Operator Algebras Generated by Left Invertibles

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Abstract. Operator algebras generated by partial isometries and their adjoints form the basis for some of the most well studied classes of C*-algebras. Motivated by questions from linear equations in Hilbert spaces (frame theory), we instigate a research program on concrete operator algebras that arise from directed graphs. In this paper, we consider the norm-closed operator algebra generated by a left invertible $T$ together with its Moore-Penrose inverse $T^\dagger$. In the isometric case, $\mathfrak{A}_T$ is a representation of the Toeplitz algebra. Of particular interest is the case when $T$ is analytic in the sense of Shimorin. We show that $T$ is analytic if and only if $T^\ast$ is Cowen-Douglas. When $T$ is analytic with Fredholm index $-1$, the algebra $\mathfrak{A}_T$ contains the compact operators, and any two such algebras are boundedly isomorphic if and only if they are similar.

1. Introduction

For a Hilbert space $\mathcal{H}$, $y \in \mathcal{H}$ and $T \in B(\mathcal{H})$, the equation

$$Tx = y$$

has a solution whenever $T$ is left invertible. For any left inverse $L$ of $T$, $x = Ly$ is a solution. More generally, if $T$ has closed range, $T$ has a pseudo left inverse. Concretely, if $T \in B(\mathcal{H})$ has closed range, restricting to $\ker(T)^\perp$ we obtain a new operator that is left invertible. Picking a left inverse $L$ for $T|_{\ker(T)^\perp}$, we find that $x = Ly$ is a solution.

In most applications, there is a canonical choice for the left inverse $L$. Generally, this is the Moore-Penrose inverse, denoted $T^\dagger$. It is the unique left inverse such that $\ker(T^\dagger) = \text{ran}(T)^\perp$. To construct $T^\dagger$, one considers $T$ mapping $\ker(T)^\perp$ to $\text{ran}(T)$. From this perspective, $T$ is bijective, and thus boundedly invertible by the open mapping theorem. The unique inverse, once extended by zero, is $T^\dagger$. For instance, if $T$ is an isometry then $T^\dagger = T^\ast$, the adjoint of $T$.

One subject where closed range operators are frequently used is frame theory. A sequence $\{f_n\}$ in a Hilbert space $\mathcal{H}$ is said to be a frame if there exists constants $0 < A \leq B$ such that

$$A\|x\|^2 \leq \sum_n |\langle x, f_n \rangle|^2 \leq B\|x\|^2$$

for all $x \in \mathcal{H}$. Associated to each Hilbert space frame $\{f_n\}$ is a (canonical) dual frame $\{g_n\}$. Using this dual frame, one can reconstruct elements $f$ of the Hilbert space $\mathcal{H}$ in an analogous way to orthonormal basis:

$$f = \sum_{n \geq 1} \langle f, g_n \rangle f_n$$

It is easy to see that orthonormal bases are frames, but not all frames need be orthogonal, norm one, or even contain a unique set of elements. A frame does not enforce the rigidity of inner products that an orthonormal basis does - allowing for variation between individual frame elements (rather than just 0 or 1). The flexibility of the definition has engendered applications across signal processing and harmonic analysis. For instance, frames may be constructed for particular features of a problem that are precluded from orthogonal bases due to their requirement for linear independence. Further, this extra redundancy helps to protect signals from degradation, ensuring...
that the effects of erasures are minimized. The looseness of the structure allows one to construct the analog of frames for structures that don’t necessarily come equipped with suitable generalization of an orthonormal basis [1], [2]. Indeed, certain classes of Hilbert C* Modules and Banach spaces possess frames [1], [2]. For more on basics of frame theory, see [3], [4], [5].

Many questions concerning Hilbert space frames are inexorably tied to questions of closed ranged and left invertible operators. All frames for closed subspaces of a Hilbert spaces arise as the application of a closed range operator \( T \) on a frame for \( \mathcal{H} \). Properties of frames, such as rigidity and perturbation results also depend on the analysis of closed range operators [6], [7]. With this groundwork of closed ranged operators and frame theory laid, we now describe our interest in generating concrete operator algebras.

A well-studied class of operator algebras are C*-algebras generated by partial isometries. For example graph algebras, C*-algebras generated partial isometries subject to constraints of a directed graph, reside here.

Representations of graph C*-algebras arising from a directed graph may be described as follows. Given a directed graph, choose a Hilbert space for each vertex of the graph (with suitable dimension depending on loops of the graph). After direct summing each of these spaces together, we obtain a Hilbert space \( \mathcal{H} \) where each vertex corresponds to an orthogonal summand of \( \mathcal{H} \). By choosing orthonormal sequences for each of these closed, orthogonal subspaces of \( \mathcal{H} \), we then pick partial isometries that map one summand to another, subject to the Cuntz-Krieger relations [8].

The partial isometric generators of concrete graph algebras preserve orthonormal basis (that is, up to the kernels of the partial isometries). The representation of the graph C*-algebra can be viewed as encoding the dynamics of walks on the graph. A natural extension of these concrete graph C*-algebras is one that encodes the dynamics of a frame. One arrives at such an extension by replacing partial isometries and their adjoints with closed ranged operators and their Moore-Penrose inverses. As discussed above, the closed range operators preserve frame theoretic quantities. In the isometric case, the Moore-Penrose inverse is the adjoint. Hence, such operator algebras reduce to the C*-algebraic case when the generators are isometric. Therefore, this class of operator algebras is the natural generalization of concrete graph C*-algebras that integrates frame theory. This leads us to the following general program:

**Program.** Given a set of operators with closed range and their Moore-Penrose inverses construct the norm-closed algebra subject to the constraints of a directed graph. What is the structure of these algebras?

The focus of this paper is on one particular class of examples in within this program. Consider the following directed graph \( \Gamma \):

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    v1       v2
     □
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It is well known that the graph C*-algebra associated to \( \Gamma \) is isomorphic to the Toeplitz algebra \( \mathcal{T} \) [8]. As a concrete operator algebra, \( \mathcal{T} \) is the C*-algebra generated by \( T = M_z \) on the Hardy space \( H^2(T) \). The graph C*-algebra representations associated to \( \Gamma \) can be described as follows. Let \( \mathcal{H}_i \) represent the Hilbert space associated to vertex \( v_i \), and \( T_1 : \mathcal{H}_1 \to \mathcal{H}_2 \), \( T_2 : \mathcal{H}_2 \to \mathcal{H}_2 \) be chosen (partial) isometries. Since \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \), and \( \text{ran}(T_1) \oplus \text{ran}(T_2) = \mathcal{H}_2 \), we have that \( T := T_1 \oplus T_2 \) defines an isometry with Fredholm index equal to the dimension of \( \mathcal{H}_1 \). Thus, the representations can be succinctly written as \( C^*(T) \) for some isometry \( T \).

The same argument can be applied to the operator algebras described above. Concretely, choose \( T_1 : \mathcal{H}_1 \to \mathcal{H}_2 \), \( T_2 : \mathcal{H}_2 \to \mathcal{H}_2 \) closed ranged operators with orthogonal ranges summing to \( \mathcal{H}_2 \).
Then \( T_2 := T_1 \oplus T_2 \) is left invertible. The associated operator algebra can be expressed as 
\[
\mathfrak{A}_T := \overline{\text{Alg}(T, T^\dagger)}
\]
where the closure is in the operator norm. The goal of this paper is to analyze the structure of the operator algebras \( \mathfrak{A}_T \).

If \( T \) is an isometry, then \( T^\dagger = T^* \). If \( T \) is purely isometric (no unitary summand) with Fredholm index \(-1\), then \( T \) is unitarily equivalent to \( M_2 \) on \( H^2(\mathbb{T}) \). Hence, \( \mathfrak{A}_T \) is the Toeplitz algebra \( \mathcal{T} \). This representation is particularly nice, as every operator \( A \in \mathcal{T} \) can be uniquely represented as a compact perturbation of a Toeplitz operator with continuous symbol. The purpose of this paper is to understand the following question:

**Question 1.** To what extent do the elements of \( \mathfrak{A}_T \) have the form “Toeplitz operator plus compact”?

The paper is organized as follows. Section two is dedicated to operator theoretic properties of left invertible operators, and elementary observations about \( \mathfrak{A}_T \). We discover that if the Fredholm index of \( T \) is finite, \( \mathfrak{A}_T \) has a heuristic description as compact perturbations of “Laurent series” in \( T \). We justify that in order to make any serious progress understanding the rich structure of \( \mathfrak{A}_T \), we need to restrict ourselves to a subclass of left invertible operators, known as analytic operators. Given an open subset \( \Omega \) of \( \mathbb{C} \), a positive integer \( n \) and \( R \in \mathcal{B}(\mathcal{H}) \), we say that \( R \) is of Cowen-Douglas class \( n \), written \( R \in B_n(\Omega) \) if

1. \( \Omega \subset \sigma(R) \)
2. \( (R - \lambda)\mathcal{H} = \mathcal{H} \) for all \( \lambda \in \Omega \)
3. \( \dim(\ker(R - \lambda)) = n \) for all \( \lambda \in \Omega \).
4. \( \bigvee_{\lambda \in \Omega} \ker(R - \lambda) = \mathcal{H} \)

In the third section, we connect analyticity of \( T \) to the class of Cowen-Douglas operators. Notably, we prove the following result:

**Theorem A.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be left invertible operator with Fredholm index \( \text{ind}(T) = -n \), for a positive integer \( n \). Then the following are equivalent:

1. \( T \) is analytic
2. \( T^\dagger \) (the Cauchy Dual of \( T \)) is analytic
3. There exists \( \epsilon > 0 \) such that \( T^\dagger \in B_n(\Omega) \) for \( \Omega = \{z : |z| < \epsilon\} \)
4. There exists \( \epsilon > 0 \) such that \( T^\dagger \in B_n(\Omega) \) for \( \Omega = \{z : |z| < \epsilon\} \)

This result has several applications. First, it gives a model for representing \( T \) in the sense that \( T \) is unitarily equivalent to \( M_2 \) on a reproducing kernel Hilbert space of analytic functions. If \( T \) is an isometry, the Wold decomposition lets us decompose \( T \) into a direct sum of Fredholm index \(-1\) isometries (and a unitary). A corollary of Theorem A is that we cannot reduce our study to the case where the Fredholm index of \( T \) is \(-1\). However, Theorem A allows us to analyze the isomorphism classes of \( \mathfrak{A}_T \) in the case when the Fredholm index of \( T \) is \(-1\).

In section four, we focus on the case when the index of \( T \) is \(-1\). Here, we determine the conditions for two such algebras to be isomorphic, establishing our main theorem. It gives a rather rigid structure on bounded isomorphisms between the algebras \( \mathfrak{A}_T \):

**Theorem B.** Let \( T_i, i = 1, 2 \) be left invertibles (analytic with \( \text{ind}(T_i) = -1 \)) and \( \mathfrak{A}_i = \mathfrak{A}_{T_i} \). Suppose that \( \phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2 \) a bounded isomorphism. Then there exists some invertible \( V \in \mathcal{B}(\mathcal{H}) \) such that \( \phi(A) = VAV^{-1} \) for all \( A \in \mathfrak{A}_1 \).

In particular, this theorem shows that all bounded isomorphisms are completely bounded, and reduces the isomorphism problem to a similarity orbit problem. We remark that this question arises from Cowen-Douglas theory. Using the results of Jiang and others on \( K_0 \) groups of Cowen-Douglas operators, we complete the classification in this case.
In the fifth section, we investigate a class of illustrative examples arising from the theory of subnormal operators. If $S$ is a subnormal operator, we let $N = mne(S)$ denote the minimal normal extension of $S$, and $\sigma_{ap}(S)$ denote the approximate point spectrum of $S$. We show that this class of operators can be described by the heuristic of compact perturbations of Toeplitz operators with Laurent series:

**Theorem C.** Let $S$ be an analytic left invertible, $\text{ind}(S) = -1$, essentially normal, subnormal operator with $N := mne(S)$ such that $\sigma(N) = \sigma_{ap}(S)$. Let $\mathcal{B}$ be the uniform algebra generated by the functions $z$ and $z^{-1}$ on $\sigma_r(S)$. Then

$$\mathfrak{A}_S = \{T_f + K : f \in \mathcal{B}, K \in H(H)\}.$$  

Moreover, the representation of each element as $T_f + K$ is unique.

2. **Properties of Left Invertible Operators and $\mathfrak{A}_T$**

Throughout, $T$ will be a left (but not right) invertible operator on a Hilbert space $H$, and $T^\dagger = (T^*T)^{-1}T^*$ will denote the Moore-Penrose inverse of $T$. All left invertible operators considered will be assumed to be not invertible unless otherwise stated.

We are interested in a sub-class of left invertible operators called analytic. Within the class of isometric operators, these are the pure isometries. We will see that an analytic operator induces a pair of wandering vectors for $T$ and $T^\dagger$ that will be instrumental to the analysis of $\mathfrak{A}_T$. We conclude this section by demonstrating that, for general left invertible operators, you cannot arrive at a direct analog of the Wold decomposition. Specifically, you cannot decompose $T$ as a direct sum of an analytic operator and an invertible operator.

2.1. **Basic Properties of Left Invertible Operators.** We begin by listing some elementary algebraic properties of $T$ and $T^\dagger$. We state the following propositions without proof, as they are well known:

**Proposition 2.1.** For $T \in \mathcal{B}(H, H)$, the following are equivalent:

i. $T$ is left-invertible
ii. $T^*$ is right-invertible
iii. $T^*T$ is invertible
iv. $T$ is injective and has closed range
v. $T$ is bounded below, that is, there exists a $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in H$

**Proposition 2.2.** Let $T \in \mathcal{B}(H)$ be left invertible and $L$ a left inverse. Then there exists an $A \in \mathcal{B}(H)$ such that

$$L = T^\dagger + A(I - TT^\dagger).$$

**Proposition 2.3.** Given any left invertible $T \in \mathcal{B}(H)$, the following hold:

i. $TT^\dagger$ is the (orthogonal) projection onto $\text{ran}(T)$
ii. $I - TT^\dagger$ is the (orthogonal) projection onto $\text{ran}(T)^\perp$
iii. $\ker(T^\dagger) = \text{ran}(T)^\perp = \ker(T^*)$
iv. $\text{ran}(T^\dagger) = \text{ran}(T^*)$.

Next, we discuss a few simple results concerning norms and left invertibility.

**Lemma 2.4.** Let $T \in \mathcal{B}(H)$ be left invertible. For any left inverse $L$ of $T$, and for all $x \in H$, we have $\|Lx\| \geq \|T^\dagger x\|$. Hence, $\|L\| \geq \|T^\dagger\|$.

**Proof.** By Proposition 2.2, there exists an $A \in \mathcal{B}(H)$ such that $L = T^\dagger + A(I - TT^\dagger)$. Using this representation and Proposition 2.3, we see that if $x \in \ker(T^\dagger)$,

$$Lx = T^\dagger x + A(I - TT^\dagger)x = Ax.$$
Thus, \( \|Lx\| \geq \|T^\dagger x\|. \) Similarly, if \( x \in \ker(T^\dagger) = \ran(T) \), we have \( x = Ty \) for some \( y \in \mathcal{H} \).

Hence,
\[
Lx = T^\dagger Ty + A(I - TT^\dagger)Ty = y = T^\dagger x.
\]

So, \( \|Lx\| = \|T^\dagger x\| \).

\[\square\]

**Lemma 2.5.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be left invertible. If \( S \in \mathcal{B}(\mathcal{H}) \) satisfies \( \|T - S\| < \|T^\dagger\|^{-1} \), then \( S \) is also left invertible.

**Proof.** By Lemma 2.4
\[
\|T^\dagger S - I\| = \|T^\dagger(S - T)\| \leq \|T^\dagger\||S - T| < 1.
\]

Therefore, \( T^\dagger S \) is invertible. Hence, \( (T^\dagger S)^{-1} T^\dagger \) is a left inverse of \( S \).

\[\square\]

We now establish some notation. If \( T \in \mathcal{B}(\mathcal{H}) \), let \( \sigma(T) \) and \( \sigma_e(T) \) denote the spectrum and the essential spectrum respectively. The **Fredholm domain of \( T \)** is
\[
\rho_F(T) := \mathbb{C} \setminus \sigma_e(T) = \{ \lambda : T - \lambda \text{ is Fredholm} \}.
\]

For \( \lambda \in \rho_F(T) \), the function
\[
\text{ind}(T - \lambda) = \text{nul}(T - \lambda) - \text{nul}(T - \lambda)^*
\]
is well-defined integer, called the **index**. It is well known that the index is constant on each component of \( \rho_F(T) \). The next result follows from Proposition 2.3.

**Proposition 2.6.** If \( T \) is left invertible, then \( 0 \in \sigma(T) \), and \( 0 \notin \sigma_e(T) \). Indeed, \( T \) is Fredholm with \( \text{ind}(T) = -\dim(\ker(T^\dagger)) \).

We will always use \( \mathcal{E}_T := \ran(T)^\perp \). If \( T \) is understood, we simply write \( \mathcal{E} \). That is,
\[
\mathcal{E} := \ran(T)^\perp = \ker(T^\dagger) = \ker(T^*).
\]

For isometric operators, \( T^n \mathcal{E} \perp T^m \mathcal{E} \) for all \( n \neq m \). This is not true for general left invertible operators, even though \( \mathcal{E} \) is perpendicular to the range of \( T \). However, it is true that \( \ker((T^\dagger)^n) = \bigvee_{k=0}^{n-1} T^k \mathcal{E} \):

**Proposition 2.7.** Let \( T \) be left invertible, and \( P = I - TT^\dagger \) be the projection onto \( \mathcal{E} \). Then for each \( n \geq 1 \), we have
\[
(I - T^n T^\dagger)^n = \sum_{k=0}^{n-1} T^k PT^\dagger^k.
\]

Consequently,
\[
\ker((T^\dagger)^n) = \bigvee_{k=0}^{n-1} T^k \mathcal{E}.
\]

**Proof.** By a telescopic sum, \( I - T^n T^\dagger^n = \sum_{k=0}^{n-1} T^k PT^\dagger^k \). To prove the set equality, suppose \( x \in \bigvee_{k=0}^{n-1} T^k \mathcal{E} \). Then it follows immediately that \( T^\dagger^n x = 0 \). On the other hand, if \( x \in \ker((T^\dagger)^n) \), then by equation (1),
\[
x = (I - T^n T^\dagger^n)x = \sum_{k=0}^{n-1} T^k PT^\dagger^k x.
\]

Since \( PT^\dagger^k x \in \mathcal{E} \) for all \( k \), it follows that \( x \in \bigvee_{k=0}^{n-1} T^k \mathcal{E} \).

\[\square\]
2. Basic Properties of $\mathfrak{A}_T$. In this subsection, we note two ways in which left invertible operators are close to invertible. This is done first by taking a particular quotient of $\mathfrak{A}_T$, and then by looking at a dilation.

Let $\mathcal{J}$ denote the commutator ideal of $\mathfrak{A}_T$. That is, $\mathcal{J}$ is the two sided ideal generated by the commutators of $\mathfrak{A}_T$, closed in the norm topology. In other words, $\mathcal{J}$ is the smallest ideal of $\mathfrak{A}_T$ such that $\mathfrak{A}_T / \mathcal{J}$ is commutative. Note that the projection $P = I - TT^\dagger = T^\dagger T - TT^\dagger \in \mathcal{J}$.

We prove that when the dimension of $\ker(T^*)$ is finite, $\mathcal{J} \subset \mathcal{K}(\mathcal{H})$. We then show that $\mathfrak{A}_T / \mathcal{J}$ consists of formal Laurent polynomials, namely polynomials in $z$ and $z^{-1}$. Moreover $T$ may also be dilated to an invertible, allowing us to identify $\mathfrak{A}_T$ as the corner of the algebra generated by this invertible. Combining these results allows one to heuristically describe $\mathfrak{A}_T$ as sums of compact operators and Laurent series. We begin this section with a simple observation that will be used throughout the paper:

**Lemma 2.8.** Let $T$ be a left invertible operator. Then $\mathfrak{A}_T \subset C^*(T)$.

**Proof.** This follows from the fact that for left invertible operators, $T^\dagger = (T^*T)^{-1}T^*$.

This paper will largely be concerned with the case when $\dim(\mathcal{E}) < \infty$. In particular, we will have much to say when the Fredholm index of $T$ is $-1$. We have the following result about the commutator ideal of $\mathfrak{A}_T$.

**Lemma 2.9.** Let $T$ be left invertible. If $\dim(\mathcal{E}) < \infty$, then $\mathcal{J} \subset \mathcal{K}(\mathcal{H})$.

**Proof.** Let $X = \sum_{n,m=0}^{N} a_{n,m} T^n T^\dagger^m$ and $Y = \sum_{k,l=0}^{m} b_{k,l} T^k T^\dagger^l$. If we can show that the commutator $XY - YX$ is finite rank, then it will follow that by taking limits $\mathcal{J} \subset \mathcal{K}(\mathcal{H})$. To this end, notice that

$$XY = \sum_{n,m=0}^{N} \sum_{k,l=0}^{M} a_{n,m} b_{k,l} T^n T^\dagger^m T^k T^\dagger^l.$$

Similarly, we have that

$$YX = \sum_{n,m=0}^{N} \sum_{k,l=0}^{M} a_{n,m} b_{k,l} T^k T^\dagger^l T^n T^\dagger^m.$$

Now consider $XY - YX$. By matching the coefficients $a_{n,m} b_{k,l}$ in the two sums, it suffices to show that

$$T^n T^\dagger^m T^k T^\dagger^l - T^k T^\dagger^l T^n T^\dagger^m$$

is compact for each non-negative integer $n, m, k, l$. Notice $T^n T^\dagger^m T^k T^\dagger^l = T^{n+k-m} T^\dagger^l$ if $m \leq k$, and $T^n T^\dagger^m T^k T^\dagger^l$ otherwise. Likewise, $T^k T^\dagger^l T^n T^\dagger^m = T^{n+k-l} T^\dagger^m$ if $l \leq n$ and $T^k T^\dagger^l T^n T^\dagger^m$ otherwise. By the preceding remark, the expression $T^n T^\dagger^m T^k T^\dagger^l - T^k T^\dagger^l T^n T^\dagger^m$ can be simplified depending on the values of $n, m, k$ and $l$. This leaves us with eight total cases to check. For example, two cases are for $m \geq k$ and $l \geq n$. By above,

$$T^n T^\dagger^m T^k T^\dagger^l - T^k T^\dagger^l T^n T^\dagger^m = T^n T^\dagger^{l+m-k} - T^k T^\dagger^{l+m-n}.$$

This leaves us with two sub-cases: either $n \leq k$ or $k \leq n$. If $n \leq k$, we have

$$T^n T^\dagger^{l+m-k} - T^k T^\dagger^{l+m-n} = T^n (I - T^k T^\dagger^{k-n}) T^\dagger^{l+m-k}.$$

By Proposition 2.7 $I - T^k T^\dagger^{k-n}$ is a sum of finite rank operators, and thus, $T^n T^\dagger^{l+m-k} - T^k T^\dagger^{l+m-n}$ is finite rank. The case when $k \leq n$ is the same. The other six cases are similar. □

Because of the previous lemma, we will always assume that the index of $T$ is a negative integer. For emphasis, we explicitly state that here:
Assumption. Henceforth, our left invertible operators $T$ will satisfy $\text{ind}(T) = -n$ for some positive integer $n$.

We now investigate the quotient of $\mathfrak{A}_T$ by the commutator ideal $\mathcal{J}$. Let $\pi$ denote the canonical map $\pi : \mathfrak{A}_T \rightarrow \mathfrak{A}_T / \mathcal{J}$. As $P = I - TT^\dagger$ is in $\mathcal{J}$, it follows that $\pi(T)$ is invertible with inverse $\pi(T^\dagger)$. Hence, $\mathfrak{A}_T / \mathcal{J}$ is a commutative Banach algebra (in fact, operator algebra [9]) generated by the invertible $\pi(T)$ and its inverse $\pi(T^\dagger)$.

If $\mathcal{B}$ is a commutative Banach algebra, let $\Omega(\mathcal{B})$ denote the character space of $\mathcal{B}$. That is, $\Omega(\mathcal{B})$ is the set of homomorphisms of $\mathcal{B}$ into $\mathbb{C}$ equipped with the weak topology. We have the following:

Lemma 2.10. Let $\mathcal{B}$ be a commutative unital Banach algebra generated by an invertible $x$ and its inverse $x^{-1}$. Then $\Omega(\mathcal{B})$ is homeomorphic to $\sigma(x)$.

By the previous lemma, the Gelfand map provides a norm decreasing homomorphism of

$$\Gamma : \mathfrak{A}_T / \mathcal{J} \rightarrow C(\sigma(\pi(T))).$$

For each $\lambda \in \sigma(\pi(T))$, let $z : \sigma(\pi(T)) \rightarrow \mathbb{C}$ represent the inclusion function. Namely, $z(\lambda) = \lambda$ for all $\lambda \in \sigma(\pi(T))$. Then $z$ is invertible by construction, with inverse $z^{-1}(\lambda) := \lambda^{-1}$ for all $\lambda \in \sigma(\pi(T))$. Under the Gelfand identification, $\pi(T) \mapsto z$ and $\pi(T^\dagger) \mapsto z^{-1}$ on $\sigma(\pi(T))$. Consequently, $z$ and $z^{-1}$ generate the image of $\mathfrak{A}_T / \mathcal{J}$ under $\Gamma$. In this sense, $\mathfrak{A}_T / \mathcal{J}$ consists of Laurent polynomials centered at zero.

A few comments are necessary at this point. First, the Gelfand map need not have closed range, and thus, $\Gamma(\mathfrak{A}_T / \mathcal{J})$ may not be complete. Moreover, $\Gamma$ may not even be injective. However, since $\mathfrak{A}_T / \mathcal{J}$ is generated by $\pi(T)$ and $\pi(T^\dagger) = \pi(T)^{-1}$, it follows that $z$ (and therefore $z^{-1}$) are non-zero. As $\Gamma$ is norm decreasing, we do have that every function in the range of $\Gamma$ is a Laurent series in $z$ and $z^{-1}$.

It will be shown in a future section that when the Fredholm index of $T$ is $-1$, $\mathcal{J} = \mathcal{H}(\mathcal{H})$. In some cases, this furnishes a rather detailed analysis of the quotient. Consider the following example:

Example 2.11. Suppose that $T$ were a left invertible, irreducible, essentially normal operator and $\pi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) / \mathcal{H}(\mathcal{H})$. Suppose further that $\mathcal{J} = \mathcal{H}(\mathcal{H})$. By Proposition 2.7, $\pi(T)$ is invertible, with inverse $\pi(T^\dagger)$. Since $\mathfrak{A}_T \subset C^*(T)$, we must have that in the Calkin algebra $\pi(\mathfrak{A}_T) \subset \pi(C^*(T))$. Since $T$ is irreducible, and since $P = I - TT^\dagger$ is a compact operator inside $C^*(T)$, necessarily $C^*(T)$ must contain all the compacts. Since $T$ is essentially normal, we can apply the Gelfand map to get

$$\pi(C^*(T)) = C^*(T) / \mathcal{H}(\mathcal{H}) \cong C(\sigma_e(T)).$$

Consequently, $\pi(\mathfrak{A}_T) \cong \overline{\text{Alg}}(z, z^{-1})$ as functions on $\sigma_e(T)$. That is, $\pi(\mathfrak{A}_T)$ is the uniform subalgebra of $C(\sigma_e(T))$ generate by Laurent series centered at zero. We will return to this example in a later section.

We just analyzed how quotienting by the commutator ideal results in $T$ becoming invertible. As a consequence, the quotient can be pictured as series in $z$ and $z^{-1}$ over a set $X \subset \mathbb{C}$. Next, we observe that if $T \in \mathcal{B}(\mathcal{H})$ is left invertible, then it dilates to an invertible. This will allow us to succinctly describe $\text{Alg}(T, T^\dagger)$. Let $P = I - TT^\dagger$. Then the operator $W \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ given by

$$W = \begin{pmatrix} T^\dagger & 0 \\ P & T \end{pmatrix}$$

is invertible, with inverse given by

$$W^{-1} = \begin{pmatrix} T & P \\ 0 & T^\dagger \end{pmatrix}.$$
Let \( Q_1 \) and \( Q_2 \) denote the projections onto \( \mathcal{H}_1 := \mathcal{H} \oplus 0 \) and \( \mathcal{H}_2 := 0 \oplus \mathcal{H} \) respectively. Then by construction \( T = W \mid_{\mathcal{H}_2} \) and \( T^\dagger = Q_2 W^{-1} \mid_{\mathcal{H}_2} \). Furthermore, for each \( n \),

\[
W^n = \begin{pmatrix} T^n & D_n \\ 0 & T^{n\dagger} \end{pmatrix}
\]

\[
W^{-n} = \begin{pmatrix} T^n & D_n \\ 0 & T^{n\dagger} \end{pmatrix}
\]

where \( D_n := \sum_{k=0}^{n-1} T^k PT^{n-1-k} \). Since \( \dim(\mathcal{H}) < \infty \) by assumption, \( D_n \) is a finite rank operator for each \( n \). Furthermore, for every \( n \), \( T^n = W^n \mid_{\mathcal{H}_2} \) and \( T^{n\dagger} = Q_2 W^{-n} \mid_{\mathcal{H}_2} \). It therefore follows that \( \text{Alg}(T, T^\dagger) = Q_2 \text{Alg}(W \mid_{\mathcal{H}_2}, W^{-1} \mid_{\mathcal{H}_2}) \). Now, a straightforward calculation reveals the following:

\[
\begin{align*}
Q_2 W^{-n} Q_1 W^m \mid_{\mathcal{H}_2} &= 0 \\
Q_2 W^{-n} Q_2 W^m \mid_{\mathcal{H}_2} &= T^{n\dagger} T^m \\
Q_2 W^m Q_1 W^{-n} \mid_{\mathcal{H}_2} &= D_m D_n \\
Q_2 W^m Q_2 W^{-n} \mid_{\mathcal{H}_2} &= T^m T^{n\dagger}.
\end{align*}
\]

(2)

Since \( \text{Alg}(T, T^\dagger) = Q_2 \text{Alg}(W \mid_{\mathcal{H}_2}, W^{-1} \mid_{\mathcal{H}_2}) \), the operators appearing in Equation (2) span \( \text{Alg}(T, T^\dagger) \). Moreover using Equation (2) we have

\[
D_m D_n + T^m T^{n\dagger} = Q_2 W^m W^{-n} \mid_{\mathcal{H}_2} = Q_2 W^{m-n} \mid_{\mathcal{H}_2} = \begin{cases} T^{m-n} & \text{if } m > n \\ T^{n-m} & \text{else.} \end{cases}
\]

Thus, \( T^m T^{n\dagger} \) is equal to some power of a generator, up to the finite rank perturbation \( D_m D_n \). Furthermore,

\[
T^{n\dagger} T^m = Q_2 W^{-n} W^m \mid_{\mathcal{H}_2} = \begin{cases} T^{m-n} & \text{if } m > n \\ T^{n-m} & \text{else.} \end{cases}
\]

Consequently, every operator \( A \) in \( \text{Alg}(T, T^\dagger) \) may be “simplified” to an operator of the form

\[
F + \sum_{k=0}^N a_k T^k + \sum_{l=1}^M b_l T^{l\dagger},
\]

where \( F \) is some finite rank operator. Hence, the dense subalgebra \( \text{Alg}(T, T^\dagger) \) are finite rank operators plus Laurent polynomials in \( T \) and \( T^\dagger \). We record this result here for future reference:

**Proposition 2.12.** Let \( T \) be a left invertible operator with \( \text{ind}(T) = -n \) for some positive integer \( n \). If \( A \in \text{Alg}(T, T^\dagger) \), then \( A \) may be written as

\[
A = F + \sum_{k=0}^N a_k T^k + \sum_{l=1}^M b_l T^{l\dagger}
\]

where \( F \) is a finite rank operator.

Recall that \( P = I - TT^\dagger \) is in the commutator ideal \( \mathcal{J} \). Hence by the preceding, all the finite rank operators \( F \) from this construction are in the commutator ideal. Combining these two coarse descriptions of \( \mathfrak{A}_T \), we arrive at our heuristic for \( \mathfrak{A}_T \):

**Heuristic 2.13.** The algebra \( \mathfrak{A}_T \) is compact perturbations of Laurent series centered at zero.

Ideally, we would like a canonical representation of \( T \) as multiplication by \( z \) on some reproducing kernel Hilbert space. If we further have \( T^\dagger \) represented as multiplication by \( z^{-1} \), then \( \mathfrak{A}_T \) could be further described as compact perturbations of multiplication operators with symbols Laurent series. This turns out to be the case for special class of operators, which we call analytic. We will expand on this particular topic in our discussion of Cowen-Douglas operators.
2.3. The Wold Decomposition. Much of the model theory and elementary properties of left invertible operators draws its inspiration from isometric operators. Isometries are a tractable class of operators due to the celebrated Wold decomposition. For future notational considerations, we state the Wold Decomposition here:

**Theorem 2.14 (Wold Decomposition for Isometries).** Let $S$ is an isometry on $\mathcal{H}$. Define

\[
\mathcal{H}_1 := \bigcap_{n \geq 1} S^n \mathcal{H}, \\
\mathcal{H}_A := \bigvee_{n \geq 0} S^n \mathcal{E}.
\]

Then $\mathcal{H}_1$ and $\mathcal{H}_A$ are reducing for $T$, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_A$, $S \mid_{\mathcal{H}_1}$ is a unitary and $S \mid_{\mathcal{H}_A}$ is a unilateral shift of rank $n$.

In other words, all isometries decompose the Hilbert space into two orthogonal, reducing subspaces for $S$. On $\mathcal{H}_1$, the isometry $S$ is invertible, and hence, a unitary. On $\mathcal{H}_A$, the isometry is purely isometric. The isometric summand yields an analytic model. Concretely, $S \mid_{\mathcal{H}_A}$ is unitarily equivalent to the operator of multiplication by $z$ on a reproducing kernel Hilbert space of analytic functions. For a general left invertible operator $T \in \mathcal{B}(\mathcal{H})$, one would like to arrive at a similar type of decomposition. We make the following definition:

**Definition 2.15.** Given a left invertible $T \in \mathcal{B}(\mathcal{H})$, we define:

\[
\mathcal{H}_1 := \bigcap_{n \geq 1} T^n \mathcal{H}, \\
\mathcal{H}_A := \bigvee_{n \geq 0} T^n \mathcal{E}.
\]

As a caution to the reader, $\mathcal{H}_1$ and $\mathcal{H}_A$ need not be reducing. However, $\mathcal{H}_1$ and $\mathcal{H}_A$ are clearly invariant subspaces for $T$. Moreover, $\mathcal{H}_1$ is invariant for $T^\dagger$ and $T \mid_{\mathcal{H}_1}$ is invertible, with inverse $T^\dagger \mid_{\mathcal{H}_1}$. We shall show that $T \mid_{\mathcal{H}_A}$ acts like a shift, not on an orthonormal basis, but on a more general basis. This will be discussed below.

For some isometries, the Wold-decomposition is trivial. For example, the unilateral shift on $\ell^2(\mathbb{N})$ is purely isometric since the subspace $\mathcal{H}_1 = 0$. This leads us to the following definition:

**Definition 2.16 ([10]).** An operator $T \in \mathcal{B}(\mathcal{H})$ is analytic if $\mathcal{H}_1 = 0$.

The terminology analytic is appropriate because we show that when a left invertible operator is analytic, then $T$ is unitarily equivalent to $M_z$ on a reproducing kernel Hilbert space of analytic functions.

There is no Wold-type decomposition for $T$ with regards to the spaces $\mathcal{H}_1$ and $\mathcal{H}_A$. See Example 2.27 below. However, Shimorin in [10] observed that there is almost a Wold-type decomposition. This decomposition is related to a canonical left invertible operator associated to $T$, called the Cauchy dual of $T$:

**Definition 2.17 ([10]).** Given a left invertible operator $T$, the Cauchy dual of $T$, denoted $T'$, is the left invertible given by

\[
T' := T(T^*T)^{-1} = T^\dagger^\ast.
\]

**Proposition 2.18.** Let $T$ be a left invertible operator, and $T'$ its Cauchy dual.

i. $T'$ is left invertible with Moore-Penrose inverse $T'^\dagger = T^*$

ii. $\mathcal{E}' := \ker((T')^*) = \ker(T^\dagger) = \ker(T^*) = \mathcal{E}$

iii. $\text{ind}(T') = \text{ind}(T)$

**Proof.** It is clear from the definition that $T'$ is left invertible with $T^*$ a left inverse. That $T'^\dagger = T^*$ follows from a simple computation:

\[
T'^\dagger = (T'^*T')^{-1}T'^* = (T^\dagger T^*)^{-1}T^\dagger = (T^*T)T^\dagger = T^*.
\]

The remaining observations now follow. \qed

□
For the Cauchy dual \( T' \), we define the analogous invariant subspaces:

\[
\mathcal{H}'_I := \bigcap_{n \geq 1} T^n \mathcal{H} \\
\mathcal{H}'_A := \bigvee_{n \geq 1} T^n \mathcal{E}.
\]

We now explain why the terminology of Cauchy dual is sensible. While one cannot hope to arrive at a decomposition \( \mathcal{H} = \mathcal{H}'_I \oplus \mathcal{H}'_A \), there is a duality between the spaces \( \mathcal{H}'_I, \mathcal{H}'_A \) and \( \mathcal{H}_A, \mathcal{H}_A' \).

**Proposition 2.19** ([10], Prop 2.7). Let \( T \) be a left invertible operator. Then

\[
\mathcal{H} = \mathcal{H}'_I \oplus \mathcal{H}'_A = \mathcal{H}'_I' \oplus \mathcal{H}_A.
\]

This duality is key in analyzing \( \mathfrak{A}_T \). Frequently one has to leverage information between \( T \) and \( T' \) (or \( T'^* \)) in order to prove theorems about \( \mathfrak{A}_T \). The first example of this is the construction of a Schauder bases used throughout the subsequent analysis.

2.4. Basis and Dual Basis. We now explore how \( T \mid_{\mathcal{H}_A} \) acts as a shift on a general basis. This will be done by showing if \( T \) is an analytic left invertible, then it endows the Hilbert space with a type of basis analogous to that of a (Hamel) basis for a vector space, called a **Schauder basis**.

**Definition 2.20.** A Banach space \( X \) is said to have a **Schauder basis** if there exists a sequence \( \{x_n\} \) of \( X \) such that for every element \( x \in X \), there is a unique sequence of scalars \( \alpha_n \) such that

\[
x = \sum_{n \geq 0} \alpha_n x_n
\]

where the above sum is converging in the norm topology of \( X \). Alternatively, \( \{x_n\} \) is a Schauder basis if and only if

i. \( \text{span}\{x_n\} = X \)
ii. \( \sum \alpha_n x_n = 0 \) if and only if \( \alpha_n = 0 \) for all \( n \).

Recall that a subspace \( \mathcal{E} \) is said to be a **wandering subspace** for an operator \( T \in \mathfrak{B}(\mathcal{H}) \) if for each \( n \in \mathbb{N} \), \( \mathcal{E} \perp T^n \mathcal{E} \) [11]. In the case of isometric operators, one further has \( T^n \mathcal{E} \perp T^m \mathcal{E} \) for each \( n, m \in \mathbb{N} \) with \( n \neq m \).

Let \( T \) be an analytic left invertible operator, and \( L \) be a left inverse of \( T \). The next result shows that \( \mathcal{E} = \ker(T^*) \) is a wandering subspace for \( T \) and \( L^* \). The invariant subspace generated in this fashion in the whole Hilbert space. Thus, the orbit of \( T \) and \( L^* \) on \( \ker(T^*) \) give rise to a Schauder basis:

**Theorem 2.21.** Let \( T \) be an analytic left invertible operator with \( \text{ind}(T) = -n \) for some positive integer \( n \). Let \( \{x_i, 0\}_{i=1}^n \) be an orthonormal basis for \( \ker(T^*) \), and \( L \) be a left inverse of \( T \). Then

i. \( x_{i,j} := T^j x_{i,0}, i = 1, \ldots, n, j = 0, 1, \ldots \) is a Schauder basis for \( \mathcal{H} \)
ii. \( x'_{i,j} := (L^*)^j x_{i,0}, i = 1, \ldots, n, j = 0, 1, \ldots \) is a Schauder basis for \( \mathcal{H} \).

**Proof.** We will only prove the case when \( \text{ind}(T) = -1 \). The general case is no more complicated, but simply requires extra notation for bookkeeping. In this case, \( \ker(T^*) = \text{span}\{x_0\} \) for some norm one element \( x_0 \in \mathcal{H} \).

The proof will proceed as follows. First we will show that the wandering space for \( T' := T'^* \) produces a Schauder basis. Then we show that the orbit of \( T \) will produce a Schauder basis, which will allow us to conclude that for any left invertible \( L \), the orbit of \( L^* \) yields a Schauder basis.

Since \( T \) is analytic, by Proposition 2.19 we have that

\[
\mathcal{H} = \mathcal{H}'_A = \bigvee_{j \geq 0} T^j \ker(T^*).
\]

Let \( x'_j := T^j x_0 \) for \( j = 0, 1, \ldots \). Then by construction, \( T' x'_j = x'_{j+1} \) and

\[
T^{*m} x'_j = \begin{cases} 
0 & \text{if } m > j \\
x'_{j-m} & \text{if } m \leq j
\end{cases}
\]
Notice that \( \{x'_j\} \) is a Schauder basis. Indeed by \([3]\), \( \text{span}\{x'_j\} = \mathcal{H} \). Furthermore, if \( \sum_{j \geq 0} a_j x'_j = 0 \), then
\[
0 = (I - TT^\dagger)T^m \left( \sum_{j \geq 0} a_j x'_j \right) = (I - TT^\dagger) \left( \sum_{j \geq m} a_j x'_{j-m} \right) = a_m x_0.
\]
Thus, \( a_j = 0 \) for all \( j \). Therefore \( \{x'_j\} \) form a Schauder basis.

We now show that \( x_j := T^j x_0 \) is a Schauder basis. Indeed, let \( \mathcal{K} \) be the closed subspace of \( \mathcal{H} \) given by \( \mathcal{K} := \text{span}_{j \geq 1} \{x_j\} \). Suppose that \( z \perp \mathcal{K} \). Then by above, \( z \) has a unique expansion in the Schauder basis \( x'_j \). Say, \( z = \sum_{j \geq 0} b_j x'_j \). Thus,
\[
0 = \langle z, x_m \rangle = \langle T^{sm} z, x_0 \rangle = \langle T^{sm} z, (I - TT^\dagger)x_0 \rangle = \langle (I - TT^\dagger)T^{sm} z, x_0 \rangle = b_m.
\]
Hence, \( b_j = 0 \) for all \( j \), so \( z = 0 \). Therefore, \( \mathcal{K} \) is dense in \( \mathcal{H} \). But since \( \mathcal{K} \) is closed, \( \mathcal{K} = \mathcal{H} \).

Now suppose that \( \sum_{j \geq 0} c_j x_j = 0 \). Then the exact same argument appearing in equation \((4)\) with \( T^{sm} \) replaced with \( T'^{sm} \) shows \( c_j = 0 \) for all \( j \).

Finally, suppose \( L \) is any left inverse of \( T \). Let \( y_j = L^* j x_0 \). Replacing the roles of \( x_j \) with \( y_j \) and \( x'_j \) with \( x_j \) in the preceding paragraph, one concludes that \( y_j \) is a Schauder basis for \( \mathcal{H} \).

**Corollary 2.22.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be left invertible. Then \( T \) is analytic if and only if \( T' \) is analytic.

**Proof.** If \( T \) is analytic, then by Theorem 2.21 \( \mathcal{H}_A = \mathcal{H} \). Hence \( \mathcal{H}'_A = 0 \). The converse statement is identical. \( \square \)

Theorem 2.21 illustrates how to construct Schauder bases for \( \mathcal{H} \) using an analytic left invertible operator \( T \) and its Cauchy dual. We reserve the notation of Theorem 2.21 for these bases. Concretely, we make the following definition:

**Definition 2.23.** Let \( T \) be an analytic left invertible operator and \( L \) be a left inverse of \( T \). Fix an orthonormal basis \( \{x_{i,0}\}_{i=1}^n \) for \( \mathcal{E} = \ker(T^*) \). Then
\[
\begin{align*}
x_{i,j} &:= T^j x_{i,0} \\
x'_{i,j} &:= L^* j x_{i,0}.
\end{align*}
\]
We refer to the Schauder basis \( \{x_{i,j}\} \) in Equation \((5)\) as the **basis of \( T \) with respect to \( \{x_{i,0}\}_{i=1}^n \)**. Similarly, we refer to the basis \( \{x'_{i,j}\} \) as the **dual basis of \( T \) with respect to \( \{x_{i,0}\}_{i=1}^n \) and \( L \)**.

If no mention is made to the choice of left inverse \( L \), it is assumed that \( L = T^\dagger \). While the above definition depends on the choice of orthonormal basis \( \{x_{i,0}\}_{i=1}^n \) for \( \mathcal{E} \), we will usually refer to each as the **basis of \( T \) and dual basis of \( T \) without reference**.

By definition of a Schauder basis, for each \( f \in \mathcal{H} \), there exists a unique sequences of scalars \( \{\alpha_{i,j}\} \) and \( \{\alpha'_{i,j}\} \) such that
\[
f = \sum_{j \geq 0} \sum_{i=1}^n \alpha_{i,j} x_{i,j} = \sum_{j \geq 0} \sum_{i=1}^n \alpha'_{i,j} x'_{i,j}.
\]
Naturally, one would like to have a relationship between \( \{\alpha_{i,j}\} \) or \( \{\alpha'_{i,j}\} \) in terms of the element \( f \in \mathcal{H} \). We have the following useful characterization:

**Proposition 2.24.** For each \( f \in \mathcal{H} \), we have the following expansions:
\[
f = \sum_{j \geq 0} \sum_{i=1}^n \langle f, x'_{i,j} \rangle x_{i,j} = \sum_{j \geq 0} \sum_{i=1}^n \langle f, x_{i,j} \rangle x'_{i,j}.
\]
Proof. Suppose that \( f = \sum_{j=0}^{n-1} x_{i,j} \). Then for each \( m \geq 0 \),
\[
\langle f, x_{i,m} \rangle = \langle T^m f, x_{i,0} \rangle = \alpha_{i,m}
\]
since \( \{x_{i,0}\} \) is an orthonormal basis for \( \ker(T^*) \). The same argument shows that if we expand \( f \) in terms of the dual basis of \( T \) as \( f = \sum_{j=0}^{n-1} x_{i,j} x_{i,j}' \), then \( \alpha_{i,m}' = \langle f, x_{i,m} \rangle \).
\[\Box\]

Corollary 2.25. The basis of \( T \) is bi-orthogonal to the dual basis of \( T \). That is, \( \langle x_{i,m}, x_{i,j}' \rangle = \delta_{i,m} \delta_{j,m} \)

Proof. By Proposition 2.24 we have that
\[
x_{i,m} = \sum_{j \geq 0} \sum_{\sigma} \langle x_{i,m}, x_{i,j}' \rangle x_{i,j}.
\]
However by definition, Schauder bases have a unique expansion in terms of the basis. Hence, \( \langle x_{i,m}, x_{i,j}' \rangle = 0 \) unless \( i = l \) and \( j = m \).
\[\Box\]

Briefly, we would like to caution the reader about the order of basis and dual basis of \( T \). A convergent series \( \sum_{n=0}^{\infty} x_n \) in a Banach space \( X \) is said to be unconditionally convergent if for every permutation \( \sigma \) of \( \mathbb{N} \), the series \( \sum_{n=0}^{\infty} x_{\sigma(n)} \) converges. Otherwise, the series is said to be conditionally convergent. A Schauder basis \( \{x_n\} \) in a Banach space \( X \) is said to be a unconditional basis if the series expansion \( x = \sum_{n=0}^{\infty} \alpha_n x_n \) is unconditional for every \( x \in X \). Otherwise, the basis is said to be conditional. Examples of unconditional bases for Hilbert spaces include orthonormal bases, and more generally, frames.

Unfortunately, all infinite dimensional Banach spaces with a basis must have conditional bases \([12]\). What is worse, verifying that a basis is unconditional is, in general, a very difficult task. Explicit constructions of conditional bases exist for Hilbert spaces. Indeed, there is a class of examples for \( L^2(\mathbb{T}) \) of the form \( \{e^{2\pi int}\phi(t)\}_{n \in \mathbb{Z}} \) for some \( \phi \in L^2(\mathbb{T}) \) (See \([13]\) Example 11.2). From the author’s perspective, it is not clear when the basis and dual basis of \( T \) are unconditional. Fortunately, this will not affect our analysis in any serious way. Trivially we have at a minimum the following:

Proposition 2.26. Let \( T \) be an analytic left invertible with \( \text{ind}(T) = -n \) for some \( 1 \leq n < \infty \). Then for any permutation \( \sigma \) of \( \{1, \ldots, n\} \), we have
\[
\sum_{j \geq 0} \sum_{i=1}^{n} \alpha_{i,j} x_{i,j} = \sum_{i=1}^{n} \sum_{j \geq 0} \alpha_{i,j} x_{i,j} = \sum_{i=1}^{n} \sum_{j \geq 0} \alpha_{\sigma(i),j} x_{\sigma(i),j} = \sum_{j \geq 0} \sum_{i=1}^{n} \alpha_{\sigma(i),j} x_{\sigma(i),j}
\]
whenever the sum converges. Consequently, \( \sum_{j \geq 0} \sum_{i=1}^{n} \alpha_{i,j} x_{i,j} \) converges if and only if \( \sum_{j \geq 0} \alpha_{i,j} x_{i,j} \) converges for each \( i = 1, \ldots, n \).

Since we are interested in the case when \( \text{ind}(T) \) is a negative integer, the above proposition fits into the purview of our study. This remark is useful when we construct a canonical model for \( T \) as multiplication by \( z \) on a reproducing kernel Hilbert space of analytic functions in Section 3. In order to conduct a more thorough analysis of \( \mathfrak{H}_T \), we will later consider the case when \( \text{ind}(T) = -1 \). In the next subsection, we discuss the ways in which a left invertible can fail to have a Wold-type decomposition.

2.5. Failure of Wold-Type Decomposition for Left Invertibles. We have mentioned that for a general left invertible operator, one cannot hope to reconstruct a exact replica of the Wold decomposition. Namely, it is not the case that \( \mathcal{H} = \mathcal{H}_I \oplus \mathcal{H}_A \). Indeed, their sum can not only fail to be orthogonal, but \( \mathcal{H}_I + \mathcal{H}_A \) may not be equal to \( \mathcal{H} \). We have the following example:

Example 2.27. Let \( \mathcal{H} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{Z}) \), and define \( T \in \mathcal{B}(\mathcal{H}) \) as
\[
T = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}
\]
where $A$ is the unilateral shift on $\ell^2(\mathbb{N})$, $C$ is the bilateral shift on $\ell^2(\mathbb{Z})$, and $B : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{Z})$ is the inclusion map given by

$$B((a_n)_{n \geq 1}) = (\ldots, 0, a_1, a_2, \ldots)$$

where the $\hat{\cdot}$ symbol denotes the entry in the zeroth slot. Let $\{e_n\}_{n=1}^\infty$ and $\{f_n\}_{n=-\infty}^\infty$ denote the standard orthonormal basis for $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ respectively.

In order to compute the subspaces $\mathcal{H}_I$ and $\mathcal{H}_A$ above, we will first need to analyze $T^n$. Note that

$$T^n = \begin{pmatrix} A^n & 0 \\ D_n & C^n \end{pmatrix}$$

where $D_n := \sum_{k=0}^{n-1} C^k B A^{n-1-k}$. By construction, $D_n e_m = n f_{m+n-1}$. Therefore, $D_n = n C^{n-1} B$, so

$$T^n = \begin{pmatrix} A^n & 0 \\ n C^{n-1} B & C^n \end{pmatrix}$$

Notice that if $x \oplus y \in \mathcal{H}$, then

$$T^n\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^n x \\ n C^{n-1} B x + C^n y \end{pmatrix}$$

We now show that

$$\mathcal{H}_I = 0 \oplus \ell^2(\mathbb{Z}).$$

Indeed, suppose that $x \oplus y \in \mathcal{H}_I$. Then for each $n \in \mathbb{N}$ exists a sequence $x_n \in \ell^2(\mathbb{N})$ and $y_n \in \ell^2(\mathbb{Z})$ such that $T^n(x_n \oplus y_n) = x \oplus y$. By equation (6), we must have $A^n x_n = x$. But since the unilateral shift $A$ is analytic, if follows that $x = 0$ so that $\mathcal{H}_I \subseteq 0 \oplus \ell^2(\mathbb{Z})$. On the other hand, suppose $y \in \ell^2(\mathbb{Z})$. Since the bilateral shift $C$ is invertible, $C^n$ is invertible for all $n \in \mathbb{Z}$. Thus, for all $n$ there exists $y_n \in \ell^2(\mathbb{Z})$ such that $C^n y_n = y$. Hence, $0 \oplus y \in \mathcal{H}_I$, demonstrating equality.

Next we compute $\mathcal{H}_A$. Notice that

$$T^* = \begin{pmatrix} A^* & B^* \\ 0 & C^* \end{pmatrix}$$

where $B^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$ is the projection onto the coordinates greater than zero. Consequently, if $x \oplus y \in \mathcal{H}$,

$$T^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^* x + B^* y \\ C^* y \end{pmatrix}.$$ 

If $x \oplus y \in \ker(T^*)$, then since $C^*$ is invertible, it follows that $y = 0$. Consequently, $x \in \ker(A^*) = \operatorname{span}\{e_1\}$. Therefore, $\mathcal{E} = \ker(T^*) = \ker(A^*) \oplus 0 = \operatorname{span}\{e_1\} \oplus 0$. Now, by equation (6),

$$T^n\begin{pmatrix} e_1 \\ 0 \end{pmatrix} = \begin{pmatrix} e_{n+1} \\ n f_n \end{pmatrix}.$$ 

As a result, we have that

$$\operatorname{span}_{0 \leq n \leq N}\{T^n(e_1 \oplus 0)\} = \left\{ \left( \sum_{n=0}^N \alpha_n e_{n+1} \right) \oplus \left( \sum_{n=1}^N \alpha_n f_n \right) : \alpha_0, \ldots, \alpha_N \in \mathbb{C} \right\}.$$ 

Now because

$$\left\| \sum_{n=0}^N \alpha_n e_{n+1} \right\|^2 + \left\| \sum_{n=1}^N \alpha_n f_n \right\|^2 = \left\| \sum_{n=0}^N \alpha_n e_{n+1} \right\|^2 + \left\| \sum_{n=1}^N \alpha_n f_n \right\|^2 = |\alpha_0|^2 + |\alpha_{N+1}|^2 + \sum_{n=1}^N (1+n^2)|\alpha_n|^2$$

it follows that

$$\mathcal{H}_A = \left\{ \left( \sum_{n=0}^\infty \alpha_n e_{n+1} \right) \oplus \left( \sum_{n=1}^\infty \alpha_n f_n \right) : \sum_{n=1}^\infty (1+n^2)|\alpha_n|^2 < \infty \right\}.$$
With $\mathcal{H}_A$ computed, we now remark that $\mathcal{H}_1 = 0 \oplus l^2(\mathbb{Z})$ is not orthogonal to $\mathcal{H}_A$. Nevertheless, $\mathcal{H}_1 \cap \mathcal{H}_A = 0$. This is clear by the form of $\mathcal{H}_A$ and $\mathcal{H}_1$.

Finally, we remark that $\mathcal{H}_1 + \mathcal{H}_A$ is dense in $\mathcal{H}$, but not closed. To see this, note that $0 \oplus f_n \in 0 \oplus l^2(\mathbb{Z}) = \mathcal{H}_1$ for all $n$. By equation (17), it follows that $\{c_n \oplus 0\}_{n \geq 0} \subset \mathcal{H}_1 + \mathcal{H}_A$. Since $\{0 \oplus f_n\}_{n \in \mathbb{Z}} \subset \mathcal{H}_1$, it follows that $\mathcal{H}_1 + \mathcal{H}_A$ is dense in $\mathcal{H}$. However, $\mathcal{H}_1 + \mathcal{H}_A \neq \mathcal{H}$, as $\mathcal{H}_1 + \mathcal{H}_A$ is not closed. Indeed, if we let $z = ((1 + n^2)^{-1}) \oplus 0$, then $z \in \mathcal{H}$ but $z \notin \mathcal{H}_1 + \mathcal{H}_A$. This concludes the example.

The above example turns out to be generic. If $T \in \mathcal{B}(\mathcal{H})$ is left invertible, then $\mathcal{H}_1 + \mathcal{H}_A$ is dense in $\mathcal{H}$ with $\mathcal{H}_1 \cap \mathcal{H}_A = 0$. To show this, we establish a few simple results.

**Proposition 2.28.** Let $T \in \mathcal{B}(\mathcal{H})$ be left invertible. Consider the decomposition $\mathcal{H} = \mathcal{H}_A' \oplus \mathcal{H}_1$ afforded by Proposition 2.19. Then with respect to this decomposition,

$$T = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

with $A$ analytic left invertible, and $C$ invertible.

**Proof.** That the operator $C = T \mid_{\mathcal{H}_1}$ is invertible is clear. Let $Q$ be the projection onto $\mathcal{H}_1'$. To show that $A = QT \mid_{\mathcal{H}_1'}$ is left invertible, we show that $A^*$ is right invertible. Indeed, notice that $\mathcal{H}_A'$ is invariant under $T'$, and that

$$T^* = \begin{pmatrix} A^* & B^* \\ 0 & C^* \end{pmatrix}.$$ 

Thus, if $x \in \mathcal{H}_A'$, we have

$$A^*(T'x) = T^*(T'x) = x$$

since $T^*T' = I$. That $A$ is analytic follows from the orthogonality of the decomposition. To see this, observe

$$T^n = \begin{pmatrix} A^n & 0 \\ * & C^n \end{pmatrix}.$$ 

Hence, $A^n = QT^n \mid_{\mathcal{H}_1'}$. Now,

$$\bigcap A^n \mathcal{H}_A' = \bigcap QT^n \mathcal{H}_A' \subset Q \left( \bigcap T^n \mathcal{H} \right) = Q \mathcal{H}_1 = 0.$$

**Lemma 2.29.** Suppose that $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and

$$T = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

with $A$ analytic left invertible, and $C$ invertible. Then $\mathcal{H}_1 = 0 \oplus \mathcal{H}_2$, $\ker(T^*) = \ker(A^*) \oplus 0$, and $\mathcal{H}_1 \cap \mathcal{H}_A = 0$.

**Proof.** First, we remark that

$$T^n = \begin{pmatrix} A^n & 0 \\ D_n & C^n \end{pmatrix}$$

where $D_n$ is an operator whose formula is not relevant for the remainder of the proof. If $x \oplus y \in \bigcap T^n \mathcal{H}$, then there exists $x_n, y_n$ such that

$$T^n \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} A^n x_n \\ D_n x_n + C^n y_n \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

Since $A$ is analytic, it follows that $x = 0$. Thus, $\bigcap T^n \mathcal{H} \subset 0 \oplus \mathcal{H}_2$. Conversely, given $y \in \mathcal{H}_2$, since $C^n$ is invertible, there exists $y_n$ such that $C^n y_n = y$. So, $T^n(0 \oplus y_n) = 0 \oplus y$. It follows that $\bigcap T^n \mathcal{H} = 0 \oplus \mathcal{H}_2$. 


Concerning the intersection of $\mathcal{H}_I$ and $\mathcal{H}_A$, notice that

$$T^* = \begin{pmatrix} A^* & B^* \\ 0 & C^* \end{pmatrix}.$$ 

Since $C^*$ is invertible, it follows that $x \oplus y \in \ker(T^*)$ if and only if $y = 0$ and $x \in \ker(A^*)$. Thus, $\mathcal{E} = \ker(A^*) \oplus 0$. Consequently if $x_0 \in \ker(A^*)$, $\mathcal{H}_A$ is densely spanned by elements of the form

$$T^n \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} A^n x_0 \\ D_n x_0 \end{pmatrix}.$$ 

Since $A$ is analytic, $A^n x_0$ form a Schauder basis for $\mathcal{H}_A$ by Theorem (2.21). As a result, $0 \oplus y \in \mathcal{H}_A$ if and only if $y = 0$. \hfill $\square$

**Corollary 2.30.** Given a left invertible operator $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{H}_I + \mathcal{H}_A$ is dense in $\mathcal{H}$ with $\mathcal{H}_I \cap \mathcal{H}_A = 0$.

**Proof.** Lemma (2.29) established that $\mathcal{H}_I \cap \mathcal{H}_A = 0$. All that remains to be shown is that $\mathcal{H}_I + \mathcal{H}_A$ is dense in $\mathcal{H}$. To this end, consider the decomposition $\mathcal{H} = \mathcal{H}'_A \oplus \mathcal{H}_I$. Write,

$$T = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}.$$ 

Let $x_0 \oplus 0 \in \ker(T^*) = \ker(A^*) \oplus 0$, so that

$$T^n \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} A^n x_0 \\ D_n x_0 \end{pmatrix}$$

as before. Given that $0 \oplus (-D_n x_0) \in 0 \oplus \mathcal{H}_I$, we have

$$T^n \begin{pmatrix} x_0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -D_n x_0 \end{pmatrix} = \begin{pmatrix} A^n x_0 \\ 0 \end{pmatrix} \in \mathcal{H}_A + \mathcal{H}_I.$$ 

Since $A$ is an analytic left invertible on $\mathcal{H}'_A$, $A^n x_0$ is a Schauder basis for $\mathcal{H}'_A$. It follows that the closure of $\mathcal{H}_A + \mathcal{H}_I$ contains $\mathcal{H}'_A$ and $\mathcal{H}_I$, and therefore is dense in $\mathcal{H} = \mathcal{H}'_A \oplus \mathcal{H}_I$. \hfill $\square$

### 3. Cowen-Douglas Operators - The Analytic Model

In the late 70s, Cowen and Douglas discovered that operators possessing an open set of eigenvalues can be associated with a particular hermitian holomorphic bundle [14], [15]. These operators, now called Cowen-Douglas operators, could in some cases be completely classified by simple geometric properties. For example, when the rank of the bundle is one, the curvature serves as a complete set of unitary invariants [15].

Cowen-Douglas operators have played an important role in operator theory, serving as a bridge between operator theory and complex geometry. The definition is rigid enough to allow for classification based on local spectral data. However, the definition is also flexible enough to allow for rich examples - including many backward weighted shifts and adjoints of some subnormal operators. Recall the definition of Cowen-Douglas operators:

**Definition 3.1.** Given an open subset $\Omega$ of $\mathbb{C}$ and a positive integer $n$, we say that $R$ is of **Cowen-Douglas class n**, and write $R \in B_n(\Omega)$ if

i. $\Omega \subset \sigma(R)$

ii. $(R - \lambda)\mathcal{H} = \mathcal{H}$ for all $\lambda \in \Omega$

iii. $\dim(\ker(R - \lambda)) = n$ for all $\lambda \in \Omega$.

iv. $\bigvee_{\lambda \in \Omega} \ker(R - \lambda) = \mathcal{H}$
Thus if \( R \in B_n(\Omega) \), then \( R \) contains an open set of eigenvalues such that each eigenspace has dimension \( n \), and the span of these eigenspaces is dense in \( \mathcal{H} \). If \( R \in B_n(\Omega) \), then the map

\[
\lambda \mapsto \ker(\lambda I - R), \quad \lambda \in \Omega
\]

yields a hermitian holomorphic vector bundle of dimension \( n \) over \( \Omega \). We denote this bundle by \( E_R \). It is known that \( E_R \) provides a complete set of unitary invariants for operators in the Cowen-Douglas class \([14]\). This approach to Cowen-Douglas theory highlights the beautiful connections that exist between complex geometry and operator theory.

Equivalently, given \( R \in B_n(\Omega) \), we can represent \( R \) as the adjoint of multiplication by \( z \) on a reproducing kernel Hilbert space. The approach of this paper more closely follows this model. We will outline this construction below, and connect it to our work on bases in Section 2. For more information about Cowen-Douglas operators, see \([14], [16], [17]\).

3.1. Analytic Left Invertibles and Cowen-Douglas Operators. The connection between Cowen-Douglas operators and left invertibles is found in the following:

**Theorem A.** Let \( T \in B(\mathcal{H}) \) be left invertible operator with \( \text{ind}(T) = -n \), for \( n \geq 1 \). Then the following are equivalent:

i. \( T \) is analytic

ii. \( T^* \) is analytic

iii. There exists \( \epsilon > 0 \) such that \( T^* \in B_n(\Omega) \) for \( \Omega = \{ z : |z| < \epsilon \} \)

iv. There exists \( \epsilon > 0 \) such that \( T^1 \in B_n(\Omega) \) for \( \Omega = \{ z : |z| < \epsilon \} \)

Theorem A is a cornerstone result for this work. It serves two fundamental roles. First, Theorem A allows us to leverage the powerful machinery associated with Cowen-Douglas operators into classifying the algebras \( A_T \). Second, it provides us with a desirable canonical model. Concretely, by the discussion above, Theorem A allows us to represent \( T \) as multiplication by \( z \) restricted to a reproducing kernel Hilbert space of analytic functions.

To help illuminate this relationship, we will take a constructive approach to proving Theorem A. This will also connect to our results on Schauder bases from Section 2. We prove the implication (3) implies (1) after stating the following lemma due to Li:

**Lemma 3.2** ([18] Cor. 2.5). Let \( \Theta \) be an open subset of \( \mathbb{C} \) and \( S \in B_m(\Theta) \). Then for any fixed \( \mu_0 \in \Theta \),

\[
\bigvee_{k \geq 1} \ker(S - \mu_0)^k = \mathcal{H}.
\]

Moreover, if \( \Omega \subset \mathbb{C} \) is open, \( \lambda_0 \in \Omega \), \( n \) is a positive integer, and \( S \in B(\mathcal{H}) \) satisfies

i. \( \Omega \subset \sigma(S) \)

ii. \( (R - \lambda)\mathcal{H} = \mathcal{H} \) for all \( \lambda \in \Omega \)

iii. \( \dim(\ker(S - \lambda)) = n \) for all \( \lambda \in \Omega \).

iv. \( \bigvee_{k \geq 1} \ker(S - \lambda_0)^k = \mathcal{H} \).

Then \( R \in B_n(\Omega) \).

**Corollary 3.3.** Let \( T \in B(\mathcal{H}) \), \( n \in \mathbb{N} \), \( \epsilon > 0 \) and \( \Omega = \{ z : |z| < \epsilon \} \). If \( T^* \in B_n(\Omega) \), then \( T \) is an analytic, left invertible operator with \( \text{ind}(T) = -n \).

**Proof.** By assumption, \( 0 \in \Omega \subset \sigma(T^*) \). By condition (2) of the definition of Cowen-Douglas operators, \( T^* \) is onto. Since \( T^* \) has closed range, it follows from the closed range theorem that \( T \) also has closed range. Moreover, since \( T^* \) is onto, \( T \) must be injective. Therefore, \( T \) is left invertible. By Proposition 2.18 and condition (3) of Cowen-Douglas operators, we have \( \text{ind}(T) = \text{ind}(T^*) = -n \).
Thus, all that remains to be shown is that $T$ is analytic. By lemma 3.2, $\mathcal{H} = \bigvee_{k \geq 1} \ker(T^{*k})$. Therefore,

$$0 = \left( \bigvee_{k \geq 1} \ker(T^{*k}) \right) \perp = \bigcap_{k \geq 1} \ker(T^{*k}) \perp = \bigcap_{k \geq 1} \text{ran}(T^k).$$

Next we show that if $T$ is an analytic left invertible, then $T^* \in \mathcal{B}_n(\Omega)$. This will be done in several steps. First, we will show that $T^*$ possess an open set $\Omega$ of eigenvalues. We establish some notation for the open set $\Omega$ that will appear in the implication (1) implies (3) of Theorem A.

### Definition 3.4.

Suppose $T$ is an analytic left invertible operator. We define

$$\Omega_T := \{ z \in \mathbb{C} : |z| < \|T^\dagger\|^{-1} \}.$$  

Notice that by Lemma 2.5 if $Y \in \mathcal{B}(\mathcal{H})$ such that $\|Y\| < \|T^\dagger\|^{-1}$, then $T + Y$ is also left invertible. One also has $\text{ind}(T + Y) = \text{ind}(T)$. In particular, if $\lambda \in \Omega_T$, then $T + \lambda$ is left invertible with the same Fredholm index as $T$. We also have the following:

### Lemma 3.5.

Let $T$ be an analytic left invertible operator with $\text{ind}(T) = -n$ for some $n \geq 1$. Then for all $\lambda \in \Omega_T$, the operator $I - \lambda T^\dagger$ is invertible with

$$(I - \lambda T^\dagger)^{-1} = \sum_{j \geq 0} \lambda^j T^{\dagger j}.$$  

**Proof.** As $|\lambda| < \|T^\dagger\|^{-1}$ and $T^\dagger = T^{*^\dagger}$, the operator $\lambda T^\dagger$ has norm less than 1. \hfill $\square$

### Lemma 3.6.

Let $T$ be an analytic left invertible operator with $\text{ind}(T) = -n$ for some positive integer $n$. Let $\{x_{0,i}\}_{i=1}^n$ be an orthonormal basis for $\ker(T^*)$, and $x_{i,j}^* = T_j^i x_{0,i}$ be the dual basis of $T$ with respect to $T^\dagger$. Then for every $\lambda \in \Omega_T$,

$$x_{\lambda} := \sum_{i=1}^n \sum_{j \geq 0} \lambda^j x_{i,j}^*$$

is well defined. Furthermore, the map $\gamma : \Omega_T \to \mathcal{H}$ via $\gamma(\lambda) := x_{\lambda}$ is analytic.

**Proof.** By Lemma 3.5 $I - \lambda T^\dagger$ is invertible. Thus,

$$(I - \lambda T^\dagger)^{-1} \left( \sum_{i=1}^n x_{0,i} \right) = \sum_{j \geq 0} \lambda^j T^{\dagger j} \left( \sum_{i=1}^n x_{0,i} \right) = \sum_{i=1}^n \sum_{j \geq 0} \lambda^j x_{i,j}^* = x_{\lambda}$$

exists for each $\lambda \in \Omega_T$. Since the map $\lambda \mapsto (I - \lambda T^\dagger)^{-1}$ is well defined and analytic on $\Omega_T$, we have $\gamma(\lambda) = x_{\lambda}$ is analytic. \hfill $\square$

In light of these observations, we make the following definition:

### Definition 3.7.

Given an analytic left invertible $T$ with $\text{ind}(T) = -n$ for some positive integer $n$, let $\Omega_T$ be as in Definition 3.4. Let $\{x_{0,i}\}_{i=1}^n$ be an orthonormal basis for $\ker(T^*)$, and $x_{i,j}^* = T_j^i x_{0,i}$ be the dual basis of $T$ with respect to $T^\dagger$. We define

$$x_{\lambda} := \sum_{i=1}^n \sum_{j \geq 0} \lambda^j x_{i,j}^*.$$  

### Proposition 3.8.

Let $T$ be an analytic left invertible operator with $\text{ind}(T) = -n$ for some positive integer $n$. Let $\Omega_T$ be as in Definition 3.4. Then $T^* \in B_n(\Omega_T)$.
Proof. We first show $\Omega_T \subset \sigma_p(T^*)$, the point spectrum of $T^*$. This follows by the definition of $x_\lambda$ in Definition 3.7. To see this, let $\lambda \in \Omega_T$ and let $\{x_{i,0}\}_{i=1}^n$ be an orthonormal basis for $\ker(T^*)$. By Lemma 3.6, $x_\lambda$ is well defined. Since $T^*$ is the Moore-Penrose inverse of $T$, $T^*(T^*)^n = (T')^{n-1}$. Therefore,

$$(T^*)^n x_\lambda = \lambda x_\lambda.$$  

Hence, $x_\lambda$ is an eigenvector for $T^*$ with eigenvalue $\lambda$. By the remarks above, if $\lambda \in \Omega_T$, then $T - \lambda$ is left invertible with Fredholm index $-n$. Therefore, each eigenspace $\ker(T^* - \lambda)$ is $n$-dimensional for each $\lambda \in \Omega_T$.

Lastly, if we choose $\lambda_0 = 0$, then

$$\ker(T^* - \lambda_0)^k = \ker((T^*)^k) = \text{ran}T^{k\perp} = \left( \bigcap_{j=0}^k T^j \mathcal{H} \right)^\perp.$$  

Since $T$ is analytic, it follows that $\bigvee_{k \geq 1} \ker((T^*)^k) = \mathcal{H}$. By Lemma 3.2 we have that $T^* \in B_n(\Omega)$. □

We have thus shown that statements (1) and (3) of Theorem A are equivalent. However, when paired with Corollary 2.22 we see that $T^\dagger$ must also be Cowen-Douglas. This completes the proof of Theorem A.

One consequence of Theorem A is a reformulation of the definition of $\mathfrak{A}_T$ and the operator algebra generated by a Cowen-Douglas operator and a particular right inverse. Indeed, recall that $\mathfrak{A}_T$ is defined by

$$\mathfrak{A}_T := \overline{\text{Alg}} \{T, T^\dagger\}.$$  

If $\varepsilon > 0$, $\Omega = \{z : |z| < \varepsilon\}$, and $R \in B_n(\Omega)$, then by definition $R$ is right invertible. There exists a canonical right inverse of $R$, which we denote by $T$, such that $\text{ran}(T) = \ker(R)^\perp$. By construction, $T$ is left invertible, and $R = T^\dagger$, the Moore-Penrose inverse of $T$. Thus, we arrive at an equivalent viewpoint of study:

**Corollary 3.9.** Let $R \in B_n(\Omega)$ for $\Omega = \{z : |z| < \varepsilon\}$. If $T$ is the right inverse of $R$ such that $\text{ran}(T) = \ker(R)^\perp$, then $T$ is an analytic left invertible operator with $R = T^\dagger$. Hence,

$$\mathfrak{A}_T = \overline{\text{Alg}} \{T, R\}.$$  

### 3.2. The Associated Reproducing Kernel Hilbert Space

As previously remarked, the general theory of Cowen-Douglas operators allows one to represent $T$ as multiplication by $z$ on a reproducing kernel Hilbert space of analytic functions over $\Omega$. This construction is highlighted here. We then connect this model to the Schauder bases associated to $T$ and $T'$ discussed in Section 2. First, let us establish some notation. Given a set $G \subset \mathbb{C}$, let $G^* := \{x : \lambda \in G\}$. Notice that $\Omega_T = \Omega_T$ as a set.

Suppose $R \in B_n(\Omega)$. Then the bundle $E_R$ has (many choices of) spanning holomorphic cross sections $\gamma : \Omega \to \mathcal{H}$ [17]. That is, $\gamma(\lambda)$ is in the fiber over $\lambda$ (namely, $\ker(R - \lambda)$), and the span of these fibers is dense in $\mathcal{H}$. A consequence is that $R$ is unitarily equivalent to multiplication by $z$ on a collection of analytic functions over $\Omega^*$. Indeed, given such a $\gamma$ and $f \in \mathcal{H}$, define an analytic function $\hat{f}_\gamma \in H(\Omega^*)$ as follows:

$$\hat{f}_\gamma(\lambda) = (f, \gamma(\lambda)) \quad \lambda \in \Omega^*.$$  

Define a linear map $U_\gamma : \mathcal{H} \to H(\Omega^*)$ via $U_\gamma f = \hat{f}_\gamma$. Let $\mathcal{K}_\gamma$ be the range of $U_\gamma$, and define an inner product on $\mathcal{K}_\gamma$ by $\langle \hat{f}_\gamma, g_\gamma \rangle = (f, g)$. By construction, $U_\gamma$ is a unitary taking $\mathcal{H} \to \mathcal{K}_\gamma$. Moreover, $\mathcal{K}_\gamma$ is a reproducing kernel Hilbert space over the set $\Omega^*$. The reproducing kernel at
Let $M_z$ denote the operator of multiplication by the indeterminate $z$. That is, for each $\lambda \in \Omega^*$, $M_z(\hat{f}_\gamma)(\lambda) = \lambda \hat{f}_\gamma(\lambda)$. Since $\overline{\lambda} \in \Omega$, it follows from the definition Cowen-Douglas operators that $\overline{\lambda}$ is an eigenvalue for $R$. Consequently, $U_\gamma$ intertwines $M_z$ on $\mathcal{H}_\gamma$ and $R^*$ on $\mathcal{H}$. Indeed for all $f \in \mathcal{H}$, \[ (U_\gamma R^*) f)(\lambda) = (\overline{R^* f}, \gamma) = \langle R^* f, \gamma(\overline{\lambda}) \rangle = \langle f, R\gamma(\overline{\lambda}) \rangle = \langle f, \overline{\lambda}\gamma(\overline{\lambda}) \rangle = (M_z u_\gamma f)(\lambda). \] Thus, we have $U_\gamma R^* = M_z U_\gamma$, so $R^*$ is unitarily equivalent to $M_z$ on $\mathcal{H}_\gamma$. 

In our current study of analytic left invertible operators, Theorem A says that $T^* \in B_n(\Omega_T)$. Therefore, equation (9) tells us that $T$ is unitarily equivalent to $M_z$ on $\mathcal{H}_\gamma$. Furthermore, $\Omega_T = \Omega_T^*$ as sets, so for ease of notation, we consider the functions in $\mathcal{H}_\gamma$ on $\Omega_T$. We record this as a corollary.

**Corollary 3.10.** Let $T$ be an analytic, left invertible operator with $\text{ind}(T) = -n$ for some positive integer $n$. Then $T$ is unitarily equivalent to multiplication by $z$ on a reproducing kernel Hilbert space of analytic functions on $\Omega_T^* = \Omega_T$.

A natural question one might ask is, “What are the functions in $\mathcal{H}_\gamma$?”. The answer will depend on the choice of analytic Section $\gamma$ described above. We will describe a salient representation $U_\gamma$ that blends together the Cowen-Douglas theory with the basis theory developed in Section 2.

Let $\gamma(\lambda) = x_{\lambda}$ as defined in Definition 3.7. That is, fix an orthonormal basis $\{i_0\}_{i=1}^n$ for $\ker(T^*)$. For each $\lambda \in \Omega_T$, \[ \gamma(\lambda) := x_{\lambda} = \sum_{i=1}^n \sum_{j \geq 0} \lambda^j x_{i,j}^i \] where $x_{i,j}^i = T^{ij} x_{i,0}$. For each $f \in \mathcal{H}$ and $\lambda \in \Omega_T$, we have by Equation (8) \[ \hat{f}(\lambda) = \langle f, x_{\lambda} \rangle = \sum_{i=1}^n \sum_{j \geq 0} \lambda^j \langle f, x_{i,j}^i \rangle \] where here we have repressed the subscript $\gamma$. The reproducing kernel Hilbert space associated with this choice of analytic section will be simply denoted $\mathcal{H}$. We store this information in a definition:

**Definition 3.11.** Given an analytic left invertible $T$, let $\Omega_T$ be as in Definition 3.4. Let $\{x_{i,0}\}_{i=1}^n$ be an orthonormal basis for $\ker(T^*)$. For each $\lambda \in \Omega_T$, set \[ x_{\lambda} := \sum_{i=1}^n \sum_{j \geq 0} \lambda^j x_{i,j}^i \] and for each $f \in \mathcal{H}$, \[ \hat{f}(\lambda) := \langle f, x_{\lambda} \rangle = \sum_{i=1}^n \sum_{j \geq 0} \lambda^j \langle f, x_{i,j}^i \rangle. \] Let $\mathcal{H}$ denote the reproducing kernel Hilbert space of functions $\hat{f}$ arising from equation (10) with inner product $\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$. The representation of $T$ as $M_z$ on $\mathcal{H}$ is called the **canonical representation of $T$.**

The terminology canonical is fitting for the above representation. In the canonical representation, the basis elements associated to $T$ become the functions $z^{i}$. That is, if $k = 1, \ldots n$, then $\tilde{x}_{k,i}(\lambda) = \lambda^i$.
for each $\lambda \in \Omega$. This follows directly by Corollary 2.25

\[ \hat{x}_{k,i}(\lambda) = \sum_{i=1}^{n} \sum_{j \geq 0} \lambda^j \langle x_{k,l}, x_{i,j}' \rangle = \lambda^i \]

for every $\lambda \in \Omega_T$. Unfortunately, this shows that as functions, $\hat{f}$ can equal $\hat{g}$ even though $f \neq g$. However, the way in which this arises is rather artificial.

To see why, for each $i = 1, \ldots, n$ set $x_{\lambda,i}$ to be one part of the summand above. That is, $x_{\lambda,i} := \sum_{j \geq 0} \lambda^j x_{i,j}'$ for each $i = 1, \ldots, n$. Put $\hat{f}_i(\lambda) := \langle f, x_{\lambda,i} \rangle = \sum_{j \geq 0} \lambda^j \langle f, x_{i,j}' \rangle$. Then if $f \neq g$ in $\mathcal{H}$, there exists some $i = 1, \ldots, n$ such that $\hat{f}_i \neq \hat{g}_i$ as functions. To see this, recall that by Proposition 2.24, we have $f = \sum_{i=0}^{n} \sum_{j \geq 1} \langle f, x_{i,j} \rangle x_{i,j}$ and $g = \sum_{j \geq 0} \sum_{i=1}^{n} \langle g, x_{i,j} \rangle x_{i,j}$. Suppose that to the contrary, $\hat{f}_i = \hat{g}_i$ for each $i = 1, \ldots, n$. Then for each fixed $i$ and all $\lambda \in \Omega_T$, we have

\[ \hat{f}_i(\lambda) = \sum_{j \geq 0} \lambda^j \langle f, x_{i,j}' \rangle = \sum_{j \geq 0} \lambda^j \langle g, x_{i,j}' \rangle = \hat{g}_i(\lambda). \]

In particular, evaluating at $\lambda = 0$ we see that $\langle f, x_{i,0}' \rangle = \langle g, x_{i,0}' \rangle$. By Proposition 2.20 $\hat{f}_i$ and $\hat{g}_i$ are analytic for each $i$. Taking derivatives and evaluating at $\lambda = 0$ we see that $\langle f, x_{i,1}' \rangle = \langle g, x_{i,1}' \rangle$. Repeating for all $j \geq 0$, we find that $\langle f, x_{i,j}' \rangle = \langle g, x_{i,j}' \rangle$ for all $j$. Since this holds for each fixed $i$, we must have $f = g$. In particular, the above argument shows that if $n = 1$, then $\hat{f} = \hat{g}$ as functions if and only if $f = g$ in $\mathcal{H}$.

Recall that in general, the reproducing kernel at $\lambda$ is given by $k_{\lambda} = \gamma(\lambda)$. Since $\gamma(\lambda) = x_{\lambda}$ for the canonical representation, the reproducing kernel $K : \Omega^2 \to \mathbb{C}$ for $\hat{\mathcal{H}}$ takes on the following form:

\[ K(\lambda, \mu) = \langle x_{\lambda}, x_{\mu} \rangle = \left\langle \sum_{k=1}^{n} \sum_{i=1}^{n} \lambda^i (x_{k,l}'), \sum_{i=1}^{n} \lambda^i (x_{i,j}') \right\rangle = \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{l \geq 0} \sum_{j \geq 0} \lambda^i (x_{k,l}', x_{i,j}') \]

where by Proposition 2.26 convergence does not depend on the order of the four sums. The kernel is analytic in $\lambda$, and co-analytic in $\mu$ by construction.

Under the canonical representation, $T^\dagger$ becomes “division by $z$”. To make this precise, first we remark that unitary equivalence preserves Moore-Penrose inverses, so that $(U_j T U_j^\dagger) = (M_j)^\dagger$. Furthermore, the functions inside $\ker(M_j^\dagger)$ (the span of $x_{i,0}'$, $i = 1, \ldots, n$) are the constant functions, while $\text{ran}(M_z) = \ker(M_z)^\perp$ consists of functions of the form $z^j g$. As $M_z^\dagger M_z = I$, it follows that either $M_z^\dagger \hat{f} = 0$ (if $\hat{f}$ is constant) or $M_z^\dagger \hat{f} = z^{-1} \hat{f}$ otherwise.

Expanding on this computation, suppose that $\hat{f} \in \hat{\mathcal{H}}$ is a polynomial of degree $k$. Consider the action of $M_z^n$ on $\hat{f}$. As $\hat{f}(\lambda) = \sum_{j=0}^{k} a_j \lambda^j$, it suffices to understand how $M_z^j$ interacts with functions $z^j(\lambda) = \lambda^j$. By construction, $M_z^n(z^j(\lambda))$ is equal to 0 if $n \geq j$ and $z^{i-n}$ otherwise.

For emphasis, the operator $M_z$ of division by $z$ is not well defined on $\hat{\mathcal{H}}$ since $0 \in \Omega$ and $\hat{\mathcal{H}}$ contains the constant functions. Yet $M_z$ is well defined as a map from $\text{ran}(M_z) = \ker(M_z)^\perp$ to $\hat{\mathcal{H}}$. By the above computation, $M_z^\dagger$ is $M_z$ on $\ker(M_z)^\perp$. Hence, $T^\dagger$ is $M_z$ wherever the operator $M_z$ is well defined, and 0 otherwise. This can be succinctly written as

\[ M_z^\dagger = M_z 1 \]

where $Q_1$ is the projection onto $\ker(M_z)^\perp$. More generally for each $n$, we have that

\[ M_z^n = M_z^n Q_n \]

where $Q_n$ is the projection onto $\ker(M_z^n)^\perp$. 
This model gives intuition into the structure of $\mathfrak{A}_T$. By Proposition 2.12, $\text{Alg}(M_z, M_z^\dagger)$ consists of operators of the form
\[
F + \sum_{k=0}^{N} a_k M_z^k + \sum_{l=1}^{M} b_l M_z^l = F + \sum_{k=0}^{N} a_k M_z^k + \sum_{l=1}^{M} b_l M_{z^{-l}} Q_l,
\]
where $F$ is a finite rank operator. One could combine via linearity the “analytic” component of the above sum to get
\[
F + M \sum_{k=0}^{N} a_k z^k + \sum_{l=1}^{M} b_l M_{z^{-l}} Q_l.
\]
In some sense, the “principal part” may also be combined into a single multiplication operator. Unfortunately, this is not done as effortlessly. We do have that
\[
\hat{f} \perp \ker(T^M),
\]
for all $\hat{f} \in \ker(T^M)^\perp$, the sum of the principal pieces combine into a single multiplication operator. That is,
\[
\left( \sum_{l=1}^{M} b_l M_{z^{-l}} Q_l \right) \hat{f}(\lambda) = \sum_{l=1}^{M} b_l \frac{\hat{f}(\lambda)}{\lambda^l} = \left( M \sum_{l=1}^{M} b_l z^{-l} \hat{f} \right)(\lambda).
\]
However, this fails on $\ker(T^M)$, as some operators in the principal part have kernels contained in $\ker(T^M)$. Concretely, if $\hat{f}$ is perpendicular to $\ker(T^M)$ but not perpendicular to $\ker(T^{M+1})$, then
\[
\left( \sum_{l=1}^{M} b_l M_{z^{-l}} Q_l \right) \hat{f}(\lambda) = \sum_{l=1}^{L} b_l \frac{\hat{f}(\lambda)}{\lambda^l} = \left( M \sum_{l=1}^{L} b_l z^{-l} \hat{f} \right)(\lambda).
\]

This discussion demonstrates that we have a canonical analytic model to represent $\mathfrak{A}_T$. It is the norm limit of finite rank operators plus multiplication operators that have “Laurent” polynomials as symbols.

**Heuristic 3.12.** If $T$ is an analytic left invertible operator, then the algebra $\mathfrak{A}_T$ is compact perturbations of multiplication operators with symbols Laurent series centered at zero.

To some extent, a converse statement is true as well. That is, if $T$ is $M_z$ on a reproducing kernel Hilbert space of analytic functions (with extra properties), then $T$ is an analytic left invertible operator.

Recall that our interest will ultimately be in the case when the Fredholm index of $T$ is $-1$. In this case, the functions in $\mathcal{H}$ satisfy a nice factorization property that was studied by Richter [19]. Richter he investigated Banach spaces that satisfied the following axioms: Let $\Omega \subset \mathbb{C}$ be open and connected, and $\mathcal{B}$ be a Banach space of analytic functions on $\Omega$ that satisfy

i. The functional of evaluation at $\lambda$ is continuous for all $\lambda \in \Omega$

ii. If $f \in \mathcal{B}$, then $zf \in \mathcal{B}$

iii. If $f \in \mathcal{B}$ and $f(\lambda) = 0$, then there exists a $g \in \mathcal{B}$ such that $(z - \lambda)g = f$.

Note that if a Hilbert space $\mathcal{H}$ satisfies the above axioms, the first condition requests $\mathcal{H}$ be a reproducing Kernel Hilbert space. The second condition says that $\mathcal{H}$ is invariant under multiplication, and combined with the first says that $M_z$ is bounded. The final condition is equivalent to asking that $M_z - \lambda$ is bounded below for every $\lambda \in \Omega$.

It is easy to see that if $T$ is an analytic left invertible with $\text{ind}(T) = -1$, then the reproducing kernel Hilbert space $\mathcal{H}$ will satisfy these axioms. In [19] it is shown that a Hilbert space satisfies the axioms if and only if the Hilbert space arises from a Cowen-Douglas operator:

**Proposition 3.13 ([19] - Thm 2.10).** Let $\Omega \subset \mathbb{C}$ be connected and open, and $R \in \mathcal{B}(\mathcal{H})$. Then $R \in B_1(\Omega^*)$ if and only if there is a RKHS $\mathcal{H}$ on $\Omega$ such that $R^*$ is unitarily equivalent to $M_z \in \mathcal{B}(\mathcal{H})$. 

3.3. Reduction of Index. Suppose that $T$ is an analytic (pure) isometry with Fredholm index $-n$ for $n \geq 2$. Then $T$ can be decomposed as a direct sum of pure isometries $T_i$ each with Fredholm index $-1$. This decomposition is clearly unique up to unitary equivalence. A similar, though much weaker statement is true for general analytic left invertible operators. We require some terminology.

Definition 3.14. An operator $R \in \mathcal{B}(H)$ is strongly irreducible if there is no nontrivial idempotent in $\{R\}^\prime$, the commutant of $R$. Equivalently, $R$ is strongly irreducible if $XRX^{-1}$ is an irreducible operator for every invertible operator $X$. We denote the set of all strongly irreducible operators by $(SI)$.

Clearly, strong irreducibility is a similarity invariant. Moreover, it follows by definition that $R \in (SI)$ if and only if $R^* \in (SI)$.

Strongly irreducible operators play an important role in single operator theory. They serve as fundamental building blocks for bounded operators. In finite dimensions, every operator may be written (up to unitary equivalence) as the direct sum of irreducible operators. These irreducible operators are the Jordan blocks of a matrix. Strongly irreducible operators play an approximate replacement for Jordan blocks in an infinite dimensional setting [20]. To help make this more precise, we have the following definition:

Definition 3.15. A sequence $\{E_j\}_{j=1}^l$, $1 \leq l \leq \infty$ of nonzero idempotents on $H$ is called a spectral family if

i. there exists an invertible operator $X \in \mathcal{B}(H)$ such that $\{XE_jX^{-1}\}$ are pairwise orthogonal projections
ii. $\sum_{j=1}^l E_j = I$

Furthermore, if $R \in \mathcal{B}(H)$, then the spectral family is a strongly irreducible decomposition of $R$ if

iii. $E_jR = RE_j$ for all $j$
iv. $R \mid \text{ran}(E_j) \in (SI)$.

In other words, $R$ has a strongly irreducible decomposition if $R$ is the topological direct sum strongly irreducible operators. Equivalently, $R$ is similar to the orthogonal direct sum of strongly irreducible operators. We denote this by $R \sim \oplus_{j=1}^l R_j$.

In finite dimensions, Jordan canonical forms force each matrix to have a unique SI decomposition up to similarity. This is not the case for operators in $\mathcal{B}(H)$. Not every operator in $\mathcal{B}(H)$ has a strongly irreducible decomposition. Moreover, even if an operator has a strongly irreducible decomposition, it may not be unique [21]. Therefore, we make the following definition:

Definition 3.16. Let $R \in \mathcal{B}(H)$, and $\mathcal{E} = \{E_j\}_{j=1}^{l_1}$ and $\mathcal{E}' = \{E'_j\}_{j=1}^{l_2}$ be two strongly irreducible decompositions of $R$. We say the two SI decompositions $\mathcal{E}$ and $\mathcal{E}'$ are similar if

i. $l_1 = l_2 = l$
ii. there exists an invertible operator $X \in \{R\}^\prime$, the commutant of $R$, such that $XE_jX^{-1} = E'_j$

If $R$ has a strongly irreducible decomposition, we say that $R$ has a unique strongly irreducible decomposition up to similarity if any two of the decompositions are similar.

There is an extensive amount of work relating strongly irreducible decompositions of operators to K-theory [22, 20, 23, 21]. We will mention some of these results in Section 4.3. Of particular interest to us in the present are the following deep results due to Y. Cao, J. Fang and C. Jiang:

Theorem 3.17 [21] - Thm. 5.5.12. Each operator in $S \in \mathcal{B}_1(\Omega)$ is strongly irreducible. Moreover for any $n$, if $R \in \mathcal{B}_n(\Omega)$, then $R$ has a unique SI decomposition up to similarity. Furthermore, $R \sim \oplus_{j=1}^m R_j$ where $R_j \in (SI) \cap B_{n_j}(\Omega)$ and $\sum_{j=1}^m n_j = n$.

Corollary 3.18. Let $T$ be an analytic left invertible operator with $\text{ind}(T) = -n$ for some $1 \leq n < \infty$. Then $T \sim \oplus_{j=1}^m T_j$ where $T_j$ are analytic, $\sum_{j=1}^m \text{ind}(T_j) = -n$ and $T_j \in (SI)$. 
Proof. By Theorem \ref{thm:abstraction} \( T^* \in B_n(\Omega) \) for some disc \( \Omega \) centered at the origin. Therefore by Theorem \ref{thm:abstraction} \( T^* \sim \oplus_{j=1}^n R_j \) where each \( R_j \in (SI) \cap B_{n_j}(\Omega) \) and \( \sum_{j=1}^m n_j = n \). By another application of Theorem \ref{thm:abstraction} \( T_j := R_j^* \) are analytic left invertibles that satisfy \( \sum_{j=1}^m \text{ind}(T_j) = -n \). Since \( R_j \in (SI) \) and strong irreducibility is preserved under taking adjoints, \( T_j \in (SI) \).

Theorem \ref{thm:abstraction} states that operators in the Cowen-Douglas class have a rather rigid structure up to similarity. Without loss of generality, we may assume that if \( R \in B_n(\Omega) \), then \( R = \oplus_{j=1}^m R_j \) where \( R_j \in (SI) \cap B_{n_j}(\Omega) \) where \( \sum_{j=1}^m n_j = n \). This decomposition suggests that in order to understand \( \mathcal{A}_n \), we should first study the analytic left invertible operators that are strongly irreducible. In particular, we should study the analytic left invertible operators with Fredholm index \(-1\).

In the isometric case, \( T^* \in B_n(\Omega) \) decomposes to a direct sum of \( n \) strongly irreducible operators in \( B_1(\Omega) \). This turns out to not be the case in general. Notice that if \( R \in B_n(\Omega) \cap (SI) \), then it cannot be further decomposed as a direct sum. Indeed, suppose to the contrary that \( R \in B_n(\Omega) \cap (SI) \) and \( R \sim \oplus_{k=1}^n R_k \) with \( R_k \in B_1(\Omega) \). By Theorem \ref{thm:abstraction} each operator in \( B_1(\Omega) \) is strongly irreducible. Hence, \( R \) would have two strongly irreducible decompositions that are dissimilar. But Theorem \ref{thm:abstraction} states that all Cowen-Douglas operators have a unique SI decomposition up to similarity, contradicting the assumption that \( R \in B_n(\Omega) \cap (SI) \) and \( R \sim \oplus_{k=1}^n R_k \).

Thus, if there exists left invertible operators with \( T^* \in B_n(\Omega) \cap (SI) \) for \( n \geq 2 \), it would not be possible to decompose \( T \) as a direct sum of left invertibles with Fredholm index \(-1\). This is unfortunately the case, as the following example outlines:

**Example 3.19.** Let \( \epsilon > 0 \), \( \Omega = \{ \lambda : |\lambda| < \epsilon \} \), and \( \mathcal{H} \) be the Sobolev space \( W^{2,2}(\Omega) \). For completeness, recall the definition of \( W^{2,2}(\Omega) \). If \( dm \) denotes the planar Lebesgue measure, then \( W^{2,2}(\Omega) \) consists of the \( f \in L^2(\Omega, dm) \) such that the first and second order distributional partial derivatives of \( f \) belong to \( L^2(\Omega, dm) \).

Let \( M_z \) be multiplication by the independent variable on \( \Omega \). The constant function \( 1 \) is a cyclic vector for \( M_z \). Let \( R(\Omega) \) denote the algebra generated by rational functions with poles off of \( \overline{\Omega} \), and denote \( \mathcal{B} = R(\Omega)1 \subseteq \mathcal{H} \). Define \( T_z \in \mathcal{B}(\mathcal{H}) \) via \( T_z := M_z |_{\mathcal{H}} \). Then we have the following:

**Lemma 3.20** \((\ref{thm:abstraction}) - \text{Prop. 3.2})\( \). \( T_z \) is a left invertible operator with \( \text{ind}(T_z) = -1 \).

Now for any \( n \in \mathbb{N} \), define \( M_n \in \mathcal{B}(\oplus_{j=1}^n \mathcal{H}) \) via

\[
M_n := \begin{pmatrix}
T_z & 0 & 0 & \ldots & 0 \\
1 & T_z & 0 & \ldots & 0 \\
0 & 1 & T_z & \ldots & 0 \\
& & & \ddots & \\
0 & 0 & 0 & \ldots & T_z
\end{pmatrix}.
\]

It can be shown \((\ref{thm:abstraction}) - \text{Thm 3.5})\( ) that \( M_n^* \in B_n(\Omega) \) and is strongly irreducible, finishing our example.

The previous example illustrates a general result about Cowen-Douglas operators. Namely, Cowen-Douglas of rank \( n \) operators take the form of upper triangular operators of size \( n \):

**Theorem 3.21** \((\ref{thm:abstraction}) - \text{Thm 1.49})\( ). Let \( R \in B_n(\Omega) \) for \( 1 \leq n < \infty \). Then there exists \( n \) operators \( R_1, \ldots, R_n \) such that \( R_i \in B_1(\Omega) \) and

\[
R = \begin{pmatrix}
R_1 & * & * & * \\
R_2 & * & * & * \\
& & \ddots & \vdots \\
& & & R_n
\end{pmatrix}
\]

with respect to some decomposition \( \mathcal{H} = \oplus_{i=1}^n \mathcal{H}_i \).
Corollary 3.22. If $T$ is an analytic left invertible with $\text{ind}(T) = -n$ for $1 \leq n < \infty$, then there exists $n$ analytic left invertibles $T_1, \ldots, T_n$ such that $\text{ind}(T_i) = -1$ and

$$T = \begin{pmatrix} T_1 & * & T_2 & * & \cdots & \cdots \\ * & T_3 & * & T_4 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \\ * & * & \cdots & T_n \end{pmatrix}$$

with respect to some decomposition $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$.

Corollary 3.22 further emphasizes the need to analyze analytic left invertible operators with $\text{ind}(T) = -1$. We showed above that we can always decompose $T$ into a direct sum of strongly irreducible pieces. The strongly irreducible blocks have the form of lower triangular operators. If $T$ is decomposed as in Corollary 3.22 then $T_n = T \mid \mathcal{H}_n$ and $T_n$ is an analytic left invertible operator with $\text{ind}(T_n) = -1$. If we are to gain any insight into a general $\mathfrak{A}_T$, it is mandatory to understand the index $-1$ case first. This analysis will be taken up next section.

4. The Algebra $\mathfrak{A}_T$

As remarked earlier, given a left invertible $T$, we view $\mathfrak{A}_T$ as a natural generalization of the concrete $C^*$-algebra generated by an isometry. By the Wold-decomposition, we can always reduce an isometry to its purely (analytic) isometric component. If the Fredholm index of the analytic isometry is $-n$, then this isometry is unitarily equivalent to a direct sum of $n$ unilateral shift operators. Hence, in order to analyze the $C^*$-algebra generated by an isometry, it is important to first understand the $C^*$-algebra generated by an analytic isometry of Fredholm index $-1$.

The preceding sections showed that, in general, we cannot reduce to either of these assumptions (analytic or $\text{ind}(T) = -1$) as we could in the isometric case. Example 2.27 demonstrated that $T$ cannot be decomposed as a direct sum of an analytic operator and an invertible operator. Furthermore, Example 3.19 shows that even if an operator is analytic, it cannot be reduced to the index $-1$ case. Nevertheless, there is a summand on which $T$ will be analytic. Similar statements may be made about strong irreducibility and the Fredholm index. Under the assumption of analytic, Theorem A implies that $T^*$ is Cowen-Douglas. Corollary 3.22 tells us that, in this case, $T$ may be written as a triangular operator where each element on the diagonal is a pure left invertible of index $-1$.

Although we cannot reduce to the case of analytic or index $-1$, the epistemological viewpoint of the author is that an important first step in understanding $\mathfrak{A}_T$ is simplifying to this case. We therefore make the following minimality assumptions on $T$:

Assumption.

i. The Fredholm index: $\text{ind}(T) = -1$

ii. Analytic: $\bigcap T^n \mathcal{H} = 0$

If $T$ is an analytic isometry of Fredholm index $-1$, an elegant representation for $C^*(T)$ is obtained by considering the representation $T = M_z$ on the Hardy space $H^2(\mathbb{T})$. Of course, $T = C^*(M_z)$ is the Toeplitz algebra. The analyticity ensures that the basis associated to $M_z$ (which is an orthonormal basis) spans the Hilbert space. The Fredholm index guarantees that $T$ will be an irreducible $C^*$-algebra, which contains a compact $(I - TT^*)$, and therefore all the compacts. Furthermore, one discovers that each element of $T$ may be uniquely written as $T_f + K$ for some $f \in C(\mathbb{T})$ and $K \in \mathcal{K}(\mathcal{H})$.

The general case is similar. That is, if $T$ is an analytic, left invertible operator with Fredholm index $-1$, then $\mathfrak{A}_T$ contains the compact operators. As a consequence, we will determine the isomorphism classes of $\mathfrak{A}_T$. 
It is worth remarking that since $\mathfrak{A}_T$ is a concrete operator algebra, it belongs to many reasonable categories. A priori, it is not clear which choice of morphism one should consider (bounded, completely bounded, etc.). Fortunately, all reasonable choices are equivalent. It will be shown that such two algebras are boundedly isomorphic if and only if the isomorphism is implemented by an invertible. This will bring us to analyze the similarity orbit of $T$. For Cowen-Douglas operators, the similarity orbit has been extensively studied. We will leverage these results into our analysis of the study of $\mathfrak{A}_T$.

4.1. The Compact Operators. In this section, we show that if $T$ is analytic left invertible with $\text{ind}(T) = -1$, then $\mathfrak{A}_T$ contains the compact operators. Our approach is to show that, more generally $\overline{\text{Alg}}(T, L)$ contains the compact operators for any left invertible $L$. This will allow us to conclude that $\overline{\text{Alg}}(T, L) = \mathfrak{A}_T$ for any left invertible $L$. First, let us establish some notation.

Fix a left inverse $L$ of $T$. We set $F_{0,0} = I - TT^\dagger$. That is, $F_{0,0}$ is the projection onto $\ker(T^\dagger)$. We define

$$F_{n,m} := T^n(I - TT^\dagger)L^m$$

for each $n, m \in \mathbb{Z}_{\geq 0}$. For $x, y, z \in H$ we use $\theta_{x,y}$ to denote the rank one operator $z \mapsto \langle z, y \rangle x$.

Recall from Section 2 the Schauder basis and dual basis associated to $T$ and $L$. Notice that since $\text{ind}(T) = -1$, we have a simplified notation. Concretely, let $x_0 \in \ker(T^*)$ be have norm equal to 1. Then $\text{span}\{x_0\} = \ker(T^*)$. Denote the Schauder basis of $T$ and dual basis $T$ (with respect to $L$) via $x_n := T^n x_0$ and $x'_n := (L^*)^n x_0$. Then by definition, $I - TT^\dagger$ is the projection $\theta_{x_0,x_0}$. So for each $n, m$ and $x \in H$,

$$F_{n,m}(x) = T^n(I - TT^\dagger)L^m(x) = T^n(\langle L^m(x), x_0 \rangle x_0) = \langle x, x'_m \rangle x_n.$$ 

That is, $F_{n,m}$ is the rank one operator $\theta_{x_n,x'_m}$. We denote

$$\mathcal{K}_T := \overline{\text{span}}\{F_{n,m}\}_{n,m \geq 1}.$$ 

As $F_{n,m} \in \text{Alg}(T, L)$, $\mathcal{K}_T \subset \overline{\text{Alg}}(T, L)$. Furthermore, the $F_{n,m}$ are rank one operators for each $n,m$; and so $\mathcal{K}_T \subset \mathcal{K}(H)$. Our previous work on Schauder bases allows us to conclude that $\mathcal{K}_T = \mathcal{K}(H)$.

Theorem 4.1. Let $T \in \mathcal{B}(H)$ be an analytic, left invertible with $\text{ind}(T) = -1$, and $L$ be a left inverse of $T$. Then $\mathcal{K}(H) = \mathcal{K}_T$. Thus, $\overline{\text{Alg}}(T, L)$ contains the algebra of compact operators $\mathcal{K}(H)$.

Proof. Let $y, z \in H$. Since $\overline{\text{span}}\{x_n\} = \mathcal{H} = \overline{\text{span}}\{x'_n\}$, there exists a sequence of sums in $x_n$ and $x'_n$ converging to $y$ and $z$ respectively. It follows that the rank one operator $\theta_{y,z}$ is a norm limit of the span of the $\{F_{n,m}\}$ by simple estimates. Thus, $\mathcal{K}_T$ contains all the rank one operators. Since $\mathcal{K}_T$ is normclosed by definition, $\mathcal{K}_T \supset \mathcal{K}(H)$. Since $\mathcal{K}_T \subset \mathcal{K}(H)$, we have $\mathcal{K}_T = \mathcal{K}(H)$.

A consequence Theorem 4.1 is that the definition of $\mathfrak{A}_T$ is not dependent on the choice of left inverse.

Corollary 4.2. Let $T \in \mathcal{B}(H)$ be left invertible (analytic with $\text{ind}(T) = -1$), and $L$ be a left inverse of $T$. Then $\mathfrak{A}_T = \overline{\text{Alg}}(T, L)$.

Proof. By Proposition 2.2 each left inverse $L$ of $T$ has the form

$$L = T^\dagger + A(I - TT^\dagger)$$

for some $A \in \mathcal{B}(H)$. Thus, each left inverse of $T$ differs from $T^\dagger$ by a compact operator. By Theorem 4.1 $\overline{\text{Alg}}(T, L)$ contains $\mathcal{K}(H)$, and therefore $T^\dagger$. So $\overline{\text{Alg}}(T, L) \subset \mathfrak{A}_T$. Reversing the argument, $\overline{\text{Alg}}(T, L) = \mathfrak{A}_T$. \qed
Recall that an ideal $\mathcal{K}$ of a Banach Algebra $\mathfrak{A}$ is said to be essential if it has non-trivial intersection with all non-zero ideals of $\mathfrak{A}$. Alternatively, if $A \in \mathfrak{A}$ and $A \mathcal{K} = 0$, then $A = 0$. In the next section, we investigate the morphisms between algebras of the form $\mathfrak{A}_T$. An important result required in subsequent analysis is the following:

**Proposition 4.3.** The compact operators $\mathcal{K}(\mathcal{H})$ are an essential ideal of $\mathfrak{A}_T$. In fact, $\mathcal{K}(\mathcal{H})$ is contained in any closed ideal of $\mathfrak{A}_T$.

**Proof.** Let $\mathfrak{J}$ be a non-zero closed two sided ideal of $\mathfrak{A}_T$, and $A \in \mathfrak{J}$ be non-zero. Then there is some $x \in \mathcal{H}$ such that $\|Ax\| = 1$. Fix $y \in \mathcal{H}$, and let $B := \theta_{y,A(x)}$. Then $B(A(x)) = y$. Thus for all $h \in \mathcal{H}$, we have

$$BA\theta_{x,x}A^*B^*(h) = BA(\langle h, BA(x) \rangle x) = \langle h, y \rangle y = \theta_{y,y}(h).$$

Since $\mathcal{K}(\mathcal{H}) \subseteq \mathfrak{A}_T$, it follows that the rank one operators $B$ and $\theta_{x,x}A^*B^*$ are in $\mathfrak{A}_T$. Since $A \in \mathfrak{J}$ and $\mathfrak{J}$ is an ideal, we must have that $\theta_{y,y}$ is inside of $\mathfrak{J}$. Thus for any $w, z \in \mathcal{H}$, $\theta_{w,z} = \theta_{w,y}\theta_{y,z}$ is in $\mathfrak{J}$, so $\mathfrak{J}$ contains all the finite rank operators, and thus contains $\mathcal{K}(\mathcal{H})$. \qed

4.2. Isomorphisms of $\mathfrak{A}_T$. Now that we have established that the compact operators $\mathcal{K}(\mathcal{H}) \subseteq \mathfrak{A}_T$ as a minimal ideal, we may indulge ourselves with identifying the isomorphism classes of $\mathfrak{A}_T$. We will show that if $T_1$ and $T_2$ are two left invertible operators, then $\mathfrak{A}_{T_1}$ is boundedly isomorphic to $\mathfrak{A}_{T_2}$ if and only if the algebras are similar. This will be done by looking at how the bounded isomorphism behaves on the compact operators.

An interesting fact about bounded homomorphisms of C*-algebras is that they necessarily have closed range. Indeed, we have the following observation due to Pitts:

**Theorem 4.4.** (Theorem 2.6). Suppose $\mathfrak{A}$ is a C*-algebra and $\phi: \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a bounded homomorphism. Let $\mathfrak{J} = \ker \phi$. Then there exists a real number $k > 0$ such that for each $n \in \mathbb{N}$, and $R \in M_n(\mathfrak{A})$,

$$kd \text{dist}(R, M_n(\mathfrak{J})) \leq \|\phi_n(R)\|.$$  

**Corollary 4.5.** If $\phi: \mathcal{K}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is a bounded monomorphism, then there exists a real number $k$ such that

$$k\|R\| \leq \|\phi(R)\|.$$  

That is, $\phi$ has closed range.

Given an invertible operator $V \in \mathcal{B}(\mathcal{H})$, we define $\text{Ad}_V : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ via $\text{Ad}_V(T) = VTV^{-1}$. As previously mentioned, to fully analyze $\mathfrak{A}_T$, we need to determine which category we are working in. On the one hand, we can view $\mathfrak{A}_T$ as an operator algebra, with our morphisms being completely bounded homomorphisms. On the other hand, we may want to simply view $\mathfrak{A}_T$ as a Banach algebra, where the morphisms are bounded homomorphisms. Fortunately, Theorem 4.1 forces the monomorphisms of these two categories to coincide:

**Theorem B.** Let $T_i, i = 1, 2$ be left invertibles (analytic with $\text{ind}(T_i) = -1$) and $\mathfrak{A}_i = \mathfrak{A}_{T_i}$. Suppose that $\phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ is a bounded isomorphism. Then $\phi = \text{Ad}_U$ for some invertible $V \in \mathcal{B}(\mathcal{H})$.

**Proof.** Let $\phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ be a bounded isomorphism. A brief outline of the proof is as follows. We first show that $\phi \mid \mathcal{K}(\mathcal{H})$ is similar to a *-automorphism of $\mathcal{K}(\mathcal{H})$. It is well known that all *-automorphisms of $\mathcal{K}(\mathcal{H})$ have the form $\text{Ad}_U$ for some unitary operator $U$. We then use the fact that $\phi$ restricted to an essential ideal has the form $\text{Ad}_V$ to conclude that it must be equal to $\text{Ad}_V$ on all of $\mathfrak{A}_1$. The details are as follows.

Note that $\phi \mid \mathcal{K}(\mathcal{H}) : \mathcal{K}(\mathcal{H}) \rightarrow \mathfrak{A}_2 \subset \mathcal{B}(\mathcal{H})$ is a bounded representation of the compact operators. It can be shown that every bounded representation of the compact operators is similar to a *-representation (more generally, every bounded representation of a nuclear C*-algebra is similar to a *-representation [23]). Let $W \in \mathcal{B}(\mathcal{H})$ be the invertible that conjugates $\phi \mid \mathcal{K}(\mathcal{H})$ to a *-representation $\psi$. That is, $\phi(u) = W\psi(u)W^{-1}$ for every $u \in \mathcal{K}(\mathcal{H})$. 
Now let us consider the *-representation \( \psi \). Note that \( \psi : \mathcal{K}(\mathcal{H}) \to W^{-1}\mathfrak{A}_2W \). The map \( \text{Ad}_{W^{-1}} : \mathfrak{A}_2 \to W^{-1}\mathfrak{A}_2W \) carries \( \mathcal{K}(\mathcal{H}) \) to \( \mathcal{K}(\mathcal{H}) \). Since every ideal of \( W^{-1}\mathfrak{A}_2W \) has the form \( W^{-1}\mathfrak{I}_2W \) for \( \mathfrak{I} \) an ideal of \( \mathfrak{A}_2 \), it follows that \( \mathcal{K}(\mathcal{H}) \) is minimal in \( W^{-1}\mathfrak{A}_2W \). Therefore, we must have that \( \mathcal{K}(\mathcal{H}) \subseteq \psi(\mathcal{K}(\mathcal{H})) \).

Now, \( \mathcal{K}(\mathcal{H}) \) is equal to the closed span of the rank one projections on \( \mathcal{H} \). As a result, if we can show that each rank one projection \( p \) gets sent to another rank one projection under \( \psi \), then \( \psi(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H}) \), yielding equality.

To this end, let \( p \) be a rank one projection, and \( p' = \psi(p) \). If \( p' \) is not rank one, then there exists a non-zero projection \( q' \) properly contained under \( p' \). Since \( \psi(\mathcal{K}(\mathcal{H})) \) contains \( \mathcal{K}(\mathcal{H}) \), there exists a projection \( q \in \mathcal{K}(\mathcal{H}) \) such that \( \psi(q) = q' \). Regarding \( \psi \) mapping from \( \mathcal{K}(\mathcal{H}) \) to \( \psi(\mathcal{K}(\mathcal{H})) \), \( \psi \) is a *-isomorphism and hence invertible. \( \psi^{-1} \) is of course also a *-isomorphism, and therefore a positive map. Hence, if \( q' < p' \), then \( q < p \), which is absurd since \( p \) was rank one. Thus, \( \psi(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H}) \), so that \( \mathcal{K}(\mathcal{H}) = \psi(\mathcal{K}(\mathcal{H})) \).

What we have just shown is that \( \phi \vert_{\mathcal{K}(\mathcal{H})} \) is similar to a *-automorphism \( \psi \) of \( \mathcal{K}(\mathcal{H}) \). Every *-automorphism of \( \mathcal{K}(\mathcal{H}) \) is of the form \( \text{Ad}_U \) for some unitary operator \( U \). Hence, we have that

\[
\phi \vert_{\mathcal{K}(\mathcal{H})} = \text{Ad}_W \psi = \text{Ad}_W \text{Ad}_U = \text{Ad}_V
\]

where \( V = UW \). We now show that \( \phi = \text{Ad}_V \). To do this, first note that for all \( A \in \mathfrak{A}_1 \) and \( K \in \mathcal{K}(\mathcal{H}) \),

\[
\phi(A)\phi(K) = \phi(AK) = \psi(AK) = \text{Ad}_V(AK) = \text{Ad}_V(A)\text{Ad}_V(K).
\]

So it follows that

\[
(\phi(A) - \text{Ad}_V(A))\text{Ad}_V(K) = 0
\]

for each \( K \in \mathcal{K}(\mathcal{H}) \). Cycling over all \( K \in \mathcal{K}(\mathcal{H}) \), we see that

\[
(\phi(A) - \text{Ad}_V(A))\mathcal{K}(\mathcal{H}) = 0.
\]

Since \( \mathcal{K}(\mathcal{H}) \) is essential in \( \mathfrak{A}_2 \), we have that \( \phi(A) = \text{Ad}_V(A) \). \( \square \)

Theorem 4.2 is a harsh rigidity statement about classification. Indeed, \( \mathfrak{A}_1 \) is boundedly isomorphic to \( \mathfrak{A}_2 \) if and only if the algebras are similar. Consequently, if we wish to delineate these operator algebras into isomorphism classes, we need to understand the similarity orbit of left invertible operators. We define the following notation for the similarity orbit:

\[
\mathcal{S}(T) := \{ VTV^{-1} : V \in \mathcal{B}(\mathcal{H}) \text{ is invertible} \}.
\]

In classifying the algebra \( \mathfrak{A}_T \), we do not need to keep track of the similarity orbit of the Moore-Penrose inverse. Indeed, suppose \( T \) is left invertible with Moore-Penrose inverse \( T^\dagger \), \( V \) is an invertible operator, and \( T_2 := VTV^{-1} \). Then \( L_2 := VT^\dagger V^{-1} \) is a left inverse of \( T_2 \). By Corollary 4.2, \( \text{Alg}(T_2, L_2) = \mathfrak{A}_T \). Therefore to identify the isomorphism class of \( \mathfrak{A}_T \), we may disregard \( \mathcal{S}(T^\dagger) \). Hence, we pose the following question:

**Question 2.** If \( T \) is left invertible (analytic, \( \text{ind}(T) = -1 \)), what is \( \mathcal{S}(T) \)?

In general, it is impossible to completely classify the similarity orbit of an operator. However, analytic left invertible operators have added structure that aid in this analysis. By Theorem 4.1 if \( T \) is analytic, \( T^* \in \mathcal{B}_1(\Omega) \) for a disc \( \Omega \) centered at the origin. Clearly if we could identify \( \mathcal{S}(T^*) \), then we would know \( \mathcal{S}(T) \). Fortunately, similarity orbits of Cowen-Douglas operators have been extensively studied \([15, 16, 23, 18, 17]\). The similarity orbit of Cowen-Douglas operators can be completely described by \( K \)-theoretic means. We will highlight these results in the next subsection.

While the question of addressing the similarity orbit is paramount to a complete classification of our algebras \( \mathfrak{A}_T \), it is not sufficient. More explicitly, suppose \( T_1 \) and \( T_2 \) are left invertible operators with \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) isomorphic. Let \( V \) be the invertible that implements the isomorphism between
\( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \), and let \( T_3 := VT_1V^{-1} \) and \( L_3 := VT_1^TV^{-1} \). Notice \( L_3 \) is a left inverse of \( T_3 \) and that 
\[ \mathfrak{A}_T(T_3, L_3) = \mathfrak{A}_2. \]
By Corollary \[ \mathfrak{A}_3 = \mathfrak{A}_2. \]

One would therefore be tempted to reduce to the case where \( T_2 = T_3 = \text{Ad}_V(T_1) \). However, it turns out that not every left invertible \( S \in \mathfrak{A}_T \) will satisfy \( \mathfrak{A}_S = \mathfrak{A}_T \). Consider the following example:

**Example 4.6.** We will construct a left invertible operator \( T \) inside the Toeplitz algebra \( \mathcal{T} \) such that \( \mathfrak{A}_T \neq \mathcal{T} \). Consider the Hardy space \( H^2(\mathbb{T}) \). Let \( \phi_0 \in C(\mathbb{T}) \) be given by

\[ \phi_0(z) := \exp\left( \frac{\pi i}{2} (z - 1)z \right) \]

for all \( z \in \mathbb{T} \). Then \( \phi_0(1) = 1 \) and \( \phi_0(-1) = -1 \). Let \( \epsilon_n(z) = z^n \). Define \( \phi := M_{\epsilon_1} \phi_0 \). Then \( \phi \)

satisfies \( \phi(1) = \phi(-1) = 1 \). Recall the following facts about invertible functions on \( C(\mathbb{T}) \) and their associated Toeplitz operators:

**Theorem 4.7** ([26] Lem. 3.5.14, Thm. 3.5.15). Let \( \phi \in C(\mathbb{T}) \) be invertible. Then

i. There exists a unique integer \( n \) such that \( \phi = \epsilon_n e^\psi \) some \( \psi \in C(\mathbb{T}) \)

ii. If \( \phi = \epsilon_n e^\psi \), then the winding number is \( \text{wn}(\phi) = n \)

iii. We have \( \text{ind}(T_{\phi_0}) = -\text{wn}(\phi) \)

iv. \( T_{\phi} \) is invertible if and only if the winding number is zero if and only if \( \phi = e^\psi \) some \( \psi \in C(\mathbb{T}) \)

By Theorem 4.7 the winding number \( \text{wn}(\phi) = 1 \), so \( \text{ind}(T_{\phi}) = -1 \). Since both \( \epsilon_1 \) and \( \phi_0 \) belong to \( H^\infty(\mathbb{T}) \) we have that \( T_{\epsilon_1} \) and \( T_{\phi_0} \) commute, so the Toeplitz operator \( T_{\phi} \) factors:

\[ T_{\phi} = T_{\epsilon_1} T_{\phi_0} = T_{\epsilon_1} T_{\phi_0}. \]

Also by Theorem 4.7 \( T_{\phi_0} \) is invertible. The point-wise inverse of \( \phi_0 \) is also continuous on \( \mathbb{T} \). Therefore, the Toeplitz operator \( T_{\phi} \) is left invertible with left inverse

\[ L = T_{\phi_0}^{-1} T_{\epsilon_1} = T_{\phi_0}^{-1} T_{\epsilon_1}^* \in \mathcal{T}. \]

Moreover, since \( T_{\epsilon_1} \) and \( T_{\phi_0} \) commute, we have \( (T_{\phi})^n = T_{\epsilon_1} T_{\phi_0}^n \). Since \( T_{\phi_0}^n \) is invertible, \( T_{\phi_0}^n H^2(\mathbb{T}) = H^2(\mathbb{T}) \). Consequently,

\[ \bigcap T_{\phi}^n H^2(\mathbb{T}) = \bigcap T_{\epsilon_1} H^2(\mathbb{T}) = 0 \]

so \( T_{\phi} \) is analytic. Recall that \( \mathfrak{A}_T \subset C^*(T) \) for any left invertible \( T \). We remark that \( C^*(T_{\phi}) \neq \mathcal{T} \).

This follows from the following result due to Coburn:

**Lemma 4.8** ([27] Cor. 6.3). If \( \phi \) is in the disc algebra, then \( C^*(T_{\phi}) = \mathcal{T} \) if and only if \( \phi \) is injective.

It is shown in [27] that \( C^*(T_{\phi})/\mathfrak{I} \mathcal{H} \) is isomorphic to continuous functions on \( \mathbb{T}/\sim \), where \( \sim \) is an equivalence relation identifying all points \( z, w \in \mathbb{T} \) such that \( \phi(z) = \phi(w) \). Since \( \phi(1) = \phi(-1) \), it follows by the above lemma that \( \mathfrak{A}_T \leq C^*(T_{\phi}) \neq \mathcal{T} \). This concludes our example.

What the above example demonstrates is that not every left invertible operator in \( \mathfrak{A}_T \) generates \( \mathfrak{A}_T \). Therefore, determining the similarity orbit is not sufficient to delineate the isomorphism classes of \( \mathfrak{A}_T \). Concretely, suppose \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) are generated by \( T_1 \) and \( T_2 \) respectively. To determine if \( \mathfrak{A}_1 \) is isomorphic to \( \mathfrak{A}_2 \), it is not sufficient to verify that \( \mathfrak{A}_2 \) possesses an operator \( T_3 \) similar to \( T_1 \). This would demonstrate that \( \mathfrak{A}_1 \) is isomorphic to a subalgebra of \( \mathfrak{A}_2 \). If one wanted \( \mathfrak{A}_1 \) to be isomorphic to \( \mathfrak{A}_2 \), it is necessary to show that \( T_3 \) also generates \( \mathfrak{A}_2 \). With this caveat emphasized, we spend the next section investigating the similarity orbit of our class of left invertible operators.
4.3. The Similarity Orbit of \( T \). If \( T \) is an analytic left invertible operator with \( \text{ind}(T) = -1 \), then by Theorem 4.10 \( T^* \in B_1(\Omega) \) for \( \Omega = \{ \lambda : |\lambda| < \epsilon \} \). Therefore, classifying \( S(T) \) is equivalent to classifying the similarity orbit of Cowen-Douglas operators over a small disc centered at the origin. The problem of identifying when two Cowen-Douglas operators are similar is a classic one. In Cowen and Douglas’ original work, they show that two operators \( R_1, R_2 \in B_1(\Omega) \) are unitarily equivalent if and only if the curvature on the associated hermitian holomorphic vector bundles are equal [13]. However, they did not completely classify the similarity orbit. Various authors have since worked on this problem, successfully describing the similarity orbit of Cowen-Douglas operators.

In [28], Jiang showed that if \( R \) is Cowen-Douglas, then \( \{R\}'/\text{rad}(\{R\}') \) is commutative, where \( \{R\}' \) is the commutant of \( R \) and \( \text{rad}(\{R\}') \) is the radical of \( \{R\}' \). This allowed Jiang to describe the similarity orbit of strongly irreducible Cowen-Douglas operators using the \( K_0 \)-group of the commutant algebra. Later, Jiang, Guo, and Ji gave a similarity classification of all Cowen-Douglas operators using the commutant [21]. The result for Cowen-Douglas operators with Fredholm index 1 is as follows:

**Theorem 4.9** [21 - Prop. 5.1.7]. Let \( A, B \in B_1(\Omega) \). Then \( A \) is similar to \( B \) if and only if
\[
K_0(\{A \oplus B\}') \cong \mathbb{Z}.
\]

This result and its generalizations solve the question of the similarity orbit, and therefore, the isomorphism problem for \( \mathfrak{A}_T \) from the last section. As stated, Theorem 4.9 is a rather difficult theorem to verify. Luckily in [28], Jiang provided the following theorem which concretely identifies the requirements on the isomorphism between the \( K_0 \) groups generated by \( A \) and \( B \):

**Theorem 4.10** [28 - Thm 4.4]. Two strongly irreducible Cowen-Douglas operators \( A \) and \( B \) are similar if and only if there is a group isomorphism \( \alpha : K_0(\{A\}') \rightarrow K_0(\{B\}') \) satisfying the following:

i. \( \alpha(\vee(\{A\}')) = \vee(\{B\}'), \) where \( \vee(\{A\}) \) is the equivalence classes of the semi-group \( \text{Proj}(M_\infty(\{A\}')) \)

ii. \( \alpha([I_{\{A\}'}]) = [I_{\{B\}'}], \) where \( I_{\{A\}'} \) is the equivalence class associated to the identity in \( \text{Proj}(M_\infty(\{A\}')) \)

iii. there exists non-zero idempotents \( p \in M_\infty(\{A\}') \) and \( q \in M_\infty(\{B\}') \) such that \( \alpha([p]) = [q] \) and \( p \) is equivalent to \( q \) in \( M_\infty(\{A \oplus B\}') \).

5. Example - Subnormal, Essentially Normal Operators

We now turn to an important class of nontrivial examples of \( \mathfrak{A}_T \). These examples will involve the theory of subnormal operators, and so, we will begin with a brief refresher of basic results required. We will then discuss a result due to Olin, Thomson, Keough and McGuire, describing the \( C^* \)-algebra generated by a subnormal, essentially normal, irreducible operator [29 30 31]. Using this framework, we characterize the algebras \( \mathfrak{A}_S \) for \( S \) a subnormal, essentially normal left invertible operator.

5.1. Basics of Subnormal Operators. In this subsection, we briefly recall some of the basics of subnormal operators. We begin with a definition:

**Definition 5.1.** An operator \( S \in \mathcal{B}(\mathcal{H}) \) is called subnormal if there exists a Hilbert space \( \mathcal{H} \) such that \( \mathcal{H} \supset \mathcal{H} \) and a normal operator \( N \in \mathcal{B}(\mathcal{H}) \) such that

i. \( N \mathcal{H} \subset \mathcal{H} \)

ii. \( S = N \mid_{\mathcal{H}} \)

Such a normal operator \( N \) is called a normal extension of \( S \). The operator \( N \) is said to be a minimal normal extension if \( \mathcal{H} \) has no proper subspace reducing \( N \) and containing \( \mathcal{H} \).
It can be shown that any two minimal normal extensions of a subnormal operator \( S \) are unitarily equivalent. Thus, we usually refer to the minimal normal extension, and denote it by \( N := \text{mne}(S) \).

Classic examples of a subnormal operators are the Toeplitz operators \( T_f \) on \( H^2(\mathbb{T}) \) for \( f \in L^2(\mathbb{T}) \). The minimal normal extension is given by \( M_f \) on \( L^2(\mathbb{T}) \) (for \( f \) non-constant). It is not hard to see that all subnormal operators have this form. We make the following definition:

**Definition 5.2.** Let \( S \in \mathcal{B}(\mathcal{H}) \) be a subnormal operator, and \( N = \text{mne}(S) \in \mathcal{B}(\mathcal{H}) \). If \( \mu \) is a scalar-valued spectral measure associated to \( N \), and \( f \in L^\infty(\sigma(N), \mu) \), we define the Toeplitz operator \( T_f \in \mathcal{B}(\mathcal{H}) \) via

\[
T_f := P(f(N)) |_{\mathcal{H}}
\]

where \( P \) is the orthogonal projection of \( \mathcal{H} \) onto \( \mathcal{H} \).

Clearly in the case when \( S \) is the unilateral shift, the above are the Toeplitz operators on \( H^2(\mathbb{T}) \). For any subnormal operator \( S \), we have that \( T_z = S \), and that \( T_{z^n} T_{z^m} = T_{z^{n+m}} \). Consequently, \( \{T_f : f \in C(\sigma(N))\} \subset C^*(S) \). We remark that, while the map from \( L^\infty(\sigma(N), \mu) \) to \( \mathcal{B}(\mathcal{H}) \) via \( f \mapsto T_f \) is positive and norm decreasing, it is not multiplicative.

Before we state the results about the C*-algebra generated by subnormal operators, we need a result about the relationship between the spectrum of \( S \) and the spectrum of \( N \).

**Proposition 5.3** (\cite{32}). Let \( S \) be a subnormal operator with \( N = \text{mne}(S) \). Then the following inclusions hold:

\[
\partial \sigma(S) \subseteq \sigma_{ap}(S) \subseteq \sigma_{ap}(N) = \sigma(N) \subseteq \sigma(S)
\]

where \( \sigma_{ap}(S) \) is the approximate point spectrum of \( S \).

5.2. **The C*-Algebra Generated By Subnormal Operator.** In this section, we highlight some C*-algebraic results about subnormal operators due to Olin, Thomson, Keough and McGuire. We remark that every operator in \( C^*(S) \) has the form of “Toeplitz operator plus compact”. We can leverage results about \( C^*(S) \) to gain valuable insight into the structure of \( \mathfrak{A}_S \). In particular, we show that \( \mathfrak{A}_S \) consists of the operators with symbols in the uniform algebra generated by Laurent polynomials.

If \( N \) is a normal operator, there is a natural identification of \( C^*(N) \) with \( C(\sigma(N)) \) given by the Gelfand transform. There is also an intimate connection between the C*-algebra generated by a subnormal operator \( S \) and its minimal normal extension \( N \). When \( S \) is the unilateral shift, the commutative C*-algebra \( C^*(N) \) appears in the symbols of the Toeplitz operators. Under the added assumptions of irreducibility and essential normality, the same is true for subnormal operators:

**Theorem 5.4** (\cite{29} \cite{30} \cite{31}). If \( S \) is an irreducible, subnormal, essentially normal operator, then

i. \( \sigma_{ap}(S) = \sigma_e(S) \)

ii. For each \( f, g \in C(\sigma(N)) \), we have

a. \( T_f \in \mathcal{K}(\mathcal{H}) \) if and only if \( f \) vanishes on \( \sigma_e(S) \)

b. \( \|T_f + \mathcal{K}(\mathcal{H})\| = \|f\|_{\sigma(S)} \)

c. \( T_{fg} - T_f T_g \in \mathcal{K}(\mathcal{H}) \)

d. \( \sigma_e(T_f) = f(\sigma_e(S)) \)

iii. Every element of \( C^*(S) \) can be written as a sum of a Toeplitz operator and compact:

\[
C^*(S) = \{T_f + K : f \in C(\sigma(N)), K \in \mathcal{K}(\mathcal{H})\}.
\]

Moreover, if \( \sigma(N) = \sigma_{ap}(S) \), then each element has \( A \in C^*(S) \) has a unique representation of the form \( T_f + K \). If \( \sigma(N) \neq \sigma_{ap}(S) \), \( A \) may be expressed as \( A = T_{f_1} + K_1 = T_{f_2} + K_2 \), where \( f_1 |_{\sigma_e(S)} = f_2 |_{\sigma_e(S)} \).

We now make a simple connection between spectral data of the operators appearing in Theorem 5.4 and left invertibility.
Lemma 5.5. Let \( S \) be a subnormal operator with \( N = mne(S) \). If \( N \) is invertible, then \( S \) is left invertible with \( L = T_{z^{-1}} \) a left inverse. If \( \sigma(N) = \sigma_{ap}(S) \), then \( S \) is left invertible if and only if \( N \) is invertible.

Proof. If \( N \) is invertible, then the Toeplitz operator \( T_{z^{-1}} = P(N^{-1}) |_{\mathcal{H}} \) is well defined. Since \( N \) is a normal extension of \( S \), we have for each \( x \in \mathcal{H} \)

\[
T_{z^{-1}}Sx = T_{z^{-1}}(Nx) = P(N^{-1}Nx) = Px = x.
\]

If \( \sigma(N) = \sigma_{ap}(S) \), then \( S \) is left invertible implies \( 0 \not\in \sigma_e(S) = \sigma(N) \).

With the basic theory of subnormal operators outlined, we can now describe the structure of \( \mathfrak{A}_S \) for a prototypical class of subnormal operators.

Theorem C. Let \( S \) be an analytic left invertible, \( \text{ind}(S) = -1 \), essentially normal, subnormal operator with \( N := mne(S) \) such that \( \sigma(N) = \sigma_{ap}(S) \). Let \( \mathcal{B} \) be the uniform algebra generated by the functions \( z \) and \( z^{-1} \) on \( \sigma_e(S) \). Then

\[
\mathfrak{A}_S = \{ T_f + K : f \in \mathcal{B}, K \in \mathcal{H}(\mathcal{H}) \}.
\]

Moreover, the representation of each element as \( T_f + K \) is unique.

Proof. By Lemma 5.5 \( L := T_{z^{-1}} \) is a left inverse of \( S \). By Corollary 4.2 \( \mathfrak{A}_S \) is the norm-closed subalgebra of \( C^*(S) \) generated by \( T_z \) and \( T_{z^{-1}} \). Since \( S \) is analytic, it is strongly irreducible, and hence, irreducible. Therefore by Theorem 5.4 each element of \( \mathfrak{A}_S \) has a unique representation as \( T_f + K \) for some \( f \in C(\sigma(N)) \) and \( \sigma(N) = \sigma_{ap}(S) = \sigma_e(S) \). Moreover, \( L^n = T_{z^{-n}} + K \) for some compact operator \( K \). Since \( \mathfrak{A}_S \) contains the compacts, it follows that \( T_{z^k} \in \mathfrak{A}_S \) for each \( k \in \mathbb{Z} \). Hence, for each \( p \in \text{Alg}(z, z^{-1}) \), we have that \( T_p \in \mathfrak{A}_S \). Using this information, we now show that \( \mathfrak{A}_S = \{ T_f + K : f \in \mathcal{B}, K \in \mathcal{H}(\mathcal{H}) \} \). To do this, it suffices to show that \( T_f \in \mathfrak{A}_S \) if and only if \( f \in \mathcal{B} \).

First, suppose that \( T_f \in \mathfrak{A}_S \) for some \( f \in C(\sigma(N)) \). Then for every \( \epsilon > 0 \), there exists a Laurent polynomial \( p \in \text{Alg}(z, z^{-1}) \) such that \( \| T_f - T_p \| < \epsilon \). By Theorem 5.4

\[
\epsilon > \| T_f - T_p \| = \| T_{f-p} \| \geq \| T_{f-p} + \mathcal{H}(\mathcal{H}) \| = \| f - p \|.
\]

Hence, \( f \in \mathcal{B} \). For the other inclusion, suppose to the contrary that \( f \in \mathcal{B} \) but \( T_f \notin \mathfrak{A}_S \). Then there exists a \( \delta > 0 \) such that for each \( p \in \text{Alg}(z, z^{-1}) \) and \( K \in \mathcal{H}(\mathcal{H}) \), we have \( \| T_f - (T_p + K) \| > \delta \). In particular, this should hold for any \( p \) such that \( \| f - p \| < \delta/2 \). Hence

\[
\delta \leq \inf_{K \in \mathcal{H}(\mathcal{H})} \| T_f - (T_p + K) \| = \| T_{f-p} + \mathcal{H}(\mathcal{H}) \| = \| f - p \| < \delta/2,
\]

which is absurd. Hence, \( T_f \) must be in \( \mathfrak{A}_S \), completing the proof.

Notice that in Theorem C, we can drop the requirement that \( \sigma(N) = \sigma_{ap}(S) \), so long as the minimal normal extension is invertible. In this case however, one will lose the uniqueness of the representation \( T_f + K \) as discussed in Theorem 5.4. As a corollary to Theorem C we get a description of \( \mathfrak{A}_T \) for analytic Toeplitz operators on \( H^2(\mathbb{T}) \) with Fredholm index \(-1\).

Corollary 5.6. Let \( g \) be an analytic function on \( \mathbb{T} \) and \( X = \text{ran}(g) \). If \( T = T_g \) and \( \text{wn}(g) = -1 \), then \( T \) is an analytic left invertible operator if and only if \( 0 \notin X \). Moreover, if \( \mathcal{B} \) is the uniform algebra generated by \( z \) and \( z^{-1} \) on \( X \), then

\[
\mathfrak{A}_T = \{ T_f + K : f \in \mathcal{B}, K \in \mathcal{H}(H^2(\mathbb{T})) \}.
\]

The hypotheses of Theorem C are natural, but numerous. This is to guarantee that we \( S \) remain within our current focus of study. We remark that even if \( S \) is left invertible, irreducible, subnormal, essentially normal operator, it need not be analytic. Indeed, a result by Qing shows that every Cowen-Douglas operator is cyclic [35]. An operator \( R \in \mathcal{B}(\mathcal{H}) \) is said to be cyclic if there exists
a $x \in \mathcal{H}$ such that $\{R^n x\}_{n=0}^{\infty}$ is norm dense in $\mathcal{H}$. While all Cowen-Douglas operators must be cyclic, the adjoints of general subnormal operators need not be cyclic. A long standing problem posed by Deddens and Wogen asked which subnormal operators had cyclic adjoints. Feldman answered this question in \cite{34}. A subnormal operator is said to be pure if it has no nontrivial normal summand. Every subnormal operator can be decomposed as $S = S_p \oplus N$, where $S_p$ is pure and $N$ is normal. The general cyclicity result is as follows:

**Theorem 5.7** (Feldman \cite{34}). If $S = S_p \oplus N$ is a subnormal operator, then $S^*$ is cyclic if and only if $N$ is cyclic. In particular, pure subnormal operators have cyclic adjoints.

Having a cyclic vector clearly is not sufficient for an operator to be Cowen-Douglas. However, Theorem 5.7 is a condition of necessity. Thomson showed in \cite{35} that if $S$ is a pure, cyclic subnormal operator, then $S^*$ is Cowen-Douglas. However, as far the author is aware, there is no known elementary equivalence to guarantee $S^*$ is Cowen-Douglas.

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