DYNAMICAL AND VARIATIONAL PROPERTIES OF THE NLS-δs′ EQUATION ON THE STAR GRAPH

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Abstract. We study the nonlinear Schr¨ odinger equation with δs′ coupling of intensity
β ∈ R \ {0} on the star graph Γ consisting of N half-lines. The nonlinearity has the form
g(u) = |u|^{p-1}u, p > 1. In the first part of the paper, under certain restriction on β, we
prove the existence of the ground state solution as a minimizer of the action functional
Sω on the Nehari manifold. It appears that the family of critical points which contains a
ground state consists of N profiles (one symmetric and N − 1 asymmetric). In particular,
for the attractive δs′ coupling (β < 0) and the frequency ω above a certain threshold, we
managed to specify the ground state.

The second part is devoted to the study of orbital instability of the critical points. We
prove spectral instability of the critical points using Grillakis/Jones Instability Theorem.
Then orbital instability for p > 2 follows from the fact that data-solution mapping
associated with the equation is of class C^2 in H^1(Γ). Moreover, for p > 5 we complete
and concertize instability results showing strong instability (by blow up in finite time)
for the particular critical points.

1. Introduction

We consider a star graph Γ consisting of a central vertex v and N ≥ 2 infinite half-
lines attached to it. We identify Γ with the disjoint union of the intervals I_j = (0, ∞),
j = 1, . . . , N, augmented by the central vertex v = 0. The function on Γ is defined by
\begin{align*}
v : Γ \to \mathbb{C}^N, \quad v = (v_j)_{j=1}^N, \quad v_j : (0, ∞) \to \mathbb{C}.
\end{align*}

We denote by v_j(0) and v'_j(0) the limits of v_j(x) and v'_j(x) as x → 0+.

The principal object of this study is the following focusing nonlinear Schrödinger equation
on Γ with δs′ coupling:
\begin{align}
\begin{cases}
    \begin{align*}
    i∂_t u(t, x) &= -\Delta_β u(t, x) - |u(t, x)|^{p-1} u(t, x), & (t, x) \in \mathbb{R} \times Γ, \\
    u(0, x) &= u_0(x),
    \end{align*}
\end{cases}
\end{align}

where \( \beta \in \mathbb{R} \setminus \{0\} \), \( p > 1 \), \( u : \mathbb{R} \times Γ \to \mathbb{C}^N \), and \( (-\Delta_β v)(x) = (-v''_j(x)) \), \( x > 0 \), with
\begin{align}
\text{dom}(\Delta_β) = \{ v \in H^1(Γ) : v'_1(0) = \ldots = v'_N(0), \quad \sum_{j=1}^N v_j(0) = \beta v'_1(0) \}.
\end{align}

2010 Mathematics Subject Classification. Primary: 35Q55; Secondary: 35Q40.
Key words and phrases. δs′ coupling, ground state, Nonlinear Schrödinger equation, orbital stability,
spectral instability, star graph.
The Schrödinger operator \(-\Delta_{\beta}\) with \(\delta'_s\) coupling has the precise interpretation as the self-adjoint operator on \(L^2(\Gamma)\) uniquely associated (by the KLMN Theorem [42, Theorem X.17]) with the closed semibounded quadratic form \(F_{\beta}\) defined on \(H^1(\Gamma)\) by

\[
F_{\beta}(v) = \|v'\|_2^2 + \frac{1}{\beta} \left| \sum_{j=1}^{N} v_j(0) \right|^2.
\]

Similarly to the case of \(\delta'\) coupling on the line (see [12]), the \(\delta'_s\) coupling on \(\Gamma\) has a high energy scattering behavior that can be reproduced through scaling limits of scatterers, the so-called spiked-onion graphs. These are obtained replacing \(\Gamma\) by the graph \(\Gamma_N(n, \ell)\), where every pair of half-line endpoints is connected by \(n\) links of length \(\ell\) (see [21]). The corresponding variables run through the interval \([-\ell/2, \ell/2]\), and it is assumed that the coupling at each graph vertex is \(\delta\) (see formula (1.3)). One assumes that \(\ell \to 0, n \to \infty\) while keeping the product \(n\ell\) fixed.

In [18] the authors proposed an interpretation of the Hamiltonian \(-\Delta_{\beta}\) as a norm resolvent limit of the operator \(H(\varepsilon)\) when \(\varepsilon \to 0^+\). The operator \(H(\varepsilon)\) acts as minus second derivative on each edge of the graph \(\Gamma_\varepsilon\) consisting of \(\Gamma\) with additional vertices of degree two at each half-line, all at the same distance \(\varepsilon > 0\) from \(v = 0\). The wave functions satisfy the \(\delta\) coupling conditions at \(v\) and at new vertices of degree two. The intensity of \(\delta\) couplings depends on \(\beta\) and \(\varepsilon\).

The operator \(-\Delta_{\beta}\) belongs to \(N^2\)-parametric family of self-adjoint extensions of the minimal symmetric operator

\[
(-\Delta_{\text{min}}v)(x) = (-v''(x))_{j=1}^{N}, \quad x > 0, \quad v = (v_j)_{j=1}^{N},
\]

\[
\text{dom}(\Delta_{\text{min}}) = \{ v \in H^2(\Gamma) : v_1(0) = \ldots = v_N(0) = 0, \quad v'_1(0) = \ldots = v'_N(0) = 0 \}.
\]

The whole family of self-adjoint conditions naturally arising at the vertex \(v = 0\) of \(\Gamma\) has the description (see [18]):

\[
(U - I)v(0) + i(U + I)v'(0) = 0,
\]

where \(v(0) = (v_j(0))_{j=1}^{N}, v'(0) = (v'_j(0))_{j=1}^{N}\), \(U\) is an arbitrary unitary \(N \times N\) matrix, and \(I\) is the \(N \times N\) identity matrix. In our case \(U = I - \frac{2}{N-\beta^2}I\), where \(I\) is \(N \times N\) matrix whose all entries equal one. It is a difficult problem to understand which of self-adjoint conditions are physically relevant (self-adjointness is just a necessary physical requirement to ensure conservation of the probability current at the vertex).

Among all the possible matching conditions, the most used are the Kirchhoff ones:

\[
v_1(0) = \ldots = v_N(0), \quad \sum_{j=1}^{N} v'_j(0) = 0.
\]

Justifications of the Kirchhoff conditions on different types of metric graphs have been obtained in many physical experiments involving wave propagation in thin waveguides and quantum nanowires. Namely, they appear when multi-dimensional models are approximated by differential operators on graphs. Moreover, when imposing the Kirchhoff conditions, transmission and reflection coefficients are independent of the momentum.
These arguments led to the fact that the Kirchhoff conditions have been assumed as the most natural, and hence they become the most widely studied. However, it is not clear whether these conditions suit different physical models (especially in the presence of non-trivial localized interactions near junctions). For instance, by choosing different values of the thickness parameters vanishing at the same rate, it was shown in [36] that generalized Kirchhoff boundary conditions (or “weighted” Kirchhoff conditions) can also arise in the asymptotic limit. Moreover, in [19, 24] the authors suggested that some other conditions could be more satisfactory from the point of view of invariance laws. However, it is worth mentioning that $\delta'_s$ coupling cannot be achieved in a purely geometrical way, by squeezing a system of branching tubes with the same topology as the graph (see [18] and references therein). Moreover, it still cannot be obtained using approximations involving potentials scaled in the usual way. Nevertheless, $\delta'_s$ coupling being modeled by complicated enough geometric scatterers, is likely to have something in common with the real world when one replaces simple junctions by regions of a nontrivial topology (see [21, Section 4]).

The systematic study of nonlinear evolution equations on metric graphs dates back to [38]. The nonlinear PDEs on graphs, mostly the nonlinear Schrödinger equation (NLS), have been studied in the past decade in the context of existence, stability, and propagation of solitary waves (see [40] for the references). Two fields where NLS equation appears as a preferred model are optics of nonlinear Kerr media and dynamics of Bose-Einstein condensates involving application to graph-like structures. For example, in nonlinear optics one studies arrays of planar self-focusing waveguides and propagation in variously shaped fiber optics devices, such as Y-junctions, H-junctions (see, for instance, [15, 20]).

The extensive study of existence of ground states (as minimizers of energy functional under fixed mass) for the NLS models on metric graphs was carried out in the presence of the Kirchhoff conditions at the vertices of the graph (see [4, 5] and references therein). The authors analyzed the problem for metric graphs with a finite number of vertices and at least one half-line. The existence and the stability of solitary waves for different types of graphs with the Kirchhoff conditions at the vertices were treated in numerous papers [31, 33, 39, 43]. It is worth mentioning recent paper [41], where the authors explored the variational methods and the analytical theory for differential equations in order to construct the standing waves for the quintic NLS with Kirchhoff conditions on the tadpole graph.

Rigorous study of the NLS models on graphs in presence of impurities is related to a so-called $\delta$ coupling. On $\Gamma$ it is defined by:

$$v_1(0) = \ldots = v_N(0), \quad \sum_{j=1}^N v_j'(0) = \alpha v_1(0), \quad \alpha \in \mathbb{R} \setminus \{0\}.$$  

The $\delta$ coupling is the most studied non-Kirchhoff condition [2, 3, 7, 13, 16, 25, 26, 34]. In [2, 3], for $\alpha < 0$, the ground state solution (as a minimizer of action functional and energy functional, respectively) has been identified with the $N$-tail state only in presence of sufficiently strong attractive interaction and sufficiently small mass, respectively. Besides the $N$-tail symmetric state, the NLS with $\delta$ coupling on $\Gamma$ admits the family of asymmetric states which are constituted by tail- and bump-components on the edges of $\Gamma$. Extensive study of their orbital stability was made in [7, 8, 34].
If we decide to drop the continuity condition, the next more general class of self-adjoint conditions which seems natural for applications consists of those which are locally permutation invariant at each vertex. The $\delta'_s$ coupling belongs to this class. Observe that it is a natural generalization of the symmetrized version of $\delta'$ coupling on the line (see [6] for the comprehensive treatment of this coupling). Systematic investigation of the NLS model with the $\delta'$ coupling on the line appears in [1]. The authors prove the existence of the minimizer of the action functional $S_\omega$ on the Nehari manifold for attractive coupling. It appears that the $\delta'$ coupling gives rise to a much richer structure of the family of ground states, including a pitchfork bifurcation with symmetry breaking. More precisely, there exists a critical value $\omega^*$ of frequency such that if $\omega < \omega^*$, then there is a single ground state and it is an odd function; if $\omega > \omega^*$, then there exist two non-symmetric ground states (contrarily to what happens in the case of the $\delta$ coupling on the line, where all ground states are even functions). This complex picture of the ground states originates from the fact that an associated energy space includes functions discontinuous at zero. The investigation of orbital stability of the odd ground state in the case of repulsive $\delta'$ coupling was carried out in [9].

In this paper we generalize and extend the results obtained in [1,9] for the NLS model on $\Gamma$. The study of NLS-$\delta'_s$ equation on $\Gamma$ was initiated in [7]. In the case of attractive coupling, the authors investigated orbital stability for standing wave $e^{\omega t} \phi(x)$ with the $N$-tail profile $\phi$ (see [7, Theorem 1.2]). Mathematically, the main advantage of studying $\delta'_s$ coupling is the existence of an explicit nontrivial family of soliton profiles to be described below.

The present manuscript has four principal parts. Firstly, we give detailed proof on well-posedness of (1.1) in $H^1(\Gamma)$ (which is the energy space). In particular, in Proposition 2.1 we show that for $1 < p < 5$ global well-posedness holds, and for $p > 2$ data-solution mapping associated with the equation is of class $C^2$ in $H^1(\Gamma)$ (which is crucial for our proof of orbital instability). Additionally, we have shown the regularity result for $p > 2$ and $u_0 \in \text{dom}(\Delta_\beta)$ (see Proposition 2.3). This regularity result is important for the proof of virial identity (2.23).

Secondly, we deal with the existence of the ground states as minimizers of the action functional $S_\omega$ restricted to the Nehari manifold (see (3.2)). In Theorem 3.1 we prove that for attractive and sufficiently weak $\delta'_s$ coupling the minimizer exists. The principal step in the proof is to compare our minimization problem with the one for $\beta = \infty$. This problem involves the technique of symmetric rearrangements on $\Gamma$ elaborated in [2]: that is, we are able to reduce the problem to the half-line. It is worth mentioning that in the space of symmetric functions $H^1_{\text{eq}}(\Gamma)$ the minimizer exists for any $\beta \in \mathbb{R} \setminus \{0\}$ (Lemma 3.3). Furthermore, for even $N$ and $\beta < 0$, the minimizer always exists in $H^1_{\frac{\beta}{2}}(\Gamma)$ as well (Lemma 3.4). This situation is analogous to the case of the real line considered in [1] (indeed, in this case $\Gamma$ can be identified with $\frac{N}{2}$ copies of $\mathbb{R}$). In particular, for $\omega \leq \frac{p+1}{p-1} \frac{N^2}{\beta}$ the minimizer is given by the symmetric profile $\phi_\beta$, and for $\omega > \frac{p+1}{p-1} \frac{N^2}{\beta}$ it coincides with the asymmetric profile $\phi_{\frac{N}{2}}$ (with equal first $\frac{N}{2}$ entries and last $\frac{N}{2}$ entries).

Thirdly, we are looking for the candidates to be the minimizers, i.e. critical points to $S_\omega$ of the form $e^{i\theta} \phi(x)$, where $\phi(x)$ is a real-valued profile. It appears that for $\omega > \frac{p+1}{p-1} \frac{N^2}{\beta}$ the whole family of such critical points consists of $N$ profiles modulo permutations of the
edges of $\Gamma$ (one is symmetric $\phi_\beta$ and $N - 1$ are asymmetric profiles $\phi_k$). We conjecture that for $\omega$ below $\frac{p+1}{q-1} \frac{N^2}{\beta^q}$ the symmetric profile $\phi_\beta$ is the minimizer, while in Theorem 4.4 we managed to prove that for $\omega > \frac{p+1}{q-1} \frac{N^2}{\beta^q}$ the minimizer is given by the asymmetric tail-profile $\phi_1$. Notice that in [1] the authors proved minimizing property of the asymmetric profile by direct comparison of the values of the action functional (at symmetric and asymmetric profile). Our proof is completely different. We use the Implicit Function Theorem and the fact that the Morse index of $S_\omega''(\phi_1)$ equals 1.

Lastly, we study instability properties of the family of the critical points mentioned above. Namely, using the Grillakis/Jones Instability Theorem (see [28, 30]), we have proved spectral instability (see Theorem 5.23) of the asymmetric critical points $\phi_k$ for $\beta < 0, k \geq 2$ and $\beta > 0, N - k \geq 4$. For $p > 2$, using $C^2$ regularity of the mapping data-solution and applying the abstract result from [29], we have shown orbital instability of $\phi_k$. This abstract result states the nonlinear instability of a fixed point of a nonlinear mapping having the linearization $L$ of spectral radius $r(L) > 1$. To apply the Grillakis/Jones approach we need to estimate the Morse index of two self-adjoint in $L^2(\Gamma)$ operators associated with $S_\omega''(\phi_k)$. These estimates were obtained in Proposition 5.25 by using the generalization of the Sturm theory elaborated for $\Gamma$ in [32, 34].

Applying the Grillakis/Jones Instability Theorem to the symmetric profile $\phi_\beta$, in Theorem 5.2 we obtain partial generalization of instability results [1, Theorem 6.13] and [9, Theorem 1.1]. Finally, we concertize the instability results by claiming strong instability (by blow up) of the symmetric profile $\phi_\beta$ in the supercritical case $p > 5$ (see Theorem 5.19 and 5.20). The proof essentially uses variational characterization of $\phi_\beta$ in $H^1_{eq}(\Gamma)$ and virial identity (2.23).

The paper has the following structure. In Section 2 we prove well-posedness of problem (1.1) in $H^1(\Gamma)$ and in $\text{dom}(\Delta_\beta)$. Section 3 is devoted to the proof of the existence of a ground state solution. In Section 4 we find the family of critical points that contains ground states. In Section 5 we provide an extensive study of the instability of standing wave solutions associated with the mentioned critical points. Namely, Subsection 5.1 is devoted to the case of symmetric profile $\phi_\beta$, while in Subsection 5.4 we study instability of the asymmetric profiles $\phi_k$.

**Notation.** The natural Hilbert space associated to the Laplace operator $-\Delta_\beta$ is $L^2(\Gamma)$, which is defined as $L^2(\Gamma) = \bigoplus_{j=1}^N L^2(\mathbb{R}^+)$. The inner product in $L^2(\Gamma)$ is given by

$$(u,v)_2 = \text{Re} \sum_{j=1}^N (u_j,v_j)_{L^2(\mathbb{R}^+)}, \quad u = (u_j)_{j=1}^N, \quad v = (v_j)_{j=1}^N.$$ 

The space $L^q(\Gamma)$, for $1 \leq q \leq \infty$, is defined as prime sum as well, and $\| \cdot \|_q$ stands for its norm. By $H^1(\Gamma) = \bigoplus_{j=1}^N H^1(\mathbb{R}^+)$ and $H^2(\Gamma) = \bigoplus_{j=1}^N H^2(\mathbb{R}^+)$ we define the Sobolev spaces. In what follows we will use the notation $H^1_l$ for $H^1(\Gamma)$. The duality pairing between $(H^1(\Gamma))'$ and $H^1(\Gamma)$ is denoted by $\langle \cdot, \cdot \rangle_{(H^1)^* \times H^1}$. 
Below we will run into the following subspaces of $H^1(\Gamma)$:

\[
H_{\text{eq}}^1(\Gamma) = \{ v \in H^1(\Gamma) : v_1(x) = \ldots = v_N(x), x > 0 \}, \\
H_{\frac{1}{2}}^1(\Gamma) = \{ v \in H^1(\Gamma) : v_1(x) = \ldots = v_N(x), v_{N+1}(x) = \ldots = v_N(x), x > 0 \}.
\]

By $C_j, C_j(\cdot), K_j(\cdot), j \in \mathbb{N}$, and $C(\cdot)$ we will denote some positive constants. Let $L$ be a self-adjoint operator in $L^2(\Gamma)$, then $n(L)$ will stand for the number of negative eigenvalues of $L$ counting multiplicity (Morse index).

2. Well-posedness

In this section we study well-posedness of problem (1.1). The proposition below states local well-posedness of (1.1) in $H^1(\Gamma)$.

**Proposition 2.1.** Let $u_0 \in H^1(\Gamma)$ and $p > 1$. Then the following assertions hold.

(i) There exist $T = T(u_0) > 0$ and a unique weak solution $u(t) \in C([0, T], H^1(\Gamma)) \cap C^1([0, T], (H^1(\Gamma))')$ of problem (1.1).

(ii) Problem (1.1) has a maximal solution defined on an interval $[0, T_{H^1})$, and the following “blow-up alternative” holds: either $T_{H^1} = \infty$ or $T_{H^1} < \infty$ and

\[
\lim_{t \to T_{H^1}} \|u(t)\|_{H^1} = \infty.
\]

(iii) For each $T_0 \in (0, T)$ the mapping $u_0 \in H^1(\Gamma) \mapsto u(t) \in C([0, T_0], H^1(\Gamma))$ is continuous. In particular, for $p > 2$ this mapping is at least of class $C^2$.

(iv) The conservation of energy and charge holds: for $t \in [0, T_{H^1})$

\[
E(u(t)) = \frac{1}{2} F_\beta(u(t)) - \frac{1}{p+1} \|u(t)\|_{p+1}^{p+1} = E(u_0), \quad \|u(t)\|_2^2 = \|u_0\|_2^2.
\]

**Proof.** The proof is based on the ideas of the one of [17, Theorem 4.10.1].

**Step 1.** To start, we mention several useful technical facts. Firstly, observe that $g(u) = |u|^{p-1}u \in C^1(\mathbb{C}, \mathbb{C})$ (i.e. $\operatorname{Im}(g)$ and $\operatorname{Re}(g)$ are $C^1$-functions of $\operatorname{Re} u, \operatorname{Im} u$) for $p > 1$. This implies inequalities (see [17, Lemma 4.10.2])

\[
\|g(u)\|_{H^1} \leq C_1(M) \|u\|_{H^1}, \\
\|g(u) - g(v)\|_2 \leq C_2(M) \|u - v\|_2,
\]

for all $u, v \in H^1(\Gamma)$ such that $\|u\|, \|v\| \leq M$, and

\[
\|g(u) - g(v)\|_{H^1} \leq C_3(M) \left[ \|u - v\|_{H^1} + \varepsilon_M \left( \|u - v\|_2 \right) \right]
\]

for all $u, v \in H^1(\Gamma)$ such that $\|u\|_{H^1}, \|v\|_{H^1} \leq M$, where $\varepsilon_M(s) \to 0$ as $s \downarrow 0$.

Secondly, we show the inequality

\[
\|e^{i\Delta\beta t}v\|_{H^1} \leq C\|v\|_{H^1}.
\]

Let $m > \frac{N^2}{\beta}$. It is known that $\inf \sigma(-\Delta_\beta) = \begin{cases} 0, & \beta \geq 0, \\ -\frac{N^2}{\beta^2}, & \beta < 0. \end{cases}$ Then the operator $-\Delta_\beta + m$ is positive. Remark that $H^1(\Gamma) = \text{dom}(F_\beta) = \text{dom}((-\Delta_\beta + m)^{1/2})$. Using
$L^2$-unitarity of $e^{i\Delta_\beta t}$, we obtain for $v \in H^1(\Gamma)$
\[
F_\beta(v) + m\|v\|_2^2 = \left((-\Delta_\beta + m)^{1/2}v, (-\Delta_\beta + m)^{1/2}v\right)_2 \\
= \left(e^{i\Delta_\beta t}(-\Delta_\beta + m)^{1/2}v, e^{i\Delta_\beta t}(-\Delta_\beta + m)^{1/2}v\right)_2 \\
= \left((-\Delta_\beta + m)^{1/2}e^{i\Delta_\beta t}v, (-\Delta_\beta + m)^{1/2}e^{i\Delta_\beta t}v\right)_2 = F_\beta(e^{i\Delta_\beta t}v) + m\|e^{i\Delta_\beta t}v\|_2^2.
\]
Using equivalence of $H^1$-norm and $F_\beta(v) + m\|v\|_2^2$ (see [11, Lemma 4.13]), we get
\[
C_2\|e^{i\Delta_\beta t}v\|_{H^1}^2 \leq F_\beta(e^{i\Delta_\beta t}v) + m\|e^{i\Delta_\beta t}v\|_2^2 = F_\beta(v) + m\|v\|_2^2 \leq C_1\|v\|_{H^1}^2,
\]
and (2.4) follows easily. Analogously one proves that $e^{i\Delta_\beta t}$ is continuous in $H^1(\Gamma)$. Indeed, let $t_n \to t$, then
\[
C_2\| (e^{i\Delta_\beta t} - e^{i\Delta_\beta t_n})v\|_{H^1}^2 \leq F_\beta ((e^{i\Delta_\beta t} - e^{i\Delta_\beta t_n})v) + m\| (e^{i\Delta_\beta t} - e^{i\Delta_\beta t_n})v\|_2^2 \\
= \left(e^{i\Delta_\beta t} - e^{i\Delta_\beta t_n})(-\Delta_\beta + m)^{1/2}v, (e^{i\Delta_\beta t} - e^{i\Delta_\beta t_n})(-\Delta_\beta + m)^{1/2}v\right)_2 \\
= 2\|(-\Delta_\beta + m)^{1/2}v\|_2^2 - \left((e^{-i\Delta_\beta(t-t_n)} + e^{i\Delta_\beta(t_n-t)}(-\Delta_\beta + m)^{1/2}v, (-\Delta_\beta + m)^{1/2}v\right)_2 \\
\to 0 \quad \text{as} \quad n \to \infty
\]

*Step 2.* Existence of solution follows by a fixed point argument. Given $M, T > 0$ to be chosen later, we set
\[
E_1 = \{ u \in L^\infty((0,T),H^1(\Gamma)) : \|u\|_{L^\infty((0,T),H^1(\Gamma))} \leq M \}, \quad d_1(u,v) = \| u - v \|_{L^\infty((0,T),L^2(\Gamma))}.
\]
By [17, Theorem 1.2.5], $(E_1,d_1)$ is a complete metric space. We now consider $\mathcal{H}$ defined by
\[
\mathcal{H}(u)(t) = e^{i\Delta_\beta t}u_0 + G(u)(t), \quad G(u)(t) = i \int_0^t e^{i\Delta_\beta(t-s)}g(u(s))ds
\]
for all $u \in E_1$ and $t \in (0,T)$. We note that if $u \in L^\infty((0,T),H^1(\Gamma))$, then $g(u) \in L^\infty((0,T),H^1(\Gamma))$ by (2.2). Using (2.4) and (2.2), for every $u \in E_1$ we obtain
\[
\|\mathcal{H}(u)(t)\|_{L^\infty((0,T),H^1(\Gamma))} \leq C\|u_0\|_{H^1} + TC\|g(u)\|_{L^\infty((0,T),H^1(\Gamma))} \leq C\|u_0\|_{H^1} + CTC_1(M)M.
\]
Furthermore, it follows from (2.2) that for $u,v \in E_1$
\[
\|\mathcal{H}(u)(t) - \mathcal{H}(v)(t)\|_{L^\infty((0,T),L^2(\Gamma))} \leq TC_2(M)\|u - v\|_{L^\infty((0,T),L^2(\Gamma))}.
\]
Therefore, we see that if
\[
M = 2C\|u_0\|_{H^1}, \quad CTC_1(M) \leq \frac{1}{2}, \quad TC_2(M) < 1,
\]
then $\mathcal{H}$ is a strict contraction of $(E_1,d_1)$. Thus, it has a fixed point $u(t)$ which is a solution to (1.1). Finally, let $t, t_n \in [0,T]$ and $t_n \to t$, then
\[
\|G(u)(t) - G(u)(t_n)\|_{H^1} \leq \int_0^t \left\| (e^{i\Delta_\beta(t-s)} - e^{i\Delta_\beta(t_n-s)})g(u(s)) \right\|_{H^1} ds + \int_t^{t_n} \left\| e^{i\Delta_\beta(t_n-s)}g(u(s)) \right\|_{H^1} ds.
\]
By continuity of $e^{i\Delta_\beta t}$ in $H^1(\Gamma)$, we conclude $G(u) \in C([0,T],H^1(\Gamma))$, therefore $u(t) \in C([0,T],H^1(\Gamma))$. 

Uniqueness follows by the Gronwall lemma. Indeed, let \( u_1, u_2 \) be two solutions of (1.1) and
\[
\bar{M} = \sup_{t \in [0, T]} \max \{ \| u_1(t) \|_{H^1}, \| u_2(t) \|_{H^1} \},
\]
then, by (2.2), one gets
\[
\| u_1(t) - u_2(t) \|_2 \leq C_2(\bar{M}) \int_0^t \| u_1(s) - u_2(s) \|_2 ds.
\]
The proof of the blow-up alternative follows standardly by a bootstrap argument.

Step 3. We show continuous dependence. Let \( u_0 \in H^1(\Gamma) \) and consider \( \{ u_0^n \}_{n \in \mathbb{N}} \subset H^1(\Gamma) \) such that \( u_0^n \xrightarrow{n \to \infty} u_0 \) in \( H^1(\Gamma) \). Assume that \( u^n \) is the maximal solution corresponding to the initial value \( u_0^n \).

Since \( \| u_0^n \|_{H^1} \leq 2 \| u_0 \|_{H^1} \) for \( n \) sufficiently large, we deduce from Step 2 that there exists \( T = T (\| u_0 \|_{H^1}) \) such that \( u \) and \( u^n \) are defined on \( [0, T] \) for \( n \geq n_0 \), and
\[
\| u \|_{L^\infty((0, T), H^1(\Gamma))} + \sup_{n \geq n_0} \| u^n \|_{L^\infty((0, T), H^1(\Gamma))} \leq 6C \| u_0 \|_{H^1}.
\]
The second estimate in (2.2) yields that there exists \( C_3 = C_3 (\| u_0 \|_{H^1}) \) such that
\[
\| u(t) - u^n(t) \|_2 \leq \| u_0 - u_0^n \|_2 + C_3 \int_0^t \| u(s) - u^n(s) \|_2 ds.
\]
We then conclude, by the Gronwall lemma,
\[
\| u(t) - u^n(t) \|_2 \leq \| u_0 - u_0^n \|_2 e^{TC_3} \xrightarrow{n \to \infty} 0.
\]
Therefore, from (2.3) and (2.5) it follows that there exist \( C_4 = C_4 (\| u_0 \|_{H^1}) \) and \( \varepsilon_n \downarrow 0 \) such that
\[
\| u(t) - u^n(t) \|_{H^1} \leq \varepsilon_n + C \| u_0 - u_0^n \|_{H^1} + C_4 \int_0^t \| u(s) - u^n(s) \|_{H^1} ds
\]
for all \( t \in [0, T] \). Applying again Gronwall’s lemma, we get the result.

Step 4. We show that for \( p > 2 \) the mapping data-solution is of class \( C^2 \).

We interpret the solution \( u(t) \) to (1.1) as the fixed point of the mapping \( \mathcal{H}(u)(t) \) defined above in the metric space \((E_2, d_2)\):
\[
E_2 = \{ u \in C \left( [0, T], H^1(\Gamma) \right) : \| u \|_{C([0, T], H^1(\Gamma))} \leq M \}, \quad d_2(u, v) = \| u - v \|_{C([0, T], H^1(\Gamma))}.
\]
Observe that \( g(u) \in C^2 (\mathbb{C}, \mathbb{C}) \), which implies
\[
\| g(u) - g(v) \|_{H^1} \leq C_3(M) \| u - v \|_{H^1} \quad \text{for} \quad \| u \|_{H^1} \leq M, \quad \| v \|_{H^1} \leq M.
\]
By (2.2), for every \( u \in E_2 \)
\[
\| \mathcal{H}(u)(t) \|_{C([0, T], H^1(\Gamma))} \leq C \| u_0 \|_{H^1} + TC \| g(u) \|_{C([0, T], H^1(\Gamma))} \leq C \| u_0 \|_{H^1} + CTC_1(M)M.
\]
Furthermore, it follows from (2.6) that, if \( u, v \in E_2 \), then
\[
\| \mathcal{H}(u)(t) - \mathcal{H}(v)(t) \|_{C([0, T], H^1(\Gamma))} \leq CTC_3(M) \| u - v \|_{C([0, T], H^1(\Gamma))}.
\]
Let
\[
M = 2C \| u_0 \|_{H^1}, \quad CTC_1(M) \leq \frac{1}{2}, \quad CTC_3(M) < 1,
\]
then \( \mathcal{H} \) is a contraction of \((E_2, d_2)\).
Consider the mapping
\[
\mathcal{J} : B(u_0, u(t)) \subset H^1(\Gamma) \times C([0, T], H^1(\Gamma)) \longrightarrow C([0, T], H^1(\Gamma)),
\]
\[
\mathcal{J}(v_0, v(t)) = v(t) - e^{i\Delta_{\beta}t}v_0 - i \int_0^t e^{i\Delta_{\beta}(t-s)}g(v(s))ds.
\]
Here \(B(u_0, u(t))\) is an open neighborhood of \((u_0, u(t))\). It is obvious that \(\mathcal{J}(u_0, u(t)) = 0\). One has
\[
D_{v(t)}\mathcal{J}(u_0, u(t))h(t) = h(t) - i \int_0^t e^{i\Delta_{\beta}(t-s)}[\partial_v g(u)h + \partial_{\overline{u}}g(u)\overline{h}]\,(s)ds = (I - A)h(t),
\]
with \(Ah(t) = i \int_0^t e^{i\Delta_{\beta}(t-s)}[\partial_v g(u)h + \partial_{\overline{u}}g(u)\overline{h}]\,(s)ds\). From \(CTC_1(M) \leq \frac{1}{2}\) we conclude
\[
\|Ah(t)\|_{C([0, T], H^1(\Gamma))} \leq \frac{1}{2}\|h(t)\|_{C([0, T], H^1(\Gamma))},
\]
and hence \(D_{v(t)}\mathcal{J}(u_0, u(t))\) is invertible on \(C([0, T], H^1(\Gamma))\) (as a linear operator on a real Banach space), and consequently it is a bijection. Therefore, by the Implicit Function Theorem, we conclude the existence of an open neighborhood \(B(u_0)\) of \(u_0\) and a unique function \(f : B(u_0) \longrightarrow C([0, T], H^1(\Gamma))\) such that \(\mathcal{J}(v_0, f(v_0)) = 0\) for all \(v_0 \in B(u_0)\), i.e.
\[
f(v_0) = e^{i\Delta_{\beta}t}v_0 + i \int_0^t e^{i\Delta_{\beta}(t-s)}g(f(v_0))(s)ds.
\]
Hence \(f(v_0)\) is the solution to (1.1) corresponding to the initial value \(v_0\). Finally, since \(g(u)\) is of class \(C^2\) for \(p > 2\), we get that \(f\) is \(C^2\) mapping.

The proof of conservation laws (2.1) might be obtained involving the regularization procedure analogous to the one introduced in the proof of [17, Theorem 3.3.5].

**Remark 2.2.** For \(1 < p < 5\) problem (1.1) is globally well-posed in \(H^1(\Gamma)\). This follows from [17, Theorem 3.4.1] (in particular, see formula (3.4.1)). Namely, it is sufficient to observe
\[
\begin{align*}
\|u\|_{p+1}^2 - \frac{1}{2} \sum_{j=1}^N u_j(0)^2 & \leq C_1\|u'\|_{2^p/2^1}^2 \|u\|_2^2 + \varepsilon \|u'\|_2^2 + C_2\|u\|_2^2 \\
& \leq 2\varepsilon \|u'\|_2^2 + C_3\|u\|_2^{2^p/2^{p+1}} + C_2\|u\|_2^2 \leq 2\varepsilon \|u\|_{H^1}^2 + C(\|u_0\|_2).
\end{align*}
\]
The above estimate follows from the conservation of charge, estimate
\[
\begin{align*}
\frac{1}{2} \sum_{j=1}^N v_j(0) & \leq \frac{N^2}{\beta} \|v\|_{\beta}^2 \leq C\|v'\|_{2}^2 \|v\|_2 \leq \varepsilon \|v'\|_2^2 + C_\varepsilon \|v\|_2^2,
\end{align*}
\]
the Gagliardo-Nirenberg inequality for \(v \in H^1(\Gamma)\)
\[
\|v\|_{r} \leq C\|v'\|_2^2 \|v\|_{q}^{1-\alpha}, \quad r, q \in [2, +\infty], \quad r \geq q, \quad \alpha = \frac{2}{r} + q\,(1 - q/r),
\]
and the Young inequality \(ab \leq \delta a^\gamma + C_b b'^\gamma, \quad \frac{1}{\gamma} + \frac{1}{\gamma'} = 1, q, q' > 1, a, b \geq 0\). Observe that the key point is that \(q = \frac{4}{p-1} > 1\) for \(1 < p < 5\).
Below we prove the regularity of solution to (1.1) when \( u_0 \in \text{dom}(\Delta_\beta) \). Given the quantity

\[ 0 < m := 1 - 2 \inf \sigma(-\Delta_\beta) < \infty, \]

we introduce the norm \( \|v\|_\beta := \|(-\Delta_\beta + m)v\|_2 \) that endows \( \text{dom}(\Delta_\beta) \) with the structure of a Hilbert space. Observe that this norm for any real \( \beta \) is equivalent to \( H^2 \)-norm on the graph. Indeed,

\[
\|v\|_\beta^2 = \|v''\|_2^2 + m^2\|v\|_2^2 + 2m\|v'\|_2^2 + \frac{2m}{\beta} \left| \sum_{j=1}^{N} v_j(0) \right|^2.
\]

Due to the choice of \( m \) and inequality (2.9), we get

\[ C_1\|v\|_{H^2(\Gamma)}^2 \leq \|v''\|_2^2 + m\|v\|_2^2 \leq \|v\|_\beta^2 \leq C_2\|v\|_{H^2(\Gamma)}^2. \]

In what follows we will use the notation \( D_\beta = (\text{dom}(\Delta_\beta), \| \cdot \|_\beta) \).

**Proposition 2.3.** Let \( p > 2 \) and \( u_0 \in \text{dom}(\Delta_\beta) \). Then there exists \( T > 0 \) such that problem (1.1) has a unique strong solution \( u(t) \in C\left([0,T],D_\beta \right) \cap C^1\left([0,T],L^2(\Gamma)\right) \). Moreover, problem (1.1) has a maximal solution defined on an interval of the form \([0,T_\beta)\), and the following “blow-up alternative” holds: either \( T_\beta = \infty \) or \( T_\beta < \infty \) and

\[ \lim_{t \to T_\beta} \|u(t)\|_\beta = \infty. \]

**Proof.** The proof is analogous to the one of [25, Theorem 2.3] (for the NLS-\( \delta \) equation with \( p > 4 \)) with a few modifications concerning the restriction \( p > 2 \).

Let \( T > 0 \) to be chosen later. We will use the notation

\[ X_\beta = C\left([0,T],D_\beta \right) \cap C^1\left([0,T],L^2(\Gamma)\right), \]

and equip the space \( X_\beta \) with the norm

\[ \|u(t)\|_{X_\beta} = \sup_{t \in [0,T]} \|u(t)\|_\beta + \sup_{t \in [0,T]} \|\partial_t u(t)\|_2. \]

Consider

\[ E = \left\{ u(t) \in X_\beta : u(0) = u_0, \|u(t)\|_{X_\beta} \leq M \right\}, \]

where \( M \) is a positive constant to be chosen later as well. It is obvious that \((E,d)\) is a complete metric space with the metric \( d(u,v) = \|u - v\|_\beta \).

Recall

\[ H(u)(t) = e^{i\Delta_\beta t}u_0 + \mathcal{G}(u)(t) = e^{i\Delta_\beta t}u_0 + i \int_0^t e^{i\Delta_\beta (t-s)} g(u(s))ds, \quad g(u) = |u|^{p-1}u. \]

**Step 1.** We will show that \( H : E \to X_\beta \).

1. It is known that \( \text{dom}(\Delta_\beta) = \{ v \in L^2(\Gamma) : \lim_{h \to 0} h^{-1}(e^{i\Delta_\beta h} - I)v \text{ exists} \} \). Obviously \( w(t) := e^{i\Delta_\beta t}u_0 \in \text{dom}(\Delta_\beta) \) and \( w(t) \in C\left([0,T],D_\beta \right) \). Moreover, \( \partial_t w(t) = i\Delta_\beta e^{i\Delta_\beta t}u_0 = i\Delta_\beta w(t) \), then \( \partial_t w(t) \in C\left([0,T],L^2(\Gamma)\right) \).

2. The inclusion \( \mathcal{G}(u)(t) \in C^1\left([0,T],L^2(\Gamma)\right) \) follows rapidly. Indeed, \([17, \text{Lemma 4.8.4}]\) implies that \( \partial_t g(u(t)) \in L^1\left((0,T),L^2(\Gamma)\right) \), and the formula

\[ \partial_t \mathcal{G}(u)(t) = i e^{i\Delta_\beta t}g(u(0)) + i \int_0^t e^{i\Delta_\beta (t-s)} \partial_s g(u(s))ds \]

holds.
from the proof of [17, Lemma 4.8.5] induces \( G(u)(t) \in C^1 ([0, T], L^2(\Gamma)) \).

3. Below we will show that \( G(u)(t) \in C ([0, T], D_\beta) \). First, we need to prove that \( G(u)(t) \in \text{dom}(\Delta_\beta) \). Second inequality in (2.2) implies \( g(u(t)) \in C ([0, T], L^2(\Gamma)) \), then for \( t \in [0, T] \) and \( h \in (0, T - t] \) we get

\[
\frac{e^{i\Delta_\beta h} - I}{h} G(u)(t) = \frac{1}{h} \int_0^t e^{i\Delta_\beta (t+h-s)} g(u(s)) ds - \frac{1}{h} \int_0^t e^{i\Delta_\beta (t-s)} g(u(s)) ds.
\]

(2.11)

Letting \( h \to 0 \), by the Mean Value Theorem, we arrive at \( i\Delta_\beta G(u)(t) = \dot{G}(u)(t) - g(t) \), i.e. we obtain the existence of the limit in (2.11), therefore, \( G(u)(t) \in \text{dom}(\Delta_\beta) \). This is still true for \( t = T \) since operator \( -\Delta_\beta \) is closed. Note that we have used differentiability of \( G(u)(t) \) proved above.

It remains to prove the continuity of \( G(u)(t) \) in \( \beta \)-norm. We will use the integration by parts formula (see [25, Proposition A.1] with \( H = -\Delta_\beta \))

\[
G(u)(t) = \int_0^t e^{i\Delta_\beta (t-s)} g(u(s)) ds = -i(-\Delta_\beta + m)^{-1} g(u(t)) + i e^{i\Delta_\beta t} (-\Delta_\beta + m)^{-1} g(u(0))
\]

\[+ m(-\Delta_\beta + m)^{-1} \int_0^t e^{i\Delta_\beta (t-s)} g(u(s)) ds + i(-\Delta_\beta + m)^{-1} \int_0^t e^{i\Delta_\beta (t-s)} \partial_s g(u(s)) ds.\]

It is easily seen that

\[
\partial_s g(u(s)) = \partial_u g(u) \partial_s u + \partial_{\pi g(u)} \partial_s \pi.
\]

(2.13)

Let \( t_n, t \in [0, T] \) and \( t_n \to t \). By (2.12) and (2.13), we deduce

\[
\|G(u)(t) - G(u)(t_n)\|_\beta \leq \|g(u(t)) - g(u(t_n))\|_2 + m \int_0^t \|\left(e^{i\Delta_\beta (t-s)} - e^{i\Delta_\beta (t_n-s)}\right) g(u(s))\|_2 ds
\]

\[+ \int_0^t \|\left(e^{i\Delta_\beta (t-s)} - e^{i\Delta_\beta (t_n-s)}\right) (\partial_s g(u) \partial_s u) (s)\|_2 ds + \int_t^{t_n} \|\left(e^{i\Delta_\beta (t_n-s)} \partial_s g(u) \partial_s u \right) (s)\|_2 ds
\]

\[+ \int_0^t \| \left(e^{i\Delta_\beta (t-s)} - e^{i\Delta_\beta (t_n-s)}\right) (\partial_{\pi g(u)} \partial_s \pi) (s)\|_2 ds + \int_t^{t_n} \|\left(e^{i\Delta_\beta (t_n-s)} \partial_{\pi g(u)} \partial_s \pi \right) (s)\|_2 ds.\]

(2.14)

Therefore, using (2.2),(2.14), unitarity and continuity properties of \( e^{i\Delta_\beta t} \), we obtain continuity of \( G(u)(t) \) in \( D_\beta \).

Step 2. Now our aim is to choose \( T \) in order to guarantee invariance of \( E \) for the mapping \( \mathcal{H} \), i.e. \( \mathcal{H} : E \to E \).

1. By (2.13), one has

\[
g(u(t)) = \int_0^t (\partial_u g(u) \partial_s u + \partial_{\pi g(u)} \partial_s \pi) (s) ds + g(u(0)).
\]

(2.15)
Let \( u(t) \in E \) and \( t \in [0, T] \). Using the Sobolev embedding, (2.12), (2.15), and equivalence of \( \beta \)- and \( H^2 \)-norm, we obtain

\[
\| \mathcal{H}(u)(t) \|_\beta \leq \| e^{i\Delta_\beta t} u_0 \|_\beta + \int_0^t \| e^{i\Delta_\beta (t-s)} g(u(s)) ds \|_\beta \leq \| u_0 \|_\beta 
\]

\[
+ \| \int_0^t (\partial_u g(u) \partial_s u + \partial_{\pi g}(u) \partial_s \tilde{u})(s) ds + g(u(0)) \|_2 
\]

\[
+ \| g(u(0)) \|_2 + m \int_0^t \| g(u(s)) \|_2 ds + \int_0^t \| (\partial_u g(u) \partial_s u)(s) \|_2 ds + \int_0^t \| (\partial_{\pi g}(u) \partial_s \tilde{u})(s) \|_2 ds 
\]

\[
\leq \| u_0 \|_\beta + C_1 \| u_0 \|_\beta^p + C_2 \int_0^t \| u \|_p^{-1} \| \partial_s u(s) \|_2 ds + C_3 \int_0^t \| u \|_p^{-1} \| u(s) \|_2 ds 
\]

\[
\leq \| u_0 \|_\beta + C_1 \| u_0 \|_\beta^p + K_1(M)TM^p. 
\]

2. Below we will estimate \( \| \partial_t \mathcal{H}(u)(t) \|_2 \). Observe that

\[
\| \partial_t e^{i\Delta_\beta t} u_0 \|_2 = \| i\Delta_\beta u_0 \| \leq \| u_0 \|_\beta. 
\]

Using (2.10), (2.17), we obtain the estimate

\[
\| \partial_t \mathcal{H}(u)(t) \|_2 \leq \| u_0 \|_\beta + \| g(u(0)) \|_2 + \int_0^t \| (\partial_u g(u) \partial_s u)(s) \|_2 ds + \int_0^t \| (\partial_{\pi g}(u) \partial_s \tilde{u})(s) \|_2 ds 
\]

\[
\leq \| u_0 \|_\beta + C_1 \| u_0 \|_\beta^p + C_2 \int_0^t \| u \|_p^{-1} \| \partial_s u(s) \|_2 ds \leq \| u_0 \|_\beta + C_1 \| u_0 \|_\beta^p + K_2(M)TM^p. 
\]

Finally, combining (2.16) and (2.18), we arrive at

\[
\| \mathcal{H}(u)(t) \|_X \leq 2 \| u_0 \|_\beta + 2C_1 \| u_0 \|_\beta^p + (K_1(M) + K_2(M))TM^p. 
\]

We now let

\[
\frac{M}{2} = \left( 2 \| u_0 \|_\beta + 2C_1 \| u_0 \|_\beta^p \right). 
\]

By choosing \( T \leq \frac{1}{2(K_1(M) + K_2(M))M^{p-1}} \), we get

\[
\| \mathcal{H}(u)(t) \|_X \leq M, 
\]

therefore, \( \mathcal{H} : E \to E \).

Step 3. Now we will choose \( T \) to guarantee that \( \mathcal{H} \) is a strict contraction on \( (E, d) \). First, since \( g(u) \) is of class \( C^2 \), we get

\[
| \partial_u g(v) - \partial_u g(w) | \leq K_3(M) | v - w |, \quad | \partial_{\pi g}(v) - \partial_{\pi g}(w) | \leq K_3(M) | v - w | 
\]

for \( | v |, | w | \leq M \).
Let $v, w \in E$. From (2.2), (2.12), (2.15), (2.19) it follows that
\[
\|\mathcal{H}(v)(t) - \mathcal{H}(w)(t)\|_\beta = \int_0^t e^{i\Delta(t-s)} \left[ g(v(s)) - g(w(s)) \right] ds \|_\beta
\leq m \int_0^t \|g(v(s)) - g(w(s))\|_2 ds + 2 \int_0^t \|\partial_\beta g(v)\partial_\beta v - \partial_\beta g(w)\partial_\beta w\|_2 ds
+ 2 \int_0^t \|\partial_\beta g(v)\partial_\beta \overline{v} - \partial_\beta g(w)\partial_\beta \overline{w}\|_2 ds \leq mC_2(M)T\|v - w\|_2
\tag{2.20}
\]
\[
+ 2 \int_0^t \|\partial_\beta g(v)\partial_\beta v\|_2 ds + 2 \int_0^t \|\partial_\beta g(w)\partial_\beta v\|_2 ds
+ 2 \int_0^t \|\partial_\beta g(v)\partial_\beta \overline{v}\|_2 ds + 2 \int_0^t \|\partial_\beta g(w)\partial_\beta \overline{w}\|_2 ds
\leq mC_2(M)T\|v - w\|_2 + 4K_3(M)T\|\partial_\beta v\|_\infty\|v - w\|_2 + 4TP\|w\|^{p-1}_\infty\|\partial_\beta v - \partial_\beta w\|_2
\leq TK_4(M)\|v - w\|_{X_\beta}.
\]

2. To get the contraction property of $\mathcal{H}$, we need to estimate $L^2$-part of $X_\beta$-norm of $\mathcal{H}(v)(t) - \mathcal{H}(w)(t)$. From (2.10) we deduce
\[
\|\partial_\beta \mathcal{H}(v)(t) - \partial_\beta \mathcal{H}(w)(t)\|_2 \leq \int_0^t \| \partial_\beta g(v(s)) - \partial_\beta g(w(s))\|_2 ds.
\tag{2.21}
\]
Using (2.13), (2.19), from (2.21) we get
\[
\|\partial_\beta \mathcal{H}(v)(t) - \partial_\beta \mathcal{H}(w)(t)\|_2 \leq K_5(M)T\|v - w\|_{X_\beta},
\tag{2.22}
\]
and finally from (2.20), (2.22) we obtain
\[
\|\mathcal{H}(v)(t) - \mathcal{H}(w)(t)\|_{X_\beta} \leq (K_4(M) + K_5(M))T\|v - w\|_{X_\beta}.
\]
Thus, for
\[
T < \min \left\{ \frac{1}{2(K_1(M) + K_2(M))M^{p-1}}, \frac{1}{K_4(M) + K_5(M)} \right\}
\]
the mapping $\mathcal{H}$ is the strict contraction of $(E, d)$. Therefore, by the Banach fixed point theorem, $\mathcal{H}$ has a unique fixed point $u \in E$ which is a solution of (1.1).

Uniqueness of the solution follows in a standard way by Gronwall’s lemma. The blow-up alternative can be shown by a bootstrap argument. \hfill \Box

**Remark 2.4.** For $1 < p < 5$, problem (1.1) is globally well-posed in $D_\beta$. By formula (3.4.2) in the proof of [17, Theorem 3.4.1] (see also formula (2.8) in Remark 2.2), we obtain that $\|u(t)\|_{H^1}$ is uniformly bounded. In particular, $\|u(t)\|_\infty \leq C(\|u_0\|_2, E(u_0)), t \in \mathbb{R}$. Then from (2.16), (2.18) we get
\[
\|u(t)\|_{X_\beta} \leq 2\|u_0\|_\beta + 2C_1\|u_0\|_\beta^p + 2C_2 \int_0^t \|u\|^{p-1}_\infty \|\partial_\beta u(s)\|_2 ds + C_3 \int_0^t \|u\|^{p-1}_\infty \|u(s)\|_2 ds
\leq C_1(\|u_0\|_\beta) + C_2(\|u_0\|_2, E(u_0)) \int_0^t \|u(s)\|_{X_\beta} ds, \quad t \in [0, T].
\]
By Gronwall’s lemma, we get
\[ \|u(t)\|_{X_\beta} \leq C_1(\|u_0\|_\beta)e^{TC_2(\|u_0\|_2,E(u_0))}. \]

Therefore, using a standard bootstrap argument, one can show the global existence.

We end this section mentioning that analogously to [25, Proposition 2.5], the following “virial identity” result holds.

**Lemma 2.5.** Assume that \( u_0 \in \Sigma(\Gamma) = \{ v \in H^1(\Gamma) : xv \in L^2(\Gamma) \} \) and \( u(t) \) is the corresponding maximal solution to (1.1). Then \( u(t) \in C([0,T_{H^1}), \Sigma(\Gamma)) \), and the function
\[ f(t) := \int_\Gamma x^2|u(t,x)|^2\,dx = \|xu(t)\|_2^2 \]
belongs to \( C^2[0,T_{H^1}) \). Moreover,
\[ (2.23) \quad f'(t) = 4 \Im \int_\Gamma x\overline{u}_t u \,dx \quad \text{and} \quad f''(t) = 8P(u(t)), \quad t \in [0,T_{H^1}), \]
where
\[ (2.24) \quad P(v) = \|v'\|_2^2 - \frac{1}{2\beta} \left| \sum_{j=1}^N v_j(0) \right|^2 - \frac{p-1}{2(p+1)} \|v\|_{p+1}^{p+1}, \quad v \in H^1(\Gamma). \]

3. Existence of the ground state

By a standing wave of (1.1), we mean a solution of the form \( e^{i\omega t}\phi(x) \), where \( \omega \in \mathbb{R} \) and \( \phi \) is a solution of the stationary equation
\[ (3.1) \quad -\Delta_\beta \phi + \omega \phi - |\phi|^{p-1} \phi = 0. \]
The above equation is the Euler-Lagrange equation of the action functional on \( H^1(\Gamma) \)
\[ S_\omega(v) := \frac{1}{2}F_\beta(v) + \frac{\omega}{2}\|v\|_2^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1}. \]
The action \( S_\omega \) is unbounded from below. Nevertheless, it is bounded from below on the so-called natural (or Nehari) manifold \( \{ v \in H^1(\Gamma) : I_\omega(v) = 0 \} \), where
\[ I_\omega(v) := F_\beta(v) + \omega\|v\|_2^2 - \|v\|_{p+1}^{p+1}. \]
Note that \( I_\omega(v) = \langle S'_\omega(v), v \rangle_{H^1 \times H^1} \), therefore, the Nehari manifold contains all the solutions to the stationary equation (3.1). We consider the minimization problem on the Nehari manifold
\[ (3.2) \quad d_\omega = \inf \{ S_\omega(v) : v \in H^1(\Gamma) \setminus \{0\}, \ I_\omega(v) = 0 \}. \]
Our first result states the existence of the minimizer for \( d_\omega \).

**Theorem 3.1.** Let \( p > 1 \) and \( \omega > \frac{N^2}{p^2} \). Then there exists \( \beta^* < 0 \) such that problem (3.2) admits a solution for any \( \beta \in (\beta^*, -\frac{N}{\sqrt{\omega}}) \).
Proof. Step 1. Let
\[ S_\omega^\infty(v) = \frac{1}{2} \| v'\|_2^2 + \omega \| v\|_2^2 - \frac{1}{p+1} \| v\|_{p+1}^{p+1}, \quad I_\omega^\infty(v) = \| v'\|_2^2 + \omega \| v\|_2^2 - \| v\|_{p+1}^{p+1}. \]

Index \( \infty \) means that formally we assume \( \beta = \infty \).

Firstly, we show that
\[
d_\omega^\infty = \frac{1}{2} \left( \frac{p+1}{2} \right)^{\frac{p+3}{p-1}} \omega^{\frac{p+3}{p-1}} \int_0^1 (1 - t^2)^{\frac{p+1}{p-1}} dt.
\]

Let \( v \in H^1(\Gamma) \) and \( v^* \in H^1_{eq}(\Gamma) \) be its symmetric rearrangement (see [2, Appendix A] for the definition). We are going to substitute the minimizing problem on \( H^1(\Gamma) \) by the one on \( H^1_{eq}(\Gamma) \). Notice that
\[
\| v\|_2 = \| v^*\|_2, \quad \| v\|_{p+1} = \| v^*\|_{p+1}, \quad \| v'\|_2 \geq \frac{2}{N} \| v^*\|_2.
\]

Since \( I_\omega^\infty(v) \geq \frac{4}{N} \| v^*\|_2^2 - \| v^*\|_{p+1}^{p+1} - \| v^*\|_2^2 \), we get
\[
d_\omega^\infty \geq \inf \left\{ \frac{p-1}{2(p+1)} \| v\|_{p+1}^{p+1}, v \in H^1_{eq}(\Gamma) \setminus \{0\}, \frac{1}{N} \| v'\|_2^2 - \| v\|_{p+1}^{p+1} + \omega \| v\|_2^2 \leq 0 \right\} : = I_1.
\]

Taking the rescaling \( \lambda^{1/2} v(\lambda x) \) of \( v(x) \in H^1_{eq}(\Gamma) \) with \( \lambda = \left( \frac{N}{2} \right)^{\frac{4}{5-p}} \), one arrives at
\[
I_1 = \left( \frac{N}{2} \right)^{\frac{2(p-1)}{5-p}} \inf \left\{ \frac{p-1}{2(p+1)} \| v\|_{p+1}^{p+1}, v \in H^1_{eq}(\Gamma) \setminus \{0\}, \frac{2}{N} \| v'\|_2^2 - \| v\|_{p+1}^{p+1} + \omega \| v\|_2^2 \leq 0 \right\}
\]
\[
= \left( \frac{N}{2} \right)^{\frac{2(p-1)}{5-p}} \inf \left\{ \frac{p-1}{2(p+1)} \| v\|_{p+1}^{p+1}, v \in H^1_{eq}(\Gamma) \setminus \{0\}, \| v\|_2^2 - \| v\|_{p+1}^{p+1} + \omega \left( \frac{2}{N} \right)^{\frac{2(p-1)}{5-p}} \| v\|_2^2 \leq 0 \right\}
\]
\[
= N \left( \frac{N}{2} \right)^{\frac{2(p-1)}{5-p}} \inf \left\{ \frac{p-1}{2(p+1)} \| v\|_{L^{p+1}(\mathbb{R}^+)}^{p+1}, v \in H^1(\mathbb{R}^+) \setminus \{0\}, \| v'\|_{L^2(\mathbb{R}^+)}^2 - \| v\|_{L^{p+1}(\mathbb{R}^+)}^{p+1} + \omega \left( \frac{2}{N} \right)^{\frac{2(p-1)}{5-p}} \| v\|_{L^2(\mathbb{R}^+)}^2 \leq 0 \right\} : = I_2.
\]

Let \( \{ v_n \} \) be a minimizing sequence for \( I_2 \). Then, taking even continuation of \( v_n \) onto the whole line, we obtain
\[
I_2 \geq N \left( \frac{N}{2} \right)^{\frac{2(p-1)}{5-p}} \inf \left\{ \frac{p-1}{2(p+1)} \| v\|_{L^{p+1}(\mathbb{R})}^{p+1}, v \in H^1(\mathbb{R} \setminus \{0\}), \| v'\|_{L^2(\mathbb{R})}^2 - \| v\|_{L^{p+1}(\mathbb{R})}^{p+1} + \omega \left( \frac{2}{N} \right)^{\frac{2(p-1)}{5-p}} \| v\|_{L^2(\mathbb{R})}^2 \leq 0 \right\}.
\]

By [1, Lemma 4.4],
\[
\inf \left\{ \frac{p-1}{2(p+1)} \| v\|_{L^{p+1}(\mathbb{R})}^{p+1}, v \in H^1(\mathbb{R} \setminus \{0\}), \| v'\|_{L^2(\mathbb{R})}^2 - \| v\|_{L^{p+1}(\mathbb{R})}^{p+1} + \omega \| v\|_{L^2(\mathbb{R})} \leq 0 \right\}
\]
\[
= \frac{p-1}{2(p+1)} \| v\|_{L^{p+1}(\mathbb{R})}^{p+1} = \frac{1}{2} \left( \frac{p+1}{2} \right)^{\frac{p+1}{p-1}} \omega^{\frac{p+1}{p-1}} \int_0^1 (1 - t^2)^{\frac{p}{p-1}} dt.
\]
where $\phi_\omega$ is the solution to the equation $-\phi''(x) + \omega \phi(x) - |\phi(x)|^{p-1}\phi(x) = 0$, $x \in \mathbb{R}$, given by

$$
\phi_\omega(x) = \left\{ \frac{(p+1)\omega}{2} \text{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2} x \right) \right\}^{\frac{1}{p-1}}.
$$

Comparing the last line of (3.4) and (3.5), we get (substituting $\omega$ by $\omega \left( \frac{2}{N} \right)^{\frac{2(p-1)}{p}}$)

$$
d_\omega^\infty \geq \frac{N}{2} \left( \frac{N}{2} \right)^{\frac{2(p-1)}{p}} \frac{(p+1)}{2} \left( \frac{(p+1)}{2} \right)^{\frac{2}{p-1}} \left[ \omega \left( \frac{2}{N} \right)^{\frac{2(p-1)}{p}} \right]^{\frac{p+1}{2(p-1)}} \int_0^1 (1-t^2)^{\frac{2}{p-1}} dt
$$

$$
= \frac{1}{2} \left( \frac{p+1}{2} \right)^{\frac{2}{p-1}} \omega^{\frac{p+1}{2(p-1)}} \int_0^1 (1-t^2)^{\frac{2}{p-1}} dt.
$$

Let $\tilde{\phi}(x) = (\tilde{\phi}_j(x))_{j=1}^N$ with $\tilde{\phi}_j(x) = \left\{ \begin{array}{ll}
\phi_\omega, & j = 1 \\
0, & j \neq 1.
\end{array} \right.$ It is easily seen that $I_\omega^\infty(\tilde{\phi}) = 0$ and

$$
S_\omega^\infty(\tilde{\phi}) = \frac{1}{2} \left( \frac{p+1}{2} \right)^{\frac{2}{p-1}} \omega^{\frac{p+1}{2(p-1)}} \int_0^1 (1-t^2)^{\frac{2}{p-1}} dt,
$$

hence (3.3) holds.

**Step 2.** We prove that $d_\omega > 0$. Since $\omega > \frac{N^2}{\beta^2} = -\inf \sigma(-\Delta_\beta)$, then $F_\beta(v) + \omega \|v\|^2_2$ is equivalent to $H^1$-norm (see [11, Lemma 4.13]). Let $v \in H^1(\Gamma) \setminus \{0\}$ satisfy $I_\omega(v) = 0$, then

$$
\|v\|_{p+1}^{p+1} = F_\beta(v) + \omega \|v\|^2_2.
$$

Summarizing the above, by the Sobolev embedding, we obtain

$$
\|v\|_{p+1}^2 \leq C_1 \|v\|^2_{H^1} \leq C_2 \left( F_\beta(v) + \omega \|v\|^2_2 \right) = C_2 \|v\|_{p+1}^{p+1},
$$

therefore $C_2^{\frac{1}{p+1}} \leq \|v\|_{p+1}$. Taking the infimum over $v$, we get $d_\omega > 0$.

**Step 3.** We introduce $\beta^*$ such that for $\beta^* < \beta$ one gets $d_\omega < d_\omega^\infty$. Consider $\phi_\beta = (\tilde{\phi}_j)_{j=1}^N$, where

$$
\tilde{\phi}_\beta(x) = \left\{ \frac{(p+1)\omega}{2} \text{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2} x - \tanh^{-1} \left( \frac{N}{\beta \sqrt{\omega}} \right) \right) \right\}^{\frac{1}{p-1}}.
$$

It is easy to check that $I_\omega(\phi_\beta) = 0$ (since $\phi_\beta$ satisfies (3.1)) and

$$
S_\omega(\phi_\beta) = \frac{N}{2} \left( \frac{p+1}{2} \right)^{\frac{2}{p-1}} \omega^{\frac{p+1}{2(p-1)}} \int_{\Phi(\beta, \omega)}^1 (1-t^2)^{\frac{2}{p-1}} dt,
$$

then

$$
S_\omega(\phi_\beta) < d_\omega^\infty \iff N \int_{\Phi(\beta, \omega)}^1 (1-t^2)^{\frac{2}{p-1}} dt < \int_0^1 (1-t^2)^{\frac{2}{p-1}} dt.
$$

Observe that $f(|\beta|) = N \int_{\Phi(\beta, \omega)}^1 (1-t^2)^{\frac{2}{p-1}} dt$ is an increasing function on $(\frac{N}{\sqrt{\omega}}, \infty)$, and $f((\frac{N}{\sqrt{\omega}}, \infty)) = \left( 0, N \int_0^1 (1-t^2)^{\frac{2}{p-1}} dt \right)$. Therefore, there exists $\beta^* < -\frac{N}{\sqrt{\omega}}$ such that

$$
N \int_{\Phi(\beta, \omega)}^1 (1-t^2)^{\frac{2}{p-1}} dt = \int_0^1 (1-t^2)^{\frac{2}{p-1}} dt.
Hence, for $\beta^* < \beta < -\frac{N}{\sqrt{\omega}}$ we get
\[ d_\omega \leq S_\omega(\phi_\beta) < d_\omega^\infty. \]

**Step 4.** Let
\[ \tilde{d}_\omega := \inf \left\{ \frac{p-1}{2(p+1)} (F_\beta(v) + \omega \|v\|^2_2) : v \in H^1(\Gamma) \setminus \{0\}, I_\omega(v) \leq 0 \right\}. \]
We prove that $\tilde{d}_\omega = d_\omega$. It is obvious that $\tilde{d}_\omega \leq d_\omega$. Let $v \in H^1(\Gamma) \setminus \{0\}$ and $I_\omega(v) < 0$. Put
\[ \lambda_1 := \left( \frac{F_\beta(v) + \omega \|v\|^2_2}{\|v\|^{p+1}_{p+1}} \right)^{\frac{1}{p+1}}. \]
Then, since $I_\omega(\lambda_1v) = 0$ and $0 < \lambda_1 < 1$ (this follows from the behavior of the function $g(\lambda) = I_\omega(\lambda v)$), we have
\[ d_\omega \leq S_\omega(\lambda_1v) = \frac{p-1}{2(p+1)} (F_\beta(\lambda_1 v) + \|\lambda_1 v\|^2_2) = \frac{p-1}{2(p+1)} \lambda_1^2 (F_\beta(v) + \|v\|^2_2) < \frac{p-1}{2(p+1)} (F_\beta(v) + \|v\|^2_2). \]
Thus, we obtain $d_\omega \leq \tilde{d}_\omega$.

**Step 5.** Let $\{v_n\} \subset H^1(\Gamma) \setminus \{0\}$ be a minimizing sequence for $d_\omega$. As long as
\[ S_\omega(v_n) = \frac{p-1}{2(p+1)} (F_\beta(v_n) + \omega \|v_n\|^2_2) = \frac{p-1}{2(p+1)} \|v_n\|^{p+1}_{p+1} \rightarrow d_\omega, \]
the sequence $\{v_n\}$ is bounded in $H^1(\Gamma)$. Hence there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and $v_0 \in H^1(\Gamma)$ such that $\{v_{n_k}\}$ converges weakly to $v_0$ in $H^1(\Gamma)$. We may assume that $v_{n_k} \neq 0$ and define
\[ \lambda_k = \left( \frac{\|v'_{n_k}\|^2_2 + \omega \|v_{n_k}\|^2_2}{\|v_{n_k}\|^{p+1}_{p+1}} \right)^{\frac{1}{p+1}}. \]
Notice that $\lambda_k > 0$ and $I_\omega^\infty(\lambda_k v_{n_k}) = 0$. Therefore, by Step 3 and the definition of $d_\omega^\infty$, we obtain
\[ d_\omega < d_\omega^\infty \leq \frac{p-1}{2(p+1)} \|\lambda_k v_{n_k}\|^{p+1}_{p+1} = \lambda_k^{p+1} \frac{p-1}{2(p+1)} \|v_{n_k}\|^{p+1}_{p+1} \quad \text{for all } k \in \mathbb{N}. \]
Furthermore, by $I_\omega(v_{n_k}) = 0$, (3.8), and the weak convergence, we get
\[ \lim_{k \to \infty} \lambda_k = \lim_{k \to \infty} \left( \frac{\|v_{n_k}\|^{p+1}_{p+1} - \frac{1}{\beta} \sum_{j=1}^N (v_{n_k})_j(0)}{\|v_{n_k}\|^{p+1}_{p+1}} \right)^{\frac{1}{p+1}} = \left( \frac{d_\omega - \frac{p-1}{2(p+1)} \frac{1}{\beta} \sum_{j=1}^N (v_0)_j(0)}{d_\omega} \right)^{\frac{1}{p+1}}. \]
Taking the limit in (3.9), we obtain $d_\omega < \lim_{k \to \infty} \lambda_k^{p+1} d_\omega$. Since, by Step 2, $d_\omega > 0$, we arrive at $\lim_{k \to \infty} \lambda_k > 1$, and consequently $\frac{1}{\beta} \sum_{j=1}^N (v_0)_j(0) < 0$. Thus, $v_0 \neq 0$. By the weak convergence, we obtain
\[ \lim_{k \to \infty} \left\{ (F_\beta(v_{n_k}) - F_\beta(v_{n_k} - v_0)) + \omega \left( \|v_{n_k}\|^2_2 - \|v_{n_k} - v_0\|^2_2 \right) \right\} = F_\beta(v_0) + \omega \|v_0\|^2_2. \]
Therefore, by the Brezis-Leib lemma [14],
\[
\lim_{k \to \infty} I_\omega(v_{n_k}) - I_\omega(v_{n_k} - v_0) = \lim_{k \to \infty} -I_\omega(v_{n_k} - v_0) = I_\omega(v_0).
\]
Since \( v_0 \neq 0 \), then the right-hand side of (3.10) is positive. Hence it follows from (3.8) and (3.10) that
\[
\frac{p-1}{2(p+1)} \lim_{k \to \infty} \left( F_\beta(v_{n_k} - v_0) + \omega \|v_{n_k} - v_0\|_2^2 \right) < \frac{p-1}{2(p+1)} \lim_{k \to \infty} \left( F_\beta(v_{n_k}) + \omega \|v_{n_k}\|_2^2 \right) = d_\omega.
\]
Then, by Step 4 (using that \( d_\omega = \tilde{d}_\omega \)), we have \( I_\omega(v_{n_k} - v_0) > 0 \) for \( k \) large enough. Thus, since \( -I_\omega(v_{n_k} - v_0) \to I_\omega(v_0) \), we obtain \( I_\omega(v_0) \leq 0 \). By \( d_\omega = \tilde{d}_\omega \) and the weak lower semicontinuity of norms, we conclude
\[
d_\omega \leq \frac{p-1}{2(p+1)} \left( F_\beta(v_0) + \omega \|v_0\|_2^2 \right) \leq \frac{p-1}{2(p+1)} \lim_{k \to \infty} \left( F_\beta(v_{n_k}) + \omega \|v_{n_k}\|_2^2 \right) = d_\omega.
\]
Therefore, from (3.10) we get
\[
\lim_{k \to \infty} F_\beta(v_{n_k} - v_0) + \omega \|v_{n_k} - v_0\|_2^2 = 0,
\]
and, consequently, we have \( v_{n_k} \to v_0 \) in \( H^1(\Gamma) \) and \( I_\omega(v_0) = 0 \).

\[\textbf{Remark 3.2.}\] The restriction \( \beta \in (\beta^*, -\frac{N}{\sqrt{\omega}}) \) seems to be non optimal at least for \( \omega > \frac{p+1}{p-1} \frac{N^2}{\beta^*} \). For example, for even \( N \) one gets \( S(\phi_N) < S(\phi_\beta) \), where \( \phi_N \) is defined in Theorem 4.2 (the proof repeats the last part of the proof of [1, Theorem 5.3]). Thus, comparison of \( d_\omega^\infty \) with \( S(\phi_N) \) might guarantee the existence of the minimizer for some interval larger than \( (\beta^*, -\frac{N}{\sqrt{\omega}}) \).

Above we proved the existence of the minimizer for sufficiently weak attractive \( \delta_\omega \) coupling. Nevertheless, below we show that the minimizer exists for any \( \beta \in \mathbb{R} \setminus \{0\} \) and \( \beta < 0 \) in two particular cases. The first case is related with \( H^1_{eq}(\Gamma) \), and the second case deals with \( H^1_{rad}(\Gamma) \) (see two Lemmas below).

**Lemma 3.3.** Let \( \beta \in \mathbb{R} \setminus \{0\} \). We set
\[
d_\omega^{eq} = \inf \left\{ S_\omega(v) : v \in H^1_{eq}(\Gamma) \setminus \{0\}, \ I_\omega(v) = 0 \right\}.
\]
Then \( d_\omega^{eq} = S_\omega(\phi_\beta) \), where \( \phi_\beta = (\tilde{\phi}_\beta)^{N}_{j=1} \) is defined by (3.7).

**Proof.** Notice that
\[
d_\omega^{eq} = N d_\omega^{half} = \frac{N}{2} d_\omega^{line},
\]
where
\[
d_\omega^{half} = \inf \left\{ \left. \frac{1}{2} \|v\|_2^2 + \omega \|v\|_{L^2(\mathbb{R}^+)}^2 - \frac{1}{p+1} \|v\|_{L^{p+1}(\mathbb{R}^+)}^{p+1} + \frac{N}{2\beta} |v(0)|^2 \ : \ v \in H^1(\mathbb{R}) \setminus \{0\} \right\},
\]
\[
d_\omega^{line} = \inf \left\{ \left. \frac{1}{2} \|v\|_2^2 + \omega \|v\|_{L^2(\mathbb{R})}^2 - \frac{1}{p+1} \|v\|_{L^{p+1}(\mathbb{R})}^{p+1} + \frac{N}{\beta} |v(0)|^2 \ : \ v \in H^1_{rad}(\mathbb{R}) \setminus \{0\} \right\}.
\]
From the results by [22, 23] one gets
\[
\phi_{\beta,0}^\text{line} = \frac{1}{2} \|\tilde{\phi}_\beta\|_{L^2(\mathbb{R})}^2 + \frac{\omega}{2p} \|\tilde{\phi}_\beta\|_{L^2(\mathbb{R})}^2 - \frac{1}{p+1} \|\tilde{\phi}_\beta\|_{L^{p+1}(\mathbb{R})}^{p+1} + \frac{N}{p} |\tilde{\phi}_\beta(0)|^2.
\]

\[\square\]

**Lemma 3.4.** Let \( \beta < 0 \), \( N \) be even, and let the profile \( \phi_{\frac{N}{2}} \) be defined in Theorem 4.2. Set
\[
(3.12) \quad d_{\omega}^N = \inf \left\{ S_\omega(v) : v \in H^1_{\frac{N}{2}}(\Gamma) \setminus \{0\}, \ I_\omega(v) = 0 \right\}.
\]

Then
(i) for \( \frac{N^2}{2p} < \omega \leq \frac{p+1}{p-1} \omega \) the solution to (3.12) is given by \( \phi_\beta \);
(ii) for \( \omega > \frac{p+1}{p-1} \omega \) the solution to (3.12) is given by \( \phi_{\frac{N}{2}} \).

The proof of the above lemma follows from [1, Theorem 4.1, Theorem 5.3], observing that for \( v \in H^1_{\frac{N}{2}}(\Gamma) \)
\[
S_\omega(v) = \frac{N}{4} \left[ \|v_1'\|_{L^2(\mathbb{R}^+)}^2 + \|v_N'\|_{L^2(\mathbb{R}^+)}^2 + \omega (\|v_1\|_{L^2(\mathbb{R}^+)}^2 + \|v_N\|_{L^2(\mathbb{R}^+)}^2) \right. \\
+ \left. \frac{N}{2\beta} |v_1(0) + v_N(0)|^2 - \frac{2}{p+1} (\|v_1\|_{L^{p+1}(\mathbb{R}^+)} + \|v_N\|_{L^{p+1}(\mathbb{R}^+)}^2) \right].
\]

In other words, in \( H^1_{\frac{N}{2}}(\Gamma) \) minimizing problem (3.12) is “equivalent” to \( \frac{N}{2} \) copies of the minimizing problem on the line.

**Remark 3.5.** In [11] the NLS equation with the \( \delta \) coupling and decaying potential \( V(x) \) on \( \Gamma \)
\[
i \partial_t u(t) = -\Delta_\gamma u(t) + V(x)u(t) - |u(t)|^{p-1}u(t)
\]
has been considered. Here \( \gamma < 0 \) and \( (-\Delta_\gamma v)(x) = (-v''_j(x)) \) with
\[
\text{dom}(\Delta_\gamma) = \left\{ v \in H^1(\Gamma) : v_1(0) = \ldots = v_N(0), \ \sum_{j=1}^N v'_j(0) = \gamma v_1(0) \right\}.
\]

It was shown that under the assumptions
\[
V(x) = (V_j(x))^N_{j=1} \in L^1(\Gamma) + L^\infty(\Gamma), \quad \lim_{x \to \infty} V_j(x) = 0,
\]
(3.13) \[
\int_{\mathbb{R}^+} V_j(x) |\varphi(x)|^2 dx < 0, \quad \varphi(x) \in H^1(\mathbb{R}^+) \setminus \{0\}
\]
there exist \( 0 < \omega_0 = -\inf \sigma(-\Delta_\gamma + V) \) and \( \gamma^* < 0 \) such that for \( \omega > \omega_0 \) and \( \gamma < \gamma^* \) problem (3.2) admits a solution (in definition of \( S_\omega \) one needs to substitute quadratic form \( F_\beta \) by the quadratic form of the operator \( -\Delta_\gamma + V) \).

Under assumptions (3.13), one may consider
\[
i \partial_t u(t) = -\Delta_\beta u(t) + V(x)u(t) - |u(t)|^{p-1}u(t).
\]
It can be proven analogously that for \( \omega > -\inf \sigma(-\Delta_\gamma + V) \) and \( \beta \in (\beta^*, -\frac{N}{\sqrt{\omega}}) \) problem (3.2) has a solution \( \phi_{\beta,V} \) (with \( S_\omega(v) \) substituted by \( S_\omega(v) + \frac{1}{2}(Vv,v)_2 \)).
4. Identification of the ground states

In this section we describe explicitly the solutions to (3.2). In what follows we will use the notation

\[(4.1) \quad \phi_a(x) = \left\{ \frac{(p+1)\omega}{2} \sech^2 \left( \frac{(p-1)\sqrt{\omega}}{2}(x+a) \right) \right\}^{\frac{1}{p-1}}, \quad a \in \mathbb{R}.\]

**Proposition 4.1.** Let \( \omega > \frac{N^2}{p^2} \). Assume that the solution of problem (3.2) exists, then it is given by the critical point of \( S_\omega \). Moreover, each critical point of \( S_\omega \) has the form \( \phi = e^{i\theta} \phi_{x_j} \), with \( \theta \in \mathbb{R} \), where \( |x_j| = \frac{2}{(p-1)\sqrt{\omega}} \tanh^{-1}(t_j) \) and \( t_j, j = 1, \ldots, N, \) satisfy the system

\[(4.2) \quad \left\{ \begin{array}{l}
t_1^{p-1} - t_1^{p+1} = \ldots = t_N^{p-1} - t_N^{p+1}, \\
\sum_{j=1}^N t_j^{-1} = |\beta| \sqrt{\omega}.
\end{array} \right.\]

In particular, for \( \beta > 0 \) (\( \beta < 0 \)) all the constants \( x_j \) are negative (positive).

**Proof.** Let \( \phi \) be a minimizer. Since \( I_\omega(\phi) = 0 \), we have

\[(4.3) \quad \langle I'_\omega(\phi), \phi \rangle_{(H^1)^* \times H^1} = 2 \left( F_\beta(\phi) + \omega \| \phi \|^2 \right) - (p+1) \| \phi \|^{p+1} = -(p-1) \| \phi \|^{p+1} < 0.\]

There exists a Lagrange multiplier \( \mu \in \mathbb{R} \) such that \( S'_\omega(\phi) = \mu I'_\omega(\phi) \). Furthermore, since

\[\mu \langle I'_\omega(\phi), \phi \rangle_{(H^1)^* \times H^1} = \langle S'_\omega(\phi), \phi \rangle_{(H^1)^* \times H^1} = I_\omega(\phi) = 0,\]

then, by (4.3), \( \mu = 0 \). Hence \( S'_\omega(\phi) = 0 \). As in the proof of [1, Proposition 5.1], one can show that \( S'_\omega(\phi) = 0 \) is equivalent to (3.1) and \( \phi \in \text{dom}(\Delta_\beta) \). The most general \( L^2(\mathbb{R}^+) \)-solution to

\[-\phi'' + \omega \phi - |\phi|^{p-1} \phi = 0\]

is given by \( \sigma \phi_a(x), |\sigma| = 1 \) (see (4.1)), therefore, \( \phi = (e^{i\theta} \phi_{x_j})_{j=1}^N \). Since \( \phi \) is the minimizer of \( S_\omega \), one easily concludes \( e^{i\theta_j} = \ldots = e^{i\theta_N} \). From (1.2) we get

\[(4.4) \quad \left\{ \begin{array}{l}
\tanh \left( \frac{p-1}{p} \sqrt{\omega} x_j \right) \cosh \frac{p-1}{p} \left( \frac{p-1}{p} \sqrt{\omega} x_j \right) + 1 = 0, \quad j = 1, \ldots, N - 1, \\
\sum_{j=1}^N \frac{t_j^{-1}}{\cosh \frac{p-1}{p} \left( \frac{p-1}{p} \sqrt{\omega} x_j \right)} = -\beta \sqrt{\omega} \frac{\tanh \left( \frac{p-1}{p} \sqrt{\omega} x_1 \right)}{\cosh \frac{p-1}{p} \left( \frac{p-1}{p} \sqrt{\omega} x_1 \right)}. \end{array} \right.\]

Introducing \( t_j = \tanh \left( \frac{p-1}{p} \sqrt{\omega} x_j \right) \) and observing that \( \cosh^{-2} \left( \frac{p-1}{p} \sqrt{\omega} x_j \right) = 1 - \tanh^2 \left( \frac{p-1}{p} \sqrt{\omega} x_j \right) \), we arrive at (4.2). To complete the proof, notice that from the first equation in (4.4) it follows that \( x_j \) have the same sign. \( \square \)

The next theorem gives a precise description of the family of the critical points of \( S_\omega \) containing the ground state.

**Theorem 4.2.** Let \( \omega > \frac{p+1}{p} N^2, N \geq 2, \beta \in \mathbb{R} \setminus \{0\} \). Then (up to permutation of the edges of the \( \Gamma \) and rotation), the family of critical points of \( S_\omega \) consists of \( N \) profiles. The first \( N - 1 \) profiles are given by \( \phi_k = (\phi_{x_1}, \ldots, \phi_{x_k}, \phi_{x_{k+1}}), k = 1, \ldots, N - 1, \) where
\[ |x_1| = \frac{2}{(p-1)\sqrt{\omega}} \tanh^{-1}(t_1), \quad |x_N| = \frac{2}{(p-1)\sqrt{\omega}} \tanh^{-1}(t_N), \quad \text{and} \quad 0 < t_1 < t_N < 1 \text{ satisfy the system} \]

(4.5)

\[
\begin{aligned}
& t_1^{p-1} - t_1^{p+1} = t_N^{p-1} - t_N^{p+1} \\
& kt_1^{-1} + (N-k)t_N^{-1} = |\beta|\sqrt{\omega}.
\end{aligned}
\]

The \(N\)th critical point of \(S_\omega\) is given by the symmetric profile \(\phi_\beta\).

**Proof.** Notice that the first line of (4.2) might be rewritten as

\[ f(t_1) = \ldots = f(t_N), \quad \text{where} \quad f(t) := t^{p-1} - t^{p+1} : [0,1) \to \mathbb{R}. \]

We have \(f([0,1)) = [0, (\frac{p-1}{2})^{\frac{p+1}{2}} / (\frac{p+1}{2})^{\frac{p+1}{2}}]\), and the maximum \(M = (\frac{p-1}{2})^{\frac{p-1}{2}} / (\frac{p+1}{2})^{\frac{p+1}{2}}\) is attained at the unique extremum point \(t_{\text{max}} = \sqrt{\frac{p-1}{p+1}}\). By monotonicity properties of \(f\), we conclude that each \(t_j\) may take only two possible values: \(t_1, t_N\) (without loss of generality), where \(0 < t_1 < t_{\text{max}} < t_N < 1\) and \(f(t_1) = f(t_N)\).

Now, assuming that the number of \(t_j\) that take value \(t_1\) (resp. \(t_N\)) is \(k \in \{1,\ldots,N-1\}\) (resp. \(N-k\)), we get the second line of system (4.5).

Then expressing \(t_1\) from the second line of (4.5), by the first line, we get that \(t_N\) satisfies the equation \(w(t_N) = 0\), where

\[ w(x) = \frac{x^2(a^2k^{p-1} - k^{p+1}) - 2a(N-k)k^{p-1}x + (N-k)^2k^{p-1}}{(ax - N + k)^{p+1}} + x^2 - 1, \quad a = |\beta|\sqrt{\omega}. \]

Arguing as in the proof of [1, Lemma 5.2], we can prove that for any \(a > N\sqrt{\frac{p+1}{p-1}}\), there exists a unique root of \(w(x)\) in \((\frac{N}{a},1]\). It is sufficient to observe that:

- \(w\left(\frac{N}{a}\right) = 0\),
- \(w'(\frac{N}{a}) = \frac{1}{ak}((p+1)N^2 - a^2(p-1)) < 0\),
- \(w(1) = \frac{1}{(a-N+k)^{p+1}}((a-N+k)^2 - k^2) > 0\),
- \(w''(x) = \frac{-(p-1)p(p+1)(a^2-k^2)x^2 + A_wx + C_w}{(ax - N + k)^{p+4}}\), where \(A_w, C_w\) are real constants. Thus, \(w''(x) < 0\) for \(x\) large enough.

To end the proof, we observe that assuming \(k = N\), we get the solution \(\phi_\beta\).

\[ \square \]

**Remark 4.3.** When \(N \geq 3\), it seems that approach by [1] cannot be applied to prove that system (4.5) has a unique solution \(\phi_\beta\) for \(\frac{N^2}{p^2} < \omega \leq \frac{p+1}{p-1}N^2\). However, it still can be shown that the solution to (4.5) with \(t_1 < t_N\) for \(k > N - k\) does not exist (it is sufficient to observe \(g^2(t_1) < \left(\frac{k}{N-k}\right)^2 g^2(t_N)\) in [1, Formula (5.25)]). We conjecture that for \(\frac{N^2}{p^2} < \omega \leq \frac{p+1}{p-1}N^2\), the profile \(\phi_\beta\) is the solution to (3.2).

We finish this section with the following important result which we will prove at the end of Subsection 5.4.

**Theorem 4.4.** Let \(\beta < 0, \omega > \frac{p+1}{p-1}N^2\), and assume that solution to problem (3.2) exists, then it is given by \(\phi_1\).
5. Orbital and Spectral Instability

The definition of the orbital stability involves the symmetry of the concrete Hamiltonian system in study. Since equation (1.1) is rotationally symmetric, we define orbital stability as follows.

**Definition 5.1.** The standing wave solution \( u(t) = e^{i \omega t} \phi(x) \) is said to be orbitally stable in \( H^1(\Gamma) \) by the flow of equation (1.1) if for any \( \varepsilon > 0 \) there exists \( \eta > 0 \) with the following property. If \( u_0 \in H^1(\Gamma) \) satisfies \( \| u_0 - \phi \|_{H^1} < \eta \), then the solution \( u(t) \) of (1.1) with \( u(0) = u_0 \) exists for any \( t \in \mathbb{R} \) and

\[
\sup_{t \in \mathbb{R}} \inf_{\theta \in \mathbb{R}} \| u(t) - e^{i\theta} \phi \|_{H^1} < \varepsilon.
\]

Otherwise, the standing wave \( u(t) = e^{i \omega t} \phi(x) \) is said to be orbitally unstable in \( H^1(\Gamma) \).

The definition of the spectral instability involves the concept of the linearization of (1.1) around the profile of the standing wave. After making necessary technical steps we give precise Definition 5.4.

5.1. Instability analysis for symmetric profile \( \phi_\beta \) via Grillakis/Jones approach.

We begin this subsection with statement of one of the main results.

**Theorem 5.2.** Let \( p > 1 \), \( N \geq 2 \), and \( \phi_\beta \) be defined by (3.7). If either \( \beta < 0 \) and \( \omega > \frac{p+1}{p-1} \frac{N^2}{\beta^2} \) or \( \beta > 0 \) and \( \frac{N^2}{\beta^2} < \omega < \frac{p+1}{p-1} \frac{N^2}{\beta^2} \), then \( e^{i \omega t} \phi_\beta \) is spectrally unstable. Moreover if, additionally, \( p > 2 \), then \( e^{i \omega t} \phi_\beta \) is orbitally unstable.

**Remark 5.3.** (i) In the case \( N = 2 \), the result of the above theorem was shown in [9, Theorem 1.1] and [1, Theorem 6.13] (for \( \beta > 0 \) and \( \beta < 0 \), respectively). Moreover, our result extends [9, Theorem 1.1] for the case \( p \in (3,5) \).

(ii) In [7, Theorem 1.2] can be found some orbital stability/instability results for \( \beta < 0 \). In particular, it was shown that for \( \frac{N^2}{\beta^2} < \omega < \frac{p+1}{p-1} \frac{N^2}{\beta^2} \) and \( 1 < p \leq 5 \) we have orbital stability of \( e^{i \omega t} \phi_\beta \). Orbital instability was shown for \( \omega > \frac{p+1}{p-1} \frac{N^2}{\beta^2} \), \( 1 < p \leq 5 \), and even \( N \).

Moreover, for \( p > 5 \) and \( \omega \neq \frac{p+1}{p-1} \frac{N^2}{\beta^2} \) we proved that there exists \( \omega^* > \frac{N^2}{\beta^2} \) such that \( e^{i \omega t} \phi_\beta \) is orbitally unstable in \( H^1(\Gamma) \) as \( \omega > \omega^* \), and it is orbitally stable in \( H^1_{eq}(\Gamma) \) as \( \omega < \omega^* \).

To define the concept of spectral instability, we linearize equation (1.1) around \( \phi_\beta \). Firstly, we put \( u(t) = e^{i \omega t} (\phi_\beta + v(t)) \). Observing that \( S_\omega \) is a \( C^2 \) functional, \( S'_\omega(\phi_\beta) = 0 \), and equation (1.1) has the form

\[
i \partial_t u(t) = E'(u(t)),
\]

we get

\[
\partial_t v(t) = -i S''_\omega(\phi_\beta) v(t) + O(\|v(t)\|_{H^1}^2).
\]

It is standard to verify that for \( u \in H^1(\Gamma) \)

\[
S''_\omega(\phi_\beta) u = -\Delta u + \omega u - (\phi_\beta)^{p-1} u - (p-1)(\phi_\beta)^{p-1} \text{Re}(u).
\]
Here the operator $-\tilde{\Delta}_\beta$ is understood in the following sense: since the bilinear form

$$t_\beta(u_1, u_2) = (u_1', u_2')_2 + \beta \Re \left( \sum_{j=1}^N u_{1,j}(0) \sum_{j=1}^N \overline{u_{2,j}(0)} \right)$$

is bounded on $H^1(\Gamma)$, there exists the unique bounded operator $\tilde{\Delta}_\beta : H^1(\Gamma) \to (H^1(\Gamma))'$ such that $t_\beta(u_1, u_2) = \langle -\tilde{\Delta}_\beta u_1, u_2 \rangle_{(H^1)' \times H^1}$. Notice that the bilinear form:

$$b_\beta(u, v) = \langle S''_\omega(\phi_\beta) u, v \rangle_{(H^1)' \times H^1} = \Re \left[ \frac{1}{\beta} \sum_{i,j=1}^N u_i(0) \overline{v_j(0)} \right] + \int_\Gamma \left( u \overline{v'} + \omega u \overline{v} - (\phi_\beta)^{p-1} u \overline{v} - (p-1)(\phi_\beta)^{p-1} \Re(u) \overline{v} \right) dx$$

is closed, densely defined, and bounded from below. Then, by the Representation Theorem \cite[Chapter VI, Theorem 2.1]{35}, we can associate with $b_\beta$ self-adjoint in $L^2(\Gamma)$ (with the real scalar product) operator

$$L^\beta u = -\Delta_\beta u + \omega u - (\phi_\beta)^{p-1} u - (p-1)(\phi_\beta)^{p-1} \Re(u), \quad \dom(L^\beta) = \dom(\Delta_\beta).$$

We define a natural injection $\mathfrak{J} \in B(H^1(\Gamma), (H^1(\Gamma))')$ by

$$\langle \mathfrak{J} u, v \rangle_{(H^1)' \times H^1} = (u, v)_2, \quad \text{for all } u, v \in H^1(\Gamma).$$

Then we have for $u \in \dom(\Delta_\beta)$

$$\langle S''_\omega(\phi_\beta) u, v \rangle_{(H^1)' \times H^1} = (L^\beta u, v)_2 = \langle \mathfrak{J} L^\beta u, v \rangle_{(H^1)' \times H^1}.$$

It is obvious that $S''_\omega(\phi_\beta)$ is the extension of $\mathfrak{J} L^\beta$.

**Definition 5.4.** The standing wave $e^{i\omega t} \phi_\beta(x)$ is said to be spectrally unstable if there exist $\lambda$ with $\Re \lambda > 0$ and $w \in \dom(\Delta_\beta)$ such that

$$-iS''_\omega(\phi_\beta)w = \lambda \mathfrak{J} w.$$

The notion of spectral instability is particularly important since frequently its presence leads to nonlinear instability.

**Remark 5.5.** Notice that spectral instability implies that 0 is unstable solution to the linearized equation

$$i \partial_t v(t) = S''_\omega(\phi_\beta)v(t)$$

in the sense of Lyapunov.

The search of $\lambda$ with $\Re \lambda > 0$ might be simplified essentially identifying $L^2_\mathbb{R}(\Gamma)$ with $L^2_\mathbb{R}(\Gamma) \oplus L^2_\mathbb{R}(\Gamma)$. Namely, take $u = u_1 + i u_2, \quad u_1, u_2 \in L^2_\mathbb{R}(\Gamma)$. Then one gets

$$L^\beta u = L^\beta_1 u_1 + i L^\beta_2 u_2, \quad \dom(L^\beta) = \dom(\Delta_\beta),$$

$$L^\beta_1 v = -\Delta_\beta v + \omega v - (\phi_\beta)^{p-1} v, \quad L^\beta_2 v = -\Delta_\beta v + \omega v - (\phi_\beta)^{p-1} v.$$

Thus, the operator $L^\beta$ in $L^2_\mathbb{R}(\Gamma) \oplus L^2_\mathbb{R}(\Gamma)$ is interpreted as

$$L^\beta = \begin{pmatrix} L^\beta_1 & 0 \\ 0 & L^\beta_2 \end{pmatrix}. $$
Observe that the operator $J$ of multiplication by $-i$ in $L^2_R(\Gamma) \oplus L^2_R(\Gamma)$ acts as
\[
J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\]
Hence we arrive at the equivalence for $w \in \text{dom}(\Delta_\beta)$
\[
3w - iS_\omega(\phi_\beta)w = \lambda w \quad \iff \quad \begin{pmatrix} 0 & L_2^\beta \\ -L_1^\beta & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad w = w_1 + iw_2.
\]
Our idea is to apply the approach by [28] for proving that the operator \( \begin{pmatrix} 0 & L_2^\beta \\ -L_1^\beta & 0 \end{pmatrix} \) has a positive eigenvalue. To do that, we first need to study spectral properties of $L_1^\beta$ and $L_2^\beta$.

**Proposition 5.6.** Let $\beta \in \mathbb{R} \setminus \{0\}$, $\omega > \frac{N^2}{p-1}$, and $\tilde{\phi}_\beta$ be defined by (3.7). Then the following assertions hold.

(i) $L_2^\beta \geq 0$ for $\beta < 0$, and $\ker(L_2^\beta) = \text{span}\{\phi_\beta\}$ for any $\beta$.

(ii) If $\omega \neq \frac{N^2}{p-1}$, then $\ker(L_1^\beta) = \{0\}$, while, for $\omega = \frac{N^2}{p-1}$, the kernel of $L_1^\beta$ is given by $\ker(L_1^\beta) = \text{span}\{\tilde{\phi}_1, \ldots, \tilde{\phi}_{N-1}\}$, where
\[
\tilde{\phi}_j = (0, \ldots, 0, \tilde{\phi}_\beta, -\tilde{\phi}_\beta, 0, \ldots, 0).
\]

(iii) $\sigma_{\text{ess}}(L_1^\beta) = \sigma_{\text{ess}}(L_2^\beta) = [\omega, \infty)$.

**Proof.** (i), (ii) Positivity of $L_2^\beta$ for $\beta < 0$ and item (ii) were proven in [7, Proposition 3.24]. The identity $\ker(L_2^\beta) = \text{span}\{\phi_\beta\}$ follows from the fact that the only decaying solution to
\[
-v'' + \omega v - (\phi_\beta)^{p-1}v = 0
\]
on $\Gamma$ is $\phi_\beta$ (up to a constant). Indeed, to show the equality $\ker(L_2^\beta) = \text{span}\{\phi_\beta\}$, we note that any $v = (v_j)_{j=1}^N \in \ker(L_2^\beta)$ has the form $v = (c_j\tilde{\phi}_\beta)_{j=1}^N$. From the inclusion $v \in \text{dom}(L_2^\beta)$ we get $c_1 = \ldots = c_N$.

(iii) Consider the self-adjoint operator $-\Delta_\infty = \bigoplus_{j=1}^N h_\infty$, where
\[
(h_\infty v)(x) = -v''(x), \quad x > 0, \quad \text{dom}(h_\infty) = \left\{ v \in H^2_0(\mathbb{R}^+) : v'(0) = 0 \right\}.
\]
Therefore, $\sigma_{\text{ess}}(-\Delta_\infty) = \sigma_{\text{ess}}(h_\infty) = [0, \infty)$. Notice that the operators $-\Delta_\beta$ and $-\Delta_\infty$ are self-adjoint extensions of the symmetric operator
\[
(-\Delta_0 v)(x) = (-v''(x))_{j=1}^N, \quad x > 0, \quad v = (v_j)_{j=1}^N,
\]
\[
\text{dom}(-\Delta_0) = \left\{ v \in H^2(\Gamma) : v_1'(0) = \ldots = v_N'(0) = 0, \sum_{j=1}^N v_j(0) = 0 \right\}.
\]
The operator $-\Delta_0$ has equal deficiency indices $n_{\pm}(-\Delta_0) = \dim \ker(-\Delta_0^* \mp i) = 1$, therefore, by Krein’s resolvent formula, the operator $(-\Delta_\beta - \lambda)^{-1} - (-\Delta_\infty - \lambda)^{-1}$, $\lambda \in$
The operator of multiplication by \( \sigma \), is of rank one (see [6, Appendix A, Theorem A.2]). Then, by
Weyl's theorem [42, Theorem XIII.14], \( \sigma_{\text{ess}}(-\Delta_\beta + \omega) = \sigma_{\text{ess}}(-\Delta_\infty + \omega) = [\omega, \infty) \).
The operator of multiplication by \((\phi_\beta)^{p-1}\) is relatively \((-\Delta_\beta + \omega)\)-compact, therefore
\( \sigma_{\text{ess}}(L_1^\beta) = \sigma_{\text{ess}}(L_2^\beta) = [\omega, \infty) \) (see Corollary 2 of [42, Theorem XIII.14]).

Further we study the number of negative eigenvalues of \( L_1^\beta \) and \( L_2^\beta \).

**Proposition 5.7.** (i) Let \( \beta < 0 \), then \( n(L_2^\beta) = 0 \). Moreover,
1) if \( \omega \leq \frac{p+1}{p-1} N^2 \), then \( n(L_1^\beta) = 1 \);
2) if \( \omega > \frac{p+1}{p-1} N^2 \), then \( n(L_1^\beta) = N \).
(iii) Let \( \beta > 0 \), then \( n(L_2^\beta) = N - 1 \). Moreover,
1) if \( \omega < \frac{p+1}{p-1} N^2 \), then \( n(L_1^\beta) = 2N - 1 \);
2) if \( \omega \geq \frac{p+1}{p-1} N^2 \), then \( n(L_1^\beta) = N \).

**Proof.** We use a generalization of the Sturm theory for the star graph elaborated in [32,34].
Firstly, we calculate \( n(L_1^\beta) \).

**Step 1.** Suppose that \( \lambda < 0 \) is an eigenvalue of \( L_1^\beta \) with an eigenvector \( u_\lambda = (v_j^\lambda)_{j=1}^N \in \text{dom}(L_1^\beta) \): \( L_1^\beta u_\lambda = \lambda u_\lambda \). Then, denoting \( a = \frac{2}{(p-1)\sqrt{\omega}} \tan^{-1} \left( \frac{N}{\beta \sqrt{\omega}} \right) \), we have

\[-(v_j^\lambda)' + \omega v_j^\lambda - \frac{p(p+1)\omega}{2} \operatorname{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2} (x-a) \right) v_j^\lambda = \lambda v_j^\lambda, \quad x \in (0, \infty).\]

Thus, \( v_j^\lambda = c_j u(x-a) \), where \( u(x) \) is the solution (on the line) to

\[-u'' + \omega u - \frac{p(p+1)\omega}{2} \operatorname{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2} x \right) u = \lambda u.\]

By [34, Lemma 4.1], \( u(x) \) is a \( C^1 \) function of \( \lambda \) for any fixed \( x_0 \in \mathbb{R} \), and

\[
\lim_{\lambda \to -\infty} \frac{u'(x_0)}{u(x_0)} = -\infty, \quad \lim_{x \to +\infty} u(x)e^{\sqrt{\omega-\lambda}x} = 1.
\]

The coefficients \( c_j \) satisfy the system

\[
(5.4) \quad c_1 u'(-a) = \ldots = c_N u'(-a), \quad \sum_{j=1}^N c_j u(-a) = \beta c_N u'(-a).
\]

The determinant of the matrix associated with the system

\[
\begin{pmatrix}
  u'(-a) & -u'(-a) & 0 & \ldots & 0 \\
  u(-a) & 0 & -u'(-a) & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  u'(-a) & 0 & 0 & \ldots & -u'(-a) \\
  u(-a) & u(-a) & u(-a) & \ldots & u(-a) - \beta u'(-a) \\
\end{pmatrix}
\]

is given by

\[
(5.5) \quad D = (u'(-a))^N \left[ Nu(-a) - \beta u'(-a) \right].
\]
Hence we conclude that \( \lambda \) is an eigenvalue of \( L^\beta \) if, and only if, either \( u'(-a) = 0 \) (and multiplicity is \( N - 1 \) in this case) or \( Nu(-a) - \beta u'(-a) = 0 \).

Notice that two terms cannot be zero simultaneously since \( u(-a) = u'(-a) = 0 \) implies \( u \equiv 0 \).

**Step 2.** We analyze negative zeroes of \( D \). Notice that \( D \) can be expressed as
\[
D = -\beta(u(-a, \lambda))^N F(\lambda)^{N-1} (F(\lambda) - \frac{N}{\beta}),
\]
where
\[
F(\lambda) := \frac{u'(-a, \lambda)}{u(-a, \lambda)}; \quad (-\infty, 0] \to \mathbb{R}.
\]

By [34, Lemma 4.1, Lemma 4.5, Remark 4.6], we get:
- \( \lim_{\lambda \to -\infty} F(\lambda) = -\infty \);
- if \( \beta > 0 \), then there exists a unique pole \( \lambda_* \in (-\infty, 0] \) of \( F(\lambda) \) and
\[
\lim_{\lambda \to \lambda_*^-} F(\lambda) = +\infty, \quad \lim_{\lambda \to \lambda_*^+} F(\lambda) = -\infty,
\]
moreover, \( F(\lambda) \) is increasing on \( (-\infty, \lambda_*) \) and \( (\lambda_*, 0] \);
- if \( \beta < 0 \), then \( F(\lambda) \) increases and is of class \( C^1 \) on \( (-\infty, 0) \).

It is easily seen that \( u(x, 0) = \phi'_\omega(x) \), where \( \phi_\omega(x) \) is defined by (3.6), and
\[
F(0) - \frac{N}{\beta} = \frac{\beta \omega}{2N} \left( \frac{(p+1)N^2}{\beta^2 \omega} - (p-1) \right) - \frac{N}{\beta} = \frac{\beta \omega (p-1)}{2N} \left( \frac{N^2}{\beta^2 \omega} - 1 \right).
\]

Since \( \omega > \frac{N^2}{\beta^2 \omega} \), we get that \( F(0) > \frac{N}{\beta} \) for \( \beta < 0 \), and \( F(0) < \frac{N}{\beta} \) for \( \beta > 0 \). Then, using properties of \( F(\lambda) \) listed above, we conclude that there exists a unique \( \tilde{\lambda} < 0 \) such that \( F(\tilde{\lambda}) = \frac{N}{\beta} = 0 \). Moreover, we have (see the graph of \( F(\tilde{\lambda}) \) for the different values of \( \beta \) and \( \omega \) on Figures 1-4 below):

- \( \beta > 0 \) \quad \Rightarrow \quad \begin{cases} F(0) > 0 & \text{for } \omega \leq \frac{p+1}{p-1} \frac{N^2}{\beta^2}, \\ F(0) \leq 0 & \text{for } \omega \geq \frac{p+1}{p-1} \frac{N^2}{\beta^2}. \end{cases}
- \( \beta < 0 \) \quad \Rightarrow \quad \begin{cases} F(0) \leq 0 & \text{for } \omega \leq \frac{p+1}{p-1} \frac{N^2}{\beta^2}, \\ F(0) > 0 & \text{for } \omega > \frac{p+1}{p-1} \frac{N^2}{\beta^2}. \end{cases}

Figure 1: Graph of \( F(\lambda) \) for \( \beta > 0 \) and \( \omega < \frac{p+1}{p-1} \frac{N^2}{\beta^2} \)

Figure 2: Graph of \( F(\lambda) \) for \( \beta > 0 \) and \( \omega \geq \frac{p+1}{p-1} \frac{N^2}{\beta^2} \)
Again, using properties of $F(\lambda)$, we conclude that:

- $\beta < 0 \Rightarrow$ there exists a unique negative zero of $F(\lambda)$ for $\omega > \frac{p+1}{p-1} \frac{N^2}{\beta^2}$, and for $\omega \leq \frac{p+1}{p-1} \frac{N^2}{\beta^2}$ we do not have negative zeroes of $F(\lambda)$.

- $\beta > 0 \Rightarrow$ there exists a unique negative zero of $F(\lambda)$ for $\omega \geq \frac{p+1}{p-1} \frac{N^2}{\beta^2}$, and for $\omega < \frac{p+1}{p-1} \frac{N^2}{\beta^2}$ we have two negative zeroes of $F(\lambda)$.

Finally, summarizing and noticing that negative zeros of $F(\lambda)$ are zeros of the determinant $D$ of multiplicity $N - 1$, we conclude:

- $\beta < 0 \Rightarrow$ for $\omega \leq \frac{p+1}{p-1} \frac{N^2}{\beta^2}$ we have $n(L_1^\beta) = 1$, and for $\omega > \frac{p+1}{p-1} \frac{N^2}{\beta^2}$ we have $n(L_1^\beta) = N$.

- $\beta > 0 \Rightarrow$ for $\omega < \frac{p+1}{p-1} \frac{N^2}{\beta^2}$ we have $n(L_1^\beta) = 2N - 1$, and for $\omega \geq \frac{p+1}{p-1} \frac{N^2}{\beta^2}$ we have $n(L_1^\beta) = N$.

**Step 3.** We calculate $n(L_2^\beta)$ for $\beta > 0$. Analogously to the previous case, the number of negative eigenvalues of $L_2^\beta$ is determined by the number of zeroes of determinant (5.5), where $u(x)$ is the solution to

$$-u'' + \omega u - \frac{(p+1)\omega}{2} \text{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2} x \right) u = \lambda u.$$ 

In this case, the function $F(\lambda)$ increases and is of class $C^1$ on $(-\infty, 0]$, and $\lim_{\lambda \to -\infty} F(\lambda) = -\infty$. The absence of the poles follows from the Sturm oscillation theorem since $u(x, 0) = \phi_\omega(x)$ defined by (3.6) (which is nonzero function). The rest of the properties follow analogously (see [34, Lemma 4.5]). Observing that $F(0) = \frac{N}{\beta}$, we get the existence of a unique negative zero of $F(\lambda)$, therefore, $n(L_2^\beta) = N - 1$. 

To prove orbital instability part in Theorem 5.2, we apply the following abstract result by Henry, Perez, and Wreszinski [29].

**Theorem 5.8.** Let $X$ be a Banach space and let $U \subset X$ be an open set containing 0. Suppose that $T : U \to X$ is such that $T(0) = 0$ and, for some $p > 1$ and $L \in B(X)$ with spectral radius $r(L) > 1$, the relation holds

$$\|T(x) - Lx\|_X = O(\|x\|^p_X) \quad \text{as} \quad x \to 0.$$ 

Then 0 is unstable as a fixed point of $T$: that is, there is $\eta_0 > 0$ such that for all $\varepsilon > 0$, there are $N_0 \in \mathbb{N}$ and $x_0$ with $\|x_0\|_X \leq \varepsilon$ such that $\|T^{N_0}(x_0)\|_X \geq \eta_0$. Moreover, if $\gamma$ is a $C^1$ curve of fixed points of $T$ such that $0 \in \gamma$, then $\gamma$ is unstable.
The proof of the above theorem might be found in [29, Section 2]. Actually on practice the corollary below appears to be useful.

**Corollary 5.9.** Let $X$ be a Banach space, $\phi \in X$, and let $\tilde{U} \subset X$ be an open set containing $\phi$. Suppose that $\tilde{T} : \tilde{U} \to X$ is $C^2$ mapping satisfying $\tilde{T}(\phi) = \phi$. If there is an element $\mu \in \sigma\left(\tilde{T}'(\phi)\right)$ with $|\mu| > 1$, then $\phi$ is an unstable fixed point of $\tilde{T}$. Moreover, if $\gamma$ is a $C^1$ curve of fixed points of $\tilde{T}$ such that $\phi \in \gamma$, then $\gamma$ is unstable.

**Proof.** The proof was established in [10, Corollary 3.1]. We repeat it for convenience of the reader.

Let $x \in U := \{y - \phi : y \in \tilde{U}\}$, and $T(x) := \tilde{T}(x + \phi) - \phi$. Then, we have $T(0) = \tilde{T}(\phi) - \phi = 0$, and $1 < |\mu| \leq r(\tilde{T}'(\phi))$. By the Taylor formula, $T(x) = T(0) + T'(0)x + O\left(\|x\|_X^2\right) = \tilde{T}'(\phi)x + O\left(\|x\|_X^2\right)$ as $x \to 0$. Therefore, by Theorem 5.8 with $L = \tilde{T}'(\phi)$, there is $\eta_0 > 0$ such that for all $\varepsilon > 0$ there exist $N_0 \in \mathbb{N}$ and $y_0$ with $\|y_0 - \phi\|_X \leq \varepsilon$ such that $\|\tilde{T}^{N_0}(y_0) - \phi\|_X \geq \eta_0$. To conclude, one observes that, by the last assertion of Theorem 5.8, we get the existence of $\eta_0 > 0$ such that for all $\varepsilon > 0$ there exist $N_0 \in \mathbb{N}$ and $y_0$ with $\|y_0 - \phi\|_X \leq \varepsilon$ such that $\text{dist}(\tilde{T}^{N_0}(y_0), \gamma) \geq \eta_0$. □

**Proof of Theorem 5.2. Step 1.** Firstly, we comment on the proof of spectral instability. Observe that, by (5.3), it is sufficient to prove that there exist $\lambda > 0$ and $w_1, w_2 \in \text{dom}(\Delta_\beta)$ such that

$$
\begin{pmatrix}
0 & L_2^\beta \\
-L_1^\beta & 0
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix}
= \lambda
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix}.
$$

[28, Theorem 1.2] states that the number $I(L_1^\beta, L_2^\beta)$ of positive $\lambda$ satisfying (5.6) is estimated by

$$n(PL_1^\beta) - n(PL_2^\beta)^{-1} \leq I(L_1^\beta, L_2^\beta),$$

where $P$ is the orthogonal projection on $\ker(L_2^\beta) = \{v \in L^2(\Gamma) : (v, \phi_\beta) = 0\}$ (here we assume that $L^2(\Gamma)$ is endowed with the usual complex scalar product).

By spectral properties of $L_1^\beta, L_2^\beta$ proved in Proposition 5.6 and 5.7, we conclude the existence of such $\lambda > 0$. In particular, for $\beta$ and $\omega$ from the statement of Theorem 5.2 one gets $I(L_1^\beta, L_2^\beta) \geq N - 1$.

**Step 2.** To show orbital instability for $p > 2$ we use Corollary 5.9. Let $u(t, x)$ be a solution of (1.1). Set $u(t, x) = e^{i\omega t}v(t, x)$, then $v(t, x)$ satisfies

$$i\partial_t v(t, x) = -\Delta_\beta v(t, x) + \omega v(t, x) - |v(t, x)|^{p-1}v(t, x) = S'_\omega(v(t, x)).$$

Define $\tilde{T} : H^1(\Gamma) \to H^1(\Gamma)$ by $\tilde{T}(y) = v_0(\frac{\omega}{\omega'}, x)$, where $v_0(t, x)$ is a solution of (5.8) with initial value $v_0(0, x) = y(x) \in H^1(\Gamma)$. Notice that $\tilde{T}(\phi_\beta) = \phi_\beta$ (since $\phi_\beta$ is the equilibrium point for (5.8)). By Proposition 2.1-(iii), $\tilde{T}$ is $C^2$ mapping for $p > 2$. It is easily seen that $\tilde{T}'(\phi_\beta)z = v_0(\frac{2\omega}{\omega'}, x)$, where $v_0(t, x)$ is the solution of the initial-value problem

$$\partial_t v(t, x) = -iS'_\omega(\phi_\beta)v(t, x), \quad v(0, x) = z(x).$$

By Step 1, we conclude that there exists $\lambda > 0$ such that $\mu = e^\lambda \in \sigma(\tilde{T}'(\phi_\beta))$ (hence $|\mu| > 1$). Thus, by Corollary 5.9, $\phi_\beta$ is nonlinearly unstable, that is, there is $\eta_0 > 0$ such that
for all $\varepsilon > 0$ there exist $N_0$ and $y_0$ with $\|y_0 - \phi_\beta\|_{H^1} \leq \varepsilon$ such that $\|\hat{T}^{N_0}(y_0) - \phi_\beta\|_{H^1} \geq \eta_0$, or

$$\|v_{y_0}(\frac{2\pi}{\omega}N_0) - \phi_\beta\|_{H^1} = \|u_{y_0}(\frac{2\pi}{\omega}N_0) - \phi_\beta\|_{H^1} \geq \eta_0,$$

where $u_{y_0}(t, x)$ is the solution to (1.1) with initial value $y_0$.

Finally, observe that $\gamma = \{e^{i\theta}\phi_\beta : \theta \in [0, 2\pi]\}$ is $C^1$ curve of fixed points of $\hat{T}$. Then $\gamma$ is nonlinearly unstable as well, that is, there is $\eta_0 > 0$ such that for all $\varepsilon > 0$ there exist $N_0$ and $y_0$ with $\|y_0 - \phi_\beta\|_{H^1} \leq \varepsilon$ such that

$$\inf_{\theta \in [0, 2\pi]} \|\hat{T}^{N_0}(y_0) - e^{i\theta}\phi_\beta\|_{H^1} = \inf_{\theta \in [0, 2\pi]} \|v_{y_0}(\frac{2\pi}{\omega}N_0) - e^{i\theta}\phi_\beta\|_{H^1} \geq \eta_0.$$

Thus, $e^{i\omega t}\phi_\beta$ is orbitally unstable. \hfill $\square$

**Remark 5.10.** The idea of the proof of [28, Theorem 1.2] is to consider the solutions of the problem

$$\left(PL_1^\beta + \lambda^2P(L_2^\beta)^{-1}\right)v = 0, \quad v \in \ker(L_2^\beta)\perp,$$

instead of the ones of (5.6). The above equation appears naturally noticing that (5.6) is equivalent to $\{L_2^\beta w_2 = \lambda w_1, L_1^\beta w_1 = -\lambda w_2\}$. Since $\text{ran}(L_2^\beta) \perp \ker(L_2^\beta)$, we get $w_1 \in \ker(L_2^\beta)\perp$. Hence $w_2 = \lambda(L_2^\beta)^{-1}w_1$ and $L_1^\beta w_1 + \lambda^2(L_2^\beta)^{-1}w_1 = 0$. The projection $P$ serves to fit the problem into the Hilbert space $\ker(L_2^\beta)\perp$.

**Remark 5.11.** Consider the NLS-$\delta$ equation on the line:

$$i\partial_t u(t, x) = -\partial_x^2 u(t, x) - \gamma \delta(x) u(t, x) - |u(t, x)|^{p-1} u(t, x),$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\gamma \in \mathbb{R} \setminus \{0\}$. Equation (5.9) has the standing wave solution $u(t, x) = e^{i\omega t} \phi_\gamma(x)$, where

$$\phi_\gamma(x) = \left\{ \frac{1}{2} \frac{(p+1)\omega}{2} \text{sech}^2 \left( \frac{\omega - \sqrt{\omega^2 - \omega^2}}{2} |x| \right) + \text{arctanh} \left( \frac{\omega}{2\sqrt{\omega^2 - \omega^2}} \right) \right\}^{\frac{1}{p-1}}.$$

Extensive study of the orbital stability of $e^{i\omega t} \phi_\gamma(x)$ was made in [22, 23, 37]. The only case which was left open was for $3 < p < 5$, $\gamma < 0$, and $\omega = \omega_2 > \frac{\sqrt{2}}{4}$, where $\partial_\omega \|\phi_\gamma\|_2^{\omega=\omega_2} = 0$. Using the arguments from the proof of Theorem 5.2, one may prove that $e^{i\omega t} \phi_\gamma(x)$ is orbitally unstable in that case. Indeed, in [37, Lemma 12] it was shown that $n(L_1^\gamma) = 2$. Observing that $n(L_2^\gamma) = 0$ and $\ker(L_2^\gamma) = \text{span}\{\phi_\gamma\}$ (the operators $L_1^\gamma, L_2^\gamma$ are the counterparts of $L_1^\beta, L_2^\beta$, by (5.7), we get $I(L_1^\gamma, L_2^\gamma) \geq 1$. Using the fact that the mapping data-solution $u_0 \in H^1(\mathbb{R}) \mapsto u(t) \in C([0, T_0], H^1(\mathbb{R}))$ for (5.9) is at least of class $C^2$ for $p > 2$, we get the result.

5.2. Stability analysis for symmetric profile $\phi_\beta$ via Grillakis-Shatah-Strauss (GSS) approach in the case $\beta > 0$. The case of $\beta < 0$ was studied in the framework of Grillakis-Shatah-Strauss approach [7]. In this subsection we complete the results of the previous subsection having repulsive $\delta'$ coupling.
Theorem 5.12. Let \( \beta > 0, \omega > \frac{N^2}{\beta^2}, \omega \neq \frac{p + 1}{p - 1} \frac{N^2}{\beta^2}, \) and \( N \geq 2, \) then

(i) for \( 1 < p \leq 3 \) the standing wave \( e^{i\omega t} \varphi _{\beta} \) is orbitally stable in \( H^1_{eq}(\Gamma) \);

(ii) for \( 3 < p < 5 \) there exists \( \omega^* > \frac{N^2}{\beta^2} \) such that the standing wave \( e^{i\omega t} \varphi _{\beta} \) is orbitally unstable in \( H^1(\Gamma) \) as \( \omega < \omega^* \) and orbitally stable in \( H^1_{eq}(\Gamma) \) as \( \omega > \omega^* \);

(iii) for \( p \geq 5 \) the standing wave \( e^{i\omega t} \varphi _{\beta} \) is orbitally unstable in \( H^1(\Gamma) \).

Let \( \mathcal{R} : H^1(\Gamma) \to (H^1(\Gamma))^\prime \) be the Riesz isomorphism. Principal role in GSS approach is played by the spectral properties of the operator \( \mathcal{R}^{-1} S''_\omega(\varphi _{\beta}) : H^1(\Gamma) \to H^1(\Gamma) \). We denote \( \mathcal{L}_\beta := \mathcal{R}^{-1} S''_\omega(\varphi _{\beta}). \) Since \( S''_\omega(\varphi _{\beta}) : H^1(\Gamma) \to (H^1(\Gamma))^\prime \) is bounded, the operator \( \mathcal{L}_\beta : H^1(\Gamma) \to H^1(\Gamma) \) is bounded and self-adjoint:

\[
(\mathcal{L}_\beta u, v)_{H^1} = (S''_\omega(\varphi _{\beta})u, v)_{(H^1)^\prime \times H^1} = (S''_\omega(\varphi _{\beta})v, u)_{(H^1)^\prime \times H^1} = (u, \mathcal{L}_\beta v)_{H^1}, \quad u, v \in H^1(\Gamma).
\]

Above we have used symmetry of \( S''_\omega(\varphi _{\beta}) \) and the fact that \( H^1(\Gamma) \) is the real Hilbert space.

One of the crucial assumptions of GSS theory is the particular set of spectral properties of \( \mathcal{L}_\beta \) (see [27, Assumption 3]). The following proposition links spectral properties of the operator \( \mathcal{L}_\beta \) and operator \( \mathcal{L}_\beta \) associated with the bilinear form \( (S''_\omega(\varphi _{\beta})u, v)_{(H^1)^\prime \times H^1} \) in \( L^2(\Gamma) \) (for the proof see [44, Lemma 4.5]).

**Proposition 5.13.** Let operator \( \mathcal{L}_\beta \) be defined by (5.1), then

\[
\ker(\mathcal{L}_\beta) = \ker(L_\beta), \quad n(\mathcal{L}_\beta) = n(L_\beta), \quad \text{and} \quad \inf \sigma_{ess}(\mathcal{L}_\beta) > 0 \implies \inf \sigma_{ess}(L_\beta) > 0.
\]

By (5.2), we conclude that to characterize spectral properties of \( \mathcal{L}_\beta \) it is sufficient to study spectral properties of \( L_1^\beta, L_2^\beta \).

**Lemma 5.14.** Let \( \beta > 0, \omega > \frac{N^2}{\beta^2}, \) then \( n(L_1^\beta) = 1 \) and \( L_2^\beta \geq 0 \) in \( L^2_{eq}(\Gamma) = \{ v \in L^2(\Gamma) : v_1(x) = \ldots = v_N(x) \} \).

**Proof.** To prove \( n(L_1^\beta) = 1 \), notice that in \( L^2_{eq}(\Gamma) \) conditions (5.4) turn into \( Nu(-a) = \beta u'(-a) \), i.e. \( \lambda < 0 \) is an eigenvalue of \( L_1^\beta \) iff \( F(\lambda) = \frac{N}{\beta} \). By the properties of \( F(\lambda) \) for \( \beta > 0 \) and since \( F(0) < \frac{N}{\beta} \), we conclude that there exists a unique \( \lambda_{eq} < 0 \) such that \( F(\lambda_{eq}) = \frac{N}{\beta} \).

Identity \( n(L_2^\beta) = 0 \) follows by the same reasoning and by the fact that \( F(\lambda) \) is the increasing function of class \( C^1 \) on \( (-\infty, 0] \) (see Step 3 in the proof of Proposition 5.7). Finally, by \( \sigma_{ess}(L_2^\beta) = [\omega, \infty) \), we get positivity of \( L_2^\beta \). \( \square \)

By Proposition 5.13, we get \( n(\mathcal{L}_\beta) = 1 \), while, by Proposition 5.6, \( \ker(\mathcal{L}_\beta) = \text{span}\{i\varphi _{\beta}\} \) for \( \omega \neq \frac{p + 1}{p - 1} \frac{N^2}{\beta^2} \), and \( \sigma_{ess}(\mathcal{L}_\beta) = [\omega, \infty) \). Thus, Assumption 3 in [27] holds for \( \omega \neq \frac{p + 1}{p - 1} \frac{N^2}{\beta^2} \), and from [27, Theorem 3], we get:

**Theorem 5.15.** The standing wave solution \( e^{i\omega t} \varphi _{\beta} \) is orbitally stable in \( H^1_{eq}(\Gamma) \) if, and only if, \( \partial^2_{\omega^2} S_{\omega}(\varphi _{\beta}) > 0 \).

Below we study the sign of \( \partial^2_{\omega^2} S_{\omega}(\varphi _{\beta}) \).
Proposition 5.16. Let $\omega > \frac{N^2}{\beta^2}$, $\beta > 0$, and $J(\omega) = \partial_{\omega}^2 S_\omega (\phi_\beta)$.

(i) If $1 < p \leq 3$, then $J(\omega) > 0$.

(ii) If $3 < p < 5$, then there exists $\omega^* > \frac{N^2}{\beta^2}$ such that $J(\omega^*) = 0$, and $J(\omega) < 0$ for $\omega < \omega^*$, while $J(\omega) > 0$ for $\omega > \omega^*$.

(iii) If $p \geq 5$, then $J(\omega) < 0$.

Proof. Notice that due to $S'_\omega (\phi_\beta) = 0$, we get $J(\omega) = \partial_{\omega} \| \phi_\beta \|_2^2$. Recall that $\phi_\beta = (\tilde{\phi}_\beta)_{j=1}^N$ is defined by (3.7). Then, via change of variables, we get

$$\int_0^\infty \tilde{\phi}_\beta^2(x)dx = \left( \frac{p+1}{2} \right)^\frac{2}{p-1} 2\omega \frac{2^\frac{2}{p-1}-\frac{1}{2}}{p-1} \int_{-\frac{N}{\beta \sqrt{\omega}}}^1 (1-t^2)^\frac{2}{p-1} dt.$$ 

The last equality yields

$$J(\omega) = C \omega^{\frac{7-5p}{3(p-1)}} J_1(\omega), \quad C = \frac{N}{p-1} \left( \frac{p+1}{2} \right)^\frac{2}{p-1},$$

where

$$J_1(\omega) = \frac{5-p}{p-1} \int_{-\frac{N}{\beta \sqrt{\omega}}}^1 (1-t^2)^\frac{2}{p-1} dt - \frac{N}{\beta \sqrt{\omega}} \left( 1 - \frac{N^2}{\beta^2 \omega} \right)^{\frac{3-p}{p-1}}.$$ 

Thus,

$$J_1(\omega) = -\frac{N}{\beta \omega^{3/2}} \frac{3-p}{p-1} \left[ \left( 1 - \frac{N^2}{\beta^2 \omega} \right)^{\frac{3-p}{p-1}} + \frac{N^2}{\beta^2 \omega} \left( 1 - \frac{N^2}{\beta^2 \omega} \right)^{-\frac{2(p-2)}{p-1}} \right].$$

It is easily seen that $J_1'(\omega) < 0$ as $1 < p < 3$, and $J_1'(\omega) > 0$ as $p > 3$. Observe that for $p < 5$

$$\lim_{\omega \to \infty} J_1(\omega) = \frac{5-p}{p-1} \int_0^1 (1-t^2)^\frac{3-p}{p-1} dt =: a_0 > 0, \quad \lim_{\omega \to \frac{N^2}{\beta^2}} J_1(\omega) = \left\{ \begin{array}{ll}
2a_0, & p \in (1,3], \\
-\infty, & p \in (3,5).
\end{array} \right.$$ 

By the above properties of $J_1(\omega)$, we conclude $J_1(\omega) > 0$ for $1 < p \leq 3$, and there exists $\omega^* > \frac{N^2}{\beta^2}$ such that $J_1(\omega^*) = 0$, and $J_1(\omega) < 0$ for $\omega < \omega^*$, while $J_1(\omega) > 0$ for $\omega > \omega^*$. Observing that $J_1(\omega) < 0$ for $p \geq 5$, we conclude the proof. \qed

Now Theorem 5.12 follows from Proposition 5.16 and Theorem 5.15.

Remark 5.17. Let $N$ be even and $\beta < 0$. Using the fact that $\phi_{\frac{N}{2}}$ is the minimizer of problem (3.12) for $\omega > \frac{p+1}{p-1} \frac{N^2}{\beta^2}$, we get from [1, Proposition 6.11] (using GSS approach) the following stability/instability result:

(i) If $1 < p \leq 5$, then for any $\omega > \frac{p+1}{p-1} \frac{N^2}{\beta^2}$, the standing wave $e^{i\omega t} \phi_{\frac{N}{2}}$ is orbitally stable in $H^1_{\frac{N}{2}}(\Gamma)$.

(ii) If $p > 5$, then

1) there exists $\omega_1$ such that $e^{i\omega t} \phi_{\frac{N}{2}}$ is orbitally stable in $H^1_{\frac{N}{2}}(\Gamma)$ for $\frac{p+1}{p-1} \frac{N^2}{\beta^2} < \omega < \omega_1$;

2) there exists $\omega_2 \geq \omega_1$ such that $e^{i\omega t} \phi_{\frac{N}{2}}$ is orbitally unstable in $H^1_{\frac{N}{2}}(\Gamma)$ (and consequently in $H^1(\Gamma)$) for $\omega > \omega_2$. 

5.3. Strong instability results for $\phi_\beta$ and $\phi_N$. In this subsection we complete the previous results introducing particular type of the orbital instability by blow up. The formal definition is the following.

**Definition 5.18.** Standing wave $e^{i\omega t} \phi(x)$ is strongly unstable if for any $\varepsilon > 0$ there exists $u_0 \in H^1(\Gamma)$ such that $\|u_0 - \phi(x)\|_{H^1} < \varepsilon$ and the solution $u(t)$ of (1.1) with $u(0) = u_0$ blows up in finite time.

The strong instability results are the next two theorems.

**Theorem 5.19.** Let $\beta > 0$, $\omega > \frac{N^2}{\beta^2}$, $N \geq 2$, and $p \geq 5$, then the standing wave $e^{i\omega t} \phi_\beta(x)$ is strongly unstable in $H^1(\Gamma)$.

**Theorem 5.20.** Let $\beta < 0$, $p > 5$, $\omega > \frac{N^2}{\beta^2}$. Then the following assertions hold.

(i) Let $N \geq 2$. Assume that $\hat{\xi}(p) \in (0, 1)$ is a unique solution to

$$
\frac{(p-5)N}{2} \int_1^1 (1 - s^2)^{\frac{p-1}{2}} ds = \xi(1 - \xi^2)^{\frac{p-1}{2}}, \quad (0 < \xi < 1),
$$

and define $\omega_3 = \omega_3(p, \beta) = \frac{N^2}{\beta^2 \hat{\xi}(p)}$. Then the standing wave solution $e^{i\omega t} \phi_\beta(x)$ is strongly unstable in $H^1(\Gamma)$ for all $\omega \in [\omega_3, \infty)$.

(ii) Let $N$ be even, then there exists $\omega_4 > \frac{p+1}{p-1} \frac{N^2}{\beta^2}$ such that $e^{i\omega t} \phi_N(x)$ is strongly unstable in $H^1(\Gamma)$ for all $\omega \in [\omega_4, \infty)$.

**Remark 5.21.** Observe that Theorem 5.20-(ii) completes orbital instability result of Remark 5.17(ii). The principal ingredients are virial identity (2.23) and Lemmas 3.3, 3.4(ii).

**Remark 5.22.** Analogously to [11, Theorem 1.4] (i.e. using virial identity and variational characterization) one can prove for the ground state solution $\phi_{\beta,V}$ of equation (3.14) the following result. Assume that $p > 5$, $\beta \in (\beta^*, -\frac{N}{\sqrt{\omega}})$, $\omega > - \inf \sigma(-\Delta_\beta + V)$, and $V(x)$ satisfies assumptions (3.13) and $x V'(x) \in L^1(\Gamma) + L^\infty(\Gamma)$, then condition $\partial^2 E(\phi_{\beta,V})_{|_{\lambda=1}} < 0$ implies orbital instability of the standing wave solution $e^{i\omega t} \phi_{\beta,V}(x)$ of (3.14) in $H^1(\Gamma)$.

5.4. Instability analysis for the asymmetric profiles $\phi_k$. The principal result of this subsection is the following theorem.

**Theorem 5.23.** Let $p > 1$, $\omega > \frac{p+1}{p-1} \frac{N^2}{\beta^2}$, and let the profiles $\phi_k$ be defined in Theorem 4.2 for $k = 1, \ldots, N - 1$.

(i) If $\beta < 0$, then $e^{i\omega t} \phi_k$ is spectrally unstable for $k \geq 2$.

(ii) If $\beta > 0$, then $e^{i\omega t} \phi_k$ is spectrally unstable for $N - k \geq 4$.

If, additionally, $p > 2$, then in all the cases mentioned above we have orbital instability.

To prove the above theorem we repeat algorithm from the previous subsection, that is, firstly we linearize (1.1) at $\phi_k$. We denote the operators analogous to $L^\beta_j$ by $L^\phi_k$, and $\text{dom}(L^\phi_k) = \text{dom}(\Delta_\beta)$. The proof of the proposition below repeats the one of Proposition 5.6-(i), (ii).
Next we study the Morse index of $L^{\phi_k}$.

**Proposition 5.25.** Suppose that $\omega > \frac{(p+1)N^2}{p-1} \beta^2$ and $N \geq 2$. Then

(i) for $\beta > 0$ we have $n(L^{\phi_k}_2) \leq N - 1$ and $n(L^{\phi_k}_1) \geq 2N - k - 3$;

(ii) for $\beta < 0$ we have $L^{\phi_k}_2 \geq 0$ and $n(L^{\phi_k}_1) \geq k$.

**Proof.** Mainly we will use the ideas from the proof of Proposition 5.7.

**Step 1.** Suppose that $\lambda < 0$ is an eigenvalue of $L^{\phi_k}_1$ with an eigenvector $v^\lambda = (v_j^\lambda) \in \text{dom}(L^{\phi_k}_1)$: $L^{\phi_k}_1 v^\lambda = \lambda v^\lambda$. Then, denoting

\[
a_1 = -\text{sign}(\beta) \frac{2}{(p-1)\sqrt{\omega}} \tanh^{-1}(t_1), \quad a_N = -\text{sign}(\beta) \frac{2}{(p-1)\sqrt{\omega}} \tanh^{-1}(t_N),
\]

and rearranging edges of $\Gamma$ (putting $k$ edges corresponding to $t_1$ at the beginning), we get for $x > 0$

\[
(v_j^\lambda)'' + \omega v_j^\lambda - \frac{p(p+1)\omega}{2} \text{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2} (x + a_j) \right) v_j^\lambda = \lambda v_j^\lambda, \quad j = 1, \ldots, k,
\]

\[
(v_j^\lambda)'' + \omega v_j^\lambda - \frac{p(p+1)\omega}{2} \text{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2} (x + a_N) \right) v_j^\lambda = \lambda v_j^\lambda, \quad j = k + 1, \ldots, N.
\]

Thus, $v_j^\lambda = \begin{cases} c_j u(x + a_1), & j = 1, \ldots, k, \\ c_j u(x + a_N), & j = k + 1, \ldots, N, \end{cases}$ where $u(x)$ is the solution on the line to

\[-u'' + \omega u - \frac{p(p+1)\omega}{2} \text{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2} x \right) u = \lambda u.
\]

The coefficients $c_j$ satisfy the system

\[c_1 u'(a_1) = \ldots = c_k u'(a_1) = c_{k+1} u'(a_N) = \ldots = c_N u'(a_N),
\]

\[\sum_{j=1}^k c_j u(a_1) + \sum_{j=k+1}^N c_j u(a_N) = \beta c_N u'(a_N).
\]

The determinant of the matrix associated with the system

\[
\begin{pmatrix}
u'(a_1) & -u'(a_1) & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
u'(a_1) & 0 & -u'(a_1) & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
u'(a_1) & 0 & 0 & \ldots & -u'(a_N) & 0 & \ldots & 0 & 0 \\
u'(a_1) & 0 & 0 & \ldots & 0 & -u'(a_N) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
u'(a_1) & 0 & 0 & \ldots & \cdot & \cdot & \ldots & \cdot & \cdot \\
u'(a_1) & 0 & 0 & \ldots & \cdot & \cdot & \ldots & \cdot & \cdot \\
u'(a_1) & 0 & 0 & \ldots & \cdot & \cdot & \ldots & \cdot & \cdot \\
u'(a_1) & 0 & 0 & \ldots & \cdot & \cdot & \ldots & \cdot & \cdot \\
u'(a_1) & 0 & 0 & \ldots & \cdot & \cdot & \ldots & \cdot & \cdot \\
\end{pmatrix}
\]
is given by

\[(5.10)\]
\[
D = (u'(a_1))^{k-1} (u'(a_N))^{N-k-1} \left[ ku(a_1)u'(u_N) + (N-k)u(a_N)u'(a_1) - \beta u'(a_N)u'(a_1) \right].
\]

Hence we conclude that \( \lambda \) is an eigenvalue of \( L^k \) if, and only if, either \( u'(a_1) = 0 \) (with multiplicity \( k - 1 \)), or \( u'(a_N) = 0 \) (with multiplicity \( N - k - 1 \)), or \( ku(a_1)u'(a_N) + (N-k)u(a_N)u'(a_1) - \beta u'(a_N)u'(a_1) = 0 \).

We rewrite \( D \) as

\[(5.11)\]
\[
D = \left( F_1(\lambda)u(a_1) \right)^{k-1} \left( F_N(\lambda)u(a_N) \right)^{N-k-1} \left[ ku(a_1)u'(a_N) + (N-k)u(a_N)u'(a_1) - \beta u'(a_N)u'(a_1) \right],
\]

where

\[
F_1(\lambda) := \frac{u'(a_1, \lambda)}{u(a_1, \lambda)} : (-\infty, 0] \to \mathbb{R}, \quad F_N(\lambda) := \frac{u'(a_N, \lambda)}{u(a_N, \lambda)} : (-\infty, 0] \to \mathbb{R}.
\]

By [34, Lemma 4.1, Lemma 4.5, Remark 4.6], we get for \( j = 1, N \):

- \( \lim_{\lambda \to -\infty} F_j(\lambda) = -\infty \).
- If \( \beta > 0 \), then there exists a unique pole \( \lambda^j_+ \in (-\infty, 0] \) and

\[
\lim_{\lambda \to \lambda^j_+} F_j(\lambda) = +\infty, \quad \lim_{\lambda \to \lambda^j_-} F_j(\lambda) = -\infty,
\]

moreover, \( F_j(\lambda) \) is increasing on \((-\infty, \lambda^j_+)\) and \((\lambda^j_-, 0]\).

- If \( \beta < 0 \), then \( F_j(\lambda) \) increases and is of class \( C^1 \) on \((-\infty, 0]\).

As in the previous case, observing that \( u(x, 0) = \phi'_\omega(x) \), we get

\[
F_j(0) = \frac{\sqrt{\omega}}{2 \text{sign}(\beta)(p+1)t_j} \left( t^2_j - \frac{p-1}{p+1} \right).
\]

Since \( t_1 < t_N \), we conclude:

- \( \beta < 0 \) \( \Rightarrow \) \( \begin{cases} F_1(0) > 0, \\ F_N(0) < 0. \end{cases} \)
- \( \beta > 0 \) \( \Rightarrow \) \( \begin{cases} F_1(0) < 0, \\ F_N(0) > 0. \end{cases} \)

Using the properties of \( F_j(\lambda) \), we obtain:

- \( \beta < 0 \) \( \Rightarrow \) there exists a unique \( \lambda^1_1 \) such that \( F_1(\lambda^1_1) = 0 \), and \( F_N(\lambda) \) does not have negative zeroes.
- \( \beta > 0 \) \( \Rightarrow \) there exists a unique \( \lambda^1_1 < 0 \) (with \( \lambda^1_1 < \lambda^1_2 \)) such that \( F_1(\lambda^1_1) = 0 \), and \( F_N(\lambda) \) has two negative zeroes \( \lambda^N_1, \lambda^N_2 < 0 \) (with \( \lambda^N_1 < \lambda^N_2 < \lambda^N_2 \)).

Now let \( \beta < 0 \) and \( \lambda^1_1 \) be a unique negative zero of \( F_1(\lambda) \). Consider the function

\[
\tilde{F}(\lambda) = \frac{k}{F_1(\lambda)} + \frac{N-k}{F_N(\lambda)} : (-\infty, \lambda^1_1) \to \mathbb{R}.
\]

From the properties of \( F_1(\lambda) \) and \( F_N(\lambda) \), we conclude that \( \tilde{F}(\lambda) \) is decreasing and

\[
\lim_{\lambda \to -\infty} \tilde{F}(\lambda) = 0-, \quad \lim_{\lambda \to \lambda^1_1} \tilde{F}(\lambda) = -\infty.
\]
Therefore, there exists $\tilde{\lambda} < \lambda_1$ such that $\tilde{F}(\tilde{\lambda}) = \beta$, and the last term in $D$ is zero at $\tilde{\lambda}$ (see the graph of the function $\tilde{F}(\lambda)$ on Figure 5).

![Figure 5: Graph of $\tilde{F}(\lambda)$ for $\beta < 0$ on $(-\infty, \lambda_1)$](image1)

![Figure 6: The case of $\beta > 0$ and $L_2^{\phi_k}$. Solid line is the graph of $1/F_N(\lambda)$, and the dashed line is the graph of $1/F_1(\lambda)$](image2)

Summarizing the above, by (5.11), we deduce:

- $\beta < 0 \implies n(L_2^{\phi_k}) \geq k$.
- $\beta > 0 \implies n(L_1^{\phi_k}) \geq 2N - k - 3$.

**Step 2.** We calculate $n(L_2^{\phi_k})$ for $\beta > 0$. Analogously to the previous case, the number of negative eigenvalues of $L_2^{\phi_k}$ is determined by the number of zeroes of the determinant (5.10), where $u(x)$ is the solution to

$$-u'' + \omega u - \frac{(p + 1)\omega}{2} \text{sech}^2 \left( \frac{(p - 1)\sqrt{\omega}}{2} x \right) u = \lambda u.$$  

In this case, the function $F_j(\lambda), j = 1, N$, increases and is of class $C^1$ on $(-\infty, 0]$ and $\lim_{\lambda \to -\infty} F_j(\lambda) = -\infty$ (this also holds for $\beta < 0$). The absence of the poles follows from the Sturm oscillation theorem since $u(x, 0) = \phi_\omega$. Observe that $F_j(0) = \sqrt{\omega}t_j > 0$, then there exists a unique $\lambda_1^j < 0$ such that $F_j(\lambda_1^j) = 0$. Without loss of generality we may assume that $\lambda_1^j < \lambda_1^N$. Consider

$$\tilde{F}(\lambda) = \frac{k}{F_1(\lambda)} + \frac{N - k}{F_N(\lambda)} : (-\infty, \lambda_1^1) \cup (\lambda_1^1, \lambda_1^N) \cup (\lambda_1^N, 0] \to \mathbb{R}.$$  

The function is decreasing on each interval of its domain, moreover it satisfies (see the graphs of the functions $1/F_1(\lambda)$ and $1/F_N(\lambda)$ on Figure 6):

$$\lim_{\lambda \to -\infty} \tilde{F}(\lambda) = 0-, \quad \lim_{\lambda \to \lambda_1^-} \tilde{F}(\lambda) = -\infty, \quad \lim_{\lambda \to \lambda_1^+} \tilde{F}(\lambda) = +\infty,$$

$$\lim_{\lambda \to \lambda_1^N^-} \tilde{F}(\lambda) = -\infty, \quad \lim_{\lambda \to \lambda_1^N^+} \tilde{F}(\lambda) = +\infty, \quad \tilde{F}(0) = \frac{1}{\sqrt{\omega}} \left( \frac{k}{t_1} + \frac{N - k}{t_N} \right) = \beta.$$  

From the above properties of $\tilde{F}(\lambda)$ it follows that there might exist $\lambda = (\lambda_1^1, \lambda_1^N)$ such that $\tilde{F}(\lambda) = \beta$, and intervals $(-\infty, \lambda_1^1)$ and $(\lambda_1^N, 0]$ do not contain zeroes of $\tilde{F}(\lambda) - \beta$. Therefore, taking into account multiplicity of $\lambda_1^1$, we obtain $n(L_2^{\phi_k}) \leq N - 1$.

For $\beta < 0$ the identity $n(L_2^{\phi_k}) = 0$ follows observing that $F_j(0) = -\sqrt{\omega}t_j < 0$ and $F_j(\lambda)$ is increasing on $(-\infty, 0]$. In particular, there does not exist any negative zero of $\tilde{F}(\lambda) - \beta$.  


Indeed, \( \tilde{F}(\lambda) \) is decreasing on \((-\infty, 0]\) and
\[
\lim_{\lambda \to -\infty} \tilde{F}(\lambda) = 0, \quad \tilde{F}(0) = -\frac{1}{\sqrt{\omega}} \left( \frac{k}{t_1} + \frac{N-k}{t_N} \right) = \beta.
\]
Recalling \( \sigma_{\text{ess}}(L_2^{\phi_k}) = [\omega, \infty) \), we get the positivity of \( L_2^{\phi_k} \).

\[ \square \]

**Remark 5.26.** For \( \beta < 0 \) and symmetric profile \( \phi_\beta \), positivity of the operator \( L_2^\beta \) was proven in [7, Proposition 3.24] using the Jensen inequality for the function \( f(x) = x^2 \).

**Proof of Theorem 5.23.** As in the previous case, the principal ingredient of the proof is to check the existence of \( \lambda > 0 \) and \( w_1, w_2 \in \text{dom}(\Delta_\beta) \) such that
\[
(5.12) \quad \begin{pmatrix} 0 & L_2^{\phi_k} \\ -L_1^{\phi_k} & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.
\]
By [28, Theorem 1.2] and the spectral properties of \( L_1^{\phi_k}, L_2^{\phi_k} \) proved in Proposition 5.24 and 5.25, we conclude the existence of such \( \lambda > 0 \) for \( k \) and \( \beta \) satisfying the statement of Theorem 5.23. In particular, by [28, Theorem 1.2], the number \( I(L_1^{\phi_k}, L_2^{\phi_k}) \) of \( \lambda > 0 \) satisfying (5.12) is estimated by
\[
(5.13) \quad n(PL_1^{\phi_k}) - n(PL_2^{\phi_k})^{-1} \leq I(L_1^{\phi_k}, L_2^{\phi_k}),
\]
where \( P \) is the orthogonal projection on \( \ker(L_2^{\phi_k}) = \{ v \in L^2(\Gamma) : (v, \phi_k)_2 = 0 \} \). As before, we assume that \( L^2(\Gamma) \) is endowed with the usual complex scalar product. From inequality (5.13) and Proposition 5.25 we get:

- \( \beta < 0 \) \( \Rightarrow \) \( I(L_1^{\phi_k}, L_2^{\phi_k}) \geq k - 1 - 0 = k - 1 \).
- \( \beta > 0 \) \( \Rightarrow \) \( I(L_1^{\phi_k}, L_2^{\phi_k}) \geq 2N - k - 3 - 1 - N + 1 = N - k - 3 \).

Observe that “\(-1\)” in the estimates above appears due to the fact that \( P \) is the orthogonal projection onto the subspace of co-dimension 1. The proof of the orbital instability for \( p > 2 \) repeats the one in the case of the profile \( \phi_\beta \).

\[ \square \]

**Remark 5.27.** It is impossible to apply the methods used to prove [25, Theorem 1.3 and Theorem 1.4] to obtain some strong instability results for profiles \( \phi_k(x) \) different from \( \phi_\beta(x) \) and \( \phi_{\frac{N}{2}}(x) \) since we do not have variational characterization of them.

We finish this paper by

**Proof of Theorem 4.4.** By Proposition 4.1, the solution to (3.2) is a critical point of \( S_\omega \). On the other hand, by Theorem 4.2, the set of critical points is given by \( \phi_\beta, \phi_k, k = 1, \ldots, N - 1 \). We call the minimizer \( \phi \). Then we get for \( v = v_1 + iv_2 \in H^1(\Gamma) \)
\[
(5.14) \quad \langle S'_\omega(\phi)v, v \rangle_{H^1 \times H^1} = t_1^\phi(v_1) + t_2^\phi(v_2),
\]
where \( t_j^\phi, j = 1, 2 \), is the quadratic form associated with the self-adjoint operator \( L_j^\phi \). Using the Implicit Function Theorem (see, for instance, the proof of [22, Lemma 29]), it can be proven that \( \langle S''_\omega(\phi)v, v \rangle_{H^1 \times H^1} \geq 0 \) on the subspace \( I' = \{ v \in H^1(\Gamma) : I'_\omega(\phi)v = 0 \} \) of codimension one. Thus, we have \( t_j^\phi \geq 0 \) on the subspace of codimension one. Indeed, for the real-valued function \( w \in I' \) we have \( t_1^\phi(w) = \langle S''_\omega(\phi)w, w \rangle_{H^1 \times H^1} \). Therefore,
$L_1^\phi$ has at most one negative eigenvalue. In fact, it has one negative eigenvalue since $(L_1^\phi, \phi)_2 = -(p - 1)\|\phi\|_{p+1}^{p+1} < 0$. By Proposition 5.7-(i), 1) and Proposition 5.25-(ii), the only candidate for the minimizer is $\phi_1$, and $n(L_1^\phi) = 1$.

Acknowledgement I would like to thank the anonymous reviewer whose valuable comments and suggestions helped to improve this manuscript.

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