Uniformity in $C^*$-algebras

by

Adam Wegert (Kraków)

Abstract. We introduce a notion of uniform structure on the set of all representations of a given separable, not necessarily commutative $C^*$-algebra $\mathfrak{A}$ by introducing a suitable family of metrics on the set of representations of $\mathfrak{A}$ and investigate its properties. We define the noncommutative analogue of the notion of the modulus of continuity of an element in a $C^*$-algebra and we establish its basic properties. We also deal with morphisms of $C^*$-algebras by defining two notions of uniform continuity and show their equivalence.

Introduction. The famous theorem of Gelfand and Naimark establishes a one-to-one correspondence between compact topological spaces and unital, commutative $C^*$-algebras. This correspondence is well behaved in the sense that it is in fact a natural equivalence between the categories of compact topological spaces and of unital, commutative $C^*$-algebras. Therefore various topological properties of spaces may be translated into algebraic properties of the corresponding $C^*$-algebra. The philosophy of noncommutative topology is to translate topological properties of spaces into the language of algebras and check whether the assumption of commutativity of a given algebra is necessary; if not, then one can state the definition in the context of noncommutative algebras and think that the underlying noncommutative space possesses the given topological property. In particular one can try to do this in the context of metric structure. This was first done by Connes in a more specific setting—see [7] and [8]—and led him to the notion of spectral triple which is a far-reaching generalisation of the notion of a Riemannian manifold. Inspired by his ideas, Rieffel [23, 24] introduced the notion of a compact quantum metric space, which was later generalised to the locally compact setting [18]. In these papers a suitable metric is defined on the space of states of a given $C^*$-algebra, resembling the Kantorovich formula in

2010 Mathematics Subject Classification: Primary 46L05.
Key words and phrases: modulus of continuity, irreducible representation, uniform continuity.

Received 6 August 2018; revised 23 February 2019.
Published online 16 December 2019.

DOI: 10.4064/sm180806-18-3
the theory of optimal transport [15]. Our aim is to investigate this noncommutative metric aspect; our approach is in the spirit of Rieffel but instead of working with the state space, we choose to work with representations. This allows us to define the notion of compact $C^*$-algebra. We show that every $n$-subhomogeneous, or more generally, shrinking (see Definition 2.3) $C^*$-algebra is compact. Although subhomogeneous $C^*$-algebras were already characterised in the sixties [28], recently another elegant description in terms of the topological category of m-towers was obtained in [20]. From that paper also comes the definition of a shrinking $C^*$-algebra, which admits analogous characterisations in terms of (solid, pointed) towers. One can hope that the class of all compact $C^*$-algebras will admit similar descriptions. Our approach is also flexible enough to provide a meaningful concept of uniform continuity in the noncommutative context (i.e. what it means for a $*$-homomorphism between $C^*$-algebras to be uniformly continuous).

1. Notation and terminology. In this section we recall standard definitions just to fix notation. All the vector spaces considered will be over the field $\mathbb{C}$ of complex numbers. $C^*$-algebras will be usually denoted by $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ etc. For an algebra $\mathfrak{A}$, $\mathfrak{Z}(\mathfrak{A})$ denotes its centre. When two objects $X, Y$ (in a given category) are isomorphic we write $X \cong Y$ (the category should be clear from the context). In the context of (unital) $C^*$-algebras, an isomorphism is understood as a (unital) bijective, $*$-preserving homomorphism (which is automatically isometric). For an element $x$ in a unital $C^*$-algebra we denote by $r(x)$ its spectral radius defined as $r(x) = \sup \{ |\lambda| : \lambda \in \sigma(x) \}$ where $\sigma(x)$ denotes the spectrum of $x$. We denote by $S(\mathfrak{A})$ the state space of $\mathfrak{A}$ and by $\mathcal{P}(\mathfrak{A})$ the subspace of pure states. $\mathcal{H}$ will stand for a (usually separable) Hilbert space, $\mathcal{B}(\mathcal{H})$ for the algebra of all bounded operators on $\mathcal{H}$ and $\mathcal{U}(\mathcal{H})$ for the group of unitary operators on $\mathcal{H}$. We let $I$ denote the identity operator—occasionally we will write $I_\mathcal{H}$ to indicate which space this operator acts on. For $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{K})$, we write $S \oplus T$ for the operator in $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ defined by $(S \oplus T)(x, y) := (Sx, Ty)$. For a family $(\mathfrak{A}_i)_{i \in \mathcal{I}}$ of $C^*$-algebras, $\prod_{i \in \mathcal{I}} \mathfrak{A}_i$ will denote the set

$$\left\{ (a_i)_{i \in \mathcal{I}} : a_i \in \mathfrak{A}_i \text{ for } i \in \mathcal{I}, \sup_{i \in \mathcal{I}} \|a_i\| < \infty \right\}.$$  

With the pointwise operations and the supremum norm, it becomes a $C^*$-algebra. If all $\mathfrak{A}_i$’s are unital, then so is $\prod_{i \in \mathcal{I}} \mathfrak{A}_i$. On the other hand, $\bigoplus_{i \in \mathcal{I}} \mathfrak{A}_i$ will denote the direct sum of the algebras $\mathfrak{A}_i$, i.e. the set of all $(a_i)_{i \in \mathcal{I}}$ vanishing at infinity (meaning that for each $\varepsilon > 0$ there is a finite set $\mathcal{I}_\varepsilon \subset \mathcal{I}$ such that $\|a_i\| < \varepsilon$ for all $i \in \mathcal{I} \setminus \mathcal{I}_\varepsilon$) with pointwise operations. For a nonunital $C^*$-algebra $\mathfrak{A}$ we denote by $\mathfrak{A}^+$ the unitization of $\mathfrak{A}$. Whenever we have a
short exact sequence of $C^*$-algebras of the form

\[(1.1) \quad 0 \to \mathcal{A}' \xrightarrow{\varphi} \mathcal{A} \xrightarrow{\psi} \mathcal{A}'' \to 0\]

we will say that $\mathcal{A}$ is an extension of $\mathcal{A}''$ by $\mathcal{A}'$. By a representation of a $C^*$-algebra (on a Hilbert space $\mathcal{H}$) we mean a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, and if $\mathcal{A}$ is unital we usually assume that $\pi(1) = I$. The space on which $\mathcal{A}$ acts via $\pi$ will occasionally be denoted by $\mathcal{H}_\pi$.

Given two representations $\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, we denote by $\pi_1 \oplus \pi_2$ their direct sum defined by $\pi_1 \oplus \pi_2 : \mathcal{A} \ni a \mapsto \pi_1(a) \oplus \pi_2(a) \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Similarly we define the direct sum $\bigoplus_{i \in I} \pi_i$ of an arbitrary family $\{\pi_i\}_{i \in I}$ of representations. In particular, if $\pi_i = \pi$ for all $i \in I$ and $|I| = \alpha$, then we denote $\bigoplus_{i \in I} \pi_i$ by $\alpha \odot \pi$. Occasionally we use the same notation for operators in Hilbert spaces. On the set of all representations of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ we can define point-norm convergence as follows: if $((\pi_\sigma)_\sigma)$ is a net of representations of $\mathcal{A}$ on $\mathcal{H}$ we declare that $\pi_\sigma \to \pi$ in the point-norm topology if $\pi_\sigma(a) \xrightarrow{\|\cdot\|} \pi(a)$ for any $a \in \mathcal{A}$. In other words, it is the topology generated by the family of mappings $\pi \mapsto \|\pi(a)\|$, $a \in \mathcal{A}$. We also consider the compact-open topology: given a net $((\pi_\sigma)_\sigma)$ of representations, $\pi_\sigma \to \pi$ in the compact-open topology if $\max_{a \in L} \|\pi_\sigma(a) - \pi(a)\| \to 0$ for any compact set $L \subset \mathcal{A}$. In other words, this is the topology of convergence on compacta. For any two unital $C^*$-algebras $\mathcal{A}$, $\mathcal{B}$ we denote by $\text{Hom}(\mathcal{A}, \mathcal{B})$ the set of all unital $*$-homomorphisms $\mathcal{A} \to \mathcal{B}$.

2. Preliminaries. A well known result of Gelfand and Najmark establishes a one-to-one correspondence between unital commutative $C^*$-algebras and compact (Hausdorff) topological spaces. In the case of nonunital algebras one has to deal with spaces which are locally compact. This correspondence is in fact functorial and allows one to translate topological properties into algebraic language and vice versa. In the table below we gather some basic correspondences between topological and algebraic notions:

| Topology       | Algebra                |
|----------------|------------------------|
| Point          | Character              |
| Closed set     | Ideal                  |
| Embedding      | Epimorphism            |
| (Continuous) Surjection | Monomorphism        |
| Homeomorphism  | Automorphism            |
| Disjoint sum   | Direct sum              |
| Cartesian product | Tensor product       |
| Connectedness  | Lack of nontrivial projections |
| Probability measure | State               |
This is the reason to think about general (not necessarily commutative) $C^*$-algebras as noncommutative topological spaces. This philosophy is a part of the much more general program called noncommutative geometry (see e.g. \cite{8, 12, 16} and also \cite{13, 14, 27} for the noncommutative analogue of measure theory, i.e. von Neumann algebras). For our purposes we recall one more correspondence in the above spirit:

**Theorem 2.1.** Let $X$ be a compact Hausdorff space. Then the following conditions are equivalent:

1. the algebra $C(X)$ is separable;
2. $X$ is metrisable.

For a general $C^*$-algebra the classical spectrum $\hat{\mathfrak{A}}$ (the set of nonzero characters) may be empty. There are several natural candidates for a generalisation of $\hat{\mathfrak{A}}$ for arbitrary $\mathfrak{A}$. One candidate is the space of pure states on $\mathfrak{A}$ (such states exist in abundance). Another is the space of all primitive ideals of $\mathfrak{A}$ (i.e. kernels of irreducible representations), or finally the space of (unitary equivalence classes of) irreducible representations of $\mathfrak{A}$. However, for a generic $C^*$-algebra, the space of classes of irreducible representations has a poor topology (not even $T_0$). For this reason we choose to work with genuine representations (instead of unitary equivalence classes).

**Definition 2.2.** Let $n$ be a fixed positive integer. A $C^*$-algebra $\mathfrak{A}$ is called $n$-homogeneous if $\dim \mathcal{H}_\pi = n$ for every irreducible representation $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_\pi)$. If instead of this equality we have the inequality $\dim \mathcal{H}_\pi \leq n$ then $\mathfrak{A}$ is called $n$-subhomogeneous or $n$-SH for short. If there is some $n \in \mathbb{N}$ such that $\mathfrak{A}$ is $n$-homogeneous (resp. $n$-subhomogeneous) then $\mathfrak{A}$ is called homogeneous (resp. subhomogeneous, or SH for short).

Homogeneous $C^*$-algebras were characterised by Fell \cite{11} in 1961 and also by Tomiyama and Takesaki. The description uses fibre bundles; an alternative approach can be found in \cite{19}. On the other side, subhomogeneous $C^*$-algebras were characterised in 1966 in \cite{28}. An alternative approach in terms of the special category of proper towers was obtained in \cite{20}. From that paper also comes the following definition:

**Definition 2.3.** A $C^*$-algebra $\mathfrak{A}$ is called shrinking if it is residually finite-dimensional and satisfies the following condition: if $(\pi_n)_{n \in \mathbb{N}}$ is a sequence of finite-dimensional irreducible representations of $\mathfrak{A}$ with $\dim \mathcal{H}_{\pi_n} \to \infty$ then $\lim_{n \to \infty} \|\pi_n(a)\| \to 0$ for any $a \in \mathfrak{A}$.

Recall that a $C^*$-algebra $\mathfrak{A}$ is called residually finite-dimensional ($RFD$ for short) if the set of its finite-dimensional representations separates the points of $\mathfrak{A}$.
It follows from the GNS construction that each subhomogeneous $C^*$-algebra is residually finite-dimensional and directly from the definition each subhomogeneous $C^*$-algebra is shrinking; the converse is not true in general. However, it is true for unital $C^*$-algebras, since $\|\pi(1)\| = \|I_{\mathcal{H}_\pi}\| = 1$ for any irreducible representation $\pi$. Note also that any irreducible representation of a shrinking $C^*$-algebra is necessarily finite-dimensional.

2.1. Concave moduli of continuity. We propose a noncommutative analogue of modulus of continuity, for an element in a general $C^*$-algebra. First we make the following definition:

**Definition 2.4.** Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces. A function $\omega : [0, \infty) \to [0, \infty)$ is called a modulus of (uniform) continuity for $f : X \to Y$ if $\omega(0) = 0$, $\omega$ is continuous, nondecreasing, concave and

$$
d_Y(f(x_1), f(x_2)) \leq \omega(d_X(x_1, x_2)) \quad \text{for all } x_1, x_2 \in X.
$$

We set

$$
\Omega = \{ \omega : [0, \infty) \to [0, \infty) : \omega \text{ is continuous, concave, nondecreasing and } \omega(0) = 0 \}.
$$

A function $f : X \to \mathbb{R}$ admits a concave modulus of continuity iff $f$ is the uniform limit of Lipschitz functions. This holds in particular if $(X, d)$ is compact.

**Remark 2.5.** Concavity of $\omega$ is meant as satisfying the weak inequality

$$
t \omega(x) + (1 - t) \omega(y) \leq \omega(tx + (1 - t)y), \quad x, y > 0, \ t \in [0, 1];
$$

in particular we allow constant functions to be concave, so the zero function could serve as a modulus of continuity (for a constant function).

If $f \in C(X)$ is uniformly continuous then we can consider its minimal modulus of continuity $\omega_f \in \Omega$. Then the mapping $f \mapsto \omega_f$ has the following properties:

- $\omega_{f+g} \leq \omega_f + \omega_g$;
- $\omega_{cf} = |c| \omega_f$;
- $\omega_{-f} = \omega_f$;
- $\omega_{fg} \leq \|f\| \omega_g + \|g\| \omega_f$.

For further considerations in which we will define the analogue of modulus of continuity for an element in a (unital, separable) not necessarily commutative $C^*$-algebra we recall the following result (see [1]):

**Theorem 2.6 (Aronszajn–Panitchpakdi).** Let $f : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $f(0) = 0$. Then the following conditions are equivalent:
(1) there is a concave, nondecreasing, continuous function \( \omega : [0, \infty) \to [0, \infty) \) with \( \omega(0) = 0 \) satisfying \( f \leq \omega \);
(2) there is a subadditive, nondecreasing, continuous function \( \omega : [0, \infty) \to [0, \infty) \) with \( \omega(0) = 0 \) satisfying \( f \leq \omega \);
(3) \( \lim_{t \to 0^+} f(t) = 0 \) and \( \limsup_{t \to \infty} f(t)/t < \infty \).

**Corollary 2.7.** A function \( \omega \) as above exists if \( f \) is nondecreasing, bounded and \( f(0) = \lim_{t \to 0^+} f(t) \).

### 2.2. Ascoli theorem and some remarks about convergence.

Since we will investigate the notion of compactness in the context of \( C^* \)-algebras, we recall the classical Ascoli–Arzelà theorem:

**Theorem 2.8.** Let \( X, Y \) be metric spaces and assume that \( X \) is compact. Then a set \( K \subset C(X, Y) \) is relatively compact (in the uniform topology) if and only if \( K \) is equicontinuous and pointwise relatively compact.

Suppose that \( (X, d_X) \) and \( (Y, d_Y) \) are compact metric spaces. On the set \( C(X, Y) \) of all continuous mappings \( X \to Y \) we can consider the topology of pointwise convergence. Then for a net \( (u_s)_s \) in \( C(X, Y) \) we have \( u_s \to u \) pointwise if and only if for every function \( f \in C(Y) \) we have \( f \circ u_s \to f \circ u \) pointwise. Indeed, if \( u_s(x) \to u(x) \) for every \( x \in X \) then \( f(u_s(x)) \to f(u(x)) \) for every \( f \in C(Y) \). Conversely, assume that for every \( f \in C(Y) \) and every \( x \in X \) we have \( f(u_s(x)) \to f(u(x)) \) and suppose that for some \( x_0 \in X \) we have \( u_s(x_0) \to u(x_0) \). Passing to a subsequence (as we can, since \( Y \) is compact) we can assume that there is some \( y \in Y \), \( y \neq u(x_0) \), such that \( u_s(x_0) \to y \). But then for every \( f \in C(Y) \) we conclude \( f(u_s(x_0)) \to f(y) \) and \( f(u_s(x_0)) \to f(u(x_0)) \), hence \( f(u(x_0)) = f(y) \). Since continuous functions separate the points, we conclude that \( u(x_0) = y \), a contradiction.

On \( C(X, Y) \) we can also consider the topology of uniform convergence; then \( u_n \Rightarrow u \) (where \( \Rightarrow \) means uniform convergence) if and only if \( f \circ u_n \Rightarrow f \circ u \) for every \( f \in C(Y) \). Indeed, let \( u_n \Rightarrow u \) and \( f \in C(Y) \). Since \( Y \) is compact, \( f \) is uniformly continuous, so we can fix \( \varepsilon > 0 \) and find \( \delta > 0 \) such that for \( y_1, y_2 \in Y \) satisfying \( d_Y(y_1, y_2) < \delta \) we have \( |f(y_1) - f(y_2)| < \varepsilon \). Let \( s_0 \) be such that \( d_Y(u_s(x), u(x)) < \delta \) for \( s > s_0 \), uniformly with respect to \( x \); then for \( s > s_0 \),

\[
|f(u_s(x)) - f(u(x))| < \varepsilon
\]

uniformly with respect to \( x \). Thus \( f \circ u_n \Rightarrow f \circ u \). Conversely, assume that \( f \circ u_n \Rightarrow f \circ u \) for any \( f \in C(Y) \) but \( u_n \not\Rightarrow u \). Then (see for example [17]) \( u_n(x_n) \to u(x) \) for some \( x \in X \) and some sequence \( x_n \to x \). Since \( Y \) is compact we can assume that \( u_n(x_n) \to y \) for some \( y \) and thus \( f(u_n(x_n)) \to f(y) \), but from \( f \circ u_n \Rightarrow f \circ u \) we also have \( f(u_n(x_n)) \to f(u(x)) \). As \( f \) was arbitrary, we have \( y = u(x) \), a contradiction.
In the classical context the uniform convergence \( u_n \Rightarrow u \) is more natural than the condition \( f \circ u_n \Rightarrow f \circ u \) for every \( f \in C(Y) \). However, the latter is better suited to noncommutative generalisations: Each \( u \in C(X,Y) \) determines a \(*\)-homomorphism \( u^* : C(Y) \to C(X) \) by \( u^*(f) = f \circ u \). Then the uniform convergence \( f \circ u_n \Rightarrow f \circ u \) is equivalent to the point-norm convergence of the sequence of \(*\)-homomorphisms \( u_n^* \to u^* \).

**Remark 2.9.** If we define \( u_n \to u \) to mean \( \|u_n^* - u^*\| \to 0 \), we get convergence in the discrete topology. Indeed, suppose that \( u,v \in C(X,Y) \) and \( u \neq v \). Then there is some \( x \in X \) such that \( u(x) \neq v(x) \) and one can choose \( f \in C(Y) \) such that \( \|f\| \leq 1 \) and \( f(u(x)) = 0, f(v(x)) = 1 \), which shows that \( \|u^*(f) - v^*(f)\| \geq 1 \). Thus any convergent sequence \( (u_n)_n \) would be eventually constant. Therefore we will not consider the convergence in the operator norm on \( \text{Hom}(\mathcal{A}, \mathcal{B}) \).

3. **Motivation.** Let \( X \) be a compact, metrisable space and let \( d \) be a metric on \( X \) inducing the topology. Let \( R := \text{diam} X > 0 \) be the (finite) diameter of \( X \) (we assume that \( X \) contains at least two distinct points). Set

\[
E := \{ f : X \to [0,R] : f \text{ is a contraction with respect to } d \}.
\]

The set \( E \) has the following properties:

1. \( E \) is compact: indeed, \( E \) is closed, \( E \) is an equicontinuous family of functions and directly from the definition it is pointwise bounded. Hence the Ascoli–Arzelà theorem shows that \( E \) is compact.

2. \( E \) separates the points of \( X \): indeed, it suffices to consider the functions \( \{f_x\}_{x \in X} \subset E \) defined by \( f_x(y) := d(x,y) \).

3. For any \( x,y \in X \) we have

\[
d(x,y) = \sup_{f \in E} |f(x) - f(y)|.
\]

Indeed, for \( f \in E \) we have \( |f(x) - f(y)| \leq d(x,y) \), so \( \sup_{f \in E} |f(x) - f(y)| \leq d(x,y) \). On the other hand for \( f_x = d(x,\cdot) \) we obtain equality.

Conversely, assume that \( E \) satisfies conditions (1) and (2) and define \( d(x,y) \) by the formula in (3). This indeed defines a metric:

- \( d(x,y) \) is finite since for fixed \( x,y \) the mapping \( C(X) \ni f \mapsto |f(x) - f(y)| \in \mathbb{R} \) is continuous, hence bounded on the compact set \( E \).
- The condition \( d(x,y) = 0 \) is equivalent to \( f(x) = f(y) \) for \( f \in E \), so from (2) we have \( x = y \).
- Symmetry is obvious and the triangle inequality follows from subadditivity of suprema.
What is more, the topology of $d$ coincides with the original topology on $X$: if $d(x_n, x) \to 0$, i.e. $\sup_{f \in E} |f(x_n) - f(x)| \to 0$, then $f(x_n) \to f(x)$ for every $f \in E$. Assume that, on the contrary, $x_n \not\to x$; then from (sequential) compactness of $X$ there is $y \in X$, $y \neq x$, such that $x_{n_k} \to y$ for some subsequence $(x_{n_k})_k$. Therefore $f(x_{n_k}) \to f(x)$ and also $f(x_{n_k}) \to f(y)$. Since $E$ separates the points of $X$, we have $x = y$, a contradiction.

Conversely, assume $x_n \xrightarrow{\text{top } X} x$. Compactness of $E$ guarantees (uniform) equicontinuity, i.e.

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in X, f \in E \quad (\rho(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$$

where $\rho$ is any metric giving the topology of $X$. Thus fix $\varepsilon > 0$, choose $\delta > 0$ as above and let $n_0 \in \mathbb{N}$ be large enough so that $\rho(x_n, x) < \delta$ for $n \geq n_0$. Then using equicontinuity, for any $f \in E$ we would have $|f(x_n) - f(x)| < \varepsilon$ and hence $d(x_n, x) < \varepsilon$.

We have shown how to recover the metric in algebraic terms. This idea is in the spirit of the work of Connes [7] and Rieffel [23, 24]: in the work of Connes the right framework is that of a spectral triple $(\mathfrak{A}, \mathcal{H}, D)$ where $\mathfrak{A}$ is a $*$-algebra represented on a Hilbert space $\mathcal{H}$ together with an unbounded, self-adjoint operator $D$ with compact resolvent such that the commutators $[D, a]$ are bounded for $a \in \mathfrak{A}$. A typical example of a commutative spectral triple comes from a spin manifold $M$ where $D$ is the Dirac operator (acting on $L^2$-spinors). In this example one can recover the geodesic distance between points $x, y \in M$ by the celebrated formula

$$d(x, y) = \sup \{ |\delta_x(f) - \delta_y(f)| : \|[D, f]\| \leq 1 \}$$

where we view $x, y$ as states $\delta_x, \delta_y$ acting on the space of (smooth) functions on $M$. One can check that the condition $\|[D, f]\| \leq 1$ means exactly that $f$ is a Lipschitz function with Lipschitz constant $\leq 1$—thus the similarity to our approach is visible. On the other hand, Rieffel works with the so called Lipschitz seminorm from the start. However, both authors when going into the noncommutative setting choose to work with the space of states, while we will work with representations.

As already seen, for a $C^*$-algebra of the form $C(X)$, separability is equivalent to the metrisability of $X$, thus we will be mainly interested in separable algebras. Those algebras may be characterised as follows:

**Fact 3.1.** For a unital $C^*$-algebra $\mathfrak{A}$ the following conditions are equivalent:

- $\mathfrak{A}$ is separable;
- there exists a compact set $K$ generating $\mathfrak{A}$, i.e. satisfying $C^*(K \cup \{1\}) = \mathfrak{A}$ (where $C^*(L)$ is by definition the smallest $C^*$-algebra containing $L$).
Proof. Suppose that $K$ is as above. Then $K$ is separable (as a topological space), hence there exists a countable dense set $\{y_n\}_{n \in \mathbb{N}} \subset K$. The smallest unital C$^*$-algebra containing $K$ is the closure (in the norm topology) of the set of all elements of the form $p(x_1, \ldots, x_k, x_1^*, \ldots, x_k^*)$ where $x_1, \ldots, x_k \in K$, $k \in \mathbb{N}$ and $p$ is a polynomial in $2k$ free (noncommuting) variables. Then the set of all elements of the form $q(y_1, \ldots, y_k, y_1^*, \ldots, y_k^*)$ where $k \in \mathbb{N}$ and $q$ is a polynomial of $2k$ free variables with (complex) rational coefficients is countable and dense in $\mathfrak{A}$.

Conversely, assume that $\mathfrak{A}$ is separable. Choose a countable dense set $\{a_n\}_{n \in \mathbb{N}}$ in the (closed) unit ball in $\mathfrak{A}$ and put $K := \{0\} \cup \{a_n/n : n \in \mathbb{N}\}$.

Since $\{a_n\}_{n \in \mathbb{N}}$ is bounded, $K$ consists of (one) convergent sequence together with its limit, therefore is compact. Moreover we have span $K = \text{span}\{a_n : n \in \mathbb{N}\}$ and this set is dense in the unit ball of $\mathfrak{A}$ so being a vector space, it is dense in the whole $\mathfrak{A}$. Therefore it generates $\mathfrak{A}$. $\blacksquare$

4. Uniform structure on the set of representations

4.1. Metric structure. Let $\mathfrak{A}$ be a separable, unital C$^*$-algebra and $K$ be a compact set which generates $\mathfrak{A}$ (we already know that such a $K$ exists). Fix two (unital) $*$-representations $\pi_1, \pi_2 : \mathfrak{A} \to \mathfrak{B}(\ell^2)$ and put

$$d_K(\pi_1, \pi_2) := \sup_{a \in K} \|\pi_1(a) - \pi_2(a)\|.$$  

(4.1)

Denote $\text{Rep}(\mathfrak{A}) := \{\pi : \mathfrak{A} \to \mathfrak{B}(\ell^2) : \pi \text{ is a } *\text{-representation, } \pi(1) = I\}$. The choice of $\ell^2$ as a representation space is due to the following reasons:

- since $\mathfrak{A}$ is separable, it can be faithfully represented on a separable Hilbert space;
- any two infinite-dimensional separable Hilbert spaces are unitarily equivalent, but $\ell^2$ has the property that for any $n \in \mathbb{N}$ we have the natural embedding $\mathbb{C}^n \hookrightarrow \ell^2$. In other words, for a finite-dimensional representation $\pi$ on $\mathbb{C}^n$, $\mathbb{K}_0 \odot \pi = \bigoplus_{n \in \mathbb{N}} \pi$ is a representation on $\ell^2$.

**FACT 4.1.** The formula (4.1) defines a metric on $\text{Rep}(\mathfrak{A})$.

**Proof.** $d_K$ has finite values:

$$\sup_{a \in K} \|\pi_1(a) - \pi_2(a)\| \leq \sup_{a \in K}(\|\pi_1(a)\| + \|\pi_2(a)\|) \leq 2 \sup_{a \in K} \|a\| < \infty$$

since $K$ is bounded.

If $d_K(\pi_1, \pi_2) = 0$ then $\pi_1(a) = \pi_2(a)$ for every $a \in K$. But $K$ generates $\mathfrak{A}$ and $\pi_j$, $j = 1, 2$, preserve all algebraic operations; therefore $\pi_1(x) = \pi_2(x)$ for each $x \in \mathfrak{A}$ and $\pi_1 = \pi_2$.

Other conditions from the definition of metric are obvious (for the triangle inequality use subadditivity of suprema). $\blacksquare$
Remarks 4.2. The argument above shows not only that \(d_K\) has finite values but also that \((\text{Rep}(\mathcal{A}), d_K)\) is bounded and \(\text{diam}(\text{Rep}(\mathcal{A})) \leq 2\text{diam}(K)\). Moreover the same argument applies when \(K\) is only bounded, but if we allow all possible bounded sets \(K\) then the topologies of the metrics \(d_K\) may differ. Soon we will see that this is not the case for a compact set \(K\).

On the other hand, if \(K\) is bounded but does not generate \(\mathcal{A}\) then \(d_K\) will only be a pseudometric.

Directly from the definition we obtain the following properties:

- \(K_1 \subset K_2 \Rightarrow d_{K_1} \leq d_{K_2}\),
- \(d_{K_1 \cup K_2} = \max(d_{K_1}, d_{K_2})\),
- \(d_{K_1 \cap K_2} \leq \min(d_{K_1}, d_{K_2})\).

Theorem 4.3. \((\text{Rep}(\mathcal{A}), d_K)\) is a complete metric space.

Proof. Let \((\pi_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \((\text{Rep}(\mathcal{A}), d_K)\). Since \(\sup_{a \in K} \|\pi_n(a) - \pi_m(a)\| \to 0\), for each \(a \in K\) we have \(\|\pi_n(a) - \pi_m(a)\| \to 0\). Therefore we obtain Cauchy sequences \((\pi_n(a))_{n \in \mathbb{N}}\) for each \(a \in K\). Let \(\mathcal{A}_0\) denote the \(*\)-algebra generated by \(K \cup \{1\}\). We claim that \((\pi_n(x))_{n \in \mathbb{N}}\) is a Cauchy sequence for each \(x \in \mathcal{A}_0\). Each \(x \in \mathcal{A}_0\) is of the form

\[
x = p(a_1, \ldots, a_k, a_1^*, \ldots, a_k^*)
\]

for some \(k \in \mathbb{N}\), \(a_1, \ldots, a_k \in K\) and a polynomial \(p\) in \(2k\) free variables. Then

\[
\|\pi_n(x) - \pi_m(x)\| = \|p(a_1, \ldots, a_k, a_1^*, \ldots, a_k^*) - p(a_1, \ldots, a_k, a_1^*, \ldots, a_k^*)\| = \|p(\pi_n(a_1), \ldots, \pi_n(a_k), \pi_n(a_1)^*, \ldots, \pi_n(a_k)^*) - p(\pi_m(a_1), \ldots, \pi_m(a_k), \pi_m(a_1)^*, \ldots, \pi_m(a_k)^*)\| \to 0
\]

since every polynomial is uniformly continuous on compact sets.

Now if \(x \in \mathcal{A}\) then \(x = \lim_{k \to \infty} x_k\) for some sequence \((x_k)_{k \in \mathbb{N}}\) \(\subset \mathcal{A}_0\). Choosing \(k \in \mathbb{N}\) sufficiently large we obtain

\[
\|\pi_n(x) - \pi_m(x)\| \leq \|\pi_n(x) - \pi_n(x_k)\| + \|\pi_n(x_k) - \pi_m(x_k)\| + \|\pi_m(x_k) - \pi_m(x)\| \\
\leq 2\|x - x_k\| + \|\pi_n(x_k) - \pi_m(x_k)\|
\]

which can be made arbitrarily small. Therefore for every \(x \in \mathcal{A}\) the sequence \((\pi_n(x))_{n \in \mathbb{N}}\) is convergent. We define

\[
\pi(x) := \lim_{n \to \infty} \pi_n(x).
\]

In this manner we obtain a \(*\)-representation of \(\mathcal{A}\) (since it is the point-norm limit of \(*\)-representations). From the Cauchy condition with respect to \(d_K\) we have

\[
\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{N} \ \forall n, m > N_0 \ \forall a \in K \ \|\pi_n(a) - \pi_m(a)\| < \varepsilon.
\]

It suffices to let \(m\) go to \(\infty\) in (4.2) to get

\[
\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{N} \ \forall n > N_0 \ \ d_K(\pi_n, \pi) \leq \varepsilon.
\]
**Theorem 4.4.** The topology of $d_K$ coincides with the point-norm topology and the compact-open topology.

**Proof.** Suppose that $\pi_n \xrightarrow{d_K} \pi$. Then $\pi_n(a) \to \pi(a)$ for each $a \in K$. As above, we check that $\pi_n(x) \to \pi(x)$ for each $x \in \mathcal{A}_0$, where $\mathcal{A}_0$ is the $\ast$-algebra generated by $K$. Then for each $x \in \mathcal{A}$ we have $x = \lim_{k \to \infty} x_k$ for some $(x_k)_k \subset \mathcal{A}_0$, hence, as before,

$$
\|\pi_n(x) - \pi(x)\| \leq \|\pi_n(x) - \pi_n(x_k)\| + \|\pi_n(x_k) - \pi(x_k)\| + \|\pi(x_k) - \pi(x)\| \\
\leq 2\|x - x_k\| + \|\pi_n(x_k) - \pi(x_k)\| \to 0.
$$

Conversely, suppose that $\pi_\sigma(x) \to \pi(x)$ for every $x \in \mathcal{A}$. In particular this holds for each $a \in K$. Since $K$ is compact, for fixed $\varepsilon > 0$ there exists a finite $\varepsilon/3$-net $\{a_1, \ldots, a_N\} \subset K$. In other words, for every $a \in K$ there is $i \in \{1, \ldots, N\}$ such that $\|a - a_i\| \leq \varepsilon/3$. Since the set $\{a_1, \ldots, a_N\}$ is finite, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have

$$
\|\pi_n(a_i) - \pi(a_i)\| \leq \varepsilon/3
$$

for every $i = 1, \ldots, N$. Fix $a \in K$ and choose $i_0 \in \{1, \ldots, N\}$ such that $\|a - a_{i_0}\| \leq \varepsilon/3$. We obtain

$$
\|\pi_n(a) - \pi(a)\| \leq \|\pi_n(a) - \pi_n(a_{i_0})\| + \|\pi_n(a_{i_0}) - \pi(a_{i_0})\| + \|\pi(a_{i_0}) - \pi(a)\| \\
\leq 2\|a - a_{i_0}\| + \|\pi_n(a_{i_0}) - \pi(a_{i_0})\| \leq \varepsilon
$$

uniformly with respect to $a$. Thus $d_K(\pi_n, \pi) = \sup_{a \in K} \|\pi_n(a) - \pi(a)\| \to 0$.

We have shown that point-norm convergence is equivalent to convergence in $d_K$. In the proof that point-norm convergence implies convergence in $d_K$ we did not use the fact that $K$ generates $\mathcal{A}$; the same argument applies for any compact set $L \subset \mathcal{A}$, so point-norm convergence implies convergence in the compact-open topology. Since the converse is always valid, all three topologies: point-norm, compact-open and the topology of $d_K$ coincide. ■

**Remark 4.5.** A priori we do not know whether the point-norm topology is metrisable (in particular, whether every point has a countable neighborhood system), hence we use nets instead of ordinary sequences. The same remark applies to the compact-open topology.

With every $x \in \mathcal{A}$ we can associate $\hat{x} : \text{Rep(}\mathcal{A}\text{)} \to \mathcal{B}(\ell^2)$ defined by

$$
(4.3) \quad \hat{x}(\pi) := \pi(x).
$$

Then

$$
(4.4) \quad d_K(\pi_1, \pi_2) = \sup_{a \in K} \|\hat{a}(\pi_1) - \hat{a}(\pi_2)\|.
$$

**Theorem 4.6.** For every $x \in \mathcal{A}$ the mapping $\hat{x}$ is uniformly continuous (with respect to $d_K$).
Proof. First pick \( a \in K \). Then
\[
\| \hat{a}(\pi_1) - \hat{a}(\pi_2) \| = \| \pi_1(a) - \pi_2(a) \| \leq \sup_{a \in K} \| \pi_1(a) - \pi_2(a) \| = d_K(\pi_1, \pi_2),
\]
thus \( \hat{a} \) is contractive (Lipschitz with Lipschitz constant 1). We have the following facts:

- a linear combination of Lipschitz functions is Lipschitz,
- if \( \hat{a} \) is Lipschitz then so is \( \hat{a}^* \) (with the same Lipschitz constant) since
\[
\| \hat{a}^*(\pi_1) - \hat{a}^*(\pi_2) \| = \| \pi_1(a^*) - \pi_2(a^*) \| = \| (\pi_1(a) - \pi_2(a))^* \|
\]
\[
= \| \pi_1(a) - \pi_2(a) \| = \| \hat{a}(\pi_1) - \hat{a}(\pi_2) \|,
\]
- if \( \hat{a}, \hat{b} \) are Lipschitz (and bounded) then \( \hat{a} \hat{b} \) is also Lipschitz.

It follows that for \( x \in \mathcal{A}_0 \), where \( \mathcal{A}_0 \) is the \(*\)-algebra generated by \( K \cup \{1\} \), the functions \( \hat{x} \) are Lipschitz, hence uniformly continuous. Now take any \( x \in \mathcal{A} \) and express it as \( x = \lim_{k \to \infty} x_k \) where \( x_k \in \mathcal{A}_0 \). Fix \( \varepsilon > 0 \), choose \( k \in \mathbb{N} \) large enough that \( \|x - x_k\| < \varepsilon/3 \) and put \( \delta := \varepsilon/(3L_k) \), where \( L_k \) is the Lipschitz constant for \( \hat{x}_k \). If \( d_K(\pi_1, \pi_2) < \delta \), we can estimate
\[
\| \hat{\pi}(\pi_1) - \hat{\pi}(\pi_2) \| = \| \pi_1(x) - \pi_2(x) \|
\]
\[
\leq \| \pi_1(x) - \pi_1(x_k) \| + \| \pi_1(x_k) - \pi_2(x_k) \| + \| \pi_2(x_k) - \pi_2(x) \|
\]
\[
\leq 2\|x - x_k\| + \| \pi_1(x_k) - \pi_2(x_k) \| = 2\|x - x_k\| + \| \hat{x}_k(\pi_1) - \hat{x}_k(\pi_2) \|
\]
\[
\leq 2\varepsilon/3 + L_kd_K(\pi_1, \pi_2) < \varepsilon. \]

The fact that \( \pi \) is a \(*\)-representation (i.e. a \(*\)-homomorphism) may be expressed in terms of the mapping \( x \mapsto \hat{x} \): if we define, on the set of all continuous functions defined on \( \text{Rep}(\mathcal{A}) \), the \(*\)-algebra structure in a natural manner (where all operations are defined pointwise), then the mapping \( x \mapsto \hat{x} \) will become a \(*\)-homomorphism. However, it is not possible to define a norm on the set of all continuous functions on \( \text{Rep}(\mathcal{A}) \) in a natural manner to obtain a \( C^*\)-algebra structure:

**Example 4.7.** The space \( \text{Rep}(\mathcal{A}) \) is almost always nonseparable: for example let \( \mathcal{A} := \mathbb{C} \oplus \mathbb{C} \) and fix a projection \( P \in \mathfrak{B}(\ell^2) \). Define \( \pi_P \) as follows:
\[
\pi_P(1, 0) = P, \quad \pi_P(0, 1) = I - P
\]
and extend it linearly on all \( \mathcal{A} \). As \( P \) is a projection, \( \pi_P \) is indeed a \(*\)-representation. Then for two distinct projections \( P \) and \( Q \) we obtain
\[
1 \leq \|P - Q\| = \|\pi_P(1, 0) - \pi_Q(1, 0)\| \leq d_K(\pi_P, \pi_Q)
\]
for every compact generating set \( K \) containing \((1, 0)\). Since within \( \mathfrak{B}(\ell^2) \) one can find uncountably many pairwise distinct (orthogonal) projections we see that \( \text{Rep}(\mathcal{A}) \) is not separable. It also follows that \( \text{Rep}(\mathcal{A}) \) is not compact, since a compact metric space is always separable. In particular,
continuous functions on \( \text{Rep}(\mathfrak{A}) \) need not be bounded and we cannot consider the supremum norm on \( C(\text{Rep}(\mathfrak{A})) \).

Remark 4.8. One can define a weaker topology on \( \text{Rep}(\mathfrak{A}) \) in the following way: a net \( \{\pi_s\}_{s \in S} \) converges to \( \pi \) if for any \( a \in \mathfrak{A} \) and \( \xi, \eta \in \ell^2 \),
\[
\langle \pi_s(a) \xi, \eta \rangle \to \langle \pi(a) \xi, \eta \rangle.
\]
In other words, this is the topology defined by the family of maps \( \{p_{a,\xi,\eta} : a \in \mathfrak{A}, \xi, \eta \in \ell^2\} \) where \( p_{a,\xi,\eta}(\pi) := |\langle \pi(a) \xi, \eta \rangle| \).

Then \( \text{Rep}(\mathfrak{A}) \) with this topology becomes separable or even a Polish space (see [2, Chapter 4] where \( \text{Rep}(\mathfrak{A}) \) is defined in a slightly different manner).

Now suppose we have two distinct compact, generating sets \( K, L \subset \mathfrak{A} \).
The topologies of the metrics \( d_K \) and \( d_L \) both coincide with the point-norm topology. This means that \( d_K, d_L \) are equivalent. But as we have already seen, \( \text{Rep}(\mathfrak{A}) \) is rarely compact, so we cannot immediately infer that these two metrics are uniformly equivalent (i.e. the identity mapping is uniformly continuous in both directions). But it turns out that this is indeed the case:

**Theorem 4.9.** Let \( K, L \) be as above. Then the metrics \( d_K \) and \( d_L \) are uniformly equivalent.

**Proof.** It suffices to prove that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( \pi, \pi' \in \text{Rep}(\mathfrak{A}) \) the condition \( d_K(\pi, \pi') < \delta \) implies \( d_L(\pi, \pi') < \varepsilon \) (the roles of \( d_K \) and \( d_L \) are symmetric).

So fix \( \varepsilon > 0 \) and consider an \( \varepsilon / 6 \)-net \( \{a'_1, \ldots, a'_N\} \) for \( L \), so \( \forall a' \in L \exists i \in \{1, \ldots, N\} \parallel a' - a'_i \parallel < \varepsilon / 6 \). As \( K \) generates \( \mathfrak{A} \), for every \( a_i, \ i = 1, \ldots, N \), we can choose \( x_i \) from the \( * \)-algebra generated by \( K \) such that \( \parallel a_i - x_i \parallel < \varepsilon / 6 \). Then
\[
\forall a' \in L \exists i \in \{1, \ldots, N\} \parallel a' - x_i \parallel \leq \parallel a' - a'_i \parallel + \parallel a'_i - x_i \parallel < \varepsilon / 3.
\]
Therefore for any \( a' \in L \),
\[
\parallel \pi(a') - \pi'(a') \parallel \leq \parallel \pi(a') - \pi(x_i) \parallel + \parallel \pi(x_i) - \pi'(x_i) \parallel + \parallel \pi'(x_i) - \pi'(a') \parallel
\leq 2 \parallel a' - x_i \parallel + \parallel \pi(x_i) - \pi'(x_i) \parallel < 2 \varepsilon / 3 + \parallel \pi(x_i) - \pi'(x_i) \parallel.
\]
It now suffices to take \( \delta \) small enough that
\[
d_K(\pi, \pi') < \delta \implies \parallel \pi(x_i) - \pi'(x_i) \parallel < \varepsilon / 3.
\]
Such a choice is possible since there are only finitely many \( x_i \) and each \( \hat{x}_i \) is uniformly continuous (with respect to \( d_K \)).

**4.2. Moduli of continuity**

**Definition 4.10.** Fix a compact set \( K \) generating \( \mathfrak{A} \). A set \( L \subset \mathfrak{A} \) is called **equicontinuous** if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( \pi, \pi' \in \text{Rep}(\mathfrak{A}) \) satisfying \( d_K(\pi, \pi') < \delta \) we have \( \parallel \pi(x) - \pi'(x) \parallel < \varepsilon \) for every \( x \in L \).
Note that if $K_1, K_2$ are compact generating sets and $L \subset \mathfrak{A}$ is equicontinuous with respect to $d_{K_1}$ then it is also equicontinuous with respect to $d_{K_2}$. Indeed, fix $\varepsilon > 0$ and choose $\delta_1 > 0$ which is “good” for equicontinuity of $L$ with respect to $d_{K_1}$. Since the metrics $d_{K_1}$ and $d_{K_2}$ are uniformly equivalent, there exists $\delta_2 > 0$ such that for any representations $\pi, \pi'$ we have the implication $d_{K_2}(\pi, \pi') < \delta_2 \Rightarrow d_{K_1}(\pi, \pi') < \delta_1$. Thus $\delta_2$ is “good” for equicontinuity of $L$ with respect to $d_{K_2}$. In other words, the equicontinuity does not depend on the choice of the compact generating set $K$.

Fix an equicontinuous and bounded set $L \subset \mathfrak{A}$. Define $f^K_L : \{0, \infty\} \to [0, \infty)$ by

$$f^K_L(t) := \sup\{|\hat{a}(\pi) - \hat{a}(\pi')| : \pi, \pi' \in \text{Rep}(\mathfrak{A}), d_K(\pi, \pi') \leq t, a \in L\}. $$

As a direct consequence of the definition we have:

- if $L_1 \subset L_2$ then $f^K_{L_1} \leq f^K_{L_2}$,
- if $K_1 \subset K_2$ then $f^K_L \geq f^K_{K_1}$.

We will show that $f^K_L$ satisfies all the conditions of Corollary 2.7.

First, for every $a \in L$ we have

$$|\hat{a}(\pi) - \hat{a}(\pi')| \leq f^K_L(d_K(\pi, \pi')).$$

Since $L$ is equicontinuous, we have

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall \pi, \pi' \in \text{Rep}(\mathfrak{A}) \quad \forall a \in L \quad (d_K(\pi, \pi') < \delta \Rightarrow |\hat{a}(\pi) - \hat{a}(\pi')| < \varepsilon),$$

so if $d_K(\pi, \pi') \to 0$ then $|\hat{a}(\pi) - \hat{a}(\pi')| \to 0$ uniformly with respect to $a \in L$. This shows that $\lim_{t \to 0^+} f^K_L(t) = 0$.

Further, since $L$ is bounded, we infer that

$$|\hat{a}(\pi) - \hat{a}(\pi')| = |\pi(a) - \pi'(a)| \leq 2\|a\| \leq 2 \text{diam } L$$

independently of the choice of $a \in L$ and of $\pi, \pi'$, hence $f^K_L$ is bounded (by 2 diam $L$).

Moreover $f^K_L$ is nondecreasing, since if $t_1 < t_2$ then in the definition of $f^K_L(t_2)$ we just take supremum over a larger set.

We have shown that $f^K_L$ satisfies all conditions of Corollary 2.7, so there exists a concave, continuous, nondecreasing function $\tilde{\omega}^K_L$ satisfying $\tilde{\omega}^K_L(0) = 0$ and $f^K_L \leq \tilde{\omega}^K_L$. We put $\omega^K_L(t) := \inf\{\omega(t) : \omega \in \Omega, \omega \geq f^K_L\}$ (this definition is correct since the relevant set is nonempty). In particular for $L = \{a\}$ we use the notation $f^K_a, \tilde{\omega}^K_a$ and $\omega^K_a$. Let $K, K'$ be compact generating sets. We already know that $d_K$ and $d_{K'}$ are uniformly equivalent. However, the following stronger result is valid:

**Theorem 4.11.** There exists a concave, continuous, nondecreasing function $\omega$ satisfying $\omega(0) = 0$ and

$$d_{K'} \leq \omega \circ d_K.$$
Proof. Define $\omega = \omega^K_{K'} \in \Omega$; then $f^K_{K'} \leq \omega$. In particular, for any $\pi, \pi'$ in $\text{Rep}(\mathfrak{A})$ we have

$$f^K_{K'}(d_K(\pi, \pi')) \leq \omega(d_K(\pi, \pi')).$$

However,

$$f^K_{K'}(d_K(\pi, \pi')) = \sup\{\|\hat{\omega}_{\pi_1} - \hat{\omega}_{\pi_2}\| : a \in K'\}$$

$$= \sup\{\|\hat{\omega}_{\pi} - \hat{\omega}_{\pi'}\| : a \in K'\} = d_K(\pi, \pi'),$$

therefore $d_K' \leq \omega \circ d_K$. □

**Theorem 4.12.** In the above notation, $\omega^K_a \leq \omega^K_a \circ \omega^K_{K'}$.

**Proof.** It suffices to show that the function $\omega^K_a \circ \omega^K_{K'}$ belongs to $\Omega$ and dominates $f^K_{a'}$. The composition of continuous/concave/nondecreasing functions has the same property and $\omega^K_a \circ \omega^K_{K'}(0) = \omega^K_a(0) = 0$. So it remains to show the appropriate inequality. We claim that

$$f^K_{a'} \leq f^K_a \circ f^K_{a'},$$

which is enough to end the proof, since then $f^K_{a'} \leq \omega^K_a \circ \omega^K_{K'}$ and the right hand side belongs to the set over which we take the infimum in the definition of $\omega^K_{a'}$.

For the proof of (4.6) we rewrite both sides as follows:

$L(t) = \sup\{\|\hat{\omega}_{\pi} - \hat{\omega}_{\pi'}\| : d_K(\pi, \pi') \leq t\}$,

$R(t) = f^K_a (\sup\{\|\hat{\omega}_{\pi} - \hat{\omega}_{\pi'}\| : b \in K, d_K(\pi, \pi') \leq t\}$

$$= \sup\{\|\hat{\omega}_{\pi} - \hat{\omega}_{\pi'}\| : d_K(\pi, \pi') \leq \sup\{\|\hat{\omega}_{\pi} - \hat{\omega}_{\pi'}\| : b \in K, d_K(\pi, \pi') \leq t\}\}.$$

The set over which we take the supremum on the left hand side is contained in the set on the right hand side: if $d_K(\pi, \pi') \leq t$ then

$$d_K(\pi, \pi') = \sup\{\|\hat{\omega}_{\pi} - \hat{\omega}_{\pi'}\| : b \in K\}$$

$$\leq \sup\{\|\hat{\omega}_{\pi} - \hat{\omega}_{\pi'}\| : b \in K, d_K(\pi, \pi') \leq t\}. \Box$$

**Remark 4.13.** The above argument is valid if we replace $\{a\}$ by any equicontinuous, bounded set $L$. In other words,

$$\omega^K_{K'} \leq \omega^K_L \circ \omega^K_{K'}.$$  

**Properties 4.14.** If $K$ is a compact set generating $\mathfrak{A}$ and $a, b \in \mathfrak{A}, \lambda \in \mathbb{C}$ then:

1. $\omega^K_{a+\lambda 1} = \omega^K_a = \omega^K_a$,
2. $\omega^K_a = 0 \iff a \in \mathbb{C} 1$,
3. $\omega^K_{a\lambda} = |\lambda| \omega^K_a$,
4. $\omega^K_{a+b} \leq \omega^K_a + \omega^K_b$,
5. $\omega^K_a \leq \|a\| \omega^K_b + \|b\| \omega^K_b$.
Proof. (1) For any $\pi, \pi'$ we have
\[
\|\pi(a^*) - \pi'(a^*)\| = \|(\pi(a) - \pi'(a))^*\| = \|\pi(a) - \pi'(a)\|,
\]
\[
\|\pi(a + \lambda 1) - \pi'(a + \lambda 1)\| = \|\pi(a) + \lambda I - \pi'(a) - \lambda I\| = \|\pi(a) - \pi'(a)\|
\]
which implies that $f_{a+\lambda 1}^K = f_a^K = f_{a^*}^K$, giving (1).

(2) If $a = \lambda 1$ then obviously $\omega_a^K = 0$. Conversely, assume $\omega_a^K = 0$; then $\pi(a) = \pi'(a)$ for any $\pi, \pi' \in \text{Rep}(\mathfrak{A})$. Fix $\pi$ and put $\pi' := U^*\pi(\cdot)U$ where $U \in \mathfrak{B}(\ell^2)$ is a unitary. We have $\pi(a) = U^*\pi(a)U$ and $U\pi(a) = \pi(a)U$. Since every operator in $\mathfrak{B}(\ell^2)$ is a linear combination of (at most) four unitaries (see e.g. [25]), we have $\pi(a) \in \mathfrak{Z}(\mathfrak{B}(\ell^2)) = \mathbb{C}I$. It follows that for any $\pi \in \text{Rep}(\mathfrak{A})$ there exists $\lambda_\pi \in \mathbb{C}$ such that $\pi(a) = \lambda_\pi I$. Therefore, for a faithful $\pi_0 \in \text{Rep}(\mathfrak{A})$ we have
\[
\pi_0(a) = \lambda_{\pi_0} I = \pi_0(\lambda_{\pi_0} 1),
\]
and thus $a = \lambda_{\pi_0} 1$ (and in fact the constant $\lambda_\pi$ does not depend on the choice of representation).

(3) Since
\[
\sup\{\|[\pi(\lambda a) - \pi'(\lambda a)] : d_K(\pi, \pi') \leq t\} = \sup\{\lambda_1 \|[\pi(a) - \pi'(a)] : d_K(\pi, \pi') \leq t\} = \|\lambda\| \sup\{\|[\pi(a) - \pi'(a)] : d_K(\pi, \pi') \leq t\}
\]
we get $f_{a_\lambda}^K = |\lambda| f_a^K$, which gives (3).

(4) We have
\[
\sup\{\|[\pi(a + b) - \pi'(a + b)] : d_K(\pi, \pi') \leq t\} \leq \sup\{\|[\pi(a) - \pi'(a)] + \|[\pi(b) - \pi'(b)] : d_K(\pi, \pi) \leq t\} \leq \sup\{\|[\pi(a) - \pi'(a)] : d_K(\pi, \pi') \leq t\} + \sup\{\|[\pi(b) - \pi'(b)] : d_K(\pi, \pi') \leq t\}
\]
therefore $f_{a+b}^K \leq f_a^K + f_b^K \leq \omega_a^K + \omega_b^K$. Since the sum of two nondecreasing/continuous/concave and vanishing at 0 functions has the same property, we obtain $\omega_{a+b}^K \leq \omega_a^K + \omega_b^K$.

(5) We have
\[
\|[\pi(ab) - \pi'(ab)]\| = \|[\pi(a)\pi(b) - \pi'(a)\pi'(b)]\| \leq \|[\pi(a)\pi(b) - \pi(a)\pi'(b)]\| + \|[\pi(a)\pi'(b) - \pi'(a)\pi'(b)]\| \leq \|a\|\|[\pi(b) - \pi'(b)]\| + \|b\|\|[\pi(a) - \pi'(a)]\|
\]
thus $f_{ab}^K \leq \|a\| f_b^K + \|b\| f_a^K \leq \|a\|\omega_b^K + \|b\|\omega_a^K$ and as before, $\|a\|\omega_b^K + \|b\|\omega_a^K \in \Omega$, which ends the proof.

The modulus of continuity $\omega_a^K$ constructed above can be identified with the minimal modulus of continuity (cf. Definition 2.4) for the function $\hat{a}$:

**Theorem 4.15.** For $a \in \mathfrak{A}$, $\omega_a^K = \omega_{\hat{a}}^K$. 

Uniformity in $C^*$-algebras

Proof. To prove $\omega^K_a \leq \omega^K_{\hat{a}}$ it is enough to show that $f^K_a \leq \omega^K_{\hat{a}}$. Fix $t \geq 0$ and $\pi, \pi' \in \text{Rep}(\mathcal{A})$ such that $d^K(\pi, \pi') \leq t$. Then

$$\|\pi(a) - \pi'(a)\| = \|\hat{a}(\pi) - \hat{a}(\pi')\| \leq \omega^K_{\hat{a}}(d^K(\pi, \pi')) \leq \omega^K(t).$$

Taking the supremum in (4.8) we get $f^K_a(t) \leq \omega^K_{\hat{a}}(t)$.

On the other hand,

$$\|\pi(a) - \pi'(a)\| \leq f^K_a(d^K(\pi, \pi')) \leq w^K_{\hat{a}}(d^K(\pi, \pi')),$$

and therefore, by minimality, $\omega^K_{\hat{a}} \leq \omega^K_a$. ■

5. Compactness and Ascoli property

5.1. Definitions, properties and examples. Let $\pi$ be an irreducible representation of a separable $C^*$-algebra $\mathcal{A}$. Then either $\mathcal{H}_\pi$ is finite-dimensional or $\mathcal{H}_\pi$ is separable infinite-dimensional. Therefore using a suitable unitary, we can assume that $\mathcal{H}_\pi = \ell^2$ or $\mathcal{H}_\pi = \mathbb{C}^n$. In the latter case, we can consider $\pi^\infty := \mathbb{N}_0 \odot \pi$. By a slight abuse of terminology (when this does not lead to confusion) $\pi^\infty$ will be called irreducible if $\pi$ was irreducible—in this manner all irreducible representations of $\mathcal{A}$ can be realised as elements of $\text{Rep}(\mathcal{A})$.

We put

$$\Sigma_f(\mathcal{A}) := \{\pi^\infty : \pi \text{ is irreducible, finite-dimensional}\},$$

$$\Sigma_\infty(\mathcal{A}) := \{\pi : \mathcal{A} \to \mathcal{B}(\ell^2) : \pi \text{ is irreducible}\},$$

$$\Sigma(\mathcal{A}) := \Sigma_f(\mathcal{A}) \cup \Sigma_\infty(\mathcal{A}),$$

where the closure is taken with respect to the topology of $d_K$ for some compact generating set $K$ (or equivalently, in the point-norm topology). For convenience set

$$\Sigma^0_f(\mathcal{A}) = \{\pi^\infty : \pi \text{ is irreducible, finite-dimensional}\},$$

$$\Sigma^0_\infty(\mathcal{A}) = \{\pi : \mathcal{A} \to \mathcal{B}(\ell^2) : \pi \text{ is irreducible}\}$$

and also $\Sigma^0(\mathcal{A}) := \Sigma^0_f(\mathcal{A}) \cup \Sigma^0_\infty(\mathcal{A})$. Then $\Sigma(\mathcal{A})$ becomes a complete subspace of $\text{Rep}(\mathcal{A})$ and $\Sigma^0(\mathcal{A})$ is a dense subset of $\Sigma(\mathcal{A})$.

**Definition 5.1.** A $C^*$-algebra $\mathcal{A}$ is called compact if $\Sigma(\mathcal{A})$ is a compact topological space.

**Theorem 5.2.** If $\mathcal{A}$ is a unital, commutative, separable $C^*$-algebra then the spaces $\Sigma(\mathcal{A})$ and $\hat{\mathcal{A}}$ are homeomorphic (where $\hat{\mathcal{A}}$ denotes the classical spectrum of $\mathcal{A}$).

**Proof.** One checks that the mapping $\hat{\mathcal{A}} \ni \omega \mapsto \mathbb{N}_0 \odot \omega \in \Sigma(\mathcal{A})$ defines a homeomorphism. ■

In particular, (unital, separable) commutative $C^*$-algebras are compact.
Examples 5.3. (1) Let $\mathfrak{A} = C(X)$ be a commutative, separable $C^*$-algebra and let $d$ be a fixed metric on $X$. Every irreducible representation $\pi$ of $\mathfrak{A}$ is one-dimensional, hence of the form $\pi(a) = \omega(a)$ where $\omega$ is a character of $\mathfrak{A}$. Then $\pi(a) = \delta_x(a)$ for some $x \in X$. Identifying $\pi \simeq \pi^\infty (=: \pi_x)$ where $\pi^\infty \in \Sigma(\mathfrak{A})$ we have

$$\pi_x(a) = \delta_x(a)I_{\ell^2}.$$ 

Therefore, if $\pi_x, \pi_y \in \Sigma(\mathfrak{A})$ correspond to $x, y \in X$ then

$$\|\pi_x(f) - \pi_y(f)\| = \|f(x)I - f(y)I\| = |f(x) - f(y)|.$$ 

So if we take

$$K := \{f : X \to [0, \text{diam } X] : f \text{ is a contraction with respect to } d\},$$

we get $d_K(\pi_x, \pi_y) = d(x,y)$. In other words, the metric spaces $(X,d)$ and $(\Sigma(\mathfrak{A}), d_K)$ are isometric. Thus the pair $(\Sigma(\mathfrak{A}), d_K)$, where $K \subset \mathfrak{A}$ is a compact generating set for a unital, separable $C^*$-algebra $\mathfrak{A}$, may be thought of as a generalisation of the (classical) spectrum of a (unital, separable) $C^*$-algebra, treated as a metric space.

Note also that if we drop the assumption of compactness of $K$ then the topology of $d_K$ may (drastically) differ from the point-norm topology. As an example, let $K$ be the unit ball in $\mathfrak{A}$. Then for $x, y \in X, x \neq y$, we can always find $f \in K$ such that $f(x) = 0$ and $f(y) = 1$. Thus $d_K(\pi(x), \pi(y)) \geq 1$, which implies that the irreducible representations in $\Sigma(\mathfrak{A})$ form a discrete set.

(2) If $\mathfrak{A}, \mathfrak{B}$ are any $C^*$-algebras then the set $\text{Irr}(\mathfrak{A} \oplus \mathfrak{B})$ of irreducible representations of $\mathfrak{A} \oplus \mathfrak{B}$ may be identified with $\text{Irr}(\mathfrak{A}) \sqcup \text{Irr}(\mathfrak{B})$ in the following way: if $\pi : \mathfrak{A} \oplus \mathfrak{B} \to \mathfrak{B}(\mathcal{H})$ is an irreducible representation then either there exists an irreducible representation $\pi_1 : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ satisfying $\pi(a,b) = \pi_1(a)$ or there exists an irreducible representation $\pi_2 : \mathfrak{B} \to \mathfrak{B}(\mathcal{H})$ such that $\pi(a,b) = \pi_2(b)$. It follows that $\Sigma(\mathfrak{A} \oplus \mathfrak{B}) = \Sigma(\mathfrak{A}) \sqcup \Sigma(\mathfrak{B})$, which shows that the direct sum of compact $C^*$-algebras is also compact.

One can also show that if $\mathfrak{A} \oplus \mathfrak{B}$ is compact then so are $\mathfrak{A}$ and $\mathfrak{B}$. In fact, a more general statement is valid:

**Theorem 5.4.** If $\mathfrak{A}$ and $\mathfrak{B}$ are $C^*$-algebras, $\mathfrak{A}$ is compact and $\alpha : \mathfrak{A} \to \mathfrak{B}$ is a $*$-epimorphism then $\mathfrak{B}$ is also compact.

**Proof.** Define $\alpha^* : \Sigma(\mathfrak{B}) \to \Sigma(\mathfrak{A})$ by $\alpha^*(\rho) := \rho \circ \alpha$. Then $\alpha^*$ has values in $\Sigma(\mathfrak{A})$, since if $\rho$ is irreducible, then $(\rho \circ \alpha(\mathfrak{A}))' = (\rho(\mathfrak{B}))' = C I$, thus $\rho \circ \alpha$ is also irreducible; from the continuity of $\alpha^*$ (in the point-norm topology) we conclude that $\alpha^*(\rho) \in \Sigma(\mathfrak{A})$ for $\rho \in \Sigma(\mathfrak{B})$.

Take a compact set $K$ generating $\mathfrak{A}$; then $\alpha(K)$ is also compact and generates $\mathfrak{B}$. Therefore, for $\rho, \rho' \in \Sigma(\mathfrak{B})$,
Uniformity in $C^*$-algebras

\[ d_K(\alpha^*(\rho), \alpha^*(\rho')) = \sup\{\|\rho(\alpha(a)) - \rho'(\alpha(a))\| : a \in K\} \]
\[ = \sup\{\|\rho(b) - \rho'(b)\| : b \in \alpha(K)\} = d_{\alpha(K)}(\rho, \rho'). \]

Thus we obtain an isometric embedding \((\Sigma(\mathfrak{B}), d_{\alpha(K)}) \hookrightarrow (\Sigma(\mathfrak{A}), d_K)\), so \(\Sigma(\mathfrak{B})\) is compact. \(\blacksquare\)

**Corollary 5.5.** If \(\mathfrak{A}\) is a compact \(C^*\)-algebra and \(\mathfrak{I} \subset \mathfrak{A}\) is an ideal then \(\mathfrak{A}/\mathfrak{I}\) is also compact.

All \(C^*\)-algebras considered so far were unital, therefore the question whether \(\mathfrak{I}\) is compact is not well posed. To deal with it we introduce the following:

**Definition 5.6.** A nonunital \(C^*\)-algebra \(\mathfrak{A}\) is called **compact** if \(\mathfrak{A}^+\) is compact.

**Remark 5.7.** If \(\mathfrak{A}\) already has a unit we can still adjoin the unit to \(\mathfrak{A}\) and consider \(\mathfrak{A}^+\). In that case we have an isomorphism \(\mathfrak{A}^+ \cong \mathfrak{A} \oplus \mathbb{C}\) (see e.g. \([29]\)), thus \(\Sigma(\mathfrak{A}^+) = \Sigma(\mathfrak{A}) \sqcup \{\delta^\infty\}\) where \(\delta(x, \lambda) = \lambda\) for \(x \in \mathfrak{A}\) and \(\lambda \in \mathbb{C}\). Therefore \(\mathfrak{A}\) is compact if and only if \(\mathfrak{A}^+\) is compact.

With the above definition we have the following:

**Fact 5.8.** If \(\mathfrak{I} \subset \mathfrak{A}\) is an ideal in a unital, compact \(C^*\)-algebra \(\mathfrak{A}\) then \(\mathfrak{I}\) is also compact.

**Proof.** If \(\pi\) is an irreducible representation for \(\mathfrak{I}\) then \(\pi\) can be uniquely extended to a representation of \(\mathfrak{A}\) (which is obviously still irreducible). In other words, every irreducible representation of \(\mathfrak{I}\) is the restriction of an irreducible representation of \(\mathfrak{A}\); this gives a surjection \(\text{Irr}(\mathfrak{A}) \ni \pi \mapsto \pi|_{\mathfrak{I}} \in \text{Irr}(\mathfrak{I}) \cup \{0\}\), and after adjoining a unit to \(\mathfrak{I}\), the surjection \(\text{Irr}(\mathfrak{A}) \ni \pi \mapsto \pi|_{\mathfrak{I}^+} \in \text{Irr}(\mathfrak{I}^+)\). Thus we get the restriction mapping which can be viewed as a mapping \(\text{Rep}(\mathfrak{A}) \to \text{Rep}(\mathfrak{I}^+)\) and obviously it is continuous in the point-norm topology—therefore the above remarks yield a surjection \(\Sigma(\mathfrak{A}) \to \Sigma(\mathfrak{I}^+)\). Since \(\Sigma(\mathfrak{A})\) is compact, so are \(\Sigma(\mathfrak{I}^+)\) and \(\mathfrak{I}\). \(\blacksquare\)

The above proofs show that any epimorphism of \(C^*\)-algebras determines an embedding at the level of \(\Sigma(-)\) spaces; on the other hand, an inclusion of an ideal in a \(C^*\)-algebra produces an epimorphism at the level of \(\Sigma(-)\) spaces.

A rich class of compact \(C^*\)-algebras is provided by unital, subhomogeneous \(C^*\)-algebras:

**Theorem 5.9.** If \(\mathfrak{A}\) is an \(N\)-subhomogeneous \(C^*\)-algebra then \(\mathfrak{A}\) is compact.

**Proof.** For a natural number \(n\) define

\[ X_n := \{\pi : \mathfrak{A} \to M_n : \pi\ \text{is a unital } *\text{-representation}\}. \]
We equip $X_n$ with the topology of pointwise convergence. We claim that $X_n$ is compact. To see this, consider the mapping

$$\iota : X_n \ni \pi \mapsto (\pi(a))_{a \in \mathfrak{A}} \in \prod_{a \in \mathfrak{A}} B(0, ||a||) \subset \prod_{a \in \mathfrak{A}} M_n$$

where the codomain is equipped with the product topology. This mapping is obviously injective and for $\{\pi_i\}_i \subset X_n$ we have $\pi_i \to \pi$ if and only if $\iota(\pi_i) \to \iota(\pi)$—therefore the mapping is an embedding. Moreover, this embedding is closed: Assume that $z_i \to z$ where $z_i \in \iota(X_n)$; thus $z_i = (\pi_i(a))_{a \in \mathfrak{A}}$. Let $z = (z_a)_{a \in \mathfrak{A}}$; then $\pi_i(a) \to z_a$ for every $a \in \mathfrak{A}$. This implies that the mapping $\mathfrak{A} \ni a \mapsto z_a$ becomes a unital $\ast$-representation; for example,

$$z_{ab} = \lim_i \pi_i(ab) = \lim_i (\pi_i(a)\pi_i(b)) = \lim_i \pi_i(a)\lim_i \pi_i(b) = z_az_b$$

(the analogous argument applies to other algebraic operations). Denote by $\pi$ the representation obtained above; then $\pi_i \to \pi$ and hence $z = \iota(\pi) \in \iota(X_n)$. From the Tikhonov theorem the cartesian product $\prod_{a \in \mathfrak{A}} B(0, ||a||)$ is compact, thus $\iota(X_n)$ is also compact (as a closed subset) and hence $X_n$ is compact as well.

Since $\mathfrak{A}$ is $N$-subhomogeneous, we have $\text{Irr}(\mathfrak{A}) \subset \bigsqcup_{n=1}^N X_n$ and, as $\pi \mapsto \mathbb{N}_0 \circ \pi$ is an embedding, the set $\mathbb{N}_0 \circ \bigsqcup_{n=1}^N X_n$ is a compact subset of $\text{Rep}(\mathfrak{A})$ (containing $\Sigma^0(\mathfrak{A})$). Taking the closure we get $\Sigma(\mathfrak{A}) \subset \mathbb{N}_0 \circ \bigsqcup_{n=1}^N X_n$, hence $\Sigma(\mathfrak{A})$ is a compact space and $\mathfrak{A}$ is a compact $C^\ast$-algebra.

The above result still holds for all shrinking $C^\ast$-algebras:

**Theorem 5.10.** If $\mathfrak{A}$ is a shrinking $C^\ast$-algebra then $\mathfrak{A}$ is compact (usually as a nonunital $C^\ast$-algebra).

**Proof.** Let $\mathfrak{B} = \mathfrak{A}^+$. Since $\text{Rep}(\mathfrak{B})$ and $\Sigma(\mathfrak{B})$ are metrisable spaces, compactness is equivalent to sequential compactness. Moreover, as $\mathfrak{B}$ is shrinking, $\Sigma^0_\infty(\mathfrak{B}) = \emptyset$ and thus $\Sigma^0(\mathfrak{B}) = \Sigma^0_f(\mathfrak{B})$ and $\Sigma(\mathfrak{B}) = \overline{\Sigma^0_f(\mathfrak{B})}$. Therefore it suffices to check sequential compactness for $\Sigma^0_f(\mathfrak{B})$. So take a sequence $(\pi_n)_n \subset \text{Irr}(\mathfrak{A})$; if the (numerical) sequence $(\dim \mathcal{H}_{\pi_n})_n$ has a bounded subsequence then this subsequence is contained in $\bigsqcup_{n=1}^N X_n$ for some $N \in \mathbb{N}$, where $X_n$ are as in the proof of Theorem 5.9. Thus as in the proof of Theorem 5.9 this subsequence has a further subsequence which is convergent; call it $(\pi_{nk_m})_m$. Then the sequence $(\pi_{nk_m}^\infty)_m$ is convergent in $\Sigma(\mathfrak{B})$. On the other hand, if $\dim \mathcal{H}_{\pi_n} \to \infty$ then $\pi_n^\infty \to \delta^\infty$ where $\delta(a + \lambda 1) = \lambda$ is the one-dimensional (irreducible) representation: indeed, for $a \in \mathfrak{A}$ we have

$$||\pi_n^\infty(a + \lambda 1) - \delta^\infty(a + \lambda 1)|| = ||\pi_n^\infty(a)|| \to 0.$$
these notions can be defined once again using not all representations but only
those lying in $\Sigma(\mathfrak{A})$. Obviously, if $L \subset \mathfrak{A}$ is equicontinuous in the sense of
the definition which uses $\text{Rep}(\mathfrak{A})$, then it is also equicontinuous in the sense
of the definition using $\Sigma(\mathfrak{A})$; similarly, all moduli of continuity defined using
$\Sigma(\mathfrak{A})$ will be estimated by those defined using $\text{Rep}(\mathfrak{A})$. All the proofs stay
the same when we replace $\text{Rep}(\mathfrak{A})$ by $\Sigma(\mathfrak{A})$; some difficulties arise in the
context of morphisms between $C^*$-algebras. The reason is that irreducibility
does not behave functorially: to be more precise, if $\pi$ is an irreducible repre-
sentation and $\alpha$ is a $*$-homomorphism between $C^*$-algebras then $\pi \circ \alpha$ need
not be irreducible. Note that if we define the notion of equicontinuity and the
modulus of continuity using only irreducible representations (without taking
the closure) then we obtain equivalent definitions to those obtained with the
use of $\Sigma(\mathfrak{A})$. Thus the distinction between the uniform structure defined with
the aid of $\text{Irr}(\mathfrak{A})$ or $\Sigma(\mathfrak{A})$ and with the aid of $\Sigma(\mathfrak{A})$ and $\text{Rep}(\mathfrak{A})$ is irrelevant.
We will describe how these two are related in the commutative case in
Section 7. We will also give an interpretation of $\text{Rep}(\mathfrak{A})$ at the end of the
paper.

**Definition 5.11.** We say that a $C^*$-algebra $\mathfrak{A}$ has the **Ascoli property**
if the following condition holds: a subset $L \subset \mathfrak{A}$ is relatively compact if and
only if

- $\text{(Asc1)}$ $L$ is bounded,
- $\text{(Asc2)}$ $L$ is pointwise relatively compact, i.e. for any irreducible representa-
tion $\pi$ the set $\{\pi(a) : a \in L\}$ is relatively compact in $\| \cdot \|,$
- $\text{(Asc3)}$ $L$ is equicontinuous.

We say that $\mathfrak{A}$ has the **strong Ascoli property** if the following condition holds:
a subset $L \subset \mathfrak{A}$ is relatively compact if and only if

- $\text{(sAsc1)}$ $L$ is bounded,
- $\text{(sAsc2)}$ $L$ is equicontinuous.

**Remarks 5.12.** Directly from the definition, the strong Ascoli property
implies the Ascoli property.

Obviously the essence of the above definition is that those conditions are
sufficient for relative compactness. This is because every relatively compact
set has all these properties: Boundedness and pointwise relative compactness
are obvious. For the equicontinuity note that if $L$ is relatively compact then
we can consider a compact generating set $K$ such that $\overline{L} \subset K$. Then for any
$x \in L$ and $\pi, \pi' \in \text{Rep}(\mathfrak{A})$ we have

$$\| \pi(x) - \pi'(x) \| \leq \sup_{y \in L} \| \pi(y) - \pi'(y) \| \leq \sup_{y \in K} \| \pi(y) - \pi'(y) \| = d_K(\pi, \pi'),$$

which proves equicontinuity with respect to $\text{Rep}(\mathfrak{A})$ and therefore also with
respect to $\Sigma(\mathfrak{A})$. 
Examples 5.13. (1) Suppose that $\mathfrak{A}, \mathfrak{B}$ have the Ascoli property with respect to $\Sigma(-)$. We will show that $\mathfrak{A} \oplus \mathfrak{B}$ also has the Ascoli property. If $K_1, K_2$ are compact generating sets for $\mathfrak{A}, \mathfrak{B}$ respectively, then $K := K_1 \times \{0\} \cup \{0\} \times K_2$ is compact and generates $\mathfrak{A} \oplus \mathfrak{B}$. Suppose that $L \subset \mathfrak{A} \oplus \mathfrak{B}$ is bounded, pointwise relatively compact and equicontinuous (with respect to $d_K$) and denote by $p_1 : \mathfrak{A} \oplus \mathfrak{B} \to \mathfrak{A}$ and $p_2 : \mathfrak{A} \oplus \mathfrak{B} \to \mathfrak{B}$ the natural projections. Then obviously $p_i(L), i = 1, 2,$ are bounded. For every irreducible representation $\pi$ of $\mathfrak{A}$ we can consider $\tilde{\pi}(a, b) := \pi(a)$; then $\tilde{\pi}$ is irreducible and the set $\tilde{\pi}(L) = \pi(p_1(L))$ is relatively compact. The same argument applies to $p_2$. Fix $\varepsilon > 0$ and choose $\delta > 0$ from the definition of equicontinuity of $L$ such that for any $\pi, \pi' \in \text{Irr}(\mathfrak{A} \oplus \mathfrak{B})$ such that $d_K(\pi, \pi') < \delta$ we have $\|\pi(a, b) - \pi'(a, b)\| < \varepsilon$ for $a, b \in L$. Note that if $\pi_1, \pi_2 \in \text{Irr}(\mathfrak{A})$ and $\tilde{\pi}_1, \tilde{\pi}_2$ denote natural extensions of $\pi_1, \pi_2$ to $\mathfrak{A} \oplus \mathfrak{B}$ then the following holds:

$$d_{K_1}(\pi_1, \pi_2) = \sup\{\|\pi_1(a) - \pi_2(a)\| : a \in K_1\}$$

$$= \sup\{\|\tilde{\pi}_1(a, b) - \tilde{\pi}_2(a, b)\| : (a, b) \in K\} = d_{\tilde{K}}(\tilde{\pi}_1, \tilde{\pi}_2),$$

therefore the same $\delta$ is good for the equicontinuity of $p_1(L)$ (for $p_2$ we argue similarly). From the fact that $\mathfrak{A}$ and $\mathfrak{B}$ have the Ascoli property we conclude that $p_1(L), p_2(L)$ are relatively compact, thus so is $p_1(L) \oplus p_2(L)$ and also $L \subset p_1(L) \oplus p_2(L)$. This proves that $\mathfrak{A} \oplus \mathfrak{B}$ has the Ascoli property with respect to $\Sigma(-)$. A similar argument can be adapted to show that if $\mathfrak{A}, \mathfrak{B}$ have the Ascoli property with respect to $\text{Rep}(-)$ then so does $\mathfrak{A} \oplus \mathfrak{B}$.

(2) Consider $\mathfrak{A} = \mathfrak{K}(l^2)^+$, the unital $C^*$-algebra generated by the compact operators. Then $\mathfrak{A}$ is not compact: Let $K$ be a compact generating set for $\mathfrak{A}$, containing $\{n^{-1}P_n : n \in \mathbb{N}\}$ where $P_n$ is the projection onto $\text{span}\{e_n\}$ where $e_n$ is the $n$th vector of the canonical basis of $l^2$. Consider the unitaries $U_{n,m}$ defined on the orthonormal basis by

$$U_{n,m}(e_k) = \begin{cases} e_k, & k \neq n \text{ and } k \neq m, \\ e_n, & k = m, \\ e_m, & k = n. \end{cases}$$

They satisfy $U_{n,m} = U_{n,m}^* = U_{n,m}^{-1}$. Consider the irreducible representations

$$\pi_{n,m}(A) := U_{n,m}AU_{n,m}^{-1} = U_{n,m}AU_{n,m}^{-1}.$$ 

Then for any distinct $m, m' > 1$ we infer

$$\pi_{1,m}(P_1)e_m = U_{1,m}P_1U_{1,m}e_m = U_{1,m}P_1e_1 = U_{1,m}e_1 = e_m,$$

$$\pi_{1,m'}(P_1)e_m = U_{1,m'}P_1U_{1,m'}e_m = U_{1,m'}P_1e_m = 0.$$ 

Thus

$$d_K(\pi_{1,m}, \pi_{1,m'}) \geq \||\pi_{1,m}(P_1) - \pi_{1,m'}(P_1)\| \geq 1,$$

hence the sequence $(\pi_{1,n})_{n \geq 2}$ does not have a convergent subsequence.
The key in the above argument is the fact that we can consider equivalent, but still distinct representations: the algebra \( R(\ell^2) \) has up to unitary equivalence exactly one irreducible representation (see e.g. [2] Corollaries from Theorem 1.4.4.)

It can be shown that \( \Sigma(\mathcal{A}) \) is not only noncompact, but not even locally compact. Indeed, every irreducible representation \( \pi \in \text{Rep}(\mathcal{A}) \) is of the form \( \pi_U(\cdot) = U(\cdot)U^{-1} \) for some \( U \in \mathcal{U}(\ell^2) \) (except \( \pi = \delta^\infty \)). Thus \( \Sigma^0(\mathcal{A}) \setminus \{\delta^\infty\} \) is the image of \( \mathcal{U}(\ell^2) \) through the mapping \( U \mapsto \pi_U \); since \( \mathcal{U}(\ell^2) \) is connected, so is \( \Sigma^0(\mathcal{A}) \setminus \{\delta^\infty\} \) and therefore also \( \Sigma(\mathcal{A}) \). Suppose that \( \Sigma(\mathcal{A}) \) is locally compact; then from Aleksandrov’s theorem (being a connected metric space) it is necessarily separable and therefore \( \Sigma \)-compact. For \( U \in \mathcal{U}(\ell^2) \) and \( \xi \in \ell^2 \) we have \( UP_1U^*\xi = P_{Ue_1}\xi \) where \( P_{Ue_1} \) is the projection onto \( \text{span}\{Ue_1\} \). As \( U \) runs over all unitaries, \( Ue_1 \) runs over all unit vectors in \( \ell^2 \). Consider the space \( \Sigma(\mathcal{A}) \times \mathbb{R} \) which is again \( \sigma \)-compact. Then the space \( X \) defined as the (continuous) image of \( \Sigma(\mathcal{A}) \times \mathbb{R} \) through the map \( (\pi, t) \mapsto t\pi(P_1)\xi \) (where \( \xi \) is some fixed unit vector) is also \( \sigma \)-compact; but one can show that \( X \) contains an open set (of all vectors not orthogonal to \( \xi \)) and hence it contains some open ball. It remains to note that the ball \( B \) in \( \ell^2 \) is not \( \sigma \)-compact: if \( B = \bigcup_{n=1}^\infty K_n \) where \( K_n \) are compact, then from Baire’s theorem one of \( K_n \)’s must have nonempty interior, which is a contradiction. This argument shows that \( \Sigma(\mathcal{A}) \) is not locally compact.

But \( \mathcal{A} \) still has the Ascoli property: if \( L \) is pointwise relatively compact and \( \iota : \mathcal{A} \rightarrow \mathcal{B}(\ell^2) \) is the (irreducible) identity representation, then \( L = \iota(L) \) is relatively compact.

(3) Let \( S : \ell^2 \rightarrow \ell^2 \) be the unilateral shift operator given by \( S(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots) \). Then \( S \) is a nonunitary isometry, in particular it is not normal. Thus the \( C^* \)-algebra \( \mathcal{A} := C^*(S) \) generated by \( S \) is noncommutative. It is not so hard to show that \( \mathcal{A} \) contains all compact operators; it turns out that \( \mathcal{A} \) is an extension of \( C(\mathbb{T}) \) by \( R(\ell^2) \), i.e. we have the following short exact sequence of \( C^* \)-algebras (see e.g. [6]):

\[
0 \rightarrow R(\ell^2) \rightarrow \mathcal{A} \rightarrow C(\mathbb{T}) \rightarrow 0.
\]

Therefore, as the identity representation of \( \mathcal{A} \) uniquely extends the irreducible identity representation of \( R(\ell^2) \) on \( \mathcal{B}(\ell^2) \), it is also irreducible, so \( \mathcal{A} \) has the Ascoli property. Moreover for a compact generating set for \( \mathcal{A} \) one can take \( K_1 = K \cup \{S\} \) where \( K \) is defined as in the previous example; then as before we conclude that \( \mathcal{A} \) is not compact (or one can use Fact 5.8 together with Example 5.13(2)).

Remark 5.14. The above example can be modified to get a discrete sequence not only in the norm but also in the SOT topology (weaker than
the norm topology). It suffices to take for example

\[ U_n(e_k) = \begin{cases} 
  e_k, & k \neq 1 \text{ and } k \neq n, \\
  \frac{1}{\sqrt{2}}(e_1 - e_n), & k = n, \\
  \frac{1}{\sqrt{2}}(e_1 + e_n), & k = 1.
\]  

We would like to generalize the argument given above and show that if \( \mathfrak{A} \) is compact then every irreducible representation of \( \mathfrak{A} \) is finite-dimensional. For our further purposes we introduce the following:

**Definition 5.15.** For \( T \in \mathcal{B}(\mathcal{H}) \) we define its *orbit* by

\[ \text{Orb}(T) := \{ UTU^{-1} : U \in \mathfrak{U}(\mathcal{H}) \}. \]

**Theorem 5.16.** Let \( \mathcal{H} \) be an infinite-dimensional Hilbert space and let \( T \in \mathcal{B}(\mathcal{H}) \). The following conditions are equivalent:

1. \( \overline{\text{Orb}(T)} \) is compact in \( \| \cdot \| \);
2. \( \overline{\text{Orb}(T)} \) is SOT-compact;
3. there exists \( \lambda \in \mathbb{C} \) such that \( T = \lambda I \).

**Proof.** If \( T = \lambda I \) then \( \overline{\text{Orb}(T)} \) consists of one element.

Thus assume that \( T \) is not of the form \( \lambda I \). Then there exists \( \xi \in \mathcal{H} \) such that \( T\xi \) and \( \xi \) are linearly independent; we can assume that \( \| \xi \| = 1 \). Put

\[ e_1 := \xi, \quad e_2 := \alpha \xi + \beta T\xi, \]

where \( \alpha, \beta \in \mathbb{C} \) are such that \( \| e_2 \| = 1 \) and \( e_1 \perp e_2 \). In particular \( \beta \neq 0 \) and thus \( T e_1 = ae_1 + be_2 \) for some \( a, b \in \mathbb{C} \), \( b \neq 0 \). We can complete \( \{ e_1, e_2 \} \) to an orthonormal system \( \{ e_n \}_{n \in \mathbb{N}} \). Let \( U_n \) be the unitary exchanging \( e_2 \) and \( e_n \). Then indeed \( U_n^* = U_n = U_n^{-1} \) and \( U_n T U_n e_1 = ae_1 + be_n \), whence

\[ \| (U_n T U_n^{-1} - U_m T U_m^{-1}) e_1 \| = \| be_n - be_m \| = \sqrt{2} \| b \| > 0. \]

This shows that \( \text{Orb}(T) \) cannot be relatively SOT-compact.

The rest follows from the fact that the SOT topology is weaker than the norm topology. ■

**Remarks 5.17.** (1) Of course if \( \dim \mathcal{H} < \infty \) then the norm topology and the SOT topology coincide. Moreover, as \( \text{Orb}(T) \) is always a bounded set, \( \overline{\text{Orb}(T)} \) being finite-dimensional, is compact for every \( T \in \mathcal{B}(\mathcal{H}) \). Note also that the mapping

\[ \mathfrak{U}(\mathcal{H}) \ni U \mapsto UTU^{-1} \in \mathcal{B}(\mathcal{H}) \]

is continuous and maps the compact topological group \( \mathfrak{U}(\mathcal{H}) \) onto \( \text{Orb}(T) \). In particular \( \text{Orb}(T) \) is automatically closed.

(2) The above proof shows slightly more: if \( T \) is not of the form \( \lambda I \) then there exists \( \xi \in \mathcal{H} \) such that \( \text{Orb}(T)\xi \subset \mathcal{H} \) is not relatively compact. But
in fact we have the following result, whose proof resembles the proof of the Banach–Alaoglu theorem (see also e.g. the proof of Theorem 5.9):

**Theorem 5.18.** A set $S \subset \mathcal{B}(\mathcal{H})$ is relatively SOT-compact if and only if for any $\xi \in \mathcal{H}$ the set $S\xi$ is relatively compact in $(\mathcal{H}, \| \cdot \|)$.

(3) If $\dim \mathcal{H} = \infty$ and $\mathcal{H}$ is separable, and if $T \in \mathcal{B}(\mathcal{H})$ is normal, then the following result is true (see e.g. [26, Theorem 1.1] and references therein):

**Theorem 5.19.** With the above assumptions, the following conditions are equivalent:

- $S \in \text{Orb}(T)$,
- $S$ is normal, $\sigma(S) = \sigma(T)$ and for any isolated eigenvalue $\lambda \in \sigma(S)$,
  $$\dim \ker(T - \lambda I) = \dim \ker(S - \lambda I).$$

Thus in general $\text{Orb}(T)$ need not be closed.

(4) There exist several different topologies on $\mathcal{B}(\mathcal{H})$: ultraweak (also called $\sigma$-weak), ultrastrong (also called $\sigma$-strong) and also $\ast$-ultrastrong and $\ast$-ultraweak (see e.g. [4], [25]). It turns out that on $\mathcal{U}(\mathcal{H})$ all these topologies coincide and are equal to the SOT (or equivalently WOT) topology (see [4, Prop. I.3.2.9]). However the closures with respect to these topologies may differ:

- $\overline{\mathcal{U}(\mathcal{H})} = \mathcal{U}(\mathcal{H})$,
- $\overline{\mathcal{U}(\mathcal{H})}^{\text{SOT}} = \{ T \in \mathcal{B}(\mathcal{H}) : T \text{ is an isometry} \}$,
- $\overline{\mathcal{U}(\mathcal{H})}^{\text{WOT}} = \{ T \in \mathcal{B}(\mathcal{H}) : \| T \| \leq 1 \}$.

**Theorem 5.20.** If $\mathfrak{A}$ is a (separable) compact $C^*$-algebra and $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ is an irreducible representation then $\dim \mathcal{H}_\pi < \infty$.

**Proof.** For $U \in \mathcal{U}(\mathcal{H}_\pi)$ denote $\pi_U := U \pi U^{-1}$. This is an irreducible representation. Since $\mathfrak{A}$ is compact, $\Sigma(\mathfrak{A})$ is a compact space, in particular the family $\{ \pi_U(a) : U \in \mathcal{U}(\mathcal{H}_\pi) \}$ is relatively compact for each $a \in \mathfrak{A}$. But if we put $A := \pi(a)$ then this family is equal to $\text{Orb}(A)$, so $\dim \mathcal{H}_\pi < \infty$.

**Corollary 5.21.** If $\mathfrak{A}$ is a separable, infinite-dimensional and simple $C^*$-algebra then $\mathfrak{A}$ has the Ascoli property and is not compact.

**Proof.** Since $\mathfrak{A}$ is simple, every (nonzero) representation is faithful, in particular every irreducible representation is faithful, and therefore $\mathfrak{A}$ has the Ascoli property. On the other hand, since $\mathfrak{A}$ is infinite-dimensional, it does not have a finite-dimensional (faithful) irreducible representation and therefore it cannot be compact.

**Example 5.22.** We will describe a nontrivial example of a simple infinite-dimensional $C^*$-algebra, defined via generators and relations. This algebra is called a noncommutative torus or an (irrational) rotation algebra and is
usually denoted by $A_\theta$, where $\theta \in [0,1]$ is an (irrational) parameter. The algebra $A_\theta$ is defined as the universal unital $C^*$-algebra generated by two unitaries $u,v$ satisfying $vu = e^{2\pi i\theta}uv$. Universality is understood as follows: whenever $\mathfrak{A}$ is another unital $C^*$-algebra containing two unitaries $U,V \in \mathfrak{A}$ which satisfy $VU = e^{2\pi i\theta}UV$, there is a morphism $\alpha : A_\theta \to \mathfrak{A}$ such that $\alpha(u) = U$ and $\alpha(v) = V$. If $\theta = 0$ we obtain the $C^*$-algebra isomorphic to the algebra $C(\mathbb{T}^2)$; this justifies the name “noncommutative torus”. For $\theta = p/q$ (where $p,q$ are relatively prime, $q > 0$) there exists a (complex) vector bundle $E$ of rank $q$ such that $A_\theta \cong \Gamma(\mathbb{T}^2, \text{End}(E))$ (see e.g. [16, Prop. 1.1.1]). Then $A_\theta$, although noncommutative, is quite close to a commutative algebra (the precise notion is *Morita equivalence*—see e.g. [22, Chapter 3]). Thus the most interesting case is when $\theta$ is irrational. Then $A_\theta$ is a simple algebra (see [9, Thm. VI.1.4]). $A_\theta$ can be represented on $L^2(S^1)$ as follows:

$$(Uf)(t) = e^{2\pi t}f(t), \quad Vf(t) = f(t + \theta)$$

where the circle $S^1$ is viewed as $\mathbb{R}/\mathbb{Z}$. In the light of the above results, $A_\theta$ (for $\theta$ irrational) is not compact but has the Ascoli property. More about $A_\theta$ and its various incarnations can be found in [16].

**Example 5.23.** There are examples of unital $C^*$-algebras which are noncompact but have the property that every irreducible representation is finite-dimensional. To give an example, put $\mathfrak{A} := \bigoplus_{n=1}^\infty M_{2^n}$ and consider the ideal $\mathfrak{J} := \bigoplus_{n=1}^\infty M_{2^n} \subset \mathfrak{A}$. Define also $\mathfrak{F} = \{ (A_n)_{n=1}^\infty \in \mathfrak{A} : A_1$ is a diagonal matrix, $A_{n+1} = A_n \oplus A_n, \ n \in \mathbb{N} \}$; then $\mathfrak{F}$ is a commutative $C^*$-algebra, isomorphic to $\mathbb{C} \oplus \mathbb{C}$. It follows that $\mathfrak{J} \cap \mathfrak{F} = 0$. Let $\mathfrak{A}_0 = \mathfrak{F} + \mathfrak{J}$. Since $\mathfrak{F}$ is finite-dimensional, $\mathfrak{A}_0$ is closed in $\mathfrak{A}$, hence $\mathfrak{A}_0$ is also a $C^*$-algebra. We will show that $\mathfrak{A}_0$ is not compact but has every irreducible representation finite-dimensional.

We have the following short exact sequence of $C^*$-algebras:

$$0 \to \mathfrak{J} \to \mathfrak{A}_0 \to \mathfrak{F} \to 0.$$ 

Since $\mathfrak{J} = \bigoplus_{n=1}^\infty M_{2^n}$, we have $\text{Irr}(\mathfrak{J}) = \bigsqcup_n \text{Irr}(M_{2^n})$. But $\text{Irr}(M_k) = \{ U(\cdot)U^{-1} : U \in \mathfrak{U}_n \}$, so every irreducible representation of $\mathfrak{J}$ is finite-dimensional. As $\mathfrak{F}$ is of finite dimension, so is $\mathfrak{F}$. Since $\mathfrak{A}_0$ is an extension of $\mathfrak{F}$ by $\mathfrak{J}$, it also has the property of having only finite-dimensional irreducible representations.

However $\mathfrak{A}_0$ is not compact: To see this fix $n \in \mathbb{N}$ and for $U \in \mathfrak{B}(\mathbb{C}^{2^n})$ consider the representation given by $\pi_{n,U}(a) := UA_nU^{-1}$ where $a = (A_n)_{n \in \mathbb{N}}$, $A_n \in M_{2^n}(\mathbb{C})$. Viewed as a representation of $\mathfrak{J}$, $\pi_{n,U}$ is irreducible. Therefore defining $\pi_{n,U}$ by the same formula but on the whole $\mathfrak{A}_0$ we also get an irreducible representation. Thus $\rho_n := \pi_0 \circ \pi_{n,U} \in \Sigma(\mathfrak{A}_0)$. Consider the matrix $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $V^2 = I$, so $V$ is a unitary and $V^{-1} = V$. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $VAV^{-1} = \begin{bmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{bmatrix}$. 
Consider unitary operators on $\mathbb{C}^{2^n}$ defined by
\[
U_n = I_2 \oplus \cdots \oplus I_2 \oplus V
\]
and take the associated irreducible representations $\pi_n, U_n$. For a compact set $K$ generating $\mathfrak{A}_0$, take any compact set containing $a = (A, A \oplus A, A \oplus A \oplus A, \ldots) \in \mathfrak{F}$ where for example $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Assume that $n < m$. Then
\[
d_K(\rho_n, \rho_m) \geq \|\rho_n(a) - \rho_m(a)\|
\]
\[
= \|\mathbb{R}_0 \circ U_n(A^{\oplus 2^{n-1}})U_n - \mathbb{R}_0 \circ U_m(A^{\oplus 2^{m-1}})U_m\|
\]
\[
= \|\mathbb{R}_0 \circ ((0_{2^n-2} \oplus B) \oplus (0_{2^n-2} \oplus B) \oplus \cdots \oplus (0_{2^n-2} \oplus B) \oplus 0_{2^n})\|
\]
\[
= \|(0_{2^n-2} \oplus B)\| = \|B\|
\]
where $B = VAV^{-1} - A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$. In this way we obtain an infinite discrete subset of $\Sigma(\mathfrak{A}_0)$, thus $\mathfrak{A}_0$ cannot be compact.

Remark 5.24. The ideal $I$ above is an example of a shrinking (nonunital) $C^*$-algebra, therefore it is compact. Obviously $\mathfrak{F}$ is also compact (being commutative or finite-dimensional). Therefore the above example shows that compactness is not preserved by taking extensions; in fact it also shows that the property of being shrinking is not closed under extensions.

To prove the relation between compactness and a strong Ascoli property, we need

Lemma 5.25. Suppose that $X, Y$ are metric spaces, $X$ is compact and $Y$ is complete, and let $K \subset C(X,Y)$ be an equicontinuous family. Then the set
\[
F := \{ x \in X : K(x) \text{ relatively compact} \}
\]
is closed where $K(x) := \{ f(x) : f \in K \}$.

Proof. Since $Y$ is complete, relative compactness is equivalent to total boundedness, so $F = \{ x \in X : K(x) \text{ totally bounded} \}$. Fix $x \in F$ and $\varepsilon > 0$. From equicontinuity there exists $\delta > 0$ such that for any $f \in K$ and $a, b \in X$ such that $d_X(a, b) < \delta$ we have $d_Y(f(a), f(b)) < \varepsilon/2$. As $x \in F$, there is $a \in F$ with $d_X(x, a) < \delta$. Since $a \in F$, $K(a)$ is completely bounded, so there exist $z_1, \ldots, z_N \in Y$ with
\[
K(a) \subset \bigcup_{j=1}^{N} B(z_j, \varepsilon/2).
\]
We claim that
\[
(5.1) \quad K(x) \subset \bigcup_{j=1}^{N} B(z_j, \varepsilon).
\]
Let \( f \in K \); as \( d_X(x,a) < \delta \) we have \( d_Y(f(x),f(a)) < \varepsilon/2 \); moreover \( f(a) \in K(a) \) so there is \( j \in \{1, \ldots, N\} \) such that \( f(a) \in B(z_j, \varepsilon/2) \). Consequently,

\[
d_Y(f(x),z_j) \leq d_Y(f(x),f(a)) + d_Y(f(a),z_j) < \varepsilon/2 + \varepsilon/2 = \varepsilon,
\]

which shows \([5.1]\). In this way we obtain an \( \varepsilon \)-net in \( Y \) for \( K(x) \), which shows that \( x \in F \). \( \blacksquare \)

**Corollary 5.26.** Under the notation and assumptions above, if the set \( K(x) \) is relatively compact for every \( x \) from some dense set \( D \subset X \) then the same is true for every \( x \in X \).

**Theorem 5.27.** If \( \mathfrak{A} \) is a compact \( C^* \)-algebra then \( \mathfrak{A} \) has the strong Ascoli property.

**Proof.** Let \( L \subset \mathfrak{A} \) be bounded and equicontinuous. For every \( x \in L \) consider \( \hat{x} \) as a mapping \( \hat{x} : \Sigma(\mathfrak{A}) \to \mathfrak{B}(\ell^2) \). Then it is a (uniformly) continuous mapping, defined on a compact metric space (and with values in a metric space), so we can use the (classical) Ascoli theorem. We check that the assumptions of the theorem are fulfilled:

1. Since \( L \) is equicontinuous, the family \( \{ \hat{x} : x \in L \} \) is equicontinuous.
2. Let \( \pi \in \Sigma(\mathfrak{A}) \) be of the form \( \mathfrak{N}_0 \odot \pi' \) where \( \pi' \) is irreducible. As \( \mathfrak{A} \) is a compact \( C^* \)-algebra, \( \pi' \) is finite-dimensional, say \( \pi' : \mathfrak{A} \to M_n \). Thus

\[
\{ \hat{x}(\pi) : x \in L \} = \{ \pi(x) : x \in L \} = \{ \mathfrak{N}_0 \odot \pi'(x) : x \in L \}.
\]

Clearly the mapping \( M_n \ni A \mapsto \mathfrak{N}_0 \odot A \in \mathfrak{B}(\ell^2) \) is an isometry, thus it is enough to show that \( \{ \pi'(x) : x \in L \} \) is relatively compact—but it is a bounded subset of \( \mathbb{C}^n \).

Since \( \Sigma_f(\mathfrak{A}) \) is dense in \( \Sigma(\mathfrak{A}) = \Sigma(\mathfrak{A}) \), the set \( \pi(L) \) is relatively compact for any \( \pi \in \Sigma(\mathfrak{A}) \) (Corollary 5.26). Therefore from the Ascoli theorem the set \( \{ \hat{x} : x \in L \} \) is relatively compact. As \( x \mapsto \hat{x} \) is an isometry (see e.g. the proof of Theorem 5.37 below), \( L \) is also relatively compact. \( \blacksquare \)

**Remark 5.28.** Note that without knowing Theorem 5.20 (and Corollary 5.26) one can show quite similarly the weaker result that every compact \( C^* \)-algebra has the Ascoli property.

### 5.2. Pointwise and uniform convergence

**Definition 5.29.** We say that a net \((a_s)_s \subset \mathfrak{A}\) tends to \( a \in \mathfrak{A} \) **pointwise** if for any representation \( \pi \in \Sigma(\mathfrak{A}) \) we have \( \| \pi(a_s) - \pi(a) \| \to 0 \).

**Remarks 5.30.** It is clear that if \( \mathfrak{A} = C(X) \) is a commutative \( C^* \)-algebra then every representation belonging to \( \Sigma(\mathfrak{A}) \) is automatically irreducible and of the form \( f \mapsto f(x) \) for some \( x \in X \). Therefore the above defined convergence is just the ordinary pointwise convergence. On the other hand, if in the above definition we require convergence for all elements in Rep(\( \mathfrak{A} \))
Uniformity in $C^*$-algebras

instead of $\Sigma(\mathfrak{A})$ then we get the definition of norm convergence: this is simply due to the fact that we could consider a faithful representation.

The above convergence comes from a locally convex topology, determined by the family of seminorms defined by $a \mapsto \|\pi(a)\|$, $\pi \in \Sigma(\mathfrak{A})$.

All $C^*$-algebra operations (linear operations, multiplication, involution) are continuous with respect to this topology—for example, to check the continuity of multiplication note that if $a_s \to a$, $b_s \to b$ pointwise and $\pi \in \Sigma(\mathfrak{A})$, then

$$\|\pi(a_s b_s) - \pi(ab)\| \leq \|\pi(a_s)\pi(b_s) - \pi(a)\pi(b)\| + \|\pi(a)\pi(b_s) - \pi(a)\pi(b)\|$$

$$\leq \|\pi(a_s) - \pi(a)\| \cdot \|\pi(b_s)\| + \|\pi(a)\| \|\pi(b_s) - \pi(b)\| \to 0$$
as the net $(\|\pi(b_s)\|)_s$ is bounded for sufficiently large $s$.

In the context of (not necessarily commutative) compact $C^*$-algebras we have the following version of the classical Dini theorem:

**Theorem 5.31.** Let $\mathfrak{A}$ be a compact, unital $C^*$-algebra and $(a_n)_n$ an increasing sequence converging pointwise to $a$. Then $a_n \to a$ in norm.

**Proof.** Denote $b_n := a - a_n$; then $b_n \geq 0$ and $b_n \to 0$. Fix $\varepsilon > 0$ and let $K_n = \{ \pi \in \Sigma(\mathfrak{A}) : \|\pi(b_n)\| < \varepsilon \}$. As $(\pi(b_n))_n$ is a decreasing sequence of nonnegative elements, $(\|\pi(b_n)\|)_n$ is decreasing, thus $(K_n)_n$ is an increasing sequence of open sets. From our assumption we have $\Sigma(\mathfrak{A}) = \bigcup_{n=1}^{\infty} K_n$. Since $\Sigma(\mathfrak{A})$ is compact, there exists $N \in \mathbb{N}$ such that $K_n = \Sigma(\mathfrak{A})$ for every $n > N$. Thus if $n > N$, then for any $\pi \in \Sigma(\mathfrak{A})$ we have $\|\pi(b_n)\| < \varepsilon$, so $\sup_{\pi \in \Sigma(\mathfrak{A})} \|\pi(b_n)\| = \|b_n\| < \varepsilon$, which shows the convergence in norm. $\blacksquare$

Next we describe the relation between the above convergence and weak convergence (in the sense of Banach space theory). We will need the following result (see for example [21]):

**Theorem 5.32 (Choquet).** Let $E$ be a Banach space and $K$ a compact, convex and metrisable subset of $E^*$. Then any $\psi \in K$ is of the form

$$\psi(x) = \int_{\text{Ex}(K)} \varphi(x) d\mu(\varphi)$$

for some probability measure $\mu$ on the set $\text{Ex}(K)$ of extreme points of $K$.

**Lemma 5.33.** If $(x_n)_n \subset \mathfrak{A}$ is a bounded sequence, then $x_n \to 0$ weakly if and only if $\varphi(x_n) \to 0$ for every pure state $\varphi$.

**Proof.** Suppose that $\varphi(x_n) \to 0$ for every $\varphi \in \mathcal{P}(\mathfrak{A})$. If we define $f_n : \mathcal{P}(\mathfrak{A}) \to \mathbb{C}$ by $f_n(\varphi) = \varphi(x_n)$ then all $f_n$'s are $*$-weakly continuous and $\|f_n\| = \|x_n\|$, thus $(f_n)_n$ is uniformly bounded. Our assumption says that $f_n \to 0$ pointwise, thus from the Lebesgue theorem, for every probability measure $\mu$ on $\mathcal{P}(\mathfrak{A})$ we have $\int_{\mathcal{P}(\mathfrak{A})} f_n(\varphi) d\mu(\varphi) = \int_{\mathcal{P}(\mathfrak{A})} \varphi(x_n) d\mu(\varphi) \to 0$. 


Since the set $\mathcal{S}(\mathfrak{A})$ of all states is $*$-weakly compact, convex and metrisable (if $\mathfrak{A}$ is separable), from the Choquet theorem every $\psi \in \mathcal{S}(\mathfrak{A})$ is of the form

$$
\psi(x) = \int \varphi(x) d\mu(\varphi)
$$

for some probability measure $\mu$ on $\mathcal{P}(\mathfrak{A})$. Thus $\psi(x_n) \to 0$ for any state $\psi$. Since every continuous linear functional on $\mathfrak{A}$ is a linear combination of (at most four) states, $x_n \to 0$ weakly.

**Theorem 5.34.** Let $\mathfrak{A}$ be a separable, unital $C^*$-algebra and $(x_n)_n \subset \mathfrak{A}$ be a bounded sequence. The following conditions are equivalent:

1. $x_n \to 0$ weakly;
2. $\pi(x_n) \xrightarrow{WOT} 0$ for any irreducible representation $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$;
3. $\pi(x_n) \xrightarrow{WOT} 0$ for any representation $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$.

**Proof.** Assume that $x_n \to 0$ weakly, fix an arbitrary representation $\pi$ and vectors $\xi, \eta \in \mathcal{H}_\pi$. Define $\varphi : \mathfrak{A} \to \mathbb{C}$ by

$$
\varphi(x) := \langle \pi(x)\xi, \eta \rangle.
$$

Then $\varphi$ is a continuous linear functional, thus $\varphi(x_n) \to 0$, which shows that $\pi(x_n) \xrightarrow{WOT} 0$.

Now assume that $\pi(x_n) \xrightarrow{WOT} 0$ for any irreducible representation $\pi$, and fix $\varphi \in \mathcal{P}(\mathfrak{A})$. Then $\varphi$ gives rise to an irreducible representation $\pi_{\varphi} : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_\varphi)$ such that

$$
\varphi(x) = \langle \pi_{\varphi}(x)\xi_{\varphi}, \xi_{\varphi} \rangle
$$

where $\xi_{\varphi} \in \mathcal{H}_{\varphi}$ is a (unit) cyclic vector. From our assumption $\langle \pi(x_n)\xi_{\varphi}, \xi_{\varphi} \rangle \to 0$, so $\varphi(x_n) \to 0$, hence from the lemma above, $x_n \to 0$ weakly.

Clearly if every irreducible representation of $\mathfrak{A}$ is finite-dimensional, then for such a representation $\pi$ the WOT-convergence $\pi(x_n) \to 0$ is equivalent to convergence in norm and we get

**Corollary 5.35.** If $\mathfrak{A}$ is a unital CCR-algebra then for a bounded sequence $(x_n)_n \subset \mathfrak{A}$ the following conditions are equivalent:

- $x_n \to 0$ pointwise,
- $x_n \to 0$ weakly.

In particular, the above equivalence takes place when $\mathfrak{A}$ is a compact $C^*$-algebra.

We can also consider the following convergence:

**Definition 5.36.** We say that a net $(a_s)_s$ converges to $a$ uniformly if $\|\pi(a_s) - \pi(a)\| \to 0$ uniformly with respect to $\pi \in \Sigma(\mathfrak{A})$. 
Directly from the definition we see that the uniform convergence $a_n \to a$ is just the uniform convergence of $\hat{a}_n$ to $\hat{a}$ as continuous functions $\Sigma(\mathcal{A}) \to \mathcal{B}(\ell^2)$. In particular, this convergence comes from some metric, so we can restrict our attention to ordinary sequences instead of nets. It turns out that this convergence coincides with norm convergence:

**Theorem 5.37.** A sequence $(a_n)_n \subset \mathcal{A}$ converges to $a \in \mathcal{A}$ uniformly iff $\|a_n - a\| \to 0$.

*Proof.* From the formula $\|x\| = \sup\{\|\pi(x)\| : \pi \in \text{Rep}(\mathcal{A})\} = \sup\{\|\pi(x)\| : \pi \in \text{Irr}(\mathcal{A})\}$ (see [10, 2.7.1 and 2.7.3]) it follows that

$$\|x\| = \sup\{\|\pi(x)\| : \pi \in \Sigma(\mathcal{A})\},$$

hence the theorem follows. ■

### 6. Uniform continuity of morphisms

**6.1. Two equivalent approaches.** Let $X, Y$ be compact metric spaces and $\alpha : C(X) \to C(Y)$ be a $*$-homomorphism (preserving the units). Then $\alpha$ induces a continuous mapping $u : Y \to X$ such that $\alpha(f) = f \circ u$. Then for any $y_1, y_2 \in Y$,

$$|\alpha(f)(y_1) - \alpha(f)(y_2)| = |f(u(y_1)) - f(u(y_2))| \leq \omega_f(d_X(u(y_1), u(y_2))) \leq \omega_u(w_u(d_Y(y_1, y_2)))$$

where $\omega_f, \omega_u$ are (minimal) moduli of continuity for $f$ and $u$ (resp.). Hence if $\omega_{\alpha(f)}$ denotes the minimal modulus of continuity for $\alpha(f) \in C(Y)$, then

$$\omega_{\alpha(f)} \leq \omega_f \circ \omega_u.$$

For noncommutative considerations assume that $\mathcal{A}, \mathcal{B}$ are separable, unital $C^*$-algebras and $K, L$ are compact generating sets for $\mathcal{A}$ and $\mathcal{B}$ (resp.). The argument given above serves as a motivation for the following:

**Definition 6.1.** A $*$-homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ is called uniformly continuous if there exists $\omega \in \Omega$ such that

$$\omega_{\alpha(a)}^L \leq \omega_a^K \circ \omega$$

for any $a \in \mathcal{A}$.

We denote $\omega_{\alpha}^{K,L} = \inf\{\omega \in \Omega : \omega_{\alpha(a)}^L \leq \omega_a^K \circ \omega, \ a \in \mathcal{A}\}$.

**Fact 6.2.** The definition above does not depend on the choice of compact sets $K, L$ generating $\mathcal{A}$ and $\mathcal{B}$.

*Proof.* Suppose that $\alpha : (\mathcal{A}, K) \to (\mathcal{B}, L)$ is a uniformly continuous $*$-homomorphism and $K', L'$ are other compact sets generating $\mathcal{A}, \mathcal{B}$ (resp.). It follows that $\omega_{\alpha(a)}^L \leq \omega_a^K \circ \omega_{\alpha}^{K,L}$ and

$$\omega_{\alpha(a)}^L \leq \omega_{\alpha(a)}^L \circ \omega_{\alpha}^{L'} \leq \omega_a^K \circ \omega_{\alpha}^{K,L} \circ \omega_{\alpha}^{L'} \leq \omega_a^K \circ \omega_a^{K'} \circ \omega_{\alpha}^{K,L} \circ \omega_{\alpha}^{L'},$$

(6.1)
which proves the uniform continuity of \( \alpha \) as a mapping \( (\mathfrak{A}, K') \to (\mathfrak{B}, L') \): in fact we get 
\[
\omega_{a\alpha}^{K', L'} \leq \omega_{K'}^L \circ \omega_a^{K, L} \circ \omega_{L'}^L. \]

If \( \alpha : \mathfrak{A}_1 \to \mathfrak{A}_2 \) and \( \beta : \mathfrak{A}_2 \to \mathfrak{A}_3 \) are two *-homomorphisms of \( C^* \)-algebras and \( K_i, i = 1, 2, 3 \), are compact sets generating \( \mathfrak{A}_i \) then
\[
\omega_{\beta\alpha(a)}^{K_3} \leq \omega_{\alpha(a)}^{K_2} \circ \omega_\beta \leq \omega_{\alpha(a)}^{K_1} \circ (\omega_a \circ \omega_\beta),
\]
which shows that \( \omega_{\beta\alpha} \leq \omega_a \circ \omega_\beta \). In particular, the composition of uniformly continuous morphisms is again uniformly continuous.

Theorem 4.12 shows that the identity morphism \( \text{id} : (\mathfrak{A}, K) \to (\mathfrak{A}, K') \) is uniformly continuous. In fact a more general statement is valid:

**Theorem 6.3.** If \( \alpha : \mathfrak{A} \to \mathfrak{B} \) is a *-homomorphism then \( \alpha \) is uniformly continuous.

**Proof.** Fix compact sets \( K, L \) generating \( \mathfrak{A} \) and \( \mathfrak{B} \) respectively and let \( a_0 \in \mathfrak{A} \). We claim that for \( t \geq 0 \),
\[
(6.2) \quad f_{\alpha(a_0)}^L(t) \leq f_{\alpha(K)}^L(t).
\]
Fix \( \tau_1, \tau_2 \in \text{Rep}(\mathfrak{B}) \) such that \( d_L(\tau_1, \tau_2) \leq t \) and put \( \pi_i := \tau_i \circ \alpha \in \text{Rep}(\mathfrak{A}), i = 1, 2 \). Then
\[
sup\{\|\tau_1(\alpha(a)) - \tau_2(\alpha(a))\| : a \in K\} \leq sup\{\|\tau(\alpha(a)) - \tau'(\alpha(a))\| : a \in K, d_L(\tau, \tau') \leq t\}.
\]
The left hand side is \( d_K(\pi_1 \circ \alpha, \pi_2 \circ \alpha) = d_K(\pi_1, \pi_2) \) and the right hand side is \( sup\{\|\tau(b) - \tau'(b)\| : b \in \alpha(K), d_L(\tau, \tau') \leq t\} = f_{\alpha(K)}^L(t) \). Therefore if \( d_L(\tau_1, \tau_2) \leq t \) then
\[
(6.3) \quad d_K(\pi_1, \pi_2) \leq f_{\alpha(K)}^L(t).
\]
Now the left hand side of (6.2) is \( sup\{\|\tau(\alpha(a_0)) - \tau'(\alpha(a_0))\| : d_L(\tau, \tau') \leq t\} \), while the right hand side is \( sup\{\|\pi(a_0) - \pi'(a_0)\| : d_K(\pi, \pi') \leq f_{\alpha(K)}^L(t)\} \). Thus (6.3) shows that the set over which we take the supremum on the left hand side of (6.2) is contained in the corresponding set on the right hand side, and for \( \pi_i, \pi_\alpha \) as above we have \( \|\pi_1(a_0) - \pi_2(a_0)\| = \|\pi_\alpha(a_0) - \tau_2(\alpha(a_0))\| \).

It suffices to take the suprema in (6.2) to conclude \( \omega_{a_0}^L \circ \omega_a^L \leq \omega_a^L \circ \omega_a^L \).

In particular \( \omega_{\alpha(a_0)}^{K, L} \leq \omega_{\alpha(K)}^{L} \).

We can also consider an alternative definition:

**Definition 6.4.** A *-homomorphism \( \alpha : \mathfrak{A} \to \mathfrak{B} \) is called uniformly continuous if the mapping
\[
\alpha^* : (\text{Rep}(\mathfrak{B}), d_L) \ni \pi \mapsto \pi \circ \alpha \in (\text{Rep}(\mathfrak{A}), d_K)
\]
is uniformly continuous as a mapping between metric spaces.
If $\alpha$ is an epimorphism we can also consider the mapping from $\Sigma(\mathcal{B})$ to $\Sigma(\mathcal{A})$ and require it to be uniformly continuous.

Note that if $\alpha : \mathcal{A} \to \mathcal{B}$ is an epimorphism then $\alpha^*$ is a monomorphism as a mapping $\Sigma(\mathcal{B}) \to \Sigma(\mathcal{A})$ as well as $\text{Rep}(\mathcal{B}) \to \text{Rep}(\mathcal{A})$. If $(\pi_n)_n \subset \text{Rep}(\mathcal{B})$ converges to $\pi \in \text{Rep}(\mathcal{B})$ in the point-norm topology (thus in the topology of $d_K$ for any compact generating set $K$) then $\pi_n(b) \to \pi(b)$ for every $b \in \mathcal{B}$. Thus if $\alpha : \mathcal{A} \to \mathcal{B}$ is a *-homomorphism then in particular for every $a \in \mathcal{A}$ we have $\pi_n(\alpha(a)) \to \pi(\alpha(a))$ and thus $\alpha^*(\pi_n) = \pi_n \circ \alpha \to \pi \circ \alpha = \alpha^*(\pi)$ in the point-norm topology. Therefore $\alpha^*$ is always a continuous mapping.

If $\alpha : \mathcal{A} \to \mathcal{B}$ and $\beta : \mathcal{B} \to \mathcal{C}$ are *-homomorphisms of $C^*$-algebras then $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$, thus the composition of uniformly continuous *-homomorphisms is still a uniformly continuous *-homomorphism. In fact, uniform continuity is an automatic property of every *-homomorphism:

**Theorem 6.5.** If $\alpha : \mathcal{A} \to \mathcal{B}$ is a *-homomorphism then $\alpha$ is uniformly continuous.

**Proof.** First note that the above definition does not depend on the choice of compact generating sets $K$ and $L$: if $K' \subset \mathcal{A}$ and $L' \subset \mathcal{B}$ are other such sets then from Theorem 4.9 the metrics $d_K$ and $d_{K'}$ as well as $d_L$ and $d_{L'}$ are uniformly equivalent. Therefore the uniform continuity of $\alpha^*$ does not depend on the choice of the metrics $d_K, d_L$. Thus let $K \subset \mathcal{A}$ and $L \subset \mathcal{B}$ be compact generating sets and assume that $\alpha(K) \subset L$. Then for $\pi, \pi' \in \text{Rep}(\mathcal{B})$,

$$d_K(\alpha^*(\pi), \alpha^*(\pi')) = d_K(\pi \circ \alpha, \pi' \circ \alpha) = \sup_{a \in K} \|\pi(\alpha(a)) - \pi'(\alpha(a))\|$$

$$\leq \sup_{b \in L} \|\pi(b) - \pi'(b)\| = d_L(\pi, \pi'),$$

which shows that $\alpha^*$ is uniformly continuous. ■

Thus for an arbitrary *-homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ the mapping

$$\alpha^* : (\text{Rep}(\mathcal{B}), d_L) \to (\text{Rep}(\mathcal{A}), d_K)$$

has a modulus of continuity, denoted by $\tilde{\omega}_{\alpha}^{K,L}$. Then

$$d_K(\pi \circ \alpha, \pi' \circ \alpha) \leq \tilde{\omega}_{\alpha}^{K,L}(d_L(\pi, \pi')),$$

$\pi, \pi' \in \text{Rep}(\mathcal{B}).$

**Fact 6.6.** In the above notation, $\omega_{\alpha}^{K,L} = \tilde{\omega}_{\alpha}^{K,L}.

**Proof.** For $\omega_{\alpha}^{K,L} \leq \tilde{\omega}_{\alpha}^{K,L}$ it suffices to show that $f_{\alpha(a)}^L \leq f_{\alpha(a)}^K \circ \tilde{\omega}_{\alpha}^{K,L}$ for any $a \in \mathcal{A}$; it is thus enough to prove that for $t \geq 0$,

$$\sup_{d_L(\pi, \pi') \leq t} \|\pi(\alpha(a)) - \pi'(\alpha(a))\|$$

$$\leq \sup\{\|\pi(a) - \pi'(a)\| : d_K(\pi, \pi') \leq \tilde{\omega}_{\alpha}^{K,L}(t)\}.$$
Take $\pi, \pi' \in \text{Rep}(\mathcal{B})$ satisfying $d_L(\pi, \pi') \leq t$. Then putting $\tau := \pi \circ \alpha$ and $\tau' := \pi' \circ \alpha$ we infer from (6.4) that
\[ d_K(\tau, \tau') \leq \tilde{\omega}_\alpha^K L(d_L(\pi, \pi')). \]
Hence $\tau, \tau'$ belong to the set over which we take the supremum on the right hand side of (6.5), which proves (6.5).

For the opposite inequality it is enough to show that for $\pi, \pi' \in \text{Rep}(\mathcal{B})$,
\[ d_K(\pi \circ \alpha, \pi' \circ \alpha) \leq \omega^K_L(d_L(\pi, \pi')). \]
If $a_0 \in K$ and $\pi, \pi' \in \text{Rep}(\mathcal{A})$ satisfy $d_K(\pi, \pi') \leq t$ then $\|\pi(a_0) - \pi'(a_0)\| \leq t$ and taking the supremum over all such representations we obtain $f^K_{a_0}(t) \leq t$, thus for any $t \geq 0$ we conclude $\omega^K_{a_0}(t) \leq t$. Thus for any $a \in K$ and any $\pi, \pi' \in \text{Rep}(\mathcal{B})$,
\[ \|\pi(\alpha(a)) - \pi'(\alpha(a))\| \leq \omega^K_{\alpha(a)}(d_L(\pi, \pi')) \leq (\omega^K_\alpha \circ \omega^K_{\alpha L})(d_L(\pi, \pi')), \]
and taking the supremum over $a \in K$ we get (6.6).

The above result shows that both definitions of uniform continuity produce the same modulus of continuity for a $\ast$-homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$. Thus we are allowed to write $\omega^K_{\alpha L}$ or briefly $\omega_\alpha$ when it is understood which compact generating sets $K, L$ we have chosen.

### 6.2. Convergence of morphisms.

On the set $\text{Hom}(\mathcal{A}, \mathcal{B})$ we can consider:

- the point-norm topology: in this topology, $\alpha_s \to \alpha$ iff $\alpha_s(a) \xrightarrow{\|\|} \alpha(a)$ for every $a \in \mathcal{A}$,
- compact-open topology: in this topology $\alpha_s \to \alpha$ iff for any compact set $K \subset \mathcal{A}$ we have
\[ \sup_{a \in K} \|\alpha_s(a) - \alpha(a)\| \to 0. \]

We can also define a topology using a metric in a similar fashion to what we did for $\text{Rep}(\mathcal{A})$: we choose a compact generating set $K \subset \mathcal{A}$ and put
\[ d_K(\alpha, \beta) := \sup\{\|\alpha(a) - \beta(a)\| : a \in K\} \quad \text{for} \quad \alpha, \beta \in \text{Hom}(\mathcal{A}, \mathcal{B}). \]

**FACT 6.7.** The formula (6.7) defines a metric on $\text{Hom}(\mathcal{A}, \mathcal{B})$.

**Proof.** The proof is similar to the proof of Fact 4.1. 

Moreover the topology of $d_K$ on $\text{Hom}(\mathcal{A}, \mathcal{B})$ behaves similarly to the case of $\text{Rep}(\mathcal{A})$:

**THEOREM 6.8.** The topology of $d_K$ coincides with the point-norm topology and with the compact-open topology.

**Proof.** See the proof of Theorem 4.4. 

Remark 6.9. We can also define the convergence $\alpha_n \to \alpha$ using the condition $\alpha_n^* \to \alpha^*$. As for any $x \in \mathcal{A}$ we have $\|x\| = \sup \{\|\pi(x)\| : \pi \text{ an irreducible representation}\} = \sup \{\|\pi(x)\| : \pi \in \text{Rep}(\mathcal{A})\}$, we get

$$
\sup_{\pi \in \Sigma(\mathcal{B})} d_K(\pi \circ \alpha, \pi \circ \beta) = \sup_{a \in K} \sup_{\pi \in \Sigma(\mathcal{B})} \|\pi(\alpha(a)) - \pi(\beta(a))\| = d_K(\alpha, \beta)
$$

thus the convergence so defined coincides with point-norm convergence (whether we view $\alpha$ as acting between $\Sigma(\mathcal{B})$’s or $\text{Rep}(\mathcal{B})$’s).

Finally, we can show that the space $(\text{Hom}(\mathcal{A}, \mathcal{B}), d_K)$ is complete using a similar proof to that for $(\text{Rep}(\mathcal{A}), d_K)$. The above results do not need the assumption of $\mathcal{B}$ separable—in particular for $\mathcal{B} = \mathcal{B}(\ell^2)$ we get our previous results for representations.

If $\mathcal{A}, \mathcal{B}$ are commutative $C^*$-algebras then $\mathcal{A} \cong C(X)$ and $\mathcal{B} \cong C(Y)$ for some compact spaces $X, Y$. Then $\text{Hom}(\mathcal{A}, \mathcal{B})$ can be identified with $C(Y, X)$ (all continuous mappings from $Y$ to $X$). Therefore we can think of elements of $\text{Hom}(\mathcal{A}, \mathcal{B})$ as continuous mappings between “noncommutative” (nonexistent) compact spaces. In such circumstances we can ask about an analogue of the Ascoli theorem, i.e. about necessary and sufficient conditions for a given family $\mathcal{F} \subset \text{Hom}(\mathcal{A}, \mathcal{B})$ to be (pre)compact. We will use the notation $\mathcal{F}(a) := \{\alpha(a) : \alpha \in \mathcal{F}\} \subset \mathcal{B}$.

Theorem 6.10. Let $\mathcal{A}, \mathcal{B}$ be unital separable $C^*$-algebras and assume that $\mathcal{F} \subset \text{Hom}(\mathcal{A}, \mathcal{B})$. The following conditions are equivalent:

- The family $\mathcal{F}$ is precompact (in the point-norm topology).
- For every $a \in \mathcal{A}$ the family $\mathcal{F}(a) \subset \mathcal{B}$ is precompact.

Proof. Since for every $a \in \mathcal{A}$ the mapping $\mathcal{F} \ni \alpha \mapsto \alpha(a) \in \mathcal{F}(a) \subset \mathcal{B}$ is continuous, if $\mathcal{F}$ is precompact then so is $\mathcal{F}(a)$.

Conversely, suppose that for every $a \in \mathcal{A}$ the family $\mathcal{F}(a) \subset \mathcal{B}$ is precompact. Consider the mapping $\iota : \mathcal{F} \ni \alpha \mapsto (\alpha(a))_{a \in \mathcal{A}} \in \prod_{a \in \mathcal{A}} \mathcal{F}(a)$

where in the codomain we consider the point-norm topology (the same as the product topology). Then $\iota$ is an embedding. Similarly to the proof of Theorem 5.9 we conclude that $\iota$ is closed. As $\mathcal{F}(a)$ is compact for each $a \in \mathcal{A}$, so is the product $\prod_{a \in \mathcal{A}} \mathcal{F}(a)$. Thus the closed subset $\iota(\mathcal{F}) = \mathcal{F}$ is compact as well and so $\mathcal{F}$ is precompact.

Corollary 6.11. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital, separable $C^*$-algebras and let $\mathcal{F} \subset \text{Hom}(\mathcal{A}, \mathcal{B})$. 

• If $\mathcal{B}$ has the Ascoli property then $F$ is relatively compact if and only if for every $a \in \mathcal{A}$ the set $F(a)$ is bounded, pointwise relatively compact and equicontinuous.

• If $\mathcal{B}$ has the strong Ascoli property then $F$ is relatively compact if and only if for every $a \in \mathcal{A}$ the set $F(a)$ is bounded and equicontinuous.

On $\text{Hom}(\mathcal{A}, \mathcal{B})$ we can also define a topology using the following notion of convergence: $\alpha_s \to \alpha$ if for any $\pi \in \Sigma(\mathcal{B})$ we have $\pi \circ \alpha_s \to \pi \circ \alpha$ in the point-norm topology. In other words, this is the pointwise convergence $\alpha_s \to \alpha^*$ where $\alpha_s^* : \Sigma(\mathcal{B}) \to \text{Rep}(\mathcal{A})$. If $\alpha_n \to \alpha$ in the point-norm topology then for every $a \in \mathcal{A}$ we get $\alpha_n(a) \to \alpha(a)$ and for every $\pi \in \Sigma(\mathcal{B})$ we have $\pi(\alpha_n(a)) \to \pi(\alpha(a))$. The converse need not hold in general but in some cases it does hold, as the following examples show:

**Examples 6.12.** (1) Let $\mathcal{A}_1 = C(X)$ and $\mathcal{A}_2 = C(Y)$ be commutative $C^*$-algebras and $\alpha_n, \alpha : C(Y) \to C(X)$ be $*$-homomorphisms. Then there exist continuous mappings $u_n, u : X \to Y$ such that $\alpha_n(f) = f \circ u_n$ and $\alpha(f) = f \circ u$. Then the point-norm convergence $\alpha_n^* \to \alpha^*$ is equivalent to $\|f \circ u_n - f \circ u\|_\infty \to 0$ (and this is equivalent to the uniform convergence $u_n \to u$). On the other hand, the convergence defined above is equivalent to the pointwise convergence $f \circ u_n \to f \circ u$ (which is equivalent to the pointwise convergence $u_n \to u$).

(2) Let $\mathcal{A} = \mathbb{R}(\ell^2)^+$. Then the identity representation is irreducible and thus point-norm convergence is equivalent to the above defined convergence.

Finally, we can also consider the uniform convergence $\alpha_n^* \to \alpha^*$—but this yields nothing new, since it is equivalent to point-norm convergence (or convergence in the metric $d_K$):

$$
\sup_{\pi \in \Sigma(\mathcal{B})} d_K(\alpha_n^*(\pi), \alpha^*(\pi)) = \sup_{\pi \in \Sigma(\mathcal{B})} \sup_{a \in K} \|\pi(\alpha_n(a)) - \pi(\alpha(a))\|
= \sup_{a \in K} \|\alpha_n(a) - \alpha(a)\| = d_K(\alpha_n, \alpha).
$$

7. Comparison of moduli of continuity with respect to $\Sigma(-)$ and $\text{Rep}(-)$. Let $(X, d)$ be a metric space with diameter $R > 0$ and let

$$
K := \{f : X \to [-R/2, R/2] : \text{Lip}(f) \leq 1\}
$$

where $\text{Lip}(f)$ denotes the Lipschitz constant for $f$. Equivalently one can put $K := \{f : X \to \mathbb{R} : \text{Lip}(f) \leq 1, f(a) = 0 \text{ for some } a\}$. Then $d(x, y) = \sup_{f \in K} |f(x) - f(y)|$. For $f : X \to \mathbb{R}$ we see that $f$ is a contraction, i.e. $\text{Lip}(f) \leq 1$, if and only if there exists $c \in \mathbb{R}$ such that $f - c \in K$. Thus for a real valued function $f$ the condition $\text{Lip}(f) \leq s$ is equivalent to the existence of $c \in \mathbb{R}$ such that $(1/s)f - c \in K$. 
Now take an arbitrary \( f \in C(X; \mathbb{R}) \). The (classical) modulus of continuity \( \omega := \omega_f \) for \( f \) coincides with the modulus of continuity \( \omega_f^\Sigma \) for \( f \) viewed as an element in a \( C^* \)-algebra, constructed using irreducible representations, and we have \( \omega_f = \omega_f^\Sigma \leq \omega_f^\text{Rep} \) where \( \omega_f^\text{Rep} \) is the modulus of continuity for \( f \) constructed using all representations in \( \text{Rep}(C(X)) \). We will show that indeed equality holds for a real-valued function. We will need the so called Fenchel transform (also called convex conjugate). Consider a finite-dimensional Euclidean space \( E \) with inner product \( \langle \cdot, \cdot \rangle \) and functions \( h : E \to [-\infty, \infty] \).

We define the Fenchel conjugate \( h^* : E \to [-\infty, \infty] \) of \( h \) by

\[
h^*(\varphi) = \sup_{x \in E} (\langle \varphi, x \rangle - h(x)).
\]

If \( h \) is not identically \(+\infty\) then \( h^*(\varphi) > -\infty \) for every \( \varphi \in E \). One can check that \( h^* \) is always a convex function and \( h^{**} \leq h \) (see [5, remarks preceding Thm. 4.2.1]). It is natural to ask when \( h = h^{**} \) or in other words when \( h = g^* \) for some function \( g \). The answer is provided by the following theorem (see [5, Thm. 4.2.1]):

**Theorem 7.1 (Fenchel–Moreau).** Let \( h : E \to (-\infty, \infty] \) be any function. The following conditions are equivalent:

1. \( h^{**} = h \),
2. \( h = g^* \) for some function \( g : E \to (-\infty, \infty] \),
3. \( h \) is convex and the set \( \{(x, t) \in E \times (-\infty, \infty] : h(x) \leq t\} \) is closed,
4. for any \( x \in E \), \( h(x) = \sup \{\alpha(x) : \alpha \leq h, \alpha \text{ an affine function}\} \).

For \( s \geq 0 \) put

\[
\delta_f(s) = \frac{1}{2} \sup_{t \geq 0} (\omega_f(t) - st).
\]

**Remark 7.2.** One can show that \( \delta_f(s) \) is the distance from \( f \) to the set of all Lipschitz functions with Lipschitz constant \( \leq s \), that is, \( \delta_f(s) = \inf \{\|f - u\|_\infty : \text{Lip}(u) \leq s\} \).

Using the Fenchel transform, one can obtain the relation between \( \delta_f \) and the (classical) modulus of continuity for \( f \):

**Theorem 7.3.** With the above notation and assumption,

\[
\omega_f(t) = \inf_{s \geq 0} (2\delta_f(s) + st).
\]

**Proof.** Denote \( \omega := \omega_f \), \( \delta := \delta_f \) and

\[
\alpha(t) := \begin{cases} -\omega(-t) & \text{if } t \leq 0, \\ +\infty & \text{if } t > 0, \end{cases}
\]

and also extend \( \delta \) by putting \( \delta(s) = +\infty \) for \( s < 0 \). Then \( \alpha \) is a convex
function. Moreover, for $s \geq 0$ we have
\[
\alpha^*(s) = \sup_{t \in \mathbb{R}} (st - \alpha(t)) = \sup_{t \leq 0} (st - \alpha(t)) = \sup_{t \leq 0} (st + \omega(-t)) = \sup_{t \geq 0} (\omega(t) - st) = 2\delta(s)
\]
(for $s < 0$ we also have $\alpha^*(s) = 2\delta(s)$ since both sides are infinite). Hence $\alpha$ satisfies the assumptions of the Fenchel–Moreau theorem: indeed, $\alpha$ takes finite values on a closed interval and is continuous there, thus the set $\{(t, u) \in \mathbb{R} \times (-\infty, \infty) : \alpha(t) \leq u\}$ is closed. Hence from the Fenchel–Moreau theorem, $\alpha^{**} = \alpha$. Therefore
\[
\alpha(s) = (2\delta(s))^* = \sup_{t \in \mathbb{R}} (st - 2\delta(t)) = \sup_{t \geq 0} (st - 2\delta(t)),
\]
hence for $s \leq 0$ we get
\[
\omega(-s) = -\sup_{t \geq 0} (st - 2\delta(t)) = \inf_{t \geq 0} (2\delta(t) - st).
\]
Substituting $-s$ for $s \geq 0$ we get the desired result. \]

A function realising the distance in the formula for $\delta$ is
\[
f_s := \delta_f(s) + \inf_{y \in X} (f(y) + sd(\cdot, y)).
\]
Then $\text{Lip}(f_s) \leq s$ (indeed, $\text{Lip}(sd(\cdot, y)) \leq s$; taking a translation and then an infimum does not affect the Lipschitz constant), therefore there exists $c \in \mathbb{R}$ such that $(1/s)f_s - c \in K$. Moreover $\|f - f_s\|_{\infty} = \delta_f(s)$, since
\[
f(x) - f_s(x) \leq \delta_f(s) \iff f(x) - \delta_f(s) - \inf_{y \in X} \{f(y) + sd(x, y)\} \leq \delta_f(s)
\]
\[
\iff f(x) + \sup_{y \in X} \{-f(y) - sd(x, y)\} \leq 2\delta_f(s)
\]
\[
\iff \sup_{y \in X} \{f(x) - f(y) - sd(x, y)\} \leq 2\delta_f(s);
\]
but the last inequality is satisfied since
\[
f(x) - f(y) - sd(x, y) \leq |f(x) - f(y)| - sd(x, y)
\]
\[
\leq \omega(d(x, y)) - sd(x, y) \leq 2\delta_f(s)
\]
and it suffices to take the supremum.

Conversely,
\[
f_s(x) - f(x) \leq \delta_f(s) \iff \delta_f(s) + \inf_{y \in X} \{f(y) + sd(x, y)\} - f(x) \leq \delta_f(s)
\]
\[
\iff \inf_{y \in X} \{f(y) - f(x) + sd(x, y)\} \leq 0;
\]
but for $y = x$ we get $f(y) - f(x) + sd(x, y) = 0$, thus the infimum of this expression is $\leq 0$. We have shown that $|f(x) - f_s(x)| \leq \delta_f(s)$, hence
\[ \| f - f_s \| \leq \delta_f(s). \] As \( \delta_f \) is the infimum of \( \| f - u \| \) where \( \text{Lip}(u) \leq s \), we get \( \| f - f_s \| = \delta_f(s) \).

Now let \( \pi, \pi' \in \text{Rep}(C(X)) \); in this case
\[
\| \pi(f_s) - \pi'(f_s) \| = s \left\| \pi \left( \frac{1}{s} f_s - c \right) - \pi' \left( \frac{1}{s} f_s - c \right) \right\| \leq sd_K(\pi, \pi'),
\]
hence
\[
\| \pi(f) - \pi'(f) \| \leq \| \pi(f) - \pi(f_s) \| + \| \pi(f_s) - \pi'(f_s) \| + \| \pi'(f_s) - \pi'(f) \|
\leq \| f - f_s \| + sd_K(\pi, \pi') + \| f - f_s \| = 2\delta(s) + sd_K(\pi, \pi').
\]
Fixing \( t \geq 0 \) and taking the supremum gives
\[
\sup\{ \| \pi(f) - \pi'(f) \| : \pi, \pi' \in \text{Rep}(C(X)), d_K(\pi, \pi') \leq t \}
\leq \inf_{s \geq 0} (2\delta(s) + st) = \omega(t)
\]
which yields \( \omega_f^{\text{Rep}} = \omega \). Thus for a real valued function it does not matter whether we consider its modulus of continuity defined using all representations or using only irreducible ones.

In the general (complex-valued) case, for \( f \in C(X) \) with \( f = u + iv \) where \( u, v \) are real-valued, we obtain
\[
\omega_f = \omega_{u+iw} \leq \omega_u + \omega_v,
\]
\[
\omega_u = \omega((f + f^*)/2) \leq \sqrt{\omega_f/2 + \omega_f}/2 = \omega_f.
\]
Similarly
\[
\omega_v = \omega((f - f^*)/(2i)) \leq \omega_f/2 + \omega_f/2 \leq \omega_f,
\]
from which it follows that
\[
(7.1) \quad \omega_f \leq \omega_f^{\text{Rep}} \leq \omega_u^{\text{Rep}} + \omega_v^{\text{Rep}} = \omega_u + \omega_v \leq 2\omega_f.
\]

**8. Interpretation of Rep(−).** We have heavily used the space \( \text{Rep}(\mathcal{A}) \) in our considerations, so let us briefly discuss its interpretation in the commutative case where \( \mathcal{A} = C(X) \). It turns out that
\[
\text{Rep}(C(X)) = \{ \text{spectral measures on } X \}.
\]
We briefly explain how we understand the above identification: Let \( \pi : C(X) \to \mathcal{B}(H) \) be a *-representation and let \( \xi, \eta \in H \). Consider the mapping
\[
\varphi_{\xi,\eta} : C(X) \ni f \mapsto \langle \pi(f)\xi, \eta \rangle \in \mathbb{C}.
\]
This is a bounded linear functional: \( \| \varphi_{\xi,\eta} \| \leq \| \xi \| \| \eta \| \), thus from the Riesz representation theorem there exists exactly one (regular, Borel) measure \( \mu_{\xi,\eta} \) such that
\[
\langle \pi(f)\xi, \eta \rangle = \int_X f \, d\mu_{\xi,\eta}
\]
and \( \| \varphi_{\xi,\eta} \| = \| \mu_{\xi,\eta} \| (= |\mu_{\xi,\eta}|(X)) \leq \| \xi \| \| \eta \| \). Fix a Borel set \( A \) and consider the mapping

\[ (\xi, \eta) \mapsto \mu_{\xi,\eta}(A). \]

Then it is a sesquilinear, bounded form on \( \mathcal{H} \), so it corresponds to the unique (bounded) operator which we denote by \( E(A) \). By a direct calculation we check that

- \( 0 \leq E(A) \leq I \),
- \( E(X) = I \),
- \( E(\bigcup_{n=1}^{\infty} A_n) = \text{WOT-} \sum_{n=1}^{\infty} E(A_n) \) for a sequence \( \{A_n\}_{n \in \mathbb{N}} \) of pairwise disjoint Borel sets.

A property which is nontrivial is \( E(A \cap B) = E(A)E(B) \). This is true for all sets which are measurable with respect to the \( \sigma \)-algebra generated by the closed \( G_\delta \) sets. In the metrisable case every closed set is automatically \( G_\delta \), so the equality holds for all sets which are measurable with respect to the \( \sigma \)-algebra generated by all closed sets, i.e. for all Borel sets. In the general, not necessarily metrisable case, one can use the regularity of the measures \( \mu_{\xi,\eta} \) together with the fact that for any pair of sets \( K, U \) such that \( K \) is closed, \( U \) is open and \( K \subset U \) one can find a closed \( G_\delta \) set \( F \) such that \( K \subset F \subset U \).

In this context it is worth mentioning the formula due to Kantorovich which allows one to extend the metric from the space \( X \) to the space of all probability measures on \( X \). In more detail, if \( (X,d) \) is a metric space then one can extend the metric \( d \) to the set \( \text{Prob}(X) \) of all regular, Borel, probability measures on \( X \) using the formula

\[ \tilde{d}(\mu, \nu) = \sup \left\{ \left| \int_X f \, d\mu - \int_X f \, d\nu \right| : f : X \to \mathbb{R}, f \text{ is a contractive map} \right\}. \]

We are using here the identification \( x \simeq \delta_x \) where

\[ \delta_x(A) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \]

Then

\[ \left| \int_X f \, d\delta_x - \int_X f \, d\delta_y \right| = |f(x) - f(y)|, \]

hence \( \tilde{d}(\delta_x, \delta_y) = d(x, y) \), thus indeed we get an extension of \( d \). It is not hard to show that \( \tilde{d} \) indeed defines a metric. Moreover one can describe the topology induced by \( \tilde{d} \):

**Theorem 8.1.** The topology induced by the metric \( \tilde{d} \) coincides with the weak-\( \ast \) topology.

**Proof.** The proof is analogous to the proof of Theorem 4.4. \( \blacksquare \)
What is more, the mapping \((X,d) \mapsto (\text{Prob}(X), \tilde{d})\) is functorial: any continuous map \(f : (X,d_X) \to (Y,d_Y)\) induces \(f_* : (\text{Prob}(X), d_X) \to (\text{Prob}(Y), d_Y)\) via transport of measure, \(f_*(\mu) = \mu \circ f^{-1}\). Those pushforwards satisfy \((f \circ g)_* = f_* \circ g_*\). If \((X,d)\) is an infinite metric space then \((\text{Prob}(X), \tilde{d})\) is homeomorphic to the Hilbert cube (this follows from Keller’s theorem, see e.g. [3, Theorem 3.1]).

9. Further problems. Finally let us briefly discuss the questions which are natural and for which we did not give the answer.

We have investigated various properties of compact \(C^*\)-algebras: in particular we have seen that the class of compact \(C^*\)-algebras is closed under taking ideals, direct sums and quotients. It is natural to ask the following:

PROBLEM 9.1. Is it true that if \(\mathfrak{A}\) is a compact \(C^*\)-algebra (unital and separable) and \(\mathfrak{B} \subset \mathfrak{A}\) is a \(C^*\)-subalgebra then \(\mathfrak{B}\) is compact as well?

Assume that \(K\) is a compact set generating a \(C^*\)-algebra \(\mathfrak{A}\). Then for \(a \in \mathfrak{A}\) we can consider the moduli of continuity \(\omega^K_a\) and define the set \(\tilde{K} = \{a \in \mathfrak{A} : \omega^K_a(t) \leq t, t \in \mathbb{R}_{\geq 0}\}\). Then obviously \(K \subset \tilde{K}\), so in particular \(\tilde{K}\) also generates \(\mathfrak{A}\). Obviously \(\tilde{K}\) is not bounded since for \(\lambda \in \mathbb{C}\) we have \(\omega^K_{\lambda 1} = 0\). But still we can define \(K' := \tilde{K} \cap B\) where \(B\) is the (closed) ball with radius \(\text{diam} \, K\). In this way we obtain a bounded generating set, hence it makes sense to consider the metric \(d_{K'}\).

PROBLEM 9.2. What is the topology of \(d_{K'}\)?

REMARK 9.3. Note that obviously \(K'\) is equicontinuous and bounded, thus if \(\mathfrak{A}\) has the strong Ascoli property then \(K'\) is compact and \(d_{K'}\) is uniformly equivalent to \(d_K\). In particular this will hold for compact \(C^*\)-algebras.

In our considerations we have defined the notion of pointwise convergence using the family of seminorms \(a \mapsto \|\pi(a)\|\) where \(\pi \in \Sigma(\mathfrak{A})\). We can also consider only those seminorms which come from irreducible representations (or more precisely representations of the form \(\mathbb{K}_0 \odot \pi\) where \(\pi\) is irreducible). A natural question is whether these two notions coincide:

PROBLEM 9.4. Is it possible to construct a net \((a_s)_s \subset \mathfrak{A}\) such that for any irreducible representation \(\pi \in \text{Irr}(\mathfrak{A})\) we get \(\pi(a_s) \to 0\) but \((a_s)_s\) does not converge to 0 pointwise?

Finally it would be interesting to know what the set \(\Sigma(\mathfrak{A})\) may look like for various \(C^*\)-algebras. For example we have already seen that if \(\mathfrak{A}\) is an \(N\)-subhomogeneous \(C^*\)-algebra then \(\Sigma(\mathfrak{A})\) contains at most \(N\)-dimensional representations (more precisely, representations of the form \(\mathbb{K}_0 \odot \pi\) where \(\pi\) is at most \(N\)-dimensional).
Problem 9.5. Is it true that if $\mathfrak{A}$ is a unital CCR-algebra then it may happen that $\Sigma(\mathfrak{A})$ contains a representation which is not of the form $\aleph_0 \otimes \pi$ where $\pi$ is finite-dimensional?

References

[1] N. Aronszajn and P. Panitchpakdi, *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. 6 (1956), 405–439.
[2] W. Arveson, *An Invitation to $C^*$-algebras*, Springer, New York, 1976.
[3] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, PWN, Warszawa, 1975.
[4] B. Blackadar, *Operator Algebras. Theory of $C^*$-algebras and von Neumann Algebras*, Springer, Berlin, 2006.
[5] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization*, Springer, 2000.
[6] L. Coburn, *The $C^*$-algebra generated by an isometry*, Bull. Amer. Math. Soc. 73 (1967), 722–726.
[7] A. Connes, *Compact metric spaces, Fredholm modules and hyperfiniteness*, Ergodic Theory Dynam. Systems 9 (1989), 207–220.
[8] A. Connes, *Noncommutative Geometry*, Academic Press, London, 1994.
[9] K. R. Davidson, *$C^*$-algebras by Example*, Amer. Math. Soc., Providence, RI, 1991.
[10] J. Dixmier, *$C^*$-algebras*, North-Holland, Amsterdam, 1977.
[11] J. M. G. Fell, *The structure of algebras of operator fields*, Acta Math. 106 (1961), 233–280.
[12] J. M. Gracia-Bondia, H. Figueroa and J. Varilly, *Elements of Noncommutative Geometry*, Birkhäuser, Boston, 2001.
[13] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Vol. 1*, Academic Press, New York, 1983.
[14] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Vol. 2*, Academic Press, London, 1986.
[15] L. Kantorovich, *A new method of solving of some classes of extremal problems*, Dokl. Akad. Nauk SSSR 28 (1940), 211–214.
[16] M. Khalkhali, *Basic Non-Commutative geometry*, Eur. Math. Soc., 2009.
[17] K. Kuratowski, *Topology, Vol. 1*, PWN, Warszawa, 1966.
[18] F. Latrémoilière, *Quantum locally compact metric spaces*, J. Funct. Anal. 264 (2013), 362–402.
[19] P. Niemiec, *Elementary approach to homogeneous $C^*$-algebras*, Rocky Mountain J. Math. 45 (2015), 1591–1630.
[20] P. Niemiec, *Models for subhomogeneous $C^*$-algebras*, arXiv:1310.5595 (2013).
[21] R. R. Phelps, *Lecture Notes on Choquet’s Theorem*, Springer, Berlin, 2001.
[22] I. Raeburn and D. Williams, *Morita Equivalence and Continuous Trace $C^*$-algebras*, Math. Surveys Monogr. 60, Amer. Math. Soc., Providence, RI, 1998.
[23] M. Rieffel, *Compact quantum metric spaces*, in: Operator Algebras, Quantization and Noncommutative Geometry, Contemp. Math. 365, Amer. Math. Soc., Providence, RI, 2004, 315–330.
[24] M. Rieffel, *Metrics on state spaces*, Documenta Math. 4 (1999), 559–600.
[25] S. Sakai, *$C^*$-algebras and $W^*$-algebras*, Springer, Berlin, 1971.
D. Sherman, *Unitary orbits of normal operators in von Neumann algebras*, J. Reine Angew. Math. 605 (2007), 95–132.

M. Takesaki, *Theory of Operator Algebras I*, Springer, Berlin, 2002.

N. B. Vasil’ev, *$C^*$-algebras with finite-dimensional irreducible representations*, Russian Math. Surveys 21 (1966), 137–155.

N. E. Wegge-Olsen, *K-theory and $C^*$-algebras. A Friendly Approach*, Oxford Univ. Press, Oxford, 1993.

Adam Wegert
Faculty of Applied Mathematics
AGH University of Science and Technology
al. Mickiewicza 30
30-059 Kraków, Poland
E-mail: a_wegert@o2.pl