EXPONENTIAL GAPS IN THE LENGTH SPECTRUM

EMMANUEL SCHENCK

Abstract. We present a separation property for the gaps in the length spectrum of a compact Riemannian manifold with negative curvature. In arbitrary small neighborhoods of the metric for some suitable topology, we show that there are negatively curved metrics with a length spectrum exponentially separated from below. This property was previously known to be false generically.

1. Introduction

The Anosov property of the geodesic flow on a compact manifold with negative curvature provides a great abundance of closed geodesics: the counting function satisfies

\[ \# \{ \gamma : \ell(\gamma) \leq T \} \asymp \frac{e^{ht_{\text{top}}T}}{2ht_{\text{top}}T}, \quad T \to +\infty, \]

where \( h_{\text{top}} > 0 \) denotes the topological entropy of the flow and \( \ell(\gamma) \) the length of a closed, unoriented geodesic \( \gamma \). We refer the reader to [Mar69, MS04, PP90] for comprehensive studies of periodic orbits for hyperbolic systems.

For generic metrics in negative curvature, distinct closed geodesics have distinct lengths [Abr70]: as a result, the length spectrum \( \mathcal{L} \) contains exponentially many points in an interval \( I(T) \) of fixed size centered at \( T \to +\infty \). This suggests naturally to study the distribution of nearby lengths in \( \mathcal{L} \) in this asymptotic limit: see [Dol98, Ana00, PS01, PS12] and references given there for refinements of the counting estimate (1.1).

The length spectrum is also of particular interest when studying the connexions between the geometry of \( M \) and the spectrum of its Laplace-Beltrami operator \( \Delta_g \) for the metric \( g \): if closed geodesics are isolated and non-degenerate in the sense of Morse, it is well known since [CdV73, Cha74, DG75] that the singular support of the tempered distribution \( \text{Trace}(e^{it\sqrt{\Delta_g}}) \) is given by the period spectrum \( \mathcal{P} = \{ k\ell, \ell \in \mathcal{L}, k \in \mathbb{N} \} \). Such distributional traces have been used widely since then in various direct and inverse spectral problems.

A box principle used with equation (1.1) shows that in \( \mathcal{L} \cap I(T) \), there are many gaps of size at least \( e^{-ht_{\text{top}}T} \) when \( T \) is large. However, nothing excludes a priori some over-exponential clustering in the length spectrum: in this case it would not be possible to control the gaps uniformly from below. Some interesting results concerning this question have been obtained by Dolgopyat and Jakobson in [DJ16]. To precise the problem of estimating the gaps in \( \mathcal{L} \) from below, we will say that the length spectrum is \textit{exponentially separated} if there exist \( C, \nu > 0 \) with the property
that
\[(1.2) \quad \forall \ell, \ell' \in \mathcal{L} \text{ with } \ell \neq \ell', \quad |\ell - \ell'| \geq C e^{-\nu_k \max(\ell, \ell')}.
\]

In [DJ16] it is established that the length spectrum of hyperbolic manifolds is exponentially separated as soon as the fundamental group have algebraic generators. The authors note that it is always the case for finite volume hyperbolic manifolds of dimension \(n \geq 3\), see [GR70]. They also show that for compact hyperbolic surfaces, \((1.2)\) is true for a dense set in the corresponding Teichmüller space.

In variable negative curvature, Theorem 4.1 in [DJ16] gives a rather different picture: \((1.2)\) is false for a \(G_\delta\)-dense set of metrics for the \(C^k\)-topology, \(k > 3\). Hence metrics with a length spectrum which is not exponentially separated are topologically generic, and then dense.

In this note, we provide a complementary result: in variable negative curvature, the set of metrics satisfying \((1.2)\) is also dense for the \(C^k\)-topology, with \(k \geq 2\). To state our result more precisely, we assume that \((M, g_0)\) is a closed manifold of dimension \(\geq 2\) with negative curvature, and that \(g_0\) is of class \(C^r\) with \(r \geq 2\). Write the bounds of the sectional curvature \(K(g_0)\) as
\[-k_0 \leq K(g_0) \leq -k_1, \quad k_0, k_1 > 0.
\]

Fix some integer \(k \in [2, r]\), and for \(\varepsilon_0 > 0\) define the set of Riemannian metrics
\[
\mathcal{M}_k(\varepsilon_0) \overset{\text{def}}{=} \{g : \|g - g_0\|_{C^k} < \varepsilon_0\}.
\]

Let us define also
\[
- K_0 = \inf_{g \in \mathcal{M}_k(\varepsilon_0)} K(g), \quad - K_1 = \sup_{g \in \mathcal{M}_k(\varepsilon_0)} K(g), \quad \kappa = \sqrt{|K_0|}, \quad h = h_{\text{top}} \sqrt{1 + \varepsilon_0}.
\]

The constant \(\varepsilon_0\) will always be fixed and chosen small enough so that \(K_1 > 0\), which is possible as the sectional curvature is a continuous function of the metric in the \(C^2\)-topology. Our main result can be stated as follows:

**Theorem 1.** Let \(M\) be a closed Riemannian manifold equipped with a \(C^r\) metric \(g_0\) with negative curvature, \(r \geq 2\). Take \(k \in [2, r]\) and \(\varepsilon_0 > 0\) as above. For some arbitrary \(\varepsilon > 0\), define
\[
\nu_k = (k + 2)h + (k + 1)\kappa + \varepsilon.
\]

There is a metric \(g \in \mathcal{M}_k(\varepsilon_0)\) and \(C = C(g_0, \varepsilon_0, \varepsilon, k) > 0\) such that for any distinct \(\ell, \ell' \in \mathcal{L}_g\),
\[
|\ell - \ell'| \geq C e^{-\nu_k \max(\ell, \ell')}.\]

Let us point out that if \(k\) is fixed with \(k < r\), the metric \(g\) given by our proof of Theorem 1 is not \(C^{k'}\) for \(r \geq k' > k\). This is due to the fact that \(g\) is obtained as a limit of \(C^r\) metrics \((g_n)_{n \in \mathbb{N}}\) with diverging \(C^{k'}\) norm if \(k' > k\). Improving the regularity of \(g\) without increasing \(\nu_k\), if possible, would probably require a new approach than the one presented here.

Some fine control over the gaps in the length spectrum is in particular important for two well-known spectral problems: precise error terms for Weyl's laws on
a negatively curved manifold \cite{JP07, JPT07}, and lower bounds for the asymptotic distribution of resonances of the Laplacian on non-compact manifolds with hyperbolic trapped sets \cite{GZ99, Pet02}. In both cases, one has to study accurately the contributions of potentially many periodic orbits of the geodesic flow to a semiclassical trace formula, which involves the distributional trace we alluded to above. Periodic orbits are subsets of the unit tangent bundle $T^1M$, and it turns out that they can be exponentially isolated there: this is a consequence of the expansivity of the geodesic flow, see also Section 2 below. This allows to overcome a lack of control of the gaps in the length spectrum to establish a trace formula by microlocalizing near each closed geodesic in the unit cotangent bundle – see \cite{JPT07}. Such formulae involve sums of the form

\[
\sum_{\gamma} \sum_{k \in \mathbb{N}} A_{\gamma,k} \cos(\lambda k \ell(\gamma)) \mathbb{1}_{J(T)}(k \ell(\gamma)), \quad T \leq \mathcal{O}(\log \lambda), \; \lambda \to +\infty,
\]

where $A_{\gamma,k} > 0$ is related to the Poincaré map of $\gamma$, and here $J(T) \subset [0, T]$ is some interval. Obtaining lower bounds for the above sum is one of the main difficulties that stand in the way of improved Weyl laws for eigenvalues (or resonances). Such estimates requires in particular the control of the oscillating terms which are directly connected with the distribution of the gaps in the length spectrum.

2. Separation of orbits in phase space

Let us start this section by gathering some standard facts and notations about the geodesic flow $\Phi^t : T^1M \to T^1M$. In the following, closed geodesics will always be unoriented. The set of closed geodesics for $(M, g)$ with $g \in \mathcal{M}(\varepsilon_0)$ will be denoted by $\mathcal{C}_g$. If $I \subset \mathbb{R}_+$ is an interval, we will also make use of the sets

\[
\mathcal{C}_g(I) = \{ \gamma \in \mathcal{C}_g : \ell(\gamma) \in I \}, \quad \mathcal{C}_g(T) = \mathcal{C}_g([0, T]), \; T > 0.
\]

If $\rho \in T^1M$ and $t \in \mathbb{R}$, we write $\rho(t) \overset{\text{def}}{=} \Phi^t(\rho)$. By $\gamma$, we will denote one of the two possible lifts of $\gamma$ in $T^1M$, namely $(\gamma(t), \pm \dot{\gamma}(t))_{t \in \mathbb{R}}$ where $t \mapsto \gamma(t)$ is any parameterization of $\gamma$ by arc-length.

We equip $T^1M$ with the Sasaki metric and write $d_S$ for the induced distance. A fundamental estimate coming from the analysis of the Green subbundles \cite{Ebe73} together with Rauch’s comparison theorem and (1.3) yields to

\[
(2.1) \quad d_S(\Phi^t(\rho), \Phi^t(\rho')) \leq \kappa_0 e^{\kappa |t|} d_S(\rho, \rho'), \quad \kappa_0 \overset{\text{def}}{=} \sqrt{1 + \kappa},
\]

for all $\rho, \rho' \in T^1M$ and $t \in \mathbb{R}$.

For any $\gamma_0 \in \mathcal{C}_{g_0}$, the curve $\gamma_0 \subset M$ is still a closed path for $g$: since $g \in \mathcal{M}_k(\varepsilon_0)$ is negatively curved, there is a unique $g$-geodesic $\gamma$ in the same free homotopy class. Hence, there is a well defined, bijective map

\[
(2.2) \quad f_{\gamma_0 \to g} : \begin{cases} \mathcal{C}_{g_0} \to \mathcal{C}_g \\ \gamma_0 \to \gamma. \end{cases}
\]
Furthermore, the $g$-length of $\gamma = f_{g_0 \to g}(\gamma_0)$ satisfies
\begin{equation}
\frac{\ell_{g_0}(\gamma_0)}{\sqrt{1 + \varepsilon_0}} \leq \ell_g(\gamma) \leq \ell_{g_0}(\gamma_0)\sqrt{1 + \varepsilon_0}.
\end{equation}

Also, by a theorem of Klingenberg, $K(g) < 0$ implies that the injectivity radius $r_{\text{inj}}(g)$ is determined by the shortest closed geodesic,
\begin{equation}
r_{\text{inj}}(g) = \frac{1}{2} \min\{\ell(\gamma), \gamma \in \mathcal{C}_g\}.
\end{equation}

Hence (2.3) implies
\begin{equation}
\forall g \in \mathcal{M}_k(\varepsilon_0), \quad r_{\text{inj}}(g) \geq \frac{1}{\sqrt{1 + \varepsilon_0}} r_{\text{inj}}(g_0).
\end{equation}

In the following, we fix a number $r_m > 0$ such that $r_m < (1 + \varepsilon_0)^{-1/2} r_{\text{inj}}(g_0)$. Finally, if $\gamma \in \mathcal{C}_g$, we define the tubular open neighborhood of a lift $\gamma \in T^1 M$ by
\begin{equation}
\Theta^\varepsilon_{\gamma} \overset{\text{def}}{=} \{\rho \in T^1 M, \ d_S(\rho, \gamma) < \varepsilon\}.
\end{equation}

The purpose of this section is to prove the following exponential separation result in $T^1 M$:

**Proposition 2.** Let $g \in \mathcal{M}_k(\varepsilon_0)$. There is $\varepsilon_0 > 0$ depending only on $g_0, \varepsilon_0$ such that any distinct $\beta, \gamma \in \mathcal{C}_g(T)$ satisfy
\begin{equation}
\Theta^\varepsilon_{\beta} e^{-2\varepsilon T} \cap \Theta^\varepsilon_{\gamma} e^{-2\varepsilon T} = \emptyset,
\end{equation}
where $\beta, \gamma$ denote any lifts of $\beta, \gamma$ in $T^1 M$.

We begin with a lemma where the lengths of the geodesics are restricted to a fixed interval.

**Lemma 3.** Let $g \in \mathcal{M}_k(\varepsilon_0)$. For any distinct $\gamma, \gamma' \in \mathcal{C}_g(T)$ with $|\ell(\gamma) - \ell(\gamma')| < r_m/2$, we have
\begin{equation}
\Theta^\delta e^{-\kappa T} \cap \Theta^\delta e^{-\kappa T} = \emptyset,
\end{equation}
where $\gamma, \gamma'$ denote any lifts of $\gamma, \gamma'$ in $T^1 M$ and $\delta = \frac{r_m}{2\kappa \varepsilon_0}$.

**Proof.** We borrow the method from [JPT07] and argue by contradiction. Let $\gamma, \gamma' \in \mathcal{C}_g(T)$ be two distinct geodesics with $|\ell(\gamma) - \ell(\gamma')| < r_m/2$ and take two lifts
\begin{equation}
\gamma(t) = \Phi^t(z), \quad \gamma'(t) = \Phi^t(z'), \quad z, z' \in T^1 M, \ t \in \mathbb{R}.
\end{equation}

Suppose that $\Theta^\delta e^{-\kappa T} \cap \Theta^\delta e^{-\kappa T} \neq \emptyset$ where $\delta$ is as in the lemma. Without loss of generality, we can assume that $d_S(z, z') < \delta e^{-\kappa T}$. From (2.1), we have $d_S(z(t), z'(t)) < r_m/2$ for $|t| \leq T$, and this implies $d_g(\gamma(t), \gamma'(t)) < r_m/2$. If we reparameterize $\gamma$ by setting
\begin{equation}
\beta(s) \overset{\text{def}}{=} \gamma(ls/l'), \quad 0 \leq s \leq l',
\end{equation}
we have by the triangle inequality
\begin{equation}
\forall t \in [0, l'], \ d_g(\gamma'(t), \beta(t)) \leq r_m/2 + (t - t_{l'/l}) < r_m/2 + r_m/2.
\end{equation}
Since $r_m < r_{\text{inj}}(g)$, this means that there is an homotopy by geodesic segments between the closed curves $\gamma'$ and $\beta$. But $\beta$ being a reparameterization of $\gamma$, this implies that we would have two distinct closed geodesics in the same free homotopy class. \qed

**Proof of Proposition.** The preceding lemma can be generalized for $\beta, \gamma \in \mathcal{C}_g(T)$ as follows. Before, let us recall a classical fact for hyperbolic flows which is at the basis of the Anosov Closing and Shadowing lemmas. Denote the dynamical ball

$$B^l_S(\rho_0, r) \overset{\text{def}}{=} \{ \rho \in T^1M : \sup_{0 \leq s \leq t} d_S(\Phi^s(\rho), \Phi^s(\rho_0)) < r \}.$$ 

Suppose that $\rho_0 \in T^1M$ is a periodic point of period $\tau$ for the geodesic flow. There exist a constant $\sigma(g) > 0$, depending continuously on the metric in the $C^2$-topology such that if $\Sigma_{\sigma(g)}(\rho_0)$ is a transversal section of the flow at $\rho_0$ with radius $\sigma(g)$, then for all $\rho \in \Sigma_{\sigma(g)}(\rho_0) \cap B^l_S(\rho_0, \sigma(g))$ we have $\Phi^\tau(\rho) \neq \rho$. Namely, $\rho_0$ is the unique periodic point of period $\tau$ in $\Sigma_{\sigma(g)}(\rho_0)$ sufficiently close to $\rho_0$ for the dynamical distance. This follows from a standard fixed point argument using hyperbolicity and a sequence of quasi-linear $C^1$ approximations of the flow near the orbit $\rho_0(t)$ for $0 \leq t \leq \tau$, see [KH95] Chapters 6 and 18.

Since we consider $g \in \mathcal{M}_k(\varepsilon_0)$ with $k \geq 2$, we can define

$$\sigma \overset{\text{def}}{=} \inf_{g \in \mathcal{M}_k(\varepsilon_0)} \sigma(g) > 0$$

to get a constant uniform with respect to $g \in \mathcal{M}_k(\varepsilon_0)$.

Consider now distinct $\beta, \gamma \in \mathcal{C}_g(T)$ and assume for instance that

$$T \geq l \overset{\text{def}}{=} \ell(\gamma) \geq \ell(\beta) \overset{\text{def}}{=} l'.$$

Fix some $\varepsilon_0 > 0$ to be chosen sufficiently small later (in function of $g_0, \varepsilon_0$), and take two lifts $\beta, \gamma \subset T^1M$ of these closed geodesics. Suppose that there are $z \in \gamma$, $z' \in \beta$ such that $d_S(z, z') < \varepsilon_0 e^{-2\kappa T}$. From (2.1), we have

$$d_S(\Phi^t(z), z') = d_S(\Phi^t(\gamma), \Phi^t(z')) \leq \varepsilon_0 \kappa_0 e^{-\kappa T}.$$ 

Hence the point $\Phi^t(z)$ stays exponentially close to $\Phi^t(z')$ for $t \in [0, l']$. Moreover, (2.5)

$$d_S(\Phi^t(z), z) \leq d_S(\Phi^t(z), z') + d_S(z', z) \leq 2\varepsilon_0 \kappa_0 e^{-\kappa T}.$$

Let $\Sigma_\sigma(z)$ be a Poincaré section of the flow of size $\sigma$ centered at $z$. Without loss of generality, we can assume that $z' \in \Sigma_\sigma(z)$. Equation (2.5) implies that for $\varepsilon_0$ small enough depending on $g_0, \varepsilon_0$ via $\kappa, \sigma$, the point $\Phi^t(z)$ belongs to a flow box with section $\Sigma_\sigma(z)$: there is a unique $s \in \mathbb{R}$ such that

$$\Phi^{t+s}(z) = \zeta \in \Sigma_\sigma(z)$$

where $d_S(\zeta, z) \leq C \varepsilon_0 e^{-\kappa T}$, $|s| \leq C \varepsilon_0 e^{-\kappa T}$. Here $C = C(g_0, \varepsilon_0) \geq 1$ can be chosen uniformly for $g \in \mathcal{M}_k(\varepsilon_0)$ since the distance $d_S$ varies continuously with $g$.

But now, $\zeta$ and $z$ are on the same periodic orbit, so they are periodic with period $l$ and both belong to $\Sigma_\sigma(z)$. Also, using (2.1) one more time, we have

$$d_S(\zeta, z) \leq C \varepsilon_0 e^{-\kappa T} \Rightarrow \sup_{[0,l]} d_S(\Phi^t(\zeta), \Phi^t(z)) \leq C \varepsilon_0 \kappa_0.$$
Therefore, if \( \epsilon_0 \) is small enough, the right hand side of the previous equation is \( < \sigma \) and we have \( \zeta \in B^g_S(z, \sigma) \cap \Sigma_g(z) \). From the remark above, this forces \( \zeta = z \). Up to shrink \( \epsilon_0 \) further, this finally implies
\[
\varepsilon = l - l' \leq C \epsilon_0 e^{-\kappa T} < r_m/2.
\]
Hence we can apply Lemma 3, in contradiction with \( d_S(z, z') < \epsilon_0 e^{-2\kappa T} < \delta e^{-\kappa T} \).

3. Almost intersections of closed geodesics

In this section we prove the key tool needed in the proof of Theorem 1. If \( x \in M \) and \( r > 0 \), we denote by \( B_g(x, r) \subset M \) an open ball of radius \( r \) centered at \( x \) for the metric \( g \).

**Proposition 4.** Let \( g \in \mathcal{M}_k(\varepsilon_0) \). Fix \( \varepsilon > 0 \) and take \( \alpha = 2\kappa + h + \varepsilon \). There is \( T_0 > 0 \) depending only on \( g_0, \varepsilon_0, \varepsilon \) such that for \( T \geq T_0 \) and any \( \beta \in \mathcal{C}_g(T) \), there exists \( z \in \beta \) with
\[
\forall \gamma \in \mathcal{C}_g(T) \setminus \beta, \quad B_g(z, e^{-\alpha T}) \cap \gamma = \emptyset,
\]
and moreover, \( B_g(z, e^{-\alpha T}) \cap \beta \) consists in a unique geodesic segment of \( \beta \) centered at \( z \).

The main idea is as follow: since two closed geodesics \( \beta, \gamma \in \mathcal{C}_g(T) \) are separated by \( \epsilon_0 e^{-2\kappa T} \) when lifted in \( T^1M \), it means that if \( \beta \) and \( \gamma \) are very close somewhere in \( M \), the angle “between” their tangent vectors must be bounded below. It follows that the two geodesics cannot stay close to each other in \( M \) for a long time. Proposition 4 below is a quantitative version of this observation.

3.1. Local divergence of orbits separated in phase space.** The next proposition is a simple application of the Toponogov comparison theorem [Kar89] to study the local divergence of two geodesics close from each other in \( M \), but separated when lifted in \( T^1M \).

**Lemma 5.** Assume that \( g \in \mathcal{M}_k(\varepsilon_0) \). Let \( \alpha \geq 2\kappa + h \) and \( (x, \xi), (x', \xi') \in T^1M \). Suppose that
\[
d_g(x, x') < e^{-\alpha T}, \quad d_S((x, \pm \xi), (x', \xi')) \geq \epsilon_0 e^{-2\kappa T},
\]
and denote by \( I_x \) the following geodesic segment:
\[
I_x = \{ \pi \circ \Phi^t(x, \xi), |t| \leq r_m/2 \} \subset M, \quad \pi : T^1M \to M.
\]
There are \( T_0, C_1, C_2 > 0 \) depending only on \( g_0, \varepsilon_0, \varepsilon \) such that if \( T \geq T_0 \) and \( |t'| \leq r_m/2 \),
\[
d_g(\pi \circ \Phi^t(x', \xi'), I_x, r_m/2) \geq \max\{C_1 |t'| e^{-2\kappa T} - C_2 e^{-\alpha T}, 0\}.
\]

**Proof.** Write \( \rho = (x, \xi), \rho' = (x', \xi') \). We consider first the case where \( x = x' \). Since \( g \) has negative curvature, the ball \( B_g(x, r_m/2) \) is convex and the geodesic triangle defined by the points
\[
\{x, \pi(\rho(t)), \pi(\rho'(t'))\}
\]
is entirely contained in this ball for \( |t|, |t'| \leq r_m/2 \). Write \( \angle(\xi, \xi') \in [0, \pi] \) for the angle between \( \xi \) and \( \xi' \) measured with the metric \( g \). From (3.2), we can assume
without loss of generality (up to change $\xi \to -\xi$) that $\angle(\xi, \xi') \in [0, \pi/2]$. From the property of the Sasaki metric, we have precisely $d_S(\rho, \rho') = |\angle(\xi, \xi')|$ since $x = x'$. Hence (3.2) yields to

$$d(3.4)$$

$$\epsilon_0 e^{-2\kappa T} \leq d_S(\rho, \rho') = |\angle(\xi, \xi')|.$$ 

Let now $\{A, B, C\}$ be an Euclidian triangle with $AB = |t|$, $AC = |t'|$ and

$$\angle(AB, AC) = \angle(\xi, \xi').$$

Since $|t|, |t'| \leq r_m/2$, the Toponogov comparison theorem for negative curvature together with (3.4) implies that

$$\epsilon_0 \frac{1}{2} |t'| \epsilon_0 e^{-2\kappa T} \leq |t'| \sin \angle(\xi, \xi') | \leq |BC| \leq d_g(\pi(\rho(t)), \pi(\rho'(t'))) \sin \angle(\xi, \xi').$$

Note that minimizing on $t \in [-r_m/2, r_m/2]$ gives (3.3) with $C_2 = 0$.

We move on now to the case where $x \neq x'$ with $d_g(x, x') < e^{-\alpha T}$. Consider in $T^1M$ the curve $(c(t), v(t))_{0 \leq t \leq d_g(x, x')}$ where $c(t)$ is a minimizing geodesic connecting $x'$ to $x$ and $v(t)$ is the parallel transport of $\xi'$ to $x$ by the definition of the Sasaki metric, this gives

$$d_S(\rho', \tilde{\rho}) = d_g(x, x'), \quad d_S(\rho, \tilde{\rho}) = |\angle(\xi, \xi')|.$$ From (3.2), the triangle inequality and the fact that $\alpha \geq 2\kappa + h$, we get

$$e^{-2\kappa T} (\epsilon_0 - e^{-\alpha T}) \leq |\angle(\xi, \xi')|.$$ Hence for $T$ large enough depending only on $g_0, \epsilon_0$ via $\epsilon_0, h$, we have $|\angle(\xi, \xi')| \geq \epsilon_0 e^{-2\kappa T}/2$. From this observation, we proceed as for (3.5) : assuming without loss of generality that $\angle(\xi, \xi) \in [0, \pi/2]$ we have

$$\frac{1}{4} |t'| \epsilon_0 e^{-2\kappa T} \leq d_g(\pi(\rho(t)), \pi(\tilde{\rho}(t')))) \leq d_g(\pi(\rho(t)), \pi(\rho'(t'))) + d_g(\pi(\rho'(t'))), \pi(\tilde{\rho}(t')))) \leq d_g(\pi(\rho(t)), \pi(\rho'(t'))) + \epsilon_0 e^{\kappa |t'|} e^{-\alpha T}.$$ 

But $|t'| \leq r_m/2$, so

$$\frac{\epsilon_0}{4} |t'| e^{-2\kappa T} - \epsilon_0 e^{\kappa r_m/2} e^{-\alpha T} \leq d(\pi(\rho(t)), \pi(\rho'(t'))),$$

and we conclude the proof by minimizing on $t \in [-r_m/2, r_m/2]$ and setting $C_1 = \epsilon_0/4$, $C_2 = \epsilon_0 e^{\kappa r_m/2}$.

Given two geodesics $\beta, \gamma \in \mathcal{C}_g(T)$, the preceding proposition will enable us to estimate the total length of pieces of these geodesics which are close to each other.
3.2. Intersections and almost-intersections. Roughly speaking, finding $z$ in Proposition 4 can be done if we are able to find a sufficiently large piece of $\beta \in \mathcal{C}_g(T)$ that “avoids” both all other geodesics in $\mathcal{C}_g(T)$, and all the rest of $\beta$. This suggests to study the situations where two given closed geodesics intersect, or more generally come close to each other.

Let $\beta, \gamma \in \mathcal{C}_g(T)$ be two closed geodesics. It is essentially well known that their intersection number $i(\beta, \gamma) \in \mathbb{N}$ grows quadratically with $T$:

$$i(\beta, \gamma) = O_{r_{\text{nj}}(g)}(T)^2.$$ (3.6)

This is a consequence of the fact that if $\beta, \gamma$ are closed geodesics intersect, and the above remark shows that the topology of the manifold excludes that too many intersections points can be themselves close to each other.

In dimension 3 or more, intersections and self-intersections are marginal, so we would like to generalize (3.6) to “almost-intersections”, namely regions in $M$ where two geodesics are close to each other but without necessarily intersecting. Of course points of almost-intersection will not be countable, so we will have to consider small segments.

Before continuing further, for $\epsilon > 0$ let us denote by

$$\theta_\gamma^\epsilon \overset{\text{def}}{=} \{ x \in M \mid d_g(x, \gamma) < \epsilon \}$$

an open tubular neighborhood of $\gamma \in \mathcal{C}_g$ of size $\epsilon$. The next proposition allows to control the size of $\beta \cap \theta_\gamma^\epsilon$ when $\beta, \gamma \in \mathcal{C}_g(T)$:

**Proposition 6.** Let $\beta, \gamma \in \mathcal{C}_g(T)$. Fix $\alpha \geq 2\kappa + h$ and set

$$\epsilon \overset{\text{def}}{=} e^{-\alpha T}.$$  

There are $T_0, C_3 > 0$ depending only on $g_0, \varepsilon_0$ such that for $T > T_0$, if the set $\beta \cap \theta_\gamma^\epsilon$ is not empty, it can be covered by a finite number of geodesic segments $\{J_1, \ldots, J_n(\beta, \gamma)\} \subset \beta$ with the following properties:

- $0 \leq n(\beta, \gamma) \leq 4 \left(\frac{1}{r_m}\right)^2$,
- $|J_i| \leq C_3 e^{-(\alpha-2\kappa)T}$ for all $1 \leq i \leq n(\beta, \gamma)$.

Before giving the proof of this proposition, we need some preliminary results. We fix two parameterizations of $\beta, \gamma$ by arc-length and define the continuous maps

$$D_{\beta, \gamma} : \begin{cases} [0, \ell(\beta)] \times [0, \ell(\gamma)] \to \mathbb{R}^+ \\ (s, t) \mapsto d_g(\beta(s), \gamma(t)) \end{cases} \quad G_{\beta, \gamma} : \begin{cases} [0, \ell(\beta)] \times [0, \ell(\gamma)] \to M \times M \\ (s, t) \mapsto (\beta(s), \gamma(t)) \end{cases}.$$  

Consider an open connected component $U \subset \mathbb{R}^2$ of $D_{\beta, \gamma}^{-1}([0, \epsilon])$. Such a set $U$ exists since we assumed $\beta \cap \theta_\gamma^\epsilon \neq \emptyset$, furthermore, there is $(s_u, t_u) \in U$ which is a local
minimum of $D_{\beta,\gamma} |_{U}$. Assume that $T$ is large enough so that $e^{-\alpha T} \leq r_{m}/2$. By the convexity of the distance function in negative curvature, for $s, t$ such that $0 < |t| \leq r_{m}/2$ and $0 < |s| \leq r_{m}/2$, we have

$$d_{g}(\beta(s_{u} + s), \gamma(t_{u} + t)) > d_{g}(\beta(s_{u}), \gamma(t_{u})).$$

It follows that the local minima of $D_{\beta,\gamma}$ are isolated, and the total number of local minima of this map is at most

$$\frac{\ell(\beta) \times \ell(\gamma)}{(r_{m}/2)^{2}} \leq 4 \left( \frac{T}{r_{m}} \right)^{2}.$$ 

If $(s, t) \in [0, \ell(\beta)] \times [0, \ell(\gamma)]$ is a local minimum of $D_{\beta,\gamma}$ and $(x, y) = G_{\beta,\gamma}(s, t)$, we say that $(x, y)$ is an almost-intersection of $\beta$ and $\gamma$.

Consider now an almost intersection $(x, y) \in \beta \times \gamma$, and let us shift the origin of the parameterizations of $\beta$ and $\gamma$ so that we can write

$$(x(0), \dot{x}(0)) = (x, \xi), \quad (y(0), \dot{y}(0)) = (y, \eta).$$

Define the open segment $I_{x} = \{ \pi \circ \Phi^{t}(x, \xi) : |t| \leq r_{m}/2 \} \subset \beta$ and its open $\epsilon$-neighborhood in $M$ by

$$\theta_{\epsilon}^{x}(I_{x}) = \{ z \in M : d_{g}(z, I_{x}) < \epsilon \}.$$ 

Define similarly $I_{y}$ and $\theta_{\epsilon}^{y}(I_{y})$.

**Lemma 7.** Let $(x, y)$ be an almost intersection as above, and fix $\alpha \geq 2\kappa + h$. For $T > 0$ sufficiently large depending only on $g_{0}, \varepsilon_{0}$, there are open segments $J(x, y) \subset \beta$ containing $x$ and $J(y, x) \subset \gamma$ containing $y$ such that:

(i) $I_{x} \cap \theta_{\epsilon}^{y}(I_{y}) \subset J(x, y)$ and $I_{y} \cap \theta_{\epsilon}^{x}(I_{x}) \subset J(y, x)$,

(ii) $\max\{|J(x, y)|, |J(y, x)|\} \leq C_{3} e^{-(\alpha - 2\kappa)T}$, where $C_{3}$ depends only on $g_{0}, \varepsilon_{0}$.

In other words,

$$\forall z \in I_{x} \setminus J(x, y), \quad d_{g}(z, I_{y}) \geq \epsilon,$$

and the symmetric property is true by exchanging the roles of $x$ and $y$.

**Proof.** Since Proposition 2 ensures that $\Theta_{\gamma} e^{-2\alpha T} \cap \Theta_{\beta} e^{-2\alpha T} = \emptyset$ for any lifts $\gamma, \beta$ in $T^{1}M$, this implies

$$d_{g} \left( (x, \pm \xi), (y, \eta) \right) \geq \varepsilon_{0} e^{-2\alpha T}.$$ 

On the other hand, $d_{g}(x, y) < \epsilon$, so we are in position to apply Proposition 5, which gives readily

$$d_{g}(x(t), I_{y}) \geq C_{1}|t| e^{-2\alpha T} - C_{2} e^{-\alpha T}, \quad t \in [-r_{m}/2, r_{m}/2],$$

and the same equation holds true by exchanging the roles of $x$ and $y$. Define $C_{3} = 2C_{1}^{-1}(1 + C_{2})$ and take $T \geq T_{0}$ where $C_{3} e^{-hT_{0}} < r_{m}$. In this case, we have

$$r_{m}/2 \geq |t| \geq C_{1}^{-1}(1 + C_{2}) e^{(2\kappa - \alpha)T} \Rightarrow d_{g}(x(t), I_{y}) \geq \epsilon = e^{-\alpha T},$$

Therefore, there is a (maximal) non-empty open interval $[t_{x}^{+}, t_{x}^{-}] \subset [-r_{m}/2, r_{m}/2]$ with $|t_{x}^{+} - t_{x}^{-}| \leq C_{2} e^{-(\alpha - 2\kappa)T}$ such that

$$x(t) \in \theta_{\epsilon}^{x}(I_{y}) \Rightarrow t \in [t_{x}^{-}, t_{x}^{+}].$$
We can define an interval $[t_y^-, t_y^+]$ in the same way by permuting the roles of $x$ and $y$. The open segments with the desired properties are precisely
\[ J(x, y) \overset{\text{def}}{=} \{ x(t), \ t_x^- < t < t_x^+ \}, \quad J(y, x) \overset{\text{def}}{=} \{ y(t), \ t_y^- < t < t_y^+ \}. \]

**Proof of Proposition 6.** Call
\[ (s_1, t_1), \ldots, (s_n(\beta, \gamma), t_n(\beta, \gamma)), \quad n(\beta, \gamma) \leq 4 \left( \frac{T}{r_m} \right)^2 \]
the local minima of the function $D_{\beta, \gamma}$, and write $(x_i, y_i) = G_{\beta, \gamma}(s_i, t_i)$ the almost-intersections identified with these local minima via the parameterizations of the closed geodesics. We just need to check that if $(x, y) \in \beta \times \gamma$ is such that
\[ d_g(x, y) < \epsilon, \]
then there is $i \in [1, n(\beta, \gamma)]$ such that $x \in J(x_i, y_i)$ and $y \in J(y_i, x_i)$ where the intervals are given by the preceding lemma. This is clear if $(x, y)$ is an almost-intersection. Otherwise let $(s_x, t_y)$ be such that $G_{\beta, \gamma}(s_x, t_x) = (x, y)$, and $U \subset \left[ 0, \ell(\beta) \right] \times \left[ 0, \ell(\gamma) \right]$ be the connected component of $D_{\beta, \gamma}^{-1}(0, \epsilon)$ containing $(s_x, t_y)$. $U$ contains a local minimum $(\tilde{s}, \tilde{t})$ of $D_{\beta, \gamma}$, and since it is arc-connected (as it is locally), there is a continuous path
\[ f : [0, 1] \to \left[ 0, \ell(\beta) \right] \times \left[ 0, \ell(\gamma) \right] \]
joining $(\tilde{s}, \tilde{t})$ to $(s_x, t_y)$ which is fully contained in $U$, namely:
\[ G_{\beta, \gamma} \circ f(0) = (\tilde{x}, \tilde{y}), \quad G_{\beta, \gamma} \circ f(1) = (x, y), \quad (\tilde{x}, \tilde{y}) = G_{\beta, \gamma}(\tilde{s}, \tilde{t}), \]
and
\[ \forall t \in [0, 1], \quad d_g(G_{\beta, \gamma} \circ f(t)) < \epsilon. \]
Let us define
\[ \tilde{T} \overset{\text{def}}{=} [\tilde{s} - \frac{r_m}{2}, \tilde{s} + \frac{r_m}{2}] \times [\tilde{t} - \frac{r_m}{2}, \tilde{t} + \frac{r_m}{2}] \subset \left[ 0, \ell(\beta) \right] \times \left[ 0, \ell(\gamma) \right]. \]
Lemma 7 shows precisely that
\[ (s, t) \in \tilde{T} \setminus G_{\beta, \gamma}^{-1}(J(\tilde{x}, \tilde{y}) \times J(\tilde{y}, \tilde{x})) \Rightarrow D_{\beta, \gamma}(s, t) \geq \epsilon. \]
In view of (3.8), the continuity of $f$ and $G_{\beta, \gamma} \circ f(0) = (\tilde{x}, \tilde{y})$, this implies that
\[ G_{\beta, \gamma} \circ f([0, 1]) \subset J(\tilde{x}, \tilde{y}) \times J(\tilde{y}, \tilde{x}), \]
and this shows that $x \in J(\tilde{x}, \tilde{y})$ and $y \in J(\tilde{y}, \tilde{x})$. In particular, in each connected component of $D_{\beta, \gamma}^{-1}(0, \epsilon)$ there is a unique local minimum of $D_{\beta, \gamma}$.

We have just shown that if $x \in \beta$ is such that $d_g(x, \gamma) < \epsilon$, there is $i \in [1, n(\beta, \gamma)]$ such that $x \in J(x_i, y_i)$. Therefore, $\beta \cap \delta_{\gamma}$ is covered by
\[ U_{\beta}(\gamma) \overset{\text{def}}{=} \bigcup_{i=1}^{n(\beta, \gamma)} J(x_i, y_i) \subset \beta. \]
The proof of Proposition 6 is completed since there are at most $4 \left( T/r_m \right)^2$ terms in the above equation, and for all $i$, $|J(x_i, y_i)| \leq C_2 e^{-2\kappa}T$ from Lemma 7.
3.3. Proof of Proposition 4: case of distinct geodesics. In this section, we establish Equation (3.1). Let $\beta \in \mathcal{C}_g(T)$ where $g \in \mathcal{M}_{k}(\varepsilon_0)$. For $\varepsilon > 0$, let us choose $\alpha > 0$ such that

$$\alpha \geq 2\kappa + h + \varepsilon.$$ 

We then take $T_0 > 0$ large enough so that Propositions 2 and 6 hold true for $T \geq T_0$. From Proposition 6, we have

$$\ell(U_\beta(\gamma)) \leq 4 \left( \frac{T}{r_m} \right)^{2} \times C_3 e^{-(\alpha - 2\kappa)T}. \tag{3.9}$$

We now consider all closed geodesics $\gamma \neq \beta$ with $\beta \in \mathcal{C}_g(T)$ fixed and $\gamma \in \mathcal{C}_g(T)$. To get a uniform bound on the counting function for closed geodesics for the metric $g$, note first that (1.1) implies that there is $C_0 > 0$ depending only on $g_0$ such that

$$\# \mathcal{C}_g(T) \leq C_0 T^{-1} e^{h_{top}T} \text{ for, say, } T > r_m. \tag{2.3}$$

For $\gamma_0 \in \mathcal{C}_g_0$, let $\gamma = f_{g_0\rightarrow g}(\gamma_0)$. In view of (2.3),

$$\ell_{g_0}(\gamma_0) > T \sqrt{1 + \varepsilon_0} \Rightarrow \ell_g(\gamma) > T,$$

so $f_{g_0\rightarrow g}(\mathcal{C}_g(T)) \subset \mathcal{C}_{g_0}(T \sqrt{1 + \varepsilon_0})$ and therefore,

$$\# \mathcal{C}_g(T) \leq \# \mathcal{C}_{g_0}(T \sqrt{1 + \varepsilon_0}) \leq C_0 e^{hT} \frac{T}{T}. \tag{3.10}$$

Setting

$$U_\beta \overset{\text{def}}{=} \bigcup_{\gamma \in \mathcal{C}_g(T) \setminus \beta} U_\beta(\gamma),$$

we obtain using (3.9) that

$$\ell(U_\beta) \leq \frac{4C_3C_0T}{r_m^2} e^{-(\alpha - 2\kappa - h)T} = \mathcal{O}_{g_0,\varepsilon_0}(T e^{-\varepsilon T}) \xrightarrow{T \to \infty} 0. \tag{3.11}$$

Define

$$V_\beta = \{ x \in \beta : \forall \gamma \in \mathcal{C}_g(T) \setminus \beta, \; d_g(x, \gamma) \geq \varepsilon \}.$$ 

By construction, $\beta \setminus U_\beta \subset V_\beta$ and $U_\beta$ is a finite union of $\mathcal{O}(T e^{hT})$ open segments. Therefore, we see by a box principle along $\beta$ using (3.11) that for $T$ large enough depending only on $g_0, \varepsilon_0$ and $\varepsilon$, the set $V_\beta$ contains at least one segment $I$ of size

$$|I| \geq T^{-2} e^{-hT} \geq 2 e^{-\alpha T}. \tag{3.12}$$

We assumed $2 e^{-\alpha T} \leq r_m$, so if we choose $z \in \beta$ to be the middle of $I$, then

$$\forall \gamma \in \mathcal{C}_g(T) \setminus \beta, \quad B_g(z, \varepsilon) \cap \gamma = \emptyset,$$

and this shows (3.1). It remains to establish that $z$ can be chosen such that the ball $B_g(z, \varepsilon)$ also avoids $\beta$ except on a single geodesic segment of $\beta$ containing $z$. 

3.4. Almost-intersections of a single closed geodesic. To conclude the proof of Proposition \[4\] we now indicate how the results of the preceding sections allow us to control almost-intersections of a closed geodesic with itself. 

As above, we take \( \alpha \geq h + 2\kappa + \varepsilon \) for some \( \varepsilon > 0 \). Fix some parameterization of \( \beta \in C_g(T) \) by arc-length. Let \((x,y) \in \beta \times \beta\), and call \( t_x, t_y \) times such that \( x = \beta(t_x), y = \beta(t_y) \). We will say that the couple \((x,y)\) is an almost-intersection of \( \beta \) with itself if \( d_g(x,y) < \epsilon, t_y \neq t_x \) and \((t_x, t_y)\) is a local minimum of \( D_{\beta,\beta} : (t, t') \mapsto d_g(\beta(t), \beta(t')) \). In particular, either \((x,y)\) is a self-intersection of \( \beta \), or the segment joining \( x \) to \( y \) is not included in \( \beta \). Using as before convexity of the distance function in negative curvature, we get that there are at most \( O_r(T^2) \) such couples of almost-intersections, and arguments identical to those developed in the proof of Proposition \[2\] show that if \((x,y)\) is an almost-intersection, then 

\[
d_S((\beta(t_x), \beta(t_y)), (\beta(t_x), \beta(t_y))) \geq \epsilon_0 e^{-2\kappa T}.
\]

For \( z = \beta(t_z) \), define as before 

\[
I_z = \{ \Phi'(\beta(t_z), \beta(t_z)), |t| \leq r_m/2 \}.
\]

In particular, as in Lemma \[7\] to \((x,y)\) we can associate segments \( J(x,y) \subset I_x \) and \( J(y,x) \subset I_y \) of \( \beta \) centered at \( x \) and \( y \) respectively, of size \( O_g(z_0) (e^{-2\kappa T}) \) such that \( \forall z \in I_x \setminus J(x,y), d_g(z, I_y) \geq \epsilon \) and symmetrically when interchanging the roles of \( y \) and \( x \). Exactly as in Proposition \[6\] we can then show that the set 

\[
I_\beta = \{ z \in \beta : (B_g(z, \epsilon) \setminus I_z) \cap \beta \neq \emptyset \}
\]

can be covered by \( O_0(z_0) (T^2) \) segments of size \( O_0(z_0) (e^{-2\kappa T}) \), for \( T \geq T_0(z_0, \varepsilon_0) \). These segments can be added up to \( U_\beta \) : exactly as previously, we end up by a box principle with the fact that for \( T \) large enough depending now on \( g_0, \varepsilon_0 \) and \( \varepsilon \), there is a segment \( I \subset \beta \) such that \(|I| \geq 2 e^{-\alpha T} \) and if \( z \) denotes the middle of \( I \), we have 

\[
\forall \gamma \in C_g(T) \setminus \beta, \quad B_g(z, \epsilon) \cap \gamma = \emptyset \quad \text{and} \quad B_g(z, \epsilon) \cap \beta = I_z \cap B_g(z, \epsilon).
\]

This concludes the proof of Proposition \[4\]. We end this section by a straightforward corollary:

**Corollary 8.** Let \( g \in M(\varepsilon_0), \varepsilon > 0 \) and \( \alpha \geq 2\kappa + h + \varepsilon \) be as above. For \( T \geq T_0(g_0, \varepsilon_0, \varepsilon) \) and each \( \gamma \in C_g(T) \), there is \( z_\gamma \in \gamma \) such that 

\[
B_g(z_\gamma, \epsilon/2) \cap \left( \bigcup_{\gamma' \in C_g(T) \setminus \gamma} \theta_{\gamma'}/2 \right) = \emptyset,
\]

and \( B_g(z_\gamma, \epsilon/2) \cap \gamma \) consists in a unique geodesic segment centered at \( z_\gamma \). In particular, if \( \gamma, \gamma' \in C_g(T) \) are distinct, then \( B_g(z_\gamma, \epsilon/2) \cap B_g(z_{\gamma'}, \epsilon/2) = \emptyset \).

4. Proof of Theorem \[1\]

For \( T > 0 \) large enough, Corollary \[8\] of Proposition \[4\] allow to perturb the metric near a point of a closed geodesic in \( C_g(T) \) without changing the length of all the others in \( C_g(T) \). Before exploiting further this property to separate the length spectrum, we recall without proof a standard fact for conformal perturbations of a given metric \( g \) near a closed geodesic.
Lemma 9. Let $\gamma \in \mathcal{C}_g$ and $x \in \gamma$. Let also $B_g(x, r_0)$ be an open ball of radius $r_0 < r_{\min}(g)/2$ centered at $x$. Assume that $\gamma \in \mathcal{C}_g$ is parametrized such that $\gamma(0) = x$ and
\[
\gamma \cap B_g(x, r_0) = \{ \pi \circ \Phi^t(\gamma(0), \dot{\gamma}(0)), |t| < r_0 \}.
\]
Fix some arbitrary $\ell$. Still, let $\tilde{\ell}$ be an arbitrary length in $[\ell, \ell']$ such that $\tilde{\ell}$ is still $\|\gamma\|$-separated, we consider the next interval $[\ell', \ell'']$ and $[\ell'', \ell'''$, etc. By a box principle, in view of (3.10) there is at least one couple of distinct $g, T$ such that $g$ is parametrized such that $g(0) = \gamma(0)$, $\gamma(0)$ is still $\tilde{g}$-closed geodesic and
\[
\ell_{\tilde{g}}(\gamma) = \ell_g(\gamma) + \delta.
\]
Moreover, for any $k \geq 0$ we have
\[
\|g - \tilde{g}\|_C^k \leq C_k \delta r_0^{-(k+1)} \|g\|_C^k
\]
where $C_k$ is independent of $r_0$.

We now pass to the proof of Theorem 1. To begin with, let us fix some $\varepsilon > 0$ and choose
\[
\alpha = 2\kappa + h + \frac{\varepsilon}{2(k+1)}, \quad k \geq 2.
\]
Let $T_0(g_0, \varepsilon_0, \varepsilon, k) > 0$ be a positive number such that Proposition 4 with the above value of $\alpha$ is satisfied for $T \geq T_0$, and set $T_n = T_0 + n$, $n \in \mathbb{N}$.

For $\nu > 0$ to be defined soon below and $n \geq 1$, we will say that $\mathcal{C}_g([T_0, T_0])$ is $\nu$-separated if for all distinct $\ell, \ell' \in \mathcal{L}_g([T_0, T_0])$, we have
\[
|\ell - \ell'| \geq e^{-\nu T_0}.
\]
We proceed iteratively: once a metric $g_{n-1} \in \mathcal{M}_k(\varepsilon_0)$ ($n > 1$) is constructed such that $\mathcal{C}_g([T_0, T_{n-1}])$ is $\nu$-separated, we consider the next interval $[T_{n-1}, T_n]$ and build a metric $g_n$ from $g_{n-1}$ such that $\mathcal{C}_g([T_0, T_n])$ is $\nu$-separated.

To do so, consider $[T_{n-1}, T_n]$ and $n \geq 1$. Denote the set of closed geodesics with length in $[T_{n-1}, T_n]$ by
\[
\mathcal{C}_g([T_{n-1}, T_n]) = \{ \gamma_1, \ldots, \gamma_{\mu_n} \}.
\]
By a box principle, in view of (3.10) there is at least one couple of distinct $\ell^-, \ell^+ \in \mathcal{L}_g([T_{n-1}, T_n])$ such that $\ell^+ - \ell^- \geq C_0^{-1} e^{-\kappa_{T_n}}$. Setting $\ell_i = \ell(\gamma_i)$, we order the points of $\mathcal{L}_g([T_{n-1}, T_n])$ so that
\[
T_{n-1} < \ell_1 < \cdots < \ell_m = \ell^- < \ell^+ = \ell_{m+1} < \cdots < \ell_{\mu_n} < T_n.
\]
Let us define $\varepsilon_n = e^{-\alpha T_{n+1}}/2$ and choose $z_i \in \gamma_i$ according to Corollary 8. In particular, $B_{g_{n-1}}(z_i, \varepsilon_n)$ does not intersect the open $\epsilon_n$-neighborhood of any $\gamma \in \mathcal{C}_g([T_0, T_{n+1}]) \setminus \gamma_i$, and $B_{g_{n-1}}(z_i, \varepsilon_n) \cap \gamma_i$ consists in a single geodesic segment centered at $z_i$. We are exactly in the settings of Lemma 9: in the ball $B_{g_{n-1}}(z_i, \varepsilon_n)$, we dilate the metric by a factor $(1 + \varepsilon_n^{-1} \delta_{n,i} \chi_{n,i})^2$, where $\chi_{n,i}$ plays the role of $\chi_0$ in this lemma. The constants $\delta_{n,i}$ are taken such that
\[
\delta_{n,i} \eqdef \begin{cases} i e^{-\nu T_n} & \text{if } 1 \leq i \leq m, \\ -(\mu_n - i + 1) e^{-\nu T_n} & \text{if } m+1 \leq i \leq \mu_n. \end{cases}
\]
In this way, the geodesics \((\gamma_i)_{1 \leq i \leq \mu_n}\) have their lengths dilated (for \(1 \leq i \leq m\)) or contracted (for \(m + 1 \leq i \leq \mu_n\)) to \(\tilde{\ell}_1, \ldots, \tilde{\ell}_{\mu_n}\) with
\[
\tilde{\ell}_i = \ell_i + i e^{-\nu T_n} \quad 1 \leq i \leq m,
\]
\[
\tilde{\ell}_j = \ell_j - (\mu_n - j + 1) e^{-\nu T_n}, \quad m + 1 \leq j \leq \mu_n.
\]
Define now
\[
g_n = \prod_{i=1}^{\mu_n} (1 + \frac{\delta_{n,i}}{\epsilon_n} \chi_{n,i})^2 g_{n-1} = \left( \prod_{p=1}^{n} \prod_{i=1}^{p} e^{2 \log(1 + \frac{\delta_{n,i}}{\epsilon_n} \chi_{n,i})} \right) g_0 \overset{\text{def}}{=} e^{2 F_n} g_0.
\]
To simplify the notations below, we write \(f_{n,i} \overset{\text{def}}{=} \delta_{n,i} \epsilon_n^{-1} \chi_{n,i}\). By \(C_k\), we will denote a positive constant depending only on \(g_0, \epsilon_0, \epsilon, k\) whose value may change from line to line. Equation \((3.10)\) gives
\[
\mu_n \leq C_0 e^{k T_n}/T_n,
\]
so we have
\[
\left| \frac{\delta_{n,i}}{\epsilon_n} \right| = \frac{2 C_0 e^{\alpha}}{T_n} e^{(h + \alpha - \nu) T_n}.
\]
We deduce from this equation and Lemma 9 that \(f_{n,i}\) satisfies
\[
\|f_{n,i}\|_{C^k} \leq C_k \frac{e^{-\nu h - (k+1) \alpha T_n}}{T_n}, \quad k \geq 0.
\]
For \(k \geq 1\), we have
\[
\frac{d^k}{dx^k} \log(1 + f) = \frac{P_k(f^{(k)}, \ldots, f)}{(1 + f)^{2^{k-1}}},
\]
where \(P_k\) is a polynomial of degree \(2^{k-1}\). Using \((4.3)\) and \(T_0\) large enough, we then get
\[
\|P_k(f_{n,i}^{(k)}, \ldots, f_{n,i})\|_{C^0} \leq C_k \frac{e^{-\nu h - (k+1) \alpha T_n}}{T_n}.
\]
Remark that for a given \(n\), Corollary 8 ensures that \(\text{supp} f_{n,i} \cap \text{supp} f_{n,j} = \emptyset\) if \(j \neq i\). This yields to
\[
\| \sum_{i=1}^{\mu_n} \log(1 + f_{n,i})\|_{C^k} \leq C_k \frac{e^{-\nu h - (k+1) \alpha T_n}}{T_n}, \quad k \geq 0.
\]
If the constant \(\nu\) satisfies
\[
\nu = h + (k+1) \alpha + \epsilon/2 = (k+2)h + 2(k+1)\kappa + \epsilon,
\]
we have finally
\[
\|F_n\|_{C^k} = \| \sum_{p=0}^{n} \sum_{i=1}^{\mu_p} \log(1 + f_{n,i})\|_{C^k} \leq C_k \sum_{p=0}^{\infty} \frac{e^{-(\nu h - (k+1) \alpha) T_p}}{T_p} \leq C_k \frac{e^{-(\nu h - (k+1) \alpha) T_0}}{T_0}.
\]
From the above remarks, we end up with
\[
\|(e^{2 F_n} - 1)g_0\|_{C^k} \leq C(g_0, \epsilon_0, \epsilon, k) \frac{e^{-(\nu h - (k+1) \alpha) T_0}}{T_0}, \quad k \geq 2.
\]
In particular, if \( T_0 = T_0(g_0, \varepsilon_0, \varepsilon, k) \) is large enough, the right hand side of the previous equation is < \( \varepsilon_0 \) and we have \( g_n \in \mathcal{M}_k(\varepsilon_0) \) for all \( n \in \mathbb{N} \).

By construction, the metric perturbation at step \( n \) avoids any \( \varepsilon_n \)-neighborhood of \( \mathcal{C}_{g_n}([T_{n-1}, T_n]) \) and \( \mathcal{C}_{g_n}([T_n, T_{n+1}]) \) : as a result, closed \( g_{n-1} \)-geodesics with length in \([T_0, T_{n-1}] \cup [T_n, T_{n+1}]\) remain unchanged for \( g_n \). This means that:

\[
(4.7) \quad \mathcal{C}_{g_{n-1}}(T_{n-1}) \subset \mathcal{C}_{g_n}(T_{n-1}) \quad \text{and} \quad \mathcal{C}_{g_{n-1}}([T_n, T_{n+1}]) \subset \mathcal{C}_{g_n}([T_n, T_{n+1}]).
\]

To have equalities, it remains to check that closed \( g_{n-1} \)-geodesics with length > \( T_{n+1} \) are not mapped by \( f_{g_{n-1}} \to g_n \)-closed geodesics with length in \([0, T_n] \) — recall equation (2.2). To see this, let \( \gamma \in \mathcal{C}_{g_{n-1}}([T_{n+1}, +\infty]) \), and write \( \tilde{\gamma} = f_{g_{n-1}}(\gamma) \). From (2.3), we have

\[
T_{n+1} \leq \ell_{g_n}(\tilde{\gamma}) \leq \ell_{g_{n-1}}(\gamma) \sqrt{1 + \Delta_n}, \quad \Delta_n \overset{\text{def}}{=} \| g_n - g_{n-1} \|_{C^0}.
\]

On the other hand, equations (4.10) and (4.6) give

\[
0 \leq \Delta_n \leq \frac{2C_0 e^\alpha}{T_n} e^{(h + \alpha - \nu)T_n} \| g_{n-1} \|_{C^0} \leq \frac{e^{-(\nu - \alpha - h)T_n}}{T_n} C(g_0, \varepsilon_0, \varepsilon, k).
\]

In particular, since \( k \geq 2 \), (4.10) gives \( \nu \geq h + 3\alpha \). In view of the fact that \( \sqrt{1 + \Delta_n} \leq 1 + C\Delta_n \) for some \( C > 0 \), if \( T_0 \) is sufficiently large we will get from the previous equation \( C\Delta_n T_n < 1 \), and then

\[
\frac{T_{n+1}}{\sqrt{1 + \Delta_n}} \geq \frac{T_{n+1}}{1 + C\Delta_n} > T_n.
\]

This means that \( \ell_{g_n}(\tilde{\gamma}) > T_n \) and we can conclude that \( \mathcal{L}_{g_n}(\gamma) [T_{n-1}, T_n] \) is precisely \( \tilde{\ell}_1, \ldots, \tilde{\ell}_{\mu_n} \).

It is easily verified that \( \mathcal{C}_{g_n}(\mathcal{L}_{g_n}(\gamma) [T_{n-1}, T_n]) \) is \( \nu \)-separated : we have indeed \( \tilde{\ell}_{i+1} - \tilde{\ell}_i = \tilde{\ell}_{i+1} - \ell_i + e^{-\nu T_n} \) for \( i \leq m - 1 \) or \( i \geq m + 1 \). Using (3.10) and (4.5) we also get

\[
\tilde{\ell}_{m+1} - \tilde{\ell}_m \geq C_0^{-1} e^{-h T_n} - 2\mu_n e^{-\nu T_n} \geq \frac{1}{2} C_0^{-1} e^{-h T_n} > e^{-\nu T_n}
\]

for \( T_0 \) large enough. As noted above, the closed \( g_{n-1} \)-geodesics with length \( \leq T_{n-1} \) have not been modified, so \( \mathcal{C}_{g_n}(\mathcal{L}_{g_n}(\gamma)) \) is \( \nu \)-separated if \( \mathcal{C}_{g_{n-1}}([T_0, T_{n-1}]) \) was.

It is now straightforward to check that we can start the above process at some time \( T_0 = T_0(g_0, \varepsilon_0, \varepsilon, k) \) and get a sequence of metrics \((g_n)_{n \in \mathbb{N}} \in \mathcal{M}_k(\varepsilon_0)\) with

\[
g_n \overset{n \to \infty}{\to} g_\infty, \quad g_\infty \overset{\text{def}}{=} \prod_{n=1}^{\mu_n} \prod_{i=1}^{\delta_n} (1 + \frac{\delta_n}{\epsilon_n} \chi_{n+i}^2) g_0, \quad \| g_0 - g_\infty \|_{C^k} < \varepsilon_0.
\]

By construction, \( \mathcal{C}_{g_\infty}([T_0, +\infty]) \) is \( \nu \)-separated. The conclusion of Theorem 4 now follows readily.

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Laboratoire d’Analyse, Géométrie et Applications, Université Paris 13, CNRS UMR 7539, 93430 Villetaneuse, France.

E-mail address: schenck@math.univ-paris13.fr