Abstract

We provide textual evidence on divisibility and primality in the ancient Vedic texts of India. Concern with divisibility becomes clear from the listing of all the fifteen pairs of divisors of the number 720. The total number of pairs of divisors of 10,800 is also given. The motivation behind finding the divisors was the theory that the number of divisors of a certain periodic process is related to the count associated with some other periodic process. For example, 720 (days and nights of the year) has 15 pairs of divisors, and this was related to the 15 days of the waxing and waning of the moon. Numbers that have no divisors appeared to have been used to symbolize the “transcendent” that is beyond periodicity and change.

1 Introduction

The Vedas are early texts from India, conservatively dated to a long period beginning in late third or early second millennium BC, which are either in verse as hymns, or as prose commentaries on ritual or as aphorisms dealing with various branches of knowledge. Because of their obscure style, they have not been studied carefully for their mathematical knowledge. These texts have many ill-understood mathematical allusions and they speak of large numbers at many places.
The Vedic literature is to be found in several layers starting with the four Vedas (Ṛgveda, Yajurveda, Sāmaveda, and Atharvaveda), Brāhmaṇa texts, Āranyakas, Upaniṣads, and Śūtras. Some of the materials is arranged in collections called Sāmhitās. The Yajurveda (17.2) gives a sequence of powers of 10 going to $10^{12}$. In the Vedic book called Śatapatha Brāhmaṇa (ŚB) [11], there is a sequence (12.3.2) speaking of different successive divisions of the year that amounts to $10,800 \times 15^6$ parts. Elsewhere the number of stars is given as $1.08 \times 10^7$. Other numbers are used symbolically.

A famous verse from the Isa Upaniṣad speaks of “fullness” from which “fullness” arises and if “fullness” is subtracted from it “fullness” remains, which indicates that the Vedic authors had the intuition of the mathematical idea of infinity. Elsewhere, there is explicit mention of infinity as being uncountable.

The texts have stories that have a mathematical basis. For example, the Maitrāyani Śaṁhitā 1.5.8 [1],[2] has the story of Manu with ten wives, who have one, two, three, four, five, six, seven, eight, nine, and ten sons, respectively. The one son allied with the nine sons, and the two sons allied with the eight, and so on until the five sons were left by themselves. They asked the father for help, and he gave them each a samidh, or “oblation-stick,” which they used to defeat all of the other sons.

Since the ten sons did not ally with anyone, and the pairing of the others, excepting the five left over, is in groups of ten, the counting is in the base 10 system. In this mathematical story, the sticks help make the five stronger than the other 50. Perhaps this happens because each stick has a power of 10, and therefore the 5 now have a total power of 55 which vanquishes the 50. This could imply knowledge of the place value system if one conjectures that each oblation-stick is in the higher place value so that $50+5=55$.

In ŚB 10.4.2, the total number of syllables in the Ṛgveda is taken to be 432,000, equal to the number of muhūrtas in 40 years (each muhūrta = 48 minutes). The Yajurveda and the Sāmaveda taken together are taken to be another 432,000 syllables. Several years ago, I showed that many other numbers in the Vedic texts also have an astronomical basis [3]-[8]. At that time I did not raise the question whether the Vedic sages were aware of properties of numbers beyond their use for numerical calculations.

While doing that research I was aware that an early index to the Ṛgveda due to Śaumaka [9] states that the number of words in the book is 153,826, and this number is twice 76,913, which is a prime number. Since the mantras
are in verses of two lines, one might expect that the number 76,913 is the more basic one and it was deliberately chosen. Likewise, the total number of verses in the Ṛgveda is 10522 = 2 \times 5261, and 5261 is prime. But these two examples of primality could just be accidental, unless one could show that the Vedic authors actually knew this concept.

It appears that in the Vedic texts one can distinguish between numbers that are derived from observed phenomena and others that are ideal, or have an abstract basis. It is the latter numbers that are likely to be prime. It is due to the assumption of connections, bandhu-, between the astronomical, the physical, and the elemental, which is central to Vedic thought, that many numbers are astronomical and related to the motions of the sun, moon, and the planets. Other numbers show up where the narrative transcends astronomy or physical structure and describes the Puruṣa, the Cosmic Man, as in yantras. These are non-astronomical and may be prime or have large prime factors.

This article presents evidence supporting an early tradition of systematic examination of properties of numbers. We begin with the background on mathematical knowledge in the Vedic times. Then we present examples of the counting of all the divisors of a number and the motivation for finding numbers that had a specified number of divisors. Although we do not have evidence from the texts speaking directly of the idea of primality, we suggest that indirect evidence supports the knowledge of this idea.

2 The Mathematics of the Vedic Times

The Brāhmaṇas and the Śulbasūtras [10] give account of early Vedic mathematics. Apart from concerns of geometry and astronomy the Śatapatha Brāhmaṇa (ŚB) [11] deals with the question of all the divisors of a number. It also provides approximations to π [12]. For general surveys of Indian mathematics, see [13]-[16].

The Śulbasūtras give geometric solutions of linear equations in a single unknown. They also deal with quadratic equations of the forms \(ax^2 = c\) and \(ax^2 + bx = c\). Baudhāyana’s Śulbasūtra gives a remarkable approximation to \(\sqrt{2}\):
\[
\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34} = \frac{577}{408}
\]

This is accurate to five decimal places. It is intriguing that Baudhāyana felt the need to add the last term in the expansion because without that the approximation is still valid to three decimal places and excellent for geometric constructions.

The motivation for the mathematics of the Śulbasūtras is the solution to problems of altar construction. Some of these constructions are squaring a circle (derived from the equivalence of the circular earth altar and the square sky altar) and construction of a geometric design of a larger size by increasing the dimensions. Since in the actual construction of such altars the accuracy of the above expansion of \(\sqrt{2}\) would not have been noticed, it is clear that Baudhāyana was interested in mathematical problems and properties of numbers.

The Śulbasūtras belong to the Vedāṅgas, or supplementary texts of the Vedas. Although they are part of the Kalpa Sūtras, which deal with ritual, their importance stems from the constructions they provide for building geometric altars. Their contents, written in the condensed sūtra style, include geometrical propositions and problems related to rectilinear figures and their combinations and transformations, squaring the circle, as well as arithmetical and algebraic solutions to these problems. The root śulb means measurement, and the word “śulba” means a cord, rope, or string.

The extant Śulbasūtras belong to the schools of the Yajurveda. The most important Śulba texts are the ones by Baudhāyana, Āpastamba, Kātyāyana, and Mānava. They have been generally assigned to the period 800 to 500 B.C., although they are likely to be older. Baudhāyana’s text is the oldest, and he is believed to have belonged to the Andhra region. Baudhāyana begins with units of linear measurement and then presents the geometry of rectilinear figures, triangles, and circles, and their transformations from one type to another using differences and combinations of areas. An approximation to the square root of 2 and to \(\pi\) are next given.

Then follow constructions for various kinds of geometric altars in the shapes of the falcon (both rectilinear and with curved wings and extended tail), kite, isosceles triangle, rhombus, chariot wheel with and without spokes,
square and circular trough, and tortoise.

In the methods of constructing squares and rectangles, several examples of Pythagorean triples are provided. It is clear from the constructions that both the algebraic and the geometric aspects of the so-called Pythagorean theorem were known. This knowledge precedes its later discovery in Greece. The other theorems in the Śulba include:

- The diagonals of a rectangle bisect each other.
- The diagonals of a rhombus bisect each other at right angles.
- The area of a square formed by joining the middle points of the sides of a square is half of the area of the original one.
- A quadrilateral formed by the lines joining the middle points of the sides of a rectangle is a rhombus whose area is half of that of the rectangle.
- A parallelogram and rectangle on the same base and within the same parallels have the same area.
- If the sum of the squares of two sides of a triangle is equal to the square of the third side, then the triangle is right-angled.

A variety of constructions are listed. Some of the geometric constructions in these texts are based on algebraic solutions of simultaneous equations, both linear and quadratic. It appears that geometric techniques were often used to solve algebraic problems.

The Śulbas are familiar with fractions. Algebraic equations are implicit in many of their rules and operations. For example, the quadratic equation and the indeterminate equation of the first degree are a basis of the solutions presented in the constructions.

The Śulba geometry was used to represent astronomical facts. The altars that were built according to the Śulba rules demonstrated knowledge of the lunar and the solar years.

To consider the larger background of Indian science requires an understanding of its cosmology [17]-[21]. Seidenberg [22],[23] examined the geometry and mathematics of the Satapatha Brāhmaṇa and he argued that the
philosophy that equivalent altars were to have equal areas led to the pos-
ing of basic problems of geometry leading to results such as the Pythagoras
theorem. A conservative chronology has placed the final form of this book
to 1000-800 B.C.; new archaeological discoveries suggest that this book may
be a thousand years older. There is also a specific astronomical reference in
the book, namely that the Pleiades do not swerve from the east, that was
true only in late third millennium B.C. Corroborating evidence comes from
the text, since it claims to belong to a period when the Sarasvati river had
recently dried up. New research shows that this drying up took place around
2000 B.C.

The Vedic Indians were also interested in metres and music and related
mathematical problems [24], [25]. The earliest codified Vedic astronomy is
given in Lagadha’s Vedāṅga Jyotīṣa [26].

3 Number Divisors

The number 360, the days in the year, forms a starting point in the design of
the altars of ŚB that were used to represent various astronomical facts about
the year.

ŚB 10.4.2.1-18 points out that 720, the nights and days in the nominal
year, has exactly 15 factors that are smaller than the companion. The text
is speaking the year as Prajāpati (the Lord of the people) represented by 720
bricks who is considering the various divisors of the number [11]:

He divided his body into two: there were three hundred and
sixty bricks in the one, and as many in the other; he did not
succeed [in finding all the divisors].

He made himself three bodies, and in each there were three
times eighty (240) bricks; he did not succeed.

....

He made himself six bodies of a hundred and twenty brick
each; he did succeed. He did not divide himself sevenfold.

He made himself eight bodies of ninety bricks each; he did not
succeed.

....
He made himself twenty bodies of thirty-six brick each; he did not succeed. He did not divide himself either twenty-one fold, or twenty-two fold, or twenty-three fold.

He made himself twenty-four bodies of thirty bricks each. There he stopped, at the fifteenth, and because he stopped at the fifteenth arrangement [having found all the divisors], there are fifteen forms of the waxing, and fifteen forms of the waning (moon).

These divisors are thus stated to be: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24. The numbers in this sequence which do not divide 720 (that is the numbers 7, 11, 13, 14, 17, 19, 21, 22, 23) are also explicitly mentioned.

The next passage claims that 24, the largest of these numbers, represents the number of half months in the year, which is a reiteration of $15 \times 24 = 360$.

This implies the thesis that the moon’s circuit around the earth is 30 days because 720, the number of days and nights in the year, has exactly 30 different divisors. In other words, numbers are used to relate a characteristic of the motion of the sun to another related to the moon’s orbit around the earth. An astronomical fact is “explained” based on an abstract numerical property.

It appears that the authors did not rather consider the number 360, which has 12 divisors, which could have also been taken to correspond to the 12 months of the year, yielding, in turn, the 30-day duration of the month, because the largest of these divisors which is less than its companion, 18, has no direct correspondence with the basic facts of the year.

The concern with the number of divisors implies finding out what numbers do not have divisors, or are prime. We know, that for number $n = \prod p_i^{a_i}$, the number of divisors $d(n)$ is given by:

$$d(n) = (a_1 + 1)(a_2 + 1)$$

The pairs of these divisors, in the manner of counting by the Vedic authors, is $\delta(n) = d(n)/2$.

The Vedic authors were also interested in the largest divisor, whose companion is smaller than itself. If this divisor is called $\nu(n)$, we have $\nu(720) = 24$. 

7
3.1 Divisors of sixty and the 6-day week

One would expect that paralleling a justification of the thirty day month based on the number of factors of 720, an argument would have been used to define divisions of the month. The argument would look at the 60 days and nights of the month and determine the number of divisors that, paralleling the procedure in ŚB, are less than the companion. These divisors are 1, 2, 3, 4, 5, and 6. This suggests a week of six days. We do get references to the six-day week called ṣaḍaha in the Vedic texts. Five ṣaḍahas made a month. This defined a symmetry with the year of five seasons and the yuga of five years.

Although it has often been assumed that the seven-day week was a later innovation, it is quite possible that it was the older tradition and that the six-day week got mentioned in the texts because of the “theory” behind its derivation. The seven-day week is a more natural, system because it divides the lunar month into four equal parts.

The number 7 plays an important number in Vedic cosmology since it appears in conjunction with the name of the entire country, Saptapáta Sarthi, with the additional ideas of seven rivers, seven continents, seven islands, seven mountains, seven rishis (the Pleiades), seven musical notes, and seven worlds.

3.2 The division of the year into 10,800 muhūrtas

ŚB 10.4.2.36 speaks of the division of the year into 10,800 muhūrtas. It is further stated that the divisors of this number go into 30 arrangements, or 30 pairs of divisors. Therefore, \( \delta(10,800) = 30 \), or \( d(10,800) = 60 \). This is indeed correct:

\[
d(10,800) = d(2^4 \times 3^3 \times 5^2) = (4 + 1)(3 + 1)(2 + 1) = 60
\]

The discovery of the 30 pairs of divisors of 10,800 suggests that this came by searching divisors of several suitable astronomical numbers for one that had precisely 30 pairs of divisors. This indicates that the Vedic authors knew that number of divisors varied and that some numbers had no divisors.

The text relates the 30 pairs of divisors of 10,800 to the 30 nights of the month.
4 Astronomical and Non-astronomical Numbers

Astronomical numbers in the Vedic texts are related to the 360 tithis of the lunar year, the 365 or 366 days of the solar year (or 371 or 372 tithis), the 27 or 28 nakṣatras, the 29 days of the month, numbers related to the divisions of the year, the planet orbits, synchronization periods for the lunar-solar motions or planet motions, and so on. Another astronomical number is 108, the distance to the sun or the moon in terms of multiples of their respective diameters [3], and it figures in the earliest temple design [27]. This number is also related to the nakṣatra year of 324 days (12 “months” of 27 lunar days).

Astronomical numbers are generally highly composite. The best examples are the cosmic cycle numbers, such as the longest cycle of 311,040,000 million years.

Other numbers, which appear non-astronomical, may actually have an astronomical basis. For example, the 33 gods are likely related to the count of 27 nakṣatras, five planets, and the moon. Or consider Āyurvedic physiology where the number of joints in the body (marma) is taken to be 107, a prime number. Its primality may not be the reason for its choice, because the 108 disks of the sun from the earth may have been taken to be mirrored in 108 links from the feet to the crown, and these 108 links will then have 107 joints.

Some of the non-astronomical origin numbers may not be actual counts, but rather ideal counts and, therefore, the choice of the number as a prime becomes significant.

A yantra is a representation of the universe. The earliest yantras, in the shape of buildings, have been excavated [28] in North Afghanistan and they date to 2000 BC. Some people see in Atharvaveda 10.2.31-32, which speaks of a eight-wheeled, nine-doored impregnable stronghold of the gods, the earliest textual reference to a yantra. The famous Śri Yantra, created out of 9 juxtaposed triangles, leads to 43 smaller triangles [29] Whether the primality of 43 is significant here, we cannot tell.

Another description, that is almost certainly that of a yantra, is found in the Śvetāśvatara Upaniṣad 1.4: “The Wheel of Brahman has one felly, a triple tire, sixteen end-parts, fifty spokes with twenty counter-spokes, and six sets of eight; whose one rope is manifold; (which moves on three different roads;
and whose illusion arises from two causes).” The definition of this yantra is complete with the description of the rope, the remainder describes two characteristics of it. The one rope (pāśa) is the central focus, the bindu which holds it together; the six sets of eight appear to be the 6 central triangles, three of which are pointed upward and three are pointed downward. The total count here is $1 + 1 + 16 + 50 + 20 + 48 + 1 = 137$, a prime number.

5  A Prime Number from Ideal Physiology

In Vedic thought, the human body is taken to mirror the universe, and it is supposed to have cyclic processes that are synchronized to that of the sun, the moon and other luminaries. The nādīs, nerves, are taken to be the web within the body where impressions are stored. It is also believed that by concentration one can transcend these impressions and achieve union with Brahman.

An early count of the nādīs is given in the Brhadāranyaka Upaniṣad 2.1.19. It speaks of 72,000 nādīs called hitāḥ that spread from the heart. The Chāndogya U. 8.6.6. speaks of 101 nādīs from the heart.

The Praśna U. 3.6 has the complete description where it is stated that there are 101 chief nādīs, each with 100 branch nādīs, each of which in turn has 72,000 tributary nādīs.

This makes a total of $101 + 101 \times 100 + 101 \times 100 \times 72,000 = 727,210,101$. This is equal to $101 \times 7,200,101$, each of which factor is a prime number.

The prime number 7,200,101 is the number of nādīs emerging from each of the 101 chief nādīs of the body.

6  Conclusions

Vedic texts are written in a style that presents several difficulties to the reader uninitiated in its cosmology. The reader needs to understand that narratives on ritual have a subtext that deals with astronomy and/or physiology because of the assumed equivalences between the microworld and the macroworld. These texts are associated with a numerical mysticism in which the many (numbers) emerge from the one (number) which is why properties of numbers were of interest to the Vedic authors.
We have presented evidence that the Vedic authors connected the number of divisors to certain periodic processes. The texts provide examples of systematic calculation of the divisors, suggesting that they were aware that other numbers do not have divisors excepting one and the number itself. The matter of counting of divisors is given prominence in the Śatapatha Brāhmaṇa.

Multiplication tables as well as Pythagorean triples have been obtained from tablets belonging to the Old Babylonian period of about 1700 BC [30]. But these lists do not present a systematic derivation of any number-theoretic property. The examination of the evidence from the Vedas is, therefore, important in the history of ideas.

The primality of the count of the nādis, the word and verse counts in the Rgveda, and the yantra counts could be coincidental. But the fact that the Vedic authors were counting divisors suggests that these numbers were deliberately chosen.

Summarizing, we show that the number of divisors of numbers such as 720 and 10,800 was systematically calculated. The concern with a complete list of divisors of a number suggests knowledge of primality although it cannot be the proof of that knowledge.

Acknowledgement. I thank A. Raghuram for comments on an earlier version of the paper.

References

1. S.D. Satvalekar (ed.), Maitrayani Samhita. Government Press, Bombay, 1941.

2. M.D. Pandit, Mathematics as known to the Vedic Samhitas. Sri Satguru Publications, Delhi, 1993.

3. S. Kak. The astronomy of the Vedic altars and the Rgveda. *Mankind Quarterly*, 33, 43-55, 1992.

4. S. Kak. The astronomy of the Vedic altars. *Vistas in Astronomy*, 36, 117-140, 1993.

5. S. Kak, The structure of the Rgveda. *Indian Journal of History of Science*, 28, 71-79, 1993.
6. S. Kak, From Vedic science to Vedanta. *Adyar Library Bulletin*, 59, 1-36, 1995.

7. S. Kak, Archaeoastronomy and literature. *Current Science*, 73, 624-627, 1997.

8. S. Kak, Time, space and structure in ancient India. Presented at the Conference on Sindhu-Sarasvati Valley Civilization: A Reappraisal, Loyola Marymount University, Los Angeles, February 21 & 22, 2009; arXiv:0903.3252

9. A.A. Macdonell, *Kātyāyana’s Sarvānukramaṇī of the Rigveda*. Clarendon Press, Oxford, 1886.

10. S.N. Sen and A.K. Bag, *The Śulbasutras*. Indian National Science Academy, New Delhi, 1983.

11. J. Eggeling (tr.), *The Satapatha Brahmana*. Charles Scribner’s Sons, New York, 1900; http://www.sacred-texts.com/hin/sbr/index.htm

12. S. Kak, Three old Indian values of pi. *Indian Journal of History of Science*, 32, 307-314, 1997.

13. B. Datta and A.N. Singh, *History of Hindu Mathematics*. Asia Publishing House, Bombay, 1962.

14. C.N. Srinivasan, *The History of Ancient Indian Mathematics*. The World Press, Calcutta, 1967.

15. G.G. Joseph, *The Crest of the Peacock, non-European roots of Mathematics*. Princeton University Press, Princeton, 2000.

16. I.G. Pearce, *Indian Mathematics: Redressing the balance*. 2002. http://www-groups.dcs.st-and.ac.uk/history/Projects/Pearce/index.html

17. S. Kak, The Indus tradition and the Indo-Aryans. *Mankind Quarterly*, 32, 195-213, 1992.

18. S. Kak, Greek and Indian cosmology: review of early history. In *The Golden Chain*. G.C. Pande (ed.). CSC, New Delhi, 2005; arXiv: physics/0303001.
19. S. Kak, Indian physics: outline of early history. arXiv: physics/0310001.

20. S. Kak, *The Gods Within*. Munshiram Manoharlal, New Delhi, 2002.

21. S. Kak, *The Architecture of Knowledge*. CSC, Delhi, 2004.

22. A. Seidenberg. The ritual origin of geometry. *Archive for History of Exact Sciences*, 1, 488-527, 1962.

23. A. Seidenberg. The origin of mathematics. *Archive for History of Exact Sciences*, 18, 301-342, 1978.

24. S. Kak, The golden mean and the physics of aesthetics. *Foarm Magazine*, 5, 73-81, 2006; arXiv:physics/0411195

25. K.D. Dvivedi and S.L. Singh, *The Prosody of Pingala*. Vishwavidyalaya Prakashan, Varanasi, 2008.

26. S. Kak, The astronomy of the age of geometric altars. *Quarterly Journal of the Royal Astronomical Society*, 36, 385-396, 1995.

27. S. Kak, The axis and the perimeter of the temple. *Kannada Vrinda Seminar Sangama* 2005 held at Loyola Marymount University in Los Angeles, November 19, 2005; arXiv:0902.4850

28. V.I. Sarianidi, *Drevnie zemeldel'tsy Afganistana*. Moskva, 1977.

29. Kulaichev, A.P., “Śriyantra and its mathematical properties,” *Indian Journal of History of Science*, 19, 279-292, 1983.

30. O. Neugebauer. *The Exact Sciences In Antiquity*. Dover Publications, New York, 1969.