Revisiting Normalized Gradient Descent: Fast Evasion of Saddle Points

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Abstract—The note considers normalized gradient descent (NGD), a natural modification of classical gradient descent (GD) in optimization problems. A serious shortcoming of GD in non-convex problems is that GD may take arbitrarily long to escape from the neighborhood of a saddle point. This issue can make the convergence of GD arbitrarily slow, particularly in high-dimensional non-convex problems where the relative number of saddle points is often large. The paper focuses on continuous-time descent. It is shown that, contrary to standard GD, NGD escapes saddle points “quickly.” In particular, it is shown that (i) NGD “almost never” converges to saddle points and (ii) the time required for NGD to escape from a ball of radius \( r \) about a saddle point \( x^* \) is at most \( 5\sqrt{\kappa}r \), where \( \kappa \) is the condition number of the Hessian of \( f \) at \( x^* \). As an application of this result, a global convergence-time bound is established for NGD under mild assumptions.

I. INTRODUCTION

Given a differentiable function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \), the canonical first-order optimization procedure is the method of gradient descent (GD). In continuous-time, GD is defined by the differential equation

\[
\dot{x} = -\nabla f(x) \tag{1}
\]

and in discrete-time, GD is defined by the difference equation

\[
x_{n+1} = x_n - \alpha_n \nabla f(x_n), \tag{2}
\]

where \( \{\alpha_n\}_{n \geq 1} \) is some step-size sequence. The discrete-time GD process (2) is merely a sample and hold (or Euler) discretization of the differential equation (1), and the properties of solutions of (1) and (2) are closely related [1]–[3].

The normalized gradient \( \nabla f(x) \|\nabla f(x)\| \) preserves the direction of the gradient but ignores magnitude. Because \( \nabla f(x) \|\nabla f(x)\| \) does not vanish near saddle points, the intuitive expectation (corroborated by evidence [6]) is that NGD should not slow down in the neighborhood of saddle points and should therefore escape “quickly.”

In this note, our goal is to elucidate the key differences between GD and NGD and, more importantly, give rigorous theoretical justification to the intuition that NGD “escapes saddle points quickly.” We will focus, in this work, on continuous-time descent. From the control perspective this may be seen as extending the seminal work of [14] by characterizing saddle-point behavior of NGD. From the optimization perspective, focusing on continuous-time dynamics allows us to more easily characterize the fundamental properties of NGD using a wealth of available analysis tools and follows in the spirit of recent works studying optimization processes through the lens of differential equations [17], [18].

We have three main results, which we state informally here:

**Main Result 1 (Theorem 9).** Our first main result is to show that NGD can only converge to saddle points from a

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we show that the time spent in the measure zero. It holds from all initial conditions except, possibly, some set with Lebesgue behavior of NGD, including demonstrating non-convergence from [14] primarily in that we characterize the saddle-point used to develop distributed gradient coordination algorithms that converge to critical points of $f$.

More precisely, we show that the maximum amount of time a trajectory of NGD can spend in a ball of radius $r > 0$ about a (non-degenerate) saddle point $x^*$ is $5\sqrt{\kappa}r$, where $\kappa$ is the condition number of the Hessian of $f$ at $x^*$ (see Theorem 1[1]).

We note that this result is independent of the dimension of the problem. In contrast to this, the saddle-point escape time of GD (i.e., the maximum amount of time a trajectory of GD may take to leave a ball of radius $r$ about a saddle point) is always infinite, independent of the function $f$, the particular saddle point $x^*$, or the dimension of the problem. (See Theorem 1[1] for a precise definition of saddle-point escape time and Remark 6 for a discussion of GD saddle-point escape time.) This is precisely the issue which causes GD to perform poorly in high-dimensional problems with many saddle points.

While a characterization of saddle-point escape time such as Theorem 1[1] is essential in understanding how SGD can mitigate the problem of saddle-point slowdown in high dimensional optimization [5], the issue is challenging to study due to the discontinuity in the right-hand side of (3). In particular, the system is not amenable to classical analytical techniques. We prove Theorem 1[1] by studying the rate of “potential energy dissipation” (to use an analogy from physics) of SGD near saddle points. The methods used are flexible and can be applied to a variety of discontinuous dynamical systems (see Remark 15 and proof of Proposition 21).

Main Result 3 (Corollary 20): As our final main result, using the local saddle-point analysis noted above (Theorem 1[1]) we provide a simple global bound on the convergence time of SGD under mild assumptions on $f$.

Literature Review: Continuous-time SGD dynamics were first introduced by Cortes [14] in the context of distributed multi-agent coordination. In [14], it was shown that SGD converges to critical points of $f$ in finite time and this result was used to develop distributed gradient coordination algorithms that achieve a desired task in finite time. Our results differ from [14] primarily in that we characterize the saddle-point behavior of SGD, including demonstrating non-convergence to saddle points and providing a strong characterization of saddle-point escape time. Furthermore, our results differ from [14] in that (i) our results show that SGD almost always converges to local minima rather than just the set of critical points of $f$ and (ii) [14] considered only local bounds on the convergence time of SGD to local minima. Because we characterize the saddle point behavior of SGD, our results enable global bounds on the convergence time of SGD to minima of non-convex functions (see Corollary 20).

Discrete-time SGD was first introduced by Nesterov [15] and variants have received increasing attention in the optimization and machine learning communities [16], [19]–[21]. The problem of coping with saddle points in non-convex optimization has received significant recent attention (see [5]–[10] and references therein). Of particular relevance to the present work are results dealing with first-order methods. Recent work along these lines includes the following. The work [7] shows that the classical stable manifold theorem implies that gradient descent only converges to minima. The work [9] shows that, even with random initialization, discrete-time GD can take exponential time to escape saddle points. The work [6] showed that noisy discrete-time GD converges to a local minimum in a polynomial number of iterations. Our work differs from [6] primarily in that we investigate the role of normalization of the dynamics (rather than noise injection) as a means of accelerating escape from saddle points.

The use of normalization in GD has also been studied in [16] where it was shown that discrete-time SGD with noise injection can outperform GD with noise injection in terms of dimensional dependence and the number of iterations required to reach the basin of a local minimum. Numerical simulations of discrete-time noisy SGD and comparisons with discrete-time noisy GD in several problems of interest were also performed in [16]. Our work differs from [16] in that we study the continuous-time deterministic SGD dynamics (which may be viewed as the mean dynamics of the noise-injected discrete-time SGD [16] as the step size is brought to zero), we characterize the stable-manifold for these dynamics near saddle points, and we explicitly characterize the saddle-point escape time.

The work [22] improved on the dimensional dependence of the results of [6] and [16], showing that GD with noise injection can reach the basin of a local minimum in a number of iterations with only polylog dependence on dimension. Our work differs from [22] in that we again study the underlying continuous dynamics and perform an explicit local analysis of the dynamics near saddle points. We demonstrate that the local saddle point escape time of SGD can be bounded independent of dimension (Theorem 1[1]). Moreover, because we show that SGD is a path-length reparametrization of GD, our results also have implications for classical GD. In particular, Theorem 1[1] together with Proposition 8 shows that a classical GD trajectory can have at most length $5\sqrt{\kappa}r$ (where $\kappa$ is the condition number of the Hessian of $f$ at $x^*$) before it must exit a ball of radius $r$ about a saddle point.

Organization: Section 1[1] sets up notation. Section 3[1] presents a simple example illustrating the salient features of GD and SGD near saddle points. Section 4[1] studies the
structural relationship between GD and NGD and presents Theorem 9 which shows generic non-convergence to saddle points. Section VI presents Theorem 11 which gives the saddle-point escape-time bound for NGD. Section VII presents a simple global convergence-time bound for NGD (Corollary 20). The proofs of all results are deferred to Section VII.

II. PRELIMINARIES

Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is a twice differentiable function. We use the following notation:

- $\nabla f(x)$ denotes the gradient of $f$ at $x$
- $D^2 f(x)$ denotes the Hessian of $f$ at $x$
- Given a set $S \subset \mathbb{R}^d$, the closure of $S$ is given by $\text{cl}(S)$ and the boundary of $S$ is given by $\partial S$
- $\mathcal{L}^d$, $d \geq 1$ denotes the $d$-dimensional Lebesgue measure
- $B_r(x)$ denotes the ball of radius $r$ about $x \in \mathbb{R}^d$
- $\|\cdot\|$ denotes the Euclidean norm
- $d(\cdot, \cdot)$ denotes Euclidean distance
- $\mathbf{x}$ is shorthand for $\frac{d}{dt}\mathbf{x}(t)$
- Given $C > 0, |D^2 f(x)| < C$ means that $|\frac{\partial^2 f(x)}{\partial x_i \partial x_j} | < C, i, j, k = 1, \ldots, d$
- For $A \in \mathbb{R}^{n \times n}$, $\sigma(A)$ denotes the spectrum of $A$
- $|\lambda_{\max}(A)| := \min\{|\lambda| : \lambda \in \sigma(A)\}$
- $|\lambda_{\max}(A)| := \max\{|\lambda| : \lambda \in \sigma(A)\}$
- The condition number of $A$ is given by $\frac{|\lambda_{\max}(A)|}{|\lambda_{\min}(A)|}$
- $\text{diag}(\lambda_1, \ldots, \lambda_d)$ gives a $d \times d$ matrix with $\lambda_1, \ldots, \lambda_d$ on the diagonal

We say that a saddle point $x^*$ of $f$ is non-degenerate if $D^2 f(x^*)$ is non-singular.

For $k \geq 1$, let $C^k$ denote the set of all functions from $\mathbb{R}^d$ to $\mathbb{R}$ that are $k$-times continuously differentiable. Unless otherwise specified, we will follow the notation throughout the paper.

Assumption 1. The objective function $f$ is of class $C^2$.

We say that a continuous mapping $\mathbf{x} : I \to \mathbb{R}^d$, over some interval $I = [0, T)$, $0 < T \leq \infty$, is a solution to an ODE with initial condition $x_0$ if $\mathbf{x} \in C^1$, $\mathbf{x}$ satisfies the ODE for all $t \in I$, and $\mathbf{x}(0) = x_0$.

Under assumption 1 there exists a unique solution to $\mathbf{x} \in C^1$ which exists on the interval $I = \mathbb{R}$ for every initial condition. A solution $\mathbf{x}$ to $\mathbf{x} \in C^1$ with initial condition $x_0$ satisfying $\nabla f(x_0) \neq 0$, will have a unique solution on some maximal interval of existence $I = [0, T)$, where $T$ is dependent on $x_0$ (see 23 for a formal definition of the maximal interval of existence). Practically, for solutions of (3) the maximal interval of existence is the maximal time interval for which a solution $\mathbf{x}$ does not intersect with a critical point of $f$. When we refer to a solution of (3) we mean the solution defined over its maximal interval of existence.

Remark 2 (Filippov solutions). We note that one can handle the discontinuity in the right hand side of (3) by considering solutions of the associated Filippov differential inclusion 14, 24. In order to keep the presentation simple and broadly accessible we have elected to avoid this approach and instead consider solutions only on intervals on which they are classically defined. Practically, the main differences between the two approaches are that (i) solutions in the classical sense cease to exist when they reach a saddle point or local minimum whereas Filippov solutions remain well defined at these points, and (ii) Filippov solutions may not be differentiable at times when solutions reach or depart from critical points. In particular, Filippov solutions to (3) may sojourn indefinitely at saddle points (and local maxima) of $f$ and remain at non-degenerate minima of $f$ once reached. Our results and analysis extend readily to solutions in this sense modulo minor technical modifications.

The following two definitions are standard from classical ODE theory.

Definition 3 (Orbit of an ODE). Let $x(t)$ be the solution of some ODE on the interval $[0, T)$. Assume that $x(0) = x_0$ and that $[0, T)$ is the maximal interval on which $x(t)$ is the unique solution of the ODE with initial value $x_0$ (here $T = \infty$ is permitted). Then the orbit corresponding to the initial condition $x_0$ is defined to be the set $\gamma_{x_0} := \{ x \in \mathbb{R}^d : \mathbf{x}(t) = x \forall t \in [0, T) \}$.

We note that $\gamma_{x_0}$ in the above definition is often referred to as a forward orbit; to simplify nomenclature, we will refer to it simply as an orbit.

Given a differentiable curve $\mathbf{x} : [0, T) \to \mathbb{R}^d$, the arc length of $\mathbf{x}$ at time $t < T$ is given by $L(t) := \int_0^t |\dot{\mathbf{x}}(s)|\, ds$, and we let $L(T) := \lim_{t \to T} L(t)$.

Definition 4 (Arc-Length Reparametrization). Suppose $\mathbf{x} : [0, T) \to \mathbb{R}^d$ is a differentiable curve in $\mathbb{R}^d$ with arc length at time $t$ given by $L(t)$. We say that $\tilde{\mathbf{x}} : I = [0, L(T)] \to \mathbb{R}^d$ is an arc-length reparametrization of $\mathbf{x}(t)$ if there holds $\mathbf{x}(t) = \tilde{\mathbf{x}}(L(t))$ for all $t \in [0, T)$.

We say that a property holds for almost every element in a set $A \subseteq \mathbb{R}^d$, $d \geq 1$, if the subset of $A$ where the property fails to hold has $\mathcal{L}^d$-measure zero. Likewise, we say that a property holds for almost every solution of an ODE if the property holds for solutions starting from almost every initial condition.

III. SADDLE-POINT BEHAVIOR OF GD AND NGD: EXAMPLES AND INTUITION

A. Saddle Points and GD

The following simple example illustrates the behavior of GD near saddle points.

Example 5. Suppose the objective function is given by

$$f(x) = \frac{1}{2}x^TAx, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and note that the origin is a saddle point of $f$. The associated GD dynamics (1) reduce to a simple linear system of the form

$$\frac{d}{dt}x(t) = -Ax(t)$$

with solution $x(t) = e^{-At}x_0$, for initial condition $x(0) = x_0 \in \mathbb{R}^2$.

By classical linear systems theory we see that solutions of this system will only converge to the origin if they start with
initial conditions in the stable eigenspace of \(-A\), which is given by \(E_s := \{x = (x^1,x^2) \in \mathbb{R}^2 : x^2 = 0\}\). Note that this is a set of initial conditions with Lebesgue measure zero.

Let \(r > 0\) and consider the following question: What is the maximum amount of time that a solution of \(\dot{x} = f(x)\) may spend in a ball of radius \(r > 0\) about the origin? It is straightforward to verify that trajectories not converging to 0 may take arbitrarily long to leave \(B_r(0)\), and so the time it could potentially take to escape saddle points is unbounded. Indeed, note that for \(\varepsilon \in (0,r)\), a trajectory of \(\dot{x} = f(x)\) starting on \(\partial B_r(0)\) which enters \(B_r(0)\) must spend at least time \(-r \log(\varepsilon)\) inside the \(r\)-ball before it may enter the \(\varepsilon\) ball.

These same basic properties generalize to GD in higher dimensional systems: Solutions of GD may only converge to a saddle point from a set of initial conditions with measure zero, but the time required to escape neighborhoods of the saddle is always infinite. This is made precise in the following remark.

**Remark 6** (Saddle-Point Escape Time of GD). Informally, given a function \(f\), a saddle point \(x^*\) of \(f\), and an \(r > 0\) to which we refer to the “saddle-point escape time” of an optimization process as the maximum amount of time a trajectory which does not converge to \(x^*\) may spend in a ball of radius \(r\) about \(x^*\). In GD, the saddle point escape time is always infinite.

That is, for arbitrary objective function \(f\), saddle point \(x^*\), and radius \(r > 0\) there holds

\[
\sup_{x_0 \in \partial B_r(x^*)} L^1 \left( \left\{ t \in [0,\infty) : x_{x_0}(t) \in B_r(x^*) \right\} \right) = \infty,
\]

where \(x_{x_0}\) is the solution of \(\dot{x} = f(x)\) with initial condition \(x_0\).

This is precisely the issue which causes GD to perform poorly in high-dimensional problems with many saddle points. In this paper we will see that NGD significantly mitigates this issue—rather than having an infinite saddle-point escape time, the saddle point escape time of NGD is at most \(5\sqrt{kr}\), where \(k\) is the condition number of \(D^2 f(x^*)\).

**B. Saddle Points and NGD**

We will now consider the behavior of NGD near the saddle point in the above example.

In order to better understand this issue, it is helpful to characterize the relationship between GD and NGD. In Section IV, we will see that GD and NGD are closely linked—the two systems are “topologically equivalent” \([23]\) and solutions of NGD are merely arc-length reparameterizations of GD solutions (see Definition 2). In practical terms this means that if one considers orbits of NGD and GD starting from the same initial condition \(x_0 \in \mathbb{R}^d\), the orbits generated by the two systems are identical (see Definition 3). The solutions of each system only vary in how quickly they move along the common orbit. In particular, since NGD always “moves with speed 1” (i.e., \(\|\dot{x}(t)\| = 1, \forall t \geq 0\)) the length of an arc generated by NGD up to time \(t\) is precisely \(t\) (this is what it means to be an arc-length reparameterization). As an important result of this characterization, we will see that NGD “almost never” converges to saddle points (see Theorem 2).

While a solution of GD may move arbitrarily slowly as it passes near a saddle point, a solution of NGD starting at the same initial condition will move along the same orbit with constant speed, not slowing near the saddle point. This is illustrated in Fig. 1.

Consider NGD with \(f\) as defined in Example 5 (see (5)). Given the simple linear structure of the corresponding GD ODE (7) it is straightforward to verify that the arc-length of any trajectory of GD (or equivalently NGD) intersecting \(B_r(0)\) is upper bounded by \(2r\) and hence the maximum time a trajectory of NGD may spend in \(B_r(0)\) is \(2r\) (see Fig. 1).

This simple example may be generalized to higher dimensions. Let \(f : \mathbb{R}^d \to \mathbb{R}, d \geq 2\) be given by \(f(x) = x^T Ax\), with \(A = \text{diag}(\lambda_1,\ldots,\lambda_d)\) with \(|\lambda_i| = 1\) for all \(i = 1,\ldots,d\), and at least one \(\lambda_i > 0\) and one \(\lambda_i < 0\). Given the simple structure of the corresponding GD ODE \(\dot{x} = -Ax\), it is straightforward to show that the arc-length of any trajectory of GD intersecting \(B_r(0)\) (and hence the amount of time spent by NGD in \(B_r(0)\)) is upper bounded by \(2r\), independent of the dimension \(d\).

Note that in this example, the condition number of \(D^2 f(0)\) is 1. In general, as the condition number increases, the time spent by NGD in \(B_r(0)\) may increase. Theorem 11 captures this relationship for general \(f\) (satisfying Assumption 1).

**Remark 7.** We note that the bound that will be established in Theorem 11 is conservative. In particular, suppose \(f : \mathbb{R}^d \to \mathbb{R}\) is quadratic of the form \(f(x) = x^T Ax\), with \(A \in \mathbb{R}^{d \times d}\) diagonal and non-singular. Then one can show that time spent by a trajectory of NGD in \(B_r(0)\) is at most \(2\sqrt{dr}\). This bound holds even as the condition number of \(D^2 f(0)\) is brought to \(\infty\). Thus, while an ill-conditioned saddle point can slow the escape time of NGD, this example suggests that in the worst case as the condition number is brought to \(\infty\), the time spent by NGD in \(B_r(x^*)\) about a saddle point \(x^*\) can be bounded by \(C\sqrt{dr}\), where \(C > 0\) is some universal constant independent of dimension and condition number. An in-depth investigation of this issue is outside the scope of this note.

**IV. NGD: Structural Properties and Generic Convergence to Local Minima**

The following proposition establishes the basic structural relationship between GD and NGD.

**Proposition 8.** Let \(x(t)\) and \(\tilde{x}(t)\) be solutions of (1) and (3) respectively, with the same initial condition \(x_0\), over maximal intervals \([0,T]\) and \([0,\bar{T}]\) respectively. Then \(\tilde{x}(t)\) is an arc length reparametrization of \(x(t)\), and \(\tilde{x}(t) = x(s(t))\) for some strictly increasing function \(s(t)\), with \(s(0) = 0\) and \(s(\bar{T}) = T\).

This result means that (classical) solutions of (1) and (3) starting at the same initial condition have identical orbits (see

4This is shown by bounding the arc length of the corresponding linear GD ODE \(\dot{x} = -Ax\). Intuitively, if \(A\) is well conditioned, then trajectories of the ODE passing near 0 travel along a “direct route” to and away from 0. If \(A\) is ill conditioned, then trajectories of the ODE travel a “Manhattan route” to and away from 0, with movement tangential to the stable eigenspace of \(A\) occurring along only one stable eigenvector at a time.
It follows from Propositions 8 and 9 that solutions of NGD exist and are unique for almost every initial condition. We note that both of these results follow as elementary applications of classical ODE theory (See Section VII).

We also note that this issue (generic non-convergence to saddle points, as in Proposition 9) was considered for discrete-time GD (2) in the recent work [7]. Addressing the question of “stable manifold” theorems for the discrete analog of (3) will be a subject of future work.

V. Fast Escape From Saddle Points

The following theorem gives our main result regarding fast escape from saddle points. The theorem provides a simple estimate on the amount of time that trajectories of NGD can spend near saddle points.

**Theorem 11** (Saddle-Point Escape Time). Let $C > 4$ and suppose $x^*$ is a non-degenerate saddle point of $f$. Then for all $r > 0$ sufficiently small, any trajectory of (3) that does not reach or converge to $x^*$ can spend time at most $C\sqrt{kr}$ in the ball $B_r(x^*)$, where $k$ is the condition number of $D^2 f(x^*)$. That is, if $x_{x_0}$ is a solution to (3) with initial condition $x_0$ and maximal interval of existence $[0, T_{x_0})$, $T_{x_0} \leq \infty$, and $x^* \notin \text{cl}(γ_{x_0}^*)$, then

$$L^1 \left( \{ t \in [0, T_{x_0}) : x_{x_0}(t) \in B_r(x^*) \} \right) \leq C \sqrt{kr}.$$

We recall that by Theorem 9, solutions of (3) can only reach or converge to saddle points from a set of initial conditions with measure zero, hence the theorem hold for solutions starting from almost every initial condition.

In order to underscore the significance of this result, we recall that the saddle-point escape time of GD (i.e., the time required to escape a ball of radius $r > 0$ about a saddle point) is infinite, independent of $f$, $d$, $x^*$, and $r$ (see Remark 6), which causes GD to perform poorly in problems with many saddle points. In contrast to this, Theorem 11 shows that trajectories of NGD always escape a ball of radius $r$ within time $5\sqrt{kr}$.

Furthermore, we recall that Proposition 8 showed that orbits of GD and NGD coincide. Thus, away from saddle points (where GD is generally “well behaved”) GD and NGD behave in an essentially identical manner in that they follow identical trajectories and the velocity of each can be bounded from below.

A few remarks are now in order.

**Remark 12** (Values of constant $C$). The above theorem holds with the constant $C$ set to any value strictly greater than 4. The proof of the estimate in the theorem utilizes several Taylor series approximations. There is a tradeoff inherent in this proof technique—as $C$ approaches 4, the range of permissible values of $r > 0$ where the Taylor approximation (and hence, the theorem) is applicable shrinks to zero. For clarity of presentation and to emphasize the key features of (3) in other words, the dynamical systems defined by (4) and (3) are topologically equivalent [23] with the concomitant homeomorphism given by the identity.

6As in the introduction, to emphasize the key features of this result we fix the constant $C$ to be 5 here. Of course, the theorem holds for the constant $C$ fixed to any value strictly greater than 4. See Remark 12 for more details.
this result we find it convenient to simply fix the constant to
be 5 in the abstract and introduction. See Proposition 27 and
proof thereof for more details.

**Remark 13** (Permissible values of \( r \)). The range of values of \( r > 0 \) where Theorem 22 holds depends both on the
constant \( C \) and the magnitude of higher order derivatives
near the saddle point \( x^* \). In particular, the result holds so
long as the Taylor estimates \([10, 11] \) used in the proof are valid. If one assumes that \( f \) is more than twice differentiable and assumes bounds on the magnitude of the higher order
derivatives near \( x^* \), then the radius where these estimates hold can
be bounded, and a more precise statement can be made about
the permissible values of \( r \) in Theorem 22. For example, if one assumes that \( |D^3 f(x)| < \hat{C} \) is uniformly bounded for
some \( \hat{C} > 0 \) then Theorem 22 holds for all \( r \in (0, \hat{r}) \), where
\( \hat{r} = 6C^{-1/2}\hat{C}^{-1}|\lambda_{max}(D^2 f(x^*))| \left( \frac{C(3\hat{C}+2)}{6C} + \frac{1}{\Gamma} \right) \), and where \( \kappa \) is the condition number of \( D^2 f(x^*) \). This is verified by
confirming that the Taylor estimates \([10, 11] \) used in the proof of Proposition 27 are valid in the ball \( B_\hat{r}(0) \), \( \hat{r} = \kappa^{1/2}r \), for values of \( r \) in this range.

**Remark 14** (Non-Applicability of the Hartman-Grobman Theorem). The Hartman-Grobman theorem from classical
differential equations states that near non-degenerate saddle
points one can construct a homeomorphism mapping the
trajectories of a non-linear ODE to trajectories of the associated
linearized system \([23] \). It is simple to show that Theorem 22 holds when \( f \) is quadratic (and hence the associated
NGD system is topologically identical to a linear system); see Section II-A. Thus, one might expect Theorem 22 to hold for
general (non-quadratic) \( f \) by the Hartman-Grobman theorem.
However, the homeomorphisms constructed in the Hartman-
Grobman theorem are in general not smooth, and so will not
preserve trajectory length, and cannot be used to prove a bound such as Theorem 22. Instead one must resort to more
analytical techniques to study path length; see the proof of
Proposition 27 below.

**Remark 15** (Theorem Proof Technique). Here, the key idea
of the proof of Theorem 22 relies on establishing a differential
inequality between the “potential” \( f \) and the “potential
dissipation rate” \( \frac{d}{dt}f(x(t)) \). The methods are flexible, and
may be applicable to other non-smooth settings. In a previous
work \([26] \) the authors utilized similar techniques to study non-
smooth dynamics in game-theoretical problems.

### VI. A GLOBAL CONVERGENCE-TIME BOUND

We will now use the above results to prove a simple corollary bounding the maximum amount of time that trajectories
can take to reach local minima under (3).

We will make the following assumptions.

**Assumption 16.** The function \( f \) is of class \( C^3 \) and
\( |D^3 f(x)| \leq \hat{C} \) uniformly for all \( x \in \mathbb{R}^d \), for some \( \hat{C} > 0 \).

This assumption ensures that there exists a single \( r > 0 \)
such that Proposition 27 holds within a ball of radius \( r \) about
every critical point (see Remark 13).

Next we assume a uniform bound on the magnitude of eigenvalues of the Hessian at critical points.

**Assumption 17.** There exist constants \( |\lambda_{max}|, |\lambda_{min}| > 0 \) such that for every critical point \( x^* \) of \( f \) there holds \( |\lambda| \leq |\lambda_{max}| \) for all \( \lambda \in \sigma(D^2 f(x^*)) \).

The next assumption ensures that at any point \( x \in \mathbb{R}^d \),
either the gradient of \( f \) at \( x \) is large (guaranteeing fast local
improvement of descent techniques), or \( x \) is close to a critical
point.

**Assumption 18.** Fix \( C > 4 \). Assuming Assumptions 16 and
27 hold, let \( r > 0 \) be chosen so that Theorem 22 (or equivalently,
Proposition 27) holds with constant \( C \) at every critical point
and so that \( r \leq \frac{|\lambda_{min}|}{C} \). Furthermore, assume that there exists a constant \( \nu > 0 \) such that for all \( x \in \mathbb{R}^d \) either
\[ \|x - x^*\| < r, \text{ with } \|\nabla f(x^*)\| = 0 \text{ or } \|\nabla f(x)\| > \nu. \]

**Assumptions 17** and **18** together are similar to the strict
saddle property assumed in \([6, 16] \). The main difference is
that here we assume a uniform (lower) bound on the minimum-
magnitude eigenvalue of the Hessian at all critical points rather
than just saddle points, and we assume a uniform (upper)
bound on the maximum-magnitude eigenvalue of the Hessian
at all critical points. The final assumption ensures that a
descent process will eventually converge to some point rather
than expanding out infinitely. This assumption is naturally
satisfied, for example, if \( f \) is coercive (i.e., \( f(x) \to \infty \) as
\( \|x\| \to \infty \)).

**Assumption 19.** There exists an \( R > 0 \) such that trajectories
of (3) that begin in \( B_R(0) \) remain in \( B_R(0) \) for all \( t \geq 0 \).

Let \( R > 0 \) be as in Assumption 19 and let
\[ M := \sup_{x \in B_R(0)} |f(x)|, \quad (7) \]

Note that since \( f \) is continuous, \( M < \infty \).

The following result gives a simple estimate on the amount of
time the dynamics (3) will take to reach a local minimum.

**Corollary 20.** Suppose that every saddle point \( x^* \) of \( f \) is
non-degenerate and that Assumptions 16, 17 hold. Then for
almost every initial condition inside \( B_R(0) \), solutions of (3)
will converge to a local minimum in at most time \( 2M\nu^{-1} +
C\sqrt{\frac{|\lambda_{max}|}{|\lambda_{min}|}}(R+\nu)^d \) \( \frac{r}{r - \nu} \), where \( C > 4 \) is the constant in Assumption 18.

### VII. PROOFS OF MAIN RESULTS

We now present the proofs of the results found in Sections
IV – VII.

We begin by presenting the proofs of Propositions 8 and
Theorem 9 which follow from elementary applications of
classical ODE theory.

**Proof of Proposition 8**. Given a solution \( x \) to (1), one can
reparametrize the trajectory by arc length, i.e., \( \tilde{x}(t) = x(L(t)) \),
and \( \|D_x \tilde{x}(t)\| = 1 \). Using the chain rule we find that \( \frac{d}{dt} \tilde{x}(t) = -\frac{\nabla f(\tilde{x}(t))}{\|\nabla f(\tilde{x}(t))\|} \). Since the solutions are classical, uniqueness of
solutions for ODE gives us that \( \tilde{x} = \hat{x} \) must be equal. \( \square \)
Proof of Theorem 9. We begin by proving part (i) of the theorem. Solutions to (1) which converge to such a saddle point are contained within a stable manifold, i.e. a smooth surface of at most dimension $n - 1$. Such a surface will be a set with Lebesgue measure zero. Proof and details of such a result may be found in [23]. The result then follows from Proposition 8.

Part (ii) of the theorem follows from the fact that if all saddle points are non-degenerate, then all saddle points are isolated. Hence, the set of saddle points is countable. By part (i) of the theorem, the union of the stable manifolds for all saddle points is a set with Lebesgue measure zero. □

The following proposition proves Theorem 11. The proposition is stated in slightly more general terms than Theorem 11 in order to account for the behavior of NGD near minima as well as saddle points.

Proposition 21. Let $C > 4$, let $x^* \in \mathbb{R}^d$ be a non-degenerate critical point of $f$, and let $x(t)$ be a solution of (3) with arbitrary initial condition $x_0 \neq x^*$ and maximal interval of existence $[0, T_{x_0})$. For all $r > 0$ sufficiently small, the time spent by $x(t)$ in $B_r(x^*) \setminus \{x^*\}$ is bounded according to

$$\mathcal{L}^1 \left( \{ t \geq 0, T_{x_0} : x(t) \in B_r(x^*) \setminus \{x^*\} \} \right) \leq C \sqrt{\kappa r},$$

where $\kappa = \frac{\lambda_{\min}(D^2f(x^*))}{\lambda_{\max}(D^2f(x^*))}$.

Proof. Without loss of generality, assume $x^* = 0$ and let $H := D^2f(0)$. For $x \in \mathbb{R}^d$ define $\tilde{d}(x) := \sqrt{x^T H x}$, where $|B| := \sqrt{B^T B}$ for a square matrix $B$. The function $\tilde{d}$ will be a convenient modified distance for the proof. For convenience in notation, throughout the proof we use the shorthand $\lambda_{\max} := \lambda_{\max}(D^2f(x^*))$ and $\lambda_{\min} := \lambda_{\min}(D^2f(x^*))$.

Note that for $a \geq 0$ we have the following relationships

\begin{align}
\|x\| \leq \frac{a}{\sqrt{\lambda_{\max}}} & \implies \tilde{d}(x) \leq a \quad (8) \\
\tilde{d}(x) \leq a & \implies \|x\| \leq \frac{a}{\sqrt{\lambda_{\min}}}.
\end{align}

By Taylor’s theorem and the non-degeneracy of $x^*$, for any $C_2 > \frac{1}{2}$ there exists a neighborhood of 0 such that

$$|f(x) - f(0)| \leq C_1 \tilde{d}(x)^2. \quad (10)$$

Using the chain rule we see that along the path $x(t)$, the potential changes as

$$\frac{d}{dt} f(x(t)) = -\|\nabla f(x(t))\|.$$ 

Let $C_2 < 1$ be arbitrary. Again using Taylor’s theorem and the non-degeneracy of $H$, for $x(t)$ in a neighborhood of 0 we have that

\begin{align}
\|\nabla f(x(t))\| & \geq C_2 \|H x(t)\| \\
& = C_2 \|H^{1/2} H^{1/2} x(t)\| \\
& \geq C_2 \sqrt{\lambda_{\min}} \|H^{1/2} x(t)\| \\
& = C_2 \sqrt{\lambda_{\min}} \tilde{d}(x(t)),
\end{align}

where $\lambda_{\min}$ denotes the magnitude of the smallest-magnitude eigenvalue of $H$. In turn

$$-\frac{d}{dt} f(x(t)) \geq C_2 \sqrt{\lambda_{\min}} \tilde{d}(x(t)). \quad (12)$$

Let $\hat{r} > 0$ be such that the estimates (10) and (11) hold inside the closed ball $B_{\hat{r}}(0)$. Suppose that $x(t) \in B_{\hat{r}}(0)$ for $t \in [t_1, t_2]$. Letting $e(t) := \tilde{d}(x(t))$ and integrating (12) gives

$$f(x(t_1)) - f(x(t_2)) \geq C_2 \sqrt{\lambda_{\min}} \int_{t_1}^{t_2} e(s) ds.$$ 

Let $r := \kappa^{-\frac{1}{2}} \hat{r}$. Suppose $\eta \leq \sqrt{\lambda_{\max}} r$ and note that by (9), $\tilde{d}(x) \leq \eta$ implies that $x \in B_{r}(0)$. Furthermore, suppose $e(t) \leq \eta$ for some $t \geq 0$, and let $t_0$ be the first time where $e(t) \leq \eta$. Let $t_3$ be the last time when $e(t) = \eta$; i.e., $t_3 = \sup\{t \in [0, \infty) : e(t) \leq \eta\}$. If $t_3 = \infty$, then in an abuse of notation we let $f(\infty(x)) = \lim_{t \to \infty} f(x(t))$, where we note that the limit exists since $f(x(t))$ is monotone non-increasing in $t$. It follows that

$$f(x(t_0)) - f(x(t_3)) \geq \int_{t_0}^{t_3} -\frac{d}{ds} f(x(s)) ds \geq \int_{e(s) \leq \eta} -\frac{d}{ds} f(x(s)) ds \geq C_2 \sqrt{\lambda_{\min}} \int_{e(s) \leq \eta} e(s) ds,$$

where we use the fact that $\frac{d}{dt} f(x(t)) \leq 0$, and the previous inequality on subintervals where $e(\cdot) \leq \eta$. Adding and subtracting $f(0)$ to the left hand side above and using (10) we obtain

$$\frac{2C_1}{C_2 \sqrt{\lambda_{\min}}} \eta^2 \geq \int_{e(s) \leq \eta} e(s) ds.$$ 

Markov’s inequality [27] then gives

$$\mathcal{L}^1 \left( \{ s : \eta \geq e(s) \geq \frac{\eta}{2} \} \right) \leq \frac{\eta}{4C_1 \sqrt{\lambda_{\min}}} \eta^2 \leq \frac{\eta C_2 \sqrt{\lambda_{\min}} \eta^2}{C_2 \sqrt{\lambda_{\min}}} \eta.$$ 

We can iteratively apply this inequality to obtain

$$\mathcal{L}^1 \left( \{ s : \eta \geq e(s) > 0 \} \right) = \sum_{i=0}^{\infty} \mathcal{L}^1 \left( \{ s : \eta \geq e(s) \geq \frac{\eta}{2^{i+1}} \} \right) \leq \sum_{i=0}^{\infty} \frac{4C_2 \eta}{C_2 \sqrt{\lambda_{\min}} 2^i} \leq \frac{8C_1}{C_2 \sqrt{\lambda_{\min}}} \eta.$$ 

(13)
By [3] we see that \( \{ s : 0 < \|x(s)\| \leq r \} \subset \{ s : 0 < \hat{d}(x(s)) \leq \sqrt{\|\lambda_{\text{max}}r\|} \} \). Letting \( \eta = \sqrt{\|\lambda_{\text{max}}r\|} \) in [13], and letting \( C := \frac{\eta c_1}{c_2} \), we get

\[
L^1 \left( \{ s : 0 < \|x(s)\| \leq r \} \right) \\
\leq L^1 \left( \{ s : 0 < \hat{d}(x(s)) \leq \sqrt{\|\lambda_{\text{max}}r\|} \} \right) \leq C \sqrt{\|\lambda_{\text{max}}\| \|\lambda_{\text{min}}\| \|r\|},
\]

where we recall that \( r = \kappa^{-1/2} \hat{r} \) and \( \hat{r} \) is the radius of the ball where (10) and (11) hold and is dependent on \( C_1 \) and \( C_2 \). Since \( C_1 > 1/2 \) and \( C_2 < 1 \) were arbitrary, the constant \( C \) may be brought arbitrarily close to 4 with the range of permissible values of \( r \) changing accordingly with the choice of \( C_1 \) and \( C_2 \). This proves the desired result.

**Proof of Corollary 20** First, we claim that critical points must be separated by a distance of at least \( 2 \hat{r} \). Let \( x^* \) be a critical point. Then

\[
\nabla f(x) = \int_0^1 D^2 f((1-s)x^* + sx)(x - x^*) \, ds \\
= \int_0^1 D^2 f(x^*) \, ds + \int_0^1 \int_0^s D^3_{x-x^*} f((1-\tau)x^* + \tau x)(x - x^*) \, d\tau \, ds,
\]

where \( D^3_{x-x^*} \) we mean the matrix representing the third derivative evaluated in the direction \( x - x^* \). We can then bound

\[
|\nabla f(x)| \geq |\lambda_{\text{min}}| |x - x^*| - \frac{\hat{C}}{2} |x - x^*|^2
\]

where \( \hat{C} \) is the bound on our third derivatives. Note that by Assumption [18] we have \( 2r \leq 2\|\lambda_{\text{max}}\| \). Thus we see that for any \( x \in B_{2\|\lambda_{\text{max}}\|}(x^*) \) we have \( \nabla f(x) \neq 0 \). Hence critical points must be separated by a distance of at least \( 2 \hat{r} \).

Now, let \( x(t) \) be a classical solution of (3) (which, by Theorem 9 holds for a.e. solution of (3)). Let \([t_1, t_2] = I \) be the maximal interval of existence for this classical solution. Our goal is to prove that \((t_2 - t_1)\) can be bounded uniformly.

To this end, we divide \( I \) into two subsets, \( I_c, I_0 \), where \( I_c \)

- are the times where \( |x(t) - x^*| \leq r \) for some critical point \( x^* \), and \( I_0 \) are points where \( |\nabla f(x(t))| \geq \nu \).

Using the chain rule we see that \( \frac{d}{dt} f(x(t)) = -\|\nabla f(x(t))\| \). By Assumption 19 [7] we have \( |f(x(t))| < M \) along any trajectory of (3) starting in \( B_\hat{r}(0) \). Thus, we immediately have that \( |I_0| < 2Mv^{-1} \).

Let \( \kappa = \frac{\|\lambda_{\text{max}}\|}{\|\lambda_{\text{min}}\|} \). By Proposition 21 we can spend at most \( C\sqrt{\hat{r}} \) near any particular critical point. Since critical points are separated by at least distance \( 2 \hat{r} \), we can cover all the critical points with disjoint balls of radius \( r \). By then estimating the volume, the total number of critical points within distance \( R \) of the origin is at most \( \frac{(R+\hat{r})^d}{\pi^d} \). Thus we find that \( |I_c| < C\sqrt{\hat{r}} \frac{(R+\hat{r})^d}{\pi^d} \).

In summary, we find that \( |I| \leq 2Mv^{-1} + C\sqrt{\hat{r}} \frac{(R+\hat{r})^d}{\pi^d} \). This implies that classical trajectories can be of length at most \( Mv^{-1} + C\sqrt{\hat{r}} \frac{(R+\hat{r})^d}{\pi^d} \). Since a.e. initial condition does not reach any saddle point, almost every initial condition will converge to a local minimizer of \( f \) in \( 2Mv^{-1} + C\sqrt{\hat{r}} \frac{(R+\hat{r})^d}{\pi^d} \) time. This concludes the proof.

**REFERENCES**

[1] H. K. Khalil, “Nonlinear systems,” Prentice-Hall, New Jersey, vol. 2, no. 5, pp. 5–1, 1996.
[2] M. Benaim, “A dynamical system approach to stochastic approximations,” SIAM Journal on Control and Optimization, vol. 34, no. 2, pp. 437–472, 1996.
[3] J. Stoer and R. Bulirsch, Introduction to numerical analysis. Springer Science & Business Media, 2013, vol. 12.
[4] S. Boyd and L. Vandenberghe, Convex optimization. Cambridge university press, 2004.
[5] Y. N. Dauphin, R. Pascanu, C. Gulcehre, K. Cho, S. Ganguli, and Y. Bengio, “Identifying and attacking the saddle point problem in high-dimensional non-convex optimization,” in Advances in neural information processing systems, 2014, pp. 2933–2941.
[6] R. Ge, F. Huang, C. Jin, and Y. Yuan, “Escaping from saddle points—online stochastic gradient for tensor decomposition,” in Conference on Learning Theory, 2015, pp. 797–842.
[7] J. D. Lee, M. Simchowitz, M. I. Jordan, and B. Recht, “Gradient descent only converges to minimizers,” 2016, pp. 1246–1257.
[8] A. Wibisono, A. C. Wilson, and M. I. Jordan, “A variational perspective on accelerated methods in optimization,” Proceedings of the National Academy of Sciences, 2016, preprint.
[9] S. S. Du, C. Jin, J. D. Lee, M. I. Jordan, B. Poczos, and A. Singh, “Gradient descent can take exponential time to escape saddle points,” Conference Neural and Information Processing Systems, 2017.
[10] S. C. Kim, K. Levy, “The power of normalization: faster evasion of saddle points,” arXiv preprint, arxiv:1903.01434, 2017.
[11] ——, “Complete dictionary recovery over the sphere,” in Sampling Theory and Applications (SampThy), 2015 International Conference on. IEEE, 2015, pp. 407–410.
[12] J. Cortés, “Finite-time convergent gradient flows with applications to network consensus,” Automatica, vol. 42, no. 11, pp. 1993–2000, 2006.
[13] Y. Nesterov, “Minimization methods for nonsmooth convex and quasiconvex functions,” Matekon, no. 29, pp. 519–531, 1984.
[14] K. Levy, “The power of normalization: faster evasion of saddle points,” arXiv preprint, arxiv:1611.04831, 2016.
[15] W. Su, S. Boyd, and E. Candes, “A differential equation for modeling newtroses accelerated gradient method: Theory and insights,” in Advances in neural Information Processing Systems, 2014, pp. 2510–2518.
[16] W. Krichene, A. Bayen, and P. L. Bartlett, “Accelerated mirror descent in continuous and discrete time,” in Advances in neural information processing systems, 2015, pp. 2845–2853.
[17] K. C. Kiwiel, “Convergence and efficiency of subgradient methods for quasiconvex minimization,” Mathematical programming, vol. 90, no. 1, pp. 1–25, 2001.
[18] I. V. Konnov, “On convergence properties of a subgradient method,” Optimization Methods and Software, vol. 18, no. 1, pp. 53–62, 2003.
[19] E. Hazan, K. Levy, and S. Shalev-Shwartz, “Beyond convexity: Stochastic quasiconvex optimization,” in Advances in Neural Information Processing Systems, 2015, pp. 1594–1602.
[20] C. Jin, R. Ge, P. Netrapalli, S. M. Kakade, and M. I. Jordan, “How to escape saddle points efficiently,” arXiv preprint, arxiv:1703.00887, 2017.
[21] L. Perko, Differential equations and dynamical systems. Springer Science & Business Media, 2013, vol. 7.
[22] A. F. Filippov, “Differential equations with discontinuous right-hand side,” Mat. Sb. (N.S.), vol. 51 (93), pp. 99–128, 1960.
[23] J. Sun, Q. Qu, and J. Wright, “When are nonconvex problems not scary?” arXiv preprint, arxiv:1510.06996, 2015.
[24] B. Swenson, R. Murray, and S. Kar, “On best-response dynamics in potential games,” to appear in SIAM J. on Control and Optimization. arXiv preprint, arxiv:1707.06465, 2017.
[25] H. Federer, Geometric measure theory. Springer, 2014.