Non-Gaussianities from ekpyrotic collapse with multiple fields

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Abstract. We compute the non-Gaussianity of the curvature perturbation generated by ekpyrotic collapse with multiple fields. The transition from the multi-field scaling solution to a single-field dominated regime converts initial isocurvature field perturbations to an almost scale invariant comoving curvature perturbation. In the specific model of two fields, $\phi_1$ and $\phi_2$, with exponential potentials, $-V_i \exp(-c_i \phi_i)$, we calculate the bispectrum of the resulting curvature perturbation. We find that the non-Gaussianity is dominated by non-linear evolution on super-Hubble scales and hence is of the local form. The non-linear parameter of the curvature perturbation is given by $f_{NL} = 5c_j^2/12$, where $c_j$ is the exponent of the potential for the field which becomes subdominant at late times. Since $c_j^2$ must be large, in order to generate an almost scale invariant spectrum, the non-Gaussianity is inevitably large. Thus, the model is strongly constrained by observational bounds on the spectral index and non-Gaussianity.

Keywords: cosmology with extra dimensions, cosmological perturbation theory, string theory and cosmology

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1. Introduction

The existence of an almost scale invariant spectrum of primordial curvature perturbations on large scales, with an approximately Gaussian distribution, is one of the most important observations that any model of the early universe should explain. The inflationary scenario offers a possible explanation in terms of vacuum fluctuations of scalar fields during an accelerated expansion preceding the standard hot big bang, though attempts to embed such a scenario within a fundamental theory such as superstring/M-theory may require some degree of fine-tuning (see [1] for a recent review).

The ekpyrotic scenario [2] (see also [3, 4]) is one alternative approach where the large scale perturbations are generated from vacuum fluctuations during a collapse phase driven by a scalar field with a steep, negative exponential potential. It was shown, however, that even though the Bardeen potential acquires a scale invariant spectrum during ekpyrotic collapse, the comoving curvature perturbation has a steep blue spectrum in the original ekpyrotic scenario driven by a single scalar field [5]. If the contracting pre-big bang phase is connected to the expanding hot big bang phase through a regular four-dimensional bounce, then we expect the comoving curvature perturbation to remain constant for adiabatic perturbations on large scales [6–8], which means that the growing mode of curvature perturbations in the expanding phase also acquires a steep blue spectrum.

One way to avoid this is to consider non-adiabatic perturbations, which require two or more fields [9, 10]. Recently, there has been progress in generating a scale invariant spectrum for curvature perturbations in the ekpyrotic scenario with...
more than one field, which we will refer to as the new ekpyrotic scenario [11]–[13]. If these fields have steep negative exponential potentials, there exists a scaling solution where the energy densities of the fields grow at the same rate during the collapse [14,15]. In this multi-field scaling solution background, the isocurvature field perturbations have an almost scale invariant spectrum [14], owing to a tachyonic instability in the isocurvature field. The multi-field scaling solution in the new ekpyrotic scenario can be shown to be an unstable saddle point in the phase space and the stable late-time attractor is the old ekpyrotic collapse dominated by a single field [16].

The existence of a tachyonic instability raises questions about initial conditions in the new ekpyrotic scenario [17], but the transition from the multi-field scaling solution to the single-field dominated solution also provides a mechanism to automatically convert the initial isocurvature field perturbations about the multi-field scaling solution into comoving curvature perturbations about the late-time attractor [18]. In this case, the final amplitude of the comoving curvature perturbation is determined by the Hubble scale at the transition and if this parameter and the initial conditions are set appropriately, the prediction for the primordial curvature perturbations from this scenario is compatible with an almost scale invariant spectrum (see [11]–[13], [19] for other mechanisms to convert the initial scale invariant isocurvature perturbations into curvature perturbations).

Recently, the non-Gaussianity of the distribution of primordial curvature perturbations in the inflationary scenario has been extensively studied by many authors (see e.g. [20] for a review). Measurements of the non-Gaussianity already provide important constraints on specific models of the early universe and such measurements will continue to improve in the near future, for instance with the Planck satellite [21]. Since the non-Gaussian signal from single-field, slow-roll inflation is suppressed by slow-roll parameters to an undetectable level [22], if non-Gaussianities are detected then this would rule out many models of inflation.

Thus, as a natural extension of the study performed in [16,18], in this paper we compute the non-Gaussianity of the primordial curvature perturbations generated from the contracting phase of the multi-field new ekpyrotic cosmology. We adopt the same specific model as the previous analysis of [16,18] where the scale invariant curvature perturbation is generated by the transition from a scaling solution, with two fields driving the collapse, to a single-field dominated regime. We assume that the ekpyrotic collapse is subsequently converted to expansion by a regular bounce, during which the comoving curvature perturbation is conserved on large scales and the curvature perturbation generated during the collapse is thus directly related to the amplitude of the observed primordial density perturbation.

This paper is organized as follows. In section 2 we briefly review the background dynamics. In section 3 we summarize the statistical properties of linear and non-linear perturbations. The \( \delta N \)-formalism is introduced which is used to compute the primordial curvature perturbation. In section 4 we study linear perturbations while in section 5 we generalize this study to non-linear perturbations and quantify the expected non-Gaussianity. In section 6 we draw our conclusions. In three appendices we review previous results for the linear fluctuations of the isocurvature field during the ekpyrotic phase and calculate its intrinsic non-Gaussianity using the interaction Hamiltonian, as well as presenting a numerical check of our analytical results.
2. Homogeneous dynamics

We first review the background dynamics of the fields in the new ekpyrotic cosmology with multiple scalar fields. During the ekpyrotic collapse the contraction of the universe is assumed to be described by a 4D Friedmann equation in the Einstein frame with \( n \) scalar fields with negative exponential potentials

\[
3H^2 = V + \sum_{j}^{n} \frac{1}{2} \dot{\phi}_j^2 ,
\]

where

\[
V = - \sum_{j}^{n} V_j e^{-c_j \phi_j} ,
\]

and we take \( V_1 > 0 \) and set \( 8\pi G \) equal to unity.

From now on, for simplicity, we concentrate our attention on the case of two fields. In this case, it will be easier to work in terms of new variables \[16\],

\[
\varphi = \frac{c_2 \phi_1 + c_1 \phi_2}{\sqrt{c_1^2 + c_2^2}} , \quad \chi = \frac{c_1 \phi_1 - c_2 \phi_2}{\sqrt{c_1^2 + c_2^2}} ,
\]

corresponding to a fixed rotation in field space. The potential given by equation (2) can then be simply rewritten as \[14, 16, 23\]

\[
V = -U(\chi)e^{-c\varphi} ,
\]

where

\[
\frac{1}{c^2} = \sum_{j}^{n} \frac{1}{c_j^2} ,
\]

and the potential for the orthogonal field is given by

\[
U(\chi) = V_1 e^{-(c_1/c_2)\chi} + V_2 e^{(c_2/c_1)\chi} ,
\]

which has a minimum at

\[
\chi = \chi_0 \equiv \frac{1}{\sqrt{c_1^2 + c_2^2}} \ln \left( \frac{c_1^2 V_1}{c_2^2 V_2} \right) .
\]

If we expand \( U(\chi) \) in equation (6) about its minimum we obtain

\[
U(\chi) = U_0 \left[ 1 + \frac{c_2^2 - c_1^2}{2} (\chi - \chi_0)^2 + \frac{\tilde{c} c^2}{6} (\chi - \chi_0)^3 + \cdots \right] ,
\]

where

\[
\tilde{c} \equiv \frac{c_2^2 - c_1^2}{\sqrt{c_1^2 + c_2^2}} .
\]

The multi-field scaling solution corresponds to the classical solution along this minimum \( \chi = \chi_0 \), while \( \varphi \) is rolling down the exponential potential. The explicit form of
the multi-field scaling solution is given as
\begin{equation}
a = (-t)^p,
\end{equation}
where \( p = \sum_j 2/c_j^2 = 2/c^2 \).

The potential for \( \chi \) has a negative mass squared around \( \chi = \chi_0 \),
\begin{equation}
m^2_\chi \equiv \frac{\partial^2 V}{\partial \chi^2} = c^2 V < 0,
\end{equation}
and thus \( \chi \) represents the instability direction. Furthermore, the \( \chi \) field evolution is non-linear, with the cubic interaction being given by
\begin{equation}
V^{(3)} \equiv \frac{\partial^3 V}{\partial \chi^3} = \tilde{c} m^2_\chi,
\end{equation}
which becomes important when we consider the non-Gaussianity later in this paper.

If the initial condition for \( \chi \) is slightly different from \( \chi_0 \) or \( \dot{\chi} \) is not zero, then \( \chi \) starts rolling down the potential and the solution approaches a single-field dominated scaling solution. The explicit form of the single-field dominated scaling solution is given as
\begin{equation}
a = (-t)^{p_j},
\end{equation}
where \( p_j = 2/c_j^2 \). In this paper, we consider the case in which the background evolves from the multi-field scaling solution to the \( \phi_2 \) dominated scaling solution without loss of generality.

### 3. Statistical correlators

Here we briefly summarize the statistical properties of the scalar field fluctuations during ekpyrotic collapse. Then, in order to link these to the observable primordial curvature perturbation we discuss the \( \delta N \)-formalism.

In the two-field new ekpyrotic cosmology, the isocurvature fluctuations acquired by the field \( \chi \) during the multi-field scaling regime, play a crucial role to generate a scale invariant spectrum of perturbations. On the other hand, the fluctuations of the field \( \varphi \) are negligible on large scales, because of its very blue spectral tilt [11]–[13]. Thus, in the following we neglect \( \delta \varphi \) fluctuations.

In linear perturbation theory, when interactions are neglected, the free field fluctuations \( \delta \chi \) are Gaussian. Their power spectrum \( P_\chi \) can be defined by
\begin{equation}
\langle \delta \chi_{k_1} \delta \chi_{k_2} \rangle \equiv (2\pi)^3 \delta^3(k_1 + k_2) \frac{2}{k_1^3} P_\chi(k_1),
\end{equation}
where the angle brackets denote an ensemble average.
If the cubic self-interaction in equation (8) is taken into account, $\delta \chi$ is no longer Gaussian and the first signal of non-Gaussianity comes from the three-point correlation function. Similarly to equation (16), the bispectrum of $\delta \chi, B_\chi$, is defined by

$$\langle \delta \chi_{k_1} \delta \chi_{k_2} \delta \chi_{k_3} \rangle \equiv (2\pi)^3 \delta^3 \left( \sum_j k_j \right) B_\chi(k_1, k_2, k_3).$$

To characterize the bispectrum one can also define the non-linear parameter of the field fluctuation, $f_{\chi, NL}$, as

$$6 \frac{f_{NL}}{5} \equiv \prod_j \frac{k_j^3}{4\pi^4} \frac{B_\chi}{P_\chi}.$$  

When the intrinsic non-Gaussianity is local in real space this parameter is $k$ independent and $\delta \chi$ can be written as

$$\delta \chi = \delta \chi_L + \frac{3}{5} f_{NL} \delta \chi_L^2.$$  

where $\delta \chi_L$ is the linear and Gaussian part of the field fluctuations.

To relate the non-Gaussianity of the scalar field fluctuations to observations, we need to calculate the three-point functions of the comoving curvature perturbation $\zeta$. In order to do that, we can use the $\delta N$- formalism \[24\]–\[29\]. In the $\delta N$- formalism, the comoving curvature perturbation $\zeta$ evaluated at some time $t = t_f$ coincides with the perturbed expansion integrated from an initial flat hypersurface at $t = t_i$, to a final uniform density hypersurface at $t = t_f$, with respect to the background expansion, i.e.,

$$\zeta(t_f, x) \simeq \delta N(t_f, t_i, x) \equiv N(t_f, t_i, x) - N(t_f),$$

with

$$N(t_f, t_i, x) \equiv \int_{t_i}^{t_f} \mathcal{H}(x, t) \, dt, \quad N(t_f) \equiv \int_{t_i}^{t_f} H(t) \, dt,$$

where $\mathcal{H}(x, t)$ is the inhomogeneous Hubble expansion. We can calculate $\delta N$ on large scales using the homogeneous equations of motion, in the assumption that the local expansion on sufficiently large scales behaves like a locally homogeneous and isotropic universe, according to the so-called ‘separate universe’ approach \[30,31,6\]. This allows us to compute the full non-linear curvature perturbation in the large scale limit. Note that we leave the initial time $t_i$ unspecified and we are free to identifying it with the time of Hubble crossing $t = t_\ast$, i.e. the time when a mode $k$ exits the Hubble radius during inflation, $k = aH$, or with a later time.

We will chose the initial time $t_i$ to be during the multi-field scaling regime. Furthermore, since $\varphi$ is unperturbed, $\delta N$ can be expanded in series of the initial field fluctuations $\delta \chi_i$. Retaining only terms up to second order, we obtain

$$\delta N = N_{\chi_i} \delta \chi_i + \frac{1}{2} N_{\chi_i \chi_i} (\delta \chi_i)^2,$$

where $N_{\chi}$ denotes the derivative of $\chi$ with respect to $\chi$.

Now we can convert the higher-order information about the initial field fluctuations into the statistical properties of the observed primordial curvature perturbations. The
power spectrum of the comoving curvature perturbation $\zeta$, $P_\zeta$, is defined as
\begin{equation}
\langle \zeta_{k_1} \zeta_{k_2} \rangle \equiv (2\pi)^3 \delta^{(3)}(k_1 + k_2) \frac{2\pi^2}{k_1^3} P_\zeta(k_1).
\end{equation}
(23)

At lowest order, from equations (16), (22) and (23), $P_\zeta$ is expressed as
\begin{equation}
P_\zeta = N_{\chi_i}^2 P_{\chi_i}.
\end{equation}
(24)

Note that $N_{\chi_i}$ is independent of wavenumber $k$ and hence the scale dependence of the primordial spectrum, $\Delta n \equiv d \ln P_\zeta / d \ln k$, is given by the spectral tilt of the field fluctuations, $\Delta \chi$, on the initial hypersurface (given in equation (A.12) in appendix A).

The bispectrum of the curvature perturbation $\zeta$, which includes the first signal of non-Gaussianity, is defined as
\begin{equation}
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \equiv (2\pi)^3 \delta^{(3)} \left( \sum_j k_j \right) B_\zeta(k_1, k_2, k_3),
\end{equation}
(25)

where the left-hand side of equation (25) can be evaluated by the $\delta N$-formalism using Wick’s theorem,
\begin{equation}
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = N_{\chi_i}^3 \langle \delta \chi_{k_1} \delta \chi_{k_2} \delta \chi_{k_3} \rangle + \frac{1}{2} N_{\chi_i}^2 N_{\chi_l \chi_i} \langle \delta \chi_{k_1} \delta \chi_{k_2} \delta \chi_{k_3} \rangle + \text{perms}. \tag{26}
\end{equation}

In the above equation, a star $\star$ denotes the convolution and we have neglected correlators higher than the four-point ones.

Observational limits on the non-Gaussianity of the primordial curvature perturbations are usually given on the non-linear parameter $f_{NL}$ defined by [22]
\begin{equation}
\frac{6}{5} f_{NL} \equiv \frac{\prod_j k_j^3}{\sum_j k_j^3 4\pi^2 P_\zeta^2} B_\zeta.
\end{equation}
(27)

If the non-Gaussianity is local, one can write $\zeta$ as
\begin{equation}
\delta N = \zeta_L + \frac{3}{5} f_{NL} \zeta_L^2,
\end{equation}
(28)

where $\zeta_L$ is a Gaussian variable.

To compute the bispectrum of the curvature perturbation one can use the $\delta N$-formalism and after some manipulations, from equations (25) and (26) one finds
\begin{equation}
B_\zeta(k_1, k_2, k_3) = N_{\chi_i}^3 B_{\chi_i}(k_1, k_2, k_3) + 4\pi^2 \sum_j \frac{k_j^3}{\prod_j k_j^3} N_{\chi_i \chi_j} N_{\chi_i \chi_k}^2 B_{\chi_i}(k_1, k_2, k_3),
\end{equation}
(29)

where $B_{\chi_i}(k_1, k_2, k_3)$ is the bispectrum of the scalar field evaluated at $t = t_i$. Together with equation (27) the non-linear parameter $f_{NL}$ becomes
\begin{equation}
f_{NL} = \frac{f_{NL}^{\chi_i}}{N_{\chi_i}} + \frac{5}{6} \frac{N_{\chi_i \chi_j}}{N_{\chi_i}^2}, \tag{30}
\end{equation}

where we have used equations (19) and (27). The first term on the right-hand side of equation (30),
\begin{equation}
f_{NL}^{(3)} = \frac{f_{NL}^{\chi_i}}{N_{\chi_i}}, \tag{31}
\end{equation}

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comes from the intrinsic three-point correlation functions of the field fluctuations and contains also the non-Gaussianity of quantum origin generated on small scales, i.e., inside the Hubble radius during the ekpyrotic collapse described by the multi-field scaling solutions. The second term on the right-hand side of equation (30),

\[ f_{NL}^{(4)} \equiv \frac{5}{6} \frac{N_{\chi_i \chi_i}}{N_{\chi_i}^2}, \]  

(32)
is completely momentum independent and local in real space, because it is due to the evolution of non-linearities outside the Hubble radius during the multi-field ekpyrotic collapse. Note that the splitting in \( f_{NL}^{(3)} \) and \( f_{NL}^{(4)} \) depends on the time \( t_i \).

4. Linear curvature perturbation

The power spectrum of \( \chi \) is given by (see [18] and equation (A.13) in appendix A)

\[ P_\chi(k) = \epsilon^2 \left( \frac{H}{2\pi} \right)^2, \]  

(33)
in the limit of large \( \epsilon \), where the fast-roll parameter \( \epsilon \) is defined as \( \epsilon \equiv -\dot{H}/H^2 = \epsilon^2/2 = 1/p \), and it is constant for the multi-field scaling solution. Thus, in this limit, \( P_\chi \) has a scale invariant spectrum.

On the other hand, as shown in [18] in the fast-roll limit and assuming an instantaneous transition from the multi-field scaling solution to the single-field \( \phi_2 \) dominated scaling solution, the power spectrum of the final (after the transition) curvature perturbations becomes

\[ P_\zeta = \frac{\epsilon^2}{c_1^2 + c_2^2} \left| \frac{H_T}{2\pi} \right|^2, \]  

(34)
where \( H_T \) is the Hubble parameter evaluated at the transition time \( t = t_T \). The scalar spectral index of \( \zeta \) is

\[ \Delta n = 4 \left( \frac{1}{c_1^2} + \frac{1}{c_2^2} \right). \]  

(35)
Note that it is always positive, i.e., the power spectrum is always blue.

Let us now interpret the result above in the light of the \( \delta N \)-formalism. We consider the situation in which \( \chi_i \) is perturbed on the \( t = t_i \) hypersurface, while \( H_i \) assumes on this hypersurface a constant value. This is justified by the fact that the \( t = t_i \) hypersurface is flat and since \( \chi \) is an isocurvature field its fluctuations do not affect the local Hubble expansion. Furthermore, we assume that the transition into the single-field dominated scaling solution at the time \( t = t_T \), happens instantaneously on the hypersurface \( \chi = \chi_T = \text{const} \), where \( H_T \) is perturbed.

Under these assumptions, the expansion \( N \) defined by equation (21) can be split into

\[ N = \int_{t_i}^{t_T} H \, dt + \int_{t_T}^{t_i} H \, dt, \]  

(36)
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where $t_f$ is set sufficiently later than the transition time $t_T$. In equation (36), the first integral is over the multi-field scaling evolution and the last integral is over the $\phi_2$ dominated phase.

Since the multi-field scaling solution is characterized by equation (10), the first term on the right-hand side of equation (36) can be expressed as $-(1/\epsilon) \ln(H_i/H_T)$. Similarly, since the single-field dominated scaling solution is characterized by equation (14), the second term becomes $-(1/\epsilon_2) \ln(H_T/H_i)$, where $\epsilon_2 = c_2^2/2$. Then, for a fixed $t_i$ and $t_f$, the expansion $N$ can be expressed as

$$N = \frac{2}{c_1^2} \ln |H_T| + \text{const},$$

which depends only on the parameter $c_1$, besides the transition time $t_T$.

During the multi-field scaling regime, the linear evolution equation of $\chi$ on large scales is given by

$$\ddot{\chi} + 3H \dot{\chi} + m_\chi^2 \chi = 0,$$

where the mass of $\chi$ is defined in equation (12). During the multi-field scaling the evolution of $\chi$ is dominated by the tachyonic mass and thus $3H \dot{\chi} \ll m_\chi^2 \chi$. With $m_\chi^2 = -2/t^2$ (see appendix A), one finds $\chi \propto 1/t \propto H$, and thus

$$H_T = H_i \frac{\chi_T}{\chi_i}.$$  

Using this relation one can derive $N$ with respect to $\chi_i$ and obtain

$$N_{\chi_i} = -\frac{2}{c_1^2 \chi_i} = -\frac{2}{c_1^2 \chi_T} \frac{H_T}{H_i}.$$  

In particular, by using this equation and comparing equation (24) with (34), one obtains the value of $\chi_T$, i.e.,

$$\chi_T = \frac{2\sqrt{c_1^2 + c_2^2}}{c_1^2}.$$  

In appendix C, we have calculated $N_{\chi_i}$ numerically and checked equation (41).

5. Non-Gaussianities

In this section we compute the non-Gaussianity of the primordial curvature perturbation generated by the non-linear dynamics during multi-field ekpyrotic collapse. We will use the $\delta N$-formalism to calculate $\delta N(\chi)$ in the instantaneous transition approximation. We will thus consider quadratic terms in $\delta \chi_i$ in the $\delta N$-expansion (22) and, in contrast with section 4, we also allow for non-linear evolution of $\chi$ on large scales. For another example of the multi-field calculation of non-Gaussianity from fields with exponential potential see [32], even though in the context of assisted inflation.

Including the cubic self-interaction $V^{(3)}$ given in equation (13), the large scale evolution equation for $\chi$ in the multi-field scaling regime becomes

$$\ddot{\chi} + 3H \dot{\chi} + m_\chi^2 \chi = -\frac{1}{2} \xi m_\chi^2 \chi^2.$$  

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The above evolution equation can be solved perturbatively. Given the solution to the linear equation (38), i.e., $\chi L \propto H$, the growing-mode solution for $\chi$ is

$$\chi = \chi L + \frac{1}{2} \tilde{c} \chi^2 = \alpha H + \frac{1}{4} \tilde{c} \alpha^2 H^2,$$

(43)

where $\alpha$ is a constant parameter whose value distinguishes the different trajectories. Note that this is a *perturbative* result, i.e., it is valid only as long as $\tilde{c} \chi \ll 1$. However, since $\chi$ grows during the collapse, unless prevented by a bouncing phase, eventually this condition is violated.

For perturbations in the value of $\chi$ on a hypersurface of uniform $H$ we have

$$\delta \chi = \left(1 + \frac{1}{2} \tilde{c} \alpha H\right) H \delta \alpha + \frac{1}{4} \tilde{c} H^2 (\delta \alpha)^2,$$

(44)

The non-linear self-interaction of $\chi$ in equation (42) grows in time on super-Hubble scales, and thus the intrinsic non-Gaussianity of $\chi$ increases. Therefore, unless we take $t_i$ sufficiently early that we can neglect these nonlinearities, we cannot naively use equation (39), which is only valid at linear order, and its derivative with respect to $\chi$, to estimate the non-Gaussianity of the curvature perturbation. We need to work with a variable that is as close as possible to a Gaussian. It turns out that it is convenient to choose $\alpha$ as such a variable. Indeed, if we assume that $\delta \chi L$, the perturbation of the solution of the linear equation (38), is Gaussian, then also $\delta \alpha$ is a Gaussian random variable because $\delta \alpha = \delta \chi L / H$. Comparing this equation with equation (19) we find that the intrinsic non-Gaussianity of $\chi$ is of local form and time independent, and it is given by

$$f^{\chi}_{\text{NL}} = \frac{5}{12} \tilde{c}.$$

(45)

In section B of the appendix, we have checked that this result agrees with the one obtained using the approach of Maldacena [22] with an interaction Hamiltonian containing $V^{(3)} [13]$, and confirms that we can consistently assume $\delta \chi L$, and thus $\delta \alpha$, to be Gaussian. The reason for this is that the classical non-linear evolution on large scales completely dominates over the sub-Hubble non-linear interactions of quantum nature.

We will now compute the non-Gaussianity of the curvature perturbation using equations (30)–(32). The final result for $f_{\text{NL}}$ will be manifestly independent of the time we choose to evaluate $\delta \chi_i$ although the splitting into $f^{(3)}_{\text{NL}}$ and $f^{(4)}_{\text{NL}}$ is dependent on $t_i$.

Since $\delta \alpha$ can be assumed to be Gaussian, the simplest way to compute $f_{\text{NL}}$ is to calculate the $\delta N$ corresponding to the fluctuation $\delta \alpha$, i.e.,

$$\delta N = N_{\alpha} \delta \alpha + \frac{1}{2} N_{\alpha \alpha} (\delta \alpha)^2.$$

(46)

In order to compute $N_{\alpha}$ and $N_{\alpha \alpha}$ we want to use equation (37), and for this we need to know how $H_T$ varies as a function of $\alpha$ at the transition from multi-field scaling to single-field $\phi_2$ dominated scaling solution. Inverting equation (43) (to leading order in $\tilde{c} \chi$) gives

$$\alpha = \frac{\chi}{H} \left(1 - \frac{1}{4} \tilde{c} \chi\right).$$

(47)

Assuming as in the linear case that the transition corresponds to a critical value of the tachyon field $\chi = \chi_T$, on the transition surface (constant $\chi_T$) we have from (47) that $\alpha \propto H_T^{-1}$ and hence we find

$$\delta N = -\frac{2}{c_T^2} \frac{\delta \alpha}{\alpha} + \frac{1}{c_T^2} \left(\frac{\delta \alpha}{\alpha}\right)^2,$$

(48)
which means
\[ N_\alpha = -\frac{1}{\xi_1^2 c_1^2}, \quad N_{\alpha\alpha} = \frac{1}{\xi_1^2 c_1^2}. \] (49)

Taking \( \delta \alpha \) to be a Gaussian random variable and comparing with equation (28) with \( \zeta_L = -2\delta \alpha / (c_1^2 \alpha) \) we obtain the non-linear parameter for the curvature perturbation after the transition:
\[ f_{\text{NL}} = \frac{5}{6} \frac{N_{\alpha\alpha}}{N_{\alpha}^2} = \frac{5}{12} \frac{\xi_2^2}{c_1^2}. \] (50)

This is our main result. The non-Gaussianity is given in terms of \( c_1 \), where \(-V_1 \exp(-c_1 \phi_1)\) is the potential of the field \( \phi_1 \) which remains subdominant after the transition from the multi-field scaling to the single-field dominated regime. The non-Gaussianity is of local form. This is due to the fact that it is generated by the non-linear super-Hubble evolution.

Equation (50) includes also the non-linear growth of the tachyon field on large scales due to its self-interaction. In order to see this, we can compute \( f_{\text{NL}}^{(3)} \) and \( f_{\text{NL}}^{(4)} \) defined in equations (31) and (32). Thus, we have to identify the linear and non-linear dependence of \( \delta N \) on the field values \( \delta \chi_1 \) on the initial hypersurface.

Replacing equation (45) in equation (31) we obtain
\[ f_{\text{NL}}^{(3)} = \frac{5}{12} \frac{\tilde{c}}{N_{\chi_1}}. \] (51)

Furthermore, from equations (47) and (48) we have
\[ N_{\chi_1} = \frac{dN}{d\alpha \, d\chi_1} = -\frac{2}{c_1^2 \chi_1} \left( 1 - \frac{1}{4} \tilde{c} \chi_1 \right), \] (52)

and hence
\[ f_{\text{NL}}^{(3)} = -\frac{5}{24} c_1^2 \tilde{c} \chi_1 \] (53)
\[ = -\frac{5}{12} \left( c_1^2 - c_1^2 \right) \frac{H_i}{H_T}. \] (54)

where for the last equality we have replaced \( \tilde{c} \) using its definition, equation (9), expressed \( \chi_1 \) in terms of \( \chi_T \) using the linear relation equation (39), and replace \( \chi_T \) using equation (41).

Secondly we have the contribution due to
\[ N_{\chi_1 \chi_1} = \frac{d^2N}{d\alpha^2} \left( \frac{d\alpha}{d\chi_1} \right)^2 + \frac{dN}{d\alpha} \frac{d^2\alpha}{d\chi_1^2} = \frac{2}{c_1^2 \chi_1^2}. \] (55)

Note that this relation is valid to linear order in \( \tilde{c} \chi_1 \), and thus contains also the non-linear self-interaction of \( \chi \) generating non-Gaussianities after \( t_1 \). Using equation (32), this gives
\[ f_{\text{NL}}^{(4)} = \frac{5}{12} c_1^2 (1 + \frac{1}{2} \tilde{c} \chi_1) \] (56)
\[ = \frac{5}{12} c_1^2 + \frac{5}{12} \left( c_1^2 - c_1^2 \right) \frac{H_i}{H_T}. \] (57)

Both \( f_{\text{NL}}^{(3)} \) and \( f_{\text{NL}}^{(4)} \) depend upon the choice of the initial hypersurface and thus of \( \chi_1 \). However the total \( f_{\text{NL}} \) comes from the sum of the two terms, it is independent of \( t_1 \), and is given by equation (50).

In appendix C, we calculated \( N_\alpha \) and \( N_{\alpha\alpha} \) numerically and confirm that the result equation (50) obtained by the instantaneous transition and the fast-roll approximations is satisfied with good accuracy.
6. Conclusion

In this paper we have studied the non-linear evolution of perturbations in the multi-field new ekpyrotic cosmology. If one sets the model parameters and initial conditions appropriately, then the prediction for the power spectrum of curvature perturbations produced in multi-field ekpyrotic collapse can be made compatible with present observations [11]–[13], [18]. In order to distinguish this model from other early universe scenarios, such as inflation, by future observations, it is important to estimate the non-Gaussianity of the curvature perturbations.

We have studied the simplest model based on two fields with exponential potentials and considered the specific scenario in which the nearly scale invariant comoving curvature perturbation is generated by the transition from the multi-field scaling solution to the single-field dominated attractor solution. We have applied the $\delta N$-formalism, which is widely adopted to study the non-linearity of the primordial curvature perturbation from inflation [28], to ekpyrotic cosmology. We identify the non-linear curvature perturbation on uniform density hypersurfaces at late times with the perturbed local expansion, $\delta N$, with respect to an initial spatially flat hypersurface. The primordial non-Gaussianity parameter $f_{NL}$ is a sum of two contributions: $f_{NL}^{(3)}$, defined in equation (31), comes from the intrinsic three-point function of the isocurvature field perturbation $\delta \chi$ on an initial hypersurface, and $f_{NL}^{(4)}$, defined in equation (32), originates from the non-linear relation between the primordial curvature perturbation and the isocurvature field perturbations during multi-field ekpyrotic collapse. It should be emphasized that although the decomposition of $f_{NL}$ into $f_{NL}^{(3)}$ and $f_{NL}^{(4)}$ may be convenient, it is unphysical and depends upon the choice of the initial time $t_i$. However, we show that the physical quantity, the total $f_{NL}$, is independent of $t_i$.

Both the multi-field and single-field ekpyrotic solutions are power-law solutions. We find a general result that in case of a sudden transition between two power-law solutions the local expansion is only a function of the Hubble rate at the transition, $H_T$. Thus the calculation of the primordial curvature perturbation in our model reduces to finding the perturbation of $H_T$ for different trajectories in phase space, and hence the local value of the isocurvature field $\chi$ on the initial spatially flat hypersurface.

We find that after the transition to the single-field attractor solution the non-Gaussian parameter $f_{NL} = 5c_1^2/12$, where $-V_1 \exp(-c_1 \phi_1)$ is the potential of the field $\phi_1$ which becomes subdominant at late time. Since the non-Gaussianity is mainly generated by the non-linear super-Hubble evolution, it is of the local form, and the non-linear parameter is $k$ independent. We show in appendix B that the contribution of the intrinsic non-Gaussianity of the isocurvature field perturbations on sub-Hubble scales is subdominant.

We have checked our analytical results by calculating the expansion, $N$, numerically in appendix C. We confirmed that when the fast-roll parameter satisfies $400 > \epsilon > 25$ the analytic estimation is accurate within at a few % level. The discrepancy arises from the breakdown of both the fast-roll approximation and the sudden transition approximation.

Large non-Gaussianity of the primordial curvature perturbation is already strongly constrained by present observation. For example, if $c_1 = c_2 = 20$, $f_{NL}$ is already as large as 160 and this is ruled out by WMAP observations [33]. For smaller values, like $c_1 = c_2 = 15$ the fast-roll parameter for the multi-field scaling solution is about 60. Then $f_{NL}$ is approximately 40, which has not been ruled out by current data but can be detected.
by the Planck satellite [34]. However smaller values like $c_1 = c_2 = 15$ lead to larger values of the spectral index (35) for the power spectrum of primordial curvature perturbations, $\Delta n = +0.03$, while observations favor a red spectrum, $\Delta n < 0$ [33]. (Note that running of the spectral index and tensor perturbations on CMB scales are both negligible.) In our simple model with two fields with exponential potentials we thus find a tension between constraints on non-Gaussianity which require small values of the fast-roll parameter and constraints on the spectral index which require large values.

Corrections to the exact exponential potentials that we have studied in this paper have been proposed [11, 12] to produce a red spectrum of perturbations. Indeed corrections are also required to stabilize the multi-field scaling solution at early times (before observables scales exit the Hubble scale) [17]. We assume that the classical background solution starts close to the multi-field scaling solution. However, as the multi-field scaling solution is a saddle point in the phase space, we need some preceding phase that initially drives the classical background solution to the saddle point. It is important to check whether such corrections will also affect the non-linear evolution of the field perturbations during the collapse phase, which could lead to additional sources of primordial non-Gaussianity.

Modifications to the effective action are certainly required at high energies to turn contraction to expansion before the collapse phase reaches the big crunch singularity [35], [11]–[13]. In this paper, we assumed that the comoving curvature perturbation is conserved on super-Hubble scales through such a non-singular bounce, neglecting the effect of non-adiabatic perturbations which are expected to rapidly decay about the single-field attractor solution. In this case the power spectrum and the non-Gaussian parameter equation (50) that we have calculated in the single-field dominated collapse are directly related to those observed perturbations in the expanding, hot big bang phase. It is certainly possible to convert the initial isocurvature field perturbations to curvature perturbations through a different mechanism, e.g., via the bounce [11]–[13], [17]. In this case the non-linear dependence of the expansion, $\delta N$, upon the initial field perturbations may be different, but the non-linear growth on super-Hubble scales of the isocurvature field perturbations about the multi-field scaling solution is still expected to lead to a large non-Gaussianity [13,17]. Further work is required to quantify the non-Gaussianity predicted in these models.

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Appendix A. Linear field fluctuations

Here, we briefly summarize the results obtained in the previous works [11]–[14], [16] about the generation and the evolution of linear fluctuations of the $\chi$ field about the background multi-field scaling solution.

In the multi-field scaling solution $\delta \chi$ coincides with entropy field perturbation, which is automatically gauge invariant and does not couple with gravity at first order. Since
there is no coupling between the adiabatic and the entropy field perturbations, neglecting
the non-linear self-interactions of the entropy field which are negligible at sufficiently early
times, the equation of motion for $\delta \chi$ becomes simply the equation of a free massive field
in an unperturbed FRW metric,

$$
\ddot{\delta \chi} + 3H \dot{\delta \chi} + \left( \frac{k^2}{a^2} + m^2_{\chi} \right) \delta \chi = 0,
$$

where $m^2_{\chi}$ has been defined in equation (12). Introducing the rescaled field $v \equiv a \delta \chi$, and
writing the wave equation in terms of conformal time $\tau \equiv \int dt / a$, we have

$$
v_{,\tau\tau} + \left[ k^2 - \frac{a_{,\tau\tau}}{a} + m^2_{\chi} a^2 \right] v = 0,
$$

where $(\cdot \cdot)_{,\tau}$ denotes the derivative with respect to $\tau$.

For the multi-field scaling solutions, we can show that the following relations hold,

$$
aH = \frac{1}{(\epsilon - 1)\tau},
$$

$$
a_{,\tau\tau} / a = -(\epsilon - 2)a^2 H^2,
$$

$$
m^2_{\chi} = 2\epsilon(3 - \epsilon)H^2.
$$

Thus, equation (A.1) becomes

$$
v_{,\tau\tau} + \left[ k^2 + \frac{\epsilon + 3\eta_{\chi} - 2}{(\epsilon - 1)^2\tau^2} \right] v = 0,
$$

where $\eta_{\chi} \equiv m^2_{\chi} / (3H^2)$.

Using the usual Bunch–Davies vacuum state to normalize the amplitude of the fluctuations on small scales, we obtain

$$
v = \sqrt{\frac{\pi}{2}} \frac{e^{i(\nu+1/2)(\pi/2)}}{k^{1/2}} (-k\tau)^{1/2} H^{(1)}_{\nu}(-k\tau),
$$

where the order of the Hankel function is given by

$$
\nu^2 = \frac{9}{4} - \frac{3\epsilon}{(\epsilon - 1)^2}.
$$

At late times, $-k\tau \rightarrow 0$, making use of the asymptotic form of the Hankel function,

$$
H^{(1)}_{\nu}(-k\tau) \rightarrow -i \frac{\Gamma(\nu)}{\pi} \left( \frac{-k\tau}{2} \right)^{-\nu},
$$

the power spectrum of $\delta \chi$ in this limit becomes

$$
\mathcal{P}_{\chi} = C_{\nu}^2 \frac{k^2}{a^2} (-k\tau)^{1-2\nu},
$$

where $C_{\nu} \equiv 2^{\nu-3/2}\Gamma(\nu) / \pi^{3/2}$. Then the spectral tilt of the generated fluctuations is

$$
\Delta n_{\chi} \equiv \frac{\ln \mathcal{P}_{\chi}}{\ln k} = 3 - 2\nu,
$$
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and to leading order in a fast-roll expansion,

\[ \Delta n_\chi \simeq \frac{2}{\epsilon}. \] (A.12)

Thus, for a steep exponential potential we obtain a slightly blue spectrum, becoming scale invariant as \( \epsilon \to \infty \).

Approximating \( \nu \simeq 3/2 \), on large scales we can relate the amplitude of \( \delta \chi \) to \( H \) as

\[ P^{1/2}_\chi = \epsilon \frac{|H|}{2\pi}. \] (A.13)

Appendix B. Isocurvature field bispectrum

In this section we calculate the intrinsic non-Gaussianity of the field \( \chi \), using the approach of Maldacena [22], which also includes the contribution from sub-Hubble non-linear interactions. Following [22], the three-point correlator is given by

\[ \langle \delta \chi^3(t) \rangle = -i \int_{-\infty}^t dt \langle [\delta \chi^3(t), H_{\text{int}}(t')] \rangle, \] (B.1)

where \( H_{\text{int}} \) is the interaction Hamiltonian. Here we consider only the cubic interaction, which is the dominant one for the three-point function,

\[ H_{\text{int}}(t') = \int d^3 x a^3 \frac{V^{(3)}}{3!} \delta \chi^3, \] (B.2)

where \( V^{(3)} \) is given in equation (13).

Writing equation (B.1) in Fourier space,

\[ \langle \delta \chi_{k_1}(t) \delta \chi_{k_2}(t) \delta \chi_{k_3}(t) \rangle = (2\pi)^3 \delta \left( \sum_j k_j \right) \times 2 \text{Re} \left( -i \delta \chi_{k_1}(t) \delta \chi_{k_2}(t) \delta \chi_{k_3}(t) \int_{-\infty}^t dt' a^3 V^{(3)} \delta \chi^*_k(t') \delta \chi^*_k(t') \right), \] (B.3)

and using the normalized free field solution

\[ \delta \chi_k = \frac{1}{a\sqrt{2k}} e^{-ik\tau} \left( 1 - \frac{i}{k\tau} \right) \] (B.4)

valid in the limit \( p \ll 1 \), we get

\[ \langle \delta \chi_{k_1}(t) \delta \chi_{k_2}(t) \delta \chi_{k_3}(t) \rangle = -(2\pi)^3 \delta \left( \sum_j k_j \right) \frac{V^{(3)} e^2 H^4}{4H^2 \prod_j k_j^3} \]

\[ \times \text{Re} \left[ i(1 + ik_1\tau)(1 + ik_2\tau)(1 + ik_3\tau)e^{-i\sum_j k_j\tau} \int_{-\infty}^\tau \frac{d\tau'}{\tau^\delta} (1 - ik_1\tau')(1 - ik_2\tau')(1 - ik_3\tau')e^{i\sum_j k_j\tau'} \right], \] (B.5)

where we have taken out \( V^{(3)} \) from the time integral using that \( V^{(3)}/H^2 \) is constant.
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Taking into account only the leading contribution in \(-k_j \tau \ll 1\), the last two lines of the above equation yield \(\sum_j k_j^2/4\). Furthermore, using also equations (13) and (A.5) to replace \(V^{(3)}\), we finally find

\[
\langle \delta \chi_{k_1}(t) \delta \chi_{k_2}(t) \delta \chi_{k_3}(t) \rangle = (2\pi)^3 \delta \left( \sum_j k_j \right) \frac{\sum_j k_j^3 \tilde{c}}{\prod_j k_j^3} 8^{4} H^4. \tag{B.6}
\]

The intrinsic non-Gaussianity after Hubble exit is thus of the local form, as it is dominated by the super-Hubble evolution. Note that using \(\epsilon = 1/p\) and \(H = -p/t\) we can rewrite the last term in the equation above as

\[
\frac{\tilde{c}}{8} 8^{4} H^4 = \frac{\tilde{c}}{8t^4}. \tag{B.7}
\]

Thus, for \(\tilde{c} = 1/M\) we recover the result of [13]. With the definition of the power spectrum for \(\delta \chi\), equation (33), we can rewrite equation (B.6) as

\[
\langle \delta \chi_{k_1}(t) \delta \chi_{k_2}(t) \delta \chi_{k_3}(t) \rangle = (2\pi)^3 \delta \left( \sum_j k_j \right) \frac{\sum_j k_j^3}{\prod_j k_j^3} 2\pi^4 \tilde{c} P^2_{\chi}. \tag{B.8}
\]

which yields, using equation (17),

\[
B_{\chi}(k_1, k_2, k_3) = \frac{\sum_j k_j^3}{\prod_j k_j^3} 2\pi^4 \tilde{c} P^2_{\chi}. \tag{B.9}
\]

This equation is equivalent to equation (45), confirming the result found from the large scale non-linear evolution in section 5. Thus non-linearities on sub-Hubble scales are completely subdominant with respect to the non-linear classical super-Hubble evolution.

Appendix C. Numerical results

In this appendix, we check the validity of the analysis of sections 4 and 5 by numerically calculating the expansion \(\delta N\).

C.1. Phase space variables

For numerical calculations, it is more convenient to adopt phase space variables. Thus, first we summarize the background dynamics in terms of the phase space variables. These are defined as [15, 36, 37]

\[
x_j = \frac{\dot{\phi}_j}{\sqrt{6H}}, \tag{C.1}
\]

\[
y_j = \frac{\sqrt{V_j} e^{-c_j \phi_j}}{\sqrt{3H}}, \tag{C.2}
\]
and their evolution equations are given by

\[ \frac{dx_j}{dN} = -3x_j \left( 1 - \sum_k x_k^2 \right) - c_j \sqrt{\frac{3}{2}} y_j^2, \]  

(C.3)

\[ \frac{dy_j}{dN} = y_j \left( 3 \sum_k x_k^2 - c_j \sqrt{\frac{3}{2}} x_j \right), \]  

(C.4)

where \( N = \log a \) and \( j = 1, 2 \) for two-field case. The Friedmann equation gives a constraint,

\[ \sum_j x_j^2 - \sum_j y_j^2 = 1. \]  

(C.5)

Assuming \( \sum_j c_j^{-2} < 1/6 \), there are four fixed points of the system where \( dx_j/dN = dy_j/dN = 0 \),

A: \( \sum_k x_k^2 = 1, \ y_j = 0. \)  

(C.6)

\[ B_j: \ x_j = \frac{c_j}{\sqrt{6}}, \quad y_j = -\sqrt{\frac{c_j^2}{6} - 1}, \quad x_k = y_k = 0, \quad (\text{for } j \neq k), \]  

(C.7)

\[ B: \ x_j = \sqrt{\frac{6}{3p c_j}}, \quad y_j = -\sqrt{\frac{2}{c_j^2 p} \left( \frac{1}{3p} - 1 \right)}, \]  

(C.8)

where \( p = \sum_j 2/c_j^2 \). A linearized stability analysis shows that the multi-field scaling solution, \( B \), always has one unstable mode. On the other hand, the single-field dominated scaling solutions, \( B_j \), are always stable.

In the \((x_1, x_2)\) plane, the three fixed points \( B, B_1 \) and \( B_2 \) are connected by a straight line, which is given by

\[ c_2 x_1 + c_1 x_2 = \frac{c_1 c_2}{\sqrt{6}}. \]  

(C.9)

Solutions which start close to the saddle point \( B \) evolve along this line to one of the single-field dominated solutions, \( B_1 \) or \( B_2 \). As in the main text, we concentrate on the case in which the background evolves from the multi-field scaling solution \( B \) to the single-field \( \phi_2 \) dominated scaling solution \( B_2 \) for simplicity.

C.2. Numerical scheme

We first briefly summarize the numerical scheme. For the phase space variables \( x_j, y_j \) defined by equations (C.1) and (C.2), we solve the evolution equations (C.3) and (C.4), numerically. Since we concentrate on the case in which the background evolves from the multi-field scaling solution, initial conditions are characterized by a point close to \( B \) in the phase space. As in the main text, in the multi-field scaling solution background, the fluctuations of the field \( \varphi \) are negligible.
Thus, for the phase space variables, we choose the initial conditions

\[
\frac{\dot{\phi}}{\sqrt{6}H} = \frac{c_2 x_1 + c_1 x_2}{\sqrt{c_1^2 + c_2^2}} = \frac{c}{\sqrt{6}}, \tag{C.10}
\]

\[
-\frac{\dot{\chi}}{\sqrt{6}H} = \frac{c_2 x_2 - c_1 x_1}{\sqrt{c_1^2 + c_2^2}} \equiv z, \tag{C.11}
\]

where equation (C.10) is given by the background scaling solution in equations (10) and (11). This guarantees that \((x_1, x_2)\) lies on the line given by equation (C.9).

Equation (C.11) defines the quantity \(z\), which characterizes the deviation along the instability direction from \(B\). Once we fix a model, i.e. the values of \(c_1\) and \(c_2\), the initial values of \(x_1\) and \(x_2\) are determined from equations (C.10) and (C.11) for a given initial value of \(z\).

For this set of \((x_1, x_2)\), \(y_1\) is fixed so that \((x_1, x_2, y_1)\) is in the instability direction from \(B\), and then \(y_2\) is fixed from the constraint equation (C.5). Using this set of initial values \((x_1, x_2, y_1, y_2)\), we can specify a background trajectory by solving the evolution equation. The left panel of figure C.1 shows an example of the background solution. For this background, we evaluate an expansion \(\delta N\) from the hypersurface characterized by \(H_i\) to that by \(H_f\).

From equations (43), (47) and (C.11), we can see that a different choice of \(z\) corresponds to that of \(\alpha\) and \(\chi\) as

\[
z = \frac{\epsilon H}{\sqrt{6}} \left(1 + \frac{1}{2} \tilde{c}_\alpha H\right) \alpha, \tag{C.12}
\]

\[
z = \frac{\epsilon}{\sqrt{6}} \left(1 + \frac{1}{4} \tilde{c}_\chi\right) \chi. \tag{C.13}
\]

Then \(z_i\) is related with \(\chi_i\) and it determines the Hubble parameter at the transition \(H_T\) in this scheme. In order to calculate the derivatives of \(N(z_i)\) with respect to \(z_i\), we change \(z_i\) to \(H_T\).
slightly and calculate the difference of the expansion $N(z_i)$. The right panel of figure C.1 shows the effect of the change in $z_i$ on the Hubble parameter.

### C.3. Power spectrum and non-Gaussian parameter

In the following we first check numerically the linear relation equation (34). Note that from equations (C.12) and (C.13), $\delta z \propto \delta \alpha \propto \delta \chi$ holds at the linearized level. From equations (24) and (33), the power spectrum of the curvature perturbation is given by

$$P_\zeta = N_{\chi i}^2 \varepsilon^2 \left| \frac{H_i}{2\pi} \right|^2.$$  (C.14)

In our numerical scheme, from equation (C.13) we evaluate $N_{\chi i}$ as

$$N_{\chi i} = \frac{\varepsilon}{\sqrt{6}} N_{z i} = \frac{\varepsilon}{\sqrt{6}} \frac{N(z_i + \delta z_i) - N(z_i - \delta z_i)}{2\delta z_i},$$  (C.15)

which is valid for sufficiently small $\delta z_i$. From equation (34), the power spectrum of the curvature perturbations can also be expressed in terms of the scalar field fluctuation $\delta \chi$ as

$$P_\zeta = \frac{\varepsilon^2}{c_1^2 + c_2^2} \left| \frac{H_i}{2\pi} \right|^2 = \frac{\varepsilon^2}{c_1^2 + c_2^2} \left| \frac{\partial \chi}{\sqrt{|B_{\chi}|}} \right| \left| \frac{H_i}{2\pi} \right|^2.$$  (C.16)

As equation (C.16) and equation (C.14) should agree, we examine whether a quantity $q$ defined by

$$q \equiv \frac{N_{\chi i}^2 (c_1^2 + c_2^2) |\partial \chi|^2}{|\partial \chi|^2 B_{\chi}},$$  (C.17)

becomes close enough to 1. We find that $q = 1$ is satisfied within about 1% accuracy in our numerical simulations.

Next, we compute numerically $f_{NL}$. From equation (C.12), we can write down $N_{,\alpha}$ and $N_{,\alpha \alpha}$ in terms of $z$:

$$N_{,\alpha} = N_{z i} \frac{dz}{d\alpha} = \frac{\varepsilon}{\sqrt{6}} H_i \left( 1 + \frac{\sqrt{6} \tilde{c}}{\epsilon} z_i \right) N_{z i},$$  (C.18)

$$N_{,\alpha \alpha} = N_{z i} \left( \frac{dz}{d\alpha} \right)^2 + N_{z i} \frac{d^2 z}{d\alpha^2} = \frac{\varepsilon^2}{6} H_i^2 \left( 1 + \frac{2\sqrt{6} \tilde{c}}{\epsilon} z_i \right) N_{z z} + \frac{\varepsilon \tilde{c} H_i^2}{\sqrt{6}} N_{z i}.$$  (C.19)

In our numerical scheme, we evaluate $N_{z i z i}$ as

$$N_{z i z i} = \frac{N(z_i + \delta z_i) - 2N(z_i) + N(z_i - \delta z_i)}{(\delta z_i)^2}.$$  (C.20)

If the instantaneous transition approximation is valid, the non-Gaussianity parameter $f_{NL}$ is given by

$$f_{NL} = \frac{5}{6} \frac{N_{,\alpha \alpha}}{N_{,\alpha}^2}.$$  (C.21)
Table C.1. $c_1$ and $c_2$ dependence of the non-linear parameter. We also compare with the results obtained by the instantaneous transition and the fast-roll approximations. The deviation from $r = 1$ denotes the error of these approximations. We adopt $z_1 = 0.01$ and $\delta z_1 = 10^{-5}$.

| $c_1$ | $c_2$ | $f_{NL}$ | $r$   |
|-------|-------|---------|-------|
| 10    | 10    | 40.6165 | 0.974797 |
| 10    | 20    | 40.9285 | 0.982283 |
| 15    | 10    | 90.4979 | 0.965311 |
| 15    | 20    | 92.6425 | 0.988187 |
| 15    | 40    | 93.5294 | 0.997647 |
| 20    | 10    | 161.014 | 0.966082 |
| 20    | 20    | 164.976 | 0.989855 |
| 20    | 40    | 169.365 | 1.01619  |
| 40    | 10    | 639.909 | 0.959865 |
| 40    | 20    | 659.744 | 0.989617 |
| 40    | 40    | 664.996 | 0.997494 |

Thus, in order to verify the accuracy of the analytical result $f_{NL} = 5c_1^2/12$, we examine whether a quantity $r$ defined by

$$r \equiv \frac{2N_{\alpha\alpha}}{c_1^2 N_{\alpha}}$$

(C.22)

becomes close to 1 or not.

Since we checked that the choice of $\delta z_1$ does not affect the results for sufficiently small values, we use $\delta z_1 = 10^{-5}$. It was also verified that the time evolution of $\delta \chi$ can be described by that of the multi-field scaling solution up to $z_1 = 0.01$; thus, we take $z_1 = 0.01$. Using these initial conditions, we calculate $f_{NL}$ for various combinations of $c_1$ and $c_2$. It is worth noting that in terms of $c_1$ and $c_2$ the fast-roll parameter is expressed as

$$\epsilon = \frac{c_1^2 c_2^2}{2(c_1^2 + c_2^2)}.$$  

For the examples we show in table C.1, the range of the value of $\epsilon$ varies from 25 ($c_1 = c_2 = 10$) to 400 ($c_1 = c_2 = 40$). The results for $f_{NL}$ and $r$ are summarized in table C.1.

From table C.1 we can see that for these parameters, the results for $f_{NL}$ are accurately described by $f_{NL} = 5c_1^2/12$ which agrees with the result obtained analytically based on the instantaneous transition approximation and the fast-roll approximation in the main text.

We can see that the analytical results deviate from the numerical results for smaller $\epsilon$ (compare the cases with $c_1 = c_2$). This is because the fast-roll approximation becomes worse for smaller $\epsilon$. We can also see that, for a fixed $c_1$, the analytical results again deviate from the numerical results for smaller $c_2$. This is because the potential around $B$ becomes flatter along the direction to $B_2$ for smaller $c_2$, and the instantaneous transition approximation becomes worse.

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