Hybrid Moments of the Riemann Zeta-Function

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The "hybrid" moments
\[ \int_2^T |\zeta(1/2+i\lambda)|^k \left( \int_{\lambda-\Gamma}^{\lambda+\Gamma} |\zeta(1/2+i\tau)|^\ell \, d\tau \right)^m \, d\lambda, \]
where \( k, \ell, m \in \mathbb{N} \), are studied. The expected upper bound for the above expression is \( O(T^{1+\varepsilon}G^m) \). This is shown to be true for certain specific values of \( k, \ell, m \in \mathbb{N} \), and the explicitly determined range of \( G = G(T; k, \ell, m) \). The application to a mean square bound for the Mellin transform function of \( |\zeta(1/2+i\tau)|^4 \) is given.

1. Introduction

Power moments represent one of the most important parts of the theory of the Riemann zeta-function \( \zeta(s) \), defined as
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\sigma = \Re s > 1) \]
and otherwise by analytic continuation. Of particular significance are the moments on the "critical line" \( \sigma = 1/2 \), and a large literature exists on this subject (see, e.g., the monographs [1–5]). Let us define
\[ I_k(T) = \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} \, dt, \]
where \( k \in \mathbb{R} \) is a fixed, positive number. Naturally one would like to then obtain good upper bounds for \( I_k(T) \). A simple bound for \( |\zeta(1/2+it)|^k \) is (see, e.g., [4])
\[ \left| \zeta\left(\frac{1}{2} + it\right) \right|^k \ll \log t \int_{t-1}^{t+1} \left| \zeta\left(\frac{1}{2} + ix\right) \right|^\ell \, dx + 1, \]
where \( k \in \mathbb{N} \) is fixed. The use of (3) allows one to replace a power of \( |\zeta(1/2+it)| \) by its integral over a suitable (short) interval. In employing this procedure one obviously loses something, but on the other hand one gains flexibility from the fact that explicit upper bounds for \( I_k(T+G) - I_k(T-G) \) are known only in the case when \( k = 1 \) (see Lemma 5) and \( k = 2 \) (see [3]). In this way bounds for \( I_{k+\ell+m}(T) \) are reduced to the so-called "hybrid" moments of the type
\[ \int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^k \left( \int_{t-G}^{t+G} \left| \zeta\left(\frac{1}{2} + ix\right) \right|^\ell \, dx \right)^m \, dt \]
where \( k, \ell, m \in \mathbb{N} \), are assumed to be fixed and \( 1 \ll G = G(T) \ll T \). The expected bound for the expression in (4) (this is consistent with the hitherto unproved Lindelöf hypothesis that \( \zeta(1/2+it) \ll |t|^\epsilon \)) is clearly
\[ O_s(T^{1+\epsilon}G^m). \]
Here and later \( \epsilon (>0) \) denotes arbitrarily small constants, not necessarily the same ones at each occurrence, and \( a = O_s(b) \)
(same as \(a \ll \varepsilon\)) means that the implied constant depends only on \(\varepsilon\). The problem is to find, for given \(k\), \(\ell\), and \(m\), the range of

\[G = G(T; k, \ell, m)\]  

for which the integral (4) is bounded by (5), and naturally one would like the lower bound for \(G\) to be as small as possible. Note that from general results (e.g., see Ramachandra’s monograph [4]) one obtains that the expression in (4) is, for \(\log \log T \ll G \ll T\),

\[
\int_{T+G}^{T} \left| \chi \left( \frac{1}{2} + it \right) \right|^k dt \ll \varepsilon G T^{\varepsilon} \quad (T^{1/3} \ll \varepsilon = \varepsilon (T) \ll T),
\]

where for some suitable coefficients \(a_{\ell,k}\) one has

\[
P_{k}(y) = \sum_{j=0}^{k} a_{\ell,k} y^j,
\]

and \(E_k(T)\) is to be considered as the error term in (13). An extensive literature exists on \(E_k(T)\), especially on \(E_1(T) = E(T)\) (see Atkinson’s classical paper [18]), and the reader is referred to [1] for a comprehensive account. It is known that \((y = -1^{(1)}(1) = 0.5772156 \cdots\) is Euler’s constant

\[P_1(y) = y + 2y - 1 - \log(2\pi),\]

and \(P_k(y)\) is a quartic polynomial in \(y\) whose leading coefficient equals \(1/(2\pi^2)\). This was obtained in Ingham’s classical work [19]. For an explicit evaluation of all the coefficients of \(P_k(y)\) see, for example, Conrey’s work [20]. For the (conjectural) general form of \(P_k(y)\) see [9, 10]. One hopes that

\[E_k(T) = o(T) \quad (T \to \infty)\]

will hold for each fixed integer \(k \geq 1\), which implies the Lindelöf hypothesis that \(\zeta(1/2 + it) \ll |t|^\sigma\). So far (16) is known to be true only in the cases \(k = 1\) and \(k = 2\), when \(E_k(T)\) is a true error term in the asymptotic formula (13). In particular we have (see [1, 3, 4, 21]) \(E(T) \ll T^{\theta/2}\) for some \(\theta\) satisfying \(1/4 < \theta < 1/3\), and \(E(T) = \Omega_0(\sqrt{T})\). We also have \(E_2(T) = \Omega_0(\sqrt{T})\) and the bounds (see [1, 3, 4, 21])

\[E_2(T) \ll \sqrt{T} \log^3 T, \quad \int_0^T E_2^2(t) dt \ll T^2 \log^{22} T.
\]

As usual, \(f(x) = \Omega_0(g(x))\) for a given \(g(x) (> 0\) for \(x > x_0 > 0\) means that

\[
\limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0, \quad \liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0.
\]

2. Statement of Results

Before we state explicitly our results note that we have the bounds

\[
\int_{T-G}^{T} \left| \chi \left( \frac{1}{2} + it \right) \right|^k dt \ll G \log T \quad (T^{1/3} \ll G = G(T) \ll T),
\]

\[
\int_{T-G}^{T+G} \left| \chi \left( \frac{1}{2} + it \right) \right|^k dt \ll G T^{\varepsilon} \quad (T^{2/3} \ll G = G(T) \ll T).
\]
This easily follows from estimates on $E(T)$ and $E_2(T)$ mentioned at the end of the last section. It means that we can restrict ourselves to the range $G \ll T^{1/3}$ when $\ell = 2$ in (4), and to the range $G \ll T^{2/3}$ when $\ell = 4$. This will be implicitly assumed in the proofs of our theorems, which are contained in the following.

**Theorem 1.** One has

\[ \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \int_{J-T}^{J+G} \left| \zeta \left( \frac{1}{2} + ix \right) \right|^4 \, dx \, dt \ll TG \log T \]

\[ (T^\varepsilon \ll G = G(T) \ll T), \quad (20) \]

\[ \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \int_{J-T}^{J+G} \left| \zeta \left( \frac{1}{2} + ix \right) \right|^4 \, dx \, dt \ll \varepsilon T^{1+\varepsilon} G^2 \]

\[ (T^{1/10} \ll G = G(T) \ll T), \quad (21) \]

\[ \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \left( \int_{J-T}^{J+G} \left| \zeta \left( \frac{1}{2} + ix \right) \right|^4 \, dx \right)^2 \, dt \ll \varepsilon T^{1+\varepsilon} G^2 \]

\[ (T^{1/5+\varepsilon} \ll G = G(T) \ll T), \quad (22) \]

\[ \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \left( \int_{J-T}^{J+G} \left| \zeta \left( \frac{1}{2} + ix \right) \right|^4 \, dx \right)^3 \, dt \ll \varepsilon T^{1+\varepsilon} G^3 \]

\[ (T^{1/5+\varepsilon} \ll G = G(T) \ll T), \quad (23) \]

and for $T^{7/12} \log^C T \ll G = G(T) \ll T$ one has

\[ \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \left( \int_{J-T}^{J+G} \left| \zeta \left( \frac{1}{2} + ix \right) \right|^4 \, dx \right)^2 \, dt \ll TG^2 \log^g T. \]

\[ (24) \]

**Theorem 2.** One has, for $1 \ll G = G(T) \ll T$ and some $C > 0$,

\[ \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 \int_{J-T}^{J+G} \left| \zeta \left( \frac{1}{2} + ix \right) \right|^4 \, dx \, dt \ll \log^C (TG + \min (T^{5/3}, T^2 G^{-1})), \]

\[ (25) \]

\[ \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 \left( \int_{J-T}^{J+G} \left| \zeta \left( \frac{1}{2} + ix \right) \right|^2 \, dx \right)^2 \, dt \ll \varepsilon T^{1+\varepsilon} G^2 \]

\[ (T^{7/24} \ll G = G(T) \ll T), \quad (26) \]

\[ \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 \left( \int_{J-T}^{J+G} \left| \zeta \left( \frac{1}{2} + ix \right) \right|^4 \, dx \right)^2 \, dt \ll \varepsilon T^{1+\varepsilon} G^2 \]

\[ (T^{5/9} \ll G = G(T) \ll T), \quad (27) \]

To assess the strength of our results note, for example, that (3) and (25) give

\[ \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^8 \, dt \ll \log T \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 \int_{J-T}^{J+G} \left| \zeta \left( \frac{1}{2} + ix \right) \right|^4 \, dx \, dt \]

\[ + \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 \, dt \ll T^{5/2} \log^C T \quad (G = T^{1/2}). \quad (28) \]

The bound in (28), which follows easily by the Cauchy-Schwarz inequality for integrals from estimates of the fourth and twelfth moment of $|\zeta(1/2 + it)|$ (see [22] and [1, Chapter 8]), is the strongest known bound for the eighth moment of $|\zeta(1/2 + it)|$.

Our last result concerns an improvement of (12). Let $\rho$ be such a constant for which

\[ \int_{0}^{T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^8 \, dt \ll \varepsilon T^{\rho+\varepsilon} \]

holds. At present we have $1 \leq \rho \leq 3/2$. The lower bound follows from general principles (see [1, Chapter 9]). The upper bound is a consequence of (28), and its improvements would be very significant. We will prove, using (21), the following.

**Theorem 3.** If $\mathcal{Z}_2(s)$ is defined by (II) and $\rho$ is defined by (29), then

\[ \int_{1}^{T} \left| \mathcal{Z}_2 (\sigma + it) \right|^2 \, dt \ll \varepsilon T^{(4\rho + 4 - 8\varepsilon)/(3\rho - 1) + \varepsilon} \]

\[ \left( \frac{5 + \rho}{8} \leq \sigma \leq \frac{1 + \rho}{2} \right). \quad (30) \]

**Corollary 4.** One has

\[ \int_{1}^{T} \left| \mathcal{Z}_2 \left( \frac{13}{16} + it \right) \right|^2 \, dt \ll \varepsilon T^{1+\varepsilon} \quad \left( \frac{13}{16} = 0.8125 \right), \quad (31) \]

\[ \int_{1}^{T} \left| \mathcal{Z}_2 (1 + it) \right|^2 \, dt \ll \varepsilon T^{6\rho + \varepsilon} \quad \left( \frac{4}{7} = 0.571428 \cdots \right). \]

Note that (12) gives

\[ \int_{1}^{T} \left| \mathcal{Z}_2 \left( \frac{5}{6} + it \right) \right|^2 \, dt \ll \varepsilon T^{1+\varepsilon}, \quad (32) \]

\[ \int_{1}^{T} \left| \mathcal{Z}_2 (1 + it) \right|^2 \, dt \ll \varepsilon T^{3/5+\varepsilon}, \]

while (31) improves both of these bounds, since $13/16 < 5/6$ and $4/7 < 3/5$.

**3. The Necessary Lemmas**

In this section we will state some lemmas that are necessary for the proofs of our theorems. The first is an explicit formula for an integral involving $|\zeta(1/2 + it)|^4$. Theorem 2.
Lemma 5. For $T^\epsilon \leq G \Rightarrow G(T) \leq T^{1-\epsilon}$, one has

$$
\frac{1}{\sqrt{\pi G}} \int_{-\infty}^{\infty} \left| \left( \frac{1}{2} + it + iy \right) \right|^2 e^{-y/(G^2)} dy
= O \left( \log T \right)
+ \sqrt{2} \sum_{n=1}^{\infty} (-1)^n d(n) n^{-1/2} \left( \frac{T}{2\pi n} + \frac{1}{4} \right)^{1/2} - \frac{1}{2} \right)^{-1/2}
\times \exp \left( -G^2 \left( \text{arsinh} \left( \frac{\pi n}{2T} \right) \right)^2 \sin f(T,n) \right),
$$

(33)

where $d(n)$ is the number of divisors of $n$, $\text{arsinh} \ z = \log (z + \sqrt{z^2 + 1})$, and

$$
f(T,n) = 2T \text{arsinh} \left( \frac{\pi n}{2T} \right) + \sqrt{2\pi nT + \pi^2 n^2} - \frac{1}{4}\pi
= -\frac{1}{4\pi} + 2\sqrt{2\pi nT} + \frac{1}{6}\sqrt{2\pi^2 n^2} T^{-1/2}
+ a_n n^{5/2} T^{-3/2} + a_n n^{7/2} T^{-5/2} + \cdots
$$

(34)

for $1 \leq n \ll T$, where $a_{2n-1}$ are suitable constants.

Proof of Lemma 5. The proof of (33) (see also [12]) is based on Motohashi's exact formula [14] and in particular [3, Theorem 4.1]. It states that

$$
\int_{-\infty}^{\infty} \left| \left( \frac{1}{2} + it \right) \right|^2 g(t) dt
= \int_{-\infty}^{\infty} \text{Re} \left\{ \frac{g(t)}{T} \right\} dt + 2y - \log (2\pi) \ g(t) dt
+ 2\pi \text{Re} \left( \frac{g(1/2)}{T} \right) + 4 \sum_{n=1}^{\infty} d(n)
\times \int_{0}^{\infty} (y(y + 1))^{-1/2} g_c \left( \log \left( 1 + \frac{1}{y} \right) \right) \cos (2\pi ny) dy,
$$

(35)

where

$$
g_c(x) := \int_{-\infty}^{\infty} g(t) \cos (xt) dt
$$

(36)

is the cosine Fourier transform of $g(t)$. One requires the function $g(r)$ in (35) to be real-valued for $r \in \mathbb{R}$ and that there exists a large constant $A > 0$ such that $g(r)$ is regular and $\ll |r|^{-A}$ for $|3\text{nmr}| \leq A$. The choice

$$
g(t) = \frac{1}{\sqrt{\pi G}} e^{-t^2/G^2}, \quad g_c(x) = e^{-1/4(Gx)^2} \cos (Tx)
$$

(37)

is permissible, and then the integral on the left-hand side of (35) becomes (see (9)) $I_1(T,G)$. The first integral on the right-hand side of (35) is $O(\log T)$, and the second one is evaluated by the saddle-point method (see, e.g., [1, Chapter 2]). A convenient result to use is [1, Theorem 2.2 and Lemma 15.1], due originally to Atkinson [18] for the evaluation of exponential integrals $\int_{a}^{b} \phi(x) \exp(2\pi i f(x)) dx$. In the latter, only the exponential factor $\exp(-(1/4)G^2 \log (1 + 1/y))$ is missing. In the notation of [1, 18] we have that the saddle point $x_0$ (root of $f'(x) = 0$) satisfies

$$
x_0 = U - \frac{1}{2} = \left( \frac{T}{2\pi n} + \frac{1}{4} \right)^{1/2} - \frac{1}{2},
$$

(38)

and the presence of the above exponential factor makes it possible to truncate the series in (35) at $n = TG^{-2} \log T$ with a negligible error. Furthermore, in the remaining range for $n$ we have (in the notation of [18])

$$
\Phi_0 \mu_0 F_0^{3/2} \ll (nT)^{-3/4},
$$

(39)

which makes a total contribution of $O(1)$, as does error term integral in Theorem 2.2 of [1]. The error terms with $\Phi(a), \Phi(b)$ vanish for $a \to 0+$, $b \to +\infty$, and (33) follows. Finally note that by using Taylor's formula it is seen that the error made by replacing

$$
\left( \frac{T}{2\pi n} + \frac{1}{4} \right)^{1/2} - \frac{1}{2}
\exp \left( -G^2 \left( \text{arsinh} \left( \frac{\pi n}{2T} \right) \right)^2 \right)
$$

(40)

with

$$
\left( \frac{T}{2\pi n} \right)^{-1/4} \exp \left( -\frac{\pi G^2 n}{2T} \right)
$$

(41)

in (33) is $\ll 1$ for $G \gg T^{1/5} \log C \ T$.

We remark that the series in (33) can be truncated at $T^{-1/3}$. Namely, the contribution for $n \leq T^{-1/3}$ is, by trivial estimation,

$$
\ll T^{-1/4} \sum_{n \leq T^{1/3}} d(n) n^{-1/4} \ll \log T,
$$

(42)

and this is absorbed by the $O$-term in (33).

Lemma 6. Let $N$ denote the number of solutions in integers $m, n, k$ of the inequality

$$
\sqrt{m} + \sqrt{n} - \sqrt{k} \ll \delta \sqrt{M} \quad (\delta > 0)
$$

(43)

with $M' < n \leq 2M'$, $M < m \leq 2M$, $k \in \mathbb{N}$, and $M' \leq M$. Then,

$$
N \ll c \ M^{k} \left( M^2 M' \delta + (MM')^{1/2} \right).
$$

(44)

Lemma 7. Let $k \geq 2$ be a fixed integer and let $\delta > 0$ be given. Then, the number of integers $n_1, n_2, n_3, n_4$ such that $N < n_1, n_2, n_3, n_4 \leq 2N$ and

$$
| \frac{1}{k} + n_2^{1/k} - n_3^{1/k} - n_4^{1/k} | \leq \delta N^{1/k}
$$

is, for any given $\epsilon > 0$,

$$
\ll c \ N^{\epsilon} \left( N^4 \delta + N^2 \right).
$$

(46)
Lemma 6 was proved by Sargos and the author [23], while Lemma 7 is due to Robert and Sargos [24]. They represent powerful arithmetic tools which are essential in the analysis when the cube or biquadrate of exponential sums involving $\sqrt{n}$ appears.

Lemma 8. For $HU \gg T^{1+\varepsilon}$ and $T^\varepsilon \ll U \leq (1/2) \sqrt{T}$, one has

$$\int_T^{T+H} (E(x+U) - E(x))^2 \, dx$$

$$= \frac{1}{4} \pi^2 \sum \frac{d^2(n)}{n^\frac{1}{2}} \times \int_T^{T+H} x^{\frac{1}{2}} \exp\left(2\pi i U \sqrt{\frac{n}{x}} \right) - 1 \right|^2 \, dx$$

$$+ O\left(T^{1+\varepsilon} + HU^{1/2}T^\varepsilon\right).$$

This result was proved by Jutila [25]. The analogous formula also holds with $E(T)$ replaced by

$$\Delta(x) = \sum_{n \leq x} d(n) - x \left(\log x + 2\gamma - 1\right),$$

the error term in the classical Dirichlet divisor problem. From (47), Jutila deduced ($a \asymp b$ means that $a \ll b \ll a$)

$$\int_T^{2T} (E(x+U) - E(x))^2 \, dx = TU \log^3\left(\frac{\sqrt{T}}{U}\right)$$

$$\left(T^\varepsilon \ll U \leq \frac{1}{2} \sqrt{T}\right).$$

The author sharpened (49) to an asymptotic formula. Namely, it was proved in [26] that, with suitable constants $e_j (e_j > 0)$ and $T^\varepsilon \ll U \leq (1/2) \sqrt{T},$

$$\int_T^{2T} (E(x+U) - E(x))^2 \, dx = TU \sum_{j=0}^{3} e_j \log^3\left(\frac{\sqrt{T}}{U}\right)$$

$$+ O\left(T^{1/2+\varepsilon}U^2\right) + O\left(T^{1+\varepsilon}U^{1/2}\right).$$

4. The Proof of Theorem 1

We begin with the bound in (20). The left-hand side equals, by the defining relation of $E(T)$ ((13) with $k = 1$)

$$\ll GT \log^2 T + \int_T^{2T} \left|k\left(\frac{1}{2} + it\right)\right|^4 \, dt$$

$$\times \int_T^{2T} (E(t+G) - E(t-G))^2 \, dt$$

$$\ll GT \log^2 T + \left(T \log^4 T \cdot TG \log^3 T\right)^{1/2} \ll GT \log^2 T$$

(51)

for $G \gg T^\varepsilon$, as asserted. Here we used the Cauchy-Schwarz inequality for integrals and (49). Note that the upper bound in (20) is the best possible, as it coincides with the lower bound in (7). An interesting, but difficult problem, would be to obtain an asymptotic formula for the integral in (20).

To discuss (21), we first exchange the order of integration in the relevant integrals. It follows that the left-hand side of (21) does not exceed

$$\int_T^{2T+G} \left|k\left(\frac{1}{2} + ix\right)\right|^4 \left(\int_{x-G}^{x+G} \left|k\left(\frac{1}{2} + it\right)\right|^2 \, dt\right) \, dx$$

$$+ E(x+G) - E(x-G) \, dx$$

$$\ll \varepsilon GT \log^5 T + T^{547/416+\varepsilon},$$

(52)

which immediately gives the bound which is somewhat weaker than the one in (21), since $13/10 = 1.3 < 547/416 = 0.314903 \cdots$. Here, we used the sharpest known bound $E(T) \ll T^{131/416+\varepsilon}$, 131/416 = 0.314903 \cdots of Watt [27], while the related sharpest bound $\zeta(1/2 + it) \ll t^{32/205+\varepsilon}$ (32/205 = 0.15609 \cdots) is due to Huxley [28]. These results have been obtained by a variation of the so-called Bombieri-Iwaniec method for the estimation of exponential sums (see [29, 30]).

To obtain the sharper bound asserted by (21) we will use results on the moments of $E^*(t)$ (see Section 5), and hence the proof of the bound in question will be completed there.

For the proof of (22) we start from (3) which gives, for $T/2 \leq t \leq 5T/2$,

$$\left|k\left(\frac{1}{2} + it\right)\right|^2 \ll \log T \int_{t-T}^{t+T} \left|k\left(\frac{1}{2} + ix\right)\right|^2 \, dx + 1,$$

(53)

and we use the trivial inequality

$$\int_{t-G}^{t+G} \left|k\left(\frac{1}{2} + ix\right)\right|^2 \, dx \leq \int_{-G}^{G} \left|k\left(\frac{1}{2} + it + iu\right)\right|^2 \, du$$

$$\leq e \int_{-\infty}^{\infty} \left|k\left(\frac{1}{2} + it + iu\right)\right|^2 e^{-\varepsilon |a/G|^2} \, du$$

$$= \sqrt{\pi} eGL_k(t, G)$$

(54)
in the notation of (9), where \( T/2 \leq t \leq 5T/2 \), \( 1 \ll G \ll T \), and \( k \in \mathbb{N} \) is fixed. This gives

\[
\int_T^{ST/2} \left| \frac{1}{2} + it \right|^2 \left( \int_{-G}^{+G} \left| \frac{1}{2} + ix \right|^2 dx \right)^2 dt \\
\ll T^2 \text{ log } T \\n\times \left( \int_{-\infty}^{\infty} \left| \frac{1}{2} + it + iu \right|^2 e^{-\left( u/G \right)^2} du \right)^2 dt \\
+ G^2 T \text{ log }^4 T,
\]

following the proof of (20), where \( \varphi(t) (\geq 0) \) is a smooth function supported in \([T/2, 5T/2]\), such that \( \varphi(t) = 1 \) for \( T \leq t \leq 2T \) and \( \varphi(t) \ll T^{-r} \) for \( t \in \mathbb{R} \) and any \( r \in \mathbb{N} \). For \( J_1(t, T^\varepsilon) \) we use Lemma 5, writing \( \sin z = (e^{iz} - e^{-iz})/(2i) \) and integrate by parts \( \exp(2\sqrt{2m}t) \). In this way it is seen that

\[
\int_T^{ST/2} \varphi(t) \sum_{n=1}^{\infty} (-1)^n d(n) n^{-1/2} \cdots \exp (if(t,n)) \\
\times \left( \int_{-\infty}^{\infty} \cdots \right)^2 dt \\
= \int_T^{ST/2} \varphi(t) \sqrt{t} \sum_{n=1}^{\infty} \frac{i}{\sqrt{\pi n}} (-1)^n d(n) n^{-1/2} \cdots \\
\times \exp \left( -i \frac{4}{3} \pi + i \frac{1}{6} \sqrt{2\pi n} n^{3/2} \right) \\
+ \alpha g n^{5/12} t^{-3/2} + \cdots \left( \int_{-\infty}^{\infty} \cdots \right)^2 \right)^2 \right)^2 dt,
\]

Note that, for \( G = T^\varepsilon, t \asymp T \), we have

\[
\left\{ \exp \left( -G^2 \left( \arcsinh \frac{\sqrt{n}}{2t} \right)^2 \right) \right\}^2 \\
= G^2 \sqrt{\pi n/2} \exp \left( -G^2 \left( \arcsinh \frac{\sqrt{n}}{2t} \right)^2 \right) \\
= \frac{G^2 n}{t^2} \exp \left( -G^2 \left( \arcsinh \frac{\sqrt{n}}{2t} \right)^2 \right),
\]

\[
\varphi(t) \sqrt{n} \ll \frac{1}{\sqrt{T}}.
\]

and that (here \( G \neq T^\varepsilon \))

\[
\left\{ \int_{-\infty}^{\infty} \left| \frac{1}{2} + it + iu \right|^2 e^{-\left( u/G \right)^2} du \right\}^2 \\
= 2 \int_{-\infty}^{\infty} \left| \frac{1}{2} + it + iu \right|^2 e^{-\left( u/G \right)^2} du \\
\times \int_{-\infty}^{\infty} \frac{d}{dt} \left[ \frac{1}{2} + it + iu \right]^2 e^{-\left( u/G \right)^2} du.
\]

Integrating by parts, we have

\[
\int_{-\infty}^{\infty} \left\{ \frac{d}{dt} \left( \frac{1}{2} + it + iu \right) \right\}^2 e^{-\left( u/G \right)^2} du \\
= \int_{-\infty}^{\infty} \left\{ \frac{d}{du} \left( \frac{1}{2} + it + iu \right) \right\} e^{-\left( u/G \right)^2} du \\
= 2 \int_{-\infty}^{\infty} \frac{uG^{-2} \left( \frac{1}{2} + it + iu \right)^2 e^{-\left( u/G \right)^2} du.}
\]

Observe that the integrals in (60) can be truncated at \( |u| = G \log T \) with a negligible error. Therefore, after an integration by parts, we get an integral with the same type of exponential factor (i.e., \( f(t,n) \) in the exponential), but there will be in the integrand a smooth factor of the order \( \ll G^{-1} \sqrt{T/n} \). Hence, after a large number of integrations by parts it follows that the contribution of \( n \) satisfying \( n > T^{1+\varepsilon}G^{-2} \) will be negligible (i.e., less than \( T^{-A} \) for any given \( A > 0 \) and \( \varepsilon = \varepsilon(A) \)). This truncation of the series over \( n \) is the crucial point in the proof, as the ensuing expression will be quite similar to the expressions for \( J_1(t, G) \), only in the exponential factor in (57) we will have \( G = T^\varepsilon \). Thus, the proof reduces to the estimation of

\[
T^2 G^2 \int_T^{ST/2} \varphi(t) \sum_{n=1}^{\infty} \left( \sum_{r=1}^{2} \right)^2 dt,
\]

where

\[
\sum_{r=1}^{\infty} \left( \sum_{n=1}^{\infty} (-1)^n d(n) n^{-1/2} \left( \left( \frac{t}{2\pi n} \right)^{1/2} - \frac{1}{2} \right)^{-1/2} \\
\times \exp \left( -G^2 \left( \arcsinh \frac{\sqrt{n}}{2t} \right)^2 \right) \right) sin f(t,n),
\]

and \( \sum_1 \) is the same expression with \( G = T^\varepsilon \) in the exponential factor. The other two terms, which arise after the squaring of the right-hand side of (33), are clearly less difficult to deal with. Note that

\[
\sum_{m=1}^{\infty} \left( \sum_{r=1}^{\infty} \right)^2 \sum_{m=1}^{\infty} (-1)^m d(m) \cdots sin f(t,m) \\
\times \sum_{n=T^{1+\varepsilon}G^{-2}} (-1)^n d(n) \cdots sin f(t,n) \\
\times \sum_{k=T^{1+\varepsilon}G^{-2}} (-1)^k d(k) \cdots sin f(t,k),
\]
and write the sines as exponentials. For $G \geq T^{1/7+\varepsilon}$, we use Taylor's formula to remove the terms $a_jn^{7/2-5/2} + \cdots$ from all functions $f$ coming from Lemma 5. Namely, we can truncate the tails of series after sufficiently many terms to obtain a negligible error term. There remain only finitely many terms, but the exponentials are identical, so it suffices to treat the first terms only. Then, we integrate by parts many times, as was done in the previous part of the proof. Thus, we are left with sums containing the exponential
\[
e^{i(\Delta t^2 + Et^{-1/2} + Ft^{-3/2})} = e^{i\theta(t)},
\]
say, where we set
\[
\Delta := \sqrt{8\pi} \left( \sqrt{m} + \sqrt{n} - \sqrt{k} \right),
\]
\[
E := \frac{1}{6} \sqrt{2\pi^3} \left( m\sqrt{m} + n\sqrt{n} - k\sqrt{k} \right),
\]
\[
F := a_0 \left( m^2\sqrt{m} + n^2\sqrt{n} - k^2\sqrt{k} \right).
\]
Namely, the terms with
\[
\sqrt{m} + \sqrt{n} + \sqrt{k}, \quad -\sqrt{m} - \sqrt{n} - \sqrt{k}
\]
are clearly negligible by sufficiently many integrations by parts. Thus, only the combination of signs as in (65) is relevant. Here, we suppose that
\[
M'<n\leq2M', \quad M<m\leq2M, \quad K<k\leq2K,
\]
\[
M'\leq M, \quad M, N, K \geq T^{1/3}.
\]
We suppose $\Delta > 0$, since the case $\Delta < 0$ is analogous and the case $\Delta = 0$ is easy. By the first derivative test ([1, Lemma 2.1]) it is seen that the contribution is $\lesssim T^{1+\varepsilon}$ ($G \geq T^{1/7}$) if $K < AM$ or $K > BM$ with suitable positive constants $A, B$ (when $\Delta \gg \sqrt{M}$ or $\Delta \gg \sqrt{k}$), thus, we may assume $K \asymp M$. Consider now the contribution from the triplets $(m,n,k) \in \mathcal{N}^3$ satisfying $\Delta \asymp T^{-1/2}$. By using the bound (44) of Lemma 6 (with $\delta = T^{-1/2}M^{-1/2}$), it is seen that the corresponding portion of the integral in (61) is
\[
\lesssim T^{1+\varepsilon} \max_{K,M,M' \geq T^{1+\varepsilon}G^{-2},K=M} T^{-3/4}M^{-3/4} \left( M^{5/2}T^{-1/2} + M \right)
\]
\[+ T^{1+\varepsilon} \]
\[
\lesssim T^{1+\varepsilon} \max_{K,M,M' \geq T^{1+\varepsilon}G^{-2},K=M} \left( T^{-5/4}M^{7/4} + T^{-3/4}M^{1/4} \right) + T^{1+\varepsilon}
\]
\[
\lesssim T^{1+\varepsilon} \left( T^{1/2}G^{-7/2} + 1 \right) \lesssim T^{1+\varepsilon}
\]
for $G \geq T^{1/7+\varepsilon}$, as asserted.

Now we proceed analogously as was done in the author's work [31]. Suppose $\Delta \geq T^{-1/2}$. We may assume that $E > 0$, since the other case is analogous. Let
\[
T^{-1/2} \leq \Delta \leq \Delta_0,
\]
where $\Delta_0$, which will be determined later, does not depend on $m, n$, and $k$. Further suppose that
\[
2^{-1} \Delta_0 < \Delta \leq 2^{1-1} \Delta_0 \quad \left( 1 \leq j \leq J \left( \log(\Delta_0T^{1/7+\varepsilon}) \right) \right).
\]
(70)
If $|F|T^{-3/2} \ll ET^{-1/2}$ or $|F|T^{-3/2} \ll \Delta T^{1/2}$ with suitable constants, then in $e^{i\theta(t)}$ either $\Delta T^{1/2}$ or $E T^{-1/2}$ dominates in size. Hence, we can use the method of [31]. If we have $\Delta > C_1E/T$ with a sufficiently large $C_1 > 0$, then $f'(t) \gg \Delta/\sqrt{T}$ in $[T,3T]$. Also if $\Delta \geq T^{-1/2}$ and $\Delta < C_2E/T$ with a sufficiently small $C_2 > 0$, then $f'(t) \gg E T^{-3/2}$. In both cases we estimate the integral of $e^{i\theta(t)}$ by the first derivative test, and then the sum over $m, n$, and $k$ by Lemma 6.

If $\Delta \geq T^{-1/2}$ and $\Delta = E/T$, then there may exist a saddle point $t_0 = E/\Delta$ (root of $f'(t_0) = 0$) in $[T,3T]$ if $\Delta = E$. Hence, by the saddle-point method (see [10, Chapter 2]) or by the use the second derivative test, making first the change of variable $\sqrt{t} = u$, we obtain $f''(t_0) = (1/2)\Delta^2E^{-3/2} = \Delta T^{-3/2}$. Hence, by the second derivative test (see Lemma 2.2 of [1]) the corresponding portion of the integral in (61) is
\[
\lesssim \sum_{j=1}^{2} \Delta_0^{-1/2}T^{3/4}T^{-3/4}M^{-3/4} \left( M^{5/2} \Delta_0^{2-j} + M \right)
\]
\[
\lesssim T^0 \left( M^{7/4} \Delta_0^{1/2} + M^{1/4}T^{1/4} + T \right) \lesssim T^{1+\varepsilon}
\]
for $M^{7/4} \Delta_0^{1/2} \ll T$ or
\[
\Delta_0 \ll T^2M^{-7/2}.
\]
(71)
(72)
But since trivially $\Delta_0 \ll \sqrt{M}$ and $M \geq T^{1/3}$ it follows that
\[
T^2M^{-7/2} \gg T^{5/6} > \sqrt{M} \gg \Delta_0,
\]
(73)
which is needed.

Finally, if $|F|T^{-3/2} \gg ET^{-1/2}$ and $|F|T^{-3/2} \gg \Delta T^{1/2}$, then
\[
f'(t) \gg |F|T^{-5/2}, \quad \Delta \ll |F|T^{-2}.
\]
(74)
The first derivative test shows that the contribution is, since $M \ll T^{1+\varepsilon}G^{-2}$,
\[
\lesssim T^{5/2} |F|^{-1} M^{-3/4}T^{-3/4} \left( M^{5/2} |F| T^{-2} + M \right)
\]
\[
\lesssim T^{-1/4}M^{7/4} + M^{1/4}T^{7/4} |F|^{-1}
\]
\[
\lesssim T^{1+\varepsilon} \left( T^{1/2}G^{7/2} + M^{1/4}T^{3/4} |F|^{-1} \right) \lesssim T^{1+\varepsilon}
\]
for $G \geq T^{1/7+\varepsilon}$, provided that
\[
M^{1/4}T^{3/4} |F|^{-1} \ll 1.
\]
(75)
(76)
If, however, $|F| \ll M^{1/4}T^{3/4}$, then $\Delta T^2 \ll |F| \ll M^{1/4}T^{3/4}$ and this implies
\[
\Delta \ll M^{1/4}T^{-5/4} < T^{-1/2},
\]
(77)
and this case has been already dealt with. This completes the proof of (22).

The proof of (23) is similar to the proof of (22). The major difference is that, instead of (61), now we will have to bound

$$T \Delta^3 G^3 \int_{T/2}^{ST/2} \varphi(t) \sum_1 \left( \sum_2 \right)^3 dt. \tag{78}$$

We use then Hölder’s inequality to deduce that the integral in (78) does not exceed

$$\left( \int_{T/2}^{ST/2} \varphi(t) \sum_1 \left| \sum_1 \right|^4 dt \right)^{1/4} \left( \int_{T/2}^{ST/2} \varphi(t) \sum_2 \left| \sum_2 \right|^4 dt \right)^{3/4}. \tag{79}$$

Both integrals in (79) are estimated similarly. Here, we have

$$T^{(1/5)\varepsilon} \ll G = G(T) \ll T^{1/3}. \tag{80}$$

Therefore, by using Taylor’s theorem, instead of the exponential

$$e^{i(\Delta t^2 + Br^{-1/2} + Pt^{-1/2})} \tag{81}$$

we will have the simpler function $e^{i(\Delta t^2 + Re^{-1/2} + Pe^{-1/2})}$ say, where now $\Delta, R$ have a different meaning than in the proof of (22), namely,

$$\Delta = \sqrt{8\pi} \left( \sqrt{m} + \sqrt{n} - \sqrt{k} - \sqrt{\ell} \right),$$

$$R = \frac{1}{6} \sqrt{2\pi}^3 \left( m \sqrt{m} + n \sqrt{n} - k \sqrt{k} - \ell \sqrt{\ell} \right). \tag{82}$$

First note that the sum over $n$ in $\sum_i$ is split into $O(\log T)$ subsums where $M < n \leq M' \leq 2M$, with $M \ll T^{1+\varepsilon} G^{-2}$. Instead of Lemma 6 we use (46) of Lemma 7 (with $\delta = M^{-1/2}$), supposing first that $\Delta > 0$ and that $\Delta < T^{-1/2}$. Afterwards the integral is estimated trivially. The contribution to the relevant integral in (79) will be

$$\ll \varepsilon T^{1+\varepsilon} \max_{M \in T^{1/5} G^{-2}} T^{-1} M^{-1} \left( M^4 T^{2-1/2} M^{-1/2} + M^2 \right) + T^{1+\varepsilon}$$

$$\ll \varepsilon T^{1+\varepsilon} \left( T G^{-5} + 1 \right) \ll \varepsilon T^{1+\varepsilon} \tag{83}$$

for $G \gg T^{1+\varepsilon}$, as asserted.

Now suppose $\Delta > T^{-1/2}$. If $\Delta > 0$, we may also assume that $E > 0$, for otherwise all derivatives of $H(t)$ have the same sign. Let $\Delta > C_1 E/T$ for some suitable $C_1 > 0$. Then, $H'(t) \gg \Delta T^{-1/2}$, and supposing that

$$\Delta \approx 2^j T^{-1/2} \quad (j = 1, 2, \ldots), \tag{84}$$

we obtain by the first derivative test that the contribution is

$$\ll \sum_{j \geq 1} M^{-1} T^{-1} 2^{-j} \left( M^{2/2} T^{2-1/2} + M^2 \right)$$

$$\ll \varepsilon M^{5/2} T^{1+\varepsilon} + T^e \ll \varepsilon T^{1+\varepsilon}. \tag{85}$$

Similar arguing is used if $\Delta < C_2 E/T$. There remains the case when $\Delta \approx E/T$, in which case (after the substitution $u = \sqrt{t}$) it is seen that

$$|H(t)|^{-1/2} \approx \Delta T^{-3/2}. \tag{86}$$

Let, as in the proof of (22),

$$2^{-j} \Delta_0 < \Delta \ll 2^{-j} \Delta_0 \quad (1 \leq j \leq J \approx \log \left( \Delta_0 T^{1/2} \varepsilon \right)). \tag{87}$$

For $M \ll T^{1/2}$ the contribution is

$$\ll \sum_{j \in J} M^{-1} T^{-1} 2^{-j} \Delta_0^{-1/2} T^{3/4} \left( M^{7/2} - 2j \Delta_0 + M^2 \right)$$

$$\ll \varepsilon M^{5/2} T^{1+\varepsilon} \Delta_0^{1/2} + M T^e \ll \varepsilon T^{1+\varepsilon}, \tag{88}$$

for $\Delta_0 \ll T^e$, when

$$M^{5/2} T^{1+\varepsilon} \Delta_0^{1/2} \ll T \Delta_0^{1/2} \ll \varepsilon T^{1+\varepsilon} / 2. \tag{89}$$

But if $\Delta_0 > T^e$, then since $\Delta \approx E/T$, successive integrations by part of $e^{i \Delta t^2}$ show that the contribution is negligible. Let now

$$T^{1/2} \ll M \ll T^{1+\varepsilon} G^{-2} \ll T^{5/5} \quad (G \gg T^{1+\varepsilon}). \tag{90}$$

Like in the preceding case, the contribution will be

$$\ll \varepsilon M^{5/2} T^{1+\varepsilon} \Delta_0^{1/2} + M T^e \ll \varepsilon T^{1+\varepsilon}, \tag{91}$$

for

$$\Delta_0 \ll T^{5/2+\varepsilon} M^{-5}. \tag{92}$$

If (92) does not hold, then we have $\Delta_0 > T^e$, since $M > T^{1/2}$. Again we integrate by parts $e^{i \Delta t^2}$ sufficiently many times. Each time we get a factor in the integrand which is

$$\ll \frac{1}{\sqrt{\Delta T}} + \frac{T^{3/2} E}{\Delta},$$

$$\ll T^{-1/2} + T^{-1} M^{3/2} \tag{93}$$

$$\ll T^{-1/2} + T^{-1} T^{9/10} < T^{-1/10},$$

so that the contribution will be negligible. This proves (23).

To prove (24), note that the left-hand side equals (see (13) with $k = 2$)

$$\int_T^{2T} \left| \kappa \left( \frac{1}{2} + it \right) \right|^2$$

$$\times (O (G \log^4 T) + E_2 (t + G) - E_2 (t - G) \right) \right)^2 dt

\ll T G^2 \log^9 T + \left( \int_T^{2T} \left| \kappa \left( \frac{1}{2} + it \right) \right|^4 dt \int_T^{2T} E_2^2 (t) dt \right)^{1/2}

\ll T G^2 \log^9 T + T^{13/6} \log^6 T \ll TG^2 \log^{12} T. \tag{94}$$
for $G \geq T^{7/12} \log^C T$, as asserted. Here we used the bound, which follows from (17); namely,

$$
\int_0^T |E_2(t)|^4 \, dt \ll T^{2(\log^2 A - 2)} \log^C T
$$

\[ A \geq 2, \, C = 22 + 8(A - 2) \]

with $A = 4$. This completes the proof of Theorem 1.

## 5. The Proof of Theorem 2

We have first, similarly as in (94),

$$
\int_T^{2T} \left| \frac{1}{2} + it \right|^4 \int_{-G}^{+G} \left| \frac{1}{2} + ix \right|^4 \, dx \, dt
\]

\[
= \int_T^{2T} \left| \frac{1}{2} + it \right|^4 \left( O(G \log^4 T) + E_2(t + G) - E_2(t - G) \right) \, dt
\]

\[ \ll TG \log^8 T + T^{5/3} \log^C T, \]

where we used the first bound in (17). This implies that the left-hand side of (96) is $\ll (TG + T^{5/3}) \log^C T$ in the whole range $1 \leq G = G(T) \leq T$. The bound in question was actually proved by Ivić et al. [15] in connection with mean square estimates for $Z_2(s)$. Hence, the main task is to prove the other bound in (25), for which we need the second bound in (17). Note that the left-hand side of (25) is majorized by a multiple ($L = \log T$) of

$$
\int_T^{2T} \left| \frac{1}{2} + it \right|^4 \int_{-\infty}^{\infty} \left| \frac{1}{2} + it + iu \right|^4 e^{-u(G^2)} \, du \, dt
\]

\[ \ll TL^8 + \int_T^{2T} \left| \frac{1}{2} + it \right|^4 \int_{-\infty}^{\infty} \left| \frac{1}{2} + it + iu \right|^4 e^{-u(G^2)} \, du \, dt
\]

\[ \ll TL^8 + 2 \int_T^{2T} \left| \frac{1}{2} + it \right|^4 \int_{-\infty}^{\infty} E_2(t + u) e^{-u(G^2)} \, du \, dt
\]

\[ \ll TL^8 + \frac{L}{G} \int_T^{2T+GL} |E_2(u)| \left| \frac{1}{2} + it \right|^4 \, du \, dt
\]

\[ \ll L^C \left( TG + T^{5/2} + G^2 \right) \int_T^{2T+GL} |E_2(u)| \left| \frac{1}{2} + it \right|^4 \int_{-u-GL^2}^{u+GL^2} E_2(t) \, dt \, du
\]

\[ \ll L^C(TG + T^{5/2} + T^2 G^{-1})
\]

We also used the bound, which follows by the Cauchy-Schwarz inequality for integrals from the mean square bound in (17); namely,

\[ \int_0^T |E_2(t)| \, dt \ll T^{3/2} \log^C T.
\]  \[ \tag{100} \]

To complete the proof it remains to note that

\[ \left( \int_T^{2T+GL} |E_2(u)| \int_{-u-GL^2}^{u+GL^2} |E_2(t)| \, dt \, du \right)^2
\]

\[ \ll \int_T^{2T+GL} \int_{-u-GL^2}^{u+GL^2} E_2(u) \, du \int_T^{2T+GL} \left( \int_{-u-GL^2}^{u+GL^2} |E_2(t)| \, dt \right)^2 \, du
\]

\[ \ll T^2 L^C \int_T^{2T+GL} \int_{-u-GL^2}^{u+GL^2} E_2(t) \, dt \, du
\]

\[ \ll T^2 L^C \int_T^{2T+GL} \int_{-u-GL^2}^{u+GL^2} E_2(t) \, dt \, du
\]

\[ \ll T^4 L^C G^2.
\]  \[ \tag{101} \]

When we take the square root in (101) and insert the resulting bound in (97) we are left with the bound

\[ O\left( L^C \left( TG + T^{5/2} + T^2 G^{-1} \right) \right)
\]  \[ \tag{102} \]

for the left-hand side of (25). But as $T^{5/2} \leq TG$ for $G \geq T^{1/2}$ and $T^{3/2} \leq T^2 G^{-1}$ for $G \leq T^{1/2}$, this means that in the bound above the term $T^{5/2}$ may be omitted, and (25) follows. We point out yet another estimate, namely (131), for the integral in (25). This was derived for the proof of Theorem 3 and does not contain the (expected) term $TG \log^2 T$, but terms which are reasonably small when $G$ is “about” $T^{131/416}$

The proof of (26) is based on the use of (98) and the fourth moment of the function $E^*(t)$, defined by

\[ E^*(t) := E(t) - 2\pi \Delta^*(t \leq 2\pi), \]  \[ \tag{103} \]
where (see (48))
\[
\Delta^*(x) := -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x)
\]
and \(\Delta(x)\) is the error term in the Dirichlet divisor problem. The function \(E'(t)\) was investigated by several authors, including Jutila [32, 33], who introduced it and the author [12, 17, 31, 34–37]. Among other things, the author ([34–37, Part II] and [31]) proved that
\[
\int_0^T |E^*(t)|^4 \, dt \ll T^{2+\epsilon}
\]
and that ([34–37, Part IV, Corollary 2] and [31])
\[
\int_0^T (E^* (t))^4 \, dt \ll T^{7/4+\epsilon}.
\]

The advantage of working with \(E^* (t)\) instead of \(E(t)\) is that the former is, in the mean power sense, smaller than the latter (for this see [1, Chapter 15] and [34–37]).

For our proof, we need from [13] the elementary formula (cf. (44))
\[
J_1 (t, G) = \frac{2}{\sqrt{\pi} G^3} \int_{-\log T}^{\log T} x E^* (t + x) e^{-(x/G)^2} \, dx + O \left( \log^2 T \right),
\]
which is valid for \(T^5 \leq G = G(T) \leq T^{1/3}, T/2 \leq t \leq 5T/2\).

First observe that the left-hand side of (26) is majorized \((\varphi(t)\) is as in the proof of (22)) by
\[
G^2 \int_{T/2}^{ST/2} \left| \kappa \left( \frac{1}{2} + it \right) \right|^4 \varphi(t) J_1^2 (t, G) \, dt
\]
\[
= G^2 \int_{T/2}^{ST/2} \left( Q_4 (\log t) + E'_2 (t) \right) \varphi(t) J_1^2 (t, G) \, dt
\]
\[
= O \left( TG^2 \log^8 T \right) - G^2 \int_{T/2}^{ST/2} E_2 (t) \varphi(t) J_1^2 (t, G) \, dt.
\]

Namely, by the Cauchy-Schwarz inequality for integrals and the classical bound
\[
\int_0^T \left| \kappa \left( \frac{1}{2} + it \right) \right|^4 \, dt \ll T \log^4 T,
\]
we have
\[
G^2 \int_{T/2}^{ST/2} Q_4 (\log t) \varphi(t) J_1^2 (t, G) \, dt
\]
\[
\ll G \log^4 T \int_{T/2}^{ST/2} \int_0^\infty \left| \kappa \left( \frac{1}{2} + it + iu \right) \right|^4 e^{-(u/G)^2} \, du \, dt
\]
\[
\ll G \log^4 T \int_{T/2}^{ST/2} \int_{x-GL}^{x+GL} e^{-(x-\eta)^2/G^2} \, dx \, dt
\]
\[
\ll TG^2 \log^8 T.
\]

We also use \(\varphi' (t) \ll 1/T\) (the contribution of this derivative is easily handled) and (107). Hence, with suitable \(g(t)\) we obtain
\[
\left\{ J_1^2 (t, G) \right\}^2 = 2J_1 (t, G) \left\{ - \frac{2}{\sqrt{\pi} G^3} \int_{G \log T}^{G \log T} E^* (t + x) \right.
\]
\[
\times \left( 1 - \frac{2x^2}{G^2} \right) e^{-x^2/G^2} \, dx + g'(t) \right\},
\]
(III)

where \(g(t) \ll \log T\). The term \(g'(t)\) is integrated back by parts, and its contribution is easily seen to be \(\ll T^{1+\epsilon} G^2\) in the required range. The first term on the right-hand side of (III) is inserted in (108). It follows that the main contribution to the left-hand side of (108) is \((L = \log T)\) bounded by \(T^{1+\epsilon} G^2\) plus
\[
G^2 \int_{T/2}^{ST/2} |E_2 (t)| \varphi(t) G^{-5} \left( \int_{-GL}^{C} E^* (t + x) \, dx \right)^2 \, dt
\]
\[
\ll G^{-3} \left( \int_{T/2}^{ST/2} |E_2 (t)|^2 \, dt \right)^{1/2}
\]
\[
\times \left( \int_{T/2}^{ST/2} \left( \int_{-GL}^{GL} |E^* (t + x) | \, dx \right)^4 \, dt \right)^{1/2}
\]
\[
\ll TL^5 G^{-3} \left( \int_{T/2}^{ST/2} G^3 \left( \int_{-GL}^{GL} \left| E^* (u) \right| \, du \right)^4 \, dt \right)^{1/2}
\]
\[
\ll TL^5 G^{-3} \left( \int_{T/2-GL}^{ST/2+GL} G^3 \left( \int_{u-GL}^{u+GL} \, du \right) \right)^{1/2}
\]
\[
\ll T^{15/8+\epsilon} G^{-1},
\]
where we used Hölder’s inequality for integrals, (98) and (106). Since
\[
T^{15/8+\epsilon} G^{-1} \ll TG^2 \quad \text{(for } G \gg T^{7/24})\),
\]
the bound in (26) follows.

Finally it remains to prove the bound (27) of Theorem 2. We proceed similarly as in the proof of (26), and we majorize \(\int_{T/2}^{2T} \cdots \int_{T/2}^{ST/2} \varphi(t) \cdots\). Then, we majorize
\[
\left( \int_{-GL}^{t+GL} \left| \kappa \left( \frac{1}{2} + ix \right) \right|^4 \, dx \right)^2
\]
by \(G^2 J_1^2 (t, G)\) and write
\[
\left| \kappa \left( \frac{1}{2} + it \right) \right|^4 = Q_4 (\log t) + E'_2 (t),
\]
\[
\left| \kappa \left( \frac{1}{2} + it + iu \right) \right|^4 = Q_4 (\log (t + u)) + E'_2 (t + u).
\]

After this we integrate by parts \(E'_2\), using (98) and obtaining \(E_2 (t), E_2 (t + u), \) but gaining essentially a factor of \(1/G\) each
time in the process. The major contribution to the left-hand side of (27) will be \( \ll T^{1+\varepsilon} G^2 \) plus
\[
G^{-3} \int_{T/2}^{ST/2} \phi(t) |E_2(t)| \left( \int_{GL}^{GL} |E_2(t+x)|^4 e^{-ix/G^2} \, dx \right) \frac{dt}{2} \\
\ll G^{-3} \left( \int_{T/2}^{ST/2} \phi(t) |E_2(t)|^2 \, dt \right)^{1/2} \\
\times \left( \int_{T/2}^{ST/2} \left( \int_{GL}^{GL} |E_2(u)| \, du \right)^4 \frac{dt}{2} \right)^{1/2}.
\]
(116)

Now we use (95) with \( A = 4 \) and Hölder's inequality, to obtain that
\[
\int_{T/2}^{ST/2} \left( \int_{GL}^{GL} |E_2(u)| \, du \right)^4 \frac{dt}{2} \\
\ll (GL)^3 \int_{T/2}^{ST/2} \int_{GL}^{GL} |E_2(u)|^4 \, du \\
= GL^3 \int_{T/2}^{ST/2} \int_{GL}^{GL} E_2^4(u) \left( \int_{GL}^{GL} \frac{dt}{2} \right) \frac{du}{4} \\
\ll T^{10/3} G^4 \log^C T.
\]

Hence, if we insert (117) in (116) and use (98), we obtain that the expression in (116) is
\[
\ll T^{10/3} G^{-1} \log^C T \ll T^{1+\varepsilon} G^2
\]
(118)
for \( G \gg T^{5/9} \), which yields then (27) and completes the proof of Theorem 2.

It remains yet to complete the proof of (21). From (52), we have
\[
\int_{T-G}^{2T+G} \left( \int_{-G}^{G} \left( \int_{-G}^{G} \frac{dt}{2} \right) \, dx \right) \\
= \int_{T-G}^{2T+G} \left( \int_{-G}^{G} \frac{dt}{2} \right) \\
\times (O(G \log T) + E(x+G) - E(x-G)) \, dx
\]
(119)
\[
\ll T^{\varepsilon} G^e \+
\int_{T-G}^{2T+G} \left( \int_{-G}^{G} \frac{dt}{2} \right) \\
\times |E^e(x+G) - E^e(x-G)| \, dx.
\]

Here, we used the defining property of \( E^e(t) \) together with the elementary bound
\[
\Delta^*(x+G) - \Delta^*(x-G) \ll T^{1+\varepsilon} \quad (x \neq T, 1 \ll G \ll T),
\]
(120)

which is easily obtained, since
\[
\Delta^*(x) = \frac{1}{2} \sum_{n \ll x} (-1)^n d(n), \quad d(n) \ll n^{\varepsilon}. \quad (121)
\]

At this point we use Hölder’s inequality for integrals, (105) and the bound (see [1, Chapter 8])
\[
\int_T^T \left( \frac{1}{2} + it \right) \frac{dt}{2} \ll T^{9/8+\varepsilon},
\]
(122)

to deduce that the last integral in (119) is
\[
\ll \left( \int_{T/3}^{T} \left( \frac{1}{2} + it \right) \frac{dt}{2} \right)^{1/2} \left( \int_{T/3}^{T} |E^e(x)|^2 \, dx \right)^{1/2}
\]
(123)
\[
\ll T^{(9/8)-(4/5)+2(1/5)+\varepsilon} = T^{(13/10)+\varepsilon},
\]
which yields (21). For small values of \( G \), namely, for \( 1 \ll G \ll T^{1/10} \), the bound in (21) may be improved by using a more general result than (49); namely,
\[
\int_T^T (E(x + U) - E(x))^2 \, dx \approx HU \log^3 \left( \sqrt{T} \over U \right),
\]
(124)
deduced by Jutila [25] from (47), for \( HU \gg T^{1+\varepsilon} \) and \( T^e \ll U \ll (1/\sqrt{T}) \). If (124) is applied in conjunction with (52), the Cauchy-Schwarz inequality and the bound (see (28))
\[
\int_0^T |(1/2 + it)|^6 \, dt \ll T^{3+2\varepsilon},
\]
we obtain (21) with \( T^4(GT + T^{5/4}G^{1/2}) \), and
\[
T^{5/4}G^{1/2} \ll T^{13/10} \quad (1 \ll G \ll T^{1/5}).
\]
(125)

This is unconditional, but conjectures on the order of \( E(T) \) and \( E^e(T) \) would lead to further improvements, for example, the conjectural bound \( E(T) \ll T^{1+4\varepsilon} \) would replace the exponent 13/10 in (21) by 5/4.

We end this section by pointing out that one can improve (8) for the range given in Section 1. Namely, since the integral \( \int_{k}^{k} G \) in (9) can be truncated at \( u = \pm G \log T \) with an error which is \( \ll T^{-A} \) for any given \( A > 0 \), it follows that (8) is equivalent to
\[
\int_T^T \left( \int_{T-G}^{T+G} \left( \int_{T-G}^{T+G} \frac{dt}{2} \right) \, dx \right)^m \ll T^{1+\varepsilon} G^m.
\]
(126)

We proceed as in the proof of (24), using (95), to infer that the integral in (126) is (as before \( C > 0 \) is a generic constant)
\[
\ll T^m (\log T)^m + \int_{T/3}^{T} |E_2(t)|^m \, dt \\
\ll T^m (\log T)^m + T^{2+2(1/3)(m-2)} \log^C T
\]
(127)
\[
\ll T^m (\log T)^C
\]
for
\[
G \gg T^{(2m-1)/3m} \quad (m \geq 2),
\]
(128)

where incidentally \( m \) does not have to be an integer. In particular, it follows then from (128) that (8) holds for \( G \gg T^{1/2} \) \( (m = 2) \), \( G \gg T^{5/9} \) \( (m = 3) \), and \( G \gg T^{7/12} \) \( (m = 4) \). Since \( 5/9 < 4/7 \) and \( 7/12 < 3/5 \), this means that we have improved the range of \( G \) for which (8) holds when \( m = 3, 4 \).
6. The Proof of Theorem 3

The method of obtaining mean square bounds for \( \mathcal{L}_2(s) \) (see (11)) was developed in [15, 17], so that we will be fairly brief. From [15, pages 337–339], we have

\[
\int_0^T |\mathcal{L}_2(\sigma + it)|^2 dt \leq \varepsilon T^{2+\varepsilon} X^{1-2\sigma} + \log T
\]

\[
\times \sup_{T^{-\varepsilon} < K < X} \int_{T/2}^{2K} \left| \xi \left( \frac{1}{2} + ix \right) \right|^4 x^{-2\sigma} \\
\times \int_{x-G}^{x+KT^{-1}} \left| \xi \left( \frac{1}{2} + iy \right) \right|^4 dy dx dt
\]

\[
\ll_{\varepsilon} T^{2+\varepsilon} X^{1-2\sigma} + T \log T \sup_{T^{-\varepsilon} < K < X} K^{-2\sigma}
\times \int_{x-G}^{x+G} \left| \xi \left( \frac{1}{2} + iy \right) \right|^4 dy dx
\]

(129)

with \( G = KT^{\frac{1}{13}} \), \( \sigma > 1/2 \), and \( X \) a parameter to be suitably chosen. For the last integral above one could use (25) of Theorem 2. However, this would not lead to the result of Theorem 3, as we need a bound when \( G \) is “about” \( K^{1/3} \), or even slightly smaller. Thus we will proceed differently, and use the elementary inequality

\[
ab \leq \frac{1}{2} \left( a^{1/2} b^{3/2} + a^{3/2} b^{1/2} \right) \quad (a, b \geq 0)
\]

(130)

to obtain

\[
\int_{T/2}^T \left| \xi \left( \frac{1}{2} + it \right) \right|^4 dt \ll_{\varepsilon} T^\varepsilon \int_{-G}^{G+T} \left| \xi \left( \frac{1}{2} + ix \right) \right|^6 dx dt
\]

\[
\ll_{\varepsilon} \int_{T/2}^T \left( \int_{-G}^{G+T} \left| \xi \left( \frac{1}{2} + it \right) \right|^4 dx \right)^6 dt
\]

\[
\ll_{\varepsilon} \int_{T}^{T+G} \left| \xi \left( \frac{1}{2} + ix \right) \right|^6 dt dx + \left| \xi \left( \frac{1}{2} + it \right) \right|^6 dt \ll_{\varepsilon} \int_{T-G}^{T+G} \left| \xi \left( \frac{1}{2} + it \right) \right|^4 dx dt
\]

\[
\ll_{\varepsilon} \int_{T/3}^T \left| \xi \left( \frac{1}{2} + it \right) \right|^6 (O (G \log T) + E(t+G) - E(t-G)) dt
\]

\[
\ll_{\varepsilon} T^{\varepsilon + (1+\rho)/2} (G + T^{137/416}).
\]

Here we used the defining relation of \( E(T) \) together with the sharpest known bound \( E(T) \ll_{\varepsilon} T^{213/416+\varepsilon} \) (see [27]). We also used the bound for the sixth moment which follows from (29) and the Cauchy-Schwarz inequality; namely,

\[
\int_0^T \left| \xi \left( \frac{1}{2} + it \right) \right|^6 dt \leq \left( \int_0^T \left| \xi \left( \frac{1}{2} + it \right) \right|^4 dt \int_0^T \left| \xi \left( \frac{1}{2} + it \right) \right|^8 dt \right)^{1/2}
\]

(132)

We use (131) (with \( G = KT^{\varepsilon-1} \)) in (129) to obtain

\[
T \sup_{T^{-\varepsilon} < K < X} K^{-2\sigma} \int_K^{2K} \left| \xi \left( \frac{1}{2} + ix \right) \right|^4 dx
\]

\[
\times \int_{x-G}^{x+G} \left| \xi \left( \frac{1}{2} + iy \right) \right|^4 dy dx
\]

\[
\ll_{\varepsilon} T^{\varepsilon + (1+\rho)/2} \int_{-G}^{G} \left| \xi \left( \frac{1}{2} + iy \right) \right|^4 dy dx
\]

(133)

if

\[
1 + \rho + \frac{131}{832} \leq \sigma \leq \frac{3 + \rho}{4}.
\]

(134)

If (134) holds, then we obtain from (129)

\[
\int_0^T |\mathcal{L}_2(\sigma + it)|^2 dt \ll_{\varepsilon} T^\varepsilon \left( T + X^{(1+\rho)/2 - 2\sigma} + T^2 X^{1-2\sigma} \right)
\]

\[
\ll_{\varepsilon} T^\varepsilon \left( T + T^{(6+2\rho-8\sigma)/(\rho+1)} \right) \ll_{\varepsilon} T^{1+\varepsilon}
\]

(135)

for \( \sigma \geq (5 + \rho)/8 \) if we choose \( X = T^{4/(1+\rho)} \). However, for \( \rho \leq 181/104 = 1.74038 \ldots \), we have

\[
\frac{1 + \rho}{4} + \frac{131}{832} \leq \frac{5 + \rho}{8}.
\]

(136)

With \( \rho = 3/2 \), we have \( (5 + \rho)/8 = 13/16 \), and the first assertion of (31) follows. Note that from (see (29))

\[
\int_X^\infty \left| \xi \left( \frac{1}{2} + ix \right) \right|^8 x^{-1-2\sigma} dx \ll 1 \quad (\sigma > \frac{1}{2} (1 + \rho))
\]

(137)

one deduces easily that

\[
\int_T^2 \mathcal{L}_2(\sigma + it)^2 dt \ll 1 \quad (\sigma > \frac{1}{2} (1 + \rho)).
\]

(138)

Thus, the bound in (30) follows from (135), (138), and the convexity of mean values for regular functions (cf. [1, Lemma 8.3]). Setting \( \sigma = 1 \) and \( \rho = 3/2 \) in (30), we obtain the second
bound in (31). We remark that, by [28, equation (3.21)], we have
\[ \int_{T}^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^8 dt \ll T^{2\varepsilon-1} \int_0^{T^{1/4}} \left| \mathcal{Z}_2 (\sigma + it) \right|^2 dt, \]
(139)
The bound (139) links the eighth moment of $|\zeta(1/2+it)|$ to the mean square of $\mathcal{Z}_2(s)$. Thus, the second bound in (31) gives the value $\nu = 11/7$, which is close to the best known bound $\nu = 3/2$.

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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