Weakly asymptotically quasiperiodic solutions for time-dependent Hamiltonians with a view to celestial mechanics

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Abstract

In a previous work [Sca22], we consider time-dependent perturbations of a Hamiltonian vector field having an invariant torus supporting quasiperiodic solutions. Assuming the perturbation decays polynomially fast as time tends to infinity, we prove the existence of a non-autonomous KAM torus. A non-autonomous KAM torus is a time-dependent family of embedded tori converging in time to the invariant torus associated with the unperturbed system.

In the first part of the present work, we generalise the theorem in [Sca22] to the case where the time decay is weaker but always polynomial, as it appears naturally in physical problems. This requires the introduction of a generalization of the definition of non-autonomous KAM torus. The proof relies on a Nash-Moser theorem. In the second part, we apply this new theorem to the example of the planar three-body problem perturbed by a given comet coming from and going back to infinity asymptotically along a hyperbolic Keplerian orbit.

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1 Introduction

In 2015, M. Canadell and R. de la Llave [CdlL15] considered a time-dependent vector field \( X^t \) converging exponentially fast in time (when \( t \to +\infty \)) to an autonomous vector field \( X_0 \) having an invariant torus \( \varphi_0 \) supporting quasiperiodic solutions. Assuming some control on the normal dynamics to \( \varphi_0 \), they proved the existence of a non-autonomous KAM torus, generalizing a previous work of A. Fortunati and S. Wiggins [FW14]. A non-autonomous KAM torus is a time-dependent family of embedded tori that converges in time to the quasiperiodic invariant torus associated with \( X_0 \). We point out that M. Canadell and R. de la Llave do not require arithmetic conditions or non-degeneracy hypotheses.

In another paper [Sca22], we generalize this result in the particular case of time-dependent Hamiltonian vector fields. In this work, the smallness assumption on the perturbation is removed, and it decays polynomially fast in time (Canadell-de la Llave demand exponentially fast convergence). I prove this result by the implicit function theorem using another approach with respect to [CdlL15]. The central part of the proof consists of estimating the solution to the associated linearized problem. It is solved by integration thanks to a suitable change of coordinates which rectifies the dynamics on the torus. The polynomial decay in time is at the origin of new difficulties, with respect to the exponential decay in time, solved in the non-linear part.

These kinds of perturbations are not artificial. These types of systems are interesting in astronomy. The example that motivated this work is the planar three-body problem perturbed by a given comet coming from and going back to infinity asymptotically along a hyperbolic Keplerian orbit (modelled as a time-dependent perturbation). On a suitable phase space, the Hamiltonian which describes this system is a time-dependent Hamiltonian

\[
H^t = H_0 + H^t_c,
\]

where \( H_0 \) is the Hamiltonian of the planar three-body problem, and \( H^t_c \) is responsible for the interaction of the planets with the given comet. We will call \( H^t \) the Hamiltonian of the planar three-body problem plus comet (P3BP+C).

On an appropriate neighbourhood of some quasiperiodic solutions associated with \( H_0 \), the perturbation \( H^t_c \) satisfies

\[
|\nabla H^t_c| \sim \frac{1}{t^2}
\]

on a suitable norm. In this case, the polynomial decay in time is too weak, preventing us from applying the theorem proved in [Sca22]. Furthermore, the time-dependent perturbation strongly modifies the dynamics at infinity, requiring the introduction of a generalization of the definition of non-autonomous KAM torus and the formulation of another abstract theorem (Theorem A).

To this end, let us introduce the definition of \( C^\sigma \)-weakly asymptotic cylinder. Let \( B \subset \mathbb{R}^{n+m} \) be a ball centred at the origin, \( \mathcal{P} \) equal to \( \mathbb{T}^n \times \mathbb{R}^m \times B \), and \( J = [1, +\infty) \subset \mathbb{R} \) a real interval. Therefore, we denote \( q \in \mathbb{T}^n \times \mathbb{R}^m \) and \( p \in B \).

Given \( \sigma \geq 0 \) and a positive integer \( k \geq 0 \), we consider time-dependent vector fields \( X^t, X^t_0 \) of class \( C^{\sigma+k} \) on \( \mathcal{P} \), for all fixed \( t \in J \), an embedding \( \varphi_0 \) from \( \mathbb{T}^n \times \mathbb{R}^m \)
to $\mathcal{P}$ of class $C^\sigma$ and a time-dependent vector field $\gamma^t$ of class $C^\sigma$ on $\mathbb{T}^n \times \mathbb{R}^m$, for all fixed $t \in J$, such that
\begin{align}
\lim_{t \to +\infty} |X^t - X^t_0|_{C^\sigma} &= 0, \\
X_0(\varphi_0(q), t) &= \partial_q \varphi_0(q) (\bar{\omega} + \gamma(q, t)) \quad \text{for all } (q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J, \\
\lim_{t \to +\infty} |\gamma^t|_{C^\sigma} &= 0,
\end{align}
where $\bar{\omega} = (\omega, 0) \in \mathbb{R}^{n+m}$ with $\omega \in \mathbb{R}^n$, and $| \cdot |_{C^\sigma}$ is the Hölder norm. For the sake of clarity, we point out that $\partial_q \varphi(q)(\bar{\omega} + \gamma(q, t))$ is a vector of $\mathbb{R}^{2(n+m)}$ with $j$ component equal to
$$
\left( \partial_q \varphi(q)(\bar{\omega} + \gamma(q, t)) \right)_j = \partial_q \varphi(q, j) \cdot (\bar{\omega} + \gamma(q, t))
$$
for all $1 \leq j \leq 2(n+m)$ with $\varphi_0 = (\varphi_{0,1}, \ldots, \varphi_{0,2(n+m)})$. In words, $X^t - X^t_0$ converges to zero when $t \to +\infty$. Furthermore, the vector field $X_0$ has an invariant cylinder $\varphi_0$ and the restriction of $X_0$ is conjugated to the non-autonomous vector field $\bar{\omega} + \gamma$, which is a time-dependent vector field converging to $\bar{\omega}$ when $t \to +\infty$.

**Definition 1.1.** We assume that $(X, X_0, \varphi_0)$ satisfy (1.1), (1.2) and (1.3). A family of $C^\sigma$ embeddings $\varphi^t : \mathbb{T}^n \times \mathbb{R}^m \to \mathcal{P}$ is a $C^\sigma$-weakly asymptotic cylinder associated to $(X, X_0, \varphi_0)$ if there exists a time-dependent vector field $\Gamma^t$ of class $C^\sigma$ on $\mathbb{T}^n \times \mathbb{R}^m$, for all fixed $t$, such that
\begin{align}
\lim_{t \to +\infty} |\varphi^t - \varphi_0|_{C^\sigma} &= 0, \\
X(\varphi(q, t), t) &= \partial_j \varphi(q, t)(\bar{\omega} + \varphi(q, t) + \partial_i \varphi(q, t), \\
\lim_{t \to +\infty} |\Gamma^t|_{C^\sigma} &= 0,
\end{align}
for all $(q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J$. Moreover, $\varphi$ is Lagrangian if $\varphi'(\mathbb{T}^n \times \mathbb{R}^m)$ is Lagrangian for all $t \in J$.

Roughly speaking, a $C^\sigma$-weakly asymptotic cylinder is a family of embeddings $\varphi^t$ converging in time to the invariant cylinder $\varphi_0$ associated with $X_0$. Moreover, the dynamics on this family of embeddings do not converge to the motions associated with $X_0$ on $\varphi_0$ but to orbits on $\varphi_0$ generated by the time-dependent vector field $\bar{\omega} + \Gamma$, hence the term weakly. Concerning the dynamics, in order to be more precise, we introduce the definition of weakly asymptotically quasiperiodic solutions. We will see that if $\varphi^t$ is a $C^\sigma$-weakly asymptotic cylinder associated to $(X, X_0, \varphi_0)$, then we have the existence of weakly asymptotically quasiperiodic solution associated to $(X, X_0, \varphi_0)$.

**Definition 1.2.** We assume that $(X, X_0, \varphi_0)$ satisfy (1.1), (1.2) and (1.3). An integral curve $g(t)$ of $X$ is a weakly asymptotically quasiperiodic solution associated to $(X, X_0, \varphi_0)$ if there exist a time-dependent vector field $\Gamma : \mathbb{T}^n \times \mathbb{R}^m \times J \to \mathbb{R}^{n+m}$ and $q \in \mathbb{T}^n \times \mathbb{R}^m$ such that
$$
\lim_{t \to +\infty} |g(t) - \varphi_0 \circ \psi^t_{t_0, \bar{\omega} + \Gamma}(q)| = 0.
$$
We observe that taking $m = 0$, $\gamma \equiv 0$ and $\Gamma \equiv 0$ in Definition 1.1, we obtain the definition of non-autonomous KAM torus due to M. Canadell and R. de la Llave [CdlL15]. Here, we are interested in a family of embedded cylinders, not embedded tori. This is due to the application to celestial mechanics that we will study in this work.

The first part of this paper is devoted to the proof of an abstract theorem (Theorem A) that is a weaker version of the result contained in [Sca22]. As mentioned before, the slower decay in time is a severe difficulty requiring the proof of a new theorem providing weaker solutions: Theorem A proves the existence of a weakly asymptotic cylinder (rather than a non-autonomous KAM torus as in [Sca22]). The price to pay is a conclusion weaker than that in [Sca22] because we lose some information on the dynamics of the found orbits. In the second part, as an application of Theorem A, we prove the existence of weakly asymptotically quasiperiodic solutions for the Hamiltonian of the P3BP+C (Theorem B).

We want to point out that Theorem A is flexible and can be applied to many physical phenomena. Its proof contains the most original and mathematically complicated part of this paper. As mentioned before, the slower decay in time obliges us to provide substantial changes in the proof (compared to [Sca22]). We need to consider a more regular system and assume a smallness assumption on the perturbation. The proof is based on a Nash Moser theorem. The analysis of the associated linearized problem is considerably more complicated than that in [Sca22]. The slower decay in time causes a “loss of regularity” preventing us from proving this theorem by the implicit function theorem (on which relies the proof of [Sca22]).

The Nash-Moser approach, proposed initially by Nash [Nas56], finds many applications in perturbative problems. Many improvements and refinements have been developed. To mention a few, we have the results of Moser [Mos61], Zehnder [Zeh75, Zeh76], Hamilton [Ham82], Hörmander [Hø76, Hø85, Hø90], Ekeland [Eke11] and Ekeland-Sére [ES15].

In the framework of the KAM theory, a series of proofs are given by the introduction of an adapted implicit function theorem in a scale of Banach spaces, which replaces the iterative scheme inaugurated by Kolmogorov. We refer to the works of Zehnder [Zeh75, Zeh76], Herman [Bos86], Berti-Bolle [BB15] and Féjoz [Fé04, Fé17]. Concerning the dissipative case, one can see the work of Massetti [Mas19].

For more recent versions of the Nash Moser theorem with applications on PDE’s problems, we refer to the works of Berti-Corsi-Procesi [BCP15], Berti-Bolle-Procesi [BBP10] and Baldi-Haus [BH17].

2 Main results

This section is divided into two sub-sections. The first is devoted to the weaker version of the result in [Sca22], while the second is dedicated to applications in celestial mechanics concerning the existence of weakly asymptotically quasiperiodic solutions for the Hamiltonian of the P3BP+C.
2.1 The abstract theorem

We begin with the introduction of the following space of functions. We recall that $B \subset \mathbb{R}^{n+m}$ is a ball centred at the origin and $J = [1, +\infty) \subset \mathbb{R}$. Given a positive real parameter $\sigma \geq 0$, we have the following definition.

**Definition 2.1.** Let $S_\sigma$ be the space of functions $f$ defined on $\mathbb{T}^n \times \mathbb{R}^m \times B \times J$ such that $f^i \in C^\sigma(\mathbb{T}^n \times \mathbb{R}^m \times B)$, for all fixed $t \in J$, and $\partial_{(q,p)}^i f \in C(\mathbb{T}^n \times \mathbb{R}^m \times B \times J)$ for all $0 \leq i \leq |\sigma|$.

We point out that $\partial_{(q,p)}^i$ stands for the partial derivatives with respect to $(q,p)$ of order $i$ and $|\sigma|$ for the integer part of $\sigma$. Given $\sigma \geq 0$ and $l \geq 0$, for every $f \in S_\sigma$, we introduce the following norm

$$|f|_{\sigma,l} = \sup_{t \in J} |f^i|_{C^\sigma} t^l,$$

(2.1)

where $| \cdot |_{C^\sigma}$ is the Hölder norm.

Now, let $s$, $\lambda$, $\rho$, $\beta$ and $\alpha$ be positive parameters satisfying the following conditions

$$\begin{cases}
1 \leq \rho < \lambda < s, \\
s > \max \left\{ \frac{\alpha}{\alpha - 1}, \frac{\alpha}{\alpha - 1} \right\}, \\
1 < \beta < 2, \\
1 < \beta < 2, \quad \alpha > 1, \quad \rho < \frac{\lambda - \beta}{\beta^2}.
\end{cases}$$

(\#)

Given $\omega \in \mathbb{R}^n$ and real positive parameters $\delta > 0$, $\varepsilon > 0$ and $\Upsilon \geq 1$, we consider the following time-dependent Hamiltonian

$$\begin{cases}
H : \mathbb{T}^n \times \mathbb{R}^m \times B \times J \rightarrow \mathbb{R} \\
H(q,p,t) = \omega \cdot p + a(q,t) + b(q,t) : p + m(q,p,t) \cdot p^2
\end{cases}$$

(\ast)

$$\begin{align*}
& a, b_0, b_r, \partial_p^2 H \in S_{s+1} \\
& |b_0|_{2,1} < \delta, \quad |b_0|_{s+1,1} \leq \Upsilon, \\
& |a|_{\lambda+1,0} + |\partial_t a|_{\lambda,2} < \varepsilon, \quad |b_r|_{\lambda+1,1} < \varepsilon, \\
& |a|_{s+1,0} + |\partial_t a|_{s,2} \leq \Upsilon, \quad |b_r|_{s+1,1} \leq \Upsilon, \quad |\partial_p^2 H|_{s+1,0} \leq \Upsilon.
\end{align*}$$

Let $\varphi_0$ be the following trivial embedding $\varphi_0 : \mathbb{T}^n \times \mathbb{R}^m \rightarrow \mathbb{T}^n \times \mathbb{R}^m \times B$, $\varphi_0(q) = (q,0)$. Furthermore, we consider the Hamiltonian $\hat{h} : \mathbb{T}^n \times \mathbb{R}^m \times B \times J \rightarrow \mathbb{R}$ such that

$$\hat{h}(q,p,t) = \omega \cdot p + m(q,p,t) \cdot p^2.$$  

(2.2)

**Theorem A.** Let $H$ be as in (\ast) and we assume that $s$, $\lambda$, $\rho$, $\beta$ and $\alpha$ satisfy (\#). Then, for $\delta$ small enough with respect to $s$, there exists $\varepsilon_0$, depending on $\delta$, $s$, $\lambda$, $\beta$, $\alpha$ and $\Upsilon$, such that for all $\varepsilon \leq \varepsilon_0$ there exists a $C^\rho$-weakly asymptotic cylinder associated to $(X_H, X_{\hat{h}}, \varphi_0)$.

Here, $b_0$ plays the role of $\gamma$ in Definition 1.1. Concerning the proof, it relies on a version of the Nash-Moser theorem due to Zehnder [Zeh75]. The conditions (\#)
are a consequence of this theorem (Theorem [5.1]). For the sake of clarity, we will give an example of parameters satisfying (#)

\[ s > \max \left\{ 7, \lambda + \frac{7}{3} \right\}, \quad \beta = \frac{3}{2}, \quad \alpha = \frac{7}{6}, \quad \lambda > 6, \quad \rho < \frac{4}{3} - \frac{2}{3}. \]

Zehnder proved that the minimum order required by \( s \) to satisfy (#) is \( s \geq 8 \) (Corollary [5.1]). It translates into

\[ s \geq 8, \quad \beta = 1 + \frac{7}{3s}, \quad \alpha = \frac{7}{6}, \quad \lambda = 2 + \frac{14}{s}, \quad \rho = 1 \]

for the other parameters.

We want to emphasize that our proof does not work for \( C^\infty \) or real analytic Hamiltonians. We will see that we are not able to find \( C^\infty \) or holomorphic solutions to the associated linearized problem.

Concerning the smallness assumption on the perturbing terms. It seems reasonable to think that this hypothesis is not essential and, reasoning as in [Sca22], one should be able to remove it. However, on the other hand, the example of the three-body problem perturbed by a given comet is a perturbative problem. So, it is good enough for the application.

Regarding the regularity of the Hamiltonian, the Nash-Moser theorem proved by Zehnder [Zeh75] requires a strong regularity to verify the convergence of the algorithm used in the proof. Therefore, it is reasonable to think that, using a more refined version of this theorem, one should be able to prove Theorem [A] for less regular Hamiltonians. Also in this case, the Hamiltonian of the planar three-body problem plus comet is \( C^\infty \) in a suitable neighbourhood of the phase space, so this theorem is good enough.

### 2.2 Planar three-body problem plus comet

We consider three points of fixed masses \( m_0, m_1 \) and \( m_2 \) undergoing gravitational attraction in the plane and a comet of fixed mass \( m_c \). The comet comes from and goes to infinity along a hyperbolic Keplerian orbit. Therefore, the motion of the comet is a given smooth function \( c(t) \) and only the planetary system is influenced by the comet. We assume,

\[ |c(t)| \to_{t \to +\infty} \infty, \quad \frac{d}{dt} |c(t)| \to_{t \to +\infty} v > 0. \]

If the comet is on a hyperbolic Keplerian orbit, \( \frac{d}{dt} c(t) \) itself has a limit. But we will not use this stronger hypothesis.

Given \( 0 < \varepsilon < 1 \) and \( J = [1, +\infty) \), the phase space is the space

\[ \{ (x_i, y_i)_{0 \leq i \leq 2}, t \} \in (\mathbb{R}^2 \times \mathbb{R}^{2s})^3 \times J \quad \forall 0 \leq i < j \leq 2, \quad |x_i - x_j| < \varepsilon \} \quad (2.3) \]

of linear momentum covectors \((y_0, y_1, y_2)\) and position vectors \((x_0, x_1, x_2)\) of each body. The Hamiltonian of the planar three-body problem plus comet (P3BP+C)
\[
H(x, y, t) = \sum_{i=0}^{2} \frac{|y_i|^2}{2m_i} - G \sum_{0 \leq i < j \leq 2} \frac{m_im_j}{|x_i - x_j|} - G \sum_{i=0}^{2} \frac{m_im_c}{|x_i - c(t)|},
\]

where \(G\) is the universal constant of gravitation that we may suppose equal to 1. It is the sum of the Hamiltonian of the planar three-body problem \(H_0\) and the Hamiltonian of the interaction with the comet \(H_c\). Let \(\varphi_0\) be a 1-parameter family of invariant tori for \(H_0\) supporting quasiperiodic dynamics with four frequencies and \(\psi_{t_0,H}^i\) the flow at time \(t\) with initial time \(t_0\) of \(H\).

**Theorem B.** Let \(H\) be the Hamiltonian of the P3BP+C. Then, there exist constants \(C(H_0, \varphi_0)\) depending on \(H_0\), \(\varphi_0\) and \(C(H)\) depending on \(H\) such that if

\[
|c(1)| > \frac{C(H_0, \varphi_0)}{\varepsilon}, \quad v > \frac{C(H)}{\varepsilon},
\]

for \(\varepsilon\) small enough, there exists a non-empty open subset \(W \subset (\mathbb{R}^2 \times \mathbb{R}^2)^3\) such that, for all \(x \in W\), \(\psi_{t_0,H}^i(x)\) is a weakly asymptotically quasiperiodic solution associated to \((X_H, X_{H_0}, \varphi_0)\).

We will see that the constants in \(\text{(cv)}\) are specified in Lemma 6.3, Lemma 6.5 and Lemma 6.9. Concerning the existence of \(\varphi_0\), in 1963, Arnold proved the existence of quasiperiodic solutions for the Hamiltonian of the planar three-body problem [Arn63]. In this work, we follow the setting of Féjoz [Fé02], which provides more general solutions. In a rotating frame of reference, the author proves the existence of quasiperiodic orbits with three frequencies for the Hamiltonian of the planar three-body problem. Before the symplectic reduction by the symmetry of rotations, these quasiperiodic motions have one additional frequency, namely the angular speed of the simultaneous rotation of the three ellipses. Furthermore, before the symplectic reduction by the symmetry of translations, each of these invariant tori translates into a 1-parameter family of invariant tori parametrized by the center of mass of the planetary system.

### 3 Functional setting

This section is dedicated to the norm (2.1) previously defined. It satisfies some properties that are similar to that of the Hölder norm. For this reason, let us recall the definition of Hölder classes of functions \(C^\sigma\) with \(\sigma \geq 0\) and discuss some well-known properties. Let \(D\) be an open subset of \(\mathbb{R}^{2(n+m)}\). For given integers \(k \geq 0\), we denote by \(C^k(D)\) the spaces of functions \(f : D \to \mathbb{R}\) with continuous partial derivatives \(\partial^\alpha f \in C^0(D)\) for all \(\alpha \in \mathbb{N}^n\) with \(|\alpha| = \alpha_1 + \ldots + \alpha_n \leq k\). Therefore, we define the following norm

\[
|f|_{C^k} = \sup_{|\alpha| \leq k} |\partial^\alpha f|_{C^0},
\]

where \(|\partial^\alpha f|_{C^0} = \sup_{x \in D} |\partial^\alpha f(x)|\) denotes the sup norm. For \(\sigma = k + \mu\), with \(k \in \mathbb{Z}, k \geq 0\) and \(0 < \mu < 1\), the Hölder spaces \(C^\sigma(D)\) are the spaces of functions...
\( f \in C^k(D) \) such that \( |f|_{C^\sigma} < \infty \), where

\[
|f|_{C^\sigma} = \sup_{|a| \leq k} \left| \partial^a f \right|_{C^\sigma} + \sup_{|a|=k} \frac{|\partial^a f(x) - \partial^a f(y)|}{|x-y|^{\mu}}.
\]  

(3.1)

Now, to avoid a flow of constants, let \( C(\cdot) \) be constants depending on \( n + m \) and the other parameters in brackets. On the other hand, \( C \) stands for constants depending only on \( n + m \). The following proposition contains some properties of the previous norm (3.1).

**Proposition 3.1.** We consider \( f, g \in C^\sigma(D) \) with \( \sigma \geq 0 \).

1. For all \( \beta \in \mathbb{N}^n \), if \( |\beta| + s = \sigma \) then
   \[
   \left| \frac{\partial^{(\beta)}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} f \right|_{C^{\sigma}} \leq |f|_{C^\sigma}.
   \]

2. \( |fg|_{C^\sigma} \leq C(\sigma) (|f|_{C^\sigma}|g|_{C^\sigma} + |f|_{C^\sigma}|g|_{C^\sigma}) \).

Now we consider composite functions. Let \( z \) be defined on \( D_1 \subset \mathbb{R}^{2(n+m)} \) and takes its values on \( D_2 \subset \mathbb{R}^{2(n+m)} \) where \( f \) is defined. If \( \sigma \geq 1 \) and \( f \in C^\sigma(D_2) \), \( z \in C^\sigma(D_1) \) then \( f \circ z \in C^\sigma(D_1) \)

3. \( |f \circ z|_{C^\sigma} \leq C(\sigma) (|f|_{C^\sigma} |\nabla z|_{C^\sigma}^\sigma + |f|_{C^\sigma} |\nabla z|_{C^{\sigma-1}} + |f|_{C^\sigma}) \).

**Proof.** Concerning the proof, 1. is obvious. For 2., we refer to \([Hö76]\). The last is proved in \([Sc22]\). \( \square \)

The following proposition shows a series of properties for the norm (2.1) defined in the previous section.

**Proposition 3.2.** We consider \( f, g \in S_\sigma \) and positive \( l, m > 0 \).

a. For all \( \beta \in \mathbb{N}^{2n} \), if \( |\beta| + s \leq \sigma \), then
   \[
   \left| \frac{\partial^{(\beta)}}{\partial q_1^{\alpha_1} \cdots \partial q_n^{\alpha_n} \partial p_1^{\beta_1} \cdots \partial p_n^{\beta_n}} f \right|_{s,l} \leq |f|_{s,l},
   \]

b. \( |f|_{s,l} \leq |f|_{s,l+m} \),

c. \( |fg|_{s,l+m} \leq C(\sigma) (|f|_{0,l} |g|_{\sigma,m} + |f|_{s,l} |g|_{0,m}) \).

If \( \sigma \geq 1 \) and \( f, z \in S_\sigma \) then \( f \circ z \in S_\sigma \),

d. \( |f \circ z|_{s,l+m} \leq C(\sigma) (|f|_{s,l} |\nabla z|_{0,m}^\sigma + |f|_{1,l} |\nabla z|_{\sigma-1,m} + |f|_{0,l+m}) \).

**Proof.** The proof consists of a straightforward application of Proposition 3.1. Properties a. and b. are obvious, we verify the others c.

\[
|fg|_{s,l+m} = \sup_{t \in J} |fg|_{C^\sigma t^l+m} \leq C(\sigma) \sup_{t \in J} \left( |f|_{C^\sigma} |g|_{C^\sigma} + |f|_{C^\sigma} |g|_{C^\sigma} \right) t^{l+m} \leq C(\sigma) \sup_{t \in J} \left( |f|_{C^\sigma} |g|_{C^\sigma} + |f|_{C^\sigma} |g|_{C^\sigma} \right) t^{l+m} \leq C(\sigma) (|f|_{0,l} |g|_{\sigma,m} + |f|_{s,l} |g|_{0,m})
\]

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Proposition 4.1. If the flows for all $t$, then a family of $\psi$ and have a series of properties in common with non-autonomous KAM tori. Let $\psi$ for all $t$ and $n$ associated to Definition 1.1.

We recall the definition of $\psi$ and $X$ as a vector field $\gamma$ on $\mathbb{R}$, an embedding $\phi$ of class $C$ on $J$, and $\bar{\sigma}$, respectively. Let $\phi$ be equal to $C$ and $f$ from $J$ to $\mathbb{R}$. Given $\sigma = 0$ and a positive integer $k \geq 0$, we consider time-dependent vector fields $X$, $X_0$ of class $C^{\sigma+k}$ on $P$, for all fixed $t \in J$, an embedding $\varphi_0$ from $\mathbb{T} \times \mathbb{R}^m$ to $P$ of class $C^{\sigma}$ and a time-dependent vector field $\gamma$ of class $C^\sigma$ on $\mathbb{T} \times \mathbb{R}^m$, for all fixed $t \in J$, such that

\[
\lim_{t \to +\infty} |X' - X_0'|_{C^{\sigma+k}} = 0, \quad (4.1)
\]

\[
X_0(\varphi_0(q), t) = \partial_q \varphi_0(q)(\bar{\omega} + \gamma(q, t)) \quad \text{for all } (q, t) \in \mathbb{T} \times \mathbb{R}^m \times J, \quad (4.2)
\]

\[
\lim_{t \to +\infty} |\gamma'|_{C^\sigma} = 0, \quad (4.3)
\]

where $\bar{\omega} = (\omega, 0) \in \mathbb{R}^{n+m}$ with $\omega \in \mathbb{R}^n$.

**Definition** (Definition 1.1). We assume that $(X, X_0, \varphi_0)$ satisfy (4.1), (4.2), and (4.3). A family of $C^\sigma$ embeddings $\varphi^t : \mathbb{T} \times \mathbb{R}^m \to P$ is a $C^\sigma$-weakly asymptotic cylinder associated to $(X, X_0, \varphi_0)$ if there exists a time-dependent vector field $\Gamma$ of class $C^\sigma$ on $\mathbb{T} \times \mathbb{R}^m$, for all fixed $t$, such that

\[
\lim_{t \to +\infty} |\varphi^t - \varphi_0|_{C^\sigma} = 0, \quad (4.4)
\]

\[
X(\varphi(q, t), t) = \partial_q \varphi(q, t)(\bar{\omega} + \Gamma(t)) + \partial_t \varphi(q, t), \quad (4.5)
\]

\[
\lim_{t \to +\infty} |\Gamma'|_{C^\sigma} = 0, \quad (4.6)
\]

for all $(q, t) \in \mathbb{T} \times \mathbb{R}^m \times J$. Moreover, $\varphi$ is Lagrangian if $\varphi^t(\mathbb{T} \times \mathbb{R}^m)$ is Lagrangian for all $t \in J$.

We point out that Definition 1.1 is a generalization of the definition of non-autonomous KAM torus given by Canadell-de la Llave [CdlL15]. Therefore, we have a series of properties in common with non-autonomous KAM tori. Let $\psi_{t_0, X}$ and $\psi_{t_0, \bar{\omega} + \Gamma}$ be the flow at time $t$ with initial time $t_0$ of $X$ and $\bar{\omega} + \Gamma$, respectively.

**Proposition 4.1.** If the flows $\psi_{t_0, X}$ and $\psi_{t_0, \bar{\omega} + \Gamma}$ are defined for all $t$, $t_0 \in J$, then (4.5) is equivalent to

\[
\psi_{t_0, X} \circ \varphi_{t_0}(q) = \varphi^t \circ \psi_{t_0, \bar{\omega} + \Gamma}(q), \quad (4.7)
\]

for all $t, t_0 \in J$ and $q \in \mathbb{T} \times \mathbb{R}^m$. 

Proof. Following the lines of the proof of Proposition 3.1 in [Sc22], we have the claim.

As a consequence of the previous proposition, (4.5) is trivial.

**Proposition 4.2.** If $\psi_{t_0,\omega}^t$ and $\psi_{t_0,\omega+\Gamma}^t$ are defined for all $t$, $t_0 \in J$, it is always possible to find a family of embeddings satisfying (4.5).

**Proof.** We consider an embedding $\hat{\varphi} : \mathbb{T}^n \times \mathbb{R}^m \to \mathcal{P}$. Then, for all $t$, $t_0 \in J$ and $q \in \mathbb{T}^n \times \mathbb{R}^m$, we define the following family of embeddings

$$\varphi^t(q) = \psi_{t_0,\omega}^t \circ \hat{\varphi} \circ \psi_{t_0,\omega+\Gamma}^t(q).$$

The latter is a family of embeddings satisfying (4.7). Indeed, by the above definition of $\varphi^t$, we have that $\varphi_{t_0}^t(q) = \hat{\varphi}(q)$ for all $q \in \mathbb{T}^n \times \mathbb{R}^m$. Then, $\varphi^t$ satisfies (4.7) and thus (4.5).

As a consequence of (4.7), we have the following property.

**Proposition 4.3.** We assume that $\psi_{t_0,\omega}^t$ and $\psi_{t_0,\omega+\Gamma}^t$ are defined for all $t$, $t_0 \in \mathbb{R}$.

Let $T \geq 0$, if there exists a $C^\sigma$-weakly asymptotic KAM torus $\varphi^t$ defined for all $t \geq T$, then we can extend the set of definition for all $t \in \mathbb{R}$.

**Proof.** For all $q \in \mathbb{T}^n \times \mathbb{R}^m$, we consider

$$\varphi^t(q) = \begin{cases} 
\varphi^t(q) & \text{for all } t \geq T \\
\psi_{T,\omega}^t \circ \varphi^T \circ \psi_{t_0,\omega+\Gamma}^t(q) & \text{for all } t \leq T.
\end{cases}$$

The above family of embeddings verifies (4.4), (4.5) and (4.6).

Now, in order to give some information concerning the dynamics associated with a $C^\sigma$-weakly asymptotic cylinder, we recall the definition of weakly asymptotically quasiperiodic solution.

**Definition (Definition 1.2).** We assume that $(X, X_0, \varphi_0)$ satisfy (4.4), (4.5) and (4.6).

An integral curve $g(t)$ of $X$ is a weakly asymptotically quasiperiodic solution associated to $(X, X_0, \varphi_0)$ if there exist a time-dependent vector field $\Gamma : \mathbb{T}^n \times \mathbb{R}^m \times J \to \mathbb{R}^{n+m}$ and $q \in \mathbb{T}^n \times \mathbb{R}^m$ such that

$$\lim_{t \to +\infty} |g(t) - \varphi_0 \circ \psi_{t_0,\omega+\Gamma}^t(q)| = 0.$$

The following proposition is quite obvious.

**Proposition 4.4.** Let $\varphi^t$ be a $C^\sigma$-weakly asymptotic cylinder associated to $(X, X_0, \varphi_0)$. Then, for all $q \in \mathbb{T}^n \times \mathbb{R}^m$ and $t_0 \in J$,

$$g(t) = \psi_{t_0,\omega}^t \circ \varphi^t(q)$$

is a weakly asymptotically quasiperiodic solution associated to $(X, X_0, \varphi_0)$.

**Proof.** The proof is a straightforward consequence of (4.4) and (4.7).
5 Proof of Theorem A

5.1 The Nash-Moser theorem (Zehnder)

The proof of Theorem A relies on a version of the Nash-Moser theorem proved by Zehnder (see [Zeh75]). For the sake of clarity, we dedicate this section to explaining this result.

We consider a one-parameter family of Banach spaces \((\mathcal{X}^\sigma, |\cdot|_\sigma)\) for \(\sigma \geq 0\) and we assume that for all \(0 \leq \sigma' \leq \sigma < \infty\)

\[
\mathcal{X}^0 \supseteq \mathcal{X}^{\sigma'} \supseteq \mathcal{X}^\sigma \supseteq \bigcap_{\sigma \geq 0} \mathcal{X}^\sigma
\]

for all \(x \in \mathcal{X}^\sigma\). The following definition introduces the notion of \(C^\infty\)-smoothing.

**Definition 5.1.** A \(C^\infty\)-smoothing in \((\mathcal{X}^\sigma, |\cdot|_\sigma)\) is a one-parameter family \[S_\tau\] of linear mappings \[S_\tau : \mathcal{X}^0 \to \mathcal{X}^\infty\] together with constants \(C(m,d)\), for positive integers \(m\) and \(d\), satisfying the following conditions:

\[|S_\tau x|_m \leq \tau^{m-d} C(m,d) |x|_d\] (S1)

for all \(x \in \mathcal{X}^d\) and \(0 \leq d \leq m\),

\[|(S_\tau - 1)x|_d \leq \tau^{-(m-d)} C(m,d) |x|_m\] (S2)

for all \(x \in \mathcal{X}^m\) and \(0 \leq d \leq m\).

The existence of a \(C^\infty\)-smoothing implies the following well-known convexity property.

**Lemma 5.1.** We assume that \((\mathcal{X}^\sigma, |\cdot|_\sigma)\) has a \(C^\infty\)-smoothing. Then, for all \(0 \leq \lambda_1 \leq \lambda_2\), \(\alpha \in [0,1]\) and \(x \in \mathcal{X}^\lambda_2\),

\[|x|_\lambda \leq C(\alpha, \lambda_1, \lambda_2) |x|_{\lambda_1}^{1-\alpha} |x|_{\lambda_2}^\alpha\]

with \(\lambda = (1 - \alpha)\lambda_1 + \alpha \lambda_2\).

**Proof.** We refer to [Zeh75] for the proof. \(\square\)

In what follows, we report the hypotheses of the Zehnder theorem. We consider three one-parameter families of Banach spaces \((\mathcal{X}^\sigma, |\cdot|_\sigma)\), \((\mathcal{V}^\sigma, |\cdot|_\sigma)\), and \((\mathcal{Z}^\sigma, |\cdot|_\sigma)\) each with a \(C^\infty\)-smoothing, which we denote by the same letter \(S_\tau\). Let \(\mathcal{F}\) be the following functional

\[\mathcal{F} : \mathcal{X}^0 \times \mathcal{V}^0 \longrightarrow \mathcal{Z}^0\]

and we assume that

\[\mathcal{F}(u_0, v_0) = 0\]

for some \((u_0, v_0) \in \mathcal{X}^0 \times \mathcal{V}^0\). Given a positive parameter \(0 < \zeta \leq 1\), we define

\[\mathcal{O}_\zeta = \{(x,v) \in \mathcal{X}^\sigma \times \mathcal{V}^\sigma : |x - x_0|_\sigma, |v - v_0|_\sigma < \zeta\}\] (5.1)
and we consider $F : \mathcal{O}_\zeta^0 \to \mathcal{Z}^0$ to be continuous. In his work, Zehnder takes $\zeta = 1$. We will see that this does not really change the proof. But, as we shall see in the proof of Theorem [A] we need $\zeta$ to satisfy a suitable smallness condition to define the right inverse of our functional $F$.

For given $x \in \mathcal{X}^0 \cap \mathcal{O}_\zeta^0$, the aim of the Zehnder theorem is to solve the equation $F(x, v) = 0$ assuming $x$ sufficiently close to $x_0$. The author makes the following hypotheses.

**Hypotheses H.1-H.4**

**H.1 Smoothness:** We assume that $F(x, \cdot) : V^0 \to \mathcal{Z}^0$ is two times differentiable with the uniform estimate

$$|D_v F(x, v)|_0, |D^2_v F(x, v)|_0 \leq C$$

for all $(x, v) \in \mathcal{O}_\zeta^0$ and for some constant $C \geq 1$, where $D_v$ is the differential with respect to the second component.

**H.2 $F$ is uniformly Lipschitz in $\mathcal{X}^0$:** For all $(x_1, v), (x_2, v) \in \mathcal{O}_\zeta^0$,

$$|F(x_1, v) - F(x_2, v)|_0 \leq C|x_1 - x_2|_0.$$

**H.3 Existence of a right-inverse of loss $\gamma$, $1 \leq \gamma < s$ (s will be specified later):** For every $(x, v) \in \mathcal{O}_\zeta^\gamma$ there exists a linear map $\eta(x, v) : \mathcal{Z}^\gamma \to V^0$ such that, for all $z \in \mathcal{Z}^\gamma$,

$$D_v F(x, v) \circ \eta(x, v) z = z$$

$$|\eta(x, v) z|_0 \leq C|z|_\gamma.$$  \hfill (\eta 1)

Moreover, for all $\gamma \leq \sigma \leq s$, if $(x, v) \in \mathcal{O}_\zeta^\sigma \cap (\mathcal{X}^\sigma \times \mathcal{V}^\sigma)$, then the linear map $\eta : \mathcal{Z}^\sigma \to \mathcal{V}^{\sigma-\gamma}$ is well defined and if $|x - x_0|_\sigma, |v - v_0|_\sigma \leq K$, then

$$|\eta(x, v) F(x, v)|_{\sigma-\gamma} \leq C(\sigma)K.$$  \hfill (\eta 2)

**H.4 Order:** The triple $(F, x_0, v_0)$ is of order $s$, $s > \gamma \geq 1$. Here, Zehnder uses the following

**Definition 5.2.** $(F, x_0, v_0)$ is called of order $s$, $1 \leq s < \infty$, if the following three conditions are satisfies:

1. $(x_0, v_0) \in \mathcal{X}^s \times \mathcal{V}^s$,

2. $F(\mathcal{O}_\zeta^0 \cap (\mathcal{X}^\sigma \times \mathcal{V}^\sigma)) \subset \mathcal{Z}^\sigma, \quad 1 \leq \sigma \leq s$

3. there exist constants $C(\sigma), 1 \leq \sigma \leq s$, such that if $(x, v) \in (\mathcal{X}^\sigma \times \mathcal{V}^\sigma) \cap \mathcal{O}_\zeta^1$ satisfies $|x - x_0|_\sigma, |v - v_0|_\sigma \leq K$ then

$$|F(x, v)|_{\sigma} \leq C(\sigma)K.$$
Zehnder, in his paper, assumes the existence of an approximate right-inverse. The reason is that, in his works [Zeh75] and [Zeh76], he wants to apply generalized implicit function theorems to solve some small divisor problems. In particular, to prove Arnold’s normal form theorem for vector fields on the torus and the KAM theorem. In the proof of the previous theorems, the author defines a functional \( F \) which does not admit a right-inverse but just an approximate right-inverse. We do not have this problem; hence we prefer to write H.4 in this form.

**Theorem 5.1** (Zehnder). Let \( \alpha, \beta, \lambda, \rho, \gamma \) and \( s \) be positive real numbers satisfying the following set of inequalities:

\[
1 < \beta < 2, \quad 1 < \alpha, \quad 1 \leq \gamma \leq \rho < \lambda < s, \quad (5.2)
\]

\[
\lambda > \max\left\{ \frac{2\beta\gamma}{2 - \beta}, \beta(\gamma + \rho\beta) \right\} \quad (5.3)
\]

\[
s > \max\left\{ \frac{\alpha\gamma}{\alpha - 1}, \lambda + \frac{\alpha\gamma}{\beta - 1} \right\}. \quad (5.4)
\]

Let \( (F, x_0, v_0) \) be of order \( s \) and satisfy H.1-H.4 with a loss of \( \gamma \). Then there exists \( \varepsilon_0 \), depending on \( \alpha, \beta, \lambda, \gamma, s \) and \( \zeta \), such that for all \( \varepsilon \leq \varepsilon_0 \) we have the existence of an open neighborhood \( D^\lambda \subset X^\lambda \) of \( x_0 \), \( D^\lambda = \{ x \in X^\lambda : |x - x_0|^\lambda \leq \varepsilon \} \) and a mapping \( \psi : D^\lambda \to V^\rho \) such that

\[
F(x, \psi(x)) = 0, \quad x \in D^\lambda
\]

\[
|\psi(x) - v_0|_\rho \leq \zeta,
\]

where \( \zeta \) is the positive parameter defined by (5.1).

**Proof.** The statement of the theorem is slightly different from the original. As mentioned before, Zehnder considers \( \zeta = 1 \). For the sake of clarity, in what follows, we will report the parts that differ from the original proof.

The proof uses an iteration technique similar to the Newton algorithm modified by a double \( C^\infty \)-smoothing. The first is introduced in \( V^0 \) to regain the loss of derivatives \( \gamma \) at each iteration step. The second approximates elements in \( D^\lambda \subset X^\lambda \) with smoother ones to keep the loss of regularity minimal in the \( X^0 \) space. Zehnder makes great use of Lemma 5.1, he estimates the lowest norms \( |\cdot|_0 \) very carefully to keep them down and the highest norms \( |\cdot|_s \) are left to grow. Then, thanks to the aforementioned Lemma 5.1, the author estimates the intermediate norms.

Let \( \varepsilon = v \varepsilon_0 \) for some \( 0 < v \leq 1 \) and a sufficiently small parameter \( \varepsilon_0 \) to be determined later. We define

\[
D^\lambda = \{ x \in X^\lambda : |x - x_0|^\lambda \leq v \varepsilon_0 \}.
\]

Following the lines of the proof of Zehnder, we define a sequence \( \{ \phi_j \}_{j \geq 0} \) of linear mapping \( \phi_j : D^\lambda \to X^\infty \) in such a way that,

\[
\phi_0(x) = x_0
\]

\[
\phi_j(x) - x_0 = S_{\tau_j}(x - x_0),
\]

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for all \( j \geq 1 \), where \( \tau_j = Q^{\beta j} \) for some \( Q > 1 \) sufficiently large to be chosen later. Since \( \beta > 1 \) then \( \tau_j \to +\infty \) if \( j \to +\infty \). By the latter, we write \( \phi_j(x) - x \) in the following form

\[
\phi_j(x) - x = (S_{\tau_j} - 1)(x - x_0).
\]

Thanks to (S2), for all \( 0 \leq \mu < \lambda \)

\[
|\phi_j(x) - x|_\mu = |(S_{\tau_j} - 1)(x - x_0)|_\mu \leq \tau_j^{\mu - \lambda}C(\lambda, \mu)|x - x_0|_\lambda \leq \tau_j^{\mu - \lambda}C(\lambda, \mu)\varepsilon_0
\]

and taking the limit for \( j \to +\infty \), we have \( \lim_{j \to +\infty} |\phi_j(x) - x|_\mu = 0 \) for all \( 0 \leq \mu < \lambda \). We construct inductively a sequence of mapping \( \{\psi_j\}_{j \geq 0}, \psi_j : \mathcal{D}^\lambda \to \mathcal{V}^\infty \) such that

\[
\psi_0(x) = v_0 \quad \psi_{j+1}(x) - \psi_j(x) = S_{t_{j+1}}\eta(\phi_{j+1}(x), \psi_j(x))\mathcal{F}(\phi_{j+1}(x), \psi_j(x)),
\]

for all \( j \geq 0 \), with \( t_j = \tau_j^\alpha = Q^{\alpha j} \). We use two different rates of approximations, \( S_{\tau_j} \) and \( S_t \), for the family of Banach spaces \( \{(\mathcal{A}^{\sigma'}, | \cdot |_{\sigma'})\}_{\sigma \geq 0} \) and \( \{(V^{\sigma'}, | \cdot |_{\sigma'})\}_{\sigma \geq 0} \), respectively. We shall show by induction that, if \( \varepsilon_0 \) is sufficiently small with respect to \( \alpha, \beta, \lambda, \gamma, s \) and \( \zeta \), and \( x \in \mathcal{X}^\lambda \) satisfies \( |x - x_0|_\lambda \leq \varepsilon \), then the following statements \( S(d) \) hold for \( d \geq 1 \):

\[
S(d, 1) \ (\phi_d(x), \psi_d(x) \in \mathcal{O}_\zeta \cap (\mathcal{X}^\infty \times \mathcal{V}^\infty) \text{ and } |\mathcal{F}(\phi_d(x), \psi_d(x))|_0 \leq \frac{1}{2}vQ^{-\lambda \beta d}
\]

\[
S(d, 2) \ |\psi_d(x) - \psi_{d-1}(x)|_0 \leq C_vQ^{-(\lambda - \beta \gamma)d - 1},
\]

\[
S(d, 3) \ |\psi_d(x) - \psi_{d-1}(x)|_s \leq vQ^{(s - \lambda)\beta_{d+1}},
\]

for a suitable constant \( C \). The only difference compared to the proof of Zehnder is in \( S(d, 1) \), where he considers \( \zeta = 1 \). In what follows, we verify only the first part of \( S(d, 1) \). The rest follows from the original proof.

We introduce the abbreviated notation

\[
x_j = \phi_j(x) \quad \text{and} \quad v_j = \psi_j(x).
\]

We claim that \( x_j \in O_\zeta \cap \mathcal{X}^\infty \) if \( \varepsilon_0 \) is sufficiently small. We recall that \( 1 \leq \gamma < \lambda \), then, by the definition of \( x_j \) and (S1),

\[
|x_j - x_0|_\gamma = |S_{\tau_j}(x - x_0)|_\gamma \leq C|x - x_0|_\gamma \leq C|x - x_0|_\lambda
\]

for a suitable constant \( C \). Letting \( C\varepsilon_0 < \zeta \), we have the claim.

Zehnder proves the totality of the three statements \( S(d) \) by induction. Letting \( d = 1 \), \( S(1) \) follows from the smallness condition by choosing \( \varepsilon_0 \) sufficiently small with respect to \( \alpha, \beta, \lambda, \gamma, s \) and \( \zeta \). Now, we assume \( S(d) \) for \( 1 \leq d \leq j \) and we prove \( S(j + 1) \). We do not provide any details concerning the proofs of \( S(j + 1, 2), S(j + 1, 3) \) and the second part of \( S(j + 1, 1) \) because they coincide with the original proof. Therefore, it remains to verify that \( |v_{j+1} - v_0|_\gamma < \zeta \). By \( S(j + 1, 2), S(j + 1, 3) \) and Lemma 5.1,

\[
|v_{j+1} - v_j|_\gamma \leq C(\gamma, s)|v_{j+1} - v_j|_0^{1 - \frac{2}{s}}|v_{j+1} - v_j|_s^{\frac{2}{s}} \leq vCQ^{-\xi \beta j}
\]
for a suitable constant $C$ and with
\[
\xi = \lambda - \beta(\gamma + \beta \gamma) + \frac{\gamma}{s}(\lambda(\beta^2 - 1) + \beta \gamma) \\
> \lambda - \beta(\gamma + \beta \gamma) > \lambda - \beta(\gamma + \beta \rho)
\]
where we have used $\gamma \leq \rho$ and $\beta > 1$. Furthermore, by (5.3) we have that $\lambda - \beta(\gamma + \beta \rho) > 0$. This means that,
\[
|v_{j+1} - v_0|_\gamma \leq \sum_{d=0}^j |v_{d+1} - v_d|_\gamma \leq C v \sum_{d=0} Q^{-\xi \beta^d} < \zeta.
\]
for $Q$ large enough.

Zehnder distinguishes between the order of $(\mathcal{F}, x_0, v_0)$ and the smoothness assumption quantified by $\lambda$. In his work, Zehnder states that the minimal order for which we can apply the previous theorem, and thus the minimal order which assures the convergence of the algorithm, is $s \geq 8 \gamma$. Moreover, concerning the corresponding minimal smoothness assumption for $\lambda$, one has $3 \gamma < \lambda < 4 \gamma$.

**Corollary 5.1.** For all $s \geq 8 \gamma$, the following holds: let $\lambda(s) = 2 \gamma + \frac{14 \gamma^2}{s}$, there exists in $\mathcal{X}^{\lambda(s)}$ a neighborhood $\mathcal{D}^{\lambda(s)} = \{x \in \mathcal{X}^{\lambda(s)} : |x - x_0|_{\lambda(s)} \leq \varepsilon(s, \zeta)\}$ and a mapping $\psi_s : \mathcal{D}^{\lambda(s)} \to \mathcal{V}$ such that, for all $x \in \mathcal{D}^{\lambda(s)}$,
\[
\mathcal{F}(x, \psi_s(x)) = 0,
|\psi_s(x) - v_0|_\gamma \leq \zeta.
\]

**Proof.** Taking $\alpha = \frac{7}{5}$, $\beta = 1 + \frac{7}{3s}$, $\rho = \gamma$, and $\lambda = 2 \gamma + \frac{14 \gamma^2}{s}$, these numbers satisfy the inequalities (5.2) - (5.4) if $s \geq 8 \gamma$ and hence the result follows from Theorem 5.1.

**5.2 Outline of the proof of Theorem A**

We are looking for a $C^n$-weakly asymptotic cylinder $\varphi$ associated to $(X_H, X_{\tilde{h}}, \varphi_0)$, where $H$ is the Hamiltonian defined by (4), $\tilde{h}$ the Hamiltonian in (2.2), and $\varphi_0$ the trivial embedding $\varphi_0 : \mathbb{T}^n \times \mathbb{R}^m \to \mathbb{T}^n \times \mathbb{R}^m \times B$, $\varphi_0(q) = (q, 0)$. More concretely, for given $H$, we are searching for $v, \Gamma : \mathbb{T}^n \times \mathbb{R}^m \times J \to \mathbb{R}^{n+m}$ such that for all $(q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J$
\[
\varphi(q, t) = (q, v(q, t))
\]
and in such a way that $\varphi, v$ and $\Gamma$ satisfy
\[
X_H(\varphi(q, t), t) - \partial_q \varphi(q, t)(\tilde{\omega} + \Gamma(q, t)) - \partial_t \varphi(q, t) = 0, \tag{5.5}
\]
\[
\lim_{t \to +\infty} |v'|_{C^\rho} = 0, \quad \lim_{t \to +\infty} |\Gamma'|_{C^\rho} = 0,
\]
for all $(q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J$. The vector $\tilde{\omega} = (\omega, 0) \in \mathbb{R}^{n+m}$ where $\omega \in \mathbb{R}^n$ is the frequency vector introduced by (4). As mentioned above, the proof rests on
Theorem 5.1. To this end, we need to introduce a suitable functional $F$ given by (5.5). First, we introduce the following notation

$$\bar{m}(q,p,t) = \left( \int_{0}^{1} \partial^{2}_{p} H(q,\tau p,t) d\tau \right) p.$$ 

It is straightforward to verify that $\bar{m}(q,p,t)p = \partial_{p}(m(q,p,t) \cdot p^{2})$. However, concerning the definition of the functional $F$, the Hamiltonian system associated to the Hamiltonian $H$ is equal to

$$X_{H}(q,p,t) = \left( \bar{\omega} + b(q,t) + \bar{m}(q,p,t)p \right)$$

where we recall that $H$ is the Hamiltonian defined by (5). Let

$$\tilde{\varphi}(q,t) = (q,v(q,t))$$

for all $(q,t) \in \mathbb{T}^{n} \times \mathbb{R}^{m} \times J$. Then $X_{H} \circ \tilde{\varphi}$ takes the following form

$$X_{H} \circ \tilde{\varphi}(q,t) = \left( \begin{array}{c} \bar{\omega} + b(q,t) + \bar{m}(q,p,t)p \\ -\partial_{q}a(q,t) - \partial_{q}b(q,t)v(q,t) - \partial_{q}m \circ \tilde{\varphi}(q,t)v^{2}(q,t) \end{array} \right)$$

for all $(q,t) \in \mathbb{T}^{n} \times \mathbb{R}^{m} \times J$. Moreover, for all $(q,t) \in \mathbb{T}^{n} \times \mathbb{R}^{m} \times J$,

$$\partial_{q} \varphi(q,t) (\bar{\omega} + \Gamma(q,t)) + \partial_{t} \varphi(q,t) = \left( \begin{array}{c} \bar{\omega} + \Gamma(q,t) \\ \partial_{q}v(q,t)(\omega + \Gamma(q,t)) + \partial_{t}v(q,t) \end{array} \right).$$

Hence, we can rewrite (5.5) in the following form

$$\left( \begin{array}{c} \Gamma - b - (\bar{m} \circ \tilde{\varphi}) v \\ \partial_{q}a + \partial_{q}b \cdot v + \partial_{q}m \circ \tilde{\varphi} \cdot v^{2} - \partial_{q}v(\omega + \Gamma) - \partial_{t}v \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right).$$

(5.6)

The latter is composed of sums and products of functions defined on $(q,t) \in \mathbb{T}^{n} \times \mathbb{R}^{m} \times J$. We have omitted the arguments $(q,t)$ in order to achieve a more elegant form. We keep this notation for the rest of the proof. We define

$$(\nabla v) \tilde{\Omega} = (\partial_{q}v) \tilde{\omega} + \partial_{t}v$$

and over suitable Banach spaces that we will specify later, let $F$ be the following functional

$$F(a,b,m,\bar{m},v) = (\nabla v) \tilde{\Omega} + \partial_{q}v (b + (\bar{m} \circ \tilde{\varphi}) v) + \partial_{q}a + (\partial_{q}b) v + \partial_{q}m \circ \tilde{\varphi} \cdot v^{2}.$$ 

This is obtained by the second equation of (5.6), where we have replaced $\Gamma$ with $b + (\bar{m} \circ \tilde{\varphi}) v$. This is our starting point. We observe that, for all $b$, $m$ and $\bar{m}$,

$$F(0,b,m,\bar{m},0) = 0.$$ 

Therefore, we can reformulate our problem in the following form. For fixed $m$ and $\bar{m}$, for a suitable $b_{0}$ and for $(a,b)$ sufficiently close to $(0,b_{0})$, we are looking for a
function \( v: \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m} \) in such a way that \( F(a, b, m, v) = 0 \) and \( \lim_{t \to +\infty} |v^t|_{C^0} = 0 \). Once we have \( v \), we let \( \Gamma = b + (\tilde{m} \circ \tilde{\varphi}) v \) and this concludes the proof.

The most technical part of the proof consists in showing that the differential of \( F \) with respect to the variable \( v \) admits a right-inverse. Let

\[
\begin{align*}
  f &= b + (\tilde{m} \circ \tilde{\varphi}) v, \\
  g &= \partial_q b + \partial_q v (\partial_p \tilde{m} \circ \tilde{\varphi}) v + \partial_q v (\tilde{m} \circ \tilde{\varphi}) + v^T (\partial_{pq} \tilde{m} \circ \tilde{\varphi}) v + 2 (\partial_q \tilde{m} \circ \tilde{\varphi}) v,
\end{align*}
\]

where \( T \) denotes the transpose. Over suitable Banach spaces, the differential of \( F \) with respect to the variable \( v \) is equal to

\[
D_v F(a, b, m, v)v = (\nabla v) \Omega + (\partial_q v) f + g v.
\]

We will see that, assuming \( f \) and \( g \) sufficiently small, we are able to find a right inverse of the latter.

The proof of Theorem A is split up into the following four sections. The first is dedicated to introducing the special one-parameter families of Banach spaces on which the functional \( F \) is defined. The second is about the solution of the homological equation. In other words, we prove the existence of a right-inverse for the latter differential. In the third section, we verify that the functional \( F \) satisfies the hypotheses of the Nash-Moser theorem proved by Zehnder (Theorem 5.1) and in the last section, we conclude the proof.

### 5.3 Preliminary settings

Given \( \sigma \geq 0 \), we recall the following definition

**Definition** (Definition 2.1). Let \( S_\sigma \) be the space of functions \( f \) defined on \( \mathbb{T}^n \times \mathbb{R}^m \times B \times J \) such that \( f^t \in C^\sigma(\mathbb{T}^n \times \mathbb{R}^m \times B) \) for all fixed \( t \in J \) and \( \partial_i f \in C(\mathbb{T}^n \times \mathbb{R}^m \times B \times J) \) for all \( 0 \leq i \leq [\sigma] \).

We use this notation also for functions defined on \( \mathbb{T}^n \times \mathbb{R}^m \times J \). Given \( \sigma \geq 0 \) and \( l \geq 0 \), for every \( f \in S_\sigma \), we recall the definition of the following norm

\[
|f|_{\sigma,l} = \sup_{t \in J} |f^t|_{C^l}. 
\]

Some properties of this norm are contained in Proposition 3.2.

Now, we consider the following families of Banach spaces \( \{(A^\sigma, | \cdot |_\sigma)_{\sigma \geq 0}, \{(B^\sigma, | \cdot |_\sigma)_{\sigma \geq 0}, \{(V^\sigma, | \cdot |_\sigma)_{\sigma \geq 0}, \{(Z^\sigma, | \cdot |_\sigma)_{\sigma \geq 0} \text{ such that, for all } \sigma \geq 0, \}
\]

\[
\begin{align*}
A^\sigma &= \{ a: \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R} \mid a \in S_{\sigma+1} \text{ and } |a|_\sigma = |a|_{\sigma+1,0} + |\partial_i a|_{\sigma,2} < \infty \}, \\
B^\sigma &= \{ b: \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m} \mid b \in S_{\sigma+1} \text{ and } |b|_\sigma = |b|_{\sigma+1,1} < \infty \}, \\
V^\sigma &= \{ v: \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m} \mid v \in S_{\sigma+1}, (\nabla v) \tilde{\Omega} \in S_\sigma \text{ and } |v|_\sigma = \max\{|v|_{\sigma+1,1}, |(\nabla v) \tilde{\Omega}|_{\sigma,2} < \infty \}, \\
Z^\sigma &= \{ z: \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m} \mid z \in S_\sigma, \text{ and } |z|_\sigma = |z|_{\sigma,2} < \infty \}. 
\end{align*}
\]
Following the lines of what we did in [Sca22], one can prove that the previous normed spaces are Banach spaces.

It is straightforward to verify that, for all $0 \leq \sigma' \leq \sigma < \infty$,

$$A^0 \supseteq A^\sigma' \supseteq A^\sigma \supseteq A^\infty = \bigcap_{\sigma \geq 0} A^\sigma,$$  
$$B^0 \supseteq B^\sigma' \supseteq B^\sigma \supseteq B^\infty = \bigcap_{\sigma \geq 0} B^\sigma,$$

$$|a|_{\sigma'} \leq |a|_{\sigma} \quad \text{and} \quad |b|_{\sigma'} \leq |b|_{\sigma}$$

for all $a \in A^\sigma$, $b \in B^\sigma$, $v \in V^\sigma$ and $z \in Z^\sigma$.

This part aims to prove the existence of a $C^\infty$-smoothing, see Definition 5.1, for these families of Banach spaces. This is not surprising because the behaviour of these norms is very similar to that of the Hölder norms.

**Lemma 5.2.** There exists a $C^\infty$-smoothing for the latter families of Banach spaces.

**Proof.** We begin by proving the existence of a $C^\infty$-smoothing for the family of Banach spaces $\{(Z^\sigma, \| \cdot \|_\sigma) : \sigma > 0\}$. Following the lines of [Zeh75], we take a function $\tilde{s} \in C^\infty_0(\mathbb{R}^{n+m})$ vanishing outside a compact set and identically equal to 1 in a neighbourhood of 0. Let $s$ be its Fourier transform then, for all $z \in S_0$,

$$S_\tau z(q, t) = \frac{1}{\tau^{n+m}} \int_{\mathbb{R}^{n+m}} s \left( \frac{q - \vartheta}{\tau} \right) z(\vartheta, t) d\vartheta.$$  

For all fixed $t \in J$, $S_\tau z^t \in C^\infty(\mathbb{T}^n \times \mathbb{R}^m) = \bigcap_{\sigma \geq 0} C^\sigma(\mathbb{T}^n \times \mathbb{R}^m)$ (see [Zeh75]). Now, we verify that $\partial^m_\tau (S_\tau z) \in C(\mathbb{T}^n \times \mathbb{R}^m \times J)$ for all $i \geq 0$. We observe that, for every $m > 0$ and $p > 0$, there exists a constant $C(m, p) > 0$ such that

$$|\partial^m_\tau s(x)| \leq C(m, p)(1 + |x|)^{-p},$$

where $\partial^m_\tau$ stands for partial derivatives of order $m$. The claim is a consequence of the regularity of $z$ and the latter. Indeed, for all $(q_1, t_1), (q_2, t_2) \in \mathbb{T}^n \times \mathbb{R}^m \times J$ and $i \geq 0$,

$$\left| \partial^i_q (S_\tau z)(q_1, t_1) - \partial^i_q (S_\tau z)(q_2, t_2) \right|$$

$$= \left| \frac{1}{\tau^{n+m+i}} \int_{\mathbb{R}^{n+m}} \partial^i_q \left( \frac{q_1 - \vartheta}{\tau} \right) z(\vartheta, t_1) d\vartheta - \frac{1}{\tau^{n+m+i}} \int_{\mathbb{R}^{n+m}} \partial^i_q \left( \frac{q_2 - \vartheta}{\tau} \right) z(\vartheta, t_2) d\vartheta \right|$$

$$= \frac{1}{\tau^i} \int_{\mathbb{R}^{n+m}} \left| \partial^i_q s(\rho) z(q_1 - \rho t_1) - z(q_2 - \rho t_2) \right| d\rho$$

$$\leq \frac{1}{\tau^i} \int_{\mathbb{R}^{n+m}} |\partial^i_q s(\rho)| \left| \left( z(q_1 - \rho t_1) - z(q_2 - \rho t_2) \right) \right| d\rho$$

where $| \cdot |$ stands for the standard Euclidean norm and, in the last line of the latter, we did the following change of coordinates $\frac{q_i - \vartheta}{\tau} = \rho$ for $i = 1, 2$. This implies the claim.
It remains to prove \((S_1)\) and \((S_2)\). For all \(z \in \mathbb{Z}^d\), \(0 \leq d \leq m\) and fixed \(t \in J\)

\[
|S_\tau z^t|_{C^m} \leq \tau^{m-d} C(m, d)|z^t|_{C^d}
\]

(always look at \([Zeh75]\)), then

\[
|S_\tau z|_m = \sup_{t \in J} |S_\tau z^t|_{C^m} t^2 \leq \tau^{m-d} C(m, d) \sup_{t \in J} |z^t|_{C^d} t^2 = \tau^{m-d} C(m, d)|z|_d
\]

and \((S_1)\) is verified. For all \(z \in \mathbb{Z}^m\), \(0 \leq d \leq m\) and fixed \(t \in J\)

\[
|(S_\tau - 1) z^t|_{C^d} \leq \tau^{-(m-d)} C(m, d)|z^t|_{C^m}
\]

(always see \([Zeh75]\)), then

\[
|(S_\tau - 1) z|_d = \sup_{t \in J} |(S_\tau - 1) z^t|_{C^d} t^2 \leq \tau^{-(m-d)} C(m, d) \sup_{t \in J} |z^t|_{C^m} t^2 \\
\leq \tau^{-(m-d)} C(m, d)|z|_m
\]

and \((S_2)\) is also verified. This implies the existence of a \(C^\infty\)-smoothing for \(\{(\mathcal{Z}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}\). Remembering that \(S_\tau\) commutes with partial differential operators, similarly, we have the claim for the family of Banach spaces \(\{(\mathcal{A}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}\) and \(\{(\mathcal{B}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}\).

It remains to prove the existence of a \(C^\infty\)-smoothing for \(\{(|\nabla|^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}\). Similarly to the previous case, because of \(S_\tau\) commutes with partial differential operators, \(S_\tau : \mathcal{V}^0 \to \mathcal{V}^\infty\).

Now, we verify \((S_1)\) and \((S_2)\). We begin by remembering that, for all \(v \in \mathcal{V}^\sigma\),

\[
|v|_\sigma = \max\{|v|_{\sigma+1,1}, |(\nabla v)_{\bar{\Omega}}|_{\sigma,2}\}.
\]

Similarly to the previous case, for all \(v \in \mathcal{V}^d\), \(0 \leq d \leq m\) and fixed \(t \in J\)

\[
|S_\tau v^t|_{C^{m+1}} \leq \tau^{m-d} C(m, d)|v^t|_{C^{d+1}},
\]

which implies

\[
|S_\tau v|_{m+1,1} = \sup_{t \in J} |S_\tau v^t|_{C^{m+1}} t \leq \tau^{m-d} C(m, d) \sup_{t \in J} |v^t|_{C^{d+1}} t \leq \tau^{m-d} C(m, d)|v|_d.
\]

Noting that \(S_\tau\) commutes with partial differential operators, for fixed \(t \in J\),

\[
|\nabla (S_\tau v^t)_{\bar{\Omega}}|_{C^m} = |S_\tau (\nabla v^t_{\bar{\Omega}})|_{C^m} \leq \tau^{m-d} C(m, d)|\nabla v^t_{\bar{\Omega}}|_{C^d}.
\]

Multiplying both sides of the latter by \(t^2\) and taking the sup for all \(t \in J\)

\[
|\nabla (S_\tau v)_{\bar{\Omega}}|_{m,2} = \sup_{t \in J} |\nabla (S_\tau v^t)_{\bar{\Omega}}|_{C^m} t^2 \leq \tau^{m-d} C(m, d) \sup_{t \in J} |\nabla v^t_{\bar{\Omega}}|_{C^d} t^2 \\\n\leq \tau^{m-d} C(m, d)|v|_d.
\]

This implies \((S_1)\) because

\[
|S_\tau v|_m = \max\{|S_\tau v|_{m+1,1}, |\nabla (S_\tau v)_{\bar{\Omega}}|_{m,2}\} \leq \tau^{m-d} C(m, d)|v|_d.
\]
Concerning \([S2]\), for all \(v \in \mathcal{V}^m, 0 \leq d \leq m\) and fixed \(t \in J\),
\[
|(S_\tau - 1)v^t|_{C^{d+1}} \leq \tau^{-(m-d)}C(m, d)|v^t|_{C^{m+1}}.
\]

Multiplying both sides by \(t\) and taking the sup for all \(t \in J\),
\[
|(S_\tau - 1)v|_{d+1,1} = \sup_{t \in J} |(S_\tau - 1)v^t|_{C^{d+1}}t \leq \tau^{-(m-d)}C(m, d) \sup_{t \in J} |v^t|_{C^{m+1}}t \\
\leq \tau^{-(m-d)}C(m, d)|v|_{m+1,1} \leq \tau^{-(m-d)}C(m, d)|v|_m.
\]

The operator \(S_\tau\) commutes with partial differential operators, then for fixed \(t \in J\),
\[
|\nabla ((S_\tau - 1)v^t) \tilde{\Omega}|_{C^d} = |(S_\tau - 1)(\nabla v^t \tilde{\Omega})|_{C^d} \leq \tau^{-(m-d)}C(m, d)|\nabla v^t \tilde{\Omega}|_{C^m}
\]
and hence
\[
|\nabla ((S_\tau - 1)v) \tilde{\Omega}|_{d,2} = \sup_{t \in J} |\nabla ((S_\tau - 1)v^t) \tilde{\Omega}|_{C^d}t^2 \\
\leq \tau^{-(m-d)}C(m, d) \sup_{t \in J} |\nabla v^t \tilde{\Omega}|_{C^m}t^2 \\
\leq \tau^{-(m-d)}C(m, d)|\nabla v \tilde{\Omega}|_{m,2} \leq \tau^{-(m-d)}C(m, d)|v|_m.
\]

This concludes the proof of this lemma because
\[
|(S_\tau - 1)v|_d = \max\{|(S_\tau - 1)v|_{d+1,1}, |\nabla ((S_\tau - 1)v) \tilde{\Omega}|_{d,2}\} \\
\leq \tau^{-(m-d)}C(m, d)|v|_m.
\]

\[\square\]

5.4 Homological equation

We introduce some fundamental Gronwall-type inequalities that we widely use in this section.

**Proposition 5.1.** Let \(J\) be an interval in \(\mathbb{R}\), \(t_0 \in J\), and \(a, b, u \in C(J)\) continuous positive functions. If we assume that
\[
u(t) \leq a(t) + \left| \int_{t_0}^{t} b(s)u(s)ds \right|, \quad \forall t \in J
\]
then it follows that
\[
u(t) \leq a(t) + \left| \int_{t_0}^{t} a(s)b(s)e^{\int_{t_0}^{s} b(\tau)d\tau}ds \right|, \quad \forall t \in J. \quad (5.7)
\]

If \(a\) is a monotone increasing function and we assume that
\[
u(t) \leq a(t) + \int_{t_0}^{t} b(s)u(s)ds \quad \forall t \geq t_0,
\]
then we obtain the estimate
\[
u(t) \leq a(t)e^{\int_{t_0}^{t} b(s)ds}, \quad \forall t \geq t_0. \quad (5.8)
\]

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**Proof.** We refer to [Ama90] for the proof of (5.7). Concerning the other, by (5.7)

\[ u(t) \leq a(t) + \int_{t_0}^t a(s)b(s)e^{\int_s^t b(\tau)d\tau}ds \]

for all \( t \geq t_0 \). Therefore, \( a \) is a monotone increasing function, then

\[ u(t) \leq a(t) \left( 1 + \int_{t_0}^t b(s)e^{\int_s^t b(\tau)d\tau}ds \right) = a(t)e^{\int_{t_0}^t b(s)ds} \]

for all \( t \geq t_0 \).

Given \( \sigma \geq 1, \mu \geq 0 \) and \( \omega \in \mathbb{R}^n \), this section is devoted to the solution of the following equation for the unknown \( \kappa : \mathbb{T}^n \times \mathbb{R}^m \times J \to \mathbb{R}^{n+m} \)

\[
\begin{cases}
\partial_q \kappa(q,t) \left( \bar{\omega} + f(q,t) \right) + \partial_t \kappa(q,t) + g(q,t)\kappa(q,t) = z(q,t) \\
f,g,z \in \mathcal{S}_\sigma, \\
|f|_{1,1} \leq \mu, \quad |g|_{1,1} \leq \mu, \quad |z|_{\sigma,2} < \infty, \\
|f|_{\sigma,1} < \infty, \quad |g|_{\sigma,1} < \infty,
\end{cases}
\]

where \( \bar{\omega} = (\omega,0) \in \mathbb{R}^{n+m} \) and \( |\cdot|_{\sigma,1} \) is the norm defined by (2.1). The functions \( f : \mathbb{T}^n \times \mathbb{R}^m \times J \to \mathbb{R}^{n+m}, z : \mathbb{T}^n \times \mathbb{R}^m \times J \to \mathbb{R}^{n+m} \) and \( g : \mathbb{T}^n \times \mathbb{R}^m \times J \to M_{n+m} \) are given, where \( M_{n+m} \) is the set of \( (n+m) \)-dimensional matrices.

We begin with several estimates. In what follows, we will widely use the properties contained in Proposition 3.1. For this reason, we recall it. Let \( D \) be an open subset of \( \mathbb{R}^{2(n+m)} \).

**Proposition** (Proposition 3.1). We consider \( f,g \in C^\sigma(D) \) and \( \sigma \geq 0 \).

1. For all \( \beta \in \mathbb{N}^n \), if \( |\beta| + s = \sigma \) then \( |\frac{\partial^{|\beta|}}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}f|_{C^\sigma} \leq |f|_{C^\sigma} \).

2. \( |fg|_{C^\sigma} \leq C(\sigma) \left( |f|_{C^\sigma} |g|_{C^\sigma} + |f|_{C^\sigma} |g|_{C^\sigma} \right) \).

Now we consider composite functions. Let \( z \) be defined on \( D_1 \subset \mathbb{R}^{2(n+m)} \) and takes its values on \( D_2 \subset \mathbb{R}^{2(n+m)} \) where \( f \) is defined. If \( \sigma \geq 1 \) and \( f \in C^\sigma(D_2) \), \( z \in C^\sigma(D_1) \) then \( f \circ z \in C^\sigma(D_1) \).

3. \( |f \circ z|_{C^\sigma} \leq C(\sigma) \left( |f|_{C^\sigma} |\nabla z|_{C^0} + |f|_{C^1} |\nabla z|_{C^{\sigma-1}} + |f|_{C^\sigma} \right) \).

We define \( \psi^t_{t_0} \) as the flow at time \( t \) with initial time \( t_0 \) of \( F(q,t) \). We recall that \( \sigma \geq 1 \) and we have the following estimates.

**Lemma 5.3.** For all \( t, t_0 \in J \), if \( t \geq t_0 \)

\[ |\partial_q \psi^t_{t_0}|_{C^{\sigma-1}} \leq C(\sigma) \left( 1 + |f|_{\sigma,1} \ln \left( \frac{t}{t_0} \right) \right) \left( \frac{t}{t_0} \right)^{c_\sigma \mu}, \quad (5.9) \]

whereas if \( t \leq t_0 \)

\[ |\partial_q \psi^t_{t_0}|_{C^{\sigma-1}} \leq C(\sigma) \left( 1 + |f|_{\sigma,1} \ln \left( \frac{t_0}{t} \right) \right) \left( \frac{t_0}{t} \right)^{c_\sigma \mu}, \quad (5.10) \]

with a positive constant \( c_\sigma \) depending on \( n + m \) and \( \sigma \).
Before the proof of this lemma, we observe that when $\sigma = 1$ and $t \geq t_0$, by (5.9), we have

$$|\partial_q \psi_t^t|_{C^0} \leq C \left(1 + |f|_{1,1} \ln \left(\frac{t}{t_0}\right)\right) \left(\frac{t}{t_0}\right)^{c_1 \mu} \leq C \left(1 + \ln \left(\frac{t}{t_0}\right)\right) \left(\frac{t}{t_0}\right)^{c_1 \mu} \leq C \left(\frac{t}{t_0}\right)^{c_1 \mu}$$

for a suitable constant $C$ depending on $n + m$ and $\bar{c}_1 > c_1$. Similarly, when $t_0 \geq t$, $|\partial_q \psi_t^t|_{C^0} \leq C \left(\frac{t}{t_0}\right)^{\bar{c}_1 \mu}$.

**Proof.** For all $q \in \mathbb{T}^n \times \mathbb{R}^m$, we can write $\psi_t^t$ in the following form

$$\psi_t^t(q) = q + \int_{t_0}^t F^\tau \circ \psi_t^\tau(q) d\tau$$

as a consequence of the fundamental theorem of calculus. Taking the derivative with respect to $q$

$$\partial_q \psi_t^t(q) = \text{Id} + \int_{t_0}^t \partial_q \left(f^\tau \circ \psi_t^\tau(q)\right) d\tau,$$

where Id stands for the identity matrix. We assume $t \geq t_0$, then the norm $C^{\sigma-1}$ of the left-hand side of the latter can be estimated as follows

$$|\partial_q \psi_t^t|_{C^{\sigma-1}} \leq 1 + \int_{t_0}^t |f^\tau \circ \psi_t^\tau|_{C^\sigma} d\tau. \quad (5.11)$$

**Case $\sigma = 1$.** By Proposition 3.1

$$|\partial_q \psi_t^t|_{C^0} \leq 1 + C \int_{t_0}^t |f^\tau|_{C^0} d\tau + C \int_{t_0}^t |f^\tau|_{C^1} |\partial_q \psi_t^\tau|_{C^0} d\tau$$

for a suitable constant $C$. We can easily estimate the first integral of the latter

$$\int_{t_0}^t |f^\tau|_{C^0} d\tau \leq \mu \int_{t_0}^t \frac{1}{\tau} d\tau = \ln \left(\frac{t}{t_0}\right)^\mu$$

and hence

$$|\partial_q \psi_t^t|_{C^0} \leq 1 + \ln \left(\frac{t}{t_0}\right)^{c_1 \mu} + C \int_{t_0}^t |f^\tau|_{C^1} |\partial_q \psi_t^\tau|_{C^0} d\tau.$$

We know that $|f|_{1,1} \leq \mu$ and thus by Gronwall’s inequality (5.8)

$$|\partial_q \psi_t^t|_{C^0} \leq \left(1 + \ln \left(\frac{t}{t_0}\right)^{c_1 \mu}\right) e^{C \int_{t_0}^t |f^\tau|_{C^1} d\tau} \leq \left(1 + \ln \left(\frac{t}{t_0}\right)^{c_1 \mu}\right) e^{\ln \left(\frac{1}{\mu}\right)^{c_1 \mu}} \leq C \left(\frac{t}{t_0}\right)^{\bar{c}_1 \mu}$$

for a suitable constant $\bar{c}_1 \geq 1$. 22
Case \( \sigma > 1 \). Similarly to the previous case, by Proposition 3.1

\[
|\partial_q \psi_{t_0}^t|_{C^{\sigma-1}} \leq 1 + C(\sigma) \left( \int_{t_0}^t |f^r|_{C^\sigma} \, d\tau + \int_{t_0}^t |f^r|_{C^\sigma} \, \partial_q \psi_{t_0}^r \, d\tau + \int_{t_0}^t |f^r|_{C^1} |\partial_q \psi_{t_0}^r|_{C^{\sigma-1}} \, d\tau \right).
\]

We have to estimate the first two integrals. We have already calculated the first one,

\[
\int_{t_0}^t |f^r|_{C^\sigma} |\partial_q \psi_{t_0}^r|_{C^\sigma} \, d\tau \leq C(\sigma) \int_{t_0}^t \frac{|f|_{\sigma,1}}{\tau} \left( \frac{t}{t_0} \right)^{\bar{c}_1 \sigma \mu} \, d\tau.
\]

\[
\leq C(\sigma) |f|_{\sigma,1} \left( \frac{t}{t_0} \right) \int_{t_0}^t \tau^{-1} \, d\tau.
\]

In the second line of the latter, rather than calculating the integral, we prefer using the trivial estimate \( \frac{t}{t_0} \left( \frac{t}{t_0} \right)^{\bar{c}_1 \sigma \mu} \leq \left( \frac{t}{t_0} \right)^{\bar{c}_1 \sigma \mu} \) to avoid a division by \( \mu \) since we do not assume it is not zero. Thus, we can estimate \( |\partial_q \psi_{t_0}^t|_{C^{\sigma-1}} \) as follows

\[
|\partial_q \psi_{t_0}^t|_{C^{\sigma-1}} \leq 1 + C(\sigma) \ln \left( \frac{t}{t_0} \right)^{\mu} + C(\sigma) |f|_{\sigma,1} \ln \left( \frac{t}{t_0} \right) \left( \frac{t}{t_0} \right)^{\bar{c}_1 \sigma \mu} + C(\sigma) \int_{t_0}^t |f^r|_{C^1} |\partial_q \psi_{t_0}^r|_{C^{\sigma-1}} \, d\tau.
\]

Furthermore, thanks to Gronwall’s inequality (5.8)

\[
|\partial_q \psi_{t_0}^t|_{C^{\sigma-1}} \leq \left(1 + C(\sigma) \ln \left( \frac{t}{t_0} \right)^{\mu} + C(\sigma) |f|_{\sigma,1} \ln \left( \frac{t}{t_0} \right) \left( \frac{t}{t_0} \right)^{\bar{c}_1 \sigma \mu} \right) e^{C(\sigma) \int_{t_0}^t |f^r|_{C^1} \, d\tau}
\]

\[
\leq \left(1 + C(\sigma) \ln \left( \frac{t}{t_0} \right)^{\mu} + C(\sigma) |f|_{\sigma,1} \ln \left( \frac{t}{t_0} \right) \left( \frac{t}{t_0} \right)^{\bar{c}_1 \sigma \mu} \right) e^{\ln \left( \frac{t}{t_0} \right)^{C(\sigma) \mu}}
\]

\[
\leq C(\sigma) \left( \left( \frac{t}{t_0} \right)^{\mu} + |f|_{\sigma,1} \ln \left( \frac{t}{t_0} \right) \left( \frac{t}{t_0} \right)^{\bar{c}_1 \sigma \mu} \right) \left( \frac{t}{t_0} \right)^{c^\sigma \mu}
\]

for a suitable constant \( c_\sigma \geq \bar{c}_1 \sigma \). This concludes the proof when \( t \geq t_0 \). Similarly, we have the other case. \( \square \)

We can see that the constant \( c_\sigma \geq \bar{c}_1 \sigma \). This means that \( c_\sigma \) goes to infinity when \( \sigma \to \infty \).

We consider \( R : T^n \times R^m \times J \times J \to M_{n+m} \), where \( M_{n+m} \) is the set of the \((n+m)\)-dimensional matrices. For all \((q, \tau, t) \in T^n \times R^m \times J \times J\), let \( R(q, t, \tau) = \{ r_{ij}(q, t, \tau) \}_{1 \leq i,j \leq n+m} \). Therefore, we define the following family of norms

\[
|R_{t_0}^t|_{C^\sigma} = \max_{1 \leq i,j \leq n} |r_{ij}(q, t, \tau)|_{C^\sigma},
\]
for positive real parameters $s \geq 0$. We consider the following system

\[
\begin{aligned}
\dot{R}(q, t, \tau) &= -g(\psi_t^s(q), t)R(q, t, \tau) \\
R(q, \tau, \tau) &= \text{Id}
\end{aligned}
\] (R)

where $g$ is introduced in (HE). In what follows, for fixed $t, \tau \in J$, we denote $R_t^\tau(q) = R(q, t, \tau)$. The following lemma is dedicated to studying the latter system and providing proper estimations of the solutions.

**Lemma 5.4.** The latter system admits a unique solution. Moreover, for all $\tau, t \in J$ with $\tau \geq t$, letting $\tilde{R}(q, t, \tau) = R(\psi_t^0(q), t, \tau)$, we have the following estimates

\[
\begin{aligned}
|R_t^\tau|_{C^0} &\leq \left(\frac{\tau - t}{\tau - t}\right)^{c_R^0}\mu \quad (5.12) \\
|\tilde{R}_t^\tau|_{C^0} &\leq C(\sigma) \left(1 + (|f|_{\sigma, 1} + |g|_{\sigma, 1})\ln \left(\frac{\tau}{\tau - t}\right)\right) \left(\frac{\tau - t}{\tau - t}\right)^{c_R^0}\mu \quad (5.13)
\end{aligned}
\]

with a positive constant $c_R^\sigma$ depending on $n$ and $\sigma$, where $\sigma$ and $\mu$ are those defined by (HE).

Also in this case, before the proof, we observe that when $\sigma = 1$, by (5.13), we have the following estimate

\[
|\tilde{R}_t^\tau|_{C^0} \leq C \left(1 + (|f|_{1, 1} + |g|_{1, 1})\ln \left(\frac{\tau}{\tau - t}\right)\right) \left(\frac{\tau - t}{\tau - t}\right)^{c_R^0}\mu \leq C \left(\frac{\tau - t}{\tau - t}\right)^{\bar{c}_R^0}\mu
\]

for a suitable constant $C$ depending on $n + m$ and $\bar{c}_1^R > c_1^R$.

**Proof.** For all $q \in \mathbb{T}^n \times \mathbb{R}^m$, by the theorem of existence and uniqueness, a unique solution of (R) exists. It remains to prove the estimates. We begin with the first and then we verify the other. Similarly to the proof of the previous lemma, by the fundamental theorem of calculus, we can write $R$ in the following form

\[
R_t^\tau(q) = \text{Id} + \int_t^\tau (g^s \circ \psi_t^s(q)) R^s_t(q) ds
\] (5.14)

for all $q \in \mathbb{T}^n \times \mathbb{R}^m$ and $t, \tau \in J$ with $\tau \geq t$. Taking the norm $C^0$ on the left-hand side of the latter, we obtain

\[
|R_t^\tau|_{C^0} \leq 1 + \int_t^\tau |g^s \circ \psi_t^s| R^s_t|_{C^0} ds \leq 1 + C \int_t^\tau |g^s|_{C^0} |R^s_t|_{C^0} ds,
\]

for a suitable constant $C$ depending on $n + m$. Thanks to Gronwall’s inequality (5.8) and remembering that $|g|_{1, 1} \leq \mu$,

\[
|R_t^\tau|_{C^0} \leq e^{C \mu \int_t^\tau \frac{1}{s} ds} \leq \left(\frac{\tau}{\tau - t}\right)^{C \mu}.
\]
Hence, letting $c_R = C$, (5.12) is proved. It remains to prove (5.13). The composition of $R^*_t$ with $\psi^R_t(q)$ is

$$R^*_t \circ \psi^R_t(q) = R^*_t(q) = 1d + \int_t^\tau (g^s \circ \psi^s_t(q)) R^*_t(q) ds$$  \hspace{1cm} (5.15)$$

for all $q \in T^n \times \mathbb{R}^m$ and $t, \tau \in J$ with $\tau \geq t$. For $\sigma \geq 1$, we can estimate the $C^\sigma$ norm of the right-hand side of the latter as follows

$$|R^*_t|_{C^\sigma} \leq 1 + \int_t^\tau |(g^s \circ \psi^s_t)| R^*_t|_{C^\sigma} ds.$$  \hspace{1cm} (5.16)$$

First of all, we estimate the norm into the integral. To this end, we use the properties in Proposition 3.1.

$$|(g^s \circ \psi^s_t)| R^*_t|_{C^\sigma} \leq C(\sigma) (|g^s \circ \psi^s_t|_{C^1} |\tilde{R}^*_{\psi}c^*_t|_{C^0} + |g^s \circ \psi^s_t|_{C^0} |\tilde{R}^*_{\psi}c^*_t|_{C^0})$$

and, thanks to the latter and (5.16), we can estimates $|\tilde{R}^*_{\psi}c^*_t|_{C^0}$ in the following way

$$|\tilde{R}^*_{\psi}c^*_t|_{C^0} \leq 1 + C(\sigma) \int_t^\tau |g^s|_{C^\sigma} |\partial_q \psi^s_t|_{C^0} |\tilde{R}^*_{\psi}c^*_t|_{C^0} ds + C(\sigma) \int_t^\tau |g^s|_{C^1} |\partial_q \psi^s_t|_{C^{\sigma-1}} |\tilde{R}^*_{\psi}c^*_t|_{C^0} ds$$

Now, by (5.12) and Lemma 5.3, we can find upper bounds for the first three integrals on the right-hand side of the previous inequality

$$\int_t^\tau |g^s|_{C^\sigma} |\partial_q \psi^s_t|_{C^0} |\tilde{R}^*_{\psi}c^*_t|_{C^0} ds \leq C(\sigma) |g|_{\sigma,1} \int_t^\tau s^{-1} \left( \frac{T}{s} \right) \left( \frac{\varepsilon_1}{\varepsilon_0} \right) \mu ds$$

$$\leq C(\sigma) |g|_{\sigma,1} \left( \frac{T}{t} \right) \left( \frac{\varepsilon_1}{\varepsilon_0} \right) \mu$$

$$\int_t^\tau |g^s|_{C^1} |\partial_q \psi^s_t|_{C^{\sigma-1}} |\tilde{R}^*_{\psi}c^*_t|_{C^0} ds \leq C(\sigma) \int_t^\tau \frac{|g|_{1,1}}{s} \left( 1 + |f|_{\sigma,1} \ln \left( \frac{T}{s} \right) \right) \left( \frac{T}{s} \right) \left( \frac{\varepsilon_1}{\varepsilon_0} \right) \mu ds$$

$$\leq C(\sigma) \mu \int_t^\tau \frac{1}{s} \left( \frac{\varepsilon_1}{\varepsilon_0} \right) \mu ds$$

$$+ C(\sigma) |f|_{\sigma,1} \mu \ln \left( \frac{T}{t} \right) \int_t^\tau \frac{1}{s} \left( \frac{\varepsilon_1}{\varepsilon_0} \right) \mu ds$$

$$= C(\sigma) \frac{\varepsilon_1}{\varepsilon_0} \left( \left( \frac{T}{t} \right) \left( \frac{\varepsilon_1}{\varepsilon_0} \right) - 1 \right)$$

$$+ C(\sigma) \frac{\varepsilon_1}{\varepsilon_0} |f|_{\sigma,1} \ln \left( \frac{T}{t} \right) \left( \left( \frac{T}{T} \right) \left( \frac{\varepsilon_1}{\varepsilon_0} \right) - 1 \right)$$

$$\int_t^\tau |g^s|_{C^0} |\tilde{R}^*_{\psi}c^*_t|_{C^0} ds \leq \mu \int_t^\tau s^{-1} \left( \frac{\varepsilon_1}{\varepsilon_0} \right) \mu ds$$

$$= \frac{1}{\varepsilon_0} \left( \frac{T}{t} \right) \left( \frac{\varepsilon_1}{\varepsilon_0} \right) \mu.$$
Similarly to the previous lemma, in the second line of the latter, we use the trivial estimate \( (\tilde{z}_s)^{(c_1 \sigma + c_0^R) \mu} \leq (\tilde{z}_s)^{(c_1 \sigma + c_0^R) \mu} \).

Therefore, remembering that \( \tilde{c}_1 \sigma \leq c_\sigma \),

\[
|\tilde{R}_c'|_{C^\sigma} \leq 1 + C(\sigma)|g|_{\sigma,1} \ln \left( \frac{T}{t} \right) \left( \frac{c_1 \sigma + c_0^R}{c_0^R} \right)^\mu + C(\sigma) \left( \left( \frac{T}{t} \right)^{c_1 \sigma + c_0^R} - 1 \right) \\
+ C(\sigma)|f|_{\sigma,1} \ln \left( \frac{T}{t} \right) \left( \frac{c_1 \sigma + c_0^R}{c_0^R} \right)^\mu + C(\sigma) \left( \left( \frac{T}{t} \right)^{c_1 \sigma + c_0^R} - 1 \right) \\
+ C(\sigma) \int_t^T |g|^s_{C^0} |\tilde{R}_c'|_{C^\sigma} ds \\
\leq C(\sigma) \left( 1 + (|f|_{\sigma,1} + |g|_{\sigma,1}) \ln \left( \frac{T}{t} \right) \right) \left( \frac{c_1 \sigma + c_0^R}{c_0^R} \right)^\mu \\
+ \left| C(\sigma) \int_t^T |g|^s_{C^0} |\tilde{R}_c'|_{C^\sigma} ds \right|.
\]

Now, letting

\[
a(t) = C(\sigma) \left( 1 + (|f|_{\sigma,1} + |g|_{\sigma,1}) \ln \left( \frac{\sigma}{\tau} \right) \right) \left( \frac{c_1 \sigma + c_0^R}{c_0^R} \right)^\mu,
\]

we can rewrite the latter as follows

\[
|\tilde{R}_c'|_{C^\sigma} \leq a(t) + \left| C(\sigma) \int_t^T |g|^s_{C^0} |\tilde{R}_c'|_{C^\sigma} ds \right|.
\]

We observe that \( a \) is a monotone decreasing function, and thus by the more general inequality \( (5.7) \)

\[
|\tilde{R}_c'|_{C^\sigma} \leq a(t) + C(\sigma) \int_t^T a(s)|g|^s_{C^0} |\tilde{R}_c'|_{C^\sigma} |g|^s_{C^0} ds ds
\]

\[
\leq a(t) + C(\sigma) a(t) \int_t^T \frac{\mu}{s} C(\sigma)^\mu ds = a(t) + C(\sigma) a(t) \int_t^T \frac{\mu}{s} C(\sigma)^\mu ds \\
= a(t) \left( 1 + C(\sigma) \left( \frac{c_1 \sigma + c_0^R}{c_0^R} \right)^\mu - 1 \right) \leq C(\sigma) a(t) \left( \frac{c_1 \sigma + c_0^R}{c_0^R} \right)^\mu \\
\leq C(\sigma) \left( 1 + (|f|_{\sigma,1} + |g|_{\sigma,1}) \ln \left( \frac{\tau}{t} \right) \right) \left( \frac{c_1 \sigma + c_0^R}{c_0^R} \right)^\mu
\]

for a suitable \( c_0^R \geq c_\sigma + c_0^R \).

As for \( c_\sigma \) of the previous lemma, we note that the constant \( c_0^R \) goes to infinity if \( \sigma \to \infty \). To solve the homological equation, we must counter the growth of \( c_\sigma \) and \( c_\sigma \) taking \( \mu \) sufficiently small. It will be clear from the following lemma.

**Lemma 5.5 (Homological equation).** There exists a solution \( \kappa \in S_\sigma \), \((\nabla \kappa) \tilde{\Omega} \in S_{\sigma-1}\) of \( (\nabla \kappa) \tilde{\Omega} \). Moreover, letting \( c_\sigma' = \max\{c_0^R + c_\sigma, c_0^R + \epsilon_1 \sigma, \epsilon_1 \sigma, c_\sigma \} \), if

\[
\mu \leq \frac{1}{c_\sigma'} \tag{5.17}
\]

then

\[
|\kappa|_{\sigma,1} \leq C(\sigma) \left| \frac{1}{c_\sigma' \mu} \right| + C(\sigma) \left| \frac{|f|_{\sigma,1} + |g|_{\sigma,1}}{(1 - c_\sigma' \mu)^2} \right| |\tilde{z}|_{1,2}. \tag{5.18}
\]
Proof. Existence: For fixed \( t_0 \in J \), let us define the following transformation

\[
h : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{T}^n \times \mathbb{R}^m \times J
\]

\[
h(q, t) = (\psi_t^{t_0}(q), t)
\]

where \( \psi_t^{t_0} \) is the flow at time \( t \) with initial time \( t_0 \) of \( F(q, t) \) previously defined. We claim that it is enough to prove the first part of this lemma for the much simpler equation

\[
\partial_t \kappa(q, t) + g \circ h^{-1}(q, t) \kappa(q, t) = z \circ h^{-1}(q, t).
\]

(5.20)

If \( \kappa \) is a solution of the latter, then \( \kappa = \kappa \circ h \) is a solution of \([\mathcal{H} \mathcal{E}]\) and viceversa. For the sake of clarity, we prove this claim. Let \( \kappa \) be a solution of \([\mathcal{H} \mathcal{E}]\),

\[
\begin{align*}
\partial_t (\kappa \circ h^{-1}) + (g \circ h^{-1}) (\kappa \circ h^{-1}) &= \left( \partial_t \kappa \circ h^{-1} \right) \partial_t \psi_t^{t_0} + \partial_t \kappa \circ h^{-1} \\
&+ (g \circ h^{-1}) (\kappa \circ h^{-1}) \\
&= (\partial_t \kappa \circ h^{-1}) (F \circ h^{-1}) + \partial_t \kappa \circ h^{-1} \\
&+ (g \circ h^{-1}) (\kappa \circ h^{-1}) \\
&= ((\partial_t \kappa) F + \partial_t \kappa + g \kappa) \circ h^{-1} = z \circ h^{-1}.
\end{align*}
\]

Since \([\mathcal{H} \mathcal{E}]\), we have the last equality of the latter. This implies that \( \kappa \circ h^{-1} \) is a solution for \((5.20)\). Let us first show that \( \partial_t \psi_t^{t_0} F + \partial_t \psi_t^{t_0} = 0 \). We consider the following trivial equality

\[
\psi_t^{t_0} \circ \psi_t^{t_0} = q,
\]

(5.21)

for all \( t, t_0 \in J \) and \( q \in \mathbb{T}^n \). Differentiating both sides of the latter with respect to the variable \( q \in \mathbb{T}^n \), we obtain

\[
\partial_q \psi_t^{t_0} \circ \psi_t^{t_0} + \partial_q \psi_t^{t_0} = \text{Id}
\]

and by the above equation

\[
\partial_q \psi_t^{t_0} = \left( \partial_q \psi_t^{t_0} \circ \psi_t^{t_0} \right)^{-1}.
\]

(5.22)

Taking the derivative with respect to \( t \) on both sides of \((5.21)\)

\[
0 = \frac{d}{dt} \left( \psi_t^{t_0} \circ \psi_t^{t_0} \right) = \partial_q \psi_t^{t_0} \circ \psi_t^{t_0} \partial_t \psi_t^{t_0} + \partial_t \psi_t^{t_0} \circ \psi_t^{t_0}.
\]

Hence, \( \partial_t \psi_t^{t_0} \) is equal to

\[
\partial_t \psi_t^{t_0} = - \left( \partial_q \psi_t^{t_0} \circ \psi_t^{t_0} \right)^{-1} \partial_q \psi_t^{t_0} \circ \psi_t^{t_0}.
\]

(5.23)

Therefore, thanks to \((5.22)\) and \((5.23)\), we can rewrite \( \partial_q \psi_t^{t_0} F + \partial_t \psi_t^{t_0} = 0 \) in the following form

\[
\partial_q \psi_t^{t_0} (q) F(q, t) + \partial_t \psi_t^{t_0} (q) = \left( \partial_q \psi_t^{t_0} \circ \psi_t^{t_0} \right)^{-1} (F(q, t) - \partial_t \psi_t^{t_0} \circ \psi_t^{t_0} (q))
\]

for all \( t, t_0 \in J \) and \( q \in \mathbb{T}^n \). This implies the claim because

\[
F(q, t) - \partial_t \psi_t^{t_0} \circ \psi_t^{t_0} (q) = F(q, t) - F(\psi_t^{t_0} \circ \psi_t^{t_0} (q), t) = 0
\]

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for all \( t, t_0 \in J \) and \( q \in \mathbb{T}^n \).

Now, let \( \kappa \) be a solution of (5.20)

\[
\partial_q (\kappa \circ h) F + \partial_t (\kappa \circ h) + g(\kappa \circ h) = (\partial_q \kappa \circ h) (\partial_q \psi_t F + \partial_t \psi_t) + \partial_t \kappa \circ h + g(\kappa \circ h)
\]

\[
= \partial_t \kappa \circ h + g(\kappa \circ h) = 0.
\]

Hence \( \kappa \circ h \) is a solution of \((HE)\), where the last equality of the latter is a consequence of (5.20).

Now, we can reduce the proof of the first part of this lemma by studying the existence of a solution for the easier equation (5.20). For all \((q, t_0) \in \mathbb{T}^n \times \mathbb{R}^m \times J\), let \( R(q, t, t_0) \) be the unique solution of (R). Then, a solution \( \kappa \) of the above equation exists and

\[
\kappa(q, t) = R(q, t, t_0) e(q) - \int_{t_0}^t R(q, t, \tau) z \circ h^{-1}(q, \tau) d\tau
\]

with a function \( e \) defined on \( \mathbb{T}^n \times \mathbb{R}^m \).

Estimates: We choose \( e \) in such a way that

\[
e(q) = \int_{t_0}^{+\infty} R(q, t_0, \tau) z \circ h^{-1}(q, \tau) d\tau.
\]

It is well defined because by Lemma 5.4 and (5.17),

\[
\left| \int_{t_0}^{+\infty} R(q, t_0, \tau) z \circ h^{-1}(q, \tau) d\tau \right| \leq \int_{t_0}^{+\infty} |R^{t_0}_\tau| C^0 |z^\tau| C^0 d\tau
\]

\[
\leq \int_{t_0}^{+\infty} \left( \frac{\tau}{t_0} \right)^{\varphi R} \frac{|z|_{0,2}}{\tau^2} d\tau
\]

\[
= \frac{|z|_{0,2}}{t^0_{\mu} \mu} \int_{t_0}^{+\infty} \tau^{\varphi R} \mu^{-2} d\tau
\]

\[
= \frac{|z|_{0,2}}{1 - c^R_{0} \mu t_0}.
\]

Furthermore,

\[
\varphi(q, t) = \kappa \circ h(q, t) = - \int_{t}^{+\infty} R^{t^0}_\tau \circ \psi^{t^0}_\tau(q) z^\tau \circ \psi^{t^0}_\tau(q) d\tau
\]

\[
= - \int_{t}^{+\infty} R^{t^0}_\tau \circ \psi^{t^0}_t(q) z^\tau \circ \psi^{t^0}_t(q) d\tau = - \int_{t}^{+\infty} \tilde{R}^{t^0}_\tau \circ \psi^{t^0}_\tau(q) z^\tau \circ \psi^{t^0}_\tau(q) d\tau
\]

is a solution of \((HE)\) we are looking for.

The estimate (5.18) is a consequence of Proposition 3.1, Lemma 5.3, Lemma 5.4 and (5.17). For all \( t \in J \) and by Proposition 3.1, we can estimate \( |\varphi'|_{C^0} \) as follows

\[
|\varphi'|_{C^0} \leq C(\sigma) \int_{t}^{+\infty} |R^{t^0}_\tau| C^0 |z^\tau \circ \psi^{t^0}_\tau| C^0 + |\tilde{R}^{t^0}_\tau \circ \psi^{t^0}_\tau| C^0 |z^\tau| C^0 d\tau.
\]
Moreover,

\[ |z^T \circ \psi_t^T|_{C^\sigma} \leq C(\sigma) \left( |z^T|_{C^\sigma} |\partial_\psi \psi_t^T|_{C^0}^\sigma + |z^T|_{C^1} |\partial_\psi \psi_t^T|_{C^{\sigma-1}}^\sigma + |z^T|_{C^0} \right) \]

\[ |\bar{R}_t^1 \circ \psi_t^T|_{C^\sigma} \leq C(\sigma) \left( |\bar{R}_t^1|_{C^\sigma} |\partial_\psi \psi_t^T|_{C^0}^\sigma + |\bar{R}_t^1|_{C^1} |\partial_\psi \psi_t^T|_{C^{\sigma-1}}^\sigma + |\bar{R}_t^1|_{C^0} \right) \]

and replacing the latter into the above integral

\[ |\mathcal{X}^T|_{C^\sigma} \leq C(\sigma) \int_t^{+\infty} |\bar{R}_t^1|_{C^0} |z^T|_{C^\sigma} |\partial_\psi \psi_t^T|_{C^0}^\sigma d\tau + C(\sigma) \int_t^{+\infty} |\bar{R}_t^1|_{C^0} |z^T|_{C^1} |\partial_\psi \psi_t^T|_{C^{\sigma-1}}^\sigma d\tau \]

\[ + C(\sigma) \int_t^{+\infty} |\bar{R}_t^1|_{C^0} |z^T|_{C^0} d\tau + C(\sigma) \int_t^{+\infty} |\bar{R}_t^1|_{C^0} |z^T|_{C^0} d\tau. \]

Now, we have to estimate each integral on the right-hand side of the latter. First, we observe that, for all \( t \in J \) and \( x < 1 \)

\[ \int_t^{+\infty} \tau^{x-2} \ln \left( \frac{\tau}{t} \right) d\tau = \frac{1}{1-x} \int_t^{+\infty} \tau^{x-2} d\tau. \]

It is obtained by integrating by part. Then, using Lemma 5.3, Lemma 5.4, (5.17) and the latter

\[ \int_t^{+\infty} |\bar{R}_t^1|_{C^0} |z^T|_{C^\sigma} |\partial_\psi \psi_t^T|_{C^0}^\sigma d\tau \leq C(\sigma) \int_t^{+\infty} \frac{|z|_{\sigma,2}}{\tau^2} \left( \frac{\tau}{t} \right)^{\mu} \left( c_0^R + \bar{c}_1 \sigma \right)^\mu d\tau \]

\[ = C(\sigma) \frac{|z|_{\sigma,2}}{t (c_0^R + \bar{c}_1 \sigma)^\mu} \int_t^{+\infty} \tau^{(c_0^R + \bar{c}_1 \sigma)^\mu - 2} d\tau \]

\[ = C(\sigma) \frac{|z|_{\sigma,2}}{1 - (c_0^R + \bar{c}_1 \sigma)^\mu t} \]

\[ \int_t^{+\infty} |\bar{R}_t^1|_{C^0} |z^T|_{C^1} |\partial_\psi \psi_t^T|_{C^{\sigma-1}}^\sigma d\tau \leq C(\sigma) \int_t^{+\infty} \frac{|z|_{\sigma,1}}{\tau^2} \left( 1 + |f|_{\sigma,1} \ln \left( \frac{\tau}{t} \right) \right)^2 \left( c_0^R + c_\sigma \right)^\mu d\tau \]

\[ = C(\sigma) \int_t^{+\infty} \frac{|z|_{\sigma,1}}{\tau^2} \left( \frac{\tau}{t} \right)^{c_0^R + c_\sigma} d\tau \]

\[ + C(\sigma) \int_t^{+\infty} \frac{|z|_{\sigma,1}}{\tau^2} |f|_{\sigma,1} \ln \left( \frac{\tau}{t} \right) \left( \frac{\tau}{t} \right)^{c_0^R + c_\sigma} d\tau \]

\[ = C(\sigma) \frac{|z|_{\sigma,1}}{1 - (c_0^R + c_\sigma)^\mu t} \]

\[ + C(\sigma) \frac{|z|_{\sigma,1}}{1 - (c_0^R + c_\sigma)^\mu} \frac{1}{t (c_0^R + c_\sigma)^\mu} \int_t^{+\infty} \tau^{(c_0^R + c_\sigma)^\mu - 2} d\tau \]

\[ = C(\sigma) \frac{|z|_{\sigma,1}}{1 - (c_0^R + c_\sigma)^\mu} \frac{1}{(1 - (c_0^R + c_\sigma)^\mu)^2} + C(\sigma) \frac{|z|_{\sigma,1}}{1 - (c_0^R + c_\sigma)^\mu} \frac{1}{t (c_0^R + c_\sigma)^\mu} \int_t^{+\infty} \tau^{(c_0^R + c_\sigma)^\mu - 2} d\tau \]
\[
\int_0^{\infty} |R_\tau|^\sigma t \, |\zeta^T|_{C^0} d\tau \leq C \int_0^{\infty} \frac{|z|_{0,2}}{\tau^2} \left( \frac{\tau}{t} \right)^{c_0^R \mu} d\tau = C \frac{|z|_{0,2}}{1 - c_0^R \mu} \frac{1}{t}
\]
\[
\int_0^{\infty} |\tilde{R}_\tau|^\sigma t \, |\partial_\tau \psi^T|_{C^0} d\tau \leq C(\sigma) \int_0^{\infty} \left( 1 + (|f|_{\sigma,1} + |g|_{\sigma,1}) \ln \left( \frac{\tau}{t} \right) \right) \frac{|z|_{0,2}}{\tau^2} \left( \frac{\tau}{t} \right)^{c_0^B + c_1 \sigma} d\tau
\]
\[
= C(\sigma) \int_0^{\infty} \frac{|z|_{0,2}}{\tau^2} \left( \frac{\tau}{t} \right)^{c_0^B + c_1 \sigma} d\tau
\]
\[
+ C(\sigma) \left( |f|_{\sigma,1} + |g|_{\sigma,1} \right) \int_0^{\infty} \frac{|z|_{0,2}}{\tau^2} \left( \frac{\tau}{t} \right)^{c_0^B + c_1 \sigma} d\tau
\]
\[
= C(\sigma) \frac{|z|_{0,2}}{1 - (c_0^B + c_1 \sigma) \mu} \frac{1}{t} + C(\sigma) \frac{|z|_{0,2} |f|_{\sigma,1} + |g|_{\sigma,1}}{1 - (c_0^B + c_1 \sigma) \mu} \frac{1}{t}
\]

Then, thanks to the latter

\[
|\mathcal{X}|_{C^0 t} \leq C(\sigma) \left( \frac{|z|_{\sigma,2}}{1 - (c_0^R + c_1 \sigma) \mu} + \frac{|z|_{1,2}}{1 - (c_0^R + c_0) \mu} + \frac{|z|_{1,2} |f|_{\sigma,1}}{1 - (c_0^R + c_1 \sigma) \mu} \frac{1}{t} \right)
\]
\[
+ \frac{|z|_{0,2}}{1 - c_0^R \mu} + \frac{|z|_{0,2}}{1 - (c_0^B + c_1 \sigma) \mu} + \frac{|z|_{0,2} |f|_{\sigma,1} + |g|_{\sigma,1}}{1 - (c_0^B + c_1 \sigma) \mu} \frac{1}{t}
\]
\[
+ \frac{|z|_{0,2}}{1 - (c_0^R + c_1 \sigma) \mu} + \frac{|z|_{0,2} |f|_{\sigma,1}}{1 - (c_0^B + c_1 \sigma) \mu} \frac{1}{t}
\]
\[
\leq C(\sigma) \frac{|z|_{\sigma,2}}{1 - c_0^R \mu} + C(\sigma) \left( |f|_{\sigma,1} + |g|_{\sigma,1} \right) \frac{|z|_{1,2}}{1 - (c_0^B + c_1 \sigma) \mu} \frac{1}{t}
\]

for all \( t \in J \), where we recall that \( c_0^B = \max \{ c_0^B + c_1 \sigma, c_0^R + c_1 \sigma, c_0^R + c_0 \} \). Taking the sup for all \( t \in J \) on the left-hand side of the latter, we conclude the proof.  

We are not able to provide holomorphic solutions to the previous homological equation. Because we solve \((HE)\) by integration thanks to a suitable change of coordinate \((5.19)\) that depends on the flow of \( \dot{\omega} + f \). We know that \( f \) decays as \( \frac{1}{t} \). So, the derivatives of this flow diverge at least as \( \ln t \). Then, in the case of holomorphic functions defined on a suitable complex neighbourhood of the phase space, this prevents us from well defining the change of coordinate \((5.19)\).

We are not even capable of finding \( C^\infty \) solutions to \((HE)\). Because the bigger \( \sigma \) is, the more we have to take \( \mu \) small in order to solve the homological equation.
5.5 Regularity of \( \mathcal{F} \)

We begin this part by reminding the following families of Banach spaces introduced in Section 5.3: 

\[
\mathcal{A}^\sigma = \left\{ a : \mathbb{T}^m \times \mathbb{R}^m \times J \to \mathbb{R} \mid a \in S_{\sigma+1} \text{ and } |a|_{\sigma} = |a|_{\sigma+1} + |\partial_qa|_{\sigma+2} < \infty \right\}
\]

\[
\mathcal{B}^\sigma = \left\{ b : \mathbb{T}^m \times \mathbb{R}^m \times J \to \mathbb{R}^{n+m} \mid b \in S_{\sigma+1} \text{ and } |b|_{\sigma} = |b|_{\sigma+1} < \infty \right\}
\]

\[
\mathcal{V}^\sigma = \left\{ v : \mathbb{T}^m \times \mathbb{R}^m \times J \to \mathbb{R}^{n+m} \mid v \in S_{\sigma+1}, (\nabla v) \bar{\Omega} \in S_{\sigma} \right. \quad \text{and } |v|_{\sigma} = \max\{ |v|_{\sigma+1}, |(\nabla v) \bar{\Omega}|_{\sigma+2} \} < \infty \left. \right\}
\]

\[
\mathcal{Z}^\sigma = \left\{ z : \mathbb{T}^m \times \mathbb{R}^m \times J \to \mathbb{R}^{n+m} \mid z \in S_{\sigma}, \text{ and } |z|_{\sigma} = |z|_{\sigma+2} < \infty \right\}
\]

where \((\nabla v) \bar{\Omega} = (\partial_q v) \bar{\omega} + \partial_t v\). Let \( s \) and \( \Upsilon \) be the positive parameters introduced by (4), we define the following Banach space

\[
\mathcal{M} = \left\{ m : \mathbb{T}^n \times \mathbb{R}^m \times B \times J \to M_{n+m} \mid m \in S_{s+1} \text{ and } |m| = |m|_{s+1,0} \leq \Upsilon \right\}
\]

where \( M_{n+m} \) is the set of \((n+m)\)-dimensional matrices. We consider an additional family of Banach spaces \( \{ (\mathcal{X}^\sigma, | \cdot | \sigma) \}_{\sigma \geq 0} \) such that, for all \( \sigma \geq 0 \), \( \mathcal{X}^\sigma = \mathcal{A}^\sigma \times \mathcal{B}^\sigma \) and for all \( x \in \mathcal{X}^\sigma \), \( |x|_{\sigma} = \max\{ |a|_{\sigma}, |b|_{\sigma} \} \). Now, we have everything we need to define the functional \( \mathcal{F} \) more precisely. Let \( \mathcal{F} \) be the following functional

\[
\mathcal{F} : \mathcal{X}^0 \times \mathcal{M} \times \mathcal{M} \times \mathcal{V}^0 \to \mathcal{Z}^0
\]

\[
\mathcal{F}(x, m, \bar{m}, v) = (\nabla v) \bar{\Omega} + \partial_q v (b + (\bar{m} \circ \bar{\varphi}) v) + \partial_q a + (\partial_q b) v + \partial_q m \circ \bar{\varphi} \cdot v^2.
\]

Thanks to Proposition 3.2, it is straightforward to verify that \( \mathcal{F} \) is well defined. We observe that, for all \( (b, m, \bar{m}) \in \mathcal{B}^0 \times \mathcal{M} \times \mathcal{M} \), letting \( x^0 = (0, b) \),

\[
\mathcal{F}(x^0, m, \bar{m}, 0) = 0.
\]

Let \( \delta \) and \( b_0 \) be as in (4). Obviously, \( b_0 \in \mathcal{B}^0 \subset \mathcal{B}^0 \). Moreover, we recall that \( b_0 \) satisfies the following estimate

\[
|b_0|_1 < \delta, \quad |b_0|_{s+1,1} \leq \Upsilon
\]

We fix \( x_0 = (0, b_0) \). For all \( \sigma \geq 0 \) and for a suitable parameter \( 0 < \zeta < 1 \), that we will choose sufficiently small with respect to \( \delta \) in Lemma 5.6, we define the following subset of \( \mathcal{X}^\sigma \times \mathcal{V}^\sigma \)

\[
\mathcal{O}^\sigma_\zeta = \{(x, v) \in \mathcal{X}^\sigma \times \mathcal{V}^\sigma : |x - x_0|_{\sigma}, |v|_{\sigma} < \zeta \} \subset \mathcal{X}^\sigma \times \mathcal{V}^\sigma.
\]

Let \( m, \bar{m} \in \mathcal{M} \) be as in (4), we consider the following functional

\[
\mathcal{F}_{m, \bar{m}} : \mathcal{O}^0_\zeta \to \mathcal{Z}^0
\]
such that, for all \((x, v) \in \mathcal{O}_\zeta^0\),
\[
\mathcal{F}_{m, \bar{m}}(x, v) = \mathcal{F}(x, m, \bar{m}, v).
\]
It is well defined, continuous and \(\mathcal{F}_{m, \bar{m}}(x_0, 0) = 0\), where \(x_0 = (0, b_0)\) is the element of \(\mathcal{X}^0\) previously introduced. This section aims to prove that \(\mathcal{F}_{m, \bar{m}}\) satisfies hypotheses H.1-H.4 of Theorem 5.1. Then, Zehnder’s Theorem ensures the existence of a \(C^\rho\)-weakly asymptotic cylinder associated to \((X_H, X_{\bar{h}}, \varphi_0)\).

In the proof of the following lemma, we widely use the properties contained in Proposition 3.2 without specifying them each time.

**Lemma 5.6.** \(\mathcal{F}_{m, \bar{m}}\) satisfies hypotheses H.1-H.4 of Theorem 5.1

**Proof.** H.1. Smoothness: \(\mathcal{F}_{m, \bar{m}}(x, \cdot) : \mathcal{V}^0 \rightarrow \mathcal{Z}^0\) is two times differentiable with respect to the variable \(v\) and
\[
D_v \mathcal{F}_{m, \bar{m}}(x, v)\hat{v} = (\nabla \hat{v}) \hat{\Omega} + (\partial_q \hat{v}) f + g\hat{v},
\]
\[
D_{v}^2 \mathcal{F}_{m, \bar{m}}(x, v)(\hat{v}, \tilde{v}) = (\partial_q \hat{v} \tilde{f} + (\partial_q \tilde{v} f) \hat{v} + \hat{v}^T \tilde{g} \hat{v},
\]
with
\[
f = b + (\bar{m} \circ \hat{\varphi}) v,
\]
\[
g = \partial_q b + \partial_q v (\partial_m \bar{m} \circ \hat{\varphi}) + \partial_q v \bar{m} \circ \hat{\varphi} + v^T \bar{m} \circ \hat{\varphi} v + 2(\partial_q m \circ \hat{\varphi}) v,
\]
\[
\hat{f} = \bar{m} \circ \hat{\varphi} + (\partial_m \bar{m} \circ \hat{\varphi}) v,
\]
\[
\tilde{f} = (\partial_m \bar{m} \circ \hat{\varphi}) v + \bar{m} \circ \hat{\varphi},
\]
\[
\tilde{g} = \partial_q v \bar{m} \circ \hat{\varphi} + 2\partial_q v \bar{m} \circ \hat{\varphi} + v^T (\partial_m \bar{m} \circ \hat{\varphi}) v + 4(\partial_m \bar{m} \circ \hat{\varphi}) v + 2\partial_m \bar{m} \circ \hat{\varphi},
\]
where \(T\) denotes the transpose. We have to verify that
\[
|D_v \mathcal{F}_{m, \bar{m}}(x, v)|_0, |D_v^2 \mathcal{F}_{m, \bar{m}}(x, v)|_0 \leq C
\]
for all \((x, v) \in \mathcal{O}_\zeta^0\) and some \(C \geq 1\).

In the latter, we consider the norm corresponding to the Banach space \((\mathcal{Z}^0, |\cdot|_0)\). Referring to the notation introduced by (2.1), this norm coincides with \(|\cdot|_{0, 2}\). For all \(\hat{v} \in \mathcal{V}^0\),
\[
|D_v \mathcal{F}_{m, \bar{m}}(x, v)\hat{v}|_{0, 2} \leq |(\nabla \hat{v}) \hat{\Omega}|_{0, 2} + C \left(|f|_{0, 1}|\hat{v}|_{1, 1} + |g|_{0, 1}|\hat{v}|_{0, 1}\right)
\]
\[
\leq |\hat{v}|_0 (1 + C (|f|_{0, 1} + |g|_{0, 1}))
\]
where we recall that \(|\hat{v}|_0 = \max\{|\hat{v}|_{1, 1}, |(\nabla \hat{v}) \hat{\Omega}|_{0, 2}\}\). Moreover, for all \((x, v) \in \mathcal{O}_\zeta^0\) and \(m \in \mathcal{M}\),
\[
|f|_{0, 1} \leq C (|b|_{0, 1} + |v|_{0, 1} |\bar{m}||_0) \leq C (|b||_0 + |v||_0 |\bar{m}||_0) ,
\]
\[
\leq C (|b||_0 + |b - b_0||_0) + C (|v|_0 |\bar{m}||_0) \leq C(\delta + \zeta) + CY\zeta
\]
\[
|g|_{0, 1} \leq C (|b||_1, 1 + |v||_1, 1 |\bar{m}||_0, 0 + |v||_1, 1 |\bar{m}||_0, 0 + (|v||_0, 1, 2 |\bar{m}||_0, 2, 0 + |v||_0, 1 |m||_1, 0) ,
\]
\[
\leq C (|b||_0 + |v||_0 + |v||_0^2) \leq C(\delta + \zeta) + CY\zeta.
\]
This implies the claim for $D_v\mathcal{F}_{m,\tilde{m}}(x, v)$. Similarly, we have the claim for $D_v^2\mathcal{F}_{m,\tilde{m}}(x, v)$.

**H.2. $\mathcal{F}_{m,\tilde{m}}$ is uniformly Lipschitz in $X^\sigma$:** For all $(x_1, v), (x_2, v) \in \mathcal{O}_C^\sigma$, remembering that $|x_0| = \max\{|a_0|, |b_0|\}$,

$$
|\mathcal{F}_{m,\tilde{m}}(x_1, v) - \mathcal{F}_{m,\tilde{m}}(x_2, v)|_{|2} \leq |\partial_q v (b_1 - b_2) + (\partial_q a_1 - \partial_q a_2) + (\partial_q b_1 - \partial_q b_2) v|_{|2} \\
\leq C |b_1 - b_2|_{0,1} |v|_{1,1} + C (|\partial_q a_1 - \partial_q a_2|_{0,2} + |b_1 - b_2|_{1,1} |v|_{0,1}) \\
\leq C (|b_1 - b_2|_0 |v|_0 + |a_1 - a_2|_0 + |b_1 - b_2|_0 |v|_0) \\
\leq C(1 + \zeta)|x_1 - x_2|_0,
$$

which proves H.2. Now, we verify H.4 before H.3.

**H.4. Order:** The first two conditions of Definition [5.2] are satisfied, meaning $(x_0, 0) \in X^\sigma \times V^\sigma$ and $\mathcal{F}_{m,\tilde{m}} (\mathcal{O}_C^\sigma \cap (X^\sigma \times V^\sigma)) \subset \mathcal{Z}^\sigma$ for all $1 \leq \sigma \leq s$. We verify the tame estimate.

For all $1 \leq \sigma \leq s$ and $(x, v) \in \mathcal{O}_C^\sigma \cap (X^\sigma \times V^\sigma)$, we rewrite the functional $\mathcal{F}_{m,\tilde{m}}$ in the following form

$$
\mathcal{F}_{m,\tilde{m}}(x, v) = (\nabla v) \tilde{\Omega} + \partial_q v (b_0 + (b - b_0)) + (\bar{m} \circ \bar{\varphi}) v + \partial_q a \\
+ (\partial_q b_0) v + \partial_q (b - b_0) v + \partial_q m \circ \bar{\varphi} \cdot v^2.
$$

We assume $|x - x_0|_\sigma, |v|_\sigma \leq K$, then

$$
|\mathcal{F}_{m,\tilde{m}}(x, v)|_{\sigma,2} \leq |(\nabla v) \tilde{\Omega}|_{\sigma,2} + |(\partial_q v) b_0|_{\sigma,2} + |(\partial_q v) (b - b_0)|_{\sigma,2} + |\partial_q v (\bar{m} \circ \bar{\varphi}) v|_{\sigma,2} \\
+ |a|_{\sigma,2} + |(\partial_q b_0) v|_{\sigma,2} + |\partial_q (b - b_0) v|_{\sigma,2} + |\partial_q m \circ \bar{\varphi} \cdot v^2|_{\sigma,2}.
$$

We have to estimate each term on the right-hand side of the latter. The terms $|a|_{\sigma,2}$ and $|(\nabla v) \tilde{\Omega}|_{\sigma,2}$ are bounded by $K$. We recall that $|b_0|_{s+1,1} \leq \Upsilon$ and we estimate the others

$$
|b_0 (\partial_q v)|_{\sigma,2} \leq C(\sigma)|b_0|_{s+1,1} |v|_{\sigma+1,1} \leq C(\sigma)|b_0|_s |v|_\sigma \leq C(\sigma)\Upsilon K \\
|(\partial_q v) (b - b_0)|_{\sigma,2} \leq C(\sigma) (|\partial_q v|_{0,1} |b - b_0|_{\sigma,1} + |\partial_q v|_{\sigma,1} |b - b_0|_{0,1}) \\
\leq C(\sigma) (|v|_{1,1} |b - b_0|_{\sigma,1} + |v|_{\sigma+1,1} |b - b_0|_{0,1}) \\
\leq C(\sigma) (|v|_0 |b - b_0|_{\sigma} + |v|_{\sigma} |b - b_0|_0) \\
\leq C(\sigma) \zeta K \leq C(\sigma) K \\
|\partial_q v (\bar{m} \circ \bar{\varphi}) v|_{\sigma,2} \leq C(\sigma) (|\partial_q v (\bar{m} \circ \bar{\varphi})|_{\sigma,1} |v|_{0,1} + |\partial_q v (\bar{m} \circ \bar{\varphi})|_{0,1} |v|_{\sigma,1}) \\
\leq C(\sigma) |v|_0 (|\bar{m} \circ \bar{\varphi}|_{\sigma,0} |v|_0 + |\bar{m}|_{0,0} |v|_\sigma) + C(\sigma) |v|_{\sigma} |\bar{m}|_{0,0} |v|_0 \\
\leq C(\sigma) |v|_0 |\bar{m}|_{\sigma,0} (1 + |v|_\sigma^\sigma + |v|_{\sigma} |v|_0 + C(\sigma) \zeta \Upsilon |v|_\sigma \\
\leq C(\sigma) \Upsilon K \\
|\partial_q b_0)|_{\sigma,2} \leq C(\sigma)|b_0|_{s+1,1} |v|_{\sigma,1} \leq C(\sigma) \Upsilon K \\
|\partial_q (b - b_0)|_{\sigma,2} \leq C(\sigma) (|v|_{0,1} |b - b_0|_{\sigma+1,1} + |v|_{\sigma,1} |b - b_0|_{1,1}) \\
\leq C(\sigma) (|v|_0 |b - b_0|_{\sigma,1} + |v|_{\sigma} |b - b_0|_0) \\
\leq C(\sigma) \zeta K \leq C(\sigma) K
$$
$|\partial_{x} m \circ \hat{v} \cdot v_{|\sigma,2}^{2} \leq C(\sigma) \left(|(\partial_{x} m \circ \hat{v}) v|_{0,1}|v|_{\sigma,1} + |(\partial_{y} m \circ \hat{v}) v|_{\sigma,1}|v|_{0,1}\right)$

$\leq C(\sigma)|m|_{1,0}|v|_{0}v_{\sigma}$

$+ C(\sigma)|v|_{0} \left(|\partial_{y} m \circ \hat{v}|_{0,0}v|_{0,1} + |\partial_{y} m \circ \hat{v}|_{0,0}|v|_{\sigma,1}\right)$

$\leq C(\sigma)\Upsilon K + C(\sigma)|v|_{0}|m|_{\sigma+1,0}(1 + |v|_{0}^{\sigma} + |v|_{\sigma})|v|_{0}$

$\leq C(\sigma)\Upsilon K$.

Therefore, H.4 is satisfied. Now, we fix $\delta < \frac{1}{C_{\sigma}}$, where $s$ is the positive parameter defined by $[\text{4}]$ and $C_{\sigma}$ is the constant in Lemma $5.5$. Furthermore, we choose $\zeta$ depending on $\delta$ in such a way that

$$\delta + C\Upsilon \zeta < \frac{1}{c_{\delta}}$$

(5.24)

for a suitable constant $C$ depending on $n + m$. This hypothesis is crucial if we want to define a right-inverse of $F$.

**H.3. Existence of a right-inverse of loss 1**: In this part, we prove that for all $(x, v) \in O_{1}^{\sigma} \cap (X^{\sigma} \times V^{\sigma})$ with $1 \leq \sigma \leq s$, a right-inverse of loss 1 is well defined. This means that, for all $(x, v) \in O_{1}^{\sigma} \cap (X^{\sigma} \times V^{\sigma})$, there exists a liner map $\eta_{m, \bar{m}} : Z^{\sigma} \to Y^{\sigma-1}$ such that $D_{x}F_{m, \bar{m}}(x, v)\eta_{m, \bar{m}}(x, v)z = z$ for all $z \in Z^{\sigma}$. In other words, for all $z \in Z^{\sigma}$, we have to solve the following equation in the unknown $\hat{v}$

$$D_{x}F_{m, \bar{m}}(x, v)\hat{v} = (\nabla \hat{v}) \bar{\Omega} + (\partial_{q} \hat{v}) f + g \hat{v} = z,$$

(5.24)

where $f$ and $g$ are defined at the beginning of the proof of this lemma. If

$$|f|_{1,1} \leq \delta + C\Upsilon \zeta, \quad |g|_{1,1} \leq \delta + C\Upsilon \zeta,$$

(5.25)

thanks to Lemma $5.5$ and (5.24), a solution to the above equation exists. It remains to verify the estimate on $|f|_{1,1}$ and $|g|_{1,1}$.

For all $1 \leq \sigma \leq s$ and $(x, v) \in O_{1}^{\sigma},$

$$|f|_{\sigma,1} \leq |b|_{\sigma,1} + |(\bar{m} \circ \hat{v}) v|_{\sigma,1} \leq |b|_{\sigma,1} + C(\sigma) \left(|\bar{m} \circ \hat{v}|_{\sigma,0}v|_{0,1} + |\bar{m} \circ \hat{v}|_{0,0}v|_{\sigma,1}\right)$

$\leq |b|_{\sigma-1} + C(\sigma) \left(\Upsilon \left(1 + |v|_{1,1} + |v|_{\sigma,1}\right) |v|_{0,1} + \Upsilon |v|_{\sigma,1}\right)$

$\leq |b|_{\sigma-1} + C(\sigma)\Upsilon |v|_{\sigma-1}$

$$|g|_{\sigma,1} \leq |b|_{\sigma+1,1} + |\partial_{q} v (\partial_{q} \bar{m} \circ \hat{v}) v|_{\sigma,1} + |\partial_{q} v (\bar{m} \circ \hat{v})|_{\sigma,1} + |v|_{\sigma,1} \left(\partial_{pq}^{2} \bar{m} \circ \hat{v}\right)v|_{\sigma,1}$

$\leq |b|_{\sigma} + C(\sigma)\Upsilon |v|_{\sigma}.$

Taking $\sigma = 1$, we obtain

$$|f|_{1,1} \leq |b|_{0} + C\Upsilon |v|_{0} \leq |b|_{0,0} + |b - b_{0}|_{0} + C\Upsilon |v|_{0}$

$\leq (\delta + \zeta) + C\Upsilon \zeta \leq \delta + C\Upsilon \zeta$.

$$|g|_{1,1} \leq |b|_{1} + C(\sigma)\Upsilon |v|_{1} \leq |b|_{0,1} + |b - b_{0}|_{1} + C\Upsilon |v|_{0}$,$

$\leq (\delta + \zeta) + C\Upsilon \zeta$.

(5.27)

This implies the claim. The second part of this proof is dedicated to verifying $[\text{5}]$ and $[\text{6}]$. In what follows, we drop the indexes $m, \bar{m}$ from $F$ and $\eta$ to achieve a more elegant proof.
For all \((x, v) \in \mathcal{O}_\zeta^1\) and \(z \in \mathbb{Z}^1\)
\[
|\eta(x, v)z|_0 = \max\{|\eta(x, v)z|_{1,1}, |\nabla (\eta(x, v)z) \bar{\Omega}|_{0,2}\}.
\]
By Lemma 5.5 (more specifically (5.18)), \(\delta\zeta\) and the previous estimates concerning \(|f|_{1,1}\) and \(|g|_{1,1}\)
\[
|\eta(x, v)z|_{1,1} \leq C(\delta, \zeta)|z|_{1,2} = C(\delta, \zeta)|z|_1.
\]
Moreover, as a consequence of (5.24), the latter estimate, (5.26), (5.27) and \(\delta\zeta\) \[\]
\[
|\nabla (\eta(x, v)z) \bar{\Omega}|_{0,2} = |z - \partial_q (\eta(x, v)z) f - g (\eta(x, v)z)|_{0,2}
\]
\[
\leq |z|_0 + C|f|_{0,1}|\eta(x, v)z|_{1,1} + C|g|_{0,1}|\eta(x, v)z|_{0,1}
\]
\[
\leq |z|_0 + C|\eta(x, v)z|_{1,1} \leq C(\delta, \zeta)|z|_1.
\]
This implies (71) because
\[
|\eta(x, v)z|_0 = \max\{|\eta(x, v)z|_{1,1}, |\nabla (\eta(x, v)z) \bar{\Omega}|_{0,2}\} \leq C(\delta, \zeta)|z|_1.
\]
Concerning (72), for all \(1 \leq \sigma \leq s\) and \((x, v) \in \mathcal{O}_\zeta^1 \cap (\mathcal{X}^\sigma \times \mathcal{Y}^\sigma)\), we assume
\[
|x - x_0|_\sigma, |v|_\sigma \leq K
\]
and we recall that
\[
|\eta(x, v)\mathcal{F}(x, v)|_{\sigma - 1} = \max\{|\eta(x, v)\mathcal{F}(x, v)|_{\sigma, 1}, |\nabla (\eta(x, v)\mathcal{F}(x, v)) \bar{\Omega}|_{\sigma - 1, 2}\}.
\]
We shall prove that the two norms on the right-hand side of the latter are smaller or equal to \(K\) multiplied by a suitable constant. To this end, we consider the estimates of \(|f|_{\sigma, 1}\) and \(|g|_{\sigma, 1}\) calculated above
\[
|f|_{\sigma, 1} \leq |b|_{\sigma - 1} + C(\sigma)\bar{\mathcal{T}}|v|_{\sigma - 1}
\]
\[
\leq |b|_{\sigma - 1} + |b - b|_{\sigma - 1} + C(\sigma)\bar{\mathcal{T}}|v|_{\sigma - 1} \leq |b|_s + C(\sigma)\bar{\mathcal{Y}}K \leq C(\sigma)\bar{\mathcal{Y}}K
\]
\[
|g|_{\sigma, 1} \leq |b|_{\sigma} + C(\sigma)\bar{\mathcal{T}}|v|_{\sigma},
\]
\[
\leq |b|_{\sigma} + |b - b|_{\sigma} + C(\sigma)\bar{\mathcal{T}}|v|_{\sigma} \leq |b|_s + C(\sigma)\bar{\mathcal{Y}}K \leq C(\sigma)\bar{\mathcal{Y}}K,
\]
where we used that \(|b|_{s+1,1} \leq \mathcal{Y}\. Furthermore, by (5.25) and \(\delta\zeta\)
\[
|f|_{0,1} \leq 1, \quad |g|_{0,1} \leq 1.
\]
Moreover, thanks to H.4 and \(\delta\zeta\)
\[
|\mathcal{F}(x, v)|_{1,2} \leq C\bar{\mathcal{Y}}\zeta \leq 1, \quad |\mathcal{F}(x, v)|_{\sigma, 2} \leq C(\sigma)\bar{\mathcal{Y}}K.
\]
Now, by (5.18) and the above estimates
\[
|\eta(x, v)\mathcal{F}(x, v)|_{\sigma, 1} \leq C(\sigma, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \zeta)|\mathcal{F}(x, v)|_{\sigma, 2}
\]
\[
+ C(\sigma, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \zeta) |f|_{\sigma, 1} + |g|_{\sigma, 1} |\mathcal{F}(x, v)|_{1,2}
\]
\[
\leq C(\sigma, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \zeta) (|f|_{\sigma, 1} + |g|_{\sigma, 1}) |\mathcal{F}(x, v)|_{1,2}
\]
\[
\leq C(\sigma, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \zeta) K |\mathcal{F}(x, v)|_{1,2} \leq C(\sigma, \bar{\mathcal{Y}}, \bar{\mathcal{Z}}, \zeta) K
\]
and by (5.24)
\[
|\nabla (\eta(x, v)\mathcal{F}(x, v)) \bar{\Omega}|_{\sigma - 1, 2} = |\mathcal{F}(x, v) - \partial_q (\eta(x, v)\mathcal{F}(x, v)) f - g (\eta(x, v)\mathcal{F}(x, v))|_{\sigma - 1, 2}
\]
\[
\leq |\mathcal{F}(x, v)|_{\sigma, 2} + |\partial_q (\eta(x, v)\mathcal{F}(x, v)) f|_{\sigma - 1, 2}
\]
\[
+ |g (\eta(x, v)\mathcal{F}(x, v))|_{\sigma, 2}.
\]
It remains to prove that each term on the right-hand side of the latter can be estimated by \( K \) multiplied by a suitable constant. By H.4, this is true for the first term \( |\mathcal{F}(x,v)|_{\sigma,2} \). We will prove it for the others using the estimates previously verified

\[
|\partial_q (\eta(x,v)\mathcal{F}(x,v)) f|_{\sigma-1,2} \leq C(\sigma) (|\eta(x,v)\mathcal{F}(x,v)|_{\sigma,1} f|_{0,1} + |\eta(x,v)\mathcal{F}(x,v)|_{1,1} f|_{\sigma,1}) 
\leq C(\sigma, \Upsilon, \delta, \zeta) K |f|_{0,1} + C(\delta, \zeta) |\mathcal{F}(u,v)|_{1,2} f|_{\sigma,1}
\]

and similarly

\[
|g (\eta(x,v)\mathcal{F}(x,v))|_{\sigma,2} \leq C(\sigma) (|g|_{0,1} |\eta(x,v)\mathcal{F}(x,v)|_{\sigma,1} + |g|_{\sigma,1} |\eta(x,v)\mathcal{F}(x,v)|_{0,1}) 
\leq C(\sigma) (|\eta(x,v)\mathcal{F}(x,v)|_{\sigma,1} + |g|_{\sigma,1} C(\delta, \zeta) |\mathcal{F}(x,v)|_{1,2}) 
\leq C(\sigma, \Upsilon, \delta, \zeta) K + C(\delta, \zeta) \Upsilon K |\mathcal{F}(x,v)|_{1,2}
\]

This concludes the proof of H.3 and also of this lemma. \( \square \)

5.6 \textit{C}^\alpha\textit{-weakly asymptotic cylinder}

We proved that, for fixed \( m, \bar{m} \in M \) as in (\ref{eq:6.1}), the functional \( \mathcal{F}_{m,\bar{m}} \) satisfies the hypotheses of Theorem 5.1. Then, there exists \( v : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m} \) such that

\[
\varphi^t = (\text{id}, v^t)
\]

is a \( C^\alpha\)-weakly asymptotic cylinder associated to \((X_H, X_{\bar{h}}, \varphi_0)\). We recall that \( H \) is the Hamiltonian defined by (\ref{eq:4.1}), \( \bar{h} \) is the Hamiltonian in (\ref{eq:4.2}) and \( \varphi_0 \) is the trivial embedding \( \varphi_0 : \mathbb{T}^n \times \mathbb{R}^m \rightarrow \mathbb{T}^n \times \mathbb{R}^m \times B^{n+m}, \varphi_0(q) = (q, 0) \). Moreover, letting \( \Gamma = b + (\bar{m} \circ \bar{\varphi}) v \) (see Section 5.2),

\[
|v|_{\rho,1} \leq \zeta, \\
|\Gamma|_{\rho,1} \leq |b|_{\rho,1} + |(\bar{m} \circ \bar{\varphi}) v|_{\rho,1} \\
\leq |b_0|_{\rho,1} + |b - b_0|_{\rho,1} + |(\bar{m} \circ \bar{\varphi}) v|_{\rho,1} \\
\leq |b_0|_{\rho,1} + |b - b_0|_{\lambda,1} + |(\bar{m} \circ \bar{\varphi}) v|_{\rho,1} \leq |b_0|_{\rho,1} + \varepsilon + C \Upsilon \zeta
\]

for a positive constant \( C \geq 1 \) depending on \( n + m \). It remains to prove that \( \varphi^t \) is Lagrangian for all \( t \in J \). First, we observe that

\[
|\Gamma|_{1,1} \leq |b_0|_{1,1} + \varepsilon + C \Upsilon \zeta \leq \delta + \varepsilon + C \Upsilon \zeta.
\]

Letting \( \psi^t_{\bar{t}_0, \bar{\omega} + \Gamma} \) be the flow at time \( t \) with initial time \( t_0 \) of \( \bar{\omega} + \Gamma \), we have the following proposition

**Proposition 5.2.** For all \( t, t_0 \in J \), if \( t \geq t_0 \)

\[
|\partial_q \psi^t_{t_0, \bar{\omega} + \Gamma}|_{C^0} \leq C \left( \frac{t}{t_0} \right)^{C(\delta + \varepsilon + \Upsilon \zeta)}
\]

for a suitable positive constant \( C \).

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Proof. Similarly to the proof of Lemma 5.3 we have the claim. We observe that the constant $C$ in this proposition may differ from that in the estimate of $|\Gamma|_{1,1}$ previously calculated.

Let $\psi_{t_0,H}^t$ be the flow at times $t$ with initial time $t_0$ of $H$, the following lemma concludes the proof of Theorem A.

Lemma 5.7. $\varphi^0$ is Lagrangian for all $t_0 \in J$

Proof. Let $\alpha = dp \wedge dq$ be the standard symplectic form associated to $(q,p) \in \mathbb{T}^n \times \mathbb{R}^m \times B$. For all fixed $t$, $t_0 \in J$, the flow $\psi_{t_0,H}^t$ is a symplectomorphisms. This means that, for all fixed $t$, $t_0 \in J$, $(\psi_{t_0}^t)^* \alpha = \alpha$. By (4.7),

$$\psi_{t_0,H}^t \circ \varphi^0 = \varphi^{t_0+t} \circ \psi_{t_0,\omega+\Gamma}^0$$

(5.28)

and taking the pull-back with respect to the standard form $\alpha$ on both sides of the latter, we obtain

$$(\varphi^0)^* (\psi_{t_0,H}^t)^* \alpha = (\psi_{t_0}^{t_0+t})^* (\varphi^{t_0+t})^* \alpha.$$

We know that $\psi_{t_0,H}^t$ is symplectic, then letting $(\psi_{t_0}^{t_0+t})^* \alpha = \alpha$ on the left-hand side of the above equation, we have

$$(\varphi^0)^* \alpha = (\psi_{t_0}^{t_0+t})^* (\varphi^{t_0+t})^* \alpha.$$

We want to prove that, for all $q \in \mathbb{T}^n \times \mathbb{R}^m$, $((\varphi^0)^*) \alpha_q = 0$, where $((\varphi^0)^*) \alpha_q$ stands for the symplectic form calculated on $q \in \mathbb{T}^n \times \mathbb{R}^m$. The idea is to prove that, for all $q \in \mathbb{T}^n \times \mathbb{R}^m$, the limit for $t \to +\infty$ on the right-hand side of the above equation converges to zero. Therefore, we recall that $\varphi^{t_0+t}(q) = (q, v^{t_0+t}(q))$, then for all $q \in \mathbb{T}^n \times \mathbb{R}^m$

$$((\psi_{t_0,\omega+\Gamma}^t)^* (\varphi^{t_0+t})^* \alpha)_q = \sum_{1\leq i < j \leq n+m} \sum_{1 \leq k < d \leq n+m} \alpha_{i,j,k,d}(q) dq_k \wedge dq_d$$

where

$$\alpha_{i,j,k,d}(q) = \left( \partial_i v_j^{t_0+t} \circ \psi_{t_0,\omega+\Gamma}^t(q) - \partial_j v_i^{t_0+t} \circ \psi_{t_0,\omega+\Gamma}^t(q) \right) \times \left( \partial_k \psi_{t_0,\omega+\Gamma}^t(q) \partial_{kd} \psi_{t_0,\omega+\Gamma}^t(q) - \partial_{kd} \psi_{t_0,\omega+\Gamma}^t(q) \partial_k \psi_{t_0,\omega+\Gamma}^t(q) \right).$$

In the latter $\times$ stands for the usual multiplication in $\mathbb{R}$. Then, for $t > 0$ and fixed $1 \leq i < j \leq n + m$, $1 \leq k < d \leq n + m$, by Proposition 5.2

$$|\alpha_{i,j,k,d}^{t_0}|_{C^0} \leq \left| \partial_i v_j^{t_0+t} \circ \psi_{t_0,\omega+\Gamma}^t \right|_{C^0} \times \left| \partial_k \psi_{t_0,\omega+\Gamma}^t \right|_{C^0}$$

$$\leq \left| \partial_i v_j^{t_0+t} \circ \psi_{t_0,\omega+\Gamma}^t \right|_{C^0} \times \left| \partial_k \psi_{t_0,\omega+\Gamma}^t \right|_{C^0} C \leq C |\varphi^{t_0+t}|_{C^1} \left( \frac{t_0 + t}{t_0} \right)^{C(\delta + \epsilon + \Upsilon \zeta)}$$

for a suitable constant $C \geq 1$. Thanks to (2) and for $\epsilon$ small enough, taking the limit for $t \to +\infty$ on both sides of the latter, the term in the last line converges to zero. This concludes the proof of this lemma.
6 Proof of Theorem \[B\]

6.1 Quasiperiodic motions in the planar three-body problem

This part is dedicated to a very brief introduction to the work of J. Féjoz \cite{Fe02} concerning the existence, in a rotating frame of reference, of quasiperiodic motions with three frequencies for the Hamiltonian of the planar three-body problem. This result is an important element for the proof of Theorem \[B\].

In this work, the author splits the dynamic into two parts: a fast, called Keplerian dynamic, and a slow, called secular dynamic. The first describes the motion of the bodies along three ellipses as if each body underwents the attraction of only one fictitious center of attraction. The slow dynamic describes how the mutual attraction of each planet deforms these Keplerian ellipses. There is a natural splitting

$$H_0 = H_{Kep} + H_{per}$$

of the Hamiltonian when one uses the well-known Jacobi coordinates \(\{(X_i, Y_i)\}_{i=0,1,2}\).

Here, \(H_{Kep}\) is the degenerate Hamiltonian of two decoupled two-body problems and \(H_{per}\) is the perturbation.

The author defines the perturbing region contained in the direct product of the phase and parameter spaces. In this region, the Hamiltonian of the planar three-body problem is \(C^k\)-close to the dynamically degenerate Hamiltonian of two decoupled two-body problems. To this end, we introduce some notations concerning the Keplerian dynamics. For the \(i\)th fictitious body, with \(i = 1\) or \(2\), the mean longitude will be designated by \(\lambda_i\), the semi-major axis by \(a_i\), the eccentricity by \(e_i\), the "centricity" \(\sqrt{1 - e_i^2}\) by \(\epsilon_i\), the argument of the pericenter by \(g_i\), the mean motion by \(\upsilon_i\) and the difference of the arguments of the pericenters by \(\Delta_g = g_1 - g_2\).

We also introduce the well-known Poincaré coordinates \((\Lambda_i, \lambda_i, \xi_i, \eta_i)\), where we refer for example to the notes of A. Chenciner and J. Laskar \cite{Che89, Las89} or the work of J. Féjoz \cite{Fe13}.

To measure how close the outer ellipse is from the inner ellipses when they are in opposition, the author defines

$$\Delta = \max_{(\lambda_1, \lambda_2, g) \in \mathbb{T}^3} \max \left\{ \sigma_0, \sigma_1 \right\} \frac{|X_1|}{|X_2|} = \max \left\{ \sigma_0, \sigma_1 \right\} \frac{a_1(1 + e_1)}{a_2(1 - e_2)}.$$  

He assumes that \(\Delta < 1\). This means that the outer ellipse does not meet the other two, whatever the difference \(g\) of the arguments of the pericenters. Moreover, the eccentricity \(e_2\) of the outer ellipse cannot be arbitrarily close to 1. He also assumes that the eccentricity of the inner ellipses is upper bounded from 1.

Let \(\mathcal{P}\) be the reduced symplectically by translations phase space and \(\mathcal{M}\) be the space described by the three masses of the planets \(m_0, m_1\) and \(m_2\).

**Definition 6.1.** For a positive parameter \(\delta\) and a non negative integer \(k\), the perturbing region \(\Pi^k_\delta\) of parameters \(\delta\) and \(k\) is the open subset of \(\mathcal{P} \times \mathcal{M}\) defined by the following inequality

$$\max \left\{ \frac{m_2}{M_1} \left( \frac{a_1}{a_2} \right)^{\frac{3}{2}}, \frac{\mu_1 \sqrt{M}}{M_1^{\frac{3}{2}}} \left( \frac{a_1}{a_2} \right)^2 \right\} \frac{1}{\epsilon_2^{3(2+k)(1-\Delta)^{2k+1}}} < \delta,$$  

(6.1)
where \( M_1 = m_0 + m_1 \), \( M = m_0 + m_1 + m_2 \) and \( \mu_1 = \frac{m_0 m_1}{m_0 + m_1} \).

Féjoz writes in his work that this inequality is not optimal and the given powers are not meaningful. He justifies this definition by proving that, inside the perturbing region, the perturbating function is \( \delta \)-small in a suitable \( C^k \)-norm.

Concerning the case of Theorem [A] where the masses are fixed, we point out that inequality (6.1) may be satisfied merely by assuming that \( \frac{m_1}{m_2} \ll 1 \) and \( e_2 \leq \text{Cst} < 1 \). This is the so-called lunar or hierarchical regime.

In order to get rid of the degeneracy of the Keplerian Hamiltonian and hence apply the well-known KAM theorem, the secular Hamiltonian is introduced. Let \( d \) and \( k \) be suitable positive integers. On a suitable open set \( \Pi^d_k \) of \( \Pi_k \), the author proves the existence of a \( C^\infty \)-symplectomorphism \( \phi^d \) which is \( \delta \)-close to the identity in a suitable \( C^k \)-norm. The Hamiltonian \( H_0 \circ \phi^d \) can be split as follows

\[
H_0 \circ \phi^d = H^d_\pi + H^d_{\text{comp}},
\]

where \( H^d_{\text{comp}} \) is of size \( \mathcal{O}(\delta^{d+1}) \) on \( \Pi^d_k \). The secular Hamiltonian is \( H^d_\pi \), which is Pöschel integrable. It can be split into an integrable part \( H^d_{\text{int}} \) and a resonant part \( H^d_{\text{res}} \) of size \( \mathcal{O}(\delta) \). The infinite jet of \( H^d_{\text{res}} \) vanishes along a suitable Cantor set.

The previous splitting is obtained by an averaging process. It consists in averaging along the Keplerian ellipses parametrized by the mean anomalies \( \lambda_1 \) and \( \lambda_2 \) of the two fictitious Kepler problems where the Keplerian frequencies are non-resonant.

After the reduction by the symmetry of rotation and far from elliptic singularities, the phase space of the secular Hamiltonian contains a positive measure of Lagrangian diophantine invariant tori. The claim relies on a sophisticated version of KAM theorem, which is proved using a normal form theorem due to Herman. More specifically Féjoz proved the following theorem

**Theorem 6.1.** In a rotating frame of reference, there are integers \( k \geq 1 \) and \( d \geq 1 \) and real numbers \( \delta > 0 \) and \( \tau \geq 1 \) such that inside the perturbing region \( \Pi^d_k \) a positive measure of quasiperiodic Lagrangian tori of \( H^d_\pi \) survive in the dynamics of the planar three-body problem.

In what follows, we will use the previous theorem as a blackbox to prove the existence of weakly asymptotically quasiperiodic solutions associated with the Hamiltonian of the planar three-body problem plus comet (see Theorem [B]).

### 6.2 Outline of the proof of Theorem [B]

The proof relies on [Fé02] and Theorem [A]. We observe that we do not prove the existence of a weakly asymptotic cylinder but only a set of initial points giving rise to weakly asymptotically quasiperiodic solutions associated with the Hamiltonian \( H \) of the P3BP+C.

The proof is divided into five parts. The first two concern the Hamiltonian of the planar three-body problem \( H_0 \). In the first part, which we call *Splitting*, we introduce a linear symplectic change of variable \( \phi_0 \). Letting \( (X_i, Y_i)_{i=0,1,2} \) be the new coordinates,
variables, which should not be confused with the Jacobi coordinates introduced in the previous section, we can split the Hamiltonian $H_0$ in such a way that

$$H_0 \circ \phi_0(X, Y) = \frac{|Y_0|^2}{2M} + K(X_1, X_2, Y_1, Y_2),$$

where $K$ is the Hamiltonian of the planar three-body problem after the symplectic reduction by the symmetry of translations, $X_0$ is the center of mass of the planetary system, $Y_0$ is the linear momentum of the planetary system and $M = m_0 + m_1 + m_2$.

We call the second part Quasiperiodic dynamics associated to $K$. Here, Theorem 6.1 ensures the existence of Lagrangian four-dimensional invariant tori in the phase space after the symplectic reduction by the symmetry of translations. As mentioned before, Féjoz proves the existence of quasiperiodic solutions with three frequencies for the Hamiltonian of the planar three-body problem in a rotating frame of reference. The additional frequency is given by the angular speed of the simultaneous rotations of the three ellipses. Therefore, we prove the existence of a symplectic change of variables $\phi_F$ in such a way that

$$K \circ \phi_F : T^4 \times B^4 \rightarrow \mathbb{R}, \quad K \circ \phi_F(\theta, r) = c + \omega \cdot r + R_0(\theta, r) \cdot r^2$$

where $c \in \mathbb{R}$, $\omega \in \mathbb{R}^4$ and $R_0(\theta, r) \cdot r^2$ stands for the vector given twice as an argument of the symmetric bilinear form $R_0(\theta, r)$.

We lift the above symplectic transformation $\phi_F(\theta, r)$ to a symplectic transformation $\tilde{\phi}_F(\theta, \xi, r, \eta)$ defined on $T^4 \times \mathbb{R}^2 \times B^4 \times B^2$, for some balls $B^4 \subset \mathbb{R}^4$ and $B^2 \subset \mathbb{R}^2$ of small radius, such that

$$\tilde{\phi}_F(\theta, \xi, r, \eta) = (\xi, \eta, \phi_F(\theta, r)),$$

where $\xi = X_0$ and $\eta = Y_0$. Letting $\phi = \phi_0 \circ \tilde{\phi}_F$, we can rewrite the Hamiltonian of the planar three-body problem $H_0$ in the following form

$$H_0 \circ \phi : T^4 \times \mathbb{R}^2 \times B^4 \times B^2 \rightarrow \mathbb{R},$$

$$H_0 \circ \phi(\theta, \xi, r, \eta) = c + \omega \cdot r + R_0(\theta, r) \cdot r^2 + \frac{|\eta|^2}{2M}.$$

The third part of the proof is devoted to the perturbing function and, for this reason, we call it Perturbing function. We introduce a suitable open subset $\mathcal{U}$ of $T^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J$ characterized by the orbits of $H$ that stay sufficiently far from the comet. Letting

$$\tilde{\phi}(\theta, \xi, r, \eta, t) = (\phi(\theta, \xi, r, \eta), t),$$

we will see that $H_c \circ \tilde{\phi} : \mathcal{U} \rightarrow \mathbb{R}$ is well defined and satisfies good time-dependent estimates. Unfortunately, the Hamiltonian $H \circ \tilde{\phi} = H_0 \circ \phi + H_c \circ \tilde{\phi} : \mathcal{U} \rightarrow \mathbb{R}$ does not satisfy the hypotheses of Theorem A. This is because $H \circ \tilde{\phi}$ is not defined in the whole phase space $T^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J$ but only on a subset. To solve this problem, in the fourth part called Smooth extension of the perturbing function, we introduce a smooth extension $H_{ex}$ of $H_c \circ \tilde{\phi}$,

$$H_{ex} : T^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J \rightarrow \mathbb{R}$$
in such a way that $H_{ex}$ coincides with $H_c \circ \tilde{\phi}$ on a suitable subset $U_{\frac{1}{2}}^c$ of $U$, where one expects the motions to take place, and $H_{ex}$ satisfies the same estimates of $H_c \circ \tilde{\phi}$ outside $U_{\frac{1}{2}}^c$. We will see that $\tilde{H} = H_0 \circ \phi + H_{ex}$ satisfies the hypotheses of Theorem A. Letting $\varphi_0$ be the following trivial embedding

$$\varphi_0 : T^4 \times \mathbb{R}^2 \to T^4 \times \mathbb{R}^2 \times B^4 \times B^2, \quad \varphi_0(\theta, \xi, r, \eta) = (\theta, \xi, 0, 0),$$

we prove the existence of a $C^1$-weakly asymptotic cylinder $\varphi^t$ associated to $(X_{\tilde{H}}, X_{H_0 \circ \tilde{\phi}}, \varphi_0)$. Let $\psi^t_{\tilde{H},\circ}$ be the flow at time $t$ with initial time $t_0$ of $\tilde{H}$. This means that, for all $z \in \varphi^1(T^4 \times \mathbb{R}^2)$, $\psi^t_{\tilde{H},\circ}(z)$ is a weakly asymptotically quasiperiodic solution associated to $(X_{\tilde{H}}, X_{H_0 \circ \tilde{\phi}}, \varphi_0)$.

In the last part, called *Weakly asymptotically quasiperiodic solutions*, we conclude the proof of Theorem B. We define

$$B^2_{\frac{1}{2}} = \left\{ \xi \in \mathbb{R}^2 : |\xi| < \frac{\varepsilon}{6}|c(1)| \right\} \subset \mathbb{R}^2,$$

where $|c(1)|$ stands for the distance of the comet from the origin when $t = 1$. Therefore, we show that, for all $z \in \varphi^1(T^4 \times B^2_{\frac{1}{2}})$,

$$\psi^t_{\tilde{H},\circ}(z) \in U_{\frac{1}{2}}^c$$

for all $t \in J$. This is a consequence of $(\psi^t_{\tilde{H}}, |c(t)| \sim vt$ and $|\xi(t)| = |X_0(t)| \sim \ln t$.

This concludes the proof because for all $z \in \varphi^1(T^4 \times B^2_{\frac{1}{2}})$,

$$\psi^t_{\tilde{H},\circ}(z) = \psi^t_{\circ}(z)$$

for all $t \in J$. Moreover, because of $\phi$ is symplectic, for all $w \in W = \phi \circ \varphi^1(T^4 \times B^2_{\frac{1}{2}})$,

$$\psi^t_{\circ}(w)$$

is a weakly asymptotically quasiperiodic solution associated to $(X_{H}, X_{H_0}, \phi \circ \varphi_0)$.

### 6.3 Splitting

The phase space and the Hamiltonian of the planar three-body problem are respectively

$$\left\{ (x_i, y_i)_{0 \leq i \leq 2} \in (\mathbb{R}^2 \times \mathbb{R}^2)^3 \mid 0 \leq i < j \leq 2, \ x_i \neq x_j \right\},$$

and

$$H_0(x, y) = \sum_{i=0}^{2} \frac{|y_i|^2}{2m_i} - \sum_{0 \leq i < j \leq 2} \frac{m_im_j}{|x_i - x_j|}.$$

We would like to split the dynamics into the absolute motion of the center of mass and the relative motion of the three bodies. For this purpose, let us introduce the following linear symplectic change of coordinates $\phi_0$

$$\begin{align*}
X_0 &= \frac{m_0}{M}x_0 + \frac{m_1}{M}x_1 + \frac{m_2}{M}x_2 \\
X_1 &= x_0 - x_1 \\
X_2 &= x_0 - x_2 \\
Y_0 &= y_0 + y_1 + y_2 \\
Y_1 &= \frac{m_1}{M}y_0 - \frac{m_0 + m_2}{M}y_1 + \frac{m_1}{M}y_2 \\
Y_2 &= \frac{m_2}{M}y_0 + \frac{m_1}{M}y_1 - \frac{m_0 + m_2}{M}y_2
\end{align*}$$

(6.2)
three-body problem

H coordinates, if $X$ the simultaneous rotation of the three ellipses. Here, we fix $m$ the symmetry of rotations. This frequency plays the role of the angular speed of oïdic motions have one additional frequency before the symplectic reduction by Theorem B and we introduce the slice

$$
\Pi_{k+4} = \Pi_{k+4}(X_0, Y_0, \theta, r) \subset \mathcal{P}
$$

of three-dimensional invariant tori for the Hamiltonian of the planar three-body problem $H_0$ is equal to

$$
H_0 \circ \phi_0(X, Y) = \frac{|Y_0|^2}{2M} + \left(\frac{|Y_1|^2}{2\mu_1} - \frac{m_0m_1}{|X_1|}\right) + \left(\frac{|Y_2|^2}{2\mu_2} - \frac{m_0m_2}{|X_2|}\right) + \left(\frac{Y_1 \cdot Y_2}{m_0} - \frac{m_1m_2}{|X_2 - X_1|}\right)
$$

where $\mu_1 = \frac{m_0m_1}{m_0+m_1}$ and $\mu_2 = \frac{m_0m_2}{m_0+m_2}$. We observe that $H_0$ is the sum of two independent Hamiltonians. The first is responsible for the motion of the center of mass $X_0$ and the linear momentum $Y_0$. On the other hand, $K$ is the Hamiltonian of the planar three-body problem after the reduction by the symmetry of translations.

6.4 Quasiperiodic dynamics associated with $K$

For suitable integers $k \geq 1$, $d \geq 1$ and real numbers $\delta > 0$ and $\tau \geq 1$, inside the perturbing region $\Pi_{k+4}^{\delta+d(\tau+4)}$ (see Definition 6.1), Theorem 6.1 proves the existence of three-dimensional invariant tori for the Hamiltonian of the planar three-body problem $K$ in a rotating frame of reference. As mentioned above, these quasiperiodic motions have one additional frequency before the symplectic reduction by the symmetry of rotations. This frequency plays the role of the angular speed of the simultaneous rotation of the three ellipses. Here, we fix $m_0$, $m_1$ and $m_2$ as in Theorem 6.1 and we introduce the slice

$$
\Pi_{k+4}^{\delta+d(\tau+4)} = \Pi_{k+4}^{\delta+d(\tau+4)} \bigg|_{m_0, m_1, m_2} \subset \mathcal{P}
$$

where $\mathcal{P}$ is the phase space after the symplectic reduction by the symmetry by translations. In other words, $\Pi_{k+4}^{\delta+d(\tau+4)}$ is the subset of $\mathcal{P}$ obtained by $\Pi_{k+4}^{\delta+d(\tau+4)}$ once we have fixed the masses $m_0$, $m_1$ and $m_2$.

We begin this second section with the following lemma concerning the dynamics of the Hamiltonian of the planar three-body problem $K$ with the frame of reference attached to the center of mass.

**Lemma 6.1.** There exists a symplectic transformation $\phi_F$ defined on $\mathbb{T}^4 \times B^4$, where $B^4$ is a 4-dimensional ball with some small unspecified radius, with $\phi_F(\mathbb{T}^4 \times B^4) \subset \mathcal{P}$ such that the Hamiltonian $K \circ \phi_F : \mathbb{T}^4 \times B^4 \to \mathbb{R}$ can be written in the following form

$$
k \circ \phi_F(\theta, r) = c + \omega \cdot r + R_0(\theta, r) \cdot r^2
$$

for some $c \in \mathbb{R}$ and $\omega \in \mathbb{R}^4$.

**Proof.** By Theorem 6.1, there exists a 4-dimensional Lagrangian invariant torus $T \subset \Pi_{k+4}^{\delta+d(\tau+4)}$ for $K$ supporting quasiperiodic dynamics. These tori form a set of positive Lebesgue measure, but we use only one such torus.
Due to the Weinstein Lagrangian neighbourhood theorem (see e.g. McDuff-Salamon [MS17]), there exists a neighbourhood $N(T)$ of $T$ and a symplectomorphism
\[ \phi_F : T^4 \times B^4 \longrightarrow N(T) \] such that $\varphi(T^4 \times \{0\}) = T$.

We observe that $\phi_F(T^4 \times \{0\}) = T$ is a Lagrangian invariant torus for $K$. Hence
\[ K \circ \phi_F(\theta, 0) = c \]
for all $\theta \in T^4$ and a suitable constant $c \in \mathbb{R}$. Moreover, $\phi_F(T^4 \times \{0\}) = T$ support a quasiperiodic dynamics with some frequency vector $\omega \in \mathbb{R}^4$, so
\[ \partial_r (K \circ \phi_F)(\theta, 0) = \omega. \]

In order to obtain suitable coordinates for the Hamiltonian $H_0$ of the planar three-body problem, we need to lift the symplectic change of variables $\phi_F$ introduced in the previous lemma. Let $B^4 \subset \mathbb{R}^4$ and $B^2 \subset \mathbb{R}^2$ be two small balls in $\mathbb{R}^4$ and $\mathbb{R}^2$, respectively. We consider the symplectic transformation $\tilde{\phi}_F(\theta, \xi, r, \eta)$ defined on $T^4 \times \mathbb{R}^2 \times B^4 \times B^2$ in such a way that
\[ \tilde{\phi}_F(\theta, \xi, r, \eta) = (\xi, \eta, \phi_F(\theta, r)) \] (6.4)
with $\xi = X_0$ and $\eta = Y_0$. We recall that $(X_0, Y_0)$ are the center of mass and the linear momentum of the planetary system defined in the previous section (see (6.2)). Letting
\[ \phi = \phi_0 \circ \tilde{\phi}_F, \] (6.5)
we have the following lemma

**Lemma 6.2.** We can write the Hamiltonian of the planar three-body problem $H_0$ in the following form
\[ H_0 \circ \phi : T^4 \times \mathbb{R}^2 \times B^4 \times B^2 \longrightarrow \mathbb{R}, \]
\[ H_0 \circ \phi(\theta, \xi, r, \eta) = c + \omega \cdot r + R_0(\theta, r) \cdot r^2 + \frac{|\eta|^2}{2M}. \]

**Proof.** The proof of this lemma is a consequence of (6.3) and Lemma 6.1. \qed

We conclude this part dedicated to the Hamiltonian $H_0$ with a property that plays an important role in the following section. First, we observe that, for all $i = 0, 1$
\[ \sup_{(\theta, r) \in T^4 \times B^4} |X_i(\theta, r)| < \infty \]
where $X_i$ are the coordinates defined in (6.2). This is because, for all $i = 1, 2, X_i$, as a function of $(\theta, r) \in T^4 \times B^4$, is continuous and 1-periodic with respect to $\theta_j$ for all $0 \leq j \leq 4$. 

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Lemma 6.3. We assume that

$$|c(1)| > \max_{i=1,2} \left( \sup_{(\theta,r) \in \mathbb{T}^4 \times B^4} |X_i(\theta, r)| \right) \frac{3}{\varepsilon} \tag{6.6}$$

where $\varepsilon$ is the positive real parameter introduced in Theorem B. Then, for all $(\theta, r) \in \mathbb{T}^4 \times B^4$ and $i = 1, 2$

$$\frac{|X_i(\theta, r)|}{|c(t)|} < \varepsilon$$

for all $t \geq 1$.

Proof. The proof is a straightforward computation. For all $i = 1, 2$

$$\frac{|X_i(\theta, r)|}{|c(t)|} \leq \frac{\sup_{(\theta,r) \in \mathbb{T}^4 \times B^4} |X_i(\theta, r)|}{|c(1)|}$$

for all $t \geq 1$. Therefore, thanks to (6.6), we conclude the proof of this lemma. \(\square\)

6.5 Perturbing function

We recall that the motion of the comet is a given smooth function $c(t)$ satisfying

$$|c(t)| \to_t \infty, \quad \frac{d}{dt} |c(t)| \to_t \infty, \quad v > 0.$$

By the latter, there exists $t_0 \gg 0$ such that

$$\frac{v}{2} \leq \frac{d}{dt} |c(t)| \leq 2v \tag{6.7}$$

for all $t \geq t_0$. At the risk of replacing $t$ by $t + t_0 - 1$, we can take $t_0 = 1$. Furthermore we have the following proposition.

Proposition 6.1. If

$$|c(1)| > \frac{1}{\varepsilon}, \quad v > \frac{2}{\varepsilon}, \tag{6.8}$$

then

$$\sup_{t \geq 1} \frac{t}{|c(t)|} < \varepsilon. \tag{6.9}$$

Proof. By (6.7) and the fundamental theorem of calculus

$$|c(1)| + \frac{v}{2}(t - 1) \leq |c(t)| \leq |c(1)| + 2v(t - 1) \tag{6.10}$$

for all $t \geq 1$. Thanks to the latter, we can estimate $\frac{t}{|c(t)|}$ as follows

$$\frac{t}{|c(t)|} \leq \frac{1 + (t - 1)}{|c(1)| + \frac{v}{2}(t - 1)}.$$
for all $t \geq 1$. Thanks to (6.8), the claim is true for $t = 1$. Now, we suppose that there exists $t_0 > 1$ such that
\[
1 + (t_0 - 1) \geq \varepsilon. \tag{6.8}
\]
We can rewrite the latter in the following form
\[
1 - \varepsilon|c(1)| \geq \left(\frac{\varepsilon}{2}v - 1\right)(t_0 - 1)
\]
and this is a contradiction because, by (6.8), $1 - \varepsilon|c(1)| < 0$ and $\left(\frac{\varepsilon}{2}v - 1\right)(t_0 - 1) > 0$.

Concerning the Hamiltonian of P3BP+C, we introduce a suitable neighbourhood $\mathcal{U}$ of $\mathbb{T}^4 \times \{0\} \times B^4 \times B^2 \times J \subset \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J$ where we expect the motions of interest to take place.

For all fixed $t \in J$, we define
\[
B_t^2 = \{ \xi \in \mathbb{R}^2 : \frac{|\xi|}{|c(t)|} < \frac{\varepsilon}{3} \}. \tag{6.11}
\]
We consider $\mathcal{U}$ as the following subset of $\mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J$,
\[
\mathcal{U} = \bigcup_{t \in J} \left( \mathbb{T}^4 \times B_t^2 \times B^4 \times B^2 \times \{t\} \right). \tag{\mathcal{U}}
\]
Let
\[
\tilde{\phi}(\theta, \xi, r, \eta, t) = (\phi(\theta, \xi, r, \eta), t), \tag{6.12}
\]
where $\phi$ is the symplectic transformation defined by (6.5). This part aims to prove that the Hamiltonian $H_c \circ \tilde{\phi} : \mathcal{U} \to \mathbb{R}$ satisfies suitable estimates. First, let us show that $H_c \circ \tilde{\phi} : \mathcal{U} \to \mathbb{R}$ is well defined.

**Lemma 6.4.** For all $t \in J$ and $(\theta, \xi, r, \eta) \in \mathbb{T}^4 \times B_t^2 \times B^4 \times B^2$,
\[
\frac{|x_i(\theta, \xi, r)|}{|c(t)|} < \varepsilon
\]
for all $i = 0, 1, 2$.

**Proof.** The proof is a straightforward computation. Because of (6.2) and Lemma 6.1 we can rewrite the cartesian coordinates $(x_0, x_1, x_2)$ as follows
\[
\begin{align*}
x_0(\theta, \xi, r) &= \xi + \frac{m_1}{M}X_1(\theta, r) + \frac{m_2}{M}X_2(\theta, r) \\
x_1(\theta, \xi, r) &= \xi - \frac{m_0^* + m_2}{M}X_1(\theta, r) + \frac{m_2}{M}X_2(\theta, r) \\
x_2(\theta, \xi, r) &= \xi + \frac{m_1}{M}X_1(\theta, r) - \frac{m_0^* + m_1}{M}X_2(\theta, r),
\end{align*}
\tag{6.13}
\]
for all $t \in J$ and $(\theta, \xi, r, \eta) \in \mathbb{T}^4 \times B_t^2 \times B^4 \times B^2$. By the latter, (\mathcal{U}) and Lemma 6.3 for all $t \in J$ and $(\theta, \xi, r, \eta) \in \mathbb{T}^4 \times B_t^2 \times B^4 \times B^2$
\[
\frac{|x_0(\theta, \xi, r)|}{|c(t)|} \leq \frac{|\xi| + \frac{m_1}{M}|X_1(\theta, r)| + \frac{m_2}{M}|X_2(\theta, r)|}{|c(t)|} \\
&\leq \frac{|\xi|}{|c(t)|} + \frac{m_1}{M} \frac{|X_1(\theta, r)|}{|c(t)|} + \frac{m_2}{M} \frac{|X_2(\theta, r)|}{|c(t)|} \\
&\leq \frac{|\xi|}{|c(t)|} + \frac{|X_1(\theta, r)|}{|c(t)|} + \frac{|X_2(\theta, r)|}{|c(t)|} < \varepsilon
\]
\[45\]
for all \( t \in J \). We recall that \( M = m_0 + m_1 + m_2 \), which implies \( \frac{m_i}{M} \leq 1 \) for all \( 0 \leq i \leq 2 \). Similarly, we have the claim for \( x_1(\theta, \xi, r) \) and \( x_2(\theta, \xi, r) \).

The previous lemma ensures that \( H_c \circ \tilde{\phi} \) is well defined on \( U \). Moreover, by (6.13), it is straightforward to verify that \( H_c \circ \tilde{\phi} \) does not depend on the variable \( \eta \). The following lemma provides time-dependent estimations for \( H_c \).

**Lemma 6.5.** If \( |c(1)| > \frac{1}{\varepsilon} \) and \( v > \frac{2}{\varepsilon} \), then, for all \( k \in \mathbb{Z} \) with \( k \geq 0 \) and a suitable constant \( C(k) \) depending on \( k \),

\[
\sup_{t \geq 1} |H^t_{k}|_{C^k} < C(k) M m_c \varepsilon,
\]

\[
\sup_{t \geq 1} |\partial_x H^t_{k}|_{C^k} t^2 < C(k) M m_c \varepsilon.
\]

For all \( t \in J \), the above norms \( |\cdot|_{C^k} \) are taken on \( \phi(T^4 \times B^2 \times B^2 \times B^4) \), where \( \phi \) is defined by (6.5).

**Proof.** For all \( t \in J \) and \((\theta, \xi, r, \eta) \in T^4 \times B^2 \times B^2 \times B^4 \), let us rewrite the Hamiltonian \( H_c \circ \tilde{\phi} \) in the following form

\[
H_c \circ \tilde{\phi}(\theta, \xi, r, \eta, t) = \sum_{i=0}^{2} \frac{m_i m_c}{|x_i(\theta, \xi, r) - c(t)|}.
\]

For the rest of this proof, \( x_i = x_i(\theta, \xi, r) \) for all \( 0 \leq i \leq 2 \). We drop the coordinates \((\theta, \xi, r)\) to obtain a more elegant form.

Using Legendre polynomials, we have that

\[
\frac{1}{|x_i - c(t)|} = \frac{1}{c(t)} \sum_{n \geq 0} \mathcal{P}_n(\cos x_i c(t)) \left( \frac{|x_i|}{|c(t)|} \right)^n
\]

\[
= \frac{1}{c(t)} \left( 1 + \sum_{n \geq 1} \mathcal{P}_n(\cos x_i c(t)) \left( \frac{|x_i|}{|c(t)|} \right)^n \right)
\]

(6.14)

for all \( 0 \leq i \leq 2 \), where \( \mathcal{P}_n \) means the \( n \)th Legendre polynomial. This expansion hold if \( \frac{|x_i|}{|c(t)|} < 1 \) and this prerequisite is verified by Lemma 6.4.

Now, we can conclude the proof of this lemma. We recall that \( C(\cdot) \) means constants depending on some parameters indicated in brackets. Therefore, thanks to (6.14), Lemma 6.4 and Proposition 6.1, for all \( t \in J \) and \((x, y) = (x_0, x_1, x_2, y_0, y_1, y_2) \in \phi(T^4 \times B^2 \times B^2 \times B^2) \),

\[
|H^t_{k}(x)| = \sum_{i=0}^{2} \frac{m_i m_c}{|x_i - c(t)|}
\]

\[
= \sum_{i=0}^{2} \frac{m_i m_c}{c(t)} \left( 1 + \sum_{n \geq 1} \mathcal{P}_n(\cos x_i c(t)) \left( \frac{|x_i|}{|c(t)|} \right)^n \right)
\]

\[
\leq \sum_{i=0}^{2} \frac{m_i m_c}{c(t)} \left( 1 + \sum_{n \geq 1} \varepsilon^n \right) < C M m_c \varepsilon.
\]
In the last line, \( t \geq 1 \) implies \( \sup_{t \geq 1} \frac{1}{|c(t)|} \leq \sup_{t \geq 1} \frac{1}{|c(t)|} \) and, letting \( 0 < \varepsilon \leq \frac{1}{2} \), we can estimate \( (1 + \sum_{n \geq 1} \varepsilon^n) \) with a suitable constant \( C \). Taking the sup on \( \phi(T^4 \times B^2 \times B^4 \times B^2) \) and then for all \( t \in J \) on the left-hand side of the latter, we obtain

\[
\sup_{t \geq 1} |H_t^l|_{C^0} < CMm_c \varepsilon. \tag{6.15}
\]

Now, for all \( 0 \leq i \leq 2, k \geq 1, t \in J \) and \((x, y) = (x_0, x_1, x_2, y_0, y_1, y_2) \in \phi(T^4 \times B^2 \times B^4 \times B^2)\)

\[
|\partial_x^k H(x)|t^2 \leq C(k) \frac{m_i m_c t^2}{|x_i - c(t)|^{k+1}}
\]

\[
= C(k)m_i m_c \frac{t^2}{|c(t)|^{k+1}} \left( 1 + \sum_{n \geq 1} P_n(\cos x_i c(t)) \left( \frac{|x_i|}{|c(t)|} \right)^n \right)^{k+1}
\]

\[
\leq C(k)m_i m_c \frac{t^{k+1}}{|c(t)|^{k+1}} \left( 1 + \sum_{n \geq 1} P_n(\cos x_i c(t)) \left( \frac{|x_i|}{|c(t)|} \right)^n \right)^{k+1}
\]

\[
\leq C(k)\varepsilon^{k+1} m_i m_c \left( 1 + \sum_{n \geq 1} \varepsilon^n \right)^{k+1} \leq \varepsilon^{k+1} C(k)m_i m_c,
\]

where \( t \geq 1 \) and \( k + 1 \geq 2 \) imply \( t^2 \leq t^{k+1} \) in the third line. Similarly to the previous case, because of \( 0 < \varepsilon \leq \frac{1}{2} \), we can estimate \( (1 + \sum_{n \geq 1} \varepsilon^n)^{k+1} \) by a suitable constant depending on \( k \). Similarly to the previous case, taking the max for all \( 0 \leq i \leq 2 \) on the left-hand side of the latter we have

\[
\sup_{t \geq 1} |\partial_x^k H_t^l|_{C^0} t^2 < C(k)Mm_c \varepsilon^{k+1}. \tag{6.16}
\]

Now, thanks to (6.15), (6.16) and remembering the definition of Hölder’s norm (3.1), we conclude the proof of this lemma. \( \square \)

### 6.6 Smooth extension of the perturbing function

This section is dedicated to the introduction of an appropriate smooth extension of \( H_c \circ \hat{\phi} \). First, we need to introduce a suitable subset of \( \mathcal{U} \). For all fixed \( t \in J \), we define

\[
B^2_t/2 = \{ \xi \in \mathbb{R}^2 : \frac{|\xi|}{|c(t)|} < \frac{\varepsilon}{6} \} \subset B^2_t. \tag{6.17}
\]

Let \( \mathcal{U}_2 \) be the following subset of \( \mathcal{U} \),

\[
\mathcal{U}_2 = \bigcup_{t \in J} (T^4 \times (B^2_t/2) \times B^4 \times B^2 \times \{t\}) \subset \mathcal{U}. \tag{U_2}
\]

**Lemma 6.6.** There exists

\[
H_{ex} : T^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J \rightarrow \mathbb{R}
\]

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such that $H_{ex}$ does not depend on $\eta$ and

$$H_{ex}|_{U_{\frac{1}{2}}} = H_c \circ \tilde{\phi}. \quad (6.18)$$

Moreover, If $|c(1)| > \frac{1}{\varepsilon}$ and $v > \frac{2}{\varepsilon}$, then for all $k \in \mathbb{Z}$ with $k \geq 0$ and some constants $C(k)$ depending on $k$, we have the following estimates

$$\sup_{t \geq 1} |H_{ex}^l|_{C^k} < C(k) M_m c \varepsilon, \quad (6.19)$$
$$\sup_{t \geq 1} |\partial_{(\theta,\xi,r)} H_{ex}^l|_{C^k t^2} < C(k) M_m c \varepsilon, \quad (6.20)$$

where the previous norms are taken on $T^4 \times \mathbb{R}^2 \times B^4 \times B^2$.

Before the proof, we have some comments. First, for the sake of clarity, (6.18) means that, for all $(\theta, \xi, r, \eta, t) \in U_{\frac{1}{2}}$, $H_{ex}(\theta, \xi, r, \eta, t) = H_c \circ \tilde{\phi}(\theta, \xi, r, \eta, t)$. Furthermore, we observe that the constants in the last estimates may differ from those in Lemma 6.5 and they depend on the chosen extension.

**Proof.** For all fixed $t \in J$, we consider the following family of functions

\[
\begin{aligned}
\rho^t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2, \\
\rho^t &\in C_0^\infty(\mathbb{R}^2), \\
\rho^t(\xi) &= \xi \text{ for all } |\xi| \leq \varepsilon |c(t)|, \\
\rho^t(\xi) &= 0 \text{ for all } |\xi| \geq \varepsilon |c(t)| / 3.
\end{aligned}
\]

Hence, we define the following map

$$\pi : T^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J \rightarrow T^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J$$
$$\pi(\theta, \xi, r, \eta, t) = (\theta, \rho^t(\xi), r, \eta, t).$$

It is straightforward to verify that

$$\pi \left( T^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J \right) \subset \mathcal{U}$$
$$\pi|_{U_{\frac{1}{2}}} = \text{Id}.$$

Now, we define

$$H_{ex} = H_c \circ \tilde{\phi} \circ \pi. \quad (6.21)$$

Obviously, (6.18) is satisfied because $\pi|_{U_{\frac{1}{2}}} = \text{Id}$. This proves the first part of this lemma. Concerning the second part, we observe that, for all fixed $t \in J$ and $k \in \mathbb{Z}$ with $k \geq 0$

$$|\rho^t|_{C^k} \leq \frac{C(k)}{(\varepsilon |c(t)|)^k} \leq C(k) \left( \frac{t}{\varepsilon |c(t)|} \right)^k \leq C(k)$$

for a suitable constant $C(k)$ depending on $k$. We point out that the second inequality of the previous estimate is due to $t \geq 1$ and the last is a consequence of Proposition 6.1. Thanks to the latter, one can see that

$$\sup_{t \in J} |\nabla \pi^t|_{C^k} < \infty.$$
Now, we have everything we need to prove the second part of this lemma. We begin with the case $k = 0$. Let $B^6 = B^4 \times B^2 \subset \mathbb{R}^4 \times \mathbb{R}^2$. For all $t \in J$, we observe that by (6.21) and Lemma 6.5

$$|H^t_{ex}|_{C^0(T^4 \times \mathbb{R}^2 \times B^6)} = \left| \left( H_c \circ \tilde{\phi} \circ \pi \right)^t \right|_{C^0(T^4 \times \mathbb{R}^2 \times B^6)} \leq |H^t_c|_{C^0(\phi(T^4 \times B^2 \times B^6))} \leq CMm_c \varepsilon$$

where $B^2_t$ is defined by (6.11). For the sake of clarity, we have specified the domain where the Hölder norms are taken. Now, concerning the case $k \geq 1$, for all $t \in J$

$$|H^t_{ex}|_{C^k(T^4 \times \mathbb{R}^2 \times B^6)} = \left| \left( H_c \circ \tilde{\phi} \circ \pi \right)^t \right|_{C^k(T^4 \times \mathbb{R}^2 \times B^6)} \leq |H^t_c|_{C^k(\phi(T^4 \times B^2 \times B^6))} + |H^t_c|_{C^k(\phi(T^4 \times B^2 \times B^6))} |\nabla \pi^t|_{C^{k-1}(T^4 \times \mathbb{R}^2 \times B^6)} + |H^t_c|_{C^k(\phi(T^4 \times B^2 \times B^6))} |\nabla \pi^t|_{C^{k-1}(T^4 \times \mathbb{R}^2 \times B^6)} \leq C \left( k, \sup_{t \in J} |\nabla \pi^t|_{C^{k-1}(T^4 \times \mathbb{R}^2 \times B^6)} \right) Mm_c \varepsilon.$$

We note that the first inequality (second line) is due to property 3. of Proposition 3.1 while the last line is a consequence of Lemma 6.5. This proves (6.19). Similarly, one can prove (6.20).

The rest of this section is devoted to showing that the Hamiltonian

$$\tilde{H} : T^4 \times \mathbb{R}^2 \times B^4 \times B^2 \longrightarrow \mathbb{R},$$

$$\tilde{H} = H_0 \circ \phi + H_{ex}$$

satisfies the hypothesis of Theorem [A], where $H_0$ is the Hamiltonian of the planar three-body problem. Obviously, by the previous lemma

$$\tilde{H}|_{U_1^2} = H \circ \tilde{\phi}.$$  \hspace{1cm} (6.22)

First, let us recall some definitions introduced at the beginning of this chapter. Let $\sigma \geq 0$ be a positive real parameter

**Definition.** Let $S_\sigma$ be the space of functions $f$ defined on $T^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J$ such that $f^t \in C^\sigma(T^4 \times \mathbb{R}^2 \times B^4 \times B^2)$ for all fixed $t \in J$ and $\partial_{(x,p)} f \in C(T^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J)$ for all $0 \leq i \leq |\sigma|$.

For every $f \in S_\sigma$, we define the following norm

$$|f|_{\sigma,l} = \sup_{t \in J} |f^t|_{C^\sigma} t^l$$

for a real positive parameter $l$. Obviously, for all $t \in J$, the above norm $| \cdot |_{C^\sigma}$ is taken on $T^4 \times \mathbb{R}^2 \times B^4 \times B^2$.
Lemma 6.7. We can rewrite $H_{\text{ex}}$ in the following form

$$H_{\text{ex}}(\theta, \xi, r, t) = a(\theta, \xi, t) + b(\theta, \xi, t) \cdot r + R_c(\theta, \xi, r, t) \cdot r^2,$$

(6.23)

for all $(\theta, \xi, r, t) \in \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times J$. Moreover, for all $k \in \mathbb{Z}$ with $k \geq 1$

$$|a|_{k+1,0} + |\partial_{(\theta, \xi)}a|_{k,2} < C(k)Mm \epsilon,$$

$$|b|_{k+1,2} < C(k)Mm \epsilon,$$

$$|\partial_r^2 H_{\text{ex}}|_{k+1,2} < C(k)Mm \epsilon,$$

for some constants $C(k)$ depending on $k$.

Proof. Expanding $H_{\text{ex}}$ in a small neighborhood of $r = 0$

$$H_{\text{ex}}(\theta, \xi, r, \eta, t) = H_{\text{ex}}(\theta, \xi, 0, t) + \partial_r H_{\text{ex}}(\theta, \xi, 0, t) \cdot r + \int_0^1 (1 - \tau) \partial_r^2 H_{\text{ex}}(\theta, \xi, \tau r, t) d\tau \cdot r^2$$

and letting

$$a(\theta, \xi, t) = H_{\text{ex}}(\theta, \xi, 0, t),$$

$$b(\theta, \xi, t) = \partial_r H_{\text{ex}}(\theta, \xi, 0, t)(\theta, \xi, 0, \eta, t),$$

$$R_c(\theta, \xi, r, t) = \int_0^1 (1 - \tau) \partial_r^2 H_{\text{ex}}(\theta, \xi, \tau r, t) d\tau$$

we prove the first part of this lemma. The second part is a straightforward consequence of Lemma 6.6. More specifically,

$$|a|_{k+1,0} \leq \sup_{t \geq 1} |H^t_{\text{ex}}|_{C^{k+1}}$$

$$|\partial_{(\theta, \xi)}a|_{k,2} \leq \sup_{t \geq 1} |\partial_{(\theta, \xi)}H^t_{\text{ex}}|_{C^{k+1}t^2},$$

$$|b|_{k+1,2} \leq \sup_{t \geq 1} |\partial_r H^t_{\text{ex}}|_{C^{k+1}t^2},$$

$$|\partial_r^2 H_{\text{ex}}|_{k+1,2} \leq \sup_{t \geq 1} |\partial_r^2 H^t_{\text{ex}}|_{C^{k+1}t^2}$$

and thanks to (6.19) and (6.20) we conclude the proof of this lemma.

Summarizing the contents of the previous sections, we conclude this part with the following lemma

Lemma 6.8. We can rewrite $\tilde{H}$ in the following form

$$\tilde{H} : \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J \rightarrow \mathbb{R}$$

$$\tilde{H}(\theta, \xi, r, \eta, t) = c + \omega \cdot r + a(\theta, \xi, t) + b(\theta, \xi, t) \cdot r + m(\theta, \xi, r, t) \cdot \left( \frac{r}{\eta} \right)^2$$

where

$$m(\theta, \xi, r, t) \cdot \left( \frac{r}{\eta} \right)^2 = R_c(\theta, \xi, r, t) \cdot r^2 + R_0(\theta, r) \cdot r^2 + \frac{|\eta|^2}{2M}.$$
We note that $c$, $\omega$ and $R_0$ are defined in Lemma 6.1, while $a$, $b$ and $R_c$ are introduced in Lemma 6.4 and $M = m_0 + m_1 + m_2$. Moreover, for all $k \in \mathbb{Z}$ with $k \geq 1$,

\[
|a|_{k+1,0} + |\partial (a, \xi) a|_{k,2} < C(k) M m_\epsilon, \quad (6.24)
\]

\[
|b|_{k+1,2} < C(k) M m_\epsilon, \quad (6.25)
\]

\[
|\partial^2_{(r,\eta)} \tilde{H}|_{k+1,0} < C(k, M, m_c, \epsilon, |\partial (x,y) H_0 \circ \phi|_{C^{k+2}}, |\nabla \phi|_{C^{k+2}}), \quad (6.26)
\]

for some constants $C(k)$ depending on $k$ and $C(k, M, m_c, \epsilon, |\partial (x,y) H_0 \circ \phi|_{C^{k+2}}, |\nabla \phi|_{C^{k+2}})$ depending on $k$, $M$, $m_c$, $|\partial (x,y) H_0 \circ \phi|_{C^{k+2}}$ and $|\nabla \phi|_{C^{k+2}}$.

**Proof.** The first part of this lemma is due to Lemmas 6.2 and Lemma 6.7. Concerning the estimates, the first two are proved in Lemma 6.7. It remains to prove the last one. First, we claim that for all $k \in \mathbb{Z}$ with $k \geq 1$, the symplectic change of coordinates $\phi$ satisfies

\[
|\nabla \phi|_{C^k} < \infty, \quad (6.27)
\]

where $\nabla \phi = (\nabla \phi_1, ..., \nabla \phi_b)$ is the transposed of the Jacobian of $\phi$. We will use this notation also for $\nabla \phi_0$ and $\nabla \phi_F$. By Proposition 3.1 concerning the properties of the Hölder norms and recalling that $\phi = \phi_0 \circ \phi_F$ (see (6.5))

\[
|\nabla \phi|_{C^k} = |\nabla \left( \phi_0 \circ \phi_F \right)|_{C^k} = |\nabla \phi_0 \circ \phi_F \left( \nabla \phi_F \right)^T|_{C^k} \leq C(k)|\nabla \phi_0|_{C^k}|\nabla \phi_F|_{C^k} \left( |\nabla \phi_F|_{C^{k+1}} + |\nabla \phi_F|_{C^0} + 1 \right),
\]

where $T$ stands for the transpose of a matrix. We know that $\phi_0$ is a linear transformation (see (6.2)), then $|\nabla \phi_0|_{C^k} < \infty$. On the other hand, $\phi_F$ is $C^\infty$- function, 1-periodic with respect to $\theta_i$ for all $0 \leq i \leq 4$, the variables $r$ vary on a bounded subspace and it is the identity with respect to $(\xi, \eta)$ (see Lemma 6.1 and (6.4)). This implies $|\nabla \phi_F|_{C^k} < \infty$ and hence the claim.

Now, concerning the inequality (6.26), by Lemma 6.7

\[
|\partial^2 H_{ex}|_{k+1,2} < C(k) M m_\epsilon.
\]

Now, we have to estimate $|\partial^2 (H_0 \circ \phi)|_{k+1,0}$. By the chain rule, (6.27) and properties 2. and 3. of Proposition 3.1

\[
|\partial^2 (H_0 \circ \phi)|_{C^{k+1}} \leq C \left( |\partial (x,y) H_0 \circ \phi|_{C^{k+1}} |\partial^2 \phi|_{C^{k+1}} + |\partial^2 (x,y) H_0 \circ \phi|_{C^{k+1}} |\partial \phi|_{C^{k+1}}^2 \right) \leq C(k, |\nabla \phi|_{C^{k+2}}) |\partial (x,y) H_0 \circ \phi|_{C^{k+2}}.
\]

We observe that $\partial (x,y) H_0 \circ \phi$ does not depend on $\xi$. Moreover, it is $C^\infty$, 1-periodic with respect to $\theta_i$ for all $0 \leq i \leq 4$ and the variable $(r, \eta)$ vary on $B^4 \times B^2$ that is bounded. This implies $|\partial (x,y) H_0 \circ \phi|_{C^{k+2}} < \infty$.

Now summarizing the previous estimates we obtain

\[
|\partial^2_{(r,\eta)} \tilde{H}|_{k+1,0} \leq |\partial^2 H_{ex}|_{k+1,0} + |\partial^2 (H_0 \circ \phi)|_{k+1,0} \leq C(k, M, m_\epsilon, |\partial (x,y) H_0 \circ \phi|_{C^{k+2}}, |\nabla \phi|_{C^{k+2}}).
\]

\[\square\]
6.7 Weakly asymptotically quasiperiodic solutions

In the previous section, for $k$ sufficiently large, Lemma 6.8 ensures that $\tilde{H}$ satisfies the hypotheses of Theorem A. Then, letting $\varphi_0 : T^4 \times R^2 \to T^4 \times R^2 \times B^4 \times B^2$ be the trivial embedding $\varphi_0(\theta, \xi) = (\theta, \xi, 0, 0)$, for $\varepsilon$ small enough, there exist $v_1 : T^4 \times R^2 \times J \to R^4$ and $v_2 : T^4 \times R^2 \times J \to R^2$ such that

$$\varphi^t(\theta, \xi) = (\theta, \xi, v_1^t(\theta, \xi), v_2^t(\theta, \xi))$$

(6.28)

is a $C^1$-weakly asymptotic cylinder associated to $(X_H, X_{H_{0\phi}}, \varphi_0)$. This means that there exist $\Gamma_1 : T^4 \times R^2 \times J \to R^4$ and $\Gamma_2 : T^4 \times R^2 \times J \to R^2$ such that, letting $\Gamma = (\Gamma_1, \Gamma_2)$ and $v = (v_1, v_2)$, for all $(\theta, \xi, t) \in T^4 \times R^2 \times J$

$$X_{H_{0\phi}}(\varphi(\theta, \xi, t), t) = \partial_{(\theta, \xi)} \varphi(\theta, \xi, t)(\omega + \Gamma(\theta, \xi, t)) + \partial_t \varphi(\theta, \xi, t),$$

(6.29)

and

$$|v|_{1,1} \leq \zeta, \quad |\Gamma|_{1,1} \leq \varepsilon + C \zeta$$

(6.30)

for a suitable constant $C$ and a small parameter $0 < \zeta \leq 1$ depending on $k$ and $\left| \partial_{(r, \eta)} \tilde{H} \right|_{k+1,0}$.

We recall that $B^2_1/2$ is defined in (6.17). Letting $\psi_{1,H}^t$ be the flow at time $t$ with initial time 1 of $H$, we have the following lemma.

Lemma 6.9. We assume

$$v > \frac{12 \left(1 + \tilde{C}\right)}{\varepsilon}$$

(6.31)

Then, for all $w \in \mathcal{W} = \phi \circ \varphi^1 (T^4 \times (B^2_1/2))$, $\psi_{1,H}^t(w)$ is a weakly asymptotically quasiperiodic solution associated to $(X_{H}, X_{H_{0\phi}}, \varphi_0)$.

Proof. Let $\psi_{1,\tilde{H}}^t$ be the flow at time $t$ with initial time 1 of $\tilde{H}$. Thanks to Proposition 4.4, we know that, for all $(\theta, \xi) \in T^4 \times R^2$, $\psi_{1,\tilde{H}}^t \circ \varphi^1(\theta, \xi)$ is a weakly asymptotically quasiperiodic solution associated to $(X_{\tilde{H}}, X_{H_{0\phi}}, \varphi_0)$.

Now, for all $(\theta, \xi) \in T^4 \times (B^2_1/2)$, we define

$$(\theta_1^t(\theta, \xi), \xi_1^t(\theta, \xi), r_1^t(\theta, \xi), \eta_1^t(\theta, \xi)) = \psi_{1,\tilde{H}}^t \circ \varphi^1(\theta, \xi)$$

for all $t \in J$. We begin by proving that, for all $(\theta, \xi) \in T^4 \times (B^2_1/2)$,

$$\psi_{1,\tilde{H}}^t \circ \varphi^1(\theta, \xi) \in \mathcal{U}_{2}$$

for all $t \in J$, where $\mathcal{U}_{2}$ is defined in (U_{2}). This is equivalent to show that, for all $(\theta, \xi) \in T^4 \times (B^2_1/2)$

$$(r_1^t(\theta, \xi), \eta_1^t(\theta, \xi)) \in B^4 \times B^2, \quad \left| \xi_1^t(\theta, \xi) \right| \leq \frac{\varepsilon}{6}$$

(6.32)

for all $t \in J$. Let $\psi_{1,\omega+\Gamma}^t$ be the flow at time $t$ with initial time 1 of $\omega + \Gamma$. We know that (6.29) is equivalent to

$$\psi_{1,\tilde{H}}^t \circ \varphi^1 = \varphi^t \circ \psi_{1,\omega+\Gamma}^t$$

(6.33)
for all \( t \in J \). This implies that, for all \((\theta, \xi) \in T^4 \times (B_1^2/2)\)
\[
(\theta_t^j(\theta, \xi), \xi_t^j(\theta, \xi), r_t^j(\theta, \xi), \eta_t^j(\theta, \xi)) = \psi_{1,\bar{H}}^t \circ \varphi^t(\theta, \xi) = \varphi^t \circ \psi_{1,\omega+1}^t(\theta, \xi)
\]
for all \( t \in J \). However, we observe that \( \varphi^t \) is the identity with respect to \((\theta, \xi)\) (see (6.28)). Then, thanks to the special form of \( \varphi^t \)
\[
r_t^j(\theta, \xi) = v_t^j(\theta_t^j(\theta, \xi), \xi_t^j(\theta, \xi)), \quad \eta_t^j(\theta, \xi) = v_t^j(\theta_t^j(\theta, \xi), \xi_t^j(\theta, \xi))
\]
for all \( t \in J \) and \((\theta, \xi) \in T^4 \times (B_1^2/2)\). Now, by (6.30)
\[
|v_t^j(\theta, \xi)| \leq v_1^j |C| \leq \frac{\zeta}{t} \leq \zeta, \quad |\eta_t^j(\theta, \xi)| \leq |v_2^j| |C| \leq \frac{\zeta}{t} \leq \zeta
\]
for all \( t \in J \). This proves the first part of (6.32). Concerning the second part,
thanks to (6.28) and (6.33), \( \xi_t^j(\theta, \xi) \) is the unique solution of the following system
\[
\begin{align*}
\dot{\xi}_t^j(\theta, \xi) &= \Gamma_2(\theta_t^j(\theta, \xi), \xi_t^j(\theta, \xi), t) \\
\zeta_t^j(\theta, \xi) &= \xi
\end{align*}
\]
where \( \dot{\xi}_t^j(\theta, \xi) \) stands for the derivative with respect to \( t \) of \( \xi_t^j(\theta, \xi) \). \( \Gamma_2 \) is defined by (6.29) and \( \xi \in (B_1^2/2) \). Now, thanks to (6.30) and the latter
\[
|\xi_t^j(\theta, \xi)| \leq |\xi_1^j(\theta, \xi)| + \int_1^t |\Gamma^c| |C| |c(t)| \leq |\xi| + (\varepsilon + C\zeta) \ln t \leq |\xi| + (1 + \bar{C}) \ln t
\]
for all \( t \in J \). We claim that
\[
\frac{|\xi_t^j(\theta, \xi)|}{|c(t)|} \leq \frac{|\xi| + (1 + \bar{C}) \ln t}{|c(1)| + \frac{\varepsilon}{2} (t - 1)} < \frac{\varepsilon}{6}
\]
for all \( t \in J \). It is true for \( t = 1 \). However, if we suppose the existence of \( t_0 > 1 \) in such a way that
\[
\frac{|\xi| + (1 + \bar{C}) \ln t_0}{|c(1)| + \frac{\varepsilon}{2} (t_0 - 1)} \geq \frac{\varepsilon}{6},
\]
then we can rewrite the latter in the following way
\[
|\xi| - \frac{\varepsilon}{6} |c(1)| \geq \frac{\varepsilon}{12} v(t_0 - 1) - (1 + \bar{C}) \ln t_0.
\]
We observe that this is a contradiction because \( \xi \in (B_1^2/2) \) implies \( |\xi| - \frac{\varepsilon}{6} |c(1)| < 0 \) and by (6.31) one can prove \( \frac{\varepsilon}{12} v(t - 1) - (1 + \bar{C}) \ln t > 0 \) for all \( t \in J \). This concludes the proof of (6.32). Then, for all \((\theta, \xi) \in (T^4 \times (B_1^2/2))\),
\[
\psi_{1,\bar{H}}^t \circ \varphi^t(\theta, \xi) \in U_{\frac{1}{2}}
\]
for all \( t \in J \). Hence, thanks to Lemma 6.6, for all \((\theta, \xi) \in (T^4 \times (B_1^2/2))\),
\[
\psi_{1,\bar{H}}^t \circ \varphi^t(\theta, \xi) = \psi_{1,\omega+1}^t \circ \varphi^t(\theta, \xi)
\]
for all \( t \in J \).
Now, for all $w \in \mathcal{W} = \phi \circ \varphi^1 (\mathbb{T}^4 \times (B^2_1/2))$ there exists $(\theta, \xi) \in \mathbb{T}^4 \times (B^2_1/2)$ such that $w = \phi \circ \varphi^1 (\theta, \xi)$. We know that $\phi$ is symplectic, then by (6.33) and the latter, we can rewrite $\psi_{1,H}^t (w)$ in the following way

$$\psi_{1,H}^t (w) = \psi_{1,H}^t \circ \phi \circ \varphi^1 (\theta, \xi) = \phi \circ \psi_{1,H}^t \circ \varphi^1 (\theta, \xi)$$

Moreover, for all $t \in J$

$$\left| \psi_{1,H}^t (w) - \phi \circ \varphi_0 \circ \psi_{1,\omega+\Gamma}^t (\theta, \xi) \right| \leq \left| \phi \circ \varphi^t \circ \psi_{1,\omega+\Gamma}^t (\theta, \xi) - \phi \circ \varphi_0 \circ \psi_{1,\omega+\Gamma}^t (\theta, \xi) \right|$$

$$\leq \left| \phi \circ \varphi^t - \phi \circ \varphi_0 \right|_{C^0}$$

$$\leq C |\nabla \varphi|_{C^1} |\varphi^t - \varphi_0|_{C^0}$$

for a suitable constant $C$. Therefore, reminding that $|\nabla \varphi|_{C^1} < \infty$ and taking the limit for $t \to +\infty$, thanks to (6.30), we conclude the proof of this lemma.

This concludes the proof of Theorem B.

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