ONE-POINT EXTENSIONS OF LOCALLY COMPACT PARACOMPACT SPACES

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Abstract. A space $Y$ is called an extension of a space $X$ if $Y$ contains $X$ as a dense subspace. Two extensions of $X$ are said to be equivalent if there is a homeomorphism between them which fixes $X$ point-wise. For two (equivalence classes of) extensions $Y'$ and $Y''$ of $X$ let $Y' \leq Y''$ if there is a continuous function of $Y'$ into $Y$ which fixes $X$ point-wise. An extension $Y$ of $X$ is called a one-point extension if $Y \setminus X$ is a singleton. An extension $Y$ of $X$ is called first-countable if $Y$ is first-countable at points of $Y \setminus X$. Let $\mathcal{P}$ be a topological property. An extension $Y$ of $X$ is called a $\mathcal{P}$-extension if it has $\mathcal{P}$.

In this article, for a given locally compact paracompact space $X$, we consider the two classes of one-point Čech-complete $\mathcal{P}$-extensions of $X$ and one-point first-countable locally-$\mathcal{P}$ extensions of $X$, and we study their order-structures, by relating them to the topology of a certain subspace of the outgrowth $\beta X \setminus X$. Here $\mathcal{P}$ is subject to some requirements and include $\sigma$-compactness and the Lindelöf property as special cases.

1. Introduction

A space $Y$ is called an extension of a space $X$ if $Y$ contains $X$ as a dense subspace. If $Y$ is an extension of $X$ then the subspace $Y \setminus X$ of $Y$ is called the remainder of $Y$. Extensions with a one-point remainder are called one-point extensions. Two extensions of $X$ are said to be equivalent if there exists a homeomorphism between them which fixes $X$ point-wise. This defines an equivalence relation on the class of all extensions of $X$. The equivalence classes will be identified with individuals when this causes no confusion. For two extensions $Y$ and $Y'$ of $X$ we let $Y \leq Y'$ if there exists a continuous function of $Y'$ into $Y$ which fixes $X$ point-wise. The relation $\leq$ defines a partial order on the set of extensions of $X$ (see Section 4.1 of [16] for more details). An extension $Y$ of $X$ is called first-countable if $Y$ is first-countable at points of $Y \setminus X$, that is, $Y$ has a countable local base at every point of $Y \setminus X$. Let $\mathcal{P}$ be a topological property. An extension $Y$ of $X$ is called a $\mathcal{P}$-extension if it has $\mathcal{P}$. If $\mathcal{P}$ is compactness then $\mathcal{P}$-extensions are called compactifications.

This work was mainly motivated by our previous work [9] (see [1], [7], [8], [11], [12] and [13] for related results) in which we have studied the partially ordered set of one-point $\mathcal{P}$-extensions of a given locally compact space $X$ by relating it to the topologies of certain subspaces of its outgrowth $\beta X \setminus X$. In this article we continue our studies by considering the classes of one-point Čech-complete $\mathcal{P}$-extensions and one-point first-countable locally-$\mathcal{P}$ extensions of a given locally compact paracompact space $X$. The topological property $\mathcal{P}$ is subject to some requirements and

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include $\sigma$-compactness, the Lindelöf property and the linearly Lindelöf property as special cases.

We review some of the terminology, notation and well-known results that will be used in the sequel. Our definitions mainly come from the standard text [3] (thus, in particular, compact spaces are Hausdorff, etc.). Other useful sources are [5] and [16].

The letters $I$ and $\mathbb{N}$ denote the closed unit interval and the set of all positive integers, respectively. For a subset $A$ of a space $X$ we let $\text{cl}_X A$ and $\text{int}_X A$ denote the closure and the interior of $A$ in $X$, respectively. A subset of a space is called clopen if it is simultaneously closed and open. A zero-set of a space $X$ is a set of the form $Z(f) = f^{-1}(0)$ for some continuous $f : X \to I$. Any set of the form $X \setminus Z$, where $Z$ is a zero-set of $X$, is called a cozero-set of $X$. We denote the set of all zero-sets of $X$ by $Z(X)$ and the set of all cozero-sets of $X$ by $\text{Coz}(X)$.

For a Tychonoff space $X$ the Stone-Čech compactification of $X$ is the largest (with respect to the partial order $\leq$) compactification of $X$ and is denoted by $\beta X$. The Stone-Čech compactification of $X$ can be characterized among all compactifications of $X$ by either of the following properties:

- Every continuous function of $X$ to a compact space is continuously extendible over $\beta X$.
- Every continuous function of $X$ to $I$ is continuously extendible over $\beta X$.
- For every $Z, S \in Z(X)$ we have $\text{cl}_{\beta X}(Z \cap S) = \text{cl}_{\beta X} Z \cap \text{cl}_{\beta X} S$.

A Tychonoff space is called zero-dimensional if it has an open base consisting of its clopen subsets. A Tychonoff space is called strongly zero-dimensional if its Stone-Čech compactification is zero-dimensional. A Tychonoff space $X$ is called Čech-complete if its outgrowth $\beta X \setminus X$ is an $F_\sigma$ in $\beta X$. Locally compact spaces are Čech-complete, and in the realm of metrizable spaces $X$, Čech-completeness is equivalent to the existence of a compatible complete metric on $X$.

Let $\mathcal{P}$ be a topological property. A topological space $X$ is called locally-$\mathcal{P}$ if for every $x \in X$ there exists an open neighborhood $U_x$ of $x$ in $X$ such that $\text{cl}_X U_x$ has $\mathcal{P}$.

A topological property $\mathcal{P}$ is said to be hereditary with respect to closed subsets if each closed subset of a space with $\mathcal{P}$ also has $\mathcal{P}$. A topological property $\mathcal{P}$ is said to be preserved under finite (closed) sums of subspaces if a Hausdorff space has $\mathcal{P}$, provided that it is the union of a finite collection of its (closed) $\mathcal{P}$-subspaces.

Let $(P, \leq)$ and $(Q, \leq)$ be two partially ordered sets. A mapping $f : (P, \leq) \to (Q, \leq)$ is said to be an order-homomorphism (anti-order-homomorphism, respectively) if $f(a) \leq f(b)$ ($f(b) \leq f(a)$, respectively) whenever $a \leq b$. An order-homomorphism (anti-order-homomorphism, respectively) $f : (P, \leq) \to (Q, \leq)$ is said to be an order-isomorphism (anti-order-isomorphism, respectively) if $f^{-1} : (Q, \leq) \to (P, \leq)$ (exists and) is an order-homomorphism (anti-order-homomorphism, respectively). Two partially ordered sets $(P, \leq)$ and $(Q, \leq)$ are called order-isomorphic (anti-order-isomorphic, respectively) if there exists an order-isomorphism (anti-order-isomorphism, respectively) between them.
2. Motivations, notations and definitions

In this article we will be dealing with various sets of one-point extensions of a given topological space $X$. For the reader’s convenience we list these sets all at the beginning.

Notation 2.1. Let $X$ be a topological space. Denote

- $\mathcal{E}(X) = \{Y : Y$ is a one-point Tychonoff extension of $X\}$
- $\mathcal{E}^*(X) = \{Y \in \mathcal{E}(X) : Y$ is first-countable at $Y \setminus X\}$
- $\mathcal{E}^C(X) = \{Y \in \mathcal{E}(X) : Y$ is Čech-complete\}$
- $\mathcal{E}^K(X) = \{Y \in \mathcal{E}(X) : Y$ is locally compact\}$

and when $P$ is a topological property

- $\mathcal{E}_P(X) = \{Y \in \mathcal{E}(X) : Y$ has $P\}$
- $\mathcal{E}_{local-P}(X) = \{Y \in \mathcal{E}(X) : Y$ is locally-$P$.\}$

Also, we may use notations which are obtained by combinations of the above notations, e.g.

$\mathcal{E}_{local-P}^*(X) = \mathcal{E}_P^*(X) \cap \mathcal{E}_{local-P}(X)$.

Definition 2.2 (10). For a Tychonoff space $X$ and a topological property $P$, let

$\lambda_P X = \bigcup \{\text{int}_\beta X \text{cl}_\beta C : C \in \text{Coz}(X) \text{ and cl}_X C \text{ has } P\}$.

Definition 2.3 (14). We say that a topological property $P$ satisfies Mrówka’s condition (W) if it satisfies the following: If $X$ is a Tychonoff space in which there exists a point $p$ with an open base $B$ for $X$ at $p$ such that $X \setminus B$ has $P$ for each $B \in B$, then $X$ has $P$.

Mrówka’s condition (W) is satisfied by a large number of topological properties; among them are (regularity $+$) the Lindelöf property, paracompactness, metacompactness, subparacompactness, the para-Lindelöf property, the $\sigma$-para-Lindelöf property, weak $\theta$-refinability, $\theta$-refinability (or submetacompactness), weak $\delta\theta$-refinability, $\delta\theta$-refinability (or the submeta-Lindelöf property), countable paracompactness, $[\theta, \kappa]$-compactness, $\kappa$-boundedness, screenability, $\sigma$-metacompactness, Dieudonné completeness, $N$-compactness (15), realcompactness, almost realcompactness and zero-dimensionality (see 10, 12 and 13 for proofs and 2, 17 and 18 for definitions).

In 11 we have obtained the following result.

Theorem 2.4 (11). Let $X$ and $Y$ be locally compact locally-$P$ non-$P$ spaces where $P$ is either pseudocompactness or a closed hereditary topological property which is preserved under finite closed sums of subspaces and satisfies Mrówka’s condition (W). The following are equivalent:

1. $\lambda_P X \setminus X$ and $\lambda_P Y \setminus Y$ are homeomorphic.
2. $(\mathcal{E}_P(X), \leq)$ and $(\mathcal{E}_P(Y), \leq)$ are order-isomorphic.
3. $(\mathcal{E}_K(X), \leq)$ and $(\mathcal{E}_K(Y), \leq)$ are order-isomorphic.
4. $(\mathcal{E}_K^C(X), \leq)$ and $(\mathcal{E}_K^C(Y), \leq)$ are order-isomorphic, provided that $X$ and $Y$ are moreover strongly zero-dimensional.

There are topological properties, however, which do not satisfy the assumption of Theorem 2.4 (σ-compactness, for example, does not satisfy Mrówka’s condition (W); see 10). The purpose of this article is to prove the following version of Theorem 2.4. Specific topological properties $P$ which satisfy the requirements of
Theorem 2.5 below are σ-compactness, the Lindelöf property and the linearly Lindelöf property. Note that in Theorem 3.19 of [9] we have shown that conditions (1) and (3) of Theorem 2.5 are equivalent, if \( P \) is σ-compactness, and in Theorem 3.21 of [9] we have shown that conditions (1) and (2) of Theorem 2.5 are equivalent, if \( P \) is the Lindelöf property. Thus, in some sense, Theorem 2.5 generalizes Theorems 3.19 and 3.21 of [9], and at the same time, brings them under a same umbrella.

**Theorem 2.5.** Let \( X \) and \( Y \) be locally compact paracompact spaces and let \( P \) be a closed hereditary topological property of compact spaces which is preserved under finite sums of subspaces and coincides with σ-compactness in the realm of locally compact paracompact spaces. The following are equivalent:

1. \( \lambda_P X \setminus X \) and \( \lambda_P Y \setminus Y \) are homeomorphic.
2. \( (\mathcal{E}^c_P(X), \leq) \) and \( (\mathcal{E}^c_P(Y), \leq) \) are order-isomorphic.
3. \( (\mathcal{E}^*_{local-P}(X), \leq) \) and \( (\mathcal{E}^*_{local-P}(Y), \leq) \) are order-isomorphic.

We now introduce some notation which will be widely used in this article.

**Notation 2.6.** Let \( X \) be a Tychonoff space. For a subset \( A \) of \( X \) denote

\[
A^* = \text{cl}_\beta X \setminus A.
\]

In particular, \( X^* = \beta X \setminus X \).

**Remark 2.7.** Note that the notation given in Notation 2.6 can be ambiguous, as \( A^* \) can mean either \( \beta A \setminus A \) or \( \text{cl}_\beta X \setminus A \). However, since for \( C^* \)-embedded subsets these two notions coincide, this will not cause any confusion.

**Definition 2.8 ([7]).** For a Tychonoff space \( X \), let

\[
\sigma X = \bigcup \{ \text{cl}_\beta X H : H \subseteq X \text{ is } \sigma \text{-compact} \}.
\]

**Notation 2.9.** Let \( X \) be a locally compact paracompact non-compact space. Then \( X \) can be represented as

\[
X = \bigoplus_{i \in I} X_i
\]

for some index set \( I \), with each \( X_i \) for \( i \in I \), being \( \sigma \)-compact and non-compact (see Theorem 5.1.27 and Exercise 3.8.C of [3]). For any \( J \subseteq I \) denote

\[
X_J = \bigcup_{i \in J} X_i.
\]

Thus, using the notation of Notation 2.6, we have

\[
X_J^* = \text{cl}_\beta X \left( \bigcup_{i \in J} X_i \right) \setminus X.
\]

**Remark 2.10.** Note that in Notation 2.9 the set \( X_J^* \) is homeomorphic to \( \beta X_J \setminus X_J \), as \( \text{cl}_\beta X_J \) is homeomorphic to \( \beta X_J \) (see Corollary 3.6.8 of [3]). Thus, when \( J \) is countable (since \( X_J \) is \( \sigma \)-compact and locally compact) \( X_J^* \) is a zero-sets in \( \text{cl}_\beta X_J \) (see 1B of [19]). But \( \text{cl}_\beta X_J \) is clopen in \( \beta X \), as \( X_J \) is clopen in \( X \) (see Corollary 3.6.5 of [3]) therefore, \( X_J^* \) is a zero-sets in \( \beta X \). Also, note that with the notation given in Notation 2.6 we have

\[
\sigma X = \bigcup \{ \text{cl}_\beta X_J : J \subseteq I \text{ is countable} \}.
\]

Note that \( \sigma X \) is open in \( \beta X \) and it contains \( X \).
3. Partially ordered set of one-point extensions as related to topologies of subspaces of outgrowth

In Lemma 3.3 we establish a connection between one-point Tychonoff extensions of a given space $X$ and compact non-empty subsets of its outgrowth $X^*$. Lemma 3.3 (and its preceding lemmas) is known (see e.g. [12]). It is included here for the sake of completeness.

**Lemma 3.1.** Let $X$ be a Tychonoff space and let $C$ be a non-empty compact subset of $X^*$. Let $T$ be the space which is obtained from $\beta X$ by contracting $C$ to a point $p$. Then the subspace $Y = X \cup \{p\}$ of $T$ is Tychonoff and $\beta Y = T$.

**Proof.** Let $q : \beta X \to T$ be the quotient mapping. Note that $T$ is Hausdorff, and thus, being a continuous image of $\beta X$, it is compact. Also, note that $Y$ is dense in $T$. Therefore, $T$ is a compactification of $Y$. To show that $\beta Y = T$, it suffices to verify that every continuous $h : Y \to I$ is continuously extendable over $T$. Let $h : Y \to I$ be continuous. Let $G : \beta X \to I$ continuously extend $hq|(X \cup C) : X \cup C \to I$ (note that $\beta(X \cup C) = \beta X$, as $X \subseteq X \cup C \subseteq \beta X$; see Corollary 3.6.9 of [3]). Define $H : T \to I$ such that $H|(\beta X \setminus C) = G|(\beta X \setminus C)$ and $H(p) = h(p)$. Then $H|Y = h$, and since $Hq = G$ is continuous, the function $H$ is continuous.

**Notation 3.2.** Let $X$ be a Tychonoff space and let $Y \in E(X)$. Denote by

$$\tau_Y : \beta X \to \beta Y$$

the (unique) continuous extension of $\text{id}_X$.

**Lemma 3.3.** Let $X$ be a Tychonoff space and let $Y = X \cup \{p\} \in E(X)$. Let $T$ be the space which is obtained from $\beta X$ by contracting $\tau_Y^{-1}(p)$ to the point $p$, and let $q : \beta X \to T$ be the quotient mapping. Then $T = \beta Y$ and $\tau_Y = q$.

**Proof.** We need to show that $Y$ is a subspace of $T$. Since $\beta Y$ is also a compactification of $X$ and $\tau_Y|X = \text{id}_X$, by Theorem 3.5.7 of [3], we have $\tau_Y(X^*) = \beta Y \setminus X$. For an open subset $W$ of $\beta Y$, the set $q(\tau_Y^{-1}(W))$ is open in $T$, as $q^{-1}(q(\tau_Y^{-1}(W))) = \tau_Y^{-1}(W)$ is open in $\beta X$. Therefore

$$Y \cap W = Y \cap q(\tau_Y^{-1}(W))$$

is open in $Y$, when $Y$ is considered as a subspace of $T$. For the converse, note that if $V$ is open in $T$, since

$$Y \cap V = Y \cap (\beta Y \setminus \tau_Y(\beta X \setminus q^{-1}(V)))$$

and $\tau_Y(\beta X \setminus q^{-1}(V))$ is compact and thus closed in $\beta Y$, the set $Y \cap V$ is open in $Y$ in its original topology. By Lemma 3.1 we have $T = \beta Y$. This also implies that $\tau_Y = q$, as $\tau_Y, q : \beta X \to \beta Y$ are continuous and coincide with $\text{id}_X$ on the dense subset $X$ of $\beta X$.

**Lemma 3.4.** Let $X$ be a Tychonoff space. Let $Y_i \in E(X)$, where $i = 1, 2$, and denote by $\tau_i = \tau_{Y_i} : \beta X \to \beta Y_i$ the continuous extension of $\text{id}_X$. The following are equivalent:

1. $Y_1 \subseteq Y_2$.
2. $\tau_2^{-1}(Y_2 \setminus X) \subseteq \tau_1^{-1}(Y_1 \setminus X)$. 


Proof. Let \( Y_i = X \cup \{ p_i \} \) where \( i = 1, 2 \). (1) implies (2). Suppose that (1) holds. By definition, there exists a continuous \( f : Y_2 \to Y_1 \) such that \( f|X = \text{id}_X \). Let \( f_{\beta} : \beta Y_2 \to \beta Y_1 \) continuously extend \( f \). Note that the continuous functions \( f_{\beta}, \tau_1 : \beta X \to \beta Y_1 \) coincide with \( \text{id}_X \) on the dense subset \( X \) of \( \beta X \), and thus \( f_{\beta} \tau_2 = \tau_1 \). Note that \( X \) is dense in \( \beta Y_i \) (where \( i = 1, 2 \)), as it is dense in \( Y_i \), and therefore, \( \beta Y_i \) is a compactification of \( X \). Since \( f_{\beta}|X = \text{id}_X \), by Theorem 3.5.7 of [3], we have \( f_{\beta}(\beta Y_2 \setminus X) = \beta Y_1 \setminus X \), and thus \( f_{\beta}(p_2) \in \beta Y_1 \setminus X \). But \( f_{\beta}(p_2) = f(p_2) \), which implies that \( f_{\beta}(p_2) \in Y_1 \setminus X = \{ p_1 \} \). Therefore
\[
\tau_2^{-1}(p_2) \leq \tau_2^{-1}(f_{\beta}^{-1}(f_{\beta}(p_2)))
\]
\[
= (f_{\beta} \tau_2)^{-1}(f_{\beta}(p_2)) = \tau_1^{-1}(f_{\beta}(p_2)) = \tau_1^{-1}(p_1).
\]

(2) implies (1). Suppose that (2) holds. Let \( f : Y_2 \to Y_1 \) be defined such that \( f(p_2) = p_1 \) and \( f|X = \text{id}_X \). We show that \( f \) is continuous, this will show that \( Y_1 \subseteq Y_2 \). Note that by Lemma 3.4, the space \( \beta Y_2 \) is the quotient space of \( \beta X \) which is obtained by contracting \( \tau_2^{-1}(p_2) \) to a point, and \( \tau_2 \) is its corresponding quotient mapping. Thus, in particular, \( Y_2 \) is the quotient space of \( X \cup \tau_2^{-1}(p_2) \), and therefore, to show that \( f \) is continuous, it suffices to show that \( f \tau_2|(X \cup \tau_2^{-1}(p_2)) \) is continuous. We show this by verifying that \( f \tau_2(t) = \tau_1(t) \) for each \( t \in X \cup \tau_2^{-1}(p_2) \). This obviously holds if \( t \in X \). If \( t \in \tau_2^{-1}(p_2) \), then \( \tau_2(t) = p_2 \), and thus \( f \tau_2(t) = p_1 \). But since \( t \in \tau_2^{-1}(\tau_2(t)) \), we have \( t \in \tau_1^{-1}(p_1) \), and therefore \( \tau_1(t) = p_1 \). Thus \( f \tau_2(t) = \tau_1(t) \) in this case as well. \( \square \)

Lemma 3.5. Let \( X \) be a Tychonoff space. Define a function
\[
\Theta : (\mathcal{E}(X), \leq) \to (\{ C \subseteq X^* : C \text{ is compact} \}\setminus \{ \emptyset \}, \subseteq)
\]
by
\[
\Theta(Y) = \tau_Y^{-1}(Y \setminus X)
\]
for any \( Y \in \mathcal{E}(X) \). Then \( \Theta \) is an anti-order-isomorphism.

Proof. To show that \( \Theta \) is well-defined, let \( Y \in \mathcal{E}(X) \). Note that since \( X \) is dense in \( Y \), the space \( X \) is dense in \( \beta Y \). Thus \( \tau_Y : \beta X \to \beta Y \) is onto, as \( \tau_Y (\beta X) \) is a compact (and therefore closed) subset of \( \beta Y \) and it contains \( X = \tau_Y(X) \). Thus \( \tau_Y^{-1}(Y \setminus X) \neq \emptyset \). Also, since \( \tau_Y|X = \text{id}_X \) we have \( \tau_Y^{-1}(Y \setminus X) \subseteq X^* \), and since the singleton \( Y \setminus X \) is closed in \( \beta Y \), its inverse image \( \tau_Y^{-1}(Y \setminus X) \) is closed in \( \beta X \), and therefore it is compact. Now we show that \( \Theta \) is onto, Lemma 3.4 will then complete the proof. Let \( C \) be a non-empty compact subset of \( X^* \). Let \( T \) be the quotient space of \( \beta X \) which is obtained by contracting \( C \) to a point \( p \). Consider the subspace \( Y = X \cup \{ p \} \) of \( T \). Then \( Y \in \mathcal{E}(X) \), and thus, by Lemma 3.1 we have \( \beta Y = T \). The quotient mapping \( q : \beta X \to T \) is identical to \( \tau_Y \), as it coincides with \( \text{id}_X \) on the dense subset \( X \) of \( \beta X \). Therefore
\[
\Theta(Y) = \tau_Y^{-1}(p) = q^{-1}(p) = C.
\]
\( \square \)

Notation 3.6. For a Tychonoff space \( X \) denote by
\[
\Theta_X : (\mathcal{E}(X), \leq) \to (\{ C \subseteq X^* : C \text{ is compact} \}\setminus \{ \emptyset \}, \subseteq)
\]
the anti-order-isomorphism defined by
\[
\Theta_X(Y) = \tau_Y^{-1}(Y \setminus X)
\]
for any \( Y \in \mathcal{E}(X) \).
Lemmas 3.7 and 3.8 below are known results (see [9]).

**Lemma 3.7.** Let $X$ be a Tychonoff space. For a $Y \in \mathcal{E}(X)$ the following are equivalent:

1. $Y \in \mathcal{E}^*(X)$.
2. $\Theta_X(Y) \in \mathcal{E}(\beta X)$.

**Proof.** Let $Y = X \cup \{ p \}$. (1) implies (2). Suppose that (1) holds. Let $\{ V_n : n \in \mathbb{N} \}$ be an open base at $p$ in $Y$. For each $n \in \mathbb{N}$, let $V'_n$ be an open subset of $\beta Y$ such that $Y \cap V'_n = V_n$, and let $f_n : \beta Y \to \mathbb{I}$ be continuous and such that $f_n(p) = 0$ and $f_n(\beta Y \setminus V'_n) \subseteq \{ 1 \}$. Let

$$Z = \bigcap_{n=1}^{\infty} Z(f_n) \in \mathcal{E}(\beta Y).$$

We show that $Z = \{ p \}$. Obviously, $p \in Z$. Let $t \in Z$ and suppose to the contrary that $t \neq p$. Let $W$ be an open neighborhood of $p$ in $\beta Y$ such that $t \notin \text{cl}_{\beta Y} W$. Then $Y \cap W$ is an open neighborhood of $p$ in $Y$. Let $k \in \mathbb{N}$ be such that $V_k \subseteq Y \cap W$. We have

$$t \in Z(f_k) \subseteq V'_k \subseteq \text{cl}_{\beta Y} V'_k \subseteq \text{cl}_{\beta Y} (Y \cap V'_k) = \text{cl}_{\beta Y} (Y \cap W) \subseteq \text{cl}_{\beta Y} W$$

which is a contradiction. This shows that $t = p$ and therefore $Z \subseteq \{ p \}$. Thus $\{ p \} = Z \in \mathcal{E}(\beta Y)$, which implies that $\tau^{-1}_Y(p) \in \mathcal{E}(\beta X)$.

(2) implies (1). Suppose that (2) holds. Let $\tau^{-1}_Y(p) = Z(f)$ where $f : \beta X \to \mathbb{I}$ is continuous. Note that by Lemma 3.8 the space $\beta Y$ is obtained from $\beta X$ by contracting $\tau^{-1}_Y(p)$ to $p$ with $\tau_Y : \beta X \to \beta Y$ as the quotient mapping. Then for each $n \in \mathbb{N}$ the set $\tau_Y(f^{-1}([0,1/n]))$ is an open neighborhood of $p$ in $\beta Y$. We show that the collection

$$\{ Y \cap \tau_Y(f^{-1}([0,1/n])) : n \in \mathbb{N} \}$$

of open neighborhoods of $p$ in $Y$ constitutes an open base at $p$ in $Y$, this will show (1). Let $V$ be an open neighborhood of $p$ in $Y$. Let $V'$ be an open subset of $\beta Y$ such that $Y \cap V' = V$. Then $p \in V'$ and thus

$$\bigcap_{n=1}^{\infty} f^{-1}([0,1/n]) = Z(f) = \tau^{-1}_Y(p) \subseteq \tau^{-1}_Y(V').$$

By compactness we have $f^{-1}([0,1/k]) \subseteq \tau^{-1}_Y(V')$ for some $k \in \mathbb{N}$. Therefore

$$Y \cap \tau_Y(f^{-1}([0,1/k])) \subseteq Y \cap \tau_Y(f^{-1}([0,1/k])) \subseteq Y \cap \tau_Y(\tau^{-1}_Y(V')) \subseteq Y \cap V' = V.$$

□

**Lemma 3.8.** Let $X$ be a locally compact space. For a $Y \in \mathcal{E}(X)$ the following are equivalent:

1. $Y \in \mathcal{E}^C(X)$.
2. $\Theta_X(Y) \in \mathcal{E}(X^*)$. 
Proof. Let $Y = X \cup \{p\}$. (1) implies (2). Suppose that (1) holds. Then $Y^*$ is an $F_\sigma$ in $\beta Y$. Let $Y^* = \bigcup_{n=1}^\infty K_n$ where each $K_n$ is closed in $\beta Y$ for $n \in \mathbb{N}$. Then

$$X^* = \tau_{\beta Y}^{-1}(p) \cup \bigcup_{n=1}^\infty K_n$$

(recall that $\beta Y$ is the quotient space of $\beta X$ which is obtained by contracting $\tau_{\beta Y}^{-1}(p)$ to $p$ and $\tau_Y$ is its quotient mapping; see Lemma 3.3). For each $n \in \mathbb{N}$, let $f_n : \beta X \to I$ be continuous and such that $f_n(\tau_{\beta Y}^{-1}(p)) = \{0\}$ and $f_n(K_n) \subseteq \{1\}$.

Let $f = \sum_{n=1}^\infty f_n/2^n$. Then $f : \beta X \to I$ is continuous and

$$\tau_{\beta Y}^{-1}(p) = Z(f) \cap X^* \in \mathcal{F}(X^*).$$

(2) implies (1). Suppose that (2) holds. Let $\tau_{\beta Y}^{-1}(p) = Z(g)$ where $g : X^* \to I$ is continuous. Then, using Lemma 3.3, we have

$$Y^* = X^* \setminus \tau_{\beta Y}^{-1}(p) = X^* \setminus Z(g) = g^{-1}([0,1]) = \bigcup_{n=1}^\infty g^{-1}([1/n,1])$$

and each set $g^{-1}([1/n,1])$, for $n \in \mathbb{N}$, being closed in $X^*$, is compact (note that since $X$ is locally compact, $X^*$ is compact) and thus closed in $\beta Y$. Therefore, $Y^*$ is an $F_\sigma$ in $\beta Y$, that is, $Y$ is Čech-complete. □

The following lemma justifies our requirement on $\mathcal{P}$ in Theorem 3.16. We simply need $\lambda_{\mathcal{P}}X$ to have a more familiar structure.

Lemma 3.9. Let $\mathcal{P}$ be a topological property which is preserved under finite closed sums of subspaces. The following are equivalent:

1. The topological property $\mathcal{P}$ coincides with $\sigma$-compactness in the realm of locally compact paracompact spaces.
2. For every locally compact paracompact space $X$ we have
$$\lambda_{\mathcal{P}}X = \sigma X.$$  

Proof. (1) implies (2). Suppose that (1) holds. Let $X$ be a locally compact paracompact space. Assume the notation of Notation 2.9. Let $J \subseteq I$ be countable. Then $X_J$ is $\sigma$-compact and thus (since it is also locally compact and paracompact) it has $\mathcal{P}$. Note that $X_J$ is clopen in $X$ thus it has a clopen closure in $\beta X$, therefore

$$\text{cl}_{\beta X}X_J = \text{int}_{\beta X}\text{cl}_{\beta X}X_J \subseteq \lambda_{\mathcal{P}}X$$

that is, $\sigma X \subseteq \lambda_{\mathcal{P}}X$. To see the reverse inclusion, let $C \in \text{Coz}(X)$ be such that $\text{cl}_X C$ has $\mathcal{P}$. Then (since $\text{cl}_X C$ being closed in $X$ is also locally compact and paracompact) $\text{cl}_X C$ is $\sigma$-compact. Therefore

$$\text{int}_{\beta X}\text{cl}_{\beta X}C \subseteq \text{cl}_{\beta X}C \subseteq \sigma X$$

which shows that $\lambda_{\mathcal{P}}X \subseteq \sigma X$. Thus $\lambda_{\mathcal{P}}X = \sigma X$.

(2) implies (1). Suppose that (2) holds. Let $X$ be a locally compact paracompact space. By assumption we have $\lambda_{\mathcal{P}}X = \sigma X$. We verify that $X$ has $\mathcal{P}$ if and only
if $X$ is $\sigma$-compact. Assume the notation of Notation 2.9. Suppose that $X$ has $\mathcal{P}$. Then $\lambda_{\mathcal{P}}X = \beta X$ and thus $\sigma X = \beta X$. Now, by compactness, we have

$$\beta X = \text{cl}_{\beta X}X_{J_1} \cup \cdots \cup \text{cl}_{\beta X}X_{J_n}$$

for some $n \in \mathbb{N}$ and some countable $J_1, \ldots, J_n \subseteq I$. Therefore

$$X = X_{J_1} \cup \cdots \cup X_{J_n}$$

is $\sigma$-compact. For the converse, suppose that $X$ is $\sigma$-compact. Then $\sigma X = \beta X$ and (since $\lambda_{\mathcal{P}}X = \sigma X$) we have $\beta X = \lambda_{\mathcal{P}}X$. Thus, by compactness, we have

$$\beta X = \text{int}_{\beta X}\text{cl}_{\beta X}C_1 \cup \cdots \cup \text{int}_{\beta X}\text{cl}_{\beta X}C_n$$

for some $n \in \mathbb{N}$ and some $C_1, \ldots, C_n \in Coz(X)$ such that $\text{cl}_{\beta X}C_i$ has $\mathcal{P}$ for any $i = 1, \ldots, n$. Now, using our assumption, the space

$$X = \text{cl}_{\beta X}C_1 \cup \cdots \cup \text{cl}_{\beta X}C_n$$

being a finite union of its closed $\mathcal{P}$-subspaces, has $\mathcal{P}$.

\textbf{Lemma 3.10.} Let $X$ be a locally compact paracompact space and let $\mathcal{P}$ be a closed hereditary topological property of compact spaces which is preserved under finite sums of subspaces and coincides with $\sigma$-compactness in the realm of locally compact paracompact spaces. For a $Y \in \mathcal{E}(X)$ the following are equivalent:

1. $Y \in \mathcal{E}_{\mathcal{P}}^{\mathcal{G}}(X)$.
2. $\Theta_X(Y) \in \mathcal{E}(X^*)$ and $\beta X \setminus \lambda_{\mathcal{P}}X \subseteq \Theta_X(Y)$. 

Thus, in particular

$$\Theta_X(\mathcal{E}_{\mathcal{P}}^{\mathcal{G}}(X)) = \{ Z \in \mathcal{E}(X^*) : \beta X \setminus \lambda_{\mathcal{P}}X \subseteq Z \} \setminus \{ \emptyset \}.$$

\textit{Proof.} Let $Y = X \cup \{ p \}$. (1) implies (2). Suppose that (1) holds. By Lemma 3.8 we have $\tau_{\mathcal{P}}^{-1}(p) \in \mathcal{E}(X^*)$. Note that by Lemma 3.9 we have $\lambda_{\mathcal{P}}X = \sigma X$. Let $t \in \beta X \setminus \sigma X$ and suppose to the contrary that $t \notin \tau_{\mathcal{P}}^{-1}(p)$. Let $f : \beta X \to I$ be continuous and such that $f(t) = 0$ and $f(\tau_{\mathcal{P}}^{-1}(p)) = \{ 1 \}$. Since $\tau_{\mathcal{P}}(f^{-1}([0, 1/2]))$ is compact, the set

$$T = X \cap f^{-1}([0, 1/2]) = Y \cap f^{-1}(f^{-1}([0, 1/2]))$$

being closed in $Y$, has $\mathcal{P}$. But $T$, being closed in $X$, is locally compact and paracompact, and thus, having $\mathcal{P}$, it is $\sigma$-compact. Therefore, by the definition of $\sigma X$ we have $\text{cl}_{\beta X}T \subseteq \sigma X$. But since

$$t \in f^{-1}([0, 1/2]) \subseteq \text{cl}_{\beta X}f^{-1}([0, 1/2]) \subseteq \text{cl}_{\beta X}(X \cap f^{-1}([0, 1/2])) \subseteq \text{cl}_{\beta X}(X \cap f^{-1}([0, 1/2])) = \text{cl}_{\beta X}T$$

we have $t \in \sigma X$, which contradicts the choice of $t$. Thus $t \in \tau_{\mathcal{P}}^{-1}(p)$ and therefore $\beta X \setminus \sigma X \subseteq \tau_{\mathcal{P}}^{-1}(p)$.

(2) implies (1). Suppose that (2) holds. Note that since $X$ is locally compact, the set $X^*$ is closed in (the normal space) $\beta X$ and thus, since $\tau_{\mathcal{P}}^{-1}(p) \in \mathcal{E}(X^*)$ (using the Tietze-Urysohn Theorem) we have $\tau_{\mathcal{P}}^{-1}(p) = Z \cap X^*$ for some $Z \in \mathcal{E}(\beta X)$. Note that by Lemma 3.9 we have $\lambda_{\mathcal{P}}X = \sigma X$. Now, since $\beta X \setminus \sigma X \subseteq \tau_{\mathcal{P}}^{-1}(p) \subseteq Z$
we have \( \beta X \setminus Z \subseteq \sigma X \). Therefore, assuming the notation of Notation 2.9 (since \( \beta X \setminus Z \), being a cozero-set in \( \beta X \), is \( \sigma \)-compact) we have

\[
\beta X \setminus Z \subseteq \bigcup_{n=1}^{\infty} \operatorname{cl}_{\beta X} X_{J_n} \subseteq \operatorname{cl}_{\beta X} X_J
\]

where \( J_1, J_2, \ldots \subseteq I \) are countable and \( J = J_1 \cup J_2 \cup \cdots \). But

\[
Y = \tau_Y(Z) \cup (X \setminus Z) \subseteq \tau_Y(Z) \cup X_J
\]

and thus we have

\[
Y = \tau_Y(Z) \cup X_J.
\]

(3.1)

Now, since \( X_J \) has \( \mathcal{P} \), as it is \( \sigma \)-compact (and being closed in \( \beta X \), it is locally compact and paracompact) and \( \tau_Y(X) \) has \( \mathcal{P} \), as it is compact, from (3.1) it follows that the space \( Y \), being a finite union of its \( \mathcal{P} \)-subspaces, has \( \mathcal{P} \). The fact that \( Y \) is Čech-complete follows from Lemma 3.3.

The following generalizes Lemma 3.18 of [9].

**Lemma 3.11.** Let \( X \) be a locally compact paracompact space and let \( \mathcal{P} \) be a closed hereditary topological property of compact spaces which is preserved under finite sums of subspaces and coincides with \( \sigma \)-compactness in the realm of locally compact paracompact spaces. For a \( Y \in \mathcal{E}(X) \) the following are equivalent:

1. \( Y \in \mathcal{E}^*_{\operatorname{local-}\mathcal{P}}(X) \).
2. \( \Theta_X(Y) \in \mathcal{F}(\beta X) \) and \( \Theta_X(Y) \subseteq \lambda_{\mathcal{P}} X \).

Thus, in particular

\[
\Theta_X(\mathcal{E}^*_{\operatorname{local-}\mathcal{P}}(X)) = \{ Z \in \mathcal{F}(\beta X) : Z \subseteq \lambda_{\mathcal{P}} X \} \backslash \{ \emptyset \}.
\]

**Proof.** Let \( Y = X \cup \{ p \} \). (1) implies (2). Suppose that (1) holds. Since \( Y \in \mathcal{E}^*(X) \), by Lemma 3.3, we have \( \tau_Y^{-1}(p) \in \mathcal{F}(\beta X) \). Let \( \tau_Y^{-1}(p) = Z(f) \) for some continuous \( f : \beta X \to I \). Since \( Y \) is locally-\( \mathcal{P} \), there exists an open neighborhood \( V \) of \( p \) in \( Y \) such that \( \operatorname{cl}_Y V \) has \( \mathcal{P} \). Let \( V' \) be an open subset of \( \beta Y \) such that \( Y \cap V' = V \). Then \( p \in V' \), and thus since

\[
\bigcap_{n=1}^{\infty} f^{-1}([0,1/n]) = Z(f) = \tau_Y^{-1}(p) \subseteq \tau_Y^{-1}(V')
\]

by compactness, we have \( f^{-1}([0,1/k]) \subseteq \tau_Y^{-1}(V') \) for some \( k \in \mathbb{N} \). Now, for each \( n \geq k \), since

\[
Y \cap \tau_Y(f^{-1}([0,1/n]) \setminus f^{-1}([0,1/(n+1)]))
\]

\[
\subseteq Y \cap \tau_Y(f^{-1}([0,1/k])) \subseteq Y \cap \tau_Y(\tau_Y^{-1}(V')) \subseteq Y \cap V' = V \subseteq \operatorname{cl}_Y V
\]

the set

\[
K_n = X \cap (f^{-1}([0,1/n]) \setminus f^{-1}([0,1/(n+1)]))
\]

being closed in \( \operatorname{cl}_Y V \), has \( \mathcal{P} \), and therefore (since being closed in \( X \) it is locally compact and paracompact) it is \( \sigma \)-compact. (It might be helpful to recall that by
Lemma 3.3 the space $\beta Y$ is obtained from $\beta X$ by contracting $\tau_Y^{-1}(p)$ to $p$ with $\tau_Y$ as its quotient mapping.) Thus, the set

$$X \cap f^{-1}([0, 1/k]) = \bigcup_{n=k}^{\infty} K_n$$

is $\sigma$-compact, and therefore, by the definition of $\sigma X$, we have

$$\text{cl}_{\beta X}(X \cap f^{-1}([0, 1/k])) \subseteq \sigma X.$$  

But

$$Z(f) \subseteq f^{-1}([0, 1/k]) \subseteq \text{cl}_{\beta X}f^{-1}([0, 1/k]) = \text{cl}_{\beta X}(X \cap f^{-1}([0, 1/k])) \subseteq \text{cl}_{\beta X}(X \cap f^{-1}([0, 1/k]))$$

from which it follows that $\tau_Y^{-1}(p) \subseteq \sigma X$. Finally, note that by Lemma 3.9 we have $\lambda_P X = \sigma X$.

(2) implies (1). Suppose that (2) holds. By Lemma 3.7 we have $Y \in \mathcal{E}^*(X)$. Therefore, it suffices to verify that $Y$ is locally-$\mathcal{P}$. Also, since by assumption $X$ is locally compact, it is locally-$\mathcal{P}$, as $\mathcal{P}$ is assumed to be a topological property of compact spaces. Thus, we need only to verify that $p$ has an open neighborhood in $Y$ whose closure in $Y$ has $\mathcal{P}$. Let $f : \beta X \to I$ be continuous and such that $Z(f) = \tau_Y^{-1}(p)$. Then since

$$\bigcap_{n=1}^{\infty} g^{-1}([0, 1/n]) = Z(f) \subseteq \lambda_P X$$

by compactness (and since $\lambda_P X$ is open in $\beta X$) we have $g^{-1}([0, 1/k]) \subseteq \lambda_P X$ for some $k \in \mathbb{N}$. Note that by Lemma 3.9 we have $\lambda_P X = \sigma X$. Assume the notation of Notation 2.9. By compactness, we have

$$g^{-1}([0, 1/k]) \subseteq \text{cl}_{\beta X}X_{J_1} \cup \cdots \cup \text{cl}_{\beta X}X_{J_n} = \text{cl}_{\beta X}X_J$$

where $n \in \mathbb{N}$, the sets $J_1, \ldots, J_n \subseteq I$ are countable and $J = J_1 \cup \cdots \cup J_n$. The set $X \cap g^{-1}([0, 1/k]) \subseteq X_J$, being closed in the latter ($\sigma$-compact space) is $\sigma$-compact, and therefore (since being closed in $X$, it is locally compact and paracompact) it has $\mathcal{P}$. Let

$$V = Y \cap \tau_Y\left(g^{-1}([0, 1/k])\right).$$

Then $V$ is an open neighborhood of $p$ in $Y$. We show that $\text{cl}_Y V$ has $\mathcal{P}$. But this follows, since

$$\text{cl}_Y V \subseteq Y \cap \tau_Y\left(g^{-1}([0, 1/k])\right) = \left(X \cap \tau_Y\left(g^{-1}([0, 1/k])\right)\right) \cup \{p\} = \left(X \cap g^{-1}([0, 1/k])\right) \cup \{p\}$$

and the latter, being a finite union of its $\mathcal{P}$-subspaces (note that the singleton $\{p\}$, being compact, has $\mathcal{P}$) has $\mathcal{P}$, and thus, its closed subset $\text{cl}_Y V$, also has $\mathcal{P}$. □

Lemmas 3.12, 3.13 are from [8].

**Lemma 3.12.** Let $X$ be a locally compact paracompact space. If $Z \in \mathcal{Z}(\beta X)$ in non-empty then $Z \cap \sigma X \neq \emptyset$
Proof. Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence in \( \sigma X \). Assume the notation of Notation 2.9. Then \( \{x_n : n \in \mathbb{N}\} \subseteq \text{cl}_{\beta X} X_J \) for some countable \( J \subseteq I \). Therefore \( \{x_n : n \in \mathbb{N}\} \) has a limit point in \( \text{cl}_{\beta X} X_J \subseteq \sigma X \). Thus \( \sigma X \) is countably compact, and therefore is pseudocompact, and \( \nu(\sigma X) = \beta(\sigma X) = \beta X \) (note that the latter equality holds, as \( X \subseteq \sigma X \subseteq \beta X \)). The result now follows, as for any Tychonoff space \( T \), any non-empty zero-set of \( \nu T \) meets \( T \) (see Lemma 5.11 (f) of [10]).

Lemma 3.13. Let \( X \) be a locally compact paracompact space. If \( Z \in \mathcal{Z}(X^*) \) is non-empty then \( Z \cap \sigma X \neq \emptyset \).

Proof. Let \( S \in \mathcal{Z}(\beta X) \) be such that \( S \cap X^* = Z \) (which exists, as \( X^* \) is closed in (the normal space) \( \beta X \), as \( X \) is locally compact, and thus, by the Tietze-Urysohn Theorem, every continuous function from \( X^* \) to \( I \) is continuously extendible over \( \beta X \)). By Lemma 3.12 we have \( S \cap \sigma X \neq \emptyset \). Suppose that \( S \cap (\sigma X \setminus X) = \emptyset \). Then \( S \cap \sigma X = X \cap S \). Assume the notation of Notation 2.9. Let \( J = \{i \in I : X_i \cap S \neq \emptyset\} \). Then \( J \) is finite. Note that since \( X_J \) is clopen in \( X \), it has a clopen closure in \( \beta X \). Now

\[ T = S \cap (\beta X \setminus \text{cl}_{\beta X} X_J) \in \mathcal{Z}(\beta X) \]

misses \( \sigma X \), and therefore, by Lemma 3.12 we have \( T = \emptyset \). But this is a contradiction, as \( Z = S \cap (\beta X \setminus \sigma X) \subseteq T \). This shows that

\[ Z \cap (\sigma X \setminus X) = S \cap (\sigma X \setminus X) \neq \emptyset. \]

Lemma 3.14. Let \( X \) be a locally compact paracompact space. For any \( S, T \in \mathcal{Z}(X^*) \), if \( S \cap \sigma X \subseteq T \cap \sigma X \) then \( S \subseteq T \).

Proof. Suppose to the contrary that \( S \setminus T \neq \emptyset \). Let \( s \in S \setminus T \). Let \( f : \beta X \to I \) be continuous and such that \( f(s) = 0 \) and \( f(T) \subseteq \{1\} \). Then \( Z(f) \cap S \) is non-empty, and thus by Lemma 3.13 it follows that \( Z(f) \cap S \neq \emptyset \). But this is not possible, as

\[ Z(f) \cap S \cap \sigma X \subseteq Z(f) \cap T = \emptyset. \]

The following lemma is from [9].

Lemma 3.15. Let \( X \) and \( Y \) be locally compact spaces. The following are equivalent:

1. \( X^* \) and \( Y^* \) are homeomorphic.
2. \( (\mathcal{E}^C(X), \leq) \) and \( (\mathcal{E}^C(Y), \leq) \) are order-isomorphic.

Proof. This follows from the fact that in a compact space the order-structure of the set of its all zero-sets (partially ordered with \( \subseteq \)) determines its topology.

The proof of the following theorem is essentially a combination of the proofs we have given for Theorems 3.19 and 3.21 in [9] with the appropriate usage of the preceding lemmas. The reasonably detailed proof is included here for the reader’s convenience.

Theorem 3.16. Let \( X \) and \( Y \) be locally compact paracompact (non-compact) spaces and let \( \mathcal{P} \) be a closed hereditary topological property of compact spaces which is preserved under finite sums of subspaces and coincides with \( \sigma \)-compactness in the realm of locally compact paracompact spaces. The following are equivalent:

1. \( \lambda_{\mathcal{P}} X \setminus X \) and \( \lambda_{\mathcal{P}} Y \setminus Y \) are homeomorphic.
(2) \( (\mathcal{E}_{\sigma}^C(X), \leq) \) and \( (\mathcal{E}_{\sigma}^C(Y), \leq) \) are order-isomorphic.
(3) \( (\mathcal{E}_{\text{local-p}}^C(X), \leq) \) and \( (\mathcal{E}_{\text{local-p}}^C(Y), \leq) \) are order-isomorphic.

Proof. Let
\[
X = \bigoplus_{i \in I} X_i \quad \text{and} \quad Y = \bigoplus_{j \in J} Y_j
\]
for some index sets \( I \) and \( J \) with each \( X_i \) and \( Y_j \) for \( i \in I \) and \( j \in J \) being \( \sigma \)-

compact and non-compact. We will use notation of Notation 2.9 and Remark 2.10
without reference. Note that by Lemma 3.9 we have \( \lambda p X = \sigma X \) and \( \lambda p Y = \sigma Y \).

Let
\[
\omega \sigma X = \sigma X \cup \{\Omega\} \quad \text{and} \quad \omega \sigma Y = \sigma Y \cup \{\Omega\}
\]
denote the one-point compactifications of \( \sigma X \) and \( \sigma Y \), respectively.

(1) implies (2). Suppose that (1) holds. Suppose that either \( X \) or \( Y \), say \( X \), is \( \sigma \)-

compact. Then \( \sigma Y \setminus Y \) is compact, as it is homeomorphic to \( \sigma X \setminus X = X^* \), and
the latter is compact, as \( X \) is locally compact. Thus
\[
\sigma Y \setminus Y = Y_{H_1}^* \cup \cdots \cup Y_{H_n}^* = Y_{H}^*
\]
where \( n \in \mathbb{N} \), the sets \( H_1, \ldots, H_n \subseteq J \) are countable and
\[
H = H_1 \cup \cdots \cup H_n.
\]
Now, if there exists some \( u \in J \setminus H \), then since \( Y_u \cap Y_H = \emptyset \) we have
\[
\text{cl}_{\beta Y} Y_u \cap \text{cl}_{\beta Y} Y_H = \emptyset.
\]
Therefore \( \text{cl}_{\beta Y} Y_u \subseteq Y \), contradicting the fact that \( Y_u \) is non-compact. Thus \( J = H \)
and \( Y \) is \( \sigma \)-compact. Therefore \( \sigma Y \setminus Y = Y^* \). Note that by Lemmas 3.8 and 3.10
we have \( \mathcal{E}_{\sigma}^C(X) = \mathcal{E}^C(X) \) and \( \mathcal{E}_{\sigma}^C(Y) = \mathcal{E}^C(Y) \). The result now follows from
Lemma 3.19.

Suppose that \( X \) and \( Y \) are non-\( \sigma \)-compact. Let \( f : \sigma X \setminus X \rightarrow \sigma Y \setminus Y \)
denote a homeomorphism. We define an order-isomorphism
\[
\phi : (\Theta_X(\mathcal{E}_{\sigma}^C(X)), \leq) \rightarrow (\Theta_Y(\mathcal{E}_{\sigma}^C(Y)), \leq).
\]
Since \( \Theta_X \) and \( \Theta_Y \) are anti-order-isomorphisms, this will prove (2). Let \( D \in \Theta_X(\mathcal{E}_{\sigma}^C(X)) \). By Lemma 3.10 we have \( D \in \mathcal{P}(X^*) \) and \( \beta X \setminus X \subseteq D \). Since \( X^* \setminus D \subseteq X^* \), being a cozero-set in \( X^* \) is \( \sigma \)-compact, there exists a countable
\( G \subseteq I \) such that \( X^* \setminus D \subseteq X^*_G \). Now, since \( D \cap X^*_G \in \mathcal{P}(X^*_G) \), we have
\[
f(D \cap X^*_G) \in \mathcal{P}(f(X^*_G)).
\]
Since \( X^*_G \) is open in \( \sigma X \setminus X \), its homeomorphic image \( f(X^*_G) \) is open in \( \sigma Y \setminus Y \), and
thus, is open in \( Y^* \). But \( f(X^*_G) \) is compact, as it is a continuous image of a compact space, and therefore, \( f(X^*_G) \) is clopen in \( Y^* \). Thus
\[
f(D \cap X^*_G) \cup (Y^* \setminus f(X^*_G)) \in \mathcal{P}(Y^*).
\]
Let
\[
\phi(D) = f(D \cap (\sigma X \setminus X)) \cup (\beta Y \setminus \sigma Y).
\]
Note that since
\[
f(D \cap (\sigma X \setminus X)) = f((D \cap X^*_G) \cup ((\sigma X \setminus X) \setminus X^*_G)) = f(D \cap X^*_G) \cup ((\sigma Y \setminus Y) \setminus f(X^*_G))
\]
we have
\[
\phi(D) = f(D \cap (\sigma X \setminus X)) \cup (\beta Y \setminus \sigma Y)
\]
\[
= f(D \cap X^*_G) \cup ((\sigma Y \setminus Y ) \setminus f(X^*_G)) \cup (\beta Y \setminus \sigma Y)
\]
\[
= f(D \cap X^*_G) \cup (Y^* \setminus f(X^*_G))
\]
which shows that \( \phi \) is well-defined. The function \( \phi \) is clearly an order-homomorphism. Since \( \sigma X \setminus X \) also is a homeomorphism, as above, it induces an order-homomorphism
\[
\psi : (\Theta_Y (\sigma_Y (Y)), \subseteq) \rightarrow (\Theta_X (\sigma_X (X)), \subseteq)
\]
which is defined by
\[
\psi(D) = f^{-1}(D \cap (\sigma Y \setminus Y)) \cup (\beta Y \setminus \sigma Y)
\]
for any \( D \in \Theta_Y (\sigma_Y (Y)) \). It is now easy to see that \( \psi = \phi^{-1} \), which shows that \( \phi \) is an order-isomorphism.

(2) implies (1). Suppose that (2) holds. Suppose that either \( X \) or \( Y \), say \( X \), is \( \sigma \)-compact (and non-compact). Then \( \sigma X = \beta X \), and thus, by Lemmas 3.8 and 3.10 we have \( \sigma_X (X) = \sigma_X (X) \). Suppose that \( Y \) is non-\( \sigma \)-compact. Note that \( X \), being paracompact and non-compact, is non-pseudocompact (see Theorems 3.10.21, 5.1.5 and 5.1.20 of [3]) and therefore, \( X^* \) contains at least two elements, as almost compact spaces are pseudocompact (see Problem 5U (1) of [16]; recall that a Tychonoff space \( T \) is called almost compact if \( \beta T \setminus T \) has at most one element). Thus, there exist two disjoint non-empty zero-sets of \( X^* \) corresponding to two elements in \( \sigma_X (X) \) with no common upper bound in \( \sigma_X (X) \). But this is not true, as \( \sigma_X (X) \) is order-isomorphic to \( \sigma_Y (Y) \), and any two elements in the latter have a common upper bound in \( \sigma_Y (Y) \). (Note that since \( Y \) is non-\( \sigma \)-compact, the set \( \beta Y \setminus \sigma Y \) is non-empty, and by Lemma 3.10 the image of any element in \( \sigma_Y (Y) \) under \( \Theta_Y \) contains \( \beta Y \setminus \sigma Y \).) Therefore, \( Y \) also is \( \sigma \)-compact and by Lemmas 3.8 and 3.10 we have \( \sigma_Y (Y) = \sigma_X (X) \). Now, since \( \sigma Y = \beta Y \), the result follows from Lemma 3.10.

Next, suppose that \( X \) and \( Y \) are both non-\( \sigma \)-compact. We show that the two compact spaces \( \omega X \setminus X \) and \( \omega Y \setminus Y \) are homeomorphic, by showing that their corresponding sets of zero-sets (partially ordered with \( \subseteq \)) are order-isomorphic. Since \( \Theta_X \) and \( \Theta_Y \) are anti-order-isomorphisms, condition (2) implies the existence of an order-isomorphism
\[
\phi : (\Theta_X (\sigma_X (X)), \subseteq) \rightarrow (\Theta_Y (\sigma_Y (Y)), \subseteq)
\]
We define an order-isomorphism
\[
\psi : (\mathcal{P}(\omega X \setminus X), \subseteq) \rightarrow (\mathcal{P}(\omega Y \setminus Y), \subseteq)
\]
as follows. Let \( Z \in \mathcal{P}(\omega X \setminus X) \). Suppose that \( \Omega \in Z \). Then, since \( (\omega X \setminus X) \setminus Z \) is a cozero-set in (the compact space) \( \omega X \setminus X \), it is \( \sigma \)-compact. Thus \( (\omega X \setminus X) \setminus Z \subseteq X^*_G \) for some countable \( G \subseteq I \). Since \( X^*_G \) is clopen in \( X^* \), we have
\[
(Z \setminus \Omega) \cup (\beta X \setminus \sigma X) = (Z \setminus X^*_G) \cup (X^* \setminus X^*_G) \in \mathcal{P}(X^*).
\]
In this case, we let
\[
\psi(Z) = (\phi((Z \setminus \Omega)) \cup (\beta X \setminus \sigma X)) \cup (\beta Y \setminus \sigma Y) \cup \{\Omega\}.
\]
Now, suppose that \( \Omega \notin Z \). Then \( Z \subseteq \sigma X \setminus X \), and therefore \( Z \subseteq X^{\ast} \) for some countable \( G \subseteq I \), and thus, using this, one can write

\[
(3.2) \quad Z = X^{\ast} \setminus \bigcup_{n=1}^{\infty} Z_n \text{ where } \beta X \setminus \sigma X \subseteq Z_n \in \mathcal{P}(X^{\ast}) \text{ for any } n \in \mathbb{N}.
\]

In this case, we let

\[
\psi(Z) = Y^{\ast} \setminus \bigcup_{n=1}^{\infty} \phi(Z_n).
\]

We check that \( \psi \) is well-defined. Assume the representation given in (3.2). Since \( Y^{\ast} \setminus \phi(Z_n) \subseteq \sigma Y \) for all \( n \in \mathbb{N} \), there exists a countable \( H \subseteq J \) such that \( Y^{\ast} \setminus \phi(Z_n) \subseteq Y^*_H \) for all \( n \in \mathbb{N} \).

**Claim 3.17.** For a \( Z \in \mathcal{P}(\omega \sigma X \setminus X) \) with \( \Omega \notin Z \) assume the representation given in (3.2). Let \( H \subseteq J \) be countable and such that \( Y^{\ast} \setminus \phi(Z_n) \subseteq Y^*_H \) for all \( n \in \mathbb{N} \). Let \( A \) be such that \( \phi(A) = Y^{\ast} \setminus Y^*_H \). Then

\[
Y^{\ast} \setminus \bigcup_{n=1}^{\infty} \phi(Z_n) = \phi(A \cup Z) \setminus \phi(A).
\]

**Proof of the claim.** Suppose that \( y \in Y^{\ast} \) and \( y \notin \phi(Z_n) \) for all \( n \in \mathbb{N} \). If \( y \notin \phi(A \cup Z) \setminus \phi(A) \), then since \( y \notin \phi(Z) \) we have \( y \notin \phi(A \cup Z) \). Therefore, there exists some \( B \in \mathcal{P}(Y^{\ast}) \) containing \( y \) such that \( B \cap \phi(A \cup Z) = \emptyset \) and \( B \cap \phi(Z_n) = \emptyset \) for all \( n \in \mathbb{N} \). Let \( C \) be such that \( \phi(C) = B \cup \phi(A \cup Z) \), and let \( S_n \) for any \( n \in \mathbb{N} \), be such that

\[
\phi(S_n) = \phi(C) \cap \phi(Z_n) = (B \cup \phi(A \cup Z)) \cap \phi(Z_n) = \phi(A \cup Z) \cap \phi(Z_n).
\]

Since \( A \subseteq Z_n \), as \( \phi(A) \subseteq \phi(Z_n) \) and \( Z \cap Z_n = \emptyset \), we have \( A \cap Z = \emptyset \), which implies that

\[
(A \cup Z) \cap Z_n = (A \cap Z_n) \cup (Z \cap Z_n) = A
\]

for all \( n \in \mathbb{N} \). Clearly \( S_n \subseteq (A \cup Z) \cap Z_n \), as by above \( \phi(S_n) \subseteq \phi(A \cup Z) \) and \( \phi(S_n) \subseteq \phi(Z_n) \) for any \( n \in \mathbb{N} \). Thus, \( \phi(S_n) \subseteq \phi(A) \) for all \( n \in \mathbb{N} \). But since \( \phi(A) \subseteq \phi(Z_n) \), we have \( \phi(A) \subseteq \phi(S_n) \), and therefore

\[
\phi(C \cap Z_n) \subseteq \phi(C) \cap \phi(Z_n) = \phi(S_n) \subseteq \phi(A)
\]

for any \( n \in \mathbb{N} \). This implies that \( C \cap Z_n \subseteq A \) for all \( n \in \mathbb{N} \). Thus

\[
C \setminus Z = C \cap \bigcup_{n=1}^{\infty} Z_n = \bigcup_{n=1}^{\infty} (C \cap Z_n) \subseteq A.
\]

Therefore \( C \subseteq A \cup Z \) and we have \( B \subseteq \phi(C) \subseteq \phi(A \cup Z) \), which is a contradiction, as \( B \cap \phi(A \cup Z) = \emptyset \). This shows that

\[
Y^{\ast} \setminus \bigcup_{n=1}^{\infty} \phi(Z_n) \subseteq \phi(A \cup Z) \setminus \phi(A).
\]

Now, suppose that \( y \in \phi(A \cup Z) \setminus \phi(A) \). Suppose to the contrary that \( y \in \phi(Z_n) \) for some \( n \in \mathbb{N} \). Then

\[
y \in \phi(Z_n) \cap \phi(A \cup Z) = \phi(D)
\]
for some $D$. Clearly $D \subseteq Z_n$ and $D \subseteq A \cup Z$, as $\phi(D) \subseteq \phi(Z_n)$ and $\phi(D) \subseteq \phi(A \cup Z)$. This implies that

$$D \subseteq Z_n \cap (A \cup Z) = (Z_n \cap A) \cup (Z_n \cap Z) = Z_n \cap A \subseteq A$$

and thus $y \in \phi(A)$, as $\phi(D) \subseteq \phi(A)$, which is a contradiction. This proves the claim.

Now, suppose that

$$S = X^* \setminus \bigcup_{n=1}^{\infty} S_n \quad \text{and} \quad Z = X^* \setminus \bigcup_{n=1}^{\infty} Z_n$$

are two representations for some $Z \in \mathcal{P}(\omega \sigma X \setminus X)$ with $\Omega \notin Z$ such that each $S_n, Z_n \in \mathcal{P}(X^*)$ contains $\beta X \setminus \sigma X$ for $n \in \mathbb{N}$. Choose a countable $H \subseteq J$ such that

$$Y^* \setminus \phi(S_n) \subseteq Y^*_H \quad \text{and} \quad Y^* \setminus \phi(Z_n) \subseteq Y^*_H$$

for all $n \in \mathbb{N}$. Then, by the claim, we have

$$Y^* \setminus \bigcup_{n=1}^{\infty} \phi(S_n) = \phi(A \cup Z) \setminus \phi(A) = Y^* \setminus \bigcup_{n=1}^{\infty} \phi(Z_n)$$

where $A$ is such that $\phi(A) = Y^* \setminus Y^*_H$. This shows that $\psi$ is well-defined. Next, we show that $\psi$ is an order-isomorphism. Suppose that $S, Z \in \mathcal{P}(\omega \sigma X \setminus X)$ and $S \subseteq Z$. We consider the following cases.

**Case 1:** Suppose that $\Omega \in S$. Then $\Omega \in Z$, and clearly

$$\psi(S) = (\phi\left(\{S\setminus\{\Omega\}\} \cup (\beta X \setminus \sigma X)\right) \setminus \{\Omega\}) \cup \{\Omega\}$$

$$\subseteq (\phi\left(\{Z\setminus\{\Omega\}\} \cup (\beta X \setminus \sigma X)\right) \setminus \{\Omega\}) \cup \{\Omega\} = \psi(Z).$$

**Case 2:** Suppose that $\Omega \notin S$ but $\Omega \in Z$. Let

$$E = \phi\left(\{Z\setminus\{\Omega\}\} \cup (\beta X \setminus \sigma X)\right)$$

and let

$$S = X^* \setminus \bigcup_{n=1}^{\infty} S_n$$

where each $S_n \in \mathcal{P}(X^*)$ contains $\beta X \setminus \sigma X$ for $n \in \mathbb{N}$. Clearly $Y^* \setminus E \subseteq \sigma Y$. Let $H \subseteq J$ be countable and such that $Y^* \setminus \phi(S_n) \subseteq Y^*_H$ for all $n \in \mathbb{N}$ and $Y^* \setminus E \subseteq Y^*_H$. By the claim, we have $\psi(S) = \phi(A \cup S) \setminus \phi(A)$, where $\phi(A) = Y^*_H \setminus Y^*_H$. Since $Y^*_H \setminus Y^*_H \subseteq E$, we have

$$A \subseteq (Z \setminus \{\Omega\}) \cup (\beta X \setminus \sigma X).$$

Now

$$\psi(S) = \phi(A \cup S) \setminus \phi(A) \subseteq \phi(A \cup S) \subseteq \phi\left(\{Z\setminus\{\Omega\}\} \cup (\beta X \setminus \sigma X)\right)$$

which implies that

$$\psi(S) \subseteq (\phi\left(\{Z\setminus\{\Omega\}\} \cup (\beta X \setminus \sigma X)\right) \setminus \{\Omega\}) \cup \{\Omega\} = \psi(Z).$$

**Case 3:** Suppose that $\Omega \notin Z$. Then $\Omega \notin S$. Let

$$S = X^* \setminus \bigcup_{n=1}^{\infty} S_n \quad \text{and} \quad Z = X^* \setminus \bigcup_{n=1}^{\infty} Z_n$$
where each \( S_n, Z_n \in \mathcal{G}(X^*) \) contains \( \beta X \setminus \sigma X \) for \( n \in \mathbb{N} \). Clearly,

\[
S = S \cap Z = (X^* \setminus \bigcup_{n=1}^{\infty} S_n) \cap (X^* \setminus \bigcup_{n=1}^{\infty} Z_n) = X^* \setminus \bigcup_{n=1}^{\infty} (S_n \cup Z_n)
\]

and thus, since \( \phi(Z_n) \subseteq \phi(S_n \cup Z_n) \) for all \( n \in \mathbb{N} \), it follows that

\[
\psi(S) = Y^* \setminus \bigcup_{n=1}^{\infty} \phi(S_n \cup Z_n) \subseteq Y^* \setminus \bigcup_{n=1}^{\infty} \phi(Z_n) = \psi(Z).
\]

Note that since

\[
\phi^{-1} : (\Theta_Y(\mathcal{G}^*(Y)), \subseteq) \to (\Theta_X(\mathcal{G}^*(X)), \subseteq)
\]

also is an order-isomorphism, as above, it induces an order-isomorphism

\[
\gamma : (\mathcal{G}(\omega \sigma Y \setminus Y), \subseteq) \to (\mathcal{G}(\omega \sigma X \setminus X), \subseteq)
\]

which is easy to see that \( \gamma = \psi^{-1} \). Therefore, \( \psi \) is an order-isomorphism. It then follows that there exists a homeomorphism \( f : \omega \sigma X \setminus X \to \omega \sigma Y \setminus Y \) such that \( f(Z) = \psi(Z) \), for any \( Z \in \mathcal{G}(\omega \sigma X \setminus X) \). Now since for each countable \( G \subseteq I \) we have

\[
f(X_G) = \psi(X_G) \subseteq \sigma Y \setminus Y
\]

it follows that \( f(\sigma X \setminus X) = \sigma Y \setminus Y \). Thus \( \sigma X \setminus X \) and \( \sigma Y \setminus Y \) are homeomorphic.

(1) implies (3). Suppose that (1) holds. Suppose that either \( X \) or \( Y \), say \( X \), is \( \sigma \)-compact. Then \( \sigma X = \beta X \) and thus, arguing as in part (1) \( \Rightarrow \) (2), it follows that \( Y \) also is \( \sigma \)-compact. Therefore, \( \sigma Y = \beta Y \). Note that by Lemmas 3.7 and 3.11 we have \( \mathcal{G}^*_{\text{local}-p}(X) = \mathcal{G}^*(X) \) and since \( X^* \in \mathcal{G}(\beta X) \) (as \( X \) is \( \sigma \)-compact and locally compact; see 1B of [19]) by Lemmas 3.7 and 3.8 we have \( \mathcal{G}^*(X) = \mathcal{G}^C(X) \). Thus \( \mathcal{G}^*_{\text{local}-p}(X) = \mathcal{G}^C(X) \) and similarly \( \mathcal{G}^*_{\text{local}-p}(Y) = \mathcal{G}^C(Y) \). The result now follows from Lemma 3.11.

Suppose that \( X \) and \( Y \) are non-\( \sigma \)-compact. Let \( f : \sigma X \setminus X \to \sigma Y \setminus Y \) be a homeomorphism. We define an order-isomorphism

\[
\phi : (\Theta_X(\mathcal{G}^*_{\text{local}-p}(X)), \subseteq) \to (\Theta_Y(\mathcal{G}^*_{\text{local}-p}(Y)), \subseteq)
\]

as follows. Let \( Z \in \Theta_X(\mathcal{G}^*_{\text{local}-p}(X)) \). By Lemma 3.11 we have \( Z \in \mathcal{G}(\beta X) \) and \( Z \subseteq \sigma X \setminus X \). Thus \( Z \subseteq X_G^G \) for some countable \( G \subseteq I \). Now \( f(Z) \in \mathcal{G}(\sigma Y \setminus Y) \) and since \( f(Z) \) is compact, as it is a continuous image of a compact space, it follows that \( f(Z) \subseteq Y_H^G \) for some countable \( H \subseteq J \). Therefore \( f(Z) \in \mathcal{G}(Y_H^G) \) and then \( f(Z) \in \mathcal{G}(\text{cl}_{\beta Y} Y_H) \). Since \( \text{cl}_{\beta Y} Y_H \) is clopen in \( \beta Y \) we have \( f(Z) \in \mathcal{G}(\beta Y) \). Define

\[
\phi(Z) = f(Z).
\]

It is obvious that \( \phi \) is an order-homomorphism. If we let

\[
\psi : (\Theta_Y(\mathcal{G}^*_{\text{local}-p}(Y)), \subseteq) \to (\Theta_X(\mathcal{G}^*_{\text{local}-p}(X)), \subseteq)
\]

be defined by

\[
\psi(Z) = f^{-1}(Z)
\]

for any \( Z \in \Theta_Y(\mathcal{G}^*_{\text{local}-p}(Y)) \), then \( \psi = \phi^{-1} \) which shows that \( \phi \) is an order-isomorphism.

(3) implies (1). Suppose that (3) holds. Suppose that either \( X \) or \( Y \), say \( X \), is \( \sigma \)-compact (and non-compact). Then \( \sigma X = \beta X \), and thus, by Lemmas 3.7 and 3.11 we have \( \mathcal{G}^*_{\text{local}-p}(X) = \mathcal{G}^*(X) \). Therefore, since \( X^* \in \mathcal{G}(\beta X) \) the set \( \mathcal{G}^*_{\text{local}-p}(X) \) has the smallest element (namely, its one-point compactification \( \omega X \)).
Thus $\mathcal{E}_{\text{local-}P}(Y)$ also has the smallest element; denote this element by $T$. Then, for each countable $H \subseteq J$ we have

$$Y_H^* \in \Theta_Y(\mathcal{E}_{\text{local-}P}(Y))$$

and therefore $\sigma Y \setminus Y \subseteq \Theta_Y(T)$. By Lemma 3.14 (with $\Theta_Y(T)$ and $Y^*$ as the zero-sets in its statement) we have $Y^* \subseteq \Theta_Y(T)$. This implies that $Y^* \in \mathcal{Z}(\beta Y)$ which shows that $Y$ is $\sigma$-compact. Thus $\sigma Y = \beta Y$, and by Lemmas 3.7 and 3.11 we have $\mathcal{E}_{\text{local-}P}(Y) = \mathcal{E}(Y)$. Therefore, in this case (and since by Lemmas 3.7 and 3.8 we have $\mathcal{E}(X) = \mathcal{E}(X)$ and $\mathcal{E}(Y) = \mathcal{E}(Y)$) the result follows from Lemma 3.15.

Next, suppose that $X$ and $Y$ are both non-$\sigma$-compact. Since $\Theta_X$ and $\Theta_Y$ are both anti-order-isomorphisms, there exists an order-isomorphism

$$\phi : (\Theta_X(\mathcal{E}_{\text{local-}P}(X)), \subseteq ) \to (\Theta_Y(\mathcal{E}_{\text{local-}P}(Y)), \subseteq ).$$

We extend $\phi$ by letting $\phi(\emptyset) = \emptyset$. We define a function

$$\psi : (\mathcal{Z}(\omega\sigma X \setminus X), \subseteq ) \to (\mathcal{Z}(\omega\sigma Y \setminus Y), \subseteq )$$

and verify that it is an order-isomorphism. Let $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$ with $\Omega \in Z$. Since $Z \subseteq X_G^*$ for some countable $G \subseteq I$, we have $Z \in \mathcal{Z}(\beta X)$, and therefore

$$Z \in \Theta_X(\mathcal{E}_{\text{local-}P}(X)) \cup \{\emptyset\}.$$ 

In this case, let

$$\psi(Z) = \phi(Z).$$

Now, suppose that $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$ and $\Omega \in Z$. Then $(\omega\sigma X \setminus X) \setminus Z$ is a cozero-set in $\omega\sigma X \setminus X$, and we have

$$Z = (\omega\sigma X \setminus X) \setminus \bigcup_{n=1}^{\infty} Z_n$$

where $Z_n \in \mathcal{Z}(\omega\sigma X \setminus X)$ for any $n \in \mathbb{N}$.

Thus, as above, it follows that

$$Z_n \in \Theta_X(\mathcal{E}_{\text{local-}P}(X)) \cup \{\emptyset\}$$

for any $n \in \mathbb{N}$. We verify that

$$\bigcup_{n=1}^{\infty} \phi(Z_n) \in \text{Coz}(\omega\sigma Y \setminus Y).$$

To show this, note that since $\phi(Z_n) \subseteq \sigma Y \setminus Y$ there exists a countable $H \subseteq J$ such that $\phi(Z_n) \subseteq Y_H^*$ for all $n \in \mathbb{N}$.

**Claim 3.18.** For a $Z \in \mathcal{Z}(\omega\sigma X \setminus X)$ with $\Omega \in Z$ assume the representation given in (3.3). Let $H \subseteq J$ be countable and such that $\phi(Z_n) \subseteq Y_H^*$ for all $n \in \mathbb{N}$. Let $A$ be such that $\phi(A) = Y_H^*$. Then

$$\phi(A \cap Z) = \phi(A) \setminus \bigcup_{n=1}^{\infty} \phi(Z_n).$$
Proof of the claim. For each \( n \in \mathbb{N} \), since \( A \cap Z \cap Z_n = \emptyset \), we have \( \phi(A \cap Z) \cap \phi(Z_n) = \emptyset \), as otherwise, \( \phi(A \cap Z) \) and \( \phi(Z_n) \) will have a common lower bound in \( \Theta_Y(\ell^{*}_{\text{local-p}}(Y)) \), that is, \( \phi(A \cap Z) \cap \phi(Z_n) \), whereas \( A \cap Z \) and \( Z_n \) do not have. Also \( \phi(A \cap Z) \subseteq \phi(A) \). Therefore
\[
\phi(A \cap Z) \subseteq \phi(A) \setminus \bigcup_{n=1}^{\infty} \phi(Z_n).
\]
To show the reverse inclusion, let \( y \in \phi(A) \) be such that \( y \notin \phi(Z_n) \) for all \( n \in \mathbb{N} \). There exists some \( B \in \mathcal{Z}(\beta Y) \) such that \( y \in B \) and \( B \cap \phi(Z_n) = \emptyset \) for all \( n \in \mathbb{N} \). If \( y \notin \phi(A \cap Z) \), then there exists some \( C \in \mathcal{Z}(\beta Y) \) such that \( y \in C \) and \( C \cap \phi(A \cap Z) = \emptyset \). Let \( D = \phi(A) \cap B \cap C \) and let \( E \) be such that \( \phi(E) = D \). For each \( n \in \mathbb{N} \), since \( \phi(E) \cap \phi(Z_n) = \emptyset \), we have \( E \cap Z_n = \emptyset \), and thus \( E \subseteq Z \). On the other hand, since \( \phi(E) \subseteq \phi(A) \) we have \( E \subseteq A \), and therefore \( E \subseteq A \cap Z \). Thus \( \phi(E) \subseteq \phi(A \cap Z) \), which implies that \( \phi(E) = \emptyset \), as \( \phi(E) \subseteq C \). This contradiction shows that \( y \in \phi(A \cap Z) \), which proves the claim.

Let \( A \) be such that \( \phi(A) = Y_N \). Now, \( \phi(A \cap Z) \in \mathcal{Z}(\omega \sigma Y \setminus Y) \), as \( \phi(A \cap Z) \subseteq \phi(A) \). By the claim we have

\[
(\omega \sigma Y \setminus Y) \setminus \bigcup_{n=1}^{\infty} \phi(Z_n) = \left( \phi(A) \setminus \bigcup_{n=1}^{\infty} \phi(Z_n) \right) \cup ((\omega \sigma Y \setminus Y) \setminus \phi(A))
\]

\[
= \phi(A \cap Z) \cup ((\omega \sigma Y \setminus Y) \setminus \phi(A)) \in \mathcal{Z}(\omega \sigma Y \setminus Y)
\]

and (3.4) is verified. In this case, we let

\[
\psi(Z) = (\omega \sigma Y \setminus Y) \setminus \bigcup_{n=1}^{\infty} \phi(Z_n).
\]

Next, we show that \( \psi \) is well-defined. Assume that

\[
Z = (\omega \sigma X \setminus X) \setminus \bigcup_{n=1}^{\infty} S_n
\]

with \( S_n \in \mathcal{Z}(\omega \sigma X \setminus X) \) for all \( n \in \mathbb{N} \), is another representation of \( Z \). We need to show that

\[
(3.5) \quad \bigcup_{n=1}^{\infty} \phi(Z_n) = \bigcup_{n=1}^{\infty} \phi(S_n).
\]

Without any loss of generality, suppose to the contrary that there exists some \( m \in \mathbb{N} \) and \( y \in \phi(Z_m) \) such that \( y \notin \phi(S_n) \) for all \( n \in \mathbb{N} \). Then there exists some \( A \in \mathcal{Z}(\beta Y) \) such that \( y \in A \) and \( A \cap \phi(S_n) = \emptyset \) for all \( n \in \mathbb{N} \). Consider

\[
A \cap \phi(Z_m) \in \Theta_Y(\ell^{*}_{\text{local-p}}(Y)).
\]

Let \( B \) be such that \( \phi(B) = A \cap \phi(Z_m) \). Since \( \phi(B) \subseteq A \) we have \( \phi(B) \cap \phi(S_n) = \emptyset \) from which it follows that \( B \cap S_n = \emptyset \) for all \( n \in \mathbb{N} \). But \( B \subseteq Z_m \), as \( \phi(B) \subseteq \phi(Z_m) \), and we have

\[
B \subseteq \bigcup_{n=1}^{\infty} Z_n = \bigcup_{n=1}^{\infty} S_n
\]

which implies that \( B = \emptyset \). But this is a contradiction, as \( \phi(B) \neq \emptyset \). Therefore (3.5) holds, and thus \( \psi \) is well-defined. To prove that \( \psi \) is an order-isomorphism, let \( S, Z \in \mathcal{Z}(\omega \sigma X \setminus X) \) and \( S \subseteq Z \). The case when \( S = \emptyset \) holds trivially. Assume that \( S \neq \emptyset \). We consider the following cases.
Case 1: Suppose that $\Omega \notin Z$. Then $\Omega \notin S$ and we have
\[ \psi(S) = \phi(S) \subseteq \phi(Z) = \psi(Z). \]

Case 2: Suppose that $\Omega \notin S$ but $\Omega \in Z$. Let
\[ Z = (\omega \sigma X \setminus X) \bigcup_{n=1}^{\infty} Z_n \]
with $Z_n \in \mathcal{P}(\omega \sigma X \setminus X)$ for all $n \in \mathbb{N}$. Then, since $S \subseteq Z$ we have $S \cap Z_n = \emptyset$, and therefore $\phi(S) \cap \phi(Z_n) = \emptyset$ for all $n \in \mathbb{N}$. Thus
\[ \psi(S) = \phi(S) \subseteq (\omega \sigma Y \setminus Y) \bigcup_{n=1}^{\infty} \phi(Z_n) = \psi(Z). \]

Case 3: Suppose that $\Omega \in S$. Then $\Omega \in Z$. Let
\[ S = (\omega \sigma X \setminus X) \bigcup_{n=1}^{\infty} S_n \text{ and } Z = (\omega \sigma X \setminus X) \bigcup_{n=1}^{\infty} Z_n \]
where $S_n, Z_n \in \mathcal{P}(\omega \sigma X \setminus X)$ for all $n \in \mathbb{N}$. Therefore
\[ S = S \cap Z = ((\omega \sigma X \setminus X) \bigcup_{n=1}^{\infty} S_n) \cap ((\omega \sigma X \setminus X) \bigcup_{n=1}^{\infty} Z_n) \]
\[ = (\omega \sigma X \setminus X) \bigcup_{n=1}^{\infty} (S_n \cup Z_n). \]

Thus, since $\phi(Z_n) \subseteq \phi(S_n \cup Z_n)$ for all $n \in \mathbb{N}$, we have
\[ \psi(S) = (\omega \sigma Y \setminus Y) \bigcup_{n=1}^{\infty} \phi(S_n \cup Z_n) \subseteq (\omega \sigma Y \setminus Y) \bigcup_{n=1}^{\infty} \phi(Z_n) = \psi(Z). \]

This shows that $\psi$ is an order-homomorphism. To show that $\psi$ is an order-isomorphism, we note that
\[ \phi^{-1} : (\Theta_Y(\mathcal{E}_{\text{local}}^*(Y)), \subseteq) \to (\Theta_X(\mathcal{E}_{\text{local}}^*(X)), \subseteq) \]
is an order-isomorphism. Let
\[ \gamma : (\mathcal{P}(\omega \sigma Y \setminus Y), \subseteq) \to (\mathcal{P}(\omega \sigma X \setminus X), \subseteq) \]
be the induced order-homomorphism which is defined as above. Then it is straightforward to see that $\gamma = \psi^{-1}$, that is, $\psi$ is an order-isomorphism. This implies the existence of a homeomorphism $f : \omega \sigma X \setminus X \to \omega \sigma Y \setminus Y$ such that $f(Z) = \psi(Z)$ for every $Z \in \mathcal{P}(\omega \sigma X \setminus X)$. Therefore, for any countable $G \subseteq I$, since $X_G^* \in \mathcal{P}(\omega \sigma X \setminus X)$, we have
\[ f(X_G^*) = \psi(X_G^*) = \phi(X_G^*) \subseteq \sigma Y \setminus Y. \]
Thus $f(\sigma X \setminus X) \subseteq \sigma Y \setminus Y$, which shows that $f(\Omega) = \Omega'$. Therefore $\sigma X \setminus X$ and $\sigma Y \setminus Y$ are homeomorphic. \qed

Example 3.19. The Lindelöf property and the linearly Lindelöf property (besides $\sigma$-compactness itself) are examples of topological properties $\mathcal{P}$ satisfying the assumption of Theorem 3.16. To see this, let $X$ be a locally compact paracompact space. Assume a representation for $X$ as in Notation 3.16. Recall that a Hausdorff space $X$ is said to be linearly Lindelöf if provided that every linearly ordered (by set inclusion $\subseteq$) open cover of $X$ has a countable subcover, equivalently, if every
For the converse, note that if $X$ is an uncountable subset of $X$ has a complete accumulation point in $X$. (Recall that a point $x \in X$ is called a complete accumulation point of a set $A \subseteq X$ if for every neighborhood $U$ of $x$ in $X$ we have $|U \cap A| = |A|$.) Note that if $X$ is non-$\sigma$-compact then (using the notation of Notation 2.9) the set $I$ is uncountable. Let $A = \{x_i : i \in I\}$ where $x_i \in X_i$ for each $i \in I$. Then $A$ is an uncountable subset of $X$ without (even) accumulation points. Thus $X$ cannot be linearly Lindelöf as well. For the converse, note that if $X$ is not linearly Lindelöf, then, obviously, $X$ is not Lindelöf, and therefore, is non-$\sigma$-compact, as it is well-known that $\sigma$-compactness and the Lindelöf property coincide in the realm of locally compact paracompact spaces (this fact is evident from the representation given for $X$ in Notation 2.9).

Theorem 3.16 might leave the impression that $(\mathcal{E}_P^C(X), \leq)$ and $(\mathcal{E}_{\sigma_{\mathcal{P}}}(X), \leq)$ are order-isomorphic. The following is to settle this, showing that in most cases this indeed is not going to be the case.

**Theorem 3.20.** Let $X$ be a locally compact paracompact (non-compact) space and let $\mathcal{P}$ be a closed hereditary topological property of compact spaces which is preserved under finite sums of subspaces and coincides with $\sigma$-compactness in the realm of locally compact paracompact spaces. The following are equivalent:

1. $X$ is $\sigma$-compact.
2. $(\mathcal{E}_P^C(X), \leq)$ and $(\mathcal{E}_{\sigma_{\mathcal{P}}}(X), \leq)$ are order-isomorphic.

**Proof.** Since $X$ is locally compact, the set $X^*$ is closed in (the normal space) $\beta X$ and thus, using the Tietze-Urysohn Theorem, every zero-set of $X^*$ is extendible to a zero-set of $\beta X$. Now if $X$ is $\sigma$-compact (since $X$ is also locally compact) we have $X^* \in \mathcal{Z}(\beta X)$ and therefore every zero-set of $X^*$ is a zero-set of $\beta X$. Note that $X_{\mathcal{P}} = \sigma X = \beta X$. Thus using Lemmas 3.10 and 3.11 we have

$$\Theta_X(\mathcal{E}_P^C(X)) = \mathcal{Z}(X^*) \setminus \{\emptyset\} = \Theta_X(\mathcal{E}_{\sigma_{\mathcal{P}}}(X))$$

from which it follows that

$$\mathcal{E}_P^C(X) = \mathcal{E}_{\sigma_{\mathcal{P}}}(X).$$

If $X$ is non-$\sigma$-compact, then any two elements of $\mathcal{E}_P^C(X)$ has a common upper bound while this is not the case for $\mathcal{E}_{\sigma_{\mathcal{P}}}(X)$. To see this, note that by Lemma 3.10 the set $\Theta_X(\mathcal{E}_P^C(X))$ is closed under finite intersections (note that the finite intersections are non-empty, as they contain $\beta X \setminus \sigma X$ and the latter is non-empty, as $X$ is non-$\sigma$-compact) while there exist (at least) two elements in $\Theta_X(\mathcal{E}_{\sigma_{\mathcal{P}}}(X))$ with empty intersection; simply consider $X^*_i$ and $X^*_j$ for some distinct $i, j \in I$ (we are assuming the representation for $X$ given in Notation 2.9).

We conclude this article with the following.

**Project 3.21.** Let $X$ be a (locally compact paracompact) space and let $\mathcal{P}$ be a (closed hereditary) topological property (of compact spaces which is preserved under finite sums of subspaces and coincides with $\sigma$-compactness in the realm of locally compact paracompact spaces). Explore the relationship between the order structures of $(\mathcal{E}_P^C(X), \leq)$ and $(\mathcal{E}_{\sigma_{\mathcal{P}}}(X), \leq)$.

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