THE SHEAF OF TWISTED CHEREDNIK ALGEBRAS AS A UNIVERSAL FILTERED FORMAL
DEFORMATION

ALEXANDER VITANOV

ABSTRACT. The notion of sheaves of (twisted) global Cherednik algebras \( \mathcal{H}_{t,c,\psi,X,G} \) attached to a global quotient orbifold \( X/G \) was introduced and first studied by Pavel Etingof in 2004. For the special case of an affine variety \( X \) with a finite group \( G \) acting on it, he proved that when \( t = 1 \) and \( c,\psi \) are formal parameters, \( \mathcal{H}_{1,c,\psi,X,G} \) is a universal formal deformation of the smash product \( \mathcal{D}_X \rtimes G \) on \( X/G \) of the sheaf of differential operators \( \mathcal{D}_X \) on \( X \) and the group \( G \). The main result of this note is a generalization of Etingof’s proof to the case of a global quotient orbifold \( X/G \), where \( X \) is a generic smooth analytic or algebraic variety. We prove that in these cases \( \mathcal{H}_{1,c,\psi,X,G} \) with formal parameters \( c \) and \( \psi \) is a universal filtered formal deformation of \( \mathcal{D}_X \rtimes G \).

First, we construct quasi-isomorphisms between the Hochschild (co)chain complex of \( \mathcal{D}_X \rtimes G \) and the \( G \)-invariant part of the direct sum of sheaves of holomorphic de Rham differential forms on cotangent bundles of the fixed point submanifolds in \( X \) associated to cyclic subgroups of \( G \). By means of them we compute the hypercohomology of the Hochschild (co)chain complex of \( \mathcal{D}_X \rtimes G \). Combining these quasi-isomorphisms with results from the theory of filtered algebraic extensions of sheaves of filtered associative algebras we compute the space \( \mathcal{E} / \mathcal{F} (\mathcal{D}_X \rtimes G) \) of isomorphism classes of filtered infinitesimal deformations of \( \mathcal{D}_X \rtimes G \). Finally, we compute the dimension of \( \mathcal{E} / \mathcal{F} (\mathcal{D}_X \rtimes G) \) and consequently prove that the sheaf of twisted global Cherednik algebras on \( X/G \) is a universal filtered deformation of \( \mathcal{D}_X \rtimes G \) in the global case.

CONTENTS

1. Introduction 1
   1.1. Outline of the main results 2
   1.2. Notation 3
2. Deformation Theory of Sheaves of Filtered Associative Algebras 3
   2.1. Square-zero extensions of associative algebras and infinitesimal deformations 3
   2.2. Square-zero extensions of sheaves of filtered algebras and infinitesimal deformations 5
3. Universal deformation of \( \mathcal{D}_X \rtimes G \) 8
   3.1. Sheaves of Calabi-Yau algebras 8
   3.2. Trace density morphisms 11
   3.3. The space of filtered infinitesimal deformations of \( \mathcal{D}_X \rtimes G \) 18
Acknowledgement 24
References 24

1. I

In the pioneering work [Eti04] Pavel Etingof proposed a broad generalization of the notion of a rational Cherednik algebra associated to a complex representation \( h \) of a finite group \( G \) to a smooth algebraic or analytic variety \( X \) with an action by a finite group \( G \) of automorphisms. He introduced a sheaf of twisted Cherednik algebras \( \mathcal{H}_{t,c,\psi,X,G} \) on the quotient orbifold \( X/G \), where \( t \) is a complex number, \( c \) are class functions on the set of complex reflections in \( G \) and \( \psi \in H^2(X, \Omega_X^{\geq 1}$/G$, and showed that much of the standard theory of rational Cherednik algebras possesses a natural generalization to the global case. For example, one can derive a global notion of a “Calogero-Moser space” for \( X \) by means of the center of the sheaf of Cherednik algebras. From the point of view of deformation theory the central result in his work is [Eti04, Theorem 2.23] according to which the sheaf of twisted Cherednik algebras \( \mathcal{H}_{t,c,\psi,X,G} \) with formal parameters \( c \) and \( \psi \) on \( X/G \), where \( X \) is a complex affine variety, is a universal formal deformation of the skew-group algebra \( \mathcal{D}_X \rtimes G \). Etingof’s proof relies solely on the fact that affine varieties are also \( D \)-affine thanks to which the space of infinitesimal deformation of \( \mathcal{D}_X \rtimes G \) is isomorphic to the second Hochschild cohomology group of the associative algebra of global sections \( \Gamma(X, \mathcal{D}_X \rtimes G) \). This reduces the global statement
to the well-known linear case. Unfortunately, as smooth analytic and algebraic varieties are not $D$-affine in general, one cannot identify $\mathcal{D}_X \times G$ with $\Gamma(X, \mathcal{D}_X \times G)$. That is the reason why the correct version of [Eti04, Theorem 2.23] in the non-affine global context demands a fully sheaf-theoretic approach. Even though some version of [Eti04, Theorem 2.23] in the case of generic smooth algebraic and analytic varieties has always been expected to be true until now, a proper generalization of that result and a corresponding proof thereof have lacked. The current note closes this longstanding gap providing necessary mathematical tools along with a final proof of the fact that the sheaf of twisted Cherednik algebras arises as a universal formal filtered deformation of $\mathcal{D}_X \times G$ in the global algebraic and analytic case. We briefly elaborate on the main results of the note in the ensuing subsection.

1.1. Outline of the main results. Section 2 is devoted to the definition of the space $\text{Def}(\Lambda)$ of first-order deformations of a sheaf of filtered associative algebras $\Lambda$. The naive identification of this space with the set $\text{exal}(\Lambda, \Lambda)$ of locally trivial square-zero extensions of $\Lambda$ by the $\Lambda - \Lambda$ bimodule $\Lambda$ is not meaningful for the purposes of deformation theory because, as noted to the author by Pavel Etingof and Valery Lunts, in many crucial cases such as for instance $\Lambda = \mathcal{D}_X$ on elliptic curve $X$ the set $\text{exal}(\Lambda, \Lambda)$ becomes infinite-dimensional. For sheaves of algebras with infinite dimensional space of infinitesimal formal deformations it is not clear how to define a meaningful notion of a universal formal deformation. However, it turns out that in the special case of sheaves of filtered associative algebras one can restrict itself to a subspace $\text{exal}(\mathcal{M}, \Lambda)_f$ of filtered square-zero extensions which in many cases including the relevant for our aims case $\Lambda = \mathcal{D}_X \times G$ has the advantage of being finite-dimensional. The main result in this section is Theorem 2.10 whose proof is an imitation of [Gra61, Theorem 3.2] for the case of filtered Hochschild cochains.

**Theorem (A).** There is an isomorphism of $\mathbb{C}$-vector spaces between $\text{exal}_f(\Lambda, \mathcal{M})$ and the second hypercohomology group $H^2(X, \sigma_{\geq 1} \mathcal{O}_X^*(\mathcal{D}_X \times G, \mathcal{D}_X \times G))$ of the filtered Hochschild cochain complex $\mathcal{C}_*(\Lambda, \mathcal{M})$ of $\Lambda$ with values in $\mathcal{M}$ brutally truncated at 1.

Section 2 is the core of the paper and it is mostly devoted to the explicit computation of the hypercohomology group $H^2(X, \sigma_{\geq 1} \mathcal{O}_X^*(\mathcal{D}_X \times G, \mathcal{D}_X \times G))$ in Theorem (A). Subsection 2.1 is by and large a rehash of the well-known theory of Calabi-Yau algebras. Here we contribute a definition of sheaves of Calabi-Yau algebras. The main result is Proposition 3.3 in which we give an explicit locally free resolution of $\text{Sym}^*(\mathcal{O}_X)$ in terms of left $\text{Sym}^*(\mathcal{O}_X)$-modules which we use to prove that $\mathcal{D}_X$ and $\mathcal{D}_X \times G$ are sheaves of Calabi-Yau algebras of dimension 2 $\text{dim}(X)$ (cf. Proposition 3.6 and Corollary 3.7) and later to prove that the canonical inclusion of the complex of filtered cochains of $\mathcal{D}_X \times G$ in the Hochschild cochain complex of $\mathcal{D}_X \times G$ is a quasi-isomorphism (cf. Proposition 5.13). In Subsection 3.2 we begin by defining a quasi-isomorphism from the Hochschild chain complex of $\mathcal{D}_X \times G$ to the $G$-invariant part of the direct sum of holomorphic de Rham complexes of cotangent bundles, shifted by the dimension of the total space of the bundle, over the strata in $X$ associated to the cyclic subgroups in $G$. The quasi-isomorphism is an adaptation of a similar construction in [Vit19] to the holomorphic setup involving trace density morphisms for $\frac{H}{\Gamma} \times G$. The construction of this quasi-isomorphism relies on a technique, developed in [Vit19], which realizes the sheaf of untwisted Cherednik algebras $\mathcal{H}_{1,c,0,X,G}$ $(c$ is not necessarily formal) in terms of gluing of special sheaves $\pi_0^\text{coor} \mathcal{O}_{\text{bat}(\mathcal{O}_X^* \circ \mathcal{O}_{X'}^H \circ \mathcal{O}_{X''}^H)}$ of flat sections of algebra bundles over connected components of fixed point submanifolds $X^H_i$ restricted to open and dense subsets $X^H_{j_i}$, called strata of $X$ associated to $H$ in $G$. This presentation of the sheaf of untwisted Cherednik algebras yields a natural projection map from $\mathcal{H}_{1,c,0,X,G}$ to the sheaf $\pi_0^\text{coor} \mathcal{O}_{\text{bat}(\mathcal{O}_X^* \circ \mathcal{O}_{X'}^H \circ \mathcal{O}_{X''}^H)}$ for every subgroup $H$ of $G$ by means of which we can use an Engeli-Felder type construction [EF08] along the lines of [RT12] and later [Vit20] to define morphism

$$\chi^H_i : \mathcal{C}_*(\mathcal{D}_X \times G) \to (j_i^H \pi_i^H \Omega_{T^*X^H_i}^{2n-2l_i^H} \star G)^G$$

where $\Omega_{T^*X^H}^*$ is the complex of holomorphic differential forms on the cotangent bundle $\pi^H_i : T^*X^H_i \to X^H$ to the connected component $X^H_i$ with codimension $l_i^H$ of the fixed point set $X^H$ and $j_i^H$ is the canonical inclusion of $X^H_i$ in $X$. Using the Calabi-Yau property of $\mathcal{D}_X \times G$, proven earlier, we arrive at a similar map for the Hochschild cochain complex of $\mathcal{D}_X \times G$

$$\mathcal{X}^H_i : \mathcal{C}_*(\mathcal{D}_X \times G, \mathcal{D}_X \times G) \to (j_i^H \pi_i^H \Omega_{T^*X^H_i}^{2n-2l_i^H} \star G)^G.$$  

Let $\langle g \rangle$ be the cyclic subgroup of $G$ generated by the element $g$ in $G$. One of the central results in the whole note is the following theorem (cf. Theorem 3.11 and Corollary 3.12).
Theorem (B). For every choice of a linearly independent trace $\phi^{i}_{(g)}$ of $\mathfrak{D}_{X}^{g} \times (g)$ and for every family of holomorphic Maurer-Cartan forms $\{\omega_{a}\}$ on the cotangent bundle $T^{*}X_{\Gamma}^{g}$ of the connected component of the fixed point submanifold $X_{l_{g}}^{g}$ with codimension $l_{g}$ with values in $\mathfrak{a}^{(g)}_{n \to l_{g}^{g}}$, the maps

\begin{equation}
\oplus_{i,g \in G}X_{l_{g}}^{g} : \mathfrak{g}(\mathfrak{D}_{X} \times \mathfrak{G}) \rightarrow \left( \oplus_{i,g \in G}j_{i}^{g} \pi_{l_{g}}^{g} \Omega_{T^{*}X_{\Gamma}^{g}}^{\mathfrak{g}} \right)^{G}
\end{equation}

\begin{equation}
\oplus_{i,g \in G}X_{l_{g}}^{g} : \mathfrak{g}(\mathfrak{D}_{X} \times \mathfrak{G}, \mathfrak{D}_{X} \times \mathfrak{G}) \rightarrow \left( \oplus_{i,g \in G}j_{i}^{g} \pi_{l_{g}}^{g} \Omega_{T^{*}X_{\Gamma}^{g}}^{\mathfrak{g}} \right)^{G}
\end{equation}

are quasi-isomorphisms. Moreover, the induced morphisms at the level of homology and cohomology sheaves are canonical.

By means of quasi-isomorphism \((2)\) from Theorem (B) we compute in Corollary 3.15 the space of infinitesimal formal deformations $\mathfrak{Def}(\mathfrak{D}_{X} \times \mathfrak{G})_{f}$ of $\mathfrak{D}_{X} \times \mathfrak{G}$. With this at hand we finally arrive at the main theorem 3.16 of the paper. In this note we show how to prove the ensuing result for analytic varieties but all of the presented results can be proven in the case of smooth algebraic varieties in a similar fashion.

Theorem (Main Theorem). Let $X$ be a smooth algebraic variety or a smooth analytic variety equipped with a finite subgroup $G \subset \text{Aut}(X)$ acting faithfully on $X$. The sheaf of twisted Cherednik algebras $\mathfrak{H}_{k,b,s(X),G}$ on the quotient orbifold $X/G$ is a universal formal filtered deformation of the skew-group algebra $\mathfrak{D}_{X} \times \mathfrak{G}$.

1.2. Notation. From now on until the end of this paper $X$ will denote a complex analytic manifold of dimension $\dim_{\mathbb{C}}X = n$ and equipped with an action by a finite group $G$ of holomorphic automorphisms. A complex reflection is an element $g \in G$ such that the fixed point submanifold $X^{g}$ has a connected component of codimension 1. Throughout the note we do not insist that $G$ contains complex reflections.

2. Deformation Theory of Sheaves of Filtered Associative Algebras

First, we quickly review well-known aspects of the theory of extension of associative algebras and its relation to formal deformations.

2.1. Square-zero extensions of associative algebras and infinitesimal deformations. Let $k$ be a field of characteristic zero. In the following we denote by $\Lambda$ an associative $k$-algebra with a unit and by $\Lambda^{\text{op}}$ the opposite $k$-algebra. Let $\Lambda^{\epsilon} := \Lambda \otimes_{k} \Lambda^{\text{op}}$ denote the enveloping algebra of $\Lambda$. In the ensuing we use that every left $\Lambda^{\epsilon}$-module is a left $\Lambda^{\epsilon}$-module.

Definition 2.1. An extension of $\Lambda$ is a $k$-algebra $\Gamma$ together with an $k$-algebra epimorphism $\sigma : \Gamma \rightarrow \Lambda$.

An extension $\sigma : \Gamma \rightarrow \Lambda$ is called square-zero extension of $\Lambda$ if the kernel of $\sigma$, $\ker(\sigma)$, satisfies $\ker(\sigma)^{2} = 0$. In that case the left $\Gamma^{\text{op}}$-module $\ker(\sigma)$ acquires a well-defined structure of a left $\Lambda^{\epsilon}$-module.

Definition 2.2. Let $A$ be a left $\Lambda^{\epsilon}$-module. A square-zero extension of $\Lambda$ by $A$ is a short exact sequence of $k$-vector spaces

\[ \zeta : 0 \rightarrow A \xrightarrow{\chi} \Gamma \xrightarrow{\sigma} \Lambda \rightarrow 0 \]

such that $\Gamma$ is a k-algebra, $\sigma : \Gamma \rightarrow \Lambda$ is a square-zero extension of $\Lambda$ and for $A$, considered as a left $\Gamma^{\text{op}}$-module by pullback along $\sigma$, the $k$-vector space isomorphism $\chi : A \rightarrow \ker(\sigma)$ is also an isomorphism of $\Gamma^{\text{op}}$-modules.

Two square-zero extensions $\zeta : 0 \rightarrow A \xrightarrow{\chi} \Gamma \xrightarrow{\sigma} \Lambda \rightarrow 0$ and $\zeta' : 0 \rightarrow A \xrightarrow{\chi'} \Gamma' \xrightarrow{\sigma'} \Lambda \rightarrow 0$ of $\Lambda$ by a $\Lambda^{\epsilon}$-bimodule $A$ are said to be equivalent if there is a $k$-algebra morphism $\beta : \Gamma \rightarrow \Gamma'$ such that the diagram

\[ \begin{array}{ccc}
0 & \rightarrow & A \\
\chi & \downarrow & \chi' \\
\downarrow & & \downarrow \\
\Gamma & \rightarrow & \Lambda \\
\beta \circ \sigma & & \sigma' \\
\end{array} \]

commutes. By the 5-Lemma, if such a morphism $\beta$ exists, it is an isomorphism of $k$-algebras. A square-zero extension $\zeta : 0 \rightarrow A \xrightarrow{\chi} \Gamma \xrightarrow{\sigma} \Lambda \rightarrow 0$ is $k$-split if there is a $k$-linear map $\theta : \Lambda \rightarrow \Gamma$ such that $\sigma \circ \theta = \text{id}_\Lambda$. The splitting map $\theta$ defines a $k$-linear isomorphism $\Lambda \oplus A \cong \Gamma$. The pull-back of the multiplication in $\Gamma$ to $\Lambda \oplus A$ yields a product on
the direct sum given by \((\lambda_1, a_1)(\lambda_2, a_2) = (\lambda_1\lambda_2, \lambda_1 a_2 + a_1 \lambda_2 + \mu(\lambda_2, \lambda_2))\) for all \((\lambda_1, a_1), (\lambda_2, a_2) \in \Gamma\), where 
\(\mu(\lambda_1, \lambda_2) = \theta(\lambda_1 \lambda_2) - \theta(\lambda_1)\theta(\lambda_2)\) is a Hochschild 2-cocycle in \(HH^2(\Lambda, A)\) called the factor set of the k-split square-zero extension. The so-defined product gives an isomorphism of \(k\)-algebras \(\Gamma \to \Lambda \oplus A\) defining an equivalence of square-zero extensions. A square-zero extension of \(\Lambda\) by \(A\) is called split if there is a \(k\)-algebra morphism \(\theta : \Lambda \to \Gamma\) such that \(\sigma \circ \theta = \text{id}_\Lambda\). In particular, every split extension is \(k\)-split. All split extensions of \(\Lambda\) by a left \(\Lambda^e\)-module \(A\) are equivalent to the semi-direct sum

\[0 \to A \to \Lambda \oplus A \to \Lambda\]

where \(\Lambda \oplus A\) is a \(k\)-algebra with respect to the product \((\lambda_1, a_1)(\lambda_2, a_2) = (\lambda_1\lambda_2, \lambda_1 a_2 + a_1 \lambda_2)\). The set of isomorphism classes of extensions of \(\Lambda\) by \(A\) is denoted by \(\text{Ext}_k(\Lambda, A)\). The Brauer sum of short exact sequences gives \(\text{Ext}_k(\Lambda, A)\) the structure of a \(k\)-vector space with zero element the isomorphism class of split extensions of \(\Lambda\) by \(A\). Let \(t\) be a central element in \(\Lambda\). We recall the definition of a first-order deformation of \(\Lambda\).

**Definition 2.3 (Definition 3.10, [DMZ07]).** Let \(R\) be an augmented unital ring with an augmentation \(\epsilon : R \to k\). A \(R\)-deformation of \(\Lambda\) is an associative \(R\)-algebra \(B\) together with a \(k\)-algebra isomorphism \(B \otimes_k R \cong \Lambda\).

In this note we are primarily interested in infinitesimal (first-order) deformations which are \(k[t]/(t^2)\)-deformations in the sense of the above definition. These deformations can be equivalently characterized using Hochschild cohomology.

**Proposition 2.4.** Given a \(k[t]/(t^2)\)-deformation \(B\) of \(\Lambda\), there is a unique Hochschild 2-cocycle \(\mu_1 \in \text{Hom}_k(\Lambda \otimes_k \Lambda, \Lambda)\) such that the \(k[t]/(t^2)\)-vector space \(\Lambda \otimes_k k[t]/(t^2)\), equipped with the product \(\lambda_1 \ast \lambda_2 = \lambda_1 \lambda_2 + \mu_1(\lambda_1, \lambda_2)t \mod (t^2)\), becomes a \(k[t]/(t^2)\)-algebra isomorphic to \(B\) as \(k[t]/(t^2)\)-algebras.

**Proof.** Proof is verbatim identical to the proof of Theorem 3.15 in [DMZ07].

Two first-order deformations \(B\) and \(B'\) of \(\Lambda\) are said to be equivalent if the corresponding unique Hochschild 2-cocycles \(\mu\) and \(\mu'\) are cohomologous, e.g. \([\mu] = [\mu']\) in \(HH^2(\Lambda, \Lambda)\). Denote by \(\text{Def}(\Lambda)\) the set of equivalence classes of infinitesimal deformations \(B\) of \(\Lambda\). We have the following easy to prove but crucial identity.

**Lemma 2.5. There is a one-to-one correspondence between the sets \(\text{Ext}_k(\Lambda, \Lambda)\) and \(\text{Def}(\Lambda)\).**

**Proof.** Suppose \(\zeta : 0 \to \Lambda \xrightarrow{\times} \Gamma \xrightarrow{\sigma} \Lambda \to 0\) is a representative of an isomorphism class in \(\text{Ext}_k(\Lambda, \Lambda)\). Since per assumption \(k\) is a field, \(\zeta\) is \(k\)-split. That means that it is equivalent to a square-zero extension

\[\zeta : 0 \to \Lambda t \xrightarrow{\times} \Lambda \oplus \Lambda t \xrightarrow{\sigma} \Lambda \to 0\]

where we accounted that \(\Lambda \cong \Lambda t\) as \(\Lambda^e\)-bimodules. Consequently, for all \(\lambda_1, \lambda_2 \in \Lambda\) we have

\[(\lambda_1, 0)(\lambda_2, 0) = (\lambda_1 \lambda_2, \mu(\lambda_1, \lambda_2))\]

where \(\mu\) is a Hochschild 2-cocycle factor set. By Proposition 2.4, the factor set \(\mu\) determines a unique \(k[t]/(t^2)\)-algebra structure on \(\Lambda \oplus \Lambda t \cong \Lambda \otimes_k k[t]/(t^2)\) which is an infinitesimal deformation of \(\Lambda\). Any other representative of the isomorphism class of \(\zeta\) is equivalent to \(\zeta\) and thus induces a cohomologous factor set. Hence, there is a well-defined map

\[\text{Ext}_k(\Lambda, \Lambda) \to \text{Def}(\Lambda)\]

Conversely, assume that \(B\) is a \(k[t]/(t^2)\)-deformation of \(\Lambda\). Again by Proposition 2.4, it is uniquely described by a Hochschild 2-cocycle \(\mu \in \text{Hom}_k(\Lambda \otimes_k \Lambda, \Lambda)\) which according to Theorem 3.1 in [Mac75] corresponds to the factor set of a square-zero extension \(\zeta\) of \(\Lambda\) by \(\Lambda\). An equivalent \(k[t]/(t^2)\)-deformation \(B'\) of \(\Lambda\) is per definition given by a cohomologous Hochschild 2-cocycle \(\mu' \in \text{Hom}_k(\Lambda \otimes_k \Lambda, \Lambda)\) which in turn defines a zero-square extension of \(\Lambda\) by \(\Lambda\) which is equivalent to \(\zeta\). Thus, there is a well-defined map

\[\text{Def}(\Lambda) \to \text{Ext}_k(\Lambda, \Lambda)\]

such that it and the map 3 are inverses of each other.

The bijectivity between \(\text{Ext}_k(\Lambda, \Lambda)\) and \(\text{Def}(\Lambda)\) equips the later with the structure of a \(k\)-vectors space.
2.2. **Square-zero extensions of sheaves of filtered algebras and infinitesimal deformations.** The material in the following section is mostly inspired by the content in [Gra61]. Let $\Lambda$ be a sheaf of associative $\mathbb{C}$-algebras on a (complex) manifold $X$ an let $\mathcal{A}$ be a left $\Lambda^e$-module.

**Definition 2.6.** A square-zero extension of $\Lambda$ by $\mathcal{A}$ is a sheaf of $\mathbb{C}$-algebras $\Gamma$ together with an exact sequence of sheaves of $\mathbb{C}$-vector spaces

\[
0 \to \mathcal{A} \xrightarrow{i} \Gamma \xrightarrow{p} \Lambda \to 0
\]

in which $p$ is a square-zero extension of sheaves of $\mathbb{C}$-algebras as in Definition 2.1 satisfying the properties of Definition 2.2.

Two algebra extensions $\Gamma$ and $\Gamma'$ of $\Lambda$ by a $\Lambda^e$-module $\mathcal{A}$ are said to be equivalent if there is a homomorphism of sheaves of $\mathbb{C}$-algebras $k : \Gamma \to \Gamma'$ such that the diagram

\[
\begin{array}{ccc}
0 & \to & \mathcal{A} \\
\downarrow & & \downarrow \\
\Gamma & \xrightarrow{k} & \Gamma'
\end{array}
\begin{array}{ccc}
\downarrow & & \downarrow \\
\Lambda & \xrightarrow{\sigma} & \Lambda
\end{array}
\begin{array}{ccc}
\downarrow & & \downarrow \\
\Lambda & \to & 0
\end{array}
\]

commutes. In that case, $k$ is an isomorphism of sheaves of $\mathbb{C}$-algebras by the 5-Lemma. Unlike the case of square-zero extensions of associative $k$-algebras, discussed in the previous section, in general not every zero-square extension of a sheaf of algebras $\Lambda$ by a sheaf of $\Lambda^e$-modules splits as a short exact sequence of sheaves of $\mathbb{C}$-vector spaces. Such extensions are of no interest for the study of deformation theory because local infinitesimal deformations of the product of a sheaf of algebras always arise from (local) splitting morphisms of sheaves of vector spaces. That is why we shall restrict our attention to a special subclass of so-called *locally trivial* square-zero extensions of sheaves of algebras.

**Definition 2.7.** A square-zero extension $0 \to \mathcal{A} \xrightarrow{\chi} \Gamma \xrightarrow{\sigma} \Lambda \to 0$ of $\Lambda$ by $\mathcal{A}$ is called locally trivial if there is an open cover $\{U_\alpha\}$ of $X$ such that for each $U_\alpha$, the short exact sequence of sheaves of $\mathbb{C}$-vector spaces

\[
0 \to \mathcal{A}|_{U_\alpha} \xrightarrow{\chi} \Gamma|_{U_\alpha} \xrightarrow{\sigma} \Lambda|_{U_\alpha} \to 0
\]

is $\mathbb{C}|_{U_\alpha}$-split, i.e. there exists a $\mathbb{C}|_{U_\alpha}$-linear homomorphism $j_\alpha : \Lambda|_{U_\alpha} \to \Gamma|_{U_\alpha}$ such that $\sigma \circ j_\alpha = \text{id}_{\Lambda|_{U_\alpha}}$.

A square-zero extension $0 \to \mathcal{A} \xrightarrow{i} \Gamma \xrightarrow{\sigma} \Lambda \to 0$ of $\Lambda$ by $\mathcal{A}$ is split if it admits a morphism of sheaves of algebras $j : \Lambda \to \Gamma$ such that $\sigma \circ j = \text{id}_\Gamma$. In particular, a split square-zero extension is locally trivial. We denote the set of isomorphism classes of locally-trivial square-zero algebra extensions of $\Lambda$ by $\mathcal{A}$ by $\text{exal}(\Lambda, \mathcal{A})$. It is a $\mathbb{C}$-vector space with respect to the Baer sum and the isomorphism class of split square-zero extensions of $\Lambda$ by $\mathcal{A}$ as the zero element.

For the study of twisted differential operators, it makes sense to restrict our attention to the subclass of sheaves of filtered associative $\mathbb{C}$-algebras. In the remainder of this section $\Lambda$ will be equipped with an increasing filtration $\Lambda_0 \subseteq \Lambda_1 \subseteq \cdots \subseteq \Lambda_n \subseteq \cdots$ which is exhaustive, i.e. $\Lambda = \cup_{i=0}^{\infty} \Lambda_i$. Similarly, the $\Lambda^e$-module $\mathcal{A}$ will be endowed with an increasing filtration $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n \subseteq \cdots$ with $\mathcal{A}_m \cdot \mathcal{A} \subseteq \mathcal{A}_{m+n}$ which is exhaustive, i.e. $\mathcal{A} = \cup_{i=0}^{\infty} \mathcal{A}_i$.

**Definition 2.8.** A filtered square-zero extension of $\Lambda$ by $\mathcal{A}$ is a square-zero extension

\[
0 \to \mathcal{A} \xrightarrow{i} \Gamma \xrightarrow{p} \Lambda \to 0
\]

such that $\Gamma$ is a sheaf of $\mathbb{C}$-algebras with an increasing exhaustive filtration and $i$ and $p$ are filtered morphisms of $\mathbb{C}$-vector spaces.

**Definition 2.9.** A filtered square-zero extension $0 \to \mathcal{A} \xrightarrow{\chi} \Gamma \xrightarrow{\sigma} \Lambda \to 0$ of $\Lambda$ by $\mathcal{A}$ is locally trivial if there is an open cover $\{U_\alpha\}$ of $X$ such that for each $U_\alpha$, the short exact sequence

\[
0 \to \mathcal{A}|_{U_\alpha} \xrightarrow{\chi} \Gamma|_{U_\alpha} \xrightarrow{\sigma} \Lambda|_{U_\alpha} \to 0
\]

has a filtration-preserving $k$-linear homomorphism $j_\alpha : \Lambda|_{U_\alpha} \to \Gamma|_{U_\alpha}$ such that $\sigma \circ j_\alpha = \text{id}_{\Lambda|_{U_\alpha}}$. 


A filtered square-zero extension $0 \to \mathcal{A} \xrightarrow{i} \Gamma \xrightarrow{p} \Lambda \to 0$ of $\mathcal{A}$ is split if it admits a morphism of sheaves of $\mathbb{C}$-algebras $\phi : \mathcal{A} \to \Gamma$ such that $p \circ \phi = \text{id}_\Gamma$. In particular, a split filtered square-zero extension is locally trivial. Two filtered square-zero algebra extensions $\Gamma$ and $\Gamma'$ of $\Lambda$ by a $\Lambda^e$-module $\mathcal{A}$ are said to be equivalent if there is a filtered homomorphism of sheaves of $\mathbb{C}$-algebras $k : \Gamma \to \Gamma'$ such that the diagram

$$
0 \longrightarrow \mathcal{A} \xrightarrow{i} \Gamma \xrightarrow{k} \Lambda' \longrightarrow 0
$$

commutes. The subspace of $\text{exal}(\Lambda, \mathcal{A})$ comprised of filtered locally trivial square-zero extensions of $\Lambda$ by $\mathcal{A}$ is denoted by $\text{exal}_f(\Lambda, \mathcal{A})$. Denote by

$$(6) \quad \hat{\mathcal{C}}^* (\Lambda, \mathcal{A}) := \mathcal{A} \otimes_\mathbb{C} \Lambda^\otimes \cdot$$

the bounded below Hochschild chain complex of the sheaf of algebras $\Lambda$ with coefficients in the $\Lambda - \Lambda$-bimodule $\mathcal{A}$ and by

$$(7) \quad \hat{\mathcal{C}}^* (\Lambda, \mathcal{A}) := \text{Hom}_{\Lambda^e} (\Lambda^\otimes + 2, \mathcal{A}) \cong \text{Hom}_{\mathbb{C}} (\Lambda^\otimes, \mathcal{A}) ,$$

where $\mathcal{A}^e := \mathcal{A} \otimes_\mathbb{C} \mathcal{A}^{op}$, the bounded below complexes of continuous Hochschild cochains of $\Lambda$ with values in $\mathcal{A}$, respectively. Accordingly, we denote by $\mathcal{H}^*(\Lambda, \mathcal{A})$ and $\mathcal{H}^*(\Lambda, \mathcal{A})$ the homology sheaf of (6) and the cohomology sheaf of (7), respectively. If we view $\Lambda^\otimes + 2$ as an acyclic bar resolution of $\Lambda$ as a left $\Lambda^e$-module, $\mathcal{H}^*(\Lambda, \mathcal{A})$ and $\mathcal{H}^*(\Lambda, \mathcal{A})$ can be expressed in terms of left and right derived functors

$$(8) \quad \mathcal{H}^* (\Lambda, \mathcal{A}) = \text{Tor}^\Lambda (\mathcal{A}, \Lambda) = \mathcal{H}_* (\mathcal{A} \otimes_\Lambda \Lambda)$$

$$(9) \quad \mathcal{H}^* (\Lambda, \mathcal{A}) = \text{Ext}^\Lambda (\mathcal{A}, \Lambda) = \mathcal{H}^* (R \text{Hom}_{\Lambda^e} (\Lambda, \mathcal{A})) .$$

In case that $\Lambda$ and $\mathcal{A}$ are equipped with an increasing filtration, there is a natural increasing filtration on the $n$-fold tensor product $\Lambda^\otimes n$ for every integer $n \geq 0$ by

$$F^p (\Lambda^\otimes n) = \oplus_{i_1 + \cdots + i_n = p} F^{i_1} \Lambda \otimes \cdots \otimes F^{i_n} \Lambda.$$ 

The restriction of the Hochschild cochains of $\Lambda$ with values in $\mathcal{A}$ to filtration preserving cochains yields a subcomplex $\hat{\mathcal{C}}^*_f (\Lambda, \mathcal{A})$ of $\hat{\mathcal{C}}^* (\Lambda, \mathcal{A})$ which is equipped with a canonical decreasing filtration

$$\hat{\mathcal{C}}^*_f (\Lambda, \mathcal{A}) = F^0 \hat{\mathcal{C}}^*_f (\Lambda, \mathcal{A}) \supseteq F^1 \hat{\mathcal{C}}^*_f (\Lambda, \mathcal{A}) \supseteq \cdots \supseteq F^n \hat{\mathcal{C}}^*_f (\Lambda, \mathcal{A}) \supseteq \cdots$$

given by

$$F^p \hat{\mathcal{C}}^*_f (\Lambda, \mathcal{A}) := \left\{ f \in \text{Hom}_{\mathcal{A}} (\mathcal{A}^\otimes n, \mathcal{A}) : f (F^q (\Lambda^\otimes n)) \subseteq F^{q-p} \mathcal{A} \text{ for every } q \right\}$$

for every $n \geq 0$. We denote the cohomology sheaf of $\hat{\mathcal{C}}^*_f (\Lambda, \mathcal{A})$ by $\mathcal{H}^*_f (\Lambda, \mathcal{A})$.

**Theorem 2.10.** There is an isomorphism of $\mathbb{C}$-vector spaces between $\text{exal}_f (\Lambda, \mathcal{A})$ and $\mathbb{H}^2 (X, \sigma_{\geq 1} \mathcal{C}^*_f (\Lambda, \mathcal{A}))$.

**Proof.** The proof is verbatim a repetition of the proof of the first half of Theorem 3.2 in [Gra61] adapted to the filtered case. Nevertheless, for the sake of completeness, we shall provide a detailed proof.

Let $K^{**} := \hat{C}^* (\mathcal{U}, \hat{\mathcal{C}}^*_f (\Lambda, \mathcal{A}))$ be the Čech double complex associated to $\mathcal{U}$ :
in which $\delta$ is the Čech differential and $d$ is the standard Hochschild differential with $\delta d - d\delta = 0$ whose total complex $\text{Tot}^\bullet(K^{**})$ has a differential $D'_\bullet := \delta_{p,q} + (-1)^p d_{p,q}$ in bidegree $(p, q)$ and differential $D'_{tot} = \sum_{p+q=n} \delta_{p,q} + (-1)^p d_{p,q}$ in total degree $n$. Let $0 \to \mathcal{A} \xrightarrow{i} \Gamma \xrightarrow{p} \Lambda \to 0$ be an element in $\text{exal}_f(\Lambda, \mathcal{A})$. Then by definition there is an open cover $\mathcal{U} = \{U_a\}$ of $X$ together with filtered morphisms of sheaves of vector spaces $j_\alpha : \Lambda|_{U_a} \to \Gamma|_{U_a}$ for every $\alpha$. On intersection $U_{\alpha\beta} := U_\alpha \cap U_\beta$, define the filtered map $h_{\alpha\beta} := j_\beta - j_\alpha$. As $p \circ h_{\alpha\beta} = 0$, it follows that $\text{Im}(h_{\alpha\beta}) = \text{Im}(i) \cong \mathcal{A}$. Hence, abusing notation we obtain a morphism of sheaves of $\mathbb{C}$-vector spaces $h_{\alpha\beta} : \Lambda|_{U_{\alpha\beta}} \to \mathcal{A}|_{U_{\alpha\beta}}$. Let $\delta$ denote the Čech differential in the Čech complex $\text{C}^\bullet(\mathcal{U}, \mathcal{F}_1(\Lambda, \mathcal{A}))$. Then,

$$
\delta_{1,1}(h_{\alpha\beta}) = (j_\gamma - j_\beta) - (j_\gamma - j_\alpha) + (j_\beta - j_\alpha) = 0
$$

implies that $h_{\alpha\beta}$ is a Čech 1-cocycle in $\text{C}^1(\mathcal{U}, \mathcal{F}_1(\Lambda, \mathcal{A}))$. Define a map $f_\alpha : \Lambda|_{U_a} \otimes \mathbb{C} \Lambda|_{U_a} \to \mathcal{A}|_{U_a}$ as the composition

$$
\Lambda|_{U_a} \otimes \mathbb{C} \Lambda|_{U_a} \xrightarrow{j_\alpha \otimes c_{j_\alpha}} \Gamma|_{U_a} \otimes \mathbb{C} \Gamma|_{U_a} \xrightarrow{\text{mult}} \Gamma|_{U_a} \xrightarrow{q_\alpha} \mathcal{A}|_{U_a}
$$

where $\text{mult}$ denotes the product in $\Gamma$ and $q_\alpha := \text{id}_\Gamma - j_\alpha \circ p$. The associativity of the product in $\Gamma$ implies that

$$
d_{0,2}f_\alpha = 0.
$$

As explained in the proof of [Gras1 Theorem 3.2], substituting in the definition of $f_\alpha$, the identities $j_\beta = j_\alpha - h_{\alpha\beta}$, $q_\beta = q_\beta + h_{\alpha\beta} \circ p$ and $p \circ \text{mult} = \text{mult}(p \otimes p)$ yields

$$
(\delta_{0,2}f)_\alpha = f_\beta - f_\alpha = h_{\beta\alpha} - h_{\alpha\beta} = d_{1,1}h_{\alpha\beta}
$$

which together with Equality (10) and Equality (11) implies $D'_{tot}(f_\alpha \oplus h_{\alpha\beta}) = (\delta_{1,1} - d_{1,1})h_{\alpha\beta} + (\delta_{0,2} + d_{0,2})f_\alpha = 0$.

This means that $f_\alpha \oplus h_{\alpha\beta}$ is a 2-cocycle in $\text{Tot}^\bullet(F^1K^{**})$. Suppose $0 \to \mathcal{A} \xrightarrow{i'} \Gamma' \xrightarrow{p'} \Lambda \to 0$ is an equivalent extension such that the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{A} \\
& \xrightarrow{i} & \Gamma \\
& & \xrightarrow{p} \Lambda \\
& & \xrightarrow{p'} \Gamma'
\end{array}
$$

commutes and with associated induced 2-cocycle $h'_{\alpha\beta} \oplus f'_\beta$ in $\text{Tot}^\bullet(F^1K^{**})$. If we identify $\mathcal{A}$ with the images of the identity inclusions $i(\mathcal{A})$ and $i'(\mathcal{A})$ then $k|_{\mathcal{A}} = \text{id}_\mathcal{A}$. Let $j_\alpha$ and $j'_\alpha$ be the corresponding filtered splitting homomorphisms of the filtered square-zero extensions. Set $t_\alpha := j'_\alpha - k \circ j_\alpha$. Then we have,

$$
p' \circ t_\alpha = \text{id}_\Lambda|_{U_a} - \text{id}_\Lambda|_{U_a} = 0
$$

which implies $\text{Im}(t_\alpha) \cong \text{Im}(i') \cong \mathcal{A}$. Hence, abusing notation this map extends to a linear morphism $t_\alpha : \Lambda|_{U_a} \to \mathcal{A}|_{U_a}$, i.e. $t_\alpha \in \text{C}^0(\mathcal{U}, \mathcal{F}_1(\Lambda, \mathcal{A}))$. Let us as before $h_{\alpha\beta} := j_\beta - j_\alpha$, and $h'_{\alpha\beta} := j'_\beta - j'_\alpha$. Note that $k \circ h_{\alpha\beta} = h_{\alpha\beta}$. Then, we have,

$$
h'_{\alpha\beta} - h_{\alpha\beta} = h'_{\alpha\beta} - k \circ h_{\alpha\beta} = (j'_\beta - k \circ j_\beta) - (j'_\alpha - k \circ j_\alpha) = \delta(t_\alpha).
$$

Consider the difference of $(0,2)$-cochains $f_\alpha$ and $f'_\alpha$

$$
(f'_\alpha - f_\alpha)(\lambda_1, \lambda_2) = q_\alpha \circ \text{mult}(j'_\alpha(\lambda_1) \otimes j'_\beta(\lambda_2)) - q_\alpha \circ \text{mult}(j_\alpha(\lambda_1) \otimes j_\beta(\lambda_2))
$$

$$
= (\text{id}_\Gamma - j_\alpha \circ p')'(j'_\alpha(\lambda_1)j'_\beta(\lambda_2)) - (\text{id}_\Gamma - j_\alpha \circ p)(j_\alpha(\lambda_1)j_\beta(\lambda_2))
$$

$$
= j'_\alpha(\lambda_1)j'_\beta(\lambda_2) - j'_\alpha(\lambda_1)j_\beta(\lambda_2) + j_\alpha(\lambda_1)j_\beta(\lambda_2) - j_\alpha(\lambda_1)j_\beta(\lambda_2)
$$

$$
= (j'_\alpha(\lambda_1)j'_\beta(\lambda_2) - j'_\alpha(\lambda_1)k_j(\lambda_2) + j'_\alpha(\lambda_1)k_j(\lambda_2) + (j'_\alpha(\lambda_1)j_\beta(\lambda_2) - k_j(\lambda_1)j_\beta(\lambda_2)) - j_\alpha(\lambda_1)j_\beta(\lambda_2) - j'_\alpha(\lambda_1)k_j(\lambda_2) + j_\alpha(\lambda_1)j_\beta(\lambda_2))
$$

$$
= j'_\alpha(\lambda_1)k_j(\lambda_2) + t_\alpha(\lambda_1)k_j(\lambda_2) - t_\alpha(\lambda_1)\lambda_2
$$

$$
= \lambda_1 \cdot t_\alpha(\alpha_2) - t_\alpha(\lambda_1)\lambda_2 + t_\alpha(\alpha_1) \cdot \lambda_2
$$
where we used that $k|_{\mathcal{A}} = \text{id}_{\mathcal{A}}$ and in the second to the last line we used that $j_{\alpha}(\lambda_1)j_{\alpha}(\lambda_1) = \lambda_1 \cdot \alpha \cdot \lambda_2$ for all sections $\lambda_1, \lambda_2$ of $\Lambda$ and every section $\alpha$ of $\mathcal{A}$ due to the fact that the extension is by definition square-zero. Identities (12) and (13) imply that both 2-cocycles $h_{\alpha, \beta} + f_\alpha$ and $h'_{\alpha, \beta} + f'_\alpha$ differ by a coboundary $D_{t_{\alpha}} = (\delta + d)t_{\alpha}$ in $\text{Tot}^*(F^1 K^{**})$. Hence, there is a well-defined mapping

$$
\text{exal}_f(\Lambda, \mathcal{A}) \rightarrow \hat{H}^2(\mathcal{U}, \tilde{\mathcal{E}}(\Lambda, \mathcal{A})).
$$

Conversely, suppose $f_\alpha + h_{\alpha, \beta}$ is a 2-cocycle in $\text{Tot}^*(F^1 K^{**})$. Then we can define a locally trivial filtered square-zero $\mathbb{C}$-vector space extension of $\Lambda$ by $\mathcal{A}$ as laid out in [Gra61] Proposition 3.1. Let $\Gamma$ be the sheaf defined by

$$
\bigcup_{\alpha}(\Lambda \oplus \mathcal{A})|_{U_\alpha} \bigg/ \left\{ (\lambda, a)|_{U_\alpha} \sim (\lambda, a + h_{\alpha, \beta}(\lambda))|_{U_\alpha} : (\lambda, a)|_{U_\alpha} \in (\Lambda \oplus \mathcal{A})|_{U_\alpha} \right\}
$$

The product in $\Gamma|_{U_\alpha}$ is given by $(\lambda_1, a_1)|_{U_\alpha} \cdot (\lambda_2, a_2)|_{U_\alpha} = (\lambda_1 \lambda_2, \lambda_1 a_2 + a_1 \lambda_2 + f_\alpha(\lambda_1, \lambda_2))|_{U_\alpha}$. The well-definition of the product on intersections $U_{\alpha, \beta}$ follows from the compatibility relation for $f_\alpha$. With this multiplication the above construction becomes in fact an algebra extension. Assume that $h'_{\alpha, \beta} + f'_\alpha$ is a cohomologous 2-cocycle in $\text{Tot}^*(F^1 K^{**})$. Then, there is by definition a $(0, 1)$-cochain $t_{\alpha}$ such that $h'_{\alpha, \beta} + f'_\alpha - h_{\alpha, \beta} - f_\alpha = D_{t_{\alpha}}$. Consequently, we can write for the associated locally trivial filtered square-zero extension $\Gamma'$ of $\Lambda$ by $\mathcal{A}$

$$
\bigcup_{\alpha}(\Lambda \oplus \mathcal{A})|_{U_\alpha} \bigg/ \left\{ (\lambda, a)|_{U_\alpha} \sim (\lambda, a + h_{\alpha, \beta}(\lambda) + \delta(t_{\alpha}(\lambda)))|_{U_\alpha} : (\lambda, a)|_{U_\alpha} \in (\Lambda \oplus \mathcal{A})|_{U_\alpha} \right\}
$$

We define a morphism $k : \Gamma \rightarrow \Gamma'$ of sheaves of $\mathbb{C}$-modules by $(\lambda, a)|_{U_\alpha} \mapsto (\lambda, a + t_{\alpha}(\lambda))|_{U_\alpha}$. This map is in fact a morphism of sheaves of $\mathbb{C}$-algebras since we have

$$
k((\lambda_1, a_1)|_{U_\alpha} \cdot (\lambda_2, a_2)|_{U_\alpha}) = k((\lambda_1 \lambda_2, \lambda_1 a_2 + a_1 \lambda_2 + f_\alpha(\lambda_1, \lambda_2))|_{U_\alpha})
$$

$$
= (\lambda_1 \lambda_2, \lambda_1 a_2 + a_1 \lambda_2 + f_\alpha(\lambda_1, \lambda_2) + t_{\alpha}(\lambda_1 \lambda_2))|_{U_\alpha}
$$

$$
= (\lambda_1 \lambda_2, \lambda_1 a_2 + a_1 \lambda_2 + dt_{\alpha}(\lambda_1, \lambda_2) + t_{\alpha}(\lambda_1 \lambda_2) + f'_\alpha(\lambda_1, \lambda_2))|_{U_\alpha}
$$

$$
= (\lambda_1, a_1 + t_{\alpha}(a_1))|_{U_\alpha} \cdot (\lambda_2, a_2 + t_{\alpha}(a_2))|_{U_\alpha}
$$

$$
= k((\lambda_1, a_1)|_{U_\alpha})k((\lambda_2, a_2)|_{U_\alpha}).
$$

As evidently $k \circ i = i'$ and $p' \circ k = p$, this morphism defines an equivalence relation between the extensions $\Gamma$ and $\Gamma'$. Thus, we obtain a well-defined map

$$
\hat{H}^2(\mathcal{U}, \tilde{\mathcal{E}}(\Lambda, \mathcal{A})) \rightarrow \text{exal}_f(\Lambda, \mathcal{A}).
$$

It is evident that the morphisms (14) and (15) are the inverses of each other. As direct limits preserve exactness, we conclude $\text{exal}_f(\Lambda, \mathcal{A}) \cong \hat{H}^2(X, \tilde{\mathcal{E}}(\Lambda, \mathcal{A}))$. The claim follows from the fact that $X$ is paracompact by assumption.

In the spacial case of $\mathcal{A} = \Lambda$, inspired by the isomorphism in Lemma 2.3 we define the space of first-order deformations $\text{Def}(\Lambda)$ of $\Lambda$ as $\text{exal}_f(\Lambda, \Lambda)$ and the space of filtered first-order deformations $\text{Def}_f(\Lambda)$ of $\Lambda$ by $\text{exal}_f(\Lambda, \Lambda)$.

3. Universal deformation of $\mathcal{D}_X \rtimes G$

3.1. Sheaves of Calabi-Yau algebras. Calabi-Yau algebras were introduced and first studied in [Gin06]. These algebras arise naturally in the theory of non-commutative deformation of spaces and possess a number of valuable properties. In this note we are interested in them because of the duality between Hochschild homology and cohomology of Calabi-Yau algebras which enables us to derive a cohomological holomorphic version of Engeli-Felder’s trace density morphism. Let us recall the definition of a Calabi-Yau algebra from [Gin06].

**Definition 3.1 (Definition 3.2.3, [Gin06]).** An associative $k$-algebra is a Calabi-Yau algebra of dimension $d$ if $\Lambda$ has a finitely-generated projective $\Lambda - \Lambda$-bimodule resolution and $\text{HH}^\bullet(\Lambda, \Lambda \otimes_k \Lambda) \cong \Lambda[-d]$ as a graded $\Lambda - \Lambda$-bimodule.

As stated earlier Calabi-Yau algebras admit a so-called van den Berg duality between Hochschild homology and cohomology which is precisely formulated in the ensuing theorem.
Theorem 3.2 ([2v98]). Let $\Lambda$ be a Calabi-Yau associative $\mathbb{C}$-algebra of dimension $d$. Then for every left $\Lambda^e$-module $M$ there is a canonical isomorphism $\mathrm{HH}_i(\Lambda, M) \cong \mathrm{HH}^{d-i}(\Lambda, M)$.

The above concepts admits a natural generalization to sheaves of associative $\mathbb{C}$-algebras.

Definition 3.3. A sheaf of associative $\mathbb{C}$-algebras $\Lambda$ on a complex space $Y$ is Calabi-Yau of dimension $d$ provided that it admits a locally free left $\Lambda^e$ resolution and $\mathcal{E}xt^\Lambda_{\mathcal{A}}(\Lambda, \Lambda \otimes \mathbb{C} \Lambda) \cong \Lambda[-d]$.

Let $\mathcal{P} \to \Lambda$ denote the finitely-generated projective left $\Lambda^e$ resolution of $\Lambda$. It is evident that for every $y \in Y$ the stalk $\Lambda_y$ is a Calabi-Yau algebra in the sense of Definition 3.1. Indeed, we have

$$
\mathcal{H} \mathcal{H}^\Lambda_{\mathcal{A}}(\Lambda_y, \Lambda_y \otimes \mathbb{C} \Lambda_y) = \mathcal{E}xt^\Lambda_{\mathcal{A}}(\mathcal{P}_y, \Lambda \otimes \mathbb{C} \Lambda)_y
$$

This taken together with Theorem 3.2 in turn delivers the following immediate result.

Lemma 3.4. Let $\Lambda$ be a Calabi-Yau sheaf of associative algebras of dimension $d$ on a complex space $Y$ and let $\mathcal{A}$ be a left $\Lambda^e$-module. Then $\mathcal{H} \mathcal{H}_{\mathcal{A}}(\Lambda, \mathcal{A}) \cong \mathcal{H} \mathcal{H}^{d-1}_{\mathcal{A}}(\Lambda, \mathcal{A})$.

Later we shall need the following fact about the $\mathcal{O}_Y$-algebra $\mathcal{S}ym^\Lambda(\mathcal{T}_X)$.

Proposition 3.5. The sheaf of $\mathcal{O}_Y$-algebras $\mathcal{S}ym^\Lambda(\mathcal{T}_X)$ is a sheaf of Calabi-Yau $\mathbb{C}$-algebras of dimension $2n$.

Proof. We start by constructing a locally free resolution of $\mathcal{S}ym^\Lambda(\mathcal{T}_X)$ of length $2n$ in terms of left $\mathcal{S}ym^\Lambda(\mathcal{T}_X)^e$-modules. Put $Y = X \times X$ with the projection $p_1 : Y \to X$ on the first term and $p_2 : Y \to X$ on the second term of $Y$. Let $\delta : X = Y, \sigma \mapsto (\sigma, \sigma)$ be the diagonal embedding with a diagonal $\Delta := \text{Im}(\delta)$. Since $X$ is a complex analytic manifold, $\Delta$ is closed and hence analytic in $Y$. It, therefore, defines a sheaf of ideals $\mathcal{I}_\Delta \subset \mathcal{O}_Y$ with a zero set $\Delta$. Let $W$ be an open set in the product topology of $Y$ and let $f_1, \ldots, f_n, g_1, \ldots, g_n \in \mathcal{I}_\Delta|W$. Let $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be local coordinates on $W$. Set a local section $s|_W := \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n g_j \frac{\partial}{\partial y_j}$ of the tangent sheaf $\mathcal{T}_Y$. It induces a $\mathcal{O}_{Y|W}$-module homomorphism

$$
s^\ast : \mathcal{T}_Y^{\ast}|_W \to \mathcal{O}_{Y|W} \quad \omega \mapsto s^\ast(\omega) := \langle \omega, s \rangle.
$$

where $\langle \cdot, \cdot \rangle$ is the pairing of 1-forms and vector fields. From the definition of $s^\ast$ it is evident that $\text{Im}(s^\ast) = \mathcal{I}_\Delta$ which identifies $\Delta$ with the zero submanifold of $s$. By the standard theory of Koszul resolutions of zero submanifolds (cf. [FT89]) we get a resolution

$$
\bigwedge^{2n} \mathcal{T}_Y^{\ast} \to \cdots \to 1 \bigwedge^{1} \mathcal{T}_Y^{\ast} \to \mathcal{O}_Y \to \mathcal{O}_Y/\mathcal{I}_\Delta
$$

of $\mathcal{O}_Y/\mathcal{I}_\Delta$ of length $2n$ on $Y$. Denote the external tensor product $p_1^\ast \mathcal{S}ym^\Lambda(\mathcal{T}_X) \otimes_{\mathcal{O}_Y} p_2^\ast \mathcal{S}ym^\Lambda(\mathcal{T}_X)$ of $\mathcal{S}ym^\Lambda(\mathcal{T}_X)$ by $\mathcal{S}ym^\Lambda(\mathcal{T}_X \boxtimes \mathcal{T}_X)$. It is per definition a locally free $\mathcal{O}_Y$-module. Hence, it is a flat module over $\mathcal{O}_Y$. Ergo, tensoring the exact sequence (16) with $\mathcal{S}ym^\Lambda(\mathcal{T}_X)$ over $\mathcal{O}_Y$ yields an exact sequence of $\mathcal{O}_Y$-modules

$$
\bigwedge^{2n} \mathcal{S}ym^\Lambda(\mathcal{T}_X \boxtimes \mathcal{T}_X) \to \cdots \to \bigwedge^{1} \mathcal{S}ym^\Lambda(\mathcal{T}_X \boxtimes \mathcal{T}_X)
$$

Let $j_{\Delta} : \Delta \to Y$ be the closed embedding and let $\ast|_{\Delta}$ denote the corresponding restriction of sheaves to the closed analytic submanifold $\Delta$. As the inverse image of $j_{\Delta}$ is an exact functor, we get an exact sequence of $\mathcal{O}_{Y|\Delta}$-modules

$$
j_{\Delta}^{-1}(\mathcal{S}ym^\Lambda(\mathcal{T}_X \boxtimes \mathcal{T}_X)) \otimes_{\mathcal{O}_{Y|\Delta}} \bigwedge^{2n} \mathcal{T}_{Y|\Delta}^{\ast} \to \cdots \to \bigwedge^{1} j_{\Delta}^{-1}(\mathcal{S}ym^\Lambda(\mathcal{T}_X \boxtimes \mathcal{T}_X)) \otimes_{\mathcal{O}_{Y|\Delta}} \mathcal{T}_{Y|\Delta}^{\ast}
$$

1A complex analytic manifold and an orbifold are special examples of a complex space.
where \( \theta_\Delta := j_\Delta^{-1}(\theta_Y/\mathcal{I}_Y) \). The last term in Sequence (19) can be rewritten as
\[
j_\Delta^{-1}(\text{Sym}^* \mathcal{T}_X \otimes \text{Sym}^* \mathcal{T}_X) \otimes \Theta_{\nu_1\Delta} \otimes \Theta_\Delta = \\
\theta_{\nu_1\Delta} \otimes (p_1\delta^{-1}_\epsilon \otimes c(p_2\delta^{-1}_\epsilon)(p_1\delta^{-1}_\epsilon) \otimes (\text{Sym}^*(\mathcal{T}_X) \otimes \Theta_{\nu_1\Delta} \otimes \Theta_\Delta = \\
\approx \theta_{\nu_1\Delta} \otimes (p_1\delta^{-1}_\epsilon \otimes c(p_2\delta^{-1}_\epsilon) \otimes \text{Sym}^*(\mathcal{T}_X \otimes (p_1\delta^{-1}_\epsilon \otimes p_2^{-1} \mathcal{T}_X)) \otimes \Theta_{\nu_1\Delta} \otimes \Theta_\Delta \\
\approx \text{Sym}^*(\mathcal{T}_{\nu_1\Delta}) \otimes \Theta_{\nu_1\Delta} \otimes \Theta_\Delta \\
\approx \text{Sym}^*(\mathcal{T}_\Delta)
\]
where \( \mathcal{T}_\Delta \) is the tangent sheaf on the diagonal submanifold \( \Delta \). As an isomorphism of abelian categories between \( \Theta_{\nu_1\Delta} \) and \( \Theta_X \) the \( \Theta_\Delta \)-module pullback \( \delta^* \) is exact, too. This way, applying \( \delta^* \) on (19) and plugging (20) in (19), we obtain an exact sequence of \( \Theta_\Delta \)-modules
\[
\delta^*(j_\Delta^{-1}(\text{Sym}^* \mathcal{T}_X \otimes \text{Sym}^* \mathcal{T}_X) \otimes \Theta_{\nu_1\Delta} \otimes \Theta_\Delta) \to \delta^*(\text{Sym}^*(\mathcal{T}_\Delta)).
\]
which accounting for \( \delta^*(j_\Delta^{-1}(\text{Sym}^* \mathcal{T}_X \otimes \text{Sym}^* \mathcal{T}_X) \otimes \Theta_{\nu_1\Delta} \otimes \Theta_\Delta) \approx \text{Sym}^*(\mathcal{T}_X) \otimes \text{Sym}^*(\mathcal{T}_X) \) becomes the same as
\[
\text{Sym}^*(\mathcal{T}_X) \otimes \text{Sym}^*(\mathcal{T}_X) \otimes \Theta_{\nu_1\Delta} \otimes \Theta_\Delta \to \text{Sym}^*(\mathcal{T}_X).
\]
The exact sequences of locally free \( \Theta_X \)-modules (21), respectively (22) can be seen as the desired Koszul type resolution of \( \text{Sym}^*(\mathcal{T}_X) \) of length \( 2n \) in terms of locally free left \( \text{Sym}^*(\mathcal{T}_X) \)-modules. The Calabi-Yau property follows consequently from
\[
\text{Hom}^*_{\text{Sym}^*(\mathcal{T}_X)}(\text{Sym}^*(\mathcal{T}_X), \text{Sym}^*(\mathcal{T}_X) \otimes \text{Sym}^*(\mathcal{T}_X)) := \\
H^*(\text{Hom}_{\text{Sym}^*(\mathcal{T}_X)}(\text{Sym}^*(\mathcal{T}_X), \text{Sym}^*(\mathcal{T}_X) \otimes \text{Sym}^*(\mathcal{T}_X)) \\
\approx H^*(\text{Sym}^*(\mathcal{T}_X) \otimes \text{Sym}^*(\mathcal{T}_X) \otimes \Theta_{\nu_1\Delta} \otimes \Theta_\Delta \\
= \text{Sym}^*(\mathcal{T}_X)[-2n]
\]
which completes the proof of the statement in the proposition. \( \square \)

With the help of Proposition (3.5) we show that the sheaf of holomorphic differential operators \( \mathcal{D}_X \) on a complex analytic manifold \( X \) is Calabi-Yau of dimension \( 2n \).

**Proposition 3.6.** \( \mathcal{D}_X \) is a sheaf of Calabi-Yau algebras of dimension \( 2n \).

**Proof.** Since \( \mathcal{D}_X \) is a filtered quantization of \( \text{Sym}^*(\mathcal{T}_X) \), the exact sequence (21) from Proposition 3.5 can be rewritten in the form
\[
\delta^*(\text{Gr}_* (j_\Delta^{-1}(\mathcal{D}_X \otimes \mathcal{D}_X)) \otimes \Theta_{\nu_1\Delta} \otimes \Theta_\Delta) \to \delta^*(\text{Gr}_*(\mathcal{D}_\Delta)).
\]
It is a known result from homological algebra that for any filtered complex \( C_* \), exactness of the associated graded of \( C_* \) implies exactness of \( C_* \). Hence,
\[
\delta^*(j_\Delta^{-1}(\mathcal{D}_X \otimes \mathcal{D}_X) \otimes \Theta_{\nu_1\Delta} \otimes \Theta_\Delta) \to \delta^*(\mathcal{D}_\Delta)
\]
is an exact sequence of locally free \( \Theta_\Delta \)-modules. In particular, this acyclic complex is the wanted Koszul type resolution of length \( 2n \) of \( \mathcal{D}_X \) in terms of locally free left \( \mathcal{D}_X \)-modules. The Calabi-Yau property follows in an analogous manner to Proposition 3.5 \( \square \)

An immediate consequence of Proposition 3.6 and Proposition 3.5 is the following result.

**Corollary 3.7.** The sheaves \( (\mathcal{D}_X)^G \) and \( \mathcal{D}_X \rtimes G \) as well as their corresponding associated graded \( \text{Sym}^*_{\mathcal{D}_X}(\mathcal{T}_X)^G \) and \( \text{Sym}^*_{\mathcal{D}_X}(\mathcal{T}_X) \rtimes G \) on \( X/G \) are sheaves of Calabi-Yau \( \mathbb{C} \)-algebras of dimension \( 2 \dim_\mathbb{C} X \).
**Proof.** Taking the invariants with respect to the action of $G$ is an exact functor $(\cdot)^G : \mathcal{Bimod}(\mathcal{D}_X) \to \mathcal{Bimod}(\mathcal{D}_X^G)$. Hence, applying $(\cdot)^G$ on (25) yields a locally free resolution of $(\mathcal{D}_X)^G$ on $X/G$ in terms of $(\mathcal{D}_X)^G - (\mathcal{D}_X)^G$-bimodules of length $2n$. With the proper locally free left $\mathcal{D}_X^G$-module resolution of $\mathcal{D}_X$ at hand we obtain

$$\mathbb{Ext}^\bullet_{\mathcal{D}_X^G}(\mathcal{D}_X)^G, (\mathcal{D}_X)^G \cong \mathcal{C}(\mathcal{D}_X)^G = (\mathcal{D}_X)^G[-2n]$$

which implies that $(\mathcal{D}_X)^G$ is Calabi-Yau of dimension $2n$. The Morita equivalence between $(\mathcal{D}_X)^G$ and $\mathcal{D}_X \rtimes G$ implies the latter statement. The proof of the claim for the associated graded is literally identical. □

**Corollary 3.8.** Let $\Theta$ be a Hochschild chain $2n$-cycle of $\mathcal{D}_X \rtimes G$. Then the morphism of cochain complexes

$$\mu : \mathcal{C}^\bullet(\mathcal{D}_X \rtimes G, \mathcal{D}_X \rtimes G) \to \mathcal{C}_{2n-*}(\mathcal{D}_X \rtimes G, \mathcal{D}_X \rtimes G)$$

$$f \mapsto \Theta \cap f$$

is a quasi-isomorphism, where $\mathcal{C}_{2n-*}(\mathcal{D}_X \rtimes G, \mathcal{D}_X \rtimes G)$ is viewed as cochain complex by inverting degrees.

**Proof.** It follows from the Calabi-Yau property of $\mathcal{D}_X \rtimes G$ together with the fact that $\mu_*$ is injective and the cohomology groups are finite-dimensional. □

### 3.2. Trace density morphisms.

#### 3.2.1. Generalized trace density morphism for $\mathcal{D}_X$.

Let in the following $(\mathcal{A}_{X,C}, d_{dR})$ denote the de Rham complex of sheaves of smooth complex-valued differential forms with differential $d_{dR}$, and $(\mathcal{A}_{X}^{\dagger}, d_{dR})$ denote the complex of sheaves of holomorphic differential forms on $X$ with differential $d_{dR}$.

We start by adapting Engeli-Felder’s *trace density map* constructed in [EF08] to the holomorphic setting. In the holomorphic setting the term *trace density* is not justified because then the images of these maps are hypercohomology classes which cannot easily be integrated like de Rham cohomology classes in the complex valued smooth case. Therefore, we use the vague term *generalized trace density morphism* to describe maps from the Hochschild (co)chain complex of $\mathcal{D}_X$ to the complex of sheaves of holomorphic de Rham differential forms on $T^* X$. We shall continue using the term *generalized trace density morphism* when dealing later in the next subsection with similar maps for $\mathcal{D}_X \rtimes G$.

Let $\Omega$ be the $\text{Aut}_\mathcal{D}_X$-invariant $W_n$-valued holomorphic connection 1-form on the Harish-Chandra $(W_n, \text{Aut}_\mathcal{D}_X)$-torsor $\pi_{\text{coor}} : X_{\text{coor}} \to X$ of formal coordinates on $X$ as defined in [BR73]. This induces a flat holomorphic connection $d + [\Omega, \cdot]$ on $X_{\text{coor}} \times \hat{\mathcal{D}}_n$. By means of Gelfand-Kazhdan formal geometry one shows that there is an isomorphism of sheaves between $\mathcal{D}_X$ and the sheaf $\pi_{\text{coor}}^* \mathcal{O}_{\text{flat}}(X_{\text{coor}} \times \hat{\mathcal{D}}_n)$ of flat sections of the algebra bundle $X_{\text{coor}} \times \hat{\mathcal{D}}_n$. For the purposes of a definition of a generalized trace density map it is more convenient to express $\mathcal{D}_X$ at least locally in terms of flat sections of a bundle over the cotangent bundle $T^* X$ and not over $X_{\text{coor}}$. We do that in the following way. The definition of $X_{\text{coor}}$ as an inverse limit of a projective system of of $\text{GL}_{n}$-equivariant holomorphic submersions $X_k \to X_{k-1}$ yields a $\text{GL}_{n}(\mathbb{C})$-equivariant holomorphic map $X_{\text{coor}} \to \text{Fr}(X)$ where $\text{Fr}(X)$ is the frame bundle of $X$ and corresponds to $X_1$ in the inverse system. Let $U = \{U_\alpha\}$ be an open cover of $X$ which trivializes the tangent bundle, hence, the cotangent bundle $\pi : T^* X \to X$ and the frame bundle $\text{Fr}(X) \to X$, too. On a small enough $U_\alpha \in \mathcal{U}$ there is always a local $\text{GL}_{n}(\mathbb{C})$-equivariant holomorphic section $s_\alpha : \text{Fr}(X)|_{U_\alpha} \to X_{\text{coor}}|_{U_\alpha}$. The pullback of $\Omega$ along $s_\alpha$ yields a $\text{GL}_{n}(\mathbb{C})$-equivariant holomorphic 1-form $\omega_\alpha := s_\alpha^* \Omega$ on $\text{Fr}(X)|_{U_\alpha}$ with values in $W_n$ which satisfies the Maurer-Cartan equation. It descends to a holomorphic $W_n$-valued Maurer-Cartan 1-form on $U_\alpha$, which for brevity we also denote by $\omega_\alpha$, and induces a holomorphic flat connection $\nabla_\alpha$ on the associated bundle $D_\alpha := \text{Fr}(X)|_{U_\alpha} \times \text{GL}_{n}(\mathbb{C}) \to U_\alpha$. We take finally the pullback holomorphic bundle $\pi^* D_\alpha \to T^* X|_{U_\alpha}$ which is flat with respect to $\pi^* \nabla_\alpha$. The direct image under $\pi$ of the sheaf of pullback horizontal sections of $\pi^* D_\alpha$ with respect to $\pi^* \nabla_\alpha$ is isomorphic to $\pi^* \mathcal{O}_{\text{flat}}(X_{\text{coor}} \times \hat{\mathcal{D}}_n)|_{U_\alpha}$ and hence its sections are in one-to-one correspondence with holomorphic differential operators in $\mathcal{D}_X|_{U_\alpha}$. The pullback of $\omega_\alpha \in \Omega^1_X(U_\alpha)$ along the holomorphic projection $\pi$ induces a holomorphic 1-form on $T^* X|_{U_\alpha}$ satisfying the Maurer-Cartan equation which again by abuse of notation we denote by $\omega_\alpha$.

Adhering to the notation in [EF08] we shall denote $k$-chains of the normalized Hochschild chain complex of an associative algebra $A$ by $(a_0, a_1, \cdots, a_k)$. Recall the nontrivial Feigin-Felder-Schoikhet normalized Hochschild 2n-cocycle $\tau_{2n}$ of $\mathcal{D}_n$ with values in the dual bimodule $\mathcal{D}_n^*$ which naturally extends to a linear form on $\hat{C}_G(\mathcal{D}_n)^* := \mathcal{D}_n^{\otimes 2n+1}$. It is $\text{GL}_n(\mathbb{C})$-basic meaning that it is invariant under the adjoint action of $\text{GL}_n(\mathbb{C})$ on $\mathcal{D}_n$ and for any
Now, consider on every open set \( U \) forms under a change of trivialization according to
\[
\alpha(x) = \frac{\text{Ad}(g^{-1}_{\alpha})\omega_\alpha + g^{-1}_{\alpha}dg_{\alpha\beta}}\text{.}
\]
Furthermore, every flat section \( \tilde{D}_\alpha \) of \( \mathcal{D} \) over \( U \), representing a unique differential operator \( D_\alpha \) in \( \mathcal{D}(U_\alpha) \) transforms under a change of trivialization according to
\[
\tilde{D}_\alpha = \text{Ad}(g^{-1}_{\alpha})\tilde{D}_\alpha\text{.}
\]
Now, consider on every open set \( U \) the following composition of morphisms
\[
C_\rho(\mathcal{D}(U_\alpha)) \hookrightarrow C_\rho(\Gamma\mathfrak{fl}(\pi^{-1}(U_\alpha), \pi^*\mathcal{D}_\alpha)) \hookrightarrow C_\rho((\Omega^*_{T^*X}(\pi^{-1}(U_\alpha), \pi^*\mathcal{D}_\alpha), \tilde{\nabla}_\alpha)) \hookrightarrow C_\rho((\Omega^*_{T^*X}(\pi^{-1}(U_\alpha), \tilde{\nabla}_\alpha), d_{\text{dR}} + [\omega_\alpha, \cdot]) \to C_\rho((\Omega^*_{T^*X}(\pi^{-1}(U_\alpha), \tilde{\nabla}_\alpha), d_{\text{dR}})) \to (26)
\]
where \( \tilde{\nabla}_\alpha \) is the covariant derivative induced by \( \nabla_\alpha \), with corresponding connecting morphisms given by
\[
(D_0, \ldots, D_p) \mapsto (\tilde{D}_0, \ldots, \tilde{D}_p) \mapsto (\tilde{D}_0, \ldots, \tilde{D}_p) \mapsto \sum_{k=0}^{2n} (-1)^k(\tilde{D}_0, \ldots, \tilde{D}_p) \times (\omega_\alpha)^k \mapsto \\
\tau_{2n}(\sum_{k=0}^{2n} (-1)^k(\tilde{D}_0, \ldots, \tilde{D}_p) \times (\omega_\alpha)^k) \mapsto (\omega_\alpha)^k.
\]
Each of the above morphism are discussed in a greater detail in [Ram11, Section 3] This composition gives a map between complexes of sheaves over \( U_\alpha \)
\[
\chi_\alpha : \mathfrak{g}^\bullet(\mathcal{D}(U_\alpha)|_{U_\alpha}) \longrightarrow \pi_\ast\Omega^*_{T^*X} \big|_{U_\alpha}
\]
\[
(D_0, \ldots, D_n) \mapsto \sum_{k=0}^{2n} (-1)^k\tau_{2n}(\tilde{D}_0, \ldots, \tilde{D}_n) \times (\omega_\alpha)^k
\]
where \((\omega_\alpha)^k\) denotes the normalized Hochschild k-chain \( (\omega_\alpha, \ldots, \omega_\alpha) \) (k copies of \( \omega_\alpha \)). We now want to show that the map \( \chi_\alpha \) extends to a map \( \chi \) over all of \( X \). To that aim, consider the following easy to prove binomial formula for the shuffle product
\[
(a + b)^k = \sum_{j=0}^{k} \binom{k}{j} (a)^{k-j} \times (b)^j
\]
which we invoke in the ensuing computation. After a change of trivialization \( \chi_\alpha \) changes as
\[
\chi_\beta(D_0, \ldots, D_p) = \sum_{k=0}^{2n} (-1)^k\tau_{2n}(\tilde{D}_0, \ldots, \tilde{D}_p) \times (\omega_\beta)^k
\]
\[
= \sum_{k=0}^{2n} (-1)^k\rho^\mathfrak{g}_{\chi}(\text{Ad}(g^{-1}_{\alpha})\tilde{D}_0, \ldots, \text{Ad}(g^{-1}_{\alpha})\tilde{D}_p) \times (\text{Ad}(g^{-1}_{\alpha})\omega_\alpha - g^{-1}_{\alpha}dg_{\alpha\beta})^k
\]
\[
= (-1)^{n+p}2^{-p}\tau_{2n}(\text{Ad}(g^{-1}_{\alpha})\tilde{D}_0, \ldots, \text{Ad}(g^{-1}_{\alpha})\tilde{D}_p) \times (\text{Ad}(g^{-1}_{\alpha})\omega_\alpha - g^{-1}_{\alpha}dg_{\alpha\beta})^{2n-p}
\]
\[
= (-1)^{n+p}2^{-p}2^{2n-p}\tau_{2n}(\text{Ad}(g^{-1}_{\alpha})\tilde{D}_0, \ldots, \text{Ad}(g^{-1}_{\alpha})\tilde{D}_p) \times \sum_{j=0}^{2n-p} (-1)^j(g^{-1}_{\alpha}dg_{\alpha\beta})^{2n-p-j}
\]
\[
= (-1)^{n+p}2^{-p}2^{2n-p}\tau_{2n}(\text{Ad}(g^{-1}_{\alpha})\tilde{D}_0, \ldots, \text{Ad}(g^{-1}_{\alpha})\tilde{D}_p) \times \sum_{j=0}^{2n-p} (-1)^j(g^{-1}_{\alpha}dg_{\alpha\beta})^{2n-p-j}
\]
This map is an isomorphism if and only if it is an isomorphism on sections at a point \( \{ \alpha \} \). Let \( C \) be a morphism of complexes of sheaves on sections where in the second and the fourth line we use the commutativity of direct limits with the cohomology functor, in the second to the last line we invoke the fact that the homotopy equivalence of the holomorphic cotangent bundle and the base space and in the third line we invoke the homotopy equivalence of the Hochschild chain complex by inverting degrees, i.e. \( \hat{\mathcal{C}} \). Therefore, at the level of cohomology sheaves the morphism \( \chi \) of cochain complexes of sheaves on the whole of \( X \) by definition is

\[
\chi: \mathcal{H}_\bullet(\mathcal{D}_X) \to \pi_* \Omega^{2n-*}_{T^*X}
\]

(28)

where in the second to the last line we made use of the fact that the Feigin-Felder-Scholkhet 2n-cocycle \( \tau_{2n} \) is \( \text{GL}_n(\mathbb{C}) \)-basic. This demonstrates that the map is independent on the choice of trivialization and hence well defined as a morphism of complexes of sheaves on \( \{ U_\alpha \} \). Consequently, the family of maps \( (U_\alpha, \chi_\alpha) \) extends to a morphism

\[
\chi: \mathcal{H}_\bullet(\mathcal{D}_X) \to \pi_* \Omega^{2n-*}_{T^*X}
\]

(27)

of cochain complexes of sheaves on the whole of \( X \) if we consider the Hochschild chain complex \( \mathcal{H}_\bullet(\mathcal{D}_X) \) as a cochain complex by inverting degrees, i.e. \( \mathcal{C}_\bullet(\mathcal{D}_X, \mathcal{D}_X) = \mathcal{H}_\bullet(\mathcal{D}_X, \mathcal{D}_X) \). For any choice of local holomorphic sections \( \{ s_\alpha \} \) on the different members of \( \mathcal{H} \) the corresponding induced cochain morphisms \( (28) \) are homotopic. Therefore, at the level of cohomology sheaves the morphism \( \chi_\alpha \) is canonical. Moreover, we can prove the ensuing proposition.

**Proposition 3.9.** The morphism \( (28) \) is a quasi-isomorphism.

*Proof.* The stalk of the cohomology sheaf \( \mathcal{H}^\bullet(\pi_* \Omega^{2n-*}_{T^*X}) \) at \( x \in X \) by definition is

\[
\mathcal{H}^\bullet(\pi_* \Omega^{2n-*}_{T^*X})_x \cong \mathcal{H}^\bullet(\pi_* \Omega^{2n-*}_{T^*X, \pi(x,v)})
\]

\[
\cong \lim_{\pi^{-1}(U) \ni (x,v)} \mathcal{H}^\bullet(\Omega^{2n-*}_{T^*X}(\pi^{-1}(U)))
\]

\[
\cong \lim_{x \supseteq U \ni x} \mathcal{H}^\bullet(\Omega^{2n-*}_{X}(U))
\]

\[
\cong \mathcal{H}^\bullet(\Omega^{2n-*}_{X}(U))
\]

\[
\cong \mathcal{H}^\bullet(af^{2n-*}_{X,C} x)
\]

\[
= \begin{cases} 
\mathbb{C}, & \text{if } \bullet = 2n \\
0, & \text{otherwise}
\end{cases}
\]

where in the second and the fourth line we use the commutativity of direct limits with the cohomology functor, in the third line we invoke the homotopy equivalence of the holomorphic cotangent bundle and the base space and in the second to the last line we invoke the fact that \( af^{2n-*}_{X,C} \) and \( \Omega^*_{X} \) are both (injective) resolutions of the constant sheaf \( \mathbb{C}_X \). Consider the induced map on the cohomology sheaves

\[
\chi_*: \mathcal{H}^\bullet(\mathcal{D}_X, \mathcal{D}_X) \to \mathcal{H}^\bullet(\pi_* \Omega^{2n-*}_{T^*X}).
\]

This map is an isomorphism if and only if it is an isomorphism on stalks. Let \( (x_1, \ldots, x_n) \) be the local coordinates at a point \( x \in X \). From Theorem 1 in \([\text{Wod57}]\) it follows that for \( x \in X \),

\[
\mathcal{H}^\bullet(\mathcal{D}_X, \mathcal{D}_X)_x \cong \mathcal{H}^\bullet(\mathcal{D}^\bullet(\mathcal{D}_X, \mathcal{D}_X, \mathcal{D}_X, x))
\]

\[
\cong HH_\bullet(\mathcal{D}_X, x)
\]

\[
\cong \lim_{x \supseteq U \ni x} HH_\bullet(\mathcal{D}_X(U))
\]

\[
\cong \begin{cases} 
\lim_{x \supseteq U \ni x} \mathbb{C} \cdot \{ c_X(U) \}, & \text{if } \bullet = 2n \\
0, & \text{otherwise}
\end{cases}
\]

\[
\cong \begin{cases} 
\mathbb{C} \cdot \{ c_{X,x,x} \}, & \text{if } \bullet = 2n \\
0, & \text{otherwise}
\end{cases}
\]

(29)
where \( c_X(U) := \sum_{\pi \in S_{2n}} \text{sgn}(\pi)(1, u_{x(1)}, \ldots, u_{x(2n)}) \) with \( u_{2j-1} = \partial x_j, u_{2j} = x_j \) is the generator of \( \text{HH}_{2n}(D_X(U)) \).

By the normalization property of \( \tau_{2n} \), for every \( x \in X \), we have on the generator \( c_{X,x} \) of \( \text{HH}_{2n}(\mathcal{D}_{X,x}) \)

\[
\chi_{s,x}(c_{X,x}) = [1]
\]

which implies that the induced morphism

\[
\chi_{s,x} : \mathcal{H}^{-\bullet}(\mathcal{D}_{X}, \mathcal{D}_{X}) \to \mathcal{H}^{-\bullet}(\pi_{2n}^{\mathbb{C}^{n}}\cdot)\]

is a non-trivial map between one-dimensional stalks of cohomology sheaves. Hence, \( \chi_{s,x} \) is an isomorphism. Ergo, \( \chi_{s} \) is an isomorphism of cohomology sheaves which concludes the proof.

3.2.2. Generalized trace density morphism for the skew group ring

which implies that the induced morphism

\[
\chi_{s,x} : \mathcal{H}^{-\bullet}(\mathcal{D}_{X}, \mathcal{D}_{X}) \to \mathcal{H}^{-\bullet}(\pi_{2n}^{\mathbb{C}^{n}}\cdot)
\]

By the normalization property of \( c \) where \( x \)

that by Cartan's Lemma apart from \( \mathcal{D}_{X,x} \), is more subtle. As explained in [Vit19, Proposition 5.4], \( \mathcal{D}_{X,x} \) is analytic within \( U_\alpha \), ergo \( U_\alpha \cap X^g_\beta \) is open in \( X \). Recall that by Cartan’s Lemma apart from \( X^g_\beta \), the slice \( U_\alpha \) intersects only strata associated to subgroups \( L \) of \( K \). This fact combined with [Vit19, Corollary B.9] yields for every slice \( U_\beta \) contained in \( U_\alpha \setminus X^g_\beta \) the following composition of

\[
p : \mathcal{D}_{X} \times G \to j_{(g)}\pi_{(\beta)}^\text{coor} \mathcal{O}_{\text{flat}}(\mathcal{N}_{(\beta)}\times \mathcal{A}^{(g)}_{(\beta)\cdot})
\]
algebra morphisms
\[ \mathcal{D}_X(\text{ind}^G_U \alpha) \times G \cong \mathcal{D}_X(\text{ind}^K_U \beta) \times K \cong \mathcal{D}_X(U_\alpha) \times K \]
\[ \cong \lim_{\substack{U_\beta \text{ is centered at } X_L \cap U_\gamma}} \mathcal{D}_X(\text{ind}^L_U \beta) \times K \cong \prod_{U_\beta \text{ is centered at } U_\gamma \cap X^i_{(g)}} \{ (s) \in \mathcal{D}_X(U_\beta) \times \langle g \rangle : \text{res}_{U_\beta}^U(s) = s' \text{ for } U_\beta \subseteq U_\gamma \}
\]
(31) \[ \to \hat{j}^g_* \pi^\text{coor}_* \mathcal{O}_\text{flat}(\mathcal{A}^\text{coor} | X^{i}_{(g)} \times \mathcal{A}^g | n_{-t_y} U_\gamma) \]

The surjectivity of the fifth arrow follows from the definition of a projective limit and the fact that L-invariant slices \( U_\beta \) with \( L \) not a subgroup of \( \langle g \rangle \), do not contain \( \langle g \rangle \)-invariant slices. The last arrow in (31) is described as follows. By the gluing conditions in [Vit19, Section 6.2, Section 6.2], each element in \( \mathcal{D}_X(U_\beta) \times \langle g \rangle \) is uniquely represented by a pair \((t|U_\beta \cap X^i_{(g)}), s|U_\beta \cap X^i_{(g)} \) in \( \mathcal{D}_X(U_\beta \cap X^i_{(g)}) \times \langle g \rangle \) and \( \pi^\text{coor}_* \mathcal{O}_\text{flat}(\mathcal{A}^\text{coor} | X^i_{(g)} \times \mathcal{A}^g | n_{-t_y} U_\gamma) \) respectively. For all \( \beta \), the collection of sections \{ \( s|U_\beta \cap X^i_{(g)} \) \} coincide on every open set \( U_\gamma \cap X^i_{(g)} \), contained in \( U_\beta \cap U_\beta' \cap X^i_{(g)} \). Because of that, by the uniqueness axiom of sheaves the collection of sections \( s|U_\beta \cap X^i_{(g)} \) coincide on the whole of every intersection \( U_\beta \cap U_\beta' \cap X^i_{(g)} \) and the gluing axiom of sheaves imply the existence of a unique section \( s|U_\alpha \cap X^i_{(g)} \) in \( \pi^\text{coor}_* \mathcal{O}_\text{flat}(\mathcal{A}^\text{coor} | X^i_{(g)} \times \mathcal{A}^g | n_{-t_y} U_\gamma) \) for the purposes of our work we need to extend the morphism (30) to the whole of \( X^g \).

**Lemma 3.10.** The map (30) uniquely extends to a morphism of sheaves
\[ \tilde{p} : \mathcal{D}_X \times G \to \hat{j}^g_* \pi^\text{coor}_* \mathcal{O}_\text{flat}(\mathcal{A}^\text{coor} \times \mathcal{A}^g | n_{-t_y} U_\gamma) \]

**Proof.** We distinguish two cases: 1) the codimension of the stratum \( X^i_{(g)} \) is equal to or bigger than 1, 2) the stratum \( X^i_{(g)} \) is the principal (dense and open) stratum \( \hat{X} \) in \( X \).

1) Let \( U_\alpha \) be \( K \)-invariant linear slice centered on a stratum \( X^j \) contained in \( X^j_{(g)} \) as above. By Hartog’s Theorem the third arrow in (31) is bijective. Hence, the images of the morphisms (31) and
\[ \mathcal{D}_X(U_\alpha \setminus X^i_K) \times \langle g \rangle \cong \lim_{\substack{U_\beta \text{ is centered at } X_L \cap U_\gamma}} \mathcal{D}_X(\text{ind}^L_U \beta) \times \langle g \rangle \]
\[ \to \prod_{U_\beta \text{ is centered at } U_\gamma \cap X^i_{(g)}} \{ (s) \in \mathcal{D}_X(U_\beta) \times \langle g \rangle : \text{res}_{U_\beta}^U(s) = s' \} \to \pi^\text{coor}_* \mathcal{O}_\text{flat}(\mathcal{A}^\text{coor} | X^{i}_{(g)} \times \mathcal{A}^g | n_{-t_y} U_\gamma) \]

coincide. Thus, the preimage \( \hat{s}_1 \) in \( \mathcal{D}_X(W_x \setminus X^y_{(g)}) \times \langle g \rangle \) of every section \( \hat{s} \) in the image of (31) is non-empty. Furthermore, as the codimension of \( X^i_{(g)} \) is at least 2 in \( X \), it follows \( \mathcal{D}_X(W_x \setminus X^i_{(g)}) \times \langle g \rangle \cong \mathcal{D}_X(W_x \times \langle g \rangle) \) by Hartog’s Theorem. By [Vit19, Theorem 6.3] \( \hat{s}_1 \) determines a unique section \( \hat{s}_2 \) of \( \hat{j}^H_* \pi^\text{coor}_* \mathcal{O}_\text{flat}(\mathcal{A}^\text{coor} \times \mathcal{A}^g | n_{-t_y} U_\gamma) \) with \( \hat{s}_2|W_x \times X^i_{(g)} = \hat{s} \). By the identity theorem, all sections \( \hat{s}_2 \) coincide on the open subset \( U_\alpha \cap X^i_{(g)} \) of \( U_\alpha \cap X^i_{(g)} \), ergo, on \( U_\alpha \cap X^i_{(g)} \). This gives a well-defined extension \( \bar{p} \).

2) Assume \( X^i_{(g)} = \hat{X} \). Assume \( \text{codim}(X^i_{K}) = 1 \). Here, we only consider the case of a basic open sets \( \text{ind}^G_U \alpha \) with \( U_\alpha \) centered on \( X^i_K \). Each element \( D_k \) in \( H_1(L, U_\alpha, K) \) is uniquely represented by a pair of sections \( \hat{u}_\alpha = (t|U_\alpha \cap X^i_K, s|U_\alpha \cap X^i_K) \) of \( \pi^\text{coor}_* \mathcal{O}_\text{flat}(\mathcal{A}^\text{coor} \times \mathcal{O}_L(U_\alpha \cap X^i_K) \times K \) and \( \pi^\text{coor}_* \mathcal{O}_\text{flat}(\mathcal{A}^\text{coor} \times \mathcal{A}^g | n_{-t_y} U_\gamma) \) satisfying the gluing conditions from [Vit19, Section 6.1]. Setting \( \hat{c}(Y, s) = 0 \) for all complex reflections \( s \) in \( K \), the pair \( \hat{u}_\alpha \) describes a unique element in \( \mathcal{D}_X(U_\alpha) \times K \). If we set \( \hat{c} = \hat{X} \prod X^i_K \), the same element \( D_k \) uniquely determines flat section \( \hat{r}_k \) in \( \pi^\text{coor}_* \mathcal{O}_\text{flat}(\mathcal{A}^\text{coor} \times \mathcal{O}_L(U_\alpha) \times K \). Clearly, the restriction maps satisfy \( \text{res}_{U_\alpha \cap X^i_K}^U_{U_\alpha \cap X^i_K}(\hat{r}_k) = \text{res}_{U_\alpha \cap X^i_K}^U_{U_\alpha \cap X^i_K}(\hat{u}_\alpha) = t|U_\alpha \cap X^i_K \). Hence, \( \hat{r} \) is the unique flat section in \( \pi^\text{coor}_* \mathcal{O}_\text{flat}(\mathcal{A}^\text{coor} \times \mathcal{O}_L(U_\alpha) \times K) \) which is assigned to \( D_k \) and extends \( t|U_\alpha \cap X^i_K \). This gives the wanted extension \( \bar{p} \). We further extend (30) step by step to the union of \( X \) with all strata of codimension 1. Extensions beyond codimension 1 follow from Hartog’s theorem. □
Let \( \mathcal{U} = \{ U_\alpha \} \) be a family of slices at points \( x \) on \( X^\theta \) with stabilizers \( H \supseteq \langle g \rangle \), which trivialize \( TX|_{X^\theta} \), and the sets \( W_\alpha = U_\alpha \cap X^\theta \) form an open cover of \( X^\theta \). We pullback \( \omega \) along a locally holomorphic section \( s_\alpha \) of \( \mathcal{A}^\text{coor} \) over \( W_\alpha = U_\alpha \cap X^\theta \) to a \( \mathcal{A}^\langle g \rangle_{n-t_y^g} \)-valued holomorphic 1-form \( \omega_\alpha := s_\alpha^* \omega \) on \( W_\alpha \) satisfying the Maurer-Cartan equation in a similar fashion to Section 3.2.1. A further pullback of \( \omega_\alpha \) along \( \pi^+_\alpha \) yields a holomorphic Maurer-Cartan form on \( T^*X^\theta|_{W_\alpha} \) with values in \( \mathcal{A}^\langle g \rangle_{n-t_y^g} \), as desired. For each linearly independent trace \( \phi \) of \( \mathcal{A}^\langle g \rangle_{n-t_y^g} \) we define a GL\(_n-t_y^g(\mathbb{C}) \times Z\)-basic normalized \( 2n-2t_y^g \) Hochschild cocycle \( \psi_{2n-2t_y^g} \) of \( \mathcal{A}^\langle g \rangle_{n-t_y^g} \) with values in the dual bimodule \( \mathcal{A}^{\langle g \rangle^*}_{n-t_y^g} \) by

\[
\psi_{2n-2t_y^g}(a_1 \otimes b_1, \ldots, a_{2n-2t_y^g} \otimes b_{2n-2t_y^g}) := \tau_{2n-2t_y^g}(a_1, \ldots, a_{2n-2t_y^g})\phi(b_1 \cdots b_{2n-2t_y^g}).
\]

This cocycle extends to a linear form on \( C_{2n-2t_y^g+1}(\mathcal{A}^\langle g \rangle_{n-t_y^g}) \), which we again denote by \( \psi_{2n-2t_y^g} \). With the help of it we can locally define on \( \text{ind}^G_U U_\alpha \) an analogous to (26) composition of maps which yields the wanted generalized trace density

\[
\chi^g_i : \hat{\mathcal{E}}(\mathcal{D}X \times G)|_{\text{ind}^G_U U_\alpha} \to \left( j_{\text{ind}^G_U U_\alpha}^{-1} \mathcal{A}^g \right)^{2n-2t_y^g}.
\]

(32)

This map can be seen as a trace density morphism of the Hochschild chain complex of \( \mathcal{D}X \times G \). We can define similar maps for all strata \( X^\theta_H \) in \( X \) but we are only interested in trace density morphism to strata associated to cyclic subgroups of \( G \).

**Theorem 3.11.** For every choice of a linearly independent trace \( \phi^i_{(g)} \) of \( \mathcal{D}^i \) \( \times \langle g \rangle \) and for every family of holomorphic Maurer-Cartan forms \( \{ \omega_\alpha \} \) on the cotangent bundle \( T^*X^\theta \) of \( X^\theta \) with values in \( \mathcal{A}^\langle g \rangle_{n-t_y^g} \) the map

\[
\oplus_{i,g \in G} \chi^g_i : \hat{\mathcal{E}}(\mathcal{D}X \times G) \to \left( \oplus_{i,g \in G} j_{\text{ind}^G_U U_\alpha}^{-1} \mathcal{A}^g \right)^{2n-2t_y^g}
\]

is a quasi-isomorphism. Moreover, the induced morphism at the level of homology sheaves is canonical.

**Proof.** The zeroth Hochschild cohomology group of \( \mathcal{D}X \times \langle g \rangle \) with values in the dual bimodule \( \mathcal{D}X \times \langle g \rangle^* \) defines the space of traces on \( \mathcal{D}X \times \langle g \rangle \). From [AFL00] Proposition 3.1] we infer the isomorphism

\[
\tau : \text{HH}^0(\mathcal{D}X \times \langle g \rangle, \mathcal{D}X \times \langle g \rangle^*) \cong \left( \oplus_{k=1}^{\text{ord}(g)} \text{HH}^0(\mathcal{D}X \times \langle g \rangle^g k^k g \langle g \rangle) \right)
\]

where \( \text{ord}(g) \) is the order of the generator \( g \) in \( G \). As each group \( \text{HH}^0(\mathcal{D}X \times \langle g \rangle^g k^k g \langle g \rangle) \) is spanned by the \( g^k \)-twisted trace \( \text{Tr}_{g^k} \), defined in [Fed00], we have for the image of every trace \( \phi^i_{(g)} \) of \( \mathcal{D}X \times \langle g \rangle \), associated to the normal bundle to \( X^\theta_H \) under \( \tau \)

\[
\tau(\phi^i_{(g)}) = \sum_{k=1}^{\text{ord}(g)} \lambda_k \text{Tr}_{g^k}(\cdot)
\]

where for each \( k \), the constant \( \lambda_k \) is a complex number, including possibly zero. On the other hand, invoking the fact that \( \mathcal{D}X \times G \) and \( \mathcal{D}X \) are sheaves of Calabi-Yau algebras of dimension \( 2n \) we have the natural identification

\[
\mathcal{H} : \mathcal{A} \otimes G \cong \left( \mathcal{H} \mathcal{A} \otimes G \right)^G
\]
which combined with the induced map of (32) on homology yields the map

\[
\oplus_{g,i} \chi_{i} g \circ \mathfrak{s}^{-1} : (\oplus_{g \in G} \mathcal{H}_* (\mathcal{D}_X, \mathcal{D}_X g)) G \to (\oplus_{i, g \in G} H^\bullet_* (j_{i}^g, \pi_{i}^g, \Omega_{T^* X^g})) G
\]

where \( \chi_{i} g \circ \mathfrak{s}^{-1} \) is the induced map on the homology we get with \( \mathfrak{s}^{-1} \) quasi-isomorphism if and only if for every \( \bar{x} \) in \( X/G \) where we dropped the index \( i \) on the side of (32) and \( Z_{H}(g) \) is the centralizer group of \( g \) in \( H \). Consequently, we get for the stalk of \( \mathcal{D}_X \times G \) at \( \bar{x} \)

\[
(\mathcal{D}_X \times G)_\bar{x} = \lim_{\mathfrak{p}^{-1}(\bar{V}) \to \mathfrak{p}^{-1}(\bar{x})} \mathcal{D}_X(p^{-1}(\bar{V})) \times G = \mathcal{D}_X|_{\text{ind}_H^G(x)} \times G
\]

where \( \text{ind}_H^G \{ x \} := \bigsqcup_{x \in G/H} \{ x \} \) is the set of conjugacy classes of \( g \) in \( H \). Morphism (32) is a quasi-isomorphism if and only if for every \( \bar{x} \) in \( X/G \), the stalk of (33) is an isomorphism. For the left hand side of the induced map on the homology we get

\[
\mathcal{H}_* (\mathcal{D}_X \times G)_\bar{x} \cong HH_* ((\mathcal{D}_X \times G)_\bar{x})
\]

\[
\cong \left( \oplus_{g, i} HH_* (\mathcal{D}_X, \mathcal{D}_X g) \right) G
\]

\[
\cong \left( \oplus_{i} HH_* (\mathcal{D}_X, \mathcal{D}_X g) \right) G
\]

\[
\cong \left( \mathcal{C} G \otimes \mathcal{H}_* \oplus_{g \in G} HH_* (\mathcal{D}_X, \mathcal{D}_X g) \right) G
\]

\[
\cong \left( \oplus_{g \in G} HH_* (\mathcal{D}_X, \mathcal{D}_X g) \right) H
\]

\[
\cong \mathcal{C} H(g) \in \text{Conj}(H) \left( HH_* (\mathcal{D}_X, \mathcal{D}_X g) \right) Z_{H}(g)
\]

where \( \text{Conj}(H) \) is the set of conjugacy classes of \( H \), \( C_{H}(g) \) is the conjugacy class of \( g \) in \( H \) and \( Z_{H}(g) \) is the centralizer group of \( g \) in \( H \), respectively. Similarly, we get for the stalk of the homology sheaf at \( \bar{x} \) on the right hand side of (32)

\[
\left( \oplus_{i, g, i} H^\bullet_* (j_{i}^g, \pi_{i}^g, \Omega_{T^* X^g}) \right) \cong \mathcal{C} H(g) \in \text{Conj}(H) \left( H^\bullet_* (\mathcal{D}_X, \mathcal{D}_X g) \right) Z_{H}(g)
\]

\[
\cong \mathcal{C} H(g) \in \text{Conj}(H) \left( H^\bullet_* (\mathcal{D}_X, \mathcal{D}_X g) \right) Z_{H}(g)
\]

\[
\cong \mathcal{C} H(g) \in \text{Conj}(H) \left( H^\bullet_* (\mathcal{D}_X, \mathcal{D}_X g) \right) Z_{H}(g)
\]

where we dropped the index \( i \), because a point cannot simultaneously be on more than one connected component of a fixed point submanifold, and in the third line we used the holomorphic Poincaré’s Lemma and the last isomorphism follows from the fact that \( \mathfrak{a}_* X^g \) and \( \Omega_{X^g}^\bullet \) are both resolutions of the constant sheaf \( C_X \). We know from [FT10] that for every \( g \) in \( G \), the cohomology group \( HH_* (\mathcal{D}_X, \mathcal{D}_X g) \) is spanned by the cocycle

\[
c_{2n-2l_{i}^g, x} := \sum_{\sigma \in S_{2n-2l_{i}^g}} 1 \otimes u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(2n-2l_{i}^g)}
\]

with \( u_{2j-1} = \partial_{x_{j}} \) and \( u_{2j} = \partial_{x_{j}} \). Then, the normalisation property of the Feigin-Felder-Schoikht cocycle implies

\[
\chi_{i} g, x (\mathfrak{s}^{-1}(c_{2n-2l_{i}^g, x})) = (-1)^{|n-l_{i}^g|} \psi_{2n-2l_{i}^g}(\mathfrak{s}^{-1}(c_{2n-2l_{i}^g, x}))
\]

\[
= (-1)^{|n-l_{i}^g|} \sum_{k=1}^{\text{ord}(g)} \lambda_{k} \tau_{2n-2l_{i}^g}(c_{2n-2l_{i}^g, x}) \Tr g^{k}(1)
\]
which means that for each \( g, \chi^i_{(g), x} \circ \sigma_x^{-1} \) is a non-zero map between the generators of the 1-dimensional homology groups \( HH_*(\mathcal{D}_X, \mathcal{D}_X, g) \) and \( H^{2n-2q}_*(\mathcal{A}_{X, (g), c, x}) \). Hence, it is an isomorphism. Thus, \( \chi^i_{(g), x} \) and correspondingly the direct sum \( \oplus_{i, g \in G} \chi^i_{(g), x} \) are invertible maps. Ergo, Morphism (32) is a quasi-isomorphism. Moreover, by normalization the stalk of the map (32) can be made independent on the particular choice of a trace \( \phi^i_{(g)} \). Ergo, Morphism (32) induces a canonical isomorphism at the level of homology.

Composing the quasi-isomorphisms from Corollary 3.8 and (32) one obtains a new quasi-isomorphism which can be interpreted as a generalized trace density morphism for the Hochschild cochain complex of \( \mathcal{D}_X \times G \). To avoid repetition we leave the standard proof of the next result to the interested reader.

**Corollary 3.12.** There is a quasi-isomorphism

\[
\mathcal{X} : \mathcal{E}^\bullet(\mathcal{D}_X \times G, \mathcal{D}_X \times G) \to \left( \oplus_{i, g \in G} j_{i, x}^g \pi_{X, x}^g \mathcal{O}_{T_X}^{0, n - i} \right)^G.
\]

The induced morphism at the level of cohomology sheaves is canonical.

**3.3. The space of filtered infinitesimal deformations of** \( \mathcal{D}_X \times G \). In the case of the Calabi-Yau algebras \( (\mathcal{D}_X)^G \) and \( \mathcal{D}_X \times G \) the inclusion of the subcomplex of filtration preserving Hochschild cochains into the complex of all Hochschild cochains is a quasi-isomorphism. We only show this for \( \mathcal{D}_X \times G \) as the proof for \( (\mathcal{D}_X)^G \) is analogous.

**Proposition 3.13.** The canonical inclusion \( i : \mathcal{E}^\bullet_{ij}(\mathcal{D}_X \times G, \mathcal{D}_X \times G) \to \mathcal{E}^\bullet_{ij}(\mathcal{D}_X \times G, \mathcal{D}_X \times G) \) is a quasi-isomorphism.

**Proof.** Denote by \( X^{\times k} \) the \( k \)-fold Cartesian product of \( X \). Let \( \delta_k : X \to X^{\times k}, x \mapsto (x, \ldots, x) \) be the diagonal embedding of \( X \) in \( X^{\times k} \). Recall from the definition of the exterior tensor product that \( \delta_k^*(\mathcal{D}_X \times G) \cong \mathcal{D}_X^{\otimes k} \) and \( \delta_k^* \mathcal{E}^\bullet(\mathcal{D}_X \times G) \cong \mathcal{E}^\bullet(\mathcal{D}_X \times G) \). By definition the nonnegative decreasing filtration of \( \mathcal{E}^\bullet_{ij}(\mathcal{D}_X \times G, \mathcal{D}_X \times G) \) results in a first quadrant spectral sequence \( E_{pq} \) with zero sheet

\[
E_{0q}^{pq} := \text{Gr}^p \mathcal{E}_{ij}^{p+q}(\mathcal{D}_X \times G, \mathcal{D}_X \times G)
\]

\[
\cong \{ F \in \text{Hom}_C(\text{Gr}^q(\mathcal{D}_X \times G), \text{Gr}^{q-p}(\mathcal{D}_X \times G)) \mid \text{for all } q \geq p \text{ and all } n \geq 0 \}
\]

\[
\cong \{ F \in \text{Hom}_C(\text{Gr}^q(\delta_k^*(\mathcal{D}_X \times G) \times \cdots \times G) \times \mathcal{D}_X \times G, \text{Gr}^{q-p}(\mathcal{D}_X \times G) \times \mathcal{D}_X \times G) \mid \text{for all } q \geq p \text{ and all } n \geq 0 \}
\]

\[
\cong \{ F \in \text{Hom}_C(\delta_k^* \mathcal{E}^\bullet(\mathcal{D}_X \times G), \mathcal{D}_X \times G) \times \mathcal{D}_X \times G, \text{Gr}^{q-p}(\mathcal{D}_X \times G) \times \mathcal{D}_X \times G) \mid \text{for all } q \geq p \text{ and all } n \geq 0 \}
\]

\[
\cong \{ F \in \text{Hom}_C(\mathcal{E}^\bullet(\mathcal{D}_X \times G) \times \mathcal{D}_X \times G) \times \mathcal{D}_X \times G, \text{Gr}^{q-p}(\mathcal{D}_X \times G) \times \mathcal{D}_X \times G) \mid \text{for all } q \geq p \text{ and all } n \geq 0 \}
\]

\[
\cong \mathcal{E}^\bullet(\mathcal{D}_X \times G) \times \mathcal{D}_X \times G
\]

with \( p + q = n \) where in the third and the fifth line we used the definition of a topologically completed tensor product, and \( [\mathbf{.]}, q \) denotes the homogeneous part of (homological) degree \( q \) of the respective graded algebra. To shorten in the following the notation denote by \( \Delta_G \) the image of the diagonal homomorphism \( \Delta : G \to G \times G \). The first sheet of the spectral sequence is accordingly becomes

\[
E_1^{pq} = \text{H}^p(\mathcal{E}_{ij}^{pq}(\mathcal{D}_X \times G, \mathcal{D}_X \times G))
\]

\[
\cong \left[ \text{H}^p(\mathcal{E}^\bullet(\mathcal{D}_X \times G, \mathcal{D}_X \times G)) \right]_p
\]

\[
\cong \left[ \mathcal{E}_{ij}^{\mathcal{E}^\bullet(\mathcal{D}_X \times G, \mathcal{D}_X \times G)} \right]_p
\]

\[
\cong \left[ (\oplus_{g \in G} \mathcal{E}_{ij}^{\mathcal{E}^\bullet(\mathcal{D}_X \times G, \mathcal{D}_X \times G)})^G \right]_p
\]

18
\[ \begin{align*}
\cohomologicalSequence_{p+q}(\Sym^\bullet(\mathcal{F}_X), \Sym^\bullet(\mathcal{F}_X)g) \\
\cong \left( \bigoplus_{g \in G} \left[ \mathcal{H}^{p+q}(\Sym^\bullet(\mathcal{F}_X), \Sym^\bullet(\mathcal{F}_X)g) \right]_p \right)^G \\
\cong \left( \bigoplus_{g \in G} \left[ \mathcal{H}^{2n-p-q}(\Sym^\bullet(\mathcal{F}_X), \Sym^\bullet(\mathcal{F}_X) \cdot g) \right]_p \right)^G
\end{align*} \]

where in the fourth line we used [KS94, Proposition 2.2.9] and in the last line we used that \( \Sym^\bullet(\mathcal{F}_X) \) is a sheaf of Calabi-Yau algebras with dimension \( 2n \) as per Proposition [55]. We proceed as in [DE05, Proposition 8] to further simplify \( E_1^{pq} \). There is a natural holomorphic splitting of the holomorphic cotangent bundle \( T^*X = T^*X^g \oplus N^g \), where \( N^g \) is the normal bundle to the fixed point submanifold \( X^g \). With the help of the resolution (32) of \( \Sym^\bullet(\mathcal{F}_X) \), we get for the homology sheaf of \( \mathcal{G}_*(\Sym^\bullet(\mathcal{F}_X), \Sym^\bullet(\mathcal{F}_X) \cdot g) \)

\[ \mathcal{H}_*(\Sym^\bullet(\mathcal{F}_X), \Sym^\bullet(\mathcal{F}_X) \cdot g) = \mathcal{O}_{\mathcal{P}}^*(\mathcal{F}_X) \cdot g, \Sym^\bullet(\mathcal{F}_X) \]

where the complex in last line of (35) is equipped with the vanishing differential and \( l_2 \) is the complex codimension of \( X^g \) in \( X \). The step from the second to the last line to the last line in (35) is a sheaf theoretic version of [Ann05, Proposition 4]. Making use of the relation \( \Omega^1_Y \cong \mathcal{F}_Y \), we get

\[ E_1^{pq} = \left( \bigoplus_{g \in G} \Sym^\bullet(\mathcal{F}_X) \otimes_{\delta^{-1}\mathcal{O}_{\mathcal{P}}(\mathcal{F}_X)} \delta^{-1} \mathcal{F}_Y^{*\Delta} \right)^G. \]

As \( E_1^{pq} \) is a first quadrant spectral sequence, it converges to \( \mathcal{H}_*(\mathcal{O}_{\mathcal{P}}(\mathcal{F}_X) \times G, \mathcal{O}_{\mathcal{P}}(\mathcal{F}_X) \times G) \). On the other hand the spectral cohomological sequence \( E_1^{pq} \) collapses on the \( p \)-axis, as shown in the picture below.

Accounting for the fact that the differential \( d_1^{pq} : E_1^{pq} \to E_1^{p+1,q} \) is the one of cohomological degree \(-1\), obtained from the Koszul resolution of \( \Sym^\bullet(\mathcal{F}_X) \), we obtain

\[ E_2^{pq} := \left( \bigoplus_{g \in G} \mathcal{H}^{-p}(\Sym^\bullet(\mathcal{F}_X) \otimes_{\delta^{-1}\mathcal{O}_{\mathcal{P}}(\mathcal{F}_X)} \delta^{-1} \mathcal{F}_Y^{*\Delta}) \right)^G. \]

Combining the convergence of \( E_1^{pq} \) with (37) yields

\[ E_{\infty}^{pq} = \left( \bigoplus_{g \in G} \mathcal{H}^{-p}(\Sym^\bullet(\mathcal{F}_X) \otimes_{\delta^{-1}\mathcal{O}_{\mathcal{P}}(\mathcal{F}_X)} \delta^{-1} \mathcal{F}_Y^{*\Delta}) \right)^G \cong \mathcal{H}_*(\mathcal{O}_{\mathcal{P}}(\mathcal{F}_X) \times G, \mathcal{O}_{\mathcal{P}}(\mathcal{F}_X) \times G). \]

On the other hand, the filtration of \( \mathcal{O}_{\mathcal{P}}(\mathcal{F}_X) \times G \) induces a bounded below and exhaustive filtration on the Hochschild \( n \)-axis, as shown in the picture below.
According to the classical convergence theorem [Wei94, Theorem 5.5.1] this spectral homological sequence converges to \( \mathcal{HH}_{p+q}(\mathcal{D}_X \times G) \cong \mathcal{HH}^{2n-p-q}(\mathcal{D}_X \times G, \mathcal{D}_X \times G) \) where the last isomorphism derives from the fact that \( \mathcal{D}_X \times G \) is a sheaf of Calabi-Yau algebras of dimension 2n. Obviously, \( E^\infty_{pq} \) collapses on the \( p \)-axis. Hence,

\[
E^\infty_{pq} = \left( \bigoplus_{g \in G} H^p(\text{Sym}^* (\mathcal{T}^g_X) \otimes \delta^{-1} \mathcal{T}^g_Y) \right)^G \cong \mathcal{HH}^{2n-p}(\mathcal{D}_X \times G, \mathcal{D}_X \times G)
\]

From Isomorphism (38) and Isomorphism (39) we infer that \( \mathcal{HH}^p(\mathcal{D}_X \times G, \mathcal{D}_X \times G) \cong \mathcal{HH}^p(\mathcal{D}_X \times G, \mathcal{D}_X \times G) \) for every \( p \leq 2n \). Since the map \( i \) is the canonical inclusion, at the level of cohomologies \( i_* \) remains injective. We leave it to the reader to convince himself that the cohomology group sheaves of the complex of sheaves are actually finite dimensional which implies that \( i_* \) is an isomorphism, as desired.

\[\square\]

**Proposition 3.14.** \( \hat{H}^2(\mathcal{U}, \sigma_{\geq 1} \mathcal{C}_1^\infty(\mathcal{D}_X \times G, \mathcal{D}_X \times G)) \cong \left( \hat{H}^2(\mathcal{U}, \sigma_{\geq 1} \mathcal{C}_1^\infty(\mathcal{D}_X \times G, \mathcal{D}_X \times G)) \right) \cong \bigoplus_{\text{codim}(\mathcal{X}^g_{\sigma})=1} \hat{H}^0(\mathcal{U}, j^g_{\pi_{\sigma}^g \mathcal{O}_{\mathcal{T}_X}^g}) \).

**Proof.** We abuse notation by denoting an open cover of the orbifold \( X/G \) and its preimage in the \( G \)-equivariant topology of \( X \) by \( \mathcal{U} \). Let for the sake of generality \( \mathcal{C}^* \) denote an arbitrary complex of sheaves. Let \( \mathcal{C}^* \) be the \( \check{\text{C}} \)ech double complex thereof. Its total complex \( T^\infty := \text{Tot}^\infty \mathcal{C}^*(\mathcal{U}, \mathcal{L}) \) has a natural decreasing filtration by the second degree

\[
F^pT^\infty := \bigoplus_{n \geq p} \mathcal{C}^i(\mathcal{U}, \mathcal{L}^j)
\]

which determines a short exact sequence

\[
0 \to F^pT^\infty \to F^{p-1}T^\infty \to \frac{F^{p-1}T^\infty}{F^pT^\infty} \to 0
\]

for every \( p \). Note the simple but important relation

\[
F^pT^\infty \cong \bigoplus_{n \geq p} \mathcal{C}^n \mathcal{C}^i(\mathcal{U}, \mathcal{L}^j)
\]

\[
= \bigoplus_{n \geq p} \mathcal{C}^n \mathcal{C}^i(\mathcal{U}, \mathcal{L}^j)
\]

\[
= \mathcal{C}^i(\mathcal{U}, \mathcal{L}^j).
\]

(40)

The above defined filtration (40) applied to the total complex of the \( \check{\text{C}} \)ech double complex of \( \left( \bigoplus_{\text{codim}(\mathcal{X}^g_{\sigma})=1} \mathcal{C}^\infty(\mathcal{U}, \mathcal{L}^j) \right) \) yields in account of (42) the short exact sequence of complexes

\[
0 \to \left( \text{Tot}^\infty \mathcal{C}^*(\mathcal{U}, \sigma_{\geq 1} \mathcal{C}^\infty(\mathcal{D}_X \times G)) \right) \to \left( \hat{H}^2(\mathcal{U}, \sigma_{\geq 1} \mathcal{C}^\infty(\mathcal{D}_X \times G)) \right) \to 0.
\]

In turn it induces a long exact sequence of \( \check{\text{C}} \)ech hypercohomology groups

\[
\cdots \to \left( \hat{H}^2(\mathcal{U}, \sigma_{\geq 1} \mathcal{C}^\infty(\mathcal{D}_X \times G)) \right) \to \left( \hat{H}^1(\mathcal{U}, \sigma_{\geq 1} \mathcal{C}^\infty(\mathcal{D}_X \times G)) \right) \to \left( \hat{H}^1(\mathcal{U}, \sigma_{\geq 1} \mathcal{C}^\infty(\mathcal{D}_X \times G)) \right) \to \cdots
\]

(43)

where \( \partial \) denotes the so-called connecting morphism. A decomposition in terms of short exact sequences yields in degree 2 the following short exact sequence

\[
0 \to \left( \hat{H}^1(\mathcal{U}, \sigma_{\geq 1} \mathcal{C}^\infty(\mathcal{D}_X \times G)) \right) \to \left( \hat{H}^2(\mathcal{U}, \sigma_{\geq 1} \mathcal{C}^\infty(\mathcal{D}_X \times G)) \right) \to \cdots
\]
In a similar fashion the short exact sequence $0 \to \mathcal{C}_X \xrightarrow{i_*} \pi_*\mathcal{O}_{T^*X} \xrightarrow{p_*} \pi_*\mathcal{O}_{T^*X}/\mathcal{C}_X \to 0$ induces the long exact sequence

$$\cdots \to \hat{H}^1(\mathcal{U}, \mathcal{C}_X) \xrightarrow{i_*} \hat{H}^1(\mathcal{U}, \pi_*\mathcal{O}_{T^*X}) \xrightarrow{p_*} \hat{H}^1(\mathcal{U}, \pi_*\mathcal{O}_{T^*X}/\mathcal{C}_X) \xrightarrow{\partial} \hat{H}^2(\mathcal{U}, \mathcal{C}_X) \xrightarrow{i_*} \hat{H}^2(\mathcal{U}, \pi_*\mathcal{O}_{T^*X}) \xrightarrow{p_*} \hat{H}^2(\mathcal{U}, \pi_*\mathcal{O}_{T^*X}/\mathcal{C}_X) \to \cdots$$

in which by abuse of notation $\partial$ again denotes the connecting morphism. It induces the short exact sequence

$$(45) \quad 0 \to \frac{\hat{H}^1(\mathcal{U}, \pi_*\mathcal{O}_{T^*X})}{\ker(i_*)} \xrightarrow{\partial} \hat{H}^1(\mathcal{U}, \pi_*\mathcal{O}_{T^*X}/\mathcal{C}_X) \to 0.$$

The pullback of the zero section $s_0 : X \to T^*X$ defines a cochain map of sheaves $s_0^* : \pi_*\Omega_{T^*X}^\bullet \to \Omega_X^\bullet$. By definition, $s_0^* \circ \pi^* = \text{id}$. On the other hand a holomorphic homotopy operator $K : \pi_*\Omega_{T^*X}^n \to \Omega_X^{n-1}$ can be constructed following verbatim Chapter 4 in [BT82] by means of which it can be shown that $\pi^* \circ s_0^*$ is cochain homotopic to the identity of $\pi_*\Omega_{T^*X}^\bullet$. Thus, $s_0^*$ is a quasi-isomorphism. Ergo, $\hat{H}^\bullet(\mathcal{U}, \pi_*\Omega_{T^*X}^\bullet) \cong \hat{H}^\bullet(\mathcal{U}, \Omega_X^\bullet)$. Consequently, from the fact that $\Omega_X^\bullet$ is a resolution of $\mathcal{C}_X$ we infer that $\text{Im}(\mathcal{P}_n) \cong \text{Im}(i_*)$ and $\ker(\mathcal{P}_n) \cong \ker(i_*)$, respectively. The isomorphism $\pi_*\mathcal{O}_{T^*X}/\mathcal{C}_X \cong \pi_*\Omega_{T^*X,cl}^\bullet$ yields a morphism of Čech cochain complexes $\kappa : \hat{C}^\bullet(\mathcal{U}, \pi_*\mathcal{O}_{T^*X}/\mathcal{C}_X) \to \text{Tot}^\bullet \hat{C}^\bullet(\mathcal{U}, \sigma_{\geq 1}\pi_*\Omega_{T^*X})[1]$ given by

$$f \mod \mathcal{C} \mapsto (-1)^nf_{dR}(f)$$

for every $f \mod \mathcal{C} \in \hat{C}^n(\mathcal{U}, \pi_*\mathcal{O}_{T^*X}/\mathcal{C}_X)$. Indeed, let $D$ be the differential in $\text{Tot}^\bullet \hat{C}^\bullet(\mathcal{U}, \pi_*\Omega_{T^*X})$. Then,

$$\tilde{D}_n := (-1)^nD_n[1] := \sum_{p+q=n+1} \delta_{dR} p^q = (n+1)^n d_{dR}(\delta_n(f))$$

is the differential in degree $n$ of the shifted total complex $\text{Tot}^\bullet \hat{C}^\bullet(\mathcal{U}, \sigma_{\geq 1}\pi_*\Omega_{T^*X})$. Then, for every $f \mod \mathcal{C} \in \hat{C}^n(\mathcal{U}, \pi_*\mathcal{O}_{T^*X}/\mathcal{C}_X)$ we have

$$\kappa(\delta_n(f \mod \mathcal{C})) = (-1)^n d_{dR}(\delta_n(f))$$

Subsequently, we fuse the short exact sequence (44) for the special case $G = \{\text{id}_G\}$ with the short exact sequence (45) in the single diagram:

$$
\begin{array}{c}
0 \to \frac{\hat{H}^1(\mathcal{U}, \pi_*\mathcal{O}_{T^*X})}{\ker(i_*)} \xrightarrow{\partial} \hat{H}^1(\mathcal{U}, \pi_*\mathcal{O}_{T^*X}/\mathcal{C}_X) \xrightarrow{i_*} \hat{H}^2(\mathcal{U}, \sigma_{\geq 1}\pi_*\Omega_{T^*X}) \to 0
\end{array}
$$

which is commutative. Indeed, given an element $f \in \hat{C}_1(\mathcal{U}, \pi_*\mathcal{O}_{T^*X})$, for every $\alpha^0 \in \hat{C}_1(\mathcal{U}, \pi_*\Omega_{T^*X})$, $f = f + \alpha^0 \in \text{Tot}^1 \hat{C}(\mathcal{U}, \pi_*\Omega_{T^*X})$ satisfies $\mathcal{P}(\alpha) = f$. Then, we have

$$\kappa_\bullet \circ p_*(f) = \kappa_\bullet(f \mod \mathcal{C})$$

Similarly, by virtue of the above and the isomorphism $\hat{H}^1(\mathcal{U}, \pi_*\Omega_{T^*X}^\bullet) \cong \hat{H}^1(\mathcal{U}, \mathcal{C}_X)$, we have

$$\mathcal{F}_\bullet \circ \kappa_\bullet(f \mod \mathcal{C}) = [D_1(\alpha)]$$
By the 5-Lemma the linear morphism $\kappa_*$ is in fact a linear isomorphism. This coupled to the fact that $\pi_*\mathcal{O}_{T \times X}/\mathcal{O}_X \to \pi_*\Omega^2_{T \times X}[1]$ is a resolution of $\pi_*\mathcal{O}_{T \times X}/\mathcal{O}_X$ implies consequently

\begin{equation}
\hat{H}^2(\mathcal{U}, \sigma_{\geq 1} \pi_* \mathcal{O}_{T \times X}) \equiv \hat{H}^1(\mathcal{U}, \pi_*\mathcal{O}_{T \times X}/\mathcal{O}_X) \equiv \hat{H}^2(\mathcal{U}, \pi_*\Omega^2_{T \times X}) \equiv \hat{H}^2(\mathcal{U}, \Omega^2_{T \times X}).
\end{equation}

The morphisms $\mathcal{X}$ and $\mathcal{X}'_{\geq 1} := \sigma_{\geq 1}(\mathcal{X})$ induce correspondingly a quasi-isomorphism

\begin{equation}
\mathcal{X} : \text{Tot}^* \left( C^* (\mathcal{U}, \mathcal{E}^* (\mathcal{D}_X \times G, \mathcal{D}_X \times G)) \right) \to \left( \oplus_{i,g} \text{Tot}^* \left( C^* (\mathcal{U}, j_{i,g}^0 \pi_* \mathcal{O}^{\bullet-2i}_{T \times X}^\mathcal{F}) \right) \right)^G
\end{equation}

and a morphism\footnote{The brutally truncated map $\mathcal{X}'_{\geq 1}$ is not a quasi-isomorphism in general.}

\begin{equation}
\mathcal{X}'_{\geq 1} : \text{Tot}^* \left( C^* (\mathcal{U}, \sigma_{\geq 1} \mathcal{E}^* (\mathcal{D}_X \times G, \mathcal{D}_X \times G)) \right) \to \left( \oplus_{i,g} \text{Tot}^* \left( C^* (\mathcal{U}, \sigma_{\geq 1} j_{i,g}^0 \pi_* \mathcal{O}^{\bullet-2i}_{T \times X}^\mathcal{F}) \right) \right)^G,
\end{equation}

respectively.

Let $K^{**}$ be the Čech double complex of $\mathcal{E}^* (\mathcal{D}_X \times G, \mathcal{D}_X \times G)$ associated to the cover $\mathcal{U}$:

\begin{equation}
\begin{array}{ccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\delta & \delta & \delta & \delta & \delta & \delta & \delta \\
\mathcal{C}_j^{i+1}(\mathcal{U}, \mathcal{E} \times G) & \mathcal{C}_j^i(\mathcal{U}, \mathcal{E} \times G) & \mathcal{C}_j^{i+1}(\mathcal{U}, \mathcal{E} \times G) & \mathcal{C}_j^i(\mathcal{U}, \mathcal{E} \times G) & \mathcal{C}_j^{i+1}(\mathcal{U}, \mathcal{E} \times G) & \mathcal{C}_j^i(\mathcal{U}, \mathcal{E} \times G) & \mathcal{C}_j^{i+1}(\mathcal{U}, \mathcal{E} \times G) \\
\mathcal{C}_j^i(\mathcal{U}, \mathcal{E} \times G) & \mathcal{C}_j^i(\mathcal{U}, \mathcal{E} \times G) & \mathcal{C}_j^i(\mathcal{U}, \mathcal{E} \times G) & \mathcal{C}_j^i(\mathcal{U}, \mathcal{E} \times G) & \mathcal{C}_j^i(\mathcal{U}, \mathcal{E} \times G) & \mathcal{C}_j^i(\mathcal{U}, \mathcal{E} \times G) & \mathcal{C}_j^i(\mathcal{U}, \mathcal{E} \times G) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\delta & \delta & \delta & \delta & \delta & \delta & \delta \\
\end{array}
\end{equation}

in which $\delta$ denotes the Čech differential, $d$ is the standard Hochschild differential with $\delta d - d \delta = 0$ and in which by definition we have $\mathcal{E}^0(\mathcal{D}_X \times G, \mathcal{D}_X \times G) = \text{Hom}_C(\mathcal{C}_X, \mathcal{E} \times G) \cong \mathcal{E} \times G$. The total complex $\text{Tot}^* (K^{**})$ has a differential $D' = \delta + (-1)^p d$ in bidegree $(p, q)$.

The natural filtration \footnote{The brutally truncated map $\mathcal{X}'_{\geq 1}$ is not a quasi-isomorphism in general.} of $\text{Tot}^* (K^{**})$ yields in accordance with \footnote{The brutally truncated map $\mathcal{X}'_{\geq 1}$ is not a quasi-isomorphism in general.} and \footnote{The brutally truncated map $\mathcal{X}'_{\geq 1}$ is not a quasi-isomorphism in general.} the short exact sequence

\begin{equation}
0 \to \text{Tot}^* \left( C^* (\mathcal{U}, \sigma_{\geq 1} \mathcal{E}^* (\mathcal{D}_X \times G, \mathcal{D}_X \times G)) \right) \xrightarrow{\mathcal{X}'} \text{Tot}^* (K^{**}) \xrightarrow{\mathcal{X}''} C^* (\mathcal{U}, \mathcal{E} \times G) \to 0.
\end{equation}

The canonical quasi-isomorphism $i : \mathcal{E}^* (\mathcal{D}_X \times G, \mathcal{D}_X \times G) \to \mathcal{E}^* (\mathcal{D}_X \times G, \mathcal{D}_X \times G)$ from Thereom\footnote{The brutally truncated map $\mathcal{X}'_{\geq 1}$ is not a quasi-isomorphism in general.} induces a natural inclusion of Čech double complexes

\begin{equation}
i : C^* (\mathcal{U}, \mathcal{E}^* (\mathcal{D}_X \times G, \mathcal{D}_X \times G)) \to C^* (\mathcal{U}, \mathcal{E}^* (\mathcal{D}_X \times G, \mathcal{D}_X \times G))
\end{equation}

which yields an isomorphism at the level of Čech hypercohomologies. Combining \footnote{The brutally truncated map $\mathcal{X}'_{\geq 1}$ is not a quasi-isomorphism in general.} with the quasi-isomorphism \footnote{The brutally truncated map $\mathcal{X}'_{\geq 1}$ is not a quasi-isomorphism in general.}, we obtain the maps

\begin{equation}
p := \mathcal{X} \circ i
\end{equation}

\begin{equation}np_{\geq 1} := \mathcal{X}'_{\geq 1} \circ i
\end{equation}

With the help of $p$ we arrive from \footnote{The brutally truncated map $\mathcal{X}'_{\geq 1}$ is not a quasi-isomorphism in general.} at the long exact sequence of Čech hypercohomology groups

\begin{equation}
\cdots \to \hat{H}^1(\mathcal{U}, \sigma_{\geq 1} \mathcal{E}^* (\mathcal{D}_X \times G)) \xrightarrow{\mathcal{X}'} \left( \hat{H}^1(\mathcal{U}, \pi_* \Omega^*_{T \times X}) \right)^G \xrightarrow{\mathcal{X}''} \left( \hat{H}^1(\mathcal{U}, \pi_* \mathcal{O}_{T \times X}) \right)^G \\
\xrightarrow{\mathcal{X}'} \hat{H}^2(\mathcal{U}, \sigma_{\geq 1} \mathcal{E}^* (\mathcal{D}_X \times G, \mathcal{D}_X \times G)) \xrightarrow{\mathcal{X}'} \left( \hat{H}^2(\mathcal{U}, \pi_* \Omega^*_{T \times X}) \oplus \oplus_{\text{codim}(\mathcal{X}^p)} \hat{H}^0(\mathcal{U}, j_{i,g}^0 \pi_* \mathcal{O}^*_{T \times X}^\mathcal{F}) \right)^G \\
\xrightarrow{\mathcal{X}'} \left( \hat{H}^2(\mathcal{U}, \pi_* \mathcal{O}_{T \times X}) \right)^G \to \cdots
\end{equation}
Comparison between the long exact sequence $\text{(43)}$ and $\text{(48)}$ delivers $\ker(\mathcal{R}_*^1) = \ker(\mathcal{R}_*^2)$. The last fact allows us to extract from the above long exact sequence the following short exact sequence

$$0 \to \left( \overline{\text{H}}^1(\mathcal{U}, \pi, \mathcal{O}_X) \right)^G \xrightarrow{\mathcal{J}'} \overline{\text{H}}^2(\mathcal{U}, \sigma \geq 2 \mathcal{O}_X^* \otimes \mathcal{D}_X \times G) \xrightarrow{\mathcal{J}'} \ker(\mathcal{R}_*) \to 0. \quad \text{(50)}$$

By virtue of these morphisms we can now merge the short exact sequences $\text{(44)}$ and $\text{(50)}$ in the ensuing diagram

$$0 \rightarrow \frac{\overline{\text{H}}^1(\mathcal{U}, \pi, \mathcal{O}_X^* \times X^G)}{\text{Im}(\mathcal{R}_*)} \xrightarrow{\partial'} \frac{\overline{\text{H}}^2(\mathcal{U}, \sigma \geq 2 \mathcal{O}_X^* \otimes \mathcal{D}_X \times G)}{\text{Im}(\mathcal{R}_*)} \xrightarrow{\mathcal{J}'} \ker(\mathcal{R}_*) \rightarrow 0. \quad \text{(51)}$$

In order to show that diagram $\text{(51)}$ commutes, consider the diagram

$$0 \rightarrow \text{Tot}^* \left( C^* \left( \mathcal{U}, \sigma \geq 2 \mathcal{O}_X^* \otimes \mathcal{D}_X \times G \right) \right) \xrightarrow{\mathcal{J}'} \text{Tot}^* (K^*) \xrightarrow{\mathcal{J}'} C^* \left( \mathcal{U}, \sigma \geq 2 \mathcal{O}_X^* \otimes \mathcal{D}_X \times G \right) \rightarrow 0. \quad \text{(52)}$$

The left-hand square of the diagram commutes because $\mathcal{J}$ and $\mathcal{J}'$ are the canonical inclusions. From this we immediately infer that the right-hand square of diagram $\text{(51)}$ is commutative, too. On the other hand the right-hand square in diagram $\text{(52)}$ commutes, since for any $\beta \in \text{Tot}^*(K^*)$, we have

$$\mathcal{P} \circ \mathcal{J}'(\beta) = \mathcal{P}(\beta) \mod \text{Im}(\mathcal{J}')$$

$$= \mathcal{P}(\beta) \mod \text{Im}(\mathcal{P} \circ \mathcal{J}')$$

$$= \mathcal{P}(\beta) \mod \text{Im}(\mathcal{J} \circ \mathcal{P} \circ \mathcal{J}')$$

$$= \mathcal{J} \circ \mathcal{P}(\beta). \quad \text{(53)}$$

where in the third line we used the commutativity of the left square in Diagram $\text{(52)}$. Finally, let the cohomology class $[f]$ be an arbitrary representative in $\frac{\overline{\text{H}}^1(\mathcal{U}, \pi, \mathcal{O}_X^* \times X^G)}{\text{Im}(\mathcal{R}_*)}$. There is an element $\alpha \in \text{Tot}^*(K^*)$ such that $\mathcal{R}_*^1(\alpha) = f$. Then, invoking the definition of the connecting morphism $\partial'$, we obtain

$$\mathcal{P}_{\geq 1} \circ \partial'([f]) = \mathcal{P}_{\geq 1} \circ [\mathcal{J}^{-1}_2 D_1(\alpha)]$$

$$= [\mathcal{P}_{\geq 1} \circ D_1(\alpha)]$$

$$= [D_1(\mathcal{P}_{\geq 1}(\alpha))]$$

$$= [\mathcal{J}^{-1}_2 \circ D_1(\mathcal{P}_{\geq 1}(\alpha))]$$

$$= \partial \circ \mathcal{P}([f])$$

where in the second line we used that $\mathcal{J}'$ is the canonical inclusion map and $D_1(\alpha) \in \text{Im}(\mathcal{J}_2^1)$, in the third the fact that $\mathcal{P}_{\geq 1}$ is a map of cocaoin complexes and in the last line we used Equality $\text{(53)}$. We conclude that the left-hand side of diagram $\text{(51)}$ commutes, whence the whole diagram is commutative. Then by the 5-Lemma, the map $\mathcal{P}_{\geq 1}$ is a linear isomorphism. This combined with Isomorphism $\text{(46)}$ proves the claim.

Theorem $\text{(2.10)}$ coupled with Proposition $\text{(3.14)}$ yields the following important isomorphism.

**Corollary 3.15.** $\text{Def}(\mathcal{D}_X \times G)_f \cong \overline{\text{H}}^2(X, \Omega^1_X \otimes \mathcal{D}_X \times G) \oplus \bigoplus_{\text{codim}_\mathbb{C}(X^G_{\mathbb{C}}) = 1} \overline{\text{H}}^0(X^G_{\mathbb{C}}, \mathbb{C})^G$ as $\mathbb{C}$-vector spaces.

---

Since $f$ is by definition a cocycle, $D_1(\alpha) \in \ker(\mathcal{R}_*^1) = \text{Im}(\mathcal{J}_2^1)$. 

---

23
Proof. The only non-trivial part in the statement is the identification of $\check{H}^0(X, j_{(g)}^* \pi_{0, g}^* \Omega_{T^* X^g})$ with $H^0(X^g, \mathbb{C})$. We demonstrate that now. Let $\mathcal{U}$ denote a $G$-invariant open cover of $X$ and let the notation $X^g_i \cap \mathcal{U}$ on $X^g_i$ denote the induced open cover on $X^g_i$. With this, we have

$$\check{H}^i(X, j_{(g)}^* \pi_{0, g}^* \Omega_{T^* X^g}) = \lim_{\mathcal{U}} \check{H}^i(\mathcal{U}, j_{(g)}^* \pi_{0, g}^* \Omega_{T^* X^g}) = \lim_{\mathcal{U}} \check{H}^i(X^g_i \cap \mathcal{U}, \pi_{0, g}^* \Omega_{T^* X^g}) \equiv \check{H}^i(X^g_i \cap \mathcal{U}, \mathcal{A}^{\mathbf{X}^g}_i, \mathbb{C}) \equiv H^i(X^g_i, \mathcal{A}^{\mathbf{X}^g}_i, \mathbb{C}) \equiv H^i(X^g, \mathbb{C})$$

where in the third isomorphism we applied Poincare’s Lemma and the fact that $\mathcal{A}^{\mathbf{X}^g}_i$ and $\Omega_{X^g_i}$ are both resolutions of $\mathbb{C}$.

We arrive at the main theorem of the paper.

**Theorem 3.16.** Let $X$ be a smooth algebraic variety or a smooth analytic variety equipped with a finite subgroup $G \subset \text{Aut}(X)$ acting faithfully on $X$. The sheaf of twisted Cherednik algebras $\mathcal{H}^{1, c, \psi, X, G}_{X, X^g, c}$ on the quotient orbifold $X^g$ with formal $c$ and $\psi$ is a universal formal filtered deformation of the sheaf of filtered skew-group algebra $\mathcal{D}_X \rtimes G$.

**Proof.** The claim follows from the fact that the dimension of the parameter space $\{(\psi, c)\}$ is the same as the dimension of $\text{Def}(\mathcal{D}_X \rtimes G)$.

---

**Acknowledgement**

I would like to express my gratitude to the hospitality of the Department of Mathematics at MIT and the excellent research conditions which enabled my work at first place. I would like to extend my sincere gratitude to Prof. Pavel Etingof for multiple valuable comments and suggestions which he generously provided in the course of my work on this note and which substantially improved its quality. In particular, I thank him for giving me the idea to use filtration preserving Hochschild cochains- a trick which "unlocked" the whole chain of arguments in the note. I also thank Prof. Giovanni Felder, Prof. Ajay Ramadoss, Prof. Valery Lunts and Dr. Konstantin Jacob for a useful exchange and valuable comments on various topics. This work is financially supported by a *Swiss National Science Foundation Early Postdoc.Mobility Fellowship* number 188014.

---

**References**

[AFLS00] J. Alev, M.A. Farinati, T. Lambre, and A.L. Solotar. “Homologie des invariants d’une algÈbre de Weyl sous l’action d’un groupe fini”. In: *Journal of Algebra* 232.2 (2000), pp. 564–577. ISSN: 0021-8693.

[Ann05] Rina Anno. “Multiplicative structure on the Hochschild cohomology of crossed product algebras”. In: *arXiv preprint math/0511396* (2005).

[BR73] I. N. Bernshtein and B. I. Rozenfe’ld. “Homogeneous spaces of infinite-dimensional Lie algebras and the characteristic classes of foliations”. In: *Uspehi Mat. Nauk* 28.4(172) (1973), pp. 103–138. ISSN: 0042-1316.

[BT82] Raoul Bott and Loring W Tu. *Differential forms in algebraic topology*. Springer, 1982.

[DE05] Vasily Dolgushev and Pavel Etingof. “Hochschild cohomology of quantized symplectic orbifolds and the Chen-Ruan cohomology”. In: *International Mathematics Research Notices* 2005.27 (2005), pp. 1657–1688. doi: [10.1155/IMRN.2005.1657](https://doi.org/10.1155/IMRN.2005.1657)

[DMZ07] Martin Doubek, Martin Markl, and Petr Zima. "Deformation theory (lecture notes)". In: *Archivum mathematicum* 43.5 (2007), pp. 333–371.

[EF08] Markus Engeli and Giovanni Felder. “A Riemann-Roch-Hirzebruch formula for traces of differential operators”. In: *Ann. Sci. Éc. Norm. Supér.(4) 41.4* (2008), pp. 621–653.

[Eti04] Pavel Etingof. “Cherednik and Hecke algebras of varieties with a finite group action”. In: *arXiv preprint math/0406499* (2004).

[Fed00] Boris Fedosov. "On G-Trace and G-Index in Deformation Quantization". In: *Letters in Mathematical Physics* 52.1 (Apr. 2000), pp. 29–49. ISSN: 1573-0530. doi: [10.1023/A:1007601802484](https://doi.org/10.1023/A:1007601802484)
[FT10] Giovanni Felder and Xiang Tang. “Equivariant Lefschetz number of differential operators”. In: *Mathematische Zeitschrift* 266.2 (2010), pp. 451–470.

[Gin06] Victor Ginzburg. “Calabi-yau algebras”. In: *arXiv preprint math/0612139* (2006).

[Gra61] John W. Gray. “Extensions of sheaves of algebras”. In: *Illinois J. Math.* 5.1 (Mar. 1961), pp. 159–174. doi: 10.1215/ijm/1255629652

[KS94] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*. Vol. 292. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original. Springer-Verlag, Berlin, 1994, pp. x+512. ISBN: 3-540-51861-4.

[Mac75] Saunders MacLane. *Homology*. Classics in Mathematics. Springer-Verlag, 1975.

[Ram11] Ajay Ramadoss. “Integration of cocycles and Lefschetz number formulae for differential operators”. In: *Symmetry, Integrability and Geometry: Methods and Applications* 7.0 (2011), pp. 1–26.

[RT12] Ajay Ramadoss and Xiang Tang. “Hochschild (Co) homology of the Dunkl Operator Quantization of $\mathbb{Z}_2$-singularity”. In: *International Mathematics Research Notices* 2012.9 (2012), pp. 2123–2162.

[Van98] Michel Van den Bergh. “A relation between Hochschild homology and cohomology for Gorenstein rings”. In: *Proceedings of the American Mathematical Society* 126.5 (1998), pp. 1345–1348.

[Vit19] Alexander Vitanov. “The sheaf of Cherednik algebras from the point of view of formal geometry”. In: *arXiv* (Jan. 2019).

[Vit20] Alexander Vitanov. “Trace densities and Index Theorems for the sheaf of Cherednik algebras”. In: *arXiv* (June 2020).

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994. doi: 10.1017/CBO9781139644136

[Wod87] Mariusz Wodzicki. “Cyclic homology of differential operators”. In: *Duke Math. J.* 54.2 (1987), pp. 641–647. doi: 10.1215/S0012-7094-87-05426-3

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave., Cambridge, MA 02139, USA

E-mail address: avitanov@mit.edu