Moments of Products of Elliptic Integrals

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Recall that $pF_q$ denotes the generalised hypergeometric series,

$$pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$
Recall that $pF_q$ denotes the generalised hypergeometric series,

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They provide a framework for much of binomial sums, special functions, etc.

$2F_1$’s enjoy many transformations.
**Definition**

The **complete elliptic integral of the first kind** is given by

\[
K(x) := \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x^2 \sin^2 t}}
\]

\[
= \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - x^2 t^2)}}
\]

\[
= \frac{\pi}{2} \, \text{hypergeometric}_1 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \\ x^2 \end{array} \right).
\]

So \( K(1/\sqrt{2}) = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}} \).
The complete elliptic integral of the second kind is given by

\[ E(x) := \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2 t} \, dt \]

\[ = \int_0^1 \frac{\sqrt{1 - x^2 t^2}}{\sqrt{1 - t^2}} \, dt \]

\[ = \frac{\pi}{2} \, \text{F}_1 \left( -\frac{1}{2}, \frac{1}{2} \mid x^2 \right). \]

We let \( x' := \sqrt{1 - x^2} \), and \( K'(x) := K(x') \), \( E'(x) := E(x') \).
Basic properties

$K$ and $E$ are entangled by

$$\frac{dE}{dx} = \frac{E - K}{x},$$  
$$\frac{dK}{dx} = \frac{E - (1 - x^2)K}{x(1 - x^2)},$$  
$$\frac{\pi}{2} = EK' + E'K - KK' \quad \text{(Legendre).}$$
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\begin{align*}
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\frac{dK}{dx} &= \frac{E - (1 - x^2)K}{x(1 - x^2)}, \\
\frac{\pi}{2} &= E K' + E' K - K K' \quad \text{(Legendre)}.
\end{align*}
\]

There is a third one, \( \Pi \), but it is expressible as integrals of the first and second kinds.
Why do we care?

- Reason 1:

  Applications of complete elliptic integrals include geometry, physics, mechanics, electrodynamics, statistical mechanics, astronomy, geodesy, geodesics on conics, and magnetic field calculations. – Wolfram Functions
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- Reason 2: Studied by Wallis, Landen, Fagnano; Euler, Lagrange, Legendre; Gauss, Jacobi.
  “A glance at the average history of mathematics shows that mathematicians are remarkably incompetent historians.” – T. W. Körner.
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  *Applications of complete elliptic integrals include geometry, physics, mechanics, electrodynamics, statistical mechanics, astronomy, geodesy, geodesics on conics, and magnetic field calculations.* — *Wolfram Functions*

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- Perimeter of lemniscate \((r^2 = 2 \cos 2t)\) and *ellipse.*
They help us evaluate integrals

The area of an ellipse \( x^2 + \frac{y^2}{b^2} = 1 \) is very easy to find, but the perimeter is non-elementary.

\[
P = \int_0^1 \sqrt{1 + (y')^2} \, dx = \int_0^1 \sqrt{\frac{1 - (1 - b^2)x^2}{1 - x^2}} \, dx = E'(b).
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\]

The period of an ideal pendulum in a frictionless medium attached to a massless, inelastic string.

\[
T'' + \frac{g}{L} \sin T = 0, \quad T \gg 0.
\]

Period is \( 4\sqrt{\frac{L}{g}} K(\sin(a/2)) \). When \( a \to 0 \), we recover \( 2\pi \sqrt{L/g} \).

Note \( g \approx \pi^2 \), and also when \( a = \pi \).
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Used to integrate \(R(t, \sqrt{P(t)})\), where \(R\) is rational and \(P\) is cubic or quartic.
Digression on names

“The beginning of wisdom is to call things by their right names.” – Confucius

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But physicists also give a lot of “things” names...
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But physicists also give a lot of “things” names...

“Many of us still feel more comfortable with a function if we have an explicit formula to look at... Some primitive people believe that if you know a man’s name, then you have power over him. It is the same principle.” – Underwood Dudley
$K$ and $E$ have extremely important and nice properties of their own, and are very easy to compute.

$$K(x') = \frac{\pi}{2\text{AGM}(1, x)}. \tag{1}$$
Nice connections

\( K \) and \( E \) have extremely important and nice properties of their own, and are very easy to compute.

\[
K(x') = \frac{\pi}{2 \text{AGM}(1, x)}. \tag{1}
\]

\[
K(k) = \frac{\pi}{2} \frac{\theta_2^2(q)}{\theta_3(q)^2}, \quad \text{where } k = \frac{\theta_2(q)^2}{\theta_3(q)^2}.
\]

\[
\frac{K(k')}{K(k)} = -\frac{\log q}{\pi}.
\]

Ties in with functional equations of \( \theta \); singular values, modularity and \( \theta \) evaluations.

\[
\theta_2(q) = \sum_n q^{(n+1/2)^2}, \quad \theta_3(q) = \sum_n q^{n^2}.
\]

Elliptic integrals are also inverses of elliptic functions.
From (1),

\[
\frac{\pi}{2} \frac{1}{K'(x)} = M(1, x) = M\left(\frac{1 + x}{2}, \sqrt{x}\right) = \frac{1 + x}{2} M\left(1, \frac{2\sqrt{x}}{1 + x}\right)
\]

\[
= \frac{\pi}{2} \frac{1 + x}{2} \frac{1}{K((1 - x)/(1 + x))}.
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\]
\[
= \frac{\pi}{2} \frac{1 + x}{K((1 - x)/(1 + x))}.
\]

\[
K'(x) = \frac{2}{1 + x} K \left( \frac{1 - x}{1 + x} \right), \tag{2}
\]
\[
K(x) = \frac{1}{1 + x} K \left( \frac{2\sqrt{x}}{1 + x} \right). \tag{3}
\]

These useful formulae equivalent to $\cos^2 t K(\sin 2t) = K(\tan^2 t)$. 

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\[
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\]

These useful formulae equivalent to \( \cos^2 t K(\sin 2t) = K(\tan^2 t) \).

(1) gives a fast way to calculate log. Thus, via Newton’s method, all elementary functions can be computed efficiently via \( K \).
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1 Introduction

2 Random Walks

3 Moments
   1 Integral
   2 Integrals
   2 Complementary Integrals

4 More results
   3 Integrals
   Integration by Parts
   Open Questions
Connections to random walks

Let $W_n(s)$ be the $s$th moment of the distance from the origin of an $n$ step uniform random walk on the plane.
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Our research showed that

$$W_4(1) = \frac{16}{\pi^3} \int_0^1 (1 - 3x^2)K'(x)^2 \, dx,$$

$$W_4(-1) = \frac{4}{\pi^3} \int_0^1 K'(x)^2 \, dx.$$

Along the way, we found

$$2 \int_0^1 K(x)^2 \, dx = \int_0^1 K'(x)^2 \, dx.$$
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\]

Along the way, we found

\[
2 \int_0^1 K(x)^2 \, dx = \int_0^1 K'(x)^2 \, dx.
\]

Now easily proven: set \( x = (1 - t)/(1 + t) \) on the left, then use (2).
Organised search

David Bailey performed the following large scale search:

1. Given \((d_1, d_2)\), compute integrals on \((0, 1)\) to 1500 digits and store in a list, using all integrands of the form

\[ x^{i_0} K^{i_1} K'^{i_2} E^{i_3} E'^{i_4}, \]

where \(0 \leq i_0 \leq d_2\) and \(i_1 + i_2 + i_3 + i_4 = d_1\).

2. Use integer relation program PSLQ on sets of \(x = 40\) integrals, search for linear relations. When one is found, replace a linearly dependent integral by another in the list, repeat. Reduce \(x\) and keep finding relations. This produces a basis.

3. Write every integral in terms of the basis.
Example output

For \((d_1, d_2) = (2, 0)\):

\[
\begin{array}{cccccc}
K^2 & KE & E^2 & KK' & EK' & KE' \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & K^2 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & KE \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & E^2 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & KK' \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & EK' \\
-2 & 0 & 0 & 0 & 0 & 0 & 1 & K'^2 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & KE' \\
0 & 0 & 0 & -1 & 2 & 1 & -3 & EE' \\
2 & -4 & 3 & 0 & 0 & 0 & -1 & K'E' \\
6 & -16 & 12 & 0 & 0 & 0 & -3 & E'^2 \\
\end{array}
\]
For \((d_1, d_2) = (2, 1)\):

| \(K^2\) | \(xK^2\) | \(KE\) | \(xKE\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) | \(K^2\) | \(xK^2\) | \(KE\) | \(xKE\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | \(K^2\) | \(xK^2\) | \(KE\) | \(xKE\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | \(K^2\) | \(xK^2\) | \(KE\) | \(xKE\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | \(K^2\) | \(xK^2\) | \(KE\) | \(xKE\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | -1 | \(K^2\) | \(xK^2\) | \(KE\) | \(xKE\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | -2 | 2 | 4 | 0 | 0 | 0 | 0 | -3 | \(E^2\) | \(xE^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | 1 | 0 | -3 | 0 | 0 | 0 | 0 | 2 | \(E^2\) | \(xE^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | -1 | \(K^2\) | \(xK^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | -1 | \(K^2\) | \(xK^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | -1 | \(E^2\) | \(xE^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | \(K^2\) | \(xK^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | \(K^2\) | \(xK^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | \(xK^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) | \(K^2\) |
| 0 | 0 | 0 | 0 | -1 | 2 | 1 | -4 | 1 | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) | \(E^2\) | \(xE^2\) |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) | \(E^2\) | \(xE^2\) |
| 0 | 0 | 0 | 0 | 0 | -2 | 1 | 4 | -3 | \(E^2\) | \(xE^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | -4 | 4 | \(E^2\) | \(xE^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| -2 | 2 | 2 | -4 | 0 | 0 | 0 | 0 | 1 | \(K^2\) | \(xK^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | -1 | \(K^2\) | \(xK^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| -6 | 8 | 8 | -16 | 0 | 0 | 0 | 0 | 3 | \(E^2\) | \(xE^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
| 0 | 1 | 0 | -3 | 0 | 0 | 0 | 0 | 2 | \(E^2\) | \(xE^2\) | \(KE'\) | \(xKE'\) | \(KK'\) | \(xKK'\) | \(EK'\) | \(xEK'\) |
For \((d_1, d_2) = (3, 0)\):

|   | \(K^3\) | \(KE^2\) | \(E^3\) | \(KEK'\) | \(E^2K'\) | \(EK'^2\) |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| -5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -10 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | -18 | 0 | 0 |
| -115 | 192 | -96 | 72 | 36 | 0 | 0 |
| 115 | -192 | 96 | 0 | -144 | 0 | 144 |
| 0 | 0 | 0 | 0 | 0 | 0 | -2 |
| -10 | 96 | -48 | 0 | 0 | -27 | 36 |
| 10 | 0 | 0 | 0 | 0 | 0 | -9 |
| 20 | -48 | 24 | 0 | 0 | -9 | 18 |
| 10 | -60 | 30 | 0 | 0 | 9 | -18 |
| 25 | -64 | 32 | 0 | 0 | 0 | -12 |
| -185 | 576 | -288 | 0 | 0 | 0 | 48 |
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Methodology

Elementary techniques:

1. Interchange order of summation and integration.
2. Change the variable $x$ to $x'$.
3. Change the variable followed by quadratic transformations.
4. Use a Fourier series.
5. Apply Legendre’s relation.
6. Differentiate then integrate by parts.
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With experimental tools:

PSLQ, ISC, OEIS, gfun, Gosper’s algorithm, Sister Celine’s method, Wolfram Functions...
Classical results

The moments of $K, E$ are expressible in terms of Catalan’s constant $G$; the moments of $K', E'$ are expressible in terms of $\pi^2$. 
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\[
\int_0^1 x^{u-1}(1 - x)^{v-1} \binom{a, 1 - a}{b} \, dx
\]

\[
= \sum_{n=0}^{\infty} \frac{(a)_n (1 - a)_n}{(b)_n \, n!} \Gamma(u) \Gamma(v) \Gamma(u + v) \binom{a, 1 - a, u}{b, u + v, 1}.
\]
Hence,

\[
\int_0^1 x^m x^n K(x) \, dx = \frac{\pi}{4} \frac{\Gamma\left(\frac{1}{2}(m + 1)\right) \Gamma\left(\frac{1}{2}(n + 2)\right)}{\Gamma\left(\frac{1}{2}(m + n + 3)\right)} \, 3F_2 \left(\frac{1}{2}, \frac{1}{2}, \frac{m+1}{2} \left| \frac{1}{2}(m + n + 3) \right| 1\right),
\]

\[
\int_0^1 x^m x^n E(x) \, dx = \frac{\pi}{4} \frac{\Gamma\left(\frac{1}{2}(m + 1)\right) \Gamma\left(\frac{1}{2}(n + 2)\right)}{\Gamma\left(\frac{1}{2}(m + n + 3)\right)} \, 3F_2 \left(-\frac{1}{2}, \frac{1}{2}, \frac{m+1}{2} \left| \frac{1}{2}(m + n + 3) \right| 1\right).
\]
Closed forms

Hence,

\[
\int_0^1 x^m x^m K(x) \, dx = \frac{\pi}{4} \frac{\Gamma\left(\frac{1}{2}(m + 1)\right)\Gamma\left(\frac{1}{2}(n + 2)\right)}{\Gamma\left(\frac{1}{2}(m + n + 3)\right)} \, 3F_2 \left(\frac{1}{2}, \frac{1}{2}, \frac{m+1}{2} \left| \frac{1}{1}, \frac{m+n+3}{2} \right| 1\right),
\]

\[
\int_0^1 x^m x^m E(x) \, dx = \frac{\pi}{4} \frac{\Gamma\left(\frac{1}{2}(m + 1)\right)\Gamma\left(\frac{1}{2}(n + 2)\right)}{\Gamma\left(\frac{1}{2}(m + n + 3)\right)} \, 3F_2 \left(-\frac{1}{2}, \frac{1}{2}, \frac{m+1}{2} \left| \frac{1}{1}, \frac{m+n+3}{2} \right| 1\right).
\]

By \( x \mapsto x' \), we get moments of \( K' \) and \( E' \), which are in fact \( 2F_1 \)'s and can be summed by Gauss’ theorem.

**Dixon’s theorem** applies to special \( 3F_2 \)'s, e.g.

\[
\int_0^1 x' E(x) \, dx = \frac{1}{48\pi} \Gamma(1/4)^4.
\]
Interchange order

To recap:

\[
\begin{align*}
\int_0^1 x^n K'(x) \, dx &= \frac{\pi \Gamma\left(\frac{1}{2}(n+1)\right)^2}{4 \Gamma\left(\frac{1}{2}(n+2)\right)^2}, \\
\int_0^1 x^n E'(x) \, dx &= \frac{\pi \Gamma\left(\frac{1}{2}(n+3)\right)^2}{2(n+1) \Gamma\left(\frac{1}{2}(n+2)\right) \Gamma\left(\frac{1}{2}(n+4)\right)}.
\end{align*}
\]
Interchange order

To recap:

\[
\int_0^1 x^n K'(x) \, dx = \frac{\pi \Gamma(\frac{1}{2}(n + 1))^2}{4 \Gamma(\frac{1}{2}(n + 2))^2},
\]

\[
\int_0^1 x^n E'(x) \, dx = \frac{\pi \Gamma(\frac{1}{2}(n + 3))^2}{2(n + 1) \Gamma(\frac{1}{2}(n + 2))\Gamma(\frac{1}{2}(n + 4))}.
\]

\[
K(x) = \sum_{k=0}^{\infty} \frac{\Gamma(k + 1/2)^2}{\Gamma(k + 1)^2} \frac{x^{2k}}{2},
\]

\[
E(x) = \sum_{k=0}^{\infty} -\frac{\Gamma(k - 1/2)\Gamma(k + 1/2)}{\Gamma(k + 1)^2} \frac{x^{2k}}{4}.
\]

So we can find \( \int_0^1 x^n K(x)K'(x) \, dx \) etc.
Closed forms

For instance,

\[
\int_0^1 x^n K(x) K'(x) \,dx = \frac{\pi^2}{8} \frac{\Gamma(\frac{1}{2}(n + 1))^2}{\Gamma(\frac{1}{2}(n + 2))^2} 4F3 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{n+1}{2}, \frac{n+1}{2} \\ 1, \frac{n+2}{2}, \frac{n+2}{2} \end{array} \bigg| 1 \right).
\]
For instance,

\[
\int_0^1 x^n K(x) K'(x) \, dx = \frac{\pi^2}{8} \frac{\Gamma\left(\frac{1}{2} (n + 1)\right)^2}{\Gamma\left(\frac{1}{2} (n + 2)\right)^2} \binom{4}_{3} F_{3} \left( \begin{array}{c}
\frac{1}{2}, \frac{1}{2}, \frac{n+1}{2}, \frac{n+1}{2} \\
1, \frac{n+2}{2}, \frac{n+2}{2}
\end{array} \middle| 1 \right).
\]

For odd \( n \), the \( n \)th moment of \( KK' \) is a rational multiple of \( \pi^3 \), and the \( n \)th moment of \( K'E, KE', EE' \) is a rational multiple of \( \pi^3 + \frac{\pi}{4(n+1)} \).
For instance,

\[ \int_0^1 x^n K(x)K'(x) \, dx = \frac{\pi^2}{8} \frac{\Gamma\left(\frac{1}{2}(n + 1)\right)^2}{\Gamma\left(\frac{1}{2}(n + 2)\right)^2} \quad _4F_3\left(\begin{array}{c}
\frac{1}{2}, \frac{1}{2}, 
\frac{n+1}{2}, \frac{n+1}{2}
\
1, 
\frac{n+2}{2}, \frac{n+2}{2}
\end{array} \mid 1\right). \]

For odd \( n \), the \( n \)th moment of \( KK' \) is a rational multiple of \( \pi^3 \), and the \( n \)th moment of \( K'E, KE', EE' \) is a rational multiple of \( \pi^3 + \frac{\pi}{4(n+1)} \).

The second claim follows from the first and Legendre’s relation.
For instance,

\[ \int_0^1 x^n K(x) K'(x) \, dx = \frac{\pi^2 \Gamma\left(\frac{1}{2}(n + 1)\right)^2}{8 \cdot \Gamma\left(\frac{1}{2}(n + 2)\right)^2} \quad \text{for odd } n, \]

\[ 2n^3 g(n + 1) - (2n - 1)(2n^2 - 2n + 1)g(n) + 2(n - 1)^3 g(n - 1) = 0. \]

This contiguous relation can be proven using Gosper’s algorithm.
Experiment

\[ h(n) := \pi^3 16^{n+1} g(n + 1) \] matched entry A036917 of the OEIS (same recursion).

The OEIS tells us that

\[ h(n) = \sum_{k=0}^{n} \binom{2n - 2k}{n - k} \frac{(2k)^2}{n} = \frac{16^n \Gamma(n + 1/2)^2}{\pi \Gamma(n + 1)^2} \quad \binom{4}{3} \left( \begin{array}{c} -n, -n, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2} - n, \frac{1}{2} - n, 1 \end{array} \right). \]

The first equality is routine to check using Sister Celine’s method.
$h(n) := \pi^3 16^{n+1} g(n + 1)$ matched entry A036917 of the OEIS (same recursion).
The OEIS tells us that

$$h(n) = \sum_{k=0}^{n} \left( \frac{2n - 2k}{n - k} \right)^2 \left( \frac{2k}{k} \right)^2 = \frac{16^n \Gamma(n + 1/2)^2}{\pi \Gamma(n + 1)^2} _4F_3 \left( \begin{array}{c} -n, -n, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2} - n, \frac{1}{2} - n, 1 \end{array} \middle| 1 \right).$$

The first equality is routine to check using Sister Celine’s method.

The ogf for $h(n)$ is simply

$$\sum_{n=0}^{\infty} h(n)t^n = \frac{4}{\pi^2} K(4\sqrt{t})^2,$$

again easy to prove using the series for $K$. 

Hence,

\[ \int_{0}^{1} \frac{x}{1 - t^2 x^2} K(x) K'(x) \, dx = \frac{\pi}{4} K(t)^2. \]
Hence,

\[ \int_0^1 \frac{x}{1 - t^2 x^2} K(x)K'(x) \, dx = \frac{\pi}{4} K(t)^2. \]

It follows that

\[ \int_0^1 \frac{2}{x} K(x)K'(x)(K(x) - E(x)) \, dx = \int_0^1 K(x)^2 E'(x) \, dx, \]
\[ \int_0^1 \frac{-\log(1 - x^2)}{x} K(x)K'(x) \, dx = \frac{7}{8} \pi \zeta(3). \]

Tantalisingly close to the product of 3 integrals (missing a $\sqrt{\cdot}$).
Hence,
\[ \int_0^1 \frac{x}{1 - t^2 x^2} K(x) K'(x) \, dx = \frac{\pi}{4} K(t)^2. \]

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There are ‘meta’-reasons why the product of two elliptic integrals can be integrated (though hard), e.g. Mellin convolution, transform and Mellin-Barnes integral (Meijer G-function). It seems there is no equivalent for products of 3 functions (c.f. Bessel).
Zudilin’s theorem

We cannot get something ‘deep’ purely by elementary techniques. Here we use a special case of Zudilin’s theorem.
We cannot get something ‘deep’ purely by elementary techniques. Here we use a special case of Zudilin’s theorem.

\[
\int_0^1 \int_0^1 \int_0^1 \frac{x^{h_2-1} y^{h_3-1} z^{h_4-1} (1-x)^{h_0-h_2-h_3} (1-y)^{h_0-h_3-h_4} (1-z)^{h_0-h_4-h_5}}{(1-x(1-y(1-z))))^{h_1}} \, dx \, dy \, dz
\]

\[
= \frac{\Gamma(h_0 + 1) \prod_{j=2}^{4} \Gamma(h_j) \prod_{j=1}^{4} \Gamma(h_0 + 1 - h_j - h_{j+1})}{\prod_{j=1}^{5} \Gamma(h_0 + 1 - h_j)} \times 7F6\left(\begin{array}{c}
h_0/2, 1 + h_0/2, h_1, h_2, h_3, h_4, h_5 \\
h_0, 1 + h_0/2, h_1, 1 + h_0 - h_2, 1 + h_0 - h_3, 1 + h_0 - h_4, h + h_0 - h_5
\end{array} \mid 1 \right).\]
We cannot get something ‘deep’ purely by elementary techniques. Here we use a special case of Zudilin’s theorem.

\[
\begin{align*}
\int_0^1 \int_0^1 \int_0^1 \frac{x^{h_2-1}y^{h_2-1}z^{h_4-1}(1-x)^{h_0-h_2-h_3}(1-y)^{h_0-h_3-h_4}(1-z)^{h_0-h_4-h_5}}{(1-x(1-y(1-z)))^{h_1}} \, dx \, dy \, dz
\end{align*}
\]

\[
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\]

We use

\[
\begin{align*}
\int_0^1 \frac{dx}{\sqrt{x(1-x)(a-x)}} &= \frac{2}{\sqrt{a}} K\left(\frac{1}{\sqrt{a}}\right), \quad \int_0^1 \frac{\sqrt{a-x}}{x(1-x)} \, dx = 2\sqrt{a} E\left(\frac{1}{\sqrt{a}}\right), \\
\int_a^1 \frac{dy}{\sqrt{y(1-y)(y-a)}} &= 2K'(\sqrt{a}), \quad \int_a^1 \frac{\sqrt{y}}{\sqrt{(1-y)(y-a)}} \, dy = 2E'(\sqrt{a}).
\end{align*}
\]
Manipulations

Using the above, we have, for instance,

\[
\int_0^1 E'(y)^2 \, dy = \frac{1}{2} \int_0^1 \int_{a^2}^1 \sqrt{\frac{y}{(1-y)(y-a^2)}} E(\sqrt{1-a^2}) \, da \, dy
\]

\[
= \frac{1}{4} \int_0^1 \int_0^1 \sqrt{\frac{y}{(1-y)z(1-z)}} E(\sqrt{1-yz}) \, dy \, dz
\]

\[
= \frac{1}{8} \int_0^1 \int_0^1 \int_0^1 \sqrt{\frac{y(1-yz)}{(1-y)z(1-z)}} \sqrt{\frac{1}{1-yz-x}} \frac{1}{x(1-x)} \, dx \, dy \, dz
\]

\[
= \frac{1}{8} \int_0^1 \int_0^1 \int_0^1 \sqrt{\frac{y(1-x(1-y(1-z)))}{x(1-x)(1-y)z(1-z)}} \, dx \, dy \, dz.
\]

The second equality comes from the change of variable \(a^2 \mapsto yz\); the fourth from \(z \mapsto 1 - z\).
Now apply the theorem,

\[
\int_0^1 x^n E'(x)^2 \, dx = \frac{2^{4n}(n + 1)^3(n + 3)^2}{16(n + 2)^3(n + 4)} \frac{\Gamma \left( \frac{1}{2}(n + 1) \right)^8}{\Gamma(n + 1)^4} \times 7F_6 \left( \begin{array}{c} -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{n+3}{2}, \frac{n+3}{2}, \frac{n+7}{4} \\ 1, \frac{n+3}{4}, \frac{n+2}{2}, \frac{n+4}{2}, \frac{n+4}{2}, \frac{n+6}{2} \end{array} \right| 1 \right).
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Now apply the theorem,

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\]

This also gives all the odd moments of \(K^2, E^2, KE\), by using

\[
\int_0^1 x^{2n+1} K(x)^a E(x)^b K'(x)^c E'(x)^d \, dx = \int_0^1 x(1-x^2)^n K'(x)^a E'(x)^b K(x)^c E(x)^d \, dx.
\]

E.g. \(\int_0^1 x K'(x)^2 \, dx = \frac{7}{4} \zeta(3)\), \(\int_0^1 x^3 K(x)^2 \, dx = \frac{1}{8} (2 + 7 \zeta(3))\).
Now apply the theorem,

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E.g. \(\int_0^1 x K'(x)^2 \, dx = \frac{7}{4} \zeta(3)\), \(\int_0^1 x^3 K(x)^2 \, dx = \frac{1}{8}(2 + 7\zeta(3))\).

When \(n\) is odd, the \(n\)th moment of the above functions is expressible as \(a + b\zeta(3), a, b \in \mathbb{Q}\) (via partial fractions).
If we write
\[ \frac{f(x)}{(x - a)^n} = \frac{A_1}{(x - a)} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_n}{(x - a)^n}, \]
then \( A_n = f(a), A_{n-1} = f'(a)/1!, \ldots, A_1 = f^{(n-1)}(a)/(n - 1)! \).

In particular, the coefficient of \( 1/(x - a) \) in \( f(x)/g(x) \) is \( f(a)/g'(a) \).

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- Analytic continuation, orthogonal polynomials, special functions, \ldots
- Forward differences, rational roots theorem, Descartes’ rule of sign, \ldots
- Experimental maths.
- Turning up to seminars.
Beukers’ integrals

The triple integrals above look like the integrals used by Beukers to prove the irrationality of $\zeta(3)$. 
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Beukers showed that

$$\int_{[0,1]^3} \frac{(x(1-x)y(1-y)z(1-z))^n}{(1-(1-xy)z)^{n+1}} = \frac{A_n + B_n \zeta(3)}{d_n^3},$$

where $A_n, B_n, d_n$ are integers and $d_n < 3^n$. 
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It is easy to bound the integral, hence

$$0 < \frac{|A_n + B_n\zeta(3)|}{d_n^3} < 3(\sqrt{2} - 1)^{4n}.$$
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Hence \( 0 < |A_n + B_n \zeta(3)| < \left( \frac{4}{5} \right)^n \), implying irrationality.
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It is easy to bound the integral, hence

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Hence $0 < |A_n + B_n \zeta(3)| < \left(\frac{4}{5}\right)^n$, implying irrationality.

Our integrals also produce $\zeta(3)$, but the bound for the integrand is too poor.
Fourier series for $K$

$$K(\sin t) = \sum_{n=0}^{\infty} \frac{\Gamma(n + 1/2)^2}{\Gamma(n + 1)^2} \sin((4n + 1)t).$$

Proof:
Fourier series for $K$

\[ K(\sin t) = \sum_{n=0}^{\infty} \frac{\Gamma(n + 1/2)^2}{\Gamma(n + 1)^2} \sin((4n + 1)t). \]

Proof:

- Only the coefficients of \( \sin(2n + 1)t \) are non-zero by symmetry. Let \( x = \cos t \), the coefficients are

\[ \frac{4}{\pi} \int_{0}^{1} K'(x) \frac{\sin((2n + 1)t)}{\sin t} \, dt. \]
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- The fraction in the integrand is the $(2n)$th Chebyshev polynomial of the 2nd kind:

\[ U_{2n}(x) = \sum_{k=0}^{n} (-1)^k \binom{2n - k}{k} (2x)^{2n-2k}. \]
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- Interchange sum and integral, use moments of $K'$. 

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- Interchange sum and integral, use moments of $K'$.
- Transform the resulting $\mathbf{3} F_2$ and use Saalschütz’s theorem.
Similarly (though not readily found in the literature),

\[
E(\sin t) = \sum_{n=0}^{\infty} \frac{\Gamma(n + 1/2)^2}{2\Gamma(n + 1)^2} \sin((4n + 1)t)
\]
\[
+ \sum_{n=0}^{\infty} \frac{(n + 1/2)\Gamma(n + 1/2)^2}{2(n + 1)\Gamma(n + 1)^2} \sin((4n + 3)t).
\]
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\]

Using Parseval’s theorem etc, integrals involving \( K(\sin t)^2 \) usually evaluated in terms of \( \,_{4}F_{3} \)'s.
Similarly (though not readily found in the literature),

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E(\sin t) = \sum_{n=0}^{\infty} \frac{\Gamma(n + 1/2)^2}{2\Gamma(n + 1)^2} \sin((4n + 1)t)
+ \sum_{n=0}^{\infty} \frac{(n + 1/2)\Gamma(n + 1/2)^2}{2(n + 1)\Gamma(n + 1)^2} \sin((4n + 3)t).
\]

Using Parseval’s theorem etc, integrals involving \(K(\sin t)^2\) usually evaluated in terms of \(4F_3\)’s.

However, using the odd moments,

\[
\int_0^{\pi/2} K(\sin t)^2 \sin 4t \, dt = -2.
\]
Quadratic transform

Quadratic transformation (3) used on $K'^m$ gives

$$\int_0^1 K'(x)^n \, dx = 2 \int K(x)^n (1 + x)^{n-2} \, dx.$$
Quadratic transform

Quadratic transformation (3) used on $K'^m$ gives

$$\int_0^1 K'(x)^n \, dx = 2 \int K(x)^n (1 + x)^{n-2} \, dx.$$ 

Use (3) on $xK(x)^3$, we get

$$\int_0^1 2(1 - x)K(x)^3 \, dx = \int_0^1 xK(x)^3 \, dx.$$
Quadratic transform

Quadratic transformation (3) used on $K'^m$ gives

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Use (3) on $xK(x)^3$, we get

$$\int_0^1 2(1 - x)K(x)^3 \, dx = \int_0^1 xK(x)^3 \, dx.$$ 

Combining, we deduce

$$\int_0^1 K'(x)^3 \, dx = \frac{10}{3} \int_0^1 K(x)^3 \, dx = 5 \int_0^1 xK(x)^3 \, dx = 5 \int_0^1 xK'(x)^3 \, dx.$$
Legendre’s relation

We can multiply Legendre’s relation by a function and integrate:

\[
\int_0^1 3E'(x)K'(x)K(x) - K(x)K'(x)^2 \, dx = \frac{\pi^3}{8},
\]

\[
\int_0^1 3E(x)K(x)K'(x) - 2K(x)^2K'(x) \, dx = \pi G,
\]

\[
\int_0^1 2E'(x)K(x)^2 - E(x)K(x)K'(x) \, dx = \pi G,
\]

\[
\int_0^1 2xE'(x)K(x)^2K'(x) - xK(x)^2K'(x)^2 \, dx = \frac{\pi^4}{32}.
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Legendre’s relation

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\[\int_{0}^{1} 3E'(x)K'(x)K(x) - K(x)K'(x)^2 \, dx = \frac{\pi^3}{8},\]

\[\int_{0}^{1} 3E(x)K(x)K'(x) - 2K(x)^2K'(x) \, dx = \pi G,\]

\[\int_{0}^{1} 2E'(x)K(x)^2 - E(x)K(x)K'(x) \, dx = \pi G,\]

\[\int_{0}^{1} 2xE'(x)K(x)^2K'(x) - xK(x)^2K'(x)^2 \, dx = \frac{\pi^4}{32}.\]

Unfortunately, we cannot uncouple these.
Integration by parts

\[
\int_0^1 (1 - x^2)^n \frac{d}{dx} \left( x^k K(x)^a E(x)^b K'(x)^c E'(x)^d \right) \, dx
\]

\[
= \int_0^1 2nx(1 - x^2)^{n-1} x^k K(x)^a E(x)^b K'(x)^c E'(x)^d \, dx.
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E.g. using \( K'^2 \), we get
\[
\int_0^1 2K'(x)E'(x) - (1 - x^2)K'(x)^2 \, dx = 0.
\]
Integration by parts

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\[= \int_0^1 2nx(1 - x^2)^{n-1} x^k K(x)^a E(x)^b K'(x)^c E'(x)^d \, dx.\]

E.g. using \( K'' \), we get \( \int_0^1 2K'(x)E'(x) - (1 - x^2)K'(x)^2 \, dx = 0.\)

We obtain:

\[\int_0^1 (n + 2)x^{n-1} E(x)^2 - 2x^{n-1} E(x)K(x) \, dx = 1,\]

\[\int_0^1 \left( nx^{n-1} E(x)K(x) - (n + 2)x^{n+1} E(x)K(x) \right) + x^{n-1} E(x)^2 - x^{n-1} K(x)^2 + x^{n+1} K(x)^2 \right) \, dx = 0,\]

\[\int_0^1 2x^{n-1} E(x)K(x) + (n - 2)x^{n-1} K(x)^2 - nx^{n+1} K(x)^2 \, dx = 0.\]
From these we obtain recursions: with $K_n := \int_0^1 x^n K(x)^2 \, dx,$

$$(n + 1)^3 K_{n+2} - 2n(n^2 + 1)K_n + (n - 1)^3 K_{n-2} = 2.$$
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\[
(n + 1)^3 K_{n+2} - 2n(n^2 + 1)K_n + (n - 1)^3 K_{n-2} = 2.
\]

With \( E_n := \int_0^1 x^n E(x)^2 \, dx \),

\[
(n+1)(n+3)(n+5)E_{n+2} - 2(n^3 + 3n^2 + n + 1)E_n + (n-1)^3 E_{n-2} = 8,
\]

plus others.
Recursion

From these we obtain recursions: with $K_n := \int_0^1 x^n K(x)^2 \, dx$,

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plus others.

Found by linear algebra or PSLQ. Give alternative proofs to $\pi^3, \zeta(3)$ results.
Towards the even moments

There are only five moments that we do not have closed forms of:

\[ E(x)^2, \ x^2E(x)^2, \ E(x)K(x), \ x^2E(x)K(x), \ x^2K(x)^2, \]

together they generate all the even moments.
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together they generate all the even moments.

Unfortunately, we can only prove four equations connecting them (by parts).

The last equation we need happens to be the only unproven entry from the \((2, 0)\) and \((2, 1)\) tables:

\[
\int_0^1 2K(x)^2 - 4E(x)K(x) + 3E(x)^2 - K'(x)E'(x) \, dx \overset{?}{=} 0.
\]
Linearly related products

\[
\begin{align*}
\int_0^1 K(x)^2 \, dx &= \frac{1}{2} \int_0^1 K'(x)^2 \, dx \\
&= \int_0^1 K'(x)^2 \frac{x}{x'} \, dx \\
&= \int_0^1 K(x)K'(x)x' \, dx \\
&= \int_0^1 2K(x)E(x) \frac{x}{x + 1} \, dx \\
&= \frac{2}{\pi} \int_0^1 \frac{\text{arcsin} \, x}{\sqrt{1 - x^2}} K(x)K'(x) \, dx \\
&= \frac{4}{\pi} \int_0^1 \text{arctanh}(x)K(x)K'(x) \, dx.
\end{align*}
\]
E.g. the derivative for $K(x)^3$ gives

$$\int_0^1 K(x)^3 - 3K(x)^2 E(x) \, dx = 0.$$
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Any two integrals in each of the following groups are related by a rational factor:

$$K(x)^3, K'(x)^3, xK(x)^3, xK'(x)^3, K(x)^2 E(x), K'(x)^2 E'(x);$$

$$K(x)K'(x)^2, K(x)^2 K'(x), xK(x)K'(x)^2, xK(x)^2 K'(x).$$
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Any two integrals in each of the following groups are related by a rational factor:

$$K(x)^3, K'(x)^3, xK(x)^3, xK'(x)^3, K(x)^2 E(x), K'(x)^2 E'(x);$$

$$K(x)K'(x)^2, K(x)^2 K'(x), xK(x)K'(x)^2, xK(x)^2 K'(x).$$

But we can’t link the two groups.
Conjecture

Amazingly, the ISC gives

\[ \int_0^1 K'(x)^3 \, dx = 2K \left( \frac{1}{\sqrt{2}} \right)^4 = \frac{\Gamma(1/4)^8}{128\pi^2}. \]
Amazingly, the ISC gives

\[
\int_0^1 K'(x)^3 \, dx = 2K \left( \frac{1}{\sqrt{2}} \right)^4 = \frac{\Gamma(1/4)^8}{128\pi^2}.
\]

C.f. \( \int_0^1 \frac{K'(x)}{\sqrt{1-t^2x^2}} \, dx \), which at \( t = 1 \) gives \( K(1/\sqrt{2})^2 \).

Equivalently,

\[
\sum_{n=0}^{\infty} \frac{8}{(2n+1)^2} 4F_3 \left( \begin{array}{c}
\frac{1}{2}, \frac{1}{2}, n + 1, n + 1 \\
1, n + \frac{3}{2}, n + \frac{3}{2}
\end{array} \right| 1 \right) \quad \text{equals} \quad \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)^4}{\Gamma(n+1)^4} 4F_3 \left( \begin{array}{c}
\frac{1}{2}, \frac{1}{2}, -n, -n \\
1, \frac{1}{2} - n, \frac{1}{2} - n
\end{array} \right| 1 \right) = \frac{\Gamma(1/4)^8}{24\pi^4}.
\]
Conjecture

Amazingly, the ISC gives

$$\int_0^1 K'(x)^3 \, dx = 2K \left( \frac{1}{\sqrt{2}} \right)^4 = \frac{\Gamma(1/4)^8}{128\pi^2}.$$ 

C.f. $\int_0^1 \frac{K'(x)}{\sqrt{1-t^2x^2}} \, dx,$ which at $t = 1$ gives $K(1/\sqrt{2})^2$.

Equivalently,

$$\sum_{n=0}^{\infty} \frac{8}{(2n+1)^2} \frac{\Gamma(n+1/2)^4}{\Gamma(n+1)^4} \binom{\frac{1}{2}, \frac{1}{2}, n+1, n+1}{1, n+\frac{3}{2}, n+\frac{3}{2}} \binom{1}{1} = \frac{\Gamma(1/4)^8}{24\pi^4}.$$ 

Some sort of WZ method?
Many other relations proven using basic techniques.
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We echo these sagely words:

“it seems to be more and more the case as experimental computational tools improve, our ability to discover outstrips our ability to prove.” – Jon Borwein & David Bailey
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Thank you!