Matrix-valued Bratu equation associated with the symmetric domains of type BDI and CI

Hiroto Inoue*

May 26, 2020

Abstract

A formulation of the exponential matrix solution of the matrix-valued Bratu equation is given, based on the structure of the symmetric domain of type BDI. Moreover, an analog for the symmetric domain of type CI is given.

Contents

1 Introduction

2 Proof for exponential matrix solution
  2.1 Block-Gauss decomposition
  2.2 Lagrangian calculus

3 Expression in symmetric domains
  3.1 Siegel domain
  3.2 Bounded domain of type BDI and power series solution

4 Analog to symmetric domain of type CI
  4.1 Siegel domain $D(J; K)$ of type CI

5 Remarks

Notation Throughout this article, $\text{Sym}^+_n(\mathbb{R})$ is the set of $n$-dimensional positive-symmetric matrices and $\text{M}_{n,r}(\mathbb{R})$ is the set of $(n \times r)$ real matrices.

1 Introduction

We consider the matrix-valued Bratu equation

$$(h'(s)h(s)^{-1})' = (a'a)h(s)^{-1}, \quad a \in \text{M}_{n,r}(\mathbb{R})$$

(1.1)

for a $\text{Sym}^+_n(\mathbb{R})$-valued function $h(s)$ of $s \in \mathbb{R}$ introduced in [1]. The solution of the initial value problem of the Bratu equation (1.1) was given in [1] using a specific exponential matrix.

In this article, we give a geometrical formulation of this solution based on the geometrical structure of the symmetric domains $G/K$. In detail, we express the exponential matrix solution of the equation (1.1) by the defining function of the symmetric domain of type BDI according to Cartan’s classification, ref. [2, 3]. Moreover, we give an analog associated with the symmetric domain of type CI, as Table 1.

Table 1: Matrix-valued differential equations associated to symmetric domains

| domain | condition for $\Delta(w, e^x w)$ | equation of $\Delta(e^x w, e^x w)^{-1}$ |
|--------|----------------------------------|----------------------------------------|
| BDI    | $(a'a)^2 + O(s^3)$               | $(h'h^{-1})' = (a'a)h^{-1}$            |
| CI     | None                             | $(h'h^{-1})' = (c'h^{-1})^2$           |

2 Proof for exponential matrix solution

2.1 Block-Gauss decomposition

Manifold $\Omega(J)$ We define a submanifold $\Omega(J)$ in the cone $\text{Sym}^+_n(\mathbb{R})$ by

$$\Omega(J) = \{ G \in \text{Sym}^+_n(\mathbb{R}) : JGJ = G^{-1} \},$$

$$J = \begin{pmatrix} 0 & 0 & I_n \\ 0 & I_r & 0 \\ I_n & 0 & 0 \end{pmatrix}.$$  

Setting the set $V(J)$ by

$$V(J) = \{ B \in \text{Sym}^+_n(\mathbb{R}) : JBJ = -B \},$$

we can consider the following map:

$$V(J) \ni B \mapsto \exp B \in \Omega(J).$$

We see that any element $B \in V(J)$ has the following form:

$$B = B(b, a, c) := \begin{pmatrix} b & a & c \\ a & 0 & -a \\ c & -a & -b \end{pmatrix}, \quad b \in \text{Sym}^+_n(\mathbb{R}), \quad a \in \text{M}_{n,r}(\mathbb{R}), \quad c \in \text{Alt}_n(\mathbb{R}).$$

For an element $B \in V(J)$, we define an $\Omega(J)$-valued function $G(s; B)$ by

$$G(s; B) := \exp(sB(b, a, c)) \quad (s \in \mathbb{R}).$$
We define the following sets of matrices:

\[ G(J) = \{ G \in \text{GL}_{2n+r} \mathbb{R); JGJ = JG^{-1} \} \cong O(n + r, n), \]

\[ N(J) = \left\{ N \in G(J); N = \begin{pmatrix} I_n & 0 & 0 \\ \ast & I_r & 0 \\ \ast & \ast & I_n \end{pmatrix} \right\}, \]

\[ N = \begin{pmatrix} I_n & 0 & 0 \\ n_1 & I_r & 0 \\ n_2 & -n_1 & I_n \end{pmatrix}, n_2 + n_1 = -n_1 n_1 \}

\[ \Omega(J) = \left\{ A \in \Omega(J); A = \begin{pmatrix} 0 & 0 & 0 \\ \ast & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \]

\[ A = \begin{pmatrix} h & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & h^{-1} \end{pmatrix}, h = h. \]

**Proposition 2.1** (Block-Gauss decomposition for \( \Omega(J) \)).

The following map is bijective:

\[ N(J) \times \Omega(J) \rightarrow \Omega(J) \]

\[ (N, A) \rightarrow NA^tN. \]

We denote the variable change defined through this decomposition by

\[ G \rightarrow N, A \rightarrow h, n_1, n_2, \]

or simply by \( G \rightarrow h \).

Now we state the exponential matrix solution in [I2020] as follows.

**Proposition 2.2.** For \( B = B(h, a, 0) \in V(J) \), define a \( \text{Sym}_n^+(\mathbb{R}) \)-valued function \( \tilde{h}(s) \) by the variable change

\[ G(s; B) \rightarrow \tilde{h}(s). \]

Then, \( \tilde{h}(s) \) satisfies the matrix-valued Bratu equation:

\[ (\tilde{h}'(s) \tilde{h}(s)^{-1})' = (a' a) \tilde{h}(s)^{-1}. \]

In the rest of this section, we give an alternative proof for this proposition based on the Lagrangian calculus.

### 2.2 Lagrangian calculus

**Lemma 2.1.** Let \( G = G(s) \) be an \( \Omega(J) \)-valued function. Under the variable change

\[ G \rightarrow N, A \rightarrow h, n_1, n_2, \]

the following equation holds:

\[ \frac{1}{2} \text{tr}((G'G^{-1})^2) = \frac{1}{2} \text{tr}((A'A^{-1})^2) + \text{tr}(A'A^{-1}N^{-1}N^tN'^{-1}N') \]

\[ = \text{tr}((h'h^{-1})^2) + 2 \text{tr}(hn'h_1 + tr(h'nmh)), \]

\[ m = n_1'n_1 + n_2' = -m. \]

**Proof.** To prove this lemma, we use the following orthogonal relation.

**Lemma 2.2.** Define a subset of \( M = M_{2n+r}(\mathbb{R}) \) as

\[ M_{\leq 0} = \left\{ X \in M; X = \begin{pmatrix} \ast & 0 & 0 \\ \ast & \ast & \ast \end{pmatrix} \right\}, \]

\[ M_{< 0} = \left\{ X \in M; X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \]

Then, it holds that

\[ \text{tr}(X_1X_2) = 0 \]

for any \( X_1 \in M_{\leq 0}, X_2 \in M_{< 0}. \)

Inserting \( G = NA^tN \) into \( G'G^{-1} \), we get

\[ G'G^{-1} = NA^tN N^{-1} \]

\[ = NA^tN N^{-1}A^{-1}N^{-1} + N^tA^{-1}N^{-1} \]

\[ \rightarrow X_1 + X_2 + X_3. \]

Here we see that

\[ X_1 \in \text{Ad}(N)^tM_{\leq 0}, X_2 \in \text{Ad}(N)^tM_{\leq 0} \cap M_{\leq 0}, X_3 \in M_{< 0}. \]

Then, by Lemma 2.2, we get

\[ \frac{1}{2} \text{tr}((G'G^{-1})^2) = \text{tr}(X_1X_3) + \frac{1}{2} \text{tr}(X_2X_2). \]

Inserting

\[ A'A^{-1} = \begin{pmatrix} h'h^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -h^{-1}h' \end{pmatrix}, \]

\[ N^{-1}N' = \begin{pmatrix} n_1' & 0 & 0 \\ 0 & 0 & 0 \\ m & -n_1' & 0 \end{pmatrix}, m = n_1'n_1 + n_2', \]

we calculate as

\[ \text{tr}(X_1X_3) = \text{tr}(A'A^{-1}N^{-1}A^{-1}N^{-1}N') \]

\[ = 2 \text{tr}(hn'h_1) + \text{tr}(h'nmh), \]

\[ \frac{1}{2} \text{tr}(X_2X_2) = \frac{1}{2} \text{tr}((A'A^{-1})^2) = \text{tr}((h'h^{-1})^2). \]

Therefore, we get the Lagrangian as

\[ \frac{1}{2} \text{tr}((G'G^{-1})^2) = \text{tr}((h'h^{-1})^2) + 2 \text{tr}(hn'h_1) + \text{tr}(h'nmh). \]

Next we proceed to derive the Euler-Lagrange equation. For the preparation, we deal with the constrain condition for \( G(s) \) as follows.
Lemma 2.3. For the Lagrangian
\[
\mathcal{L} = \frac{1}{2} \text{tr}((G'G^{-1})^2) + \text{tr}((JG'G-I)\lambda) + \text{tr}(G-G'\mu)
\]
for GL(n, \mathbb{R})-valued function G, \lambda, \mu, the Euler-Lagrange equation reads
\[
\begin{cases}
(G'G^{-1})' = 0, \\
G(\mu - \mu)G + J(\mu - \mu)J = 0.
\end{cases}
\]

Proof. By the property of the matrix trace, we have the equation
\[
\text{tr}(JG_1JG_2) = \text{tr}(G_1 \cdot JG_2\lambda) = \text{tr}(G_2 \cdot JG_1J\lambda).
\]
Then, we can derive the Euler-Lagrange equation as
\[
G' = (G'G^{-1}G)'G^{-1} = JG\lambda J + J'GJ\lambda, \\
\lambda = GJ'G = J, \\
\mu = G - G = 0.
\]
Applying the map \(X = X - J'XJ\) to the both sides of the first equation, we get
\[
2(G'G^{-1})' = G(\mu - \mu) + J(\mu - \mu)JG^{-1}
\]
\[
\Leftrightarrow 2(G'G^{-1})'G = G(\mu - \mu)G + J(\mu - \mu)J.
\]
We see that the left hand side is a symmetric matrix and the right hand side is a skew-symmetric matrix. Therefore, they turn to be 0.

\[\square\]

Derivation of Euler-Lagrange equation. Now we derive the Euler-Lagrange equation for our Lagrangian
\[
\mathcal{L} = \mathcal{L}(h, n_1, n_2, h', n'_1, n'_2)
\]
\[
= \text{tr}((h'_{-1})^2) + 2 \text{tr}(hn'_1n'_1) + \text{tr}(h'mhm),
\]
\[
m = n_1n'_1 + n_2.
\]
The result is written as follows:
\[
\text{variable } x: \quad \text{E-L eq. } \frac{d}{ds}\frac{\partial\mathcal{L}}{\partial x'} - \frac{\partial\mathcal{L}}{\partial x} = 0
\]
\[
h: \quad (h'h_{-1})' = hn'_1n'_1 + h'mhm,
\]
\[
n_1: \quad 2hn'_1 + h'mhn'_1 = hnmn'_1,
\]
\[
n_2: \quad (h'mh)' = 0.
\]
From the third equation, we get
\[
h'mh = \tilde{c} : \text{constant matrix, } \quad \tilde{c} \in \text{Alt}_n(\mathbb{R}).
\]
Thus, the above equations become
\[
h: \quad (h'h_{-1})' = hn'_1n'_1 - ch^{-1}ch^{-1},
\]
\[
n_1: \quad 2(hn'_1)' + 2(\tilde{c}n_1)' = 0.
\]
From the second equation, we get
\[
hn'_1 + cn_1 = \tilde{a} : \text{constant matrix.}
\]
Especially, in the case \(\tilde{c} = 0\), we get \(hn'_1 = \tilde{a}\).

Inserting this into the first equation, we obtain the matrix-valued Bratu equation
\[
h: \quad (h'h_{-1})' = (\tilde{a}'\tilde{a})h^{-1}.
\]
Next we identify the constant matrices \(\tilde{c}, \tilde{a}\).
Defining the function \(G(s)\) by the relation \(G \rightarrow h, n_1, n_2\), we have that
\[
G'G^{-1} = \begin{pmatrix} * & h'n_1 - hnmn_1 & h'mh \\ * & * & * \\ * & * & *
\end{pmatrix} = \begin{pmatrix} * & \tilde{a} & \tilde{c} \\ * & * & * \\ * & * & *
\end{pmatrix}
\]
On the other hand, \(G(s)\) satisfies the original E-L equation
\[
G'G^{-1} = (\text{constant matrix}).
\]
This shows that \(G(s)\) is given as
\[
G = G(s; B(\tilde{b}, \tilde{a}, \tilde{c})), \quad \tilde{b} \in \text{Sym}_n(\mathbb{R}).
\]
We summarize the above calculation in the following table:

| \(G(s; B(b, a, c))\) | \(h(s), n_1(s), n_2(s)\) |
|---------------------|---------------------|
| \((G'G^{-1})' = 0\) | \(hn'_1 + cn_1 = a\) |
| \(\frac{1}{2} \text{tr}((G'G^{-1})^2)\) | \(hn'_1 + n_2h = c\) |

Theorem 2.1. The equation
\[
G'(s)G(s)^{-1} = B(b, a, c)
\]
for an \(\Omega(J)\)-valued function \(G(s)\) is expressed as
\[
\begin{cases}
(h'h_{-1})' = hn'_1n'_1 - ch^{-1}ch^{-1}, \\
chn'_1 + cn_1 = a, \\
hn'_1 + (n'_1 + n'_2)h = c
\end{cases}
\]
by the variable change \(G(s) \rightarrow h(s), n_1(s), n_2(s)\). Especially, in the case \(c = 0\), it is equivalent to the matrix-valued Bratu equation
\[
(h'h_{-1})' = (a'\tilde{a})h^{-1}.
\]
The exponential matrix solution (Proposition 2.2) follows from this theorem.
3 Expression in symmetric domains

Considering a realization of $\Omega(J)$ as a symmetric domain, we express

- the condition $c = 0$ for $B = B(b, a, c),$
- the variable change $G \to h$ by the block-Gauss decomposition
in Proposition 2.2 in terms of the structure of symmetric domains.

3.1 Siegel domain of type BDI

Siegeldomain $D(J)$ of type BDI For the set

$$M = \left\{ U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \in M_{2n+r,n}(\mathbb{R}); \text{rank}U = n \right\},$$

we consider the group actions

$$\text{GL}(2n + r, \mathbb{R}) \curvearrowright M \curvearrowleft \text{GL}(n, \mathbb{R})$$

by the matrix multiplication. For the quotient

$$D := M/\text{GL}(n, \mathbb{R}) = \left\{ u = [U] = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}; U \in M \right\},$$

we equip it with the induced topology to consider $D$ as a domain. We let

$$J = \begin{pmatrix} 0 & 0 & I_n \\ 0 & I_r & 0 \\ I_n & 0 & 0 \end{pmatrix}$$

and define a subdomain $D(J) \subset D$ by

$$M(J) = \left\{ U \in M; \text{rank}UJU > 0 \right\},$$

$$D(J) := \left\{ u = [U] \in D; U \in M(J) \right\}.$$

Lemma 3.1 (Realization as a Siegel domain). The domain $D(J)$ is the following set:

$$D(J) = \left\{ u = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \in D; -U_1 - U_1 - U_2U_2 > 0 \right\}.$$ We define a subdomain $\Sigma \subset D$ (the Shilov boundary of $D(J)$) by

$$\Sigma = \left\{ u = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \in D; -U_1 - U_1 - U_2U_2 = 0 \right\}.$$ We take two points $u_0, w \in D$ as

$$u_0 = [U_0] \in D(J), \quad w = [W] \in \Sigma,$$

$$U_0 = \begin{pmatrix} -I_n \\ 0 \\ I_n \end{pmatrix}, \quad W = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ For two points $u_1, u_2 \in D(J)$ expressed as

$$u_j = [U_j], \quad U_j = \begin{pmatrix} U_{j,1} \\ U_{j,2} \\ U_{j,3} \end{pmatrix} \quad (j = 1, 2),$$

we define $\Delta(u_1, u_2) \in M_n(\mathbb{R})$ by

$$\Delta(u_1, u_2) := t(U_1 \Gamma_1^{-1})J(U_2 \Gamma_2^{-1}),$$

$$\Gamma_j = \Gamma WU_j = U_{j,3} \quad (j = 1, 2).$$

Expression of $D(J)$ as a symmetric space $G/K$
Consider the action $G(J) \curvearrowright D(J)$ defined by

$$g \cdot u = [gU] \quad (g \in G(J), u = [U] \in D(J)).$$

We also define the following subgroups:

$$\mathcal{P}(J) = \left\{ g \in \mathcal{G}(J); g = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\},$$

$$\mathcal{K}(J) = \left\{ g \in \mathcal{G}(J); \quad \Gamma J = \Gamma^{-1} \right\} \cong O(n + r) \times O(n),$$

$$\mathcal{G}(J)_{u_0} = \left\{ g \in \mathcal{G}(J); \quad gu_0 = u_0 \right\}.$$

Lemma 3.2. The following assertions hold:

1. $\mathcal{G}(J) = \mathcal{P}(J)\mathcal{K}(J).$
2. $\mathcal{G}(J)_{u_0} = \mathcal{K}(J).$
3. The action $\mathcal{G}(J) \curvearrowright D(J)$ is transitive, and it holds that

$$D(J) \cong SO_0(n + r, n)/SO(n + r) \times SO(n).$$

Proof. 2 Let

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & 0 & -I_n \\ 0 & \sqrt{2}I_r & 0 \\ I_n & 0 & I_n \end{pmatrix},$$

and

$$v_0 := Au_0 = \begin{pmatrix} I_n \\ 0 \\ 0 \end{pmatrix}, \quad K := AJ^TA = \begin{pmatrix} I_n & 0 \\ 0 & -I_{n+r} \end{pmatrix},$$

$$\mathcal{G}(K) = \mathcal{A}\mathcal{G}(J)^TA.$$ For $g \in \text{GL}(2n + r, \mathbb{R})$, we see the equivalence

$$gv_0 = v_0 \iff g = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$ Thus, for $g \in \mathcal{G}(K)$, it holds the equivalence

$$gv_0 = v_0 \iff g = t \cdot g^{-1}.$$ Therefore, for $g \in \mathcal{G}(J)$,

$$gu_0 = u_0 \iff g = t \cdot g^{-1}.$$

\[\square\]
Lemma 3.3. Isomorphism $F : \Omega(J) \sim \rightarrow D(J)$ We define a map $F : \Omega(J) \rightarrow D$ by

$$F(G) = \begin{bmatrix} G_3 - I_n \\ G_2 \\ h \end{bmatrix}, \quad G = \begin{pmatrix} h & * \\ G_2 & * \\ G_3 & * \end{pmatrix} \in \Omega(J),$$

and a map $\pi : \Omega(J) \rightarrow \text{Sym}_n^\times(\mathbb{R})$ by

$$\pi(G) = h, \quad G = \begin{pmatrix} h & * \\ G_2 & * \\ G_3 & * \end{pmatrix} \in \Omega(J).$$

We define an action $G(J) \curvearrowright \Omega(J)$ by

$$g \cdot G = t^gGt^{-1} (g \in G(J), G \in \Omega(J)).$$

Lemma 3.3. 1. For any $g \in G(J), G \in \Omega(J)$, it holds that

$$F(g \cdot G) = gF(G).$$

2. It holds that $F(\Omega(J)) \subset D(J)$, and $F : \Omega(J) \rightarrow D(J)$ is bijective.

3. For any $G \in \Omega(J)$, it holds that

$$\frac{1}{2} \pi(G) = \Delta(F(G), F(G))^{-1}.$$

By this lemma, we obtain the following commutative diagram:

$$\begin{array}{ccc}
\Omega(J) & \xrightarrow{F} & D(J) \\
\downarrow{\pi} & \circ & \downarrow{\Delta(\cdot, \cdot)^{-1}} \\
\text{Sym}_n^\times(\mathbb{R}) & \xrightarrow{\text{Sym}_n^\times(\mathbb{R})} & \text{Sym}_n^\times(\mathbb{R})
\end{array}$$

Proof. 1. First, we prove for the case $G = I_{2n+r}$. Take an arbitrary $g \in G(J)$ and express it as

$$g = pk, \quad p \in \mathcal{P}(J), \quad k \in K(J)$$

Then, from the relation $t^p = JpJ$ we get

$$p \cdot I_{2n+r} = t^p^{-1}p^{-1}$$

Thus, its image under the map $F$ is

$$\Delta(F(G), F(G)) = \begin{pmatrix} -p_1 - p_2 \\ p_3 \\ 0 \end{pmatrix} = \begin{pmatrix} -p_1 + t^p_3 \\ t^p_2 \\ t^p_1 \end{pmatrix}.$$
Proof. Let
\[ \Gamma_1 = \gamma W W = I_n, \quad \Gamma(s) = \gamma W e^{sB} W, \]
\[ \Phi(s) = \gamma W J e^{sB} W. \]
Then, we have
\[ \Delta(w, e^{sB}w) = \gamma(WT_1^{-1})J(e^{sB}W\Gamma(s)^{-1}) = \Phi(s)\Gamma(s)^{-1}. \]
By the definition, it is easily seen that
\[ \Phi(s) = -cs + (a'u - bc + cb)\frac{s^2}{2} + O(s^3), \]
\[ \Gamma(s)^{-1} = I_n + bs + O(s^2). \]
Then, we obtain the expansion
\[ \Delta(w, e^{sB}w) = -cs + (a'u - bc - cb)\frac{s^2}{2} + O(s^3). \]

Proposition 3.1. For \( B \in \mathbb{P}(J) \), assume the Taylor expansion
\[ \Delta(w, e^{sB}w) = (a'u)\frac{s^2}{2} + O(s^3), \quad a \in M_{n,r}(\mathbb{R}) \]
at \( s = 0 \). Define a \( \text{Sym}_n^+(\mathbb{R}) \)-valued function \( h(s) \) by
\[ h(s) = \Delta(e^{sB}u_0, e^{sB}u_0)^{-1}. \]
Then, \( h(s) \) satisfies the matrix-valued Bratu equation;
\[ (h'h^{-1})' = 2(a'u)h^{-1}. \]
Proof. From the assumption, the element \( B \) has the form \( B = B(b, a, 0) \). Since
\[ F^{-1}(e^{sB}u_0) = e^{-2sB} =: G(s), \]
the function \( \tilde{h}(s) := \pi(G(s)) \) satisfies the matrix-valued Bratu equation
\[ (\tilde{h}'\tilde{h}^{-1})' = 4(a'u)\tilde{h}^{-1}. \]
Therefore, the function
\[ h(s) := \Delta(e^{sB}u_0, e^{sB}u_0)^{-1} = \frac{1}{2} \tilde{h}(s) \]
satisfies the equation
\[ (h'h^{-1})' = 2(a'u)h^{-1}. \]

3.2 Bounded domain of type BDI and power series solution

Bounded domain \( D(L) \) of type BDI
Let
\[ L = \begin{pmatrix} I_n & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & -I_n \end{pmatrix}. \]
We define a subdomain \( D(L) \subset D \) by
\[ M(L) = \{ U \in M; \ 'UUU > 0 \}, \]
\[ D(L) := \{ u = [U] \in D; U \in M(L) \}. \]
We notice the relation \( L = AJ'A \) with a matrix
\[ A = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & 0 & -I_n \\ 0 & \sqrt{2}I_n & 0 \\ I_n & 0 & I_n \end{pmatrix}. \]

Lemma 3.5. 1. It holds the following bijection:
\[ D(J) \sim D(L) : u \mapsto Au \]
2. (Realization as a bounded domain) \( D(L) \) is described as
\[ D(L) = \left\{ v = \begin{pmatrix} I_n \\ V_2 \\ V_3 \end{pmatrix} \in D; \ I_n - V_2V_2 - V_3V_3 > 0 \right\}. \]
Next, we take two points \( v_0, z \) as
\[ v_0 = [V_0] \in D(L), \quad z = [Z] \in A\Sigma, \]
\[ V_0 = -\sqrt{2} \begin{pmatrix} I_n \\ 0 \\ 0 \end{pmatrix} = AU_0, \quad Z = \frac{1}{\sqrt{2}} \begin{pmatrix} -I_n \\ 0 \\ I_n \end{pmatrix} = AW. \]
For two points \( v_1, v_2 \in D(L) \) expressed as
\[ v_j = [V_j], \quad V_j = \begin{pmatrix} V_{j,1} \\ V_{j,2} \\ V_{j,3} \end{pmatrix} \quad (j = 1, 2), \]
we define \( \psi(v_1, v_2) \in M_n(\mathbb{R}) \) by
\[ \psi(v_1, v_2) = (V_1\Gamma_1^{-1})L(V_2\Gamma_2^{-1}), \]
\[ \Gamma_j = ZV_j = (V_{j,3} - V_{j,1})/\sqrt{2} \quad (j = 1, 2). \]

Lemma 3.6. For any \( u_1, u_2 \in D(J) \), it holds that
\[ \psi(Au_1, Au_2) = \Delta(u_1, u_2). \]
By above lemma, we have the following commutative diagram:
\[ \text{Sym}_n^+(\mathbb{R}) \]
We define the following functions of \( u \):

\[ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^n. \]

With the relation

\[ \Gamma_j = \psi(AU_j) = \mathcal{W}U_j, \]

we calculate as

\[ \psi(Au_1, Au_2) = \psi(\Gamma_1, \Gamma_2) \]

\[ = (U_1^{-1} J U_2^{-1}) \]

\[ = \Delta(u, u). \]

Proof. Take arbitrary \( u_1, u_2 \in D(J) \) and express them as

\[ u_j = [U_j] \quad (j = 1, 2). \]

\[ \Gamma_j = \psi(AU_j) = \mathcal{W}U_j, \]

we calculate as

\[ \psi(Au_1, Au_2) = \psi(\Gamma_1, \Gamma_2) \]

\[ = (U_1^{-1} J U_2^{-1}) \]

\[ = \Delta(u, u). \]

Proof. From the singular value decomposition, we get the equation

\[ B = P \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} tP, \quad P := \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}. \]

Then, we can calculate its exponential as

\[ e^{sB}V_0 = P \begin{pmatrix} \operatorname{ch}(s) & 0 \\ 0 & \operatorname{sh}(s) \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & \operatorname{ch}(s) \end{pmatrix} \begin{pmatrix} q \operatorname{sh}(s) \delta_q \nu & \nu \end{pmatrix}. \]

Putting

\[ \Gamma := psh(s) - \operatorname{ch}(s) \delta_q, \quad e(s) = \Gamma q, \]

we obtain that

\[ \psi(e^{sB}V_0, e^{sB}V_0) = \psi(e^{sB}V_0, e^{sB}V_0) \]

\[ = 2 \Gamma^{-1} \Gamma^{-1} = 2e(s)^{-1}e(s)^{-1}. \]

Corollary 3.1. \( h(s) \) has the following power series expansion:

\[ h(s) = \frac{1}{2} I_n + \frac{1}{2} \sum_{n=1}^{\infty} (q \sigma^{2n} q + p \sigma^{2n} p) \frac{s^{2n}}{(2n)!} \]

\[ - \frac{1}{2} \sum_{n=1}^{\infty} (p \sigma^{2n-1} q + q \sigma^{2n-1} p) \frac{s^{2n-1}}{(2n - 1)!}. \]

Proof. It is shown by the direct calculation:

\[ \frac{1}{2} e(s)^te(s) \]

\[ = \frac{1}{2} (-q \quad p) \begin{pmatrix} \operatorname{ch}(s)^2 & \operatorname{ch}(s) \operatorname{sh}(s) \\ \operatorname{sh}(s) \operatorname{ch}(s) & \operatorname{sh}(s)^2 \end{pmatrix} (-q \quad p) \]

\[ = \frac{1}{4} (-q \quad p) \begin{pmatrix} \operatorname{ch}(2s) + I_n & \operatorname{sh}(2s) \\ \operatorname{sh}(2s) & \operatorname{ch}(2s) - I_n \end{pmatrix} \]

\[ = \frac{1}{2} (2I_n + q \operatorname{ch}(2s) - I_n)q + p(\operatorname{ch}(2s) - I_n)p \]

\[ - q \operatorname{sh}(2s)q - p \operatorname{sh}(2s)q. \]

Proposition 3.3. For \( B \in p(L) \), define a function

\[ h(s) := \psi(e^{sB}V_0, e^{sB}V_0)^{-1}. \]

Then, \( h(s) \) has the following expression:

\[ h(s) = \frac{1}{2} e(s)^te(s), \quad e(s) = psh(s) - \operatorname{ch}(s). \]
4 Analog to symmetric domain of type CI

**Manifold \( \Omega(J) \)** We define a submanifold \( \Omega(J) \) in the cone \( \text{Sym}^+_{2n}(\mathbb{R}) \) by

\[
\Omega(J) = \{ G \in \text{Sym}^+_{2n}(\mathbb{R}); \ JGJ = G^{-1} \},
\]

\[
J = \begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix}.
\]

Setting the set \( V(J) \) by

\[
V(J) = \{ B \in \text{Sym}_{2n}(\mathbb{R}); \ JBJ = -B \},
\]

we can consider the following map:

\[
V(\cdot) \ni B \mapsto \exp B \in \Omega(J).
\]

We see that any element \( B \in V(J) \) has the following form:

\[
B = B(b, c) := \begin{pmatrix} b & c \\ -c & -b \end{pmatrix}, \quad b, c \in \text{Sym}_n(\mathbb{R}).
\]

For an element \( B \in V(J) \), we define an \( \Omega(J) \)-valued function \( G(s; B) \) by

\[
G(s; B) := \exp(sB(b, c)) \quad (s \in \mathbb{R}).
\]

We define the following sets of matrices:

\[
\mathcal{G}(J) = \{ G \in \text{GL}_{2n}(\mathbb{R}); \ JGJ = G^{-1} \} \cong \text{Sp}(n, \mathbb{R}),
\]

\[
\mathcal{N}(J) = \left\{ N \in \mathcal{G}(J); \ N = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} \right\},
\]

\[
= \left\{ N = \begin{pmatrix} I_n & 0 \\ n_1 & I_n \end{pmatrix}, \ n_1 = h^{-1} \right\},
\]

\[
\mathcal{N}_0(J) = \left\{ A \in \mathcal{G}(J); \ A = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \right\},
\]

\[
= \left\{ A = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \ h = h \right\}.
\]

**Proposition 4.1** (Block-Gauss decomposition for \( \Omega(J) \)).

The following map is bijective:

\[
\mathcal{N}(J) \times \mathcal{N}_0(J) \ni (N, A) \mapsto \Omega(J).
\]

We denote the variable change defined through this decomposition by

\[
G \mapsto N, A \mapsto h, n_1,
\]

or simply by \( G \mapsto h \).

**Lemma 4.1.** Under the variable change

\[
G \mapsto N, A \mapsto h, n_1,
\]

it holds that

\[
\frac{1}{2} \text{tr}((G'G^{-1})^2) = \frac{1}{2} \text{tr}((A'A^{-1})^2 + \text{tr}(A'N'^{-1}A^{-1}N^{-1}N')) = \text{tr}((h'h^{-1})^2 + \text{tr}(h'n_1h'n_1')).
\]

By the direct derivation of the Euler-Lagrange equation, we obtain the following analog of the matrix-valued Bratu equation.

**Theorem 4.1.** The equation

\[
G'(s)G(s)^{-1} = B(b, c)
\]

for an \( \Omega(J) \)-valued function \( G(s) \) is expressed as

\[
(h'(s)h(s)^{-1})' = (ch(s)^{-1})^2
\]

by the variable change \( G(s) \rightarrow h(s), n_1(s) \).

4.1 Siegel domain \( D(J; K) \) of type CI

Similarly as the type of BDI, we put

\[
M = \{ U = \begin{pmatrix} U_1 & U_2 \\ I_n & 0 \end{pmatrix} \in M_{2n, n}(\mathbb{C}); \ rank U = n \}
\]

and consider the actions

\[
\text{GL}(2n, \mathbb{C}) \curvearrowleft M \curvearrowright \text{GL}(n, \mathbb{C})
\]

by the matrix multiplication. Put the quotient space

\[
D := M/\text{GL}(n, \mathbb{C}) = \left\{ u = [U] = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}; U \in M \right\}
\]

and regard it as a domain. We put

\[
J = \begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}
\]

and define a subdomain \( D(J; K) \) by

\[
M(J; K) = \{ U \in M; \ U^*JU > 0, \ UJKU = 0 \},
\]

\[
D(J; K) := \{ u = [U] \in D; \ U \in M(J; K) \}.
\]

**Lemma 4.2** (Realization as a Siegel upper half space).

\( D(J; K) \) is the following set:

\[
D(J; K) = \left\{ u = \begin{pmatrix} U_1 \\ I_n \end{pmatrix}; \ \frac{1}{i} (U_1 - U_1^*) > 0, \ U_1 = U_1 \right\}
\]

\[
\cong \text{Sp}(n, \mathbb{R})/U(n).
\]

We define a subdomain \( \Sigma \subset D \) (Shilov boundary of \( D(J; K) \)) by

\[
\Sigma = \left\{ u = \begin{pmatrix} U_1 \\ I_n \end{pmatrix}; \ \frac{1}{i} (U_1 - U_1^*) = 0, \ U_1 = U_1 \right\}.
\]

We take two points \( u_0, w \) as

\[
u_0 = [U_0] \in D(J; K), \quad w = [W] \in \Sigma,
\]

\[
U_0 = \begin{pmatrix} iI_n \\ I_n \end{pmatrix}, \quad W = \begin{pmatrix} 0 \\ I_n \end{pmatrix}.
\]

For two points \( u_1, u_2 \in D(J; K) \) expressed as

\[
u_j = [U_j], \quad U_j = \begin{pmatrix} U_{j,1} \\ U_{j,2} \end{pmatrix} \quad (j = 1, 2),
\]
we define \( \Delta(u_1, u_2) \in M_n(\mathbb{C}) \) by
\[
\Delta(u_1, u_2) = (U_1 \Gamma_1^{-1})^* J (U_2 \Gamma_2^{-1}),
\]
\[
\Gamma_j = \{ WU_j = U_{j,2} \quad (j = 1, 2) \}.
\]

Put
\[
\mathfrak{g}(J) = \{ X \in \mathfrak{gl}(2n + r, \mathbb{R}); XJ + J'X = 0 \} 
\cong \mathfrak{sp}(n, \mathbb{R}),
\]

\[
p(J) = \{ X \in \mathfrak{g}(J); X = 'X \} 
\]
\[
= \{ B \in \mathfrak{p}(J); B = \begin{pmatrix} b & c \\ c & -b \end{pmatrix}, \quad b, c \in \text{Sym}_n(\mathbb{R}) \}.
\]

**Proposition 4.2** (Analog to type CI). For \( B \in \mathfrak{p}(J) \), define \( \text{Sym}_n^+(\mathbb{R}) \)-valued function \( h(s) \) by
\[
h(s) = \Delta(e^{sB}u_0, e^{sB}u_0)^{-1}.
\]

Then, \( h(s) \) satisfies the equation
\[
(h'h_1^{-1})'(ch^{-1})^2.
\]

We can define an action \( \mathcal{G}(J) \rhd D(J; K) \) by
\[
g \cdot u = [gU] \quad (g \in \mathcal{G}(J), u = [U] \in D(J; K)).
\]

We also define subgroups:
\[
\mathcal{P}(J) = \{ g \in \mathcal{G}(J); \quad g = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \},
\]
\[
\mathcal{K}(J) = \{ g \in \mathcal{G}(J); \quad \mathcal{G}(J) \cap \mathcal{K}(J) \}
\]
\[
\mathcal{G}(J)_{0n} = \{ g \in \mathcal{G}(J); \quad g_0 = u_0 \}.
\]

**Lemma 4.3.** The following assertions hold.
1. \( \mathcal{G}(J) = \mathcal{P}(J) \mathcal{K}(J) \).
2. \( \mathcal{G}(J)_{0n} = \mathcal{K}(J) \).
3. The action \( \mathcal{G}(J) \rhd D(J; K) \) is transitive and the following expression holds:
\[
D(J; K) \cong \text{Sp}(n, \mathbb{R})/U(n).
\]

**Proof of the proposition** Define a manifold
\[
\Omega(J) = \{ g \in \text{Herm}_{2n}(\mathbb{C}); \quad gJg^* = J, gK'g = K \}
\]
\[
= \{ g \in \text{Sym}_{2n}^+(\mathbb{R}); \quad gJ'g = J \}
\]

and a map \( \mathcal{F} : \Omega(J) \to D \) by
\[
\mathcal{F}(G) = \begin{pmatrix} iI_n - G_2 \\ h \end{pmatrix}, \quad G = \begin{pmatrix} h & * \\ G_2 & * \end{pmatrix} \in \Omega(J; K),
\]

and a map \( \pi : \Omega(J) \to \text{Sym}_n^+(\mathbb{R}) \) by
\[
\pi(G) = h, \quad G = \begin{pmatrix} h \\ G_2 \end{pmatrix} \in \Omega(J; K).
\]

We define an action \( \mathcal{G}(J) \rhd \Omega(J) \) by
\[
g \cdot G = \mathcal{G}(J) g^{-1} Gg^{-1} \quad (g \in \mathcal{G}(J), \quad G \in \Omega(J)).
\]

**Lemma 4.4.** The following assertions hold.
1. \( F(g \cdot G) = gF(G) \) for any \( g \in \mathcal{G}(J), \quad G \in \Omega(J) \).
2. \( F(\Omega(J)) \subset D(J) \) and the map \( \mathcal{F} : \Omega(J) \to D(J) \) is bijective.
3. For any \( G \in \Omega(J) \), it holds that
\[
\frac{1}{2} \pi(G) = \Delta(F(G), F(G))^{-1}.
\]

From the above lemma, we have the following commutative diagram:
\[
\begin{array}{ccc}
\Omega(J) & \xrightarrow{\mathcal{F}} & D(J; K) \\
\downarrow{\pi} & \nearrow{\Delta(\cdot, \cdot)^{-1}} \\
\text{Sym}_n^+(\mathbb{R}) & \end{array}
\]

**Proof.** Each \( G \in \Omega(J) \) is expressed as
\[
G = \mathcal{G}(J) g^{-1} = \begin{pmatrix} h & t_1 h \\ t_1 h & I_n + t_1 t h n_1 \end{pmatrix}.
\]
\[
\mathcal{F}(G) = \begin{pmatrix} t_1 h & 0 \\ 0 & h^{-\frac{1}{2}} \end{pmatrix} \in \mathcal{G}(J).
\]

We also see that
\[
g_0 u_0 = \begin{pmatrix} I & -n_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} h^\frac{1}{2} & 0 \\ 0 & h^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I \\ n_1 \end{pmatrix} \]
\[
= \begin{pmatrix} i h^{-1} - n_1 \\ I_n \end{pmatrix}.
\]

1. First, we show in the case \( G = I_{2n} \). Take an arbitrary \( g \in \mathcal{G}(J) \) and express it as
\[
g = pk, \quad p \in \mathcal{P}(J), \quad k \in \mathcal{K}(J),
\]
\[
p = \begin{pmatrix} p_1 & -p_2 \\ 0 & p_1 \end{pmatrix}.
\]

Then, using the relation \( p^{-1} = J^{-1}pJ \), we get
\[
p \cdot I_{2n} = \mathcal{G}(J) p^{-1} G_1 p^{-1} = \begin{pmatrix} p_1 & -p_2 \\ 0 & p_1 \end{pmatrix} \begin{pmatrix} p_1 & 0 \\ 0 & p_1 \end{pmatrix} = \begin{pmatrix} p_1^{-1} & p_2 \\ 0 & p_1 \end{pmatrix}.
\]

Therefore,
\[
\mathcal{F}(p \cdot I_{2n}) = \begin{pmatrix} i n_1 - p_2 p_1^{-1} \\ p_1 \end{pmatrix} = \begin{pmatrix} i p_1 - p_2 \\ p_1 \end{pmatrix}.
\]

On the other hand, we see that
\[
p \mathcal{F}(I_{2n}) = p \begin{pmatrix} i n_1 \\ I_n \end{pmatrix} = \begin{pmatrix} i p_1 - p_2 \\ p_1 \end{pmatrix}.
\]
and then $\mathcal{F}(p \cdot I_{2n}) = p \mathcal{F}(I_{2n})$. Therefore,

$$\mathcal{F}(g \cdot I_{2n}) = \mathcal{F}(p \cdot I_{2n}) = pu_0 = pku_0 = g \mathcal{F}(I_{2n}).$$

Next, we show in the general case $g \in G(J), G \in \Omega(J)$. Express an arbitrary $G \in \Omega(J)$ as

$$G = g_0 \cdot I_{2n}, \quad g_0 \in G(J).$$

Then we see that

$$\mathcal{F}(g \cdot G) = \mathcal{F}(gg_0 \cdot I_{2n}) = gg_0 \mathcal{F}(I_{2n}) = pu_0 = pku_0 = g \mathcal{F}(I_{2n}).$$

3. Take an arbitrary $G \in \Omega(J)$ and express it as

$$G = \begin{pmatrix} h & iG_2 \\ G_2 & * \end{pmatrix}, \quad iG_2h - hG_2 = 0.$$ Then we see that

$$\mathcal{F}(G) = [U], \quad U = \begin{pmatrix} iI_n - G_2 \\ h \end{pmatrix}.$$ Putting

$$\Gamma = \Psi WU = h,$$

we see that

$$\Delta(\mathcal{F}(G), \mathcal{F}(G)) = (UT^{-1})^* J(UT^{-1})$$

$$= \frac{1}{i} \left\{ (ih^{-1} - G_2h^{-1}) - (ih^{-1} - G_2h^{-1})^* \right\}$$

$$= 2h^{-1} > 0.$$

\[\square\] **Proof of Proposition 4.2.** From the definition, we have

$$\mathcal{F}^{-1}(e^{sB}u_0) = e^{-2sB} =: G(s).$$

Then, the function $\hat{h}(s) := \pi(G(s))$ satisfies

$$(\hat{h}'h^{-1})' = 4(ch^{-1})^2.$$ Therefore, the function

$$h(s) := \Delta(e^{sB}u_0, e^{sB}u_0)^{-1} = \frac{1}{2} \hat{h}(s)$$

satisfies

$$(h'h^{-1})' = (ch^{-1})^2.$$ \[\square\]

5 **Remarks**

We gave individual calculations for each symmetric domain of type BDI and CI. The same calculation, which consists of the Lagrangian calculation and the realization of symmetric domains, is likely applicable to other symmetric domains classified by Cartan, ref [2, 3]. For this purpose, it is natural to expect a unified way to treat general symmetric domains.

References

[1] Inoue, H., Matrix-valued Bratu equation and the exact solution of its initial value problem, Int. Jour. Math. Ind., published on 12 Mar. 2020.

[2] Helgason, S., Differential geometry and symmetric spaces, Academic Press, 1962.

[3] Pyatetskii-Shapiro, I. I., Automorphic functions and the geometry of classical domains, Gordon and Breach Science Publishers, 1969.