Three-dimensional quasi-Tonks gas in a harmonic trap

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We analyze the macroscopic dynamics of a Bose gas in a harmonic trap with a superimposed two-dimensional optical lattice, assuming a weak coupling between different lattice sites. We consider the situation in which the local chemical potential at each lattice site can be considered as that provided by the Lieb-Liniger solution. Due to the weak coupling between sites and the form of the chemical potential, the three-dimensional ground-state density profile and the excitation spectrum acquire remarkable properties different from both 1D and 3D gases. We call this system a quasi-Tonks gas. We discuss the range of applicability of this regime, as well as realistic experimental situations where it can be observed.

Recent developments in low-dimensional trapping \textsuperscript{1}, loading of optical lattices \textsuperscript{2} \textsuperscript{3}, and external modification of the interparticle interactions \textsuperscript{4}, open fascinating perspectives towards the achievement of strongly-interacting systems in cold atomic gases. In this sense, a remarkable example is provided by the recent observation of the superfluid to Mott-Insulator (MI) transition in cold bosonic gases in optical lattices \textsuperscript{5}. Other strongly-interacting systems in cold atomic gases have been recently considered theoretically, as the case of large scattering length \textsuperscript{6}, rapidly rotating Bose gases \textsuperscript{7}, and one-dimensional systems in cold atomic gases have been recently considered experimentally \textsuperscript{8}, \textsuperscript{9}, \textsuperscript{10}, \textsuperscript{11}, \textsuperscript{12}, \textsuperscript{13}, \textsuperscript{14}.

One-dimensional systems differ significantly when compared to 3D ones. Most remarkably, in 1D gases the interactions become relatively more important the more dilute the gas is. In particular, for very low densities or a large value of the s-wave scattering length $a$, the system enters the so-called Tonks-Girardeau (TG) regime, in which the bosons acquire fermionic properties \textsuperscript{10}, \textsuperscript{11}. For finite interactions, the problem of delta-interacting 1D homogeneous Bose gases was exactly solved by Lieb and Liniger (LL) by means of the Bethe Ansatz \textsuperscript{12}. A combination of the LL solution, the local density approximation, and hydrodynamic equations have been recently employed in the analysis of different properties of trapped 1D Bose gases, as the ground-state density \textsuperscript{13}, the excitation spectrum \textsuperscript{14}, the non self-similar expansion in a 1D guide \textsuperscript{15}, and the correlation and coherence properties \textsuperscript{16}. The latter has also been recently analyzed by means of quantum Monte Carlo methods \textsuperscript{17}.

The accomplishment of strongly-interacting 1D gases requires tight transversal trapping, low atom numbers, and possibly the increase of $a$ via Feshbach resonances \textsuperscript{18} \textsuperscript{16} \textsuperscript{15} \textsuperscript{17}. In this sense, 2D optical lattices are specially favorable, since the on-site transversal confinement can be very strong. For sufficiently intense lattices, the tunneling rate could be made small enough to ensure that each lattice site behaves as an independent 1D experiment. This paper analyzes the situation in which the tunneling is non negligible, but sufficiently weak to assume the 1D character of the local chemical potential. In this regime, which we call the quasi-Tonks regime, we predict that the combination of tunneling and 1D local chemical potential results in important modifications of the 3D ground-state density and the excitations.

In the following we consider a gas of $N$ bosons in a cylindrically-symmetric harmonic trap, with axial (radial) frequency $\omega_z$ ($\omega_z$). We assume a superimposed periodic potential provided by an optical lattice of the form $V_i(x) + V_j(y) = V_0(\sin(qx/\lambda) + \sin(qy/\lambda))$, where $q = 2\pi/\lambda$ is related to the laser wavelength $\lambda$, and $V_0$ is the lattice amplitude. The lattice has a periodicity $d = \pi/q = \lambda/2$. From a macroscopic point of view it is convenient to introduce the effective mass, $m^*$, to describe the effects of the lattice on the dynamics. The value of $m^*$ can be obtained from $\partial^2 E_0(\partial k)^2 = \hbar^2/m^*$, where $E_0(k)$ is the dispersion law corresponding to the lowest energy band. In tight-binding regime, $m^*$ can be related with the tunneling rate $J$, as $m/m^* = \pi^2 J/E_R$, where $E_R = \hbar^2 q^2/2m$ is the recoil energy, $J = -\int w(x)(-\hbar^2 \nabla^2 z_j/2m + V_i(x))w_i+1(x)dx$, and $\{w_i\}$ are the Wannier functions for the lowest band.

We first consider the case of a single isolated lattice site, which can be well approximated as a dilute gas of $N_1$ bosons confined in a very elongated harmonic trap with radial and axial frequencies $\omega_\rho$ and $\omega_z$ ($\omega_\rho \gg \omega_z$). The transversal confinement is strong enough to fulfill that the interaction energy per particle is smaller than the zero-point energy $\hbar \omega_\rho$ of the transversal trap. In this way, the transversal dynamics is effectively “frozen” and the system can be considered 1D. The Hamiltonian describing the physics of the delta-interacting 1D gas is

$$\hat{H}_{1D} = \hat{H}_{1D}^0 + \sum_{j=1}^{N_1} \frac{m_\omega z_j^2}{2} + g_{1D} \sum_{i=1}^{N_1-1} \sum_{j=i+1}^{N_1} \delta(z_i - z_j)$$

where

$$\hat{H}_{1D}^0 = \frac{\hbar^2}{2m} \sum_{j=1}^{N_1} \frac{\partial^2}{\partial z_j^2} + g_{1D} \sum_{i=1}^{N_1-1} \sum_{j=i+1}^{N_1} \delta(z_i - z_j)$$

is the Hamiltonian in absence of the harmonic trap, and $g_{1D} = -2\hbar^2/m_1$. The scattering problem under one-dimensional constraints was analyzed in detail by Olshanii \textsuperscript{16}, and it is characterized by the 1D scattering...
length $a_{1D} = (-a_p^2/2a)[1 - C(a/a_p)]$, with $a$ the 3D scattering length, $a_p = \sqrt{2\hbar/m\omega_p}$ the oscillator length in the radial direction, and $C = 1.4603 \ldots$. For the thermodynamic limit, a 1D gas at zero temperature with linear density $n$, is characterized by an energy per particle, which can be obtained from the corresponding LL integral equations \[12\]. Assuming that the density varies sufficiently slowly, at each point $z$ the gas can be considered in local equilibrium, and the local energy per particle is provided by the LL equations for the density $n(z)$. We call this approach Local Lieb-Liniger (LLL) model.

Although, strictly speaking, the LLL approach is only valid for 1D systems, for sufficiently low $J$, the virtual processes involving transitions into nearest sites, should just result in a small correction to the 1D scattering of the order of $J/\mu$, where $\mu$ is the chemical potential. Therefore, if $J/\mu \ll 1$, the actual local chemical potential can be well approximated by that obtained from the LLL approach. If the tunneling becomes very small, even for large values of $N_1$ the system could enter into the MI regime. However, this regime typically demands an extremely small value of $J/\mu$. Summarizing, in what we called the quasi-Tonks regime, $1/N_1 \lesssim J/\mu \ll 1$, the LLL chemical potential can be employed, and at the same time the tunneling cannot be neglected.

Assuming a sufficiently slow variation of the gas density, the dynamics can be well described by means of the corresponding macroscopic hydrodynamic equations \[18\]:

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0 \tag{3}
\]

\[m^* \frac{\partial v_x}{\partial t} + \frac{\partial}{\partial x}(K + V + U) = 0 \tag{4}
\]

\[m^* \frac{\partial v_y}{\partial t} + \frac{\partial}{\partial y}(K + V + U) = 0 \tag{5}
\]

\[m^* \frac{\partial v_z}{\partial t} + \frac{\partial}{\partial z}(K + V + U) = 0 \tag{6}
\]

where

\[K = \frac{m^*}{2} v_x^2 + \frac{m^*}{2} v_y^2 + \frac{m}{2} v_z^2 \tag{7}
\]

\[V = \frac{m}{2} \omega_1^2 (x^2 + y^2) + \frac{m}{2} \omega_z^2 z^2 \tag{8}
\]

\[U = \mu(n) = \mu_{1D}(d^2 n) \tag{9}
\]

where $n(x, y, z)$ is the macroscopic 3D density and $\mathbf{v} = (v_x, v_y, v_z)$ is the velocity field. Note that all the information about the lattice is contained in $m^*$, and in the local chemical potential $\mu(n)$. The use of the hydrodynamic equations is justified by the superfluidity of the quasi-1D tubes \[19\] and by the avoidance of the MI phase. In the equation \[9\] the local chemical potential of the 3D system is related to the 1D chemical potential, $\mu_{1D}$, provided by the LLL approach. In order to obtain $\mu_{1D}$ the value of $a_{1D}$ must be obtained for each site. The latter demands the knowledge of the effective oscillator length $a_p$ associated with the on-site radial confinement. Since the latter can be very large, a Gaussian Ansatz can be assumed for the on-site radial wave function. In this way, we can calculate $a_p$ by minimizing the radial energy

\[
\frac{E(a_p)}{E_R} = \frac{1}{q^2 a_p^2} + \frac{V_0}{E_R} \left(1 - e^{-q^2 a_p^2}\right). \tag{10}
\]

In Eq. \[10\] we have neglected the interaction energy, which for $V_0/E_R > 3$ does not provide any significant contribution. Once known the value of $a_{1D}$, we evaluate $\mu_{1D}(n)$ by numerically solving the corresponding LL integral equations.

Figure 1: Density profile as a function of the dimensionless radius $\tilde{r}^2 = \sum_i (r_i/R_i)^2$ (see text). The solid, dotted and dashed lines correspond to $\log A = -3, 0, 3$, respectively. The resulting fit with a power law dependence $(1 - \tilde{r}^2)^s$ provides $s = 0.54, 0.72, 0.91$, respectively, being an indicator of the transition from the MF regime to the strongly-interacting one.

By imposing the equilibrium conditions $\partial n/\partial t = v_i = 0$, we obtain the equation of state

\[\mu[n_0(x, y, z)] = \mu_T - V(x, y, z), \tag{11}\]

where $n_0$ is the equilibrium density, and $\mu_T$ is the chemical potential of the system. Inverting \[11\], one obtains the expression for $n_0$. Imposing the normalization to the number of particles $N$ we obtain:

\[A = \frac{|a_{1D}|^4 d^2 N}{2 a_1^2 a_0^4} = 4\pi \xi^3 \int_0^1 t^2 \tilde{\mu}_{1D}(t) d^3 \xi = A(t^2) dt. \tag{12}\]

where $\xi = 2\mu m|a_{1D}|^2/\hbar^2$. From Eq. \[12\] we observe that, similarly to the strict 1D case \[6\], \[9\], \[10\], the ground state of the system is completely characterized by a single parameter $A$. For $A > 1$ the mean-field (MF) regime is retrieved, whereas the TG is found for $A \ll 1$. In Eq. \[12\], $\tilde{\mu}_{1D}(x) = 2m|a_{1D}|^2 \mu_{1D}/\hbar^2$, is a function of the dimensionless density $\eta = |a_{1D}|^2 n$. Fig. \[a\] shows the density as a function of the dimensionless radius $\tilde{r}$, where $\tilde{r}^2 = \sum_i (r_i/R_i)^2$, with $R_i$ the Thomas-Fermi radii.
In order to compare the different cases, we re-scale the density with respect to the central one. Nevertheless, we stress that the values of $R_i$ and the central density depend on the parameters, and in particular on the interaction regime.

In the following we calculate the frequencies for the lowest excitations of a quasi-Tonks gas. In particular, we evaluate the breathing and quadrupole modes, by using the Ansatz $n = n_0 (r_i/b_i) / \prod_{j} b_j$, $v_i = (b_i/b_i)r_i$, which constitutes an exact solution of the continuity equation \cite{BEC}. Multiplying the Euler equations by $n x_i$ and integrating, one obtains

$$
\dot{b}_x + \ddot{\omega}_1^2 b_x + \ddot{\omega}_2^2 F( \prod_{j=x,y,z} b_j) = 0 \quad (13)
$$

$$
\dot{b}_y + \ddot{\omega}_1^2 b_y + \ddot{\omega}_2^2 F( \prod_{j=x,y,z} b_j) = 0 \quad (14)
$$

$$
\dot{b}_z + \ddot{\omega}_1^2 b_z + \ddot{\omega}_2^2 b_z F( \prod_{j=x,y,z} b_j) = 0, \quad (15)
$$

where we have employed the effective frequencies $\ddot{\omega}_1 = \ddot{\omega}_2^2 m/m^*$ and defined the function

$$
F(\rho) = \frac{1}{m \ddot{\omega}_1^2 N(x_i^2)_0} \int n_0 x_i \rho \frac{\partial}{\partial x_i} \mu (\rho^{-1} n_0) d^3 r. \quad (16)
$$

Using the previously calculated equilibrium solution, one obtains that $F(\rho)$ is independent of the particular choice of the coordinate $i$, and that $F(1) = -1$. Note that the function $F$ only depends on $A$, which, as previously discussed, completely characterizes the static properties of the system. Linearizing around the equilibrium solution $b_i = 1$, one obtains the frequencies of the breathing mode ($\ddot{\omega}_B$), the $m = 0$ quadrupole mode ($\ddot{\omega}_{Q1}$) and the $m = 2$ quadrupole one ($\ddot{\omega}_{Q2}$):

$$
\ddot{\omega}_B = \frac{1}{2} \left( \ddot{\omega}_1^2 (2 + 2f) + \ddot{\omega}_2^2 (2 + f) + \sqrt{(\ddot{\omega}_1^2 (2 + 2f) + \ddot{\omega}_2^2 (2 + f))^2 - 8 \ddot{\omega}_1^2 \ddot{\omega}_2^2 (2 + 3f)} \right) \quad (17)
$$

$$
\ddot{\omega}_{Q1} = \frac{1}{2} \left( \ddot{\omega}_1^2 (2 + 2f) + \ddot{\omega}_2^2 (2 + f) - \sqrt{(\ddot{\omega}_1^2 (2 + 2f) + \ddot{\omega}_2^2 (2 + f))^2 - 8 \ddot{\omega}_1^2 \ddot{\omega}_2^2 (2 + 3f)} \right) \quad (18)
$$

$$
\ddot{\omega}_{Q2} = 2 \ddot{\omega}_1^2 \quad (19)
$$

In the expression for the frequencies we employ the parameter $f = F'(1)$, which is an universal function of the parameter $A$. We show this dependence in Fig. 2. The parameter $f$ ranges from 2, at the TG limit, to 1, for the MF regime. In this two limiting cases the function $F$ can be obtained analytically, being $F = \rho^{-2}$ (TG) and $F = \rho^{-1}$ (MF).

The quasi Tonks regime is not only achievable for realistic conditions, but, actually, it is expected to be the case for typical parameters in ongoing experiments, as those of Fig. 3, which shows the dependence of the frequencies \cite{BEC, BEC} on the lattice amplitude $V_0/E_R$. We stress that the system enters the MI regime for large values of $V_0/E_R$ (around 40 for the case of Fig. 3). If this is the case, as discussed above, our macroscopic hydrodynamic approach fails, and the system becomes a set of independent 1D gases. On the other hand, for decreasing values of $V_0/E_R$ the system abandons the quasi-Tonks regime (which for the case of Fig. 3 occurs at $V_0/E_R \simeq 15$), and the condition $J/\mu \approx 1$ is reached. This constitutes an additional intermediate cross-dimensional regime, in which the gas is 3D, but the local chemical potential is not that expected for a 3D Bose gas. For this regime, contrary to the case of the quasi-Tonks gas, the local chemical poten-
In particular, we have shown that both the ground-state current typical experimental conditions. In this cross- 

that this regime should be the one encountered under 

tical lattice, can present significant tunneling, and at the 

quasi-Tonks regime, in which a gas confined in a 2D op- 

[10] 

ily revealed in experiments, by either observing the size 

pled Luttinger liquids [20]. 

We thank M. Lewenstein, P. Öhberg, L. P. Pitaevskii, G. V. Shlyapnikov, S. Stringari, and the experimental Quantum Optics group of the University of Han- nover for stimulating discussions. We acknowledge sup- port from Deutsche Forschungsgemeinschaft SFB 407 and SPP1116, the RTN Cold Quantum Gases, IST Pro- gram EQUIP, ESF PESC BEC2000+, EPSRC, and the Ministero dell’Istruzione, dell’Università e della Ricerca (MIUR). The authors wish to thank the Alexander von Humboldt Foundation, the Federal Ministry of Education and Research and the ZIP Programme of the German Government.

Figure 3: Frequencies of the breathing mode (solid), quadrupole 1 (dotted) and quadrupole 2 (dashed) as a function of $V_0/E_R$, for the particular case of $^{87}$Rb atoms, $N = 2 \times 10^3$, $\omega_\perp = 2\pi \times 4$Hz, $\omega_\parallel = 2\pi \times 40$Hz, and $d = 0.5\mu$m. 

properties and the excitation spectrum are significantly different than the expected results for a harmonically confined BEC.

We thank M. Lewenstein, P. Öhberg, L. P. Pitaevskii, G. V. Shlyapnikov, S. Stringari, and the experimental Quantum Optics group of the University of Han- nover for stimulating discussions. We acknowledge sup- port from Deutsche Forschungsgemeinschaft SFB 407 and SPP1116, the RTN Cold Quantum Gases, IST Pro- gram EQUIP, ESF PESC BEC2000+, EPSRC, and the Ministero dell’Istruzione, dell’Università e della Ricerca (MIUR). The authors wish to thank the Alexander von Humboldt Foundation, the Federal Ministry of Education and Research and the ZIP Programme of the German Government.

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