Multiple Convex Objects Image Segmentation via Proximal Alternating Direction Method of Multipliers

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Abstract

This paper focuses on the issue of image segmentation with convex shape prior. Firstly, we use binary function to represent convex object(s). The convex shape prior turns out to be a simple quadratic inequality constraint on the binary indicator function associated with each object. An image segmentation model incorporating convex shape prior into a probability-based method is proposed. Secondly, a new algorithm is designed to solve involved optimization problem, which is a challenging task because of the quadratic inequality constraint. To tackle this difficulty, we relax and linearize the quadratic inequality constraint to reduce it to solve a sequence of convex minimization problems. For each convex problem, an efficient proximal alternating direction method of multipliers is developed to solve it. The convergence of the algorithm follows some existing results in the optimization literature. Moreover, an interactive procedure is introduced to improve the accuracy of segmentation gradually. Numerical experiments on natural and medical images demonstrate that the proposed method is superior to some existing methods in terms of segmentation accuracy and computational time.

Keywords: Image segmentation, convexity prior, proximal ADMM, interactive procedure

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1. Introduction

Image segmentation is one of the fundamental topics in the fields of image processing and computer vision, which aims to partition an image domain into several meaningful sub-regions. Over the past decades, numerous approaches have been proposed in the literature, e.g., the Mumford-Shah (MS) model [1], the piecewise constant MS or Chan-Vese model [2], the geodesic or snake model [3]. These models usually need to minimize an energy function consisting of a region force term and a boundary force term, in which, the region force term measures the similarity of the image intensity in each sub-region and the boundary force term penalizes the nonsmoothness of the sub-region boundary curve. Meanwhile, various approaches have been proposed for these models, such as piecewise smoothness, piecewise constant, and maximum of likelihood of intensity distribution (e.g., Gaussian mixed model [4, 5, 6]) for the region force term, and the length [7], Euler-elastics length [8, 9] and nonlocal length [10] for the boundary penalty term.

The existing models are capable of yielding correct results for most of the images except for the cases contained heavy noises, illumination biases, occlusions and distortions, etc. To address this issue, a natural idea is to introduce shape prior to improve segmentation accuracy. Therefore, image segmentation models with shape priors, such as connectivity [11], star shape [12, 13] and convexity [14, 15], have attracted much attention over the past few years. In this paper, we consider the segmentation of objects with convexity prior because the convexity of object(s) is an important hue for human vision [16, 17]. At the first place, we note that the computational characterization method for convex regions is the core for the developments of segmentation with convexity prior. These methods can be categorized in two classes, i.e., the boundary or curvature method and the region method. A quick review is presented as follows.

The boundary method is based on the fact that the convexity of a region is equivalent to the nonnegativity of its boundary curvature. Actually, image segmentation with curvature regularization has been widely studied in the lit-
erature to retain fine details \cite{18}. In order to compute the curvature efficiently, the object boundary is usually embedded in a so-called level set function \cite{13}. As far as we know, the convexity of objects is firstly incorporated in an iterative segmentation procedure by eliminating negative curvature computed by the associated level set function \cite{19}. Then, the integral of absolute curvature is studied in the literature to remain the convexity of region of interest \cite{20, 21}. This approach is developed by using the associated signed distance function (SDF, a special level set function) to characterize the convexity of the region of interest \cite{15}, where the Laplacian of the SDF is sufficient to guarantee the convexity. In fact, the Laplacian is nothing but the curvature of all the level set curves including the boundary, and this method is generalized to multiple convex objects representation and convex ring region. This method is also developed for 3D convex region representation and applied to convex hull computation of some given points \cite{22}.

Some region based methods use the definition of convex shapes. A method based on the definition of convexity is proposed in \cite{14}, where all 1-0-1 configurations on all lines at the image domain are penalized with 0 (resp. 1) to the associated pixels in the object (resp. background). Later, this method is extended to multiple convex objects segmentation case \cite{23}. It should be emphasized that the minimization problem in \cite{23} is NP-hard and it only can be solved approximately by interior-point method. Another region based method is proposed in \cite{24}, where the authors employ the fact that the line segment of the two ends of any curve in a convex region is in the convex region as well.

Although different models have been proposed in the last few years, it is very difficult to solve the associated problems. For example, the minimization problem in the region methods mentioned above is NP-hard and has only an approximated solution can be obtained. The minimization problem involved in the level set function method has a nonsmooth and nonconvex curvature term \cite{25}, and an energy function with a nonconvex constraint coming from signed distant function must be minimized and hence a forth-order partially differential equation must be solved, which is computationally intensive.
Very recently, we proposed a novel method for convex shape characterization in our short paper [26]. This method is similar to the level set function method reviewed above except a binary function is used to compute the curvature of the boundary curve. For this type of method, the convexity of a region of interest is equivalently characterized by a set of quadratic constraint on the indicator function \( o(x) = 1 \) for \( x \) in the region and 0 otherwise, that is

\[
[b_r \ast o](x)(1 - o(x)) \leq \frac{1}{2}[1 - o(x)], \quad \forall r \geq 0,
\]

where 's' denotes the convolution operation of two functions, and \( b_r \) is a positive radial function defined on a disc with radius \( r \) and center \((0, 0)\) such that \( \int_{\mathbb{R}^2} b_r(x)dx = 1 \). Based on this method, our group incorporates convex shape prior into a U-net for image segmentation [27].

The main purpose of this paper is to extend and improve the method in [26] in the sense that it has the ability to address multiple convex objects segmentation issue efficiently. Even more, we develop an efficient iterative algorithm based on linearization technique so that the considered model is divided into a series of convex minimization problems. For each problem, a proximal alternating direction method of multipliers (ADMM) [28] is employed to solve it which is very simple and efficient since closed form solutions for all the subproblems can be obtained. Finally, we also design an interactive technique to improve segmentation accuracies step by step according to additional subscribed labels by users. As mentioned in [26], the main computational cost lies in the convolution operations for the convex shape constraints on the objects. In our algorithm, only a few of \( b_r \)'s with changed radii \( r \)'s are used in the iteration procedure to reduce computation cost.

To numerically demonstrate the practical performance of our proposed algorithm, extensive numerical experiments on a large number of real images consisting of convex object(s) are conducted. The numerical results illustrate that our proposed algorithm can maintain the convexity of objects(s) of interest, and the segmentation accuracies are improved. Comparing with the paper [26], we extend it in the following aspects.
We generalize the method to multiple convex objects characterization.

Proximal ADMM is employed to solve the resulting linearized minimization problems. Not only the numerical efficiency is higher than the algorithm in [26], but also the convergence of our proposed algorithm is guaranteed theoretically by selecting some proper semi-definite terms [28].

Interactive technique is developed to improve the segmentation accuracies by incorporating additional user-subscribed labels on the objects and background.

The rest of this paper is organized as follows. Section 2 introduces some basic concepts and preliminary results in image segmentation and reviews the schemes and theoretical results on the proximal ADMM. Our model for multiple convex objects segmentation is presented in Section 3. Section 4 is devoted to the proximal ADMM for the convex minimization problems resulted from the original model. Besides, the convergence result is also included in this section.

Numerical experiments and some performance comparisons are demonstrated in Section 5 to show the superiority of the propose algorithm. Section 6 concludes this paper and discusses some interesting research topics for further research.

2. Some preliminary results

In this section, we will introduce some notations, and present a brief review on the proximal ADMM for convex composite minimization problem.

2.1. Some notations

Let $\Omega \subset \mathbb{R}^2$ be an image domain and $D$ be a subregion of $\Omega$. The indicator function $o(x)$ of $D$ is defined as

$$o(x) = \begin{cases} 
1 & x \in D, \\
0 & x \in D^c,
\end{cases} \quad (1)$$

where $D^c = \Omega \setminus D$ and $x = (x_1, x_2)^T \in \Omega$. Let $B_r(x)$ be the circle centered at $x$ with radius $r > 0$, and $b_r : \mathbb{R}^2 \to \mathbb{R}$ be a positive integrable radial function defined on $B_r(0)$, i.e., $b_r(||x||) = b_r(||y||)$ if $||x|| = ||y||$ and $b_r(x) = 0$.
if \( \|x\| > r \), and \( \int_{B_r(0)} b_r(x) dx = 1 \), where \( \|\cdot\| \) is the Euclidean norm. For any given two vector-valued functions \( p(x) := (p_1(x), p_2(x), \cdots, p_n(x))^\top \) and \( q(x) := (q_1(x), q_2(x), \cdots, q_n(x))^\top \), the symbol \( p \circ q \) represents the point-wise product, that is,
\[
p \circ q := p(x) \circ q(x) = (p_1(x) q_1(x), p_2(x) q_2(x), \cdots, p_n(x) q_n(x))^\top,
\]
and the convolution between \( b_r(x) \) and \( p \) is defined as (broadcast)
\[
(b_r * p)_i(x) := \int_{\mathbb{R}^2} b_r(x - y) \cdot p_i(y) dy, \quad i = 1, 2, \ldots, n.
\]

### 2.2. Brief review on proximal ADMM

Let \( \mathcal{X}, \mathcal{Y} \) and \( \mathcal{Z} \) be real finite dimensional Euclidean spaces with inner product \( \langle \cdot, \cdot \rangle \) and the induced norm \( \|\cdot\| \). Consider the following convex minimization problem
\[
\min_{x,y} \quad h(x) + g(y)
\]
\[\text{s.t.} \quad Ax + By = c,
\]
where \( h : \mathcal{X} \to (-\infty, +\infty] \) and \( g : \mathcal{Y} \to (-\infty, +\infty] \) are closed proper convex functions, \( A : \mathcal{X} \to \mathcal{Z} \) and \( B : \mathcal{Y} \to \mathcal{Z} \) are linear operators and their adjoints are denoted by \( A^\top \) and \( B^\top \), respectively, and \( c \in \mathcal{Z} \) is a given data. Starting from \((x^0, y^0, z^0) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}\), the proximal ADMM of \([28]\) for \((3)\) takes the following iterative framework
\[
\begin{align*}
x^{j+1} &= \arg \min_{x \in \mathcal{X}} h(x) + \langle z^j, Ax \rangle + \frac{\mu}{2} \|Ax + By^j - c\|^2 + \frac{1}{2}\|x - x^j\|^2_S, \\
y^{j+1} &= \arg \min_{y \in \mathcal{Y}} g(y) + \langle z^j, By \rangle + \frac{\mu}{2} \|Ax^{j+1} + By - c\|^2 + \frac{1}{2}\|y - y^j\|^2_T, \\
z^{j+1} &= z^j + \tau \mu (Ax^{j+1} + By^{j+1} - c),
\end{align*}
\]
where \( z \in \mathcal{Z} \) is a multiplier, \( \mu > 0 \) is a penalty parameter, \( \tau \in (0, (1+\sqrt{5})/2) \) is a step length, \( S \) and \( T \) are two self-adjoint positive semi-definite linear operators, and \( \|x\|^2_S := \langle Sx, x \rangle \) (similarly for \( \|y\|^2_T \)). A vector \((\bar{x}, \bar{y}) \in \mathcal{X} \times \mathcal{Y}\) is said to be a solution of \((3)\) if there exists a \( \bar{z} \in \mathcal{Z} \) such that the following Karush-Kuhn-Tucker (KKT) system is satisfied
\[
0 \in A^\top \bar{z} + \partial h(\bar{x}), \quad 0 \in B^\top \bar{z} + \partial g(\bar{y}), \quad \text{and} \quad A\bar{x} + B\bar{y} = c.
\]
where $\partial h$ and $\partial g$ denote the sub-differentials of $h$ and $g$, respectively. Under some proper assumptions, the sequence $\{(x^j, y^j, z^j)\}$ generated by (1) converges to an accumulation point $\{(\bar{x}, \bar{y}, \bar{z})\}$ such that $\{(\bar{x}, \bar{y})\}$ is an optimal solution of (3). For more details on the proof, one can refer to [28, Theorem B].

3. Method for multiple convex objects segmentation

In this section, we will present the characterization method for single convex object, and extend it to multiple convex objects characterization firstly. Then, a model for object(s) segmentation with convex prior is proposed by incorporating the characterization method into a probability-based image segmentation model.

In our previous short paper [26], we prove the following characterization theorem for single convex object.

**Theorem 3.1.** Let $o$ be an indicator function of a region $D \subset \mathbb{R}^2$. Then $D$ is convex if and only if

$$\left( b_r \ast o \right)(1 - o) \leq \frac{1}{2}(1 - o).$$

(6)

For the case of multiple convex objects, this characterization method can be easily extended as Theorem 3.2.

**Theorem 3.2.** Let $u_i$ be the indicator function of region $D_i \in \mathbb{R}^2$ for $i = 1, \ldots, P$. Denote $\hat{u}(x) := (u_1(x), u_2(x), \ldots, u_P(x))^\top$. Then, $D_1, D_2, \ldots, D_P$ are convex if and only if for all $r > 0$

$$\mathcal{C}_r(\hat{u}) := \left( \hat{u} - 1 \right) \circ (b_r \ast \hat{u}) - \frac{1}{2} (\hat{u} - 1) \geq 0,$$

(7)

where $b_r$ is a positive integrable radial function defined previously.

**Proof 1.** From Theorem 3.1, we know that the convexity of the $i$-th region $D_i$ can be characterized as

$$(u_i - 1)(b_r \ast u_i) - \frac{1}{2} (u_i - 1) \geq 0.$$  

(8)

Reformulating both sides of this inequality as a vector, we can obtain the assertion of this theorem.
For a given image $I(x)$ defined on $\Omega$, we assume the foreground consists of $P$ regions of interest, named $D_i$ for $i = 1, \ldots, P$, and the background is denoted as $D_0 := \Omega \setminus \bigcup D_i$ hereafter. The segmentation model for multiple objects with convex shape prior can be stated as follows

\[
\min_{\{D_i\}_{i=0}^P} \sum_{i=0}^P \int_{D_i} f_i(x) dx + \sum_{i=1}^P \lambda_i |\partial D_i|, \quad \text{s.t. } D_i \text{ being convex, } (i = 1, 2, \ldots, P),
\]

(9)

where $f_i(x)$ is a function (will be given later) to measure the similarity of $I(x)$ in $D_i$ and $\lambda_i > 0$ is a trade-off parameter, and $|\partial D_i|$ is the boundary length of $D_i$. Recalling that $u_i$ is an indicator function defined on $D_i$, from \cite{29, 30, 31}, we have

\[
|\partial D_i| \approx \mathcal{L}_\sigma(u_i) := \sqrt{\frac{\pi}{\sigma}} \int_{\Omega} u_i(x) \big[ G_\sigma * (1 - u_i) \big](x) dx
\]

(10)

with $0 < \sigma \ll 1$, where $G_\sigma$ is the Gaussian kernel function defined as

\[
G_\sigma(x) = \frac{1}{4\pi\sigma} \exp\left( - \frac{||x||^2}{2\sigma^2} \right),
\]

(11)

and

\[
\big[ G_\sigma * (1 - u_i) \big](x) = \int_{\Omega} G_\sigma(x - y)(1 - u_i(y)) dy
\]

from the definition of convolution in \cite{2}.

According Theorem 3.2 and \cite{10}, the model \cite{9} can be formulated as the following minimization problem

\[
\min_{u_i \in \tilde{U}} \int_{\Omega} \sum_{i=0}^P f_i u_i dx + \sum_{i=0}^P \lambda_i \mathcal{L}_\sigma(u_i) \quad \text{s.t. } \mathcal{C}_r(\hat{u}) = 0, \quad i = 1, 2, \ldots, P,
\]

(12)

where $\lambda_0 = 0$ and $\tilde{U} = \{ u_i | u_i \in \{0,1\}, \sum_{i=0}^P u_i = 1 \}$. Clearly, solving (12) is a challenging task because of the quadratic constraints on $u_i(i = 1, 2 \cdots, P)$ and $u_i \in \tilde{U}(i = 0, 1, \cdots, P)$.

In order to reduce the difficulty of this minimization problem, we relax the constraint set $\tilde{U}$ as $\hat{U} := \{ u_i | u_i \geq 0, \sum_{i=0}^P u_i = 1 \}$. We will show in the numerical experiment section that this relaxation is enough to produce high accuracy segmentation results. For convenience, we denote $f := (f_0, f_1, \cdots, f_P)^T$, $\lambda :=
\( (\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_P) ^\top \), and \( u := (u_0, u_1, \cdots, u_P) ^\top \). By the definition of \( L_\sigma(\cdot) \) in (10), the problem (12) can be reformulated as

\[
\begin{aligned}
& \min_{u \in U} \langle f, u \rangle + \sqrt{\frac{\pi}{\sigma}} \langle u, \lambda \odot G_\sigma \ast (1 - u) \rangle \\
& \text{s.t. } C_r(\hat{u}) \geq 0,
\end{aligned}
\]  

(13)

where \( \langle f, u \rangle := \int_\Omega \sum_{i=0}^P f_i u_i \, dx \) and \( C_r(\cdot) \) is defined in (7).

As for the similarity function \( f_i \), we adopt a probability-based method used in the literature [32, 33]. For \( x \) in the domain of image \( I \), let the probability of \( x \) belonging to the region \( D_i \) be \( p_i(x | I) \). Then, the similarity function \( f_i \) is computed by

\[
 f_i(x) := -\ln p_i(x | I).
\]  

(14)

There are many methods to estimate the probability \( p_i(x | I) \) in the literature [34, 35]. Here, we employ the one in [26] which is listed as follows for completeness. This method is based on some given labels on the foreground and background. These labels are necessary for low quality images to obtain high accuracy segmentation. Let \( R_i \) be the set of points with known labels for \( D_i \). Then, the probability density function \( p_i(x | I) \) is estimated by

\[
p_i(x | I) = \frac{\exp(-d_i(x, I(x); R_i))}{\sum_{i=0}^P \exp(-d_i(x, I(x); R_i))},
\]  

(15)

where

\[
d_i(x, I(x); R_i) = \begin{cases} 
  \sum_{y \in R_i} \| I(y) - I(x) \|, & i = 0, \\
  \sum_{y \in R_i} \left[ \| I(y) - I(x) \| + \omega \| x - y \|^2 \right], & i = 1, \ldots, P,
\end{cases}
\]  

(16)

in which \( \omega > 0 \) is a given parameter.

4. Numerical algorithm

Although the continuous minimization problem (13) is simpler than the original one (12), it is still difficult to solve because there are quadratic functions in the objective function and constraint. In this section, we develop an efficient proximal ADMM to seek its approximate solution by using linearization
technique. Furthermore, an interactive method is proposed to improve segmentation accuracies by subscribing some additional labels if segmentation result is not satisfactory using the initial labels.

4.1. Linearization

To alleviate the quadratic terms in (13), in this subsection, we linearize them to obtain a linearly constrained convex minimization problem. Let \( \tilde{u}^k \) be an estimation of the minimizer. The boundary length term in (13) can be approximated by using first-order Taylor expansion at \( \tilde{u}^k \), that is

\[
\langle u, \lambda \odot G_\sigma * (1 - u) \rangle \approx \langle u, \lambda \odot G_\sigma * (1 - 2\tilde{u}^k) \rangle + \langle \tilde{u}^k, \lambda \odot G_\sigma * \tilde{u}^k \rangle.
\] (17)

For given \( \hat{u}^k := (u^k_1, \ldots, u^k_P) \), the constraint \( C_r(\hat{u}) > 0 \) can be linearized as follows

\[
C_r(\hat{u}) = (\hat{u} - 1) \odot (b_r \ast \hat{u}) - \frac{1}{2}(\hat{u} - 1)
\]

\[
\approx \hat{u}^k \odot b_r \ast \hat{u}^k + 2(\hat{u} - \hat{u}^k) \odot b_r \ast \hat{u}^k - 1 \odot b_r \ast \hat{u} - \frac{1}{2} \hat{u} + \frac{1}{2}
\]

\[
= T(\hat{u}) + \frac{1}{2} - \hat{u}^k \odot b_r \ast \hat{u}^k,
\]

where \( T(\hat{u}) := 2\hat{u} \odot b_r \ast \hat{u}^k - \frac{1}{2} \hat{u} - b_r \ast \hat{u} \). Since \( T \) is a linear operator, there must be a matrix \( \hat{A}_r \) for all \( r > 0 \) such that

\[
T(\hat{u}) = \hat{A}_r \hat{u}
\] (18)

By ignoring the constant term in (17), the model (13) can be rewritten as

\[
\min_{u \in U} \langle f, u \rangle + \sqrt{\frac{\pi}{\sigma}} \langle u, \lambda \odot G_\sigma * (1 - 2\tilde{u}^k) \rangle
\]

\[
s.t. \hat{A}_r \hat{u} \geq \hat{c}_r,
\] (19)

where \( \hat{c}_r := \hat{u}^k \odot b_r \ast \hat{u}^k - \frac{1}{2} \) is a \( P \)-dimensional vector. In order to minimize (19) efficiently, we only choose a few radial functions \( b_r, (e = 1, 2 \cdot \cdot \cdot, E) \) for the constraint in (19), the details and effectiveness of which will be discussed in the
numerical experiments section. Therefore, let

\[ \hat{A} = \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \vdots \\ \hat{A}_E \end{bmatrix}, \quad \hat{c} = \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \vdots \\ \hat{c}_E \end{bmatrix}, \]

and the linear inequality constraint can be expressed as \( Au \geq c \) if by setting

\[ A := \begin{bmatrix} 0 \\ \hat{A} \end{bmatrix} \quad \text{and} \quad c := \begin{bmatrix} 0 \\ \hat{c} \end{bmatrix}. \]  \hspace{1cm} (20)

At last, we can reformulate (19) as

\[
\begin{align*}
\min_{u,v} & \quad \langle f, u \rangle + \sqrt{\frac{\pi}{\sigma}} \langle u, \lambda \odot G_{\rho} \ast (1-2\tilde{u}^k) \rangle + \delta_{\Omega}(u) + \delta_{\mathbb{R}^P_{+}}(u) + \delta_{\mathbb{R}^P_{+}}(v) \\
\text{s.t.} & \quad Au - v = c,
\end{align*}
\]  \hspace{1cm} (21)

where \( v \) is called a remaining variable and \( \mathbb{R}^P_{+} \) denotes the first quadrant. The symbol \( \delta_C(x) \) is called characterization function of \( C \), i.e., \( \delta_C(x) = 0 \) if \( x \in C \) and \( +\infty \) otherwise.

We will present some numerical techniques to the model (21) for the purpose of making it easier to follow. Noting that the set of points with known labels for foreground, denoted by \( R_i \) (\( i = 1, 2, \ldots, P \)), is often nonconvex and disconnected, we use its convex hull \( \text{conv}(R_i) \) instead of \( R_i \) as the set of labels, which means that we should constrain \( u_i(x) = 1 \) if \( x \in \text{conv}(R_i) \) in the model. As for the initialization of \( u \), we set \( u_i(i = 1, 2, \ldots, P) \) is the indicator function of \( \text{conv}(R_i) \) and \( u_0 = 1 - \sum_{i=1}^{P} u_i \). Define a narrow belt surrounding the current estimated boundary

\[ S_\rho(\tilde{u}^k) := \{ x \mid \| b_{r_0} * \tilde{u}^k(x) - \tilde{u}^k(x) \| \geq \rho \}, \]  \hspace{1cm} (22)

where \( \rho, r_0 > 0 \) are user-specified parameters, and we only update \( u \) on \( S_\rho(\tilde{u}^k) \) when minimizing (21). We then update \( u \) if it is located in this belt only.
For two points \( w \) and \( u \), we measure their closeness by the relative error defined as follows

\[
R(w, u) := \frac{\int_{\Omega} \|u - w\| \, dx}{\int_{\Omega} \|u\| \, dx}.
\]

(23)

In our numerical implementations, we will compute the relative error every 100 loops, and terminate the iteration if \( Ru^k := R(u^{k-100}, u^k) \) is smaller than a given tolerance. The main reason to terminate the iteration based on this criterion is to avoid early stopping.

At last, we make enough preparations to design the following iterative Algorithm 1 for the model (21).

**Algorithm 1** (Algorithm for problem (13))

Step 0. Input an image \( I \) containing \( P \) objects of interest, \( R_i \) and trade-off parameter \( \lambda_i \) for \( i = 0, \ldots, P \). Choose proper \( r_e(e = 0, 1, 2, \ldots, E) \) for radial functions \( b_{re} \). Set \( \epsilon > 0, \rho > 0, 0 < \sigma \ll 1, \omega > 0, Ru^0 := +\infty \) and \( k := 0 \).

Step 1. Initialization: Compute \( f_i \) via (14)-(16), and let \( \tilde{u}_0^i \) be the convex hull of \( R_i \), and \( u_0^0 = 1 - \sum_{i=1}^P u_i^0 \).

Step 2. While \( Ru^k > \epsilon \), do the following steps:

Step 2.1 Compute \( A \) and \( c \) via (20);

Step 2.2 Find an approximate solution \( (\tilde{u}^{k+1}, \tilde{v}^{k+1}) \) of problem (21) by Algorithm 2;

Step 2.3 Update \( \tilde{u}^{k+1} \) if it lies in the narrow belt \( S_\rho(\tilde{u}^k) \) defined in (22);

Step 2.4 Compute \( Ru^{k+1} = (\tilde{u}^{k+1}, \tilde{v}^{(k+1)-100}) \) if \( \text{mod} (k + 1, 100) = 0 \);

Step 2.5 \( k := k + 1 \).

Step 3. Let \( \bar{u} := \tilde{u}^k \) and do an iterative procedure by Algorithm 3 if necessary.

Step 4. Output \( \bar{u} \).
4.2. Proximal ADMM for (21)

In this subsection, we propose an efficient proximal ADMM to solve the linearized convex minimization problem (21) which should dominate the efficiency of Algorithm 1.

Let $\mu > 0$ be a augmented parameter. The augmented Lagrangian function associated with (21) is given as follows

$$L_\mu(u, v; z) = \langle f, u \rangle + \sqrt{\frac{\mu}{\sigma}} \langle u, \lambda \odot G_\sigma \ast (1 - 2u^k) \rangle + \delta_U(u) + \delta_{\mathbb{R}^{EP+1}}(v) + \langle z, Au - v - c \rangle + \frac{\mu}{2} \|Au - v - c\|^2,$$

(24)

where $z \in \mathbb{R}^{EP+1}$ is a multiplier. Using the proximal ADMM reviewed in (4), we can obtain the following iterative framework to minimize (24)

$$\begin{align*}
    u^{j+1} & = \arg \min_{u \in U} L_\mu(u, v^j; z^j) + \frac{1}{2} \|u - u^j\|^2_S, \\
    v^{j+1} & = \arg \min_{v} L_\mu(u^{j+1}, v; z^j) + \frac{1}{2} \|v - v^j\|^2_T, \\
    z^{j+1} & = z^j + \tau \mu (Au^{j+1} - v^{j+1} - c),
\end{align*}$$

(25)

where $\tau \in (0, (1 + \sqrt{5})/2)$, and $S$ and $T$ are two selected self-adjoint positive semi-definite matrices defined subsequently to guarantee the convergence of the iteration procedure.

Each step in the iteration (25) has close form solution by selecting some proper $S$ and $T$, which means that we can minimize (21) efficiently. Let $\alpha > 0$ be a positive scalar such that the matrix $\alpha I - \mu A^T A$ be positive semi-definite. Firstly, fixing $v^j$ and $z^j$ and choosing $S := \alpha I - \mu A^T A$, we can obtain the
solution to \(u\)-subproblem in (25) as follows

\[
\begin{align*}
  u^{j+1} &= \arg \min_u \delta_U(u) + \langle u, f \rangle + \sqrt{\frac{\pi}{\sigma}} \langle u, \lambda \odot G_\sigma \ast (1 - 2\tilde{u}^k) \rangle + \langle z^j, Au \rangle \\
  &\quad + \frac{\mu}{2} \|Au - v^j - c\|_2^2 + \frac{1}{2} \|u - u^j\|_S^2 \\
  &= \arg \min_u \delta_U(u) + \alpha \|u\|_2^2 \\
  &\quad + \langle u, f + \sqrt{\frac{\pi}{\sigma}} \lambda \odot G_\sigma \ast (1 - 2\tilde{u}^k) - \left(\mu A^\top \xi^j + (\alpha I - \mu A^\top) u^j\right) \rangle \\
  &= \arg \min_u \delta_U(u) + \alpha \frac{1}{2} \|u - \xi^j\|_2^2 \\
  &= \Pi_U(\xi^j),
\end{align*}
\]

where \(\xi^j := c + v^j - z^j/\mu\) and

\[
\xi^j := \frac{1}{\alpha} \left(\mu A^\top \tilde{u}^k + (\alpha I - \mu A^\top) u^j - f - \sqrt{\frac{\pi}{\sigma}} \lambda \odot G_\sigma \ast (1 - 2\tilde{u}^k)\right),
\]

and the operation \(\Pi_U(\xi^j)\) denotes a projection of \(\xi^j\) on the simplex \(U\), which can be numerical implementation easily using the existing methods (e.g., [36]).

Secondly, fixing \(v = v^{j+1}\) and choosing \(T := 0\), we can obtain the solution to the \(v\)-subproblem in (25) as follows

\[
\begin{align*}
  v^{j+1} &= \arg \min_v \delta_{R_{P+1}}(v) + \frac{\mu}{2} \|v - Au^{j+1} - c - z^j/\mu\|_2^2 \\
  &= \Pi_{R_{P+1}}(Au^{j+1} - c + z^j/\mu) \\
  &= \max\{0, Au^{j+1} - c + z^j/\mu\}.
\end{align*}
\]

Using the latest \(u^{j+1}\) and \(v^{j+1}\), the multiplier \(z\) is then updated immediately.

In light of above analysis, the algorithm of proximal ADMM for the linearized problem (21) can be summarized as Algorithm 2.

At the end of this section, it is worth listing some remarks to make this algorithm more easier to follow. Firstly, it is easy to see that the considered problem (21) fills into the framework of the convex minimization problem (3), which means that the Algorithm 2 is well-defined. Secondly, it is about the stopping criterion of the iterative process which may influence the numerical performance of Algorithm 2 to a certain degree. Ideally, Algorithm 2 should
Algorithm 2 (Proximal ADMM for problem (21))

Step 0. Choose positive scalars $\mu > 0$, $\tau \in (0, (1 + \sqrt{5})/2)$, and $\alpha > 0$ such that $S$ be positive and semi-definite, a staring point $(u^0, v^0, z^0)$ such that $u^0 = \tilde{u}^k$. Let $j := 0$. 

Step 1. While ‘not convergence’, do

Step 1.1 Compute $\xi^j$ via (26) and then compute $u^{j+1} = \Pi_U(\xi^j)$;

Step 1.2 Compute $v^{j+1} = \max\{0, Au^{j+1} - c + z^j/\mu\}$;

Step 1.3 Compute $z^{j+1} = z^j + \tau\mu(Au^{j+1} - v^{j+1} - c)$.

Step 1.4 $j := j + 1$.

Step 2. Let $(\tilde{u}^{k+1}, \tilde{v}^{k+1}) := (u^{j+1}, v^{j+1})$ and output $(\tilde{u}^{k+1}, \tilde{v}^{k+1})$.

be terminated at a point when the KKT system (5) is met. But in order to make Algorithm 1 derive some medium accuracy solutions faster, we stop the iteration early when the relative error of two successive points is smaller than a given tolerance. Details on this stopping criterion are omitted here because it is out of the scope of this paper. The last issue is about the convergence of Algorithm 2. Because the iterative framework is actually an implementation of the proximal ADMM in [28] reviewed in Subsection 2.2, its convergence can be guaranteed under some mild conditions. For the completeness of this paper, we list the convergence theorem of Algorithm 2. One may refer to [28, Theorem B] for more details on its proof.

Theorem 4.1. Let the sequence $\{(w^j, v^j, z^j)\}$ be generated by Algorithm 2 from an initial point $\{(w^0, v^0, z^0)\}$. Assume that $\tau \in (0, (1 + \sqrt{5})/2)$ and $\alpha$ is chosen such that $S$ is positive semi-definite, then, the sequence $\{(w^j, v^j, z^j)\}$ converges to the accumulation point $\{(\bar{u}, \bar{v}, \bar{z})\}$ such that $(\bar{u}, \bar{v})$ is the solution of the primal problem (21).

It should be noted that, for the vast majority of images, Algorithm 1 has
the ability to yield correct segmentation results, but for low quality images, it is difficult to completely extract the object of interest. In order to handle this issue, we employ an interactive procedure to improve the segmentation accuracy by using some additional labels. In other words, assuming that $\bar{u}$ is an unsatisfactory segmentation result produced by Algorithm 1 using of the initial input labels, we manually subscribe some more labels on the incorrect segmentation regions and then run Algorithm 1 again. This process may be repeated until an acceptable segmentation result is obtained. The interactive procedure is described as Algorithm 3.

### Algorithm 3 An interactive procedure

**Step 0.** Input an unsatisfied segmentation result $\bar{u}$.

**Step 1.** While ‘not satisfied’, do the following steps:

1. **Step 1.1** Add labels to $R_i$ for $i = 0, 1, 2, \ldots, P$;

2. **Step 1.2** Run Algorithm 1 to modify $\bar{u}$;

**Step 2.** Output $\bar{u}$.

### 5. Numerical experiments

In this section, we conduct a series of experiments on various images to verify the effectiveness of the proposed model and the efficiency of the employed algorithm, and do performance comparisons with the 1-0-1 method [14, 37]. For a given digital image $I \in \mathbb{R}^{m \times n \times d}$ with $d = 1$ and 3 for gray and color image, respectively, we view it as a continuous image $I$ defined on $[0, m - 1] \times [0, n - 1]$ with meshsize equaling to 1. All experiments are run on a PC with Inter(R) Core(TM) i7-6700 CPU @ 3.40GHz 3.40GHz and 8.00GB memory using Matlab2016a.

All the parameters in algorithms and model [21] are used the follow default values if they are not specifically given. The boundary length penalty param-
eters $\lambda_0 = 0$ and $\lambda_1 = \lambda_2 = \ldots = \lambda_P = 2$, $\sigma = 0.01$ for boundary length approximation, the weight parameter $\omega = 0.1$ for the region force computation, $\rho = 0.1$ and $r_0 = 3$ for belt $S_\rho$ and . We choose the augmented parameter $\mu = 1$ in \cite{24}, and set $\alpha = 1 + \mu$ to formulate the linear operator $S$. The tolerance $\epsilon = 10^{-3}$ in Algorithm 1 and the step length $\tau = 1$ in Algorithm 2. As for the radial function $b_r$, $b_r(x) = 0$ if $\|x\| > r$, and $b_r(x) = 1/r$ if $\|x\| \leq r$ is selected in all experiments for simplicity. In addition, only three radial functions are used for $k$-th step with radii $\text{mod}(l + 1, 14), \text{mod}(l + 3, 14), \text{mod}(l + 5, 14)$ and $l := 2\text{mod}([k/100], 7)$ in Algorithm 1 where mod denotes the remainder operation.

In order to measure the accuracy of segmentation, shape-distance between segmentation result and ground truth is computed. Assuming that $D'$ is the segmentation result and $D$ is the ground truth, the shape-distance between them is defined as

$$S\text{-dist}(D', D) := \max \left\{ \max_{x \in D} \min_{y \in D'} \|x - y\|_2, \max_{y \in D'} \min_{x \in D} \|y - x\|_2 \right\} \frac{2\sqrt{\mathcal{A}(D)}/\pi}{\sqrt{\mathcal{A}(D')}}.
$$

(27)

where $\mathcal{A}$ denotes the area of $D$. In fact, this type of distance \cite{27} is also known as the Hausdorff distance between $D$ and $D'$ \cite{38, 39}.

5.1. Influence of the choice of $r$

It is obvious that the value of $r$ in the model \cite{9} is very important to keep the convexity of object(s). In order to investigate its influence and provide a guidance for users in practical applications, we do some experiments on an image with different value configurations of $r$. In this test, we consider three types of combination of $r$, that is, $(1, 2, 3, 4, 5)$, $(1, 4, 7, 10, 13)$, $(1, 12, 23, 34, 45)$, and report the results from left to right in Fig. 1. Observing the images in Fig. 1, we find that the smaller $rs$ (say $(1, 2, 3, 4, 5)$) cannot eliminate the nonconvexity of smooth boundary if the curvature is near zero. The larger $rs$ (say $(11, 12, 23, 34, 45)$) cannot suppress the small oscillations of boundary, which in turn lead to nonconvex result. But the moderate values (say $(1, 4, 7, 10, 13)$) can
overcome these defects, and have the ability to yield convex result. Besides, the running time for each case is about 7s, 15s, and 87s, respectively. This might not be surprising in light of that a larger $r$ may result in higher computational cost in the convolution operation to guarantee convexity. Therefore, one could attempt to choose some small $r$’s and then increase them gradually to remain the objects’ convexity.

Figure 1: Influence of the choice of $r$: The radius configurations for the results from left to right are $(1, 2, 3, 4, 5), (1, 4, 7, 10, 13)$ and $(1, 12, 23, 34, 45)$.

5.2. Experiments on images with a single object

In this part, we do experiments on an image set consisting of 32 images and do performance comparisons with the 1-0-1 method [14]. Particularly, we only care about the task of segmenting a single object on each image. The case of multi-objects segmentation will be considered subsequently. These 32 images are chosen from the real data set available at [http://vision.csd.uwo.ca/code/](http://vision.csd.uwo.ca/code/).

The objects contained in each image are convex which can be seen clearly in Fig. 2. In the first test, we perform our method by using their default labels and then compare the results with those obtained by the direct 1-0-1 method and gradual 1-0-1 method in [14]. In the direct 1-0-1 method, the convexity penalty parameter is chosen as 10, and in the gradual 1-0-1 method, the penalty parameters vary from 0.0001, 0.001, 0.01, 0.1, 1 to 10. For more details on other parameters’ settings, one can refer to [14]. In the second test, we add some new labels manually, and set the penalty parameter by default as 10 in the direct 1-0-1 method. Unfortunately, in this case, we find that the gradual 1-0-1 method
cannot work any more using the downloaded codes. The new labels at each image are shown in Fig. 2 for background and foreground with red and blue color respectively, and parts of results attained by each method are listed in Fig. 3. In order to observe the performance each algorithm more clearly, we also report the shape-distances of the images in Table 1. In this table, the symbol D5 (resp. D11) represents the direct 1-0-1 method with $5 \times 5$ (resp. $11 \times 11$) stencils, and G5 (resp. G11) represents the gradual 1-0-1 method with $5 \times 5$ (resp. $11 \times 11$) stencils. Besides, the first column 'Image' denotes the image's name in this real data set, the column 'Ours' denotes our proposed method in this paper. Finally, taking the first three images as examples, in Fig. 4 we also show the convergence behavior of Algorithm 1 and the total number of points that violate the convex constraint as the iteration moves forward.

Figure 2: Test images with manual new labels: The labels for background and foreground are marked in red and blue color, respectively.
Figure 3: Results comparison: Results of D11 and the proposed method using the downloaded (new subscribed) labels are shown in 1st and 3rd rows (2nd and 4th rows).

Table 1: Shape-distances of the results yielded by the 1-0-1 method and our proposed method, where the best (minimal) values of each image using the downloaded and new labels are bold.

| Image | D5     | D11    | G5     | G11    | Ours  | D11    | Ours  |
|-------|--------|--------|--------|--------|-------|--------|-------|
| img1  | 0.1101 | 0.0881 | 0.1101 | 0.0881 | 0.0934| 0.0908 | 0.0881|
| img2  | 0.0516 | 0.0608 | 0.0644 | 0.0573 | 0.0955| 0.1672 | 0.0453|
| img3  | 0.0856 | 0.0751 | 0.0856 | 0.0751 | 0.0438| 0.0950 | 0.0425|
| img4  | 0.0890 | 0.0834 | 0.0890 | 0.0834 | 0.0556| 0.0013 | 0.0393|
| img5  | 0.2393 | 0.2426 | 1.5600 | 0.1671 | 0.1269| 0.0736 | 0.0568|
| img6  | 0.0504 | 0.0504 | 0.0504 | 0.0504 | 0.0480| 0.1014 | 0.0374|
| Image  | Default labels | New labels |
|--------|----------------|------------|
|        | D5  | D11 | G5  | G11 | Ours   | D11 | Ours   |
| img7   | 0.0699 | 0.1127 | 0.0753 | **0.0559** | 0.1582 | 0.0442 | **0.0419** |
| img8   | **0.1079** | **0.1079** | **0.1079** | **0.1079** | 0.2006 | 0.0752 | **0.0532** |
| img9   | 1.7346 | 0.0939 | 0.0992 | 0.0939 | **0.0470** | 0.0782 | **0.0583** |
| img10  | 2.3249 | 0.1311 | 4.7102 | 0.1158 | **0.0738** | 0.1761 | **0.0738** |
| img11  | 0.0507 | 0.507 | 0.0455 | 0.0455 | **0.0378** | 0.1320 | **0.0267** |
| img12  | 5.6305 | 0.1903 | 5.7053 | 0.1682 | **0.1009** | 0.2128 | **0.1064** |
| img13  | 1.1002 | 0.1002 | 0.1042 | 0.0957 | **0.0617** | 0.1034 | **0.0406** |
| img14  | 0.2791 | 0.3396 | **0.2027** | 0.2618 | 0.2791 | 0.0959 | **0.0468** |
| img15  | 0.0543 | 0.0568 | 0.0512 | **0.0454** | 0.1456 | 0.0523 | **0.0201** |
| img16  | 0.1810 | 0.1697 | 0.1750 | 0.1655 | **0.1421** | 0.1057 | **0.0547** |
| img17  | 0.1441 | 0.1441 | 0.1064 | 0.1139 | **0.0594** | 0.1546 | **0.0425** |
| img18  | 0.1018 | 0.1018 | 0.1018 | 0.1018 | **0.0580** | 0.1026 | **0.0496** |
| img19  | 0.2252 | **0.1957** | 0.2002 | 0.2002 | 0.2543 | **0.0937** | 0.1263 |
| img20  | 0.1475 | **0.1240** | 0.1475 | **0.1240** | 0.2060 | 0.1263 | **0.0750** |
| img21  | 0.1397 | 0.1297 | 0.1696 | 0.1637 | **0.1197** | 0.1060 | **0.0726** |
| img22  | **0.1438** | 0.1458 | **0.1438** | 0.1458 | 0.1458 | 0.1696 | **0.0698** |
| img23  | 0.1545 | 0.1492 | 0.1430 | 0.1430 | **0.0922** | 0.0809 | **0.0543** |
| img24  | 0.1460 | 0.1460 | 0.1460 | 0.1460 | **0.0836** | 0.0529 | 0.0561 |
| img25  | 0.1360 | 0.1452 | 0.1241 | 0.1241 | **0.0827** | 0.1619 | **0.0517** |
| img26  | 0.1367 | 0.1331 | 0.1164 | 0.1164 | **0.0566** | 0.1140 | **0.0566** |
| img27  | 0.1317 | 0.0986 | 0.0913 | 0.0881 | **0.0782** | 0.1250 | **0.0748** |
| img28  | 1.4986 | 0.1560 | 0.1182 | 0.1194 | **0.0492** | 0.1182 | **0.0418** |
| img29  | 0.1310 | 0.1217 | 0.1310 | 0.1217 | **0.0419** | 0.0504 | 0.0529 |
| img30  | 0.1419 | 0.1385 | 0.1419 | 0.1373 | **0.0872** | 0.1429 | **0.0858** |
| img31  | 0.0675 | 0.0735 | 0.0492 | **0.0482** | 0.0965 | 0.0386 | **0.0273** |
| img32  | 0.1773 | 0.1263 | 0.1773 | 0.1263 | **0.0609** | 0.0754 | **0.0467** |
| avg    | 0.4620 | 0.1276 | 0.4795 | 0.1155 | **0.1026** | 0.1045 | **0.0567** |
Observing the results in Table 1, we conclude that the proposed method performs better than the 1-0-1 methods (D5, D11, G5 and G11) for most of the cases and also in the average case. On the 1-0-1 method, the gradual 1-0-1 method seems slightly better than the direct method with the same stencils, and the performances of the direct and gradual methods are improved when the number of stencils becomes bigger. When we turn our attention to the performance of our method, we observe that, for some special tested images, our method performs not so well as we expect. We guess that the labels may influence the performance of the proposed method. Hence, we subscribe new labels to further investigate the performance of our method and the 1-0-1 method. Because D11 outperforms D5, only D11 is performed with the new labels. Observing the last two columns in Table 1 which are from the new labels in Fig. 2, we obtain that the performance of the proposed method is improved significantly, but performance of D11 is improved slightly. Additionally, we note that the average computing time of the proposed method using these new labels is 59.5 s, but the D11 method is a little faster which requires 52.8 s. In summary, these experiments illustrate that the proposed method outperforms the direct 1-0-1 method [14] in the sense of yielding higher accurate results within almost the same time. Additionally, from Fig. 3 which is used to visually demonstrate
the accuracy of each method, we see that the proposed method can preserve
the convexity of the objects, but the 1-0-1 method cannot, which further means
that the progress of the proposed method based on these new labels is evident.

Finally, from the left plot in Fig. 4 we find that, for each case, the energy
function decreases monotonically until arriving at its minimum value, which
shows that the proposed algorithm possesses convergence property. From the
right plot, we conclude that the number of violated points reduces quickly within
the first 700 steps, and then remains a relatively small level to the end.

5.3. Experiments on images with multiple convex objects

In this subsection, we do some experiments on images containing multiple
convex objects of interest. The images include natural scene images and medical
images as shown in Fig. 5. In this test, as in [37], we also use the default
penalty parameters $10$ and $11 \times 11$ stencils. The ground truths of the concerned
objects are drawn out manually. In this test, we also run the 1-0-1 method for
performance comparisons, and their results are visually listed in Figs. 6 and 7,
respectively. Besides, we list the shape-distances of the results to the ground
truths and the computing time in Table 2.

Comparing each image in Figs. 6 and 7 we clearly observe that the proposed
method can segment and simultaneously preserve the convexity of the objects
almost exactly, but the 1-0-1 method cannot. Taking the top-left image in Fig.
7 as an example, the 1-0-1 method fails to catch both convex objects of interests.
Taking the second image in the first row as another example, the 1-0-1 method
cannot extract the objects accurately. The fault of the 1-0-1 method can be
observed for other images as well. When turning our attention to Table 2 we
see that for all the tested cases, the accuracy of segmentation yielded by our
method is better than the 1-0-1 method in the sense of obtaining smaller shape-
distances values. For these tested images, the average shape-distances for our
method is 0.05809, which is one half of the one obtained by the 1-0-1 method
with $5 \times 5$ and $11 \times 11$ stencils. When we focus on the computing time listed
in the last three columns in Table 2 we obtain that our proposed method is
more efficient than the 1-0-1 method, and it is at least 4 times faster than D11. This experiment illustrates that our proposed method is superior to the 1-0-1 method in the sense of segmentation accuracy and computation efficiency.

Figure 5: Test images with labels. The labels on background are marked in red, and the labels on foreground are marked in other colors.

Figure 6: The segmentation results of the images illustrated in Fig. 5 by the proposed method.
Figure 7: The segmentation results of the images in Fig. 5 by direct 1-0-1 method with 5 × 5 (1st and 2nd row) and 11 × 11 (3rd and 4th row) stencils.
Table 2: Shape-distances and the computing time consumed by the 1-0-1 and the proposed method.

| Image | Shape-distances   | Time (seconds) |
|-------|------------------|----------------|
|       | D5   | D11   | Ours | D5   | D11   | Ours |
| img1  | 0.8560 | 0.1088 | 0.0815 | 72   | 118   | 112  |
| img2  | 0.1266 | 0.1266 | 0.0654 | 311  | 1101  | 90   |
| img3  | 0.0469 | 0.0497 | 0.0332 | 43   | 187   | 84   |
| img4  | 0.0283 | 0.0334 | 0.0196 | 121  | 589   | 26   |
| img5  | 0.1651 | 0.1741 | 0.1551 | 30   | 135   | 172  |
| img6  | 0.1108 | 0.0944 | 0.0519 | 145  | 848   | 189  |
| img7  | 0.1681 | 0.1552 | 0.0408 | 167  | 612   | 198  |
| img8  | 0.0534 | 0.0630 | 0.0454 | 21   | 308   | 76   |
| img9  | 0.3648 | 0.2777 | 0.0413 | 713  | 1716  | 491  |
| img10 | 0.0761 | 0.0798 | 0.0467 | 210  | 1051  | 108  |
| average| 0.19961 | 0.11627 | 0.05809 | 183.3 | 666.5 | 154.6 |

5.4. Experiments by interactive procedure

In order to show the benefits and ability of the interactive procedure, we employ it to segment some challenging medical images as shown in Fig. 8. These images are challenging for most of existing methods because their edges are weak or cluttered, or the intensities of the objects are not uniform. To tackle these difficulties, labels should be given properly and suitably. Hence, some additional labels should be subscribed to improve the segmentation accuracy step by step when the result is not satisfactory. The labels and results are shown in Fig. 8 where the labels for background are marked in purple, and the initial foreground labels are marked in red and the additional labels at different steps are marked in different colors.
Comparing the segmentation results, we can observe that segmentation accuracy using the initial labels is low, which means that the initial labels marked in red are not enough. Then, we subscribe some new labels which marked in green, and list the segmentation results in the third row. Clearly, the accuracy of the segmentation results are improved significantly. Later, we subscribe some more new labels marked in blue, and report the segmentation results in the last row, which are improved further. For more details, let’s take the left abdomen image as an instance. Boundary of the object of interest is very weak.
and the intensity on its background is nonuniform strongly. Using the initial labels (red), the proposed method cannot extract the object accurately which can be seen from the first image in the second row. With more and more labels added, the segmentation result is improved step by step until an accurate segmentation result is obtained. We should note that, the breast image listed in the fourth column is more challenging because of the cluttered textures and edges. Therefore, the result only based on the initial labels is far from satisfactory. When some additional labels are subscribed properly, our proposed method is also able to segment them successfully. In summary, this demonstrates that the interactive procedure is capable of dealing with the issue of challenging medical images segmentation by adding proper labels gradually.

6. Summary and outlook

In this paper, we proposed a model for object(s) segmentation with convexity shape prior, and developed a numerical algorithm for the involved minimization problem arisen from the model. To tackle the difficulties from convex shape constraints, we relaxed and linearized these constraints to obtain a linearly-constrained nonconvex minimization problem, and then solved them by the proximal ADMM in the optimization literature. In addition, we also introduced an interactive procedure such that accuracies of the results were improved gradually if more labels were subscribed. It should be emphasized that, although the proposed method merely yielded an approximate solution of the developed model, the interactive procedure along with some additional labels could remedy these defects so that higher accurate segmentation results were obtained. Numerical experiments on various images verified the effectiveness and efficiency of the proposed method, and the superiority to the 1-0-1 methods \cite{14, 37} in the terms of segmentation accuracy and computation time. Last but not least, it is worthy of testing against some machine learning framework method to further evaluate the performance of our method. This is an interesting topic for further research.
Acknowledgments

This work was supported by the Programs for Science and Technology Development of Henan Province (192102310181, 212102310305), RG(R)-RC/17-18/02-MATH, HKBU 12300819, NSF/RGC Grant N-HKBU214-19, ANR/RGC Joint Research Scheme (A-HKBU203-19), RC-FNRA-IG/19-20/SCI/01, and National Natural Science Foundation of China (No.11971149).

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