Baryonic Torii

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Abstract

We study a Skyrme-type model with a potential term motivated by Bose-Einstein condensates (BECs), which we call the BEC Skyrme model. We consider two flavors of the model, the first is the Skyrme model and the second has a sixth-order derivative term instead of the Skyrme term; both with the added BEC-motivated potential. The model contains toroidally shaped Skyrmions and they are characterized by two integers $P$ and $Q$, representing the winding numbers of two complex scalar fields along the toroidal and poloidal cycles of the torus, respectively. The baryon number is $B = PQ$. We find stable Skyrmion solutions for $P = 1, 2, 3, 4, 5$ with $Q = 1$, while for $P = 6$ and $Q = 1$ it is only metastable. We further find that configurations with higher $Q > 1$ are all unstable and split into $Q$ configurations with $Q = 1$. 

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I. INTRODUCTION

Half a century has passed since Skyrme proposed [1] that Skyrmions characterized by the topological charge $\pi_3(S^3) \simeq \mathbb{Z}$ describe nucleons in the pion effective field theory or the chiral Lagrangian [2], where the Skyrme term, i.e. quartic in derivatives, is needed to stabilize Skyrmions against shrinkage. Although nucleons are now known to be bound states of quarks, the idea of the Skyrme model is still attractive. In fact, the Skyrme model is still valid as a low-energy description of QCD, has only a small number of parameters and is, for instance, used also in holographic QCD [3, 4].

Meanwhile in condensed matter physics, considerable efforts have been made recently to realize stable 3-dimensional Skyrmions in two-component Bose-Einstein condensates (BECs) [5–8] (see Ref. [9] for a review of two-component BECs). In Ref. [8], the creation of a Skyrmion is proposed to be a consequence of the annihilation of a brane and an anti-brane [10]. At strong coupling, these systems reduce to the SU(2) principal chiral model, but the existence of Skyrmions is elusive due to the lack of the Skyrme term (or an even higher order derivative term) [48]. One interesting feature in these systems is that a potential term, breaking the SU(2) symmetry is present, which deforms the (would-be) Skyrmion to the shape of a torus [5]. Consequently, the Skyrmion can be interpreted [5, 8, 11] as a vorton [12–15], that is, a vortex ring in the first component with the second component flowing inside said ring.

In this paper, we consider a Skyrme-like model with a potential term in the form $V = m^2|\phi_1|^2|\phi_2|^2$ which was introduced in our previous papers [16, 17] and is motivated by two-component BECs [5, 6, 8], where we use a notation of two complex scalar fields $\phi_1(x)$ and $\phi_2(x)$ with the constraint $|\phi_1|^2 + |\phi_2|^2 = 1$ along the lines of two-component BECs. For higher-derivative terms needed to stabilize Skyrmion, we consider either the conventional fourth-order derivative term, i.e. the Skyrme term or a sixth-order derivative term, which is the baryon charge density squared (see, e.g. Refs. [17–19]); for a short-term notation we will call the first case the 2+4 model and the second case the 2+6 model. We construct stable Skyrmions which were elusive in two-component BECs in the absence of the Skyrme term or other higher-order derivative terms, and find that they take the shape of a torus as two-component BECs. We find that the most general solutions are characterized by two integers $P$ and $Q$, representing the winding numbers of the scalar fields $\phi_1$ and $\phi_2$ along the toroidal and poloidal cycles of the torus, respectively, and show that the baryon number or the Skyrmion number of $\pi_3(S^3) \simeq \mathbb{Z}$ is $B = PQ$ (which is also known...
as the linking number). We explicitly construct stable Skyrmion solutions with $P = 1, 2, 3, 4, 5$ and $Q = 1$, yielding the baryon numbers $B = 1, 2, 3, 4, 5$. We also construct the $P = 6, Q = 1$ solution and find that it is metastable, i.e. is energetically prone to decay into two $B = 3$ objects. This turns out to be the case for both the 2+4 and the 2+6 model.

The energy and baryon charge distributions of the configuration of $P = 1$ are spherically symmetric in the 2+4 model, whereas in the 2+6 model it is a deformed ball (with a hint of a torus-like shape). The configurations with $P > 1$ are all of toroidal shapes (for both models). This is in contrast to the conventional Skyrmions (i.e. without our BEC-motivated potential) for which the configuration of $B = 1$ is spherically symmetric, that of $B = 2$ is toroidal, and those of $B > 1$ have energy distributions with some point symmetry. We compare our $B = 2$ solutions in the 2+4 and 2+6 models to those of the conventional model (i.e. without the BEC-motivated potential), and find that the energy distribution of the solution in the 2+6 model is a surface of a torus while the energy distributions of the solutions in the 2+4 model and the conventional model are solid torii, i.e. filled torii.

Although the classification of our solutions is given by the integers $P$ and $Q$, we find that configurations with $Q > 1$ are unstable, that is, a configuration with $(P, Q)$ decays into $Q$ copies of the $(P, 1)$ configuration.

We also note that our configurations can be identified as global analogues of vortons [12–14], that is, twisted closed global vortex strings as in two-component BECs [49]. While vortices in this model are global vortices so that straight vortices have divergent energy per unit length, a closed string has finite energy because of cancellation of vorticity. A vortex in the field $\phi_1$ traps the field $\phi_2$ in its core and has the U(1) phase modulus of $\phi_2$. The integers $Q$ and $P$ are identified with the winding numbers of the vortex of the $\phi_1$ field and of the $\phi_2$ field along the ring inside the vortex core, respectively. The identification of the Skyrmions with global vortex rings also explains why configurations with higher $Q > 1$ are unstable. This is because $Q$ is the winding number of the vortex in the field $\phi_1$, and a global vortex with higher winding is unstable to decay as two global vortices repel each other.

This paper is organized as follows. In Sec. II, we present our model and explain the symmetries and topological structures of the model. In Sec. III, we construct a domain wall and a global vortex which serve as constituents of the torus. Finally, in Sec. IV, we construct toroidal Skyrmions which are the strings wrapped up on a circle and we further study their stability. Sec. V is devoted to a summary and discussions. In Appendix A, we show that solutions with $P = 1, 2$ and $Q = 2$ are
unstable to decay into two configurations of $P = 1, 2$ and $Q = 1$. In Appendix B, we compare our $B = 2$ solutions in the 2+4 and 2+6 models and that in the conventional models (i.e. without the BEC-motivated potential).

II. A SKYRME-LIKE MODEL WITH BEC-MOTIVATED POTENTIAL

We consider the SU(2) principal chiral model with the addition of the Skyrme term and a sixth-order derivative term in $d = 3 + 1$ dimensions. In terms of the SU(2)-valued field $U(x) \in SU(2)$, the Lagrangian which we are considering is given by

$$L = \frac{f_\pi^2}{16} \text{tr} (\partial_{\mu} U^\dagger \partial^{\mu} U) + L_4 + L_6 - V(U),$$

where we use the mostly-negative metric and the higher-derivative terms are given by

$$L_4 = \frac{\kappa}{32e^2} \text{tr} ([U^\dagger \partial_{\mu} U, U^\dagger \partial_{\nu} U]^2),$$

$$L_6 = \frac{c_6}{36e^4 f_\pi^2} (\epsilon^{\mu\nu\rho\sigma} \text{tr} [U^\dagger \partial_{\nu} U U^\dagger \partial_{\rho} U U^\dagger \partial_{\sigma} U])^2.$$

The symmetry of the Lagrangian for $V = 0$ is $\tilde{G} = \text{SU}(2)_L \times \text{SU}(2)_R$ acting on $U$ as $U \rightarrow U' = g_L U g_R^\dagger$. The requirement of a finite-energy configuration, however, spontaneously breaks this symmetry down to $\tilde{H} \simeq \text{SU}(2)_{L+R}$, which in turn acts as $U \rightarrow U' = gU g^\dagger$ so that the target space is $\tilde{G}/\tilde{H} \simeq \text{SU}(2)_{L-R}$. The conventional potential term, i.e. the pion mass term, is $V = m_\pi^2 \text{tr} (21_2 - U - U^\dagger)$, which breaks the symmetry $\tilde{G}$ to SU(2)$_{L+R}$ explicitly.

In this paper, it will prove convenient to use the following notation where we express the field $U$ in terms of two complex scalar fields, $\phi^T = (\phi_1(x), \phi_2(x))$, as

$$U = \begin{pmatrix} \phi_1 - \phi_2^* \\ \phi_2 \phi_1^* \end{pmatrix},$$

subject to the constraint

$$\det U = |\phi_1|^2 + |\phi_2|^2 = 1.$$
which we can write the static Lagrangian density as
\begin{equation}
-\mathcal{L} = \frac{1}{2} \partial_i \phi^\dagger \partial_i \phi + \frac{\kappa}{4} \left[ (\partial_i \phi^\dagger \partial_i \phi)^2 - \frac{1}{4} (\partial_i \phi^\dagger \partial_j \phi + \partial_j \phi^\dagger \partial_i \phi)^2 \right] + \frac{c_6}{4} \left( \epsilon^{ijk} \phi^\dagger \partial_i \phi \partial_j \phi^\dagger \partial_k \phi \right)^2
+ V(\phi, \phi^*). \tag{6}\end{equation}

The full symmetry \( \tilde{G} \) is not manifest in terms of \( \phi \), where only SU(2)\( _L \) is manifest but SU(2)\( _R \) is not. The U(1) subgroup generated by \( \sigma_3 \) in SU(2)\( _R \), however, is manifest and acts on \( \phi \) as \( \phi \to e^{i\alpha} \phi \), constituting a U(2) group with SU(2)\( _L \).

The target space (the vacuum manifold with \( m = 0 \)) \( M \simeq SU(2) \simeq S^3 \) has a nontrivial homotopy group
\begin{equation}
\pi_3(M) = \mathbb{Z}, \tag{7}\end{equation}
which admits Skyrmions as usual. The baryon number (the Skyrme charge) of \( B \in \pi_3(S^3) \) is defined as
\begin{align}
B &= -\frac{1}{24\pi^2} \int d^3x \ \epsilon^{ijk}\text{tr} \left( U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U \right) \\
&= \frac{1}{24\pi^2} \int d^3x \ \epsilon^{ijk}\text{tr} \left( U^\dagger \partial_i U \partial_j U^\dagger \partial_k U \right) \\
&= \frac{1}{4\pi^2} \int d^3x \ \epsilon^{ijk} \phi^\dagger \partial_i \phi \partial_j \phi^\dagger \partial_k \phi. \tag{8}\end{align}

Instead of the conventional potential term, we consider here a potential term motivated by two-component Bose-Einstein condensates (BECs), given by
\begin{equation}
V(\phi, \phi^*) = \frac{m^2}{8} \left[ 1 - (\phi^3 \sigma_3 \phi)^2 \right] = \frac{1}{2} m^2 |\phi_1|^2 |\phi_2|^2; \tag{9}\end{equation}
see Appendix of Ref. \[16\] for a relation to BECs. With this potential, the full symmetry \( \tilde{G} \) is explicitly broken down to
\begin{equation}
G = U(1) \times O(2) \simeq U(1)_0 \times [U(1)_3 \times (\mathbb{Z}_2)_{1,2}]. \tag{10}\end{equation}
Here, each group is defined as

\begin{align}
U(1)_0 & : \phi \rightarrow e^{i\alpha} \phi, \\
U(1)_3 & : \phi \rightarrow e^{i\beta \sigma_3} \phi, \\
(Z_2)_{1,2} & : e^{i(\pi/2)\sigma_{1,2}} \phi
\end{align}

where $U(1)_3$ acts on $\mathbb{Z}_2$ so that they are a semi-direct product denoted by $\rtimes$. The vacua of the potential in Eq. (9) are

\begin{align}
\sqcup & : \phi^T = (e^{i\alpha}, 0), \\
\otimes & : \phi^T = (0, e^{i\beta}),
\end{align}

and the unbroken symmetry $H$ is

\begin{align}
H_{\sqcup} = U(1)_{0-3} : & \phi \rightarrow e^{i\alpha} e^{-i\alpha \sigma_3} \phi, \\
H_{\otimes} = U(1)_{0+3} : & \phi \rightarrow e^{i\alpha} e^{+i\alpha \sigma_3} \phi,
\end{align}

for the $\sqcup$ and the $\otimes$ vacuum of Eq. (14), respectively. Therefore, the vacuum manifold (or the moduli space of vacua) is given by

\[ M \simeq G/H = \frac{U(1)_0 \times [U(1)_3 \rtimes (Z_2)_{1,2}]}{U(1)_{0\pm3}} \simeq SO(2)_{0\mp3} \rtimes (Z_2)_{1,2} = O(2). \]

The nontrivial homotopy groups of the vacuum manifold are

\[ \pi_0(M) = \mathbb{Z}_2, \quad \pi_1(M) = \mathbb{Z}, \]

admitting domain walls and vortices, respectively.

By means of the Hopf map $\vec{n} = \phi^\dagger \vec{\sigma} \phi$, the principal chiral SU(2) model can be mapped to the O(3) nonlinear sigma model with $\vec{n}^2 = 1$ or equivalently the $\mathbb{C}P^1$ model. The potential term in Eq. (9) is mapped to $V = \frac{m^2}{8} (1 - n_3^2)$, which is referred to as the Ising-type potential in ferromagnets [20]. The $\mathbb{C}P^1$ model with the same potential is often called the massive $\mathbb{C}P^1$ model [21, 24]. This map can be obtained by coupling a U(1) gauge field to $\phi$ with common U(1) charges and subsequently taking the strong gauge coupling limit $e \rightarrow \infty$. 
III. DOMAIN WALLS AND VORTICES

In this section, we will review the constituents which will be used in the next section in modified
or compactified forms.

A. Domain walls

In $d = 1+1$ dimensions, a (n anti-)kink solution interpolating between the two vacua in Eq. (14)
is given by

$$\phi_T = \frac{1}{\sqrt{1 + e^{\pm 2m(x-X)}(e^{i\alpha}, e^{\pm m(x-X)+i\beta})}}, \quad (18)$$

with $X \in \mathbb{R}$ being the translational modulus of the kink. Here $\alpha$ and $\beta$ are not moduli of the
kink but moduli of the vacua in Eq. (14). Note that this solution is (statically) exact in the form
given above, even in the presence of the Skyrme or sixth-order derivative term (this can easily
be understood as the Skyrme (sixth-order derivative) term is nonzero only when a solution non-
trivially depends on two (three) spatial coordinates). Once waves on top of this static solution is
considered, the higher-order derivative terms must be taken into account; see e.g. Ref. [25].

In the static case, the kink can trivially be extended to a domain line in $d = 2 + 1$ dimensions
and to a domain wall in $d = 3 + 1$ dimensions, with a one- and two-dimensional world volume,
respectively.

By the Hopf map, the solution (18) is mapped to a kink in the massive $\mathbb{C}P^1$ model [21, 22, 26].
In that case, the phase difference $\beta - \alpha$ becomes a modulus of the kink.

In the $(3 + 1)$-dimensional case, we can think of our toroidal objects in Sec. IV as a domain
wall wrapped up on a torus with its $S^1$ moduli twisted in both world-volume directions. It will,
however, prove convenient to take a different point of view, as we shall see, namely to consider
first a vortex string which is then wrapped up on a circle. In the next subsection we therefore
review the (global) vortex.

B. Vortices

In $d = 2+1$ dimensions the model allows for global vortices. The vortices of $\phi_1$ trap or localize
$\phi_2$ in their cores and they carry a U(1) modulus being the phase of $\phi_2$. 

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We will now review the global vortex in the nonlinear sigma model with the potential (9), see [16]. The vortex can be constructed using the following Ansatz

$$\phi^T = (\sin f(r)e^{i\varphi+i\alpha}, \cos f(r)e^{i\beta}) ,$$

(19)

where \(r \in [0, \infty), \varphi \in [0, 2\pi)\) are polar coordinates in a plane. The constant, \(\alpha\), can be absorbed by a redefinition of the coordinate \(\varphi\), while the constant \(\beta\) is a U(1) modulus. This simplifies the Lagrangian density to [16]

$$-L = \frac{1}{2} f_r^2 + \frac{1}{2r^2} \sin^2 f + \frac{\kappa}{2r^2} \sin^2 (f) f_r^2 + \frac{1}{8} m^2 \sin^2(2f) ,$$

(20)

for which the equation of motion reads [16]

$$f_{rr} + \frac{1}{r} f_r - \frac{1}{2r^2} \sin 2f + \frac{\kappa}{r^2} \sin^2 f \left( f_{rr} - \frac{1}{r} f_r \right) + \frac{\kappa}{2r^2} \sin(2f) f_r^2 - \frac{1}{4} m^2 \sin 4f = 0 .$$

(21)

The boundary conditions for the vortex system are given by

$$f(0) = 0, \quad f(\infty) = \frac{\pi}{2} .$$

(22)

We show numerical solutions in Fig. [1] for \(m = 1, 4\) and \(\kappa = 0, 1\). By the Hopf map, they can (topologically) be mapped to lumps.

In \(d = 3 + 1\) dimensions, these vortices are extended to vortex strings or cosmic strings. They are global analogues of Witten’s superconducting strings [27]. We may call them superflowing cosmic strings. Once extended to \((3 + 1)\)-dimensional spacetime, the strings bear a U(1) modulus, which we can parametrize as

$$\phi^T = (\sin f(r)e^{i\varphi}, \cos f(r)e^{i\zeta}) ,$$

(23)

In the next section we will compactify these strings on a circle which requires a nontrivial twist of the modulus \(\zeta\).
FIG. 1: Vortex profiles and energy densities for solutions without the Skyrme term $\kappa = 0$ (blue curve) and with the Skyrme term $\kappa = 1$ (dotted red curve) for $m = 1$ (left panels) and $m = 4$ (right panels).

IV. TOROIDAL SKYRMIONS IN 3 + 1 DIMENSIONS

In this section we will consider a closed vortex string, i.e. the vortex string wound up on a circle and thus forming a torus-like object. Such a closed vortex string is unstable unless its U(1) modulus is twisted along the string (viz. it is topologically trivial otherwise).

In the final configuration, the U(1) modulus is twisted $P$ times along the toroidal ($\alpha$) cycle of the torus and the global string winds $Q$ times “along” the poloidal ($\beta$) cycle of the torus; see Fig.2.

The torus-shaped solution requires us to study the full partial differential equation (PDE) numerically, for which we will use the relaxation method on a cubic square lattice. Because of the topological nature of the objects we study, it is sufficient to employ Neumann conditions on the boundary of the lattice whereas the initial condition is very important. For the initial configuration we will use the following Ansatz

$$\phi^T = \left(\sin \left[\cos^{-1}\{\sin f(r) \sin \theta\}\right] e^{iQ \tan^{-1}(\tan f(r) \cos \theta)}, \cos \left[\cos^{-1}\{\sin f(r) \sin \theta\}\right] e^{iP\phi}\right),$$

(24)
FIG. 2: The two cycles of the torus. The toroidal and poloidal cycles are denoted by $\alpha$ and $\beta$, respectively. The $\bigcirc$ and $\otimes$ denote the vacua in Eq. (14), respectively. The U(1) modulus is twisted $P$ and $Q$ times along the cycles $\alpha$ and $\beta$, respectively.

where $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$ and $f(r)$ is an appropriately chosen monotonically decreasing function satisfying the boundary conditions

\[ f(r \to 0) \to \pi, \quad f(r \to \infty) \to 0. \] (25)

The baryon number (Skyrme charge) of $\pi_3(S^3) \simeq \mathbb{Z}$ for the configuration given in Eq. (24) is

\[ B = \frac{1}{4\pi^2} \int d^3x \epsilon^{ijk} \partial_i \phi \partial_j \phi^\dagger \partial_k \phi = \frac{1}{2\pi^2} \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin \theta PQ f'(r) \sin^2 f(r) \]

\[ = - \frac{PQ}{\pi} \int_0^\infty dr \partial_r [f(r) - \sin f(r) \cos f(r)] \]

\[ = PQ. \] (26)

Although we seemingly have two quantum numbers to dial in the configuration, it will prove convenient to think about the winding number $Q$ as that of the global vortex. This may suggest that $Q > 1$ will be unstable as global vortices repel with a force $\sim 1/d$, where $d$ (here) is the separation distance between two strings. We confirm this expectation by numerically solving the equations and find for a wide range of parameters that for $Q > 1$, the relaxation method always splits up the object into $Q$ individual strings; each with a $P$-wound U(1) phase. For details, see Appendix A.

We can therefore study the numerical solutions with baryon number $B = P$, for which the
Ansatz (24) reduces to

$$
\phi^T = \left( \cos f(r) + i \sin f(r) \cos \theta, \sin f(r) \sin \theta e^{iP\phi} \right).
$$

This is exactly the axially symmetric generalization of the hedgehog Ansatz and this is just what we need (note that for Skyrmions without our BEC-motivated potential, this Ansatz is only appropriate for $B = 1, 2$ while for $B > 2$ the axial symmetry no longer yields the minimum-energy configuration). We will study two cases in turn; in the first we turn on only the fourth-order derivative term, i.e. $\kappa = 1$ and $c_6 = 0$ while in the second case we switch off the fourth-order but use the sixth-order derivative term, i.e. $\kappa = 0$ and $c_6 = 1$. We will call them the 2+4 model and the 2+6 model, respectively.

In Figs. 3, 4 and 5 we show solutions for case of the 2+4 model ($\kappa = 1$ and $c_6 = 0$) with mass $m = 4$. In Fig. 3 we show the 3-dimensional isosurfaces at half the maximum value of the baryon charge density. The color scheme used is chosen such that the U(1) phase, $\arg \phi_2$, is mapped to the hue while the lightness is given by the absolute value of the imaginary part of the vortex condensate: $|\Im(\phi_1)|$. In Figs. 4 and 5 we show the baryon charge density and energy density, at two different cross sections cutting through the origin of the torus, respectively. In this case, they are practically identical, which means that the energy density is located where the baryon charge is.

As a check on our numerical precision, we calculate the baryon charge density and integrate it numerically, see Table I. As already explained, our Skyrmionic torii are only stable for $Q = 1$, but to study whether they are stable for higher $P > 1$, we need to compare the energy of the configurations. In Table I we calculate the energy per $B = P$ and find that the energy drops for the first four torii, viz. $P = 1, 2, 3, 4$, but then it starts to increase slightly. The increase is so small that the $P = 5$ solution is still energetically stable (also taking into account the numerical accuracy) while $P = 6$ is only metastable [50]. That is, the energy of the $P = 6$ solution is larger than two times that of the $P = 3$ solution and hence it is bound to decay. Here we have not studied the potential barrier for the decay and thus cannot calculate its life time.

Next we will turn to the case of the 2+6 model, i.e. with only sixth-order derivative terms ($\kappa = 0$ and $c_6 = 1$) and again with a mass of $m = 4$. Numerical solutions are shown in Figs. 6, 7 and 8. As in the previous case, we show the 3-dimensional isosurfaces of the baryon charge density at half the maximum value in Fig. 6. In Figs. 7 and 8 we show the baryon charge density and energy
\((P, Q) = (1, 1)\)  \hspace{1cm} (P, Q) = (2, 1)  \hspace{1cm} (P, Q) = (3, 1) \\
\hspace{1cm} (P, Q) = (4, 1)  \hspace{1cm} (P, Q) = (5, 1)  \hspace{1cm} (P, Q) = (6, 1) \\

**FIG. 3:** Isosurfaces showing the solutions for the 2+4 model, i.e. for \(\kappa = 1\) and \(c_6 = 0\), at constant baryon charge density equal to half its maximum value. The color represents the phase of the scalar field \(\phi_2\) and the lightness is given by \(|\Im(\phi_1)|\). The calculations are done on an \(81^3\) cubic lattice with the relaxation method.

**TABLE I:** Numerically integrated baryon charge and energy (mass) for the solutions in the 2+4 model. Stability is observed for the first five solutions whilst \(P = 6\) is only energetically metastable.

| \(B\) | \(B_{\text{numerical}}\) | \(E_{\text{numerical}} / B\) |
|---|---|---|
| 1 | 0.9995 | 93.3151 ± 0.0297 |
| 2 | 1.9994 | 85.2782 ± 0.0223 |
| 3 | 2.9985 | 84.0152 ± 0.0200 |
| 4 | 3.9981 | 83.6919 ± 0.0516 |
| 5 | 4.9959 | 84.1664 ± 0.0312 |
| 6 | 5.9921 | 84.7335 ± 0.0204 |

density, respectively, at two different cross sections cutting the torus through the origin. Notice that the energy densities for these solutions are somewhat more complex that their respective baryon
FIG. 4: Baryon charge density for solutions in the 2+4 model, i.e. with $\kappa = 1$ and $c_6 = 0$, at $xz$ slices (for $y = 0$) and $xy$ slices (for $z = 0$). $yz$ slices are omitted as they are identical to the $xz$ slices by rotational symmetry of the torus. The calculations are done on an $81^3$ cubic lattice with the relaxation method.

charge densities. This is one difference between the 2+6 model and the 2+4 model. The second difference is that in this case, the torus shape is vaguely visible already for $P = 1$, whereas for the previous case $P = 1$ has (unbroken) spherical symmetry. Let us also comment on the circular shape of the torus for the $(P, Q) = (6, 1)$ solution along the toroidal direction in Fig. 6; this flattening out of the circle as not aligned with the lattice, but is at almost 45 degrees to the lattice axis. Since the small $P$ solutions do posses almost perfect circular symmetry, we believe that this is not a lattice effect, but instead signals metastability of the string: for high enough $B = P$ the string wants to collapse and break up. The same effect can also be observed in the $(P, Q) = (6, 1)$ solution in Fig. 8 on the $xy$ slice where the energy density displays four distinct wave tops around
\[(P, Q) = (1, 1)\]
\[(P, Q) = (2, 1)\]
\[(P, Q) = (3, 1)\]
\[(P, Q) = (4, 1)\]
\[(P, Q) = (5, 1)\]
\[(P, Q) = (6, 1)\]

FIG. 5: Energy density for solutions in the 2+4 model, i.e. with \(\kappa = 1\) and \(c_0 = 0\), at \(xz\) slices (for \(y = 0\)) and \(xy\) slices (for \(z = 0\)). \(yz\) slices are omitted as they are identical to the \(xz\) slices by rotational symmetry of the torus. The calculations are done on an \(81^3\) cubic lattice with the relaxation method.

We again check the numerical precision by numerically evaluating the total baryon charge, see Table II. As for the stability of the higher \(P > 1\) solutions, we numerically evaluate the energy (mass) of the solutions and again find that the energy decreases as \(P\) is increased, for the first few solutions, but this time only for the first three \(P = 1, 2, 3\) and then it starts to increase slightly. The first five solutions are all energetically \textit{stable} while \(P = 6\) is only metastable.
FIG. 6: Isosurfaces showing the solutions for the 2+6 model, i.e. for \( \kappa = 0 \) and \( \epsilon_6 = 1 \), at constant baryon charge density equal to half its maximum value. The color represents the phase of the scalar field \( \phi_2 \) and the lightness is given by \( |\Im(\phi_1)| \). The calculations are done on an \( 81^3 \) cubic lattice with the relaxation method.

TABLE II: Numerically integrated baryon charge and energy (mass) for the solutions in the 2+6 model. Stability is observed for the first five solutions whilst \( P = 6 \) is only energetically metastable.

| \( B \) | \( B_{\text{numerical}} \)          | \( E_{\text{numerical}} / B \) |
|--------|-----------------------------------|----------------------------------|
| 1      | 0.9999 100.8613 ± 0.0410           |                                  |
| 2      | 1.9998 89.7184 ± 0.0532            |                                  |
| 3      | 2.9995 87.3095 ± 0.1871            |                                  |
| 4      | 3.9981 87.5179 ± 0.0721            |                                  |
| 5      | 4.9970 87.5560 ± 0.0901            |                                  |
| 6      | 5.9939 88.1414 ± 0.1145            |                                  |
\[(P, Q) = (1, 1)\]
\[(P, Q) = (2, 1)\]
\[(P, Q) = (3, 1)\]
\[(P, Q) = (4, 1)\]
\[(P, Q) = (5, 1)\]
\[(P, Q) = (6, 1)\]

FIG. 7: Baryon charge density for solutions in the 2+6 model, i.e. with \(\kappa = 0\) and \(c_6 = 1\), at \(xz\) slices (for \(y = 0\)) and \(xy\) slices (for \(z = 0\)). \(yz\) slices are omitted as they are identical to the \(xz\) slices by rotational symmetry of the torus. The calculations are done on an \(81^3\) cubic lattice with the relaxation method.

V. SUMMARY AND DISCUSSION

We have studied Skyrmion solutions in the BEC Skyrme model, which is a Skyrme model with the potential term motivated by two-component BECs. We have constructed stable Skyrmion solutions for \(P = 1, 2, 3, 4, 5\) and \(Q = 1\), yielding the baryon numbers \(B = 1, 2, 3, 4, 5\) as well as a metastable solution for \(P = 6\) and \(Q = 1\) \((B = 6)\). We suspect that higher baryon charged solutions will all be metastable. The energy and baryon charge distributions of the configurations with \(P > 1\) are all of toroidal shape. They are vortex rings of the field \(\phi_1\), with the field \(\phi_2\) trapped in their cores, where the phase of the field \(\phi_2\) winds \(P\) times along the ring. We have found that configurations with charge \((P, Q)\) decay into \(Q\) rings of charge \((P, 1)\). This string splitting can be
understood as repulsion of global vortex strings.

In two-component BECs, one can introduce a Rabi oscillation term $\gamma(\phi_1(x)^*\phi_2(x) + c.c.)$, known as a Josephson term in superconductors, in the Lagrangian. Introduction of this term deforms the Skyrmions inside a domain wall [23, 26, 28, 29]. What deformation this term introduces for toroidal Skyrmions in the BEC Skyrme model remains as a future problem. On the other hand, if we introduce the potential term $V \sim \phi_1 + \phi_1^*$ [16], our configurations will become $P$ sine-Gordon kinks on a vortex ring, which is a $(3 + 1)$-dimensional analogue of Ref. [30], in which sine-Gordon kinks on a domain wall ring was constructed in $2 + 1$ dimensions.

Two-component BECs are known to admit a stable composite soliton, viz. a D-brane soliton,
that is, a domain wall on which vortices end from both sides [31], originally found in the massive \( \mathbb{C}P^1 \) model [32, 33]. The (BEC) Skyrme model discussed in this paper has the same potential term and is expected to admit such a D-brane soliton. A configuration made of a domain wall and an anti-domain wall stretched by lump-strings in the massive \( \mathbb{C}P^1 \) model was considered in Ref. [34], in which it was discussed that such a configuration is unstable to decay, resulting in the creation of Hopfions. Therefore, the same mechanism should work also in the (BEC) Skyrme model discussed in this paper creating Skyrmions from brane annihilation, as was discussed for two-component BECs [3].

The Bogomol’nyi-Prasad-Sommerfield (BPS) Skyrme model, proposed recently [18], consists of only the sixth-order derivative term as well as appropriate potentials. This model admits exact solutions with compact support. By choosing the potential of the BEC Skyrme model in this paper, we may be able to construct exact solutions of Skyrmions with toroidal shape.

The Skyrmions with the charge \((P,Q)\) are related through the Hopf map to \((P,Q)\) Hopfions [35, 36] in the Ising Faddeev-Skyrme (FS) model [34], that is, the FS model [37, 38] with an Ising-type potential term admitting two discrete vacua. The domain wall in the BEC Skyrme model is mapped to a domain wall with a U(1) modulus interpolating between these two vacua [21, 22, 25], and a global vortex is mapped to a lump or baby Skyrmion [39, 40]. This model also admits a twisted domain-wall tube with the U(1) modulus twisted along the cycle of the tube [30] as a baby-Skyrmion string. The original FS model without said potential term is known to admit Hopfions, i.e. solitons with Hopf charge \(\pi_3(S^2) \simeq \mathbb{Z} [14, 38, 41–45]\), and, in particular, Hopfions with Hopf charge 7 or higher were found to have knot structures [43–45]. The \((P,Q)\) Hopfions in the Ising FS model are not knots but toroidal domain walls characterized by two integers \((P,Q)\), where the U(1) modulus of the domain wall is twisted \(P\) and \(Q\) times along the toroidal and poloidal cycles of the torus, respectively. In this case, some configurations with \(Q > 1\) were found to be stable [36], unlike our case of Skyrmions for which all configurations for \(Q > 1\) are unstable. This is because there is no repulsion between lumps.

If we consider compactifying space to \(\mathbb{R}^2 \times S^1\) we have another solution in addition to the one studied here, in which the vortex string extends along the \(S^1\) direction and has \(P\) twists on its U(1) modulus. The corresponding solution for the case of the Hopfion was discussed in Ref. [46]. Skyrmions in the conventional model on \(S^2 \times S^1\) was discussed in Ref. [47].
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Appendix A: String splitting for $Q > 1$

In this section we show that the relaxation of the $(P, Q) = (P, 2)$ torus splits into two separate $(P, Q) = (P, 1)$ objects for $P = 1, 2$. For concreteness we carry out the calculations in the 2+6 model ($\kappa = 0$ and $c_6 = 1$). In Figs. 9 and 10 are shown the $(1, 2) \rightarrow 2 \times (1, 1)$ and $(2, 2) \rightarrow 2 \times (2, 1)$ string splittings as function of relaxation time $\tau$, respectively.

Appendix B: Comparison of torus and Skyrmion

In this section we will compare the case of $(P, Q) = (2, 1)$ and thus baryon number 2 and $m = 4$, where the Skyrmion is a torus, with the case of $m = 0$, which is just the normal $B = 2$ Skyrmion and also in the form of a torus. We will make the comparison for both the 2+4 model and the 2+6 model. In Figs. 11 and 12 are shown the comparison for the 2+4 and 2+6 models, respectively. For the 2+4 model, the main difference is the size (and in turn the total mass) of the two solutions. For the 2+6 model, differences are evident both in the baryon charge density slices (middle row) and the energy density slices (bottom row). For the BEC Skyrmion in the 2+6 model, the torus is more hollow with respect to its potential-less counterpart.

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FIG. 9: Isosurfaces showing an initial configuration with \((P, Q) = (1, 2)\) \((B = 2)\) in 2+6 model \((\kappa = 0, c_6 = 1\) and \(m = 4)\) which after some finite relaxation time splits the Skyrmion into two separate Skyrmions of charge one, i.e. \((P, Q) = (1, 1)\). The isosurfaces show constant baryon charge density equal to half its maximum value. The color represents the phase of the scalar field \(\phi_2\) and the lightness is given by \(|\Im(\phi_1)|\). The calculation is carried out on an 81\(^3\) cubic lattice with the relaxation method.

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FIG. 10: Isosurfaces showing an initial configuration with \((P, Q) = (2, 2)\) in 2+6 model \((\kappa = 0, c_6 = 1\) and \(m = 4\) which after some finite relaxation time splits the Skyrmion into two separate Skyrmions of charge two, i.e. \((P, Q) = (2, 1)\). The isosurfaces show constant baryon charge density equal to half its maximum value. The color represents the phase of the scalar field \(\phi_2\) and the lightness is given by \(|\Im(\phi_1)|\). The calculation is carried out on an \(81^3\) cubic lattice with the relaxation method.
FIG. 11: Comparison between the BEC Skyrmion in the 2+4 model ($m = 4$) on the left and the normal Skyrmion ($m = 0$) on the right. From top to bottom is shown the isosurface of the baryon density at half maximum, the baryon density at $xz$ and $xy$ slices through the origin of the torus, and finally similar energy density slices.

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FIG. 12: Comparison between the BEC Skyrmion in the 2+6 model \((m = 4)\) on the left and the “normal” Skyrmion in the 2+6 model \((m = 0)\) on the right. From top to bottom is shown the isosurface of the baryon density at half maximum, the baryon density at \(xz\) and \(xy\) slices through the origin of the torus, and finally similar energy density slices.

\[
m = 4, B^{\text{numerical}} = 1.9998
\]
\[
m = 0, B^{\text{numerical}} = 1.9834
\]
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[50] The question of stability may also depend on the coefficients of the higher-derivative terms and the mass.