POLAR DECOMPOSITION OF THE WIENER MEASURE: SCHWARZIAN THEORY VERSUS CONFORMAL QUANTUM MECHANICS

V. V. Belokurov* † and E. T. Shavgulidze*

We find an explicit form of the polar decomposition of the Wiener measure and obtain an equation relating functional integrals in conformal quantum mechanics to functional integrals in the Schwarzian theory. Using this relation, we evaluate some nontrivial functional integrals in the Schwarzian theory and also find the fundamental solution of the Schrödinger equation in imaginary time in the model of conformal quantum mechanics.

Keywords: Wiener measure, functional integral over the group of diffeomorphisms

DOI: 10.1134/S004057791909006X

1. Introduction

In recent years, the Schwarzian theory has become extremely popular. It is given by the action

\[ I = -\frac{1}{\sigma^2} \int_{S^1} \left[ S_\varphi(t) + 2\pi^2 (\varphi'(t))^2 \right] dt, \]

where \( S_\varphi(t) = (\varphi''(t)/\varphi'(t))' - (\varphi''(t)/\varphi'(t))^2/2 \) is the Schwarzian derivative and \( \varphi(t) \) is an orientation-preserving \( \varphi'(t) > 0 \) diffeomorphism of the unit circle \( \varphi \in \text{Diff}^1_+ (S^1) \).

The Schwarzian action turns out to be the effective action in the quantum mechanical model of Majorana fermions with a random coupling constant (Sachdev–Ye–Kitaev model) arising in the holographic description of the Jackiw–Teitelboim dilaton gravity, in open string theory, and in some other models [1]–[35].

The unusual universality of the Schwarzian action is a consequence of its \( SL(2, \mathbb{R}) \) invariance. At the same time, not only the Schwarzian theory has an \( SL(2, \mathbb{R}) \)-invariant action. This symmetry is manifested in conformal quantum mechanics [36], Liouville quantum mechanics [37]–[39], and some other models that are used to describe the near-horizon geometry of a Reissner–Nordström black hole and AdS\(_2\)/CFT\(_1\) duality, gravity near a spacelike singularity (see, e.g., [40]–[45]), and other physical problems in which a universal regime is attained. Therefore, attempts [10], [19]–[25] to connect the Schwarzian theory in one way or another to other conformally invariant quantum mechanical models are quite natural.

In this case, a hope for the possibility of replacing Schwarzian action (1) with a simpler conformal action can arise. But the relation between these theories, as is seen in what follows, is more complicated.

*Lomonosov Moscow State University, Moscow, Russia, e-mail: vbelokurov@yandex.ru, shavgulidze@bk.ru.
†Institute for Nuclear Research of the Russian Academy of Sciences, Moscow, Russia.
Here, to relate theories to each other, we propose to study the corresponding functional integration measures.

The measures

\[ \mu_\sigma(d\varphi) = \exp \left\{ \frac{1}{\sigma^2} \int_0^1 S_\varphi(t) \, dt \right\} \, d\varphi \]  

(2)
on the group \( \text{Diff}_1^1(S^1) \) and

\[ \mu_\sigma(d\varphi) = \frac{1}{\sqrt{\varphi'(0)\varphi'(1)}} \exp \left\{ \frac{1}{\sigma^2} \int_0^1 S_\varphi(\tau) \, d\tau + \frac{1}{\sigma^2} \left[ \varphi''(0) - \varphi''(1) \right] \right\} \, d\varphi \]  

(3)
on the group \( \text{Diff}_1^1([0, 1]) \) and \( \text{Diff}_3^3(S^1) \) (\( \text{Diff}_3^3([0, 1]) \)). Quasi-invariance means that under the action of the subgroup \( G = \text{Diff}_3^3(S^1) \) (\( G = \text{Diff}_3^3([0, 1]) \)), the measure \( \mu_\sigma(d\varphi) \) transforms into itself multiplied by a function \( R_\sigma(h) \) parameterized by the elements of the subgroup \( g \in G : \mu(d(g \circ h)) = R_\sigma(h) \mu(dh) \).

Using the quasi-invariance of the measure \( \mu_\sigma(d\varphi) \), we evaluated functional integrals for the partition function and correlation functions in the Schwarzian theory explicitly and derived the general rules of functional integration in Schwarzian-type theories [49]–[51].

The Wiener measure on the infinite-dimensional space \( C_+([0, 1]) \) of continuous positive functions is also quasi-invariant under the action of \( \text{Diff}_3^3([0, 1]) \) [48], [52] (also see [53]).

Here (see Sec. 2 and 3), we establish that the two measures are related by

\[ w_\sigma(dx) = e^{-\sigma^2/8\rho^2}(\varphi'(0)\varphi'(1))^{3/4} \mu_{2\sigma/\rho}(d\varphi) \, d\rho, \]

where \( x \in C_+([0, 1]), \varphi \in \text{Diff}_3^3([0, 1]) \), and \( 0 < \rho < +\infty \), which we call the “polar decomposition of the Wiener measure.” The corresponding polar decomposition of the measure on the circle has the form

\[ w_\sigma(dx) = e^{-\sigma^2/8\rho^2} \mu_{2\sigma/\rho}(d\varphi) \, d\rho, \]

\( x \in C_+(S^1) \), \( \varphi \in \text{Diff}_3^3(S^1) \), \( 0 < \rho < +\infty \).

In Sec. 4, for \( SL(2, \mathbb{R}) \)-invariant functionals \( \Phi \), we obtain the equation

\[ \int_{C'(S^1)/SL(2, \mathbb{R})} \Phi(x) \exp \left\{ -\frac{1}{2\sigma^2} \int_{S^1} \left[ (x'(t))^2 - 2\pi^2 x^2(t) + \frac{2g}{x^2(t)} \right] \, dt \right\} \, dx = \]

\[ = \int_0^{\infty} \exp \left\{ -\left( \frac{\sigma^2}{8} + \frac{g}{\rho^2} \right) \frac{1}{\rho^2} \right\} \, d\rho \times \]

\[ \times \int_{\text{Diff}_3^3(S^1)/SL(2, \mathbb{R})} \Phi(x(\rho, \varphi)) \exp \left\{ \frac{\rho^2}{4\sigma^2} \int_{S^1} [S_\varphi(t) + 2\pi^2 (\varphi'(t))^2] \, dt \right\} \, d\varphi. \]

It relates functional integrals in conformal quantum mechanics to the corresponding functional integrals in the Schwarzian theory and allows significantly improving the technique of functional integration to evaluate nontrivial functional integrals in both theories.

In Sec. 5, we apply the above equation in conformal quantum mechanics. Using already evaluated functional integrals in the Schwarzian theory, we find the fundamental solution of the Schrödinger equation in imaginary time in the quantum mechanical model with a Calogero-type potential. In Sec. 6, we present concluding remarks.
2. Stratification of the space $C_+([0, 1])$ and polar decomposition of the Wiener measure

We consider a Wiener measure with the standard deviation $\sigma$ on the space of continuous positive functions $x(t)$ on the interval $[0, 1]$ with arbitrary values $x(0)$ and $x(1)$ independent of each other. It is formally written as

$$w_\sigma(dx) = \exp\left(-\frac{1}{2\sigma^2} \int_0^1 (x'(t))^2 \, dt\right) \, dx. \quad (5)$$

Measure (5) is quasi-invariant under the action of the group of diffeomorphisms $Diff^3_+([0, 1])$ on $C_+([0, 1])$ \cite{48, 52},

$$x \mapsto fx, \quad (fx)(t) = x(f^{-1}(t)) \frac{1}{\sqrt{(f^{-1}(t))'}}. \quad x \in C_+([0, 1]), \quad f \in Diff^3_+([0, 1]). \quad (6)$$

The integral

$$\frac{1}{\rho^2} = \int_0^1 \frac{1}{x^2(t)} \, dt \quad (7)$$

is invariant under action (6) of the group $Diff^3_+([0, 1])$.

We define $\varphi \in Diff^3_+([0, 1])$ by the equation $\varphi^{-1}(t) = \rho^2 \int_0^t d\tau / x^2(\tau)$, and $x(t)$ is then expressed in terms of $\rho$ and $\varphi(t)$: $x(t) = \rho / \sqrt{(\varphi^{-1}(t))'}$. In this case, we have $\int_0^1 x^2(t) \, dt = \rho^2 \int_0^1 (\varphi'(\tau))^2 \, d\tau$.

Therefore, there exists a one-to-one correspondence $(\rho, \varphi) \leftrightarrow x$, and the space $C_+([0, 1])$ is stratified into orbits with different values of the invariant $\rho$.

For a smooth $x(t)$ ($x \in C_+^1([0, 1])$) and a thrice-differentiable $\varphi(t)$ ($\varphi \in Diff^3_+([0, 1])$), we also obtain

$$-\frac{1}{2\sigma^2} \int_0^1 (x'(t))^2 \, dt = \frac{\rho^2}{4\sigma^2} \left\{ \int_0^1 S_\varphi(\tau) \, d\tau + \left[ \frac{\varphi''(0)}{\varphi'(0)} - \frac{\varphi''(1)}{\varphi'(1)} \right] \right\}. \quad (8)$$

Hence, for the Wiener measure on the space $C_+([0, 1])$, we have the polar decomposition

$$w_\sigma(dx) = P_\sigma(\rho)(\varphi'(0)\varphi'(1))^{3/4} \mu_{2\sigma/\rho}(d\varphi) \, d\rho. \quad (9)$$

The normalizing factor $P(\rho)$ (to be evaluated below) determines the relative weight of the input to the measure from the path $x(t)$ with the definite value $1/\rho^2$ of invariant (7).
After change (4), Eq. (10) becomes

\[ \mathcal{P}_\sigma(\tilde{\rho}) \frac{2}{\tilde{\rho}} \int_{C_\mu([0,1])} \delta(\xi(1)) \frac{1}{\int_0^1 e^x(\tau) \, d\tau} w_{2\sigma/\rho}(d\xi), \]

and it is easy to find if we differentiate the identity

\[ \int_{C_\mu([0,1])} \delta(\xi(1)) \exp\left\{-\frac{2\beta^2}{\kappa^2(\beta + 1)} \int_0^1 e^x(\tau) \, d\tau \right\} w_{\kappa}(d\xi) = \frac{1}{\sqrt{2\pi\kappa}} \exp\left\{-\frac{1}{2\kappa^2}(2\log(\beta + 1))^2 \right\}, \]

proved in [50]. Transforming the \( \delta \)-function as

\[ \delta\left( \tilde{\rho} - \left( \int_0^1 \frac{dt}{x^2(t)} \right)^{-1/2} \right) = \frac{2}{\tilde{\rho}^3} \delta\left( \frac{1}{\tilde{\rho}^2} - \int_0^1 \frac{dt}{x^2(t)} \right), \]

we rewrite (9) in the form

\[ \frac{1}{\sqrt{2\pi\sigma}} \mathcal{P}_\sigma(\tilde{\rho}) = \frac{2}{\tilde{\rho}^3} e^{-\sigma^2/8\tilde{\rho}^2} \int_{C_+([0,1])} \delta\left( \frac{1}{\tilde{\rho}^2} - \int_0^1 \frac{dt}{x^2(t)} \right) \delta(x(0) - x(1)) w_{\sigma}(dx). \]  

(11)

To evaluate this functional integral, we consider the Fourier transform of the first \( \delta \)-function in the integrand in (11), represent the Wiener measure as \( w_{\sigma}(dx) = \exp\{-1/2\sigma^2\} \int_0^1 (x'(t))^2 \, dt \, dx \), and change \( y(t) = x(t)/\sigma \). As the result, the functional integral in the right-hand side of (11) becomes

\[ \int_{C_+([0,1])} \delta\left( \frac{1}{\tilde{\rho}^2} - \int_0^1 \frac{dt}{x^2(t)} \right) \delta(x(0) - x(1)) w_{\sigma}(dx) = \]

\[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda e^{i\lambda/\tilde{\rho}^2} \int_{C_+([0,1])} \exp\left\{-\frac{1}{2} \int_0^1 \left[ (y'(t))^2 + \frac{2i\lambda}{\sigma^2 y^2(t)} \right] dt \right\} \delta(y(0) - y(1)) \, dy. \]  

(12)

Also rescaling the parameters \( \tilde{\lambda} = \sigma^2\lambda \) and \( \tilde{\rho}^2 = \sigma^2\eta^2 \), we rewrite (12) as

\[ \frac{\sigma^2}{2\pi} \int_{-\infty}^{+\infty} d\lambda e^{i\lambda/\eta^2} \int_{C_+([0,1])} \exp\left\{-\frac{1}{2} \int_0^1 \left[ (y'(t))^2 + \frac{2i\lambda}{\eta^2 y^2(t)} \right] dt \right\} \delta(y(0) - y(1)) \, dy. \]  

(13)

The integral over \( C_+([0,1]) \) in (13) is just \( \text{Tr} e^{-A_0} \), i.e., the functional integral for the partition function in the Calogero model given by the Euclidean action

\[ A_0(g) = \frac{1}{2} \int_0^1 \left[ (y'(t))^2 + 2g \frac{1}{y^2(t)} \right] dt, \]  

(14)

although with an imaginary coupling constant \( g = i\lambda \).

We note that the appearance of the action of conformal quantum mechanics is closely connected with reducing the Wiener measure to orbit (7).

Because action (14) has the continuous spectrum, we first consider the regularized action

\[ A_\omega(g) = \frac{1}{2} \int_0^1 \left[ (y'(t))^2 + \omega^2 y^2(t) + 2g \frac{1}{y^2(t)} \right] dt, \]  

(15)

calculate the Fourier transform

\[ \frac{\sigma^2}{2\pi} \int_{-\infty}^{+\infty} d\lambda e^{i\lambda/\rho^2} \int_{C_+([0,1])} e^{-A_\omega(g, \lambda)\delta(y(0) - y(1))} \, dy, \]

1327
and then let the parameter \( \omega \) tend to zero.

The solution of the quantum problem for the action of a quantum oscillator with inverse-square potential (15) is well known. The wave eigenfunctions \( \psi_n(x) \) (\( \psi_n(0) = 0 \)) form a basis in the Hilbert space of functions square integrable on the half-axis \( 0 < x < +\infty \) (see, e.g., [54]) with the energy levels

\[
E_n = \frac{\omega}{2} \left( 1 + \sqrt{2g + \frac{1}{4}} \right) + 2n\omega, \quad n = 0, 1, \ldots
\]

Therefore, the partition function for the theory with action (15) is

\[
\int_{\mathcal{C}_+([0,1])} e^{-A_\omega \delta(y(0) - y(1))} dy = \text{Tr} e^{-A_\omega} = \sum_0^\infty e^{-E_n} = \exp \left( \frac{\omega}{2} \left[ 1 - \sqrt{2g + \frac{1}{4}} \right] \right) \frac{1}{2\sinh \omega}.
\]

The polar decomposition also holds for the Wiener measure on the space \( \text{Diff}_+^1(0,1) \).

To find integral (16), we first substitute \( \xi = 2i\lambda + 1/4 \), divide the integral into two parts,

\[
\int_{\Re \xi = 0} = \int_{\Re \xi = 0, \Im \xi > 0} + \int_{\Re \xi = 0, \Im \xi < 0} = \int_{\Re \xi \leq 0, \Im \xi = 0} + \int_{\Re \xi > 0, \Im \xi = 0},
\]

and substitute \( \xi = i\sqrt{z} \) and \( \xi = -i\sqrt{z} \) in the respective first and second integrals. Taking the limit \( \omega \rightarrow 0 \) in the above equations, we obtain

\[
\mathcal{P}_\sigma(\rho) = e^{-\sigma^2/8\rho^2}
\]

from (11).

We have thus proved the polar decomposition of the Wiener measure:

\[
w_\sigma(dx) = e^{-\sigma^2/8\rho^2} (\varphi'(0)\varphi'(1))^{3/4} \mu_{2\sigma/\rho}(d\varphi) d\rho,
\]

\[
x \in \mathcal{C}_+([0,1]), \quad \varphi \in \text{Diff}_+^1([0,1]), \quad 0 < \rho < +\infty.
\]

The polar decomposition also holds for the Wiener measure on the space \( \mathcal{C}_+(S^1) \). But in this case, to evaluate the factor \( \mathcal{P}(\rho) \), we must somehow normalize the measure. For example, we can parameterize the circle of unit length \( S^1 \) by the interval \([0,1]\) and consider the measure on \( \mathcal{C}_+(S^1) \) as the measure on \( \mathcal{C}_+([0,1]) \) with the ends of the interval “glued”: \( x(0) = x(1) \). In this case, it has the form

\[
w_\sigma(dx) = e^{-\sigma^2/8\rho^2} \mu_{2\sigma/\rho}(d\varphi) d\rho, \quad x \in \mathcal{C}_+(S^1), \quad \varphi \in \text{Diff}_+^1(S^1), \quad 0 < \rho < +\infty.
\]

4. Relation between the functional integrals in the Schwarzian theory and in conformal quantum mechanics

From polar decomposition (19) of the Wiener measure, we have the equality for the functional integrals

\[
\int_{\mathcal{C}_+(S^1)} F(x) \exp \left\{ -\frac{1}{2\sigma^2} \int_{S^1} \left[ \frac{(x'(t))^2}{2} - 2\pi^2 x^2(t) + \frac{2g}{x^2(t)} \right] dt \right\} dx =
\]

\[
= \int_0^{+\infty} d\rho \exp \left\{ -\left( \frac{\sigma^2}{8} + \frac{g}{\sigma^2} \right) \frac{1}{\rho^2} \right\} \times
\]

\[
\times \int_{\text{Diff}_+^1(S^1)} F(x(\rho, \varphi)) \exp \left\{ \frac{\rho^2}{4\sigma^2} \int_{S^1} [S_\varphi(t) + 2\pi^2 (\varphi'(t))^2] \right\} d\varphi.
\]
In particular, for $F(x) = \Phi(x) \exp\left\{\frac{-\beta^2}{2\sigma^2} \int_{\mathbb{R}} x^2(t) \, dt\right\}$ with a sufficiently well-behaved $\Phi(x)$, the integrals are well defined. Therefore, we have an obvious relation between conformal quantum mechanics and the Schwarzian theory.

If the functional $\Phi$ is $SL(2, \mathbb{R})$ invariant, then it is reducible to the orbits. In this case, the integrals can be factored with the result

$$\int_{C_+(S^1)/SL(2, \mathbb{R})} \Phi(x) \exp\left\{-\frac{1}{2\sigma^2} \int_{S^1} \left[(x'(t))^2 - 2\pi^2 x^2(t) + \frac{2g}{x^2(t)}\right] dt\right\} \, dx =$$

$$= \int_0^{+\infty} dp \exp\left\{-\left(\frac{\sigma^2}{8} + \frac{g}{\sigma^2}\right) \frac{1}{p^2}\right\} \times$$

$$\times \int_{Diff^+_1(S^1)/SL(2, \mathbb{R})} \Phi(x(\rho, \varphi)) \exp\left\{\frac{\rho^2}{4\sigma^2} \int_{S^1} [S_\varphi(t) + 2\pi^2 (\varphi'(t))^2] \, dt\right\} \, d\varphi. \quad (20)$$

Evaluating the functional integral in conformal quantum mechanics, we thus find the corresponding functional integral in the Schwarzian theory. Conversely, the technique of functional integration over the group of diffeomorphisms allows evaluating nontrivial functional integrals in conformal quantum mechanics.

We consider a simple example. We set $g = 0$ and $\sigma = 1$ and take $\Phi(x) = \delta(k^2 - 4 \int_{S^1} dt/x^2(t))$. The right-hand side of (20) defines the partition function in the Schwarzian theory,

$$Z_{Schw}(\kappa) = \frac{1}{\sqrt{2\pi \kappa}} \int_{Diff^+_1(S^1)/SL(2, \mathbb{R})} e^{-I} \, d\varphi.$$ 

The left-hand side of (20) leads to the result [55] $Z_{Schw}(\kappa) = e^{2\pi^2/\kappa^2}/\sqrt{2\pi \kappa^3}$ previously obtained in [49] by direct functional integration in the Schwarzian theory.

In the same way, functional integrals in conformal quantum mechanics can be used to evaluate other Schwarzian functional integrals.

5. An example of applying the polar decomposition in conformal quantum mechanics

In [55], we used the exact solutions of the quantum oscillator model with the Calogero potential to evaluate the functional integral giving the partition function in the Schwarzian theory.

Polar decomposition (18) of the Wiener measure can also be used in the opposite direction. Namely, we can use already evaluated functional integrals of the Schwarzian theory to perform nontrivial functional integration in conformal quantum mechanics. We thus find the fundamental solution of the Schrödinger equation in imaginary time in a quantum mechanical model with a Calogero-type potential.

We consider the heat conduction equation of the form

$$\frac{\partial}{\partial \tau} \psi_g(q, \tau) = \left(\frac{1}{2} \frac{\partial^2}{\partial q^2} - \frac{g}{q^2}\right) \psi_g(q, \tau), \quad q > 0,$$

or, in other words, the Schrödinger equation in imaginary time with the potential $V(q) = g/q^2$.

To exclude paths with negative values of $x(\tau)$ in the subsequent functional integrals, we impose the boundary condition $\psi_g(q = 0, \tau) = 0$ and integrate over the space $C_+(0, \ell)$.

Let the initial condition be $\psi_g(q, 0) = \psi_0(q) = \delta(q(0) - q_0), q_0 > 0$.

The fundamental solution of the Cauchy problem is given by the functional integral $\psi_g(t, q; q_0) = \int_{C_+(0, \ell)} \delta(x(0) - q_0) \delta(x(t) - q) \exp\left\{-\int_0^t \left(g/y^2(\tau)\right) \, d\tau\right\} w_1(dx)$. After the change of variables $\tau = \tau \tau, y(\tau) = x(\tau \tau), \tau \in [0, 1]$, it becomes

$$\psi_g(q, t; q_0) = \int_{C_+(0, 1)} \delta(\sqrt{\tau}y(0) - q_0) \delta(\sqrt{\tau}y(1) - q) \exp\left\{-\int_0^1 \frac{g}{y^2(\tau)} \, d\tau\right\} \sqrt{\tau} w_1(dy).$$
We now use polar decomposition (18) of the Wiener measure and obtain
\[
\psi_g(q, t; q_0) = \frac{1}{\sqrt{t}} \int_0^{+\infty} dp e^{-(g+1/8)/\rho^2} \times \\
\times \int_{D_2/(0,1)} \delta \left( \rho \sqrt{\phi'(0)} - \frac{q_0}{\sqrt{t}} \right) \delta \left( \rho \sqrt{\phi'(1)} - \frac{q}{\sqrt{t}} \right) (\phi'(0) \phi'(1))^{3/4} \mu_{2/\rho}(d\varphi).
\]
Introducing the special functional integral
\[
E_\sigma(u, v) = \int_{D_2/(0,1)} \delta(\phi'(0) - u) \delta(\phi'(1) - v) \mu_\sigma(d\varphi),
\]
we can rewrite it as
\[
\psi_g(q, t; q_0) = \frac{1}{\sqrt{t}} \int_0^{+\infty} dp e^{-(g+1/8)/\rho^2} \times \\
\times \int_0^{+\infty} du \int_0^{+\infty} dv \delta \left( \rho \sqrt{u} - \frac{q_0}{\sqrt{t}} \right) \delta \left( \rho \sqrt{v} - \frac{q}{\sqrt{t}} \right) (uv)^{3/4} E_{2/\rho}(u, v) = \\
= \frac{4}{t^2} (q_0 q)^{5/2} \int_0^{+\infty} dp \frac{1}{\rho^2} e^{-(g+1/8)/\rho^2} E_{2/\rho} \left( \frac{q_0^2 + q^2}{4 \rho^4} \right). \tag{22}
\]

In [50], we performed the functional integration in (21) and represented the function \(E_\sigma(u, v)\) in the form of the ordinary integral
\[
E_\sigma(u, v) = \left( \frac{2}{\pi \sigma^2} \right)^{3/2} \frac{1}{\sqrt{uv}} \exp \left\{ \frac{2}{\sigma^2} (-u - v) \right\} \times \\
\times \int_0^{+\infty} \exp \left\{ -\frac{2}{\sigma^2} (2 \sqrt{uv} \cosh \theta + \theta^2 - \pi^2) \right\} \sin \left( \frac{4\pi \theta}{\sigma^2} \right) \sinh \theta d\theta. \tag{23}
\]
With (23) taken into account, Eq. (22) is transformed into
\[
\psi_g(q, t; q_0) = \frac{1}{4} \frac{4}{\sqrt{2\pi}} \left( \frac{q_0 q}{2\pi} \right)^{3/2} e^{-(q_0^2 + q^2)/(2\pi)} \int_0^{+\infty} dp \frac{1}{\rho^2} e^{-(g+1/8)/\rho^2} \times \\
\times \int_{-\infty}^{+\infty} d\theta \sinh \theta e^{-(q_0 q / t) \cosh \theta} \left[ e^{\rho^2(\theta-i\pi)^2/2} - e^{\rho^2(\theta+i\pi)^2/2} \right].
\]

Substituting \(z = \theta - i\pi\) or \(z = \theta + i\pi\), we rewrite the integral over \(\theta\) as
\[
\lim_{R \to +\infty} \left[ \int_{-R+i\pi}^{+R+i\pi} f(z) dz - \int_{-R-i\pi}^{+R-i\pi} f(z) dz \right] = \\
= \lim_{R \to +\infty} \left[ \int_{R-i\pi}^{R+i\pi} f(z) dz - \int_{-R-i\pi}^{-R+i\pi} f(z) dz \right] = 2i \lim_{R \to +\infty} \int_{-\pi}^{+\pi} f(R + i\tau) d\tau,
\]
where \(f(z) = \sinh z e^{(q_0 q / t) \cosh z} e^{-\rho^2 z^2/2}\). Hence, we have
\[
\psi_g(q, t; q_0) = \frac{2}{i^2} \left( \frac{q_0 q}{2\pi} \right)^{3/2} e^{-(q_0^2 + q^2)/(2\pi)} \int_0^{+\infty} dp \frac{1}{\rho^2} e^{-\rho^2 t^2/\rho^2} \times \\
\times \lim_{R \to +\infty} \int_{-\pi}^{+\pi} e^{(q_0 q / t) \cosh(R+i\tau)} e^{-\rho^2(R+i\tau)^2/2} \sinh(R + i\tau) d\tau. \tag{24}
\]
We now change the order of integration and integrate over \( \rho \). After the substitution \( \zeta = \cosh(R + i\tau) \), we take the limit \( R \to +\infty \). As a result, (24) becomes

\[
\psi_g(q, t; q_0) = \frac{(q_0 q)^{3/2}}{\pi t^2} \frac{1}{\sqrt{2g + 1/4}} e^{-(q_0^2 + q^2)/2t} \left[ \int_{\Gamma_2} \chi_g(\zeta) d\zeta - \int_{\Gamma_3} \chi_g(\zeta) d\zeta \right],
\]

where \( \chi_g(\zeta) = e^{(q_0 q/t)\zeta} e^{-\sqrt{2g+1/4}\text{arch}\zeta} \) and \( \Gamma_2 \) and \( \Gamma_3 \) are the lower and the upper edges of the cut along the real axis in the complex plane of \( \zeta \) from \(-\infty \) to 1.

We note that for \(-1 < x < 1\), \( \text{arch}(x - i0) = -i \arccos x \) and \( \text{arch}(x + i0) = i \arccos x \), and for \(-\infty < x < -1\), \( \text{arch}(x - i0) = \log(|x| + \sqrt{x^2 - 1}) - i\pi \) and \( \text{arch}(x + i0) = \log(|x| + \sqrt{x^2 - 1}) + i\pi \).

After the corresponding reorganization of the integrals in (25) and obvious substitutions, we obtain the fundamental solution of the Cauchy problem in the final form

\[
\psi_g(q, t; q_0) = \mathcal{F}_g(q, t; q_0) \sin \left( \pi \sqrt{2g + 1/4} \right) \int_0^{+\infty} e^{-(q_0 q/t)\cosh \theta} e^{-\theta \sqrt{2g+1/4} \sinh \theta} d\theta + \\
+ \mathcal{F}_g(q, t; q_0) \int_0^{\pi} e^{(q_0 q/t)\cos \tau} \sin \left( \pi \sqrt{2g + 1/4} \right) \sin \tau d\tau,
\]

(26)

where

\[
\mathcal{F}_g(q, t; q_0) = \frac{(q_0 q)^{3/2}}{\pi t^2} \frac{1}{\sqrt{2g + 1/4}} e^{-(q_0^2 + q^2)/2t}.
\]

(27)

For \( g = 0 \), the integrals can be evaluated explicitly. In this case, \( \chi_0(\zeta) = e^{(q_0 q/t)\zeta} e^{-\text{arch} \zeta/2} \). Representing \( e^{-\text{arch} \zeta/2} = \cosh(-\text{arch} \zeta/2) - \sinh(-\text{arch} \zeta/2) = \sqrt{\zeta + 1/2} - \sqrt{\zeta - 1/2} \) and substituting \( \xi = i\sqrt{\zeta + 1/2} \) and \( \eta = i\sqrt{\zeta - 1/2} \) in integrals (25), we obtain the expected result

\[
\psi_0(q, t; q_0) = \frac{1}{\sqrt{2\pi t}} e^{-\sigma^2/8t^2} e^{3\xi(1)/4} \frac{1}{\int_0^1 e^{\xi(\tau)} d\tau} \frac{1}{\int_0^1 e^{\eta(\tau)} d\tau} w_{2\sigma/\rho}(d\xi) d\rho.
\]

(28)

6. Concluding remarks

We have demonstrated that because there exists Diff\(^3\)-invariant (7), there is a one-to-one correspondence \( C_+ ([0, 1]) \leftrightarrow (0, +\infty) \times \text{Diff}_+^1([0, 1]) \), and the Wiener measure can be written as (18). Factor (17) determines the relative weight of the paths \( x(t) \) with a definite value of invariant (7).

Having in mind an analogy with the form of the Riemann–Lebesgue measure on the two-dimensional plane in polar coordinates, we call representation (18) the “polar decomposition of the Wiener measure.” Elements of the group \( \varphi \in \text{Diff}_+^1 \) play the role of angles, and values of the invariant \( \rho \) correspond to lengths of radius vectors.

Equation (20) relating Schwarzian functional integrals to functional integrals in conformal quantum mechanics allows choosing the most successful strategy of functional integration.

After substitution (4), we can rewrite polar decomposition (18) of the Wiener measure as

\[
w_{\sigma}(dx) = e^{-\sigma^2/2t\rho^2} \frac{1}{(\int_0^1 e^{\xi(t)} d\tau)^{3/2}} w_{2\sigma/\rho}(d\xi) d\rho.
\]

In [51], we already noted the apparent violation of the Markov property by the function \( \varphi(t) \). Although \( x(t) \) and \( \xi(t) \) are both Wiener processes, the Markov behavior of \( x(t) \) with respect to the time \( t \) of “its own world” obviously does not imply its Markov behavior with respect to the time \( \tau \) of the “shadow world” \( \xi(t) \), and vice versa.
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