Russell on Weyl’s unified field theory

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In 1918, H. Weyl proposed a unified theory of gravity and electromagnetism based on a generalization of Riemannian geometry. With hindsight we now could say that the theory carried with it some of the most original ideas that inspired the physics of the twentieth century. In a book published in 1927, Bertrand Russell devoted an entire chapter to explain and give a critical appraisal of Weyl’s theory. We briefly revisit the text written by Russell, who gave one of the first philosophical approaches to Weyl’s ideas.

I. INTRODUCTION

Weyl’s attempt at unifying gravity and electromagnetism came to light in 1918 [1]. With hindsight we now could say that the theory carried with it some of the most original ideas that inspired the physics of the twentieth century. Indeed, it gave us a clue of how is possible, by extending already known geometrical frameworks, to achieve unification of the gravitational and non-gravitational physical fields, the latter being viewed as fundamental ingredients of the spacetime geometry. In addition to this innovative result, his search for hidden symmetries of space-time, somehow implied by the new geometry contained the germs of the modern gauge theories of elementary particles [2].

Although Weyl’s theory was not considered by Einstein to constitute a viable physical theory, the powerful and elegant ideas put forward by the publication of Weyl’s paper survived and now constitutes a constant source of inspiration for new proposals, particularly in the domain of the so-called “modified gravity theories” [3].

Despite Einstein’s objections, Weyl’s unified theory attracted the attention of some eminent contemporary physicists of Weyl, among whom we can quote Pauli, Eddington, London, and Dirac [4]. However, the great majority of theoretical physicists in the first decades of the twentieth century remained completely unaware of Weyl’s work. Therefore, it would not be expected the general public of the time to have any possibility of accessing Weyl’s theory unless some influential and well informed science populariser decided to write about it. It turned out that an author with that profile did appear in scene and happened to be no one else than the renowned British philosopher and mathematician Bertrand Russell, who was one of the best public writer of the time.

In his book “The Analysis of Matter”, published in 1927, Russell dedicates an entire chapter devoted to explain and give a critical appraisal of Weyl’s theory [5]. The book was one of the earliest and best philosophical investigations of the physics of relativity and quantum mechanics, written by someone who managed to master the physics, mathematics and psychology of his time. This is particularly true with respect to theoretical physics, a subject about what Russell had a special interest and was always aware of the latest developments.

The paper is organised as follows. In Section 2, we give a very brief account of the essentials of Weyl’s theory. We then proceed to Section 3 to review Einstein’s argument and examine in more detail the assumptions implicitly made therein. In Section 4, we discuss Russell’s appraisal of Weyl’s theory and how he saw some of the problems posed by Weyl’s concerning the actual meaning of the process of measuring time (or length) in the context of space-time theories that still need to be clarified. We conclude with some remarks in Section 5.

II. WEYL’S THEORY BRIEFLY EXPLAINED

Weyl’s theory is essentially based upon a modification of the geometry assumed to model the space-time of the general theory of relativity, namely, Riemannian geometry. It is perhaps one of the simplest generalization of the latter, the only modification being that the length of a vector field may change when parallel-transported along a curve in the manifold [4], this change being regulated by a one-form field $\sigma$ defined on the space-time manifold. Let us briefly explain this new geometric property by considering a closed curve $\alpha$ in the space-time manifold. If $L_0$ and
\( L \) denotes, respectively, the values of the initial and the final length of a vector parallel-transported along \( \alpha \), then it follows that

\[
L = L_0 e^{\oint \sigma \alpha \, dx^\alpha}.
\]

From this assumption, Weyl showed that the field \( \sigma \) possesses amazing similarities with the electromagnetic 4-potential vector field, and that was the way he found to geometrise electrodynamics.

### III. EINSTEIN’S OBJECTION

As is well known, in an appendix to Weyl’s paper, Einstein set forth a serious objection to the theory. In his critique, Einstein argued that the theory predicts the existence of the so-called “second clock effect” [7]. According to Einstein, in a space-time ruled by Weyl geometry the existence of sharp spectral lines in the presence of an electromagnetic field would not be possible since atomic clocks would depend on their past history. Einstein argued that this predicted effect is a logical consequence of Weyl’s theory, insofar as in a Weyl space-time the length of a vector is not held constant by parallel transport, and this, in turn, would imply that the clock rate of atomic clocks, measured by some periodic physical process, would be path dependent.

In order to examine Einstein’s objection to Weyl’s unified theory, let us first recall two of the hypotheses on which the argument is based. They can be stated as follows:

**H1)** The proper time \( \Delta \tau \) measured by a clock travelling along a curve \( \alpha = \alpha(\lambda) \) is given as in general relativity, namely, by the (Riemannian) prescription

\[
\Delta \tau = \frac{1}{c} \int [g(V, V)]^{1/2} d\lambda = \frac{1}{c} \int [g_{\mu\nu} V^\mu V^\nu]^{1/2} d\lambda,
\]

where \( V \) denotes the vector tangent to the clock’s world line and \( c \) is the speed of light. This supposition is known as the clock hypothesis and clearly assumes that the proper time only depends on the instantaneous speed of the clock and on the metric field.

**H2)** The fundamental clock rate of clocks (in particular, atomic clocks) is to be associated with the (Riemannian) length \( L = \sqrt{g(\Upsilon, \Upsilon)} \) of a certain vector \( \Upsilon \). As a clock moves in space-time \( \Upsilon \) is parallel-transported along its worldline from a point \( P_0 \) to a point \( P \), hence \( L = L_0 e^{\oint \sigma \alpha \, dx^\alpha} \), \( L_0 \) and \( L \) indicating the duration of the clock rate of the clock at \( P_0 \) and \( P \), respectively.

### IV. RUSSELL’S COMMENT ON THE PROBLEM OF LENGTH MEASUREMENT

The key point of Einstein’s objection, namely, the existence of the second clock effect relies entirely on the assumption that the physical process of measuring length (or time) is well defined. This very subtle question did not go unnoticed by Russell. In fact, in the chapter he wrote on general relativity he starts by pointing out that he agrees with Weyl in that the measurement of lengths of vectors at points which are not infinitesimally close should be carried out by direct comparison. However, according to Russell the physical process of doing this measurement is not the same of the purely mathematical process of parallel transport of vectors. As far as we are doing pure mathematics, this question does not appear. Nevertheless, here we are concerned with an experimental science, and thus it seems that the question is preliminary to any interpretation of the theory and, ipso facto, should be addressed properly. (Of course, by length we mean here length in the sense of the space-time manifold, so that it includes the notion, for instance, of proper time and clock rates).

We then see that two assumptions are implicit in the argument used by Einstein against Weyl’s theory. The first, as we have already mentioned, refers to the assignment of a vector the length of which gives the clock rate of a clock carried by an observer. This assumption is also made by J. Ehlers, F. Pirani and A. Schild (EPS) in a well known paper, in which the authors starting from a few plausible physical hypothesis conclude that the geometry of space-time should be given by an integrable version of Weyl geometry [8]. The second assumption concerns the identification of the real physical transport of a clock with the purely mathematical operation of parallel transport of a vector, and here we meet Russell’s critical objection. At this point, we should mention Weyl’s unease with both the first and second assumptions mentioned above. Indeed, he pointed out that the physics behind the mechanism of a real clock was not known, and this lack of knowledge does not allow us to accept the “geometrization” of clock rates, as we can infer from Weyl’s words [1]:
At first glance it might be surprising that according to the purely close-action geometry, length transfer is non-integrable in the presence of an electromagnetic field. Does this not clearly contradict the behaviour of rigid bodies and clocks? The behaviour of these measurement instruments, however, is a physical process whose course is determined by natural laws and as such has nothing to do with the ideal process of ‘congruent displacement of spacetime distance’ that we employ in the mathematical construction of the spacetime geometry. The connection between the metric field and the behaviour of rigid rods and clocks is already very unclear in the theory of Special Relativity if one does not restrict oneself to quasi-stationary motion. Although these instruments play an indispensable role in praxis as indicators of the metric field, (for this purpose, simpler processes would be preferable, for example, the propagation of light waves), it is clearly incorrect to define the metric field through the data that are directly obtained from these instruments.

Russell starts his critical analysis of Weyl’s theory by drawing a clear distinction between a real physical transport of a clock and the mathematical procedure of parallel transport. He provides a detailed and rigorous discussion on the problem of length measurement in general relativity, after which he remarks that the theory raises problems which bring us naturally to Weyl’s relativistic theory of electromagnetism. In a certain sense, in order to avoid Einstein’s objection, Weyl takes a stand regarding Russell’s comment about the need to distinguish the ideal (or mathematical) process of congruent transference of lengths from the real behaviour of measuring rods and clocks.

At this point, it is worth mentioning that Dirac, who revived Weyl’s theory some years later, also argued that the time interval measured by atomic clocks need not to be identified with the length of timelike vectors, as proposed by Einstein. In other words, it is questionable whether clock rates are correctly modelled by the length of a certain timelike vector, in complete accord with Russell.

However, in his critique of Weyl’s theory he goes further by saying that the theory has not been expressed with the logical purity that is to be desired, chiefly because it is not prefaced by any clear account of what is to be understood by “measurement”.

V. FINAL REMARKS

From the above we see that Russell’s objections are deep and preliminary to any serious attempt to give Weyl’s theory a precise physical meaning, and therefore are at the very heart of the debate between Einstein and Weyl. And, as we can see, all this discussion is at the root of the question whether the second clock effect exists or not. As far as we know, neither theoretical calculations nor any experimental attempt at measuring the magnitude of the predicted second clock effect had been carried out until very recently. In fact, it is still not known by experience whether atomic clocks, which in principle may define units of time, are integrably transported.

Incidentally, it should be pointed out that in Weyl’s parallel transport the ratio between the length of two vector transported along the same path is constant, according to the law \( L = L_0 e^{\frac{1}{2} \int \sigma \, ds} \). Therefore, we see that the “first clock effect” (i.e. the twin paradox) is not affected by Weyl geometry, since the standard unit of time follows the same variation as the total time elapsed along the path.

Finally, despite the controversy concerning its physical viability, firstly raised by Einstein, and also the philosophical difficulties pointed out by Russell against Weyl theory, we believe that the essential features of Weyl’s theory remain untouched. As some authors have put it: "Weyl geometrical theory contains a suggestive formalism and may still have the germs of a future fruitful theory.” Weyl’s theory was virtually the first robust attempt at unifying an important part of theoretical physics: gravity and electromagnetism. It is undeniable that the quest for unification of physics continues, now more than ever. In this respect it is worth remembering Russell’s words written in his essay of 1927: It is not chimerical to hope that a unified treatment of the whole of physics may be possible before many years have passed.

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1 As Russel points out, in essence parallel transport tries to adapt the concept of "rigid" transportation of geometrical objects to curved spaces.
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