Quantum Bounded Query Complexity

Harry Buhrman
Quantum Computing and Advanced Systems Research, C.W.I. Amsterdam*
buhrman@cwi.nl

Wim van Dam
Centre for Quantum Computation, University of Oxford†
Quantum Computing and Advanced Systems Research, C.W.I. Amsterdam*
wimvdam@qubit.org

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Abstract

We combine the classical notions and techniques for bounded query classes with those developed in quantum computing. We give strong evidence that quantum queries to an oracle in the class NP does indeed reduce the query complexity of decision problems. Under traditional complexity assumptions, we obtain an exponential speedup between the quantum and the classical query complexity of function classes.

For decision problems and function classes we obtain the following results:

• $P_{||}^{NP[2^k]} \subseteq EQP_{||}^{NP[k]}$
• $P_{||}^{NP[2^{k+1}-2]} \subseteq EQP_{||}^{NP[A]}$
• $FP_{||}^{NP[2^{k+1}-2]} \subseteq FEQP_{||}^{NP[2^k]}$
• $FP_{||}^{NP} \subseteq FEQP_{||}^{NP[O(\log n)]}$

For sets $A$ that are many-one complete for PSPACE or EXP we show that $FP^A \subseteq FEQP^A[1]$.
Sets $A$ that are many-one complete for PP have the property that $FP^A_{||} \subseteq FEQP^A[1]$. In general we prove that for any set $A$ there is a set $X$ such that $FP^A \subseteq FEQP^X[1]$, establishing that no set is superterse in the quantum setting.

1 Introduction

The query complexity of a function is the minimum number of queries (to some oracle) that are needed to compute one value of this function. With bounded query complexity we look at the set of functions that can be calculated if we put an upper bound on the number of queries that we allow the computer to ask the oracle. This notion has been extensively studied both in the resource bounded setting [1, 3, 4, 7, 8, 9, 13, 31, 33, 36] and in the recursive setting [11, 12].

This notion and its variants has lead to a series of techniques and tools that are used throughout complexity theory.

In this paper we combine some of the bounded query notions with quantum computation. The main goal of the paper is to further—as was done by Fortnow and Rogers [27]—the incorporation of quantum computation into complexity theory. We feel that the synthesis of quantum computation

*Quantum Computing and Advanced Systems Research, C.W.I., P.O. Box 94079, NL–1098 GB Amsterdam, The Netherlands.
†Centre for Quantum Computation, Clarendon Laboratory, University of Oxford, Parks Road, Oxford, OX1 3PU, United Kingdom.
and classical complexity theory serves two purposes. First, it is important to know the limits of feasible quantum computation and this can be done by incorporating it into the framework of classical computation. Second, the insights of quantum computation can be useful for classical complexity theory in turn.

We start out with the class of sets (or decision problems) that are computable in polynomial time with bounded queries to a set in NP. We consider the setting where the queries are adaptive (i.e., a query may depend on the answers to previous ones), as well as where they are non-adaptive. Classically, it is known that any decision problem that can be solved in polynomial time with $k$ adaptive queries to a set in NP (the class $P^{NP|k}$) can also be solved with $2^k-1$ non-adaptive queries (the class $P^{NP|2^k-1}$, where "||" indicates the parallel or non-adaptive queries), and vice-versa.

In other words: $P^{NP|k} = P^{NP|2^k-1}$. Moreover, there is strong evidence that this trade-off is optimal in the sense that every non-adaptive class $P^{NP|k}$ is different for different values of $k$. For example if $P^{NP|2} \subseteq P^{NP|1}$, then the polynomial hierarchy collapses [1] (see also [7, 23]).

The natural quantum analogue of $P$ is the class $EQP$, which stands for exact quantum polynomial time. This is the class of sets or decision problems that is computable in polynomial time with a quantum computer that makes no errors (i.e., is exact). In this paper we will see that if we allow the query machine to make use of quantum mechanical effects such as superposition and interference the situation changes. In the non-adaptive case we will show that $2k$ classical queries can be simulated with only $k$ non-adaptive ones on a quantum computer and in the adaptive case we show how to simulate $2^{k+1} - 2$ classical queries with only $k$ quantum queries. Hence

$$P^{NP|2^k} \subseteq EQP^{NP[k]} \quad \text{and} \quad P^{NP|2^{k+1}-2} \subseteq EQP^{NP[k]}.$$  

In particular it follows from this result that $P^{NP|2} \subseteq EQP^{NP[1]}$ (see also [22]).

In order to prove these results we combine the classical mind-change technique [9] with the one query version (see [20]) of the first quantum algorithm developed by David Deutsch [23].

Next, we turn our attention to functions that are computable with bounded queries to a set in NP. Compared to the decision problems there is probably no nice trade-off between adaptive and non-adaptive queries for functions. This is because the following is known [13]: for any $k$ the inclusion $FP^{NP|k} \subseteq FP^{NP|k-1}$ implies that $P = NP$. Moreover, if $FP^{NP} \subseteq FP^{NP[O(\log n)]}$ then the polynomial time hierarchy collapses [8, 14, 22].

When the adaptive query machine is a quantum computer, things are different and we seem to get a trade-off between adaptiveness and query complexity. We show the following:

$$FP^{NP|2^{k+1}-2} \subseteq FEQP^{NP|2^k} \quad \text{and} \quad FP^{NP} \subseteq FEQP^{NP[O(\log n)]}.$$  

Here $FEQP^{NP[k]}$ is the class of functions that is computable by an exact quantum Turing machine that runs in polynomial time and is allowed to make $k$ queries to a set in NP. The proofs of these results use our previous results on decision problems and a quantum algorithm developed by Deutsch-Jozsa [24] and Bernstein-Vazirani [15].

Using the same ideas we are able to show that for any set $A$ there exists a set $X$ such that $FP^A \subseteq FEQP^{X[1]}$, establishing that no set is ‘superterse’. Also because the complexity of $X$ is not much harder than that of $A$ (the problem $X$ is Turing reducible to $A$), we get quite general theorems for complete sets of complexity classes.

For a complexity class $\mathcal{C}$ that is closed under Turing reductions, and a problem $A \in \mathcal{C}$ that is many-one complete for the class $\mathcal{C}$, the inclusion $FP^\mathcal{C} \subseteq FEQP^A[1]$ is proven. This holds in particular for the set $\text{QBF}$ of the true quantified Boolean formulae which is a PSPACE complete problem, and the complete sets for the class $\text{EXP}$. If $\mathcal{C}$ is a class that is closed under truth-table reductions, then it holds that $FP^\mathcal{C} \subseteq FEQP^A[1]$. The Theta levels of the polynomial hierarchy and PP are examples of such classes.

The ingredients for all our results are standard quantum algorithms combined with well known techniques from complexity theory. Nevertheless we feel that this combination gives a new point of view on the nature of bounded query classes and the structure of complete sets in general.
2 Preliminaries

2.1 Classical computing

We assume the reader to be familiar with basic notions of complexity theory such as the various complexity classes and types of reducibility as can be found in many textbooks in the area [28, 29]. The essentials for this article are mentioned below.

For a set (decision problem) $A$ we will identify $A$ with its characteristic function. Hence for a string $x$ we have $A(x) \in \{0, 1\}$, and $A(x) = 1$ if and only if $x \in A$. A class $C$ consists of a set of decision problems. A problem $A$ is many-one, or $\leq_{m}$-complete for a class $C$ if for any problem $B \in C$, there exists a polynomial function or "Karp-reduction" $\tau$ such that $x \in B$ if and only if $\tau(x) \in A$. The typical example of such a complete problem is SAT (the set of satisfiable Boolean formulae) which is $\leq_{m}$-complete for the class NP. The class FP indicates the set of functions that can be calculated on a polynomial time, deterministic Turing machine.

An oracle Turing machine is non-adaptive, if it can produce a list of all of the oracle queries it is going to make before it makes the first query. For any set $A$, the elements of the class $P^{A[k]}$ (FP$^{A[k]}$) are the languages (functions) that are computable by polynomial time Turing machines that accesses the oracle $A$ at most $k$ times on each input. The class $P^{A[k]}$ and FP$^{A[k]}$ allow only non-adaptive access to $A$. The notation $P^{\Sigma P[\tau(n)]}$ is used to indicate algorithms that might require $q(n)$ oracle calls, where $q$ is a function of the input size $n$.

The class NP can be generalised by defining the polynomial time hierarchy. We start with the definition $\Sigma^p_0 = P$ and then for the higher levels continue in an inductive fashion with $\Sigma^p_{i+1} = \text{NP}^{\Sigma^p_i}$ for $i = 1, 2, \ldots$ Many complexity theorists conjecture that this polynomial time hierarchy is infinite, i.e., $\Sigma^p_{i+1} \neq \Sigma^p_i$ for all $i$.

A class $C$ of languages is closed under Turing (truth-table) reduction if any decision problem that can be solved with a polynomial time Turing machine and (non-adaptive) queries to a set in $C$, is itself also an element of $C$. Examples of such classes are PSPACE, EXP, and the Delta levels $\Delta^p_{i+1} = \text{P}^{\Sigma^p_i}$ of the polynomial time hierarchy. The classes PP and $\Theta^P_{i+1} = \text{P}^{\Sigma^p_i}$ (Theta levels of the polynomial hierarchy) are for example closed under this truth-table-reduction.

2.2 Quantum computing

In this section we define quantum oracle Turing Machines. For an introduction to quantum computing see for example the survey by Berthiaume [16].

A qubit is a superposition $\alpha_0|0\rangle + \alpha_1|1\rangle$ of both values of a classical bit. The complex values $\alpha_0$ and $\alpha_1$ are the amplitudes of the quantum state, and they obey the normalisation restriction: $|\alpha_0|^2 + |\alpha_1|^2 = 1$.

The tensor or Kronecker-product is used to describe a system of several qubits. The combination of two qubits is thus calculated by

$$|x\rangle \otimes |y\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle) = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle.$$

Consequently, a register of $n$ qubits is a superposition $|\psi\rangle$ of all $2^n$ strings of $n$ classical bits, written

$$|\psi\rangle = \sum_{i\in\{0,1\}^n} \alpha_i |i\rangle.$$

If we measure the quantum register $|\psi\rangle$ in the standard (i.e. classical) basis, we will observe one and only one of the basis states $|i\rangle$ with probability $|\alpha_i|^2$. Hence we must have $\sum_{i\in\{0,1\}^n} |\alpha_i|^2 = 1$ (the normalisation restriction). After measuring $|\psi\rangle$ and observing $|i\rangle$, we say that the superposition $|\psi\rangle$ "has collapsed" to the new state $|i\rangle$.

If we do not observe a state, quantum mechanics tells us that it will evolve unitarily, as this is the only evolution that respects the normalisation restriction. Unitarity means that the vector of
amplitudes is transformed according to a linear operator that preserves the unit norm. This can be viewed as a rotation in the complex, finite Hilbert space of dimension $2^n$. A unitary operator $U$ always has an inverse $U^{-1}$ which equals its conjugate transpose $U^\dagger$.

An example of a one-qubit operation is the Hadamard transform $H$ which is defined by

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

The effect of the $n$-fold tensor product of $H^{\otimes n}$ on $n$ qubits is now calculated by the linearity of quantum mechanics:

$$H^{\otimes n} \sum_{i \in \{0,1\}^n} \alpha_i |i\rangle = \sum_{i \in \{0,1\}^n} \alpha_i \cdot H^{\otimes n} |i\rangle.$$  

This shows that the evolution of the superposition $|\psi\rangle$ is fully determined by the evolution of its basis states $|i\rangle$. We will use this transformation $H^{\otimes n}$ in Section 4.

A quantum Turing machine’s transition function can be described by a unitary matrix $M$ with complex entries. The computation of $t$ time-steps will then correspond to the application of the matrix product $M^t$ to the initial configuration $|j\rangle$. At the end of the computation we measure the state that has evolved in this manner and accept $j$ if some designated bit is 1 and otherwise reject the input $j$.

The class EQP is defined as those sets that can be computed by a quantum Turing machine that runs in polynomial time and accepts every string $j$ with probability 1 or 0. Likewise, we define the class of functions FEQP as the class of functions that can be computed by some quantum Turing machine that runs in polynomial time. The output of the Turing machine may now be several bits.

We model oracle computation as follows (see also [14]). An oracle Turing machine has some special query tape, and at some point in the computation the Turing machine may go into a special pre-query state to make a query to the oracle set $A$. Suppose the query tape contains the state $|i\rangle|b\rangle$ ($i$ represents the query and $b$ is a bit). The result of this operation is that after the call the machine will go into a special state called the post-query state and that the query tape has changed into $|i\rangle |A(i) \oplus b\rangle$, where $\oplus$ is the exclusive or. We will denote this unitary operation by $U_A$. Note that $U_A$ only changes the contents of the special query tape $b$, and leaves all the other registers unchanged.

As with classical oracle computation, we make the distinction between adaptive and non-adaptive quantum oracle machines. We call a quantum oracle machine non-adaptive if on every computation path a list of all the oracle queries (on this path) is generated before the first query is made.

The class $\text{EQP}^{A[k]}$ are the sets recognised by an exact quantum Turing machine that runs in polynomial time and makes at most $k$ queries to the oracle for $A$. Again, we define classes like $\text{EQP}^{[q(n)]}$, $\text{FEQP}^{A[q(n)]}$, and $\text{FEQP}^{A[q(n)]}$, for non-adaptive decision, adaptive function, and non-adaptive function classes respectively (with $q(n)$ a function that gives an upper bound on the number of queries and $n$ the size of the input string).

## 3 Decision Problems

In this section we will investigate the extra power that a polynomial time, exact quantum computer yields compared to classical deterministic computation when querying a set in the class $\text{NP}$. In the case of deterministic computation the following equality between adaptive and non-adaptive queries to $\text{NP}$ is well known.
Theorem 1 [4, 7, 17]

1. For any \( k \geq 0 \) we have \( P^{NP[2^k-1]} = P^{NP[k]} \).

2. For any polynomial \( q(n) > 1 \) the equality \( P^{NP[q(n)]} = P^{NP[O(\log(q(n))] \) holds.

Proof: Both items are proved in a similar way which has two parts. The first part shows that computing a function in \( P^{NP[2^k-1]} \) can be reduced to computing the parity of \( 2^k-1 \) other queries to NP. The second part then proceeds by showing that using binary search one can compute the parity of \( 2^k-1 \) NP-queries with \( k \) adaptive queries to sat. On the other hand, it is trivial to see that any computation with \( k \) adaptive queries can be simulated exhaustively with \( 2^k-1 \) non-adaptive oracle calls. \( \Box \)

There is also strong evidence that the above trade-off is tight (see \( [10, 31] \)). It follows for example that if \( P^{NP[2]} = P^{NP[1]} \) then the polynomial hierarchy collapses \( [31] \). (See \( [17] \) for the latest developments with respect to this question.)

Perhaps surprisingly the situation changes when the query machine is quantum mechanical. David Deutsch [23] developed a quantum algorithm to compute the parity of any two Boolean variables in one query with higher probability than a classical (randomised) algorithm can. Cleve et al. [20] showed how to make this procedure exact.

Theorem 2 [20, 23] Let \( f : \{0, 1\} \rightarrow \{0, 1\} \). There exists an exact quantum algorithm that computes the parity bit \( f(0) \oplus f(1) \) with one query to the function \( f \). This algorithm works in constant time.

Proof: For simplicity we only describe what is happening to the states that get effected by the oracle query. Construct the following initial state:

\[
|\text{Initial} \rangle = \frac{1}{2}(|0 \rangle + |1 \rangle) \otimes (|0 \rangle - |1 \rangle).
\] (1)

Next, make the only query to \( f \) depending on the value of the first bit. Note that \( f \) will thus be queried in superposition for both \( f(0) \) and \( f(1) \). Applying \( f \) establishes the following evolution on the two qubits:

\[
|i \rangle \otimes |b \rangle \rightarrow |i \rangle \otimes |b \oplus f(i) \rangle.
\]

This results in the following outcome when applied to the initial state:

\[
\frac{(-1)^{f(0)}}{2}(|0 \rangle + |1 \rangle) \otimes (|0 \rangle - |1 \rangle) \quad \text{if} \quad f(0) = f(1)
\]

\[
\frac{(-1)^{f(0)}}{2}(|0 \rangle - |1 \rangle) \otimes (|0 \rangle - |1 \rangle) \quad \text{if} \quad f(0) \neq f(1).
\]

Which means that if we apply a Hadamard transformation to the first register, we obtain

\[
|\text{Final} \rangle = (-1)^{f(0)}|f(0) \oplus f(1) \rangle \otimes (|0 \rangle - |1 \rangle).
\]

Hence observing the first bit yields the correct answer \( f(0) \oplus f(1) \). \( \Box \)

Using this procedure we will now show that a quantum Turing machine can compute decision problems with half the number of non-adaptive queries.

Theorem 3 For any \( k \geq 0 \) we have the inclusion \( P^{NP[2k]} \subseteq EQP^{NP[k]} \).

Proof: Without loss of generality we will assume that the queries are made to sat, and that the predicate that is computable with \( 2k \) queries to sat is \( f(x) \). Let \( \psi_1, \psi_2, \ldots, \psi_{2k} \) be the queries that the computation of \( f(x) \) makes. We will use the proof technique of Theorem 3 (also
called mind-change technique) which enables us to compute \( f(x) \) by calculating the single bit \( \text{SAT}(\phi_1) \oplus \cdots \oplus \text{SAT}(\phi_{2k}) \). Here the new formulae \( \phi_1, \ldots, \phi_{2k} \) can be computed in polynomial time from \( \psi_1, \ldots, \psi_{2k}, f \), and \( x \), but without having to consult \( \text{SAT} \).

Next, we use Theorem 3 to compute the parity \( \text{SAT}(\phi_i) \oplus \text{SAT}(\phi_{i+1}) \) for odd \( i \) \((1 \leq i < 2k)\) with \( k \) non-adaptive queries to \( \text{SAT} \). Finally we compute the parity of these answers, thus obtaining the necessary information for calculating \( f(x) \). \( \square \)

**Corollary 1** \( P^{NP[2]} \subseteq EQP^{NP[1]} \) (see [2]).

We do not know whether this is tight. It would be interesting to either improve this result to \( P^{NP[2]} \subseteq EQP^{NP[1]} \) or to show as a consequence of this that the polynomial time hierarchy collapses.

Theorem 3 relates adaptive query classes to non-adaptive ones, thereby establishing an exponential gain in the number of queries \((2^k - 1)\) versus \( k \) queries. We will now show how to use the Deutsch trick to do even slightly better than that in the quantum case.

**Theorem 4** \( P^{NP[2k+1-2]} \subseteq EQP^{NP[k]} \) for any \( k \geq 0 \).

**Proof:** The proof is by induction on \( k \). For \( k = 1 \) we have back the situation of Corollary 1. Let the predicate \( f(x) \) be computable with \( 2^{k+1} - 2 \) non-adaptive queries to \( \text{SAT} \). As in the proof of Theorem 3 we reduce the \( 2^{k+1} - 2 \) queries \( \psi_i \) that \( f(x) \) makes, to the calculation of the parity-bit \( \text{SAT}(\phi_1) \oplus \cdots \oplus \text{SAT}(\phi_{2k+1-2}) \). Next, we construct \( 2^{k+1} - 2 \) new formulae \( \chi_1, \ldots, \chi_{2^{k+1}-2} \) according to:

\[
\chi_i \text{ is satisfiable } \iff |\{\phi_1, \ldots, \phi_{2^{k+1}-2}\} \cap \text{SAT}| \geq i.
\]

The construction of each such \( \chi_i \) can be done in polynomial time. To see this, consider the non-deterministic polynomial time Turing machine \( M \) that on input \( \langle i, \phi_1, \ldots, \phi_{2^{k+1}-2} \rangle \), accepts if and only if it can find for \( i \) of the formulae a satisfying assignment. Cook and Levin [21, 32] —proving that \( \text{SAT} \) is \( \leq^P \text{NP} \)-complete for \( \text{NP} \)—showed that any polynomial time non-deterministic Turing machine computation \( M(x) \) in polynomial time can be transformed into a formula that is satisfiable if and only if \( M(x) \) has an accepting computation. Let \( \chi_i \) be the result of this Cook-Levin reduction.

Note the following two properties of those formulae \( \chi_i \):

1. The parity \( \text{SAT}(\phi_1) \oplus \cdots \oplus \text{SAT}(\phi_{2^{k+1}-2}) \) is the same as the parity \( \text{SAT}(\chi_1) \oplus \cdots \oplus \text{SAT}(\chi_{2^{k+1}-2}) \).
2. For every \( i \) we have \( \text{SAT}(\chi_i) \geq \text{SAT}(\chi_{i+1}) \).

Now we are ready to make the first query. We compute the parity of \( \chi_{2^k-1} \) and \( \chi_{2^k-1 + 2^k-1} \). This can be done in one query using Theorem 3. By doing this we have at the cost of one query reduced the question of computing the parity of \( 2^{k+1} - 2 \) formulae to computing the parity of \( 2^k - 2 \). These we can solve using \( k - 1 \) queries using the induction hypothesis. To see observe the following.

For convenience set \( a = 2^{k-1} \) and \( b = 2^{k-1} + 2^k - 1 \).

Suppose the parity of \( \chi_{a} \) and \( \chi_{b} \) is odd. Hence \( \chi_1, \chi_2, \ldots, \chi_{a} \) are all satisfiable and \( \chi_{b}, \ldots, \chi_{2^{k+1}-2} \) all un-satisfiable (using property 2 above). Also note that \( a \) is even, so the parity of \( \chi_1, \ldots, \chi_{2^{k+1}-2} \) is the same as the parity of \( \chi_{a+1}, \ldots, \chi_{b-1} \) (these are \( 2^k - 2 \) many formulae).

On the other hand assume that the parity of \( \chi_{a} \) and \( \chi_{b} \) is even. This means (again using property 2 above) that \( \chi_{a}, \ldots, \chi_{b} \) are all either satisfiable or un-satisfiable and hence have even parity. So again the question reduces to the parity of the remaining formulae: \( \chi_{1}, \ldots, \chi_{a-1} \) and \( \chi_{b+1}, \ldots, \chi_{2^{k+1}-2} \). Which happen to be \( 2^k - 2 \) many formulae. \( \square \)

We do not know if it is possible to do better than this. In essence the above technique seems to boil down to searching in an ordered list. Buhrman and De Wolf [18] show that this cannot be done faster than \( \sqrt{\log n / \log(\log n)} \), which was improved by Fahri et al. [23] to \( \log(n) / 2 \log(\log(n)) \) and later by Ambainis [2] to a lower bound of \( 1/12 \log n - O(1) \) queries. Recent results by Farhi et al. [20] seem to suggest a reduction in the query complexity by a factor of two. But it is not clear if their exact quantum algorithm for ‘insertion into an ordered list’ translates correctly into our setting.
4 Function Classes

Now we turn our attention to function classes where the algorithm can output bit strings rather than single bits. We will see that in this scenario the difference between classical and quantum computation becomes more eminent.

4.1 Functions computable with queries to an oracle in NP

We start out by looking at functions that are computable with queries to a complete set for the class \( \text{NP} \). Classically the situation is not as well understood as the class of decision problems. There is strong evidence that the analogue of Theorem 1 is not true.

**Theorem 5** The following holds for the classical, exact computation of functions:

1. If for some \( k \geq 0 \) we have \( \text{FP}^{\text{NP}[k+1]} \subseteq \text{FP}^{\text{NP}[k]} \), then \( \text{P} = \text{NP} \).

2. If for all polynomials \( q(n) \) (with \( n \) the size of the input string): \( \text{FP}^{\text{NP}[q(n)]} \subseteq \text{FP}^{\text{NP}[O(\log n)]} \), then \( \text{NP} = \text{R} \) (and the polynomial hierarchy collapses) \([3,7,23]\).

When we allow the adaptive query machine to be quantum mechanical the picture becomes again quite different. We will see that in this scenario the difference between classical and quantum computation becomes more eminent.

The core of the algorithm is the following observation. The \( n \)-fold Hadamard transform \( H^{\otimes n} \) (see Section 2.2) does the following when applied to a basis state of \( n \) bits:

\[
H^{\otimes n}|a_1a_2\cdots a_n\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{(x,a)} |x\rangle.
\]  

Since the Hadamard transform is its own inverse we have also the other direction:

\[
H^{\otimes n} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{(x,a)} |x\rangle = |a_1a_2\cdots a_n\rangle.
\]

So, if we are able to obtain the state of Equation 3, then we can extract the \( n \)-bit string \( a \) out of it by simply applying \( H^{\otimes n} \) to it. This state however can be obtained with one application of the function \( f \) as follows:

\[
U_f \frac{1}{\sqrt{2^n+1}} \sum_{x \in \{0,1\}^n} |x\rangle(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2^n+1}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle(|0\rangle - |1\rangle).
\]

References:
- [3]
- [7]
- [23]
Now observe that the last qubit is always in state \((|0⟩ - |1⟩)\). Using the definition of \(f\) we can rewrite this state to:

\[
\frac{1}{\sqrt{2^{n+1}}} \sum_{x \in \{0,1\}^n} (-1)^{(x,a)} |x⟩ \otimes (|0⟩ - |1⟩).
\]

Let us turn back now to our setting of bounded query classes. Using the above quantum tricks we can show the following.

**Theorem 6** For exact function calculation with the use of an oracle in NP it holds that

1. \(\text{FP}^{\text{NP}[2^{k+1}-2]} \subseteq \text{FEQP}^{\text{NP}[2k]}\) for any \(k \geq 0\).
2. \(\text{FP}^{\text{NP}[\|]} \subseteq \text{FEQP}^{\text{NP}[O(\log n)]}\).

**Proof:** Fix \(k \geq 0\), the input \(z\) of length \(m\) and let \(g\) be the function in \(\text{FP}^{\text{NP}[\|]}\). Suppose that \(g(z) = (a_1 \cdots a_n) = a\) with \(n = mc\) for some \(c\) depending on \(g\). The goal is to obtain the following state:

\[
|\text{Output}⟩ = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{(x,a)} |x⟩.
\]

(6)

Since with this state one application of \(H^\otimes n\) will give us \(a = g(z)\) (see Equation 3). Similar to Equation 3 we can obtain this state if we had access to a function \(f\) with the same property as the one in Equation 2.

The goal thus is to transform the function we have access to—\(\text{SAT}\) in our case—into one that resembles the one in Equation 2. The way to do this is to make use of a quantum subroutine. Observe the following: the binary function \(f_z(x) = (x,a)\) is in \(\text{P}^{\text{SAT}[2^{k+1}-2]}\) because we can first compute \(g(z) = a\) with \(2^{k+1} - 2\) queries to \(\text{SAT}\) and then determine \((x,a)\). By Theorem 4 this function is computable in \(\text{EQP}^{\text{SAT}[k]}\). Hence, when we use this adaptive EQP algorithm in superposition we have the desired function \(f\). There is however one problem with this approach. The algorithm that comes out of Theorem 4 leaves several of the registers in states depending on the input \(x\) and \(\text{SAT}\). For example the algorithm that computes the parity of two function calls in one generates a phase of \((-1)^{1}\) depending on the value of the first function call (see Equation 2). These changes in registers and phase shifts obstruct our base quantum machine and as a consequence the sum computed in Equation 3 does not work out the way we want (i.e., the interference pattern is different and terms do not cancel out as nice as before.)

The solution to this kind of ‘garbage’ problem is as follows:

1. Compute \(f_z(x)\) with \(k\) queries to \(\text{SAT}\).
2. Copy the outcome onto an extra auxiliary qubit (by setting the auxiliary bit \(b\) to the exclusive OR of \(b\) and the outcome).
3. Reverse the computation of \(f_z(x)\) making another \(k\) queries to \(\text{SAT}\).

Observe that when we compute \(f_z(x)\) in this way, all the phase changes and registers are reset and are in the same state as before computing \(f\), except for the auxiliary qubit that contains the answer. Since the subroutine was exact (i.e., in EQP) the answer bit is a classical bit and will not interfere with the rest of the computation. Note (see Section 2) that this corresponds exactly to one oracle call to \(f\). Thus we simulated 1 call to \(f\) with \(2k\) queries to \(\text{SAT}\) and hence have established a way of producing the desired state of Equation 3.

The second part of the theorem is proved in a similar way now using part 2 of Theorem 1. □
4.2 Terseness, and other complexity classes

The quantum techniques described above are quite general and can be applied to sets outside of NP. Classically the following question has been studied (see [8] for more information). For any set $A$ define the function $F^n_A(x_1, \ldots, x_n) = (A(x_1) \cdots A(x_n))$ which is an $n$ bit vector telling which of the $x_i$’s is in $A$ and which ones are not. A basic question now is: how many queries to $A$ do we need to compute $F^n_A$? Sets for which $F^n_A$ can not be computed with less than $n$ queries to $A$ (i.e., $F^n_A \not\subseteq FP^{F[n-1]}$) are called $P$-terse. We call the decision problem $A P$-superterse if $F^n_A \not\subseteq FP^{X[n-1]}$ for any set $X$. The next theorem shows that the notion $P$-superterse is not useful in the quantum setting.

Theorem 7 For any set $A$ there exists a set $X$ such that for all $n$ we have $F^n_A \subseteq FEQP^{X[1]}$.

Proof: Let $X$ be the following set: $X = \{(z_1 \cdots z_n, x) | (F^n_A(z_1, \ldots, z_n), x) \equiv 1 \text{ mod } 2\}$. Using the same approach as the proof of Theorem 6 it is not hard to see that $F^n_A$ can be computed relative $X$ with only a single query. □

Using the same idea we can prove the following general theorem about oracles for complexity classes other than NP.

Theorem 8 Let $C$ be a complexity class and the set $A \leq_m^p$-complete for $C$.

1. If $C$ is closed under $\leq_T^p$-reductions then $FP^C = FP^A \subseteq FEQP^A[1] = FEQP^C[1]$.

2. If $C$ is closed under $\leq_L^p$-reductions then $FP^I_C = FP^I_A \subseteq FEQP^A[1] = FEQP^C[1]$.

Proof: Let $f$ be the function we want to compute relative to $A$. Without loss of generality we assume that $|f(z)| = |z|^c$ for some $c$ depending on $f$. As before we construct the following set:

$$X = \{(z, y) | (f(z), y) \equiv 1 \text{ mod } 2, \text{ and } |y| = |z|^c\}.$$

As in Theorem 6 it follows that $f(z)$ is computable with one quantum query to $X$. Since $C$ is closed under $\leq_T^p$-reductions and $X \leq_T^p A$, it follows that $X \in C$. Furthermore, since $A$ is $\leq_m^p$-complete for $C$ it also follows that $X \leq_m^p A$. Thus the quantum query can be made to $A$ itself instead of $X$. The proof of the second part of the theorem is analogous to the first. □

This last theorem gives us immediately the following two corollaries about quantum computation with oracles for some known complexity classes.

Corollary 2

$$FP^{PSPACE} \subseteq FEQP^{PSPACE[1]}$$

$$FP^{EXP} \subseteq FEQP^{EXP[1]}$$

$$FP^{\Delta_p} \subseteq FEQP^{\Delta_p[1]}$$

for the Delta levels $\Delta_p$ in the polynomial time hierarchy.

Corollary 3

$$FP^{PP} \subseteq FEQP^{PP[1]}$$

$$FP^{\Theta_p} \subseteq FEQP^{\Theta_p[1]}$$

with $\Theta_p \equiv P^{\Sigma_p}$.

The first corollary holds in particular for $A = QBF$ (the set of true quantified Boolean formulae) which is PSPACE-complete. Observe also that the situation is quite different in the classical setting, since for EXP-complete sets the above is simply not true.
5 Conclusions and Open Problems

We have combined techniques from complexity theory with some of the known quantum algorithms. In doing so we showed that a quantum computer can compute certain functions with fewer queries than classical deterministic computers. Many question however remain. Is it possible to get trade-off results between the adaptive class EQP^{NP[k]} and the non-adaptive EQP_{NP}^{NP[2^{k-1}]} for quantum machines? Are the results we present here optimal? (Especially the recent results on exact searching in an ordered list [26] deserve further analysis as they seem to suggest a reduction of the quantum query complexity of Theorems 4 and 6 by a factor of two.)

What can one deduce from the assumption that P^{NP} ⊆ EQP^{NP[1]}? Is it true that for any set A we have P^{A} ⊆ EQP^{A[1]} or are there sets where this is not true? A random set would be a good candidate where more than 1 quantum query is necessary.

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