Current-loops, phase transitions, and the Higgs mechanism in Josephson-coupled multi-component superconductors

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The \(N\)-component London \(U(1)\) superconductor is expressed in terms of integer-valued supercurrents. We show that the inclusion of inter-band Josephson couplings introduces monopolies in the current fields, which convert the phase transitions of the charge-neutral sector to crossovers. The monopolies only couple to the neutral sector, and leave the phase transition of the charged sector intact. The remnant non-critical fluctuations in the neutral sector influence the one remaining phase transition in the charged sector, and may alter this phase transition from a \(3D\) XY inverted phase transition into a first-order phase transition depending on what the values of the gauge-charge and the inter-component Josephson coupling are. This preemptive effect becomes more pronounced with increasing number of components \(N\), since the number of charge-neutral fluctuating modes that can influence the charged sector increases with \(N\). We also calculate the gauge-field correlator, and by extension the Higgs mass, in terms of current-current correlators. We show that the onset of the Higgs-mass of the photon (Meissner-effect) is given in terms of a current-loop blowout associated with going into the superconducting state as the temperature of the system is lowered.

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I. INTRODUCTION

Models with multiple \(U(1)\) condensates coupled by a vector potential are relevant to a variety of condensed matter systems. The number of possible interactions between the individual condensates make the models much more complex than single-band systems. Multiple, individually conserved condensates are applicable to systems of low temperature atoms, such as hydrogen under extreme pressures\textsuperscript{1–6} and as effective models of easy-plane quantum anti-ferromagnets\textsuperscript{7,8}. Superconductors with multiple superconducting bands, such as MgB\textsubscript{2}\textsuperscript{9–11} and iron pnictides\textsuperscript{12} may also be described by a model of multiple \(U(1)\) condensates, but in these systems the individual condensates are not conserved. Inter-band Josephson couplings must always be included, as they cannot \textit{a priori} be excluded on symmetry grounds.

Ginzburg-Landau models of \(N\)-component superconductors in the London limit host a rich variety of interesting phenomena\textsuperscript{13–16}. Each condensate supports topological vortex line defects, which represent disorder in the condensate ordering field. When the condensates are coupled through a gauge field, the vortices carry magnetic flux quanta, and may be bound into composite vortices with \(\pm 2\pi\) phase windings in multiple condensates\textsuperscript{17}. It turns out that this gives rise to composite superfluid modes that do not couple to the gauge field, even though their constituent vortices interact via the gauge field. In addition to the superfluid modes, there will be a single charged mode which is coupled by the gauge field. This causes the \(N\)-component model without Josephson interactions to have \(N - 1\) superfluid phase transitions and a single superconducting phase transition\textsuperscript{17}. For certain values of the gauge charge these transitions will interfere in a non-trivial way, causing the transitions to merge in a single first-order transition\textsuperscript{18,19}.

The question of the nature of the phase transitions present in Josephson-coupled multiband superconductors is of considerable interest. Symmetry arguments dictate that the inclusion of the Josephson coupling breaks the \([U(1)]^N\) symmetry down to \(U(1)\), at any strength. The Josephson term locks the superfluid modes so that the phase transition in the neutral sector is replaced by a crossover\textsuperscript{17}, while the phase transition in the gauge-coupled sector is expected to remain. If this transition remains continuous, it is expected to be in the inverted \(3D\) XY universality class\textsuperscript{17}. A recent study has observed a first order transition in this model for weak Josephson coupling\textsuperscript{20}, suggesting a subtle interplay between the two length scales dictated by the Josephson length and the magnetic field penetration depth. A schematic phase diagram is shown in Fig. 1 for the two-component case. This is based on arguments provided in this work, and supports the numerical results obtained in recent numerical studies\textsuperscript{20}. Also of note is multiband superconductors with frustrated inter-band couplings, which is \(U(1) \times \mathbb{Z}_2\) symmetric. These systems have been shown to have a single first-order transition in three dimensions from a symmetric state into a state that breaks both \(U(1)\) and \(\mathbb{Z}_2\) symmetry for weak values of the gauge field coupling. For stronger values of the charge, the transitions split\textsuperscript{21,22}.

In this paper we present an alternate approach to the multiband superconductor which has certain advantages over standard formulations, allowing further analytical insights to be made. In particular, we are able reconcile the different results for the character of the phase transition in the charged sector found in Refs. \textsuperscript{17} and \textsuperscript{20} in the presence of interband Josephson-couplings. By applying a character expansion\textsuperscript{23,24} to the action, we replace the phases of the order parameter with integer-current fields. These currents are the actual supercurrents of the model. Section II presents the details and basic properties of the multiband superconductor in the London limit. In Section III A we present the character expansion, apply it to the model with no Josephson coupling, and compare the resulting representation to the original model. We apply the character expansion to the multiband superconductor with Josephson couplings in Section III B and discuss it in the light of the current representation. In Section IV we
present the calculation of the Higgs mass in terms of current-correlators. We present our conclusions in Section V.

II. STANDARD REPRESENTATION OF THE MODEL

We consider a model of $N$ bosonic complex matter fields in three dimensions. The matter fields are given by $\psi_\alpha(r) = |\psi_\alpha(r)| \exp i\theta_\alpha(r)$, interacting through the electromagnetic vector potential, $A(r)$. We also allow inter-band Josephson couplings of the matter fields. In the general case, this is described by a partition function

$$Z = \int \mathcal{D}A \left( \prod_\alpha \int \mathcal{D}\psi_\alpha \right) e^{-S},$$

where the action is

$$S = \beta \int d^3r \left\{ \frac{1}{2} \sum_\alpha \left( |\nabla - ieA(r)| \psi_\alpha(r) \right)^2 + V(\{|\psi_\alpha(r)|\}) + \frac{1}{2} (\nabla \times A(r))^2 \right.$$ 

$$- \sum_{\alpha<\beta} \lambda_{\alpha,\beta} |\psi_\alpha(r)| \psi_\beta(r) \cos (\theta_\alpha(r) - \theta_\beta(r)) \right\}. \quad (2)$$

The potential $V$ contains terms that are powers of $|\psi_\alpha|$. At this point we employ the phase-only, or London, approximation and choose all bare stiffnesses, $|\psi_\alpha|$, equal to unity. Hence, $V$ is an unimportant constant. We will also focus on equal couplings between all bands, i.e. $\lambda_{\alpha,\beta} = \lambda \forall \alpha, \beta$. The action is then given by

$$S = \beta \int d^3r \left\{ \frac{1}{2} \sum_\alpha \left( \nabla \theta_\alpha(r) - eA(r) \right)^2 + \frac{1}{2} (\nabla \times A(r))^2 \right.$$ 

$$- \lambda \sum_{\alpha<\beta} \cos (\theta_\alpha(r) - \theta_\beta(r)) \right\}. \quad (3)$$

We regularize this action on a cubic lattice of size $L^3$ by defining the fields on a discrete set of coordinates $r_\mu \in \{1, \ldots, L\}$, that is $\theta_\alpha(r) \rightarrow \theta_{r,\alpha}$ and $A(r) \rightarrow A_r$. On the lattice, the action reads

$$S = \beta \sum_r \left\{ - \sum_{\mu,\alpha} \cos (\Delta_\mu \theta_{r,\alpha} - eA_{r,\mu}) + \frac{1}{2} (\Delta A_r)^2 \right.$$ 

$$- \lambda \sum_{\alpha<\beta} \cos (\theta_{r,\alpha} - \theta_{r,\beta}) \right\}. \quad (4)$$

Here, we use the cosine function to represent the kinetic term of the continuum Hamiltonian in a way that preserves the periodic nature of the phases. Alternatively, one may arrive at Eq. (4) by directly replacing the derivatives in Eq. (2) with the gauge invariant forward difference,

$$(\nabla - ieA(r))\psi_\alpha(r) \rightarrow \psi_{r+\mu,\alpha} e^{-ieA_r} - \psi_{r,\alpha}, \quad (5)$$
and then taking the London limit as described above. We discuss the two-dimensional case in Appendix B.

In the formulation of Eq. (4) with \( \lambda = 0 \), the model is known\(^{17,19} \) to have one phase transition from a normal state to a superconducting state in one composite degree of freedom, and \( N-1 \) phase transitions from a normal fluid to a superfluid in the remaining degrees of freedom. The reason for this division into one superconducting and \( N-1 \) superfluid degrees of freedom becomes apparent when one correctly identifies the relevant combinations of the phase fields. The part of the continuum action describing the coupling between the phases and the gauge field is

\[
S' = \beta \int dr \left\{ \frac{1}{2} \sum_{\alpha} (\nabla \theta_{\alpha}(r) - eA(r))^2 \right\}.
\]

(6)

This can be rewritten into\(^{17} \)

\[
S' = \beta \int dr \left\{ \frac{1}{2N} \left( \sum_{\alpha} \nabla \theta_{\alpha}(r) - NeA(r) \right)^2 + \frac{1}{2N} \sum_{\alpha < \beta} \nabla (\theta_{\alpha} - \theta_{\beta})^2 \right\}.
\]

(7)

Hence, the phase combination \( \sum_{\alpha} \theta_{\alpha} \) will couple to the gauge field, and is identified as the single charged mode, while all other combinations \( \theta_{\alpha} - \theta_{\beta} \) do not couple, and are neutral. Note that for \( N = 1 \) only the charged mode remains. Two important points need to be emphasized. Firstly, the composite variables are not compact in the same sense that the individual phases are. This means that the composite variables do not support topological defects by themselves, only composite topological defects. Secondly, the last term in the action of Eq. (7) has \( N(N-1)/2 \) terms. Therefore, one may not interpret the phase differences \( \theta_{\alpha} - \theta_{\beta} \) as independent degrees of freedom. This is because of the multiple connectedness of the physical space, fluctuations in a single individual phase induce fluctuations in \( N-1 \) composite neutral modes, as well as in the charged mode.

In the present form, with \( \lambda = 0 \) and \( e \) sufficiently large, this model is known to have one phase transition in the inverted 3dXY-universality class, and \( N-1 \) transitions in the 3dXY-universality class at a higher temperature\(^{17,19} \). These transitions correspond to proliferations of the composite charged mode and the composite neutral modes, respectively. If the charge is lowered, the charged and neutral transitions will approach each other in temperature. When they merge, the proliferation of neutral vortices will trigger proliferation of the charged mode. Consequently, the \( N \) phase transitions collapse into a single first-order transition. This interplay between the charged and neutral sector has been coined a *preemptive* phase transition\(^{25} \), and has been verified numerically in two-component systems in the absence of inter-component Josephson-coupling in several detailed large-scale Monte Carlo simulations\(^{18,19,25} \).

In the following Section, we reformulate the model in terms of integer-valued current fields, considering first the case with zero Josephson-coupling and then move on to include Josephson coupling. The first case is useful to consider in connecting the results of previous works mentioned above to the current-formulation.

## III. CURRENT REPRESENTATION OF THE MODEL

### A. Zero intercomponent Josephson coupling

The basis of the expansion used is a character expansion\(^{23,24} \),

\[
e^{\beta \cos \gamma} = \sum_{b=-\infty}^{\infty} I_b(\beta) e^{ib\gamma},
\]

where \( I_b(\beta) \) are the modified Bessel functions of integer order. We apply this to the terms \( \exp \beta \cos(\Delta \theta_{r,\alpha} - eA_r) \) for each value of \( r, \mu, \) and \( \alpha \). This introduces integer vector fields \( b_{r,\alpha} \) representing supercurrents. In fact, the integer vector fields will be the actual physical supercurrents of the system\(^{24} \). The low-temperature phase is characterized by a state with proliferated current-loops on all length scales, while the high-temperature phase only features small current-loops.

By applying Eq. (8) to the partition function with Eq. (4) as the action, and integrating out the phases and the gauge field, details of which may be found in Appendix A, we arrive at the partition function

\[
Z = \sum_{\{b,n\}} \prod_{r,\alpha} \delta_{\Delta \cdot b_{r,\alpha}} \prod_{r,\mu,\alpha} I_{b_{r,\mu}}(\beta) \prod_{r,r'} e^{-\frac{\beta}{2} \sum_{\alpha,\beta} b_{r,\alpha} b_{r',\beta} D(r-r')},
\]

where \( (r) \) has been verified numerically in two-component systems in the absence of inter-component Josephson-coupling in several detailed large-scale Monte Carlo simulations. The basis of the expansion used is a character expansion\(^{23,24} \),

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\]

This is a model of \( N \) current fields, with contact intra-component interactions parametrized by the Bessel functions, and long-range intra- and inter-component interactions originating with the gauge-field fluctuations, \( D(r-r') \). The constraint \( \Delta \cdot b_{r,\alpha} = 0 \) forces the currents, \( b_{r,\alpha} \) to form closed loops, and implies a non-analytical behavior of each individual component, and an associated phase transition.

In the current language, the interpretation of the phase transitions explained in the previous Section is as follows. Consider first a single component model. In the high temperature state, only the lowest term in the Bessel-function expansion will contribute, and only small loops of supercurrents will be present in the system. As the temperature is lowered all orders of the expansion contribute, and the integer currents will proliferate, filling the system with loops of supercurrent. In the low temperature state all \( b \)-fields have proliferated. As temperature is increased, the proliferated current loops in the charged sector will collapse. Only the neutral superfluid currents fill the system, and the state is therefore a metallic superfluid\(^{15} \). As temperature is raised further the superfluid currents collapse as well, and the system is in the normal metallic state.
B. Non-zero intercomponent Josephson couplings

The expansion of Eq. (8) may also be applied to the Josephson term. The expansion is only valid when the argument of the cosine is expanded around zero, the present formulation is therefore not valid for any ground state which does not fulfill this requirement. In particular, if the Josephson coupling is negative and sufficiently strong, the phase differences will be locked to nonzero values\textsuperscript{21,22}. For \( N = 2 \) the phases are locked to \( \pi \), while for \( N = 3 \) the ground state of the three phases may form a star-pattern with an accompanying \( Z_2 \) symmetry associated with the two possible chiralities of the star\textsuperscript{21,22}. These cases are not covered by the current-loop formulation derived from the character-expansion Eq. (8). While the above arguments do not constrain us to only consider all Josephson couplings equal, we may limit our considerations to the case \( \lambda_{\alpha\beta} = \lambda > 0 \) without loss of generality in the present discussion. Having universal \( \lambda_{\alpha\beta} \) will not allow for any additional physics than simply having unequal strength of the individual phase lockings, when they are constrained to be all positive.

Applying the expansion introduces an additional \( N(N - 1)/2 \) integer fields \( m_{r,\alpha,\beta} \). After expanding both the kinetic terms and the Josephson terms, the partition function reads

\[
Z = \int \mathcal{D}A \left( \prod_{\alpha} \int \mathcal{D}\theta_{\alpha} \right) \times \prod_{r,\alpha} \sum_{\beta=-\infty}^{\infty} I_{b_{r,\mu,\alpha}}(\beta) e^{ib_{r,\mu,\alpha}(\Delta\theta_{\alpha} - eA_{r,\mu})} \times \prod_{r,\alpha<\beta} \sum_{m_{r,\alpha,\beta}=-\infty}^{\infty} I_{m_{r,\alpha,\beta}}(\beta\lambda) e^{im_{r,\alpha,\beta}(\theta_{\alpha} - \theta_{\beta})} \times \prod_{r} e^{-\frac{\beta}{2}((\Delta A_r)^2) (10)}
\]

The effect of the Josephson coupling becomes apparent when we integrate out the phase fields. The divergences of the \( b \) fields will no longer be constrained to zero, but may take any finite integer value, determined by the value of the \( m \) fields. The new constraints read

\[
\Delta \cdot b_{r,\alpha} = \sum_{\beta \neq \alpha} m_{r,\alpha,\beta} \forall \alpha, r;
\]

where we have defined \( m_{r,\alpha,\beta} = -m_{r,\beta,\alpha} \). The gauge-term is not coupled directly to the \( m \) fields, and we may integrate it out in the same fashion as before. The resulting partition function is

\[
Z = \sum_{\{b,m\}} \prod_{r,\alpha} \delta^{\Delta} b_{r,\alpha} \cdot \sum_{\beta \neq \alpha} m_{r,\alpha,\beta} \prod_{r,\mu,\alpha} I_{b_{r,\mu,\alpha}}(\beta) \prod_{r,\alpha<\beta} I_{m_{r,\alpha,\beta}}(\beta\lambda) \prod_{r,r'} e^{-\frac{\beta}{2\pi} \sum_{\alpha,\beta} b_{r,\alpha} \cdot b_{r',\beta} D(r-r')} (12)
\]

C. Monopoles and phase transitions

The effect of the \( m \) fields is to introduce monopoles into the closed loops of \( b \) currents. A current of a particular component (color) may now terminate at any site. However, this termination must always be accompanied by a current of another color originating at the same site. Termination of a current of one component, and the appearance of a current of another component at the same site represents an excitation of \( \pm 1 \) in \( m \). An important observation is that if one adds the constraints, we have

\[
\sum_{\alpha} \Delta \cdot b_{r,\alpha} = 0 \forall r. (13)
\]

This reflects the color changing event stated above, the total current when summing over all colors is conserved at all sites. It also shows that there is a particular combination of currents, the sum of all components, which will be divergence-free. The net effect of the Josephson coupling, pictorially, is to chop up the closed currents of the individual components and glue them together into closed loops that may change color on any site.

We may expand the partition function first in terms of \( m \) fields, and then in terms of \( \lambda \), by using the Bessel-function representation

\[
I_{\nu}(z) = \left( \frac{z}{2} \right)^{\nu} \sum_{k=0}^{\infty} \frac{z^{2k}}{k!(\nu+k)!} (14)
\]

This demonstrates that the partition function consists of a single term with zero divergence on all sites, which we know has one or more phase transitions from a superconducting superfluid state into a non-superconducting normal fluid, and many terms where the divergence of \( b_{r,\alpha} \) is finite on any number of sites.

Let us now consider two limits, and assume \( e \) is large, so that there is no preemptive effect for \( \lambda = 0 \). For \( \lambda = 0 \), it is evident that only \( m = 0 \) will contribute, and we are left with only divergenceless terms, and hence the behaviour described previously. The other limit is \( \lambda \to \infty \). In this case we must examine the asymptotic form of the Bessel functions, which to leading order in the argument is

\[
I_{m}(z) \sim \frac{e^{z}}{\sqrt{2\pi z}}, (15)
\]

i.e. independent of \( m \), and the monopole field will fluctuate strongly, causing the zero-divergence constraint on each component to be removed. The only remaining constraint on the current fields pertains to the composite current \( \sum_{\alpha} b_{\alpha} \), which is divergence-free. The interpretation of this is that the phase transitions in the \( N - 1 \) superfluid modes are converted to crossovers by the Josephson coupling, while the single superconducting mode still undergoes a genuine phase transition. The neutral crossover will be far removed from the charged phase transition in this limit, and the remaining fluctuations in the neutral sector will be almost completely suppressed. There is no possibility of any interference between the sectors, and
Therefore no preemptive phase transition. The phase transition in the charged sector will therefore be in the universality class of the inverted 3DXY phase transition.

For intermediate and small values of $\lambda$, the effect of the Josephson coupling on the interplay between the charged and neutral sectors is quite subtle in the present formulation, and will be discussed in the following section.

D. Charged and neutral currents

We start with the action where the phase sum and phase differences have been separated, Eq. (7), with a Josephson coupling included. To simplify the notation, we introduce composite fields $\Theta \equiv \sum_\alpha \theta_\alpha$ and $\theta_{\alpha\beta} \equiv \theta_\alpha - \theta_\beta$. The lattice action then reads

$$ S = \beta \sum_r \left\{ - \sum_\mu \cos (\Delta_\mu \Theta_r - Ne A_{r,\mu}) - \sum_{\mu, \alpha < \beta} \cos (\Delta_\mu \theta_{r,\alpha\beta}) - \lambda \sum_{\alpha < \beta} \cos (\theta_{r,\alpha\beta}) + \frac{1}{2} (\Delta \times A_r)^2 \right\}. $$

(16)

One may arrive at this form by defining the composite fields in Eq. (7), then use the Villain approximation on the original action of Eq. (4), rewrite the resulting action into one with the composite fields, then reverse the Villain approximation.

In Eq. (16), there is one charged mode and $N(N - 1)/2$ neutral modes, while the original theory has $N$ degrees of freedom. There is therefore an excess of $(N - 1)(N - 2)/2$ degrees of freedom. (Note that there are no redundant modes for $N = 1$ and $N = 2$). Therefore, not all of the phase differences are independent when $N > 2$. Consider the case $N = 3$, where one may form the phase differences $\theta_1 - \theta_2$, $\theta_2 - \theta_3$ and $\theta_1 - \theta_3$, but $\theta_1 - \theta_2 = (\theta_1 - \theta_3) + (\theta_2 - \theta_3)$. It suffices to include the phase differences $\theta_{12}$ and $\theta_{23}$.

This may be generalized to arbitrary $N$. Identify all $\theta_{\alpha\beta}$ where

$$ \{(\alpha, \beta) | \alpha \in \{1, \ldots, N - 1\} \land \beta = \alpha + 1\}. $$

(17)

Then, all $\theta_{\alpha\beta}$ where

$$ \{(\alpha, \beta) | \alpha \in \{1, \ldots, N - 2\} \land \beta \in \{\alpha + 2, \ldots, N\}\} $$

(18)

may be constructed by adding up the intermediate phase differences, that is $\theta_{\alpha\beta} = \theta_{\alpha,\alpha+1} + \theta_{\alpha+1,\alpha+2} + \cdots + \theta_{\beta-1,\beta}$. With this in mind, we may write out the partition function in terms of the charged and neutral modes

$$ Z = \int D\Theta \left( \prod_{\alpha < \beta} \int D\theta_{\alpha\beta} \right) \times \left( \prod_{\alpha=1}^{N-1} \prod_{\beta=\alpha+2}^{N-1} \delta (\theta_{\alpha\beta} - \sum_{\eta=\alpha}^{\beta-1} \theta_{\eta,\eta+1}) \right) e^S $$

(19)

where $S$ is the action of Eq. (16).

As an illustration, we perform the character expansion on the model where the charged and neutral sectors have been separated, for the special cases $N = 2$ and $N = 3$.

For $N = 2$ there are no redundant variables, and we have the two composite variables $\Theta \equiv \theta_1 + \theta_2$ and $\theta \equiv \theta_1 - \theta_2$. Using the identity Eq. (8), and integrating out the phases and gauge field, we obtain

$$ Z = \sum_{\{B, B_m\}} \prod_r \delta_{\Delta, B_{r,0}} \delta_{\Delta, B_{r,m_r}} \prod_{r,\mu} I_{B_{r,\mu}}(\beta) I_{B_{r,\mu}}(\beta) \prod_r I_{m_r}(\beta\lambda) \prod_{r,r'} \exp \left\{ - \frac{(Ne)^2}{2\beta} B_r \cdot B_{r'} D(r - r') \right\}. $$

(20)

Here, $B$ is the charged current field associated with $\Theta$, while $B_m$ is the neutral current field associated with $\theta$.

In this formulation, it is immediately clear that the model features two integer vector-field degrees of freedom, one which has long-range interactions mediated by the gauge field, and one with contact interactions. The neutral current field has its constraint removed by the $m$-field, while the charged field is still constrained to be divergenceless. Hence, the model will feature a single phase transition in the charged sector driven by the collapse of closed loops of charged currents, while the transition of the neutral sector is converted to a crossover by the complete removal of constraints on $B$.

Let us consider this in a bit more detail. In Eq. (20), we may perform the summation over the fields $m \in Z$. Since we have that $\Delta \cdot B_{r} \in Z$ as well, the summation over the $m$’s will guarantee that the constraint is satisfied for some value of $m$, such that the summation over $m$ effectively removes the constraints on $\Delta \cdot B_{r}$. Hence, we have $\sum_{\{m\}} \delta_{\Delta, B_{r,m}} \prod_r I_{m_r}(\beta\lambda) = \prod_r I_{\Delta, B_{r}}(\beta\lambda)$, with no constraints on $\Delta \cdot B_{r}$. We may thus perform the now unconstrained summation of the field $B_{r}$, namely

$$ \sum_{\{B\}} \left( \prod_{r,\mu} I_{B_{r,\mu}}(\beta) \right) \left( \prod_r I_{\Delta, B_{r}}(\beta\lambda) \right) = F(\beta, \lambda), $$

(21)

where $F$ is an analytic function of its arguments. This may be seen by mapping the left hand side of Eq. (21) to a Villain model, using the approximation

$$ \frac{I_b(x)}{I_0(x)} \approx \frac{1}{|b|!} e^{\log(\beta/2)|b|}. $$

(22)

This may be rewritten as a gaussian provided $\beta$ is sufficiently small so that contributions $|b| > 1$ are small,

$$ \frac{I_b(x)}{I_0(x)} \approx e^{-\frac{|b|^2}{2\lambda}}. $$

(23)

where $\lambda'$ is a renormalized coupling constant, and we find

$$ F(\beta, \lambda) = \sum_{\{B\}} \left( \prod_r \exp \frac{-B_{r,\mu}^2}{2\beta'} \right) \left( \prod_r \exp \frac{-(\Delta \cdot B_{r})^2}{2\lambda'} \right), $$

(24)
Since there are no constraints $B$, this demonstrates that Eq. (21) essentially is a discrete Gaussian theory, and the neutral sector therefore does not suffer any phase transition. This point may be further corroborated by going back to the formulation of Eq. 16. The neutral sector of the action is seen to be identical to that of an XY spin-model in an external magnetic field, with field strength $\lambda$. Any $\lambda \neq 0$ converts the phase transition, from a low-temperature ferromagnetic state to a high-temperature paramagnetic state, into to a crossover from an ordered to a disordered system. Note also that in the limit $\lambda = 0$, the Bessel function will revert to $I_\Delta \delta^0 = \delta \Delta \delta_0$, and the non-analytical constraint is re-introduced.

We emphasize that although the above argument utilized a Villain-approximation to the Bessel-functions, the conclusion that the phase-transition is wiped out in the neutral sector by introducing monopoles (Josephson-coupling) does not depend on this approximation. At any rate, a Villain-approximation to the XY-model does not change the symmetry of the problem or the character of phase transitions. What is crucial is the inclusion of Josephson-couplings converts the phase transition in the neutral sector to a crossover in exactly the same way that the ferromagnetic-paramagnetic phase transition in the 3DXY model is converted to a crossover by the inclusion of a magnetic field coupling linearly to the $XY$-spins, cf. Eq 16. This leaves only a phase-transition in the charged sector, but it does not completely suppress fluctuations in the neutral sector. It merely cuts the fluctuations off on a length-scale given by the Josephson-length $1/\lambda$, thereby preventing them from becoming critical. As temperature is increased, the neutral sector approaches its crossover region, with increasingly large fluctuations. At the same time, the charged sector approaches its putative inverted-3dXY fixed point. Provided that the crossover region of the neutral sector and the fixed point of the charged sector are sufficiently close, the fluctuations in both sectors may still strongly influence each other, and a first-order preemptive phase transition may still occur in the charged sector. This is consistent with recent numerical work, which observed a first order phase transition in multi-band superconductors with weak Josephson-coupling in Monte-Carlo simulations using the original $U(1)$ phases.

For $N = 3$ we must consider carefully the redundant variable, $\vartheta_{13} = \vartheta_{12} + \vartheta_{23}$. The partition function, prior to integration of the phases and the gauge field reads

$$Z = \int D\varTheta \left( \prod_{\alpha < \beta} \int D\vartheta_{\alpha \beta} \right) \delta(\vartheta_{13} - \vartheta_{12} - \vartheta_{23}) \prod_{r,\mu} \sum_{B_{\alpha \beta}} I_{B_{\alpha \beta}}(\varTheta) e^{iB_{\alpha \beta}(\varTheta_{\alpha} - N e A_{\alpha \beta})} \prod_{r,\mu,\alpha \beta} I_{B_{\alpha \beta}}(\varTheta) e^{iB_{\alpha \beta}(\varTheta_{\alpha} - N e A_{\alpha \beta})} \prod_{r,\alpha < \beta} \sum_{m_{\alpha \beta}} I_{m_{\alpha \beta}}(\varTheta) e^{i m_{\alpha \beta}(\varTheta_{\alpha} - N e A_{\alpha \beta})} \prod_{r} e^{-\frac{1}{2}(\Delta A_{\alpha \beta})^2}. \quad (26)$$

Again, $B$ is the charged current associated with $\varTheta$, while $B_{\alpha \beta}$ are the neutral currents associated with $\vartheta_{\alpha \beta}$. The $\delta$-function is included to account for the redundancy of the composite phase representation.

We now proceed with the integration of phases and gauge field, taking care to integrate out the redundant phase first. The partition function may then be written as

$$Z = \sum_{\{B_{\alpha \beta}\}} \prod_{r} \delta \Delta \delta_{\alpha \beta} \prod_{r,\alpha \beta} I_{B_{\alpha \beta}}(\varTheta) \prod_{r} \delta \Delta \delta_{\alpha \beta} \prod_{r} \delta \Delta \delta_{\alpha \beta} \prod_{r} I_{B_{\alpha \beta}}(\varTheta) \prod_{r,\alpha < \beta} \exp \left\{ -\frac{(Ne)^2}{2\beta} B_{\alpha \beta} \cdot B_{\alpha \beta} D(r - r') \right\}. \quad (27)$$

This is a model of a single gauge coupled supercurrent $B$ which are constrained to form closed loops, and three superfluid currents $B_{12}, B_{23}$ and $B_{13}$ which are not constrained to
form closed loops. The three superfluid currents are not independent, as is seen from the two constraints on them. As in the case $N = 2$, the summation over the $m$-fields may be performed, eliminating the constraints on the fields $\mathcal{B}_{\nu,\mu,\alpha\beta}$, after which the unconstrained summation over these fields may be performed. As for $N = 2$, this yields multiplicative analytic factors in the partition function, and the phase transitions in the neutral sectors will be converted to crossovers. Given that the crossovers in the neutral sectors and the charged fixed point have sufficient overlap, the system may still feature a single preemptive first-order phase transition arising from the interplay between the charged and neutral modes. Furthermore, the inclusion of the additional degree of freedom enhances the combined fluctuations of the neutral mode at a given Josephson coupling, $\lambda$, and therefore strengthens the preemptive first-order transition. This is consistent with the results of recent numerical work\textsuperscript{20}.

E. Preemptive effect and current-loop interactions

In this subsection, we discuss further the preemptive scenario discussed above, interpreting it in terms of renormalizations of current-current interactions. This provides a dual picture to the physical picture of the first-order phase transition presented in Ref. 20.

The preemptive phase transition may be understood in the current-loop picture by considering the effect of the monopoles on the neutral counter-flowing current sector (facilitated by the presence of monopoles, i.e. Josephson coupling), and how this in turn influences the interaction between the charged co-flowing currents which interact via the fluctuating gauge-field.

Consider first the current-loop excitations allowed by Eq. (12) for the case $N = 2$. The lowest order configurations in the individual fields are closed loops of a single color. On top of these one may add monopoles, such that one has closed loops that change color twice before completing a closed loop. The presence of the Josephson coupling also allows for small dumbbells of counter-flowing currents with a monopole at one end and an anti-monopole at the other end. The gauge field will bind loops of co-flowing currents together, creating small loops of both colors flowing in the same direction. At high temperatures, the co-flowing currents only form small closed loops, and the system is non-superconducting. Barring any influence from the neutral sector, they will proliferate in an inverted $3DXY$-transition\textsuperscript{20} at some critical temperature. If the charge, or the Josephson coupling, is sufficiently strong, there will be no significant fluctuations in the neutral sector that may influence this. The co-flowing current loops simply proliferate in a background of only tightly bound counter-flowing currents, with which they do not interact at all. The only way they can interact is if a counter-flowing composite current locally dissociates into individual currents on length scales below the Josephson length, which needs to be large enough. This will not happen if either the Josephson coupling is sufficiently strong, or if the charge is sufficiently large so that the charged transition is separated sufficiently from the neutral crossover.

Figs. 2 and 3 show simple representations of current configurations as the transition occurs in the two scenarios. For simplicity the illustration is given in two spatial dimensions. In Fig. 2, we show the case of having a sufficiently strong Josephson coupling. A generic snapshot of a single loop of charged current is shown, represented by two co-flowing red and green lines, surrounding a gas of tightly bound pieces of counter-flowing neutral currents. As there are no individual red or green lines, there will be no interactions between the loop of composite charged current and the small pieces of composite neutral current, and hence no renormalization of the interactions in the charged sector. The loops of charged current will therefore proliferate in an inverted $3DXY$-transition\textsuperscript{20} as the temperature is lowered. In Fig. 3, the situation is different. Here, the Josephson coupling is sufficiently low, or alternatively the Josephson length is sufficiently large, so that the individual pieces of current may undergo local dissociations of the tightly bound counter-flowing configurations. These individual pieces of currents, represented by only red or green lines, will interact with the loop of charged current, and may therefore influence the proliferation of composite charged current loops.

The current loops are dual objects to vortex loops. It is known that there is a precise correspondence between the sign of vortex interactions and the character of the phase transition in superconductors. Namely, attractive interactions between vortices leads to a first-order phase transition, while repulsive vortex interactions lead to second order phase transition\textsuperscript{20,27}. Therefore, an alternative natural way of interpreting the preemptive first-order phase-transition in the dual picture, is that neutral counter-flowing currents on the co-flowing charged currents screen or oversee the interactions between the latter, effectively changing the sign of the interactions between charged current-segments.

With reference to Fig. 3, we elaborate briefly on how the configurations depicted there may cause attractive interactions between composite charged current-segments. Note that the screening is accounted for entirely by removing all tightly bound counter-flowing currents, leaving only the closed color-changing loops. The relevant screening fluctuations are therefore complicated collective phase-fluctuations amounting to inserting closed color-changing loops in the problem. Loops which interact attractively with the composite charged current-segments will have a larger Boltzmann weight in the dual action than those that attract repulsively, and they will therefore dominate the configurations where many closed current-changing (originating with tightly bound counter-flowing currents) are present. This attraction may cause an effective attraction between the charged composite current segments, via the attraction to the closed current-changing loops. An identical physical picture holds when working with the dual objects to the currents, namely vortices.

To summarize, the basic mechanism causing a first-order phase transition is the influence of partial decomposition of composite neutral currents on the interaction between charged composite currents, equivalently the influence of partial decomposition of composite neutral vortices on the interaction
between composite charged vortices. These pictures are particular dual manifestations of the general concept of a preemptive first order phase transition. In such a transition, a putative second order phase transition associated with proliferation of topological defects in a given order parameter, is converted to a first order phase transition preemptively by strong fluctuations (not necessarily critical) in some other field.

**IV. CURRENT CORRELATIONS AND THE HIGGS MECHANISM**

The defining characteristic of the inverted 3DXY-transition in the charged sector is a spontaneous $U(1)$ gauge-symmetry breaking associated with the gauge field $A$ becoming massive as the system crosses the transition point of the metallic state into the superconducting state. In this section, we investigate how the onset of the mass $m_A$ of the photon (the Higgs mass), which is equivalent to the Meissner effect of the superconductor, comes about as result of a non-analytic change in the infrared properties of the current-correlations of the system. $m_A$ is found from the limiting form of the gauge-field correlation function

$$\langle A_{q}^{\mu} A_{-q}^{\nu} \rangle \sim \frac{1}{q^2 + m_A^2}.$$  \hspace{1cm} (28)

To calculate $\langle A_{q}^{\mu} A_{-q}^{\nu} \rangle$, we consider the action of the charged sector given on the form Eq. (A7) before integrating out the gauge field, and insert source terms source $J_q$,

$$S_J = \sum_{q} \left[ \frac{ie}{2} \sum_{\alpha} b_{q,\alpha} \cdot A_{-q} + \frac{ie}{2} \sum_{\alpha} b_{-q,\alpha} \cdot A_{q} + \frac{\beta}{2} |Q_q|^2 A_q \cdot A_{-q} + \frac{1}{2} (J_q \cdot A_{-q} + J_{-q} \cdot A_q) \right].$$ \hspace{1cm} (29)
which in turn may be written on form

\[
S_J = \sum_q \left[ (A_q + \frac{1}{2} (J_q + i e \sum_\alpha b_{q,\alpha}) D_q^{-1}) D_q \times (A_{-q} + \frac{1}{2} (J_{-q} + i e \sum_\alpha b_{-q,\alpha}) D_{-q}^{-1} \right] \nonumber \\
+ \frac{1}{4} (J_q + i e \sum_\alpha b_{q,\alpha}) D_q^{-1} \nonumber \\
\times (J_{-q} + i e \sum_\beta b_{-q,\beta}) \right]. 
\] (30)

Here, \( D_q = \beta |Q_q|^2 / 2 \) as before. After shifting and integrating the gauge field, we have

\[
S_J = -\sum_q \left[ \frac{1}{2 \beta |Q_q|^2} \left( J^{\mu}_q P^{\mu}_T J^{\nu}_q - c^2 \sum_{\alpha\beta} b_{q,\alpha} b_{-q,\beta} \right) \right. \\
+ \left. i e \sum_\alpha \left( J^{\mu}_{-q} b_{q,\alpha} + J^{\nu}_q b_{-q,\alpha} \right) \right], 
\] (31)

where repeated indices are summed over, and \( P^{\mu\nu}_T \) is the transverse projection operator

\[
P^{\mu\nu}_T = \delta^{\mu\nu} - \frac{Q^\mu_q Q^\nu_q}{|Q_q|^2} 
\] (32)

The gauge-field correlator is then given by

\[
\langle A^{\mu}_q A^{\nu}_{-q} \rangle = \frac{1}{Z_0} \frac{\delta^2 Z_J}{\delta J^{\mu}_q \delta J^{\nu}_{-q}} \bigg|_{J=0} = \frac{1}{Z_0} \sum_{(b,m)} \prod_{r,\alpha} \delta b_{r,\alpha} \sum_{\beta,\alpha} \epsilon_{\alpha\beta} m_{r,\alpha,\beta} \prod_{r,\alpha < \beta} I_{b_{r,\alpha,\beta}} (\beta) \prod_{r,\alpha < \beta} I_{m_{r,\alpha,\beta}} (\beta) \times \left( \delta^2 S_J \frac{\delta S_J}{\delta J^{\mu}_q} \delta S_J \frac{\delta S_J}{\delta J^{\nu}_q} \right) e^{-S_J} \bigg|_{J=0} 
\] (33)

where we have defined \( \langle B_q \cdot B_{-q} \rangle = \langle \sum_{\alpha\beta} b_{q,\alpha} \cdot b_{-q,\beta} \rangle \)

The effective gauge field mass is given by the zero momentum limit of the inverse propagator,

\[
m_A^2 = \lim_{q \to 0} \frac{2}{\beta (A_q A_{-q})} 
\] (38)

As is seen from Eq. (37), the relevant combination of current-field correlators is the superconducting current, while charge-neutral currents do not appear in the expression. The current-correlator may be interpreted as the helicity modulus, which at a charged fixed point has a non-analytic behavior of the term proportional to \( q^2 \). We expect the leading behavior to be\(^28\)

\[
\lim_{q \to 0} \frac{e^2}{2 \beta} \langle B_q \cdot B_{-q} \rangle \sim \begin{cases} 
(1 - C_2(T))q^2, & T > T_C. \\
q^2 - C_3(T)q^{2+\eta_A}, & T = T_C. \\
q^2 - C_4(T)q^4, & T < T_C.
\end{cases} 
\] (39)

The result given above is dual to an expression for the gauge-mass in terms of correlation function of topological defects of the superconducting order, i.e. vortices\(^17,28,29\), since vor-
tices are dual objects to the currents $b$. In $3D$, it is known that the dual of a superfluid is a superconductor, and vice versa\textsuperscript{17,24,28,29}. Therefore, the above result for the current-correlator of a superconductor features the same behavior as the vortex-vortex correlator at a neutral fixed point, since a neutral fixed point in the original theory is a charged fixed point in the dual theory. Here, $C_2$ is the helicity modulus of the system, $C_3$ is a critical amplitude, and $C_4$ is essentially the inverse mass of the gauge-field. The physical interpretation of $\lambda > 0$ is that when this quantity is zero, there are no long-range correlations of current-loops in the system, i.e. there are no supercurrents threading the entire system which is therefore normal metallic. Conversely, when $T < T_c$, this correlator is non-zero. There are supercurrents threading the entire system, which is therefore superconducting. When $T > T_c$, the gauge mass will be zero in the long wavelength limit. When $T < T_c$, however, the factors of $q^2$ will cancel, and the gauge correlator obtains a finite expectation value, and hence a mass. The Higgs-mechanism (Meissner effect) in an $N$-component superconductor is therefore a result of a blowout of closed loops of charged currents as the temperature is lowered through the phase transition. Conversely, the transition to the normal state is driven by a collapse of closed current loops, which is dual to a blowout of closed vortex loops. In either way of looking at the problem, the Higgs-mechanism is fluctuation driven.

Note that the above result is valid for any number of components $N \geq 1$, and any value of the Josephson coupling $\lambda \geq 0$.

The preemptive scenario described in the previous section impacts the temperature-dependence of the Higgs-mass at the transition from the superconducting to the normal metallic state. The mass vanishes continuously in an inverted $3DXY$ phase transition if the value of the gauge-charge is large enough for the preemptive scenario to be ruled out for any $\lambda$, including $\lambda = 0$. For small enough gauge-charge, such that fluctuations in the neutral sector strongly affect fluctuations in the charged sector, and vice versa, the preemptive effect comes into play. In that case, the Higgs-mass vanishes discontinuously at the phase transition.

\section{V. CONCLUSION}

We have formulated an $N$-component London superconductor with intercomponent Josephson couplings as a model of $N$ integer-current fields $b_{\alpha}$ and $N(N - 1)/2$ monopole fields, $m_{\alpha, \beta}$. These monopoles allow supercurrents of a particular condensate component to be converted to a supercurrent of a different component, i.e. currents may change "color" at any site. For zero Josephson coupling, $\lambda$, only configurations where all the monopole fields are zero contribute, and the model reverts to an $N$-component gauge-coupled 3DXY-model. This model is known to have either i) $N - 1$ transitions in the $XY$-universality class and a single phase transition in the inverted $XY$-universality class, or ii) a single preemptive first-order phase transition for intermediate values of the charge. For any $\lambda > 0$, the $N - 1$ phase transitions in the neutral sector are converted to crossovers. In the limit $\lambda \rightarrow \infty$, all orders of monopole excitations will contribute. This effectively removes the constraints $\Delta \cdot b_{\alpha} = 0$ on each individual component. There is only one particular composite mode, $\sum_{\alpha} b_{\alpha}$, which is still divergenceless, and which thus features a phase transition. This transition is known to be in the inverted 3DXY-universality class for $\lambda = 0$. For small, but finite $\lambda$, fluctuations in the neutral sector are still substantial although the phase transitions are all converted to crossovers. These charge-neutral non-critical fluctuations nonetheless substantially influence the putative critical fluctuations in the charged sector, particularly at temperatures close to the $\lambda = 0$ 3DXY critical point. This converts the inverted $3DXY$ critical point into a first-order phase-transition via a preemptive effect. The degree to which the charge-neutral fluctuations influence the fluctuations in the charged sector for small $\lambda$, increases with the number of composite charge-neutral fluctuating modes. In the parameter regime $(e, \lambda)$ where one may have a preemptive effect, the first-order character of the superconductor-normal metal phase transition will therefore be more pronounced with increasing $N$.

As a byproduct of our analysis, we have recast the onset of the photon Higgs-mass in the superconductor (Meissner effect) in terms of a blowout of current loops associated with the onset of superconductivity. This analysis goes beyond mean-field theory and takes all critical fluctuations of the theory into account. The description giving the onset of the Higgs-mass of the photon in terms of a current-loop blowout going into the superconducting state as temperature is lowered, is dual to the description of the vanishing of the Higgs-mass of the photon in terms of vortex-loop blowout going into the normal metallic state as the temperature is increased.

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\section*{Appendix A: The character expansion}

We apply the expansion

\begin{equation}
\sum_{\beta = -\infty}^{\infty} I_\beta e^{ib\gamma} = e^{b \cos \gamma} = \sum_{\beta = -\infty}^{\infty} I_\beta e^{ib\gamma}, \quad (A1)
\end{equation}
to the cosine terms of Eq. (4), with $\lambda = 0$. This gives the action

$$Z = \int \mathcal{D}A \left( \prod_\alpha \int \mathcal{D}\theta_\alpha \right)$$

$$\times \prod_{r,\mu,\alpha} \sum_{\beta} I_{b_{r,\mu,\alpha}}(\beta) e^{i b_{r,\mu,\alpha} (\Delta_r \theta_{r,\alpha} - e A_{r,\mu})}$$

$$\times \prod_r e^{-\frac{\beta}{2} (\Delta \times A_r)^2} \quad (A2)$$

By performing a partial integration of each phase component, $\theta_{r,\alpha}$, we move the lattice derivative from the phase to the integer field $b$ in the first term. Then we factorize the terms dependent on the phases on each lattice site, which may then be integrated separately.

$$Z_\theta = \prod_{r,\alpha} \int_0^{2\pi} d\theta_{r,\alpha} e^{-i \theta_{r,\alpha} (\sum_\mu \Delta_{r,\mu} b_{r,\mu,\alpha})}. \quad (A3)$$

This constrains the $b$-fields to have zero divergence,

$$\Delta \cdot b_{r,\alpha} = 0 \quad \forall \ r, \alpha. \quad (A4)$$

The partition function then reads

$$Z = \int \mathcal{D}(A) \sum_{\{b\}} \prod_r \prod_\alpha \delta b_{r,\alpha,0} \prod_{r,\mu,\alpha} I_{b_{r,\mu,\alpha}}(\beta)$$

$$\prod_r e^{-\frac{i}{2} \sum_\alpha b_{r,\alpha} \cdot \Delta A_r + \frac{\beta}{2} (\Delta \times A_r)^2} \quad (A5)$$

This represents $N$ integer-current fields which must form closed loops individually, coupled by a single gauge field, $A$.

The next step is to integrate out the gauge degrees of freedom. To this end we Fourier transform the action

$$S = \sum_r \left[ i e \sum_\alpha b_{r,\alpha} \cdot A_r + \frac{\beta}{2} (\Delta \times A_r)^2 \right] \quad (A6)$$

into

$$S = \sum_q \left[ \frac{i e}{2} \sum_\alpha b_{q,\alpha} \cdot A_{-q} + \frac{i e}{2} \sum_\alpha b_{-q,\alpha} \cdot A_q$$

$$+ \frac{\beta}{2} (Q_q \times A_q)(Q_{-q} \times A_{-q}) \right] \quad (A7)$$

Here, we have symmetrized the $b \cdot A$-term, and $Q_q$ is the Fourier representation of the lattice differential operator, $\Delta$. We can further simplify the expression by choosing the gauge $\Delta \cdot A_r = 0$, which translates to $Q_q \cdot A_q = 0$ in Fourier space. This reduces the last term to $\beta |Q_q|^2 A_q \cdot A_{-q} / 2$, where $|Q_q|^2 = \sum_\mu (2 \sin q_\mu / 2)^2$. Now we complete the squares in $A_q$, to facilitate the Gaussian integration

$$S = \sum_q \left[ \left( A_q + \frac{i e}{2} \sum_\alpha b_{q,\alpha} D_q^{-1} \right) D_q$$

$$\times \left( A_{-q} + \frac{i e}{2} \sum_\alpha b_{-q,\alpha} D_{-q}^{-1} \right) + \frac{e^2}{4} \left( \sum_\alpha b_{q,\alpha} \right) D_q^{-1} \left( \sum_\beta b_{-q,\beta} \right) \right] \quad (A8)$$

where $D_q = \beta |Q_q|^2 / 2$. Now we can shift and integrate out the gauge field, $A_q$, which leaves us with

$$S = \sum_q \frac{e^2}{2 \beta |Q_q|^2} \left( \sum_\alpha b_{q,\alpha} \right) \cdot \left( \sum_\beta b_{-q,\beta} \right) \quad (A9)$$

or in real space

$$S = \sum_{r,r'} \frac{e^2}{2 \beta} \left( \sum_\alpha b_{r,\alpha} \right) \cdot \left( \sum_\beta b_{r',\beta} \right) D(r - r'). \quad (A10)$$

Here, $D(r - r')$ is the Fourier transform of $1 / |Q_q|^2$. Inserting this into the action, we arrive at

$$Z = \sum_{\{b_{r,0}\}} \prod_{r,\alpha} \delta b_{r,\alpha,0} \prod_{r,\mu,\alpha} I_{b_{r,\mu,\alpha}}(\beta)$$

$$\prod_r e^{-\frac{\beta}{2} \sum_\alpha \sum_{\beta} b_{r,\alpha} b_{r,\beta} D(r - r')} \quad (A11)$$

which is Eq. (9)

### Appendix B: Two-dimensional multiband superconductors

In a thin-film superconductor, the effective magnetic penetration depth is inversely proportional to the film thickness. Hence, in a two-dimensional system, the magnetic penetration depth becomes infinite, and the effective charge of the charge carriers become zero. This effectively freezes out the gauge-field fluctuations of the interior of the film, in turn eliminating the long-range gauge-field mediated vortex-vortex interactions. In this case the relevant lattice action will be

$$S = -\beta \sum_r \sum_{\mu,\alpha} \cos (\Delta_{r,\mu} \theta_{r,\alpha})$$

$$- \beta \lambda \sum_{r,\alpha < \beta} \cos (\theta_{r,\alpha} - \theta_{r,\beta}). \quad (B1)$$

That is, it is effectively a neutral condensate.

We may apply the character expansion of Eq. (8) to Eq. (B1), which results in the partition function

$$Z = \sum_{\{b_{r,0}\}} \prod_{r,\alpha} \delta b_{r,\alpha,0} \sum_{\beta,\alpha} m_{r,\alpha,\beta}$$

$$\prod_{r,\mu,\alpha} I_{b_{r,\mu,\alpha}}(\beta) \prod_{r,\alpha < \beta} I_{m_{r,\alpha,\beta}}(\beta \lambda). \quad (B2)$$
This is of course very similar to Eq. (12), with the differences being as follows. The integer-current field, $b_r$, is now a two-component vector, as is naturally the position vector, $r$, and the gauge-field mediated interaction has disappeared.

We may apply the same reasoning to Eq. (B2) as we did in the main text. There will be a single mode, $\sum_\alpha b_r$, which is divergenceless, and $N(N - 1)/2$ modes with finite divergence. The only difference now in the two-dimensional case is the lack of gauge-field mediated interactions in the divergenceless mode. Hence, the single remaining phase transition is expected to be a Kosterlitz-Thouless transition from a two-dimensional superfluid to a normal fluid. This prediction could be verified in Monte-Carlo simulations, as the partition function of Eq. (B2) is particularly well suited for worm-type algorithms.

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