A semismooth Newton-proximal method of multipliers for 
ℓ₁-regularized convex quadratic programming

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Abstract

In this paper we present a method for the solution of ℓ₁-regularized convex quadratic optimization problems. It is derived by suitably combining a proximal method of multipliers strategy with a semi-smooth Newton method. The resulting linear systems are solved using a Krylov-subspace method, accelerated by appropriate general-purpose preconditioners, which are shown to be optimal with respect to the proximal parameters. Practical efficiency is further improved by warm-starting the algorithm using a proximal alternating direction method of multipliers. We show that the method achieves global convergence under feasibility assumptions. Furthermore, under additional standard assumptions, the method can achieve global linear and local superlinear convergence. The effectiveness of the approach is numerically demonstrated on L₁-regularized PDE-constrained optimization problems.

1 Introduction

In this paper we consider convex optimization problems of the following form:

\[
\min_{x \in \mathbb{R}^n} \, c^\top x + \frac{1}{2} x^\top Q x + g(x) + \delta_K(x), \quad \text{s.t. } Ax = b,
\]

where \( c \in \mathbb{R}^n \), \( Q \succeq 0 \in \mathbb{R}^{n \times n} \), \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), and \( g(x) = \|Dx\|_1 \), with \( D \succeq 0 \) and diagonal. Without loss of generality, we assume that \( m \leq n \). Furthermore, \( K := \{ x \in \mathbb{R}^n : l \leq x \leq u \} \), for some arbitrary (possibly unbounded) vectors \( l \leq u \). Finally, \( \delta_K(\cdot) \) is an indicator function for the set \( K \), with \( \delta_K^*(\cdot) \) denoting its Fenchel conjugate, that is:

\[
\delta_K(x) = \begin{cases} 0, & \text{if } x \in K \\ \infty, & \text{otherwise} \end{cases}, \quad \delta_K^*(x^*) = \sup_{x \in \mathbb{R}^n} \{ (x^*)^\top x - \delta_K(x) \}.
\]

Notice that problem (P) can accommodate instances where sparsity is sought in some appropriate dictionary (i.e. in that case \( D \) would be a general rectangular matrix). Indeed, this can be done by appending some additional linear equality constraints in (P), making the ℓ₁ regularization separable (e.g. see [21, Sections 3–5]). Using Fenchel duality, we can easily verify (see Appendix A.1) that the dual of (P) can be written as

\[
\min_{x \in \mathbb{R}^n, y \in \mathbb{R}^m, z \in \mathbb{R}^n} b^\top y - \frac{1}{2} x^\top Q x - \delta_K^*(z) - g^*(A^\top y - c - Qx - z).
\]

Throughout the paper we make use of the following blanket assumption.

Assumption 1. Problems (P) and (D) are both feasible.
If $D = 0$, from [9, Proposition 2.3.4] we know that Assumption 1 implies that there exists a primal-dual triple $(x^*, y^*, z^*)$ solving $(P)–(D)$. Now if the primal-dual pair $(P)–(D)$ is feasible, it must remain feasible for any positive semi-definite $D$, since in this case $(P)$ can be written as a convex quadratic problem by appending appropriate (necessarily feasible) linear equality and inequality constraints (e.g. as in [21, 51]). Thus, Assumption 1 suffices to guarantee that the solution set of $(P)–(D)$ is non-empty.

There are numerous applications that require the solution of problems of the form of $(P)$. Indeed, $(P)$ can model linear and convex quadratic programming instances, regularized (group) lasso instances (often arising in signal or image processing and machine learning, e.g. see [14, 62, 69]), as well as sub-problems arising from the linearization of a nonlinear (possibly non-convex or non-smooth) problem (such as those arising within sequential quadratic programming [10] or globalized proximal Newton methods [38, 39]). Furthermore, various optimal control problems can be tackled in the form of $(P)$, such as those arising from $L^1$-regularized partial differential equation (PDE) optimization, assuming that a discretize-then-optimize strategy is adopted (e.g. see [70]). Given the diversity of applications, most of which require a highly-accurate solution of problems of the form of $(P)$, the construction of efficient, scalable, and robust solvers has attracted a lot of attention.

In particular, there is a plethora of first-order methods capable of finding an approximate solution to $(P)$. For example, one could employ proximal (sub-)gradient-based schemes (e.g. see [6, 7]), or splitting schemes (e.g. see [11, 19, 22]). However, while such solution methods are very general, easy to implement, and require very little memory, they are usually attractive when trying to find only an approximate solution, not exceeding 2- or 3-digits of accuracy. If a more accurate solution is needed, then one has to resort to a second-order approach, or, in general, an approach that utilizes second-order information.

There are three major classes of second-order methods for problems of the form of $(P)$. Those include globalized (smooth, semi-smooth, quasi or proximal) Newton methods (e.g. see [13, 32, 43, 45, 46, 50, 52, 56, 65]), variants of the proximal point method (e.g. see [20, 24, 25, 38, 39, 41, 49, 59]), or interior point methods (IPMs) applied to a reformulation of $(P)$ (e.g. see [21, 28, 31, 51]). Most globalized Newton-like approaches or proximal point variants studied in the literature are developed for composite programming problems in which either \( g(x) = 0 \) (e.g. see [13, 20, 30, 36, 41]) or \( K = \mathbb{R}^n \) (e.g. see [24, 33, 40, 46, 50]). Nonetheless, more recently there have been developed certain globalized Newton-like schemes, specialized to the case of $L^1$-regularized PDE-constrained optimization (see [43, 52]), in which the $\ell_1$ term as well as the box constraints in $(P)$ are both explicitly handled. We should notice, however, that globalized Newton-like schemes applied to $(P)$ need additional assumptions on the smooth Hessian matrix $Q$, as well as the constraint matrix $A$, since otherwise, the stability of the related Newton linear systems, arising as sub-problems, might be compromised. Under certain assumptions, superlinear convergence of Newton-like schemes is observed “close to a solution”, and global linear convergence can be achieved via appropriate line-search or trust-region strategies (e.g. see the developments in [13, 16, 32, 36, 46, 66] and the references therein). On the other hand, interior point methods can readily solve problems of the form of $(P)$ in a polynomial number of steps ([21, 28, 31, 51]), and stability of the associated Newton systems can be guaranteed by means of algorithmic regularization (which can be interpreted as the application of a proximal point method, see [2, 29, 53]). Nevertheless, the resulting linear systems arising within IPMs are of larger dimensions as compared to those arising within pure Newton-like or proximal approaches, since $(P)$ needs to be appropriately reformulated into a smooth problem. Furthermore, IPM linear systems have significantly worse conditioning compared
to linear systems arising within Newton-like or proximal-Newton methods.

The potential stability issues of the linear systems arising within Newton-like schemes can be alleviated by combining Newton-like methods with proximal point variants. In practice, solvers based on the proximal point method can achieve superlinear convergence, given that their penalty parameters increase to infinity at a suitable rate (see for example the developments in [40, 41, 59, 60, 73]). For problems of the form of (P)–(D), the sub-problems arising within proximal methods are non-smooth convex optimization instances, and are typically solved by means of semi-smooth Newton strategies. The resulting linear systems that one has to solve are better conditioned than their (possibly regularized) interior-point counterparts (e.g. see [8, 21, 28, 31, 71]), however, convergence is expected to be slower, as the method does not enjoy the polynomial worst-case complexity of interior-point methods. Nevertheless, these better conditioned linear systems can in certain cases allow one to achieve better computational and/or memory efficiency.

In this paper, we employ a proximal method of multipliers (PMM) using a semi-smooth Newton (SSN) strategy for solving the associated sub-problems, in order to efficiently deal with problems of the form of (P)–(D). The SSN linear systems are approximately solved by means of Krylov subspace methods, using appropriate general-purpose preconditioners. Unlike most proximal point methods given in the literature (e.g. see the primal approaches in [38, 39, 49], the dual approaches in [40, 41, 73] or the primal-dual approaches in [30, 20, 25, 59]), the proposed method is introducing proximal terms for each primal and dual variable of the problem, and this results in linear systems which are easy to precondition and solve, within the semi-smooth Newton method. Additionally, we explicitly deal with each of the two non-smooth terms of the objective in (P). This also contributes to the simplification of the resulting SSN linear systems, while it paves the way for generalizing this approach to a much wider class of problems. We show that global convergence is guaranteed with the minimal assumption of primal and dual feasibility, since our discussion is restricted to instances having a convex objective, the smooth part of which is quadratic. Moreover, we show that global linear and local superlinear convergence holds under standard assumptions.

Furthermore, we note that while most proximal Newton-like methods proposed in the literature allow inexactness in the solution of the associated Newton linear systems, the development of general-purpose preconditioners for them is lacking. Indeed, aside from the work in [52] which is specialized to the case of $L^1$-regularized PDE constrained optimization, most proximal Newton-like schemes utilizing Krylov subspace methods, do so without employing any preconditioner (e.g. see [13, 41, 40, 49]). Drawing from the interior point literature, and by suitably specializing the preconditioning approach given in [8, 31], we propose general-purpose positive definite preconditioners that are robust with respect to the penalty parameters of the PMM, and thus their behaviour does not deteriorate as the method approaches optimality. The positive definiteness of the preconditioners allows one to employ symmetric Krylov subspace solvers such as the minimal residual method (MINRES), [48]. Unlike non-symmetric Krylov solvers, the memory requirements of MINRES are very reasonable and do not depend on the number of Krylov iterations.

Additionally, the method deals with general box constraints and thus there is no need for introducing auxiliary variables to deal with upper and lower bounds separately, something that is required when employing conic-based solvers. As a result, the associated linear systems solved within SSN have significantly smaller dimensions, compared to linear systems arising within interior point methods suitable for the solution of $\ell_1$-regularized convex quadratic problems (e.g. see [21, 31, 51]), potentially making the proposed approach a more attractive alternative for large-scale instances, such as those arising from
the optimal control of PDEs. Finally, we propose an efficient warm-starting strategy based on a proximal alternating direction method of multipliers to further improve the efficiency of the approach at a low computational cost. We provide numerical evidence to demonstrate that the proposed method is efficient and robust when applied to $\ell_1$-regularized problems arising from PDE-constrained optimization.

To summarize, in Section 2 we derive a proximal method of multipliers and discuss its convergence properties. Then, in Section 3 we present a globally convergent semismooth Newton method used to approximately solve the PMM sub-problems. Furthermore, we propose general-purpose preconditioners for the associated SSN linear systems and analyze their effectiveness. In Section 4 we present a warm-starting strategy for the method. Then, in Section 5, the overall approach is extensively tested on certain partial differential equation optimization problems. Finally, we derive some conclusions in Section 6.

**Notation**

Given a vector $x$ in $\mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm. Given a closed set $\mathcal{K} \subset \mathbb{R}^n$, $\Pi_{\mathcal{K}}(x)$ denotes the Euclidean projection onto $\mathcal{K}$, that is $\Pi_{\mathcal{K}}(x) := \arg \min \{\|x - z\| : z \in \mathcal{K}\}$. Given an arbitrary rectangular matrix $A$, $\sigma_{\text{max}}(A)$ (resp. $\sigma_{\text{min}}(A)$) denotes its maximum (resp. minimum) singular value. For an arbitrary square matrix $B$, $\lambda_{\text{max}}(B)$ (resp. $\lambda_{\text{min}}(B)$) denotes its maximum (resp. minimum) eigenvalue. Given a closed set $\mathcal{K}$, and a positive definite matrix $R$, we write $\text{dist}(z, D) := \inf_{x \in \mathcal{K}} \|z - x\|_R$. If $R = I$, we assume that $\text{dist}_I(z, \mathcal{K}) \equiv \text{dist}(z, \mathcal{K})$. Given an index set $D$, $|D|$ denotes its cardinality. Given a rectangular matrix $A \in \mathbb{R}^{m \times n}$ and an index set $B \subseteq \{1, \ldots, n\}$, we denote the columns of $A$, the indices of which belong to $B$, as $A_B$. Given a square matrix $Q \in \mathbb{R}^{n \times n}$, we denote the subset of columns and rows of $Q$, the indices of which belong to $B$, as $Q_{(B,B)}$. Furthermore, we denote by $\text{Diag}(Q)$ the diagonal matrix, with diagonal elements equal to those of $Q$. Similarly, we write $\text{Off}(Q)$ to denote the square matrix with off-diagonal elements equal to those of $Q$ and zero diagonal.

### 2 A primal-dual proximal method of multipliers

In what follows, we derive the proximal augmented Lagrangian penalty function corresponding to the primal problem (P). Using the latter, we derive a primal-dual PMM for solving the pair (P)-(D).

We begin by deriving the Lagrangian associated to (P). To that end, let us define the function $\varphi(x) = c^\top x + \frac{1}{2} x^\top Q x + g(x) + \delta_{\mathcal{K}}(x) + \delta_{\{0\}}(b - Ax)$. Next, we use the dualization strategy (proposed in [61, Chapter 11]). That is, we define the function $\hat{\varphi}(x, u', w') = c^\top x + \frac{1}{2} x^\top Q x + g(x) + \delta_{\mathcal{K}}(x + w') + \delta_{\{0\}}(b - Ax + u')$, for which it holds that $\varphi(x) = \hat{\varphi}(x, 0, 0)$. Then, the Lagrangian associated to (P) can be computed as:

$$\ell(x, y, z) := \inf_{u', w'} \left\{ \hat{\varphi}(x, u', w') - y^\top u' - z^\top w' \right\}$$

$$= c^\top x + \frac{1}{2} x^\top Q x + g(x) - \sup_{w'} \left\{ z^\top w' - \delta_{\mathcal{K}}(x + w') \right\}$$

$$- \sup_{u'} \left\{ y^\top u' - \delta_{\{0\}}(b - Ax + u') \right\}$$

$$= c^\top x + \frac{1}{2} x^\top Q x + g(x) + z^\top x - \delta_{\mathcal{K}}^*(z) - y^\top (Ax - b),$$

where we used the definition of the Fenchel conjugate. Before deriving the augmented Lagrangian associated to (P), we introduce some necessary notation as well as relations.
that will be used later. Firstly, given a convex function \( p : \mathbb{R}^n \to \mathbb{R} \), we define
\[
\text{prox}_p(u) := \arg \min_x \left\{ p(x) + \frac{1}{2} \| u - x \|^2 \right\}.
\]
Then, given some positive constant \( \beta \), it holds that (Moreau Identity, see [47]):
\[
\text{prox}_{\beta p}(u') + \beta \text{prox}_{\beta^{-1} p'}(\beta^{-1} u') = u'.
\] (2.1)

Finally, we have that (e.g. see [35, Equation 2.2])
\[
\frac{1}{2} \| \text{prox}_p(x) \|^2 + p^*(\text{prox}_p(x)) = \frac{1}{2} \| x \|^2 - \| x - \text{prox}_p(x) \|^2 - p(\text{prox}_p(x)).
\] (2.2)

Given a penalty parameter \( \beta > 0 \), the augmented Lagrangian corresponding to (P) is derived as:
\[
\mathcal{L}_\beta(x; y, z) := \sup_{u', w} \left\{ \ell(x, u', w') - \frac{1}{2\beta} \| u' - y \|^2 - \frac{1}{2\beta} \| w' - z \|^2 \right\}
\]
\[
= c^T x + \frac{1}{2} x^T Q x + g(x) - \inf_{u'} \left\{ u'^T (Ax - b) + \frac{1}{2\beta} \| u' - y \|^2 \right\}
\]
\[
- \inf_{w'} \left\{ - w'^T x + \delta^T_{\mathcal{K}}(w') + \frac{1}{2\beta} \| w' - z \|^2 \right\}
\]
\[
= c^T x + \frac{1}{2} x^T Q x + g(x) - y^T (Ax - b) + \frac{\beta}{2} \| Ax - b \|^2 + x^T (\text{prox}_{\beta \delta^T_{\mathcal{K}}}(z + \beta x))
\]
\[
- \delta^T_{\mathcal{K}}\left( \text{prox}_{\beta \delta^T_{\mathcal{K}}}(z + \beta x) \right) - \frac{1}{1\beta} \| \text{prox}_{\beta \delta^T_{\mathcal{K}}}(z + \beta x) - z \|^2,
\] (2.3)

where we used the fact that if \( p_1(u') = p_2(u') + r^T x \), where \( p_1(\cdot) \) and \( p_2(\cdot) \) are two closed convex functions, and \( r \) is a vector, then \( \text{prox}_{p_1}(u') = \text{prox}_{p_2}(u' - r) \). We continue by applying the Moreau identity (2.1) to obtain:
\[
\mathcal{L}_\beta(x; y, z) = c^T x + \frac{1}{2} x^T Q x + g(x) - y^T (Ax - b) + \frac{\beta}{2} \| Ax - b \|^2
\]
\[
+ x^T \left( (z + \beta x) - \beta \text{prox}_{\beta^{-1} \delta^T_{\mathcal{K}}}(\beta^{-1} z + x) \right) - \delta^T_{\mathcal{K}}\left( \text{prox}_{\beta \delta^T_{\mathcal{K}}}(z + \beta x) \right)
\]
\[
- \frac{1}{1\beta} \| \beta x - \beta \text{prox}_{\beta^{-1} \delta^T_{\mathcal{K}}}(\beta^{-1} z + x) \|^2
\] (2.4)

Finally, we write:
\[
R := \left( \delta^T_{\mathcal{K}}\left( \text{prox}_{\beta \delta^T_{\mathcal{K}}}(z + \beta x) \right) + \frac{1}{1\beta} \| \beta \text{prox}_{\beta^{-1} \delta^T_{\mathcal{K}}}(\beta^{-1} z + x) \|^2 \right)
\]
\[
= \frac{1}{\beta} \left( \beta \delta^T_{\mathcal{K}}\left( \text{prox}_{\beta \delta^T_{\mathcal{K}}}(z + \beta x) \right) + \frac{1}{1\beta} \| \beta \text{prox}_{\beta^{-1} \delta^T_{\mathcal{K}}}(\beta^{-1} z + x) \|^2 \right)
\]
\[
= \frac{1}{\beta} \left( \frac{1}{2} \| z + \beta x \|^2 - \| z + \beta x - \beta \text{prox}_{\beta^{-1} \delta^T_{\mathcal{K}}}(\beta^{-1} z + x) \|^2 \right)
\]
\[
- \beta \delta^T_{\mathcal{K}}\left( \text{prox}_{\beta^{-1} \delta^T_{\mathcal{K}}}(\beta^{-1} z + x) \right)
\]
\[
= \frac{1}{1\beta} \left( \| z \|^2 + \beta^2 \| x \|^2 + 2\beta z^T x - \| z + \beta x - \beta \Pi_{\mathcal{K}}(\beta^{-1} z + x) \|^2 \right),
\]
where we used (2.2), along with the fact that \((\beta \delta^*_K)(x) = \beta \delta_K(b^{-1}x)\), while \(\text{prox}_{(\beta \delta^*_K)}(x) = \beta \text{prox}_{\beta^{-1} \delta_K}(\beta^{-1}x)\). To finally derive the augmented Lagrangian, we substitute \(R\) in the last line of (2.4), which yields:

\[
L_\beta(x; y, z) = c^\top x + \frac{1}{2} x^\top Q x + g(x) - y^\top (A x - b) + \frac{\beta}{2} \|A x - b\|^2 - \frac{1}{2\beta}\|z\|^2 + \frac{1}{2\beta}\|z + \beta x - \beta \Pi_K(\beta^{-1}z + x)\|_F^2.
\] (2.5)

Assume that at an iteration \(k \geq 0\) of the algorithm, we have the estimates \((x_k, y_k, z_k)\) as well as the penalty parameters \(\beta_k, \rho_k\), such that \(\rho_k \equiv \frac{2}{\tau_k}\), where \(\{\tau_k\}\) is a non-increasing positive sequence, i.e. \(\tau_k > 0\) for all \(k \geq 0\). We begin by defining the following continuously differentiable function:

\[
\phi(x) \equiv \phi_{\beta_k, \rho_k}(x; x_k, y_k, z_k) := L_{\beta_k}(x; y_k, z_k) - g(x) + \frac{1}{2\rho_k}\|x - x_k\|^2.
\]

Using the previous notation, the minimization of the proximal augmented Lagrangian function can be written as \(\min_x \psi(x) := \phi(x) + g(x)\), and thus we need to find \(x^*\) such that

\[
(\nabla \phi(x^*))^\top (x - x^*) + g(x) - g(x^*) \geq 0, \quad \forall \, x \in \mathbb{R}^n.
\]

To that end, we observe that

\[
\nabla \phi(x) = c + Q x - A^\top y_k + \beta_k A^\top (A x - b) + (z_k + \beta_k x) - \beta_k \Pi_K(\beta_k^{-1} z_k + x) + \rho_k^{-1}(x - x_k).
\]

By introducing the variable \(y = y_k - \beta_k (A x - b)\) the optimality conditions of \(\min_x \psi(x)\) can be written as

\[
(0, 0) \in F_{\beta_k, \rho_k}(x, y),
\] (2.6)

where

\[
F_{\beta_k, \rho_k}(x, y) := \{(u', v') : u' \in r_{\beta_k, \rho_k}(x, y) + \partial g(x), \quad v' = A x + \beta_k^{-1}(y - y_k) - b\},
\]

\[
r_{\beta_k, \rho_k}(x, y) := c + Q x - A^\top y + (z_k + \beta_k x) - \beta_k \Pi_K(\beta_k^{-1} z_k + x) + \rho_k^{-1}(x - x_k).
\]

Notice that problem (2.6) admits a unique solution (since, as we show later in Proposition 1, it corresponds to a single-valued proximal operator). We now describe the primal-dual PMM in Algorithm PD-PMM.

Notice that we allow step (2.7) to be computed inexactly. In Section 2.1 we will provide precise conditions on this error sequence guaranteeing that Algorithm PD-PMM can achieve global linear and local superlinear convergence. Furthermore, notice that the characterization of \(\text{dist}(0, F_{\beta_k, \rho_k}(x, y))\) follows from the definition of \(F_{\beta_k, \rho_k}(x, y)\) as well as from the definition of \(\text{dist}(x, A)\) for some closed convex set \(A\). This connection is established in the Appendix for completeness (see Appendix A.2). Finally, we note that the condition in (2.7) can be evaluated expeditiously, since we assume that \(g(x) = \|D x\|_1\), for some positive semi-definite and diagonal matrix \(D\), and hence its sub-gradient is explicitly known.
Algorithm PD-PMM Primal-dual proximal method of multipliers

Input: \((x_0, y_0, z_0) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n\), \(\beta_0, \beta_{\infty}, \tau_{\infty} > 0\), and a non-increasing sequence \(\{\tau_k\}_{k=0}^{\infty}\) such that \(\tau_k \geq \tau_{\infty} > 0\) for all \(k \geq 0\).

Choose a sequence of positive numbers \(\{\epsilon_k\}\) such that \(\epsilon_k \to 0\).

for \((k = 0, 1, 2, \ldots)\) do

Find \((x_{k+1}, y_{k+1})\) such that:

\[
\text{dist} \left(0, F_{\beta_k \rho_k} (x_{k+1}, y_{k+1})\right) \leq \epsilon_k,
\]

where letting \(\hat{r} = r_{\beta_k \rho_k} (x_{k+1}, y_{k+1})\) we have

\[
\text{dist} \left(0, F_{\beta_k \rho_k} (x_{k+1}, y_{k+1})\right) = \left\| \begin{bmatrix} \hat{r} + \Pi_{\partial (g(x_{k+1}))} (-\hat{r}) \\ Ax_{k+1} + \beta_k^{-1} (y_{k+1} - y_k) - b \end{bmatrix} \right\|.
\]

Set

\[
z_{k+1} = (z_k + \beta_k x_{k+1}) - \beta_k \Pi_{\mathcal{K}} (\beta_k^{-1} z_k + x_{k+1}).
\]

Update

\[
\beta_{k+1} \not\to \beta_{\infty} \leq \infty, \quad \rho_{k+1} = \frac{\beta_{k+1}}{\tau_{k+1}}.
\]

end for

return \((x_k, y_k, z_k)\).

2.1 Convergence analysis

In this section we provide conditions on the error sequence \(\{\epsilon_k\}\) in (2.7) that guarantee the convergence of Algorithm PD-PMM, potentially at a global linear or local superlinear rate. We note that the analysis is immediately derived by the analyses in [41, Section 2] and [18, Section 10] (or by an extension of the analyses in [59, 60]) after connecting Algorithm PD-PMM to an appropriate proximal point iteration. To that end, let us define the maximal monotone operator \(T_{\ell}: \mathbb{R}^{2n+m} \rightrightarrows \mathbb{R}^{2n+m}\), associated to (P)–(D):

\[
T_{\ell}(x, y, z) := \{(u', v', w') : v' \in Qx + c - A^T y + z + \partial g(x),
\]

\[
\quad u' = Ax - b, \quad w' + x \in \partial \delta_{\mathcal{K}}^* (z)\}
\]

\[
= \{(u', v', w') : v' \in Qx + c - A^T y + z + \partial g(x),
\]

\[
\quad u' = Ax - b, \quad z \in \partial \delta_{\mathcal{K}} (x + w')\},
\]

where the second equality follows from [18, Lemma 5.7]. Obviously, the inverse of this operator can be written as

\[
T_{\ell}^{-1}(u', v', w') := \arg \max_{y, z} \min_x \{\ell(x, y, z) + u'^T x - v'^T y - w'^T z\}.
\]

Notice that Assumption 1 implies that \(T_{\ell}^{-1}(0) \neq \emptyset\). Following the result in [41], we note that \(T_{\ell}\) is in fact a polyhedral multifunction (see [57] for a detailed discussion on the properties of such multifunctions). In light of this property of \(T_{\ell}\) we note that the following holds.

Lemma 2.1. For any \(r > 0\), there exists \(\kappa > 0\) such that

\[
\text{dist}(p, T_{\ell}^{-1}(0)) \leq \kappa \text{dist}(0, T_{\ell}(p)), \quad \forall \ p \in \mathbb{R}^{2n+m}, \text{ with dist}(p, T_{\ell}^{-1}(0)) \leq r.
\]
Proof. The reader is referred to [41, Lemma 2.4] as well as [57].

Next, let us define the sequence of positive definite matrices \( \{R_k\}_{k=0}^{\infty} \) with \( R_k := \tau_k I_n \oplus I_m \oplus I_n \), for all \( k \geq 0 \), where \( \tau_k \) is defined in Algorithm PD-PMM and \( \oplus \) denotes the direct sum of two matrices. Using this sequence, we can define the single-valued and non-expansive proximal operator \( P_k : \mathbb{R}^{2n+m} \mapsto \mathbb{R}^{2n+m} \), associated to (2.9):

\[
P_k := (R_k + \beta_k T_k)^{-1} R_k. \tag{2.12}
\]

In particular, under our assumptions on the matrices \( R_k \), we have that (e.g. see [25, 41]) for all \( (u_1, v_1, w_1), (u_2, v_2, w_2) \in \mathbb{R}^{2n+m} \), the following inequality (non-expansiveness) holds

\[
\|(u_1, v_1, w_1) - P_k(u_2, v_2, w_2)\|_R \leq \|(u_1, v_1, w_1) - (u_2, v_2, w_2)\|_R. \tag{2.13}
\]

Obviously, we can observe that if \( (x^*, y^*, z^*) \in T_k^{-1}(0) \), then \( P_k(x^*, y^*, z^*) = (x^*, y^*, z^*) \).

We are now able to connect Algorithm PD-PMM with the proximal point iteration produced by (2.12).

**Proposition 1.** Let \( \{(x_k, y_k, z_k)\}_{k=0}^{\infty} \) be a sequence of iterates produced by Algorithm PD-PMM. Then, for every \( k \geq 0 \) we have that

\[
\|(x_{k+1}, y_{k+1}, z_{k+1}) - P_k(x_k, y_k, z_k)\|_R \leq \frac{\beta_k}{\min \{\sqrt{\tau_k}, 1\}} \text{dist}(0, F_{\beta_k, \rho_k}(x_{k+1}, y_{k+1})). \tag{2.14}
\]

**Proof.** Firstly, let us define the pair

\[
(\hat{u}, \hat{v}) := (r_{\beta_k, \rho_k}(x_{k+1}, y_{k+1}) + \Pi_{\partial g(x_{k+1})}(-r_{\beta_k, \rho_k}(x_{k+1}, y_{k+1})), Ax_{k+1} + \beta_k^{-1}(y_{k+1} - y_k) - b).
\]

We observe that given a sequence produced by Algorithm PD-PMM, we have

\[
\begin{bmatrix}
\hat{u} \\
\hat{v} \\
0
\end{bmatrix} + \beta_k^{-1} \begin{bmatrix}
\tau_k(x_k - x_{k+1}) \\
y_k - y_{k+1} \\
z_k - z_{k+1}
\end{bmatrix} \in T_k(x_{k+1}, y_{k+1}, z_{k+1}). \tag{2.15}
\]

To show this, we firstly notice that

\[
\begin{bmatrix}
\hat{u} \\
\hat{v} \\
0
\end{bmatrix} + \beta_k^{-1} \begin{bmatrix}
\tau_k(x_k - x_{k+1}) \\
y_k - y_{k+1} \\
z_k - z_{k+1}
\end{bmatrix} \in \begin{bmatrix}
Q x_{k+1} + c - A^\top y_{k+1} + z_{k+1} + \partial g(x_{k+1}) \\
Ax_{k+1} - b
\end{bmatrix},
\]

where we used the definition of the \((\hat{u}, \hat{v})\) as well as (2.8). It remains to show that \( \beta_k^{-1}(z_k - z_{k+1}) \in -x_{k+1} + \partial \delta_K^*(z_{k+1}) \). Alternatively, from the second equality in (2.9), we need to show that \( z_{k+1} \in \partial \delta_K(x_{k+1} + \beta_k^{-1}(z_k - z_{k+1})) \). To that end, we characterize the subdifferential of \( \partial \delta_K(\cdot) \). By convention we have that \( \partial \delta_K(\bar{x}) = \emptyset \) if \( \bar{x} \notin K \). Hence, assume that \( \bar{x} \in K \). Then, we obtain

\[
\partial \delta_K(\bar{x}) = \{ \bar{z} \in \mathbb{R}^n : \bar{z}^\top (\bar{x} - \bar{x}) \leq \delta_K(\bar{x}), \ \forall \ \bar{x} \in \mathbb{R}^n \}.
\]

By inspection, we fully characterize the latter component-wise, for any \( i \in \{1, \ldots, n\} \) as follows

\[
\partial \delta_{\{l_i, u_i\}}(\bar{x}) = \begin{cases} 
\{0\} & \bar{x}_i \in (l_i, u_i), \\
(-\infty, 0] & \bar{x}_i = l_i, \\
[0, \infty) & \bar{x}_i = u_i.
\end{cases}
\]
From (2.8) we have that \( z_{k+1} = z_k + \beta_k x_{k+1} - \beta_k \Pi_K(\beta_k^{-1} z_k + x_{k+1}) \). Proceeding component-wise, if \((\beta_k^{-1} z_{k,i} + x_{k+1,i}) \in (l_i, u_i)\), then \( z_{k+1,i} = 0 \), i.e.

\[
0 = z_{k+1,i} \in \partial \delta_{l_i,u_i}(x_{k+1,i} + \beta_k^{-1}(z_{k,i} + z_{k+1,i})) = \partial \delta_{l_i,u_i}(\beta_k^{-1} z_{k,i} + x_{k+1,i}).
\]

If \((\beta_k^{-1} z_{k,i} + x_{k+1,i}) \leq l_i\), then \( z_{k+1,i} \leq 0 \), and from the previous characterization we obtain that \( z_{k+1,i} \in \partial \delta_{K}(l_i) \). Finally, if \((\beta_k^{-1} z_{k,i} + x_{k+1,i}) \geq u_i\), we obtain that \( z_{k+1,i} \geq 0 \) and thus \( z_{k+1,i} \in \partial \delta_{K}(u_i) \). This shows that (2.15) holds.

Next, by appropriately re-arranging (2.15) we obtain

\[
\begin{bmatrix}
x_{k+1} \\
y_{k+1} \\
z_{k+1}
\end{bmatrix} = P_k \left( R_k^{-1} \begin{bmatrix} \hat{u} \\ \hat{v} \\ 0 \end{bmatrix} + \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} \right),
\]

where \( P_k \) is defined in (2.12). Subtracting both sides by \( P_k(x_k, y_k, z_k) \), taking norms, using the non-expansiveness of \( P_k \) (see (2.13)), and noting that \( \text{dist}(0, F_{\rho_k}(x_{k+1}, y_{k+1})) = \| (\hat{u}, \hat{v}) \| \) (see Appendix A.2), yields (2.14) and concludes the proof.

Now that we have established the connection of Algorithm PD-PMM with the proximal point iteration governed by the operator \( P_k \) defined in (2.12), we can directly provide conditions on the error sequence in (2.7), to guarantee global (possibly linear) and potentially local superlinear convergence of Algorithm PD-PMM. To that end, we will make use of certain results, as reported in [41, Section 2] (and also found in [18, Sections 9–10]). Firstly, we provide the global convergence result for the algorithm.

**Theorem 2.2.** Let Assumption 1 hold. Let \( \{(x_k, y_k, z_k)\}_{k=0}^{\infty} \) be generated by Algorithm PD-PMM. Furthermore, assume that we choose a sequence \( \{\epsilon_k\}_{k=0}^{\infty} \) in (2.7), such that

\[
\epsilon_k \leq \min\{\sqrt{\beta_k}, 1\}, \quad 0 \leq \beta_k, \quad \sum_{k=0}^{\infty} \delta_k < \infty. \tag{2.16}
\]

Then, \( \{(x_k, y_k, z_k)\}_{k=0}^{\infty} \) is bounded and converges to a primal-dual solution of (P)-(D).

**Proof.** The proof is omitted since it is a direct application of [41, Theorem 2.3], which is a direct extension of [60, Theorem 1]. See also the more general developments in [18, Section 9.2].

Next, we discuss local superlinear convergence of Algorithm PD-PMM, which is again given by direct application of the results in [18, 41]. To that end, let \( r > \sum_{k=0}^{\infty} \delta_k \), where \( \delta_k \) is defined in (2.16). Then, from Lemma 2.1 we know that there exists \( \kappa > 0 \) associated with \( r \) such that

\[
\text{dist}((x, y, z), T_r^{-1}(0)) \leq \kappa \text{dist}(0, T_r(x, y, z)), \tag{2.17}
\]

for all \((x, y, z) \in \mathbb{R}^{2n+m}\) such that \( \text{dist}((x, y, z), T_r^{-1}(0)) \leq r \).

**Theorem 2.3.** Let Assumption 1 hold. Furthermore, assume that \((x_0, y_0, z_0)\) satisfies \( \text{dist}_{R_0}((x_0, y_0, z_0), T_r^{-1}(0)) \leq r - \sum_{k=0}^{\infty} \delta_k \), where \( \delta_k \) is defined in (2.16). Let also \( \kappa \) be given as in (2.17) and assume that we choose a sequence \( \{\epsilon_k\}_{k=0}^{\infty} \) in (2.7) such that

\[
\epsilon_k \leq \min\{\sqrt{\beta_k}, 1\} \min\{\delta_k, \delta_k'\| (x_{k+1}, y_{k+1}, z_{k+1}) - (x_k, y_k, z_k) \|_{R_k} \}, \tag{2.18}
\]
where $0 \leq \delta_k$, $\sum_{k=0}^{\infty} \delta_k < \infty$, and $0 \leq \delta_k' < 1$, $\sum_{k=0}^{\infty} \delta_k' < \infty$. Then, for all $k \geq 0$ we have that

\[
\text{dist}_R(x_{k+1}, y_{k+1}, z_{k+1}, T_{\ell}^{-1}(0)) \leq \mu_k \text{dist}_R(x_k, y_k, z_k, T_{\ell}^{-1}(0))
\]

(2.19)

where

\[
\mu_k := (1 - \delta_k')^{-1} \left( \delta_k' (1 + \delta_k') \frac{\kappa \gamma_k}{\sqrt{\beta_k^2 + \kappa^2 \gamma_k^2}} \right),
\]

with $\gamma_k := \max\{\tau_k, 1\}$. Finally,

\[
\lim_{k \to \infty} \mu_k = \mu_\infty := \frac{\kappa \gamma_\infty}{\sqrt{\beta_\infty^2 + \kappa^2 \gamma_\infty^2}} \quad (\mu_\infty = 0, \text{ if } \beta_\infty = \infty),
\]

where $\gamma_\infty = \max\{\tau_\infty, 1\}$.

**Proof.** The proof is omitted since it follows by direct application of [41, Theorem 2.5] (see also [60, Theorem 2]).

**Remark 1.** Following [41, Remarks 2, 3], we can choose a non-increasing sequence $\{\delta_k'\}_{k=0}^{\infty}$ and a large enough $\beta_0$ such that $\mu_0 < 1$, which in turn implies that $\mu_k \leq \mu_0 < 1$, yielding a global linear convergence of both $\text{dist}((x_k, y_k, z_k), T_{\ell}^{-1}(0))$ as well as $\text{dist}_R((x_k, y_k, z_k), T_{\ell}^{-1}(0))$.

Furthermore, one can mirror the analysis in [41, Section 3.3], which indicates that Algorithm PD-PMM can have a finite termination property assuming that the associated PMM sub-problems are solved accurately enough.

### 3 Semismooth Newton method

In this section we employ a semi-smooth Newton (SSN) method to solve problem (2.7) in Algorithm PD-PMM. More specifically, given the estimates $(x_k, y_k, z_k)$ as well as the penalties $\beta_k$, $\rho_k$, we apply SSN to approximately solve (2.6). Given any bounded positive penalty $\zeta_k > 0$, the optimality conditions in (2.6) can equivalently be written as

\[
\hat{F}_{\beta_k, \rho_k, \zeta_k}(x, y) := \begin{bmatrix} x - \text{prox}_{\zeta_k g}(x - \zeta_k r_{\beta_k, \rho_k}(x, y)) \\ \zeta_k (Ax + \beta_k^{-1}(y - y_k) - b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

(3.1)

which follows from the properties of the $\text{prox}_{\zeta_k g}(\cdot)$ operator. We set $x_0 = x_k$, $y_0 = y_k$, and at every iteration $j$ of SSN, we approximately solve a system of the following form:

\[
M_k \begin{bmatrix} dx \\ dy \end{bmatrix} = -\hat{F}_{\beta_k, \rho_k, \zeta_k}(x_k, y_k),
\]

(3.2)

where $M_k \in \mathcal{M}_k$, with

\[
\mathcal{M}_k := \left\{ M = \begin{bmatrix} M_1 & M_2 \\ \zeta_k A & \beta_k^{-1} I_m \end{bmatrix} \in \mathbb{R}^{(n+m) \times (n+m)} : \right. \\
M_1 = \left( I - \hat{B}_{k_j}(\tilde{u}_{k_j}) \right) + \zeta_k \hat{B}_{k_j}(\tilde{u}_{k_j}) H(x_k, y_k), \\
H(x_k, y_k) \in \partial F(x_k, y_k) \right\},
\]

(3.3)

\[
\hat{B}_{k_j}(\tilde{u}_{k_j}) \in \partial F(\text{prox}_{\zeta_k g}(\tilde{u}_{k_j})), \quad \tilde{u}_{k_j} = x_k - \zeta_k r_{\beta_k, \rho_k}(x_k, y_k),
\]

\[
M_2 = -\zeta_k \hat{B}_{k_j}(\tilde{u}_{k_j}) A^T.
\]
The symbol $\partial^C_w(\cdot)$ denotes the Clarke subdifferential of a function (see [17]) with respect to $x$, which can be obtained as the convex hull of the Bouligand subdifferential ([17, 63]). We know that the Clarke subdifferential is a Newton derivative (see [18, Chapter 13]), as we have that $r_{\beta_k,\rho_k}(\cdot, y)$ and $g(\cdot)$ are piecewise continuously differentiable (i.e. PC$^1$) and regular functions (see [18]).

Using [18, Theorem 14.7] (see also [18, Example 14.9]), we obtain that for any $i \in \{1, \ldots, n\}$:

$$
\partial^C_{w_i}(\Pi_{[l_i, u_i]}(w_i)) = \begin{cases}
\{1\}, & \text{if } w_i \in (l_i, u_i), \\
\{0\}, & \text{if } w_i \notin [l_i, u_i], \\
[0, 1], & \text{if } w_i \in \{l_i, u_i\}.
\end{cases}
$$

Furthermore, since $g(x) = \|Dx\|_1$, where $D$ positive semi-definite and diagonal, we have

$$(\text{prox}_{\hat{\zeta}_i g}(w))_i = \max \left\{ |w_i| - \zeta_k D(i,i), 0 \right\} \text{sign}(w_i),$$

where $\text{sign}(\cdot)$ represents the sign of a scalar, and

$$(\partial^C_w(\text{prox}_{\hat{\zeta}_i g}(w)))_i = \begin{cases}
\{1\}, & \text{if } |w_i| > \zeta_k D(i,i), \text{ or } D(i,i) = 0, \\
\{0\}, & \text{if } |w_i| < \zeta_k D(i,i), \\
[0, 1], & \text{if } |w_i| = \zeta_k D(i,i).
\end{cases}$$

In order to complete the derivation of the SSN, we need to define a primal-dual merit function, based on which a backtracking line-search method will be employed to ensure that SSN is globally convergent. To that end, we write the resulting primal-dual sub-problem as an $\ell_1$-regularized convex instance, by using a generalized primal-dual augmented Lagrangian merit function (e.g. see [30]). Thus we obtain:

$$
\hat{\psi}(x, y) = \hat{\phi}(x, y) + g(x), \quad \hat{\phi}(x, y) := \phi(x) + \frac{\beta_k}{2} \|Ax + \beta_k^{-1}(y - y_k) - b\|^2,
$$

and the SSN sub-problem can be expressed as $\min_{x,y} \hat{\psi}(x, y)$. If $g(x) = 0$, then this smooth primal-dual merit function can be used to globalize the SSN. For properties as well as an analysis of this merit function, we refer the reader to [4, 30]. However, in the non-smooth case we have to resort to a different globalization strategy. Here we use the following merit function to globalize the SSN:

$$
\Theta(x, y) := \left\| \hat{F}_{\rho_k, \hat{\zeta}_k} (x_k, y_k) \right\|^2.
$$

(3.4)

This function is very often employed when globalizing SSN schemes applied to non-smooth equations of the form of (3.1) (also known as the natural map, e.g. see [56, 58]) by means of line-search. Indeed, its directional derivatives can be computed easily, assuming that the Bouligand subdifferential is exploited (see for example the analyses in [13, 32, 33, 45, 55] and the references therein). Algorithm SSN outlines a globalized and locally superlinearly convergent semismooth Newton method for the approximate solution of (2.7). We assume that the associated linear systems are approximately solved by means of a Krylov subspace method. An analysis of the effect of errors arising from the use of Krylov methods within SSN applied to nonsmooth equations can be found in [13].

Global convergence and local superlinear convergence of Algorithm SSN has been established multiple times in the literature, and we refer the reader to [13, 33, 45, 55] and the references therein for more details.
Algorithm SSN Semismooth Newton method

Input: Let $\epsilon_k > 0$, $\eta_1 \in (0, 1)$, $\eta_2 \in (0, 1]$, $\mu \in (0, \frac{1}{2})$, $\delta \in (0, 1)$, $\zeta_k > 0$.

Set: $x_{k0} = x_k$, $y_{k0} = y_k$.

for $(j = 0, 1, 2, \ldots)$ do

Compute $M_{kj} \in \mathcal{M}_{kj}$, where $\mathcal{M}_{kj}$ is defined in (3.3).

Solve:

$$M_{kj} \left[ \frac{d_x}{d_y} \right] \approx -\tilde{F}_{\beta_k, \rho_k, \zeta_k} \left( x_{kj}, y_{kj} \right),$$

such that $\left\| M_{kj} \left[ \frac{d_x}{d_y} \right] + \tilde{F}_{\beta_k, \rho_k, \zeta_k} \left( x_{kj}, y_{kj} \right) \right\| \leq \min \left\{ \eta_1, \left\| \tilde{F}_{\beta_k, \rho_k, \zeta_k} \left( x_{kj}, y_{kj} \right) \right\|^{1+\eta_2} \right\}$. 

(Line Search) Set $\alpha_j = \delta^{m_j}$, where $m_j$ is the first non-negative integer for which:

$$\Theta \left( x_{kj} + \delta^{m_j} d_x, y_{kj} + \delta^{m_j} d_y \right) \leq (1 - 2\mu\delta^{m_j}) \Theta \left( x_{kj}, y_{kj} \right)$$

$$x_{k,j+1} = x_{kj} + \alpha_j d_x, \quad y_{k,j+1} = y_{kj} + \alpha_j d_y.$$ 

if $(\text{dist} \left( 0, F_{\beta_k, \rho_k} \left( x_{kj}, y_{kj} \right) \right) \leq \epsilon_k)$ then

return $(x_{k,j+1}, y_{k,j+1})$.

end if

end for

At this point we should mention other alternatives to the merit function given in (3.4).

In particular, there has been an extensive literature on the globalization of semismooth Newton methods for the solution of nonsmooth equations. Indeed, there have been developed approaches based on trust-region strategies (e.g. see [1, 16, 23, 44, 56]), as well as line-search strategies based on smooth penalty functions (e.g. see the developments on the forward-backward envelope (FBE) [49, 50, 65, 66] or developments based on the proximal point method [24]). We have chosen to employ (3.4) since it is well-studied in the literature, it is simple to implement, and performs quite well for the problems studied in this paper.

3.1 Approximate solution of the SSN linear systems

We note that the major bottleneck of the previously presented inner-outer scheme, is the approximate solution of the associated linear systems in (3.2). As one can observe, Algorithm SSN does not require an exact solution. In turn this allows us to utilize preconditioned Krylov subspace solvers for the efficient solution of such systems.

Let $k \geq 0$ be an arbitrary iteration of Algorithm PD-PMM, and $j \geq 0$ an arbitrary iteration of Algorithm SSN. Firstly, let us notice that any element $B_{kj} \in \partial \mathcal{C} \left( r_{\beta_k, \rho_k} \left( x_{kj}, y_{kj} \right) \right)$ yields a Newton derivative (see [18, Theorem 14.8]). The same applies for any element $\hat{B}_{kj} \in \partial \mathcal{C} \left( \text{prox}_{\zeta_k, \hat{a}} \left( \hat{a}_{kj} \right) \right)$. Thus, we can choose $B_{kj}$, $\hat{B}_{kj}$ to improve computational efficiency. To that end, we set $B_{kj}$ as a diagonal matrix with

$$B_{kj,(i,i)} := \begin{cases} 1, & \text{if } \beta_k^{-1} z_{k,i} + x_{k,i} \in (l_i, u_i), \\ 0, & \text{otherwise}, \end{cases} \quad (3.5)$$

and $\hat{B}_{kj}$ as the following diagonal matrix:

$$\hat{B}_{kj,(i,i)} := \begin{cases} 1, & \text{if } |\hat{u}_{kj}| > \zeta_k D_{(i,i)}, \text{ or } D_{(i,i)} = 0, \\ 0, & \text{otherwise}, \end{cases} \quad (3.6)$$
where $\hat{u}_{kj}$ is defined in (3.3). Given (3.5), we can now explicitly write (3.2), for inner-outer iteration $k_j$ in the following saddle-point form

$\begin{bmatrix}
-G_{kj} & \zeta_k \hat{B}_j A^T \\
\zeta_k A & \zeta_k \beta_k^{-1} I_m
\end{bmatrix} M_{kj} \begin{bmatrix}
x_{kj} \\
y_k
\end{bmatrix} = \begin{bmatrix}
x_k - \text{prox}_{\zeta_k g} (\hat{u}_{kj}) \\
\zeta_k \left( b - A x_k - \beta_k^{-1} (y_k - y_k) \right)
\end{bmatrix},$

(3.7)

where

$G_{kj} := \left( I_n - \hat{B}_{kj} \right) + \zeta_k \hat{B}_{kj} H_{kj},$

and

$H_{kj} := Q + (\beta_k + \rho_k^{-1}) I_n - \beta_k B_{kj}.$

Driven from the definition of $\hat{B}_{kj}$ in (3.6), let us define the following index sets

$\hat{B}_j := \{ i \in \{1, \ldots, n\} : \hat{B}_{kj} = 1 \}, \quad \hat{N}_j := \{1, \ldots, n\} \setminus \hat{B}_j.$

Observe that $A \hat{B}_{kj} = \begin{bmatrix} A \hat{B}_j & 0 \end{bmatrix} \mathcal{P}^\top$, where $\mathcal{P}$ is an appropriate permutation matrix. Furthermore, we write:

$\mathcal{P}^\top G_{kj} \mathcal{P} = \begin{bmatrix}
\zeta_k H_{kj,(\hat{B}_j,\hat{B}_j)} & \zeta_k H_{kj,(\hat{B}_j,\hat{N}_j)} \\
0 & I_{\hat{N}_j}
\end{bmatrix}.$

Using the introduced notation, we re-write (3.7) as

$\begin{bmatrix}
-\zeta_k H_{kj,(\hat{B}_j,\hat{B}_j)} & -\zeta_k H_{kj,(\hat{B}_j,\hat{N}_j)} & \zeta_k A^T \hat{B}_j \\
0 & I_{\hat{N}_j} \\
\zeta_k A \hat{B}_j & \zeta_k A \hat{N}_j & \zeta_k \beta_k^{-1} I_m
\end{bmatrix} \begin{bmatrix}
d_{x,\hat{B}_j} \\
d_{x,\hat{N}_j} \\
d_y
\end{bmatrix} = \begin{bmatrix}
\mathcal{P}^\top - I_m \\
\mathcal{P}^\top I_m
\end{bmatrix} \begin{bmatrix}
\hat{F}_{\beta_k,\rho_k,\zeta_k} (x_k, y_k)
\end{bmatrix}.$

From the second block equation of the previous system, we obtain

$d_{x,\hat{N}_j} = - \left( x_{kj} - \text{prox}_{\zeta_k g} (\hat{u}_{kj}) \right)_{\hat{N}_j}.$

Thus, system (3.7) is reduced to the following saddle-point system

$\begin{bmatrix}
-H_{kj,(\hat{B}_j,\hat{B}_j)} & A^T \hat{B}_j \\
A \hat{B}_j & \beta_k^{-1} I_m
\end{bmatrix} \begin{bmatrix}
d_{x,\hat{B}_j} \\
d_{x,\hat{N}_j}
\end{bmatrix} = \begin{bmatrix}
\zeta_k^{-1} \left( x_k - \text{prox}_{\zeta_k g} (\hat{u}_{kj}) \right)_{\hat{B}_j} + H_{kj,(\hat{B}_j,\hat{N}_j)} d_{x,\hat{N}_j} \\
\left( b - A x_k - \beta_k^{-1} (y_k - y_k) \right)_{\hat{N}_j} d_{x,\hat{N}_j}
\end{bmatrix}.$

(3.8)

the coefficient matrix of which is symmetric quasi-definite (see [68] for a detailed discussion on symmetric quasi-definite matrices).

Next, we would like to construct an effective preconditioner for $\hat{M}_{kj}$. To that end, we define

$\hat{M}_{kj} := \begin{bmatrix}
\hat{H}_{kj,(\hat{B}_j,\hat{B}_j)} & 0 \\
0 & \left( A \hat{B}_j E_{kj} A^T \hat{B}_j + \beta_k^{-1} I_m \right)
\end{bmatrix},$

(3.9)
with $\tilde{H}_{kj, (\tilde{B}_j, \tilde{B}_j)} := \text{Diag} \left( H_{kj, (\tilde{B}_j, \tilde{B}_j)} \right)$ and $E_{kj} \in \mathbb{R}^{|\tilde{B}_j| \times |\tilde{B}_j|}$ the diagonal matrix defined as

$$E_{kj, (i,i)} := \begin{cases} \tilde{H}_{kj, (\tilde{B}_j, \tilde{B}_j)}^{-1} & \text{if } \beta_k^{-1} z_{k,i} + x_{kj, \tilde{B}_j,i} \in (l_{\tilde{B}_j,i}, u_{\tilde{B}_j,i}), \\ 0, & \text{otherwise}, \end{cases}$$

where $\tilde{B}_{j,i}$ denotes the $i$-th index of this index set, following the order imposed by the permutation matrix $\mathcal{P}$.

The preconditioner in (3.9) is an extension of the preconditioner proposed in [8, 31] for the solution of linear systems arising from the application of a regularized interior point method to convex quadratic programming problems. Being a diagonal matrix, $E_{kj}$ yields a sparse approximation of the Schur complement of the saddle-point matrix in (3.9). This sparse approximation is then used to construct a positive definite block-diagonal preconditioner (i.e. $\tilde{M}_{kj}$), which can be used within a symmetric Krylov solver, like the minimal residual (MINRES) method (see [48]). In order to invert the $(2,2)$ block of $\tilde{M}_{kj}$, one can employ a Cholesky decomposition.

In what follows, we analyze the spectral properties of the preconditioned matrix $(\tilde{M}_{kj})^{-1} \tilde{M}_{kj}$. To that end, let

$$\tilde{S}_{kj} := \left( A_{\tilde{B}_j} \tilde{H}_{kj, (\tilde{B}_j, \tilde{B}_j)}^{-1} A_{\tilde{B}_j}^\top + \beta_k^{-1} I_m \right), \quad \tilde{S}_{kj} := \left( A_{\tilde{B}_j} E_{kj} A_{\tilde{B}_j}^\top + \beta_k^{-1} I_m \right).$$

In the following lemma, we bound the eigenvalues of the preconditioned matrix $\tilde{S}_{kj}^{-1} \tilde{S}_{kj}$. This is subsequently used to analyze the spectrum of $\tilde{M}_{kj}^{-1} \tilde{M}_{kj}$.

**Lemma 3.1.** Given two arbitrary iterates $k$ and $j$ of Algorithms PD-PMM and SSN, respectively, we have that

$$1 \leq \lambda \leq 1 + \sigma_{\text{max}}^2(A) \left( \frac{1}{1 + \beta_k^{-2} \tau_k} \right),$$

where $\lambda \in \lambda \left(\tilde{S}_{kj}^{-1} \tilde{S}_{kj}\right)$, and $\beta_{\infty}$, $\tau_{\infty}$ are defined in Algorithm PD-PMM.

**Proof.** Consider the preconditioned matrix $\tilde{S}_{kj}^{-1} \tilde{S}_{kj}$, and let $(\lambda, u)$ be its eigenpair. Then, $\lambda$ must satisfy the following equation:

$$\lambda = \frac{u^\top (A_B B_B A_B^\top + \beta_k^{-1} I_m + A_N D_N A_N^\top) u}{u^\top (A_B B_B A_B^\top + \beta_k^{-1} I_m) u},$$

where $B = \{ i \in \tilde{B}_j : B_{kj,(i,i)} = 1 \}$, $N = \tilde{B}_j \setminus B$, $D_B = \tilde{H}_{kj,(B,B)}^{-1}$, $D_N = \tilde{H}_{kj,(N,N)}^{-1}$. The above equality holds since $\tilde{H}_{kj}$ is a diagonal matrix, and $E_{kj,(i,i)} = 0$ is zero for every $i$ such that $\tilde{B}_{j,i} \notin B$ (indeed, see the definition in (3.10)). Hence, from positive semi-definiteness of $Q$, we derive

$$1 \leq \lambda = 1 + \frac{u^\top (A_N D_N A_N^\top) u}{u^\top (A_B B_B A_B^\top + \beta_k^{-1} I_m) u} \leq 1 + \beta_k \sigma_{\text{max}}^2(A_N) \left( \beta_k + \rho_k^{-1} \right)^{-1}$$

$$\leq 1 + \sigma_{\text{max}}^2(A) \left( \frac{1}{1 + \beta_k^{-2} \tau_k} \right)$$

$$\leq 1 + \sigma_{\text{max}}^2(A) \left( \frac{1}{1 + \beta_{\infty}^{-2} \tau_{\infty}} \right),$$

where we used that $\rho_k = \beta_k / \tau_k$, $\beta_k \leq \beta_{\infty}$, and $\tau_k \geq \tau_{\infty}$. \hfill $\square$
Given Lemma 3.1, we are now able to invoke [8, Theorem 3] to characterize the spectral properties of the preconditioned matrix \( \tilde{M}_{kj}^{-1}M_{kj} \). Before proceeding with the theorem, let us introduce some notation. Let

\[
\begin{align*}
S_{kj} & := \left( \frac{1}{2} S_{kj} \right)^{-1/2}, \\
H_{kj} & := \tilde{H}_{kj,(\tilde{E}_{kj})}^{1/2} H_{kj,(\tilde{E}_{kj})}^{-1/2} \tilde{H}_{kj,(\tilde{E}_{kj})}^{-1/2}, \\
\alpha_{NE} & := \lambda_{\min}(S_{kj}), \\
\beta_{NE} & := \lambda_{\max}(S_{kj}), \\
\alpha_H & := \lambda_{\min}(H_{kj}), \\
\beta_H & := \lambda_{\max}(H_{kj}),
\end{align*}
\]

Notice that Lemma 3.1 yields upper and lower bounds for \( \alpha_{NE} \) and \( \beta_{NE} \). From the definition of \( H_{kj} \) we can also obtain that \( \alpha_H \leq 1 \leq \beta_H \) (see [8]). We are now ready to state the spectral properties of the preconditioned matrix \( \tilde{M}_{kj}^{-1}M_{kj} \).

**Theorem 3.2.** Let \( k, j \) be some arbitrary iteration counts of Algorithms PD-PMM and SSN, respectively. Then, the eigenvalues of \( \tilde{M}_{kj}^{-1}M_{kj} \) lie in the union of the following intervals:

\[
I_- := [-\beta_H - \sqrt{\beta_{NE}} - \alpha_H], \\
I_+ := \left[ \frac{1}{1 + \beta_H}, 1 + \sqrt{\beta_{NE}} - 1 \right].
\]

**Proof.** We omit the proof, which follows by direct application of [8, Theorem 3]. \( \square \)

**Remark 2.** By combining Lemma 3.1 with Theorem 3.2, we can observe that the eigenvalues of the preconditioned matrix \( \tilde{M}_{kj}^{-1}M_{kj} \) are not deteriorating as \( \beta_k \to \infty \). In other words, the preconditioner is robust with respect to the penalty parameters \( \beta_k, \rho_k \) of Algorithm PD-PMM. Furthermore, our choices of \( B_{kj}, \tilde{B}_{kj} \) in (3.5) and (3.6) respectively, serve the purpose of further sparsifying the preconditioner in (3.9), thus potentially further sparsifying its Cholesky decomposition.

**Remark 3.** For problems solved within this work, a diagonal approximation of the Hessian (within the preconditioner) seems sufficient to deliver very good performance. Indeed, this is the case for a wide range of problems. However, in certain instances, one might consider non-diagonal approximations of the Hessian. In that case, the preconditioner in (3.9) can be readily generalized and analyzed, following the developments in [31, Section 3].

## 4 Warm-starting

Following the developments in [41, 73], we would like to find a starting point \((x_0, y_0, z_0)\) for Algorithm PD-PMM that is relatively close to the solution of (P)–(D), since then we can expect to observe early linear convergence of the algorithm. To that end, we employ a proximal alternating direction method of multipliers (pADMM; e.g. see [22, 72]) to find an approximate solution of (P)–(D). To do so, we reformulate (P) by introducing an artificial variable \( w \), as follows:

\[
\min_{x \in \mathbb{R}^n, w \in \mathbb{R}^n} c^\top x + \frac{1}{2} x^\top Q x + g(w) + \delta_K(w), \quad \text{s.t. } Ax = b, \quad w - x = 0. \quad (P')
\]

Given a penalty \( \sigma > 0 \), we associate the following augmented Lagrangian to \((P')\)

\[
\hat{L}_\sigma(x, w, y_1, y_2) := c^\top x + \frac{1}{2} x^\top Q x + g(w) + \delta_K(w) - y_1^\top (Ax - b) - y_2^\top (w - x) + \frac{\sigma}{2} ||Ax - b||^2 + \frac{\sigma}{2} ||w - x||^2.
\]
Let an arbitrary positive definite matrix $R_x$ be given, and assume the notation $\|x\|_{R_x}^2 = x^T R_x x$. We now provide (in Algorithm pADMM) a proximal ADMM for the approximate solution of (P’).

**Algorithm pADMM proximal ADMM**

**Input:** $\sigma > 0$, $R_x > 0$, $\gamma \in \left(0, \frac{1 + \sqrt{5}}{2}\right)$, $(x_0, w_0, y_{1,0}, y_{2,0}) \in \mathbb{R}^{3n+m}$.

for $(k = 0, 1, \ldots)$ do

\[
\begin{align*}
    w_{k+1} &= \arg \min_w \left\{ \hat{L}_\sigma (x_k, w, y_{1,k}, y_{2,k}) \right\} \equiv \Pi_{\mathcal{K}} \left( \text{prox}_{\sigma^{-1}g} \left(x_k + \sigma^{-1}y_{2,k}\right) \right).
    \\
    x_{k+1} &= \arg \min_x \left\{ \hat{L}_\sigma (x, w_{k+1}, y_{1,k}, y_{2,k}) + \frac{1}{2} \|x - x_k\|_{R_x}^2 \right\}.
    \\
    y_{1,k+1} &= y_{1,k} - \gamma \sigma (Ax_{k+1} - b).
    \\
    y_{2,k+1} &= y_{2,k} - \gamma \sigma (w_{k+1} - x_{k+1}).
\end{align*}
\]

end for

Let us notice that under certain standard assumptions on (P’), Algorithm pADMM is able to achieve linear convergence (see [22]). This holds even in cases where $R_x$ is not positive definite; the reader is referred to [22, 37] for specific conditions on the nature of the indefiniteness of $R_x$. Nevertheless, here we assume that $R_x$ is positive definite and in particular, we employ it as a means of reducing the memory requirements of this approach. More specifically, given some constant $\sigma > 0$, such that $\sigma I_n - \text{Off}(Q) > 0$, we define

\[
    R_x = \sigma I_n - \text{Off}(Q),
\]

where $\text{Off}(B)$ denotes the matrix with zero diagonal and off-diagonal elements equal to the off-diagonal elements of $B$.

Notice that the first, third and fourth steps of Algorithm pADMM are trivial to solve, since we have assumed that $g(x) = \|Dx\|_1$, which is a proximable function (i.e. a convex function, the proximal operator of which can be computed expeditiously). Thus, the main computational bottleneck lies in the solution of the second sub-problem.

In order to efficiently solve the second step of Algorithm pADMM, we merge it with the subsequent dual updates (in steps three and four). This yields the following system of equations:

\[
\begin{bmatrix}
-\gamma (Q + R_x) & A^T & -I_n \\
A & \frac{1}{\gamma \sigma} I_m & 0 \\
-I_n & 0 & \frac{1}{\gamma \sigma} I_n
\end{bmatrix}
\begin{bmatrix}
x \\
y_1 \\
y_2
\end{bmatrix} = 
\begin{bmatrix}
\gamma (c - R_x x_k) + (1 - \gamma) (A^T y_{1,k} - y_{2,k}) \\
\frac{1}{\gamma \sigma} y_{1,k} \\
\frac{1}{\gamma \sigma} y_{2,k} - w_{k+1}
\end{bmatrix}.
\]

Assuming we have sufficient memory, the previous system can be solved by means of a $LDL^T$ factorization, since the coefficient matrix is symmetric quasi-definite (see [3, 68]). The benefit of this approach is that a single factorization can be utilized for all iterations of Algorithm pADMM. If the available memory is not sufficient, or the problem under consideration is structured (e.g. its data matrices belong to an appropriate structured matrix sequence), one might attempt to solve the previous system using a symmetric solver like MINRES ([48]) or CG ([34]). In this case, a preconditioner for either MINRES or CG would have to be computed only once (see for example the solver in [54]).

Finally, once an approximate solution $(\bar{x}, \bar{w}, \bar{y}_1, \bar{y}_2)$ is retrieved, we set the starting point of Algorithm PD-PMM as $(x_0, y_0, z_0) = (\bar{x}, \bar{y}_1, z)$, where

\[
    z = \bar{y}_2 - \Pi_{\partial g(\bar{w})} (\bar{y}_2).
\]
Indeed, an optimal primal-dual solution of \((P')\) is such that \(\tilde{y}^*_u \in \partial g(\tilde{w}^*) + \partial \delta_{\mathcal{K}}(\tilde{w}^*)\), thus the characterization of \(z\) in Algorithm PD-PMM can be obtained as in the Appendix (see Appendix A.3).

5 Applications to PDE-constrained optimization

In this section, we test the proposed methodology on some optimization problems with partial differential equation (PDE) constraints. We note that various other applications would be suitable for the presented method, however, we observe that the approach is especially efficient when applied to PDE optimization instances, and thus we focus on such problems.

We consider optimal control problems of the following form:

\[
\begin{align*}
\min_{y,u} & \quad J(y(x), u(x)), \\
\text{s.t.} & \quad Dy(x) + u(x) = g(x), \\
& \quad u_a(x) \leq u(x) \leq u_b(x),
\end{align*}
\]

(5.1)

where \((y, u) \in H^1(\Omega) \times L^2(\Omega), J(y(x), u(x))\) is a convex functional defined as

\[
J(y(x), u(x)) := \frac{1}{2} \|y - \bar{y}\|_{L^2(\Omega)}^2 + \frac{\alpha_1}{2} \|u\|_{L^1(\Omega)}^2 + \frac{\alpha_2}{2} \|u\|_{L^2(\Omega)}^2,
\]

(5.2)

\(D\) denotes some linear differential operator associated with the differential equation, \(x\) is a 2-dimensional spatial variable, and \(\alpha_1, \alpha_2 \geq 0\) denote the regularization parameters of the control variable. Other variants of the convex functional \(J(y, u)\) are possible, including measuring the state misfit and/or the control variable in different norms, as well as alternative weightings within the cost functionals.

The problem is considered on a given compact spatial domain \(\Omega\), where \(\Omega \subset \mathbb{R}^2\) has boundary \(\partial \Omega\), and is equipped with Dirichlet boundary conditions. The algebraic inequality constraints are assumed to hold a.e. on \(\Omega\). We further note that \(u_a\) and \(u_b\) may take the form of constants, or functions in spatial variables, however we restrict our attention to the case where these represent constants.

We solve problem (5.1) via a discretize-then-optimize strategy. In particular, we employ the Q1 finite element discretization implemented in IFISS\(^1\) (see [26, 27]). This yields a sequence of \(l_1\)-regularized convex quadratic programming problems, in the form of \((P)\). We note that the discretization of the smooth parts of problem (5.1) follows a standard Galerkin approach (e.g. see [67]), while the \(L^1\) term is discretized by the nodal quadrature rule as in [64, 70] (an approximation that achieves a first-order convergence–see [70]). In what follows, we consider two classes of state equations (i.e. the equality constraints in (5.1)): the Poisson’s equation, as well as the convection–diffusion equation.

Before proceeding with the experiments, let us mention certain implementation details. The solver is written in MATLAB and the code can be found on GitHub\(^2\). The experiments were run on MATLAB 2019a, on a PC with a 2.2GHz Intel core i7 processor (hexa-core), 16GM RAM, using the Windows 10 operating system. The warm-starting mechanism proposed in Section 4 is allowed to run for at most 400 iterations, and is terminated if it reaches a 3-digit accurate solution. Its associated linear systems are solved using a single call to the \texttt{ldl} decomposition of MATLAB. In order to accelerate the convergence.

\(^1\)https://personalpages.manchester.ac.uk/staff/david.silvester/ifiss/default.htm
\(^2\)https://github.com/spougkakiotis/SSN_PMM
of the SSN solver, we employ the following heuristic: at the first SSN iteration we accept a full step, without employing line-search. Then, every subsequent iteration follows exactly the developments in Section 3. This is because the PMM estimates/penalties are updated before every first SSN iteration, and thus we expect an increase of the magnitude of (3.4) after taking the Newton step. In turn, line-search would force very small step-lengths, significantly slowing down the algorithm. As we observe in practice, this heuristic does not prevent the algorithm from converging rapidly. Any linear system solved within SSN-PMM is solved using preconditioned MINRES, and the (2,2) block of the preconditioner (given in (3.9)) is inverted using MATLAB’s chol function.

The penalty parameters of PMM are tuned as follows: we initially set $\beta = 10^2$ and $\rho = 5 \cdot 10^2$. At the end of each call to SSN, we increase them at a suitable rate. In particular, these are increased more rapidly if the dual or primal infeasibilities, respectively, have sufficiently decreased. If not, we increase the penalties more conservatively. The termination criteria of the implemented approach are given in Appendix A.3. All other implementation details follow exactly the developments in Sections 2, 3.

5.1 Poisson optimal control

Let us first consider two-dimensional $L^1/L^2$-regularized Poisson optimal control problems. The problem is posed on $\Omega = (0,1)^2$. Following [31, 51], we set constant control bounds $u_a = -2$, $u_b = 1.5$, and set the desired state as $\bar{y} = \sin(\pi x_1)\sin(\pi x_2)$. In Table 1, we fix the $L^2$ regularization parameter to the value $\alpha_2 = 10^{-2}$, the tolerance to $\text{tol} = 10^{-5}$, and we present the runs of the method for varying $L^1$ regularization (i.e. $\alpha_1$) as well as grid size. We report the size of the resulting discretized problems, the value of $\alpha_1$, the number of PMM, SSN and MINRES iterations (with average MINRES iterations per solve in parenthesis), the total number of factorizations of the associated preconditioners, as well as the total time to convergence. As we will see in practice, it is often the case that the preconditioner used in a previous SSN iteration needs not be altered in a subsequent one. Thus, we report the overall number of factorizations employed by the algorithm. Finally, notice that in the case where $\alpha_1 = 0$, the problem is a standard convex quadratic program, and thus we employ a smooth line-search using $\hat{\phi}(x,y)$ as a merit function (see Section 3).

Next, we fix $\alpha_1 = 10^{-4}$ and $n = 1.32 \cdot 10^5$, and we vary the $L^2$ regularization parameter (i.e. $\alpha_2$) as well as the tolerance. The results are collected in Table 2.

We can draw several observations from the results in Tables 1, 2. Firstly, we should note that the algorithm is very efficient for finding a solution to relatively high accuracy (i.e. $\text{tol} = 10^{-5}$). Furthermore, we can see that the solver exhibits a level of robustness with respect to various parameters of the problem under consideration (except for the $L^1$ regularization parameter which affected the performance of the solver), and is always reliable. We should notice at this point that the implementation is rather aggressive, since we allow at most 8 SSN iterations per PMM sub-problem, and thus we are able to observe such a good efficiency. We should also notice that the method scales very well with the size of the problem, and the memory requirements are very reasonable, allowing the method to solve large-scale instances on a personal computer. Finally, when requesting a low-accuracy solution (i.e. $\text{tol} = 10^{-3}$), we observe that the second-order solver is barely needed, as the starting point yielded by Algorithm pADMM is already very close to such a solution.
Table 1: Poisson control: varying grid size and $L^1$ regularization ($tol = 10^{-5}$, $\alpha_2 = 10^{-2}$).

| $\alpha_1$ | $\alpha_2$ | Iterations | Factorizations | Time (s) |
|------------|------------|-------------|----------------|----------|
| $10^{-2}$  | $10^{-3}$  | PMM | 11 | 33 | 370 (11.21) | 24 | 2.97 |
| $10^{-4}$  | $10^{-4}$  | SSN | 11 | 32 | 296 (9.25) | 22 | 2.87 |
| $10^{-6}$  | $10^{-6}$  | MINRES (Avg.) | 11 | 31 | 288 (9.29) | 22 | 2.81 |

| $n$ | $\alpha_1$ | Iterations | Factorizations | Time (s) |
|-----|------------|-------------|----------------|----------|
| $8.45 \times 10^8$ | $10^{-2}$  | PMM | 15 | 41 | 526 (12.83) | 31 | 15.00 |
| $3.32 \times 10^4$ | $10^{-2}$  | SSN | 15 | 39 | 356 (9.13) | 29 | 12.38 |
| $1.32 \times 10^5$ | $10^{-2}$  | MINRES (Avg.) | 15 | 39 | 355 (9.10) | 30 | 12.72 |

Table 2: Poisson control: varying accuracy and $L^2$ regularization ($n = 1.32 \times 10^5$, $\alpha_1 = 10^{-4}$).

| $\alpha_2$ | $\alpha_3$ | Iterations | Factorizations | Time (s) |
|------------|------------|-------------|----------------|----------|
| $10^{-2}$  | $10^{-3}$  | PMM | 2 | 2 | 24 (12.00) | 2 | 32.52 |
| $10^{-4}$  | $10^{-4}$  | SSN | 2 | 2 | 24 (12.00) | 2 | 33.06 |
| $10^{-6}$  | $10^{-6}$  | MINRES (Avg.) | 2 | 2 | 24 (12.00) | 2 | 31.17 |

5.2 Convection–diffusion optimal control

At this point we consider the optimal control of the convection–diffusion equation, i.e. $\epsilon \Delta y + w \nabla y = u$, on the domain $\Omega = (0,1)^2$, where $w$ is the wind vector given by $w = [2x_2(1-x_1)^2,-2x_1(1-x_2)^2]^T$, with control bounds $u_a = -2$, $u_b = 1.5$ and free state (e.g. see [51, Section 5.2]). Once again, the problem is discretized using $Q_1$ finite elements, employing the Streamline Upwind Petrov-Galerkin (SUPG) upwinding scheme.
implemented in [12]. We define the desired state as $\tilde{y} = \exp(-64((x_1 - 0.5)^2 + (x_2 - 0.5)^2))$ with zero boundary conditions. The diffusion coefficient $\epsilon$ is set as $\epsilon = 0.02$. As before, we set the $L^2$ regularization parameter $\alpha_2$ as $\alpha_2 = 10^{-2}$ and fix the tolerance to $\text{tol} = 10^{-6}$. We present the runs of the method with different $L^1$ regularization values (i.e. $\alpha_1$) and with increasing grid size. The results are collected in Table 3.

**Table 3: Convection–diffusion control: varying grid size and $L^1$ regularization ($\text{tol} = 10^{-6}$, $\alpha_2 = 10^{-2}$).**

| $n$         | $\alpha_1$ | Iterations | Factorizations | Time (s) |
|-------------|-------------|------------|----------------|----------|
|             |             | PMM SSN MINRES (Avg.) |              |          |
| $8.45 \cdot 10^3$ | $10^{-3}$  | 16 115 4,378 (38.07) | 64           | 16.56    |
|             | $10^{-4}$  | 16 88 3,462 (39.34) | 30           | 12.25    |
|             | $10^{-5}$  | 16 48 1,487 (30.98) | 25           | 6.09     |
|             | 0          | 16 48 1,128 (23.50) | 24           | 5.40     |
| $3.32 \cdot 10^4$ | $10^{-3}$  | 17 109 3,659 (33.53) | 69           | 65.71    |
|             | $10^{-4}$  | 17 57 2,019 (35.42) | 30           | 34.68    |
|             | $10^{-5}$  | 17 36 959 (26.64)  | 25           | 20.27    |
|             | 0          | 17 36 746 (20.72)  | 25           | 17.13    |
| $1.32 \cdot 10^5$ | $10^{-3}$  | 18 84 2,720 (32.38) | 61           | 230.09   |
|             | $10^{-4}$  | 18 28 793 (28.32)  | 22           | 81.13    |
|             | $10^{-5}$  | 18 23 516 (22.43)  | 21           | 67.29    |
|             | 0          | 18 23 477 (20.74)  | 21           | 66.10    |
| $5.26 \cdot 10^5$ | $10^{-3}$  | 16 27 613 (22.70)  | 26           | 235.21   |
|             | $10^{-4}$  | 16 17 323 (19.00)  | 17           | 136.58   |
|             | $10^{-5}$  | 16 17 322 (18.94)  | 17           | 136.17   |
|             | 0          | 16 17 324 (19.06)  | 16           | 139.83   |

We then set $\alpha_1 = 10^{-3}$, $n = 1.32 \cdot 10^5$, $\text{tol} = 10^{-6}$, and run the method with varying $L^2$ regularization as well as diffusion coefficient $\epsilon$. The results are collected in Table 4. There are several observations that can be drawn from the previous results. Firstly, we can observe that for both problem classes, the method is robust with respect to the $L^2$ regularization parameter. On the other hand, there is a slight dependence on the $L^1$ regularization, although the method manages to solve efficiently instances for a wide range of values for this parameter. For the convection–diffusion problem, we also observe a slight dependence on the diffusion coefficient (as expected). Importantly, the behaviour of the algorithm does not deteriorate as the problem dimensions are increased, and as expected, the preconditioner is robust with the problem size as well as all the problem parameters. Finally, we can observe that the method is able to find accurate solutions consistently and very efficiently, making it a competitive solver for PDE constrained optimization instances.

6 Conclusions

In this paper we derived a proximal method of multipliers that employs a semismooth Newton method for the solution of the associated sub-problems, suitable for $\ell_1$-regularized convex quadratic instances. We have shown that the method converges globally under very mild assumptions, while it can potentially achieve a global linear and local superlinear convergence rate. The linear systems within SSN are solved using the preconditioned
minimal residual method, and the proposed preconditioner is cheap to invert and exhibits very good behaviour and robustness with respect to the PMM parameters. The efficiency of the method is further improved by using a warm-starting strategy based on a proximal alternating direction method of multipliers. The proposed approach has been extensively tested on certain PDE-constrained optimization problems, and the computational evidence has been provided to demonstrate that it is efficient and reliable.

A Appendix

A.1 Derivation of the dual problem

In what follows we derive the dual of (P). To that end, we note from Section 2 that the Lagrangian associated to (P) is

$$\ell(x, y, z) = c^T x + \frac{1}{2} x^T Q x + g(x) + z^T x - \delta^*_K(z) - y^T (A x - b).$$

Let $$f(x) = c^T x + \frac{1}{2} x^T Q x$$. We proceed by deriving the dual problem as

$$\inf_x \{\ell(x, y, z)\} = \inf_x \left\{ f(x) + g(x) + z^T x - y^T A x \right\} + y^T b - \delta^*_K(z)$$

where in the second and fourth equalities we used the definition of the convex conjugate, while in the third equality we used a property of the infimal convolution, i.e. \((f + g)^*(x) = \inf_{x'} \{f^*(x - x') + g^*(x')\}\) (see [5, Proposition 13.21]). However, from the definition of \(f(\cdot)\) we have

$$f^*(A^T y - z - x') = \frac{1}{2} x^T Q x + \delta_0 \left(c + Q x - A^T y + z + x'\right).$$

By substituting this, and by eliminating variable \(x'\) we obtain (D).
A.2 Characterization of \( \text{dist}(0, F_{\beta_k,p_k}(x,y)) \)

Let an arbitrary pair \((x,y)\) be given, and define

\[
F_{\beta_k,p_k}(x,y) := \{(u',v') : u' \in r_{\beta_k,p_k}(x,y) + \partial g(x), : v' = Ax + \beta_k^{-1}(y - y_k) - b\},
\]

where \(r_{\beta_k,p_k}(x,y)\) is defined in Section 2. Then, using the definition of \(\text{dist}(x,A)\), where \(A\) is a closed convex set, we have

\[
\text{dist}(0, F_{\beta_k,p_k}(x,y)) = \left\| \left[ \text{prox}_{f}(0) \right] - \left[ Ax + \beta_k^{-1}(y - y_k) - b \right] \right\|
\]

where \(\hat{f}(w) = \delta_{\partial g(x)}(w - r_{\beta_k,p_k})\), and \(\delta_{\partial g(x)}(\cdot)\) is an indicator function of the sub-differential of \(g(\cdot)\). Then, we note that \(\text{prox}_{f}(0) = \text{prox}_{\delta_{\partial g(x)}(-r_{\beta_k,p_k}) + r_{\beta_k,p_k}}\). A direct evaluation of this proximal operator yields the characterization used in Algorithm PD-PMM.

A.3 Termination criteria

Let us derive the optimality conditions for \((P)-(D)\), which are used to construct termination criteria for Algorithm PD-PMM. To that end, using the Lagrangian associated to \((P)\), i.e.

\[
\ell(x,y,z) = c^\top x + \frac{1}{2}x^\top Qx + g(x) + z^\top x - \delta^*_K(z) - y^\top (Ax - b),
\]

we can write the optimality conditions for \((P)-(D)\) as

\[
0 \in c + Qx - A^\top y + z + \partial g(x), \quad 0 = Ax - b, \quad 0 \in x - \partial \delta^*_K(z).
\]

However, given a closed proper convex function \(f(\cdot)\), the condition \(w \in \partial f(x)\) can equivalently be written as \(x = \text{prox}_f(x + w)\). Furthermore, \(x \in \partial f^*(w) \iff w \in \partial f(x)\). Then, the optimality conditions for \((P)-(D)\) can be re-written as

\[
\begin{align*}
x &= \text{prox}_{g}\left(x - c - Qx + A^\top y - z\right), \\
Ax &= b, \quad x = \Pi_K(x + z),
\end{align*}
\]

and the termination criteria for Algorithm PD-PMM (given a tolerance \(\epsilon > 0\)) can be summarized as

\[
\begin{align*}
\frac{\|x - \text{prox}_{g}\left(x - c - Qx + A^\top y - z\right)\|}{1 + \|c\|} &\leq \epsilon, \quad \frac{\|Ax - b\|}{1 + \|b\|} \leq \epsilon, \quad \frac{\|x - \Pi_K(x + z)\|}{1 + \|x\| + \|z\|} \leq \epsilon. \\
\end{align*}
\]

From the reformulation of \((P)\) given in \((P')\), the termination criteria of Algorithm pADMM are as follows (upon noting that the variables of the algorithm are \((x,y_1,y_2)\))

\[
\begin{align*}
\frac{\|c + Qx - A^\top y_1 + y_2\|}{1 + \|c\|} &\leq \epsilon, \quad \frac{\|(Ax - b, w - x)\|}{1 + \|b\|} \leq \epsilon, \quad \frac{\|w - \Pi_K\left(\text{prox}_g\left(x + w + y_2\right)\right)\|}{1 + \|w\| + \|y_2\|} \leq \epsilon.
\end{align*}
\]

References

[1] Z. Akbari, R. Yousefpour, and M. R. Peyghami, A new nonsmooth trust region algorithm for locally Lipschitz unconstrained optimization problems, Journal of Optimization Theory and Applications, 164 (2015), pp. 733-754, https://doi.org/10.1007/s10957-014-0534-6.
A. Altman and J. Gondzio, Regularized symmetric indefinite systems in interior point methods for linear and quadratic optimization, Optimization Methods and Software, 11 (1999), pp. 275–302, https://doi.org/10.1080/10556789908805754.

E. D. Andersen, J. Gondzio, C. Mészáros, and X. Xu, Implementation of interior point methods for large scale linear programming, in Interior Point Methods in Mathematical Programming, T. Terlaky, ed., Kluwer Academic Publishers, 1996, pp. 189–252, https://doi.org/10.1007/978-1-4613-3449-1_6.

P. Armand and R. Omheni, A globally and quadratically convergent primal–dual augmented Lagrangian algorithm for equality constrained optimization, Optimization Methods and Software, 32 (2017), pp. 1–21, https://doi.org/10.1080/10556788.2015.1025401.

H. H. Bauschke and P. L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, CMS Books in Mathematics, Springer, New York, NY, 2011, https://doi.org/10.1007/978-1-4419-9467-7.

A. Beck, First-Order Methods in Optimization, MOS-SIAM Series on Optimization, SIAM & Mathematical Optimization Society, Philadelphia, 2017, https://doi.org/10.1137/1.9781611974997.

A. Beck and M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, SIAM Journal on Imaging Sciences, 2 (2009), pp. 183–202, https://doi.org/10.1137/080716542.

L. Bergamaschi, J. Gondzio, A. Martínez, J. W. Pearson, and S. Pougkakiotis, A new preconditioning approach for an interior point-proximal method of multipliers for linear and convex quadratic programming, Numerical Linear Algebra with Applications, 28 (2020), p. e2361, https://doi.org/10.1002/nla.2361.

D. P. Bertsekas, A. Nedic, and E. Ozdaglar, Convex Analysis and Optimization, Athena Scientific, 2003.

P. T. Boggs and J. W. Tolle, Sequential quadratic programming for large-scale nonlinear optimization, Journal of Computational and Applied Mathematics, 124 (2000), pp. 123–137, https://doi.org/10.1016/S0377-0427(00)00429-5.

S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, Distributed optimization and statistical learning via the alternating direction method of multipliers, Foundations and Trends in Machine Learning, 3 (2010), pp. 1–122, https://doi.org/10.1561/2200000016.

A. N. Brooks and T. J. R. Hughes, Streamline upwind/Petrov–Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier–Stokes equations, Computer Methods in Applied Mechanics and Engineering, 32 (1982), pp. 199–259, https://doi.org/10.1016/0045-7825(82)90071-8.

J. Chen and L. Qi, Globally and superlinearly convergent inexact Newton-Krylov algorithms for solving nonsmooth equations, Numerical Linear Algebra with Applications, 17 (2010), pp. 155–174, https://doi.org/10.1002/nla.673.
[14] S. S. Chen, D. L. Donoho, and M. A. Saunders, *Atomic decomposition by basis pursuit*, SIAM Review, 43 (2001), pp. 129–159, [https://doi.org/10.1137/S003614450037906X](https://doi.org/10.1137/S003614450037906X).

[15] T. Chen, F. E. Curtis, and D. P. Robinson, *FaRSA for ℓ1-regularized convex optimization: local convergence and numerical experience*, Optimization Methods and Software, 33 (2018), pp. 396–415, [https://doi.org/10.1080/10556788.2017.1415336](https://doi.org/10.1080/10556788.2017.1415336).

[16] C. Christof, H. C. De Los Reyes, and C. Meyer, *A nonsmooth trust-region method for locally Lipschitz functions with applications to optimization problems constrained by variational inequalities*, SIAM Journal on Optimization, 30 (2020), pp. 2163–2196, [https://doi.org/10.1137/18M1164925](https://doi.org/10.1137/18M1164925).

[17] F. Clarke, *Optimization and Nonsmooth Analysis*, Classics in Applied Mathematics, John Wiley and Sons, New York, 1990, [https://doi.org/10.1137/1.9781611971309](https://doi.org/10.1137/1.9781611971309).

[18] C. Clason and T. Valkonen, *Introduction to Nonsmooth Analysis and Optimization*, 2020, [https://arxiv.org/abs/arXiv:1912.08672](https://arxiv.org/abs/arXiv:1912.08672).

[19] P. L. Combettes and J. C. Pesquet, *Proximal splitting methods in signal processing*, in Fixed-Point Algorithms for Inverse Problems in Science and Engineering, H. Bauschke, R. Burachik, P. Combettes, V. Elser, D. Luke, and H. Wolkowicz, eds., vol. 49 of Springer Optimization and Its Applications, Springer, New York, NY, 2011, [https://doi.org/10.1007/978-1-4419-9569-8_10](https://doi.org/10.1007/978-1-4419-9569-8_10).

[20] A. De Marchi, *On a primal-dual Newton proximal method for convex quadratic programs*, Computational Optimization and Applications, (2022), [https://doi.org/10.1007/s10589-021-00342-y](https://doi.org/10.1007/s10589-021-00342-y).

[21] V. De Simone, D. di Serafino, J. Gondzio, S. Pougkakiotis, and M. Viola, *Sparse approximations with interior point methods*, 2021, [https://arxiv.org/abs/arXiv:2102.13608](https://arxiv.org/abs/arXiv:2102.13608). Accepted at SIAM Review.

[22] W. Deng and W. Yin, *On the global and linear convergence of the generalized alternating direction method of multipliers*, Journal of Scientific Computing, 66 (2016), pp. 889–916, [https://doi.org/10.1007/s10915-015-0048-x](https://doi.org/10.1007/s10915-015-0048-x).

[23] J. E. Dennis, S.-B. B. Li, and R. A. Tapia, *A unified approach to global convergence of trust region methods for nonsmooth optimization*, Mathematical Programming, 68 (1995), pp. 319–346, [https://doi.org/10.1007/BF01585770](https://doi.org/10.1007/BF01585770).

[24] N. K. Dhingra, S. Z. Khong, and M. R. Jovanović, *A second order primal-dual algorithm for nonsmooth convex composite optimization*, in 2017 IEEE 56th Annual Conference on Decision and Control (CDC), 2017, pp. 2868–2873, [https://doi.org/10.1109/CDC.2017.8264075](https://doi.org/10.1109/CDC.2017.8264075).

[25] J. Eckstein, *Nonlinear proximal point algorithms using Bregman functions, with applications to convex programming*, Mathematics of Operations Research, 18 (1993), pp. 202–226, [https://doi.org/10.1287/moor.18.1.202](https://doi.org/10.1287/moor.18.1.202).

[26] H. C. Elman, A. Ramage, and D. J. Silvester, *Algorithm 866: IFISS, a Matlab toolbox for modelling incompressible flow*, ACM Transactions on Mathematical Software, 33 (2007), p. 14, [https://doi.org/10.1145/1236463.1236469](https://doi.org/10.1145/1236463.1236469).
[27] H. C. Elman, A. Ramage, and D. J. Silvester, *IFISS: A computational laboratory for investigating incompressible flow problems*, SIAM Review, 52 (2014), pp. 261–273, https://doi.org/10.1137/120891393.

[28] K. Fountoulakis, J. Gondzio, and P. Zhlobich, *Matrix-free interior point method for compressed sensing problems*, Mathematical Programming Computation, 6 (2014), pp. 1–31, https://doi.org/10.1007/s12532-013-0063-6.

[29] M. P. Friedlander and D. Orban, *A primal-dual regularized interior-point method for convex quadratic programs*, Mathematical Programming Computation, 4 (2012), pp. 71–107, https://doi.org/10.1007/s12532-012-0036-2.

[30] P. E. Gill and D. P. Robinson, *A primal–dual augmented Lagrangian*, Computational Optimization and Applications, 15 (2012), pp. 1–25, https://doi.org/10.1007/s10589-010-9339-1.

[31] J. Gondzio, S. Pougkakiotis, and J. W. Pearson, *General-purpose preconditioning for regularized interior point methods*, 2021, https://arxiv.org/abs/arXiv:2107.06822.

[32] S.-P. Han, J.-S. Pang, and N. Rangaraj, *Globally convergent Newton methods for nonsmooth equations*, Mathematics of Operations Research, 17 (1992), pp. 586–607, https://doi.org/10.1287/moor.17.3.586.

[33] E. Hans and T. Raasch, *Global convergence of damped semismooth Newton methods for ℓ1 Tikhonov regularization*, Inverse Problems, 31 (2015), p. 025005, https://doi.org/10.1088/0266-5611/31/2/025005.

[34] M. R. Hestenes and E. Stiefel, *Method of conjugate gradients for solving linear systems*, Journal of Research of the National Bureau of Standards, 49 (1952), pp. 409–436.

[35] J.-B. Hiriart-Urruty, J.-J. Strodiot, and V. H. Nguyen, *Generalized Hessian matrix and second-order optimality conditions for problems with C^{1,1} data*, Applied Mathematics and Optimization, 11 (1984), pp. 43–56, https://doi.org/10.1007/BF01442169.

[36] K. Ito and K. Kunisch, *On a semi-smooth Newton method and its globalization*, Mathematical Programming, 118 (2009), pp. 347–370, https://doi.org/10.1007/s10107-007-0196-3.

[37] F. Jiang, Z. Wu, and X. Cai, *Generalized ADMM with optimal indefinite proximal term for linearly constrained convex optimization*, Journal of Industrial & Management Optimization, 16, pp. 835–856, https://doi.org/10.3934/jimo.2018181.

[38] C. Kanzow and T. Lechner, *Globalized inexact proximal Newton-type methods for nonconvex composite functions*, Computational Optimization and Applications, 78 (2021), pp. 377–410, https://doi.org/10.1007/s10589-020-00243-6.

[39] J. D. Lee, Y. Sun, and M. A. Saunders, *Proximal Newton-type methods for minimizing composite functions*, SIAM Journal on Optimization, 24 (2014), pp. 1420–1443, https://doi.org/10.1137/130921428.
[40] X. Li, D. Sun, and K. C. Toh, *A highly efficient semismooth Newton augmented Lagrangian method for solving Lasso problems*, SIAM Journal on Optimization, 28 (2018), pp. 433–458, https://doi.org/10.1137/16M1097572.

[41] X. Li, D. Sun, and K. C. Toh, *An asymptotically superlinearly convergent semismooth Newton augmented Lagrangian method for linear programming*, SIAM Journal on Optimization, 30 (2020), pp. 2410–2440, https://doi.org/10.1137/19M1251795.

[42] Z. Lu and X. Chen, *Generalized conjugate gradient methods for $\ell_1$ regularized convex quadratic programming with finite convergence*, Mathematics of Operations Research, 43 (2017), pp. 275–303, https://doi.org/10.1287/moor.2017.0865.

[43] F. Mannel and A. Rund, *A hybrid semismooth quasi-Newton method for nonsmooth optimal control with PDEs*, Optimization and Engineering, 22 (2021), pp. 2087–2125, https://doi.org/10.1007/s11081-020-09523-w.

[44] F. Mannel and A. Rund, *A hybrid semismooth quasi-Newton method for nonsmooth optimal control with PDEs*, Optimization and Engineering, 22 (2021), pp. 2087–2125, https://doi.org/10.1007/s11081-020-09523-w.

[45] J. Martínez and L. Qi, *Inexact Newton methods for solving nonsmooth equations*, Journal of Computational and Applied Mathematics, 60 (1995), pp. 127–145, https://doi.org/10.1016/0377-0427(94)00088-I.

[46] A. Milzarek and M. Ulbrich, *A semismooth Newton method with multidimensional filter globalization for $l_1$-optimization*, SIAM Journal on Optimization, 24 (2014), pp. 298–333, https://doi.org/10.1137/120892167.

[47] J. J. Moreau, *Proximité et dualité dans un espace Hilbertien*, Bulletin de la Société Mathématique de France, 93 (1965), pp. 273–299, https://doi.org/10.24033/bsmf.1625.

[48] C. C. Paige and M. A. Saunders, *Solution of sparse indefinite systems of linear equations*, SIAM Journal on Numerical Analysis, 12 (1975), pp. 617–629, https://doi.org/10.1137/0712047.

[49] P. Patrinos and A. Bemporad, *Proximal Newton methods for convex composite optimization*, in 52nd IEEE Conference on Decision and Control, 2013, pp. 2358–2363, https://doi.org/10.1109/CDC.2013.6760233.

[50] P. Patrinos, L. Stella, and A. Bemporad, *Forward-backward truncated Newton methods for convex composite optimization*, 2014, https://arxiv.org/abs/arXiv:1402.6655.

[51] J. W. Pearson, M. Porcelli, and M. Stoll, *Interior-point methods and preconditioning for PDE-constrained optimization problems involving sparsity terms*, Numerical Linear Algebra with Applications, 27 (2019), p. e2276, https://doi.org/10.1002/nla.2276.

[52] M. Porcelli, V. Simoncini, and M. Stoll, *Preconditioning PDE-constrained optimization with $l^1$-sparsity and control constraints*, Computers & Mathematics with Applications, 74 (2017), pp. 1059–1075, https://doi.org/10.1016/j.camwa.2017.04.033.
[53] S. Pougkakiotis and J. Gondzio, An interior point-proximal method of multipliers for convex quadratic programming, Computational Optimization and Applications, 78 (2021), pp. 307–351, https://doi.org/10.1007/s10589-020-00240-9.

[54] S. Pougkakiotis, J. W. Pearson, S. Leveque, and J. Gondzio, Fast solution methods for convex quadratic optimization of fractional differential equations, SIAM Journal on Matrix Analysis and Applications, 41 (2020), pp. 1443–1476, https://doi.org/10.1137/19M128288X.

[55] L. Qi, Convergence analysis of some algorithms for solving nonsmooth equations, Mathematics of Operations Research, 18 (1993), pp. 227–244, https://doi.org/10.1287/moor.18.1.227.

[56] W. Quoyang and A. Milzarek, A trust region-type normal map-based semismooth Newton method for nonsmooth nonconvex composite optimization, 2021, https://arxiv.org/abs/arXiv:2106.09340v1.

[57] S. M. Robinson, Some continuity properties of polyhedral multifunctions, in Mathematical Programming at Oberwolfach, H. König, B. Korte, and K. Ritter, eds., vol. 14 of Mathematical Programming Studies, Springer, Berlin, Heidelberg, 1981, pp. 206–214, https://doi.org/10.1007/BFb0120929.

[58] S. M. Robinson, Normal maps induced by linear transformations, Mathematics of Operational Research, 17 (1992), pp. 691–714, https://doi.org/10.1287/moor.17.3.691.

[59] R. T. Rockafellar, Augmented Lagrangians and applications of the proximal point algorithm in convex programming, Mathematics of Operations Research, 1 (1976), pp. 97–116, https://doi.org/10.1287/moor.1.2.97.

[60] R. T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM Journal on Control and Optimization, 14 (1976), pp. 877–898, https://doi.org/10.1137/0314056.

[61] R. T. Rockafellar and R. J. B. Wets, Variational Analysis, vol. 317 of Grundlehren der mathematischen Wissenschaften, Springer-Verlag Berlin Heidelberg, 1998, https://doi.org/10.1007/978-3-642-02431-3.

[62] L. I. Rudin, S. Osher, and E. Fatemi, Nonlinear total variation based noise removal algorithms, Physica D: Nonlinear Phenomena, 60 (1992), pp. 259–268, https://doi.org/10.1016/0167-2789(92)90242-F.

[63] A. Shapiro, On concepts of directional differentiability, Journal of Optimization Theory and Applications, 66 (1990), pp. 477–487, https://doi.org/10.1007/BF00940933.

[64] X. Song, B. Chen, and B. Yu, An efficient duality-based approach for PDE-constrained sparse optimization, Computational Optimization and Applications, 69 (2018), pp. 461–500, https://doi.org/10.1007/s10589-017-9951-4.

[65] L. Stella, A. Themelis, and P. Patrinos, Forward-backward quasi-Newton methods for nonsmooth optimization problems, Computational Optimization and Applications, 67 (2017), pp. 443–487, https://doi.org/10.1007/s10589-017-9912-y.
A. Themelis, L. Stella, and P. Patrinos, *Forward-backward envelope for the sum of two nonconvex functions: Further properties and nonmonotone line-search algorithms*, SIAM Journal on Optimization, 28 (2018), pp. 2274–2303, https://doi.org/10.1137/16M1080240.

F. Tröltzsch, *Optimal Control of Partial Differential Equations: Theory, Methods and Applications*, vol. 112 of Graduate Studies in Mathematics, American Mathematical Society, 2010, https://doi.org/10.1090/gsm/112.

R. J. Vanderbei, *Symmetric quasidefinite matrices*, SIAM Journal on Optimization, 5 (1993), pp. 100–113, https://doi.org/10.1137/0805005.

V. N. Vapnik, *Statistical Learning Theory*, John Wiley & Sons, New York, 1998.

G. Wachsmuth and D. Wachsmuth, *Convergence and regularization results for optimal control problems with sparsity functional*, ESAIM: Control, Optimisation and Calculus of Variations, 17 (2011), pp. 858–886, https://doi.org/10.1051/cocv/2010027.

R. A. Waltz, J. L. Morales, J. Nocedal, and D. Orban, *An interior algorithm for nonlinear optimization that combines line search and trust region steps*, Mathematical Programming, 107 (2006), pp. 391–408, https://doi.org/10.1007/s10107-004-0560-5.

H. Wang and A. Banerjee, *Bregman alternating direction method of multipliers*, in Advances in Neural Information Processing Systems, Z. Ghahramani, M. Welling, C. Cortes, N. Lawrence, and K. Q. Weinberger, eds., vol. 27, Curran Associates, Inc., 2014, https://proceedings.neurips.cc/paper/2014/file/ad71c82b22f4f65b9398f76d8be4c615-Paper.pdf.

X. Y. Zhao, D. F. Sun, and K. C. Toh, *A Newton-CG augmented Lagrangian method for semidefinite programming*, SIAM Journal on Optimization, 20 (2010), pp. 1737–1765, https://doi.org/10.1137/080718206.