Higher Order Thermal Corrections to Photon Self Energy

October 2, 2018

Mahnaz Q. Haseeb\textsuperscript{1} and Samina S. Masood\textsuperscript{2}

\textsuperscript{1}Physics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan.
\textsuperscript{2}Department of Physics, University of Houston Clear Lake, Houston TX 77058

Abstract

We investigate temperature behavior up to two loop level in QED in the background heat bath using real time formalism. The thermal correction to the coupling constant in QED at low temperature are presented up to the two loop level. It is observed that the removal of singularities at the two loop level is only possible if the integrations of loop momenta are done in a specific order. Some of the possible applications of the results are also indicated.

1 Motivations

It is now well-known that the temperatures were too high to ignore thermal contributions to particle processes in the very early universe. Quantum field theory at finite temperature is being studied for more than 50 years to investigate these effects. Techniques for calculations in many body systems were originally developed in condensed matter physics\cite{1} and were used to describe a large ensemble of multi-particle interactions in a thermal background. The techniques of high temperature field theory are being used to understand the phase transitions in cosmology and were adopted in quantum field theories in late seventies. They are extensively used to describe the phase transitions due to symmetry breaking after the Hot Big Bang and in tracing the history of the early universe. The methods can be utilized not only in particle physics but also to investigate various aspects of nuclear matter and plasma physics.

Some of our results can be applied to study the physical systems where temperatures are high enough to include the background contributions. In most of the cases the density effects are also nonignorable and these techniques can easily be generalized to incorporate them.
The early universe provides a very good example for studying hot plasmas. Right after the big bang, the universe was impossibly hot and dense. It rapidly expanded and cooled. Around $t \leq 100 \text{ sec}$, $T = 10^9 K$ [2], the universe was cool enough for neutrons and protons to combine to form Deuterium, then Helium and traces of Lithium (primordial nucleosynthesis). For the next few $10^5$ years it was still too hot for electrons to form atoms. The universe was filled with hot plasma of electrons and nuclei, bathed in photons constantly interacting with both, such as the interior of a star.

Similar type of nucleosynthesis occurs in the stellar cores at very high temperatures. Specifically, the cores of supermassive stars like supernova and neutron stars are very hot and dense. When the high mass stars exhaust their Helium fuel they have enough gravitational energy to heat up to $6 \times 10^8 K$ [3]. Cores of neutron stars, red giants and white dwarfs are composed of extremely dense plasmas ($\rho = 10^6 \text{--} 10^{15} \text{ g/cm}^3$). The neutrinos and axions emission rates in these stars require the incorporation of thermal background [4]. A tremendous amount of energy is released in a supernova. The only supernova in modern time, visible to the naked eye, was detected on Feb. 23, 1987 and is known as SN1987A. It emitted more than $10^{10}$ times as much visible light as the Sun for over one month and temperatures as high as $2 \times 10^{11} K$ were reached.

There is a series of different type of fusion reactions in stars leading to luminous supergiants. When helium fusion ceases in the core, gravitational compression increases the core’s temperature above $6 \times 10^8 K$ at which carbon can fuse into neon and magnesium. As the core reaches $1.5 \times 10^8 K$, oxygen begins fusing into silicon, phosphorous, sulfur, and others. At $2.7 \times 10^8 K$, silicon begins fusing into iron. This process essentially stops with the creation of iron and a catastrophic implosion of the entire star initiates.

The quark gluon plasma is the form of matter at transition temperatures $T_c = 100 \text{--} 200 \text{ MeV}$. The hot and dense environment in quark gluon plasma and the studies of its prospective reproduction in nucleus-nucleus collisions require the methods of thermal field theory for detailed investigations. With the increased feasibility of creation of quark gluon plasma in heavy ion collisions, the methods developed in this theory got their specific relevance in non-perturbative QCD at finite temperature as well.

Some of the methods of quantum field theory are very useful to understand the material properties and the transport theories. These mathematical techniques can thus be used to study some of the applications of condensed matter physics [5]. In the next section we describe the finite temperature corrections to vacuum polarization tensor. Sections III and IV give the one loop and higher order corrections respectively. Finally Sec. V comprises of discussion of the results.

2 Finite Temperature Effects

The main idea of finite temperature field theory is to use the approach of the usual quantum field theory. Matsubara [6] was first to develop thermal field
theory by incorporating a purely imaginary time variable in the evolution operator. In Euclidean space the covariance breaks and time is included as an imaginary parameter. The imaginary time domain is finite and periodic because of which the energy integrations are converted into summations over the discrete Matsubara frequencies. The presence of discrete energies along with the particle distribution functions destroys the covariance of the theory.

The important contributions by Schwinger [7], Mills [8], and Keldysh [9] led to the development of a formalism based upon the choice of a contour in the complex plane. This is called the real time formalism. In the real-time formalism, an analytical continuation of the energies along with Wick’s rotation restores covariance in Minkowski space at the expense of Lorentz invariance. The breaking of Lorentz invariance leads to the non-commutative nature of the gauge theories [10]. The covariance is incorporated through the 4-component velocity of the background heat bath \( u^\mu = (1, 0, 0, 0) \). In a heat bath the particles are in constant interaction with the thermal surroundings. Implementing these interactions is straightforward as is done in vacuum field theory. The temperature is included through the statistical distribution functions of the particles.

Umezawa and coworkers [11] independently worked on a different approach called Thermofield dynamics that also gives the same results. In this formalism, the propagators are taken in the form of \( 2 \times 2 \) matrices. Field theory at finite temperature is renormalizable since the presence of the Boltzmann factor in the thermal corrections cuts off any ultraviolet divergence. Choosing the suitable counter terms, as in vacuum, can eliminate them. The infrared divergences are inherent in almost all perturbation theories, whether at zero or finite temperature. KLN theorem [12] demonstrates that singularities appearing at intermediate stages of the calculation cancel in the final state physical result.

Quantum electrodynamics (QED) is the simplest and most successful gauge theory. The behavior of QED at finite temperatures serves as a model for the determination of background effects in other physical theories - the electroweak theories as well as quantum chromodynamics. In the real time formalism, the tree level fermion propagator in Feynman gauge in momentum space is [13]

\[
S_\beta(p) = (p^\mu - m)[\frac{i}{p^2 - m^2 + i\varepsilon} - 2\pi\delta(p^2 - m^2)n_F(E_p)], \tag{1}
\]

where

\[
n_F(E_p) = \frac{1}{e^{\beta(p.u)}/2 + 1}, \tag{2}
\]

is the Fermi-Dirac distribution function with \( \beta = \frac{1}{T} \). The boson propagator is

\[
D_\beta^{\mu\nu}(p) = \left[\frac{i}{k^2 + i\varepsilon} - 2\pi\delta(k^2)n_B(k)\right], \tag{3}
\]

with

\[
n_B(E_k) = \frac{1}{e^{\beta(k.u)} - 1}. \tag{4}
\]
3 One Loop Corrections

At the one loop level, Feynman diagrams are calculated in the usual way by substituting these propagators in place of the ones in vacuum. The Lorentz invariance breaking and conserving terms remain separate at the one loop level since the propagators comprise of temperature dependent (hot) terms added to temperature independent (cold) terms. This effect has been studied in detail and established at the one-loop level [14]. The renormalization of QED in this scheme [15] has already been checked in detail at the one loop level for all temperatures and chemical potentials.

The thermal background effects are incorporated through the radiative corrections. In finite temperature electrodynamics, electric fields are screened due to the interaction of the photon with the thermal background of charged particles. The physical processes take place in a heat bath comprising of hot particles and antiparticles instead of vacuum. The exact state of all these background particles is unknown since they continually fluctuate between real and virtual configurations. The net statistical effects of the background fermions and bosons enter in the theory through the fermion and boson distributions respectively.

The electric permittivity and the magnetic susceptibility of the medium are modified by incorporating the thermal background effects. At low temperatures, \( T \ll m_e \) (\( m_e \) is the electron mass), the hot fermions contribution in background is suppressed and only the hot photons contribute from the background heat bath. The vacuum polarization tensor in order \( \alpha \) does not acquire any hot corrections from the photons in the heat bath. This is because of the absence of self-interaction of photons in QED.

The thermal mass is generated radiatively. The mass shift that enters into physical quantities acts as a kinematical cut off, in the production rate of light weakly coupled particles from the heat bath. The effective mass corresponds to the fact that in the heat bath, the propagation of particles is altered by their continuous interactions with the medium.

4 Higher Order Corrections

The higher order loop corrections are required to get predictions on perturbative behavior at finite temperature. At the higher-loop level, the loop integrals involve a combination of cold and hot terms which appear due to the overlapping propagator terms in the matrix element. In such situations, specific techniques are needed, even at the two loop level, to solve them. Higher loops get analytically even more complicated. In the hot terms there appear overlapping divergent terms. The removal of such divergences is already shown at the two loop level [16] for electron self energy.
We restrict ourselves to the low-temperatures, for simplicity, to prove the renormalizability of QED at the two-loop level through the order by order cancellation of singularities. The \( \delta(0) \) type pinch singularities also appear in Minkowski space. The problem of pinch singularities has been resolved in thermofield dynamics by doubling the degrees of freedom [11]. The particle propagators become \( 2 \times 2 \) matrices whose 1-1 elements correspond to the usual thermal propagators. We used an alternative technique to get rid of these \( \delta(0) \) type pinch singularities through the identity [17]

\[
i \pi \left[ \delta(k^2) \right]^2 = -\frac{\delta(k^2)}{k^2 + i\varepsilon} - \frac{1}{2} \delta'(k^2). \tag{5}
\]

In this technique, the results depend on the order of doing the hot and cold integrations. As an illustration of the difference of the order of integration, we compare the results of one of the terms. Consider the singular terms when the hot loop in Fig. (1a) is evaluated before the cold one, we simply get

\[
g^{\mu\nu} \Pi^{\sigma}_{\mu\nu}(p, T) = \frac{\alpha^2 T^2}{3} (1 - \frac{2}{\varepsilon}), \tag{6}
\]

whereas, in the same term, the evaluation of the hot loop after the cold one gives

\[
g^{\mu\nu} \Pi^{\sigma}_{\mu\nu}(p, T) = \left. -\frac{\alpha^2}{\pi^2} \frac{A^2 T^2}{3 \varepsilon} - \frac{\pi^2 T^2}{5} - \frac{2T^3}{5m^2} \varepsilon (3|p| + \frac{49}{3} p_0) + \frac{52p_0^2 T^4}{5m^4 |p|} \right]. \tag{7}
\]

The justification of this specific order is the fact that the temperature dependent part corresponds to the contribution of real background particles on mass-shell and incorporates thermal equilibrium. The breaking of Lorentz invariance changes these conditions for the cold integrals. We have checked that the renormalization can only be proven with the preferred order of integrations, i.e., if covariant hot integrals are evaluated before the cold ones. At the higher loop level the vacuum polarization contribution is non zero, even at low temperature. The calculations are simplified if the temperature dependent integrations are performed before the temperature independent ones. The temperature independent loops can then be integrated using the standard techniques of Feynman parametrization and dimensional regularization as in vacuum [18]. At the two loop level, the vertex type corrections to the virtual electron in Fig. (1b) vanish and the self energy type corrections to the electron loop in Fig. (1a) contribute.

The presence of the statistical effects of photons modifies the vacuum polarization and hence the electron charge. This leads to the changes in the electromagnetic properties of the hot medium even at low temperatures. We have calculated the electromagnetic coupling constant in QED in the photon background up to the second order. The longitudinal and the transverse components of the vacuum polarization tensor are

\[
\Pi_L(p, T) = -\frac{p^2}{|p|^2} \epsilon^{\mu\nu}\Pi_{\mu\nu}(p, T) = \frac{2\alpha^2 T^2 p^2}{3|p|^2} \left( 1 + \frac{p_0^2}{2m^2} \right), \tag{8}
\]
Figure 1:

\[ \Pi_T(p, T) = -\frac{1}{2} \Pi_L(p, T) - g^{\mu\nu} \Pi_{\mu\nu}^a(p, T) = \frac{\alpha^2 T^2}{3} \left[ \frac{1}{2} - \frac{p^2}{|p|^4} \left( 1 + \frac{p_0^2}{2m^2} \right) \right]. \] (9)

respectively. These components of the vacuum polarization tensor can then be used to determine the electromagnetic properties of a medium with hot photons.

5 Results

The electron mass and charge renormalization has been obtained from the two loop self-energy for electrons and photons respectively in Ref. [19]. The renormalizability of the self mass of electron is proven through the order by order cancelation of singularities at both loop levels. The charge renormalization constant of QED, the electron charge renormalization up to the order \( \alpha^2 \) can be expressed as

\[ Z_3 = 1 - \frac{\alpha}{3\pi\varepsilon} + \frac{\alpha^2 T^2}{6m^2}. \] (10)

The second term in the above equation, the first order contribution is calculated in detail in Ref. [14]. The corresponding value of the QED coupling constant is now

\[ \alpha_R = \alpha(T = 0) \left( 1 - \frac{\alpha}{3\pi\varepsilon} + \frac{\alpha^2 T^2}{6m^2} \right). \] (11)

The results [16] are an explicit proof of renormalizability of QED up to the two-loop level. They also estimate the temperature dependent modification in the electromagnetic properties of a medium. This helps to evaluate the decay rates and the scattering cross-sections of particles in such a medium. These results can be applied to check the abundance of light elements in primordial nucleosynthesis, baryogenesis and leptogenesis. If the background magnetic fields are also incorporated then one can look for applications to neutron stars, supernovae, red giants, and white dwarfs.

References
1. A. L. Fetter and J. L. Walecka, *Quantum Theory of Many Particle Systems*, (McGraw-Hill, New York, 1971).

2. E. Kolb, M. S. Turner, *Early Universe*, D. A. Dicus, et.al., Phys. Rev. **D26**, 2694 (1982); J. L. Primack and M. Sher, Nucl. Phys. B209, 372 (1982); *ibid* B222, 517(E) (1983); [http://www.damtp.cam.ac.uk/user/gr/public/bb.history.html](http://www.damtp.cam.ac.uk/user/gr/public/bb.history.html).

3. Bradley W. Carroll and Dale A. Ostlie, *Introduction to Modern Astrophysics*, (Benjamin Cummings, 1996).

4. Stuart Shapiro and Saul Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars*, (John Wiley and Sons, 1983).

5. D. A. Kirzhnits, *Field Theoretical Methods in Many Body Systems* (Pergamon, Oxford, 1967); E. M. Lifshitz and L. P. Pitaevskii, *Course on Theoretical Physics - Physical Kinetics* (Pergamon Press, New York).

6. T. Matsubara, Prog. Theor. Phys. **14** (1955) 351.

7. J. Schwinger, J. Math. Phys. **2** (1961) 407

8. R. Mills, *Propagators for Many Particle Systems* (Gordon and Breach, New York, 1969).

9. L. V. Keldysh, Sov. Phys. **20** (1964) 1018.

10. C. Brouder, A. Frabetti [hep-ph/0011161](http://arxiv.org/abs/hep-ph/0011161) and F. T. Brandt, Ashok Das, J. Frenkel, Phys. Rev. **D65** (2002) 085017.

11. H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North Holland, Amsterdam, 1982); A. Das, *Finite Temperature Field Theory* (World Scientific, Singapore, 1997) and several other references.

12. T. Kinoshita, J. Math. Phys. **3** (1962) 650.

13. J. F. Donoghue and B. R. Holstein, Phys. Rev. **D28** (1983) 340; E. Braaten and R. D. Pisarski, Nucl. Phys. **B337** (1990) 567; R. Kobes, Phys. Rev. **D42** (1990) 562; L. Dolan and R. Jackiw, Phys. Rev. **D9** (1974) 3320; P. Landsman and Ch G. Weert, Phys. Rep. **145** (1987) 141 and the references therein.

14. K. Ahmed and Samina Saleem (Masood), Phys. Rev. **D35** (1987) 1861; *ibid* (1987) 4020 and several other papers refered therein; J. F. Donoghue, B. R. Holstein, and R. W. Robinett, Ann. Phys. (N.Y.) **164**, 233 (1985); K. Ahmed and Samina S. Masood, Ann. Phys. (N.Y.) **207** (1991) 460; Samina Masood, Phys. Rev. **D44**, (1991) 3943 and references therein.

15. Samina S. Masood, Phys. Rev. **D47** (1993) 648; *ibid* Phys. Rev. **D36** (1987) 2602.
16. Mahnaz Qader (Haseeb), Samina S. Masood, and K. Ahmed, Phys. Rev. D44 (1991) 3322; ibid Phys. Rev. D46, (1992) 5633 and references therein.

17. L. R. Mohan, Phys. Rev. D14 (1976) 2670.

18. See for example: C. Itzykson and J. B. Zuber, Quantum Field Theory (McGraw- Hill Inc., 1990).

19. Samina S. Masood and Mahnaz Q. Haseeb hep-ph/0406079; Samina S. Masood and Mahnaz Q. Haseeb hep-ph/0612136.