Multiple Exchange Property for 
M$^\natural$-concave Functions and Valuated Matroids

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Abstract

The multiple exchange property for matroid bases is generalized for valuated matroids and M$^\natural$-concave set functions. The proof is based on the Fenchel-type duality theorem in discrete convex analysis. The present result has an implication in economics: The strong no complementarities (SNC) condition of Gul and Stacchetti is in fact equivalent to the gross substitutes (GS) condition of Kelso and Crawford.

Keywords: discrete convex analysis; matroid; exchange property; combinatorial optimization

1 Introduction

Discrete convex analysis \cite{Fujishige2005, Murota2008} offers a general framework of discrete optimization, combining the ideas from submodular/matroid theory and convex analysis. It has found applications in many different areas \cite{Murota2008}, including systems analysis \cite{Fujishige2008}, inventory theory in operations research \cite{Andersson2010}, and mathematical economics and game theory \cite{Fujishige2008, Murota2008, Murota1988}. The interaction between discrete convex analysis and mathematical economics was initiated by \cite{Fujishige1995} (see also \cite{Murota2008}, Chapter 11) and accelerated by the crucial observation of Fujishige–Yang \cite{Fujishige1995} that M$^\natural$-concavity (see Section 2 for the definition) is equivalent to the gross substitutability (GS) of Kelso–Crawford \cite{Kelso1985}. \cite{Murota2008} is a detailed recent survey on the relation between M$^\natural$-concavity and (GS).

In matroid theory, one of the classical results \cite{Aigner1979, Murota1994, Oxley1992} says that the basis family of a matroid enjoys the multiple exchange property: For two bases $X$ and $Y$ in a matroid and a subset $I \subseteq X \setminus Y$, there exists a subset $J \subseteq Y \setminus X$ such that $(X \setminus I) \cup J$ and $(Y \setminus J) \cup I$ are both bases. As a quantitative version of this, we may naturally consider the multiple exchange property for a set function $f$: For two subsets $X, Y$ and a subset $I \subseteq X \setminus Y$, there exists $J \subseteq Y \setminus X$ such that $f(X) + f(Y) \leq f((X \setminus I) \cup J) + f((Y \setminus J) \cup I)$.

The objective of this paper is to establish this multiple exchange property for M$^\natural$-concave functions and valuated matroids. The results are described in Section 2 and the proof, based on the Fenchel-type duality theorem in discrete convex analysis, is given in Section 3. Our result settles an old question in economics: The strong no complementarities (SNC) condition of Gul and Stacchetti \cite{Gul1991} is in fact equivalent to the gross substitutes condition. This is discussed in Section 4. Section 5 offers a proof to the fact that the multiple exchange property characterizes M$^\natural$-concavity.

2 Results

Let $N$ be a finite set, say, $N = \{1, 2, \ldots, n\}$. For a function $f : 2^N \rightarrow \mathbb{R} \cup \{-\infty\}$, $\text{dom} \ f$ denotes the effective domain of $f$, i.e., $\text{dom} \ f = \{X \mid f(X) > -\infty\}$. 

A function \( f : 2^N \to \mathbb{R} \cup \{-\infty\} \) with \( \text{dom} \ f \neq \emptyset \) is called \( M^f\text{-concave} \) \cite{16,19}, if, for any \( X, Y \in \text{dom} \ f \) and \( i \in X \setminus Y \), we have (i) \( X - i \in \text{dom} \ f \), \( Y + i \in \text{dom} \ f \) and

\[
f(X) + f(Y) \leq f(X - i) + f(Y + i),
\]

(2.1)
or (ii) there exists some \( j \in Y \setminus X \) such that \( X - i + j \in \text{dom} \ f \), \( Y + i - j \in \text{dom} \ f \) and

\[
f(X) + f(Y) \leq f(X - i + j) + f(Y + i - j).
\]

(2.2)

Here we use short-hand notations \( X - i = X \setminus \{i\}, Y + i = Y \cup \{i\}, X - i + j = (X \setminus \{i\}) \cup \{j\}, \) and \( Y + i - j = (Y \cup \{i\}) \setminus \{j\} \). This property is referred to as the exchange property. The exchange property can be expressed more compactly as:

\[
\text{(M\(^f\text{-EXC}\)) [Exchange property] } \text{For any } X, Y \subseteq N \text{ and } i \in X \setminus Y, \text{ we have }
\]

\[
f(X) + f(Y) \leq \max_{j \in Y \setminus X} \left( f(X - i) + f(Y + i), \max_{j \in Y \setminus X} f(X - i + j) + f(Y + i - j) \right),
\]

(2.3)

where \((-\infty) + a = a + (-\infty) = (-\infty) + (-\infty) = -\infty \) for \( a \in \mathbb{R}, -\infty \leq -\infty \), and a maximum taken over an empty set is defined to be \(-\infty \).

In this paper we are concerned with the multiple exchange property:

\[
\text{(M\(^f\text{-EXC}_m\)) [Multiple exchange property]} \text{For any } X, Y \subseteq N \text{ and } I \subseteq X \setminus Y, \text{ there exists } J \subseteq Y \setminus X \text{ such that } f(X) + f(Y) \leq f((X \setminus I) \cup J) + f((Y \setminus J) \cup I), \text{ i.e., }
\]

\[
f(X) + f(Y) \leq \max_{J \subseteq Y \setminus X} f((X \setminus I) \cup J) + f((Y \setminus J) \cup I).
\]

(2.4)

**Theorem 2.1.** An \( M^f\text{-concave} \) function \( f : 2^N \to \mathbb{R} \cup \{-\infty\} \) has the multiple exchange property (M\(^f\text{-EXC}_m\)).

*Proof.* The proof is given in Section\cite{3}. \( \square \)

As an immediate corollary we obtain the multiple exchange property for the maximizers. The set of the maximizers of \( f \) is denoted by \( \text{arg max} \ f \).

**Theorem 2.2.** Let \( f : 2^N \to \mathbb{R} \cup \{-\infty\} \) be an \( M^f\text{-concave} \) function. For any \( X, Y \in \text{arg max} \ f \) and \( I \subseteq X \setminus Y \), there exists a subset \( J \subseteq Y \setminus X \) such that \( (X \setminus I) \cup J \in \text{arg max} \ f \) and \( (Y \setminus J) \cup I \in \text{arg max} \ f \).

The concept of valuated matroid due to of Dress–Wenzel \cite{3,4} (see also \cite{15} Chapter 5) is defined in terms of an exchange property similar to (M\(^f\text{-EXC}\)). A function \( f : 2^N \to \mathbb{R} \cup \{-\infty\} \) with dom \( f \neq \emptyset \) is called a valuated matroid, if, for any \( X, Y \subseteq N \) and \( i \in X \setminus Y \), it holds that

\[
f(X) + f(Y) \leq \max_{j \in Y \setminus X} f(X - i + j) + f(Y + i - j).
\]

(2.5)

A valuated matroid is nothing but an \( M^f\text{-concave} \) function \( f \) such that dom \( f \) consists of equi-cardinal subsets, i.e., \(|X| = |Y|\) for any \( X, Y \in \text{dom} \ f \). In this case, dom \( f \) forms the basis family of a matroid on \( N \).

As a corollary of Theorem\cite{2,1} we obtain the following.

**Theorem 2.3.** A valuated matroid \( f \) has the multiple exchange property (M\(^f\text{-EXC}_m\)) with \(|J| = |I|\).

\footnote{In economics \cite{9}, the multiple exchange property (M\(^f\text{-EXC}_m\)) is called “strong no complementarities property (SNC)”. See Section\cite{4}.}
This theorem contains, as a special case, the multiple exchange theorem for matroid bases due to Brylawski [11], Greene [8] and Woodall [25]; see also [13, 14, 21].

**Theorem 2.4 ([11][8][25]).** Let $X$ and $Y$ be bases in a matroid, and let $I \subseteq X \setminus Y$. Then there exists a subset $J \subseteq Y \setminus X$ such that $(X \setminus I) \cup J$ and $(Y \setminus J) \cup I$ are both bases.

The converse of Theorem 2.1 should be intuitively obvious, but a formal proof is needed. We have to assure that for $I = \{i\}$ in $(M^\circ\text{-EXC}_m)$ there exists $J$ with $|J| \leq 1$.

**Proposition 2.5.** $(M^\circ\text{-EXC}_m)$ implies $(M^\circ\text{-EXC})$.

*Proof.* We provide two proofs in this paper. The first, given in Section 4 for an economic implication, is rather indirect, relying essentially on some (generalizations of) known results in economics. The second proof, given in Section 5, is more straightforward, relying on basic results in matroid theory and discrete convex analysis.

From Proposition 2.5 and Theorem 2.1 we can obtain a characterization of $M^\circ$-concave functions in terms of the multiple exchange property.

**Theorem 2.6.** A function $f : 2^N \to \mathbb{R} \cup \{\infty\}$ is $M^\circ$-concave if and only if it has the multiple exchange property $(M^\circ\text{-EXC}_m)$.

## 3 Proof of Theorem 2.1

In this section we give a proof to the main theorem, Theorem 2.1. Let $f : 2^N \to \mathbb{R} \cup \{\infty\}$ be an $M^\circ$-concave function, $X, Y \in \text{dom } f$ and $I \subseteq X \setminus Y$. We shall prove

$$f(X) + f(Y) \leq \max_{J \subseteq Y \setminus I} \{f((X \setminus I) \cup J) + f((Y \setminus J) \cup I)\}. \quad (3.1)$$

Our proof is based on the Fenchel-type duality theorem in discrete convex analysis.

With the notations

$$C = X \cap Y, \quad X_0 = X \setminus Y = X \setminus C, \quad Y_0 = Y \setminus X = Y \setminus C,$$

$$f_1(J) = f((X \setminus I) \cup J) = f(X_0 \setminus I) \cup C \cup J) \quad (J \subseteq Y_0),$$

$$f_2(J) = f((Y \setminus J) \cup I) = f(I \cup C \cup (Y_0 \setminus J)) \quad (J \subseteq Y_0),$$

the inequality (3.1) is rewritten as

$$f(X) + f(Y) \leq \max_{J \subseteq Y_0} \{f_1(J) + f_2(J)\}. \quad (3.2)$$

Both $f_1$ and $f_2$ are $M^\circ$-concave set functions on $Y_0$.

Consider the (convex) conjugate functions of $f_1$ and $f_2$ given by

$$g_1(q) = \max_{J \subseteq Y_0} \{f_1(J) - q(J)\} \quad (q \in \mathbb{R}^{Y_0}),$$

$$g_2(q) = \max_{J \subseteq Y_0} \{f_2(J) - q(J)\} \quad (q \in \mathbb{R}^{Y_0}),$$

where $q(J) = \sum_{j \in J} q_j$. For any $J \subseteq Y_0$ and $q \in \mathbb{R}^{Y_0}$, we have

$$f_1(J) + f_2(J) = (f_1(J) - q(J)) + (f_2(J) + q(J))$$

$$\leq \max_{J \subseteq Y_0} \{f_1(J) - q(J)\} + \max_{J \subseteq Y_0} \{f_2(J) + q(J)\}$$

$$= g_1(q) + g_2(-q).$$

3
The Fenchel-type duality theorem in discrete convex analysis [16, Theorem 8.21 (1)] asserts that there exist \( J \) and \( q \) for which the above inequality holds in equality, i.e.,

\[
\max_{J \subseteq Y_0} f_1(J) + f_2(J) = \min_{q \in \mathbb{R}^{Y_0}} (g_1(q) + g_2(-q)).
\]  

(3.3)

Note that \( \text{dom } g_1 = \text{dom } g_2 = \mathbb{R}^{Y_0} \) and the assumption in [16, Theorem 8.21 (1)] is satisfied.

The desired inequality (3.2) follows from (3.3) and Lemma 3.1 below.

**Lemma 3.1.** For any \( q \in \mathbb{R}^{Y_0} \), we have \( g_1(q) + g_2(-q) \geq f(X) + f(Y) \).

**Proof.** Let \( g \) be the (convex) conjugate function of \( f \), i.e.,

\[
g(p) = \max_{Z \subseteq N} f(Z) - p(Z) \quad (p \in \mathbb{R}^N).
\]

By the conjugacy theorem in discrete convex analysis ([17, Theorem 3.4], [16, Theorems 8.4, (8.10)]), \( g \) is a polyhedral \( L \)-convex function. In particular, it is submodular:

\[
g(p) + g(p') \geq g(p \lor p') + g(p \land p') \quad (p, p' \in \mathbb{R}^N),
\]

(3.4)

where \( p \lor p' \) and \( p \land p' \) denote the vectors of component-wise maximum and minimum, i.e.,

\[
(p \lor p')_i = \max(p_i, p'_i), \quad (p \land p')_i = \min(p_i, p'_i).
\]

For a vector \( q \in \mathbb{R}^{Y_0} \) we define \( p^{(1)}_1, p^{(2)}_1 \in \mathbb{R}^N \) by

\[
p^{(1)}_1 = \begin{cases} 
q_i \quad (i \in Y_0), \\
-M \quad (i \in X_0 \setminus I), \\
+M \quad (i \in I), \\
-M \quad (i \in C), \\
+M \quad (i \in N \setminus (X \cup Y)), 
\end{cases} \quad p^{(2)}_1 = \begin{cases} 
q_i \quad (i \in Y_0), \\
+M \quad (i \in X_0 \setminus I), \\
-M \quad (i \in I), \\
-M \quad (i \in C), \\
+M \quad (i \in N \setminus (X \cup Y)), 
\end{cases}
\]

where \( M \) is a sufficiently large positive number. Then we have

\[
\begin{align*}
g_1(q) &= \max_{J \subseteq Y_0} f((X_0 \setminus I) \cup C \cup J) - q(J) \\
&= g(p^{(1)}) - M(|X_0 \setminus I| + |C|), \\
g_2(-q) &= \max_{J \subseteq Y_0} f(I \cup C \cup (Y_0 \setminus J)) + q(J) \\
&= \max_{K \subseteq Y_0} f(I \cup C \cup K) - q(K) + q(Y_0) \\
&= g(p^{(2)}) - M(|I| + |C|) + q(Y_0).
\end{align*}
\]

By adding these two and using submodularity (3.4) of \( g \), we obtain

\[
g_1(q) + g_2(-q) = g(p^{(1)}) + g(p^{(2)}) - M(|X| + |C|) + q(Y_0)
\]

\[
\geq g(p^{(1)} \lor p^{(2)}) + g(p^{(1)} \land p^{(2)}) - M(|X| + |C|) + q(Y_0).
\]

(3.5)

Since

\[
(p^{(1)} \lor p^{(2)})_i = \begin{cases} 
q_i \quad (i \in Y_0), \\
+M \quad (i \in X_0 \setminus I), \\
+M \quad (i \in I), \\
-M \quad (i \in C), \\
+M \quad (i \in N \setminus (X \cup Y)), 
\end{cases} \quad (p^{(1)} \land p^{(2)})_i = \begin{cases} 
q_i \quad (i \in Y_0), \\
-M \quad (i \in X_0 \setminus I), \\
-M \quad (i \in I), \\
-M \quad (i \in C), \\
+M \quad (i \in N \setminus (X \cup Y)), 
\end{cases}
\]

we obtain

\[
\max_{J \subseteq Y_0} f_1(J) + f_2(J) = \min_{q \in \mathbb{R}^{Y_0}} (g_1(q) + g_2(-q)).
\]
we have
\[ g(p^{(1)} \lor p^{(2)}) \geq f(Y) - q(Y_0) + M[C], \quad (3.6) \]
\[ g(p^{(1)} \land p^{(2)}) \geq f(X) + M[X]. \quad (3.7) \]

The substitution of (3.6) and (3.7) into (3.5) yields the desired inequality
\[ g_1(q) + g_2(-q) \geq f(X) + f(Y). \]

\[ \square \]

**Remark 3.1.** Among several different proofs known for the multiple exchange property of matroid bases (Theorem 2.4), the proofs of Woodall [25] and McDiarmid [14] are based on minimax duality formulas for matroid rank functions (matroid union/intersection theorems). Our proof of Theorem 2.1 generalizes this idea to $M^2$-concave functions. Note that the matroid union/intersection theorems are special cases of the Fenchel-type duality theorem for $M^2$-concave functions [16, Section 8.2.3], [18, Section 5].

**Remark 3.2.** The above proof shows that the subset $J$ in ($M^2$-EXC$_m$) can be computed in polynomial time by an adaptation of the valuated matroid intersection algorithm [13, Chapter 5].

### 4 An Implication in Economics

For a vector $p \in \mathbb{R}^N$ we define
\[ D(p|f) = \arg \max_X \{f(X) - p(X) \mid X \subseteq N\}, \quad (4.1) \]
where $p(X) = \sum_{i \in X} p_i$. In economic applications where $f$ denotes a utility (valuation) function over indivisible goods, $p$ is interpreted as the vector of prices and $D(p) = D(p|f)$ represents the demand correspondence. We use notation $f[−p]$ for the function defined by $f[−p]|X = f(X) - p(X)$ for $X \subseteq N$.

Kelso and Crawford [11] introduced the following property for $f : 2^N \to \mathbb{R} \cup \{-\infty\}$, which turned out to be the key property in discussing economies with indivisible goods:

- **(GS)** [Gross Substitutes property] For any vectors $p$ and $q$ with $p \leq q$ and $X \in D(p|f)$, there exists $Y \in D(q|f)$ such that $\{i \in X \mid p_i = q_i\} \subseteq Y$.

Gul and Stacchetti [9] considered the following three properties:

- **(SI)** [Single Improvement property] For any $p \in \mathbb{R}^N$, if $X \notin D(p|f)$, there exists $Y \subseteq N$ such that $|X \backslash Y| \leq 1$, $|Y \backslash X| \leq 1$, and $f[-p](X) < f[-p](Y)$,

- **(NC)** [No Complementarities property] For any $p \in \mathbb{R}^N$, if $X, Y \in D(p|f)$ and $I \subseteq X \backslash Y$, there exists $J \subseteq Y \backslash X$ such that $(X \backslash I) \cup J \in D(p|f)$,

- **(SNC)** [Strong No Complementarities property] For $X, Y \subseteq N$ and $I \subseteq X \backslash Y$, there exists $J \subseteq Y \backslash X$ such that $f(X) + f(Y) \leq f((X \backslash I) \cup J) + f((Y \backslash J) \cup I)$.

They showed that (NC) and (SI) are equivalent to (GS), and these (mutually equivalent) conditions are implied by (SNC). Subsequently, Fujishige and Yang [7] pointed out that (GS) is equivalent to ($M^2$-EXC) for $M^2$-concavity. These results are summarized schematically here as:

\[ (\text{SNC}) \implies (\text{NC}) \iff (\text{GS}) \iff (\text{SI}) \iff (M^2\text{-EXC}). \quad (4.2) \]

\[ ^2\text{To be precise, Kelso and Crawford [11] as well as Gul and Stacchetti [9] and Fujishige and Yang [7] treat the case of } f : 2^N \to \mathbb{R}, \text{ with } \text{dom } f = 2^N. \text{ It can be verified that (4.2)} \text{ is true for } f : 2^N \to \mathbb{R} \cup \{-\infty\}. \]
Since (SNC) and \((M^\natural-\text{EXC}_m)\) are mathematically the same, and \((M^\natural-\text{EXC}_m)\) follows from \((M^\natural-\text{EXC})\) by Theorem 2.1, we now see that the above five properties are in fact equivalent:

\[(\text{SNC}) \iff (\text{NC}) \iff (\text{GS}) \iff (\text{SI}) \iff (M^\natural-\text{EXC}).\] (4.3)

In this context it would be natural to consider the following simultaneous version of (NC):

\[\text{(NCsim)} \quad \text{For any } p \in \mathbb{R}^N, \text{ if } X, Y \in D(p|f) \text{ and } I \subseteq X \setminus Y, \text{ there exists } J \subseteq Y \setminus X \text{ such that } (X \setminus I) \cup J \in D(p|f) \text{ and } (Y \setminus J) \cup I \in D(p|f).\]

Obviously, \((\text{SNC}) \Rightarrow (\text{NCsim})\) and \((\text{NCsim}) \Rightarrow (\text{NC})\). Hence \((\text{NCsim})\) is also equivalent to \((\text{GS})\).

We conclude this section by stating the equivalence of all the six properties as a theorem.

**Theorem 4.1.** For a function \(f : 2^N \to \mathbb{R} \cup \{-\infty\}\), we have the following equivalence:

\[(M^\natural-\text{EXC}_m) = (\text{SNC}) \iff (\text{NCsim}) \iff (\text{NC}) \iff (\text{GS}) \iff (\text{SI}) \iff (M^\natural-\text{EXC}).\]

## 5 A Direct Proof of Proposition 2.5

The proof of “\((M^\natural-\text{EXC}_m) \Rightarrow (M^\natural-\text{EXC})\)” consists of the following propositions, which refer to the following conditions for a set family \(F \subseteq 2^N:\)

\[\text{(B}^\natural\text{-EXC)} \quad \text{For any } X, Y \in F \text{ and } i \in X \setminus Y, \text{ we have (i) } X - i \in F, Y + i \in F \text{ or (ii) there exists some } j \in Y \setminus X \text{ such that } X - i + j \in F, Y + j \in F,\]

\[\text{(B}^\natural\text{-EXC}_m) \quad \text{For any } X, Y \in F \text{ and } I \subseteq X \setminus Y, \text{ there exists } J \subseteq Y \setminus X \text{ such that } (X \setminus I) \cup J \in F \text{ and } (Y \setminus J) \cup I \in F.\]

**Proposition 5.1.** If \(f\) satisfies \((M^\natural-\text{EXC}_m)\), then \(\text{dom } f\) satisfies \((B^\natural\text{-EXC}_m)\).

*Proof.* This is obvious

**Proposition 5.2.** For a set family \(F\), \((B^\natural\text{-EXC}_m) \Rightarrow (B^\natural\text{-EXC}).\)

*Proof.* The proof is given in Section 5.1

**Proposition 5.3.** If \(\text{dom } f\) satisfies \((B^\natural\text{-EXC})\), then \((M^\natural-\text{EXC}_m) \Rightarrow (M^\natural-\text{EXC}).\)

*Proof.* The proof is given in Section 5.2

### 5.1 Proof of Proposition 5.2

We make use of the following proposition (Tardos [24, Theorem 2.3], Murota–Shioura [19, Remark 5.2]), where

\[\text{(B}^\natural\text{-EXC}_+) \quad \text{For any } X, Y \in F \text{ and } i \in X \setminus Y, \text{ both (a) and (b) hold, where (a) (i) } X - i \in F \text{ or (ii) } X - i + j \in F \text{ for some } j \in Y \setminus X;\]

\[(\text{b) (i) } Y + i \in F \text{ or (ii) } Y + i - k \in F \text{ for some } k \in Y \setminus X.\]

**Proposition 5.4.** The following three conditions are equivalent.

(i) \(F\) is a generalized matroid.

(ii) \(F\) satisfies \((B^\natural\text{-EXC}_+).\)

(iii) \(F\) satisfies \((B^\natural\text{-EXC}).\)

With this equivalence, the statement “\((B^\natural\text{-EXC}_m) \Rightarrow (B^\natural\text{-EXC})\)” in Proposition 5.2 is immediate from the following.
Proposition 5.5. \((\text{B}^3\text{-EXC}_m) \implies (\text{B}^3\text{-EXC}_\pm)\).

Proof. We prove by contradiction. Suppose that \((\text{B}^3\text{-EXC}_\pm)\) fails for some \(X, Y \in \mathcal{F}\), and take such \((X, Y)\) with \(|X\Delta Y| = |X \setminus Y| + |Y \setminus X|\) minimum. There exists \(i \in X \setminus Y\) such that

\[ (F): \quad \text{(a) or (b) fails for } (X, Y, i) \quad \text{(including: both (a) and (b) fail).} \]

By \((\text{B}^3\text{-EXC}_m)\) there exists \(J \subseteq Y \setminus X\) such that \((X - i) \cup J \in \mathcal{F}\) and \((Y \setminus J) + i \in \mathcal{F}\). Choose such \(J\) with \(|J|\) minimum. If \(|J| = 0\), we have (a-i) and (b-i), which contradicts (F). If \(|J| = 1\), say, \(J = \{j\}\), we have (a-ii) and (b-ii), which contradicts (F).

Suppose that \(|J| \geq 2\). Take \(j \in J\).

- \((\text{B}^3\text{-EXC}_m)\) for \((X - i) \cup J, X, j)\) yields
  - [Case X0: \((X - i) \cup (J - j) \in \mathcal{F}\) and \(X + j \in \mathcal{F}\)] or [Case X1: \(X \cup (J - j) \in \mathcal{F}\) and \(X - i + j \in \mathcal{F}\)].

- \((\text{B}^3\text{-EXC}_m)\) for \((Y, (Y \setminus J) + i, j)\) yields
  - [Case Y0: \(Y - j \in \mathcal{F}\) and \(Y \setminus (J - j) + i \in \mathcal{F}\)] or [Case Y1: \(Y + i - j \in \mathcal{F}\) and \(Y \setminus (J - j) \in \mathcal{F}\)].

We have four cases to consider: X0–Y0, X1–Y1, X1–Y0 and X0–Y1.

- Case X0–Y0: We have \((X - i) \cup (J - j) \in \mathcal{F}\) and \(Y \setminus (J - j) + i \in \mathcal{F}\), contradicting the minimality of \(|J|\).
- Case X1–Y1: We have \(X - i + j \in \mathcal{F}\) and \(Y + i - j \in \mathcal{F}\), and hence (a-ii) and (b-ii). A contradiction to (F).
- Case X1–Y0: By Case X1, we have \(X - i + j \in \mathcal{F}\), i.e., (a-ii). Let \(X' = X \cup (J - j) \in \mathcal{F}\) and note that \(|X'\Delta Y| < |X\Delta Y|\) by \(|J| \geq 2\). Then \((X', Y)\) must satisfy \((\text{B}^3\text{-EXC}_{\pm})\). Since \(i \in X' \setminus Y\), \((\text{B}^3\text{-EXC}_{\pm})\) for \((X', Y, i)\) shows that (b-i) \(Y + i \in \mathcal{F}\) or (b-ii) \(Y + i - k \in \mathcal{F}\) for some \(k \in Y \setminus X'\). Since \(k \in Y \setminus X' \subseteq Y \setminus X\), this means that (b) holds for \((X, Y, i)\). A contradiction to (F).
- Case X0–Y1: (A similar argument as in Case X1–Y0): By Case Y1, we have \(Y + i - j \in \mathcal{F}\), i.e., (b-ii). Let \(Y' = Y \setminus (J - j) \in \mathcal{F}\) and note that \(|X\Delta Y'| < |X\Delta Y|\) by \(|J| \geq 2\). Then \((X, Y')\) must satisfy \((\text{B}^3\text{-EXC}_{\pm})\). Since \(i \in X \setminus Y'\), \((\text{B}^3\text{-EXC}_{\pm})\) for \((X, Y', i)\) shows that (a-i) \(X - i \in \mathcal{F}\) or (a-ii) \(X - i + k \in \mathcal{F}\) for some \(k \in Y' \setminus X\). Since \(k \in Y' \setminus X \subseteq Y \setminus X\), this means that (a) holds for \((X, Y, i)\). A contradiction to (F).

In all cases we have reached a contradiction, which was caused by our initial assumption that \((\text{B}^3\text{-EXC}_{\pm})\) fails for some \((X, Y)\). Therefore, \((\text{B}^3\text{-EXC}_{\pm})\) must be true for all \((X, Y)\). \(\square\)

5.2 Proof of Proposition 5.5

To prove “\((\text{M}^3\text{-EXC}_m) \implies (\text{M}^3\text{-EXC})\)” under the assumption of \((\text{B}^3\text{-EXC})\) for \text{dom} \(f\), we use the following local characterization of \(\text{M}^3\)-concavity ([16], [19]).

Theorem 5.6. A set function \(f : 2^N \to \mathbb{R} \cup \{-\infty\}\) is \(\text{M}^3\)-concave if and only if \text{dom} \(f\) satisfies \((\text{B}^3\text{-EXC})\) and \(f\) satisfies the following three conditions:

\[
\begin{align*}
 f(X + i + j) + f(X) &\leq f(X + i) + f(X + j) \quad (\forall X \subseteq N, \forall i, j \in N \setminus X, i \neq j), \\
 f(X + i + j) + f(X + k) &\leq \max \{f(X + i + k) + f(X + j), f(X + j + k) + f(X + i)\} \\
 &\quad (\forall X \subseteq N, \forall i, j, k (\text{distinct}) \in N \setminus X), \\
 f(X + i + j) + f(X + k + l) &\leq \max \{f(X + i + k) + f(X + j + l), f(X + j + k) + f(X + i + l)\} \\
 &\quad (\forall X \subseteq N, \forall i, j, k, l (\text{distinct}) \in N \setminus X).
\end{align*}
\]
We shall derive (5.1), (5.2), and (5.3) from \(M^\sharp - \text{EXC}_m\). First, (5.1) follows from \(M^\sharp - \text{EXC}_m\) applied to \((X + i + j, X, i)\), where \(J = \emptyset\) is the unique possibility. Second, (5.2) follows from \(M^\sharp - \text{EXC}_m\) applied to \((X + i + j, X + k, i)\), where \(J = \emptyset\) or \(J = \{k\}\) is possible.

To derive (5.3) we introduce notation \(M^\sharp - \text{EXC}_m(X, Y, I)\) to mean \(M^\sharp - \text{EXC}_m\) for \((X, Y, I)\), i.e.,

\[
(M^\sharp - \text{EXC}_m(X, Y, I)) \quad \text{For any } X, Y \subseteq N \text{ and } I \subseteq X \setminus Y, \text{ there exists } J \subseteq Y \setminus X \text{ such that}
\]

\[
f(X) + f(Y) \leq f((X \setminus I) \cup J) + f((Y \setminus J) \cup I)
\]

with abbreviation of \(M^\sharp - \text{EXC}_m(X + i + j, X + k, i)\) to \(\text{EXC}_m(i, j, k, i)\), etc. We also use short-hand notation \(g(i) = f(X + i), \ g(ij) = f(X + i + j)\) and \(g(ijk) = f(X + i + j + k)\). In (5.3) we may assume \(i = 1, \ j = 2, \ k = 3, \ l = 4\). To prove (5.3) by contradiction, suppose that (5.3) fails, i.e.,

\[
\max \{g(13) + g(24), g(23) + g(14)\} < g(12) + g(34).
\] (5.4)

Under the assumption (5.4), \(\text{EXC}_m(12, 34, 2)\) yields

\[
g(12) + g(34) \leq \max \{g(11) + g(234), g(2) + g(134)\}.
\] (5.5)

Without loss of generality, we may assume

\[
g(2) + g(134) \leq g(1) + g(234).
\]

\(\text{EXC}_m(234, 1, 4)\) yields

\[
g(1) + g(234) \leq \max \{g(23) + g(14), g(4) + g(123)\} = g(4) + g(123).
\]

\(\text{EXC}_m(123, 4, 2)\) yields

\[
g(4) + g(123) \leq \max \{g(24) + g(13), g(2) + g(134)\} = g(2) + g(134).
\]

Therefore,

\[
g(2) + g(134) = g(1) + g(234) = g(4) + g(123).
\]

Since we have symmetry \(3 \leftrightarrow 4\), we obtain

\[
g(1) + g(234) = g(2) + g(134) = g(3) + g(124) = g(4) + g(123) =: \alpha.
\] (5.6)

By (5.4) and (5.5) we have

\[
\max \{g(13) + g(24), g(23) + g(14)\} < g(12) + g(34) \leq \alpha.
\] (5.7)

\(\text{EXC}_m(123, 1, 3), \ \text{EXC}_m(124, 2, 4), \ \text{EXC}_m(234, 3, 4),\) and \(\text{EXC}_m(134, 4, 3)\) yield, respectively,

\[
g(1) + g(123) \leq g(13) + g(12),
g(2) + g(124) \leq g(12) + g(24),
g(3) + g(234) \leq g(34) + g(23),
g(4) + g(134) \leq g(14) + g(34).
\]

By adding these four inequalities and using (5.6) we obtain

\[
4\alpha \leq 2\{g(12) + g(24)\} + \{g(13) + g(24)\} + \{g(14) + g(23)\}.
\] (5.8)

This contradicts (5.7). Thus (5.3) is proved.
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References

[1] T. H. Brylawski: Some properties of basic families of subsets, Discrete Mathematics, 6 (1973), 333–341
[2] V. Danilov, G. Koshevoy, and K. Murota: Discrete convexity and equilibria in economies with indivisible goods and money, Mathematical Social Sciences, 41 (2001), 251–273.
[3] A. W. M. Dress and W. Wenzel: Valuated matroid: A new look at the greedy algorithm, Applied Mathematics Letters, 3 (1990), 33–35.
[4] A. W. M. Dress and W. Wenzel: Valuated matroids, Advances in Mathematics, 93 (1992), 214–250.
[5] S. Fujishige: Submodular Functions and Optimization, 2nd ed., Elsevier, Amsterdam, 2005.
[6] S. Fujishige and A. Tamura: A two-sided discrete-concave market with possibly bounded side payments: An approach by discrete convex analysis, Mathematics of Operations Research, 32 (2007), 136–155.
[7] S. Fujishige and Z. Yang: A note on Kelso and Crawford’s gross substitutes condition, Mathematics of Operations Research, 28 (2003), 463–469.
[8] C. Greene: A multiple exchange property for bases, Proc. American Mathematical Society 39 (1973), 45–50.
[9] F. Gul and E. Stacchetti: Walrasian equilibrium with gross substitutes, Journal of Economic Theory, 87 (1999), 95–124.
[10] Y. T. Ikebe, Y. Sekiguchi, A. Shioura and, A. Tamura: Stability and competitive equilibria in multi-unit trading networks with discrete concave utility functions, Japan Journal of Industrial and Applied Mathematics, 32 (2015), 373–410.
[11] A. S. Kelso, Jr., and V. P. Crawford: Job matching, coalition formation, and gross substitutes, Econometrica, 50 (1982), 1483–1504.
[12] F. Kojima, A. Tamura and M. Yokoo: Designing matching mechanisms under constraints: An approach from discrete convex analysis. The 7th International Symposium on Algorithmic Game Theory, Patras, 2014; https://mpra.ub.uni-muenchen.de/56189/
[13] J. P. S. Kung: Basis-exchange properties, in: N. White, ed., Theory of Matroids, Cambridge University Press, London, 1986, Chapter 4, 62–75.
[14] C. J. H. McDiarmid: An exchange theorem for independence structures, Proc. American Mathematical Society 47 (1975), 513–514.
[15] K. Murota: Matrices and Matroids for Systems Analysis, Springer, Berlin, 2000.
[16] K. Murota: Discrete Convex Analysis, SIAM, Philadelphia, 2003.
[17] K. Murota. Recent developments in discrete convex analysis, in: W. J. Cook, L. Lovász, and J. Vygen (eds.), Research Trends in Combinatorial Optimization, Springer, Berlin, 2009, pp. 219–260.

[18] K. Murota: Submodular function minimization and maximization in discrete convex analysis, RIMS Kokyuroku Bessatsu, B23 (2010), 193–211.

[19] K. Murota and A. Shioura: M-convex function on generalized polymatroid, Mathematics of Operations Research, 24 (1999), 95–105.

[20] K. Murota and A. Tamura: Application of M-convex submodular flow problem to mathematical economics, Japan Journal of Industrial and Applied Mathematics, 20 (2003), 257–277.

[21] A. Schrijver: Combinatorial Optimization—Polyhedra and Efficiency, Springer, Heidelberg, 2003.

[22] A. Shioura and A. Tamura: Gross substitutes condition and discrete concavity for multi-unit valuations: a survey, Journal of Operations Research Society of Japan, 58 (2015), 61–103.

[23] D. Simchi-Levi, X. Chen and J. Bramel: The Logic of Logistics: Theory, Algorithms, and Applications for Logistics Management, 3rd ed. Springer, New York, 2014.

[24] É. Tardos: Generalized matroids and supermodular colourings, in: A. Recski and L. Lovász (eds.), Matroid Theory, Colloquia Mathematica Societatis János Bolyai, 40, North-Holland, Amsterdam, 1985, 359–382.

[25] D. R. Woodall: An exchange theorem for bases of matroids, Journal of Combinatorial Theory, B16 (1974), 227–228.