Integrable Gradient Flows 
and Morse Theory *

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Abstract

Examples of Morse functions with integrable gradient flows on some classical 
Riemannian manifolds are considered. In particular, we show that a generic height 
function on the natural embeddings of classical Lie groups and certain symmetric 
spaces is a perfect Morse function, i.e. has as many critical points as the homology 
requires, and the corresponding gradient flow can be described explicitly. This gives 
an explicit cell decomposition and geometric realization of the homology for such 
a manifold. As another application of the integrable Morse functions we give an 
elementary proof of Vassiljev’s theorem on the flag join of Grassmannians.

Introduction.

Morse theory based on a very natural idea till now remains one of the most attractive 
fields in the modern topology. We mention, for example, the papers by S. P. Novikov on a 
multivalued analogue of the Morse theory and those by A.T.Fomenko and his collaborators 
on the Morse theory for integrable Hamiltonian systems (see e.g. [1]).

In the present paper we are not concerned with those generalizations but consider 
Morse theory in its classical aspects. It is well-known (see e.g. [1, 2]) that it allows one to 
extract a topological information about a manifold $M^n$ from the knowledge of the critical 
points of some smooth function $f : M^n \rightarrow \mathbb{R}$. If one knows in addition the behaviour of 
the gradient flow of the function $f$, it provides a cell decomposition of the manifold $M^n$.

Note that for a given metric the gradient flow

$$\dot{x} = \text{grad } f(x)$$

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*This work was partially supported by Russian Foundation for Fundamental Investigations (grant 
94-01-01444), Soros International Science Foundation (grants MD8000, M3Z000) and INTAS grant (93- 
0166).
is a system of ordinary nonlinear differential equations, which in general does not admit an explicit integration. The critical points are defined by the system of nonlinear equations (in algebraic situation - by algebraic one)

$$\text{grad } f(x) = 0,$$  \hspace{1cm} (0.2)

which is also difficult to solve in general case.

We will say that $f$ is an **integrable Morse function** on a Riemannian manifold $M$ if: first, $f$ is a Morse function (or, more generally, Morse-Bott function) and, second, the equations (0.1) and (0.2) are “integrable”. In the present paper we will mean the integrability in the naive sense as a possibility of explicit solution leaving apart more serious discussions of this notion for gradient flows.

We should note that in the literature there exist already examples of gradient flows which could be naturally considered as integrable. For example, the Toda lattice can be interpreted as a gradient flow (see [3, 4]). Once more remarkable example of an integrable gradient system was recently pointed out by S. P. Novikov in connection with the problem of “isoperiodical” deformations for finite-gap potentials and related problems of topological field theory [5, 6]. It has a feature that the corresponding metric is not positively definite. For discussions of the integrability of the algebraic equations (0.2) and examples of such systems we refer to [7, 8].

Such examples naturally appear in the theory of integrable systems. For instance, the stationary points of the mechanical system with a potential $U(x)$ are nothing else but the critical points of $U(x)$:

$$\text{grad } U(x) = 0.$$  

It is natural to expect that for an integrable system the corresponding stationary problem also has an integrable nature.

The starting point for us was another example arised from the theory of integrable systems with discrete time [9, 10, 11]. Such systems are defined by a “Lagrangian” which is a function $L$ on the product $Q^{2n} = M^n \times M^n$. A sequence of points from $M^n$ $X = \{x_k\}, \ k \in \mathbb{Z}$, is called a solution of the discrete system with Lagrangian $L$ if $X$ is a critical point of the functional

$$S(X) = \sum_{k \in \mathbb{Z}} L(x_k, x_{k+1}).$$  \hspace{1cm} (0.3)

In the papers [3, 10, 11] it was shown that in the case when $M^n = SO(N)$ and $L = \text{Tr}(XJY^t)$, $J = J^t$ is a symmetric matrix, one has an integrable discrete version of the multidimensional top’s dynamics. Such a form of the Lagrangian is uniquely determined by the following conditions:

1) symmetry $L(X, Y) = L(Y, X)$;
2) left-invariance $L(gX, gY) = L(X, Y), \ g \in SO(N)$;
3) bilinearity with respect to $X$ and $Y$ considered in the standard matrix realization.

Note that because of the left-invariance $L(X, Y)$ depends only on $\omega = Y^{-1}X : L(X, Y) = L(Y^{-1}X, Y^{-1}Y) = L(Y^{-1}X, I) = F(Y^{-1}X)$, where $F$ is the corresponding function on the group,

$$F(\omega) = \text{Tr}(J\omega).$$  \hspace{1cm} (0.4)
The critical points of the function $F$ correspond to the stationary solutions of the discrete top's equations.

It turns out that the function (0.4) and its analogues on the unitary and symplectic groups [11] can be considered as examples of integrable Morse functions on the corresponding group space with biinvariant (Cartan - Killing) metric. We discuss them in details in the first and second sections of the present paper. As an application we get a geometrical representation of the homology classes of these groups found by V. A. Vassiljev in [12]. For example, in the case of the unitary group we have the following isomorphism:

$$H_i(U(n)) \cong \bigoplus_{k=0}^{n} H_{i-k^2}(G_k(C^n)), \quad (0.5)$$

where $G_k(C^n)$ is the Grassmannian manifold of complex $k$-dimensional subspaces in $C^n$.

Let us define for an arbitrary Schubert cell $\alpha$ in $G_k(C^n)$ the following cycle $[\alpha]$ in $H_*(U(n))$. For each point in $\alpha$, that is for a $k$-dimensional plane in $C^n$, consider the set of all unitary operators preserving this plane and acting on its orthogonal complement as the reflection $(-I)$ (in the paper [12] the trivial action on the complement is considered; this difference is not essential). The union of all such operators over all points from $\alpha$ forms a cycle $[\alpha]$. Vassiljev’s theorem [12, Theorem 1] states that the set of all cycles $[\alpha]$ corresponding to all Schubert cells of all Grassmannians $G_k(C^n)$ gives a basis in the integer homology of the unitary group, which is reflected by equation (0.5). As V. A. Vassiljev told us this result was found also by M. Mahowald, so we will call equation (0.5) Vassiljev-Mahowald decomposition. We will show that it can be obtained easily from the Morse theory for the function (0.4).

We consider also the general height functions on symmetric embeddings of certain symmetric spaces like Grassmannians, Lagrangian Grassmannians, the spaces of complex and quaternion structures and show that they are perfect Morse functions with integrable gradient flows. For the Grassmannians the corresponding cell decomposition coincides with the classical Schubert decomposition.

We should note that in a special case $J = \text{diag}(1, 0, \ldots, 0)$ function (1.4) was considered and used for the investigation of the topology of classical Lie groups by L. S. Pontrjagin in the paper [13].

It is interesting that the consideration of another integrable Morse function allows us to simplify essentially the proof of the following remarkable result from the same Vassiljev’s paper [12]. Let the Grassmannians $G_1(V), G_2(V), \ldots, G_{n-1}(V)$, where $V$ is an $n$-dimensional vector space over $k = \mathbb{R}, \mathbb{C}$ or quaternions $\mathbb{H}$, be embedded into a Euclidean space $\mathbb{R}^N$ of a large dimension $N$ in a general position. The union $\Theta_n(k)$ of all the $(n-2)$-dimensional simplexes with vertices $(V_1, V_2, \ldots, V_{n-1})$, where the subspaces $V_i \in G_i(V)$ form a complete flag $F$, that is $V_1 \subset V_2 \subset \ldots \subset V_{n-1}$, endowed with the induced topology we will call the flag join of the Grassmannians.

The theorem 5 from [12] says that this space $\Theta_n(k)$ is homeomorphic to the sphere of certain dimension. In the simplest nontrivial case $\Theta_3(\mathbb{R})$ can be identified with the quotient space of $\mathbb{C}P^2$ by complex conjugation and, therefore, Vassiljev’s result could be considered as higher-dimensional generalization of the Kuiper-Massey theorem stating that the last space is homeomorphic to $S^4$. The proof given in [12] is based on some deep results of the modern topology.
In the third paragraph of the present paper we give an elementary proof of even a little more strong, smooth, version of this theorem. The idea is the following. Let’s consider the set of the symmetric (Hermitian) matrices satisfying
\[
\begin{align*}
\text{Tr } X &= 0 \\
\text{Tr } X^2 &= 1.
\end{align*}
\]
This is a sphere of the dimension we need. Consider the function \( f = \text{Tr } X^3 \) on it. Its critical points form \((n - 1)\) smooth submanifolds isomorphic to \( G_1(V), \ldots, G_{n-1}(V) \). The corresponding gradient flow preserves the flag of the eigenspaces of a matrix \( X \) and determines in a natural way the decomposition of our sphere into “flag” simplexes \( \sigma_F \), giving the equivalence with the flag join we need. It is interesting to note that the evolution of the eigenvalues of matrix \( X \) is described by the generalized Volterra chain (see. [14]).

Our paper does not cover the whole variety of the applications of the integrable Morse functions. Looking at the history of the Morse theory from this point of view one can find a number of other remarkable examples. For instance, the well-known proof of the Bott periodicity [1, 2] uses the functionals corresponding to integrable systems on the classical Lie groups. In the original Bott’s proof [3] it was the action functional on the loop space corresponding to the geodesic flow on the group which is obviously integrable, while in Fomenko’s version [5] the Dirichlet functional was used, whose extremals are known in the mathematical physics as the main chiral fields, the equations of which were integrated by the inverse scattering method only in 70-th (Zakharov and Mikhailov) [14, 17]. Of course, in both cases the integrability was used only to find explicitly some partial solutions, but anyway this also illustrates our approach to Morse theory.

From this point of view it seems to be important that the equations of the famous self-dual Yang-Mills fields can be interpreted as the gradient flow of the Chern-Simons functional (see e.g. [18]). It is well-known that with these equations are connected the recent remarkable achievements in topology (Donaldson, Witten, Seiberg).

We note also an interesting paper by C. Tomei [19] in which the Toda lattice is used for studying topology of manifold of isospectral Jacoby matrices.

1. **Height function on a symmetric space as a perfect Morse function.**

We start with consideration of the height functions on the classical Lie groups. Let \( G \) be one of the following compact Lie groups: \( O(n) \), \( U(n) \), \( Sp(n) \). Here \( Sp(n) \) is the group of quaternionic \( n \times n \) matrices \( X \) such that \( X^{-1} = X^T \). Sometimes it is called also the unitary symplectic group because it can be realized as the intersection of the complex symplectic group \( Sp(2n, \mathbb{C}) \) with \( U(2n) \).

Consider an arbitrary height function on \( G \) embedded in a natural way into the matrix space \( M(n, k) \), \( k = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \), supplied with the standard scalar product
\[
(X, Y) = \text{Re} \text{Tr}(X^*Y).
\]

Here for \( X \in M(n, k) \) by \( X^* \) we denote the matrix \( \overline{X}^T \). Such a function has the form:
\[
f_A(X) = \text{Re} \text{Tr}(AX), \tag{1.1}
\]
for some matrix \( A \in M(n, k) \).
Proposition 1.1 The critical points of the function $f_A$ are the matrices $X \in G$ that satisfy $AX = (AX)^*$. The gradient flow for the function $f_A$ in the Cartan-Killing metric on the group $G$ has the form

$$\dot{X} = A^* - XAX.$$ 

(1.2)

**Proof.** Recall that the Cartan-Killing metric on the group $G$ is the pullback of the Euclidean structure on the matrix space $M(n,k)$ under the embedding. The right hand side of the equation (1.2) is the result of the projection of the vector $A^*$, the gradient of the function $f_A$ in the ambient space $M(n,k)$, onto the tangent space $T_X G: X((X^{-1}A^*) - (X^{-1}A^*)^*) = A^* - XAX$, since $X^{-1} = X^*$ for $X \in G$. The last expression vanishes iff $AX = (AX)^*$.\[\leadsto\]

**Corollary 1.1** The gradient flow of the function $F_A$ on the group $G$ solves the problem of the polar decomposition for the matrix $A$.

Indeed, let

$$A = JQ,$$

where $J = J^*$ is a positive definite matrix and $Q \in G$, is the polar decomposition of the non-degenerate matrix $A$. Then for any $X$ from $G$

$$\Re \text{Tr}(JX) \leq \Re \text{Tr}(J) = \Re \text{Tr}(AQ^*),$$

so, $X = Q^*$ is the maximum point of the function $f_A$ on $G$. It is not difficult to show (see also below) that the other critical points are not local maxima. Therefore, for almost all initial points $X(0) = X_0 \in G$ for the equation (1.2) we will have $X(t) \to Q^* (t \to \infty)$.

**Remark 1.** It is not necessary to take as a starting point an element of the group $G$. For example, if we set $X_0 = 0$ then solving (1.2) we will always have $X(t) \to Q^* (t \to \infty)$.

To investigate the structure of the set of the critical points of the function $f_A$, its gradient flow, and the corresponding cell decomposition of the group $G$ it is sufficient to consider only diagonal positive matrices $A = \text{diag}(a_1, \ldots, a_n)$ since any nondegenerate matrix can be reduced to such a form by means of suitable left and right actions of $G$ corresponding to some rotation of the ambient space $M(n,k)$.

Let’s consider first the case when $A = I$, $I$ is the identity matrix: $f_I(X) = \Re \text{Tr} X$. It is easy to prove the following result (see for the details e.g. [20]).

**Proposition 1.2** 1) The set of the critical points of the function $f_I$ on $G$ consists of the involutions: $X^2 = I$ and, therefore, can be naturally identified with the disjoint union of the Grassmannian manifolds $G_{n,m}(k)$, $m = 0, 1, \ldots, n$. Namely, to an arbitrary subspace $V \subset k^n$ is related the involutive operator $S_V$, the reflection with respect to $V$:

$$S_V|_V = I$$

$$S_V|_{V^\perp} = -I,$$

and all the involutions can be represented in such a way;
2) The separatrices of the gradient flow coming from the point $S_V \in G$ fill the domain with closure isomorphic to $O(m)$, $U(m)$, or $Sp(m)$ correspondingly, $m = \dim V$, that consists of the operators $X$ such that $X \bigg|_{V^\perp} = -I$;

3) the index of a critical point $S_V$, $\dim V = m$, equals to

$$\text{ind}_{S_V} f_A = \begin{cases} \frac{1}{2}m(m-1), & k = \mathbb{R} \\ m^2, & k = \mathbb{C} \\ m(2m+1), & k = \mathbb{H} \end{cases}.$$  

Let’s consider now the case when all the eigenvalues of the matrix $A$ are different.

**Theorem 1.1** Let $A = \text{diag}(a_1, \ldots, a_n)$, $0 < a_1 < \ldots < a_n$. Then

1) $f_A$ is a Morse function on the group $G$;

2) the critical points of the function $f_A$ are the matrices of the form $X = \text{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$;

3) the index of such a critical point equals to

$$\text{ind}_X (f_A) = \sum_{k=1}^{n} \delta_{\varepsilon_k, 1}((\dim_{\mathbb{R}} k)k - 1);$$  (1.3)

4) the function $f_A$ has the minimal possible number of critical points among all Morse functions on $G$.

**Example.** For the matrix

$$A = \text{diag}(d - 1, 2d - 1, \ldots, nd - 1), \quad d = \dim_{\mathbb{R}} k$$

the value of the function $f_A$ at a critical point coincides with the signature of the Hessian at this point. This implies that the critical points of the same index lie on the same level and the greater index the greater level. Such functions are called sometimes Morse-Smale functions.

**Proof.** We begin with the item 2). According to proposition 1.1, if $X$ is a critical point of $f_A$ then $XAX = A = X^*AX^* = X^{-1}AX^{-1}$, therefore, $A^2 = XAX^{-1}$ and the matrices $X$ and $A^2$ commute. It follows from the diagonal form of the matrix $A$ that such matrix $X$ should be diagonal too.

Thus, $X_0 = \text{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_i = \pm 1$. Let’s compute the second derivative of the function $f_A$ at the point $X_0$. Consider the exponential map $B \mapsto X_0 \exp(B)$, where $B \in T_{X_0}G$. The diagonal and off-diagonal elements of the matrix $B$ can be taken as the coordinates in a small neighbourhood of the point $X_0$ under that map. We have:

$$f_A(X) - f_A(X_0) = \text{Re Tr } A(X - X_0) = \text{Re Tr } AX_0 (\exp B - I) =$$

$$= \text{Re Tr } AX_0 (B + \frac{1}{2}B^2) + o(\|B\|^2) = \sum_{i,j=1}^{n} a_i \varepsilon_i b_{ij} + o(\|B\|^2) =$$

$$= \sum_{n \geq j \geq i \geq 1} (a_i \varepsilon_i + a_j \varepsilon_j) |b_{ij}|^2 + o(\|B\|^2).$$
Since $a_j > a_i \geq 0$ for $j > i$ one has $\text{sgn}(a_i \varepsilon_i + a_j \varepsilon_j) = \text{sgn}(a_j \varepsilon_j)$. Thus $|b_{ij}|^2$, $j \geq i$, is contained in $d^2 f$ with the negative sign if $\varepsilon_j = 1$ and with the positive one if $\varepsilon_j = -1$. This implies the items 1) and 3).

The item 4) follows from the Morse inequalities \[1\] valid for an arbitrary Morse function $f$:

$$m_j \geq b_j,$$

where $m_j$ is the number of the critical points of index $j$ of the function $f$ and $b_j$ is the $j$-th Betti number. For the function $f_A$ a simple calculation shows that the number $m_j$ coincides with the corresponding Betti number (in the case $G = O(n)$ one should consider the homology groups over $\mathbb{Z}_2$). The Betti numbers of the classical Lie groups have been calculated first by L. S. Pontrjagin in \[13\], one can find them also e. g. in \[21\]. Thus the Morse inequalities \[1.4\] for $f_A$ are in fact equalities and this completes the proof.\>

A Morse function with the minimal numbers of the critical points is called a perfect Morse function. We will give now a geometrical proof of this property for the height functions, which does not use the information about the homology groups of $G$ in a more general situation.

Let $M$ be a symmetric space embedded in Euclidean Space $E$. Associate with each point $x \in M$ the reflection $S_x$ of the space $E$ with respect to the normal subspace at the point $x$:

$$S_x(x) = x, \quad dS_x\big|_{T_x M} = -I, \quad dS_x\big|_{(T_x M)^\perp} = I.$$ 

We will call an embedding $M \to E$ symmetric if all such reflections $S_x$ map $M$ into itself.

The classical groups $G$ with the natural embeddings give the examples of such symmetric spaces: $M = G, E = M(n, k), S_X(Y) = XY^*X$. Among other symmetric embeddings\[4\] we mention the following:

1. Grassmann manifolds $M = G_{n,m}(k)$ embedded in the space of symmetric (Hermitian) matrices $n \times n$ as the set of the reflections with respect to the corresponding subspaces, $S_X(Y) = XYX$.

2. Lagrangian Grassmannians, consisting of the Lagrangian subspaces in $\mathbb{R}^{2n} = \mathbb{C}^n$: $M = U(n)/O(n)$. It is embedded into the space of complex symmetric matrices as the subset of the unitary matrices, $X = X^T, X^{-1} = X^*$. The reflection $S_X$ has the form: $S_X(Y) = XYX$.

3. The symmetric space of complex structures in $\mathbb{R}^{2n}$: $M = O(2n)/U(n)$. One can realise it as the set of the skew-symmetric orthogonal matrices in the Euclidean space of all skew-symmetric matrices, $S_X(Y) = -XYX$.

4. The space of quaternionic structures in $\mathbb{C}^{2n}$: $M = U(2n)/Sp(n)$, realised as the set of the skew-symmetric unitary matrices in the space of all skew-symmetric matrices, $S_X(Y) = -XYX$.

5. Symmetric space $M = Sp(n)/U(n)$ can be embedded into the space of skew-Hermitian quaternion matrices as the intersection with $Sp(n)$, $S_X(Y) = -XYX$.

The restriction to $M$ of a linear function on $E$ will be referred to as a height function. Among them the Morse functions form an open everywhere dense set. For example, the function $f_A$ on the classical Lie group $G$ is a Morse function iff the eigenvalues of

\[1\] All such embeddings are classified, see \[28\] and Note added in proof below.
the positive symmetric (Hermitian) matrix $J$ from the polar decomposition $A = JQ$ are pairwise different (see the calculation above).

**Theorem 1.2** A Morse height function on a symmetric embedding of symmetric space $M$ is a perfect Morse function.

**Proof.** Notice first of all that by a small variation of such a function $h$ which does not change the number and indeces of the critical points we can always reduce to the case when all the critical values of the function $h$ are different and the gradient separatrix surfaces outgoing from the critical points intersect the incoming ones transversally. In particular, the separatrices outgoing from a critical point of index $k$ never hit into critical the points of indeces $\geq k$ and the set of such separatrices outgoing from a point of index $k$ and incoming to a point of index $(k - 1)$ is finite.

Such a function $h\big|_M$ provides a cell decomposition of the manifold $M$ in the following way. With each critical point of index $k$ we associate a $k$-dimensional cell which interior is the union of this critical point and all outgoing gradient separatrices. The incidence number of two cells corresponding to critical points of indeces $k$ and $(k - 1)$ is defined as the number of gradient separatrices outgoing from the first point and hitting the second one taken with some sign depending on the orientation of the cells, which is not essential for us.

Let $x_0$ be one of the critical points of $h\big|_M$. Since $dh(T_{x_0}M) = 0$ the function $h$ is invariant under the reflection $S_{x_0} : S_{x_0}^* h = h$. Thus the set of the critical points of the function $h\big|_M$ is mapped to itself under such reflection. Since there is no more than one critical point on the same level of $h$ all the critical points are fixed under $S_{x_0}$.

Consider now the cell complex $C_*$ defined by the function $h\big|_M$. The groups $C_k$ of this complex consist of the formal sums of the critical points of index $k$. Let points $x$ and $y$ have indeces $k$ and $(k - 1)$ correspondingly. As it was shown before, $S_x(y) = y$. Since $dS_x\big|_{T_xM} = -I$ $S_x$ acts freely on the set of the gradient separatrices going from $x$ to $y$ and from $S_x^2 = I$ it follows that the number of these trajectories is even.

Thus all the incidence numbers of the complex $C_*$ are even and boundaries of all elements of the complex $C_* \otimes \mathbb{Z}_2$ are 0. According to the Morse theory, the homology of this complex coincides with $H_*(M, \mathbb{Z}_2)$. The fact that the differential of the complex is trivial implies that Morse inequalities turn out to be equalities and the number of critical points of the function $h\big|_M$ is the minimal possible one among all Morse functions.

**Remark 2.** It turns out that for the height functions on the symmetric spaces the gradient separatrices never hit a critical point with “wrong” index, so we have a correct cell decomposition for all Morse height functions (see the next section).

**Remark 3.** The height functions turn out to be perfect Morse functions not only for symmetric spaces but also for some other classical manifolds. For instance, this is the case for the Stiefel manifolds $V_{n,k}$ naturally embedded into the space of $n \times k$ matrices and for the flag varieties realized as the set of symmetric (Hermitian) matrices with given spectrum.
The functions considered in the previous section have the following remarkable property: their gradient flows can be linearized by an appropriate change of variables and, therefore, can be integrated explicitly.

Consider first the gradient flow of the function \( f_A(X) = \text{Tr} AX \), \( A = A^* \) on the group \( G = O(n), U(n), \text{Sp}(n) \):

\[
\dot{X} = A - XAX. \tag{2.1}
\]

Let

\[
Y = (I - X)(I + X)^{-1}
\]

be the Cayley transform of matrix \( X \). It is known that the Cayley transform is involutive and provides a bijection between skew-symmetric (skew-hermitian) and orthogonal (unitary, symplectic) matrices not having \(-1\) as an eigenvalue.

**Lemma 2.1** ([22]) \textit{The Cayley transform linearizes the flow (2.1):}

\[
\dot{Y} = - (AY + YA). \tag{2.2}
\]

**Proof.**

\[
\dot{Y} = \left((I - X)(I + X)^{-1}\right)^* = -\dot{X}(I + X)^{-1} - (I - X)(I + X)^{-1}\dot{X}(I + X)^{-1} =
\]

\[
= -(I + Y)\dot{X}(I + X)^{-1} = -\frac{1}{2}(I + Y)(A - XAX)(I + Y) =
\]

\[
= -\frac{1}{2}(I + Y)\left(A - (I + Y)^{-1}(I - Y)A(I - Y)(I + Y)^{-1}\right)(I + Y) =
\]

\[
= -\frac{1}{2}\left((I + Y)A(I + Y) - (I - Y)A(I - Y)\right) = -(AY + YA). \blacktriangleleft
\]

**Proposition 2.1** \textit{The equation (2.1) can be solved explicitly for arbitrary initial data \( X(0) = X_0 \). The solution has the form:}

\[
X(t) = \left( \sinh(At) + \cosh(At)X_0 \right)\left( \cosh(At) + \sinh(At)X_0 \right)^{-1}, \tag{2.3}
\]

where \( \sinh \) and \( \cosh \) denote the standard hyperbolic functions of a matrix.

**Proof.** For an initial matrix \( X_0 \) not having \(-1\) as an eigenvalue the equation (2.3) is obtained by means of the Cayley transform from the solution of the equation (2.2):

\[
Y(t) = \exp(-At)Y_0 \exp(-At),
\]

with the initial point \( Y_0 = (I - X_0)(I + X_0)^{-1} \). The set of the matrices not having \(-1\) in the spectrum is everywhere dense in \( G \), so the formula (2.3) gives the solution for all \( X_0 \) it has a sense.

The function \( \text{th} x = \sinh x / \cosh x \) takes values less than 1 for all real \( x \), hence the matrix \( \cosh(At) + \sinh(At)X_0 = \cosh(At)\left(I + \text{th}(At)X_0\right) \) is non-degenerate for all \( X_0 \) from \( G \). \blacktriangleleft

**Proposition 2.2** \textit{If the matrices \( A_1 \) and \( A_2 \) are symmetric (Hermitian) and \( A_1A_2 = A_2A_1 \) then the gradient flows of the functions \( f_{A_1} \) and \( f_{A_2} \) on the group \( G \) commute.}
Proof.

\[ \left[ \text{grad} f_{A_1}, \text{grad} f_{A_2} \right]_X = \left( \nabla_{\text{grad} f_{A_1}} (\text{grad} f_{A_2}) - \nabla_{\text{grad} f_{A_2}} (\text{grad} f_{A_1}) \right)_X = \]

\[ = \nabla_{A_1 - XA_1X} (A_2 - XA_2X) - \nabla_{A_2 - XA_2X} (A_1 - XA_1X) = \]

\[ = -(A_1 - XA_1X)A_2X - XA_2(A_1 - XA_1X) + \]

\[ + (A_2 - XA_2X)A_1X + XA_1(A_2 - XA_2X) = \]

\[ = (A_2A_1 - A_1A_2)X + X(A_1A_2 - A_2A_1). \]

The last expression vanishes for commuting matrices \( A_1 \) and \( A_2 \).

Corollary 2.1 Grassmann manifolds \( G_{n,m}(k) \) embedded into \( G \) are invariant under the gradient flow \((2.1)\). The restricted flow coincides with the gradient flow of the function \( f_{A} \big|_{G_{n,m}(k)} \) with respect to the induced metric on \( G_{n,m}(k) \).

Proof. Since the gradient flows of functions \( f_I \) and \( f_A \) commute the last flow should preserve the set of the first one’s stable points. Note that \( G_{n,m}(k) \) is a connected component of the critical point set of the function \( f_I \) (see. proposition 1.2). The restricted flow coincides with the gradient flow of the function \( f_{A} \big|_{G_{n,m}(k)} \) with respect to the induced metric on \( G_{n,m}(k) \) because of the following general result.

Proposition 2.3 Suppose that a submanifold \( N \) of a Riemannian manifold \( M \) is invariant under the gradient flow of some function \( f \) on \( M \). Then the restricted flow on \( N \) is the gradient one with respect to the restricted function \( f \big|_N \) and induced Riemannian metric on \( N \).

Proof. It is easy to see that in general case the gradient flow of the restricted function \( f \big|_N \) coincides with the orthogonal projection of grad \( f \) in the induced metric to the tangent subspace of \( N \).

It is easy to check that all the symmetric spaces listed in the previous section are invariant under the gradient flows of the height functions on the corresponding Lie groups. Indeed, for the Lagrangian Grassmannians \( LG_n = U(n)/O(n) \) this follows immediately from the flow equation \((2.1)\). For the spaces of complex and quaternionic structures \( O(2n)/U(n) \) and \( U(2n)/Sp(n) \) one can see it from the equations \((1.2)\):

\[ \dot{X} = A^* - XAX = -(A + XAX) = -\dot{X}^T \]

since \( A^T = -A \) and \( X^T = -X \). Similarly, for spaces \( Sp(n)/U(n) \) \( A^* = -A \), \( X^* = -X \),

\[ \dot{X} = A^* - XAX = -(A + XAX) = -\dot{X}^*. \]

Theorem 2.1 The gradient flows of the height functions on the classical Lie groups and the embedded symmetric spaces described above can be integrated explicitly.
Proof. Using an appropriate shift by an orthogonal (unitary, symplectic) matrix one can always reduce the equations of the gradient flow to the form (2.1) and then apply the explicit formula (2.3).

For the Grassmannians one can give another explicit formula for the gradient flow. Let us determine an element \( V \in G_{n,m}(k) \) by \( n \times m \) matrix \( Z \) with the columns forming a basis in the subspace \( V \). Then the operator of the reflection with respect to the subspace \( V \) has the form:

\[
X = 2Z(Z^*Z)^{-1}Z^* - I.
\]

Lemma 2.2 If \( Z(t) \) is changing according to the equation

\[
\dot{Z} = AZ,
\]

then \( X(t) \) satisfies (2.1).

Proof. Indeed,

\[
\dot{X} = \left(2Z(Z^*Z)^{-1}Z^* - I\right) = 2\dot{Z}(Z^*Z)^{-1}Z^* + 2Z(Z^*Z)^{-1}\dot{Z}^* - 2Z(Z^*Z)^{-1}(\dot{Z}^*Z + Z^*\dot{Z})(Z^*Z)^{-1}Z^* = 2AZ(Z^*Z)^{-1}Z^* + 2Z(Z^*Z)^{-1}Z^*A - 4Z(Z^*Z)^{-1}Z^*AZ(Z^*Z)^{-1}Z^* = A - XAX.\]

Proposition 2.4 The evolution of a subspace \( V \in G_{n,m}(k) \) under the gradient flow of the function \( f_A \) on the Grassmannian \( G_{n,m}(k) \) is given by

\[
V(t) = \exp(At)V.
\]

Let’s consider now the cell decompositions determined by our Morse functions. Let \( A = \text{diag}(a_1, \ldots, a_n) \), \( 0 < a_1 < a_2 < \ldots < a_n \). As a simple corollary of the the previous description of the gradient flow one has

Proposition 2.5 The cell decomposition defined by the function \( f_A \) on the Grassmannian \( G_{n,m}(k) \) coincides with the classical Schubert decomposition.

The following statement describes the corresponding cell decomposition of the group \( G \), which turns out to be a small modification of the one considered by V. Vassiljev [12].

Theorem 2.2 The cells of the classical Lie group \( G \) determined by the Morse function \( f_A \) correspond bijectively to the Schubert cells of Grassmannians \( G_{n,m}(k) \), \( m = 0, 1, \ldots, n \). The interior of the cell corresponding to a Schubert cell \( \sigma \) consists of the operators such that the eigenspace corresponding to the eigenvalue \(-1\) has the orthogonal complement belonging to \( \sigma \).
Proof. As follows from the proposition [1] all the critical points of the function $f_A$ lie on the Grassmannians embedded into $G$. This gives the bijectivity. It is easy to check that the dimensions of the cells coincide with the indeces of the corresponding critical points. Thus only the invariance of the cells under the gradient flow has to be proven. Let $X(t)$ be a gradient trajectory. Denote by $V_-(t)$ the eigenspace of the operator $X(t)$ with the eigenvalue $-1$. We state that

$$V_-(t) = \exp(-At)(V(0)),$$

which implies

$$(V_-(t))^\perp = \exp(At)(V(0))^\perp.$$

Indeed,

$$\left. \left( \cosh(At) + \sinh(At)X(0) \right) \right|_{V_-(0)} = \left. \left( \cosh(At) - \sinh(At) \right) \right|_{V_-(0)} = \exp(-At)\left|_{V_-(0)} \right.$$

This means that

$$\left( \cosh(At) + \sinh(At)X(0) \right)^{-1}\exp(-At)V_-(0) = V_-(0).$$

Now taking into account (2.3) we get:

$$X(t)\left|_{\exp(-At)V_-(0)} \right. =$$

$$= \left. \left( \sinh(At) + \cosh(At)X(0) \right) \left( \cosh(At) + \sinh(At)X(0) \right)^{-1} \right|_{\exp(-At)V_-(0)} =$$

$$= \left. \left( \sinh(At) - \cosh(At) \right) \exp(At) \right|_{\exp(-At)V_-(0)} = -I\left|_{\exp(-At)V_-(0)} \right.$$

Thus $\exp(-At)(V(0)) \subset V(t)$. In a similar way $V(t) \subset \exp(-At)(V(0))$. This completes the proof.$\triangleright$

We see that the cell decomposition is common for a big family of the Morse functions $f_A$. This is a particular case of the following general result.

**Theorem 2.3** Any two functions with commuting gradient flows from a connected family of Morse functions on a compact Riemannian manifold $M$ determine the same cell decomposition of $M$.

Proof. From the commutativity it follows that the gradient flow of the first function preserves the stationary set of the second gradient flow. But since they are Morse functions this is a discrete set and, therefore, both functions have the same critical points. For given point $x$ the cell containing this point corresponds to the critical point which is a limit at the plus infinity of the gradient trajectory starting from $x$ (see above). It is clear that for functions close enough we have the same limit and, therefore, the same cell decomposition. Now the theorem follows from the connectness of the family.$\triangleright$

This allows one to know the cell decomposition for such a family if one knows it for a particular function from it. For example, in the family $f_A$ we can choose

$$A = \text{diag}(d-1, 2d-1, \ldots, nd-1), \quad d = \dim_{\mathbb{R}} k$$

when the corresponding $f_A$ is a Morse-Smale function (see section 1).

As a corollary of the previous results we have the Vassiljev-Mahowald decomposition.
Theorem 2.4 ([12]) For the homology of the classical Lie groups one has the following isomorphisms:

\[ H_i(O(n), \mathbb{Z}_2) \cong \bigoplus_{k=0}^{n} H_{i-k(k+1)/2}^{\lbrace G_{n,k}^{\text{R}} \rbrace} \mathbb{Z}_2 \]
\[ H_i(U(n), \mathbb{Z}) \cong \bigoplus_{k=0}^{n} H_{i-k^{2}}^{\lbrace G_{n,k}^{\text{C}} \rbrace} \mathbb{Z} \]
\[ H_i(Sp(n), \mathbb{Z}) \cong \bigoplus_{k=0}^{n} H_{i-k(2k+1)}^{\lbrace G_{n,k}^{\text{H}} \rbrace} \mathbb{Z}. \]

In a similar way the following result can be proved.

Theorem 2.5 For the homology of Lagrangian Grassmannians, spaces of complex and quaternion structures and spaces \( Sp(n)/U(n) \) one has the following decompositions:

\[ H_i(U(n)/O(n), \mathbb{Z}_2) \cong \bigoplus_{k=0}^{n} H_{i-k(k+1)}^{\lbrace G_{n,k}^{\text{R}} \rbrace} \mathbb{Z}_2 \]
\[ H_i(O(2n)/U(n), \mathbb{Z}) \cong \bigoplus_{k=0}^{n} H_{i-k(k-1)}^{\lbrace G_{n,k}^{\text{C}} \rbrace} \mathbb{Z} \]
\[ H_i(U(2n)/Sp(n), \mathbb{Z}) \cong \bigoplus_{k=0}^{n} H_{i-2k^{2}}^{\lbrace G_{n,k}^{\text{H}} \rbrace} \mathbb{Z} \]
\[ H_i(Sp(n)/U(n), \mathbb{Z}) \cong \bigoplus_{k=0}^{n} H_{i-k(k+1)}^{\lbrace G_{n,k}^{\text{C}} \rbrace} \mathbb{Z}. \]

For Lagrangian Grassmannians the cell decomposition determined by a suitable height function turns out to coincide with one sometimes called Schubert decomposition. It could be described as follows (see [23]): for given Schubert cell \( \alpha \) in \( G_{n,k}^{\text{R}} \) we define the cell \( [\alpha] \subset L\text{G}_n \) as the set of all Lagrangian subspaces \( L \in \mathbb{C}^n \) with a standard symplectic structure such that the projection \( V \) of \( L \) to \( \mathbb{R}^n \subset \mathbb{C}^n \) belong to \( \alpha \). Indeed, let \( U = X + iY \) be a unitary matrix, which columns represent a basis in \( L \). One has \( (X + iY)(X^T - iY^T) = I \) and, therefore, \( XX^T + YY^T = I \) and \( YY^T - XX^T = 0 \). This implies that the symmetric unitary matrix \( UU^T \) acts as \( -I \) on the orthogonal complement of \( V \) because \( UU^T = (XX^T - YY^T) + i(YY^T + XX^T) \) and \( XX^T v = 0 \) for all \( v \in V^\perp \).

Cell decomposition of the space \( Sp(n)/U(n) \) can be constructed in a way analogous to Lagrangian Grassmannians. Geometrically the space \( Sp(n)/U(n) \) can be realized as the set of complex \( n \)-dimensional subspaces in \( \mathbb{H}^n \), on which the restriction of bilinear skew-symmetric function \( \Omega(u, v) = (uj, v) \) is trivial, where \( \langle \cdot, \cdot \rangle \) is the hermitian scalar product in \( \mathbb{C}^n \). To any such a subspace \( L \) a matrix \( U = X + jY \) from \( Sp(n) \), where \( X, Y \in M(n, \mathbb{C}) \), corresponds columns of which make up a basis in \( L \). Matrix \( UiU^* \) is skew-Hermitian (as a quaternionian one) and the operator \( -iUiU^* \) acts as \( -I \) on \( V^\perp \), where \( V \subset \mathbb{C}^n \) is the subspace spanned over \( \mathbb{C} \) by the columns of matrix \( X \).

For the space of complex structures \( CS_n \) the gradient cell decomposition implying isomorphism mentioned above is constructed in the following way. Let’s fix some complex structure \( J_0 \subset CS_n \), that is an orthogonal operator in \( \mathbb{R}^{2n} \), satisfying the relation \( J_0^2 = -I \). Then for a Schubert cell \( \alpha \subset G_{n,k}^{\text{C}} \) we define \( [\alpha] \) as the union of all complex structures.

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\(^2\)We would like to quote D. B. Fuchs [23]: “The manifold \( U(n)/O(n) \) has a convenient cell decomposition which also could be referred to as Schubert one although it was invented by Arnold.”
preserving $V$ and coinciding with $J_0$ on $V^\perp$ for all $V \in \alpha$. In the quaternion case we have a similar construction.

We would like to mention that one can prove other decomposition formulas using a suitable Morse-Bott height functions with critical submanifolds. For instance, using the gradient flow of the height function $f_A$ with diagonal $A$ having only two different eigenvalues we can prove the following decomposition formula for the homology of Grassmannians.

**Theorem 2.6** The following isomorphisms hold for arbitrary $n = n_1 + n_2$:

\[
H_i(G_{n,k}(\mathbb{R}), \mathbb{Z}_2) \cong \bigoplus_{k_1+k_2=k} H_{i-(n_1-k_1)k_2}(G_{n_1,k_1}(\mathbb{R}) \times G_{n_2,k_2}(\mathbb{R}), \mathbb{Z}_2)
\]

\[
H_i(G_{n,k}(\mathbb{C}), \mathbb{Z}) \cong \bigoplus_{k_1+k_2=k} H_{i-2(n_1-k_1)k_2}(G_{n_1,k_1}(\mathbb{C}) \times G_{n_2,k_2}(\mathbb{C}), \mathbb{Z})
\]

\[
H_i(G_{n,k}(\mathbb{H}), \mathbb{Z}) \cong \bigoplus_{k_1+k_2=k} H_{i-4(n_1-k_1)k_2}(G_{n_1,k_1}(\mathbb{H}) \times G_{n_2,k_2}(\mathbb{H}), \mathbb{Z}).
\]

A flow similar to ours was used by V. M. Buchstaber [24, Lemma 3.7] in theory of characteristic classes.

3. Flag join of Grassmannians and an integrable gradient flow on the sphere

In this section we will give an elementary proof of the following geometrical result of V. Vassiljev. Let’s consider a general embedding of the Grassmannians $G_{n_1}(k), G_{n_2}(k), \ldots, G_{n,n-1}(k)$ into a Euclidean space of a large dimension. The union of the simplexes with the vertices $V_1 \in G_{n_1}(k), V_2 \in G_{n_2}(k), \ldots, V_{n-1} \in G_{n,n-1}(k)$ forming a complete flag: $V_1 \subset V_2 \subset \cdots \subset V_{n-1}$, with the induced topology we will call the flag join of Grassmannians and denote following to Vassiljev by $\Theta_n(k)$.

**Theorem 3.1 (V. Vassiljev [12, Theorem 5])** The flag join of Grassmannians is isomorphic to the sphere of the appropriate dimension:

\[
\Theta_n(\mathbb{R}) \cong S^{\frac{n(n-1)}{2}+n-2}
\]

\[
\Theta_n(\mathbb{C}) \cong S^{n(n-1)+n-2}
\]

\[
\Theta_n(\mathbb{H}) \cong S^{2n(n-1)+n-2}.
\]

To prove this let’s consider the function $f = \frac{1}{3} \text{Tr} X^3$ on the set of symmetric (Hermitian) $n \times n$ matrices $X$ satisfying the conditions:

\[
\begin{align*}
\text{Tr} X &= 0 \\
\text{Tr} X^2 &= 1.
\end{align*}
\]

(3.1)

This is the sphere $S^N$ of dimension

\[
N = n - 2 + (\dim_{\mathbb{R}} k) \frac{n(n-1)}{2}.
\]
Proposition 3.1 The set of the critical points of the function $f$ on the sphere $S^N$ is a disjoint union of the Grassmann manifolds $G_{n,1}(k), G_{n,2}(k), \ldots, G_{n,n-1}(k)$ smoothly embedded into $S^N$ in such a way that to a subspace $V \in G_{n,m}(k)$ corresponds the matrix $X = X(V)$ such that
\[
X\bigg|_V = -\sqrt{\frac{n-m}{nm}} I, \\
X\bigg|_{V^\perp} = \sqrt{\frac{m}{n(n-m)}} I.
\] (3.2)

Proof. Functions $\text{Tr} X$, $\text{Tr} X^2$, $\text{Tr} X^3$ in the space of symmetric (Hermitian) matrices have gradients:
\[
\begin{align*}
\text{grad}(\text{Tr} X) &= I, \\
\text{grad}(\text{Tr} X^2) &= 2X, \\
\text{grad}(\text{Tr} X^3) &= 3X^2.
\end{align*}
\]

The critical points of the function $f$ are those $X$ for which matrices $I$, $X$, $X^2$ are dependent. This holds iff $X$ has no more than two different eigenvalues. Since $\text{Tr} X = 0$, $\text{Tr} X^2 = 1$ such a matrix $X$ has exactly two different eigenvalues $\mu_1$ and $\mu_2$, $\mu_1 < 0 < \mu_2$. Let $m$ and $(n - m)$ be their multiplicity. It is easy to find from (3.1) that
\[
\mu_1 = -\sqrt{\frac{n-m}{nm}}, \quad \mu_2 = \sqrt{\frac{m}{n(n-m)}}.
\]

An operator with such a spectrum is uniquely determined by its eigenspace corresponding to $\mu_1$.\)

Lemma 3.1 The gradient flow of the function $f$ has the form:
\[
\dot{X} = X^2 - f(X)X - \frac{1}{n}I. \quad (3.3)
\]

Proof. The right hand side of the equation (3.3) is the orthogonal projection of the vector $X^2$ to the tangent space of our $S^N$ with the normal vectors $X$ and $I$:
\[
X^2 - \frac{(X^2, X)}{(X, X)} X - \frac{(X^2, I)}{(I, I)} I = X^2 - \frac{\text{Tr} X^3}{\text{Tr} X^2} X - \frac{\text{Tr} X^2}{\text{Tr} I} I = X^2 - (\text{Tr} X^3)X - \frac{1}{n} I.
\]

It is well-known that it gives the gradient vector of the restricted function with respect to the induced metric.\)

Proposition 3.2 The gradient flow of the function $f$ preserves the eigenspaces and the anharmonic ratios of the eigenvalues of the matrix $X$.

Proof. The first statement follows from the fact that the left hand side of the equality (3.3) is a polynomial of $X$ with scalar coefficients. The dynamics of the eigenvalues is determined by a vector field of the form $(a\lambda^2 + b\lambda + c)\frac{\partial}{\partial \lambda}$, where $(a, b, c) \neq (0, 0, 0)$, which is the same for all of them. But it is well-known that such a vector field generates a one-parametric group of the projective transformations of the line.\)

Let
\[
\lambda_1(X) \leq \lambda_2(X) \leq \ldots \leq \lambda_n(X) \quad (3.4)
\]
be the eigenvalues of the matrix $X$,

$$V_1(X), V_2(X), \ldots, V_n(X)$$

be the corresponding eigenspaces. Let’s define the eigenflag of $X$ as

$$F(X) = \{U_0 \subset U_1 \subset \ldots \subset U_n\}, \quad (3.5)$$

where

$$U_0(X) = \{0\}, \quad U_k(X) = \bigoplus_{i=1}^{k} V_i(X), \quad 1 \leq k \leq n.$$ 

If some eigenvalues are equal to each other the corresponding eigenflag will be incomplete.

Proposition 3.2 gives the set of integrals of the flow $(3.3)$, consisting of the eigenflag of $X$ and the anharmonic ratios of the eigenvalues. This set is complete in the sense that if we fix the values of the integrals we will have a trajectory.

Indeed, a symmetric (Hermitian) matrix is completely determined by its spectrum and the eigenflag. Given the anharmonic ratios the spectrum is defined up to a projective transformation. The group of projective transforms of the line is 3-dimensional, therefore, the conditions

$$\sum_{i=1}^{n} \lambda_i = 0$$

$$\sum_{i=1}^{n} \lambda_i^2 = 1,$$ 

equivalent to $(3.4)$, determine a 1-dimensional family.

For two flags $F_1$ and $F_2$ we will write $F_1 \leq F_2$ if all the subspaces of $F_1$ are contained also in $F_2$.

**Lemma 3.2** For a given flag $F = \{U_0 = \{0\} \subset U_1 \subset U_2 \subset \ldots \subset U_{k+1} = k^n\}$ the matrices $X$ satisfying $(3.1)$ and such that $F(X) \leq F$ form a $(k-1)$-dimensional simplex with the vertices corresponding to the subspaces $U_1, \ldots, U_k$ according to $(3.2)$. The ratios

$$a_i = \frac{\lambda_{i+1}(X) - \lambda_i(X)}{\lambda_n(X) - \lambda_1(X)},$$ 

$(3.7)$

are the barycentric coordinates on this simplex.

**Proof.** For simplicity we give the proof only for the case of complete flag $F$. In this case we can assume without loss of generality that the subspaces $U_1, \ldots, U_n$ are $\langle e_1 \rangle, \langle e_1, e_2 \rangle, \ldots, \langle e_1, \ldots, e_n \rangle$, where $e_1, \ldots, e_n$ is the standard basis in $k^n$. Then the condition $F(X) \leq F$ is equivalent to the statement that $X$ is a diagonal real matrix with the increasing elements at the diagonal: $X_{11} \leq X_{22} \leq \ldots \leq X_{nn}$.

So, there is one-to-one correspondence between the set of matrices $X \in S^N$ satisfying $F(X) \leq F$ and the set of sequences $\lambda_1 \leq \ldots \leq \lambda_n$, satisfying $(3.4)$. We should only check that the functions $(3.7)$ can be considered as the barycentric coordinates on this set.

Indeed, by construction we have

$$\sum_{i=1}^{n-1} a_i = 1,$$

$$a_i \geq 0 \quad \text{for all } i = 1, \ldots, n - 1.$$ 

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The sequence \( \{\lambda_i\} \) is defined by \( \{a_i\} \) from (3.7) uniquely up to an affine transform. The normalization conditions (3.6) together with the increasing property (3.4) determine it uniquely.

Summarizing, we have the following

**Theorem 3.2** The gradient flow of the function \( f = \frac{1}{3}trX^3 \) on the sphere in the space of symmetric matrices \( X \): \( trX = 0, \ trX^2 = 1 \), is integrable and gives a decomposition of this sphere into a union of simplexes. The vertices of such a simplex belong to the Grassmannians embedded into the sphere as the critical set of \( f \) and form a flag.

Concerning the integrability we would like to mention that the gradient flow of the function \( f \) in the barycentric variables \( a_i \) after a suitable change of time has the form of the *generalised Volterra chain* (see [14]):

\[
\frac{d}{d\tau} a_i = a_i \left( \sum_{k=i-n}^{i-1} a_k - \sum_{l=i+1}^{i+n} a_l \right),
\]

where we set \( a_i \) with \( i \leq 0 \) and \( i > n \) to be zero.

Indeed, from (3.3) we have:

\[
\dot{\lambda}_i = \lambda_i^2 - I_3 \lambda_i - \frac{1}{n}, \text{ where } I_3 = \sum_{k=1}^{n} \lambda^3.
\]

Hence

\[
\dot{a}_i = \left( \frac{\lambda_{i+1} - \lambda_1}{\lambda_n - \lambda_1} \right) \dot{\lambda}_i + \left[ \frac{\dot{\lambda}_n - \dot{\lambda}_1}{(\lambda_n - \lambda_1)^2} \right] (\lambda_n - \lambda_1) = a_i \left( \frac{\lambda_{i+1} - \lambda_i}{\lambda_n - \lambda_1} \right)
\]

\[
= a_i \left( \frac{\lambda_{i+1}^2 - I_3 \lambda_{i+1} - (\lambda_i^2 - I_3 \lambda_i)}{\lambda_{i+1} - \lambda_i} - \frac{\lambda_n^2 - I_3 \lambda_n - (\lambda_1^2 - I_3\lambda_1)}{\lambda_n - \lambda_1} \right)
\]

\[
= a_i (\lambda_{i+1} + \lambda_i - \lambda_n - \lambda_1) = (\lambda_n - \lambda_1) a_i \left( \sum_{k=1}^{i-1} a_k - \sum_{l=i+1}^{n-1} a_l \right).
\]

Let’s change now the time variable on the gradient flow trajectory: \( d\tau = (\lambda_n - \lambda_1) dt \). The equation will take the form:

\[
\frac{d}{d\tau} a_i = a_i \left( \sum_{k=1}^{i-1} a_k - \sum_{l=i+1}^{n-1} a_l \right),
\]

which coincides with (3.8) if one set \( a_0 = a_{-1} = a_{-2} = \ldots = a_n = a_{n+1} = a_{n+2} = \ldots = 0 \).

In this case the generalized Volterra chain can be easily explicitly integrated. To do this let us change variables (cf. [14]):

\[
b_i = \sum_{k=1}^{i} a_i = \frac{\lambda_{i+1} - \lambda_1}{\lambda_n - \lambda_1}.
\]

We will have:

\[
\frac{d}{d\tau} b_i = b_i \frac{\lambda_{i+1} + \lambda_1 - \lambda_n - \lambda_1}{\lambda_n - \lambda_1} = b_i (b_i - 1),
\]

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hence

\[ b_i = \frac{1}{1 - c_i e^t}, \]

where the constants \( c_i \) are determined by the initial data: \( c_i = 1 - b_i(0)^{-1} \). The last formula is in a good agreement with our previous considerations.

As a corollary we have the following strengthened version of Vassiljev’s theorem.

**Theorem 3.3** Flag join of the Grassmannians \( \Theta_n(k) \) admits the smooth structure of the standard sphere for which all the embeddings \( G_{n,m} \hookrightarrow \Theta_n(k) \) are smooth.

**Acknowledgements.** We would like to thank V. M. Buchstaber for pointing out the references [20, 24]. One of us (A. V.) is grateful to J. Moser and P. Santini for useful discussions and to Forschungsinstitut für Mathematik (ETH, Zurich) for the hospitality during the summer term of 1995.

**Note added in proof.** Recently L. Nicolaescu attracted our attention to the papers [25, 26, 27], where the properties of the height Morse functions on a class of so-called R-symmetric spaces are investigated. These spaces turn out to be precisely the ones for which embeddings with the property we demanded in section 1 do exist [28]. It is interesting that the same embeddings are remarkable from another geometrical point of view: they are minimal in the sense of total curvature (see [29, 30]). The whole list of R-symmetric spaces besides the examples we mentioned in section 1 consists of \( M = Sp(n)/U(n) \) and exceptional spaces \( M = E_6/\text{Spin}(10) \times T, F_4/\text{Spin}(9) \) or Cayley projective plane, \( E_7/E_6 \times T \) and \( E_6/F_4 \) (see [29]). It would be interesting to investigate the integrability of the gradient flows for the height functions on the corresponding embeddings. We are very grateful to L.Nicolaescu and E.Leuzinger for pointing out these remarkable papers to us.

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