Determining the Quark Mixing Matrix
From CP-Violating Asymmetries

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Abstract

If the Standard Model explanation of CP violation is correct, then measurements of CP-violating asymmetries in B meson decays can in principle determine the entire quark mixing matrix.

According to the Standard Model (SM), CP violation arises from the fact that in the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix, some of the elements are not real. This picture of CP violation will be incisively tested in neutral B meson decays, where some of the CP-violating asymmetries can yield theoretically clean information on the phases of various products of CKM elements [1].

While the SM does not predict the elements of the CKM matrix, V, in detail, it does
require that $V$ be unitary. This requirement is, as we shall see, a very powerful constraint. It implies, among other things, that any pair of columns, or any pair of rows, of $V$ be orthogonal. Thus, assuming that there are three generations of quarks, so that $V$ is a $3 \times 3$ matrix, we have the six orthogonality conditions

$$
\begin{align*}
\sum_{\alpha=1}^{3} V_{\alpha i} V^*_{\alpha j} &= 0 , \ i \neq j, \\
\sum_{i=1}^{3} V_{\alpha i} V^*_{\beta i} &= 0 , \ \alpha \neq \beta.
\end{align*}
$$

(1)

Here and hereafter, Greek subscripts run over the up-type quarks $u, c$ and $t$, while Latin ones run over the down-type quarks $d, s$ and $b$. It is often useful to picture each of Eqs. (1) as the statement that the “unitarity triangle” in the complex plane whose sides are the terms in the equation is closed. The six unitarity triangles corresponding to Eqs. (1) are depicted, somewhat schematically, in Fig. 1. We refer to each of these triangles by naming the columns or rows whose orthogonality it represents. As shown in Fig. 1, in the $ds$, $sb$, $uc$ and $ct$ triangles, one leg is known empirically to be short compared to the other two, so that the angle opposite the short leg is small.

Apart from an extra $\pi$ and a possible minus sign, each of the angles in the unitarity triangles is just the relative phase of the two adjacent legs. Let

$$
\omega_{ij}^{\alpha\beta} \equiv \arg(V_{\alpha i} V^*_{\alpha j}/V_{\beta i} V^*_{\beta j}),
$$

with $\alpha \neq \beta$ and $i \neq j$, be the relative phase of the leg involving the up-type quark $\alpha$ (the “$\alpha$ leg”) and the leg involving the up-type quark $\beta$ (the “$\beta$ leg”) in the $ij$ column triangle. Since $\arg(V_{\alpha i} V^*_{\alpha j}/V_{\beta i} V^*_{\beta j}) = \arg(V_{\alpha i} V^*_{\beta i}/V_{\alpha j} V^*_{\beta j})$, $\omega_{ij}^{\alpha\beta}$ is also the relative phase of the $i$ and $j$ legs in the $\alpha\beta$ row triangle. At most, four of the $\omega_{ij}^{\alpha\beta}$ can be independent, since four parameters, usually taken to be three mixing angles and one complex phase, are sufficient to fully determine $V$ [2]. Indeed, it is easy to show [3] that, mod $2\pi$, any $\omega_{ij}^{\alpha\beta}$ is a simple linear combination of, for example, the four phases $\omega_{tu}^{bd}$, $\omega_{ct}^{bd}$, $\omega_{tu}^{sb}$ and $\omega_{ct}^{sb}$. Let us now demonstrate that the phase of any phase-convention-independent product of CKM elements (that is, the quantity probed by the CP asymmetry in any neutral $B$ decay) is a linear combination of these four phases, with integer coefficients. To show this, it is convenient to work in the phase convention where all elements of $V$ are real and positive,
except for $V_{ud}$, $V_{us}$, $V_{cd}$ and $V_{cs}$. In this phase convention,

\[
\begin{align*}
\omega_{tu}^{bd} & \equiv \arg(V_{tb}V_{td}^*/V_{ub}V_{ud}^*) = \arg(V_{ud}) \\
\omega_{tu}^{sb} & \equiv \arg(V_{ts}V_{tb}^*/V_{us}V_{ub}^*) = -\arg(V_{us}) \\
\omega_{ct}^{bd} & \equiv \arg(V_{cb}V_{cd}^*/V_{tb}V_{td}^*) = -\arg(V_{cd}) \\
\omega_{ct}^{sb} & \equiv \arg(V_{cs}V_{cb}^*/V_{ts}V_{tb}^*) = \arg(V_{cs}).
\end{align*}
\]

(3)

Now, suppose $P$ is some phase-convention-independent product of CKM elements. In our present phase convention, the phase of $P$, $\omega$, is given by

\[
\omega = n_{ud} \arg(V_{ud}) + n_{us} \arg(V_{us}) + n_{cd} \arg(V_{cd}) + n_{cs} \arg(V_{cs}).
\]

(4)

Here, $n_{\alpha i}$ is the number of factors of $V_{\alpha i}$ appearing in $P$, minus the number of factors of $V_{\alpha i}^*$. From Eqs. (4) and (3), $\omega$ is given in the present phase convention by

\[
\omega = n_{ud} \omega_{tu}^{bd} - n_{us} \omega_{tu}^{sb} - n_{cd} \omega_{ct}^{bd} + n_{cs} \omega_{ct}^{sb}.
\]

(5)

That is,

\[
\omega = \sum_{q=1}^{4} n_q \omega_q,
\]

(6)

where the $\omega_q$ are the phases $\omega_{tu}^{bd}$, etc., appearing in Eq. (5), and the $n_q$ are integers which are known for any given CKM product $P$. Since $\omega$ is phase-convention-independent, and, as one may easily verify, so are the $\omega_{ij}^{\alpha\beta}$, the relation (6) must hold in any phase convention.

We see that the four phases $\omega_q$ form a complete set of variables for the description of CP violation in $B$ decay. Moreover, through Eq. (6) the phases $\omega$ that are probed in the $B$ experiments are related very simply to the $\omega_q$. In contrast, when $V$ is treated exactly, the phases $\omega$ are quite complicated functions of the quark mixing angles and complex phase factor often used to parametrize $V$. Thus, it appears useful to think of measurements of CP violation in $B$ decays as probes of the variables $\omega_q$.

Imagine that through observation of CP-violating asymmetries in $B$ decays we have determined the four phases $\omega_q$. Let us show that from these phases we can reconstruct the entire CKM matrix!

To determine the phases of the CKM elements from the $\omega_q$, we must first choose a phase convention, because the phases of individual $V_{\alpha i}$ change with quark-field phase
redefinitions, while the $\omega_q$ do not. Let us adopt the phase convention defined just before Eqs. (3). In this convention, the only $V_{\alpha i}$ which are not real and positive are the four whose phases are given in terms of the $\omega_q$ by Eqs. (3). Thus, all nontrivial phases in $V$ are determined by the $\omega_q$.

To see how the magnitudes of the CKM elements are determined by the $\omega_q$, let us first note that, according to the law of sines, the ratio between the $\alpha$ and $\beta$ legs of the $ij$ triangle is given by

$$\frac{|V_{\alpha i} V_{\alpha j}^*|}{|V_{\beta i} V_{\beta j}^*|} = \frac{\sin \omega_{ij}^\beta \gamma}{\sin \omega_{ij}^\alpha \gamma}.$$  \hspace{1cm} (7)

Here and in the following relation, $\alpha\beta\gamma$ is some cyclic permutation of $uct$, and $ijk$ of $dsb$. Applying the law of sines to the $jk$ and $ki$ triangles as well, we find that

$$\frac{|V_{\alpha i}|^2}{|V_{\beta i}|^2} = \frac{|V_{\alpha i} V_{\alpha j}^*|}{|V_{\beta i} V_{\beta j}^*|} \frac{|V_{\beta j} V_{\beta k}^*|}{|V_{\alpha j} V_{\alpha k}^*|} \frac{|V_{\alpha k} V_{\alpha i}^*|}{|V_{\beta k} V_{\beta i}^*|}$$
$$= \frac{\sin \omega_{ij}^\beta \gamma}{\sin \omega_{ij}^\alpha \gamma} \frac{\sin \omega_{jk}^\gamma \alpha}{\sin \omega_{jk}^\beta \gamma} \frac{\sin \omega_{ki}^\gamma \alpha}{\sin \omega_{ki}^\beta \gamma}.$$  \hspace{1cm} (8)

The phase angles in the column triangles appearing on the right-hand side of this relation are all known once the $\omega_q$ are known. Thus, given the $\omega_q$, Eq. (8) with $\alpha = c$ and $\beta = u$ determines $|V_{ci}|^2/|V_{ui}|^2 \equiv a_i$, and with $\alpha = t$ and $\beta = u$ determines $|V_{ti}|^2/|V_{ui}|^2 \equiv b_i$. If we then impose the unitarity constraint on the $i^{th}$ column of $V$,

$$|V_{ui}|^2 + |V_{ci}|^2 + |V_{ti}|^2 = 1,$$  \hspace{1cm} (9)

we obtain $|V_{ui}|^2$:

$$|V_{ui}|^2 = \frac{1}{1 + a_i + b_i}.$$  \hspace{1cm} (10)

From $a_i$ and $b_i$, the remaining magnitudes in the $i^{th}$ column, $|V_{ci}|^2$ and $|V_{ti}|^2$, then follow immediately. In this way, the phases $\omega_q$ determine the magnitudes of all the elements of $V$.

It is worth emphasizing the crucial role played here by unitarity. Naively, one might guess that, by determining all the interior angles in the unitarity triangles, the phases $\omega_q$ would fix the shapes of these triangles, but not their sizes. There would then be nothing to set the scale for the magnitudes of individual CKM elements. However, in reality this scale is set by unitarity via Eq. (9).
Decay Mode | Quantity Determined by Observed CP Asymmetry
--- | ---
$B_d \rightarrow \pi^+\pi^-$ | $\sin 2\omega_{tu}^{bd}$
$B_d \rightarrow \Psi K_s$ | $\sin 2\omega_{ct}^{bd}$
$B^\pm \rightarrow DK^\pm$ | $\sin^2(\omega_{uc}^{bd} + \omega_{uc}^{ds})$
$\bar{B}_s \rightarrow D^\pm K^\mp$ | $\sin^2(\omega_{uc}^{bd} - 2\omega_{ct}^{sb} + \omega_{uc}^{ds})$
$\bar{B}_s \rightarrow \Psi \phi$ | $\sin 2\omega_{ct}^{sb}$

Table 1. An illustrative complete set of $B$ decay experiments [6], [7].

Since the four $\omega_q$ fully determine $V$, and we know four independent parameters are required to do that, it is now obvious that the $\omega_q$ are independent parameters. Attempts to find relations among them within the SM would prove fruitless. Of course, by using the linear relations that permit us to express some of the nine $\omega_{ij}^{\alpha\beta}$ in terms of others [3], we can replace the four independent $\omega_q$ of Eq. (5) by any other set of four independent phases in the unitarity triangles [4].

Is it feasible for $B$ decay experiments to determine the four independent phases $\omega_q$, and through them the entire CKM matrix? To explore this question, it is illuminating to choose the four independent $\omega_q$, not as the phases which appear in Eq. (5), but as

$$\omega_1 \equiv \omega_{tu}^{bd}, \quad \omega_2 \equiv \omega_{ct}^{bd}, \quad \omega_3 \equiv \omega_{ct}^{sb}, \quad \omega_4 \equiv \omega_{uc}^{ds}. \quad (11)$$

Referring to Fig. 1, and recalling the rough magnitudes of the CKM elements summarized by the Wolfenstein approximation [5], we see that $\omega_1$ and $\omega_2$ may both be large (i.e., far from 0 and $\pm \pi$), but $\omega_3 = \pm(\pi - \epsilon)$ with $\epsilon \lesssim 0.05$ radians, and $\omega_4 = \pm(\pi - \epsilon')$ with $\epsilon' \lesssim 0.003$ radians. In Table 1, we list for purposes of illustration a set of $B$ decay modes whose study could in principle determine all four $\omega_q$ of Eq. (11), with no ambiguities.

As Table 1 illustrates, the quantities determined by CP asymmetries in $B$ decays are not precisely the phases of products of CKM elements, but trigonometric functions of these phases. These functions leave the phases themselves discretely ambiguous. However, after a lengthy but straightforward analysis [3], it is found that, together, the measurements in
Table 1, supplemented by the constraint that the angles in any triangle add up to \( \pi \) [8], determine the four phases (11) without ambiguities [9], [10].

Clearly, the determination of the larger phases \( \omega_1, \omega_2 \) and \( \omega_3 \) may be feasible, but, given that \( \epsilon' \lesssim 0.003 \), the determination of \( \omega_4 \) would be extremely difficult, if not impossible. For example, to determine \( \omega_4 \equiv \omega_{uc}^{ds} \) from the decays in Table 1, we would first have to use the first two of these decays to very accurately fix \( \omega_{tu}^{bd} \) and \( \omega_{ct}^{bd} \), from which \( \omega_{uc}^{bd} \) follows. We would then have to use the decays \( B^\pm \to DK^\pm \to (K^+K^-)K^\pm \) to determine \( \sin^2(\omega_{uc}^{bd} + \omega_{uc}^{ds}) \) to better than \( \pm 0.001! \) Suppose, then, that \( \omega_4 \) proves to be beyond reach. Can we learn interesting things about the CKM matrix, and in particular about the magnitudes of its elements, from a knowledge of \( \omega_1, \omega_2 \) and \( \omega_3 \) alone? Indeed we can. Neglecting \( \epsilon \) and \( \epsilon' \) compared to \( \omega_1 \) and \( \omega_2 \), we find from Eq. (8) that

\[
\frac{|V_{ub}|^2}{|V_{cb}|} \simeq \frac{\sin \beta \sin \epsilon}{\sin \alpha \sin \gamma}.
\]

(12)

Here, \( \alpha \equiv \pi - |\omega_1|, \beta \equiv \pi - |\omega_2|, \) and \( \gamma = \pi - \alpha - \beta \) are the same (positive) interior angles of the \( bd \) triangle commonly denoted by these symbols in the literature [1]. Similarly, from the analogue of Eq. (8) for the ratio of CKM elements in one row, we find that

\[
\frac{|V_{td}|^2}{|V_{ts}|} \simeq \frac{\sin \gamma \sin \epsilon}{\sin \alpha \sin \beta}.
\]

(13)

Since \( |V_{cb}| \) is known and \( |V_{ts}| \simeq |V_{cb}| \) [11], Eqs. (12) and (13) determine \( |V_{ub}| \) and \( |V_{td}| \) in terms of CP-violating angles. This is interesting, because these very small CKM elements are difficult to determine in other ways. Note that their determination via Eqs. (12) and (13) has the advantage of being free of theoretical hadronic uncertainties [12], apart from those needed to fix \( |V_{cb}| \).

From the analogue of Eq. (8) for CKM elements in one row, we find the relation

\[
\frac{|V_{us}|^2}{|V_{ud}|} \simeq \frac{\sin \alpha \sin \epsilon}{\sin \beta \sin \gamma},
\]

(14)

expressing the square of the Cabibbo angle in terms of CP-violating angles. The Cabibbo angle is, of course, very well known, so this relation can serve as a good test of the SM explanation of CP violation.

Conceptually, it is very interesting that the entire CKM matrix can in principle be determined by CP-violating \( B \)-decay asymmetries alone. This implies that these asymmetries can serve as an incisive probe of the structure of the matrix responsible, according
to the SM, for CP violation. Perhaps these asymmetries can even be a practical source of significant information on $|V_{ub}/V_{cb}|$ and $|V_{td}/V_{ts}|$.

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1. References

[1] For reviews, see, for example, Y. Nir and H.R. Quinn in “B Decays,” ed. S. Stone (World Scientific, Singapore, 1992), p. 362; I. Dunietz, *ibid* p. 393.

[2] See, for example, C. Jarlskog in “CP Violation,” ed. C. Jarlskog (World Scientific, Singapore, 1989), p. 3; J. Rosner in “B Decays,” ed. S. Stone (World Scientific, Singapore, 1992), p. 312.

[3] A more complete account of our results will be presented elsewhere.

[4] Just as four independent phases $\omega_q$ fully determine $V$, so do four independent magnitudes $|V_{\alpha i}|$ plus one sign which fixes the orientation of the unitarity triangles. See C. Jarlskog, Ref. [2].

[5] L. Wolfenstein, *Phys. Rev. Lett.* **51** (1983) 1945.

[6] The decays $B_{\pm} \to DK_{\pm}$, followed by $D \to K^+K^-$, cleanly yield CKM phase information even though they involve charged parent $B$ mesons. See M. Gronau and D. Wyler, *Phys. Lett.* **265B** (1991) 172.

[7] The decays $(B_s^{0} \to \Psi \phi$ are expected to have a CP asymmetry of only a few percent, but a relatively large branching ratio of $\sim 10^{-3}$, facilitating their study. See I. Dunietz, Fermilab preprint Fermilab-Conf-93/90-T, to appear in the Proceedings of the Workshop on $B$ Physics at Hadron Accelerators, Snowmass, Colorado, June 21-July 2, 1993.

[8] We thank Paris Sphicas for pointing out that this constraint can help resolve ambiguities.
[9] The decays $\overline{B}_s \to D_s^{\pm} K^{\mp}$ need be used only to resolve a discrete ambiguity. This same discrete ambiguity can also be resolved by determining $\cos 2\omega_{tu}^{bd}$ through study of the decays $\overline{B}_d \to \rho \pi$. How these decays determine this quantity is explained in A.E. Snyder and H.R. Quinn, *Phys. Rev.* **D48** (1993) 2139.

[10] Ambiguities remain for special, isolated values of the phases, but such values are unlikely.

[11] See, for example, J. Rosner, Ref. [2].

[12] If one is willing to use $|V_{cd}|$ as an input, then one can also find $|V_{ub}/V_{cb}|$ using the relation $|V_{ub}/V_{cb}|/|V_{cd}/V_{ud}| = \sin \beta/\sin \alpha$, obtained by applying the law of sines to the $bd$ triangle. If one is further willing to assume $|V_{tb}| \simeq 1$, then one can find $|V_{td}/V_{cb}|$ using the relation $|V_{td}V_{tb}|/|V_{cd}V_{cb}| = \sin \gamma/\sin \alpha$, obtained similarly.

Figure Caption:

The unitarity triangles. To the left of each triangle is indicated the pair of columns, or of rows, whose orthogonality this closed triangle expresses.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9403341v1