The Abraham–Minkowski momentum controversy is the outwardly visible symptom of an inconsistency in the use of the energy–momentum tensor in the case of a plane quasimonochromatic field in a simple linear dielectric. We show that the Gordon form of the electromagnetic momentum is conserved in a thermodynamically closed system. We regard conservation of the components of the four-momentum in a thermodynamically closed system as a fundamental property of the energy–momentum tensor. Then the first row and column of the energy–momentum tensor is populated by the electromagnetic energy density and the Gordon momentum density. We derive new electromagnetic continuity equations for the electromagnetic energy and momentum that are based on the Gordon momentum density. These continuity equations can be represented in the energy–momentum tensor using a material four-divergence operator in which temporal differentiation is performed with respect to \( \frac{ct}{n} \).

I. INTRODUCTION

The energy–momentum tensor is a concise way to represent the conservation properties of a flow field. Simple in concept, the form of the energy–momentum tensor for the field in a dielectric has been at the center of the century-long Abraham–Minkowski momentum controversy [1, 2]. The tensor that was proposed by Minkowski [3] in 1908 is not symmetric and consequently does not conserve angular momentum [4]. Abraham [5] subsequently proposed a symmetric tensor at the expense of a phenomenological force. The original point of contention of the Abraham–Minkowski momentum controversy was whether the Abraham momentum density

\[
  g_A = \frac{(E \times H)}{c}
\]

(1.1)

or the Minkowski momentum density

\[
  g_M = \frac{(D \times B)}{c}
\]

(1.2)

provides the correct description of the momentum of light in a linear medium. The absence of an experimental decision allowed the debate to persist until the 1960s, when resolution of the Abraham–Minkowski dilemma was provided by Penfield and Haus [6], based on earlier work by Møller [7], who showed that the issue is undecidable because neither the spatial integral of the Abraham momentum density nor the spatial integral of the Minkowski momentum density is the total momentum of the closed system. Likewise, the total energy–momentum tensor of the closed system is not the Abraham energy–momentum tensor and it is not the Minkowski energy–momentum tensor.

What, then, is the total momentum and the total energy–momentum tensor? Here, we adopt the definition of the total momentum as the momentum quantity that is conserved in a thermodynamically closed system. This approach has the advantage of obtaining the total momentum directly from global conservation principles and neatly avoids any issues regarding the ill-defined roles of the Abraham and Minkowski momentums. We construct a thermodynamically closed system consisting of a homogeneous dielectric block illuminated by a quasimonochromatic field at normal incidence through an antireflection-coating. In this closed system, the total energy and total momentum are conserved quantities and are well-defined by virtue of being conserved. We find that the total energy is the spatial integral of the electromagnetic energy density

\[
  \rho_e = \frac{1}{2} (n^2 E^2 + B^2)
\]

(1.3)

and that the total momentum is the spatial integral of the Gordon [8] momentum density

\[
  g_G = \frac{(n E \times B)}{c}.
\]

(1.4)
The four-momentum density is therefore \( g_\alpha = (\rho_\alpha/c, g_\alpha G) \). We then show that the elements of the first row of the energy–momentum tensor can be chosen to be energy and momentum densities that satisfy the requirement for conservation of the components of the four-momentum. By doing so, the four-divergence of the energy–momentum tensor produces an energy continuity equation that is incompatible with Poynting’s theorem. Retaining the conservation properties of the total energy–momentum tensor, we derive new electromagnetic continuity equations. We define a material four-divergence operator in which the temporal differentiation is performed with respect to \( ct/n \) and obtain a traceless symmetric energy–momentum tensor. The first-row and first-column components of the tensor are the densities of conserved energy and momentum quantities and the material four-divergence of the tensor generates the new electromagnetic continuity equations in terms of the electromagnetic energy density and the Gordon total momentum density.

\[ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \frac{\partial}{\partial t} \frac{\partial}{\partial \tau} \]

**II. THE TOTAL ENERGY AND TOTAL MOMENTUM**

The first step toward obtaining the total energy and total momentum is to define a thermodynamically closed free-body system. We consider a stationary dielectric block illuminated at normal incidence from vacuum by a plane quasimonochromatic field. The dielectric has a gradient-index antireflection coating that allows radiation pressure on the vacuum/dielectric interface to be neglected. Then the refractive index \( n \) for a linear, isotropic, transparent dispersion-negligible dielectric is real and time-independent. The dielectric is homogeneous and occupies a finite region of 3-dimensional space so spatial variation of the refractive index, \( n = n(r) \), is limited to the transition region of the coating. The quasimonochromatic electromagnetic field is represented by the vector potential

\[
\mathbf{A}(\mathbf{r}, t) = \frac{1}{2} \left( \mathbf{\hat{A}}(\mathbf{r}, t)e^{-i(\omega_d t - k_d \cdot \mathbf{r})} + \mathbf{\hat{A}}^*(\mathbf{r}, t)e^{i(\omega_d t - k_d \cdot \mathbf{r})} \right)
\]

where \( \mathbf{\hat{A}} \) is a slowly varying function of \( \mathbf{r} \) and \( t \), \( \omega_d \) is the center frequency of the field, \( k_d = (n\omega_d/c)e_k \), and \( e_k \) is a unit vector in the direction of propagation. Figure 1 shows the envelope of the incident field \( \mathbf{A}_i(z) = (\mathbf{\hat{A}}(z, t_0) \cdot \mathbf{\hat{A}}^*(z, t_0))^{1/2} \) about to enter the dielectric through a gradient-index antireflection coating. Figure 2 shows a time-domain numerical solution of the wave equation at a later time when the refracted field \( \mathbf{A}_r(z) = (\mathbf{\hat{A}}(z, t_1) \cdot \mathbf{\hat{A}}^*(z, t_1))^{1/2} \) is entirely inside the dielectric. As shown in Fig.2, the amplitude of the refracted field is \( \mathbf{\hat{A}}_r = \mathbf{\hat{A}}_i/\sqrt{n} \) and the spatial extent of the refracted pulse is \( w_r = w_i/n \) in terms of the width \( w_i \) of the incident pulse. It can be shown that this linear result is quite general in terms of the refractive index, as well as the width and amplitude of the field.

For a stationary dielectric, the electromagnetic energy is

\[
U = \int_\sigma \frac{1}{2} (n^2E^2 + B^2) \, dv,
\]

where the integration extends over all space \( \sigma \). For the example shown in Figs. 1 and 2, there is no significant field outside the rectangular pulse. Then, the incident energy at \( t = t_0 \),

\[
U(t_0) = \frac{\omega_d^2w_i}{2c^2} A_i^2,
\]

is equal to the refracted energy at \( t = t_1 \),

\[
U(t_1) = \frac{n^2\omega_d^2w_r}{2c^2} A_r^2 = \frac{n^2\omega_d^2(w_i/n)}{2c^2} \left( \frac{A_i}{\sqrt{n}} \right)^2 = U(t_0).
\]

The field has been averaged on a scale that is long compared to an optical period, but short compared to \( t_1 - t_0 \), accounting for a factor of one-half. It can be demonstrated, in a similar manner, that the electromagnetic energy at any time \( t > t_0 \) is equal to the incident energy. The temporal invariance of the electromagnetic energy makes \( U \), Eq. (2.2), the conserved energy of the closed system.
Conservation of the electromagnetic energy is all that is needed to show that the Gordon momentum

\[ \mathbf{G}_G = \int_\sigma \frac{n\mathbf{E} \times \mathbf{B}}{c} \, dv, \]  

(2.5)

obtained by spatially integrating the Gordon momentum density [8], Eq. (1.4), is the total momentum of our closed system. For monochromatic radiation, where \( \mathbf{B} = n\mathbf{E} \), we can write the momentum as

\[ \mathbf{G}_G = \int_\sigma \frac{n^2 \mathbf{E}^2}{c} \hat{\mathbf{e}}_k \, dv = \int_\sigma \frac{1}{2} \left( n^2 \mathbf{E}^2 + \mathbf{B}^2 \right) \frac{\hat{\mathbf{e}}_k}{c} \, dv = \frac{U}{c} \hat{\mathbf{e}}_k \]  

(2.6)

in the direction of the propagation unit vector \( \hat{\mathbf{e}}_k \). If the total energy \( U \) is temporally invariant, and therefore conserved, then so is \( \mathbf{G}_G = (U/c) \hat{\mathbf{e}}_k \). Alternately, we can show that the momentum balance

\[ \mathbf{G}_G(t_1) = \frac{n^2 \omega_d^2 w_i}{2c^3} A_r^2 \hat{\mathbf{e}}_k = \frac{n^2 \omega_d^2 (w_i/n)}{2c^3} \left( \frac{A_i}{\sqrt{n}} \right)^2 \hat{\mathbf{e}}_k = \mathbf{G}_G(t_0) \]  

(2.7)

is temporally invariant [9]. The momentum formula, Eq. (2.5), was originally derived in 1973 by Gordon [8]. Although there are some issues with Gordon’s derivation, temporal invariance is decisive. Hence, the total momentum in our closed system is given by Gordon’s formula, Eq. (2.5), and the total momentum density is the Gordon momentum density \( g_G \), Eq. (1.4) [9].

### III. THE ENERGY–MOMENTUM TENSOR

The energy and momentum continuity equations of a thermodynamically closed system can be combined to form a tensor differential equation and the energy–momentum tensor is the central element of this formalism. While the energy–momentum tensor formalism is straightforward, it has not been successful in application to classical continuum electrodynamics. Now that we have identified the total momentum by global conservation principles [9], we can use the apparatus of energy–momentum tensor theory to derive the energy–momentum tensor. However, it is not that simple. The Abraham–Minkowski momentum controversy arises from an incompatibility between two of the properties of the energy–momentum tensor.

The energy–momentum tensor has a number of important properties. Here, we employ the summation convention where identical indices on the same side of the equation are summed over, Greek indices belong to \((0, 1, 2, 3)\), and Roman indices run from 1 to 3. Then, the four main properties of the energy–momentum tensor \( T^{\alpha\beta} \) are: i) The four-divergence operator \( \partial_\alpha = (c^{-1}\partial_t, \partial_x, \partial_y, \partial_z) \) applied to the rows of the tensor generates continuity equations

\[ \partial_\alpha T^{\alpha\beta} = 0 \]  

(3.1)

for the electromagnetic energy and the components of the momentum. ii) Conservation of angular momentum requires diagonal symmetry

\[ T^{\alpha\beta} = T^{\beta\alpha} . \]  

(3.2)

iii) The array has a vanishing trace

\[ T^\alpha_\alpha = 0 \]  

(3.3)

corresponding to massless particles [10]. iv) The components of the four-momentum

\[ U = \int_\sigma dv T^{00} \]  

(3.4a)

\[ G^i = \frac{1}{c} \int_\sigma dv T^{0i} \]  

(3.4b)
are conserved in an unimpeded flow. Conditions iv) and ii) dictate the elements of the first row and first column of the energy-momentum tensor, such that

\[
T^\alpha{}_{\beta} = \begin{bmatrix}
\rho_e & c g_{G1} & c g_{G2} & c g_{G3} \\
\frac{c g_{G1}}{W} & W_{11} & W_{12} & W_{13} \\
\frac{c g_{G2}}{W} & W_{21} & W_{22} & W_{23} \\
\frac{c g_{G3}}{W} & W_{31} & W_{32} & W_{33}
\end{bmatrix},
\]

(3.5)

where the elements of \( W \) are yet to be specified. Applying condition i) to the first row of the array in Eq. (3.5), we have

\[
\frac{1}{c} \frac{\partial \rho_e}{\partial t} = \nabla \cdot (n E \times B),
\]

(3.6)

which is incompatible with the Poynting theorem

\[
\frac{\partial \rho_e}{\partial t} = \nabla \cdot c (E \times H).
\]

(3.7)

Alternatively, we can use Poynting’s theorem and the momentum continuity equation

\[
\frac{\partial}{\partial t} \left( \frac{D \times B}{c} \right) + \nabla \cdot W = 0
\]

(3.8)

to populate the tensor

\[
T^\alpha{}_{\beta} = \begin{bmatrix}
\rho_e & c g_{A1} & c g_{A2} & c g_{A3} \\
\frac{c g_{M1}}{W} & W_{11} & W_{12} & W_{13} \\
\frac{c g_{M2}}{W} & W_{21} & W_{22} & W_{23} \\
\frac{c g_{M3}}{W} & W_{31} & W_{32} & W_{33}
\end{bmatrix},
\]

(3.9)

where

\[
W_{ij} = -E_i D_j - B_i B_j + \frac{1}{2} (E \cdot D + B \cdot B) \delta_{ij}
\]

(3.10)
is the Maxwell stress tensor. The resulting Minkowski energy-momentum tensor violates condition iv), in addition to condition ii). Because we are in a regime of negligible dispersion, we can rewrite the momentum continuity equation, Eq. (3.8), as

\[
\frac{\partial}{\partial t} \left( \frac{E \times H}{c} \right) + \nabla \cdot W = (1 - n^2) \frac{\partial}{\partial t} \left( \frac{E \times B}{c} \right)
\]

(3.11)

and obtain the Abraham energy-momentum tensor,

\[
T^\alpha{}_{\beta} = \begin{bmatrix}
\rho_e & c g_{A1} & c g_{A2} & c g_{A3} \\
\frac{c g_{A1}}{W} & W_{11} & W_{12} & W_{13} \\
\frac{c g_{A2}}{W} & W_{21} & W_{22} & W_{23} \\
\frac{c g_{A3}}{W} & W_{31} & W_{32} & W_{33}
\end{bmatrix}
\]

(3.12)

Our condition i) becomes

\[
\partial_\alpha T^{\alpha\beta} = f^A_\alpha.
\]

(3.13)

However, the Abraham force \( f^A_\alpha = (0, c(1 - n^2)\partial \phi_A) \) is a source or sink of momentum such that momentum is not conserved in a closed system, also in violation of condition iv).

We have demonstrated an inconsistency in the energy-momentum tensor formulation of classical continuum electrodynamics. Specifically we have demonstrated that all of the properties of the energy-momentum tensor
cannot be simultaneously satisfied for electrodynamic fields in a dielectric, even if the system is thermody-
namically closed. The elements of the first row of the energy–momentum tensor can be chosen such that the
four-divergence of the first row is the Poynting theorem or they can be chosen to satisfy conservation of the
components of the four-momentum, but not both.

In order to fully resolve the Abraham–Minkowski controversy, we must make the properties of the energy–
momentum tensor self-consistent when applied to electromagnetic radiation in matter. Conservation of the
momentum four-vector is paramount for the total energy–momentum tensor of a thermodynamically closed
system. Likewise, conservation of angular momentum is required and we affirm our tensor properties iv) and
ii). Then the first row and column of the tensor are populated by the densities of the electromagnetic energy
density and the Gordon total momentum density as shown in Eq. (3.5). These densities, integrated over space,
correspond to the conserved total quantities of energy and momentum as shown in Eq. (3.4). Property iii) is
not in question.

The problem that remains is a situation in which the four-divergence of the total energy–momentum tensor,
property i), is not consistent with the electromagnetic continuity equations, and Poynting’s theorem in particular.
We can multiply Poynting’s theorem by $n(r)$ and use a vector identity to commute the refractive index with
the divergence operator to obtain a continuity equation

$$\frac{n}{c} \frac{\partial \rho_e}{\partial t} + \nabla \cdot (n \mathbf{E} \times \mathbf{B}) = \frac{\nabla n}{n} \cdot (n \mathbf{E} \times \mathbf{B}).$$

(3.14)

The second-order energy continuity equation, Eq. (3.14), can be written as two first-order equations

$$\frac{n}{c} \frac{\partial (n \mathbf{E})}{\partial t} = (\nabla \times \mathbf{B})$$

(3.15a)

$$\frac{n}{c} \frac{\partial \mathbf{B}}{\partial t} = - (\nabla \times (n \mathbf{E})) + \frac{\nabla n}{n} \times (n \mathbf{E}).$$

(3.15b)

Then, for a homogeneous dielectric, we can drop the term $\nabla n \times \mathbf{E}$ and combine Eqs. (3.15) to obtain the energy
and momentum continuity equations

$$\frac{n}{c} \frac{\partial}{\partial t} \left[ \frac{1}{2} (n^2 \mathbf{E}^2 + \mathbf{B}^2) \right] + \nabla \cdot (n \mathbf{E} \times \mathbf{B}) = 0$$

(3.16a)

$$\frac{n}{c} \frac{\partial}{\partial t} (n \mathbf{E} \times \mathbf{B}) + \nabla \cdot \mathbf{W} = 0.$$  

(3.16b)

The elements of $\mathbf{W}$ are the elements of the stress tensor

$$W_{ij} = -nE_i nE_j - B_i B_j + \frac{1}{2} (n \mathbf{E} \cdot n \mathbf{E} + \mathbf{B} \cdot \mathbf{B}) \delta_{ij}.$$  

(3.17)

We define a material four-divergence operator

$$\tilde{\partial}_\alpha = \left( \frac{n}{c} \frac{\partial}{\partial t}, \partial_x, \partial_y, \partial_z \right)$$

(3.18)

and replace property i) with

$$\tilde{\partial}_\alpha T^{\alpha \beta} = 0$$

(3.19)

that generates Eqs. (3.16). Then the tensor, Eq. (3.5) is a traceless, symmetric energy momentum tensor whose
material four-divergence generates the new electromagnetic energy and momentum continuity equations. It
should be noted that the new energy continuity equation, Eq. (3.16a), is mathematically equivalent to Poynting’s
theorem, Eq. (3.7) because the two are related by a vector identity. Likewise the new momentum continuity
equation, Eq. (3.16b), is mathematically equivalent to the momentum continuity equation, Eq. (3.8) if the
radiation is sufficiently monochromatic and sufficiently far from any material resonances that dispersion can be
neglected.
IV. SUMMARY

A transparent linear dielectric block in free space illuminated by a quasimonochromatic field can be configured as an isolated system. However, the partial reflection of the field at the surface causes a change in the momentum of the field and the resulting radiation pressure accelerates the block. In principle, we could write a complete set of equations of motion at the microscopic level, but the effects of radiation pressure are not treated in a complete way at the level of the macroscopic Maxwell equations. In order to derive a theory of minimum complexity, we assumed a simple dielectric defined to be linear, isotropic, homogeneous, transparent, and dispersionless with negligible electrostrictive and magnetostrictive effects. The block is stationary in free space and radiation pressure on the antireflection coating is negligible. For this system, the elements of the energy–momentum tensor

\[
T^{\alpha\beta} = \begin{bmatrix}
\rho_c & c g G_1 & c g G_2 & c g G_3 \\
c g G_1 & W_{11} & W_{12} & W_{13} \\
c g G_2 & W_{21} & W_{22} & W_{23} \\
c g G_3 & W_{31} & W_{32} & W_{33}
\end{bmatrix}
\]  

(4.1)

are the electromagnetic energy density \( \rho_e = (n^2 E^2 + B^2)/2 \), the Gordon momentum density \( g G = (nE \times B)/c \) and the Maxwell stress tensor, Eq. (3.10), in the form

\[
W_{ij} = -nE_i nE_j - B_i B_j + \frac{1}{2} (n^2 E \cdot E + B \cdot B) \delta_{ij}.
\]

The properties of the energy–momentum tensor are: i) The material four-divergence operator

\[
\bar{\partial}_\alpha = \left( \frac{n}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)
\]

applied to the rows of the tensor generates continuity equations

\[
\bar{\partial}_\beta T^{\alpha\beta} = 0
\]

(4.3)

for the electromagnetic energy and momentum. The energy continuity equation and the momentum continuity equation are mathematically equivalent to Poynting’s theorem and the momentum continuity equation for a linear, isotropic, homogeneous, transparent, dispersionless dielectric. ii) Conservation of angular momentum requires diagonal symmetry

\[
T^{\alpha\beta} = T^{\beta\alpha}.
\]

(4.4)

iii) The array has a vanishing trace

\[
T^\alpha_\alpha = 0
\]

(4.5)

corresponding to massless particles [4] [10]. iv) The components of the four-momentum

\[
U = \int_\sigma dv T^{00}
\]

(4.6a)

\[
G^i = \frac{1}{c} \int_\sigma dv T^{0i}
\]

(4.6b)

are conserved [4] in an unimpeded flow.

[1] For a recent review, see: Pfeifer, R. N. C., Nieminen, T. A., Heckenberg, N. R., and Rubinsztein-Dunlop, H., “Colloquium: Momentum of an electromagnetic wave in dielectric media,” Rev. Mod. Phys. 79, 1197–1216 (2007).
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Fig. 1. A quasimonochromatic field incident on a linear medium with a gradient-index anti-reflection coating. The shaded region in the profile of the index of refraction.

Fig. 2. The quasimonochromatic field of Fig. 1 after it has propagated into the linear medium.