An adaptive MOR method for vibro-acoustic analysis of dynamic systems with viscoelastic damping

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Abstract. Viscoelastic materials have the capability of dissipating vibrational energy which makes them very effective in the passive control of structural vibration and noise radiation. Efficient modeling- and simulation strategies for vibro-acoustic systems with these frequency-dependent damping materials are necessary. One way of doing that is to utilize the finite element (FE) method supported by a damping model that describes the frequency-dependency of the complex elastic modulus. However, in many applications, very large-scale FE models are required to obtain reliable predictions, which makes full-order system evaluation often intractable. Model order reduction (MOR), however, cannot be directly applied to models including viscoelastic materials due to the frequency-dependent property which exactly makes that the equations of motion are not of a standard second-order form as that for regular FE models. In this paper, a transformation technique based on Taylor’s theorem is introduced to treat the non-standard form of the equations of motion. After transforming the problem into a second-order system equation with an associated remainder term, an adaptive MOR procedure is applied to reduce the computational complexity of the vibro-acoustic systems. A numerical example is provided to demonstrate the simplicity and efficiency of the proposed approach.

1. Introduction

Over the past years, the steadily growing requirements on the vibro-acoustic performance of products as well as comfort considerations have made the interior acoustic behavior into an important criterion in many industries, for instance, automobiles, underwater vehicles, aircraft, aerospace and office partitions etc. In this framework, the vibro-acoustic coupling phenomenon is significant and will become even more important in the course of the next decades with the increasing of restrictive legal regulations regarding noise emission levels. In order to achieve the desired vibro-acoustic performance, the application of viscoelastic damping technologies in the passive control of structural vibration can be very effective [1]. For example, free-layer damping treatments are widely bonded to the metal panels in vehicle industry to reduce the response to external loads. The underlying physical mechanisms behind the viscoelastic materials are typically quite complex; efficient constitutive mathematical models able to simply approximate their frequency-dependent behavior bring a lot of convenience. The Fractional Derivative (FD) model [2, 3, 4] provides a suitable description of viscoelastic materials with
only four phenomenological parameters. This makes the FD model an interesting option to formulate the mathematical model of the damping.

In order to evaluate the performance of products, the reliable and efficient methodologies are required for the solution of complex vibro-acoustic system with viscoelastic damping. One of the most commonly used numerical methods in this field is the finite element (FE) method. However, because of the 3D nature of acoustics and the frequency-dependent damping, the number of elements in the structural-acoustic model grows rapidly with increasing problem size (and also with increasing frequency). This apparently limits original full-order system evaluation due to computational constraints. As a result, model order reduction (MOR) approaches which attempt to accurately approximate the original model using a similar one with much smaller size have been developed to speed up the numerical simulations. For coupled vibro-acoustic problems, several MOR approaches have been applied to enhance the computational efficiency in frequency-domain analysis [5, 6]. However, these MOR methods proposed in literature give satisfactory results for models with frequency-independent parameters, but for systems with strongly frequency-dependent damping as will be discussed herein, their applicability is not straightforward. In Ref. [7], an adaptive MOR technique based on Taylor’s theorem and the second-order Arnoldi algorithm (SOAR-algorithm) [8, 9] for large-scale FE dynamical systems with frequency-dependent damping is proposed, where two different orthonormal bases of the second-order Krylov subspace are obtained simultaneously. The difference between two reduced-order models (ROMs) projected from these two orthonormal bases is used to estimate the exact error between the full-order model (FOM) and ROM such that the evaluation of the FOM can be avoided. Moreover, the order for each expansion point and the number of expansion points required for convergence can be determined in an adaptive manner. This study can be considered as an extension of the adaptive MOR technique for structural vibration analysis to vibro-acoustic coupling analysis. To demonstrate the accuracy and efficiency of ROM as a surrogate of FOM, a numerical example is investigated, considering constrained-layer damping treatment.

2. Finite element modeling

Application of the FE method to a coupled vibro-acoustic system with a time-harmonic external load leads to a set of second-order ordinary differential equations of the form

\[
\begin{align*}
\left( s^2 M + s D + K(s) \right) X &= F \\
H(s) &= L^T X
\end{align*}
\]

where the complex variable \( s = 2\pi j f \), \( j \) is the imaginary unit satisfying \( j^2 = -1 \) and \( f \) is the frequency in Hz. \( M, D \) and \( K(s) \in \mathbb{C}^{N \times N} \) are the mass, damping and stiffness matrices. \( F \in \mathbb{C}^N \) is the external loading vector and \( X \in \mathbb{C}^N \) is the nodal solution vector. \( N \) is the dimension of the full-order system and is the sum of the total number of structural degrees of freedom (DOFs) \( N_u \) and acoustic DOFs \( N_p \). \( L \) is an output measurement matrix that is only used to pick certain DOFs in this work. \( H(s) \) is the transfer function describing the input-output behaviour of the large-scale vibro-acoustic system.

Coupled vibro-acoustic systems are often decomposed into two interacted components, i.e. the elastic structure and acoustic fluid domain. Specifically, when using the displacement in the structure and sound pressure in the fluid to describe the properties of the coupled system, Eq. (1) can be further split as follows

\[
\begin{align*}
\left( s^2 M - \rho C_c^T M_p \right) + s \left[ D_u \ 0 \right] + \left[ K_u(s) \ C_c \right] \right) \begin{bmatrix} u \\
p \end{bmatrix} &= \begin{bmatrix} F_u \\
P_p \end{bmatrix},
\end{align*}
\]

where subscripts \( u \) and \( p \) denote that the matrices are related to the structural and acoustic part of the system, respectively. \( C_c \in \mathbb{C}^{N_u \times N_p} \) is the global structural-acoustic coupling matrix.
and ρ is the fluid density. The vectors $\mathbf{u} \in \mathbb{C}^{N_u}$ and $\mathbf{p} \in \mathbb{C}^{N_p}$ contain the nodal displacements in the structural domain and the nodal pressures in the fluid domain.

The damping matrices $\mathbf{D}_u$ and $\mathbf{D}_p$ considered in this work are modeled as Rayleigh damping to describe internal structural damping, in which case these are determined by a mass-proportional term and a stiffness-proportional term of the corresponding subsystems. However, it should be pointed out that the Rayleigh damping is usually negligible in comparison with the damping due to the viscoelastic materials, and thus $\mathbf{D}_u$ and $\mathbf{D}_p$ will be ignored in the rest of this paper without loss of generality. Furthermore, the FE discretisation of the structural part consisting of an elastic structure (with typically frequency-independent elastic properties) and a viscoelastic damping material (with frequency-dependent elastic properties) results in the following

\[
\begin{pmatrix}
  \alpha \\
  \beta
\end{pmatrix}
\begin{bmatrix}
  \mathbf{M}_u & 0 \\
  -\rho \mathbf{C}_c^T & \mathbf{M}_p
\end{bmatrix}
+ \begin{bmatrix}
  \mathbf{K}_e + \bar{G}(s)\mathbf{K}_v \\
  0
\end{bmatrix}
\begin{bmatrix}
  \mathbf{C}_c \\
  \mathbf{K}_p
\end{bmatrix}
\]
\[
\begin{bmatrix}
  \mathbf{u} \\
  \mathbf{p}
\end{bmatrix}
= \begin{bmatrix}
  \mathbf{F}_u \\
  \mathbf{F}_p
\end{bmatrix},
\]

in which $\mathbf{K}_e$ is the stiffness matrix with respect to the elastic substructure, $\mathbf{K}_v$ is the stiffness matrix associated with the viscoelastic substructure and evaluated for a unit shear modulus. $G(s)$ is the complex shear modulus. The FD model, characterized by four parameters, has been shown to be well suited for representing the frequency-dependent characteristics of viscoelastic materials over a wide frequency range [3, 4]. The corresponding $\bar{G}(s)$ of such FD model can be written as follows:

\[
\bar{G}(s) = \frac{G_0 + G_\infty (s\tau)^\alpha}{1 + (s\tau)^\alpha},
\]

in which $G_0$ is the dynamic modulus at zero frequency, i.e. the static modulus of elasticity, $G_\infty$ is the high frequency asymptote of the dynamic modulus. $\tau$ is the relaxation time. $\alpha$ is fractional-order time derivative relating stress and strain fields. If $\alpha = 0$, a pure spring is obtained while $\alpha = 1$ represents a pure dashpot.

3. MOR via SOAR-algorithm

In principle, Eq. (1) can be directly used for solving the steady-state harmonic responses of the FOM at each frequency. However, such a procedure can be impractical when dealing with large-scale FE models. This fact motivates the use of MOR technique. Most of the moment-matching MOR methods are proposed for the frequency-independent system which is a special case in this paper. In order to extend their applicability to the frequency-dependent system, the complex shear modulus $\bar{G}(s)$ should be firstly handled. According to Taylor’s theorem, we expand $\bar{G}(s)$ at a given point $s_0$:

\[
\bar{G}(s) = \bar{G}(s_0) + G'(s_0)(s - s_0) + \frac{G''(s_0)}{2}(s - s_0)^2 + R_2(s),
\]

where $R_2(s)$ denotes the remainder term. In general, $\lim_{s \to s_0} R_2(s) = 0$. Note that $s_0$ can be an arbitrary value, but needs to be fixed. In order to comply with the actual physical meaning ($s = 2\pi j f$), the choice of a purely imaginary value for $s_0$ is preferred. Then, substituting Eq. (5) into Eq. (3), it can be easily derived that

\[
\begin{pmatrix}
  \alpha \\
  \beta
\end{pmatrix}
\begin{bmatrix}
  \mathbf{M}_u & 0 \\
  -\rho \mathbf{C}_c^T & \mathbf{M}_p
\end{bmatrix}
+ \begin{bmatrix}
  \mathbf{K}_e + \bar{G}(s)\mathbf{K}_v \\
  0
\end{bmatrix}
\begin{bmatrix}
  \mathbf{C}_c \\
  \mathbf{K}_p
\end{bmatrix}
\]
\[
\begin{bmatrix}
  \mathbf{u} \\
  \mathbf{p}
\end{bmatrix}
= \begin{bmatrix}
  \mathbf{F}_u \\
  \mathbf{F}_p
\end{bmatrix}.
\]

It should be noted that the $s_0$ dependency of $G$ and its derivatives is omitted for simplicity. Also, Eq. (6) can be rewritten in the compact form

\[
[s^2 \mathbf{M} + \mathbf{D} + \bar{K} + \bar{K}_r] \mathbf{X} = \mathbf{F}.
\]
In practice, we are often interested in the approximation of Eq. (7) around a selected expansion point \( s_0 \neq 0 \). In this case, Eq. (7) is equivalent to the following
\[
[(s - s_0)^2 \mathbf{M} + (s - s_0) \mathbf{D} + \mathbf{K} + \mathbf{K}_r] \mathbf{X} = \mathbf{F},
\]
where \( \mathbf{M} = \mathbf{M}, \mathbf{D} = 2s_0 \mathbf{M} + \mathbf{D} \) and \( \mathbf{K} = s_0^2 \mathbf{M} + s_0 \mathbf{D} + \mathbf{K} \). In this work, instead of spanning a higher-order Krylov subspace, we just use \( \tilde{\mathbf{M}}, \tilde{\mathbf{D}} \) and \( \mathbf{K} \) to construct an orthonormal matrix whose columns span the second-order Krylov subspace. The memory saving template of SOAR-algorithm which was introduced by Bai and Su, see [8, 9] and references therein, is selected to achieve this. By means of the SOAR-algorithm, we can generate an orthonormal and frequency-independent basis \( \mathbf{Q} \in \mathbb{C}^{N \times N} \) with \( n \ll N \)
\[
\text{span}\{\mathbf{Q}\} = \mathcal{G}_n(\mathbf{A}, \mathbf{B}; \mathbf{r}_0) = \text{span}\{\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_{n-1}\}
\]
where \( \mathbf{A} = -\mathbf{K}^{-1} \tilde{\mathbf{D}}, \mathbf{B} = -\mathbf{K}^{-1} \tilde{\mathbf{M}} \) and starting vector \( \mathbf{r}_0 = \mathbf{K}^{-1} \mathbf{F} \). The processing time for the calculation of the inverse of the matrix \( \mathbf{K} \) needs to be accelerated. Since \( \mathbf{K} \) generated from FE model is a sparse matrix, a factorized form of \( \mathbf{K} \), such as an LU factorization, should be made available before the iterative solution of the orthonormal basis \( \mathbf{Q} \) for computational efficiency. The vectors \( \mathbf{r}_i \) are obtained from the recurrence relation
\[
\begin{aligned}
\mathbf{r}_1 &= \mathbf{A} \mathbf{r}_0, \\
\mathbf{r}_i &= \mathbf{A} \mathbf{r}_{i-1} + \mathbf{B} \mathbf{r}_{i-2}, \quad \text{for} \; i \geq 2
\end{aligned}
\]
Such a subspace \( \mathcal{G}_n(\mathbf{A}, \mathbf{B}; \mathbf{r}_0) \) is called an \( n \)th second-order Krylov subspace. Furthermore, this second-order Krylov subspace spanned by the nonzero columns of \( \mathbf{Q} \) can be used as the projection subspace to define a reduced system about the selected expansion point \( s_0 \). For the sake of derivation convenience, \( \mathbf{Q} \) can be given by
\[
\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_u \\ \mathbf{Q}_p \end{bmatrix}
\]
where \( \mathbf{Q}_u \in \mathbb{C}^{N_u \times N} \) is the part of the orthonormal basis that corresponds to the structural DOFs and \( \mathbf{Q}_p \in \mathbb{C}^{N_p \times N} \) is the part corresponding to the acoustic DOFs. The projection of the full-order system in Eq. (8) onto the subspace \( \mathcal{G}_n(\mathbf{A}, \mathbf{B}; \mathbf{r}_0) \) yields the reduced-order system of the same form as
\[
[(s - s_0)^2 \tilde{\mathbf{M}} + (s - s_0) \tilde{\mathbf{D}} + \mathbf{K} + \mathbf{K}_r] \mathbf{X} = \mathbf{F},
\]
with
\[
\begin{aligned}
\mathbf{M}_n &= \mathbf{Q}_u^H \left[ \mathbf{M}_u + \frac{G'' \mathbf{K}_v}{2} \right] \mathbf{Q}_u - \rho \mathbf{Q}_p^H \mathbf{C}_c^T \mathbf{Q}_u + \mathbf{Q}_p^H \mathbf{M}_p \mathbf{Q}_p, \\
\tilde{\mathbf{D}}_n &= 2s_0 \mathbf{M}_n + \mathbf{Q}_u^H \left[ G' \mathbf{K}_v - G'' s_0 \mathbf{K}_c \right] \mathbf{Q}_u, \\
\mathbf{K}_n &= s_0^2 \mathbf{M}_n + s_0 \mathbf{Q}_u^H \left[ G' \mathbf{K}_v - G'' s_0 \mathbf{K}_c \right] \mathbf{Q}_u \\
&+ \mathbf{Q}_u^H \left[ \mathbf{K}_e + G \mathbf{K}_v - G' s_0 \mathbf{K}_c + \frac{G''}{s_0^2} \mathbf{K}_v \right] \mathbf{Q}_u + \mathbf{Q}_u^H \mathbf{C}_c \mathbf{Q}_p + \mathbf{Q}_p^H \mathbf{K}_p \mathbf{Q}_p, \\
\mathbf{F}_n &= \mathbf{Q}_u^H \mathbf{F}_u + \mathbf{Q}_p^H \mathbf{F}_p, \\
\mathbf{K}_r &= R_2(s) \mathbf{Q}_u^H \mathbf{K}_e \mathbf{Q}_u
\end{aligned}
\]
in which the superscript \( H \) designates the complex conjugate transpose. The fundamental idea behind the formula \( \mathbf{X} \approx \mathbf{Q} \mathbf{X}_n \) can be regarded as to approximate the vector \( \mathbf{X} \) of the original full system by another vector \( \mathbf{X}_n \) constrained to stay in the subspace \( \mathcal{G}_n \) spanned by \( \mathbf{Q} \). Furthermore, by simple algebraic manipulation, Eq. (12) can be expressed as:
\[
[(s^2 \mathbf{M}_n + \mathbf{K}_n(s))] \mathbf{X}_n = \mathbf{F}_n,
\]
\[\text{(14)}\]
with the transfer function of the reduced system

$$H_n(s) = L_n^H \left[ s^2 M_n + K_n(s) \right]^{-1} F_n,$$

(15)

where $L_n = Q^H L$, $M_n = Q^H M Q$, $K_n(s) = Q^H K(s) Q$ and $F_n = Q^H F$. Obviously, the Taylor approximation about the selected expansion point $s_0$ is only used to generate the matrix triplet $(\tilde{K}, \tilde{D}, \tilde{M})$ such that the second-order Krylov subspace $G_n(A, B; \mathbf{r}_0)$ based on a pair of square matrices $A$ and $B$ and a vector $\mathbf{r}_0$ can be constructed. Note that the transformed matrix triplet $(\tilde{K}, \tilde{D}, \tilde{M})$ is adopted to compute the orthonormal basis $Q$ of the projection subspace $G_n$, but the original system in Eq. (3) is directly projected onto the same subspace $G_n$. Moreover, although the equation of motion in Eq. (1) is not the standard second-order dynamic system due to the presence of viscoelastic treatments, the subspace $G_n$ spanned by $Q$ can still be used as the projection subspace similarly to conventional second-order problems with constant properties. Without the truncation of Eq. (5) from infinite order to finite order, the ROM in Eq. (14) holds the same form as the original FOM in Eq. (1), which means that the inherent frequency dependence of the system is preserved in the projection, and is therefore also present in the ROM.

4. Adaptive procedure

In the previous section, a framework for the MOR of vibro-acoustic system with viscoelastic materials was presented. Since obtaining better accuracy and smaller model size are the objectives in the reduction process, important open problems are how many expansion points should be selected and how large the orthonormal basis for each expansion point should be in order to build the ROM, preferably in an adaptive manner. Of course, their offline construction cannot be too computationally expensive. This requires information about the total reduction error. Recall from Section 3 that there are two main contributions in the overall reduction error. One is the approximation error of the frequency-dependence in generating the frequency-independent orthonormal basis, which originates from truncating the Taylor series for the construction of the second-order Krylov projection subspace. The second error is the reduction error in the classical sense, which originates from projecting the FOM onto the projection subspace to define a ROM. Since the analysis of the FOM is to be avoided, one must resort to the approximation of the transfer function error between the FOM and the ROM.

In this section, an adaptive process to iteratively enrich two ROMs by increasing their order and expansion points in the complex-frequency domain is developed. These two different ROMs are compared to check each other’s accuracy until the pre-specified termination tolerance of error is satisfied. Algorithm 1 shows the pseudocode of this procedure, which is based on Taylor’s theorem and the SOAR-algorithm. At line 2 of the algorithm, the ‘Level’ indicates how many times the expansion points are added and at every Level the order of the orthonormal basis for each expansion point is determined. For more details about this process, we refer the readers to Refs. [7, 10] and references therein.

5. Numerical results and discussion

In this section, we present a numerical example to demonstrate the potential of the proposed MOR strategy for dimension reduction of a large-scale vibro-acoustic system with viscoelastic materials. All our routines are implemented in MATLAB R2018a. The system matrices $K_e$, $K_v$, $M$, $F$ and $L$ are extracted from COMSOL Multiphysics 5.3 with MATLAB. All numerical tests are performed on a Windows Machine with Intel Core(TM) i7-6560U CPU at 2.20GHz and 8GB of RAM.

The example comprises the frequency response analysis of a coupled vibro-acoustic system with order $N=27110$, which comes from a FE model of a cavity backed flat plate with
Algorithm 1 Adaptive Arnoldi-algorithm for frequency-dependent damping [10]

1: Choose the initial order \( m \), step size \( h \) and maximum order \( \text{MaxOrder} \) allowed for the ROM with respect to each expansion point. Set the inner-loop and outer-loop termination tolerances \( \text{tol} \) and \( \text{Tol} \), and choose two initial expansion points \( s_{j0}^1 \) and \( s_{j0}^2 \).

2: for Level = 1, 2, ... do
   3:   for \( j \) = 1, 2 do
      4:      Use Taylor’s theorem to handle \( \bar{G}(s) \) at given \( s_{j0}^j \):
      5:      \[ \bar{G}(s) = G(s_{j0}^j) + G'(s_{j0}^j)(s - s_{j0}^j) + \frac{G''(s_{j0}^j)}{2}(s - s_{j0}^j)^2 + R_2(s). \]
      6:      Calculate the transformed matrix triplet \( (\tilde{K}, \tilde{D}, \tilde{M}) \).
      7:      Run the SOAR-algorithm to generate an orthonormal and frequency independent basis \( Q_m \in \mathbb{C}^{N \times m} \) of the second-order Krylov subspace \( G_m(A, B; r_0) \).
      8:      Compute \( K_m(s) = Q_m^H K(s) Q_m \), \( M_m = Q_m^H M Q_m \), \( F_m = Q_m^H F \) and \( L_m = Q_m^H L \).
      This defines a ROM about the selected expansion point \( s_{j0}^j \).
      9:      Calculate the transfer functions \( H_j^m(s) \) and \( H_{j+1}^m(s) \), respectively.
      10: if \( \max\left( \frac{|H_{j}^m(s) - H_{j+1}^m(s)|}{|H_{j}^m(s)|} \right) < \text{tol} \) or \( (m + h) > \text{MaxOrder} \) then
          11:         Break
      12:      else
          13:         Increase the order \( m \) by every \( h \) steps. Iterate steps 6-8 again until the prescribed inner-loop stopping criteria is satisfied.
      14:   end if
   15:   end for
   16: if \( \max\left( \frac{|H_{m+h}^1(s) - H_{m+h}^2(s)|}{|H_{m+h}^1(s)|} \right) < \text{Tol} \) then
          17:         Break
      18: else
      19:         Two new expansion points are selected and then separately added to the two sets of expansion points of the previous Level in the MOR procedure.
      20: end if
   21: end for

Table 1. Material properties.

| Material   | Density (kg/m\(^3\)) | Young’s Modulus (GPa) | Poisson’s ratio | Sound speed (m/s) |
|------------|------------------------|-----------------------|----------------|------------------|
| air        | 1.225                  | -                     | -              | 343              |
| steel      | 7800                   | 210                   | 0.3            | -                |
| aluminum   | 2700                   | 70                    | 0.3            | -                |
| viscoelastic| 1400                  | \( E(s) = 2(1 + \nu)\bar{G}(s) \) | \( \nu = 0.45 \) | -                |

constrained-layer damping treatment, as depicted in Fig. 1, along with the dimensions of this system. All walls of the cavity which are not in contact with the plate are assumed to be acoustically rigid. The clamped base plate is excited using a unit point force \( F \) at position (0.2 m, 0.2 m).

The modeling approach adopted here is to use classical shell elements for the base structure and thin foil layer, and standard solid elements for the viscoelastic core and the acoustic cavity. In such case, shell offsets which are defined as the distance from the shells midsurface to the reference surface are necessary to be coincident with the solid element. In this work, the material properties of the acoustic medium (air), the base structure (steel), the viscoelastic core and its constraining layer (aluminum) are given in Table 1. The material parameters of the considered
FD damping model can be referred to Table 2. The considered frequency range is from $f_{\text{min}} = 1$ Hz to $f_{\text{max}} = 400$ Hz with a 1 Hz frequency step. Note that this considered frequency range covers the first 27 uncoupled structural modes of the clamped base plate and the first 2 uncoupled acoustic modes of the rigid-walled cavity.

Two expansion points $s^{(1)}_0 = 2\pi j \ast 160$ for the ROM 1 and $s^{(2)}_0 = 2\pi j \ast 240$ for the ROM 2 around $2\pi j f_{\text{mid}} = 2\pi j \ast 200$ are selected to perform the SOAR-algorithm. The empirical values are taken to be: $m = h = 20$, $tol = 1e^{-2}$, $Tol = 1e^{-3}$ and $MaxOrder = 400$.

| Table 2. FD model parameters. |
|--------------------------------|
| $G_0$ (Pa) | $G_\infty$ (Pa) | $\tau$ (s) | $\alpha$ |
| 3.102e5 | 2.62e7 | 4.713e-5 | 0.7351 |

Fig. 2 shows the direct transfer function $H(s)$ at location (0.2 m, 0.2 m) on the plate together with its approximations $H_n(s)$ obtained with the proposed MOR technique. The true relative error for these two ROMs and the estimated error using the proposed error indicator are shown at the second plot of Fig. 2a and Fig. 2b, in which the blue dashed line denotes the set tolerance of $Tol$. It is clear from Fig. 2 that the suggestion on the choice of purely imaginary expansion points offers good local approximation property around the chosen frequencies (such as 160 Hz and 240 Hz). Furthermore, the estimated relative error almost follows the larger of the two true relative error curves, which means that once the estimated error has been reduced to satisfy a prescribed threshold, the two true relative error are naturally less than the threshold. The estimated relative error in Fig. 2a clearly indicates that next Level calculation has to be carried out and the values of expansion points that are automatically determined via the proposed error indicator are given along with the figure (see the circle with magenta rim). After each 40 iterations, we find that two final reduced-order systems of orders $n = 240$ and $n = 159$ are sufficient for the desired accuracy (i.e. less than the out-loop threshold $Tol = 1e^{-3}$). Note that $n = 159$ is not the multiple of $m = h = 20$. This is because that the combination of the orthonormal bases of different expansion points needs to remove the linearly dependent columns between them. Obviously, the ROM with $n = 159$ should be selected for further application due to the lower dimensions.

In this example, the CPU time required to solve the transfer function of the FOM is $t_f = 4141.25$ seconds and the elapsed time of the present approach is $t_r = 96.60$ seconds,
For ROM 1: \( n = 200 \)
\[ \text{New } s_0^{(1)} = 2\pi j \ast 282 \]

For ROM 2: \( n = 120 \)
\[ \text{New } s_0^{(2)} = 2\pi j \ast 57 \]

Figure 2. Transfer function and relative error of the FOM and two ROMs: (a) Level 1; (b) Level 2.
where the matrix-vector multiplication operations at step 6 of the algorithm 1 accounts for the largest percentage. In this case, the processing time for the LU-decomposition of the $\tilde{K}$ does not cost the most of the calculation time $t_r$, which is different from the Ref. [7] due to the much higher dimension of the system matrices $N$. The computational benefit of the MOR technique is apparent.

In Fig. 3, the pressure amplitude spectra of the FOM and two ROMs at location (0.2 m, 0.5 m, 0.2 m) inside the cavity are illustrated. Once again, although the preferred choice of the reduced order and expansion points changes with the problems to be studied, we have demonstrated that the automatic generation of reduced compact models is feasible by the straightforward

For ROM 1: $n = 400$
New $s_0^{(1)} = 2\pi j \times 252$

For ROM 2: $n = 120$
New $s_0^{(2)} = 2\pi j \times 54$

For ROM 1: $n = 440$
New $s_0^{(1)} = 2\pi j \times 370$

For ROM 2: $n = 199$
New $s_0^{(2)} = 2\pi j \times 370$
For ROM 1: $n = 476$
For ROM 2: $n = 238$

Figure 3. Transfer function and relative error of the FOM and two ROMs: (a) Level 1; (b) Level 2; (c) Level 3.

For ROM 1: $n = 128$
For ROM 2: $n = 109$

Figure 4. Transfer function and relative error of the FOM and two ROMs.

implementation of the proposed strategy. According to that figure, finally, two ROMs that can represent the FOM over a wide frequency range are achieved at $n = 476$ and $n = 238$, respectively.

The total time of the present approach is $t_r = 281.13$ seconds, while the calculation time
for the FOM is \( t_f = 4162.86 \) seconds. Of course, the elapsed time for obtaining the transfer function of the ROM is depend on the selection of the initial parameters. For example, when \( m = h = 10, \, tol = 1e^{-1} \), we find that two ROMs of orders \( n = 128 \) and \( n = 109 \) are sufficient for the desired accuracy, see Fig. 4. The elapsed time for this configuration is \( t_r = 71.43 \) seconds with Level = 4. This is due to the smaller step size \( h \) and inner-loop termination tolerance \( tol \), the required order of the orthonormal basis for each expansion point to satisfy the inner-loop threshold (i.e. less than \( tol \)) is decreased. In order to achieve the final accuracy requirement (i.e. less than \( Tol \)), more expansion points (i.e. Level = 4) are needed to meet the multipoint moment-matching property.

6. Conclusions

This paper presents an efficient MOR strategy for the frequency response analysis of vibro-acoustic model consisting of an interior acoustic cavity and an elastic flat plate covered by a constrained-layer viscoelastic damping. The FD model for describing the frequency-dependent shear modulus of the viscoelastic material is considered. A simple strategy based on a local Taylor’s theorem is proposed to transform the complex shear modulus into second-order polynomial with associated remainder term and then the transformed matrix triplet is used to compute a frequency-independent and orthonormal basis of the second-order Krylov subspace by using the SOAR-algorithm. Moreover, this frequency-independent basis is utilized in the projection of the full, frequency-dependent system to construct a reduced, frequency-dependent system. This way the inherent frequency dependence of the vibro-acoustic system is preserved in the projection, and is therefore also present in the ROM. In support, the dimension of the reduced system can be determined in an adaptive manner due to the proposed relative error indicator.

We have validated that the presented error indicator provides a cheap tool to ensure sufficient accuracy for the large-scale vibro-acoustic system, and hence it is a suitable and achievable convergence criterion. The simplicity and high-efficiency of the proposed overall strategy are obvious. With this promising strategy, vibro-acoustic simulation can be effectively sped up, especially in the presence of damping materials.

In view of the convergence theory, it would be worthwhile to further derive the explicit theoretical justification or rigorous error bound on the approximation error of the ROMs in the desired range of frequency. Another important extension of this work is about how to optimally choose the initial parameters.

Acknowledgments

The first author gratefully acknowledges the financial support from China Scholarship Council (CSC NO. 201606230074).

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