We propose a quantitative test for the validity of the semi-classical approximation in gravity, namely that the solutions to the semi-classical equations should be stable to linearized perturbations, in the sense that no gauge invariant perturbation should become unbounded in time. We show that a self-consistent linear response analysis of these perturbations based upon an invariant effective action principle involves metric fluctuations about the mean semi-classical geometry and brings in the two-point correlation function of the quantum energy-momentum tensor in a natural way. The properties of this correlation function are discussed and it is shown on general grounds that it contains no state-dependent divergences and requires no new renormalization counterterms beyond those required in the leading order semi-classical approximation.

**I. INTRODUCTION**

There are many well known difficulties that arise when attempting to combine quantum field theory and general relativity into a full quantum theory of gravity. Almost certainly, a consistent quantum theory at the Planck scale requires a fundamentally different set of principles from those of classical general relativity, in which even the concept of spacetime itself is likely to be radically altered. Yet over a very wide range of distance scales, from that of the electroweak interactions ($10^{-16}$ cm) to cosmology ($10^{27}$ cm), the basic framework of a spacetime metric theory obeying general coordinate invariance is assumed to be valid, and receives phenomenological support both from the successes of flat space quantum field theory at the lower end of this scale and classical general relativity at its upper end. Hence whatever the full quantum theory of gravity entails, it should reduce to an effective low energy field theory on this very broad range of some 33 orders of magnitude of distance [1,2].

To the extent that quantum effects are relevant at all in gravitational phenomena within this range of scales one would expect to be able to apply semi-classical techniques to the low energy effective theory. For the purposes of this paper the quantization of matter fields in curved spacetime backgrounds is what we mean by the semi-classical approximation to gravity. Thus the spacetime metric $g_{ab}$ is treated as a classical $c$-number field and its quantum fluctuations are neglected. An interesting question then arises: Can a quantitative criterion for the validity of this semi-classical approach be given? One would like to have some test of its validity, preferably within the semi-classical framework itself.

To be clear we should distinguish what we mean in this paper by the semi-classical approximation to gravity from the ordinary loop expansion, which is sometimes also called semi-classical. In the ordinary loop expansion of the effective action $\bar{h}$ is the formal loop expansion parameter. As a result both the matter and gravitational fields are treated on exactly the same footing. However the technical issues involved in even defining a one-loop effective action for gravitons that respects both linearized gauge and background field coordinate invariance are difficult enough to have impeded progress in the standard loop expansion in gravity [3, 4]. An unambiguous definition of the corresponding conserved and gauge invariant energy-momentum tensor for gravitons on an arbitrary curved spacetime has not yet been given. Apart from such technical shortcomings an ordinary loop expansion is physically ill-suited to many applications that have been and are likely to be of interest in semi-classical gravity, such as particle creation in the early universe, or black hole radiance, where quantum effects significantly affect the background geometry after some period of time. This is because when quantum effects significantly affect the classical geometry the loop expansion breaks down.
The semi-classical approximation to gravity we discuss in this paper treats the matter fields as quantum but the spacetime metric as classical. This asymmetric treatment can be justified formally by replicating the number of matter fields \( N \) times and taking the large \( N \) limit of the quantum effective action for the matter fields in an arbitrary background metric \( g_{ab} \). Since no assumption of the weakness or perturbative nature of the metric is assumed, the large \( N \) expansion is able to address problems in which gravitational effects on the matter are strong and the matter fields can have a significant effect on the classical geometry in turn. The absence of quantum gravitational effects in the lowest order large \( N \) approximation also means that most of the technical obstacles arising from the quantum fluctuations of the geometry are avoided. General coordinate invariance is assured, provided only that the matter effective action is regularized and renormalized in a manner which respects this invariance \( [3] \). In that case the quantum expectation value of the matter energy-momentum tensor \( \langle T_{ab} \rangle \) is necessarily conserved.

Assuming that the classical energy-momentum tensor for the matter field(s) vanishes (an assumption that may easily be relaxed if necessary), the unrenormalized form of the semi-classical backreaction equations take the form

\[
G_{ab} + \Lambda g_{ab} = 8\pi G_N \langle T_{ab} \rangle.
\]

Here \( G_{ab} \) is the Einstein tensor, \( \Lambda \) is the cosmological constant (which may be taken to be zero in some applications), \( G_N \) is Newton’s constant, and \( \langle T_{ab} \rangle \) is the expectation value of the energy-momentum tensor operator of the quantized matter field(s). Among the technical issues that must be confronted is the renormalization of the expectation value of \( T_{ab} \), a quartically divergent composite operator in \( d = 4 \) spacetime dimensions. Hence renormalization of its expectation value requires the introduction of fourth order counterterms in the effective (low energy) action, modifying the geometric terms on the left side of eq. (1.1) \( [2] \). This renormalization of very high momenta, ultraviolet divergences must be performed in a way that is consistent with general coordinate invariance and leaves undisturbed the vanishing of the covariant divergence of both sides of eq. (1.1).

Once a renormalized semi-classical theory has been defined, one possible route to investigating its validity is to compare calculations in a theory of quantum gravity with similar semi-classical calculations. Since a full quantum theory is lacking, this has been done only in some simplified models of quantum gravity. Large quantum gravity effects were found in three-dimensional models by Ashtekar \( [6] \) and Beetle \( [8] \). In four dimensions Ford considered the case of graviton production in a linearized theory of quantum gravity on a flat space background and compared the results with the production of gravitational waves in semi-classical gravity \( [9] \). He found that they were comparable when the renormalized energy-momentum (connected) correlation function,

\[
\langle T_{ab}(x)T_{cd}(y) \rangle_{\text{con}} \equiv \langle T_{ab}(x)T_{cd}(y) \rangle - \langle T_{ab}(x) \rangle \langle T_{cd}(y) \rangle
\]

satisfied the condition

\[
\langle T_{ab}(x)T_{cd}(y) \rangle_{\text{con}} \ll \langle T_{ab}(x) \rangle \langle T_{cd}(y) \rangle.
\]

The limits of validity of the semi-classical approximation have also been studied without making reference to a specific model of quantum gravity. Kuo and Ford \( [10] \) proposed that a measure of how strongly the semi-classical approximation is violated can be given by how large the quantity,

\[
\Delta_{abcd}(x,y) \equiv \frac{\langle T_{ab}(x)T_{cd}(y) \rangle_{\text{con}}}{\langle T_{ab}(x) \rangle \langle T_{cd}(y) \rangle}
\]

is, where it is assumed that the expectation values in this expression are suitably renormalized. They suggested that the expectation values be normal ordered, which is strictly correct only in a flat spacetime. Later authors used other renormalization schemes. It is important to note that eq. (1.4) is coordinate dependent, since both the numerator and denominator are tensor quantities. The situation is complicated further by the regularization and renormalization issues that arise in defining the quantities appearing in this expression. Kuo and Ford \( [11] \) computed the quantity

\[
\Delta(x) \equiv \frac{\langle T_{00}(x)T_{00}(x) \rangle_{\text{con}}}{\langle T_{00}(x) \rangle^2}
\]

for a free scalar field in flat space for several states including the Casimir vacuum. They found that it vanishes in a coherent state, whereas in many other cases, including the Casimir vacuum, it is of order unity.

\[\text{Page 2}\]
Wu and Ford [11] computed the radial flux component of eq. (1.4) in the case of an evaporating black hole far from the event horizon. They found that it was of order unity over time scales comparable to the black hole mass, but that it averages to zero over much larger times. In a normal ordering prescription they found state dependent divergent terms, which should not be present if all divergences are local and therefore absorbable into purely geometric UV counterterms.

Phillips and Hu [12] used zeta function regularization to compute $\Delta(x)$ with the denominator replaced by the quantity $(T_{00}(x))^2$, for a free scalar field in some curved spacetimes having Euclidean sections. They also computed $\Delta(x)$ for a scalar field in flat space in the Minkowski vacuum state using both point splitting and a smearing operator to remove the divergences [13]. For the flat space calculation they found that $\Delta(x)$ depends on the direction the points are split, but that it is of order unity regardless of how the points are split. They used their results to criticize the Kuo-Ford conjecture and to suggest that the criteria for the validity of the semi-classical approximation should depend on the scale at which the system is being probed.

Although it is somewhat unclear what the dimensionless small parameter which controls the inequality (1.4) is, Ford’s initial work and these subsequent discussions draw attention to the importance of the higher point correlation functions of the energy-momentum tensor. It is quite clear in qualitative terms that if the higher point connected correlation functions of $T_{ab}$ are large (in an appropriate sense to be determined), it cannot be correct to neglect them completely, as the semi-classical equations (1.1) certainly do.

The energy-momentum correlation function $\langle T_{ab}(x)T_{cd}(y) \rangle_{\text{con}}$ has been computed in some cases. Carlitz and Willey [14] computed it for a scalar field in a two dimensional spacetime with a moving boundary. Roura and Verdaguer [15] give its expression for a massless minimally coupled scalar field in de Sitter spacetime in the case that the points are spacelike separated and geodesically connected. Wu and Ford [16] showed that in the simple case of radiation exerting a force on a mirror, the quantum fluctuations in the radiation pressure are due to a state-dependent cross term in the energy-momentum correlation function.

In the same paper [16] Wu and Ford also addressed the Kuo-Ford conjecture and the above mentioned criticism of it by Phillips and Hu. They stated that the conjecture is incomplete because it does not address the effect of divergent state-dependent terms. They suggested that any criterion for the validity of the semi-classical approximation should be a non-local one that involves integrals over the world lines of test particles. They also argued that the question of whether the semi-classical approximation is valid depends on the specifics of a given situation including the scales being probed and the choice of initial quantum state. Although technical problems such as renormalization and coordinate invariance complicate matters, this body of previous work suggests that the correlation function $\langle T_{ab}(x)T_{cd}(y) \rangle_{\text{con}}$ should play an important role in determining the validity of the semi-classical approximation. However, the proper context for incorporating and making use of the information contained in this correlation function remains somewhat in question.

In this paper we propose an unambiguous quantitative criterion for the validity of the semi-classical approximation, namely that solutions to the semi-classical equations should be stable against linearized perturbations. According to standard linear response theory [17] we show that the linearized equations for the perturbed metric depend on the two-point correlation function of the energy-momentum tensor evaluated in the semi-classical metric $g_{ab}$. The correlation function in this case consists of the retarded commutator of two energy-momentum tensor operators, and hence the perturbations are manifestly causal. Moreover the UV divergences in this correlation function are exactly those required to renormalize the semi-classical theory itself. This ensures that no state dependent divergences occur. Finally, gauge dependence of the solutions to the linear response equations is easily studied within the framework of linearized coordinate transformations of the semi-classical background, so that ambiguities related to quantities such as (1.4) do not arise.

Another advantage of this criterion for the validity of the semi-classical approximation is that it lies strictly within the context of that approximation. In this way one avoids problems such as gauge invariance of the energy-momentum tensor for gravitons, that inevitably appear if one tries to go beyond the semi-classical approximation and include quantum effects due to the gravitational field.

To understand qualitatively the role of the two-point correlation function in the validity of the semi-classical approximation, it is helpful to consider the physical analogy between semi-classical gravity and semi-classical electromagnetism. The connected correlation function (1.2) measures the gravitational vacuum polarization which contributes to the proper self-energy of the linearized graviton fluctuations around the background metric, just as the current two-point correlation function, $\langle j^a(x)j^b(y) \rangle$, measures the electromagnetic vacuum polarization which contributes to the proper self-energy of the photon [18]. Hence if these polarization effects are significant, the gravitational fluctuations of the metric must be taken into account in some form and the semi-classical approxi-
information, with or without backreaction, has certainly broken down, at least in the form specified by eq. (1.1), where all fluctuations of the metric have been ignored. Moreover, in quantum electromagnetism we know exactly how to take these fluctuation effects into account, namely by scattering and interaction Feynman diagrams involving the photon propagator \[18\]. These processes are important not only in scattering between a few particles at high energies but also in low energy processes in hot or dense plasmas. Analogous statements should be applicable to gravity whenever quantum correlations are important. Thus, if the linear response validity criterion is not satisfied, then there will be no avoiding the technical difficulties of quantizing the gravitational field, even if we seek to understand only the infrared behavior of a semi-classical approximation to the effective theory far below the Planck scale.

In the next section the properties of the large \(N\) semi-classical approximation in gravity and its renormalization within the covariant effective action framework are reviewed. In Section III the linear response theory for the semi-classical backreaction equations is introduced. The form of the two-point correlation function for the energy-momentum tensor that appears in the linear response equations is given and its properties and renormalization are discussed. Then our proposal for a necessary condition for the validity of the semi-classical approximation is presented. Finally some possible applications of our criterion to the study of quantum effects in cosmological and black hole spacetimes are suggested.

II. SEMI-CLASSICAL GRAVITY AND RENORMALIZATION

The most direct route to the semi-classical equations \([13]\) is via the effective action method in the large \(N\) limit. We consider the specific example of \(N\) non-interacting scalar fields. Generalizations to interacting fields and fields of other spin are straightforward, but as they are not required to expose the main elements of the stability criterion, we shall treat only this simplest case in detail.

The classical action for one scalar field is
\[
S_m[\Phi, g] = -\frac{1}{2} \int d^4 x \sqrt{-g} \left[ \left( \nabla_a \Phi \right) g^{ab} \left( \nabla_b \Phi \right) + m^2 \Phi^2 + \xi R \Phi^2 \right].
\]
where \(\nabla_a\) denotes the covariant derivative in the metric \(g_{ab}\), \(\xi\) the dimensionless curvature coupling, and \(R\) the scalar curvature. The path integral over the free scalar field \(\Phi\) is Gaussian and may be performed formally by inspection, i.e.,
\[
\int [D\Phi] \exp \left( \frac{i}{\hbar} S_m[\Phi, g] \right) = \exp \left( -\frac{1}{2} \text{Tr} \log G^{-1}[g] \right) \equiv \exp \left( \frac{i}{\hbar} S^{(1)}_{\text{eff}}[g] \right),
\]
where
\[
G^{-1}[g] \equiv -\Box + m^2 + \xi R
\]
is the inverse propagator of the scalar field in the background metric \(g_{ab}\), and the (generally non-local) functional
\[
S^{(1)}_{\text{eff}}[g] = \frac{i\hbar}{2} \text{Tr} \log G^{-1}[g]
\]
may be regarded as the effective action in this metric. It contains an explicit factor of \(\hbar\) and records the quantum effects of the free scalar field in the arbitrary curved background metric \(g_{ab}\). No assumption about smallness of the metric deviations from flat spacetime or any other preferred spacetime has been made.

The expectation value of the energy-momentum tensor of the quantum matter field in this background can be formally obtained by the variation
\[
\langle T_{ab} \rangle = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} S^{(1)}_{\text{eff}}[g].
\]
By Noether's theorem, this (unrenormalized) expectation value is covariantly conserved, provided that the effective action \(S^{(1)}_{\text{eff}}[g]\) is invariant under general coordinate transformations. However, \(\langle T_{ab} \rangle\) is divergent and requires a careful UV regularization and renormalization procedure consistent with this invariance \([14]\).

In physical terms the UV regularization and renormalization mean that the theory may not be trustworthy at infinitely short time and distance scales, but that the lack of information about the
physics at those small scales may be absorbed into a finite number of parameters in the effective low energy theory at much larger scales. Since the effective lagrangian and energy-momentum tensor have canonical scale dimension \( d \) (in \( d \) spacetime dimensions), the number of parameters is given by the number of local coordinate invariant scalars up to dimension \( d \) (in \( d \) dimensions). In \( d = 4 \) dimensions these are the parameters of the Einstein-Hilbert action plus the coefficients of the two independent fourth order invariants \( R^2 \) and \( C_{abcd}C^{abcd} \), where \( R \) is the scalar curvature and \( C_{abcd} \) is the Weyl tensor. Thus we shall consider the total low energy effective gravitational action

\[
S_{\text{eff}}[g] = S_{\text{eff}}^{(1)}[g] + \frac{1}{16\pi G_N} \int d^4 x \sqrt{-g} (R - 2\Lambda) + \frac{1}{2} \int d^4 x \sqrt{-g} \left( \alpha R^2 + \beta C_{abcd}C^{abcd} \right),
\]

(2.6)

with arbitrary dimensionless constants \( \alpha \) and \( \beta \). Renormalization means that \( G_N, \Lambda, \alpha, \) and \( \beta \) may be chosen to depend on the UV cutoff (introduced to regulate the divergences in the one-loop term \( S_{\text{eff}}^{(1)}[g] \)) in such a way as to cancel those divergences and render the total \( S_{\text{eff}}[g] \) independent of the cutoff. Hence the four parameters of the local geometric terms (up to fourth order derivatives of the metric which are \( a \) priori independent of \( h \)) must be considered as parameters of the same order as the corresponding divergent terms in \( S_{\text{eff}}^{(1)}[g] \), which from eq. (2.4) is first order in \( h \). Formally this may be justified by considering \( N \) identical copies of the matter field, so that \( S_{\text{eff}}^{(1)}[g] \) is replaced by \( NS_{\text{eff}}^{(1)}[g] \) and \( G_N^{-1}, \Lambda/G_N, \alpha, \) and \( \beta \) are rescaled by a factor of \( N \). In this way all the terms in eq. (2.6) are now of the same order in \( N \) as \( N \to \infty \).

This formal rescaling by \( N \) is carried out at the level of the generating functional \( S_{\text{eff}}[g] \) of connected \( n \)-point vertices (which are the inverse of \( n \)-point Green’s functions) rather than the Green’s functions themselves. Therefore, it has the net effect of resumming the quantum effects contained in the one-loop diagrams of the matter field(s) to all orders in the metric \( g_{ab} \). The large \( N \) expansion and its relationship to the standard loop expansion have been extensively studied in both \( \Phi^4 \) theory and electrodynamics (both scalar and spinor QED) in flat space \([18]\). The QED case is most analogous to the present discussion with the classical vector potential \( A_\mu \) replaced by the metric \( g_{ab} \). The large \( N \) approximation (2.6) is also invariant under changes in the ultraviolet renormalization scale (by definition of the UV cutoff dependence of the local counterterms), and is equivalent to the UV renormalization group (RG) improved one-loop approximation.

In the present case of a free field theory in curved spacetime, it is the large \( N \), RG improved one-loop approximation that is necessary to derive the semi-classical equations (1.1) with back-reaction, for only in such a resummed loop expansion can the one-loop quantum effects of \( T_{ab} \) influence the nominally classical background metric \( g_{ab} \). As mentioned in the previous section, in the ordinary (unimproved) loop expansion the quantum fluctuations of the matter can make at most small corrections to the background metric. The large \( N \) approximation also explicitly preserves the covariance properties of the theory, since it can be derived from an invariant action functional (2.6). The divergences in \( T_{ab} \) are in one-to-one correspondence with the local counterterms in the action \( S_{\text{eff}}[g] \), whose variations with respect to \( g_{ab} \) produce, in addition to the terms in the classical Einstein equations, the fourth order tensors

\[
\begin{align*}
(1) \quad H_{ab} &= -\frac{\delta}{\sqrt{-g} \, \delta g^{ab}} \int d^4 x \sqrt{-g} \, R^2 = 2g_{ab} \Box R + 2 \nabla_a \nabla_b R + 2RR_{ab} - \frac{g_{ab}}{2} R^2, \\
(2) \quad H_{ab} &= \frac{\delta}{\sqrt{-g} \, \delta g^{ab}} \int d^4 x \sqrt{-g} \, C_{abcd}C^{abcd} = 4\nabla^c \nabla^d C_{acbd} + 2R_{cd}C_{acbd}.
\end{align*}
\]

(2.7)

Hence the variation of the effective action (2.6) gives the equations of motion for the spacetime metric for zero expectation value of the free scalar field \( \Phi \)

\[
\alpha \, (1) \, H_{ab} + \beta \, (2) \, H_{ab} + \frac{1}{8\pi G_N} \left( G_{ab} + \Lambda g_{ab} \right) = \langle T_{ab} \rangle_R,
\]

(2.8)

where \( \langle T_{ab} \rangle_R \) is the renormalized expectation value of the energy-momentum tensor of the scalar field. Non-zero values of \( \langle \Phi \rangle \) and self-interactions are easily taken into account in the large \( N \) approximation as well \([18]\).

It is worth emphasizing that the UV renormalization of the energy-momentum tensor and the covariant form of the equations of motion (2.8) are justified by formal appeal to an underlying covariant action principle (2.4) whose variation they are. Although particular regularization and renormalization procedures, such as non-covariant point-splitting or adiabatic subtraction, may break explicit covariance, the result must be of the form (2.8) with a covariantly conserved \( \langle T_{ab} \rangle_R \) or the procedure does not correspond to the addition of local counterterms up to dimension \( d = 4 \).
in the effective action, as required by the general principles of renormalization theory. Thus the renormalization of the effective action \( (2.6) \) suffices in principle to renormalize the equations of motion and its higher variations, a fact we make use of in the next section.

In all cases the large \( N \) approximation is equivalent to a Gaussian path integration for the quantum matter fields, in which the spacetime metric and gravitational degrees of freedom have been treated as \( c \)-numbers, coupled only to the expectation value of the energy-momentum tensor through \( (1.3) \). This energy-momentum tensor expectation value can be expressed as a coincident limit of local derivatives of the one-loop matter Green’s function \( G[g](x, x) \) in the background metric \( g_{ab} \). That is, it requires solving the differential equation \( G^{-1}[g] \circ G[g] = 1 \) or more explicitly

\[
\left( -\Box + m^2 + \xi R \right) G[g](x, x') = \frac{\delta^4(x, x')}{\sqrt{-g}},
\]

concurrently with the semi-classical backreaction equation \( (2.8) \). It is the exact solution of this equation without any perturbative re-expansion of \( G[g] \), and the resulting self-consistent solution of eq. \( (2.8) \) for the metric \( g_{ab} \) that constitutes the principal non-perturbative RG improved feature of the large \( N \) limit.

The equations of motion \( (2.8) \) which are the original eqs. \( (1.3) \) modified by the additional terms required by the UV renormalization of \( \langle T_{ab} \rangle \) are fourth order in derivatives of the metric. As is known from the general theory of differential equations, if the order of the equations is changed by adding higher derivative terms, the solutions of the modified equations fall into two classes, viz., those that approach the solutions of the lower order equations as \( \alpha, \beta \to 0 \), and those which are singular in that limit. The latter class of solutions are not present in the lower order theory and physically correspond to those solutions which vary on Planck length and time scales (in order for the higher derivative terms to be of the same order as the lower derivative terms). Such solutions are clearly not trustworthy in a low energy effective theory which breaks down at the Planck scale. Furthermore, we are not interested in the stability issue raised by the presence of such rapid variations in time and space. Quite to the contrary, we would like to study the validity of the semi-classical approximation at scales far below the Planck scale, where this approximation is at least not invalid \( a \ priori \). In this regime the effects of the higher order local terms in eq. \( (2.8) \) can be handled by an approach similar to that proposed by Simon \[19\] and Parker and Simon \[20\].

### III. LINEAR RESPONSE AND THE STABILITY CRITERION

Just as the first variation of the action functional (including that for the matter field) with respect to \( g_{ab} \) gives Euler-Lagrange equations for the mean value of the metric, its second variation gives information about the fluctuations of the metric about its mean value. The second variation of the action functional is equivalent to the first variation of the semi-classical Einstein equations, namely

\[
\delta \left[ \alpha (1) H_{ab} + \beta (C) H_{ab} + \frac{1}{8\pi G_N} (G_{ab} + \Lambda g_{ab}) \right] = \delta \langle T_{ab} \rangle_R,
\]

where

\[
\Pi_{ab}^{(ret)}(x, x') \equiv -i\theta(t, t') \left\langle \left[ T_{ab}(x), T_{cd}(x') \right] \right\rangle = -\frac{\delta^2 S_{eff}^{(1)}[g]}{\delta g^{ab}(x)\delta g_{cd}(x')},
\]

is the retarded gravitational polarization operator of the matter fields in the curved background \( g_{ab} \). This retarded operator corresponds to the causal boundary conditions of the effective action functional \( S_{eff}^{(1)}[g] \), which are explicitly enforced by the Schwinger-Keldysh closed time path (CTP) contour of the time integration \[22\]. The linearized fluctuation \( \delta g_{ab}(x) \) obeys an integro-differential equation \( (3.3) \) in which the integral depends only on the past of \( x \), due to the causal boundary conditions, and which involves the two-point correlation function of the matter energy-momentum tensor. According to the general principles of linear response analysis, this retarded correlation function is evaluated in the background geometry of the leading order solution of the semi-classical equations \( (2.8) \).

Before discussing the stability criterion we make some additional technical remarks. First, the polarization operator \( \Pi_{ab}^{(ret)}(x, x') \) has local divergences involving derivatives of \( \delta^{(4)}(x, x') \) when
the spacetime points \(x\) and \(x'\) coincide. Each of these divergences is proportional to one of the local
tensor variations on the left side of eq. (3.1). To see this note that if the UV renormalization
is performed at the level of the effective action (2.6) then all variations of the renormalized effective
action are necessarily finite. Hence all the local UV divergences may be removed by adjusting the
coefficients \(\alpha, \beta, G_N\), and \(\Lambda\) in the effective action, and there are no state dependent divergences in
either \(\langle T_{ab}\rangle_R\) or \(\Pi_{ab}^{\text{ret}} \delta^d (x, x')\).

Next, since the polarization operator is determined by the second variation of the same effective
action that determines the energy-momentum tensor, it also obeys the covariant form

\[
\nabla^a \Pi_{ab}^{\text{ret}} \delta^d (x, x') = \nabla_{x'} \Pi_{ab}^{\text{ret}} \delta^d (x, x') = 0 .
\]

Finally, the equations (3.1) are covariant in form and therefore are non-unique up to linearized
coordinate (gauge) transformations,

\[
\delta g_{ab} \to \delta g_{ab} + \nabla_a X_b + \nabla_b X_a ,
\]

for any vector field \(X_a\). Singular gauge transformations in the initial data for \(\delta g_{ab}\) are certainly
not allowed, and some care is required to decide whether time dependent linearized gauge trans-
formations which grow in time without bound are allowed or not. Since the action principle is
fundamental to the present approach, any transformation of the form (3.4) for which the action
(2.6) is not invariant (due to boundary or surface terms) is not a true invariance and should be
excluded from the set of allowable gauge transformations of the linear response equations (3.1).

We are now in a position to state our stability criterion for the the semi-classical approximation.
A necessary condition for the validity of the large \(N\) semi-classical equations of motion (2.3) is that
the linear response equations (3.1) should have no solutions with finite non-singular initial data for
which any linearized gauge invariant scalar quantity grows without bound. Such a quantity must
be constructed only from the linearized metric perturbation \(\delta G_{ab}\) and its derivatives, and it must be
invariant under allowed gauge transformations of the kind described by eq. (3.1).

The existence of any solutions to the linear response equations with unbounded growth in time,
that cannot be removed by an allowed linearized gauge transformation (3.4), implies that the in-
fluence of the growing gravitational fluctuations on the semi-classical background geometry are large
and must be taken into account in the evolution of the background itself. That is to say, if the
gravitational fluctuations around the background grow, even if they were initially small, then the
leading order semi-classical equations (2.3), which neglect these fluctuations, must eventually break
down.

To this point in time a consistent inclusion of gravitational fluctuations in the dynamical evolution
of the background metric has not been attempted, even in the most symmetric and cosmologically
relevant cases, namely the Robertson-Walker background. However, if the linearized solutions
show any growing modes then such self-consistent inclusion of the effect of the gravitational fluctu-
ations beyond the leading order semi-classical approximation would be required.

If the linear response equation (3.1) has physical unstable solutions at space and time scales
determined by the semi-classical background metric, which are very far from the Planck scale,
then the analysis should be reliable. An instability in the low energy infrared effective theory
means that it is the semi-classical solution to eq. (2.3) that must be modified by taking these
infrared gravitational fluctuations into account self-consistently, rather than abandoning the entire
framework of a spacetime metric description of gravity.

One important application of this criterion is to de Sitter spacetime. A number of different
arguments leads to the conclusion that de Sitter spacetime is not the ground state of a quantum
theory of gravity with a cosmological term \(22\). In fact, the two-point correlation function of
the energy-momentum tensor for a scalar field was estimated in \(23\) and argued to contribute to
a gauge invariant growing mode on the horizon time scale. This proposition could be tested by
detailed calculation of the two-point correlation function of the energy-momentum tensor and the
solutions of the linear response equations (3.3).

A second relevant application of the criterion is to black hole spacetimes. Ever since the discovery
of black hole radiance it has been recognized that the quantum behavior of black holes is qualitatively
different from the classical analogs at long times, since semi-classical black holes decay at late times
while classical black holes are absolutely stable. In the Hartle-Hawking state \(24\) one can construct
a static solution to the semi-classical equations (2.3) that is close to the classical one near the
horizon \(27\). However, the stability of this self-consistent solution has not been investigated.
The validity criterion proposed in this paper provides a clear test for the stability of the self-
consistent solutions in both the black hole and de Sitter cases which we plan to investigate in future
publications.
ACKNOWLEDGMENTS

P. R. A. would like to thank T-8, Los Alamos National Laboratory for its hospitality, as well as
J. Donoghue, L. Ford, B.L. Hu, and N. Phillips for helpful conversations. This work was supported
in part by grant numbers PHY-9800971 and PHY-0070981 from the National Science Foundation.
It was also supported in part by contract number W-7405-ENG-36 from the Department of Energy.

[1] B.L. Hu, Physica A 158, 399 (1989).
[2] J.F. Donoghue, Phys. Rev. D 50, 3874 (1994).
[3] B.S. DeWitt, in Les Houches 1985, Proceedings, Architecture Fundamental Interactions at Short Distances, Vol. 2, 1023-1057.
[4] B.S. DeWitt and C. Molina-París, Mod. Phys. Lett. A 13, 2475 (1998).
[5] E. Tomboulis, Phys. Lett. B 70, 361 (1977).
[6] N.D. Birrell and P.C.W. Davies, Quantum Fields in Carved Space (Cambridge University Press, Cambridge, England, 1982), and references therein.
[7] A. Ashtekar, Phys. Rev. Lett. 77, 4864 (1996).
[8] C. Beetle, Adv. Theor. Math. Phys. 2, 471 (1998).
[9] L.H. Ford, Ann. Phys. (N.Y.) 144, 238 (1982).
[10] C.-I. Kuo and L.H. Ford, Phys. Rev. D 47, 4510 (1993).
[11] C.-H. Wu and L.H. Ford, Phys. Rev. D 60, 104013 (1999).
[12] N.G. Phillips and B.L. Hu, Phys. Rev. D 55, 6123 (1997).
[13] B.L. Hu and N.G. Phillips, Int. J. Theor. Phys. 39, 1817 (2000);
N.G. Phillips and B.L. Hu, Phys. Rev. D 62, 084017 (2000).
[14] R.D. Carlitz and R.S. Willey, Phys. Rev. D 36, 2327 (1987).
[15] A. Roura and E. Verdaguer, Int. J. Theor. Phys. 38, 3123 (1999).
[16] C.-H. Wu and L.H. Ford, Phys. Rev. D 64, 045010 (2001); e-print gr-qc/0102063.
[17] P.C. Martin and J. Schwinger, Phys. Rev. 115, 1342 (1959).
[18] F. Cooper, S. Habib, Y. Kluger, E. Mottola, J. P. Paz, and P.R. Anderson, Phys. Rev. D 50, 2848 (1994);
F. Cooper, S. Habib, Y. Kluger, and E. Mottola, Phys. Rev. D 55, 6471 (1997).
[19] J.Z. Simon, Phys. Rev. D 41, 3720 (1990); ibid. 43, 3308 (1991).
[20] L. Parker and J.Z. Simon, Phys. Rev. D 47, 1339 (1993).
[21] J. Schwinger, J. Math. Phys. 2, 407 (1961);
L.V. Keldysh, Zh. Eksp. Teor. Fiz. 47, 1515 (1964) [Sov. Phys. JETP 20, 1018 (1965)];
K.-C. Chou, Z.-B. Su, B.-L. Hao, and L. Yu, Phys. Rep. 118, 1 (1985).
[22] E. Mottola, Phys. Rev. D 31, 754 (1985); ibid. 33, 1616 (1986);
P.O. Mazur and E. Mottola, Nucl. Phys. B 278, 694 (1986);
I. Antoniadis and E. Mottola, J. Math. Phys. 32, 1037 (1991);
E. Mottola, J. Math. Phys. 36, 2470 (1995).
[23] E. Mottola, Phys. Rev. D 33, 2136 (1986).
[24] J. B. Hartle and S. W. Hawking, Phys. Rev. D 13, 2188 (1976)
[25] J.W. York, Jr., Phys. Rev. D 31, 775 (1985).
[26] D. Hochberg, T.W. Kephart, and J.W. York, Jr., Phys. Rev. D 48, 479 (1993).
[27] P.R. Anderson, W.A. Hiscock, J. Whitesell, and J.W. York Jr., Phys. Rev. D 50, 6427 (1994).