Counterexamples of the Geometrization Conjecture

Sze Kui Ng
Department of Mathematics, Hong Kong Baptist University, Hong Kong
Email: szekuing@hotmail.com

Abstract
In this paper we propose counterexamples to the Geometrization Conjecture and the Elliptization Conjecture.

Mathematics Subject Classification: 57M50, 57M27, 57N10, 57N16.

1 A counterexample of the Geometrization Conjecture

A version of the Thurston’s Geometrization Conjecture states that if a closed (oriented and connected) 3-manifold is irreducible and atoroidal, then it is geometric in the sense that it can either have a hyperbolic geometry or have a spherical geometry [1][2][3]. In this paper we propose counterexamples to this conjecture by using the Dehn surgery method of constructing closed 3-manifolds [4][5].

Let $K_{1RT}$ denote the right trefoil knot with framing 1. Let $K_r^E$ denote the figure-eight knot with framing $r$ where $r = \frac{p}{q}$ is a rational number ($p$ and $q$ are co-prime integers) such that $r > 4$.

We then consider a Dehn surgery on the framed link $L = K_{1RT} \cup K_r^E$ where the linking is of the simplest Hopf link type.

We have that the Dehn surgery on $K_{1RT}$ gives the Poincaré sphere $M_{1RT}$ which is with spherical geometry and with a finite nontrivial fundamental group [1][2][4][6][7]. Then the Dehn surgery on $K_r^E$ gives a hyperbolic manifold $M_r^E$ [1][2][6][7]. We want to show that the 3-manifold $M_L$ obtained from surgery on $L$ is irreducible and atoroidal, and is not geometric. From this we then have that $M_L$ is a counterexample of the Geometrization Conjecture.

Let us first show that $M_L$ is irreducible and atoroidal. From [9] we have the following quantum invariant $\mathcal{W}(K_{1RT})$ of $M_{1RT}$:

$$\mathcal{W}(K_{1RT}) = R^2 R_1^{-1} R_2 W(C_1) W(C_2)$$

(1)

where the indexes of the $R$-matrices $R_1$ and $R_2$ are 1 and $-1$ respectively (These $R$-matrices are the monodromies of the Knizhnik-Zamolodchikov equation; the notation $W(K)$ denotes the generalized Wilson loop of a knot $K$ and is a quantum representation of $K$ [9]). Thus the indexes of $R_1$ and $R_2$ are nonzero and are different. In [9] we call this property as the maximal non-degenerate property.

Now let us consider the manifold $M_L$. Since $K_{1RT}$ and $K_r^E$ both have the maximal non-degenerate property we have that there is no degenerate degree of freedom for the quantum representation of $M_L$ by using the link $L$. From this we have that $L$ is a minimal link for the
Deln surgeries obtaining $M_L$ [9] (We shall later give more explanations on the definition of minimal link and the related theorems on the classification of 3-manifolds by quantum invariant of 3-manifolds). It follows that the quantum invariant of $M_L$ is given by the quantum representation of $L$ and is of the following form:

$$\check{\pi}(L) = P_L \check{\pi}(K_{RT}^1) \check{\pi}(K_E^r)$$

(3)

where $P_L$ denotes the linking part of the representation of $L$.

In this quantum invariant [3] of $M_L$ we have that $\check{\pi}(K_{RT}^1)$ and $\check{\pi}(K_E^r)$ representing $K_{RT}^1$ and $K_E^r$ respectively are independent of each other and that the framed knots $K_{RT}^1$ and $K_E^r$ are independent of each other in the sense that the framed knots $K_{RT}^1$ and $K_E^r$ do not wind each other in the form as described by the second Kirby move [4][8].

We have that the quantum invariant [3] of $M_L$ uniquely represents $M_L$ because $L$ is minimal (We shall explain this point in the next section). This means that there are no nontrivial symmetry transforming it to another representation of $M_L$ with two framed knots such that their quantum representations are different from the two quantum representations $\check{\pi}(K_{RT}^1)$ and $\check{\pi}(K_E^r)$ in [3].

Let us then first show that $M_L$ is irreducible. Since the quantum invariant [3] of $M_L$ uniquely represents $M_L$ and represents topological properties of $M_L$ we have that the linking part $P_L$ of [3] is a topological property of $M_L$ and thus cannot be eliminated. From this linking of $W(K_{RT}^1)$ and $\check{\pi}(K_E^r)$ in [3] we have that the invariant [3] of $M_L$ cannot be written as a free product form $\check{\pi}(K_{RT}^1) \check{\pi}(K_{RT}^2)$ or two unlinked framed knots $K_{RT}^1$ and $K_{RT}^2$ where each $\check{\pi}(K_{RT}^i), i = 1, 2$ gives a closed 3-manifold. From this we have that $M_L$ cannot be written as a connected sum of two closed 3-manifolds. This shows that $M_L$ is irreducible.

Then we want to show that $M_L$ is atoroidal. Since the toroidal property of a 3-manifold $M$ is about the existence of an infinite cyclic subgroup $Z \oplus Z$ in $\pi_1(M)$ and is a property derived from closed curves in $M$ only we have that this toroidal property is derived from framed knots only since framed knots are closed curves for constructing 3-manifolds. Now since $L$ is minimal we have that the representation [3] uniquely represents $M_L$ and thus it gives all the topological properties of $M_L$. From this we have that if $M_L$ has the toroidal property then this property can only be derived from the two framed knot components $K_{RT}^1$ and $K_E^r$. Now we have that the 3-manifolds $M_{RT}^1$ and $M_E^r$ are both atoroidal and that the fundamental group of $M_{RT}^1$ is finite [1][2][6][7]. Thus the two framed knot components $K_{RT}^1$ and $K_E^r$ do not give the toroidal property of $M_L$. This shows that $M_L$ does not have the toroidal property. Thus $M_L$ is atoroidal.

Let us explicitly compute the fundamental group $\pi_1(M_L)$ of $M_L$ to give another proof for that $M_L$ is atoroidal. We have that $L = K_{RT}^1 \cup K_E^r$ is of the Hopf link type. Thus by a computation similar to the computation of the link group of the Hopf link which is a direct product of the two knot groups of the two unknotted forming the Hopf link we have that the fundamental group $\pi_1(M_L)$ of $M_L$ is a direct product of the fundamental groups $\pi_1(M_{RT}^1)$ and $\pi_1(M_E^r)$:

$$\pi_1(M_L) = \pi_1(M_{RT}^1) \ast \pi_1(M_E^r)$$

(4)

where $\pi_1(M_{RT}^1) \ast \pi_1(M_E^r)$ denotes the direct product of the fundamental groups $\pi_1(M_{RT}^1)$ and $\pi_1(M_E^r)$. Now since the 3-manifolds $M_{RT}^1$ and $M_E^r$ are both atoroidal and that the fundamental group $\pi_1(M_{RT}^1)$ is finite we have that $\pi_1(M_L)$ does not contain a subgroup of the form $Z \oplus Z$. This shows that $M_L$ does not have the toroidal property. Thus $M_L$ is atoroidal.

Now since the quantum invariant [3] uniquely represents $M_L$ we have that the two components $\check{\pi}(K_{RT}^1)$ and $\check{\pi}(K_E^r)$ are topological properties of $M_L$. Then since $\check{\pi}(K_{RT}^1)$ (or $K_{RT}^1$) gives spherical geometry property to $M_L$ and $\check{\pi}(K_E^r)$ (or $K_E^r$) gives hyperbolic geometry property to $M_L$ we have that $M_L$ is not geometric. Indeed, since the two independent components $\check{\pi}(K_{RT}^1)$ and $\check{\pi}(K_E^r)$ of [3] represent the manifolds $M_{RT}^1$ and $M_E^r$ respectively (and thus represent the fundamental groups $\pi_1(M_{RT}^1)$ and $\pi_1(M_E^r)$ of $M_{RT}^1$ and $M_E^r$ respectively) we have that the fundamental group $\pi_1(M_L)$ of $M_L$ contains the direct product $\pi_1(M_{RT}^1) \ast \pi_1(M_E^r)$ of the fundamental groups $\pi_1(M_{RT}^1)$ and $\pi_1(M_E^r)$. Now let $\tilde{M_L}$ denote the universal covering space of $M_L$. Then we have that $\pi_1(M_L)$ acts isometrically on $\tilde{M_L}$. Now since $\pi_1(M_{RT}^1)$ of the Poincaré sphere $M_{RT}^1$ is not a
subgroup of the isometry group of the hyperbolic geometry $H^3$ and $\pi_1(M_E)$ is not a subgroup of
the isometry group of the spherical geometry $S^3$ we have that $\pi_1(M_{RT}) \ast \pi_1(M_E)$ is not a subgroup of
the isometry group of $H^3$ and is not a subgroup of the isometry group of $S^3$. Thus $\pi_1(M_L)$ is
not a subgroup of the isometry group of $H^3$ and is not a subgroup of the isometry group of $S^3$.
It follows that $M_L$ is not the hyperbolic geometry $H^3$ and is not the spherical geometry $S^3$. This
shows that $M_L$ is not geometric, as was to be proved. Now since $M_L$ is irreducible and atoroidal
and is not geometric we have that $M_L$ is a counterexample of the Geometrization Conjecture.

2 Minimal link and classification of closed 3-manifolds

In this section we give more explanations on the definition of minimal link and the related theorems
on the classification of closed 3-manifolds by quantum invariant used in the above counterexample.

We have the following theorem of one-to-one representation of 3-manifolds obtained from framed
knots $K^\mp$ [9]:

**Theorem 1** Let $M$ be a closed (oriented and connected) 3-manifold which is constructed by a
Dehn surgery on a framed knot $K^\mp$ where $K$ is a nontrivial knot and $M$ is not a lens space. Then
we have the following one-to-one representation of $M$:

$$\overline{W}(K^\mp) : = R^{2p}R_1^{-m}R_2^{-am}W(C_1)W(C_2)$$

(5)

where $m \neq 0$ ($m$ is also denoted by $m_1$ in [9]) is the index of a nontrivial knot (which may or may
not be the knot $K$ such that $M$ is also obtained from this knot by Dehn surgery) and $am \neq 0$ is
an integer related to $m, p$ and $q$ such that $am \neq m$ (Thus [9] is with the maximal non-degenerate
property).

We remark that if $M$ is a lens space we can also define a similar quantum invariant $\overline{W}(K^\mp)$
for $M$ which however is not of the above maximal non-degenerate form [9].

Let us then consider a 3-manifold $M$ which is obtained from a framed link $L$ with the minimal
number $n$ of component knots where $n \geq 2$ (where the minimal number $n$ means that if $M$ can
also be obtained from another framed link then the number of component knots of this framed
link must be $\geq n$). In this case we call $L$ a minimal link of $M$. From the generalized second Kirby
moves (which generalizes second Kirby move from integer to rational number [9] and for simplicity
we shall call them again as the second Kirby moves) we may suppose that $L$ is in the form that

the components $K_i^{-m}$, $i = 1, ..., n$ of $L$ do not wind each other in the form described by the second
Kirby move. In this case we say that this minimal $L$ is in the form of maximal non-degenerate
state where the degenerate property is from the winding of one component knot with the other
component knot by the second Kirby moves. Thus this $L$ has both the minimal and maximal
property as described. Then we want to find a one-to-one representation (or invariant) of $M$ from
this $L$. Let us write $W(L)$, the generalized Wilson loop of $L$, in the following form [9]:

$$W(L) = P_L \prod_i W(K_i^{-m})$$

(6)

where $P_L$ denotes a product of $R$-matrices acting on a subset of $\{W(K_i), W(K_{ic}), i = 1, ..., n\}$
where $W(K_i^{-m})$ are independent (This is from the form of $L$ that the component knots $K_i$ are
independent in the sense that they do not wind each other by the second Kirby moves). Then we
consider the following representation (or invariant) of $M$:

$$\overline{W}(L) : = P_L \prod_i \overline{W}(K_i^{-m})$$

(7)

where we define $\overline{W}(K_i^{-m})$ by [9] and they are independent. We then have the following theorem:
**Theorem 2** Let \( M \) be a closed (oriented and connected) 3-manifold which is constructed by a Dehn surgery on a minimal link \( L \) with the minimal number \( n \) of component knots (and with the maximal property). Then we have that (7) is a one-to-one representation (or invariant) of \( M \).

**Proof.** We want to show that (7) is a one-to-one representation (or invariant) of \( M \). Let \( L' \) be another framed link for \( M \) which is also with the minimal number \( n \) (and with the maximal property). Then we want to show \( \mathbf{W}(L) = \mathbf{W}(L') \).

For the case \( n = 1 \) this is true by the above theorem for manifolds \( M \) obtained from minimal framed knot \( K_{E}^{\pm} \).

Let us consider \( n \geq 2 \). Since the components of \( L \) do not wind each other as described by the second Kirby move we have that the components of \( L \) are independent of each other. Thus there is no nontrivial homeomorphism changing these components \( \mathbf{W}(K_{i}^{\pm}) \) except those homeomorphisms involving the second Kirby moves for the winding of the components of \( L \) with each other. Then under the second Kirby moves we have that the components of \( L \) wind each other and thus will reduce the independent degree of freedom to be less than \( n \). Thus to restore the degree of freedom to \( n \) these homeomorphisms must also contain the first Kirby moves of adding unknots with framing \( \pm 1 \). In this case these unknots can be deleted and thus \( L \) is not minimal and this is a contradiction. Thus there is no nontrivial homeomorphism changing the components \( \mathbf{W}(K_{i}^{\pm}) \) of \( \mathbf{W}(L) \) except those homeomorphisms consist of only the second Kirby moves for the winding of the components of \( L \) with each other.

Now suppose that \( \mathbf{W}(L) \neq \mathbf{W}(L') \). Then there exists nontrivial homeomorphism of changing \( L \) to \( L' \) for changing the components \( \mathbf{W}(K_{i}^{\pm}) \) of \( \mathbf{W}(L) \) to the components of \( \mathbf{W}(L') \). This is impossible since there are no nontrivial homeomorphism for changing these components \( \mathbf{W}(K_{i}^{\pm}) \) except those homeomorphisms consist of only the second Kirby moves for the winding of the components of \( L \) with each other. Thus \( \mathbf{W}(L) = \mathbf{W}(L') \).

Thus we have that (7) is a one-to-one representation (or invariant) of \( M \), as was to be proved.

As a converse to the above theorem let us suppose that the representation (7) uniquely represents \( M_{L} \) in the sense that there are no nontrivial symmetry transforming the \( n \) independent components of \( \mathbf{W}(L) \) to other \( n \) independent components of \( \mathbf{W}(L') \) where the link \( L' \) also gives the manifold \( M_{L} \). Then from the above proof we see that the link \( L \) is a minimal (and maximal) link for obtaining \( M_{L} \).

**Remark.** Let \( L \) be a minimal (and maximal) framed link. Then from the above proof we have that the components of \( L \) are independent of each other in the sense that if we transform a component framed knot of \( L \) to an equivalent framed knot by a homeomorphism then the other components of \( L \) are not affected by this transformation.

Now let us consider the framed link \( L = K_{RT}^{1} \cup K_{E}^{-1} \) in the above section. We have that the knot components \( K_{RT}^{1} \) and \( K_{E}^{-1} \) of \( L \) do not wind each other in the form as described by the second Kirby move. Thus we have that their corresponding quantum invariants \( \mathbf{W}(K_{RT}^{1}) \) and \( \mathbf{W}(K_{E}^{-1}) \) are independent. Then \( \mathbf{W}(K_{RT}^{1}) \) and \( \mathbf{W}(K_{E}^{-1}) \) are in the maximal non-degenerate form which is invariant under all homeomorphisms except the second Kirby moves which are excluded (Indeed for \( \mathbf{W}(K_{RT}^{1}) \) there is a homeomorphism transforming \( K_{RT}^{1} \) to \( K_{E}^{-1} \). Then the informations of these two frame knots are included in \( \mathbf{W}(K_{RT}^{1}) \) and thus \( \mathbf{W}(K_{RT}^{1}) \) is invariant under this homeomorphism. Then since \( \mathbf{W}(K_{RT}^{1}) \) is in the maximal non-degenerate form there are no degenerate degree of freedoms for other homeomorphisms except the second Kirby moves which reduce the degree of freedom of \( L \). Similarly for \( \mathbf{W}(K_{E}^{-1}) \). Thus \( L \) is a minimal (and maximal) link of \( M_{L} \) and the representation (7) is the quantum invariant of \( M_{L} \).
A counterexample of the Elliptization Conjecture

The above counterexample of the Geometrization Conjecture is with an infinite fundamental group. Let us in this section propose a counterexample which is with a finite fundamental group to the Geometrization Conjecture. This example is then also a counterexample of the Thurston’s Elliptization Conjecture which states that if a closed (oriented and connected) 3-manifold is irreducible and atoroidal and is with a finite fundamental group then it is geometric in the sense that it can have a spherical geometry \[ W \].

Let us consider a Dehn surgery on the framed link \( L = K_{RT}^1 \cup K_{RT}^1 \) where the linking \( \cup \) is of the simplest Hopf link type. We want to show that the 3-manifold \( M_L \) obtained from this surgery is a counterexample of the Elliptization Conjecture.

As similar to the above example we have that this \( L \) is minimal and the 3-manifold \( M_L \) is uniquely represented by the following quantum invariant:

\[
\overline{W}(L) = P_L \overline{W}(K_{RT}^1) \overline{W}(K_{RT}^1)
\]

where \( P_L \) denotes the linking part of the representation of \( L \).

Then as similar to the above example we have that this 3-manifold \( M_L \) is irreducible and atoroidal. Let us then show that \( M_L \) is with a finite fundamental group and is not geometric. Since the quantum invariant \( \overline{W} \) uniquely represents \( M_L \) we have that the two components \( \overline{W}(K_{RT}^1) \) are topological properties of \( M_L \). Then we have that the fundamental group \( \pi_1(M_L) \) of \( M_L \) contains the direct product \( \pi_1(M_{RT}) \ast \pi_1(M_{RT}) \).

Further as similar to the above example because \( L \) is of the Hopf link type we have that \( \pi_1(M_L) = \pi_1(M_{RT}) \ast \pi_1(M_{RT}) \). Now since the fundamental group \( \pi_1(M_{RT}) \) is finite we have that the fundamental group \( \pi_1(M_L) \) is also finite.

Now let \( \tilde{M}_L \) denote the universal covering space of \( M_L \). Then we have that \( \pi_1(M_L) \) acts isometrically on \( \tilde{M}_L \). We want to show that \( \tilde{M}_L \) is not the 3-sphere \( S^3 \). Suppose this is not true. Then since \( \pi_1(M_L) \) contains and equals to the direct product \( \pi_1(M_{RT}) \ast \pi_1(M_{RT}) \) we have that the direct product \( \pi_1(M_{RT}) \ast \pi_1(M_{RT}) \) is a subgroup of the isometry group of \( S^3 \). Now since \( S^3 \) is a fully isotropic manifold containing no boundary (\( S^3 \) is closed) there is no way to distinguish two identical but independent subgroups \( \pi_1(M_{RT}) \) of the isometry group of \( S^3 \). From this we have that the direct product \( \pi_1(M_{RT}) \ast \pi_1(M_{RT}) \) can only act on \( S^3 \times S^3 \) where each \( \pi_1(M_{RT}) \) acts on a different \( S^3 \) and cannot act on the same \( S^3 \) such that \( \pi_1(M_{RT}) \ast \pi_1(M_{RT}) \) acts on \( S^3 \) (Comparing to the hyperbolic case we have that the direct product of two subgroups of the isometry group of the hyperbolic geometry \( H^3 \) may act on \( H^3 \) since \( H^3 \) has nonempty boundary which can be used to distinguish two identical but independent subgroups of the isometry group of \( H^3 \)). Thus the direct product \( \pi_1(M_{RT}) \ast \pi_1(M_{RT}) \) is not a subgroup of the isometry group of \( S^3 \) (We can also prove this statement by the fact that \( \pi_1(M_{RT}) \) is a nonabelian subgroup of the rotation group \( O(4) \) which is the isometry group of \( S^3 \)). Indeed since \( \pi_1(M_{RT}) \) is nonabelian it must act on a space with dimension \( \geq 3 \). Thus \( \pi_1(M_{RT}) \ast \pi_1(M_{RT}) \) must act on a space with dimension \( \geq 6 \). Now \( O(4) \) can only act on a space with dimension \( 4 \) we have that \( \pi_1(M_{RT}) \ast \pi_1(M_{RT}) \) is not a subgroup of \( O(4) \). This is a contradiction. This contradiction shows that \( M_L \) is not the 3-sphere \( S^3 \). Thus \( M_L \) is not geometric. Now since \( M_L \) is irreducible and atoroidal and is with finite fundamental group and is not geometric we have that \( M_L \) is a counterexample of the Elliptization Conjecture.

References

[1] W. Thurston, The geometry and topology of 3-Manifolds, Princeton University, 1978.

[2] W. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. 6 (1982), 357-381.

[3] A. Casson and D. Jungreis, Convergence groups and Seifert fibered 3-manifolds, Invent. Math. 118 441-456 (1994).
[4] D. Rolfsen, *Knots and links*, 2nd edn, Publish or Perish (1990).

[5] W.B.R. Lickorish. *A representation of orientable combinatorial 3-manifolds*. Ann. of Math. 76 531-538 (1962).

[6] A. Hatcher and W. Thurston, *Incompressible surfaces in 2-bridge knot complements*, Inv. Math. 79 (1985) 225-246.

[7] M. Brittenham and Y.-Q. Wu, *The classification of exceptional Dehn surgeries on 2-bridge knots*, Comm. Anal. Geom. 9 (2001) 97-113.

[8] R. Kirby. *A calculus for framed links in $S^3$*, Invent. Math. 45 35-56 (1978).

[9] S. K. Ng, *Quantum invariant of 3-manifolds and Poincaré Conjecture*, [math.QA/0008103](http://arxiv.org/abs/math.QA/0008103).