SOLA INVERSIONS FOR THE CORE STRUCTURE OF SOLAR-TYPE STARS

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ABSTRACT

The Subtractive Optimally Localized Averages (SOLA) method, developed and extensively used in helioseismology, is applied to artificial data to obtain measures of the sound speed inside a solar-type star. In contrast to inversion methods which fit models to some aspect of the data, methods such as SOLA provide an honest assessment of what can truly be resolved using seismic data, without introducing additional assumptions such as that the space of admissible stellar models is small. The resulting measures obtained from SOLA inversion can subsequently be used to eliminate putative stellar models. Here we present results of experiments to test the reliability of SOLA inferences using solar models and models of solar-type stars.

Key words: Stars: structure – Stars: oscillations

1. INTRODUCTION

Seismology of solar-type stars is expected in the not-too-distant future to provide information of relevance for understanding stellar structure. We need to develop tools that will allow us to use the full potential of the information provided by the seismic observations of other stars.

Early attempts at inverting artificial sets of low-degree modes have given mixed results. Gough & Kosovichev (1993) and Roxburgh et al. (1998) inverted the frequencies calculated for a 1.1 M⊙ and a 0.8 M⊙ star, respectively, to obtain very encouraging results. More recently, Berthomieu et al. (2001) carried out a careful analysis of the results to be expected in inferences of solar-like oscillations, taking into account the expected errors in the COROT observations and mode amplitudes in a 1.45 M⊙ star; they concluded that reasonably well-localised inversion is possible in the core of such a star. On the other hand, Basu et al. (2001) failed in their attempts at inverting a set of low-degree modes of a solar-type star.

One of the main differences between these efforts was the mode set used for the inversions. The Roxburgh et al. work used a very optimistic set: it is unlikely that all those modes can be observed. The Gough & Kosovichev mode set was somewhat more conservative. Basu et al. (2001) used two conservative mode sets. The modes sets and errors used in the different works are summarized in Table 1. In addition, unlike Gough & Kosovichev and Roxburgh et al., Basu et al. tried to invert for the sound speed c using density ρ as the second variable, instead of inverting for u = p/ρ, p being the pressure, with the helium abundance Y as the second variable. It is known that because density kernels (at fixed c²) have much larger amplitudes than Y kernels (at fixed u), a c inversion is more difficult than a u inversion. The Basu et al. (2001) inversions were done under the assumption that the mass, or radius, or both of these properties of the test star was not known very well and hence this introduced additional uncertainty. Also we believe that there were numerical errors in the kernels used.

In this work we attempt to understand the differences in the results, in particular we study the effect of the mode set and errors on the results obtained. The mode sets used are the same as those used by Basu et al. (2001); the mode set of Gough & Kosovichev (henceforth the GK set) is used for comparison. In order to be able to compare with older results we try to invert for u rather than c² which is the variable of choice in solar inversions. To avoid additional uncertainties due to uncertainty in the radius or the mass of the star under consideration, in this work we confine our attention to solar models. Finally, we note that the assumed errors are much larger than is typical in solar frequencies.

2. MODELS AND INVERSION TECHNIQUES

Inversions for stellar structure are based on linearizing the equations of stellar oscillations around a known reference model. The relative differences in u (i.e. δu/u) and the difference in Y (δY) between a star and a reference stellar model can be related to the differences in the frequencies of the star and the model (δω_i/ω_i),

$$
\frac{\delta \omega_i}{\omega_i} = \int K_{u,Y}(r)dr + \int K_{Y,u}(r)dr + F_{\text{surf}}(\omega_i),
$$

where r is the normalized distance to the centre. The index i numbers the multiplets (n, l). The kernels K_{u,Y} and K_{Y,u} are known functions of the reference model. The term in F_{\text{surf}}(\omega_i) is the contribution from the uncertainties in the near-surface region (e.g. Christensen-Dalsgaard & Berthomieu 1991); here Q_i is the mode inertia, normalized by the inertia of a radial mode of the same frequency. In general, the right-hand side of equation (1) may also
satisfy

\[ \sum \psi F \]

by assuming that on the inferred \( \delta u/u \), which measures the influence of the contribution from \( \delta Y \).

The goal of the inversions is to obtain a localized averaging kernel, while suppressing the contributions from the cross term and the surface term in the linear combination in equation (4), and limiting the error in the solution. Also, \( K(r_0, r) \) must have unit integral with respect to \( r \). If this can be achieved, then

\[ \left\langle \frac{\delta u}{u} (r_0) \right\rangle \simeq \int K(r_0, r) \frac{\delta u}{u} \, dr \]

defines a proper average of \( \delta u/u \).

Here we use the Subtractive Optimally Localized Averages (SOLA) method (Pijpers & Thompson 1992, 1994) to determine the inversion coefficients such that the averaging kernel is an approximation to a given target \( T(r_0, r) \). Details on the application of the SOLA technique to structure inversion were given by, e.g., Rabello-Soares, Basu & Christensen-Dalsgaard (1999).

The reference model used in this work is the solar Model S of Christensen-Dalsgaard et al. (1996). This is a standard solar model.

The test model (the proxy star) we use is model MIX of Basu, Pinsonneault & Bahcall (2000). This is a non-standard model, with an artificially mixed core. This model was selected because the differences in \( u \) with respect to the reference model are large, somewhat along the lines of what we expect for stars other than the Sun due to uncertainties in their ages.

### Table 1. Mode sets considered and assumed standard deviation of errors.

| Paper               | Mode set           | Assumed errors |
|---------------------|--------------------|----------------|
| Gough & Kosovichev  | \( l = 0, 1, n = 10-30 \) | 0.3 \( \mu \)Hz |
| (GK set)            | \( l = 2, n = 9-29 \)       |                |
| Basu et al.         | \( l = 0, n = 11-27 \) | 0.1 \( \mu \)Hz |
| Set 1               | \( l = 1, n = 12-28 \)       |                |
| Basu et al.         | \( l = 0, n = 14-32 \) | 0.3 \( \mu \)Hz |
| Set 2               | \( l = 1, n = 13-29 \)       |                |
| \( \star \)         | \( l = 2, n = 15-30 \)       |                |
| Roxburgh et al.     | \( l = 1, 2, n = 1-30 \) | 0.3 \( \mu \)Hz |
|                     | \( l = 3, n = 1-29 \)       |                |

*For Set 2, various errors are considered (see Fig. 5).

contain a term \( \frac{1}{2} \delta \ln(M/R^3) \) to absorb scaling of the frequencies with stellar mass \( M \) and radius \( R \) according to \((GM/R^3)^{1/2}\).

For linear inversion methods, the solution at a given point \( r_0 \) is determined by a set of inversion coefficients \( c_i(r_0) \), such that the inferred value of, say, \( \delta u/u \) is

\[ \left\langle \frac{\delta u}{u} (r_0) \right\rangle = \sum_i c_i(r_0) \frac{\delta \omega_i}{\omega_i}. \] (2)

From the corresponding linear combination of equations (1) it follows that the solution is characterized by the averaging kernel, obtained as

\[ K(r_0, r) = \sum_i c_i(r_0) K_{i, Y}(r), \] (3)

and also by the cross-term kernel:

\[ C(r_0, r) = \sum_i c_i(r_0) K_{i, Y}(r), \] (4)

which measures the influence of the contribution from \( \delta Y \) on the inferred \( \delta u/u \).

The surface term in equation (3) may be suppressed by assuming that \( F_{\text{surf}} \) can be expanded in terms of polynomials \( \psi \), and constraining the inversion coefficients to satisfy \( \sum \psi = 0, \lambda = 0, 1, ..., \Lambda. \) (Däppen et al. 1991).

The goal of the inversions is to obtain a localized averaging kernel, while suppressing the contributions from the cross term and the surface term in the linear combination in equation (4), and limiting the error in the solution. Also, \( K(r_0, r) \) must have unit integral with respect to \( r \). If this can be achieved, then

\[ \left\langle \frac{\delta u}{u} (r_0) \right\rangle \simeq \int K(r_0, r) \frac{\delta u}{u} \, dr \] (5)

defines a proper average of \( \delta u/u \).

3. Results

3.1. Comparing \( c^2 \) and \( u \) inversions

![Figure 1. Averaging kernels for sound-speed inversions (continuous) and \( u \) inversions (dashed) for a few target radii. The mode set used was Set 2 with a uniform error of 0.3 \( \mu \)Hz.](image)

Fig. 1 shows a comparison of averaging kernels obtained for \( c^2 \) and \( u \) inversions. The data set used was Set 2, with uniform errors of 0.3 \( \mu \)Hz. Note that the \( c^2 \) averaging kernels have a lot of surface structure, which has a detrimental effect on the inversion results. The mode set needs to be expanded in order to get a cleaner averaging kernel. Fig. 2 shows a comparison of the cross-term kernels obtained for \( c^2 \) and \( u \) inversions. One can see that with the mode set used (Set 2), we cannot successfully suppress the cross term in a \( c^2 \) inversion. In contrast, the cross term for the \( u \) inversions is very small.
3.2. Effect of mode set

Fig. 3 shows a comparison of the averaging kernels obtained for $u$ inversions using mode sets Set 1, Set 2 and GK. A uniform error of 0.3 $\mu$Hz was assumed for all sets. Note that the Set 1 averaging kernels have structure at the surface. Set 2 and GK kernels are much cleaner. Set 1 also results in very large cross-term kernels. It should be noted that MOLA inversions can give somewhat better averaging kernels.

Fig. 4 shows the inversion results for the three sets. The horizontal error bars are an indication of the resolution of the inversion. One can see that the Set 1 results have very large errors, as well as poor resolution. Set 2 and GK give very good results for a substantial portion of the core.

3.3. Effect of mode errors

A reduction in mode errors helps the inversions. The effects can be felt in two ways: one could improve the resolution, while keeping the error on the solution the same, or, one could keep the averaging kernel the same and decrease the error on the solution. Fig. 5 shows the inversion results for Set 2 for three different error-distributions: (1) uniform errors of 0.3 $\mu$Hz, (2) uniform errors of 0.1 $\mu$Hz, and (3) an error-distribution which mimics the distribution of errors in solar oscillations frequencies. The distribution was scaled to be around 0.1 $\mu$Hz at low frequency (see Fig. 6 for the error distribution). The reduction of errors gives a more dramatic improvement for Set 1 results. If the errors are reduced by a factor of 5, the surface structure of the averaging kernels can be removed and the cross-term kernels can be made much smaller.

4. Discussion

In common with other investigations of the resolving power of the low-degree modes expected from observations of...
Figure 5. Results of $u$ inversion with Set 2. Different error distributions were used for the inversions whose results are shown in the different panels.

Figure 6. The distribution of errors used for the inversions shown in Fig. 5(c). The errors were scaled from the error distribution of solar oscillation frequencies so that the low-frequency part of the distribution is around 0.1 $\mu$Hz.

Solar-like oscillations in distant stars, we find that with realistic assumptions about the expected mode set and observational errors it is possible to achieve some limited resolution in the core of the model. With our present proxy star this, for example, shows clear evidence for the core mixing as reflected in the steep rise in $\delta u / u$ towards the centre.

The results depend strongly on the choice of variables in the inversion. For helioseismic inversion, a common choice is to express the inverse problem in terms of adiabatic sound speed and density. With the limited set of modes available in the stellar case, the quality of the inversion for the sound speed suffers from the need to suppress the contribution from the density. A preferable choice of variables is the pair $(u, Y)$; here the contribution from $Y$ is small and essentially confined to the outer layers, where helium is partly ionized.

Not surprisingly, the results show substantial dependence on the choice of mode set, although the precise manner in which this happens is not obvious and needs further study. It is important here also to include more realistic estimates of the expected properties of the modes and observations, as has been attempted by Berthomieu et al. (2001).

In analyses of evolved stars it is possible that modes with partial g-mode character may be observed; some evidence for this may already have been found in the subgiant $\eta$ Bootis (e.g. Christensen-Dalsgaard, Bedding & Kjeldsen 1995). Such modes would have a highly beneficial impact on inversions for the stellar core, because the averaging kernels could exploit the localized trapping of the eigenfunctions.

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