Finite temperature corrections in 2d integrable models

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Abstract

We study the finite size corrections for the magnetization and the internal energy of the 2d Ising model in a magnetic field by using transfer matrix techniques. We compare these corrections with the functional form recently proposed by Delfino and LeClair-Mussardo for the finite temperature behaviour of one-point functions in integrable 2d quantum field theories. We find a perfect agreement between theoretical expectations and numerical results. Assuming the proposed functional form as an input in our analysis we obtain a relevant improvement in the precision of the continuum limit estimates of both quantities.

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1 Introduction

Despite the impressive progress of these last years, 2d quantum field theories (QFT) still provide many interesting open problems. One of these is the finite temperature behaviour of correlators. Besides its experimental and theoretical relevance, this issue is particularly important when these theories are studied by means of numerical simulations. In fact choosing a finite temperature setting corresponds, in the euclidean formulation of the theory, to compactify the (imaginary) time direction on a circle whose circumference $R$ coincides with the inverse temperature. Thus the finite temperature corrections become, in this framework, finite size corrections in the cylinder geometry which, as it is well known, play a crucial role in the extraction of continuum limit expectation values from numerical simulations.

While for a generic 2d QFT finding a functional form for the finite temperature correction seems a hopeless task, recently two interesting proposals by Delfino [1] and LeClair and Mussardo [2] (based on a previous work by LeClair and collaborators [3]) appeared in the literature to address this problem in the simpler case of integrable models. In particular, in [1, 2], the authors studied the finite temperature behaviour of one-point correlation functions. General arguments suggest that, outside the critical point, any one-point function evaluated on a finite size lattice $\langle \Phi \rangle_R \to \infty$ should approach its infinite volume limit $\langle \Phi \rangle_{R=\infty}$ with an exponential decay of the type

$$\langle \Phi \rangle_R \sim \langle \Phi \rangle_{R=\infty} + Ce^{-\frac{\xi}{2}} + \cdots$$

(1)

$\xi$ being the correlation length (i.e. the inverse of the lowest mass of the theory). However with these considerations nothing can be said on the constant $C$ and the higher order terms in the above equation.

The main achievement of [1, 2] was to show that the constant $C$ is indeed universal and to predict its value. At the same time they were able to give an explicit expression for higher order corrections.

The two proposals [1] and [2], have different theoretical starting points and indeed the predictions for these higher order corrections turn out to be different in the two approaches (for a critical comparison and a discussion of these differences see [4]). However, if one looks at the first few orders in a low temperature expansion (those which do not involve multi-particle form factors) the results coincide.
The aim of this paper is to test these proposals in a particular integrable model: the 2d Ising model in a magnetic field. This model has recently been the subject of several theoretical and numerical studies [4]-[8]. In particular, in the numerical analyses (performed using transfer matrix techniques [4] or montecarlo simulations [7, 8]) a great attention was devoted to the analysis of finite size correction. However, despite this careful treatment these correction were the major source of uncertainty in the final results. Our goal in this paper is thus twofold: first we shall use the numerical results for finite size lattices to test the proposals of [1, 2]. Second, we shall see if, assuming the functional form for the finite size corrections proposed in [1, 2] we can improve the precision of the continuum limit results for the one-point correlation functions.

This paper is organized as follows. In sect.2 we shall briefly discuss the model, the observables and the proposal of [1, 2]. Sect.3 will be devoted to a discussion of our numerical test and to a comparison with [1, 2]. Finally sect.4 will be devoted to some concluding remarks.

2 Ising model in a magnetic field

The Ising model in a magnetic field is defined by the partition function

$$Z = \sum_{\sigma_i = \pm 1} e^{\beta(\sum_{\langle n,m \rangle} \sigma_n \sigma_m + H \sum_n \sigma_n)}$$

(2)

where the field variable $\sigma_n$ takes the values $\{\pm 1\}$; $n \equiv (n_0, n_1)$ labels the sites of a square lattice size $L_0$ and $L_1$ in the two directions and lattice spacing $a$. $\langle n, m \rangle$ denotes nearest neighbour sites on the lattice. In the following we shall treat asymmetrically the two directions. We shall denote $n_0$ as the compactified “time” coordinate and $n_1$ as the space one. The number of sites of the lattice will be denoted by $N \equiv L_0 L_1$. The lattice extent in the transverse direction will be denoted as $R \equiv L_0$. This length will play a major role in the following. In fact our goal will be to describe the $R$ dependence of the expectation values of the spin and energy operators of the theory.

\footnote{Since the lattice spacing will play no role in the following we shall set $a = 1$ in the rest of the paper.}
In order to select only the magnetic perturbation, the coupling $\beta$ must be fixed to its critical value

$$\beta = \beta_c = \frac{1}{2} \log (\sqrt{2} + 1) = 0.4406868...$$

by defining $h_l = \beta_c H$ we end up with

$$Z(h_l) = \sum_{\sigma_i = \pm 1} e^{\beta_c \sum_{(n,m)} \sigma_m \sigma_m + h_l \sum_n \sigma_n}.$$  \hspace{1cm} (3)

In the continuum limit the model is described by the action:

$$A = A_0 + h \int d^2x \sigma(x).$$  \hspace{1cm} (4)

where $\sigma(x)$ is the perturbing operator and the perturbing field is the magnetic field $h$.

In this paper we want to use results obtained in the continuum theory to describe the finite size scaling behaviour of the lattice model. In general this would require a precise definition of the relations between lattice and continuum quantities like for instance that between $h_l$ which appears in eq.(3) and the perturbing field $h$. However, as we shall see below, we shall only be interested in adimensional ratios in which all these normalizations cancel out. Thus we shall neglect this problem in the following. The interested reader can find a careful treatment of this issue in [6].

2.1 Lattice operators

As a first step of our analysis let us define the lattice analogous of the spin and energy operators of the continuum theory\footnote{Strictly speaking these lattice operators do not correspond exactly to the continuum ones but are instead linear combinations of all possible relevant and irrelevant operators of the continuum theory with compatible symmetry properties with respect to the $Z_2$ symmetry of the model (odd for the spin operator and even for the energy one). However near the critical point this linear combination is dominated by the corresponding relevant operator ($\sigma$ for $\sigma_l$ and $\epsilon$ for $\epsilon_l$) and the only remaining freedom will be a conversion constant relating the continuum and lattice versions of the two operators. As mentioned above we shall neglect in the following these normalization constants.}. The simplest choices for these lattice analogous are
• Spin operator

\[ \sigma_l(x) \equiv \sigma_x \]  

(5)

i.e. the operator which associates to each site of the lattice the value of the spin in that site.

• Energy operator

\[ \epsilon_l(x) \equiv \frac{1}{4} \sigma_x \left( \sum_{y \text{ n.n. } x} \sigma_y \right) - \epsilon_b \]  

(6)

where the sum runs over the four nearest neighbour sites \( y \) of \( x \). \( \epsilon_b \) represents a constant “bulk” term which we shall discuss below \[ \text{[3]} \].

The index \( l \) indicates that these are the lattice discretizations of the continuous operators. We shall denote in the following the normalized sum over all the sites of these operators simply as

\[ \sigma_l \equiv \frac{1}{N} \sum_x \sigma_l(x) \quad \epsilon_l \equiv \frac{1}{N} \sum_x \epsilon_l(x) \quad . \]  

(7)

In the following we shall be interested in the expectation value of these lattice operators. More precisely we shall be interested in:

• Magnetization

The magnetization per site \( M(h_l) \) defined as

\[ M(h_l) \equiv \frac{1}{N} \frac{\partial}{\partial h_l} \left( \log Z(h_l) \right) |_{\beta = \beta_c} = \frac{1}{N} \langle \sum_i \sigma_i \rangle \]  

(8)

which implies

\[ M(h_l) = \langle \sigma_l \rangle \quad . \]  

(9)

\[ ^3\text{As a matter of fact, for technical reasons, in our Transfer Matrix calculations we evaluated only the “time-like” part of the action i.e. } \sum_{n_0,n_1} \sigma_{(n_0,n_1)} \sigma_{(n_0+1,n_1)}. \text{ While, obviously, this choice makes no difference in the thermodynamic limit, it might have some effect for finite values of } R. \text{ We shall further discuss this point in sect.3.1 below.} \]
• Internal Energy

The internal energy density $\hat{E}(h_l)$ defined as

$$\hat{E}(h_l) \equiv \frac{1}{2N} \langle \sum_{(n,m)} \sigma_n \sigma_m \rangle.$$  (10)

It is important to stress that in this case one also has to take into account a bulk analytic contribution (as it happens also for the free energy itself) $E_b(h_l)$ which is an even function of $h_l$. Let us define $\epsilon_b \equiv E_b(0)$. The value of $E_b(0)$ can be easily evaluated (for instance by using Kramers-Wannier duality) to be $\epsilon_b = \frac{1}{\sqrt{2}}$. Let us define $E(h_l) \equiv \hat{E}(h_l) - \epsilon_b$, we have

$$E(h_l) = \frac{1}{2N} \langle \sum_{(n,m)} \sigma_n \sigma_m \rangle - \frac{1}{\sqrt{2}}.$$  (11)

Hence we have

$$E(h_l) = \langle \epsilon_l \rangle.$$  (12)

In the following we shall in particular be interested in the ratios

$$\frac{\langle \sigma_l \rangle_R}{\langle \sigma_l \rangle_{R=\infty}} \quad \frac{\langle \epsilon_l \rangle_R}{\langle \epsilon_l \rangle_{R=\infty}}.$$  (13)

$\langle \Phi_l \rangle_R$ being the mean value on a lattice with transverse extent $R$ of the lattice operator $\Phi_l$. Since in the ratio all the normalization constants cancel out, we can identify the two ratios with the analogous ones evaluated in the continuum theory, with the continuum operators.

$$\frac{\langle \Phi_l \rangle_R}{\langle \Phi_l \rangle_{R=\infty}} = \frac{\langle \Phi \rangle_R}{\langle \Phi \rangle_{R=\infty}}.$$  (14)

Notice that to perform this identification it is mandatory to eliminate the bulk correction as we did in eq.(11)

2.2 Critical behaviour

The critical behaviour of magnetization and internal energy can be easily obtained by means of standard renormalization group methods. One finds

$$M \propto |h|^d/\gamma_n - 1.$$  (15)
\[ E \propto |h|^{(d-y_t)/y_h} \quad (16) \]

where \( y_h, y_t \) are the magnetic and thermal RG-exponents respectively.

From the exact solution of the Ising model at the critical point we know that \( y_h = \frac{15}{8} \) and \( y_t = 1 \). Inserting these values in the above expressions we find

\[ M \propto |h|^{\frac{1}{15}} \quad (17) \]

\[ E \propto |h|^{\frac{8}{15}} \quad (18) \]

From the conformal field theory (CFT) description of the Ising model at the critical point it is possible to construct also the \( h \) dependence of the following terms in the two scaling functions (17) and (18). A detailed analysis can be found in [6]. We shall make use of this result in the following.

### 2.3 S-matrix results

In 1989 A. Zamolodchikov [9] suggested that the scaling limit of the Ising Model in a magnetic field could be described by a scattering theory which contains eight different species of self-conjugated particles \( A_a, a = 1, \ldots, 8 \) with masses

\[
\begin{align*}
m_2 &= 2m_1 \cos \frac{\pi}{5} = (1.6180339887..) m_1, \\
m_3 &= 2m_1 \cos \frac{\pi}{30} = (1.9890437907..) m_1, \\
m_4 &= 2m_2 \cos \frac{7\pi}{30} = (2.4048671724..) m_1, \\
m_5 &= 2m_2 \cos \frac{2\pi}{15} = (2.9562952015..) m_1, \\
m_6 &= 2m_2 \cos \frac{\pi}{30} = (3.2183404585..) m_1, \\
m_7 &= 4m_2 \cos \frac{\pi}{5} \cos \frac{7\pi}{30} = (3.8911568233..) m_1, \\
m_8 &= 4m_2 \cos \frac{\pi}{5} \cos \frac{2\pi}{15} = (4.7833861168..) m_1,
\end{align*}
\]  

(19)

where \( m_1(h) \) is the lowest mass of the theory. which coincides with the inverse of the (exponential) correlation length. After Zamolodchikov’s paper several
other interesting results were obtained, ranging from the explicit values of various critical amplitudes \([1, 11, 12]\) to the values of the overlap amplitudes of correlators \([13, 14]\). All these predictions have been tested with numerical simulations both in the 1d Ising quantum chain \([15]\), in the dilute \(A_3\) IRF (Interaction Round a Face) model \([16]\) (which is another realization of the scaling Ising model in a magnetic field) and directly in the 2d lattice Ising model \([6, 7, 8, 17, 18]\) and in all cases a full agreement between S-matrix predictions and numerical results was found.

2.4 Delfino and LeClair-Mussardo proposals

One of the most interesting features of the approach discussed in \([1]\) is that, by using the form-factor technology the author was able to give a very explicit and compact expression for the first few orders of the finite size corrections both for \(\langle \sigma \rangle\) and \(\langle \epsilon \rangle\) for the Ising model in a magnetic field. This result will be of great importance for our analysis.

According to \([1]\) a generic one point function \(\langle \Phi \rangle_R\) evaluated on a cylinder of transverse size \(R\) approaches exponentially its asymptotic value \(\langle \Phi \rangle_{R=\infty}\) with the following law.

\[
\frac{\langle \Phi \rangle_R}{\langle \Phi \rangle_{R=\infty}} = 1 + \frac{1}{\pi} \sum_{i=1}^{3} A_i^\Phi K_0(m_i R) + O(e^{-2m_1 R}) \tag{20}
\]

where \(m_1, m_2\), and \(m_3\) are the first three masses (the only ones below the lowest pair creation threshold) of the Zamolodchikov’s solution discussed above and \(K_0\) denotes the zeroth order modified Bessel function.

The major result of \([1]\) was to show that the \(A_i^\Phi\) constants are indeed universal and can be evaluated exactly in the framework of the S-matrix description of the model. Their values in the case of the two operators in which we are interested here are given in tab. \([1]\). As mentioned in the introduction these results turn out to agree at this order with those obtained by LeClair and Mussardo in \([2]\).

3 Numerical test

In order to test the above prediction we performed a numerical study of the finite size corrections by using the results of the transfer matrix analysis dis-
Table 1: The universal amplitudes entering the expansion (20) for the Ising model in a magnetic field.

| $\Phi$ | $\sigma$          | $\varepsilon$          |
|--------|-------------------|------------------------|
| $A_1^\Phi$ | $-8.0999744..$   | $-17.893304..$         |
| $A_2^\Phi$ | $-21.206008..$   | $-24.946727..$         |
| $A_3^\Phi$ | $-32.045891..$   | $-53.679951..$         |

cussed in [6] to which we refer for further details on the algorithm and on the raw data. We only recall here some general informations on the experiment. We studied the Ising model for the 19 different values of the external magnetic field listed in tab.2. For each choice of $h$ we studied lattices with values of $R$ ranging from 10 to 21. For each values of $h$ and each lattice size we evaluated the mean value of the magnetization and internal energy (see however the footnote below eq.(6)). We show in tab.3 a typical sample of our data. An important ingredient of eq.(20) are the values of the first three masses of the theory. In principle these masses could be evaluated directly from the S-matrix solution, using the suitable normalization constants to match with the lattice discretization and then imposing the standard $h$ dependence dictated by the RG analysis. However it is important to stress that at the level of precision of our analysis, these three masses show rather large corrections to scaling for the values of $h$ that we study. To avoid large systematic errors in the analysis of the finite size correction it is mandatory to take into account these corrections (they are discussed in great detail in [6]). An alternative route, which turns out to be much simpler, is to evaluate these masses, when possible, directly from the transfer matrix calculation. For the present analysis we used this second route. We report for completeness these values in tab.2 and refer to [6] for a detailed discussion of this table.

3.1 Analysis of the data

We performed a two steps analysis

1] First of all, for each value of $h$ we fitted the values of $<\sigma>$ and $<\epsilon>$ as a function of $R$, according to eq.(20): The results are reported in tab.4. Here are some comments on the fits:
Table 2: The first three masses.

| $h$  | $1/m_1$                   | $1/m_2$                   | $1/m_3$                   |
|------|---------------------------|---------------------------|---------------------------|
| 0.20 | 0.59778522553(1)          | 0.37795775263(1)          | 0.310888(1)               |
| 0.19 | 0.6138448719(1)           | 0.38765653507(1)          | 0.318578(1)               |
| 0.18 | 0.63134670477(1)          | 0.39818995529(1)          | 0.326940(1)               |
| 0.17 | 0.65037325706(1)          | 0.40968266918(1)          | 0.336077(2)               |
| 0.16 | 0.67120940172(1)          | 0.42228634593(5)          | 0.34615(3)                |
| 0.15 | 0.69415734924(1)          | 0.4361773124(1)           | 0.357209(3)               |
| 0.14 | 0.71959442645(1)          | 0.4516985381(4)           | 0.369548(4)               |
| 0.13 | 0.74798884641(1)          | 0.4688779288(2)           | 0.38338(1)                |
| 0.12 | 0.77987154161(1)          | 0.488342470(1)            | 0.3990(1)                 |
| 0.11 | 0.81637015277(1)          | 0.510513817(1)            | 0.4168(1)                 |
| 0.10 | 0.85239135695(5)          | 0.5360654(1)              | 0.4374(5)                 |
| 0.09 | 0.9071039295(1)           | 0.5659287(6)              | 0.4624(5)                 |
| 0.08 | 0.965123997(1)            | 0.60144(1)                | 0.492(1)                  |
| 0.075| 0.998514180(1)            | 0.62189(1)                | 0.508(1)                  |
| 0.066| 1.067300500(2)            | 0.66405(5)                | 0.543(1)                  |
| 0.055| 1.17524158(3)             | 0.7305(1)                 |                           |
| 0.044| 1.322589(6)               | 0.82(1)                   |                           |
| 0.033| 1.54057(2)                |                           |                           |
| 0.022| 1.91(1)                   |                           |                           |

- We always kept the asymptotic value of $<\sigma>$ and $<\epsilon>$ as a free parameter.
- The error of the input data was always of the order of $10^{-12}$. We started by fitting all the values of $R$ and then eliminated them one by one, starting from the smallest ones, until we reached a reduced $\chi^2$ equal or smaller than 1. Then we accepted the result of the fit and stopped the analysis. These acceptable $\chi^2$’s could be achieved for $h \geq 0.075$ keeping only the first term in eq.(20). For $h = 0.066.., 0.055.., 0.044..$ we had to introduce also the second mass, and finally for $h = 0.033.., 0.022..$ all the three terms were needed. The values of the masses which could not be directly

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4This trend is due to the fact that we studied, for each value of $h$, the same range of values of $R$ while $m_1$ decreases as $h$ goes to 0. This means that the argument of the first
Table 3: Magnetization and Internal Energy at $h = 0.075$

| $R$ | $M$           | $E$           |
|-----|---------------|---------------|
| 10  | 0.884069072534 | 0.122792065846 |
| 11  | 0.884096308760 | 0.122797848782 |
| 12  | 0.884105802401 | 0.122799890318 |
| 13  | 0.884109130761 | 0.122800613003 |
| 14  | 0.884110302840 | 0.122800869443 |
| 15  | 0.884110717053 | 0.122800960633 |
| 16  | 0.884110863864 | 0.122800993122 |
| 17  | 0.884110916027 | 0.122801004716 |
| 18  | 0.884110934601 | 0.122801008861 |
| 19  | 0.884110942262 | 0.122801010344 |
| 20  | 0.884110943594 | 0.122801010875 |
| 21  | 0.884110944441 | 0.122801023168 |

evaluated via transfer matrix (the empty spaces in tab.2) were evaluated using the S-matrix results and keeping into account the proper corrections to scaling.

- For all the values of $h$ the asymptotic values that we obtained for $<\sigma>$ and $<\epsilon>$ from these fits are compatible within the errors with those quoted in [6]. In general the present estimates are more precise. This gain in precision becomes larger and larger as $h$ decreases. For instance for $h = 0.075$ we found

$$E = 0.122801011162(5) \quad M = 0.884110944919(2) \quad (21)$$

to be compared with the values reported in [6]

$$E = 0.1228010112(1) \quad M = 0.88411094491(1) \quad (22)$$

with an improvement of more than one order of magnitude in precision.

Let us stress that both this improvement and the fact that all our results agree within the errors (which means typically an agreement within 10 significative digits as in eq.s(21) and (22) ) with Bessel function in eq.(20) (i.e. $m_1 R$) decreases as $h \to 0$ thus allowing to detect, within the errors of our data also higher order terms of the equation.
those of ref [6] are rather impressive evidences of the correctness of eq. (20). In fact, as it can be seen by looking at tab.3 the finite size corrections which eq. (20) is able to describe are much larger (five order of magnitude!) than the uncertainties of the data: the corrections are of the order of $10^{-7}$ while the precision of the input data is $10^{-12}$.

- For all the values of $h$ we obtained stable and reliable values for $A^{\phi}_{1}$. These values are reported in tab.4. The same was not true for $A^{\phi}_{2}$. This is not strange. In all the similar analyses performed in [6] it was impossible to have reliable estimates for the amplitudes of the subleading exponents due to the large systematic deviations induced by the uncertainties in the leading corrections.

Looking at the data of tab.4 one can see that there are rather large corrections to scaling. This is true in particular for $A^{\phi}_{1}$ (for which it is possible that the rather large magnitude of the correction is a consequence of our asymmetric definition for the internal energy, see footnote below eq. (6)). In any case it is clear that in order to test the predicted values for $A^{\sigma}_{1}$ and $A^{\epsilon}_{1}$ it is mandatory to discuss these deviations. To this end we performed a second level of analysis

2] The aim of this second level of analysis is to use the data reported in tab.4 to reach the scaling limit values for $A^{\sigma}_{1}$ and $A^{\epsilon}_{1}$. This problem is exactly the same that we had to face in [6] to extract the continuum limit values of the critical amplitudes and thus we use here the same method developed and discussed in sect.6 of [6] to which we refer the interested reader for further details.

We used as fitting functions

$$A^{\sigma}_{1}(h_{l}) = A^{\sigma}_{1}(1 + b_{1,\sigma}|h_{l}|^{\frac{16}{15}} + b_{2,\sigma}|h_{l}|^{\frac{22}{15}}) \quad (23)$$

5] The results of [6] were obtained with an iterative method (see sect.5.1 of ref.[6] and sect.3.1.2 of [17] for a discussion) based on the general assumptions on the finite size behaviour of one point functions summarized in eq.(1). In particular the discussion of sect.3.1.2 of [17] shows that the agreement with the present results is not a case but is due to the fact that the iterative method actually mimics the exact behaviour of eq.(20). It would be nice to see if this iterative algorithm keeps its predicting power even when the theory is not integrable and no exact prediction can be deduced from a S-matrix analysis.
(24)

\[ A_1'(h_l) = A_1'(1 + b_{1,\sigma}|h_l|^\frac{8}{15} + b_{2,\epsilon}|h_l|^\frac{16}{15}) \]

Differently from the cases studied in [6] keeping only two terms in the scaling function is enough due to the large errors in the input data. The final results are

\[ 8.090 < |A_1'| < 8.125 \]
\[ 17.0 < |A_1'| < 18.1 \]

which perfectly agree with the predictions of [1, 2] reported in tab.1

Then (similarly to what we did in [6]) we tried to extract the dominant correction to scaling by assuming as fixed inputs the values for \( A_1^\Phi \) reported in tab.1 and performing again the fits. The results are:

\[ 0.56 < b_{1,\sigma} < 0.66 \]
\[ -0.96 < b_{1,\epsilon} < -0.82 \]

As it also happens for the energy itself (see [6]) the leading correction to scaling in the energy sector is rather large and negative in sign.

### 3.2 An attempt to obtain \( A_2^\Phi \)

It is impossible to extract the \( A_2^\Phi \) by directly fitting. Moreover we cannot improve the situation by using the known continuum limit values of \( \langle \Phi \rangle \) and \( A_1^\Phi \) due to the large correction to scaling whose uncertainty is of the same order of magnitude of the terms that we would like to observe. The only possible way out is to construct a combination of the input data which exactly eliminates \( \langle \Phi \rangle \) and \( A_1^\Phi \). This can be easily done by combining the values of \( \Phi \) measured at three different values of the lattice size \( R_1, R_2, R_3 \).

The combination is the following:

\[ A_2^\Phi = \frac{\pi}{\langle \Phi \rangle_{R=\infty}} \frac{[\Phi(R_1) - \Phi(R_2)]\Delta_1(1, 3) - [\Phi(R_1) - \Phi(R_3)]\Delta_1(1, 2)}{\Delta_2(1, 2)\Delta_1(1, 3) - \Delta_2(1, 3)\Delta_1(1, 2)} \]

(29)

where

\[ \Delta_i(i, j) \equiv [K_0(m_iR_i - K_0(m_iR_j))]. \]

(30)

The result obtained using as input data those for \( h = 0.022034 \ldots \) for the magnetization are reported in tab.4. The quality of the result is rather good.
and is compatible with the prediction for this constant reported in tab. It
Notice however that the quality of the results becomes worse and worse as
$h$ increases (they are already almost completely unstable at $h = 0.055085...$)
moreover no result can be obtained for $\epsilon$ even at $h = 0.022034....$ This can
be easily understood. It due to the fact that the absolute value of $<\epsilon>$ is
much smaller than that of $<\sigma>$ and as a consequence the relative errors
(which are those which generate the observed instabilities) are larger. We
have checked that in our range of values of $h$ higher contributions in the
expansion of eq.(20) essentially give no effect.

### Table 4: Results of the fits.

| $h$   | $-A^2_0$     | $-A^2_1$    |
|-------|--------------|-------------|
| 0.20  | 9.186(1)     | 9.14(1)     |
| 0.19  | 9.1185(15)   | 9.625(15)   |
| 0.18  | 9.0365(15)   | 9.935(15)   |
| 0.17  | 8.9735(15)   | 10.24(1)    |
| 0.16  | 8.912(1)     | 10.55(1)    |
| 0.15  | 8.843(2)     | 10.97(2)    |
| 0.14  | 8.770(4)     | 11.44(3)    |
| 0.13  | 8.730(2)     | 11.665(10)  |
| 0.12  | 8.660(5)     | 12.035(10)  |
| 0.11  | 8.612(5)     | 12.34(4)    |
| 0.10  | 8.562(2)     | 12.64(2)    |
| 0.09  | 8.512(2)     | 12.95(2)    |
| 0.08  | 8.460(2)     | 13.35(5)    |
| 0.075 | 8.435(4)     | 13.52(5)    |
| 0.066103019026467 | 8.389(2) | 13.83(5) |
| 0.055085849188723  | 8.337(2) | 14.27(5)   |
| 0.044068679350978   | 8.288(2) | 14.65(5)   |
| 0.033051509513233   | 8.237(4) | 15.15(10)  |
| 0.022034339675489    | 8.206(20)| 15.68(15)  |

4 Concluding remarks

Let us briefly summarize our main results.
Our results nicely agree with the functional form for the finite size behaviour of the one-point functions proposed by Delfino and LeClair-Mussardo. We are also able to give a good estimate of the first correction to scaling terms, both for $\langle \sigma \rangle$ and for $\langle \epsilon \rangle$ and a rough but reliable estimate of the $A_2^\sigma$ for the magnetization.

Unfortunately we are not able to discriminate between the two proposals. This would require precisions which are definitely outside the range of our transfer matrix methods.

The asymptotic values for the energy and the magnetization obtained implementing eq. (20) are more precise that those that quoted in [6] of more than one order of magnitude. As the critical point is approached.

Table 5: Tentative estimate for $A_2^\sigma$.

| $R_1$ | $R_2$ | $R_3$ | $-A_2^\sigma$ |
|-------|-------|-------|---------------|
| 15    | 16    | 17    | -22.3477682808347 |
| 15    | 16    | 18    | -22.3151718797824 |
| 15    | 16    | 19    | -22.2853020505359 |
| 15    | 16    | 20    | -22.2593381627093 |
| 15    | 17    | 18    | -22.2646875236694 |
| 15    | 17    | 19    | -22.2261047816289 |
| 15    | 17    | 20    | -22.1918745162029 |
| 15    | 18    | 19    | -22.1652771489821 |
| 15    | 18    | 20    | -22.1211891807013 |
| 15    | 19    | 20    | -22.0505359976403 |
| 16    | 17    | 18    | -22.1917546952317 |
| 16    | 17    | 19    | -22.1416354502675 |
| 16    | 17    | 20    | -22.0965339005886 |
| 16    | 18    | 19    | -22.0640813462388 |
| 16    | 18    | 20    | -22.0064100955253 |
| 16    | 19    | 20    | -21.9155756522890 |
| 17    | 18    | 19    | -21.9520757204636 |
| 17    | 18    | 20    | -21.8778373894290 |
| 17    | 19    | 20    | -21.7630532199712 |
| 18    | 19    | 20    | -21.5973234560565 |
this improvement in precision increases.

- The agreement between the present results and those of [6], besides being a further strong evidence of the correctness of the proposal of [1] and [2], also points out the power of the algorithm proposed in [6] to deal with the finite size correction. This is an important result since this algorithm is not constrained to integrable theories.

It would be interesting to extend this analysis to other integrable models which could offer examples in which it could be simpler to discriminate between the two proposals of [1] and [2] and to compare them with other existing approaches (say, for instance, [19]). It would also be important to extend the analysis to models which could be more easily accessible for comparison with experimental results (see for instance the results of [20, 21] on the Haldane-Gapped spin chains). At the same time it would be interesting to extend the present analysis to two-point functions, for which very precise numerical results exist for the 2d Ising model in a magnetic field [7, 8]. This would be a rather important test since it has been recently claimed [22, 23] that the extension to two-point functions of the proposals [1] and [2] could fail in the case of interacting theories.

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