FINITE RAMSEY DEGREES AND FRAÎSSÉ EXPANSIONS WITH THE RAMSEY PROPERTY

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Abstract. By a result of Zucker, every Fraïssé structure $F$ for which the elements of $\text{Age}(F)$ have finite Ramsey degrees admits a Fraïssé precompact expansion $F^*$ whose age $\text{Age}(F^*)$ has the Ramsey property. While the original method uses dynamics in spaces of ultrafilters, the purpose of the present short note is to provide a different proof, based on classical tools from Fraïssé theory.

1. Introduction

Recent works in structural Ramsey theory have largely been influenced by the paper [KPT05] of Kechris-Pestov-Todorcevic which exhibited a strong connection with topological dynamics. When studying it, it became apparent that of particular interest are those Fraïssé structures $F$ whose elements of $\text{Age}(F)$ have a finite Ramsey degree. (For background and notations, see Section 2.) On the surface, this property is only a slight weakening of the now classical Ramsey property introduced by Nešetřil and Rödl. However, practice suggests that it could be substantially less restrictive, and knowing exactly to which extent this is so remains open, despite the recent progress made on the model theoretic side (for example, Evans proved that $\omega$-categoricity of $F$ is not sufficient [Eva15]), on the combinatorial side (see [HN10] by Hubička and Nešetřil, where the scope of the main technique to prove the Ramsey property is made broader still) and on the dynamical side (see [Zuc16] by Zucker, [MNT16] by Melleray-Tsankov-Nguyen Van Thé and [BM16] by Ben Yaacov-Melleray-Tsankov).

One method to make sure that a Fraïssé structure $F$ has finite Ramsey degrees is to construct a precompact expansion $F^*$ so that $\text{Age}(F^*)$ has the Ramsey property. It turns out that every Fraïssé structure with finite Ramsey degrees arises that way:

Theorem (Zucker [Zuc16]). Let $F$ be a Fraïssé structure. TFAE:

i) Every element of $\text{Age}(F)$ has a finite Ramsey degree.

ii) The structure $F$ admits a Fraïssé precompact expansion $F^*$ whose age has the Ramsey property, the expansion property relative to $\text{Age}(F)$, and is made of rigid elements.

The original proof of this result uses topological dynamics in spaces of ultrafilters. The purpose of the present short note is to provide a different proof, based on classical tools from Fraïssé theory as well as some additional results from [NVT13].

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2. Background and notation

The purpose of this section is to introduce the terminology that will be used in order to show the main result of the paper. It is based on [NVT13], which itself rests on [KPT05]. The reader will be assumed to have some familiarity with Fraïssé theory. Given a first order language $L$, the age of a countable $L$-structure $F$ is the class $\text{Age}(F)$ of all of its finite substructures (up to isomorphism). The structure $F$ itself is a Fraïssé structure when it is countable, every finite subset of $F$ generates a finite substructure of $F$ ($F$ is locally finite), and every isomorphism between finite substructures of $F$ extends to an automorphism of $F$ ($F$ is ultrahomogeneous).

Given structures $A$, $B$, $C$ in that language, write $A \cong B$ when $A$ and $B$ are isomorphic, and define the set of all copies of $A$ in $C$ as

$$\left( \frac{C}{A} \right) := \{ \tilde{A} \subset C : \tilde{A} \cong A \}.$$ 

Given a class $\mathcal{K}$ of $L$-structures, $A \in \mathcal{K}$ has a finite Ramsey degree (in $\mathcal{K}$) when there exists an integer $t(A)$ such that for every $k \in \mathbb{N}$, every $B \in \mathcal{K}$, there exists $C \in \mathcal{K}$ such that for every map $\chi : \left( \frac{C}{A} \right) \to [k] := \{0, 1, ..., k-1\}$ (usually referred to as a $k$-coloring of $\left( \frac{C}{A} \right)$) there is $B' \in \left( \frac{C}{B} \right)$ such that $\chi$ takes at most $t(A)$-many colors on $\left( \frac{B'}{A} \right)$.

When $\mathcal{K} = \text{Age}(F)$ for some Fraïssé structure $F$, this is equivalent to the fact that for every $k$-coloring $\chi$ of $\left( \frac{F}{A} \right)$, there is $B' \in \left( \frac{F}{B} \right)$ such that $\chi$ takes at most $t(A)$-many colors on $\left( \frac{B'}{A} \right)$. When in addition $t(A) = 1$ for every $A \in \mathcal{K}$, $\mathcal{K}$ has the Ramsey property.

Consider now an expansion $L^*$ of $L$ with at most countably many relational symbols. Say that an expansion $\mathcal{K}^*$ of $\mathcal{K}$ in $L^*$ is precompact when any $A \in \mathcal{K}$ only has finitely many expansions in $\mathcal{K}^*$. Similarly, say that a Fraïssé structure $F^*$ is a precompact expansion of $F$ when $\text{Age}(F^*)$ is a precompact expansion of $\text{Age}(F)$. Finally, say that $\mathcal{K}^*$ satisfies the expansion property relative to $\mathcal{K}$ if, for every $A \in \mathcal{K}$, there exists $B \in \mathcal{K}$ such that every expansion of $A$ in $\mathcal{K}^*$ embeds in every expansion of $B$ in $\mathcal{K}^*$.

3. Proof of the main result

With the terminology of the previous section in mind, we are now ready to provide a proof of Zucker’s theorem using classical techniques. The implication $ii) \Rightarrow i)$ is obvious so we concentrate on $i) \Rightarrow ii)$. The first step makes use of an ingenious technique due to Križ in [Kri91]:

**Definition 1.** Let $A, B \in \text{Age}(F)$.

For $t \in \mathbb{N}$, say that $B$ is $t$-$A$-Ramsey when for every $k \in \mathbb{N}$, and every $k$-coloring $\chi$ of $\left( \frac{B}{A} \right)$, there exists $\tilde{B} \in \left( \frac{B}{A} \right)$ such that $\chi$ takes at most $t$-many values on $\left( \frac{\tilde{B}}{A} \right)$.

For $E$ an equivalence relation on $\left( \frac{B}{A} \right)$, say that $B$ is $E$-$A$-Ramsey when for every $k \in \mathbb{N}$, $k$-coloring $\chi$ of $\left( \frac{B}{A} \right)$, there exists an embedding $b : B \to F$ such that:

$$\forall \tilde{A}_0, \tilde{A}_1 \in \left( \frac{B}{A} \right) \quad \tilde{A}_0 E \tilde{A}_1 \Rightarrow \chi(b(\tilde{A}_0)) = \chi(b(\tilde{A}_1))$$

**Lemma 1.** Let $A, B \in \text{Age}(F)$, $t \in \mathbb{N}$. Assume that $B$ is $t$-$A$-Ramsey. Then $B$ is $E$-$A$-Ramsey for some equivalence relation $E$ on $\left( \frac{B}{A} \right)$ with at most $t$-many classes.
Proof. Let $\mathcal{E}$ denote the (finite) set of all equivalence relations on $(\mathcal{A})$ with at most $t$-many classes. We assume that $\mathcal{B}$ is not $E$-$\mathcal{A}$-Ramsey for any $E \in \mathcal{E}$, and show that $\mathcal{B}$ is not $t$-$\mathcal{A}$-Ramsey.

By assumption, for every $E \in \mathcal{E}$, there exists $k_E \in \mathbb{N}$ and $\chi_E : (\frac{\mathcal{F}}{\mathcal{A}}) \to [k_E]$ witnessing that $\mathcal{B}$ is not $E$-$\mathcal{A}$-Ramsey, i.e. for every embedding $b : \mathcal{B} \to \mathcal{F}$, there exist $\tilde{\mathcal{A}}_0 \mathcal{E} \tilde{\mathcal{A}}_1$ in $(\frac{\mathcal{F}}{\mathcal{A}})$ such that $\chi_E(b(\tilde{\mathcal{A}}_0)) \neq \chi_E(b(\tilde{\mathcal{A}}_1))$.

Let $\chi : \frac{\mathcal{F}}{\mathcal{A}} \to \prod_{E \in \mathcal{E}} [k_E]$ be the finite coloring defined by $\chi(\tilde{\mathcal{A}}) = (\chi_E(\tilde{\mathcal{A}}))_{E \in \mathcal{E}}$.

We claim that for every $\tilde{\mathcal{B}} \in (\frac{\mathcal{F}}{\mathcal{A}})$, $\chi$ takes at least $(t+1)$-many values on $(\frac{\mathcal{B}}{\mathcal{A}})$. Let $b : \mathcal{B} \to \mathcal{F}$ an embedding such that $\tilde{\mathcal{B}} = b(\mathcal{B})$. Define $E_b$ on $(\frac{\mathcal{B}}{\mathcal{A}})$ by:

$$\chi(b(\tilde{\mathcal{A}}_0)) = \chi(b(\tilde{\mathcal{A}}_1))$$

By construction of $\chi$, $E_b$ disagrees with each $E \in \mathcal{E}$ on at least one pair of elements of $(\frac{\mathcal{B}}{\mathcal{A}})$. It follows that $E_b \notin \mathcal{E}$, as required.

Writing $t(\mathcal{A})$ the Ramsey degree in $\text{Age}(\mathcal{F})$, it follows that every $\mathcal{B} \in \text{Age}(\mathcal{F})$ is $E_{\mathcal{A}}^{\mathcal{B}}$-$\mathcal{A}$-Ramsey for some equivalence relation $E_{\mathcal{A}}^{\mathcal{B}}$ on $(\frac{\mathcal{B}}{\mathcal{A}})$ with at most $t(\mathcal{A})$-many classes.

Lemma 2. For every $\mathcal{A} \in \text{Age}(\mathcal{F})$, there is an equivalence relation $E_{\mathcal{A}}$ on $(\frac{\mathcal{F}}{\mathcal{A}})$ with at most $t(\mathcal{A})$-many classes such that every finite substructure $\mathcal{B}$ of $\mathcal{F}$ is $(E_{\mathcal{A}} \upharpoonright \mathcal{B})$-$\mathcal{A}$-Ramsey. (where $(E_{\mathcal{A}} \upharpoonright \mathcal{B})$ denotes the restriction of $E_{\mathcal{A}}$ to $\mathcal{B}$.)

Proof. Write $\mathcal{F} = \{x_k : k \in \mathbb{N}\}$ and let $\mathcal{B}_n$ denote the substructure of $\mathcal{F}$ supported by $\{x_k : k \leq n\}$. By Lemma 1 each $\mathcal{B}_n$ is $E_{\mathcal{A}}^{\mathcal{B}_n}$-$\mathcal{A}$-Ramsey for some equivalence relation $E_{\mathcal{A}}^{\mathcal{B}_n}$ on $(\frac{\mathcal{B}_n}{\mathcal{A}})$ with at most $t(\mathcal{A})$-many classes. Observe that if $m \leq n$, then $\mathcal{B}_m$ is also $(E_{\mathcal{A}}^{\mathcal{B}_n} \upharpoonright \mathcal{B}_m)$-$\mathcal{A}$-Ramsey. Thus, the set $\{(\mathcal{B}_m, E_{\mathcal{A}}^{\mathcal{B}_n} \upharpoonright \mathcal{B}_m) : m \leq n\}$ becomes a finitely branching tree when equipped with the relation

$$(\mathcal{B}, E) \leq (\mathcal{C}, F) \iff F \upharpoonright (\frac{\mathcal{B}}{\mathcal{A}}) = E$$

By König’s lemma, this tree admits an infinite branch, whose union is of the form $(\mathcal{F}, E_{\mathcal{A}})$. The relation $E_{\mathcal{A}}$ as required.

From that point on, the proof makes use of several results from [NVT13]. For each $\mathcal{A} \in \text{Age}(\mathcal{F})$, we now add predicates, $P_{\mathcal{A},0},...,P_{\mathcal{A},t(\mathcal{A})-1}$, one for each equivalence class of $E_{\mathcal{A}}$. Write $\tilde{\mathcal{F}}$ for the family $(P_{\mathcal{A},i})_{\mathcal{A},i}$ where $\mathcal{A}$ ranges over $\text{Age}(\mathcal{F})$ and $i < t(\mathcal{A})$. Note that every element of $\text{Age}(\mathcal{F})$ only has finitely many expansions in $\text{Age}(\mathcal{F}, \tilde{\mathcal{F}})$. Next, let $<^*$ be a linear ordering on $\mathcal{F}$, and consider the logic action of $\text{Aut}(\mathcal{F})$ on the compact space

$$\prod_{\mathcal{A} \in \text{Age}(\mathcal{F}) \ \ \ \ \ \ \ \ \ \ \ \ i < t(\mathcal{A})} 2^{[\mathcal{F}]} \times \text{LO}(\mathcal{F})$$

Pick $(\tilde{\mathcal{P}}^*, <^*) \in \text{Aut}(\mathcal{F}) \cdot (\tilde{\mathcal{F}}, <)$ with minimal orbit closure in that space. Then, $<^*$ is a linear ordering on $\mathcal{F}$, and writing $\tilde{\mathcal{F}}^* = (P_{\mathcal{A},i}^*)_{\mathcal{A},i}$, the family $(P_{\mathcal{A},i}^*)_{i < t(\mathcal{A})}$ is a partition of $(\frac{\mathcal{F}}{\mathcal{A}})$ for every $\mathcal{A} \in \text{Age}(\mathcal{F})$. Let $\mathcal{K}^* := \text{Age}(\mathcal{F}, \tilde{\mathcal{P}}^*, <^*)$.

Lemma 3. Let $\mathcal{A} \in \text{Age}(\mathcal{F})$. Then $\mathcal{A}$ has at least $t(\mathcal{A})$-many expansions in $\mathcal{K}^*$.
Proof. Let $B \in \text{Age}(F)$ witness the fact that $t(A)$ is the Ramsey degree of $A$ in $\text{Age}(F)$. This means that there is $\chi : \binom{F}{A} \to [k]$ taking at least $t(A)$-many values on $\binom{B}{A}$ whenever $B \in \binom{F}{A}$. Let $\tilde{B} \in \binom{F}{A}$. Because $\tilde{F}^* \in \text{Aut}(F) \cdot \tilde{P}$, we have $\text{Age}(F, \tilde{F}^*) \subset \text{Age}(F, \tilde{P})$ (see [NVT13 Proposition 6]). It follows that if $E$ denotes the equivalence relation induced on $\binom{B}{A}$ by $(P_{\tilde{A},i})_{i \in H(A)}$, then $\tilde{B}$ is $E$-$A$-Ramsey. Hence, there is an embedding $b : \tilde{B} \to F$ such that the value of $\chi(b(\tilde{A}))$ depends only on the $E$-class of $\tilde{A}$ in $\binom{\tilde{B}}{A}$. By choice of $\chi$, it follows that $\binom{\tilde{B}}{A}$ is made up of at least $t(A)$ many $E$-classes. In other words, at least $t(A)$-many expansions of $A$ appear in the substructure of $(F, \tilde{P})$ supported by $\tilde{B}$. This is valid for every copy $\tilde{B}$ of $B$ in $F$, so every expansion of $B$ in $\text{Age}(F, \tilde{P})$ contains at least $t(A)$-many expansions of $A$. As a result, this is also the case for every expansion of $B$ in $K^*$, and $A$ has at least $t(A)$-many expansions in $K^*$.

Clearly, $K^*$ has the hereditary property and the joint embedding property. It has the expansion property relative to $\text{Age}(F)$ because $(\tilde{F}^*, \prec^*)$ has minimal orbit closure (see [NVT13 Theorem 4]). By the previous lemma, every $A \in \text{Age}(F)$ has a finite Ramsey degree in $\text{Age}(F)$ whose value is at most the number of non-isomorphic expansions of $A$ in $K^*$. According to [NVT13 Proposition 8], these conditions guarantee that $K^*$ has the Ramsey property, and is therefore a Fraïssé class. As such, it has a Fraïssé limit, which we write $F^*$. To finish the proof, it suffices to show that $F^*$ is an expansion of $F$.

Lemma 4. The structure $F^*$ is an expansion of $F$.

Proof. It suffices to show that the reduct $F^* \models L$ of $F^*$ to the language $L$ of $F$ is Fraïssé. Indeed, $\text{Age}(F^* \models L) = \text{Age}(F)$, so $F^* \models L = F$ will follow.

To show that $F^* \models L$ is Fraïssé, following [KPT05 5.2], it suffices to show that $K^*$ has the following reasonability property: For every embedding $\pi : A \to B$ between elements of $\text{Age}(F)$, and every expansion $A^*$ of $A$ in $K^*$, there exists an expansion $B^*$ of $B$ in $K^*$ so that $\pi$ induces an embedding from $A^*$ to $B^*$.

Let us first check that this condition is sufficient. Let $\pi : A \to B$ be an inclusion embedding between finite substructures of $F^* \models L$, and $\varphi : A \to F^* \models L$ be an embedding. Then $\varphi(A)$ supports a substructure of $F^*$, and this can be pulled back to $A$ to define an expansion $A^*$ of $A$ in $K^*$. By reasonability, expand $B$ to $B^* \in K^*$ so that $\pi$ induces an embedding from $A^*$ to $B^*$, as required. Because $F^*$ is Fraïssé, $\pi$ extends to $B^*$, which induces $\varphi : B \to F$ extending $\varphi$.

We now show that the reasonability property holds. Let $\pi : A \to B$ be an embedding between elements of $\text{Age}(F)$, and $A^*$ an expansion of $A$ in $K^*$. Because $K^* = \text{Age}(F, \tilde{F}^*, \prec^*)$, we can find a substructure of $(F, \tilde{F}^*, \prec^*)$ isomorphic to $A^*$; call it $A_0^*$, and let $\varphi : A^* \to A_0^*$ be an isomorphism. Then $\varphi$ induces an isomorphism from $A$ to $A_0^* \models L$, and by ultrahomogeneity of $F$, there is $g \in \text{Aut}(F)$ such that $g \models A = \varphi$. Then $g(B)$ supports in $(F, \tilde{F}^*, \prec^*)$ an expansion $B^*$ of $B$. Pulling this back to $B$ via $g$, we see that $B^*$ is as required.

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