Epidemics on networks with preventive rewiring

Frank Ball
Frank.Ball@nottingham.ac.uk

University of Nottingham

Conference on Branching Processes and Applications, Angers, 24 May 2023

Joint work with Tom Britton (Stockholm University)

Ball, F. and Britton T. (2022) *Random Struct. Alg.* **61**, 250-297.
Network epidemic models

Random graph of possible contacts

Spread epidemic on graph. Interest often focussed on effect of properties of the random graph on disease dynamics.

In this talk we analyse a model with adaptive dynamics, in which susceptibles may rewire edges away from infective neighbours.
Model with preventive rewiring

- Population of size $n$ socially structured by an Erdős-Rényi graph $G(n, \frac{\mu}{n})$. Between each of the $\binom{n}{2}$ pairs of distinct nodes an edge is present independently with probability $\frac{\mu}{n}$, where $\mu > 1$ so giant component exists for large $n$.

- Markovian SIR (susceptible $\rightarrow$ infective $\rightarrow$ recovered) epidemic model with infection rate $\lambda$ between neighbours and recovery rate $\gamma$.

- If a susceptible individual has an infective neighbour then that edge is rewired (to an individual chosen uniformly at random from the other $n - 2$ individuals in the population) at rate $\omega$.

- Equivalently, an infective warns their neighbours independently at rate $\omega$ and warned susceptibles rewire such edges.

- Initially one infective and all other individuals susceptible.

(Jiang et al. (2019), cf. Britton et al. (2016), Leung et al. (2018))
Approximating branching process $\mathcal{B}$

- The process of infectives in the initial phase of an epidemic can be approximated by a branching process $\mathcal{B}$ in which
  - the lifetime of an individual $\sim \text{Exp}(\gamma)$;
  - at birth an individual is assigned $\text{Po}(\mu)$ infective edges;
  - an individual drops each infective edge independently at rate $\omega$ and infects down them independently at rate $\lambda$;
  - when an individual infects down an edge, a new individual is born and the edge is dropped.

- The basic reproduction number $R_0$ for the epidemic is given by the offspring mean of $\mathcal{B}$, viz.

$$R_0 = \mu \frac{\lambda}{\lambda + \omega + \gamma}.$$
Threshold theorem

Theorem 1  Let $T^{(n)}$ be the final size of the epidemic (i.e. the total number infected) and $T$ be the total progeny of the branching process $B$.

(a) $T^{(n)} \xrightarrow{p} T$ as $n \to \infty$.

(b) $\lim_{n \to \infty} P(T^{(n)} \geq \log n) = P(T = \infty)$.

(c) Suppose $R_0 > 1$. Then there exists $\tau' = \tau'(\mu, \lambda, \gamma, \omega) > 0$ such that

$$\lim_{n \to \infty} P(n^{-1} T^{(n)} \geq \tau'|T^{(n)} \geq \log n) = 1.$$ 

We say that a major epidemic occurs if $T^{(n)} \geq \log n$.

In the limit $n \to \infty$, a major epidemic occurs with non-zero probability if and only if $R_0 = \frac{\mu \lambda}{\lambda + \omega + \gamma} > 1$.

If all other parameters are held fixed, $R_0 > 1 \iff \lambda > \lambda_C = \frac{\gamma + \omega}{\mu - 1}$. 

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Epidemics on networks with preventive rewiring – p.5
Final outcome of SIR model with rewiring

Final fraction infected in SIR model with rewiring on Erdős-Rényi graph with 5 initial infectives. Vertical line shows value of $\lambda (= \lambda_C)$ so that $R_0 = 1$. Figure based on Jiang et al. (2019), Figure 4.
Construction of nearly-exact SIR model

Key ideas of construction

- Construct epidemic and partial network simultaneously.
- Only consider edges from infectives that are connected to susceptibles.
- Only keep track of the number of susceptible-susceptible rewired edges and not the individuals involved.
Construction of nearly-exact SIR model

Let $S(t)$, $I(t)$ and $W(t)$ be the numbers of susceptibles, infectives and susceptible-susceptible rewired edges at time $t$ and let $S(t)$ be the set of susceptibles at time $t$.

When an individual is infected it acquires $\text{Po}(\mu_n S(t)/n)$ infectious edges. where $\mu_n = \mu (1 - \frac{\mu}{n})^{-1}$.

Infectious edges send warnings (and the infective loses the edge) at rate $\omega$. When warning occurs, the edge is “rewired" to a susceptible, infective or recovered with probabilities $\frac{S(t)-1}{n-2}$, $\frac{I(t)-1}{n-2}$ and $\frac{n-S(t)-I(t)}{n-2}$. If the rewire is to

- a susceptible then $W(t) \rightarrow W(t) + 1$;
- an infective then that infective gains an infectious edge;
- a recovered then nothing further happens.
Construction of nearly-exact SIR model

- Each infectious edge transmits infection at rate $\lambda$. Then the edge is dropped, $(S(t), I(t)) \to (S(t) - 1, I(t) + 1)$ and
- an individual ($i_0$ say), chosen uniformly at random from $S(t-)$ is infected and $S(t) \to S(t) \setminus \{i_0\}$;
- each other infectious edge is dropped independently with probability $\frac{1}{S(t-)}$;
- individual $i_0$ acquires $R \sim \text{Bin} \left( W(t-), \frac{2}{S(t-)} \right)$ further (rewired) infectious edges and $W(t) \to W(t) - R$.

- Infectives recover (and lose any remaining infectious edges) at rate $\gamma$.
- Construction is fully faithful to the original model if there is no multiple edge.
- $\liminf_{n \to \infty} P(\text{no multiple edge}) > 0$, so convergence in probability results transfer from construction to original model.
Construction of nearly-exact SIR model

For \( t \geq 0 \) and \( j = 0, 1, \ldots \), let \( I_j^{(n)}(t) \) be the number of infectives with \( j \) infectious edges at time \( t \).

Then \( X^{(n)} = \{ (S^{(n)}(t), I_0^{(n)}(t), I_1^{(n)}(t), \ldots, W^{(n)}(t)) : t \geq 0 \} \) is a density-dependent continuous-time Markov chain, with an infinite-dimensional state space, \( E^{(n)} \) say, so the LLN in Ethier and Kurtz (1986) cannot be applied.

Let \( X^{(n)}(t) = (S^{(n)}(t), I^{(n)}(t), I_E^{(n)}(t), W^{(n)}(t)) \), where
\[
I_E^{(n)}(t) = \sum_{j=0}^{\infty} jI_j^{(n)}(t)
\]
is the total number of infectious edges at time \( t \).

Can apply Darling and Norris (2008), Theorem 4.1, to obtain a WLLN for \( \{ X^{(n)}(t) : t \geq 0 \} \).
Weak law of large numbers

Theorem 2  Suppose $n^{-1} X^{(n)}(0) \xrightarrow{p} x(0)$ as $n \to \infty$, where $i(0) > 0$ and $i_E(0) > 0$. Then, for any $t_0 > 0$,

$$\sup_{0 \leq t \leq t_0} \left| n^{-1} X^{(n)}(t) - x(t) \right| \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty,$$

where $x(t) = (s(t), i(t), i_E(t), w(t))$ is the solution of the ODE

$$\frac{ds}{dt} = -\lambda i_E,$$

$$\frac{di}{dt} = \lambda i_E - \gamma i,$$

$$\frac{di_E}{dt} = \lambda i_E \left[ \mu s + 2 \frac{w}{s} - 1 - \frac{i_E}{s} \right] - \gamma i_E - \omega i_E (1 - i),$$

$$\frac{dw}{dt} = \omega i_E s - 2 \lambda i_E \frac{w}{s},$$

having initial condition $x(0) = (s(0), i(0), i_E(0), w(0))$. 
Illustration of WLLN

100 simulated realisations of trajectories of fraction infected in SIR model with $\mu = 5$, $\lambda = 1.5$, $\gamma = 1$, $\omega = 4$ ($R_0 = 1.1538$) and 1% initially infective. Also shown is the deterministic fraction $i(t)$ (solid curve) and the mean of the stochastic trajectories (dashed curve).
Illustration of WLLN

Upper row: 1% initially infective. Lower row: 5% initially infective
Let $\zeta^{(n)} = \inf\{t \geq 0 : I_E^{(n)}(t) = 0\}$, then the final size $T^{(n)}$ of the epidemic is given by $T^{(n)} = n - S^{(n)}(\zeta^{(n)})$.

To study $T^{(n)}$ it is fruitful to consider the following random time-scale transformation of $X^{(n)}$ (cf. Watson (1980) and Janson et al. (2014)).

Let $\xi = (n^S, n_0^I, n_1^I, \ldots, n^W)$ be a typical element of the state space $E^{(n)}$ of $X^{(n)}$ and $n^E = \sum_{k=0}^{\infty} kn_k^I$ and $\tilde{X}^{(n)}$ be the process with jump rates

$$\tilde{q}^{(n)}(\xi, \xi') = q^{(n)}(\xi, \xi')/(\lambda n^{-1} n^E) \quad (\xi, \xi' \in E^{(n)}, \xi \neq \xi').$$

The distribution of final size is invariant to this time transformation. We use $\tilde{X}^{(n)} = \{(\tilde{X}^{(n)}(t), \tilde{I}^{(n)}(t), \tilde{I}_E^{(n)}(t), \tilde{W}^{(n)}(t)) : t \geq 0\}$ to analyse $T^{(n)}$. 
The time-transformed deterministic approximation to $n^{-1} \tilde{X}^{(n)}$ is

\[
\frac{d\tilde{s}}{dt} = -1,
\]
\[
\frac{d\tilde{i}}{dt} = 1 - \frac{\gamma}{\lambda} \frac{\tilde{i} \tilde{i}_E}{s},
\]
\[
\frac{d\tilde{i}_E}{dt} = \mu s + 2 \tilde{w} \frac{\tilde{i}_E}{s} - 1 - \frac{\tilde{i}_E}{s} - \frac{\gamma}{\lambda} - \frac{\omega}{\lambda} (1 - \tilde{i}),
\]
\[
\frac{d\tilde{w}}{dt} = \frac{\omega s}{\lambda} - 2 \frac{\tilde{w}}{\tilde{s}}.
\]

Final fraction infected $\tau = 1 - \tilde{s}(\tilde{\zeta})$, where $\tilde{\zeta} = \inf\{t > 0 : \tilde{i}_E(t) = 0\}$. (Note $\tilde{\zeta} < \infty$, unlike $\zeta = \inf\{t > 0 : i_E(t) = 0\}$.)

Problems owing to this system not being Lipschitz in the neighbourhood of $\tilde{i}_E = 0$:

- Darling and Norris (2008) Theorem 4.1 cannot be applied.
- For epidemics with few initial infectives, $\tau$ depends on $\lim_{t \downarrow 0} \frac{\tilde{i}(t)}{\tilde{i}_E(t)}$. 
Discontinuity at threshold $\lambda = \lambda_C$

Consider modifications which bound the epidemic process with rewiring:

- a lower bounding process, in which if a susceptible rewire an edge from one infective to another infective then the edge is dropped;
- an upper bounding process, in which if a susceptible rewire an edge from an infective to a recovered individual then the edge to the infective is retained.

Both modifications have the same approximating branching process $B$, $R_0$ and $\lambda_C$ as the original process, and yield time-transformed deterministic models for $(\tilde{s}(t), \tilde{i}_E(t), \tilde{w}(t))$ that are closed and Lipschitz.

In a time transformed deterministic model, $\tilde{i}_E'(0) = 0 \iff \lambda = \lambda_C$. The final size is discontinuous (continuous) at $\lambda = \lambda_C$ if $\tilde{i}_E''(0) > 0$ ($< 0$) when $\lambda = \lambda_C$. 
Discontinuity at threshold $\lambda = \lambda_C$

Theorem 3  Suppose that $R_0 > 1$.

(a) Suppose that $\omega > \gamma$ and $\mu > \frac{2\omega}{\omega - \gamma}$. Then there exists $\tau_0 = \tau_0(\mu, \gamma, \omega) > 0$ such that, conditional upon a major epidemic,

$$\lim_{n \to \infty} P(n^{-1}T^{(n)} > \tau_0) = 1 \quad \text{for all } \lambda > \lambda_C.$$

(b) Suppose that $2\omega \leq \gamma$ or $\mu \leq \frac{3\omega}{2\omega - \gamma}$. Then, for all $a > 0$, there exists $\lambda_1 > \lambda_C$ such that, conditional upon a major epidemic,

$$\lim_{n \to \infty} P(n^{-1}T^{(n)} < a) = 1 \quad \text{for all } \lambda \in (\lambda_C, \lambda_1).$$
**Discontinuity at threshold** $\lambda = \lambda_C$

**Theorem 3** Suppose that $R_0 > 1$.

(a) Suppose that $\omega > \gamma$ and $\mu > \frac{2\omega}{\omega - \gamma}$. Then there exists $\tau_0 = \tau_0(\mu, \gamma, \omega) > 0$ such that, conditional upon a major epidemic,

$$\lim_{n \to \infty} P(n^{-1}T^{(n)} > \tau_0) = 1 \quad \text{for all } \lambda > \lambda_C.$$

(b) Suppose that $2\omega \leq \gamma$ or $\mu \leq \frac{3\omega}{2\omega - \gamma}$. Then, for all $a > 0$, there exists $\lambda_1 > \lambda_C$ such that, conditional upon a major epidemic,

$$\lim_{n \to \infty} P(n^{-1}T^{(n)} < a) = 1 \quad \text{for all } \lambda \in (\lambda_C, \lambda_1).$$

**Theorem 3'** (Chen, Hou and Yao (2022)). Theorem 3(b) holds if $\omega \leq \gamma$ or $\mu \leq \frac{2\omega}{\omega - \gamma}$. 
Final outcome of epidemic

Suppose that $R_0 > 1$. For $\varepsilon \in (0, 1)$, let $x^\varepsilon(t) = (s^\varepsilon(t), i^\varepsilon(t), i_E^\varepsilon(t), w^\varepsilon(t))$ be the solution of the deterministic model with $x^\varepsilon(0) = (1 - \varepsilon, \varepsilon, L^{-1}\varepsilon, 0)$, where $L = \frac{\lambda}{\lambda(\mu-1)-\omega}$, and $\tau = 1 - \lim_{\varepsilon \downarrow 0} s^\varepsilon(\infty)$.

$L$ is the almost sure limit of $I(t)/I_E(t)$ as $t \to \infty$ in the approximating branching process, conditional upon non-extinction.

**Conjecture 1** Conditional upon a major epidemic,

$$n^{-1} T^{(n)} \xrightarrow{p} \tau \quad \text{as} \quad n \to \infty.$$  

Proved in Chen, Hou and Yao (2022) when $\omega \leq \gamma$ or $\mu \leq \frac{2\omega}{\omega-\gamma}$, i.e. when there is not a discontinuity at the threshold $\lambda = \lambda_C$. 

1,000 simulations of final size of SIR epidemic when \( n = 10,000, \mu = 5, \gamma = 1, \alpha = 1 \) and varying \( \lambda; \ \omega = \frac{3}{2} \) in the left panel and \( \omega = 4 \) in the right panel. Each simulation was started with 5 infectives. Solid curves show limiting fraction infected predicted by Conjecture 1.
Suppose removal rate $\gamma = 0$ so infectives remain so forever, and $I^{(n)}(t) = n - S^{(n)}(t)$ and $i(t) = 1 - s(t)$ for all $t \geq 0$.

Time-transformed ODE for $(\tilde{s}(t), \tilde{i}_E(t), \tilde{w}(t))$ is Lipschitz and admits a closed-form solution.

**Theorem 4**  
(a) Suppose $R_0 > 1$. Then conditional upon a major epidemic,  

$$n^{-1}T^{(n)} \xrightarrow{p} \tau \quad \text{as} \ n \to \infty,$$

where $\tau = \tau_{SI}(\mu, \lambda, \omega)$ is the unique solution in $(0, 1)$ of

$$1 - \tau = \exp \left( -\frac{\tau(\mu \lambda + \omega)}{\lambda + 2\omega(1 - \tau)} \right).$$

(b) Provided $\omega > 0$, $\tau_{SI}(\mu, \lambda, \omega) \to \tau_0(\mu)$ as $\lambda \downarrow \lambda_C$ ($= \frac{\omega}{\mu-1}$), where

$$\tau_0(\mu) > 0 \iff \mu > \frac{3}{2}.$$
\[ \tau_0(\mu) = \lim_{\lambda \downarrow \lambda_C} \tau_{SI}(\mu, \lambda, \omega) \]
Dependence of final size on $\omega$

Recall that $R_0 = \frac{\mu \lambda}{\lambda + \omega}$. Fix $\mu > 1$, $\lambda > 0$ and let $\omega_C = (\mu - 1)\lambda$. Then

$$R_0 > 1 \iff \omega \in [0, \omega_C).$$

Let $\mu_0 (\approx 1.7564)$ be the unique solution in $[1, \infty)$ of $2\mu = e^{\mu - \frac{1}{2}}$. Then for $\omega \in [0, \omega_C)$,

$$\tau_{SI}(\mu, \lambda, \omega) \begin{cases} 
\text{decreases with } \omega & \text{if } \mu < \mu_0 \\
\text{constant with } \omega & \text{if } \mu = \mu_0 \\
\text{increases with } \omega & \text{if } \mu > \mu_0
\end{cases} \text{ rewiring beneficial, neutral, harmful.}$$

Note $\tau_{SI}(\mu, \lambda, 0) = \text{size of giant component of Erdős-Rényi graph } G(n, \frac{\mu}{n})$ for all $\lambda > 0$. 

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Epidemics on networks with preventive rewiring – p.23
Final size $\tau_{SI}(\mu, \lambda, \omega)$ when $\lambda = 1$

Four regimes are: (a) $1 < \mu \leq 1.5$; (b) $1.5 < \mu < \mu_0$; (c) $\mu = \mu_0$; and (d) $\mu > \mu_0$, where $\mu_0 > 1$ solves $2\mu = e^{\mu - \frac{1}{2}}$. 

Epidemics on networks with preventive rewiring – p.24
Rewiring only to susceptibles

Suppose that when a susceptible rewires an edge away from an infective, they rewire to an individual chosen uniformly at random from the other susceptibles.

The deterministic approximation becomes

\[
\begin{align*}
\frac{ds}{dt} &= -\lambda i_E, \\
\frac{di}{dt} &= \lambda i_E - \gamma i, \\
\frac{di_E}{dt} &= \lambda i_E \left[ \mu s + 2 \frac{w}{s} - 1 - \frac{i_E}{s} \right] - \gamma i_E - \omega i_E, \\
\frac{dw}{dt} &= \omega i_E - 2\lambda i_E \frac{w}{s}.
\end{align*}
\]

The equations for \((s, i_E, w)\) form a closed system.
Rewiring only to susceptibles - final size

- The time transformed deterministic model for \((\tilde{s}(t), \tilde{i}_E(t), \tilde{w}(t))\) is Lipschitz. Its solution with initial condition \((\tilde{s}(0), \tilde{i}_E(0), \tilde{w}(0)) = (1, 0, 0)\) is

\[
\tilde{s}(t) = 1 - t, \quad \tilde{i}_E(t) = \tilde{s}(t)\tilde{g}(\tilde{s}(t)), \quad \tilde{w}(t) = \frac{\omega \alpha}{\lambda} \tilde{s}(t)(1 - \tilde{s}(t)).
\]

where

\[
\tilde{g}(s) = \left(1 + \frac{\gamma - \omega}{\lambda}\right) \log \tilde{s} + \left(\mu - \frac{2\alpha}{\lambda}\right)(1 - \tilde{s}).
\]

- Note that \(\tilde{i}_E(t) = 0 \iff \tilde{s}(t) = 0\) or \(\tilde{g}(\tilde{s}(t)) = 0\).

- The equation \(\tilde{g}(s) = 0\) has 0 or 1 solution in \((0, 1)\). If it has 0 solution then, in the model in real time, the final fraction susceptible \(s(\infty) = 0\), otherwise it is given by the solution of \(\tilde{g}(s) = 0\) in \((0, 1)\).
Theorem 5  Suppose that \( R_0 = \frac{\mu \lambda}{\lambda + \omega + \gamma} > 1 \). Then, conditional upon a major epidemic,

\[
n^{-1} T^{(n)} \xrightarrow{p} \tilde{\tau} = \tilde{\tau}(\mu, \lambda, \gamma, \omega) \quad \text{as } n \to \infty,
\]

where

(a) if \( \mu (\gamma - \omega) + 2\omega \geq 0 \) then, for all \( \lambda > \lambda_C \), \( \tilde{\tau} \) is given by the unique solution in \((0, 1)\) of \( \tilde{g}(1 - x) = 0 \), and is continuous at \( \lambda = \lambda_C \);

(b) if \( \mu (\gamma - \omega) + 2\omega < 0 \) then \( \tilde{\tau} = 1 \), for \( \lambda_C < \lambda \leq \omega - \gamma \), and \( \tilde{\tau} \) is given by the unique solution in \((0, 1)\) of \( \tilde{g}(1 - x) = 0 \), for \( \lambda > \omega - \gamma \).
1,000 simulations of final size of SIR epidemic with rewiring only to susceptibles when $n = 10,000$, $\mu = 2.5$, $\gamma = 1$, $\alpha = 1$ and varying $\lambda$; $\omega = 4$ in the left panel and $\omega = 10$ in the right panel. Each simulation was started with 10 infectives. Solid curves show limiting fraction infected predicted by Theorem 5.
Concluding comments

- All results generalise to the model in which warned susceptibles rewired the edge with probability $\alpha \in (0, 1)$ and dropped it otherwise.

- Approximating deterministic model is equivalent to a pair-approximation model.

- Extension to other network models, e.g. configuration model (see Yao and Durrett (2022) for SI model).

- $R_0 < 1$ may not prevent a large epidemic unless $n$ is very large.
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