A piece of Victor Katsnelson’s mathematical biography

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Abstract

We give an overview of several works of Victor Katsnelson published in 1965–1970, and pertaining to the complex and harmonic analysis and the spectral theory.

A preamble

As a mathematician, Victor Katsnelson was raised within a fine school of function theory and functional analysis, which was blossoming in Kharkov starting the second half of 1930s. He studied in the Kharkov State University in 1960-1965. Among his teachers were Naum Akhiezer, Boris Levin, Vladimir Marchenko. That time he became acquainted with Vladimir Matsaev whom Victor often mentions as one of his teachers. In 1965 Katsnelson graduated with the master degree, Boris Levin supervised his master thesis. Since then and till 1990, he teaches at the Department of Mathematics and Mechanics of the Kharkov State University. In 1967 he defends the PhD Thesis “Convergence and Summability of Series in Root Vectors of Some Classes of Non-Selfadjoint Operators” also written under Boris Levin guidance. Until he left Kharkov in the early 1990s, Katsnelson remained an active participant of the Kharkov function theory seminar run on Thursdays by Boris Levin and Iossif Ostrovskii. His talks, remarks and questions were always interesting and witty.

Already in the 1960s Victor established himself among the colleagues as one of the
finest Kharkov mathematicians of his generation, if not the finest one. Nevertheless, he was not appointed as a professor and was never allowed to travel abroad.

Most of Katsnelson’s work pertain to the spectral theory of functions and operators. I will touch only a handful of his results, mostly published in 1965–1970, that is, at the very beginning of his mathematical career. A big portion of his works written in Kharkov appeared in the local journal “Function Theory, Functional Analysis and Their Applications” and were never translated in English. Today, the volumes of this journal are available at http://dspace.univer.kharkov.ua/handle/123456789/43.

In this occasion, let me mention two wonderful books carefully written by Katsnelson [18, 19]. They exist only as manuscripts, and curiously, both have “Part I” in their titles, though, as far as I know, no continuations appeared. In both books mathematics interlaces with interesting historical comments. Last but not least, let me also mention an extensive survey of Issai Schur’s works in analysis written jointly by Dym and Katsnelson [7].

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1 A Paley-Wiener-type theorem

The paper [14] was, probably, the first published work of Katsnelson. Therein, he studied the following question raised by Boris Levin. Given a convex compact set $K \subset \mathbb{C}$ with the boundary $\Gamma = \partial K$, let $L^2(\Gamma)$ be the $L^2$-space of function on $\Gamma$ with respect to the Lebesgue length measure. How to characterize entire functions $F$ represented by the Laplace integral

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} f(w) e^{wz} \, dw,$$

(1.1)
with \( f \in L^2(\Gamma) \)?

In the case when \( K = \Gamma \) is an interval, the answer is provided by the classical Paley-Wiener theorem. In this case, it is convenient to assume that \( \Gamma \subset i\mathbb{R} \). Then we can rewrite (1.1) as follows

\[
F(z) = \frac{1}{2\pi i} \int_a^b f(w)e^{wz} \, dw = \frac{1}{2\pi} \int_a^b \varphi(t)e^{itz} \, dt, \quad \varphi \in L^2(a, b),
\]

and, by the Paley-Wiener theorem, a necessary and sufficient condition for this representation with some \( a < b \) is that \( F \) is an entire function of exponential type (EFET, for short) and \( F \in L^2(\mathbb{R}) \).

Now, assume that the convex compact \( K \) is not an interval, that is, is a closure of its interior, and put \( \Omega_K = \mathbb{C} \setminus K \). Note that the Laplace transform of \( F \) coincides with the Cauchy integral of \( f \):

\[
\int_0^\infty F(z)e^{-\lambda z} \, dz = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{\lambda - w} \, dw.
\]

The RHS is analytic in \( \Omega_K \), vanishes at infinity, and belongs to the Smirnov space \( E^2(\Omega_K) \), which can be defined, for instance, as the closure in \( L^2(\Gamma) \) of analytic functions in \( \Omega_K \), continuous up to the boundary, and vanishing at infinity. Thus, Levin’s question can be reformulated as follows: Given a convex compact set \( K \) with non-empty interior, find a complete normed space \( B_K \) of EFET such that the Laplace integral \( L \) defined in (1.1) gives a bounded bijection \( E^2(\Omega_K) \to B_K \). Note that the representation (1.1) yields that all functions \( F \in B_K \) have the growth bound

\[
|F(re^{i\theta})| \leq C(\Gamma) \|f\|_{L^2(\Gamma)} \exp\left[\max_{w \in K} \Re(we^{i\theta})r\right] = C(\Gamma) \|f\|_{L^2(\Gamma)} e^{h_K(-\theta)r},
\]

where \( h_K(\theta) \) is the supporting function of \( K \).

The first result in that direction is due to Levin himself who considered in [24, Appendix I, Section 3] the case when \( K \) is a convex polygon and noticed that in this case the answer is a straightforward consequence of the classical version of the Paley-Wiener theorem. Then, M. K. Liht [26] considered the case when \( K \) is a disk centered at the origin and of radius \( h \). He showed that in this case one can take \( B_K \).
being a Bargmann-Fock-type space, which consists of entire functions $F$ satisfying
\[ \int_0^\infty \int_{-\pi}^\pi |F(re^{i\theta})|^2 e^{-2hr} \sqrt{r} \, dr \, d\theta < \infty. \]

The starting point of Katsnelson’s work \cite{Katsnelson} was a remark that a more accurate version of the Liht argument yields an isometry
\[ \int_{\Gamma} |f|^2 |dw| = \int_0^\infty \int_{-\pi}^\pi |F(re^{i\theta})|^2 e^{-2hr} \rho(hr) \, dr \, d\theta, \]
where
\[ \rho(r) = 2r \int_0^\infty \frac{e^{-2tr}}{\sqrt{2t + t^2}} \, dt. \]
Then, Katsnelson proves that representation (1.1) yields a uniform bound
\[ \sup_{|\theta| \leq \pi} \int_0^\infty |F(re^{i\theta})|^2 e^{-2hK(-\theta)r} \, dr \leq C(\Gamma) \|f\|_{L^2(\Gamma)}. \]

The proof is based on the following lemma close in the spirit to known estimates due to Gabriel and Carlson.

**Lemma 1.1.** Suppose that $K$ is a convex compact set, $\Gamma = \partial K$, $\Omega_K = \mathbb{C} \setminus K$, and $f \in E^2(\Omega_K)$. Then, for any supporting line $\ell$ to $\Gamma$,
\[ \int_{\ell} |f|^2 |dw| \leq C(\Gamma) \int_\Gamma |f|^2 |dw|. \]
The constant on the RHS does not depend on $\ell$ and $f$.

One can modify Levin’s question replacing the space $E^2(\Omega_K)$ by another space of functions analytic in $\Omega_K$. If functions in that space do not have boundary values on $\Gamma$, then one needs to replace the integral over $\Gamma$ on the RHS of (1.1) by the contour integral
\[ \frac{1}{2\pi i} \int_\gamma f(w)e^{\omega z} \, dw, \]
where $\gamma$ is a simple closed contour in $\Omega_K$, which contains $K$ in its interior. This integral is called the Borel transform of $f$. It acts on the Taylor coefficients as follows:
\[ f(w) = \sum_{n \geq 0} \frac{a_n}{n^{n+1}} \mapsto F(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n. \]
One of possible modifications of Levin’s question was considered in [14]. Kat-
snelson introduces the weight
\[ \rho_\Gamma(w) = \frac{1}{|w - a_1(w)| + |w - a_2(w)|}, \]
where \(a_j(w), j = 1, 2\), are supporting points for the line supporting to \(\Gamma\) that passes through \(w \in \Omega_K\) (the weight \(\rho_\Gamma(w)\) is not defined when \(w\) belongs to the supporting line to \(\Gamma\) that has a common segment with \(\Gamma\)). The last result proven in [14] is a curious isometry
\[ \int_0^\infty \int_{-\pi}^\pi |F(re^{i\theta})|^2 e^{-2h_K(-\theta)r} \, dr \, d\theta = \frac{1}{2\pi} \int_{\Omega_K} |f(w)|^2 \rho_\Gamma(w) \, d\sigma(w), \]
where \(\sigma\) is the Lebesgue area measure.

Works of Liht and Katsnelson had follow-ups. In [30], Lyubarskii extended Liht’s theorem to convex compact sets \(K\) with smooth boundary. The decisive word was said by Lutsenko and Yulmukhametov. In [29] they proved that the Laplace integral \(L\) defines an isomorphism between \(E^2(\Omega_K)\) and a space of EFET such that
\[ \int_0^\infty \int_{-\pi}^\pi |F(re^{i\theta})|^2 \frac{dr \, d\Delta(\theta)}{K(re^{i\theta})}, \]
where
\[ K(z) = \|e^{wz}\|^2_{E^2(\Omega_K)} = \int_G e^{2\text{Re}(wz)} |dw|, \]
and \(d\Delta(\theta) = (h''(-\theta) + h(-\theta))d\theta\) (understood as a distribution). One of the novelties in their work is the fact that the identity map provides an isomorphism between the Smirnov space \(E^2(\Omega_K)\) and the space of analytic functions in \(\Omega_K\) vanishing at infinity, with finite Dirichlet-type integral
\[ \iint_{\Omega_K} |f'(w)|^2 \text{dist}(w, \Gamma) \, d\sigma(w) < \infty. \]
The proof of that fact relies on Lemma 1.1. We mention that Yulmukhametov together with his pupils and collaborators proved several other non-trivial results

\footnote{Understood as an isomorphism between Banach spaces}
related to Levin’s question (see, for instance, [28, 13]) and that Lindholm [27] extended the Lutsenko-Yulmukhametov theorem to analytic functions of several complex variables.

2 Riesz bases of eigenvectors of non-selfadjoint operators

One of the central questions in the spectral theory is the expansion in eigenfunctions (more generally, of root vectors) of non-selfadjoint operators. It originates in the theories of ordinary and partial differential equations and of integral equations. In the middle of the 1960s the corresponding completeness problem was already understood relatively well, first of all, due to the pioneering works by Keldysh and Matsaev. A portion of their works can be found in the classical Gohberg-Krein book [10], another portion became available later in [22] and in [31, 32]. The situation with convergence of the series of eigenfunctions was understood much less clear. Though a few results, due to Glazman, Mukminov, and Markus, were known (all of them were summarized in [10, Chapter VI]), no general methods existed until in [15] Katsnelson discovered a novel approach to the Riesz basis property of eigenfunctions of arbitrary contractions and dissipative operators. His approach is based on a deep result of Carleson pertaining to the interpolation by bounded analytic functions in the unit disk.

We start with some definitions. First, we remind the notion of Riesz basis of a system of subspaces \( (X_k) \) of a Hilbert space \( H \). The details can be found in [10, Chapter VI]. In the case when all subspaces \( (X_k) \) are one-dimensional, this notion reduces to the usual notion of the Riesz basis of vectors in \( H \).

Let \( (X_k) \) be a collection of linear subspaces of \( H \), and \( X \) be the closure of their linear span. The subspaces \( (X_k) \) form a basis in \( X \) if any vector \( x \in X \) has a unique decomposition into a convergent series \( x = \sum_k x_k, \quad x_k \in X_k \). To simplify notation, we assume that the linear span of the subspaces \( (X_k) \) is dense in \( H \), i.e., that \( X = H \). Let \( P_k \) be projectors on \( X_k \). Then the system \( (X_k) \) forms a basis if and only if \( P_k P_j = 0 \) for all \( k \neq j \).
\( \delta_{kj}P_k \), and \( \sup_n \left\| \sum_{k=1}^n P_k \right\| < \infty \). The subspaces \((X_k)\) form an orthogonal basis if all \(P_k\)s are orthogonal projectors, that is, for any \(x\), \(\|x\|^2 = \|P_k x\|^2 + \|(1-P_k)x\|^2\). The subspaces \((X_k)\) form a Riesz basis if there exists an invertible operator \(A\) from \(H\) onto \(H\) such that subspaces \((AX_k)\) form an orthogonal basis. Gelfand’s theorem \([10, \text{Chapter VI, } \S \ 5]\) says that a basis of subspaces \((X_k)\) is a Riesz basis if and only if it remains a basis after any permutation of its elements.

In \([15]\) Katsnelson studies the question when the collection of root subspaces of a non-selfadjoint operator is a Riesz basis in the closure of its linear span. He considers two general classes of non-selfadjoint operators, contractions and dissipative operators. A linear operator \(T\) on a Hilbert space \(H\) is called a contraction if \(\|T\| \leq 1\). Given an eigenvalue \(\lambda \in \overline{D}\), the linear space

\[
X(\lambda) = \bigcup_{n \geq 1} \text{Ker}[(T - \lambda I)^n]
\]

is called the root subspace corresponding to \(\lambda\). The eigenvalue \(\lambda\) has finite order if there exists a positive integer \(m\) such that

\[
X(\lambda) = \bigcup_{1 \leq n \leq m} \text{Ker}[(T - \lambda I)^n] = \text{Ker}[(T - \lambda I)^m].
\]

The least value \(m\) is called the order \(m(\lambda)\) of the eigenvalue \(\lambda\). The following theorem is the main result of \([15]\).

**Theorem 2.1.** Let \((\lambda_k)\) be some eigenvalues of a contraction \(T\), let \((X_k)\) be the corresponding root subspaces, and let \((m_k)\) be the orders of \((\lambda_k)\). Suppose that

\[
\inf_j \prod_{k \neq j} \left| \frac{\lambda_k - \lambda_j}{1 - \lambda_j \lambda_k} \right|^{m_j m_k} \geq \delta > 0,
\]

Then the system of root subspaces \((X_k)\) forms a Riesz basis in the closure of its linear span.

In \([15]\) Katsnelson only sketches the proof of this result, some details can be found in Nikolskii’s survey paper \([36, \S \ 3]\). Here are the main steps of the proof.
First, Katsnelson observes that when in the assumptions of Theorem 2.1 only those $\lambda_k$ that lie in the open unit disk matter, while the unitary part of the operator $T$ can be discarded. He also assumes that the linear span of the root subspaces $X(\lambda_k)$ is dense in $H$ (otherwise, he considers the restriction of $T$ on the closure of this linear span). Keeping in mind Gelfand’s theorem, it suffices to find projectors $P_j: H \to X(\lambda_j)$ such that $P_jX(\lambda_k) = \{0\}$ for $j \neq k$, and

$$\sup_j \left\| \sum_{j \in J} P_j \right\| < \infty,$$

where the supremum is taken over all finite subsets $J$ of the set of all indices $j$.

Fix a finite set $J$. Suppose that we succeeded to find an analytic in the unit disk function $f_J$ such that $f_J(\lambda_j) = 1$ for $j \in J$, $f_J(\nu) = 0$ for $j \notin J$ and $0 \leq \nu \leq m(\lambda_j) - 1$, and $\sup_{D} |f_J| \leq M$, with a constant $M$ independent of $J$. Suppose momentarily that the function $f_J$ is analytic on a neighbourhood of the closed unit disk (that is, that the set $(\lambda_j)$ is finite). Then, by the F. Riesz operator calculus [38, Chapter IX], $f_J(T)$ is well-defined and, by von Neumann’s theorem [38, Section 153, Theorem A], $\|f_J(T)\| \leq M$.

At the next step, Katsnelson again uses a piece of the F. Riesz operator calculus. The projectors $P_j$ can be defined by the contour integrals

$$P_j = \frac{-1}{2\pi i} \int_{C_j} (T - \zeta I)^{-1} d\zeta,$$

where $C_j$ is a circumference of a small radius which separates the point $\lambda_j$ from the rest of the spectrum and traversed counterclockwise. Whence,

$$f_J(T) = \frac{-1}{2\pi i} \int_T f_J(\zeta)(T - \zeta I)^{-1} d\zeta = \sum_{j \in J} P_j,$$

and therefore,

$$\left\| \sum_{j \in J} P_j \right\| = \|f_J(T)\| = \sup_{D} |f_J| \leq M.$$

To get rid of the assumption that the function $f_J$ is analytic on the neighbourhood of the closed unit disk, Katsnelson applies a classical result due to Pick and
Schur, which says that *given a bounded analytic function* \( f \) *in the unit disk and given a finite set of points* \( \Lambda \subset \mathbb{D} \), *there exists a rational function* \( R \) *which interpolates* \( f \) *at* \( \Lambda \), *that is,* \( R(\lambda) = f(\lambda), \ \lambda \in \Lambda \), *and* \( \max_{\mathbb{D}} |R| = \sup_{\mathbb{D}} |f| \), *see, for instance, [8, Corollary IV.1.8].

At the final step, Katsnelson deduces the existence of the analytic function \( f_J \) with the properties as above from Carleson’s “0 − 1-interpolation theorem”, which, in turn, was the main step in his solution to the corona problem [5, Theorem 2]. \( \square \)

This chain of arguments discovered in [15] had a significant impact on works of many mathematicians, notably from the Saint Petersburg school, cf. Nikolski-Pavlov [34, 35] (apparently, Nikolskii and Pavlov rediscovered some of Katsnelson’s results), Treil [40, 41], Vasyunin [45], see also [37, Lectures IX and X].

Katsnelson also notes that condition (2.1) in Theorem 2.1 cannot be weakened. Given a sequence \( (\lambda_k) \subset \mathbb{D} \) satisfying the Blaschke condition \( \sum_k (1 - |\lambda_k|) < \infty \) and such that

\[
\inf_j \prod_{k \neq j} \left| \frac{\lambda_k - \lambda_j}{1 - \lambda_j \bar{\lambda}_k} \right|^{m_j m_k} = 0,
\]

he brings a simple construction (the idea of which, according to [15], is due to Matsaev) of a contraction \( T \) such that

(i) \( (\lambda_k) \) are simple eigenvalues of \( T \) and the whole spectrum of \( T \) coincides with \( (\lambda_k) \),

and

(ii) the eigenvalues of \( T \) are complete in \( \mathbb{H} \) but are not uniformly minimal\(^2\).

Furthermore, the operator \( I - T^*T \) is one-dimensional.

Among other results brought in [15], there is a version of Theorem 2.1 for dissipative operators, i.e., the operators \( A \) such that \( \text{Im} \langle Ax, x \rangle \geq 0 \), for any \( x \) in the domain of \( A \). This version is reduced to Theorem 2.1 by an application of the Caley transform \( A \mapsto (A - il)(A + il)^{-1} \).

\(^2\) A system of vectors \( \{x_n\} \) in the Hilbert space \( \mathbb{H} \) is called *uniformly minimal* if there exists \( \delta > 0 \) such that for all \( n \) the distance between \( x_n \) and the linear span of \( \{x_k: k \neq n\} \) is at least \( \delta \).
3 Series of simple fractions

Let $C_0(\mathbb{R})$ be the Banach space of complex-valued continuous functions on $\mathbb{R}$, tending to zero at infinity, equipped with the uniform norm $\|f\| = \sup_{\mathbb{R}} |f|$. Fix finite subsets in the upper and lower half-planes $\{z_k\}_{1 \leq k \leq n} \subset \mathbb{C}_+$ and $\{w_k\}_{1 \leq k \leq m} \subset \mathbb{C}_-$ and denote by $E_+ = E_+ (w_1, \ldots, w_m)$, $E_- = E_- (z_1, \ldots, z_n)$ the subspaces in $C_0(\mathbb{R})$ generated by the simple fractions $\{1/(t - w_k) : 1 \leq k \leq m\}$ and $\{1/(t - z_k) : 1 \leq k \leq n\}$. The functions in $E_+$ are analytic on $\mathbb{C}_+$, the functions in $E_-$ are analytic on $\mathbb{C}_-$. Furthermore, $E_+ \cap E_- = \emptyset$ and the sum $E = E_+ + E_-$ is a direct one, i.e., for any function $f \in E$, there exists a unique decomposition $f = f_+ + f_-$ with $f_+ \in E_+$ and $f_- \in E_-$. In [16] Katsnelson estimates the norms of the projectors $P_\pm = P_\pm (z_1, \ldots, z_n; w_1, \ldots, w_m)$ from $E$ onto the corresponding subspace $E_\pm$. The main result of that work is the following theorem:

**Theorem 3.1.** There exists a positive numerical constant $C$ such that

\[
\|P_\pm\| \leq C \min(m, n) (m + n).
\]

The main point in this theorem is that the upper bound it gives does not depend on the positions of $z_k$s and $w_k$s.

Note that in the space $L^2(\mathbb{R})$, by one of the versions of the Paley-Wiener theorem, the functions analytic in the upper and lower half-planes are orthogonal to each other, which makes the corresponding projectors orthogonal. In view of this remark, it is quite natural that the proof of Theorem [16] uses the Fourier transform. The proof is nice and not too long and the reader can find its details in [16]. Bochtejn and Katsnelson bring in [2] a counterpart of Theorem 3.1 for the unit circle $\mathbb{T}$ instead of the real line $\mathbb{R}$.

Likely, the upper bound given in Theorem 3.1 is not sharp. In [16], Katsnelson conjectures that, for $m = n$, the sharp upper bound should be $C \log(n + 1)$, and, as far as we know, this conjecture remains open till today. As a supporting evidence towards this conjecture, he brings the following result.
Theorem 3.2. Given two finite sets of points \( \{z_k\}_{1 \leq k \leq n} \subset \mathbb{C}_+ \) and \( \{w_k\}_{1 \leq k \leq n} \subset \mathbb{C}_- \), consider the functions

\[
g_+(t) = \sum_{k=1}^{n} \frac{1}{t - w_k}, \quad g_-(t) = \sum_{k=1}^{n} \frac{1}{t - z_k},
\]

and let \( g(z) = g_+(z) + g_-(z) \). Then

\[
\max_{\mathbb{R}} |g_\pm| \leq C \log(n + 1) \cdot \max_{\mathbb{R}} |g|.
\]

with a positive numerical constant \( C \).

A simple example shows that the order of growth of the RHS cannot be improved. Put

\[
g_+(t) = \sum_{k=1}^{n} \frac{1}{t + ik}, \quad g_-(t) = \sum_{k=1}^{n} \frac{1}{t - ik}.
\]

Then

\[
g(t) = g_+(t) + g_-(t) = \sum_{k=1}^{n} \frac{2t}{t^2 + k^2},
\]

and

\[
\max_{\mathbb{R}} |g| \leq \int_{0}^{\infty} \frac{2t}{t^2 + x^2} \, dx = \pi,
\]

while

\[
\max_{\mathbb{R}} |g_-| = |g_-(0)| = \sum_{k=1}^{n} \frac{1}{k} = \log n + O(1).
\]

The proof of Theorem 3.2 is short and elegant (and accessible to undergraduate students). As a byproduct of that proof he obtains

Theorem 3.3. Let \( P \) be a polynomial of degree \( n \geq 2 \) such that

\[
\sup_{\mathbb{R}} \left| \frac{P'}{P} \right| \leq M.
\]

Then \( P \) has no zeroes in the strip

\[
|\text{Im} \, z| \leq \frac{c}{M \log n},
\]

where \( c \) is a positive numerical constant.
Slightly earlier a similar estimate was obtained by Gelfond [9]. Gelfond’s proof was rather different (and more involved). The question about the size of the strip around the real axis free of zeroes of \( P \) was raised by Gorin in [11], first results in that direction were obtained by him and then by Nikolaev in [33]. The final word in this question was said by Danchenko, who proved in [6] that under assumption of Theorem 3.3 the polynomial \( P \) has no zeroes in the strip

\[
| \text{Im } z | \leq \frac{c}{M} \cdot \frac{\log \log n}{\log n},
\]

and that the order of decay of the RHS cannot be improved.

Twenty five years later, Katsnelson returned in [20, 21] to the linear spans of simple fraction but from a different point of view. That time his work was motivated by Potapov’s results on factorization of \( J \)-contractive matrix functions.

Let \( m \) be the Lebesgue measure on the unit circle \( \mathbb{T} \), and \( w: \mathbb{T} \to [0, \infty] \) be an \( m \)-integrable weight, satisfying the Szegő condition

\[
\int_{\mathbb{T}} \log w \, dm > -\infty. \tag{3.1}
\]

By \( \text{PCH}^2(w) \) Katsnelson denotes the Hilbert space of functions \( f \) analytic on \( \mathbb{C} \setminus \mathbb{T} \) and satisfying the following conditions

(a) The restriction of \( f \) onto \( \mathbb{D}_+ \) and \( \mathbb{D}_- \) belongs to the Smirnov class, i.e., \( \log_+ |f| \) has positive harmonic majorants both in \( \mathbb{D}_+ \) and \( \mathbb{D}_- \).

(b) The boundary values of \( f|_{\mathbb{D}_+} \) and \( f|_{\mathbb{D}_-} \) coincide \( m \)-a.e. on \( \mathbb{T} \), that is,

\[
\lim_{r \uparrow 1} f(rt) = \lim_{r \downarrow 1} f(rt) \quad (=: f(t)) \quad m \text{-a.e. on } \mathbb{T}.
\]

(Conditions (a) and (b) together provide the so called pseudocontinuation property of the function \( f \).)

(c)

\[
\|f\|_w^2 = \int_{\mathbb{T}} |f|^2 w \, dm < \infty.
\]
Note that whenever $w^{-1} \in L^1(m)$ the space $PCH^2(w)$ is trivial, i.e., contains only constant functions $f$. Indeed, convergence of the integrals
\[ \int_T |f|^2 w \, dm < \infty, \quad \int_T \frac{dm}{w} < \infty \]
yields that $f \in L^1(m)$, and then, by a version of the removable singularity theorem that goes back to Carleman, the function $f$ is entire, and since it is bounded, by Liouville’s theorem it is a constant function.

Given a set of points $S \subset \mathbb{D}_+ \cup \mathbb{D}_-$ satisfying the Blaschke condition
\[ \sum_{\lambda \in S \cap \mathbb{D}_+} (1 - |\lambda|) < \infty, \quad \sum_{\lambda \in S \cap \mathbb{D}_-} (1 - |\lambda|^{-1}) < \infty, \tag{3.2} \]
denote by $R(S;w)$ the closure of the linear span of the simple fractions $\{(t-\lambda)^{-1}\}_{\lambda \in S}$ together with the constant functions in the space $L^2(w)$. Let $S = S_1 \supset S_2 \supset \ldots$ be a chain of sets such that $\bigcap_n S_n = \{\emptyset\}$.

The starting point of Katsnelson’s work [20] is the inclusion $\bigcap_n R(S_n;w) \subset PCH^2(w)$, which follows from classical results of Tumarkin [42], see also [43, 44]. Katsnelson observes that this inclusion might be a strict one, that is, generally speaking, not every function in the space $PCH^2(w)$ can be approximated in $L^2(w)$ by a sequence of functions $r_n \in R(S_n;w)$. The main result of [20] is the following approximation theorem.

**Theorem 3.4.** For any non-negative $m$-integrable function $w$ on $\mathbb{T}$ satisfying the Szegő condition (3.1), there exists a set $S \subset \mathbb{D}_+ \cup \mathbb{D}_-$ satisfying the Blaschke condition (3.2) such that $\bigcap_n R(S_n;w) = PCH^2(w)$.

In [21] Katsnelson extends this result to a more general approximation scheme by simple fractions with poles at a given table of points in $\mathbb{C} \setminus \mathbb{T}$. In [23] Kheifets used Katsnelson’s construction to answer a question raised by Sarason.
4 Spectral radius of hermitian elements in Banach algebras and the Bernstein inequality

One of the most important properties of EFET is the classical Bernstein inequality, which states that if \( F \) is an entire function of exponential type \( \sigma \), then
\[
\sup_{\mathbb{R}} |F'| \leq \sigma \sup_{\mathbb{R}} |F|,
\]
and the equality sign attains if and only if \( F(z) = c_1 \cos \sigma z + c_2 \sin \sigma z \). Different proofs, deep extensions, and various applications of the Bernstein inequality can be found in the books [1, 24, 25] and in the survey paper [12]. Interestingly, Bernstein’s inequality is also closely related to the theory of Banach algebras.

An element \( a \) of a Banach algebra \( A \) is called hermitian if \( \|e^{it\lambda}a\| = 1 \) for every \( t \in \mathbb{R} \). For instance, hermitian elements of the algebra of all bounded operators in a Hilbert space are self-adjoint operators. Another, more special, example is the differentiation operator \( D = \frac{1}{i} \frac{d}{dx} \) considered in various Banach spaces of EFET equipped with some translation-invariant norm, in which case the exponent \( e^{itD} \) is realized by the translation. It is well-known that the operator norm of a self-adjoint operator in a Hilbert space coincides with its spectral radius. Making use of the Bernstein inequality, Katsnelson proved in [17] the following result, which, independently (and more or less simultaneously), was also found by Browder [4] and Sinclair [39].

**Theorem 4.1.** For every hermitian element in a Banach algebra, the norm coincides with the spectral radius.

Moreover, as both Katsnelson and Browder observed, this result is equivalent to the Bernstein inequality, that is, the latter follows from the former, applied to the differentiation operator \( D \) in the Bernstein space \( B_\sigma \) of EFET at most \( \sigma \) bounded on the real axis and equipped with the uniform norm.

The proof of Theorem 4.1 is short and elegant: Let \( a \) be a hermitian element in a Banach algebra \( A \). Take an arbitrary linear functional \( \varphi \in A^* \) with the unit norm,
and consider the EFET
\[ F(z) \overset{\text{def}}{=} \varphi(e^{iz}) = \sum_{n \geq 0} \varphi(a^n) \frac{z^n}{n!}. \]

Applying, first, the formula, which expresses the exponential type of an entire function via its Taylor coefficients, then a crude estimate of \( n! \), and then Gelfand’s formula for the spectral radius, we estimate the exponential type of \( F \):
\[
\sigma_F = \frac{1}{e} \limsup_{n \to \infty} n \left( \frac{\varphi(a^n)}{n!} \right)^{1/n} \\
\leq \frac{1}{e} \limsup_{n \to \infty} n \left( \frac{\|a^n\|}{n!} \right)^{1/n} = \limsup_{n \to \infty} \frac{\|a^n\|^{1/n}}{1/n} = \rho(a),
\]
where \( \rho(a) \) denotes the spectral radius of \( a \). Since the element \( a \) is hermitian, we have \( |F(x)| = |\varphi(e^{ix})| \leq \|e^{ix}\| = 1 \), whence, by the Bernstein inequality,
\[
|\varphi(a)| = |F'(0)| \leq \sigma_F \sup_{\mathbb{R}} |F| \leq \rho(a),
\]
and then, by the Hahn-Banach theorem, \( \|a\| \leq \rho(a) \). This completes the proof of Theorem 4.1 since the converse inequality \( \rho(a) \leq \|a\| \) is obvious.

The proof of the result which goes in the opposite direction is also quite simple. Let \( D \) be the differentiation operator in the Bernstein space \( \mathcal{B}_\sigma \) of EFET at most \( \sigma \) bounded on \( \mathbb{R} \). As we have already mentioned, the exponential function \( e^{Dt} \) acts on \( \mathcal{B}_\sigma \) as the translation by \( t \), so \( D \) is a hermitian operator in \( \mathcal{B}_\sigma \). To evaluate the spectral radius of \( D \), we need to estimate from above the norms \( \|D^n\| \), that is, \( \sup_{\mathbb{R}} |F^{(n)}|, F \in \mathcal{B}_\sigma \). By Cauchy’s estimate for the derivatives of analytic functions, combined with the bound \( |F(x+w)| \leq e^{\sigma|w|} \sup_{\mathbb{R}} |F| \) valid for any \( F \in \mathcal{B}_\sigma \), we obtain
\[
|F^{(n)}(x)| \leq n! r^{-n} e^{\sigma r} \|F\| \text{ for any } r > 0 \text{ and any } x \in \mathbb{R}.
\]
Optimising the RHS, we get
\[
|F^{(n)}(x)| \leq n! \exp[n - n \log n + n \log \sigma] \|F\|, \text{ that is, } \|D^n\| \leq n! \exp[n - n \log n + n \log \sigma],
\]
and finally, \( \rho(D) = \lim \|D^n\|^{1/n} = \sigma \). Thus, for any function \( F \in \mathcal{B}_\sigma \) and any \( x \in \mathbb{R} \), we have
\[
|F'(x)| = |(DF)(x)| \leq \|D\| \cdot \|F\| = \rho(D) \|F\| \leq \sigma \|F\|,
\]
proving the Bernstein inequality.

In this context, it is also worth mentioning that a bit later Bonsall and Crabb \[3\] found a simple direct proof of Theorem 4.1 which yields another proof of the Bernstein inequality. Their proof is based on the following lemma, which is a simple exercise on the functional calculus in Banach algebras:

**Lemma 4.2.** Let \(a\) be a hermitian element in a Banach algebra with \(\rho(a) < \pi/2\). Then, \(a = \arcsin(\sin a)\).

Now, Theorem 4.1 follows almost immediately. Proving Theorem 4.1 it suffices, assuming that \(a\) is an arbitrary hermitian element with \(\rho(a) < \pi/2\), to show that \(\|a\| \leq \pi/2\). Let \(c_n\) be the Taylor coefficient of the function \(z \mapsto \arcsin z, |z| \leq 1\). The values \(c_n\) are positive and their sum equals \(\arcsin(1) = \pi/2\). By Lemma 4.2, \(\|a\| \leq \sum_{n \geq 1} c_n \|\sin a\|^n\). Since the element \(a\) is hermitian, \(\|\sin a\| \leq 1\), and therefore, \(\|a\| \leq \sum_{n \geq 1} c_n = \pi/2\). \(\square\)

In the reference list, referring to the papers in Russian published in journals translated from cover to cover, we mention only the translations. Today, the original Russian versions of these papers can be found at the Math-Net.Ru site (http://www.mathnet.ru).

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