An extension of the Maxwell–Boltzmann, Bose–Einstein and Fermi–Dirac distributions by the Caputo fractional derivative

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In order to generalize the Maxwell–Boltzmann (MB), Bose–Einstein (BE), and Fermi–Dirac (FD) distributions to fractional order, we start with the thermodynamical equation, \( \partial U/\partial \beta = -aU - bU^2 \), with \( \beta = 1/k_B T \) and parameters \( a > 0 \) and \( b \), which is equivalent to the equation proposed by Planck in 1900. Setting \( R = 1/U \) and \( x = a(\beta - \beta_0) \), we obtain the linear partial differential equation \( \partial R/\partial x = R + b/a \) from the thermodynamical equation. Then, the Caputo fractional derivative of order \( 0 < p < 1 \) is introduced in place of the partial derivative of \( x \). We obtain the fractional MB, BE, and FD distributions, where the exponential function, \( e^x \), is replaced by the Mittag-Leffler function, \( E_p(x^p) \). The behaviors of the fractional FD distribution are examined.

I. INTRODUCTION

As an extension of quantum statistics, Wu interpolated the statistical weights of Bose–Einstein (BE), and Fermi–Dirac (FD) statistics [1,2] as

\[
W = \frac{[G + (N - 1)(1 - g)]}{N![G - gN - (1 - g)]},
\]

(1)

where \( N \) is the number of identical particles occupying a group of \( G \) states, and \( 0 \leq g \leq 1 \). Weight \( W \) corresponds to the distribution of bosons if \( g = 0 \), and of fermions if \( g = 1 \). In general, with \( G_j \) being the number of levels and, \( N_j \) being the number of particles in the \( j \)th \( (j = 1, 2, \ldots) \) group, and under the assumption that each particle in the \( j \)th group has the same energy, \( \varepsilon_j \), the grand partition function is given by

\[
Z = \sum_{\{N_j\}} W(\{N_j\}) \exp\left[-\beta \sum_j \left(N_j \varepsilon_j - \mu \right) \right],
\]

where \( \beta = 1/k_B T \) is the inverse temperature ( \( k_B \) is the Boltzmann constant) and \( \mu \) is the chemical potential. The average number, \( n_j \), is determined from the equation \( \partial \ln Z/\partial n_j = 0 \), where \( n_j = N_j/G_j [1,2] \).

Ertik et al. [3] introduced the Caputo fractional derivative of order \( 0 < p < 1 \) in place of the partial derivative of \( n_j \), and proposed the fractional generalized distribution,

\[
n_j(g, p) = \frac{1}{E_p(x_j) + (2g - 1)},
\]

(2)

where \( x_j = \beta(\varepsilon_j - \mu) \), and \( E_p(x_j) \) is the Mittag-Leffler (ML) function defined by

\[
E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha + 1)}, \quad \alpha > 0.
\]

They analyzed the NASA COBE data [5] using a modified Planck distribution Eq. (2) with Eq. (3) and \( q = 0 \).

On the other hand, in theories of stochastic processes, fractional calculus is introduced: the fractional derivative has been used to generalize the diffusion equation (DE) [10], the Fokker–Planck equation (FPE) [11, 12], and others to fractional order. When the partial derivative of time is replaced by the Riemann–Liouville derivative in the DE or FPE, the ML function appears in the solutions [10, 12].

In 1900, in order to derive the blackbody radiation law, Planck adopted a differential equation [13, 17] for thermodynamical relations which was equivalent to the following equation [16, 17]:

\[
\frac{\partial U}{\partial \beta} = -aU - bU^2,
\]

(4)

where \( U \) is the energy density, \( \beta = 1/k_B T \) and, \( a > 0 \) and \( b \) are parameters. Making use of ordinary calculus, we obtain the following expression for \( U(x) \):

\[
U(x) = \frac{a/b}{e^{x/a} - 1}, \quad x = a(\beta - \beta_0).
\]

(5)

We are interested in applying fractional calculus to extending the BE distribution. However, if we introduce a fractional derivative in place of the partial derivative of \( \beta \) in Eq. (4), it is very difficult to obtain an analytical solution, because Eq. (4) is nonlinear in function \( U \). Putting \( U = 1/R \) in Eq. (4), we obtain the linear partial differential equation,

\[
\frac{\partial R}{\partial x} = R + \frac{b}{a}.
\]

(6)

In a previous paper [18], we introduced the Riemann–Liouville fractional derivative into Eq. (6) in place of the partial derivative of \( x \):

\[
a_0 D_x^p R(x) = R(x) + b/a, \quad x = a(\beta - \beta_0) > 0.
\]

(7)

The Riemann–Liouville fractional derivative of function \( R(x) \) of order \( p \) \((m - 1 \leq p < m)\) for \( m = 1, 2, \ldots \) is
defined \[\text{as}\] as
\[aD^p_x R(x) = \frac{1}{\Gamma(m-p)} \left( \frac{d}{dx} \right)^m \int_0^x (x - \tau)^{m-p-1} R(\tau) d\tau.\] (8)
If \(p = m - 1\) in Eq. (8), we have \(aD^p_x R(x) = \frac{d^{m-1}}{dx^{m-1}} R(x)\).

The solution to Eq. (7), we can only obtain \(c\) where \(c = \frac{a}{b} \{E_p(x^p) - 1\}\) and start from Eq. (7), we can only obtain \(p\)

\[R(x) = \frac{b}{a} \{E_p(x^p) - 1\} + \sum_{k=0}^{m-1} c_k x^{p-k-1} E_{p,k}(x^p),\]
\[m - 1 < p < m,\] (9)

where \(c_k = aD^{p-k-1}_x R(x)|_{x=0}\). In Eq. (9), \(E_{\alpha,\beta}(x)\) denotes the generalized ML function \[\sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha + \beta)}, \quad \alpha, \beta > 0.\] (10)

Note that when \(0 < p < 1\) \((m = 1)\), \(R(0)\) in Eq. (9) diverges unless \(c_0 = 0\), and that when \(1 < p < 2\) \((m = 2)\), \(R(0)\) diverges unless \(c_1 = 0\). Therefore, only when the boundary conditions \(c_0 = 0\) and \(c_1 = 0\) are satisfied, do we obtain the solution to Eq. (7):
\[R(x) = \frac{b}{a} \{E_p(x^p) - 1\}, \quad 0 < p < 2,\] (11)

where \(R(0) = 0\). Then we are led to a fractional BE distribution,
\[U(x) = \frac{1}{R(x)} = \frac{a/b}{E_p(x^p) - 1}.\] (12)

In the limit where \(p \to 1\), function \(U(x)\) approaches the BE distribution, as \(\lim_{p \to 1} E_p(x^p) = e^x\).

| TABLE I. Comparison of ordinary calculus and the Riemann–Liouville calculus, where \(x = a(\beta - \beta_0) = h\nu/k_BT - \mu\). |
|-----------------|-----------------|-----------------|
| Approach        | BE              | FD              |
| Eq. (3) or (9)  | \(a/b = 1\)    | \(a = \frac{h\nu}{e - 1}\) |
| Riemann-Liouville fractional calculus | \(a/b = 1\)    | \(a = \frac{h\nu}{e - 1}\) |
| Eq. (7)         | \(R(x) = E_p(x^p) - 1\) | Unknown |

The results of Eqs. (3) and (7) are compared in Table I. As long as we introduce the Riemann–Liouville fractional derivative, and start from Eq. (7), we can only obtain the fractional BE distribution. Therefore, we would like to investigate whether we can get fractional Maxwell–Boltzmann (MB) and FD distributions in addition to the fractional BE distribution if the Caputo fractional derivative is introduced instead of the Riemann–Liouville fractional derivative in Eq. (9). See Appendix A.

The outline of this study is as follows: in section II, the Caputo fractional derivative is introduced into Eq. (9), and the solutions are obtained, from which derive the fractional MB, BE, and FD distributions. The behaviors of the fractional FD distribution are examined. In section III, those distributions are expanded around \(p = 1\) under the condition that \(|p - 1| << 1\) in order to investigate the deviation of each fractional distribution from the original. Section IV is devoted to concluding remarks. In order to concretely demonstrate the difference among the Riemann–Liouville fractional derivative, its fractional integral, and the Caputo derivative, calculated results of three typical functions are compared in appendix A.

II. APPLICATION OF THE CAPUTO FRACTIONAL DERIVATIVE TO EQ. (9)

The Caputo fractional derivative \[\text{of function } R(x) \text{ for } m = 1, 2, \ldots \text{ is defined as}
\[\frac{\text{d}^m}{\text{d}x^m}(R(x)) = \frac{1}{\Gamma(m-p)} \int_0^x (x - \tau)^{m-p-1} R^{(m)}(\tau) d\tau,
\]
\[m - 1 < p < m,\] (13)

where \(R^{(m)}(\tau) = \left(\frac{d}{d\tau}\right)^m R(\tau)\). From Eq. (13), we obtain
\[\lim_{p \to m} \frac{\text{d}^m}{\text{d}x^m}(R(x)) = R^{(m)}(x).\]

Then, instead of Eq. (9), we consider the following equation:
\[\frac{\text{d}^m}{\text{d}x^m}(R(x)) = R(x) + b/a.\] (14)

The Laplace transform, \(\tilde{R}(s)\), of the function \(R(x)\) is defined as
\[\tilde{R}(s) = L[R(x); s] = \int_0^\infty e^{-sx} R(x) dx.\]

The Laplace transform of \[\text{of Eq. (14)}\] is given by
\[L\left[\frac{\text{d}^m}{\text{d}x^m}(R(x)); s\right] = \frac{1}{\Gamma(m - p)} L[x^{m-p-1}; s] L[R^{(m)}(x); s] = s^p \tilde{R}(s) - \sum_{k=0}^{m-1} s^{k-n} R^{(m-k-1)}(0).\]

Thus, we obtain the following equation from Eq. (14),
\[\tilde{R}(s) = \frac{b}{as(s^p - 1)} + \sum_{k=0}^{m-1} R^{(m-k-1)}(0) \frac{s^{k-\nu}}{s^p - 1}.\] (15)

In order to obtain the solution \(R(x)\) from Eq. (15), we use the following formula \[\text{of } L[x^{\beta-1} E_{\alpha,\beta}(\gamma x^\alpha); s] = \frac{s^{\alpha-\beta}}{s^\alpha - \gamma}, \quad Re(s) > |a|^{1/\alpha}.\] (16)
Then the solution $R(x)$ is given by

$$R(x) = \frac{b}{a} E_{p,p+1}(x^p) + \sum_{k=0}^{m-1} R^{(m-k-1)}(0) x^{m-k-1} E_{p,m-k}(x^p),$$

$$m - 1 < p < m.$$  \hfill (17)

Using the identity $E_{p,p+1}(x) = E_p(x) - 1$ and $E_{p,1}(x) = E_p(x)$, we have

$$R(x) = \frac{b}{a} \{E_p(x^p) - 1\} + R(0) E_p(x^p),$$

$$0 < p < 1 \ (m = 1),$$  \hfill (18)

or

$$R(x) = \frac{b}{a} \{E_p(x^p) - 1\} + R(0) E_p(x^p) + R^{(1)}(0) x E_{p,2}(x^p),$$

$$1 < p < 2 \ (m = 2).$$  \hfill (19)

Here we consider the following three cases.

(i) At first, we set $b = 0$ in Eq. (13). Furthermore, if $R(0) = 1/c \neq 0$ and $R^{(1)}(0) = 0$, $R(x)$ is given by

$$R(x) = (1/c) E_p(x^p)$$

for $0 < p < 2$. Then, function $U(x)$ is the fractional MB distribution,

$$U(x) = \frac{1}{R(x)} = \frac{c}{E_p(x^p)}, \quad 0 < p < 2. \quad (20)$$

(ii) If $R(0) = 0$ and $R^{(1)}(0) = 0$ in Eqs. (18) and (19), the solution $R(x)$ is given by

$$R(x) = \{b/a\} [E_p(x^p) - 1]$$

for $0 < p < 2$. Then, the function $U(x)$ becomes the fractional BE distribution,

$$U(x) = \frac{1}{R(x)} = \frac{a/b}{E_p(x^p) - 1}, \quad 0 < p < 2. \quad (21)$$

(iii) If $R(0) = -2b/a$ and $R^{(1)}(0) = 0$, the solution is

$$R(x) = -(b/a) [E_p(x^p) + 1]$$

for $0 < p < 2$. Then, the function $U(x)$ becomes the fractional FD distribution, for $x \geq 0$,

$$U(x) = \frac{1}{R(x)} = \frac{-a/b}{E_p(x^p) + 1}, \quad 0 < p < 2. \quad (22)$$

For $x < 0$, an imaginary component appears in $U(x)$, which is written as

$$-\frac{b}{a} U(x) = 1 + \sum_{k=0}^{\infty} \frac{|x|^{pk} \cos(pk \pi)}{\Gamma(pk + 1)} - i \sum_{k=0}^{\infty} \frac{|x|^{pk} \sin(pk \pi)}{\Gamma(pk + 1)}$$

$$= \left(1 + \sum_{k=0}^{\infty} \frac{|x|^{pk} \cos(pk \pi)}{\Gamma(pk + 1)} \right)^2 + \left(\sum_{k=0}^{\infty} \frac{|x|^{pk} \sin(pk \pi)}{\Gamma(pk + 1)} \right)^2.$$

$$0 < p < 2. \quad (23)$$

In Fig. 1 these fractional FD distributions with order $p$ are compared with those from [20, 22].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The order $p$ dependencies of the fractional FD distribution, Eq. (22), with $x = \beta(\varepsilon - \mu)$ and $\mu = 1$, and the inverse temperatures $\beta$ are fixed. As $x < 0$, the real part of Eq. (23) is used.}
\end{figure}

III. BEHAVIORS OF EQUATIONS (20), (21), AND (22) AROUND $p = 1$

In order to investigate the behaviors of the fractional MB, BE, and FD distributions (Eqs. (20), (21), and (22), respectively), around $p = 1$, we approximate those distributions under the assumption that $|p - 1| << 1$.

(i) The fractional MB distribution, Eq. (20), for $x > 0$ is
where $\psi(z)$ denotes the digamma function, which is defined as the logarithmic derivative of the Gamma function, $\psi(z) = d \ln \Gamma(z)/dz$.

(ii) The fractional BE distribution, Eq. (21), is expanded for $x > 0$ as,

$$
\frac{1}{E_p(x^p)-1} \simeq \frac{1}{e^x-1} + \frac{p-1}{(e^x-1)^2} \times \sum_{k=0}^{\infty} \left[ \frac{kx^k}{\Gamma(k+1)} \left\{ \psi(k+1) - \ln x \right\} \right].
$$

(iii) Similarly, the fractional FD distribution, Eq. (22), for $x > 0$ is expanded as

$$
\frac{1}{E_p(x^p)+1} \simeq \frac{1}{e^x+1} + \frac{p-1}{(e^x+1)^2} \times \sum_{k=0}^{\infty} \left[ \frac{kx^k}{\Gamma(k+1)} \left\{ \psi(k+1) - \ln x \right\} \right].
$$

One of these formulas was utilized for the analysis of COBE data [18] (see also [23]).

### IV. CONCLUDING REMARKS

1) As far as we know, by introducing the Riemann–Liouville fractional derivative into Eq. (6) (which is equivalent to the celebrated differential equation proposed by M. Planck), one can only obtain the fractional BE distribution [18].

2) On the other hand, if the Caputo fractional derivative is introduced into Eq. (6) instead of the Riemann–Liouville fractional derivative, we can arrive at fractional MB, BE, and FD distributions by the use of appropriate initial conditions.

3) The difference among the solutions of the Riemann–Liouville fractional derivative, Eq. (14), and of Caputo fractional derivative, Eq. (15), is the following: the boundary conditions of Eq. (9), $c_k = 0 \mathcal{D}^{p-k-1} R(x) |_{x=0}$, $m - 1 < p < m$, $k = 0, 1, \ldots, m - 1$, are of non-integral order. Those conditions are not independent of the value of $R(0)$, in general. On the other hand, the initial conditions of Eq. (15) are composed of the integral order derivatives $R^{(k)}(0)$, $k = 0, 1, \ldots, m - 1$, and are the same for the differential equation with integral order $m$. They are not subject to any other constraints.

4) The fractional BE, MB and FD distributions are given by the original BE, MB, and FD distributions, respectively, with their exponential functions, $e^x$, replaced by the ML function, $E_p(x^p)$, where $p$ indicates the order of the Caputo fractional derivative.

### Appendix A: Comparison of the Riemann–Liouville fractional calculus and the Caputo fractional derivative

From the definition of the Riemann–Liouville fractional derivative, Eq. (8), and that of the Caputo fractional derivative, Eq. (13), the following relation is obtained [7]:

$$
_0^C \mathcal{D}_x^p f(x) = _0^C \mathcal{D}_x^p f(x) + \sum_{k=0}^{m-1} \frac{x^{k-p}}{\Gamma(k-p+1)} f^{(k)}(0).
$$

The Riemann–Liouville fractional integral for $m - 1 < p < m$ $(m = 1, 2, \ldots)$ is defined as

$$
_0^C \mathcal{D}_x^{-p} f(x) = \frac{1}{\Gamma(m-p)} \int_0^x (x-\tau)^{m-p-1} f(\tau) d\tau.
$$

Then, the Caputo derivative is expressed by the use of the Riemann–Liouville fractional derivative and integral, Eqs. (8) and (A2), as

$$
_0^C \mathcal{D}_x^p f(x) = \frac{1}{\Gamma(m-p)} \int_0^x (x-\tau)^{m-p-1} f(m)(\tau) d\tau,
$$

where

$$
_0^C \mathcal{D}_x^p x^k = 0 \text{ for } k = 1, 2, \ldots, m - 1 \text{ and } m \geq 2.
$$

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TABLE II. Comparison of the Riemann–Liouville fractional calculus and the Caputo fractional derivative for three typical functions, where $\mu$ and $a \neq 0$ are constant, $m - 1 \leq p < m$, and $\nu = m - p > 0$ for $m = 1, 2, \ldots$.

| | | | |
|---|---|---|---|
| | $\frac{d^p}{dx^p} f(x)$ | $i_0^D_p f(x)$ | $0_i^D_p f(x)$ |
| $x^\mu$, $\mu > -1$ | $\frac{1}{\Gamma(\mu + 1)} x^{\mu-p}$ | $\frac{1}{\Gamma(\mu + 1)} x^{\mu+p}$ | $\frac{1}{\Gamma(\mu + 1)} x^{\mu+p}$ |
| $e^{ax}$, $a \neq 0$ | $x^{-p} E_{1,1-p}(ax)$ | $x^{p} E_{1,1+p}(ax)$ | $x^{p} E_{1,1+p}(ax)$ |
| $\ln x$ | $\frac{\Gamma(\nu+1)}{\Gamma(\nu)} (\ln x + \psi(1) - \psi(\nu + 1))$ | $\frac{\Gamma(\nu+1)}{\Gamma(\nu)} (\ln x + \psi(1) - \psi(\nu + 1))$ | $\frac{\Gamma(\nu+1)}{\Gamma(\nu)} (\ln x + \psi(1) - \psi(\nu + 1))$ |

a) the Riemann–Liouville fractional derivative $0^D_p f(x)$ and fractional integral $0_i^D_p f(x)

b) the Caputo fractional derivative, $C^D_p f(x)$

| $f(x)$ | $\frac{d^p}{dx^p} f(x)$ | remarks |
|---|---|---|
| $x^\mu$, $\mu > -1$ | $\frac{1}{\Gamma(\mu + 1)} x^{\mu-p}$ | $\mu$ : not integer, or $\mu \geq m$ |
| | $0$ | $\mu = 0, 1, \ldots, m - 1$ |
| $e^{ax}$, $a \neq 0$ | $x^{-p} E_{1,1-p}(ax) - \sum_{j=0}^{m-1} \frac{(ax)^j}{\Gamma(j+p+1)}$ | |
| $\ln x$ | $\frac{\Gamma(m) \Gamma(m+1) x^{\nu-m}}{\Gamma(\nu+1) (e^{ax} - 1)}$ | $\nu :$ diverge for $m \geq 1$ |

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