Existence and Uniqueness of Solutions for Nonlinear Katugampola Fractional Differential Equations

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Abstract: The present paper deals with the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations with Katugampola fractional derivative. The main results are proved by means of Guo-Krasnoselskii and Banach fixed point theorems. For applications purposes, some examples are provided to demonstrate the usefulness of our main results.

AMS Subject Classification: 34A08, 34A37.
Keywords and Phrases: Fractional equation; Fixed point theorems; Boundary value problem; Existence; Uniqueness.

1. Introduction

The differential equations of fractional order are generalizations of classical differential equations of integer order. They are increasingly used in a variety of fields such as fluid flow, control theory of dynamical systems, signal and image processing, aerodynamics, electromagnetics, probability and statistics, (Samko et al. 1993 [18], Podlubny 1999 [17], Kilbas et al. 2006 [9], Diethelm 2010 [3]) books can be checked as a reference.

Boundary value problem of fractional differential equations is recently approached by various researchers ([1], [8], [19], [20]).

In [20], Bai and L used some fixed point theorems on cone to show the existence and multiplicity of positive solutions for a Dirichlet-type problem of the nonlinear fractional differential equation:

\[
\begin{cases}
D_0^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\
u(0) = u(1) = 0,
\end{cases}
\]
where $D^\alpha_{0+} u$ is the standard Riemann Liouville fractional derivative of order $1 < \alpha \leq 2$ and $f : [0,1] \times [0, \infty) \to [0, \infty)$ is continuous function.

In a recent work [8], Katugampola studied the existence and uniqueness of solutions for the following initial value problem:

$$\begin{align*}
&\begin{cases}
\rho D^\alpha_{0+} u (t) = f (t, u (t)) , \quad \alpha > 0, \\
D^k u (0) = u^{(k)}_0 , \quad k = 1, 2, ..., m - 1,
\end{cases}
\end{align*}$$

where $m = \lfloor \alpha \rfloor$, $\rho D^\alpha_{0+}$ is the Caputo-type generalized fractional derivative, of order $\alpha$, and $f : G \to \mathbb{R}$ is a given continuous function with:

$$G = \left\{ (t,u) : t \in [0,h^*], \quad \left| u - \sum_{k=0}^{m-1} \frac{t^k u^{(k)}_0}{k!} \right| \leq K, \ K, h^* > 0 \right\}.$$ 

This paper focuses on the existence and uniqueness of solutions for a nonlinear fractional differential equation involving Katugampola fractional derivative:

$$\rho D^\alpha_{0+} u (t) + \beta f (t, u (t)) = 0, \quad 0 < t < T, \tag{1.1}$$

supplemented with the boundary conditions:

$$u (0) = 0, \quad u (T) = 0, \tag{1.2}$$

where $\beta \in \mathbb{R}$, and $\rho D^\alpha_{0+}$, for $\rho > 0$, presents Katugampola fractional derivative of order $1 < \alpha \leq 2$, $f : [0,T] \times [0, \infty) \to [h, \infty)$ is a continuous function, with finite positive constants $h, T$.

2. Background materials and preliminaries

In this section, some necessary definitions from fractional calculus theory are presented. Let $\Omega = [0,T] \subset \mathbb{R}$ be a finite interval.

As in [9], let us denote by $X^p_c [0,T]$, $(c \in \mathbb{R}, 1 \leq p \leq \infty)$ the space of those complex-valued Lebesgue measurable functions $y$ on $[0,T]$ for which $\|y\|_{X^p_c} < \infty$ is defined by

$$\|y\|_{X^p_c} = \left( \int_0^T |s^c y (s)|^p \frac{ds}{s} \right)^{\frac{1}{p}} < \infty,$$

for $1 \leq p < \infty$, $c \in \mathbb{R}$, and

$$\|y\|_{X^\infty_c} = \text{ess sup}_{0 \leq t \leq T} [t^c |y (t)|], \quad (c \in \mathbb{R}).$$

**Definition 2.1** (Riemann-Liouville fractional integral [9]). The left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $y : [0,T] \to \mathbb{R}$ is given by:

$$\begin{align*}
^{RLT}_{0+} y (t) &= \frac{1}{\Gamma (\alpha)} \int_0^t (t - s)^{\alpha-1} y (s) \, ds, \quad t \in [0,T],
\end{align*}$$

where $\Gamma (\alpha) = \int_0^{+\infty} e^{-s} s^{\alpha-1} \, ds$, is the Euler gamma function.
Definition 2.2 (Riemann-Liouville fractional derivative [9]). The left-sided Riemann Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : [0, T] \rightarrow \mathbb{R}$ is given by:

$$RLD_0^\alpha y(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t - s)^{n - \alpha - 1} y(s) \, ds, \quad t \in [0, T], \quad n = \lfloor \alpha \rfloor + 1,$$

Definition 2.3 (Hadamard fractional integral [9]). The left-sided Hadamard fractional integral of order $\alpha > 0$ of a continuous function $y : [0, T] \rightarrow \mathbb{R}$ is given by:

$$HI_0^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left( \log \frac{t}{s} \right)^{\alpha - 1} y(s) \frac{ds}{s}, \quad t \in [0, T].$$

Definition 2.4 (Hadamard fractional derivative [9]). The left-sided Hadamard fractional derivative of order $\alpha > 0$ of a continuous function $y : [0, T] \rightarrow \mathbb{R}$ is given by:

$$H^D_0^\alpha y(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \left( \log \frac{t}{s} \right)^{n - \alpha - 1} y(s) \frac{ds}{s}, \quad t \in [0, T], \quad n = \lfloor \alpha \rfloor + 1,$$

if the integral exist.

A recent generalization in 2011, introduced by Udita Katugampola [6], combines the Riemann-Liouville fractional integral and the Hadamard fractional integral into a single form (see [9]), the integral is now known as Katugampola fractional integral, it is given in the following definition:

Definition 2.5 (Katugampola fractional integral [6]). The left-sided Katugampola fractional integral of order $\alpha > 0$ of a function $y \in X_p [0, T]$ is defined by:

$$^{\rho}I_0^\alpha y(t) = \rho^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_0^t \left( \frac{t}{s} \right)^{\alpha - 1} y(s) \frac{ds}{s}, \quad \rho > 0, \quad t \in [0, T]. \quad (2.1)$$

Similarly, we can define right-sided integrals [6]-[7], [9].

Definition 2.6 (Katugampola fractional derivatives [7]). Let $\alpha, \rho \in \mathbb{R}^+$, and $n = \lfloor \alpha \rfloor + 1$. The Katugampola fractional derivative corresponding to the Katugampola fractional integral (2.1) are defined for $0 \leq t \leq T \leq \infty$ by:

$$^{\rho}D_0^\alpha y(t) = \left( \frac{d}{dt} \right)^n \left( \frac{d}{dt} \right)^{\alpha - n - 1} y(t) = \rho^{\alpha-n+1} \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_0^t \left( \frac{t}{s} \right)^{\alpha - n - 1} y(s) \frac{ds}{s}. \quad (2.2)$$
Theorem 2.7 ([7]). Let $\alpha, \rho \in \mathbb{R}^+$, then

\[
\lim_{\rho \to 1^+} (\mathcal{D}_0^\alpha y)(t) = \mathcal{R} \mathcal{L} (\mathcal{D}_0^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, ds,
\]

\[
\lim_{\rho \to 0^+} (\mathcal{D}_0^\alpha y)(t) = \mathcal{H} (\mathcal{D}_0^\alpha y)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y(s) \, ds,
\]

\[
\lim_{\rho \to 0^+} (\mathcal{D}_0^\alpha y)(t) = \mathcal{R} \mathcal{L} (\mathcal{D}_0^\alpha y)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y(s) \, ds.
\]

Remark. As an example, for $\alpha, \rho > 0$, and $\mu > -\rho$, we have

\[
\mathcal{D}_0^\alpha t^\mu = \frac{\rho^{\alpha-1} \Gamma \left( 1 + \frac{\mu}{\rho} \right)}{\Gamma \left( 1 - \alpha + \frac{\mu}{\rho} \right)} t^{\mu-\alpha\rho}.
\]

In particular

\[
\mathcal{D}_0^\alpha t^{\rho(\alpha-m)} = 0, \text{ for each } m = 1, 2, \ldots, n.
\]

For $\mu > -\rho$, we have

\[
\mathcal{D}_0^\alpha t^\mu = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( \frac{t^{1-\rho} d}{dt} \right)^n \int_0^t s^{\rho+\mu-1} (t^\rho - s^\rho)^{n-\alpha-1} \, ds
\]

\[
= \frac{\rho^{\alpha-n}}{\Gamma(n-\alpha)} \left( \frac{t^{1-\rho} d}{dt} \right)^n t^{\rho(n-\alpha)+\mu} \int_0^1 \tau^\mu (1 - \tau)^{n-\alpha-1} \, d\tau
\]

\[
= \frac{\rho^{\alpha-n}}{\Gamma(n-\alpha)} B \left( n-\alpha, 1 + \frac{\mu}{\rho} \right) \left( \frac{t^{1-\rho} d}{dt} \right)^n t^{\rho(n-\alpha)+\mu}
\]

\[
= \frac{\rho^{\alpha-n} \Gamma \left( 1 + \frac{\mu}{\rho} \right)}{\Gamma \left( 1 + n - \alpha + \frac{\mu}{\rho} \right)} \left( \frac{t^{1-\rho} d}{dt} \right)^n t^{\rho(n-\alpha)+\mu}.
\]

Then

\[
\mathcal{D}_0^\alpha t^\mu = \frac{\rho^{\alpha-1} \Gamma \left( 1 + \frac{\mu}{\rho} \right)}{\Gamma \left( 1 + n - \alpha + \frac{\mu}{\rho} \right)} \left[ n - \alpha + \frac{\mu}{\rho} \right] \left[ n - \alpha - 1 + \frac{\mu}{\rho} \right] \cdots \left[ 1 - \alpha + \frac{\mu}{\rho} \right] t^{\mu-\alpha\rho}.
\]

As

\[
\Gamma \left( 1 + n - \alpha + \frac{\mu}{\rho} \right) = \left[ n - \alpha + \frac{\mu}{\rho} \right] \left[ n - \alpha - 1 + \frac{\mu}{\rho} \right] \cdots \left[ 1 - \alpha + \frac{\mu}{\rho} \right] \Gamma \left( 1 - \alpha + \frac{\mu}{\rho} \right),
\]

we get

\[
\mathcal{D}_0^\alpha t^\mu = \frac{\rho^{\alpha-1} \Gamma \left( 1 + \frac{\mu}{\rho} \right)}{\Gamma \left( 1 - \alpha + \frac{\mu}{\rho} \right)} t^{\mu-\alpha\rho}.
\]
In case \( m = \alpha - \frac{\mu}{\rho} \), it follows from (2.4), that
\[
\rho D_{0}^{\alpha} \rho^{(\alpha-m)} = \rho^{\alpha-1} \frac{\Gamma (\alpha - m + 1)}{\Gamma (n - m + 1)} (n - m) (n - m - 1) \cdots (1 - m) t^{-\rho m}.
\]
So, for \( m = 1, 2, \ldots, n \), we get
\[
\rho D_{0}^{\alpha} \rho^{(\alpha-m)} = 0.
\]
Similarly, for all \( \alpha, \rho > 0 \), we have:
\[
\rho I_{0}^{\alpha} + \rho^{\alpha} = \rho^{-\alpha} \Gamma \left(1 + \frac{\mu}{\rho}\right)t^{\mu + \alpha}, \forall \mu > -\rho.
\] (2.5)

By \( C[0, T] \), we denote the Banach space of all continuous functions from \([0, T]\) into \( \mathbb{R} \) with the norm:
\[
\|y\| = \max_{0 \leq t \leq T} |y(t)|.
\]

**Remark.** Let \( p \geq 1, \ c > 0 \) and \( T \leq (pc)^{\frac{1}{p}} \). For all \( y \in C[0, T] \), note that
\[
\|y\|_{X_{p}^{c}} = \left( \int_{0}^{T} |s^{c} y(s)|^{p} \frac{ds}{s} \right)^{\frac{1}{p}} \leq \left( \|y\|^{p} \int_{0}^{T} s^{pc-1} ds \right)^{\frac{1}{p}} = \frac{T^{c}}{(pc)^{\frac{1}{p}}} \|y\|,
\]
and
\[
\|y\|_{X_{\infty}^{c}} = \text{ess sup}_{0 \leq t \leq T} [t^{c} |y(t)|] \leq T^{c} \|y\|,
\]
which imply that \( C[0, T] \hookrightarrow X_{p}^{c}[0, T] \), and
\[
\|y\|_{X_{p}^{c}} \leq \|y\|_{\infty}, \text{ for all } T \leq (pc)^{\frac{1}{p}}.
\]

We express some properties of Katugampola fractional integral and derivative in the following result.

**Theorem 2.8** ([6]-[7]-[8]).

Let \( \alpha, \beta, \rho, c \in \mathbb{R} \), be such that \( \alpha, \beta, \rho > 0 \). Then, for any \( y \in X_{p}^{c}[0, T] \), where \( 1 \leq p \leq \infty \), we have:

- **Index property:**
  \[
  \rho I_{0}^{\alpha} \rho I_{0}^{\beta} y(t) = \rho I_{0}^{\alpha+\beta} y(t), \text{ for all } \alpha, \beta > 0,
  \]
  \[
  \rho D_{0}^{\alpha} \rho D_{0}^{\beta} y(t) = \rho D_{0}^{\alpha+\beta} y(t), \text{ for all } 0 < \alpha, \beta < 1.
  \]

- **Inverse property**
  \[
  \rho D_{0}^{\alpha} \rho I_{0}^{\alpha} y(t) = y(t), \text{ for all } \alpha \in (0, 1).
  \]
From Definitions 2.5 and 2.6, and Theorem 2.8, we deduce that

\[
\rho_{T_0^+} \left( t^{1-\rho} \frac{d}{dt} \right) \rho_{T_0^+} y(t) = \int_0^t s^{\rho-1} \left( s^{1-\rho} \frac{d}{ds} \right) \rho_{T_0^+} y(s) \, ds
\]

\[= \int_0^t \frac{d}{ds} \rho_{T_0^+} y(s) \, ds\]

\[= \left[ \frac{1}{\rho \Gamma (\alpha + 1)} \int_0^s \tau^{\rho-1} (t^{\rho} - \tau^{\rho})^\alpha y(\tau) \, d\tau \right]_0^t\]

\[= \rho_{T_0^+} y(t).\]

Consequently

\[
\left( t^{1-\rho} \frac{d}{dt} \right) \rho_{T_0^+} y(t) = \rho_{T_0^+} y(t) , \forall \alpha > 0. \quad (2.6)
\]

**Definition 2.9** ([4]). Let \( E \) be a real Banach space, a nonempty closed convex set \( P \subset E \) is called a cone of \( E \) if it satisfies the following conditions:

(i) \( u \in P, \lambda \geq 0 \), implies \( \lambda u \in P \).

(ii) \( u \in P, -u \in P \), implies \( u = 0 \).

**Definition 2.10** ([2]). Let \( E \) be a Banach space, \( P \in C(E) \) is called an equicontinuous part if and only if

\[
\forall \varepsilon > 0, \exists \delta > 0, \forall u, v \in E, \forall A \in P, \|u - v\| < \delta \Rightarrow \|A(u) - A(v)\| < \varepsilon.
\]

**Theorem 2.11** (Ascoli-Arzel [2]). Let \( E \) be a compact space. If \( A \) is an equicontinuous, bounded subset of \( C(E) \), then \( A \) is relatively compact.

**Definition 2.12** (Completely continuous [4]). We say \( A : E \to E \) is completely continuous if for any bounded subset \( P \subset E \), the set \( A(P) \) is relatively compact.

The following fixed-point theorems are fundamental in the proofs of our main results.

**Lemma 2.13** (Guo-Krasnosel’skii fixed point theorems [12]). Let \( E \) be a Banach space, \( P \subset E \) a cone, and \( \Omega_1, \Omega_2 \) two bounded open balls of \( E \) centered at the origin with \( \Omega_1 \subset \Omega_2 \). Suppose that \( A : P \cap (\Omega_2 \setminus \Omega_1) \to P \) is a completely continuous operator such that either

(i) \( \|Ax\| < \|x\|, x \in P \cap \partial \Omega_1 \) and \( \|Ax\| \geq \|x\|, x \in P \cap \partial \Omega_2 \), or

(ii) \( \|Ax\| \geq \|x\|, x \in P \cap \partial \Omega_1 \) and \( \|Ax\| \leq \|x\|, x \in P \cap \partial \Omega_2 \),

holds. Then \( A \) has a fixed point in \( P \cap (\Omega_2 \setminus \Omega_1) \).

**Theorem 2.14** (Banach’s fixed point [5]). Let \( E \) be a Banach space, \( P \subset E \) a nonempty closed subset. If \( A : P \to P \) is a contraction mapping, then \( A \) has a unique fixed point in \( P \).
3. Main results

In the sequel, \( T, p \) and \( c \) are real constants such that

\[
p \geq 1, \quad c > 0, \quad \text{and} \quad T \leq (pc)^{\frac{1}{p}}.
\]

Now, we present some important lemmas which play a key role in the proofs of the main results.

**Lemma 3.1.** Let \( \alpha, \rho \in \mathbb{R}^+ \). If \( u \in C [0, T] \), then:

(i) The fractional equation \( ^\rho D_0^\alpha u(t) = 0 \), has a solution as follows:

\[
u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)} + \cdots + C_n t^{\rho(\alpha-n)}, \quad \text{where} \quad C_m \in \mathbb{R}, \quad \text{with} \quad m = 1, 2, \ldots, n.
\]

(ii) If \( ^\rho D_0^\alpha u \in C [0, T] \) and \( 1 < \alpha \leq 2 \), then:

\[
^\rho T_0^\alpha \, ^\rho D_0^\alpha \, u(t) = u(t) + C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)}, \quad \text{for some} \quad C_1, C_2 \in \mathbb{R}. \quad (3.1)
\]

**Proof.** (i) Let \( \alpha, \rho \in \mathbb{R}^+ \). From remark 2, we have:

\[
^\rho D_0^\alpha u(t) = 0, \quad \text{for each} \quad m = 1, 2, \ldots, n.
\]

Then, the fractional differential equation \( ^\rho D_0^\alpha u(t) = 0 \), admits a solution as follows:

\[
u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)} + \cdots + C_n t^{\rho(\alpha-n)}, \quad C_m \in \mathbb{R}, \quad m = 1, 2, \ldots, n.
\]

(ii) Let \( ^\rho D_0^\alpha u \in C [0, T] \) be the fractional derivative \((2.2)\) of order \( 1 < \alpha \leq 2 \). If we apply the operator \( ^\rho T_0^\alpha \) to \( ^\rho D_0^\alpha u(t) \) and use Definitions 2.5, 2.6, Theorem 2.8 and property (2.6), we get

\[
^\rho T_0^\alpha \, ^\rho D_0^\alpha u(t) = \left( \frac{d}{dt} \right)^{\rho(\alpha-1)} \frac{d}{dt} \left[ \frac{\rho^{-\alpha}}{\Gamma(\alpha + 1)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} \, s^\rho \, dT_0^{\alpha-\alpha} u(s) \, ds \right]
\]

\[
= \left( \frac{d}{dt} \right)^{\rho(\alpha-1)} \frac{\rho^{-\alpha}}{\Gamma(\alpha + 1)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} \, s^\rho \, \left[ \left( s^{1-\rho} \frac{d}{ds} \right)^2 T_0^{\alpha-\alpha} u(s) \right] \, ds
\]

\[
= \left( \frac{d}{dt} \right)^{\rho(\alpha-1)} \frac{\rho^{-\alpha}}{\Gamma(\alpha + 1)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} \, s^\rho \, \left[ \left( s^{1-\rho} \frac{d}{ds} \right) T_0^{\alpha-\alpha} u(s) \right] \, ds
\]

\[
= \left( \frac{d}{dt} \right)^{\rho(\alpha-1)} \frac{\rho^{-\alpha}}{\Gamma(\alpha + 1)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} \, s^\rho \, \left[ (t^\rho - s^\rho)^{-\alpha} \left( s^{1-\rho} \frac{d}{ds} \right) T_0^{\alpha-\alpha} u(s) \right]_0^t
\]

\[
+ \alpha \rho \int_0^t s^\rho \, (t^\rho - s^\rho)^{-\alpha-1} \, s^{1-\rho} \, \left( s^{1-\rho} \frac{d}{ds} \right) T_0^{\alpha-\alpha} u(s) \, ds.
\]
From (2.6), we have

$$\left( s^{1-\rho} \frac{d}{ds} \right)^{\rho \mathcal{I}_{0^+}^{2-\alpha} u(s)} = \rho \mathcal{I}_{0^+}^{1-\alpha} u(s).$$ \hspace{1cm} (3.2)$$

On the other hand, from (2.2), we have

$$\left( s^{1-\rho} \frac{d}{ds} \right)^{\rho \mathcal{I}_{0^+}^{2-\alpha} u(s)} = \left( s^{1-\rho} \frac{d}{ds} \right)^{1} \rho \mathcal{I}_{0^+}^{1-(\alpha-1)} u(s) = \rho \mathcal{D}_{0^+}^{\alpha-1} u(s).$$ \hspace{1cm} (3.3)$$

Then

$$\rho \mathcal{I}_{0^+}^{\alpha} \rho \mathcal{D}_{0^+}^{\alpha} u(t) = t^{1-\rho} \frac{d}{dt} \left( \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{0}^{t} (t^\rho - s^\rho)^{\alpha-1} \frac{d}{ds} \rho \mathcal{I}_{0^+}^{2-\alpha} u(s) ds \right)$$

$$- \rho^{1-\alpha} \rho \mathcal{I}_{0^+}^{1-\alpha} u(0^+) t^{\rho(\alpha-1)},$$

where

$$\psi = t^{1-\rho} \frac{d}{dt} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left( [(t^\rho - s^\rho)^{\alpha-1} \rho \mathcal{I}_{0^+}^{2-\alpha} u(s)]_0^t + \rho (\alpha - 1) \int_{0}^{t} s^{\rho-1} (t^\rho - s^\rho)^{\alpha-2} \rho \mathcal{I}_{0^+}^{2-\alpha} u(s) ds \right)$$

$$= t^{1-\rho} \frac{d}{dt} \left( \frac{\rho^{2-\alpha}}{\Gamma(\alpha - 1)} \int_{0}^{t} s^{\rho-1} (t^\rho - s^\rho)^{\alpha-2} \rho \mathcal{I}_{0^+}^{2-\alpha} u(s) ds \right)$$

$$- \rho^{1-\alpha} \rho \mathcal{I}_{0^+}^{2-\alpha} u(0^+) t^{\rho(\alpha-1)} \right)$$

$$= t^{1-\rho} \frac{d}{dt} \left( \rho \mathcal{I}_{0^+}^{\alpha-1} \rho \mathcal{D}_{0^+}^{\alpha-2} u(t) - \rho^{1-\alpha} \rho \mathcal{I}_{0^+}^{2-\alpha} u(0^+) \frac{1}{\Gamma(\alpha)} t^{\rho(\alpha-1)} \right)$$

$$= t^{1-\rho} \frac{d}{dt} \left( \rho \mathcal{D}_{0^+}^{\alpha-1} u(t) - \rho^{1-\alpha} \rho \mathcal{I}_{0^+}^{2-\alpha} u(0^+) \frac{1}{\Gamma(\alpha)} t^{\rho(\alpha-1)} \right)$$

Finally, for $1 < \alpha \leq 2$, we have:

$$\rho \mathcal{I}_{0^+}^{\alpha} \rho \mathcal{D}_{0^+}^{\alpha} u(t) = u(t) - \rho^{1-\alpha} \rho \mathcal{I}_{0^+}^{1-\alpha} u(0^+) \frac{1}{\Gamma(\alpha)} t^{\rho(\alpha-1)} - \rho^{2-\alpha} \rho \mathcal{I}_{0^+}^{2-\alpha} u(0^+) \frac{1}{\Gamma(\alpha - 1)} t^{\rho(\alpha-2)}. \hspace{1cm} (3.4)$$

As

$$\rho \mathcal{I}_{0^+}^{\alpha} t^\mu = \frac{\rho^{1-\alpha}}{\Gamma(1 + \alpha + \frac{\mu}{\rho})} t^{\mu + \alpha}, \quad \forall \mu > -\rho,$$
we use (3.2), (3.3), to prove that

\[
\rho^\alpha \beta^{\alpha-1} \left[ C_1 t^{\rho(\alpha-1)} \right] = C_1 \frac{\rho^{(\alpha-1)} \Gamma \left( 1 + \frac{\rho(\alpha-1)}{\rho} \right) t^{\rho(\alpha-1)+(\alpha-1)\rho} = C_1 \rho^{\alpha-1} (\alpha), \quad (3.5)
\]

\[
\rho^\alpha \beta^{\alpha-1} \left[ C_2 t^{\rho(\alpha-2)} \right] = C_2 \rho^{\alpha-2} t^{\rho(\alpha-2)} = C_2 \rho^{\alpha-2} t^{\rho((\alpha-1)-1)} = 0, \quad (3.6)
\]

for some \( C_1, C_2 \in \mathbb{R} \), and

\[
\rho^\alpha \beta^{\alpha-1} \left[ C_1 t^{\rho(\alpha-1)} \right] = C_1 \frac{\rho^{(\alpha-1)} \Gamma \left( 1 + \frac{\rho(\alpha-2)}{\rho} \right) t^{\rho(\alpha-1)+(\alpha-2)\rho} = C_1 \rho^{\alpha-2} (\alpha) t^\rho \quad (3.7)
\]

\[
\rho^\alpha \beta^{\alpha-1} \left[ C_2 t^{\rho(\alpha-2)} \right] = C_2 \frac{\rho^{(\alpha-2)} \Gamma \left( 1 + \frac{\rho(\alpha-2)}{\rho} \right) t^{\rho(\alpha-2)+(\alpha-2)\rho} = C_2 \rho^{\alpha-2} (\alpha-1) \quad (3.8)
\]

Then, for \( u(t) = C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)} \), we have respectively:

\[
\rho^\alpha \beta^{\alpha-1} u(0^+) = \rho^\alpha \beta^{\alpha-1} \left[ C_1 t^{\rho(\alpha-1)} \right] (0^+) + \rho^\alpha \beta^{\alpha-1} \left[ C_2 t^{\rho(\alpha-2)} \right] (0^+) = C_1 \rho^{\alpha-1} (\alpha), \quad (3.9)
\]

\[
\rho^\alpha \beta^{\alpha-1} u(0^+) = \rho^\alpha \beta^{\alpha-1} \left[ C_1 t^{\rho(\alpha-1)} \right] (0^+) + \rho^\alpha \beta^{\alpha-1} \left[ C_2 t^{\rho(\alpha-2)} \right] (0^+) = C_2 \rho^{\alpha-2} (\alpha-1) \quad (3.10)
\]

From (3.4), (3.5), (3.6), (3.7), (3.8), (3.9) and (3.10) we get (3.1). \[Q.E.D.\]

In the following lemma, we define the integral solution of the boundary value problem (1.1)-(1.2).

**Lemma 3.2.** Let \( \alpha, \rho \in \mathbb{R}^+ \), be such that \( 1 < \alpha \leq 2 \). We give \( \rho \mathcal{D}^\alpha_{a^+} u \in C[0, T] \), and \( f(t, u) \) is a continuous function. Then the boundary value problem (1.1)-(1.2), is equivalent to the fractional integral equation

\[
u(t) = \beta \int_0^T G(t, s) f(s, u(s)) \, ds, \quad t \in [0, T],
\]

where

\[
G(t, s) = \begin{cases}
\frac{\beta^{\alpha-1} \rho^{\alpha-1}}{\Gamma(\alpha)} [\frac{\rho^{\alpha}}{T^\rho (T^\rho - s^\rho)]^{\alpha-1} - (T^\rho - s^\rho)]^{\alpha-1}, & 0 \leq s \leq t \leq T, \\
\frac{\beta^{\alpha-1} \rho^{\alpha-1}}{\Gamma(\alpha)} [\frac{\rho^{\alpha}}{T^\rho (T^\rho - s^\rho)]^{\alpha-1}, & 0 \leq t \leq s \leq T, \quad (3.11)
\end{cases}
\]

is the Green’s function associated with the boundary value problem (1.1)-(1.2).

**Proof.** Let \( \alpha, \rho \in \mathbb{R}^+ \), be such that \( 1 < \alpha \leq 2 \). We apply Lemma 3.1 to reduce the fractional equation (1.1) to an equivalent fractional integral equation. It is easy to
prove the operator $\mathcal{T}_0^{\alpha+}$ has the linearity property for all $\alpha > 0$ after direct integration. Then by applying $\mathcal{T}_0^{\alpha+}$ to equation (1.1), we get
\[
\mathcal{T}_0^{\alpha+} \mathcal{D}_0^{\beta}, u(t) + \beta \mathcal{D}_0^{\alpha}, f(t, u(t)) = 0.
\]
From Lemma 3.1, we find for $1 < \alpha \leq 2$,
\[
\mathcal{T}_0^{\alpha+} \mathcal{D}_0^{\alpha}, u(t) = u(t) + C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)},
\]
for some $C_1, C_2 \in \mathbb{R}$. Then, the integral solution of the equation (1.1) is:
\[
u(t) = \frac{\beta \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t s^{\rho-1} f(s, u(s)) ds - C_1 t^{\rho(\alpha-1)} - C_2 t^{\rho(\alpha-2)}.
\]
(3.12)
The conditions (1.2) imply that:
\[
\begin{aligned}
u(0) &= 0 = 0 - 0 - \lim_{t \to 0} C_2 t^{\rho(\alpha-2)} \quad \Rightarrow \quad C_2 = 0, \\
u(T) &= 0 = -\frac{\beta \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T s^{\rho-1} f(s, u(s)) ds - C_1 T^{\rho(\alpha-1)} \quad \Rightarrow \quad C_1 = -\frac{\beta \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T s^{\rho-1} f(s, u(s)) ds.
\end{aligned}
\]
The integral equation (3.12) is equivalent to:
\[
u(t) = \frac{\beta \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t s^{\rho-1} f(s, u(s)) ds + \frac{\beta t^{\rho(\alpha-1)} \rho^{1-\alpha}}{T^{\rho(\alpha-1)} \Gamma(\alpha)} \int_0^T s^{\rho-1} f(s, u(s)) ds.
\]
Therefore, the unique solution of problem (1.1)-(1.2) is:
\[
u(t) = \beta \int_0^t \rho^{1-\alpha} s^{\rho-1} \left[ \frac{T^{\rho} (T^\rho - s^\rho)}{T^\rho} \right]^{\alpha-1} - (T^\rho - s^\rho)^{\alpha-1} \frac{f(s, u(s)) ds}{\Gamma(\alpha)}
\]
\[
+ \beta \frac{T^{\rho-1}}{\Gamma(\alpha)} \left[ \frac{T^{\rho} (T^\rho - s^\rho)}{T^\rho} \right]^{\alpha-1} f(s, u(s)) ds
\]
\[
= \beta \int_0^T G(t, s) f(s, u(s)) ds.
\]
The proof is complete. \hfill \Box

3.1. Application of Guo-Krasnosel’skii fixed point theorem

In this part, we assume that $\beta > 0$ and $0 < \rho \leq 1$. We impose some conditions on $f$, which allow us to obtain some results on existence of positive solutions for the boundary value problem (1.1)-(1.2).

We note that $u(t)$ is a solution of (1.1)-(1.2) if and only if:
\[
u(t) = \beta \int_0^T G(t, s) f(s, u(s)) ds, \quad t \in [0, T].
\]
Now we prove some properties of the Green’s function $G(t, s)$ given by (3.11).
Lemma 3.3. Let $1 < \alpha \leq 2$ and $0 < \rho \leq 1$, then the Green’s function $G(t, s)$ given by (3.11) satisfies:

1. $G(t, s) > 0$ for $t, s \in (0, T)$.
2. $\max_{0 \leq t \leq T} G(t, s) = G(s, s)$, for each $s \in [0, T]$.
3. For any $t \in [0, T]$,
   \[ G(t, s) \geq b(t) G(s, s), \text{ for any } T \frac{s}{8} \leq s \leq T \text{ and some } b \in C[0, T]. \] (3.13)

Proof. (1) Let $1 < \alpha \leq 2$ and $0 < \rho \leq 1$. In the case $0 < t \leq s < T$, we have:
\[
\frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[ \frac{t^\rho}{T^\rho} (T^\rho - s^\rho) \right]^{\alpha-1} > 0.
\]
Moreover, for $0 < s \leq t < T$, we have $\frac{t^\rho}{T^\rho} < 1$, then $\frac{t^\rho}{T^\rho} s^\rho < s^\rho$ and $t^\rho - \frac{t^\rho}{T^\rho} s^\rho > t^\rho - s^\rho$,
thus
\[
t^\rho - \frac{t^\rho}{T^\rho} s^\rho = \frac{t^\rho}{T^\rho} (T^\rho - s^\rho) > t^\rho - s^\rho \Rightarrow \left[ \frac{t^\rho}{T^\rho} (T^\rho - s^\rho) \right]^{\alpha-1} - (t^\rho - s^\rho)^{\alpha-1} > 0,
\]
which imply that $G(t, s) > 0$ for any $t, s \in (0, T)$.

(2) To prove that
\[
\max_{0 \leq t \leq T} G(t, s) = G(s, s) = \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[ \frac{s^\rho}{T^\rho} (T^\rho - s^\rho) \right]^{\alpha-1}, \forall s \in [0, T],
\] (3.14)
we choose
\[
g_1(t, s) = \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[ \frac{t^\rho}{T^\rho} (T^\rho - s^\rho) \right]^{\alpha-1},
g_2(t, s) = \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[ \frac{t^\rho}{T^\rho} (T^\rho - s^\rho) \right]^{\alpha-1}.
\]
Indeed, we put $\max_{0 \leq t \leq T} G(t, s) = G(t^*, s)$, where $0 \leq t^* \leq T$. Then, we get for some $0 < t_1 < t_2 < T$, that
\[
\max_{0 \leq t \leq T} G(t, s) = \begin{cases} 
g_1(t^*, s), & s \in [0, t_1], 
g_2(t^*, s), & s \in [t_1, t_2], 
g_1(t^*, s), & s \in [t_2, T], \end{cases}
\]
where $r \in [t_1, t_2]$, is the unique solution of equation
\[
g_1(t^*, s) = g_2(t^*, s) \Leftrightarrow t^* = s,
\]
which shows the equality (3.14) .

(3) In the following, we divide the proof into two-part, to show the existence $b \in C[0, T]$ , such that

$$G(t, s) \geq b(t) G(s, s) , \text{ for any } \frac{T}{8} \leq s \leq T .$$

(i) Firstly, if $0 \leq t \leq s \leq T$ , we see that $\frac{G(t, s)}{G(s, s)}$ is decreasing with respect to $s$. Consequently

$$\frac{G(t, s)}{G(s, s)} = \left[ \frac{\rho}{\rho} (T^p - s^p) \right]^{\alpha - 1} = \left( \frac{t}{s} \right)^{\rho(\alpha - 1)} \geq \left( \frac{t}{T} \right)^{\rho(\alpha - 1)} = b_1(t) , \forall t \in [0, s] .$$

(ii) In the same way, if $0 \leq s \leq t \leq T$ , we have $\frac{\rho}{\rho} < \frac{t^p}{t^p} \leq 1 , \left( \frac{t^p}{t^p} \right)^{\alpha - 2} \geq 1 , \forall \alpha \in (1, 2] , \text{ and}

$$G(t, s) = \frac{\rho^{1 - \alpha} s^{\alpha - 1}}{\Gamma(\alpha)} \left[ \frac{t^p}{t^p} (T^p - s^p) \right]^{\alpha - 1} - \left( t^p - s^p \right)^{\alpha - 1} \right]$$

$$= \frac{(\alpha - 1) \rho^{1 - \alpha} s^{\alpha - 1}}{\Gamma(\alpha)} \int_{t^p - s^p}^{s^p} T^p - s^p \tau^{\alpha - 2} d\tau$$

$$\geq \frac{(\alpha - 1) \rho^{1 - \alpha} s^{\alpha - 1}}{\Gamma(\alpha)} \left( \frac{t^p}{t^p} \right)^{\alpha - 2} (T^p - s^p)^{\alpha - 2} \left( \frac{t^p}{T^p} (T^p - s^p) - (t^p - s^p) \right)$$

As $0 < \rho \leq 1$ , we get

$$T^p - t^p = \rho \int_t^T T^p - s^p \geq \rho \int_{T - t}^T \tau^{\alpha - 2} d\tau$$

Therefore

$$\frac{G(t, s)}{G(s, s)} \geq \frac{(\alpha - 1) \rho^{1 - \alpha} s^{\alpha - 1}}{\Gamma(\alpha)} \left[ \frac{t^p}{t^p} (T^p - s^p) \right]^{\alpha - 1} \frac{T^p - s^p}{s^p (T^p - t^p)} \geq \frac{(\alpha - 1) s^p (T^p - t^p)}{T^p (T^p - s^p)} \left( \frac{T^p}{s^p} \right)^{\alpha - 1}$$

Finally, for $s \in \left[ \frac{T}{8}, t \right]$ , we have:

$$\frac{G(t, s)}{G(s, s)} \geq \frac{(\alpha - 1) (T - t)}{8T} = b_2(t) .$$
It is clear that \( b_1(t) \) and \( b_2(t) \) are positive functions, it is enough to choose:

\[
b(t) = \begin{cases} 
\frac{1}{T} t^{(\alpha - 1)}, & \text{for } t \in [0, \bar{t}], \\
\frac{1}{T} \frac{t^{(\alpha - 1)}}{(T-t)^{\alpha}}, & \text{for } t \in [\bar{t}, T], 
\end{cases}
\]

(3.15)

where \( \bar{t} \in (0, T) \) is the unique solution of the equation \( b_1(t) = b_2(t) \). We see that

\[
b(t) \leq \bar{b} = b(\bar{t}) = \left( \frac{\bar{t}}{T} \right)^{(\alpha - 1)} \frac{(\alpha - 1)(T-\bar{t})}{8T} < 1 \text{ for all } t \in [0, T].
\]

Finally, we have \( \forall s \in \left[ \frac{T}{8}, T \right] \),

\[
G(t, s) \geq b(t) G(s, s), \ \forall t \in [0, T].
\]

The proof is complete.

\[\square\]

**Lemma 3.4.** Let \( 1 < \alpha \leq 2 \) and \( 0 < \rho \leq 1 \), then there exists a positive constant

\[
\lambda = 1 + \frac{8^{\rho \alpha} L (\alpha + 1) [8^{\rho \alpha} - (8^\rho - 1)^{\alpha}]}{h (8^\rho - 1)^{\alpha} [8^{\rho (\alpha + 1)} + 8^{\rho (\alpha - 1)} (\alpha - 1)(8^\rho - 1)]}, \text{ for some } h, L > 0,
\]

such that

\[
\int_0^T G(s, s) f(s, u(s)) \, ds \leq \lambda \int_0^T G(s, s) f(s, u(s)) \, ds.
\]

(3.16)

**Proof.** As \( f(t, u(t)) \geq h \), for any \( t \in [0, T] \), we get

\[
\int_0^T G(s, s) f(s, u(s)) \, ds \geq h \int_0^T \frac{1-\alpha s^{\rho - 1}}{\Gamma(\alpha)} \left[ \frac{s^\rho}{T^\rho} (T^\rho - s^\rho) \right]^{\alpha-1} \, ds
\]

\[
\geq -\frac{h}{\alpha \rho s^\rho T^\rho \Gamma(\alpha)} \int_0^T s^{\rho (\alpha - 1)} \left[ -\rho s^{\rho - 1} (T^\rho - s^\rho)^{\alpha-1} \right] \, ds.
\]

The integral by part gives:

\[
\int_0^T G(s, s) f(s, u(s)) \, ds \geq \frac{h \frac{T^{\rho (\alpha - 1)}}{\rho s^{\rho (\alpha - 1)}} (T^\rho - s^\rho) \alpha + \rho (\alpha - 1) \int_0^T s^{\rho (\alpha - 1)} (T^\rho - s^\rho)^{\alpha} \, ds}{\rho \rho^\rho T^\rho \Gamma(\alpha + 1)}
\]

\[
\geq \frac{h \left[ \frac{T^\rho}{\rho s^{\rho (\alpha - 1)}} (T^\rho - s^\rho)^{\alpha} + \rho (\alpha - 1) \int_0^T s^{\rho (\alpha - 2)} (T^\rho - s^\rho)^{\alpha} \, ds \right]}{\rho \rho^\rho T^\rho \Gamma(\alpha + 1)}
\]

\[
\geq \frac{h \left[ \frac{T^\rho}{\rho s^{\rho (\alpha - 1)}} (T^\rho - s^\rho)^{\alpha} - \frac{\alpha + 1}{\rho^\rho} \int_0^T s^{\rho (\alpha - 1)} (T^\rho - s^\rho)^{\alpha} \, ds \right]}{\rho \rho^\rho T^\rho \Gamma(\alpha + 1)}
\]

\[
\geq \frac{h T^\rho (8^\rho - 1)^{\alpha}}{\rho \rho^\rho 8^{\rho \alpha} \Gamma(\alpha + 1)} \left[ 8^{\rho (\alpha + 1)} + 8^{\rho (\alpha - 1)} (\alpha - 1)(8^\rho - 1) \right].
\]
Then
\[\frac{\rho^\alpha 8^\alpha \Gamma (\alpha + 1)}{h^T(8^\rho - 1)^\alpha} \left[ \frac{8^\alpha (\alpha + 1)}{8^\rho (\alpha + 1) + 8^\rho (\alpha - 1) (8^\rho - 1)} \right] \int_0^T G (s, s) f (s, u (s)) \, ds \geq 1.\]

(3.17)

On the other hand, if \( \max_{0 \leq t \leq T} f (t, u) \) is bounded for \( u \in [0, \infty) \), then there exists \( L_0 > 0 \), such that
\[ |f (t, u (t))| \leq L_0, \quad \forall t \in [0, T].\]

In the similar way, if \( \max_{0 \leq t \leq T} f (t, u) \) is unbounded for \( u \in [0, \infty) \), then there exists \( M_0 > 0 \), such that
\[ \sup_{0 \leq u \leq M_0} \max_{0 \leq t \leq T} |f (t, u (t))| \leq L_1, \quad \text{for some } L_1 > 0.\]

In all cases, for \( L = \max \{ L_0, L_1 \} \), we have:
\[ \int_0^T G (s, s) f (s, u (s)) \, ds \leq L \int_0^T G (s, s) \, ds \leq \frac{LT^\alpha [8^\alpha - (8^\rho - 1)^\alpha]}{8^\rho \rho^\alpha \Gamma (\alpha + 1)}.\]

From (3.17), we get
\[
\begin{align*}
\int_0^T G (s, s) f (s, u (s)) \, ds &= \int_0^T G (s, s) f (s, u (s)) \, ds + \int_0^T G (s, s) f (s, u (s)) \, ds \\
&\leq \int_0^T G (s, s) f (s, u (s)) \, ds + \frac{LT^\alpha [8^\alpha - (8^\rho - 1)^\alpha]}{\rho^\alpha 8^\rho \Gamma (\alpha + 1)} \\
&\leq \int_0^T G (s, s) f (s, u (s)) \, ds \\
&\quad + \frac{LT^\alpha [8^\alpha - (8^\rho - 1)^\alpha]}{\rho^\alpha 8^\rho \Gamma (\alpha + 1)} \times \frac{\rho^\alpha 8^\rho \Gamma (\alpha + 1)}{h^T(8^\rho - 1)^\alpha} \\
&\quad \times \left[ \frac{8^\rho (\alpha + 1) + 8^\rho (\alpha - 1) (8^\rho - 1)}{8^\rho \Gamma (\alpha + 1) + 8^\rho (\alpha - 1) (8^\rho - 1)} \right] \\
&\quad \times \int_0^T G (s, s) f (s, u (s)) \, ds \\
&\leq \lambda \int_0^T G (s, s) f (s, u (s)) \, ds.
\end{align*}
\]

Let us define the cone \( P \) by:
\[ P = \left\{ u \in C [0, T] \mid u (t) \geq \frac{b(t)}{\lambda} \| u \|, \quad \forall t \in [0, T] \right\}. \quad (3.18) \]
Lemma 3.5. Let \( A : P \to C[0,T] \) be an integral operator defined by:

\[
A_u(t) = \beta \int_0^T G(t,s) f(s, u(s)) \, ds,
\]

(3.19)

equipped with standard norm

\[
\|A u\| = \max_{0 \leq t \leq T} |A u(t)|.
\]

Then \( A(P) \subset P \).

**Proof.** For any \( u \in P \), we have from (3.13), (3.16) and (3.18), that

\[
A_u(t) = \beta \int_0^T G(t,s) f(s, u(s)) \, ds \geq \beta b(t) \int_0^T G(s,s) f(s, u(s)) \, ds
\]

\[
\geq \frac{\beta b(t)}{\lambda} \int_0^T G(s,s) f(s, u(s)) \, ds
\]

\[
\geq \frac{b(t)}{\lambda} \max_{0 \leq t \leq T} \left( \beta \int_0^T G(t,s) f(s, u(s)) \, ds \right)
\]

\[
\geq \frac{b(t)}{\lambda} \|A u\|, \forall t \in [0,T].
\]

Thus \( A(P) \subset P \). The proof is complete. \( \square \)

Lemma 3.6. \( A : P \to P \) is a completely continuous operator.

**Proof.** In view of continuity of \( G(t,s) \) and \( f(t,u) \), the operator \( A : P \to P \) is a continuous.

Let \( \Omega \subset P \) be a bounded. Then there exists a positive constant \( M > 0 \), such that:

\[
\|u\| \leq M, \forall u \in \Omega.
\]

By choice

\[
L = \sup_{0 \leq u \leq M} \max_{0 \leq t \leq T} |f(t,u)| + 1.
\]

In this case, we get \( \forall u \in \Omega \),

\[
|A u(t)| = \left| \beta \int_0^T G(t,s) f(s, u(s)) \, ds \right| \leq \beta \int_0^T |G(t,s) f(s, u(s))| \, ds
\]

\[
\leq \beta L \int_0^T G(s,s) \, ds \leq \frac{\beta L}{\rho^{\alpha-1} \Gamma(\alpha)} \int_0^T s^{\rho-1} (T^\rho - s^\rho)^{\alpha-1} \, ds
\]

\[
\leq \frac{\beta L T^{\alpha \rho}}{\rho^{\alpha} \Gamma(\alpha + 1)}.
\]
Thus (a) If $\delta \in \Omega$, where $\Omega$, we divide the proof into three cases.

Then $\forall u \in \Omega$, and $t_1, t_2 \in [0, T]$, where $t_1 < t_2$, and $t_2 - t_1 < \delta$, we find $|Au(t_2) - Au(t_1)| < \varepsilon$.

Consequently, for $0 \leq s \leq t_1 < t_2 \leq T$, we have:

$$G(t_2, s) - G(t_1, s) = \frac{\mu_{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)} \left[ t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)} \right] \left( \frac{T^\rho - s^\rho}{T^\rho} \right)^{\alpha-1}$$

$$< \frac{\mu_{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)} \left[ t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)} \right] \left( \frac{T^\rho - s^\rho}{T^\rho} \right)^{\alpha-1}$$

In the same way, for $0 \leq t_1 \leq s < t_2 \leq T$ or $0 \leq t_1 < t_2 \leq s \leq T$, we have:

$$G(t_2, s) - G(t_1, s) < \frac{\mu_{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)} \left[ t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)} \right].$$

Then

$$|Au(t_2) - Au(t_1)| = \left| \beta \int_0^T [G(t_2, s) - G(t_1, s)] f(s, u(s)) \, ds \right|$$

$$\leq \beta L \int_0^T |G(t_2, s) - G(t_1, s)| \, ds$$

$$< \beta L \int_0^T \frac{\mu_{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)} \left[ t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)} \right] \, ds$$

$$< \frac{\beta L \mu_{1-\alpha}}{\Gamma(\alpha)} \left[ t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)} \right] \left[ \frac{1}{\rho} T^\rho \right].$$

Finally

$$|Au(t_2) - Au(t_1)| < \frac{\beta L T^\rho}{\rho^\alpha \Gamma(\alpha)} \left[ t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)} \right].$$

In the following, we divide the proof into three cases.

(a) If $\delta \leq t_2 - t_1 < T$, we have:

$$\delta \leq t_2 - t_1 \iff t_2^{\rho(\alpha-2)} - t_1^{\rho(\alpha-2)} \leq \delta^{\rho(\alpha-2)},$$

and $t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)}$. Thus

$$t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)} - t_1 t_2^{\rho(\alpha-2)} - t_2 t_1^{\rho(\alpha-2)} = t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)} \leq \delta^{\rho(\alpha-2)}.$$

Thus
In similar way
\[ t_2^{\rho(a-1)} - t_1^{\rho(a-1)} = t_2^\rho \left( t_2^{\rho(a-2)} - t_1^{\rho(a-2)} \right) < t_2^\rho t_2^{\rho(a-2)} - t_1^\rho t_1^{\rho(a-2)} = t_2^\rho (t_2^{\rho(a-2)} - t_1^{\rho(a-2)}) \]
\[ < \delta^{\rho(a-2)} (t_2^\rho - t_1^\rho) < \delta^{\rho(a-1)}. \]

Then, the inequality (3.20) gives:
\[ |Au(t_2) - Au(t_1)| < \frac{\beta LT^\rho}{\rho^a \Gamma(\alpha)} \left[ t_2^{\rho(a-1)} - t_1^{\rho(a-1)} \right] < \frac{\beta LT^\rho}{\rho^a \Gamma(\alpha)} \delta^{\rho(a-1)} \]
\[ < \frac{\beta LT^\rho}{\rho^a \Gamma(\alpha)} \left( \frac{\rho^a \Gamma(\alpha)}{T^p \beta L} \frac{1}{\rho^a - 1} \right) \delta^{\rho(a-1)} \]
\[ < \varepsilon. \tag{3.21} \]

(b) If \( t_1 \leq \delta < t_2 < 2\delta \), we have:
\[ t_1 \leq \delta < t_2 \Leftrightarrow t_2^{\rho(a-2)} < \delta^{\rho(a-2)} \leq t_1^{\rho(a-2)}, \]
and
\[ t_2^{\rho(a-1)} - t_1^{\rho(a-1)} = t_2^\rho \left( t_2^{\rho(a-2)} - t_1^{\rho(a-2)} \right) < t_2^\rho \delta^{\rho(a-2)} - t_1^\rho \delta^{\rho(a-2)} \]
\[ < \delta^{\rho(a-2)} (t_2^\rho - t_1^\rho) < \delta^{\rho(a-1)}. \]
Also, we find the same result (3.21).

(c) If \( t_1 < t_2 \leq \delta \), we have:
\[ |Au(t_2) - Au(t_1)| < \frac{\beta LT^\rho}{\rho^a \Gamma(\alpha)} \left[ t_2^{\rho(a-1)} - t_1^{\rho(a-1)} \right] < \frac{\beta LT^\rho}{\rho^a \Gamma(\alpha)} \delta^{\rho(a-1)} \]
\[ < \varepsilon. \]

By the means of the Ascoli-Arzel Theorem 2.11, we have \( A : P \to P \) is completely continuous. \( \square \)

We define some important constants
\[ F_0 = \lim_{u \to 0^+} \max_{t \in [0, T]} f(t, u), \quad F_\infty = \lim_{u \to +\infty} \max_{t \in [0, T]} f(t, u), \]
\[ f_0 = \lim_{u \to 0^+} \min_{t \in [0, T]} f(t, u), \quad f_\infty = \lim_{u \to +\infty} \min_{t \in [0, T]} f(t, u), \]
\[ \omega_1 = \int_0^T G(s, s) \, ds, \quad \omega_2 = \frac{b}{\kappa} \int_0^T G(s, s) \, b(s) \, ds. \]
Assume that \( \frac{1}{\omega_1 F_0} = 0 \) if \( f_\infty \to \infty \), \( \frac{1}{\omega_1 F_0} = \infty \) if \( F_0 \to 0 \), \( \frac{1}{\omega_2 f_0} = 0 \) if \( f_0 \to \infty \), and \( \frac{1}{\omega_1 F_\infty} = \infty \) if \( F_\infty \to 0 \).
Theorem 3.7. If $\omega_2 f_\infty > \omega_1 F_0$ holds, then for each:

$$\beta \in \left( (\omega_2 f_\infty)^{-1}, (\omega_1 F_0)^{-1} \right),$$

the boundary value problem (1.1)-(1.2) has at least one positive solution.

Proof. Let $\beta$ satisfies (3.22) and $\varepsilon > 0$, be such that

$$\left( (f_\infty - \varepsilon) \omega_2 \right)^{-1} \leq \beta \leq \left( (F_0 + \varepsilon) \omega_1 \right)^{-1}.$$  

From the definition of $F_0$, we see that there exists $r_1 > 0$, such that

$$f(t, u) \leq (F_0 + \varepsilon) u, \ \forall t \in [0, T], \ 0 < u \leq r_1.$$  

Consequently, for $u \in P$ with $\|u\| = r_1$, we have from (3.23) , (3.24), that

$$\|A u\| \geq \beta \int_0^T G(t, s) f(s, u(s)) ds \geq \beta \int_0^T (F_0 + \varepsilon) u(s) ds \geq \beta (F_0 + \varepsilon) \|u\| \omega_1 \geq \|u\|.$$  

Therefore, for $u \in P$ with $\|u\| = r_2 = \max \{2r_1, r_3\}$, we have from (3.23), (3.26), that

$$\|A u\| \geq \beta \int_0^T G(t, s) f(s, u(s)) ds \geq \beta \int_0^T \bar{b}(\bar{t}) G(s, s) f(s, u(s)) ds \geq \beta \int_0^T \bar{b}(\bar{t}) G(s, s) [(f_\infty - \varepsilon) u(s)] ds, \ \forall \bar{t} \in [0, T].$$  

By definition of $P$ in (3.18), we have:

$$\|A u\| \geq \frac{\beta \bar{b}(f_\infty - \varepsilon)}{\lambda^2} \|u\| \int_0^T G(s, s) b(s) ds \geq \beta (f_\infty - \varepsilon) \|u\| \omega_2 \geq \|u\|.$$
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If we set \( \Omega_2 = \{ u \in C[0,T] : \|u\| < r_2 \} \), then

\[ \|Au\| \geq \|u\|, \text{ for } u \in P \cap \partial \Omega_2. \] (3.27)

Now, from (3.25), (3.27), and Lemma 2.13, we guarantee that \( A \) has a fix point \( u \in P \cap (\Omega_2 \setminus \Omega_1) \) with \( r_1 \leq \|u\| \leq r_2 \). It is clear that \( u \) is a positive solution of (1.1)-(1.2). The proof is complete. \( \square \)

**Theorem 3.8.** If \( \omega_2 f_0 > \omega_1 F_\infty \) holds, then for each:

\[ \beta \in \left( (\omega_2 f_0)^{-1}, (\omega_1 F_\infty)^{-1} \right), \] (3.28)

the boundary value problem (1.1)-(1.2) has at least one positive solution.

**Proof.** Let \( \beta \) satisfies (3.28) and \( \varepsilon > 0 \), be such that

\[ ((f_0 - \varepsilon) \omega_2)^{-1} \leq \beta \leq ((F_\infty + \varepsilon) \omega_1)^{-1}. \] (3.29)

From definition of \( f_0 \), we see that there exists \( r_1 > 0 \), such that

\[ f(t,u) \geq (f_0 - \varepsilon) u, \forall t \in [0,T], 0 < u \leq r_1. \]

Further, if \( u \in P \) with \( \|u\| = r_1 \), then similar to the proof’s second part of Theorem 3.7, we can get that \( \|Au\| \geq \|u\| \). Then, if we choose \( \Omega_1 = \{ u \in C[0,T] : \|u\| < r_1 \} \), thus

\[ \|Au\| \geq \|u\|, \text{ for } u \in P \cap \partial \Omega_1. \] (3.30)

Next, and by definition of \( F_\infty \), we may choose \( R_1 > 0 \), such that

\[ f(t,u) \leq (F_\infty + \varepsilon) u, \text{ for } u \geq R_1. \] (3.31)

We consider two cases:

1) If \( \max_{0 \leq t \leq T} f(t,u) \) is bounded for \( u \in [0,\infty) \). Then, there exists some \( L > 0 \), such that

\[ f(t,u) \leq L, \text{ for all } t \in [0,T], u \in P. \]

Let us denote by \( r_3 = \max \{2r_1, \beta L \omega_1\} \), if \( u \in P \) with \( \|u\| = r_3 \), then

\[ \|Au\| = \max_{0 \leq t \leq T} \left| \beta \int_0^T G(t,s) f(s,u(s)) ds \right| \leq \beta L \int_0^T G(s,s) ds = \beta L \omega_1 \leq r_3 = \|u\|. \]

Hence,

\[ \|Au\| \leq \|u\|, \text{ for } u \in \partial P_{r_3} = \{ u \in P : \|u\| \leq r_3 \}. \] (3.32)

2) If \( \max_{0 \leq t \leq T} f(t,u) \) is unbounded for \( u \in [0,\infty) \), then there exists some \( r_4 = \max \{2r_1, R_1\} \), such that

\[ f(t,u) \leq \max_{0 \leq t \leq T} f(t,r_4), \text{ for all } 0 < u \leq r_4, t \in [0,T]. \]
Let $u \in P$ with $\|u\| = r_4$. Then, from \eqref{eq:3.30}, \eqref{eq:3.31}, we have:

$$
\|Au\| = \max_{0 < t < T} \left| \beta \int_0^T G(t, s) f(s, u(s)) \, ds \right| \leq \beta \int_0^T G(s, s) (F_\infty + \varepsilon) u(s) \, ds \\
\leq \beta (F_\infty + \varepsilon) \|u\| \int_0^T G(s, s) \, ds = \beta (F_\infty + \varepsilon) \|u\| \omega_1 \\
\leq \|u\|.
$$

Thus, \eqref{eq:3.32} is also true for $u \in \partial P_{r_4}$.

In both cases 1 and 2, if we set $\Omega_2 = \{u \in C[0, T] : \|u\| < r_2 = \max \{r_3, r_4\}\}$, then

$$
\|Au\| \leq \|u\|, \text{ for } u \in P \cap \partial \Omega_2. \tag{3.33}
$$

Now, from \eqref{eq:3.30}, \eqref{eq:3.33}, and Lemma 2.13, we guarantee that $A$ has a fix point $u \in P \cap (\Omega_2 \setminus \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$. It is clear that $u$ is a positive solution of \eqref{eq:1.1}-(\ref{eq:1.2}). The proof is complete. \hfill \square

**Theorem 3.9.** Suppose there exists $r_2 > r_1 > 0$, such that

$$
\sup_{0 \leq u \leq r_2} \max_{0 \leq t \leq T} f(t, u) \leq \frac{r_2}{\beta \omega_1}, \text{ and } \inf_{0 \leq u \leq r_1} f(t, u) \geq \frac{r_1}{\beta \lambda \omega_2} b(t), \forall t \in [0, T]. \tag{3.34}
$$

Then, the boundary value problem \eqref{eq:1.1}-(\ref{eq:1.2}) has a positive solution $u \in P$, with $r_1 \leq \|u\| \leq r_2$.

**Proof.** Choose $\Omega_1 = \{u \in C[0, T] : \|u\| < r_1\}$. Then, for $u \in P \cap \partial \Omega_1$, we get

$$
\|Au\| \geq \beta \int_0^T G(t, s) f(s, u(s)) \, ds \geq \beta \int_0^T b(t) G(s, s) f(s, u(s)) \, ds \\
\geq \frac{\beta b}{\lambda} \int_0^T G(s, s) \sup_{0 \leq u \leq r_1} f(s, u(s)) \, ds \geq \frac{\beta b}{\lambda} \int_0^T G(s, s) \frac{r_1}{\beta \lambda \omega_2} b(s) \, ds \\
\geq r_1 = \|u\|.
$$

On the other hand, choose $\Omega_2 = \{u \in C[0, T] : \|u\| < r_2\}$. Then, for $u \in P \cap \partial \Omega_2$, we get

$$
\|Au\| = \max_{0 < t < T} \left| \beta \int_0^T G(t, s) f(s, u(s)) \, ds \right| \leq \beta \int_0^T G(s, s) \sup_{0 \leq u \leq r_2} \max_{0 \leq t \leq T} f(s, u(s)) \, ds \\
\leq \beta \int_0^T G(s, s) \frac{r_2}{\beta \omega_1} \, ds = r_2 = \|u\|.
$$

Now, from Lemma 2.13, we guarantee that $A$ has a fix point $u \in P \cap (\Omega_2 \setminus \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$. It is clear that $u$ is a positive solution of \eqref{eq:1.1}-(\ref{eq:1.2}). The proof is complete. \hfill \square
3.2. Application of Banach fixed point theorem

In this part, we assume that \( \beta \in \mathbb{R} \) and \( \rho > 0 \), and \( f : [0, T] \times [0, \infty) \to [0, \infty) \) satisfies the conditions:

(H1) \( f(t, u) \) is Lebesgue measurable function with respect to \( t \) on \([0, T]\),

(H2) \( f(t, u) \) is continuous function with respect to \( u \) on \( \mathbb{R} \).

**Theorem 3.10.** Assume (H1), (H2) hold, and there exists a constant \( \sigma > 0 \), such that

\[
|f(t, u) - f(t, v)| \leq \sigma |u - v|, \text{ for almost every } t \in [0, T], \text{ and all } u, v \in C[0, T].
\]

(3.35)

If

\[
|\beta| < \frac{\rho^\alpha \Gamma(\alpha + 1)}{\sigma T^\alpha \rho}.
\]

(3.36)

Then, there exists a unique solution of the boundary value problem (1.1)-(1.2) on \([0, T]\).

**Proof.** Assume that \( |\beta| < \frac{\rho^\alpha \Gamma(\alpha + 1)}{\sigma T^\alpha \rho} \), and consider the operator \( A : C[0, T] \to C[0, T] \) defined by (3.19) as follows

\[
Au(t) = \beta \int_0^T G(t, s) f(s, u(s)) \, ds.
\]

We shall show that \( A \) is a contraction mapping. In fact, for any \( u, v \in C[0, T] \), we have

\[
|Au(t) - Av(t)| = \left| \beta \int_0^T G(t, s) [f(s, u(s)) - f(s, v(s))] \, ds \right|
\leq |\beta| \sigma \int_0^T G(s, s) |u(s) - v(s)| \, ds,
\]

then

\[
\|Au - Av\| \leq |\beta| \sigma \|u - v\| \int_0^T G(s, s) \, ds
\leq \frac{|\beta| \sigma T^\alpha \rho}{\rho^\alpha \Gamma(\alpha + 1)} \|u - v\|.
\]

(3.37)

This imply from (3.37) that \( A \) is a contraction operator. As a consequence of Theorem 2.14, by Banach’s contraction principle [5], we deduce that \( A \) has a unique fixed point which is the unique solution of the problem (1.1)-(1.2) on \([0, T]\). \( \square \)
4. Examples

In this section, we present some examples to illustrate the usefulness of our main results.

Example 1. Consider the following boundary value problem

\[
\begin{aligned}
\{ & 1D_0^\frac{3}{2} u(t) + \beta (1 + t) u(t) \ln (1 + u(t)) = 0, \quad t \in [0, 1], \\
& u(0) = u(1) = 0.
\end{aligned}
\] (4.1)

Set \( \beta > 0 \) any finite positive real number, and

\[ f(t, u) = (1 + t) u \ln (1 + u). \]

In this case, the function \( f \) is jointly continuous for any \( t \in [0, 1] \), and any \( u > 0 \).

We get

\[
\begin{aligned}
F_0 &= \lim_{u \to 0^+} \max_{t \in [0,T]} \frac{f(t,u)}{u} = 0^+, \\
\infty &= \lim_{u \to +\infty} \min_{t \in [0,T]} \frac{f(t,u)}{u} = \infty.
\end{aligned}
\]

On the other hand, we get

\[
\omega_1 = \int_0^1 G(s,s) ds = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^1 \sqrt{s(1-s)} ds = \frac{1}{\frac{3}{2} \sqrt{\pi}} = \frac{\sqrt{\pi}}{4},
\]

and

\[
b(t) = \begin{cases} 
\sqrt{t} & \text{for } t \in [0, \bar{t}], \\
\frac{1}{16} t & \text{for } t \in [\bar{t}, 1].
\end{cases}
\]

Then

\[
\omega_2 = \frac{\bar{b}}{\lambda^2 \Gamma\left(\frac{3}{2}\right)} \left[ \int_0^\bar{t} \sqrt{1-s} ds + \frac{1}{16} \int_{\bar{t}}^1 \sqrt{s(1-s)^\frac{3}{2}} ds \right] \simeq \frac{\bar{b} \sqrt{\pi}}{128 \lambda^2}.
\]

Where \( \bar{t} \approx 0, 003876 \ldots \) and \( \bar{b} \approx 0, 062258 \ldots \) and the choice of \( \lambda \) depends directly by choice of \( r_1, r_2 \) in (3.25), (3.27).

Because \( \omega_1, \omega_2 > 0 \), two finite constants for any choice of \( 0 < r_1 < r_2 < \infty \). We have always:

\[
\frac{1}{\omega_2 f} = 0, \quad \text{and} \quad \frac{1}{\omega_1 F_0} = \infty.
\]

Then, the condition (3.22) is satisfied for any \( 0 < \beta < \infty \).

It follows from Theorem 3.7 that the problem (4.1) has at least one solution.

Example 2. Consider

\[
\begin{aligned}
\{ & 1D_0^\frac{3}{2} u(t) + \beta (1 + t) u(t) \exp\left(\frac{1}{u(t)} - [u(t)]^2\right) = 0, \quad t \in [0, 1], \\
u(0) = u(1) = 0.
\end{aligned}
\] (4.5)
Existence of Solutions for Katugampola Fractional Differential Equations

Set \( \beta > 0 \) any finite positive real number, and

\[
f(t, u) = (1 + t) u \exp \left( \frac{1}{u} - u^2 \right).
\]

Clearly, for any \( t \in [0, 1] \) and any \( u > 0 \), the function \( f \) is jointly continuous.

Here, we have:

\[
f_0 = \lim_{u \to 0^+} \min_{t \in [0, T]} \frac{f(t, u)}{u} = \infty, \quad F_\infty = \lim_{u \to +\infty} \max_{t \in [0, T]} \frac{f(t, u)}{u} = 0^+.
\]

Also, we find the same function \( b(t) \) in (4.3), and same constant \( \omega_1, \omega_2 \) respectively in (4.2), (4.4).

The choice of \( \lambda > 1 \) depends directly by choice of \( r_1, r_2 \) in (3.30), (3.33).

Because \( \omega_1, \omega_2 > 0 \), two finite constants for any choice of \( 0 < r_1 < r_2 < \infty \). We have always:

\[
\frac{1}{\omega_2 f_0} = 0, \quad \text{and} \quad \frac{1}{\omega_1 F_\infty} = \infty.
\]

Then, the condition (3.28) is satisfied for any \( 0 < \beta < \infty \).

It follows from Theorem 3.8 that the problem (4.5) has at least one solution.

**Example 3.** Consider the following boundary value problem

\[
\begin{cases}
1D^{\frac{3}{2}}_{0^+} u(t) + \frac{(1+t)(1+u(t))}{\sqrt{\pi}} = 0, & t \in [0, 1]. \\
u(0) = u(1) = 0.
\end{cases}
\]

Set \( \beta = \frac{1}{\sqrt{\pi}} \), and

\[
f(t, u) = (1 + t) (1 + u).
\]

The function \( f \) is jointly continuous for any \( t \in [0, 1] \) and any \( u > 0 \).

We find the same function \( b(t) \) in (4.3), such that \( 0 \leq b(t) < 1 \), and

\[
\omega_1 = \int_0^1 G(s,s) \, ds = \frac{\sqrt{\pi}}{4}.
\]

Choosing \( r_1 = \frac{1}{10} < r_2 = 2 \). Then, for all \( t \in [0, 1] \), we have:

\[
h = 1 \leq f(t, u) \leq 6 = L.
\]

In this case

\[
\lambda = 1 + \frac{8^\alpha L (\alpha + 1) [8^\alpha - (8^\rho - 1)^\alpha]}{h (8^\rho - 1)^\alpha [8^\rho (\alpha + 1) + 8^\rho (\alpha - 1) (\alpha - 1) (8^\rho - 1)]}
\]

\[
= 1 + \frac{8^\frac{3}{2} \times 6 \times \frac{5}{2} \times \left( 8^\frac{3}{2} - 7^\frac{3}{2} \right)}{7^\frac{3}{2} \times (8 \times \frac{5}{2} + \sqrt{8 \times \frac{5}{2}})}
\]

\[
= 3, 517426 \ldots
\]
Then
\[
\omega_2 \simeq \frac{b\sqrt{\pi}}{128\lambda^2} \simeq \frac{0.062258 \times \sqrt{\pi}}{128 \times 3,517426^2} \simeq \frac{3.9313\sqrt{\pi}}{10^5}.
\]

It remains to show that the conditions in (3.34), which is
\[
\sup_{0 \leq u \leq r_2} \max_{0 \leq t \leq T} f(t, u) = 6 \leq \frac{r_2}{\beta \omega_1} \simeq 8,
\]
and
\[
\inf_{0 \leq u \leq r_1} f_3(t, u) = 1 + t \geq \frac{r_1}{\beta \lambda \omega_2} b(t) \simeq 0.72317 \times b(t), \forall t \in [0, 1].
\]
Are satisfied. It follows from Theorem 3.9 that the problem (4.6) has at least one solution.

**Example 4.** Let
\[
\begin{aligned}
\frac{4}{3} D_{0^+}^{3} u(t) + \frac{\cos(t)[2+|u(t)|]}{\sqrt{2}\cos(t)+\sin(t)[1+|u(t)|]} &= 0, \quad t \in [0, \frac{\pi}{4}], \\
u(0) &= u\left(\frac{\pi}{4}\right) = 0.
\end{aligned}
\]

Set \(\beta = \frac{1}{\pi}\) and
\[
f(t, u) = \frac{\cos(t)[2+|u|]}{\sqrt{2}\cos(t)+\sin(t)[1+|u|]}, \quad t \in \left[0, \frac{\pi}{4}\right], \quad u, v \in \mathbb{R}.
\]

As \(\sin(t), \cos(t)\) are continuous positive functions \(\forall t \in \left[0, \frac{\pi}{4}\right]\), the function \(f\) is jointly continuous. For any \(u, v \in \mathbb{R}\) and \(t \in \left[0, \frac{\pi}{4}\right]\), we have \(\sqrt{2} \leq \cos(t) \leq 1\), and \(0 \leq \sin(t) \leq \frac{\sqrt{2}}{2}\), then
\[
|f(t, u) - f(t, v)| = \left| \frac{\cos(t)[2+|u|]}{\sqrt{2}\cos(t)+\sin(t)[1+|u|]} - \frac{\cos(t)[2+|v|]}{\sqrt{2}\cos(t)+\sin(t)[1+|v|]} \right|
\]
\[
= \left| \frac{\cos(t)}{\sqrt{2}\cos(t)+\sin(t)} \left( \frac{2 + |u|}{2} - \frac{2 + |v|}{2} \right) \right|
\]
\[
\leq \left| |u| - |v| \right| \leq |u - v|.
\]

Hence, the condition (3.35) is satisfied with \(\sigma = 1\). It remains to show that the condition (3.36)
\[
0 < \beta = \frac{1}{\pi} \simeq 0.318309 \ldots < \frac{\rho^\alpha \Gamma(\alpha + 1)}{\sigma T^{\alpha \rho}} = \frac{2\frac{\pi}{3} \times \Gamma\left(\frac{5}{2}\right)}{\frac{\pi}{4}} \simeq 0.921317 \ldots
\]
is satisfied. It follows from Theorem 3.10 that the problem (4.7) has a unique solution.
5. Conclusion

In this paper we have discussed the existence and the uniqueness of solutions for a class of nonlinear fractional differential equations with a boundary value, by using the properties of Guo-Krasnosel’skii and Banach fixed point theorems. The used differential operator is developed by Katugampola, which generalizes the Riemann-Liouville and the Hadamard fractional derivatives into a single form.

Acknowledgments

The authors are deeply grateful to the referees and the editors for their kind comments on improving the presentation of this paper.

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DOI: 10.7862/rf.2019.3

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Received 03.08.2018 Accepted 29.12.2018