Rubio de Francia Extrapolation Theorems
for Quasi-Monotone Functions

Arun Pal Singh\textsuperscript{1}, Rahul Panchal\textsuperscript{2}, Pankaj Jain\textsuperscript{3} and Monika Singh\textsuperscript{4}

\textsuperscript{1} Department of Mathematics, Dyal Singh College (University of Delhi), Lodhi Road, Delhi - 110003, INDIA Email: arunpalsingh@dsc.du.ac.in

\textsuperscript{2} Department of Mathematics, University of Delhi, Delhi - 110007, INDIA Email: drrpanchal0@gmail.com

\textsuperscript{3} Department of Mathematics, South Asian University, Akbar Bhawan, Chanakya Puri, New Delhi - 110021, INDIA Email: pankaj.jain@sau.ac.in; pankajkrjain@hotmail.com

\textsuperscript{4} Department of Mathematics Lady Shri Ram College for Women (University of Delhi), Lajpat Nagar, Delhi - 110024, INDIA Email: monikasingh@lsr.du.ac.in

Abstract
We prove Rubio de Francia extrapolation results in Lebesgue and grand Lebesgue spaces for quasi monotone functions with $QB_{\beta,p}$ weights. The extrapolation in Lebesgue spaces with the weight class $QB_{\beta,\infty}$ has also been investigated. As an application, we characterize the boundedness of the Hardy averaging operator for quasi monotone functions in the grand Lebesgue spaces.

2010 AMS Subject Classification. 26D10, 26D15, 46E35.
Key words and Phrases. Rubio de Francia extrapolation; grand Lebesgue space; $QB_{\beta,p}$-weights; Hardy averaging operator, quasi-monotone functions.

1 Introduction
We shall denote by $\mathcal{M}$, the set of all measurable functions defined and finite almost everywhere (a.e.) on $\mathbb{R}^+$. Also, $\mathcal{M}^+ \subset \mathcal{M}$ and $\mathcal{M}_1^+ \subset \mathcal{M}^+$ will denote, respectively, the cones of non-negative and non-negative non-increasing (↓) functions in $\mathcal{M}$. By a weight $w$, we mean a function in $\mathcal{M}^+$ which is locally integrable as well. For a weight $w$ and $1 \leq p < \infty$, denote by $L^p_w$, the weighted Lebesgue space consisting of all $f \in \mathcal{M}$ such that

$$\|f\|_{L^p_w} := \left( \int_0^\infty |f|^p w \right)^{1/p} < \infty.$$  

A weight $w$ is said to be in the Muckenhoupt class $A_p$, $1 < p < \infty$, if

$$[w]_{A_p} := \sup_J \left( \frac{1}{|J|} \int_J w \right) \left( \frac{1}{|J|} \int_J w^{-p'/p} \right)^{p-1} < \infty,$$

and in class $A_1$, if

$$[w]_{A_1} := \sup_J \sup_{x \in J} \frac{W(J)}{w(x)|J|} < \infty,$$

where $W(J) = \sup_{x \in J} w(x)$. 

ArXiv:2202.02544v1 [math.FA] 5 Feb 2022
where supremum is taken over all non-degenerate intervals \( J \subset \mathbb{R}^+ \), \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( W(J) := \int_J w(x)dx \). The weight class \( A_p \) is found to be useful in so many ways. It characterizes the boundedness of the maximal operator \([24]\) and Riesz potential \([9]\) in Lebesgue spaces. Moreover, this class also characterizes the boundedness of these operators in grand Lebesgue spaces \([7, 21]\).

Another beauty of the \( A_p \)-class of weights can be realized via the celebrity extrapolation result of J.L. Rubio de Francia \([25]\), which asserts that if a sublinear operator \( T \) is bounded on \( L^p_w \) for every \( w \in A_{p_0} \) \((p_0 \geq 1, \) fixed) with the constant of inequality depending only on \([w]_{A_{p_0}}\), then for every \( 1 < p < \infty \), \( T \) is bounded on \( L^p_w \) for every \( w \in A_p \). This extrapolation result was further re-investigated and explored by many people (see \([6]\) and the references therein) and now it is known that the operator \( T \) has no role to play. In fact, it is known that if \((f, g)\) is a pair of non-negative measurable functions such that for some \( 1 \leq p_0 < \infty \), the inequality

\[
\int_0^\infty f^{p_0}(x)w(x)dx \leq C \int_0^\infty g^{p_0}(x)w(x)dx
\]

holds for every \( w \in A_{p_0} \) with constant \( C \) depending on \([w]_{A_{p_0}}\), then for every \( 1 < p < \infty \), the inequality

\[
\int_0^\infty f^p(x)w(x)dx \leq C \int_0^\infty g^p(x)w(x)dx
\]

holds for every \( w \in A_p \) with constant \( C \) depending on \([w]_{A_p} \).

This theory has been generalized to \( A_\infty \)-weights also, see \([5]\). In \([4]\), Carro and Lorente established a parallel extrapolation theory for a pair of functions from \( \mathcal{M}^+ \) in the framework of \( B_p \)-class of weights: A weight \( w \) is said to belong to the class \( B_p \) \((p > 0)\) if there exists a constant \( C > 0 \) such that the inequality

\[
\int_r^\infty \left( \frac{r}{x} \right)^p w(x)dx \leq C \int_0^r w(x)dx
\]

holds for every \( r > 0 \). Like the \( A_p \)-class of weights, the weight class \( B_p \) is also an important class of weights. It characterizes the boundedness of the Hardy averaging operator

\[
Hf(x) := \frac{1}{x} \int_0^x f(t)dt
\]

in \( L^p_w \) spaces for \( f \in \mathcal{M}^+ \) (see \([1, 26]\)) and also in grand Lebesgue spaces (defined in Section 3) for \( f \in \mathcal{M}^+_m \) \([13, 21]\). These characterizations, in fact, are equivalent to the boundedness of the maximal operator, respectively in, Lorentz space \( \Lambda^p(w) \) \([2]\) and grand Lorentz space \( \Lambda^p(w) \) \([13]\).

In this paper, we consider quasi non-increasing functions, the class of such functions being denoted by \( Q_\beta \): A function \( f \in \mathcal{M}^+ \) is said to belong to \( Q_\beta \), \( \beta \in \mathbb{R} \), if \( x^{-\beta}f(x) \) is non-increasing. Clearly \( \mathcal{M}^+_m = Q_0 \). For the functions \( f \in Q_\beta \), Bergh, Burenkov and Persson \([3]\) investigated Hardy’s inequality with power type weights, while for general weights it has been proved in \([16]\) that the inequality

\[
\left( \int_0^\infty \left( \frac{1}{x} \int_0^x f(t)dt \right)^p w(x)dx \right) \leq C \int_0^\infty f^p(x)w(x)dx, \quad 1 \leq p < \infty
\]

holds for all \( f \in Q_\beta \) if and only if \( w \in Q_{\beta, 1} \), \( \beta > -1 \), i.e.,

\[
\int_r^\infty \left( \frac{r}{x} \right)^p w(x)dx \leq C \int_0^r \left( \frac{x}{r} \right)^{\beta p} w(x)dx, \quad r > 0. \tag{1.1}
\]

Note that for \( \beta = 0 \), the weight class \( QB_{\beta, p} \) reduces to the class \( B_p \). In the present paper, we define a variant of the class \( QB_{\beta, p} \) to be denoted by \( \hat{Q}B_{\beta, p} \), and prove the extrapolation results for this class of weights, as well as for the weight class

\[
QB_{\beta, \infty} := \bigcup_{p>0} QB_{\beta, p}.
\]
Further, we prove the extrapolation result for quasi-monotone functions in the frame of grand Lebesgue spaces. As an application, we prove the boundedness of the Hardy averaging operator for quasi-monotone functions in the grand Lebesgue spaces. Our results generalize the extrapolation results of Carro and Lorente [11] and Meskhi [22]. Throughout, all the functions used in this paper are assumed to be non-negative and measurable.

2 Extrapolation results in Lebesgue spaces

For $p > 0$, we say that a weight $w \in QB_{\beta,\psi,p}$ if

\[ \int_{r}^{\infty} \left( \frac{\Psi(r)}{\Psi(x)} \right)^{p} w(x) \, dx \leq C \int_{0}^{r} \left( \frac{\Psi(x)}{\Psi(r)} \right)^{\beta p} w(x) \, dx, \quad r > 0 \tag{2.1} \]

for some constant $C > 0$ and $\Psi(x) := \int_{0}^{x} \psi(t) \, dt$, where $\psi$ is a non-negative, non-increasing locally integrable function, i.e., $\psi \in L_{1}^{\text{loc}}$. For $\psi \equiv 1$, the weight class $QB_{\beta,\psi,p}$ reduces to the class $QB_{\beta,p}$.

In [16], the class $QB_{\beta,\psi,p}$ was used to characterize the boundedness of the operator

\[ S_{\psi}f(x) := \frac{1}{\Psi(x)} \int_{0}^{x} f(t) \psi(t) \, dt \]

on the cone of functions $f \in Q_{\beta}$. Precisely, the following was proved:

**Theorem A** [16]. Let $p \geq 1$ and $-1 < \beta \leq 0$. Then the inequality

\[ \int_{0}^{\infty} (S_{\psi}f)^{p}(x)w(x) \, dx \leq C' \int_{0}^{\infty} f^{p}(x)w(x) \, dx \]

holds for all $f \in Q_{\beta}$ if and only if $w \in QB_{\beta,\psi,p}$, where $C' = \frac{C+1}{\beta+1}$ and $C$ is as in (2.1).

We define $QB_{\beta,\psi,p}$-constant for a weight $w \in QB_{\beta,\psi,p}$ as follows

\[ [w]_{QB_{\beta,\psi,p}} := \inf \left\{ D : \int_{r}^{\infty} \left( \frac{\Psi(r)}{\Psi(x)} \right)^{p} w(x) \, dx \leq (D - 1) \int_{0}^{r} \left( \frac{\Psi(x)}{\Psi(r)} \right)^{\beta p} w(x) \, dx, \quad r > 0 \right\}. \tag{2.2} \]

**Remark 2.1.** Note that

1. $[w]_{QB_{\beta,\psi,p}} > 1$.
2. For $-1 < \beta \leq 0$ and $p \leq q$, we have $QB_{\beta,\psi,p} \subset QB_{\beta,\psi,q}$.

We begin with the following:

**Lemma 2.2.** Let the function $\varphi$ be non-decreasing ($\uparrow$) defined on $(0, \infty)$, $f, g \in Q_{\beta}$ ($\beta > -1$), $0 < p_{0} < \infty$, $\psi \in L_{1}^{\text{loc}}$ be $\downarrow$ and $\lim_{x \to \infty} \Psi(x) = \infty$. Suppose that for each $w \in QB_{\beta,\psi,p_{0}}$, the inequality

\[ \int_{0}^{\infty} f(x)w(x) \, dx \leq \varphi([w]_{QB_{\beta,\psi,p_{0}}}) \int_{0}^{\infty} g(x)w(x) \, dx \]

holds. Then for every $0 < \varepsilon < p_{0}(\beta + 1)$ and $t > 0$, the following inequality holds:

\[ \int_{0}^{t} f(s)(\Psi(s))^{p_{0}-1-\varepsilon} \psi(s) \, ds \leq \varphi \left( \frac{p_{0}(\beta + 1)}{\varepsilon} \right) \int_{0}^{t} g(s)(\Psi(s))^{p_{0}-1-\varepsilon} \psi(s) \, ds. \]
Proof. Let \( v \in \mathcal{M}_+^1 \). Set \( w(x) = v(x)(\Psi(x))^{p_0-1-\epsilon}\psi(x) \) so that \( w \in L^1_{loc} \). We claim that \( w \in QB_{\beta,\psi,p_0} \). Indeed, we have

\[
\left(\Psi(r)^{\beta+1}\right)^{p_0} \int_r^\infty \frac{w(x)}{\Psi(x)^{p_0}} \, dx = \left(\Psi(r)^{\beta+1}\right)^{p_0} \int_r^\infty v(x)(\Psi(x))^{-1-\epsilon}\psi(x) \, dx
\]

\[
\leq \frac{v(r)}{\epsilon} (\Psi(r))^{(\beta+1)-\epsilon}
\]

\[
= \frac{(p_0(\beta+1)-\epsilon)}{\epsilon} v(r) \int_0^r (\Psi(x))^{p_0(\beta+1)-1-\epsilon}\psi(x) \, dx
\]

\[
\leq \frac{p_0(\beta+1)}{\epsilon} \int_0^r v(x)(\Psi(x))^{p_0(\beta+1)-1-\epsilon}\psi(x) \, dx
\]

\[
\leq p_0(\beta+1) \int_0^r (\Psi(x))^{p_0}\psi(x) \, dx.
\]

The assertion now follows on taking \( v(x) = \chi_{(0,\epsilon]}(x) \) and using the fact that \( [w]_{QB_{\beta,\psi,p_0}} \leq \frac{p_0(\beta+1)}{\epsilon} \).

**Definition 2.3.** For a given \( \beta > -1 \), a weight function \( w \) is said to be in the class \( \hat{QB}_{\beta,p} \) if

(i) \( w \in QB_{\beta,p} \); and

(ii) there exists \( 0 < \epsilon < p(\beta+1) \) such that \( w \in QB_{\beta,p-\epsilon} \).

**Remark 2.4.** The class \( \hat{QB}_{\beta,p} \) in Definition 2.3 [16] is reasonably defined. In view of Lemma 2.3 [16], it is clear that for \( \beta \geq 0 \), \( \hat{QB}_{\beta,p} = QB_{\beta,p} \). We prove below that for \( -1 < \beta < 0 \), the power weights belong to the class \( \hat{QB}_{\beta,p} \). It is of interest if the same can be proved for general weights as well.

**Lemma 2.5.** Let \( 1 \leq p < \infty \), \( -1 < \beta < 0 \) and \( \alpha \in \mathbb{R} \). If \( x^\alpha \in QB_{\beta,p} \), then there exists \( 0 < \epsilon < p(\beta+1) \) such that \( x^\alpha \in QB_{\beta,p-\epsilon} \).

**Proof.** Since \( x^\alpha \in QB_{\beta,p} \), we have that

\[
\int_r^\infty \left(\frac{r}{x}\right)^p x^\alpha \, dx \leq C \int_0^r \left(\frac{x}{r}\right)^{\beta p} x^\alpha \, dx, \quad r > 0
\]

which holds if and only if

\[
-\beta p - 1 < \alpha < p - 1.
\]

Choose \( \epsilon > 0 \) such that \( 0 < \epsilon < p - \alpha - 1 \). Clearly, \( 0 < \epsilon < p(\beta+1) \). Now, using the estimates (2.3) and (2.4) at appropriate places, we obtain

\[
\int_r^\infty \left(\frac{r}{x}\right)^{p-\epsilon} x^\alpha \, dx = \frac{r^{\alpha+1}}{p-\epsilon-\alpha-1}
\]

\[
= \frac{p-\alpha - 1}{p-\epsilon-\alpha-1} \int_r^\infty \left(\frac{r}{x}\right)^p x^\alpha \, dx
\]

\[
\leq C \left(\frac{p-\alpha - 1}{p-\epsilon-\alpha-1}\right) \int_0^r \left(\frac{x}{r}\right)^{\beta p} x^\alpha \, dx
\]

\[
= K \left(\frac{\beta(p-\epsilon)+\alpha+1}{\alpha+\beta p+1}\right) \int_0^r \left(\frac{x}{r}\right)^{\beta(p-\epsilon)} x^\alpha \, dx
\]
i.e., \( x^\alpha \in QB_{\beta,p-\varepsilon} \) with the constant
\[
C^x := K \left( \frac{\beta(p-\varepsilon) + \alpha + 1}{\alpha + \beta p + 1} \right),
\]
where \( K = C \left( \frac{p-\varepsilon}{p-\alpha-\varepsilon} \right) \) and \( C \) is as in [23].

**Remark 2.6.** For \(-\beta p - 1 < \alpha < p - 1\), from Lemma [2.5] and [2.2], it follows that
\[
[x^\alpha]_{QB_{\beta,p-\varepsilon}} \leq C^x + 1 = C \left( \frac{p - \alpha - 1}{p - \varepsilon - \alpha - 1} \right) \left( \frac{\beta(p-\varepsilon) + \alpha + 1}{\alpha + \beta p + 1} \right) + 1.
\]

We now prove the first main extrapolation theorem:

**Theorem 2.7.** Let \( \varphi \uparrow \) be defined on \((0, \infty)\), \((f,g)\) be a pair of functions such that \( f, g \in Q_\beta, -1 < \beta \leq 0 \) and \( 1 \leq p_0 < \infty \). Suppose that for every \( w \in QB_{\beta,p_0} \), the inequality
\[
\int_0^\infty f^{p_0}(x)w(x)dx \leq \varphi([w]_{QB_{\beta,p_0}}) \int_0^\infty g^{p_0}(x)w(x)dx
\]
holds. Then for all \( p_0 \leq p < \infty \) and all \( w \in QB_{\beta,p} \), the following holds:
\[
\int_0^\infty f^p(x)w(x)dx \leq C \int_0^\infty g^p(x)w(x)dx,
\]
where
\[
C = \inf_{0 < \varepsilon < p_0(\beta+1)} [w]_{QB_{\beta,(p_0-\varepsilon)}} \left[ \frac{1}{\beta+1} \left( \frac{p_0(\beta + 1) - \varepsilon}{p_0 - \varepsilon} \right) \varphi \left( \frac{p_0(\beta + 1)}{\varepsilon} \right) \right]^{p/p_0}.
\]

**Proof.** The case \( \beta = 0 \) is just Theorem 2.1 of [4]. So, we assume that \(-1 < \beta < 0\).

Let \( p_0 \leq p < \infty \), \( w \in QB_{\beta,p} \) and \( 0 < \varepsilon < p_0(\beta + 1) \). Clearly the function \( h(x) := x^{-\beta}f(x) \) is \( \downarrow \). Note that
\[
\int_0^\infty f^p(x)w(x)dx = \int_0^\infty h^p(x)w(x)x^{\beta p}dx. \tag{2.5}
\]
Since \( h \) is decreasing, we have
\[
h^{p_0}(x) \leq \left( \frac{p_0(\beta + 1) - \varepsilon}{p_0 - \varepsilon} \right)^{p/p_0} \int_0^x h^{p_0}(s)s^{p_0(\beta+1)-\varepsilon-1}ds
\]
which together with (2.5) and Lemma 2.2 (for \( \psi \equiv 1 \)) gives
\[
\int_0^\infty f^p(x)w(x)dx \leq \left( \frac{p_0(\beta + 1) - \varepsilon}{p_0 - \varepsilon} \right)^{p/p_0} \int_0^\infty \left( \frac{p_0 - \varepsilon}{p_0 - \varepsilon} \right)^{p/p_0} \int_0^x f^{p_0}(s)g^{p_0-1-\varepsilon}ds w(x)dx
\]
\[
\leq \left( \frac{p_0(\beta + 1) - \varepsilon}{p_0 - \varepsilon} \right)^{p/p_0} \varphi \left( \frac{p_0(\beta + 1)}{\varepsilon} \right)^{p/p_0} \int_0^\infty \left( \frac{p_0 - \varepsilon}{p_0 - \varepsilon} \right)^{p/p_0} \int_0^x g^{p_0}(s)s^{p_0(\beta+1)-\varepsilon-1}ds w(x)dx
\]
\[
= \gamma \int_0^\infty \left( \frac{p_0 - \varepsilon}{p_0 - \varepsilon} \right)^{p/p_0} \int_0^x g^{p_0}(s)s^{p_0-1-\varepsilon}ds w(x)dx
\]
\[
= \gamma \int_0^\infty (S_\psi g^{p_0}(x))^{p/p_0} w(x)dx, \tag{2.6}
\]
where \( \psi(s) = s^{p_0-1-\varepsilon} \) and
\[
\gamma = \left( \frac{p_0(\beta + 1) - \varepsilon}{p_0 - \varepsilon} \right)^{p/p_0} \varphi \left( \frac{p_0(\beta + 1)}{\varepsilon} \right)^{p/p_0}.
\]
Now, since \( w \in \hat{Q}B_{\beta,p} \), by definition, there exists \( \tilde{\varepsilon} > 0 \) such that \( w \in QB_{\beta,p-\tilde{\varepsilon}} \). It is sufficient to take \( \varepsilon \) so that \( p - \tilde{\varepsilon} = (p_0 - \varepsilon) \frac{p}{p_0} \) or \( \varepsilon = \frac{p_0}{p} \cdot \tilde{\varepsilon} \). Then \( w \in QB_{\beta,(p_0-\varepsilon)\frac{p}{p_0}} \), which gives that for all \( r > 0 \) the following inequality holds:

\[
\int_r^\infty \left( \frac{p}{p_0} \right) \frac{(p_0-\varepsilon)}{p_0} w(x)dx \leq (A - 1) \int_0^r \left( \frac{x}{r} \right)^{\beta(p_0-\varepsilon)} \frac{p_0}{p} w(x)dx,
\]

or

\[
\int_r^\infty \left( \frac{\Psi(r)}{\Psi(x)} \right) \frac{p}{p_0} w(x)dx \leq (A - 1) \int_0^r \left( \frac{\Psi(x)}{\Psi(r)} \right)^{\beta(p/p_0)} w(x)dx,
\]

with \( \psi(s) = s^{p_0-1-\varepsilon} \), which by Theorem A holds if and only if

\[
\int_0^\infty (S_\psi g^{p_0}(x))^{p/p_0} w(x)dx \leq \frac{A}{(\beta + 1)^{p/p_0}} \int_0^\infty g^p(x)w(x)dx,
\]

where \( A = [w]QB_{\beta,(p_0-\varepsilon)\frac{p}{p_0}} = [w]QB_{\beta,p-\tilde{\varepsilon}} \).

Consequently, Lem. 2.6 gives

\[
\int_0^\infty f^p(x)w(x)dx \leq \frac{\gamma A}{(\beta + 1)^{p/p_0}} \int_0^\infty g^p(x)w(x)dx = K \int_0^\infty g^p(x)w(x)dx,
\]

where

\[
K = [w]QB_{\beta,(p_0-\varepsilon)\frac{p}{p_0}} \left[ \left( \frac{p(p_0+1)-\varepsilon}{p_0(p_0-\varepsilon)} \right)^{\varphi \left( \frac{p(p_0+1)}{p_0-\varepsilon} \right)} \right]^{p/p_0}.
\]

Since \( \varepsilon \in (0,p_0(\beta + 1)) \) is arbitrary, taking infimum over all such \( \varepsilon \), the assertion follows.

In view of the Remark 2.7 (for \( \psi = 1 \)), following the definition of the class \( B_{\infty} \), we define the class \( QB_{\beta,\infty} \) as

\[
QB_{\beta,\infty} := \bigcup_{p>0} QB_{\beta,p}
\]

and we also define

\[
[w]QB_{\beta,\infty} := \inf \{ [w]QB_{\beta,p} : w \in QB_{\beta,p}, p > 0 \}.
\]

Similarly, we define

\[
QB_{\beta,\psi,\infty} := \bigcup_{p>0} QB_{\beta,\psi,p}
\]

and

\[
1 \leq [w]QB_{\beta,\psi,\infty} := \inf \{ [w]QB_{\beta,\psi,p} : w \in QB_{\beta,\psi,p}, p > 0 \}.
\]

We prove the following:

**Lemma 2.8.** Let \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) be \( \uparrow \), \( v \in L^{1}_{\text{loc}} \) be \( \downarrow \), \( -1 < \beta \leq 0 \) and \( \alpha > -1 \). Then the function \( w \) defined by

\[
w(x) = \Psi^\alpha(x)\psi(x)v(x)
\]

belongs to the class \( QB_{\beta,\psi,\infty} \).

**Proof.** Let \( 0 < r < \infty \) be arbitrary and choose \( p_0 \) such that \( \alpha + 1 < p_0 < -\frac{1}{\beta}(\alpha + 1) \). Then we have

\[
\int_r^\infty \left( \frac{\Psi^{\beta+1}(r)}{\Psi(x)} \right)^{p_0} w(x)dx = (\Psi(r))^{(\beta+1)p_0} \int_r^\infty (\Psi(x))^{\alpha-p_0} \psi(x)v(x)dx
\]

\[
\leq \frac{1}{(p_0 - \alpha - 1)}(\Psi(r))^{\beta p_0 + \alpha + 1}v(r)
\]

6
\[
\begin{aligned}
\leq \left( \frac{\beta p_0 + \alpha + 1}{p_0 - \alpha - 1} \right) \int_0^r \Psi(x) \beta p_0 + \alpha \psi(x) v(x) dx \\
= \left( \frac{\beta p_0 + \alpha + 1}{p_0 - \alpha - 1} \right) \int_0^r \Psi(x) \beta p_0 w(x) dx
\end{aligned}
\]

and the assertion follows. Moreover, \([w]_{QB_{\beta,\psi,\infty}} \leq \frac{\beta p_0 + \alpha + 1}{p_0 - \alpha - 1} + 1.\]

Below, we prove an extrapolation result for \(QB_{\beta,\psi,\infty}\) class of weights.

**Theorem 2.9.** Let \(\varphi\) be \(\uparrow\) defined on \((0, \infty), -1 < \beta \leq 0, (f, g)\) be a pair of functions such that \(f, g \in Q_{\beta}\) and \(0 < p_0 < \infty\). Suppose that for every weight \(w \in QB_{\beta,\infty}\), the inequality

\[
\int_0^\infty f^{p_0}(t)w(t)dt \leq \varphi([w]_{QB_{\beta,\infty}}) \int_0^\infty g^{p_0}(t)w(t)dt
\]

holds. Then for every \(p_0 \leq p < \infty\) and \(w \in QB_{\beta,\infty}\) the following holds

\[
\int_0^\infty f^p(t)w(t)dt \leq K \int_0^\infty g^p(t)w(t)dt,
\]

with

\[
K = \inf_{\alpha > -1} [w]_{QB_{\beta}} \left( \frac{\varphi(1)}{\beta + 1} \right)^{p/p_0}.
\]

**Proof.** For \(s > 0\) and \(\alpha > -1\), consider the following

\[
\tilde{w}(t) = \chi_{(0,s)}(t)t^\alpha.
\]

Clearly by Lemma 2.8 \(\tilde{w} \in QB_{\beta,\infty}\). Then, in view of Remark 2.4 and Lemma 2.8, we have

\[
1 \leq [\tilde{w}]_{QB_{\beta,\psi,\infty}} \leq \lim_{p_0 \to \infty} \frac{\beta p_0 + \alpha + 1}{p_0 - \alpha - 1} + 1 = \beta + 1 \leq 1
\]

and consequently, in view of (2.7) the following holds

\[
\int_0^s f^{p_0}(t)t^\alpha dt \leq \varphi(1) \int_0^s g^{p_0}(t)t^\alpha dt. \tag{2.8}
\]

Since \(t^{-\beta}f(t)\) is \(\downarrow\), we find that

\[
f^{p_0}(t) = \frac{\alpha + 1}{t^{\alpha + 1}} \int_0^t f^{p_0}(s)s^\alpha ds
\]

\[
= \frac{\alpha + 1}{t^{\alpha + 1}} \int_0^t (t^{-\beta}f(t))p_0 t^{\beta p_0} s^\alpha ds
\]

\[
\leq \frac{\alpha + 1}{t^{\alpha + 1}} \int_0^t (s^{-\beta}f(s))p_0 t^{\beta p_0} s^\alpha ds
\]

\[
= \frac{\alpha + 1}{t^{\alpha + 1}} \int_0^t f^{p_0}(s) \left( \frac{t}{s} \right)^{\beta p_0} s^\alpha ds
\]

\[
\leq \frac{\alpha + 1}{t^{\alpha + 1}} \int_0^t f^{p_0}(s)s^\alpha ds
\]

which in view of (2.7) gives

\[
\int_0^\infty f^p(t)w(t)dt \leq \int_0^\infty \left( \frac{\alpha + 1}{t^{\alpha + 1}} \int_0^t f^{p_0}(s)s^\alpha ds \right)^{p/p_0} w(t)dt
\]

\[
\leq \varphi(1)^{p/p_0} \int_0^\infty \left( \frac{\alpha + 1}{t^{\alpha + 1}} \int_0^t g^{p_0}(s)s^\alpha ds \right)^{p/p_0} w(t)dt.
\]
\[ \varphi(1)^{p/p_0} \int_0^\infty (S_{w}g^{p_0}(t))^{p/p_0} w(t) dt, \tag{2.9} \]

with \( \psi(s) = s^{\alpha}. \)

Now, let \( w \in QB_{\beta,\infty}. \) Then there exists \( q > 0 \) such that \( w \in QB_{\beta,q}. \) We can choose \( \alpha > -1 \) such that \( q = (\alpha + 1) \frac{p}{p_0}. \) Then \( w \in QB_{\beta,(\alpha+1)\frac{p}{p_0}} \), which in view of \( (1.1) \) implies that the following holds for all \( r > 0 \):

\[
\int_r^\infty \left( \frac{r}{t} \right)^{(\alpha+1) \frac{p}{p_0}} w(t) dt \leq (C - 1) \int_0^r \left( \frac{t}{r} \right)^{\beta(\alpha+1) \frac{p}{p_0}} w(t) dt
\]

or, equivalently

\[
\int_r^\infty \left( \frac{\psi(r)}{\psi(t)} \right)^{p/p_0} w(t) dt \leq (C - 1) \int_0^r \left( \frac{\psi(t)}{\psi(r)} \right)^{\beta p/p_0} w(t) dt
\]

with \( \psi(s) = s^{\alpha}. \) But the last inequality, in view of Theorem A, holds if and only if

\[
\int_0^\infty (S_{w}g^{p_0}(t))^{p/p_0} w(t) dt \leq \frac{C}{(\beta + 1)^{p/p_0}} \int_0^\infty g^{p}(t)w(t) dt, \tag{2.10}
\]

where \( C = [w]QB_{\beta,(\alpha+1)\frac{p}{p_0}}. \) Now \( (2.9) \) and \( (2.10) \) give that

\[
\int_0^\infty f^{p}(t)w(t) dt \leq [w]QB_{\beta,(\alpha+1)\frac{p}{p_0}} \left( \frac{\varphi(1)}{\beta + 1} \right)^{p/p_0} \int_0^\infty g^{p}(t)w(t) dt,
\]

so that on taking the infimum over all \( \alpha > -1, \) the assertion follows.

\[ \square \]

3 Extrapolation results in grand Lebesgue spaces

In this section, we shall prove a version of the extrapolation result (Theorem \( 2.7 \)) in the framework of grand Lebesgue spaces defined on finite intervals, which without any loss of generality is taken as \( I = (0,1). \)

Let \( 0 < p < \infty \) and \( -1 < \beta < \infty. \) We say that a weight function \( w \) on \( I \) belongs to the class \( QB_{\beta,p}(I) \) if there exists a constant \( C > 0 \) such that the inequality:

\[
\int_r^1 \left( \frac{r}{t} \right)^p w(t) dt \leq C \int_0^r \left( \frac{t}{r} \right)^{\beta p} w(t) dt
\]

holds for all \( 0 < r \leq 1. \) Also, for \( 0 < r \leq 1, \) we set

\[
[w]QB_{\beta,p}(I) := \inf \left\{ C > 1 : \int_r^1 \left( \frac{r}{t} \right)^p w(t) dt \leq (C - 1) \int_0^r \left( \frac{t}{r} \right)^{\beta p} w(t) dt \right\}.
\]

It can be seen that if \( 0 < p < \infty \) and \( w \in QB_{\beta,p}(I), \) then the function \( \tilde{w} = w\chi_I \in QB_{\beta,p} \) and

\[
[w]QB_{\beta,p}(I) = [\tilde{w}]QB_{\beta,p}.
\]

Following the arguments used in Lemma \( 2.8 \) we can prove:

Lemma 3.1. Let \(-1 < \beta \leq 0 \) and \( 1 \leq p < \infty. \) If \( x^\alpha \in QB_{\beta,p}(I), \) then there exists \( 0 < \varepsilon < p(\beta + 1) \) such that \( x^\alpha \in QB_{\beta,p-\varepsilon}(I). \)

Definition 3.2. For a given \(-1 < \beta < \infty, \) a weight function \( w \in \tilde{Q}B_{\beta,p}(I) \) if

(i) \( w \in QB_{\beta,p}(I); \) and

(ii) there exists \( 0 < \varepsilon < p(\beta + 1) \) such that \( w \in QB_{\beta,p-\varepsilon}(I). \)
Remark 3.3. It can be checked that for $-1 < \beta \leq 0$, the power weights $x^\alpha \in QB_{\beta, p}(I)$ if and only if $-\beta p - 1 < \alpha < p - 1$. Then, in view of Lemma 3.1, the class $QB_{\beta, p}(I)$ is reasonably defined.

It is seen that Theorem 2.7 can be modified for the interval $I$. We state it formally for later purpose.

Theorem 3.4. Let $\varphi \uparrow$ be defined on $\mathbb{R}^+$ and $(f, g)$ be a pair of functions such that $f, g \in Q_{\beta}(I), \ -1 < \beta \leq 0$. Let $1 \leq p_0 < \infty$ and that for every weight function $w \in QB_{\beta, p_0}(I)$, the inequality
\[
\int_0^1 f^{p_0}(x)w(x)dx \leq \varphi \left( |w|_{QB_{\beta, p_0}(I)} \right) \int_0^1 g^{p_0}(x)w(x)dx
\]
holds. Then for every $p_0 \leq p < \infty$ and every $w \in \hat{QB}_{\beta, p}(I)$, the following inequality holds:
\[
\int_0^1 f^p(x)w(x)dx \leq K'(p) \int_0^1 g^p(x)w(x)dx,
\]
where
\[
K'(p) := \inf_{0<\delta<\beta(p_0+1)} \left( \frac{1}{\beta+1} \left( \frac{p_0(\beta+1) - \delta}{p_0 - \delta} \right) \varphi \left( \frac{p_0(\beta+1)}{\delta} \right) \right)^{p/p_0}.
\]

In this section, we shall prove Theorem 2.7 in the framework of grand Lebesgue spaces $L^p, \beta(I)$ which consist of all measurable functions $f$ finite a.e. on $I$ for which
\[
\|f\|_{L^p, \beta(I)} := \sup_{0<\varepsilon<p-1} \left( \varepsilon^\beta \int_0^1 |f(t)|^{p-\varepsilon}dt \right)^{1/(p-\varepsilon)} < \infty.
\]

These spaces without weight have been defined in [10], which in fact, were initially defined for $\theta = 1$ by Iwaniec and Sbordone [12] and later have been generalized, studied and applied by several people in different directions. We refer to [15] and the references therein. For some very recent updates on grand Lebesgue spaces, we mention [16, 17, 18, 19, 20].

We now prove the following:

Theorem 3.5. Let $\theta > 0$, $\varphi$ be a non-negative $\uparrow$ function defined on $(0, \infty), \ -1 < \beta \leq 0, 1 < p_0 < \infty$ and $(f, g)$ be a pair of functions such that $f, g \in Q_{\beta}(I)$. Suppose that for every $w \in QB_{\beta, p_0}(I)$, the following inequality holds:
\[
\int_0^1 f^{p_0}(x)w(x)dx \leq \varphi \left( |w|_{QB_{\beta, p_0}(I)} \right) \int_0^1 g^{p_0}(x)w(x)dx.
\]
Then for every $p : p_0 \leq p < \infty$ and every $w \in \hat{QB}_{\beta, p}(I)$, the inequality
\[
\|f\|_{L^p, \beta(I)} \leq C^* \|g\|_{L^{p, \beta}(I)}
\]
holds with
\[
C^* = \inf_{0<\sigma<p-1} \left[ \max \left\{ 1, p^\sigma \frac{\theta}{p-\sigma} \left( W(I) + 1 \right)^{\frac{p-1-\sigma}{p-\sigma}} \right\} \sup_{0<\varepsilon<\sigma} (K'(p-\varepsilon))^{\frac{1}{p-\varepsilon}} \right].
\]

Proof. Let $w \in \hat{QB}_{\beta, p}(I)$, then by definition $w \in QB_{\beta, p}(I)$, and there exists $0 < \xi < p(\beta + 1)$ such that $w \in QB_{\beta, p-\xi}(I)$. Take $\sigma = \min\{\xi, p - p_0\}$. Clearly $0 < \sigma < p - 1$, so, by Remark 2.1, $w \in QB_{\beta, p-\sigma}(I)$. Let $\varepsilon \in (0, \sigma)$. Then, in view of the fact that $QB_{\beta, p} \subset \hat{QB}_{\beta, q}$ for $p < q$, we have $w \in QB_{\beta, p-\varepsilon}(I)$. Therefore, by Theorem 3.4 we have
\[
\int_0^1 f^{p-\varepsilon}(x)w(x)dx \leq K'(p-\varepsilon) \int_0^1 g^{p-\varepsilon}(x)w(x)dx, \tag{3.1}
\]
where

\[ K'(p - \varepsilon) := \inf_{0 < \delta < \frac{1}{p_0}} \left[ w \right]_{QB_{\beta, p_0 - \delta}}^{p - \varepsilon} \left( \frac{1}{\beta + 1} \left( \frac{p_0 (\beta + 1) - \delta}{p_0 - \delta} \right) \varphi \left( \frac{p_0 (\beta + 1)}{\delta} \right) \right]^{p - \varepsilon/p_0}. \]

Now, for \( \sigma < \varepsilon < p - 1 \), using H"older’s inequality with the indices \( \frac{p - \sigma}{p - \varepsilon} \) and \( \frac{p - \varepsilon}{p - \sigma} \), we obtain

\[
\| f \|_{L_{w}^{p}(I)} = \left( \int_{0}^{1} f^{p - \varepsilon}(x)w(x)dx \right)^{1/p - \varepsilon}
\leq \left( \int_{0}^{1} f^{p - \sigma}(x)w(x)dx \right)^{1/p - \sigma}(W(I))^{\frac{p - \varepsilon}{p - \sigma}}(W(I) + 1)^{\frac{p - 1 - \sigma}{p - \sigma}}.
\]

Now in view of (3.1) and (3.2), we get

\[
\| f \|_{L_{w}^{p}(I)} = \max \left\{ \sup_{0 < \varepsilon \leq \sigma} \frac{\varepsilon}{p - \varepsilon} \| f \|_{L_{w}^{p - \varepsilon}}, \sup_{\sigma < \varepsilon < p - 1} \frac{\varepsilon}{p - \varepsilon} \| f \|_{L_{w}^{p - \varepsilon}} \right\}
\leq \max \left\{ \sup_{0 < \varepsilon \leq \sigma} \frac{\varepsilon}{p - \varepsilon} \| f \|_{L_{w}^{p - \varepsilon}}, \sup_{\sigma < \varepsilon < p - 1} \frac{\varepsilon}{p - \varepsilon} \| f \|_{L_{w}^{p - \sigma}}(W(I) + 1)^{\frac{p - 1 - \sigma}{p - \sigma}} \right\}
\leq \max \left\{ 1, p^{\theta} \sigma^{\frac{\theta}{p - \sigma}}(W(I) + 1)^{\frac{p - 1 - \sigma}{p - \sigma}} \right\} \sup_{0 < \varepsilon \leq \sigma} \frac{\varepsilon}{p - \varepsilon} \| f \|_{L_{w}^{p - \varepsilon}}
\leq \max \left\{ 1, p^{\theta} \sigma^{\frac{\theta}{p - \sigma}}(W(I) + 1)^{\frac{p - 1 - \sigma}{p - \sigma}} \right\} \sup_{0 < \varepsilon \leq \sigma} \frac{\varepsilon}{p - \varepsilon} (K'(p - \varepsilon))^{\frac{1}{p - \varepsilon}} \| g \|_{L_{w}^{p}(I)}
\leq \max \left\{ 1, p^{\theta} \sigma^{\frac{\theta}{p - \sigma}}(W(I) + 1)^{\frac{p - 1 - \sigma}{p - \sigma}} \right\} \sup_{0 < \varepsilon \leq \sigma} (K'(p - \varepsilon))^{\frac{1}{p - \varepsilon}} \| g \|_{L_{w}^{p}(I)}^{L_{w}^{p}(I)}
\leq \max \left\{ 1, p^{\theta} \sigma^{\frac{\theta}{p - \sigma}}(W(I) + 1)^{\frac{p - 1 - \sigma}{p - \sigma}} \right\} \sup_{0 < \varepsilon \leq \sigma} \{ (K'(p - \varepsilon))^{\frac{1}{p - \varepsilon}} \| f \|_{L_{w}^{p}(I)}^{L_{w}^{p}(I)} \}
\]

where \( C^* = c(p, \theta, \sigma) \sup_{0 < \varepsilon \leq \sigma} (K'(p - \varepsilon))^{\frac{1}{p - \varepsilon}} \) and

\[
c(p, \theta, \sigma) = \max \left\{ 1, p^{\theta} \sigma^{\frac{\theta}{p - \sigma}}(W(I) + 1)^{\frac{p - 1 - \sigma}{p - \sigma}} \right\}.
\]

The proof is completed. \( \square \)

4 Application

We provide an application of the extrapolation result proved in the previous section to characterize the boundedness of the Hardy averaging operator \( H \) between weighted grand Lebesgue spaces \( L_{w}^{p}(I) \) for quasi-monotone functions. We prove the following:

**Theorem 4.1.** Let \( 1 < p < \infty \), \(-1 < \beta \leq 0 \) and \( \theta > 0 \). The inequality

\[
\| Hf \|_{L_{w}^{p}(I)} \leq C\| f \|_{L_{w}^{p}(I)}^{L_{w}^{p}(I)} \]

holds for all \( f \in Q_{\beta}(I) \) if and only if \( w \in Q_{B_{\beta,p}}(I) \).

**Proof.** Let us first assume that \( w \in Q_{B_{\beta,p}}(I) \). Note that if \( f \in Q_{\beta}(I) \), then for \( 0 < t \leq s \) and \( \alpha \in I \), we have that

\[
t^{-\beta} f \left( \frac{\alpha t}{s} \right) \geq s^{-\beta} f(\alpha)
\]
using which we get that

\[ s^{-\beta} Hf(s) \leq \frac{1}{s} \int_0^s t^{-\beta} f \left( \frac{\alpha t}{s} \right) \, d\alpha \]

\[ = t^{-\beta - 1} \int_0^t f(z) \, dz \]

\[ = t^{-\beta} Hf(t) \]

i.e., \( Hf \in Q_\beta(I) \).

Further, on taking \( \psi \equiv 1 \) in a modified form of Theorem A, and considering the functions \( f \) defined on \( I \) instead of \( (0, \infty) \), we see that the inequality

\[ \int_0^1 (Hf(x))^p w(x) \, dx \leq C \int_0^1 f^p(x) w(x) \, dx \]

holds. Now, in view of Theorem [3.5] the inequality (4.1) holds.

Conversely, assume that the inequality (4.1) holds. Consider the test function \( f_r(x) = x^\beta \chi_{(0,r)}(x) \) for \( 0 < r < 1 \). Then

\[
\|f_r\|_{L^p(I)} = \sup_{0 < r < p - 1} \left( \varepsilon^\theta \int_0^r x^{\beta(p-\varepsilon)} w(x) \, dx \right)^{\frac{1}{p-\varepsilon}} \\
= \max \left\{ \sup_{0 < \varepsilon < \sigma} \varepsilon^\theta \|f_r\|_{L^{p-\varepsilon}}, \sup_{\sigma < \varepsilon < p-1} \varepsilon^\theta \|f_r\|_{L^{p-\varepsilon}} \right\},
\]

where \( \sigma \) is chosen such that \( 0 < \sigma < \min\{(\beta + 1)p, p - 1\} \). Now, for \( \sigma < \varepsilon < p - 1 \), taking the conjugate indices \( \frac{p-\sigma}{p-\varepsilon} \) and \( \frac{\sigma}{\varepsilon-\sigma} \), on using Hölder’s inequality we obtain

\[
\|f_r\|_{L^{p-\varepsilon}} \leq \left( \int_0^r x^{\beta(p-\varepsilon)} w(x) \, dx \right)^{\frac{1}{p-\varepsilon}} (W(I))^{\frac{\varepsilon}{p-\varepsilon}} (W(I) + 1)^{\frac{p-1-\varepsilon}{p-\varepsilon}}. \tag{4.2}
\]

Thus, on using (4.2) and an argument from [22], Theorem 3.1, we have

\[
\|f_r\|_{L^p(I)} \leq \max \left\{ 1, p^\theta \sigma^{-\theta} (W(I) + 1)^{\frac{p-1-\varepsilon}{p-\varepsilon}} \right\} \sup_{0 < \varepsilon < \sigma} \varepsilon^\theta \|f_r\|_{L^{p-\varepsilon}} \\
= \max \left\{ 1, p^\theta \sigma^{-\theta} (W(I) + 1)^{\frac{p-1-\varepsilon}{p-\varepsilon}} \right\} \varepsilon^\theta \|f_r\|_{L^{p-\varepsilon}} \\
= C_1 \left( \varepsilon^\theta \int_0^r x^{\beta(p-\varepsilon)} w(x) \, dx \right)^{\frac{1}{p-\varepsilon}}, \tag{4.3}
\]

for some \( 0 < \varepsilon_r \leq \sigma \), where \( C_1 := \inf_{0 < \sigma < (\beta + 1)p} \max \left\{ 1, p^\theta \sigma^{-\theta} (W(I) + 1)^{\frac{p-1-\varepsilon}{p-\varepsilon}} \right\} \). Further, note that

\[
\int_0^1 (Hf_r(x))^{p-\varepsilon} w(x) \, dx \geq \int_r^1 (Hf_r(x))^{p-\varepsilon} w(x) \, dx = \left( \frac{r^{\beta+1}}{\beta + 1} \right)^{p-\varepsilon} \int_r^1 \frac{w(x)}{x^{p-\varepsilon}} \, dx
\]

so that

\[
\|Hf_r\|_{L^p(I)} \geq \frac{r^{\beta+1}}{\beta + 1} \sup_{0 < \varepsilon < p - 1} \left( \varepsilon^\theta \int_r^1 \frac{w(x)}{x^{p-\varepsilon}} \, dx \right)^{\frac{1}{p-\varepsilon}}
\]
\[
\geq \frac{r^{\beta+1}}{\beta+1} \left( \varepsilon_r \theta \int_r^1 \frac{w(x)}{x^{p-\varepsilon_r}} dx \right)^{\frac{1}{p-\varepsilon_r}}.
\]

The above estimate together with (4.3), and the assumption that (4.1) holds, gives that

\[
\frac{r^{\beta+1}}{\beta+1} \left( \varepsilon_r \theta \int_r^1 \frac{w(x)}{x^{p-\varepsilon_r}} dx \right)^{\frac{1}{p-\varepsilon_r}} \leq CC_1 \left( \varepsilon_r \theta \int_0^r x^{\beta(p-\varepsilon_r)} w(x) dx \right)^{\frac{1}{p-\varepsilon_r}}.
\]

Therefore,

\[
\int_r^1 \left( \frac{r}{x} \right)^{p-\varepsilon_r} w(x) dx \leq (CC_1(\beta+1))^{p-\varepsilon_r} \int_0^r \left( \frac{x}{r} \right)^{\beta(p-\varepsilon_r)} w(x) dx
\]

\[
\leq (CC_1(\beta+1) + 1)^p \int_0^r \left( \frac{x}{r} \right)^{\beta(p-\varepsilon_r)} w(x) dx.
\]

Thus, \( w \in QB_{\beta,p-\varepsilon_r}(I) \), where \( 0 < \varepsilon_r < (\beta+1)p \). Consequently, \( w \in QB_{\beta,p}(I) \) and hence \( w \in \hat{QB}_{\beta,p}(I) \)

Acknowledgment. The first author acknowledges the MATRICS Research Grant No. MTR/2019/000783 of SERB, Department of Science and Technology (DST), India. Also, the second author acknowledges the research fellowship award No.: 09/045(1716)/2019-EMR-I of Council of Scientific and Industrial Research (CSIR), INDIA.

Conflict of interest statement. On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] K.F. Andersen, Weighted generalized Hardy inequalities for non-increasing functions, Can. J. Math., 43 (1991), 1121–1135.

[2] M.A. Arino and B. Muckenhoupt, Maximal functions on classical Lorentz spaces and Hardy’s inequality with weights for non-increasing functions, Trans. Amer. Math. Soc., 320 (1990), 727–735.

[3] J. Bergh, V. Burenkov and L.E. Persson, Best constants in reversed Hardy’s inequalities for quasimonotone functions, Acta Sci. Math. (Szeged), 59 (1994), 221-239.

[4] M.J. Carro and M. Lorente, Rubio de Francia’s extrapolation theorem for \( B_p \) weights, Proc. Amer. Math. Soc., 138 (2010), 629–640.

[5] D.V. Cruze-Uribe, J.M. Martell and C. Pérez, Extrapolation from \( A_\infty \) weights and applications, J. Func. Anal., 213 (2004), 412–439.

[6] D.V. Cruze-Uribe, J.M. Martell and C. Pérez, Weights, Extrapolation and the Theory of Rubio de Francia, Birkhäuser, 2011.

[7] A. Fiorenza, B. Gupta and P. Jain, The maximal theorem for weighted grand Lebesgue spaces, Studia Math., 188 (2008), 123–133.

[8] A. Fiorenza and V. Kokilashvili, Nonlinear harmonic analysis of integral operators in weighted grand Lebesgue spaces and applications, Ann. Funct. Anal., 2017, 1-13. doi.org/10.1215/20088752-2017-0056

[9] J. García-Cuerva and A.E. Gatto, Boundedness properties of fractional integral operators associated to non-doubling measures, Studia Math., 162 (2004), 245—261.
[10] L. Greco, T. Iwaniec and C. Sbordone, *Inverting the p-harmonic operator*, Manuscripta Math. 92 (1997), 249-258.

[11] T. Hagverdi, *On Stability of Bases Consisting of Perturbed Exponential Systems in Grand Lebesgue Spaces*, J. Contemp. Appl. Math., 2 (2021), 81-92.

[12] T. Iwaniec, C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Ration. Mech. Anal. 119 (1992), 129-143.

[13] P. Jain and S. Kumari, *On grand Lorentz spaces and the maximal operator*, Georgian Math. J., 19 (2012), 235-246.

[14] P. Jain, A. Molchanova, M. Singh and S. Vodopyanov, *On grand Sobolev spaces and pointwise description of Banach function spaces*, Nonlinear Analysis, 202 (2021), 1-17. doi.org/10.1016/j.na.2020.112100.

[15] P. Jain, M. Singh and A.P. Singh, *Recent trends in grand Lebesgue spaces* in Function Spaces and Inequalities, Springer Proceedings in Mathematics and Statistics, Volume 206 (2015), 137-159, New Delhi, India.

[16] P. Jain, M. Singh and A.P. Singh, *Hardy-type integral inequalities for quasi-monotone functions*, Georgian Math. J., 24 (2016), 523-533.

[17] V. M. Kokilashvili, *Weighted grand Lebesgue space with a mixed norm and integral operators*, Doklady Mathematics, 100 (2019), 549–550.

[18] V. Kokilashvili, and A. Meskhi, *Extrapolation in Grand Lebesgue Spaces with $A_\infty$ Weights*, Mathematical Notes, 104 (2018), 518–529.

[19] V. Kokilashvili and A. Meskhi, *On integral operators in weighted grand Lebesgue spaces of Banach-valued functions*, Math Meth Appl Sci., 2020, 1–17. doi.org/10.1002/mma.6779

[20] L. Maligranda, *Weighted inequalities for monotone functions*, Collect. Math. 48 (1997), 687-700.

[21] A. Meskhi, *Criteria for the boundedness of potential operators in grand Lebesgue spaces*, Proc. A. Razmadze Math. Inst., 169 (2015), 119–132.

[22] A. Meskhi, *Weighted criteria for the Hardy transform under the $B_p$ condition in grand Lebesgue spaces and some applications*, J. Math Scs., 178 (2011), 622-636.

[23] A. Molchanova, *A note on the continuity of minors in grand Lebesgue spaces*, J. Fixed Point Theory Appl., 2019, pp.1-13. doi.org/10.1007/s11784-019-0686-y

[24] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., 165 (1972), 207-226.

[25] J.L. Rubio de Francia, *Factorization theory and $A_p$ weights*, Amer. J. Math., 106 (1984), 533–547.

[26] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math., 96 (1990), 145–158.