Two finiteness theorem for (a,b)-modules.

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Summary.

We prove the following two results

1. For a proper holomorphic function \( f : X \to D \) of a complex manifold \( X \) on a disc such that \( \{ df = 0 \} \subset f^{-1}(0) \), we construct, in a functorial way, for each integer \( p \), a geometric (a,b)-module \( E^p \) associated to the (filtered) Gauss-Manin connexion of \( f \).
   This first theorem is an existence/finiteness result which shows that geometric (a,b)-modules may be used in global situations.

2. For any regular (a,b)-module \( E \) we give an integer \( N(E) \), explicitely given from simple invariants of \( E \), such that the isomorphism class of \( E/b^{N(E)}E \) determines the isomorphism class of \( E \).
   This second result allows to cut asymptotic expansions (in powers of \( b \)) of elements of \( E \) without loosing any information.

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1 Introduction.

The following situation is frequently met: we consider a vector space $E$ of multi-valued holomorphic functions (possibly with values in a complex finite dimensional vector space $V$) with finite determination on a punctured disc around 0 in $\mathbb{C}$, stable by multiplication by the variable $z$ and by ”primitive”. These functions are determined by their formal asymptotic expansions at 0 of the type

$$\sum_{(\alpha,j)\in A \times [0,n]} c_{\alpha,j}(z) \cdot \frac{(\log z)^j}{j!}$$

where $n$ is a fixed integer, where $A$ is a finite set of complex numbers whose real parts are, for instance, in the interval $]-1,0[$, and where the $c_{\alpha,j}$ are in $\mathbb{C}[[z]] \otimes V$. Define $e(\alpha,j) = z^\alpha \cdot \frac{(\log z)^j}{j!}$ and $E(\alpha,n) := \oplus_{j=0}^n (\mathbb{C}[[z]] \otimes V).e(\alpha,j)$. Then for each $\alpha$, $E(\alpha,n)$ is a free $\mathbb{C}[[z]]$-module of rank $(n+1) \cdot \dim V$ which is stable by $b := \int_0^z$. To be precise, $b$ is defined by induction on $j \geq 0$ by the ”obvious” formulas :

$$b[e(\alpha,0)] = \frac{e(\alpha+1,0)}{\alpha+1} \quad \text{and for} \quad j \geq 1$$

$$b[e(\alpha,j)] = \frac{e(\alpha+1,j)}{\alpha+1} - \frac{1}{\alpha+1} b[e(\alpha,j-1)].$$

So we have an inclusion $E \subset \bigoplus_{\alpha \in A} E(\alpha)$ which is compatible with $a := \times z$ and by $b$. Note that each $E(\alpha)$ is also a free finite rank $\mathbb{C}[[b]]$-module which is stable by $a$.

Let us assume now that $E$ is also a $\mathbb{C}[[b]]$-module which is stable by $a$. Then $E$ has to be free and of finite rank over $\mathbb{C}[[b]]$. Our aim in this situation is to understand and to describe the relations between the coefficients $c_{\alpha,j}$ of the asymptotic expansions of elements in $E$. This leads to construct ”invariants” associated to the given $E$.

As in the ”geometric” situations we consider the complex numbers $\alpha \in A$ are rational numbers, a change of variable of type $t := z^{1/N}$, with $N \in \mathbb{N}^*$, allows to reduce the situation to the case where all $\alpha$ are 0. Then one can use the ”nilpotent” operator $\frac{d}{dt}$ and the nilpotent part of the monodromy $e_{\alpha,j} \rightarrow e_{\alpha,j-1}$ to construct some filtrations in order, in the good cases, to build a Mixte Hodge structure on a finite dimensional vector space associated to $E$. For instance, this is the case in A.N. Varchenko’s description[V. 80] of the Mixte Hodge structure built by J. Steenbrink [St. 76] on the cohomology of the Milnor fiber of an holomorphic function with an isolated singularity at the origin of $\mathbb{C}^{n+1}$.

But it is clear that we loose some information in this procedure. The point of view which is to consider $E$ itself as a left module on the (non commutative) algebra

$$\tilde{A} := \{\sum_{\nu \geq 0} P_\nu(a).b^\nu\}$$
where the $P_{\nu}$ are polynomials with complex coefficients is richer. This is evidenced by M. Saito result [Sa. 91].

The aim of the first part of this article is to build in a natural way, for any proper holomorphic function $f : X \rightarrow D$ of a complex manifold $X$, assumed to be smooth outside its $0$–fiber $X_0 := f^{-1}(0)$, a regular (geometric) $(a,b)$-module for each degree $p \geq 0$, which represent a filtered version of the Gauss-Manin connexion of $f$ at the origin. This result is in fact a finiteness theorem which is a first step to refine the limite Mixte Hodge structure in this situation. It is interesting to remark that no Kähler assumption is used in this construction of these geometric $(a,b)$-modules. This obviously shows that $(a,b)$-modules are basic objects and that they are important not only in the study of local singularities of holomorphic functions but more generally in complex geometry. So it is interesting to have some tools in order to compute them.

This is precisely the aim of the second part of this paper. We prove a finiteness result which gives, for a regular $(a,b)$-module $E$, an integer $N(E)$, bounded by simple numerical invariants of $E$, such that you may cut the asymptotic expansions (in powers of $b$) of elements of $E$ without any lost of information on the structure of the $(a,b)$-module $E$. It is well known that the formal asymptotic expansions for solutions of a regular differential system always converge, and also that such an integer exists for any meromorphic connexion in one variable (see [M.91] proposition 1.12). But it is important to have an effective bound for such an integer easily computable from simple invariants of the $(a,b)$-module structure of $E$.

2 The existence theorem.

2.1 Preliminaries.

Here we shall complete and precise the results of the section 2 of [B.07]. The situation we shall consider is the following : let $X$ be a connected complex manifold of dimension $n + 1$ and $f : X \rightarrow \mathbb{C}$ a non constant holomorphic function such that $\{x \in X/ df = 0\} \subset f^{-1}(0)$. We introduce the following complexes of sheaves supported by $X_0 := f^{-1}(0)$

1. The formal completion ”in $f$” $(\hat{\Omega}^\bullet, d^\bullet)$ of the usual holomorphic de Rham complex of $X$.

2. The sub-complexes $(\hat{K}^\bullet, d^\bullet)$ and $(\hat{I}^\bullet, d^\bullet)$ of $(\hat{\Omega}^\bullet, d^\bullet)$ where the subsheaves $\hat{K}^p$ and $\hat{I}^{p+1}$ are defined for each $p \in \mathbb{N}$ respectively as the kernel and the image of the map

$$\wedge df : \hat{\Omega}^p \rightarrow \hat{\Omega}^{p+1}$$

given par exterior multiplication by $df$. We have the exact sequence

$$0 \rightarrow (\hat{K}^\bullet, d^\bullet) \rightarrow (\hat{\Omega}^\bullet, d^\bullet) \rightarrow (\hat{I}^\bullet, d^\bullet)[+1] \rightarrow 0. \quad (1)$$
Note that \( \hat{K}^0 \) and \( \hat{I}^0 \) are zero by definition.

3. The natural inclusions \( \hat{I}^p \subset \hat{K}^p \) for all \( p \geq 0 \) are compatible with the differential \( d \). This leads to an exact sequence of complexes

\[
0 \to (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to (\hat{K}/\hat{I}^\bullet, d^\bullet) \to 0.
\]

4. We have a natural inclusion \( f^*(\hat{\Omega}^1_C) \subset \hat{K}^1 \cap \text{Ker} \, d \), and this gives a subcomplex (with zero differential) of \( (\hat{K}^\bullet, d^\bullet) \). As in [B.07], we shall consider also the complex \( (\hat{K}^\bullet, d^\bullet) \) quotient. So we have the exact sequence

\[
0 \to f^*(\hat{\Omega}^1_C) \to (\hat{K}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to 0.
\]

We do not make the assumption here that \( f = 0 \) is a reduced equation of \( X_0 \), and we do not assume that \( n \geq 2 \), so the cohomology sheaf in degree 1 of the complex \( (\hat{K}^\bullet, d^\bullet) \), which is equal to \( \hat{K}^1 \cap \text{Ker} \, d \) does not coincide, in general, with \( f^*(\hat{\Omega}^1_C) \). So the complex \( (\hat{K}^\bullet, d^\bullet) \) may have a non zero cohomology sheaf in degree 1.

Recall now that we have on the cohomology sheaves of the following complexes \( (\hat{K}^\bullet, d^\bullet), (\hat{I}^\bullet, d^\bullet), ([\hat{K}/\hat{I}]^\bullet, d^\bullet) \) and \( f^*(\hat{\Omega}^1_C), (\hat{K}^\bullet, d^\bullet) \) natural operations \( a \) and \( b \) with the relation \( a.b - b.a = b^2 \). They are defined in a naïve way by

\[
a := \times f \quad \text{and} \quad b := \wedge df \circ d^{-1}.
\]

The definition of \( a \) makes sens obviously. Let me precise the definition of \( b \) first in the case of \( \mathcal{H}^p(\hat{K}^\bullet, d^\bullet) \) with \( p \geq 2 \) : if \( x \in \hat{K}^p \cap \text{Ker} \, d \) write \( x = d\xi \) with \( \xi \in \hat{\Omega}^{p-1} \) and let \( b[x] := [df \wedge \xi] \). The reader will check easily that this makes sens. For \( p = 1 \) we shall choose \( \xi \in \hat{\Omega}^0 \) such that \( \xi = 0 \) on the smooth part of \( X_0 \) (set theoretically). This is possible because the condition \( df \wedge d\xi = 0 \) allows such a choice : near a smooth point of \( X_0 \) we can choose coordinates such \( f = x_0 \) and the condition on \( \xi \) means independance of \( x_1, \ldots, x_n \). Then \( \xi \) has to be (set theoretically) locally constant on \( X_0 \) which is locally connected. So we may kill the value of such a \( \xi \) along \( X_0 \).

The case of the complex \( (\hat{I}^\bullet, d^\bullet) \) will be reduced to the previous one using the next lemma.

**Lemme 2.1.1** For each \( p \geq 0 \) there is a natural injective map

\[
\bar{b} : \mathcal{H}^p(\hat{K}^\bullet, d^\bullet) \to \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)
\]

which satisfies the relation \( a.\bar{b} = \bar{b}.(a + b) \). For \( p \neq 1 \) this map is bijective.
Proof. Let \( x \in \hat{K}^p \cap \text{Ker} \, d \) and write \( x = d\xi \) where \( \xi \in \hat{\Omega}^{p-1} \) (with \( \xi = 0 \) on \( X_0 \) if \( p = 1 \)), and set \( \tilde{b}(x) := [df \wedge \xi] \in \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \). This is independent on the choice of \( \xi \) because, for \( p \geq 2 \), adding \( d\eta \) to \( \xi \) does not modify the result as \([df \wedge d\eta] = 0\). For \( p = 1 \) remark that our choice of \( \xi \) is unique.

This is also independent of the the choice of \( x \) in \([x] \in \mathcal{H}^p(\hat{K}^\bullet, d^\bullet) \) because adding \( \theta \in \hat{K}^{p-1} \) to \( \xi \) does not change \( df \wedge \xi \).

Assume \( \tilde{b}(x) = 0 \) in \( \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \); this means that we may find \( \alpha \in \hat{\Omega}^{p-2} \) such that \( df \wedge \xi = df \wedge d\alpha \). But then, \( \xi - d\alpha \) lies in \( \hat{K}^{p-1} \) and \( x = d(\xi - d\alpha) \) shows that \([x] = 0\). So \( \tilde{b} \) is injective.

Assume now \( p \geq 2 \). If \( df \wedge \eta \) is in \( \hat{I}^p \cap \text{Ker} \, d \), then \( df \wedge d\eta = 0 \) and \( y := d\eta \) lies in \( \hat{K}^p \cap \text{Ker} \, d \) and defines a class \([y] \in \mathcal{H}^p(\hat{K}^\bullet, d^\bullet) \) whose image by \( \tilde{b} \) is \([df \wedge \eta]\). This shows the surjectivity of \( \tilde{b} \) for \( p \geq 2 \).

For \( p = 1 \) the map \( \tilde{b} \) is not surjective (see the remark below).

To finish the proof let us to compute \( \tilde{b}(a[x] + b[x]) \). Writing again \( x = d\xi \), we get
\[
a[x] + b[x] = [f.d\xi + df \wedge \xi] = [d(f.\xi)]
\]
and so
\[
\tilde{b}(a[x] + b[x]) = [df \wedge f.\xi] = a.\tilde{b}([x])
\]
which concludes the proof. \( \blacksquare \)

Denote by \( i : (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \) the natural inclusion and define the action of \( b \) on \( \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \) by \( b := \tilde{b} \circ \mathcal{H}^p(i) \). As \( i \) is \( a \)-linear, we deduce the relation \( a.b - b.a = b^2 \) on \( \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \) from the relation of the previous lemma.

The action of \( a \) on the complex \(([\hat{K}/\hat{I}]^\bullet, d^\bullet) \) is obvious and the action of \( b \) is zero.

The action of \( a \) and \( b \) on \( f^\ast(\hat{\Omega}^1_\mathbb{C}) \simeq E_1 \otimes \mathbb{C}_{X_0} \) are the obvious one, where \( E_1 \) is the rank 1 \((a,b)\)-module with generator \( e_1 \) satisfying \( a.e_1 = b.e_1 \) (or, if you prefer, \( E_1 := \mathbb{C}[[z]] \) with \( a := z \), \( b := \int_0^z \) and \( e_1 := 1 \)).

Remark that the natural inclusion \( f^\ast(\hat{\Omega}^1_\mathbb{C}) \hookrightarrow (\hat{K}^\bullet, d^\bullet) \) is compatible with the actions of \( a \) and \( b \). The actions of \( a \) and \( b \) on \( \mathcal{H}^1(\hat{K}^\bullet, d^\bullet) \) are simply induced by the corresponding actions on \( \mathcal{H}^1(\hat{K}^\bullet, d^\bullet) \).

Remark. The exact sequence of complexes (1) induces for any \( p \geq 2 \) a bijection
\[
\partial^p : \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \to \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)
\]
and a short exact sequence
\[
0 \to \mathbb{C}_{X_0} \to \mathcal{H}^1(\hat{I}^\bullet, d^\bullet) \xrightarrow{\partial^1} \mathcal{H}^1(\hat{K}^\bullet, d^\bullet) \to 0 \tag{\@}
\]
because of the de Rham lemma. Let us check that for \( p \geq 2 \) we have \( \partial^p = (\tilde{b})^{-1} \) and that for \( p = 1 \) we have \( \partial^1 \circ \tilde{b} = Id \). If \( x = d\xi \in \hat{K}^p \cap \text{Ker} \, d \) then \( \tilde{b}(x) = [df \wedge \xi] \) and \( \partial^p [df \wedge \xi] = [d\xi] \). So \( \partial^p \circ \tilde{b} = Id \) \( \forall p \geq 0 \). For \( p \geq 2 \) and
$df \wedge \alpha \in \hat{I}^p \cap \text{Ker } d$ we have $\partial^p[df \wedge \alpha] = [d\alpha]$ and $\tilde{b}[d\alpha] = [df \wedge \alpha]$, so $\tilde{b} \circ \partial^p = Id$. For $p = 1$ we have $\tilde{b}[d\alpha] = [df \wedge (\alpha - \alpha_0)]$ where $\alpha_0 \in \mathbb{C}$ is such that $\alpha|_{X_0} = \alpha_0$. This shows that in degree 1 $\tilde{b}$ gives a canonical splitting of the exact sequence (\@).

2.2 $\tilde{A}$–structures.

Let us consider now the $\mathbb{C}$–algebra

$$\tilde{A} := \{ \sum_{\nu \geq 0} P_{\nu}(a)b^\nu \}$$

where $P_\nu \in \mathbb{C}[z]$, and the commutation relation $a.b - b.a = b^2$, assuming that left and right multiplications by $a$ are continuous for the $b$–adic topology of $\tilde{A}$.

Define the following complexes of sheaves of left $\tilde{A}$–modules on $X$:

$$(\Omega'^\bullet[[b]], D^\bullet) \text{ and } (\Omega''\bullet[[b]], D^\bullet)$$

where

$$\Omega'^\bullet[[b]] := \sum_{j=0}^{+\infty} b^j.\omega_j \text{ with } \omega_0 \in \hat{K}^p$$

$$\Omega''\bullet[[b]] := \sum_{j=0}^{+\infty} b^j.\omega_j \text{ with } \omega_0 \in \hat{I}^p$$

$$D(\sum_{j=0}^{+\infty} b^j.\omega_j) = \sum_{j=0}^{+\infty} b^j.(d\omega_j - df \wedge \omega_{j+1})$$

$$a. \sum_{j=0}^{+\infty} b^j.\omega_j = \sum_{j=0}^{+\infty} b^j.(f.\omega_j + (j - 1).\omega_{j-1}) \text{ with the convention } \omega_{-1} = 0$$

$$b. \sum_{j=0}^{+\infty} b^j.\omega_j = \sum_{j=1}^{+\infty} b^j.\omega_{j-1}$$

It is easy to check that $D$ is $\tilde{A}$–linear and that $D^2 = 0$. We have a natural inclusion of complexes of left $\tilde{A}$–modules

$$\tilde{i} : (\Omega''\bullet[[b]], D^\bullet) \to (\Omega'^\bullet[[b]], D^\bullet).$$

Remark that we have natural morphisms of complexes

$$u : (\hat{I}^\bullet, d^\bullet) \to (\Omega''\bullet[[b]], D^\bullet)$$

$$v : (\hat{K}^\bullet, d^\bullet) \to (\Omega'^\bullet[[b]], D^\bullet)$$

and that these morphisms are compatible with $i$. More precisely, this means that we have the commutative diagram of complexes

$$\begin{array}{ccc}
(\hat{I}^\bullet, d^\bullet) & \xrightarrow{u} & (\Omega''\bullet[[b]], D^\bullet) \\
\downarrow i & & \downarrow \tilde{i} \\
(\hat{K}^\bullet, d^\bullet) & \xrightarrow{v} & (\Omega'^\bullet[[b]], D^\bullet)
\end{array}$$
The following theorem is a variant of theorem 2.2.1. of [B. 07].

**Théorème 2.2.1** Let $X$ be a connected complex manifold of dimension $n+1$ and $f : X \to \mathbb{C}$ a non constant holomorphic function with the following condition:

$$\{ x \in X / df = 0 \} \subset f^{-1}(0).$$

Then the morphisms of complexes $u$ and $v$ introduced above are quasi-isomorphisms. Moreover, the isomorphisms that they induce on the cohomology sheaves of these complexes are compatible with the actions of $a$ and $b$.

This theorem builds a natural structure of left $\tilde{A}$–modules on each of the complex $(\tilde{K}^\bullet, d^\bullet), (\tilde{I}^\bullet, d^\bullet), ([\tilde{K}/\tilde{I}]^\bullet, d^\bullet)$ and $f^*(\tilde{\Omega}_C^1), (\tilde{K}^\bullet, d^\bullet)$ in the derived category of bounded complexes of sheaves of $\mathbb{C}$–vector spaces on $X$.

Moreover the short exact sequences

$$0 \to (\tilde{I}^\bullet, d^\bullet) \to (\tilde{K}^\bullet, d^\bullet) \to ([\tilde{K}/\tilde{I}]^\bullet, d^\bullet) \to 0 \quad (2)$$

and

$$0 \to f^*(\tilde{\Omega}_C^1) \to (\tilde{K}^\bullet, d^\bullet), (\tilde{I}^\bullet, d^\bullet) \to (\tilde{K}^\bullet, d^\bullet) \to 0 \quad (3)$$

are equivalent to short exact sequences of complexes of left $\tilde{A}$–modules in the derived category.

**Proof.** We have to prove that for any $p \geq 0$ the maps $\mathcal{H}^p(u)$ and $\mathcal{H}^p(v)$ are bijective and compatible with the actions of $a$ and $b$. The case of $\mathcal{H}^p(v)$ is handled (at least for $n \geq 2$ and $f$ reduced) in prop. 2.3.1. of [B.07]. To seek completeness and for the convenience of the reader we shall treat here the case of $\mathcal{H}^p(u)$.

First we shall prove the injectivity of $\mathcal{H}^p(u)$. Let $\alpha = df \wedge \beta \in \hat{I}^p \cap \text{Ker} d$ and assume that we can find $U = \sum_{j=0}^{+\infty} b^j.u_j \in \Omega^{0p-1}[b]$ with $\alpha = DU$. Then we have the following relations

$$u_0 = df \wedge \zeta, \quad \alpha = du_0 - df \wedge u_1 \quad \text{and} \quad du_j = df \wedge u_{j+1} \quad \forall j \geq 1.$$

For $j \geq 1$ we have $[du_j] = b[du_{j+1}]$ in $\mathcal{H}^p(\tilde{K}^\bullet, d^\bullet)$; using corollary 2.2. of [B.07] which gives the $b$–separation of $\mathcal{H}^p(\tilde{K}^\bullet, d^\bullet)$, this implies $[du_j] = 0, \forall j \geq 1$ in $\mathcal{H}^p(\tilde{K}^\bullet, d^\bullet)$. For instance we can find $\beta_1 \in \hat{K}^{p-1}$ such that $du_1 = d\beta_1$. Now, by de Rham, we can write $u_1 = \beta_1 + d\xi_1$ for $p \geq 2$, where $\xi_1 \in \hat{\Omega}^{p-2}$. Then we conclude that $\alpha = -df \wedge d(\xi_1 + \zeta)$ and $[\alpha] = 0$ in $\mathcal{H}^p(\tilde{I}^\bullet, d^\bullet)$.

For $p = 1$ we have $u_0 = 0, du_1 = 0$ so $[\alpha] = [-df \wedge d\xi_1] = 0$ in $\mathcal{H}^1(\tilde{I}^\bullet, d^\bullet)$.

We shall now show that the image of $\mathcal{H}^p(u)$ is dense in $\mathcal{H}^p(\Omega^\bullet'''[b], D^\bullet)$ for its $b$–adic topology. Let $\Omega := \sum_{j=0}^{+\infty} b^j.\omega_j \in \Omega^{0p}[b]$ such that $D\Omega = 0$. The following relations holds $d\omega_j = df \wedge \omega_{j+1}$, $\forall j \geq 0$ and $\omega_0 \in \hat{I}^1$. The corollary 2.2. of [B.07] again allows to find $\beta_j \in \hat{K}^{p-1}$ for any $j \geq 0$ such that $d\omega_j = d\beta_j$. Fix $N \in \mathbb{N}^\star$.

We have

$$D(\sum_{j=0}^{N} b^j.\omega_j) = b^N.d\omega_N = D(b^N.\beta_N)$$
and $\Omega_N := \sum_{j=0}^N b^j.\omega_j - b^N.\beta_N$ is $D$-closed and in $\Omega^{op}[[b]]$. As $\Omega - \Omega_N$ lies in $b^N.\mathcal{H}^p(\Omega^{**}[[b]], D^\bullet)$, the sequence $(\Omega_N)_{N \geq 1}$ converges to $\Omega$ in $\mathcal{H}^p(\Omega^{**}[[b]], D^\bullet)$ for its $b$–adic topology. Let us show that each $\Omega_N$ is in the image of $\mathcal{H}^p(u)$.

Write $\Omega_N := \sum_{j=0}^N b^j.\omega_j$. The condition $D\Omega_N = 0$ implies $dw_N = 0$ and $dw_{N-1} = df \wedge w_N = 0$. If we write $w_N = dv_N$ we obtain $d(w_{N-1} + df \wedge v_N) = 0$ and $\Omega_N - D(b^N.v_N)$ is of degree $N - 1$ in $b$. For $N = 1$ we are left with $w_0 + b.w_1 - (df \wedge v_1 + b.dv_1) = w_0 + df \wedge v_1$ which is in $\hat{I}^p \cap Ker d$ because $dw_0 = df \wedge dv_1$.

To conclude it is enough to know the following two facts

i) The fact that $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ is complete for its $b$–adic topology.

ii) The fact that $Im(\mathcal{H}^p(u)) \cap b^N.\mathcal{H}^p(\Omega^{**}[[b]], D^\bullet) \subset Im(\mathcal{H}^p(u) \circ b^N)$ $\forall N \geq 1$.

Let us first conclude the proof of the surjectivity of $\mathcal{H}^p(u)$ assuming i) and ii).

For any $[\Omega] \in \mathcal{H}^p(\Omega^{**}[[b]], D^\bullet)$ we know that there exists a sequence $(\alpha_N)_{N \geq 1}$ in $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ with $\Omega - \mathcal{H}^p(u)(\alpha_N) \in b^N.\mathcal{H}^p(\Omega^{**}[[b]], D^\bullet)$. Now the property ii) implies that we may choose the sequence $(\alpha_N)_{N \geq 1}$ such that $[\alpha_{N+1}] - [\alpha_N]$ lies in $b^N.\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. So the property i) implies that the Cauchy sequence $([\alpha_N])_{N \geq 1}$ converges to $[\alpha] \in \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. Then the continuity of $\mathcal{H}^p(u)$ for the $b$–adic topologies coming from its $b$–linearity, implies $\mathcal{H}^p(u)([\alpha]) = [\Omega]$.

The compatibility with $a$ and $b$ of the maps $\mathcal{H}^p(u)$ and $\mathcal{H}^p(v)$ is an easy exercise.

Let us now prove properties i) and ii).

The property i) is a direct consequence of the completion of $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ for its $b$–adic topology given by the corollary 2.2. of [B.07] and the $b$–linear isomorphism $\tilde{b}$ between $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ and $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ constructed in the lemma 2.1.1. above.

To prove ii) let $\alpha \in \hat{I}^p \cap Ker d$ and $N \geq 1$ such that

$$\alpha = b^N.\Omega + DU$$

where $\Omega \in \Omega^{op}[[b]]$ satisfies $D\Omega = 0$ and where $U \in \Omega^{op-1}[[b]]$. With obvious notations we have

$$\alpha = du_0 - df \wedge u_1$$

$$\ldots$$

$$0 = du_j - df \wedge u_{j+1} \quad \forall j \in [1, N - 1]$$

$$\ldots$$

$$0 = \omega_0 + du_N - df \wedge u_{N+1}$$

which implies $D(u_0 + b.u_1 + \cdots + b^N.u_N) = \alpha + b^N.du_N$ and the fact that $du_N$ lies in $\hat{I}^p \cap Ker d$. So we conclude that $[\alpha] + b^N.[du_N]$ is in the kernel of $\mathcal{H}^p(u)$ which is 0. Then $[\alpha] \in b^N.\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. $\blacksquare$
Remark. The map
\[ \beta : (\Omega[[b]]^\bullet, D^\bullet) \to (\Omega''[[b]]^\bullet, D^\bullet) \]
defined by \( \beta(\Omega) = b\Omega \) commutes to the differentials and with the action of \( b \). It induces the isomorphism \( \tilde{b} \) of the lemma 2.1.1 on the cohomology sheaves. So it is a quasi-isomorphism of complexes of \( \mathbb{C}[[b]] \)-modules.

To prove this fact, it is enough to verify that the diagram
\[ \xymatrix{ (\hat{K}^\bullet, d^\bullet) \ar[r]^v \ar[d]^{\hat{b}} & (\Omega'[[b]]^\bullet, D^\bullet) \ar[d]^\beta \\
(\hat{I}^\bullet, d^\bullet) \ar[r]^u & (\Omega''[[b]]^\bullet, D^\bullet) } \]
induces commutative diagrams on the cohomology sheaves.

But this is clear because if \( \alpha = d\xi \) lies in \( \hat{K}^p \cap \text{Ker } d \) we have \( D(b.\xi) = b.d\xi - df \wedge \xi \) so \( H^p(\beta \circ H^p(v)([\alpha])) = H^p(u) \circ H^p(\tilde{b})([\alpha]) \) in \( H^p(\Omega''[[b]]^\bullet, D^\bullet) \).

2.3 The existence theorem.

Let us recall some basic definitions on the left modules over the algebra \( \tilde{A} \).

Définition 2.3.1 An \((a,b)\)-module is a left \( \tilde{A} \)-module which is free and of finite rank on the commutative sub-algebra \( \mathbb{C}[[b]] \) of \( \tilde{A} \).

An \((a,b)\)-module \( E \) is

1. local when \( \exists N \in \mathbb{N} \) such that \( a^N.E \subset b.E \);
2. simple pole when \( a.E \subset b.E \);
3. regular when it is contained in a simple pole \((a,b)\)-module;
4. geometric when it is contained in a simple pole \((a,b)\)-module \( E^\sharp \) such that the minimal polynomial of the action of \( b^{-1}.a \) on \( E^\sharp/b.E^\sharp \) has its roots in \( \mathbb{Q}^+ \).

We shall give more details and examples of \((a,b)\)-modules in the section 3.

Now let \( E \) be any left \( \tilde{A} \)-module, and define \( B(E) \) as the \( b \)-torsion of \( E \), that is to say
\[ B(E) := \{ x \in E / \exists N \quad b^N.x = 0 \}. \]

Define \( A(E) \) as the \( a \)-torsion of \( E \) and
\[ \hat{A}(E) := \{ x \in E / \mathbb{C}[[b]].x \subset A(E) \}. \]

Remark that \( B(E) \) and \( \hat{A}(E) \) are sub-\( \tilde{A} \)-modules of \( E \) but that \( A(E) \) is not stable by \( b \).

Définition 2.3.2 A left \( \tilde{A} \)-module \( E \) is small when the following conditions hold

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1. $E$ is a finite type $\mathbb{C}[[b]]$-module;

2. $B(E) \subset \hat{A}(E)$;

3. $\exists N / a^N \hat{A}(E) = 0$;

Recall that for $E$ small we have always the equality $B(E) = \hat{A}(E)$ and that this complex vector space is finite dimensional. The quotient $E/B(E)$ is an (a,b)-module called the associate (a,b)-module to $E$.

Conversely, any left $\hat{A}$-module $E$ such that $B(E)$ is a finite dimensional $\mathbb{C}$-vector space and such that $E/B(E)$ is an (a,b)-module is small.

The following easy criterium to be small will be used later:

**Lemme 2.3.3** A left $\hat{A}$-module $E$ is small if and only if the following conditions hold:

1. $\exists N / a^N \hat{A}(E) = 0$;

2. $B(E) \subset \hat{A}(E)$;

3. $\cap_{m \geq 0} b^m \cdot E \subset \hat{A}(E)$;

4. $\text{Ker } b$ and $\text{Coker } b$ are finite dimensional complex vector spaces.

As the condition 3) in the previous lemma has been omitted in [B.07] (but this does not affect the results of this article because this lemma was used only in a case where this condition 3) was satisfied, thanks to proposition 2.2.1. of loc. cit.), we shall give the (easy) proof.

**Proof.** First the conditions 1) to 4) are obviously necessary. Conversely, assume that $E$ satisfies these four conditions. Then condition 2) implies that the action of $b$ on $\hat{A}(E)/B(E)$ is injective. But the condition 1) implies that $b^{2N} = 0$ on $\hat{A}(E)$ (see [B.06]). So we conclude that $\hat{A}(E) = B(E) \subset \text{Ker } b^{2N}$ which is a finite dimensional complex vector space using condition 4) and an easy induction.

Now $E/B(E)$ is a $\mathbb{C}[[b]]$-module which is separated for its $b$-adic topology.

The finiteness of $\text{Coker } b$ now shows that it is a free finite type $\mathbb{C}[[b]]$-module concluding the proof.

**Définition 2.3.4** We shall say that a left $\hat{A}$-module $E$ is geometric when $E$ is small and when it associated (a,b)-module $E/B(E)$ is geometric.

The main result of this section is the following theorem, which shows that the Gauss-Manin connexion of a proper holomorphic function produces geometric $\hat{A}$-modules associated to vanishing cycles and nearby cycles.
Theorem 2.3.5 Let \( X \) be a connected complex manifold of dimension \( n + 1 \) where \( n \in \mathbb{N} \), and let \( f : X \to D \) be an non constant proper holomorphic function to an open disc \( D \) in \( \mathbb{C} \) with center \( 0 \). Let us assume that \( df \) is nowhere vanishing outside of \( X_0 := f^{-1}(0) \).

Then the \( \mathbb{A} \)-modules

\[
H^j(X, (\hat{K}^\bullet, d^\bullet)) \quad \text{and} \quad H^j(X, (\hat{I}^\bullet, d^\bullet))
\]

are geometric for any \( j \geq 0 \).

In the proof we shall use the \( C^\infty \) version of the complex \( (\hat{K}^\bullet, d^\bullet) \). We define \( K^p_\infty \) as the kernel of \( \wedge df : C^\infty(D^* \to C^\infty(D^*) \) where \( C^\infty \) denote the sheaf of \( C^\infty - \) forms on \( X \) of degree \( j \), let \( \hat{K}^p_\infty \) be its formal \( f \)-completion and \( (\hat{K}^\bullet_\infty, d^\bullet) \) the corresponding de Rham complex.

The next lemma is proved in [B.07] (lemma 6.1.1.)

Lemma 2.3.6 The natural inclusion

\[
(\hat{K}^\bullet, d^\bullet) \hookrightarrow (\hat{K}^\bullet_\infty, d^\bullet)
\]

is a quasi-isomorphism.

Remark. As the sheaves \( \hat{K}^\bullet_\infty \) are fine, so we have a natural isomorphism

\[
H^p(X, (\hat{K}^\bullet_\infty, d^\bullet)) \simeq H^p(\Gamma(X, \hat{K}^\bullet_\infty), d^\bullet).
\]

Let us denote by \( X_1 \) the generic fiber of \( f \). Then \( X_1 \) is a smooth compact complex manifold of dimension \( n \) and the restriction of \( f \) to \( f^{-1}(D^*) \) is a locally trivial \( C^\infty \) bundle with typical fiber \( X_1 \) on \( D^* = D \setminus \{0\} \), if the disc \( D \) is small enough around \( 0 \). Fix now \( \gamma \in H_p(X_1, \mathbb{C}) \) and let \( (\gamma_s)_{s \in D^*} \) the corresponding multivalued horizontal family of \( p \)-cycles \( \gamma_s \in H_p(X_s, \mathbb{C}) \). Then for \( \omega \in \Gamma(X, \hat{K}^p_\infty \cap \text{Ker } d) \) define the multivalued holomorphic function

\[
F_\omega(s) := \int_{\gamma_s} \frac{\omega}{df}.
\]

Let now

\[
\Xi := \sum_{\alpha \in \mathbb{Q} \cap [1, 0], j \in [0, n]} \mathbb{C}[[s]] s^\alpha \frac{(\text{Logs})^j}{j!}.
\]

This is an \( \mathbb{A} \)-modules with \( a \) acting as multiplication by \( s \) and \( b \) as the primitive in \( s \) without constant. Now if \( F_\omega \) is the asymptotic expansion at \( 0 \) of \( F_\omega \), it is an element in \( \Xi \), and we obtain in this way an \( \mathbb{A} \)-linear map

\[
\text{Int} : H^p(X, (\hat{K}^\bullet, d^\bullet)) \to H^p(X_1, \mathbb{C}) \otimes_{\mathbb{C}} \Xi.
\]

To simplify notations, let \( E := H^p(X, (\hat{K}^\bullet, d^\bullet)) \). Now using Grothendieck theorem [G.66], there exists \( N \in \mathbb{N} \) such that \( \text{Int}(\omega) \equiv 0 \), implies \( a^N[\omega] = 0 \) in \( E \).
As the converse is clear we conclude that \( \hat{A}(E) = \ker(\text{Int}) \). It is also clear that \( B(E) \subset \ker(\text{Int}) \) because \( \Xi \) has no \( b \)--torsion. So we conclude that \( E \) satisfies properties 1) and 2) of the lemma \[2.3.3\].

The property 3) is also true because of the regularity of the Gauss-Manin connexion of \( f \).

**End of the proof of theorem \[2.3.5\].** To show that \( E := H^p(X, (\hat{K}^\bullet, d^\bullet)) \) is small, it is enough to prove that \( E \) satisfies the condition 4) of the lemma \[2.3.3\].

Consider now the long exact sequence of hypercohomology of the exact sequence of complexes

\[
0 \to (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to ([\hat{K}/\hat{I}]^\bullet, d^\bullet) \to 0.
\]

It contains the exact sequence

\[
\mathbb{H}^{p-1}(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet)) \to \mathbb{H}^p(X, (\hat{I}^\bullet, d^\bullet)) \to \mathbb{H}^p(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet)) \]

and we know that \( b \) is induced on the complex of \( \tilde{A} \)--modules quasi-isomorphic to \( (\hat{K}^\bullet, d^\bullet) \) by the composition \( i \circ \tilde{b} \) where \( \tilde{b} \) is a quasi-isomorphism of complexes of \( \mathbb{C}[[b]] \)--modules. This implies that the kernel and the cokernel of \( \mathbb{H}^p(i) \) are isomorphic (as \( \mathbb{C} \)--vector spaces) to \( \ker b \) and \( \text{coker} \ b \) respectively. Now to prove that \( E \) satisfies condition 4) of the lemma \[2.3.3\] it is enough to prove finite dimensionality for the vector spaces \( \mathbb{H}^j(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet)) \) for all \( j \geq 0 \).

But the sheaves \( [\hat{K}/\hat{I}]^j \simeq [\ker df/\text{Im} \; df]^j \) are coherent on \( X \) and supported in \( X_0 \). The spectral sequence

\[
E_2^{p,q} := H^q(H^p(X, [\hat{K}/\hat{I}]^\bullet), d^\bullet)
\]

which converges to \( \mathbb{H}^j(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet)) \), is a bounded complex of finite dimensional vector spaces by Cartan-Serre. This gives the desired finite dimensionality.

To conclude the proof, we want to show that \( E/B(E) \) is geometric. But this is an easy consequence of the regularity of the Gauss-Manin connexion of \( f \) and of the Monodromy theorem, which are already incoded in the definition of \( \Xi \) : the injectivity on \( E/B(E) \) of the \( \tilde{A} \)--linear map \( \text{Int} \) implies that \( E/B(E) \) is geometric. Remark now that the piece of exact sequence above gives also the fact that \( \mathbb{H}^p(X, (\hat{I}^\bullet, d^\bullet)) \) is geometric, because it is an exact sequence of \( \tilde{A} \)--modules.

**3 Basic properties.**

**3.1 Definition and examples.**

First recall in a more naïve way the definition of an \( (a,b) \)--module.

**Définition 3.1.1** An \( (a,b) \)--module \( E \) is a free finite type \( \mathbb{C}[[b]] \)--module with a \( \mathbb{C} \)--linear endomorphism \( a : E \to E \) which is continuous for the \( b \)--adic topology of \( E \) and satisfies \( ab - ba = b^2 \).

The rank of \( E \), denote by \( \text{rank}(E) \), will be the rank of \( E \) as a \( \mathbb{C}[[b]] \)--module.
Remarks.

1. Let \((e_1, \ldots, e_k)\) a \(\mathbb{C}[[b]]\)-basis of a free finite type \(\mathbb{C}[[b]]\)-module. Then choosing arbitrarily elements \((\varepsilon_1, \ldots, \varepsilon_k)\) and defining \(a.e_j = \varepsilon_j \forall j \in [1,k]\) gives an \((a,b)\)-module: the commutation relation implies that \(a.b^n = b^n.a + n.b^{n+1}\) so \(a\) is defined on \(\sum_{j=1}^k \mathbb{C}[b].e_j\). The continuity assumption gives its (unique) extension.

2. There is a natural \((a,b)\)-module associated to every algebraic linear differential system (see [B.95] p.42)

\[ Q(z). \frac{dF}{dz} = M(z).F(z), \quad Q \in \mathbb{C}[z], \quad M \in \text{End}(\mathbb{C}^n) \otimes \mathbb{C} \mathbb{C}[z]. \]

In the sequel of this article we shall mainly consider regular \((a,b)\)-modules (see definition recalled below). To try to convince the reader that the "general" \((a,b)\)-module structure is interesting, let me quote the following result, which is quite elementary in the regular case, but which is not so easy in general.

**Théorème 3.1.2** ([B.95] th.1bis p.31) Let \(E\) be an \((a,b)\)-module. Then the kernel and cokernel of "a" are finite dimensional.

This result implies a general finiteness theorem for extensions of \((a,b)\)-modules (see [B.95] and also section 1.3).

**Définition 3.1.3** We shall say that an \((a,b)\)-module \(E\) has a simple pole when the inclusion \(a.E \subset b.E\) is satisfied.

This terminology comes from the terminology of meromorphic connexions (see for instance [D.70]).

**Example.** For any \(\lambda \in \mathbb{C}\) define the simple pole rank 1 \((a,b)\)-module \(E_\lambda\) as \(E := \mathbb{C}[[b]].e_\lambda\) where "a" is defined by the relation \(a.e_\lambda = \lambda.b.e_\lambda\). \(\Box\)

As an introduction to our second theorem, the reader may solve the following exercice by direct computation.

**Exercice.** For any \(S \in \mathbb{C}[[b]]\) show that the simple pole \((a,b)\)-module defined by \(E := \mathbb{C}[[b]].e_S\) and \(a.e_S = b.S(b).e_S\) is isomorphic to \(E_\lambda\) with \(\lambda = S(0)\) (hint: begin by looking for \(\alpha_1 \in \mathbb{C}\) such that \((a - S(0).b)(e + \alpha_1.b.e) \in b^3.E\). \(\Box\)

For a simple pole \((a,b)\)-module, the linear map \(b^{-1}.a : E \to E\) is well defined and induces an endomorphism \(f := b^{-1}.a : E/b.E \to E/b.E\). For any \(\lambda \in \mathbb{C}\) we shall denote by \(\lambda_{\text{min}}\) the smallest eigenvalue of \(f\) which is in \(\lambda + \mathbb{Z}\). Then for
\[ \lambda = \lambda_{\text{min}} - k \text{ with } k \in \mathbb{N}^* \text{ the bijectivity of the map } f - \lambda \text{ on } E/b.E \text{ implies easily its bijectivity on } E \text{ (see the exercise above). It gives then the equality} \]

\[ (a - \lambda.b).E = b.E. \]

Using this remark, it is not difficult to prove the following result from [B.93] (prop.1.3. p.11) that we shall use later on.

**Proposition 3.1.4** Let \( E \) be a simple pole \((a,b)\)-module, and let \( \lambda \in \mathbb{C} \) and \( \kappa \in \mathbb{N} \) such that \( \lambda - \kappa \leq \lambda_{\text{min}} \). If \( y \in E \) satisfies \( (a - \lambda.b).y \in b^{\kappa+2}.E \) then there exists an unique \( \tilde{y} \in E \) such that \( (a - \lambda.b).\tilde{y} = 0 \) and \( \tilde{y} - y \in b^{\kappa+1}.E \).

An easy consequence of this proposition is that for an eigenvalue \( \lambda \) of \( f \) such that \( \lambda = \lambda_{\text{min}} \) there always exists a non zero \( x \in E \) such that \( (a - \lambda.b).x = 0 \). This gives an embedding of \( E_\lambda \) in \( E \). Remark also that if \( E \) is a non zero simple pole \((a,b)\)-module, such a \( \lambda \) always exists. This leads to a rather precise description a of "general" simple pole \((a,b)\)-module (see [B.93] th. 1.1 p.15).

**Définition 3.1.5** An \((a,b)\)-module \( E \) is regular when its saturation by \( b^{-1}.a \) in \( E[b^{-1}] \) is finitely generated on \( \mathbb{C}[[b]] \).

We shall denote \( E^\sharp \) this saturation. It is a simple pole \((a,b)\)-module and it is the smallest simple pole \((a,b)\)-module containing \( E \) in the sense that for any \((a,b)\)-linear morphism \( j : E \rightarrow F \) where \( F \) is a simple pole \((a,b)\)-module, there exists a unique \((a,b)\)-linear extension \( j^\sharp : E^\sharp \rightarrow F \) of \( j \).

It is easy to show that a regular \((a,b)\)-module of rank 1 is isomorphic to some \( E_\lambda \) for some \( \lambda \in \mathbb{C} \). The classification of rank 2 regular \((a,b)\)-module is not so obvious. We recall it here for a later use.

**Proposition 3.1.6** (see [B.93] prop.2.4 p. 34) The list of rank 2 regular \((a,b)\)-modules is, up to isomorphism, the following :

1. \( E_\lambda \oplus E_\mu \) for \( (\lambda, \mu) \in \mathbb{C}^2/\mathfrak{S}_2 \).

2. For any \( \lambda \in \mathbb{C} \) and any \( n \in \mathbb{N} \) let \( E_\lambda(n) \) be the simple pole \((a,b)\)-module with basis \((x, y)\) such that

\[ a.x = (\lambda + n).b.x + b^{n+1}.y \text{ and } a.y = \lambda.b.y. \]

3. For any \( (\lambda, \mu) \in \mathbb{C}^2/\mathfrak{S}_2 \) let \( E_{\lambda,\mu} \) the rank 2 regular \((a,b)\)-module with basis \((y, t)\) such that

\[ a.y = \mu.b.y \text{ and } a.t = y + (\lambda - 1).b.t. \]
4. For any $\lambda \in \mathbb{C}$, any $n \in \mathbb{N}^*$ and any $\alpha \in \mathbb{C}^*$ let $E_{\lambda, \lambda-n}(\alpha)$ be the rank 2 regular $(a,b)$-module with basis $(y,t)$ such that

$$a.y = (\lambda - n).b.y \quad \text{and} \quad a.t = y + (\lambda - 1)b.t + \alpha.b^n.y$$

Note that the first two cases are simple pole $(a,b)$-modules.

The saturation by $b^{-1}.a$ in case 3. is generated by $b^{-1}.y$ and $t$ as a $\mathbb{C}[[b]]$-module.

To conclude this first section, let me recall also the theorem of existence of Jordan-Hölder sequences for regular $(a,b)$-module, which will be useful in the induction in the proof of our result.

**Théorème 3.1.7** (see [B.93] th. 2.1 p.30) For any regular rank $k$ $(a,b)$-module $E$ there exists a sequence of sub-$(a,b)$-modules

$$0 = E^0 \subset E^1 \subset \cdots \subset E^{k-1} \subset E^k = E$$

such that for any $j \in [1,k]$ the quotient $E^j/E^{j-1}$ is isomorphic to $E_{\lambda_j}$. Moreover we may choose for $E^1$ any normal\footnote{normal means $E^1 \cap b.E = b.E^1$, so that $E/E^1$ is again free on $\mathbb{C}[[b]]$.} rank 1 sub-$(a,b)$-module of $E$.

The number $\alpha(E) := \sum_{j=1}^{k} \lambda_j$ is independant of the choice of the Jordan-Hölder sequence. It is given by the following formula

$$\alpha(E) = \text{trace}(b^{-1}.a : E^s/b.E^s \rightarrow E^s/b.E^s) + \dim_{\mathbb{C}}(E^s/E).$$

### 3.2 The regularity order.

**Définition 3.2.1** Let $E$ be a regular $(a,b)$-module. We define the **regularity order** of $E$ as the smallest integer $k \in \mathbb{N}$ such that the inclusion

$$a^{k+1}.E \subset \sum_{j=0}^{k} a^j.b^{k-j+1}.E$$

(reg.)

is valid. We shall note this integer $\text{or}(E)$.

We define also the **index** $\delta(E)$ of $E$ as the smallest integer $m \in \mathbb{N}$ such that $E^s \subset b^{-m}.E$.

**Remarks.**

i) The $(a,b)$-module $E$ has a simple pole if and only iff $\text{or}(E) = 0$. 


ii) The inclusion (reg.) implies that \((b^{-1}.a)^{k+1}.E \subset \Phi_k(E) := \sum_{j=0}^{k} (b^{-1}.a)^j.E\) and this implies that \(\Phi_k(E)\) is stable by \(b^{-1}.a\). So \(\Phi_k(E)\) is a simple pole \((a,b)\)-module contained in \(b^{-k}.E \subset E[b^{-1}]\). This implies clearly the regularity of \(E\).

For \(k = or(E)\) we have \(E^\sharp = \Phi_k(E) \subset b^{-k}.E\). So we have \(\delta(E) \leq or(E)\).

iii) As the quotient \(b^{-k}.E/E\) is a finite dimensional \(\mathbb{C}\)-vector space, the quotient \(E^\sharp/E\) is always a finite dimensional \(\mathbb{C}\)-vector space.

The remark iii) shows that for a regular \((a,b)\)-module \(E\) there always exists a simple pole sub-(\(a,b\))-module of \(E\) which is a finite codimensional vector space in \(E\). This comes from the fact that for \(k = \delta(E)\) we have \(b^k.E^\sharp \subset E\) and that \(b^k.E^\sharp\) has a simple pole.

**Example.** The inequality \(\delta(E) \leq or(E)\) may be strict for \(or(E) \geq 2\). For instance the \((a,b)\)-module of rank 3 with \(\mathbb{C}[\lceil b\rceil]\)-basis \(e_1, e_2, e_3\) with \(a.e_1 = e_2, a.e_2 = b.e_3, a.e_3 = 0\) has index 1 and regularity order 2 : an easy computation gives that a \(\mathbb{C}[\lceil b\rceil]\)-basis for \(E^\sharp\) is given by \(e_1, b^{-1}.e_2, b^{-1}.e_3\), and that a \(\mathbb{C}[\lceil b\rceil]\)-basis for \(E + b^{-1}.a.E\) is given by \(e_1, b^{-1}.e_2, e_3\).

**Définition 3.2.2** Let \(E\) be a regular \((a,b)\)-module. The **biggest simple pole sub-(a,b)-module of \(E\)** exists and has finite \(\mathbb{C}\)-codimension in \(E\). We shall note it \(E^b\).

In general, for \(k = \delta(E)\) the inclusion \(b^k.E^\sharp \subset E^b\) is strict. For instance this is the case for \(E_{\lambda,\mu} \oplus E_{\nu}\).

**Lemme 3.2.3** Let \(E\) be a regular \((a,b)\)-module. The smallest integer \(m\) such we have \(b^m.E \subset E^b\) is equal to \(\delta(E)\).

**Proof.** Let \(k := \delta(E)\). Then \(b^k.E^\sharp\) is a simple pole sub-(\(a,b\))-module of \(E\). So we have \(b^k.E \subset b^k.E^\sharp \subset E^b\). Conversely, an inclusion \(b^m.E \subset E^b\) gives \(E \subset b^{-m}.E^b\). As \(b^{-m}.E^b\) has a simple pole this implies \(E^\sharp \subset b^{-m}.E^b \subset b^{-m}.E\). So \(\delta(E) \leq m\).

**Examples.** In the case 3 of the proposition\(^{3.1.6}\) \(E^b\) is generated as a \(\mathbb{C}[\lceil b\rceil]\)-module by \(y\) and \(b.t\), so \(E^b = b.E^\sharp\).

In case 4 we have also \(E^b = b.E^\sharp\).

**Lemme 3.2.4** Let \(E\) be a regular \((a,b)\)-module. For any exact sequence of \((a,b)\)-modules

\[
0 \rightarrow E' \rightarrow E \xrightarrow{\pi} E'' \rightarrow 0
\]

we have \(or(E'') \leq or(E) \leq rank(E') + or(E'')\).

As a consequence, the order of regularity of \(E\) is at most \(rank(E) - 1\) for any regular non zero \((a,b)\)-module.
**Proof.** The inequality \( \text{or}(E'') \leq \text{or}(E) \) is trivial because an inequality

\[
a^{k+1}.E \subset \sum_{j=0}^{k} a^j b^{k-j+1}.E
\]

implies the same for \( E'' \) and, by definition, the best such integer \( k \) is the order of regularity.

The crucial case is when \( E' \) is of rank 1. So we may assume that \( E' \cong E_\lambda \) for some \( \lambda \in \mathbb{C} \) (see [3.1.7] or [B.93] prop.2.2 p.23). Let \( k = \text{or}(E'') \). Then the inclusion

\[
a^{k+1}.E'' \subset \sum_{j=0}^{k} a^j b^{k-j+1}.E''
\]

implies that

\[
a^{k+1}.E \subset \sum_{j=0}^{k} a^j b^{k-j+1}.E + b^l.E_\lambda
\]

for some \( l \in \mathbb{N} \). In fact we can take for \( l \) the smallest integer such that the generator \( e_\lambda \) of \( E_\lambda \) (defined up to \( \mathbb{C}^* \) by the relation \( a.e_\lambda = \lambda.b.e_\lambda \)) satisfies \( b^l.e_\lambda \in \Psi_k = \sum_{j=0}^{k} a^j b^{k-j+1}.E \).

Remark that this integer \( l \geq 0 \) is well defined because \( b^{k+1}.E_\lambda \in \Psi_k \). Moreover, as \( \Psi_k \) is a \( \mathbb{C}[[b]] \)-submodule of \( E \), \( b^l.e_\lambda \in \Psi_k \) implies \( b^l.E_\lambda \subset \Psi_k \).

Now, thanks to (2) we have

\[
a^{k+2}.E \subset \sum_{j=0}^{k} a^{j+1} b^{k+1-j}.E + a.b^l.E_\lambda
\]

which gives

\[
a^{k+2}.E \subset \sum_{j=0}^{k+1} a^j b^{k+2-j}.E
\]

because \( a.b^l.E_\lambda = b.b^l.E_\lambda \subset b.\Psi_k \).

This proves that \( \text{or}(E) \) is at most \( k+1 = \text{or}(E'') + \text{rank}(E') \).

Assume now that our inequality is proved for \( E' \) of rank \( p-1 \) and consider an exact sequence (*) with \( \text{rank}(E') = p \geq 2 \). Let \( E_\lambda \subset E' \) be a normal rank 1 sub-(a,b)-module of \( E' \) (see [3.1.7] or [B.93] prop.2.2 p.23 for a proof of the existence of such sub-(a,b)-module) and consider the exact sequence of (a,b)-modules (using the fact that \( E_\lambda \) is also normal in \( E \); see lemma 2.5 of [B.93])

\[
0 \to E'/E_\lambda \to E/E_\lambda \to E'' \to 0
\]

Using the induction hypothesis and the rank 1 case we get

\[
\text{or}(E') \leq \text{or}(E/E_\lambda) + 1 \leq p - 1 + \text{or}(E'') + 1 = p + \text{or}(E'')
\]

Now using an easy induction (or a Jordan-Hölder sequence for \( E \)) we obtain \( \text{or}(E) \leq \text{rank}(E) - 1 \) for any regular \( E \). \( \blacksquare \)
Remark. In the situation of the previous lemma we have $\delta(E') \leq \delta(E)$. This is a consequence of the obvious inclusion $(E')^2 \subset E'[b^{-1}] \cap E^2$; assume that $x \in E'[b^{-1}] \cap E^2$; then, for $k := \delta(E)$ we have $b^k.x \in E'[b^{-1}] \cap E$ so that $b^{N+k}.x \in E'$ for $N$ large enough. As $E/E'$ has no $b$-torsion, we conclude that $b^k.x \in E'$. So our initial inclusion implies $\delta(E') \leq k$. \hfill $\square$

3.3 Duality.

In this section we consider more carefully the associative and unitary $\mathbb{C}$–algebra

$$\tilde{A} := \left\{ \sum_{n=0}^{\infty} P_n(a).b^n \mid P_n \in \mathbb{C}[z] \right\}$$

with the commutation relation $a.b - b.a = b^2$, and such that the left and right multiplications by $a$ are continuous for the $b$–adic topology of $\tilde{A}$.

The right structure as a commuting left-structure on $\tilde{A}$.

There exits an unique $\mathbb{C}$–linear (bijective) map $\theta : \tilde{A} \to \tilde{A}$ with the following properties

i) $\theta(1) = 1, \theta(a) = a, \theta(b) = -b$;

ii) $\theta(xy) = \theta(y).\theta(x) \quad \forall x, y \in \tilde{A}$.

iii) $\theta$ is continuous for the $b$–adic topology of $\tilde{A}$

The uniqueness is an easy consequence of iii) and the fact that the conditions i) and ii) implies $\theta(b^p.a^q) = (-1)^p.a^q.b^p \quad \forall p, q \in \mathbb{N}$. Existence is then clear from the explicit formula deduced from this remark.

We define a new structure of left $\tilde{A}$–module on $\tilde{A}$, called the $\theta$–structure and denote by $x.\square$, by the formula

$$x.y = y.\theta(x).$$

It is easy to see that this new left-structure on $\tilde{A}$ commutes with the ordinary one and that with this $\theta$–structure $\tilde{A}$ is still free of rank one as a left $\tilde{A}$–module.

Définition 3.3.1. Let $E$ be a (left) $\tilde{A}$–module. On the $\mathbb{C}$–vector space $\text{Hom}_{\tilde{A}}(E, \tilde{A})$ we define a left $\tilde{A}$–module structure using the $\theta$–structure on $\tilde{A}$. Explicitly this means that for $\varphi \in \text{Hom}_{\tilde{A}}(E, \tilde{A})$ and $x \in \tilde{A}$ we let

$$\forall e \in E \quad (x.\varphi)(e) := x.\varphi(e) = \varphi(e).\theta(x).$$

We obtain in this way a left $\tilde{A}$–module that we shall still denote $\text{Hom}_{\tilde{A}}(E, \tilde{A})$.

\footnote{remark that for each $k \in \mathbb{N}$ $b^k.\tilde{A} = \tilde{A}.b^k$}
It is clear that $E \to Hom_{\tilde{A}}(E, \tilde{A})$ is a contravariant functor which is left exact in the category of left $\tilde{A}$–modules. As every finite type left $\tilde{A}$–module has a resolution of length $\leq 2$ by free finite type modules (see [B.95] cor.2 p.29), we shall denote by $Ext^i_{\tilde{A}}(E, \tilde{A}), i \in [0, 2]$ the right derived functors of this functor. They are finite type left $\tilde{A}$–modules when $E$ is finitely generated because $\tilde{A}$ is left noetherian (see [B.95] prop.2 p.26).

Any $(a,b)$-module is a left $\tilde{A}$–module. They are characterized by the existence of special simple resolutions.

**Lemma 3.3.2** Let $M$ be a $(p, p)$ matrix with entries in the ring $\mathbb{C}[[b]]$. Then the left $\tilde{A}$–linear map $\text{Id}_p.a - M : \tilde{A}^p \to \tilde{A}^p$ given by

\[^tX := (x_1, \cdots, x_p) \to ^tX(Id_p.a - M)\]

is injective. Its cokernel is the $(a,b)$-module $E$ given as follows:

$E$ has a $\mathbb{C}[[b]]$ base $e := (e_1, \cdots, e_p)$ and $a$ is defined by the two conditions

1. $a.e := M(b).e$ ;
2. the left action of $a$ is continuous for the $b$–adic topology of $E$.

Any $(a,b)$-module is obtained in this way and so, as a $\tilde{A}$–left-module, has a resolution of the form

\[0 \to \tilde{A}^p \to \tilde{A}^p \to E \to 0.\]  

(\text{@})

**Proof.** First remark that for $x \in \tilde{A}$ the condition $x.a \in b.\tilde{A}$ implies $x \in b.\tilde{A}$. Now let us prove, by induction on $n \geq 1$, that, for any $(p, p)$ matrix $M$ with entries in $\mathbb{C}[[b]]$ the condition $^tX(Id_p.a - M) = 0$ implies $^tX \in b^n.\tilde{A}^p$.

For $n = 1$ this comes from the previous remark. Let assume that the assertion is proved for $n \geq 1$ and consider an $X \in \tilde{A}^p$ such that $^tX(Id_p.a - M) = 0$. Using the induction hypothesis we can find $Y \in \tilde{A}^p$ such that $X = b^n.Y$. Now we obtain, using $a.b^n = b^n.a + nb^{n+1}$ and the fact that $\tilde{A}$ has no zero divisor, the relation

\[^tY(Id_p.a - (M + n.Id_p,b)) = 0\]

and using again our initial remark we conclude that $Y \in b.\tilde{A}^p$ so $X \in b^{n+1}.\tilde{A}^p$.

So such an $X$ is in $\cap_{n \geq 1} b^n.\tilde{A}^p = (0)$.

The other assertions of the lemma are obvious. \hfill \blacksquare

We recall now a construction given in [B.95] which allows to compute more easily the vector spaces $Ext^i_{\tilde{A}}(E, F)$ when $E, F$ are $(a,b)$-modules

**Définition 3.3.3** Let $E, F$ two $(a,b)$-modules. Then the $\mathbb{C}[[b]]$–module $Hom_{b}(E, F)$ is again a free and finitely generated $\mathbb{C}[[b]]$–module. Define on it an $(a,b)$-module structure in the following way.
1. First change the sign of the action of \( b \). So \( S(b) \in \mathbb{C}[[b]] \) will act as \( \bar{S}(b) = S(-b) \).

2. Define \( a \) using the linear map \( \Lambda : \text{Hom}_b(E, F) \to \text{Hom}_b(E, F) \) given by \( \Lambda(\varphi)(e) = \varphi(a.e) - a.\varphi(e) \).

We shall denote \( \text{Hom}_{a,b}(E, F) \) the corresponding \((a,b)\)-module.

The verification that \( \Lambda(\varphi) \) is \( \mathbb{C}[[b]] \)-linear and that \( \Lambda \cdot \bar{b} \cdot \Lambda = \bar{b}^2 \) are easy (and may be found in [B.95] p.31).

**Remark.** In loc. cit. we defined the \((a,b)\)-module structure on \( \text{Hom}_{a,b}(E, F) \) with opposite signs for \( a \) and \( b \). The present convention is better because it fits with the usual definition of the formal adjoint of a differential operator: \( z^* = z \) and \( (\partial/\partial z)^* = -\partial/\partial z \).

The following lemma is also proved in loc.cit.

**Lemme 3.3.4** Let \( E, F \) two \((a,b)\)-modules. Then there is a functorial isomorphism of \( \mathbb{C} \)-vector spaces

\[
H^i\left(\text{Hom}_{a,b}(E, F) \xrightarrow{a} \text{Hom}_{a,b}(E, F)\right) \to \text{Ext}^i_{\tilde{A}}(E, F) \quad \forall i \geq 0.
\]

Here the map \( a \) of the complex \( \text{Hom}_{a,b}(E, F) \xrightarrow{a} \text{Hom}_{a,b}(E, F) \) is equal to the \( \Lambda \) defined above which is, by definition, the operator “\( a \)” of the \((a,b)\)-module \( \text{Hom}_{a,b}(E, F) \).

Now the following corollary of the lemma 3.3.2 gives that the two natural ways of defining the dual of an \((a,b)\)-module give the same answer.

**Corollaire 3.3.5** Let \( E \) an \((a,b)\)-module. There is a functorial isomorphism of \((a,b)\)-modules between the following two \((a,b)\)-modules constructed as follows:

1. \( \text{Ext}^1_{\tilde{A}}(E, \tilde{A}) \) with the \( \tilde{A} \)-structure defined by the \( \theta \)-structure of \( \tilde{A} \).
2. \( \text{Hom}_{a,b}(E, E_0) \) where \( E_0 := \tilde{A}/\tilde{A}.a \).

**Proof.** Using a free resolution \((\oplus)\) of \( E \) deduced from a \( \mathbb{C}[[\bar{b}]] \)-basis \( e := (e_1, \cdots, e_p) \) we obtain, by the previous lemma, an exact sequence

\[
0 \to \tilde{A}^p \xrightarrow{(Id_p, a^{-1}M)} \tilde{A}^p \to \text{Ext}^1_{\tilde{A}}(E, \tilde{A}) \to 0. \quad (\oplus)
\]

of left \( \tilde{A} \)-modules where \( \tilde{A}^p \) is endowed with its \( \theta \)-structure. Writing the same exact sequence with the ordinary left-module structure of \( \tilde{A}^p \) gives

\[
0 \to \tilde{A}^p \xrightarrow{(Id_p, a^{-1}M)} \tilde{A}^p \to \text{Ext}^1_{\tilde{A}}(E, \tilde{A}) \to 0. \quad (\oplus \text{ bis})
\]
where \( {\tilde{\mathcal{M}}}'(b) := {\mathcal{M}}'(-b) \).

Denote by \( e^* := (e^*_1, \ldots, e^*_p) \) the dual basis of \( \text{Hom}_{\mathbb{C}[|b|]}(E, E_0) \). By definition of the action of \( a \) on \( \text{Hom}_{a,b}(E E_0) \) we get, if \( \omega \) is the class of 1 in \( E_0 \):

\[
(a.e^*_i(e_j) = e^*_i(a.e_j) - a.e^*_i(e_j) = e^*_i \left( \sum_{h=1}^p m_{j,h},e_h \right) - a.\delta_{i,j}.\omega = {\tilde{m}}_{j,i}.\omega
\]

because \( a.\omega = 0 \) in \( E_0 \), and the definition of the action of \( b \) on \( \text{Hom}_{a,b}(E, E_0) \).

So we have \( a.e^* = {\tilde{\mathcal{M}}}.e^* \) concluding the proof. \( \blacksquare \)

Définition 3.3.6 For any \((a,b)\)-module \( E \) the dual of \( E \), denoted by \( E^* \), is the \((a,b)\)-module \( \text{Ext}^1_{\mathcal{A}}(E, \mathcal{A}) \simeq \text{Hom}_{a,b}(E, E_0) \).

Of course, for any \( \mathcal{A} \)-linear map \( f : E \to F \) between two \((a,b)\)-modules we have an \( \mathcal{A} \)-linear "dual" map \( f^* : F^* \to E^* \).

It is an easy consequence of our previous description of \( \text{Ext}^1_{\mathcal{A}}(E, \mathcal{A}) \) that we have a functorial isomorphism \( (E^*)^* \to E \).

Examples.

1. For each \( \lambda \in \mathbb{C} \) we have \((E_\lambda)^* \simeq E_{-\lambda}\).
2. For \((\lambda, \mu) \in \mathbb{C}^2 \) we have \( E_{\lambda,\mu}^* \simeq E_{-\mu+1,-\lambda+1} \).
3. Let \( E \) be the rank two simple pole \((a,b)\)-module \( E_1(0) \) defined by \( a.e_1 = b.e_1 + b.e_2 \) and \( a.e_2 = b.e_2 \). Then its dual is isomorphic to \( E_{-1}(0) \).

It is also an elementary exercise to show the following isomorphisms:

\[
E_1(0) \simeq \mathbb{C}[[z]] \oplus \mathbb{C}[[z]].\text{Log}z \quad \text{and} \quad E_{-1}(0) \simeq \mathbb{C}[[z]] \frac{1}{z^2} \oplus \mathbb{C}[[z]].\text{Log}z \frac{1}{z^2}
\]

with \( a := xz \) and \( b := \int_0^z \).

Proposition 3.3.7 For any exact sequence of \((a,b)\)-modules

\[
0 \to E' \xrightarrow{u} E \xrightarrow{v} E'' \to 0
\]

we have an exact sequence of \((a,b)\)-modules

\[
0 \to (E'')^* \xrightarrow{v^*} E^* \xrightarrow{u^*} (E')^* \to 0.
\]

If \( E \) is a simple pole \((a,b)\)-module, \( E^* \) has a simple pole.

For any regular \((a,b)\)-module \( E \) its dual \( E^* \) is regular. Moreover, if \( E^b \) and \( E^z \) are respectively the biggest simple pole submodule of \( E \) and the saturation of \( E \) by \( b^{-1}.a \) in \( E[b^{-1}] \), we have

\[
(E^z)^* \simeq (E^*)^b \quad \text{and} \quad (E^b)^* \simeq (E^*)^z.
\]
Proof. The first assertion is a direct consequence of the vanishing of $\text{Ext}_i^\theta(E, \tilde{A})$ for $i = 0, 2$, for any (a,b)-module and the long exact sequence for the "Ext".

The condition that $E$ has a simple pole is equivalent to the fact that for any chosen basis $e$ of $E$ the matrix $M$ has its coefficients in $b, \tilde{A} = \tilde{A}b$. Then this remains true for $i = \tilde{M}$.

To prove the regularity of $E^*$ when $E$ is regular, we shall use induction on the rank of $E$. The rank 1 case is obvious because we have a simple pole in this case. Assume that the assertion is true for rank $< p$ and consider a rank $= p$ regular (a,b)-module $E$. Using the theorem 3.1.7 we have an exact sequence of (a,b)-modules

$$0 \rightarrow E_\lambda \rightarrow E \rightarrow F \rightarrow 0$$

where $F$ is regular of rank $p - 1$. This gives a short exact sequence

$$0 \rightarrow F^* \rightarrow E^* \rightarrow E_{-\lambda} \rightarrow 0$$

and the regularity of $F^*$ and of $E_{-\lambda}$ implies the regularity of $E^*$.

Now the inclusions $E^b \subset E \subset E^\sharp$ gives exact sequences

$$0 \rightarrow \text{Ext}_1^\theta(E/E^b, \tilde{A}) \rightarrow E^* \rightarrow (E^b)^* \rightarrow \text{Ext}_2^\theta(E/E^b, \tilde{A}) \rightarrow 0$$

and the next lemma will show that the $\text{Ext}_1^\theta(V, \tilde{A}) = 0$ for any $\tilde{A}$-module which is a finite dimensional vector space, and also the finiteness (as a vector space) of $\text{Ext}_2^\theta(V, \tilde{A})$. This implies that we have, for any regular (a,b)-module, the inclusions

$$E^* \subset (E^b)^* \quad \text{and} \quad (E^\sharp)^* \subset E^*.$$

They imply, thanks to the fact that $(E^b)^*$ and $(E^\sharp)^*$ have simple poles,

$$(E^*)^\sharp \subset (E^b)^* \quad \text{and} \quad (E^\sharp)^* \subset (E^*)^b.$$

But the inclusion $(E^*)^b \subset E^*$ gives

$$E = (E^*)^* \subset ((E^*)^b)^* \subset ((E^\sharp)^*)^* = E^\sharp$$

and the minimality of $E^\sharp$ gives $((E^*)^b)^* = E^\sharp$ because $((E^*)^b)^*$ has a simple pole and contains $E$. Dualizing again gives $(E^\sharp)^* \simeq (E^*)^b$. The last equality is obtained in a similar way from $E^* \subset (E^*)^\sharp$.

Lemma 3.3.8 Let $V$ be an $\tilde{A}$-module of finite dimension over $\mathbb{C}$. Then we have $\text{Ext}_i^\theta(V, \tilde{A}) = 0$ for $i = 0, 1$ and $\text{Ext}_2^\theta(V, \tilde{A})$ is again an $\tilde{A}$-module (via the $\theta$-structure of $\tilde{A}$) which is a finite dimensional vector space. Moreover it has the same dimension than $V$ and there is a canonical $\tilde{A}$-module isomorphism

$$\text{Ext}_2^\theta(\text{Ext}_2^\theta(V, \tilde{A}), \tilde{A}) \simeq V.$$
PROOF. We begin by proving the first assertion of the lemma for the special case
\( V_{\lambda} := \tilde{A}/\tilde{A}(a - \lambda) + \tilde{A}.b \) for any \( \lambda \in \mathbb{C} \). Let us show that we have the free resolution
\[
0 \to \tilde{A} \xrightarrow{\alpha} \tilde{A}^2 \xrightarrow{\beta} \tilde{A} \to V_{\lambda} \to 0
\]
where \( \alpha(x) := (x.b, -x.(a - b - \lambda)) \), \( \beta(u, v) := u.(a - \lambda) + v.b \). The map \( \alpha \)
is clearly injective and \( \beta(\alpha(x)) = x.(b.a - \lambda.b - (a - b - \lambda).b) = 0 \). If we have \( \beta(u, v) = 0 \)
then \( u \in \tilde{A}.b \); let \( u = x.b \). Then we get
\[
x.(a - b - \lambda).b + v.b = 0 \quad \text{and so} \quad v = -x.(a - b - \lambda).
\]
This gives the exactness of our resolution.
Now the \( \text{Ext}^i(V_{\lambda}, \tilde{A}) \) are given by the cohomology of the complex
\[
0 \to \tilde{A} \xrightarrow{\beta^*} \tilde{A}^2 \xrightarrow{\alpha^*} \tilde{A} \to 0.
\]
The map \( \beta^*(x) = ((a - \lambda).x, b.x) \) and \( \alpha^*(u, v) = b.u - (a - b - \lambda).v \) are \( \tilde{A} \)-linear for
the \( \theta \)-structure of \( \tilde{A} \). Clearly \( \beta^* \) is injective and \( \alpha^*(\beta^*(x)) \equiv 0 \). If \( \alpha^*(u, v) = 0 \)
set \( v = b.y \) and conclude that \( u = (a - \lambda).y \). This gives the vanishing of the \( \text{Ext}^i \)
for \( i = 0, 1 \). The \( \text{Ext}^2 \) is the cokernel of \( \beta^* \) which is easily seen to be isomorphic
to \( V_{\lambda} \).
Consider now any finite dimensional \( \tilde{A} \)-module \( V \) over \( \mathbb{C} \). We make an induction
on \( \text{dim}_\mathbb{C}(V) \) to prove the vanishing of the \( \text{Ext}^i \) for \( i = 0, 1 \) and the assertion on
the dimension of the \( \text{Ext}^2 \).
The \( \text{dim} V = 1 \) case is clear because reduced to the case \( V = V_{\lambda} \) for some \( \lambda \in \mathbb{C} \).
Assume that the case \( \text{dim} V = p \) is proved, for \( p \geq 1 \) and consider some \( V \)
with \( \text{dim} V = p + 1 \). Then \( \text{Ker} b \) is not \( \{0\} \) and is stable by \( a \). Let \( \lambda \in \mathbb{C} \)
an eigenvalue of \( a \) acting on \( \text{Ker} b \). Then a eigenvector generates in \( V \) a sub-
\( \tilde{A} \)-module isomorphic to \( V_{\lambda} \).
The exact sequence of \( \tilde{A} \)-modules
\[
0 \to V_{\lambda} \to V \to W \to 0
\]
where \( W := V/V_{\lambda} \) has dimension \( p \) allows us to conclude, looking at the long
exact sequence of \( \text{Ext} \).
The last assertion follows from the remark that we produce a free resolution of
\( \text{Ext}^2(V, \tilde{A}) \) by taking \( \text{Hom}_{\tilde{A}}(-, \tilde{A}) \) of a free (length two, see [B.97]) resolution of
\( V \) because of the already proved vanishing of the \( \text{Ext}^i \) for \( i = 0, 1 \). Doing this again
gives back the initial resolution (remark that we use here that the \( \theta \circ \theta \)-structure
on \( \text{Hom}_{\tilde{A}}(\tilde{A}, \tilde{A}), \tilde{A} \) is the usual left structure on \( \tilde{A} \)).

Corollaire 3.3.9 For a simple pole \((a, b)\) module \( E \) denote by \( S(E) \) the spectrum of \( b^{-1}.a \) acting on \( E/b.E \). Then we have
\[
S(E^*) = -S(E).
\]
Proof. We make an induction on the rank of $E$. In rank 1 the result is clear because we have $E \cong E_{\lambda}$ for some $\lambda \in \mathbb{C}$, and $S(E_{\lambda}) = \{\lambda\}$. But we know that $E_{\lambda}^* = E_{-\lambda}$. Assume the assertion proved for any rank $p \geq 1$ simple pole $(a,b)$-module, and consider $E$ with rank $p + 1$. Using theorem 3.1.7 there exists $\lambda \in \mathbb{C}$ and an exact sequence $(a,b)$-modules

$$0 \to E_{\lambda} \to E \to F \to 0$$

where $\text{rank}(F) = p$ and where $F$ has a simple pole (because a quotient of a simple pole $(a,b)$-module has a simple pole!). The exact sequence of vector spaces

$$0 \to E_{\lambda}/b.E_{\lambda} \to E/b.E \to F/b.F \to 0$$

shows that $S(E) = S(F) \cup \{\lambda\}$. Now proposition 3.3.7 gives the exact sequence

$$0 \to F^* \to E^* \to E_{-\lambda} \to 0$$

which implies, as before, $S(E^*) = S(F^*) \cup \{-\lambda\}$. The induction hypothesis $S(F^*) = -S(F)$ allows to conclude. ■

Lemme 3.3.10 For any pair of $(a,b)$-modules $E$ and $F$ there is a canonical isomorphism of vector spaces

$$D : Ext^1_A(E, F) \to Ext^1_A(F^*, E^*)$$

associated to the correspondance between 1-extensions (i.e. short exact sequences)

$$(0 \to F \to G \to E \to 0) \to (0 \to E^* \to G^* \to F^* \to 0).$$

Proof. We have a obvious isomorphism of $\mathbb{C}[[b]]$-modules:

$$I : Hom_b(E, F) \to Hom_b(Hom_b(F, E_0), Hom_b(E, E_0)) \cong Hom_b(F^*, E^*)$$

because $E_0 \cong \mathbb{C}[[b]]$ as a $\mathbb{C}[[b]]$-module. But recall that $Ext^1_A(E, F)$ (resp. $Ext^1_A(F^*, E^*)$) is the cokernel of the $\mathbb{C}$-linear map "$a$" defined on $Hom_b(E, F)$ by the formula

$$(a.\varphi)(x) = \varphi(a.x) - a.\varphi(x)$$

So it is enough to check that the isomorphism $I$ commutes with "$a$" in order to get an isomorphism between the cokernels of "$a$" in these two spaces. Let $\varphi \in Hom_b(E, F)$ and $\xi \in F^*$. Then $I(\varphi)(\xi) = \varphi \circ \xi$. So, for $x \in E$ we have (using $\Lambda$ to avoid too many "$a$"

$$\Lambda(I(\varphi))(\xi) = I(\varphi)(a.\xi) - a.(I(\varphi)(\xi))$$

$$\Lambda(I(\varphi))(\xi)(x) = (\varphi \circ \xi)(a.x) - a.\xi(\varphi(x)) - (\xi(\varphi(a.x)) - a.\xi(\varphi(x)))$$

$$= [((\Lambda(\varphi)) \circ \xi)](x) = I(\Lambda(\varphi))(x).$$

3but be careful with the $b \to \bar{b}$!
So \( \Lambda \circ I = I \circ \Lambda \). The map \( I \) gives an isomorphism of complexes

\[
\begin{array}{ccc}
\text{Hom}_{a,b}(E, F) & \overset{\Lambda}{\longrightarrow} & \text{Hom}_{a,b}(E, F) \\
\downarrow I & & \downarrow I \\
\text{Hom}_{a,b}(F^*, E^*) & \overset{\Lambda}{\longrightarrow} & \text{Hom}_{a,b}(F^*, E^*)
\end{array}
\]

and this conclude the proof, using lemma 3.3.4. \( \square \)

For an \((a,b)\)-module \( E \) and an integer \( m \in \mathbb{N} \) it is clear that \( b^m.E \) is again an \((a,b)\)-module. This can be generalize for any \( m \in \mathbb{C} \).

**Définition 3.3.11** For any \((a,b)\)-module \( E \) and any complex number \( m \in \mathbb{C} \) define the \((a,b)\)-module \( b^m.E \) as follows: as an \( \mathbb{C}[b] \)-module we let \( b^m.E \simeq E \simeq \mathbb{C}[b]^{\text{rank}(E)} \); the operator \( a \) is defined as \( a = a + m.b.1 \).

Precisely, this means that if \( (e_1, \ldots, e_k) \) is a \( \mathbb{C}[b] \)-basis of \( E \) such that we have \( a.e = M(b).e \) where \( M \in \text{End}(C^p) \otimes \mathbb{C}[b] \), the \((a,b)\)-module \( b^m.E \) admit a basis, denote by \( (b^m.e_1, \ldots, b^m.e_k) \), such that the operator \( a \) is defined by the relation \( a.(b^m.e) := (M(b) + m.b.1.d.b_k).(b^m.e) \).

Remark that for \( m \in \mathbb{N} \) this notation is compatible with the preexisting one, because of the relation \( a.b^m = b^m.(a + m.b) \).

For any \( m \in \mathbb{N} \) there exists a canonical \((a,b)\)-morphism

\[
b^m.E \rightarrow E
\]

which is an isomorphism of \( b^m.E \) on \( \text{Im}(b^m : E \rightarrow E) \). But remark that the map \( b^m : E \rightarrow E \) is not \( a \)-linear (but the image is stable by \( a \)).

For any \( m \in \mathbb{N} \) there is also a canonical \((a,b)\)-morphism

\[
E \rightarrow b^{-m}.E
\]

which induces an isomorphism of \( E \) on \( \text{Im}(b^m : b^{-m}.E \rightarrow b^{-m}.E) \). So we may write, via this canonical identification, \( b^m.(b^{-m}.E) = E \).

It is easy to see that for any \( m, m' \in \mathbb{C} \) we have a natural isomorphism

\[
b^{m'}.(b^m.E) \simeq b^{m+m'}.E \quad \text{and also} \quad b^0.E \simeq E.
\]

**Remark.** It is easy to show that for any \( m \in \mathbb{C} \) there exists an unique \( \mathbb{C} \)-algebra automorphism

\[
\eta_m : \tilde{A} \rightarrow \tilde{A} \quad \text{such that} \quad \eta(1) = 1, \eta(b) = b \quad \text{and} \quad \eta(a) = a + m.b.
\]

Using this automorphism, one can define a left \( \tilde{A} \)-module \( b^m.F \) for any left \( \tilde{A} \)-module \( F \) and any \( m \in \mathbb{C} \). This is, of course compatible with our definition in the context of \((a,b)\)-modules. \( \square \)

The behaviour of the correspondence \( E \rightarrow b^m.E \) by duality is given by the following easy lemma; the proof is left as an exercice.
Lemme 3.3.12 For any (a,b)-module $E$ and any $m \in \mathbb{C}$ there is natural (a,b)-isomorphism

$$(b^m.E)^* \rightarrow b^{-m}.E^*.$$ 

The following corollary of the lemma 3.2.3 and the proposition 3.3.7 allows to show that duality preserves the index.

Lemme 3.3.13 Let $E$ be a regular (a,b)-module. Then we have $\delta(E^*) = \delta(E)$.

Proof. By definition $\delta(E)$ is the smallest integer $k \in \mathbb{N}$ such that $E^k \subset b^{-k}.E$. Now $E^k \subset b^{-m}.E$ implies by duality that $b^m.E^* \subset (E^*)^k$. So, by lemma 3.2.3 we have $m \geq \delta(E^*)$. This proves that $\delta(E) \leq \delta(E^*)$ and we obtain the equality by symmetry. □

Remark. Duality does not preserve the order of regularity : in the example given before the definition 3.2.2 we have $or(E) = 2$ and $or(E^*) = 1$. □

Let us conclude this section by an easy exercize.

Exercice. For any (a,b)-modules $E, F$ and any $\lambda \in \mathbb{C}$ there are natural (a,b)-isomorphisms

1. $b^\lambda.E_\mu \simeq E_{\lambda+\mu}$.

2. $b^\lambda.Hom_{a,b}(E, F) \simeq Hom_{a,b}(b^{-\lambda}.E, F) \simeq Hom_{a,b}(E, b^\lambda.F)$.

3. Then deduce from the previous isomorphisms that $Hom_{a,b}(E, E_\lambda) \simeq b^{-\lambda}.E^*$, and $Ext_{a,b}^1(E, E_\lambda) \simeq E^*/(a + \lambda.b).E^*$.

3.4 Width of a regular (a,b)-module.

Notation. For a complex number $\lambda$ we shall note by $\bar{\lambda}$ is class in $\mathbb{C}/\mathbb{Z}$. We shall order elements in each class modulo $\mathbb{Z}$ by its natural order on real parts. □

Définition 3.4.1 Let $E$ be a regular (a,b)-module and let $\bar{\lambda} \in \mathbb{C}/\mathbb{Z}$. We define the following complex numbers :

$$\bar{\lambda}_{min}(E) := \inf \{\lambda \in \bar{\lambda}/\exists \text{ a non zero morphism } E_\lambda \rightarrow E\}$$

$$\bar{\lambda}_{max}(E) = \sup \{\lambda \in \bar{\lambda}/\exists \text{ a non zero morphism } E \rightarrow E_\lambda\}$$

$$L_\lambda(E) = \bar{\lambda}_{max}(E) - \bar{\lambda}_{min}(E) \in \mathbb{Z}$$

$$L(E) = \sup \{\bar{\lambda} \in \mathbb{C}/\mathbb{Z} / L_\lambda(E)\}$$
with the following conventions:

\[
\inf\{\emptyset\} = +\infty, \quad \sup\{\emptyset\} = -\infty \quad \text{and} \\
-\infty - \lambda = -\infty \quad \forall \lambda \in ]-\infty, +\infty[ \\
+\infty - \lambda = +\infty \quad \forall \lambda \in [-\infty, +\infty[.
\]

We shall call \( L(E) \) the width of \( E \).

**Remarks.**

1. A non zero morphism \( E_\lambda \to E \) is necessarily injective. Either its image is a normal submodule in \( E \) or there exists an integer \( k \geq 1 \) and a morphism \( E_{\lambda-k} \to E \) whose image is normal an contains the image of the previous one.

2. In a dual way, a non zero morphism \( E \to E_\lambda \) has an image equal to \( b^k.E_\lambda \cong E_{\lambda+k} \), where \( k \in \mathbb{N} \).

3. A non zero morphism \( E_\lambda \to E_\mu \) implies that \( \lambda \) lies in \( \mu + \mathbb{N} \). It is possible that for some \( E \) we have \( \tilde{\lambda}_{\max}(E) < \tilde{\lambda}_{\min}(E) \). For instance this is the case for the rank 2 regular \((a,b)\)-module \( E_{\lambda,\mu} \) from [3.1.6]. So the width of a regular but not simple pole \((a,b)\)-module is not necessarily a non negative integer.

4. Let \( E \) and \( F \) be regular \((a,b)\)-modules. If there is a surjective morphism \( E \to F \) then for all \( \tilde{\lambda} \in \mathbb{C}/\mathbb{Z} \) we have \( \tilde{\lambda}_{\max}(E) \geq \tilde{\lambda}_{\max}(F) \). If there is an injective morphism \( E' \to E \) then for all \( \tilde{\lambda} \in \mathbb{C}/\mathbb{Z} \) we have \( \tilde{\lambda}_{\min}(E) \leq \tilde{\lambda}_{\min}(E') \).

5. Every submodule of \( E \) isomorphic to \( E_\lambda \) is contained in \( E_\lambda^b \). So we have \( \tilde{\lambda}_{\min}(E) = \tilde{\lambda}_{\min}(E^b) \), for every regular \((a,b)\)-module \( E \) and every \( \tilde{\lambda} \in \mathbb{C}/\mathbb{Z} \).

6. In a dual way, every morphism \( E \to E_\lambda \) extends uniquely to a morphism \( E^* \to E_{\lambda}^* \) with the same image. So for every regular \((a,b)\)-module \( E \) and every \( \tilde{\lambda} \in \mathbb{C}/\mathbb{Z} \), we get \( \tilde{\lambda}_{\max}(E) = \tilde{\lambda}_{\max}(E^*) \). □

**Lemme 3.4.2**

1. Let \( E \) a simple pole \((a,b)\)-module and let \( S(E) \) denotes the spectrum of the linear map \( b^{-1}.a : E/b.E \to E/b.E \), we have

\[
\tilde{\lambda}_{\min}(E) = \inf\{\lambda \in S(E) \cap \tilde{\lambda}\} \quad \text{and} \quad \tilde{\lambda}_{\max}(E) = \sup\{\lambda \in S(E) \cap \tilde{\lambda}\} \quad (@)
\]

2. For any regular \((a,b)\)-module \( E \) we have

\[
(-\lambda)_{\max}(E^*) = -\tilde{\lambda}_{\min}(E) \quad \text{and} \quad (-\lambda)_{\min}(E^*) = -\tilde{\lambda}_{\max}(E).
\]

This implies \( L_{-\tilde{\lambda}}(E^*) = L_{\tilde{\lambda}}(E) \quad \forall \tilde{\lambda} \in \mathbb{C}/\mathbb{Z} \), and so \( L(E^*) = L(E) \).

3. For any regular \((a,b)\)-module \( E \) and any \( \tilde{\lambda} \in \mathbb{C}/\mathbb{Z} \) we have equivalence between

\[
\tilde{\lambda}_{\min}(E) \neq +\infty \quad \text{and} \quad \tilde{\lambda}_{\max}(E) \neq -\infty.
\]
Proof. Let \( E \) be a simple pole \((a,b)\)-module. We have already seen (in proposition 3.1.4) that if \( \lambda \in S(E) \) is minimal in its class modulo 1, there exists a non zero \( x \in E \) such that \( a.x = \lambda.b.x \). This implies that \( \tilde{\lambda}_{\text{min}} \leq \inf \{ \lambda \in S(E) \cap \tilde{\lambda} \} \). But the opposite inequality is obvious, so the first part of (\@) is proved.

Using corollary 3.3.9 and the result already obtained for \( E^\ast \) gives
\[
(-\lambda)_{\text{min}}(E^\ast) = \inf \{ -\lambda \in S(E^\ast) \cap (-\tilde{\lambda}) \} = -\sup \{ \lambda \in S(E) \cap \tilde{\lambda} \}.
\]
So for \( \mu = \sup \{ \lambda \in S(E) \cap \tilde{\lambda} \} \) we have an exact sequence of \((a,b)\)-modules
\[
0 \to E_{-\mu} \to E^\ast \to F \to 0
\]
and by duality, a surjective map \( E \to E_{\mu} \). This implies \( \tilde{\lambda}_{\text{max}} \geq \mu \). As, again, the opposite inequality is obvious, the second part of (\@) is proved.

Let us prove now the relations in 2.

Remark first that these equalities are true for a simple pole \((a,b)\)-module because of (\@) and corollary 3.3.9.

For any regular \((a,b)\)-module \( E \) we know that
\[
\tilde{\lambda}_{\text{min}}(E) = \tilde{\lambda}_{\text{min}}(E^b) = \inf \{ \lambda \in S(E^b) \cap \tilde{\lambda} \}
\]
and
\[
(-\lambda)_{\text{max}}(E^\ast) = (-\tilde{\lambda})_{\text{max}}((E^\ast)^\sharp).
\]
But we have
\[
(-\lambda)_{\text{max}}((E^\ast)^\sharp) = \sup \{ -\lambda \in S((E^\ast)^\sharp) \cap (-\tilde{\lambda}) \} = -\inf \{ \lambda \in S((E^\ast)^\sharp) \cap (-\tilde{\lambda}) \}
\]
because \( (E^\ast)^\sharp \) has a simple pole, using corollary 3.3.9. So we obtain
\[
(-\lambda)_{\text{max}}(E^\ast) = -\tilde{\lambda}_{\text{min}}(E^b) = -\tilde{\lambda}_{\text{min}}(E)
\]
because \( (E^\ast)^\sharp \) has a simple pole, using proposition 3.3.7. The second relation is analogous.

The equivalence in 3 is obvious in the simple pole case using (\@).

The general case is an easy consequence using \( E^b, E^\sharp : E_{\lambda_{\text{min}}(E^\sharp)} \neq +\infty \) so is \( \tilde{\lambda}_{\text{min}}(E^\sharp) \) because \( E \subset E^\sharp \). Then \( \tilde{\lambda}_{\text{max}}(E^\sharp) \neq -\infty \) and so is \( \lambda_{\text{max}}(E) \). The converse is analogous using \( E^b \).

\( \blacksquare \)

Remarks.

1. If \( E \) has a simple pole, we have \( L_{\lambda}(E) \geq 0 \) or \( L_{\lambda}(E) = -\infty \) for any \( \tilde{\lambda} \) in \( \mathbb{C}/\mathbb{Z} \). So \( L(E) \) is always \( \geq 0 \).

2. In cases 1 and 2 of the proposition 3.1.6 the formula (\@) gives the values of \( \tilde{\lambda}_{\text{min}} \) and \( \tilde{\lambda}_{\text{max}} \) for any \( \tilde{\lambda} \in \mathbb{C}/\mathbb{Z} \).

For the remaining cases we can compute these numbers using the fact that we already know the corresponding \( E^b \) and \( E^\sharp \) and the remark 5 and 6 before the preceeding lemma. \( \blacksquare \)
Proposition 3.4.3 Let $E$ be a regular $(a,b)$-module and let $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$. Assume that $\lambda = \tilde{\lambda}_{\min}(E) < +\infty$. Consider an exact sequence of $(a,b)$-modules

$$0 \to E_\lambda \to E \xrightarrow{\pi} F \to 0.$$ 

Then we have for all $\tilde{\mu} \in \mathbb{C}/\mathbb{Z}$ the inequality

$$L_{\tilde{\mu}}(F) \leq L_{\tilde{\mu}}(E) + 1. \quad (i)$$

Proof. As $\tilde{\mu}_{\max}(F) \leq \tilde{\mu}_{\max}(E)$ for any $\mu \in \mathbb{C}$ it is enough to prove that we have $\tilde{\mu}_{\min}(E) \leq \tilde{\mu}_{\min}(F) + 1$ for all $\tilde{\mu} \in \mathbb{C}/\mathbb{Z}$.

Let begin by the case of $\tilde{\mu} = \tilde{\lambda}$. We want to show the inequality

$$\tilde{\lambda}_{\min}(F) \geq \lambda - 1 \quad (ii)$$

Let $E_{\lambda - d} \hookrightarrow F$ with $d \geq 0$. The rank 2 $(a,b)$-module $G := \pi^{-1}(E_{\lambda - d})$ is contained in $E$, so $\lambda = \tilde{\lambda}_{\min}(G)$. We have the exact sequence of $(a,b)$-modules

$$0 \to E_\lambda \to \pi^{-1}(E_{\lambda - d}) \xrightarrow{\pi} E_{\lambda - d} \to 0.$$ 

Now let us compare $G$ with the list in proposition 3.1.6.

If $G$ is in case 1, we have $E_{\lambda - d} \subset G$ so $d = 0$ because $\lambda = \tilde{\lambda}_{\min}(G)$.

If $G$ is in case 2, we have $\lambda - d = \lambda + n$ with $n \in \mathbb{N}$, so $d = 0$.

If $G$ is in case 3, we have $G \simeq E_{\lambda, \lambda + k}$ with $k \in \mathbb{N}$. Then the theorem 3.1.7 gives $2\lambda - d = 2\lambda + k - 1$ and so $d = 1 - k \leq 1$.

If $G$ is in case 4, we have $G \simeq E_{\lambda, \lambda + n}(\alpha)$. Again theorem 3.1.7 gives $2\lambda - d = 2\lambda + n - 1$ so $d = 1 - n \leq 0$ because $n \in \mathbb{N}^\ast$. So $d = 0$.

We conclude that we always have $d \leq 1$ and this proves (ii).

For $\tilde{\mu} \neq \tilde{\lambda}$ let us prove now the following inequality :

$$\tilde{\mu}_{\min}(F) \leq \tilde{\mu}_{\min}(E) \leq \tilde{\mu}_{\min}(F) + 1. \quad (iii)$$

Consider an injective morphism $E_\mu \to E$ with $\mu = \tilde{\mu}_{\min}(E)$. The restriction of $\pi$ to $E_\mu$ is injective and so it gives $\tilde{\mu}_{\min}(E) \geq \tilde{\mu}_{\min}(F)$. Assume now that we have an injective morphism $E_{\mu'} \hookrightarrow F$ with $\mu' = \tilde{\mu}_{\min}(F)$, and consider the rank 2 $(a,b)$-module $\pi^{-1}(E_{\mu'})$. Using the proposition 3.1.6 where only cases 1 or 3 are possible now, it can be easily check that (iii) is satisfied.

Remarks.

1. In the situation of the previous proposition we have either $\tilde{\lambda}_{\min}(E) \geq \tilde{\lambda}_{\max}(E)$ or $\tilde{\lambda}_{\max}(E) = \tilde{\lambda}_{\max}(F)$: Assume that we have $\lambda < \lambda' := \tilde{\lambda}_{\max}(E)$. Then there exists a surjective morphism $q : E \to E_{\lambda'}$, and, as the restriction of $q$ to $E_{\lambda'}$ is zero, the map $q$ can be factorized and gives a surjective morphism $\bar{q} : F \to E_{\lambda'}$. So we get $\tilde{\lambda}_{\max}(E) \leq \tilde{\lambda}_{\max}(F)$, and the desired equality thanks to the preceeding lemma.

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2. We shall use later that in the situation of the previous proposition we have the inequality $\tilde{\lambda}_{\text{max}}(F) \leq \lambda + L(E)$.

**Corollaire 3.4.4** In the situation of the previous proposition we have the inequality $L(E) + \text{rank}(E) \geq L(F) + \text{rank}(F)$. So this integer is always positive for any nonzero regular $(a,b)$-module.

**Proof.** As the rank 1 case is obvious, an easy induction on the rank of $E$ using the propositions 3.1.4 and 3.4.3 gives the proof. □

**Examples.**

1. The $(a,b)$-module

   

   $$J_k(\lambda) := \mathcal{A}/\mathcal{A}.(a - (\lambda + k - 1).b)(a - (\lambda + k - 2).b) \cdots (a - \lambda.b)$$

   which has rank $k$, satisfies $\lambda_{\text{max}} = \lambda$ and $\lambda_{\text{min}} = \lambda + k - 1$. So its width is $L(J_k(\lambda)) = -k + 1$.

   To understand easily the $(a,b)$-module $J_k(\lambda)$ the reader may use the following alternative definition of it: there is a $\mathbb{C}[[b]]$-basis $(e_1, \cdots, e_k)$ in which the action of “$a$” is given by

   $$a.e_1 = e_2 + \lambda.b.e_1, \quad a.e_2 = e_3 + (\lambda + 1).b.e_2, \cdots, a.e_k = (\lambda + k - 1).b.e_k.$$ 

2. The rank 2 $(a,b)$-module $E_\lambda \oplus E_{\lambda+n}$ has width $n$. This shows that, despite the fact that the width is always bigger than $-\text{rank}(E) + 1$, the width may be arbitrarily big, even for a rank 2 regular $(a,b)$-module. □

4 Finite determination of regular $(a,b)$-modules.

4.1 Some more preliminaries.

**Lemme 4.1.1** Let $E$ be a regular $(a,b)$-module of index $\delta(E) = k$. For $N \geq k + 1$ the quotient map $q_N : E \to E/b^N.E$ induces a bijection between simple pole sub-$(a,b)$-modules $F$ containing $b^k.E^z$ and sub $\mathcal{A}$-modules $\mathcal{F} \subset E/b^N.E$ satisfying the following two conditions

   i) $a.\mathcal{F} \subset b.\mathcal{F}$;

   ii) $b^k.E^z/b^N.E \subset \mathcal{F}$.

**Proof.** It is clear that if $F$ is a simple pole sub-$(a,b)$-module of $E$ containing $b^k.E^z$ the image $\mathcal{F} := q_N(F)$ is a $\mathcal{A}$-submodule of $E/b^N.E$ such that i) and ii) are fulfilled. Conversely, if a $\mathcal{A}$-submodule $\mathcal{F}$ satisfies i) and ii), let $F := q_N^{-1}(\mathcal{F})$. Of course, $F$ is a sub-$(a,b)$-module of $E$ and contains $b^k.E^z$. The only point to see is that $F$ has a simple pole. If $x \in F$ then $a.q_N(x) \in b.\mathcal{F}$ so $a.x \in b.F + b^N.E$. As $N \geq k + 1$ we may write $a.x = b.y + b.z$ with $y \in F$ and $z \in b^{N-1}.E \subset b^k.E^z \subset F$. This completes the proof. □
Remarks.

1. we may replace \( b^k.E^\sharp \) by \( b^k.E \) in the second condition imposed on \( F \) and \( \mathcal{F} \): if a simple pole \((a,b)\)-submodule \( F \) contains \( b^k.E \) it contains \( b^k.E^\sharp \) by definition of \( E^\sharp \). This allows to avoid the use of \( E^\sharp \) in the previous lemma.

2. The biggest \( \mathcal{F} \) satisfying i) and ii) corresponds to \( E^b \). So we may recover \( E^b \) from the quotient \( E/b^N.E \) for \( N \geq \delta(E) + 1 \).

\[ \Box \]

Corollaire 4.1.2 Let \( E \) be a regular \((a,b)\)-module of order of regularity \( k \). Fix \( N \geq k + 1 \) and assume that we has an isomorphism of \( \hat{A} \)-modules

\[ \varphi : E/b^N.E \rightarrow E'/b^N.E' \]

where \( E' \) is an \((a,b)\)-module. Then \( E' \) is regular and has order of regularity \( k \). Moreover we have the equality \( \varphi(E^b/b^N.E) = (E')^b/b^N.E' \).

**Proof.** As \( k \) is the order of regularity of \( E \) we have \( a^{k+1}.E \subset \sum_{j=0}^{k} a^j.b^{k-j+1}E \).

The inequality \( N \geq k + 1 \) gives \( a^{k+1}.E/b^N.E \subset \sum_{j=0}^{k} a^j.b^{k-j+1}E/b^N.E \), and this is also true for \( E'/b^N.E' \), and implies \( a^{k+1}.E' \subset \sum_{j=0}^{k} a^j.b^{k-j+1}E' \). So the order of regularity of \( E' \) is at most \( k \). We conclude that it is exactly \( k \) by symetry. The last statement comes from the second remark above, as \( \text{or}(E) \geq \delta(E) \). \[ \Box \]

4.2 Finite determination for a rank one extension.

Lemme 4.2.1 Let \( E \) be an \((a,b)\)-module et fix a complex number \( \lambda \). There exists \( N(E, \lambda) \in \mathbb{N} \) such that for any \( N \geq N(E, \lambda) \) we have the following inclusion : 

\[ b^N.E \subset (a-\lambda.b).E. \]

**Proof.** With the \( b \)-adic topology, \( E \) is a Frechet space. The \( \mathbb{C} \)-linear map \( a-\lambda.b : E \rightarrow E \) is continuous. The finiteness theorem of [B.95], theorem 1.bis p.31 gives that the kernel and cokernel of this map are finite dimensional vector spaces. So the subspace \((a-\lambda.b).E \) is closed in \( E \). This statement corresponds to the equality 

\[ \cap_{N \geq 0} [(a-\lambda.b).E + b^N.E] = (a-\lambda.b).E \quad (\@) \]

But the images of the subspaces \( b^N.E \) in the finite dimensional vector space \( E/(a-\lambda.b).E \) is a decreasing sequence. So it is stationnary, and, as the intersection is \( \{0\} \) thanks to \((\@)\), the result follows. \[ \Box \]

Proposition 4.2.2 Let \( F \) be an \((a,b)\)-module and \( \lambda \) a complex number. Consider a short exact sequence of \((a,b)\)-modules

\[ 0 \rightarrow E_\lambda \xrightarrow{\alpha} E \xrightarrow{\beta} F \rightarrow 0 \quad (@@) \]
where \( E_\lambda := \tilde{A}/\tilde{A}.(a - \lambda b) \). Then, for any \( N \geq N(F^*, -\lambda) \), the extension \((@@)\) is uniquely determined by the following extension of \( \tilde{A}-\)modules which are finite dimensional vectors spaces

\[
0 \to E_\lambda/b^N.E_\lambda \overset{\alpha}{\to} E/b^N.E \overset{\beta}{\to} F/b^N.F \to 0 \quad (@@_N)
\]

obtained from \((@@)\) by ’quotient by \( b^N\).

**Comments.** This statement needs some more explanations. Denote by \( K_N \) the kernel of the obvious map (forget ”a”)

\[
ob_N : Ext^1_\tilde{A}(F/b^N.F, E_\lambda/b^N.E_\lambda) \to Ext^1_b(F/b^N.F, E_\lambda/b^N.E_\lambda)
\]

where \( Ext^1_b(-, -) \) is a short notation for \( Ext^1_{\mathbb{C}[b]}(-, -) \). The short exact sequence correspondance \((@@) \to (@@_N)\) gives a map

\[
\delta_N : Ext^1_\tilde{A}(F, E_\lambda) \to Ext^1_\tilde{A}(F/b^N.F, E_\lambda/b^N.E_\lambda)
\]

whose range lies in \( K_N \), because the \( \mathbb{C}[b] \)-exact sequence \((@@)\) is split as \( F \) is \( \mathbb{C}[b] \)-free, and so is the exact sequence \((@@_N)\). The precise signification of the previous proposition is that for \( N \geq N(F^*, -\lambda) \) the map \( \delta_N \) is a \( \mathbb{C} \)-linear isomorphism between the vector spaces \( Ext^1_\tilde{A}(F, E_\lambda) \) and \( K_N \).

**Proof.** As a first step to realize the map \( \delta_N \) let us consider the following commutative diagramm of complex vector spaces, deduced from the exact sequences of \( \tilde{A} \)-modules:

\[
\begin{array}{ccc}
0 & \to & E_{\lambda+N} \to E_\lambda \to E_\lambda/b^N.E_\lambda \to 0 \\
0 & \to & b^N.F \to F \to F/b^N.F \to 0
\end{array}
\]

\[
\begin{array}{ccccccc}
Ext^1(F/b^N.F, E_{\lambda+N}) & \to & Ext^1(F, E_{\lambda+N}) & \to & Ext^1(b^N.F, E_{\lambda+N}) \\
\downarrow & & \downarrow & & \downarrow \\
Ext^1(F/b^N.F, E_\lambda) & \to & Ext^1(F, E_\lambda) & \to & Ext^1(b^N.F, E_\lambda) \\
\downarrow & & \downarrow & & \downarrow \\
Ext^1(F/b^N.F, E_\lambda/b^N.E_\lambda) & \to & Ext^1(F, E_\lambda/b^N.E_\lambda) & \to & Ext^1(b^N.F, E_\lambda/b^N.E_\lambda)
\end{array}
\]

We have the following properties:

1. The surjectivity of the map \( \beta \) is consequence of the vanishing of the vector space \( Ext^2_\tilde{A}(F, E_{\lambda+N}) \) thanks to the proposition 3.3.7.
2. The vanishing of the composition $u \circ v$ is consequence of lemma 3.3.4 and of the fact that the restriction map

$$\text{Hom}_b(F, E_\lambda) \to \text{Hom}_b(b^N . F, E_\lambda) \to \text{Hom}_b(b^N . F, E_\lambda / b^N . E_\lambda)$$

is obviously zero.

3. So the map $w$ is zero and $\gamma$ is surjective.

4. The kernel of $\gamma$ is given by the image of the injective map

$$\partial : \text{Hom}_{\tilde{A}}(b^N . F, E_\lambda / b^N . E_\lambda) \hookrightarrow \text{Ext}^1_{\tilde{A}}(F / b^N . F, E_\lambda / b^N . E_\lambda).$$

This is a consequence of the vanishing of the map

$$\text{Ext}^0_{\tilde{A}}(F, E_\lambda / b^N . E_\lambda) \to \text{Ext}^0_{\tilde{A}}(b^N . F, E_\lambda / b^N . E_\lambda).$$

Let us show now that for $N \geq N(F^*, -\lambda)$ the map $\alpha$ is zero. Using again the isomorphisms given by the lemma 3.3.4, $\alpha$ is induced by the obvious map

$$\text{Hom}_b(F, b^N . E_\lambda) \to \text{Hom}_b(F, E_\lambda),$$

whose image is $b^N . \text{Hom}_b(F, E_\lambda)$. Denote respectively by $G$ and $H$ the $(a,b)$-modules given by $\text{Hom}_b(F, b^N . E_\lambda)$ and $\text{Hom}_b(F, E_\lambda)$ with the action of $"a"$ defined by $\Lambda$ (see 3.3.4). Then we have the following commutative diagramm

$$\begin{array}{ccc}
G & \to & H \\
\downarrow & & \downarrow \\
G/a.G & \cong & H/a.H \\
\downarrow & & \downarrow \\
\text{Ext}^1_{\tilde{A}}(F, b^N . E_\lambda) & \alpha & \cong \text{Ext}^1_{\tilde{A}}(F, E_\lambda)
\end{array}$$

and the image of $i$ is $b^N . H$. So the map $\alpha$ will be zero as soon as $b^N . H \subset a.H$ and this is fulfilled for $N \geq N(H, 0) = N(F^*, -\lambda)$. This last equality coming from the isomorphisms

$$H/a.H \cong \text{Ext}^1_{\tilde{A}}(F, E_\lambda) \cong \text{Ext}^1_{\tilde{A}}(E_{-\lambda}, F^*) \cong F^*/(a + \lambda.b).F^*$$

see the exercise concluding section 3.3.

Consider now the commutative diagramm

$$\begin{array}{ccc}
0 & \to & \text{Hom}_{\tilde{A}}(b^N . F, E_\lambda / b^N . E_\lambda) \\
\downarrow \phi_N & & \downarrow \phi_N \\
0 & \to & \text{Ext}^1_{\tilde{A}}(F / b^N . F, E_\lambda / b^N . E_\lambda) \\
\downarrow \delta_N & & \downarrow \delta_N \\
\text{Hom}_b(b^N . F, E_\lambda / b^N . E_\lambda) & \cong & \text{Ext}^1_b(F / b^N . F, E_\lambda / b^N . E_\lambda)
\end{array}$$

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The surjectivity of $\beta$ implies that the map $i \circ \gamma$ is surjective (we know that the extensions in the image of $\delta_N$ comes from $K_N$, so $\delta_N$ factors in $\tilde{\delta}_N \circ i$).

We have $i(K_N) \cap Im(\partial_N) = (0)$ because $ob_N$ is injective on $Im(\partial_N)$.

So $i$ induces an isomorphism of vector spaces from $K_N$ to

$$Ext^1_{\mathcal{A}}(F/b^N.F, E_\lambda/b^N.E_\lambda)/Im(\partial_N) \overset{\gamma}{\sim} Ext^1_{\mathcal{A}}(F, E_\lambda/b^N.E_\lambda) \overset{\beta^{-1}}{\sim} Ext^1_{\mathcal{A}}(F, E_\lambda).$$

This completes the proof. \hfill \blacksquare

We shall need some bound for the integer $N(F^*, -\lambda)$ which appears in the previous proposition for the proof of our theorem.

**Lemme 4.2.3** Let $G$ be a regular $(a,b)$-module and let $\mu \in \mathbb{C}$. A sufficient condition on $N \in \mathbb{N}$ in order to have the inclusion $b^N.G \subset (a - \mu.b).G$ is

$$N \geq \mu - \tilde{\mu}_{\min}(G) + \delta(G) + 2.$$

**Proof.** As we know that $\tilde{\mu}_{\min}(G^b) = \tilde{\mu}_{\min}(G)$, for $M \in \mathbb{N}$, the assumption $M > \mu - \tilde{\mu}_{\min}(G)$ implies that $(a - (\mu - M).b).G^b = b.G^b$ (see the remark before proposition 3.1.4). By definition of the index of $G$ we have $b^{\delta(G)}.G \subset G^b$.

Combining both gives

$$b^{M+\delta(G)+1}.G \subset b^M.(a - (\mu - M).b).G = (a - \mu.b).b^M.G \subset (a - \mu.b).G.$$

Now let $N = M + \delta(G) + 1$; a sufficient condition on the integer $N$ is now $N \geq \mu - \tilde{\mu}_{\min}(G) + \delta(G) + 2$. \hfill \blacksquare

**Corollaire 4.2.4** A sufficient condition for $N \geq N(F^*, -\lambda)$ in the situation of prop. 4.2.2 in the regular case is that $N \geq \text{or}(E) + L(E) + \text{rank}(E) + 1$.

Remark that the inequality $L(E) + \text{rank}(E) \geq 1$ for any non zero regular $E$ implies that we have $\text{or}(E) + L(E) + \text{rank}(E) + 1 \geq \text{or}(E) + 2$.

**Proof.** We apply the previous lemma with $F^* = G$ and $\mu = -\lambda = -\tilde{\lambda}_{\min}(E)$.

The conclusion comes now from the following facts:

1. $-(-\lambda)_{\min}(F^*) = \tilde{\lambda}_{\max}(F) \leq \lambda + L(E)$ this last inequality is proved in 3.4.3
2. $\delta(F^*) = \delta(F) \leq \text{or}(F) \leq \text{or}(E)$ proved in 3.3.13 and 3.2.4 \hfill \blacksquare

### 4.3 The theorem

**Théorème 4.3.1** Let $E$ be a regular $(a,b)$-module. There exists an integer $N(E)$ such that for any $(a,b)$-module $E'$, any integer $N \geq N(E)$ and any $\mathcal{A}$--isomorphism

$$\varphi : E/b^N.E \rightarrow E'/b^N.E'$$

there exists an unique $\mathcal{A}$--isomorphism $\Phi : E \rightarrow E'$ inducing the given $\varphi$.

Moreover the choice $N(E) = N_0(E) := \text{or}(E) + L(E) + \text{rank}(E) + 1$ is possible.
Remarks.

1. It is easy to see that for a rank 1 regular (a,b)-module the integer 2 is the best possible.

2. In our final lemma 4.3.2 we show that the integer given in the theorem is optimal for the rank \( k \) (a,b)-module \( J_k(\lambda) \), (defined in the lemma), for any \( k \in \mathbb{N}^* \).

3. For the rank 2 (a,b)-modules \( E_{\lambda,\mu} \) the integer given by the theorem is \( or(E) + L(E) + 2 + 1 = 3 \) is again optimal, as it can be shown in the same manner that in our final lemma.

4. For the rank 2 simple pole (a,b)-module \( E_{\lambda}(0) \) the integer given by the theorem is 3 = \( L(E) + rank(E) + 1 \) and the best possible is 2 : the action of \( b^{-1}.a \) on \( E/b.E \) which is determined by \( E/b^2.E \) characterizes this rank 2 regular (a,b)-module in the classification given in proposition 3.1.6.

5. For the (a,b)-module \( E \) associated to an holomorphic germ at the origine of \( \mathbb{C}^{n+1} \) with an isolated singularity we have the uniform bounds \( or(E) \leq n + 1 \) and \( L(E) \leq n \) so the choice \( N(E) = 2n + \mu + 2 \) is always possible, where \( \mu \) is the Milnor number (equal to the rank). \( \square \)

Proof. We shall make an induction on the rank of \( E \). So we shall assume that the result is proved for a rank \( p - 1 \) (a,b)-module and we shall consider a regular (a,b)-module \( E \) of rank \( p \geq 1 \), an (a,b)-module \( E' \), an integer \( N \geq N_0(E) \) and an \( \tilde{A} \)-isomorphism \( \varphi \) as in (1). From 4.1.2 we know that \( E' \) is then regular and has order of regularity \( or(E') = or(E) \).

Choose now a complex number \( \lambda \) which is minimal modulo \( \mathbb{Z} \) such there exists an exact sequence of (a,b)-module (so \( \lambda = \lambda_{\min}(E) \) with the terminology of §1.3)

\[
0 \to E_{\lambda} \overset{\alpha}{\to} E \overset{\beta}{\to} F \to 0. \tag{2}
\]

This exists from theorem 3.1.7. The (a,b)-module \( F \) has rank \( p - 1 \) and from 4.2.4 we have \( N_0(E) \geq N(F^*, -\lambda) \). So we know from 4.2.2 that the extension (2) is determined by the extension

\[
0 \to E_{\lambda}/b^N.E_{\lambda} \overset{\alpha_N}{\to} E/b^N.E \overset{\beta_N}{\to} F/b^N.F \to 0. \tag{2_N}
\]

Now, using the \( \tilde{A} \)-isomorphism \( \varphi \) we obtain an injective \( \tilde{A} \)-linear map

\[
j_N : E_{\lambda}/b^N.E_{\lambda} \hookrightarrow E'/b^N.E'.
\]

Using the proposition 3.1.4 with the fact that \( N \geq or(E') + 2 \) there exists a unique normal inclusion \( j : E_{\lambda} \hookrightarrow E' \) inducing \( j_N \).
Define \( F' := E'/j(E_\lambda) \). Then \( F' \) is a rank \( p - 1 \) \((a,b)\)-module and the exact sequence

\[
0 \rightarrow E_\lambda \xrightarrow{j} E' \rightarrow F' \rightarrow 0
\]

induced the extension \((2_N)\). Using the induction hypothesis, because the inequalities \( or(E) \geq or(F) \) from 4.1.2 and \( L(E) + rank(E) \geq L(F) + rank(F) \) from 3.4.4 implies \( N_0(E) \geq N_0(F) \), we have a unique isomorphism \( \Psi : F \rightarrow F' \) compatible with the one induced by \( \varphi \) between \( F/bN.F \) and \( F'/bN.F' \). Using 4.2.2, 4.2.4 and the inequality \( N_0(E) \geq N_0(F^*, -\lambda) \) we have a unique isomorphism of extensions

\[
0 \rightarrow E_\lambda \xrightarrow{\alpha} E \xrightarrow{\beta} F \xrightarrow{\Phi} 0
\]

concluding the proof. ■

**Lemma 4.3.2** Let \( E := J_k(\lambda) \) the rank \( k \) \((a,b)\)-module defined by the \( \mathbb{C}[[b]] \)-basis \( e_1, \cdots, e_k \) and by the following relations

\[
a.e_j = (\lambda + j - 1).b.e_j + e_j+1 \quad \forall j \in [1, k]
\]

with the convention \( e_{k+1} = 0 \). We have \( \delta(E) = or(E) = k - 1, L(E) = -k + 1 \). The integer \( or(E) + L(E) + rank(E) + 1 = k + 1 \) is the best possible for the theorem.

**Proof.** It is easy to see that the saturation \( E^* \) is generated by \( e_1, b^{-1}.e_2, \cdots, b^{-k+1}.e_k \). This gives the equality \( \delta(E) = or(E) = k - 1 \).

Assume that we have an inclusion \( E_\mu \rightarrow E \) such that \( e_\mu \notin b.E \). Then there exists \( (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \setminus \{0\} \) such that

\[
a.(\sum_{h=1}^{k} \alpha_h.e_h) = \mu.b.(\sum_{h=1}^{k} \alpha_h.e_h) + b^2.E.
\]

Then we obtain

\[
\sum_{h=1}^{k} \alpha_h.((\lambda + j - 1).b.e_h + e_{h+1}) = \sum_{h=1}^{k} \alpha_h.\mu.b.e_h + b^2.E
\]

and so \( \alpha_1 = \cdots = \alpha_{k-1} = 0 \) and we conclude that \( \mu = \lambda + k - 1 \).

An easy computation shows that \( J_k(\lambda)^* = J_k(-\lambda - 2k + 2) \) and so we have \( \lambda_{\text{max}} = \lambda \).

So \( L(E) = -k + 1 \).

Now we shall prove that the integer \( k + 1 \) is the best possible in the theorem 4.3.1 for \( E = J_k(\lambda) \) by giving a regular \((a,b)\)-module \( F \) such that \( F/b^k.E \simeq E/b^k.E \) and not isomorphic to \( E \).
Let consider the rank \( k \) \((a,b)\)-module \( F \) defined by \( \sum_{j=1}^{k} C[[b]].e_j \) with the following relations
\[
a.e_j = (\lambda + j - 1).b.e_j + e_{j+1} \quad \forall j \in [1, k]
\]
\[
a.e_k = (\lambda + k - 1).b.e_k + \sum_{h=1}^{k-1} \alpha_h.b^{k-h+1}.e_h
\]

Then define, for \( \beta_1, \cdots, \beta_{k-1} \in \mathbb{C} \),
\[
\varepsilon := e_k + \sum_{j=1}^{k-1} \beta_j.b^{k-j}.e_j.
\]

We have
\[
a.\varepsilon := (\lambda + k - 1).b.e_k + \sum_{h=1}^{k-1} \alpha_h.b^{k-h+1}.e_h + \sum_{j=1}^{k-1} \beta_j.\left[(\lambda + j - 1).b.e_j + e_{j+1}\right] + (k - j).b^{k-j+1}.e_j
\]
\[
a.\varepsilon := (\lambda + k - 1).b.e_k + \sum_{h=1}^{k-1} \left(\alpha_h + \beta_h.(\lambda + k - 1) + \beta_{h-1}\right).b^{k-h+1}.e_h
\]

Let now choose \( \beta_1, \cdots, \beta_{k-1} \) such that we have
\[
\alpha_h + \beta_h.\left(\lambda + k - 1\right) + \beta_{h-1} = (\lambda + k - 1 + \beta_{k-1}).\beta_h \quad \forall h \in [1, k - 1]
\]

with the convention \( \beta_0 = 0 \). We obtain the system of equations
\[
\alpha_h + \beta_{h-1} = \beta_{k-1}.\beta_h \quad \forall h \in [1, k - 1].
\]

This implies, assuming \( \beta_{k-1} \neq 0 \), that \( \beta_{k-1} \) is solution of the equation
\[
x^k = \alpha_{k-1}.x^{k-2} + \cdots + \alpha_2.x + \alpha_1.
\]

Now choose \( \alpha_2 = \cdots = \alpha_{k-1} = 0 \) and \( \alpha_1 := \rho^k \) with \( \rho \in ]0, 1[ \). Then choose \( \beta_j = \rho^{k-j} \quad \forall j \in [1, k - 1] \). It is clear that the corresponding \( F_\rho \) satisfies \( F/b^k.F \cong E/b^k.E \) as \( a.e_k = e_k + \rho.b^k.e_1 \) in \( F_\rho \). But the relation \( a.\varepsilon = (\lambda + k - 1 + \rho^k).b.\varepsilon \) with \( \varepsilon \neq 0 \) shows that \( F_\rho \) cannot be isomorphic to \( J_k(\lambda) \).
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