UNIFIED TREATMENT OF MULTISYMPLECTIC 3-FORMS IN DIMENSION 6

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ABSTRACT. On a 6-dimensional real vector space $V$ there are three types of multisymplectic 3-forms. We present in this paper a unified treatment of these three types. Forms of each type represent a subset of $\Lambda^3 V^*$. In two cases they are open subsets, in the third one it is a submanifold of codimension 1. We study the geometry of these subsets.

0. INTRODUCTION

We shall consider a 6-dimensional real vector space $V$. Let us recall that a multisymplectic 3-form on $V$ is a 3-form $\omega$ such that the associated homomorphism $\kappa : V \rightarrow \Lambda^2 V^*$, $\kappa v = \iota_v \omega = \omega(v, \cdot, \cdot)$ is injective. We denote $\Lambda^3_{ms} V^*$ the subset of $\Lambda^3 V^*$ consisting of all multisymplectic forms. It is easy to see that $\Lambda^3_{ms} V^*$ is an open subset. The natural action of $GL(V)$ on $\Lambda^3 V^*$ preserves $\Lambda^3_{ms} V^*$. It is well known that under this action $\Lambda^3_{ms} V^*$ decomposes into three orbits (see e.g. [D], [H]). Two of them are open orbits, the third one is a submanifold of codimension 1. As representatives of these orbits we can take the following 3-forms. (We choose a basis $e_1, \ldots, e_6$ of $V$, and we denote $\alpha_1, \ldots, \alpha_6$ the corresponding dual basis.)

(1) $\omega_+ = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6$,
(2) $\omega_- = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6$,
(3) $\omega_0 = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6$.

The open set containing the form $\omega_+$ ($\omega_-$) we shall denote $U_+$ ($U_-$), and the codimension 1 submanifold containing $\omega_0$ we shall denote $U_0$. There is also another possible characterization of these orbits. Namely, for any 3-form $\omega$ we define

$\Delta^2(\omega) = \{ v \in V; (\iota_v \omega) \wedge (\iota_v \omega) \} = 0$.

In other words, the subset $\Delta^2(\omega) \subset V$ consists of all vectors $v \in V$ such that the 2-form $\iota_v \omega$ is decomposable. A computation shows that

$\Delta^2(\omega_+) = [e_1, e_2, e_3] \cup [e_4, e_5, e_6]$,
$\Delta^2(\omega_-) = \{0\}$,
$\Delta^2(\omega_0) = [e_1, e_2, e_3]$.

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We find easily that

1. \( \omega \in U_+ \) if and only if \( \Delta^2(\omega) \) consists of the union of two transversal 3-dimensional subspaces.
2. \( \omega \in U_- \) if and only if \( \Delta^2(\omega) = \{0\} \).
3. \( \omega \in U_0 \) if and only if \( \Delta^2(\omega) \) is a 3-dimensional subspace.

We consider now a multisymplectic 3-form \( \omega \), and we choose a nonzero 6-form \( \theta \) on \( V \). It is easy to see that there exists a unique endomorphism \( Q : V \to V \) such that

\[(\ast) \quad (t_v \omega) \wedge \omega = \iota_{Qv} \theta.\]

We shall now study the form of the endomorphism \( Q \).

1. The Product Case

Let us assume that \( \omega \in U_+ \). Then \( \Delta^2(\omega) = V_3' \cup V_3'' \), where \( V_3' \) and \( V_3'' \) are transversal 3-dimensional subspaces. Our main aim in this case is to prove that the endomorphism \( Q \) is a product structure, i.e., it satisfies \( Q^2 = I \), and its associated subspaces are the subspaces \( V_3' \) and \( V_3'' \).

If \( v \in V_3', v \neq 0 \) then applying \( t_v \) to \( \ast \), we get

\[0 = (t_v \omega) \wedge (t_v \omega) = t_v \iota_{Qv} \theta,\]

which shows that the vectors \( v \) and \( Qv \) are linearly dependent. This means that there is a function \( \lambda_1 : V_3' - \{0\} \to \mathbb{R} \) such that \( Qv = \lambda_1(v)v \) for every \( v \in V_3' - \{0\} \).

It is easy to see that the function \( \lambda_1 \) is constant. Namely, taking two linearly independent vectors \( v_1, v_2 \in V_3' \), we get

\[\lambda_1(v_1 + v_2)v_1 + \lambda_1(v_1 + v_2)v_2 = Q(v_1 + v_2) = Q(v_1) + Q(v_2) = \lambda_1(v_1)v_1 + \lambda_1(v_2)v_2,\]

which implies that \( \lambda(v_1) = \lambda(v_2) \). Consequently, we have \( Qv = \lambda_1v \) for every \( v \in V_3' \). Similarly we find that there is a constant \( \lambda_2 \) such that \( Qv = \lambda_2v \) for every \( v \in V_3'' \). Now, we are going to prove that \( \lambda_1 + \lambda_2 = 0 \). We shall need the following lemma.

1.1. Lemma. If \( \omega \in U_+ \), \( v' \in V_3' \) and \( v'' \in V_3'' \), then \( \iota_{v'} \iota_{v''} \omega = 0 \).

Proof: The lemma is obvious for the form \( \omega_+ \). But then it holds for every form \( \omega \in U_+ \).

Let us take two vectors \( v' \in V_3' \) and \( v'' \in V_3'' \), \( v' \neq 0 \), \( v'' \neq 0 \). We have

\[(t_{v'} \omega) \wedge \omega = \iota_{Qv'} \theta = \lambda_1 t_{v'} \theta.\]

Applying \( t_{v''} \) to the above equation, we get

\[(t_{v''} t_{v'} \omega) \wedge \omega + (t_{v'} \omega) \wedge (t_{v''} \omega) = \lambda_1 t_{v''} t_{v'} \theta \]
\[(t_{v'} \omega) \wedge (t_{v''} \omega) = \lambda_1 t_{v''} t_{v'} \theta.\]

Along the same lines we get

\[(t_{v''} \omega) \wedge \omega = \iota_{Qv''} \theta = \lambda_2 t_{v''} \theta \]
\[(t_{v''} t_{v'} \omega) \wedge \omega + (t_{v'} \omega) \wedge (t_{v''} \omega) = \lambda_2 t_{v''} t_{v'} \theta \]
\[(t_{v'} \omega) \wedge (t_{v''} \omega) = \lambda_2 t_{v''} t_{v'} \theta.\]
From the last two results we obtain
\[ 0 = (t_{v'}\omega) \wedge (t_{v''}\omega) - (t_{v''}\omega) \wedge (t_{v'}\omega) = \lambda_1 t_{v''} t_{v'} \theta - \lambda_2 t_{v''} t_{v'} \theta = (\lambda_1 + \lambda_2) t_{v''} t_{v'} \theta, \]
which implies \( \lambda_1 + \lambda_2 = 0 \). We set now \( \lambda = \lambda_1 = -\lambda_2 \). Obviously \( \lambda \neq 0 \). Otherwise we would have \( \Delta^2(\omega) = V \), which is a contradiction. Further, we get \( Q^2 = \lambda^2 I \).

Now we can see that the automorphisms
\[ S_+ = \frac{1}{\lambda} Q \text{ and } S_- = -\frac{1}{\lambda} Q \text{ satisfy } S_+^2 = I \text{ and } S_-^2 = I, \]

i.e. they define product structures on \( V \), and \( S_- = -S_+ \). Setting
\[ \theta_+ = \lambda \theta, \quad \theta_- = -\lambda \theta, \]
we get
\[ (t_{v'}\omega) \wedge \omega = t_{S_+ v} \theta_+, \quad (t_{v''}\omega) \wedge \omega = t_{S_+ v} \theta_- \]

In the sequel we shall denote \( S = S_+ \) and \( \theta = \theta_+ \). The same results which are valid for \( S_+ \) hold also for \( S_- \).

1.2. Lemma. If \( v' \in V_3' \), \( v' \neq 0 \), then the kernel \( K(t_{v'}\omega) \) of the 2-form \( t_{v'}\omega \) equals to \( [v', V_3''] \). If \( v'' \in V_3'' \), \( v'' \neq 0 \), then the kernel \( K(t_{v''}\omega) \) of the 2-form \( t_{v''}\omega \) equals to \( [v'', V_3''] \).

Proof. If \( v' \in V_3' \), \( v' \neq 0 \), then the 2-form \( t_{v'}\omega \) is a nonzero decomposable form. Consequently \( \dim K(t_{v'}\omega) = 4 \). Obviously \( v' \in K(t_{v'}\omega) \), and by virtue of Lemma 1.1 also any vector from \( V_3'' \) belongs to \( K(t_{v'}\omega) \). This proves that \( K(t_{v'}\omega) = [v', V_3'] \). The second assertion follows along the same lines.

1.3. Lemma. For any \( v \in V \) there is \( t_{S_+ v} t_{v'} \omega = 0 \).

Proof. Let us assume that \( S|V_3' = I \) and \( S|V_3'' = -I \). Then for arbitrary \( v = v' + v'' \) with \( v' \in V_3' \) and \( v'' \in V_3'' \) we have
\[ t_{S_+ v} t_{v'} \omega = t_{S(v' + v'')} t_{v'} \omega = t_{v'} \omega = t_{v'} t_{v'} \omega = 2 t_{v'} t_{v'} \omega = 0. \]

1.4. Proposition. There exists a unique (up to the sign) product structure \( S \neq I \) on \( V \) such that the form \( \omega \) satisfies the relation
\[ \omega(Sv_1, v_2, v_3) = \omega(v_1, Sv_2, v_3) = \omega(v_1, v_2, Sv_3) \quad \text{for any } v_1, v_2, v_3 \in V. \]

Proof. We shall prove first that the product structure \( S \) defined above satisfies this relation. According to the above lemma we have \( t_{v'} t_{S_+ v} \omega = 0 \) for any \( v \in V \). Therefore we have
\[ 0 = \omega(S(v_1 + v_2), v_1 + v_2, v_3) = \omega(Sv_1, v_2, v_3) + \omega(Sv_2, v_1, v_3), \]
which implies
\[ \omega(Sv_1, v_2, v_3) = \omega(v_1, Sv_2, v_3). \]
The second equality now easily follows. Obviously, the opposite product structure \( -S \) satisfies the same relation. It remains to prove that there is no other product
structure with the same property. Let \( \tilde{S} \) be another product structure with the above property. Then there is a unique automorphism \( A : V \to V \) such that \( \tilde{S} = SA. \) We have then

\[
\omega(v_1, \tilde{S}v_2, v_3) = \omega(v_1, v_2, \tilde{S}v_3)
\]

\[
\omega(Sv_1, Av_2, v_3) = \omega(v_1, v_2, SAv_3)
\]

\[
\omega(Sv_1, Av_2, v_3) = \omega(Sv_1, v_2, Av_3)
\]

\[
(\iota_Sv_1)(Av_2, v_3) = (\iota_Sv_1)(v_2, Av_3).
\]

Because \( S \) is an automorphism we get the equality

\[
(\iota_v(\omega)(Av_2, v_3) = (\iota_v(\omega)(v_2, Av_3).
\]

Let us take a vector \( v'_1 \in V'_3. \) Then for any \( v'_2 \in V'_3 \) we have

\[
0 = (\iota_v(\omega)(Av'_2, v'_1) = (\iota_{v'}(\omega)(v'_2, Av'_1). 
\]

Because \( v'_2 \) is arbitrary, we can see that \( Av'_1 \) belongs to the kernel \( K(\iota_v(\omega). \) This means that there is \( \lambda(v'_1) \in \mathbb{R} \) and \( v'' \in V''_3' \) such that \( Av'_1 = \lambda(v'_1)v'_1 + v'' \). Now we can easily see that there is \( \lambda \in \mathbb{R} \) and a homomorphism \( \varphi : V'_3 \to V''_3' \) such that

\[
Av'_1 = \lambda v'_1 + \varphi v'_1
\]

for every \( v'_1 \in V'_3. \) Similarly we find \( \mu \in \mathbb{R} \) and a homomorphism \( \psi : V''_3' \to V'_3 \) such that

\[
Av''_1 = \mu v''_1 + \psi v''_1
\]

for every \( v''_1 \in V''_3'. \) Taking a fixed \( v'_2 \in V'_3 \) and arbitrary \( v'_1, v''_1 \in V''_3' \), we get

\[
(\iota_v(\omega)(Av'_2, v''_1) = (\iota_{v'}(\omega)(v'_2, v''_1))
\]

\[
(\iota_v(\omega)(\varphi v'_2, v''_1) = 0,
\]

\[
(\iota_{v'}(\omega)(v'_1, v''_1) = 0.
\]

For any \( v'_1, v'_3 \in V'_3 \) we have by virtue of Lemma 1.1

\[
(\iota_{\varphi v'_1}(\omega)(v'_1, v'_3) = 0, \quad (\iota_{\varphi v'_3}(\omega)(v'_1, v'_3) = 0,
\]

which together with the preceding result shows that \( \iota_{\varphi v'_1}(\omega = 0. \) The form \( \omega \) is multisymplectic and consequently \( \varphi = 0. \) We have thus shown that \( \varphi = 0. \) Similarly we find that \( \psi = 0. \) This proves that \( AV'_3 \subset V'_3, \quad AV''_3 \subset V''_3 \) and that \( A|V'_3 = \lambda I, A|V''_3 = \mu I. \) Because \( \tilde{S}^2 = I, \) we find easily that \( \lambda = \pm 1 \) and \( \mu = \pm 1. \) Now the proof easily follows.

2. The complex case

In this section we present only the relevant results. Proofs can be found in [PV].

Let \( \omega \) be a 3-form on \( V \) such that \( \Delta^2(\omega) = 0 \). This means that for any \( v \in V, \)

\[
v \neq 0 \text{ there is } (\iota_v(\omega) \wedge (\iota_v(\omega) \neq 0. \text{ This implies that } \text{rank}(\iota_v(\omega) \geq 4. \text{ On the other hand obviously } \text{rank}(\iota_v(\omega) \leq 4. \text{ Consequently, for any } v \neq 0 \text{ rank}(\iota_v(\omega) = 4. \text{ Thus the kernel } K(\iota_v(\omega) \text{ of the 2-form } \iota_v(\omega) \text{ has dimension 2. Moreover } v \in K(\iota_v(\omega). \)

We have

\[
(\iota_v(\omega) \wedge \omega = \iota_Qv \theta.
\]

If \( v \neq 0 \) then \( (\iota_v(\omega) \wedge \omega \neq 0, \) and this shows that \( Q \) is an automorphism. It is also obvious that if \( v \neq 0, \) then the vectors \( v \) and \( Qv \) are linearly independent (apply \( \iota_v \) to the last equality).
2.1. Lemma. For any $v \in V$ there is $t_{Qv,\omega} = 0$, i.e. $Qv \in K(\iota_v, \omega)$.

This lemma shows that if $v \neq 0$, then $K(\iota_v, \omega) = [v, Qv]$. Applying $t_{Qv}$ to the equality $(\iota_v, \omega) \wedge \omega = t_{Qv, \theta}$ and using the last lemma we obtain easily the following result.

2.2. Lemma. For any $v \in V$ there is $(\iota_v, \omega) \wedge (\iota_{Qv}, \omega) = 0$.

Lemma 2.1 shows that $v \in K(\iota_{Qv}, \omega)$. Because $v$ and $Qv$ are linearly independent, we can see that $K(\iota_{Qv}, \omega) = [v, Qv]$.

It can be proved that there is $\lambda \in \mathbb{R}$ such that $Q^2 = -\lambda^2 I$. We can now see that the automorphisms

$$J_+ = \frac{1}{\lambda} Q \quad \text{and} \quad J_- = -\frac{1}{\lambda} Q$$

satisfy $J_+^2 = -I$ and $J_-^2 = -I$, i.e. they define complex structures on $V$, and $J_- = -J_+$. Setting

$$\theta_+ = \lambda \theta, \quad \theta_- = -\lambda \theta$$

we get

$$(\iota_v, \omega) \wedge \omega = \iota_{J_+, \theta} v, \quad (\iota_v, \omega) \wedge \omega = \iota_{J_-, \theta} v.$$

In the sequel we shall denote $J = J_+$ and $\theta = \theta_+$. The same results which are valid for $J_+$ hold also for $J_-$.  

2.3. Lemma. There exists a unique (up to the the sign) complex structure $J$ on $V$ such that the form $\omega$ satisfies the relation

$$\omega(Jv_1, v_2, v_3) = \omega(v_1, Jv_2, v_3) = \omega(v_1, v_2, Jv_3) \quad \text{for any } v_1, v_2, v_3 \in V.$$

3. The tangent case

Let us assume that $\omega \in U_0$. We denote $V_0 = \Delta^2(\omega)$. If $v \in V_0$, $v \neq 0$, then applying $\iota_v$ to (*), we get

$$0 = (\iota_v, \omega) \wedge (\iota_v, \omega) = \iota_v t_{Qv, \theta},$$

which shows again that the vectors $v$ and $Qv$ are linearly dependent. Consequently, there exists a function $\lambda : V_0 - \{0\} \to \mathbb{R}$ such that $Qv = \lambda(v)v$ for any $v \in V_0 - \{0\}$.

It is easy to see that this function is constant. We shall need the following two lemmas.

3.1. Lemma. For any $\alpha \in V^*$ we have $(\iota_v, \omega) \wedge \omega \wedge \alpha = -\alpha(Qv)\theta$.

Proof. For a fixed $\alpha \in V^*$ there exists a unique $l_\alpha \in V^*$ such that

$$(\iota_v, \omega) \wedge \omega \wedge \alpha = l_\alpha(v)\theta.$$ 

Hence we get

$$(\iota_{Qv}, \theta) \wedge \alpha = l_\alpha(v)\theta$$

and

$$t_{Qv}(\theta \wedge \alpha) - \alpha(Qv)\theta = l_\alpha(v)\theta$$

which finishes the proof.
3.2. Lemma. Let $\alpha \in V^*$ be such that $\alpha|V_0 = 0$. Then we have $(\iota_v \omega) \wedge \omega \wedge \alpha = 0$.

Proof. The formula can be verified for the form $\omega_0$ by a direct computation. But then it must be true for any 3-form $\omega \in U_0$.

Using these two lemmas, we get for any 1-form $\alpha$ with $\alpha|V_0 = 0$

$$0 = (\iota_v \omega) \wedge \omega \wedge \alpha = -\alpha(Qv)\theta,$$

which shows that $\alpha(Qv) = 0$. We have thus proved that for any $v \in V$ we have $Qv \in V_0$, i.e. $\text{im } Q \subset V_0$. Further, for any $v \in V$ we have $Q^2v = Q(Qv) = \lambda Qv$. This shows that the endomorphism $Q$ satisfies the equation

$$Q(Q - \lambda I) = 0.$$

Our next aim is to prove that the above constant $\lambda$ is zero. Let us assume on the contrary that $\lambda \neq 0$. Then there are subspaces $R_0, R_\lambda \subset V$ such that

$$V = R_0 \oplus R_\lambda, \quad Q|R_0 = 0, Q|R_\lambda = \lambda I.$$

Obviously, both these subspaces are nontrivial. $R_0 \neq 0$ because $\text{ker } Q \subset R_0$, and $R_\lambda \neq 0$ because $R_\lambda \supset V_0$. On the other hand for any $v \in R_0$ we have

$$\begin{align*}
(\iota_v \omega) \wedge \omega &= \iota_{Qv} \theta = 0 \\
(\iota_v \omega) \wedge (\iota_{Fv} \omega) &= 0.
\end{align*}$$

This shows that $v \in V_0$. Consequently, we get the inclusion $R_0 \subset V_0 \subset R_\lambda$, which is a contradiction. We have thus proved that $\lambda = 0$ and that $Q^2 = 0$. Because for every $v \notin V_0$ we have $Qv \neq 0$ (otherwise we would have $v \in V_0$), it is easy to see that $\text{im } Q = \text{ker } Q = V_0$. The endomorphisms $Q$ satisfying $Q^2 = 0$ are in differential geometry usually called tangent structures, and very often they are denoted by $T$. But because we would have here already too many $T$’s, we have decided to introduce the notation $F = Q$. We shall call the endomorphism $F$ tangent structure. Let us remark that when speaking about tangent structure, we always assume that $F^2 = 0$ and $\text{im } F = \text{ker } F$.

3.3. Lemma. For any $v \in V$ we have $\iota_v \circ \iota_{Fv} \omega = 0$.

Proof. We start with the equality

$$(\iota_v \omega) \wedge \omega = \iota_{Fv} \theta.$$

Applying $\iota_{Fv}$ we get

$$\begin{align*}
(\iota_{Fv} \iota_v \omega) \wedge \omega + (\iota_v \omega) \wedge (\iota_{Fv} \omega) &= 0 \\
-(\iota_v \circ \iota_{Fv} \omega) \wedge \omega + (\iota_v \omega) \wedge (\iota_{Fv} \omega) &= 0 \\
-\iota_v (\iota_{Fv} \omega \wedge \omega) + 2(\iota_v \omega) \wedge (\iota_{Fv} \omega) &= 0.
\end{align*}$$

Applying $\iota_v$ we have

$$\begin{align*}
(\iota_v \omega) \wedge (\iota_{Fv} \omega) &= 0.
\end{align*}$$

If the 1-form $\iota_v \circ \iota_{Fv} \omega$ were not zero, then it would exist a 1-form $\sigma$ such that $\iota_v \omega = \sigma \wedge \iota_{Fv} \omega$, and we would get

$$(\iota_v \omega) \wedge (\iota_v \omega) = \sigma \wedge (\iota_v \circ \iota_{Fv} \omega) \wedge \sigma \wedge (\iota_v \circ \iota_{Fv} \omega) = 0$$

for every $v \in V$, which is a contradiction.
3.4. Lemma. For any three vectors $v_1, v_2, v_3 \in V$ we have

$$\omega(Fv_1, v_2, v_3) = \omega(v_1, Fv_2, v_3) = \omega(v_1, v_2, Fv_3).$$

Proof. By virtue of Lemma 3.3 we have

$$0 = \omega(v_1 + v_2, F(v_1 + v_2), v_3) = \omega(v_1, Fv_2, v_3) + \omega(v_2, Fv_1, v_3),$$

which implies

$$\omega(Fv_1, v_2, v_3) = \omega(v_1, Fv_2, v_3).$$

The rest of the proof is easy.

Let us notice that the construction of the tangent structure $F$ depends on the choice of the 6-form $\theta$. Any other nonzero 6-form is a nonzero real multiple $a\theta$ and the relevant construction gives the tangent structure $(1/a)F$. In other words, the 3-form $\omega \in U_0$ determines a tangent structure up to a nonzero real multiple.

We shall now show another possibility how to obtain these tangent structures. It is easy to see that if $v, v'$ are two vectors from the subspace $V_0(\omega_0) = [e_1, e_2, e_3]$, then $\iota_{v} \iota_{v'} \omega_0 = 0$. Consequently, we have the following lemma.

3.5. Lemma. Let $\omega \in U_0$. Then for any two vectors $v, v' \in V_0 = \Delta^2(\omega)$ we have $\iota_{v} \iota_{v'} \omega = 0$.

3.6. Lemma. Let $R_3 \subset V$ be a 3-dimensional subspace such that for any two vectors $v, v' \in R_3$ there is $\iota_{v} \iota_{v'} \omega = 0$. Then $R_3 = \text{im } F$.

Proof. Let $v, v' \in R_3$. Then we have

$$(\iota_{v'} \omega) \wedge \omega = \iota_{Fv'} \theta$$

$$(\iota_{v} \iota_{v'} \omega) \wedge \omega + (\iota_{v'} \omega) \wedge (\iota_{v} \omega) = \iota_{v} \iota_{Fv'} \theta$$

$$(\iota_{v'} \omega) \wedge (\iota_{v} \omega) = \iota_{v} \iota_{Fv'} \theta.$$

Because the left hand side of this equality is symmetric with respect to $v$ and $v'$, we have

$$\iota_{v} \iota_{Fv'} \theta = \iota_{v'} \iota_{Fv} \theta$$

$$\theta(Fv', v, \ldots, v') = \theta(Fv, v', \ldots, v')$$

$$\theta(Fv, v', \ldots, v') = -\theta(v, Fv', \ldots, v')$$

for any two vectors $v, v' \in R_3$.

Let us assume first that $R_3 \cap \text{im } F$ is 0-dimensional. Then, taking a basis $v_1, v_2, v_3 \in R_3$, we get a basis $v_1, v_2, v_3, Fv_1, Fv_2, Fv_3$ of $V$, and consequently we have $\theta(v_1, v_2, v_3, Fv_1, Fv_2, Fv_3) \neq 0$. We take the vectors $v_1, v_2, v_3, v_1, Fv_2, Fv_3$. Applying the last formula, we get

$$0 \neq \omega(Fv_1, v_2, v_3, v_1, Fv_2, Fv_3) = -\omega(v_1, Fv_2, v_3, v_1, Fv_2, Fv_3) = 0,$$

which is a contradiction.
Next, let us assume that \( R_3 \cap \text{im} F \) is 1-dimensional. Obviously \( FR_3 \) is 2-dimensional. Then there are two possibilities. (1) Either \( FR_3 \supset R_3 \cap \text{im} F \). Then there are vectors \( v_1, v_2 \in R_3 \) such that \( v_1, v_2, Fv_1 \) is a basis of \( R_3 \). Then we can find a vector \( v_3 \) such that \( v_1, v_2, Fv_1, Fv_2, Fv_3 \) is a basis of \( V \). Taking the vectors \( v_1, v_2, v_1, v_3, Fv_2, Fv_3 \) and applying the above formula, we get

\[
0 \neq \theta(Fv_1, v_2, v_1, v_3, Fv_2, Fv_3) = -\theta(v_1, Fv_2, v_1, v_3, Fv_2, Fv_3) = 0,
\]

which is a contradiction. (2) Or \( (FR_3) \cap (R_3 \cap \text{im} F) = 0 \). Then we can take a basis of \( R_3 \) in the form \( v_1, v_2, Fv_3 \), and we can complete it to a basis \( v_1, v_2, Fv_3, Fv_1, Fv_2, v_3 \) of \( V \). This time we take the vectors \( v_1, v_2, Fv_3, v_1, Fv_2, v_3 \) and we apply the same formula.

\[
0 \neq \theta(Fv_1, v_2, Fv_3, v_1, Fv_2, v_3) = -\theta(v_1, Fv_2, Fv_3, v_1, Fv_2, v_3) = 0,
\]

which is again a contradiction.

It remains to consider the case when \( R_3 \cap \text{im} F \) is 2-dimensional. Then there are again two possibilities. (1) Either \( (FR_3) \cap (R_3 \cap \text{im} F) \neq 0 \). Then we can take a basis of \( R_3 \) in the form \( v_1, Fv_1, Fv_2 \), and we can complete it to a basis \( v_1, Fv_1, Fv_2, Fv_2, v_3, Fv_3 \). We take the vectors \( v_1, v_2, v_1, v_3, Fv_2, Fv_3 \) and we apply again the formula.

\[
0 \neq \theta(Fv_1, v_2, v_1, v_3, Fv_2, Fv_3) = -\theta(v_1, Fv_2, v_1, v_3, Fv_2, Fv_3) = 0,
\]

which is a contradiction. (2) Or \( (FR_3) \cap (R_3 \cap \text{im} F) = 0 \). Then we take a basis of \( R_3 \) in the form \( v_1, Fv_2, Fv_3 \), and we complete it to a basis \( v_1, Fv_2, Fv_3, Fv_1, Fv_2, v_3 \). Then, taking the vectors \( v_1, Fv_2, Fv_3, v_1, v_2, v_3 \) we get in the same way as above

\[
0 \neq \omega(Fv_1, Fv_2, Fv_3, v_1, v_2, v_3) = -\omega(v_1, F^2v_2, Fv_3, v_1, v_2, v_3) = 0,
\]

and we get again a contradiction. In this way we have proved that \( R_3 = \text{im} F \).

**3.7. Lemma.** Let \( \tilde{F}: V \to V \) be a tangent structure (i. e. an endomorphism satisfying \( \tilde{F}^2 = 0 \) and \( \text{im} \tilde{F} = \ker \tilde{F} \)) such that

\[
\omega(\tilde{F}v_1, v_2, v_3) = \omega(v_1, \tilde{F}v_2, v_3) = \omega(v_1, v_2, \tilde{F}v_3).
\]

Then \( \text{im} \tilde{F} = \text{im} F \).

**Proof.** It suffices to prove that the 3-dimensional subspace \( \text{im} \tilde{F} \) has the property described in the preceding lemma. Any two vectors \( v, v' \in \text{im} \tilde{F} \) can be expressed in the form \( v = \tilde{F} w, v' = \tilde{F} w' \). Then we have

\[
\iota_v t_{v'} \omega = t_{\tilde{F}w} t_{\tilde{F}w'} \omega = \omega(\tilde{F}w', \tilde{F}w, \cdot) = \omega(\tilde{F}^2 w', w, \cdot) = 0.
\]

**3.8. Proposition.** Let \( \omega \in U_0 \). Then there exists (up to a nonzero multiple) a unique tangent structure \( F \) such that

\[
\omega(Fv_1, v_2, v_3) = \omega(v_1, Fv_2, v_3) = \omega(v_1, v_2, Fv_3)
\]

for all \( v_1, v_2, v_3 \in V \).
Proof. Let $F$ and $\tilde{F}$ be two tangent structures with the above property. We introduce on $V$ two 3-forms by setting

$$\sigma_F(v_1, v_2, v_3) = \omega(Fv_1, v_2, v_3), \quad \sigma_{\tilde{F}}(v_1, v_2, v_3) = \omega(\tilde{F}v_1, v_2, v_3).$$

Because by virtue of the preceding lemma $V_0 = \ker F = \ker \tilde{F}$, it is obvious that if $v \in V_0$, then $\iota_v \sigma_F = 0$ and $\iota_v \sigma_{\tilde{F}} = 0$. This implies that there exist two 3-forms $s_F$ and $s_{\tilde{F}}$ on $V/V_0$ such that

$$\sigma_F = \pi^* s_F, \quad \sigma_{\tilde{F}} = \pi^* s_{\tilde{F}},$$

where $\pi : V \to V/V_0$ is the projection. The tangent structures $F$ and $\tilde{F}$ induce isomorphisms

$$f : V/V_0 \to V_0, \quad \tilde{f} : V/V_0 \to V_0.$$

We denote $A : V/V_0 \to V/V_0$ the automorphism $A = f^{-1}\tilde{f}$. For any three vectors $v_1, v_2, v_3 \in V$ we find

$$s_F(\pi v_1, \pi v_2, \pi v_3) = \sigma_F(v_1, v_2, v_3) = \omega(Fv_1, v_2, v_3) = \omega(\tilde{F}v_1, v_2, v_3).$$

We remind that the last term makes sense because $\tilde{f}v_1 \in V_0$. Further we have

$$\omega(\tilde{f}v_1, v_2, v_3) = \omega(fA\pi v_1, v_2, v_3).$$

Let us choose an element $w_1 \in V$ such that $\pi w_1 = A\pi v_1$. Then we get

$$\omega(fA\pi v_1, v_2, v_3) = \omega(f\pi w_1, v_2, v_3) = \omega(Fw_1, v_2, v_3) = \sigma_F(w_1, v_2, v_3) = s_F(A\pi v_1, \pi v_2, \pi v_3).$$

Proceeding in this way we obtain the relations

$$s_F(\pi v_1, \pi v_2, \pi v_3) = s_F(A\pi v_1, \pi v_2, \pi v_3),$$

$$s_F(\pi v_1, \pi v_2, \pi v_3) = s_F(\pi v_1, A\pi v_2, \pi v_3),$$

$$s_F(\pi v_1, \pi v_2, \pi v_3) = s_F(\pi v_1, \pi v_2, A\pi v_3),$$

and the relation

$$s_F(A\pi v_1, \pi v_2, \pi v_3) = s_F(\pi v_1, A\pi v_2, \pi v_3) = s_F(\pi v_1, \pi v_2, A\pi v_3).$$

Because the 3-form $s_F$ is nontrivial and because the homomorphism $\kappa : V \to \Lambda^2 V^*$ induces an isomorphism $\kappa_0 : V_0 \to \Lambda^2(V/V_0)^*$, we can see that for any 2-form $\alpha$ on $V/V_0$ and any two vectors $z_1, z_2 \in V/V_0$ we have

$$\alpha(Az_1, z_2) = \alpha(z_1, A z_2).$$

Let now $z \in V/V_0$ be arbitrary, and let us take 1-forms $\beta_1, \beta_2 \in (V/V_0)^*$ such that $\beta_1(z) = \beta_2(z) = 0$. We shall consider the 2-form $\beta_1 \wedge \beta_2$. For any vector $z' \in V/V_0$ we have

$$(\beta_1 \wedge \beta_2)(Az, z') = (\beta_1 \wedge \beta_2)(z, Az') = 0,$$

which shows that there is $\lambda(z) \in \mathbb{R}$ such that $Az = \lambda(z)z$. Moreover, it can be easily seen that the function $\lambda(z)$ is a nonzero constant. We thus get $A = \lambda I$ and this finishes the proof.

Choosing a nonzero 3-form $\eta \in \Lambda^3(V/V_0)^*$, we can define an isomorphism $V/V_0 \to \Lambda^2(V/V_0)^*$ by $w \mapsto \iota_w \eta$. Similarly, the monomorphism $\kappa : V \to \Lambda^2 V^*$, $\kappa v = \iota_v \omega$ induces an isomorphism $\kappa_0 : V_0 \to \Lambda^2(V/V_0)^*$. We take now the following chain of homomorphisms

$$V \xrightarrow{\kappa} V/V_0 \to \Lambda^2(V/V_0)^* \xrightarrow{\kappa_0^{-1}} V_0.$$

We denote this composition by $C$. 

9
3.9. Lemma. The homomorphism $C$ is a tangent structure satisfying $C^2 = 0$, \( \text{im} \, C = \ker C \) and the relation
\[
\omega(Cv_1, v_2, v_3) = \omega(v_1, Cv_2, v_3) = \omega(v_1, v_2, Cv_3)
\]
for every $v_1, v_2, v_3 \in V$.

Proof. Let us take any tangent structure $F$ with the above properties, and let us define a 3-form $\sigma_F(v_1, v_2, v_3) = \omega(Fv_1, v_2, v_3)$ as before. There is a unique 3-form $s_F$ on $V/V_0$ such that $\sigma_F = \pi^* s_F$, where $\pi : V \to V/V_0$ is the projection. Obviously, fixing a volume form $\mathrm{a}$ there is a nonzero $a \in \mathbb{R}$ such that $\eta = as_F$. For any $v, v', v'' \in V$ we have
\[
\omega(Cv, v', v'') = \eta(\pi v, \pi v', \pi v'') = as_F(\pi v, \pi v', \pi v'') = a\omega(Fv, v', v'') = \omega(aFv, v', v''),
\]
which shows that $C = aF$. This finishes the proof.

4. Orbit of forms of the product type

This is the orbit $U_+$, which represents an open submanifold in $\Lambda^3 V^*$. We take a point $\zeta \in U_+$. For the tangent space at this point we have $T_\zeta U_+ = \Lambda^3 V^*$. Obviously, fixing a volume form $\theta_0$ on $V$, we can choose for each $\zeta \in U_+$ an appropriate volume form $\theta(\zeta)$ (out of the two differing by the sign) such that $\theta(\zeta) = a\theta_0$ with $a > 0$. This means that we choose at the same time at each point $\zeta \in U_+$ a product structure $P(\zeta) \in \text{Aut}(V)$. In other words, we can consider over $U_+$ a trivial vector bundle $V$ with the fiber $V$, and on this vector bundle we have a tensor field $P$ of type $(1, 1)$ satisfying $P^2 = I$, $\dim \ker(P - I) = 3$, and $\dim \ker(P + I) = 3$. Our aim is to define a product structure on $T_\zeta U_+$. We shall try to define such a product structure by the formula
\[
(P(\zeta)\Omega)(v_1, v_2, v_3) = a\Omega(Pv_1, Pv_2, Pv_3) + b[\Omega(Pv_1, Pv_2, v_3) + \Omega(Pv_1, v_2, Pv_3) + \Omega(v_1, Pv_2, Pv_3)] + c[\Omega(Pv_1, v_2, v_3) + \Omega(v_1, Pv_2, v_3) + \Omega(v_1, v_2, Pv_3)] + d\Omega(v_1, v_2, v_3)
\]
for any $\Omega \in T_\zeta U_+$. Here $P$ denotes $P(\zeta)$. It is a matter of computation to prove

4.1. Proposition. $P(\zeta)$ satisfies $P(\zeta)^2 = I$ if and only if the quadruple $(a, b, c, d)$ is equal to one of the following 16 quadruples

\[
\begin{align*}
(\pm 1, 0, 0, 0), & \quad (\pm \frac{1}{2}, 0, \mp \frac{1}{2}, 0), \quad (0, \pm \frac{1}{2}, 0, \mp \frac{1}{2}), \quad (0, 0, 0, \pm 1) \\
(\frac{1}{4}, \frac{1}{4}, -\frac{3}{4}), & \quad (\frac{1}{4}, \frac{1}{4}, 1, -\frac{3}{4}), \quad (-\frac{1}{4}, -\frac{1}{4}, 1, -\frac{3}{4}), \quad (-\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{3}{4}) \\
3 & \quad 1, -1, 1, \quad 3 & \quad 1, 1, 1, \quad 3 & \quad 1, 1, 1, \quad 3 & \quad 1, 1, 1.
\end{align*}
\]

We can define subbundles
\[
\mathcal{V}_1 = \ker(P - I), \quad \mathcal{V}_2 = \ker(P + I)
\]
satisfying $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$. This decomposition enables to introduce in the standard way forms of type $(r, s)$. We denote by the symbol $\mathcal{D}^{r,s}$ the subbundle of the bundle $\Lambda^s\mathcal{V}$ consisting of forms of type $(r, s)$. Now, it is obvious that the tangent bundle of $U_+$ can be expressed as a direct sum of four subbundles (distributions)

$$TU_+ = \mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3},$$

where $\dim \mathcal{D}^{3,0} = \dim \mathcal{D}^{0,3} = 1$, $\dim \mathcal{D}^{2,1} = \dim \mathcal{D}^{1,2} = 9$. Let us denote $\pi_1 : \mathcal{V} \to \mathcal{V}_1$ and $\pi_2 : \mathcal{V} \to \mathcal{V}_2$ the projections. If $\zeta \in U_+$, we can define vectors $\zeta_1, \zeta_2 \in T\zeta U_+$ by the formulas

$$\zeta_1 = \pi_1(\zeta|_{\mathcal{V}_1}), \quad \zeta_2 = \pi_2(\zeta|_{\mathcal{V}_2}).$$

Now we can define vector fields $\omega, \omega_1$ and $\omega_2$ on $U_+$ by $\omega_\zeta = \zeta, \omega_1 \zeta = \zeta_1$ and $\omega_2 \zeta = \zeta_2$. Obviously, $\omega = \omega_1 + \omega_2$.

To each quadruple $(a, b, c, d)$ from Proposition 4.1 there correspond a product structure $\mathcal{P}$ and a subbundle $\mathcal{V}_1 = \ker(\mathcal{P} - I)$. Routine considerations show that the correspondence $(a, b, c, d) \mapsto \mathcal{V}_1$ is the following one.

$$(1, 0, 0, 0) \mapsto \mathcal{D}^{3,0} \oplus \mathcal{D}^{1,2} \quad (-1, 0, 0, 0) \mapsto \mathcal{D}^{2,1} \oplus \mathcal{D}^{0,3}$$

$$(\frac{1}{2}, 0, -\frac{1}{2}, 0) \mapsto \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3} \quad (\frac{1}{2}, 0, \frac{1}{2}, 0) \mapsto \mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1}$$

$$(0, \frac{1}{2}, 0, -\frac{1}{2}) \mapsto \mathcal{D}^{3,0} \oplus \mathcal{D}^{0,3} \quad (0, -\frac{1}{2}, 0, \frac{1}{2}) \mapsto \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}$$

$$(0, 0, 0, 1) \mapsto \mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3} \quad (0, 0, 0, -1) \mapsto 0$$

$$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}) \mapsto \mathcal{D}^{3,0} \quad (\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}) \mapsto \mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2}$$

$$(\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4}) \mapsto \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3} \quad (-\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{3}{4}) \mapsto \mathcal{D}^{0,3}$$

$$(\frac{3}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}) \mapsto \mathcal{D}^{1,0} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3} \quad (\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}) \mapsto \mathcal{D}^{1,2}$$

$$(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}) \mapsto \mathcal{D}^{1,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{0,3} \quad (\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}) \mapsto \mathcal{D}^{2,1}$$

In the sequel we are going to investigate the integrability of all these distributions. Our first result is easy because the distributions $\mathcal{D}^{3,0}$ and $\mathcal{D}^{0,3}$ are 1-dimensional.

**4.2. Proposition.** The distribution $\mathcal{D}^{3,0}$ ($\mathcal{D}^{0,3}$) is generated by the vector field $\omega_1$ ($\omega_2$). The distributions $\mathcal{D}^{3,0}$ and $\mathcal{D}^{0,3}$ are integrable.

Now we shall introduce on $U_+$ a flat connection $\nabla$, which is the restriction of the canonical connection on the vector space $\Lambda^2\mathcal{V}^*$. Notice that for any vector field $\Omega$ on $U_+$ we have $\nabla_\Omega \omega = \Omega$. We shall need the following three lemmas.

**4.3. Lemma.** Let $\tilde{\Omega}$ be a vector field on $U_+$ belonging to $\mathcal{D}^{3,0}$ ($\mathcal{D}^{2,1}$, $\mathcal{D}^{1,2}$, $\mathcal{D}^{0,3}$). Further, let $\Omega$ be arbitrary vector field on $U_+$. Then

$$\nabla_\Omega \tilde{\Omega} \in \mathcal{D}^{3,0} \oplus \mathcal{D}^{2,1} \oplus \mathcal{D}^{1,2} \oplus \mathcal{D}^{0,3}.$$

**Proof.** Let $\Theta$ be a section of the trivial vector bundle $\mathcal{V}^*$ over $U_+$. Then for any vector field $\Omega$ on $U_+$ we have

$$\nabla_\Omega \Theta \in \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}.$$

Now the assertion of the lemma easily follows.
4.4. Lemma. If $\Omega$ belongs to the distribution $D^{3,0}$ ($D^{0,3}$), then we have

$$\nabla_\Omega \omega_1 = \Omega, \quad \nabla_\Omega \omega_2 = 0 \quad (\nabla_\Omega \omega_1 = 0, \quad \nabla_\Omega \omega_2 = \Omega).$$

If $\Omega$ belongs to the distribution $D^{2,1}$ ($D^{1,2}$), then we have again

$$\nabla_\Omega \omega_1 = \Omega, \quad \nabla_\Omega \omega_2 = 0 \quad (\nabla_\Omega \omega_1 = 0, \quad \nabla_\Omega \omega_2 = \Omega).$$

Proof. We start with the equality $\omega_1 + \omega_2 = \omega$. If $\Omega$ belongs to $D^{3,0}$, then applying $\nabla_\Omega$ to this equality we get

$$\nabla_\Omega \omega_1 + \nabla_\Omega \omega_2 = \Omega$$

$$(\nabla_\Omega \omega_1)^{3,0} + (\nabla_\Omega \omega_1)^{2,1} + (\nabla_\Omega \omega_2)^{1,2} + (\nabla_\Omega \omega_2)^{0,3} = \Omega,$$

where the superscripts denote the corresponding component. Because $\Omega$ belongs to $D^{3,0}$ we obtain the first assertion. The remaining assertions follow along the same lines.

4.5. Lemma. A vector field $\Omega$ belongs to the distribution $D^{3,0} \oplus D^{2,1}$ ($D^{1,2} \oplus D^{0,3}$) if and only if

$$\nabla_\Omega \omega_2 = 0 \quad (\nabla_\Omega \omega_1 = 0).$$

Proof. If $\Omega$ belongs to $D^{3,0} \oplus D^{2,1}$ we know that the above condition is satisfied. Conversely, let us assume that the condition is satisfied. We have

$$\Omega = \Omega^{3,0} + \Omega^{2,1} + \Omega^{1,2} + \Omega^{0,3},$$

and we get

$$0 = \nabla_\Omega \omega_2 = \nabla_{\Omega^{3,0}} \omega_2 + \nabla_{\Omega^{2,1}} \omega_2 + \nabla_{\Omega^{1,2}} \omega_2 + \nabla_{\Omega^{0,3}} \omega_2 =$$

$$= \nabla_{\Omega^{1,2}} \omega_2 + \nabla_{\Omega^{0,3}} \omega_2 = \Omega^{1,2} + \Omega^{0,3},$$

which finishes the proof.

4.6. Proposition. The distributions $D^{3,0} \oplus D^{2,1}$ and $D^{1,2} \oplus D^{0,3}$ are integrable.

Proof. Let two vector fields $\Omega, \tilde{\Omega}$ belong to the distribution $D^{3,0} \oplus D^{2,1}$. Then we have $\nabla_\Omega \omega_2 = \nabla_{\tilde{\Omega}} \omega_2 = 0$, and we obtain

$$\nabla_{[\Omega, \tilde{\Omega}]} \omega_2 = \nabla_\Omega \nabla_{\tilde{\Omega}} \omega_2 - \nabla_{\tilde{\Omega}} \nabla_\Omega \omega_2 = 0$$

because the connection $\nabla$ is flat. Along the same lines we can prove the integrability of the distribution $D^{1,2} \oplus D^{0,3}$.

The following lemma is obvious.

4.7. Lemma. A vector field $\Omega$ belongs to the distribution $D^{2,1} \oplus D^{1,2}$ if and only if $\Omega \wedge \omega = 0$. 

12
4.8. Proposition. The distribution \( D^{2,1} \oplus D^{1,2} \) is not integrable.

Proof. Let \( \Omega \) and \( \hat{\Omega} \) lie in \( D^{2,1} \) and \( D^{1,2} \), respectively. Then we have \( \Omega \wedge \omega = 0 \) and \( \hat{\Omega} \wedge \omega = 0 \). Hence we obtain

\[
(\nabla_{\Omega} \hat{\Omega}) \wedge \omega + \hat{\Omega} \wedge \Omega = 0, \quad (\nabla_{\hat{\Omega}} \Omega) \wedge \omega + \Omega \wedge \hat{\Omega} = 0.
\]

Subtracting these two equalities, we have

\[
[\Omega, \hat{\Omega}] \wedge \omega = 2\Omega \wedge \hat{\Omega}.
\]

Now it suffices to choose \( \Omega \) and \( \hat{\Omega} \) in such a way that \( \Omega \wedge \hat{\Omega} \neq 0 \) at some point \( \zeta \in U_+ \). Then it is obvious that the bracket \([\Omega, \hat{\Omega}]\) does not lie in \( D^{2,1} \oplus D^{1,2} \).

4.9. Proposition. The distributions \( D^{2,1} \) and \( D^{1,2} \) are integrable.

Proof. Let \( \Omega \) and \( \hat{\Omega} \) be two vector fields lying in \( D^{2,1} \). Proceeding in the same way as in the proof of preceding lemma we find again

\[
[\Omega, \hat{\Omega}] \wedge \omega = 2\Omega \wedge \hat{\Omega}.
\]

But this time \( \Omega \wedge \hat{\Omega} = 0 \), which shows that \([\Omega, \hat{\Omega}]\) lies in \( D^{2,1} \oplus D^{1,2} \). Moreover, we have

\[
\nabla_{[\Omega, \hat{\Omega}]} \omega_2 = \nabla_{\Omega} \nabla_{\hat{\Omega}} \omega_2 - \nabla_{\hat{\Omega}} \nabla_{\Omega} \omega_2 = 0,
\]

which shows that \([\Omega, \hat{\Omega}]\) lies in \( D^{2,1} \).

4.10. Proposition. There is \([\omega_1, \omega_2] = 0\) and the distribution \( D^{3,0} \oplus D^{0,3} \) is integrable.

Proof. We have

\[
\nabla_{[\omega_1, \omega_2]} \omega_1 = \nabla_{\omega_1} \nabla_{\omega_2} \omega_1 - \nabla_{\omega_2} \nabla_{\omega_1} \omega_1 = 0 - \nabla_{\omega_1} \omega_1 = 0,
\]

which shows that \([\omega_1, \omega_2]\) lies in \( D^{1,2} \oplus D^{0,3} \). Along the same lines we can show that \([\omega_1, \omega_2]\) lies in \( D^{3,0} \oplus D^{2,1} \). This implies that \([\omega_1, \omega_2] = 0\) and that the distribution \( D^{3,0} \oplus D^{0,3} \) is integrable.

4.11. Proposition. For any vector field \( \Omega \) lying in \( D^{1,2} \) \((D^{2,1})\) the vector field \([\omega_1, \Omega]\) \(([\omega_2, \Omega])\) lies again in \( D^{1,2} \) \((D^{2,1})\). Consequently the distributions \( D^{3,0} \oplus D^{1,2} \) and \( D^{2,1} \oplus D^{0,3} \) are integrable.

Proof. Let us assume that \( \Omega \) lies in \( D^{1,2} \). Then we have

\[
\nabla_{[\omega_1, \Omega]} \omega_1 = \nabla_{\omega_1} \nabla_{\Omega} \omega_1 - \nabla_{\Omega} \nabla_{\omega_1} \omega_1 = 0 - \nabla_{\Omega} \omega_1 = 0,
\]

which proves that \([\omega_1, \Omega]\) lies in \( D^{1,2} \oplus D^{0,3} \). Because \( \Omega \) lies in \( D^{1,2} \), there is \( \Omega \wedge \omega = 0 \). Applying \( \nabla_{\omega_1} \) to this equality we find that

\[
0 = (\nabla_{\omega_1} \Omega) \wedge \omega + \Omega \wedge \nabla_{\omega_1} \omega = (\nabla_{\omega_1} \Omega) \wedge \omega + \Omega \wedge \omega_1.
\]

Obviously \( \Omega \wedge \omega_1 = 0 \), and this shows that \( \nabla_{\omega_1} \Omega \) lies in \( D^{2,1} \oplus D^{1,2} \). But we can immediately see that

\[
[\omega_1, \Omega] = \nabla_{\omega_1} \Omega - \nabla_{\Omega} \omega_1 = \nabla_{\omega_1} \Omega.
\]

Consequently \([\omega_1, \Omega]\) lies not only in \( D^{1,2} \oplus D^{0,3} \), but also in \( D^{2,1} \oplus D^{1,2} \). This implies that \([\omega_1, \Omega]\) lies in \( D^{1,2} \).
4.12. Proposition. The distributions $D^{3,0} \oplus D^{2,1} \oplus D^{0,3}$ and $D^{3,0} \oplus D^{1,2} \oplus D^{0,3}$ are integrable. The distributions $D^{3,0} \oplus D^{2,1} \oplus D^{1,2}$ and $D^{2,1} \oplus D^{1,2} \oplus D^{0,3}$ are not integrable.

Proof. The first assertion is easy to prove. Therefore, let us consider the distribution $D^{3,0} \oplus D^{2,1} \oplus D^{1,2}$. We shall take the same vector fields $\Omega$ lying in $D^{2,1}$ and $\tilde{\Omega}$ lying in $D^{1,2}$ as in the proof of Proposition 4.8. Then we have

$$[\Omega, \tilde{\Omega}] \wedge \omega_1 = (\nabla_\Omega \tilde{\Omega}) \wedge \omega_1 - (\nabla_{\tilde{\Omega}} \Omega) \wedge \omega_1 =$$

$$= \nabla_\Omega (\Omega \wedge \omega_1) - \tilde{\Omega} \wedge (\nabla_{\Omega} \omega_1) - \nabla_{\tilde{\Omega}} (\Omega \wedge \omega_1) + \Omega \wedge (\nabla_{\tilde{\Omega}} \omega_1) =$$

$$= -\tilde{\Omega} \wedge \Omega = \Omega \wedge \tilde{\Omega}.$$

At the same point $\zeta \in U_+$ as in the proof of Proposition 4.8 we have $\Omega_\zeta \wedge \tilde{\Omega}_\zeta \neq 0$, which shows that $[\Omega, \tilde{\Omega}]_\zeta \neq 0$. This proves that the distribution under consideration is not integrable.

We can summarize our results.

4.13. Proposition. The distributions

$$D^{3,0} \oplus D^{2,1}, \quad D^{3,0} \oplus D^{1,2}, \quad D^{3,0} \oplus D^{0,3}, \quad D^{2,1} \oplus D^{0,3}, \quad D^{1,2} \oplus D^{0,3}$$

$$D^{3,0} \oplus D^{2,1} \oplus D^{0,3}, \quad D^{3,0} \oplus D^{1,2} \oplus D^{0,3}$$

are integrable. The distributions

$$D^{2,1} \oplus D^{1,2}, \quad D^{3,0} \oplus D^{2,1} \oplus D^{1,2}, \quad D^{2,1} \oplus D^{1,2} \oplus D^{0,3}$$

are not integrable.

4.14. Remark. Requiring $\dim \ker(\mathcal{P} - I) = \dim \ker(\mathcal{P} + I) = 10$ we have only four possibilities how to define a product structure $\mathcal{P}$. It is easy to see that these product structures correspond to the quadruples

$$(1, 0, 0, 0), \quad (-1, 0, 0, 0), \quad \left(\frac{1}{2}, 0, -\frac{1}{2}, 0\right), \quad \left(-\frac{1}{2}, 0, \frac{1}{2}, 0\right).$$

Because all the distributions associated with these projectors are integrable, in all these cases the Nijenhuis tensor $[\mathcal{P}, \mathcal{P}] = 0$.

5. Orbit of forms of the complex type

Here we shall study the orbit $U_-$, which also represents an open submanifold in $\Lambda^3 V^*$. Taking a point $\zeta \in U_-$, we have $T_\zeta U_- = \Lambda^3 V^*$. Fixing again a volume form $\theta_0$ on $V$, we can choose for each $\zeta \in U_-$ an appropriate volume form $\theta(\zeta)$ (out of the two differing by the sign) such that $\theta(\zeta) = a\theta_0$ with $a > 0$. This enables us to choose at each point $\zeta \in U_-$ a complex structure $J(\zeta) \in \text{Aut}(V)$. In other words, this time we have on the trivial vector bundle $\mathcal{V}$ a tensor field $J$ of type $(1,1)$
satisfying $J^2 = -I$. We shall again try to define a complex structure on $T_\xi U_-$ by the formula

\[
(J(\zeta)\Omega)(v_1, v_2, v_3) = a\Omega(Jv_1, Jv_2, Jv_3) + \\
b[\Omega(Jv_1, Jv_2, v_3) + \Omega(Jv_1, v_2, Jv_3) + \Omega(v_1, Jv_2, Jv_3)] + \\
c[\Omega(Jv_1, v_2, v_3) + \Omega(v_1, Jv_2, v_3) + \Omega(v_1, v_2, Jv_3)] + \\
d\Omega(v_1, v_2, v_3)
\]

for any $\Omega \in T_\xi U_-$.  

5.1. Proposition. $J(\zeta)$ satisfies $J(\zeta)^2 = -I$ if and only if the quadruple $(a, b, c, d)$ is equal to one of the following 4 quadruples

$(\pm 1, 0, 0, 0), \quad (\pm \frac{1}{2}, 0, \pm \frac{1}{2}, 0)$.  

The proof is a simple computation and will be omitted. We shall denote

\[
(J_1(\zeta)\Omega)(v_1, v_2, v_3) = \Omega(J(\zeta)v_1, J(\zeta)v_2, J(\zeta)v_3) \\
(J_2(\zeta)\Omega)(v_1, v_2, v_3) = \frac{1}{2}\Omega(J(\zeta)v_1, J(\zeta)v_2, J(\zeta)v_3) + \\
\frac{1}{2}\Omega(J(\zeta)v_1, v_2, v_3) + \Omega(v_1, J(\zeta)v_2, v_3) + \Omega(v_1, v_2, J(\zeta)v_3)].
\]

The mapping $\zeta \in U_- \mapsto J_1(\zeta)$ (resp. $\zeta \in U_- \mapsto J_2(\zeta)$) defines an almost complex structure $J_1$ (resp. $J_2$) on the orbit $U_-$.  

5.2. Proposition. The almost complex structure $J_2$ is integrable.

Proof. We denote again by $\nabla$ the canonical connection on $\Lambda^3 V^*$. Let $\Omega$ and $\hat{\Omega}$ be two vector fields on $U_-$. Applying $\nabla_{\hat{\Omega}}$ to the identity $J^2 = -I$, we get

\[
(\nabla_{\hat{\Omega}}J)J + J(\nabla_{\hat{\Omega}}J) = 0.
\]

Further, we shall use the identity

\[
\omega(Jv_1, v_2, v_3) = \omega(v_1, Jv_2, v_3),
\]

and apply to it the covariant derivative $\nabla_{\hat{\Omega}}$. We obtain

\[
\hat{\Omega}(Jv_1, v_2, v_3) + \omega((\nabla_{\hat{\Omega}}J)v_1, v_2, v_3) = \hat{\Omega}(v_1, Jv_2, v_3) + \omega(v_1, (\nabla_{\hat{\Omega}}J)v_2, v_3).
\]

Substituting now $Jv_2$ instead of $v_2$ and $(\nabla_{\hat{\Omega}}J)v_3$ instead of $v_3$, we get the relation

\[
\hat{\Omega}(Jv_1, Jv_2, (\nabla_{\hat{\Omega}}J)v_3) = \\
-\hat{\Omega}(v_1, v_2, (\nabla_{\hat{\Omega}}J)v_3) - \omega((\nabla_{\hat{\Omega}}J)v_1, Jv_2, (\nabla_{\hat{\Omega}}J)v_3) - \omega(Jv_1, (\nabla_{\hat{\Omega}}J)v_2, (\nabla_{\hat{\Omega}}J)v_3).
\]
Similarly we obtain the relations

\[
\tilde{\Omega}(Jv_1, (\nabla_{\Omega} J)v_2, Jv_3) = \\
-\tilde{\Omega}(v_1, (\nabla_{\Omega} J)v_2, v_3) - \omega((Jv_1, (\nabla_{\tilde{\Omega}} J)v_2, (\nabla_{\Omega} J)v_3) - \omega((\nabla_{\Omega} J)v_1, (\nabla_{\Omega} J)v_2, Jv_3), \\
\tilde{\Omega}(\nabla_{\Omega} J)v_1, Jv_2, Jv_3) = \\
-\tilde{\Omega}((\nabla_{\Omega} J)v_1, v_2, v_3) - \omega((\nabla_{\Omega} J)v_1, (\nabla_{\Omega} J)v_2, Jv_3) - \omega((\nabla_{\Omega} J)v_1, Jv_2, (\nabla_{\Omega} J)v_3).
\]

Let us compute now

\[
2(\nabla_{\Omega}(J\tilde{\Omega}))(v_1, v_2, v_3) = 2\nabla_{\Omega}((J\tilde{\Omega}))(v_1, v_2, v_3) = \nabla_{\Omega}(\tilde{\Omega}(Jv_1, Jv_2, Jv_3) + \\
+[(\tilde{\Omega}(Jv_1, v_2, v_3) + \tilde{\Omega}(v_1, Jv_2, v_3 + \tilde{\Omega}(v_1, v_2, Jv_3)]) = (\nabla_{\Omega} \tilde{\Omega})(Jv_1, Jv_2, Jv_3) + \\
+[(\nabla_{\Omega} \tilde{\Omega})(Jv_1, v_2, v_3) + (\nabla_{\Omega} \tilde{\Omega})(Jv_1, Jv_2, v_3) + (\nabla_{\Omega} \tilde{\Omega})(Jv_1, v_2, Jv_3) + \\
+\tilde{\Omega}(\nabla_{\Omega} J)v_1, Jv_2, Jv_3) + \tilde{\Omega}(Jv_1, (\nabla_{\Omega} J)v_2, Jv_3) + \tilde{\Omega}(Jv_1, Jv_2, (\nabla_{\Omega} J)v_3) + \\
+\tilde{\Omega}(\nabla_{\Omega} J)v_1, Jv_2, (\nabla_{\Omega} J)v_3) + \tilde{\Omega}(v_1, (\nabla_{\Omega} J)v_2, v_3) + \tilde{\Omega}(v_1, v_2, (\nabla_{\Omega} J)v_3) = \\
= 2(\nabla_{\Omega} \tilde{\Omega})(v_1, v_2, v_3) - \\
-\omega((\nabla_{\Omega} J)v_1, (\nabla_{\Omega} J)v_2, Jv_3) - \omega((\nabla_{\Omega} J)v_1, Jv_2, (\nabla_{\Omega} J)v_3) - \\
-\omega((\nabla_{\Omega} J)v_1, (\nabla_{\Omega} J)v_2, (\nabla_{\Omega} J)v_3) - \omega((\nabla_{\Omega} J)v_1, (\nabla_{\Omega} J)v_2, Jv_3) - \\
-\omega((\nabla_{\Omega} J)v_1, Jv_2, (\nabla_{\Omega} J)v_3) - \omega(Jv_1, (\nabla_{\Omega} J)v_2, (\nabla_{\Omega} J)v_3).
\]

Here we have used the previous relations. Let us notice that the expression consisting of the last six terms is symmetric with respect to \(\Omega\) and \(\tilde{\Omega}\). Consequently we obtain

\[
\nabla_{\Omega}(J\tilde{\Omega}) - \nabla_{\tilde{\Omega}}(J\Omega) = J(\nabla_{\Omega} \tilde{\Omega} - \nabla_{\tilde{\Omega}} \Omega) = J[\Omega, \tilde{\Omega}].
\]

Writing \(J\Omega\) instead of \(\Omega\), we get

\[
\nabla_{J\Omega}(J\tilde{\Omega}) = -\nabla_{\tilde{\Omega}} \Omega + J[J\Omega, \tilde{\Omega}].
\]

Interchanging \(\Omega\) and \(\tilde{\Omega}\) we get the relation

\[
\nabla_{J\tilde{\Omega}}(J\Omega) = -\nabla_{\Omega} \tilde{\Omega} + J[J\tilde{\Omega}, \Omega].
\]

Substracting these last two relations we obtain

\[
[J\Omega, J\tilde{\Omega}] = [\Omega, \tilde{\Omega}] + J[J\Omega, \tilde{\Omega}] - J[J\tilde{\Omega}, \Omega] = 0,
\]

which shows that the Nijenhuis tensor \([J, J] = 0\).

**5.3. Remark.** The almost complex structure \(J_2\) was introduced in quite different way by N. Hitchin in [H]. He also proved the integrability and some other properties of \(J_2\).
6. Orbit of forms of the tangent type

Here we shall investigate the last orbit $U_0$, which represents a submanifold of codimension 1 in $\Lambda^3 V^*$. Let $\zeta \in U_0$ be arbitrary point, and let us denote $V_0(\zeta) = \Delta^2(\zeta)$. We shall introduce three subspaces $D_i(\zeta) \subset V$, $i = 1, 2, 3$ in the following way:

$$D_i(\zeta) = \{ \Omega \in T\zeta U_0; \Omega(v_1, v_2, v_3) = 0 \text{ if the vectors } v_1, \ldots, v_i \text{ belong to } V_0(\zeta) \}.$$ 

It is easy to verify that $\dim D_1 = 1$, $\dim D_2 = 10$, $\dim D_3 = 19$. Moreover, it is obvious that

$$D_1 \subset D_2 \subset D_3.$$

We describe first the tangent spaces to the orbit $U_0$. It is obvious that the projection

$$\pi_\zeta : GL(6, \mathbb{R}) \to U_0, \quad \pi_\zeta(\varphi) = \varphi^* \zeta$$

admits a smooth local section $\sigma$ defined on an open neighborhood $W$ of $\zeta$ and such that $\sigma(\zeta) = 1$. For any $\omega \in W$ we have then

$$\omega = \sigma(\omega)^* \zeta.$$

Let $\gamma : (-\varepsilon, \varepsilon) \to W$ be a smooth curve such that $\gamma(0) = \zeta$. We have then

$$\gamma(t) = \sigma(\gamma(t))^* \zeta$$

$$\gamma(t)(v_1, v_2, v_3) = \zeta(\sigma(\gamma(t))v_1, \sigma(\gamma(t))v_2, \sigma(\gamma(t))v_3),$$

where $v_1, v_2, v_3 \in V$ are arbitrary. Differentiating the last equality at $t = 0$, we get

$$\Omega(v_1, v_2, v_3) = \zeta(Av_1, v_2, v_3) + \zeta(v_1, Av_2, v_3) + \zeta(v_1, v_2, Av_3),$$

where $\Omega = (d/dt)_{t=0} \gamma(t)$ and $A = (d/dt)_{t=0} \sigma(\gamma(t))$.

6.1. Proposition. There is $T\zeta U_0 = D_3(\zeta)$.

Proof. If $\Omega \in T\zeta U_0$, then according to the above formula there is $\Omega \in D_3(\zeta)$ because $\zeta(v, v', v'') = 0$ if two entries belong to $V_0(\zeta)$. We have therefore $T\zeta U_0 \subset D_3(\zeta)$. Because $\dim T\zeta U_0 = 19$ and $\dim D_3(\zeta) = 19$, we get $T\zeta U_0 = D_3(\zeta)$.

It is obvious that it makes no sense to use in the future the notation $D_3(\zeta)$. The following lemma can be easily verified for the form $\omega_0$. But then it necessarily holds for any form $\zeta \in U_0$.

6.2. Lemma. There is

$$D_2(\zeta) = \{ \Omega \in T\zeta U_0; \Omega \wedge (\iota_v \zeta) = 0 \text{ for every } v \in V_0(\zeta) \} =$$

$$= \{ \Omega \in T\zeta U_0; \Omega \wedge \beta \wedge \beta' = 0 \text{ for any } \beta, \beta' \in V^* \text{ such that } \beta|V_0(\zeta) = \beta'|V_0(\zeta) = 0 \}.$$

On $U_0$ we have the trivial 6-dimensional vector bundle $\mathcal{V}$ with fiber $V$, and we can define a 3-dimensional vector subbundle $V_0$ whose fiber at $\zeta$ is $V_0(\zeta)$. We denote $\mathcal{W}$ the 3-dimensional quotient vector bundle $\mathcal{V}/V_0$. Moreover, assigning to each point $\zeta \in U_0$ the vector space $D_i(\zeta)$, we obtain over $U_0$ a vector bundle $D_i$, $i = 1, 2$. 

17
In other words we have two distributions $D_1 \subset D_2 \subset TU_0$. Furthermore, we have on $U_0$ an everywhere non-zero vector field $\omega$ defined by the formula $\omega_\zeta = \zeta$, i.e. assigning to a point $\zeta \in U_0$ the vector $\zeta$. This vector field $\omega$ lies in the distribution $D_2$. It is easy to see that the 1-dimensional distribution $I$ generated by the vector field $\omega$ and the 1-dimensional distribution $D_1$ are transversal.

Fixing a volume form $\theta_0 \in \Lambda^6V^*$, we get for each $\zeta \in U_0$ a tangent structure $F(\zeta)$. Namely, this tangent structure can be determined by the formula

$$(\iota_v \zeta) \wedge \zeta = \iota_{F(\zeta)v} \theta_0.$$ 

For any 3-form $\Omega \in \Lambda^3V^*$ we can then define

$$(D_{F(\zeta)} \Omega)(v_1, v_2, v_3) = \Omega(F(\zeta)v_1, v_2, v_3) + \Omega(v_1, F(\zeta)v_2, v_3) + \Omega(v_1, v_2, F(\zeta)v_3).$$

It is obvious that if $\Omega \in T_\zeta U_0$, then also $D_{F(\zeta)} \Omega \in T_\zeta U_0$. Consequently, on $T_\zeta U_0$ we can define an endomorphism $\mathcal{N}(\zeta)$ by the formula $\mathcal{N}(\zeta) = D_{F(\zeta)}$. In this way we get on $U_0$ a tensor field $\mathcal{N}$ of type $(1,1)$. It is easy to see that $\mathcal{N}^3 = 0$.

Our main aim in this section will be to prove the following proposition.

**6.3. Proposition.** On $U_0$ we have the following chain of distributions:

$$\text{im}\,\mathcal{N}^2 \subset \ker\,\mathcal{N} \subset \text{im}\,\mathcal{N} \subset \ker\,\mathcal{N}^2,$$

where $\text{im}\,\mathcal{N}^2 = D_1$ and $\text{im}\,\mathcal{N} = D_2$. The distributions $\text{im}\,\mathcal{N} \cap \ker\,\mathcal{N}$ and $\text{im}\,\mathcal{N}$ are integrable. The distribution $\ker\,\mathcal{N}^2$ is not integrable.

**6.4. Remark.** If $A \in \text{End}(V)$ is arbitrary we can define $D_A \Omega$ for any $\Omega \in \Lambda^kV^*$ by the formula

$$(D_A \Omega)(v_1, \ldots, v_k) = \sum_{i=1}^k \Omega(v_1, \ldots, v_{i-1}, Av_i, v_{i+1}, \ldots, v_k).$$

It is well known that $D_A$ is a derivation on the graded algebra $\Lambda^*V^*$.

We shall first investigate the subspace $\text{im}\,\mathcal{N}^2$. Let $\Omega \in \text{im}\,\mathcal{N}^2(\zeta)$. If $\Omega = \mathcal{N}^2(\zeta)\tilde{\Omega}$, then we have

$$\Omega(v_1, v_2, v_3) = 2(\tilde{\Omega}(Fv_1, Fv_2, v_3) + \tilde{\Omega}(Fv_1, v_2, Fv_3) + \tilde{\Omega}(v_1, Fv_2, Fv_3)),$$

where $F = F(\zeta)$. It is easy to see that if one of the entries $v_1, v_2, v_3$ belongs to $V_0(\zeta)$, then $\Omega(v_1, v_2, v_3) = 0$, or in other words, $\Omega \in \mathcal{D}_1(\zeta)$. Because obviously $\text{im}\,\mathcal{N}^2 \neq 0$, we get easily the following lemma. (Notice that $\dim\text{im}\,\mathcal{N}^2 = 1$.)

**6.5. Proposition.** There is $\text{im}\,\mathcal{N}^2 = D_1$ and $\text{im}\,\mathcal{N}^2 \subset \ker\,\mathcal{N}$. The distribution $\text{im}\,\mathcal{N}^2 \subset TU_0$ is integrable.

Next, we shall consider the subspace $\mathcal{D}_2(\zeta)$. It is obvious that for any $\Omega \in \mathcal{D}_2(\zeta)$ the correspondence $v \in V_0(\zeta) \mapsto \iota_v \Omega$ defines a homomorphism

$$\kappa_\Omega : V_0(\zeta) \to \Lambda^2W(\zeta)^*.$$

18
We have obvious formulas
\[ \kappa_{\Omega+\hat{\Omega}} = \kappa_{\Omega} + \kappa_{\hat{\Omega}}, \quad \kappa_{a\Omega} = a\kappa_{\Omega} \]
for any \( \Omega, \hat{\Omega} \in \mathcal{D}_2(\zeta) \) and any \( a \in \mathbb{R} \). Using the isomorphism \( \kappa_{\zeta} : V_0(\zeta) \to \Lambda^2 W(\zeta)^* \), we can define a homomorphism
\[ k_\Omega : \mathcal{D}_2(\zeta) \to \text{End}(V_0(\zeta)), \quad k_\Omega(v) = \kappa_{\zeta}^{-1}\kappa_{\Omega}(v). \]
It is easy to see that \( \ker k_\Omega = \mathcal{D}_1(\zeta) \). Consequently, we get a homomorphism
\[ K_\Omega : \mathcal{D}_2(\zeta)/\mathcal{D}_1(\zeta) \to \text{End}(V_0(\zeta)). \]
Because \( \dim \mathcal{D}_2(\zeta)/\mathcal{D}_1(\zeta) = \dim \text{End}(V_0(\zeta)) = 9 \), we can see that \( K_\Omega \) is an isomorphism.

6.6. Proposition. There is \( \text{im} \mathcal{N} = \mathcal{D}_2 \) and \( \dim \text{im} \mathcal{N} = 10 \).

Proof. If \( \Omega = \mathcal{N}(\zeta)\hat{\Omega} \), where \( \hat{\Omega} \in T_\zeta U_0 \), we have
\[ \Omega(v_1, v_2, v_3) = \hat{\Omega}(F(\zeta)v_1, v_2, v_3) + \hat{\Omega}(v_1, F(\zeta)v_2, v_3) + \hat{\Omega}(v_1, v_2, F(\zeta)v_3), \]
and it is obvious that \( \Omega \in \mathcal{D}_2(\zeta) \). This shows that \( \text{im} \mathcal{N} \subset \mathcal{D}_2 \).

Conversely, let us assume that \( \Omega \in \mathcal{D}_2(\zeta) \). We choose a basis \( v_1, v_2, v_3 \) of \( V_0(\zeta) \), and we denote \( \pi(\zeta) : V \to W(\zeta) \) the projection. Because \( \Omega \in \mathcal{D}_2(\zeta) \), there exist 2-forms \( \Omega_1, \Omega_2, \Omega_3 \in \Lambda^2 W(\zeta)^* \) such that
\[ \iota_v \Omega = \pi(\zeta)^* \Omega_i, \quad i = 1, 2, 3. \]
Let us take now 1-forms \( \beta_1, \beta_2, \beta_3 \in V^* \) such that \( \beta_i(v_j) = \delta_{ij} \). We shall consider a 3-form
\[ \hat{\Omega} = \sum_{i=1}^3 \beta_i \wedge \pi(\zeta)^* \Omega_i. \]
Now we can easily see that \( \iota_v(\Omega - \hat{\Omega}) = 0 \) for any \( v \in V_0(\zeta) \), or in other words \( \Omega - \hat{\Omega} \in \mathcal{D}_1 = \text{im} \mathcal{N}^2 \). This means that there is a 3-form \( \hat{\Omega} \in T_\zeta U_0 \) such that \( \Omega - \hat{\Omega} = \Lambda^2(\zeta)\hat{\Omega} \).

Let us consider the monomorphism \( \pi(\zeta)^* : \Lambda^* W(\zeta)^* \to \Lambda^* V^* \). It is easy to see that \( \pi(\zeta)^* W(\zeta)^* \) has a basis \( D_{F(\zeta)} \beta_1, D_{F(\zeta)} \beta_2, D_{F(\zeta)} \beta_3 \), and that
\[ D_{F(\zeta)}^2 \beta_1 = D_{F(\zeta)}^2 \beta_2 = D_{F(\zeta)}^2 \beta_3 = 0. \]
It is obvious that any 2-form \( \Omega' \in \pi(\zeta)^* \Lambda^2 W(\zeta)^* \) belongs to \( \text{im} D_{F(\zeta)}^2 \). Consequently, we can find 2-forms \( \Omega'_1, \Omega'_2, \Omega'_3 \) such that
\[ \pi(\zeta)^* \Omega_i = D_{F(\zeta)}^2 \Omega'_i. \]
We have then

\[ \hat{\Omega} = \sum_{i=1}^{3} \beta_i \wedge \pi(\xi)^* \tilde{\Omega}_i = \sum_{i=1}^{3} \beta_i \wedge D^2_{F(\xi)} \Omega'_i = \]

\[ = \sum_{i=1}^{3} D_{F(\xi)}(\beta_i \wedge D_{F(\xi)} \Omega'_i) - \sum_{i=1}^{3} D_{F(\xi)}(\Omega'_i \wedge \beta_i) = \]

\[ = D_{F(\xi)} \sum_{i=1}^{3} (\beta_i \wedge D_{F(\xi)} \Omega'_i) - D_{F(\xi)}(\beta_i \wedge \Omega'_i). \]

Now we can see that \( \Omega \in \text{im} \mathcal{N}(\xi) \), which finishes the proof.

6.7. Proposition. There is the inclusion \( \ker \mathcal{N} \subset \text{im} \mathcal{N} \).

Proof. Let \( \Omega \in \ker \mathcal{N}(\xi) \). Then we have (we write \( F \) instead of \( F(\xi) \))

\[ \Omega(Fv_1, v_2, v_3) + \Omega(v_1, Fv_2, v_3) + \Omega(v_1, v_2, Fv_3) = 0. \]

Using this relation we get

\[ \Omega(Fv_1, Fv_2, v_3) = -\Omega(v_1, F^2 v_2, v_3) - \Omega(v_1, F v_2, F v_3) = -\Omega(v_1, F v_2, F v_3) \]

\[ \Omega(Fv_1, v_2, v_3) = -\Omega(F^2 v_1, v_2, v_3) - \Omega(Fv_1, v_2, F v_3) = -\Omega(Fv_1, v_2, F v_3) \]

Adding these two relations, we obtain

\[ 2\Omega(Fv_1, Fv_2, v_3) = -\Omega(v_1, F v_2, F v_3) - \Omega(F v_1, v_2, F v_3), \]

\[ \Omega(Fv_1, Fv_2, v_3) = -\Omega(Fv_1, v_2, v_3) - \Omega(Fv_1, v_2, F v_3) = -\Omega(Fv_1, v_2, F v_3) = \]

\[ = -\frac{1}{2} D^2_{F} \Omega(v_1, v_2, v_3) = 0, \]

which shows that \( \Omega \in D_2(\xi) \).

6.8. Proposition. Let \( \xi \in U_0 \). Then \( \Omega \in T_2 U_0 \) belongs to \( \ker \mathcal{N}^2 \) if and only if \( \xi \wedge \Omega = 0 \). Moreover \( \dim \ker \mathcal{N}^2 = 18 \).

Proof. Let us choose vectors \( v, v', v'' \in V \) such that \( Fv, Fv', Fv'', v, v', v'' \) is a basis of \( V \). (We denote for simplicity \( F = F(\xi) \).) We shall consider the value \( (\xi \wedge \Omega)(Fv, Fv', Fv'', v, v', v'') \). (We recall that \( \zeta(w, w', \cdot) = 0 \) if \( w, w' \in V_0(\xi), \zeta(w, Fw, \cdot) = 0 \) for any \( w \in V \), and \( \Omega|V_0(\xi) = 0 \).) We get

\[ (\xi \wedge \Omega)(Fv, Fv', Fv'', v, v', v'') = \zeta(Fv, v', v'') \Omega(Fv', Fv'', v) + \]

\[ + \zeta(Fv', v, v'') \Omega(Fv, Fv'', v') + \zeta(Fv'', v, v') \Omega(Fv, Fv', v'') = \]

\[ = \zeta(Fv, v', v'')[\Omega(Fv, Fv', v'') + \Omega(Fv, Fv', v'') + \Omega(v, Fv', Fv'')]. \]

Because \( \zeta(Fv, v', v'') \neq 0 \) the first assertion easily follows. Now it is obvious that \( \dim \ker \mathcal{N}^2 = 18 \).
6.9. Proposition. There is \( \text{im} N \subset \ker N^2 \).

Proof. If \( \Omega \in \text{im} N(\zeta) = D_2(\zeta) \) then obviously \( \zeta \wedge \Omega = 0 \).

On the trivial vector bundle \( V \) with fiber \( V \) over \( U_0 \) we introduce a linear connection \( \nabla \). For any vector field \( \Omega \) on \( U_0 \) and any section \( S \) of \( V \) we define \( \nabla_\Omega S = \Omega S \). Obviously, \( \nabla \) induces a linear connection on every exterior power \( \Lambda^k V^* \), which will be denoted by the same symbol. It is obvious that the same formula \( \nabla_\Omega S = \Omega S \), where \( S \) is a section of the trivial vector bundle \( V \) with fiber \( V \) over \( \Lambda^3 V^* \), extends the connection \( \nabla \) to the whole vector space \( \Lambda^3 V^* \). The connection \( \nabla \) induces again a linear connection on the vector bundle \( \Lambda^k V^* \), which will be denoted again by the symbol \( \nabla \). Let \( \Omega_1 \) and \( \Omega_2 \) be (local) vector fields on \( U_0 \), and let \( \tilde{\Omega}_1 \) and \( \tilde{\Omega}_2 \) be their (local) extensions. Because the connection \( \nabla \) is flat, we have \( \nabla_{\Omega_1} \tilde{\Omega}_2 - \nabla_{\Omega_2} \tilde{\Omega}_1 = [\tilde{\Omega}_1, \tilde{\Omega}_2] \). Restricting this formula to the submanifold \( U_0 \), we obtain the formula

\[
\nabla_{\Omega_1} \Omega_2 - \nabla_{\Omega_2} \Omega_1 = [\Omega_1, \Omega_2],
\]

which will be needed in the sequel.

6.10. Lemma. Let \( S \) be a section of the subbundle \( V_0 \), and let \( \Omega \) be a vector field on \( U_0 \) lying in \( \text{im} N \). Then \( \nabla_{\Omega S} \) is also a section of the subbundle \( V_0 \).

Proof. Because \( S \) is a section of the subbundle \( V_0 \), we have the relation \( (\iota_S \omega) \wedge \omega = 0 \). Applying to this relation \( \nabla_{\Omega S} \), we obtain

\[
(\iota_{\nabla_{\Omega S} \omega}) \wedge \omega + (\iota_S \Omega) \wedge \omega + (\iota_S \omega) \wedge \Omega = 0.
\]

It is easy to see that the second term vanishes. The last term vanishes by virtue of Lemma 6.3. Consequently, we obtain \( (\iota_{\nabla_{\Omega S} \omega}) \wedge \omega = 0 \), which shows that \( \nabla_{\Omega S} \) is a section of \( V_0 \).

6.11. Remark. The previous lemma shows that the connection \( \nabla \) on \( V \) induces a partial connection on \( V_0 \), which we shall denote by the same symbol. This partial connection determines the covariant derivative \( \nabla_{\Omega S} \) only for vector the fields \( \Omega \) lying in \( \text{im} N \). This partial connection induces a partial connection on the vector bundle \( \mathcal{W} \) and on any exterior power of the vector bundles \( V_0 \) and \( \mathcal{W} \). Moreover, if \( \tilde{\Omega} \) is a vector field on \( U_0 \) (i.e., a section of \( \Lambda^3 V^* \) such that \( \tilde{\Omega}|_V = 0 \)), then for any vector field \( \Omega \) lying in \( \text{im} N \) and any three sections \( S_1, S_2, S_3 \) of \( V_0 \) we have

\[
\hat{\Omega}(S_1, S_2, S_3) = 0 \quad \nabla_{\Omega}(\hat{\Omega}(S_1, S_2, S_3)) = 0
\]

\[
(\nabla_{\omega} \hat{\Omega})(S_1, S_2, S_3) + \hat{\Omega}(\nabla_{\Omega} S_1, S_2, S_3) + \hat{\Omega}(S_1, \nabla_{\Omega} S_2, S_3) + \hat{\Omega}(S_1, S_2, \nabla_{\Omega} S_3) = 0
\]

\[
(\nabla_{\Omega} \hat{\Omega})(S_1, S_2, S_3) = 0,
\]

which shows that the partial connection \( \nabla \) induces a partial connection (again denoted by the same symbol) on \( TU_0 \). Because the original connection on \( V \) is flat, we have for any two vector fields \( \Omega \) and \( \hat{\Omega} \) lying in \( \text{im} N \)

\[
\nabla_{\Omega} \hat{\Omega} - \nabla_{\hat{\Omega}} \Omega = [\Omega, \hat{\Omega}].
\]
6.12. Proposition. The distribution im\(\mathcal{N}\) is integrable.

Proof. According to Proposition 6.6 there is im\(\mathcal{N} = \mathcal{D}_2\). Let us take two vector fields \(\hat{\Omega}, \hat{\Omega}\) lying in \(\mathcal{D}_2\), and three sections \(S_1, S_2, S_3\) of \(\mathcal{V}\) such that \(S_1\) and \(S_2\) lie in \(\mathcal{V}_0\). Then we have

\[
(\nabla_{\hat{\Omega}}\hat{\Omega})(S_1, S_2, S_3) = \nabla_{\hat{\Omega}}(\hat{\Omega}(S_1, S_2, S_3)) - \hat{\Omega}(\nabla_{\hat{\Omega}}S_1, S_2, S_3) - \hat{\Omega}(S_1, \nabla_{\hat{\Omega}}S_2, S_3) - \hat{\Omega}(S_1, S_2, \nabla_{\hat{\Omega}}S_3) = 0
\]

according to Lemma 6.10. This shows that \(\nabla_{\hat{\Omega}}\hat{\Omega}\) lies in \(\mathcal{D}_2\). Now, it is obvious that \([\hat{\Omega}, \hat{\Omega}] = \nabla_{\hat{\Omega}}\hat{\Omega} - \nabla_{\hat{\Omega}}\hat{\Omega}\) lies in \(\mathcal{D}_2\).

6.13. Proposition. \(\ker\mathcal{N} = \{\Omega \in \text{im}\mathcal{N}; \text{Tr} k(\Omega) = 0\}\) and \(\dim \ker\mathcal{N} = 9\).

Proof. We shall denote for simplicity \(V_0 = V_0(\xi), F = F(\xi), W = W(\xi), \pi = \pi(\xi)\). Let us notice first that for each endomorphism \(A \in \text{End}(V_0)\) there exists an endomorphism \(B \in \text{End}(V)\) (not uniquely determined) such that

\[
AF = FB \quad \text{and} \quad BV_0 \subset V_0.
\]

Moreover, any endomorphism \(B\) with these properties induces an endomorphism \(\hat{B} \in \text{End}(W)\) and \(\text{Tr} \hat{B} = \text{Tr} A\).

Let us take now a 3-form \(\Omega \in \text{im}\mathcal{N}(\xi) = \mathcal{D}_2(\xi)\). We have

\[
(\mathcal{N}(\xi)\Omega)(v, v', v'') = \Omega(Fv, v', v'') + \Omega(v, Fv', v'') + \Omega(v, v', Fv'').
\]

It is easy to see that \(\mathcal{N}(\xi)\Omega \in \mathcal{D}_1(\xi)\), and consequently there exists a uniquely determined 3-form \(\hat{\Omega} \in \Lambda^3\mathcal{W}^*\) such that \(\mathcal{N}(\xi)\Omega = \pi^*\hat{\Omega}\). Similarly, there is a 3-form \(\hat{\zeta} \in \Lambda^3\mathcal{W}^*\) such that \(\mathcal{N}(\xi)\zeta = \pi^*\hat{\zeta}\). We recall that the homomorphism \(\pi^* : \Lambda^3\mathcal{W}^* \to \Lambda^3\mathcal{V}^*\) is a monomorphism. Consequently \(\hat{\zeta} \neq 0\).

Let us take now \(A = k(\xi)\). Obviously for any \(v, v', v'' \in V\) we have

\[
\zeta(AFv, v', v'') = \Omega(Fv, v', v''),
\]

\[
\zeta(v, AFv', v'') = \Omega(v, Fv', v''),
\]

\[
\zeta(v, v', AFv'') = \Omega(v, v', Fv'').
\]

Then we get

\[
(\mathcal{N}(\xi)\Omega)(v, v', v'') = \Omega(Fv, v', v'') + \Omega(v, Fv', v'') + \Omega(v, v', Fv'') =
\]

\[
= \zeta(AFv, v', v'') + \zeta(v, AFv', v'') + \zeta(v, v', AFv'') =
\]

\[
= (1/3)[\zeta(FBv, v', v'') + \zeta(FBv, v'', v') + \zeta(FBv, v', v'') + \zeta(Bv, Fv', v'') + \zeta(Bv, Fv'', v') + \zeta(Bv, Fv', v'')] +
\]

\[
= (1/3)[\zeta(Fv, Fv', v'') + \zeta(v, Fv', v'') + \zeta(v, Fv', Fv'')] +
\]

\[
= (1/3)[\zeta(Fv, v', v'') + \zeta(Fv, v'', v') + \zeta(v, v', Fv'')] =
\]

\[
= (1/3)\zeta(\hat{B}[v], [v'], [v'']) + (1/3)\zeta([v], \hat{B}[v'], [v'']) + (1/3)\zeta([v], [v'], \hat{B}[v'']) =
\]

\[
= (1/3)\text{Tr}(\hat{B})\zeta([v], [v'], [v'']) = (1/3)\text{Tr}(A)\zeta(v, v', v''),
\]

which shows that \(\mathcal{N}(\xi)\Omega = 0\) if and only if \(\text{Tr}(A) = 0\).
6.14. Lemma. Let $M$ be a differentiable manifold, and let $\xi$ be an $n$-dimensional differentiable vector bundle over $M$ endowed with a linear connection $\nabla$. Let $A$ be an endomorphism of the vector bundle $\xi$, i.e. a section of the vector bundle $\xi^* \otimes \xi$. Then for any vector field $X$ on $M$ we have

$$\text{Tr}(\nabla_X A) = X \text{Tr}(A).$$

Proof. Let us choose (at least locally) a non-zero $n$-form $\varepsilon$ on $\xi$. Then for any vector fields $X_1, \ldots, X_n$ we have

$$\sum_{i=1}^n \varepsilon(X_1, \ldots, X_{i-1}, AX_i, X_{i+1}, \ldots, X_n) = \text{Tr}(A) \cdot \varepsilon(X_1, \ldots, X_n).$$

Let $X$ be a vector field on $M$. Applying $\nabla_X$ to the above equality, we obtain

$$\sum_{i=1}^n \varepsilon(X_1, \ldots, X_{i-1}, (\nabla_X A)X_i, X_{i+1}, \ldots, X_n) = (X \text{Tr}(A)) \cdot \varepsilon(X_1, \ldots, X_n),$$

which implies the desired equality.

6.15. Proposition. The distribution $\ker N$ is integrable.

Proof. Let $\Omega$ and $\tilde{\Omega}$ be two vector fields lying in the distribution $\ker N$. We denote $A = k_\Omega$ and $\tilde{A} = k_{\tilde{\Omega}}$. According to the previous result there is $\text{Tr}(A) = \text{Tr}(\tilde{A}) = 0$. For any section $S$ of $\mathcal{V}_0$ and any constant sections $S', S''$ of $\mathcal{V}$ we have

$$\omega(AS, S', S'') = \Omega(S, S', S''), \quad \omega(\tilde{A}S, S', S'') = \tilde{\Omega}(S, S', S'').$$

Applying $\nabla_{\Omega}$ to the second equality we obtain

$$(\nabla_{\Omega} \omega)(\tilde{A}S, S', S'') + \omega((\nabla_{\Omega} \tilde{A})S, S', S'') = (\nabla_{\Omega} \tilde{\Omega})(S, S', S''),$$

which shows that $k_{[\Omega, \tilde{\Omega}]} = [A, \tilde{A}] + \nabla_{\Omega} \tilde{A} - \nabla_{\tilde{\Omega}} A$. On any integral submanifold of the distribution $\text{im} \mathcal{N}$ we have

$$\text{Tr}([A, \tilde{A}] + \nabla_{\Omega} \tilde{A} - \nabla_{\tilde{\Omega}} A) = 0 + \Omega \text{Tr}(\tilde{A}) - \tilde{\Omega} \text{Tr}(A) = 0.$$

This finishes the proof.
6.16. Proposition. The distribution ker\( N^2 \) is not integrable.

Proof. Let \( \Omega \) and \( \tilde{\Omega} \) be two vector fields on \( U_0 \) lying in ker\( N^2 \). We shall apply the vector field \( \Omega \) to the relation \( \tilde{\Omega} \wedge \omega = 0 \). We get

\[
(\nabla_\Omega \tilde{\Omega}) \wedge \omega + \tilde{\Omega} \wedge \Omega = 0.
\]

Interchanging \( \Omega \) and \( \tilde{\Omega} \) and subtracting the two relations, we obtain

\[
[\Omega, \tilde{\Omega}] \wedge \omega + \tilde{\Omega} \wedge \Omega - \Omega \wedge \tilde{\Omega} = 0
\]

\[
[\Omega, \tilde{\Omega}] \wedge \omega = 2\Omega \wedge \tilde{\Omega}.
\]

Let us choose now vectors \( \alpha, \tilde{\alpha} \in T_{\omega_0}U_0 \) as follows:

\[
\alpha = \alpha_1 \wedge \alpha_2 \wedge \alpha_5, \quad \tilde{\alpha} = \alpha_3 \wedge \alpha_4 \wedge \alpha_6.
\]

It is easy to verify that \( \Omega, \tilde{\Omega} \in \text{ker} N^2(\omega_0) \). We choose now vector fields \( \Omega \) and \( \tilde{\Omega} \) in such a way that they lie in ker\( N^2 \) and \( \Omega_{\omega_0} = \alpha \) and \( \tilde{\Omega}_{\omega_0} = \tilde{\alpha} \). According to the above formula we have then

\[
[\Omega, \tilde{\Omega}]_{\omega_0} \wedge \omega_0 = 2\alpha \wedge \tilde{\alpha} \neq 0,
\]

which shows that the vector field \( [\Omega, \tilde{\Omega}] \) does not lie in ker\( N^2 \).

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24