Abstract

The first aim of this paper is to introduce and study symmetric (Bi)Hom–Leibniz algebras, which are left and right Leibniz algebras. We discuss $\alpha^k \beta^l$-generalized derivations, $\alpha^k \beta^l$-quasi-derivations and $\alpha^k \beta^l$-quasi-centroid of (Bi)Hom-Leibniz algebras and colour BiHom-Leibniz algebras. The second aim is to define a new type of BiHom–Lie algebras satisfies the following hierarchy

$\{ \text{BiHom-Lie type } B_1 \} \supseteq \beta = \text{id} \{ \text{Hom-Lie} \} \supseteq \alpha = \text{id} \{ \text{Lie} \}$. 

Moreover, define representations and a cohomology of symmetric BiHom–Leibniz algebras of type $B_1$.

Introduction

J.L. Loday introduced, In 1993, Leibniz algebras which are a generalization of Lie algebras [28, 29]. They are defined by a bilinear bracket which is no longer skew-symmetric. More precisely, A left Leibniz algebra over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $A$ with a $\mathbb{K}$-bilinear map $[\cdot, \cdot] : A \times A \to A$ satisfying

$$[[a, b], c] = [[a, b], c] + [b, [a, c]],$$  

(0.1)

for all $a, b, c \in A$. 

This property means that, for each $a$ in $A$, the adjoint endomorphism of $A$, $ad_a = [a, \cdot]$ is a derivation of $(A, [\cdot, \cdot])$.

Similarly, a right Leibniz algebra is defined by the identity :

$$[[a, b], c] = [[a, b], c] + [a, [b, c]],$$  

(0.2)
which means for each \( a \in A \), the map \( x \mapsto [x, a] \) is a derivation of \( A \). A left or right Leibniz algebra in which bracket \([\cdot, \cdot]\) is skew-symmetric (alternating, if \( K \) is of characteristic 2) is a Lie algebra. Notice from (1) that a left Leibniz algebra \( A \) satisfies, for all \( a, x, y \in A \) \( [a, [x, y]] = -[a, [y, x]] \), and dually, a right Leibniz algebra satisfies \([[[x, y], a] = -[[y, x], a]. Symmetric Leibniz algebras satisfy
\[
[a, [x, y]] = -[[x, y], a].
\] (0.3)

In the last years, the theory of Leibniz algebras has been extensively studied. Many results on Lie algebras have been generalized to the case of Leibniz algebras (\[5, 9, 10, 14, 16, 17, 19, 20, 35\]). More recently, several papers deal with a so-called symmetric Leibniz algebras, which are left and right Leibniz algebras.

A superalgebra is a \( Z_2 \)-graded algebra \( A = A_0 \oplus A_1 \) (that is, if \( a \in A_\alpha, b \in A_\beta, \alpha, \beta \in \mathbb{Z}_2 = \{0, 1\} \), then, \( ab \in A_{\alpha+\beta} \)). A Lie superalgebra is a superalgebra \( A \) with an operation \([\cdot, \cdot]\) satisfying the following identities:

\[
[a, b] = (-1)^{|a||b|} [b, a] \quad \text{(Skew-supersymmetry)}
\]

\[
[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|} [b, [a, c]] \quad \text{(Super Jacobi identity)}
\]

for all homogeneous elements \( a, b, c \in A_0 \cup A_1 \).

Leibniz superalgebras appeared as an extension of Leibniz algebras (see \[11, 23\]), in a similar way than Lie superalgebras generalize Lie algebras, motivated in part for its applications in Physics. Color Lie (super)algebras, originally introduced in \[36, 37\], can be seen as a direct generalization of Lie (super)algebras. Indeed, the latter are defined through antisymmetric (commutator) or symmetric (anticommutator) products, although for the former the product is neither symmetric nor antisymmetric and is defined by means of a commutation factor. This commutation factor is equal to \( \mp 1 \) for (super)Lie algebras and more general for arbitrary color Lie (super)algebras. As happened for Lie superalgebras, the basic tool to define color Lie (super)algebras is a grading determined by an abelian group. The latter, besides defining the underlying grading in the structure, moreover, provides a new object known as commutation factor.

Hom-algebra structures are given on linear spaces by products twisted by linear maps. Hom-Lie algebras and general quasi-Hom-Lie and quasi-Lie algebras were introduced by Hartwig, Larsson and Silvestrov as algebras embracing Lie algebras, super and color Lie algebras and their quasi-deformations by twisted derivations. In \[40\], the authors gives a systematic exploration of other possibilities to define Hom-type algebras. In \[21\], the authors introduced a generalized algebraic structure endowed with two
commuting multiplicative linear maps, called BiHom-algebras. These algebraic structures include BiHom-associative algebras, BiHom-Lie algebras and BiHom-Leibniz algebras. In some particular cases, when the two linear maps are the same, BiHom-algebras led to Hom-algebras.

In this paper, we study symmetric (super)Hom-Leibniz, (colour)BiHom-Leibniz, BiHom-Leibniz algebras of type $B_1$ and $B_2$. Recall from [21], that a BiHom-Lie algebra is a 4-tuple $(L, [\cdot, \cdot], \alpha, \beta)$ where $L$ is a $K$-linear space, $\alpha, \beta : L \to L$ are linear maps and $[\cdot, \cdot] : L \times L \to L$ is a bilinear map, satisfying the following conditions, for all $x, y, z \in L$:

\begin{align*}
\alpha \circ \beta &= \beta \circ \alpha \quad (0.4) \\
\alpha ([x, y]) &= [\alpha(x), \alpha(y)] \quad \text{and} \quad \beta ([x, y]) = [\beta(x), \beta(y)] \quad (0.5) \\
[\beta(x), \alpha(y)] &= -[\beta(y), \alpha(x)], \quad \text{(BiHom-skew-symmetry)} \quad (0.6) \\
\bigwedge_{x,y,z} [\beta^2(x), [\beta(y), \alpha(z)]] &= 0. \quad \text{(BiHom-Jacobi condition)} \quad (0.7)
\end{align*}

Obviously, there is a hierarchy of algebras.

\{ left Leibniz \} \cup \{ right Leibniz \} \supseteq \{ symmetric Leibniz \} \supseteq \{ Lie \} \quad (0.8)

and

\{ Hom-Lie \} \supseteq_{\alpha=id_L} \{ Lie \}. \quad (0.9)

Hence, one recovers Lie algebras properties from Hom-Lie algebras when stating $\alpha = id_L$. It turns out that the hierarchy (0.9) doesn’t hold for BiHom-Lie algebras. This led us to construct special types of BiHom-algebras that retains this property. We introduce BiHom-Lie algebras of type $B_1$, where (0.6) and (0.7) are replaced by $[\beta(x), \beta^2(y)] = -[\beta(y), \beta^2(x)]$ and $\bigwedge_{x,y,z} [\alpha(x), [\beta(y), \beta^2(z)]] = 0$. Then

\{ BiHom-Lie type $B_1$ \} \supseteq_{\beta=id} \{ Hom-Lie \} \supseteq_{\alpha=id} \{ Lie \}.

A BiHom-Lie algebra in the usual sense should be referred to be ”BiHom-Lie algebra of type $B_2$”. With BiHom-Lie algebras of type $B_2$, we have

\{ Hom-Lie type $I_2$ \} \supseteq_{\alpha=id} \{ Lie \} and \{ BiHom-Lie type $B_2$ \} \supseteq_{\beta=id_L} \{ Lie \},

where, the Hom-Lie algebra of type $I_2$ is given by the skew-symmetric bilinear bracket satisfying $\bigwedge_{x,y,z} [x, [y, \alpha(z)]] = 0$.

Throughout the article, we mean by a (Bi)Hom-Leibniz algebra a left or right or symmetric (Bi)Hom-Leibniz algebra.

The paper is organized as follows. In Section 1, we recall definitions and some key constructions of Hom-Leibniz (super)algebras and give some
example of symmetric Hom-Leibniz (super)algebras. Moreover we introduce the concept of centroid and quasicentroid for Hom-Leibniz superalgebra (left or right or symmetric) and study some of their properties. In Section 2, we give some constructions of Hom-Lie algebras by BiHom-Lie algebras and conversely. Also, we provide a construction of BiHom-Leibniz algebras \( L(\alpha, \beta) = (L, \{\cdot, \cdot\}, \alpha, \beta) \) from a Leibniz algebras \((L, [\cdot, \cdot])\). We also give some basic definitions, properties of Ideals of BiHom-Leibniz algebras. This section also includes the concept of generalized derivations of a BiHom-Leibniz algebras and some properties. Section 3 is dedicated to BiHom-Leibniz colour algebras, we give a BiHom-Lie colour algebra \((A, [\cdot, \cdot], \alpha, \beta, \varepsilon)\) from an associative colour algebra \((A, \mu, \varepsilon)\). We construct color BiHom-Leibniz algebras starting from two even centroids of Leibniz or BiHom-Leibniz colour algebras. In Section 4, we define BiHom-Lie and BiHom-Leibniz algebras of type \(B_1\), we study representations of symmetric BiHom-Leibniz algebras of type \(B_1\), and give their cohomology. We show that any extension of a symmetric BiHom-Leibniz algebras of type \(B_1\), are controlled by the second cohomology with respect its corresponding representation.

1 Symmetric Hom-Leibniz (super)algebras

In this section, we define the symmetric Hom-Leibniz (super)algebras generalizing the well known Leibniz algebras given in [4, 17] and we gives a few examples of the symmetric Hom-Leibniz (super)algebras. We also study some properties of centroids of Hom-Leibniz superalgebras.

1.1 Symmetric Hom-Leibniz algebras

Definition 1.1. [22, 3, 31] A Hom-Lie algebra is a triple \((G, [\cdot, \cdot], \alpha)\) consisting of a \(K\) vector space \(G\), a bilinear map \([\cdot, \cdot] : G \times G \to G\) and a \(K\)-linear map \(\alpha : G \to G\) satisfying

\[
[x, y] = -[y, x], \quad \text{(skew-symmetry)} \tag{1.1}
\]

\[
\bigcirc_{x, y, z} [\alpha(x), [y, z]] = 0, \quad \text{(Hom–Jacobi identity)} \tag{1.2}
\]

for all \(x, y, z \in G\). The two condition lead to the following identities.

\[
[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + [\alpha(y), [x, z]], \quad \tag{1.3}
\]

\[
[\alpha(x), [y, z]] = [[x, y], \alpha(z)] - [[x, z], \alpha(y)] \quad \tag{1.4}
\]

for all \(x, y, z \in G\).

Now, we define left and right Hom-Leibniz algebras.
Definition 1.2. A left (resp. right) Hom-Leibniz algebra is a $K$-vector space $L$ equipped with a bracket operation $[\cdot, \cdot]$ and a linear map $\alpha$ that satisfy the equation (1.3) (resp. (1.4)).

Obviously, a Hom-Lie algebra is a left and right Hom-Leibniz algebra. If $\alpha = id_L$, then a left (resp. right) Hom-Leibniz algebra becomes a left (resp. right) Leibniz algebra. A left (resp. right) Hom-Leibniz algebra is a Hom-Lie algebra if and only if $[x, x] = 0, \forall x \in L$.

Definition 1.3. A triple $(L, [\cdot, \cdot], \alpha)$ is called a symmetric Hom-Leibniz algebra if it is a left and a right Hom-Leibniz algebra.

Example 1.1. Let $(e_1, e_2)$ be a basis of vector space $L$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ a matrix of linear map $\alpha$ with respect to this basis. In the following table we give all possible cases for $[\cdot, \cdot]$ to be a left Hom-Leibniz algebra

| Left Hom-Leibniz bracket | Remark |
|--------------------------|--------|
| $[e_1, e_1] = xe_2, [e_1, e_2] = ye_2, [e_2, e_1] = [e_2, e_2] = 0$ | $L$ is multiplicative $\iff y = 0$
$L$ is symmetric $\iff xy = 0$ |
| $[e_1, e_1] = [e_1, e_2] = 0, [e_2, e_1] = ce_1, [e_2, e_2] = de_1$ | $L$ is multiplicative $\iff c = 0$
$L$ is symmetric $\iff c = 0$ |
| $[e_1, e_1] = ae_1 + xe_2, [e_1, e_2] = [e_2, e_1] = -\frac{a}{x} [e_1, e_1], [e_2, e_2] = \left( \frac{a}{x} \right)^2 [e_1, e_1]$ | $L$ is multiplicative $\iff a = 0$
$L$ is symmetric $\iff a = 0$ |
| $[e_1, e_1] = [e_2, e_2] = 0, [e_1, e_2] = -[e_2, e_1] = be_1 + ye_2$ | $L$ is multiplicative $\iff y = 0$
$L$ is a Hom-Lie algebra |

Example 1.2. Let $(x_1, x_2, x_3)$ be a basis of 3-dimensional space $G$ over $K$. Define a bilinear bracket operation on $G \otimes G$ by

$$
[x_1 \otimes x_3, x_1 \otimes x_3] = x_1 \otimes x_1 \\
[x_2 \otimes x_3, x_1 \otimes x_3] = x_2 \otimes x_1 \\
[x_2 \otimes x_3, x_2 \otimes x_3] = x_2 \otimes x_2.
$$

The others brackets are equal to 0. For any linear map $\alpha$ on $G$, the triple $(G \otimes G, [\cdot, \cdot], \alpha \otimes \alpha)$ is not a Hom-Lie algebra but it is a symmetric Hom-Leibniz algebra.

In the following examples, we construct Hom-Leibniz algebras on a vector space $L \otimes L$ starting from a Lie or a Hom-Lie algebra $L$.

Proposition 1.4. [26] For any Lie algebra $(G, [\cdot, \cdot])$, the bracket

$$
[x \otimes y, a \otimes b] = [x, [a, b]] \otimes y + x \otimes [y, [a, b]]
$$

defines a Leibniz algebra structure on the vector space $G \otimes G$.  

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Proposition 1.5. Let \((L, [\cdot, \cdot])\) be a Leibniz algebra and \(\alpha : L \to L\) be a Leibniz algebra endomorphism. Then \((L, [\cdot, \cdot], \alpha)\) is a Hom-Leibniz algebra, where \([x, y]_\alpha = [\alpha(x), \alpha(y)]\).

Using Proposition 1.5 and Proposition 1.4 we obtain the following result.

Proposition 1.6. Let \((G, [\cdot, \cdot]')\) be a Lie algebra and \(\alpha : G \to G\) be a Lie algebra endomorphism. We define on \(G \otimes G\) the following bracket

\[ [x \otimes y, a \otimes b] = [\alpha(x), [\alpha(a), \alpha(b)]]' \otimes \alpha(y) + \alpha(x) \otimes [\alpha(y), [\alpha(a), \alpha(b)]]' \]

on \(G \otimes G\). Then \((G \otimes G, [\cdot, \cdot], \alpha)\) is a right Hom-Leibniz algebra.

Remark 1.7. In order to get a left Hom-Leibniz algebra.

1.2 Centroids and derivations of Hom-Leibniz superalgebra

The concept of centroids and derivation of Leibniz algebras is introduced in [24]. Left Leibniz superalgebras, originally were introduced by Dzhumadil’daev in [15], can be seen as a direct generalization of Leibniz algebras. The left Hom-Leibniz superalgebras is introduced in [8]. In this section, we introduce the notion of right and symmetric Hom-Leibniz superalgebras. Moreover, we introduce the concept of centroids and derivation of Hom-Leibniz superalgebras.

Let \(V\) be a vector superspace over a field \(K\) that is a \(\mathbb{Z}_2\)-graded vector space with a direct sum \(V = V_0 \oplus V_1\). The elements of \(V_j, j \in \mathbb{Z}_2\), are said to be homogenous of parity \(j\). The parity of a homogeneous element \(x\) is denoted by \(|x|\). The space \(\text{End}(V)\) is \(\mathbb{Z}_2\)-graded with a direct sum \(\text{End}(V) = (\text{End}(V))_0 \oplus (\text{End}(V))_1\) where \(\text{End}(V))_j = \{f \in \text{End}(V)/f(V_i) \subset V_{i+j}\}. The elements of \((\text{End}(V))_j\) are said to be homogenous of parity \(j\).

Definition 1.8. (see [1, 2]) A Hom-Lie superalgebra is a triple \((G, [\cdot, \cdot], \alpha)\) consisting of a superspace \(G\), an even bilinear map \([\cdot, \cdot] : G \times G \to G\) and an even superspace homomorphism \(\alpha : G \to G\) satisfying

\[
[x, y] = (-1)^{|x||y|}[y, x],
\]

\[
\circ_{x,y,z} \ (-1)^{|x||z|} [\alpha(x), [y, z]] = 0
\]

for all homogeneous element \(x, y, z\) in \(G\).
Proposition 1.9. Let $L$ be a superspace and define the following vector subspace $\Omega$ of $\text{End}(L)$ consisting of linear maps $u$ on $L$ as follows:

$$\Omega = \{ u \in \text{End}(L) / u \circ \alpha = \alpha \circ u \}.$$ 

and the map

$$\tilde{\alpha} : \Omega \to \Omega; \quad \tilde{\alpha}(u) = \alpha \circ u.$$ 

Then $(\Omega, [\cdot, \cdot]')$ (resp. $(\Omega, [\cdot, \cdot]'_\alpha$) is a Lie (resp. Hom-Lie) superalgebra with the bracket $[u, v]' = uv - (-1)^{|u||v|}vu$ for all $u, v \in \Omega$.

Remark 1.10. The Hom-Lie algebras $(\Omega, [\cdot, \cdot]'_\alpha)$ is not necessarily multiplicative.

Definition 1.11. (see [2]) Let $(G, [\cdot, \cdot], \alpha)$ be a Hom-Lie superalgebra and $V = V_0 \oplus V_1$ be an arbitrary vector superspace. Let $\beta \in \mathcal{G}l(V)$ be an arbitrary even linear self-map on $V$ and $[\cdot, \cdot]_V : G \times V \to V$ an even bilinear map. The triple $(V, [\cdot, \cdot]_V, \beta)$ is called a module on the Hom-Lie superalgebra $G = G_0 \oplus G_1$ if the even bilinear map $[\cdot, \cdot]_V$ satisfies

$$[[x, y], \beta(v)]_V = [\alpha(x), [y, v]_V]_{\alpha(y), [x, v]_V} - (-1)^{|x||y|}[\alpha(y), [x, v]_V]$$

for all homogeneous elements $x, y \in G$ and $v \in V$.

Definition 1.12. Let $(L, [\cdot, \cdot], \alpha)$ a triple consisting of a superspace $L$, an even bilinear map $[\cdot, \cdot] : L \times L \to L$ and an even superspace homomorphism $\alpha : L \to L$. Then, $(L, [\cdot, \cdot], \alpha)$ is called :

(i) a left Hom-Leibniz superalgebra if it satisfies

$$[[x, y], \alpha(z)] = [[x, y], \alpha(z)] + (-1)^{|x||y|}[\alpha(y), [x, z]].$$

(ii) a right Hom-Leibniz superalgebra if it satisfies

$$[[x, y], \alpha(z)] = [[x, y], \alpha(z)] - (-1)^{|y||z|}[x, z], \alpha(y)] \quad \forall x, y, z \in L_0 \cup L_1.$$ 

The following proposition provides a method to construct a left Hom-Leibniz superalgebra by a module of Hom-Lie superalgebra.

Proposition 1.13. Let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie superalgebra and $(V, [\cdot, \cdot]_V, \beta)$ a $L$-module. Let $\varphi : V \to L$ be an even linear map satisfying $\varphi([x, v]_V) = [x, \varphi(v)]$ and $\varphi \circ \beta = \alpha \circ \varphi$. Then one can define a left Hom-Leibniz superalgebra $(V, [\cdot, \cdot]'_V, \beta)$ as follows: $[u, v]'_V = [\varphi(u), v]_V$. 

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Now, we define the symmetric Hom-Leibniz superalgebra.

**Definition 1.14.** If \((\mathcal{L}, [\cdot, \cdot], \alpha)\) is a left and a right Leibniz algebra, then \(\mathcal{L}\) is called a symmetric Leibniz algebra.

**Proposition 1.15.** A triple \((\mathcal{L}, [\cdot, \cdot], \alpha)\) is a symmetric Hom-Leibniz superalgebra if and only if

\[
[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + (-1)^{|x||y|} [\alpha(y), [x, z]]; \\
[\alpha(y), [x, z]] = -(-1)^{|x|+|z|} [x, \alpha(z)] + \alpha(y) + (-1)^{|y|} [\alpha(x), \alpha(z)];
\]

for all \(x, y, z \in \mathcal{L}_0 \cup \mathcal{L}_1\).

**Example 1.3.** Let \(\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1\) be a 3-dimensional superspace, where \(\mathcal{L}_0\) is generated by \(e_1, e_2\) and \(\mathcal{L}_1\) is generated by \(e_3\). The product is given by

\[
[e_1, e_1] = ae_1 + xe_2; \\
[e_2, e_2] = [e_2, e_1] = -\frac{a}{x} [e_1, e_1]; \\
[e_2, e_1] = \left(\frac{a}{x}\right)^2 [e_1, e_1]; \\
[e_3, e_3] = \frac{d}{x} [e_1, e_1]; \\
[e_1, e_3] = [e_3, e_1] = [e_3, e_2] = [e_2, e_3] = 0.
\]

We consider the homomorphism \(\alpha: \mathcal{L} \rightarrow \mathcal{L}\) defined by the matrix

\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \mu
\end{pmatrix}
\]

with respect to basis \((e_1, e_2, e_3)\). Then \((\mathcal{L}, [\cdot, \cdot], \alpha)\) is a symmetric Hom-Leibniz superalgebra.

**Definition 1.16.** A \(\alpha^k\)-derivation of a Hom-Leibniz superalgebra \((\mathcal{L}, [\cdot, \cdot], \alpha)\) is a homogeneous linear map \(D \in \Omega\) satisfying

\[
D([x, y]) = [D(x), \alpha^k(y)] + (-1)^{|D||x|} [\alpha^k(x), D(y)]
\]

for all \(x, y, z \in \mathcal{L}_0 \cup \mathcal{L}_1\). The set of all derivation of a Hom-Leibniz superalgebra \(\mathcal{L}\) is denoted by \(\text{Der}(\mathcal{L}) = \bigoplus_{k \geq 0} \text{Der}_{\alpha^k}(\mathcal{L})\).

**Proposition 1.17.** Let \((\mathcal{L}, [\cdot, \cdot], \alpha)\) be a left (resp. right) Hom-Leibniz superalgebra. For any \(a \in L\) satisfying \(\alpha(a) = a\), define \(ad_k(a) \in \text{End}(L)\) (resp. \(Ad_k(a) \in \text{End}(L)\))

\[
ad_k(a)(x) = [a, \alpha^k(x)]
\]

respectively

\[
Ad_k(a)(x) = (-1)^{|a||x|} [\alpha^k(x), a], \quad \forall x \in L.
\]

Then \(ad_k(a)\) (resp. \(Ad_k(a)\)) is an \(\alpha^{k+1}\)-derivation of the left (resp. right) Hom-Leibniz algebra \(L\).
Definition 1.18. Let \((L, [\cdot, \cdot], \alpha)\) be a Hom-Leibniz superalgebra. Then the \(\alpha^k\)-centroid of \(L\) denoted as \(C_{\alpha^k}(L)\) is defined by
\[
C_{\alpha^k}(L) = \{ d \in \Omega \mid d([x, y]) = [d(x), \alpha^k(y)] = (-1)^{|d||x|}[\alpha^k(x), d(y)], \forall x, y \in L_0 \cup L_1 \}.
\]
Denote by \(C(L) = \bigoplus_{k \geq 0} C_{\alpha^k}(L)\) the centroid of \(L\).

Definition 1.19. Let \((L, [\cdot, \cdot], \alpha)\) be a Hom-Leibniz superalgebra. Then the \(\alpha^k\)-centroid of \(L\) denoted as \(C_{\alpha^k}(L)\) is defined by
\[
C_{\alpha^k}(L) = \{ d \in \Omega \mid d([x, y]) = [d(x), \alpha^k(y)] = (-1)^{|d||x|}[\alpha^k(x), d(y)], \forall x, y \in L_0 \cup L_1 \}.
\]
Denote by \(C(L) = \bigoplus_{k \geq 0} C_{\alpha^k}(L)\) the centroid of \(L\).

Definition 1.20. Let \((L, [\cdot, \cdot], \alpha)\) be a Hom-Leibniz superalgebra and \(d \in \text{End}(\mathcal{L})\). Then \(d\) is called a \(\alpha^k\)-central derivation, if \(d \in \Omega\) and
\[
D([x, y]) = [D(x), \alpha^k(y)] = (-1)^{|D||x|}[\alpha^k(x), D(y)] = 0.
\]
The set of all central derivations of \(L\) is denoted by \(Z\text{Der}(L) = \bigoplus_{k \geq 0} Z\text{Der}_{\alpha^k}(L)\).

In the following of this section we study the structure of the centroids and derivations of Hom-Leibniz superalgebras.

Lemma 1.21. Let \(L\) be a Hom-Leibniz superalgebra. Let \(d \in \text{Der}_{\alpha^k}(L)\) and \(\Phi \in C_{\omega^l}(L)\) then

(i) \(\Phi \circ d\) is an \(\alpha^{k+l}\)-derivation of \(L\).

(ii) \([\Phi, d]\) is \(\alpha^{k+l}\)-centroid of \(L\).

(iii) \(d \circ \Phi\) is an \(\alpha^{k+l}\)-centroid if and only if \(\Phi \circ d\) is a \(\alpha^{k+l}\)-central derivation.

(iv) \(d \circ \Phi\) is an \(\alpha^{k+l}\)-derivation only if only \([d, \Phi]\) is a \(\alpha^{k+l}\)-central derivation.

Theorem 1.22. Let \(L\) be a Hom-Leibniz superalgebra. Then
\[
Z\text{Der}_{\alpha^k}(L) = \text{Der}_{\alpha^k}(L) \cap C_{\alpha^k}(L).
\]

Proof. The proof of previous theorem is similar to the case Leibniz algebra given in [24].

Proposition 1.23. Let \((L, [\cdot, \cdot], \alpha)\) be a Hom-Leibniz superalgebra. Then the following statements hold:

(i) \(\text{ad}(L) \subset \text{Der}(L) \subset \Omega\), where \(\text{ad}(L)\) denotes the superalgebra of inner derivations of \(L\).

(ii) \(\text{ad}(L), \text{Der}(L)\) and \(C(L)\) are Lie (resp. Hom-Lie) subsuperalgebras of \((\Omega, [\cdot, \cdot])\) (resp. \((\Omega, [\cdot, \cdot]'), \hat{\alpha}\))
2 BiHom-Leibniz algebras

In this section, we recall the notion of BiHom-Lie algebras and then give some relations between it and the Hom-Lie algebras. Moreover, we introduce the notion of symmetric BiHom-Leibniz algebras. We introduce the definitions and give some properties related to ideals of BiHom–Leibniz algebras and we extend the concept of \((\alpha, \beta, \gamma)\)-derivations of Lie algebras introduced in [33] to BiHom–Leibniz case.

2.1 BiHom-Lie algebras

We first recall the definition of Hom-Lie algebra of type \(I_3\) and then we give some connections between they and BiHom-Lie algebras.

**Definition 2.1.** [40] A Hom-Lie algebra of type \(I_3\) is defined by replacing, in Definition 1.1, equation (1.5) by

\[ \triangle_{x,y,z} [x, [y, \alpha(z)]] = 0. \] (2.1)

**Definition 2.2.** [27, 39] A BiHom-Lie algebra over \(K\) is a 4-tuple \((L, [\cdot, \cdot], \alpha, \beta)\) where \(L\) is a \(K\)-vector space, \(\alpha, \beta : L \to L\) are linear maps and \([\cdot, \cdot] : L \times L \to L\) is a bilinear map, satisfying the following conditions, for all \(x, y, z \in L\):

\[
\alpha \circ \beta = \beta \circ \alpha; \\
[\beta(x), \alpha(y)] = -[\beta(y), \alpha(x)] \quad \text{(BiHom-skew-symmetry);} \\
[\beta^2(x), [\beta(y), \alpha(z)]] + [\beta^2(y), [\beta(z), \alpha(x)]] + [\beta^2(z), [\beta(x), \alpha(y)]] = 0 \quad \text{(BiHom-Jacobi condition).}
\]

In particular, if \(\alpha\) and \(\beta\) preserves the bracket, then we call \((L, [\cdot, \cdot], \alpha, \beta)\) a multiplicative BiHom-Lie algebra.

We recover Hom-Lie algebra when we have \(\beta = id_L\) and \(\beta\) is a bijective. If \((L, [\cdot, \cdot], \alpha)\) is a Hom-Lie algebra of type \(I_3\) and \(\alpha\) is in the centroid of \(L\) (i.e. \(\alpha([x, y]) = [\alpha(x), y]\)), then \((L, [\cdot, \cdot], \alpha, id_L)\) is a BiHom-Lie algebra.

In the following of this subsection, we establish a connection between BiHom-Lie algebra and (original) Hom-Lie algebra.

**Proposition 2.3.** The 4-tuple \((L, [\cdot, \cdot], \alpha, \alpha)\) is a BiHom-Lie algebra if and only if the triple \((\alpha(L), [\cdot, \cdot], \alpha)\) is a Hom-Lie algebra.

**Proposition 2.4.** If \((L, [\cdot, \cdot], \alpha, \beta)\) is a BiHom-Lie algebra and we define the map \([\cdot, \cdot'] : L \times L \to L, [x, y'] = [\beta(x), \alpha(y)]\), for all \(x, y \in L\), then \((L, [\cdot, \cdot'], \alpha \beta)\) is a Hom-Lie algebra.
2.2 BiHom-Leibniz algebras

Inspired by [4], the definition of generalized derivations of Lie algebras (see [18]) and the definition of twisted derivations (see [13]), we introduce the concept of BiHom-Leibniz algebra.

Let a 4-tuple \((L, [\cdot, \cdot], \alpha, \beta)\) be a 4-tuple consisting of a vector space \(L\), a bilinear map \([\cdot, \cdot]: L \times L \to L\) and two linear maps \(\alpha, \beta: L \to L\) such that \(\alpha \circ \beta = \beta \circ \alpha\), \(\alpha ([x, y]) = [\alpha(x), \alpha(y)]\) and \(\beta ([x, y]) = [\beta(x), \beta(y)]\). Let \(\Delta_{L, L}(L)\) denote the set of triples \((f, f', f'')\) with \(f, f', f'' \in \text{End}(L)\) such that \([f(x), \alpha^k \beta^l(y)] + [\alpha^k \beta^l(x), f''([x, y])] = f''([x, y])\). For all \(a \in L\) define the endomorphisms \(L_a, R_a\) of \(L\) by \(L_a(x) = [a, x], R_a(x) = [x, a]\).

**Definition 2.5.** (i) If \((L_{\beta(x)}, L_{\alpha(x)}, L_{\alpha \beta(x)}) \in \Delta_{0,1}(L), \forall x \in L\), i.e

\[
[\alpha \beta(x), [y, z]] = [\beta(x), [y, \beta(z)]] + [\beta(y), [\alpha(x), z]], \forall x, y, z \in L,
\]

then \((L, [\cdot, \cdot], \alpha, \beta)\) is called a left BiHom-Leibniz algebra.

(ii) If \((R_{\beta(z)}, R_{\alpha(z)}, R_{\alpha \beta(z)}) \in \Delta_{1,0}(L), \forall z \in L\), i.e

\[
[[x, y], \alpha \beta(z)] = [[x, \beta(z)], \alpha(y)] + [\alpha(x), [y, \alpha(z)]], \forall x, y, z \in L,
\]

then \((L, [\cdot, \cdot], \alpha, \beta)\) is called a right BiHom-Leibniz algebra.

**Proposition 2.6.** Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a left (respectively right) BiHom-Leibniz algebra. Then

\[
[[\beta(x), \alpha(y)], \alpha \beta(z)] = -[[\beta(y), \alpha(x)], \alpha \beta(z)],
\]

respectively

\[
[\alpha \beta(x), [[\beta(y), \alpha(z)]], = -[\alpha \beta(x), [\beta(z), \alpha(y)]],
\]

for all \(x, y, z \in L\).

**Definition 2.7.** If \((L, [\cdot, \cdot], \alpha, \beta)\) is a left and a right BiHom-Leibniz algebra, then \(L\) is called a symmetric BiHom-Leibniz algebra.

**Proposition 2.8.** Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a left BiHom-Leibniz algebra. Then \((L, [\cdot, \cdot], \alpha, \beta)\) is a symmetric BiHom-Leibniz algebra if and only if

\[
[[\beta(y), \alpha(x)], \alpha(z)] = -[[\beta(x), \beta(z)], \alpha(y)],\tag{2.2}
\]

for all \(x, y, z \in L\).
Proposition 2.9. If \((L, [\cdot, \cdot])\) is a symmetric Leibniz algebra and \(\alpha, \beta : L \to L\) are two commuting morphisms of Leibniz algebras, and we define the map \(\{\cdot, \cdot\} : L \times L \to L, \{x, y\} = [\alpha(x), \beta(y)]\), for all \(x, y \in L\), then \((L, \{\cdot, \cdot\}, \alpha, \beta)\) is a symmetric BiHom-Leibniz algebra, called the Yau twist of \(L\) and denoted by \(L_{(\alpha, \beta)}\).

Proof. Since \((L, [\cdot, \cdot])\) is a left Leibniz algebra, there \(L_{(\alpha, \beta)}\) is a left BiHom-Leibniz algebra (see [27]). It remains to show the equality \(2.2\) is satisfied:

\[
\{\beta(x), \{\alpha(x), \alpha(y)\}\} = \{\beta(x), [\alpha^2(x), \alpha\beta(y)]\}.
\]

Since \((L, [\cdot, \cdot])\) is a symmetric Leibniz algebra, we have

\[
[a, [b, c]] = [[b, c], a] \quad \text{(see [4])}.
\]

Therefore, we have

\[
\{\beta(x), \{\alpha(x), \alpha(y)\}\} = -[[\beta\alpha^2(x), \alpha\beta^2(y)], \alpha\beta(x)]
\]

\[
= -\{[\beta\alpha(x), \beta^2(y)], \alpha(x)\}
\]

\[
= -\{\{\beta(x), \beta(y)\}, \alpha(x)\}.
\]

By Proposition 2.8, we deduce that \(L_{(\alpha, \beta)}\) is a symmetric BiHom-Leibniz algebra. ■

The following results gives a way to construct Hom-Leibniz algebra starting from a BiHom-Leibniz algebra.

Proposition 2.10. Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a BiHom-Leibniz algebra. Define the bilinear map \(\{\cdot, \cdot\} : L \times L \to L, \{x, y\} = [\beta(x), \alpha(y)]\), for all \(x, y \in L\). Then \((L, \{\cdot, \cdot\}, \alpha\beta)\) is a Hom-Leibniz algebra.

Proposition 2.11. Let \(L\) be a vector space, \([\cdot, \cdot] : L \times L \to L\) a bilinear map, \(\alpha, \beta : L \to L\) two commuting linear maps such that \(\alpha([x, y]) = [\alpha(x), \alpha(y)]\) and \(\beta([x, y]) = [\beta(x), \beta(y)]\), for all \(x, y \in L\). Define the bilinear map \(\{\cdot, \cdot\} : L \times L \to L, \{x, y\} = [\beta(x), \alpha(y)]\), for all \(x, y \in L\). Then:

\((\alpha(L), [\cdot, \cdot], \alpha, \beta)\) is a BiHom-Lie algebra if and only if \((L, \{\cdot, \cdot\}, \alpha\beta)\) is a symmetric Hom-Leibniz algebra.

2.3 Ideals of BiHom-Leibniz algebras

In this subsection we extend Ideals of Hom-Leibniz algebras introduced in [6] to BiHom-Leibniz algebras.
Definition 2.12. Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a BiHom–Leibniz algebra. A vector subspace \(H\) of \(L\) is called a BiHom–Leibniz subalgebra of \((L, [\cdot, \cdot], \alpha, \beta)\) if \(\alpha(H) \subseteq H, \beta(H) \subseteq H\) and \([H, H] \subseteq H\). In particular, a BiHom–Leibniz subalgebra \(H\) is said to be a two-sided ideal if \([h, l], [l, h] \in H\) for all \(l \in L, h \in H\). If only one relation holds, then we call \(H\) a right (or left) ideal.

If \((H, [\cdot, \cdot], \alpha_H, \beta_H)\) is a two-sided ideal, then the quotient \(L/H\) is endowed with a BiHom–Leibniz algebra structure, naturally induced from the bracket on \(L\).

The commutator of two-sided ideals \(H\) and \(K\) of a BiHom–Leibniz algebra \(L\), denoted by \([H, K]\), is the BiHom–Leibniz subalgebra of \(L\) spanned by the brackets \([h, k]\) and \([k, h]\) for all \(h \in H, k \in K\).

The following lemma can be readily checked.

Lemma 2.13. Let \(H\) and \(K\) be two-sided ideals of a BiHom–Leibniz algebra \((L, [\cdot, \cdot], \alpha, \beta)\). The following statements hold:

(a) \(H \cap K\) and \(H + K\) are two-sided ideals of \(L\);

(b) \([H, K] \subseteq H \cap K\);

(c) \([H, K]\) is a two-sided ideal of \(H\) and \(K\). In particular, \([L, L]\) is a two-sided ideal of \(L\);

(d) \(\alpha(L)\) and \(\beta(L)\) are BiHom–Leibniz subalgebras of \(L\);

(e) If \(L\) is a left (resp. right) BiHom–Leibniz algebra and \(\beta\) (resp. \(\alpha\)) is surjective, then \([H, K]\) is an ideal of \(L\);

(f) If \(\alpha\) and \(\beta\) are surjective, then \([H, K]\) is a two-sided ideal of \(L\).

In the following, we extend some result of Ideals of Leibniz algebras introduced in [4] to BiHom-Leibniz case:

Proposition 2.14. Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a left (resp. right) BiHom–Leibniz algebra and \(I_L\) the vector space \(I_L\) spanned by the set \(\{[x, x] \mid x \in L\}\). If \(\beta\) (resp. \(\alpha\)) is a surjective homomorphism, then, \(I_L\) is an ideal of \(L\). Moreover, \([I_L, L] = 0\) (resp. \([L, I_L] = 0\)).

It is clear that \(L\) is a BiHom–Lie algebra if and only if \(I_L = \{0\}\). Therefore, the quotient algebra \(L/I_L\) is a BiHom–Lie algebra.

Proposition 2.15. If \((L, [\cdot, \cdot], \alpha, \beta)\) is a symmetric BiHom–Leibniz algebra, then the two-sided ideal \(\left(L^2 = [L, L], [\cdot, \cdot]/L^2\times L^2, \alpha/_{L^2}, \beta/_{L^2}\right)\) is a BiHom–Lie algebra.
Definition 2.16. Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a BiHom–Leibniz algebra. The sub-

space \(Z(L) = \{x \in L \mid [x, y] = 0 = [y, x], \forall y \in L\}\) of \(L\) is said to be the
center of \(L\). Note that if \(\alpha\) and \(\beta\) are surjective, then \(Z(L)\) is a two-sided
ideal of \(L\).

2.4 \((\lambda, \mu, \gamma)\)- derivations

Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a BiHom–Leibniz algebra. We set

\[\Omega = \{f \in \text{End}(L) \mid f \circ \alpha = \alpha \circ f, f \circ \beta = \beta \circ f\}\].

Definition 2.17. For any integer \(k, l\), a linear map \(D: L \to L\) is called an
\(\alpha^k\beta^l\)-derivation of the BiHom–Leibniz algebra \((L, [\cdot, \cdot], \alpha, \beta)\), if \(D \in \Omega\) and

\[D([x, y]) = [D(x), \alpha^k\beta^l(y)] + [\alpha^k\beta^l(x), D(y)],\]

for all \(x, y \in L\). Denote by \(\text{Der}(L) = \bigoplus_{0 \leq k, l} \text{Der}_{\alpha^k\beta^l}(L)\) the set of derivations
of the BiHom–Leibniz algebra \((L, [\cdot, \cdot], \alpha, \beta)\).

Proposition 2.18. Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a left BiHom–Leibniz algebra and
\(a \in L\). Define respectively \(D, D', D''\) by

\[D(x) = [\alpha\beta(a), \alpha^k\beta^l(x)], \quad D'(x) = [\beta(a), \alpha^k\beta^l(x)], \quad D''(x) = [\alpha(a), \alpha^k\beta^l(x)].\]

Then

(i) \(\alpha \circ D' = D \circ \alpha, \beta \circ D'' = D \circ \beta;\)

(ii) \([D(x), \alpha^k\beta^l(y)] + [\alpha^k\beta^l(x), D'(y)] = D''([x, y]);\)

(iii) If \(\alpha^2(a) = \alpha(a) = \beta(a),\) then \(D\) is a \(\alpha^k\beta^{l+1}\)-derivation of the left
BiHom–Leibniz algebra \(L\).

Proposition 2.19. Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a symmetric BiHom–Leibniz algebra
and \(a \in L\). For any \(a \in L\), define respectively \(ad_{kl}(a), Ad_{kl}(a) \in \text{End}(L)\) by

\[ad_{kl}(a)(x) = [a, \alpha^k\beta^l(x)], \quad Ad_{kl}(a)(x) = [\alpha^k\beta^l(x), a], \forall x \in L.\]

Then

(i)

\[\alpha \circ ad_{kl}(a) = ad_{kl}(\alpha(a)) \circ \alpha; \quad \beta \circ ad_{kl}(a) = ad_{kl}(\beta(a)) \circ \beta;\]

\[\alpha \circ Ad_{kl}(a) = Ad_{kl}(\alpha(a)) \circ \alpha; \quad \beta \circ Ad_{kl}(a) = Ad_{kl}(\beta(a)) \circ \beta;\]
(ii) Let \( \alpha, \beta \) be elements of \( K \) for all \( x, y \).

\[
\text{Ad}_x(\alpha \beta(a))([x, y]) = -\text{ad}_{k+1, l-1}(\beta^2(a))([x, y]);
\]
\[
\text{ad}_x(\alpha \beta(a))([x, y]) = -\text{Ad}_{k-1, l+1}(\alpha^2(a))([x, y]);
\]
\[
\text{Ad}_x(\alpha \beta(a))([x, y]) = [\text{Ad}_x(\beta(a))(x), \alpha^{k+1} \beta^l(y)] - [\alpha^{k+1} \beta^l(x), \text{ad}_{k+1, l-1}(\beta(a))(x)].
\]

(iii) If \( \alpha \beta(a) = \alpha(a) = \beta(a) \), \( \text{Ad}_x, \text{ad}_x \) and \( \text{ad}_x \) are \( \alpha^k \beta^l \)-derivation of the symmetric BiHom-Leibniz algebra \( L \).

**Definition 2.20.** Let \( (L, [\cdot, \cdot], \alpha, \beta) \) be a BiHom–Leibniz algebra and \( \lambda, \mu, \gamma \) be elements of \( K \). A linear map \( d \in \Omega \) is a \( \lambda, \mu, \gamma \)-\( \alpha^k \beta^l \)-derivation of \( L \) if for all \( x, y \in L \) we have

\[
\lambda d([x, y]) = \mu [d(x), \alpha^k \beta^l(y)] + \gamma [\alpha^k \beta^l(x), d(y)].
\]

We denote the set of all \( \lambda, \mu, \gamma \)-derivations by \( \text{Der}^{(\lambda, \mu, \gamma)}(L) = \bigoplus_{0 \leq k, l} \text{Der}^{(\lambda, \mu, \gamma)}(L) \).

**Definition 2.21.** Let \( (L, [\cdot, \cdot], \alpha, \beta) \) be a BiHom–Leibniz algebra over a field \( K \) (\( \text{char} \ K \neq 2 \)) and \( \lambda, \mu, \gamma \in K \). Then for \( \text{Der}^{(\lambda, \mu, \gamma)}(L) \) we fix the followings particular cases:

(a) \( \text{Der}^{(1,1)}_{\alpha^k \beta^l}(L) = \text{Der}^{\alpha^k \beta^l}(L) \);

(b) \( \text{Der}^{(1,0)}_{\alpha^k \beta^l}(L) = \{d \in \Omega \mid d([x, y]) = [d(x), \alpha^k \beta^l(y)]\} \), is called the \( \alpha^k \beta^l \)-left-centroid of \( L \);

(c) \( \text{Der}^{(0,1)}_{\alpha^k \beta^l}(L) = \{d \in \Omega \mid d([x, y]) = [\alpha^k \beta^l(x), d(y)]\} \), is called the \( \alpha^k \beta^l \)-right-centroid of \( L \);

(d) \( \text{Der}^{(1,0)}_{\alpha^k \beta^l}(L) \cap \text{Der}^{(0,1)}_{\alpha^k \beta^l}(L) \), is called the \( \alpha^k \beta^l \)-centroid of \( L \) and it is noted \( C_{\alpha^k \beta^l}(L) \);

(e) \( \text{Der}^{(0,1,-1)}_{\alpha^k \beta^l}(L) \cap \text{Der}^{(1,1,-1)}_{\alpha^k \beta^l}(L) \)

\[
= \{d \in \Omega \mid d([x, y]) = 0 = [d(x), \alpha^k \beta^l(y)] = [\alpha^k \beta^l(x), d(y)] \}
\]

is called the \( \alpha^k \beta^l \)-central derivation of \( L \) and it is noted \( Z\text{der}_{\alpha^k \beta^l}(L) \);

(f) \( \text{Der}^{(0,1,-1)}_{\alpha^k \beta^l}(L) \) \{d \in \Omega \mid [d(x), \alpha^k \beta^l(y)] = [\alpha^k \beta^l(x), d(y)]\} \), is called the \( \alpha^k \beta^l \)-quasi-centroid of \( L \) and it is noted \( QC_{\alpha^k \beta^l}(L) \).
Now, we consider the subspace
\[ \text{IDer}_{\alpha^k\beta^l}(L) = \{ d \in \Omega \mid \lambda d([x,u]) = \mu [d(x), \alpha^k\beta^l(u)] + \gamma [\alpha^k\beta^l(x), d(u)], \forall x, u \in L, u \in L^2 \}. \]

**Theorem 2.22.** Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a symmetric BiHom–Leibniz algebra such that the maps \(\alpha\) and \(\beta\) are surjective. Let \(\lambda, \mu, \gamma\) be elements of \(\mathbb{K}\).

(a) If \(\lambda \neq 0\) and \(\mu^2 \neq \gamma^2\). Then \(\text{IDer}_{\alpha^k\beta^l}(L) = \text{IDer}_{\alpha^k\beta^l}(1,0)(L)\).

(b) If \(\lambda \neq 0\), \(\mu \neq 0\) and \(\gamma = -\mu\). Then \(\text{IDer}_{\alpha^k\beta^l}(L) = \text{IDer}_{\alpha^k\beta^l}(1,0)(L)\).

(c) If \(\lambda \neq 0\), \(\mu = \gamma\) and \(\mu \neq 0\). Then \(\text{IDer}_{\alpha^k\beta^l}(L) = \text{IDer}_{\alpha^k\beta^l}(1,1)(L)\).

(d) If \(\lambda \neq 0\), \(\mu = \gamma = 0\). Then \(\text{IDer}_{\alpha^k\beta^l}(L) = \text{IDer}_{\alpha^k\beta^l}(0,0)(L)\).

(e) If \(\lambda = 0\) and \(\mu^2 \neq \gamma^2\). Then \(\text{IDer}_{\alpha^k\beta^l}(L) = \text{IDer}_{\alpha^k\beta^l}(0,0)(L)\).

(f) If \(\lambda = 0\) and \(\mu = \gamma\). Then \(\text{IDer}_{\alpha^k\beta^l}(L) = \text{IDer}_{\alpha^k\beta^l}(0,1)(L)\).

(g) If \(\lambda = 0\) and \(\mu = -\gamma\). Then \(\text{IDer}_{\alpha^k\beta^l}(L) = \text{IDer}_{\alpha^k\beta^l}(0,1,-1)(L)\).

**Proof.** Let \(x \in L, u = [y, z] \in L^2\). Since \(\alpha\) and \(\beta\) are surjectives there exists \(a, v \in L\) so that \(x = \beta(a), u = \alpha(v) = [\alpha(y), \alpha(z)]\). It follows from (2.2) that \([\beta(a), \alpha(v)] = -[\beta(v), \alpha(a)]\). The rest of proof is similar to the one of case of Lie algebras given in [33].

**Corollary 2.23.** Let \((L, [\cdot, \cdot], \alpha, \beta)\) be a BiHom–Lie algebra such that the maps \(\alpha\) and \(\beta\) are surjective. For any \(\lambda, \mu, \gamma \in \mathbb{K}\) there exists \(\delta \in \mathbb{K}\) such that the subspace \(\text{Der}_{\alpha^k\beta^l}(L)\) is equal to one of the four following subspaces:

(a) \(\text{Der}_{\alpha^k\beta^l}(5,0,0)(L)\);

(b) \(\text{Der}_{\alpha^k\beta^l}(6,1,1)(L)\);

(c) \(\text{Der}_{\alpha^k\beta^l}(6,1,-1)(L)\);

(d) \(\text{Der}_{\alpha^k\beta^l}(5,0,1)(L)\).

### 3 BiHom-Leibniz colour algebras

Let \(\Gamma\) be an abelian group. A vector space \(\mathcal{A}\) is said to be \(\Gamma\)-graded, if there is a family \((\mathcal{A}_\gamma)_{\gamma \in \Gamma}\) of vector subspace of \(\mathcal{A}\) such that \(\mathcal{A} = \oplus_{\gamma \in \Gamma} \mathcal{A}_\gamma\). An element \(x \in \mathcal{A}\) is said to be homogeneous of degree \(\gamma\) if \(x \in \mathcal{A}_\gamma\). We denote by \(\mathcal{H}(\mathcal{A})\) the set of all the homogeneous elements of \(\mathcal{A}\).
Definition 3.1. A map $\varepsilon: \Gamma \times \Gamma \rightarrow K^*$ is called a skewsymmetric bicharacter on $\Gamma$ if the following identities hold, for all $a, b, c \in \Gamma$

(i) $\varepsilon(a, b)\varepsilon(b, a) = 1$;
(ii) $\varepsilon(a, b + c) = \varepsilon(a, b)\varepsilon(a, c)$;
(iii) $\varepsilon(a + b, c) = \varepsilon(a, c)\varepsilon(b, c)$.

If $x$ and $y$ are two homogeneous elements of degree $\gamma$ and $\gamma'$ respectively, then we shorten the notation by writing $\varepsilon(x, y)$ instead of $\varepsilon(\gamma, \gamma')$.

3.1 BiHom-Lie colour algebras

In the following we summarize definitions of BiHom–Lie and BiHom-associative color algebraic structures generalizing the well known Hom–Lie and Hom–associative color algebras (See [42]).

Definition 3.2. (see [25]) A BiHom-Lie colour algebra over a field $K$ is a 5-tuple $(A, \mu, \varepsilon, \alpha, \beta)$ consisting of a $\Gamma$-graded vector space $A$, an even bilinear mapping $\mu: A \times A \rightarrow A$ (i.e $\mu(A_a, A_b) \subset A_{a+b}$ for all $a, b \in \Gamma$), a bicharacter $\varepsilon: A \times A \rightarrow K^*$ and two even homomorphism $\alpha, \beta: A \rightarrow A$ such that for all $x, y, z \in H(A)$ we have

\[
\alpha \circ \beta = \beta \circ \alpha
\]
\[
\alpha ([x, y]) = [\alpha(x), \alpha(y)] \quad \text{and} \quad \beta ([x, y]) = [\beta(x), \beta(y)]
\]
\[
[\beta(x), \alpha(y)] = -\varepsilon(x, y) [\beta(y), \alpha(x)], \quad (\varepsilon-\text{BiHom-skew-symmetry})
\]
\[
\alpha \circ \beta = \beta \circ \alpha,
\]
\[
\beta(\alpha(x)), \alpha(y)] = \varepsilon(x, y) [\beta(\alpha(x)), \alpha(y)] = 0. \quad (\varepsilon-\text{BiHom-Jacobi condition})
\]

Definition 3.3 (BiHom-associative colour algebras). (see [25]) A BiHom-associative colour algebra over $K$ is a 5-tuple $(A, \mu, \varepsilon, \alpha, \beta)$ consisting of a $\Gamma$-graded vector space $A$, an even bilinear mapping $\mu: A \times A \rightarrow A$ (i.e $\mu(A_a, A_b) \subset A_{a+b}$ for all $a, b \in \Gamma$), a bicharacter $\varepsilon: A \times A \rightarrow K^*$ and two even homomorphism $\alpha, \beta: A \rightarrow A$ such that $\alpha \circ \beta = \beta \circ \alpha$ and for all $x, y, z \in A$ we have $\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \beta(z))$.

In particular, if $\alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y))$ and $\beta(\mu(x, y)) = \mu(\beta(x), \beta(y))$, we call it multiplicative BiHom-associative colour algebra.

Inspired by [27], we give the following constructions of BiHom-associative and BiHom-Lie algebras starting by an ordinary associative colour algebra.

Proposition 3.4. Let $(A, \mu, \varepsilon)$ be an ordinary associative colour algebra and let $\alpha, \beta: A \rightarrow A$ two commuting even linear maps such that $\alpha(\mu(x, y)) =$
First we introduce a definition of BiHom–Leibniz colour algebra.

### 3.2 BiHom-Leibniz colour algebras

We consider a 5-tuple \((L, [\cdot, \cdot], \varepsilon, \alpha, \beta)\), consisting of a \(\Gamma\)-graded vector space \(L\), an even bilinear map \([\cdot, \cdot]: L \times L \to L\) (i.e. \([L_a, L_b] \subseteq L_{a+b}\) for all \(a, b \in \Gamma\)), a bicharacter \(\varepsilon: L \times L \to \mathbb{K}^*\) and two even homomorphism \(\alpha, \beta: L \to L\) such that for all homogeneous elements \(x, y, z\) we have \(\alpha \circ \beta = \beta \circ \alpha\), \(\alpha ([x, y]) = [\alpha(x), \alpha(y)]\) and \(\beta ([x, y]) = [\beta(x), \beta(y)]\).

**Definition 3.5.** The 5-tuple \((L, [\cdot, \cdot], \varepsilon, \alpha, \beta)\) is said to be

(i) left BiHom-Leibniz colour algebra if for any homogeneous elements \(x, y, z \in L\) the so-called left Leibniz identity

\[
[\alpha \beta(x), [y, z]] = [[\beta(x), y], \beta(z)] + \varepsilon(x, y) [\beta(y), [\alpha(x), z]]
\]

holds.

(ii) right BiHom-Leibniz colour algebra if for any homogeneous elements \(x, y, z \in L\) if it satisfies the identity

\[
[[x, y], \alpha \beta(z)] = \varepsilon(y, z) [[x, \beta(z)], \alpha(y)] + [\alpha(x), [y, \alpha(z)]].
\]

**Remark 3.6.**

1. If \(\Gamma = \{1\}\) and \(\varepsilon(x, y) = 1\), for all \(x, y \in L\) then \((L, [\cdot, \cdot], \varepsilon)\) is a left (resp. right) Leibniz colour algebra if and only if \((L, [\cdot, \cdot], Id_L, Id_L, \varepsilon)\) is a left (resp. right) BiHom-Leibniz colour algebra.

2. If \(\Gamma = \mathbb{Z}_2\) and \(\varepsilon(x, y) = (-1)^{|x||y|}\), for all \(x, y \in L\) then \((L, [\cdot, \cdot], \varepsilon)\) is a left (resp. right) Leibniz superalgebra if and only if \((L, [\cdot, \cdot], Id_L, Id_L, \varepsilon)\) is a left (resp. right) BiHom-Leibniz colour algebra.

3. If \(\alpha\) is surjective then \((L, [\cdot, \cdot], \alpha, \varepsilon)\) is a left (resp. right) Hom-Leibniz colour algebra if and only if \((L, [\cdot, \cdot], \alpha, \alpha, \varepsilon)\) is a left (resp. right) BiHom-Leibniz colour algebra.
Proposition 3.7. If \((L, [\cdot, \cdot], \varepsilon, \alpha, \beta)\) is a left (resp. right) BiHom-Leibniz colour algebra, then
\[
\left[ [\beta(x), \alpha(y)], \alpha\beta(z) \right] = -\varepsilon(x,y) \left[ [\beta(y), \alpha(x)], \alpha\beta(z) \right], \quad \forall x,y,z \in L.
\]
Respectively
\[
\left[ \alpha\beta(z), [\beta(x), \alpha(y)] \right] = -\varepsilon(x,y) \left[ \alpha\beta(z), [\beta(y), \alpha(x)] \right], \quad \forall x,y,z \in L,
\]
for all homogeneous elements \(x,y,z \in L\).

Definition 3.8. If \((L, [\cdot, \cdot], \varepsilon, \alpha, \beta)\) is a left and a right BiHom-Leibniz colour algebra, then \(L\) is called a symmetric BiHom-Leibniz colour algebra.

Proposition 3.9. Let \((L, [\cdot, \cdot], \varepsilon, \alpha, \beta)\) be a left BiHom-Leibniz colour algebra. Then, \((L, [\cdot, \cdot], \varepsilon, \alpha, \beta)\) is a symmetric BiHom-Leibniz colour algebra if and only if
\[
[\beta(y), [\alpha(x), \alpha(z)]] = -\varepsilon(y,x+z) \left[ [\beta(x), \beta(z)], \alpha(y) \right], \quad \forall x,y,z \in L.
\]

In the following we construct BiHom–Leibniz colour algebras involving elements of the centroid of colour Leibniz algebras. Let \((L, [\cdot, \cdot], \varepsilon)\) be a Leibniz colour algebra. An endomorphism \(\alpha \in \text{End}(L)_\gamma\) of degree \(d\) is said to be an element of degree \(\gamma\) of the centroid if \(\alpha([x,y]) = [\alpha(x),y] = \varepsilon(\gamma,x)[x,\alpha(y)]\) for all \(x,y \in \mathcal{H}(L)\). The centroid of \(L\) of degree \(\gamma\) is defined by
\[
C_\gamma(L) = \{ \alpha \in \text{End}(L) \mid \alpha([x,y]) = [\alpha(x),y] = \varepsilon(\gamma,x)[x,\alpha(y)] \}.
\]

Proposition 3.10. Let \((L, [\cdot, \cdot], \varepsilon)\) be a Leibniz colour algebra and let \(\alpha, \beta : L \times L \to L\) two commuting even linear maps such that \(\alpha^2 = \alpha\) and \(\beta^2 = \beta\). Set for \(x,y \in \mathcal{H}(L)\), \([x,y] = [\beta(x), \alpha(y)]\). Then \((L, [\cdot, \cdot], \varepsilon, \alpha, \beta)\) is a BiHom-Leibniz colour algebra.

Proof. Since \(\alpha\) and \(\beta\) are even, then \(\varepsilon(\gamma,x) = 1\). Therefore, \([x,y] = [\beta(x), \alpha(y)] = \alpha\beta([x,y])\). Suppose that \((L, [\cdot, \cdot], \varepsilon)\) is a Leibniz colour algebra. We have
\[
\alpha([x,y]) = \alpha^2 \beta([x,y]) = \alpha^4 \beta([x,y]) = \alpha^2 \beta([\alpha(x), \alpha(y)]) = \alpha \beta([\alpha(x), \alpha(y)]) = \{\alpha(x), \alpha(y)\}.
\]
Similarly,
\[
\beta([x,y]) = \alpha^2 \beta([x,y]) = \alpha^4 \beta([x,y]) = \alpha \beta([\beta(x), \beta(y)]) = \{\beta(x), \beta(y)\}.
\]
1. If \((L, [\cdot, \cdot], \varepsilon)\) is a left Leibniz colour lgebra:
\[
\{\alpha\beta(x), \{y, z\}\} = \{\alpha\beta(x), \alpha\beta([y, z])\} = \alpha\beta([\alpha\beta(x), \alpha\beta([y, z])]) = \alpha^3\beta^3([x, [y, z]]).
\]
\[
\{\{\beta(x), y\}, \beta(z)\}\} + \varepsilon(x, y) \{\beta(y), \{\alpha(x), z\}\}
= \alpha^2\beta^2[[\beta(x), y], \beta(z)] + \varepsilon(x, y)\alpha^2\beta^2[\beta(y), [\alpha(x), z]]
= \alpha^2\beta^3[[x, y], z] + \varepsilon(x, y)\alpha^2\beta^3[y, [x, z]]
= \alpha^3\beta^3([x, [y, z]])
= \{\alpha\beta(x), \{y, z\}\}
\]

Then, \((L, [\cdot, \cdot], \varepsilon, \alpha, \beta)\) be a left BiHom-Leibniz colour lgebra.

2. If \((L, [\cdot, \cdot], \varepsilon)\) is a symmetric Leibniz colour lgebra:
\[
\{\beta(x), \{\alpha(y), \alpha(z)\}\} = \alpha^2\beta^2[\beta(x), [\alpha(y), \alpha(z)]] = \alpha^4\beta^3([x, [y, z]]) = \alpha\beta([x, [y, z]])
\]
\[
-\varepsilon(x, y + z) \{\{\beta(y), \beta(z)\}, \alpha(x)\} = -\varepsilon(x, y + z)\alpha^2\beta^2[[\beta(y), \beta(z)], \alpha(x)]
= -\varepsilon(x, y + z)\alpha^3\beta^4([x, [y, z]])
= -\varepsilon(x, y + z)\alpha\beta([y, z], [x])
= \alpha\beta([x, [y, z]])
= \{\beta(x), \{\alpha(y), \alpha(z)\}\}.
\]

Then, \((L, [\cdot, \cdot], \varepsilon, \alpha, \beta)\) is a symmetric BiHom-Leibniz colour lgebra.

3. If \((L, [\cdot, \cdot], \varepsilon)\) is a right Leibniz colour lgebra: Reasoning similarly as above proves that
\[
\{\{x, y\}, \alpha\beta(z), \} = \alpha\beta([x, [y, z]])
\]
and
\[
\varepsilon(x, z) \{\{x, \beta(z)\}, \alpha(y)\} + \{\alpha(x), \{y, \alpha(z)\}\} = \alpha\beta([x, \varepsilon(z)], y) + [x, [y, z]]
\]

Therefore, \((L, [\cdot, \cdot], \varepsilon, \alpha, \beta)\) is a right BiHom-Leibniz colour lgebra.

**Definition 3.11.** Let \((L, [\cdot, \cdot], \varepsilon, \alpha, \beta)\) be a Leibniz colour lgebra. The set \(C_r(L)\) consisting of linear map \(d\) of degree \(\gamma\) with the property
\[
d \circ \alpha = \alpha \circ d, \quad d \circ \beta = \beta \circ d
\]
\[
d([x, y]) = [d(x), y] = \varepsilon(\gamma, x)[x, d(y)], \quad \forall x, y \in \mathcal{H}(L),
\]
is called the centroids of \(L\).
In the following proposition, we construct colour BiHom- Leibniz algebras starting from a colour BiHom- Leibniz algebra and an two even element in its centroid.

**Proposition 3.12.** Let \((L, [\cdot, \cdot], \varepsilon, \alpha, \beta)\) be a colour BiHom- Leibniz algebra, and \(\theta, \theta'\) be two even element in the centroid of \(L\) satisfies \(\theta \circ \theta' = \theta' \circ \theta\), \(\theta^2 = \theta\) and \(\theta'^2 = \theta'\). Set for \(x, y \in \mathcal{H}(L)\), \([x, y]^{\theta'} = [\theta'(x), y]\). Then \((L, [\cdot, \cdot], \varepsilon, \theta \circ \alpha, \theta \circ \beta)\) and \((L, [\cdot, \cdot]^{\theta'}, \varepsilon, \theta \circ \alpha, \theta \circ \beta)\) are colour BiHom- Leibniz algebras.

### 4 Representations and cohomology of (Bi)Hom-Leibniz algebras of type \(B_1\)

First, we define a new type of BiHom-Lie algebras. We call it BiHom-Lie algebras of type \(B_1\). With the BiHom-Lie algebra of type \(B_1\) we have the following hierarchy of algebras:

\[
\{\text{BiHom-Lie type } B_1\} \supseteq \{\text{Hom-Lie}\} \supseteq \{\text{Lie}\}
\]

For the rest of this article, we mean by \((L, [\cdot, \cdot], \alpha, \beta)\) a 4-tuple consisting of \(\mathbb{K}\)-linear space \(L\), two linear maps \(\alpha, \beta : L \to L\) and a bilinear map \([\cdot, \cdot] : L \times L \to L\), satisfying the following conditions

\[
\alpha \circ \beta = \beta \circ \alpha, \quad \alpha ([x, y]) = [\alpha(x), \alpha(y)] \quad \text{and} \quad \beta ([x, y]) = [\beta(x), \beta(y)]
\]

for all \(x, y, z \in L\).

**Definition 4.1.** The 4-tuple \((L, [\cdot, \cdot], \alpha, \beta)\) is a BiHom-Lie algebra algebra of type \(B_1\) if

\[
[\beta(x), \beta^2(y)] = -[\beta(y), \beta^2(x)],
\]

\[
\bigcirc_{x,y,z} [\alpha(x), [\beta(y), \beta^2(z)]] = 0.
\]

for all \(x, y, z \in L\).

**Remark 4.2.** Obviously, a BiHom-Lie algebra \((L, [\cdot, \cdot], \alpha, \beta)\) of type \(B_1\) for which \(\beta = \text{Id}_L\) is just a Hom-Lie algebra \((L, [\cdot, \cdot], \alpha)\).

**Definition 4.3.** The 4-tuple \((L, [\cdot, \cdot], \alpha, \beta)\) is is called a left (resp. right) BiHom-Leibniz algebra of type \(B_1\) if it satisfies the identity

\[
[\alpha(x), [\beta(y), \beta(z)]] = [[x, \beta(y)], \alpha(z)] + [\alpha(y), [\beta(x), \beta(z)]
\]

respectively

\[
[[x, y], \alpha(z)] = [[x, z], \alpha(y)] + [\alpha(x), [y, \beta(z)]].
\]
Proposition 4.4. If \((L, [\cdot, \cdot], \alpha, \beta)\) is a left BiHom-Leibniz algebra of type \(B_1\), then
\[\,[y, \beta(x)], \alpha(z)\, = \, -\,[x, \beta(y)], \alpha(z)\,\]

Proposition 4.5. If \((L, [\cdot, \cdot], \alpha, \beta)\) is a right BiHom-Leibniz algebra of type \(B_1\), then
\[\,[\alpha(x), [z, \beta(y)]]\, = \, -\,[\alpha(x), [y, \beta(z)]]\,\]

Definition 4.6. \((L, [\cdot, \cdot], \alpha, \beta)\) is called a symmetric BiHom-Leibniz algebra of type \(B_1\) if it is a left and a right BiHom-Leibniz algebra of type \(B_1\).

Proposition 4.7. A 4-tuple \((L, [\cdot, \cdot], \alpha, \beta)\) is a symmetric BiHom-Leibniz algebra of type \(B_1\) if and only if it satisfies
\[\,[x, y], \alpha(z)\, = \, [x, z], \alpha(y)\, + \, [\alpha(x), [y, \beta(z)]];\]
\[\,[\alpha(y), [\beta(x), \beta(z)]\, = \, -\,[x, z], \alpha\beta(y)]\,.

4.1 Representations of BiHom-Leibniz algebras of type \(B_1\)

Lie algebra cohomology was introduced by Chevalley and Eilenberg [7]. For Hom-Lie algebra, the cohomology theory has been given by [32, 38]. A cohomology of BiHom-Lie algebras were introduced and investigated in [43]. We refer the reader to [29, 30, 34, 12] for more information about Leibniz representations and Leibniz cohomologies.

In the following we define a representations of (Bi)Hom–Leibniz algebras of type \(B_1\) and the corresponding coboundary operators. We show that one can obtain the direct sum symmetric (Bi)Hom–Leibniz algebras \((L \oplus V, [\cdot, \cdot], f, \alpha + \alpha_V, \beta + \beta_V)\) of type \(B_1\).

Definition 4.8. Let \(V\) be a vector space, \(\alpha_V, \beta_V \in \text{End}(V)\) and \(r, l: \mathcal{L} \to \text{End}(V)\) be two linear maps satisfying
\[\,\alpha_V \circ l(x) = l(\alpha(x)) \circ \alpha_V; \quad \alpha_V \circ r(x) = r(\alpha(x)) \circ \alpha_V;\]
\[\,\beta_V \circ l(x) = l(\alpha(x)) \circ \beta_V; \quad \beta_V \circ r(x) = r(\alpha(x)) \circ \beta_V;\]

(i) If \((L, [\cdot, \cdot], \alpha, \beta)\) is a left BiHom-Leibniz algebra of type \(B_1\), then we say that \((r, l)\) is a left representation of \(L\) in \(V\) if for all \(x, y \in L, v \in V:\)
\[\,l([x, \beta(y)]) \circ \alpha_V = l(\alpha(x)) \circ l(\beta(y)) \circ \beta_V - l(\alpha(y)) \circ l(\beta(x)) \circ \beta_V;\]
\[\,r \circ \beta([x, y]) \circ \alpha_V = \alpha(l(x)) \circ r(\beta(y)) \circ \beta_V - r(\alpha(y)) \circ l(x) \circ \beta_V;\]
\[\,r \circ \beta([x, y]) \circ \alpha_V = r(\alpha(y)) \circ r(\beta(x)) + r(\alpha(x)) \circ r(\beta(y)) \circ \beta_V.\]
(ii) If $L$ is a right BiHom-Leibniz algebra of type $B_1$, then we say that $(r, l)$ is a right representation of $L$ in $V$ if for all $x, y \in L$, $v \in V$:

\[
l([x, y]) \circ \alpha_V = l(\alpha(x)) \circ l(y) \circ \beta_V + r(\alpha(y)) \circ l(x);
\]

\[
l([x, y]) \circ \alpha_V = r(\alpha(y)) \circ l(x) - l(\alpha(x)) \circ r(\beta(y));
\]

\[
r([x, \beta(y)]) \circ \alpha_V = r(\alpha(y)) \circ r(x) - r(\alpha(x)) \circ r(y).
\]

**Example 4.1.** Let $(L, [\cdot, \cdot], \alpha, \beta)$ be a left (resp. right) BiHom-Leibniz algebra of type $B_1$. Then, we consider the maps $\text{ad}_k: \mathcal{L} \to \text{End}(\mathcal{L})$ and $\text{Ad}_{k,l}: \mathcal{L} \to \text{End}(\mathcal{L})$ Defined by by $\text{ad}_{k,l}(a)(x) = [x, \alpha^k \beta^l(a)]$, $\text{Ad}_{k,l}(a)(x) = [\alpha^k \beta^l(a), x]$, $\forall x \in L$. Therefore, $(\text{ad}_{k,l}, \text{Ad}_{k,l})$ is a left representation (resp. a right representation ) of $L$ in $L$ called the left (resp. right) $\alpha^k \beta^l$-adjoint representation of $L$.

In the rest of this article, if $r, l: \mathcal{L} \to \text{End}(V)$, we denote $r(x)(v) = [v, x]_V$ and $l(x)(v) = [x, v]_V$, $\forall x \in L$, $v \in V$ and if $(r, l)$ is a representation of $L$ we say $(V, [\cdot, \cdot]_V, \alpha_V, \beta_V)$ is a representation of $L$ (or a $L$-module).

**Proposition 4.9.** Let $(L, [\cdot, \cdot], \alpha, \beta)$ be a left BiHom-Leibniz algebra of type $B_1$ and $(V, [\cdot, \cdot]_V, \alpha_V, \beta_V)$ be a left representation of $L$. Then

\[
[[v, \beta(x)]_V, \alpha(y)]_V = -[[x, \beta_V(v)]_V, \alpha(y)]_V
\]

for all $x, y \in L$, $v \in V$.

**Proposition 4.10.** Let $(L, [\cdot, \cdot], \alpha, \beta)$ be a right BiHom-Leibniz algebra of type $B_1$ and $(V, [\cdot, \cdot]_V, \alpha_V, \beta_V)$ be a right representation of $L$. Then

\[
[\alpha(x), [y, \beta_V(v)]_M] = -[[\alpha(x), \alpha(y)]_M]
\]

for all $x, y \in L$, $v \in V$.

**Definition 4.11.** Let $(L, [\cdot, \cdot], \alpha, \beta)$ be a left (resp. right) BiHom-Leibniz algebra of type $B_1$. Let $V$ be a vector space, $\alpha_V, \beta_V \in \text{End}(V)$ and $r, l: \mathcal{L} \to \text{End}(V)$ be two linear maps. Then, we say that $(r, l)$ is a representation of $L$ in $V$ if $(r, l)$ is a left and a right representation of $L$ in $V$.

**Definition 4.12.** Let $(L, [\cdot, \cdot], \alpha, \beta)$ be a BiHom-Leibniz algebra of type $B_1$. A representation $(V, [\cdot, \cdot]_V, \alpha_V, \beta_V)$ is called:

- (i) trivial if $[x, \beta_V(v)]_V = [v, \beta(x)]_V = 0$, $\forall v \in V, x \in L$.

- (ii) adjoint if $(V, [\cdot, \cdot]_V, \alpha_V, \beta_V) = (L, [\cdot, \cdot], \alpha, \beta)$.
Proposition 4.13. Let $(L, [\cdot, \cdot], \alpha, \beta)$ be a symmetric BiHom-Leibniz algebra of type $B_1$. A 4-tuples $(V, [\cdot, \cdot]_V, \alpha_V, \beta_V)$ is a symmetric representation of $V$ if and only if.

\[
\alpha_V([v, x]_V) = [\alpha(v), \alpha_V(v)]_V; \quad \alpha_V([v, x]_V) = [\alpha_V(v), \alpha(x)]_V;
\]

\[
\beta_V([v, x]_V) = [\beta(x), \beta_V(v)]_V; \quad \beta_V([v, x]_V) = [\beta_V(v), \beta(x)]_V;
\]

\[
[[v, x]_V, \alpha(y)]_V = [[v, y]_V, \alpha(x)]_V + [\alpha_V(v), [x, \beta(y)]_V]_V;
\]

\[
[[x, y], \alpha_V(v)]_V = [[x, v]_V, \alpha(y)]_V + [\alpha(x), [y, \beta_V(v)]_V]_V;
\]

\[
[[x, v]_V, \alpha(y)]_V = [[x, y], \alpha_V(v)]_V + [\alpha(x), [v, \beta(y)]_V]_V;
\]

\[
[\alpha(x), [\beta(y), \beta_V(v)]_V] = -[[y, \beta_V(v)], \alpha\beta(x)]_V;
\]

\[
[\alpha(x), [\beta_V(v), \beta(y)]_V] = -[[v, y]_V, \alpha\beta(x)]_V;
\]

\[
[\alpha(x), [\beta(y), \beta_V(v)]_V] = -[[y, v]_V, \alpha\beta(x)]_V.
\]

Remark 4.14. If $(V, [\cdot, \cdot]_V, \alpha_V, \beta_V)$ is a symmetric representation of a symmetric BiHom-Leibniz algebra $(L, [\cdot, \cdot], \alpha, \beta)$ of type $B_1$. Let us consider two linear maps $\text{Ad}_{n,m}, \text{ad}_{n,m}: \mathcal{L} \rightarrow \text{End}(V)$ defined by

\[
\text{ad}_{n,m}(x)(v) = [x, \alpha^n\beta^m(v)], \quad \text{Ad}_{n,m}(x)(v) = [\alpha^n\beta^m(v), x], \quad \forall x \in L, v \in V.
\]

Then $(\text{Ad}_{n,m}, \text{ad}_{n,m})$ is symmetric representation of $L$ in $V$.

4.2 Cohomology of symmetric BiHom-Leibniz algebras of type $B_1$

Let $(L, [\cdot, \cdot], \alpha, \beta)$ be a symmetric BiHom-Leibniz algebra of type $B_1$ and $(V, [\cdot, \cdot]_V, \alpha_V, \beta_V)$ be a symmetric representation of $V$. Denote

\[
C^k(L, V) = \text{Hom}(L^k, V), \quad k \geq 0,
\]

and

\[
C^k_{(\alpha, \alpha_V)}(L, V) = \{ f \in C^k(L, V) \mid f \circ \alpha = \alpha_V \circ f \}.
\]

Let $\delta^k : C^k_{(\alpha, \alpha_V)}(L, V) \rightarrow C^{k+1}_{(\alpha, \alpha_V)}(LV)$ be a $k$-Homomorphism defined by

\[
\delta^k_{n,m}(f)(x_0, \ldots, x_k) = \sum_{0 < t \leq k} (-1)^{t+1} f([x_0, x_1], \ldots, \hat{x}_t, \ldots, x_k)
\]

\[
+ \sum_{0 < s < t \leq k} (-1)^{s+t-1} f(\alpha(x_0), \ldots, \alpha(x_{s-1}), [x_s, \beta(x_t)], \alpha(x_{s+1}), \ldots, \hat{x}_t, \ldots, x_k)
\]
Two extensions $ϕ$ that phism classes of extensions of $L$ are equivalent if there is an isomorphism $⊕$. The pair $(\oplus_{k>0} C_{\alpha,\alpha'})^k(L, V), \{δ^k\}_{k>0}$ defines a chomology complex, that is $δ^k ◦ δ^{k-1} = 0$.

- The $k$-cocycles space is defined as $Z^k(L, V) = \ker δ^k$.
- The $k$-coboundary space is defined as $B^k(L, V) = \Im δ^{k-1}$.
- The $k^{th}$ cohomology space is the quotient $H^k(L, V) = Z^k(L, V)/B^k(L, V)$.

If $(ad_{n,m}, Ad_{n,m})$ is a symmetric adjoint representation of $L$. Then any 1-cocycle $f ∈ Z^1(L, L)$ is called a $α^nβ^m$-derivation of $L$.

4.3 Extensions of BiHom-Leibniz algebras of type $B_1$

In this section we extend extensions theory of Leibniz algebras introduced in [30] to BiHom-Leibniz algebras of type $B_1$ case.

An extension of a BiHom-Leibniz algebras $(L, [, ], α, β)$ of type $B_1$ by $L$-module $(V, [, ], α_V, β_V)$ is an exact sequence

$$0 → (V, α_V, β_V) → (L, α, β) → 0$$

satisfying $α ◦ i = i ◦ α_V, β ◦ i = i ◦ β_V, α ◦ π = π ◦ α and β ◦ π = π ◦ β$. We say that the extension is central if $[G, i(V)]_β = 0$.

Two extensions

$$0 → (V_k, α_{V_k}, β_{V_k}) → (L_k, α_k, β_k) → (L, α, β) → 0 \quad (k = 1, 2)$$

are equivalent if there is an isomorphism $φ → (L_1, α_{11}) → (L_2, α_2, β_2)$ such that $φ ◦ i_1 = i_2 and π_2 ◦ φ = π_1$. One denote by $Ext(L, V)$ the set of isomorphism classes of extensions of $L$ by $V$. In the sequel, we assume that $i(V)$ is of finite codimension in $L$.

Let $f ∈ C^2_{(\alpha,\alpha')} (L, V)$. Assume that $L \cap V = \{0\}$ and we consider the direct sum $L = L ⊕ V$ with the following bracket

$$[(x, u), (y, v)]_L = ([x, y], [x, v]_V + [u, y)_V + f(x, y)); \quad ∀x, y ∈ L, u, v ∈ V.$$ Define the linear maps $α, β: L → L$ by $α(x, v) = (α(x), α_V(v))$ by $β(x, v) = (β(x), β_V(v))$.
Lemma 4.15. With the above notations, the 4-tuple $(\tilde{L}, [\cdot, \cdot]_{L}, \tilde{\alpha}, \tilde{\beta})$ is a symmetric BiHom-Leibniz of type $B_1$ if and only if $f$ is a 2-cocycle (i.e. $\delta^2(f) = 0$).

Theorem 4.16. For any symmetric BiHom-Leibniz of $L$ type $B_1$ and any symmetric representation $V$ of $L$, there is a natural bijection

$$\text{Ext}(L, V) \cong H^2(L, V).$$

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