This paper is not intended as a sweeping survey on the topic of spectral theory and PDE as this would exceed any reasonable bounds. Rather, we intend to outline some of the recent work by the author and Joachim Krieger on the topic of stable manifolds for unstable PDEs. More specifically, we intend to give some ideas of how to prove a recent result by Krieger and the author for the quintic wave equation \( \Box \psi - \psi^5 = 0 \) in \( \mathbb{R}^3 \).

This is the focusing, \( H^1 \)-critical nonlinear wave equation (NLW). There exist stationary solutions \( \phi(x, a) \) of this equation given by:

\[
\phi(x, a) = (3a)^{\frac{5}{3}}(1 + a|x|^2)^{-\frac{1}{2}}, \quad a > 0
\]

The question is what happens if the initial data of NLW is a small perturbation of \( \phi \). To answer this question, we first linearize around \( \phi \). The ansatz \( \psi(x, t) = \phi(x, a) + u(x, t) \) leads to the equation

\[
\partial_{tt} u - \Delta u + Vu = N(u, \phi),
\]

with \( V = -5\phi^4 \) and a nonlinear term

\[
N(u, \phi) = 10u^2\phi^3 + 10u^3\phi^2 + 5u^4\phi + u^5
\]

Let \( H = -\Delta + V \). It is known that the spectrum of such a Schrödinger operator on the interval \( (0, \infty) \) is purely absolutely continuous. Also, \( H \) has a simple negative eigenvalue, say \( -k^2 \) (the ground-state). Indeed,

\[
\inf_{f \in H^1, \|f\|_2 = 1} \langle Hf, f \rangle \leq \langle H\phi, \phi \rangle = -4 \int_{\mathbb{R}^3} \phi^6 \, dx < 0
\]

The presence of negative spectrum implies that the static solutions \( \phi(\cdot, a) \) are linearly (exponentially) unstable. We remark that by Agmon’s estimate, we know that the ground-state function \( g \), which satisfies \( Hg = -k^2 g \), decays exponentially. In addition, \( g > 0 \), and \( g \) is radial. Because of symmetries, \( H \) has a zero mode. More precisely, by the dilation and translation symmetries

\[
H\partial_a \phi(\cdot, a) = 0, \quad H\nabla \phi(\cdot, a) = 0
\]

It is not an easy task to show that \( H \) has exactly one negative eigenvalue (which then must be simple), and no other zero modes than those four which we have specified. Numerically, this has been verified by means of the Birman-Schwinger method (the numerics show that the Birman-Schwinger operator has exactly five

---

The author was partially supported by the NSF grant DMS-0300081 and a Sloan Fellowship. This article is a much expanded version of some lectures which the author held at the MSRI in Berkeley during August 2005. He wishes to thank Jeremy Marzuola for taking and typing notes and the MSRI and the organizers of the dispersive PDE meeting for the opportunity to deliver these lectures.
eigenvalues \geq 1 \text{ counted with multiplicity. This corresponds precisely to the four independent eigenfunctions and the resonance function).}

However, if we restrict \( H \) to the invariant subspace of radial functions, then the determination of the spectrum of this restricted operator is elementary. Indeed, \( Hf = Ef \) for radial \( f = f(r) \) is the same as \( -(\frac{d^2}{dr^2} + V)rf(r) = Ef(r) \). In other words, we are reduced to an eigenvalue problem for the half-line operator \( \tilde{H} = -\frac{d^2}{dx^2} + V \) with a Dirichlet condition at \( r = 0 \). In our case, the function \( r\partial_r \phi(r,a) \) has a unique positive zero and solves \( \tilde{H}(r\partial_r \phi(r,a)) = 0 \) (moreover, it is bounded and therefore a resonance function for \( \tilde{H} \)). By Sturm theory, it follows that there is a unique negative eigenvalue which is simple. It agrees with \(-k^2\) from above. Hence, the operator \( \tilde{H} \) has a one-dimensional exponential instability (in the radial setting) and the evolution of the wave equation driven by \( H \) has a codimension one stable manifold. Our main theorem states that this remains true (in some sense) on the nonlinear level.

**THEOREM 1**

There exists a \( \delta > 0 \) such that the following statement holds. Let \( B_\delta(0) \) be the ball centered at \( 0 \) of size \( \delta \) in \( H^3_{rad}(\mathbb{R}^3) \times H^2_{rad}(\mathbb{R}^3) \). There exists a codimension 1 Lipschitz manifold \( \Sigma \) which divides \( B_\delta(0) \setminus \Sigma \) into two connected components and such that if \((u_1,u_2) \in \Sigma \cap B_\delta(0)\), then the Cauchy problem for NLW with data

\[
(\psi(\cdot,0),\partial_t \psi(\cdot,0)) = (\phi(\cdot,1) + u_1, u_2)
\]

has a global solution \( \psi(x,t) \). Moreover, there is the representation

\[
\psi(x,t) = \phi(x,a(\infty)) + u(x,t)
\]

where \( |a(\infty) - 1| \lesssim \delta \), and \( \|u(\cdot,t)\|_\infty \lesssim \delta(t)^{-1} \) as well as

\[
(u,\partial_t u) = (v,\partial_t v) + o_{\dot{H}^1 \times L^2}(1) \quad \text{as} \ t \to \infty
\]

with \( \Box v = 0 \) and \((v,\partial_t v)(0) \in \dot{H}^1 \times L^2 \).

One motivation for this problem came from a numerics paper by Bizon, Chmaj, and Tabor which appeared in *Nonlinearity* in 2004. These are physicists who are mainly interested in the Einstein equations of general relativity, but proposed the quintic NLW as a model case. They predicted numerically that the stable manifold should exist, at least in a small neighborhood of \( \phi \). Moreover, Bizon et al. made conjectures about the behavior above and below the manifold (one should lead to blow-up, the other to dispersion).

Another motivation for studying the quintic NLW came from the author’s work on the focusing nonlinear Schrödinger equation (NLS), which we now describe. Consider the equation

\[
i\partial_t \Psi + \Delta \Psi + |\Psi|^{2\sigma} \Psi = 0 \text{ in } \mathbb{R}^3,
\]

for \( \sigma < 2 \) (where 2 is the \( H^1 \) critical exponent in \( \mathbb{R}^3 \). We also have \( \sigma_{crit} = \frac{2}{3} = \frac{2}{3} \), which is the \( L^2 \) critical exponent). The Hamiltonian and the charge are conserved functionals:
\[ H(\Psi(t)) = \int \left( \frac{1}{2} |\nabla \Psi|^2 - \frac{1}{2(\sigma+1)} |\Psi|^{2(\sigma+1)} \right) dx \]

\[ Q(\Psi(t)) = \frac{1}{2} \int |\Psi|^2 dx. \]

For \( \sigma < \frac{2}{d} \), we get global existence in \( H^1 \) using the Gagliardo-Nirenberg inequality and the above conservation laws. For \( \sigma \geq \frac{2}{d} \), we in fact can have blow-up in finite time. This result is easy to see simply by looking at the so-called \textit{virial identity}. This approach is known as Glassey’s argument and is based on the functional

\[ \int |x|^2 |\Psi(t, x)|^2 dx. \]

Indeed, if \( \sigma \geq 2/d \), then

\[ \frac{d^2}{dt^2} \int |x|^2 |\Psi(t, x)|^2 dx \leq 16 H(\Psi(0)) \]

Thus, any solution of negative energy must blow up in finite time (at least in the sense of the \( H^1 \) norm). Indeed, at some point the positive quantity above disappears and the solution must blow-up. Strictly speaking, to apply the virial identity, we need to work in the space \( \langle x \rangle^{-1} H^1 \) (it suffices to assume that \( \Psi(0) \in \langle x \rangle^{-1} H^1 \)).

There exist stationary state solutions for NLS as well. Make the ansatz \( \Psi(t, x) = e^{it\alpha^2} \phi \). Then \( \phi \) satisfies the equation

\[ (\alpha^2 - \Delta) \phi = |\phi|^{2\sigma} \phi. \]

Berestycki and Lions showed using variational methods that there exist solutions to such equations that are positive, radial, and exponentially decaying. In many cases, uniqueness of said ground-state solutions has been proved by Coffman, Kwong, and McLeod. \( \phi \) must satisfy the Derrick–Pohozaev identities and for monomial nonlinearities we have a scaling property, namely \( \phi(x, \alpha) = \alpha^{\frac{4}{\sigma}} \phi(\alpha x, 1) \).

There is a group of natural symmetries acting on NLS. There is the modulation symmetry \( \Psi \to e^{i\gamma} \Psi \) and the Galilei transforms

\[ G(t) = e^{i(-v^2 t + x \cdot v)} e^{-i p \cdot (y + 2tv)}, \]

for \( p = -i \frac{\partial}{\partial x} \). These are the unique transformations that satisfy

\[ G(t)e^{i\Delta} = e^{it\Delta} G(0) \]

with \( G(0) = e^{ix \cdot v} e^{-ip \cdot y} \). Note that this latter transform at time \( t = 0 \) is the Galilei transform from classical mechanics, i.e., \( x \mapsto x + y, \ p \mapsto p + v \). Finally, we have dilation symmetries from the scaling relation above for ground state solutions. In total, we have a \( 2d + 2 \) vector of symmetry parameters

\[ \pi = (\gamma, v, y, \alpha). \]

The relevance of these symmetries can be seen in the following theorem of Michael Weinstein.

**THEOREM 2**

For any \( \epsilon \), there exists a \( \delta \) such that if

\[ \inf_{\gamma, y} \|\Psi(\cdot, 0) - e^{i\gamma} \phi(\cdot + y, \alpha)\|_{H^1} < \delta, \]
then
\[
\inf_{\gamma, y} \| \Psi(x, t) - e^{i\gamma} \phi(\cdot + y, \alpha) \|_{H^1} < \epsilon.
\]

It is essential to mod out the symmetries in the second inequality, since a small perturbation of the ground-state \( \phi \) (which unperturbed has zero momentum) will impart a nonzero (but very small) momentum on it. Consequently, the soliton will have moved a large distance in finite (but large) time. In a similar fashion, the phase will also change by size one in finite time.

Weinstein’s theorem expresses what is known as orbital stability of the ground-state solutions. During the late 1980s Grillakis, Shatah, and Strauss developed a fundamental theory of orbital stability of standing waves generated by symmetries. Their theory applies to general Hamiltonian equations which are invariant under the action of Lie groups. In particular, it gives the dichotomy \( \sigma < 2/d \) and \( \sigma \geq 2/d \) for the focusing NLS, but applies to a much wider range of examples.

Note that the conjectured instability of excited state solutions is largely open. An excited state is a solution of
\[
-\Delta \phi + \alpha^2 \phi = |\phi|^{2\sigma} \phi
\]
which changes sign. Important work has been done in that direction by T. P. Tsai and H. T. Yau for NLS with a potential. Generally speaking, NLS with a potential or Hartree equations is a fascinating field to which we cannot do justice here, see for example the work of Fröhlich et. al.

Berestycki and Cazenave showed that for \( \sigma \geq \frac{2}{d} \), blow-up can occur for arbitrarily small perturbations of ground states. This is easy for \( \sigma = \frac{2}{d} \) using Glassey’s argument from above and the fact that \( \mathcal{H}((1 + \epsilon)\phi) < 0 \) for \( \epsilon > 0 \) since \( \mathcal{H}(\phi) = 0 \).

Now, let us discuss the spectral analysis of linearized operators for NLS. Set \( \psi = e^{i\alpha^2 \phi} R + \) and linearize. Note that we are assuming \( \alpha \) does not change with time here, but we do not let that concern us at this point. We end up with an equation
\[
i \partial_t \begin{pmatrix} R \\ \overline{R} \end{pmatrix} + H \begin{pmatrix} R \\ \overline{R} \end{pmatrix} = N,
\]
for \( N \) a nonlinear term quadratic in \( R \) and
\[
H = \begin{pmatrix}
\Delta - \alpha^2 + (\sigma + 1)\phi^{2\sigma} & \sigma\phi^{2\sigma} \\
-\sigma\phi^{2\sigma} & -\Delta + \alpha^2 - (\sigma + 1)\phi^{2\sigma}
\end{pmatrix}.
\]
If \( \phi = 0 \), we have a self-adjoint operator with purely absolutely continuous spectrum given by \( (-\infty, -\alpha^2] \cup [\alpha^2, \infty) \). If we do turn on \( \phi \), then this set turns into the essential spectrum of \( H \) by Weyl’s criterion. In addition, there is an eigenvalue at zero due to the symmetries. Indeed, writing NLS as \( \mathcal{N}(\Psi_\pi) = 0 \) where \( \Psi_\pi \) indicates the symmetries given by the vector \( \pi \) acting on \( \Psi \), we conclude that \( \mathcal{N}(\Psi_\pi) \partial_\pi \Psi_\pi = 0 \). Since clearly \( H(\Psi_\pi) = \mathcal{N}(\Psi_\pi) \) by definition of the linearized operator, it follows that \( \partial_\pi \Psi_\pi \) is a zero energy state of \( H \). In the NLS case, they are (generalized) eigenfunctions with zero eigenvalue (but for the quintic NLW from above, there is a resonance and not just an eigenvalue at zero). If we look at \( R = u + iv \) and write \( H \) in terms of \( u, v \), then we obtain the equivalent matrix operator
\[
H = \begin{pmatrix} 0 & iL_- \\ -iL_+ & 0 \end{pmatrix},
\]
where

\[ L_- = -\Delta + \alpha^2 - \phi^{2\sigma} \]

and

\[ L_+ = -\Delta + \alpha^2 - (2\sigma + 1)\phi^{2\sigma}. \]

Clearly, \( L_-(\phi) = 0 \) and \( L_+ (\nabla \phi) = 0 \). Hence, we have

\[
\begin{pmatrix}
0 \\
\phi
\end{pmatrix},
\begin{pmatrix}
\nabla \phi \\
0
\end{pmatrix} \in \text{Ker } (H).
\]

Differentiating \( L_- = 0 \) in \( \alpha \) yields

\[ L_+ (\partial_\alpha \phi) = -2\alpha \phi. \]

Therefore,

\[ H \begin{pmatrix}
\partial_\alpha \phi \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
2i\alpha \phi
\end{pmatrix}. \]

In fact, it is easy to see that

\[
\begin{pmatrix}
0 \\
x\phi
\end{pmatrix},
\begin{pmatrix}
\partial_\alpha \phi \\
0
\end{pmatrix} \in \text{Ker } (H^2).
\]

This shows that the eigenvalue 0 is of geometric multiplicity at least \( d + 1 \) and of algebraic multiplicity at least \( 2d + 2 \), but we actually have equality here (at least if \( \sigma \neq \frac{2}{d} \)) due to a result of M. Weinstein from 1986. So, \( H \) is truly a non-self adjoint operator with a nontrivial Jordan form at zero energy. In the critical case \( \sigma = \frac{2}{d} \) the dimension of the root space at zero increases to \( 2d + 4 \) due to the extra pseudo-conformal symmetry.

Let us now analyze the operator \( e^{itH} \). Using the \( L_- , L_+ \) version of \( H \), we see that \( Hf = Ef \) implies

\[ H^2 \begin{pmatrix}
f_1 \\
f_2
\end{pmatrix} = \begin{pmatrix}
L_- L_+ & 0 \\
0 & L_+ L_-
\end{pmatrix} \begin{pmatrix}
f_1 \\
f_2
\end{pmatrix} = E^2 \begin{pmatrix}
f_1 \\
f_2
\end{pmatrix}. \]

Since \( L_- \phi = 0 \) and \( \phi > 0 \) it follows that \( L_- \) is a non-negative operator with \( \phi \) as a ground state. By work of M. Weinstein, \( L_+ \) has exactly one negative eigenvalue and the kernel is spanned by the derivatives \( \partial_j \phi \). From the above argument, since \( E^2 f_1 \) is in the range of \( L_- \), we know that \( f_1 \perp \phi \). Therefore, we have

\[ \sqrt{L_-} L_+ \sqrt{L_-} f_1 = E^2 \tilde{f}_1, \]

for \( \tilde{f}_1 = L_-^{-\frac{1}{2}} f_1 \). Thus, \( E^2 \) is real as an eigenvalue of a self-adjoint operator. The dichotomy of linear stability or instability is decided by the question: When is \( E^2 < 0 \) ? In order to answer this, look at

\[
\inf \frac{\langle \sqrt{L_-} L_+ \sqrt{L_-} f, f \rangle}{\langle f, f \rangle} < 0
\]

\[ \iff \inf_{f \perp \phi, \|f\|_2 = 1} \langle L_+ \tilde{f}, \tilde{f} \rangle < 0. \]

Denote this latter infimum as \( \mu_0 \) and assume that \( \mu_0 < 0 \). One can show that the minimum is attained at some function \( f_0 \). Using Lagrange multipliers, we have

\[ L_+ f_0 = \mu_0 f_0 + \lambda \phi, \quad f_0 \perp \phi, \quad \mu_0 = \langle L_+ f_0, f_0 \rangle. \]

If \( \lambda = 0 \), then \( \mu_0 \) is a negative eigenvalue, hence the ground state of \( L_+ \) and this contradicts the orthogonality relation. So, we have \( f_0 = (L_+ - \mu_0)^{-1} \phi \). Consider the function

\[ h(\mu) = \langle (L_+ - \mu)^{-1} \phi, \phi \rangle. \]
Note that this goes to $-\infty$ for $\mu$ tending to the unique negative eigenvalue of $L_+$ and $h(0)$ makes sense since the kernel of $L_+$ is orthogonal to $\phi$ (remember it is $\nabla \phi$).

We know $h'(\mu) > 0$ as it is $\| (L_+ - \mu)^{-1} \phi \|_2^2$. Hence, $h(0) > 0$ iff $\mu_0 < 0$. But,

\[
h(0) = -\frac{1}{2\alpha} \langle \partial_\alpha \phi, \phi \rangle > 0,
\]

Finally, one checks from scaling that $\partial_\alpha \| \Psi_\alpha \|_2^2 < 0 \iff \sigma > \frac{2}{3}$. This (up to the equality sign) is the well-known instability condition for NLS. We now state our first stable manifold theorem for focusing, cubic NLS in three dimensions. It is conditional on the assumption that there are no imbedded eigenvalues in the essential spectrum of the linearized operator.

**THEOREM 3**

(Schlag - 2004)

Consider NLS with $\sigma = 1$ in $\mathbb{R}^3$. Assume that the linearized operator does not have any imbedded eigenvalues in its essential spectrum. Then there exists $\delta > 0$ such that the following holds: Let $B_\delta(0) \subset W^{1,1} \cap W^{1,2}$ be the ball centered at 0 of radius $\delta$. There exists a codimension one Lipschitz graph $\Sigma \subset B_\delta(0)$ which divides $B_\delta(0) \setminus \Sigma$ into two connected components and so that for initial data in $\Psi(\cdot, 0) \in \Sigma + \phi(\cdot, 1)$, there exists a global solution

\[
\Psi(x, t) = W(x, t) + R(x, t),
\]

with $\| \Psi(\cdot, t) \|_\infty \lesssim \delta(t)^{-\frac{2}{3}}, R = e^{it\Delta} \hat{R} + o_{L^2}(1)$. We have

\[
W(x, t) = e^{i\theta(t, x)} \phi(x - y(t), \alpha(t)),
\]

\[
\theta(t, x) = -\int_0^t v^2(s) dx + v(t)x + \gamma(t)
\]

and

\[
y(t) = 2 \int_0^t v(s) ds + y_0(t).
\]

Finally, the vector $\pi(t) := (\gamma(t), v(t), y_0(t), \alpha(t))$ approaches a terminal vector $\pi(\infty)$ which is $\delta$ close to the initial vector $\pi(0) = (0, 0, 0, 1)$.

It is clear that we cannot have stability for initial data in the entire ball $\phi + B_\delta(0)$. In fact, $\sigma = 1$ is in the supercritical range for which Berestycki and Cazenave have exhibited arbitrarily small perturbations of $\phi$ that lead to finite-time blow-up. On the other hand, we do not understand the general behavior of solutions with data in $\phi + B_\delta(0) \setminus \Sigma$.

We remark that the method of proof does not extend to the entire supercritical range $\sigma > 2/3$. More precisely, an important gap property for the operators $L_\pm$ (which we defined above) breaks down below some value $\sigma_* = 0.91396 \ldots$. The gap property here refers to $L_+$ and $L_-$ having neither an eigenvalue in $[0, \alpha^2]$ nor a resonance at the edge $\alpha^2$. This was shown in some recent numerical work by Laurent Demanet and the author. This numerical work also establishes that the gap property does hold above $\sigma_*$, and thus in particular for $\sigma = 1$ (which is essential for the proof of the theorem).
For the proof of the theorem, we follow the basic strategy of the modulation theory of Soffer and Weinstein. More precisely, we seek an ODE for 
\[
\pi(t) = (\gamma, y, v, \alpha)(t)
\]
and a PDE for \(R\). The ODE is constructed in such a way that the non-dispersive nature of the root-space of \(H\) is eliminated. Note that the dimension of the root-space is 8, as explained above, and \(\pi(t)\) has eight components. This agreement is of course not only essential but also no coincidence, as \(\pi\) has as many components as there are symmetries, and the dimension of the root-space must also agree with the number of these symmetries.

Of fundamental importance are the following estimates on the linearized operator:

\[
\| e^{itH}P_s f \|_\infty \lesssim t^{-\frac{3}{2}} \| f \|_1,
\]

\[
\| e^{itH}P_s f \|_2 \lesssim \| f \|_2
\]

where \(P_s\) projects onto the stable part of the spectrum (i.e., it is the identity minus the Riesz projection onto the discrete spectrum). The proof of these bounds not only requires that there are no imbedded eigenvalues in the essential spectrum, but also the following properties:

- zero is the only point in the discrete spectrum
- the edges \(\pm \alpha^2\) are neither eigenvalues nor resonances

We prove these facts by modifying some arguments of Galina Perelman for the case \(d = 1\) and \(\sigma = 2\). It is at this point that the aforementioned numerical work by Demanet and the author becomes essential.

In one dimension a corresponding result was proven by Joachim Krieger and the author in the entire supercritical range \(\sigma > 2\). This result was \emph{completely unconditional} and also did not rely on any numerics. Rather, one can prove all the necessary spectral properties by means of the scattering theory established by Buslaev and Perelman, as well as by invoking the explicit form of the ground state in \(d = 1\) (the latter is needed to prove that there are no imbedded eigenvalues). The linear estimates needed for \(d = 1\) are

\[
\| e^{itH}P_s f \|_\infty \lesssim t^{-\frac{3}{2}} \| f \|_1
\]

\[
\| (x)^{-1} e^{itH}P_s f \|_\infty \lesssim t^{-\frac{3}{2}} \| (x)f \|_1
\]

The latter bound is somewhat curious, in so far as it does not hold in the free case \(H = -\partial^2\). Rather, it exploits that the edges of the essential spectrum are not resonances (which of course fails in the free case). Heuristically, one can arrive at the second bound as follows. Suppose the scalar operator \(-\partial^2 + V\) does not have a resonance at zero energy. This is well-known to be equivalent to the fact that the distorted Fourier basis \(\{ e(x, \xi) \}_{\xi \in \mathbb{R}}\) vanishes when \(\xi = 0\). Now write the evolution relative to this basis:

\[
e^{itH}P_s f(x) = \int e^{it\xi^2} e(x, \xi) \langle f, e(\cdot, \xi) \rangle d\xi.
\]

By the vanishing of \(e(\cdot, 0)\), we integrate by parts in \(\xi\) gaining a factor of \(t^{-1}\). However, this evidently comes at the expense of a factor of \(x\). Hence, we obtain the \(t^{-\frac{3}{2}}\) decay by invoking the dispersive decay on top of the \(t^{-1}\) gain, leading to the above bound. On a more heuristic level, we remark that compact data \(f\) should be thought of as having traveled distance \(t\) after time \(t\). Due to the weight \((x)^{-1}\) this
will lead to a gain of $t^{-1}$ on top of the dispersive decay of $t^{-\frac{7}{6}}$. The point of not having a resonance at the edge is to avoid having some mass "being stuck" around the origin. This would of course invalidate the heuristic idea of transport.

The possibility of obtaining improved decay bounds at the expense of weights was apparently discovered by Murata in the 1980s. It applies only to those cases where the free operator has a zero energy resonance, i.e., when $d = 1, 2$. In dimension two, Murata discovered an improvement of $(\log t)^{-2}$ over the free dispersive decay.

We remark that the presence of these weights leads to substantial technical problems in the proof of the 1-d stable manifold theorem (as compared to the 3-d result).

Finally, we would like to point the reader to the second stable manifold paper with Krieger which deals with the pseudo conformal blow-up solutions for critical NLS in $d = 1$. The $t^{-\frac{7}{6}}$ bound again plays a crucial role, but this is not the venue to explain the blow-up argument in any detail.

As stated above, we desire to understand stability of the focusing, quintic NLW

$$\Box \psi - \psi^{5} = 0$$

around the special solutions $\phi(\cdot, a)$. Recall that Aubin showed that there exists $\phi = \phi(\cdot, a) > 0$ such that $-\Delta \phi(\cdot, a) - \phi^{5}(\cdot, a) = 0$. Explicitly, with $a > 0$

$$\phi(x, a) = (3a)^{\frac{1}{5}}(1 + a|x|^{2})^{-\frac{1}{2}}$$

Note from the formula above that $\phi(x, a) \sim \frac{1}{|x|}$ and also $\partial_{a} \phi(x, a) \sim \frac{1}{|x|}$. In particular, neither $\phi$ nor $\partial_{a} \phi$ belong to $L^{2}(\mathbb{R}^{3})$. We also remark that it is essential for us to work with the quintic, i.e., $H^{1}$-critical equation for a number of reasons. For example, by a result of Gidas and Spruck, there are no positive solutions of $\Delta \phi + \phi^{p} = 0$ in $\mathbb{R}^{3}$ if $p < 5$.

We wish to study radial perturbations to the initial data of the form

$$\begin{pmatrix}
\psi(0) \\
\partial_{t} \psi(0)
\end{pmatrix} = \begin{pmatrix}
\phi(\cdot, 1) \\
0
\end{pmatrix} + \begin{pmatrix}
f_{1} \\
f_{2}
\end{pmatrix}.$$ 

As mentioned before, Bizon, Chmaj, and Tabor, as well as Szpak, studied this problem numerically. They tested initial data $(f_{1}, f_{2})$ from a one-parameter family of functions (e.g., Gaussians with changing variance) and observed that there is a unique point on such a curve of data at which there is long-time stability. This point splits the curve into two connected components (i.e., intervals). One interval led to instability (or possibly blow-up), whereas another led to scattering to a free wave. By letting this curve of initial data vary, they then observed that the unique stability point sweeps out a codimension one stable manifold on which the family $\phi(\cdot, a)$ acts as a one-dimensional attractor.

Our theorem establishes the existence of such a codimension one stable manifold. However, at this point we cannot describe the behavior off the manifold. We now state the theorem, which already appeared above, in full detail.

**THEOREM 4**
(Krieger-Schlag - 2005)
Let $R > 1$. Denote

$$X_{R} = \{(f_{1}, f_{2}) \in H^{3}_{rad} \times H^{2}_{rad}, \text{ supp}(f_{j}) \subset B(0, R)\}.$$
Define
\[ \Sigma_0 = \{(f_1, f_2) \in X_R : \langle k(\cdot, 1)f_1 + f_2, g(\cdot, 1) \rangle_{L^2} = 0 \}, \]
where \(-k^2\) is the unique negative eigenvalue of the linearized Hamiltonian and \(g\) is the associated ground-state eigenfunction. There exists \(\delta > 0\) such that if \((f_1, f_2) \in B_\delta(0) \subset \Sigma_0\), then there is a real number \(h(f_1, f_2)\) such that
\[ |h(f_1, f_2)| \lesssim \|(f_1, f_2)\|_{X_R}^2, \]
\[ |h(f_1, f_2) - h(\tilde{f}_1, \tilde{f}_2)| \lesssim \delta \| (f_1, f_2) - (\tilde{f}_1, \tilde{f}_2) \|_{X_R} \]
for \((f_1, f_2), (\tilde{f}_1, \tilde{f}_2) \in B_\delta(0)\)

and
\[
\begin{cases}
\Box \psi - \psi^5 = 0 \\
\left( \begin{array}{c}
\psi(\cdot, 0) \\
\partial_t \psi(\cdot, 0)
\end{array} \right) = \left( \begin{array}{c}
\phi(\cdot, 1) \\
0
\end{array} \right) + \left( \begin{array}{c}
f_1(\cdot) + h(f_1, f_2)g(\cdot, 1) \\
f_2(\cdot)
\end{array} \right)
\end{cases}
\]
has a global solution. Moreover, this global solution admits the representation
\[ \psi(\cdot, t) = \phi(\cdot, a(\infty)) + v(\cdot, t) \]
where
\[ \|v(\cdot, t)\| \lesssim \delta(t)^{-1}; \]
\[ \| (v, \partial_t v) - (\tilde{v}, \partial_t \tilde{v}) \|_{H^1 \times L^2} \to 0 \]
as \(t \to \infty\) where \(\Box \tilde{v} = 0\) and \((\tilde{v}, \partial_t \tilde{v}) \in H^1 \times L^2\).

The radial assumption is more than a mere convenience. It assures that the bulk term remains at the origin. In other words, in case of nonradial perturbations we would need to allow \(\phi\) to move away from the origin, i.e., \(\phi(X(t, x), a(t))\) where \(X(x, t)\) preserves the Lorentz invariance. In the same spirit, recall that the eigenspace at zero is generated by the functions \(\nabla \phi\) (an expression of translation invariance), whereas the resonance at zero results from the dilation invariance. The latter is the only allowed symmetry in the radial case.

The map
\[ B_\delta(0) \to H^2_{rad} \times H^2_{rad} : (f_1, f_2) \mapsto (f_1(\cdot) + h(f_1, f_2)g(\cdot, 1), f_2) \]
traces out a (small) manifold \(\Sigma\). By our estimates, \(\Sigma\) can be written as a Lipschitz graph and \(\Sigma_0\) is the tangent plane to the desired manifold \(\Sigma\) (because of the quadratic bound on \(h\)).

For a general Schrödinger operator \(H = -\Delta + V\) in \(\mathbb{R}^3\) the spectrum changes under the scaling \(V \to \lambda^2 V(\lambda x)\) by multiplication by \(\lambda^2\). By inspection, \(\phi(\lambda x, a) = \lambda^{-2} \phi(x, \lambda^2 a)\). Hence, our linearized operators
\[ H(a) = -\Delta - 5\phi^4(\cdot, a) \]
have the property that the spectrum depends on \(a\) via scaling. In particular, this is true of \(k\) (and of course \(g(\cdot, a)\) depends on \(a\) in a similar fashion).

To prove the theorem, we make the ansatz \(\psi(t, x) = \phi(x, a(t)) + u(t, x)\) with some path that satisfies \(a(0) = 1\) (we will in fact also need that \(a(0) = 0\) as well as \(|a(t)| \lesssim \delta(t)^{-2}\)). Then we plug this ansatz into our NLW which yields
\[ \partial_t u - \Delta u - 5\phi^4(\cdot, a(t))u(t) = -\partial_t \phi(\cdot, a(t)) + N(\phi(\cdot, a(t)), u) \]
with the nonlinearity
\[ N(u, \phi) = 10u^2 \phi^3 + 10u^3 \phi^2 + 5u^4 \phi + u^5 \]
Let us define $V(a) = -5 \phi^4(\cdot, a)$. We want to eliminate the time dependence in $V$, so we look at $V(a(\infty))$ and define
\[ \Box_{V(a(\infty))} u = \partial_t u - \Delta u - 5 \phi(\cdot, a(\infty)) u. \]
Thus, the linearized equation becomes
\[ \Box_{V(a(\infty))} u = -\partial_t \phi(\cdot, a(t)) + N(\phi(\cdot, a(t)), u) + (V(\cdot, a(\infty)) - V(\cdot, a(t))) u =: (*) \]
Before we say more on how to analyze this equation, let us look at the Schrödinger operator $H(a) = -\Delta + V(\cdot, a)$. The following relations were already noted above:
\[
H(a) \partial_a \phi = 0 \text{ from dilation invariance} \\
H(a) \nabla \phi = 0 \text{ from translation invariance.}
\]
Since $\partial_a \phi \notin L^2$ we have a resonance at the point 0 of the spectrum.

Let us discuss the spectrum of the above operator (on the radial subspace). First of all, since $V(a)$ decays like $|x|^{-4}$, a result of Agmon and Kato implies that there is no singular continuous spectrum as well as no imbedded eigenvalues in the continuous spectrum. We also know that the absolutely continuous spectrum is the same as that of $-\Delta$, namely $[0, \infty)$. Hence, we can decompose $L^2_{rad} = L^2_{pp} \oplus L^2_{ac}$. The pure-point part $L^2_{pp}$ is a one-dimensional space consisting of span$\{g\}$. Since $g$ is a ground-state solution, it decays exponentially. This decomposition is blind to the resonance. In particular, there is no $L^2$ projection associated with the resonance function. We shall say more about the characterization of resonances in terms of the perturbed resolvent in the next lecture. The basic principle is as follows: Look at $(H(a) - z^2)^{-1}$. As $z \to 0$ in $\text{Im } z > 0$, this can either approach a bounded or an unbounded operator in a suitable weighted $L^2$ topology. Due to the work of Jensen and Kato from 1978, we know that this operator can be Laurent expanded in the form
\[
(H(a) - z^2)^{-1} = \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + O(1).
\]
Clearly, $-c_2$ must equal the orthogonal projection onto the zero energy eigenspace, whereas $c_{-1}$ has contributions from both the resonance and the eigenfunctions at zero. For now, let us not concern ourselves with respect to which topology this is valid. Let us rather note that in our application we not only eliminate $c_{-2}$ but also that $c_{-1}$ is solely due to the zero energy resonance (recall the restriction to the radial subspace).

We now return to a more detailed description of the main ansatz. Rewrite the linearized equation as a Hamiltonian system:
\[
\partial_t U = J H_{\infty} U + \begin{pmatrix} 0 \\ (*) \end{pmatrix},
\]
where
\[
U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
\[
H_{\infty} = \begin{pmatrix} H(a(\infty)) & 0 \\ 0 & 1 \end{pmatrix}.
\]
It is of course necessary to understand the propagator $e^{tJH_\infty}$, and therefore also the spectrum of the operator $JH_\infty$. To this end, note that if
\[
JH_\infty \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = E \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},
\]
then $H(a(\infty))f_1 = -E^2 f_1$.

The conclusion here is that $\text{spec}(JH_\infty) = \sqrt{-\text{spec}H(a(\infty))} = i\mathbb{R} \cup \{ \pm k \}$. Moreover, both $\pm k$ are simple eigenvalues with eigenfunctions $G_{\pm}$.

Define $P_-$ and $P_+$ to be the Riesz projections associated with the eigenvalues $-k$ and $k$, respectively. This can be done in the standard way by taking contour integrals of the resolvent. More importantly, they turn out to be very explicit rank one operators with kernels defined by the ground state function and eigenvalue. Decompose the full solution $U(\cdot, t)$ as
\[
U(\cdot, t) = n_+(t)G_+ + n_-(t)G_- + \tilde{U}(\cdot, t),
\]
where $(I - P_+ - P_-)U = \tilde{U}$. Then, we get the following hyperbolic system of ODEs in the plane (with $k = k(\cdot, a(\infty))$):
\[
\begin{pmatrix} \dot{n}_+(t) \\ \dot{n}_-(t) \end{pmatrix} + \begin{pmatrix} -k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} n_+(t) \\ n_-(t) \end{pmatrix} = \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \in L^\infty_t(0, \infty).
\]

Now let us find the unique line in the plane with the property that initial data chosen on it lead to bounded solutions:
\[
n_+(t) = e^{tk}n_+(0) + \int_0^t e^{k(t-s)}F_+(s) \, dx
\]
\[
e^{-tk}n_+(t) \to 0 = n_+(0) + \int_0^\infty e^{-ks}F_+(s) \, ds
\]
\[
\Rightarrow n_+(t) = -\int_t^\infty e^{k(t-s)}F_+(s) \, ds.
\]

It follows from the final relation that $n_+(t)$ inherits the decay of $F_+(t)$ (e.g. any decay like $t^{-\beta}$ with $\beta > 0$) provided $n_+(0)$ is chosen as in (1). Thus, (1) is precisely the stability condition that we use to define the (unique) correction to the initial data which leads to bounded $U$ at each step of the Banach iteration. Summing up all these corrections then yields the function $h(f_1, f_2)$. Next, by the explicit form of the projections $P_\pm$ one checks that
\[
\tilde{U} = \begin{pmatrix} \tilde{u} \\ \partial_t \tilde{u} \end{pmatrix}
\]
where $\tilde{u} = P_{g(\cdot, a(\infty))}u$. Hence, we can go back from the Hamiltonian system to the wave equation
\[
\square V(\cdot, a(\infty))\tilde{u} = P_{g(\cdot, a(\infty))}(\ast),
\]
\[
\tilde{u}(0) = P_{g(\cdot, a(\infty))}(f_1 + h(f_1, f_2)g(\cdot, 1)),
\]
\[
\partial_t \tilde{u}(0) = P_{g(\cdot, a(\infty))}f_2
\]
where $(\ast)$ stands for the right-hand side of the linearized equation, see above. As a starting point for bounding the solutions of this wave-equation, we show that for
some constant $c_0 \neq 0$

$$
(2) \quad \frac{\sin \left( t \sqrt{\frac{H(a(\infty))}{H(a(\infty))}} \right)}{\sqrt{H(a(\infty))}} \mathcal{P}_{g^1, a(\infty)} f = c_0 \partial_a \phi(\cdot, a(\infty)) \otimes \partial_a \phi(\cdot, a(\infty)) f + S_\infty(t) f,
$$

where the first operator is due to the resonance (and therefore must be rank one) and $\|S_\infty(t) f\|_\infty \lesssim \|t\|^{-1} \|f\|_{W^{1,1}}$. If we choose $f$ to be a Schwartz function, say, then the first operator here is well-defined (despite the fact that $\partial_a \phi \not\in L^2$) and it is given explicitly as

$$
c_0 \partial_a \phi(x, a) \int \partial_a \phi(y, a) f(y) \, dy.
$$

Let us give a naive reason for the constant term which the resonance produces in the sine-evolution. First, the sine-evolution operator acting on $\nabla \phi$ (a zero eigenfunction) would give a growth rate of $t$ (just Taylor-expand). The dispersive part decays like $t^{-1}$. Naively, the resonance should be half-way in between and grow something like $t^0$. This is actually the case. In our proof, we show also that the cosine evolution operator is dispersive.

We continue our discussion of the stable manifold theorem for NLW. As mentioned previously, we make the ansatz

$$
(3) \quad \psi(t, x, t) = \phi(t, a(t)) + u(x, t)
$$

The theorem is proved by solving for $a$ and $u$ globally in time in such a way that $u$ disperses and $a(t)$ converges to a limit which is $\delta$-close to $a(0) = 1$. Note, however, that we cannot prove energy scattering for $u$ to a free wave, which explains why we stated the scattering for a different decomposition of $u$. To understand this better, one checks from the Duhamel formula\footnote{This was pointed out by Herbert Koch} for the nonlinear solution $\psi$ that for any $t_2 > t_1 > 0$

$$
\psi(t, t_2) - \psi(t, t_1) \in L^2
$$

Since

$$
\psi(t, t_2) - \psi(t, t_1) = \phi(t, t_2) - \phi(t, t_1) + u(t, t_2) - u(t, t_1)
$$

and $\phi(t, t_2) - \phi(t, t_1) \not\in L^2$ we see that

$$
u(t, t_2) - u(t, t_1) \not\in L^2
$$

On the other hand, if $u$ were to scatter to a free wave, then $\partial_t u \in L^2$ and consequently $u(t, t_2) - u(t, t_1) \in L^2$. The truth is that $u$ scatters to a free energy solution up to a small, decaying piece which has the same behavior at spatial infinity as $\partial_a \phi(\cdot, a)$. In fact, our proof shows that

$$
(u, \partial_t u) = (\tilde{u}, \partial_t \tilde{u}) + (0, -\tilde{u}(t, a) \phi(\cdot, a(t))) + o_{H^1 \times L^2}(1)
$$

as $t \to \infty$ where $(\tilde{u}, \partial_t \tilde{u})$ is a free wave with $(\tilde{u}, \partial_t \tilde{u})(0) \in H^1 \times L^2$. Consequently, if we set

$$
v(t, t) = u(t, t) + \phi(t, a(t)) - \phi(t, a(\infty))
$$

then $v$ scatters and

$$
\psi(t, t) = \phi(t, a(\infty)) + v(t, t)
$$

as claimed in our theorem. The ansatz modifies the profile of $\phi$ at spatial infinity instantaneously. While this is convenient technically, it may be less so physically, since perturbations should propagate at finite speed. The fact that we modify $\phi$
outside of the light-cone of course forces us to absorb the change in that region by means of $u$. This is exactly why $u$ does not scatter. It is natural to ask for an ansatz which does modify the profile of $\phi$ only inside the light-cone. Thus, the question is as follows: can we write the stable solutions as

$$\psi(\cdot, t) = \phi(\cdot, a(t))\chi_1(\cdot, t) + \phi(\cdot, 1)\chi_2(\cdot, t) + u(\cdot, t)$$

where $\chi_1$ is a cut-off inside the light cone and $\chi_2$ is a cut-off outside the light-cone, and $u$ disperses and scatters in the energy space? It remains to be seen if our proof can accommodate this ansatz.

One of our goals for this lecture is to explain how to obtain the ODE for $a(t)$. This hinges on the linear estimates that we derived for the wave equation with a potential. During the past twelve years there has been much activity on dispersive and Strichartz estimates for the wave equation with a potential. Specifically, we would like to mention the work by Beals, Strauss, Cuccagna, Georgiev, d’Ancona, Pierfelice, Visciglia, and Yajima. However, these authors either assume that the potential is positive, or small, or that zero energy is neither an eigenvalue nor a resonance. In addition, varying degrees of smoothness and decay are assumed. Since we are dealing with a negative potential that leads to a zero energy resonance, none of these results apply to our problem.

However, for the Schrödinger equation with a potential there has been recent work of a similar flavor to what is needed here. Let us mention the following result from 2004 (there is a matrix version of this result from 2005 which is relevant to the linearized NLS, see above).

**THEOREM 5**
(Erdogan-Schlag)
For any Hamiltonian $H = -\Delta + V$ with a real-valued potential $V(x)$ decaying like $\langle x \rangle^{-\beta}$, $\beta > 12$, we have

$$\| (e^{itH}P_c - t^{-\frac{1}{2}}F_t)f \|_\infty \lesssim t^{-\frac{3}{2}} \| f \|_1,$$

where $F_t$ is an operator for which $\sup_t \| F_t \|_{L^1 \to L^\infty} < \infty$.

Furthermore, if zero is not an eigenvalue but only a resonance, then $F_t$ is rank one, and one can give an explicit expression for $F_t$. Independently, Yajima proved a similar result which required less decay of $V$ and he gave an explicit expression for $F_t$ in all cases.

As mentioned before, a unifying approach to resonances and/or eigenvalues at zero energy is given by the Laurent expansion of the perturbed resolvent around zero. To this end, we again recall the Jensen, Kato formula

$$(-\Delta + V - z^2)^{-1} = \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + \ldots,$$

where we can incorporate the non-singular terms into $O(1)$ relative to a suitable weighted $L^2$ operator norm. To understand the relevance of this expansion, let us consider the free case $V = 0$ in dimensions one and three.
Example 1
Let \( d = 1 \) and \( V = 0 \). We then have, for all \( \text{Im} \ z > 0 \),
\[
\left(-\frac{d^2}{dx^2} - z^2\right)^{-1}(x,y) = \frac{e^{iz|x-y|}}{2i z} - \frac{1}{2iz} + \frac{e^{i|z-x-y|} - 1}{2iz} = \frac{c-1}{z} + O(1).
\]
Hence, \( c-1 = \frac{1}{2} 1 \otimes 1 \). This tells us that 1 is a resonance function. Indeed, it is clear that \( y(x) = 1 \) is a solution of \( -y''(x) = 0 \). More generally, we say that zero is a resonance of \( -\frac{d^2}{dx^2} + V \) provided there is a bounded, nonzero, solution of
\[
\left(-\frac{d^2}{dx^2} + V\right)f = 0
\]
The reason for \( L^\infty \) lies with the mapping properties of the free resolvent. Also, if \( V \) decays sufficiently fast, then zero energy is never an eigenvalue (this is specific to \( d = 1 \)).

Example 2
For \( d = 3 \) and \( V = 0 \), we have for \( \text{Im} \ z > 0 \)
\[
\left(-\Delta - z^2\right)^{-1}(x,y) = \frac{e^{iz|x-y|}}{4\pi|x-y|}.
\]
Hence, we see that there is no singular term in \( z \) and hence no resonance at 0. As in the case \( d = 1 \) there is a characterization of resonances as solutions of \( Hf = 0 \).
Indeed, we say that zero is a resonance iff there is \( f \in L^{2,-\sigma} \setminus L^2 \) for \( \sigma > \frac{1}{2} \)
\( (-\Delta + V)f = 0 \). In particular, \( f(x) = 1 \) is not a resonance in \( d = 3 \). To understand this definition better, let us write \( (-\Delta + V)f = 0 \) equivalently as \( f + (-\Delta)^{-1}Vf = 0 \).
Explicitly,
\[
f(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} V(y) f(y) dy = -\frac{1}{4\pi} \int_{\mathbb{R}^3} V(y) f(y) dy + O(\langle x\rangle^{-2}).
\]
This shows us two things:
- Since \( |x|^{-1} : L^{2,\alpha_1} \to L^{2,-\alpha_2}(\mathbb{R}^3) \) as convolution operator provided \( \alpha_1, \alpha_2 > 1/2 \) and \( \alpha_1 + \alpha_2 > 1 \), we see that necessarily
  \( f \in L^{2,-\sigma}(\mathbb{R}^3) \) \( \forall \sigma > \frac{1}{2} \)
  at least if \( |V(x)| \lesssim \langle x \rangle^{-2-\epsilon} \). This explains the definition of a resonance function.
- In fact, in that case \( f \) decays like \( |x|^{-1} \) at infinity iff \( \int Vf(y) dy \neq 0 \) (the resonant case), whereas \( f \) is an eigenfunction decaying like \( |x|^{-2} \) at infinity iff \( \int Vf(y) dy = 0 \).

It is of course important to understand how to obtain Laurent expansions of the perturbed resolvent. This was done by Jensen and Kato around 1978, but a more transparent approach was recently found by Jensen and Nenciu. Let us state the main lemma from their paper.
LEMMA 3
(Jensen-Nenciu - 2001)
Let $H$ be a Hilbert space, and suppose that bounded operators

$$A(z) = A_0 + zA_1(z) : H \to H$$

are given for all $z \in F \subset \mathbb{C} \setminus \{0\}$. In addition, assume that $A_1(z)$ is uniformly bounded in $z$, and that $A_0^* = A_0$. Suppose that zero is an isolated point of the spectrum of $A_0$ and let

$$S = \text{Proj}(\text{Ker}A_0)$$

be of finite, positive rank. Then $(A_0 + S)^{-1}$ exists, and thus $(A(z) + S)^{-1}$ also exists for small $z$. Define

$$B(z) = \frac{1}{z}(S - S(A(z) + S)^{-1}S)$$

Then $A(z)$ is invertible iff $B(z)$ is invertible on $SL^2$, and in that case we have

$$A(z)^{-1} = (A(z) + S)^{-1} + \frac{1}{z}(A(z) + S)^{-1}SB(z)^{-1}S(A(z) + S)^{-1}.$$

Note that $B(z)$ is uniformly bounded in the operator norm as $z \to 0$. Indeed,

$$S - S(A_0 + S)^{-1}S = 0$$

due to the self-adjointness of $A_0$.

The connection between this lemma and resolvent expansions is furnished by the symmetric resolvent identity: Let $V = v^2U$ were $U = \text{sgn}V$. Then

$$R_V(z) := (-\Delta + V - z^2)^{-1} = R_0(z) - R_0(z)v(U + vR_0(z)v)^{-1}vR_0(z)$$

for all $\text{Im} \; z > 0$. Let us set

$$A(z) = U + vR_0(z)v = U + vR_0(0)v + z\frac{v(R_0(z) - R_0(0)v)}{z} =: A_0 + zA_1(z)$$

If $A_0$ is invertible, then the Laurent expansion of $R_V(z)$ around $z = 0$ has no singular terms. The converse is also true. Thus zero energy is neither an eigenvalue nor a resonance iff $A_0$ is invertible. Using Fredholm theory (and the compactness of $vR_0(0)v$ on $L^2$), we conclude that $A_0$ is invertible iff there is no nonzero $L^2$ solution of

$$(U + v(-\Delta)^{-1}v)f = 0$$

This in turn is equivalent to the existence of certain solutions $h$ to

$$(-\Delta + V)h = 0$$

Indeed, formally set $h = (-\Delta)^{-1}vf$.

In our application to NLW, $A_0$ is therefore not invertible. We apply the Jensen, Nenciu lemma and check that the $B(z)$ is invertible. Since $S$ is rank-one in this case, this invertibility is equivalent with the nonvanishing of a number. Not surprisingly, this number exactly turns out to be $\int V(y)\partial_\alpha \phi(y,a)dy \neq 0$, see above. Hence, the Jensen, Nenciu lemma allows us to perform the Laurent expansion with relatively explicit coefficients. This latter feature is most relevant for understanding the evolution operators of the perturbed wave equation. Indeed, by the spectral calculus,

$$\sin(t\sqrt{H}) P_e = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\sin(t\lambda)}{\lambda} R_V(\lambda)\lambda d\lambda.$$
In order to derive dispersive estimates we follow the usual path of splitting the integration here into $|\lambda| > \epsilon$ and $|\lambda| < \epsilon$ for some $\epsilon$ small enough. It is common knowledge that dispersive estimates for perturbed operators belong to the realm of a zero energy theory. The point here is that energies different from zero cannot destroy the free rate of decay. Only zero energy can do that. Nevertheless, the regime $|\lambda| > \epsilon$ is non-trivial in particular with respect to the amount of decay of $V$ required to make this work. Recall that we have exactly $|x|^{-4}$ decay in NLW. To appreciate the difficulty of lowering the decay requirements on $V$ see Goldberg’s paper [Gol]. At any rate, in this regime we work with the (finite) Born expansion

$$R_V = R_0 - R_0 V R_0 + R_0 V R_0 V R_0 \pm \cdots \pm R_0 V \cdots V R_0 V \cdots V R_0$$

Plugging the terms here not involving $R_V$ into the right-hand side of (4) leads to completely explicit expressions for the kernels due to the explicit nature of $R_3$ in $\mathbb{R}^3$. It is therefore possible, with some work, to obtain the desired dispersive bounds on those kernels. For the remainder term, which still involves $R_V$, we rely on the so-called limiting absorption principle due to Agmon. This refers to the fact that the (perturbed) resolvent remains bounded on weighted $L^2$ spaces. For further details on not small energies, we refer the reader to our paper.

For energies $|\lambda| < \epsilon$, we expand the resolvent $R_V$ as in the Jensen, Nenciu lemma. The singular term ultimately leads to the non-decaying part $c_0 \partial_a \phi(\cdot, a) \otimes \partial_a \phi(\cdot, a)$ whereas we show in our paper that the nonsingular terms yield a dispersive operator $\mathcal{S}(t)$, see (2) above. Dispersion here refers to bounds in $L^\infty$. It is an open question whether or not $\mathcal{S}(t)$ satisfies Strichartz estimates. For this see the author’s work on Littlewood-Paley theory and the distorted Fourier transform.

To conclude these lectures, let us turn to the question of how to determine the ODE for $a(t)$. To this end let us recall that we write the solution vector in the form

$$\begin{pmatrix} u(\cdot, t) \\ \partial_t u(\cdot, t) \end{pmatrix} = n_+(t)G_+ + n_-(t)G_- + \tilde{u}(\cdot, t)$$

As explained before, generally speaking $n_+$ is an exponentially unstable mode. It needs to be stabilized by means of the stability condition (1). This is the origin of the codimension one condition. On the other hand, $n_-$ is exponentially decaying. Finally, the remainder $\tilde{u}$ satisfies the equation

$$\begin{aligned}
\partial_t \tilde{u} + H(a(\infty))\tilde{u} &= 0 \\
P_{g_n}[-\partial_t \phi(\cdot, a(t)) + (V(\cdot, a(\infty)) - V(\cdot, a(t)))u(\cdot, t) + N[u(\cdot, t), \phi(\cdot, a(t))]]
\end{aligned}$$

It is our goal to show that $\tilde{u}$ (and therefore also $u$) disperses, but this is non-obtus because of the non-dispersive term in (2). This is precisely where $a(t)$ comes in, the point being that we will collect all non-decaying terms and observe that they are all multiples of $\psi := \partial_a \phi(\cdot, a(\infty))$. To give a flavor of this, let us consider the contribution of the first term in (2) to the Duhamel formula. It is (writing $H$ instead of $H(a(\infty))$ for simplicity)

$$- \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_{g_n} \partial_s \hat{a}(s) \partial_a \phi(\cdot, a(s)) \, ds$$
We could use the representation \( \Box \) here. However, this is very misguided since it is impossible to let
\[
c_0 \psi \otimes \psi
\]
act on \( \partial_\alpha \phi(\cdot, a(s)) \) (because of \( \partial_\alpha \phi \not\in L^2(\mathbb{R}^3) \)). Instead, we proceed as follows:
\[
- \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} P_{g=\infty} \partial_\alpha \phi(\cdot, a(s)) ds = \hat{a}(0) \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_{g=\infty} \partial_\alpha \phi(\cdot, a(0)) + \int_0^t \hat{a}(s) \cos((t-s)\sqrt{H}) P_{g=\infty} \partial_\alpha \phi(\cdot, a(s)) ds
\]
(6)
Now observe that
\[
\frac{\sin(t\sqrt{H})}{\sqrt{H}} P_{g=\infty} \psi = t\psi, \quad \cos(t\sqrt{H})\psi = \psi
\]
Using these relations and the fact that (for \( a_1, a_2 \in (1/2, 2) \))
\[
\partial_\alpha \phi(\cdot, a_1) = (a_2/a_1)^2 \partial_\alpha \phi(\cdot, a_2) + O(|a_1 - a_2| x^{-3})
\]
we reduce \( \partial_\alpha \phi(\cdot, a(s)) \) to \( \psi \) by means of the scalar factor \( (a(\infty)/a(s))^{1/2} \). This shows that we need \( \hat{a}(0) = 0 \), and also allows us to pull out \( \psi \) from the integral above with a factor depending on \( a(t) \). The remainder which comes from the \( O(|x|^{-3}) \)-term turns out to be dispersive. We refer the reader to our paper for further details.

References

[Agm] Agmon, S. Spectral properties of Schrödinger operators and scattering theory. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), no. 2, 151–218.
[Bea] Beals, M. Optimal \( L^\infty \) decay for solutions to the wave equation with a potential. Comm. Partial Differential Equations 19 (1994), no. 7-8, 1319–1369.
[BeaStr] Beals, M., Strauss, W. \( L^p \) estimates for the wave equation with a potential. Comm. Partial Differential Equations 18 (1993), no. 7-8, 1365–1397.
[BerCaz] Berestycki, H., Cazenave, T. Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires. (French. English summary) [Instability of stationary states in nonlinear Schrödinger and Klein-Gordon equations] C. R. Acad. Sci. Paris Sér. I Math. 293 (1981), no. 9, 489–492.
[BerLio] Berestycki, H., Lions, P. L. Existence d’ondes solitaires dans les problèmes nonlinéaires du type Klein-Gordon. C.R. Acad. Sci. 288 (1979), no. 7, 395–398.
[BizChmTab] Bizoń, P., Chmaj, T., Tabor, Z. On blowup for semilinear wave equations with a focusing nonlinearity. Nonlinearity 17 (2004), no. 6, 2187–2201.
[BusPer1] Buslaev, V. S., Perelman, G. S. Scattering for the nonlinear Schrödinger equation: states that are close to a soliton. (Russian) Algebra i Analiz 4 (1992), no. 6, 63–102; translation in St. Petersburg Math. J. 4 (1993), no. 6, 1111–1142.
[BusPer2] Buslaev, V. S., Perelman, G. S. On the stability of solitary waves for nonlinear Schrödinger equations. Nonlinear evolution equations, 75–98, Amer. Math. Soc. Transl. Ser. 2, 164, Amer. Math. Soc., Providence, RI, 1995.
[BusPer3] Buslaev, V. S., Perelman, G. S. Nonlinear scattering: states that are close to a soliton. (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 200 (1992), Kraev. Zadachi Mat. Fiz. Smezh. Voprosy Teor. Funktsii. 24, 38–50, 70, 187; translation in J. Math. Sci. 77 (1995), no. 3, 3161–3169.
[CazLio] Cazenave, T., Lions, P.-L. em Orbital stability of standing waves for some nonlinear Schrödinger equations. Comm. Math. Phys. 85 (1982), 549–561.
[Cof] Coffman, C. V. Uniqueness of positive solutions of \( \Delta u - u + u^3 = 0 \) and a variational characterization of other solutions. Arch. Rat. Mech. Anal. 46 (1972), 81–95.
[Cuc] Cuccagna, S. On the wave equation with a potential. Comm. Partial Differential Equations 25 (2000), no. 7-8, 1549–1565.
[DemSch] Demanet, L., Schlag, W. Numerical verification of a gap condition for linearized NLS, preprint 2005.

[ErdSch1] Erdoğan, M. B., Schlag, W. Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: I, Dynamics of PDE, vol. 1, no. 4 (2004), 359–379.

[ErdSch2] Erdoğan, M. B., Schlag, W. Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: II, preprint 2005.

[FroGusJonSig] Fröhlich, J., Gustafson, S., Jonsson, B. L. G., Sigal, I. M. Solitary wave dynamics in an external potential. Comm. Math. Phys. 250 (2004), no. 3, 613–642.

[FroTsaYau] Fröhlich, J., Tsai, T. P., Yau, H. T. On the point-particle (Newtonian) limit of the non-linear Hartree equation. Comm. Math. Phys. 225 (2002), no. 2, 223–274.

[GeoVis] Georgiev, V., Visciglia, N. Decay estimates for the wave equation with potential. Comm. Partial Differential Equations 28 (2003), no. 7-8, 1325–1369.

[Gol] Goldberg, M. Dispersive bounds for the three-dimensional Schrödinger equation with almost critical potential, preprint 2004, to appear in GAFA.

[GolSch] Goldberg, M., Schlag, W. Dispersive estimates for Schrödinger operators in dimensions one and three. Comm. Math. Phys. 251 (2004), no. 1, 157–178.

[Grif] Grillakis, M. Analysis of the linearization around a critical point of an infinite dimensional Hamiltonian system. Comm. Pure Appl. Math. 41 (1988), no. 6, 747–774.

[GrShaStr1] Grillakis, M., Shatah, J., Strauss, W. Stability theory of solitary waves in the presence of symmetry. I. J. Funct. Anal. 74 (1987), no. 1, 160–197.

[GrShaStr2] Grillakis, M., Shatah, J., Strauss, W. Stability theory of solitary waves in the presence of symmetry. II. J. Funct. Anal. 94 (1990), 308–348.

[JenKat] Jensen, A., Kato, T. Spectral properties of Schrödinger operators and time-decay of the wave functions. Duke Math. J. 46 (1979), no. 3, 583–611.

[JenNon] Jensen, A., Nenciu, G. A unified approach to resolvent expansions at thresholds. Rev. Math. Phys. 13 (2001), no. 6, 717–754.

[KriSch1] Krieger, J., Schlag, W. Stable manifolds for all monic supercritical NLS in one dimension, preprint 2005.

[KriSch2] Krieger, J., Schlag, W. Non-generic blow-up solutions for the critical focusing NLS in 1-d, preprint 2005.

[KriSch3] Krieger, J., Schlag, W. On the focusing critical semi-linear wave equation, preprint 2005.

[Kwo] Kwong, M. K. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. Arch. Rat. Mech. Anal. 65 (1979), 243–266.

[Lev] Levine, H. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + F(u)$. Trans. Amer. Math. Soc. 192 (1974), 1–21.

[McLSer] McLeod, K., Serrin, J. Nonlinear Schrödinger equation. Uniformity of positive solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$. Arch. Rat. Mech. Anal. 99 (1987), 115–145.

[MerZaa1] Merle, F., Zaag, H. On growth rate near the blowup surface for semilinear wave equations. Int. Math. Res. Not. 2005, no. 19, 1127–1155.

[MerZaa2] Merle, F., Zaag, H. Determination of the blow-up rate for a critical semilinear wave equation. Math. Ann. 331 (2005), no. 2, 395–416.

[MerZaa3] Merle, F., Zaag, H. Determination of the blow-up rate for the semilinear wave equation. Amer. J. Math. 125 (2003), no. 5, 1147–1164.

[Per1] Perelman, G. Some Results on the Scattering of Weakly Interacting Solitons for Nonlinear Schrödinger Equations in “Spectral theory, microlocal analysis, singular manifolds”, Akad. Verlag (1997), 78–137.

[Per2] Perelman, G. On the formation of singularities in solutions of the critical nonlinear Schrödinger equation. Ann. Henri Poincaré 2 (2001), no. 4, 605–673.

[Per3] Perelman, G. Asymptotic stability of multi-soliton solutions for nonlinear Schrödinger equations. Comm. Partial Differential Equations 29 (2004), no. 7-8, 1051–1095.

[Pie] Pierfelice, V. Decay estimate for the wave equation with a small potential. preprint 2003, to appear.

[RodSch] Rodnianski, I., Schlag, W. Time decay for solutions of Schrödinger equations with rough and time-dependent potentials. Invent. Math. 155 (2004), 451–513.
[RodSchSof1] Rodnianski, I., Schlag, W., Soffer, A. Dispersive Analysis of Charge Transfer Models, Comm. Pure Appl. Math. 58 (2005), no. 2, 149–216.

[RodSchSof2] Rodnianski, I., Schlag, W., Soffer, A. Asymptotic stability of N-soliton states of NLS, preprint 2003

[Sch1] Schlag, W. Dispersive estimates for Schrödinger operators in dimension two, Comm. Math. Phys. 257 (2005), 87–117.

[Sch2] Schlag, W. Stable manifolds for an orbitally unstable NLS, preprint 2004.

[Sch3] Schlag, W. A remark on Littlewood-Paley theory for the distorted Fourier transform, preprint 2005.

[Sha] Shatah, J. Stable standing waves of nonlinear Klein-Gordon equations. Comm. Math. Phys. 91 (1983), no. 3, 313–327.

[ShaStr] Shatah, J., Strauss, W. Instability of nonlinear bound states. Comm. Math. Phys. 100 (1985), no. 2, 173–190.

[SofWei1] Soffer, A., Weinstein, M. Multichannel nonlinear scattering for nonintegrable equations. Comm. Math. Phys. 133 (1990), 119–146

[SofWei2] Soffer, A., Weinstein, M. Multichannel nonlinear scattering, II. The case of anisotropic potentials and data. J. Diff. Eq. 98 (1992), 376–390.

[Str1] Strauss, W. Nonlinear wave equations. CBMS Regional Conference Series in Mathematics, 73. AMS, Providence, RI, 1989.

[Str2] Strauss, W. Existence of solitary waves in higher dimensions. Comm. Math. Phys. 55 (1977), 149–162.

[SulSul] Sulem, C., Sulem, P.-L. The nonlinear Schrödinger equation. Self-focusing and wave collapse. Applied Mathematical Sciences, 139. Springer-Verlag, New York, 1999.

[Szp] Szpak, N. Relaxation to intermediate attractors in nonlinear wave equations, Theoretical and Mathematical Physics, Vol. 127 (2001), no. 3, pp. 817-826.

[TsaYau] Tsai, T.-P., Yau, H.-T. Stable directions for excited states of nonlinear Schrödinger equations. Comm. Partial Differential Equations 27 (2002), no. 11-12, 2363–2402.

[Wei1] Weinstein, M. I. Modulational stability of ground states of nonlinear Schrödinger equations. SIAM J. Math. Anal. 16 (1985), no. 3, 472–491.

[Wei2] Weinstein, M. I. Lyapunov stability of ground states of nonlinear dispersive evolution equations. Comm. Pure Appl. Math. 39 (1986), no. 1, 51–67.

[Yaj1] Yajima, K. The Wk,p-continuity of wave operators for Schrödinger operators. J. Math. Soc. Japan 47 (1995), no. 3, 551–581.

[Yaj2] Yajima, K. Dispersive estimate for Schrödinger equations with threshold resonance and eigenvalue, preprint 2004, to appear in Comm. Math. Phys.

Department of Mathematics, University of Chicago, 5734 South University Ave., Chicago, IL 60637, USA

E-mail address: schlag@math.uchicago.edu