On the Erdős–Hajnal problem in the case of 3-graphs

Danila Cherkashin*

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Abstract

Let \( m(n, r) \) denote the minimal number of edges in an \( n \)-uniform hypergraph which is not \( r \)-colorable. For the broad history of the problem see [9]. It is known [3] that for a fixed \( n \) the sequence

\[
\frac{m(n, r)}{r^n}
\]

has a limit.

The only trivial case is \( n = 2 \) in which \( m(2, r) = \binom{r+1}{2} \). In this note we focus on the case \( n = 3 \). First, we compare the existing methods in this case and then improve the lower bound.

1 Introduction

A hypergraph \( H = (V, E) \) consists of a finite set of vertices \( V \) and a family \( E \) of the subsets of \( V \), which are called edges. A hypergraph is called \( n \)-uniform if every edge has size \( n \). A vertex \( r \)-coloring of a hypergraph \( H = (V, E) \) is a map from \( V \) to \( \{1, \ldots, r\} \). A coloring is proper if there is no monochromatic edges, i.e., any edge \( e \in E \) contains two vertices of different color. The chromatic number of a hypergraph \( H \) is the smallest number \( \chi(H) \) such that there exists a proper \( \chi(H) \)-coloring of \( H \). Let \( m(n, r) \) be the minimal number of edges in an \( n \)-uniform hypergraph with chromatic number more than \( r \).

We are interested in the case when \( n \) is much smaller than \( r \) (see [9] for general case and related problems). Erdős and Hajnal [6] introduced problems on determining \( m(n, r) \) and related quantitites.

1.1 Upper bounds

Erdős conjectured [5] that

\[
m(n, r) = \binom{r(n-1)+1}{n},
\]

for \( r > r_0(n) \), that is achieved on the complete hypergraph.

However Alon [2] disproved the conjecture for \( n \geq 13 \) by using the estimate

\[
m(n, r) < \min_{a \geq 0} T(r(n+a-1)+1, n+a, n),
\]

where the Turán number \( T(v, k, n) \) is the smallest number of edges in an \( n \)-uniform hypergraph on \( v \) vertices such that every induced subgraph on \( k \) vertices contains an edge (see [10] for a survey). Using the same inequality with better bounds on Turán numbers Akolzin and Shabanov [1] showed that

\[
m(n, r) < Cn^3 \ln n \cdot r^n.
\]

Also Alon conjectured that the sequence \( m(n, r)/r^n \) has a limit which was proved by Cherkashin and Petrov [3]. Denote the corresponding limit be \( L_n \). In this paper we are interested in estimates on \( L_3 \). The best known upper bound follows from the complete hypergraph:

\[
L_3 \leq \frac{4}{3}.
\]

*National Research University Higher School of Economics, Soyuza Pechatnikov str., 16, St. Petersburg, Russian Federation
1.2 Lower bounds

There are several ways to show an inequality of type $m(n, r) \geq c(n)r^n$ (i.e. $L_n \geq c(n)$). Note that Erdős–Hajnal conjecture implies in particular that

$$L_n = \frac{(n-1)^n}{n!}.$$ 

Alon [2] suggested to color vertices of an $n$-uniform hypergraph in $a < r$ colors uniformly and indepen-dently, and then recolor a vertex in every monochromatic edge in unused color. The expected number of monochromatic edges is

$$|E| \cdot a^{1-n}.$$ 

Note that we have $r - a$ remaining colors, and we can color $n - 1$ vertices in each unused color such that no new monochromatic edge appears. Summing up, if

$$|E| < a^{n-1}(r-a)(n-1)$$

then a hypergraph $H = (V, E)$ has a proper $r$-coloring. Substituting $a = \left\lfloor \frac{n-1}{n} \right\rfloor$, we get

$$m(n, r) \geq (n-1) \left( \left\lfloor \frac{r}{n} \right\rfloor \frac{n-1}{n} r \right)^{n-1}.$$ 

This method gives $L_3 \geq 8/27 = 0.296 \ldots$.

Another way is due to Pluhár [8]. He introduced the following useful notion. A sequence of edges $a_1, \ldots, a_r$ is an $r$-chain if $|a_i \cap a_j| = 1$ if $|i - j| = 1$ and $a_i \cap a_j = \emptyset$ otherwise; it is an ordered $r$-chain if $i < j$ implies that every vertex of $a_i$ is not bigger than any vertex of $a_j$ (with respect to a certain fixed linear ordering on $V$).

Pluhár’s theorem states that existence of an order on $V$ without ordered $r$-chains is equivalent to $r$-colorability of $H = (V, E)$. Let us prove a lower bound on $m(n, r)$ via this theorem. Consider a random order on the vertex set. Note that the probability of an $r$-chain to be ordered is

$$\frac{[(n-1)]!(n-2)!r^{-2}}{(n-1)r+1)!}.$$ 

From the other hand, the number of $r$-chains is at most $2|E|^r/r!$ since every set of $r$ edges generates at most 2 chains. So if

$$2 \frac{|E|^r}{r!} \frac{[(n-1)]!(n-2)!r^{-2}}{(n-1)r+1)!} < 1,$$

then we have a proper $r$-coloring of $H$. After taking $r$-root and some calculations we have

$$m(n, r) > c\sqrt{n}r^n,$$

and in particular $L_3 \geq 4/e^3 = 0.199 \ldots$.

Combining two previous arguments with Cherkashin–Kozik approach [4] Akolzin and Shabanov [1] proved that

$$m(n, r) \geq c \frac{n}{\ln r} r^n,$$

without explicit bounds on $c$. We show that this method gives the bound $L_3 \geq 0.205 \ldots$ in Section 3.

Cherkashin and Petrov [3] suggested an approach, based on the evaluation of the inverse function, to show that the sequence $m(n, r)/r^n$ has a limit. Denote by $f(N)$ the maximal possible chromatic number of an $n$-uniform hypergraph with $N$ edges. Also $f(0) = 1$ by agreement. The function $f$ non-strictly increases and satisfies

$$m(n, r) = \min\{N : f(N) > r\}.$$ 

Therefore $m(n, r) \sim Cr^n$ if and only if $f(N) \sim (N/C)^{1/n}$. The following lemmas were proved in [3].
Lemma 1. For any $N > 0$ and any positive integer $p$ we have
\[ f(N) \leq \max_{a_1+a_2+\ldots+a_p \leq N/p^{n-1}} f(a_1) + f(a_2) + \ldots + f(a_p). \]

Lemma 2. Denote $c_n = (1 - 2^{1-n})^{-n}$. For any $M > 0$ the inequality
\[ f(N) \leq N^{1/n} \cdot \max_{M \leq a < c_n M} f(a) \cdot a^{-1/n} \]
holds for all $N \geq M$.

It is known that $f(0) = 1$, $f(1) = \ldots = f(6) = 2$, $f(7) = \ldots = f(26) = 3$ (see [1]). Lemmas [1], [2] and computer calculations was used to get
\[ L_3 \geq 0.324 \ldots. \]

One more way to get a (very weak) bound $L_n \geq n^{-n}$ appeared in the same paper [3]. It is based on the straightforward induction on $r$: it was shown that there is a large independent set, so we can color it in color $r$ and apply the inductive assumption.

The contribution of the paper is the following theorem, which is proved by refining Pluhar approach via inducibility arguments.

**Theorem 1.**
\[ L_3 \geq \frac{4}{e^2} = 0.54 \ldots. \]

**Structure of the paper.** In Section 2 we show how to apply inducibility to the chain argument and proof Theorem [1]. In Section 3 we find the constant in Akolzin–Shabanov theorem for $n = 3$ and show that even if we apply Theorem [2] to the corresponding part of the proof, the constant will be still worse than in Theorem [1].

## 2 Inducibility tool

**Theorem 2.** Suppose $H = (V, E)$ is a hypergraph. Then it has at most
\[ \frac{|E|}{2} \left( \frac{|E|}{r - 1} \right)^{r-1} \]
r-chains.

We need a notion of inducibility. Denote by $I(G, H)$ the number of induced subgraphs in $G$, isomorphic to $H$. The following simple bound was proved by Pippenger and Golumbic.

**Lemma 3** (Pippenger–Golumbic [7]). Let $G$ be a graph on $n$ vertices. Then
\[ I(G, P_r) \leq \frac{n}{2} \left( \frac{n}{r-1} \right)^{r-1}. \]

It turns out that the bound is close to optimal.

**Example 1.** Let $n$ be the $k$-th power of $r + 1$, $r > 2$. Consider the blow-up of $C_{r+1}$ where the same construction is placed inside the blow-up of each vertex. Every $r$-chain has vertices only in the different copies. Hence the number of chains is
\[ \sum_{i=1}^{k} \frac{n^r}{(r+1)^{r-1}}. \]
Proof of Lemma 3. Let $X(q, l)$ denote the largest possible number of ways of sequentially choosing $q$ objects $w_0, w_1, \ldots, w_{q-1}$ from among $l$ objects, subject to rules whereby the set of objects that are eligible to be chosen as $w_i$ depends only on the previous choices $w_0, w_1, \ldots, w_{i-1}$, and whereby no object that is eligible to be chosen as $w_i$ will be eligible to be chosen as $w_j$ for any $i + 1 \leq j \leq q - 1$. Clearly, $X(0, l) = 1$. If $q > 0$, let $m$ denote the number of objects eligible to be chosen as $w_0$. For any choice of $w_0$, the remaining $q - 1$ objects can be chosen in at most $X(q - 1, l - m)$ ways. Thus

$$X(q, l) \leq \max_{1 \leq m \leq l} mX(q - 1, l - m) \leq \max_{1 \leq m \leq l} mX(q - 1, l - m).$$

From these relations, we obtain

$$X(q, l) \leq \left( \frac{l}{q} \right)^q.$$ 

by induction on $q$: the base $q = 1$ is obvious. To prove the step it is enough to maximize the right-hand side of

$$X(q, l) \leq \max_{1 \leq m \leq l} m \left( \frac{l - m}{q - 1} \right)^{q-1}.$$ 

Taking derivative, we get the maximum at $m = l/q$, and we are done.

Also there are $n$ ways to choose the first vertex. Obviously, it is an upper bound on the number of induced $r$-paths, multiplied by 2, because every path is counted twice.

Proof of Theorem 2. Consider an auxiliary graph $G = (E, F)$ with vertex set is the edge set of $H$ and edges connect pairs of graph vertices, which intersect (as hyperedges) on exactly one vertex. The number of $r$-chains is at most the number of induced graphs $P_r$ in $G$, because every $r$-chain forms induced $P_r$ (note that the reverse consequence is wrong, because non-edge in $G$ can mean that the corresponding hyperedges have large intersection, which is impossible in $r$-chain). Hence, Lemma 3 finishes the proof.

Proof of Theorem 7. Let us try to color by the Pluhar greedy algorithm. Recall that the probability of an $r$-chain to be ordered is

$$\frac{[(n-1)!]^2[(n-2)!]^r-2}{((n-1)r+1)!} = \frac{4}{(2r+1)!}.$$ 

Using Theorem 2 we get that if

$$\frac{|E|^r}{2(r-1)^{r-1}(2r+1)!} < 1,$$

than hypergraph is $r$-colorable. Summing up,

$$L_3 \geq \lim_{r \to \infty} \sqrt[2]{\frac{(2r+1)!}{2(r-1)^{r-1}}} \frac{1}{r^3} = \frac{4}{e^2}.$$ 

3 Analysis of the Akolzin–Shabanov proof

We rewrite the proof from 1 to get optimal constant in the case $n = 3$.

First, for every vertex $v$ introduce the weight $w(v)$ as randomly (accordingly to the uniform distribution and independently) chosen number from $[0, 1]$. Fix parameters $p \in [0, 1]$, $a < r$. An edge $e$ is called bad if

$$\max_{v \in e} w(v) - \min_{v \in e} w(v) \leq \frac{1-p}{a};$$

otherwise it is called good.

The coloring algorithm is the following. First we color a (random) subhypergraph, consisting of all good edges, in $a$ colors via Pluhar approach; then we color (or recolor) some vertices from bad edges in unused $r-a$ colors. If Pluhar approach succeed (i.e. there is no ordered $a$-chains) and we have at most $(n-1)(r-a)$ bad edges, then the algorithm return a proper $r$-coloring. Let us evaluate the probability of success.
Lemma 4 (Akolzin–Shabanov [1]).

\[
P[e \text{ is bad}] = \left(\frac{1-p}{a}\right)^{n-1} \left(\frac{1-p}{a} + n \left(1 - \frac{1-p}{a}\right)\right) \leq n \left(\frac{1-p}{a}\right)^{n-1} = 3 \left(\frac{1-p}{a}\right)^2.
\]

Let \(C(A_1, \ldots, A_a)\) denote the event that all the edges \(A_j\) are good and \((A_1, \ldots, A_a)\) is an ordered \(a\)-chain.

Lemma 5 (Akolzin–Shabanov [1]).

\[
P[C(A_1, \ldots, A_a)] \leq a^{-a(n-2)} \left(\frac{p^{a-1}}{(a-1)!}\right) = a^{-a} \left(\frac{p^{a-1}}{(a-1)!}\right).
\]

By Theorem 2 we have at most \(|E|/a\) \(a\)-chains. Define \(c = |E|/r^3\); we need

\[
\left(\frac{|E|}{a}\right)^a a^{-a} \frac{p^{a-1}}{(a-1)!} = \left((1 + o(1)) \frac{|E| p e}{a^3}\right)^a = \left((c + o(1)) \frac{r^3 p e}{a^3}\right)^a < 1.
\]

Also we need at most \((n-1)(r-a) = 2(r-a)\) bad edges:

\[
P[X > 2(r-a)] \leq \frac{1}{2(r-a)} \frac{3(1-p)^2|E|}{a^2} < 1.
\]

Define \(x = r/a\). Then we need \(cx^3pe < 1\) and

\[
\frac{3c(1-p)x^3}{2(x-1)} < 1.
\]

Computer simulations gives that for \(p = 0.741\) and \(x = 1.05\) the algorithm with \(c = 0.42\) returns a proper coloring with positive probability, which implies \(L_3 \geq 0.42\) . . .

If we simply follow the initial proof, the required inequalities are

\[
\begin{align*}
    cx^3pe^2 < 1 & \quad \text{and} \quad \frac{3c(1-p)x^3}{2(x-1)} < 1.
\end{align*}
\]

So pure Akolzin–Shabanov approach gives \(L_3 \geq 0.205\) . . . Both constants are worse than in Theorem 1.

4 Open problems

- First, recall that the Erdős conjecture is still open in the range \(3 \leq n \leq 12\).
- Also it is natural to ask if \(m(n, r)\) is regular on the first variable, i.e.

\[
\lim_{n \to \infty} \frac{m(n+1, r)}{m(n, r)} = r?
\]

- In the proof of Theorem 2 we consider an auxiliary graph \(G\). The problem is to describe the set of graphs, which may be achieved from an \(r\)-chromatic \(n\)-uniform hypergraph.

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