Spin and Madelung fluid

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Starting from the Pauli current we obtain the decomposition of the non-relativistic local velocity in two parts: one parallel and the other orthogonal to the momentum. The former is recognized to be the “classical” part, that is the velocity of the center-of-mass, and the latter the “quantum” one, that is the velocity of the motion in the center-of-mass frame (namely, the internal “spin motion” or Zitterbewegung). Inserting the complete expression of the velocity into the kinetic energy term of the classical non-relativistic (i.e., Newtonian) Lagrangian, we straightforwardly derive the so-called “quantum potential” associated to the Madelung fluid. In such a way, the quantum mechanical behaviour of particles appears to be strictly correlated to the existence of spin and Zitterbewegung.

I. VARIATIONAL APPROACHES TO THE MADELUNG FLUID

As is well-known, the Lagrangian for a non-relativistic (NR) scalar particle can be assumed to be:

\[ \mathcal{L} = \frac{i\hbar}{2} (\psi^* \partial_t \psi - (\partial_t \psi^*) \psi) - \frac{\hbar^2}{2m} \nabla \psi^* \nabla \psi - U \psi^* \psi \]  

(1)

where \( U \) is the potential due to the external forces, the other symbols meaning as usual. Taking the variations with respect to \( \psi, \psi^* \), (i.e. working out the Euler-Lagrange equations), we get the Schrödinger equations for \( \psi^* \) and \( \psi \), respectively.

The most general scalar wavefunction \( \psi \in \mathbb{C} \) may be factorized as follows:

\[ \psi = \sqrt{\rho} e^{i\phi} \]  

(2)

where \( \rho, \phi \in \mathbb{R} \). By this position, eq.(1) becomes:

\[ \mathcal{L} = - \left[ \partial_t \varphi + \frac{1}{2m} (\nabla \varphi)^2 + \frac{\hbar^2}{8m} \left( \frac{\nabla \rho}{\rho} \right)^2 + U \right] \rho. \]  

(3)

Taking the variations with respect to \( \rho \) and \( \varphi \) we obtain\(^1,2\) the two well-known equations for the so-called Madelung\(^3\) fluid which, taken together, are equivalent to the Schrödinger equation, i.e.:

\[ \partial_t \varphi + \frac{1}{2m} (\nabla \varphi)^2 + \frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 - \frac{\Delta \rho}{\rho} \right] + U = 0, \]  

(4)

where

\[ \frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 - \frac{\Delta \rho}{\rho} \right] = -\frac{\hbar^2}{2m} \frac{\Delta |\psi|}{|\psi|}, \]  

(5)

is often called “quantum potential”;

\[ \partial_t \rho + \nabla \cdot (\rho \nabla \varphi/m) = 0. \]  

(6)

Eqs.(4),(6) are the Hamilton–Jacobi and the continuity equations for the probabilistic fluid respectively, and constitute the “hydrodynamical” formulation of the Schrödinger theory. Usually, they are not obtained by the above variational approach, but by inserting position (2) directly into the Schrödinger equation and subsequently separating away the real and imaginary parts. This second way of proceeding does obey merely to mathematical requirements, and does not gives any physical insight of the Madelung fluid. On the contrary, our variational approach can provide us with a physical interpretation of the non-classical terms appearing in Eqs.(3-4).

The early physical interpretation of quantum potential was forwarded by de Broglie’s pioneering theory of the \textit{pilot wave}\(^4\); in the fifties, Bohm\(^5\) revisited and completed de Broglie’s approach in a systematical way. Sometimes Bohm’s theoretical formalism is referred to as the “Bohm formulation of Quantum Mechanics”, alternative and complementary to the Heisenberg (matrices and Hilbert spaces), Schrödinger (wave-functions), and Feynman (path integrals) ones. From Bohm’s up to present days, several conjectures about the origin of that mysterious quantum potential have been put forth, by postulating “subquantal” forces, the presence of ether, and so on. Particularly important are the derivations of the Madelung fluid within the \textit{stochastic mechanics} framework. In such theories, the origin of the non-classical term (5) appears as substantially \textit{kinematical}. In fact to the classical \textit{drift} (or \textit{translational}) velocity \( p/m \), it is added therein a non-classical, stochastic \textit{diffusion} velocity (either of \textit{markovian}\(^6\) or not \textit{markovian}\(^2\) type). By adopting markovian–brownian assumptions, the Hamilton–Jacobi eq.(4) is obtained in the form of a “generalized”Newton equation \( F = ma; \) the continuity equation comes out instead from the simple sum of the “forward” and “backward” Fokker–Planck equations \(^7\).

In the present paper we shall correlate, at variance

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with the above theories, quantum potential with the spinning nature of the elementary particles constituting matter. The starting point is the existence of the so-called Zitterbewegung (ZBW)\textsuperscript{[7–13]} expected to enter any spinning particle theories. As is well-known, ZBW is nothing but the spin motion, expected to exist for spinning particles. A spinning particle endowed with ZBW appears as an extended-like object, so that the non-classical component of the global velocity is actually related to the “internal” motion [i.e. to the motion observed in the center-of-mass frame (CMF), which is the one where \( p = 0 \) by definition]. The existence of an “internal” motion is denounced, besides by the mere presence of spin, by the remarkable fact that, according to the standard Dirac theory, the particle momentum \( p \) is not parallel to the velocity: \( \mathbf{v} \neq \frac{\mathbf{p}}{m} \); moreover, in the free case, while \( [\mathbf{p}, \mathbf{H}] = 0 \) so that \( p \) is a conserved quantity, quantity \( \mathbf{v} \) is not a constant of the motion: \( [\mathbf{v}, \mathbf{H}] \neq 0 \) (\( \mathbf{v} = \alpha \mathbf{v} \) being the usual vector matrix of Dirac theory). Consequently, a decomposition for the global motion, quite analogous to the one above seen, in classical plus non-classical terms, comes out in two famous relativistic quantum-mechanical procedures: namely, in the Gordon decomposition\textsuperscript{[14]} of the Dirac current, and in the decompositions of the Dirac velocity and Dirac position operators proposed by Schrödinger in his pioneering works\textsuperscript{[7]} As we are working in a NR framework, let us recall that in the literature about ZBW\textsuperscript{[9–13]} it is recognized that the above decomposition for the velocity holds also in the NR limit, i.e., for small velocities of the CM \( |\mathbf{p}| \rightarrow 0 \). In such a way, besides spin and the related intrinsic magnetic moment, also another “spin effect”, ZBW, does not vanish in the NR theory. Therefore also the Schrödinger electron, being endowed with a ZBW motion, does actually show its spinning nature, and is not a “scalar” particle (as often assumed). As a matter of fact, when constructing atoms (in the usual NR framework), we have necessarily to introduce “by hand” the Pauli exclusion principle which is related, as known, to spin; and in the Schrödinger equation the Planck constant \( \hbar \) implicitly denounces the presence of spin. All that will be further probed in the next section. For the moment let us explicitly notice that assuming ZBW is equivalent\textsuperscript{[11,12]} to splitting the motion variables as follows (the dot meaning derivation with respect to time)

\[
\mathbf{x} \equiv \xi + \mathbf{X} ; \quad \dot{\mathbf{x}} \equiv \mathbf{v} = \mathbf{w} + \mathbf{V} , \tag{7}
\]

where \( \xi \) and \( \mathbf{w} \equiv \dot{\xi} \) describe the motion of the CM in the chosen reference frame, whilst \( \mathbf{X} \) and \( \mathbf{V} \equiv \dot{\mathbf{X}} \) describe the internal motion referred to the CMF. From an electrodynamical point of view, the conserved electric current is associated to the helical trajectories of the electrical charge (i.e. to \( \mathbf{x} \)), whilst the center of the particle coulombian field is associated to the geometrical centers of such trajectories (i.e. to \( \mathbf{w} \)). As a consequence, it is the charge which performs the total motion, while the CM undergoes the mean motion only.

Going back to Lagrangian (3), it is now possible, starting by the above assumptions, to attempt an interpretation of the non-classical term \( \frac{\hbar^2}{8m} \left( \frac{\nabla \rho}{\rho} \right)^2 \) appearing therein. Indeed, the first term in the r.h.s. of eq. (3) represents, apart from the sign, the total energy

\[
\partial_t \varphi = -E ; \tag{8}
\]

whereas the second term is recognized to be the kinetic energy \( \mathbf{p}^2/2m \) of the CM, if one assumes that

\[ \mathbf{p} = \nabla \varphi . \tag{9} \]

The third term, that gives origin to quantum potential, may be instead interpreted as the kinetic energy in the CMF, that is the internal energy due to the ZBW motion, provided that we re-write it in the following form:

\[
\frac{\hbar^2}{8m} \left( \frac{\nabla \rho}{\rho} \right)^2 \rightarrow \frac{1}{2} m \mathbf{V}^2 , \tag{10}
\]

\[
\mathcal{L} \rightarrow - \left[ \partial_t \varphi + \frac{1}{2m} (\nabla \varphi)^2 + \frac{1}{2} m \mathbf{V}^2 + U \right] \rho . \tag{11}
\]

Eq. (11) actually implies

\[
|\mathbf{V}| = \frac{\hbar}{2m \rho} \left| \nabla \rho \right| . \tag{12}
\]

At this point it is easily realized that in Lagrangian (11) the sum of the two kinetic energy terms, \( \mathbf{p}^2/2m \) and \( \frac{1}{2} m \mathbf{V}^2 \), is nothing but a mere application of the well-known König theorem.

In the next section we shall show that assumption (12) can be easily obtained from the NR analogue of the so-called Gordon decomposition, that is to say from the well-known Pauli current\textsuperscript{[15]}, together with a constraint derived from the “hydrodynamics” of the Dirac equation in the NR limit.

## II. SPIN AND QUANTUM POTENTIAL

During the last thirty years Hestenes\textsuperscript{[13]} did systematically employ the Clifford algebras language in the description of the geometrical, kinematical and hydrodynamical (i.e., field) properties of spinning particles, both in relativistic and NR frameworks. He applied the Clifford formalism to Dirac and to Schrödinger–Pauli theories. In the small-velocity limit of the Dirac equation, or directly from Pauli’s, Hestenes got the following decomposition of the particle velocity field:

\[
\mathbf{v} = \frac{\mathbf{p} - e \mathbf{A}}{m} + \frac{\mathbf{\nabla} \times (e \mathbf{s})}{m \rho} \tag{13}
\]

where the light speed \( c \) is assumed equal to 1; \( \rho \) is the before-seen quantity; \( e \) is the electric charge; \( \mathbf{A} \) is the
external electromagnetic vector potential; \( p \) is the local momentum, \( p \equiv \rho^{-1} \frac{m}{2} [(\psi^\dagger \psi) - \psi^\dagger \nabla \psi]; \) and \( s \) is the local spin vector, \( s \equiv \rho^{-1} \psi^\dagger \bar{s} \psi, \) where \( \bar{s} \) is the spin operator usually represented by the Pauli matrices as:

\[
\hat{s} = \frac{\hbar}{2}(\sigma_x; \sigma_y; \sigma_z). \tag{14}
\]

In this way, the internal ZBW velocity reads:

\[
V \equiv \frac{\nabla \times (ps)}{m \rho}. \tag{15}
\]

As a particular case, the Schrödinger one arises; namely, when no external magnetic field is present (\( A = 0 \)) and the local spin vector has no precession, \( s \) is constant in time and uniform in space. In this case, we can explicitate the previous equation as follows

\[
V = \frac{\nabla \rho \times s}{m \rho}. \tag{16}
\]

As said above, we can notice that, even in the Schrödinger theoretical framework, ZBW does not vanish (except for the unrealistic case of plane waves, i.e., for the \( p \)-eigenfunctions, for which not only \( s \), but also \( \rho \) is constant, so that \( \nabla \rho = 0 \)). Notice also that the continuity equation \((6) \partial_\rho + \nabla \cdot (p \rho/m) = 0 \) can be still re-written in the usual form, namely \( \partial_\rho + \nabla \cdot (\rho \nabla) = 0. \) In fact, since \( \nabla \times s = 0 \), we have \( \nabla \cdot (\rho \nabla) = \frac{1}{m} \nabla \cdot (\nabla \rho \times s) \equiv \frac{1}{m} \nabla \cdot (\nabla \times (p \rho)) \). Therefore \( \nabla \cdot (\rho \nabla) \), being the divergence of a rotor, is identically equal to zero: as a consequence from Eq. (13) we get \( \nabla \cdot (\rho \nabla) = \nabla \cdot (\rho \nabla). \)

By the ordinary tensor language, \textit{without employing Clifford algebras}, we will now show that the decomposition (13) is easily obtained from the familiar expression of the so-called Pauli current (that is, from the Gordon decomposition of the Dirac current in the NR limit\cite{15});

\[
j = \frac{i \hbar}{2m} [(\psi^\dagger \psi) - \psi^\dagger \nabla \psi] - \frac{eA}{m} \psi^\dagger \psi + \frac{1}{m} \nabla \times (\psi^\dagger \bar{s} \psi). \tag{17}
\]

A spinning NR particle can be described through a Pauli 2-component spinor \( \Phi \):

\[
\psi \equiv \sqrt{\rho} \Phi \tag{18}
\]

where \( \Phi \), if we want to have \( |\psi| = \rho \), has to obey the normalization constraint

\[
\Phi^\dagger \Phi = 1. \tag{19}
\]

Exploiting the expressions for \( s \) and \( p \) introduced at the beginning of this section and inserting the factorization (18) into the above expression (17) gives just the equation:

\[
j \equiv \rho \nu = \rho \frac{p - eA}{m} + \frac{\nabla \times (ps)}{m} \tag{20}
\]

which is nothing but equation (13).

The Schrödinger subcase (i.e., as above said, the case with local spin vector constant and uniform) corresponds to \textit{spin eigenstates}, and then we have to require a wavefunction factorizable into the product of a “non-spin” part \( \sqrt{\rho} e^{i\varphi} \) (scalar) and of a “spin” part \( \chi \) (Pauli spinor):

\[
\psi \equiv \sqrt{\rho} e^{i\varphi} \chi, \tag{21}
\]

\( \chi \) being constant with regard to time and space. Now we have \( p \equiv \nabla \varphi \) [i.e. eq. (9)], \( s \equiv \chi^\dagger \bar{s} \chi = \text{constant} \), and, as seen above, \( V \) will be given by eq. (16).

Another equivalent way of getting out the decomposition given by eq. (20) is the one we have recently followed in ref.\cite{16}. In that paper we proposed a new NR velocity operator endowed with ZBW, starting from which we just obtained the velocity field found above.

Let us finally derive, as promised, equation (12). Because of the following, known mathematical property of the square of the vector product between two generic vectors \( a \) and \( b \):

\[
(a \times b)^2 = a^2 b^2 - (a \cdot b)^2, \tag{22}
\]

we have

\[
V^2 = \left( \frac{\nabla \rho \times s}{m \rho} \right)^2 = \frac{(\nabla \rho)^2 s^2 - (\nabla \rho \cdot s)^2}{(m \rho)^2}. \tag{23}
\]

Let us now insert in equation (23) the NR limit of a constraint found by Hestenes in his hydrodynamical formulation of the Dirac theory. Being \( \beta \) the Takabayasi angle\cite{17}, Hestenes derived from the Dirac equation, by means of Clifford algebras, the following relation:

\[
\nabla \cdot (\rho s) = -m \rho \sin \beta. \tag{24}
\]

In the NR limit where \( \beta \approx 0 \) (so that only the two positive-energy components do not vanish in the standard Dirac bispinor), equation (24) reduces to

\[
\nabla \cdot (\rho s) = 0. \tag{25}
\]

In the Schrödinger case \( s = \text{const} . \), so that \( \nabla \cdot s = 0; \) then, we can write

\[
\nabla \rho \cdot s = 0. \tag{26}
\]

This result can also be derived also in the ordinary algebraic (tensorial) formalism. In fact the negative energy component, the so-called “small” component of the Dirac (bi)spinor \( \chi \), may be written as follows\cite{15}:

\[
\chi = \frac{\hbar^2}{4m^2} [\sigma \cdot \nabla \varphi],
\]

where \( \varphi \) is the positive-energy “large” component of the Dirac spinor. Because, as well known, \( \chi \sim \varphi/c \), and consequently \( p = \varphi^\dagger \varphi + \chi^\dagger \chi \sim \varphi^\dagger \varphi \), after few manipulations and approximations we can see that from the smallness
of $\chi$ follows the smallness of $\nabla \rho \cdot s$. Putting eq. (26) into eq. (23), we easily get

$$V^2 = s^2 \left( \frac{\nabla \rho}{mp} \right)^2;$$

(27)

then, since $|s| = h/2$, we are finally able to deduce just eq. (12)

$$|V| = \frac{h}{2} \frac{\nabla \rho}{m \rho}.$$

Let us remark that, after having inserted eq. (27) into Lagrangian (11), the Hamilton–Jacobi and the Schrödinger equations can be re-written:

$$\partial_t \varphi + \frac{1}{2m} (\nabla \varphi)^2 + \frac{s^2}{m} \left[ \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 - \frac{\Delta \rho}{\rho} \right] + U = 0, \quad (28)$$

$$- \frac{2s^2}{m} \Delta \psi = E\psi. \quad (29)$$

Note that the quantity $2|s|$ replaces $\hbar$, the latter quantity appearing no longer; in a way we might say that it is more appropriate to write $\hbar = 2|s|$, rather than $|s| = h/2 \ldots$! With regard to these results, let us recall that in a recent work of ours,[18], even the celebrated de Broglie relation $E = \hbar \omega$ is subjected to a “spinorial” interpretation, so that it is therein substantially “re-written” as $E = 2|s|\omega$.

### III. CONCLUSIONS

We have first achieved a Gordon-like NR decomposition of the local velocity by the ordinary tensorial language. Secondly, we have derived quantum potential, no longer within the traditional stochastic framework, but (without the ad hoc postulates and the a priori assumptions characterizing stochastic quantum mechanics) by relating in a natural way the non-classical energy term to ZBW and spin. Being quantum potential the “zero point energy” of our probabilistic fluid (in that it is a residual energy, not vanishing even when the CM is at rest and the external fields are absent), it is quite natural to interpret it as the ZBW kinetic energy. The quantum indetermination, to which the zero point energy is strictly related, results therefore to be connected to the existence of ZBW. All this carries further evidence that the quantum behaviour of microsystems may be a direct consequence of the existence of spin. In fact, when $s = 0$ we consequently have a vanishing quantum potential in the Hamilton–Jacobi equation, which becomes then totally classical and Newtonian. In this way we are induced to conjecture that no really elementary quantum scalar particles exist, but that such particles are always constituted by spinning objects endowed with ZBW[24] and up to present no contrary experimental evidence has been found.

Finally, we want to recall recent theoretical approaches where the phase space of the system results to be extended—with consequent additional terms in the hamiltonian structure—in a way very close to the one followed in the present work. We refer to the so-called “semiquantum” dynamics[19–21], treating the quantum fluctuations of (classically) chaotic systems by means of essentially classical formalisms, as, e.g., the “gaussian wave-packet dynamics” or the “time-dependent variational principle”. Both in the latter approach and in another more recent method[21] based on first-order “quantum corrections” to the classical equations of motion, the “semi-quantum extended potential” appearing in the global effective hamiltonian contains, in addition to the usual classical energy terms, a “centrifugal” barrier endowed with an angular momentum $\hbar/2$. This last energy term—common to every physical system and resulting from the minimum uncertainty condition—does really correspond to the ZBW kinetic energy which, as we have shown, is at the origin of the quantum potential. On the other hand the separation, employed in the quoted papers, of the canonical variables in classical/centroid plus quantum/fluctuations variables is fully analogous to our decomposition in translational plus spin components. From this point of view, our approach and our results may be of some usefulness for the physical interpretation of those formalisms, which are rigorous but of essentially analytical character, and in which nothing is said above the spin origin of, e.g., the quantum corrections or of the quantum “chaos suppression”. Therefore it is possible to think that, after further investigations, all the ad hoc assumptions of those theories—from the extension of phase space to the “squeezed coherent state” assumption, and so on—may be understood on the grounds of physical requirements related to the spinning nature of quantum systems.

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[21] B. Sundaram and P.W. Milonni: Phys. Rev. E51 (1995) 791.

[22] “Generalized” in that the acceleration a is defined by means of “forward” and “backward” time derivatives.

[23] Hereafter, every quantity is a local or field quantity: v ≡ v(x; t); p ≡ p(x; t); s ≡ s(x; t); and so on.

[24] As, e.g., pion composed by quarks.