Coprime partitions and Jordan totient functions

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Abstract

We show that while the number of coprime compositions of a positive integer \( n \) into \( k \) parts can be expressed as a \( \mathbb{Q} \)-linear combinations of the Jordan totient functions, this is never possible for the coprime partitions of \( n \) into \( k \) parts. We also show that the number \( p'_k(n) \) of coprime partitions of \( n \) into \( k \) parts can be expressed as a \( \mathbb{C} \)-linear combinations of the Jordan totient functions, for \( n \) sufficiently large, if and only if \( k \in \{2,3\} \) and in a unique way. Finally we introduce some generalizations of the Jordan totient functions and we show that \( p'_k(n) \) can be always expressed as a \( \mathbb{C} \)-linear combinations of them. Our methods allow to get manageable formulas also for the number \( p_k(n) \) of the partitions of \( n \) into \( k \) parts.

Keywords: coprime compositions; coprime partitions; generalized Jordan totient functions.

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1 Introduction

The study of partitions with a fixed number $k$ of parts, satisfying some coprimality condition [5], has revealed to be very fruitful for analysing the normal covering number $\gamma(S_n)$ of the symmetric group $S_n$ ([6]), that is, the smallest number of conjugacy classes of proper subgroups needed to cover $S_n$. If $\sigma \in S_n$ and $k$ is the number of orbits of $\langle \sigma \rangle$ on $\{1, \ldots, n\}$ then the unordered list $p(\sigma) = [x_1, \ldots, x_k]$ of the sizes $x_i$ of those orbits is a partition of $n$ into $k$ parts called the type of $\sigma$. Now, by a basic result of group theory, two permutations are conjugate if and only if they have the same type. Thus the conjugates of some subgroups $H_1, \ldots, H_s$ cover $S_n$ if and only if for every partition $p$ of $n$ there exists $H_i$ containing at least a permutation of type $p$.

We emphasize that the problem of determining the normal covering number of a finite group arises from Galois theory and is linked to the investigation of integer polynomials having a root modulo $p$, for every prime number $p$, see [3], [7] and [12]. Fortunately, in order to efficiently bound $\gamma(S_n)$, it is not necessary to deal with partitions into $k$ parts for every possible $k$ and the focus is on $k = 2, 3, 4$ (see [4]).

Recently, using a deep knowledge about partitions into three parts Bubboloni, Praeger and Spiga [6] have shown that, for $n$ even, $\gamma(S_n) \geq \frac{n}{2} \left(1 - \sqrt{1 - \frac{4}{\pi^2}}\right) - \frac{\sqrt{17}}{2} \frac{n^{3/4}}{4}$. Similar results about $S_n$ for $n$ odd remain undone and the research could greatly benefit from a deeper knowledge about partitions into four parts, especially those satisfying suitable coprimality conditions. To start with, one should manage well at least the exact number $p'_3(n)$ of coprime partitions of $n$ into four parts. This initial and somewhat narrow motivation inspired the present paper. Looking to the case $k = 4$, we immediately realized that many considerations could be indeed carried on for every $k \geq 2$, shedding light both on the expression of the number $p_k(n)$ of partitions of $n$ into $k$ parts and on the number $p'_k(n)$ of coprime partitions of $n$ into $k$ parts.

Our idea is, on one hand, to get those expressions as linear combinations of classic number theoretic functions and, on the other hand, to develop a method which could lead to an effective computation of $p_k(n)$ and $p'_k(n)$. Let $J_i$ denote the Jordan totient function of degree $i \geq 0$. In [2] it is proved that $p'_3(n) = \frac{J_3(n)}{12}$ holds for $n \geq 4$. It is also clear that $p'_2(n) = \frac{J_1(n)}{2}$ holds for $n \geq 3$. So one can ask if similar results could hold for every $k$. We show that those two situations are pure miracles, because $p'_k(n)$ is a $\mathbb{C}$-linear combination of the Jordan totient functions for $n$ sufficiently large just in those cases (Theorem 2). The feeling is that the class of the Jordan totient functions is too restrictive and some generalizations of them are needed. We
consider then three generalizations which are finely linked among them: the Jordan root totient function, the Jordan modulo totient function and the Jordan-Dirichlet totient functions (Section 1). The first two are original, while the third is introduced in [9] to investigate the values of cyclotomic polynomial at the roots of unity. Fortunately, the Jordan-Dirichlet totient functions admit a very easy and manageable formula. We show that \( p_k' \) is a \( \mathbb{C} \)-linear combination of the Jordan root totient functions (Theorem 1). Relying on the partial fraction decomposition of the generating function of \( p_k(n) \) we show how to determine the coefficients in such \( \mathbb{C} \)-linear combination and then we show how to deduce the expression of the Jordan root totient functions involved. To that purpose the idea is to split a Jordan root totient function in a \( \mathbb{C} \)-linear combination of Jordan modulo totient functions, which in turn can be determined by suitable Jordan-Dirichlet totient functions, choosing some particular Dirichlet characters. Our concrete approach is proposed for \( k \in \{2, 3, 4\} \) but it is immediately understood how to use it in general.

We close noticing that the use of generalizations of Jordan totient functions is present in the very recent research. For instance in [11], Moree et al. introduce the Jordan totient quotients of weight \( w \) in order to study the average of the normalized derivative of cyclotomic polynomials.

2 Basic facts

2.1 Notation

Let \( n \in \mathbb{N} \). We denote by \( \Omega(n) \) the total number of prime factors of \( n \) and by \( \omega(n) \) the number of distinct prime factors of \( n \) where \( \Omega(1) = \omega(1) = 0 \). For \( n \geq 2 \) we define \( \delta(n) = \text{lcm}\{m \in \mathbb{N} : m < n\} \). As usual, \( \phi \) denotes the Euler's totient function, \( \mu \) the Möbius function and \( \zeta \) the zeta-Riemann function. For \( k \in \mathbb{N} \cup \{0\} \), set \( [k] = \{n \in \mathbb{N} : n \leq k\} \) and \( [k]_0 = \{n \in \mathbb{N} \cup \{0\} : n \leq k\} \). In particular, \( [0] = \emptyset \) while \( [0]_0 = \{0\} \). A function \( T : \mathbb{N} \cup \{0\} \to \mathbb{C} \) is called an integer periodic function if it is periodic with period \( m \in \mathbb{N} \). Given a sequence \( (a_n)_{n \geq k} \) of complex numbers for some \( k \geq 0 \), its generating function is the formal power series \( \sum_{n \geq k} a_n z^n \). We set \( D = \{z \in \mathbb{C} : |z| < 1\} \) and \( \overline{D} = \{z \in \mathbb{C} : |z| \leq 1\} \). For \( n \in \mathbb{N} \) we consider the group of \( n \)-roots of unity \( U_n = \{z \in \mathbb{C} : z^n = 1\} \). As is well known \( U_n \) is cyclic with \( \phi(n) \) generators called primitive \( n \)-roots of unity. Among the primitive \( n \)-roots of unity we call \( e^{2\pi i/n} \) the principal \( n \)-root of unity. Every \( \omega \in U \) is called a root of unity. Note that \( U \) is a group and that \( 1 \in U_n \) for all \( n \in \mathbb{N} \). We
refer to 1 as the trivial root of unity. If \( P(X) \in \mathbb{C}[X] \), we denote its degree by \( \deg(P) \).

### 2.2 The Jordan totient functions and their generalizations

Throughout this section, let \( k \) be a non-negative integer. We first recall the basic properties of the *Jordan totient function* \( J_k : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\} \) of degree \( k \). For every \( n \in \mathbb{N} \), by definition, we have

\[
J_k(n) := \sum_{d|n} d^k \mu(n/d).
\]

Note that \( J_k \) is a Dirichlet convolution of multiplicative functions, and thus it is a multiplicative function. Note that

\[
J_0(n) = \sum_{d|n} \mu(n/d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}
\]

is the neutral element with respect to the Dirichlet \(*\)-product of arithmetic functions.

The values of \( J_k(n) \) for \( k \geq 1 \) can be easily computed in terms of the prime divisors of \( n \) by the formula

\[
J_k(n) = n^k \prod_{p|n} \left( 1 - \frac{1}{p^k} \right),
\]

which makes clear that

\[
J_1(n) = \sum_{d|n} d\mu(n/d) = \phi(n)
\]

and that \( \phi(n) \) divides \( J_k(n) \) for all \( k \geq 1 \) and all \( n \in \mathbb{N} \). Moreover, for \( k \geq 2 \), \( J_k(n)/n^k \) belongs to the interval \( (1/\zeta(k), 1) \) and \( \zeta(k) \) is a positive real number. In particular, \( J_k(n) \) has the same order of magnitude of \( n^k \). For \( k = 1 \), \( J_1(n)/n \in (0, 1) \) and, unlike for the case \( k \geq 2 \), this ratio can be arbitrarily small. Anyway, for every \( k \geq 0 \) and every \( n \in \mathbb{N} \), we have \( J_k(n) \leq n^k \) and thus

\[
J_k(n) = O(n^k).
\]

For the scope of our paper it is fundamental to define some variations of the Jordan totient functions.
We define, for $\omega$ a root of unity, the $\omega$-Jordan totient function of degree $k$, $J_{k,\omega} : \mathbb{N} \to \mathbb{C}$ associating, with every $n \in \mathbb{N}$,

$$J_{(k,\omega)}(n) := \sum_{d|n} \omega^d d^k \mu(n/d).$$

(6)

We call those functions the Jordan root totient functions. Note that we are generalizing the Jordan totient functions because $J_{(k,1)} = J_k$. However those functions are not multiplicative when $\omega \neq 1$.

We define next, for every $m \in \mathbb{N}$ and $j \in [m-1]_0$ the Jordan modulo totient functions of degree $k$, $J_{j,m}^k : \mathbb{N} \to \mathbb{C}$ associating with every $n \in \mathbb{N}$,

$$J_{j,m}^k(n) := \sum_{d|n, d \equiv j \pmod{m}} d^k \mu(n/d).$$

(7)

Note that those functions cannot be interpreted as convolutions of multiplicative functions because the sum is not extended to all the divisors of $n$. In particular, in general, they are not multiplicative. Since $J_{0,1}^k = J_k$ the Jordan modulo totient functions are a generalization of the Jordan totient functions too.

It is immediately observed that the Jordan root totient functions are $\mathbb{C}$-linear combinations of the Jordan modulo totient functions. More precisely, consider $J_{(k,\omega)}$ for some $\omega \in U$ and some $k \in \mathbb{N} \cup \{0\}$. Let $m$ be the minimum positive integer such that $\omega \in U_m$. Then, for every $n \geq 1$, we have

$$J_{(k,\omega)}(n) = \sum_{j=0}^{m-1} \omega^j \sum_{d|n, d \equiv j \pmod{m}} d^k \mu(n/d) = \sum_{j=0}^{m-1} \omega^j J_{j,m}^k(n).$$

(8)

Thus $J_{(k,\omega)} = \sum_{j=0}^{m-1} \omega^j J_{j,m}^k$.

We recall finally a definition from [9]. For $\chi$ a Dirichlet character, the function $J_k(\chi; \cdot) : \mathbb{N} \to \mathbb{C}$ is defined by associating, with every $n \in \mathbb{N}$,

$$J_k(\chi; n) := \sum_{d|n} \chi(d)d^k \mu(n/d).$$

(9)

We call those functions the Jordan-Dirichlet totient functions. Since $\chi$ is totally multiplicative, the function $J_k(\chi; \cdot)$ is a Dirichlet convolution of multiplicative functions, and thus it is a multiplicative function. Note that if $1$ is the unique Dirichlet character modulo 1 (called the trivial character),
that is the function \(1(x) = 1\) for every \(x \in \mathbb{Z}\), we have \(J_k(1; \cdot) = J_k\). Recall that 1 is the only character assuming value different from 0 in 0.

Thus the \(J_k(\chi; \cdot)\) are a generalization of the Jordan totient function \(J_k\).

The values \(J_k(\chi; n)\) can be explicitly computed when \(\chi\) is assigned (see, for example, [9, Lemma 6]). Moreover, the Jordan-Dirichlet totient functions are \(C\)-linear combinations of the Jordan modulo totient functions.

**Lemma 1.** Let \(k\) be a non-negative integer and \(\chi\) be a Dirichlet character modulo \(m\), for \(m\) a positive integer. Write \(n \in \mathbb{N}\) as \(n = \prod_{p \mid n, p \text{ prime}} \chi(p)\).

Then

\[
(i) \quad J_k(\chi; n) = n^k \prod_{p \mid n, p \text{ prime}} \chi(p)^{c_p - 1} \left(\chi(p) - \frac{1}{p^k}\right). \tag{10}
\]

\[
(ii) \quad \text{If } (n, m) = 1, \text{ then}
J_k(\chi; n) = n^k \chi(n) \prod_{p \mid n, p \text{ prime}} \left(1 - \frac{1}{\chi(p)p^k}\right). \tag{11}
\]

\[
(iii) \quad J_k(\chi; \cdot) = \sum_{j=0}^{m} \chi(j) J_{k,m}^j. \tag{12}
\]

**Proof.** (i) Using that \(\chi\) is totally multiplicative, we have

\[
J_k(\chi; p^e) = \sum_{d \mid p^e} \chi(d)d^{e} \mu(p^e/d) = -\chi(p)^{c_p-1}p^{k(c_p-1)} + \chi(p^{c_p})p^{kc_p}
\]

\[
= -\chi(p)^{c_p-1}p^{k(c_p-1)} + \chi(p)^{c_p}p^{kc_p} = \chi(p)^{c_p-1}p^{kc_p} \left(\chi(p) - \frac{1}{p^k}\right).
\]

Hence, by the multiplicativity of \(J_k(\chi; \cdot)\), we obtain

\[
J_k(\chi; n) = \prod_{p \mid n, p \text{ prime}} J_k(\chi; p^{e_p}) = \prod_{p \mid n, p \text{ prime}} \chi(p)^{c_p-1}p^{kc_p} \left(\chi(p) - \frac{1}{p^k}\right).
\]

\[= n^k \prod_{p \mid n, p \text{ prime}} \chi(p)^{c_p-1} \left(\chi(p) - \frac{1}{p^k}\right).\]
(ii) Since \((n, m) = 1\) we have that, for every prime \(p\) dividing \(n\), \(\chi(p) \neq 0\) holds. Thus the result follows immediately by (i) using again that \(\chi\) is totally multiplicative.

(iii) Since the Dirichlet characters modulo \(m\) are periodic of period \(m\), we have

\[
J_k(\chi; n) = \sum_{d \mid n} \chi(d)d^k \mu(n/d) = \sum_{j=1}^{m} \sum_{d \equiv j \pmod{m}} \chi(j)d^k \mu(n/d)
\]

\[
= \sum_{j=1}^{m} \chi(j)J^j,m_k(n).
\]

\(\square\)

Recall that there are exactly \(\phi(m)\) Dirichlet characters modulo \(m\) so that, once \(m\) is fixed, the equalities in (12) give \(\phi(m)\) independent linear equations in the \(m\) variables \(J^j,m_k\), for \(j \in [m-1]_0\) with vanishing coefficient for the \(j\) such that \((j, m) > 1\). From those one can easily find the expression for \(J^j,m_k\), with \((j, m) = 1\), in terms of the \(J_k(\chi; \cdot)\). In fact, by the orthogonality relations for characters, we have

\[
J^j,m_k = \frac{1}{\phi(m)} \sum_{\chi} \chi(j)J_k(\chi; \cdot).
\] (13)

Moreover the cases with \((j, m) > 1\) can be easily managed and are always obtained as integer multiple of the functions \(J_k(m')\) where \((m', m) = 1\) (see Section [8] and particularly the proof of Lemma [3]).

### 2.3 Compositions and partitions

Let \(k \in \mathbb{N}\). A \(k\)-composition of \(n \in \mathbb{N}\) is an ordered \(k\)-tuple \(x = (x_1, \ldots, x_k)\) where, for every \(j \in [k]\), \(x_j \in \mathbb{N}\) and \(\sum_{j=1}^k x_j = n\). Let \(c_k(n)\) be the number of \(k\)-compositions of \(n\). Then \(c_k(n) = 0\) for all \(n < k\) and, it is well known that, for every \(n \geq k\), we have

\[
c_k(n) = \binom{n-1}{k-1} = \frac{(n-1) \cdots (n-k+1)}{(k-1)!}.
\] (14)

Consider now the corresponding polynomial

\[
C_k(X) := \frac{(X-1) \cdots (X-k+1)}{(k-1)!} = \sum_{i=0}^{k-1} a_i X^i \in \mathbb{Q}[X],
\] (15)
and note that $c_k(n) = C_k(n)$ holds, not only for $n \geq k$ but for all $n \geq 1$ because any positive integer less than $k$ is a root of $C(X)$. Thus,

$$c_k(n) = \sum_{i=0}^{k-1} a_i n^i, \quad \text{for all} \quad n \geq 1. \quad (16)$$

We call $C(X)$ the $k$-composition polynomial. Note that, for $i \in [k - 1]_0$, $a_i \in \mathbb{Q}$ depend on $k$ and are easily computable. For example, $a_{k-1} = \frac{1}{(k-1)!}$ so that

$$c_k(n) = \frac{1}{(k-1)!} n^{k-1} + O(n^{k-2}). \quad (17)$$

The generating function of $c_k(n)$ is well known ([10, page 44]) and given by

$$\sum_{n \geq 1} c_k(n) z^n = \frac{z^k}{(1 - z)^k}. \quad (18)$$

The above equality can be obviously rewritten in terms of the $k$-composition polynomial, as

$$\sum_{n \geq 1} C_k(n) z^n = \frac{z^k}{(1 - z)^k}. \quad (19)$$

A $k$-partition of $n \in \mathbb{N}$ is an unordered $k$-tuple $x = [x_1, \ldots, x_k]$ where, for every $j \in [k]$, $x_j \in \mathbb{N}$ and $n = \sum_{j=1}^{k} x_j$. Both for compositions and for partitions $x$, the numbers $x_1, \ldots, x_k$ are called the terms of $x$. Let $p_k(n)$ be the number of $k$-partitions of $n$. Again, we have $p_k(n) = 0$ for all $n < k$ but, unlike for compositions, there is no handy formula for $p_k(n)$ similar to (14) valid for all $k$ and $n$. For $k = 2, 3$ the formulas for $p_k(n)$ are well known (see, for example, [1, page 81]).

The generating function of $p_k(n)$ is instead quite simple and given by ([10, page 45])

$$\sum_{n \geq 1} p_k(n) z^n = \frac{z^k}{(1 - z)(1 - z^2) \cdots (1 - z^k)}. \quad (20)$$

For what concerns the asymptotic, it is well known that

$$p_k(n) = \frac{1}{k!(k-1)!} n^{k-1} + O(n^{k-2}). \quad (21)$$

Partitions and compositions are strictly linked and in many occasions one deduces formulas from the ones starting from those for the other one. But
dealing with partitions is considerably less easy than dealing with compositions and formulas become more complicated.

A \(k\)-composition (a \(k\)-partition) of \(n\) is called coprime provided that \(\gcd(x_1, \ldots, x_k) = 1\) or, equivalently, if \(\gcd(x_1, \ldots, x_k, n) = 1\). We denote with \(c'_k(n)\) and with \(p'_k(n)\) the number of coprime \(k\)-compositions and \(k\)-partitions respectively. It is immediate to check that \(c_k(n) = \sum_{d|n} c'_k(n/d)\) as well as \(p_k(n) = \sum_{d|n} p'_k(n/d)\). Hence, by Möbius inversion, we also have

\[ c'_k(n) = \sum_{d|n} \mu(n/d) c_k(d) \quad (22) \]

and

\[ p'_k(n) = \sum_{d|n} \mu(n/d) p_k(d). \quad (23) \]

### 3 Coprime \(k\)-compositions and asymptotics

Since it is well known that

\[ J_k(n) = |\{(x_1, \ldots, x_k) \in \mathbb{N}^k : \forall i \in [k], \ 1 \leq x_i \leq n, \ \gcd(x_1, \ldots, x_k, n) = 1\}|, \]

the role of the Jordan totient functions in describing the number of coprime compositions or partitions is reasonably expected. We start with the easy case of compositions and discussing the asymptotic behaviour of both coprime compositions and partitions. Part (i) and (ii) in the following proposition are not a novelty. For instance they appear in [13, page 2]. We reprove briefly them, for completeness.

**Proposition 1.** Let \(k \in \mathbb{N}\) and write \(c_k(n) = \sum_{i=0}^{k-1} a_in^i\). Then the following facts hold:

(i) For every \(n \geq 1\), we have \(c'_k(n) = \sum_{i=0}^{k-1} a_i J_i(n)\). In particular, \(c'_k(n)\) is a \(\mathbb{Q}\)-linear combination of the Jordan totient functions.

(ii) For \(k \geq 2\), we have

\[ c_k(n) = \frac{1}{(k-1)!} J_{k-1}(n) + O(n^{k-2}). \]

(iii) For \(k \geq 2\), we have

\[ p_k(n) = \frac{1}{k!(k-1)!} J_{k-1}(n) + O(n^{k-2}). \]
Proof. (i) By (16), for every \( n \geq 1 \), we have \( c_k(n) = \sum_{i=0}^{k-1} a_in^i \). Hence, recalling the definition (1) and using (22), we get

\[
c_k'(n) = \sum_{d|n} \mu(n/d) c_k(d) = \sum_{d|n} \mu(n/d) \sum_{i=0}^{k-1} a_id^i = \sum_{i=0}^{k-1} a_i I_i(n).
\]

(ii) It follows immediately by (i) and by (14), since \( a_{k-1} = \frac{1}{(k-1)!} \).

(iii) Let \( \mathcal{E}_k(n)_* \) be the set of \( k \)-compositions of \( n \) with two or more equal parts and put \( c_k(n)_* = |\mathcal{E}_k(n)_*| \). Let also \( c_k'(n)_* \) be the number of coprime \( k \)-compositions of \( n \) with two or more equal parts. Similarly, let \( \mathcal{P}_k(n)_* \) be the set of \( k \)-partitions of \( n \) with two or more equal parts, \( p_k(n)_* = |\mathcal{P}_k(n)_*| \) and \( p_k'(n)_* \) be the number of coprime \( k \)-partitions of \( n \) with two or more equal parts. We achieve our goal in three steps.

**Step 1.** We show that \( p_k(n)_* = O(n^{k-2}) \) and that \( p_k'(n)_* = O(n^{k-2}) \).

For every partition \( \mathbf{p} = [x_1, \ldots, x_k] \in \mathcal{P}_k(n) \) with \( x_1 \leq x_2 \leq \cdots \leq x_k \), we let \( d := |\{x_1, \ldots, x_k\}| \). Clearly we have \( 1 \leq d \leq k \). Moreover, if \( \mathbf{p} \in \mathcal{P}_k(n)_* \), then we have \( 1 \leq d \leq k - 1 \). Let \( i_1, i_2, \ldots, i_d \) be the multiplicities of the smallest part, second smallest part of \( \mathbf{p} \) and so on.

Putting \( y_1 = x_1 \), \( y_2 = x_{i_1+1} \), \( \ldots \), \( y_d = x_{i_1+i_2+\cdots+i_{d-1}+1} \) for the actual distinct parts of \( \mathbf{p} \) in increasing order, we then have \( \sum_{j=1}^{d} i_j y_j = n \) with \( 1 \leq y_j \leq n \) and \( i_j \geq 1 \) for all \( j \in [d] \). Since \( i_1 + i_2 + \cdots + i_d = k \), we have that \( (i_1,\ldots,i_d) \in \mathcal{C}_d(k) \). Consider then the map

\[
i : \mathcal{P}_k(n) \to \bigcup_{d=1}^{k} \mathcal{C}_d(k)
\]

defined, for every \( \mathbf{p} \in \mathcal{P}_k(n) \), by \( i(\mathbf{p}) := (i_1, \ldots, i_d) \) and call \( i(\mathbf{p}) \) the vector of multiplicities of \( \mathbf{p} \). Let \( I := i(\mathcal{P}_k(n)_*) \) and note that

\[
|I| \leq \sum_{d=1}^{k-1} \mathcal{C}_d(k) =: C_k.
\]

We find an upper bound for the size of \( \mathcal{P}_k(n)_* \) looking to a uniform bound for the sizes of the preimages by \( i \) of the compositions in \( I \). Let \( (i_1,\ldots,i_d) \in I \),
for some $d \in [k-1]$. Then
\[
|i^{-1}((i_1, \ldots, i_d))| = |\{p \in \mathfrak{P}_k(n)_*: i(p) = (i_1, \ldots, i_d)\}|
= \left|\left\{(y_1, \ldots, y_d) \in \mathbb{N}^d : y_d = \frac{1}{i_d} \left( n - \sum_{j=1}^{d-1} i_j y_j \right), y_1 < \cdots < y_d \leq n \right\}\right|
\leq n^{d-1} \leq n^{k-2}.
\]

Thus, by (24), we get $p_k(n)_* = |\mathfrak{P}_k(n)_*| \leq C_k n^{k-2}$. Thus, $p_k(n)_* = O(n^{k-2})$.

Since, trivially, we have $p'_k(n)_* \leq p_k(n)_*$, we also have $p'_k(n)_* = O(n^{k-2})$.

So, in light of Step 1, it suffices to get an asymptotic for the number of coprime $k$-partitions with distinct parts.

**Step 2.** We show that $c_k(n)_* = O_k(n^{k-2})$ and $c'_k(n)_* = O_k(n^{k-2})$.

We have that $c_k(n)_* \leq k! p_k(n)_*$ as well as $c'_k(n)_* \leq k! p_k(n)_*$, so that $c_k(n)_* = O(n^{k-2})$ and $c'_k(n)_* = O(n^{k-2})$ immediately follow from Step 1.

**Step 3.** The conclusion.

We have
\[
\frac{c'_k(n) - c_k(n)_*}{k!} = p'_k(n) - p_k(n)_*,
\]
and thus, by Steps 1 and 2 and by (17), we get
\[
p'_k(n) = O(n^{k-2}) + \frac{c'_k(n) - O(n^{k-2})}{k!} = \frac{c'_k(n)}{k!} + O(n^{k-2}).
\]

By (ii), we then get
\[
p'_k(n) = \frac{J_{k-1}(n)}{(k-1)!k!} + O(n^{k-2}).
\]

The above proposition gives, among other things, an easy formula for calculating $c'_k(n)$ in terms of the prime divisors of $n$. For instance, by (14), we have
\[
c_2(n) = n - 1, \quad c_3(n) = \frac{n^2 - 3n + 2}{2} \quad \text{and} \quad c_4(n) = \frac{n^3 - 6n^2 + 11n - 6}{6}.
\]  

Thus, by Proposition (1)(i), we get for every $n \geq 1$,
\[
c'_2(n) = J_1(n) - J_0(n),
\]  

(25)
\[ c'_4(n) = \frac{1}{6} J_3(n) - J_2(n) + \frac{11}{6} J_1(n) \]
\[ = \frac{1}{6} n^3 \prod_{p|n} \left( 1 - \frac{1}{p^3} \right) - n^2 \prod_{p|n} \left( 1 - \frac{1}{p^2} \right) + \frac{11}{6} \varphi(n) \]
\[ = \frac{\varphi(n)}{6} \left[ n^2 \prod_{p|n} \left( 1 + \frac{1}{p^3} \right) - 6n \prod_{p|n} \left( 1 + \frac{1}{p^2} \right) + 11 \right]. \quad (28) \]

One can wonder if similar easy formulas hold for partitions too, just adapting the coefficients of the Jordan totient functions. In [2, Theorem 1.1, Theorem 2.2] it is shown that
\[ p'_2(n) = \frac{J_1(n)}{2}, \quad \text{for all } n \geq 3 \]
and
\[ p'_3(n) = \frac{J_2(n)}{12}, \quad \text{for all } n \geq 4. \]

These two formulas could seem encouraging. However in [2] it is observed that the situation becomes very complicated for \( k \geq 4 \) and no information is given for the general approach. Our paper is, among other things, a contribution to explain in which sense and why complications do arise.

Since \( p'(2) \neq \frac{J_1(2)}{2} \) as well as \( p'_3(3) \neq \frac{J_2(3)}{12} \), the limitations on \( n \) in (29) and in (30) cannot be eliminated but, in principle, one cannot exclude that the small cases for \( n \) could be included in a more rich formula involving as terms other Jordan totient functions.

Inspired to (29)-(30), we consider three problems:

**Problem 1.** Determine the \( k \geq 2 \) such that \( p'_k(n) \) is a \( \mathbb{C} \)-linear combination of the Jordan totient functions, in the whole domain \( n \geq 1 \).

**Problem 2.** Determine the \( k \geq 2 \) such that \( p'_k(n) \) is a \( \mathbb{C} \)-linear combination of the Jordan totient functions, in a domain \( n \geq N \) for some suitable \( N \in \mathbb{N} \) depending on \( k \).

**Problem 3.** Determine the \( k \geq 2 \) such that \( p'_k(n) \) is a \( \mathbb{C} \)-linear combination of the Jordan root totient functions.
4 Generating functions of polynomial and periodic sequences

In order to address Problems 1-3 we need to explore first the nature of the generating functions of polynomial and periodic sequences.

Lemma 2. (i) Let \( k \geq 1, s \geq 0 \) be integers. Then, we have

\[
\frac{1}{(1-z)^k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} z^n \tag{31}
\]

and

\[
\sum_{n \geq 0} n^s z^n = \frac{Q_s(z)}{(1-z)^{s+1}} \tag{32}
\]

for a polynomial \( Q_s(X) \in \mathbb{Q}[X] \) such that \( \deg(Q_s) \leq s \) and \( Q_s(1) = s! \).

(ii) Let \( F \) be a subfield of \( \mathbb{C} \), \( P(X) \in F[X] \) be a polynomial of degree \( t \geq 0 \) and \( T \) be an integer periodic function of period \( m \in \mathbb{N} \), with values in \( F \). Let \( k \geq 0 \). Then, we have

(a) \[
\sum_{n \geq k} P(n)z^n = \frac{Q(z)}{(1-z)^{t+1}} \tag{33}
\]

for some polynomial \( Q(X) \in F[X] \) such that \( \deg(Q) \leq t + k \). If \( P \) is not the zero polynomial, then \( Q(1) \neq 0 \).

(b) \[
\sum_{n \geq k} T(n)z^n = \frac{R(z)}{1-z^m} \tag{34}
\]

for some polynomial \( R(X) \in F[X] \) such that \( \deg(R) \leq m - 1 + k \).

For \( z \in D \) the expansions \( (31)-(34) \) represent numerical equalities in \( \mathbb{C} \).

Proof. (i) We use induction on \( k \). If \( k = 1 \) the equality is simply the well known \( \frac{1}{1-z} = \sum_{n \geq 0} z^n \). Assume that

\[
\frac{1}{(1-z)^k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} z^n \tag{35}
\]

holds for some \( k \geq 1 \) and prove it for \( k + 1 \).
Deriving both sides of equality (35), we get

\[
\frac{d}{dz} \left( \frac{1}{(1-z)^k} \right) = \frac{k}{(1-z)^{k+1}}
\]

and

\[
\frac{d}{dz} \left( \sum_{n \geq 0} \binom{n+k-1}{k-1} z^n \right) = \sum_{n \geq 1} \binom{n+k-1}{k-1} n z^{n-1} = \sum_{n \geq 0} \binom{n+k}{k} (n+1) z^n.
\]

Thus,

\[
\frac{1}{(1-z)^{k+1}} = \sum_{n \geq 0} \binom{n+k}{k} \frac{(n+1)}{k} z^n = \sum_{n \geq 0} \binom{n+k}{k} z^n.
\]

In order to get the second formula, we use induction on \(m\). If \(m = 0\) then the result is immediate using \(Q_0(z) = 1\). Assume that

\[
\sum_{n \geq 0} n^j z^n = \frac{Q_j(z)}{(1-z)^{j+1}}
\]

holds for all \(j \leq m - 1\) and show it for \(m\), where \(m \geq 1\). From (31), we have

\[
\frac{1}{(1-z)^{m+1}} = \sum_{n \geq 0} \binom{n+m}{m} z^n = \sum_{n \geq 0} \frac{\prod_{j=1}^{m} (n+j)}{m!} z^n.
\]

The factor \(\frac{\prod_{j=1}^{m} (n+j)}{m!}\) is a polynomial in \(n\) with rational positive coefficients. Its degree is \(m\) and the leading coefficient is \(\frac{1}{m!}\). Let \(b_j \in \mathbb{Q}\), for \(j \in [m-1]_0\) be such that

\[
\frac{\prod_{j=1}^{m} (n+j)}{m!} = \frac{n^m}{m!} + \sum_{j=0}^{m-1} b_j n^j.
\]

Then, by (37), we get

\[
\frac{1}{(1-z)^{m+1}} = \sum_{n \geq 0} \left( \frac{n^m}{m!} + \sum_{j=0}^{m-1} b_j n^j \right) z^n = \frac{1}{m!} \sum_{n \geq 0} n^m z^n + \sum_{j=0}^{m-1} b_j \sum_{n \geq 0} n^j z^n.
\]
By the inductive hypothesis, it follows that
\[
\sum_{n \geq 0} n^m z^n = m! \left( \frac{1}{(1-z)^{m+1}} - \sum_{j=0}^{m-1} b_j \frac{Q_j(z)}{(1-z)^{j+1}} \right) = m! \frac{1 - \sum_{j=0}^{m-1} b_j Q_j(z) (1-z)^{m-j}}{(1-z)^{m+1}}.
\]

Now it is enough to define \( Q_m(z) = m! (1 - \sum_{j=0}^{m-1} b_j Q_j(z) (1-z)^{m-j}) \), in order to reach
\[
\sum_{n \geq 0} n^m z^n = \frac{Q_m(z)}{(1-z)^{m+1}}.
\]

Since every \( Q_j(z) \) is a polynomial of degree at most \( j \), we have that \( Q_m(z) \) is a polynomial of degree at most \( m \). Of course we also have \( Q_m(1) = m! \).

(ii) Since any sum for \( n \geq k \) can be written as the sum for \( n \geq 0 \) minus the same sum for \( n \in [k-1]_0 \), the results follows immediately from those for \( k = 0 \). Thus, we prove \((33)\) and \((34)\) only for \( k = 0 \).

(a) If \( P \) is the zero polynomial we reach \((33)\) considering the zero polynomial \( Q \). Assume next that \( P \) is not the zero polynomial. Then we write \( P(n) = \sum_{j=0}^t b_j n^j \), where \( b_j \in F \) for \( j \in [t]_0 \) and \( b_t \neq 0 \). By \((32)\), we have
\[
\sum_{n \geq 0} P(n) z^n = \sum_{j=0}^t b_j \left( \sum_{n \geq 0} n^j z^n \right) = \sum_{j=0}^t b_j Q_j(z) = \sum_{j=0}^{t} b_j Q_j(z) (1-z)^{t-j} \sum_{n \geq 0} \frac{(1-z)^j}{(1-z)^{j+1}}.
\]

Define then \( Q(z) := \sum_{j=0}^{t} b_j Q_j(z) (1-z)^{t-j} \) and note that this is a polynomial with coefficients in \( F \) with degree less or equal to \( t \) and such that \( Q(1) = b_t! \neq 0 \).

(b) By the periodicity of \( T \), we have that \( T(j + km) = T(j) \) holds for all \( j \in [m-1]_0 \) and \( k \in \mathbb{N} \cup \{0\} \). Hence,
\[
\sum_{n \geq 0} T(n) z^n = \sum_{j=0}^{m-1} T(j) \sum_{k \geq 0} z^{j+km} = \sum_{j=0}^{m-1} T(j) z^j \sum_{n \geq 0} z^{mn} = \sum_{j=0}^{m-1} T(j) z^j \frac{1}{1-z^m}
\]
and we reach \((34)\), just by setting \( R(X) := \sum_{j=0}^{m-1} T(j) X^j \).

Finally, note that all the established equalities surely hold as equalities in \( \mathbb{C} \) for \( |z| < 1 \) because \( \sum_{n \geq 0} z^n = \frac{1}{1-z} \) holds in \( \mathbb{C} \) for \( |z| < 1 \) and that fact is the starting point of our arguments.
5 Coprime partitions and Jordan root totient functions

We start this section with a classical result which describes the nature of the complex sequences having as generating function a rational function. We greatly inspire to [14], collecting some ideas in there. However since no statement in [14] perfectly fits our needs, for the sake of clarity and completeness we state an explicit result giving also the short proof.

**Proposition 2.** Let \((a_n)_{n \geq 0}\) be a sequence in \(\mathbb{C}\). Assume that there exist \(P(X), Q(X) \in \mathbb{C}[X]\), with \((P(X), Q(X)) = 1\), \(P(X)\) not constant, \(P(0) = 1\) and \(\deg(Q) < \deg(P)\) such that

\[
\sum_{n \geq 0} a_n z^n = \frac{Q(z)}{P(z)}.
\]

Let \(\alpha_1^{-1}, \ldots, \alpha_s^{-1} \in \mathbb{C}^*\), \(s \geq 1\), be the distinct roots of \(P(z)\) and \(b_j \geq 1\) be the multiplicity of \(\alpha_j^{-1}\), for \(j \in [s]\). Then, for every \(j \in [s]\), there exists \(C_j(X) \in \mathbb{C}[X]\) of degree \(b_j - 1\), such that

\[
a_n = \sum_{j=1}^{s} C_j(n) \alpha_j^n,
\]

for all \(n \geq 0\).

**Proof.** Let \(c_i \in \mathbb{C}\), for \(i \in [b]\), be such that \(P(X) = \sum_{i=0}^{b} c_i X^i\), where \(b = \sum_{j=1}^{s} b_j = \deg(P)\). Then, since \(P(0) = 1\), we have

\[
P(z) = c_b \prod_{j=1}^{s} (z - \alpha_j^{-1})^{b_j} = \frac{c_b(-1)^b}{\prod_{j=1}^{s} \alpha_j^{b_j}} \prod_{j=1}^{s} (1 - \alpha_j z)^{b_j} = \prod_{j=1}^{s} (1 - \alpha_j z)^{b_j}.
\]

Note that, since \((P(X), Q(X)) = 1\), \(b_j\) is the order of the pole \(\alpha_j^{-1}\) in \(\frac{Q(z)}{P(z)}\). Since \(\deg(Q) < \deg(P)\), then partial fraction expansion yields

\[
\frac{Q(z)}{P(z)} = \sum_{j=1}^{s} \sum_{i=1}^{b_j} \frac{r_{ji}}{(1 - \alpha_j z)^i}
\]

for suitable \(r_{ji} \in \mathbb{C}\), for \(j \in [s]\) and \(i \in [b_j]\). Observe that \(r_{jb_j} \neq 0\) because otherwise the order of the pole \(\alpha_j^{-1}\) in \(\frac{Q(z)}{P(z)}\) would be less than \(b_j\). By Lemma
2(i), we then get

\[
\frac{Q(z)}{P(z)} = \sum_{j=1}^{s} \sum_{i=1}^{b_j} \sum_{n=0}^{\infty} r_{ji} \left( \frac{n+i-1}{i-1} \right) \alpha_j^n z^n
\]

\[
= \sum_{n \geq 0} \sum_{j=1}^{s} \left( \sum_{i=1}^{b_j} r_{ji} \left( \frac{n+i-1}{i-1} \right) \right) \alpha_j^n z^n
\]

\[
= \sum_{n \geq 0} \sum_{j=1}^{s} C_j(n) \alpha_j^n z^n,
\]

where \( C_j(n) = \sum_{i=1}^{b_j} r_{ji} \left( \frac{n+i-1}{i-1} \right) \) is a complex polynomial. Since \( \left( \frac{n+i-1}{i-1} \right) \) is a polynomial in \( n \) of degree \( i-1 \) and we have observed that \( r_{jb_j} \neq 0 \), it is finally clear that \( \deg(C_j) = b_j - 1 \).

Then we have

\[
\sum_{n \geq 0} a_n z^n = \sum_{n \geq 0} \sum_{j=1}^{s} C_j(n) \alpha_j^n z^n
\]

so that a comparison of the coefficient of \( z^n \) gives \( a_n = \sum_{j=1}^{s} C_j(n) \alpha_j^n \), for all \( n \geq 0 \).

We are now ready to solve Problem 3.

**Theorem 1.** Let \( k \geq 2 \). Then the following facts hold:

(i) If \( k \in \{2, 3\} \) then there exist \( P(X) \in \mathbb{Q}[X] \) and \( T \) an integer periodic function such that

\[
p_k(n) = P(n) + T(n), \quad \text{for all} \quad n \geq 1.
\]

\( T \) is of period 2 for \( k = 2 \), and of period 6 for \( k = 3 \).

(ii) \( p_k' \) is a \( \mathbb{C} \)-linear combination of the Jordan root totient functions.

**Proof.** Consider the sequence \((p_k(n))_{n \geq 1}\). By (20), we have

\[
\sum_{n \geq 1} p_k(n) z^n = \frac{z^k}{(1-z)(1-z^2) \cdots (1-z^k)},
\]

so that Proposition 2 applies, with \( P(z) = (1-z)(1-z^2) \cdots (1-z^k) \) and \( Q(z) = z^k \). The roots of \( P(z) \) are the elements of the set \( V = \cup_{m=1}^{k} U_m \) and, since \( V \) is closed under inversion, we have that \( V \) coincides with the
set of the inverses of the roots of $P$, that we need to consider for applying Proposition 2. Let $1 = \omega_1, -1 = \omega_2, \ldots, \omega_s$ be the distinct elements of $V$ and let $b_j \geq 1$ be the multiplicity of $\omega_j$, for $j \in [s]$. Note that $s = \sum_{i=1}^{k} \phi(i)$, because

$$V = \bigcup_{m=1}^{2\pi i n_j} \{ z \in U_m : z \text{ is primitive} \}.$$ 

Moreover, if $\omega_j = e^{\pi i/3}$ for some positive integers $u_j, v_j$ such that $v_j \leq k$ and $(u_j, v_j) = 1$, then $b_j = \lfloor k/v_j \rfloor$. In particular, we have $b_1 = k$ and $b_2 = \lfloor k/2 \rfloor$. Thus, by Proposition 2, we get

$$p_k(n) = \sum_{j=1}^{s} C_j(n)\omega_j^n, \quad \text{for all } n \geq 1 \quad (39)$$

for suitable $C_j(X) \in \mathbb{C}[X]$ of degree $b_j - 1$, $j \in [s]$.

(i) If $k = 2$, then equality (39) becomes

$$p_2(n) = C_1(n) + C_2(n)(-1)^n \quad (40)$$

with $\deg(C_1) = 1$ and $\deg(C_2) = 0$. Since the polynomial $C_2$ is constant and not the zero polynomial, once set $T(n) := C_2(n)(-1)^n$, it is immediately observed that $T$ is a periodic function of period 2.

If $k = 3$, then equality (39) becomes

$$p_3(n) = C_1(n) + C_2(n)(-1)^n + C_3(n)\omega^n + C_4(n)\omega^2n$$

where $\omega = e^{2\pi i/3} = -1 + \sqrt{3}/2$, $\deg(C_1) = 2$ and $\deg(C_j) = 0$, for $j \in \{2, 3, 4\}$. Let $T(n) := C_2(n)(-1)^n + C_3(n)\omega^n + C_4(n)\omega^2n$. Since the polynomials $C_j$ are constant and not the zero polynomials for all $j \in \{2, 3, 4\}$, we deduce that $T$ is periodic of period 6.

(ii) By (39) and by (40), we have

$$p_k'(n) = \sum_{d|n} \sum_{j=1}^{s} C_j(d)\omega_j^d \mu(n/d) = \sum_{j=1}^{s} \sum_{d|n} C_j(d)\omega_j^d \mu(n/d). \quad (41)$$

Since the $C_j$ are polynomials, it is clear that every term $\sum_{d|n} C_j(d)\omega_j^d \mu(n/d)$ is a $\mathbb{C}$-linear combination of Jordan root totient functions. Thus, by (41), also $p_k'$ is a $\mathbb{C}$-linear combination of the Jordan root totient functions.  

\[\Box\]
6 Partitions, polynomials and periodic functions

Proposition 3. (i) Let $k \geq 2$. Then there exists no $P(X) \in \mathbb{C}[X]$ such that
\[ p_k(n) = P(n), \quad \text{for all} \quad n \geq 1. \]

(ii) Let $k \geq 4$. Then there exists no $P(X) \in \mathbb{C}[X]$ and no integer periodic function $T$ such that
\[ p_k(n) = P(n) + T(n), \quad \text{for all} \quad n \geq 1. \]

Proof. (i) Assume, by contradiction, that there exists $P(X) \in \mathbb{C}[X]$ of degree $t \geq 0$ such that $p_k(n) = P(n)$ for all $n \geq 1$. By Lemma 2(ii), we have
\[ \sum_{n \geq 1} P(n)z^n = \frac{Q(z)}{(1 - z)^{t+1}}, \]
for some $Q(X) \in \mathbb{C}[X]$ and the above expansion is valid for the complex numbers $z \in D$. On the other hand, by (20), we also have
\[ \sum_{n \geq 1} p_k(n)z^n = \frac{z^k}{(1 - z)(1 - z^2) \cdots (1 - z^k)}, \quad (42) \]
with the above expansion also valid in $D$. Thus, we get the equality
\[ \frac{z^k}{(1 - z)(1 - z^2) \cdots (1 - z^k)} = \frac{Q(z)}{(1 - z)^{t+1}} \]
for all the complex numbers $z \in D$. This implies that the functions on the two sides must have the same poles on $D$. But this is not the case because clearly $-1$ is a pole for the left side function but not for the right side function.

(ii) Assume, by contradiction, that there exists $P(X) \in \mathbb{C}[X]$ of degree $t \geq 0$ and $T$ a periodic function of period $m \in \mathbb{N}$ such that
\[ p_k(n) = P(n) + T(n), \quad \text{for all} \quad n \geq 1. \]

Passing to the generating functions in the above equality one gets now, by (42) and by Lemma 2(ii),
\[ \frac{z^k}{(1 - z)(1 - z^2) \cdots (1 - z^k)} = \frac{Q(z)}{(1 - z)^{t+1}} + \frac{R(z)}{1 - z^m} \]
for suitable $Q(X), R(X) \in \mathbb{C}[X]$. Such equality is valid in $D$, hence the functions on the two sides must have the same poles, with the same multiplicities, on $\overline{D}$. Since $k \geq 4$, the pole $-1$ has multiplicity at least 2 for the left side function. But the poles different from 1, for the right side function are the roots of $1 - z^m$ and thus appear with multiplicity 1.

7 Coprime partitions and Jordan totient functions

We are now in position for solving the Problems 1 and 2.

Corollary 1. Let $k \geq 2$. Then $p'_k(n)$ is not a $\mathbb{C}$-linear combination of the Jordan totient functions.

Proof. Assume the contrary. Then there exist $s \in \mathbb{N}$ and $c_i \in \mathbb{C}$, for all $i = 0, \ldots, s$, such that

$$p'_k(n) = \sum_{i=0}^{s} c_i J_i(n), \quad \text{for all } n \geq 1.$$  

Writing the above relation for all $d \mid n$ and using the Möbius inversion, we then get for every $n \geq 1$,

$$p_k(n) = \sum_{d \mid n} p'_k(d) = \sum_{d \mid n} \sum_{i=0}^{s} c_i J_i(d) = \sum_{i=0}^{s} c_i \sum_{d \mid n} J_i(d) = \sum_{i=0}^{s} c_i n^i = P(n),$$  

where $P(X) = \sum_{i=0}^{s} c_i X^i \in \mathbb{C}[X]$, against Proposition 3(i).

Theorem 2. Let $k \geq 2$. Let $N \in \mathbb{N}$ be minimum such that, for $n \geq N$, $p'_k(n)$ is a $\mathbb{C}$-linear combination of the Jordan totient functions. Then one of the following cases hold:

(i) $k = 2$, $N = 3$ and $p'_2(n) = \frac{J_1(n)}{2}$;

(ii) $k = 3$, $N = 4$ and $p'_3(n) = \frac{J_2(n)}{12}$.

In particular, if $p'_k(n)$ is definitely a $\mathbb{C}$-linear combination of the Jordan totient functions, then it is a rational multiple of $J_{k-1}(n)$ and $k \in \{2, 3\}$.

Proof. Surely $p'_k(n)$ cannot be definitely a multiple of $J_0(n)$. Thus, there exists $s \in \mathbb{N}$ and $c_i \in \mathbb{C}$, for $i \in [s]_0$, with $c_s \neq 0$, such that

$$p'_k(n) = \sum_{i=0}^{s} c_i J_i(n), \quad \text{for all } n \geq N.$$  

20
As a consequence of Corollary 1, we have that $N \geq 2$. Thus if $n \geq N$, we also have $n \geq 2$ and so $J_0(n) = 0$. Hence, whatever $c_0$ is in (44), we can surely guarantee the same equality adopting $c_0 = 0$. Let then

$$p'_k(n) = \sum_{i=1}^{s} c_i J_i(n), \quad \text{for all } n \geq N.$$  \hspace{1cm} (45)

and $P(X) = \sum_{i=1}^{s} c_i X^i \in \mathbb{C}[X]$ be the corresponding polynomial. Note that $\deg(P) = s$.

Define the function $f^k_N : \mathbb{N} \to \mathbb{C}$ given, for every $n \in \mathbb{N}$, by

$$f^k_N(n) = \sum_{d|n \atop d < N} p'_k(d) - \sum_{i=1}^{s} c_i J_i(d).$$  \hspace{1cm} (46)

We claim that

$$p_k(n) = P(n) + f^k_N(n), \quad \text{for all } n \geq 1.$$  \hspace{1cm} (47)

Indeed, by (45) and Möbius inversion, for every $n \geq 1$, we have

$$p_k(n) = \sum_{d|n \atop d < N} p'_k(d) = \sum_{d|n \atop d < N} p'_k(d) + \sum_{d|n \atop d \geq N} p'_k(d)$$

$$= \sum_{d|n \atop d < N} p'_k(d) + \sum_{d|n \atop d \geq N} \sum_{i=1}^{s} c_i J_i(d) = \sum_{d|n \atop d < N} p'_k(d) + \sum_{i=1}^{s} c_i \sum_{d|n \atop d \geq N} J_i(d)$$

$$= \sum_{d|n \atop d < N} p'_k(d) + \sum_{i=1}^{s} c_i \left( \sum_{d|n \atop d < N} J_i(d) - \sum_{d|n \atop d \geq N} J_i(d) \right)$$

$$= f^k_N(n) + \sum_{i=1}^{s} c_i n^i = f^k_N(n) + P(n).$$

We next claim that

$f^k_N$ is periodic of integer period $m := \delta(N).$  \hspace{1cm} (48)
Namely, since every $d < N$ divides $m$, we have for all $\ell \in \mathbb{N}$,

$$
\begin{align*}
  f^k_N(n + \ell m) &= \sum_{d | n + \ell m, \quad \ell < N} \left( p'_k(d) - \sum_{i=1}^{s} c_i J_i(d) \right) \\
  &= \sum_{d | n, \quad \ell < N} \left( p'_k(d) - \sum_{i=1}^{s} c_i J_i(d) \right) = f^k_N(n).
\end{align*}
$$

By (47) and (48), we then have that $p_k$ is the sum of a polynomial and of an integer periodic function. By Proposition 3 (i) this rules out $k \geq 4$, so that $k \in \{2, 3\}$.

Now, by (21) and by (47), we get

$$
  f^k_N(n) + P(n) = \frac{1}{k!(k-1)!} n^{k-1} + O(n^{k-2}).
$$

By (48), $\frac{f^k_N(n)}{n^{k-1}}$ tends to 0 as $n$ goes to infinity. Thus, $\frac{P(n)}{n^{k-1}}$ tends to $\frac{1}{k!(k-1)!}$ as $n$ goes to infinity, which implies $s = \deg(P) = k - 1$ and $c_{k-1} = \frac{1}{k!(k-1)!}$. In particular, we have $P(X) = \sum_{i=1}^{k-1} c_i X^i$.

If $k = 2$, this gives $P(X) = \frac{X}{2}$ and (15) becomes $p'_2(n) = \frac{J_1(n)}{2}$ for $n \geq N$. We know from (29) that that equality holds for $N = 3$. Since it does not hold for $N = 2$, we deduce that the minimum $N$ for which it holds is $N = 3$.

If $k = 3$, we get $P(X) = c_1 X + \frac{X^2}{12}$ and (15) becomes

$$
  p'_3(n) = c_1 J_1(n) + \frac{J_2(n)}{12} \quad \text{for all } n \geq N. \quad (49)
$$

By (30), we know that

$$
  p'_3(n) = \frac{J_2(n)}{12} \quad \text{for all } n \geq 4. \quad (50)
$$

Note that equality (50) does not hold for $n = 3$. Computing $p'_3(4)$ in (49) and (50), we get $c_1 J_1(4) = 0$, which implies $c_1 = 0$. It follows that (50) gives the only possible expression of $p'_3(n)$ as a C-linear combination of the Jordan totient functions for $n \geq N$, with minimum $N$, and such minimum is $N = 4$. \qed
8 Computation of some generalized Jordan totient functions

In the last two sections of the paper we illustrate how to explicitly find the polynomials $C_j$ of (41) relying on the generating function of $p_k(n)$. This allows us to represent $p'_k$ as a $\mathbb{C}$-linear combination of Jordan root functions. Next we explain how explicitly compute the Jordan modulo totient functions in which those Jordan root functions split, making use of the Jordan-Dirichlet totient functions. We limit ourselves to treat $k \in \{2, 3, 4\}$. Anyway the general method should be clear.

In this section, we gather together all the computations which we will need later. They illustrate very well how to connect the diverse generalizations of the Jordan totient functions in order to obtain one from the other. For this reason they are of interest in themselves. In the next section we will examine separately $p'_2, p'_3$ and $p'_4$.

Lemma 3. Write $n = 3^b m_1$ with $3, m_1 = 1$. Then

$$J^{1,3}_0(n) = \begin{cases} 
0 & \text{if } \exists \ p \equiv 1 \pmod{3}, \ p \mid m_1; \\
(-1)^\Omega(n) 2^{\omega(m_1) - 1} & \text{if } b \geq 2; \\
\text{otherwise.} 
\end{cases} \quad (51)$$

Proof. For shortness, write $f(n)$ instead of $J^{1,3}_0(n)$. Then

$$f(n) = \sum_{d \mid n, \ (d \equiv 1 \pmod{3})} \mu(n/d).$$

If $d \equiv 1 \pmod{3}$ and $d \mid n$, then $d \mid m_1$. Thus, $3^b \mid n/d$ over all such divisors $d$ and $n/d = 3^b(m_1/d)$ with $3^b$ and $m_1/d$ coprime. Thus, by the multiplicativity of the $\mu$ function, we get

$$f(n) = \sum_{d \equiv 1 \pmod{3}} \mu(n/d) = \sum_{d \mid m_1} \mu(3^b)\mu(m_1/d) = \mu(3^b)f(m_1). \quad (52)$$

Thus, if $b \geq 2$ we have $f(n) = 0$. Hence, it suffices to study $f(m_1)$. Let $\chi$ be the unique non principal Dirichlet character modulo $3$. Then $\chi(k) = 1$ if $k \equiv 1 \pmod{3}$, $\chi(k) = -1$ if $k \equiv 2 \pmod{3}$ and $\chi(k) = 0$ if $\gcd(k, 3) > 1$. It is easily seen that

$$f(m_1) = \frac{1}{2} \sum_{d \mid m_1} (\chi(d) + 1)\mu(m_1/d) = \frac{1}{2} \sum_{d \mid m_1} \chi(d)\mu(m_1/d) + \frac{1}{2} \sum_{d \mid m_1} \mu(m_1/d).$$
Since \( m_1 \) is coprime to 3, by (11), we get for \( m_1 > 1, \)
\[
 f(m_1) = \frac{1}{2} \sum_{d \mid m_1} \chi(d)\mu(m_1/d) = \frac{1}{2} \chi(m_1) \prod_{p \mid m_1} \left(1 - \frac{1}{\chi(p)}\right)
 = \frac{1}{2} \chi(m_1) \prod_{p \equiv 1 \pmod{3}} (1 - 1) \prod_{p \equiv 2 \pmod{3}} (1 + 1)
 = \begin{cases} 
 0 & \text{if } p \mid m_1 \text{ for some } p \equiv 1 \pmod{3}; \\
 (-1)^{\Omega(m_1)2\omega(m_1)-1} & \text{if } p \equiv 2 \pmod{3} \text{ for all } p \mid m_1.
\end{cases}
\]
Thus, by (52), taking into consideration that \( f(1) = 1, \) the formula for \( f(n) \)
immediately follows. \( \square \)

**Lemma 4.** Write \( n = 2^a m \) where \( m \) is odd. Then

(i)
\[
 J_{(0,-1)}(n) = \begin{cases} 
 -1 & \text{if } n = 1; \\
 2 & \text{if } n = 2; \\
 0 & \text{if } n > 2.
\end{cases}
\]

(ii)
\[
 J_{(1,-1)}(n) = \begin{cases} 
 -\varphi(n) & \text{if } a = 0; \\
 \varphi(n) & \text{if } a \geq 2; \\
 3\varphi(n) & \text{if } a = 1.
\end{cases}
\]

(iii)
\[
 J_{(0,i^k)}(n) = \begin{cases} 
 i^k & \text{if } n = 1; \\
 -1 - i^k & \text{if } n = 2; \\
 0 & \text{if } \exists p \equiv 1 \pmod{4}, p \mid m; \\
 2 & \text{if } n = 4; \\
 0 & \text{if } a \geq 3 \text{ or } a = 2 \text{ and } m > 1; \\
 i^k(-1)^{\Omega(n)2\omega(m)} & \text{otherwise},
\end{cases}
\]
for \( k = 1, 3. \)

(iv) Let \( n = 3^b m_1 \) with \( \gcd(m_1, 3) = 1, \) and denote by \( \omega \) the principal 3-root of 1. Then
\[
 J_{(0,\omega^k)}(n) = \begin{cases} 
 \omega^k & \text{if } n = 1; \\
 -\omega^k + 1 & \text{if } n = 3; \\
 0 & \text{if } \exists p \equiv 1 \pmod{3}, p \mid m_1; \\
 0 & \text{if } b \geq 2; \\
 (\omega^k - \omega^{2k})(-1)^{\Omega(n)2\omega(m_1)-1} & \text{otherwise},
\end{cases}
\]
for $k = 1, 2$.

Proof. (i) For $n = 1$ and $n = 2$ one makes a direct computation. For $n \geq 3$, note that

$$\sum_{d|m} \mu(m/d) = \sum_{d|m} \mu(2^a(m/d)) = \mu(2^a) \sum_{d|m} \mu(m/d). \quad (53)$$

If $m = 1$, then $a \geq 2$ and thus $\mu(2^a) = 0$ so that, by (53), we get

$$\sum_{d|m, \ odd} \mu(n/d) = 0.$$  

If $m > 1$, then $m \geq 3$ so that

$$\sum_{d|m, \ odd} \mu(m/d) = 0.$$  

and, by (53), we again get

$$\sum_{d|m, \ odd} \mu(n/d) = 0.$$  

It follows that

$$0 = \sum_{d|n, \ odd} \mu(n/d) = \sum_{d|n, \ even} \mu(n/d) + \sum_{d|n, \ odd} \mu(n/d) = \sum_{d|n} \mu(n/d).$$  

Hence,

$$J_{(0,-1)}(n) = \sum_{d|n} (-1)^d \mu(n/d) = - \sum_{d|n, \ odd} \mu(n/d) + \sum_{d|n, \ even} \mu(n/d) = 0.$$  

(ii) We start again with the odd $d$’s getting

$$\sum_{d|n, \ odd} d \mu(n/d) = \sum_{d|m} d \mu(2^a(m/d)) = \mu(2^a) \sum_{d|m} d \mu(m/d) = \mu(2^a) \phi(m). \quad (54)$$  

The above calculation proves (ii) if $a = 0$. If $a \geq 2$, the right–hand side above is zero. Hence, we have

$$\phi(n) = \sum_{d|n} d \mu(n/d) = \sum_{d|n, \ even} d \mu(n/d) + \sum_{d|n, \ odd} d \mu(n/d) = \sum_{d|n} d \mu(n/d),$$  

so that we also have

$$J_{(1,-1)}(n) = \sum_{d|n} (-1)^d d \mu(n/d) = \sum_{d|n, \ even} d \mu(n/d) - \sum_{d|n, \ odd} d \mu(n/d) = \phi(n).$$  

Finally, if $a = 1$, we have $n = 2m$ and then every even divisor of $2m$ is of
the form $2d$ for $d \mid m$. Thus,

$J_{(1,-1)}(n) = \sum_{d \mid 2m, d \text{ even}} d\mu(2m/d) - \sum_{d \mid 2m, d \text{ odd}} d\mu(2m/d) = \sum_{d \mid m} (2d)\mu(2m/2d) + \sum_{d \mid m} d\mu(m/d) = 2 \sum_{d \mid m} d\mu(m/d) + \phi(m) = 3\phi(m) = 3\phi(n)$.

(iii) The function $f_k(n) = i^{k(n-1)}$ defined for odd $n$ and extended multiplicatively to all positive integers by putting $f_k(n) = 0$ for even $n$, is totally multiplicative. Indeed, if $m, n$ are both odd, then $f_k(mn) = i^{k(mn-1)}$ and $f_k(m)f_k(n) = i^{k(m-1)}i^{k(n-1)} = i^{k(m+n-2)}$ and then the equality

$f_k(mn) = f_k(m)f_k(n)$

is equivalent to

$i^{k(mn-1)} = i^{k(m+n-2)}$,

which is equivalent to

$1 = i^{k(mn-m-n+1)} = i^{k(m-1)(n-1)}$,

which holds since both $m - 1$ and $n - 1$ are even. If instead at least one between $m$ and $n$ is even, then $mn$ is even, so that $f_k(mn) = 0 = f_k(m)f_k(n)$.

So,

$\sum_{d \mid n} i^{kd}\mu(n/d) = i^k\mu(2^\alpha)\sum_{d \mid m} i^{k(d-1)}\mu(m/d) = i^k\mu(2^\alpha)(f_k*\mu)(m)$, (55)

and $f_k*\mu$ is multiplicative. If $m = p^\lambda$, with $p$ an odd prime and $\lambda \geq 1$, then

$(f_k*\mu)(p^\lambda) = \sum_{d \mid p^\lambda} i^{k(d-1)}\mu(p^\lambda/d) = -i^{k(p^\lambda-1)} + i^{k(p^\lambda-1)}$

\[
= \begin{cases} 
0 & \text{if } p \equiv 1 \pmod{4}; \\
1 + (-1)^{k+1} & \text{if } p \equiv 3 \pmod{4}, 2 \nmid \lambda; \\
(-1)(1 + (-1)^{k+1}) & \text{if } p \equiv 3 \pmod{4}, 2 \mid \lambda.
\end{cases}
\]

So, if $k = 1, 3$, we get that $(f_k*\mu)(p^\lambda)$ equals 0 when $p \equiv 1 \pmod{4}$ and equals $2(-1)^\lambda$ if $p \equiv 3 \pmod{4}$. We thus get that for $m > 1$,

$(f_k*\mu)(m) = \begin{cases} 
0 & \text{if } p \equiv 1 \pmod{4} \text{ for some } p \mid m; \\
(-1)^{\Omega(m)}2^\omega(m) & \text{if } p \equiv 3 \pmod{4} \text{ for all } p \mid m.
\end{cases}$ (56)
If $a = 0$, then $n = m$ is odd and thus, by (55), we get

$$J_{(0,i^k)}(n) = \sum_{d|n} i^k d \mu(n/d) = i^k (f_k * \mu)(m),$$

and this is $i^k$ if $n = m = 1$, 0 if $m > 1$ and $p \mid m$ for some prime number $p \equiv 1 \pmod{4}$ and $i^k (-1)^{\Omega(m)} 2^{\omega(m)} = i^k (-1)^{\Omega(m)} 2^{\omega(m)}$, otherwise.

If $a \geq 2$, then by (55), the sum over the divisors $d$ of $n$ which are odd is zero since $\mu(2^a) = 0$. Thus, the given sum is concentrated on the even divisors and we have

$$J_{(0,i^k)}(n) = \sum_{d|n/2} (-1)^d \mu((n/2)/d)$$

for $k = 1, 3$. Moreover, by (i) and the fact that $n/2 \geq 2$, this last sum is zero unless $n/2 = 2$ in which case it is 2.

Let finally $a = 1$, so that $n = 2m$. We compute that the given sum is $-1 - i^k$ for $n = 2$. Now assume $m > 1$. In this case, by (55), we have

$$\sum_{d|n} i^k d \mu(n/d) = -i^k (f_k * \mu)(m)$$

and, by (56), this is zero unless all prime factors of $m$ are congruent to 3 modulo 4 in which case it is $-i^k (-1)^{\Omega(m)} 2^{\omega(m)} = i^k (-1)^{\Omega(m)} 2^{\omega(m)}$. As for the even divisors, these are of the form $2d$ for some $d \mid m$, and we get

$$\sum_{d|n} i^k d \mu(n/d) = \sum_{d|2m} i^k 2d \mu(2m/2d) = \sum_{d|m} (-1)^d \mu(m/d),$$

and, by (i), this last sum is 0 since $m \geq 3$.

(iv) We have

$$J_{(0,\omega,i^k)}(n) = \sum_{d|n} \omega d \mu(n/d)$$

$$= \omega^k \sum_{d|n \text{ (mod 3)}} \mu(n/d) + \omega^2 k \sum_{d|n \text{ (mod 3)}} \mu(n/d) + \sum_{d|n \text{ (mod 3)}} \mu(n/d)$$

$$:= \omega^k S_1(n) + \omega^2 k S_2(n) + S_0(n), \quad (57)$$
where, for shortness, we have set $S_j(n) := J^{j,3}_0(n)$, for $j \in \{0, 1, 2\}$. Thus, we need to compute $S_j(n)$, for $j \in \{0, 1, 2\}$.

The easiest one is $S_0$. If $3 \nmid n$, we obviously have that $S_0(n) = 0$. If $3 \mid n$, that is $b \geq 1$, we instead have, by (2):

$$S_0(n) = \sum_{d \mid n \atop 3 \nmid d} \mu(n/d) = \sum_{d \mid n/3} \mu((n/3)/d) = \begin{cases} 1 & \text{if} \quad n = 3; \\ 0 & \text{if} \quad n > 3. \end{cases}$$

So, $S_0(n)$ is always 0 except if $n = 3$ when it is 1. As for $S_1$, $S_2$, we write

$$S_1(n) = \sum_{d \mid n \atop d \equiv 1 \pmod{3}} \mu(n/d) = \mu(3^b) \sum_{d \mid m_1 \atop d \equiv 1 \pmod{3}} \mu(m_1/d) = \mu(3^b)S_1(m_1),$$

and similarly $S_2(n) = \mu(3^b)S_2(m_1)$. By (51), we have

$$S_1(m_1) = \begin{cases} 1 & \text{if} \quad m_1 = 1; \\ 0 & \text{if} \quad \exists p \equiv 1 \pmod{3}, p \mid m_1; \\ (-1)^{\Omega(m_1)} 2^{\omega(m_1) - 1} & \text{otherwise}. \end{cases}$$

Since

$$S_1(m_1) + S_2(m_1) = \sum_{d \mid m_1} \mu(m_1/d)$$

is 1 for $m_1 = 1$ and 0 for $m_1 > 1$, we get that

$$S_2(m_1) = \begin{cases} 0 & \text{if} \quad m_1 = 1; \\ 0 & \text{if} \quad \exists p \equiv 1 \pmod{3}, p \mid m_1; \\ (-1)^{\Omega(m_1)} 2^{\omega(m_1) - 1} & \text{otherwise}. \end{cases}$$

Thus, by (57), get that

$$J_{(0,\omega^k)}(n) = \begin{cases} \omega^k & \text{if} \quad n = 1; \\ -\omega^k + 1 & \text{if} \quad n = 3; \\ 0 & \text{if} \quad \exists p \equiv 1 \pmod{3}, p \mid m_1; \\ 0 & \text{if} \quad b \geq 2; \\ (\omega^k - \omega^2)^k (-1)^{\Omega(n)} 2^{\omega(m_1) - 1} & \text{otherwise}. \end{cases}$$
9 Partitions and coprime partitions into \( k \) parts, for \( k \in \{2, 3, 4\} \)

9.1 The case of 2 parts

By (20), we have

\[
\sum_{n \geq 0} p_2(n) z^n = \frac{z^2}{(1-z)(1-z^2)}.
\]

Partial fraction expansion gives

\[
\frac{z^2}{(1-z)(1-z^2)} = \frac{z^2}{(1-z)^2(1+z)} = -\frac{3}{4(1-z)} + \frac{1}{2(1-z)^2} + \frac{1}{4(1+z)}.
\]

Hence, using Lemma 2 (i), we get

\[
\sum_{n \geq 0} p_2(n) z^n = -\frac{3}{4} \sum_{n \geq 0} z^n + \frac{1}{2} \sum_{n \geq 0} (n+1) z^n + \frac{1}{4} \sum_{n \geq 0} (-1)^n z^n
\]

\[
= \sum_{n \geq 0} \left( \frac{2n-1}{4} + \frac{(-1)^n}{4} \right) z^n
\]

and thus

\[
p_2(n) = \frac{2n-1}{4} + \frac{(-1)^n}{4}.
\]

This is, of course, a reedition of the obvious \( p_2(n) = \lfloor \frac{n}{2} \rfloor \), which has the advantage of putting in evidence the nature of \( p_2(n) \) as a sum of a polynomial and of a periodic function of period 2.

Since \( p_2(n) = \sum_{d|n} p_2'(d) \), by Möbius inversion we get, for every \( n \geq 1 \),

\[
p_2'(n) = \frac{1}{2} \sum_{d|n} d \mu(n/d) - \frac{1}{4} \sum_{d|n} \mu(n/d) + \frac{1}{4} \sum_{d|n} (-1)^d \mu(n/d)
\]

\[
= \frac{1}{2} J_1(n) - \frac{1}{4} J_0(n) + \frac{1}{4} J_{(0,-1)}(n).
\]

By Lemma 4(i),

\[
\frac{1}{4} J_{(0,-1)}(n) = \begin{cases} 
-1/4 & \text{if } n = 1; \\
1/2 & \text{if } n = 2; \\
0 & \text{if } n > 2.
\end{cases}
\]

Note that if \( n \geq 3 \), then both \( J_0(n) \) and \( J_{(0,-1)}(n) \) vanish in (59) so that \( p_2'(n) = \frac{J_1(n)}{2} \), which gives (29).
9.2 The case of 3 parts

By \((20)\), we have

\[
\sum_{n \geq 0} p_3(n) z^n = \frac{z^3}{(1 - z)(1 - z^2)(1 - z^3)}.
\]

Partial fraction expansion gives

\[
\frac{z^2}{(1 - z)(1 - z^2)(1 - z^3)} = \frac{z^3}{(1 - z)^3(1 + z)(1 + z + z^2)}
\]

\[
= -\frac{1}{72(1 - z)} - \frac{1}{4(1 - z)^2} + \frac{1}{6(1 - z)^3} - \frac{1}{8(1 + z)} + \frac{2 + z}{9(1 + z + z^2)}.
\]

Let \(\omega = e^{2\pi i/3} = \frac{-1 + \sqrt{3}}{2}\) be the principal 3-root of 1. Decomposing in \(\mathbb{C}\) and using Lemma 2(i), we get

\[
\frac{2 + z}{9(1 + z + z^2)} = \frac{1}{9(\omega - \bar{\omega})} \left( \frac{2\omega + 1}{1 - \omega z} - \frac{2\bar{\omega} + 1}{1 - \bar{\omega} z} \right)
\]

\[
= \sum_{n \geq 0} \left( \frac{(2\omega + 1)\omega^n - (2\bar{\omega} + 1)\omega^n}{9(\omega - \bar{\omega})} \right) z^n.
\]

Using repeatedly Lemma 2(i) and after elementary simplifications we then get

\[
\sum_{n \geq 0} p_3(n) z^n = \sum_{n \geq 0} \left( \frac{n^2}{12} - \frac{7}{72} \frac{(-1)^n}{8} + \frac{(2\omega + 1)\omega^n - (2\bar{\omega} + 1)\omega^n}{9(\omega - \bar{\omega})} \right) z^n.
\]

Thus, for every \(n \geq 1\), we have

\[
p_3(n) = \frac{n^2}{12} - \frac{7}{72} \frac{(-1)^n}{8} + \frac{(2\omega + 1)\omega^n - (2\bar{\omega} + 1)\omega^n}{9(\omega - \bar{\omega})}.
\]  
(60)

The above equality puts in evidence the nature of \(p_3(n)\) as a sum of a polynomial and of a periodic function of period 6 and give \((39)\) for \(k = 3\).

By Möbius inversion we then get, for every \(n \geq 1\),

\[
p_3'(n) = \frac{1}{12} J_2(n) - \frac{7}{72} J_0(n) - \frac{1}{8} J_{(0,-1)}(n) + V(n),
\]  
(61)
where

\[
V(n) = \frac{1}{9(\omega - \overline{\omega})} \left( (2\omega + 1) \sum_{d|n} \omega^{2d} \mu(n/d) - (2\omega^2 + 1) \sum_{d|n} \omega^d \mu(n/d) \right)
\]

\[
= \frac{2\omega + 1}{9(\omega - \overline{\omega})} J_{(0,\omega^2)}(n) - \frac{2\omega^2 + 1}{9(\omega - \overline{\omega})} J_{(0,\omega)}(n)
\]

Thus, we see the way in which \(p_3'(n)\) is a \(\mathbb{C}\)-linear combination of Jordan root functions. By Lemma 4(i), we have

\[
-\frac{1}{8} J_{(0,-1)}(n) = \begin{cases} 
 1/8 & \text{if } n = 1; \\
 1/4 & \text{if } n = 2; \\
 0 & \text{if } n \geq 3.
\end{cases}
\]

Moreover, by Lemma 4(iv), \(V(n)\) can be explicitly computed. Indeed, recalling that \(\omega\) satisfies \(1 + \omega + \omega^2 = 0\) it is easily checked that, for every \(n \geq 4\), \(V(n) = 0\) holds. More precisely, we have

\[
V(n) = \begin{cases} 
 -1/9 & \text{if } n = 1; \\
 0 & \text{if } n = 2; \\
 1/3 & \text{if } n = 3; \\
 0 & \text{if } n \geq 4.
\end{cases}
\]

Since \(J_0(n), J_{(0,-1)}(n)\) and \(V(n)\) vanish for all \(n \geq 4\), we deduce that \(p_3'(n) = \frac{J_2(n)}{12}\) holds for all \(n \geq 4\), which gives (30).

### 9.3 The case of 4 parts

By (20), we have

\[
\sum_{n \geq 0} p_4(n)z^n = \frac{z^4}{(1 - z)(1 - z^2)(1 - z^3)(1 - z^4)}.
\]

Partial fraction expansion gives

\[
\frac{z^4}{(1 - z)(1 - z^2)(1 - z^3)(1 - z^4)} = \frac{1}{(1 - z)^4(1 + z)^2(1 + z^2)(1 + z + z^2)}
\]

\[
= \frac{-13 + 38z - 13z^2}{288(1 - z)^4} + \frac{1}{32(1 + z)^2} + \frac{1}{8(1 + z^2)} - \frac{1}{9(1 + z + z^2)}
\]

\[
= \frac{1}{288(1 - z)^2} - \frac{12}{288(1 - z)^3} + \frac{12}{288(1 - z)^4} + \frac{1}{32(1 + z)^2} + \frac{1}{8(1 + z^2)}
\]

\[
- \frac{1}{9(1 + z + z^2)}.
\]

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Considering the principal 3-root of 1, \( \omega = e^{2\pi i/3} = \frac{-1 + \sqrt{3}}{2} \), we have

\[
\frac{1}{8(1 + z^2)} = \frac{1}{16} \left( \frac{1}{1 - iz} + \frac{1}{1 + iz} \right) = \sum_{n \geq 0} \left( \frac{i^n + (-i)^n}{16} \right) z^n
\]

and

\[
-\frac{1}{9(1 + z + z^2)} = -\frac{1}{9(\omega - \bar{\omega})} \left( \frac{\omega}{1 - \omega z} - \frac{\bar{\omega}}{1 - \bar{\omega} z} \right) = -\sum_{n \geq 0} \left( \frac{\omega^{n+1} - \bar{\omega}^{n+1}}{9(\omega - \bar{\omega})} \right) z^n.
\]

Next, by Lemma 2(i), we have

\[
\frac{1}{(1 + z)^2} = \sum_{n \geq 0} (-1)^n (n + 1) z^n, \quad \frac{1}{(1 - z)^2} = \sum_{n \geq 0} (n + 1) z^n,
\]

\[
\frac{1}{(1 - z)^3} = \sum_{n \geq 0} \left( \frac{n + 2}{2} \right) z^n, \quad \frac{1}{(1 - z)^4} = \sum_{n \geq 0} \left( \frac{n + 3}{3} \right) z^n.
\]

Hence, we get

\[
p_4(n) = \frac{12}{288} \binom{n + 3}{3} - \frac{12}{288} \binom{n + 2}{2} - \frac{13}{288} (n + 1) + \left( \frac{(-1)^n(n + 1)}{32} \right)
\]

\[+ \left( \frac{i^n + (-i)^n}{16} \right) - \left( \frac{\omega^{n+1} - \bar{\omega}^{n+1}}{9(\omega - \bar{\omega})} \right).
\]

Simplifying we obtain the expression of \( p_4(n) \), for every \( n \geq 1 \):

\[
p_4(n) = \frac{n^3}{144} + \frac{n^2}{48} - \frac{n}{32} - \frac{13}{288} \binom{(-1)^n(n + 1)}{32} \]

\[+ \left( \frac{i^n + (-i)^n}{16} \right) - \left( \frac{\omega^{n+1} - \bar{\omega}^{n+1}}{9(\omega - \bar{\omega})} \right). \tag{63}
\]

The above equality is (39) for \( k = 4 \) and exhibits \( p_4(n) \) as the sum of a polynomial, a periodic function of period 12 and the further term

\[
\frac{(-1)^n(n + 1)}{32}
\]

which is neither of polynomial type nor periodic.

The expression of \( p'_4(n) \), for every \( n \geq 1 \), follows as usual:

\[
p'_4(n) = \frac{J_3(n)}{144} + \frac{J_2(n)}{48} - \frac{J_1(n)}{32} - \frac{13J_0(n)}{288} + S_1(n) + S_2(n) + S_3(n), \tag{64}
\]
where

\[ S_1(n) = \frac{1}{32} \sum_{d \mid n} (-1)^d (d + 1)\mu(n/d) = \frac{1}{32} J_{(1,-1)}(n) + \frac{1}{32} J_{(0,-1)}(n); \]

\[ S_2(n) = \frac{1}{16} \sum_{d \mid n} (i^d + i^{3d})\mu(n/d) = \frac{1}{16} J_{(0,i)}(n) + \frac{1}{16} J_{(0,i^3)}(n); \]

\[ S_3(n) = -\frac{1}{9(\omega - \overline{\omega})} \sum_{d \mid n} (\omega^{d+1} - \overline{\omega}^{d+1})\mu(n/d) \]

\[ = -\frac{\omega}{9(\omega - \overline{\omega})} J_{(0,\omega)}(n) + \frac{\overline{\omega}}{9(\omega - \overline{\omega})} J_{(0,\overline{\omega})}(n). \]

Hence, equality (64) shows \( p'_4(n) \) as a \( \mathbb{C} \)-linear combination of the Jordan root totient functions.

From the computations made in Section 8, writing \( n = 3^b m_1 \) with \( \text{gcd}(m_1, 3) = 1 \), we see that

\[
S_1(n) = \frac{1}{32} \begin{cases}
-2 & \text{if } n = 1; \\
5 & \text{if } n = 2; \\
-\phi(n) & \text{if } n \equiv 1 \pmod{2}, n > 1; \\
3\phi(n) & \text{if } 2 \mid n, n > 2; \\
\phi(n) & \text{if } 4 \mid n,
\end{cases}
\]

\[
S_2(n) = \frac{1}{16} \begin{cases}
0 & \text{if } n = 1; \\
0 & \text{if } n = 2; \\
-2 & \text{if } n = 4; \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
S_3(n) = -\frac{1}{9} \begin{cases}
-1 & \text{if } n = 1; \\
2 & \text{if } n = 3; \\
0 & \text{if } \exists p \equiv 1 \pmod{3}, p \mid m_1; \\
0 & \text{if } b \geq 2; \\
-(\omega^{d(n)} - \overline{\omega}^{d(n)})^{2^{\omega(m_1)-1}} & \text{otherwise}.
\end{cases}
\]

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