Dissipative Control and Observation of Linear Time-Delay Systems: Full State Feedback
Qian Feng

Abstract—The paper is the first step of a series dealing with the dissipative control and observation problems of continuous-time linear systems with general delays of finite length via the Krasovskiı̆ functional (KF) approach. Our model imposes no limits to the delay numbers at the states, inputs, regulated outputs or controller, where the distributed delays (DDs) can contain any number of square-integrable functions over the delay intervals. We introduce the notion of equivalent decompositions which can simultaneously factorize or approximate any function in the DDs. Moreover, the decomposition approach allows the construction of complete-type KFs whose integral kernels can contain any number of differentiable and linearly independent functions, supported by the use of integral inequalities based on the least square principle. The solutions to the problem are expressed in terms of matrix inequalities summarized in several theorems/corollaries and iterative algorithms, which can be used together as a whole to compute controller gains without requiring nonlinear solvers. Two numerical examples are tested to show the effectiveness of the proposed methodologies.

Index Terms—Continuous-Time Linear Delay Systems; Pointwise and Distributed Delays; Equivalent Decomposition; Dissipativity; Krasovskiı̆ Functionals and Integral Inequalities.

I. INTRODUCTION

Generally speaking, two types of delays, pointwise and distributed delays (DDs), have been utilized to model transport, propagation or aftereffects in practical dynamical systems. The nature of a pointwise-delay $x(t - \tau)$ is explained in [1] which can be denoted by a transport equation with boundary conditions. Meanwhile, delays can be created by transporting media with more complex structures. A DD is denoted by an integral $\int_{-\tau}^{0} F(\tau)x(t + \tau)d\tau$ over a delay interval $[-\tau, 0]$ with a matrix-valued function $F(\cdot)$, which takes into account a segment of the past dynamics's information. Systems with both pointwise and DDs have many applications such as modeling biological processes [2] and chemical reaction networks [3], etc. In fact, the general form of a linear delay operator, denoted via a Lebesgue-Stieltjes integral [4], can be expressed as a summation of pointwise and distributed delays in general. Therefore, it is ideal to consider both types of delays in a method for time-delay systems.

Most methodologies for linear time-delay systems (LTDSs) are carried out in the time or frequency domain, where real or complex analysis is applied. To the best of the author's knowledge, the newest trend of frequency-domain-based methods can be found in [5] and [6]. These works are predominantly nourished by the recent development of non-smooth optimization algorithms [7], [8]. However, no DDs have been considered by the above works, which is attributable to the obstacles in handling the Laplace transform of $\int_{-\tau}^{0} F(\tau)x(t + \tau)d\tau$ since $\int_{-\tau}^{0} F(\tau)e^{\tau s}d\tau$ may not have a closed-form expression in the frequency domain.

For time-domain approaches [9], [10], the construction of Krasovskiı̆ functionals (KF) has been proven as an effective solution for the stability analysis and stabilization of LTDSs [11], [12], [13], [14] supported by efficient numerical algorithms for semidefinite programming (SDPs) [15], [16]. For a comprehensive collection of the existing literature on this topic, see the monographs in [9]. In contrast to the Lyapunov approach for an LTI system, the KF approach could only establish sufficient conditions where the induced conservatism is largely based on the generality of the predetermined form of KFs [10] and the integral inequalities [12] utilized to construct them. Because more general KFs [11], [13] have been increasingly adopted for the reduction of conservatism, we may not directly use congruent transformations to form

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convex controller synthesis conditions from the original stability analysis condition. Finally, a very interesting method is proposed in [17] for the stabilization of LTDSs with DDs, which can be considered as a combination of both time and frequency domain approaches based on the concept of smoothed spectral abscissa [18], [19] and delay Lyapunov matrix [20], [21].

Nevertheless, it is safe to say that there are no effective solutions in the literature for the control and observation of LTDSs with both pointwise and general DDs, especially if there is an unlimited number of delays. Even if we only consider the case of stability analysis, most existing KF approaches impose restrictions on the structure of state space parameters [22] or DD kernels [12] or the number of DD kernels and delays [11], [13]. The method in [17] requires the computation of the delay Lyapunov matrix and its derivatives. It has not been elaborated in [17] how the computation can be carried out for an LTDS with general DDs or non-commensurate delays. Finally, the linear quadratic optimal control [23], [24] approach (infinite time horizon) for the stabilization of LTDSs require finding solutions for infinite-dimensional algebraic Riccati equations. However, the Riccati equations are solved via finite dimensional approximations [25], whose numerical results do not guarantee the stability the closed-loop system mathematically.

The aim of this work is to establish an efficient optimization-based solution to the dissipative controller synthesis problem of a very general class of LTDSs via the KF approach. The system’s model contains an unlimited number of pointwise and general DDs at the states, inputs, outputs, where the DDs can contain any number of $L^2$ functions over bounded intervals. The solutions to the synthesis problem are obtained by solving convex SDPs without asking for nonlinear solvers. Finally, the method is extended to construct controllers with general delays, when no delays exist at the input.

The main items and contributions of this paper are summarized as follows:

- The control problem researched in this work has not been considered by existing literature according to the author’s best knowledge. The generality of our LTDS with a quadratic supply rate function has important research significance as it can cover many LTDSs related control problems in an engineering setting. This is primarily due to the incorporation of an unlimited number of pointwise and $L^2$ DDs, as the system can be regarded as a realization of the general LTDS denoted by a Lebesgue-Stieltjes integral.
- The notion of equivalent decomposition is proposed to handle the $L^2$ DD kernels, which significantly generalizes the approximation scheme in [13]. Unlike [13] where all $L^2$ DD kernels are approximated by a restricted class of differentiable functions, the proposed approach allows users to decide which $L^2$ kernels are factorized directly and which are approximated by any number of differentiable and linearly independent functions (DALIFs) with their derivatives. Moreover, it also allows one to construct KFs with integral kernels containing any number of DALIFs, which can be totally independent of the functions inside of the DDs. Finally, the KF is constructed with the integral inequalities developed by the author in [13] based on the least square principle. The use of all the aforementioned instruments can minimize the induced conservatism of the proposed synthesis approach.
- Two theorems and an iterative algorithm are proposed as the solution to the dissipative synthesis problem. The second theorem is derived from convexifying the bilinear matrix inequality (BMI) in the first theorem via Projection Lemma [26], without weakening the matrix parameters of the KFs. Moreover, the first theorem can be solved by the proposed iterative algorithms initiated by a feasible solution of the second theorem. Hence our method does not require the use of nonlinear SDP solvers.
- The method has also been extended to compute controllers with general delays when delays do not exist at the system’s input. This is a crucial contribution since static state controllers may not be sufficient for LTDSs in terms of stability or performance.
- Due to the connection between LTDSs and other types of systems, it has been shown that advanced treatment of LTDSs with general DDs [11], [12], [13] can lead to efficient solutions of many engineering problems such as networked control system [27] and PDE-ODE coupled system [28]. As a result, the proposed methods can serve as a blueprint for the future development of new solutions of real-world and engineering problems.
The organization of the rest of the paper is outlined as follows. Preliminaries are first presented in Section II concerning the derivation of the closed-loop system and the notion of equivalent decomposition. The main results concerning dissipative static controller synthesis are set out in Sections III, whereas Section IV summarizes the extended method for designing controllers with delays. Finally, the computation results of two numerical examples are presented in Section V prior to the final conclusion. Note that we place some important lemmas and proofs in the appendices.

**Notation**

Throughout this work, we use notations \( \mathcal{Y}^{\mathcal{X}} := \{ f(\cdot) : \mathcal{X} \ni x \mapsto f(x) \in \mathcal{Y} \} \) and \( \mathbb{R}_{\geq a} := \{ x \in \mathbb{R} : x \geq a \} \) and \( \mathcal{S}^{n} := \{ X \in \mathbb{R}^{n \times n} : X = X^{\top} \} \). Moreover, the space of continuous and differentiable functions is \( C(\mathcal{X} ; \mathbb{R}^{n}) := \{ f(\cdot) \in (\mathbb{R}^{n})^{\mathcal{X}} : f(\cdot) \text{ is continuous on } \mathcal{X} \} \) and \( C^{k}(\mathcal{I} ; \mathbb{R}^{n}) := \{ f(\cdot) \in C(\mathcal{I} ; \mathbb{R}^{n}) : \frac{df(x)}{dx} \in C(\mathcal{I} ; \mathbb{R}^{n}) \} \).

The open-loop LTDS

In this paper, we deal with an LTDS in the form of

\[
\dot{x}(t) = \sum_{i=0}^{\nu} A_{i} x(t - r_{i}) + \sum_{i=1}^{\nu} \int_{t-r_{i}}^{t} \tilde{A}_{i}(\tau)x(t + \tau)d\tau \\
+ \sum_{i=0}^{\nu} B_{i} u(t - r_{i}) + \sum_{i=1}^{\nu} \int_{t-r_{i}}^{t} \tilde{B}_{i}(\tau)u(t + \tau)d\tau + D_{1} w(t)
\]

and rules in the programming language of Matlab®. We assume \( I_{0} = [0,0[, \; O_{0,m} = [0,m] \) and \( \text{Col}^{\nu}_{i=1} x_{i} = [0,m] \) if \( n < 1 \), where \( [0,m],[m,0] \) are empty matrices with an appropriate column dimension \( m \in \mathbb{N} \) based on specific contexts.

**II. Preliminaries**

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### B. Chemical Reaction Networks

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\]
Remark 3. The quadratic supply rate function in (2) is based on the research in [31], capable of featuring numerous performance criteria such as
- $\mathcal{L}^2$ gain performance: $J_1 = -\gamma I_m, \bar{J} = I_m, J_2 = O_{m,q}, J_3 = \gamma I_q$ with $\gamma > 0$
- Strict Passivity: $J_1 < 0, \bar{J} = O_m, J_2 = I_m, J_3 = O_m$
- Sector constraints: $J_1 = \bar{J} = -I_m, J_2 = -\frac{1}{2}(\alpha + \beta)$, $J_3 = \alpha\beta I_m$ with $m = q$.

B. Equivalent Decompositions of DDs

Since the DD matrices in (3) are infinite-dimensional, including them in a synthesis (stability) condition will lead to infinite-dimensional optimization constraints. To circumvent this obstacle, we introduce the notion of equivalent decompositions in this paper, which can parameterized any DD in (3) with matrices of finite dimensions.

**Proposition 1.** The conditions in (3) are true if and only if there exist $f_i(\cdot) \in C^1(I_i; \mathbb{R}^d), \varphi_i(\cdot) \in \mathcal{L}^2(I_i; \mathbb{R}^d)$, $g_i(\cdot) \in \mathcal{L}^2(I_i; \mathbb{R}^\nu)$ and $M_i \in \mathbb{R}^{d \times \kappa_e}, A_i \in \mathbb{R}^{\kappa_e \times \kappa_n}, B_i \in \mathbb{R}^{\kappa_n \times \kappa_p}, C_i \in \mathbb{R}^{\kappa_n \times \kappa_e}, \bar{B}_i \in \mathbb{R}^{\kappa_p \times \kappa_n}$ such that

$$
\bar{A}_i(\tau) = \bar{A}_i(g_i(\tau) \otimes I_n), \quad \bar{B}_i(\tau) = \bar{B}_i(g_i(\tau) \otimes I_p), \quad \bar{C}_i(\tau) = \bar{C}_i(g_i(\tau) \otimes I_n), \quad \bar{B}_i(\tau) = \bar{B}_i(g_i(\tau) \otimes I_p),
$$

$$
\frac{df_i(\tau)}{d\tau} = M_i \tilde{f}_i(\tau), \quad \tilde{f}_i(\tau) = \left[ \varphi_i(\tau), \frac{\varphi_i(\tau)}{f_i(\tau)} \right],
$$

$$
G_i := \int_{I_i} g_i(\tau)g_i^\top(\tau) d\tau > 0, \quad g_i(\tau) = \left[ \varphi_i(\tau), \frac{\varphi_i(\tau)}{f_i(\tau)} \right]
$$

hold for all $i \in \mathbb{N}_\nu$ and $\tau \in I_i$, where $\kappa_i = d_i + \delta_i + \mu_i$, $\kappa_e = d_i + \delta_i$, with $d_i \in \mathbb{N}$ and $\delta_i; \mu_i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, the derivatives in (6) at the boundaries of $I_i$ are one-sided. Finally, the conclusion of this proposition is always true for the case of $\mu_i = 0$.

**Proof.** See Appendix A. \qed

Remark 4. The matrix inequalities in (7) indicate that the functions at each row of $g_i(\cdot)$ are linearly independent [32] in a Lebesgue sense over $I_i$ for all $i \in \mathbb{N}_\nu$, where $G_i$ in (7) are the Gramian matrices of $g_i(\cdot)$.

**Remark 5.** Proposition 1 suffers no conservatism. From Appendix A, we know that any $f_i(\cdot) \in C^1(I_i; \mathbb{R}^d)$ can be used for the conditions in (6) even if no functions in $f_i(\cdot)$ are included by the DDs in (3). This is because one can add an unlimited number of new functions to $f_i(\cdot)$, $\varphi_i(\cdot)$ and (6) can still be satisfied with some $M_i$. As long as all the kernels in (3) are "covered" by some functions in $g_i(\cdot)$, then the constant matrices in Proposition 1 can always be constructed.

III. DISSIPATIVE FULL STATE FEEDBACK CONTROLLER (DSFC) DESIGN

A. Problem Formulation

Inspired by the state variable $z(t, \tau)$ in [33], let $\chi(t, \theta) = [x(t + \theta)]_{i=1}^\nu \in \mathbb{R}^{\nu\tau}$ with $\theta \in [-1, 0]$ and $\dot{\theta} = \tau_{i-1} - \tau_{i-1}$. Now employ a static full state controller $u(t) = Kx(t), K \in \mathbb{R}^{\nu \times \nu}$ to (1) with Proposition 1, then the closed-loop system (CLS) is denoted by

$$
\dot{x}(t) = (A_0 + B_0K) x(t) + ([A_i + B_iK])^\nu t \chi(t, -1)
$$

$$
+ \sum_{i=1}^\nu \int_{I_i} \left( \bar{A}_i + \bar{B}_i (I_{\kappa_i} \otimes K) \right) G_i(\tau) x(t + \tau) d\tau + D_1 w(t)
$$

$$
x(t) = (C_0 + \mathcal{B}_0K) x(t) + ([C_i + \mathcal{B}_iK])^\nu t \chi(t, -1)
$$

$$
+ \sum_{i=1}^\nu \int_{I_i} \left( \bar{C}_i + \bar{B}_i (I_{\kappa_i} \otimes K) \right) G_i(\tau) x(t + \tau) d\tau + D_2 w(t)
$$

$$
\forall \theta \in \mathcal{J}, \quad x(t_0 + \theta) = \psi(\tau)
$$

where $G_i(\tau) = (g_i(\tau) \otimes I_n)$ and the decompositions of DDs are attained via the identities

$$
\forall i \in \mathbb{N}_\nu, \quad (g_i(\tau) \otimes I_p) K = (g_i(\tau) \otimes I_p) (1 \otimes K)
$$

$$
= I_{\kappa_i} g_i(\tau) \otimes K I_n = (I_{\kappa_i} \otimes K) (g_i(\tau) \otimes I_n). \quad (9)
$$

The functions $\varphi_i(\cdot)$ and $f_i(\cdot)$ are separated in $g_i(\cdot)$ because they will receive different mathematical treatment. Specifically, $\phi_i(\cdot)$ is always approximated by $\tilde{f}_i(\tau)$ via

$$
\phi_i(\tau) = \Gamma_i F_i^{-1} \tilde{f}_i(\tau) + \epsilon_i(\tau), \quad \tau \in [-r_{i-1}, -r_{i-1}] = I_i
$$

with $F_i = \int_{I_i} \tilde{f}_i(\tau) \tilde{f}_i^\top(\tau) d\tau > 0$ and

$$
\mathbb{R}^{\nu \times \nu} \ni \Gamma_i := \int_{I_i} \phi_i(\tau) \tilde{f}_i^\top(\tau) d\tau
$$

(11)

for all $i \in \mathbb{N}_\nu$ based on the application of least-square approximation. (See [34, page 182] for the expression of the approximation). Note that $\epsilon_i(\tau) = \phi_i(\tau) - \Gamma_i F_i^{-1} \tilde{f}_i(\tau)$ defines the error, and $F_i > 0$ always holds because of (7).

Moreover, we utilize $\mathbb{S}^{\nu \times \nu} \ni \Sigma_i := \int_{I_i} \epsilon_i(\tau) \tilde{f}_i^\top(\tau) d\tau =

\int_{I_i} \left( \phi_i(\tau) - \Gamma_i F_i^{-1} \tilde{f}_i(\tau) \right) \left( \phi_i(\tau) - \Gamma_i F_i^{-1} \tilde{f}_i(\tau) \right)^\top d\tau

= \int_{I_i} \phi_i(\tau) \tilde{f}_i^\top(\tau) d\tau - \Sigma_i \left( \int_{I_i} \phi_i(\tau) \tilde{f}_i^\top(\tau) d\tau F_i^{-1} \Gamma_i \right)

+ \Gamma_i F_i^{-1} \int_{I_i} \tilde{f}_i(\tau) \tilde{f}_i^\top(\tau) d\tau F_i^{-1} \Gamma_i

= \int_{I_i} \phi_i(\tau) \tilde{f}_i^\top(\tau) d\tau - \Gamma_i F_i^{-1} \Gamma_i \quad (12)
to measure the approximation error, where \( E_i > 0 \) always holds due to [13, eq.(18)].

Remark 6. Equations (10)–(12) are well defined with \( \mu_i = 0, \phi_i(\cdot) = \mathbb{I}_{0 \times 1} \) which corresponds to the case that no functions are approximated in \( g_i(\cdot) \). Such a case is always usable with Proposition 1 since an unlimited number of linearly independent \( L^2 \) functions can be added to \( \varphi_i(\cdot) \). This shows the advantage of using empty matrices, as the cases of \( \phi_i(\tau) = \mathbb{I}_{0 \times 1} \) and \( \phi_i(\tau) \neq \mathbb{I}_{0 \times 1} \) can be treated with a unified framework. In conclusion, Proposition 1 allows users to decide which \( L^2 \) functions in (3) are approximated (the ones in \( \phi_i(\cdot) \)) by \( \hat{f}(\cdot) \) and which are factorized directly (the ones in \( \varphi_i(\cdot) \)).

Remark 7. Let \( \delta_i = 0 \) in Proposition 1, then (10)–(13) with \( \nu = 1 \) becomes identical to the approximation scheme in [13]. However, the absence of \( \varphi_i(\cdot) \) severely limits the generality of the approximator \( f_i(\cdot) \). This is because \( \frac{df_i(\tau)}{d\tau} = M_f f_i(\tau) \) cannot be satisfied by all differentiable functions for some \( M \in \mathbb{R}^{d_i \times d_i} \). As a result, Proposition 1 is significantly more general than the approach in [13].

Now by (6) and the approximation in (10), we have

\[
g_i(\tau) = \begin{bmatrix} \phi_i(\tau) \\ \tilde{f}_i(\tau) \end{bmatrix} = \begin{bmatrix} \Gamma_i f_i^{-1} \tilde{f}_i(\tau) \\ \tilde{f}_i(\tau) \end{bmatrix} + \begin{bmatrix} \varepsilon_i(\tau) \\ \mathbf{0}_{\infty} \end{bmatrix},
\]

\[
\tilde{f}_i = \begin{bmatrix} \Gamma_i f_i^{-1} \\ I_{\infty i} \end{bmatrix} \in \mathbb{R}^{\kappa_i \times \kappa_i}, \quad \tilde{I}_i = \begin{bmatrix} I_{\mu_i} \\ O_{\infty_i \mu_i} \end{bmatrix} \in \mathbb{R}^{\kappa_i \times \mu_i},
\]

which further gives the identity

\[
\forall i \in \mathbb{N}_+, \quad (I_{\kappa_i \times K}) (g_i(\tau) \otimes I_n) = (I_{\kappa_i \times K}) \left( \tilde{f}_i(\tau) + \tilde{I}_i \varepsilon_i(\tau) \otimes I_n \right)
= (I_{\kappa_i \times K}) \left( \Gamma_i \tilde{f}_i(\tau) + \tilde{I}_i \varepsilon_i(\tau) \otimes I_n \right)
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= \left( \tilde{f}_i(\tau) + \tilde{I}_i \varepsilon_i(\tau) \right) \otimes I_n
= \left( \Gamma_i \tilde{f}_i(\tau) + \Gamma_i \tilde{I}_i \varepsilon_i(\tau) \right) \otimes I_n
\]

via (9) and (92), where \( \tilde{f}_i(\tau) = \left( \tilde{f}_i(\tau) \otimes I_n \right) \) and \( E_i(\tau) = (\varepsilon_i(\tau) \otimes I_n) \). Using (13)–(14) and (92) to the DDs in (8), we can conclude that

\[
\left( \tilde{A}_i + \tilde{B}_i (I_{\kappa_i \times K}) \right) (g_i(\tau) \otimes I_n)
= \left( \tilde{A}_i \left( \tilde{f}_i(\tau) \otimes I_n \right) + \tilde{B}_i (I_{\kappa_i \times K}) \right) \left( \tilde{f}_i(\tau) + \tilde{I}_i \varepsilon_i(\tau) \otimes I_n \right)
= \left( \tilde{A}_i \left( \tilde{f}_i(\tau) \otimes I_n \right) + \tilde{B}_i (I_{\kappa_i \times K}) \right) \left( \tilde{f}_i(\tau) \otimes I_n \right)
= \left( \tilde{A}_i \left( \tilde{f}_i(\tau) \otimes I_n \right) + \tilde{B}_i (I_{\kappa_i \times K}) \right) \left( \sqrt{f_i^{-1}} \tilde{f}_i(\tau) \otimes I_n \right)
\]

for all \( i \in \mathbb{N}_+ \), where \( \Gamma_i = \sqrt{f_i^{-1}} \).

Now by using the relations in (16)–(17) with (92), the DDs in (8) can be further denoted as

\[
\sum_{i=1}^{\nu} \int_{I_i} \left( \tilde{A}_i + \tilde{B}_i (I_{\kappa_i \times K}) \right) (g_i(\tau) \otimes I_n) x(t + \tau) d\tau
= \left[ \tilde{A}_i \left( \tilde{f}_i(\tau) \otimes I_n \right) + \tilde{B}_i (I_{\kappa_i \times K}) \right]_{i=1}^\nu \xi(t)
+ \left[ \tilde{A}_i \left( \tilde{I}_i \sqrt{f_i} \otimes I_n \right) + \tilde{B}_i (I_{\kappa_i \times K}) \right]_{i=1}^\nu \zeta(t)
\]

for all \( i \in I_i \), where \( \Gamma_i = \sqrt{f_i} \).

Note that \( \left( \bigoplus_{i=1}^\nu X_i \otimes I_n \right) = \bigoplus_{i=1}^\nu \left( X_i \otimes I_n \right) \) for all \( X_i \in \mathbb{C}^{p_i \times q_i} \), hence the above form can be written as \( \bigoplus_{i=1}^\nu X_i \otimes I_n \) by dropping the parenthesis.

Now using (18)–(19) to (8) produces

\[
\dot{x}(t) = \left( A + B_1 [I_{\beta \times K} \otimes O_{\eta_q}] \right) \dot{\theta}(t),
\]

\[
z(t) = \left( C + B_2 [I_{\beta \times K} \otimes O_{\eta_q}] \right) \dot{\theta}(t), \quad \forall t \geq t_0
\]

with \( t_0 \) and \( \psi(\cdot) \) in (1), where \( \beta = 1 + \nu + \kappa \) with \( \kappa = \sum_{i=1}^{\nu} \kappa_i \) and \( \kappa_i = d_i + \delta_i + \mu_i \), and

\[
A = \left[ [A_i]_{i=0}^\nu \right] = \left[ \tilde{A}_i \left( \tilde{f}_i(\tau) \otimes I_n \right) \right]_{i=1}^\nu \quad \cdots
\]

\[
B_1 = \left[ [B_i]_{i=0}^\nu \right] = \left[ \tilde{B}_i (I_{\kappa_i \times K}) \right]_{i=1}^\nu \quad \cdots
\]

\[
C = \left[ [C_i]_{i=0}^{\nu} \right] = \left[ \tilde{C}_i \left( \tilde{f}_i(\tau) \otimes I_n \right) \right]_{i=1}^\nu \quad \cdots
\]
B_2 = \left[ [\mathcal{B}_i]_{i=0}^\nu \left[ \mathcal{R}_i \left( \tilde{I}_i \otimes I_p \right) \right] \right]_{i=1}^\nu \\
\vartheta(t) = \text{Col}[x(t), \chi(t), -1, \xi(t), \zeta(t)]
(24)
(25)

B. Main Results on the DSFC

To verify the stability and dissipativity of the CLS in (20) with (2), the following lemma and definition are presented.

**Lemma 1.** Let \( w(t) \equiv 0_q \) in (20) and all delay values be given, then the trivial solution \( x(t) \equiv 0_n \) of (20) is uniformly asymptotically (exponentially) stable with any \( \psi(\cdot) \in C(\mathcal{J}; \mathbb{R}^n) \) if there exist \( \epsilon_1; \epsilon_2; \epsilon_3 > 0 \) and a differentiable function \( v : C(\mathcal{J}; \mathbb{R}^n) \to \mathbb{R} \) with \( v(0_n(\cdot)) = 0 \) such that

\[
\epsilon_1 \| \psi(0) \|^2_2 \leq v(\psi(\cdot)) \leq \epsilon_2 \| \psi(\cdot) \|^2_\infty
\]

\[
\forall \tau \geq t_0, \quad \frac{d}{dt} v(x_t(\cdot)) - \epsilon_3 \| x(t) \|^2_2 \leq 0
\]

for any \( \psi(\cdot) \in C(\mathcal{J}; \mathbb{R}^n) \) in (20), where \( \| \psi(\cdot) \|^2_2 := \sup_{\tau \leq 0} \| \psi(\tau) \|^2_2 \). Furthermore, \( x_t(\cdot) \) defined by

\[
\forall \tau \geq t_0, \forall \theta \in \mathcal{J}, \quad x_t(\theta) = x(t+\theta) \text{ in which } x : [t_0 - \nu, \infty) \to \mathbb{R}^n \text{ satisfies (20) with } w(t) \equiv 0_q.
\]

**Proof.** See Corollary 1 in [35].

**Definition 1.** The system in (20) with \( s(z(t), w(t)) \) in is said to be dissipative if there exists a differentiable function \( v : C(\mathcal{J}; \mathbb{R}^n) \to \mathbb{R}^+ \) such that

\[
\forall \tau \geq t_0, \quad \frac{d}{dt} v(x_t(\cdot)) - s(z(t), w(t)) \leq 0
\]

with \( t_0 \in \mathbb{R}, \tau(t) \) and \( w(t) \) in (20). Moreover, \( x_t(\cdot) \) defined by

\[
\forall \tau \geq t_0, \forall \theta \in \mathcal{J}, \quad x_t(\theta) = x(t+\theta) \text{ with } x(t) \text{ satisfying (20)}.
\]

If (28) holds, then we have

\[
\forall \tau \geq t_0, \quad v(x_t(\cdot)) - v(x_t(\cdot)) \leq \int_{t_0}^{\tau} s(z(\theta), w(\theta))d\theta
\]

based on the fundamental theorem of Lebesgue integration, which now is in line with the original definition of dissipativity in [36], given \( v(x_t(\cdot)) \) is well defined for almost all \( t \geq t_0 \).

Next, the main results on DSFC are presented in Theorem 1–2 and Algorithm 1, where Theorem 2 is proposed as a convexification of Theorem 1 which can be further solved by Algorithm 1.

**Theorem 1.** Let all the parameters in Proposition 1 be given, then the CLS in (20) with the supply rate function in (2) is dissipative, and the trivial solution of (20) with \( w(t) \equiv 0_q \) is uniformly asymptotically (exponentially) stable if there exist a controller gain \( K \in \mathbb{R}^{p \times n} \) and matrix parameters \( P_1 \in \mathbb{S}^n, P_2 \in \mathbb{R}^{n \times q}, P_3 \in \mathbb{S}^q \) with \( q = nd, d = \sum d_i \) and \( Q_i \in \mathbb{S}^n \) such that

\[
\begin{align*}
\Phi &= \left[ P_1 \quad P_2 \quad P_3 \right] \\
\Xi &= \left[ \begin{array}{ccc}
\Phi \otimes I_n & O_{n, (n + q + m)} \\end{array} \right] + \Xi \otimes (-J_1) \quad (40)
\end{align*}
\]

Finally, the number of unknown variables is \((0.5d^2 + 0.5d + \nu + 0.5)n^2 + (0.5d + 0.5 + \nu + p)n \in O(d^2n^2)\).
Proof. The proof of Theorem 1 is based on the construction of the complete type Krasovskiĭ functional

\[ v(\mathbf{x}_i(t)) = \eta^T(t) \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} \eta(t) + \sum_{i=1}^\nu \int_{I_i} \mathbf{x}^T(t + \tau) Q_i \mathbf{x}(t + \tau) d\tau + \sum_{i=1}^\nu \int_{I_i} (\tau + r_i) \mathbf{x}^T(t + \tau) R_i \mathbf{x}(t + \tau) d\tau \]

where \( \mathbf{x}_i(\cdot) \) follows the same definition in (28), and \( P_1 \in \mathbb{S}^n, P_2 \in \mathbb{R}^n \times \mathbb{R}^n, P_3 \in \mathbb{S}^o \) and \( Q_i \in \mathbb{S}^n, R_i \in \mathbb{S}^n \) and

\[ \eta(t) := \text{Col} [x(t), \xi(t)] \]

with \( F_i = \int_{I_i} f_i(\tau) \, d\tau, \) \( \forall i \in \mathbb{N}_u. \) Note that \( \sqrt{F_i^{-1}} \) are well defined and unique because of (7).

From \( \chi(t, \tau) = [x(t + \hat{r}_i \tau - r_i)]_{i=1}^\nu, \) we have

\[ \sum_{i=1}^\nu \frac{d}{dt} \int_{I_i} \mathbf{x}^T(t + \tau) Q_i \mathbf{x}(t + \tau) d\tau = \sum_{i=1}^\nu \mathbf{x}^T(t - r_i - 1) Q_i \mathbf{x}(t - r_i - 1) - \sum_{i=1}^\nu \mathbf{x}^T(t - r_i) Q_i \mathbf{x}(t - r_i) = \mathbf{X}^T(t, 0) Q \mathbf{X}(t, 0) - \mathbf{X}^T(t, -1) Q \mathbf{X}(t, -1) \]

\[ \sum_{i=1}^\nu \frac{d}{dt} \int_{I_i} (\tau + r_i) \mathbf{x}^T(t + \tau) R_i \mathbf{x}(t + \tau) d\tau = \sum_{i=1}^\nu \mathbf{f_i}^T(t - r_i - 1) R_i \mathbf{x}(t - r_i - 1) - \sum_{i=1}^\nu \int_{I_i} \mathbf{x}^T(t + \tau) R_i \mathbf{x}(t + \tau) d\tau = \mathbf{X}^T(t, 0) R \Lambda \mathbf{X}(t, 0) - \sum_{i=1}^\nu \int_{I_i} [s_i] R_i \mathbf{x}(t + \tau) d\tau \]

where \( Q = \oplus_{i=1}^\nu Q_i, \) and \( R = \oplus_{i=1}^\nu R_i \) with \( \Lambda \) in (37).

Given the relations in (42)-(44), differentiating (weak derivative) \( v(\mathbf{x}_i(\cdot)) \) along the trajectory of (20) and consider \( s(z(t), w(t)) \) in (2) produces

\[ \frac{\partial^2}{\partial t^2} \geq 0, \quad v(\mathbf{x}_i(\cdot)) - s(z(t), w(t)) = \mathbf{S}'(\mathbf{A} + \mathbf{B}_1 (\mathbf{I}_\beta \otimes K) \oplus \mathbf{O}_q) \mathbf{S}' + \sum_{i=1}^\nu \int_{I_i} \mathbf{x}^T(t + \tau) R_i \mathbf{x}(t + \tau) d\tau \]

where \( \mathbf{A}, \mathbf{B}_1, \mathbf{C} \) and \( \mathbf{B}_2 \) in (45) are defined in the statements of Theorem 1. Note that \( \hat{I} \) and \( \hat{\mathbf{F}} \) in (37)-(38) are obtained by the identities

\[ \int_{I_i} \mathbf{F}^{-1}_i f_i(\tau) \otimes \mathbf{I}_n \mathbf{x}(t + \tau) d\tau = \int_{I_i} \left( \sum_{i=1}^\nu \int_{I_i} \mathbf{f}_i(\tau - r_i) \otimes \mathbf{I}_n \right) \mathbf{X}(t, \tau) d\tau \]

Note that also the parameters \( \mathbf{A}, \mathbf{B}_1, \mathbf{C} \) and \( \mathbf{B}_2 \) in (45) are given in (21)-(24).
Assume (31) is true, apply (97) with \( \varpi(\tau) = 1 \) and \( g_i(\tau) = \phi_i(\tau), f_i(\tau) = f_i(\tau) \), \( i \in \mathbb{N}_\nu \) to the integral terms 
\[ \sum_{i=1}^\nu \int_{I_i} x^T(t + \tau) R_i x(t + \tau) d\tau \]
for any \( \phi(\cdot) \in C(\mathcal{J}; \mathbb{R}^n) \) in (20), where (53) is derived via the property of \( \forall X \in \mathbb{S}_+^n, \exists \lambda > 0 : \forall x \in \mathbb{R}^n \setminus \{0\}, x^T (\lambda I_n - X) x > 0 \) and (97) with \( \varpi(\tau) = 1 \) and \( f_i(\tau) = \sqrt{F_i^{-1}} f_i(\tau) \). Consequently, (53) shows that there exists \( \epsilon_2 > 0 \) such that \( v(\cdot) \) in (41) satisfies (26).

Now we show that if (30)–(31) are feasible, then \( v(\cdot) \) in (41) satisfies (26) with some \( \epsilon_1; \epsilon_2 > 0 \). Applying (97) to (41) with \( \varpi(\tau) = 1 \), \( g_i(\cdot) = 0 \), and \( f_i(\tau) = \sqrt{F_i^{-1}} f_i(\tau) \) produces

\[ \int_{I_i} x^T(t + \tau) Q_i x(t + \tau) d\tau \geq 0 \]

for any \( \phi(\cdot) \in C(\mathcal{J}; \mathbb{R}^n) \) in (20) with \( u(t) = 0 \). Note that \( x_i(\cdot) \) in (52) is defined in (27). As a result, if (31) and \( \Psi - \Sigma^T J_i^{-1} J_i^T \Sigma < 0 \) are feasible, then \( v(\cdot) \) in (41) satisfies (27)–(28). Finally, applying the Schur complement to \( \Psi - \Sigma^T J_i^{-1} J_i^T \Sigma < 0 \) with (31) and \( J_i^{-1} \) yields (32). Hence we have proved that if (31)–(32) are feasible, then there exists \( \epsilon_3 > 0 \) such that \( v(\cdot) \) in (41) satisfies (27)–(28).

Now we start to show that there exist \( \epsilon_1; \epsilon_2 > 0 \) such that \( v(\cdot) \) in (41) satisfies (26) if (30)–(31) are feasible. Consider \( v(\cdot) \) in (41) with \( t = t_0 \), it follows that

\[ \exists \lambda > 0, v(x(t_0)) = v(\phi(\cdot)) - \eta^T(t_0) \lambda \eta(t_0) \]
in (41) is substantially greater than the ones in [11], [12], [13] even for the case of $\nu = 1$. Meanwhile, the inequality (96) utilized in (49) and (54) is derived based on the principle of least square approximation, thereby ensuring the induced conservatism is minimized.

Remark 10. One can select $g_i(\tau)$ considering what functions are included by the DDs in (3). Since $\varphi(\cdot)$ and $\phi(\cdot)$ are not included by (41), hence the number of unknowns in Theorem 1 is of $O(d^2 n^2)$, which depends on the dimensions of $f_i(\cdot)$. This does show that the the dimensions of $f_i(\cdot)$ can materially affect the number of unknown variables in Theorem 1.

The inequality in (32) is bilinear with respect to $K$ and $P_1$, $P_2$, which cannot be handled by convex SDP numerical solvers. Since $P_2 \in \mathbb{R}^{n \times q}$ is a non-square matrix, convexification via direct congruent transformations seems impossible without simplifying the structure of $P_2$. In the following theorem, the BMI in (32) is convexified via the application of Projection Lemma [26] with slack variables, while preserving the intact structure of $P_2 \in \mathbb{R}^{n \times q}$.

Lemma 2 (Projection Lemma). [26] Given $n; p; q \in \mathbb{N}$, $\Pi \in \mathbb{S}^n$, $P \in \mathbb{R}^{p \times n}$, $Q \in \mathbb{R}^{p \times q}$, there exists $\Theta \in \mathbb{R}^{p \times q}$ such that

\[
\Pi + P^\top \Theta^\top Q + Q^\top \Theta P \prec 0, \quad P_{\perp}^\top P_{\perp} < 0 \quad \text{and} \quad Q_{\perp}^\top Q_{\perp} < 0,
\]

where the columns of $P_{\perp}$ and $Q_{\perp}$ contain bases of null space of matrix $P$ and $Q$, respectively, which means that $P_{\perp}P_{\perp} = 0$ and $QQ_{\perp} = 0$.

Proof. Refer to [26] and [9].

Theorem 2. Given $\{\alpha_i\}_{i=1}^\beta \subset \mathbb{R}$ and the functions and parameters in Proposition 1, then the CLS in (20) with the supply rate function in (2) is dissipative and the trivial solution of (20) with $w(t) \equiv 0_q$ is uniformly asymptotically (exponentially) stable if there exists $\tilde{P}_1; X \in \mathbb{S}^n$, $\tilde{P}_2 \in \mathbb{R}^{n \times q}$, $\tilde{P}_3 \in \mathbb{S}^q$ and $\tilde{Q}_i; \tilde{R}_i \in \mathbb{S}^n$, $e = nd$ and $V \in \mathbb{R}^{p \times q}$ such that

\[
\begin{bmatrix}
\tilde{P}_1 & \tilde{P}_2 \\
\ast & \tilde{P}_3
\end{bmatrix} + \begin{bmatrix}
O_n \oplus \left( \bigoplus_{i=1}^\nu I_{d_i} \otimes \tilde{Q}_i \right)
\end{bmatrix} \succ 0, \quad \gamma^\top \begin{bmatrix}
O_n & P \\
\ast & \Phi
\end{bmatrix} \gamma = \begin{bmatrix}
J_3 - \text{Sy}(D_2^\top J_2) & D_2^\top J_1 \end{bmatrix} \prec 0
\]

where $\tilde{P}_1 = \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \end{bmatrix}$ and $\tilde{Q}_i = \bigoplus_{i=1}^\nu \tilde{Q}_i > 0$, $\tilde{R}_i = \bigoplus_{i=1}^\nu \tilde{R}_i > 0$

\[\Phi = \text{Sy} \left( \begin{bmatrix}
I_n \\
\text{Col}_{i=1}^\beta \alpha_i I_n \end{bmatrix} \left[ -X \quad \bar{\Pi} \right] + \begin{bmatrix}
O_n & \hat{\Phi}
\end{bmatrix} \right) < 0 \quad (59)\]

with $\bar{\Pi}$ in (37) and matrices

\[\Phi = \text{Sy} \left( \begin{bmatrix}
\tilde{P}_2 \\
O_{n,\nu} \\
\bar{\Pi} \tilde{P}_3 \\
O_{n,\mu+q+m}
\end{bmatrix} \right) \otimes I_n \otimes \bigoplus_{i=1}^\nu I_{\kappa_i} \otimes \hat{R}_i \\
\bigoplus_{i=1}^\nu I_{\mu_i} \otimes \hat{R}_i \bigoplus J_3 \bigoplus (-J_1)
\]

\[\text{with } \bar{\Pi} \text{ in (38) and } \bar{\Sigma} = C \left[ (I_\beta \otimes X) \oplus I_q \right] + B_2 \left[ (I_\beta \otimes V) \oplus O_q \right] \text{ and the parameters } A, B_1, B_2, C \text{ in (21)–(24). The controller gain is calculated via } K = VX^{-1}. \]

Finally, the number of unknowns in this theorem is $(0.5d^2 + 0.5d + \nu + 1)n^2 + (0.5d + 1 + \nu + p)n \in O(d^2n^2)$.

Proof. First of all, note that the inequality $\text{Sy}(P^\top \Pi) + \Phi \prec 0$ in (32) can be rewritten as

\[\text{Sy}(P^\top \Pi) + \Phi = \begin{bmatrix}
O_n & P \\
\ast & \Phi
\end{bmatrix} \begin{bmatrix}
\Pi \\
I_{\beta+q+m}
\end{bmatrix} \prec 0 \quad (60)\]

It is easy to see that the structure of (61) is similar to one of the inequalities in (56). Given that two matrix inequalities are presented in (56), thus a new matrix inequality must be constructed to utilize Lemma 2. Now by considering the structure of $\Phi$, we have

\[\gamma^\top \begin{bmatrix}
O_n & P \\
\ast & \Phi
\end{bmatrix} \gamma = \begin{bmatrix}
J_3 - \text{Sy}(D_2^\top J_2) & D_2^\top J_1 \\
\ast & \ast
\end{bmatrix} \prec 0 \quad (62)\]

where $\gamma^\top := \begin{bmatrix} O_{(q+m),(n+\beta)} & I_{q+m} \end{bmatrix}$. Since the matrix in (62) corresponds to the $2 \times 2$ block matrix at the bottom-right corner of the matrices $S_y(P^\top \Pi) + \Phi$ or $\Phi$, hence
the inequality in (62) is implied by (61) or (32). On the other hand, the following identities
\[
[ -I_n \ \Pi ] \begin{bmatrix} \Pi \\ I_{n+\beta q+m} \end{bmatrix} = O_{n \times (n \beta + q + m)} , \\
[ -I_n \ \Pi ] \perp \begin{bmatrix} \Pi \\ I_{n+\beta q+m} \end{bmatrix} , \\
[ I_{n+n\beta} \ O_{(n+n\beta),(q+m)} ] \begin{bmatrix} O_{n+n\beta),(q+m) \\ I_{q+m} \end{bmatrix}
\]
(63)
which satisfy rank \( \left[ I_{n+n\beta} \ O_{(n+n\beta),(q+m)} \right] = n + n\beta \) and rank \( \left[ -I_n \ \Pi \right] = n \) imply that \( -I_n \ \Pi \) and \( I_{n+n\beta} \ O_{(n+n\beta),(q+m)} \) can be utilized by Lemma 2 given the rank nullity theorem.

Applying Lemma 2 to (61)–(63) yields the conclusion that (61)–(62) are true if and only if \( \exists W \in \mathbb{R}^{(n+\beta \times n)} \)
\[
Sy \left[ \begin{bmatrix} I_{n+n\beta} \\ O_{(q+m),(n+n\beta)} \end{bmatrix} W \begin{bmatrix} -I_n \ \Pi \end{bmatrix} \right] + \begin{bmatrix} O_n \ P \\ * \ \Phi \end{bmatrix} < 0 .
\]
(64)
Now (64) is still bilinear due to the product between \( W \) and \( \Pi \). To convexify (64), let
\[
W = \begin{bmatrix} W \\ \text{Col}_{i=1}^{\beta} \alpha_i W \end{bmatrix}
\]
(65)
with \( W \in \mathbb{S}^n \) and \( \{ \alpha_i \} \beta_{i=1}^\beta \subset \mathbb{R} \). With (65), the inequality in (64) becomes
\[
\Theta = Sy \left[ \begin{bmatrix} W \\ \text{Col}_{i=1}^{\beta} \alpha_i W \end{bmatrix} \begin{bmatrix} -I_n \ \Pi \end{bmatrix} \right] + \begin{bmatrix} O_n \ P \\ * \ \Phi \end{bmatrix} < 0
\]
(66)
which infers (66). Note that (66) is only a sufficient condition implying (61) or (32) due to the structural constraints in (65). Note that also the invertibility of \( W \in \mathbb{S}^n \) is guaranteed by (66) since \(-2W\) is the only element at the first diagonal block of \( \Theta \).

Let \( X^T = W^{-1} \), apply congruent transformations to the matrix inequalities in (30)–(31) and (66). Then
\[
(I_{\nu} \otimes X) \ Q (I_{\nu} \otimes X) > 0 , (I_{\nu} \otimes X) \ R (I_{\nu} \otimes X) > 0 ,
\]
\[
[ (I_{1+\beta} \otimes X^T) \otimes I_{q+m} ] \Theta [ (I_{1+\beta} \otimes X) \otimes I_{q+m} ] > 0 ,
\]
(67)
which hold if and only if (30)–(31) and (66) hold. Moreover, with (92) and the definitions
\[
\begin{bmatrix} \hat{P}_1 \ \hat{P}_2 \\ * \ \hat{P}_3 \end{bmatrix} := \begin{bmatrix} P_1 \ P_2 \\ * \ \ P_3 \end{bmatrix} (I_{1+d} \otimes X) ,
\]
(68)
the inequalities in (67) can be rewritten as (57)–(58) and
\[
( I_{1+\beta} \otimes X^T ) \Theta ( I_{1+\beta} \otimes X ) \otimes I_{q+m} = \Theta = Sy \left[ \begin{bmatrix} I_n \\ \text{Col}_{i=1}^{\beta} \alpha_i I_n \end{bmatrix} \begin{bmatrix} -X \ \Pi \end{bmatrix} \right] + \begin{bmatrix} O_n \ \hat{P} \\ * \ \hat{\Phi} \end{bmatrix} < 0
\]
(69)
where \( \hat{P} = XP \left[ (I_{\beta} \otimes X) \otimes I_{q+m} \right] = \begin{bmatrix} \hat{P}_1 \ O_{n,\nu_n} \ \hat{P}_2 \hat{I} \ \ O_{n,\mu m} \ \ O_{n,q} \ O_{n,m} \end{bmatrix} \)
(70)
and \( \hat{\Pi} = \Pi [ (I_{\beta} \otimes X) \otimes I_{q+m} ] = \begin{bmatrix} A \left[ (I_{\beta} \otimes X) \otimes I_q \right] + \ B_1 \left[ (I_{\beta} \otimes X) \otimes O_q \right] \ O_{n,m} \\
A \left[ (I_{\beta} \otimes V) \otimes X \right] + \ B_1 \left[ (I_{\beta} \otimes V) \otimes O_q \right] \ O_{n,m} \end{bmatrix} \)
(71)
with \( V = KX \) and \( \hat{\Phi} \) in (60). Note that (69) is the same as (59), and the form of \( \hat{\Phi} \) in (60) is derived via the relations \( \hat{I} (I_{\alpha} \otimes X) = (I_{\alpha} \otimes X) \hat{I} \) and
\[
\left[ \hat{F} \otimes I_n \ \ O_{\varphi,(q+m)} \right] [ (I_{\beta} \otimes X) \otimes I_{q+m} ]
\]
(72)
which are derived from (92) and (94). Furthermore, since \(-2X\) is the only term at the first diagonal block of \( \Theta \), (59) holds. This is consistent with the invertibility of \( W \) is implied by (66).

As a result, we have shown the equivalence between (30)–(31) and (57)–(58). Meanwhile, it has been shown that (59) is equivalent to (66) which infers (32). Consequently, (30)–(32) are satisfied if (57)–(59) hold for some \( W \in \mathbb{S}^n \) and \( \{ \alpha_i \} \beta_{i=1}^\beta \subset \mathbb{R} \). This finishes the proof.

**Remark 11.** Though the simplification in (65) can introduce conservatism, the structure of the matrix parameters in (57) remains identical to (30). As a result, the use of Lemma 2 at (64) does not degenerate the matrix parameters of the KF in (41), thereby creating less conservatism compared to directly simplifying \( P_2 \) to convexify the BMI in (32). Finally, note that the slack variables in Theorem 2 do not increase feasibility compared to Theorem 1.
Remark 12. For the scalars \( \{ \alpha_i \}_{i=1}^{\beta} \subset \mathbb{R} \), we can assume \( \alpha_i = 0 \) for \( i = 2 \cdots \beta \) with an adjustable \( \alpha_1 \in \mathbb{R} \setminus \{0\} \). Note that \( \alpha_1 \neq 0 \) is necessary since \( \alpha_1 = 0 \) will render the \( A_0 \) related-diagonal-block in (59) infeasible.

C. Inner Convex Approximation for (32)

Though Theorem 2 provides a convex synthesis condition with given \( \{ \alpha_i \}_{i=1}^{\beta} \), the step at (65) can introduce conservatism by reducing the size of feasible region. As a result, it is preferable to have methods which can solve (32) directly. Though there are many numerical approaches for solving generic non-convex SDPs [37], here an iterative algorithm (Algorithm 1) is proposed based on the inner convex approximation scheme outlined in [38], where each iteration is a convex program. The algorithm guarantees the convergence to a local optimum and can be initiated by a feasible solution of Theorem 2. This combines the advantage of both Theorem 1 and 2.

First of all, we want to point out that the inequality in (32) is nonconvex whereas (30)–(31) remain convex even if a controller design problem is considered. Now we reformulate the inequality in (32) as

\[
\mathcal{U}(H, K) := \text{Sy} \left[ P^\top \Pi + \Phi \right] = \text{Sy} \left[ P^\top B \left[ (I_\beta \otimes K) + O_{p+m} \right] \right] + \Phi < 0 \quad (73)
\]

with \( B := \begin{bmatrix} B_1 & 0_{n,m} \end{bmatrix} \) and \( \hat{\Phi} := \text{Sy} \left( P^\top \left[ A \ 0_{n,m} \right] \right) + \Phi \), where \( P \) is given in (39), and \( A \) and \( B_1 \) are given in (21)–(22), and \( H := \begin{bmatrix} P_1 & P_2 \end{bmatrix} \) with \( P_1 \) and \( P_2 \) in Theorem 1. It is important to stress that \( \hat{\Phi} \) is convex with respect to the unknowns inside. Utilizing the results of Example 3 in [38], one can conclude that \( \Delta \left( \cdot, \tilde{G}, \tilde{N} \right) \), which is defined as

\[
S^x \ni \Delta \left( G, \tilde{G}, N, \tilde{N} \right) := [\ast] [Z \oplus (I_n - Z)]^{-1} \begin{bmatrix} G - \tilde{G} \\ N - \tilde{N} \end{bmatrix}
+ \text{Sy} \left( \tilde{G}^\top N + G^\top \tilde{N} - \tilde{G}^\top \tilde{N} \right) + T \quad (74)
\]

with \( Z \oplus (I_n - Z) > 0 \) satisfying

\[
\forall G; \tilde{G} \in \mathbb{R}^{n \times \ell}, \forall N; \tilde{N} \in \mathbb{R}^{n \times \ell}, \quad T + \text{Sy} \left( G^\top N \right) := T + \text{Sy} \left( G^\top N \right) \leq \Delta \left( G, \tilde{G}, N, \tilde{N} \right) \quad (75)
\]

is a psd-convex overestimate of \( \hat{\Delta}(G, N) = T + \text{Sy} \left[ G^\top N \right] \) with respect to the parameterization

\[
\text{Col} \left( \text{vec}(\tilde{G}), \text{vec}(\tilde{N}) \right) = \text{Col} \left( \text{vec}(G), \text{vec}(N) \right). \quad (76)
\]

Now let \( T = \hat{\Phi}, G = P \) and \( \tilde{P}_1 \in \mathbb{S}^n, \tilde{P}_2 \in \mathbb{R}^{n,dn} \)

\[
\tilde{G} = \tilde{P}_1 \quad O_{n,m} \quad \tilde{P}_2 \quad \begin{bmatrix} I_n,_{(n+q+m)} \end{bmatrix} \quad (77)
\]

\[
H = \begin{bmatrix} P_1 & P_2 \end{bmatrix}, \quad \tilde{H} := \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \end{bmatrix}, \quad N = BK, \quad K = (I_\beta \otimes K) \oplus O_{p+m}, \quad \tilde{N} = B\tilde{K}, \quad \tilde{K} = (I_\beta \otimes \tilde{K}) \oplus O_{p+m}
\]

in (74) with \( \ell := n_\beta + q + m \) and \( Z \oplus (I_n - Z) > 0 \) and \( \hat{\Phi}, H \) and \( K \) are defined in (73). Then one can obtain

\[
\mathcal{U}(H, K) = \hat{\Phi} + \text{Sy} \left[ P^\top B \left[ (I_\beta \otimes K) + O_{p+m} \right] \right] \leq S \left( H, \tilde{H}, K, \tilde{K} \right) := \hat{\Phi} + \text{Sy} \left( P^\top N + P^\top \tilde{N} - \tilde{P}^\top \tilde{N} \right) + \left[ P^\top - \tilde{P}^\top \right] \left[ N^\top - \tilde{N}^\top \right] \left[ Z \oplus (I_n - Z) \right]^{-1} \ast \quad (78)
\]

by (75), where \( S(\cdot, \hat{\Phi}, \cdot, \tilde{K}) \) is a psd-convex overestimate of \( \mathcal{U}(H, K) \) in (73) with respect to the parameterization

\[
\text{Col} \left( \text{vec}(H), \text{vec}(\tilde{K}) \right) = \text{Col} \left( \text{vec}(H), \text{vec}(K) \right). \quad (76)
\]

From (78), it is obvious that \( S \left( H, \tilde{H}, K, \tilde{K} \right) < 0 \) infers (73). Moreover, \( S \left( H, \tilde{H}, K, \tilde{K} \right) < 0 \) holds if and only if

\[
\hat{\Phi} + \text{Sy} \left( P^\top N + P^\top \tilde{N} - \tilde{P}^\top \tilde{N} \right) + \left[ P^\top - \tilde{P}^\top \right] \left[ N^\top - \tilde{N}^\top \right] \left[ Z \oplus (I_n - Z) \right]^{-1} \ast \quad (79)
\]

holds with \( N, \tilde{N} \) in (77) based on the application of the Schur complement given \( Z \oplus (I_n - Z) > 0 \). As a result, (73) is inferred by (79) which can be computed by standard SDP solvers if \( \tilde{H} \) and \( \tilde{K} \) are known.

By compiling all the aforementioned procedures according to the expositions in [38], Algorithm 1 is established where \( x \) consists of all the variables in \( P_3, Q_1, Q_2, R_1, R_2 \) in Theorem 1 and \( Z \) in (79). Furthermore, \( \rho_1, \rho_2 \) and \( \varepsilon \) are given constants for regularizations and indicating error tolerance, respectively.

Remark 13. It is worthy to point out that different convex overestimate methods [39], [40] can be utilized in (75) by replacing \( [Z(I - Z)]^{-1} \) in (74) with other terms. However, these replacement does not change the essence of inner convex approximation and the corresponding inequality (79) can be immediately constructed. Hence users have the liberty to decide which overestimate methods they want to adopt. Finally, one can apply a more general regularization scheme \( \text{tr} \left( \left[ x^\top - \bar{x}^\top \right] \text{vec} \left( A - \bar{A} \right) \text{vec} \left( K - \bar{K} \right) \right) \) with weighting matrices \( T := \bigoplus_{i=1}^3 T_i \geq 0 \) to improve the convergence of Algorithm 1, where \( T_k \geq 0 \) are known and \( \bar{x} \) is the value of \( x \) at the previous step.
Algorithm 1: An iterative solution for Theorem 1

begin
solve Theorem 2 return \( K \)
solve Theorem 1 with \( K \) return \( P_1, P_2 \)
solve Theorem 1 with \( P_1, P_2 \) return \( K \).
update \( \bar{H} \leftarrow H = [P_1 \ P_2] \), \( \bar{K} \leftarrow K \),
solve \( \min_{x, \bar{H}, \bar{K}} \text{tr}\left[\rho_1[x](H - \bar{H})\right] + \text{tr}\left[\rho_2[x](K - \bar{K})\right] \)
subject to (30)–(31), (79) with (77) and the parameters in Theorem 1, return \( H \) and \( K \)
while \( \left\| \frac{\text{vec}(H)}{\text{vec}(K)} - \frac{\text{vec}(\bar{H})}{\text{vec}(\bar{K})} \right\|_\infty \geq \varepsilon \) do
update \( \bar{H} \leftarrow H \), \( \bar{K} \leftarrow K \),
solve \( \min_{x, \bar{H}, \bar{K}} \text{tr}\left[\rho_1[x](H - \bar{H})\right] + \text{tr}\left[\rho_2[x](K - \bar{K})\right] \)
subject to (30)–(31), (79) with (77) and the parameters in Theorem 1, return \( H \) and \( K \)
end

IV. A variant scheme of DSFC

If there are no delays at the control input of (1), we can modify Theorem 1–2 and Algorithm 1 to solve a different DSFC problem where the controller contains \( \nu \) pointwise and distributed delays.

Specifically, let \( B_i = \bar{B}_i(\tau) = O_{n,p}, \mathcal{B}_i = \bar{\mathcal{B}}_i(\tau) = O_{m,p}, i \in \mathbb{N}_\nu \) in (1), which corresponds to a distributed-delay system without input delays. Now we want to construct

\[
u(t) = \sum_{i=0}^{\nu} K_i x(t - r_i) + \sum_{i=1}^{\nu} \int_{-r_{i-1}}^{-r_i} \bar{K}_i(\tau) x(t + \tau) d\tau \tag{80}\]

to stabilize the open-loop system in (1), where \( K_i \in \mathbb{R}^{p \times n}, \bar{K}_i(\cdot) \in \mathcal{L}^2(\mathcal{I}_i ; \mathbb{R}^{p \times n}) \). The above case is vitally important for the research on LTDSs, as early results in [41] have indicated that a controller with delays could be necessary in order to stabilize certain unstable LTDS.

By the proof of Proposition 1 in Appendix A, one can conclude that (3) and \( \bar{K}_i(\cdot) \in \mathcal{L}^2(\mathcal{I}_i ; \mathbb{R}^{p \times n}) \) are true if and only if (4)–(6) holds and there exist \( \mathcal{K}_i \in \mathbb{R}^{p \times n \cdot i} \) such that

\[
\forall i \in \mathbb{N}_\nu, \forall \tau \in \mathcal{I}_i, \bar{K}_i(\tau) = \mathcal{K}_i (g_i(\tau) \otimes I_n) \tag{81}\]

Now by using the above conclusion with (10)–(13) and (81), the CLS with (80) is

\[
\dot{x}(t) = (A + B_0 K) \vartheta(t), \quad z(t) = (C + \mathcal{B}_0 K) \vartheta(t),
\forall \vartheta \in \mathcal{J}, \quad x(t_0 + \tau) = \psi(\vartheta), \quad \psi(\cdot) \subset C(\mathcal{J} ; \mathbb{R}^n)
\]

\[
K = \left[ \left[ K_i \right]_{i=0}^{\nu} \left[ \mathcal{K}_i \left( \tilde{g}_i \otimes I_n \right) \right]_{i=1}^{\nu} \right] \tag{82}\]

where \( A, C, \vartheta(t) \) are given in (21)–(25).

Since the structure in (82) is similar to (20), then one can modify Theorem 1 and 2 to handle (82) accordingly. This yields the following two corollaries.

**Corollary 1.** Let all the parameters in Proposition 1 and (81) be given. Then the CLS (82) with the supply rate function in (2) is dissipative and the trivial solution of (82) with \( w(t) \equiv 0 \) is uniformly asymptotically (exponentially) stable if there exist \( P_1 \in \mathbb{S}^n, P_2 \in \mathbb{R}^{n \times e}, P_3 \in \mathbb{S}^e \) with \( 0 = n \sum_{i=1}^{\nu} d_i \), and \( Q_i, R_i \in \mathbb{S}^n, K_i, \Gamma_i \in \mathbb{R}^{p \times n}, i \in \mathbb{N}_\nu \) such that (30)–(32) hold with \( \Omega = A + B_0 K \) and \( \Sigma = C + \mathcal{B}_0 K \) where \( A \) and \( C \) are given in (21)–(23) and \( K \) is given in (82).

Finally, the number of unknown variables in Corollary 1 is \((0.5d^2 + 0.5d + \nu + 0.5)n^2 + (0.5d + 0.5 + \nu + p + \nu p + \kappa p)n \) in \( \mathcal{O}(d^2n^2) \), where \( d = \sum_{i=1}^{\nu} d_i \).

**Proof.** The corollary is prove via the substitutions \( \Omega = A + B_0 K \) and \( \Sigma = C + \mathcal{B}_0 K \) in (32).

**Corollary 2.** Given the conditions in Proposition 1 with (81) and known parameters \( \{\alpha_i\}_{i=1}^{\beta} \). Then the CLS (82) with the supply rate function in (2) is dissipative and the trivial solution of (82) with \( w(t) \equiv 0 \) is uniformly asymptotically (exponentially) stable if there exist \( \hat{P}_1, X \in \mathbb{S}^n, \hat{P}_2 \in \mathbb{R}^{n \times e}, \hat{P}_3 \in \mathbb{S}^e \) and \( \hat{Q}_i, \hat{R}_i \in \mathbb{S}^n \) and \( \hat{V}_0, \hat{V}_i \in \mathbb{R}^{p \times n}, \hat{V}_i \in \mathbb{R}^{p \times n} \), \( i \in \mathbb{N}_\nu \), such that (57)–(59) hold with

\[
\dot{\hat{H}} = \left[ A \left[ (I_{\beta} \otimes X) \oplus I_q \right] + B_0 V - O_{n,m} \right],
\dot{\hat{\Omega}} = \left[ C \left[ (I_{\beta} \otimes X) \oplus I_q \right] + \mathcal{B}_0 V \right]
\]

where \( A \) and \( C \) are given in (21),(23) and

\[
V = \left[ \left[ V_i \left( \tilde{g}_i \otimes I_n \right) \right]_{i=1}^{\nu} \right] \tag{83}\]

Moreover, the controller gains are calculated via \( K_0 = V_0 X^{-1} \) and \( K_i = V_i X^{-1} \) and \( \mathcal{K}_i = V_i \left( I_{\kappa_i} \otimes X^{-1} \right) \) for \( i \in \mathbb{N}_\nu \).

Finally, the number of unknowns is \((0.5d^2 + 0.5d + \nu + 1)n^2 + (0.5d + 1 + \nu + p + \nu p + \kappa p)n \) in \( \mathcal{O}(d^2n^2) \).
Proof. The proof is obtained based on the proof of Theorem 2. Note that the corresponding step at (71) is
\[
\tilde{H} = \begin{bmatrix} A \left( (I_\beta \otimes X) \oplus I_\theta \right) + B_0 K \left( (I_\beta \otimes X) \oplus I_\theta \right) & 0_{n,m} \\ A \left( (I_\beta \otimes X) \oplus I_\theta \right) + B_1 V & 0_{n,m} \end{bmatrix} \tag{84}
\]
with \( V \) in (83) where \( V_0 = K_0 X, V_i = K_i X \) and \( V_i = X_i \left( I_{n_i} \otimes X \right) \) for all \( i \in \mathbb{N}_p \). Note that the equality \( K \left( (I_\beta \otimes X) \oplus I_\theta \right) = V \) with \( K \) in (82) and \( V \) in (83) can be proved by the application of (92).

Corollary 1 can be solved by a modified version of Algorithm 1, as summarized in using the substitutions \( N = B_0 \begin{bmatrix} K & 0_{p \times m} \end{bmatrix}, \tilde{N} = B_0 \begin{bmatrix} \tilde{K} & 0_{p \times m} \end{bmatrix}, K \leftarrow \tilde{K} \), \( \tilde{K} \leftarrow \tilde{\mathbf{r}} \) for the condition in (79) with the parameters in Corollary 1 and the parameterization
\[
\text{Col} \begin{bmatrix} \text{vec}(\tilde{H}) \end{bmatrix}, \text{vec}(\tilde{\mathbf{r}}) = \text{Col} \begin{bmatrix} \text{vec}(H) \end{bmatrix}, \text{vec}(\mathbf{r}) \]
where \( \mathbf{K} \) is given in (82) and
\[
\tilde{K} = \left[ \left[ \tilde{K}_i \right] \right]_{i=0}^\nu \left[ \left[ K_i \left( \bar{I}_i \otimes I_n \right) \right] \right]_{i=1}^{\nu} \cdots \\
\cdots \left[ \left[ K_i \left( \bar{I}_i \sqrt{E_i} \otimes I_n \right) \right] \right]_{i=1}^{\nu} 0_{p,q} \tag{85}
\]
\[
\tilde{\mathbf{r}} = \left[ \left[ K_i \right] \right]_{i=0}^\nu \left[ \left[ X_i \right] \right]_{i=1}^{\nu}, \tilde{\mathbf{r}} = \left[ \left[ K_i \right] \right]_{i=0}^\nu \left[ \left[ X_i \right] \right]_{i=1}^{\nu}.
\]

Finally, the following diagrams can intuitively explain the relations between the proposed theorems (corollaries) and iterative algorithms, which can be used as a single package to solve the DSFC problem for (1).

V. NUMERICAL EXAMPLES

In this section, we present two numerical examples to show the effectiveness of our proposed methodologies. The examples involve using Algorithm 1-2 to solve the DSFC problem for LTDSs, thereby involving all the components of the proposed methods. All computation is carried out in Matlab© using Yalmip [42] as the optimization parser, and SDPT3, Mosek [43], [44] as the numerical solvers for SDPs.
for the supply rate function in (1) with \( r_1 = 1, r_2 = 1.7 \) and the state space matrices

\[
A_0 = \begin{bmatrix} -2 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix},
\]

\[
B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.01 \\ 0.1 \end{bmatrix}, \quad B_2 = -\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},
\]

\[
\tilde{A}_1(\tau) = \begin{bmatrix} 0.1 + 3 \sin(20\tau) & 0.8 \sin(20\tau) - 0.3 \sin(20\tau) \\ 0.3 \sin(12\tau) + 1 & 0.2 \sin(20\tau) \end{bmatrix},
\]

\[
\tilde{A}_2(\tau) = \begin{bmatrix} -10 \cos(18\tau) & 0.3 \sin(18\tau) + 1 \\ 0.1 \sin(18\tau) & -20 \cos(18\tau) \end{bmatrix},
\]

\[
\tilde{B}_1(\tau) = \begin{bmatrix} 0.01 \tau + 0.1 \sin^2(1.2\tau) + 1 + 0.1 \\ 0.1 \tau + 0.2 \sin^2(1.2\tau) + 1 \end{bmatrix},
\]

\[
\tilde{B}_2(\tau) = \begin{bmatrix} 0.2 \cos(2\tau) - 0.01 \sin(2\tau) + 0.01 \cos(0.7\tau) + 1 \\ 0.1 \cos(2\tau) + 0.02 \sin(2\tau) + 0.01 \cos(0.7\tau) + 1 \end{bmatrix},
\]

\[
C_0 = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.1 & 0 \end{bmatrix},
\]

\[
\tilde{C}_1(\tau) = \begin{bmatrix} 0.7 + \cos(20\tau) \\ 0.4 - 0.5 \sin(20\tau) \end{bmatrix}, \quad \tilde{C}_2(\tau) = \begin{bmatrix} 0.2 + \sin(18\tau) \\ 0.8 - \sin(20\tau) \end{bmatrix},
\]

\[
\tilde{B}_1(\tau) = \begin{bmatrix} 0.01 \tau + 0.1 \sin^2(2\tau) + 1 \\ 0.2 \sin^2(2\tau) \end{bmatrix},
\]

\[
\tilde{B}_2(\tau) = \begin{bmatrix} 0.2 \cos(2\tau) + 0.01 \sin(2\tau) + 0.01 \cos(0.7\tau) + 1 \\ 0.2 \sin^2(2\tau) + 0.01 \cos(0.7\tau) + 1 \end{bmatrix},
\]

\[
D_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix},
\]

with \( n = m = 2, \ p = q = 1 \). By using the numerical toolbox of the spectral method proposed in [45], it shows the nominal system is unstable. Moreover, we utilize

\[
\gamma > 0, \quad J_1 = -\gamma I_2, \quad J_2 = J_2, \quad J_3 = \gamma
\]

for the supply rate function in (2) where the scalar \( \gamma \) is the \( L_2 \) gain performance objective to be minimized.

**Remark 14.** The parameters in (86) are chosen with sufficient degree of mathematical complexity in order to illustrate the strength of the proposed method. It is important to point out that our approach can handle many practical examples such as the ones mentioned in Remark 2, whose DDs are simpler than the DDs in (86). Note that no existing methods can effectively solve the DSFC problem of an LTDS in (1) with (86) due to the complexity of the DDs with multiple non-commensurate delays and a non-Hurwitz \( A_0 \).

Assuming all system’s states can be measured, we want to find a controller gain for \( u(t) = K x(t) \) to stabilize the open-loop system (1) while minimizing the \( L_2 \) gain. Observing the functions inside of the DDs, let \( \varphi_1(\tau) = 1/(\sin^2 1.2\tau + 1) \) and \( \varphi_2(\tau) = 1/(\cos^2 0.7\tau + 1) \) and

\[
\phi_1(\tau) = \begin{bmatrix} e^{\sin(20\tau)} \\ e^{\cos(20\tau)} \end{bmatrix}, \quad \phi_2(\tau) = \begin{bmatrix} e^{\sin(18\tau)} \\ e^{\cos(18\tau)} \end{bmatrix},
\]

\[
f_1(\tau) = \begin{bmatrix} [r^T]_{i=0}^{d_1} \\ [\sin 20\tau]_{i=1}^{d_1} \end{bmatrix}, \quad f_2(\tau) = \begin{bmatrix} [r^T]_{i=0}^{d_2} \\ [\cos 20\tau]_{i=1}^{d_2} \end{bmatrix}
\]

for the parameters in Proposition 1 with

\[
M_1 = \begin{bmatrix} 0 & 0_d^T & 0 & O_{L_1} & \oplus^{\lambda_1}_{i=1} 20i \\ 0 & \oplus^{d_1}_1 & 0_d & -\oplus^{\lambda_1}_{i=1} 20i & O_{L_1} \end{bmatrix},
\]

\[
M_2 = \begin{bmatrix} 0 & 0_d^T & 0 & O_{L_2} & \oplus^{\lambda_2}_{i=1} 18i \\ 0 & \oplus^{d_2}_1 & 0_d & -\oplus^{\lambda_2}_{i=1} 18i & O_{L_2} \end{bmatrix}
\]

in (6). By (88) and (92), we can construct

\[
\hat{A}_1 = \begin{bmatrix} 0.8 & -0.3 & 0.1 & 0.1 \omega^T_{2d_1} & 3.0 \omega^T_{1\lambda_1} - 2 \\ 0 & 0 & 0 & 0 & 1.3 \omega^T_{2\lambda_1} - 2 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[
\hat{B}_1 = \begin{bmatrix} 0 & 0 & 0 & -0.1 & 0.1 & 0.1 \omega^T_{d_1 - 1 + 2\lambda_1} \\ 0 & 0 & 0 & 0.2 & 0.1 & 0 \end{bmatrix},
\]

\[
\hat{B}_2 = \begin{bmatrix} 0.01 & 0.2 & 0.1 & 0 & 0 \omega^T_{d_2 + 1 + 2\lambda_2} \\ 0.02 & 0.1 & 0 & 0 & 0 \end{bmatrix},
\]

\[
\hat{C}_1 = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.5 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}, \quad \hat{C}_2 = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.5 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix},
\]

\[
\tilde{B}_1 = \begin{bmatrix} 0.1 & 0 & -0.1 & 0 & 0.01 \omega^T_{d_1 + 2\lambda_1 - 1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[
\tilde{B}_2 = \begin{bmatrix} 0.01 & 0.2 & 0.1 & 0 & 0 \omega^T_{d_2 + 2\lambda_2 - 1} \\ 0.02 & 0.2 & 0.0 & 0 & 0 \end{bmatrix},
\]

satisfying the decompositions in (4)–(5).

**Remark 15.** The functions in (88) are chosen based on several reasons. First of all, the functions in \( \phi_i(\cdot) \) can be well approximated via appropriate trigonometric functions together with polynomials, some of which exist in the DDs in (86). On the other hand, the functions in
\( \varphi(\cdot) \) are directly factorized since it is very difficult to approximate them with \( d_i, \lambda_i \) of manageable values. As a result, the choice for (88) involves all the components proposed in Proposition 1, which balances feasibility and the implied computational complexity \( O(d^2 n^2) \) affected by the dimensions of \( f_i(\tau) \). This serves as a good example showing the advantage of Proposition 1 over the existing approaches in [11], [12], [13].

For computing the controller gain \( K \), apply Theorem 2 first to (20) with \( d_1 = d_2 = \lambda_1 = \lambda_2 = 1 \) and \( \alpha_i = 0, i = 2 \cdots \beta; \alpha_i = 5 \) and the parameters in (86)–(90), where the matrices in (10)–(13) are computed via the \( \text{vpintegra}1 \) function in Matlab®. The numerical program yields \( K = \begin{bmatrix} -1.3794 & -1.8605 \\ -1.9359 & -1.9539 \end{bmatrix} \) with \( \min \gamma = 0.8986 \), where the controller gain is used for initializing Algorithm 1.

After running Algorithm 1 for the same system, the numerical results are summarized in Table I–II, where SA stands for the Spectral Abscissa\(^c\) of the resulting CLSs with \( w(t) \equiv 0_q \), and Nols for the number of iteration in the while loop. The results clearly show that adding more functions (larger \( \lambda_1, \lambda_2 \)) to \( f_i(\cdot) \) may increase the feasibility of the synthesis conditions leading to smaller \( \min \gamma \). Moreover, it shows that using Algorithms 1 can produce controller gains with significantly better performance (\( \min \gamma \)) than Theorem 2 alone. (\( \min \gamma = 0.8986 \)) This illustrates the contribution of Algorithms 1.

For numerical simulation, we consider the CLSs in (20) with \( K = \begin{bmatrix} -1.5810 & -1.9805 \end{bmatrix} \) in Table II. Moreover, let \( t_0 = 0, x(t) = 0_q, t < 0 \), and \( \psi(\tau) = 50 30 \), \( \tau \in [-r_2, 0] \) for the initial condition, and \( w(t) = 50 \sin 10(t - 20t - 15t - 5) \) as the disturbance where \( 1(t) \) is the Heaviside step function. The simulation is performed in Simulink via the ODE solver \( \text{ode8} \) with 0.002 as the fundamental sampling time. The results are summarized in Figures 1–3 including the trajectories of the states \( x(t) \), control action \( u(t) \) and regulated outputs \( z(t) \) of the CLS. Note that all the DDs are discretized for the simulation via the trapezoidal rule with 200 sample points.

![Fig. 1: The close-loop system’s trajectories \( x(t) \)](image1.png)

![Fig. 2: The trajectory of the control action \( u(t) \)](image2.png)

### Table I: Controller gains with \( \min \gamma \) produced with \( d_1 = d_2 = \lambda_1 = \lambda_2 = 1 \)

| Controller gain | \( \begin{bmatrix} -1.5456 & -1.5395 \\ -1.9359 & -1.9539 \end{bmatrix} \) | \( \begin{bmatrix} -1.5180 & -1.5033 \\ -1.9696 & -1.9815 \end{bmatrix} \) | \( \begin{bmatrix} -1.5810 & -1.9805 \end{bmatrix} \) |
|-----------------|-----------------|-----------------|-----------------|
| \( \min \gamma \) | 0.6573 | 0.6542 | 0.6523 | 0.6509 |
| SA | -0.7223 | -0.7214 | -0.7224 | -0.7233 |
| Nols | 5 | 10 | 15 | 20 |

### Table II: Controller gains with \( \min \gamma \) produced with \( d_1 = d_2 = 1, \lambda_1 = \lambda_2 = 2 \)

| Controller gain | \( \begin{bmatrix} -1.5538 & -1.5848 \\ -1.9560 & -1.9638 \end{bmatrix} \) | \( \begin{bmatrix} -1.5870 & -1.9721 \\ -1.9805 & -1.9805 \end{bmatrix} \) | \( \begin{bmatrix} -1.5810 & -1.9805 \end{bmatrix} \) |
|-----------------|-----------------|-----------------|-----------------|
| \( \min \gamma \) | 0.6443 | 0.6398 | 0.6376 | 0.6361 |
| SA | -0.7223 | -0.7214 | -0.7224 | -0.7233 |
| Nols | 5 | 10 | 15 | 20 |

### B. DSFC for an LTDS with Controllers Delays

This subsection aims to show the advantage of adding delays to controllers when the system in (1) has no input delays.

\(^c\)All results of SA are calculated via the spectral method in [45]
Consider a system with the same parameters in subsection V-A except for $B_i = B_i(\tau) = O_{n,p}$ and $\mathfrak{B}_i = \mathfrak{B}_i(\tau) = O_{m,n}$, $\forall i \in \mathbb{N}_p$. Then the controller defined in (80)–(81) can be utilized for stabilizing the system while minimizing $\gamma$ in (87).

The procedures of computing controller gains here are entirely identical to the previous subsection apart from utilizing Algorithm 2 to the CLS in (82) supported by Corollary 1–2. Specifically, we apply Proposition 1 with (81) to the DDs in (86), (80) using the same parameters in (88) for $g_i(\cdot)$ and $M_i$. This leads to the same parameters $\tilde{A}_i, \tilde{C}_i$ in (90), whereas $\mathcal{K}_i$ in (81) are the gains to be computed.

Next, we assume $\alpha_k = 0$, $i = 2, \cdots, \beta$, $\alpha_1 = 5$ and $\alpha_1 = 50$, respectively, when Corollary 2 is applied. The numerical results produced by Algorithm 2 are summarized in Table III–VI where the resulting $\mathcal{K}_i$ are omitted due to limit space. Note that the results in Table III–IV and Table V–VI shows that adding delays to controllers can materially improve the performance of $\min \gamma$ compared to the use of $u(t) = K \bar{x}(t)$. This justifies the use of the delay structures in (80) even though it requires more resources for the controller realization.

Since the CLSs in (82) belong to the retarded type, their nominal stability is guaranteed [46] with the numerical implementation of the DDs in (81) as long as the accuracy reach a certain degree. This property ensures that the resulting controllers in (82) can be realized numerically for real-world applications.

For numerical simulation, we use the CLS with

$$K_0 = \begin{bmatrix} -9.1247 & -22.9729 \\ 0.0972 & 0.1237 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0.2429 & -0.117 \\ -9.2358 & -0.3997 -0.3585 0.1859 -0.4752 -0.8404 4.0466 -2.6485 -0.9318 -3.3031 -1.2266 -2.4909 0.1010 -0.5549 0.80751.4130 0.0620 0.148 \end{bmatrix}$$

$$\mathcal{K}_1 = \begin{bmatrix} -0.2358 -0.3997 -0.3585 0.1859 -0.4752 -0.8404 4.0466 -2.6485 -0.9318 -3.3031 -1.2266 -2.4909 0.1010 -0.5549 0.80751.4130 0.0620 0.148 \end{bmatrix}$$

This work has set out an effective solution for the DSFC problem of a general LTDS in (1) based on the application of multiple mathematical concepts including the notion of equivalent decomposition outlined in
Proposition 1. The decomposition approach circumvents the infinite dimensionality of the DDs and gives users the liberty to use different ways to handle them without theoretical conservatism. Unlike many existing methods which do not guarantee the stability of the resulting CLSs, the CLS in (20) and (82) are handled without losing any information, thereby ensuring the stability and dissipativity of the resulting CLS. Meanwhile, the generality of $f_i(\cdot)$ in (41) allows one to construct a KF much more general than the existing ones, hence leading to less conservative synthesis conditions. Because the number of delays $\nu$ is unlimited and the DDs in (1) can contain any number of $L^2$ functions, the method in this paper can be utilized as a blueprint for solving many open-problems related to the LTDS in (1) such as

- dissipative full state observer design
- dissipative full state observer-based control
- multi-objectives control and observation

Finally, one of the promising application of the proposed method is the design of performance-guaranteed predictor controllers for LTDS with both state and input delays.

**Appendix**

Some lemmas are presented here which are crucial for the derivations of the results in this paper. A novel integral inequality is also proposed to construct lower bounds for the integrals defined over $\mathcal{I}_i$ based on the idea proposed in [13].

**Lemma 3.** For all $X \in \mathbb{R}^{n \times m}$, $Y \in \mathbb{R}^{m \times p}$, $Z \in \mathbb{R}^{q \times r}$,

$$(X \otimes I_q)(Y \otimes Z) = XY \otimes Z$$

$$= XY \otimes Z I_r = (X \otimes Z)(Y \otimes I_r).$$

(92)

**TABLE V:** $\min \gamma$ produced by the Algorithm 2 with $d_1 = d_2 = \lambda_1 = \lambda_2 = 1$ and $K_i = \tilde{K}_i(\tau) = O_{p,n}$

| $\min \gamma$ | 0.5723 | 0.5669 | 0.5626 | 0.5590 |
|---------------|--------|--------|--------|--------|
| SA            | -0.7179 | -0.7113 | -0.7099 | -0.7099 |
| Nols          | 5      | 10     | 15     | 20     |

**TABLE VI:** $\min \gamma$ produced by the Algorithm 2 with $d_1 = d_2 = 1$, $\lambda_1 = \lambda_2 = 2$ and $K_i = \tilde{K}_i(\tau) = O_{p,n}$

| $\min \gamma$ | 0.5723 | 0.5669 | 0.5626 | 0.5590 |
|---------------|--------|--------|--------|--------|
| SA            | -0.7259 | -0.7319 | -0.7358 | -0.7385 |
| Nols          | 5      | 10     | 15     | 20     |

Moreover, $\forall X \in \mathbb{R}^{n \times m}$, we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \otimes X = \begin{bmatrix} A \otimes X & B \otimes X \\ C \otimes X & D \otimes X \end{bmatrix}, \quad I_n \otimes X = \bigoplus_{i=1}^{n} X.$$  (94)

We define the weighted Lebesgue function space

$$\mathcal{L}^2_{\infty} (\mathcal{K}; \mathbb{R}^d) := \left\{ \phi(\cdot) \in M(\mathcal{K}; \mathbb{R}^d) : \|\phi(\cdot)\|_{2,\infty} < \infty \right\}$$

with $d \in \mathbb{N}$ and $\|\phi(\cdot)\|_{2,\infty} := \int_{\mathcal{K}} \varpi(\tau) \phi^T(\tau) \phi(\tau) d\tau$ where $\varpi(\cdot) \in M(\mathcal{K}; \mathbb{R}_{\geq 0})$ and the function $\varpi(\cdot)$ has countably infinite or finite number of zero values. Furthermore, $\mathcal{K} \subseteq \mathbb{R} \cup \{\pm \infty\}$ and its Lebesgue measure is non-zero.

**Lemma 4.** Given $\mathcal{K}$ and $\varpi(\cdot)$ in (95) and $U \in \mathbb{S}^n_{\geq 0} := \{X \in \mathbb{S}^n : X \geq 0\}$ with $n \in \mathbb{N}$. Let $f_i(\cdot) \in \mathcal{L}^2_{\infty}(\mathcal{K}; \mathbb{R}^d)$ and
\[ g(\cdot) \in \mathcal{L}_\infty^2(\mathbb{K}; \mathbb{R}^{\lambda_i}) \text{ with } \lambda_i \in \mathbb{N} \text{ and } \lambda_i \in \mathbb{N}_0, \ i \in \mathbb{N}_\nu, \text{ in which the functions } f_i(\cdot) \text{ and } g_i(\cdot) \text{ satisfy} \]
\[ \int_{\mathbb{K}} \omega(\tau) \begin{bmatrix} g_i(\tau) \\ f_i(\tau) \end{bmatrix} \begin{bmatrix} g_i^T(\tau) & f_i^T(\tau) \end{bmatrix} d\tau > 0, \ i \in \mathbb{N}_\nu \] (96)

which implies \[ \int_{\mathbb{K}} \omega(\tau)f_i(\tau)f_i^T(\tau)d\tau > 0, \ i \in \mathbb{N}_\nu. \]

Then
\[ \int_{\mathbb{K}} \omega(\tau)x^T(\tau) \begin{bmatrix} \nu \ I_{n_i} \end{bmatrix} x(\tau)d\tau \geq [\nu] \begin{bmatrix} f_i(\cdot) \bigotimes I_{n_i} \end{bmatrix} x(\tau)d\tau \]
\[ + [\nu] \begin{bmatrix} \mu \ I_{n_i} \end{bmatrix} x(\tau)d\tau \]
\[ \geq [\nu] \begin{bmatrix} f_i(\cdot) \bigotimes I_{n_i} \end{bmatrix} x(\tau)d\tau \]

holds for all \[ x(\cdot) \in \mathcal{L}_\infty^2(\mathbb{K}; \mathbb{R}^{\nu}), \] where \[ \mathcal{F}_i = \int_{\mathbb{K}} \omega(\tau)f_i(\tau)f_i^T(\tau)d\tau > 0. \] In addition, \[ e_i(\cdot) = g_i(\cdot) \bigotimes I_{n_i} \] and \[ A_i = \int_{\mathbb{K}} \omega(\tau)g_i(\tau)f_i^T(\tau)d\tau \] for all \[ i \in \mathbb{N}_\nu. \] and \[ E_i := \int_{\mathbb{K}} \omega(\tau)e_i(\tau)e_i^T(\tau)d\tau. \]

Proof. By using the inequality in [13, eq.(17)] \[ \nu \] times, then (97) is obtained. Note that the definition of \[ \mathcal{F}_i \] here is the inverse of the one in [13]. Furthermore, \[ E_i^{-1} \] in (97) is well defined with \[ g_i(\cdot) = [0 \times 1]. \]

A. Proof of Proposition 1

Proof. First of all, it is obvious that (3) is implied by (4)–(7) because of the definitions of \[ \varphi_i(\cdot), f_i(\cdot), \phi_i(\cdot) \] and the fact that \[ C^1(\mathbb{I}_J; \mathbb{R}^{d_i}) \subset \mathcal{L}_\infty^2(\mathbb{I}_J; \mathbb{R}^{d_i}). \] So the necessity part of the statement is proved.

Now we start to prove the sufficiency part of the statement. Namely, the condition in (3) implies the existence of the parameters in Proposition 1 satisfying (4)–(7). Given any \[ f_i(\cdot) \in C^1(\mathbb{I}_J; \mathbb{R}^{d_i}), \ i \in \mathbb{N}_\nu, \ d \in \mathbb{N} \] satisfying \[ \int_{\mathbb{I}_J} f_i^T(\tau)f_i(\tau)d\tau > 0, \] one can always construct appropriate \[ \phi_i(\cdot) \text{ and } \varphi_i(\cdot) \in \mathcal{L}_\infty^2(\mathbb{I}_J; \mathbb{R}^{d_i}) \] with \[ m_i \in \mathbb{R}^{d_i \times n_i}, \] such that the conditions in (6)–(7) are satisfied. Note that \[ \int_{\mathbb{I}_J} f_i^T(\tau)f_i(\tau)d\tau > 0 \] is implied by the matrix inequalities in (7) which indicate that the functions in \[ g_i(\cdot) \] in (6) are linearly independent\(^d\) in a Lebesgue sense over \[ [-r_{i-1}, -r_{i-1}]. \] The aforementioned conclusion is true because \[ \frac{df_i(\tau)}{d\tau} \in \mathcal{C}(\mathbb{I}_J; \mathbb{R}^{d_i}) \subset \mathcal{L}_\infty^2(\mathbb{I}_J; \mathbb{R}^{d_i}) \] for all \[ i \in \mathbb{N}_\nu, \] and the dimensions of \[ \varphi_i(\tau) \text{ and } \phi_i(\tau), \ i \in \mathbb{N}_\nu, \] can be arbitrarily enlarged with more linearly independent functions. Note that \[ \varphi_i(\tau) \text{ or } \phi_i(\tau) \] can be an empty matrix \[ [0 \times 1]. \]

Now since \[ \dim(g_i(\cdot)) \text{ in (6)–(7)} \] can be arbitrarily increased, (appropriate new functions can always be added) there always exist \[ A_{i,j} \in \mathbb{R}^{n \times n}, \ C_{i,j} \in \mathbb{R}^{n \times n}, \ B_{i,j} \in \mathbb{R}^{n \times p}, \ \delta_{i,j} \in \mathbb{R}^{n \times p} \] and \[ g_i(\cdot) = \text{Col}_{j=1}^{\kappa_i} g_{i,j}(\cdot) \] in (7) for the distributed delay terms in (3) such that

\[ \bar{A}_i(\cdot) = \sum_{j=1}^{\kappa_i} A_{i,j} g_{i,j}(\cdot), \ \bar{C}_i(\cdot) = \sum_{j=1}^{\kappa_i} C_{i,j} g_{i,j}(\cdot), \]
\[ \bar{B}_i(\cdot) = \sum_{j=1}^{\kappa_i} B_{i,j} g_{i,j}(\cdot), \ \bar{\delta}_i(\cdot) = \sum_{j=1}^{\kappa_i} \delta_{i,j} g_{i,j}(\cdot) \]

\( \forall i \in \mathbb{N}_\nu, \forall \tau \in \mathbb{I}_i \) with \[ \kappa_i \in \mathbb{N}_0 \] where \[ \varphi_i(\cdot) \in \mathcal{L}_\infty^2([-r_i, -r_{i-1}]; \mathbb{R}^{d_i}), \ f_i(\cdot) \in C^1(\mathbb{I}_J; \mathbb{R}^{d_i}), \] and \[ \phi_i(\cdot) \in \mathcal{L}_\infty^2(\mathbb{I}_J; \mathbb{R}^{d_i}) \] satisfy (6)–(7) for some \[ M_i \in \mathbb{R}^{d_i \times n_i}, \ i \in \mathbb{N}_\nu. \]

Moreover, (98) can be rewritten as

\[ \bar{A}_i(\cdot) = \left[ \begin{array}{c} A_{i,j} \end{array} \right]_{j=1}^{\kappa_i} (g(\cdot) \bigotimes I_p), \ \bar{C}_i(\cdot) = \left[ \begin{array}{c} C_{i,j} \end{array} \right]_{j=1}^{\kappa_i} (g(\cdot) \bigotimes I_p), \]
\[ \bar{B}_i(\cdot) = \left[ \begin{array}{c} B_{i,j} \end{array} \right]_{j=1}^{\kappa_i} (g(\cdot) \bigotimes I_p), \ \bar{\delta}_i(\cdot) = \left[ \begin{array}{c} \delta_{i,j} \end{array} \right]_{j=1}^{\kappa_i} (g(\cdot) \bigotimes I_p) \]

\( \forall \tau \in \mathbb{I}_i \) which are in line with the decompositions in (4)–(5) by letting \[ \bar{A}_i = \left[ \begin{array}{c} A_{i,j} \end{array} \right]_{j=1}^{\kappa_i}, \ \bar{C}_i = \left[ \begin{array}{c} C_{i,j} \end{array} \right]_{j=1}^{\kappa_i}, \ \bar{B}_i = \left[ \begin{array}{c} B_{i,j} \end{array} \right]_{j=1}^{\kappa_i}, \ \bar{\delta}_i = \left[ \begin{array}{c} \delta_{i,j} \end{array} \right]_{j=1}^{\kappa_i} \text{ for all } i \in \mathbb{N}_\nu. \] Finally, the conclusion in (98) is true for the case of \( \mu_i = 0 \) or \( \delta_i = 0 \). Given all the aforementioned statements we have presented, then Proposition 1 is proved.

\[ \square \]

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\(^d\)See Theorem 7.2.10 in [32] for more information
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