Nilpotent matrices having a given Jordan type as maximum commuting nilpotent orbit*

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Abstract

The Jordan type of a nilpotent matrix is the partition giving the sizes of its Jordan blocks. We study pairs of partitions \((P,Q)\), where \(Q = \Omega(P)\) is the Jordan type of a generic nilpotent matrix \(A\) commuting with a nilpotent matrix \(B\) of Jordan type \(P\). T. Košir and P. Oblak have shown that \(Q\) has parts that differ pairwise by at least two \([KO]\). Such partitions, which are also known as “super distinct” or “Rogers-Ramanujan”, are exactly those that are stable or “self-large” in the sense that \(\Omega(Q) = Q\).

In 2012 P. Oblak formulated a conjecture concerning the cardinality of \(\Omega^{-1}(Q)\) when \(Q\) is stable with two parts, and proved some special cases. R. Zhao refined this to posit that the partitions in \(\Omega^{-1}(Q)\) for \(Q = (u,u-r)\) with \(r \geq 2\) could be arranged in an \((r-1) \times (u-r)\) table \(T(Q)\) where the entry in the \(k\)-th row and \(\ell\)-th column has \(k + \ell\) parts. We prove this Table Theorem, and then generalize the statement to propose a Box Conjecture for the set of partitions \(\Omega^{-1}(Q)\) for an arbitrary stable partition \(Q\).

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1 Introduction.

Fix an infinite field $k$, denote by $M_n(k)$ the ring of $n \times n$ matrices with entries in $k$ acting on the vector space $V = k^n$. Let $P$ be a partition of $n$ and denote by $B = J_P$ the nilpotent Jordan block matrix of partition $P$. Let $C_B = \{A \in M_n(k) \mid AB = BA\}$ be the centralizer of $B$ in $M_n(k)$, and let $N_B$ be the subvariety of nilpotent elements in $C_B$.

There has been substantial work in the last ten years studying the map $Q$ that takes $P$ to the Jordan type $Q(P)$ of a generic element of $N_B$. P. Oblak conjectured a beautiful recursive description of $Q(P)$. This conjecture appears close to resolved (see Section 5.1, Conjecture 5.2, Remark 5.6, and [BKO, Obl1, Kh1, Kh2, IKh, Bas2]).

An almost rectangular partition is one whose largest part is at most one larger than its smallest part. R. Basili introduced the invariant $r_P$, the smallest number of almost rectangular partitions whose union is $P$, and showed that $Q(P)$ has $r_P$ parts (Theorem 2.5). T. Košir and P. Oblak showed that $Q(P)$ is stable: it has parts differing pairwise by at least two (Theorem 2.7). However, even in cases where the Oblak recursive conjecture had been shown some time ago, (as $r_P = 2$ [KO], or $r_P = 3$ [Kh2]) the set $Q^{-1}(Q)$ remained mysterious. In 2012 P. Oblak made a second conjecture: when $Q$ is a stable partition with two parts, that is $Q = (u, u-r)$ with $u > r \geq 2$, then the cardinality $|Q^{-1}(Q)| = (r - 1)(u-r)$. In 2013, R. Zhao noticed an even stronger pattern in $Q^{-1}(Q)$ for such $Q$. She conjectured that there is a table $T(Q)$ of partitions $P_{k,\ell}$ where the number of parts in $P_{k,\ell}$ is $k + \ell$: see Theorem 1.1 immediately below. We here prove a precise version, the Table Theorem (Theorems 3.14, 3.25). We then propose a Box Conjecture 5.9 describing $Q^{-1}(Q)$ for arbitrary stable $Q$ (Section 5.2) and we show some special cases where $Q$ has three parts (Section 5.4).

The question, which pairs of conjugacy classes can occur for pairs of commuting matrices reduces to the case where both matrices are nilpotent. There is an extensive literature on commuting pairs of nilpotent matrices, including [GurSe, Obl1, Obl2, KO, IKh, Kh1, Kh2, Bas2] and others, some of whose results we specifically cite. Connections to the Hilbert scheme are made in [Bar, Bas1, Bl, Prem, BuEv], and commuting nilpotent orbits occur in the study of Artinian algebras [HW, Bl]. However, the study of the map $P \to Q(P)$ seems to be, surprisingly, very recent, beginning with [Bar, Bas1, Pan1, Prem]; apparently, early workers in the area were more drawn to determining commuting vector spaces of matrices of maximum dimension (see [Ma, J, SuTy] and references in the latter). There is further recent work on commuting $r$-tuples of nilpotent matrices, as [GurSe, GurNg, Si, NgSi] and these also appear
to be connected to the study of group schemes. There is much study of nilpotent orbits for Lie algebras, as [Gi, BroBru, CoMc, Pan2]; for generalizations of problems considered here to other Lie algebras than $\mathfrak{sl}_n$, see [Pan1].

P. Oblak conjectured the formula $|\Omega^{-1}(u,u-r)| = (r-1)(u-r)$ for the cardinality of the in principle known set of partitions $\mathbb{P}$ for which $\Omega(\mathbb{P}) = (u,u-r)$, where $u, r \in \mathbb{N}$ with $u > r \geq 2$\footnote{The formula has a remarkable symmetry: the proposed value for $|\Omega^{-1}((u, u-r))|$ is the same as that for $|\Omega^{-1}((u, r-1))|$. Understanding this symmetry was a goal of R. Zhao in her study of the two sets: it remains obscure to us.} Our main result is

**Theorem 1.1.** Let $Q = (u,u-r)$ where $u > r \geq 2$.

i. The cardinality $|\Omega^{-1}(Q)| = (r-1)(u-r)$ (P. Oblak conjectured this in [Ob1, p. 609, and Remark 2]).

ii. The set $\Omega^{-1}(Q)$ may be arranged as an $(r-1) \times (u-r)$ array $\mathcal{T}(Q)$ of partitions

$$P_{k,\ell} = P_{k,\ell}(Q), 1 \leq k \leq r-1, 1 \leq \ell \leq u-r$$

in a natural way, such that the number of parts of $P_{k,\ell}$ is $k + \ell$.

**Remark 1.2.** We call this the Table Theorem. Below Theorem 3.14 specifies each $P_{k,\ell}$ in $\mathcal{T}(Q)$, and shows that $\Omega(P_{k,\ell}) = Q$; Theorem 3.25 says that $\mathcal{T}(Q)$ is all of $\Omega^{-1}(Q)$. Some special cases had been shown prior to our work here: P. Oblak had shown it for $2 \leq r \leq 4$ in [Ob1]. R. Zhao in [Z] had shown the case $(u-r) = 1, 2, 3$ and also the case $u \gg r$; the latter is the case that $\mathcal{T}(Q)$ has a “normal pattern” (Corollary 3.26).

**Summary.**

In Section 2.1 we first review some results we will need; in Section 2.2 we recall the poset $\mathcal{D}_P$ associated with the nilpotent commutator $\mathcal{N}_B$ of $B = J_P$ and more particularly to a maximal commuting nilpotent subalgebra $\mathcal{U}_B$ of $\mathcal{N}_B$.

Let $Q = (u,u-r)$ with $u > r \geq 2$ and put $B = J_Q$. After dividing the partitions in $\Omega^{-1}(Q)$ into three types A, B and C, in Section 3.1 we prove, in Section 3.2 Theorem 3.14 which specifies the filling of the table $\mathcal{T}(Q)$ with partial A rows and B/C hooks. We give examples and properties of the tables in Section 3.3. In Section 3.4 we show that the table $\mathcal{T}(Q)$ is the complete inverse image of $Q$ under the map $\Omega$ (Theorem 3.25). We also obtain as a corollary the normal pattern case first shown by R. Zhao [Z].

In Section 4 we study the equations for loci $\mathfrak{F}(P_{k,\ell})$ in $\mathcal{N}_B$: these are the algebraic subsets of $\mathcal{N}_B$ parametrizing the matrices $A \in \mathcal{N}_B$ with $P_A = P_{k,\ell}(Q)$. In Section 4.1 we define the weighting on $\mathcal{U}_B$ determined by the $sl_2$-triple associated to $B$, where $B = J_P$ for an arbitrary partition $P$. In Section 4.2, where $B$ is again $J_{u,u-r}$ and so $\mathcal{U}_B = \mathcal{N}_B$, we first show that these equations of loci are $sl_2$-homogeneous (Lemma 4.2) and then give some examples. In Section 4.3, joint with M. Boij, we conjecture that the closure $\overline{\mathfrak{F}(P_{k,\ell})}$ is a complete intersection of codimension $k + \ell - 2$ in $\mathcal{N}_B$. We also propose equations for $\overline{\mathfrak{F}(P_{k,\ell})}$.

After reviewing P. Oblak’s recursive conjecture in Section 5.1, we propose in Section 5.2 a Box Conjecture for $\Omega^{-1}(Q)$, where $Q$ is an arbitrary stable partition (see Conjecture 5.9).
The stable partitions are those whose parts differ pairwise by at least two, and are called “Rogers-Ramanujan” or “super-distinct” in the partitions literature. The Box Conjecture in short states that if \( Q \) is a stable partition with \( k \) parts then its key \( S(Q) \) gives the lengths of the sides of a \( k \)-dimensional box \( B(Q) \) containing the elements of \( \Omega^{-1}(Q) \), and that \( B(Q) \) has further regularities. In Section 5.3 we note that the number \( p(a, Q) \) of partitions having a given stable diagonal hook partition \( Q \), and a given number \( a \) of parts is exactly the analogous number for \( \Omega^{-1}(Q) \), under the Box Conjecture. This shows that the Conjecture is consistent with known formulas for the number of partitions of \( n \) with \( a \) parts. In Section 5.4 we show some special cases of the Box Conjecture for \( Q \) having three parts.

We believe that this article introduces a new approach to viewing the map \( \Omega : P \to \Omega(P) \). While our methods are elementary, our results suggest interesting algebraic and geometric explanations and consequences.

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2 Preliminaries and Background.

2.1 Notation and Preliminaries.

We fix notation and summarize some concepts and results we will need. Let \( P = (p_1, \ldots, p_s) \) be a partition of the positive integer \( n \). This means that \( p_1 \geq \cdots \geq p_s > 0 \) and \( p_1 + p_2 + \cdots + p_s = n \). We denote by \( S_P \) the set of parts of \( P \), i.e. \( S_P = \{p_1, p_2, \ldots, p_s\} \). Note that \( 1 \leq |S_P| \leq s \).

Recall that the Ferrers or Young diagram of \( P \) has rows whose lengths are the parts of \( P \). We denote by \( P^\vee \) the conjugate partition to \( P \): the Ferrers diagram of \( P^\vee \) has rows the columns of the Young diagram of \( P \). We denote by \( s^k \) the partition of \( ks \) having \( k \) parts equal to \( s \); its conjugate is \( k^s \). We now introduce almost rectangular partitions, whose importance for the problem of describing the map \( P \to \Omega(P) \) was first noted by R Basili [Bas1].

Definition 2.1 (Almost Rectangular). A partition \( P = (p_1, p_2, \ldots, p_s) \) of \( n \) with \( p_1 \geq p_2 \geq \cdots \geq p_s > 0 \) is almost rectangular if \( p_1 - p_s \leq 1 \). For \( 1 \leq k \leq n \) we denote by \( \lfloor n \rfloor^k \) the unique almost rectangular partition of \( n \) having \( k \) parts. (See Figure 1.)

Write \( n = qk + r \) with \( r, q \in \mathbb{N} \) and \( 0 \leq r < k \) and put \( d = k \cdot \lfloor \frac{n}{k} \rfloor - n \). Then

\[
d = \begin{cases} 
    k - r & \text{if } r \neq 0 \\
    0 & \text{if } r = 0,
\end{cases}
\]
and we have

\[ [n]^k = (q+1)^r, q^{k-r}) = (\lceil \frac{n}{k} \rceil^{k-d}, \lfloor \frac{n}{k} \rfloor^d). \tag{2.1} \]

The regular partition of \( n \), denoted by \([n]\) or \((n)\), is the only partition of \( n \) with a single part. Given a partition \( P \) of \( n \) we denote by \( J_P \) the unique Jordan block matrix having blocks of lengths \( p_1, \ldots, p_s \). For example \( J_{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \). Given a nilpotent \( n \times n \) matrix \( A \) we denote by \( P_A \) its Jordan type; it is the partition giving the sizes of the blocks of the Jordan block matrix similar to \( A \) (we write \( J_{P_A} \sim A \)). Recall that the corank of \( A \) is \( n - \text{rank} A \), the dimension of the kernel of \( A \). We take \( A^0 = I_n \), the identity. The following result is standard.

**Lemma 2.2.** The number of parts greater than or equal to \( i \) in \( P_A \) is the difference \( \text{corank} A^i - \text{corank} A^{i-1} \). \( \tag{2.2} \)

**Lemma 2.3.** Let \( B = J_{(n)} \). Then \( P_{B^k} = [n]^k \).

**Proof.** Evidently \( B^k \) has corank \( k \). The number of parts of \( P_A \) is the corank of \( A \) so \( P_{B^k} \) has \( k \) parts. Let \( q = \lfloor \frac{n}{k} \rfloor \). Then \((B^k)^{q+1} = 0\), so no part of \( P_{B^k} \) is greater than \( q + 1 \). The Lemma follows. \( \square \)

This allows us to describe \( \Omega^{-1}(Q) \) when \( Q = (n) \) has a single part.

**Corollary 2.4.** If \( A \) is a nilpotent matrix commuting with \( J_{(n)} \) then \( P_A = [n]^k \) for some \( k \). Consequently, \( \Omega^{-1}([n]) \) is the set of almost rectangular partitions \( \{[n]^k, 1 \leq k \leq n\} \).

**Proof.** Lemma 2.3 implies that if \( B = J_{(n)} \) then \( P_{B^k} = [n]^k \), so \( \Omega([n]^k) = (n) \). The matrices \( A \) commuting with a regular nilpotent matrix \( B \) are the polynomials \( A = p(B) \) where \( p \in k[x] \). When \( p = x^k \cdot p', p' = a_0x^u + \cdots + a_0, a_0 \neq 0 \) then \( p'(B) \) is invertible, so \( A = p(B) \sim B^k \) and \( P_A = [n]^k \). \( \square \)

Arranging \( \Omega^{-1}([n]) \) in a linear table \( T([n]) \) we have

\[
\begin{array}{c|cccc}
\# \text{ parts} & 1 & 2 & \cdots & n \\
\hline
\Omega^{-1}(Q) = & \{[n] & [n]^2 & \cdots & [n]^n = 1^n\} \\
\end{array}
\] \( \tag{2.3} \)

Recall that \( r_P \) is the smallest number of almost rectangular partitions whose union is \( P \). We say that a partition \( P \) is stable if \( \Omega(P) = P \).
Theorem 2.5. ([Bas1]) The partition $Q(P)$ has $r_P$ parts.

Theorem 2.6. ([BL, Pan1]) A partition is stable if and only if its parts differ pairwise by at least two.

Theorem 2.7. ([KO]) The partition $Q(P)$ is stable.

The proof depends on showing that when $B = J_P$ and the matrix $A \in N_B$ is generic, then the Artinian ring $k[A, B]$ is Gorenstein, so in height two is a complete intersection. The Hilbert function of $k[A, B]$ is the conjugate of the partition $Q(P)$ [BI]; the characterization of the Hilbert functions of (non-graded) Artinian CI algebras of height two by F.H.S. Macaulay [Mac] implies the stability property of $Q(P)$.

Denote the partition $P$ by $(\cdots i_n\cdots)$ meaning it has $n_i$ parts of length $i$. An almost rectangular subpartition $P' = (a^n_a, (a - 1)^{n_{a-1}})$ of $P$ defines a U-chain $C_a$ in a partially ordered set associated to $P$ (Definition 2.13). For a subpartition $(a^n_a, (a - 1)^{n_{a-1}})$ of $P$ the length of the U-chain $C_a$ is

$$|C_a| = an_a + (a - 1)n_{a-1} + 2 \sum_{i>a} n_i.$$ (2.4)

Theorem 2.8. ([Obl1]) The largest part of $Q(P)$ is $\max\{|C_a| : a \in S_P\}$.

The above theorems were originally shown with some restrictions on $k$, such as $k = \mathbb{C}$. Their proofs were subsequently seen to be valid over any infinite field $k$: see [BIK, IKh].

2.2 Background: the poset $D_P$.

We now recall the poset $D_P$ associated to $P$. This poset plays an important role in understanding the map $P \to Q(P)$. For example, it is behind the proofs of Theorem 2.7 of P. Oblak and T. Košir and Theorem 2.8 of P. Oblak. The main proofs of Section 3 refer to the U-chains in the poset. However, we note for those readers less interested in this background, that the proofs there will use Equation (2.4) and Theorem 2.8 above and may be read independently of the Definition 2.9 of the poset $D_P$. We will use the poset $D_Q$ and its elementary maps in an essential way when we describe the equations for the loci of certain Jordan types in Section 4 and we review P. Oblak’s recursive conjecture in Section 5.1. For further discussion of the poset $D_P$ see [BIK, KO, Kh1, Kh2, IKh].

The poset $D_P$.

Let $P$ be a partition of $n$, and let $B = J_P$ acting on the vector space $V$. The poset $D_P$ has $n$ vertices corresponding to a basis $\mathcal{B}$ of $V$. First we recall the basis $\mathcal{B}$. We write $n_i$ for the multiplicity of the part $i$ in $P$, so $P = (\cdots, i^n_i, \cdots)$. Following [IKh] we have $V = \bigoplus_{i \in S_P} V_i$, where $V_i$ has a decomposition

$$V_i = \bigoplus_{k=1}^{n_i} V_{i,k}$$ (2.5)

into cyclic $B$-modules $V_{i,k}$, each of length $i$. The subspace $V_{i,k}$ has a cyclic vector $(1, i, k)$ and basis

$$\{(u, i, k) = B^{u-1}(1, i, k) : 1 \leq u \leq i\}.$$ (2.6)
So $V_{i,k} \cong k[x]/x^i$ as a $k[x]$-module through the action of $B$. We denote by $\mathfrak{B}$ the concatenation of the above bases for $V_{i,k}$, and by $\langle A \cdot v \mid (u, i, k) \rangle$ the component of $A \cdot v$ on the basis vector $(u, i, k)$. Fix $i$ and denote by $\mathfrak{W}_i$, the subset consisting of the cyclic vectors of $\{V_{i,k} : 1 \leq k \leq n_i\}$, that is,

$$\mathfrak{W}_i = \{(1, i, 1), (1, i, 2), \ldots, (1, i, n_i)\} \quad (2.7)$$

and by $W_i$ the span of $\mathfrak{W}_i$. Denote by $\pi_i$ the projection from the centralizer $C_B$ to $\text{Mat}_{n_i}(k)$ obtained by restricting $A \in C_B$ to $\mathfrak{W}_i$ and then projecting to $W_i$. Let

$$\pi : C_B \rightarrow \prod_i \text{Mat}_{n_i}(k) \quad (2.8)$$

be the product of the $\pi_i$. It is well known that $\pi$ is the map from $C_B$ to its semisimple part. We define a maximal nilpotent subalgebra $\mathcal{U}_B \subset \mathcal{N}_B$ by requiring the each $\pi_i(A)$ be strictly upper triangular on $W_i$:

$$\mathcal{U}_B = \{ A \in \mathcal{N}_B \mid 1 \leq u \leq v < n_i \Rightarrow \langle A \cdot (1, i, v) \mid (1, i, u) \rangle = 0 \}. \quad (2.9)$$

**Definition 2.9 (Poset $\mathcal{D}_P$).** The poset $\mathcal{D}_P$ has as vertices the basis elements $\mathfrak{B}$ for $V$: and for $v, v' \in \mathfrak{B}$, we set $v < v'$ if there is an element $A \in \mathcal{U}_B$ such that $\langle A \cdot v \mid v' \rangle \neq 0$.

The diagram $\text{Diag}(\mathcal{P})$ of a poset $\mathcal{P}$ is a directed graph for which the vertices are the elements of $\mathcal{P}$ and with an arrow $v \rightarrow v'$ if $v'$ covers $v$ (here $v'$ covers $v$ if $v < v'$ and there is no $v''$ such that $v < v'' < v'$). Recall that $\mathcal{P}_P$ is the set of integers that are parts of $P$. For $i \in \mathcal{P}_P$ we denote by $i^-$ the next smaller element of $\mathcal{P}_P$ if it exists (if $i$ is not the smallest part of $P$), and by $i^+$ the next larger element of $\mathcal{P}_P$, if it exists. For $P = (5, 4, 4, 3, 2, 2)$ where $\mathcal{P}_P = \{5, 4, 3\}$, $4^+ = 5$, $3^- = 2$.

**Definition 2.10 (Elementary Maps associated to $P$).** [BIK] Def. 2.9.

a. Vertices of $\mathcal{D}_P$. For each pair $(u, i)$ with $i \in \mathcal{P}_P$ and $1 \leq u \leq i$, there are $n_i$ vertices $\{(u, i, k), 1 \leq k \leq n_i\}$.

b. Elementary maps of $\text{End}_k(V)$. The maps defined below are zero on those basis elements of $V$ from (2.5) and (2.6) not specifically listed.

i. for $i \in \mathcal{P}_P \setminus p_s$, $\beta_i = \beta_{i,-}$ maps the vertex $(u, i, n_i)$ to $(u, i^-, 1)$, whenever $1 \leq u \leq i^-$.  

ii. for $i \in \mathcal{P}_P \setminus p_s$, $\alpha_i = \alpha_{i,-}$ maps $(u, i^-, n_i)$ to $(u + i^-, n_i)$, whenever $1 \leq u \leq i^-$.  

iii. $e_{i,k}$ maps the vertex $(u, i, k)$ to $(u, i, k + 1)$, $1 \leq u \leq i$, $1 \leq k < n_i$.  

iv. When $i \in \mathcal{P}_P$ is isolated (when neither $i - 1 \in \mathcal{P}_P$ nor $i + 1 \in \mathcal{P}_P$), the map $w_i$ sends $(u, i, n_i)$ to $(u + 1, i, 1)$ whenever $1 \leq u < i$.

**Lemma 2.11.** There is an edge $v \rightarrow v'$ in the diagram $\text{Diag}(\mathcal{D}_P)$ if and only if there exists an elementary map $\gamma$ such that $\gamma(v) = v'$.

For further discussion of $\mathcal{D}_P$ and the diagram of $\mathcal{D}_P$ see [OB1, BIK, IKh, Kh1, Kh2].

\[\text{Here we quote from [IKh] Definition 2.3.}\]
\(\beta_3 \searrow \beta_3 \searrow \alpha_3 \nearrow \alpha_3 \nearrow\) \hfill (2.10)

\(\beta_3 \searrow \beta_3 \searrow \alpha_3 \nearrow \alpha_3 \nearrow \) \hfill (2.11)

Figure 2: Diag(\(\mathcal{D}_P\)) for \(P = (3, 2)\) and \(P' = (3, 2, 2, 1)\).

Example 2.12. When \(P = (3, 2)\) and \(B = J_P\) then the algebra \(U_B\) is generated by \(\alpha_3, \beta_3\), satisfying \(\alpha_3^2 = \beta_3^2 = \beta_3 \alpha_3 = 0\). For \(P' = (3, 2, 2, 1)\) and \(B' = J_{P'}\) the algebra \(U_{B'}\) is generated by \(\alpha_3, \alpha_2, \beta_3, \beta_2\) and \(\eta_{2,1}\) (Figure 2). When \(P = (4, 2, 2, 1)\) the algebra \(U_B\) is generated by \(\beta_4, \beta_2, \alpha_4, \alpha_2, w_4\) and \(e_{2,1}\) (Figure 3).

Definition 2.13. Let \(a \in S_P\). The U-chain \(C_a\) of the poset \(\mathcal{D}_P\) is comprised of three parts:

a. the unique maximum chain through all the vertices of \(\mathcal{D}_P\) in rows of size \(a, a-1\).

b. a chain from the source vertex \((1,p_1,1)\) down to \((1,a,1)\), of length \(\sum_{i>a} n_i\).

c. a chain from the vertex \((a,a,n_a)\) to the sink vertex \((p_1,p_1,n_{p_1})\) of \(\mathcal{D}_P\).

By definition, the length \(|C_a|\) is the number of vertices in the U-chain. It satisfies \(|C_a| = an_a + (a-1)n_{a-1} + 2\sum_{i>a} n_i\) (this is equation (2.4)).

Example 2.14. For the partition \(P' = (3, 2, 2, 1)\) of Figure 2 the U-chain \(C_2\) is
\[
(1,3,1) \rightarrow (1,2,1) \rightarrow (1,2,2) \rightarrow (1,1,1) \rightarrow (2,2,1) \rightarrow (2,2,2) \rightarrow (1,3,3)
\]
given by the chain of maps (right to left) \(\alpha_3 \circ e_{21} \circ \alpha_2 \circ \beta_2 \circ e_{21} \circ \beta_3\). The U-chain \(C_3\) is
\[
(1,3,1) \rightarrow (1,2,1) \rightarrow (1,2,2) \rightarrow (1,3,2) \rightarrow (2,2,1) \rightarrow (2,2,2) \rightarrow (1,3,3)
\]
given by \(\alpha_3 \circ e_{21} \circ \beta_3 \circ \alpha_3 \circ e_{21} \circ \beta_3\).

3 The table \(\mathcal{T}(Q)\) for \(\Omega^{-1}(Q)\) when \(Q = (u, u-r)\).

In this section we determine the tables \(\mathcal{T}(Q)\) giving the complete set \(\Omega^{-1}(Q)\) for all stable partitions \(Q\) having two parts: \(Q = (u, u-r)\) with \(u > r \geq 2\). Our main results, Theorem 3.14 specifying the table \(\mathcal{T}(Q) \subset \Omega^{-1}(Q)\) and Theorem 3.25 showing completeness of the table are shown in Sections 3.2 and 3.4 respectively.
3.1 Three subsets of $\Omega^{-1}(Q)$ and their intersections.

By Theorem 2.5, $Q = \Omega(P)$ has two parts exactly when $P$ is the union of two almost rectangular partitions. Hence there are positive integers $a, b$ with $a \geq b + 2$ such that

$$P = (a^{n_a}, (a - 1)^{n_{a-1}}, b^{n_b}, (b - 1)^{n_{b-1}})$$

with $n_a, n_b > 0$. \hfill (3.1)

Here we have denoted by $n_i$ the number of parts of $P$ having length $i$.

**Definition 3.1** (Type A,B,C partitions in $\Omega^{-1}(Q)$). Let $Q = (u, u - r)$ with $u, r \in \mathbb{N}, u > r \geq 2$ and let $P \in \Omega^{-1}(Q)$ satisfy (3.1).

We say that $P$ is of type $A$ if $u = (a) \cdot n_a + (a - 1)n_{a-1}$;

We say that $P$ is of type $B$ if $u = 2n_a + 2n_{a-1} + bn_b + (b - 1)n_{b-1}$, or if $b = a - 2, n_{b-1} = 0$ and $u = 2n_a + (a - 1)n_{a-1} + bn_b$.

We say that $P$ is of type $C$ if $b = a - 2$, if each of $n_a, n_{a-1}, n_b, n_{b-1}$ is non-zero, and $u = 2n_a + (a - 1)n_{a-1} + bn_b$.

**Remark 3.2.** It is clear from Theorem 2.8 that every $P \in \Omega^{-1}((u, u - r))$ is of type A,B, or C. Note that a partition can have more than one type. When $P$ has type A then the length of the U-chain through the upper almost rectangular subpartition of $P$ is the biggest part of $Q$, and $u - r = bn_b + (b - 1)n_{b-1}$.

When $P$ has type B the length of the U-chain through the lowest almost rectangular subpartition of $P$ is the biggest part of $Q$. Then $u - r = (a - 2)n_a + (a - 3)n_{a-1}$.

When $P$ has type C the middle almost rectangular U-chain is a longest U-chain. Then $u - r = (a - 2)n_a + (b - 1)n_{b-1}$.

Figure 3: Diag$(D_P)$ and maps for $P = (4, 2, 2, 1)$. cf. [IKh] Figure 2]
Definition 3.4. If not of type A or B.

Consequently, but in the latter case we take

\[ n \text{ is a partition of } |g| \]

We define the type A: the longest U-chain is \( |P| \). Consequently, \( P \text{ is the set of integers that are parts of } P \). Here \(|S_P| \in \{2, 3, 4\} \) since \( r_P = 2 \).

Our focus here and a result we need for Theorem 3.25 is on partitions of type C that are not of type A or B.

**Definition 3.4.** If \( r_P = 2 \) then

\[ P = (a^n, (a-1)^{n-1}, b^n, (b-1)^{n-1}) \text{ for some } a, b \in \mathbb{N} \text{ with } a - b \geq 2. \] (3.2)

We define the gap \( g = a - b - 2 \) between the two almost rectangular (AR) parts of \( P \). Then \( P \) is a partition of \( n \cdot a + n_{a-1} \cdot (a-1) + n_{b-1} \cdot (b-1) \), and \( P \) has \( t(P) = n_a + n_{a-1} + n_{b-1} + n_{b-1} \) parts. When \(|S_P| = 4, r_P = 2 \), we may write the unique AR decomposition of \( P \)

\[ P = (\left\{ n \cdot a + n_{a-1} \cdot (a-1) \right\}^{n_a+n_{a-1}}, [n_b \cdot b + n_{b-1} \cdot (b-1)]^{n_b+n_{b-1}}). \] (3.3)

When \(|S_P| = 3 \) we allow \( n_{b-1} = 0 \) or \( n_{a-1} = 0 \). In the former case the gap is again \( g = a - b - 2 \), but in the latter case we take \( g = a - b - 1 \). When \(|S_P| = r_P = 2 \) then we have \( n_{b-1} = n_{a-1} = 0 \), the gap \( g = a - b - 1 \) and must satisfy \( g \geq 1 \) (else \( P \) is almost rectangular and \( r_P = 1 \)).

The following is a consequence of Equation 2.4.

**Lemma 3.5.** Let \( P \) be a partition as in Equation (3.2). The length of the top U-chain \( U_{top} = C_a \) of \( D_P \) is

\[ |U_{top}| = n_a \cdot a + n_{a-1} \cdot (a-1), \] (3.4)

while the length of the bottom U-chain \( U_{bottom} = C_b \) is

\[ |U_{bottom}| = n_b \cdot b + n_{b-1} \cdot (b-1) + 2(n_a + n_{a-1}). \] (3.5)

We have

\[ |U_{top}| - |U_{bottom}| = n_a \cdot (a - 2) + n_{a-1} \cdot (a - 3) - n_b \cdot b - n_{b-1} \cdot (b - 1). \] (3.6)

If \( b = a - 2 \) and \( n_{a-1} > 0 \), then the length of the middle U-chain \( U_{middle} = C_{a-1} \) is

\[ |U_{middle}| = n_{a-1} \cdot (a - 1) + n_{a-2} \cdot (a - 2) + 2n_a, \] (3.7)

and we have

\[ |U_{middle}| - |U_{top}| = (n_{a-2} - n_a) \cdot (a - 2), \]
\[ |U_{middle}| - |U_{bottom}| = (n_{a-1} - n_{a-3}) \cdot (a - 3). \] (3.8)

Consequently, \( P \) is of type C and not of type A or B if and only if \( b = a - 2 \) and both

\[ n_{a-1} > n_{a-3} > 0, \text{ and } n_{a-2} > n_a. \] (3.9)
Classification of case C.

Definition 3.6. Given the sequence $C = (c_1, c_2, s_1, s_2; a)$ of non-negative integers satisfying
\[ c_1, c_2 \geq 1, a \geq 4, \] (3.10)
we denote by $P_C$ the partition
\[ P_C = (a^{c_1}, (a - 1)^{c_2 + s_2}, (a - 2)^{c_1 + s_1}, (a - 3)^{c_2}) \] (3.11)
Note that
\[ P_C = \left( [(c_1 + c_2 + s_2)a - (c_2 + s_2)]^{c_1 + c_2 + s_1}, [(c_1 + c_2 + s_1)(a - 2) - c_2]^{c_1 + c_2 + s_1} \right), \] (3.12)
and that $P_C$ is a partition of
\[ n = a \cdot c_1 + (a - 1)(c_2 + s_2) + (a - 2)(c_1 + s_1) + (a - 3)c_2 
= (2a - 2) \cdot c_1 + (2a - 4) \cdot c_2 + (a - 1) \cdot s_2 + (a - 2) \cdot s_1. \] (3.13)
The number of parts of $P_C$ is $2c_1 + 2c_2 + s_1 + s_2$.

The following Lemma is a straightforward consequence of Lemma 3.5 and (3.11).

Lemma 3.7. Let the sequence $C = (c_1, c_2, s_1, s_2; a)$ satisfy (3.10). Then $P_C$ is a type C partition and
\[ \Omega(P_C) = ((c_2 + s_2)(a - 1) + (c_1 + s_1) \cdot (a - 2) + 2c_1, c_1 \cdot (a - 2) + c_2 \cdot (a - 3)). \] (3.14)
In other words, $P_C \in \Omega^{-1}((u, u - r))$ where
\[ u = (c_2 + s_2)(a - 1) + (c_1 + s_1) \cdot (a - 2) + 2c_1 \text{ and} \]
\[ r = s_1 \cdot (a - 2) + s_2 \cdot (a - 1) + 2(c_1 + c_2). \] (3.15)
Moreover, if $P$ is a partition of type C, then $P = P_C$ for some sequence $C = (c_1, c_2, s_1, s_2; a)$ satisfying (3.10). Here $P_C$ is also of type A if and only if $s_1 = 0$; and $P_C$ is also of type B if and only if $s_2 = 0$. Consequently, \{ $P_C : C = (c_1, c_2, s_1, s_2; a)$ satisfying (3.10) with $s_1 \geq 1$ and $s_2 \geq 1$ \} is the complete set of partitions that are of type C but not of type A or B.

Examples 3.30, 3.32 and 3.33 of Section 3.4 below contain partitions that are only of type C.

Corollary 3.8. If $u > r + r^2 / 8$ then every partition in $\Omega^{-1}((u, u - r))$ is of type A or of type B.

Proof. This is a straightforward calculation based on Lemma 3.7. \qed

Remark 3.9. We note the following, which we do not use or explore further in this paper. The formulas (3.11) for $P_C$ and (3.14) for $\Omega(P_C)$ are bilinear: linear in the multiplicities $(c_1, c_2, s_1, s_2)$ and linear in $a$. Thus increasing $a$ by 1 to form $P' = P_{C'} = P_C + 1$ increases each part of $P_C$ by 1, so $|P'_C| = |P_C| + t(P_C)$, but the multiplicities stay the same; it increases $\Omega(P_C) = (u, u - r)$ by $\Delta Q = (c_1 + c_2 + s_1 + s_2, c_1 + c_2)$ to form $Q + \Delta Q = Q' = (u', u' - r')$. It increases $r$ by $(s_1 + s_2)$ and what we call the key $S_Q = (r - 1, u - r)$ to $S_{Q'} = S_Q + (s_1 + s_2, c_1 + c_2)$.

By setting $a = 4$ we find the most basic partition $P_{C_0}, C_0 = (c_1, c_2, s_1, s_2; 4)$ of case C having the given multiplicities: we have that $|P_C| = |P_{C_0}| + t(a - 4)$, where $t = t(P_C)$. Finally, we note that even for $C_0$ where $a = 4$, we have that the number of parts of $P_C$ satisfies
\[ t(P_C) \leq \min\{2u/3, r\}. \] (3.16)
3.2 The table $\mathcal{T}(Q)$ for $Q = (u, u - r)$.

We show here Theorem 3.14 which describes the $(r - 1) \times (u - r)$ table $\mathcal{T}(Q)$ of elements in $\Omega^{-1}(Q)$.

**Definition 3.10 (Table invariants).** Let $Q = (u, u - r)$ with $u > r \geq 2$. For $0 \leq t < \min\{u - r, \lfloor \frac{r - 1}{2} \rfloor \}$ define

$$q_t = \left\lfloor \frac{u - r}{t + 1} \right\rfloor$$

$$d_t = (t + 1)q_t - (u - r).$$

We set $k_{-1} = 0$, and if $r \geq 3$ then for $0 \leq t < \min\{u - r, \lfloor \frac{r - 1}{2} \rfloor \}$ define

$$k_t = t + \left\lceil \frac{u - t + d_t}{q_t + 1} \right\rceil, \text{ and}$$

$$c_t = \begin{cases} 
0 & \text{if } d_t = 0, \\
\left\lceil \frac{u - 2(t + 1) + d_t}{q_t} \right\rceil - (k_t - t) & \text{if } d_t > 0.
\end{cases} \tag{3.17}$$

**Remark 3.11.** Since $0 \leq \left\lceil \frac{u - r}{t + 1} \right\rceil - \frac{u - r}{t + 1} < 1$, we have $0 \leq d_t < t + 1$. Note that $q_t$ and $d_t$ are defined in a way such that

$$[u - r]^{t+1} = ((q_t)^{t+1-d_t}, (q_t - 1)^{d_t}). \tag{3.18}$$

The invariants $k_t$ determine the rows of the table $\mathcal{T}(Q)$ that include B/C type partitions, and the invariants $c_t$ determine the columns of the table that may contain C type partitions.

Note that, using the definition of $d_t = (t + 1)q_t - (u - r)$, we can also write

$$k_t = 2t + 1 + \left\lceil \frac{r - 2(t + 1) + 1}{q_t + 1} \right\rceil. \tag{3.19}$$

The following lemma states some of the basic properties of the invariants of Definition 3.10.

**Lemma 3.12 (Relations among the table invariants).** Assume that $Q = (u, u - r)$ with $r \geq 3$ and let $t_{\max} = \min\{u - r, \lfloor \frac{r - 1}{2} \rfloor \} - 1$. Then

(a) $k_{t_{\max}} = \begin{cases} 
  u - \lfloor \frac{r - 1}{2} \rfloor - 1 & \text{if } u - r \leq \lfloor \frac{r - 1}{2} \rfloor \\
r - 1 & \text{if } u - r > \lfloor \frac{r - 1}{2} \rfloor \text{ and } r \text{ is odd} \\
r - 2 & \text{if } u - r > \lfloor \frac{r - 1}{2} \rfloor \text{ and } r \text{ is even}.
\end{cases}$

(b) If $0 \leq t' \leq t \leq t_{\max}$ then we have $k_{t'} - t' \leq k_t - t \leq \lfloor \frac{r + 1}{2} \rfloor$.

(c) The sequence $\{k_0, k_1, \ldots, k_{t_{\max}}\}$ is a strictly increasing sequence of positive integers satisfying $2 \leq k_t \leq r - 1$.
Proof. Proof of (a) First assume that \( u - r \leq \lfloor \frac{r-1}{2} \rfloor \). Then \( t_{\text{max}} = u - r - 1 \). So \( q_{t_{\text{max}}} = 1 \), \( d_{t_{\text{max}}} = 0 \), and by definition \ref{3.10} we have

\[
k_{t_{\text{max}}} = t_{\text{max}} + \left\lceil \frac{r+1}{2} \right\rceil = u - r - 1 + \left\lceil \frac{r+1}{2} \right\rceil = u - \left\lfloor \frac{r-1}{2} \right\rfloor - 1.
\]

(3.20)

Now assume that \( u - r > \lfloor \frac{r-1}{2} \rfloor \). Then \( t_{\text{max}} = \lfloor \frac{r-1}{2} \rfloor - 1 \) and by Formula \ref{3.19} we have

\[
k_{t_{\text{max}}} = \lfloor \frac{r-1}{2} \rfloor - 1 + \left\lceil \frac{r-1}{2} \right\rceil - \frac{r-2(\lfloor \frac{r-1}{2} \rfloor + 1)}{q_{t_{\text{max}}} + 1}.
\]

Thus \( \left\lfloor \frac{r-1}{2} \right\rfloor = 1 \) and therefore

\[
k_{t_{\text{max}}} = 2\left\lfloor \frac{r-1}{2} \right\rfloor.
\]

(3.21)

To complete the proof of (a) we note that \( \left\lfloor \frac{r-1}{2} \right\rfloor = \frac{r}{2} - 1 \) if \( r \) is even, and is \( \frac{r-1}{2} \) if \( r \) is odd.

Proof of (b) By equation \ref{3.19}, for \( 0 < t \leq t_{\text{max}} \) we have

\[
k_{t-1} - (t - 1) = t + \left\lceil \frac{r-2t+1}{q_{t-1}+1} \right\rceil
\]

\[
\leq t + \left\lceil \frac{r-2t+q_{t}}{q_{t}+1} \right\rceil \quad \text{since } 1 \leq q_{t} \leq q_{t-1}
\]

\[
= t + 1 + \left\lceil \frac{r-2(t+1)+1}{q_{t}+1} \right\rceil
\]

\[
= k_{t} - t
\]

Thus for \( 0 \leq t \leq t' \leq t_{\text{max}} \) we have \( k_{t} - t \leq k_{t'} - t' \). To complete the proof of part (b), it is enough to use part (a) to see that \( k_{t_{\text{max}}} - t_{\text{max}} \leq \left\lfloor \frac{r+1}{2} \right\rfloor \).

Proof of (c) By Part (b), the sequence \( \{k_{0}, k_{1}, \ldots, k_{t_{\text{max}}} \} \) is strictly increasing. By part (a) this sequence is bounded above by \( r - 1 \). To complete the proof, by Definition \ref{3.10} we have \( q_{0} = u - r, d_{0} = 0 \) so \( k_{0} = \left\lceil \frac{u}{u-r+1} \right\rceil \geq 2 \).

\[\square\]

Notation 3.13. Let \( Q = (u, u - r) \) with \( r \geq 2 \) and \( u - r > 0 \). For \( t = \min\{u - r, \lfloor \frac{r-1}{2} \rfloor \} \), we set \( k_{t} = r \). Recall that \( k_{-1} = 0 \).

The following is the first part of our main result.

Theorem 3.14 (Table Theorem, part I). Let \( Q = (u, u - r) \) with \( u > r \geq 2 \).
(a) For a non-negative integer $t$ such that $0 \leq t \leq \min\{u-r, \lfloor \frac{r-1}{2} \rfloor\}$, define the subset $A_t \subset \mathbb{N} \times \mathbb{N}$ as $A_t = \{(k, \ell) | k_{t-1} < k < k_t \text{ and } 1 \leq \ell \leq u-r-t\}$. Then for all $(k, \ell) \in A_t$, the partition $P_{k,\ell} = ([u]^{k-t}, [u-r]^{t+\ell})$ satisfies $\Omega(P_{k,\ell}) = (u,u-r)$ and is of type $A$.

(b) For a non-negative integer $t$ such that $0 \leq t < \min\{u-r, \lfloor \frac{r-1}{2} \rfloor\}$, define the subset $C_t \subset \mathbb{N} \times \mathbb{N}$ as

$$C_t = \{(k_t, \ell) | 1 \leq \ell \leq \min\{c_t, u-r-t\}\} \cup \{(k, u-r-t) | k > k_t \text{ and } \ell \leq c_t - (k-k_t)\}.$$ 

Then for all $(k, \ell) \in C_t$, the partition

$$P_{k,\ell} = ([u-r + 2(t+1)]^{t+1}, [u-2(t+1) - d_t(q_t - 1)]^{k-t+\ell-d_t-1}, (q_t - 1)^{d_t})$$ 

satisfies $\Omega(P_{k,\ell}) = (u,u-r)$ and is of type $C$ but not of type $A$ or $B$.

(c) For a non-negative integer $t$ such that $0 \leq t < \min\{u-r, \lfloor \frac{r-1}{2} \rfloor\}$, define the subset $B_t \subset \mathbb{N} \times \mathbb{N}$ as $B_t = \{(k_t, \ell) | c_t < \ell \leq u-r-t\} \cup \{(k, u-r-t) | k > k_t \text{ and } \ell > c_t - (k-k_t)\}$. Then for all $(k, \ell) \in B_t$, the partition

$$P_{k,\ell} = ([u-r + 2(t+1)]^{t+1}, [u-2(t+1)]^{k-t+\ell})$$ 

satisfies $\Omega(P_{k,\ell}) = (u,u-r)$ and is of type $B$.

Furthermore, each pair $(k, \ell) \in \mathbb{N} \times \mathbb{N}$ with $1 \leq k \leq r-1, 1 \leq \ell \leq u-r$ belongs to one and only one set $A_t, B_t$ or $C_t$ defined above. In particular there are listed above $(r-1)(u-r)$ distinct partitions $\{P_{k,\ell}\}$, each satisfying $\Omega(P_{k,\ell}) = (u,u-r)$.

The proof of Theorem 3.14 starts on p.16 after Remark 3.17

**Definition 3.15.** For $Q = (u,u-r)$ as in Theorem 3.14 we define the table $\mathcal{T}(Q)$ as the array of partitions $\{P_{k,\ell}, 1 \leq k \leq r-1, 1 \leq \ell \leq u-r\}$ from Theorem 3.14.

**Corollary 3.16 (The A partial rows and the B/C hooks of $\mathcal{T}(Q)$).** The table $\mathcal{T}(Q)$ comprises

A. **[B/C hooks]** For $0 \leq t < \min\{u-r, \lfloor \frac{r-1}{2} \rfloor\}$ the subset $B_t \cup C_t$ forms the $t$-th B/C hook of $\mathcal{T}(Q)$. The hook begins at $P_{k_t,1}$ in the $k_t$ row, has a corner at $P_{k_t,u-r-t}$, includes the rest of the $j = u-r-t$ column $\{P_{k,u-r-t}, k > k_t\}$, and ends in $P_{r-1,u-r-t} = ([u-r + 2(t+1)]^{t+1}, 1^{u-2(t+1)}) \quad (3.22)$

in the last row. The t-th B/C hook has length $u-t-1-k_t$. When $t = t_{\max}$ and $k_{t_{\max}} = r-1$ it comprises just part of the last row if; and it comprises the last portion of the first column when $t = t_{\max}$ and the hook column degree is 1 (i.e. $u-r-t_{\max} = 1$).

B. **[A rows]** For a pair $(t, k)$ consisting of a non-negative integer $t$ such that $0 \leq t \leq \min\{u-r, \lfloor \frac{r-1}{2} \rfloor\}$ and $k$ satisfying $k_{t-1} < k < k_t$ the subset $A_{t,k} \subset A_t$ consisting of the partitions $P_{k,\ell} = ([u]^{k-t}, [u-r]^{t+\ell}), 1 \leq \ell \leq u-r-t$ is the $A$ row beginning $P_{k,1} = ([u]^{k-t}, [u-r]^{t+1})$ and ending in $P_{k,u-r-t-1} = ([u]^{k-t}, 1^{u-r})$. It has length $u-r-t-1$. 14
Remark 3.17 (Table decomposition into A rows and B/C hooks). Theorem \[3.14\] and Corollary \[3.16\] show that the \((r-1) \times (u-r)\) table \(T(Q)\) is decomposed into A rows (or partial rows), and B/C hooks (or partial hooks), each beginning in the leftmost column of \(T(Q)\). The \(t\)-th B/C hook begins at \(P_{kt,1}\), \(t = 0, \ldots, t_{\text{max}}\), has a corner at \(P_{k,u-r-t}\) and descends to \(P_{r-1,u-r-t}\).

The top row of the table is comprised of type A partitions. Each subsequent type A row or partial row begins below the \(t\)-th B/C hook, and above next B/C hook \((k_t < k < k_{t+1})\), or below the last B/C hook \((k > k_{t_{\text{max}}})\): this partial A row begins at \(P_{k,1}\) and ends at \(P_{k,u-r-(t+1)}\). These rows and hooks exactly fit together to form the rectangular table \(T(Q)\). Note that a B/C-hook will be entirely horizontal if it begins in the last row of \(T(Q)\); and the last B/C hook, beginning at \(k = k_{t_{\text{max}}}\) row will be vertical if \(t_{\text{max}} = u - r - 1\) and \(t_{\text{max}} < r - 1\).

Sometimes a partition may fall into several classes: for example \(P = (a, a - 1, a - 2, a - 3)\) with \(\Omega(P) = (2a - 1, 2a - 5)\) may be regarded as types A, B, or C. However, for purposes of the labeling for the Table Theorem, type C means type C but not also A or B; also, we have given each partition \(P_{k,\ell}\) a single label A, B, C according to whether it is in an A row, or a B/C hook. These labels correspond also, as we shall see in Section 4.2 to the sets of equations defining the locus \(3(P_{k,\ell})\) of matrices \(A \in \mathcal{N}(Q)\) having \(P_A = P\).

Finally, each C entry \(P_{k,\ell}\) is preceded in its B/C hook only by other C entries, and by \(3.16\) they can occur only when \(k + \ell \leq \min\{2u/3, r\}\). However, \(P_{11,2}\) in \(T(Q), Q = (27, 3)\) (see Example \[3.22\], Table \[3.1\]) shows that a type C entry may occur in the vertical portion of a B/C hook.

Proof of Theorem \[3.14\]. Proof of (a). [Type A partitions] First let \(0 \leq t < \min\{u-r, \lfloor \frac{r-1}{2} \rfloor\}\) and suppose that \((k, \ell) \in A_t\). Then \(k_{t-1} < k < k_t\) and \(1 \leq \ell \leq u - r - t\).

By definition \(k_{t-1} - (t-1) \geq 1\). Since \(k \geq k_{t-1} + 1\), we get \(k - t \geq 1\). On the other hand,

\[
k - t \leq k_t - 1 - t = \left\lfloor \frac{u - t + d_t}{q_t} \right\rfloor - 1 \leq \frac{u - t + d_t - 1}{q_t + 1}.
\]

Thus \(\frac{u}{k-t} \geq \frac{u}{u-t+d_t-1}(q_t + 1)\). Since \(d_t < t + 1\), we get \(\frac{u}{k-t} > q_t + 1\). In particular we have

\[
\left\lfloor \frac{u}{k-t} \right\rfloor \geq q_t + 1 \text{ and } \left\lceil \frac{u}{k-t} \right\rceil \geq q_t + 2.
\]

On the other hand, since \(1 \leq \ell \leq u - r - t\), we get \(t + 1 \leq t + \ell \leq u - r\). So \(\frac{u-r}{t+\ell} \leq \frac{u-r}{t+1} \leq q_t\).

Thus \(\left\lfloor \frac{u-r}{t+\ell} \right\rfloor \leq q_t \leq \left\lfloor \frac{u}{k-t} \right\rfloor - 1\).

If \(\left\lfloor \frac{u-r}{t+\ell} \right\rfloor < \left\lfloor \frac{u}{k-t} \right\rfloor - 1\), then the poset of \(P_{k,\ell}\) contains two simple \(U\)-chains with the following lengths.

\[
|U_{\text{top}}| = u,
\]

\[
|U_{\text{bottom}}| = u - r + 2(k - t) \
\leq u - r + 2(k_t - 1 - t) \
\leq u - r + 2(\left\lceil \frac{r+1}{2} \right\rceil - 1) \quad \text{(By Lemma \[3.12b\])}
\leq u.
\]

So \(\Omega(P_{k,\ell}) = (u, u - r)\) in this case.
Now assume that \(\left\lceil \frac{u-r}{t+\ell} \right\rceil = \left\lfloor \frac{u}{k-\ell} \right\rfloor - 1 = q_t\). We can then write
\[
P_{k,\ell} = ((q_t + 2)^{n_2}, (q_t + 1)^{n_1}, q_t^{n_0}, (q_t - 1)^{n-1})
\]
such that \(n_1, n_2 > 0, n_2 + n_1 = k - t, 0 < n_0 \leq t + \ell,\) and \(n_0 + n_{-1} = t + \ell\).

By (3.23), \((k-t)(q_t + 1) + (t-d_t + 1) \leq u\). Thus \([u]^{k-t}\) has at least \(t+1-d_t\) parts of size \(q_t + 2\). So \(n_2 \geq t + 1 - d_t\).

On the other hand since \((t+1)q_t - d_t = u - r\), we get
\[
(t + \ell)q_t - d_t = u - r + (\ell - 1)q_t \geq u - r.
\]
Thus \([u - r]^{t+\ell}\) has at least \(d_t\) parts of size \(q_t - 1\). So \(n_{-1} \geq d_t\).

So the poset of \(P_{k,\ell}\) contains three simple \(U\)-chains with the following lengths.
\[
\begin{align*}
|U_{\text{top}}| &= u, \\
|U_{\text{middle}}| &= \left| \mathcal{D}_{P_{k,\ell}} \right| - (n_{-1}(q_t - 1) + n_2 q_t) \\
&\leq u + (u - r) - (d_t(q_t - 1) + (t + 1 - d_t)q_t) \\
&= u + u - r - u = u - r < u,
\end{align*}
\]
\[
\begin{align*}
|U_{\text{bottom}}| &= u - r + 2(k - t) \\
&\leq u - r + 2(k_t - 1 - t) \\
&\leq u - r + 2(\left\lceil \frac{t+1}{r} \right\rceil - 1) \quad \text{(By Lemma 3.12)} \\
&\leq u.
\end{align*}
\]

So \(\mathcal{Q}(P_{k,\ell}) = (u, u - r)\) in this case as well. This completes the proof of part (a), Type A rows of the table.

Now let \(t = \min\{u - r, \left\lfloor \frac{r-1}{2} \right\rfloor \}\). By Lemma 3.12(a), \(A_t\) is non-empty only if \(r\) is even and \(u - r > \left\lfloor \frac{r-1}{2} \right\rfloor = \frac{r}{2} - 1\). In this case, \(t = \frac{r}{2} - 1\) and \(k_{t-1} = r - 2\).

Assume that \((k, \ell) \in A_t\) in this case. Then \(k = r - 1, 1 \leq \ell \leq u - \frac{3}{2}r + 1\) and \(P_{k,\ell} = ([u]^{\frac{r}{2}}, [u - r]^{\frac{r}{2}+\ell-1})\).

Since \(t + \ell \geq t + 1 = \frac{r}{2}\),
\[
\frac{u - r}{t + \ell} \leq \frac{u - r}{\frac{r}{2}} = \frac{u}{\frac{r}{2}} - 2 = \frac{2u}{r} - 2. \tag{3.24}
\]

If \(\left\lceil \frac{u-r}{t+\ell} \right\rceil < \left\lfloor \frac{2u}{r} \right\rfloor\), then the poset of \(P_{k,\ell}\) contains two simple \(U\)-chains with the following lengths.
\[
\begin{align*}
|U_{\text{top}}| &= u, \\
|U_{\text{bottom}}| &= u - r + 2(\frac{r}{2})
\end{align*}
\]

So \(\mathcal{Q}(P_{k,\ell}) = (u, u - r)\) in this case and it is of type A (as well as type B).
If \( \left\lfloor \frac{u-r}{t+t} \right\rfloor = \left\lfloor \frac{2u}{r} \right\rfloor - 1 \), then by (3.24) we must have \( \left\lfloor \frac{2u}{r} \right\rfloor = \left\lfloor \frac{2u}{r} \right\rfloor - 1 \) and \( \ell = 1 \). Therefore

\[
P_{k,\ell} = ( (\left\lfloor \frac{2u}{r} \right\rfloor + 1)^{n_1}, (\left\lfloor \frac{2u}{r} \right\rfloor )^{n_0}, (\left\lfloor \frac{2u}{r} \right\rfloor - 1)^{n_1}, (\left\lfloor \frac{2u}{r} \right\rfloor - 2)^{n_0} )
\]
such that \( n_1 > 0 \), \( n_0 + n_1 = \frac{r}{2} \) and \( n_1 (\left\lfloor \frac{2u}{r} \right\rfloor + 1) + n_0 (\left\lfloor \frac{2u}{r} \right\rfloor ) = u \). So the poset of \( P_{k,\ell} \) contains three simple \( U \)-chains with the following lengths.

\[
\begin{align*}
|U_{top}| &= u, \\
|U_{middle}| &= n_1 (\left\lfloor \frac{2u}{r} \right\rfloor - 1) + n_0 (\left\lfloor \frac{2u}{r} \right\rfloor ) + 2n_1 \\
&= n_1 (\left\lfloor \frac{2u}{r} \right\rfloor + 1) + n_0 (\left\lfloor \frac{2u}{r} \right\rfloor ) \\
&= u, \\
|U_{bottom}| &= u - r + 2(\frac{r}{2}) \\
&= u.
\end{align*}
\]

So in this case \( Q(P_{k,\ell}) = (u, u - r) \) and it is of type \( A \) (as well as types \( B \) and \( C \)).

**Proof of (b).** [Type C partitions]. Let \( 0 \leq t < \min\{u-r, \left\lfloor \frac{u-1}{2} \right\rfloor \} \) and let \( (k, \ell) \in C_t \). Then \( k = k_t \) and \( 1 \leq \ell \leq \min\{c_t, u-r-t\} \), or \( k > k_t \), \( \ell = u-r-t \) and \( \ell \leq c_t - (k - k_t) \).

If \( c_t \leq 0 \) then \( C_t \) is empty and therefore there is nothing to prove. We assume that \( c_t > 0 \). In particular, by definition of \( c_t \), this implies that \( d_t > 0 \).

Since by assumption \( k < k_t \), using the definition of \( c_t \) we get

\[
k - t + \ell \leq k_t - t + c_t \leq \left\lfloor \frac{u - 2(t + 1) + d_t}{q_t} \right\rfloor \leq \frac{u - 2(t + 1) + d_t + q_t - 1}{q_t}.
\]

Thus

\[
(k - t + \ell - 1 - d_t) q_t \leq u - 2(t + 1) - d_t(q_t - 1) - 1 < u - 2(t + 1) - d_t(q_t - 1).
\]

On the other hand, since \( k \geq k_t \) and \( \ell \geq 1 \), by definition of \( k_t \), we have

\[
k - t + \ell - 1 \geq k_t - t = \left\lfloor \frac{u - t + d_t}{q_t + 1} \right\rfloor \geq \frac{u - t + d_t}{q_t + 1}.
\]

Thus

\[
(k - t + \ell - 1 - d_t)(q_t + 1) \geq u - t + d_t - d_t(q_t + 1)
\]

\[
= u - 2(t + 1) - d_t(q_t - 1) + (t + 2 - d_t)
\]

\[
> u - 2(t + 1) - d_t(q_t - 1) \quad \text{(Since } t + 1 > d_t). \]

So we can write

\[
[u - 2(t + 1) - d_t(q_t - 1)]^{k-t+\ell-1-d_t} = ((q_t + 1)^{n_1}, q_t^{n_0})
\]
such that \( n_1 \geq 1, \ n_0 \geq t + 2 - d_t \) and \( n_0 + n_1 = k - t + \ell - 1 - d_t \).

Thus, using \( [u - r + 2(t + 1)]^{t+1} = ((q_t + 2)^{t+1-d_t}, (q_t + 1)^{d_t}) \), we have

\[
P_{k,\ell} = ((q_t + 2)^{t+1-d_t}, (q_t + 1)^{d_t+n_1}, q_t^{n_0}, (q_t - 1)^{d_t}).
\]

(3.25)

Therefore the poset of the partition \( P_{k,\ell} \) contains three simple \( U \)-chains with the following lengths.

\[
|U_{top}| = |D_{P_{k,\ell}}| - (d_t(q_t - 1) + n_0q_t) = u + u - r - d_t(q_t - 1) - n_0q_t \\
\leq (u + u - r) - d_t(q_t - 1) - (t + 2 - d_t)q_t = u + u - r - d_tq_t - (-d_t + (t + 1)q_t) - q_t + d_tq_t \leq u - q_t < u.
\]

\[
|U_{middle}| = |D_{P_{k,\ell}}| - [(t + 1 - d_t)q_t + d_t(q_t - 1)] = (u + u - r) - [(t + 1 - d_t)q_t + d_t(q_t - 1)] = u.
\]

\[
|U_{bottom}| = |D_{P_{k,\ell}}| - [(t + 1 - d_t)(q_t + 2) + (d_t + n_1)(q_t + 1)] = u + u - r - [u - r + n_1(q_t + 1)] < u.
\]

So \( Q(P_{k,\ell}) = (u, u - r) \), as desired and \( P_{k,\ell} \) is of type C.

**Proof of (c).** [Type B partitions]. Let \( 0 \leq t < \min\{u - r, \left\lceil \frac{r - 1}{2} \right\rceil \} \) and let \( (k, \ell) \in B_t \). Then \( k = k_t \) and \( c_t < \ell \leq u - r - t \), or \( k > k_t, \ \ell = u - r - t \) and \( \ell > c_t - (k - k_t) \).

Recall \( P_{k,\ell} = ([u - r + 2(t + 1)]^{t+1}, [u - 2(t + 1)]^{k-t+\ell-1}) \).

Since \( k \geq k_t \) and \( \ell \geq 1 \), we have

\[
k - t + \ell - 1 \geq k_t - t = \left\lfloor \frac{u - t + d_t}{q_t + 1} \right\rfloor \geq \frac{u - t + d_t}{q_t + 1}.
\]

Thus

\[
(k - t + \ell - 1)(q_t + 1) \geq u - t + d_t = u - 2(t + 1) + (t + 2 - d_t) > u - 2(t + 1). \quad \text{(Since } d_t < t + 1. \text{)}
\]

Therefore \( \left\lfloor \frac{u - 2(t + 1)}{k - t + \ell - 1} \right\rfloor \leq q_t + 1. \)

**Case 1.** Let \( d_t = 0. \)

In this case \( k_t - t = \left\lfloor \frac{u - t}{q_t + 1} \right\rfloor \) and \( [u - r + 2(t + 1)]^{t+1} = ((q_t + 2)^{t+1}). \)
Case 1.1. If \( \lceil \frac{u-2(t+1)}{k-t+\ell-1} \rceil < q_t + 1 \), then the biggest part of the partition \( [u - 2(t + 1)]^{k-t+\ell-1} \) is at most \( q_t \), and therefore it is not adjacent to the parts of \( [u - r + 2(t + 1)]^{t+1} \). Thus the lengths of the simple \( U \)-chains in the poset of \( P_{k,\ell} \) are as follows.

\[
|U_{\text{top}}| = u - r + 2(t + 1) \\
\leq u - r + 2\left\lfloor \frac{r-1}{2} \right\rfloor \\
< u.
\]

\[
|U_{\text{bottom}}| = 2(t + 1) + [u - 2(t + 1)]
\]

So \( Q(P_{k,\ell}) = (u, u - r) \) and \( P_{k,\ell} \) is of type B.

Case 1.2. If \( \lceil \frac{u-2(t+1)}{k-t+\ell-1} \rceil = q_t + 1 \), then, since \( d_t = 0 \), using (3.26), we get

\[(k - t + \ell - 1)(q_t + 1) - (t + 2) \geq u - 2(t + 1).\]

Thus the partition \( [u - 2(t + 1)]^{k-t+\ell-1} \) must have at least \( t + 2 \) parts of size \( q_t \). So we can write

\[
[u - 2(t + 1)]^{k-t+\ell-1} = ((q_t + 1)^{k-t+\ell-1-n_0}, q_t^{n_0}),
\]

with \( n_0 \geq t + 2 \). Thus \( P_{k,\ell} = ( (q_t + 2)^{t+1}, (q_t + 1)^{k-t+\ell-1-n_0}, q_t^{n_0} ) \) and the lengths of the simple \( U \)-chains in the poset of \( P_{k,\ell} \) are as follows.

\[
|U_{\text{top}}| = |D_{P_{k,\ell}}| - n_0 q_t \\
\leq (u + u - r) - n_0 q_t \\
= u + u - r - (u - r) - q_t \quad \text{(Since} \ u - r = (t + 1)q_t) \\
< u.
\]

\[
|U_{\text{bottom}}| = |D_{P_{k,\ell}}| - (t + 1)q_t \\
= u + u - r - (u - r) \\
= u.
\]

So \( Q(P_{k,\ell}) = (u, u - r) \) and \( P_{k,\ell} \) is of type B.

This completes the proof in Case 1 of (c).

Case 2. Let \( d_t > 0 \).

In this case \( [u - r + 2(t + 1)]^{t+1} = ((q_t + 2)^{t+1-d_t}, (q_t + 1)^{d_t}) \). Since by assumption \( \ell > c_t - (k - k_t) \), by definition of \( c_t \), we must have

\[
k - t + \ell - 1 \geq \left\lfloor \frac{u-2(t+1)+d_t}{q_t} \right\rfloor \geq \frac{u-2(t+1)+d_t}{q_t}.
\]

Therefore

\[(k - t + \ell - 1)q_t \geq u - 2(t + 1) + d_t > u - 2(t + 1).
\]

Thus \( \lceil \frac{u-2(t+1)}{k-t+\ell-1} \rceil \leq q_t \).
Case 2.1. If \( \lceil \frac{u-2(t+1)}{k-t+\ell-1} \rceil < q_t \) then the lengths of the simple \( U \)-chains in the poset of \( P_{k,\ell} \) are as follows.

\[
\begin{align*}
|U_{\text{top}}| &= u - r + 2(t + 1) \\
&\leq u - r + 2 \left\lfloor \frac{r-1}{2} \right\rfloor \\
&< u.
\end{align*}
\]

\[
|U_{\text{bottom}}| = 2(t + 1) + [u - 2(t + 1)] = u
\]

So \( \mathcal{O}(P_{k,\ell}) = (u, u - r) \) and \( P_{k,\ell} \) is of type B.

Case 2.2. If \( \lceil \frac{u-2(t+1)}{k-t+\ell-1} \rceil = q_t \), then since by (3.27) we have

\[
(k - t + \ell - 1)q_t - d_t \geq u - 2(t + 1),
\]

the partition \( [u - 2(t + 1)]^{k-t+\ell-1} \) has at least \( d_t \) parts of size \( q_t - 1 \). So we can write

\[
[u - 2(t + 1)]^{k-t+\ell-1} = (q_t^{k-t+\ell-1-n-1}, (q_t - 1)^{n-1})
\]

with \( n-1 \geq d_t \). Thus

\[
P_{k,\ell} = ((q_t + 2)^{t+1-d_t}, (q_t + 1)^{d_t}, q_t^{k-t+\ell-1-n-1}, (q_t - 1)^{n-1})
\]

and the lengths of the simple \( U \)-chains in the poset of \( P_{k,\ell} \) are as follows.

\[
\begin{align*}
|U_{\text{top}}| &= u - r + 2(t + 1) \\
&\leq u - r + 2 \left\lfloor \frac{r-1}{2} \right\rfloor \\
&< u.
\end{align*}
\]

\[
|U_{\text{middle}}| = |\mathcal{D}_{P_{k,\ell}}| - [(t + 1 - d_t)q_t + n-1(q_t - 1)] \\
&\leq u + u - r - [(t + 1 - d_t)q_t + d_t(q_t - 1)] \\
&= u
\]

\[
|U_{\text{bottom}}| = 2(t + 1) + [u - 2(t + 1)] = u
\]

So \( \mathcal{O}(P_{k,\ell}) = (u, u - r) \) and \( P_{k,\ell} \) is of type B. [It may also be of type \( C \cap B \).]

This completes the proof in Case 2 and therefore the proof of part (c).

Filling the table \( \mathcal{T}(Q) \) and proof of Corollary [3.16] It is clear from the definition of \( A_t, B_t \) and \( C_t \) that these are disjoint sets and that the partitions \( P_{k,\ell}, 1 \leq k \leq r - 1, 1 \leq \ell \leq u - r \) defined in parts (a)-(c) are distinct, and exactly fill the \((r - 1) \times (u - r)\) table \( \mathcal{T}(Q) \). \( \square \)
3.3 Properties of the Table $\mathcal{T}(Q)$.

**Corollary 3.18** (Corner elements of $\mathcal{T}(Q), Q = (u, u - r)$). The four corners are

\[
P_{1,1} = (u, u - r) \quad P_{1,u-r} = (u, 1^{u-r})
\]

\[
P_{r-1,1} \text{ (given below)} \quad P_{r-1,u-r} = (u - r + 2, 1^{u-2})
\]

For $P_{r-1,1}$ we need to distinguish between the following three cases:

- If $u - r \leq \left\lfloor \frac{r-1}{2} \right\rfloor$ then $P_{r-1,1} = (3^{u-r}, 1^{2r-u})$, the final in a B hook (column)

- If $u - r > \frac{r-1}{2}$ and $r$ is odd, then $P_{r-1,1} = ([u - 1]^{\frac{r-1}{2}}, [u - r + 1]^{\frac{r+1}{2}})$, the first in a B hook (row)

- If $u - r > \frac{r}{2}$ and $r$ is even, then $P_{r-1,1} = ([u]^{\frac{r}{2}}, [u - r]^{\frac{r}{2}})$, the first in the last A partial row.

**Proof.** The partitions $P_{1,1}$ and $P_{1,u-r}$ are directly obtained from part (a) of Theorem 3.14. When $r > 2$, $P_{r-1,u-r}$ is the last entry of the first B/C hook of $\mathcal{T}(Q)$ so is obtained from part (b) of Theorem 3.14.

To obtain $P_{r-1,u-r}$, we use Lemma 3.12(b) to obtain $k_{t_{\text{max}}}$ in each case and then use Theorem 3.14. If $u - r \leq \left\lfloor \frac{r-1}{2} \right\rfloor$ then $t_{\text{max}} = u - r - 1$ and as shown in the proof of Lemma 3.12(b), $d_{t_{\text{max}}} = 0$ and $k_{t_{\text{max}}} = u - \left\lfloor \frac{r-1}{2} \right\rfloor$. Thus, by Theorem 3.14d (here $P_{r-1,1}$ is the final partition of the last B hook, which is vertical),

\[
P_{r-1,1} = ([u - r + 2(u - r)]^{u-r}, [u - 2(u - r)]^{2r-u}) = (3^{u-r}, 1^{2r-u}).
\]

If $u - r > \frac{r-1}{2}$ and $r$ is odd, then $t_{\text{max}} = \frac{r-1}{2} - 1$ and by Lemma 3.12(b), $k_{t_{\text{max}}} = r - 1$. Thus by Theorem 3.14b (here $P_{r-1,1}$ is the first partition in the last type B hook, which is horizontal),

\[
P_{r-1,1} = ([u - r + 2(\frac{r-1}{2})]^{\frac{r-1}{2}}, [u - 2(\frac{r-1}{2})]^{\frac{r+1}{2}}) = ([u - 1]^{\frac{r-1}{2}}, [u - r + 1]^{\frac{r+1}{2}}).
\]

Finally, if $u - r > \frac{r}{2}$ and $r$ is even, then $t_{\text{max}} = \frac{r}{2} - 2$ and by Lemma 3.12(b), $k_{t_{\text{max}}} = r - 2$. Thus by Theorem 3.14a (here $P_{r-1,1}$ is the first partition in the final type A partial row),

\[
P_{r-1,1} = ([u]^{r-1-(\frac{r}{2}-2+1)}, [u - r]^{(\frac{r}{2}-2+1)+1}) = ([u]^{\frac{r}{2}}, [u - r]^{\frac{r}{2}}).
\]

\[\square\]

**Remark 3.19.** The partition $P_{r-1,u-r} = (u - r + 2, [u-2]^{u-2})$ is the unique partition in the table satisfying $\# \text{ parts} = u - 1 = (r - 1) + (u - r)$, the maximum possible for $\Omega^{-1}(Q)$. The first case for $P_{r-1,u-r}$ occurs in Examples 3.32, 3.22, and 3.30. The two last cases for $P_{r-1,u-r}$ are seen respectively in Example 3.27 and Example 3.28.
Recall that the Bruhat partial order on partitions of \( n \) satisfies, for \( P = (p_1, p_2, \ldots, p_s), p_1 \geq p_2 \geq \ldots \geq p_s \) and \( P' = (p'_1, \ldots, p'_{s'}), p'_1 \geq \ldots \geq p'_{s'} \)
\[
P \geq P' \iff \sum_{i=1}^{u} p_i \geq \sum_{i=1}^{u} p'_i \text{ for all } u, 1 \leq u \leq \min\{s, s'\}.
\]

(3.28)

The following should be a combinatorial consequence of Theorem 3.14. We state it as a conjecture since our proof is not complete.

**Conjecture 3.20** (Bruhat order on the first column of \( T(Q) \)). Let \( Q = (u, u - r), r \geq 3 \). For \( 1 \leq k \leq r - 2 \) we have \( P_{k,1} \geq P_{k+1,1} \) in Bruhat order.

**Proof plan.** This is obvious except for transition entries from type A to type B/C, \( k = k_t - 1, 0 \leq t \leq t_{\text{max}} \), or at a type B entry \( k = k_t, 0 \leq t < t_{\text{max}} \). The simplest way to proceed seems to be to prove the geometric Conjecture 4.6, from which this follows. The combinatorial route would require comparing the ranks of \( P, A, A^2, A^3, \ldots \) for matrices \( A \) of Jordan type \( P_{k,1} \) with the corresponding ranks for \( P_{k+1,1} \).

□

Taking \( s = 1 \) in the definition of Bruhat order, this conjecture implies that the biggest part of \( P_{k,1} \) is non-increasing in the first column, a fact we use in Section 5.4. We give a proof.

**Lemma 3.21.** Fix \( Q = (u, u - r) \) with \( u > r \geq 2 \). For \( 0 < k < r \), we denote by \( p_k \) the biggest part of \( P_{k,1} \in T(Q) \). Then \( p_1, \ldots, p_{r-1} \) is a non-increasing sequence of positive integers.

**Proof.** By Theorem 3.14 it is clear that going down the first column of \( T(Q) \), the biggest part of the partitions does not increase from \( P_{k,1} \) to \( P_{k+1,1} \) as long as the kind A or B/C of the partition \( P_{k,1} \) is the same as that of \( P_{k+1,1} \). So to complete the proof, it is enough to show that for each integer \( t : 0 \leq t < \min u - r, \lfloor r - 1 \rfloor \), we have \( p_{k_t - 1} \geq p_{k_t} \geq p_{k_t + 1} \).

By Theorem 3.14 we have
\[
p_{k_t - 1} = \left\lceil \frac{u}{k_t - 1 - t} \right\rceil, \quad p_{k_t} = \left\lceil \frac{u - r}{t + 1} \right\rceil + 2, \quad \text{and} \quad p_{k_t + 1} = \left\lceil \frac{u}{k_t + 1 - (t + 1)} \right\rceil.
\]

Recall that by Definition 3.10, we have
\[
q_t = \left\lceil \frac{u - r}{t + 1} \right\rceil, \quad \text{and} \quad k_t = t + \left\lceil \frac{u - t + d_t}{q_t + 1} \right\rceil \text{ where } d_t = (t + 1)q_t - (u - r).
\]

Therefore
\[
p_{k_t - 1} = \left\lceil \frac{u}{\left\lceil \frac{u - t + d_t}{q_t + 1} \right\rceil - 1} \right\rceil, \quad p_{k_t} = q_t + 2, \quad \text{and} \quad p_{k_t + 1} = \left\lceil \frac{u}{\frac{u - t + d_t}{q_t + 1}} \right\rceil.
\]

Using the inequalities \( \frac{u - t + d_t}{q_t + 1} \leq \left\lceil \frac{u - t + d_t}{q_t + 1} \right\rceil \leq \frac{u - t + d_t}{q_t + 1} \), we get
\[
p_{k_t - 1} \geq \left\lceil \frac{u}{\frac{u - t + d_t}{q_t + 1} - 1} \right\rceil \geq \frac{u(q_t + 1)}{u - t + d_t - 1} = q_t + 1 + \frac{(t - d_t + 1)(q_t + 1)}{u - t + d_t - 1} > q_t + 1.
\]
Here to see the last inequality, we use the inequalities \( d_t < t + 1 \leq u - r \) which hold by definition of \( d_t \) and the assumption about \( t \).

Thus \( p_{k_t-1} \geq q_t + 2 = p_{k_t} \).

On the other hand,

\[
p_{k_t+1} < \frac{u}{u - t + d_t} + 1
\]

\[
\leq \frac{u(q_t+1)}{u - t + d_t} + 1
\]

\[
= q_t + 1 + \frac{(t-d_t)(q_t+1)}{u - t + d_t} + 1.
\]

Since \( d_t \geq 0 \), we get

\[(t - d_t)(q_t + 1) \leq t(q_t + 1) = (t + 1)q_t + t - q_t = u - r + dt + t - q_t = u - t + dt - (r + q_t) < u - t + dt.
\]

Thus \( p_{k_t+1} \leq q_t + 2 \). This completes the proof. \( \square \)

**Example 3.22** (Table \( \mathcal{T}(Q) \) and table invariants for \( Q = (27, 3) \)). Here \( u - r = 3 \), and \( r = 24 \).

We have

| \( q_0 \) = \( \left\lceil \frac{3}{1} \right\rceil = 3 \) | \( q_1 \) = \( \left\lceil \frac{3}{2} \right\rceil = 2 \) | \( q_2 \) = \( \left\lceil \frac{3}{3} \right\rceil = 1 \)
| --- | --- | --- |
| \( d_0 \) = 0, \( d_1 \) = 1, and \( d_2 \) = 0, | \( k_0 \) = \( \left\lceil \frac{27}{1} \right\rceil = 7 \), and \( k_1 \) = 1 + \( \left\lceil \frac{27}{3} \right\rceil = 10 \), and \( k_2 \) = 2 + \( \left\lceil \frac{25}{2} \right\rceil = 15 \), and | \( c_0 \) = 0, \( c_1 \) = 3, \( c_2 \) = 0, |

Recall that \( k_0 = 5, k_1 = 10, k_2 = 15 \) are the initial rows of type B/C hooks. By Theorem 3.14 we have

- For \( 1 \leq k \leq 6 \), and \( 1 \leq \ell \leq 3 \), we have \( P_{k,\ell} = ([27]^k, [3]^\ell) \). (Type A)
- For \( k = 7 \) and \( 1 \leq \ell \leq 3 \), we have \( P_{7,\ell} = (5, [25]^\ell+6) \). (Horizontal part of Type B hook)
- For \( 8 \leq k \leq 23 \), and \( \ell = 3 \), we have \( P_{k,3} = (5, [25]^k+2) \). (Vertical part of same B hook)
- For \( 8 \leq k \leq 9 \), and \( 1 \leq \ell \leq 2 \), we have \( P_{k,\ell} = ([27]^k, [3]^\ell+1) \). (Type A)
- For \( k = 10 \), and \( 1 \leq \ell \leq 2 \), we have \( \ell \leq c_1 = 3 \), and therefore \( P_{10,\ell} = ([7]^2, [22]^\ell+7, 1) \). (Type C, part of partial hook in blue)
The table $\mathcal{T}(Q), Q = (27, 3)$ below lists all partitions $P_{k,\ell}$, $1 \leq k \leq 23$ and $1 \leq \ell \leq 3$, with $P_{k,\ell}$ filling the position in the $k$-th row and $\ell$-th column of the table.

$$
\begin{array}{ccc}
(27, 3) & (27, [3]^2) & (27, [3]^3) \\
([27]^2, 3) & ([27]^2, [3]^2) & ([27]^2, [3]^3) \\
\vdots & \vdots & \vdots \\
([27]^6, 3) & ([27]^6, [3]^2) & ([27]^6, [3]^3) \\
(5, [25]^7) & (5, [25]^8) & (5, [25]^9) \\
([27]^8, [3]^2) & ([27]^8, [3]^3) & (5, [25]^{10}) \\
([27]^9, [3]^2) & ([27]^9, [3]^3) & (5, [25]^{11}) \\
([7]^2, [22]^8, 1) & ([7]^2, [22]^9, 1) & (5, [25]^{12}) \\
([27]^9, [3]^3) & ([7]^2, [22]^{10}, 1) & (5, [25]^{13}) \\
([27]^10, [3]^3) & ([7]^2, [23]^{12}) & (5, [25]^{14}) \\
([27]^11, [3]^3) & ([7]^2, [23]^{13}) & (5, [25]^{15}) \\
([27]^12, [3]^3) & ([7]^2, [23]^{14}) & (5, [25]^{16}) \\
([9]^3, [21]^{13}) & ([7]^2, [23]^{15}) & (5, [25]^{17}) \\
([9]^3, [21]^{14}) & ([7]^2, [23]^{16}) & (5, [25]^{18}) \\
\vdots & \vdots & \vdots \\
([9]^3, [21]^{21}) & ([7]^2, [23]^{23}) & (5, [25]^{25}) \\
\end{array}
$$

Table 3.1: Table $\mathcal{T}(Q), Q = (27, 3)$

- For $k = 11$ and $\ell = 2$, we have $\ell \leq c_1 - (11 - 10) = 2$, and therefore $P_{11,2} = ([7]^2, [22]^{10}, 1)$. (Type C, part of partial hook, in blue)

- For $12 \leq k \leq 23$ and $\ell = 2$ we have $\ell > c_1 - (k - 10)$ and therefore $P_{k,2} = ([7]^2, [23]^k)$. (Type B, partial hook in red, forming with the previous two cases a single B,C hook).

- For $11 \leq k \leq 14$ and $\ell = 1$ we have $P_{k,1} = ([27]^{k-2}, [3]^3)$. (Type A)

- For $15 \leq k \leq 23$, $P_{k,1} = ([9]^3, [21]^{k-2})$. (This Type B vertical hook is in purple).

### 3.4 Completeness of the table $\mathcal{T}(Q)$.

We show here in Theorem 3.25 that $\mathcal{T}(Q)$ is all of $\Omega^{-1}(Q)$. This completes the proof of the Table Theorem 1.1.

**Lemma 3.23.** Fix $Q = (u, u - r), u > r \geq 2$. All type C partitions $P$ satisfying $\Omega(P) = Q$ and that are not of type A or B, occur in the table $\mathcal{T}(Q)$ of Definition 3.15.
Proof. From the Theorem 3.14(b) and (3.25) the type C partition \( P_{k,l} \) satisfies
\[
P_{k,l} = ((q_t + 2)^{t+1-d_t}, (q_t + 1)^{d_t+n_1}, q_t^{n_0}, (q_t - 1)^{d_t}).
\] (3.29)

We can compare this with the complete list of Type C partitions of \( n = 2u - r \) from section 3.1 equation (3.11). From Lemma 3.14, given any sequence \( C = (c_1, c_2, s_1, s_2; a) \) of integers satisfying \( a \geq 4 \) and each of \( c_1, c_2, s_1, s_2 \geq 1 \), we have
\[
P_C = (a^{c_1}, (a - 1)^{c_2+s_2}, (a - 2)^{c_1+s_1}, (a - 3)^{c_2})
\] (3.30)
is a type C partition not of type A or B. Also, each type C partition not of type A or B occurs in this way, and \( \Psi(P_C) \) satisfies \( \Psi(P_C) = (u, u - r) \)
\[
\begin{align*}
u &= (a - 2)(c_1 + c_2 + s_1 + s_2) + c_2 + s_2 + 2c_1 \\
&= (a - 1)(c_1 + c_2 + s_1 + s_2) + c_1 - s_1; \\
u - r &= (a - 2)(c_1 + c_2) - c_2.
\end{align*}
\]
Comparing the two, we have
\[
c_1 = t + 1 - d_t, c_2 = d_t, s_1 = n_0 - (t + 1 - d_t), s_2 = n_1, a = q_t + 2, \text{ and conversely},
\]
\[
q_t = a - 2, d_t = c_2, n_1 = s_2, n_0 = c_1 + s_1, t + 1 = c_1 + c_2.
\]
The latter invariants \( (q_t, d_t, n_1, n_0, t + 1) \) determine a partition of the form C as in the proof of (b) in Theorem 3.14 provided \( c_t > 0 \). By the definition of \( k_t \) and the formula for \( u \) involving \( (a - 1) \) we have
\[
k_t = t + (c_1 + c_2 + s_1 + s_2) + \left\lfloor \frac{1 - s_1}{a - 1} \right\rfloor.
\] (3.31)

By the definition of \( c_t \) and the formula for \( u \) involving \( (a - 2) \) we have
\[
c_t = c_1 + c_2 + s_1 + s_2 + \left\lfloor \frac{s_2}{a - 2} \right\rfloor - (c_1 + c_2 + s_1 + s_2 + \left\lfloor \frac{1 - s_1}{a - 1} \right\rfloor)
\]
\[
= \left\lfloor \frac{s_2}{a - 2} \right\rfloor + \left\lfloor \frac{s_1 - 1}{a - 1} \right\rfloor;
\] (3.32)
which is always positive, as we assumed \( s_2 \geq 1 \).\(^3\) Thus we have \( P_C = P_{k,l}(Q_C) \) where \( k \geq k_t \) as above: \( k > k_t \) for the vertical portion of the B/C hook. Since \( k + \ell = \# \text{ parts of } P_C \), so \( k + \ell = 2c_1 + 2c_2 + s_1 + s_2 \), and since \( t = c_1 + c_2 - 1 \), we have for \( k = k_t \),
\[
\ell = 2c_1 + 2c_2 + s_1 + s_2 - \left( (c_1 + c_2 - 1) + (c_1 + c_2 + s_1 + s_2) - \left\lfloor \frac{s_1 - 1}{a - 1} \right\rfloor \right)
\]
\[
= 1 + \left\lfloor \frac{s_1 - 1}{a - 1} \right\rfloor \text{ (when } k = k_t).\] (3.33)

But the maximum column index \( j \) in the \( t \)-th B/C hook (here \( 0 \leq t \leq t_{\text{max}} \)) is
\[
u - r - t = (a - 4)(c_1 + c_2) + c_1 + 1.
\] (3.34)

\(^3\)Here \( c_1, c_2 \) is from Section 3.1, not the Theorem 3.14.

\(^4\)Note that \( c_t \) depends on the choice of a partition \( Q \) as well as \( t \): the partition \( Q \) changes when \( s_1, s_2 \) change in the formula.
It follows that $P_C = P_{k,\ell}$ where

$$
\ell = \begin{cases} 
1 + \left\lfloor \frac{n-1}{a} \right\rfloor & \text{if } (1 + \left\lfloor \frac{n-1}{a} \right\rfloor) < u - r - t, \\
u - r - t & \text{otherwise.}
\end{cases}
$$

(3.35)

and

$$
k = \begin{cases} 
k_1 & \text{if } \ell = (1 + \left\lfloor \frac{n-1}{a} \right\rfloor) \\
2c_1 + 2c_2 + s_1 + s_2 - \ell & \text{otherwise.}
\end{cases}
$$

(3.36)

It is easy to check from equation (3.14) of Lemma 3.7 that $k \leq r - 1$ in the latter case. This completes the proof that $P_C$ occurs in $\mathcal{T}(Q)$. □

**Example 3.24.** Let the sequence $C = (c_1, c_2, s_1, s_2; a) = (1, 2, 1, 1; 4)$, so $P_C = (4, 3^2, 2^2, 1^2)$, a type C partition of 19. Then $\mathcal{Q}(P) = (15, 4)$ and we have $t = d_2 = 2, k_2 = (2+|15-2+2)/3| = 7, \ell = 1$, so $P = P_{7,1}$ in the table $\mathcal{T}(15, 4)$. Since also $c_1 = \left\lfloor \frac{1}{2} \right\rfloor + 0 = 1$, we have that $P_{7,2}$ must be a type B partition: and in fact $P_{7,2} = ([10]^4, [9]^5)$. As we shall see, $P' = ([10]^4, [9]^4)$ is a Jordan type that can not occur as $P_A$ for any $A \in \mathcal{N}(Q)$.

Take instead $C = (1, 2, 4, 1; 4)$, then $P_C = (4, 3^3, 2^2, 1^2)$, with $\mathcal{Q}(P_C) = (21, 4)$, and we have $t = d_2 = 2, k_2 = 9, \ell = c_2 = 2$, so $P_C = P_{9,2}$. The next partition in this B-C hook is $P_{10,2} = ([10]^3, [15]^9)$ of type B, and the leading one (which must be type C) is $P_{C'} = (4, 3^3, 2^2, 1^3), C' = (1, 2, 1, 3)$ where $k_2 = 9, \ell' = 1$.

We can now show the completeness part of the Table Theorem 1.1.

**Theorem 3.25** (Part II of Table theorem). Let $Q = (u, u - r), u \geq r \geq 2$. The table $\mathcal{T}(Q)$ of Definition 3.15 comprises a complete list of the elements of $\mathcal{Q}^{-1}(Q)$.

**Proof.** That the cases A,B in the table form a complete list of cases A,B in $\mathcal{Q}^{-1}(Q)$ is evident from the proof of Theorem 3.14. Lemma 3.23 shows the completeness for type C partitions. □

**Normal pattern.**

We say that $\mathcal{T}(Q)$ has normal pattern if there are no partitions of type C but not of type A or B in $\mathcal{T}(Q)$ and if also in the first column of $\mathcal{T}(Q)$ the type A rows and type B elements strictly alternate in the first column of the table $\mathcal{T}(Q)$. R. Zhao showed that when $u >> r$ then $\mathcal{T}(Q)$ has normal pattern $[Z]$, and in particular $|\mathcal{Q}^{-1}(Q)| = (r - 1)(u - r)$ in this case. For completeness we include a proof of this normal pattern result.

**Corollary 3.26** (Normal pattern). When $u > \max\{r + r^2/8, 3r/2\}$ then $\mathcal{T}(Q)$ has normal pattern.

**Proof.** By the proof of Corollary 3.18 when $u > 3r/2$ and $r$ is odd then $t_{\max} = (r - 1)/2 - 1$ and $k_{t_{\max}} = r - 1$; when $u > 3r/2$ and $r$ is even then $t_{\max} = r/2 - 2$ and $k_{t_{\max}} = r - 2$; each is consistent with the number of B rows needed and with the type A or B/C of entry $P_{r-1,1}$ needed for $\mathcal{T}(Q)$ to have normal pattern. For normal pattern we now need $k_1 = 2t + 2$ for
each $t, 0 \leq t \leq t_{\text{max}}$. By equation (3.19) we have $k_t = 2t + 1 + \lceil \frac{r-2(t+1)+1}{q_t+1} \rceil$, so we need for each $t, 0 \leq t \leq t_{\text{max}},$

$$q_t + 1 \geq r - 2(t + 1) + 1 = r - 2t - 1.$$  \hspace{1cm} (3.37)

Since $q_t = \lceil (u - r)/(t + 1) \rceil$ it suffices to have for $0 \leq t \leq r/2$

$$\frac{u - r}{t + 1} \geq r - 2t - 2,$$

$$u \geq r + (r - 2t - 2)(t + 1).$$ \hspace{1cm} (3.38)

The expression on the right of (3.38) is maximum on the interval $t \in [0, r/2]$ at $t_0 = r/4 - 1$ and has maximum value $r + r^2/8$. This and Corollary 3.8 complete the proof. \hfill \square

Example 3.27. For $Q = (10, 7)$ the table $\mathcal{T}(Q)$ has normal pattern

\[
\begin{array}{cccccccc}
(10, 7) & (10, [7]^2) & (10, [7]^3) & (10, [7]^4) & (10, [7]^5) & (10, [7]^6) & (10, [7]^7) \\
(9, [8]^2) & (9, [8]^3) & (9, [8]^4) & (9, [8]^5) & (9, [8]^6) & (9, [8]^7) & (9, [8]^8)
\end{array}
\]

Example 3.28. For $Q = (10, 6)$ the table $\mathcal{T}((10, 6))$ also has normal pattern

\[
\begin{array}{cccccc}
(10, 6) & (10, [6]^2) & (10, [6]^3) & (10, [6]^4) & (10, [6]^5) & (10, [6]^6) \\
([10]^2, [6]^2) & ([10]^2, [6]^3) & ([10]^2, [6]^4) & ([10]^2, [6]^5) & ([10]^2, [6]^6) & ([10]^2, [6]^8)
\end{array}
\]

Some tables having non-normal pattern.

Examples 3.22 above and 3.32 3.33 below have $\mathcal{T}(Q)$ of non-normal patterns. We give a few more here, in particular to illustrate Corollary 3.18 on corner elements.

Example 3.29. Let $Q = (10, 2)$, so $u = 10, r = 8$. For readability we list here the transpose $\mathcal{T}(Q)^T$, a $(u - r) \times (r - 1)$ rectangle of partitions, in place of $\mathcal{T}(Q)$. Here we indicate in bold the transition from Case A to Case B.

\[
\begin{array}{cccccccc}
(10, 2) & (5, 5, 2) & (4, 3, 3, 2) & (4, 2^4) & (3, 3, 2, 2, 1, 1) & (3, 3, 2, 1^4) & (3, 3, 1^6) \\
(10, 1, 1) & (5, 5, 1, 1) & (4, 3, 3, 1, 1) & (4, 2^3, 1, 1) & (4, 2^2, 1^4) & (4, 2, 1^6) & (4, 1^8)
\end{array}
\]

Using the $[n]^k$ notation for the same partitions, we again write the transpose $\mathcal{T}(Q)^T$

\[
\begin{array}{cccccccc}
(10, 2) & ([10]^2, 2) & ([10]^3, 2) & (4, [8]^4) & ([6]^2, [6]^4) & ([6]^2, [6]^5) & ([6]^2, [6]^6) \\
(10, [2]^2) & ([10]^2, [2]^2) & ([10]^2, [2]^2) & (4, [8]^5) & (4, [8]^6) & (4, [8]^7) & (4, [8]^8)
\end{array}
\]

Example 3.30. [2] The example $Q = (12, 3), n = 15$ was a key example, the lowest $n$ that R. Zhao found for which $\mathcal{Q}^{-1}(Q)$ contains a type C partition. It is $P_{3,1}(12, 3) = (4, [10]^4, 1) = ([10]^3, [5]^3) = (4, 3, 3, 2, 2, 1)$, where neither the top U-chain $C_4$ nor the bottom U-chain $C_2$ of $D_P$ has the maximum length 12 of the type C U-chain $C_3$. By Lemma 3.7 above and Example 3.32, $n = 15$ is the smallest integer where type C occurs, and $P = (4, [10]^4, 1)$ is the unique type C partition for $n = 15$. 

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Observation 3.31. (Placement of a B hook in $T(Q)$, $Q = (12, 3)$). The first type B hook 

$$(u - r + 2, [u - 2]^k) = (3 + 2, [12 - 2]^k) = (5, [10]^k), 3 \leq k \leq 10$$

begins in the $k_0 = a + 1$ row where $a$ is the largest integer such that $[12]^a$ has smallest part at least $3 + 2$. Thus $a$ satisfies 

$$[12/a] \geq 3 + 2, \text{ but } [12/(a + 1)] < 3 + 2.$$ \hspace{1cm} (3.39)

Here $a = 2$ so $k_0 = 3$. The next type A row $k = a + 2 = 4$ begins $([12]^{a+1}, [3]^2)$ (instead of $([12]^{a+1}, 3)$ which is not in $\Omega^{-1}(Q)$); this A row ends early at $P_{4,2} = ([12]^{3}, [3]^{3})$, leaving a free place for 

$$P_{4,3}(Q) = (5, [10]^6)$$ \hspace{1cm} (3.40)

on the first $B$ hook. The last type A row $k = 6$ has two free places in its last columns to accommodate $([7]^2, [8]^6)$ on the second B hook and $P_{6,3}(Q) = (5, [10]^6)$ on the first B hook.

The transitions between A row and B hook, as we have seen, need not occur in an alternating even-odd or other regular pattern: rather there may be adjacencies of type A rows as here for $([12],[3]^6)$ and $([12],[3]^6)$ rows; or there may be adjacencies for distinct type B/C hooks, as for $Q = (16,5)$ where $k_3 = 9, k_4 = 10$ (Example 3.32).

The type C element $P = P_{5,1}(Q) = ([10]^3, [5]^3) = (4,3,3,2,2,1)$ here “replaces” the potential element $P' = [7]^2, [8]^4) = (4,3,2,2,2,2)$ (as $\Omega(P') = (13,2)$, not $(12,3)$). Here one dot in the Ferrers diagram of $P'$ has been shifted to form $P$.

Further tables $T(Q)$ having C entries.

We sometimes write $P \vdash n$ for “$P$ partitions $n$”.

Example 3.32. For $C = (1,1,1,1;4)$, $P_C = (4,3^2,2^2,1) = ([10]^3, [5]^2)$ and $\Omega(P_C) = (12,3)$: by Lemma 3.7 this is the smallest case of type C partition not of type A or B (See Example 3.30 and 12).

For $C' = (1,1,1,1;5)$, $P' = P_{C'} = (5,4^2,3^2,2) = ([13]^3, [8]^3)$, and $Q' = Q(P_C) + (4,2) = (16,5) \vdash 21$, $S_Q = S_Q + (2,2) = (10,5)$. Here $T(Q')$ is
The relative position of \( P' = ([13]^3, [8]^3) \) here is the same as that of the related \( P = ([10]^3, [5]^3) \) in \( T(Q), Q = (12, 3) \) (see example 3.32) simply because \( t(P') = t(P) \) and the upper left triangular portions of the two tables correspond. Here there are five upper right-corner hooks, of lengths 12, 8, 6, 3 and 1 corresponding each to an almost rectangular partition of 5, where the upper AR portion for each partition in the \( i \)-th hook is \([5]^i + 2 = 5 + 2i] \). These are the \((7, [14]^k), ([9]^2, [12]^k), ([11]^3, [10]^k), ([13]^4, [8]^k), \) and \(([15]^5, [6]^6) \) hooks.

**Example 3.33.** Consider \( Q = (18, 3) \) so \( S_Q = (14, 3) \). (Feature: two \( C \) cases in same row).

In the following table for \( T(Q) \) we have underlined the last \( A \)-row entry in the first column before a \( B \)-row; and we have put in boldface the entries that are the corners of the \( B/C \)-hooks.

| \( \mathcal{T}(Q') \) | \( \ell = 1 \) | 2 | 3 | 4 | 5 |
|------------------------|----------------|---|---|---|---|
| \( k = 1 \)           | \([16, 5]^2\)  | \([16, 5]^3\) | \([16, 5]^4\) | \([16, 5]^5\) |
| 2                      | \([16^2, 5]\)  | \([16^2, 5]^2\) | \([16^2, 5]^3\) | \([16^2, 5]^4\) |
| 3                      | \([7, 14]^3\)  | \([7, 14]^4\)  | \([7, 14]^5\)  | \([7, 14]^6\)  |
| 4                      | \([16^3, 5]^2\) | \([16^3, 5]^3\) | \([16^3, 5]^4\) | \([16^3, 5]^5\) | \([7, 14]^8\) |
| 5                      | \([13]^3, [8]^3\) | \([9]^2, [12]^5\) | \([9]^2, [12]^6\) | \([9]^2, [12]^7\) | \([7, 14]^9\) |
| 6                      | \([16^4, 5]^3\) | \([16^4, 5]^4\) | \([16^4, 5]^5\) | \([9]^2, [12]^8\) | \([7, 14]^9\) |
| 7                      | \([11^3, 10^5]\) | \([11^3, 10^6]\) | \([11^3, 10^7]\) | \([9]^2, [12]^9\) | \([7, 14]^9\) |
| 8                      | \([16^5, 5]^4\) | \([16^5, 5]^5\) | \([11^3, 10^8]\) | \([9]^2, [12]^9\) | \([7, 14]^9\) |
| 9                      | \([13]^4, [8]^6\) | \([13]^4, [8]^7\) | \([11^3, 10^9]\) | \([9]^2, [12]^9\) | \([7, 14]^8\) |
| 10                     | \([15]^5, [6]^6\) | \([13]^4, [8]^8\) | \([11^3, 10^10]\) | \([9]^2, [12]^8\) | \([7, 14]^7\) |

Note the following features of the table: there are \([r/2] = 7\) \( A \)-rows, and \( \min([r/2], u - r) = 3 \) \( B/C \)-hooks, beginning in rows \( k_0 = 5, k_1 = 7, \) and \( k_2 = 10 \) (notation \( k_t \) from (3.17) below). The two \( C \)-entries replace what would be impossible \( 7 \)-th \( B \)-row entries \([7]^2, [14]^6\) and \([7]^2, [14]^7\) (as these have higher \( \Omega(P) \) than \( (18, 3) \)). When \( r > u - r \) the last \((u - r)\)-th \( B/C \)-hook fills the last entries of the first column (and we count only one \( B/C \) hook for those entries).

We call attention to the two \( C \) entries. Following the labelling of Definition 3.6 and
Lemma 3.7 we have
\[ ([16]^5, [5]^3) = (4, 3, 3, 3, 2, 2, 1) : C_1 = (1, 1, 1, 2; a = 4) \]
\[ ([10]^3, [11]^6) = (4, 3, 3, 2, 2, 2, 2, 1) : C_2 = (1, 1, 4, 1; a = 4) \]
(3.41)

By raising \( a \) to 5, we obtain two other related type C partitions of 29 and 30, respectively,
\[ ([21]^5, [8]^3) = (5, 4, 4, 4, 3, 3, 2) C'_1 = (1, 1, 1, 2; a = 5), Q_{C'_1} = (24, 5) \vdash 29; \]
\[ ([13]^3, [17]^6) = (5, 4, 4, 3, 3, 3, 3, 2) : C'_2 = (1, 1, 4, 1; a = 5), Q_{C'_2} = (25, 5) \vdash 30. \]
(3.42)

Thus the next avatars of these two type C partitions from (3.41) in the table \( T(Q), Q = (18, 3) \), are the two type C partitions \( P_{C'_1}, P_{C'_2} \) from (3.42): the latter have different corresponding \( Q \)'s that partition different integers!

4 Equations of the table loci.

We first describe in section 4.1 the \( sl_2 \) grading on the poset \( D_P \) defined in Section 2.2. We next in section 4.2 study the equations defining the loci of table elements, showing their \( sl_2 \) homogeneity and giving an example. In Section 4.3 we propose equations for the loci.

4.1 The \( sl_2 \) triple of \( B \) and a grading on \( U_B \).

The poset \( D_P \) and the weighted Dynkin diagram for type \( A_n \).

There is an order-reversing involution on \( D_P \):
\[ \tau : D_P \rightarrow D_P, \ \tau(u, i, k) = (i + 1 - u, i, n_i + 1 - k), \]
(4.1)

whose center of symmetry for the \( i \) row is \( u = (i + 1)/2 \). The involution extends to one on \( E_B \) [BIK Definition 2.15], and evidently also to \( C_B \), and satisfies (adapted from [BIK Equation 2.24])
\[ \tau(\alpha_i) = \beta_i, \tau(w_i) = w_i, \tau(e_{i,k}) = e_{i,n_i-k}. \]
(4.2)

We define a function on the basis of \( V \):
\[ \varrho(u, i, k) = \varrho(u, i) = 2u - i - 1. \]
(4.3)

the integer giving the relative position of a vertex with respect to the center of symmetry of \( D_P \), with respect to \( \tau \). The integer \( \varrho(u, i, k) \) is also the weight of \( (u, i, k) \) from the \( sl_2 \) triple associated to \( B \) [CoMc §3.6]: the set
\[ S_{[p_1, p_2, \ldots, p_r]} = \bigcup_{1 \leq i \leq k} \{ p_i - 1, p_i - 3, \ldots, -p_i + 1 \} \]
(4.4)

there is just the union of the \( \varrho \) weights of the vertices in each row of the poset \( D_P \) (they write \( d_i \) in place of \( p_i \) for the parts of \( P \)). So the weighted Dynkin diagram of \( P \) is \( n - 1 \) vertices with the first differences in the set \( S_{[p_1, \ldots, p_r]} \) as the weights.

5This is \( \nu(u, i, k) \) in [BIK Section 2.2 and Theorem 2.13], and \( \rho(u, i, k) \) in [IKh Lemma 2.5].
In other words, \( \mathcal{D}_P \) appears to add some further information - the poset - to the weighted Dynkin diagram, but for type \( A_n \) the Dynkin diagram determines \( P \). Kostant’s Theorem [3.5.4]. Since \( P \) determines \( \mathcal{D}_P \) the information in the Dynkin diagram and in \( \mathcal{D}_P \) are equivalent for type \( A_n \).

**Example 4.1.** For \( P = (3, 2) \) we have the vertices \( \mathcal{D}_P = \left( \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \end{array} \right) \) with \( sl_2 \) grading

\[ \varrho(\mathcal{D}_P) = \begin{pmatrix} -2 & 0 & 2 \\ -1 & 1 & 0 \end{pmatrix}. \]

The related Dynkin diagram is obtained by giving 4 vertices the weights from the first difference of the sequence \( S_{[3,2]} = (-2, -1, 0, 1, 2) \) of values of \( \varrho \) on \( \mathcal{D}_P \), so here the diagram is \( (* - -* - * - *) \) where each vertex * has weight 1. We label the vertices of \( \mathcal{D}_P \)

\[ \left( \begin{array}{ccc} (1, 3, 1) \\ (2, 1, 1) \\ (2, 2, 1) \\ (3, 3, 1) \end{array} \right) = \begin{pmatrix} v_3 \\ v_5 \\ v_4 \\ v_1 \end{pmatrix}. \quad (4.5) \]

In the basis \( \mathfrak{B} \) the general element of \( \mathcal{C}_B \) is the matrix

\[ M = \begin{pmatrix} a_0 & a_1 & a_2 & g_1 & g_2 \\ 0 & a_0 & a_1 & 0 & g_1 \\ 0 & 0 & a_0 & 0 & 0 \\ 0 & g_1 & g_2 & b_0 & b_1 \\ 0 & 0 & g_1 & 0 & b_0 \end{pmatrix} = \begin{pmatrix} x_{e_1} & x_{\alpha_3\beta_3} & x_{(\alpha_3\beta_3)^2} & x_{\alpha_3} & x_{\alpha_3\beta_3\beta_3} \\ 0 & x_{e_1} & x_{\alpha_3\beta_3} & x_{\alpha_3} & 0 \\ 0 & 0 & x_{e_1} & 0 & 0 \\ 0 & x_{\beta_3} & x_{\beta_3\alpha_3\beta_3} & x_{e_2} & x_{\beta_3\alpha_3} \\ 0 & 0 & x_{\beta_3} & 0 & x_{e_2} \end{pmatrix}, \quad (4.6) \]

where \( M \in \mathcal{U}_B \) when \( a_0 = b_0 = 0 \). Thus, \( \dim \mathcal{U}_B = 7 \), and a basis \( \mathfrak{U}_B \) for \( \mathcal{U}_B \) is

\[ \mathfrak{U}_B = \{ \alpha_3, \beta_3, \alpha_3\beta_3, \beta_3\alpha_3, \beta_3\alpha_3\beta_3, \alpha_3\beta_3\alpha_3, \alpha_3\beta_3\alpha_3\beta_3 \}, \quad (4.7) \]

and \( \mathcal{C}_B = \mathfrak{U}_B \cup \{ \epsilon_1, \epsilon_2 \} \) is a basis of \( \mathcal{C}_B \).

### 4.2 Homogeneity of equations for loci of elements in \( \mathcal{T}(Q), Q = (u, u - r) \).

The centralizer \( \mathcal{C}_B \) is a Lie subalgebra of \( \mathfrak{I}(\mathcal{D}_P) \), the kernel of the linear map \( ad(B) : \mathfrak{I}(\mathcal{D}_P) \to \mathfrak{I}(\mathcal{D}_P) \). We here show that the equation for loci of elements in \( \mathcal{T}(Q) \) are homogeneous in the \( sl_2 \) grading on \( \mathcal{U}_B \) and \( \mathcal{N}_B \). We propose the conjecture that the locus of \( \mathcal{P}_{k,\ell}(Q) \) has codimension \( k + \ell - 2 \) and record some particular examples of what can occur.

We set \( B = J_Q \) and consider the locus \( \mathfrak{J}(P) \) in \( \mathcal{U}_B \) of those matrices \( A \) having a given Jordan type \( P_A = P \), especially for \( P = P_{k,\ell} \) from the table \( \mathcal{T}(Q) \). We consider the locally closed sets,

\[ \mathfrak{J}(P) = \{ A \in \mathcal{U}_B \mid P_A = P \} \subset \mathcal{U}_B \]

and their closures \( \overline{\mathfrak{J}(P)} \subset \mathcal{U}_B \).

The \( sl_2 \) homogeneity of equations for \( \overline{\mathfrak{J}(P_{k,\ell})} \).

Recall that the \( sl_2 \) weight of a vertex \( v \) of \( \mathcal{D}_Q \) is denoted \( \rho(v) \) (see (4.3)); we associate to a union \( I \) of vertices the sum of the individual weights. Given a rank sequence \( \mathcal{R} = (r_1, r_2, \ldots, r_n) \)
the Jordan type \( P(\mathcal{R}) \) for matrices \( A \mid \text{rk} A^k = r_k, 1 \leq k \leq n \) satisfies, letting \( c_k = n - r_k \), the corank of \( A^k \),
\[
c_k = n - r_k = \sum (P(\mathcal{R})_i - k)^+,
\]
which is the sum of the excess of the parts over \( k \). For example for \( n = 7, \mathcal{R} = (5, 3, 2, 1, 0, \ldots , 0) \), we have \( C = (2, 4, 5, 6, 7, \ldots , 7) \) and \( P(\mathcal{R}) = (5, 2) \).

**Lemma 4.2.** The equations of the ideal \( I(\mathcal{R}) \) defining rank conditions \( R \) in \( \mathcal{U}_B, B = J_Q \), and as well those for its radical \( \sqrt{I(\mathcal{R})} \) are \( sl_2 \) homogeneous. In particular, for each pair \((I, J), \# I = \# J = k \) of sets of \( k \) indices, the condition rank \( A_{i,j}^k \leq r_k \) is \( sl_2 \) homogeneous of degree \( \rho(I) - \rho(J) \).

**Proof.** Let \( \mathcal{R} = (r_1, r_2, \ldots , r_n) \) be a decreasing-until-zero sequence of integers. Then
\[
\mathcal{S}(P(\mathcal{R})) = \{ A \mid \text{rank} A^k \leq r_k, 1 \leq k \leq n \}.
\]
To obtain the locus in \( \mathcal{U}_B \) we impose the further condition rank \( A^k > r_k - 1, 1 \leq k \leq n \) (ruling out more special loci). Thus, the equations of \( \mathcal{S}(P(\mathcal{R})) \) arise from setting all \((r_k + 1) \times (r_k + 1)\) minors of \( A^k \) to zero, and for each \( k \) requiring that at least one such minor is non-zero. We give below two arguments, the first general facts about graded rings. The second argument is more in the spirit of the question, as it shows that each determinant of a minor of \( A \) is homogeneous. We conclude that \( \mathcal{S}(P(\mathcal{R})) \) is defined by a set of homogeneous conditions (of possibly different degrees).

**Argument 1.** The algebra \( \mathcal{U}_B \) is a \( \mathbb{Z} \)-graded ring (grading determined by the \( sl_2 \) action as in Section 4.1) so \( f = 0 \) implies that each graded component is zero. So we may regard the equations as homogeneous.

**Argument 2.** Let \( I = (i_1, i_2, \ldots , i_u), J = (j_1, j_2, \ldots , j_u) \) be increasing length-\( u \) subsequences of \( \{1, 2, \ldots , n\} \), where \( u = r_k + 1 \). The \((a, b)\) entry \( A_{i_a,j_b}^k \) of the submatrix \( A_{i,j}^k \) of \( A^k \) is the sum of monomials, each the product of variables labelled by a length \( k \) path from the vertex \( i_a \) of \( \mathcal{D}_Q \) to the vertex \( j_b \): so each monomial of the entry has the weight \( \rho(j_b) - \rho(i_a) \). Thus, each term of the minor, being a product of a choice of entries with row indices \( I \) and column indices \( J \), has weight \( \sum \rho(i_a) - \sum \rho(j_b) = \rho(I) - \rho(J) \). Thus, each equation given by \( \det A_{i,j}^k = 0 \) is \( sl_2 \) homogeneous.

That an ideal is \( sl_2 \) homogeneous implies that its radical is also homogeneous.

**Example 4.3** (Loci for \( Q = (5, 2) \)). Let \( Q = (5, 2), B = J_Q \) the corresponding Jordan block matrix of Jordan type \( Q \).

\[
\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
& & & v_5 & v_4 & v_3 & v_2 \\
& & & v_7 & v_6 & & \\
\end{array}
\]

Figure 4: Vertices of \( \mathcal{D}_Q, Q = (5, 2) \) (Ex. 4.3).
Figure 5: Diagram of $\mathcal{D}_Q$ and maps for $Q = (5, 2)$.

An element $A \in \mathcal{U}_B$ (note that since the parts of $Q$ are distinct, $\mathcal{U}_B = N_B$) satisfies, letting $C = J_{(5,1,1)}$,

$$A = a_1 B_0 + a_2 B_0^2 + a_3 B_0^3 + a_4 B_0^4 + E + E' + X$$

$$B_0 = J_{[5]} = w_5, E = g_1 \alpha_{5,2} + g_2 C \alpha_{5,2}, E' = g_1' \beta_{5,2} + g_2' C \beta_{5,2}, X = b_1 w_2. \quad (4.11)$$

or in matrix form

$$A = \begin{pmatrix}
0 & a_1 & a_2 & a_3 & a_4 & g_1 & g_2 \\
0 & 0 & a_1 & a_2 & a_3 & 0 & a \\
0 & 0 & 0 & a_1 & a_2 & 0 & 0 \\
0 & 0 & 0 & 0 & a_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g_1' & g_2' & 0 & b_1 \\
0 & 0 & 0 & 0 & g_1' & 0 & 0
\end{pmatrix}$$

When $e \neq 0$, then $P_A = (5, 2)$ if $b_1 \neq 0$, and $P_A = (5, [2]^2)$ when $b_1 = 0$.

We will need that when $a_1 = 0$ then $A^2$ has zero entries except for rows $I = (1, 2, 6)$, and columns $J = (4, 5, 7)$ where

$$A^2_{I,J} = \begin{pmatrix}
g_1 g_1' & a_2^2 + g_1 g_2' + g_1' g_2 & a_1 b_1 \\
g_1 g_1' & 0 & 0 \\
g_1 g_1' & g_1' b_1 & 0
\end{pmatrix} \quad (4.12)$$

- When $P_A = (4, \ldots)$:
  - When $a_1 = 0$ then the only length 3 ($sl_2$ weight 6) path in $\mathcal{D}_Q$ is $v_5 \to v_7 \to v_6 \to v_1$, and $A^3 = g_1 b_1 g_1' B^4$. So $g_1 b_1 g_1' \neq 0$ with $a_1 = 0$ is a necessary condition for $P_A$ to have a part 4.
  - Then $\text{rank } A = 2 + \text{rank } A'$, where $A' = A_{2,6;4,8}$, namely

$$A' = \begin{pmatrix}
a_2 & g_1 \\
g_1' & b_1
\end{pmatrix}$$

For $P_A = (4, [3]^2) = (4, 2, 1)$ the further condition is rank $A = (7 - 3) = 4$, so $\text{det } A' \neq 0$.

For $P_A = (4, [3]^{-1}) = (4, 1, 1, 1)$, the further condition is rank $A = 7 - 4 = 3$ so $\text{det } A' = 0$. Note that $\text{det } A'$ is homogeneous of weight 6 as $sl_2$ invariant.

Here for $Q = (5, 2)$ the codimension of each locus $\mathfrak{Z}(P_{k,\ell}), P_{k,\ell} \in \mathcal{T}(Q)$ is $k + \ell - 2$. But specialization is not according to neighbors in the rectangle $\mathcal{T}(Q)$. Although from the loci
\[ T(Q) = (5, 2), (5, [2]^2) \]
\[ (4, [3]^2), (4, [3]^3) \]

\[ T(Q) \] equations of loci
\[
\begin{align*}
&\{ a_1b_1 \neq 0; \\
&a_1 = 0, \ g_1b_1g'_1 \neq 0, \\
&a_1 = a_2b_1 - g_1g'_1 = 0, \\
&a_2b_1 - g_1g'_1 \neq 0; \\
&g_1b_1g'_1 \neq 0.
\end{align*}
\]

Figure 6: Equations of loci \( P_{i,j} \), \( P_{i,j} \in T(Q), Q = (5, 2) \). (Ex. [4.3]).

Remark 4.4. We note that here, and also in general for \( B = J_Q \), where \( Q \) is any stable partition, the set \( \{ A \in U_B \mid Q(A) = Q \} \) is constructible, being the finite union of locally closed subsets of \( U_B \) corresponding to certain loci \( \mathcal{Z}(P) \). The phenomenon just observed is related to the constructibility of this set for \( Q = (5, 2) \).

4.3 Proposed equations for table loci.

The following conjectures were developed in collaboration with M. Boij, using calculations in MACAULAY 2 GS. T. Košir pointed out to us that the quadratic equations that occur are polarizations of the \( 2 \times 2 \) determinant, as described in [KS2], and studied more generally for the \( k \times k \) determinant in [KS]. By “equations of the locus” of \( P \) in this section, we mean the equations of the closure of the locus in \( U_B \).

\textbf{Conjecture 4.5} (Table Locus Conjecture). Let \( B = J_Q, Q = (u, u - r), u > r \geq 2 \). The locus \( \mathcal{Z}(P_{k,l}) \subset U_B \) (with reduced structure) is an irreducible complete intersection defined by \( k + \ell - 2 \) equations, of which \( \min\{k + \ell - 2, r - 2\} \) are linear in the variables of \( U_B \) and the rest are quadrics. Each new quadric appearing on the \( k + \ell = s + r \) diagonal, \( s \geq 1 \) is the sum of \( s \) determinants of \( 2 \times 2 \) matrices of variables, each having \( sl_2 \) weight \( 2(k + \ell - 1) \). 

\textbf{Linear Equations.}

We state a precise version of the Locus Conjecture for the left hand column of \( T(Q) \), then we generalize it to those equation sets that are entirely linear \( (k + \ell \leq r) \), and for the linear portions of the equations when \( k + \ell > r \).

\textbf{Notation.} Given \( Q = (u, u - r), r \geq 2 \) we denote the vertices of \( \mathcal{D}_Q \) by \( v_{1,1}, \ldots, v_{1,u} \) from left to right in the top, longer row, and \( v_{2,1}, \ldots, v_{2,u-r} \) from left to right in the second row. We denote by \( a_i, 1 \leq i \leq u - 1 \) the map taking each vertex \( v_{1,k} \) to \( v_{1,k+i} \) for \( k \leq u - i \), and that is zero otherwise. We denote by \( b_i, 1 \leq i \leq u - r - 1 \), the map taking each vertex \( v_{2,k} \) to \( v_{2,k+o} \)}
for \( k \leq u - r - i \), and that is zero otherwise. The map \( g_i, 1 \leq i \leq u - r \) takes each \( v_{2,k} \) to \( v_{1,k+i+r-1}, 1 \leq k \leq u + 1 - r - i \), and \( g'_i, 1 \leq i \leq u - r \) takes \( v_{1,k} \) to \( v_{2,k+i}, 1 \leq k \leq u - r - i \) and each takes all other vertices to zero. See Figure 4.3.

**Conjecture 4.6** (Loci equations for the left hand column of \( \mathcal{T} \)). A. The locus of \( P_{k,1} \) in the left hand column of \( \mathcal{T}(Q) \) are a set of linear equations of the form

\[
a_1 = a_2 = \cdots a_{\mu(k)} = b_1 = b_2 = \cdots b_{\nu(k)} = 0, \tag{4.14}
\]

where \( \mu(k) + \nu(k) = k - 1 \) and \( \mu(k + 1) \geq \mu(k), \nu(k + 1) \geq \nu(k) \).

B. Let \( k_{t-1} < k < k_t \). The equations for the locus \( \overline{3}(P_{k,1}) \) of the partition \( P_{k,1} = ([u]^{k-t}[u - r]^{t+1}) \) in the left column of an A row of \( \mathcal{T}(Q) \) are

\[
a_1 = a_2 = \cdots a_{k-t-1} = 0; \ b_1 = \cdots b_{t-1} = 0. \tag{4.15}
\]

C. The entry \( P_{k,1}, k = k_t, k \leq r - 1 \), leading a \( B/C \) row/hook is either

\[
P_{k,1} = ([u - r + 2(t + 1)]^{t+1}, [u - 2(t + 1)]^{k-t}), \text{ case type } B \text{ first entry: } c_t = 0 \tag{4.16}
\]

or in the case the first entry of the B/C hook is type C.

\[
P_{k,1} = ([u - r + 2(t + 1)]^{t+1}, [u - 2(t + 1) - d_t(q_t - 1)]^{k-t-d_t}, (q_t - 1)^{d_t}). \text{ for } c_t \geq 1 \tag{4.17}
\]

and its locus is defined by the equations

\[
a_1 = a_2 = \cdots = a_{k-t-1} = 0 \text{ and } b_1 = \cdots = b_t = 0. \tag{4.18}
\]

This conjecture implies our earlier Conjecture 3.20 that for \( 1 \leq k \leq r - 2 \) the partition \( P_{k,1} > P_{k+1,1} \) in the Bruhat order.

**Conjecture 4.7** (Linear loci equations). Assume that \( 1 \leq k \leq r - 1, 1 \leq \ell \leq u - r \) and \( k + \ell \leq r \). Then the equations for the locus of \( P_{k,\ell} \) satisfy:

A. For \( P_{k,\ell} \) in a partial A row: if \( \overline{3}(P_{k,1}) \) has equations \( E(k - t - 1, t - 1) \) then \( \overline{3}(P_{k,\ell}) \) has equations \( E(k - t - 1, t + \ell - 2) \).

B. For \( P_{k,\ell} \) in a B/C hook whose left column entry is \( P_{k_t,1} \) with equations \( E(k_t - t - 1, t) \) then \( \overline{3}(P_{k,\ell}) \) has equations \( E(k + \ell - t - 2, t) \).

When \( k + \ell > r \), the linear subset of equations for the locus of \( P_{k,\ell} \) in an A row or B,C hook is the same set as the complete set for the locus of the partition \( P_{k',r-k'} \) in the same A row or B/C hook of \( \mathcal{T}(Q) \).
Quadratic equations.

Fix $Q = (u, u - r), r \geq 2$. Let $k + \ell = r + s, s > 0, \ell \leq u - r, k + \ell \leq u - 1$ and let $P_{k', r - k'}$ be in the same A row or B/C column of $\mathcal{T}(Q)$ as $P_{k, \ell}$ and set $(k_1, k_2) = (k_1(k'), k_2(k'))$ the pair of integers such that $E(k_1, k_2)$ are the equations for the locus of $P_{k', r - k'}$. We denote by

$$X_s(k') = \sum_{v=1}^{s} \det \begin{pmatrix} a_{k_1+v} & g_v \\ g'_{s+1+v} & b_{k_2+s+1-v} \end{pmatrix}$$  \hspace{1cm} (4.19)

**Conjecture 4.8.** Fix $Q = (u, u - r), r \geq 2$. Let $k + \ell = r + s, s > 0, \ell \leq u - r, k + \ell \leq u - 1$ and let $P_{k', r - k'}$ be in the same A row or B/C column of $\mathcal{T}(Q)$ as $P_{k, \ell}$. Then the equations for $3(\overline{P_{k, \ell}})$ are $E(k_1, k_2) = 0, X_1(k') = X_2(k') = \cdots = X_s(k') = 0$.

**Example 4.9.** For $Q = (5, 3)$, the locus of $P_{1, 2}$ is defined by $X_1(1) : \det \begin{pmatrix} a_1 & g_1 \\ g'_1 & b_1 \end{pmatrix} = 0$; and $P_{1, 3}$ is defined by $X_1(1) = X_2(1) = 0$, here $X_2(1) : \det \begin{pmatrix} a_1 & g_1 \\ g'_2 & b_2 \end{pmatrix} + \det \begin{pmatrix} a_2 & g_2 \\ g'_1 & b_1 \end{pmatrix} = 0$.

**Remark 4.10.** We have made some progress in showing these conjectures, which we will report in [BoIKVS]. The Ljubljana colleagues T. Košir, P. Oblak and K. Šivic have connected these equations to the spaces of jets over classical determinantal varieties studied in [KS1, KS2]. The above conjectures would show the naturality of our division of the tables $\mathcal{T}$ to the spaces of jets over classical determinantal varieties studied in [KS1, KS2].

5 The Box Conjecture.

We first state P. Oblak’s conjectured recursive process for obtaining $\mathcal{Q}(P)$ and summarize what is known in Section 5.1. We then state a Box Conjecture for $\mathcal{Q}^{-1}(Q)$ (Section 5.2). In Section 5.4 we prove the analog of Theorem 3.14 – that we can fill the box with distinct partitions in $\mathcal{Q}^{-1}(Q)$ – in the special case that the stable partition $Q = (u + s, u, u - r)$, with $r \geq 2$ and $2 \leq s \leq 4$.

5.1 The Recursive Conjecture for $\mathcal{Q}(P)$.

P. Oblak conjectured a recursive process (Definition 5.1) for $\mathcal{Q}(P)$ discussed in [BKO, Kh2, BIK]). This greatly influenced further work in the area. We state this here, as it is closely related to the Box Conjecture, to follow. Recall from Definition 2.13 that a U-chain $C_a$ of $\mathcal{D}_P$ has three parts, a maximum chain through all the vertices of parts $a, a - 1$ and two chains linking the almost rectangular sub partition $(a^{n_a}, (a - 1)^{n_{a-1}})$ to the source and sink of the poset $\mathcal{D}_P$ in the top rows. Recall also that the length $|C_a|$ satisfies equation (2.4):

$$|C_a| = an_a + (a - 1)n_{a-1} + 2 \sum_{i > a} n_i.$$

Given a partition $P$ and integer $a \in S_P$ we let $P' = P'(P, a) = P - C_a$ by which we mean the unique partition obtained by omitting the vertices of $C_a$ from $\mathcal{D}_P$ and counting the vertices...
left in each row. We have that that \( P' = (\cdot \cdot \cdot , i^{n'_i}, \cdot \cdot \cdot ) \) where the multiplicity integers \( n'_i \) for \( P'(P, a) \) satisfy

\[
n'_i = \begin{cases} 
n_i & \text{if } i \leq a - 2 \\
n_i + 2 - i & \text{if } i \geq a - 1 \end{cases}
\]

(5.1)

The poset \( D_{P'} \) is not in general a subposet of \( D_P \).

**Definition 5.1** (P. Oblak recursive process). Let \( C_a \) be a maximum length chain of \( D_P \), and that \( Ob(P'), P' = P'(P, a) \) has been chosen. Then we set \( Ob(P) = (|C_a|, Ob(P')) \). When \( P \vdash n \) is almost rectangular we take \( Ob(P) = (n) \).

**Conjecture 5.2.** (P. Oblak) The map \( P \rightarrow \Omega(P) \) satisfies \( \Omega(P) = Ob(P) \).

E.R. Gansner, C Greene, D. Kleitman, S. Poljak, M. Saks, and T. Britz and S. Fomin associate a partition \( \lambda(P) \) to any finite poset by first setting \( c_i = \# \) most number of vertices covered by \( i \) chains of \( P \), then setting \( \lambda_i(P) = c_i - c_{i-1} \), with \( c_0 = 0 \). For the poset \( D_P \), L. Khatami defined a partition \( \lambda_U(D_P) \) using U-chains in a similar way, setting \( c_{i,U}(D_P) = \max \# \) vertices covered by \( i \) U-chains. As stated above in Definition 5.1, the partition \( Ob(P) \) might not be well-defined, and originally, P. Oblak chose the largest integer \( a \) giving a maximum length chain in each step. However the second author showed

**Theorem 5.3** (Independence). (L. Khatami \[Kh1\]) The resulting partition \( Ob(P) \) of the Oblak recursive process is independent of the choice of maximal length U-chains in Definition 5.1 and is equal to \( \lambda_U(P) \).

The first and second author showed,

**Theorem 5.4.** \[IKh\] Let \( k \) be an infinite field. Then \( \Omega(P) \geq \lambda_U(D_P) \).

L. Khatami also determined the smallest part of \( \Omega(P) \) using a study of the antichains of \( D_P \) in \[Kh2\]. This, with P. Oblak’s index Theorem 2.8 implies a result we will use later,

**Theorem 5.5** (Oblak conjecture for \( r_P \leq 3 \)). The P. Oblak conjecture is true over any infinite field \( k \) when \( r_P \leq 3 \).

**Remark 5.6** (Summary of results on the Oblak recursive process). Thus, the cases \( r_P = 2 \) \[Ob1, KO, BIK, Z\] and \( r_P = 3 \) \[Kh2\] of the Oblak recursive conjecture have been known since 2012 (2008 for \( r_P = 2 \)) and the map \( \Omega : P \rightarrow \Omega(P) \) is explicit for \( r_P \leq 3 \). Theorem 5.4 of the first and second authors then showed “half” the P. Oblak recursive conjecture in all characteristics. R. Basili has proposed a proof of the P. Oblak conjecture in \[Bas2\]. We note that even a characteristic zero proof – in fact even a proof of the in principle weaker statement that \( \lambda(D_P) = \lambda_U(D_P) \) – is an entirely combinatorial issue, and in combination with Theorem 5.4 is enough to show the recursive conjecture over any infinite field.\(^8\)

\(^8\)In contrast, which pairs of Jordan types occur for \( A, B \) with \([A, B] = 0\) depends on \( \text{char } k \): see \[BrWi\], \[McN, Example 22\], and \[BIK, Example 2.18\].

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5.2 The key of a stable partition $Q$ and the Box Conjecture.

We first define the key of $Q$, which determines the shape of the conjectural box $B(Q)$ of partitions.

**Definition 5.7 (Key of $Q$).** Let $Q = (q_1, q_2, \ldots, q_t), q_1 \geq q_2 \geq \cdots$ be a stable partition as in Theorem 2.6: $q_i - q_{i-1} \geq 2$ for $i = 1, \ldots, t - 1$. We let

$$s_i = \begin{cases} q_i - q_{i+1} - 1 & \text{for } 1 \leq i \leq k - 1 \\ q_k & \text{for } i = k \end{cases}$$

(5.2)

We call the sequence $S_Q = (s_1, s_2, \ldots, s_k)$ (5.3) the key of the stable partition $Q$.

**Example 5.8.** The key of $Q = (u, u - r)$ is $S_Q = (r - 1, u - r)$. The key of $Q = (11, 6, 2)$ is $S_Q = (4, 3, 2)$; the key of $Q' = (11, 8, 3)$ is $S_{Q'} = (2, 4, 3)$.

Evidently, the keys of the stable partitions with $k$ parts run through all the ordered sequences of $k$ positive integers; and the key determines $Q$. Given the key sequence $S = (s_1, s_2, \ldots, s_k)$ of positive integers, the stable partition $Q_S$ determined by $S$, satisfies

$$Q_S = \left( \left( \sum_i s_i \right) + k - 1, \left( \sum_{i>1} s_i \right) + k - 2, \ldots, s_{k-1} + s_k + 1, s_k \right),$$

(5.4)

and is a partition of

$$n = \left( \sum_i i \cdot s_i \right) + \frac{k(k - 1)}{2}.$$ 

(5.5)

Rearranging the key of the partition $Q \vdash n$ leads to a partition $Q' \vdash n'$, where $n'$ is not usually $n$. We now give a conjecture generalizing the Table Theorem. Recall that we denote by $\mathbb{P}(N_Q)$ the projective space parametrizing nilpotent elements in the centralizer of $J_Q$ in the matrix ring $\text{Mat}_n(k)$.

**Conjecture 5.9 (Box Conjecture).** Let $Q$ be a stable partition having $k$ parts and let $S = S_Q$.

i. The cardinality $|Q^{-1}(Q)| = \prod_i s_i$.

ii. There is a $s_1 \times s_2 \times \cdots \times s_k$ array (box) $B(Q)$ of partitions

$$B(Q) = \{P_{i_1, i_2, \ldots, i_k} \mid 1 \leq i_u \leq s_u\},$$

comprising $Q^{-1}(Q)$ such that the partition $P_{i_1, i_2, \ldots, i_k}$ has $\sum_{1 \leq u \leq k} i_u$ parts.

iii. The codimension of the locus of $P_{i_1, i_2, \ldots, i_k}$ in $N_Q$ is $\sum_1^k i_u - k$. This locus is a complete intersection defined by linear and irreducible quadrics in the coordinates of $\mathbb{P}(N_Q)$.

Given Conjecture 5.2 parts (i)-(ii) of the Box Conjecture are purely combinatorial statements, that should be obtainable as consequences. Of course, a deeper understanding of $B(Q)$ and Conjecture 5.9 could very well give a new approach to showing Conjecture 5.2.
Remark 5.10. Note that when \( s_i = 1 \) there is no contribution of this part to the cardinality \(|\mathcal{Q}^{-1}(Q)|\), a fact that was known at least in the case \( S = (s_1 = s_2 = \ldots = s_{k-1} = 1, s_k) \) (see [Ob2] Theorem 4.1), also \( \mathbb{Z} \). See Example 5.11 for a listing of \( |\mathcal{Q}^{-1}(Q)| \).

We now give several examples where we have verified Conjecture 5.9 (i),(ii) for a partition \( Q \) with \( k = 3 \) parts, with no \( s_i = 1 \) (an easier case). For the first, and simplest possible we take \( S_Q = (2, 2, 2) \), so \( \mathcal{B}(Q) \) is a \( 2 \times 2 \times 2 \) array.

Example 5.11. Let \( Q = (8, 5, 2), S_Q = (2, 2, 2) \). Then \(|\mathcal{Q}^{-1}(Q)| = 8\). The two floors of \( \mathcal{B}(Q) \) are
\[
\begin{pmatrix}
(8, 5, 2) & (8, 5, 1^2) \\
(8, 4, 2, 1) & (8, 4, 1^3)
\end{pmatrix},
\begin{pmatrix}
(7, 4, 2^2) & (7, 4, 2, 1^2) \\
(7, 3^2, 1^2) & (7, 4, 1^4)
\end{pmatrix}.
\]
(5.7)

The floor at left are the partitions obtained from \( \mathcal{Q}^{-1}((5, 2)) \) by adjoining 8. The partitions in the second floor at right are in Case B, and are obtained from those partitions \( P' \) in \( \mathcal{Q}^{-1}((6, 2)) \) having no part 6 by adjoining 7. The reason is that considering \( P' = (3^2, 1^2) \) with \( \mathcal{Q}(P') = (6, 2) \) we then add two to the largest part to include the tail to the 7-level, so one has \( (8, 2) \); this leaves \( 5 = 7 - 2 \) as a new part, giving \( \mathcal{Q}((7, 3^2, 1^2)) = (8, 5, 2) \).

Example 5.12. Let \( Q = (9, 6, 3) \vdash 18, S_Q = (2, 2, 3) \). Then \(|\mathcal{Q}^{-1}(Q)| = 12\). The two floors of \( \mathcal{B}(Q) \) are
\[
\begin{pmatrix}
(9, 6, 3) & (9, 6, 2, 1) & (9, 6, 1^3) \\
(9, 5, 2^2) & (9, 5, 2, 1^2) & (9, 5, 1^4)
\end{pmatrix},
\begin{pmatrix}
(8, 5, 3, 2) & (8, 4, 3, 2, 1) & (8, 4, 3, 1^3) \\
(8, 5, 2^2, 1) & (8, 5, 2, 1^3) & (8, 5, 1^5)
\end{pmatrix}.
\]
(5.8)

These have the form
\[
\begin{pmatrix}
(9, 6, 3) & (9, 6, [3]^2) & (9, 6, [3]^3) \\
(9, 5, [4]^2) & (9, 5, [4]^3) & (9, 5, [4]^4)
\end{pmatrix},
\begin{pmatrix}
(8, 5, [5]^2) & (8, [7]^2, 3^2) & (8, [7]^2, [3]^3) \\
(8, 5, [5]^3) & (8, [5]^4) & (8, [5]^5)
\end{pmatrix}.
\]
(5.9)

The two other partitions whose keys are permutation of this \( S \) are \((9, 5, 2) \vdash 16\) corresponding to key \((3, 2, 2)\) and \((9, 6, 2) \vdash 17\) corresponding to key \((2, 3, 2)\). For \( Q = (9, 6, 2) \vdash 17 \) we have \( \mathcal{B}(Q) \)
\[
\begin{pmatrix}
(9, 6, 2) & (9, 6, 1, 1) \\
(9, 4, 2^2) & (9, 4, 2, 1^2) \\
(9, 3^2, 1^3) & (9, 4, 1^4)
\end{pmatrix},
\begin{pmatrix}
(8, 4, 3, 2) & (8, 4, 3, 1^2) \\
(8, 4, 2^2, 1) & (8, 4, 2, 1^3) \\
(8, 3^2, 1^3) & (8, 4, 1^5)
\end{pmatrix}.
\]
(5.10)

For \( Q = (9, 5, 2) \vdash 16 \) we have \( \mathcal{B}(Q) \)
\[
\begin{pmatrix}
(9, 5, 2) & (9, 5, 1, 1) \\
(9, 4, 2, 1) & (9, 4, 1^3)
\end{pmatrix},
\begin{pmatrix}
(7, 4, 3, 2) & (7, 4, 3, 1^2) \\
(7, 4, 2^2, 1) & (7, 4, 2, 1^3)
\end{pmatrix},
\begin{pmatrix}
(6, [7]^2, [3]^2) & (7, 3^2, 1^3) \\
(6, [7]^2, [3]^3) & (7, 4, 1^5)
\end{pmatrix}.
\]
(5.11)

We now give the simplest example with no \( s_i = 1 \) with \( Q \) having four parts.

Example 5.13. Let \( Q = (11, 8, 5, 2) \vdash 26, S_Q = (2, 2, 2, 2) \). Then \(|\mathcal{Q}^{-1}(Q)| = 16\), conveniently viewed with the 4-D glasses supplied to the reader. The length 8 floor of the box \( \mathcal{B}(Q) \) is obtained by adjoining 11 to each element of \( \mathcal{B}((8, 5, 2)) \) in display (5.7). The second floor, beginning with a 5-part partition, is
\[
\begin{pmatrix}
(10, 7, 4, 3, 2) & (10, 7, 4, 3, 1^2) \\
(10, 7, 4, 2^2, 1) & (10, 7, 4, 2, 1^3)
\end{pmatrix},
\begin{pmatrix}
(10, 6, 4, 3, 2, 1) & (10, 7, 3^2, 1^3) \\
(10, 6, 4, 3, 1^3) & (10, 7, 4, 1^5)
\end{pmatrix}.
\]
(5.12)
Figure 7: $\mathcal{DHL}((6,2))$: Partitions with diagonal hook lengths $(6,2)$.

## 5.3 Diagonal hook lengths and the Box Conjecture.

We here compare the count of partitions in the putative box $B(Q)$ with the count of the partitions $\mathcal{DHL}(Q)$ having diagonal hook lengths $Q$. These counts are the same, implying that the Box Conjecture is consistent with the count of all partitions of $n$.

The *rank* $k$ of a partition is the side of its Durfee square, the largest square that fits into the upper left corner of its Ferrer’s graph. Let $P$ be any partition. The *principal* or *diagonal hook-length partition* $dhl(P)$ of $P$ is the sequence $dhl(P) = (h_{11}, h_{22}, \ldots, h_{kk}), h_{kk} > 0$ \footnote{\textit{Gut}}, and it is readily seen to be stable – have parts that differ pairwise by at least two. Let $Q$ be a stable partition. We denote by $\mathcal{DHL}(Q) = dhl^{-1}(Q)$ the set of partitions $P$ of $n$ having $dhl(P) = Q$. It is easy to show that they may be arranged in a box of dimensions $S_Q$, such that the $(i_1, \ldots, i_k)$ entry has $i_1 + \cdots + i_k$ parts.

**Example 5.14.** For $Q = (6,2)$, then $S_Q = (3,2)$ and we may write $\mathcal{DHL}(Q)$ as (Figure 7)

$$
\begin{pmatrix}
(5,3) & (4,2,2) \\
(4,3,1) & (3,2,2,1) \\
(3,3,1,1) & (2,2,2,1,1)
\end{pmatrix}.
$$

The following is easy to show by counting partitions first by their (stable) diagonal hook-length partitions $Q$ – fixing $S_Q$ – and may be regarded as well-known (although we didn’t find a specific reference).

**Proposition 5.15.** The parts enumerator of the $|S_Q| = \Pi s_i$ partitions in $\mathcal{DHL}(Q)$ is the function $E_Q(t)$,

$$E_Q(t) = \frac{t^k}{(1-t)^k} (\Pi_{1 \leq i \leq k} (1 - t^{s_i})). \quad (5.14)$$
The partition generating function \( \sum_{n=1}^{\infty} p(n)q^n \) satisfies
\[
\sum_{n=1}^{\infty} p(n)q^n = \sum_{k=1}^{\infty} q^{k(k-1)/2} \left( \sum_{\#S=k} (s_1 \cdot s_2 \cdots s_k) q^{(\sum i s_i)} \right).
\]
(5.15)

The partition-parts generating function \( p(a,n) = \# \) partitions of \( n \) with \( a \) parts – satisfies
\[
\sum_{a,n=1}^{\infty} p(a,n)t^a q^n = \sum_{k=1}^{\infty} q^{k(k-1)/2} \left( \sum_{\#S=k} \left( \prod_{i=1}^{k} \left( \sum_{t^u=1}^{s_i} t^u \right) \right) q^{(\sum i s_i)} \right)
= \sum_{k=1}^{\infty} t^k \cdot q^{k(k-1)/2} \left( \sum_{\#S=k} \left( \prod_{i=1}^{k} (1 - t^{s_i}) \right) q^{(\sum i s_i)} \right).
\]
(5.16)

**Remark 5.16.** The Box Conjecture implies that the parts enumerator of \( B(Q) = \Omega^{-1}(Q) \) is the same as the parts enumerator \( E_Q(t) \) for \( \mathcal{DHL}(Q) \). Thus, counting partitions \( P \) of \( n \) by their maximum commuting Jordan type \( \Omega(P) \) gives the same sums as above in Proposition 5.15 for \( p(n) \) or \( p(a,n) \): the Box Conjecture is consistent with these known formulas. An explicit isomorphism between the two sets \( \Omega^{-1}(Q) \) and \( \mathcal{DHL}(Q) \), or even a direct count of partitions having \( r_P = k \) fixed, would contribute to showing the Box Conjecture.

### 5.4 Box Conjecture for certain stable partitions \( Q \) with three parts.

We now use a method similar to certain steps in the proof of our main Theorem 3.14 to show the first part of the Box Conjecture – filling the box – in an infinite series of cases when \( Q \) has three parts.

**Theorem 5.17.** Let \( r \geq 2, 2 \leq s \leq 4 \) and \( Q = (u+s, u, u-r) \). Then we can fill a box \( B(Q) \) of dimensions \((s-1) \times (r-1) \times (u-r)\) by partitions having \( \Omega(P) = Q \).

These cases correspond to all keys \( S = (s-1, r-1, u-r) \) with \( 1 \leq s-1 \leq 3 \). We prove this using Theorem 5.5 (the P. Oblak recursive conjecture is known for \( r_P \leq 3 \)), and methods similar to those we used in the proof of Theorem 3.14. We now split the Theorem up into cases, where, in each, we specify the entries of \( B(Q) \). That the number of entries is \( (s-1)(r-1)(u-r) \) in each case, as claimed, is easy to check.

**Lemma 5.18.** Let the key be \( S = (1, r-1, u-r) \). So \( Q = (u+2, u, u-r) \) with \( r \geq 2 \), and \( u-r \geq 1 \). Then
\[
B(Q) = \{(u+2, P_{k,\ell})|P_{k,\ell} \in \mathcal{T}((u, u-r))\}.
\]

**Proof.** For all \( P_{k,\ell} \in \mathcal{T}((u, u-r)) \), the biggest part of \( P_{k,\ell} \) is at most \( u \) so it differs from \( u+2 \) by at least 2. So we obviously get \( \Omega((u+2, P_{k,\ell})) = Q \).
\[\square\]

**Lemma 5.19.** Let the key be \( S = (2, r-1, u-r) \). So \( Q = (u+3, u, u-r) \) with \( r \geq 2 \), and \( u-r \geq 1 \). Then \( B(Q) \) consists of
\[
\{(u+3, P_{k,\ell})|P_{k,\ell} \in \mathcal{T}((u, u-r))\}
\]
and
\[ \{(u + 2, P_{k,\ell})| P_{k,\ell} \in \mathcal{T}((u + 1, u - r)) \text{ and } k \geq 2\}. \]

**Remark 5.20.** Let \( Q' \) be a stable partition with two parts. Then in \( \mathcal{T}(Q') \), the first part of the partitions in the same Type A row, or in the same B/C hook is always the same. Also, as we descend the first column of \( \mathcal{T}(Q') \), the biggest part of the partition does not increase (Lemma 3.21).

**Proof of Lemma 5.19.** Since the biggest part of every partition in \( \mathcal{T}((u, u - r)) \) is at most \( u \), it is obvious that for all \( P_{k,\ell} \in \mathcal{T}((u, u - r)) \), we get \( \Omega((u + 3, P_{k,\ell})) = Q \), as desired.

Now we consider partitions of the form \( P = (u + 2, P_{k,\ell}) \) with \( P_{k,\ell} \in \mathcal{T}(u + 1, u - r) \). First we consider \( P_{2,1} \in \mathcal{T}((u + 1, u - r)) \). Since in \( \mathcal{T}((u + 1, u - r)) \), \( k_0 = \lceil \frac{u + 1}{u - r} \rceil \), we have
\[
P_{2,1} = \begin{cases} 
(u + 1)^2, u - r & \text{if } 2r > u + 1; \\
(u - r + 2, [u - 1]^2) & \text{if } 2r \leq u + 1.
\end{cases}
\]

If \( 2r > u + 1 \). Then \( \frac{u + 1}{2} < r \) and therefore \( \lceil \frac{u + 1}{2} \rceil \leq r < u \). So for all \( P_{k,\ell} \in \mathcal{T}((u + 1, u - r)) \) with \( k \geq 2 \), the biggest part of \( P_{k,\ell} \) is less than \((u + 2) - 2\).

If \( 2r \leq u + 1 \), then for all \( P_{k,\ell} \in \mathcal{T}(u + 1, u - r) \) with \( k \geq 2 \), the biggest part of \( P_{k,\ell} \) is at most \( u - r - 2 \leq u \), which is again less than \((u + 2) - 2\).

Thus in either case the longest simple \( U \)-chain in the poset of \((u + 2, P_{k,\ell})\) has length \( u + 1 + 2 = u + 3 \) and it is the union of the longest simple \( U \)-chain in the poset of \( P_{k,\ell} \) and the first and last vertices in the \( u + 2 \) row of the poset of \((u + 2, P_{k,\ell})\). Once this \( U \)-chain is removed from the poset, the remaining simple \( U \)-chains have lengths \( u \) (left over on top) and \( u - r + 2 \) (the remaining vertices in the poset of \( P_{k,\ell} \) union the first and last remaining vertices on the top row). Thus, by the Oblak recursive process, \( \Omega((u + 2, P_{k,\ell})) = (u + 3, u, u - r) \).

**Proposition 5.21.** Let the key be \( S = (3, r - 1, u - r) \). So \( Q = (u + 4, u, u - r) \) with \( r \geq 2 \), and \( u - r \geq 1 \). Then \( B(Q) \) consists of

(i) \( \{(u + 4, P_{k,\ell})| P_{k,\ell} \in \mathcal{T}((u, u - r))\}; \)

(ii) \( \{(u + 2, P_{k,\ell})| P_{k,\ell} \in \mathcal{T}((u + 2, u - r)), k \geq 2\}; \)

(iii) \[ \{(u + 4)^2, P_{k,\ell})| P_{k,\ell} \in \mathcal{T}((u, u - r)), k \geq 2\} \quad \text{if } 2r > u + 2, \]
\[ \{(u - r + 4, P_{3,\ell})| P_{3,\ell} \in \mathcal{T}((u + 2, u - 2)), \ell \leq u - 3\} \cup \]
\[ \{(u + 4)^2, P_{k,\ell})| P_{k,\ell} \in \mathcal{T}((u, u - r)), k \geq 3, \ell \leq u - r - 1\}. \]

**Proof.** Part (i). Since the biggest part of every partition in \( \mathcal{T}((u, u - r)) \) is at most \( u \), it is obvious that for all \( P_{k,\ell} \in \mathcal{T}((u, u - r)) \), we get \( \Omega((u + 4, P_{k,\ell})) = (u + 4, u, u - r) \), as desired.

Part (ii). Our goal is to show that the biggest part of the desired partitions \( (u + 2, P_{k,\ell}) \),
namely \( u + 2 \), differs from the biggest part of \( P_{k, \ell} \in \mathcal{T}(u + 2, u - r) \) with \( k \geq 2 \), by at least 2. Since the biggest part of such a \( P_{k, \ell} \) is at most equal to the biggest part of \( P_{2,1} \in \mathcal{T}(u+2, u-r) \), it is enough to compare the biggest part of \( P_{2,1} \) and \( u + 2 \). Using the Table theorem, we see that \( P_{2,1} \) has one of the two following forms, depending on whether it is type A or B.

If \( P_{2,1} \) is type B then its biggest part is \( u - r + 2 \) which is at most \( u \) because \( r \) is at least 2. On the other hand, if \( P_{2,1} \) is type A then its biggest part is \( \lceil \frac{u + 2}{2} \rceil \). We have

\[
\left\lfloor \frac{u + 2}{2} \right\rfloor = \left\lfloor \frac{u}{2} \right\rfloor + 1 \leq \frac{u + 1}{2} + 1,
\]

which is less than \( u \), because \( u \) is at least 3. So in either case the biggest part of \( P_{2,1} \) and consequently the biggest part of \( P_{k, \ell} \) for \( P_{k, \ell} \in \mathcal{T}(u + 2, u - r) \) with \( k \geq 2 \) is at most \( u \). So in the poset \( \mathcal{D}_P \) of \( P = (u + 2, P_{k, \ell}) \) the longest \( U \)-chain is obtained by taking the longest \( U \)-chain in the poset of \( P_{k, \ell} \), which has length \( u + 2 \), and adding the first and last vertices of the top \( u + 2 \) row of \( P \). We will then be left with two other \( U \)-chains, one of length \( u \) on the top row and the other one of length \( u - r \) on the bottom row of the poset of \( P_{k, \ell} \). Thus, by the Oblak recursive process, \( \mathcal{Q}(u + 2, P_{k, \ell}) = (u + 4, u, u - r) \), as desired.

\underline{Part (iii).}

If \( r = 2 \), then there are no partitions satisfying the given conditions in this part and therefore there is nothing left to prove. We assume that \( r > 2 \). Consequently, we also have \( u > 3 \).

\textbf{Case 1.} Assume that \( 2r > u + 2 \).

Let \( P_{2,1} \in \mathcal{T}((u, u - r)) \). Then since \( 2r > u + 2 \), we get \( u > 2(u - r + 1) \), and therefore \( k_0 = \left\lceil \frac{u}{u - r + 1} \right\rceil \geq 3 \). Therefore, \( P_{2,1} = ([u]^2, u - r) \) is of type A.

Note that the biggest part of \( ([u]^2, u - r) \) is \( \left\lceil \frac{u}{2} \right\rceil \), and the smallest part of \( ([u + 4]^2, [u]^2, u - r) \) is \( \frac{u + 4}{2} \). So if \( u \) is even then the difference is 2, which implies that \( \mathcal{Q}(([u + 4]^2, [u]^2, u - r)) = Q \), as desired. Now assume that \( u \) is odd. Then

\[
([u + 4]^2, [u]^2, u - r) = \left( \frac{u + 1}{2} + 2, \frac{u + 1}{2} + 1, \frac{u + 1}{2}, \frac{u - 1}{2}, u - r \right).
\]

It is easy to see that also in this case we get \( \mathcal{Q}(([n + r + 4]^2, [n + r]^2, n)) = Q \). Since the biggest part of each other partition in \( \{ P_{k, \ell} \in \mathcal{T}((u, u - r)), k \geq 2 \} \) is less than or equal to the biggest part of \( ([u]^2, u - r) \), for all such partitions we have \( \mathcal{Q}(([u + 4]^2, P_{k, \ell})) = Q \) as well. This proves (iii) and (iv) when \( 2r > u + 2 \).

\textbf{Case 2.} Assume that \( 2r \leq u + 2 \).

Since by assumption \( u > 3 \), the set \( \{ P_{3, \ell} \in \mathcal{T}((u + 2, u - 2)) \mid k \geq 3 \) and \( \ell \leq u - 3 \} \) is \( A_{r - 1} \), the last A row in the table. Thus \( P_{3, \ell} = ([u]^2, [u - 2]^{\ell + 1}) \).

Using the assumption \( 2r \leq u + 2 \), we have \( 2(u - r + 2) \geq u + 2 \). Thus \( \left\lceil \frac{u}{2} \right\rceil \leq \frac{u + 1}{2} < u - r + 2 \). So the biggest part of \( P_{3, \ell} \) differs from \( u - r + 4 \) by more than 2. Therefore \( \mathcal{Q}((u - r + 4, P_{k, \ell})) = Q \).

Now let \( P_{k, \ell} \) be a partition in \( \mathcal{T}((u, u - r)) \) such that \( k \geq 3 \) and \( \ell \leq u - r - 1 \). Note that by assumption \( 2r \leq u + 2 \), for \( \mathcal{T}((u, u - r)) \), \( k_0 = 2 \) and

\[
\{ P_{k, \ell} \in \mathcal{T}((u, u - r)), k \geq 3, \ell \leq u - r - 1 \} = \mathcal{T}((u, u - r)) \setminus (A_0 \cup B_0).
\]
We note that the set above is empty when $r \leq 3$. So we assume that $r \geq 4$. Since by assumption $2r \leq u + 2$, we also get $u - r \geq 2$.

By definition of $k_1$ for $\mathcal{T}(u, u - r)$, we have $k_1 \geq 2(1 + 1) = 4$. So $P_{3,1}$ in the table is of type A and therefore $P_{3,1} = ([u]^2, [u - r]^2)$. Thus the biggest part of $P_{k,\ell}$, which is less than or equal to the biggest part of $P_{3,1}$, is at most $\lceil \frac{u}{2} \rceil$. So we clearly have $\mathcal{Q}(([u + 4]^2, P_{k,\ell})) = Q$ if the biggest part of $P_{k,\ell}$ is less than $\lceil \frac{u}{2} \rceil$ or $u$ is even. To complete the proof, it is enough to show that $\mathcal{Q}(([u + 4]^2, P_{3,\ell})) = Q$ when $u$ is odd as well. This is also true simply because in this case

$$([u + 4]^2, P_{3,\ell}) = (\lceil \frac{u}{2} \rceil + 2, \lceil \frac{u}{2} \rceil + 1, \lceil \frac{u}{2} \rceil, \lceil \frac{u}{2} \rceil - 1, [u - r]^2).$$

\[\Box\]

**Example 5.22.** We specify the box $B(Q)$ for $Q = (11, 7, 3)$ where the key $S_Q = (3, 3, 3)$ (See Figure 8 where we write $Q^{-1}(Q)$ for $\mathcal{T}(Q)$). Following the statement of Proposition 5.21, the first sheet in the diagram consists of partitions in part (i). The second sheet and the last row of the third sheet (with the same colors extended from the second sheet) consists of partitions in part (ii). Finally the first two rows of the third sheet are partitions in part (iii), the second case (here $r = 4$ and $u = 7$, so $2r < u + 2$). The orange hook in the third sheet is the first of the two sets listed in (iii) and the white cells on the second row are the partitions listed in the second set of (iii).

**References**

[Bar] V. Baranovsky: *The variety of pairs of commuting nilpotent matrices is irreducible*, Transform. Groups 6 (2001), no. 1, 3–8.

[Bas1] R. Basili: *On the irreducibility of commuting varieties of nilpotent matrices*. J. Algebra 268 (2003), no. 1, 58–80.

[Bas2] R. Basili: *On the maximum nilpotent orbit intersecting a centralizer in $M(n, K)$*, preprint, 2014, arXiv:math.RT/1202.3369 v.5.

[BI] R. Basili and A. Iarrobino: *Pairs of commuting nilpotent matrices, and Hilbert function*. J. Algebra 320 # 3 (2008), 1235–1254.

[BIK] R. Basili, A. Iarrobino and L. Khatami, *Commuting nilpotent matrices and Artinian Algebras*, J. Commutative Algebra (2) #3 (2010) 295–325.

[BKO] R. Basili, T. Košir, P. Oblak: *Some ideas from Ljubljana*, (2008), preprint.

[BoIKVS] M. Boij, A. Iarrobino, L. Khatami, B. Van Steirteghem: *Loci of nilpotent matrices having a given Jordan type as maximum commuting orbit*. (In preparation, 2014, a.k.a Notes on Loci Equations from Stockholm and Cambridge).

[BrWi] J.R. Britnell and M. Wildon: *On types and classes of commuting matrices over finite fields*, J. Lond. Math. Soc. (2) 83 (2011), no. 2, 470–492.
Figure 8: Box $\mathcal{B}(Q)$ for $Q = (11, 7, 3)$

[BrFo] T. Britz and S. Fomin: *Finite posets and Ferrers shapes*, Advances Math. 158 #1 (2001), 86–127.

[BroBru] J. Brown and J. Brundan: *Elementary invariants for centralizers of nilpotent matrices*, J. Aust. Math. Soc. 86 (2009), no. 1, 1–15.

[BuEv] M. Bulois, L. Evain: *Nested punctual Hilbert schemes and commuting varieties of parabolic subalgebras*, preprint, 2013, arXiv:math.RT/1306.4838.

[CoMc] D. Collingwood, W. McGovern: *Nilpotent Orbits in Semisimple Lie algebras*, Van Nostrand Reinhold (New York), (1993).

[FrPS] E. Friedlander, J. Pevtsova, A. Suslin: *Generic and maximal Jordan types*, Invent. Math. 168 (2007), no. 3, 485–522.

[Gans] E.R. Gansner: *Acyclic digraphs, Young tableaux and nilpotent matrices*, SIAM Journal of Algebraic Discrete Methods, 2(4) (1981) 429–440.

[Gi] V. Ginzburg: *Principal nilpotent pairs in a semisimple Lie algebra. I*, Invent. Math. 140 (2000), no. 3, 511–561.
[Gre] C. Greene: *Some partitions associated with a partially ordered set*, J. Combinatorial Theory Ser A 20 (1976), 69–79.

[GreKl] C. Greene and D. Kleitman: *The structure of Sperner k-families*, J. Combinatorial Theory Ser. A 20 (1976), no. 1, 41–68.

[GS] D. R. Grayson and M. E. Stillman: *Macaulay 2*, a software system for research in algebraic geometry, Available at [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/).

[GurNg] R. Guralnick and N. Ngo: *Reducibility of nilpotent commuting varieties*, arXiv:math.RT/1308.2420.

[GurSe] R. Guralnick and B.A. Sethuraman: *Commuting pairs and triples of matrices and related varieties*, Linear Algebra Appl. 310 (2000), 139–148.

[Gut] C. Gutschwager: *On principal hook length partition and Durfee sizes in skew characters*, Ann. Comb. 15 (2011) 81–94.

[HW] T. Harima and J. Watanabe: *The commutator algebra of a nilpotent matrix and an application to the theory of commutative Artinian algebras*, J. Algebra 319 (2008), no. 6, 2545–2570.

[IKh] A. Iarrobino and L. Khatami: *Bound on the Jordan type of a generic nilpotent matrix commuting with a given matrix*, J. Alg. Combinatorics, 38, #4 (2013), 947–972.

[J] N. Jacobson: *Schur’s theorems on commutative matrices*, Bull. Amer. Math. Soc. 50, (1944). 431–436.

[Kh1] L. Khatami: *The poset of the nilpotent commutator of a nilpotent matrix*, Linear Algebra Appl. 439 (2013), no. 12, 3763–3776.

[Kh2] L. Khatami: *The smallest part of the generic partition of the nilpotent commutator of a nilpotent matrix*, J. Pure Appl. Algebra 218 (2014), no. 8, 1496–1516.

[KO] T. Košir and P. Oblak: *On pairs of commuting nilpotent matrices*, Transform. Groups 14 (2009), no. 1, 175–182.

[KS1] T. Košir and B. A. Sethuraman: *Determinantal varieties over truncated polynomial rings*, J. Pure Appl. Algebra 195 (2005), no. 1, 75–95.

[KS2] T. Košir and B. A. Sethuraman: *A Groebner basis for the 2 × 2 determinantal ideal mod t²*, J. Algebra 292 (2005), 138–153.

[Mac] Macaulay F. H. S.: *On a method for dealing with the intersection of two plane curves*, Trans. A.M.S. 5 (1904), 385–400.

[Ma] A. Malcev: *Commutative subalgebras of semisimple Lie algebras*, Izvestia Ak. Nauk USSR (Russian) 9 (1945), 125–133; English: Amer. Math. Sre. Translations No. 40 (1951).
[McN] G. McNinch: *On the centralizer of the sum of commuting nilpotent elements*, J. Pure and Applied Alg. 206 (2006) # 1-2, 123–140.

[Ng] N. Ngo: *On the cohomology ring of infinitesimal group schemes*, preprint, 2014, arXiv:math.RT/1401.6820

[NgSi] N. Ngo and K. Šivic: *On varieties of commuting nilpotent matrices*, Linear Algebra Appl. 452 (2014), 237–262.

[Obl1] P. Oblak: *The upper bound for the index of nilpotency for a matrix commuting with a given nilpotent matrix*, Linear and Multilinear Algebra 56 (2008) no. 6, 701–711. Slightly revised in arXiv:math.AC/0701561

[Obl2] P. Oblak: *On the nilpotent commutator of a nilpotent matrix*, Linear Multilinear Algebra 60 (2012), no. 5, 599–612.

[Pan1] D. I. Panyushev: *Two results on centralisers of nilpotent elements*, J. Pure and Applied Algebra, 212 no. 4 (2008), 774–779.

[Pan2] D. I. Panyushev: *Nilpotent pairs, dual pairs, and sheets*, J. Algebra 240 (2001), 635–664.

[Pol] S. Poljak: *Maximum Rank of Powers of a Matrix of Given Pattern*, Proc. A.M.S., 106 #4 (1989), 1137–1144.

[Prem] A. Premet: *Nilpotent commuting varieties of reductive Lie algebras*, Invent. Math. 154 (2003), no. 3, 653–683.

[Sak] M. Saks: Some sequences associated with combinatorial structures Discrete Math. 59 (1986), no. 1-2, 135–166.

[Ši] K. Šivic: *On varieties of commuting triples II*, Linear Algebra Appl. 437(2), 461–489 (2012).

[SuTy] D. A. Suprunenko, R.I. Tyshkevich: Commutative Matrices, viii+155p. Academic Press, New York (1968).

[TA] H.W. Turnbull and A.C. Aitken: *An Introduction to the Theory of Canonical Matrices*, Dover, New York, 1961.

[Z] R. Zhao: *Commuting nilpotent matrices and Oblak’s proposed formula*, preprint, 2014.

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