Quantum wormhole and its dynamics

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Received 14 April 2004; accepted 6 May 2004
Available online 2 June 2004
Editor: M. Cvetič

Abstract

The two-dimensional dilaton gravity model is generalized to include a ghost Klein–Gordon field, i.e., a negative gravitational coupling, which supports the existence of static traversible wormhole solutions, and is semiclassically modified by adding local covariant terms of one-loop order. In the semiclassically corrected model, the black hole and the wormhole solutions are given. When a static traversible wormhole is used to transport matter or radiation, we study the back-reaction of the transported matter on the wormhole and the end state of the wormhole in the semiclassical level and compare the results with the classical case. We show that the semiclassical wormhole is stable to such back-reaction: the wormhole starts radiating when the matter arrives and finally return to a static one.

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PACS: 04.70.Bw; 04.20.-q; 04.50.+h

The respectable theoretical possibility of traversible wormholes presented by Morris and Thorne [1] has excited both experts and the public. They studied static, spherically symmetric cases in detail, and the spatial topology was the same as in the black-hole cases, but the throat or the minimal surface was preserved in time, so that observers can pass through it in either direction, traveling freely between two universes. According to the Einstein equation, such a geometry requires negative-energy matter, which was dubbed exotic matter by Morris and Thorne. This provoked renewed interest in traversible wormholes, as reviewed by Visser [2].

In the Morris–Thorne framework, the exotic matter was not modeled, but simply assumed to exist in exactly the configuration needed to support the wormhole. Furthermore, the back-reaction of the transported matter on the wormhole was ignored. If the wormhole turns out to be unstable to such a back-reaction, it will be useless for actual transport. In recent papers [3,4], a classical framework for studying dynamic wormholes was introduced. In that framework, both wormholes and black holes are locally defined by the same geometrical objects, trapping horizons, with one key difference: the black holes have achronal horizons, and the wormholes have temporal horizons, so that they are, respectively, one-way and two-way traversible. If definite dynamic wormhole processes are to be obtained, a specific model for the exotic matter must be proposed. This was done by a simple generalization of...
the Callan, Giddings, Harvey, and Strominger (CGHS) model [5] to include a ghost Klein–Gordon field with a negative kinetic sign.

In this Letter, we take a semiclassical approach of the CGHS model, including a ghost Klein–Gordon field by using Bose, Parker, and Peleg’s model [6], which was modified by adding local covariant terms of one-loop order to the original CGHS action. In this semiclassically corrected model, black hole and wormhole solutions can be given by using the radiationless static condition. When a static traversable wormhole is used to transport matter or radiation, we study the back-reaction of the transported matter on the wormhole and compare the results with the classical case. In the semiclassical model, the wormhole starts radiating when the matter arrives and finally return to a static one. Although many physicists’ instinctive reaction is that negative energy, unbounded below, will lead to instability, our model shows that the wormhole is stable to such back-reaction.

The two-dimensional dilaton gravity of Callan et al. [5] is generalized by the action

\[ \int_S \mu \left[ e^{-2\phi} \left( R + 4(\nabla \phi)^2 + 4\lambda^2 \right) - \frac{1}{2} (\nabla f)^2 + \frac{1}{2} (\nabla g)^2 \right] \]

where \( S \) is a 2-manifold, \( \mu, R, \) and \( \nabla \) are the area form, the Ricci scalar and the covariant derivative of a Lorentz 2-metric on \( S \), respectively, \( \lambda \) represents a cosmological constant, \( \phi \) is a scalar dilaton field, \( f \) is a scalar field representing matter, and \( g \) is a ghost scalar field with negative gravitational coupling. The last term is added to the CGHS action so that \( g \) can provide the negative energy densities needed to construct a wormhole [1,2,7–11].

The classical theory described by the action in Eq. (1) is most easily analyzed by choosing future-pointing null coordinates \((x^+, x^-)\) where the line element may be written as

\[ ds^2 = -2e^{2\rho} dx^+ dx^- \]

and the gauge as

\[ \rho = \phi. \]

It is also convenient to transform the dilaton field \( \phi \) to

\[ r = 2e^{-2\phi}. \]

As one can see in a previous work [3], the model naturally contains both static black holes and static wormholes. In vacuum, \( f = g = 0 \), the general solution is

\[ r = 2m - 4\lambda^2 x^+ x^- . \]

The constant \( m \) is interpreted as the mass of space–time, as described previously [12]. For \( m > 0 \), there is a static black hole with a global structure analogous to the Schwarzschild black hole (Fig. 1(a)). The case \( m = 0 \) is flat, and the case \( m < 0 \) contains an eternal naked singularity, analogous to the negative-mass Schwarzschild solution.

The general static solution with \( f = 0 \) and \( g = 2\lambda (x^+ - x^-) \) is

\[ r = a + 2\lambda^2 (x^+ - x^-)^2, \]

where the constant \( a \) is a minimal radius. The static nature can be seen explicitly by a coordinate transformation \( x^\pm = (t \pm z)/\sqrt{2} \), which puts the line element in the form

\[ ds^2 = \frac{2(dz^2 - dt^2)}{a + 4\lambda^2 z^2}. \]

If \( a > 0 \), this represents a wormhole with a global structure analogous to a Morris–Thorne wormhole, with a throat \( r = a \) at \( z = 0 \), joining two regions with \( r > a \): a \( z > 0 \) universe and a reflected \( z < 0 \) universe (Fig. 1(b)). If \( a < 0 \), there is a curvature singularity at \( r = 0 \) while if \( a = 0 \), the space–time has constant negative curvature. There is a
Fig. 1. Penrose conformal diagrams of (a) a static black hole and (b) a static wormhole.

The definition of active gravitational mass-energy \cite{12}

\[ E = \frac{1}{2} r \left( 1 - \frac{\nabla r \cdot \nabla r}{4\lambda^2 r^2} \right), \]  

which evaluates as \( r/2 \) on any trapping horizon. Thus a wormhole with a throat \( r = a \) has an effective mass \( a/2 \).

We note an important feature of both cases, the trapping horizons defined by \( \nabla r \cdot \nabla r = 0 \), or equivalently \( \partial_+ r = 0 \) or \( \partial_- r = 0 \) \cite{12}. In the black hole, they coincide with the event horizons \( r = 2m \) at \( x^- = 0 \) and \( x^+ = 0 \), respectively. In the wormhole, there is a double trapping horizon, \( \partial_+ r = \partial_- r = 0 \), at the throat \( r = a \). This illustrates how trapping horizons of different types may be used to locally define both black holes and wormholes. Locating the trapping horizons is a key feature of the analysis of dynamic situations \cite{3}.

In order to find analytic solutions, including semiclassical corrections, one can modify the classical action. We use the model modified from the original CGHS model by Bose et al. \cite{6}. We take the classical action

\[ S_0 = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[ e^{-2\phi} \left( R + 4(\nabla \phi)^2 + 4\lambda^2 \right) - \frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2 + \frac{1}{2} (\nabla g)^2 \right] \]  

and a local covariant term of one-loop order

\[ S_{\text{corr}} = \frac{N\hbar}{24\pi} \int d^2 x \sqrt{-g} \left[ (\nabla \phi)^2 - \phi R \right]. \]  

Now, the total modified action, including the one-loop Polyakov–Liouville term, is

\[ S = S_0 + S_{\text{corr}} + NS_{\text{PL}}. \]
$S_{PL}$ is the Polyakov–Liouville action \cite{13}

$$S_{PL} = -\frac{h}{96\pi} \int d^2x \sqrt{-g(x)} \int d^2x' \sqrt{-g(x')} R(x) G(x, x') R(x'),$$  \hfill (12)

where $G(x, x')$ is a Green’s function for $\nabla^2$. Here, we take the large-$N$ limit in which $h$ goes to zero while $Nh$ is held fixed. In that limit, the quantum corrections for the gravitational, dilaton and ghost fields are negligible. If the null coordinates $x^\pm$ and the conformal gauge in Eq. (2) are used, the equations of motion derived from $S$ are the same as the classical ones:

$$\partial_+ \partial_- e^{-2\rho} = \partial_+ \partial_- e^{-2\phi} = -2\lambda^2,$$
$$\partial_+ \partial_- f_i = 0,$$
$$\partial_+ \partial_- g = 0,$$  \hfill (15)

while the constraints are modified by nonlocal terms arising from the Polyakov–Liouville action. In the conformal gauge, we can use the trace anomaly of $N$ massless scalar fields $f_i$ to obtain $(T^f_{\pm\pm} = -\kappa \partial_+ \partial_- \rho$, where $\kappa = Nh/12$, and we can integrate the equation $\nabla^2 (T^f_{\mu\nu}) = 0$ to get the quantum corrections to the energy–momentum tensor of the $f_i$ fields:

$$\{T^f_{\pm\pm}\} = \kappa \left[ \partial^2 \rho - (\partial_+ \rho)^2 + t \right],$$  \hfill (16)

where $t$ are integration functions of $x^+$ and $x^-$ determined by the boundary conditions. Also, the modified constraints $\delta S/\delta g_{\pm\pm} = 0$ become

$$-\partial^2 e^{-2\phi} - T^f_{\pm\pm} + T^g_{\pm\pm} + \kappa t = 0,$$  \hfill (17)

where $T^f_{\pm\pm} = (1/2) \sum_{i=1}^N \partial_+ f_i \partial_- f_i$ and $T^g_{\pm\pm} = (1/2) (\partial_\pm g)^2$ are the contributions to the energy–momentum tensors of the $f_i$ and the $g$ fields, respectively.

In Eq. (17) as

$$r = -4\lambda^2 \chi^+ \chi^- - \int dx_1^+ \int dx_1^- \left[ T^f_{++} + T^g_{++} - \kappa t \right] - \int dx_2^- \int dx_1^- \left[ T^f_{--} + T^g_{--} - \kappa t \right]
+ a_+ x^+ + a_- x^- + b,$$  \hfill (18)

where $a_\pm$ and $b$ are constants. When $T^g_{\mu\nu} = T^f_{\mu\nu} = 0$ and $t = a_\pm = b = 0$, the linear dilaton flat space–time solution is obtained.

The choice $T^f_{\mu\nu} = T^g_{\mu\nu} = 0$, $t = a_\pm = 0$ and $b = m/\lambda$ lead to the static black-hole solutions in Eq. (5). When we do not consider ghost scalar fields, we get a static black-hole solution including semiclassical corrections. In Eq. (16), $t$ can be determined by the radiationless static condition

$$0 = \{T^f_{\pm\pm}\} = \kappa \left[ \partial^2 \rho - (\partial_+ \rho)^2 + t \right].$$  \hfill (19)

Since $\rho = \rho(x^\pm)$ in black-hole case, one can get

$$t = \left\{ \begin{array}{ll}
t(x^+) = 1/(2\chi^+)^2, \\
t(x^-) = 1/(2\chi^-)^2,
\end{array} \right.$$  \hfill (20)

where $t(x^+)$ and $t(x^-)$ are applied in $x^+$ and $x^-$ integrations, respectively in Eq. (18). Thus the static black hole solution is corrected semiclassically to

$$r = -2\lambda^2 \chi^+ \chi^- - \frac{\kappa}{2} \ln |\lambda^2 \chi^+ \chi^- | + C,$$  \hfill (21)
Fig. 2. A step pulse of positive-energy radiation is beamed through the wormhole.

where \( C \) is a constant.

Next, if we consider a nonzero ghost scalar field \( g \), which provides the negative energy densities needed to maintain a wormhole \( (G_{\pm} = (\partial_{\pm} g)^2 = \pm 2\lambda) \), and take \( F_{\pm} = a_{\pm} = 0 \) and \( b = a \), then we can obtain static wormhole solution in Eq. (6). Similarly, we can also obtain the semiclassically corrected wormhole solution with the radiationless static condition in Eq. (19). Since \( \rho = \rho(x^+, x^-) \) in the wormhole case, and we can get

\[
t = -\frac{(a/2)\lambda^2}{[a/2 + \lambda^2(x^+ - x^-)^2]^2},
\]

and the static wormhole solution, including semiclassical corrections, is

\[
r = a + 2\lambda^2(x^+ - x^-)^2 - 2\kappa\sqrt{\frac{2\lambda^2}{a}(x^+ - x^-) \tan^{-1} \sqrt{\frac{2\lambda^2}{a}(x^+ - x^-)}}.
\]

The double trapping horizon, \( \partial_{\pm} r = \partial_{\pm} r = 0 \), is located at \( x^+ = x^- \) or \( r = a \).

If a wormhole is actually used to transport a parcel or a person between two universes, the transported matter will affect the wormhole by changing the gravitational field and, as expected, making outgoing radiation caused by the anomaly of the Polyakov–Liouville action Eq. (12). In a previous work [3], the dynamical effects of such a back-reaction were studied in the classical level by using the infalling field \( f \) to model the matter or radiation.

When one consider a step pulse of positive-energy radiation (Fig. 2):

\[
F_+ = \begin{cases} 0, & x^+ < x_0, \\ \Delta, & x_0 \leq x^+ < x_1, \\ 0, & x_1 \leq x^+, \end{cases} \quad F_- = 0
\]

with \( G_{\pm} = \pm 2\lambda \). The constraints are

\[
\partial_+ \partial_- r = \begin{cases} \frac{4\lambda^2}{4\lambda^2 - \Delta^2}, & x^+ < x_0, \\ \frac{4\lambda^2}{4\lambda^2 - \Delta^2}, & x_0 \leq x^+ < x_1, \\ \frac{4\lambda^2}{4\lambda^2}, & x_1 \leq x^+, \end{cases}
\]

and the resulting solution is

\[
r = \begin{cases} a + 2\lambda^2(x^+ - x^-)^2, & x^+ < x_0, \\ a - \Delta^2 x_0^2/2 + 2\lambda^2(x^+ - x^-)^2 - (\Delta^2/2)x^+ + \Delta^2x_0x^+, & x_0 \leq x^+ < x_1, \\ a + (\Delta^2/2)(x_1^2 - x_0^2) + 2\lambda^2(x^+ - x^-)^2 - \Delta^2(x_1 - x_0)x^+, & x_1 \leq x^+. \end{cases}
\]

The locations of the trapping horizons \( \partial_+ r = 0 \) and \( \partial_- r = 0 \) are given, respectively, by

\[
x^- = \begin{cases} x^+, & x^+ < x_0, \\ x^+ - (\Delta^2/4\lambda^2)x^+ + \Delta^2x_0/(4\lambda^2), & x_0 \leq x^+ < x_1, \\ x^+ - (\Delta^2/(4\lambda^2))(x_1 - x_0), & x_1 \leq x^+, \end{cases}
\]
and $x^+ = x^-$. Thus, the double trapping horizon of the initially static wormhole bifurcates when the radiation arrives (Fig. 3). After the pulse has passed, the two trapping horizons run parallel in the $x^\pm$ coordinates, forming a nonstatic traversible wormhole. In order to maintain the wormhole classically, additional negative-energy radiation may be beamed before or after the positive-energy pulse [3].

In the semiclassical model, one should use the modified constraints in Eq. (17) and the general solution in Eq. (18). The nonlocal term $r$ is determined to be the same function as in Eq. (22) by using the initially radiationless static condition. The solutions are corrected semiclassically to

$$r = \begin{cases} 
  a + 2\kappa^2(x^+ - x^-)^2 - 2\kappa\sqrt{2\lambda^2/a}(x^+ - x^-)\tan^{-1}\sqrt{2\lambda^2/a}(x^+ - x^-), & x^+ < x_0, \\
  a - \Delta^2 x_0^2/2 + 2\kappa^2(x^+ - x^-)^2 - \Delta^2 x_0^2/2 + \Delta^2 x_0 x^+ & x_0 \leq x^+ < x_1, \\
  a - 2\kappa\sqrt{2\lambda^2/a}(x^+ - x^-)\tan^{-1}\sqrt{2\lambda^2/a}(x^+ - x^-), & x_1 \leq x^+, \\
  a - \kappa^2(4\pi^2\lambda^2)/(\Delta^2 a) + 2\kappa^2(x^+ - x^-)^2 & x_0 \leq x^+ < x_1, \\
  a - 2\kappa\sqrt{2\lambda^2/a}(x^+ - x^-)\tan^{-1}\sqrt{2\lambda^2/a}(x^+ - x^-), & x_1 \leq x^+,
\end{cases}$$

where $x_1 = x_0 + \kappa(2\pi/\Delta^2)\sqrt{2\lambda^2/a}$, which is given in order to reform the final static, traversible wormhole when the pulse is switched off. The locations of the trapping horizons $\partial_+ r = 0$ and $\partial_- r = 0$ are also changed, respectively, to

$$x^- = x^+, \\
0 = 2\kappa^2(x^+ - x^-) - (\Delta^2/2)x^+ + \Delta^2 x_0^2/2 - \kappa\sqrt{2\lambda^2/a}\tan^{-1}\sqrt{2\lambda^2/a}(x^+ - x^-) - 2\kappa\lambda^2(x^+ - x^-)/(a + 2\lambda^2(x^+ - x^-)^2), \\
x^- = x^+,$$

and $x^+ = x^-$. Thus the double trapping horizon of the initially static wormhole bifurcates when the matter field $f$ arrives (Fig. 4). While the pulse is passing, the two trapping horizons merge again, forming a static, traversible wormhole finally. The throat of the final static wormhole is shifted to $r = a - \kappa^2(4\pi^2\lambda^2/(\Delta^2 a))$ after the pulse has passed.

Except for the effect of gravitational field, this result can be considered in the view of energy. When the double trapping horizon of the initially static wormhole bifurcates to the two trapping horizons of the nonstatic wormhole, the wormhole starts radiating. One can see this by calculating $\langle T_{\mu\nu}^f(x^\pm) \rangle$ at future null infinity ($x^+ \to \infty$). Using Eqs. (16) and (19) we get

$$\langle T_{\mu\nu}^f(x^\pm) \rangle_{L_+} = \frac{\kappa}{4(x^2 - x_0 x^- + x_0^2)}.$$
where we use the condition for ingoing \( f \) matter, \( \Delta^2 = 4\lambda^2 \), which makes nonzero value of radiation at \( I_+ \). This is positive to be caused by ordinary positive \( f \) matter. As the nonstatic wormhole evaporates by emitting radiation, the bifurcated trapping horizons get together. When the horizons merge again at \( x^+ = x^- = x_1 \), the nonstatic wormhole has radiated the bulk of the ingoing energy and the unradiated mass would generate change of throat radius.

We calculate the amount of energy \( E_{\text{rad}} \) radiated by the nonstatic wormhole by integrating Eq. (30) over \( I_+ \) up to the null curve \( x^- = x_1 \).

\[
E_{\text{rad}} = \int_{x_0}^{x_1} \left\langle T_{f}^{\ -\ -}(x^-) \right\rangle dx^- = \frac{\kappa}{2\sqrt{3x_0}} \left[ \arctan \frac{x_0 + \pi \kappa \sqrt{2/(a\lambda^2)}}{\sqrt{3x_0}} - \frac{\pi}{6} \right]. \tag{31}
\]

If the wormhole energy given by Eq. (8) is defined in null or spatial infinity, energy conservation would give \( \delta M = E_{\text{in}} - E_{\text{rad}} \), where \( \delta M \) is the mass difference between the final and the initial static wormholes, and \( E_{\text{in}} \) is the ingoing energy given by

\[
E_{\text{in}} = \int_{x_0}^{x_1} \left\{ T_{f}^{\ +\ +}(x^+) \right\}_{\text{cl}} dx^+ = \frac{\Delta^2}{2} (x_1 - x_0) = \pi \kappa \sqrt{\frac{2\lambda^2}{a}}. \tag{32}
\]

If one assume that the throat radius of the initial static wormhole was very large, \( \delta M \) can be ignored and one can get \( x_0^2 = \sqrt{3\kappa}/(8\lambda^2) \) from \( E_{\text{in}} \approx E_{\text{out}} \).

In this Letter, we have studied the semiclassical approach of a two-dimensional dilaton model including a ghost Klein–Gordon field with negative gravitational coupling. In this semiclassically corrected model, the blackhole and the wormhole solutions are obtained by using a radiationless static condition. When a static traversible wormhole is used to transport matter or radiation, we describe the dynamics of the wormhole both at the classical and the semiclassical level. The classical static wormhole exhibits a type of neutral stability, neither strictly stable in that it does not return to its initial state nor strictly unstable in that there is no sudden runaway. Maintenance of the classical wormhole requires additional negative energy to balance the transported matter. In semiclassical theory, however, the stability of the wormhole has been shown; i.e., the nonstatic wormhole constructed by ingoing \( f \) matter makes radiation through the quantum corrections of \( f \) matter fields, and finally returns to its original static state without the addition of any fields.
Acknowledgements

This work is supported in part by Grant No. R01-2000-000-00015-0 from the Korea Science and Engineering Foundation.

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