Rank-metric codes and their MacWilliams identities

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Abstract

We compare the two duality theories of rank-metric codes proposed by Delsarte and Gabidulin, proving that the former generalizes the latter. We also give an elementary proof of MacWilliams identities for the general case of Delsarte rank-metric codes, in a form that never appeared in the literature. The identities which we derive are very easy to handle, and allow us to re-establish in a very concise way the main results of the theory of rank-metric codes. We study how the minimum and maximum rank of a rank-metric code relate to the minimum and maximum rank of the dual code, giving some bounds and characterizing the codes attaining them. We also study optimal anticodes in the rank metric, describing them in terms of MRD codes. In particular, we prove that the dual of an optimal anticode is an optimal anticode. Finally, as an application of our results to a classical problem in enumerative combinatorics, we derive a recursive formula for the number of $k \times m$ matrices over a finite field with given rank and $h$-trace.

Introduction

In [6] Delsarte defines rank-metric codes as sets of matrices of given size over a finite field $\mathbb{F}_q$. The distance between two matrices is given by the rank of their difference. Interpreting matrices as bilinear forms, Delsarte studies rank-metric codes as association schemes, whose adjacency algebra yields the so-called MacWilliams transform of distance enumerators of codes. The results of [6] are based on the general theory of designs and codesigns in regular semilattices developed in [5].

In coding theory, a MacWilliams identity establishes a relation between metric properties of a code and metric properties of the dual code. More generally, in the sequel by “duality theory” we mean a series of results which relate a code to the dual code. MacWilliams identities exist for several types of codes and metrics. As Gluesing-Luerssen observed in [11], association schemes provide the most general approach to MacWilliams identities, and apply to both linear and non-linear codes (see [4], [3] and [7]). On the other side, the machinery of association schemes and of the related Bose-Mesner algebras is a very elaborated mathematical tool. Several authors proved independently the MacWilliams identities for the various types of codes in less sophisticated ways.

*The author was partially supported by the Swiss National Science Foundation through grant no. 200021_150207.
A different viewpoint on MacWilliams identities for general additive codes was recently proposed by Gluesing-Luerssen in [11]. The approach is based on character theory and partitions of groups. See also [12] for a character-theoretic approach to MacWilliams identities for the rank and the Hamming metric.

Both the theory of association schemes and the approach of [11] apply to Delsarte rank-metric codes, giving MacWilliams identities in the form presented in [6]. On the other side, to the extent of our knowledge, there is no elementary proof for MacWilliams identities for the rank metric.

A different definition of rank-metric code was proposed by Gabidulin in [9]. MacWilliams identities for Gabidulin codes were obtained in [10]. There exists a natural way to associate to a Gabidulin code a Delsarte code with the same rank-metric properties. Hence Gabidulin codes can be regarded as special cases of Delsarte codes. It is however not clear in general how the duality theories of these two families of codes relate to each other. This is one of the questions that we address in this work.

Both linear Delsarte and Gabidulin codes have interesting applications. Recently it was shown how to employ them for error correction in coherent linear network coding (see e.g. [19] and the references within). Rank-metric codes also play an important role in the construction of subspace codes to be used for random linear network coding (see [21]). Finally, rank-metric codes were also proposed to secure a network coding communication system against an eavesdropper in a universal way (see [20] for details).

Motivated by these applications, in this paper we focus on linear Delsarte and Gabidulin codes. We first compare the two families of codes, proving that Delsarte codes generalize Gabidulin codes also in terms of duality theory. Then we give a short proof of MacWilliams identities for the general case of Delsarte rank-metric codes. We only employ elementary properties of the rank metric and linear algebra techniques, avoiding any sort of numerical calculation. The identities which we derive have a very convenient form, which allows us to re-establish the most important results of the theory of rank-metric codes in a very concise way. In a second part we prove some bounds that relate the minimum and maximum rank of a code to the minimum and maximum rank of the dual code, characterizing the codes which attain them. The bounds show that also the maximum rank of a code (and not only the minimum rank) deserves interest. We also investigate anticodes in the rank metric, and present a new characterization of optimal anticodes in terms of optimal codes. Then we apply such characterization to show that the dual of an optimal code is an optimal code. This result may be regarded as the analogue for anticodes of the fact that the dual of an optimal code is an optimal code. Finally, as an application of our results to a classical problem in enumerative combinatorics, we give recursive formulas for the number of rectangular matrices with given rank and h-trace. To the extent of our knowledge, a formula of this type is not available in the literature.

The layout of the paper is as follows. Section 1 contains the basic definitions and results of the theory of Delsarte rank-metric codes. In Section 2 we compare Delsarte and Gabidulin codes. In Section 3 we give an elementary proof for MacWilliams identities for the general case of Delsarte rank-metric code, and use them to establish the main results of the duality theory of such codes. In Section 4 and in the last part of Section 5 we study how the minimum and the maximum rank of a rank-metric code relate to the minimum and maximum rank of the dual code, proving some bounds on the involved parameters and characterizing the codes which attain them. Optimal anticodes in the rank metric are studied in Section 5. In Section
Derive a recursive formula for the number of rectangular matrices over \( \mathbb{F}_q \) of given rank and \( h \)-trace.

## 1 Preliminaries

Throughout this paper, \( q \) denotes a fixed prime power, and \( \mathbb{F}_q \) the finite field with \( q \) elements. We also work with positive integers \( k \) and \( m \).

**Notation 1.** We denote by \( \text{Mat}(k \times m, \mathbb{F}_q) \) the \( \mathbb{F}_q \)-vector space of \( k \times m \) matrices with entries in \( \mathbb{F}_q \). Given a matrix \( M \in \text{Mat}(k \times m, \mathbb{F}_q) \), we write \( \text{Tr}(M) \) for the trace of \( M \), and \( M_i \) for the \( i \)-th column of \( M \), i.e., the vector \( (M_{1i}, M_{2i}, \ldots, M_{ki}) \in \mathbb{F}_q^k \). The transpose of \( M \) is \( M^t \), while \( \text{rk}(M) \) denotes the rank of \( M \). The vector space generated by the columns of a matrix \( M \in \text{Mat}(k \times m, \mathbb{F}_q) \) is \( \text{colsp}(M) \subseteq \mathbb{F}_q^k \).

Let us briefly recall the setup of \cite{[6]}.

**Definition 2.** The trace product of matrices \( M, N \in \text{Mat}(k \times m, \mathbb{F}_q) \) is denoted and defined by

\[
\langle M, N \rangle := \text{Tr}(MN^t).
\]

It is easy to check that the map \( \langle \cdot, \cdot \rangle : \text{Mat}(k \times m, \mathbb{F}_q) \times \text{Mat}(k \times m, \mathbb{F}_q) \rightarrow \mathbb{F}_q \) is a scalar product (i.e., symmetric, bilinear and non-degenerate).

**Definition 3.** A (Delsarte rank-metric) code of size \( k \times m \) over \( \mathbb{F}_q \) is defined to be a vector subspace \( C \subseteq \text{Mat}(k \times m, \mathbb{F}_q) \). The minimum rank of a non-zero code \( C \) is denoted and defined by \( \text{minrk}(C) := \min\{ \text{rk}(M) : M \in C, \ \text{rk}(M) > 0 \} \), while the maximum rank of any code \( C \) is denoted and defined by \( \text{maxrk}(C) := \max\{ \text{rk}(M) : M \in C \} \). The dual of \( C \) is \( C^\perp := \{ N \in \text{Mat}(k \times m, \mathbb{F}_q) : \langle M, N \rangle = 0 \text{ for all } M \in C \} \).

The following lemma summarizes some straightforward properties of duality.

**Lemma 4.** Let \( C, D \subseteq \text{Mat}(k \times m, \mathbb{F}_q) \) be codes. We have:

- \( (C^\perp)^\perp = C \),
- \( \dim_{\mathbb{F}_q}(C^\perp) = km - \dim_{\mathbb{F}_q}(C) \),
- \( (C \cap D)^\perp = C^\perp + D^\perp \),
- \( (C + D)^\perp = C^\perp \cap D^\perp \).

Recall that for \( n \in \mathbb{N}_{\geq 1} \) the standard inner product of vectors \( (x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{F}_q^n \) is defined by \( (x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) := \sum_{i=1}^{n} x_i y_i \). It is easy to see that the trace product \( \langle \cdot, \cdot \rangle \) on \( \text{Mat}(k \times m, \mathbb{F}_q) \) and the standard inner product on \( \mathbb{F}_q^k \) relate as follows.

**Lemma 5.** Let \( M, N \in \text{Mat}(k \times m, \mathbb{F}_q) \). We have \( \langle M, N \rangle = \sum_{i=1}^{n} M_i \cdot N_i \).

**Notation 6.** Lemma 4 says in particular that the trace product \( \langle \cdot, \cdot \rangle \) on \( \text{Mat}(k \times 1, \mathbb{F}_q) \cong \mathbb{F}_q^k \) coincides with the standard inner product on \( \mathbb{F}_q^k \). Hence from now on we denote both products by \( \langle \cdot, \cdot \rangle \). We also denote by \( U^\perp \) the dual of a vector subspace \( U \subseteq \mathbb{F}_q^k \), i.e., \( U^\perp := \{ x \in \mathbb{F}_q^k : \langle u, x \rangle = 0 \text{ for all } u \in U \} \).
The following result, first proved by Delsarte, is well-known.

**Theorem 7** ([9], Theorem 5.4). Let $\mathcal{C} \subseteq \text{Mat}(k \times m, \mathbb{F}_q)$ be a non-zero code, and let $d := \min \text{rk}(\mathcal{C})$. We have
\[
\dim_{\mathbb{F}_q}(\mathcal{C}) \leq \max\{k, m\}(\min\{k, m\} - d + 1).
\]
Moreover, for any $1 \leq d \leq \min\{k, m\}$ there exists a non-zero code $\mathcal{C} \subseteq \text{Mat}(k \times m, \mathbb{F}_q)$ of minimum rank $d$ which attains the upper bound.

**Definition 8.** A code attaining the bound of Theorem 7 is said to be a **maximum rank distance** code (MRD code in short). The zero code will be also considered MRD.

**Remark 9.** Notice that $\text{Mat}(k \times m, \mathbb{F}_q)$ is a trivial example of MRD code with minimum rank 1 and dimension $km$. See [9], Section 6, for the construction of codes attaining the bound of Theorem 7 for any choice of the parameters.

**Definition 10.** Given a code $\mathcal{C}$ and an integer $i \in \mathbb{N}_{\geq 0}$ define $A_i(\mathcal{C}) := |\{M \in \mathcal{C} : \text{rk}(M) = i\}|$. The collection $(A_i(\mathcal{C}))_{i \in \mathbb{N}_{\geq 0}}$ is said to be the **rank distribution** of $\mathcal{C}$.

**Remark 11.** The minimum rank of a non-zero code $\mathcal{C} \subseteq \text{Mat}(k \times m, \mathbb{F}_q)$ is the smallest $i > 0$ such that $A_i(\mathcal{C}) > 0$. Notice that we define $A_i(\mathcal{C})$ for any $i \in \mathbb{N}_{\geq 0}$, even if we clearly have $A_i(\mathcal{C}) = 0$ for all integers $i > \min\{k, m\}$. This choice will simplify the statements in the sequel.

## 2 Delsarte and Gabidulin rank-metric codes

A different definition of rank-metric codes, proposed by Gabidulin, is the following.

**Definition 12** (see [9]). Let $\mathbb{F}_{q^m}/\mathbb{F}_q$ be a finite field extension. A **Gabidulin (rank-metric)** code of length $k$ over $\mathbb{F}_{q^m}$ is an $\mathbb{F}_{q^m}$-vector subspace $\mathcal{C} \subseteq \mathbb{F}_{q^m}$. The rank of a vector $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{F}_{q^m}$ is defined as $\text{rk}(\alpha) := \dim_{\mathbb{F}_q} \text{Span}\{\alpha_1, ..., \alpha_k\}$. The minimum rank of a Gabidulin code $\mathcal{C} \neq 0$ is $\min \text{rk}(\mathcal{C}) := \min\{\text{rk}(\alpha) : \alpha \in \mathcal{C}, \ \alpha \neq 0\}$, and the **maximum rank** of any Gabidulin code $\mathcal{C}$ is $\max \text{rk}(\mathcal{C}) := \max\{\text{rk}(\alpha) : \alpha \in \mathcal{C}\}$. The **rank distribution** of $\mathcal{C}$ is the collection $(A_i(\mathcal{C}))_{i \in \mathbb{N}_{\geq 0}}$, where $A_i(\mathcal{C}) := |\{\alpha \in \mathcal{C} : \text{rk}(\alpha) = i\}|$. The dual of a Gabidulin code $\mathcal{C}$ is denoted and defined by $\mathcal{C}^\perp := \{\beta \in \mathbb{F}_{q^m} : \langle \alpha, \beta \rangle = 0 \text{ for all } \alpha \in \mathcal{C}\}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{F}_{q^m}$.

It is natural to ask how Gabidulin and Delsarte codes relate to each other.

**Definition 13.** Let $\mathcal{G} = \{\gamma_1, ..., \gamma_m\}$ be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. The matrix associated to a vector $\alpha \in \mathbb{F}_{q^m}$ with respect to $\mathcal{G}$ is the $k \times m$ matrix $M_{\mathcal{G}}(\alpha)$ with entries in $\mathbb{F}_q$ defined by $\alpha_i = \sum_{j=1}^{m} M_{\mathcal{G}}(\alpha)_{ij} \gamma_j$ for all $i = 1, ..., k$. The Delsarte code associated to a Gabidulin code $\mathcal{C} \subseteq \mathbb{F}_{q^m}$ with respect to the basis $\mathcal{G}$ is $\mathcal{C}_{\mathcal{G}}(\mathcal{C}) := \{M_{\mathcal{G}}(\alpha) : \alpha \in \mathcal{C}\} \subseteq \text{Mat}(k \times m, \mathbb{F}_q)$.

The following result is immediate.

**Proposition 14.** Let $\mathcal{C} \subseteq \mathbb{F}_{q^m}$ be a Gabidulin code. For any basis $\mathcal{G}$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, $\mathcal{C}_{\mathcal{G}}(\mathcal{C}) \subseteq \text{Mat}(k \times m, \mathbb{F}_q)$ is a Delsarte rank-metric code with
\[
\dim_{\mathbb{F}_q}(\mathcal{C}_{\mathcal{G}}(\mathcal{C})) = m \cdot \dim_{\mathbb{F}_{q^m}}(\mathcal{C}).
\]
Moreover, $\mathcal{C}_{\mathcal{G}}(\mathcal{C})$ has the same rank distribution as $\mathcal{C}$. In particular we have $\max \text{rk}(\mathcal{C}) = \max \text{rk}(\mathcal{C}_{\mathcal{G}}(\mathcal{C}))$, and if $\mathcal{C} \neq 0$ we have $\min \text{rk}(\mathcal{C}) = \min \text{rk}(\mathcal{C}_{\mathcal{G}}(\mathcal{C}))$.  

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Remark 15. Proposition \[14\] shows that any Gabidulin code can be regarded as a Delsarte rank-metric code with the same cardinality and rank distribution. Clearly, not all Delsarte rank-metric codes arise from a Gabidulin code in this way. In fact, only a few of them do. For example, a Delsarte code \( C \subseteq \text{Mat}(k \times m, \mathbb{F}_q) \) such that \( \text{dim}_{\mathbb{F}_q}(C) \not\equiv 0 \mod m \) cannot arise from a Gabidulin code.

In the remainder of the section we compare the duality theories of Gabidulin and Delsarte codes, proving in particular that the former generalizes the latter.

**Definition 16.** Let \( \text{Trace} : \mathbb{F}_q^m \to \mathbb{F}_q \) be the \( \mathbb{F}_q \)-linear trace map given by \( \text{Trace}(\alpha) := \alpha + \alpha^q + \cdots + \alpha^{q^{m-1}} \) for all \( \alpha \in \mathbb{F}_q^m \). Bases \( G = \{ \gamma_1, ..., \gamma_m \} \) and \( G' = \{ \gamma_1', ..., \gamma_m' \} \) of \( \mathbb{F}_q^m \) over \( \mathbb{F}_q \) are said to be **orthogonal** if \( \text{Trace}(\gamma_i' \gamma_j) = \delta_{ij} \) for all \( i, j \in \{1, ..., m\} \).

The following result on orthogonal bases is well-known.

**Proposition 17** ([17], page 54). For every basis \( G \) of \( \mathbb{F}_q^m \) over \( \mathbb{F}_q \) there exists a unique orthogonal basis \( G' \).

**Theorem 18.** Let \( C \subseteq \mathbb{F}_q^m \) be a Gabidulin code, and let \( G, G' \) be orthogonal bases of \( \mathbb{F}_q^m \) over \( \mathbb{F}_q \). We have

\[
C_{G'}(C^\perp) = C_G(C)^\perp.
\]

In particular, if we set \( C := C_G(C) \), then \( C \) has the same rank distribution as \( C \), and \( C^\perp \) has the same rank distribution as \( C^\perp \).

**Proof.** Let \( G = \{ \gamma_1, ..., \gamma_m \} \) and \( G' = \{ \gamma_1', ..., \gamma_m' \} \). Take any \( M \in C_{G'}(C^\perp) \) and \( N \in C_G(C) \). There exist \( \alpha \in C^\perp \) and \( \beta \in C \) such that \( M = M_{G'}(\alpha) \) and \( N = M_G(\beta) \). According to Definition \[13\] we have

\[
0 = \langle \alpha, \beta \rangle = \sum_{i=1}^{k} \alpha_i \beta_i = \sum_{i=1}^{k} \sum_{j=1}^{m} M_{ij} \gamma_j \sum_{t=1}^{m} N_{it} \gamma_t = \sum_{i=1}^{k} \sum_{j=1}^{m} \sum_{t=1}^{m} M_{ij} N_{it} \gamma_j' \gamma_t. \tag{1}
\]

Applying the function \( \text{Trace} : \mathbb{F}_q^m \to \mathbb{F}_q \) to both members of equation (1) we get

\[
0 = \text{Trace} \left( \sum_{i=1}^{k} \sum_{j=1}^{m} \sum_{t=1}^{m} M_{ij} N_{it} \gamma_j' \gamma_t \right)
= \sum_{i=1}^{k} \sum_{j=1}^{m} \sum_{t=1}^{m} M_{ij} N_{it} \text{Trace}(\gamma_j' \gamma_t)
= \sum_{i=1}^{k} \sum_{j=1}^{m} \sum_{t=1}^{m} M_{ij} N_{it} \delta_{jt}
= \sum_{i=1}^{k} \sum_{j=1}^{m} M_{ij} N_{ij}
= \text{Tr}(MN^t)
= \langle M, N \rangle.
\]

It follows \( C_{G'}(C^\perp) \subseteq C_G(C)^\perp \). By Proposition \[14\] and Lemma \[4\] \( C_{G'}(C^\perp) \) and \( C_G(C)^\perp \) have the same dimension over \( \mathbb{F}_q \). Hence the two codes are equal. The second part of the statement easily follows from Proposition \[14\].
Remark 19. Theorem 18 shows that the duality theory of Delsarte rank-metric codes can be regarded as a generalization of the duality theory of Gabidulin rank-metric codes. In particular, we notice that all the results on Delsarte codes which we will prove in the following sections also apply to Gabidulin codes.

Remark 20. Theorem 18 does not hold in general when the bases $G$ and $G'$ are not orthogonal. Let e.g. $q = 3$, $k = m = 2$ and $\mathbb{F}_3^2 = \mathbb{F}_3[\xi]$, where $\xi$ is a root of the irreducible primitive polynomial $x^2 + 1 \in \mathbb{F}_3[x]$. The two vectors $\alpha := (\xi, 2)$ and $\beta := (\xi, 1)$ satisfy $\langle \alpha, \beta \rangle = 1 \neq 0$. On the other side, if we take $G = G' := \{1, \xi\}$ as bases of $\mathbb{F}_3^2$ over $\mathbb{F}_3$ then we have

$$M_G(\alpha) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad M_G(\beta) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$ 

Hence $\langle M(\alpha), M_G(\beta) \rangle = \text{Tr}(M_G(\alpha)(M_G(\beta))^t) = 0$.

In the remainder of the paper we focus on the general case of Delsarte rank-metric codes.

3 MacWilliams identities for rank-metric codes

In this section we give an elementary proof of certain MacWilliams identities for Delsarte rank-metric codes. MacWilliams identities for such codes were also obtained in [6] by Delsarte himself using the machinery of association schemes. The formulas which we derive are different from those of [6]. We notice also that our proof is elementary and concise, and essentially uses linear algebra and a double counting argument.

Definition 21. Let $q$ be a prime power, and let $s$ and $t$ be integers. The $q$-binomial coefficient of $s$ and $t$ is denoted and defined by

$$\left[ \begin{array}{c} s \\ t \end{array} \right]_q = \left\{ \begin{array}{ll} 0 & \text{if } s < 0, \ t < 0, \text{ or } t > s, \\ 1 & \text{if } t = 0 \text{ and } s \geq 0, \\ \prod_{i=1}^t q^{s+i+1} - q^{s+i+1} & \text{otherwise}. \end{array} \right.$$

It is well-known that this number counts the number of $t$-dimensional $\mathbb{F}_q$-subspaces of an $s$-dimensional $\mathbb{F}_q$-space. In particular we have

$$\left[ \begin{array}{c} s \\ t \end{array} \right]_q = \left[ \begin{array}{c} s \\ s-t \end{array} \right]$$

for all integers $s, t$. Since in the paper we work with a fixed prime power $q$, we omit the subscript in the sequel.

Remark 22. Given any matrices $M, N \in \text{Mat}(k \times m, \mathbb{F}_q)$ we always have $\text{colsp}(M + N) \subseteq \text{colsp}(M) + \text{colsp}(N)$. As a consequence, if $U \subseteq \mathbb{F}_q^k$ is a vector subspace, then the set of matrices $M \in \text{Mat}(k \times m, \mathbb{F}_q)$ with $\text{colsp}(M) \subseteq U$ is a vector subspace of $\text{Mat}(k \times m, \mathbb{F}_q)$.

Notation 23. We denote the vector subspace of Remark 22 by $\text{Mat}_U(k \times m, \mathbb{F}_q)$.

We start with a series of preliminary results.
Lemma 24. Let $U \subseteq \mathbb{F}_q^k$ be a subspace. We have $\dim_{\mathbb{F}_q} \text{Mat}_U(k \times m, \mathbb{F}_q) = m \cdot \dim_{\mathbb{F}_q}(U)$.

**Proof.** Let $s := \dim_{\mathbb{F}_q}(U)$. Define the $s$-dimensional space $V := \{x \in \mathbb{F}_q^k : x_i = 0 \text{ for } i > s\} \subseteq \mathbb{F}_q^k$. There exists an $\mathbb{F}_q$-isomorphism $g : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^k$ that maps $U$ into $V$. Let $G \in \text{Mat}(k \times k, \mathbb{F}_q)$ be the invertible matrix associated to $g$ with respect to the canonical basis $\{e_1, \ldots, e_k\}$ of $\mathbb{F}_q^k$, i.e.,

$$g(e_j) = \sum_{i=1}^k G_{ij} e_i \quad \text{for all } j = 1, \ldots, k.$$ 

For any matrix $M \in \text{Mat}(k \times m, \mathbb{F}_q)$ we have $g(\text{colsp}(M)) = \text{colsp}(GM)$, and it is easy to check that the map $M \mapsto GM$ is an $\mathbb{F}_q$-isomorphism $\text{Mat}_U(k \times m, \mathbb{F}_q) \rightarrow \text{Mat}_V(k \times m, \mathbb{F}_q)$. Now we observe that $\text{Mat}_V(k \times m, \mathbb{F}_q)$ is the vector space of matrices $M \in \text{Mat}(k \times m, \mathbb{F}_q)$ whose last $k - s$ rows equal zero. Hence $\dim_{\mathbb{F}_q} \text{Mat}_V(k \times m, \mathbb{F}_q) = km - m(k-s) = ms$, and the lemma follows. \hfill \square

Lemma 25. Let $U \subseteq \mathbb{F}_q^k$ be a subspace. We have $\text{Mat}_U(k \times m, \mathbb{F}_q)^\perp = \text{Mat}_{U^\perp}(k \times m, \mathbb{F}_q)$.

**Proof.** Let $N \in \text{Mat}_{U^\perp}(k \times m, \mathbb{F}_q)$ and $M \in \text{Mat}_U(k \times m, \mathbb{F}_q)$. By definition, each column of $N$ belongs to $U^\perp$, and each column of $M$ belongs to $U$. Hence by Lemma 5 we have

$$\langle M, N \rangle = \sum_{i=1}^m \langle M_i, N_i \rangle = 0.$$ 

This proves $\text{Mat}_{U^\perp}(k \times m, \mathbb{F}_q) \subseteq \text{Mat}_U(k \times m, \mathbb{F}_q)^\perp$. By Lemma 24 the two spaces of matrices $\text{Mat}_{U^\perp}(k \times m, \mathbb{F}_q)$ and $\text{Mat}_U(k \times m, \mathbb{F}_q)^\perp$ have the same dimension over $\mathbb{F}_q$. Hence they are equal. \hfill \square

Lemma 26. Let $\mathcal{C} \subseteq \text{Mat}(k \times m, \mathbb{F}_q)$ be a code, and let $U \subseteq \mathbb{F}_q^k$ be a subspace. Denote by $s$ the dimension of $U$ over $\mathbb{F}_q$. We have

$$|\mathcal{C} \cap \text{Mat}_U(k \times m, \mathbb{F}_q)| = \frac{|\mathcal{C}|}{q^{m(k-s)}} |\mathcal{C}^\perp \cap \text{Mat}_{U^\perp}(k \times m, \mathbb{F}_q)|.$$ 

**Proof.** Combining Lemma 4 and Lemma 25 we obtain

$$(\mathcal{C} \cap \text{Mat}_U(k \times m, \mathbb{F}_q))^\perp = \mathcal{C}^\perp + \text{Mat}_U(k \times m, \mathbb{F}_q)^\perp = \mathcal{C}^\perp + \text{Mat}_{U^\perp}(k \times m, \mathbb{F}_q).$$

Hence by Lemma 4 we have

$$|\mathcal{C} \cap \text{Mat}_U(k \times m, \mathbb{F}_q)| \cdot |\mathcal{C}^\perp + \text{Mat}_{U^\perp}(k \times m, \mathbb{F}_q)| = q^{km}. \quad (2)$$

On the other hand, Lemma 24 gives

$$\dim_{\mathbb{F}_q}(\mathcal{C}^\perp + \text{Mat}_{U^\perp}(k \times m, \mathbb{F}_q)) = \dim_{\mathbb{F}_q}(\mathcal{C}^\perp) + m \cdot \dim_{\mathbb{F}_q} U^\perp - \dim_{\mathbb{F}_q}(\mathcal{C}^\perp \cap \text{Mat}_{U^\perp}(k \times m, \mathbb{F}_q)),$$

and so, again by Lemma 4

$$|\mathcal{C}^\perp + \text{Mat}_{U^\perp}(k \times m, \mathbb{F}_q)| = \frac{q^{km} \cdot q^{m(k-s)}}{|\mathcal{C}| \cdot |\mathcal{C}^\perp \cap \text{Mat}_{U^\perp}(k \times m, \mathbb{F}_q)|}. \quad (3)$$

Combining equation (2) and equation (3) one easily obtains the lemma. \hfill \square
The following result is well-known, but we include it for completeness.

**Lemma 27.** Let $0 \leq t, s \leq k$ be integers, and let $X \subseteq \mathbb{F}_q^k$ be a subspace of dimension $t$ over $\mathbb{F}_q$. The number of subspaces $U \subseteq \mathbb{F}_q^k$ such that $X \subseteq U$ and $\dim_{\mathbb{F}_q}(U) = s$ is

\[
\binom{k-t}{s-t}.
\]

**Proof.** Let $\pi : \mathbb{F}_q^k \to \mathbb{F}_q^k/X$ denote the projection on the quotient vector space $\mathbb{F}_q^k$ modulo $X$. It is easy to see that $\pi$ induces a bijection between the $s$-dimensional vector subspaces of $\mathbb{F}_q^k$ containing $X$ and the $(s-t)$-dimensional subspaces of $\mathbb{F}_q^k/X$. The lemma follows from the fact that $\mathbb{F}_q^k/X$ has dimension $k-t$. \hfill \Box

**Lemma 28.** Let $C \subseteq \text{Mat}(k \times m, \mathbb{F}_q)$ be a code. Denote by $(A_i)_i$ the rank distribution of $C$. Let $0 \leq s \leq k$ be an integer. We have

\[
\sum_{U \subseteq \mathbb{F}_q^k \atop \dim_{\mathbb{F}_q}(U) = s} |C \cap \text{Mat}_U(k \times m, \mathbb{F}_q)| = \sum_{i=0}^{k} A_i \binom{k-i}{k-s}.
\]

**Proof.** Define the set $\mathcal{A}(C, s) := \{(U, M) : U \subseteq \mathbb{F}_q^k, \dim(U) = s, M \in C, \colsp(M) \subseteq U\}$. We will count the elements of $\mathcal{A}(C, s)$ in two different ways. On the one hand, using Lemma 27, we have

\[
|\mathcal{A}(C, s)| = \sum_{M \in \mathcal{C}} |\{U \subseteq \mathbb{F}_q^k, \dim(U) = s, \colsp(M) \subseteq U\}|
\]

\[
= \sum_{i=0}^{k} \sum_{M \in \mathcal{C} \atop \text{rk}(M) = i} |\{U \subseteq \mathbb{F}_q^k, \dim(U) = s, \colsp(M) \subseteq U\}|
\]

\[
= \sum_{i=0}^{k} \sum_{M \in \mathcal{C} \atop \text{rk}(M) = i} \binom{k-i}{s-i} = \sum_{i=0}^{k} A_i \binom{k-i}{s-i} = \sum_{i=0}^{k} A_i \binom{k-i}{k-s}.
\]

On the other hand,

\[
|\mathcal{A}(C, s)| = \sum_{U \subseteq \mathbb{F}_q^k \atop \dim(U) = s} |\{M \in \mathcal{C} : \colsp(M) \subseteq U\}| = \sum_{U \subseteq \mathbb{F}_q^k \atop \dim(U) = s} |C \cap \text{Mat}_U(k \times m, \mathbb{F}_q)|,
\]

and the lemma follows. \hfill \Box

Now we state the MacWilliams identities.

**Theorem 29.** Let $C \subseteq \text{Mat}(k \times m, \mathbb{F}_q)$ be a code. Let $(A_i)_i$ and $(B_j)_j$ be the rank distributions of $C$ and $C^\perp$, respectively. For any integer $0 \leq \nu \leq k$ we have

\[
\sum_{i=0}^{k-\nu} A_i \binom{k-i}{\nu} = \frac{|C|}{q^{mn}} \sum_{j=0}^{\nu} B_j \binom{k-j}{\nu-j}.
\]
Proof. Lemma \([28]\) applied to \(C\) with \(s = k - \nu\) gives
\[
\sum_{\text{dim}_{F_q}(U) = k - \nu} |C \cap \text{Mat}_U(k \times m, F_q)| = \sum_{i=0}^{k} A_i \left[ k - i \atop \nu \right].
\]
The map \(U \mapsto U^\perp\) gives a bijection between the \(\nu\)-dimensional and the \((k - \nu)\)-dimensional subspaces of \(F_q^k\). Hence we have
\[
\sum_{U \subseteq F_q^k} |C^\perp \cap \text{Mat}_U(k \times m, F_q)| = \sum_{j=0}^{k} B_j \left[ k - j \atop k - \nu \right],
\]
where the second equality follows from Lemma \([28]\) applied to the code \(C^\perp\) with \(s = \nu\).Lemma \([26]\) with \(s = k - \nu\) gives
\[
\sum_{i=0}^{k} A_i \left[ k - i \atop \nu \right] = \frac{|C|}{q^{\nu m}} \sum_{j=0}^{k} B_j \left[ k - j \atop \nu - j \right].
\]
By definition, for \(i > k - \nu\) and for \(j > \nu\) we have
\[
\left[ k - i \atop \nu \right] = \left[ k - j \atop \nu - j \right] = 0,
\]
and the theorem follows. \(\square\)

**Remark 30.** MacWilliams identities in a form similar to that of Theorem \([29]\) were recently proved for Gabidulin codes (see \([10]\), Proposition 3). The proof of \([10]\) is based on the Hadamard transform, \(q\)-products, \(q\)-derivatives and \(q\)-transforms of polynomials. By Remark \([19]\) Theorem \([29]\) generalizes \([10]\), Proposition 3.

**Remark 31.** Theorem \([29]\) can be regarded as the \(q\)-analog of \([15]\), Lemma 2.2, which yields analogous identities for the Hamming metric.

Theorem \([29]\) gives in particular recursive formulas for the rank distribution of a dual code \(C^\perp\) in terms of the rank distribution of \(C\).

**Corollary 32.** Let \(C \subseteq \text{Mat}(k \times m, F_q)\) be a code. Let \((A_i)_i\) and \((B_j)_j\) be the rank distributions of \(C\) and \(C^\perp\), respectively. For \(\nu = 0, \ldots, k\) define
\[
a^k_\nu := q^{\nu m} \sum_{i=0}^{k-\nu} A_i \left[ k - i \atop \nu \right].
\]
The \(B_j\)'s are given by the recursive formula
\[
\begin{cases}
  B_0 = 1, \\
  B_\nu = a^k_\nu - \sum_{j=0}^{\nu-1} B_j \left[ k - j \atop \nu - j \right] & \text{for } \nu = 1, \ldots, k, \\
  B_\nu = 0 & \text{for } \nu > k.
\end{cases}
\]
Proof. Clearly, \( B_0 = 1 \) and \( B_\nu = 0 \) for \( \nu > k \). For any fixed integer \( \nu \in \{1, \ldots, k\} \) Theorem 29 gives

\[
a_\nu^k = \sum_{j=0}^{\nu-1} B_j \left[ \frac{k-j}{\nu-j} \right] + B_\nu,
\]

which proves the result. \( \square \)

Theorem 29 and Corollary 32 allow us to re-establish the main results of the duality theory of rank-metric codes in a very concise way.

**Corollary 33.** The rank distribution of a code \( \mathcal{C} \) determines the rank distribution of the dual code \( \mathcal{C}^\perp \).

**Proof.** This immediately follows from Corollary 32. \( \square \)

**Remark 34.** Corollary 33 was first proved by Delsarte using the theory of association schemes. See [6], Theorem 3.3 for details.

**Example 35.** Let \( q = 3 \), \( k = 3 \), \( m = 4 \). Consider the code \( \mathcal{C} \subseteq \text{Mat}(3 \times 4, F_3) \) generated by the following three matrices:

\[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 \\
1 & 1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 \\
1 & 1 & 1 & 1
\end{bmatrix}.
\]

It can be checked that \( \dim_{F_3} \mathcal{C} = 3 \) and that the rank distribution of \( \mathcal{C} \) is \( A_0 = 1 \), \( A_1 = 2 \), \( A_2 = 0 \), \( A_3 = 24 \). If \( (B_j)_j \) denotes the rank distribution of \( \mathcal{C}^\perp \), then the recursive formula of Corollary 32 allows us to compute:

\[
B_0 = 1, \quad B_1 = 50, \quad B_2 = 3432, \quad B_3 = 16200.
\]

Notice that \( \sum_{i=0}^3 B_i = 19683 = 3^9 = |\mathcal{C}^\perp| \), as expected.

**Remark 36.** For a code \( \mathcal{C} \subseteq \text{Mat}(k \times m, F_q) \) define \( \mathcal{C}^t := \{ M^t : M \in \mathcal{C} \} \subseteq \text{Mat}(m \times k, F_q) \). Clearly, \( \mathcal{C} \) and \( \mathcal{C}^t \) have the same dimension and rank distribution. Moreover, one can check that \( (\mathcal{C}^t)^\perp = (\mathcal{C}^\perp)^t \). As a consequence, up to a transposition, without loss of generality in the sequel we will always assume \( k \leq m \) in the proofs of our results.

**Corollary 37.** If a code \( \mathcal{C} \) is MRD, then \( \mathcal{C}^\perp \) is also MRD.

**Proof.** Let \( \mathcal{C} \subseteq \text{Mat}(k \times m, F_q) \) be MRD. If \( \mathcal{C} = 0 \) or \( \mathcal{C} = \text{Mat}(k \times m, F_q) \) the result follows from Definition 8 and Remark 9. Hence we assume \( 0 < \dim_{F_q}(\mathcal{C}) < km \). Assume \( k \leq m \) without loss of generality. Denote by \( d \) the minimum rank of \( \mathcal{C} \), so that \( |\mathcal{C}| = q^{m(k-d+1)} \). Let \( (A_i)_i \) and \( (B_j)_j \) be the rank distributions of \( \mathcal{C} \) and \( \mathcal{C}^\perp \), respectively. We have \( A_0 = B_0 = 1 \) and \( A_i = 0 \) for \( 1 \leq i \leq d - 1 \). Theorem 29 with \( \nu = k - d + 1 \) gives

\[
\begin{bmatrix}
k \\
k-d+1
\end{bmatrix} = \begin{bmatrix}
k \\
k-d+1
\end{bmatrix} + \sum_{j=1}^{k-d+1} B_j \begin{bmatrix}
k-j \\
k-d+1-j
\end{bmatrix},
\]

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i.e.,
\[\sum_{j=1}^{k-d+1} B_j \begin{bmatrix} k-j \\ k-d+1-j \end{bmatrix} = 0.\]

Since \(d \geq 1\), for \(1 \leq j \leq k-d+1\) we have \(k-j \geq k-d+1-j \geq 0\), and so \(\begin{bmatrix} k-j \\ k-d+1-j \end{bmatrix} > 0\).

Hence it must be \(B_j = 0\) for \(1 \leq j \leq k-d+1\), i.e., \(\minrk(C) \geq k-d+2\). On the other hand, Theorem 7 gives \(\dim F_q(C^\perp) = m(d-1) \leq m(k - \minrk(C^\perp) + 1)\), i.e., \(\minrk(C^\perp) \leq k-d+2\).

It follows \(\minrk(C^\perp) = k-d+2\), and so \(C^\perp\) is MRD. \(\square\)

Remark 38. Corollary 37 was first proved by Delsarte using the theory of designs and codeigns in regular semilattices (6, Theorem 5.5). Theorem 29 allows us to give a short proof for the same result. Notice also that, by Remark 19, Corollary 37 generalizes the analogous result for Gabidulin codes of 9.

4 Minimum and maximum rank of a code

In this section we investigate the minimum and the maximum rank of a Delsarte code \(C\), and show how they relate to the minimum and maximum rank of its dual code \(C^\perp\). As an application, we give a recursive formula for the rank distribution of an MRD code.

Proposition 39. Let \(C \subseteq \text{Mat}(k \times m, F_q)\) be a non-zero code. We have
\[
\minrk(C^\perp) \leq \min\{k, m\} - \minrk(C) + 2.
\]

Moreover, the bound is attained if and only if \(C\) is MRD.

Proof. Assume \(k \leq m\) without loss of generality. Theorem 7 applied to the code \(C\) gives \(\dim F_q(C) \leq m(k - \minrk(C) + 1)\). The same theorem applied to \(C^\perp\) gives \(\dim F_q(C^\perp) \leq m(k - \minrk(C^\perp) + 1)\), i.e., \(\dim F_q(C) \geq m(\minrk(C^\perp) - 1)\). Hence we have
\[
m(\minrk(C^\perp) - 1) \leq \dim F_q(C) \leq m(k - \minrk(C) + 1). \tag{4}
\]

In particular, \(\minrk(C^\perp) - 1 \leq k - \minrk(C) + 1\), and the bound follows. Let us prove the second part of the statement. Assume that \(C\) is MRD, and let \(d := \minrk(C)\). We have \(\dim F_q(C) = m(k-d+1)\), and so \(\dim F_q(C^\perp) = m(d-1)\). By Corollary 37, \(C^\perp\) is also MRD, and so \(m(d-1) = m(k - \minrk(C^\perp) + 1)\). It follows \(\minrk(C^\perp) = k-d+2\). On the other side, if \(\minrk(C^\perp) = k - \minrk(C) + 2\) then both the inequalities in (4) are in fact equalities, and so \(C\) is MRD. \(\square\)

Corollary 40. The rank distribution of a non-zero MRD code \(C \subseteq \text{Mat}(k \times m, F_q)\) only depends on \(k, m\) and \(\minrk(C)\).

Proof. Assume \(k \leq m\) without loss of generality. Let \(d := \minrk(C)\), and let \((A_i)_i\) denote the rank distribution of \(C\). By Proposition 39, \(C^\perp\) has minimum rank \(k-d+2\). Hence the equations of Theorem 29 for \(0 \leq \nu \leq k-d\) reduce to
\[
\begin{bmatrix} k \\ \nu \end{bmatrix} + \sum_{i=d}^{k-\nu} A_i \begin{bmatrix} k-i \\ \nu \end{bmatrix} = \frac{|C|}{q^\nu m^\nu} \begin{bmatrix} k \\ \nu \end{bmatrix}, \quad 0 \leq \nu \leq k-d.
\]
These identities give a linear system of \( k - d + 1 \) equations in the \( k - d + 1 \) unknowns \( A_d, ..., A_k \). It easy to see that the matrix associated to the system is triangular with all 1’s on the diagonal. In particular, the solution to the system is unique. Hence \( A_d, ..., A_k \) are uniquely determined by \( k, m \) and \( d \). Since \( A_0 = 1 \) and \( A_i = 0 \) for \( 0 < i < d \) and for \( i > k \), the thesis follows.

**Remark 41.** Corollary 40 was first proved by Delsarte by computing explicitly the rank distribution of an MRD code, and then observing that the obtained formulas only depend on the parameters \( m, k, d \) (see [6], Theorem 5.6). Corollary 40 allows us to give a concise proof for the same result. The rank distribution of Delsarte MRD codes was also computed in [8] employing elementary techniques.

**Remark 42.** Using the same argument as Corollary 32 it is easy to derive a recursive formula for the rank distribution \((A_i)_i\) of a non-zero MRD code \( C \subseteq \text{Mat}(k \times m, \mathbb{F}_q)\) of given minimum rank \( d \):

\[
\begin{aligned}
A_0 &= 1, \\
A_d &= (q^m - 1) \begin{bmatrix} k \\ k - d \end{bmatrix}, \\
A_{d+\varepsilon} &= (q^{m(1+\varepsilon)} - 1) \begin{bmatrix} k \\ k - d - \varepsilon \end{bmatrix} - \sum_{i=d}^{d+\varepsilon-1} A_i \begin{bmatrix} k - i \\ k - d - \varepsilon \end{bmatrix} \quad \text{for } 1 \leq \varepsilon \leq k - d.
\end{aligned}
\]

We do not go into the details of the proof.

The following result is the analogue of Theorem 7 for the maximum rank.

**Proposition 43.** Let \( C \subseteq \text{Mat}(k \times m, \mathbb{F}_q)\) be a code. We have

\[
\dim_{\mathbb{F}_q}(C) \leq \max\{k, m\} \cdot \maxrk(C).
\]

Moreover, for any choice of \( 0 \leq D \leq \min\{k, m\} \) there exists a code \( C \subseteq \text{Mat}(k \times m, \mathbb{F}_q)\) with maximum rank equal to \( D \) and attaining the upper bound.

**Proof.** Assume \( k \leq m \) without loss of generality. Fix \( 0 \leq D \leq k \). The set of all \( k \times m \) matrices having the last \( k - D \) rows equal to zero is an example of code of maximum rank \( D \) and dimension \( mD \) over \( \mathbb{F}_q \). Now we prove the first part of the statement. Let \( C \subseteq \text{Mat}(k \times m, \mathbb{F}_q)\) be a code with \( \maxrk(C) = D \). If \( D = k \) then the bound is trivial. Hence we assume \( D \leq k - 1 \). Theorem 7 gives a code \( D \subseteq \text{Mat}(k \times m, \mathbb{F}_q)\) with \( \minrk(D) = D + 1 \) and \( \dim_{\mathbb{F}_q}(D) = m(k - D) \). We clearly have \( C \cap D = 0 \) and \( C \oplus D \subseteq \text{Mat}(k \times m, \mathbb{F}_q)\). Hence \( \dim_{\mathbb{F}_q}(C) \leq km - \dim_{\mathbb{F}_q}(D) = mD \). □

**Definition 44.** A code \( C \subseteq \text{Mat}(k \times m, \mathbb{F}_q)\) which attains the upper bound of Proposition 43 is said to be a **(Delsarte) optimal anticode**.

We conclude the section with a result that relates the minimum rank of a code with the maximum rank of the dual code.

**Proposition 45.** Let \( C \subseteq \text{Mat}(k \times m, \mathbb{F}_q)\) be a non-zero code. We have

\[
\minrk(C) \leq \maxrk(C^\perp) + 1.
\]
Proof. Assume \( k \leq m \) without loss of generality. Applying Theorem 7 to \( \mathcal{C} \) we obtain \( \dim_{\mathbb{F}_q}(\mathcal{C}) \leq m(k - \minrk(\mathcal{C}) + 1) \), while Proposition 43 applied to \( \mathcal{C}^\perp \) gives \( \dim_{\mathbb{F}_q}(\mathcal{C}^\perp) \leq m \cdot \maxrk(\mathcal{C}^\perp) \), i.e., \( \dim_{\mathbb{F}_q}(\mathcal{C}) \geq m(k - \maxrk(\mathcal{C}^\perp)) \). Hence we have
\[
m(k - \maxrk(\mathcal{C}^\perp)) \leq \dim_{\mathbb{F}_q}(\mathcal{C}) \leq m(k - \minrk(\mathcal{C}) + 1),
\]
and the thesis follows. \( \square \)

5 Optimal anticodes

In this section we provide a new characterization of optimal anticodes in terms of their intersection with MRD codes. As an application of such description, we prove that the dual of an optimal anticode is an optimal anticode.

Let us first briefly recall some notions which we will need in the sequel. See [17], Section 3.4 for details.

Definition 46. Let \( \mathbb{F}_{q^m}/\mathbb{F}_q \) be a finite field extension. A linearized polynomial \( p \) over \( \mathbb{F}_{q^m} \) is a polynomial of the form
\[
p(x) = \alpha_0 x + \alpha_1 x^{q} + \alpha_2 x^{q^2} + \cdots + \alpha_s x^{q^s}, \quad \alpha_i \in \mathbb{F}_{q^m}, \quad i = 0, \ldots, s.
\]
The degree of \( p \), denoted by \( \deg(p) \), is the smallest \( i \geq 0 \) such that \( \alpha_i \neq 0 \).

Remark 47. It is well known ([17], Theorem 3.50) that the roots of a linearized polynomial \( p \) over \( \mathbb{F}_{q^m} \) form an \( \mathbb{F}_q \)-vector subspace of \( \mathbb{F}_{q^m} \), which we denote by \( V(p) \subseteq \mathbb{F}_{q^m} \). Notice that for any linearized polynomial \( p \) we have \( \dim_{\mathbb{F}_q}(V(p)) \leq \deg(p) \).

Lemma 48. Let \( \mathcal{C} \subseteq \text{Mat}(k \times m, \mathbb{F}_q) \) be a non-zero MRD code with minimum rank \( d \), and let \( (A_i)_i \) be the rank distribution of \( \mathcal{C} \). Then \( A_{d+i} > 0 \) for all \( 0 \leq i \leq \min\{k, m\} - d \).

Proof. Assume \( k \leq m \) without loss of generality. By Corollary 10 we shall prove the lemma for a given MRD code \( \mathcal{C} \subseteq \text{Mat}(k \times m, \mathbb{F}_q) \) of our choice with minimum rank \( d \). We first construct a convenient MRD code with the prescribed parameters, and we essentially follow the construction of [13], Section V.B.

Let \( \gamma_1, \ldots, \gamma_k \in \mathbb{F}_{q^m} \) be linearly independent over \( \mathbb{F}_q \). Denote by \( \mathcal{L}(\mathbb{F}_{q^m}, k - d) \) the \( \mathbb{F}_{q^m} \)-vector space of linearized polynomials over \( \mathbb{F}_{q^m} \) of degree smaller or equal than \( k - d \). We have \( \dim_{\mathbb{F}_{q^m}} \mathcal{L}(\mathbb{F}_{q^m}, k - d) = k - d + 1 \). Let \( ev : \mathcal{L}(\mathbb{F}_{q^m}, k - d) \to \mathbb{F}_{q^m}^k \) be the evaluation map defined by \( ev(p) := (p(\gamma_1), \ldots, p(\gamma_k)) \) for any \( p \in \mathcal{L}(\mathbb{F}_{q^m}, k - d) \). Then the image of \( ev \) is a Gabidulin code \( \mathcal{C} \subseteq \mathbb{F}_{q^m}^k \) with minimum rank \( d \) and dimension \( k - d + 1 \) over \( \mathbb{F}_{q^m} \) ([13], Theorem 14). Let \( \mathcal{G} \) be any basis of \( \mathbb{F}_{q^m} \) over \( \mathbb{F}_q \). By Proposition 14 \( \mathcal{C} := \mathcal{C}_q(\mathcal{C}) \subseteq \text{Mat}(k \times m, \mathbb{F}_q) \) is a Delsarte rank-metric code with \( \dim_{\mathbb{F}_q}(\mathcal{C}) = m(k - d + 1) \) and the same rank distribution as \( \mathcal{C} \). In particular, \( \mathcal{C} \) is a non-zero MRD code with minimum rank \( d \).

Now we prove the lemma for the MRD code \( \mathcal{C} \) that we constructed. Fix \( 0 \leq \epsilon \leq k - d \). Define \( t := k - d - \epsilon \), and let \( U \subseteq \mathbb{F}_{q^m} \) be the \( \mathbb{F}_q \)-subspace generated by \( \{\gamma_1, \ldots, \gamma_t\} \). If \( t = 0 \) we set \( U \) to be the zero space. By [17], Theorem 3.52,
\[
p_U := \prod_{\beta \in U} (x - \beta) \in \mathbb{F}_{q^m}[x]
\]
is a linearized polynomial over $\mathbb{F}_q^m$ of degree $t = k - d - \varepsilon \leq k - d$. Hence $p_U \in \mathcal{L}(\mathbb{F}_q^m, k - d)$. By Proposition 14 it suffices to prove that $ev(p_U) = (p_U(\gamma_1), ..., p_U(\gamma_k))$ has rank $d + \varepsilon = k - t$. Clearly, $V(p_U) = U$. In particular we have $ev(p_U) = (0, ..., 0, p_U(\gamma_{t+1}), ..., p_U(\gamma_k))$. We will prove that $p_U(\gamma_{t+1}), ..., p_U(\gamma_k)$ are linearly independent over $\mathbb{F}_q$. Assume that there exist $a_{t+1}, ..., a_k \in \mathbb{F}_q$ such that $\sum_{i=t+1}^{k} a_i p_U(\gamma_i) = 0$, (1). Then we have $p_U \left( \sum_{i=t+1}^{k} a_i \gamma_i \right) = 0$, i.e., $\sum_{i=t+1}^{k} a_i \gamma_i \in V(p_U) = U$. It follows that there exist $a_1, ..., a_t \in \mathbb{F}_q$ such that $\sum_{i=1}^{t} a_i \gamma_i = \sum_{i=t+1}^{k} a_i \gamma_i$, i.e., $\sum_{i=1}^{t} a_i \gamma_i - \sum_{i=t+1}^{k} a_i \gamma_i = 0$. Since $\gamma_1, ..., \gamma_k$ are linearly independent over $\mathbb{F}_q$, we have $a_i = 0$ for all $i = 1, ..., k$. Hence $p_U(\gamma_{t+1}), ..., p_U(\gamma_k)$ are linearly independent over $\mathbb{F}_q$, as claimed.

**Proposition 49.** Let $0 \leq D \leq \min\{k, m\} - 1$ be an integer, and let $C \subseteq Mat(k \times m, \mathbb{F}_q)$ be an $\mathbb{F}_q$-subspace with $\text{dim}_{\mathbb{F}_q}(C) = \max\{k, m\} \cdot D$. The following facts are equivalent.

1. $C$ is an optimal anticode.
2. $C \cap D = 0$ for all non-zero MRD codes $D \subseteq Mat(k \times m, \mathbb{F}_q)$ with $\text{minrk}(D) = D + 1$.

**Proof.** If $C$ is an optimal anticode, then by Definition 44 we have $D = \maxrk(C)$. Hence if $D$ is any non-zero code with $\text{minrk}(D) = D + 1$ we clearly have $C \cap D = 0$. So $(1) \Rightarrow (2)$ is trivial. Let us prove $(2) \Rightarrow (1)$. By contradiction, assume that $C$ is not an optimal anticode. Since $\maxrk(C) \geq D$ (see Proposition 43), we must have $s := \maxrk(C) \geq D + 1$. Let $N \in C$ with $\text{rk}(N) = s$. Let $D'$ be a non-zero MRD code with $\text{minrk}(D') = D + 1$ (see Theorem 7 for the existence of such a code). By Lemma 45 there exists $A \in D'$ with $\text{rk}(A) = s$. There exist invertible matrices $P$ and $Q$ of size $k \times k$ and $m \times m$ (respectively) such that $N = PAQ$. Define $D := PD'Q := \{PMQ : M \in D'\}$. Then $D \subseteq Mat(k \times m, \mathbb{F}_q)$ is a non-zero MRD code with $\text{minrk}(D) = D + 1$ and such that $N \in C \cap D$. Since $\text{rk}(N) = s \geq D + 1 \geq 1$, $N$ cannot be the zero matrix. This contradicts Proposition 49.

The following result may be regarded as the analogue of Corollary 37 for antcodes in the rank metric.

**Theorem 50.** If $C$ is an optimal anticode, then $C^\perp$ is also an optimal anticode.

**Proof.** Let $C \subseteq Mat(k \times m, \mathbb{F}_q)$ be an optimal anticode with $D := \maxrk(C)$. Assume $k \leq m$ without loss of generality. If $D = k$ then the result is trivial. Hence from now on we assume $0 \leq D \leq k - 1$. By Definition 14 we have $\text{dim}_{\mathbb{F}_q}(C) = mD$, and so $\text{dim}_{\mathbb{F}_q}(C^\perp) = \text{m}(k - D)$. By Proposition 39 it suffices to prove that $C^\perp \cap D = 0$ for all non-zero MRD codes $D \subseteq Mat(k \times m, \mathbb{F}_q)$ with $\text{minrk}(D) = k - D + 1$. If $D$ is such an MRD code, then we have $\text{dim}_{\mathbb{F}_q}(D) = m(k - (k - D + 1) + 1) = mD < mk$. Hence, by Proposition 39 $D^\perp$ is an MRD code with $\text{minrk}(D^\perp) = k - (k - D + 1) + 2 = D + 1$. Proposition 39 gives $C \cap D^\perp = 0$. Since $\text{dim}_{\mathbb{F}_q}(C) + \text{dim}_{\mathbb{F}_q}(D^\perp) = mD + m(k - (D + 1) + 1) = mk$, it follows $C \oplus D^\perp = Mat(k \times m, \mathbb{F}_q)$. Hence by Lemma 3 we have $C^\perp \cap D = 0$, as claimed.

The following result shows how the maximum rank of a code $C$ and the maximum rank of the dual code $C^\perp$ relate to each other.

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Proposition 51. Let $C \subseteq \text{Mat}(k \times m, \mathbb{F}_q)$ be a code. We have

$$\max \text{rk}(C) \geq \min\{k, m\} - \max \text{rk}(C^\perp).$$

Moreover, the bound is attained if and only if $C$ is an optimal anticode.

Proof. Assume $k \leq m$ without loss of generality. Proposition 43 applied to $C$ gives $\dim_{\mathbb{F}_q}(C^\perp) \leq m \cdot \max \text{rk}(C^\perp)$, i.e., $\dim_{\mathbb{F}_q}(C) \geq m(k - \max \text{rk}(C^\perp))$. The same proposition applied to $C$ gives $\dim_{\mathbb{F}_q}(C) \leq m \cdot \max \text{rk}(C)$. Hence we have

$$m(k - \max \text{rk}(C^\perp)) \leq \dim_{\mathbb{F}_q}(C) \leq m \cdot \max \text{rk}(C). \tag{5}$$

In particular, $k - \max \text{rk}(C^\perp) \leq m \cdot \max \text{rk}(C)$. Given the inequalities in (5), it is easy to see that the bound is attained if and only if both $C$ and $C^\perp$ are optimal anticodes. We conclude by Theorem 50. □

6 Matrices with given rank and $h$-trace

In this section we apply Corollary 32 to classical problems in enumerative combinatorics, deriving a recursive formula for the number of $k \times m$ matrices over $\mathbb{F}_q$ with prescribed rank and $h$-trace.

Definition 52. Let $M \in \text{Mat}(k \times m, \mathbb{F}_q)$, and let $1 \leq h \leq \min\{k, m\}$ be an integer. The $h$-trace of $M$ is defined by

$$\text{Tr}_h(M) := \sum_{i=1}^{h} M_{ii}.$$

Remark 53. Since for any matrix $M$ we have $\text{Tr}_h(M) = \text{Tr}_h(M^t)$, without loss of generality in the following we only treat the case $k \leq m$. Notice also that when $k = m$ we have $\text{Tr}_h(M) = \text{Tr}(M)$. Hence the $h$-trace generalizes the trace of a matrix.

Notation 54. Given integers $1 \leq k \leq m$, $0 \leq r \leq k$ and $1 \leq h \leq k$, we denote by $n_q(k \times m, r, h)$ the number of matrices $M \in \text{Mat}(k \times m, \mathbb{F}_q)$ such that $\text{rk}(M) = r$ and $\text{Tr}_h(M) = 0$. We also denote by $n_q(k \times m, r, 0)$ the number of matrices in $\text{Mat}(k \times m, \mathbb{F}_q)$ of rank $r$.

Lemma 55. Let $1 \leq k \leq m$ and $0 \leq r \leq k$ be integers. We have

$$n_q(k \times m, r, 0) = \left[\frac{m}{r}\right] \cdot \prod_{i=0}^{r-1}(q^k - q^i).$$

Sketch of proof. For a given vector subspace $U \subseteq \mathbb{F}_q^m$ with $\dim_{\mathbb{F}_q}(U) = r$, the number of matrices $M \in \text{Mat}(k \times m, \mathbb{F}_q)$ whose row space equals $U$ is precisely the number of full-rank $r \times k$ matrices, which is $\prod_{i=0}^{r-1}(q^k - q^i)$. The thesis follows from the fact that the number of subspaces $U \subseteq \mathbb{F}_q^m$ with $\dim_{\mathbb{F}_q}(U) = r$ is $\left[\frac{m}{r}\right]$. □
Remark 56. We notice that if one has the number of matrices in $\text{Mat}(k \times m, F_q)$ of rank $r$ and zero $h$-trace, then he also has the number of matrices in $\text{Mat}(k \times m, F_q)$ of rank $r$ and $h$-trace equal to $\alpha$, for any $\alpha \in F_q$. Since the number of $k \times m$ matrices over $F_q$ of rank $r$ is given by Lemma 55, this fact is trivial when $q = 2$. On the other side, if $q > 2$ and $\alpha \neq \beta$ are non-zero elements of $F_q$, then the map $\text{Mat}(k \times m, F_q) \to \text{Mat}(k \times m, F_q)$ defined by $M \mapsto \alpha^{-1}M$ gives a bijection between the rank $r$ matrices with $h$-trace equal to $\alpha$ and the rank $r$ matrices with $h$-trace equal to $\beta$. It follows that for any $\alpha \in F_q \setminus \{0\}$ the number of matrices in $\text{Mat}(k \times m, F_q)$ with rank $r$ and $h$-trace equal to $\alpha$ is

$$
n_q(k \times m, r, 0) - n_q(k \times m, r, h) = \frac{q - 1}{q - 1},$$

where $n_q(k \times m, r, 0)$ is explicitly given by Lemma 55.

Remark 57. The usual way of computing $n_q(k \times k, k, k)$ involves the Bruhat decomposition of $\text{GL}_k(F_q)$ and the theory of $q$-analogues (see [22], Proposition 1.10.15). A different approach proposed in [14] is based on Gauss sums over finite fields and properties of the Borel subgroup of $\text{GL}_k(F_q)$. In [2] Buckheister derived a recursive description for $n_q(k \times k, r, k)$ using an elementary argument, and in [1] Bender applied the results of [2] to provide a closed formula for $n_q(k \times k, r, k)$. As Stanley observed ([22], page 100), the description of [2] is quite complicated. Delsarte computed the value of $n_q(k \times m, r, h)$ for any choice of the parameters using the theory of association schemes (see [6], page 235 and the Appendix). The following Theorem 58 provides a new recursive formula for the numbers $n_q(k \times m, r, h)$ which easily follows from Corollary 32.

**Theorem 58.** Let $1 \leq k \leq m$ and $1 \leq h \leq k$ be integers. For all $0 \leq r \leq k$ the numbers $n_q(r, h) := n_q(k \times m, r, h)$ are recursively computed by the following formulas.

$$n_q(r, h) = \begin{cases} 1 & \text{if } r = 0, \\ q^{mr-1} \left( \binom{k}{r} + (q - 1) \binom{k - h}{r} \right) - \sum_{j=0}^{r-1} n_q(j, h) \binom{k - j}{r - j} & \text{if } 1 \leq r \leq k - h, \\ q^{mr-1} \binom{k}{r} - \sum_{j=0}^{r-1} n_q(j, h) \binom{k - j}{r - j} & \text{if } k - h + 1 \leq r \leq k. \end{cases}$$

**Proof.** We fix $1 \leq h \leq k$. Let $M \in \text{Mat}(k \times m, F_q)$ be the matrix defined by

$$M_{ij} := \begin{cases} 1 & \text{if } i = j \leq h, \\ 0 & \text{otherwise.} \end{cases}$$

Let $C := \langle M \rangle \subseteq \text{Mat}(k \times m, F_q)$ be the Delsarte code generated by $M$ over $F_q$. It is easy to check that for any matrix $N \in \text{Mat}(k \times m, F_q)$ we have $\text{Tr}_h(N) = \text{Tr}(MN^t) = \langle M, N \rangle$. As a consequence, the set of matrices in $\text{Mat}(k \times m, F_q)$ with zero $h$-trace is precisely $C^\perp$. Hence, denoted by $(B_j)_j$ the rank distribution of $C^\perp$, we have $n_q(r, h) = B_r$ for all $0 \leq r \leq k$. If $(A_i)_i$ is the rank distribution of $C$, then we clearly have $A_0 = 1$, $A_h = q - 1$, and $A_i = 0$ for $i \notin \{0, h\}$. The theorem now follows from Corollary 32.

**Example 59.** Let $q = 4$, $k = 3$, $m = 4$. Theorem 32 allows us to compute all the values of $n_4(3 \times 4, r, h)$ as in Table 11.
| $h$ | $r = 0$ | $r = 1$ | $r = 2$ | $r = 3$ |
|-----|--------|--------|--------|--------|
| 1   | 1      | 2283   | 381780 | 3810240|
| 2   | 1      | 1515   | 336468 | 3856320|
| 3   | 1      | 132    | 337428 | 3855552|

Table 1: Values of $n_4(3 \times 4, r, h)$.

Acknowledgement

The author is grateful to Elisa Gorla for many useful suggestions that improved the presentation of the paper.

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