Three ideas on magnetic mass

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I. INTRODUCTION

In this talk, I shall discuss some ideas on how a magnetic mass can be dynamically generated in a nonabelian gauge theory. The ultimate goal, of course, is to understand the thermal perturbative expansion of Yang-Mills (YM) theory, at least to the same extent as the zero-temperature theory. Let me start with some simple, well known, observations about the magnetic mass.

Why do we need a magnetic mass?

It is well known that the thermal perturbative expansion of Yang-Mills theory suffers from infrared divergences. One set of such divergences are associated with the so-called hard thermal loops [1]; these are of the “electric type” in the sense that they can be cured by taking account of the electric screening mass. The hard thermal loop effective action, in fact, gives a gauge-invariant definition of the electric mass and eventually a systematic way of reorganizing perturbation theory to include the electric screening. However, the perturbative expansion also contains other, “magnetic type”, of divergences which can be eliminated only if there is screening of magnetostatic interactions, or in other words, if there is a magnetic mass [2].

How can magnetic mass arise in Yang-Mills theories?

At a very heuristic level, it is easy to see why magnetic screening is to be expected. Nonabelian gauge theories have magnetic monopole-like field configurations which are thermally excited and such a monopole-antimonopole plasma can screen magnetic interactions just as the charges of the electrical type can screen electrostatic interactions. A more detailed argument can be made as follows. In the imaginary time formalism, with Matsubara frequencies \( \omega_n = 2\pi n T \), where \( T \) is the temperature, the gauge fields have a mode expansion as

\[
A_i(\vec{x}, x^0) = \sum_n A_{i,n}(\vec{x}) \exp(2\pi i n T x^0).
\]

At high temperatures and for modes of wavelength long compared to \( 1/T \), the modes with nonzero Matsubara frequencies are unimportant and the theory reduces to the theory of the \( \omega_n = 0 \) mode, viz., a three (Euclidean) dimensional Yang-Mills theory (or a (2+1)-dimensional theory in a Wick rotated version). Yang-Mills theories in three or (2+1) dimensions are expected to have a mass gap and this is effectively the magnetic mass of the (3+1)-dimensional theory at high temperature [3]. This gives the qualitative origin and a quantitative first approximation to the magnetic mass. The \( \omega_n \neq 0 \) modes can give small corrections to this mass.

Many indirect arguments have been proposed for the existence of a mass gap in \( YM_3 \) or \( YM_{2+1} \). For example, we can consider an \( SU(2) \) gauge theory spontaneously broken to \( U(1) \). This is effectively compact electrodynamics which has a mass gap. This feature could survive in the unbroken theory if one has sufficient smoothness as the parameters are relaxed towards the unbroken phase. The recent conjectured connexion between supergravity on anti-de Sitter (AdS) space and gauge theories on the boundary of this AdS space gives another way to, at least qualitatively, argue that there should be a mass gap [4].

Why is magnetic mass important?

The individual Feynman diagrams of thermal perturbation theory are divergent and therefore, for meaningful calculations, we need the screening masses. The best definition of continuum Yang-Mills theory at zero temperature is in terms of its perturbative expansion. There is a systematic way to obtain finite meaningful results for this case. At least, term by term, the expansion is well defined; questions of convergence, Borel summability, etc. are another matter. Ultimately, apart from questions of calculability, there is the question of principle: Does thermal perturbative YM theory exist at least to the same extent as the zero temperature limit exists? This is important since it may be the only way to define thermal YM theory. (A lattice definition is more difficult for the thermal case, especially for nonequilibrium situations. If we have a well-defined continuum theory, it can be extended, with some labor, to the nonequilibrium situations.)

I shall discuss three approaches: the self-consistent resummation of perturbation theory, a nonperturbative analysis of the mass gap in \( YM_{2+1} \) (which is also very interesting in its own right) and the possibility of modifying the thermal distribution to take account of screening.
II. SELF-CONSISTENT RESUMMATION

In this approach, we reorganize the perturbation expansion by adding and subtracting a mass term to the action. The general strategy is as follows. We discuss partly complementary to mine.)

\[ S_{Y_M} + m^2 S_m - \Delta S_m \]

where \( S_{Y_M} \) is the standard Yang-Mills part of the action and \( S_m \) is a mass term which must respect gauge invariance. \( \Delta \) is a parameter which is taken to have a loop expansion, viz., \( \Delta = \Delta^{(1)} + \Delta^{(2)} + \cdots \). Calculations will be done in a loop expansion wherein \( S_{Y_M} + m^2 S_m \) is taken to be the zeroth order term. Of course suitable gauge-fixing and ghost terms are to be added to the above action as usual. The quantum effective action or the generator of the 1-particle-irreducible graphs is then calculated as

\[ \Gamma = (S_{Y_M} + m^2 S_m) + (\gamma_1 - \Delta^{(1)}) S_m + \cdots \]

where the ellipsis stands for higher loop terms, terms with field dependences other than the structure given by \( S_m \) and ghost-dependent and gauge-fixing type terms. \( \gamma_1 \) is the coefficient of \( S_m \) generated by the one-loop diagrams with the rules given by \( S_{Y_M} + m^2 S_m \), similarly \( \gamma_2 \) corresponds to the two-loop contribution and so on. The \( \gamma_i \) are functions of \( m \). The parameter \( m \) is the entire dynamically generated magnetic mass by definition, and hence, we need the corrections to vanish. Thus we need \( \gamma_1 - \Delta^{(1)} = 0 \), \( \gamma_2 - \Delta^{(2)} = 0 \), etc. These conditions determine \( \Delta \). Finally, we do not want to change the theory, only rearrange and resum various terms. Thus \( \Delta \) must itself be \( m^2 \). We thus get

\[ \gamma_1 + \gamma_2 + \cdots = m^2 \]

This is a ‘gap equation’ which determines \( m \) to the order to which the calculation is performed.

We have to choose a specific mass term to implement this procedure. As mentioned in the introduction, the \( \omega_n = 0 \) mode is the important one and hence, as a first step, we can use a three-dimensional mass term and do a four-dimensional calculation. The mass term which Alexanian and I have used is given by

\[ S_m = \int d^4 \Omega \ K(A_n, A_n) \]

where \( A_n = \frac{1}{2} A_i n_i, A_n = \frac{1}{2} \bar{A}_i \bar{n}_i, A_i \) being the gauge potentials. \( n_i \) is a (complex) three-dimensional null vector of the form \( n_i = (\cos \theta \cos \varphi - i \sin \varphi, -\cos \theta \sin \varphi + i \cos \varphi, \sin \theta) \). \( d\Omega = \sin \theta d\theta d\varphi \) and denotes integration over the angles \( n_i \). \( K(A_n, A_n) \) is related to the Wess-Zumino-Witten action as well as the eikonal for a Chern-Simons theory and is mathematically very similar to the hard thermal loop effective action. I have discussed elsewhere the full expression for \( K(A_n, A_n) \) and some of its nice geometrical properties.

Carrying out the integration over \( n_1, n_2 \),

\[ S_m = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2} A^2_i (-k) A^2_j (k) (\delta_{ij} - \frac{k_i k_j}{k^2}) + \int \frac{A^2_i A^2_j A^2_k f^{abc} V_{ijk}}{i \delta (k + q + p)} \left[ \frac{1}{k^2 q^2 - (q \cdot k)^2} \right. \]

where \( \frac{1}{k^2 q^2 - (q \cdot k)^2} \)

\[ V_{ijk}(k, q, p) = - \left( \frac{i}{6} \right) \delta (k + q + p) \left[ \frac{1}{k^2 q^2 - (q \cdot k)^2} \right. \]

\[ \times \left[ \frac{q \cdot k}{k^2 q^2 - (q \cdot k)^2} \right. k_i k_j k_k \]

\[ + \left( \frac{k \cdot (q + k)}{q + k} \right) \left( q_k q_j k_k + q_k q_i k_j + q_j q_k k_i \right) \]

\[ -(q \leftrightarrow k) \]

The self-consistent procedure then gives, to one-loop order, a gap equation with the solution

\[ m \approx 1.2 \sqrt{e^2 N / 2\pi} \]

where \( e^2 = g^2 T \), \( g \) being the \( (3+1) \)-dimensional coupling constant.

The primary advantage of this approach is that it is perturbatively implementable. However, some criticisms have also been raised about this approach. First of all, similar calculations have been done with different mass terms. Buchmuller and Philipsen have considered a mass term generated by coupling to a Higgs scalar field (A similar mass term was also suggested by Cornwall (3)). They find a value which is \( (3/4) \)-th of our value. Jackiw and Pi have used a mass term of the form \( F_{ij} (D^2)^{-1} F_{ij} \) and find a complex mass (which is somewhat worrisome) (4). More recently, they have used a mass term quadratic in the gauge potentials; the gap equation part of the calculation is then equivalent to a “unitary gauge” limit of the Buchmuller-Philipsen calculation and gives the same value for the gap. The question then arises: Does it matter whether different mass terms give different answers? Could we take the point of view that one workable mass term proves the generation of the magnetic mass? After all, the self-consistent procedure mixes up different loop orders when solving the gap equation and hence there is no reason why different mass terms have to agree to the order calculations have been performed. An optimist might say that different mass terms could agree in a full calculation. Nevertheless, the fact that one can get different values for \( m \) is worrisome.
Another potential difficulty has been pointed out by Cornwall \[10\]. The residue $Z$ at the pole of the corrected propagator in our approach is about 180. This is a big change from $Z = 1$ at the tree level and suggests that one may run into difficulties, perhaps developing a tachyonic mass, at the two-loop level. And also, the two-loop corrections are not parametrically small in terms of powers of coupling. (If they are smaller than the one-loop contribution, it has to be simply due to numerical factors.) This makes it difficult to get a systematic expansion and solution for the gap equation, a point which has been emphasized by Jackiw and Pi \[8\]. On the positive side though, there is a recent two-loop analysis which seems to show that the two-loop contribution is only about 15 - 20% of the one-loop contribution \[11\].

So where do we stand on this problem at this time? I believe there are two distinct questions here. Some of the criticisms do apply to the estimation of the value of the magnetic mass. To be more confident, one needs to have some independent nonperturbative analysis, a question to which I shall return in the next section. However, there is also the question of including magnetic screening in thermal perturbation theory. To make the expansion well defined one needs to introduce an infrared cutoff in a systematic gauge-invariant way. The specific numerical value is not important to this question of principle; nor is it important for the practical calculation of some types of processes. It is not consistent in the course of a perturbative calculation to introduce an ad hoc infrared cutoff whose value is taken from a lattice or some other nonperturbative analysis. The self-consistent approach can be useful in such a context.

Finally, there is the question of how much the higher Matsubara frequencies as well as how much electrostatic screening can modify the result. This has been analyzed in references \[12\]; see also the article by P. Petreczky in these proceedings.

There can be one other way of introducing an infrared cutoff within a perturbative expansion. This would involve changing the distribution function for the gauge bosons, a possibility which needs to be explored to see if it can provide a better systematic scheme. I shall return to this in the last section.

III. NONPERTURBATIVE ANALYSIS OF $YM_{2+1}$

As I have mentioned in the introduction, the magnetic mass may be considered as the mass gap of $YM_3$ or $YM_{2+1}$ in a Wick rotated version. Thus analysis of $YM_{2+1}$ gives another approach to the magnetic mass. Of course, the existence of a mass gap in $YM_{2+1}$ or more generally, any kind of nonperturbative analysis of this theory, is also interesting in its own right. Here I shall report on a nonperturbative analysis of $YM_{2+1}$ which we have carried out over the last three years \[13\]-\[14\]. We do a Hamiltonian analysis, starting from first principles; quite simply, what we try to do is to write the Schrödinger equation for $YM_{2+1}$ and solve it. This is notoriously difficult in field theory in general, but in our case, we can make significant progress by using some results from two-dimensional conformal field theory.

Let us consider an $SU(N)$-gauge theory in the $A_0 = 0$ gauge. The gauge potential can be written as $A_i = -it^a A^a_i$, $i = 1, 2$, where $t^a$ are hermitian $N \times N$-matrices which form a basis of the Lie algebra of $SU(N)$ with $[t^a, t^b] = if^{abc} t^c$, $\text{Tr}(t^a t^b) = \frac{1}{2} g^{ab}$. Wavefunctions for physical states must be gauge-invariant and thus they are functions on the space of all gauge potentials modulo gauge transformations, i.e., on the gauge-invariant configuration space $C$. It is therefore necessary to make a change of variables to gauge-invariant quantities. We are not just interested in a formal change of variables. We will need to calculate explicitly the Hamiltonian in terms of the new variables, the Jacobian of which needs to be explored to see if it can provide a better systematic scheme. I shall return to this in the last section.

Result 1: The volume measure on the gauge-invariant configuration space (which is the measure of integration for the inner product of wavefunctions, etc., and then construct eigenstates of the Hamiltonian. Thus we need a parametrization of the gauge fields such that these steps can be explicitly carried out. We start by combining the spatial coordinates $x_1, x_2$ into the complex combinations $z = x_1 - i x_2, \bar{z} = x_1 + i x_2$ with the corresponding components for the potential $A \equiv A_2 = \frac{1}{2} (A_1 + i A_2), \quad A \equiv A_\bar{z} = \frac{1}{2} (A_1 - i A_2) = -(A_z)\dagger$. The parametrization we use is then given by

$$A_z = -\partial_z M M^{-1}, \quad A_{\bar{z}} = M^{\dagger -1} \partial_{\bar{z}} M^{\dagger}$$

(8)

Here $M, M^{\dagger}$ are complex $SL(N, C)$-matrices. Such a parametrization is possible and is standard in many discussions of two-dimensional gauge fields. A particular advantage of this parametrization is the way gauge transformations are realized. A gauge transformation $A_1 \rightarrow A_1^{(g)} = g^{-1} A_1 g + g^{-1} \partial_1 g, \quad g(x) \in SU(N)$ is obtained by the transformation $M \rightarrow M^{(g)} = g M$. The gauge-invariant degrees of freedom are parametrized by the hermitian matrix $H = M^{\dagger} M$, which may be thought of as elements of $SL(N, C) / SU(N)$. Physical state wavefunctions are functions of $H$.

I shall now go through the main results of our analysis: details of the calculations may be found in references \[13\]-\[14\]. Some of the technical details are also explained in the article by D. Karabali in these proceedings.

In this equation, $d\mu(H)$ is the Haar measure for $SL(N, C) / SU(N)$, given explicitly by $d\mu(H) = (\text{det} r)|\delta \varphi| \text{ with } H$ parametrized in terms of a real field $\varphi^a(x)$ and $H^{-1} \delta H = \delta \varphi^a r_{ab}(\varphi) t_b$. $S(H)$ is the Wess-Zumino-Witten (WZW) action for $H$ given by
\[
S(H) = \frac{1}{2\pi} \int \text{Tr}(\partial H \partial H^{-1}) + \frac{i}{12\pi} \int e^{\mu\nu\alpha} \text{Tr}(H^{-1} \partial_\mu HH^{-1} \partial_\nu HH^{-1} \partial_\alpha H) \tag{10}
\]

This calculation of the measure of integration for the inner products does not suffer from Gribov or gauge-fixing ambiguities; for more details, see [3] and also the article by Karabali in these proceedings. (Incidentally, \(K(A_n, A_0)\) in the mass term used in the last section is the same as \(S(H)\) above with the identification \(z = x \cdot n, \bar{z} = x \cdot \bar{n}\) and with all fields being three-dimensional.)

The inner product for states \(|1\rangle\) and \(|2\rangle\), represented by the wavefunctions \(\Psi_1\) and \(\Psi_2\), is given by

\[
\langle 1|2 \rangle = \int d\mu(H) e^{2NS(H)} \Psi_1^\dagger \Psi_2 \tag{11}
\]

Observables are ultimately matrix elements of operators, i.e., they are inner products. By (11), this gives a correlator of the two-dimensional WZW model for \(H\), which is a conformal field theory. In other words, we have the exact mapping

\[
\{ \text{Matrix elements of } Y M_{2+1} \} = \{ \text{Correlators of a hermitian WZW model} \}
\]

Result 2: Normalizable wavefunctions are functions of the current \(J = (N/\pi) \partial_2 H H^{-1}\).

In other words, the wavefunctions are more restricted than being just functions of \(H\); they can only depend on \(H\) via the specific combination in \(J\). This result follows from considerations of integrable representations in the WZW theory via a mapping between unitary and hermitian models [4]. The quantity \(J\) may be considered as the gauge-invariant definition of a gluon. Since wavefunctions depend only on \(J\) by this result, we can write the Hamiltonian in terms of \(J\) and functional derivatives with respect to \(J\).

This result is also consistent with the fact that the Wilson loop operator, which provides a complete description of gauge-invariant observables, can be constructed from \(J\) alone as

\[
W(C) = \text{Tr} \rho e^{-\oint_C (A dz + \bar{A} d\bar{z})} = \text{Tr} \rho e^{(\sigma/\rho_0)} \oint_C J
\]

Result 3: The \(YM_{2+1}\) Hamiltonian is given by

\[ H = T + V \tag{13a} \]

\[
T = \frac{e^2}{2} \int E^a_i E_i^a
\]

\[
m \left[ \int u J^a(u) \frac{\delta}{\delta J^a(u)} + \int \Omega_{ab}(u, v) \frac{\delta}{\delta J^a(u)} \frac{\delta}{\delta J^b(v)} \right] \tag{13b}
\]

\[
V = \frac{1}{2e^2} \int B^a B^a = \frac{\pi}{mN} \int \bar{\partial} J_a(x) \bar{\partial} J_a(x) \tag{13c}
\]

where \(m = e^2 N/2\pi\) and

\[
\Omega_{ab}(u, v) = \frac{N}{\pi^2 (u - v)^2} - \frac{i \int_{ab} J^c(u)}{\pi (u - v)} \tag{14}
\]

Result 4: The kinetic energy operator \(T\) obviously has the ground state wavefunction \(\Phi_0 = 1\). With the inner product (11), this is normalizable. The term \(J^a(u) \frac{\delta}{\delta J^a(u)}\) in \(T\) counts the number of currents \(J\) and each \(J\) contributes \(m = e^2 N/2\pi\) to the eigenvalue. In particular

\[
T \cdot J^a = \frac{e^2 N}{2\pi} J^a \tag{15}
\]

This is again an exact result, indicating that we should expect a gap of \(m = e^2 N/2\pi\) for the gluons.

Result 5: The vacuum wavefunction \(\Psi_0\) is given by

\[
\Psi_0 = \exp \left[ -\frac{1}{2e^2} \int_{x,y} B_a(x) \left( \frac{1}{m + \sqrt{m^2 - \nabla^2}} \right)_{x,y} B_a(y) \right] + 3J - \text{and higher } J - \text{terms} \tag{16}
\]

\(\Phi_0 = 1\) is the ground state wavefunction for \(T\), but it is not an eigenstate for \(H = T + V\). By treating \(V\) perturbatively, we can obtain a \(1/e^2\) or \(1/m\)-expansion of the full vacuum wavefunction. Summing up all terms with two powers of \(J\) in \(\log \Psi_0\), we get the result 5 [15].

The first term in (16) has the correct (perturbative) high momentum limit. Thus although we started with the high \(m\) or low momentum limit, the result (16) does match onto the perturbative limit. The higher terms are also small for the low momentum limit.

Result 6: The expectation value of the Wilson loop is given, for the fundamental representation, by

\[
\langle W_F(C) \rangle = \text{constant} \ \exp \left[ -\sigma A_C \right] \tag{17}
\]

where \(A_C\) is the area of the loop \(C\) and \(\sigma\), which may be identified as the string tension, is obtained as

\[
\sqrt{\sigma} = e^2 \sqrt{\frac{N^2 - 1}{8\pi}} \tag{18}
\]

This is a prediction of our analysis starting from first principles with no adjustable parameters. Notice that the dependence on \(e^2\) and \(N\) is in agreement with large-\(N\) expectations, with \(\sigma\) depending only on the combination \(e^2 N\) as \(N \to \infty\). Formula (18) gives the values \(\sqrt{\sigma/e^2} = 0.345, 0.564, 0.772, 0.977\) for \(N = 2, 3, 4, 5\).
There are estimates for $\sigma$ based on Monte Carlo simulations of lattice gauge theory. The most recent results for the gauge groups $SU(2)$, $SU(3)$, $SU(4)$ and $SU(5)$ are $\sqrt{\sigma/e^2} = 0.335, 0.553, 0.758, 0.966$ [17]. We see that our result agrees with the lattice result to within $\sim 3\%$.

Given this result, we might venture to say that we have a viable approach to nonperturbative phenomena in $YM_{2+1}$ or $YM_3$.

A comment on the accuracy of our calculation is in order. Why is the result so good? The string tension is determined by large area loops and for these, it is the long distance part of the wavefunction which contributes significantly. In this limit, the $3J$- and higher terms in $[15]$ are small compared to the quadratic term. The latter leads to the result [18].

**Result 7**: To lowest order in a power series expansion for the current $J = (N/\pi)\partial \varphi_a a$, the Hamiltonian can be simplified as

$$\mathcal{H} \approx \frac{1}{2} \int \left[ \frac{\delta^2}{\delta \varphi_a^2}(\vec{x}) + \varphi_a(\vec{x}) (m^2 - \nabla^2) \varphi_a(\vec{x}) \right] + ... \quad (19)$$

where $\varphi_a(\vec{k}) = \sqrt{Nkk/(2\pi m)} \varphi_a(\vec{k})$, in momentum space.

In arriving at this expression we have expanded the currents and also absorbed the WZW-action part of the measure into the definition of the wavefunctions, i.e., the operator $[19]$ acts on

$$\tilde{\Psi} = e^{NS(H)} \Psi \approx e^{-\frac{\delta S}{\delta \varphi_a}(\vec{x})} \int \partial_a \partial_a^* \Psi$$

(20)

The above equation shows that the propagating particles in the perturbative regime, where the power series expansion of the current is appropriate, have a mass $m = e^2N/2\pi$. This value can therefore be identified as the magnetic mass of the gluons as given by this nonperturbative analysis.

We thus see that the self-consistent method, which amounts to selective summation of terms in the perturbative expansion gives a value $\approx 1.2(e^2N/2\pi)$ while the above analysis, which starts from a high $e^2$-approximation and then connects to the perturbative limit gives a value $e^2N/2\pi$ for the magnetic mass. The two values are quite close to each other and also agree with other estimates. I expect that the actual numerical value of the magnetic mass should be very close to this number.

Finally, I would like to give a short intuitive argument for why there is a mass gap, which may help to understand the exact calculations reported above. The crucial ingredient is the measure of integration in the inner product $[11]$. Writing $\Delta E, \Delta B$ for the root mean square fluctuations of the electric field $E$ and the magnetic field $B$, we have, from the canonical commutation rules $[E_a^i, A_b^j] = -i\delta_{ij} \delta^{ab}$, $\Delta E \Delta B \sim k$, where $k$ is the momentum variable. This gives an estimate for the energy

$$\mathcal{E} = \frac{1}{2} \left( \frac{e^2k^2}{\Delta B^2} + \frac{\Delta B^2}{e^2} \right) \quad (21)$$

For low lying states, we minimize $\mathcal{E}$ with respect to $\Delta B^2$, $\Delta B_{min}^2 \sim e^2k$, giving $\mathcal{E} \sim k$. This corresponds to the standard photon. For the nonabelian theory, this is inadequate since $\langle \mathcal{H} \rangle$ involves the factor $e^{2NS(H)}$. In fact, $\langle \mathcal{H} \rangle = \int d\mu(H) e^{2NS(H)} \frac{1}{2}(e^2E^2 + B^2/e^2)$

(22)

In terms of $B$, the WZW action goes like $S(H) \approx \left[-(N/\pi)\frac{1}{2} \int B(1/k^2)B + ... \right]$, we thus see that $B$ follows a Gaussian distribution of width $\Delta B \approx \pi k^2/N$, for small values of $k$. This Gaussian dominates near small $k$ giving $\Delta B^2 \sim k^2(\pi/N)$. In other words, even though $\mathcal{E}$ is minimized around $\Delta B^2\sim k$, probability is concentrated around $\Delta B^2 \sim k^2(\pi/N)$. For the expectation value of the energy, we then find $\mathcal{E} \sim e^2N/2\pi + O(k^2)$. Thus the kinetic term in combination with the measure factor $e^{2NS(H)}$ could lead to a mass gap of order $e^2N$. The argument is not rigorous, but captures the essence of how a mass gap arises in our formalism.

**IV. MODIFYING THE BOSE DISTRIBUTION**

I now return to the question of thermal perturbation expansion for Yang-Mills theory. The nonperturbative analysis I have outlined can give a good understanding of the origin of the mass gap and even a value for it. However, for thermal perturbative $YM$ theory, we would like a scheme for magnetic screening which is perturbatively implementable. The selfconsistent scheme has this virtue, but is rather difficult from a calculational point of view. There is also the further difficulty of the two- and higher loop contributions not being parametrically smaller. In this section I propose an alternative method. In the previous analyses, we have concentrated on the spectrum, arguing for a mass gap. The thermal distribution of gluons, which is the other essential ingredient in any thermal calculation, was unchanged. It is well known that the ‘bad’ infrared behaviour of the Bose distribution of gluons will certainly be modified. The proportionality of $\varphi_a(\vec{k})$ acts

$$\varphi_a(\vec{k}) \sim \sqrt{Nkk/(2\pi m)} \varphi_a(\vec{k})$$

The above equation shows that the propagating particles in the perturbative regime, where the power series expansion of the current is appropriate, have a mass $m = e^2N/2\pi$. This value can therefore be identified as the magnetic mass of the gluons as given by this nonperturbative analysis.

We thus see that the self-consistent method, which amounts to selective summation of terms in the perturbative expansion gives a value $\approx 1.2(e^2N/2\pi)$ while the above analysis, which starts from a high $e^2$-approximation and then connects to the perturbative limit gives a value $e^2N/2\pi$ for the magnetic mass. The two values are quite close to each other and also agree with other estimates. I expect that the actual numerical value of the magnetic mass should be very close to this number.

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The maximization of this with equal a priori probabilities for the states and subject to the condition of fixed total energy leads to the standard Bose distribution. In other words, we can determine the most probable occupation numbers \( \bar{n}_k \) by
\[
\frac{\partial}{\partial \bar{n}_k} \log W[\{\bar{n}_k\}] - \beta(\sum_k n_k \bar{n}_k - E) = 0,
\]
where the term \( \beta(\sum_k n_k \bar{n}_k - E) \) enforces the condition of fixed total energy \( E \) via the Lagrange multiplier \( \beta \) which is eventually identified as \( 1/T \). A simple way to incorporate magnetic screening is to assign an a priori probability \( (1 - e^{-\omega/m})^n \) to the state and maximize \( W = \prod_k (1 - e^{-\omega/m})^{n_k} W[\{n_k\}] \). This leads to the distribution function
\[
\bar{n}_k = \frac{1}{e^{\omega/T} - 1 + \alpha(\omega)}
\]
\[
\alpha(\omega) = \frac{e^{\omega/T}}{e^{\omega/m} - 1}
\]
(24)
\( \bar{n}_k \) has good infrared behaviour with \( \bar{n}_k \to 0 \) as \( \omega \to 0 \); further, for \( \omega \gg m \), \( \bar{n}_k \) is exponentially close to the standard Bose distribution. The above distribution can be derived from the density matrix
\[
\rho = e^{-\omega a^\dagger a/T} e^{-\omega b^\dagger b/m} e^{b^\dagger a} e^{-a^\dagger b}
\]
(25)
a, \( a^\dagger \), \( b^\dagger \), \( b \) are the usual creation-annihilation operators. \( a, a^\dagger \) refer to the gluons, \( b, b^\dagger \) are similar, but have no direct meaning; they are just a mathematical device to obtain (24) in a simple way.

At first glance, there would seem to be many ways of modifying the a priori probabilities to get magnetic screening. But actually, the choices are rather limited. The modification (24) has the properties:
1) It has infrared cutoff.
2) Modification is exponentially small in the ultraviolet, so that moment calculations are not affected.
3) It is perturbatively implementable.
4) It corresponds to a simple density matrix, so that it is calculationally simple.
5) It has a gauge-invariant generalization to an interacting theory.

Regarding the last point, a generalization of (25) is given by
\[
\rho = e^{-H_{YM}/T} e^{-\tilde{H}_{YM}/m} e^{B^\dagger A} e^{-A^\dagger B}
\]
(26)
where \( H_{YM} \) is the Yang-Mills Hamiltonian, \( \tilde{H}_{YM} \) is similar but constructed out of the \( b^\dagger, b \) operators and \( A, B \) can be defined as eigenoperators of the corresponding Hamiltonians
\[
[H_{YM}, A] = -\omega A, \quad [\tilde{H}_{YM}, B] = -\omega B
\]
(27)
\( A, B \) are a generalization of \( a, b \) to the interacting case. By doing free energy calculations, I expect to determine the parameter \( m \) variationally. These calculations are currently under way.

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