Thermodynamics with pressure and volume of 4D Gauss-Bonnet AdS Black Holes under the scalar field

Benrong Mu\textsuperscript{a,b,}\, Jing Liang\textsuperscript{a,b}\, and Xiaobo Guo\textsuperscript{c,}\,\,\,†

\textsuperscript{a}Physics Teaching and Research section, College of Medical Technology, Chengdu University of Traditional Chinese Medicine, Chengdu, 611137, PR China
\textsuperscript{b}Center for Theoretical Physics, College of Physics, Sichuan University, Chengdu, 610064, PR China and
\textsuperscript{c}Mechanical and Electrical Engineering School, Chizhou University, Chizhou, 247000, PR China

Abstract

By using the scattering of a scalar field, we discuss the thermodynamics and overcharging problem in a 4D Gauss-Bonnet AdS black hole in both the normal phase space and extended phase space. In the normal phase space, where the cosmological constant and Gauss-Bonnet parameter are fixed, the first law and the second law of thermodynamics are valid. In addiction, the black hole cannot be overcharged and the weak cosmic censorship conjecture is valid. In the extended phase space, where the cosmological constant and Gauss-Bonnet parameter are treated as the thermodynamic variables, the first law is still valid. However, the second law is indefinite. Moreover, after the scattering of the scalar field, the extremal black hole cannot be overcharged and the near-extremal black hole can be overcharged.

\textsuperscript{*}Electronic address: benrongmu@cdutcm.edu.cn
\textsuperscript{†}Electronic address: jingliang@stu.scu.edu.cn
\textsuperscript{‡}Electronic address: guoxiaobo@czu.edu.cn
I. INTRODUCTION

It is well known that space-time singularities can be formed at the end of gravitational collapse. Near the singularity, the laws of physics fail. In order to avoid physical damage caused by singularities, Penrose proposed the weak cosmic censorship conjecture (WCCC). It points out that under normal material properties and initial conditions, naked singularities cannot be formed in real physical processes, and the singularities are covered by the event horizon [1]. The WCCC plays an important part in black hole physics, and its validity is a necessary condition to ensure the predictability of the laws of physics. Although its correctness has been widely accepted, there is no complete evidence to prove its correctness. People can only test its correctness in different space-time. The Gedanken experiment designed by Wald is an effective method to test the validity of WCCC [2]. In this experiment, a particle with sufficient charge or angular momentum is thrown into a charged black hole
to test the evolution of the black hole. If the final state of the black hole evolution is no
longer a black hole, then the event horizon does not exist and the singularity is naked.
According to Wald’s work, the effectiveness of WCCC has been tested in various space-
time [3–20]. In addition, Semiz first used a complex scalar field to replace particles to test
the effectiveness of WCCC, and found that the strong Kerr-Newman black hole cannot be
overcharged [21]. There are also many papers that have verified the effectiveness of WCCC
in different black holes through this method [22–33]. In particular, RN-AdS black hole
is studied in both normal and extended phase space [34, 35]. For RN-AdS black holes,
the first law of thermodynamics and WCCC are always satisfied, while the second law of
thermodynamics is valid in normal phase space and violated in the extended phase space.
Moreover, the thermodynamics and WCCC of the high-dimensional Gauss-Bonnet AdS black
hole \((d \geq 5)\) are studied in [16].

Generally speaking, no matter is able to escape through the black hole’s event horizon.
So that external observers cannot detect any radiation from inside the black hole. From the
perspective of quantum physics, there is quantum tunnelling in black holes. From this, the
temperature of the black hole can be defined, and the black hole can be regarded as a thermal
system with Hawking temperature [36, 37]. In addition, black holes have irreducible mass,
which is a property that increases with irreversible processes [38–41]. The irreducible mass
is similar to the entropy in the thermal system and based on this similarity, the entropy
of the black hole can be obtained. This entropy is the Bekenstein-Hawking entropy of
the black hole [42, 43], which is proportional to the area of the horizon. Using these two
thermodynamic properties, temperature and entropy, the laws of thermodynamics for black
holes as a thermal system is established. The first law of black hole thermodynamics is
usually written as

\[
dM = TdS + \Phi dQ. \tag{1}
\]

Where \(M\) is the mass, \(T\) is the Hawking temperature, \(S\) is the entropy, \(\Phi\) is the electric
potential and \(Q\) is the electric charge. The mass is usually interpreted as the enthalpy [44].
It is worth noting that there is no \(PdV\) term in Eq. (1). Recently, an interesting idea has
been proposed. The cosmological constant can be explained by thermodynamic pressure.
When the cosmological constant, \(\Lambda\), is treated as the pressure of the black hole [45–50] and
the volume of the black hole is defined as the thermodynamic variable conjugate to the
pressure \([51]\), Eq. (1) is modified as
\[
dM = TdS + VdP + \Phi dQ. \tag{2}
\]
where \(P, \Lambda\) and \(V\) satisfy \(P = -\frac{A}{8\pi}, V = \left(\frac{\partial M}{\partial P}\right)_{S,Q}\).

As we all know, there are static spherically symmetric black hole solutions with \(d \geq 5\) in Gauss-Bonnet gravity \([52-56]\). When \(d = 4\), the Gauss-Bonnet term becomes a topological invariance and does not contribute to the field equation in the four-dimensional space-time, so there is no 4D Gauss-Bonnet black hole. However, in the recent work of Glavan and Lin \([57]\), it is proposed that the non-trivial solution of the four-dimensional black hole can be obtained by re-adjusting the Gauss-Bonnet coupling parameter \(\alpha\) to \(\alpha \to \alpha/(d - 4)\), and then taking the limit of \(d \to 4\). Subsequently, the black hole solution is extended to charged case in an anti-de Sitter (AdS) space \([58]\). The first law of black hole thermodynamics of 4D Gauss-Bonnet AdS black hole is
\[
dM = TdS + \varphi dQ + VdP + \mathcal{A}d\alpha. \tag{3}
\]
where \(\mathcal{A}\) is the conjugate quantity of Gauss-Bonnet parameter and is defined as
\[
\mathcal{A} = \left(\frac{\partial M}{\partial \alpha}\right)_{S,Q,P} = \frac{1}{2r_+} + 2\pi T \left(1 - 2\ln \frac{r_+}{\sqrt{\alpha}}\right). \tag{4}
\]
When taking the limit \(\alpha \to 0\), the RN-AdS will be recovered. There are many studies on the thermodynamic properties of 4D Gauss-Bonnet black holes \([59-68]\).

The present paper is organized as follows. In Sec. II we briefly review 4D Gauss-Bonnet AdS black hole solution. In Sec. III we investigate the complex scalar field in the black hole background. Next, we briefly discuss the first law and the second law of thermodynamics of the black hole in the normal and extended phase space in Sec. IV. Furthermore, the overcharging problem in the normal and extended phase space is discussed in Sec. V. Sec. VI is devoted to our discussion and conclusion.

II. 4D GAUSS-BONNET ADS BLACK HOLES

The action of the Gauss-Bonnet-Maxwell theory in a d-dimensional space-time is \([57]\)
\[
S = \frac{1}{16\pi} \int d^dx \sqrt{-g} \left[R - 2\Lambda + \frac{\alpha}{d-4} \left(r^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) - F^{\mu\nu}F_{\mu\nu}\right], \tag{5}
\]
where $\Lambda$ is the cosmological constant that relates to the AdS radius $l$ with the relation 
$$\Lambda = \frac{-(D-1)(D-2)}{2l^2},$$
$g$ is determinant of the metric tensor and $F_{\mu\nu}$ is the Maxwell field strength. Adopting the limit $d \to 4$ and solving the field equations, we obtain

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $M$ and $Q$ are the mass and electric charge of the black hole, respectively. $\alpha$ is the Gauss-Bonnet coupling constant. The metric function is written as

$$f(r) = 1 + \frac{r^2}{2\alpha} \left[ 1 - \sqrt{1 + 4\alpha \left( \frac{1}{l^2} + \frac{2M}{r^3} - \frac{Q^2}{r^4} \right)} \right],$$

In Fig. 1 we plot the metric function $f(r)$ for different situations. It can be observed from the figure that as the value of $\alpha$ increases, the value of $f(r)$ also increases. In the limit $\alpha \to 0$, the RN-AdS black hole solution will be recovered. Solving the equation $f(r) = 0$, we can obtain two solutions $r_-$ and $r_+$. The two solutions correspond to the radius of the Cauchy horizon and the event horizon. At the event horizon, the black hole mass is expressed as

$$M = \frac{\alpha l^2 + l^2 Q^2 + l^2 r_+^2 + r_+^4}{2l^2 r_+}.$$
The Hawking temperature of the black hole is

$$T = \frac{f'(r_+)}{4\pi} = \frac{-\alpha + r_+^2 + 3r_+^4}{4\pi (r_+^3 + 2\alpha r_+)}.$$  \hspace{1cm} (10)

The electric potential and the entropy take on the forms as

$$\varphi = \frac{Q}{r_+^2},$$  \hspace{1cm} (11)

$$S = \int \frac{dM}{T} = \pi r_+^2 + 4\pi \alpha \ln \frac{r_+}{\sqrt{\alpha}}.$$  \hspace{1cm} (12)

In the normal phase space, where the cosmological constant is fixed, the first law of thermodynamics takes on the form as

$$dM = TdS + \varphi dQ.$$  \hspace{1cm} (13)

In the extended phase space, we construct the first law of thermodynamics by considering the cosmological constant as dynamic pressure, and its conjugate as thermodynamic volume. In this case, the black hole mass $M$ is interpreted as the enthalpy $H$ rather than the internal energy $U$ of the system. In addition, since the cosmological constant is regarded as a thermodynamic pressure, the Smarr relationship of black hole thermodynamics can be obtained through the scaling argument. In the Gauss-Bonnet gravity, in order to satisfy the Smarr relationship, the Gauss-Bonnet coefficient should also be regarded as a dynamic quantity. Therefore, the form of the first law in extended phase space is

$$dM = TdS + \varphi dQ + VdP + A d\alpha,$$  \hspace{1cm} (14)

where $P$ is the pressure, $V$ is the conjugate of $P$, which is interpreted as volume, and $A$ is the conjugate quantity of Gauss-Bonnet coefficient $\alpha$, which are defined as

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2},$$  \hspace{1cm} (15)

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,\alpha} = \frac{4\pi r_+^3}{3},$$  \hspace{1cm} (16)

$$A = \left(\frac{\partial M}{\partial \alpha}\right)_{S,Q,P} = \frac{1}{2r_+} + 2\pi T \left(1 - 2\ln \frac{r_+}{\sqrt{\alpha}}\right).$$  \hspace{1cm} (17)

The following Smarr relation is also satisfied

$$H = 2TS + \varphi Q - 2PV + 2A\alpha.$$  \hspace{1cm} (18)
III. ENERGY AND CHARGE’S VARIATION OF THE 4D GAUSS-BONNET ADS BLACK HOLE

The charged complex scalar field $\psi$ with mass $m$ and charge $q$ satisfies

$$\left(\nabla^\mu - iqA^\mu\right)\left(\nabla_\mu - iqA_\mu\right)\psi - m^2\psi = 0,$$  \hspace{1cm} (19)

which can be written as

$$\frac{1}{\sqrt{-g}}\left(\partial_\mu - iqA_\mu\right)\left[\sqrt{-g}g^{\mu\nu}\left(\partial_\nu - iqA_\nu\right)\psi\right] - m^2\psi = 0. \hspace{1cm} (20)$$

Since space-time is static and spherically symmetric, the complex scalar field can be decomposed into

$$\psi = e^{-i\omega t}R(r)\Phi(\theta, \phi),$$  \hspace{1cm} (21)

where $\omega$ is the energy of the particle, $R(r)$ is the radial function and $\Phi(\theta, \phi)$ is spherical harmonic function. As usual, we introduce the tortoise coordinate to solve the radial equation

$$\frac{dr}{dr_*} = f. \hspace{1cm} (22)$$

Then, the radial equation in Eq. (21) is rewritten as

$$R(r) = e^{\pm i\left(\omega - \frac{q\phi}{r}\right)r_*},$$  \hspace{1cm} (23)

where $+/-$ corresponds to the solution of the outgoing/ingoing radial wave. Since we discuss the thermodynamics and validity of the WCCC by scattering of the ingoing waves at the event horizon, we focus on the ingoing wave function. The energy-momentum tensor is given by

$$T^\mu_\nu = \frac{1}{2}\left[\left(\partial^\mu - iqA^\mu\right)\psi^*\partial_\nu\psi + \left(\partial^\mu + iqA^\mu\right)\psi\partial_\nu\psi^*\right] + \delta^\mu_\nu\mathcal{L}. \hspace{1cm} (24)$$

From Eq. (24), we can get the energy flux through the event horizon is

$$\frac{dE}{dt} = \int T^t_t\sqrt{-g}d\theta d\phi = \omega(\omega - q\phi)r_+^2. \hspace{1cm} (25)$$

In addition, the charge flux can get from the energy flux \[71\]. Therefore, the charge flux is written as

$$\frac{dQ}{dt} = -\int j^\tau\sqrt{-g}d\theta d\phi = q(\omega - q\phi)r_+^2. \hspace{1cm} (26)$$
According to the conservation of energy and charge, when the complex scalar field is scattered by the black hole, the decrease in the energy and charge of the scalar field is equal to the increase in the energy and charge of the black hole. In an infinitely time interval $dt$, the changes in the mass and charge of the black hole are

$$dU = dE = \omega(\omega - q\varphi)r_+^2 dt, dQ = q(\omega - q\varphi)r_+^2 dt.$$  \hspace{1cm} (27)

**IV. THERMODYNAMICS UNDER THE SCALAR FIELD**

In this section, we investigate the black hole thermodynamics of the Einstein-Gauss-Bonnet gravity coupled to the Maxwell theory in the normal and extended phase spaces by the scattering of a scalar field. For the convenience of the later discussion, before the subsequent discussions on the thermodynamic properties, we first give the following formulas

$$\begin{align*}
\frac{\partial f}{\partial M} \bigg|_{r=r_+} &= -\frac{2}{r_+ + \frac{2\alpha}{r_+}}, \\
\frac{\partial f}{\partial Q} \bigg|_{r=r_+} &= \frac{2\varphi}{r_+ + \frac{2\alpha}{r_+}}, \\
\frac{\partial f}{\partial l} \bigg|_{r=r_+} &= -\frac{2r_+^2}{l^3} \frac{1}{1 + \frac{2\alpha}{r_+}}, \\
\frac{\partial f}{\partial \alpha} \bigg|_{r=r_+} &= \frac{1}{r_+^2 + 2\alpha}, \\
\frac{\partial f}{\partial r} \bigg|_{r=r_+} &= 4\pi T.
\end{align*}$$  \hspace{1cm} (28)

**A. Thermodynamics in the normal phase space**

In normal phase space, cosmological constant is fixed, and black holes are characterized by mass $M$ and charge $Q$. During the scattering of the scalar field, the mass $M$, the charge $Q$ and other thermodynamic variables of the black hole change due to the conservation law. Assuming that the black hole’s initial state is expressed by $(M, Q, r_+)$ and final state is expressed by $(M + dM, Q + dQ, r_+ + dr_+)$. The initial state $(M, Q, r_+)$ and the final state $(M + dM, Q + dQ, r_+ + dr_+)$ satisfy

$$f(M, Q, r_+) = f(M + dM, Q + dQ, r_+ + dr_+) = 0.$$  \hspace{1cm} (29)
The functions \( f(M, Q, r_+) \) and \( f(M + dM, Q + dQ, r_+ + dr_+) \) satisfy the following relation

\[
f(M + dM, Q + dQ, r_+ + dr_+) = f(M, Q, r_+)
\]

\[
+ \frac{\partial f}{\partial M} \bigg|_{r=r_+} dM + \frac{\partial f}{\partial Q} \bigg|_{r=r_+} dQ + \frac{\partial f}{\partial r} \bigg|_{r=r_+} dr_+.
\]

Inserting Eqs. (28) and (29) to Eq. (30), we have

\[
dM = 2\pi T \left( r_+ + \frac{2\alpha}{r_+} \right) dr_+ + \varphi dQ.
\]

(31)

Considering Eqs. (12), the above equation is modified as

\[
dM = TdS + \varphi dQ,
\]

(32)

which is the first law of thermodynamics.

In the normal phase space, the mass \( M \) is interpreted as the internal energy of the thermodynamic system. Then, the changes in the internal energy and charge of the black hole within an infinitesimal time interval \( dt \) is

\[
dM = dU = \omega(\omega - q\varphi)r_+^2 dt, dQ = q(\omega - q\varphi)r_+^2 dt.
\]

(33)

Using Eqs. (32) and (33), we have

\[
dS = \frac{r_+^2(\omega - q\varphi)^2}{T} dt \geq 0,
\]

(34)

which shows that the entropy of the black hole increases. Therefore, the second law of black hole thermodynamics is satisfied for the black hole in the normal phase space.

### B. Thermodynamics in the extended phase space

In the extended phase space, the cosmological constant is treated as the function of the pressure of the black hole. Moreover, the Gauss-Bonnet coupling constant \( \alpha \) is also treated as a new thermodynamic variable \([72, 73]\). The mass \( M \) is regarded as the enthalpy \( H \) of the thermodynamic system, not the thermodynamic energy \( U \). The relationship between thermodynamic energy and enthalpy is \([74, 75]\)

\[
M = U + PV.
\]

(35)

In this case, the variations of the energy and charge are

\[
dU = d(M - PV) = \omega(\omega - q\varphi)r_+^2 dt, dQ = q(\omega - q\varphi)r_+^2 dt.
\]

(36)
In the extended phase space, the radius $r$ changes from the initial black hole horizon radius $r_+$ to the changed horizon radius $r_+ + dr_+$ after the scattering of a scalar field. The initial and final states of the black hole are represented by $(M, Q, l, \alpha, r_+)$ and $(M + dM, Q + dQ, l + dl, \alpha + d\alpha, r_+ + dr_+)$, respectively. They satisfy

$$f (M, Q, l, \alpha, r_+) = f (M + dM, Q + dQ, l + dl, \alpha + d\alpha, r_+ + dr_+) = 0. \quad (37)$$

The relationship between $f (M, Q, l, \alpha, r_+)$ and $f (M + dM, Q + dQ, l + dl, \alpha + d\alpha, r_+ + dr_+)$ is

$$f (M + dM, Q + dQ, l + dl, \alpha + d\alpha, r_+ + dr_+) = f (M, Q, l, \alpha, r_+)$$

$$+ \frac{\partial f}{\partial M} \bigg|_{r=r_+} + \frac{\partial f}{\partial Q} \bigg|_{r=r_+} + \frac{\partial f}{\partial l} \bigg|_{r=r_+} + \frac{\partial f}{\partial \alpha} \bigg|_{r=r_+} + \frac{\partial f}{\partial r} \bigg|_{r=r_+}. \quad (38)$$

Substituting Eqs. (28) and (37) into Eq. (38), we can obtain the first law of thermodynamics of the black hole in the extended phase space

$$dM = TdS + \varphi dQ + VdP + \mathcal{A} d\alpha. \quad (39)$$

where $\mathcal{A}$ is the conjugate quantity of Gauss-Bonnet parameter and is defined as

$$\mathcal{A} = \left( \frac{\partial M}{\partial \alpha} \right)_{S, Q, P} = \frac{1}{2r_+^2} + 2\pi T \left( 1 - 2\ln \frac{r_+}{\sqrt{\alpha}} \right). \quad (40)$$

Using Eqs. (12), (18), (27), (39) and (40), we get

$$dS = \frac{\left( 1 + \frac{2\alpha}{r_+^2} \right) (\omega - q\varphi)^2 r_+^2 dt - \left( 1 + \frac{2\alpha}{r_+^2} \right) \left[ \frac{1}{2r_+^2} + 2\pi T \left( 1 - 2\ln \frac{r_+}{\sqrt{\alpha}} \right) \right] d\alpha}{\left( 1 + \frac{2\alpha}{r_+^2} \right) T - \frac{3r_+}{4\pi T^2}}. \quad (41)$$

Based on Eq. (41), for large enough $T$, the denominator is positive. On the contrary, for small enough $T$, the denominator is negative. Since $d\alpha$ is arbitrary, the sign of the numerator in Eq. (41) is indefinite. It means the second law of thermodynamics is not always satisfied for the extremal or near-extremal black hole. As shown in Fig. 2, we plot the graph of $dS$ with different values of $d\alpha$. We fix $M = 0.5, l = 1, q = \omega = 0.1, \alpha = 0.15$ and $dt = 0.0001$.

V. OVERCHARGING PROBLEM UNDER THE SCALAR FIELD

In this section, we investigate the validity of the WCCC by the scattering of a scalar field in the normal and extended phase spaces. An effective way to test the validity of WCCC is to check whether the event horizon exists after the scalar field scattering. The event horizon is
Fig. 2: The relation between $dS$ and $r_+$ which parameter values are $M = 0.5$, $l = 1$, $q = \omega = 0.1$, $\alpha = 0.15$ and $dt = 0.0001$.

determined by the function $f(r)$. In the initial state, the minimum value of $f(r)$ is negative or zero and $f(r) = 0$ has real roots. It means event horizon exists. After the scalar field scattering, the mass and charge of the black hole change during the infinitesimal time interval $dt$. Besides, the minimum value of $f(r)$ also changes. If the minimum value of $f(r)$ changes to a negative value or zero, as shown in Fig. 3(a) and 3(b), the event horizon exists. Then, the black hole cannot be overcharged and the WCCC is effective. Otherwise, as shown in Fig. 3(c), the minimum value of $f(r)$ changes to a positive value. The event horizon doesn’t exist. Consequently, the black hole is overcharged and the WCCC is ineffective. Assuming that there is a minimum value of $f(r)$ at $r = r_0$. For the near-extremal and extremal black

(a)$f(r)$ in non-extremal black holes.  (b)$f(r)$ in extremal black holes.  (c)$f(r)$ in naked singularities.

Fig. 3: Graphs of $f(r)$ for given states of the GB-AdS black holes.
holes, the minimum value is not greater than zero

\[ \delta \equiv f(r_0) \leq 0. \]  

(42)

where \( \delta = 0 \) corresponds to the extremal black hole. As before, we first give the following partial derivative formulas

\[
\begin{align*}
\frac{\partial f}{\partial M} \bigg|_{r=r_0} &= -\frac{2}{r_0 + \frac{2\alpha}{r_0} (1 - \delta)}, \\
\frac{\partial f}{\partial Q} \bigg|_{r=r_0} &= \frac{2Q}{r_0} \\
\frac{\partial f}{\partial l} \bigg|_{r=r_0} &= -\frac{2r_0^2}{(1 + \frac{2\alpha}{r_0} (1 - \delta))} \\
\frac{\partial f}{\partial \alpha} \bigg|_{r=r_0} &= \frac{(1 - \delta)^2}{r_0^2 + 2\alpha (1 - \delta)}, \\
\frac{\partial f}{\partial r} \bigg|_{r=r_0} &= 0.
\end{align*}
\]

(43)

**A. Overcharging problem in the normal phase space**

After the scattering of a scalar field, the physical parameters of the black hole change from the initial state \((M, Q, r_0)\) to the final state \((M + dM, Q + dQ, r_0 + dr_0)\). At the final state, the value of \(f(r)\) at \(r = r_0 + dr_0\) satisfies

\[
\begin{align*}
f(M + dM, Q + dQ, r_0 + dr_0) &= f(M, Q, r_0) \\
+ \frac{\partial f}{\partial M} \bigg|_{r=r_0} + \frac{\partial f}{\partial Q} \bigg|_{r=r_0} + \frac{\partial f}{\partial l} \bigg|_{r=r_0} + \frac{\partial f}{\partial \alpha} \bigg|_{r=r_0}.
\end{align*}
\]

(44)

Substituting Eqs. (33) and (43) into Eq. (44), we obtain

\[
f(M + dM, Q + dQ, r_0 + dr_0) = \delta - \frac{2r_0^2(\omega - q\varphi)(\omega - q\varphi)}{r_0 + \frac{2\alpha}{r_0} (1 - \delta)} dt.
\]

(45)

When the initial black hole is an extremal black hole, \(r_0 = r_+\) and \(\delta = 0\). Hence, the minimum value of final state metric function \(f(r)\) takes on the form as

\[
f(M + dM, Q + dQ, r_0 + dr_0) = -\frac{2r_0^2(\omega - q\varphi)^2}{r_0 + \frac{2\alpha}{r_0}} dt \leq 0.
\]

(46)

Therefore, the black hole can not be overcharged and the WCCC is valid. If \(\omega = q\varphi\), the extremal black hole still extremal black hole. If \(\omega \neq q\varphi\), the extremal black hole will change
to non-extremal black hole. When it is a near-extremal black hole, \( \delta \) is a small negative quantity and \( r_+ \) is numerically greater than \( r_0 \). Eq. (45) can be regarded as a quadratic function of \( \omega \). When \( \omega = \frac{1}{2} q Q \left( \frac{1}{r_+} + \frac{1}{r_0} \right) \), the function \( f(M + dM, Q + dQ, r_0 + dr_0) \) has the maximum value

\[
f(M + dM, Q + dQ, r_0 + dr_0)_{\text{max}} = \delta + \frac{q^2 Q^2 (r_+ - r_0)^2}{2r_0^3 + 4\alpha r_0^2 (1 - \delta)} dt.
\] (47)

We define that

\[
r_0 = r_+ (1 - \epsilon),
\] (48)

where \( 0 < \epsilon \leq 1 \). Then Eq. (47) is rewritten as

\[
f(M + dM, Q + dQ, r_0 + dr_0)_{\text{max}} = \delta - \frac{q^2 Q^2 \epsilon^2}{2r_+ (1 - \epsilon)^3 + 4\alpha (1 - \epsilon)^2 (1 - \delta)} dt.
\] (49)

The second term of the above equation can be ignored since the quantity \( \epsilon \) is infinitesimal. Hence the metric function has a minimum negative value, which indicates that the event horizon also exists in the final state. The black hole cannot be overcharged by the scattering of the scalar field. Therefore, the WCCC is valid in the near-extremal black hole.

**B. Overcharging problem in the extended phase space**

In the extended phase space, the physical parameters of the black hole change from the initial state \((M, Q, l, \alpha, r_0)\) to the final state \((M + dM, Q + dQ, l + dl, \alpha + d\alpha, r_0 + dr_0)\) after the scattering of a scalar field. At the final state, the value of \( f(r) \) at \( r = r_0 + dr_0 \) satisfies

\[
f(M + dM, Q + dQ, l + dl, \alpha + d\alpha, r_0 + dr_0)
= f(M, Q, l, \alpha, r_0) + \frac{\partial f}{\partial M} \bigg|_{r=r_0} + \frac{\partial f}{\partial Q} \bigg|_{r=r_0} + \frac{\partial f}{\partial l} \bigg|_{r=r_0} + \frac{\partial f}{\partial \alpha} \bigg|_{r=r_0} + \frac{\partial f}{\partial r} \bigg|_{r=r_0}
= \delta - \frac{2T dS}{r_0 + \frac{2\alpha}{r_0^2} (1 - \delta)} - \frac{2Q}{r_0 + \frac{2\alpha}{r_0^2} (1 - \delta)} \left( \frac{1}{r_+} - \frac{1}{r_0} \right) dQ + \frac{2}{r_0 + \frac{2\alpha}{r_0^2} (1 - \delta)} \left( \frac{r_+^3 - r_0^3}{f^3} \right) dl
- \frac{2}{r_0 + \frac{2\alpha}{r_0^2} (1 - \delta)} \left[ \frac{1}{2r_+} + 2\pi T \left( 1 - 2\ln \frac{r_+}{\sqrt{\alpha}} \right) - \frac{(1 - \delta)^2}{2r_0} \right] d\alpha.
\] (50)

Considering the extremal black hole, which implies \( r_+ = r_0, T = 0 \) and \( \delta = 0 \). Therefore, we find that the minimum value of \( f(r) \) of the final black hole becomes

\[
f(M + dM, Q + dQ, l + dl, \alpha + d\alpha, r_0 + dr_0) = 0
\] (51)
which means that the extremal black hole is still extremal. When the initial black hole is a near-extremal black hole, \( r_0 \) and \( r_+ \) do not coincide and \( \delta < 0 \). As before, we define that

\[
r_0 = r_+ (1 - \epsilon),
\]

(52)

where \( 0 < \epsilon \leq 1 \). Besides, \( f(r_+) = 0 \), \( f'(r_+) \) is very close to zero and \( f'(r_0) = 0 \). Then, Eq. (50) is rewritten as

\[
f(M + dM, Q + dQ, l + dl, \alpha + d\alpha, r_0 + dr_0)
= \delta - \frac{2T}{r_+ (1 - \epsilon) + \frac{2\alpha}{r_+ (1 - \epsilon)} (1 - \delta)} dS
+ \frac{(1 - \delta)^2 - 1 - 4\pi T r_+ \left(1 - 2ln \frac{r_+}{\sqrt{\alpha}}\right)}{r_+^2 (1 - \epsilon)^2 + 2\alpha (1 - \delta)} d\alpha
+ \frac{\epsilon}{r_+^2 (1 - \epsilon)^2 + 2\alpha (1 - \delta)} \left[2Q + 6r_+^4 (1 - \epsilon) \frac{1}{l^3} + 1 + 4\pi T r_+ \left(1 - 2ln \frac{r_+}{\sqrt{\alpha}}\right)\right] d\alpha.
\]

(53)

Since the quantity \( \epsilon \) is infinitesimal, the forth line of Eq. (53) can be ignored. Both terms in the second row are negative. However, the sign of the third line in Eq. (53) is indefinite. As shown in Fig. 4, we set \( \epsilon = 0.00000000001, \delta = -0.1, Q = 0.7, d\alpha = -0.1, \alpha = 1 \) and \( l = 1 \).

From the figure we can know that the sign of \( f(M + dM, Q + dQ, l + dl, \alpha + d\alpha, r_0 + dr_0) \) will change from negative to positive when \( r_0 \) increase. Therefore, in the extended phase space, the near-extremal black hole can be overcharged and the effectiveness of WCCC is indefinite.

Fig. 4: The relation between \( f(r_0) \) and \( r_+ \) which parameter values are \( \epsilon = 0.00000000001, \delta = -0.1, Q = 0.7, d\alpha = -0.1, \alpha = 1 \) and \( l = 1 \).
VI. DISCUSSION AND CONCLUSION

In this paper, we investigated the thermodynamics and overcharging problem in a 4D Gauss-Bonnet AdS black hole under the scalar field. First, we reviewed the solutions of the 4D Gauss-Bonnet AdS black hole. Moreover, the variations of this black hole’s energy and charge within the infinitesimal time interval are investigated. Then we tested the validity of the first and second laws of thermodynamics and the weak cosmic censorship conjectures. It is worth noting that the RN-AdS black hole solution will be recovered when we take the limit $\alpha \to 0$. In Table [I] and Table [II] we summarize and compare the results of thermodynamics and WCCC both for the 4D Gauss-Bonnet AdS black hole and RN-AdS black hole. Moreover, the detailed formulas of the first law of thermodynamics for the 4D Gauss-Bonnet AdS black hole and the RN-AdS black hole are shown in Table [III].

| Normal phase space | 1st law | 2nd law | WCCC |
|--------------------|---------|---------|------|
| GB-AdS black hole  | Satisfied | Satisfied | Satisfied in the extremal black hole and near-extremal black hole. After the scalar field scattering, the extremal black hole will change to a non-extremal black hole under certain conditions. |
| RN-AdS black hole  | Satisfied | Satisfied | Satisfied for the extremal and near-extremal black holes. After the scalar field scattering, the extremal black hole will change to a non-extremal black hole under certain conditions. |

Table I: Results for the first and second laws of thermodynamics and the overcharging problem in the normal phase space.

As shown in Table [I] and Table [II] for the 4D Gauss-Bonnet AdS black hole, the first law of thermodynamics is satisfied both in the normal and extended phase spaces after the scattering of a scalar field. However, the second law of thermodynamics are not the same in this two cases. In the normal phase space, the second law of thermodynamics is satisfied. In the extended phase space, the validity of the second law of thermodynamics is indefinite.
Table II: Results for the first and second laws of thermodynamics and the overcharging problem in the extended phase space.

| Types of black holes | 1st law in the extended phase space |
|----------------------|-------------------------------------|
| GB-AdS black hole    | \( dM = TdS + \varphi dQ + \psi dP + \left[ \frac{1}{2r_s} + 2\pi T \left( 1 - 2\ln \frac{r_s}{\sqrt{\alpha}} \right) \right] d\alpha \) |
| RN-AdS black hole    | \( dM = TdS + \varphi dQ + \psi dP \) |

Table III: Results for the first thermodynamic law under different conditions in the extended phase space.

Furthermore, the overcharging problem is considered both in the normal and extremal phase spaces. In the normal phase space, the black hole cannot be overcharged after the scattering of a scalar field and the WCCC is satisfied. In the extended phase space, the extremal black hole cannot be overcharged. However, the near-extremal black hole can be overcharged under certain conditions. Therefore, the validity of the WCCC is indefinite.

In Ref. [3], thermodynamics and overcharging problem with pressure and volume of 4D Gauss-Bonnet AdS black holes are discussed by the charged particle absorption. In Ref. [16], weak cosmic censorship conjecture with pressure and volume in the Gauss-Bonnet AdS black hole \((d > 4)\) is discussed. It is hoped that more methods can be used to study the thermodynamic properties of 4D Gauss-Bonnet AdS black holes in the future. This will prompt us to further explore the deep relationship between the thermodynamic properties of the new four-dimensional black hole and its boundary conditions, and to learn more about the black hole.
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[1] R. Penrose, “Gravitational collapse: The role of general relativity,” Riv. Nuovo Cim. 1, 252 (1969) [Gen. Rel. Grav. 34, 1141 (2002)]. doi:10.1023/A:1016578408204
[2] R. M. Wald, “Gedanken experiments to destroy a black hole,” Ann. Phys. 82, 548 (1974).
[3] S. Ying, “Thermodynamics and Weak Cosmic Censorship Conjecture of 4D Gauss-Bonnet-Maxwell Black Holes via Charged Particle Absorption,” [arXiv:2004.09480 [gr-qc]].
[4] S. Yang, J. Wan, J. Chen, J. Yang and Y. Wang, “Weak cosmic censorship conjecture for the novel 4D charged Einstein-Gauss-Bonnet black hole with test scalar field and particle,” [arXiv:2004.07934 [gr-qc]].
[5] B. Mu and J. Tao, “Minimal Length Effect on Thermodynamics and Weak Cosmic Censorship Conjecture in anti-de Sitter Black Holes via Charged Particle Absorption,” [arXiv:1906.10544 [gr-qc]].
[6] S. Q. Hu, Y. C. Ong and D. N. Page, “No evidence for violation of the second law in extended black hole thermodynamics,” Phys. Rev. D 100, no.10, 104022 (2019) doi:10.1103/PhysRevD.100.104022 [arXiv:1906.05870 [gr-qc]].
[7] K. He, X. Hu and X. Zeng, “Weak cosmic censorship conjecture and thermodynamics in quintessence AdS black hole under charged particle absorption,” Chin. Phys. C 43, no.12, 125101 (2019) doi:10.1088/1674-1137/43/12/125101 [arXiv:1906.10531 [gr-qc]].
[8] P. Wang, H. Wu and H. Yang, “Thermodynamics and Weak Cosmic Censorship Conjecture in Nonlinear Electrodynamics Black Holes via Charged Particle Absorption,” [arXiv:1904.12365 [gr-qc]].
[9] C. Liu and S. Gao, “Overcharging magnetized black holes at linear order and the weak cosmic
censorship conjecture,” [arXiv:2003.12999 [gr-qc]].

[10] X. Hu, K. He, X. Zeng and J. Wu, “Thermodynamics and weak cosmic censorship conjecture of an AdS black hole with a monopole in the extended phase space,” [arXiv:2003.06783 [gr-qc]].

[11] P. Wang, H. Wu and S. Ying, “Validity of Thermodynamic Laws and Weak Cosmic Censorship for AdS Black Holes and Black Holes in a Cavity,” [arXiv:2002.12233 [gr-qc]].

[12] P. Wang, H. Wu and H. Yang, “Thermodynamics of nonlinear electrodynamics black holes and the validity of weak cosmic censorship at charged particle absorption,” Eur. Phys. J. C 79, no.7, 572 (2019) doi:10.1140/epjc/s10052-019-7090-z

[13] D. Chen and S. Zeng, “Overcharging problem and thermodynamics in extended phase spaces,” arXiv:2003.08102 [gr-qc].

[14] X. Y. Hu, K. J. He, X. X. Zeng and J. P. Wu, “Thermodynamics and weak cosmic censorship conjecture of an AdS black hole with a monopole in the extended phase space,” Chin. Phys. C 44, no. 5, 055103 (2020). doi:10.1088/1674-1137/44/5/055103

[15] X. X. Zeng and X. Y. Hu, “Thermodynamics and weak cosmic censorship conjecture with pressure in the rotating BTZ black holes,” [arXiv:1908.03845 [gr-qc]].

[16] X. X. Zeng, X. Y. Hu and K. J. He, “Weak cosmic censorship conjecture with pressure and volume in the Gauss-Bonnet AdS black hole,” Nucl. Phys. B 949, 114823 (2019) doi:10.1016/j.nuclphysb.2019.114823 [arXiv:1905.07750 [hep-th]].

[17] X. X. Zeng and H. Q. Zhang, “Thermodynamics and weak cosmic censorship conjecture in the Kerr-AdS black hole,” [arXiv:1905.01618 [gr-qc]].

[18] Y. W. Han, M. J. Lan and X. X. Zeng, “Thermodynamics and weak cosmic censorship conjecture in (2+1)-dimensional regular black hole with nonlinear electrodynamics sources,” Eur. Phys. J. Plus 135, no. 2, 172 (2020) doi:10.1140/epjp/s13360-020-00186-1 [arXiv:1903.03764 [gr-qc]].

[19] Y. W. Han, X. X. Zeng and Y. Hong, “Thermodynamics and weak cosmic censorship conjecture of the torus-like black hole,” Eur. Phys. J. C 79, no. 3, 252 (2019) doi:10.1140/epjc/s10052-019-6771-y [arXiv:1901.10660 [hep-th]].

[20] X. X. Zeng, Y. W. Han and D. Y. Chen, “Thermodynamics and weak cosmic censorship conjecture of BTZ black holes in extended phase space,” Chin. Phys. C 43, no. 10, 105104 (2019) doi:10.1088/1674-1137/43/10/105104 [arXiv:1901.08915 [gr-qc]].

[21] I. Semiz, “Dyonic Kerr-Newman black holes, complex scalar field and cosmic censorship,”
[22] W. Hong, B. Mu and J. Tao, “Test the Weak Cosmic Censorship Conjecture in Torus-Like Black Hole under Charged Scalar Field,” arXiv:2001.09008 [physics.gen-ph].

[23] S. Yang, J. Chen, J. Wan, S. Wei and Y. Liu, “Weak cosmic censorship conjecture for a Kerr-Taub-NUT black hole with a test scalar field and particle,” Phys. Rev. D 101, no.6, 064048 (2020) doi:10.1103/PhysRevD.101.064048 arXiv:2001.03106 [gr-qc].

[24] T. Bai, W. Hong, B. Mu and J. Tao, “Weak cosmic censorship conjecture in the nonlinear electrodynamics black hole under the charged scalar field,” Commun. Theor. Phys. 72, no.1, 015401 (2020) doi:10.1088/1572-9494/ab544b

[25] D. Chen, W. Yang and X. Zeng, “Thermodynamics and weak cosmic censorship conjecture in Reissner-Nordstrøm anti-de Sitter black holes with scalar field,” Nucl. Phys. B 946, 114722 (2019) doi:10.1016/j.nuclphysb.2019.114722 arXiv:1901.05140 [hep-th].

[26] D. Chen, “Weak cosmic censorship conjecture in BTZ black holes with scalar fields,” Chin. Phys. C 44, no. 1, 015101 (2020) doi:10.1088/1674-1137/44/1/015101 arXiv:1812.03459 [gr-qc].

[27] S. Shaymatov, N. Dadhich, B. Ahmedov and M. Jamil, “Five dimensional charged rotating minimally gauged supergravity black hole cannot be over-spun and/or over-charged in non-linear accretion,” Eur. Phys. J. C 80, no. 5, 481 (2020) doi:10.1140/epjc/s10052-020-8009-4 arXiv:1908.01195 [gr-qc].

[28] B. Gwak, “Weak Cosmic Censorship Conjecture in Kerr-(Anti)-de Sitter Black Hole with Scalar Field,” JHEP 09, 081 (2018) doi:10.1007/JHEP09(2018)081 arXiv:1807.10630 [gr-qc].

[29] G. Z. Toth, “Test of the weak cosmic censorship conjecture with a charged scalar field and dyonic Kerr-Newman black holes,” Gen. Rel. Grav. 44, 2019-2035 (2012) doi:10.1007/s10714-012-1374-z arXiv:1112.2382 [gr-qc].

[30] J. Goncalves and J. Natario, “Proof of the weak cosmic censorship conjecture for several extremal black holes,” arXiv:2004.02902 [gr-qc].

[31] J. Jiang and M. Zhang, “Weak cosmic censorship conjecture in Einstein-Maxwell gravity with scalar hair,” Eur. Phys. J. C 80, no.3, 196 (2020) doi:10.1140/epjc/s10052-020-7751-y

[32] K. Duztas, “Over-spinning Kerr-Sen black holes with test fields,” Int. J. Mod. Phys. D 28, no. 02, 1950044 (2018) doi:10.1142/S0218271819500445 arXiv:1811.03452 [gr-qc].

[33] K. Duztas, “Overspinning Kerr-MOG black holes by test fields and the third law of black
hole dynamics,” Eur. Phys. J. C 80, no. 1, 19 (2020) doi:10.1140/epjc/s10052-020-7607-5 [arXiv:1907.13435 [gr-qc]].

[34] B. Gwak, “Thermodynamics with Pressure and Volume under Charged Particle Absorption,” JHEP 1711, 129 (2017) doi:10.1007/JHEP11(2017)129 [arXiv:1709.08665 [gr-qc]].

[35] B. Gwak, “Weak Cosmic Censorship with Pressure and Volume in Charged Anti-de Sitter Black Hole under Charged Scalar Field,” JCAP 1908, 016 (2019) doi:10.1088/1475-7516/2019/08/016 [arXiv:1901.05589 [gr-qc]].

[36] S. W. Hawking, “Particle Creation by Black Holes,” Commun. Math. Phys. 43, 199 (1975) Erratum: [Commun. Math. Phys. 46, 206 (1976)]. doi:10.1007/BF02345020, 10.1007/BF01608497

[37] S. W. Hawking, “Black Holes and Thermodynamics,” Phys. Rev. D 13, 191 (1976). doi:10.1103/PhysRevD.13.191

[38] D. Christodoulou, “Reversible and irreversible transformations in black hole physics,” Phys. Rev. Lett. 25, 1596 (1970). doi:10.1103/PhysRevLett.25.1596

[39] D. Christodoulou and R. Ruffini, “Reversible transformations of a charged black hole,” Phys. Rev. D 4, 3552 (1971). doi:10.1103/PhysRevD.4.3552

[40] L. Smarr, “Mass formula for Kerr black holes,” Phys. Rev. Lett. 30, 71 (1973) Erratum: [Phys. Rev. Lett. 30, 521 (1973)]. doi:10.1103/PhysRevLett.30.521, 10.1103/PhysRevLett.30.71

[41] J. M. Bardeen, “Kerr Metric Black Holes,” Nature 226, 64 (1970). doi:10.1038/226064a0

[42] J. D. Bekenstein, “Black holes and entropy,” Phys. Rev. D 7, 2333 (1973). doi:10.1103/PhysRevD.7.2333

[43] J. D. Bekenstein, “Generalized second law of thermodynamics in black hole physics,” Phys. Rev. D 9, 3292-3300 (1974) doi:10.1103/PhysRevD.9.3292

[44] D. Kastor, S. Ray and J. Traschen, “Enthalpy and the Mechanics of AdS Black Holes,” Class. Quant. Grav. 26, 195011 (2009) doi:10.1088/0264-9381/26/19/195011 [arXiv:0904.2765 [hep-th]].

[45] B. P. Dolan, “Pressure and volume in the first law of black hole thermodynamics,” Class. Quant. Grav. 28, 235017 (2011) doi:10.1088/0264-9381/28/23/235017 [arXiv:1106.6260 [gr-qc]].

[46] D. Kubiznak and R. B. Mann, “P-V criticality of charged AdS black holes,” JHEP 1207, 033 (2012) doi:10.1007/JHEP07(2012)033 [arXiv:1205.0559 [hep-th]].
[47] M. Cvetic, G. W. Gibbons, D. Kubiznak and C. N. Pope, “Black Hole Enthalpy and
 an Entropy Inequality for the Thermodynamic Volume,” Phys. Rev. D 84, 024037 (2011)
doi:10.1103/PhysRevD.84.024037 [arXiv:1012.2888 [hep-th]].

[48] E. Caceres, P. H. Nguyen and J. F. Pedraza, “Holographic entanglement entropy
and the extended phase structure of STU black holes,” JHEP 1509, 184 (2015)
doi:10.1007/JHEP09(2015)184 [arXiv:1507.06069 [hep-th]].

[49] S. H. Hendi and M. H. Vahidinia, “Extended phase space thermodynamics and P-V crit-
icality of black holes with a nonlinear source,” Phys. Rev. D 88, no. 8, 084045 (2013)
doi:10.1103/PhysRevD.88.084045 [arXiv:1212.6128 [hep-th]].

[50] J. F. Pedraza, W. Sybesma and M. R. Visser, “Hyperscaling violating black holes with spherical and hyperbolic horizons,” Class. Quant. Grav. 36, no. 5, 054002 (2019) doi:10.1088/1361-6382/ab0094 [arXiv:1807.09770 [hep-th]].

[51] B. P. Dolan, “The cosmological constant and the black hole equation of state,” Class. Quant. Grav. 28, 125020 (2011) doi:10.1088/0264-9381/28/12/125020 [arXiv:1008.5023 [gr-qc]].

[52] D. G. Boulware and S. Deser, “String Generated Gravity Models,” Phys. Rev. Lett. 55, 2656 (1985) doi:10.1103/PhysRevLett.55.2656

[53] D. L. Wiltshire, “Spherically Symmetric Solutions of Einstein-maxwell Theory With a {Gauss-
Bonnet} Term,” Phys. Lett. B 169, 36-40 (1986) doi:10.1016/0370-2693(86)90681-7

[54] R. G. Cai, “Gauss-Bonnet black holes in AdS spaces,” Phys. Rev. D 65, 084014 (2002)
doi:10.1103/PhysRevD.65.084014 [arXiv:hep-th/0109133 [hep-th]].

[55] S. Nojiri and S. D. Odintsov, “Anti-de Sitter black hole thermodynamics in higher derivative
gravity and new confining deconfining phases in dual CFT,” Phys. Lett. B 521, 87-95 (2001)
[erratum: Phys. Lett. B 542, 301 (2002)] doi:10.1016/S0370-2693(01)01186-8 [arXiv:hep-
th/0109122 [hep-th]].

[56] M. Cvetic, S. Nojiri and S. D. Odintsov, “Black hole thermodynamics and negative entropy
in de Sitter and anti-de Sitter Einstein-Gauss-Bonnet gravity,” Nucl. Phys. B 628, 295-330
(2002) doi:10.1016/S0550-3213(02)00075-5 [arXiv:hep-th/0112045 [hep-th]].

[57] D. Glavan and C. Lin, “Einstein-Gauss-Bonnet Gravity in Four-Dimensional Space-
time,” Phys. Rev. Lett. 124, no.8, 081301 (2020) doi:10.1103/PhysRevLett.124.081301
[arXiv:1905.03601 [gr-qc]].

[58] P. G. S. Fernandes, “Charged Black Holes in AdS Spaces in 4D Einstein Gauss-Bonnet Grav-
ity,” Phys. Lett. B 805, 135468 (2020) doi:10.1016/j.physletb.2020.135468 [arXiv:2003.05491 [gr-qc]].

[59] B. Eslam Panah, K. Jafarzade and S. H. Hendi, “Charged 4D Einstein-Gauss-Bonnet-AdS Black Holes: Shadow, Energy Emission, Deflection Angle and Heat Engine,” [arXiv:2004.04058 [hep-th]].

[60] K. Hegde, A. Naveena Kumara, C. L. A. Rizwan, A. K. M. and M. S. Ali, “Thermodynamics, Phase Transition and Joule Thomson Expansion of novel 4-D Gauss Bonnet AdS Black Hole,” [arXiv:2003.08778 [gr-qc]].

[61] S. W. Wei and Y. X. Liu, “Extended thermodynamics and microstructures of four-dimensional charged Gauss-Bonnet black hole in AdS space,” Phys. Rev. D 101, no.10, 104018 (2020) doi:10.1103/PhysRevD.101.104018 [arXiv:2003.14275 [gr-qc]].

[62] S. A. Hosseini Mansoori, “Thermodynamic geometry of the novel 4-D Gauss Bonnet AdS Black Hole,” [arXiv:2003.13382 [gr-qc]].

[63] D. V. Singh and S. Siwach, “Thermodynamics and P-v criticality of Bardeen-AdS Black Hole in 4D Einstein-Gauss-Bonnet Gravity,” Phys. Lett. B 808, 135658 (2020) doi:10.1016/j.physletb.2020.135658 [arXiv:2003.11754 [gr-qc]].

[64] R. A. Konoplya and A. F. Zinhailo, “Quasinormal modes, stability and shadows of a black hole in the novel 4D Einstein-Gauss-Bonnet gravity,” [arXiv:2003.01188 [gr-qc]].

[65] R. A. Konoplya and A. Zhidenko, “(In)stability of black holes in the 4D Einstein–Gauss–Bonnet and Einstein–Lovelock gravities,” Phys. Dark Univ. 30, 100697 (2020) doi:10.1016/j.dark.2020.100697 [arXiv:2003.12492 [gr-qc]].

[66] S. L. Li, P. Wu and H. Yu, “Stability of the Einstein Static Universe in 4D Gauss-Bonnet Gravity,” [arXiv:2004.02080 [gr-qc]].

[67] C. Y. Zhang, S. J. Zhang, P. C. Li and M. Guo, “Superradiance and stability of the regularized 4D charged Einstein-Gauss-Bonnet black hole,” JHEP 08, 105 (2020) doi:10.1007/JHEP08(2020)105 [arXiv:2004.03141 [gr-qc]].

[68] A. K. Mishra, “Quasinormal modes and Strong Cosmic Censorship in the novel 4D Einstein-Gauss-Bonnet gravity,” [arXiv:2004.01243 [gr-qc]].

[69] R. G. Cai, L. M. Cao, L. Li and R. Q. Yang, “P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space,” JHEP 1309, 005 (2013) doi:10.1007/JHEP09(2013)005 [arXiv:1306.6233 [gr-qc]].
[70] S. Mahish and B. Chandrasekhar, “Chaos in Charged Gauss-Bonnet AdS Black Holes in Extended Phase Space,” Phys. Rev. D 99, no.10, 106012 (2019) doi:10.1103/PhysRevD.99.106012 [arXiv:1902.08932 [hep-th]].

[71] J. D. Bekenstein, “Extraction of energy and charge from a black hole,” Phys. Rev. D 7, 949 (1973). doi:10.1103/PhysRevD.7.949

[72] S. Q. Hu, Y. C. Ong and D. N. Page, “No evidence for violation of the second law in extended black hole thermodynamics,” Phys. Rev. D 100, no.10, 104022 (2019) doi:10.1103/PhysRevD.100.104022 [arXiv:1906.05870 [gr-qc]].

[73] S. W. Wei and Y. X. Liu, “Triple points and phase diagrams in the extended phase space of charged Gauss-Bonnet black holes in AdS space,” Phys. Rev. D 90, no. 4, 044057 (2014) doi:10.1103/PhysRevD.90.044057 [arXiv:1402.2837 [hep-th]].

[74] A. Haldar and R. Biswas, “Geometrothermodynamic analysis and $P-V$ criticality of higher dimensional charged Gauss-Bonnet black holes with first order entropy correction,” Gen. Rel. Grav. 51, no. 2, 35 (2019) doi:10.1007/s10714-019-2520-7 [arXiv:1906.01970 [gr-qc]].

[75] S. Q. Lan, “Joule-Thomson expansion of charged Gauss-Bonnet black holes in AdS space,” Phys. Rev. D 98, no. 8, 084014 (2018) doi:10.1103/PhysRevD.98.084014 [arXiv:1805.05817 [gr-qc]].