$ and has $c$-bounded first variation in $B(p, r) \cap M$, we have

$$\emptyset \neq \text{spt} \|W\| \cap \partial B(p, r) = \text{spt} \|W\| \setminus \overline{B(p, r) \cap \partial B(p, r)}; \quad (3.2)$$

$$\text{spt} \|W\| \cap \overline{B(p, r)} = \bigcup_{0 < s < r} \text{spt}(\|W\| \text{L}(B(p, s)) \cap \partial B(p, s)). \quad (3.3)$$

Equation (3.3) is a consequence of [ZZ19, Lemma 6.2] (see [DT13, equation (5.6) and A.3. Proof of Lemma 5.4]). Let us fix $0 < s < t < r_0$. $V^*$ is a $c$-replacement of $V$ in $\text{Clos}(A(p, s, t) \cap M)$. By the regularity of the minimizers for the $A^c$ functional [ZZ19, Theorem 2.14; Mor03] and Theorem 3.3, there exists $\Sigma_1$, a $c$-CMC hypersurface in $A(p, s, t) \cap M$ with optimal regularity, such that $\Sigma_1 = \partial \Omega_1 \cap A(p, s, t) \cap M$ for some $\Omega_1 \in \mathcal{C}(M)$ and

$$V^* \text{L}(A(p, s, t) \cap M) = |\Sigma_1|.$$

$\mathcal{S}(\Sigma_1)$ is contained in $\cup_k \Sigma_1^{(k)}$, a countable union of $(n - 1)$-dimensional submanifolds. We choose $s_2 \in (s, t)$ such that $\partial B(p, s_2)$ intersects $\text{reg}(\Sigma_1)$ and all the $\Sigma_1^{(k)}$'s transversally. For $s_1 \in (0, s)$, let us consider $c$-replacement $V^{**}$ of $V^*$ in $\text{Clos}(A(p, s_1, s_2) \cap M)$. There exists $\Sigma_2$, a $c$-CMC hypersurface in $A(p, s_1, s_2) \cap M$ with optimal regularity, such that $\Sigma_2 = \partial \Omega_2 \cap A(p, s_1, s_2) \cap M$ for some $\Omega_2 \in \mathcal{C}(M)$ and

$$V^{**} \text{L}(A(p, s_1, s_2) \cap M) = |\Sigma_2|.$$

We need to show that $\Sigma_2$ smoothly glues with $\Sigma_1$ across $\partial(B(p, s_2) \cap M)$.  

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**Lemma 3.8.** For all $x \in \text{reg}(\Sigma_1) \cap \partial(B(p, s_2) \cap M)$, $\text{VarTan}(V^{**}, x) = \{\Theta^n(||V^{**}||, x)|T_x \Sigma_1]\}$, where $T_x \Sigma_1$ denotes the tangent space of $\Sigma_1$ at $x$ with multiplicity one.

**Proof.** Let $C \in \text{VarTan}(V^{**}, x)$. Denoting $\Lambda = \partial(B(p, s_2) \cap M)$, since $V^{**} = V^* = |\Sigma_1|$ in $A(p, s_2, t) \cap M$, we must have $C = \Theta^n(||V^{**}||, x)|T_x \Sigma_1|$ in an open half-space of $T_x M$, defined by $T_2 \Lambda$. (By the transversality assumption, $T_x \Sigma_1 \neq T_2 \Lambda$.) Using Proposition 3.6 and Lemma 3.7, we obtain $C = \Theta^n(||V^{**}||, x)|T_x \Sigma_1|$. \(\Box\)

As argued in [ZZ19, Step 2], using (3.2), one can show that

$$\text{Clos}(\Sigma_2) \cap \partial(B(p, s_2) \cap M) \subset \Sigma_1 \cap \partial(B(p, s_2) \cap M).$$

Moreover, smallness of $\text{sing}(\Sigma_1)$ and Lemma 3.8 imply that

$$\Sigma_1 \cap \partial(B(p, s_2) \cap M) = \text{Clos}(\text{reg}(\Sigma_1) \cap \partial(B(p, s_2) \cap M)) \subset \text{Clos}(\Sigma_2) \cap \partial(B(p, s_2) \cap M).$$

Hence,

$$\text{Clos}(\Sigma_2) \cap \partial(B(p, s_2) \cap M) = \Sigma_1 \cap \partial(B(p, s_2) \cap M),$$

i.e. $\Sigma_1, \Sigma_2$ glue continuously along $\partial B(p, s_2)$. Furthermore, Lemma 3.8, [Sim83, Theorem 3.2(2)] and smallness of $\text{sing}(\Sigma_1)$ also give that $||V^{**}||(\partial B(p, s_2)) = 0$, which implies, by [ZZ19, Claim 1(c) in Section 6],

$$V^{**} \llcorner (A(p, s_1, t) \cap M) = |\partial \Omega^{**}| \llcorner (A(p, s_1, t) \cap M),$$

for some $\Omega^{**} \in \mathcal{C}(M)$.

Arguing as in [ZZ19, Step 2], we can also show that $\text{reg}(\Sigma_1)$ and $\text{reg}(\Sigma_2)$ glue smoothly across $\partial(B(p, s_2) \cap M)$; we need to use Lemma 3.7 in place of the half space theorem of [HM90] and Theorem 3.3 in place of [ZZ19, Theorem 2.11]. Let us give some details. Let $\nu_1, \nu_2$ denote the outward unit normal of $\text{reg}(\Sigma_1)$ and $\text{reg}(\Sigma_2)$ respectively;

$$\Gamma = \text{reg}(\Sigma_1) \cap \partial(B(p, s_2) \cap M), \quad \mathcal{R}(\Gamma) = \mathcal{R}(\Sigma_1) \cap \Gamma, \quad \mathcal{S}(\Gamma) = \mathcal{S}(\Sigma_1) \cap \Gamma.$$

Suppose $x \in \mathcal{R}(\Gamma), y_i \in \text{reg}(\Sigma_2)$ converges to $x$. It is possible to choose $x_i \in \mathcal{R}(\Gamma)$ such that $r_i = |x_i - y_i|$ converges to 0 and hence $x_i \to x$. Using the fact that $V^{**} \llcorner (A(p, s_2, t) \cap M) = |\Sigma_1| \llcorner (A(p, s_2, t) \cap M)$, Proposition 3.6 and Lemma 3.7, one can prove that

$$\lim_{i \to \infty} (\eta_{x_i, r_i})_# V^{**} = |T_x \Sigma_1| \quad \text{in the sense of varifolds}.$$

This, along with Theorem 3.3, implies that $\eta_{x_i, r_i}(\Sigma_2 \cap B(y_i, r_i/2))$ smoothly converges to a domain in $T_x \Sigma_1$, which gives

$$\lim_{y \to x, y \in \text{reg}(\Sigma_2)} \nu_2(y) = \nu_1(x) \quad \text{uniformly in } x \text{ on compact subsets of } \mathcal{R}(\Gamma). \quad (3.4)$$
Similarly, if \( x \in \mathcal{S}(\Gamma) \) and \( y_i \in \text{reg}(\Sigma_2) \) converges to \( x \), it is possible to choose \( x_i \in \Gamma \) such that \( r_i = |x_i - y_i| \) converges to 0 and hence \( x_i \to x \). Using the fact that \( V^{**} \mathbb{L}(A(p, s_2, t) \cap M) = |\Sigma_1| \mathbb{L}(A(p, s_2, t) \cap M) \), Proposition 3.6 and Lemma 3.7, one can show that

\[
\lim_{i \to \infty} (\eta_{x_i, r_i}) \# V^{**} = \begin{cases} \left[ T_x \Sigma_1 \right] + |\tau_v T_x \Sigma_1| & \text{if } \lim inf_{i \to \infty} d_{\mathbb{R}L}(x_i, \mathcal{S}(\Gamma))/r_i = \infty; \\ 2 \left| T_x \Sigma_1 \right| & \text{if } \lim inf_{i \to \infty} d_{\mathbb{R}L}(x_i, \mathcal{S}(\Gamma))/r_i < \infty, \end{cases}
\]

where \( T_x \Sigma_1 \) denotes the tangent space of \( \Sigma_1 \) at \( x \) with multiplicity one and \( \tau_v \) denotes translation by \( v \in (T_x \Sigma_1)^\perp \subset T_x M \) (\( v \) might be \( \infty \), in which case \( \tau_v T_x \Sigma_1 = \emptyset \)). As before, this, together with Theorem 3.3, gives that

\[
\lim_{y \to x, y \notin \text{reg}(\Sigma_2)} [T_y \Sigma_2] = [T_x \Sigma_1] \quad \text{uniformly in } x \text{ on compact subsets of } \mathcal{S}(\Gamma),
\]

where \([T_y \Sigma_2]\) and \([T_x \Sigma_1]\) respectively denote the un-oriented, multiplicity one tangent spaces of \( \Sigma_2 \) and \( \Sigma_1 \). (3.4), (3.5) and elliptic regularity imply that \( \text{reg}(\Sigma_1) \) and \( \text{reg}(\Sigma_2) \) glue smoothly across \( \partial(B(p, s_2) \cap M) \). Hence, by unique continuation and by the smallness of \( \text{sing}(\Sigma_1) \) and \( \text{sing}(\Sigma_2) \), \( \Sigma_2 = \Sigma_1 \) in \( A(p, s, s_2) \cap M \).

Since we will vary \( s_1 \in (0, s) \), let us use the notation \( V^{**}_{s_1} \) and \( \Sigma_{s_1} \) instead of \( V^{**} \) and \( \Sigma_2 \). As argued in [ZZ19, Step 3], the above discussion implies that

\[
\Sigma = \{p\} \cup \bigcup_{0<s_1<s} \Sigma_{s_1}
\]

is a \( c \)-CMC hypersurface with optimal regularity in \( B(p, s_2) \) (if necessary, \( p \) can be absorbed inside \( \text{sing}(\Sigma) \)). Further, for any \( s_1 < s \),

\[
V^{**}_{s_1} \mathbb{L}(A(p, s_1, s_2) \cap M) = |\Sigma| \mathbb{L}(A(p, s_1, s_2) \cap M).
\]

Following the argument of Schoen-Simon [SS81] and De Lellis-Tasnady [DT13], we will show that \( \text{spt} \|V\| \cap B(p, s) = \Sigma \cap B(p, s) \). By the definition of \( c \)-replacement, for all \( 0 < s_1' \leq s_1 \),

\[
V^{**}_{s_1} \mathbb{L}(B(p, s_1') \cap M) = V^{**} \mathbb{L}(B(p, s_1') \cap M) = V \mathbb{L}(B(p, s_1') \cap M).
\]

Hence, using (3.3),

\[
\text{spt} \|V\| \cap \partial(B(p, s_1) \cap M) = \text{spt} \|V^{**}_{s_1}\| \cap \partial(B(p, s_1) \cap M).
\]

Moreover, by (3.6) and (3.2),

\[
\text{spt} \|V^{**}_{s_1}\| \cap \partial(B(p, s_1) \cap M) = \Sigma \cap \partial(B(p, s_1) \cap M).
\]

Therefore, combining (3.7) and (3.8),

\[
\text{spt} \|V\| \cap \partial(B(p, s_1) \cap M) = \Sigma \cap \partial(B(p, s_1) \cap M),
\]

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for all $0 < s_1 < s$. By varying $s_1 \in (0, s)$, we obtain $\text{spt} \|V\| \cap B(p, s) = \Sigma \cap B(p, s)$. (We assumed that $p \in \text{spt} \|V\|$ and by our definition of $\Sigma$, $p \in \Sigma$.)

As $p \in \text{spt} \|V\|$ is arbitrary, we have proved that $\text{spt} \|V\|$ is a $c$-CMC hypersurface with optimal regularity. By a slight abuse of notation, let us denote $\text{spt} \|V\|$ by $\Sigma$. We will show that if $p \in \mathcal{R}(\Sigma)$, there exists $r > 0$ such that $\Theta^\eta(\|V\|, y) = 1$ for all $y \in (\mathcal{R}(\Sigma) \cap B(p, r)) \setminus \{p\}$. This will imply that $V = |\Sigma|$ as $\mathcal{H}^n(\Sigma \setminus \mathcal{R}(\Sigma)) = 0$.

The argument is similar to [ZZ19, Proof of Claim 6]. Since $p \in \mathcal{R}(\Sigma)$, one can choose $0 < r < r_0$ (where $r_0$ is as chosen after the proof of Lemma 3.7), such that $B(p, r) \cap \Sigma \subset \mathcal{R}(\Sigma)$ and for all $0 < r' \leq r$, $\partial B(p, r')$ intersects $\Sigma$ transversally. Let $\rho = |p - y|$ and we consider the second replacement $V^{**}_\rho$. Since $V^{**}_\rho \mathbf{L}(A(p, \rho, s) \cap M) = |\Sigma| \mathbf{L}(A(p, \rho, s) \cap M)$ for some $s > \rho$, by the transversality assumption, Proposition 3.6 and Lemma 3.7, $\text{VarTan}(V^{**}_\rho, y) = \{|T_y \Sigma|\}$. As $V \mathbf{L}(B(p, \rho) \cap M) = V^{**}_\rho \mathbf{L}(B(p, \rho) \cap M)$, one can again use Proposition 3.6 and Lemma 3.7 to conclude that $\text{VarTan}(V, y) = \{|T_y \Sigma|\}$. Hence $\Theta^\eta(\|V\|, y) = 1$.

4 Proof of the main theorems

Proof of Theorem 1.1. The theorem will be proved in five parts.

Part 1. We choose $\delta$ such that

$$0 < \delta < \omega_{k+1} - \omega_k - c \text{Vol}(g) \quad \text{and} \quad \delta < \eta/2. \quad (4.1)$$

By the definition of $\omega_k$, there exists a map $\Phi : X \to \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ with no concentration of mass, such that $X$ is a connected cubical complex, $\Phi$ is a $k$-sweepout and

$$\sup_{x \in X} \text{M}(\Phi(x)) \leq \omega_k + \delta/2. \quad (4.2)$$

Following the argument in [Zho19], $\Phi$ is a $k$-sweepout implies that

$$\Phi^* : H^1(\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2), \mathbb{Z}_2) \to H^1(X, \mathbb{Z}_2)$$

is non-zero; hence,

$$\Phi_* : \pi_1(X) \to \pi_1(\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2))(= \mathbb{Z}_2)$$

is onto. Thus, $\ker(\Phi_*)$ is an index 2 subgroup of $\pi_1(X)$. From [Hat02, Proposition 1.36], it follows that there exists a two sheeted covering $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is connected and if

$$\pi_* : \pi_1(\tilde{X}) \to \pi_1(X),$$

$\text{im}(\pi_*) = \ker(\Phi_*)$. By [Hat02, Proposition 1.33], $\Phi$ has a lift

$$\tilde{\Phi} : \tilde{X} \to (\mathcal{C}(M), \mathcal{F}) \quad \text{such that} \quad \partial \circ \tilde{\Phi} = \Phi \circ \pi.$$
Moreover, $\Phi$ is $\mathbb{Z}_2$-equivariant, i.e. if $T : \tilde{X} \to \tilde{X}$ is the deck transformation, then for all $x \in \tilde{X}$,

$$\Phi(T(x)) = M - \Phi(x).$$

Let us recall the notation from Section 2.4.

$$S\tilde{X} = \text{suspension of } \tilde{X} = \tilde{X} \times [-1,1].$$

By the abuse of notation, an element of $S\tilde{X}$ will be denoted by a pair $(x, t)$ with $x \in \tilde{X}, t \in [-1,1]$.

$$S\tilde{X} = C_+ \cup C_-,$$

where $C_+ = \tilde{X} \times [0,1], C_- = \tilde{X} \times [-1,0].$

Since $\tilde{X}$ is connected, $S\tilde{X}$ is simply-connected. There is a free $\mathbb{Z}_2$ action on $S\tilde{X}$ given by $(x, t) \mapsto (T(x),-t)$. Let $Y$ denote the quotient of $S\tilde{X}$ with respect to this $\mathbb{Z}_2$ action and $\rho : S\tilde{X} \to Y$ be the covering map. $\tilde{X}$ naturally sits inside $S\tilde{X}$ by the map $x \mapsto (x,0)$; hence, $X$ also naturally sits inside $Y$. We will identify $\tilde{X}$ with its image under $\tilde{X} \hookrightarrow S\tilde{X}$ and $X$ with its image under $X \hookrightarrow Y$.

To illustrate the above constructions, let us consider the following example. If $X = \mathbb{RP}^k$, then $\tilde{X} = S^k, S\tilde{X} = S^{k+1}, C_+$ and $C_-$ are respectively the upper hemisphere and the lower hemisphere of $S^{k+1}, Y = \mathbb{RP}^{k+1}$.

**Part 2.** We begin with the following proposition from [AFP00]. For the shake of completeness, we also include its proof (following [AFP00]).

**Proposition 4.1 ([AFP00, Proposition 3.38]).** For an $\mathcal{H}^{n+1}$-measurable set $E \subset M$ and an open set $U \subset M$, let us use the notation

$$P(E,U) = \sup \left\{ \int_{E \cap U} \text{div} \omega \ d\mathcal{H}^{n+1} : \omega \in \mathcal{X}_c^1(U) \text{ and } \|\omega\|_\infty \leq 1 \right\}.$$  

Here $\mathcal{X}_c^1(U)$ is the space of compactly supported $C^1$ vector-fields on $U$. Suppose $E, F \in \mathcal{C}(M)$. Then

$$P(E \cap F, U) \leq P(E,U) + P(F,U).$$

Hence, in particular, $E \cap F \in \mathcal{C}(M)$.

**Proof.** By [MPP07, Proposition 1.4], there exist sequences $\{f_j\}_{j=1}^\infty, \{g_j\}_{j=1}^\infty \subset C^\infty(U)$ such that $0 \leq f_j, g_j \leq 1$ for all $j \in \mathbb{N},$

$$f_j \to \chi_E|_U \text{ and } g_j \to \chi_F|_U \text{ in } L^1(U);$$

$$P(E,U) = \lim_{j \to \infty} \int_U \|Df_j\| d\mathcal{H}^{n+1} \text{ and } P(F,U) = \lim_{j \to \infty} \int_U \|Dg_j\| d\mathcal{H}^{n+1}.$$
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By the dominated convergence theorem,

\[ f_j g_j \to \chi E \chi F |U = \chi E \cap F |U \quad \text{in} \quad L^1(U). \]

Therefore,

\[
P(E \cap F, U) \leq \liminf_{j \to \infty} \int_U \|D(f_j g_j)\| \, d\mathcal{H}^{n+1}
\leq \liminf_{j \to \infty} \int_U \|Df_j\| \, d\mathcal{H}^{n+1} + \liminf_{j \to \infty} \int_U \|Dg_j\| \, d\mathcal{H}^{n+1}
= P(E, U) + P(F, U).
\]

The map \( \tilde{\Phi} : \tilde{X} \to (C(M), \mathcal{F}) \) (defined in Part 1) can be extended to a map \( \tilde{\Psi} : S\tilde{X} \to (C(M), \mathcal{F}) \) as follows. Let us define

\[ \tilde{Z} = \{(x, t) \in S\tilde{X} : |t| \leq 1/2\} \subset S\tilde{X}. \]

Let \( \Lambda : [0, 1] \to (C(M), \mathcal{F}) \) be such that \( \Lambda(0) = M, \Lambda(1) = \emptyset, \partial \circ \Lambda : [0, 1] \to \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2) \) has no concentration of mass and

\[
\sup_{t \in [0, 1]} M(\partial \Lambda(t)) \leq W_0 + \eta/2 - \delta. \tag{4.3}
\]

We define

\[
\tilde{\Psi}'(x, t) = \begin{cases} 
\tilde{\Phi}(x) & \text{if } (x, t) \in \tilde{Z}; \\
\tilde{\Phi}(x) \cap \Lambda(2t - 1) & \text{if } (x, t) \in C_+ \setminus \tilde{Z}; \\
M - (\tilde{\Phi}(T(x)) \cap \Lambda(-1 - 2t)) & \text{if } (x, t) \in C_- \setminus \tilde{Z}.
\end{cases} \tag{4.4}
\]

By Proposition 4.1, \( \tilde{\Psi}'(x, t) \in C(M) \) for all \((x, t) \in S\tilde{X}\).

**Claim 4.2.** \( \tilde{\Psi}' : S\tilde{X} \to C(M) \) is continuous in the flat topology.

**Proof.** It is enough to prove that \((x, t) \mapsto \tilde{\Phi}(x) \cap \Lambda(2t - 1)\) is continuous in \(\mathcal{F}\) for \((x, t) \in C_+ \setminus \tilde{Z}\). We note that for arbitrary sets \(A_1, A_2, B_1, B_2,\)

\[(A_1 \cap A_2) \Delta (B_1 \cap B_2) \subset (A_1 \Delta B_1) \cup (A_2 \Delta B_2),\]

where \(A \Delta B = (A \setminus B) \cup (B \setminus A)\) denotes the symmetric difference of \(A\) and \(B\). Hence, if \((x_1, t_1), (x_2, t_2) \in C_+ \setminus \tilde{Z},\)

\[
(\tilde{\Phi}(x_1) \cap \Lambda(2t_1 - 1)) \Delta (\tilde{\Phi}(x_2) \cap \Lambda(2t_2 - 1)) \subset (\tilde{\Phi}(x_1) \Delta \tilde{\Phi}(x_2)) \cup (\Lambda(2t_1 - 1) \Delta (2t_2 - 1)),
\]

which implies

\[
\mathcal{F}
\left( (\tilde{\Phi}(x_1) \cap \Lambda(2t_1 - 1)) - (\tilde{\Phi}(x_2) \cap \Lambda(2t_2 - 1)) \right)
\leq \mathcal{F}(\tilde{\Phi}(x_1) - \tilde{\Phi}(x_2)) + \mathcal{F}(\Lambda(2t_1 - 1) - \Lambda(2t_2 - 1)), \tag{4.5}
\]

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Since $x \mapsto \Phi(x)$ and $t \mapsto \Lambda(t)$ are continuous in the flat topology, by (4.5), $(x, t) \mapsto \Phi(x) \cap \Lambda(2t - 1)$ is also continuous in the flat topology.

As $\partial \tilde{\Psi}'(x, t) = \partial \tilde{\Psi}'(T(x), -t)$, $\tilde{\Psi}'$ descends to a continuous map $\Psi' : Y \to \mathbb{Z}_n(M; F; \mathbb{Z}_2)$, i.e. $\Psi' \circ \rho = \partial \circ \tilde{\Psi}'$. Furthermore, $\Psi'|_X = \Phi$.

**Claim 4.3.** $\Psi' : Y \to \mathbb{Z}_n(M; F; \mathbb{Z}_2)$ has no concentration of mass.

**Proof.** If $E \in C(M)$ and $U \subset M$ is open, $P(E, U) = \|\partial E\|(U)$. By Proposition 4.1, for $r > 0$ and $p \in M$,

$$\sup_{y \in Y} \|\Psi'(y)\|(B(p, r)) \leq \sup_{x \in X} \|\Phi(x)\|(B(p, r)) + \sup_{t \in [0, 1]} \|\partial \Lambda(t)\|(B(p, r)).$$

This finishes the proof of the claim as the maps $\Phi$ and $\partial \circ \Lambda$ have no concentration of mass. \qed

By the interpolation theorems of Marques and Neves [MN14, Theorem 13.1, 14.1], [MN17, Theorem 3.9, 3.10], the above Claim 4.3 implies that there exists a continuous map $\Psi : Y \to \mathbb{Z}_n(M; F; \mathbb{Z}_2)$ such that $\Psi$ is homotopic to $\Psi'$ in the $F$-topology and for all $y \in Y$,

$$M(\Psi(y)) \leq M(\Psi'(y)) + \delta/2. \quad (4.6)$$

**Part 3.** We will prove that $\Psi : Y \to \mathbb{Z}_n(M; F; \mathbb{Z}_2)$, constructed above, is a $(k+1)$-sweepout.

Let $\bar{X}$ be the generator of $H^*(\mathbb{Z}_n(M; F; \mathbb{Z}_2), \mathbb{Z}_2)$ as in Section 2.3 and $\lambda = \Psi* \bar{X}$. We need to show that $\lambda^{k+1} \neq 0 \in H^{k+1}(Y, \mathbb{Z}_2)$. Since $S\bar{X}$ is simply-connected and $Y$ is the quotient of $S\bar{X}$ under the $\mathbb{Z}_2$ action, $\pi_1(Y) = \mathbb{Z}_2$. Hence, $H_1(Y, \mathbb{Z}_2) = H^1(Y, \mathbb{Z}_2) = \mathbb{Z}_2$ as well.

Let $\iota : X \to Y$ be the inclusion map. Since $\Psi'|_X = \Phi$, $\Psi$ is homotopic to $\Psi'$ in the $F$-topology and $\Phi$ is a $k$-sweepout, $\iota^* \lambda = \Phi^* \bar{X} \neq 0 \in H^1(Y, \mathbb{Z}_2)$. Therefore, $\lambda$ is the unique non-zero element of $H^1(Y, \mathbb{Z}_2)$. Hence, if $\gamma : S^1 \to Y$ is a non-contractible loop, $\lambda[\gamma] = 1$.

To prove $\lambda^{k+1} \neq 0 \in H^{k+1}(Y, \mathbb{Z}_2)$, it is enough to find $\alpha \in H_{k+1}(Y, \mathbb{Z}_2)$ such that $\lambda^{k+1}.\alpha = 1$. By [Hat02, Proof of Proposition 2B.6], there exists a map (called the transfer homomorphism) $\tau_* : H_k(X, \mathbb{Z}_2) \to H_k(\bar{X}, \mathbb{Z}_2)$, which is induced by the chain map $\tau : C_k(X, \mathbb{Z}_2) \to C_k(\bar{X}, \mathbb{Z}_2)$, defined as follows. If $u : \Delta^k \to X$, $u$ has precisely two lifts $\bar{u}_1, \bar{u}_2 : \Delta^k \to \bar{X}$. We define $\tau(u) = \bar{u}_1 + \bar{u}_2$.

As $\Phi$ is a $k$-sweepout, $(\iota^* \lambda)^k = (\Phi^* \bar{X})^k \neq 0 \in H^k(X, \mathbb{Z}_2)$. Therefore, there exists $\sigma \in H_k(X, \mathbb{Z}_2)$ such that $(\iota^* \lambda)^k.\sigma = 1$. Let $\sigma = \sum_{i=1}^I \sigma_i$, where each $\sigma_i : \Delta^k \to X$ is a singular
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$k$-chain. Let $\tau(\sigma_i) = \tilde{\sigma}_{2i-1} + \tilde{\sigma}_{2i}$, where $\tilde{\sigma}_{2i-1}$, $\tilde{\sigma}_{2i} : \Delta^k \to \tilde{X}$ are lifts of $\sigma_i$. Since $\partial(\sum_i \sigma_i) = 0$ and $\tau$ is a chain map, we have

$$\partial\left(\sum_i \tilde{\sigma}_{2i-1} + \tilde{\sigma}_{2i}\right) = 0 \quad (4.7)$$

as well. $C\Delta^k$ can be naturally identified with $\Delta^{k+1}$ such that the collapsed image of $\Delta^k \times \{1\}$ is the $(k + 1)$-st vertex $v_{k+1}$ (see Section 2.4 for the notation). Let us also recall from Part 1 that $C_+$ is a cone over $\tilde{X}$; let $y_0 \in C_+$ be the collapsed image of $\tilde{X} \times \{1\}$. For $s = 1, \ldots, 2I$, we consider

$$C\tilde{\sigma}_s : C\Delta^k(= \Delta^{k+1}) \to C\tilde{X}(= C_+); \quad C\tilde{\sigma}_s(v_{k+1}) = y_0.$$ 

Using equation (4.7), one can check that

$$\partial\left(\sum_i C\tilde{\sigma}_{2i-1} + C\tilde{\sigma}_{2i}\right) = \left(\sum_i \tilde{\sigma}_{2i-1} + \tilde{\sigma}_{2i}\right). \quad (4.8)$$

As $\tilde{\sigma}_{2i-1}$, $\tilde{\sigma}_{2i}$ are lifts of $\sigma_i$, $\rho\#\tilde{\sigma}_{2i-1} = \rho\#\tilde{\sigma}_{2i} = \sigma_i$. Hence,

$$\partial\left(\sum_i \rho\#C\tilde{\sigma}_{2i-1} + \rho\#C\tilde{\sigma}_{2i}\right) = \rho\#\left(\sum_i \tilde{\sigma}_{2i-1} + \tilde{\sigma}_{2i}\right) = 0 \in C_k(X, \mathbb{Z}_2). \quad (4.9)$$

Let us define

$$\alpha = \left[\sum_i \rho\#C\tilde{\sigma}_{2i-1} + \rho\#C\tilde{\sigma}_{2i}\right] \in H_{k+1}(Y, \mathbb{Z}_2). \quad (4.10)$$

We claim that $\lambda^{k+1}\alpha = 1$. Let $\lambda = [\ell], \ell \in C^{k+1}(Y, \mathbb{Z}_2)$, i.e. $\ell : C_{k+1}(Y, \mathbb{Z}_2) \to \mathbb{Z}_2$. By the definition of the cup product,

$$\ell^{k+1}(\rho\#C\tilde{\sigma}_{2i-1} + \rho\#C\tilde{\sigma}_{2i}) = \ell^{k}\left(\rho\#C\tilde{\sigma}_{2i-1}|_{\Delta^k}\right) \ell\left(\rho\#C\tilde{\sigma}_{2i}|_{[v_k, v_{k+1}]}\right) + \ell^{k}\left(\rho\#C\tilde{\sigma}_{2i}|_{\Delta^k}\right) \ell\left(\rho\#C\tilde{\sigma}_{2i}|_{[v_k, v_{k+1}]}\right)$$

$$= \ell^{k}(\ell(\rho\#C\tilde{\sigma}_{2i-1}|_{[v_k, v_{k+1}]} + \rho\#C\tilde{\sigma}_{2i}|_{[v_k, v_{k+1}]})]. \quad (4.11)$$

In the second equality we have used the fact that

$$C\tilde{\sigma}_{2i-1}|_{\Delta^k} = \tilde{\sigma}_{2i-1} \text{ and } C\tilde{\sigma}_{2i}|_{\Delta^k} = \tilde{\sigma}_{2i}.$$ 

We note that

$$C\tilde{\sigma}_{2i-1}|_{[v_k, v_{k+1}]} \text{ is a curve joining } \tilde{\sigma}_{2i-1}(v_k) \text{ and } \tilde{\sigma}_{2i-1}(v_{k+1}) = y_0,$$

and

$$C\tilde{\sigma}_{2i}|_{[v_k, v_{k+1}]} \text{ is a curve joining } \tilde{\sigma}_{2i}(v_k) \text{ and } \tilde{\sigma}_{2i}(v_{k+1}) = y_0.$$ 

As $\tilde{\sigma}_{2i-1}$, $\tilde{\sigma}_{2i}$ are lifts of $\sigma_i$, $T(\tilde{\sigma}_{2i-1}(v_k)) = \tilde{\sigma}_{2i}(v_k)$. Hence,

$$\left(\rho\#C\tilde{\sigma}_{2i-1}|_{[v_k, v_{k+1}]} + \rho\#C\tilde{\sigma}_{2i}|_{[v_k, v_{k+1}]})\right]$$
is a non-contractible loop in $Y$. Therefore,

$$\ell \left( \rho_# C \tilde{\sigma}_{2i-1} \big|_{[v_k,v_{k+1}]} + \rho_# C \tilde{\sigma}_{2i} \big|_{[v_k,v_{k+1}]} \right) = 1 \quad (4.12)$$

Using (4.11) and (4.12), we conclude that

$$\lambda^{k+1} \alpha = \sum_i \ell^{k+1} \left( \rho_# C \tilde{\sigma}_{2i-1} + \rho_# C \tilde{\sigma}_{2i} \right) = \sum_i \ell^k(\sigma_i) = (\ell^* \lambda)^k \sigma = 1.$$  

**Part 4.** Since $\Psi$ is a $(k+1)$-sweepout, using the argument as in Part 1, $\Psi$ has a lift

$$\tilde{\Psi} : S\tilde{X} \rightarrow (C(M), F) \quad \text{such that} \quad \partial \circ \tilde{\Psi} = \Psi \circ \rho.$$  

$\tilde{\Psi}$ is $\mathbb{Z}_2$-equivariant, i.e.

$$\tilde{\Psi}(T(x), -t) = M - \tilde{\Psi}(x, t),$$

for all $(x, t) \in S\tilde{X}$. Moreover, by the definition of the $F$-metric on $C(M)$ and $\mathbb{Z}_n(M; \mathbb{Z}_2)$, continuity of $\Psi : Y \rightarrow \mathbb{Z}_n(M; F; \mathbb{Z}_2)$ implies that $\tilde{\Psi} : S\tilde{X} \rightarrow (C(M), F)$ is also continuous. Let $\tilde{\Pi}$ be the $(S\tilde{X}, \tilde{Z})$-homotopy class of $\tilde{\Psi}$. We will show that

$$L^c(\tilde{\Pi}) > \sup_{y \in \tilde{Z}} A^c(\tilde{\Psi}(y)). \quad (4.13)$$

The proof closely follows [Zho19, Proof of Lemma 5.8]. By (4.6), (4.4) and (4.2),

$$\sup_{y \in \tilde{Z}} A^c(\tilde{\Psi}(y)) \leq \sup_{y \in \tilde{Z}} M(\partial \tilde{\Psi}(y)) \leq \sup_{y \in \tilde{Z}} M(\partial \tilde{\Phi}'(y)) + \delta/2
\leq \sup_{x \in \tilde{X}} M(\partial \tilde{\Phi}(x)) + \delta/2 \leq \omega_k + \delta. \quad (4.14)$$

Let $\{\tilde{\Psi}_i : S\tilde{X} \rightarrow (C(M), F)\}_{i=1}^{\infty}$ be an arbitrary element in $\tilde{\Pi}$. There exists $H_i : S\tilde{X} \times [0, 1] \rightarrow (C(M), F)$ such that $H_i(-, 0) = \tilde{\Psi}$, $H_i(-, 1) = \tilde{\Psi}_i$ and

$$\limsup_{i \rightarrow \infty} \sup_{y \in \tilde{Z}} \{F(H_i(y,s), \tilde{\Psi}(y)) : y \in \tilde{Z}, s \in [0, 1] \} = 0. \quad (4.15)$$

We define $\{\tilde{\Psi}_i^+ : C_+ \rightarrow (C(M), F)\}_{i=1}^{\infty}$ as follows.

$$\tilde{\Psi}_i^+(y) = \begin{cases} \tilde{\Psi}(y) & \text{if } y \in \tilde{X} ; \\
H_i(y, 2t) & \text{if } y = (x, t) \in \tilde{Z} \cap C_+ ; \\
\tilde{\Psi}_i(y) & \text{if } y \in C_+ \setminus \tilde{Z} . \end{cases} \quad (4.16)$$

$\rho|_{C_+} : C_+ \rightarrow Y$ is a quotient map. Since $\tilde{\Psi}$ is $\mathbb{Z}_2$-equivariant, $\tilde{\Psi}_i^+$ descends to a map $\Psi_i^+ : Y \rightarrow \mathbb{Z}_n(M; F; \mathbb{Z}_2)$ such that $\Psi_i^+ \circ (\rho|_{C_+}) = \partial \circ \tilde{\Psi}_i^+$. Moreover, $\Psi_i^+$ is homotopic to $\Psi$ (in
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the $F$-topology) and hence, is a $(k + 1)$-sweepout. $\Psi^*_i$ has no concentration of mass as it is continuous in the $F$-topology. Therefore,

$$\sup_{y \in C_+} M(\partial \Psi^*_i(y)) = \sup_{y \in \tilde{Y}} M(\Psi^*_i(y)) \geq \omega_{k+1}. \quad (4.17)$$

However, as noted in (4.14),

$$\sup_{y \in \tilde{Z} \cap C_+} M(\partial \tilde{\Psi}(y)) \leq \omega_k + \delta < \omega_{k+1} \quad \text{(using (4.1))};$$

hence, (4.15) and (4.16) imply that if $i$ is sufficiently large,

$$\sup_{y \in \tilde{Z} \cap C_+} M(\partial \tilde{\Psi}(y)) < \omega_{k+1} \quad (4.18)$$

as well. Since $\tilde{\Psi}_i$ coincides with $\tilde{\Psi}_i$ on $C_+ \setminus \tilde{Z}$, (4.17) and (4.18) imply that for $i$ sufficiently large,

$$\sup_{y \in C_+ \setminus \tilde{Z}} M(\partial \tilde{\Psi}_i(y)) \geq \omega_{k+1} \quad \text{implies} \quad \sup_{y \in S\tilde{X}} M(\partial \tilde{\Psi}_i(y)) \geq \omega_{k+1} - c \cdot \text{Vol}(g). \quad (4.19)$$

Since (4.19) holds for any $\{\tilde{\Psi}_i\}_{i=1}^{\infty} \in \tilde{\Pi}$, using (4.1) and (4.14) we conclude that

$$L^c(\tilde{\Pi}) \geq \omega_{k+1} - c \cdot \text{Vol}(g) > \omega_k + \delta \geq \sup_{y \in \tilde{Z}} A^c(\tilde{\Psi}(y)). \quad (4.20)$$

**Part 5.** By Theorem 3.5 and (4.13), there exists $\Omega \in C(M)$ such that $\partial \Omega$ is a closed $c$-CMC (with respect to the inward unit normal) hypersurface with optimal regularity and $A^c(\Omega) = L^c(\tilde{\Pi})$. (4.20) implies that

$$A^c(\Omega) = L^c(\tilde{\Pi}) > \omega_k + \delta > \omega_k. \quad (4.21)$$

Moreover,

$$A^c(\Omega) = L^c(\tilde{\Pi}) \leq \sup_{(x,t) \in S\tilde{X}} A^c(\tilde{\Psi}(x,t)) \leq \sup_{(x,t) \in S\tilde{X}} M(\partial \tilde{\Psi}(x,t)) \leq \sup_{(x,t) \in S\tilde{X}} M(\partial \tilde{\Psi}(x,t)) + \delta/2 \quad \text{(by (4.6))}$$

$$\leq \sup_{x \in \tilde{X}} M(\partial \tilde{\Psi}(x)) + \sup_{t \in [0,1]} M(\partial \Lambda(t)) + \delta/2 \quad \text{(by (4.4) and Proposition 4.1)}$$

$$\leq \omega_k + \delta/2 + W_0 + \eta/2 - \delta + \delta/2 \quad \text{(by (4.2) and (4.3))}$$

$$< \omega_k + W_0 + \eta.$$  

This finishes the proof of Theorem 1.1. □
Proof of Theorem 1.2. We start with the following lemma.

Lemma 4.4. Let \( \Omega_1, \Omega_2 \in C(M) \) and \( c > 0 \) such that for \( i = 1, 2 \), \( \partial \Omega_i \) is a closed \( c \)-CMC (with respect to the inward unit normal) hypersurface with optimal regularity and \( A^c(\Omega_1) \neq A^c(\Omega_2) \). Then \( \partial \Omega_1 \neq \partial \Omega_2 \).

Proof. Suppose \( \Omega_1 = \Omega_2 = \Sigma \). Since \( A^c(\Omega_1) \neq A^c(\Omega_2) \), \( \Omega_1 \neq \Omega_2 \). Hence, by the constancy theorem, \( \Omega_2 = M \setminus \Omega_1 \); this is impossible as by our hypothesis, the mean curvature vector of \( \Sigma \) points inside both \( \Omega_1 \) and \( \Omega_2 \).

Since \( \{\omega_p\}_{p=1}^\infty \) is a non-decreasing sequence of positive real numbers converging to \(+\infty\), for every \( m \in \mathbb{N} \), there exists a subsequence \( \{\omega_{p_i}\}_{i=1}^m \) such that

\[
\omega_{p_i} < \omega_{p_{i+1}} \quad \text{and} \quad \omega_{p_i} + W_0 < \omega_{p_{i+1}},
\]

for all \( i = 1, \ldots, m \). Let us define

\[
c^*(m) = \frac{1}{\operatorname{Vol}(g)} \min \{\omega_{p_{i+1}} - \omega_{p_i} : i = 1, \ldots, m\}.
\] (4.22)

If \( 0 < c < c^*(m) \), applying Theorem 1.1 for \( k = p_i, i = 1, \ldots, m \), we conclude that there exists \( \Omega_i \in C(M) \), such that \( \partial \Omega_i \) is a closed \( c \)-CMC (with respect to the inward unit normal) hypersurface with optimal regularity and

\[
\omega_{p_i} < A^c(\Omega_i) < \omega_{p_{i+1}}.
\]

Therefore, using Lemma 4.4, \( \{\partial \Omega_i\}_{i=1}^m \) are distinct \( c \)-CMC hypersurfaces.

To prove the more quantitative statement, we will use the growth estimate of the volume spectrum (2.1). From the proof of Theorem 1.1, one can obtain the following inequality.

\[
\omega_{k+1} \leq \omega_k + W_0 \quad \forall k \in \mathbb{N}.
\] (4.23)

Let us briefly repeat the argument here. We fix \( \delta > 0 \). There exists a \( k \)-sweepout \( \Phi : X \to \mathcal{Z}_n(M; F; \mathbb{Z}_2) \) with no concentration of mass such that \( \sup \{M(\Phi(x)) : x \in X\} \leq \omega_k + \delta \). Using the notation from the Proof of Theorem 1.1, Part 1, there exists a double cover \( \pi : \tilde{X} \to X \) such that \( \pi \) has a lift \( \tilde{\Phi} : \tilde{X} \to (C(M), F) \). Let \( \Lambda : [0, 1] \to (C(M), F) \) be such that \( \Lambda(0) = M, \Lambda(1) = \emptyset, \partial \circ \Lambda : [0, 1] \to \mathcal{Z}_n(M; F; \mathbb{Z}_2) \) has no concentration of mass and \( \sup \{M(\partial \Lambda(t)) : t \in [0, 1]\} \leq W_0 + \delta \). We define the map \( \tilde{\Psi} : S\tilde{X} \to (C(M), F) \) as follows (cf. (4.4)).

\[
\tilde{\Psi}(x, t) = \begin{cases}
\tilde{\Phi}(x) \cap \Lambda(t) & \text{if } (x, t) \in C_+; \\
M - (\tilde{\Phi}(\tilde{T}(x)) \cap \Lambda(-t)) & \text{if } (x, t) \in C_-.
\end{cases}
\]
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$\Psi$ descends to a map $\Psi: Y \rightarrow Z_n(M;F;Z_2)$. As proved in Part 2 (using Proposition 4.1), $\Psi$ has no concentration of mass. Moreover, as argued in Part 3, $\Psi$ is a $(k+1)$-sweepout; hence,

$$\sup \{M(\Psi(y)) : y \in Y \} \geq \omega_{k+1}. \quad (4.24)$$

Further, as estimated in Part 5 (using Proposition 4.1),

$$\sup \{M(\Psi(y)) : y \in Y \} \leq \omega_k + W_0 + 2\delta. \quad (4.25)$$

Since $\delta > 0$ is arbitrary, (4.24) and (4.25) imply (4.23).

As a consequence of (4.23), for all $N \geq \omega_1$, there exist $q, r \in \mathbb{N}$ such that $\omega_q \in [N, N + W_0)$ and $\omega_r \in [N + 2W_0, N + 3W_0)$. Using (2.1),

$$K_0 r^{n+1} \leq \omega_r < N + 3W_0 \implies r < \left(\frac{N + 3W_0}{K_0}\right)^{n+1}.$$ 

Hence, there exists $s \in \mathbb{N}$, $q \leq s \leq r$, such that

$$\omega_{s+1} - \omega_s \geq \frac{\omega_r - \omega_q}{r - q} > W_0 \left(\frac{N + 3W_0}{K_0}\right)^{-(n+1)}.$$ 

Therefore, by Theorem 1.1, if

$$0 < c\, \text{Vol}(g) \leq W_0 \left(\frac{N + 3W_0}{K_0}\right)^{-(n+1)},$$

there exists $\Omega \in \mathcal{C}(M)$, such that $\partial\Omega$ is a closed $c$-CMC (with respect to the inward unit normal) hypersurface with optimal regularity and

$$N \leq \omega_s < A^c(\Omega) < \omega_s + W_0 + (N + 3W_0 - \omega_s) = N + 4W_0.$$ 

We note that $W_0 \geq \omega_1$. Setting $N = (4i - 3)W_0$, $i = 1, \ldots, m$ in the above argument, we conclude that if

$$0 < c\, \text{Vol}(g) \leq W_0 \left(\frac{4iW_0}{K_0}\right)^{-(n+1)},$$

there exists $\Omega_i \in \mathcal{C}(M)$, such that $\partial\Omega_i$ is a closed $c$-CMC (with respect to the inward unit normal) hypersurface with optimal regularity and

$$(4i - 3)W_0 < A^c(\Omega_i) < (4i + 1)W_0.$$ 

Further, from the one parameter min-max construction of Zhou and Zhu [ZZ19], for any $c > 0$ there exists $\Omega_0 \in \mathcal{C}(M)$ such that $\partial\Omega_0$ is a closed $c$-CMC (with respect to the inward unit normal) hypersurface with optimal regularity and $A^c(\Omega_0) \leq W_0$. Therefore, if

$$0 < c\, \text{Vol}(g) \leq W_0 \left(\frac{4mW_0}{K_0}\right)^{-(n+1)}, \quad (4.26)$$

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by Lemma 4.4, there exist at least \((m + 1)\) many closed \(c\)-CMC hypersurfaces in \((M, g)\). This implies the number of closed \(c\)-CMC hypersurfaces in \((M, g)\) is at least

\[
\left\lfloor \frac{K_0}{4W_0} \left( \frac{W_0}{c\text{Vol}(g)} \right)^{\frac{1}{n+1}} \right\rfloor + 1,
\]

where for \(x \in \mathbb{R}^+\), \(\lfloor x \rfloor\) is the largest (non-negative) integer \(\leq x\) and \(K_0\) is the constant appearing in (2.1).

**Remark 4.5.** For \(m \in \mathbb{N}\), the value of \(c^*(m)\), obtained from (4.22) for an optimal subsequence \(\{\omega_{p_i}\}_{i=1}^m\), will be greater than or equal to the value, obtained from the asymptotic formula (4.26).

**Example 4.6.** Let us consider \(S^3\) equipped with the round metric \(g_0\). By the one parameter min-max construction of Zhou and Zhu [ZZ19], for any \(c \in \mathbb{R}^+\) there exists \(\Omega \in \mathcal{C}(S^3)\), such that \(\partial \Omega\) is a closed \(c\)-CMC (with respect to the inward unit normal) hypersurface and \(\mathcal{A}(\Omega) \leq W_0(S^3, g_0) = 4\pi\). As proved by Nurser [Nur16], for the metric \(g_0\), \(\omega_1 = \cdots = \omega_4 = 4\pi\) and \(\omega_5 = 2\pi^2\). Therefore, by Theorem 1.1, if

\[
0 < c < \frac{\omega_5 - \omega_4}{\text{Vol}(g_0)} = \frac{2\pi^2 - 4\pi}{2\pi^2} = 1 - \frac{2}{\pi} \approx \frac{1}{3},
\]

there exists \(\Omega' \in \mathcal{C}(S^3)\) such that \(\partial \Omega'\) is a closed \(c\)-CMC (with respect to the inward unit normal) hypersurface and \(\mathcal{A}(\Omega') > \omega_4(S^3, g_0) = 4\pi\). Thus, for the metric \(g_0\) (with the notation as in Theorem 1.2), \(c^*(2)\) can be taken to be \(1 - 2/\pi\). Moreover, the widths are continuous functions of the metric [IMN18, MNS19] and \(W_0\) coincides with \(\omega_1\) when the ambient metric has positive Ricci curvature [Zho15, Zho17]. Therefore, if \(\bar{g}\) is a small perturbation of \(g_0\), \(c^*(2)\) for \(\bar{g}\) can be taken to be close to \(1 - 2/\pi\).

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