FREE FIELD REALIZATION OF QUANTUM AFFINE SUPERALGEBRA $U_q(\hat{\mathfrak{sl}}(N|1))$

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Abstract

We construct a free field realization of the quantum affine superalgebra $U_q(\hat{\mathfrak{sl}}(N|1))$ for an arbitrary level $k \in \mathbb{C}$. 
1 Introduction

The free field approach [1] provides a powerful method to construct correlation functions of exactly solvable models. In this paper we construct a free field realization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ ($N \geq 2$) for an arbitrary level $k \in \mathbb{C}$. The level parameter $k$ plays an important role in representation theory. Free field realizations of an arbitrary level $k \in \mathbb{C}$ are completely different from those of level $k = 1$. In the case of level $k = 1$, free field realizations [2, 3, 4] have been constructed for quantum affine algebra $U_q(g)$ in many cases $g = (ADE)^{(r)}$ [4, 7], $(BC)^{(1)}$, $G_2^{(1)}$ [5, 6, 8], $\widehat{sl}(M|N)$, $osp(2|2)^{(2)}$ [9, 10, 11]. In the case of an arbitrary level $k \in \mathbb{C}$, free field realizations [12, 13, 14], have not yet been studied well for quantum affine algebra $U_q(g)$. In the case of an arbitrary level $k \in \mathbb{C}$, free field realizations have been constructed only for $U_q(\widehat{sl}(N))$ [16] and $U_q(\widehat{sl}(2|1))$ [17]. The purpose of this paper is to construct a free field realization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbb{C}$. The representation theories of the superalgebra are much more complicated than non-superalgebra and have rich structures [18, 19, 20, 21].

This paper is organized as follows. In section 2 we review the Chevalley realization of the quantum superalgebra $U_q(sl(N|1))$ [22] and the Drinfeld realization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ [23]. In section 3 we review the Heisenberg realization of quantum superalgebra $U_q(sl(N|1))$ [15] and construct a free field realization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbb{C}$. In appendix A we explain how to find the free field realization of affine $U_q(\widehat{sl}(N|1))$ from the Heisenberg realization $U_q(sl(N|1))$. In appendix B we summarize some useful formulae.

2 Quantum Affine Superalgebra $U_q(\widehat{sl}(N|1))$

In this section we review the Chevalley realization of the quantum superalgebra $U_q(sl(N|1))$ [22] and the Drinfeld realization of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ [23, 24] for $N = 2, 3, 4, \cdots$. We fix a complex number $q \neq 0, |q| < 1$. In what follows we use

\begin{align}
[x, y] &= xy - yx, \\
\{x, y\} &= xy + yx, \\
[a]_q &= \frac{q^a - q^{-a}}{q - q^{-1}}.
\end{align}
2.1 Quantum Superalgebra $U_q(sl(N\{1\}))$

Let us recall the definition of the quantum superalgebra $U_q(sl(N\{1\}))$ [22]. We set $\nu_1 = \nu_2 = \cdots = \nu_N = +, \nu_{N+1} = -$. The Cartan matrix $(A_{i,j})_{1 \leq i,j \leq N}$ of the Lie algebra $sl(N\{1\})$ is given by

$$A_{i,j} = (\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i\delta_{i,j+1} - \nu_{i+1}\delta_{i+1,j}. \quad (2.4)$$

The diagonal part is $(A_{i,i})_{1 \leq i \leq N} = (\underbrace{N-1}_{N-1}, \cdots, 2, 0)$.

**Definition 2.1** [22] The Chevalley generators of the quantum superalgebra $U_q(sl(N\{1\}))$ are

$$h_i, e_i, f_i \quad (1 \leq i \leq N). \quad (2.5)$$

Defining relations are

$$[h_i, h_j] = 0, \quad (2.6)$$

$$[h_i, e_j] = A_{i,j}e_j, \quad (2.7)$$

$$[h_i, f_j] = -A_{i,j}f_j, \quad (2.8)$$

$$[e_i, f_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \quad \text{for} \; (i, j) \neq (N, N), \quad (2.9)$$

$$\{e_N, f_N\} = \frac{q^{h_N} - q^{-h_N}}{q - q^{-1}}, \quad (2.10)$$

and the Serre relations

$$e_i e_j e_i - (q + q^{-1})e_i e_j e_i + e_j e_i e_i = 0 \quad \text{for} \; |A_{i,j}| = 1, i \neq N, \quad (2.11)$$

$$f_i f_j f_i - (q + q^{-1})f_i f_j f_i + f_j f_i f_i = 0 \quad \text{for} \; |A_{i,j}| = 1, i \neq N. \quad (2.12)$$

2.2 Quantum Affine Superalgebra $U_q(\widehat{sl}(N\{1\}))$

Let us recall the definition of the quantum affine superalgebra $U_q(\widehat{sl}(N\{1\}))$ [23]. The Cartan matrix $(A_{i,j})_{0 \leq i,j \leq N}$ of the affine Lie algebra $\widehat{sl}(N\{1\})$ is given by

$$A_{i,j} = (\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i\delta_{i,j+1} - \nu_{i+1}\delta_{i+1,j}. \quad (2.13)$$

Here we should read the suffixes $j$ of $\nu_j \mod(N+1)$, i.e. $\nu_0 = \nu_{N+1}$. Here the diagonal part is $(A_{i,i})_{0 \leq i \leq N} = (0, 2, \cdots, 2, 0)$. 

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Definition 2.2 \[23\] The Drinfeld generators of the quantum affine superalgebra \( U_q(\hat{\mathfrak{sl}}(N|1)) \) are

\[
x_{i,m}^\pm, \quad h_{i,m}, \quad c, \quad (1 \leq i \leq N, m \in \mathbb{Z}).
\]  

(2.14)

Defining relations are

\[
c : \text{central}, \quad [h_i, h_{j,m}] = 0, \tag{2.15}
\]

\[
[a_{i,m}, h_{j,n}] = \frac{[A_{i,m}]}{m} q^{c|m|} q^{-c|m|} \delta_{m+n,0} \quad (m, n \neq 0), \tag{2.16}
\]

\[
[h_i, x_j^+(z)] = \pm A_{i,j} x_j^+(z), \tag{2.17}
\]

\[
[h_{i,m}, x_j^+(z)] = \frac{[A_{i,m}]}{m} q^{-c|m|} z^m x_j^+(z) \quad (m \neq 0), \tag{2.18}
\]

\[
[h_{i,m}, x_j^-(z)] = -\frac{[A_{i,m}]}{m} z^m x_j^-(z) \quad (m \neq 0), \tag{2.19}
\]

\[
(x_1 - q^{\pm A_{i,j}} z_2) x_i^\pm(z_1) x_j^\pm(z_2) = (q^{\pm A_{i,j}} z_1 - z_2) x_j^\pm(z_2) x_i^\pm(z_1) \quad \text{for } |A_{i,j}| \neq 0, \tag{2.20}
\]

\[
x_i^\pm(z_1) x_j^\pm(z_2) = x_j^\pm(z_2) x_i^\pm(z_1) \quad \text{for } |A_{i,j}| = 0, (i, j) \neq (N, N), \tag{2.21}
\]

\[
\{x_N^+(z_1), x_N^-(z_2)\} = 0, \tag{2.22}
\]

\[
[x_i^+(z_1), x_j^-(z_2)] = \frac{\delta_{i,j}}{(q - q^{-1}) z_1 z_2} \left( \delta(q^{-c} z_1/z_2) \psi_i^+(q^{1/2} z_2) - \delta(q^c z_1/z_2) \psi_i^-(q^{-1/2} z_2) \right),
\]

\text{for } (i, j) \neq (N, N), \tag{2.23}

\[
\{x_N^+(z_1), x_N^-(z_2)\} = \frac{1}{(q - q^{-1}) z_1 z_2} \left( \delta(q^{-c} z_1/z_2) \psi_N^+(q^{1/2} z_2) - \delta(q^c z_1/z_2) \psi_N^-(q^{-1/2} z_2) \right), \tag{2.24}
\]

\[
(x_i^+(z_1) x_j^+(z_2) x_j^-(z) - (q + q^{-1}) x_i^+(z_1) x_j^+(z) x_i^+(z_2) + x_j^+(z) x_i^+(z_1) x_j^+(z_2))
\]

\[
(1 - z_1 \leftrightarrow z_2) = 0 \quad \text{for } |A_{i,j}| = 1, \quad i \neq N. \tag{2.25}
\]

where we have used \( \delta(z) = \sum_{m \in \mathbb{Z}} z^m \). Here we have used the abbreviation \( h_i = h_{i,0} \). We have set the generating function

\[
x_j^\pm(z) = \sum_{m \in \mathbb{Z}} x_{j,m}^\pm z^{-m-1}, \tag{2.26}
\]

\[
\psi_i^+(q^{1/2} z) = q^{h_i} \exp \left( (q - q^{-1}) \sum_{m > 0} h_{i,m} z^{-m} \right), \tag{2.27}
\]

\[
\psi_i^-(q^{-1/2} z) = q^{-h_i} \exp \left( -(q - q^{-1}) \sum_{m > 0} h_{i,-m} z^m \right). \tag{2.28}
\]

We changed the gauge of boson \( h_{i,m} \) from those of [23] and revised a misprint (2.22) in [23].
3 Free Field Realization

In this section we review the Heisenberg realization of $U_q(sl(N|1))$ [17] and construct a free field realization of the quantum affine superalgebra $U_q(\hat{sl}(N|1))$ for an arbitrary level $k \in \mathbb{C}$.

3.1 Heisenberg Realization

Let us recall the Heisenberg realization of quantum superalgebra $U_q(sl(N|1))$ [17]. We introduce the coordinates $x_{i,j}$, $(1 \leq i < j \leq N + 1)$ by

$$x_{i,j} = \begin{cases} z_{i,j} & (1 \leq i < j \leq N), \\ \theta_{i,j} & (1 \leq i \leq N, j = N + 1). \end{cases} \tag{3.1}$$

Here $z_{i,j}$ are complex variables and $\theta_{i,N+1}$ are the Grassmann odd variables that satisfy $\theta_{i,N+1} \theta_{i,N+1} = 0$ and $\theta_{i,N+1} \theta_{j,N+1} = -\theta_{j,N+1} \theta_{i,N+1}$, $(i \neq j)$. We introduce the differential operators $\partial_{i,j} = x_{i,j} \frac{\partial}{\partial x_{i,j}}$, $(1 \leq i < j \leq N + 1)$. We fix parameters $\lambda_i \in \mathbb{C}$, $(1 \leq i \leq N)$. We set the differential operators $H_i, E_i, F_i$, $(1 \leq i \leq N)$ by

$$H_i = \sum_{j=1}^{N} H_{i,j}, \quad E_i = \sum_{j=1}^{i} E_{i,j}, \quad F_i = \sum_{j=1}^{N} F_{i,j}. \tag{3.2}$$

Here we have set

$$H_{i,j} = \begin{cases} \nu_i \partial_{j,i} - \nu_{i+1} \partial_{j,i+1} & (1 \leq j \leq i - 1), \\ \lambda_i - (\nu_i + \nu_{i+1}) \partial_{i,i+1} & (j = i), \\ \nu_{i+1} \partial_{i+1,j+1} - \nu_i \partial_{i,j+1} & (i + 1 \leq j \leq N). \end{cases} \tag{3.3}$$

$$E_{i,j} = \frac{x_{j,i}}{x_{j,i+1}} [\partial_{j,i+1}]_q q^{\sum_{l=i+1}^{j-1} (\nu_i \partial_{l,i} - \nu_{i+1} \partial_{l,i+1})}, \tag{3.4}$$

$$F_{i,j} = \begin{cases} \nu_i \frac{x_{j,i+1}}{x_{j,i}} [\partial_{j,i}]_q q^{\sum_{l=j+1}^{i} (\nu_i \partial_{l,i+1} - \nu_{i+1} \partial_{l,i})} & (1 \leq j \leq i - 1), \\ x_{i,i+1} \left[ \lambda_i - \nu_i \partial_{i,i+1} - \sum_{l=i+2}^{N+1} (\nu_i \partial_{l,i} - \nu_{i+1} \partial_{l,i+1}) \right]_q & (j = i), \\ -\nu_{i+1} \frac{x_{i+1,j+1}}{x_{i+1,i+1}} [\partial_{i+1,j+1}]_q q^{\lambda_i + \sum_{l=j+1}^{N+1} (\nu_{i+1} \partial_{l,i+1} - \nu_i \partial_{i,i})} & (i + 1 \leq j \leq N). \end{cases} \tag{3.5}$$

Here we read $x_{i,i} = 1$ and, for Grassmann odd variables $x_{i,j}$, the expression $\frac{1}{x_{i,j}}$ stands for the derivative $\frac{1}{x_{i,j}} = \frac{\partial}{\partial x_{i,j}}$. 
Theorem 3.1  [17] A Heisenberg realization of the quantum superalgebra $U_q(sl(N|1))$ is given in the following way.

\begin{align}
  h_i & \rightarrow H_i, \\
  e_i & \rightarrow E_i, \\
  f_i & \rightarrow F_i.
\end{align}  

(3.6)  (3.7)  (3.8)

In appendix A we explain how to find the free field realization of affine $U_q(\hat{sl}(N|1))$ from this Heisenberg realization $U_q(sl(N|1))$.

3.2 Boson

Let us fix the level $c = k \in \mathbb{C}$. Let us introduce the bosons and the zero-mode operators $a^i_m$, $Q^i_m$ ($m \in \mathbb{Z}$, $1 \leq j \leq N$), $b^i_m$, $Q^i_m$, $c^i_m$, $Q^i_m$ ($m \in \mathbb{Z}$, $1 \leq i < j \leq N + 1$). The bosons $a^i_m, b^i_m, c^i_m, (m \in \mathbb{Z}_\neq 0)$ satisfy

\begin{align}
  [a^i_m, a^j_n] &= \frac{[(k + N - 1)m][A_i,j]_m q}{m}[d_{m,n,0}, \\
  [b^i_m, b^j_{n'}] &= -\nu_i \nu_j [m]_{A_i,j}^2 \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \\
  [c^i_m, c^j_{n'}] &= \nu_i \nu_j [m]_{A_i,j}^2 \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}.
\end{align}  

(3.9)  (3.10)  (3.11)

The zero-mode operators $a^i_0, Q^i_0, b^i_0, Q^i_0, c^i_0, Q^i_0$ satisfy

\begin{align}
  [a^i_0, Q^i_n] &= (k + N - 1)A_i,j, \\
  [b^i_0, Q^{i'}_{n'}] &= -\nu_i \nu_j \delta_{i,i'} \delta_{j,j'}, \\
  [c^i_0, Q^{i'}_{n'}] &= \nu_i \nu_j \delta_{i,i'} \delta_{j,j'}.
\end{align}  

(3.12)  (3.13)  (3.14)

and other commutators vanish. We impose the cocycle condition on the zero-mode operator $Q^{i,j}_b$, ($1 \leq i < j \leq N + 1$) by

\begin{align}
  [Q^{i,j}_b, Q^{i',j'}_{b}] = \delta_{j,N+1} \delta_{j',N+1} \pi \sqrt{-1} \quad \text{for} \ (i, j) \neq (i', j').
\end{align}  

(3.15)

We have the following (anti)commutation relations

\begin{align}
  \left\{ \exp \left( Q^{i,j}_b \right), \exp \left( Q^{i',j'}_{b} \right) \right\} &= 0 \quad (1 \leq i < j \leq N, 1 \leq i' < j' \leq N), \\
  \left\{ \exp \left( Q^{i,N+1}_b \right), \exp \left( Q^{i',N+1}_{b} \right) \right\} &= 0 \quad (1 \leq i \neq j \leq N).
\end{align}  

(3.16)  (3.17)

We use the following normal ordering symbol :: as follows.

\begin{align}
  :b^i_m b^{i'}_{n'}:= \begin{cases} 
    b^i_m b^{i'}_{n'} & (m < 0), \\
    b^i_{m'} b^{i'}_{m} & (m > 0),
\end{cases} & :a^i_m a^j_n:= \begin{cases} 
    a^i_m a^j_n & (m < 0), \\
    a^j_n a^i_m & (m > 0),
\end{cases} \\
  :b^i_0 Q^{i'}_b := Q^{i'}_{b} Q^{i}_b & :a^i_0 Q^i_0 := Q^i_0 a^i_0 := Q^i_0 a^i_0.
\end{align}  

(3.18)  (3.19)
The above boson structure is the straightforward generalization of those in [17]. Note that \((N - 1)\) is the dual Coxeter number. In what follows we use \(\{a^j_m(1 \leq j \leq N), b^i_m, Q^{ij}_c(1 \leq i < j \leq N)\, b^{ij}_m, Q^{ij}_c(1 \leq i < j \leq N)\}\) which is a subset of the above boson system. In what follows we use the abbreviations \(b^{ij}(z), c^{ij}(z), b^{ij}_\pm(z), a^{ij}_\pm(z)\).

\[
b^{ij}(z) = - \sum_{m \neq 0} \frac{b^{ij}_m}{|m|} z^{-m} + Q^{ij}_c + b^{ij}_0 \log z, \tag{3.20}
\]

\[
c^{ij}(z) = - \sum_{m \neq 0} \frac{c^{ij}_m}{|m|} z^{-m} + Q^{ij}_c + c^{ij}_0 \log z, \tag{3.21}
\]

\[
b_{\pm}^{ij}(z) = \pm (q - q^{-1}) \sum_{m > 0} b^{ij}_m z^{-m} \pm b^{ij}_0 \log q, \tag{3.22}
\]

\[
a_{\pm}^{ij}(z) = \pm (q - q^{-1}) \sum_{m > 0} a^{ij}_m z^{-m} \pm a^{ij}_0 \log q. \tag{3.23}
\]

### 3.3 Free Field Realization

In this section we construct a free field realization of the quantum affine superalgebra \(U_q(\widehat{sl}(N|1))\) for an arbitrary level \(k\). In [15], on the basis of the Heisenberg realization of the quantum algebra \(U_q(sl(N))\), a free field realization of the quantum affine algebra \(U_q(\widehat{sl}(N))\) was obtained. Here we try to generalize it to the quantum affine superalgebra \(U_q(\widehat{sl}(N|1))\). Detailed calculations of this trial are summarized in appendix A. We introduce the operators \(X^{\pm}_i(z), \Psi^{\pm}_i(z), (1 \leq i \leq N)\) on the Fock space as follows. For \(1 \leq i \leq N - 1\) we introduce

\[
X^+_i(z) = \frac{1}{(q - q^{-1})z} \prod_{j=1}^{i} \left( X^+_i z^{j-1}(z) - X^+_j z^{j-1}(z) \right), \tag{3.24}
\]

\[
X^+_N(z) = \sum_{j=1}^{N} X^+_N z^{j-1}(z), \tag{3.25}
\]

\[
X^-_i(z) = \frac{1}{(q - q^{-1})z} \left( \sum_{j=1}^{i-1} (X^-_i z^{j-1}(z) - X^-_j z^{j-1}(z)) + (X^-_{i,2i-1}(z) - X^-_{i,2i}(z)) \right. \\
\left. + \sum_{j=i+1}^{N-1} (X^-_{i,2j-1}(z) - X^-_{i,2j}(z)) \right) + q^{k+N-1} X^-_{i,2N-1}(z), \tag{3.26}
\]

\[
X^-_N(z) = \frac{1}{(q - q^{-1})z} \prod_{j=1}^{N} \left( -q^{j-1} X^-_N z^{j-1}(z) + q^{j-1} X^-_N z^{j-1}(z) \right), \tag{3.27}
\]

\[
\Psi^{\pm}_i(\tilde{q}^{H} z) = \exp \left( a^i_\pm (q^{i+N-1} z^{-1}) + \sum_{j=1}^{i} (b^i_{\pm} (q^{i+N-1} z^{-1}) - b^i_{\pm} (q^{i+N-1} z^{-1})) \right)
\]
For 1 \leq i \leq N - 1 and 1 \leq j \leq i we set
\[
X_{i,2j-1}^+(z) = \exp \left( (b + c)^{j,i}(q^j z) + b_{i+1}^{j,i+1}(q^{j+1} z) \right)
+ \sum_{l=1}^{j-1} (b_{i+1}^{j,i+1}(q^{j-l} z) - b_{i}^{j,i}(q^{j-l} z)) \right) :,
\]
\[
X_{i,2j}^+(z) = \exp \left( (b + c)^{j,i}(q^j z) + b_{i+1}^{j,i+1}(q^{j-1} z) \right)
+ \sum_{l=1}^{j-1} (b_{i+1}^{j,i+1}(q^{j-l} z) - b_{i}^{j,i}(q^{j-l} z)) \right) :,
\]
\[
X_{i,2j-1}^-(z) = \exp \left( a_{i}^{k,i}(q^{-k} z) + (b + c)^{j,i+1}(q^{-k-j} z) \right)
- b_{i+1}^{j,i}(q^{k-j} z) \right)
+ \sum_{l=1}^{i} (b_{i+1}^{j,i+1}(q^{-j-l+1} z) - b_{i}^{j,i}(q^{-j-l} z))
+ \sum_{l=1}^{N} (b_{i}^{j,i}(q^{-j-l} z) - b_{i+1}^{j,i+1}(q^{-j-l+1} z))
+ b_{i+1}^{j,i+1}(q^{-k-N} z) - b_{i+1,N+1}^{j,i+1}(q^{-k-N+1} z) \right) :,
\]
\[
X_{i,2j}^-(z) = \exp \left( a_{i}^{k,i}(q^{-k} z) + (b + c)^{j,i+1}(q^{-k-j} z) \right)
- b_{i+1}^{j,i}(q^{k-j} z) \right)
+ \sum_{l=1}^{i} (b_{i+1}^{j,i+1}(q^{-j-l+1} z) - b_{i}^{j,i}(q^{-j-l} z))
+ \sum_{l=1}^{N} (b_{i}^{j,i}(q^{-j-l} z) - b_{i+1}^{j,i+1}(q^{-j-l+1} z))
+ b_{i+1}^{j,i+1}(q^{-k-N} z) - b_{i+1,N+1}^{j,i+1}(q^{-k-N+1} z) \right) :,
\]
Here we have used the auxiliary bosonic operators \( X^+_{i,j}(z) \) as follows.

For 1 \leq i \leq N - 1 and 1 \leq j \leq 1 we set
\[
\Psi_{N}^{\pm}(q^{\pm\frac{N}{2}}) = \exp \left( a_{N}^{N}(q^{\pm N} \frac{N}{2} z) - b_{N}^{N}(q^{N} \frac{N}{2} z) \right) \)
\]
\[ +b_{i,N+1}^i(q^{-k-N}z) - b_{i,N+1}^{i+1}(q^{-k-N+1}z) \] 

For \(1 \leq i \leq N - 1\) we set

\[ X_{i,2i-1}^-(z) = : \exp \left( a_i^-(q^{-\frac{k+N-1}{2}}z) + (b + c)^i,i+1(q^{-k-i}z) \right. \]
\[ + \sum_{l=i+1}^{N} (b_{i,l}^{-i}(q^{-k-l}z) - b_{i+1,l}^{-i+1}(q^{-k-l+1}z)) \]
\[ +b_{i,N+1}^{-i}(q^{-k-N}z) - b_{i+1,N+1}^{-i+1}(q^{-k-N+1}z) \] \( , \) \hspace{1cm} (3.35) 

\[ X_{i,2j}^-(z) = : \exp \left( a_i^-(q^{-\frac{k+N-1}{2}}z) + (b + c)^i,i+1(q^{k-i}z) \right. \]
\[ + \sum_{l=j+1}^{N} (b_{j,i}^{i+1}(q^{k+j}z) - b_{i+1,j}^{i+1}(q^{k+j+1}z)) \]
\[ +b_{i+1,N+1}(q^{k+N}z) - b_{i,j+1,N+1}(q^{k+N+1}z) \] \( , \) \hspace{1cm} (3.36) 

For \(1 \leq i \leq N - 1\) and \(i + 1 \leq j \leq N - 1\) we set

\[ X_{i,2j-1}^-(z) = : \exp \left( a_i^-(q^{-\frac{k+N-1}{2}}z) + (b + c)^i,j+1(q^{k+j}z) \right. \]
\[ +b_{i,j+1}^{i+1,j+1}(q^{k+j}z) - (b + c)^i,j+1(q^{k+j+1}z) \]
\[ +b_{j+1,N+1}(q^{k+N}z) - b_{i+1,j+1,N+1}(q^{k+N+1}z) \] \( , \) \hspace{1cm} (3.37) 

\[ X_{i,2j}^-(z) = : \exp \left( a_i^-(q^{-\frac{k+N-1}{2}}z) + (b + c)^i,j+1(q^{k+j}z) \right. \]
\[ +b_{j+1,N+1}(q^{k+N}z) - b_{i,j+1,N+1}(q^{k+N+1}z) \] \( , \) \hspace{1cm} (3.38) 

For \(1 \leq i \leq N - 1\) we set

\[ X_{i,2N-1}^-(z) = : \exp \left( a_i^-(q^{-\frac{k+N-1}{2}}z) - b_{i+1,N+1}(q^{k+N-1}z) \right. \]
\[ -b_{i+1,N+1}(q^{k+N-1}z) + b_{i+1,N+1}(q^{k+N}z) \] \( , \) \hspace{1cm} (3.39) 

For \(1 \leq j \leq N - 1\) we set

\[ X_{N,2j-1}^-(z) = : \exp \left( a_j^N(q^{-\frac{k+N-1}{2}}z) - b_{i,j}^{i,j}(q^{k-j}z) - (b + c)^j,j,N(q^{k-j+1}z) \right. \]
\[ +b_{i,j+1,N+1}(q^{k-j}z) - b_{i,j+1,N+1}(q^{k-j+1}z) \]
Now we have introduced the bosonic operators $X_i^\pm(z)$ and $\Psi_i^\pm(z)$. The following is main result of this paper.

**Theorem 3.2** A free field realization of the quantum affine superalgebra $U_q(\widehat{sl}(N|1))$ is given in the following way.

\[
\begin{align*}
  c & \mapsto k \\
  x_i^\pm(z) & \mapsto X_i^\pm(z) \\
  \psi_i^\pm(z) & \mapsto \Psi_i^\pm(z).
\end{align*}
\]  

In other words, the above map gives a homomorphism from $U_q(\widehat{sl}(N|1))$ to the bosonic operator. Very explicitly the relation (3.46) is written as

\[
\begin{align*}
  h_{i,m} & \mapsto q^{\frac{k+k-N-1}{2}|m|}a_i^m + \sum_{l=1}^i (q^{-(k+l-1)|m|}b^l_{m}|i\rangle - q^{-(k+l)|m|}b^l_{m}|i\rangle) \\
  & \quad + \sum_{l=i+1}^N (q^{-(k+l)|m|}b^l_{m}|i\rangle - q^{-(k+l-1)|m|}b^l_{m}|i\rangle) \\
  & \quad + q^{-(k+N)|m|}b^i_{m}|N+1\rangle - q^{-(k+N-1)|m|}b^i_{m}|N+1\rangle \\
  h_{N,m} & \mapsto q^{\frac{k+k-N-1}{2}|m|}a^N_m - \sum_{l=1}^{N-1} (q^{-(k+l)|m|}b^l_{m}|N\rangle + q^{-(k+l)|m|}b^l_{m}|N\rangle).
\end{align*}
\]  

We give some comments on this realization. Upon the specialization $N = 2$, this free field realization reproduces the result for $U_q(\widehat{sl}(2|1))$ in [17]. The structure of non-superalgebra $U_q(\widehat{sl}(N))$ exists inside the superalgebra $U_q(\widehat{sl}(N|1))$. Hence the free field realizations of the currents $X_i^\pm(z)$ ($i \neq N$) for $U_q(\widehat{sl}(N|1))$ are quite similar as those for $U_q(\widehat{sl}(N))$. The free field realizations of the fermionic operators $X_{N,j}^+(z)$, $X_{N,2j-1}^-(z)$, $X_{N,2j}^-(z)$ and $X_{j,2N-1}^-(z)$ of
$U_q(\widehat{sl}(N|1))$ are completely different from those of $U_q(\widehat{sl}(N))$. The free field realization of this paper is not irreducible representation. We have to construct screening currents that commute with the currents $X^\pm_j(z)$ in order to get an irreducible representation [26, 27, 28]. We would like report this subject in the future publication. Applying the dressing method developed in [25] to this theorem, we have a free field realization of the elliptic algebra $U_{q,p}(\widehat{sl}(N|1))$.

**Proof of Theorem.** Direct calculations of the normal orderings show this theorem. The normal orderings of bosonic operators $X^\pm_{i,j}(z)$ ($i \neq N, j \neq 2N - 1$) of the superalgebra $U_q(\widehat{sl}(N|1))$ are exactly the same as those of the non-superalgebra $U_q(\widehat{sl}(N))$. Hence the proof of the relations for the bosonic operators $X^\pm_i(z)$ ($i \neq N$) is exactly the same as those of $U_q(\widehat{sl}(N))$. Let us focus our attention on the fermionic operators $X^\pm_N(z)$ that is new for the superalgebra. We show the following relations for the fermionic operators $X^\pm_N(z)$.

\[
\{X^+_N(z_1), X^-_N(z_2)\} = \frac{1}{(q - q^{-1})z_1 z_2} \left( \delta(q^k z_2 / z_1) \Psi_N^+(q^{k/2} z_2) - \delta(q^{-k} z_2 / z_1) \Psi_N^-(q^{-k/2} z_2) \right), \tag{3.49}
\]

and

\[
[X^+_N(z_1), X^-_j(z_2)] = 0 \quad \text{for } 1 \leq j \leq N - 1. \tag{3.50}
\]

First, let us show (3.49). Using the relation (B.5) in appendix B, we have

\[
\{X^+_N(z_1), X^-_N(z_2)\} = \frac{1}{(q - q^{-1})z_1 z_2} \sum_{j=1}^{N} q^{j-1} \left( -\{X^+_{N,j}(z_1), X^-_{N,2j-1}(z_2)\} + \{X^+_{N,j}(z_1), X^-_{N,2j}(z_2)\} \right).
\]

Using the relations (B.1), (B.2), (B.3) and (B.4) in appendix B, we have

\[
\{X^+_N(z_1), X^-_N(z_2)\} = \frac{1}{(q - q^{-1})z_1 z_2} \left( \delta(q^k z_2 / z_1) \Psi_N^+(q^{k/2} z_2) - \delta(q^{-k} z_2 / z_1) \Psi_N^-(q^{-k/2} z_2) \right) + \frac{1}{(q - q^{-1})z_1 z_2} \exp \left( a_+^N(q^{k+1/2} / z_2) \right) \times
\]

\[
\left\{ \sum_{j=1}^{N-1} \delta \left( \frac{q^{-k-2j} z_2}{z_1} \right) : \exp \left( -\sum_{l=1}^{j} (b_+^{l,N}(q' z_1) + b_+^{l,N+1}(q' z_1)) - \sum_{l=j+1}^{N-1} (b_+^{l,N}(q^{-k-l} z_2) + b_+^{l,N+1}(q^{-k-l} z_2)) \right) : \right\}
\]

\[
- \sum_{j=2}^{N} \delta \left( \frac{q^{-k-2j+2} z_2}{z_1} \right) : \exp \left( -\sum_{l=1}^{j-1} (b_+^{l,N}(q' z_1) + b_+^{l,N+1}(q' z_1)) - \sum_{l=j}^{N-1} (b_+^{l,N}(q^{-k-l} z_2) + b_+^{l,N+1}(q^{-k-l} z_2)) \right) : \right\}.
\]

Making the transformation $j \rightarrow j - 1$ in the first sum $\sum_{j=1}^{N-1} \delta(q^{-k-2j} z_2 / z_1)$, we see cancellations.

We have the relation (3.49).
Next, let us show (3.50). Using the relation (B.9) in appendix B, we have the following for $1 \leq j \leq N - 2$.

$$\left[ X_N^+(z_1), X_j^-(z_2) \right] = \frac{-1}{(q - q^{-1})z_2} \left[ X_{N,j}^+(z_1), X_{j,2N-3}(z_2) \right] + q^{k+N-1} \left[ X_{N,j+1}^+(z_1), X_{j,2N-1}(z_2) \right].$$

Using the relations (B.6), (B.8) in appendix B, we have

$$\left[ X_N^+(z_1), X_j^-(z_2) \right] = \delta \left( \frac{q^{k+N-j}z_2}{z_1} \right) \left( -\frac{1}{z_2} + \frac{q^{k+N-j}}{z_1} \right) \times : \exp \left( a^j_+ \left( q^{k+N-1} - \frac{1}{z_2} \right) - b^j_{+1,N+1} \left( q^{k+N-1}z_2 \right) + b_{j+1,N+1} \left( q^{k+N}z_2 \right) + (b+c)^{j,N} \left( q^{k+N-1}z_2 \right) \right) - \sum_{l=1}^{j-1} \left( a^{l,N}_+ \left( q^{k+N-j+l}z_2 \right) + b^{l,N+1}_+ \left( q^{k+N-j+l}z_2 \right) \right) : \right.$$

From the relation $\left( -\frac{1}{z_2} + \frac{q^{k+N-j}}{z_1} \right) \delta \left( \frac{q^{k+N-j}z_2}{z_1} \right) = 0$, we have

$$\left[ X_N^+(z_1), X_j^-(z_2) \right] = 0 \text{ for } 1 \leq j \leq N - 2.$$

From the relation (B.9) in appendix B, we have

$$\left[ X_N^+(z_1), X_{N-1}^-(z_2) \right] = \frac{-1}{(q - q^{-1})z_2} \left[ X_{N,N-1}^+(z_1), X_{N-1,2N-2}^-(z_2) \right] + q^{k+N-1} \left[ X_{N,N}^+(z_1), X_{N-2N-1}^-(z_2) \right].$$

Using the relations (B.7), (B.8) and the relation $\delta \left( \frac{q^{k-1}z_2}{z_1} \right) \left( -\frac{1}{z_2} + \frac{q^{k-1}}{z_1} \right) = 0$, we have

$$\left[ X_N^+(z_1), X_{N-1}^-(z_2) \right] = \delta \left( \frac{q^{k-1}z_2}{z_1} \right) \left( -\frac{1}{z_2} + \frac{q^{k-1}}{z_1} \right) \times : \exp \left( a^N_{+1} \left( q^{k+N-1} - \frac{1}{z_2} \right) - b^N_{+1,N+1} \left( q^{k+N-1}z_2 \right) + b^{N,N+1} \left( q^{k+N}z_2 \right) + (b+c)^{N-1,N} \left( q^{k+N-1}z_2 \right) \right) - \sum_{l=1}^{N-2} \left( a^{l,N}_+ \left( q^{k+l+1}z_2 \right) + b^{l,N+1}_+ \left( q^{k+l+1}z_2 \right) \right) := 0.$$

We have shown the relation (3.50). Q.E.D.

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A Replacement

In this appendix we explain how to find the free field realization of affine $U_q(sl(N|1))$ from the Heisenberg realization of $U_q(sl(N|1))$.

A.1 Basic Operator

We would like to explain the role of the basic operators

$$\exp \left( \pm b^{i,N+1}(z) \right), \exp \left( b^{i,j}(z) \pm (b + c)^{i,j}(q^{\mp 1}z) \right),$$

(A.1)

which have been used for $U_q(sl(2|1))$ [17] and $U_q(sl(2))$ [14], respectively. The basic operators $\exp \left( \pm b^{i,N+1}(z) \right) : (1 \leq i \leq N)$ satisfy the fermionic relation

$$\{ : \exp(b^{i,N+1}(z_1)) : , : \exp(-b^{i,N+1}(z_2)) : \} = \frac{1}{z_1} \delta(z_2/z_1).$$

(A.2)

The basic operators $\exp \left( \pm b^{i,N+1}(z) \right)$ : create the delta-function $\delta(z)$ and play important roles in constructions of the fermionic operators $X_{N}^{\pm}(z)$ that satisfy

$$\{X_{N}^{+}(z_1), X_{N}^{-}(z_2)\} = \frac{1}{(q - q^{-1})z_1z_2} \left( \delta(q^k z_2/z_1)\Psi_{N}^{+}(q^{\frac{k}{2}}z_2) - \delta(q^{-k} z_2/z_1)\Psi_{N}^{-}(q^{-\frac{k}{2}}z_2) \right).$$

The basic operators $\exp \left( b^{i,j}(z) \pm (b + c)^{i,j}(q^{\mp 1}z) \right) : (1 \leq i < j \leq N)$ satisfy the bosonic relations

$$\begin{align*}
&\left[ : \exp(b^{i,j}(z_1) - (b + c)^{i,j}(qz_1)) : , : \exp(b^{i,j}(z_2) + (b + c)^{i,j}(q^{-1}z_2)) : \right] \\
&= (q^{-1} - q) \delta(q^{-2}z_2/z_1) : \exp\left(b^{i,j}_{+}(z_1) + b^{i,j}_{-}(z_2)\right) : \\
&\left[ : \exp(b^{i,j}(z_1) - (b + c)^{i,j}(q^{-1}z_1)) : , : \exp(b^{i,j}(z_2) + (b + c)^{i,j}(qz_2)) : \right] \\
&= (q - q^{-1}) \delta(q^{2}z_2/z_1) : \exp\left(b^{i,j}_{+}(z_1) + b^{i,j}_{-}(z_2)\right) : .
\end{align*}$$

(A.3)

(A.4)

The basic operators $\exp \left( b^{i,j}(z) \pm (b + c)^{i,j}(q^{\mp 1}z) \right)$ : create the delta-function $\delta(z)$ and play important roles in constructions of the bosonic operators $X_{i}^{\pm}(z)$ ($i \neq N$) that satisfy

$$\left[ X_{i}^{+}(z_1), X_{j}^{-}(z_2) \right] = \frac{\delta_{i,j}}{(q - q^{-1})z_1z_2} \left( \delta(q^k z_2/z_1)\Psi_{i}^{+}(q^{\frac{k}{2}}z_2) - \delta(q^{-k} z_2/z_1)\Psi_{i}^{-}(q^{-\frac{k}{2}}z_2) \right).$$
Multiplying and adding proper operators to these basic operators (A.1), we construct the free field realization. For this purpose, the following replacement from the Heisenberg realization of $U_q(sl(N|1))$ to the free field realization of the affine $U_q(\widehat{sl}(N|1))$ gives useful information.

### A.2 Replacement

In this appendix we explain how to find the free field realization of the affine superalgebra $U_q(\widehat{sl}(N|1))$ from the Heisenberg realization of $U_q(sl(N|1))$. We make the following replacement with suitable argument.

\[
\begin{align*}
\vartheta_{i,j} & \rightarrow -b_{i,j}^\pm(z)/\log q \quad (1 \leq i < j \leq N + 1), \\
[\vartheta_{i,j}]_q & \rightarrow \begin{cases} 
\exp \left( \pm b_{i,j}^\pm(z) \right) - \exp \left( \pm b_{i,j}^\pm(z) \right) & (j \neq N + 1), \\
(q - q^{-1})z & (j = N + 1).
\end{cases} \\
x_{i,j} & \rightarrow \begin{cases} 
: \exp \left( (b + c)^{i,j}(z) \right) : & (j \neq N + 1), \\
: \exp \left( -b^{i,j}(z) \right) : \text{ or } : \exp \left( -b_{i,j}^\pm(q^\pm1z) - b^{i,j}(z) \right) : & (j = N + 1).
\end{cases} \\
\lambda_i & \rightarrow a_i^\pm(z)/\log q \quad (1 \leq i \leq N), \\
[\lambda_i]_q & \rightarrow \begin{cases} 
\exp \left( \pm a_i^\pm(z) \right) - \exp \left( \pm a_i^\pm(z) \right) & (1 \leq i \leq N).
\end{cases}
\end{align*}
\]

Taking the basic operators (A.1) into account, we gave this rule of the replacement.

From the above replacement, $H_i$ of the Heisenberg realization (3.2) is replaced as following.

\[
q^{H_i} \rightarrow \begin{cases} 
\exp \left( a_i^\pm(z) + \sum_{l=1}^i(b_{i,l}^{\pm1}(z) - b_{l,i}^\pm(z)) + \sum_{l=i+1}^N(b_{l,i}^{\pm1}(z) - b_{i,l}^\pm(z)) \right) & (1 \leq i \leq N - 1), \\
\exp \left( a_i^\pm(z) - \sum_{l=1}^{N-1}(b_{l,N}^l(z) + b_{i,N}^l(z)) \right) & (i = N).
\end{cases}
\]

There exist small gaps between the above operators (A.10) and the free field realizations $\Psi_i^\pm(z)$ (3.28), (3.29). In order to make the operators (A.10) satisfy the defining relations of $U_q(\widehat{sl}(N|1))$, we have to impose $q$-shift to variable $z$ of the operators $a_i^\pm(z)$, $b_{i,j}^\pm(z)$. For instance, we have to replace $a_i^\pm(z) \rightarrow a_i^\pm(q^{\pm1+i/2}z)$. Bridging the gap by the $q$-shift, we have the free field realizations $\Psi_i^\pm(q^{\pm1/2}z)$ (3.28), (3.29) from $q^{H_i}$.

\[
q^{H_i} \rightarrow \Psi_i^\pm(q^{\pm1/2}z) \quad (1 \leq i \leq N).
\]

The structure of non-superalgebra $U_q(sl(N))$ exists inside the superalgebra $U_q(\widehat{sl}(N|1))$. Hence the free field realizations of the currents $X_i^\pm(z)$ ($i \neq N$) for $U_q(\widehat{sl}(N|1))$ are quite similar.
as those for $U_q(\hat{sl}(N))$. Let us focus our attention on the fermionic operators $X^\pm_N(z)$ that is new for the superalgebra. Let us consider $E_N = \sum_{j=1}^N E_{N,j}$ of the Heisenberg realization (3.2). From the above replacement, we have

$$ E_{N,j} \rightarrow \exp \left( (b + c)j^N(z) + b^jN+1(z) - \sum_{l=1}^{j-1} (b^{l,jN+1}_+ + b^{l,jN+1}_-) \right) : \quad (A.12) $$

There exists an ambiguity of the replacement of $x_{j,N+1}$ in (A.7). Here we have chose the replacement $x_{j,N+1} \rightarrow \exp \left( -b^jN+1(z) \right) : (1 \leq j \leq N)$. Imposing proper $q$-shift to the variable $z$ of the operators $(b + c)j^N(z)$, $b^jN+1(z)$, $b^{j\pm j}(z)$, we have the free field realizations $X^+_{N,j}(z)$ in (3.32).

$$ E_{N,j} \rightarrow X^+_{N,j}(z) \quad (1 \leq j \leq N). \quad (A.13) $$

Let us consider $F_N = \sum_{j=1}^N F_{N,j}$ of the Heisenberg realization (3.2). From the above replacement we have

$$ F_{N,j} \rightarrow \frac{1}{(q - q^{-1})z} \times 

\times \left \{ \exp \left( -a^-_N(z) - b^jN+1(z) - (b + c)j^N(z) + \sum_{l=j+1}^{N-1} (b^{l,jN+1}_+ - b^{l,jN}_-) \right) \right \} 

\times \left \{ \begin{array}{ll}
\exp \left( -b^jN(z) - b^jN+1(z) \right) - \exp \left( -b^jN(z) - b^jN+1(z) \right) & (j \neq N), \\
\exp \left( -b^N(z) \right) \left( \exp \left( a^N(z) \right) - \exp \left( a^-_N(z) \right) \right) & (j = N).
\end{array} \right. 

(A.14) $$

There exists an ambiguity of the replacement of $x_{j,N+1}$ in (A.7). Here we have chose the replacement $x_{j,N+1} \rightarrow \exp \left( b^{j,jN+1}_+ (q^{j+1}z) - b^{j,jN+1}_- (z) \right) : (1 \leq j \leq N - 1)$ and $x_{N,N+1} \rightarrow \exp \left( -b^N(z) \right) :$. Imposing proper $q$-shift to the variable $z$ of the operators $(b + c)j^N(z)$, $b^jN+1(z)$, $b^{j\pm j}(z)$, $a^N(z)$, we have the free field realizations $X^-_{N,2j-1}(z)$, $X^-_{N,2j}(z)$ in (3.40), (3.41), (3.42) and (3.43).

$$ F_{N,j} \rightarrow \frac{-1}{(q - q^{-1})z} \left( X^-_{N,2j-1}(z) - X^-_{N,2j}(z) \right) \quad (1 \leq j \leq N). \quad (A.15) $$

Replacements for bosonic operators $X^\pm_j(z), (j \neq N)$ have already appeared in $U_q(\hat{sl}(N))$ [16]. We explained details of the replacement for the fermionic operator $X^+_N(z)$, which is new for the superalgebra.

**B Normal Orderings**

In this appendix we summarize useful relations.

For $1 \leq j \leq N$ we have

$$ \{ X^+_{N,j}(z_1), X^-_{N,2j-1}(z_2) \} = \frac{1}{q^{j-1}z_1} \delta(q^{-2j+2}z_2/z_1) $$

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\[ \times \exp \left( a^- N_q \left( q^{-k+N-1} \right)^{2z_2} z_2 \right) - \sum_{l=1}^{j-1} \left( b^l+_{N_2} \left( q^{-k-2j+l+2} \right) z_2 + b^l+_{N+1} \left( q^{-k-2j+l+2} \right) z_2 \right) \]
\[ - \sum_{l=j}^{N-1} \left( b^-_{N_2} \left( q^{-k-l} \right) z_2 + b^-_{N+1} \left( q^{-k-l} \right) z_2 \right) \right) :. \] (B.1)

Especially for \( j = 1 \) we have
\[ \left\{ X^+_N(z_1), X^-_N(z_2) \right\} = \frac{1}{z_1} \delta(q^{-k} z_2 z_1) \Psi_N(q^{-k} z_2). \] (B.2)

For \( 1 \leq j \leq N - 1 \) we have
\[ \left\{ X^+_N(z_1), X^-_{N,j}(z_2) \right\} = \frac{1}{q^{j-1} z_1} \delta(q^{-k-2j} z_2 z_1) \]
\[ \times \exp \left( a^- N_q \left( q^{-k+N-1} \right)^{2z_2} z_2 \right) - \sum_{l=1}^{j} \left( b^l+_{N_2} \left( q^{-k-2j+l} \right) z_2 + b^l+_{N+1} \left( q^{-k-2j+l} \right) z_2 \right) \]
\[ - \sum_{l=j+1}^{N-1} \left( b^-_{N_2} \left( q^{-k-l} \right) z_2 + b^-_{N+1} \left( q^{-k-l} \right) z_2 \right) \right) :, \] (B.3)

\[ \left\{ X^+_N(z_1), X^-_{N,2N}(z_2) \right\} = \frac{1}{q^{N-1} z_1} \delta(q^k z_2 z_1) \Psi^+_N(q^k z_2). \] (B.4)

Other anti-commutators relations \( \left\{ X^+_N(z_1), X^-_{N,j}(z_2) \right\} \) vanish.
\[ \left\{ X^+_N(z_1), X^-_{N,j}(z_2) \right\} = 0 \quad \text{for} \quad j \neq 2i - 1, 2i. \] (B.5)

For \( 1 \leq j \leq N - 2 \) we have
\[ \left[ X^+_N,j+1(z_1), X^-_{j,2N-3}(z_2) \right] = (q - q^{-1}) \delta(q^{k+N-j} z_2 z_1) \]
\[ \times \exp \left( a^-_N q^{k+N-1} z_2 \right) - b^l+_{N+1} \left( q^{k+N-1} z_2 \right) \]
\[ + b^l+_{N+1} \left( q^{k+N-1} z_2 \right) + (b + c)^{j,N} \left( q^{k+N-1} z_2 \right) \]
\[ - \sum_{l=1}^{j-1} \left( b^l+_{N} \left( q^{k+N-j+l} \right) z_2 + b^l+_{N+1} \left( q^{k+N-j+l} \right) z_2 \right) :. \] (B.6)

We have
\[ \left[ X^+_N(z_1), X^-_{N-1,2N-2}(z_2) \right] = (q - q^{-1}) \delta(q^{k+1} z_2 z_1) \]
\[ \times \exp \left( a^-_N q^{k+N-1} z_2 \right) - b^N+_{N+1} \left( q^{k+N-1} z_2 \right) \]
\[ + b^N+_{N+1} \left( q^{k+N-1} z_2 \right) + (b + c)^{N-1,N} \left( q^{k+N-1} z_2 \right) \]
\[ - \sum_{l=1}^{N-2} \left( b^l+_{N} \left( q^{k+l+1} \right) z_2 + b^l+_{N+1} \left( q^{k+l+1} \right) z_2 \right) :. \] (B.7)
For $1 \leq j \leq N - 1$ we have
\[
[X_{N,j}^+(z_1), X_{j,2N-1}^-](z_2) = \frac{1}{q^{j-1}z_1} \delta(q^{k-N-j}z_2/z_1)
\times \exp \left( a_j^+(q^{k-N-j}z_2) - b_j^{i+1,N+1}(q^{k-N-1}z_2) 
+ b_j^{i+1,N+1}(q^{k+N}z_2) + (b+c)^{j,N}(q^{k-N}z_2)
- \sum_{l=1}^{j-1} (b^{l,N}_+(q^{k-N-j+l}z_2) + b^{l+1,N+1}_+(q^{k+N-j+l}z_2)) \right).
\] (B.8)

Other commutation relations $[X_{N,i}^+(z_1), X_{i,j}^-](z_2)$ vanish.

\[
[X_{N,i}^+(z_1), X_{j,l}^-(z_2)] = 0 \quad \text{for} \quad (i, j, l) \neq \begin{cases} (j, j, 2N-1) & (1 \leq j \leq N - 1), \\
(j + 1, j, 2N-3) & (1 \leq j \leq N - 2), \\
(N, N - 1, 2N-2). & \end{cases}
\] (B.9)

For $1 \leq i \leq N - 1$ we have
\[
[X_{i,2}^+(z_1), X_{i,1}^-](z_2) = (q - q^{-1}) \delta(q^{-k}z_2/z_1) \Psi_i^-(q^{-k}z_2),
\] (B.10)
\[
[X_{i,2i-1}^+(z_1), X_{i,2i}^-](z_2) = -(q - q^{-1}) \delta(q^kz_2/z_1) \Psi_i^+(q^kz_2).
\] (B.11)

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