On realization of the original Weyl-Titchmarsh functions by
Shrödinger L-systems

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Dedicated with great pleasure to Henk de Snoo on the occasion of his 75-th birthday

Abstract. We study realizations generated by the original Weyl-Titchmarsh functions $m_\infty(z)$ and $m_\alpha(z)$. It is shown that the Herglotz-Nevanlinna functions ($-m_\infty(z)$) and ($1/m_\infty(z)$) can be realized as the impedance functions of the corresponding Schrödinger L-systems sharing the same main dissipative operator. These L-systems are presented explicitly and related to Dirichlet and Neumann boundary problems. Similar results but related to the mixed boundary problems are derived for the Herglotz-Nevanlinna functions ($-m_\alpha(z)$) and ($1/m_\alpha(z)$). We also obtain some additional properties of these realizations in the case when the minimal symmetric Schrödinger operator is non-negative. In addition to that we state and prove the uniqueness realization criteria for Schrödinger L-systems with equal boundary parameters. A condition for two Schrödinger L-systems to share the same main operator is established as well. Examples that illustrate the obtained results are presented in the end of the paper.

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1. Introduction

In the current paper we consider L-systems with dissipative Shr"{o}dinger operators. For the sake of brevity we will refer to these L-systems as *Shr"{o}dinger L-systems* for the rest of the manuscript. The formal definition, exposition and discussions of general and Shr"{o}dinger L-systems are presented in Sections 2 and 3. We capitalize on the fact that all Shr"{o}dinger L-systems $\Theta_{\mu,h}$ form a two-parametric family whose members are uniquely defined by a real-valued parameter $\mu$ and a complex boundary value $h$ (Im $h > 0$) of the main dissipative operator.

The focus of the paper is set on two classical objects related to a Shr"{o}dinger operator: the original Weyl-Titchmarsh function $m(\infty,z)$ and its linear-fractional transformation $m(\alpha,z)$ given by (48) that was introduced and studied in [27], [28], [17], [21]. It is well known (see [21], [17]) that $(-m(\infty,z))$ and $(1/m(\infty,z))$ as well as $(-m(\alpha,z))$ and $(1/m(\alpha,z))$ are the Herglotz-Nevanlinna functions. In Section 4 we show that the Herglotz-Nevanlinna functions $(-m(\infty,z))$ and $(1/m(\infty,z))$ can be realized as the impedance function of Shr"{o}dinger L-systems $\Theta_{0,i}$ and $\Theta_{\infty,i}$, respectively (see Theorems 3 and 4). Moreover, these two realizing L-system share the same main dissipative Shr"{o}dinger operator and are connected to Dirichlet (32) and Neumann (43) boundary problems, respectively. In Section 5 we treat the realization of the Herglotz-Nevanlinna functions $(-m(\alpha,z))$ and $(1/m(\alpha,z))$ that are linear-fractional transformations of $m(\infty,z)$ described in details in [17]. As a result we obtain a one-parametric families of realizing Shr"{o}dinger L-systems $\Theta_{\tan \alpha,i}$ and $\Theta_{(-\cot \alpha),i}$, respectively. In Section 6 we narrow down the realization results from the previous two sections to the class of Shr"{o}dinger L-systems that are based on non-negative symmetric Shr"{o}dinger operator to obtain additional properties. In particular, in Theorem 12 we describe the cases when the realizing Shr"{o}dinger L-systems are accretive. Moreover, it turns out that the quasi-kernel $\hat{A}_0$ of Re $A_{0,i}$ in the constructed realizing L-system $\Theta_{0,i}$ corresponds to the Friedrich’s extension while the quasi-kernel $\hat{A}_\infty$ of Re $A_{\infty,i}$ in $\Theta_{\infty,i}$ corresponds to the Krein-von Neumann extension of our non-negative symmetric operator $A$ only in the case when $m(\infty(-0)) = 0$. Section 7 raises and answers the uniqueness questions of realization by a Shr"{o}dinger L-systems. After giving the general definition of two equal L-systems, we state and prove the criteria for two Shr"{o}dinger L-systems to be equal. Precisely, Theorem 17 is saying that two Shr"{o}dinger L-systems with the same underlying parameters $h$ and $\mu$ are equal if and only if their impedance functions match. Then we generalize this result and establish a condition for two Shr"{o}dinger L-systems to share the same main operator. In Theorem 18 we show that two Shr"{o}dinger L-systems with the same parameter $h$ share the same main operator $T_h$ if and only if their impedance functions are connected by the Donoghue transform (88). The paper is concluded with two examples that illustrate the main results and concepts.

The present work is a further development of the theory of open physical systems conceived by M. Livšič in [18].

2. Preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\hat{A}$ be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ with inner product $(f,g)$, $f, g \in \mathcal{H}$. Any non-symmetric
operator $T$ in $\mathcal{H}$ such that

$$\hat{A} \subset T \subset \hat{A}^*$$

is called a quasi-self-adjoint bi-extension of $\hat{A}$.

Consider the rigged Hilbert space (see [1, 2]) $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, where $\mathcal{H}_+ = \text{Dom}(\hat{A}^*)$ and

$$(f,g)_+ = (f,g) + (\hat{A}^* f, \hat{A}^* g), \quad f, g \in \text{Dom}(\hat{A}^*).$$

Let $\mathcal{R}$ be the Riesz-Berezansky operator $\mathcal{R}$ (see [1, 2]) which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that $(f,g)_+ = (f,\mathcal{R} g)_+ \quad (\forall f \in \mathcal{H}_+, \ g \in \mathcal{H}_-) \quad \text{and} \quad \|\mathcal{R} g\|_+ = \|g\|_-$. Note that identifying the space conjugate to $\mathcal{H}_\mp$ with $\mathcal{H}_\mp^*$, we get that if $\mathcal{A} \in \mathcal{H}_+, \mathcal{H}_-$, then $\mathcal{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$. An operator $\mathcal{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a self-adjoint bi-extension of a symmetric operator $\hat{A}$ if $\mathcal{A} = \mathcal{A}^*$ and $\mathcal{A} \supset \hat{A}$. Let $\mathcal{A}$ be a self-adjoint bi-extension of $\hat{A}$ and let the operator $\hat{A}$ in $\mathcal{H}$ be defined as follows:

$$\text{Dom}(\hat{A}) = \{f \in \mathcal{H}_+ : \mathcal{A} f \in \mathcal{H}\}, \quad \hat{A} = \mathcal{A} | \text{Dom}(\hat{A}).$$

The operator $\hat{A}$ is called a quasi-kernel of a self-adjoint bi-extension $\mathcal{A}$ (see [20, 2, Section 2.1]). According to the von Neumann Theorem (see [2, Theorem 1.3.1]) the domain of $\mathcal{A}$, a self-adjoint extension of $\hat{A}$, can be expressed as

$$(2) \quad \text{Dom}(\hat{A}) = \text{Dom}(\hat{A}) \oplus (I + U)\mathfrak{N}_i,$$

where von Neumann’s parameter $U$ is a (·) and (·)-isometric operator from $\mathfrak{N}_i$ into $\mathfrak{N}_{i-1}$ and

$$\mathfrak{N}_{i+1} = \text{Ker} (\hat{A}^* \mp iI)$$

are the deficiency subspaces of $\hat{A}$. A self-adjoint bi-extension $\mathcal{A}$ of a symmetric operator $\hat{A}$ is called t-self-adjoint (see [2, Definition 4.3.1]) if its quasi-kernel $\hat{A}$ is self-adjoint operator in $\mathcal{H}$. An operator $\mathcal{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a quasi-self-adjoint bi-extension of an operator $T$ if $\mathcal{A} \supset T \supset \mathcal{A}$ and $\mathcal{A}^* \supset T^* \supset \mathcal{A}$. We will be mostly interested in the following type of quasi-self-adjoint bi-extensions. Let $T$ be a quasi-self-adjoint extension of $\hat{A}$ with nonempty resolvent set $\rho(T)$. A quasi-self-adjoint bi-extension $\mathcal{A}$ of an operator $T$ is called (see [2, Definition 3.3.5]) a (·)-extension of $T$ if $\mathcal{A}$ is a t-self-adjoint bi-extension of $\hat{A}$. In what follows we assume that $\hat{A}$ has deficiency indices $(1,1)$. In this case it is known [2] that every quasi-self-adjoint extension $T$ of $\hat{A}$ admits (·)-extensions. The description of all (·)-extensions via Riesz-Berezansky operator $\mathcal{R}$ can be found in [2, Section 4.3].

Recall that a linear operator $T$ in a Hilbert space $\mathcal{H}$ is called accretive [16] if $\text{Re}(T f, f) \geq 0$ for all $f \in \text{Dom}(T)$. We call an accretive operator $T$ $\beta$-sectorial [16] if there exists a value of $\beta \in (0, \pi/2)$ such that

$$(3) \quad (\cot \beta) |\text{Im}(T f, f)| \leq \text{Re}(T f, f), \quad f \in \text{Dom}(T).$$

We say that the angle of sectoriality $\beta$ is exact for a $\beta$-sectorial operator $T$ if

$$\tan \beta = \sup_{f \in \text{Dom}(T)} \frac{|\text{Im}(T f, f)|}{\text{Re}(T f, f)}.$$ 

An accretive operator is called extremal accretive if it is not $\beta$-sectorial for any $\beta \in (0, \pi/2)$. A (·)-extension $\mathcal{A}$ of $T$ is called accretive if $\text{Re}(\mathcal{A} f, f) \geq 0$ for all $f \in \mathcal{H}_+$. This is equivalent to that the real part $\text{Re} \mathcal{A} = (\mathcal{A} + \mathcal{A}^*)/2$ is a nonnegative self-adjoint bi-extension of $\hat{A}$.

The following definition is a “lite” version of the definition of L-system given for a scattering L-system with one-dimensional input-output space. It is tailored
for the case when the symmetric operator of an L-system has deficiency indices $(1, 1)$. The general definition of an L-system can be found in [2, Definition 6.3.4] (see also [3] for a non-canonical version).

**Definition 1.** An array

\[
\Theta = \begin{pmatrix}
\mathcal{A} & K & 1 \\
\mathcal{H}^+ & \mathcal{H} & \mathcal{H}^- \\
0 & \mathbb{C}
\end{pmatrix}
\]

is called an **L-system** if:

1. $T$ is a dissipative ($\text{Im}(Tf, f) \geq 0$, $f \in \text{Dom}(T)$) quasi-self-adjoint extension of a symmetric operator $A$ with deficiency indices $(1, 1)$;
2. $\mathcal{A}$ is a $(*)$-extension of $T$;
3. $\text{Im} A = KK^*$, where $K \in [\mathbb{C}, \mathcal{H}^-]$ and $K^* \in [\mathcal{H}^+, \mathbb{C}]$.

Operators $T$ and $\mathcal{A}$ are called a **main** and **state-space operators** respectively of the system $\Theta$, and $K$ is a **channel operator**. It is easy to see that the operator $\mathcal{A}$ of the system (4) is such that $\text{Im} \mathcal{A} = (\cdot, \chi)\chi$, $\chi \in \mathcal{H}^-$ and pick $Kc = c \cdot \chi$, $c \in \mathbb{C}$ (see [2]). A system $\Theta$ in (4) is called **minimal** if the operator $\mathcal{A}$ is a prime operator in $\mathcal{H}$, i.e., there exists no non-trivial reducing invariant subspace of $\mathcal{H}$ on which it induces a self-adjoint operator. Minimal L-systems of the form (4) with one-dimensional input-output space were also considered in [2].

We associate with an L-system $\Theta$ the function

\[
W_\Theta(z) = I - 2iK^*(A - zI)^{-1}K, \quad z \in \rho(T),
\]

which is called the **transfer function** of the L-system $\Theta$. We also consider the function

\[
V_\Theta(z) = K^*(\text{Re}A - zI)^{-1}K,
\]

that is called the **impedance function** of an L-system $\Theta$ of the form (4). The transfer function $W_\Theta(z)$ of the L-system $\Theta$ and function $V_\Theta(z)$ of the form (4) are connected by the following relations valid for $\text{Im} z \neq 0$, $z \in \rho(T)$,

\[
W_\Theta(z) = i[\Theta_0(z) + I]^{-1}[\Theta_0(z) - I],
\]

\[
W_\Theta(z) = (I + iV_\Theta(z))^{-1}(I - iV_\Theta(z)).
\]

An L-system $\Theta$ of the form (4) is called an **accretive L-system** ([2], [13]) if its state-space operator operator $\mathcal{A}$ is accretive, that is $\text{Re}(\mathcal{A}f, f) \geq 0$ for all $f \in \mathcal{H}^+$. An accretive L-system is called **sectorial** if the operator $\mathcal{A}$ is sectorial, i.e., satisfies (4) for some $\beta \in (0, \pi/2)$ and all $f \in \mathcal{H}^+$.

Now let us consider a minimal L-system $\Theta$ of the form (4). Let also

\[
\Theta_\alpha = \begin{pmatrix}
\mathcal{A}_\alpha & K_\alpha & 1 \\
\mathcal{H}^+ & \mathcal{H} & \mathcal{H}^- \\
0 & \mathbb{C}
\end{pmatrix}, \quad \alpha \in [0, \pi),
\]

be a one-parametric family of L-systems such that

\[
W_{\Theta_\alpha}(z) = W_\Theta(z) \cdot (-e^{2i\alpha}), \quad \alpha \in [0, \pi).
\]

The existence and structure of $\Theta_\alpha$ were described in details in [2, Section 8.3]. In particular, it was shown that $\Theta$ and $\Theta_\alpha$ share the same main operator $T$ and that

\[
V_{\Theta_\alpha}(z) = \frac{\cos \alpha + (\sin \alpha)V_\Theta(z)}{\sin \alpha - (\cos \alpha)V_\Theta(z)}.
\]
A scalar function $V(z)$ is called the Herglotz-Nevanlinna function if it is holomorphic on $\mathbb{C} \setminus \mathbb{R}$, symmetric with respect to the real axis, i.e., $V(z)^* = V(\bar{z})$, $z \in \mathbb{C} \setminus \mathbb{R}$, and if it satisfies the positivity condition $\text{Im} V(z) \geq 0$, $z \in \mathbb{C}_+$. The class of all Herglotz-Nevanlinna functions, that can be realized as impedance functions of L-systems, and connections with Weyl-Titchmarsh functions can be found in [4, 5, 12, 13] and references therein. The following definition can be found in [13]. A scalar Herglotz-Nevanlinna function $V(z)$ is a Stieltjes function if it is holomorphic in $\text{Ext}(0, +\infty)$ and

$$\frac{\text{Im}[zV(z)]}{\text{Im} z} \geq 0.$$

It is known [15] that a Stieltjes function $V(z)$ admits the following integral representation

$$V(z) = \gamma + \int_0^\infty \frac{dG(t)}{t - z},$$

where $\gamma \geq 0$ and $G(t)$ is a non-decreasing on $[0, +\infty)$ function such that $\int_0^\infty \frac{dG(t)}{t + \epsilon} < \infty$. We are going to focus on the class $S_0(R)$ (see [3, 12, 2] of scalar Stieltjes functions, whose definition is the following. A scalar Stieltjes function $V(z)$ is said to be a member of the class $S_0(R)$ if the measure $G(t)$ in representation (11) is of unbounded variation. It was shown in [4] (see also [5]) that such a function $V(z)$ can be realized as the impedance function of an accretive L-system $\Theta$ of the form (1) with a densely defined symmetric operator if and only if it belongs to the class $S_0(R)$.

3. L-systems with Schrödinger operator and their impedance functions

Let $\mathcal{H} = L_2[\ell, +\infty]$, $\ell \geq 0$, and $l(y) = -y'' + q(x)y$, where $q$ is a real locally summable on $[\ell, +\infty)$ function. Suppose that the symmetric operator

$$\begin{cases} \hat{A}y = -y'' + q(x)y \\ y(\ell) = y'(\ell) = 0 \end{cases}$$

has deficiency indices $(1,1)$. Let $D^*$ be the set of functions locally absolutely continuous together with their first derivatives such that $l(y) \in L_2[\ell, +\infty]$. Consider $\mathcal{H}_+ = \text{Dom}(A^*) = D^*$ with the scalar product

$$(y, z)_+ = \int_\ell^\infty \left( y(x)\overline{z(x)} + l(y)l(z) \right) dx, \quad y, z \in D^*.$$

Let $\mathcal{H}_+ \subset L_2[\ell, +\infty] \subset \mathcal{H}_-$ be the corresponding triplet of Hilbert spaces. Consider the operators

$$\begin{cases} T_0 y = l(y) = -y'' + q(x)y \\ h y(\ell) - y'(\ell) = 0 \end{cases}, \quad \begin{cases} T_0^* y = l(y) = -y'' + q(x)y \\ \overline{h} y(\ell) - y'(\ell) = 0 \end{cases},$$

where $\text{Im} h > 0$. Let $\hat{A}$ be a symmetric operator of the form (13) with deficiency indices $(1,1)$, generated by the differential operation $l(y) = -y'' + q(x)y$. Let also
\( \varphi_k(x, \lambda)(k = 1, 2) \) be the solutions of the following Cauchy problems:

\[
\begin{align*}
\varphi_1(\ell, \lambda) &= 0, \\
\varphi_2(\ell, \lambda) &= -1.
\end{align*}
\]

It is well known [15, 17] that there exists a function \( m_\infty(\lambda) \) introduced by H. Weyl [27, 28], for which

\[
\varphi(x, \lambda) = \varphi_2(x, \lambda) + m_\infty(\lambda) \varphi_1(x, \lambda)
\]

belongs to \( L_2[\ell, +\infty) \). The function \( m_\infty(\lambda) \) is not a Herglotz-Nevanlinna function but \((-m_\infty(\lambda))\) and \((1/m_\infty(\lambda))\) are.

Now we shall construct an L-system based on a non-self-adjoint Schrödinger operator \( T_h \) with \( \text{Im} h > 0 \). It was shown in [4, 2] that the set of all \((\ast)\)-extensions of a non-self-adjoint Schrödinger operator \( T_h \) of the form (13) in \( L_2[\ell, +\infty) \) can be represented in the form

\[
\begin{align*}
\mathcal{A}_{\mu, h} y &= -y'' + q(x)y - \frac{1}{\mu - h} [y'(\ell) - hy(\ell)] [\mu \delta(x - \ell) + \delta'(x - \ell)], \\
\mathcal{A}_{\mu, h}^{\ast} y &= -y'' + q(x)y - \frac{1}{\mu - h} [y'(\ell) - hy(\ell)] [\mu \delta(x - \ell) + \delta'(x - \ell)].
\end{align*}
\]

Moreover, the formulas (13) establish a one-to-one correspondence between the set of all \((\ast)\)-extensions of a Schrödinger operator \( T_h \) of the form (13) and all real numbers \( \mu \in [-\infty, +\infty] \). One can easily check that the \((\ast)\)-extension \( \mathcal{A}_{\mu, h} \) in (14) of the non-self-adjoint dissipative Schrödinger operator \( T_h \), \( (\text{Im} h > 0) \) of the form (13) satisfies the condition

\[
\text{Im} \mathcal{A}_{\mu, h} = \frac{\mathcal{A}_{\mu, h} - \mathcal{A}_{\mu, h}^{\ast}}{2t} = (.., g_{\mu, h}) g_{\mu, h},
\]

where

\[
(15)
g_{\mu, h} = \frac{(\text{Im} h)^{\frac{1}{2}}}{|\mu - h|} [\mu \delta(x - \ell) + \delta'(x - \ell)]
\]

and \( \delta(x - \ell), \delta'(x - \ell) \) are the delta-function and its derivative at the point \( \ell \), respectively. Furthermore,

\[
(16)
(y, g_{\mu, h}) = \frac{(\text{Im} h)^{\frac{1}{2}}}{|\mu - h|} [\mu y(\ell) - y'(\ell)],
\]

where \( y \in \mathcal{H}_+, g_{\mu, h} \in \mathcal{H}_-, \mathcal{H}_+ \subset L_2[\ell, +\infty) \subset \mathcal{H}_- \) and the triplet of Hilbert spaces discussed above.

It was also shown in [2] that the quasi-kernel \( \mathcal{A}_\xi \) of \( \text{Re} \mathcal{A}_{\mu, h} \) is given by

\[
(16)
\begin{align*}
\mathcal{A}_\xi y &= -y'' + q(x)y, \\
y'(\ell) &= \xi y(\ell),
\end{align*}
\]

where \( \xi = \frac{\mu \text{Re} h - |h|^2}{\mu - \text{Re} h} \).

Let \( E = \mathbb{C}, K_{\mu, h} c = cg_{\mu, h} \) \((c \in \mathbb{C})\). It is clear that

\[
(17)
K_{\mu, h}^{\ast} y = (y, g_{\mu, h}), \quad y \in \mathcal{H}_+,
\]

and \( \text{Im} \mathcal{A}_{\mu, h} = K_{\mu, h} K_{\mu, h}^{\ast} \). Therefore, the array

\[
(18)
\Theta_{\mu, h} = \left( K_{\mu, h} \begin{array}{cc} A_{\mu, h} & K_{\mu, h}^{\ast} \end{array} \right),
\]

where \( A_{\mu, h} \) is the operator of the form (13) and \( K_{\mu, h} \) is the operator of the form (13).
It was shown in [13] that if the parameters $\mu$ and $\xi$ are related via (16), then the two L-systems $\Theta_{\mu,h}$ and $\Theta_{\xi,h}$ of the form (13) have the following property

\begin{equation}
W_{\Theta_{\mu,h}}(z) = -W_{\Theta_{\xi,h}}(z), \quad V_{\Theta_{\mu,h}}(z) = -\frac{1}{V_{\Theta_{\xi,h}}(z)}, \quad \text{where} \quad \xi = \frac{\mu \Re h - |h|^2}{\mu - \Re h}.
\end{equation}

This result can be generalized as follows.

**Lemma 2.** Let $\Theta_{\mu,h}$ and $\Theta_{\mu(\alpha),h}$ be two L-systems of the form (13) such that

\begin{equation}
V_{\Theta_{\mu(\alpha),h}}(z) = \frac{\cos \alpha + (\sin \alpha)V_{\Theta_{\mu,h}}(z)}{\sin \alpha - (\cos \alpha)V_{\Theta_{\mu,h}}(z)}.
\end{equation}

Then

\begin{equation}
\mu(\alpha) = \frac{h(\mu - \bar{h}) + e^{2i\alpha}(\mu - h)\bar{h}}{\mu - h + e^{2i\alpha}(\mu - h)}.
\end{equation}

**Proof.** It was shown in [2] Section 8.3] that if the impedance functions $V_{\Theta_{\mu(\alpha),h}}(z)$ and $V_{\Theta_{\mu,h}}(z)$ are connected by the Donoghue transform (22) (see also (1)), then the corresponding transfer functions are related by

\begin{equation}
W_{\Theta_{\mu(\alpha),h}}(z) = (-e^{2i\alpha}) \cdot W_{\Theta_{\mu,h}}(z).
\end{equation}

Combining (22) with (13) above and setting $U = -e^{2i\alpha}$ temporarily we obtain

\begin{align*}
&\frac{\mu(\alpha) - h}{\mu(\alpha) - \bar{h}} \frac{m_\infty(z) + \bar{h}}{m_\infty(z) + \bar{h}} = U \cdot \frac{\mu - h}{\mu - \bar{h}} \frac{m_\infty(z) + \bar{h}}{m_\infty(z) + \bar{h}},
\end{align*}

or, after canceling common factors,

\begin{equation}
\frac{\mu(\alpha) - h}{\mu(\alpha) - \bar{h}} = U \cdot \frac{\mu - h}{\mu - \bar{h}}.
\end{equation}

This yields

\begin{equation}
\mu(\alpha)\mu - \mu(\alpha)\bar{h} - h\mu + |h|^2 = U\mu(\alpha)\mu - U\mu(\alpha)\bar{h} - U\bar{h}\mu + U|h|^2.
\end{equation}

Solving the above for $\mu(\alpha)$ gives

\begin{equation}
\mu(\alpha) = \frac{h\mu - |h|^2 - U\bar{h}\mu + U|h|^2}{\mu - h - U\mu + U\bar{h}}.
\end{equation}

Substituting $U = -e^{2i\alpha}$ in the above and simplifying results in (23). \qed

As one can easily see the value of $\xi$ in (21) follows from (20) if one sets $\alpha = 0$ and then $\xi = \mu(0)$. 

4. Realizations of $-m_\infty(z)$ and $1/m_\infty(z)$.

It is known [17], [18] that the original Weyl-Titchmarsh function $m_\infty(z)$ has a property that $(-m_\infty(z))$ is a Herglotz-Nevanlinna function. Hence, the question whether $(-m_\infty(z))$ can be realized as the impedance function of a Shrödinger L-system is more than relevant. The following theorem contains the answer.

**Theorem 3.** Let $\hat{A}$ be a symmetric Schrödinger operator of the form (22) with deficiency indices $(1,1)$ and locally summable potential in $\mathcal{H} = L^2[\ell, \infty)$. If $m_\infty(z)$ is the Weyl-Titchmarsh function of $\hat{A}$, then the Herglotz-Nevanlinna function $(-m_\infty(z))$ can be realized as the impedance function of a Shrödinger L-system $\Theta_{\mu,h}$ of the form (18) with

$$\mu = 0 \text{ and } h = i.$$ 

Conversely, let $\Theta_{\mu,h}$ be a Shrödinger L-system of the form (18) with the symmetric operator $\hat{A}$ such that

$$V_{\Theta_{\mu,h}}(z) = -m_\infty(z),$$

for all $z \in \mathbb{C}_\pm$ and $\mu \in \mathbb{R} \cup \{\infty\}$. Then the parameters $\mu$ and $h$ defining $\Theta_{\mu,h}$ are given by (25), i.e., $\mu = 0$ and $h = i$.

**Proof.** Let $\Theta_{\mu,h}$ be a Shrödinger L-system of the form (18) with our symmetric operator $\hat{A}$. Then its impedance function $V_{\Theta_{\mu,h}}(z)$ is determined by formula (20) for any $\mu \in \mathbb{R} \cup \{\infty\}$ and any non-real $h$. If we set $\mu = 0$ and $h = i$ in (20) we obtain

$$V_{\Theta_{0,i}}(z) = \frac{(m_\infty(z) + 0) \cdot 1}{(0 - 0) m_\infty(z) + 0 - 1} = -m_\infty(z), \quad z \in \mathbb{C}_\pm.$$

Thus, $\Theta_{0,i}$ realizes $(-m_\infty(z))$ and the first part of the theorem is proved.

Conversely, let $\Theta_{\mu,h}$ be a Shrödinger L-system of the form (18) such that

$$V_{\Theta_{\mu,h}}(z) = -m_\infty(z).$$

Then (20) implies

$$\frac{(m_\infty(z) + \mu) \text{Im } h}{(\mu - \text{Re } h) m_\infty(z) + \mu \text{Re } h - |h|^2} = -m_\infty(z),$$

for all $z \in \mathbb{C}_\pm$ and $\mu \in \mathbb{R} \cup \{\infty\}$. In particular, if we set $\mu = 0$ in the above equation we obtain

$$\frac{(m_\infty(z)) \text{Im } h}{(-\text{Re } h) m_\infty(z) - |h|^2} = -m_\infty(z), \quad z \in \mathbb{C}_\pm,$$

or, taking into account that $m_\infty(z)$ is not identical zero in $\mathbb{C}_\pm$ (see [17]),

$$\text{Im } h \frac{m_\infty(z) + |h|^2}{\text{Re } h m_\infty(z) + |h|^2} = 1, \quad z \in \mathbb{C}_\pm$$

leading to

$$\text{Re } h m_\infty(z) + |h|^2 - \text{Im } h = 0, \quad z \in \mathbb{C}_\pm.$$

Set $z = i$, then $m_\infty(i) = a - bi$, where $a$ and $b$ are real and $b > 0$. Then (25) yields

$$\text{Re } h(a - bi) + |h|^2 - \text{Im } h = 0,$$

or

$$a \text{Re } h + |h|^2 - \text{Im } h - (b \text{Re } h)i = 0.$$

Thus, the imaginary part of the above must be zero or $b \text{Re } h = 0$. Since $b > 0$ we have $\text{Re } h = 0$ yielding

$$(\text{Im } h)^2 - \text{Im } h = 0.$$
Discarding the case $\text{Im } h = 0$, we obtain $\text{Im } h = 1$. Consequently, $h = i$. Substituting this value of $h$ into \cite{26} we get
\[
\frac{m_\infty(z) + \mu}{\mu m_\infty(z) - 1} = -m_\infty(z), \quad z \in \mathbb{C}_\pm,
\]
yielding $\mu(1 + m_\infty^2(z)) = 0$. This means that either $\mu = 0$ or $-m_\infty(z) \equiv i$. The latter case is impossible as it can be seen from the asymptotic expansion of $m_\infty(z)$ (see \cite{11, 17} Chapter 2) and therefore $\mu = 0$. This proves the second part of the theorem. □

Theorem 3 above allows us to explicitly present a Shrödinger L-system whose impedance function matches $-m_\infty(z)$. This is

\[
\Theta_{0,i} = \begin{pmatrix}
A_{0,i} & K_{0,i} \\
\mathcal{H}_+ \subset L_2[\ell, +\infty) \subset \mathcal{H}_- & 1 \\
\end{pmatrix},
\]
where
\[
A_{0,i} y = -y'' + q(x)y - i[y'(\ell) - iy(\ell)]\delta'(x - \ell),
\]
\[
A_{0,i}^* y = -y'' + q(x)y + i[y'(\ell) + iy(\ell)]\delta'(x - \ell),
\]
$K_{0,i}c = cg_{0,i}$, $(c \in \mathbb{C})$ and
\[
g_{0,i} = \delta'(x - \ell).
\]
Clearly,
\[
\text{Re } A_{0,i} y = -y'' + q(x)y - y(\ell)\delta'(x - \ell),
\]
and (see also \cite{19})
\[
\left\{ \begin{aligned}
\hat{A}_{0,i} y &= -y'' + q(x)y \\
y(\ell) &= 0
\end{aligned} \right.,
\]
is the quasi-kernel of $\text{Re } A_{0,i}$ in \cite{11}. Note that \cite{23} defines a self-adjoint Shrödinger operator with Dirichlet boundary conditions. Also,
\[
V_{\Theta_{0,i}}(z) = -m_\infty(z) \quad \text{and} \quad W_{\Theta_{0,i}}(z) = \frac{1 + i m_\infty(z)}{1 - i m_\infty(z)}.
\]
Thus, $V_{\Theta_{0,i}}(z)$ and hence $-m_\infty(z)$ has the following resolvent representation (see \cite{11})
\[
-m_\infty(z) = V_{\Theta_{0,i}}(z) = \left((\text{Re } A_{0,i} - zI)^{-1}g_{0,i}, g_{0,i}\right)
\]
\[
= \left((A_{0,i} - zI)^{-1}\delta'(x - \ell), \delta'(x - \ell)\right),
\]
where $\text{Re } A_{0,i}$ is given by \cite{31}, $g_{0,i}$ by \cite{23}, and $(A_{0,i} - zI)^{-1}$ is the extended resolvent of the quasi-kernel $A_{0,i}$ in \cite{31} (see \cite{2}).

Now we can obtain a similar result for the function $1/m_\infty(z)$.

**Theorem 4.** Let $\hat{A}$ be a symmetric Schrödinger operator of the form \cite{12} with deficiency indices $(1,1)$ and locally summable potential in $\mathcal{H} = L^2[\ell, \infty)$. If $m_\infty(z)$ is the Weyl-Titchmarsh function of $\hat{A}$, then the Herglotz-Nevanlinna function $(1/m_\infty(z))$ can be realized as the impedance function of a Shrödinger L-system $\Theta_{\mu,h}$ of the form \cite{15} with
\[
\mu = \infty \quad \text{and} \quad h = i.
\]
Conversely, let $\Theta_{\mu,h}$ be a Shrödinger L-system of the form (18) with the symmetric operator $\hat{A}$ such that

$$V_{\Theta_{\mu,h}}(z) = \frac{1}{m_\infty(z)},$$

for all $z \in \mathbb{C}_\pm$ and $\mu \in \mathbb{R} \cup \{\infty\}$. Then the parameters $\mu$ and $h$ defining $\Theta_{\mu,h}$ are given by (25), i.e., $\mu = \infty$ and $h = i$.

**Proof.** Let $\Theta_{\mu,h}$ be a Shrödinger L-system of the form (18) with our symmetric operator $\hat{A}$. Once again, its impedance function $V_{\Theta_{\mu,h}}(z)$ is determined by formula (20) for any $\mu \in \mathbb{R} \cup \{\infty\}$ and any non-real $h$. If we set $h = i$ and then $\mu = \infty$ in (20) we obtain

$$V_{\Theta_{\infty,i}}(z) = \lim_{\mu \to \infty} \frac{m_\infty(z) + \mu}{\mu m_\infty(z) - 1} = \frac{1}{m_\infty(z)}, \quad z \in \mathbb{C}_\pm.$$

Thus, $\Theta_{\infty,i}$ realizes $(1/m_\infty(z))$ and the first part of the theorem is proved.

Conversely, let $\Theta_{\mu,h}$ be a Shrödinger L-system of the form (18) such that $V_{\Theta_{\mu,h}}(z) = 1/m_\infty(z)$. Then reciprocating (20) gives

$$\frac{(\mu - \text{Re} \, h) m_\infty(z) + \mu \text{Re} \, h - |h|^2}{(m_\infty(z) + \mu) \text{Im} \, h} = m_\infty(z),$$

for all $z \in \mathbb{C}_\pm$ and $\mu \in \mathbb{R} \cup \{\infty\}$. Passing to the limit in (36) when $\mu \to \infty$ yields

$$\frac{m_\infty(z) + \text{Re} \, h}{\text{Im} \, h} = m_\infty(z), \quad z \in \mathbb{C}_\pm,$$

or

$$m_\infty(z) + \text{Re} \, h = (\text{Im} \, h) m_\infty(z), \quad z \in \mathbb{C}_\pm.$$

As in the proof of Theorem 3 we set $z = i$ and $m_\infty(i) = a - bi$, where $a$ and $b$ are real and $b > 0$. Then (37) yields

$$(a - bi) + \text{Re} \, h = \text{Im} \, h (a - bi).$$

Equating real and imaginary parts on both sides gives

$$a + \text{Re} \, h = a \text{Im} \, h \quad \text{and} \quad b = b \text{Im} \, h, \quad b > 0.$$

Consequently, $\text{Im} \, h = 1$ and hence $a + \text{Re} \, h = a$ implying $\text{Re} \, h = 0$. Thus $h = i$. Substituting this value of $h$ in (36) gives

$$\frac{\mu m_\infty(z) - 1}{m_\infty(z) + \mu} = m_\infty(z), \quad z \in \mathbb{C}_\pm.$$

If we assume that $\mu$ takes any finite real value, then (38) leads to $1 + m_\infty^2(z) = 0$ for all $z \in \mathbb{C}_\pm$ or $m_\infty(z) \equiv -i$ in the upper half-plane. That is impossible (see [11, 17]) and we are reaching a contradiction with the assumption that $\mu$ is finite and real. Thus, the only option is $\mu = \infty$. This proves the second part of the theorem. \qed

Similarly to Theorem 3, Theorem 4 above allows us to explicitly present a Shrödinger L-system whose impedance function matches $1/m_\infty(z)$. This is

$$(39) \quad \Theta_{\infty,i} = \left( \mathcal{H}_{\infty,i} \in L_2(\ell^+; +\infty) \subset \mathcal{H}_- \quad K_{\infty,i} 1 \quad \mathbb{C} \right),$$

where $\mathcal{H}_{\infty,i}$ is the Hilbert space of functions $f_{\infty,i}(x)$ such that $\int_{\ell^+} |f_{\infty,i}(x)|^2 \, dx < \infty$ and $K_{\infty,i} 1$ is the operator of multiplication by $1$ in the Hilbert space $\mathcal{H}_-$.
where
\[ A_{\infty,i} y = -y'' + q(x)y - [y'(\ell) - iy(\ell)] \delta(x - \ell), \]
\[ A^*_{\infty,i} y = -y'' + q(x)y - [y'(\ell) + iy(\ell)] \delta(x - \ell), \]
\[ K_{\infty,i} c = cg_{\infty,i}, \quad (c \in \mathbb{C}) \]
and (see also (42))
\[ Re A_{\infty,i} y = -y'' + q(x)y - y'(\ell)\delta(x - \ell), \]
Clearly,
\[ Re A_{\infty,i} y = -y'' + q(x)y - y'(\ell)\delta(x - \ell), \]
and (see also (43))
\[ \begin{cases} \hat{A}_{\infty,i} y = -y'' + q(x)y \\ y'(\ell) = 0 \end{cases}, \]
is the quasi-kernel of Re \( A_{\infty,i} \) in (42). Note that (43) defines a self-adjoint Shrödinger operator with \textit{Neumann} boundary conditions.

Also,
\[ V_{\Theta_{\infty,i}}(z) = \frac{1}{m_{\infty}(z)} \quad \text{and} \quad W_{\Theta_{\infty,i}}(z) = -\frac{1 + i m_{\infty}(z)}{1 - i m_{\infty}(z)}. \]
Thus, \( V_{\Theta_{\infty,i}}(z) \) and hence \( 1/m_{\infty}(z) \) has the following resolvent representation (see (3))
\[ \frac{1}{m_{\infty}(z)} = V_{\Theta_{\infty,i}}(z) = ((Re A_{\infty,i} - zI)^{-1}g_{\infty,i},g_{\infty,i}) \]
\[ = (\hat{A}_{\infty,i} - zI)^{-1}\delta(x - \ell), \delta(x - \ell) ; \]
where Re \( A_{\infty,i} \) is given by (12), \( g_{\infty,i} \) by (11), and \( \hat{A}_{\infty,i} = (A_{\infty,i} - zI)^{-1} \) is the extended resolvent of the quasi-kernel \( A_{\infty,i} \) in (13) (see (4)).

We note that both L-systems \( \Theta_{\infty,i} \) in (2) and \( \Theta_{\infty,i} \) in (3) share the same main operator
\[ \begin{cases} T_i y = -y'' + q(x)y \\ y'(\ell) = i y(\ell) \end{cases}. \]

5. Functions \( m_\alpha(z) \)

Let \( \hat{A} \) be a symmetric operator of the form (12) with deficiency indices (1,1), generated by the differential operation \( l(y) = -y'' + q(x)y \). Let also \( \varphi_\alpha(x,z) \) and \( \theta_\alpha(x,z) \) be the solutions of the following Cauchy problems:
\[ \begin{cases} l(\varphi_\alpha) = z\varphi_\alpha \\ \varphi_\alpha(\ell,z) = \sin \alpha \\ \varphi'_\alpha(\ell,z) = -\cos \alpha \end{cases}, \quad \begin{cases} l(\theta_\alpha) = z\theta_\alpha \\ \theta_\alpha(\ell,z) = \cos \alpha \\ \theta'_\alpha(\ell,z) = \sin \alpha \end{cases}. \]
It is known (11), (19), (21) that there exists an analytic in \( \mathbb{C}_\pm \) function \( m_\alpha(z) \) for which
\[ \psi(x,z) = \theta_\alpha(x,z) + m_\alpha(z)\varphi_\alpha(x,z) \]
belongs to \( L_2[\ell, +\infty) \). It is easy to see that if \( \alpha = \pi \), then \( m_\pi(z) = m_{\infty}(z) \). The functions \( m_\alpha(z) \) and \( m_{\infty}(z) \) are connected (see (11), (21)) by
\[ m_\alpha(z) = \frac{\sin \alpha + m_{\infty}(z)\cos \alpha}{\cos \alpha - m_{\infty}(z)\sin \alpha}. \]
Indeed, for any fixed \( z \in \mathbb{C}_\pm \) and any \( \alpha \) we have that \( \psi(x, z) \) belongs to one-dimensional deficiency subspace \( \mathfrak{N}_z = \text{Ker} (\hat{A}^* - zI) \). Hence,

\[
\theta_\alpha(x, z) + m_\alpha(z)\varphi_\alpha(x, z) = C(z)(\theta_\alpha(x, z) + m_\infty(z)\varphi_\alpha(x, z)), \quad x \in [\ell, +\infty),
\]

where \( C(z) \) is independent of \( x \). Setting \( x = \ell \), then differentiating and plugging in \( x = \ell \) again, we obtain

\[
\begin{align*}
\cos \alpha + m_\alpha(z) \sin \alpha &= C(z), \\
\sin \alpha - m_\alpha(z) \cos \alpha &= -C(z)m_\infty(z).
\end{align*}
\]

Eliminating \( C(z) \) we obtain \([58]\).

We know \([14], [22]\) that for any real \( \alpha \) the function \(-m_\alpha(z)\) is a Herglotz-Nevanlinna function. Also, modifying \([58]\) slightly we obtain

\[
(49) \quad -m_\alpha(z) = \frac{\sin \alpha + m_\infty(z) \cos \alpha}{-\cos \alpha + m_\infty(z) \sin \alpha} = \frac{\cos \alpha + \frac{1}{m_\infty(z)} \sin \alpha}{\sin \alpha - \frac{1}{m_\infty(z)} \cos \alpha}.
\]

Now we are going to state and prove the realization theorem for Herglotz-Nevanlinna functions \(-m_\alpha(z)\) that is similar to Theorem \([1]\).

**Theorem 5.** Let \( \hat{A} \) be a symmetric Schrödinger operator of the form \([12]\) with deficiency indices \((1, 1)\) and locally summable potential in \( \mathcal{H} = L^2[\ell, \infty) \). If \( m_\alpha(z) \) is the function of \( \hat{A} \) described in \([14]\), then the Herglotz-Nevanlinna function \((-m_\alpha(z))\) can be realized as the impedance function of a Schrödinger L-system \( \Theta_{\mu, h} \) of the form \([18]\) with

\[
\mu = \tan \alpha \quad \text{and} \quad h = i.
\]

**Conversely,** let \( \Theta_{\mu, h} \) be a Schrödinger L-system of the form \([18]\) with the symmetric operator \( \hat{A} \) such that

\[
V_{\Theta_{\mu, h}}(z) = -m_\alpha(z),
\]

for all \( z \in \mathbb{C}_\pm \) and \( \mu \in \mathbb{R} \cup \{\infty\} \). Then the parameters \( \mu \) and \( h \) defining \( \Theta_{\mu, h} \) are given by \([22]\), i.e., \( \mu = \tan \alpha \) and \( h = i \).

**Proof.** As we have shown in Section \([1]\), the function \( 1/m_\infty(z) \) can be realized as the impedance function of the Schrödinger L-system \( \Theta_{\infty, i} \) of the form \([59]\), i.e. \( 1/m_\infty(z) = V_{\Theta_{\infty, i}}(z) \) for all \( z \in \mathbb{C}_\pm \). Consequently, \([17]\) yields

\[
(51) \quad -m_\alpha(z) = \frac{\cos \alpha + V_{\Theta_{\infty, i}}(z) \sin \alpha}{\sin \alpha - V_{\Theta_{\infty, i}}(z) \cos \alpha}, \quad z \in \mathbb{C}_\pm.
\]

But the right hand side of \([17]\) is exactly the Donoghue transform \([1]\) of the impedance function \( V_{\Theta_{\infty, i}}(z) \) and thus represents the impedance function of a Schrödinger L-system with the same main operator \( T_i \) of the form \([14]\) (see \([2]\), Section 8.3). That is,

\[
-m_\alpha(z) = V_{\Theta_{\mu, i}}(z),
\]

where \( \mu_\alpha \) is a real parameter we need to find to describe the Schrödinger L-system \( \Theta_{\mu_\alpha, i} \) realizing \(-m_\alpha(z)\). In order to do that we utilize \([22]\) once more. Substituting the value of \( h = i \) into \([22]\) and applying \([17]\) we get

\[
(52) \quad V_{\Theta_{\mu_\alpha, i}}(z) = \frac{m_\infty(z) + \mu_\alpha}{\mu_\alpha m_\infty(z) - 1} = -m_\alpha(z) = \frac{\sin \alpha + m_\infty(z) \cos \alpha}{-\cos \alpha + m_\infty(z) \sin \alpha}, \quad z \in \mathbb{C}_\pm,
\]
or, after dividing the second fraction by \( \cos \alpha \),
\[
\frac{m_\infty(z) + \mu \alpha}{\mu \alpha m_\infty(z) - 1} = \frac{m_\infty(z) + \tan \alpha}{m_\infty(z) \tan \alpha - 1}, \quad z \in \mathbb{C}_+.
\]
Solving the above for \( \mu \alpha \) leads to that either \( m_\infty(z) \equiv -i \) in \( \mathbb{C}_+ \) (which is impossible \([11], [17]\)) or
\[
(53) \quad \mu \alpha = \tan \alpha.
\]
Therefore, if we set \( \mu = \tan \alpha \) and \( h = i \) in a Shrödinger L-system \( \Theta_{\mu, h} \) of the form \((58)\), then according to the above derivations we obtain \( V_{\Theta_{\mu, h}}(z) = -m_\alpha(z) \).

Conversely, if \( \Theta_{\mu, h} \) is a Shrödinger L-system of the form \((58)\) with our symmetric operator \( \hat{A} \) such that \( V_{\Theta_{\mu, h}}(z) = -m_\alpha(z) \), then \((52)\) takes place and implies \((53)\), that is \( \mu = \mu \alpha = \tan \alpha \) as shown above. The proof is complete. \( \square \)

We note that when \( \alpha = \pi \) we obtain \( \mu \alpha = 0 \), \( m_\alpha(z) = m_\infty(z) \), and the realizing Shrödinger L-system \( \Theta_{\mu, i} \) is thoroughly described by \((53)\) in Section 4. If \( \alpha = \pi/2 \), then we get \( \mu \alpha = \infty \), \( -m_\alpha(z) = 1/m_\infty(z) \), and the realizing Shrödinger L-system \( \Theta_{\infty, i} \) is given by \((53)\) in Section 4. Assuming that \( \alpha \in (0, \pi] \) and neither \( \alpha = \pi \) nor \( \alpha = \pi/2 \) we give the description of a Shrödinger L-system \( \Theta_{\mu, i} \) realizing \(-m_\alpha(z)\) as follows.

\[
\Theta_{\tan \alpha, i} = \left( K_{\tan \alpha, i} \right. \mathbb{C}_+ \subset L_2(l, +\infty) \subset \mathbb{H}_- \left. \begin{array}{c} \hat{A}_{\tan \alpha, i} \\
\mathbb{H}_+ \subset L_2(l, +\infty) \subset \mathbb{H}_- \end{array} \right),
\]
where
\[
\begin{align*}
\hat{A}_{\tan \alpha, i} y &= l(y) - \frac{1}{\tan \alpha - i} [y'(\ell) - iy(\ell)][(\tan \alpha) \delta(x - \ell) + \delta'(x - \ell)], \\
\hat{A}^*_{\tan \alpha, i} y &= l(y) - \frac{1}{\tan \alpha + i} [y'(\ell) + iy(\ell)][(\tan \alpha) \delta(x - \ell) + \delta'(x - \ell)], \\
K_{\tan \alpha, i} &= c g_{\tan \alpha, i}, \quad (c \in \mathbb{C}) \text{ and} \\
g_{\tan \alpha, i} &= (\tan \alpha) \delta(x - \ell) + \delta'(x - \ell).
\end{align*}
\]
Clearly,
\[
(56) \quad \text{Re} \hat{A}_{\tan \alpha, i} = l(y) - (\cos^2 \alpha)[(\tan \alpha) y'(\ell) + y(\ell)][(\tan \alpha) \delta(x - \ell) + \delta'(x - \ell)],
\]
and (see also \((50)\))
\[
(57) \quad \left\{ \begin{array}{l}
\hat{A}_{\tan \alpha, i} y = -y'' + q(x) y \\
y(\ell) = -(\tan \alpha) y'(\ell)
\end{array} \right.,
\]
is the quasi-kernel of \( \text{Re} \hat{A}_{\tan \alpha, i} \) in \((57)\). Also,
\[
\begin{align*}
V_{\hat{\Theta}_{\tan \alpha, i}}(z) &= -m_\alpha(z) \\
W_{\hat{\Theta}_{\tan \alpha, i}}(z) &= \frac{\tan \alpha - i}{\tan \alpha + i} \frac{m_\infty(z) - i}{m_\infty(z) + i} = (-e^{2\alpha i}) \frac{m_\infty(z) - i}{m_\infty(z) + i}.
\end{align*}
\]
Thus, \( V_{\hat{\Theta}_{\tan \alpha, i}}(z) \) and hence \(-m_\alpha(z)\) has the following resolvent representation
\[
(58) \quad -m_\infty(z) = V_{\hat{\Theta}_{\tan \alpha, i}}(z) = (\text{Re} \hat{A}_{\tan \alpha, i} - z I)^{-1} g_{\tan \alpha, i},
\]
where \( \text{Re} \hat{A}_{\tan \alpha, i} \) is given by \((52)\), \( g_{\tan \alpha, i} \) by \((53)\), and \((\hat{A}_{\tan \alpha, i} - z I)^{-1} \) is the extended resolvent of the quasi-kernel \( \hat{A}_{\tan \alpha, i} \) in \((58)\) (see [2]).
Now we are going to state and prove the realization theorem for Herglotz-Nevanlinna functions $1/m_\alpha(z)$ that is similar to Theorem 1.

**Theorem 6.** Let $\hat{A}$ be a symmetric Schrödinger operator of the form (12) with deficiency indices $(1,1)$ and locally summable potential in $H = L^2[\ell, \infty)$. If $m_\alpha(z)$ is the function of $\hat{A}$ described in (17), then the Herglotz-Nevanlinna function $1/m_\alpha(z)$ can be realized as the impedance function of a Schrödinger L-system $\Theta_{\mu,h}$ of the form (18) with

$$\mu = -\cot \alpha \quad \text{and} \quad h = i. \quad (61)$$

Conversely, let $\Theta_{\mu,h}$ be a Schrödinger L-system of the form (18) with the symmetric operator $\hat{A}$ such that

$$V_{\Theta_{\mu,i}}(z) = \frac{1}{m_\alpha(z)}, \quad \text{for all } z \in \mathbb{C}_+ \text{ and } \mu \in \mathbb{R} \cup \{\infty\}. \quad (62)$$

Then the parameters $\mu$ and $h$ defining $\Theta_{\mu,h}$ are given by (61), i.e., $\mu = -\cot \alpha$ and $h = i$.

**Proof.** On order to prove the first part of the theorem we simply observe that the functions $(-m_\alpha(z))$ and $1/m_\alpha(z)$ are connected via similar to the middle part of (21) relation. Hence, if, according to Theorem 4, $(-m_\alpha(z))$ is realized by an L-system (13) or (see (20))

$$-m_\alpha(z) = V_{\Theta_{\tan,\alpha}}(z),$$

then $1/m_\alpha(z)$ is realized [2, Section 10.2] by an Schrödinger L-system $\Theta_{\xi,h}$ with the same parameter $h = i$ and the parameter $\xi$ related to $\mu = \tan \alpha$ by the right part of (21). This gives

$$\xi = \frac{\mu \Re h - |h|^2}{\mu - \Re h} = -\frac{1}{\tan \alpha} = -\cot \alpha.$$

Thus, $\Theta_{-\cot \alpha, i}$ realizes $1/m_\alpha(z)$.

Conversely, if $\Theta_{\mu,h}$ is a Schrödinger L-system of the form (18) with the symmetric operator $\hat{A}$ such that $V_{\Theta_{\mu,h}}(z) = 1/m_\alpha(z)$, then $\Theta_{\mu,h}$ shares the same main operator with $\Theta_{\tan,\alpha, i}$ and hence $h = i$. Moreover, (see (20) and (18))

$$V_{\Theta_{\mu,i}}(z) = \frac{\mu_\alpha m_\infty(z) + \mu m_\infty(z)}{\mu_\alpha m_\alpha(z) - 1} = \frac{1}{m_\alpha(z)} = \frac{\cos \alpha - m_\infty(z) \sin \alpha}{\sin \alpha + m_\infty(z) \cos \alpha} \quad z \in \mathbb{C}_+.$$

Solving the above for $\mu_\alpha$ leads to that either $m_\infty(z) \equiv -i$ in $\mathbb{C}_+$ (which is impossible [11, 17]) or

$$\mu = \mu_\alpha = -\cot \alpha.$$

The proof is complete. \[ \square \]

Assuming again that $\alpha \in (0, \pi)$ and $\alpha \neq \pi/2$ we give the description of a Schrödinger L-system $\Theta_{\mu_\alpha, i}$ realizing $1/m_\alpha(z)$ as follows.

$$\Theta_{(-\cot \alpha), i} = \left( \mathcal{H}_+ \subset L^2[\ell, +\infty) \subset \mathcal{H}_- \quad \begin{bmatrix} K_{(-\cot \alpha), i} & 1 \\ \mathcal{A}_{(-\cot \alpha), i} & \mathcal{C} \end{bmatrix} \right), \quad (63)$$

where

$$\mathcal{A}_{(-\cot \alpha), i} y = l(y) + \frac{1}{\cos \alpha + i} [y'(\ell) - iy(\ell)][(-\cot \alpha)\delta(x - \ell) + \delta'(x - \ell)], \quad (64)$$

$$\mathcal{A}^*_{(-\cot \alpha), i} y = l(y) + \frac{1}{\cos \alpha - i} [y'(\ell) + iy(\ell)][(-\cot \alpha)\delta(x - \ell) + \delta'(x - \ell)], \quad (65)$$

and

$$K_{(-\cot \alpha), i} y = \frac{1}{\cos \alpha + i} [y'(\ell) - iy(\ell)]\delta(x - \ell) + \delta'(x - \ell).$$
We conclude the section with the following result.

**Theorem 7.** Let \( \Theta_{\mu,i} \) be a Shrödinger L-system of the form \([18]\) with \( h = i \). Then there exists a unique value of \( \alpha \in (0, \pi] \) such that

\[
V_{\Theta_{\mu,i}}(z) = -m_\alpha(z),
\]

where \( m_\alpha(z) \) is defined in \([18]\).

**Proof.** As we did in the proof of Theorem 1 (see \([22]\)) we set

\[
V_{\Theta_{\mu,i}}(z) = \frac{m_\infty(z) + \mu}{\mu m_\infty(z) - 1} = -m_\alpha(z) = \frac{\sin \alpha + m_\infty(z) \cos \alpha}{-\cos \alpha + m_\infty(z) \sin \alpha}, \quad z \in \mathbb{C}_\pm,
\]

which leads to (see \([23]\)) \( \mu = \tan \alpha \).

Therefore, any value of \( \mu \in \mathbb{R} \cup \{ \pm \infty \} \) produces a unique value of \( \alpha \in (0, \pi] \) and thus \( m_\alpha(z) \) defined by \([18]\). Consequently, \([23]\) is true. \( \square \)

Clearly, trigonometry implies that a similar to Theorem 1 result takes place if one replaces \([20]\) with \( V_{\Theta_{\mu,i}}(z) = 1/m_\alpha(z) \).

### 6. Non-negative Schrödinger operator case

Now let us assume that \( \hat{A} \) is a non-negative (i.e., \( (\hat{A}f, f) \geq 0 \) for all \( f \in \text{Dom}(\hat{A}) \)) symmetric operator of the form \([12]\) with deficiency indices \((1,1)\), generated by the differential operation \( l(y) = -y'' + q(x)y \). The following theorem takes place.
Theorem 8 ([3], [22], [23]). Let $\hat{A}$ be a nonnegative symmetric Schrödinger operator of the form (14) with deficiency indices $(1, 1)$ and locally summable potential in $\mathcal{H} = L^2(\ell, \infty)$. Consider operator $T_h$ of the form (13). Then

1. operator $\hat{A}$ has more than one non-negative self-adjoint extension, i.e., the Friedrichs extension $A_F$ and the Krein-von Neumann extension $A_K$ do not coincide, if and only if $m_{\infty}(-0) < \infty$;
2. operator $T_h$, $(h = \hat{h})$ coincides with the Krein-von Neumann extension $A_K$ if and only if $h = -m_{\infty}(-0)$;
3. operator $T_h$ is accretive if and only if

\[ \text{Re} h \geq -m_{\infty}(-0); \]

4. operator $T_h$, $(h \neq \hat{h})$ is $\beta$-sectorial if and only if $\text{Re} h > -m_{\infty}(-0)$ holds;
5. operator $T_h$, $(h \neq \hat{h})$ is accretive but not $\beta$-sectorial for any $\beta \in (0, \frac{\pi}{2})$ if and only if $\text{Re} h = -m_{\infty}(-0)$
6. If $T_h, (\text{Im} h > 0)$ is $\beta$-sectorial, then the exact angle $\beta$ can be calculated via

\[ \tan \beta = \frac{\text{Im} h}{\text{Re} h + m_{\infty}(-0)}. \]

For the remainder of this paper we assume that $m_{\infty}(-0) < \infty$. Then according to Theorem 8 above (see also [22], [23]) we have the existence of the operator $T_h, (\text{Im} h > 0)$ that is accretive and/or sectorial. It was shown in [2] that if $T_h (\text{Im} h > 0)$ is an accretive Schrödinger operator of the form (13), then for all real $\mu$ satisfying the following inequality

\[ \mu \geq \frac{(\text{Im} h)^2}{m_{\infty}(-0) + \text{Re} h} + \text{Re} h, \]

formulas (14) define the set of all accretive $(\ast)$-extensions $\mathcal{A}_{\mu, h}$ of the operator $T_h$. Moreover, an accretive $(\ast)$-extensions $\mathcal{A}_{\mu, h}$ of a sectorial operator $T_h$ with exact angle of sectoriality $\beta \in (0, \frac{\pi}{2})$ also preserves the same exact angle of sectoriality if and only if $\mu = +\infty$ in (14) (see [4]. Theorem 3). Also, $\mathcal{A}_{\mu, h}$ is accretive but not $\beta$-sectorial for any $\beta \in (0, \frac{\pi}{2}) (\ast)$-extension of $T_h$ if and only if in (14)

\[ \mu = \frac{(\text{Im} h)^2}{m_{\infty}(-0) + \text{Re} h} + \text{Re} h, \]

(see [4. Theorem 4]). An accretive operator $T_h$ has a unique accretive $(\ast)$-extension $\mathcal{A}_{\infty, h}$ if and only if

\[ \text{Re} h = -m_{\infty}(-0). \]

In this case this unique $(\ast)$-extension has the form

\[ \mathcal{A}_{\infty, h} y = -y'' + q(x) y + \left[ h y(\ell) - y'(\ell) \right] \delta(x - \ell), \]

\[ \mathcal{A}_{\infty, h}^* y = -y'' + q(x) y + \left[ h y(\ell) - y'(\ell) \right] \delta(x - \ell). \]

Now we will see how the additional requirement of non-negativity affects the realization of functions $-m_{\infty}(z)$ and $1/m_{\infty}(z)$.

Theorem 9. Let $\hat{A}$ be a non-negative symmetric Schrödinger operator of the form (14) with deficiency indices $(1, 1)$ and locally summable potential in $\mathcal{H} = L^2(\ell, \infty)$. If $m_{\infty}(z)$ is the Weyl-Titchmarsh function of $\hat{A}$, then the L-system $\Theta_{0,i}$ of the form (28), realizing the function $(-m_{\infty}(z))$ is never accretive. The L-system

...
\(\Theta_{\infty,i}\) of the form \([3]\) realizing the function \(1/m_\infty(z)\) is accretive if and only if \(m_\infty(-0) \geq 0\).

**Proof.** First, let us consider the L-system \(\Theta_{0,i}\) of the form \([26]\) realizing the function \(-m_\infty(z)\). According to Theorem 9 we have that the main operator \(T_i\) of the form \([10]\) is accretive if and only if \([75]\) holds, that is \(0 = \Re h \geq -m_\infty(-0)\) or

\[m_\infty(-0) \geq 0.\]

Assume that \([73]\) holds (if it does not, the L-system \(\Theta_{0,i}\) can not be accretive a priori). Then, applying \([72]\) we conclude that a (\(\ast\))-extensions \(\Lambda_{0,i}\) of the operator \(T_i\) is accretive if

\[0 = \mu \geq \frac{(\Im h)^2}{m_\infty(-0) + \Re h} + \Re h = \frac{1}{m_\infty(-0)},\]

that contradicts our assumption \([73]\) since \(m_\infty(-0) < \infty\). Thus, \(\Lambda_{0,i}\) of the form \([26]\) is not accretive and hence the L-system \(\Theta_{0,i}\) of the form \([26]\) realizing the function \(-m_\infty(z)\) can not be accretive.

Now let us consider the L-system \(\Theta_{\infty,i}\) of the form \([33]\) realizing the function \(1/m_\infty(z)\). As we have shown above the main operator \(T_i\) of the form \([10]\) is accretive if and only if \([72]\) holds. Then, applying \([72]\) we conclude that a (\(\ast\))-extensions \(\Lambda_{\infty,i}\) of the operator \(T_i\) is accretive if and only if

\[+\infty = \mu \geq \frac{(\Im h)^2}{m_\infty(-0) + \Re h} + \Re h = \frac{1}{m_\infty(-0)},\]

that is always true due to \([71]\) and the assumption of our theorem. Consequently, the L-system \(\Theta_{\infty,i}\) of the form \([33]\) realizing the function \(1/m_\infty(z)\) is accretive. \(\Box\)

Now we are going to turn to functions \(m_\alpha(z)\) described by \([47]-[48]\) and associated with the non-negative operator \(\hat{A}\) above. We will see how the parameter \(\alpha\) in the definition of \(m_\alpha(z)\) affects the L-system realizing \((-m_\alpha(z))\). The following theorem answers that question.

**Theorem 10.** Let \(\hat{A}\) be a non-negative symmetric Schrödinger operator of the form \([12]\) with deficiency indices \((1, 1)\) and locally summable potential in \(\mathcal{H} = L^2(l, \infty)\) and such that \(m_\infty(-0) \geq 0\). If \(m_\alpha(z)\) is described by \([47]-[48]\), then the L-system \(\Theta_{\tan \alpha,i}\) of the form \([54]\) realizing the function \((-m_\alpha(z))\) is accretive if and only if

\[\tan \alpha \geq \frac{1}{m_\infty(-0)}.\]

**Proof.** Let \(\Theta_{\tan \alpha,i}\) be the Schrödinger L-system of the form \([54]\) realizing the function \((-m_\alpha(z))\) and \(T_i\) of the form \([10]\) be its main operator. Clearly, if \(\Theta_{\tan \alpha,i}\) is accretive, then both main operator \(T_i\) and state-space operator \(\Lambda_{\tan \alpha,i}\) of the form \([33]\) must be accretive. Then the condition (3) in the statement of Theorem 8 applied to the operator \(T_i\) says that it is accretive if and only if

\[0 = \Re h \geq -m_\infty(-0)\] or \(m_\infty(-0) \geq 0.\)

Hence, the necessary condition for \(\Theta_{\tan \alpha,i}\) to be accretive is \(m_\infty(-0) \geq 0\). Applying \([72]\) we conclude that a \(\Lambda_{\tan \alpha,i}\) is accretive if and only if

\[\tan \alpha \geq \frac{(\Im h)^2}{m_\infty(-0) + \Re h} + \Re h = \frac{1}{m_\infty(-0)}.\]
Note that if \( m_\infty(-0) = 0 \) in (76), then \( \alpha = \pi/2 \) and \( -m_\infty(z) = 1/m_\infty(z) \). From Theorem 9 we know that if \( m_\infty(-0) \geq 0 \), then \( 1/m_\infty(z) \) is realized by an accretive system \( \Theta_\infty,i \) of the form (59).

Now once we established a criteria for an L-system realizing \( (-m_\alpha(z)) \) to be accretive, we can look into more of its properties. There are two choices for an accretive L-system \( \Theta_{\tan \alpha,i} \): it is either (1) accretive sectorial or (2) accretive extremal. In the case (1) we have that \( A_{\tan \alpha,i} \) of the form (55) is \( \beta_1 \)-sectorial with some angle of sectoriality \( \beta_1 \) that can only exceed the exact angle of sectoriality \( \beta \) of \( T_i \). In the case (2) the state-space operator \( A_{\tan \alpha,i} \) is extremal (not sectorial for any \( \beta \in (0, \pi/2) \)) and is a \( (\ast) \)-extension of \( T_i \) that itself can be either \( \beta \)-sectorial or extremal. The following theorem describes all these possibilities.

**Theorem 11.** Let \( \Theta_{\tan \alpha,i} \) be the accretive L-system realizing the function \( (-m_\alpha(z)) \) described in Theorem 10. The following is true:

1. if \( m_\infty(-0) = 0 \), then there is only one accretive L-system \( \Theta_\infty,i \) realizing \( (-m_\alpha(z)) \). This L-system is extremal and its main operator \( T_i \) is extremal as well.

2. if \( m_\infty(-0) > 0 \), then \( T_i \) is \( \beta \)-sectorial for \( \beta \in (0, \pi/2) \) and
   a) if \( \tan \alpha = 1/m_\infty(-0) \), then \( \Theta_{\tan \alpha,i} \) is extremal;
   b) if \( 1/m_\infty(-0) < \tan \alpha < +\infty \), then \( \Theta_{\tan \alpha,i} \) is \( \beta_1 \)-sectorial with \( \beta_1 > \beta \);
   c) if \( \tan \alpha = +\infty \), then \( \Theta_{\infty,i} \) is \( \beta \)-sectorial.

**Proof.** (1) As we already noted above if \( m_\infty(-0) = 0 \), then \( \alpha = \pi/2 \) and \( -m_\infty(z) = 1/m_\infty(z) \). Also, the condition (5) in the statement of Theorem 8 implies that \( T_i \) is extremal since \( Re h = -m_\infty(-0) = 0 \). Thus, \( \Theta_{\tan \alpha,i} \) is extremal as well and (77) yields that \( \tan \alpha = +\infty \). Therefore, \( \Theta_{\infty,i} \) is the only accretive L-system realizing \( (-m_\alpha(z)) \) in this case.

(2) If \( m_\infty(-0) > 0 \), then the condition (4) in the statement of Theorem 8 implies that \( T_i \) is \( \beta \)-sectorial with \( \beta \in (0, \pi/2) \) and the exact angle of sectoriality \( \beta \) is given by (77) as

\[
\tan \beta = \frac{Im h}{Re h + m_\infty(-0)} = \frac{1}{m_\infty(-0)} \leq \tan \alpha,
\]

where the last inequality is due to the fact that \( \Theta_{\tan \alpha,i} \) is accretive and hence (78) takes place. If we assume (2a), then for the L-system \( \Theta_{\tan \alpha,i} \) we have

\[
\mu = \tan \alpha = \frac{1}{m_\infty(-0)} = \frac{(Im h)^2}{m_\infty(-0) + Re h} + Re h,
\]
and hence according to (73) we have that $A_{\tan, i}$ is accretive but not $\beta$-sectorial for any $\beta \in (0, \pi/2)$. Consequently, $\Theta_{\tan, i}$ is extremal.

If we assume (2b), then $\mu = \tan \alpha$ is finite but strictly greater than $\tan \beta = \frac{1}{m_{\infty}(-0)}$ and hence (72) is not true. Therefore, the accretive $A_{\tan, i}$ cannot be extremal and thus is $\beta_1$-sectorial for some $\beta_1 \in (0, \pi/2)$. Since $A_{\tan, i}$ is a ($\ast$)-extension of the $\beta$-sectorial operator $T_i$, then $\beta_1 > \beta$. Thus, $\Theta_{\tan, i}$ is $\beta_1$-sectorial with $\beta_1 > \beta$.

Our last possible option is (2c) where $\mu = \tan \alpha = +\infty$. We know (see [6, Theorem 3]) that in this case $A_{\tan, i}$ preserves the same exact angle of sectoriality as in $T_i$. As a result $\Theta_{\infty, i}$ is $\beta$-sectorial.

The proof is complete. □

Theorem 12. Let $\hat{A}$ be a non-negative symmetric Schrödinger operator of the form (12) with deficiency indices $(1, 1)$ and locally summable potential in $\mathcal{H} = L^2[\ell, \infty)$. Then:

1. the function $1/m_{\infty}(z)$ is Stieltjes if and only if $m_{\infty}(-0) \geq 0$;
2. the function $(-m_{\infty}(z))$ is never Stieltjes;
3. the function $(-m_{\alpha}(z))$ given by (48) is Stieltjes if and only if $0 < \frac{1}{m_{\infty}(-0)} \leq \tan \alpha$.

Proof. It was shown in [2, Section 9.8] that the impedance function of an L-system is Stieltjes if and only if this L-system is accretive. The rest of the proof immediately follows from Theorems 9 and 10. □

We note that the Schrödinger L-systems $\Theta_{0, i}$ of the form (28) and $\Theta_{\infty, i}$ of the form (39) that we described in Theorem 1 in this section have special properties. It was shown in [2] (see also [1]) that the quasi-kernel $\hat{A}_0$ of $\text{Re} A_{0, i}$ of the form (60) corresponds to the Friedrich’s extension while the quasi-kernel $\hat{A}_{\infty}$ of $\text{Re} A_{\infty, i}$ corresponds to the Krein-von Neumann extension of our symmetric operator $\hat{A}$ only in the case when $m_{\infty}(-0) = 0$.

7. Uniqueness of Schrödinger L-systems

We start this section with the definition of two equal L-systems of the form (4).

Definition 13. Two L-systems

$$\Theta_1 = \begin{pmatrix} A_1 & K_1 \\ H_{1+} \subset \mathcal{H} \subset H_{1-} & 1 \\ \mathcal{C} & \end{pmatrix}$$

and

$$\Theta_2 = \begin{pmatrix} A_2 & K_2 \\ H_{2+} \subset \mathcal{H} \subset H_{2-} & 1 \\ \mathcal{C} & \end{pmatrix}$$

\(^{1}\)It will be shown in an upcoming paper that if $m_{\infty}(-0) \geq 0$, then the function $(-m_{\infty}(z))$ is actually inverse Stieltjes.
are equal if $H_{1+} = H_{2+}$, $\lambda_1 = \lambda_2$, and $K_1 = K_2$.

In this section we are going to look into uniqueness issues as applied to Schrödinger L-systems of the form \( [8] \). The main question to consider is when two identical impedance functions guarantee two equal (in the sense of Definition \( [3] \)) Schrödinger L-systems they represent. Suppose

$$\Theta_1 = \Theta_{\mu_1, \lambda_1} = \begin{pmatrix} \lambda_1 & K_1 \\ H_{1+} \subset L_2(\ell, +\infty) \subset H_{1-} \\ \mathbb{C} \end{pmatrix}$$

and

$$\Theta_2 = \Theta_{\mu_2, \lambda_2} = \begin{pmatrix} \lambda_2 & K_2 \\ H_{2+} \subset L_2(\ell, +\infty) \subset H_{2-} \\ \mathbb{C} \end{pmatrix}$$

be two Schrödinger L-systems of the form \( [13] \) corresponding to two generally speaking different symmetric in \( L_2(\ell, +\infty) \) operators

$$\begin{cases} \dot{A}_1 y = -y'' + q_1(x)y \\ y(\ell) = y'(\ell) = 0 \end{cases}$$

and

$$\begin{cases} \dot{A}_2 y = -y'' + q_2(x)y \\ y(\ell) = y'(\ell) = 0 \end{cases}$$

of the form \( [2] \). Let also $m_{\infty, 1}(z)$ and $m_{\infty, 2}(z)$ be the Weyl-Titchmarsh functions of $A_1$ and $A_2$, respectively. We will see under what conditions $V_{\Theta_1}(z) = V_{\Theta_2}(z)$ would imply that $\Theta_1 = \Theta_2$. Our first result is related to functions $m_{\alpha}(z)$ of the form \( [8] \).

**Theorem 14.** Let $m_{\alpha_1}(z)$ and $m_{\alpha_2}(z)$ be the functions of the form \( [8] \) related to the operators $A_1$ and $A_2$ of the forms \( [82] \) and \( [83] \). Let also $\Theta_1 = \Theta_{\tan \alpha_1, i}$ and $\Theta_2 = \Theta_{\tan \alpha_2, i}$ be Schrödinger L-systems of the form \( [8] \) that realize the functions \(-m_{\alpha_1}(z)\) and \(-m_{\alpha_2}(z)\), respectively. If $m_{\infty, 1}(z) = m_{\infty, 2}(z)$, $(z \in \mathbb{C}_\pm)$, then

$$V_{\Theta_1}(z) = V_{\Theta_2}(z), \quad (z \in \mathbb{C}_\pm)$$

implies $\Theta_1 = \Theta_2$.

**Proof.** First we will show that the equality $m_{\infty, 1}(z) = m_{\infty, 2}(z)$, $(z \in \mathbb{C}_\pm)$ in addition to

$$V_{\Theta_1}(z) = V_{\Theta_{\tan \alpha_1, i}}(z) = V_{\Theta_2}(z) = V_{\Theta_{\tan \alpha_2, i}}(z), \quad (z \in \mathbb{C}_\pm)$$

yields that $\tan \alpha_1 = \tan \alpha_2$. We know that according to Theorem \( [4] \)

$$(-m_{\alpha_1}(z)) = V_{\Theta_{\tan \alpha_1, i}}(z)$$

and

$$(-m_{\alpha_2}(z)) = V_{\Theta_{\tan \alpha_2, i}}(z)$$

for all $z \in \mathbb{C}_\pm$. Hence,

$$m_{\alpha_1}(z) = \frac{\sin \alpha_1 + m_{\infty, 1}(z) \cos \alpha_1}{\cos \alpha_1 - m_{\infty, 1}(z) \sin \alpha_1} = m_{\alpha_2}(z) = \frac{\sin \alpha_2 + m_{\infty, 2}(z) \cos \alpha_2}{\cos \alpha_2 - m_{\infty, 2}(z) \sin \alpha_2}.$$

Since $m_{\infty, 1}(z) = m_{\infty, 2}(z) = m_{\infty}(z)$, then we have

$$\frac{\sin \alpha_1 + m_{\infty}(z) \cos \alpha_1}{\cos \alpha_1 - m_{\infty}(z) \sin \alpha_1} = \frac{\sin \alpha_2 + m_{\infty}(z) \cos \alpha_2}{\cos \alpha_2 - m_{\infty}(z) \sin \alpha_2}.$$

or, after simple calculations,

$$\left(1 + m_{\infty}(z)^2\right) \left(\sin \alpha_1 \cos \alpha_2 - \cos \alpha_1 \sin \alpha_2\right) = 0, \quad (\forall z \in \mathbb{C}_\pm).$$
First set of parentheses can not be identical zero because it leads to \(m_\infty(z) \equiv -i\), which is impossible as we explained earlier in the paper. Thus,

\[
\sin \alpha_1 \cos \alpha_2 - \cos \alpha_1 \sin \alpha_2 = \sin(\alpha_1 - \alpha_2) = 0,
\]

implying \(\alpha_1 = \alpha_2 + \pi k, k \in \mathbb{Z}\). Therefore, \(\tan \alpha_1 = \tan \alpha_2\).

Now, the fact that \(m_{\infty,1}(z) = m_{\infty,2}(z)\) allows us to use the fundamental Borg-Marchenko uniqueness theorem to conclude that the potentials \(q_1(x)\) and \(q_2(x)\) in \((\mathfrak{S})\) and \((\mathfrak{T})\) are the same. Taking into account that \(\tan \alpha_1 = \tan \alpha_2\) we have that \(\Theta_1 = \Theta_{\tan \alpha_1, i}\) and \(\Theta_2 = \Theta_{\tan \alpha_2, i}\) are equal by construction. \(\Box\)

Now we are going to state and prove a bit more general result.

**Theorem 15.** Let \(\Theta_1 = \Theta_{\mu_1, h_1}\) and \(\Theta_2 = \Theta_{\mu_2, h_2}\) be Schrödinger L-systems of the form \((\mathfrak{P})\) and \((\mathfrak{Q})\) with \(\mu_1 = \mu_2 = \mu\) and \(h_1 = h_2 = h\), respectively. If \(V_{\Theta_1}(z) = V_{\Theta_2}(z), (z \in \mathbb{C}_\pm)\), then \(m_{\infty,1}(z) = m_{\infty,2}(z), (z \in \mathbb{C}_\pm)\) and \(\Theta_1 = \Theta_2\).

**Proof.** The equality of impedance functions \(V_{\Theta_1}(z) = V_{\Theta_2}(z)\) and \((\mathfrak{Q})\) imply

\[
\frac{m_{\infty,1}(z) + \mu}{(\mu - Re h) m_{\infty,1}(z) + \mu Re h - |h|^2} = \frac{m_{\infty,2}(z) + \mu}{(\mu - Re h) m_{\infty,2}(z) + \mu Re h - |h|^2}.
\]

Then

\[
(m_{\infty,1}(z) + \mu)((\mu - Re h) m_{\infty,2}(z) + \mu Re h - |h|^2)
= (m_{\infty,2}(z) + \mu)((\mu - Re h) m_{\infty,1}(z) + \mu Re h - |h|^2),
\]

which leads to

\[
(m_{\infty,1}(z) - m_{\infty,2}(z))(\mu^2 - 2 \mu Re \mu + |h|^2) = 0, \quad (\forall z \in \mathbb{C}_\pm).
\]

Assuming that \(m_{\infty,1}(z) \neq m_{\infty,2}(z)\) in \(\mathbb{C}_\pm\) brings us to the quadratic equation in \(\mu\)

\[
\mu^2 - 2 \mu Re \mu + |h|^2 = 0.
\]

Applying the quadratic formula gives us \(\mu = Re h \pm (Im h) i\). This contradicts the fact that \(\mu\) must be real since \(Im h \neq 0\). Therefore, we arrived at a contradiction and the only logical choice is \(m_{\infty,1}(z) = m_{\infty,2}(z), (z \in \mathbb{C}_\pm)\). Now we apply the Borg-Marchenko uniqueness theorem to conclude that the potentials \(q_1(x)\) and \(q_2(x)\) in \((\mathfrak{S})\) and \((\mathfrak{T})\) are the same. Taking into account that \(\mu_1 = \mu_2\) and \(h_1 = h_2\) we have that \(\Theta_1 = \Theta_{\mu_1, h_1}\) and \(\Theta_2 = \Theta_{\mu_2, h_2}\) are equal by construction. \(\Box\)

Summarizing the above derivations we arrive at the following uniqueness criterion for Schrödinger L-systems with equal boundary parameters \(\mu\) and \(h\).

**Theorem 16.** Let \(\Theta_1 = \Theta_{\mu_1, h_1}\) and \(\Theta_2 = \Theta_{\mu_2, h_2}\) be Schrödinger L-systems of the form \((\mathfrak{P})\) and \((\mathfrak{Q})\) with \(\mu_1 = \mu_2 = \mu\) and \(h_1 = h_2 = h\), respectively. Then \(\Theta_1 = \Theta_2\) if and only if \(V_{\Theta_1}(z) = V_{\Theta_2}(z), (z \in \mathbb{C}_\pm)\).

**Proof.** In one direction the theorem is obvious. If \(\Theta_1 = \Theta_2\), then clearly Definition \((\mathfrak{R})\) implies that \(V_{\Theta_1}(z) = V_{\Theta_2}(z), (z \in \mathbb{C}_\pm)\).

In the other direction follows from Theorem \((\mathfrak{L})\). Indeed, if \(V_{\Theta_1}(z) = V_{\Theta_2}(z), (z \in \mathbb{C}_\pm)\), then according to Theorem \((\mathfrak{S})\) we have that \(m_{\infty,1}(z) = m_{\infty,2}(z), (z \in \mathbb{C}_\pm)\) and \(\Theta_1 = \Theta_2\). \(\Box\)

Now let us consider the case of Schrödinger L-systems that share the same main operator but have different impedance functions.
THEOREM 17. Let $\Theta_1 = \Theta_{\mu_1, h}$ and $\Theta_2 = \Theta_{\mu_2, h}$ be Schrödinger L-systems of the form (80) and (81) with

$$\mu_2 = \frac{\mu_1 \Re h - |h|^2}{\mu_1 - \Re h}. \tag{85}$$

If

$$V_{\Theta_1}(z) = -\frac{1}{V_{\Theta_2}(z)}, \quad (z \in \mathbb{C}_\pm), \tag{86}$$

then $m_{\infty, 1}(z) = m_{\infty, 2}(z), \quad (z \in \mathbb{C}_\pm)$ and $\Theta_1$ and $\Theta_2$ share the same main operator.

**Proof.** Formula (82) together with (21) yields

$$\frac{(m_{\infty, 1}(z) + \mu_1) \Im h}{(\mu_1 - \Re h) m_{\infty, 1}(z) + \mu_1 \Re h - |h|^2} = -\frac{(\mu_2 - \Re h) m_{\infty, 2}(z) + \mu_2 \Re h - |h|^2}{(m_{\infty, 2}(z) + \mu_2) \Im h}.$$ 

Substituting (84) into the right hand side of the above and simplifying gives us

$$\frac{m_{\infty, 1}(z) + \mu_1}{(\mu_1 - \Re h) m_{\infty, 1}(z) + \mu_1 \Re h - |h|^2} = \frac{m_{\infty, 2}(z) + \mu_1}{(\mu_1 - \Re h) m_{\infty, 2}(z) + \mu_1 \Re h - |h|^2},$$

that is exact analogue of (84) from the proof of Theorem 14. Following the corresponding steps of the proof of Theorem 15 we obtain that $m_{\infty, 1}(z) = m_{\infty, 2}(z), \quad (z \in \mathbb{C}_\pm)$. Then we apply the Borg-Marchenko uniqueness theorem [10, 20] and conclude that the potentials $q_1(x)$ and $q_2(x)$ in (82) and (83) are the same. Consequently, the L-systems $\Theta_1$ and $\Theta_2$ share the same main operator. \square

Below we provide a generalized version of Theorem 17.

THEOREM 18. Let $\Theta_1 = \Theta_{\mu, h}$ and $\Theta_2 = \Theta_{\mu_2, h}$ be Schrödinger L-systems of the form (80) and (81) with $h = h_1 = h_2, \quad \mu_1 = \mu$, and

$$\mu_2 = \mu(\alpha) = \frac{\mu - \bar{h} + e^{2i\alpha}(\mu - \bar{h})\bar{h}}{\mu - \bar{h} + e^{2i\alpha}(\mu - h).} \tag{87}$$

If

$$V_{\Theta_2}(z) = \frac{\cos \alpha + (\sin \alpha)V_{\Theta_1}(z)}{\sin \alpha - (\cos \alpha)V_{\Theta_1}(z)} \tag{88}$$

then $m_{\infty, 1}(z) = m_{\infty, 2}(z), \quad (z \in \mathbb{C}_\pm)$ and $\Theta_1$ and $\Theta_2$ share the same main operator.

Conversely, if two Schrödinger L-systems $\Theta_1 = \Theta_{\mu_1, h}$ and $\Theta_2 = \Theta_{\mu_2, h}$ of the form (80) and (81) share the same main operator, then their impedance functions $V_{\Theta_1}(z)$ and $V_{\Theta_2}(z)$ are related by (86) and $\mu_2$ and $\mu_1$ are connected with (87).

**Proof.** Our proof will be based on the method shown in the proof of Lemma 2. It was shown in [2] Section 8.3 that if the impedance functions $V_{\Theta_1}(z)$ and $V_{\Theta_1}$ are connected by the Donoghue transform (88) (see also (1) and (2)), then the corresponding transfer functions are related by

$$W_{\Theta_{\mu_2, \eta}}(z) = (-e^{2i\alpha}) \cdot W_{\Theta_{\mu_1, \eta}}(z). \tag{89}$$

Combining (88) with (88) above and setting $U = -e^{2i\alpha}$ temporarily we obtain

$$\frac{\mu_2 - h}{\mu_2 - \bar{h}} \frac{m_{\infty, 2}(z) + \bar{h}}{m_{\infty, 2}(z) + h} = U \cdot \frac{\mu - h}{\mu - \bar{h}} \frac{m_{\infty, 1}(z) + \bar{h}}{m_{\infty, 1}(z) + h}. \tag{90}$$
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Substituting the value of \( \mu_2 \) from (87) into the first factor of left hand side above and simplifying we obtain

\[
\frac{\mu_2 - h}{\mu_2 - \bar{h}} = U \cdot \frac{\mu(h - \bar{h}) + |h|^2 - h^2}{\mu(h - \bar{h}) - |h|^2 + h^2} = U \cdot \frac{\mu - h i}{\mu - h i}.
\]

Plugging (91) into the left side of (90) allows us to cancel \( U \) and obtain

\[
\mu - h i \quad \text{and} \quad \mu - \bar{h} i \cdot m_{\infty, 1}(z) + \bar{h} = \mu_\infty, 2(z) + h^2 = \mu_\infty, 1(z) + \bar{h},
\]

Performing further algebraic manipulations leads us to

\[
2 \text{Im} h \left( m_{\infty, 1}(z) - m_{\infty, 2}(z) \right) (\mu^2 - 2 \mu \text{Re} \mu + |h|^2) = 0, \quad (\forall z \in \mathbb{C}_\pm).
\]

Since \( \text{Im} h > 0 \) and, as we have shown in the proof of Theorem 15, the quadratic equation \( \mu^2 - 2 \mu \text{Re} \mu + |h|^2 = 0 \) does not have any real solutions we conclude that \( m_{\infty, 1}(z) = m_{\infty, 2}(z), (z \in \mathbb{C}_\pm) \). Now we apply the Borg-Marchenko uniqueness theorem to conclude that the potentials \( q_1(x) \) and \( q_2(x) \) in (82) and (83) are the same. Consequently, the L-systems \( \Theta_1 \) and \( \Theta_2 \) share the same main operator.

Conversely, let two Schrödinger L-systems \( \Theta_1 = \Theta_{\mu_1, h} \) and \( \Theta_2 = \Theta_{\mu_2, h} \) of the form (81) and (82) share the same main operator. Then according to [2, Theorem 8.2.3]

\[
W_{\Theta_{\mu_2, h}}(z) = (-e^{2i\alpha}) \cdot W_{\Theta_{\mu_1, h}}(z),
\]

for some \( \alpha \in (0, \pi) \). As it was shown in [2, Theorem 8.3.1], the corresponding impedance functions \( V_{\Theta_1}(z) \) and \( V_{\Theta_2}(z) \) are related by the Donoghue transform for the same value of \( \alpha \). Furthermore, applying Lemma 2 gives us connection between \( \mu_2 \) and \( \mu_1 \).

As we can see the result of Theorem 17 follows from Theorem 18 if one sets \( \alpha = \pi \) in (87) and (88).

8. Examples

We conclude this paper with a couple of simple illustrations. Consider the differential expression with the Bessel potential

\[
l_\nu = -\frac{d^2}{dx^2} + \nu^2 - \frac{1}{4}x^2, \quad x \in [1, \infty)
\]
of order \( \nu > 0 \) in the Hilbert space \( \mathcal{H} = L^2[1, \infty) \). The minimal symmetric operator

\[
\left\{ \begin{array}{l}
\dot{A} y = -y'' + \frac{\nu^2 - 1/4}{x^2} y \\
y(1) = y'(1) = 0
\end{array} \right.
\]
generated by this expression and boundary conditions has defect numbers \( (1, 1) \). Consider also the operator

\[
\left\{ \begin{array}{l}
T_h y = -y'' + \frac{\nu^2 - 1/4}{x^2} y \\
y'(1) = hy(1)
\end{array} \right.
\]
Example 1. Let \( \nu = 1/2 \). It is known \([2]\) that in this case the operator \( \hat{A} \) is nonnegative and
\[
m_\infty(z) = -i\sqrt{z}.
\]
The minimal symmetric operator in \([2]\) then becomes
\[
\begin{cases}
\hat{A} y = -y'' \\
y(1) = y'(1) = 0.
\end{cases}
\]
The main operator \( T_h \) is written for \( h = i \) in \((94)\) as
\[
\begin{cases}
T_i y = -y'' \\
y'(1) = iy(1)
\end{cases}
\]
and it will be shared by two L-systems realizing the functions \((-m_\infty(z))\) and \((1/m_\infty(z))\). Note, that this operator \( T_i \) in \((95)\) is accretive extremal (not \( \beta \)-sectorial for any \( \beta \in (0, \pi/2) \)) since \( \text{Re} h = 0 = m_\infty(0) \).

We begin by constructing an L-system \( \Theta_{0,i} \) of the form \((28)\) that realizes \((96)\)
\[
m_\infty(z) = i\sqrt{z}
\]
according to Theorem \([4]\). The quasi-kernel of the real part of the state-space operator of \( \Theta_{0,i} \) is determined by \((32)\) as follows
\[
\begin{cases}
\hat{A}_0 y = -y'' \\
y'(1) = 0.
\end{cases}
\]
Applying \((28)-(30)\) to our case we obtain
\[
\begin{cases}
\hat{A}_0 y = -y'' \\
y'(1) = iy(1)
\end{cases}
\]
where
\[
\begin{align*}
A_{0,i} y &= -y'' - i[y'(1) - iy(1)]\delta'(x - 1), \\
A_0^* y &= -y'' + i[y'(1) + iy(1)]\delta'(x - 1),
\end{align*}
\]
and
\[
K_{0,i} c = cg_{0,i}, \ (c \in \mathbb{C})
\]
\[
g_{0,i} = \delta'(x - 1).
\]
As Theorem \([4]\) states the L-system \( \Theta_{0,i} \) in \((98)\) is not accretive. Also,
\[
V_{\Theta_{0,i}}(z) = i\sqrt{z} \quad \text{and} \quad W_{\Theta_{0,i}}(z) = \frac{1 + \sqrt{z}}{1 - \sqrt{z}}.
\]

Now we build an L-system \( \Theta_{\infty,i} \) of the form \((39)\) that realizes
\[
\frac{1}{m_\infty(z)} = \frac{i}{\sqrt{z}}
\]
according to Theorem \([4]\). The quasi-kernel of the real part of the state-space operator of \( \Theta_{\infty,i} \) is given by \((13)\) and in our case is
\[
\begin{cases}
\hat{A}_{\infty,i} y = -y'' \\
y'(1) = 0.
\end{cases}
\]
Applying \((13)-(11)\) to our case we obtain
\[
\Theta_{\infty,i} = \begin{pmatrix}
A_{\infty,i} & K_{\infty,i} \\
H_+ \subset L_2[1, +\infty) & H_-
\end{pmatrix} \subset \mathbb{C},
\]
where
\[
A_{\infty,i} y = -y'' - [y'(1) - iy(1)] \delta(x - 1),
\]
\[
A_{\infty,i}^* y = -y'' - [y'(1) + iy(1)] \delta(x - 1),
\]
\[
K_{\infty,i} c = cg_{\infty,i}, \quad (c \in \mathbb{C}) \text{ and}
\]
\[
g_{\infty,i} = \delta(x - 1).
\]
By direct check using (12) one confirms that operator $A_{\infty,i}$ in (104) is accretive as follows
\[
(\Re A_{\infty,i} y, y) = ||y'(x)||_{L^2}^2 \geq 0.
\]
As Theorem 3 states the L-system $\Theta_{\infty,i}$ in (105) is extremal accretive. Also,
\[
V_{\Theta_{\infty,i}}(z) = \frac{i}{\sqrt{z}} \quad \text{and} \quad W_{\Theta_{\infty,i}}(z) = -\frac{1 + \sqrt{z}}{1 - \sqrt{z}}.
\]

**Example 2.** In this example we are going to illustrate Theorem 1 by constructing two accretive L-systems realizing functions $(-m_{\alpha}(z))$ that correspond to two endpoints of the parametric interval for $\mu = \tan \alpha$ depicted in Figure 4.

Let $\nu = 3/2$. It is known (11) that in this case
\[
m_{\infty}(z) = \frac{iz - \frac{3}{2} \sqrt{z} - \frac{3}{2} i}{\sqrt{z} + i} - \frac{1}{2} = \frac{\sqrt{z} - iz + i}{\sqrt{z} + i} = 1 - \frac{iz}{\sqrt{z} + i}
\]
and
\[
m_{\infty}(-0) = 1.
\]
The minimal symmetric operator then becomes
\[
\begin{cases}
\hat{A} y = -y'' + \frac{2}{z} y \\
y(1) = y'(1) = 0.
\end{cases}
\]
The main operator $T_h$ is written for $h = i$ in (12) as
\[
\begin{cases}
T_i y = -y'' + \frac{2}{z} y \\
y'(1) = i y(1)
\end{cases}
\]
and it will be shared by all the family of L-systems realizing functions $(-m_{\alpha}(z))$ described by (17)-(18). This operator is accretive and $\beta$-sectorial since $\Re h = 0 > -m_{\infty}(-0) = -1$ with the exact angle of sectoriality given by (see (11))
\[
\tan \beta = \frac{\Im h}{\Re h + m_{\infty}(-0)} = \frac{1}{0 + 1} = 1 \quad \text{or} \quad \beta = \frac{\pi}{4}.
\]
We will construct a family of L-systems $\Theta_{\tan \alpha,i}$ of the form (12) that realizes functions $(-m_{\alpha}(z))$ described by (17)-(18) as
\[
-m_{\alpha}(z) = \frac{\cos \alpha + \frac{1}{m_{\infty}(z)} \sin \alpha}{\sin \alpha - \frac{1}{m_{\infty}(z)} \cos \alpha} = \frac{\cos \alpha + \sqrt{z + i}}{\sqrt{z - iz + i}} \sin \alpha
\]
\[
= \frac{(\sqrt{z} - iz + i) \cos \alpha + (\sqrt{z} + i) \sin \alpha}{(\sqrt{z} - iz + i) \sin \alpha - (\sqrt{z} + i) \cos \alpha}
\]
according to Theorem 1. The quasi-kernel of the real part of the state-space operator of $\Theta_{\tan \alpha,i}$ is determined by (18) as
\[
\begin{cases}
\hat{A}_{\tan \alpha,i} y = -y'' + \frac{2}{z} y \\
y(1) = -(\tan \alpha) y'(1).
\end{cases}
\]
Using (14) we get
\[
\Theta_{\tan\alpha,i} = \begin{pmatrix} A_{\tan\alpha,i} & K_{\tan\alpha,i} \\ \mathcal{H}_+ \subset L_2[1, +\infty) \subset \mathcal{H}_- & 1 \\ \mathbb{C} \end{pmatrix},
\]
where
\[
A_{\tan\alpha,i} y = -y'' + \frac{2}{x^2} y - \frac{1}{\tan \alpha + i} [y'(1) - iy(1)][(\tan \alpha)\delta(x-1) + \delta'(x-1)],
\]
\[
A^*_{\tan\alpha,i} y = -y'' + \frac{2}{x^2} y - \frac{1}{\tan \alpha + i} [y'(1) + iy(1)][(\tan \alpha)\delta(x-1) + \delta'(x-1)],
\]
\[
K_{\tan\alpha,i} c = c g_{\tan\alpha,i}, \ (c \in \mathbb{C}) \text{ and}
\]
\[
g_{\tan\alpha,i} = (\tan \alpha)\delta(x-1) + \delta'(x-1).
\]
Also,
\[
V_{\Theta_{\tan\alpha,i}}(z) = \begin{pmatrix} -m_\alpha(z) = \frac{(\sqrt{z} - iz + i)\cos \alpha + (\sqrt{z} + i)\sin \alpha}{(\sqrt{z} - iz + i)\sin \alpha - (\sqrt{z} + i)\cos \alpha} \\ m_\infty(z) - i & 0 \\ z & e^{2\pi i} m_\infty(z) + i \end{pmatrix} = (-e^{2\pi i}) \frac{\sqrt{z} - 2i\sqrt{z} + 1 + i}{\sqrt{z} - 1 + i}.
\]
According to Theorem 1, the L-systems \(\Theta_{\tan\alpha,i}\) in (112) are accretive if
\[
1 = \frac{1}{m_\infty(-0)} \leq \tan \alpha < +\infty.
\]
In addition, as we have shown in Theorem 1 (see also Figure 1), the realizing L-system \(\Theta_{\tan\alpha,i}\) in (112) becomes accretive extremal if
\[
\mu = \tan \alpha = \frac{1}{m_\infty(-0)} = 1,
\]
and hence \(\alpha = \pi/4\) makes an extremal L-system
\[
(114) \quad \Theta_{1,i} = \begin{pmatrix} A_{1,i} & K_{1,i} \\ \mathcal{H}_+ \subset L_2[1, +\infty) \subset \mathcal{H}_- & 1 \\ \mathbb{C} \end{pmatrix},
\]
where
\[
A_{1,i} y = -y'' + \frac{2}{x^2} y - \frac{1}{1 - i} [y'(1) - iy(1)](\delta(x-1) + \delta'(x-1)],
\]
\[
A^*_{1,i} y = -y'' + \frac{2}{x^2} y - \frac{1}{1 + i} [y'(1) + iy(1)](\delta(x-1) + \delta'(x-1)],
\]
\[
K_{1,i} c = c g_{1,i}, \ (c \in \mathbb{C}) \text{ and}
\]
\[
g_{1,i} = \delta(x-1) + \delta'(x-1).
\]
It is easy to see that since in this case \(\tan \alpha = m_\infty(-0) = 1\), the quasi-kernel \(\hat{A}_{1,i}\) of \(\Re A_{1,i}\) in (111) with \(\alpha = \pi/4\) becomes the Krein-von Neumann extension of operator \(\hat{A}\) and has boundary conditions \(y(1) = -y'(1)\). Also,
\[
V_{\Theta_{1,i}}(z) = \begin{pmatrix} -m_{\pi/4}(z) = \frac{\sqrt{z} - iz + i + \sqrt{z} + i}{\sqrt{z} - iz + i - \sqrt{z} + i} = 1 - \frac{2}{\sqrt{z} + 2i} \\ m_\infty(z) - i & 0 \\ z & e^{\pi i} m_\infty(z) + i \end{pmatrix} = \frac{1 - i\sqrt{z} - 2\sqrt{z} - i}{\sqrt{z} - 1 + i}.
\]
Thus, \(\Theta_{1,i}\) in (114) represents an extremal (not \(\beta\)-sectorial for any \(\beta \in (0, \pi/2)\)) L-system with \(\pi/4\)-sectorial main operator \(T_1\) of the form (108).
Now we are going to address the other end of the parametric interval described in Theorem [1] and depicted in Figure [1]. According to part (2c) of Theorem [1], the realizing L-system $\Theta\tan\alpha,i$ in (112) preserves the angle of sectoriality and becomes $\frac{\pi}{4}$-sectorial if $\mu = \tan\alpha = +\infty$ or $\alpha = \pi/2$. Therefore the L-system

\[(\Theta_{\infty,i}) = \begin{pmatrix} A_{\infty,i} & K_{\infty,i} \\ H_+ \subset L_2[1, +\infty) \subset H_- & 1 \end{pmatrix}, \]

where

\[A_{\infty,i} = -y'' + \frac{2}{x^2}y - |y'(1) - iy(1)| \delta(x-1), \]
\[A^*_{\infty,i} = -y'' + \frac{2}{x^2}y - |y'(1) + iy(1)| \delta(x-1), \]
\[K_{\infty,i} = cg_{\infty,i}, \quad (c \in \mathbb{C}) \quad \text{and} \quad g_{\infty,i} = \delta(x-1), \]

realizes the function $-m_{\frac{\pi}{4}}(z) = 1/m_{\infty}(z)$. Also,

\[V_{\Theta_{\infty,i}}(z) = -m_{\frac{\pi}{4}}(z) = \frac{1}{m_{\infty}(z)} = \frac{\sqrt{z} + i}{\sqrt{z} - iz + i}, \]
\[W_{\Theta_{\infty,i}}(z) = -e^{\frac{i\pi}{4}} \frac{m_{\infty}(z) - i}{m_{\infty}(z) + i} = \frac{(1 - i)\sqrt{z} - iz + 1 + i}{(1 + i)\sqrt{z} - iz - 1 + i}. \]

This L-system $\Theta_{\infty,i}$ is clearly accretive according to Theorem [1] which is also independently confirmed by direct evaluation

\[(\Re A_{\infty,i} y, y) = ||y'(x)||^2_{L^2} + 2||y(x)/x||^2_{L^2} \geq 0. \]

The quasi-kernel $\hat{A}_{\infty,i}$ of $\Re A_{\infty,i}$ in (111) with $\alpha = \pi/2$ has boundary conditions $y'(1) = 0$ and is not the Krein-von Neumann extension of $\hat{A}$. Moreover, according to Theorem [1] (see also [2, Theorem 9.8.7]) the L-system $\Theta_{\infty,i}$ is $\frac{\pi}{4}$-sectorial. Taking into account that $(\Im A_{\infty,i} y, y) = |y(1)|^2$, (see formula (13)) we obtain inequality (1) with $\beta = \frac{\pi}{4}$, that is

\[(\Re A_{\infty,i} y, y) \geq |(\Im A_{\infty,i} y, y)|, \]

or

\[(\Re A_{\infty,i} y, y) \geq |(\Im A_{\infty,i} y, y)|. \]

Note that inequality (13) turns into equality on $y(x) = 1/x$ which confirms that the angle of sectoriality of the L-system $\Theta_{\infty,i}$ in (110) is exact and equals $\beta = \pi/4$. In addition, we have shown that the $\beta$-sectorial form $(T, y, y)$ defined on $\text{Dom}(T)$ can be extended to the $\beta$-sectorial form $(\hat{A}_{\infty,i} y, y)$ defined on $H_+ = \text{Dom}(\hat{A}^*)$ (see (107)–(108)) having the exact (for both forms) angle of sectoriality $\beta = \pi/4$. A general problem of extending sectorial sesquilinear forms to sectorial ones was mentioned by T. Kato in [16]. It can also be shown (see [2]) that function $-m_{\frac{\pi}{4}}(z)$ in (114) belongs to a sectorial class of Stieltjes functions introduced in [4].

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