5D $\mathcal{N} = 1$ super QFT: symplectic quivers

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Abstract

We develop a method to build new 5D $\mathcal{N} = 1$ gauge models based on Sasaki-Einstein manifolds $Y^{p,q}$. These models extend the standard 5D ones having a unitary $SU(p)_q$ gauge symmetry based on $Y^{p,q}$. Particular focus is put on the building of a gauge family with symplectic $SP(2r,\mathbb{R})$ symmetry. These super QFTs are embedded in M-theory compactified on folded toric Calabi-Yau threefolds $\hat{X}(Y^{2r,0})$ constructed from conical $Y^{2r,0}$. By using outer-automorphism symmetries of 5D $\mathcal{N} = 1$ BPS quivers with unitary $SU(2r)$ gauge invariance, we also construct BPS quivers with symplectic $SP(2r,\mathbb{R})$ gauge symmetry. Other related aspects are discussed.

Keywords: SCFT$_5$, 5D $\mathcal{N} = 1$ super QFT on a finite circle, toric threefolds based on Sasaki-Einstein manifolds, toric diagrams, BPS quivers, outer-automorphisms, folding.

1 Introduction

$\mathcal{N} = 1$ supersymmetric gauge theories in five space time dimensions (super QFT$_5$) are non renormalizable field theories with eight supercharges. They are admitted to have UV fixed points which can be deformed by relevant operators such that in the infrared they flow to 5D $\mathcal{N} = 1$ super Yang-Mills (SYM$_5$) coupled to hypermultiplets [1] [2]. A typical massive
deformation generating this type of flow is given by the SYM term $\text{tr}(F_{\mu\nu}^2)/g_Y^2$ where in 5D the inverse gauge coupling square $1/g_Y^2$ has dimension of mass. These 5D gauge theories are somehow special compared to 6D gauge theories \[3, 4, 5\] including maximally supersymmetric Yang-Mills theory believed to flow to $\mathcal{N} = (2, 0)$ supersymmetric 6D theory in the UV \[6, 7\]. In the few last years, super QFTs and their compactification, in particular on a Kaluza-Klein circle with finite radius and to 3D, have been subject to some interest in connection with their critical behaviour and specific properties of their gauge phases \[8-15\]. Though a complete classification is still lacking \[16, 5\], several examples of such gauge theories are known; and most of them can be viewed as deformations of 5D superconformal theories \[17, 18, 19\]. Simplest examples of SCFTs are given by the so-called Seiberg family possessing a rich flavor symmetry \[20\]; many others are obtained through embedding in string theory. Generally speaking, this embedding can be achieved in two interesting ways; either by using 5-brane webs in type IIB string theory \[21-26\]; or by using M-theory compactification on Calabi-Yau threefolds \[27-31\]. Below, we comment briefly on these two methods while giving some references which certainly are not the complete list since the works in this matter are abundant.

The method of $(p, q)$ 5-brane webs in type IIB string theory has led to several findings and has several features; in particular the following: First, it gives evidence for the existence of fix point of 5D gauge theories flowing to UV conformal points corresponding to collapsed webs; and as such permits to study conditions for existence of critical fix points. This web construction also indicates that not every 5D gauge theory can flow to a SCFT \[1\]; the existence of a SCFT constraints the matter content of the theory. The 5-brane method allows also to study gauge theory dualities in 5D. This is because a given SCFT can have several gauge theory deformations; thus generating different (but dual) gauge theories in infrared \[21\]. Also, the web method provides us with a tool to compute the instanton partition function that captures the BPS spectrum of the 5D theory by applying the topological vertex formalism \[32-37\]. It also allows the study the global symmetry enhancements of the SCFTs \[7, 38\] and UV-dualities \[21, 39-41\]. More interestingly, the 5-brane webs approach give a way to elaborate families of 5D gauge models with fix points closely related to quivers with SU gauge in the shape of Dynkin diagrams. By introducing an orientifold plane like O5-plane, the 5-brane webs can describe 5D super QFTs with flavors and gauge groups beyond SU(N) such as SO(N) and Sp(2N) \[42, 43\] as well as exceptional ones like G$_2$ \[34\].
Certain \((p, q)\) 5-brane webs have interpretation in terms of toric diagrams [23] although, for 5D gauge theories with a large number of flavors, they lead to non-toric Calabi-Yau geometries [44]. This brane based method is not used in this paper; it is described here as one of two approaches to study 5D \(\mathcal{N} = 1\) super QFTs underlying SCFT\(_5\). For works using this method, we refer to rich the literature in this matter; for instance [21]-[23],[45]-[50].

Regarding the M-theory method, to be used in this study, one can also list several interesting aspects showing that it is a powerful higher dimensional geometric approach. First of all, the 5D gauge theories are obtained by compactifying M-theory on Calabi-Yau threefolds (CY3) \(\hat{X}\) (resolved \(\hat{X}\)). Then, the effective prepotential \(\mathcal{F}_{5D}\) and its non trivial variations \(\delta^n \mathcal{F}_{5D}/\delta \phi^n\), characterising the Coulomb branch of the 5D super QFTs, have interesting CY3 interpretations; i.e. a geometric meaning in the internal dimensions. The \(\mathcal{F}_{5D}\) is given by the volume \(\text{vol}(\hat{X})\) while its variations—describing magnetic string tensions amongst others—are interpreted as volumes of p-cycles. Moreover, the calculation of \(\mathcal{F}_{5D}\) can be explicitly done for a wide class of \(\hat{X}\)’s; in particular for the family of toric Calabi-Yau threefolds like those based on the three following geometries: (a) The toric del Pezzo surfaces \(dP_n\) with \(n=1,2,3\); these Kahler manifolds are toric deformations of the complex projective plane \(\mathbb{P}^2\). (b) The Hirzebruch surfaces \(\mathbb{F}_n\) given by non trivial fibrations of a complex projective line \(\mathbb{P}^1\) over a base \(\mathbb{P}^1\) [51]-[54]. (c) The family \(\hat{X}(Y^{p,q})\) given by a crepant resolution of toric threefolds realised as real metric cone on Sasaki-Einstein \(Y^{p,q}\) spaces labeled by two positive integers \((p, q)\) constrained as \(p \geq q \geq 0\) [55]-[59].

In this investigation, we focus on the particular class of 5D supersymmetric \(\text{SU}(p)_q\) unitary field models based on \(\hat{X}(Y^{p,q})\) and look for a generalisation of these quantum field models to other gauge symmetries. Our interest into the Sasaki-Einstein (EM) based CY3s has been motivated by yet unexplored specific properties of \(Y^{p,q}\) and also by the objective of generalizing partial results obtained for the unitary family. In this context, recall that the toric 5D super QFTs based on \(\hat{X}(Y^{p,q})\) have unitary \(\text{SU}(p)_q\) gauge symmetries with Chern-Simons (CS) level \(q\). Thus, it is interesting to seek how to generalize these unitary gauge models based on \(\hat{X}(Y^{p,q})\) for other gauge symmetries like the orthogonal and the symplectic. As a first step in this exploration, we show in this study that the 5D unitary gauge theories based on \(Y^{p,q}\) have discrete symmetries that can be used to construct new gauge models. These finite groups come from symmetries of p-cycles inside the \(\hat{X}(Y^{p,q})\). By using specific properties of the unitary set and folding under outer-automorphisms of p-cycles, we construct a
new family of 5D SQFTs having symplectic $\text{SP}(2r, \mathbb{R})$ gauge invariance.

To undertake this study, it is helpful to recall some features of the Sasaki-Einstein based CY3:  

(i) They are toric and they extend the $\hat{X} (dP_1)$ and the $\hat{X} (\mathbb{F}_0)$. These geometries appear as two leading members in the $\hat{X} (Y^{p,q})$ family.  

(ii) They have been used in the past in the engineering of 4D supersymmetric quiver gauge theories [60]-[63]; and have been recently considered in models building of \textit{unitary} 5D $\mathcal{N} = 1$ super CFTs [64]-[68].  

(iii) Being toric, the threefolds $\hat{X} (Y^{p,q})$s and the \textit{unitary} 5D super QFTs based on them can be respectively represented by toric diagrams $\Delta_{\hat{X} (Y^{p,q})}$ and by BPS quivers $Q_{\hat{X} (Y^{p,q})}$ describing the BPS particle states of the unitary supersymmetric theory.

The toric $\Delta_{\hat{X} (Y^{p,q})}$ and the BPS $Q_{\hat{X} (Y^{p,q})}$ are particularly interesting because they play a central role in our construction; as such, we think it is useful to comment on them here. We split the properties of these objects in two types: general and specific. The general properties, which will be understood in this investigation, are as in the geometric engineering of 4D super QFTs [69]-[73]. They also concern aspects of the Sasaki-Einstein manifolds and the brane tiling algorithms (a.k.a dimer model) [74]-[83]. Some useful general aspects for this study are reported in the appendices A, B, C. The specific properties $\Delta_{\hat{X} (Y^{p,q})}$ and $Q_{\hat{X} (Y^{p,q})}$ regard their outer-automorphisms and the implementation of the Calabi-Yau condition of $\hat{X}$ as well as a previously unknown property of $\hat{X} (Y^{p,q})$ that we describe for the leading members $p = 2, 3, 4$. By trying to exhibit manifestly the Calabi-Yau condition on the toric diagram $\Delta_{\hat{X} (Y^{p,q})}$, we end up with the need to introduce a new graph representing $\hat{X} (Y^{p,q})$. This new graph is denoted like $G^G_{\hat{X} (Y^{p,q})}$ with $G$ referring either to the gauge symmetry $\text{SU}(p)$ or to $\text{SP}(2r, \mathbb{R})$. The construction of $G^G_{\hat{X} (Y^{p,q})}$ will be studied with details in this paper; to fix ideas, see eq(4.1) and the Figure 7, the Figure 8 and the Figure 9.

In the present paper, we contribute to the study of 5D $\mathcal{N} = 1$ super QFT models based on conical Sasaki-Einstein manifolds and their compactification on a circle with finite radius. Using the above mentioned discrete symmetries, we develop a method to build new 5D $\mathcal{N} = 1$ Kaluza-Klein quiver gauge models based on Sasaki-Einstein manifolds $Y^{p,q}$. For that, we first revisit properties of the internal $\hat{X} (Y^{p,q})$ geometries which are known to host gauge models with $\text{SU}(p)_q$ gauge symmetry. Then, we show that some of these Sasaki-Einstein based threefolds have non trivial discrete symmetries that exchange p-cycles in $\hat{X} (Y^{p,q})$ and which we construct explicitly. By using these finite symmetries and cycle-folding ideas, we build a new set of 5D supersymmetric gauge models based on $\hat{X} (Y^{p,q})$ having symplectic $\text{SP}(2r, \mathbb{R})$
gauge invariance; thus extending the set of unitary gauge models for this family of CY3. We also derive the associated BPS quivers encoding the data on the BPS states of the symplectic theory. We moreover show that the cycle-folding by outer-automorphisms generate super QFT models having no standard interpretation in terms of gauge phases. For a pedagogical reason, we mainly focus on the leading members of the symplectic $\text{SP}(2r, \mathbb{R})$ family; in particular on the 5D $\mathcal{N}=1$ super QFT with $\text{SP}(4, \mathbb{R})$ invariance. The first $\text{SP}(2, \mathbb{R})$ member is isomorphic to the 5D $\mathcal{N}=1$ SU(2) model of the unitary series. To achieve this goal, we (i) revisit the toric Calabi-Yau threefold $\hat{X}(Y^{4,0})$ (p=4 and q=0), hosting a lifted SU(4)$_0$ gauge symmetry; and (ii) reconsider the BPS quiver $Q^{SU_4}_{\hat{X}(Y^{4,0})}$ of the underlying with 5D $\mathcal{N}=1$ super QFT compactified on a circle with finite size. After that we develop an approach to construct toric Calabi-Yau threefolds with symplectic symmetry and a method to build the BPS quiver $Q^{\text{SP}_4}_{\hat{X}(Y^{4,0})}$ with $\text{SP}(4, \mathbb{R})$ invariance. The extension of this construction to other gauge symmetries is discussed in the conclusion section.

The organisation is as follows: In section 2, we review properties of the toric diagram $\Delta^{SU_4}_{\hat{X}(Y^{4,0})}$ of the Calabi-Yau threefold $\hat{X}(Y^{4,0})$. We show that $\Delta^{SU_4}_{\hat{X}(Y^{4,0})}$ has non trivial outer-automorphisms $H^{\text{outer}}_{\Delta^{SU_4}_{\hat{X}(Y^{4,0})}}$ having a fix point. We also show that this discrete group $H^{\text{outer}}_{\Delta^{SU_4}_{\hat{X}(Y^{4,0})}}$ can be interpreted as a parity symmetry in $\mathbb{Z}^2$ lattice. In section 3, we investigate the properties of the BPS quiver $Q^{SU_4}_{\hat{X}(Y^{4,0})}$ associated with $\Delta^{SU_4}_{\hat{X}(Y^{4,0})}$. Here we show that $Q^{SU_4}_{\hat{X}(Y^{4,0})}$ has also an outer-automorphism symmetry $H^{\text{outer}}_{Q^{SU_4}_{\hat{X}(Y^{4,0})}}$ with fix points. This outer-automorphism group has two factors given by $(\mathbb{Z}_4)_{Q^{SU_4}_{\hat{X}(Y^{4,0})}} \times (\mathbb{Z}_2^{\text{outer}})_{Q^{SU_4}_{\hat{X}(Y^{4,0})}}$. In section 4, we introduce a new diagram to represent the toric $\hat{X}(Y^{4,0})$. It is given by a graph $\mathcal{G}$ where the Calabi-Yau condition is manifestly exhibited. To avoid confusion, we denote this graph like $\mathcal{G}^{SU_4}_{\hat{X}(Y^{4,0})}$ and refer to it as the unitary CY graph of the toric $\hat{X}(Y^{4,0})$ with SU(4) gauge symmetry. To deepen the construction, we also give the unitary CY graphs $\mathcal{G}^{SU_2}_{\hat{X}(Y^{2,0})}$ and $\mathcal{G}^{SU_3}_{\hat{X}(Y^{3,0})}$ representing the toric $\hat{X}(Y^{p,0})$ with p=2 and p=3. In section 5, we construct the symplectic CY graph $\mathcal{G}^{\text{SP}_4}_{\hat{X}(Y^{4,0})}$ and the associated symplectic BPS quiver $Q^{\text{SP}_4}_{\hat{X}(Y^{4,0})}$. In section 6, we give a conclusion and make comments. In the appendix, we give useful properties on the geometric properties of the Coulomb branch of M-theory on CY3s and describe the building of BPS quivers.
2 Conical Sasaki-Einstein threefold \( \hat{X}(Y^{4,0}) \)

We begin by recalling that the Calabi-Yau threefold \( \hat{X}(Y^{4,0}) \), taken as a real metric cone over the 5d Sasaki-Einstein variety \([84, 85]\), is a toric complex 3d manifold whose toric diagram \( \Delta_{\hat{X}(Y^{4,0})}^{SU_4} \) is a finite sublattice of \( \mathbb{Z}^3 \) as in the Figure [1]. This toric \( \Delta_{\hat{X}(Y^{4,0})}^{SU_4} \) has seven points given by: (a) Four external points \( W_1, W_2, W_3, W_4 \) defining the geometry on which rests the singularity of the SU(4) gauge fiber. (b) Three internal points \( V_1, V_2, V_3 \) describing the crepant resolution of the singularity. This resolution can be imagined as an intrinsic sub-geometry of the toric \( \hat{X}(Y^{4,0}) \) to which we often refer to as the fiber geometry.

2.1 Toric diagram \( \Delta_{\hat{X}(Y^{4,0})}^{SU_4} \) and divisors of \( \hat{X}(Y^{4,0}) \)

The four above mentioned \( W_i \) points (i=1,2,3,4) of the toric diagram \( \Delta_{\hat{X}(Y^{4,0})}^{SU_4} \) can be also interpreted as associated with four non compact divisors \( D_i \) of the toric \( \hat{X}(Y^{4,0}) \). Similarly, the three internal points \( V_a \) (a = 1, 2, 3) are interpreted as corresponding to three divisors \( E_a \) of the toric \( \hat{X}(Y^{4,0}) \); but with the difference that the three \( E_a \)'s are compact complex 2d surfaces. In terms of the classes of these divisors, the Calabi-Yau condition of the toric \( \hat{X}(Y^{4,0}) \) is given by the vanishing sum; see also appendix A,

\[
\sum_{i=1}^{4} D_i + \sum_{a=1}^{3} E_a = 0 \tag{2.1}
\]

This homological condition is implemented at the level of the toric diagram by restricting the seven points of \( \Delta_{\hat{X}(Y^{4,0})}^{SU_4} \) to sit in the same hyperplane by taking the external like \( W_i = (w_i, 1) \) and the internal points as \( V_a = (v_a, 1) \) with \( w_i \) and \( v_a \) belonging to \( \mathbb{Z}^2 \). A particular realisation of the seven points of \( \Delta_{\hat{X}(Y^{4,0})}^{SU_4} \) is given by The toric diagram \( \Delta_{\hat{X}(Y^{4,0})} \)

| \( \Delta_{\hat{X}(Y^{4,0})} \) | \( w_1 \) | \( w_2 \) | \( w_3 \) | \( w_4 \) | \( v_1 \) | \( v_2 \) | \( v_3 \) |
|---|---|---|---|---|---|---|---|
| points | \((-1, 4)\) | \((0, 0)\) | \((1, 0)\) | \((0, 4)\) | \((0, 1)\) | \((0, 2)\) | \((0, 3)\) |
| divisors | \( D_1 \) | \( D_2 \) | \( D_3 \) | \( D_4 \) | \( E_1 \) | \( E_2 \) | \( E_3 \) |

Table 1: Toric data of the Calabi-Yau threefold \( \hat{X}(Y^{4,0}) \).

representing the resolved Calabi-Yau threefold \( \hat{X}(Y^{4,0}) \) is depicted by the Figure [1] where a triangulation the surface of \( \Delta_{\hat{X}(Y^{4,0})} \) is highlighted [86]. It describes the lifting of the \( A_3 \cong SU(4) \) singularity. Notice that Table [1] is the data for \((p, q) = (4, 0)\) saturating the
Figure 1: The toric diagrams $\Delta^{SU_4}_{X(Y_{4,0})}$ having four external points (two blue and two green) and three internal points (in red). These red points are associated with the lifting of the SU(4) singularity of the gauge fiber. The surface of $\Delta^{SU_4}_{X(Y_{4,0})}$ is divided into 4+4 triangles. By merging the red points into the first point, one is left with 4 triangles.

lower bound of the constraint $0 \leq q \leq 4$. For generic values of $q$ constrained like $0 \leq q \leq p$, we have the following data: This toric $\Delta^{X}_{(Y_{p,q})}$ has 3+p points and then 3+p divisors; p-1 of them are compact. They concern the divisor set $\{E_a\}_{1 \leq a \leq p-1}$. Notice also that the three internal (red) points of $\Delta^{X}_{(Y_{4,0})}$ represented by the Figure 1 form a (vertical) linear chain $A_3$ in the toric diagram with boundary points effectively given by the two (blue) external $w_2$ and $w_4$. For convenience, we rename these two particular boundary points like $w_2 = \nu_0$ and $w_4 = \nu_4$ so that the above mentioned chain $A_3$ can be put in correspondence with the standard $A_3$-geometry of the ALE space with resolved SU(4) singularity [87, 88]. With this renaming, the Table 1 gets mapped to a similar description can be done for $\Delta^{X}_{(Y_{p,q})}$. For simplicity of the presentation, we omit it. Having introduced the particular toric diagram $\Delta^{SU_4}_{X(Y_{4,0})}$ hosting an underlying unitary SU(4) gauge symmetry, we turn now to explore one of its exotic properties namely its outer-automorphism symmetries.

| $\Delta^{X}_{(Y_{p,q})}$ | $w_1$   | $w_2$   | $w_3$   | $w_4$   | $\{\nu_a\}_{1 \leq a \leq p-1}$ |
|--------------------------|---------|---------|---------|---------|-----------------------------------|
| points                   | $(-1, p-q)$ | $(0,0)$ | $(1,0)$ | $(0,p)$ | $(0,a)$                           |
| divisors                 | $D_1$   | $D_2$   | $D_3$   | $D_4$   | $\{E_a\}_{1 \leq a \leq p-1}$   |

Table 2: Toric data of $X(Y_{p,q})$ with $0 \leq q \leq p$. 
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$\check{X}(Y^{4,0})$ & A$_3$- geometry & transverse geometry \\
\hline
$\Delta_{\check{X}(Y^{4,0})}$ & $v_0$ & $v_1$ & $v_2$ & $v_3$ & $v_4$ & $w_1$ & $w_3$ \\
\hline
points & (0, 0) & (0, 1) & (0, 2) & (0, 3) & (0, 4) & (-1, 4) & (1, 0) \\
\hline
\end{tabular}
\caption{Toric data of the A$_3$- geometry within the resolved Calabi-Yau threefold $\check{X}(Y^{4,0})$.}
\end{table}

\section{2.2 Outer-automorphisms of $\Delta_{\check{X}(Y^{4,0})}^{SU_4}$}

A careful inspection of the Figure reveals that the toric diagram $\Delta_{\check{X}(Y^{4,0})}^{SU_4}$ has outer-automorphism symmetries forming a discrete group $H_{\text{outer}}^{\Delta_{\check{X}(Y^{4,0})}^{SU_4}}$. This is a finite symmetry group generated by the following transformations of the external point $w_i$ and the internal $v_a$,

$$H_{\text{outer}}^{\Delta_{\check{X}(Y^{4,0})}^{SU_4}}: w_1 \leftrightarrow w_3, \quad w_2 \leftrightarrow w_4, \quad v_a \leftrightarrow v_{4-a}$$

(2.2)

Notice that the outer-automorphisms in the gauge fiber act by exchanging the two internal $v_1 \leftrightarrow v_3$; but fix the central point $v_2$. This property is interesting; it will be used later on to engineer a new gauge fiber. By using the parametrisation $w_i = (w^x_i, w^y_i)$ and $v_a = (v^x_a, v^y_a)$, we learn that the outer-automorphism group $H_{\text{outer}}^{\Delta_{\check{X}(Y^{4,0})}^{SU_4}}$ is given by the product of two reflections like

$$H_{\text{outer}}^{\Delta_{\check{X}(Y^{4,0})}^{SU_4}} = (\mathbb{Z}_2^x)^{\Delta_{SU_4}} \times (\mathbb{Z}_2^y)^{\Delta_{SU_4}}$$

(2.3)

with

$$(\mathbb{Z}_2^x)^{\Delta_{SU_4}}: w_1^x \rightarrow -w_1^x, \quad w_3^x \rightarrow -w_3^x, \quad v_a^x \rightarrow -v_a^x$$

$$(\mathbb{Z}_2^y)^{\Delta_{SU_4}}: w_1^y \rightarrow w_3^y, \quad w_3^y \rightarrow w_1^y, \quad v_a^y \rightarrow v_{4-a}^y$$

(2.4)

Form these outer-automorphism transformations, we learn that $(\mathbb{Z}_2^x)^{\Delta_{SU_4}}$ acts trivially on the internal points $v_a$ of the A$_3$- linear chain of $\Delta_{\check{X}(Y^{4,0})}^{SU_4}$. So the group $(\mathbb{Z}_2^x)^{\Delta_{SU_4}}$ leaves invariant the A$_3$- gauge fiber within the toric Calabi-Yau $\check{X}(Y^{4,0})$. It affects only the external points $w_1$ and $w_3$ which are associated with the transverse geometry shown in the table 3. Regarding the $(\mathbb{Z}_2^y)^{\Delta_{SU_4}}$ reflection, it acts non trivially on the points of the A$_3$-chain; we have:

$$(\mathbb{Z}_2^y)^{\Delta_{SU_4}}: v_a \rightarrow v_{4-a}$$

(2.5)

Under this mirror symmetry, the A$_3$- gauge fiber has then a fix point which is an interesting feature that we want to exploit to build a new gauge fiber by using folding ideas \cite{89,90,69,70}. In this regards, recall that the $(\mathbb{Z}_2^y)^{\Delta_{SU_4}}$ action looks like a well known outer-automorphism symmetry group $\mathbb{Z}_2$ that we encounter in the folding of the Dynkin diagrams
of the finite dimensional Lie algebras $A_{2r-1}$. Here, we are dealing with the particular $A_3 \sim SU(4)$ which is just the leading non-trivial member of the $A_{2r-1}$ series. As an illustration; see the pictures of the Figure 2 describing the folding of the Dynkin diagram $A_3$ giving the Dynkin diagram of the symplectic $C_2 \simeq sp(4,\mathbb{R})$ which, thought not relevant for our present study, it is also isomorphic to $B_2 \simeq so(5)$. Recall as well that the Dynkin diagrams

Figure 2: a) The Dynkin diagram of the Lie algebra $A_3$. It has a mirror $(\mathbb{Z}_2)_{SU_4}$ outer-automorphism symmetry leaving one node fixed (in magenta color). b) The Dynkin diagram of the symplectic Lie algebra $C_2$. It is obtained by folding $A_3$ under $(\mathbb{Z}_2)_{SU_4}$.

of finite dimensional Lie algebras $g$ may be also thought of in terms of the Cartan matrices $K(g)_{ij} = \alpha_i^\vee \alpha_j$ defined by the intersection of simple roots $\alpha_i$ and co-roots $\alpha_i^\vee = \alpha_i/\alpha_i^2$. For the examples of $A_3 \simeq su(4)$ and $C_2 \simeq sp(4,\mathbb{R})$, we have the following matrices

$$K(A_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad K(C_2) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

(2.6)

Notice that the picture on the left of the Figure 2 can be put in correspondence with the internal (red) points of the $A_3$-linear chain of the Figure 1. At this level, one may ask what about toric diagrams with a $C_2$-type sub-diagram. We will answer this question later on after highlighting another property of $\Delta_{SU_4}^{X(Y^{4,0})}$. Before that, let us describe succinctly the BPS quivers associated with the toric diagram of $\hat{X}(Y^{4,0})$; and study its outer-automorphisms.

3 BPS quiver $Q^{SU_p}_{\hat{X}(Y^{p,0})}$: cases $p = 3, 4$

In this section, we investigate two examples of unitary BPS quivers namely the $Q_{SU_3}^{X(Y^{3,0})}$ and $Q_{SU_4}^{X(Y^{4,0})}$. These unitary BPS quivers are representatives of the families $Q_{SU_{2r-1}}^{X(Y^{2r-1,0})}$ and $Q_{SU_{2r}}^{X(Y^{2r,0})}$ with $r \geq 1$. They have intrinsic properties that we want to study and which will be used later on. First, we consider the quiver $Q_{SU_4}^{X(Y^{4,0})}$ with gauge symmetry $SU(4)$ as this quiver is one of the main graphs that interests us in this study. Then, we turn to the BPS
quiver $Q^{SU_3}_{\hat{X}(Y^{3,0})}$ with unitary symmetry SU(3). The $Q^{SU_3}_{\hat{X}(Y^{3,0})}$ quiver is reported here for a matter of comparison with $Q^{SU_4}_{\hat{X}(Y^{4,0})}$. The results obtained for these quivers hold as well for the families $Q^{SU_{2r-1}}_{\hat{X}(Y^{2r-1,0})}$ and $Q^{SU_{2r}}_{\hat{X}(Y^{2r,0})}$.

### 3.1 BPS quiver $Q^{SU_4}_{\hat{X}(Y^{4,0})}$

The construction of the unitary BPS quiver $Q^{SU_4}_{\hat{X}(Y^{4,0})}$ of the 5D $N=1$ super QFTs, compactified on a circle with finite size and based on $\hat{X}(Y^{4,0})$, follows from the brane tiling of the so called brane-web $\tilde{\Delta}^{SU_4}_{\hat{X}(Y^{4,0})}$ (the dual of the toric diagram $\Delta^{SU_4}_{\hat{X}(Y^{4,0})}$) by applying the fast inverse algorithm [92, 93, 94]. Up to a Seiberg-type duality transformation, the representative $Q^{SU_4}_{\hat{X}(Y^{4,0})}$ has a quiver-dimension $d_{bps}$ equals to $2 \times 3 + 2$. Then the $Q^{SU_4}_{\hat{X}(Y^{4,0})}$ has 8 elementary BPS particles that generate the BPS spectrum of the 5D super QFT. For further details; see the appendices A and B. To fix ideas, let us illustrate the numbers involved in the $d_{bps}$ dimension which for $Q^{SU_p}_{\hat{X}(Y^{p,0})}$ with $p \geq 2$ reads as follows; see also eqs (B.4-B.3),

$$d_{bps} = 2 (p - 1) + 2$$  \ (3.1)

(i) the number $3 = 4 - 1$ is precisely the rank of the SU(4) gauge fiber within the toric $\hat{X}(Y^{4,0})$. It is also the number of compact divisors $-E_1, E_2, E_3$— of the threefold $\hat{X}(Y^{4,0})$.  
(ii) The product $2 \times 3 = 6$ designates the number of the electric/magnetic charged particles. These 3+3 particles have interpretation in terms of M2- and M5-branes wrapping 2- and 4-cycles in the internal threefold $\hat{X}(Y^{4,0})$. (iii) The extra number $2=1+1$ in the $d_{bps}$-dimension of $Q^{SU_4}_{\hat{X}(Y^{4,0})}$ refers to an instanton and to the elementary Kaluza-Klein D0 brane; for more details see [64, 65, 66] and the appendix A.

The schematic structure of the BPS quiver $Q^{SU_4}_{\hat{X}(Y^{4,0})}$ is depicted by the Figure 3. It has 8 nodes \{1\}, ..., \{8\} interpreted in terms of 8 elementary BPS particles. These nodes organise into four Kronecker quivers (4 doublets of nodes) denoted like $\kappa_c = \{2c - 1, 2c\}_{1 \leq c \leq 4}$; explicitly, we have:

$$\kappa_1 = \{1, 2\} \ , \ \kappa_3 = \{5, 6\}$$

$$\kappa_2 = \{3, 4\} \ , \ \kappa_4 = \{7, 8\}$$  \ (3.2)

As shown by the Figure 3, the 8 nodes of the BPS quiver are linked by $4 \times 4 = 16$ quiver-edges $\langle j | l \rangle$ interpreted in terms of chiral superfields in the language of supersymmetric quantum mechanics (SQM) [65]. The unitary BPS quiver $Q^{SU_4}_{\hat{X}(Y^{4,0})}$ has been first considered in [66] (see figure 25-a, page 61). For later use, we re-draw the Figure 3 as depicted by the equivalent
Figure 3: BPS quiver $Q_{\hat{X}(Y^4,0)}^{SU_4}$ associated with $\hat{X}(Y^4,0)$. Its elementary BPS states are represented by the 8 nodes and are linked by 16 edges. The shape of this BPS quiver has been borrowed from the paper [66]—Figure 25-(a)—. The subdiagram with 6 blue nodes and 10 blue arrows refer to the 4D subquiver.

Figure 4: The BPS quiver is $Q_{\hat{X}(Y^4,0)}^{SU_4}$ has eight nodes organised into 4 vertical pairs $\{2a - 1; 2a\}$. In this representative chain, the first pair and the last one are given by $\{7; 8\}$. They should be identified as they concern the same nodes’ pair.
In this redrawing, we have represented the Kronecker quiver \( \{7, 8\} \) twice. This way of doing allows to think of the \( Q_{\hat{X}(Y, 0)}^{\text{SU}_4} \) as a periodic chain of Kronecker quivers with periodicity generated by a \((\mathbb{Z}_4)_{Q_{\text{SU}_4}} \) outer-automorphism symmetry acting (i) on the quiver-nodes \( \{2c - 1\}_{1 \leq c \leq 4} \) and \( \{2c\}_{1 \leq c \leq 4} \) as follows
\[
(Z_4)_{Q_{\text{SU}_4}} : \begin{cases} 
2c - 1 & \rightarrow 2c + 7 \\
2c & \rightarrow 2c + 8
\end{cases}
\] (3.3)
and (ii) on the Kronecker quivers like \( \kappa_c \rightarrow \kappa_{c+4} \). These outer-automorphisms, which act also on the oriented arrows, have no fix node and no fix arrow. They play a secondary role in our construction.

In addition to \((\mathbb{Z}_4)_{Q_{\text{SU}_4}}\), the unitary BPS quiver \( Q_{\hat{X}(Y, 0)}^{\text{SU}_3} \) has another outer-automorphism group factor namely \((\mathbb{Z}_{\text{outer}})_2^{\text{SU}_4}\). It acts as a reflection symmetry mirroring nodes and exchanging oriented arrows. Contrary to \((\mathbb{Z}_4)_{Q_{\text{SU}_4}}\), the mirror \((\mathbb{Z}_{\text{outer}})_2^{\text{SU}_4}\) has the remarkable property of fixing four quiver-nodes and the associated arrows. It acts on the Kronecker quivers as
\[
(Z_2^{\text{outer}})_{Q_{\text{SU}_4}} : \kappa_c \rightarrow \kappa_{4-c}
\] (3.4)
thus exchanging \( \kappa_1 \leftrightarrow \kappa_3 \); but fixing \( \kappa_2 \) and \( \kappa_4 \) since \( \kappa_4 \equiv \kappa_0 \) due to the periodicity property \( \kappa_c \simeq \kappa_{c+4} \); thanks to \((\mathbb{Z}_4)_{Q_{\text{SU}_4}}\). By denoting the eight nodes like \( \{1\}, \ldots, \{8\} \), the \((\mathbb{Z}_{\text{outer}})_2^{\text{SU}_4}\) group is then generated by the double transposition \( \{1\} \{5\} \circ \{2\} \{6\} \). Below, we refer to this double transposition simply as \((15)(26)\); so, we have:
\[
(Z_2^{\text{outer}})_{Q_{\text{SU}_4}} = \{ s = (15)(26) \mid s^2 = I \}
\] (3.5)
From this description, we learn two interesting things: First, the \((\mathbb{Z}_{\text{outer}})_2^{\text{SU}_4}\) is a particular subgroup of the symmetric (permutation) group \( S_8 \) of eight elements (nodes) \( \{1, \ldots, 8\} \). Second the \((\mathbb{Z}_4)_{Q_{\text{SU}_4}}\) is also a subgroup of \( S_8 \); it generated by the product of two 4-cycles as follows,
\[
(Z_4)_{Q_{\text{SU}_4}} = \{ t = (1357)(2468) \mid t^4 = I \}
\] (3.6)
So, both \((\mathbb{Z}_{\text{outer}})_2^{\text{SU}_4}\) and \((\mathbb{Z}_4)_{Q_{\text{SU}_4}}\) are subgroups of the enveloping \( S_8 \). Similar outer-automorphism groups can be written down for the family \( Q_{\hat{X}(Y^{2r, 0})}^{\text{SU}_2} \) with \( r \geq 2 \).

### 3.2 BPS quiver \( Q_{\hat{X}(Y^{3, 0})}^{\text{SU}_3} \)

Here, we study the BPS quiver \( Q_{\hat{X}(Y^{3, 0})}^{\text{SU}_3} \) and some of its outer-automorphisms in order to compare with \( Q_{\hat{X}(Y, 0)}^{\text{SU}_4} \). The BPS quiver \( Q_{\hat{X}(Y^{3, 0})}^{\text{SU}_3} \) with gauge symmetry SU(3) has a quite
similar structure as $Q^{SU_4}_{X(Y^{4,0})}$; but a different quiver dimension which is given by

$$d_{bps} = 2 (p - 1) + 2 = 2 \times 2 + 2 = 6$$

(3.7)

As such, the BPS quiver $Q^{SU_3}_{X(Y^{3,0})}$ has six nodes $\{1\}, \ldots, \{6\}$ interpreted in terms of 6 elementary BPS particles. They organise into three Kronecker quivers namely

$$\kappa_1 = \{1, 2\} \quad \kappa_2 = \{3, 4\} \quad \kappa_3 = \{5, 6\}$$

(3.8)

This BPS quiver has 12 oriented arrows as depicted by the Figure 5. Though not very important for our present study as it cannot induce a BPS quiver with symplectic gauge symmetry, notice that the quiver $Q^{SU_3}_{X(Y^{3,0})}$ has also outer-automorphism symmetries forming a group $H_{Q^{SU_3}}^{\text{outer}}$ with two factors as given below

$$H_{Q^{SU_3}}^{\text{outer}} = \left(\mathbb{Z}_2^{\text{outer}}\right)_{Q^{SU_3}} \times \left(\mathbb{Z}_3\right)_{Q^{SU_3}}$$

(3.9)

The factor $\left(\mathbb{Z}_2^{\text{outer}}\right)_{Q^{SU_3}}$ exchanges the nodes $\{1\} \leftrightarrow \{3\}$ and $\{2\} \leftrightarrow \{4\}$; but it fixes the nodes $\{5\}$ and $\{6\}$ as clearly seen on the Figure 5. In terms of the Kronecker quivers, we have $\kappa_1 \leftrightarrow \kappa_2$ but $\kappa_3 \leftrightarrow \kappa_3$. So, the outer-automorphisms of $Q^{SU_3}_{X(Y^{3,0})}$ are different from those of the quiver $Q^{SU_4}_{X(Y^{4,0})}$ which are given by Figure 4. Recall that the $H_{Q^{SU_4}}^{\text{outer}}$ has four fix nodes instead of two for $Q^{SU_3}_{X(Y^{3,0})}$; i.e: two Kronecker quivers for $H_{Q^{SU_4}}^{\text{outer}}$, against one Kronecker

![Figure 5: BPS quiver $Q^{SU_3}_{X(Y^{3,0})}$ associated with $\hat{X}(Y^{3,0})$. Its elementary BPS states are represented by the 6 nodes and are linked by 12 edges. The shape of this BPS quiver has been borrowed from the paper [66]—Figure 22-(a) page 58—. The subgraph with four blue points and 6 orientied arrows correspond to the quiver of the 4D super QFT.](image-url)
quiver for $H_{Q_{SU_3}}^{outer}$. This difference holds as well for generic quivers $Q_{X(Y^{2r},0)}^{SU_2}$ and $Q_{X(Y^{2r-1},0)}^{SU_2}$ with respective outer-automorphism groups $H_{Q_{SU_2}}^{outer}$ and $H_{Q_{SU_2}}^{outer}$.

Regarding the factor $(\mathbb{Z}_3)_{Q_{SU_3}}$, it allows to represent the quiver $Q_{X(Y^{3,0})}^{SU_3}$ as a periodic chain as depicted by the Figure 6.

Figure 6: BPS quiver $Q_{X(Y^{3,0})}^{SU_3}$ associated with $\hat{X}(Y^{3,0})$. Its elementary BPS states are represented by the 6 nodes and are linked by 12 edges. The outer-automorphism group $(\mathbb{Z}_2)_{Q_{SU_3}}^{outer}$ has no fix point.

4 Graphs $G_{\hat{X}(Y_{p,q})}$ with manifest CY condition

In this section, we introduce a new graph to deal with the toric diagram $\Delta_{\hat{X}(Y^{4,0})}^{SU_4}$ with $p=4$ representing the Calabi-Yau threefold $\hat{X}(Y^{4,0})$ with a resolved SU(4) gauge fiber. We refer to this new graph as the unitary Calabi-Yau graph and we denote it like $G_{\hat{X}(Y^{4,0})}^{SU_4}$. This graph is explicitly defined by $p - 1$ vector $q^b$ with components given by the triple intersection numbers

$$q^b_A = D_A E_b^2 \quad (4.1)$$

where the label $A = (i, a)$ with $i = 1, 2, 3, 4$, for non compact divisors $D_i$, and $a = 1, 2, 3$ for the compact $E_a$. Below, we refer to these $q^b$’s as generalised Mori-vectors. Though this CY graph $G_{\hat{X}(Y^{4,0})}^{SU_4}$ looks formally different from the toric diagram, it is in fact equivalent to it. It is just another way to deal with $\Delta_{\hat{X}(Y^{4,0})}^{SU_4}$ where the Calabi-Yau condition is manifestly exhibited. As we will show below, this is useful in looking for solutions of underlying constraint relations required by the toric threefold $\hat{X}(Y^{4,0})$.

4.1 Building the CY graph $G_{\hat{X}(Y^{4,0})}^{SU_4}$

To engineer the unitary Calabi-Yau graph $G_{\hat{X}(Y^{4,0})}^{SU_4}$ of the toric $\hat{X}(Y^{4,0})$, we start form the Calabi-Yau condition given by eq(2.1) namely $\sum_{i=1}^{4} D_i + \sum_{a=1}^{3} E_a = 0$. This constraint
relation is expressed in terms of the four non compact divisors $D_i$ and the three compact $E_a$; but it is not the only constraint that must be obeyed by the divisors. There are two other constraints that must be satisfied by the divisors. So, the seven divisors $(D_i, E_a)$ of the toric Calabi-Yau threefolds $\hat{X}(Y^{4,0})$ are subject to three basic constraints. They can be collectively expressed as 3-vector equation like

$$ \sum_{i=1}^{4} W_i D_i + \sum_{a=1}^{3} V_a E_a = 0 \quad (4.2) $$

where $W_i = (w_i, 1)$ and $V_a = (v_a, 1)$ are as in Table 1. To deal with the CY constraint eq(2.1), we bring it to a relation between triple intersection numbers $I_{ABC} = D_A . D_B . D_C$ with $D_A$ standing for the seven $(D_i, E_a)$. Multiplying formally both sides of eq(2.1) by $E_b^2 = E_b . E_b$ with $b = 1, 2, 3$, we obtain the following relationships between the triple intersection numbers,

$$ \sum_{i=1}^{4} (D_i . E_b^2) + \sum_{a=1}^{3} (E_a . E_b^2) = 0 \quad (4.3) $$

These three relationships can be put into two convenient expressions; either as

$$ \sum_{i=1}^{4} J^b_i + \sum_{a=1}^{3} T^b_a = 0 \quad (4.4) $$

where we have set $J^b_i = D_i . E_b^2$ and $T^b_a = E_a . E_b^2$; or into a more familiar form like

$$ \sum_{A=1}^{7} q^b_A = 0 \quad (4.5) $$

with $q^b_A = (J^b_i, T^b_a)$ standing three generalised Mori-vectors denoted below like $q^b$ ($b = 1, 2, 3$). The second expression is precisely the relation that we have in gauged linear sigma model (GLSM) realisation of toric Calabi-Yau threefolds [95]. Regarding the CY relation (4.5), notice that it is quite similar to the well known relation

$$ \sum_{A=0}^{r} (Q^b_A)_{ADE} = 0 \quad (4.6) $$
giving the CY condition we encounter in the study of complex 2d ADE surfaces describing the resolution of ALE spaces with ADE singularities. In this regards, recall that these complex ADE surfaces play a central role in the geometric engineering of 4D $\mathcal{N} = 2$ super QFTs from type IIA string on Calabi-Yau threefolds given by ADE geometries fibered over the complex line $\mathbb{C}$ [29, 30, 87, 96, 97, 98]. For these ADE geometries which can be imagined in terms
of orbifolds \( \mathbb{C}^2/\Gamma \) with \( \Gamma \) a discrete subgroup in \( \text{SU}(2) \), the expression the (Mori-) vectors \( Q^b_{ADE} = (Q^b_A)_{ADE} \) can be written down. For the example of the complex \( A_3 \) surface, the three Mori-vectors read as follows

\[
(Q^a)_{SU_4} = \begin{pmatrix}
1 & -2 & 1 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 \\
0 & 0 & 1 & -2 & 1
\end{pmatrix}
\]

where the Cartan matrix \( K(SU_4) \) of the Lie algebra of the \( \text{SU}(4) \) gauge symmetry appears as a square sub-matrix of the above \( (Q^a)_{SU_4} \). Recall that \( K(SU_4) \) is given by

\[
K(SU_4) = \begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -2
\end{pmatrix}
\]

For the case of the CY graph \( \mathcal{G}^{SU_4}_{X(Y^{4,0})} \) we are interested in this study, and depicted by the Figure 7, the three generalised Mori-vectors \( (q^a)_{SU_4} \) are given by

![Figure 7](image_url)

Figure 7: The graph of the CY threefolds geometry \( X(Y^{4,0}) \) exhibiting manifestly the CY condition at each internal point. This graph has three (red) compact 4-cycles \( E_1, E_2, E_3 \), each with triple self intersection \((-8)\), intersecting four non compact (blue) 4-cycles. The Calabi-Yau threefolds condition is ensured by the vanishing sum of the total charge at each red exceptional node. The underlying SQFT has an \( \text{SU}(4) \) gauge symmetry. Notice also that this graph has a remarkable outer-automorphism symmetry to used later on.
This is a 3×7 rectangular matrix that contains the square 3×3 sub-matrix \( I_a^b \) defined like \( E_a^b \) and reading as follows

\[
\begin{pmatrix}
E_1 & E_2 & E_3 \\
-8 & 2 & 0 \\
2 & -8 & 2 \\
0 & 2 & -8
\end{pmatrix}
\]

(4.10) The diagonal terms \( I_a^a \) describe precisely the triple self intersections of the compact divisors namely \( E_a^3 = -8 \). The off diagonal terms \( I_a^b \) describes the intersection between the compact divisor \( E_a \) and the compact curve \( E_2^b \).

As eqs (4.9) and (4.10) are one of the results of this study, it is interesting to comment them by describing their content and exploring their relationship with ADE Dynkin diagrams. These comments are as listed below:

(1) First, recall that \( E^3 = -8 \) is the triple self intersection of the Hirzebruch surface \( \mathbb{F}_0 \) given by a complex projective curve \( \mathbb{P}^1 \) fibered over another \( \mathbb{P}^1 \). So, eq (4.10) describes three \( (\mathbb{F}_0)_1, (\mathbb{F}_0)_2 \) and \( (\mathbb{F}_0)_3 \) intersecting transversally. The cross intersection described by (4.10) concerns \( (\mathbb{F}_0)_a^b \cdot C_b \) with \( C_b = (\mathbb{F}_0)_b^2 \) and \( a \neq b \).

(2) The quantity \( q_A^b = D_A \cdot E_b^2 \) given by (4.11) describes the graph of the toric Calabi-Yau threefolds \( \hat{X}(Y^{4,0}) \). This quantity is interesting from various views; in particular the three following:

(i) The \( q_A^b \) of the Calabi-Yau threefolds \( \hat{X}(Y^{4,0}) \) is the analogue of \( Q_A^b \) eq (4.7) associated with Calabi-Yau twofolds (CY2). Then the \( q_A^b \), concerning 4-cycles, can be imagined as a generalisation of the Mori vector \( Q_A^b \) dealing with 2-cycles. As such \( q_A^b \) and \( Q_A^b \) can be put in correspondence. This link is also supported by the fact that both \( q_A^b \) and \( Q_A^b \) are based on SU(4) and both obey the CY condition namely \( \sum q_A^b = 0 \) (4.5) and \( \sum Q_A^b = 0 \) (4.6).

(ii) As for \( q_A^b \) describing the toric \( \hat{X}(Y^{4,0}) \), with graphic representation given by the Figure 7, the \( Q_A^b \) describes also a toric CY2 surface \( \hat{Z}_{SU_4} \). This complex surface also has a graphic representation formally similar to the vertical line of the Figure 7, that is the line containing the red nodes. Recall that \( \hat{Z}_{SU_4} \) is given by the resolution of ALE space \( \mathbb{C}^2/\mathbb{Z}_4 \). The compact
part of the associated toric diagram is given by the Figure 2-a where the nodes describe three intersecting $\mathbb{CP}^1$ curves.

(iii) The above comments done for SU(4) holds in fact for the full SU($p$) family with $p \geq 2$. So, the correspondence between $q_A^b$ and $Q_A^b$ is a general property valid for SU($p$)$_0$ gauge models in 5D. This correspondence holds also for the intersection matrices $I(SU_p)_a$ and $K(SU_p)_a$ associated with the compact parts in $q_A^b$ and $Q_A^b$ respectively. However, the graph of $K_{ab}$ is just the Dynkin diagram of the Lie algebra of SU(4). In this regards, recall that we have $K_{ab} = \alpha_a^\vee \cdot \alpha_b$ where the $\alpha_a$'s stand for the simple roots and the $\alpha_a^\vee = 2\alpha_a/\alpha_a^2$ for the co-roots. Clearly for SU($p$), we have $\alpha_a^2 = 2$. The Cartan matrix $K_{ab}$ has also an interpretation in terms of intersecting 2-cycles $C^{(2)}_a$ in the second homology group $H_2$; that is $C^{(2)}_a \cdot C^{(2)}_b = -K_{ab}$.

From this description, a natural question arises. Could the intersection matrix (4.10), thought of as $I_{ab} = E_a^\vee \cdot E_b$ with $E_a^\vee = E_a^2$, also has a similar algebraic interpretation as $K_{ab} = \alpha_a^\vee \cdot \alpha_b$? For example, could $I_{ab}$ be a generalized Cartan matrix $K_{ab}^{(gen)}$ or a cousin object of $K_{ab}^{(gen)}$? In this regards, notice that like for $-K_{SU4}$, the matrix $-I_{SU4}$ is an integer matrix with positive entries on the diagonal and non positive off diagonal entries. At first sight one might suspect this matrix to be a generalised Cartan matrix. However, though it is symmetric and has a positive determinant $-\det(-I_{SU4}) > 0$, we have not found an algebraic interpretation of this matrix. It is not either a generalised Cartan matrix of Borcherds type. Progress in this direction will be reported in a future occasion.

We end this section by noticing that eq(4.9) is a particular solution of the Calabi-Yau condition (4.5). It relies on the equality

$$E_a \cdot E_b^2 = E_a^2 \cdot E_b \quad \Leftrightarrow \quad E_a^\vee \cdot E_b = E_b^\vee \cdot E_a$$

Other solutions of $\sum_A q_A^b = 0$ violating the above symmetric property can be also written down; they are omitted here.

### 4.2 Leading members of the $G_{\hat{X}(Y_{p,0})}^{SU_p}$ family

In the above subsection, we have focussed on the CY graph $G_{\hat{X}(Y_{4,0})}^{SU_4}$ given by the Figure 7 which is based on exhibiting the triple intersection numbers of the compact divisors $E_1, E_2, E_2$ amongst themselves and with the non compact $D_1, D_2, D_3, D_4$. This CY graph is however the third member of the family $G_{\hat{X}(Y_{p,0})}^{SU_p}$ with $p \geq 2$. Below, we give comments on the two
The first member of the $\mathcal{G}_{X(Y^2,0)}^{SU_2}$ family is given by $\mathcal{G}_{X(Y^2,0)}^{SU_2}(p=2)$. It has four (external) non compact divisors $D_1, D_2, D_3, D_4$; but only one internal compact divisor that we denote $E_0$. So, there is one generalised Mori-vector given by

$$
(q)_{SU_2} = \begin{pmatrix}
D_1 & D_2 & E_0 & D_3 & D_4 \\
2 & 2 & -8 & 2 & 2
\end{pmatrix}
$$

(4.12)

where the CY condition, given by the vanishing of the trace of $(q)_{SU_2}$, is manifestly exhibited. The diagram representing the CY graph $\mathcal{G}_{X(Y^2,0)}^{SU_2}$ is given by the picture on the right side of the Figure 8. On the left side of this figure, we have given the picture of the standard $A_1$ geometry of ALE space involving complex projective curves with self intersection $-2$. Notice that the Calabi-Yau threefold $\hat{X}(Y^2,0)$ is precisely $\hat{X}(\mathbb{F}_0)$, the toric threefold based on the Hirzebruch surface $\mathbb{F}_0$ which is known to have a triple self intersection $(-8)$. Notice also that this graph has outer-automorphisms given by the mirror $(\mathbb{Z}_2^x)_{\Delta SU_2} \times (\mathbb{Z}_2^y)_{\Delta SU_2}$ fixing $E_0$ and acting by the exchange $D_1 \leftrightarrow D_3$ and $D_2 \leftrightarrow D_4$.

Concerning the second member of the family namely $\mathcal{G}_{X(Y^p,0)}^{SU_p}$ with $p = 3$, it has four external non compact divisors $D_1, D_2, D_3, D_4$; but two compact divisors $E_1$ and $E_2$. For this case, there are two generalised Mori-vectors given by

$$
(q^a)_{SU_3} = \begin{pmatrix}
D_1 & D_2 & E_1 & E_2 & D_3 & D_4 \\
2 & 2 & -8 & 2 & 2 & 0 \\
2 & 0 & 2 & -8 & 2 & 2
\end{pmatrix}
$$

(4.13)
The representative CY graph \( G^{{{SU}_3}}_{X(Y^3,0)} \) is depicted by the Figure 9. Notice that the graph

\[
G^{{{SU}_3}}_{X(Y^3,0)} \hat{=} X(Y^3,0)
\]

exhibiting manifestly the Calabi-Yau condition at each internal point of the graph.

\( G^{{{SU}_3}}_{X(Y^3,0)} \) has an outer-automorphism symmetry group \( H^\text{outer}_{SU_3} \) given by \((\mathbb{Z}^2_2)_{SU_3} \times (\mathbb{Z}^y_2)_{SU_3} \); but with no fix divisor. The full \( H^\text{outer}_{SU_3} \) acts by the exchange \( E_1 \leftrightarrow E_2, D_1 \leftrightarrow D_3 \) and \( D_2 \leftrightarrow D_4 \). This \( H^\text{outer}_{SU_3} \) is a subsymmetry of \( \mathbb{Z}_6 \). It is generated by the product of three transpositions namely \( \tau \circ \tau' \circ \tau'' \) with transpositions given by \( \tau = (E_1E_2), \tau' = (D_1D_3) \) and \( \tau'' = (D_2D_4) \).

### 5 Symplectic graphs and quivers

In this section, we first build the symplectic CY graph \( G^{SP_4}_{X(Y^4,0)} \) by starting from the unitary \( G^{SU_4}_{X(Y^4,0)} \) and using folding ideas under \((\mathbb{Z}^x_2)_{SU_4} \times (\mathbb{Z}^y_2)_{SU_4} \). Then, we construct the symplectic quiver \( Q^{SP_4}_{X(Y^4,0)} \) with symplectic \( SP(4,\mathbb{R}) \) gauge symmetry by using the unitary BPS quiver \( Q^{SU_4}_{X(Y^4,0)} \) and outer-automorphisms \((\mathbb{Z}^2_2)^{\text{outer}}\) \( Q^{SU_4}_{X(Y^4,0)} \).

#### 5.1 Symplectic CY graph \( G^{SP_4}_{X(Y^4,0)} \)

We start by the toric data of \( \Delta^{SU_4}_{X(Y^4,0)} \) given by Table 3. Because these data are defined up to a global shift; we translate the points of \( \Delta^{SU_4}_{X(Y^4,0)} \) by \((0,-2)\). So the values of the \( w_i \) and \( v_a \) points —Table 3— of the previous toric diagram gets mapped to new points that we present as follows where we have set \( D_{-2} \equiv E_{-2} \) and \( D_{+2} \equiv E_{+2} \). With this parametrisation, the internal point \( v_0 = (0,0) \) is at the centre of the toric diagram. Moreover, the toric \( \Delta^{SU_4}_{X(Y^4,0)} \) is invariant under the outer-automorphism symmetry group \( H^\text{outer}_{SU_4} \cong (\mathbb{Z}^x_2)_{SU_4} \times (\mathbb{Z}^y_2)_{SU_4} \), mapping the points \( w_{\pm i}, v_0, v_{\pm a} \) into the symmetric ones namely \( w_{\mp i}, v_0, v_{\pm a} \). Because of
Table 4: Toric data exhibiting manifestly outer-automorphism symmetry

| $\Delta^\text{SU}_4_{X(Y^{4,0})}$ | $v_{-2}$ | $v_{-1}$ | $v_0$ | $v_{+1}$ | $v_{+2}$ | $w_{-1}$ | $w_{+1}$ |
|-----------------------------------|---------|---------|------|---------|---------|--------|--------|
| points                            | $(0, -2)$ | $(0, -1)$ | $(0, 0)$ | $(0, +1)$ | $(0, +2)$ | $(-1, +2)$ | $(+1, -2)$ |
| divisors                          | $E_{-2}$ | $E_{-1}$ | $E_0$ | $E_{+1}$ | $E_{+2}$ | $D_{-1}$ | $D_{+1}$ |

the property $w_{\pm i} = -w_{\pm i}$ and $v_{\pm a} = -v_{\pm a}$, the outer-automorphism $H^\text{outer}_{\Delta^\text{SU}_4}$ acts as a parity symmetry of the toric diagram,

$$H^\text{outer}_{\Delta^\text{SU}_4} : (w_{\pm i}, 0, v_{\pm a}) \rightarrow (-w_{\pm i}, 0, -v_{\pm a})$$

(5.1)

Notice that the outer-automorphism parity $H^\text{outer}_{\Delta^\text{SU}_4}$ is isomorphic to the group product $(\mathbb{Z}_2^x)_{\Delta^\text{SU}_4} \times (\mathbb{Z}_2^y)_{\Delta^\text{SU}_4}$ generated by the reflections in x- and y- directions acting as follows

$$(\mathbb{Z}_2^x)_{\Delta^\text{SU}_4} : (n_x, n_y) \rightarrow (-n_x, n_y)$$

$$(\mathbb{Z}_2^y)_{\Delta^\text{SU}_4} : (n_x, n_y) \rightarrow (n_x, -n_y)$$

$$(\mathbb{Z}_2^x)_{\Delta^\text{SU}_4} \times (\mathbb{Z}_2^y)_{\Delta^\text{SU}_4} : (n_x, n_y) \rightarrow (-n_x, -n_y)$$

(5.2)

where the $(n_x, n_y)$’s stand for the values of the external and the internal points of the toric diagram. So, the triangulated $\Delta^\text{SU}_{2r, X(Y^{2r,0})}$ is invariant under the outer-automorphism symmetry group with the central point $v_0$ being the unique fix point of $H^\text{outer}_{\Delta^\text{SU}_4}$.

By folding the CY graph $\mathcal{G}^\text{SU}_4_{X(Y^{4,0})}$ under the parity symmetry $(\mathbb{Z}_2^x)_{\Delta^\text{SU}_4} \times (\mathbb{Z}_2^y)_{\Delta^\text{SU}_4}$, we end up with a new CY graph

$$\mathcal{G}^\text{SP}_4_{X(Y^{4,0})} = \mathcal{G}^\text{SU}_4_{X(Y^{4,0})} / (\mathbb{Z}_2^x)_{\Delta^\text{SU}_4} \times (\mathbb{Z}_2^y)_{\Delta^\text{SU}_4}$$

(5.3)

having $2 + 2 = 4$ points given, up to identifications, by $w_{-1} \equiv w_{+1}$, $w_{-2} \equiv w_{+2}$; and $v_0$ as well as $v_{-1} \equiv v_{+1}$. The CY graph $\mathcal{G}^\text{SP}_4_{X(Y^{4,0})}$ is depicted by the Figure 10. The generalised Mori-vectors $\tilde{q}^1$ and $\tilde{q}^2$ associated with the symplectic CY graph $\mathcal{G}^\text{SP}_4_{X(Y^{4,0})}$ have each four components $\tilde{q}_\beta^a$. They are given by

$$\tilde{q}_\beta^a = \begin{pmatrix}
E_{\pm 2} & E_{\pm 1} & E_0 & D_{\pm 1} \\
2 & -8 & 2 & 4 \\
0 & 4 & -8 & 4
\end{pmatrix}$$

(5.4)
Figure 10: The folding CY graphs $G_{SU_4}^{\hat{X}(Y^{2r,0})}$ under parity symmetry $(\mathbb{Z}_2^x)^{\Delta_{SU_4}} \times (\mathbb{Z}_2^y)^{\Delta_{SU_4}}$. On the vertical line, one has a vertical chain that looks like Dynkin diagram of the $C_2 \simeq sp(4,\mathbb{R})$ Lie algebras. Here, we have $E_0^2.D_1 = E_1^2.D_1 = 4$ and $E_0^2.E_1 \neq E_1^2.E_0$.

The $2 \times 2$ square submatrix of the above rectangular $\tilde{q}_{\beta}^a$, associated with the triple intersections of the compact divisors $E_0$ and $E_{\pm 1}$, is given by

$$I_b^a = \begin{pmatrix} -8 & 2 \\ 4 & -8 \end{pmatrix}$$

(5.5)

Remarkably, this intersection matrix $I_b^a = E_a^2.E_b$ between the compact divisors is non symmetric. It can be put in correspondence with the non symmetric Cartan matrix $K(C_2)$ of the symplectic $C_2$ Lie algebra given by

$$K(C_2) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

(5.6)

The construction we have done for the particular $G_{SU_4}^{\hat{X}(Y^{4,0})}$ can be straightforwardly generalized to $G_{X(Y^{2r,0})}^{SP_2}$ with $r \geq 2$. The generic intersection matrix $I_{ab}$ with $a,b=1, ..., r$ has a quite similar form as (5.5). It is non symmetric $E_a^2.E_b \neq E_b^2.E_a$ and can be put in correspondence with the Cartan matrix of the symplectic Lie algebra $C_r$; see the discussion given after eq(4.10).

We end this subsection by making a comment on the folding of the family of Calabi-Yau graphs $G_{X(Y^{p,0})}^{SU_{p-1}}$ with respect to the factor $(\mathbb{Z}_2^x)^{\Delta_{SU_p}}$. It may be imagined as a partial folding in the transverse geometry represented by the points $w_1$ and $w_3$ of the toric diagram as indicated by Table 3. Recall that $(\mathbb{Z}_2^x)^{\Delta_{SU_p}}$ fixes all internal points $\upsilon_a$ of the toric diagrams.
$\Delta_{SU_{p-1}}^{\hat{X}(Y,p,0)}$ as well as the two external $w_2$ and $w_4$; but exchanges the two other external $w_1$ and $w_3$. The folded

$$\mathcal{G}_{\hat{X}(Y,p,0)}^{SU_{p-1}}/(\mathbb{Z}_2^x)_{\Delta_{SU_p}}$$

(5.7)

gives an exotic Calabi-Yau diagram; which for the example $p=3$ is given by the Figure 11. For this exotic folding, there are still two generalised Mori vectors that are associated with Figure 11: The folding CY graphs $\mathcal{G}_{\hat{X}(Y,3,0)}^{SU_3}$ under the partial outer-automorphism symmetry group $(\mathbb{Z}_2^x)_{\Delta_{SU_3}}$. Because of the folding, the two divisors $D_1$ and $D_3$ merge. Here, we have $E_1^2.D_1 = E_2^2.D_1 = 4$ and $E_2^2.E_1 = E_1^2.E_2$.

The compact divisors $E_1$ and $E_2$. These vectors are given by

$$\tilde{q}^a = \begin{pmatrix} D_2 & E_1 & E_2 & D_3 & D_4 \\ 2 & -8 & 2 & 4 & 0 \\ 0 & 2 & -8 & 4 & 2 \end{pmatrix}$$

(5.8)

The intersection matrix $\mathcal{I}_{ab} = E_a^2.E_b$ concerning the compact divisors is given by

$$\mathcal{I}_{ab} = \begin{pmatrix} -8 & 2 \\ 2 & -8 \end{pmatrix}$$

(5.9)

it is symmetric as in the unitary case.

### 5.2 BPS quiver with $SP(4,\mathbb{R})$ invariance

The BPS quiver $Q_{\hat{X}(Y,4,0)}^{SU_4}$ with $SP(4,\mathbb{R})$ gauge invariance is obtained by folding the unitary $Q_{\hat{X}(Y,4,0)}^{SU_4}$ by its outer-automorphism symmetry group $H_{Q^{SU_4}}^{outer}$ whose action on the quiver nodes and the arrows is constructed below. The BPS quiver $Q_{\hat{X}(Y,4,0)}^{SU_4}$ has 8 nodes $\{j\}$ with $j =$
1, ..., 8; and 16 oriented arrows \( j | l \) as depicted by the Figure 4. Clearly, the BPS quiver \( Q^{SU_4}_{X(Y^4, 0)} \) has a non trivial outer- automorphism symmetry group with two factors given by

\[
H_{Q^{SU_4}}^{outer} = (\mathbb{Z}_4)^{Q^{SU_4}} \times (\mathbb{Z}_2^{outer})^{Q^{SU_4}}
\]  

The factor \((\mathbb{Z}_4)^{Q^{SU_4}}\) has no fix quiver-node and no fix quiver-arrows; while \((\mathbb{Z}_2^{outer})^{Q^{SU_4}}\) has fix nodes and arrows. The transformations under \((\mathbb{Z}_4)^{Q^{SU_4}}\) are given eq(3.3); and the change under \((\mathbb{Z}_2^{outer})^{Q^{SU_4}}\) is as in eq(3.4). Thus, it is the symmetry \((\mathbb{Z}_2^{outer})^{Q^{SU_4}}\) that is important in our folding construction as it has fix nodes and arrows.

For the transformations under \((\mathbb{Z}_4)^{Q^{SU_4}}\), the nodes are transformed as follows

\[
(\mathbb{Z}_4)^{Q^{SU_4}} : \quad \{2a - 1\} \to \{2a + 7\} \quad \{2a\} \to \{2a + 8\}
\]  

(5.11)

This change indicates that the nodes \(7\) and \(8\) can be also denoted like \(\bar{1}\) and \(0\) respectively where the label \(\bar{1}\) refers to \(-1 \equiv 7\) and \(0 \equiv 8\). Regarding the action of the \((\mathbb{Z}_2^{outer})^{Q^{SU_4}}\) symmetry, we have the following transformations of the nodes

\[
(\mathbb{Z}_2^{outer})^{Q^{SU_4}} : \quad \{2a - 1\} \to \{7 - 2a\} \quad \{2a\} \to \{8 - 2a\}
\]  

(5.12)

showing that four quiver- nodes amongst the eight ones are fixed. They concern the pair \(\{3\}, \{4\}\) and the pair \(\{7\}, \{8\}\).

By folding \(Q^{SU_4}_{X(Y^4, 0)}\) with respect to \((\mathbb{Z}_2^{outer})^{Q^{SU_4}}\), we obtain a BPS quiver interpreted as the BPS quiver \(Q^{SP_4}_{X(Y^4, 0)}\) with an SP(4, \(\mathbb{R}\)) gauge symmetry. This folded BPS quiver has 6 nodes namely \(\{1, 2, 3, 4, 7, 8\}\) (the old \(\{5\}\) and \(\{6\}\) omitted due to folding). This new set of nodes can be also denoted as \(\{\bar{1}, 0, 1, 2, 3, 4\}\) where we have renamed the Kronecker quiver \(\{7, 8\}\) like \(\{\bar{1}, 0\}\). The resulting BPS quiver \(Q^{SP_4}_{X(Y^4, 0)}\) is as depicted in the Figure 12. In addition to the nodes, the folded BPS quiver has 16 oriented arrows distributed as in the Table 5 where the complex \(X_{12}^{aa}\) with \(\alpha = 1, 2\) and \(a = 1, 2\) form a quartet; and where the \(U_{\alpha}^a\) and the \(U_{\alpha}^a\) are doublets with \(U\) standing for \(X, Y\) and \(Z\). With these complex superfields, one can write down the SQM superpotential of the theory; it will not be discussed here.

### 6 Conclusion and comments

In this paper, we have developed a method to construct a new family of 5D \(\mathcal{N} = 1\) supersymmetric QFT models compactified on a circle with finite radius. This family of gauge
The models has symplectic SP(2r, R) gauge invariance and is embedded in M-theory on CY3s based on Sasaki-Einstein manifolds Ypq. Recall that gauge models engineered from M-theory on \( \hat{X} (Y^{p,q}) \) are well known; and they have unitary symmetries. So, our construction can be viewed as widening the family of unitary models based on \( \hat{X} (Y^{p,q}) \) to include the family of symplectic invariant models. To engineer this new theory, we started from the 5D \( \mathcal{N} = 1 \) super QFT, with unitary SU(4) gauge symmetry (corresponding to 2r=4), embedded in M-theory compactification on the toric Calabi-Yau threefold \( \hat{X} (Y^{4,0}) \). This complex 3d variety is a resolution of a conical singularity based on the Sasaki-Einstein manifold \( Y^{4,0} \). Then, we have proposed a graph to represent \( \hat{X} (Y^{4,0}) \) by using the numbers \( J_{iab} \) and \( I_{abc} \) given by triple intersections of the 7 divisors of \( \hat{X} (Y^{4,0}) \); four non compact \( D_i \) and three compact \( E_a \).

This new graph, denoted as \( G_{\hat{X} (Y^{4,0})} \), is given by a generalisation of the Mori-vectors of the ADE geometries of ALE spaces. It is defined by eq(4.1) and, to our knowledge, it has not been used before. We qualified the graph \( G_{\hat{X} (Y^{4,0})} \) as a unitary CY graph, first because of the unitary \( SU(4) \) symmetry of the gauge fiber within \( \hat{X} (Y^{4,0}) \); and second to distinguish it from the CY graph \( G_{\hat{X} (Y^{4,0})} \) having a symplectic \( SP(4, \mathbb{R}) \) gauge symmetry. The use of \( G_{\hat{X} (Y^{4,0})} \) has the merit to (i) highlight the CY condition of the toric \( \hat{X} (Y^{4,0}) \); (ii) extend the usual complex A3 surface describing the resolution of an ALE space with an SU(4) singular-

![Figure 12: The BPS quiver \( Q_{X(Y^{4,0})}^{SP_4} \) resulting by the folding of the \( Q_{X(Y^{4,0})}^{SU_4} \) under the mirror \( (\mathbb{Z}_2^{outer})_Q \). The nodes \{1, 2, 3, 4\} denote the elementary BPS particles associated with the rank of the C2 Lie algebra.](image-url)
ity; and (iii) to study non trivial outer-automorphisms $H^{\text{outer}}_{\Delta SU_4}$ of the toric diagram $\Delta^{SU_4}_{X(Y^{4,0})}$. The outer-automorphism group $H^{\text{outer}}_{\Delta SU_4}$ has a fixed internal point (a compact divisor); and is used to build the symplectic CY graph $G^{SU_4}_{X(Y^{4,0})}$ by using the folding $G^{SU_4}_{X(Y^{4,0})}/H^{\text{outer}}_{\Delta SU_4}$. After having set the basis for the CY graphs to represent the toric threefolds $\hat{X}$ ($Y^{p,q}$), we turned to investigating the BPS particles by constructing the symplectic BPS quiver $Q^{SP_4}_{X(Y^{4,0})}$ that is associated with the symplectic CY graph $G^{SP_4}_{X(Y^{4,0})}$. This BPS quiver is obtained by folding the unitary BPS $Q^{SU_4}_{X(Y^{4,0})}$ with respect to outer-automorphisms $(\mathbb{Z}_2^\text{outer})Q^{SU_4}$. Recall that the $Q^{SU_4}_{X(Y^{4,0})}$ has 8 nodes and 16 oriented arrows respectively describing 8 elementary BPS particles and 16 chiral superfields in SQM. The mirror symmetry $(\mathbb{Z}_2^\text{outer})Q^{SU_4}$ fixes four nodes of $Q^{SU_4}_{X(Y^{4,0})}$ and exchanges the four others. It fixes four arrows and exchanges the 12 others.

We end this conclusion by making two more comments regarding extensions of the analysis done in this paper.

The first extension concerns the building of symplectic BPS quivers $Q^{SP_{2r}}_{X(Y^{2r,0})}$ with generic rank. This is achieved by starting from the unitary quiver $Q^{SU_{2r}}_{X(Y^{2r,0})}$ with rank $2r-1$ and use folding ideas. The resulting symplectic quivers $Q^{SP_{2r}}_{X(Y^{2r,0})}$ are associated with the toric threefolds obtained by folding the unitary $Q^{SU_{2r}}_{X(Y^{2r,0})}$ with respect to the outer-automorphism group $(\mathbb{Z}_2^\text{outer})Q^{SU_{2r}}$. The quiver series $Q^{SP_{2r}}_{X(Y^{2r,0})}$ is also related to the symplectic CY graphs $G^{SP_{2r}}_{X(Y^{2r,0})}$ obtained from the folding of the unitary $G^{SU_{2r}}_{X(Y^{2r,0})}$ under the outer-automorphism symmetry $H^{\text{outer}}_{\Delta SU_{2r}}$. The explicit expression of the generalised Mori-vectors and representative graph $G^{SP_{2r}}_{X(Y^{2r,0})}$ as well as the associated quivers have been omitted for the sake of simplifying the presentation of the underlying idea.

The second extension regards 5D super QFT models, based on conical Sasaki-Einstein manifolds $Y^{p,q}$, with gauge symmetries beyond the unitary $SU(r+1)$ and the symplectic $SP(2r, \mathbb{R})$ groups. These gauge symmetries concern the orthogonal $SO(2r)$ and $SO(2r+1)$ groups; and eventually the three exceptional Lie groups $E_6$, $E_7$ and $E_8$. For 5D super QFT models with $SO(2r)$ gauge symmetry embedded in M-theory on $\hat{X}$ ($Y^{p,q}$), one needs engineering toric Calabi-Yau threefolds $\hat{X}^{p,q}(D_r)$ with an $SO(2r)$ gauge fiber. This might be nicely reached by using the technique of the CY graphs $G^{SO_{2r}}_{\hat{X}(D_r)}$ used in this study although an explicit check is still missing. This series of $G^{SO_{2r}}_{\hat{X}(D_r)}$ could be constructed by taking advantage of known results from the so-called complex $D_r$ surfaces describing the resolution of ALE space with $SO(2r)$ singularity. The family of the CY graphs $G^{SO_{2r}}_{\hat{X}(D_r)}$ might be also motivated from the correspondence between eq (1.8) and eq (4.10) for simply laced case; see also the correspon-
dence between eq(5.5) and eq(5.6) for non simply laced diagrams. If this SO(2r) study can be rigourously performed, one can also use outer-automorphisms of $G_{\hat{X}(D_r)}^{SO_{2r}}$, inherited for the Dynkin diagram of so(2r) Lie algebra, as well as the outer-automorphisms of the associated BPS $Q_{SO_{2r}}^{\hat{X}(D_r)}$ to construct 5D supersymmetric QFT models with SO(2r − 1) gauge invariance. Progress in these directions will be reported elsewhere.

Appendices

In this section, we give three appendices: A, B and C. They collect useful tools and give some details regarding the study given in this paper. In appendix A, we recall general aspects of the families of CY3s used in the geometric engineering of 5D $\mathcal{N} = 1$ super QFTs and the 5D $\mathcal{N} = 1$ super CFTs. We also describe properties of the Coulomb branch of the 5D SQFTs. In appendix B, we illustrate the derivation of the formula (3.1). In appendix C, we describe through examples the relationship between the 5D Kaluza-Klein BPS quivers and their 4D counterparts.

Appendix A

We begin by reviewing interesting aspects of M-theory compactified on a smooth non compact Calabi-Yau threefold $\hat{X}$. Then, we focus on illustrating these aspects for the class of CY3s given by $\hat{X}(Y^{p,q})$ used in present study. We also use these aspects to comment on the properties of the BPS particle and string states of the 5D gauge theory.

Two local CY3 families

Generally speaking, we distinguish two main families of local Calabi-Yau threefolds $\hat{X}$ depending on whether they have an elliptic fibration or not. These two families are used in the compactification of F-theory/M-theory/ type II strings leading respectively to effective gauge theories in 6/5/4 space time dimensions. These compactifications have received lot of interest in recent years in regards with the full classification of superconformal theories in various dimensions and their massive deformations. Because of dualities and due to the biggest 6D, the classification of 6D effective gauge theories has been conjectured to be the mother of the classifications in the lower dimensional theories. What concerns us in this
appendix is not the study of the classification issue; but rather give some mathematical tools developed there and which can also be applied to our study.

- **Family of local CY3s admitting an elliptic fibration.**

These local Calabi-Yau threefolds \( \hat{X} \) are complex 3D spaces given by the typical fibration \( E \to B \) with building blocks as: (i) \( B \) a complex 2D base; this is a Kahler surface. (ii) a complex 1D fiber \( E \) given by an elliptic curve. This genus zero curve is expressed by the Weirstrass equation

\[
E : y^2z = x^3 + fxz^2 + gz^3
\]

(A.1)

where \((x, y, z)\) are homogeneous coordinates of \( \mathbb{P}^2 \). Moreover, \( z \) is a function on the base \( B \) and \((x, y, f, g)\) are sections \( K_B^{-2}, K_B^{-3}, K_B^{-4}, K_B^{-6} \) with \( K_B \) the canonical divisor class of \( B \).

Depending on the nature of the base, one can preserve either preserve 16 supersymmetric charges for bases \( B \) type \( \mathbb{T}^2 \to \mathbb{P}^1 \); or eight supercharges in the case of bases \( B \) like for example \( \mathbb{P}^1 \times \mathbb{P}^1 \) and in general Hirzebruch surfaces \( \mathbb{F}_n \). These elliptically fibered CY3 geometries \( \hat{X} \sim E \times B \) have been used recently in the engineering of superconformal theories in dimensions bigger than 4D. Regarding the SCFTs in 4D, the classification has been obtained a decade ago by using type II strings. For the classification of the 5D SCFTs using M-theory on elliptically fibered CY3 we refer to [4]. The graphs representing these theories are intimately related with the Dynkin diagrams of affine Kac-Moody Lie algebras.

- **Family of local CY3s not elliptically fibered.**

As examples of local Calabi-Yau threefolds \( \hat{X} \), we cite the orbifolds of the complex 3-dimensional space; i.e \( \mathbb{C}^3/\Gamma \) with discrete group \( \Gamma \) contained in \( \text{SU}(3) \). These orbifolds include the conical Sasaki-Einstein threefolds \( \hat{X} (Y^{p,q}) \) we have considered in this paper. The local CY3 geometries which are not elliptically fibered are used in the engineering of massive supersymmetric QFTs. The graphs representing these theories are related with the Dynkin diagrams of ordinary Lie algebras.

In what follows, we focus on M-theory compactified on \( \hat{X} (Y^{p,0}) \) considered in this study and on the corresponding \( \text{U}(1)^{p-1} \) Coulomb branch.

**M-theory on \( \hat{X} (Y^{p,0}) \)**

The local threefolds \( \hat{X} (Y^{p,0}) \) has four non compact divisors \( \{D_i\}_{1 \leq i \leq 4} \) and p-1 compact divisors \( \{E_a\}_{1 \leq a \leq p-1} \). These divisors are not completely free; they obey some constraint relations; in particular the Calabi Yau condition of \( \hat{X} (Y^{p,0}) \). They also obey gluing properties
through holomorphic curves. The CY condition reads in terms of the divisor classes as in eq(2.1). For a generic positive integer p; it reads as follows [64]

$$
\sum_{i=1}^{4} D_i + \sum_{a=1}^{p-1} E_a = 0
$$

(A.2)

In our study, this condition has been transformed as in eq(4.1); and has been used to introduce the graphs given in section 4. Notice that the union of the compact divisors $S = \bigcup_{a=1}^{p-1} E_a$ is important in this investigation; it is a local surface made of a collection of irreducible compact holomorphic surfaces $E_a$. The irreducible holomorphic surfaces intersect each other pairwise transversally; this intersection is important and will be described below with details. Notice also that the Kahler parameters of the $E_a$'s are identified as the Coulomb branch moduli $\phi_a$; they appear in the calculation through the linear combination $\sum \phi_a E_a$ which also plays an important role in the construction.

Regarding the gluing properties of the compact divisors and their consequences; they need introducing some geometric tools of the CY3. For a shortness and self contained of the presentation, we restrict to giving only those main tools that are interesting for this study. However, we take the occasion to also describe some particular geometric objects that are relevant for the investigation of the Coulomb branch of the gauge theory. These geometric objects are introduced through the four following points (a), (b), (c) and (d).

a) Gluing the compact divisors

The compact holomorphic surfaces $\{E_a\}$ are complex surfaces in $\hat{X}$. Neighboring surfaces $E_a$ and $E_b$ are glued to each others while satisfying consistency conditions. Before giving these conditions, recall that in our study, we have solved the CY condition by thinking of the $E_a$'s as given by $(F_0)_a$. As the holomorphic surface $F_0$ is given by a projective line $\mathbb{P}^1_f$ trivially fibered over a base $\mathbb{P}^1_B$, then we have

$$
E_a = (\mathbb{P}^1_f)_a \times (\mathbb{P}^1_B)_a
$$

(A.3)

Notice that this is a particular solution of the CY on 4-cycles; it has been motivated by looking for a simple solution to exhibit the CY condition as in the Figures 7-8-9 of section 4. However, general solutions might be worked out by using other type of holomorphic compact surfaces like the Hirzebruch surfaces $(F_n)_a$ of degree n and their blow ups at generic points. To fix the ideas, we focus below on the surfaces $F_n$ and on two lattices associated with $F_n$ namely: (1) the lattice $\Lambda_l(F_n)$ of complex curves $l$ in $F_n$; and (2) the Mori cone of curves
\( M_l(\mathbb{F}_n) \); this is a particular sublattice of \( \Lambda_l(\mathbb{F}_n) \). To that purpose, recall that holomorphic curves \( l \) in the compact surface \( \mathbb{F}_n \) are generated by two basic (irreducible) curves \( e \) and \( f \). The base curve \( e \) is the zero section of the fibration; and the \( f \) is the fiber \( \mathbb{P}^1_f \). The intersection numbers of these generators are given by

\[
e^2 = -n \quad , \quad f^2 = 0 \quad , \quad e.f = 1 \quad \text{(A.4)}
\]

Before proceeding forward, notice the four following interesting aspects: (i) The positivity \( e.f \geq 0 \) captures the irreducibility property of the generators \( e \) and \( f \). In general, a given curve \( l \) belonging to \( \Lambda(l, \mathbb{F}_n) \) is said irreducible if we have \( l.e \geq 0 \) and \( l.f \geq 0 \). (ii) As far compact holomorphic curves in \( \mathbb{F}_n \) are concerned; one distinguishes two interesting curves that play an important role in the study of \( \mathbb{F}_n \). These are the curve \( h = e + nf \); and the canonical class \( K_{\mathbb{F}_n, g} = -2h + (2g - 2 + n)f \) where we have moreover figured the genus \( g \). For the case of \( \mathbb{F}_n \), we have \( K_{\mathbb{F}_n} = -2e + (n - 2)f \); it reduces for the case \( n = 0 \) to \( K_{\mathbb{F}_0} = -2e - 2f \) with triple intersection given by

\[
(\mathbb{F}_0)^3 = K_{\mathbb{F}_0}.K_{\mathbb{F}_0} = 8e.f = 8 \quad \text{(A.5)}
\]

From the above relations, we can perform several computations. For example, we have

\[
\begin{align*}
h^2 &= +n \quad , \quad h.e &= 0 \quad , \quad h.f &= 1 \\
K_{\mathbb{F}_0}.h &= -2 \quad , \quad K_{\mathbb{F}_0}.e &= 2n - 2 \quad , \quad K_{\mathbb{F}_0}.f &= -2 \\
\end{align*}
\quad \text{(A.6)}
\]

and

\[
(K_{\mathbb{F}_{n,g} + l}).l = 2g - 2 \quad \text{(A.7)}
\]

where \( g \) is the genus of \( l \). Because the genus \( g \geq 0 \), the above quantity is greater than \(-2\) due to the constraint \( 2(g - 1) \geq -2 \). (iii) Holomorphic curves in Mori cone \( M_l(\mathbb{F}_n) \) of the surface \( \mathbb{F}_n \) are given by the linear combination \( l_{n_e,n_f} = n_e e + n_f f \) with positive integers \( n_e \) and \( n_f \). These are particular curves of \( \Lambda_l(\mathbb{F}_n) \) corresponding to \( n_e \) and \( n_f \) arbitrary integers. Notice that with this notation, we have \( h = l_{1,n} \); and the particular curve \( l_{1,1} = e + f \) has a self intersection \( l_{1,1}^2 = 2 - n \). (iv) If considering several surfaces \( \mathbb{F}_{n_a} \) with \( a=1,\ldots,p-1 \); then eq(A.4) extend as follows \( e_a^2 = -n_a \) and \( f_a^2 = 0 \) as well as \( e_a.f_a = 1 \). Quite similar relationships can be written down for the holomorphic curves in \( \Lambda^a_l = \Lambda_l(\mathbb{F}_{n_a}) \) and curves in \( M^a_l = M_l(\mathbb{F}_{n_a}) \).

Returning to the gluing of curves \( l_a \) and \( l_b \) inside two compact surfaces \( S_a \) and \( S_b \); say the
divisor $E_a$ and the divisor $E_b$. It is defined by using the following restrictions

$$l_{ab} = l_a|_{E_b}, \quad l_{ba} = l_b|_{E_a} \quad (A.8)$$

and imposing some consistency conditions coming from topology and geometry. The topology requires the two curves $l_a$ and $l_b$ to be identical in the following sense: (α) if the $l_a$ is irreducible; then the $l_b$ must be also irreducible; and (β) the genera of the two curves have to be equal and positive; that is $g(l_a) = g(l_b) \geq 0$. The geometry requires moreover the volumes vol($l_{ab}$) and vol($l_{ba}$) to be equal; these volumes are computed by using the dual Kahler divisor as usual like $-J.l_{ab} = -J.l_{ba}$. Under these consistency conditions, the gluing of the two curves is thought of in terms of the identification $l_{ab} \simeq l_{ba}$ together with the CY condition that reads as follows

$$(l_{ab})^2 + (l_{ba})^2 = 2g - 2 \quad (A.9)$$

where $g$ is the genus of $l_{ab} \simeq l_{ba}$. For the solution of section 4, the surfaces $S_a$ and $S_b$ are given by $(F_0)_a$ and $(F_0)_b$; and the curves $l_{ab}$ and $l_{ba}$ may be taken as $(e_a + f_a)_b$ and $(e_b + f_b)_a$.

b) Compact holomorphic curves in $\hat{X}(Y^{p,0})$

The compact holomorphic curves $C$ in $\hat{X}(Y^{p,0})$ are 2-cycles in the local Calabi-Yau threefolds. A subset of these curves is given by the $e_a$’s and the $f_a$’s generating the curves in the divisors $E_a$ when realised in terms of $(F_0)_a$. In general, the compact curves $C$ are given by linear combinations of generators $C_\tau$ of compact holomorphic curves in $\hat{X}(Y^{p,0})$; they can be denoted like $C_n$ where $n$ is an integer vector. As we have done above for the irreducible gauge divisors $E_a = (F_0)_a$, these CY3 holomorphic curves can be expressed as integer linear combinations like

$$C_n = \sum_{\tau=1}^{d} n_\tau C_\tau \quad (A.10)$$

with $n_\tau \in \mathbb{Z}$. From this expansion, we learn: (i) the set of compact holomorphic curves in $\hat{X}(Y^{p,0})$ form a d-dimensional lattice $\Lambda_C(\hat{X})$ contained in $\mathbb{Z}^d$. (ii) In the case where all $n_\tau$ integers are positive ($n_\tau \in \mathbb{Z}^+$); the corresponding holomorphic curves belong to Mori cone $\mathcal{M}_C(\hat{X})$.

c) Curves intersecting surfaces

This is an interesting intersection product defined in the CY3. Given the two following:

(i) a holomorphic curve $l$ belonging to the Mori cone $\mathcal{M}(\hat{X})$. (ii) a holomorphic surface $S$ with canonical class $K_S$ sitting in the local Calabi-Yau threefolds $\hat{X}$. Then, the intersection
between \( l \) and \( S \) is given by
\[
l.S = (l.K_S)|_S \tag{A.11}
\]
For the interesting case where the holomorphic surface \( S \) is given by the compact divisors \( E_a \), the above intersection reads as \( (l.K_S)|_{E_a} \). The value of this intersection depends on two possibilities:

(\( \alpha \)) The case where \( l \) lives inside \( \mathcal{M}(\hat{X}) \); then, we have \( l.E_a = (l.K_S)|_{E_a} \).

(\( \beta \)) The case where \( l \) lives inside another surface; say \( S = E_b \); then we have
\[
l.E_a = (l.l_{\text{ba}})|_{E_b} \tag{A.12}
\]
where \( l_{\text{ba}} \) is the curve participating in the gluing between \( E_a \) and \( E_b \). The curve \( l_{\text{ba}} \) also sits in \( \mathcal{M}(\hat{X}) \). From these relations, we learn that the intersections \( l.E_a \) can be recovered from the intersection products on the Mori cones \( \mathcal{M}_a \).

d) Triple intersections

The triple intersections \( E_a.E_b.E_c \) of the holomorphic surfaces are numbers that can be expressed as intersection products of gluing curves inside any of the three surfaces. For that, we use the typical curves \( L_{ab} = E_a.E_b \); these intersection curves appear as irreducible curves \( l_{ab} \) from the \( E_a \) side; and as irreducible curves \( l_{ba} \) from the side of \( E_b \). The intersection of \( E_a \) and \( E_b \) is obtained as described before; that is by the identification \( l_{ba} = l_{ab} \). Similar identifications hold for the intersections of \( E_b.E_c \) and \( E_c.E_a \). By taking the intersection curve \( l_{\alpha\beta} \) as the diagonal sum of the the generators namely \( l_{\alpha\beta} = e_\alpha + f_\alpha \), we obtain
\[
E_a^3 = K_{(F_0)_a}.K_{(F_0)_a} = +8e_a.f_a = +8
\]
\[
E_{a^2}.E_b = K_{(F_0)_a}.l_{ab} = -2e_a.f_a = -2 \tag{A.13}
\]
in agreement with eq\( (4.9) \).

5D Coulomb branch and BPS states

To deal with the Coulomb branch of the 5D effective gauge theory and its BPS states, we need, in addition to the algebraic geometric objects given above, other basic quantities. One of these quantities concerns the metric \( ds^2 = \tau_{ab} d\phi^a d\phi^b \) of the Coulomb branch. It turns out \( \tau_{ab} \) derives from the effective scalar potential \( F(\phi) \) of the low energy theory; it reads as follows
\[
\tau_{ab} = \frac{\partial^2 F(\phi)}{\partial \phi^a \partial \phi^b} \tag{A.14}
\]
Given $F(\phi)$, one also has two other interesting quantities associated to it. (i) the gradient $\frac{\partial F(\phi)}{\partial \phi}$ which give the tensions $T_a$ of BPS string states. (ii) the third derivatives as $\partial^3 F(\phi)/\partial \phi^a \partial \phi^b \partial \phi^c$ giving coefficient of the Chern-Simons term $\kappa_{abc} = kd_{abc}$. The higher derivatives vanish identically because $F(\phi)$ is a cubic function. Recall that the effective potential of the 5D effective theory is exactly known; it reads as follows

\[
F = \frac{1}{2g_0^2} h_{ab} \phi^a \phi^b + \frac{\kappa}{6} d_{abc} \phi^a \phi^b \phi^c + \frac{1}{12} \left( \sum_{\text{roots } \alpha} |\alpha \cdot \phi|^3 - \sum_f \sum_{\omega \in \mathbb{R}_f} |\omega \cdot \phi + m_f|^3 \right) \quad (A.15)
\]

This function has the properties: (i) It is a cubic function of the gauge scalar field moduli $\{\phi_1, ..., \phi_r\}$ parameterising the Coulomb branch. (ii) It depends on the mass parameters $m_f$ of the 5D effective theory. (iii) It also depends on the roots $\alpha$ of Lie algebra and representations weights $\omega \in \mathbb{R}_f$ of the underlying gauge symmetry group. From the geometric view, the 5D gauge theory has an interesting description in terms of even p-cycles in the CY3. These cycles captures information whose some are presented through the four following point (a), (b), (c) and (d).

(a) Dual of the Kahler 2-form

The dual of the Kahler 2-form of $\hat{X}(Y^{p,0})$ is a divisor of the CY3 reading in terms of the generating divisors and the Coulomb branch moduli as follows

\[
J = \sum_{i=1}^{4} m^i D_i + \sum_{a=1}^{p-1} \phi^a E_a \quad (A.16)
\]

The compact complex surfaces $E_a$ are in one to one with the $U(1)$ factors of the Coulomb branch of the 5D gauge theory. In the SU(4) gauge theory studied in the paper, we have three $\phi$’s.

(b) Volume of compact even p-cycles in $\hat{X}(Y^{p,0})$

These compact cycles include: (i) the set of compact 2- cycles $C$ belonging $H_2(\hat{X})$, (ii) the set of compact 4- cycles $S$ belonging $H_4(\hat{X})$; and (iii) the 6-cycle given by $\hat{X}(Y^{p,0})$.

The volume of a compact 2-cycles $C$ is given by the intersection number

\[
Vol(C) = -J \cdot C \quad (A.17)
\]

For the particular compact holomorphic curves given by the $p - 1$ curves basic curves $C_a$ generating the Mori cone of $\hat{X}(Y^{p,0})$; we have the elementary volumes $Vol(C_a) = \nu_a$.

The volume of a compact 4-cycles $S$ is given by the intersection number

\[
Vol(S) = \frac{1}{2} J \cdot J \cdot S \quad (A.18)
\]
For the particular compact holomorphic surfaces given by the basic $p-1$ divisors $E_a$, we have the elementary volumes $Vol(E_a) = \tilde{\nu}_a$.

Finally, the volume of of $\hat{X}(Y^{p,0})$; it is given by the triple intersection number of the divisor $J$. This is the prepotential of the low energy 5D theory

$$F(\phi) = -\frac{1}{3!} J.J.J$$  \hspace{1cm} (A.19)

Notice that by putting (A.19) back into $T_a, \tau_{ab}, \kappa_{abc}$ and using (A.16), we end up with the following interpretation in terms of intersections

$$T_a \sim E_a.J.J, \quad \tau_{ab} \sim E_a.E_a.J, \quad \kappa_{abc} \sim E_a.E_b.E_c$$  \hspace{1cm} (A.20)

(c) The BPS states of the 5D theory

In this effective gauge theory, we distinguish two kinds of BPS states:

(i) Massive particle states ($M2/C$) given by M2- branes wrapping the compact holomorphic curves $C$. The masses of these particle states are given by $Vol(C)$. For the particular compact curves $C_a$; it is associated $p-1$ electrically charged elementary BPS particles given by the wrapping $M2/C_a$. The masses of these particles are given by $\nu_a$.

(ii) String states $M5/S$ arising from M5- brane wrapping the compact holomorphic surfaces $S$. The tensions of these strings are given by $Vol(S)$. For the particular compact surfaces given by the $p-1$ divisor $E_a$; it is associated $p-1$ magnetically charged elementary BPS strings $M5/E_a$ with tensions given by $\tilde{\nu}_a$.

Notice that the BPS spectrum of 5D N=1 theories include gauge instantons I in addition to the electrically charged particles and the magnetically charged monopole strings. The central charges of these particles are given by

$$Z_{elc} = \sum_{a=1}^{p-1} n_a^{(elc)} \phi_a + m_0 I, \quad Z_{mag} = \sum_{a=1}^{p-1} n_a^{(mag)} \frac{\partial F}{\partial \phi_a}$$  \hspace{1cm} (A.21)

where $n_a^{(elc)}, n_a^{(mag)}$ are integers. Notice that not every choice of these integers corresponds to the central charge of a physical state whose mass or tension has to be positive. The values of these $n$’s are obtained using BPS quivers and their mutations. Notice also that by compactifying the 5D gauge theories on a finite circle; we generate a Kaluza Klein particle states as described in the core of the paper.

(d) Dirac pairing

The intersection numbers $C_a.E_b$ of compact curves $C_a$ and compact surfaces $E_b$ describe the Dirac pairing between the BPS particles and the BPS strings.
Appendix B

Here, we consider M-theory compactified on $\hat{X} (Y^{2,0})$ with SU(2) gauge symmetry and look for the derivation of the quiver dimension $d_{bps} = 2(p - 1) + 2$ of eq(3.1). Because of the choice $p=2$, we have $d_{bps} = 4$ indicating that the BPS quiver $Q_{\hat{X}(Y^{2,0})}^{SU(2)}$ has four nodes as shown by the Figure 13(b). Recall that the BPS quiver $Q_{\hat{X}(Y^{2,0})}^{SU(2)}$ is related to the toric diagram $\Delta_{\hat{X}(Y^{2,0})}^{SU(2)}$ by the so-called fast inverse algorithm [92, 93, 94]. This algorithm involves two main steps summarized as follows:

- **Brane tiling BT**
  This step maps the toric $\Delta_{\hat{X}(Y^{2,0})}^{SU(2)}$ into a brane tiling in the 2-torus to which we refer to as $BT_{\hat{X}(Y^{2,0})}$. It uses the brane web $\hat{\Delta}_{\hat{X}(Y^{2,0})}^{SU(2)}$ (the dual of the toric diagram) to represent it by the tiling as given by the Figure 13(a). Recall that the toric graph representing $\Delta_{\hat{X}(Y^{2,0})}^{SU(2)}$ is a standard diagram; it can be drawn by using the Table 2 with $p=2$ and $q=0$. It has four external points ($n_{ext} = 4$), describing the four non compact divisors; and one internal point ($n_{int} = 1$) describing the compact divisor associated with the SU(2) gauge symmetry. For a short presentation, we have omitted this graph.

- **The BPS quiver**
  The second step maps the brane tiling $BT_{\hat{X}(Y^{2,0})}$ into the BPS quiver $Q_{\hat{X}(Y^{2,0})}^{SU(2)}$ as shown by the picture Figure 13(b). This mapping is somehow technical, we propose to illustrate the construction by giving some details.

![Figure 13: (a) The brane tiling of $\Delta_{\hat{X}(F_0)}^{SU(2)}$. (b) the BPS quiver $Q_{\hat{X}(F_0)}^{SU(2)}$ with SU(2) gauge symmetry. (c) the BPS subquiver of the 4d $N = 2$ pure SU(2)$_0$ gauge theory.](image-url)
Dimension $d_{bps}$ of the quiver $Q_{X(Y^2,0)}^{SU_2}$

The Figure 13(a) is a bipartite graph on the 2-torus with two kinds of nodes white and black. So, half of the nodes are white and the other half are black. This tiling is characterised by three positive integers $(N_W, N_E, N_F)$ related amongst others by the following relation

$$\chi_g = N_F - N_E + N_W$$  \hspace{1cm} (B.1)

where $\chi_g = 2g - 2$ is the well known Euler characteristics relation of discretized real genus-$g$ Riemann surfaces. In this relation: (a) $N_W$ is the number of nodes; (b) $N_E$ is the number of edges connecting the nodes; and (c) $N_F$ is the number of faces. This number is precisely the quiver dimension; i.e

$$N_F = d_{bps}$$  \hspace{1cm} (B.2)

Before proceeding, notice that the mapping between a brane tiling $BT_{X(S)}$ and a toric diagram $\Delta_{X(S)}$ is not unique. To a given toric diagram one may generally associate several brane tilings. So the brane tiling is a $1 \rightarrow many$. This diversity has an interpretation in terms of quiver gauge dualities of Seiberg- type. Notice also that for the 2-torus, we have $g = 1$; and then the quiver dimension can be also expressed as follows

$$N_F = N_E - N_W$$  \hspace{1cm} (B.3)

Building the quiver $Q_{X(Y^2,0)}^{SU_2}$

To build the quiver $Q_{X(Y^2,0)}^{SU_2}$ from $BT_{X(Y^2,0)}$, we one proceed in steps as follows:

(i) pick up a representative 2-torus unit cell (in green color in the Figure 13-a).

(ii) draw the corresponding BPS quiver given by the Figure 13(b) by using the following method.

• To each face $F_i$ within the 2-torus unit cell of the BT-tiling, we associate a quiver- node $\{i\}$ in the gauge quiver $Q_{X(Y^2,0)}^{SU_2}$. As there are $N_F = 4$ faces in the unit cell of $BT_{X(Y^2,0)}$, then the $Q_{X(Y^2,0)}^{SU_2}$ has four nodes $\{1;2;3;4\}$. Notice that the number $N_F$ can be presented in different, but equivalent, ways; for instance like $N_F = 2 (p - 1) + 2$, or

$$N_F = 2r - 2 + n_{ext} = 2r + f + 1$$  \hspace{1cm} (B.4)

where we have set $n_{ext} - 2 = f + 1$, and where for $SU(2)$ the rank $r=1$. The number $N_F$ is precisely the dimension $d_{bps}$ given by eq(3.1).
To each edge $E_{ij}$ of the brane tiling, separating the faces $F_i$ and $F_j$, we associate a quiver-arrow $\langle ij \rangle$ with direction determined by a traffic rule. In this rule, the circulation goes clockwise around white BT-nodes; and counter-clockwise around black BT-nodes. In the example $Q^{SU_2}_{\hat{X}(Y^2,0)}$, we have 8 quiver- arrows organized into four pairs. The are given by arrows $\langle i (i+1) \rangle$ with $i = 1, 2, 3, 4 \mod 4$.

To each BT-node in the brane tiling corresponds a superpotential monomial. So, the full superpotential $W^{SU_2}_{\hat{X}(Y^2,0)}$ associated with the BPS quiver has four monomials; this is because $N_W = 4$. For simplicity, we omit the explicit expression of $W^{SU_2}_{\hat{X}(Y^2,0)}$.

In the end of this appendix, notice that the four nodes of the quiver $Q^{SU_2}_{\hat{X}(Y^2,0)}$ are interpreted in type IIA string as the elementary BPS particles. The particles sitting at the nodes $\{1, 2\}$ correspond to the electrically $D2/\mathcal{E}_2$ and the magnetically $D4/\mathcal{E}_4$ charged BPS particles where $\mathcal{E}_n$ refers to $n$-cycles in $\hat{X}(Y^2,0)$. These two nodes form together a sub-graph of the SU(2) gauge quiver $Q^{SU_2}_{\hat{X}(Y^2,0)}$; it is given by the usual Kronecker diagram depicted by the Figure 13(c). The node $\{3\}$ is associated with the elementary instanton $I$; and the node $\{4\}$ with the elementary Kaluza-Klein D0. These two nodes form together the Kronecker sub-diagram $\{3, 4\}$.

**Appendix C**

In this appendix, we describe briefly helpful tools regarding the structure of BPS quivers in 4D $\mathcal{N} = 1$ Kaluza-Klein while focussing on those relevant aspects for our study. The material given below aims facilitating the reading of section 3. For a rigorous and abstract formulation of BPS quivers using amongst others the central charges and the Coulomb branch moduli, we refer to literature in this matter. For instance, the section 2.3 of [66] for 4D KK quivers and [99]-[101], [90], [69]-[73] for 4D.

**ADE gauge models**

A short way to introduce the 4D KK BPS quivers is to go through the well studied BPS quivers $\mathcal{Q}^G_X(4d)$ of 4D $\mathcal{N} = 2$ gauge theories with ADE gauge symmetries. The use of these 4D quivers may be motivated from various views in particular from the three following:

1. $\mathcal{Q}^G_X(4d) \subset \mathcal{Q}^G_X(5d)$. The quivers $\mathcal{Q}^G_X(4d)$, which are described below and mention in the text, appear as sub-quivers of the 4D KK BPS quivers $\mathcal{Q}^G_X(5d)$. For example, compare the two pictures of the Figure 14 with the Figures 5 and 6 in the main text. This feature,
which implies that the BPS states of 4D $\mathcal{N} = 2$ belong also to 4D KK $\mathcal{N} = 1$, can be explained by the fact that the 4D $\mathcal{N} = 2$ theory corresponds just to the zero mode of 4D Kaluza-Klein $\mathcal{N} = 1$ theory.

(2) Type II strings on CY3. The $\Omega^{G}_{X}(4d)$ quivers deal with 4D $\mathcal{N} = 2$ gauge theories with gauge symmetry $G$. These SQFTs can be remarkably embedded in type IIA string on CY3s. However, because of the relationship between type IIA and M-theory, the 4D $\mathcal{N} = 2$ theories can be also embedded in M-theory on CY3×$S^{1}$ which is the mother of 4D $\mathcal{N} = 1$ KK theory.

(3) Quiver mutations and duality. The BPS quivers have been widely employed in the case of four-dimensional $\mathcal{N} = 2$ theories. There, several techniques have been developed to handle them. Some of these techniques like quiver mutation algorithm apply also to $Q^{G}_{X}(5d)$; these mutation have an interpretation in terms of Seiberg- like duality. For an explicit study, see [102, 71, 73].

In what follows, we focus on the 4D BPS quivers of pure 4D $\mathcal{N} = 2$ gauge theories with gauge invariance $G$; say of type ADE. A general description of BPS quivers would also involve flavor matter; but for convenience, we ignore them here. The determination of the full set of BPS states of the $\mathcal{N} = 2$ SQFT is a complicated issue; but nicely formulated in terms of BPS quivers $\Omega^{ADE}_{X}(4d)$. So, the BPS quivers encode the relevant data on the BPS states of the $\mathcal{N} = 2$ SQFT. Their properties depend on the coordinates of the moduli space of the theory. Depending on the gauge coupling regime, we distinguish two sets $\{\Omega^{ADE}_{X}(4d)_{n}\}_{I,II}$ of BPS quivers termed as strong and weak chambers:

- **Strong chamber** $\{\Omega^{ADE}_{X}(4d)_{n}\}_{str}$. This is a finite set of BPS quivers describing the BPS particle states in the strong chamber. For the derivation of the full list of BPS states for ADE Lie algebras; see for instance [71] and references therein.

- **Strong chamber** $\{\Omega^{ADE}_{X}(4d)_{n}\}_{weak}$. This is an infinite set of BPS quivers describing the BPS states in the weak chamber. For a description of this set; see for instance [73].

The full content of these BPS chambers can be obtained by constructing all BPS quivers using mutation algorithm (mutation symmetry group). One of these BPS quivers is given by the so-called primitive BPS quiver denoted below like $\hat{\Omega}^{ADE}_{X}$. This is a basic BPS quiver made of the elementary BPS states. By applying the mutation algorithm on $\hat{\Omega}^{ADE}_{X}$, one generates new quivers made of BPS states given by composites of the elementary ones. By repeating this operation several times, one can generate all BPS particles of the theory. For
the strong chambers, the mutation group is finite; however is it infinite for weak chambers. There, one obtains recursive relations for the EM charges of the BPS states.

**Primitive quiver**

As far as the primitive quiver of pure gauge theories is concerned, its $2r$ BPS states have electric/magnetic (EM) charge vectors given by $b_1, ..., b_r$ and $c_1, ..., c_r$; they appear in $\hat{\mathcal{X}}_{ADE}$ as depicted by the pictures of the Figure [14] for SU(2) and SU(3) gauge groups. The integer $r$ is the rank of the Lie algebra ADE. The EM charge vectors $b_i$ and $c_i$ read in terms of the simple roots $\vec{a}_1, ..., \vec{a}_r$ of the simply laced Lie algebra of the gauge symmetry $G$ as follows

$$b_i = \begin{pmatrix} \vec{0} \\ \vec{a}_i \end{pmatrix}, \quad c_i = \begin{pmatrix} \vec{a}_i \\ -\vec{a}_i \end{pmatrix}$$

(C.1)

Recall that the simple roots $\vec{a}_i$ have the intersection

$$\vec{a}_i \cdot \vec{a}_j = K_{ij}$$

(C.2)

giving the Cartan matrix of the Lie algebra. Notice that these charge vectors can be denoted collectively like $b_i = \gamma_{2i}$ and $c_i = \gamma_{2i-1}$ with $i = 1, ..., r$. These EM charge vectors obey the Dirac pairings $\gamma_m \circ \gamma_n$ that splits as

$$\gamma_m \circ \gamma_n = \begin{pmatrix} c_i \circ c_j & c_i \circ b_j \\ b_i \circ c_j & b_i \circ b_j \end{pmatrix} \equiv A^G_0$$

(C.3)
Notice also that for simply laced Lie algebras, the primitive BPS quiver $\hat{Q}_X^{ADE}$ consists of $2r$ nodes and $3r - 2$ links as described below:

**A) nodes in $\hat{Q}_X^{ADE}$**

The $2r$ nodes of $\hat{Q}_X^{ADE}$ are as depicted by the pictures of the Figure 14 corresponding to SU(2) and SU(3). They refer to the elementary BPS states with EM charges $b_i$ and $c_i$. Notice that in the case of pure gauge theories considered in this appendix, the $b_i$’s are the charges of $r$ elementary monopoles $M_i$ while the $c_i$’s are the charges of $r$ elementary dyons $D_i$. So the elementary BPS states of the theory are given by

$$\text{Monopoles} \ : \ M_1, \ldots, M_r$$
$$\text{Dyons} \ : \ D_1, \ldots, D_r$$

(C.4)

These BPS states have interpretation in terms of D2- and D4- branes wrapping 2- and 4-cycles in the CY3. In M-theory language, they can be interpreted in terms of wrapped M2- and M5- branes.

**B) Links between nodes of $\hat{Q}_X^{ADE}$**

There are $r + (2r - 2)$ oriented links joining the nodes of $\hat{Q}_X^{ADE}$. These links are of two types as described below:

- $r$ vertical links $l_1, \ldots, l_r$ joining the nodes $(b_i, c_i)$; that is the $r$ pairs $(b_1, c_1), \ldots, (b_r, c_r)$.

They are oriented from the node $c_i$ to the node $b_i$. These links carry a charge given by the absolute value of the Dirac pairing $b_i \circ c_i$ which is equal to 2 due to the relation

$$b_i \circ c_i = \bar{a}_i \bar{a}_i = 2$$

(C.5)

Recall that the Dirac pairing of the electric magnetic charge vectors is antisymmetric $\gamma_m \circ \gamma_n = -\gamma_n \circ \gamma_m$; so we have

$$b_i \circ c_i = -c_i \circ b_i = K_{ij}$$

(C.6)

$$b_i \circ b_j = c_i \circ c_j = 0$$

(C.7)

- $2(r - 1)$ oblique links $l_{ij}$ joining two nodes of different pairs $(b_i, c_i)$ and $(b_j, c_j)$. This reduced number of links is due to the constraint eqs(C.6-C.7). So, the intersection matrix $A_0^G$ describing the primitive quiver $\hat{Q}_X^{ADE}$ is related to the Cartan matrix of the Lie algebra as follows

$$A_0^G = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix}$$

(C.8)
This construction extends to the BPS quivers with non simply laced gauge symmetries; see for instance [90, 69, 70].

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