GSE spectra in uni-directional quantum systems

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Abstract. Generically, spectral statistics of spinless systems with time reversal invariance (TRI) and chaotic dynamics are well described by the Gaussian Orthogonal ensemble (GOE). However, if an additional symmetry is present, the spectrum can be split into independent sectors which statistics depend on the type of the group’s irreducible representation. In particular, this allows the construction of TRI quantum graphs with spectral statistics, characteristic of the Gaussian Symplectic ensembles (GSE). To this end one usually has to use groups admitting pseudo-real irreducible representations. In this paper we show how GSE spectral statistics can be realized in TRI systems with simpler symmetry groups lacking pseudo-real representations. As an application, we provide a class of quantum graphs with only $C_4$ rotational symmetry possessing GSE spectral statistics.

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1. Introduction

Starting from the ground breaking work of E. Wigner, [1], Random Matrix Theory (RMT) has been extensively used to describe statistics of energy levels in complex quantum systems. But, the scope of RMT turned out to be much wider. With the initial development of numerical simulations in the 80’s it was realized that RMT can be applied equally to simple single particle systems, like quantum billiards, provided their classical dynamics are fully chaotic [2]. Accordingly, the energy level statistics of such systems fall in one of the three universality classes of RMT, given by the Gaussian Unitary
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(GUE), Gaussian Orthogonal (GOE) and Gaussian Symplectic Ensembles (GSE) of random matrices. The relevant symmetry class is determined by the system’s behavior under the time reversal operation \( T \). In the absence of any additional symmetries, the spectral statistics of systems with time reversal invariance (TRI) follow GOE if \( T^2 = 1 \) and GSE if \( T^2 = -1 \). If, on the other hand, TRI is broken, then GUE statistics are observed \([3]\).

For spin-less particles with intact TRI one always has \( T^2 = 1 \), which generically results in GOE statistics for the corresponding Hamiltonians. This, however, might change if an additional symmetry \( G \) is present in the system. In such a case the energy spectrum can be split into uncorrelated sectors in accordance to the irreducible representations of \( G \). Provided the underlying classical dynamics are chaotic, the type of representation determines the spectral statistics of the corresponding sector \([4,8]\). As a result, GSE statistics appear for systems with symmetry groups possessing pseudo-real representations. This allows, for instance, the construction of quantum graphs with GSE statistics utilizing Cayley graphs of the quaternion group \( Q_8 \), \([9,10]\). However, finite isometry groups of the Euclidean space (i.e., point groups) have no pseudo-real representation. This leaves open the question whether planar quantum billiards or even planar quantum graphs with GSE spectral statistics can be constructed.

An alternative way to obtain GSE statistics in spin-less systems has been recently suggested in \([11,12]\). The idea is that systems with broken TRI might posses GSE statistics even for simple symmetry groups, such as \( C_{2n} \), provided that half of the group elements admit an anti-unitary representation. An example of a magnetic billiard with this property is shown in fig. 1a. Here, the system’s symmetry group is generated by a \( \pi/2 \)-rotation combined with the time reversal operation \( T \). The \( C_2 \) subgroup of rotation by \( \pi \) naturally admits a unitary representation such that the whole billiard spectrum can be split into two sectors. It can then be shown on the basis of co-representation theory \([13,15]\) that the two sectors posses GOE and GSE spectral statistics, respectively \([11,12]\).

The aim of this paper is to demonstrate that the above construction can be carried over to certain types of systems with TRI, as well. Specifically, we provide an example of an unidirectional TRI billiard and graph with \( C_4 \) symmetry exhibiting GSE spectral statistics in the remainder.

2. GSE billiards

2.1. GSE billiard with broken TRI

Before considering TRI systems, let us first explain the construction of \([11]\) in some detail. The symmetry group \( G \) of the billiard in fig. 1a can be decomposed into the union

\[
G = G_+ \cup \bar{T} G_+, \tag{1}
\]

where \( G_+ \cong C_2 \) is the rotation group by \( \pi \) and \( \bar{T} = T g_{\pi/2} \) is the composition of a rotation
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by $\pi/2$ with the time reversal operation $T$. For groups admitting such a decomposition ($\mathbb{Z}_2$ graded groups) Wigner developed a co-representation theory. According to it, a co-representation $\rho(g)$ of a group element $g$ is a linear operator if $g \in G_+$ and an antilinear operator if $g \in \bar{T}G_+$. As in conventional representation theory, there are three types of irreducible co-representations, which can be distinguished by Bargmann’s indicator:

$$
\sigma_B(\rho) = \frac{2}{|G|} \sum_{g \in \bar{T}G_+} \chi_\rho(g^2),
$$

where $\chi_\rho$ is the character of $\rho$ and $g^2 \in G_+$ \cite{14}. The co-representation is real if $\sigma_B(\rho) = 1$, complex if $\sigma_B(\rho) = 0$ and quaternionic if $\sigma_B(\rho) = -1$ \cite{17}. Given a Hamiltonian with a $\mathbb{Z}_2$-graded symmetry group $G$ its spectrum can be decomposed into subspectra corresponding to the irreducible co-representations of $G$. Furthermore, for systems with chaotic dynamics Dyson’s threefold way principle states that the spectral statistics depend on the type of the co-representation. In particular, GSE spectral statistics occurs if $G$ admits a quaternionic co-representation \cite{3}. A direct calculation of Bargmann’s indicator for the group \cite{1} shows that it has two irreducible co-representations with $\sigma_B(\rho) = +1$ and $\sigma_B(\rho) = -1$, respectively \cite{11}. Thereby, the billiard in fig. \cite{1} has GOE and GSE subspectra corresponding to the above co-representations.

An alternative way to see the same result uses the $C_2$ subgroup of $G$ to split the billiard spectrum into a symmetric and an antisymmetric part. The role of the time reversal operation is then fulfilled by the antiunitary operator $\bar{T}$. It is easy to see that its square, $\bar{T}^2$, acts as 1 within the symmetric sector and as $-1$ in the antisymmetric one. Accordingly, the two sectors possess GOE and GSE spectral statistics, respectively.

2.2. GSE billiard with TRI

A prototype construction of a GSE billiard with TRI is sketched in fig.\cite{1}. The billiard is composed of four blocks (shown in fig.\cite{1b}) such that the total system has $C_4$ symmetry.

Figure 1. a) GSE billiard with $\pi/2$ rotational symmetry from \cite{11}. At four corners a magnetic field of opposite sign is added. b) A unidirectional billiard of constant width with chaotic dynamics. The two directions of motion illustrated by red and blue arrows are dynamically decoupled from each other \cite{16}. c) A unidirectional billiard with $C_4$ symmetry composed of four blocks in figure (b).
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Being of constant width each block possesses unidirectional dynamics, i.e., clockwise and anticlockwise motions are ergodically separated \[18\]. Despite of this, the dynamics are fully hyperbolic such that the Lyapunov exponent is positive almost everywhere \[16\]. The resulting total billiard possesses unidirectional dynamics as well, but the direction of motion is interchanged between neighboring blocks. Due to the last property a rotation by \(\pi/2\) of the whole billiard changes the direction of motion to the opposite one, i.e., switches between the two ergodic components.

To see the implications of the unidirectional dynamics on the quantum level it is instructive to split the Hilbert space of the system into clockwise and anticlockwise sectors. After defining a chiral basis within each sector the total Hamiltonian of the time reversal system can be cast into the form

\[
H = \begin{pmatrix} H_+ & W^\dagger \\ W & H_- \end{pmatrix}, \quad H_+ = H_-^*, \quad W = W^*,
\]

where \(H_+\) and its complex conjugate \(H_-\) act within the clockwise and anti-clockwise sectors \(\mathcal{H}_\pm\) of the Hilbert space, respectively, while \(W, W^\dagger\) couple between them. For strictly unidirectional systems (such as circle billiards) \(W = 0\) holds and their spectrum is decomposed into doubly degenerate eigenvalues provided by the spectra of \(H_+\) and \(H_-\). In the absence of any additional symmetry \(H_\pm\) has the form of a generic Hermitian matrix. Therefore the doublets of chaotic unidirectional systems possess GUE spectral statistics. But for unidirectional systems with \(C_4\) rotation symmetry as in fig. 1c, \(H_\pm\) possesses an additional constrain. Namely, under rotation by \(\pi/2\), \(H_+\) is transformed into \(H_-\). This makes the combination of \(g_{\pi/2}\) with the time reversal operation, \(T g_{\pi/2}\), a symmetry of \(H_\pm\). In other words, \(H_\pm\) stays invariant under the action of the group \((1)\). Accordingly, for strictly unidirectional systems with \(C_4\) symmetry and chaotic dynamics one might expect a spectrum of doublets distributed according to GOE statistics and quadruplets obeying GSE statistics.

For the billiards shown in fig. 1b,c unidirectionality is, in fact, weakly broken on the quantum level via diffraction effects, i.e., \(W \neq 0\) is reminiscent of singular perturbations \[19\]. In particular, this lifts the degeneracies and thereby affects the spectral statistics of the system. Nevertheless, traces of GUE statistics have been previously observed in billiards of the same type as presented in fig. 1b, \[20\]. And in the same spirit it might be expected that traces of GSE and GOE spectral statistics (depending on the sector) are recognizable in the billiard depicted in fig. 1c. To avoid these complications we illustrate the above construction for a more accessible class of systems – quantum graphs, where unidirectionality is exact.

\[\text{‡}\] Strictly speaking, for billiards of constant width a more accurate model of the system Hamiltonian \[3\] should include, in addition, a neutral part \(H_0 = H_0^\dagger\) corresponding to the sector \(\mathcal{H}_0\) of the Hilbert space spanned by bouncing ball modes \[21\]. For instance, in the circle billiards Bessel functions provide a sequence of eigenstates with zero angular momentum, i.e., they belong to \(\mathcal{H}_0\). However, the ratio between the dimensions of \(\mathcal{H}_0\) and \(\mathcal{H}_\pm\) tends to 0 in the semiclassical limit. For the sake of simplicity of exposition we omit the neutral sector from the Hamiltonian \[3\].
3. Unidirectional GSE graphs

Based on the prototype GSE billiard shown in fig. 1b we construct a family of quantum graphs with GSE spectral statistics. The basic idea is to substitute each block in fig. 1c with a quantum graph possessing unidirectional dynamics.

3.1. Unidirectional quantum graphs.

Let $\Gamma$ be a graph with $N$ bonds connecting $V$ vertices. Then the corresponding quantum graph is defined in terms of the Schrödinger equation on the bonds, i.e., free wave propagation, satisfying some boundary conditions at the respective vertices [22]. To formulate this problem in a way more suitable for our discussion we introduce the corresponding quantum map acting on directed graph bonds. To this end each bond of $\Gamma$ is equipped with two directions such that the set of $2N$ directed bonds can be split into two halves, $1_+,...,N_+$ and $1_-,...,N_-$, where $i_+$ and $i_-$ correspond to the opposite directions of the $i$'th bond of the graph $\Gamma$. We denote with $\Gamma_+$, $\Gamma_-$ the pair of graphs with the same topological structure as $\Gamma$, but composed of the directed bonds $1_+,...,N_+$ and $1_-,...,N_+$, respectively.

For a given wavenumber $k$ the quantum map is defined as a product, $S(k) = S_0 \Lambda(k)$, of the diagonal part,

$$\Lambda(k) = \text{diag}\{e^{ik\ell_1},...,e^{ik\ell_N},e^{ik\ell_1},...,e^{ik\ell_N}\},$$

depending on the bond length’s $\ell_i$ and the constant scattering matrix $S_0$ which encodes the boundary conditions at the vertices. Specifically, $S_0$ is composed of local unitary blocks $\sigma^{(v)}$, $v = 1,2,...,V$, describing the scattering processes at the vertex $v$. The quantum map $S(k)$ acts on the $2N$-dimensional vectors $(\psi_1,...,\psi_N,\psi_{N+1},...,\psi_{2N})$, where the first $N$ indices belong to $\Gamma_+$, and the remaining to $\Gamma_-$. The quantum graph spectrum, $\lambda_n = k_n^2$, is then found as the solutions $k_n,n = 1,...,\infty$, to the secular

§ As the assignment of $+$ or $-$ to a particular direction of the graph’s bonds is a matter of choice, there are $2^N$ different ways to define $\Gamma_{\pm}$ for a general graph. However, for unidirectional graphs the choices are more restricted, as detailed in the remainder of the text.
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det(I - S(k)) = 0. \hspace{1cm} (5)

The unidirectional quantum graphs were introduced in [19]. Their construction is based on a special choice of the local TRI scattering matrices,

\[ \vec{\psi}_{out} = \sigma^{(v)} \vec{\psi}_{in}, \hspace{1cm} (6) \]

which relate incoming and outgoing wave functions \( \vec{\psi}_{in}, \vec{\psi}_{out} \) at the vertex \( v \) [19]. Specifically, the matrix \( \sigma^{(v)} \) must have even dimensions \( 2M_v \times 2M_v \) such that the \( 2M_v \) indices can be separated into two subgroups – “entrances” \( \{1, 2, \ldots, M_v\} \) and “exits” \( \{1, 2, \ldots, M_v\} \). The corresponding vertex elements satisfy \( \sigma^{(v)}_{i,j} = \sigma^{(v)}_{i,j} = 0 \), for \( i, j \in 1, \ldots, \). Thereby each vertex only connects “entrances” to “exits” \((i \rightarrow j)\) and vice verse. A TRI, unidirectional graph \( \Gamma \) is then constructed by pairwise connecting exits to entrances from different vertices. Due to the dynamical separation of the two “directions of motion” at the graph vertices the resulting scattering matrix \( S(k) \) can be split into two blocks [19],

\[ S(k) = \begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix}, \hspace{1cm} S_+ = S^T, \hspace{1cm} (7) \]

where \( S_+ \) and \( S_- \) act on the \( \Gamma_+ \) and \( \Gamma_- \) parts of the vectors. Accordingly, the transition matrix \( |S_{i,j}|^2 \), for the underlying classical Markov chain, possesses a doubly degenerate largest eigenvalue 1 corresponding to two ergodic components supported on \( \Gamma_+ \) and \( \Gamma_- \), respectively.

By eq. (7) the spectrum of a TRI unidirectional quantum graph is doubly degenerate, the identical eigenvalues stem from \( S_+ \) and \( S_- \), respectively. Furthermore, in the absence of symmetries \( S_\pm \) takes on the form of a generic unitary matrix such that the doublets \( \{\lambda_n\}_{n=1}^{\infty} \) defined by eq. (5) possess GUE statistics rather than GOE, as would have been expected for a TRI systems.

3.2. Construction of GSE quantum graphs.

Similar to the GSE billiard presented in fig. 1c: its quantum graph analogue is build from four copies of a TRI unidirectional quantum graph \( \Gamma \). The procedure goes as follows: In the first step, we cut \( 2M \) bonds of the original graph \( \Gamma \) in the middle, turning it into the open graph \( \Gamma_0 \), see figs. 2a,b. Due to unidirectionality the \( 4M \) loose ends of \( \Gamma_0 \) are naturally separated into two groups of \( 2M \) “exits” and \( 2M \) “entrances”, as discussed in the previous section. We will enumerate all “entrances” by numbers from 1 to \( 2M \) (in an arbitrary order) and the respective “exits” by numbers with bars, such that \( i \) and \( i \) correspond to one and the same original bond of \( \Gamma \). Due to the unidirectionality of the original graph, the scattering matrix corresponding to the open graph \( \Gamma_0 \) satisfies \( S^{(0)}_{i,j} = S^{(0)}_{i,j} = 0 \) for \( i, j \in 1, \ldots, 2M \) (compare with the analogous property of the vertex matrices \( \sigma^{(v)} \)). In other words, \( S^{(0)} \) connects ”entrances” with ”exits” and vice versa. In the second step we divide all entering (resp. exiting) ends
into two groups \( \{1, \ldots, M\} \cup \{M+1, \ldots, 2M\} \) (resp. \( \{1, \ldots, M\} \cup \{M+1, \ldots, 2M\} \)) and connect four copies of \( \Gamma_0 \) according to the rule (see fig. 2c):

\[
i \leftrightarrow M+i, \quad \bar{i} \leftrightarrow M+i, \quad i \in \{1, \ldots, M\}.
\]

The constructed quantum graph is strictly unidirectional and its scattering matrix \( S(k) \) has the same split form (7), where \( S_{\pm} \) correspond to the two different ergodic components. On the other hand, since the same type of entries are put together, e.g., bar with bar, the directionality between neighbouring copies of \( \Gamma \) is reversed - everything entering from \( \Gamma_+ \) ends in \( \Gamma_- \) and vice versa. As a result, the rotation by \( \pi/2 \) leads to an exchange of the clockwise and anti-clockwise blocks, such that

\[
g_{\pi/2} S_{\pm} g_{\pi/2}^{-1} = S_{\mp} = S_{\pm}^T.
\]

From eq. (8) and the time reversal invariance condition, \( TST = S^T \), immediately follows that \( [S_{\pm}, T g_{\pi/2}] = 0 \), i.e., \( T g_{\pi/2} \) is the anti-unitary symmetry of \( S_{\pm} \). Repeating the arguments of section 2.2 the spectra of the \( S_{\pm} \) blocks can be split into periodic and anti-periodic sectors with respect to the rotation by \( \pi \). The resulting spectral statistics are expected to be in good agreement with the ones of doubly degenerate GOE and four-fold degenerate GSE, respectively, provided that the underlying graph \( \Gamma \) obeys (GUE) RMT predictions and the connectivity between the four copies is of sufficient rank. The topological structure of \( \Gamma \) can otherwise be arbitrary. In particular, it might also be planar.

In fig. 3b we provide an example of a GSE quantum graph, where each block \( \Gamma \) is a fully connected unidirectional graph, shown in fig. 3a. Four edges of the graph \( \Gamma \) are cut in the middle and reconnected according to the above rules. The \( 2N \times 2N \) evolution matrix (quantum map) \( S(k) \) is defined then, as in (5), with \( N = 84 \). The nearest-neighbour spacing distribution of the corresponding \( k_n \) spectrum, for the separate sectors, is presented in fig. 4. As predicted, it shows an excellent agreement with GOE and GSE statistics, respectively.
4. Conclusions

We provided an explicit example of TRI quantum graphs with GSE spectral statistics. As opposed to previous cases these systems have a simple $C_4$ symmetry group, which does not admit a pseudo-real representation. This allows (at least in principle) to realize them as planar structures. The key ingredient to our construction is the dynamical separation of the system into two ergodic components. In combination with the four-fold rotation symmetry this leads to two uncorrelated subspectra obeying the GOE and GSE statistics, respectively.

The construction can be straightforwardly extended to a general $\mathbb{Z}_2$-graded group, provided it has a pseudo-real co-representation. In particular, the $C_4$ symmetry group can be traded off to any rotation group $C_{4n}$. The corresponding graph is constructed by cyclically connecting $4n$ copies of a unidirectional graph. In such a case the spectrum of the resulting quantum graph is divided into $n$ doubly degenerate GOE subspectra and $n$ four-fold degenerate GSE subspectra. It would be of interest to explore whether an experimental microwave realization of such quantum graphs is possible.

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