Super $G$-spaces

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Abstract

We review the basic theory of super $G$-spaces. We prove a theorem relating the action of a super Harish-Chandra pair $(G_0, g)$ on a supermanifold to the action of the corresponding super Lie group $G$. The theorem was stated in [DM99] without proof. The proof given here does not use Frobenius theorem but relies on Koszul realization of the structure sheaf of a super Lie group (see [Kosz83]). We prove the representability of the stability subgroup functor.

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1 Introduction

In his seminal paper [Kost77], B. Kostant gave a complete and rigorous foundation of supergeometry, including super Lie groups. He introduced, for the first time, the notion of super Harish-Chandra pair (called Lie–Hopf algebra, in that paper) and proved the equivalence between those and super
Lie groups (see also [DM99], where the name super Harish-Chandra pair was introduced).

In this paper we review the basic aspects of the theory of smooth actions of super Lie groups on supermanifolds. The language we adopt is different than that used by Kostant in [Kost77]. In particular we use the explicit realization of the sheaf of a super Lie group in terms of the corresponding super Harish-Chandra pair, as given by Koszul in [Kosz83]. This has the advantage that many constructions become more transparent and easy to prove.

In the first sections we briefly recall the basic definitions and results on super Lie groups and super Harish-Chandra pairs. In particular we state the precise link existing between them giving an explicit construction of the equivalence of the two categories. This is the main ingredient of all subsequent results. In section 4 we recall the concept of action of a super Lie group $G$ on a supermanifold $M$, and in prop. 4.3 we establish the precise link between super Lie group actions and super Harish-Chandra pair actions (the proposition was stated without proof in [DM99]). In section 5 the notion of transitive action is analyzed and characterized both from the point of view of super Harish-Chandra pairs and from the point of view of the functor of points. In section 6 we consider the stabilizer of a supergroup action and a representability theorem for the stability group functor is given. Finally in the last section we review the construction of super homogeneous spaces.

2 Supermanifolds and super Lie groups

A supermanifold $M$ is a locally compact, second countable, Hausdorff topological space $|M|$ endowed with a sheaf $\mathcal{O}_M$ of superalgebras, locally isomorphic to $C^\infty(\mathbb{R}^p) \otimes \Lambda(\vartheta_1, \ldots, \vartheta_q)$. A morphism $\psi: M \to N$ between supermanifolds is a pair of morphisms $(|\psi|, \psi^*)$ where $|\psi|: |M| \to |N|$ is a continuous map and $\psi^*: \mathcal{O}_N \to \mathcal{O}_M$ is a sheaf morphism above $|\psi|$.

**Remark 2.1.** We will consider only smooth supermanifolds. It can be proved that in this category a morphism of supermanifolds is determined once we know the corresponding morphism of the global sections (see, for example, [Kost77] and [BBHR91]). In other words, a morphism $\psi: M \to N$ can be identified with a superalgebra map $\psi^*: \mathcal{O}_N(|N|) \to \mathcal{O}_M(|M|)$. We will tacitly use this fact several times. Moreover, in the following, we will denote with $\mathcal{O}(M)$ the superalgebra of global sections $\mathcal{O}_M(|M|)$.

Suppose now $U$ is an open subset of $|M|$ and let $\mathcal{J}_M(U)$ be the ideal generated by the nilpotent elements of $\mathcal{O}_M(U)$. It is possible to prove that
\( \mathcal{O}_M/J_M \) defines a sheaf of purely even algebras over \( |M| \) locally isomorphic to \( C^\infty(\mathbb{R}^p) \). Therefore \( \tilde{M} := (|M|, \mathcal{O}_M/J_M) \) defines a classical manifold, called the reduced manifold associated to \( M \). Analogously it is possible to prove that each supermanifold morphism \( \psi: M \to N \) determines a corresponding reduced map \( \tilde{\psi}: \tilde{M} \to \tilde{N} \). The map \( f \mapsto \tilde{f} := f + J_M(U) \), with \( f \in \mathcal{O}_M(U) \), defines the embedding \( \tilde{M} \to M \). In the following we will denote with \( ev_p(f) := \tilde{f}(p) \) the evaluation at \( p \in U \).

An important and very used tool in working with supermanifolds is the functor of points. Given a supermanifold \( M \) one can construct the functor \( M(\cdot): \text{SMan}^{\text{op}} \to \text{Set} \) from the opposite of the category of supermanifolds to the category of sets defined by \( S \mapsto M(S) := \text{Hom}(S,M) \) and called the functor of points of \( M \). In particular, for example, \( M(\mathbb{R}^{0|0}) \cong |M| \) as sets. Each supermanifold morphism \( \psi: M \to N \) defines the natural transformation \( \psi(\cdot): M(\cdot) \to N(\cdot) \) given by \( \psi(S)(x) := \psi \circ x \). Due to Yoneda’s lemma, each natural transformation between \( M(\cdot) \) and \( N(\cdot) \) arises from a unique morphism of supermanifolds in the way just described. The category of supermanifolds can thus be embedded into a full subcategory of the category \( [\text{SMan}^{\text{op}}, \text{Set}] \) of functors from the opposite of the category of supermanifolds to the category of sets. Let

\[
\mathcal{Y}: \text{SMan} \to [\text{SMan}^{\text{op}}, \text{Set}]
\]

\( M \mapsto M(\cdot) \)

denote such embedding. It is a fact that the image of \( \text{SMan} \) under \( \mathcal{Y} \) is strictly smaller than \( [\text{SMan}^{\text{op}}, \text{Set}] \). The elements of \( [\text{SMan}^{\text{op}}, \text{Set}] \) isomorphic to elements in the image of \( \mathcal{Y} \) are called representable. Supermanifolds can thus be thought as the representable functors in \( [\text{SMan}^{\text{op}}, \text{Set}] \). For all the details we refer to [Kost77, Lei80, Man97, DM99, Var04].

Super Lie groups (SLG) are, by definition, group objects in the category of supermanifolds. This means that morphisms \( \mu, i \) and \( e \) are defined satisfying the usual commutative diagrams for multiplication, inverse and unit respectively. From this, it follows easily that the reduced morphisms \( \tilde{\mu}, \tilde{i} \), and \( \tilde{e} \) endow \( \tilde{G} \) with a Lie group structure. \( \tilde{G} \) is called the reduced (Lie) group associated with \( G \). \( \tilde{G} \) acts in a natural way on \( G \). In particular, in the following, we will denote by

\[
\tau_g := \mu \circ \langle 1_G, \hat{g} \rangle \\
\ell_g := \mu \circ \langle \hat{g}, 1_G \rangle
\]

the right and left translations by the element \( g \in \tilde{G} \), respectively.\(^1\)

\(^1\)Some explanations of the notations used: given two morphisms \( \alpha: X \to Y \) and
Many classical constructions carry over to the super setting. For example it is possible to define left-invariant vector fields and to prove that they form a super Lie algebra $\mathfrak{g}$, isomorphic to the super tangent space at the identity of $G$ (see, for example, [Kost77] or [Var04]).

In the spirit of the functor of points, one can think of a SLG as a representable functor from $\text{SMan}^{\text{op}}$ to the category $\text{Grp}$ of set theoretical groups. The SLG structure imposes severe restrictions on the structure of the supermanifold carrying it. In the next section, we want to briefly discuss this point.

3 Super Harish-Chandra pairs

Definition 3.1. Suppose $(G_0, \mathfrak{g}, \sigma)$ are respectively a Lie group, a super Lie algebra and a representation of $G_0$ on $\mathfrak{g}$ such that

1. $\mathfrak{g}_0 \cong \text{Lie}(G_0)$,

2. $\sigma|_{\mathfrak{g}_0}$ is equivalent to the adjoint representation of $G_0$ on $\mathfrak{g}_0$.

$(G_0, \mathfrak{g}, \sigma)$ is called a super Harish-Chandra pair (SHCP).

Example 3.2. Let $G$ be a SLG, it is clear that we can associate to it the SHCP given by:

1. the reduced Lie group $\tilde{G}$;

2. the super Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $G$; notice that $\mathfrak{g}_0 \cong \text{Lie}(\tilde{G})$;

$\beta: X \to Z$,

$(\alpha, \beta): X \to Y \times Z$

is the morphism that composed with the projection on the first (resp. second) component gives $\alpha$ (resp. $\beta$); if $x \in \tilde{X}$, the map

$\hat{x}: T \to X$

is the constant map obtained composing the unique map $T \to \mathbb{R}^{0|0}$ with the embedding $\mathbb{R}^{0|0} \to X$ whose image is $x$. 
3. the adjoint representation of $\tilde{G}$ on $\mathfrak{g}$ given by

$$\text{Ad}(g)X := (\text{ev}_g \otimes X \otimes \text{ev}_{g^{-1}})(1 \otimes \mu^*)\mu^*$$

with $g \in \tilde{G}$ and $X \in \mathfrak{g}$ ($X$ is thought as a vector in $T_e(G)$).

**Definition 3.3.** If $(G_0, \mathfrak{g}, \sigma)$ and $(H_0, \mathfrak{h}, \tau)$ are SHCP, a morphism between them is a pair of morphisms

$$\psi_0: G_0 \longrightarrow H_0$$
$$\rho_\psi: \mathfrak{g} \longrightarrow \mathfrak{h}$$

satisfying the compatibility conditions

1. $\rho_\psi|_{G_0} \cong d\psi_0$;
2. $\rho_\psi \circ \sigma(g) = \tau(\psi_0(g)) \circ \rho_\psi$ for all $g \in G_0$.

**Example 3.4.** If $\psi: G \rightarrow H$ is a SLG morphism, then it defines the morphism between the associated SHCP given by $\psi_0 = \tilde{\psi}$ and $\rho_\psi = d\psi$.

Definitions 3.1 and 3.3 allow to define the category $\text{SHCP}$ of super Harish-Chandra pairs. Moreover the above examples show that the correspondence

$$\text{SGrp} \longrightarrow \text{SHCP}$$
$$G \longmapsto (\tilde{G}, \text{Lie}(G), \text{Ad})$$

is functorial.

The following is a crucial result in the development of the theory.

**Theorem 3.5 (B. Kostant).** The functor $(1)$ defines an equivalence of categories.

It is fundamental to notice that it is possible to give a very explicit form to the inverse functor (see Koszul’s paper [Kosz83]). We now want to briefly describe it.

---

2If $M$ is a supermanifold and $U$ is an open subset of $|M|$ we endow $\mathcal{O}_M(U)$ with the usual topology considered in [Kosz77]. As in the classical case (see for example [Gro52]), it can be proved that if $M$ and $N$ are two supermanifolds and $U \times V \subseteq M \times N$ is an open subset, $\mathcal{O}_{M \times N}(U \times V)$ can be identified with the completed projective tensor product $\mathcal{O}_M(U) \otimes \mathcal{O}_N(V)$. This fact will be used each time we will write a morphism between supermanifolds in the tensor product form. Moreover since all the maps we will consider are continuous in the given topology, we will check formulas only on decomposable elements. The reader can easily work out the details each time.
Let us preliminarily remember that the super enveloping algebra $\mathfrak{U}(\mathfrak{g})$ can be endowed with a super Hopf algebra structure (see Kostant’s paper [Kost77]). In fact it is a unital superalgebra with respect to the natural identity $1_{\mathfrak{U}(\mathfrak{g})}$ and multiplication $m_{\mathfrak{U}(\mathfrak{g})}$. Moreover the map $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ defined by $X \mapsto X \otimes 1 + 1 \otimes X$ can be extended to a comultiplication map

$$\Delta_{\mathfrak{U}(\mathfrak{g})} : \mathfrak{U}(\mathfrak{g}) \longrightarrow \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$$

in such a way that together with the counit

$$\varepsilon : \mathfrak{U}(\mathfrak{g}) \longrightarrow \mathbb{R}$$

$\mathfrak{U}(\mathfrak{g})$ becomes a super bialgebra. The antipode is finally defined as the super antiautomorphism

$$S : \mathfrak{U}(\mathfrak{g}) \longrightarrow \mathfrak{U}(\mathfrak{g})$$

$$X \mapsto -X$$

whose action on $\mathfrak{g}$ is given by $X \mapsto -X$. Clearly $X Y = (-1)^{|X||Y|} Y X$.

Suppose hence a SHCP $(G_0, \mathfrak{g}, \sigma)$ is given and notice that

1. $\mathfrak{U}(\mathfrak{g})$ is naturally a left $\mathfrak{U}(\mathfrak{g}_0)$-module;
2. $C^\infty(G_0)$ is a left $\mathfrak{U}(\mathfrak{g}_0)$ module. In fact each $X \in \mathfrak{U}(\mathfrak{g}_0)$ acts from the left on smooth functions on $G_0$ as the left invariant differential operator $\tilde{D}_X$.

Hence, for each open subset $U \subseteq G_0$, it is meaningful to consider

$$O_G(U) := \text{Hom}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), C^\infty(U))$$

where the r. h. s. is the subset of $\text{Hom}(\mathfrak{U}(\mathfrak{g}), C^\infty(U))$ consisting of $\mathfrak{U}(\mathfrak{g}_0)$-linear morphisms.

$O_G(U)$ has a natural structure of unital, commutative superalgebra. The multiplication $O_G(U) \otimes O_G(U) \to O_G(U)$ is defined by

$$\varphi_1 \cdot \varphi_2 := m_{C^\infty(G_0)} (\varphi_1 \otimes \varphi_2) \circ \Delta_{\mathfrak{U}(\mathfrak{g})}$$

and the unit is (with a mild abuse of notation) $\varepsilon$.

The following proposition and lemma are stated in Koszul’s paper [Kosz83].

---

3We recall that if $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ are super vector spaces then $\text{Hom}(V, W)$ denotes the super vector space of all linear morphisms between $V$ and $W$ with the gradation $\text{Hom}(V, W) = \text{Hom}(V_0, W_0) \oplus \text{Hom}(V_0, W_1) \oplus \text{Hom}(V_1, W_0) = \text{Hom}(V_0, W_0) \oplus \text{Hom}(V_1, W_0)$.
Proposition 3.6. \((G_0, \mathcal{O}_G)\) is a supermanifold that is globally split i.e.
\[
\mathcal{O}_G(G_0) \cong \text{Hom}(\Lambda(g_1), \mathcal{C}^\infty(G_0)) \cong \mathcal{C}^\infty(G_0) \otimes \Lambda(g_1)^*	ag{3}
\]
\(\mathcal{O}_G\) carries a natural \(\mathbb{Z}\)-gradation.

The first isomorphism in (3) is given by
\[
\varphi \mapsto \varphi \circ \gamma
\]
where \(\gamma\) is the map defined in the following useful lemma, that will be needed also in the following.

Lemma 3.7.  

- **The antisymmetrizer**
  \[
  \gamma : \Lambda(g_1) \longrightarrow \mathfrak{u}(g)
  \]
  \[
  X_1 \wedge \cdots \wedge X_p \longmapsto \frac{1}{p!} \sum_{\tau \in S_p} (-1)^{\tau} X_{\tau(1)} \cdots X_{\tau(p)}
  \]
  is a super coalgebra morphism.

- **The map**
  \[
  \tilde{\gamma} : \mathfrak{u}(g_0) \otimes \Lambda(g_1) \longrightarrow \mathfrak{u}(g)
  \]
  \[
  X \otimes Y \longmapsto X \cdot \gamma(Y)
  \]
  is an isomorphism of super left \(\mathfrak{u}(g_0)\)-modules.

Next proposition exhibits explicitly the structure of a SLG in terms of the corresponding SHCP.

Proposition 3.8. \((G_0, \mathcal{O}_G)\) is a SLG with respect to the operations
\[
\begin{align*}
\mu^*(\varphi)(X,Y) & = [\varphi((h^{-1}.X)Y)](gh) \tag{5a} \\
i^*(\varphi)(X) & = [\varphi(g^{-1}.X)](g) \tag{5b} \\
e^*(\varphi) & = [\varphi(1)](e) \tag{5c}
\end{align*}
\]
where \(X, Y \in \mathfrak{u}(g)\), \(g, h \in G_0\), \(e\) is the unit of \(G_0\) and \(g.X := \sigma(g)X\). Moreover the associated SHCP is precisely \((G_0, \mathfrak{g}, \sigma)\).

In this approach the reconstruction of a SLG morphism from a SHCP one is very natural. Suppose indeed that \((\psi_0, \rho_\psi)\) is a morphism from \((G_0, \mathfrak{g})\)
to \((H_0, h)\), and suppose \(\varphi \in \mathcal{O}_H(U)\). It is natural to define \(\psi^* (\varphi)\) through the following diagram

\[
\begin{array}{ccc}
\Omega(g) & \overset{\rho_0}{\longrightarrow} & \Omega(h) \\
\downarrow^\psi & & \downarrow^\varphi \\
C^\infty_G(\psi_0^{-1}(U)) & \overset{\psi_0^*}{\leftarrow} & C^\infty_H(U)
\end{array}
\]

It is not difficult to prove that this defines a SLG morphism with associated SHCP morphism \((\psi_0, \rho_\psi)\).

Let us finally collect a glossary of some frequently used operations in Koszul realization, completing those given in eq. (2) and (5) (notice that, since \((-1)^{|X|(|\varphi|+|Y|)} \varphi(Y X) = (-1)^{|X|} \varphi(Y X)\), it is possible to slightly simplify the form of some expressions).

| operation | formula |
|-----------|---------|
| evaluation map | \(\tilde{\varphi} = \varphi(1)\) |
| left translation | \([\ell_h^* (\varphi)](X) = \tilde{\ell}_h^* (\varphi(X))\) |
| right translation | \([r_h^* (\varphi)](X) = \tilde{r}_h^* (\varphi(h^{-1} X))\) |
| left invariant vector fields | \((D^L_\varphi)(Y) = (-1)^{|X|} \varphi(Y X)\) |
| right invariant vector fields | \([D^R_\varphi](Y) = (-1)^{|X||\varphi|} \varphi ((h^{-1} X) Y)\) |

**Example 3.9.** We consider the SLG \(G = \text{Gl}(1|1)\). Formally it can be thought as the set of invertible matrices \((x_1 \theta_1, x_2 \theta_2)\) with multiplication

\[
\begin{pmatrix}
x_1 & \theta_1 \\
\theta_2 & x_2
\end{pmatrix}
\begin{pmatrix}
y_1 & \xi_1 \\
\xi_2 & y_2
\end{pmatrix} =
\begin{pmatrix}
x_1 y_1 + \theta_1 \xi_2 & x_1 \xi_1 + \theta_1 y_2 \\
\theta_2 y_1 + x_2 \xi_2 & x_2 y_2 + \theta_2 \xi_1
\end{pmatrix}
\]

(6)

The corresponding reduced group is \(\tilde{G} = (\mathbb{R} \setminus \{ 0 \})^2\). A basis of left invariant vector fields \(\text{gl}(1|1)\) is easily recognized to be

\[
\begin{align*}
X_1 &= x_1 \frac{\partial}{\partial x_1} + \theta_2 \frac{\partial}{\partial \theta_2} \\
T_1 &= x_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial x_2}
\end{align*}
\]

\[
\begin{align*}
X_2 &= x_2 \frac{\partial}{\partial x_2} + \theta_1 \frac{\partial}{\partial \theta_1} \\
T_2 &= x_2 \frac{\partial}{\partial \theta_2} - \theta_1 \frac{\partial}{\partial x_1}
\end{align*}
\]
with commutation relations (for all $i, j = 1, 2$)

$$
[X_i, X_j] = 0 \quad [T_i, T_i] = 0 \\
[X_i, T_j] = (-1)^{i+j}T_j \quad [T_1, T_2] = -X_1 - X_2
$$

$h = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \in \tilde{G}$ acts through the adjoint representation on $\mathfrak{gl}(1|1)_1$ as follows:

$$
h.T_1 = y_1 T_1 y_2^{-1} \quad h.T_2 = y_2 T_2 y_1^{-1}
$$

Using the theory developed in the previous section, we now want to reconstruct the multiplication map of $G$ in terms of the corresponding SHCP. Introduce the linear operators

$$
\varphi_i : \Lambda(\mathfrak{gl}(1|1)_1) \longrightarrow C^\infty(\tilde{G}) \\
1 \longmapsto y_i \\
T_1, T_2, T_1 \wedge T_2 \longmapsto 0
$$

and

$$
\Phi_i : \Lambda(\mathfrak{gl}(1|1)_1) \longrightarrow C^\infty(\tilde{G}) \\
T_i \longmapsto 1 \\
1, T_j \neq i, T_1 \wedge T_2 \longmapsto 0
$$

These are going to be our coordinates on $\text{Hom}(\Lambda(\mathfrak{gl}(1|1)_1), C^\infty(\tilde{G}))$. These maps extend in a natural way to $\mathfrak{U}(\mathfrak{g}_0)$-linear maps from $\mathfrak{U}(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1)$ to $C^\infty(\tilde{G})$, which we will denote by the same letter. We denote by $\hat{\varphi}$ (resp. $\hat{\Phi}$) the composition $\varphi \circ \hat{\gamma}^{-1}$ (resp. $\Phi \circ \hat{\gamma}^{-1}$).

We want to calculate the pullbacks

$$
(\mu^*(\varphi_i))(X,Y)(g,h) := \hat{\varphi}_i\big(\hat{\gamma}^{-1}(h^{-1}.\gamma(X)\gamma(Y))\big)(gh) = \varphi_i\left(\hat{\gamma}^{-1}(h^{-1}.\gamma(X)\gamma(Y))\right)(gh) \\
(\mu^*(\Phi_i))(X,Y)(g,h) := \hat{\Phi}_i\big(\hat{\gamma}^{-1}(h^{-1}.\gamma(X)\gamma(Y))\big)(gh) = \Phi_i\left(\hat{\gamma}^{-1}(h^{-1}.\gamma(X)\gamma(Y))\right)(gh)
$$

In order to perform the computations we first need to compute the elements $\hat{\gamma}^{-1}(h^{-1}.\gamma(X)\gamma(Y))$. Next table collects them.
From this and using definitions (7) and (8), we can calculate easily the various pullbacks. Let us do it in detail in the case of $\varphi_1$. In such a case the pullback table of $(\mu^*(\varphi_1))(X,Y)((x_1,x_2),(y_1,y_2))$ is

$$
\begin{array}{cccc}
1 & T_1 & T_2 & T_1 \wedge T_2 \\
T_1 & 0 & y_2^{-1}y_1(T_1 \wedge T_2 - \frac{1}{2}(X_1 + X_2)) & -y_2^{-1}y_1(X_1 + X_2)T_1 \\
T_2 & 0 & y_2^{-1}y_1(-T_1 \wedge T_2 - \frac{1}{2}(X_1 + X_2)) & -y_2^{-1}y_1(X_1 + X_2)T_2 \\
T_1 \wedge T_2 & -\frac{1}{2}(X_1 + X_2)T_1 & \frac{1}{2}(X_1 + X_2)T_2 & \frac{1}{2}(X_1 + X_2)^2
\end{array}
$$

The link with the form of the multiplication morphism as given in eq. (6) is established by the isomorphism

$$
x_1 = \varphi_1 \left(1 + \frac{\Phi_1 \Phi_2}{2}\right)
$$

$$
x_2 = \varphi_2 \left(1 - \frac{\Phi_1 \Phi_2}{2}\right)
$$

$$
\vartheta_i = \varphi_i \Phi_i
$$

### 4 G-supermanifolds

Let $M$ be a supermanifold and let $G$ denote a SLG with multiplication, inverse and unit $\mu$, $i$ and $e$ respectively.

**Definition 4.1.** A morphism of supermanifolds

$$
a : G \times M \longrightarrow M
$$

is called an action of $G$ on $M$ if it satisfies

$$
a \circ (\mu \times 1_M) = a \circ (1_G \times a) \quad (9a)
$$

$$
a \circ \langle \dot{e}, 1_M \rangle = 1_M \quad (9b)
$$

(see footnote 1 for the notations).
If an action \( a \) of \( G \) on \( M \) is given, then we say that \( G \) acts on \( M \), or that \( M \) is a \( G \)-supermanifold.

Using the functor of points language, an action of the SLG \( G(\cdot) \) on the the supermanifold \( M(\cdot) \) is a natural transformation

\[
a(\cdot): G(\cdot) \times M(\cdot) \longrightarrow M(\cdot)
\]
such that, for each \( S \in \text{SMan} \), \( a(S): G(S) \times M(S) \rightarrow M(S) \) is an action of the set theoretical group \( G(S) \) on the set \( M(S) \).

If \( p \in \tilde{M} \) and \( g \in \tilde{G} \), we define for future use the maps

\[
a_p: G \longrightarrow M \quad a_p := a \circ (1_G, \hat{p}) \\
a^g: M \longrightarrow M \quad a^g := a \circ (\hat{g}, 1_M)
\]

that, in the functor of points notation, become

\[
a_p(S): G(S) \longrightarrow M(S) \quad a^g(S): M(S) \longrightarrow M(S)
\]

\[
g \mapsto g.\hat{p} \\
m \mapsto \hat{g}.m
\]

They obey the following relations

- \( a^g \circ a^{g^{-1}} = 1_M \) for all \( g \in \tilde{G} \)
- \( a^g \circ a_p = a_p \circ \ell_g \) for all \( g \in \tilde{G} \) and \( p \in \tilde{M} \)

As in the classical case the above relations play an important role in proving that \( a_p \) is a constant rank mapping (next lemma). In the super context, this is a more delicate result than its classical counterpart, since the concept of constant rank itself is more subtle (see [Le80]). We briefly recall it. If \( M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \) is an even \( p|q \times m|n \) matrix with entries in the sections \( \mathcal{O}(U) \) over a superdomain \( U \), then we say that \( M \) has constant rank \( r|s \) if there exist \( G_1 \in \text{Gl}_{p|q}(\mathcal{O}(U)) \) and \( G_2 \in \text{Gl}_{m|n}(\mathcal{O}(U)) \) such that \( G_1 MG_2 \) has the form \( \left( \begin{array}{cc} A' & 0 \\ 0 & D' \end{array} \right) \) with \( A' = \left( \begin{array}{cc} 1_r & 0 \\ 0 & 0 \end{array} \right) \) and \( D' = \left( \begin{array}{cc} 1_s & 0 \\ 0 & 0 \end{array} \right) \). Finally, if \( \psi: M \rightarrow N \) is a morphism between supermanifolds, we say that \( \psi \) has constant rank \( r|s \) at \( m \in \tilde{M} \), if there exists a coordinate neighborhood of \( m \) such that the super Jacobian matrix \( J_\psi \) has rank \( r|s \). We can now prove the following fundamental lemma.

**Lemma 4.2.** \( a_p \) has constant rank.

**Proof.** Let \( \mathfrak{g} \) be the super Lie algebra of \( G \) and let \( J_{a_p} \) be the Jacobian matrix of \( a_p \). Since

\[
\tilde{J}_{a_p}(g) = (\text{da}_p)_g = (\text{da}^g)_p (\text{da}_p) \epsilon (d\ell_g^{-1})_g
\]

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and $a^g$ and $\ell_g$ are isomorphisms, $\bar{J}_{ap}(g)$ has rank $\dim \mathfrak{g} - \dim \ker (da_p)_e$ for each $g \in G$. Moreover, recalling that, if $X \in \mathfrak{g}$, $D^L_X = (\mathbb{1} \otimes X)\mu^*$ and using eq. (9a) we have that, for each $X \in \ker (da_p)_e$,

\[
D^L_X a^*_p = (\mathbb{1} \otimes X)\mu^*(\mathbb{1} \otimes ev_p)a^*
\]

\[
= (\mathbb{1} \otimes X \otimes ev_p)(\mathbb{1} \otimes \mu^*)a^*
\]

\[
= (\mathbb{1} \otimes (da_p)_e(X))a^* = 0
\]

If $\{x_i, \vartheta_j\}$ and $\{y_k, \xi_l\}$ are coordinates in a neighbourhood $U$ of $e$, and in a neighbourhood $V \supseteq \bar{a}(U)$ of $p$ respectively, then

\[
J_{ap} = \begin{pmatrix}
\frac{\partial a^*_p(y_k)}{\partial x_i} & -\frac{\partial a^*_p(y_k)}{\partial \vartheta_j} \\
\frac{\partial a^*_p(\xi_l)}{\partial x_i} & \frac{\partial a^*_p(\xi_l)}{\partial \vartheta_j}
\end{pmatrix}
\]

Let $m|n = \dim \ker (da_p)_e$ and let $\{X_u\}$ and $\{T_v\}$ be bases of $\mathfrak{g}_0$ and $\mathfrak{g}_1$ such that $X_u, T_v \in \ker (da_p)_e$ for $u \leq m$ and $v \leq n$. If

\[
D^L_{X_u} = \sum_i a_{u,i} \frac{\partial}{\partial x_i} + \sum_j b_{u,j} \frac{\partial}{\partial \vartheta_j} \quad D^L_{T_v} = \sum_i c_{v,i} \frac{\partial}{\partial x_i} + \sum_j d_{v,j} \frac{\partial}{\partial \vartheta_j}
\]

(with $a_{u,i}, d_{v,j} \in \mathcal{O}_G(U)_0$ and $b_{u,j}, c_{v,i} \in \mathcal{O}_G(U)_1$) and $A = \begin{pmatrix} a_{u,i} - c_{v,i} & b_{u,j} \\ d_{v,j} & a_{u,i} \end{pmatrix}$, then the matrix

\[
J_{ap} A = \begin{pmatrix} D^L_{X_u} a^*_p(y_k) & -D^L_{T_v} a^*_p(y_k) \\
D^L_{X_u} a^*_p(\xi_l) & D^L_{T_v} a^*_p(\xi_l)
\end{pmatrix}
\]

has $m + n$ zero columns. Since $\{X_u, T_v\}$ is a linearly independent set of vectors, $A$ is invertible and so, for [Le80, lemma 2.3.8] $J_{ap}$ has constant rank in $U$ and, by translation, in all $\bar{G}$.

Since the category of SLG is equivalent to the category of SHCP, one could ask whether there is an equivalent notion of action of a SHCP on a supermanifold. The answer is affirmative and it is given in the next proposition (see also [DM99]).

**Proposition 4.3.** Suppose $G$ acts on a supermanifold $M$, then there are

1. an action

\[
a : \bar{G} \times M \longrightarrow M
\]

\[
a : = a \circ (j_{\bar{G} \rightarrow G} \times \mathbb{1}_M) \text{ of the reduced Lie group } \bar{G} \text{ on the supermanifold } M;
\]
2. a representation

\[ \rho_a : g \to \text{Vec}(M)^{op} \]

\[ X \mapsto (X \otimes 1_{\mathcal{O}(M)}) a^* \]  

(12)

of the super Lie algebra \( g \) of \( G \) on the opposite of the Lie algebra of vector fields over \( M \).

The above two maps satisfy the following compatibility relations

\[ \rho_{a|\mathfrak{g}_0}(X) = (X \otimes 1_{\mathcal{O}(M)}) a^* \quad \forall X \in \mathfrak{g}_0 \]  

(13a)

\[ \rho_a(g, Y) = (g^{-1})^* \rho_a(Y)(a^g)^* \quad \forall g \in \tilde{G}, Y \in g \]  

(13b)

Conversely, let \((\tilde{G}, \mathfrak{g})\) be the SHCP associated with \( G \) and let maps \( a \) and \( \rho \) like in points 1 and 2 above satisfying conditions (13) be given. There is a unique action \( a_\rho : G \times M \to M \) of the SLG \( G \) on \( M \) whose reduced and infinitesimal actions are the given ones. It is given by

\[ a_\rho^* : \mathcal{O}(M) \to \text{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), \mathcal{C}^\infty(\tilde{G}) \hat{\otimes} \mathcal{O}(M)) \]

\[ f \mapsto [X \mapsto (-1)^{|X|}(1 \otimes \rho(X)) a^*(f)] \]  

(14)

In analogy with the classical case, one can use super Frobenius theorem to reconstruct a local action from an infinitesimal action (12). Nevertheless it is particularly interesting that the assignment of (11) allows to avoid the use of super Frobenius theorem and makes possible an explicit reconstruction of the global action. The form of the reconstruction formula given by eq. (14) can be easily obtained as follows.

Let \( a \) be an action of \( G \) on \( M \) and let \((\mathfrak{g}, \rho_a)\) be as in prop. 1.3. If \( f \in \mathcal{O}(M) \), then

\[ a^*(f) \in \text{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), \mathcal{C}^\infty(\tilde{G})) \hat{\otimes} \mathcal{O}(M) \equiv \text{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), \mathcal{C}^\infty(\tilde{G}) \hat{\otimes} \mathcal{O}(M)) \]  

hence, using eq. (9a) and the fact that \( \rho_a \) is an antihomomorphism, for all \( X \in \mathcal{U}(\mathfrak{g}) \)

\[ a^*(f)(X) = (-1)^{|X|}[(D_X \otimes 1)a^*(\varphi)](1) \]

\[ = (-1)^{|X|}(1 \otimes \rho_a(X))(a^*(f)(1)) \]

\[ = (-1)^{|X|}(1 \otimes \rho_a(X)) a^*(f) \]  

This also proves the uniqueness statement of the theorem and it suggests how to prove the existence of the action \( a_\rho \).
Proof of prop. 4.3. Let us check that $a^*_\rho(f)$ is $\mathcal{U}(g_0)$-linear. For all $X \in \mathcal{U}(g)$ and $Z \in g_0$ we have

$$a^*_\rho(f)(ZX) = (-1)^{|X|}(1 \otimes \rho(ZX))a^*_\rho(f)$$

$$= (-1)^{|X|}(1 \otimes \rho(X))(1 \otimes Z \otimes 1)(1 \otimes a^*_\rho)g^*(f)$$

$$= (-1)^{|X|}(1 \otimes \rho(X))(1 \otimes Z \otimes 1)(\hat{\mu}^* \otimes 1)a^*_\rho(f)$$

$$= (\hat{D}_Z^2 \otimes 1)[a^*_\rho(f)(X)]$$

We now check that $a^*_\rho$ is a superalgebra morphism.

$$[a^*_\rho(f_1) \cdot a^*_\rho(f_2)](X) = m_{c^\infty(G)}[a^*_\rho(f_1) \otimes a^*_\rho(f_2)]\Delta(X)$$

$$= (-1)^{|X|}m[(1 \otimes \rho(X(1)))a^*_\rho(f_1) \otimes (1 \otimes \rho(X(2)))a^*_\rho(f_2)]$$

$$= (-1)^{|X|}(1 \otimes \rho(X))(a^*_\rho(f_1) \cdot a^*_\rho(f_2))$$

$$= a^*_\rho(f_1 \cdot f_2)(X)$$

where $f_i \in \mathcal{O}(M)$ and $X(1) \otimes X(2)$ denotes $\Delta(X)$. Concerning the “associative” property, we have that, for $X, Y \in \mathcal{U}(g)$ and $g, h \in \tilde{G}$,

$$[(\mu^* \otimes 1)a^*_\rho(f)](X, Y)(g, h) = [a^*_\rho(f)](h^{-1}.XY)(gh)$$

$$= (-1)^{|X|+|Y|+|X||Y|}\rho(Y)\rho(h^{-1}.X)(g^\rho h)^*(f)$$

$$= (-1)^{|X|+|Y|+|X||Y|}\rho(Y)(g^h)^* \rho(X)(g^\rho)^*(f)$$

$$= [(1 \otimes a^*_\rho)\Delta(a^*_\rho)](X, Y)(g, h)$$

and, finally, $(\text{ev}_{e} \otimes 1)a^*_\rho(f) = \rho(1)(g^e)^*(f) = f$. \hfill \Box

We end this section resuming example 3.9

Example 4.4. Consider again the SLG $G = \text{GL}(1|1)$ introduced in example 3.9. $G$ acts on itself by left multiplication, and, using the same notations as in the previous example, we have

1. left action of $\tilde{G}$ on $G$:

   $$\left(\begin{array}{cc} x_1 & 0 \\ 0 & x_2 \end{array}\right) \cdot \left(\begin{array}{cc} y_1 & \xi_1 \\ \xi_2 & y_2 \end{array}\right) = \left(\begin{array}{cc} x_1y_1 & x_1\xi_1 \\ x_2\xi_2 & x_2y_2 \end{array}\right)$$

2. representation of $\mathfrak{gl}(1|1)$ on the super Lie algebra $\text{Vec}(G)^{op}$:

   $$X_1 \mapsto y_1 \frac{\partial}{\partial y_1} + \xi_1 \frac{\partial}{\partial \xi_1}$$

   $$X_2 \mapsto y_2 \frac{\partial}{\partial y_2} + \xi_2 \frac{\partial}{\partial \xi_2}$$

   $$T_1 \mapsto y_2 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial y_1}$$

   $$T_2 \mapsto y_1 \frac{\partial}{\partial \xi_2} + \xi_1 \frac{\partial}{\partial y_2}$$
In this case the representation sends each element of $\mathfrak{gl}(1|1)$ into the corresponding right invariant vector field.

The action $\mu$ can be reconstructed using eq. (14); a simple calculation shows that

$$
\mu^*(x_1) = x_1 y_1 (1 + \vartheta_1 \vartheta_2) + x_1 \xi_2 \vartheta_1 \\
\mu^*(x_2) = x_2 y_2 (1 + \vartheta_1 \vartheta_2) + x_2 \xi_1 \vartheta_2 \\
\mu^*(\vartheta_1) = x_1 \xi_1 (1 + \vartheta_1 \vartheta_2) + x_1 y_2 \vartheta_1 \\
\mu^*(\vartheta_2) = x_2 \xi_2 (1 - \vartheta_1 \vartheta_2) + x_2 y_1 \vartheta_2
$$

The action $\mu$ can be reconstructed using eq. (14); a simple calculation shows that

The usual form of the multiplication map (as given in example 3.9) is obtained using the isomorphism

$$
\begin{align*}
x_1 &\mapsto x_1 (1 + \vartheta_1 \vartheta_2) \\
\vartheta_1 &\mapsto \frac{\vartheta_1}{x_1} \\
x_2 &\mapsto x_2 (1 + \vartheta_1 \vartheta_2) \\
\vartheta_2 &\mapsto \frac{\vartheta_2}{x_2}
\end{align*}
$$

5 Transitive actions

Let $M$ be a $G$-supermanifold with respect to an action $a: G \times M \to M$. Next definition is the natural generalization of the classical one.

**Definition 5.1.** Suppose $G$ acts on $M$ through $a: G \times M \to M$. We say that $a$ is transitive if there exists $p \in \tilde{M}$ such that $a_p$ (see eq. (10a)) is a surjective submersion.

**Remark 5.2.** Since $a_{g,p} = a_p \circ r_g$, if $a_p$ is submersive for one $p \in \tilde{M}$ then it is submersive for all $p \in \tilde{M}$.

Next proposition characterizes transitive actions.

**Proposition 5.3.** Suppose $M$ is a $G$-superspace, then the following facts are equivalent:

1. $M$ is transitive;
2. $a: \tilde{G} \times \tilde{M} \to \tilde{M}$ is transitive;
3. $(da_p)_e : \mathfrak{g} \to T_p(M)$ is surjective;
4. if $q$ denotes the odd dimension of $G$, then

$$
a_p(\mathbb{R}^{0|q}) : G(\mathbb{R}^{0|q}) \to M(\mathbb{R}^{0|q})
$$

is surjective;
4. the sheafification of the functor

$$\text{SMan}^{\text{op}} \rightarrow \text{Set}$$

$$S \mapsto (\text{Im} a_p)(S) := \{ a_p \circ \phi \mid \phi \in G(S) \}$$

is the functor of points of $M$.

Proof. The second statement is an immediate consequence of lemma 4.2 and previous remark.

Let us hence check the equivalence of the third with the first. If $\phi \in M(\mathcal{R}^{0|q}) = \text{Hom}(\mathcal{R}^{0|q}, M)$, let $\tilde{\phi} \in \tilde{M}$ be the image of the reduced map associated with $\phi$. It is clear that the pullback $\phi^*$ depends only on the restriction of the sections of $O_M$ to an arbitrary neighbourhood of $\tilde{\phi}$. If $a_p$ is a surjective submersion, there exists a local right inverse $s$ of $a_p$ defined in a neighbourhood of $\tilde{\phi}$. By the locality of $\phi \circ \phi$, $s \circ \phi$ is a well defined element of $G(\mathcal{R}^{0|q})$ and moreover

$$[a_p(\mathcal{R}^{0|q})](s \circ \phi) = a_p \circ s \circ \phi = \phi$$

so that $a_p(\mathcal{R}^{0|q})$ is surjective.

Suppose, conversely, $a_p(\mathcal{R}^{0|q})$ surjective. Looking at the reduced part of each morphism in $a_p(\mathcal{R}^{0|q})(G(\mathcal{R}^{0|q}))$, we have that $a_p(\mathcal{R}^{0|0}) = \tilde{a}_p : \tilde{G} \rightarrow \tilde{M}$ is surjective. As a consequence (see [KMS93, th. 5.14]), $\tilde{a}$ is a classical transitive action and $\tilde{a}_p$ is a submersion. Let now $m \in M$ and $\{x_i, \vartheta_j\}$ be coordinates in a neighbourhood $U$ of it. Consider the following element of $M(\mathcal{R}^{0|q})$ defined by the pullback

$$\varphi^* : O_M(U) \rightarrow O(\mathcal{R}^{0|q}) = \Lambda(\eta_1, \ldots, \eta_q)$$

$$x_i \mapsto \tilde{x}_i(m)$$

$$\vartheta_j \mapsto \eta_j$$

By surjectivity of $a_p(\mathcal{R}^{0|q})$, there exists $\psi \in G(\mathcal{R}^{0|q})$ such that

$$\psi^* \circ a_p^*(x_i) = \tilde{x}_i(m)$$

$$\psi^* \circ a_p^*(\vartheta_j) = \eta_j$$

and this implies that $T_m(M)_1$ is in the image of $(da_p)_\psi$. Since, by previous considerations, $\tilde{a}_p$ is a submersion, also $T_m(M)_0$ is in the image. Hence, due to lemma 4.2, we are done. For the last point see [BCF08].
Remark 5.4. In the third point of the above proposition, it is not possible to require the transitivity of \(a(S)\) for each \(S\). Indeed, in such a case, each map \(S \to M\) can be lifted to a map \(S \to G\). This in particular implies the existence of a global section of the fibration \(G \to M\) (take \(S = M\) and the identity map). This problem is solved exactly by taking the sheafification of the functor as indicated in in point 4.

6 Stabilizer

Let \(G\) be a SLG, and suppose \(M\) is a \(G\)-supermanifold; the aim of this section is to define the notion of stability subgroup and to characterize it from different perspectives.

We start recalling the definition of an equalizer. Given two objects (\(X\) and \(Y\)) and two arrows (\(\alpha\) and \(\beta\)) between them, an equalizer is a universal pair \((E, \varepsilon)\) that makes

\[
E \xrightarrow{\varepsilon} X \xrightarrow{\alpha} Y \xrightarrow{\beta} Y
\]

commuting. This means that if \(\tau: T \to X\) is such that \(\alpha \circ \tau = \beta \circ \tau\), then there exists a unique \(\sigma: T \to E\) such that \(\varepsilon \circ \sigma = \tau\). If an equalizer exists, it is unique up to isomorphism.

One can easily convince himself that the next definition mimic the classical one.

Definition 6.1. Suppose \(G\) is a SLG and \(a: G \times M \to M\) is an action of \(G\) on the supermanifold \(M\). We call stabilizer of \(p \in \tilde{M}\) the supermanifold \(G_p\) equalizing the diagram (where \(\hat{p}: G \to M\) is as in footnote 1)

\[
G \xrightarrow{\hat{p}} M
\]

It is not a priori obvious that such an equalizer exists. Next proposition shows that the definition is meaningful and characterizes the notion of stabilizer both from the point of view of the functor of points and in terms of the corresponding SHCP.

Proposition 6.2. 1. The diagram

\[
G \xrightarrow{\hat{p}} M
\]

admits an equalizer \(G_p\);
2. \( G_p \) is a sub-SLG of \( G \);

3. the functor \( S \mapsto G(S)_\hat{p} \) assigning to each supermanifold \( S \) the stabilizer of \( \hat{p} \) of the action of \( G(S) \) on \( M(S) \) is represented by \( G_p \);

4. let \( (\tilde{G}_p, \mathfrak{g}_p) \) be the SHCP associated with the stabilizer \( G_p \). Then \( \tilde{G}_p \subseteq \tilde{G} \) is the classical stabilizer of \( p \) with respect to the reduced action and \( \mathfrak{g}_p = \ker da_p \).

**Proof.** Let us put ourself in a general context and let us suppose that \( \psi: M \to N \) is a morphism of constant rank between two supermanifolds. If \( p \in \tilde{N} \), define \( I_p = \{ f \in \mathcal{O}(N) \mid \tilde{f}(p) = 0 \} \) and \( \mathcal{J}_p^\psi \) as the ideal in \( \mathcal{O}(M) \) generated by \( \psi^*(I_p) \). In this case there is a unique closed subsupermanifold \( S \) of \( M \) such that \( \mathcal{J}_p^\psi = \ker j^*_S \), where \( j^*_S \) is the pullback of the embedding \( j_S: S \to M \). This submanifold is denoted with \( \psi^{-1}(p) \) (see [Le˘ ı80, § 3.2.9]). Let \( G_p \) be \( a_p^{-1}(p) \). We are going to see that, as in classical context, \( G_p \) is the stabilizer of \( p \).

First of all we recall that, if \( A \) and \( B \) are two generic algebras and \( \alpha \) and \( \beta \) are morphisms between them, as it is easy to check, their coequalizer — the equalizer in the opposite category — is the algebra \( C = B/J \), where \( J = \langle \alpha(a) - \beta(a) \mid a \in A \rangle \) is the ideal generated by \( \{ \alpha(a) - \beta(a) \mid a \in A \} \).

Since the embedding \( j_G: G_p \to G \) is regular and closed, \( j_G^* \) is surjective (see [Le˘ ı80 § 3.2.5]). Hence \( \mathcal{O}(G_p) \cong \mathcal{O}(G)/\ker j_G^* \), and moreover

\[
\ker j_G^* = \left\langle a_p^*(f) \mid f \in I_p \right\rangle = \\
= \left\langle a_p^*(f - \tilde{f}(p)) \right\rangle = \left\langle a_p^*(f) - \hat{p}_*(f) \right\rangle = \langle a_p^*(f) - \hat{p}_*(f) \mid f \in \mathcal{O}(M) \rangle
\]

Therefore \( \mathcal{O}(G_p) \) is the coequalizer of

\[
\mathcal{O}(M) \xrightarrow{a_p} \mathcal{O}(G) \xrightarrow{j_G^*} \mathcal{O}(G_p)
\]

and hence \( G_p \) is the equalizer of

\[
G_p \xrightarrow{j_G} G \xrightarrow{a_p} M
\]

This concludes item 1. In order to prove point 2 we have to show that \( G_p \) is a sub-SLG of \( G \). Due to Yoneda lemma, if we prove item 3 also 2
is done. On the other hand item 3 can be proved easily noticing that the functor $G(\cdot)_\bar{p}$ equalizes the natural transformations

$$G(\cdot) \xrightarrow{a_p(\cdot)} M(\cdot) \xrightarrow{\bar{p}(\cdot)}$$

and, since the Yoneda embedding preserves equalizers and due to their uniqueness, $G_p(\cdot) \cong G(\cdot)_{\bar{p}}$.

Let us finally consider item 4. The first statement is clear since $\tilde{G} \cong G(R^{0|0})$ as set theoretical groups. Moreover, since, for all $f \in O(M)$, $j_{\tilde{G}_p}^* \circ a_p^p(f)$ is a constant, $\mathfrak{g}_p \subseteq \ker da_p$ and they are equal for dimension considerations.

7 Homogeneous supermanifolds

In this section we give a detailed account of homogeneous supermanifolds. Essentially all the results presented in this section are known (see [Kost77], and [FLV07]). We nevertheless spend some time in proving them since we adopt a slightly different approach through Koszul’s realization of the sheaf. This allows us to give a very explicit description of the structure sheaf of the homogeneous supermanifold (lemma 7.1 below) and to prove its local triviality without using super Frobenius theorem (proposition 7.2 below). On the other hand proposition 7.4 completely relies on [FLV07].

Let $G$ be a SLG and let $H$ be a closed sub-SLG. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the respective super Lie algebras. Let $U \subseteq \tilde{G}/\tilde{H}$ and $V \subseteq \tilde{H}\backslash\tilde{G}$ be open sets and define

$$O_{G/H}(U) := \left\{ \varphi \in O_G(\pi^{-1}(U)) \left| r_h^*(\varphi) = \varphi \quad \forall h \in \tilde{H} \right\} \right.$$  \hspace{1cm} (15)

$$O_{H\backslash G}(V) := \left\{ \varphi \in O_G(\pi^{-1}(V)) \left| \ell_h^*(\varphi) = \varphi \quad \forall h \in \tilde{H} \right\} \right.$$  \hspace{1cm} (16)

where $\pi$ denotes, for simplicity, both the projections $\tilde{G} \to \tilde{G}/\tilde{H}$ and $\tilde{G} \to \tilde{H}\backslash\tilde{G}$. Define now the morphisms

$$\mu_{G,H} : G \times H \xrightarrow{\mathbb{1}_G \times j_{H \to G}} G \times G \xrightarrow{\mu} G$$

$$\mu_{H,G} : H \times G \xrightarrow{j_{H \to G} \times \mathbb{1}_G} G \times G \xrightarrow{\mu} G$$

Next lemma shows that $O_{G/H}(U)$ (resp. $O_{H\backslash G}(V)$) can be interpreted as the set of sections over $\pi^{-1}(U)$ (resp. $\pi^{-1}(V)$) that are right (resp. left) $H$-invariant.
Lemma 7.1. $\mathcal{O}_{G/H}$ and $\mathcal{O}_{H\setminus G}$ define sheaves of superalgebras over $\tilde{G}/\tilde{H}$ and $\tilde{H}\setminus\tilde{G}$ respectively. Moreover

$$\mathcal{O}_{G/H}(U) = \{ \varphi \in \mathcal{O}_G(\pi^{-1}(U)) \mid \mu_{G,H}^*(\varphi) = \text{pr}_1^*(\varphi) \}$$

$$\mathcal{O}_{H\setminus G}(V) = \{ \varphi \in \mathcal{O}_G(\pi^{-1}(V)) \mid \mu_{H,G}^*(\varphi) = \text{pr}_2^*(\varphi) \}$$

where $\text{pr}_i$ is the projection into the $i$th factor.

Proof. The first statement is easy to establish. We only show the first equality, the proof of the second being equal.

Suppose $\varphi$ belongs to the set in the r. h. s., then

$$r_h^*(\varphi) = (1 \otimes \text{ev}_h)\mu_{G,H}^*(\varphi) = (1 \otimes \text{ev}_h)\text{pr}_1^*(\varphi) = \varphi$$

$$D_X^*\varphi = (1 \otimes X)\mu_{H,G}^*(\varphi) = (1 \otimes X)(\varphi \otimes 1) = 0$$

Conversely, suppose $\varphi \in \mathcal{O}_{G/H}(U)$, $X \in \mathfrak{U}(g)$, $Y \in \mathfrak{U}(h)$, $g \in \tilde{G}$ and $h \in \tilde{H}$, then

$$[[\mu_{G,H}^*\varphi](X,Y)](gh) = [\varphi(h^{-1}.XY)](gh)$$

$$= (-1)^{|Y|}[r_h^*(D_Y^*\varphi)](X)(g)$$

$$= \begin{cases} 
0 & \text{if } Y \not\in \mathbb{R} \\
[\varphi(X)](g) & \text{if } Y = 1 \\
[\text{pr}_1^*(\varphi)](X,Y)](g,h) & \text{if } Y = 1 
\end{cases} \qed$$

Proposition 7.2. $G/H := (\tilde{G}/\tilde{H}, \mathcal{O}_{G/H})$ and $H\setminus G := (\tilde{H}\setminus\tilde{G}, \mathcal{O}_{H\setminus G})$ are isomorphic supermanifolds.

Proof. We start by showing that $G/H$ is a supermanifold. In view of lemma 7.1 it only remains to prove the local triviality of the sheaf.

Let $\mathfrak{h}$ be the super Lie algebra of $H$ and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be a homogeneous decomposition. Moreover let $s: U \to \tilde{G}$, $U \subseteq \tilde{G}/\tilde{H}$, be a local section in a neighbourhood of $\dot{e}$ (the equivalence class of $e$). Consider the trivializing map

$$\mathcal{O}_{G/H}(U) \longrightarrow \text{Hom}(\Lambda(\mathfrak{p}_1), \mathcal{C}_G^{\infty}(\tilde{G}/\tilde{H}))(U)$$

$$\varphi \mapsto \overline{\varphi} \quad (17)$$

where, if $P \in \Lambda(\mathfrak{p}_1)$, $\dot{g} \in U$ and $\gamma$ is as in lemma 3.7

$$[\overline{\varphi}(P)](\dot{g}) := [\varphi(\gamma(P))](s(\dot{g}))$$

The bijectivity of this map can be obtained easily from the following remarks:
• each \( g \in \pi^{-1}(U) \) can be written uniquely as \( g = s(\dot{g})h(g) \), with \( h(g) \in H \);

• due to lemma 3.7 each \( X \in \mathcal{U}(g) \) can be written as \( X = X_0 \gamma(X_1) \) with \( X_0 \in \mathcal{U}(g_0) \) and \( X_1 \in \Lambda(g_1) \);

• due to \( \mathcal{U}(g_0) \)-linearity and condition \( r^*_h \varphi = \varphi \) for each \( h \in \tilde{H} \), each \( \varphi \in O_{G/H}(U) \) satisfies

\[
[\varphi(X)(g)] = [\varphi(X_0 \gamma(X_1))(s(\dot{g})h(g))] = \left[ \tilde{D}^L_{h(g),x_0} \varphi(h(g) \gamma(X_1)) \right](s(\dot{g}))
\]

• due to condition \( D^L_H \varphi = 0 \) for each \( H \in h_1 \), \( \varphi \) is determined by its value on \( \gamma(\Lambda(p_1)) \); indeed, if \( H \in h_1 \) and \( X_1 \in g_1 \),

\[
D^L_H \varphi(\gamma(X_1 \wedge \cdots \wedge X_n)) = -\varphi(\gamma(X_1 \wedge \cdots \wedge X_n \wedge H)) + \varphi(Y) = 0
\]

with

\[
Y \in (R \oplus g_0) \gamma \left( \bigoplus_{i=0}^{n-1} \Lambda^i(g_1) \right)
\]

(see [Kosz83, lemma 2.3]) and than \( \varphi \) can be calculated by induction on \( n \) once it is known on \( \gamma(\Lambda(p_1)) \);

• the condition \( D^L_H \varphi = 0 \) for each \( H \in h_0 \) does not give further restriction since \( D^L_H = \frac{d}{dt} r^*_{\exp tH} \bigg|_{t=0} \).

Moreover, since \( (\gamma \otimes \gamma) \circ \Delta_{\Lambda(p_1)} = \Delta_{\mathcal{U}(g)} \circ \gamma_{\Lambda(p_1)} \), the maps in eq. (17) is easily seen to be a superalgebra morphism. To end the first part of the proof notice that each \( g \in \tilde{G} \) acts by left translation on the sheaf \( O_G \) and that its action preserves \( O_{G/H} \). Hence \( g \) acts as an algebra isomorphism on \( O_{G/H} \), so that local triviality is proved at all points of \( \tilde{G}/\tilde{H} \).

For \( H \backslash G \) the proof is analogous. As in the classical case, in order to show that \( G/H \) and \( H \backslash G \) are isomorphic supermanifolds it is enough to consider the inverse morphism \( i \) defined by (5b). Notice indeed that \( i^* \) sends \( O_{G/H} \) to \( O_{H \backslash G} \) (and vice versa).

**Definition 7.3.** We call \( G/H \) (resp. \( H \backslash G \)) the homogeneous supermanifold of left (resp. right) invariant cosets.

Next proposition establishes some properties of the manifold \( G/H \).
Proposition 7.4. \( G/H \) has the following properties

1. \( \pi : G \to G/H \) is a submersion whose reduced part \( \tilde{\pi} : \tilde{G} \to \tilde{G}/\tilde{H} \) is the natural projection;

2. there is a unique action \( \beta \) of \( G \) on \( G/H \) such that the following diagram is commutative

\[
\begin{array}{ccc}
G \times G & \xrightarrow{\mu} & G \\
\downarrow{1_G \times \pi} & & \downarrow{\pi} \\
G \times G/H & \xrightarrow{\beta} & G/H
\end{array}
\]

If a supermanifold \( X \) exists satisfying the above properties then \( X \cong G/H \).

Proof. Essentially all assertions are consequences of the fact that

\[ \pi^* : \mathcal{O}_{G/H} \longrightarrow \mathcal{O}_G \\
\quad f \longmapsto f \]

For all details, see [FLV07]. \( \square \)

Next proposition proves that, exactly as in the classical case, each transitive supermanifold is isomorphic to a homogeneous supermanifold (see [Kost77]).

Proposition 7.5. Let \( M \) be a transitive \( G \)-supermanifold. If \( p \in \tilde{M} \) and \( G_p \) is the stabilizer of \( p \), then there exists a \( G \)-equivariant isomorphism

\[ G/G_p \longrightarrow M \]

Proof. Fix a point \( p \in \tilde{M} \) and consider the map \( a_p : G \to M \). If \( U \subseteq \tilde{M} \), notice that \( a_p^*(\mathcal{O}_M(U)) \subseteq \mathcal{O}_{G/G_p}(\tilde{a}_p^{-1}(U)) \). Indeed, if \( h \in \tilde{G}_p \) and \( X \in \mathfrak{g}_p = \text{Lie}(G_p) \), for each \( f \in \mathcal{O}_M(U) \)

\[ r_h(a_p^*(f)) = a_{h,p}^*(f) = a_p^*(f) \]

and, due to prop. 6.2

\[
D^L_X(a_p^*(f)) = (1 \otimes X)\mu^*(1 \otimes ev_p)a^*(f) \\
= (1 \otimes X)(1 \otimes a_p^*)a^*(f) \\
= (1 \otimes (da_p)_e(X))a^*(f) = 0
\]

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Hence we can define a map

$$\eta: G/G_p \longrightarrow M$$

through $$\eta^* := a^*_p$$. It is easy to see that such a map is $$G$$-equivariant:

$$a \circ (\mathbb{1}_G \times \eta) = \eta \circ \beta$$

Finally, since $$\bar{\eta}$$ is bijective and $$d\eta$$ is bijective at each point ($$da_p$$ is surjective for transitivity hypothesis and $$g_p = \ker da_p$$, so $$d\eta$$ is bijective at $$\bar{e}$$ and at each point because of the equivariance), $$\eta$$ is an isomorphism (see corollary to th. 2.16 in [Kost77]).

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