Remark on the coherent information saturating its upper bound

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Abstract

Coherent information is a useful concept in quantum information theory. It connects with other notions in data processing. In this short remark, we discuss the coherent information saturating its upper bound. A necessary and sufficient condition for this saturation is derived.

1 Coherent information inequality

The fundamental problem in quantum error correction is to determine when the effect of a quantum channel (trace-preserving completely positive map) \( \Phi \in \mathcal{T}(\mathcal{H}_B) \) acting on half of a pure entangled state can be perfectly reversed. Define the coherent information

\[
I_c(\rho, \Phi) \overset{\text{def}}{=} S(\Phi(\rho)) - S(\mathbb{1}_A \otimes \Phi(|u_\rho\rangle\langle u_\rho|)),
\]

where \(|u_\rho\rangle = \sum_j \sqrt{\lambda_j} |x_j\rangle \otimes |\lambda_j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B\) is any purification of \( \rho = \sum_j \lambda_j |\lambda_j\rangle \langle \lambda_j| \).

In general, we have

\[
I_c(\rho, \Phi) \leq S(\rho).
\]

It was shown that there exists a quantum channel \( \Psi \) (see [1]) such that

\[
I_c(\rho, \Phi) = S(\rho) \iff (\mathbb{1}_A \otimes \Psi \circ \Phi)(|u_\rho\rangle\langle u_\rho|) = |u_\rho\rangle\langle u_\rho|.
\]

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By the Stinespring dilation theorem, we may assume that

\[ \Phi(\rho) = \text{Tr}_C(U(\rho \otimes |e\rangle\langle e|)U^\dagger), \quad U \in U(\mathcal{H}_B \otimes \mathcal{H}_C), |e\rangle \in \mathcal{H}_C, \]

which indicates that

\[ 1_A \otimes \Phi(|u_\rho\rangle\langle u_\rho|) = \text{Tr}_C((1_A \otimes U)(|u_\rho\rangle\langle u_\rho| \otimes |e\rangle\langle e|)(1_A \otimes U)^\dagger) = \text{Tr}_C(|\Omega\rangle\langle \Omega|), \quad (1.4) \]

where \(|\Omega\rangle = (1_A \otimes U)(|u_\rho\rangle \otimes |e\rangle)\). Now

\[ |\Omega\rangle\langle \Omega| = (1_A \otimes U)((|u_\rho\rangle\langle u_\rho| \otimes |e\rangle\langle e|)(1_A \otimes U)^\dagger) \]

is a tripartite state in \(D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)\), it follows that

\[ \text{Tr}_C(|\Omega\rangle\langle \Omega|) = 1_A \otimes \Phi(|u_\rho\rangle\langle u_\rho|) \equiv \Omega_{AB}, \]
\[ \text{Tr}_A(|\Omega\rangle\langle \Omega|) = U(\rho \otimes |e\rangle\langle e|)U^\dagger \equiv \Omega_{BC}, \]
\[ \text{Tr}_{AC}(|\Omega\rangle\langle \Omega|) = \Phi(\rho) \equiv \Omega_B, \]

where \(\Omega_{ABC} \equiv |\Omega\rangle\langle \Omega|\). From the above expressions, it is obtained that

\[ S(\Omega_{ABC}) = 0, \]
\[ S(\Omega_B) = S(\Phi(\rho)) \]
\[ S(\Omega_{BC}) = S(\rho), \]
\[ S(\Omega_{AB}) = S((1_A \otimes \Phi)(|u_\rho\rangle\langle u_\rho|)) \]

Apparently, \(I_c(\rho, \Phi) = S(\rho) \iff S(\Phi(\rho)) = S((1_A \otimes \Phi)(|u_\rho\rangle\langle u_\rho|)) + S(\rho)\), that is,

\[ I_c(\rho, \Phi) = S(\rho) \iff S(\Omega_B) = S(\Omega_{AB}) + S(\Omega_{BC}) \iff S(\Omega_B) - S(\Omega_C) = S(\Omega_{BC}). \]

It follows from Proposition 2.2 in Appendix that this equation holds if and only if

(i) \(\mathcal{H}_B\) can be factorized into the form \(\mathcal{H}_B = \mathcal{H}_L \otimes \mathcal{H}_R\),

(ii) \(\Omega_{BC} = \rho_L \otimes |\psi\rangle\langle \psi|_{RC}\) for \(|\psi\rangle_{RC} \in \mathcal{H}_R \otimes \mathcal{H}_C\).

Hence

\[ U(\rho \otimes |e\rangle\langle e|)U^\dagger = \rho_L \otimes |\psi\rangle\langle \psi|_{RC} \implies \rho \otimes |e\rangle\langle e| = U^\dagger(\rho_L \otimes |\psi\rangle\langle \psi|_{RC})U. \]
Clearly, $\Omega_{ABC} = |\phi\rangle\langle\phi|_{AL} \otimes |\psi\rangle\langle\psi|_{RC}$. Thus

$$|u_\rho\rangle\langle u_\rho| = (1_A \otimes \Psi \circ \Phi)(|u_\rho\rangle\langle u_\rho|) = (1_A \otimes \Psi)(\Omega_{AB}) = (1_A \otimes \Psi)(|\phi\rangle\langle\phi|_{AL} \otimes \rho_R).$$

Since $|\Omega\rangle\langle\Omega| = (1_A \otimes U)(|u_\rho\rangle\langle u_\rho| \otimes |e\rangle\langle e|)(1_A \otimes U)^{\dagger}$, it follows that

$$|u_\rho\rangle\langle u_\rho| = Tr_C \left((1_A \otimes U)^{\dagger}|\Omega\rangle\langle\Omega| (1_A \otimes U)\right) = Tr_C \left((1_A \otimes U)^{\dagger}(|\phi\rangle\langle\phi|_{AL} \otimes |\psi\rangle\langle\psi|_{RC})(1_A \otimes U)\right).$$

The above equation gives that

$$(1_A \otimes \Psi)(|\phi\rangle\langle\phi|_{AL} \otimes \rho_R) = Tr_C \left((1_A \otimes U)^{\dagger}(|\phi\rangle\langle\phi|_{AL} \otimes |\psi\rangle\langle\psi|_{RC})(1_A \otimes U)\right).$$

Given the state $\Omega_{AB} = 1_A \otimes \Phi(|u_\rho\rangle\langle u_\rho|)$, the recovery procedure $\Psi$ is:

(i) preparing the state $|\psi\rangle_{RC}$ on $\mathcal{H}_R \otimes \mathcal{H}_C$; thus we have a state $|\phi\rangle\langle\phi|_{AL} \otimes |\psi\rangle\langle\psi|_{RC}$.

(ii) next performing $U^{\dagger}$; we get

$$(1_A \otimes U)^{\dagger}(|\phi\rangle\langle\phi|_{AL} \otimes |\psi\rangle\langle\psi|_{RC}) (1_A \otimes U).$$

(iii) finally discarding the fixed ancillary state $|e\rangle\langle e|$;

$$Tr_C \left((1_A \otimes U)^{\dagger}(|\phi\rangle\langle\phi|_{AL} \otimes |\psi\rangle\langle\psi|_{RC})(1_A \otimes U)\right).$$

Note that $1_A \otimes \Phi(|u_\rho\rangle\langle u_\rho|) = |\phi\rangle\langle\phi|_{AL} \otimes \rho_R$ implies that

$$\Phi(\rho) = \rho_L \otimes \rho_R.$$

This indicates that the coherent information reaches its maximal value if and only if the output state of the quantum channel $\Phi$ is a product state. Therefore we have the following theorem:

**Theorem 1.1.** Let $\rho \in D(\mathcal{H})$ and $\Phi \in T(\mathcal{H})$ be a quantum channel. The coherent information achieves its maximum, that is, $I_c(\rho, \Phi) = S(\rho)$ if and only if the following statements holds:

(i) the underlying Hilbert space can be decomposed as: $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$;

(ii) the output state of the quantum channel $\Phi$ is of a product form: $\Phi(\rho) = \rho_L \otimes \rho_R$ for $\rho_L \in D(\mathcal{H}_L), \rho_R \in D(\mathcal{H}_R)$. 

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Remark 1.2. Consider a Kraus representation of a quantum channel $\Phi \in \mathcal{T}(\mathcal{H})$ in its canonical Kraus form: $\Phi = \sum_k \text{Ad}_{M_k}$. For any $\rho \in \mathcal{D}(\mathcal{H})$, define

$$\hat{\Phi}(\rho) \equiv \sum_{i,j} \text{Tr} \left( M_i \rho M_j^\dagger \right) |i\rangle \langle j|.$$  

If $\rho$ is purified as $|u_\rho\rangle \in \mathcal{H} \otimes \mathcal{K}$ with $\dim(\mathcal{K}) \geq \dim(\mathcal{H})$, then

$$S(\hat{\Phi}(\rho)) = S \left( (\Phi \otimes 1_{L(\mathcal{K})})(|u_\rho\rangle \langle u_\rho|) \right).$$

Indeed, let $\rho = \sum_k \lambda_k |\lambda_k\rangle \langle \lambda_k|$ be its spectral decomposition,

$$|u_\rho\rangle \equiv \sum_k \sqrt{\lambda_k} |\lambda_k\rangle \otimes |\lambda_k\rangle,$$

$$|u_\rho\rangle \langle u_\rho| = \sum_{m,n} \sqrt{\lambda_m \lambda_n} |\lambda_m\rangle \langle \lambda_n| \otimes |\lambda_n\rangle \langle \lambda_m|,$$

$$|\Omega\rangle \equiv \sum_{k,i} \sqrt{\lambda_k} M_i |\lambda_k\rangle \otimes |\lambda_k\rangle \otimes |i\rangle.$$  

Thus

$$|\Omega\rangle \langle \Omega| = \sum_{m,n,i,j} \sqrt{\lambda_m \lambda_n} M_i |\lambda_m\rangle \langle \lambda_n| M_j^\dagger \otimes |\lambda_m\rangle \langle \lambda_n| \otimes |i\rangle \langle j|,$$

which implies that

$$\text{Tr}_3 \left( |\Omega\rangle \langle \Omega| \right) = \sum_{m,n,i,j} \sqrt{\lambda_m \lambda_n} M_i |\lambda_m\rangle \langle \lambda_n| M_j^\dagger \otimes |\lambda_m\rangle \langle \lambda_n|$$

$$= \Phi \otimes 1_{L(\mathcal{K})}(|u_\rho\rangle \langle u_\rho|),$$

$$\text{Tr}_{1,2} \left( |\Omega\rangle \langle \Omega| \right) = \sum_{i,j} \text{Tr}(M_i \rho M_j^\dagger) |i\rangle \langle j| = \hat{\Phi}(\rho).$$

Clearly, $S \left( (\Phi \otimes 1_{L(\mathcal{K})})(|u_\rho\rangle \langle u_\rho|) \right)$ is independent of an arbitrary purification $|u_\rho\rangle$ of $\rho$. In fact, if $|u_\rho^{(1)}\rangle$ and $|u_\rho^{(2)}\rangle$ are any two purification of $\rho$, then by Schimdt decomposition:

$$|u_\rho^{(1)}\rangle = \sum_k \sqrt{\lambda_k} |\lambda_k\rangle \otimes |x_k\rangle,$$

$$|u_\rho^{(2)}\rangle = \sum_k \sqrt{\lambda_k} |\lambda_k\rangle \otimes |y_k\rangle,$$

it is seen that there exists an isometry operator $U$ such that $U|x_k\rangle = |y_k\rangle$ for each $k$, moreover $|u_\rho^{(2)}\rangle = (1 \otimes U)|u_\rho^{(1)}\rangle$. Now $|u_\rho^{(2)}\rangle \langle u_\rho^{(2)}| = (1 \otimes U)|u_\rho^{(1)}\rangle \langle u_\rho^{(1)}| (1 \otimes U)^\dagger$, which implies that

$$(\Phi \otimes 1)(|u_\rho^{(2)}\rangle \langle u_\rho^{(2)}|) = (1 \otimes U)(\Phi \otimes 1)(|u_\rho^{(1)}\rangle \langle u_\rho^{(1)}|)(1 \otimes U)^\dagger,$$

$$S \left( (\Phi \otimes 1)(|u_\rho^{(1)}\rangle \langle u_\rho^{(1)}|) \right) = S \left( (\Phi \otimes 1)(|u_\rho^{(2)}\rangle \langle u_\rho^{(2)}|) \right).$$
2 Appendix

2.1 The saturation of the strong subadditivity inequality

Proposition 2.1 ([1]). A state \( \rho_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C) \) saturating the strong subadditivity inequality, i.e.,

\[
S(\rho_{AB}) + S(\rho_{BC}) = S(\rho_{ABC}) + S(\rho_B)
\]

if and only if there is a decomposition of system B as

\[\mathcal{H}_B = \bigoplus_j \mathcal{H}_{b_j^L} \otimes \mathcal{H}_{b_j^R}\]

into a direct (orthogonal) sum of tensor products, such that

\[
\rho_{ABC} = \bigoplus_j \lambda_j \rho_{Ab_j^L} \otimes \rho_{b_j^R C},
\]

where \( \rho_{Ab_j^L} \in D(\mathcal{H}_A \otimes \mathcal{H}_{b_j^L}) \) and \( \rho_{b_j^R C} \in D(\mathcal{H}_{b_j^R} \otimes \mathcal{H}_C) \), and \( \{\lambda_j\} \) is a probability distribution.

2.2 The saturation of Araki-Lieb inequality

The following proposition can be seen as a characterization of the saturation of Araki-Lieb inequality:

\[
|S(\rho_B) - S(\rho_C)| \leq S(\rho_{BC}). \tag{2.1}
\]

For the readers’ convenience, we copy the proof here.

Proposition 2.2 ([2]). Let \( \rho_{BC} \in D(\mathcal{H}_B \otimes \mathcal{H}_C) \). The reduced states are \( \rho_B = \text{Tr}_C(\rho_{BC}), \rho_C = \text{Tr}_B(\rho_{BC}) \), respectively. Then \( S(\rho_{BC}) = S(\rho_B) - S(\rho_C) \) if and only if

1. \( \mathcal{H}_B \) can be factorized into the form \( \mathcal{H}_B = \mathcal{H}_L \otimes \mathcal{H}_R \),
2. \( \rho_{BC} = \rho_L \otimes |\psi\rangle \langle \psi|_{RC} \) for \( |\psi\rangle_{RC} \in \mathcal{H}_R \otimes \mathcal{H}_C \).

Proof. The sufficiency of the condition is immediate. The proof of necessity is presented as follows: Assume that \( S(\rho_{BC}) = S(\rho_B) - S(\rho_C) \). The bipartite state \( \rho_{BC} \) can be purified
into a tripartite state $|\Omega_{ABC}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, where $\mathcal{H}_A$ is a reference system. Denote $\rho_{ABC} = |\Omega_{ABC}\rangle \langle \Omega_{ABC}|$. We have

$$
\text{Tr}_{AB}(\rho_{ABC}) = \rho_C, \quad \text{Tr}_{AC}(\rho_{ABC}) = \rho_B, \quad \text{Tr}_C(\rho_{ABC}) = \rho_A, \quad \text{Tr}_A(\rho_{ABC}) = \rho_{BC}.
$$

Now since $S(\rho_{ABC}) = 0$, it follows that $S(\rho_C) = S(\rho_{AB})$. Thus we have

$$
S(\rho_{AB}) + S(\rho_{BC}) = S(\rho_B) = S(\rho_B) + S(\rho_{ABC}),
$$

which, by Proposition 2.1, implies that

(i) $\mathcal{H}_B$ can be factorized into the form $\mathcal{H}_B = \bigoplus_{k=1}^K \mathcal{H}_{b_k^L} \otimes \mathcal{H}_{b_k^R}$,

(ii) $\rho_{ABC} = \bigoplus_{k=1}^K \lambda_k \rho_{A_{b_k^L}} \otimes \rho_{b_k^R_C}$ for $\rho_{A_{b_k^L}} \in D(\mathcal{H}_A \otimes \mathcal{H}_{b_k^L})$ and $\rho_{b_k^R_C} \in D(\mathcal{H}_{b_k^R} \otimes \mathcal{H}_C)$, where $\{\lambda_k\}$ is a probability distribution.

Clearly,

$$
S(\rho_{BC}) = S(\rho_B) - S(\rho_C) \implies S(\rho_A) + S(\rho_C) = S(\rho_{AC}).
$$

But

$$
S(\rho_A) + S(\rho_C) = S(\rho_{AC}) \iff \rho_{AC} = \rho_A \otimes \rho_C.
$$

From the expression

$$
\rho_{ABC} = \bigoplus_{k=1}^K \lambda_k \rho_{A_{b_k^L}} \otimes \rho_{b_k^R_C},
$$

it follows that

$$
\rho_{AC} = \sum_{k=1}^K \lambda_k \rho_{A_{b_k}} \otimes \rho_{C_{b_k}}.
$$

Combining all the facts above mentioned, we have

$$
K = 1,
$$

i.e., the statement (1) in the present theorem holds. Hence

$$
\rho_{ABC} = \rho_{AL} \otimes \rho_{RC}
$$

for $\rho_{AL} \in D(\mathcal{H}_A \otimes \mathcal{H}_L)$ and $\rho_{RC} \in D(\mathcal{H}_R \otimes \mathcal{H}_C)$, which implies that both $\rho_{AL}$ and $\rho_{RC}$ are pure states since $\rho_{ABC}$ is pure state. Therefore

$$
\rho_{BC} = \text{Tr}_A(\rho_{AL}) \otimes \rho_{RC} = \rho_L \otimes |\psi\rangle \langle \psi|_{RC}
$$

for $|\psi\rangle_{RC} \in \mathcal{H}_R \otimes \mathcal{H}_C$, i.e., the statement (2) holds. This completes the proof.
Remark 2.3. The result in Proposition 2.2 is employed to study the saturation of the upper bound of quantum discord in [3]. Later on, E.A Carlen gives an elementary proof about this result in [4].

References

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