On stability of a collinear libration point in the planar circular restricted photogravitational three-body problem in the cases of first and second order resonances

B S Bardin$^{1,2,*}$ and A N Avdyushkin$^{1,**}$

$^1$Department of Mechatronics and Theoretical Mechanics, Moscow Aviation Institute (National Research University), 4 Volokolamskoe Shosse, 125993, Moscow, Russian Federation
$^2$Mechanical Engineering Research Institute of the Russian Academy of Sciences, 4 M. Kharitonyevskiy Pereulok, 101990, Moscow, Russian Federation

*E-mail: bsbardin@yandex.ru; **e-mail: avdyushkin.a.n@yandex.ru

Abstract We deal with the planar circular photogravitational three-body problem. That is, we consider the motion of a particle under influence of gravitational and radiation forces acting from two bodies, which move in circular orbits. The stability of collinear point $L_1$, which located on the line between the bodies, is investigated. By using the method of normal forms and applying theorems of KAM theory we perform a nonlinear stability study for parameter values corresponding to the cases of first and second order resonances. Rigorous conclusions on instability and stability in the sense of Lyapunov have been obtained.

1. Introduction
The restricted photogravitational three-body problem is a good mathematical model for study of dynamics of artificial satellites or small natural celestial bodies, such as asteroids and as well as particles of cosmic dust. This model describes motion of a body, which is called particle, under influence of gravitational and radiation forces, acting from side of two large bodies, which are called primaries. The particle has infinitesimally small mass as compared with the masses of the primaries, so that it does not affect the motion of the primaries, which move in given elliptic orbits. If the orbits of primaries are circular and particle moves in the plane of primaries motion then we deal with the so-called planar circular restricted photogravitational three-body problem.

Mathematical formulation of the photogravitational three-body problem has been given in [1]. It was shown [2] that equations of particle motion admit five particular solutions which are analogs of libration points $L_i$ ($i = 1, \ldots, 5$) of classical three body problem. It is well known that in classical three body problem the libration point $L_1$ is unstable. In [3] a surprising fact was established that the presence of radiation pressure repulsive forces can lead to stability of $L_1$. The problem of stability of $L_1$ has attracted a lot of attention of researchers. In particular, the stability of $L_1$ have been studied in linear approximation [4] as well as a nonlinear stability analysis has been performed for some special values of parameters [3,5-9].

In this paper we study the stability of $L_1$ in the sense of Lyapunov for the cases of first and second order resonances, which take place in the boundaries of stability domains.
2. Mathematical formulation of the problem
Let us consider planar circular restricted photogravitational three-body problem. That is we study motion of a particle $P$ in gravitational field of two primaries $P_1$ and $P_2$, which also act on the particle with radiation pressure repulsive forces. It is supposed that the primaries move in given circular orbits. To describe motion of the particle we introduce an orthogonal coordinate system $Ox\hat{y}z$ with the origin $O$ located in the barycenter (see figure 1). The axes $Ox$ and $Oy$ lie in the plane of primaries orbits and rotate with the primaries such that axis $Ox$ passes through $P_1$ and $P_2$, axis complements the coordinate system to right hand and orthogonal one.

![Figure 1. Coordinate system.](image)

In what follow we use dimensionless coordinates $\xi, \eta$ instead of Cartesian coordinates $x, y$ of the particle $P$

$$x = R\xi, \quad y = R\eta,$$

where by $R$ we denote the constant distance between $P_1$ and $P_2$.

The motion of the particle $P$ in the plane of primaries orbits can be described by the following canonical system

$$\frac{d\xi}{dt} = \frac{\partial H}{\partial p_\xi}, \quad \frac{d\eta}{dt} = \frac{\partial H}{\partial p_\eta}, \quad \frac{dp_\xi}{dt} = -\frac{\partial H}{\partial \xi}, \quad \frac{dp_\eta}{dt} = -\frac{\partial H}{\partial \eta},$$

(2)

whose Hamiltonian reads

$$H = \frac{1}{2}(p_\xi^2 + p_\eta^2) + p_\xi \eta - p_\eta \xi - \frac{Q_1(1 - \mu)}{R_1} - \frac{Q_2\mu}{R_2},$$

$$R_1 = \sqrt{(\xi + \mu)^2 + \eta^2}, \quad R_2 = \sqrt{(\xi + \mu - 1)^2 + \eta^2}, \quad \mu = \frac{m_2}{m_1 + m_2}.$$ 

By $m_1$ and $m_2$ we denote masses of the primaries; $Q_1$ and $Q_2$ are values, which describe the so-called mass reduction caused by radiation repulsive forces [1]. If $Q_1 = 1$ ($i = 1, 2$) then the radiation forces are equal to zero and we have classical three body problem. We suppose that $Q_1 \in [0; 1]$ ($i = 1, 2$). It means that values of gravitational forces are greater than the values of the radiation pressure forces.

The equations of motion has the following solution

$$\xi = \xi^*, \quad \eta = 0, \quad p_\xi = 0, \quad p_\eta = \xi^*,$$

where $\xi^*$ is a real root of the equation

$$\xi^* = \frac{Q_1(1 - \mu)}{(\xi^* + \mu)\xi^* + \mu} - \frac{Q_2\mu}{(\xi^* + \mu - 1)\xi^* + \mu - 1} = 0,$$

(3)

such that $\xi^* \in (-\mu; 1 - \mu)$.

The solution (3) describes the collinear libration point $L_1$ located on the axis $Ox$ between the primaries (see figure 2). The libration point $L_1$ is a relative equilibrium of the particle in the rotating coordinate system $Ox\hat{y}z$. 

---

2
Let us pass to local variables \( q_i, p_i \) \((i = 1, 2)\) by means of the following formulas
\[
\xi = \xi_* + q_1, \quad \eta = q_2, \quad p_\xi = p_1, \quad p_\eta = \xi_* + p_1,
\]
and expand the Hamiltonian in the power series with respect to \( q_i, p_i \)
\[
H = \frac{1}{2} (p_1^2 + p_2^2) + p_1 q_2 - p_2 q_1 - a q_1^2 + \frac{1}{2} a q_2^2 + b q_1^3 - \frac{3}{2} b q_1 q_2^2 - c q_1^4 + 3 c q_1^2 q_2^2 - \frac{3}{8} c q_2^4 + \ldots, \tag{4}
\]
where the constant term is omitted and other coefficients reads
\[
a = \frac{Q_1 (1 - \mu)}{\vert \xi_* + \mu \vert^3} + \frac{Q_2 \mu}{\vert \xi_* + \mu - 1 \vert^3}, \tag{5}
\]
\[
b = \frac{Q_1 (1 - \mu)}{\vert \xi_* + \mu \vert^3} + \frac{Q_2 \mu}{\vert \xi_* + \mu - 1 \vert (\xi_* + \mu - 1)^3}, \tag{6}
\]
\[
c = \frac{Q_1 (1 - \mu)}{\vert \xi_* + \mu \vert^5} + \frac{Q_2 \mu}{\vert \xi_* + \mu - 1 \vert^5}. \tag{7}
\]

The problem on stability of the libration point \( L_1 \) is equivalent to the stability problem of trivial equilibrium position \( q_1 = p_1 = q_2 = p_2 = 0 \) of the canonical system with the Hamiltonian (4).

Conclusions on stability in first approximation can be obtained from an analysis of roots of characteristic equation of canonical system, whose Hamiltonian is the quadratic part of (4). Such an analysis has shown that for \( \alpha \in \left( \frac{8}{9}, 1 \right) \) the libration point \( L_1 \) is stable in the first approximation [4].

In this paper, we considered the boundary cases, when \( \alpha = \frac{8}{9} \) or \( \alpha = 1 \). These cases corresponds to low order resonances. In next sections we perform a nonlinear stability analysis and obtain the rigorous conclusions on stability of libration point \( L_1 \) in the sense of Lyapunov.

3. Stability analysis in the case of first order resonance

We start with considering the boundary case \( \alpha = 1 \), when the first order resonance takes place, i.e. one of frequencies of linear system is equal to zero. In particular, in this case the frequencies read \( \omega_1 = 1 \) and \( \omega_2 = 0 \).

Let us perform the following linear canonical change of variables
\[
q_1 = -x_1 + \frac{2\sqrt{3}}{3} y_2, \quad q_2 = \sqrt{3} x_2 - 2 y_1, \tag{8}
\]
\[
p_1 = -\sqrt{3} x_2 + y_1, \quad p_2 = x_1 - \frac{\sqrt{3}}{3} y_2,
\]
which reduces the quadratic part of the Hamiltonian (4) to the following normal form
\[
H_2 = \frac{1}{2} (x_1^2 + y_1^2) - \frac{1}{2} y_2^2. \tag{9}
\]

The change of variables (8) can be constructed, for instance, by the method described in [10,11].
Now we simplify (normalize) the cubic and quartic parts of the Hamiltonian. It can be done by means of a canonical near-identity change of variables $x_i, y_i \rightarrow u_i, v_i$. The generating function for such a change reads
\begin{equation}
S = x_i v_i + S^{(3)}(x_i, v_i) \quad (i = 1, 2),
\end{equation}
where $S^{(3)}(x_i, v_i)$ is a convergent power series, which does not include terms of orders lower than three. The Deprit-Hori method [11,12] can be used to calculate the coefficients of the generating function (10). The above change of variables reduce the Hamiltonian into the following normal form
\begin{equation}
H = H_2 + k_3 u_2^3 + k_{12} (u_1^2 + v_1^2) u_2 + A u_2^4 + B (u_1^2 + v_1^2) u_2^2 + C (u_1^2 + v_1^2)^2 + \cdots.
\end{equation}

The coefficients of normalized Hamiltonian (11) can be also found by using the above-mentioned Deprit-Hori method. If the coefficient $k_3$ in the Hamiltonian (11) is nonzero, then in accordance with the Sokol'skii’s theorem [13] the equilibrium position is unstable. The calculations have shown that in our problem with the Hamiltonian (4), the coefficients $k_3$ and $k_{12}$ are equal to zero. In this case conclusions on stability can be obtained on the basis of the terms of the fourth degree. From Sokol’skii’s theorem [13] it follows that if the inequality $A < 0$ is fulfilled, then the equilibrium $x_i = y_i = 0$ of the system with Hamiltonian (11) is stable in the sense of Lyapunov. Otherwise if $A > 0$, then the equilibrium is unstable. Our calculations have shown that the coefficients of the normalized Hamiltonian (11) read
\begin{equation}
A = \frac{27}{8} (b^2 - c), \quad B = 9 (b^2 - c), \quad C = \frac{9}{8} (b^2 - c).
\end{equation}

Now we obtain the coefficient $A$ in terms of the parameters $\xi$, and $\mu$. First, from equation (3) and the equation $\alpha = 1$, where the coefficient $\alpha$ is represented by the formula (5), we express the coefficients $Q_1$ and $Q_2$ as
\begin{equation}
Q_1 = |\xi + \mu|^3, \quad Q_2 = |\xi + \mu - 1|^3.
\end{equation}

Then, by combining formulas (6), (7), (13) and (12) we have
\begin{equation}
A = \frac{27 \mu (\mu - 1)}{8 (\xi + \mu)^2 (\xi + \mu - 1)^2}.
\end{equation}

Since $\mu \in (0; 1)$, from (14) it follow that $A < 0$. Hence, the equilibrium position is stable in the sense of Lyapunov. It yields the stability of the collinear libration point $L_1$.

4. Stability analysis in the case of second order resonance

Let us now consider the boundary case $a = \frac{8}{9}$, when the second order resonance takes place. That is the linear system has equal frequencies $\omega_1 = \omega_2 = \frac{\sqrt{3}}{3}$. In this case, by using the linear canonical change of variables
\begin{equation}
q_1 = -\frac{\sqrt{3}}{5} x_2 + \frac{\sqrt{15}}{5} y_1, \quad q_2 = \frac{\sqrt{15}}{5} x_1 - \sqrt{3} y_2, \quad p_1 = -\frac{7\sqrt{15}}{15} x_1 + \frac{2\sqrt{3}}{3} y_2, \quad p_2 = -\frac{7\sqrt{3}}{15} x_2 - \frac{2\sqrt{15}}{15} y_1,
\end{equation}
the quadratic part of the Hamiltonian (4) can be reduced to the form
\begin{equation}
H_2 = \frac{1}{2} (x_1^2 + x_2^2) + \omega (x_1 y_2 - x_2 y_1).
\end{equation}

By means of nonlinear change of variables $x_i, y_i \rightarrow u_i, v_i$ the Hamiltonian can be reduced to the following form [14]
\[ H = H_2 + (u_1^2 + v_1^2)(A(u_2^2 + v_2^2) + B(u_1v_2 - u_2v_1) + C(u_1^2 + v_2^2)) + \ldots, \]  

(17)

The calculations have shown that the coefficients of the normalized Hamiltonian read

\[ A = \frac{33291}{40000}b^2 - \frac{1161}{1600}c, \quad B = \frac{-27\sqrt{5}}{10000}(387b^2 - 335c), \quad C = \frac{93069}{25000}b^2 - \frac{3321}{1000}c. \]  

(18)

In accordance with Sokol'skii's theorem [15,16] the equilibrium position is stable in the sense of Lyapunov, if the inequality \( A > 0 \) is satisfied. Otherwise, if \( A < 0 \), then the equilibrium is unstable. To analyze the coefficient \( A \) we obtain it in terms of the parameters \( \xi \) and \( \mu \). From equation (3) and the equation \( a = \frac{8}{9} \), where \( a \) is given by formula (5), we can determine the coefficients \( Q_1 \) and \( Q_2 \)

\[ Q_1 = \frac{|\xi_\ast + \mu|^3(8\mu - \xi_\ast - 8)}{9(\mu - 1)}, \quad Q_2 = \frac{|\xi_\ast + \mu - 1|^3(8\mu - \xi_\ast)}{9\mu}. \]  

(19)

Next, we define a domain of possible values of the parameters plane \( \xi \) and \( \mu \), where the condition \( Q_i \in [0; 1] \) (\( i = 1,2 \)) is satisfied. By equating each coefficient \( Q_1 \) and \( Q_2 \), according to the formulas (19), to 0 and 1, we can get boundaries of the above domain analytically (see figure 3).

**Figure 3.** The domain of possible values of the parameters.

By combining formulas (6), (7), (19) and (18) we have

\[ A = \frac{3(9272\mu^2 - 1562\mu\xi_\ast + 263\xi_\ast^2 - 9272\mu + 781\xi_\ast + 168)}{40000(\xi_\ast + \mu)^2(\xi_\ast + \mu - 1)^2}. \]  

(20)

From equation \( A = 0 \) we get the following formulas for boundaries of the stability domains

\[ \xi_\ast = \frac{781}{263}\mu - \frac{781}{526} \pm \frac{5}{526} \sqrt{-292572\mu^2 + 292572\mu + 17329}. \]  

(21)
In figure 4 the results of stability study are shown. In regions I and II, highlighted in dark gray, the inequality $A > 0$ is satisfied, and the collinear libration point $L_1$ is stable in the sense of Lyapunov. In region III, highlighted in light gray, the inequality $A < 0$ is satisfied, and the collinear libration point $L_1$ is unstable. The boundary curve $\alpha$ separating regions I and III is given by the equation (21), where sign "+" should be taken in the right hand side. The boundary curve $\beta$ separating regions II and III is given by the equation (19), where sign "–" should be taken in the right hand side.

5. Conclusions
Let's summarize the results of the stability study. To obtain rigorous conclusions on stability of $L_1$ in the sense of Lyapunov a nonlinear analysis is necessary. Such an analysis is performed separately for non-resonant and resonant cases. In this paper we have considered the cases of first and second order resonances, which occur at the boundaries of domain of stability in first approximation.

Our study has shown that in the case of first order resonance ($a = 1$) the libration point $L_1$ is stable in the sense of Lyapunov. In the case of second order resonance ($a = 8/9$) the libration point $L_1$ can be both stable and unstable. In this case regions of instability and stability in the sense of Lyapunov have been obtained in the plane of parameters $\xi_0$ and $\mu$. The boundary curves separating the stability and instability regions were found in an explicit analytical form. The question on stability on these curves remains open. To solve it, it is necessary to perform an additional nonlinear analysis taking into account the terms of order six or higher in the Hamiltonian.

Acknowledgments
The reported study was performed at the Moscow Aviation Institute (National Research University) and funded by RFBR, projects number №20-31-90064.
References

[1] Radzievskii V V 1950 The restricted three-body problem, including radiation pressure. (in Russian) *Astron. Zh.* 27 pp 250-56

[2] Radzievskii V V 1953 Three-dimentional case of the restricted problem of three radiating and gravitating bodies. (in Russian) *Astron. Zh.* 30 pp 265-73

[3] Kunitsyn A L and Tureshbaev A T 1983 The collinear libration points in the photogravitational three-body problem. *Sov. Astron. Lett.* 9 pp 228-9

[4] Zimovshikov A S and Tkhai V N 2010 Stability diagrams for a heterogeneous ensemble of particles at the collinear libration points of the photogravitational three-body problem. *J. Appl. Math. Mech.* 74 pp 158-63

[5] Kunitsyn A L and Tureshbaev A T 1985 On the collinear libration points in the photo-gravitational three-body problem. *Celestial Mech.* 35 pp 105-12

[6] Tkhai V N and Zimovshikov A S 2009 The possible existence of cloud-like clusters of microparticles at the libration points of a binary star. *Astronomy Reports* 53 pp 552-60

[7] Tkhai N V 2012 Stability of the collinear libration points of the photogravitational three-body problem with an internal fourth order resonance. *J. Appl. Math. Mech.* 76 pp 441-5

[8] Bardin B S and Avdushkin A N 2018 Stability analysis of an equilibrium position in the photogravitational Sitnikov problem. *AIP Conf. Proc.* 1959, 040002

[9] Bardin B S and Avdushkin A N 2020 Stability of the collinear point \( L_1 \) in the planar restricted photogravitational three-body problem in the case of equal masses of primaries. *IOP Conf. Ser.: Mater. Sci. Eng.* 927 012015

[10] Markeev A P 2009 *Linear Hamiltonian Systems and Certain Stability Problems of Satellite’s Motion about Its Center of Mass* (in Russian) (Moscow, Izhevsk: Regular and Chaotic Dynamics, Institute of Computer Research) p 396

[11] Markeev A P 1978 *Libration Points in Celestial Mechanics and Space Dynamics* (in Russian) (Moscow: Nauka) p 312

[12] Giacaglia G E O 1972 *Perturbation Method in Non-Linear Systems* (New York: Springer) p 369

[13] Sokol’skii A G 1977 On stability of an autonomous Hamiltonian system with two degrees of freedom under first-order resonance. *J. Appl. Math. Mech.* 41 pp 20-8

[14] Sokol’skii A G 1975 On stability of autonomous Hamiltonian system with two degrees of freedom in the case of equal frequencies. *J. Appl. Math. Mech.* 38 pp 741–9

[15] Sokol’skii A G 1978 Proof of the stability of Lagrange solutions at a critical relation of masses. *Sov. Astron. Lett.* 4 pp 79–81

[16] Lerman L M and Markova A P 2009 On stability at the Hamiltonian Hopf bifurcation. *Regul. Chaotic Dyn.* 14 pp 148-62