RIGIDITY THEOREMS OF THE SPACE-LIKE $\lambda$-HYPERSURFACES
IN THE LORENTZIAN SPACE $\mathbb{R}^{n+1}_1$

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Abstract. In this paper, we study complete space-like $\lambda$-hypersurfaces in the Lorentzian space $\mathbb{R}^{n+1}_1$. As the result, we prove some rigidity theorems for these hypersurfaces including the complete space-like self-shrinkers in $\mathbb{R}^{n+1}_1$.

1. Introduction

For $\varepsilon = \pm 1$, let $\mathbb{E}^{n+1}_{\varepsilon}$ be the Euclidean space $\mathbb{R}^{n+1}$ (when $\varepsilon = 1$) or the Lorentzian space $\mathbb{R}^{n+1}_1$ (when $\varepsilon = -1$). The standard inner product on $\mathbb{E}^{n+1}_{\varepsilon}$ is given by:

$$\langle X, Y \rangle = X_1 Y_1 + X_2 Y_2 + \cdots + \varepsilon X_{n+1} Y_{n+1}.$$ 

Let $M$ be an immersed hypersurface in $\mathbb{E}^{n+1}_{\varepsilon}$ and $x : M^n \to \mathbb{E}^{n+1}_{\varepsilon}$ be the corresponding immersion of $M$. In this paper, we also use $x$ to denote the position vector of $M$. Thus $x$, as well as the unit normal vector $N$ and the mean curvature $H$, are taken as smooth $\mathbb{R}^{n+1}$-valued functions on $M^n$. For a suitably chosen function $s$ on $M^n$, if the position vector $x$ of $M$ satisfies

$$H - s\langle x, N \rangle := \lambda = \text{const}, \quad (1.1)$$

then $M$ is called a $\lambda$-hypersurface with the weight function $s$.

When $s \equiv 0$, the corresponding $\lambda$-hypersurfaces reduce to hypersurfaces with constant mean curvature which have been studied extensively. For example, Calabi considered in [1] the maximum space-like hypersurfaces $M^n$ in the Lorentzian space $\mathbb{R}^{n+1}_1$ and proposed some Bernstein-type problems for a nonlinear equation; For a given complete space-like hypersurface $M$ in $\mathbb{R}^{n+1}_1$, it was proved by Xin ([20]) that if the image of the Gauss map is inside a bounded subdomain of the hyperbolic $n$-space $\mathbb{H}^n$, then $M$ must be a hyperplane. A similar result was also proved earlier in ([18]) with extra assumptions. In [3], Cao, Shen and Zhu further extended the result by showing that if the image of the Gauss map lies inside a horoball of $\mathbb{H}^n$, $M$ is necessarily a hyperplane. Later, Wu ([19]) generalized the above mentioned results and proved a more general Bernstein theorem for complete space-like hypersurfaces in Lorentzian space with constant mean curvature.

If $s$ is chosen to be constant and $\lambda = 0$, $M$ is called a self-shrinker. It is known that self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow ups at a given singularity of the mean curvature flow ([2]). There are also other mathematicians who have been studying the geometries of self-shrinkers and obtained a lot of interesting theorems, including some gap theorems and rigidity theorems for complete self-shrinkers. Details of this can be found in, for example, [2], [6], [9], [10], [11], [12], [15] etc.

According to [13], the concept of $\lambda$-hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ were firstly introduced by M. Mcgonagle and J. Ross with $s = \frac{1}{2}$ ([16]); Q. Guang ([14]) also defined the $\lambda$-hypersurfaces in $\mathbb{R}^{n+1}$.
with \( s = \frac{1}{2} \) and proved a Bernstein-type theorem showing that smooth \( \lambda \)-hypersurfaces which are entire graphs and with a polynomial volume growth are necessarily hyperplanes in \( \mathbb{R}^{n+1} \).

If one takes \( s(x) = -1 \) in (1.1), the corresponding \( \lambda \)-hypersurfaces are exactly what Q.-M. Cheng and G. Wei defined and studied in [7], where the authors have successfully introduced a weighted volume functional and proved that the \( \lambda \)-hypersurfaces in the Euclidean space \( \mathbb{R}^{n+1} \) are nothing but the critical points of the above functional. Later, Cheng, Ogawa and Wei (6, 8) have obtained some rigidity and Bernstein-type theorems for these complete \( \lambda \)-hypersurface. In particular, the following result is proved:

**Theorem 1.1 ([6]).** Let \( x : M^n \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional complete \( \lambda \)-hypersurface with weight \( s = -1 \) and a polynomial area growth. Then, either \( x \) is isometric to one of the following embedded hypersurfaces:

1. the sphere \( S^n(r) \subset \mathbb{R}^{n+1} \) with radius \( r > 0 \);
2. the hyperplane \( \mathbb{R}^n \subset \mathbb{R}^{n+1} \);
3. the cylinder \( S^1(r) \times \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1} \);
4. the cylinder \( S^{n-1}(r) \times \mathbb{R} \subset \mathbb{R}^{n+1} \),

or, there exists some \( p \in M^n \) such that the squared norm \( S \) of the second fundamental form of \( x \) satisfies

\[
\left( \sqrt{S(p) - \frac{H^2(p)}{n}} + |\lambda| \frac{n - 2}{2\sqrt{n(n - 1)}} \right)^2 + \frac{1}{n} (H(p) - \lambda)^2 > 1 + \frac{n\lambda^2}{4(n - 1)}.
\]

In this paper, we consider space-like \( \lambda \)-hypersurfaces \( x : M^n \to \mathbb{R}^{n+1}_1 \) in the Lorentzian space \( \mathbb{R}^{n+1}_1 \). We first extend the definitions of \( \lambda \)-hypersurfaces and self-shrinkers to those in \( \mathbb{R}^{n+1}_1 \), and then generalize the \( \mathcal{L} \)-operator that has been effectively used by many authors (see the operators \( \tilde{\mathcal{L}} \) and \( \mathcal{L} \) defined, respectively in (2.4) and (2.13)). We shall be using these generalized operators to extend Theorem 1.1 to the complete space-like \( \lambda \)-hypersurfaces in \( \mathbb{R}^{n+1}_1 \).

Let \( a \) be a nonzero constant and denote \( \epsilon = \text{Sgn}(a\langle x, x \rangle) \) where \( \langle \cdot, \cdot \rangle \) is the Lorentzian product. We shall study \( \lambda \)-hypersurfaces in \( \mathbb{R}^{n+1}_1 \) either with weight \( s = \epsilon a \) or with weight \( s = \langle x, x \rangle \). For a given hypersurface \( M \), we always use \( S \) to denote the squared norm of the second fundamental form, and use \( A \) and \( I \) to denote the shape operator and the identity map, respectively. Then the rigidity theorems we have proved in this paper are stated as follows:

**Theorem 1.2.** Let \( x : M^n \to \mathbb{R}^{n+1}_1 \) be a complete space-like \( \lambda \)-hypersurface with \( s = \epsilon a \). Suppose that \( \langle x, x \rangle \) does not change sign and

\[
\int_{M^n} \left( \left| \nabla \left( S - \frac{H^2}{n} \right) \right| + \left| \mathcal{L} \left( S - \frac{H^2}{n} \right) \right| \right) e^{-\frac{\epsilon a(x,x)}{2}} dV_{M^n} < +\infty,
\]

where the differential operator \( \mathcal{L} \) is defined by (2.4). Then, either \( x \) is totally umbilical and thus isometric to one of the following two hypersurfaces:

1. the hyperbolic space \( \mathbb{H}^n(c) \subset \mathbb{R}^{n+1}_1 \) with an arbitrary sectional curvature \( c < 0 \);
2. the Euclidean space \( \mathbb{R}^n \subset \mathbb{R}^{n+1}_1 \),

or, there exists some \( p \in M^n \) such that, at \( p \)

\[
\left( \sqrt{S - \frac{H^2}{n}} - |\lambda| \frac{n - 2}{2\sqrt{n(n - 1)}} \right)^2 + \frac{1}{n} (H - \lambda)^2 - \frac{n\lambda^2}{4(n - 1)} + \epsilon a < 0.
\]
Theorem 1.3. Let \( x : M^n \to \mathbb{R}^{n+1}_1 \) be a complete space-like \( \lambda \)-hypersurface with \( s = \langle x, x \rangle \). Suppose that
\[
\int_{M^n} \left( \left| \nabla \left( S - \frac{H^2}{n} \right) \right| + \left| \tilde{\mathcal{L}} \left( S - \frac{H^2}{n} \right) \right| e^{-\frac{(\alpha x)^2}{4}} dV_{M^n} < +\infty, \tag{1.4} \right.
\]
where the differential operator \( \tilde{\mathcal{L}} \) is defined by \( \text{(2.14)} \). Then, either \( x \) is totally umbilical and thus isometric to one of the following two embedded hypersurfaces:

1. the hyperbolic space \( \mathbb{H}^n(c) \subset \mathbb{R}^{n+1}_1 \) with an arbitrary \( c < 0 \);
2. the Euclidean space \( \mathbb{R}^n \subset \mathbb{R}^{n+1}_1 \);

or, there exists some \( p \in M^n \) such that
\[
\left( \sqrt{S(p) - \frac{H^2(p)}{n}} - |\lambda| \frac{n-2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n} (H(p) - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} < 0. \tag{1.6} \]

Corollary 1.5 follows direct from Theorem 1.4.

Proof. Since \( \lambda = 0 \) by the definition of \( \lambda \)-hypersurface, we have
\[
\mathcal{L} \left( S - \frac{H^2}{n} \right) = \int R - \frac{1}{n} (H - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} \geq 0, \tag{1.7} \]
then one of the following two conclusions must hold:

1. \( \lambda \leq \left( \frac{n}{2} \right)^{\frac{2}{n-1}} \), and \( x \) is isometric to the hyperbolic space \( \mathbb{H}^n(-r^{-2}) \subset \mathbb{R}^{n+1}_1 \) with \( r \geq \left( \frac{n}{2} \right)^{\frac{1}{n-1}} \);
2. \( \lambda = 0 \) and \( x \) is isometric to the Euclidean space \( \mathbb{R}^n \subset \mathbb{R}^{n+1}_1 \).

Proof. If the condition \( \text{(1.7)} \) is satisfied for a hypersurface \( \mathbb{H}^n(-r^{-2}) \) with \( r > 0 \), then by the fact that \( x = rN \) we have
\[
\lambda = H - \langle x, x \rangle \langle x, N \rangle = \frac{n}{r} - r^3. \]
If follows that
\[
0 \leq \left( \sqrt{S - \frac{1}{n} H^2} - |\lambda| \frac{n-2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n} (H - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} = \frac{1}{r^2} (H^2 - 2\lambda H) = \frac{1}{r^2} (2r^4 - n). \tag{1.8} \]
Therefore \( r \geq \left( \frac{n}{2} \right)^{\frac{1}{n-1}} \) which implies directly that \( \lambda \leq \left( \frac{n}{2} \right)^{\frac{2}{n-1}} \). As for the Euclidean space \( \mathbb{R}^n \), \( \lambda = 0 \) is direct by the definition of \( \lambda \)-hypersurfaces. \( \square \)

A similar corollary of Theorem 1.2 can also be derived, which is omitted here.

Corollary 1.5. Let \( x : M^n \to \mathbb{R}^{n+1}_1 \) be a complete space-like \( \lambda \)-hypersurface with \( s = \langle x, x \rangle \). Suppose \( S - \frac{H^2}{n} \) is constant. If \( \text{(1.5)} \) and \( \text{(1.7)} \) are satisfied, then \( x \) is isometric to either the hyperbolic space \( \mathbb{H}^n(-r^{-2}) \subset \mathbb{R}^{n+1}_1 \) with \( r \geq \left( \frac{n}{2} \right)^{\frac{1}{n-1}} \) or the hyperplane \( \mathbb{R}^n \subset \mathbb{R}^{n+1}_1 \).

Proof. Since \( S - \frac{H^2}{n} \) is constant, the condition \( \text{(1.6)} \) in Theorem 1.3 is trivially satisfied. Then Corollary 1.5 follows direct from Theorem 1.4.
Remark 1.1. For the special case that \( \lambda = 0 \), that is, for the “self-shrinker” case, the following two conclusions can be easily seen from Theorem 1.4.

Theorem 1.6. Let \( x : M^n \to \mathbb{R}^{n+1} \) be a complete space-like hypersurface self-shrinker with \( s = \langle x, x \rangle \). Suppose that (1.4) and (1.5) are satisfied, then \( x \) is isometric to one of the following two embedded hypersurfaces:

1. the hyperbolic space \( \mathbb{H}^n \left( -\frac{1}{\sqrt{n}} \right) \subset \mathbb{R}^{n+1} \);
2. the Euclidean space \( \mathbb{R}^n \subset \mathbb{R}^{n+1} \).

Proof. When \( \lambda = 0 \), it is clear that (1.7) is trivially satisfied. Furthermore, for a hyperbolic space \( \mathbb{H}^n \left( -\frac{1}{\sqrt{n}} \right) \subset \mathbb{R}^{n+1} \), \( \lambda = 0 \) also implies that \( r^2 = \sqrt{n} \).

Corollary 1.7. Let \( x : M^n \to \mathbb{R}^{n+1} \) be a complete space-like self-shrinker with \( s = \langle x, x \rangle \). If \( S - \frac{H^2}{n} \) is constant and (1.5) is satisfied, then \( x \) is isometric to the either the hyperbolic space \( \mathbb{H}^n \left( -\frac{1}{\sqrt{n}} \right) \subset \mathbb{R}^{n+1} \) or the hyperplane \( \mathbb{R}^n \subset \mathbb{R}^{n+1} \).

Proof. The assumption that \( S - \frac{H^2}{n} \) is constant directly means that (1.4) is trivially satisfied.

2. Preliminaries and necessary lemmas

Firstly we fix the following convention for the ranges of indices:

\[ 1 \leq i, j, k, \cdots \leq n, \quad 1 \leq A, B, C, \cdots \leq n + 1. \]

Let \( x : M^n \to \mathbb{R}^{n+1} \) be a connected space-like hypersurface of the \((n+1)\)-dimensional Lorentzian space \( \mathbb{R}^{n+1} \) and \( \{ e_A \}^{n+1}_{A=1} \) be a local orthonormal frame field of \( \mathbb{R}^{n+1} \) along \( x \) with dual coframe field \( \{ \omega^A \}^{n+1}_{A=1} \) such that, when restricted to \( x \), \( e_1, \ldots, e_n \) are tangent to \( x \) and thus \( N := e_{n+1} \) is the unit normal vector of \( x \). Then with the connection forms \( \omega^B_A \) we have

\[ dx = \sum_i \omega^i e_i, \quad de_i = \sum_j \omega^j e_j + \omega_i^{n+1} e_{n+1}, \quad de_{n+1} = \sum_i \omega^i_{n+1} e_i. \]

By restricting these forms to \( M^n \) and using Cartan’s lemma, we have

\[ \omega^{n+1} = 0, \quad \omega_i^{n+1} = \sum_{j=1}^n h_{ij} \omega^j, \quad h_{ij} = h_{ji}, \]

where \( h_{ij} \) are nothing but the components of the second fundamental form \( h \) of \( x \), that is, \( h = \sum h_{ij} \omega^i \omega^j \).

Then the mean curvature \( H \) of \( x \) is given by \( H = \sum_{j=1}^n h_{jj} \). Denote

\[ h_{ijk} = (\nabla h)_{ijk} = (\nabla_k h)_i, \quad h_{ijkl} = (\nabla^2 h)_{ijkl} = (\nabla_i (\nabla h))_{jkl} \quad (2.1) \]

where \( \nabla \) is the Levi-Civita connection of the induced metric and \( \nabla_i := \nabla e_i \). Then the Gauss equations, Codazzi equations and Ricci identities are given respectively by

\[ R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}, \quad h_{ijk} = h_{jik}, \quad (2.2) \]

\[ h_{ijkl} - h_{ijlk} = \sum_{m=1}^n h_{im} R_{jmkl} + \sum_{m=1}^n h_{mj} R_{imkl}, \quad (2.3) \]

where \( R_{ijkl} \) are the components of the Riemannian curvature tensor. For a function \( F \) defined on \( M \), the covariant derivatives of \( F \) are denoted by

\[ F_{,i} = (\nabla F)_i = f_i F, \quad F_{,ij} = (\nabla^2 F)_{ij} = (\nabla_i (\nabla F))_j, \quad \cdots. \]
Let $\Delta$ be the Laplacian operator of the induced metric on $M^n$. In case that $\langle x, x \rangle$ does not change its sign, we can define

$$\mathcal{L}v = \Delta v - \epsilon a(x, \nabla v), \quad \forall v \in C^2(M^n),$$

(2.4)

where, for any constant $a$, $\epsilon = \text{Sgn}(a(x, x))$. Then $\mathcal{L}$ is an elliptic operator and

$$\mathcal{L}v = e^{-\frac{\epsilon a(x, x)}{2}} \text{div} (e^{-\frac{\epsilon a(x, x)}{2}} \nabla v), \quad \forall v \in C^2(M^n).$$

(2.5)

In fact, for $v \in C^2(M^n)$, we find

$$e^{-\frac{\epsilon a(x, x)}{2}} \text{div} (e^{-\frac{\epsilon a(x, x)}{2}} \nabla v)$$

$$= e^{-\frac{\epsilon a(x, x)}{2}} (\text{div}(\nabla v) + \langle \nabla e^{-\frac{\epsilon a(x, x)}{2}}, \nabla v \rangle)$$

$$= e^{-\frac{\epsilon a(x, x)}{2}} \Delta v + e^{-\frac{\epsilon a(x, x)}{2}} (-\epsilon a(x, x_i) \langle e_i, \nabla v \rangle)$$

$$= \Delta v - \epsilon a(x, \nabla v) = \mathcal{L}v.$$

Lemma 2.1 (cf. [1]). Let $x : M^n \to \mathbb{R}^{n+1}$ be a complete space-like hypersurface for which $\langle x, x \rangle$ does not change its sign. Then, for any $C^1$-function $u$ on $M^n$ with compact support, it holds that

$$\int_{M^n} u(\mathcal{L}v)e^{-\frac{\epsilon a(x, x)}{2}} \, dV_{M^n} = - \int_{M^n} \langle \nabla v, \nabla u \rangle e^{\frac{\epsilon a(x, x)}{2}} \, dV_{M^n}, \quad \forall v \in C^2(M^n).$$

(2.6)

Proof. By (2.5) we find

$$\int_{M^n} u(\mathcal{L}v)e^{-\frac{\epsilon a(x, x)}{2}} \, dV_{M^n} = \int_{M^n} u \left( e^{-\frac{\epsilon a(x, x)}{2}} \text{div} \left( e^{-\frac{\epsilon a(x, x)}{2}} \nabla v \right) \right) e^{-\frac{\epsilon a(x, x)}{2}} \, dV_{M^n}$$

$$= \int_{M^n} \text{div} \left( e^{-\frac{\epsilon a(x, x)}{2}} \nabla v \right) \, dV_{M^n}$$

$$= \int_{M^n} \left( \text{div} (ue^{-\frac{\epsilon a(x, x)}{2}} \nabla v) - \langle \nabla u, e^{-\frac{\epsilon a(x, x)}{2}} \nabla v \rangle \right) \, dV_{M^n}$$

$$= \int_{M^n} \text{div} (ue^{-\frac{\epsilon a(x, x)}{2}} \nabla v) \, dV_{M^n} - \int_{M^n} \langle \nabla u, \nabla v \rangle e^{-\frac{\epsilon a(x, x)}{2}} \, dV_{M^n}.$$

Hence there are two cases to be considered:

Case (1): $M^n$ is compact without boundary. In this case, we can directly use the divergence theorem to get

$$\int_{M^n} \text{div} \left( ue^{-\frac{\epsilon a(x, x)}{2}} \nabla v \right) \, dV_{M^n} = 0.$$

Case (2): $M^n$ is complete and noncompact. In this case, we can find a geodesic ball $B_r(o)$ big enough such that $\text{Supp } u \subset B_r(o)$. It follows that

$$\int_{M^n} \text{div} \left( ue^{-\frac{\epsilon a(x, x)}{2}} \nabla v \right) \, dV_{M^n} = \int_{B_r(o)} \text{div} \left( ue^{-\frac{\epsilon a(x, x)}{2}} \nabla v \right) \, dV_{B_r(o)}$$

$$= - \int_{\partial B_r(o)} \langle N, ue^{-\frac{\epsilon a(x, x)}{2}} \nabla v \rangle \, dV_{\partial B_r(o)} = 0.$$

It follows that

$$\int_{M^n} u(\mathcal{L}v)e^{-\frac{\epsilon a(x, x)}{2}} \, dV_{M^n} = - \int_{M^n} \langle \nabla v, \nabla u \rangle e^{-\frac{\epsilon a(x, x)}{2}} \, dV_{M^n}. \quad \square$$
Corollary 2.2. Let \( x : M^n \to \mathbb{R}^{n+1} \) be a complete space-like hypersurface. If \( u, v \) are \( C^2 \)-functions satisfying
\[
\int_{M^n} (|u\nabla v| + |\nabla u||\nabla v| + |uL_v|) e^{-\frac{\alpha(x,x)}{2}} dV_{M^n} < +\infty,
\]
then we have
\[
\int_{M^n} u(L_v) e^{-\frac{\alpha(x,x)}{2}} dV_{M^n} = - \int_{M^n} \langle \nabla u, \nabla v \rangle e^{-\frac{\alpha(x,x)}{2}} dV_{M^n}.
\]

Proof. Within this proof, we will use square brackets \([\cdot]\) to denote weighted integrals
\[
[f] = \int_{M^n} f e^{-\frac{\alpha(x,x)}{2}} dV_{M^n}.
\]
Given any \( \phi \) that is \( C^1 \)-with compact support, we can apply Lemma 2.4 to \( \phi u \) and \( v \) to get
\[
[\phi u L_v] = -[\phi \langle \nabla v, \nabla u \rangle] - [u \langle \nabla v, \nabla \phi \rangle].
\]
Now we fix one point \( o \in M \) and, for each \( j = 1, 2, \ldots \), let \( B_j \) be the intrinsic ball of radius \( j \) in \( M^n \) centered at \( o \). Define \( \phi_j \) to be one smooth cutting-off function on \( M^n \) that cuts off linearly from one to zero between \( B_j \) and \( B_{j+1} \). Since \( |\phi_j| \) and \( |\nabla \phi_j| \) are bounded by one, \( \phi_j \to 1 \) and \( |\nabla \phi_j| \to 0 \), as \( j \to +\infty \). Then the dominated convergence theorem (which applies because of (2.7)) shows that, as \( j \to +\infty \), we have the following limits:
\[
[\phi_j u L_v] \to [u L_v], \quad [\phi_j \langle \nabla v, \nabla u \rangle] \to [\langle \nabla v, \nabla u \rangle], \quad [u \langle \nabla v, \nabla \phi \rangle] \to 0.
\]
Replacing \( \phi \) in (2.10) with \( \phi_j \), we obtain the corollary.

Next we consider the case that \( s = \langle x, x \rangle \) and define
\[
\tilde{L}_v = \Delta v - \langle x, x \rangle \langle x, \nabla v \rangle, \quad \forall v \in C^2(M^n).
\]
Then, similar to (2.5), we have for all \( v \in C^2(M^n) \),
\[
e^{-\frac{(x,x)^2}{4}} \text{div} \left( e^{-\frac{(x,x)^2}{4}} \nabla v \right) = e^{-\frac{(x,x)^2}{4}} \left( e^{-\frac{(x,x)^2}{4}} \text{div}( \nabla v ) + \left( \nabla e^{-\frac{(x,x)^2}{4}} , \nabla v \right) \right) = e^{-\frac{(x,x)^2}{4}} \left( e^{-\frac{(x,x)^2}{4}} \Delta v + e^{-\frac{(x,x)^2}{4}} \left( - \frac{2(x,x)}{4} \right) \langle x, \nabla v \rangle \right) = \Delta v - \langle x, x \rangle \langle x, \nabla v \rangle = \tilde{L}_v.
\]

Lemma 2.3. If \( x : M^n \to \mathbb{R}^{n+1} \) is a complete space-like hypersurface, \( u \) is a \( C^1 \)-function with compact support, and \( v \) is a \( C^2 \)-function, then
\[
\int_{M^n} u(\tilde{L}_v) e^{-\frac{(x,x)^2}{4}} dV_{M^n} = - \int_{M^n} \langle \nabla v, \nabla u \rangle e^{-\frac{(x,x)^2}{4}} dV_{M^n}.
\]

Proof. Using (2.15) we have
\[
\int_{M^n} u(\tilde{L}_v) e^{-\frac{(x,x)^2}{4}} dV_{M^n} = \int_{M^n} u \left( e^{-\frac{(x,x)^2}{4}} \text{div} \left( e^{-\frac{(x,x)^2}{4}} \nabla v \right) \right) e^{-\frac{(x,x)^2}{4}} dV_{M^n} = \int_{M^n} u \text{div} \left( e^{-\frac{(x,x)^2}{4}} \nabla v \right) dV_{M^n} = \int_{M^n} \text{div} \left( u e^{-\frac{(x,x)^2}{4}} \nabla v \right) dV_{M^n} - \int_{M^n} \langle \nabla u, e^{-\frac{(x,x)^2}{4}} \nabla v \rangle dV_{M^n} = \int_{M^n} \text{div} \left( u e^{-\frac{(x,x)^2}{4}} \nabla v \right) dV_{M^n} - \int_{M^n} \langle \nabla u, \nabla v \rangle e^{-\frac{(x,x)^2}{4}} dV_{M^n}.
\]
(1) If \( M^n \) is compact without boundary, then by the divergence theorem,
\[
\int_{M^n} \text{div} \left( u e^{-\frac{(x,a)^2}{4}} \nabla v \right) dV_{M^n} = 0.
\]

(2) If \( M^n \) is complete and noncompact, then there exists some geodesic ball \( B_r(o) \) big enough such that \( \text{Supp} u \subset B_r(o) \). It follows that
\[
\int_{M^n} \text{div} \left( u e^{-\frac{(x,a)^2}{4}} \nabla v \right) dV_{M^n} = -\int_{\partial B_r(o)} \left\langle N, u e^{-\frac{(x,a)^2}{4}} \nabla v \right\rangle dV_{\partial B_r(o)} = 0.
\]
Therefore
\[
\int_{M^n} u(\tilde{L} v) e^{-\frac{(x,a)^2}{4}} dV_{M^n} = -\int_{M^n} \langle \nabla v, \nabla u \rangle e^{-\frac{(x,a)^2}{4}} dV_{M^n}.
\]  

Corollary 2.4. Let \( x : M^n \to \mathbb{R}^{1+n} \) be a complete space-like hypersurface. If \( u, v \) are \( C^2 \)-functions satisfying
\[
\int_{M^n} (|u\nabla v| + |\nabla u||\nabla v| + |u\tilde{L} v|) e^{-\frac{(x,a)^2}{4}} dV_{M^n} < +\infty,
\]
then we have
\[
\int_{M^n} u(\tilde{L} v) e^{-\frac{(x,a)^2}{4}} dV_{M^n} = -\int_{M^n} \langle \nabla v, \nabla u \rangle e^{-\frac{(x,a)^2}{4}} dV_{M^n}.
\]

Proof. The proof here is same as that of Corollary 2.2 and is omitted.

The following lemma is also needed in this paper:

Lemma 2.5 ([17]). Let \( \mu_1, \ldots, \mu_n \) be real numbers satisfying
\[
\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = \beta^2,
\]
with \( \beta \) a nonnegative constant. Then
\[
-\frac{n-2}{\sqrt{n(n-1)}} \beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}} \beta^3
\]
with either equality holds if and only if \( (n-1) \) of \( \mu_i \) are equal to each other.

3. Proof of the main theorems

In this section, we give the proofs of our main theorems.

(1) Proof of Theorem 1.2

Since \( H - \alpha \langle x, N \rangle = \lambda \), we have
\[
H_{i} = (\lambda + \alpha \langle x, N \rangle)_i = \alpha \langle x, N \rangle_i = \sum_k e a h_{ik} \langle x, e_k \rangle,
\]
\[
H_{ij} = \sum_k e a h_{ik} \langle x, e_k \rangle + \sum_k e a h_{ik} \langle X_j, e_k \rangle + \sum_k e a h_{ik} \langle x, e_k,j \rangle
= \sum_k e a h_{ik} \langle x, e_k \rangle + e a h_{ij} + \sum_k e a h_{ik} h_{kj} \langle x, N \rangle
= \sum_k e a h_{ik} \langle x, e_k \rangle + e a h_{ij} + \sum_k h_{ik} h_{kj} (H - \lambda).
\]
Using the Codazzi equation in (2.2) we infer
\[ \Delta H = \sum_i H_{ii} = \epsilon a \langle x, \nabla H \rangle + \epsilon a H + S(H - \lambda), \]
where \( S = \sum_{i,k} h_{ik}^2 \). It then follows that
\[ L H = \Delta H - \epsilon a \langle x, \nabla H \rangle = \epsilon a H + S(H - \lambda), \]
implying that
\[ \frac{1}{2} L H^2 = \frac{1}{2} (\Delta H^2 - \epsilon a \langle x, \nabla H^2 \rangle) \]
\[ = \frac{1}{2} \left( \sum_i (H^2)_{ii} - \epsilon a \langle x, \nabla H^2 \rangle \right) \]
\[ = \frac{1}{2} \left( 2|\nabla H|^2 + 2H \Delta H - 2\epsilon a H \langle x, \nabla H \rangle \right) \]
\[ = |\nabla H|^2 + H (\Delta H - \epsilon a \langle x, \nabla H \rangle) \]
\[ = |\nabla H|^2 + \epsilon a H^2 + S H (H - \lambda). \] (3.1)

By making use of the Ricci identities and the Gauss-Codazzi equations, we have
\[ L h_{ij} = \Delta h_{ij} - \epsilon a \langle x, \nabla h_{ij} \rangle = \sum_k h_{ki,jk} - \epsilon a \langle x, \nabla h_{ij} \rangle \]
\[ = \sum_k h_{ki,k} + \sum_{m,k} h_{mi} R_{kj}^m + \sum_{k,m} h_{km} R_{ij}^m - \epsilon a \langle x, \nabla h_{ij} \rangle \]
\[ = \sum_k h_{kk,j} + \sum_{m,k} h_{mi} R_{km}^j + \sum_{k,m} h_{km} R_{im}^j - \epsilon a \langle x, \nabla h_{ij} \rangle \]
\[ = H_{ij} - H \sum_m h_{im} h_{mj} + S h_{ij} - \epsilon a \langle x, \nabla h_{ij} \rangle \]
\[ = (\epsilon a + S) h_{ij} - \lambda \sum_k h_{ik} h_{kj}. \]

Therefore, it holds that
\[ \frac{1}{2} L S = \frac{1}{2} \left( \Delta \sum_{i,j} (h_{ij})^2 - \sum_k \epsilon a \langle x, e_k \rangle \left( \sum_{i,j} (h_{ij})^2 \right)_{ik} \right) \]
\[ = \sum_{i,j,k} h_{ij,k}^2 + (\epsilon a + S) S - \lambda \sum_{i,j,k} h_{ik} h_{kj} h_{ij} \]
\[ = \sum_{i,j,k} h_{ij,k}^2 + (\epsilon a + S) S - \lambda f_3, \] (3.2)

where
\[ f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}. \]

Let \( \lambda_i \) be the principal curvatures of \( x \) and denote
\[ \mu_i = \lambda_i - \frac{H}{n}, \quad 1 \leq i \leq n. \]

For any point \( p \in M^n \), suitably choosing \( \{e_1, e_2, \ldots, e_n\} \) around \( p \) such that \( h_{ij}(p) = \lambda_i(p) \delta_{ij} \). Then, at the given point \( p \),
\[ f_3 = \sum_i \lambda_i^3 = \sum_i \left( \mu_i + \frac{H}{n} \right)^3 = B_3 + \frac{3}{n} H B + \frac{1}{n^2} H^3, \]
where
where

\[ B = \sum_{i} \mu_{i}^{2} = S - \frac{H}{n}, \quad B_{3} = \sum_{i} \mu_{i}^{3}. \]

By a direct computation with (3.1) and (3.2), we have

\[
\frac{1}{2} \mathcal{L}B = \frac{1}{2} \mathcal{L}S - \frac{1}{n} \left( \frac{1}{2} \mathcal{L}H^{2} \right)
= \sum_{i,j,k} h_{ijk}^{2} + (\epsilon a + S)S - \lambda f_{3} - \frac{1}{n}(|\nabla H|^{2} + \epsilon a H^{2} + SH(H - \lambda))
= \sum_{i,j,k} h_{ijk}^{2} - \frac{1}{n}(|\nabla H|^{2} + (\epsilon a + S)S - \lambda f_{3} - \frac{1}{n} \epsilon a H^{2} - S(H - \lambda) \frac{H}{n})
= \sum_{i,j,k} h_{ijk}^{2} - \frac{1}{n}(|\nabla H|^{2} + (\epsilon a + B)B + \frac{H^{2} B}{n} - \lambda B_{3} - \frac{2}{n} \lambda H B.
\]

Since

\[ \sum_{i} \mu_{i} = 0, \quad \sum_{i} \mu_{i}^{2} = B, \]

we have by Lemma 2.5

\[ |B_{3}| \leq \frac{n - 2}{\sqrt{n(n - 1)}} B^{\frac{2}{n}}, \]

where the equality holds if and only if at least \( n - 1 \) of \( \mu_{i} \)'s are equal. Consequently,

\[
\frac{1}{2} \mathcal{L}B \geq \sum_{i,j,k} h_{ijk}^{2} - \frac{1}{n} |\nabla H|^{2} + (\epsilon a + B)B + \frac{1}{n} H^{2} B - |\lambda| \frac{n - 2}{\sqrt{n(n - 1)}} B^{\frac{2}{n}} - \frac{2}{n} \lambda H B
= \sum_{i,j,k} h_{ijk}^{2} - \frac{1}{n} |\nabla H|^{2} - B \left( (B + \epsilon a) + \frac{1}{n} H^{2} - |\lambda| \frac{n - 2}{\sqrt{n(n - 1)}} B^{\frac{2}{n}} - \frac{2}{n} \lambda H \right)
= \sum_{i,j,k} h_{ijk}^{2} - \frac{1}{n} |\nabla H|^{2} + B \left( \sqrt{B} - |\lambda| \frac{n - 2}{2 \sqrt{n(n - 1)}} \right)^{2} + \frac{1}{n} (H - \lambda)^{2} + \epsilon a - \frac{n \lambda^{2}}{4(n - 1)}.
\]

Because of (1.2), we can apply Corollary 2.2 to functions 1 and \( B = S - \frac{H}{n} \) to obtain

\[
0 \geq \int_{M^{n}} \left( \sum_{i,j,k} h_{ijk}^{2} - \frac{1}{n} |\nabla H|^{2} \right) e^{-\frac{\|_{e} \epsilon a \|}{2}} dV_{M^{n}} + \int_{M^{n}} B \left( \left( \sqrt{S - \frac{H^{2}}{n}} - |\lambda| \frac{n - 2}{2 \sqrt{n(n - 1)}} \right)^{2} + \frac{1}{n} (H - \lambda)^{2} - \frac{n \lambda^{2}}{4(n - 1)} + \epsilon a \right) e^{-\frac{\|_{e} \epsilon a \|}{2}} dV_{M^{n}}. \quad (3.3)
\]

On the other hand, by use of the Codazzi equations and the Schwarz inequality, we find

\[
\sum_{i,j,k} h_{ijk}^{2} = 3 \sum_{i \neq k} h_{iik}^{2} + \sum_{i} h_{i}^{2} + \sum_{i \neq j \neq k \neq i} h_{ijk}^{2}, \quad \frac{1}{n} |\nabla H|^{2} \leq \sum_{i,k} h_{iik}^{2}.
\]

So that

\[
\sum_{i,j,k} h_{ijk}^{2} - \frac{1}{n} |\nabla H|^{2} \geq 2 \sum_{i \neq k} h_{iik}^{2} + \sum_{i \neq j \neq k \neq i} h_{ijk}^{2} \geq 0, \quad (3.4)
\]

in which the equalities hold if and only if \( h_{iik} = 0 \) for any \( i, j, k \).
If $B \not\equiv 0$ and, for all $p \in M^n$, (1.3) does not hold, that is

$$\left(\sqrt{S - H^2/n} - |\lambda| \frac{n - 2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} + \epsilon a \geq 0$$

everywhere on $M^n$, then the right hand side of (3.3) is nonnegative. It then follows that

$$\sum_{i,j,k} h^2_{ijk} - \frac{1}{n} |\nabla H|^2 \equiv 0,$$

(3.5)

and

$$\left(\sqrt{S - H^2/n} - |\lambda| \frac{n - 2}{2\sqrt{n(n-1)}}\right)^2 + \frac{1}{n}(H - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} + \epsilon a \equiv 0$$

(3.6)
on where $B \not\equiv 0$. By (3.4) and (3.5), the second fundamental form $h$ of $x$ is parallel. In particular, $x$ is isoparametric and thus both $B$ and $H$ are constant. Since $B \not\equiv 0$, the equality (3.6) shows that $x$ is a complete isoparametric space-like hypersurface in $\mathbb{R}^{n+1}_1$ of exactly two distinct principal curvatures one of which is simple. It then follows by [14] and $B \not\equiv 0$ that $x$ is isometric to one of the product spaces $\mathbb{H}^{n-1}(c) \times \mathbb{R}^1$ or $\mathbb{H}^1(c) \times \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}_1$. But it is clear that, for both of these two product spaces, the function $\langle x, x \rangle$ does change its sign, contradicting the assumption. This contradiction proves that either $B \equiv 0$, namely, $x$ is totally umbilical and isometric to either of the hyperbolic $n$-space $\mathbb{H}^n(c) \subset \mathbb{R}^{n+1}_1$ and the Euclidean $n$-space $\mathbb{R}^n \subset \mathbb{R}^{n+1}_1$, or there exists some $p \in M^n$ such that (1.3) holds.

The proof of Theorem 1.2 is thus finished.

(2) Proof of Theorem 1.3:

Since the idea and method here are the same as those in the proof of Theorem 1.2, we omit the computation detail.

First, by $H - \langle x, x \rangle \langle x, N \rangle = \lambda$, we have

$$H_{,i} = 2\langle x, e_i \rangle \langle x, N \rangle + \langle x, x \rangle \sum_k h_{ik} \langle x, e_k \rangle,$$

$$H_{,ij} = 2\delta_{ij} \langle x, N \rangle + 2h_{ij} \langle x, N \rangle^2 + 2 \sum_k h_{jk} \langle x, e_i \rangle \langle x, e_k \rangle$$
$$+ 2 \sum_k h_{ik} \langle x, e_k \rangle \langle x, e_j \rangle + \sum_k h_{ikj} \langle x, x \rangle \langle x, e_k \rangle$$
$$+ \langle x, x \rangle h_{ij} + \sum_k h_{ik} h_{kj} (H - \lambda).$$

Then by using the Codazzi equation in (2.2), we find

$$\triangle H = 2n \langle x, N \rangle + 2H \langle x, N \rangle^2 + 4 \sum_{i,k} h_{ik} \langle x, e_i \rangle \langle x, e_k \rangle$$
$$+ \sum_i H_{,i} \langle x, x \rangle \langle x, e_i \rangle + H \langle x, x \rangle + S(H - \lambda).$$

Secondly, by the definition of $\tilde{L}$, we find

$$\tilde{L}H = \triangle H - \langle x, x \rangle \langle x, \nabla H \rangle$$
$$= 2n \langle x, N \rangle + 2H \langle x, N \rangle^2 + 4 \sum_{i,k} h_{ik} \langle X, e_i \rangle \langle x, e_k \rangle + H \langle x, x \rangle + S(H - \lambda),$$
implying
\[
\frac{1}{2} \hat{\mathcal{L}} H^2 = \frac{1}{2} \left( \triangle H^2 - \langle x, x \rangle \langle x, \nabla H^2 \rangle \right) = |\nabla H|^2 + 2nH \langle x, N \rangle + 2H^2 \langle x, N \rangle^2 + 4H \sum_{i,k} h_{ik} \langle x, e_i \rangle \langle x, e_k \rangle + H^2 \langle x, x \rangle + SH(H - \lambda). \tag{3.7}
\]

On the other hand
\[
\hat{\mathcal{L}} h_{ij} = \triangle h_{ij} - \langle x, x \rangle \langle x, \nabla h_{ij} \rangle 
= H_{ij} + \sum_{k,m} h_{mi} R_{kmjk} + \sum_{k,m} h_{km} R_{imjk} - \langle x, x \rangle \langle x, \nabla h_{ij} \rangle 
= H_{ij} + 2H_{ij} - H \sum_{m} h_{mi} h_{mj} - \langle x, x \rangle \langle x, \nabla h_{ij} \rangle 
= 2\delta_{ij} \langle x, N \rangle + 2h_{ij} \langle x, N \rangle^2 + 2 \sum_{k} h_{jk} \langle x, e_i \rangle \langle x, e_k \rangle 
+ \sum_{k} h_{ik} h_{ij} (H - \lambda) + Sh_{ij} - H \sum_{m} h_{mi} h_{mj} - \langle x, x \rangle \langle x, \nabla h_{ij} \rangle 
= 2\delta_{ij} \langle x, N \rangle + 2h_{ij} \langle x, N \rangle^2 + 2 \sum_{k} h_{jk} \langle x, e_i \rangle \langle x, e_k \rangle 
+ 2 \sum_{k} h_{ik} \langle x, e_i \rangle \langle x, e_j \rangle + \langle x, x \rangle h_{ij} + Sh_{ij} - \lambda \sum_{k} h_{ik} h_{kj}.
\]

It follows that
\[
\frac{1}{2} \hat{\mathcal{L}} S = \frac{1}{2} \left( \triangle \sum_{i,j} (h_{ij})^2 - \langle x, x \rangle \langle x, \nabla \left( \sum_{i,j} (h_{ij})^2 \right) \rangle \right) 
= \sum_{i,j,k} h_{ik}^2 + 2H \langle x, N \rangle + 2S \langle x, N \rangle^2 + 4 \sum_{i,j,k} h_{ij} h_{jk} \langle x, e_i \rangle \langle x, e_k \rangle 
+ \langle x, x \rangle S + S^2 - \lambda f_3, \tag{3.8}
\]

where again \( f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki} \).

Denote by \( x^\top = \langle x, e_i \rangle e_i \) be the tangential part of the position vector \( x \). Then, as in the proof of Theorem 1.2 we can choose a suitable frame field \( \{ e_1, e_2, \ldots, e_n \} \) making diagonal the second fundamental form \( h_{ij} \) around each point \( p \in M^n \), and perform a direct computation using (1.5), (3.7) and (3.8) to obtain
\[
\frac{1}{2} \hat{\mathcal{L}} B = \frac{1}{2} \hat{\mathcal{L}} S - \frac{1}{n} \left( \frac{1}{2} \hat{\mathcal{L}} H^2 \right) 
= \sum_{i,j,k} h_{ik}^2 + B \langle x, N \rangle^2 - \frac{1}{n} |\nabla H|^2 + x^\top \left( 4A^2 + BI - \frac{4HA}{n} \right) (x^\top)^t + B^2 + \frac{H^2 B}{n} - \lambda B_3 - \frac{2\lambda HB}{n} 
\geq \sum_{i,j,k} h_{ik}^2 - \frac{1}{n} |\nabla H|^2 + B^2 + \frac{H^2 B}{n} - \lambda B_3 - \frac{2\lambda HB}{n},
\]

where the assumption (1.5) has been used. Once again we use Lemma 2.5 to get
\[
|B_3| \leq \frac{n - 2}{\sqrt{n(n - 1)}} B^2,
\]
where the equality holds if and only if at least \( n - 1 \) of \( \mu_i \) are equal. It then follows that

\[
\frac{1}{2} \mathcal{E} B \geq \sum_{i,j,k} h_{ij}^2 - \frac{1}{n} |\nabla H|^2 + B^2 + \frac{H^2 B}{n} - |\lambda| \frac{n-2}{\sqrt{n(n-1)}} B^\frac{n}{2} - \frac{2}{n} \lambda HB
\]

\[
= \sum_{i,j,k} h_{ij}^2 - \frac{1}{n} |\nabla H|^2 + B \left( B + \frac{H^2}{n} - |\lambda| \frac{n-2}{\sqrt{n(n-1)}} B^\frac{n}{2} - \frac{2}{n} \lambda H \right)
\]

\[
= \sum_{i,j,k} h_{ij}^2 - \frac{1}{n} |\nabla H|^2 + B \left( \sqrt{B} - |\lambda| \frac{n-2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n} (H - \lambda)^2 - \frac{n\lambda^2}{4(n-1)}.
\]

(3.10)

Because of \([1,2]\), we can apply the Corollary 2.4 to functions 1 and \( B = S - \frac{H^2}{n} \) to find

\[
0 \geq \int_{M^n} \left( \sum_{i,j,k} h_{ij}^2 - \frac{1}{n} |\nabla H|^2 \right) e^{-\frac{(\omega_2^2)^2}{4}} dV_{M^n}
\]

\[
+ \int_{M^n} B \left( \sqrt{B} - |\lambda| \frac{n-2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n} (H - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} \right) e^{-\frac{(\omega_2^2)^2}{4}} dV_{M^n}.
\]

(3.11)

If \( B \neq 0 \) and, for all \( p \in M^n \), \([1,6]\) does not hold, that is

\[
\left( \sqrt{S(p)} - \frac{H^2(p)}{n} - |\lambda| \frac{n-2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n} (H(p) - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} \geq 0.
\]

everywhere on \( M^n \), then the right hand side of \((3.10)\) is nonnegative. It then follows that

\[
\sum_{i,j,k} h_{ij}^2 - \frac{1}{n} |\nabla H|^2 \equiv 0,
\]

(3.12)

and at points where \( B \neq 0 \)

\[
\left( \sqrt{S(p)} - \frac{H^2(p)}{n} - |\lambda| \frac{n-2}{2\sqrt{n(n-1)}} \right)^2 + \frac{1}{n} (H(p) - \lambda)^2 - \frac{n\lambda^2}{4(n-1)} \equiv 0.
\]

(3.13)

By \((3.12)\) and \((3.13)\), the second fundamental form \( h \) of \( x \) is parallel. In particular, \( x \) is isoparametric and thus both \( B \) and \( H \) are constant. Since \( B \neq 0 \), the equality \((3.12)\) shows that \( x \) is a complete isoparametric space-like hypersurface in \( \mathbb{R}^{|n+1|} \) of exactly two distinct principal curvatures one of which is simple. It then follows by \([12]\) and \( B \neq 0 \) that \( x \) is isometric to one of the product spaces \( \mathbb{H}^{n-1}(c) \times \mathbb{R}^1 \subset \mathbb{R}^{n+1} \) and \( \mathbb{H}^1(c) \times \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1} \). But it is clear that, for both of these two product spaces, the function \( \langle x, x \rangle \) is not a constant so that both \( \mathbb{H}^{n-1}(c) \times \mathbb{R}^1 \subset \mathbb{R}^{n+1} \) and \( \mathbb{H}^1(c) \times \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1} \) could not be \( \lambda \)-hypersurfaces with \( s = \langle x, x \rangle \). This contradiction proves that either \( B \equiv 0 \), namely, \( x \) is totally umbilical and isometric to either of the hyperbolic \( n \)-space \( \mathbb{H}^n(c) \subset \mathbb{R}^{n+1} \) and the Euclidean \( n \)-space \( \mathbb{R}^n \subset \mathbb{R}^{n+1} \), or there exists some \( p \in M^n \) such that \([1,6]\) holds.

The proof of Theorem \([1,3]\) is thus finished. \( \Box \)

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