A Generalized Weil Representation for the finite split orthogonal group $O_q(2n, 2n)$, $q$ odd greater than 3.

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Abstract

We construct via generators and relations a generalized Weil representation for the split orthogonal group $O_q(2n, 2n)$ over a finite field of $q$ elements. Besides, we give an initial decomposition of the representation found. We also show that the constructed representation is equal to the restriction of the Weil representation to $O_q(2n, 2n)$ for the reductive dual pair $(Sp_{2n}(\mathbb{F}_q), O_q(2n, 2n))$ and that the initial decomposition is the same as the decomposition with respect to the action of $Sp_{2n}(\mathbb{F}_q)$.

1 Introduction

Weil representations have proven to be a powerful tool in the theory of group representations. They originate from a very general construction of A. Weil ([12]), which has as a consequence the existence of a projective representation of the group $Sp(2n, K)$, $K$ a locally compact field. Weil built this representation taking advantage of the representation theory of the related Heisenberg group, as described by the Stone-von Neumann theorem in the real case ([7]). In particular, these representations have allowed to build in a universal and uniform way all irreducible complex linear representations of the general linear group of rank 2 over a finite field ([11]), and later over a local field, except in residual characteristic two ([6]).

Weil representations can be constructed in various ways. For instance, they can be constructed via Heisenberg groups, via constructions of equivariant vector bundles ([4]), via presentations or via dual pairs ([1]), as we shall see later. The method using presentations is accomplished by having a simple presentation of the group, and then defining linear operators on a suitable vector space which preserve the relations among the generators of the presentation. This idea was originally suggested by Cartier, and used successfully by Soto-Andrade, for symplectic groups $Sp(2n, \mathbb{F}_q)$ ([11]) and complex irreducible representations of $SL(2, \mathbb{F}_q)$. In the case of $Sp(2n, \mathbb{F}_q)$ he considered this group as a group "$SL(2)$" but with entries in the matrix ring $M_n(\mathbb{F}_q)$, and thus he obtained a suitable presentation for the group. In this way, he constructed the Weil representation for the symplectic groups $Sp(2n, \mathbb{F}_q)$. In the case of $Sp(4, \mathbb{F}_q)$, Soto-Andrade obtained all the irreducible representations of this group by decomposing the two Weil representations associated to the two isomorphy types of quadratic forms of rank 4 over $\mathbb{F}_q$.

This point of view was generalized and gave rise to the groups $SL^\varepsilon(2, A)$ for $(A, \ast)$ an involution ring and $\varepsilon = \pm 1 \in A$. These groups are a generalization of special linear groups $SL(2, K)$, where $K$ is a field. They were defined for $\varepsilon = -1$ by Pantoja and Soto-Andrade in [8] and generalized to $\varepsilon = 1$ in [10].

The groups $SL^\varepsilon(2, A)$ include, among others, symplectic and split orthogonal groups. For instance, if $K$ is a field, $A = M_n(\mathbb{Z})$, $\diamond$ the transposition of matrices then $SL_{\diamond = -1}^\varepsilon(2, A)$ is the symplectic group $Sp(2n, K)$ and $SL_{\diamond = 1}^\varepsilon(2, A)$ is the split orthogonal group $O_K(2n, 2n)$. Thus, these groups allow us to look at higher rank classical groups as rank two groups, considering them with coefficients in a new ring.

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The procedure used by Soto-Andrade in [11] was approached even in the case of a non semi-simple involutive ring \((A, \ast)\) with non-trivial nilpotent Jacobson radical. In fact, in [2] Gutiérrez found a Bruhat presentation and a generalized Weil representation for \(SL,q^{-1}(2, A_n)\), where \(A_n = F_q[x]/\langle x^n \rangle\).

In [3] Gutiérrez, Pantoja and Soto-Andrade build Weil representations in a very general way via generators and relations, for the groups \(SL,q(2, A)\) for which a “Bruhat” presentation analogue to the classical one holds. In this way, in order to use this method it is very important to have an adequate presentation of the group. In [9], Pantoja generalized the classical Bruhat presentation of \(SL(2, K)\) to \(SL,q^{-1}(2, A)\), when \(A\) is simple artinian ring with involution.

In this work, we construct a generalized Weil representation of finite split orthogonal groups \(O_q(2n, 2n)\), using the method described in [3]. As mentioned above, this group is naturally the group \(SL,q^{-1}(2, M_{2n}(F_q))\). However one of the results of this paper is to realize this group as a \(SL,q^{-1}(2, M_{2n}(F_q))\) group, where \(\ast\) is a certain specific involution in \(M_{2n}(F_q)\), different from \(\circ\). This allows to use the Bruhat-like presentation that is exhibited in [9] and facilitate significantly technical aspects of the construction. In fact, the result is more general, and provides an isomorphism between the groups \(SL,q^1(2, M_2(A_0))\) and \(SL,q(2, M_2(A_0))\), where \((A_0, \ast)\) is a unitary involutive ring and \(\sim\) is another involution in \(A_0\) obtained from \(\ast\).

Also, we study the structure of the associated unitary group and using this group we get an initial decomposition of the representation.

In section 6 we show compatibility of the method of Gutiérrez, Pantoja and Soto-Andrade with theory of dual pairs. For this, we prove that the representation of \(O_q(2n, 2n)\) constructed using this method is equal to the restriction of the Weil representation to \(O_q(2n, 2n)\) for the dual pair \((\text{Sp}(2, k), O_q(2n, 2n))\). Also, we prove that the initial decomposition mentioned above is the same as decomposition with respect to the action of \(\text{Sp}(2, k)\) via the Weil representation.

### 2 The groups \(SL,q_x(2, A)\).

Let \((A, \ast)\) be a unitary ring with an involution \(\ast\). We can extend the involution \(\ast\) in \(A\) to the ring \(M_2(A)\) putting

\[
T^\ast = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\ast = \begin{pmatrix} a^\ast & c^\ast \\ b^\ast & d^\ast \end{pmatrix}.
\]

We consider \(\varepsilon = \pm 1 \in A\) and \(J_\varepsilon = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \in M_2(A)\). Let us denote by \(H_\varepsilon\) the associated \(\varepsilon\)-hermitian form defined by the matrix \(J_\varepsilon\).

**Definition 2.1.** The group \(SL,q_x(2, A)\) is the set of all automorphisms \(g\) of the \(A\)-module \(M = A \times A\) such that \(H_\varepsilon \circ (g \times g) = H_\varepsilon\). In matrix form:

\[
SL,q_x(2, A) = \{ T \in M_2(A) \mid T J_\varepsilon T^\ast = J_\varepsilon \}
\]

**Remark 2.2.** In [10] it is shown that a matrix \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A)\) is in \(SL,q_x(2, A)\) if and only if the following equalities hold:

1. \(ab^\ast = -\varepsilon ba^\ast\);
2. \(cd^\ast = -\varepsilon dc^\ast\);
3. \(a^\ast c = -\varepsilon c^\ast a\);
4. \(b^\ast d = -\varepsilon d^\ast b\);
5. \(ad^\ast + \varepsilon bc^\ast = a^\ast d + \varepsilon c^\ast b = 1\).
We note that if \((A, \circ)\) is the matrix ring \(M_m(\mathbb{F}_q)\) with the transpose involution \(\circ\), then \(\text{SL}_q^{-1}(2, A)\) is the symplectic group \(\text{Sp}(2m, \mathbb{F}_q)\) defined over \(\mathbb{F}_q\). On the other hand \(\text{SL}_q^1(2, A)\) gives the split orthogonal group \(O_q(m, m)\).

In what follows we put \(\text{SL}_q^+(2, A) = \text{SL}_q^1(2, A)\), \(\text{SL}_q^{-}(2, A) = \text{SL}_q^{-1}(2, A)\) and \(\circ\) always will be the transpose involution in a matrix ring.

Let \((A_0, \circ)\) be a unitary ring with involution \(\circ\) and \(A = M_2(A_0)\). Let us consider the matrix \(J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in A^\times\), which satisfies \(J^{-1} = J^* = -J\). Using the matrix \(J\) we can define a new involution in \(A\), namely, \(a^\circ = Ja^*J^{-1}\).

Let us consider \(M_2(A)\) provided with the involutions \(\circ\) and \(\sim\) inherited from \(A\).

**Theorem 2.3.** The groups \(\text{SL}_q^+(2, A)\) and \(\text{SL}_q^-(2, A)\) are isomorphic.

**Proof 2.4.** Let \(U = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \in \text{GL}_2(A)\). A direct computation proves that \(T^\circ U = UT^\circ\) for all \(T \in M_2(A)\). It is clear that \(TJ = J^T\) if and only if \(TJ^T = JU\). Then we must show that \(J\) is equivalent to \(J^T\). In fact, the (orthogonal) matrix \(P = \begin{pmatrix} 0 & J \\ 1 & 0 \end{pmatrix} \in M_2(A)\) satisfies \(PJ^TP^* = JU\).

Although the split orthogonal group is naturally a “\(\text{SL}_+\)”-group, in practical terms it is better to look at it as a “\(\text{SL}_-\)”-group, because this fact will greatly facilitate technical aspects.

**Corollary 2.5.** The split orthogonal group \(O_q(2n, 2n)\) is isomorphic to the group \(\text{SL}_q^-(2, M_{2n}(\mathbb{F}_q))\), where the involution \(\sim\) in \(M_{2n}(\mathbb{F}_q)\) is generated by \(a^\circ = J_{2n}a^*J_{2n}^{-1}\). Let us consider the matrix \(J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{F}_q)\).

**Proof 2.6.** Taking \((A_0, \circ)\) as the involutive ring \((M_n(\mathbb{F}_q), \circ)\) and using the theorem (2.3) we get that the groups \(\text{SL}_q^+(2, M_{2n}(\mathbb{F}_q))\) and \(\text{SL}_q^-(2, M_{2n}(\mathbb{F}_q))\) are isomorphic.

## 3 A Bruhat presentation for \(\text{SL}_q^\varepsilon(2, A)\)

Let \(A\) be a unitary ring with involution \(\circ\). We will write \(A_{\varepsilon, \ast}^\circ\) to denote the set of all \(\varepsilon\)-symmetric elements in \(A\) respect to the involution \(\circ\). Namely,

\[ A_{\varepsilon, \ast}^\circ = \{ a \in A \mid a^\circ = -\varepsilon a \}. \]

In order to facilitate the notation, we put \(A_{+\ast}^\circ = A_{+, \ast}^\circ\) and \(A_{-\ast}^\circ = A_{-1, \ast}^\circ\). Let us consider

\[ h_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} (t \in A^\times), \quad w = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}, \quad u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} (s \in A_{\varepsilon, \ast}^\circ) \]

**Definition 3.1.** We will say that \(\text{SL}_q^\varepsilon(2, A)\) has a Bruhat presentation if it is generated by the above elements with defining relations:

1. \(h_t h_{t'} = h_{tt'}, \quad u_s u_{s'} = u_{s+s'}\);
2. \(w^2 = h_{\varepsilon}\);
3. \(h_t u_s = u_{s+t} h_t\);
4. \(w h_t = h_{-s} w\);
5. \(w u_{t-1} w^{-1} u_{t-1} = h_{-t}, \quad t \in A^\times \cap A_{\varepsilon, \ast}^\circ\).
Remark 3.2. Observing the last relation, we note that in order to have a Bruhat presentation for SL\(2, A\) is necessary that \(A^\times \cap A^\times_{\epsilon} \neq \emptyset\).

In [3] it is proved that if \(A\) is a simple artinian ring with involution \(\ast\) that either is infinite or isomorphic to the full matrix ring over \(F_q\) with \(q > 3\), then the group SL\(2, A\) has a Bruhat presentation.

Thus, the group SL\(2, (M_{2n}(\mathbb{F}_q))\) mentioned in Corollary (2.5) has a Bruhat presentation if \(q > 3\). We will use this fact to construct the desired representation.

4 A generalized Weil representation for the split orthogonal group \(O_q(2n, 2n)\).

In this section, our aim is to construct a Weil representation for the split orthogonal group seen as the group SL\(2, (M_{2n}(\mathbb{F}_q))\). One way to this goal is to construct the representation using the Bruhat presentation. For this purpose we will use the result that follows (\[3\]).

Let \(A\) be a ring with an involution \(\ast\). Let us suppose that the ring \(A\) is finite and that the group \(G = \text{SL}_2(2, A)\) has a Bruhat presentation. Let \(M\) be a finite right \(A\)-module and let us consider the following data:

1. A bi-additive function \(\chi : M \times M \to \mathbb{C}^\times\) and a character \(\alpha \in \hat{A}^\times\) such that for all \(x, y \in M, t \in A^\times\):
   (a) \(\chi(tx, y) = \alpha(tt^\ast)\chi(x, yt^\ast)\)
   (b) \(\chi(y, x) = \chi(-\varepsilon x, y)\)
   (c) \(\chi(x, y) = 1\) for all \(x \in M \Rightarrow y = 0\)

2. A function \(\gamma : A^\times_{\epsilon} \times A^\times_{\epsilon} \to \mathbb{C}^\times\) such that for all \(s, s' \in A^\times_{\epsilon}, x, z \in M, r \in A^\times, t \in A^\times_{\epsilon} \cap A^\times\):
   (a) \(\gamma(s + s', x) = \gamma(s, x)\gamma(s', x)\)
   (b) \(\gamma(s, xr) = \gamma(rsr^\ast, x)\)
   (c) \(\gamma(t, x + z) = \gamma(t, x)\gamma(t, z)\gamma(x, zt)\)

3. \(c \in \mathbb{C}^\times\) such that \(c^2|M| = \alpha(\varepsilon)\), and for all \(t \in A^\times_{\epsilon} \cap A^\times\) the following equality holds:

\[
\sum_{y \in M} \gamma(t, y) = \frac{\alpha(tt)}{c}
\]

Theorem 4.1. (Gutiérrez, Pantoja and Soto-Andrade, [3]) Let \(M\) be a finite right \(A\)-module. Denote \(L^2(M)\) the vector space of all complex-valued functions on \(M\), endowed with the inner product with respect to the counting measure on \(M\). Set:

1. \(\rho(h_t)(f)(x) = \overline{\alpha(t)}f(x), f \in L^2(M), t \in A^\times, x \in M;\)
2. \(\rho(u_s)(f)(x) = \gamma(s, x)f(x), f \in L^2(M), b \in A^\times_{\epsilon}, x \in M;\)
3. \(\rho(w)(f)(x) = c \sum_{y \in M} \chi(-\varepsilon x, y)f(y), f \in L^2(M), x \in M\)

(where \(\overline{\alpha}\) denotes the complex conjugate of the character \(\alpha\)). These formulas define a unitary linear representation \((L^2(M), \rho)\) of \(G\), called the generalized Weil representation of \(G\) associated to the data \((M, \alpha, \gamma, \chi)\).

Remark 4.2. Let us note that this definition contains the classic Weil representation of SL\(2(K)\), where \(K\) is a field (see [11], for instance).
In what follows, we will focus on finding the necessary data to construct a generalized Weil representation for \( O_q(2n, 2n) \).

From now on we put \( k = \mathbb{F}_q \), \( A = M_{2n}(k) \) and we consider \( q \) odd greater than 3.

We will apply Theorem (4.1) to the group \( \text{SL}_2(2, A) \cong O_q(2n, 2n) \). To do this, we recall the following fact. Let \( E \) be a finite dimensional vector space over a field \( K \). In [5], the authors describe a correspondence between the linear anti-automorphisms of \( \text{End}_K(E) \) and the equivalence classes of non degenerate bilinear forms on \( E \) modulo multiplication by a factor in \( K^\times \). Under this correspondence, \( K \)-linear involutions on \( \text{End}_K(E) \) correspond to non degenerate bilinear forms which are either symmetric or skew-symmetric. Let \( B \) be a non degenerate bilinear form. The aforementioned correspondence associates \( B \) with \( \sigma_B \), where \( \sigma_B \) is a linear anti-automorphism in \( \text{End}_K(E) \) defined by the following equality:
\[
B(f(x), y) = B(x, \sigma_B(f)y), \quad f \in \text{End}_K(E), x, y \in E.
\]
(1)

Now, let \( V \) a vector space of \( k \)-dimension 2n. We fix a basis for \( V \) in order to put \( M_{2n}(k) \cong \text{End}_k(V) \). Let \( < \, , \, > : V \times V \rightarrow k \) be the non degenerate symmetric bilinear form given by the standard dot product. We consider the non degenerate skew-symmetric bilinear form \([ \, , \, ] : V \times V \rightarrow k\), given by
\[
[x, y] = < x, yJ_{2n} >.
\]

According to the correspondence between involutions and non degenerate bilinear forms described above, the symmetric bilinear form \( < \, , \, > \) corresponds to the transpose involution \( \circ \). Similarly, the skew-symmetric bilinear form \([ \, , \, ] \) corresponds to the new involution \( \sim \). That is, for all \( x, y \in V \) and \( a \in A \):
\[
< xa, y >= < x, ya^\circ > \tag{2}
\]
\[
[xa, y] = [ x, ya^\sim ] \tag{3}
\]

Now, let \( \psi \) be a non trivial character of \( k^\times \). Using the notation above let us consider:

1. \( M \) the right \( A \)-module \( V^2 \) with the following action:
\[
(x, y)a = (xa, ya) \quad a \in A, x, y \in V.
\]

2. \( \chi : M \times M \rightarrow \mathbb{C}^\times, \chi((x, y), (v, z)) = \psi([x, z] - [y, v]).\)

3. \( \alpha \) the trivial character of \( A^\times \).

4. \( \gamma : A^\times \times M \rightarrow \mathbb{C}^\times, \gamma(u, (x, y)) = \psi([ux, y]).\)

Lemma 4.3. For all \( u \in A^\times \cap A^\times \cap M \), the map \( Q_u : V^2 \rightarrow k \) given by \( Q_u((x, y)) = [xu, y] \) is a non degenerate split quadratic form. Furthermore, for \( u, u' \in A^\times \cap A^\times \cap M \), the quadratic forms \( Q_u \) and \( Q_{u'} \) are equivalent.

Proof 4.4. Let \( \lambda \in k, (x, y), (v, z) \in V^2 \). Clearly \( Q_u(\lambda(x, y)) = \lambda^2 Q_u((x, y)) \). We will prove that
\[
B((x, y), (v, z)) = Q_u((x + v, y + z)) - Q_u((x, y)) - Q_u((v, z))
\]
is a symmetric non degenerate bilinear form. We have:
\[
B((x, y), (v, z)) = [xu + vu, y + z] - [xu, y] - [vu, z] = [xu, z] + [vu, y].
\]
Now;

\[ B((x, y) + (r, t), (v, z)) = [(x + r)u, z] + [vu, (y + t)] \]
\[ = [xu, z] + [ru, z] + [vu, y] + [vu, t] \]
\[ = B((x, y), (v, z)) + B((r, t), (v, z)) \]

\[ B(\lambda(x, y), (v, z)) = [\lambda xu, z] + [vu, \lambda y] \]
\[ = \lambda[xu, z] + \lambda[vu, y] \]
\[ = \lambda B((x, y), (v, z)). \]

Then, \( B \) is a symmetric bilinear form.

Let us suppose that \( B((x, y), (v, z)) = 0 \) for all \((v, z) \in V^2\). If we choose \( v = 0 \), then \([xu, z] = 0\) for all \( z \in V \). Since \( \begin{bmatrix} x & z \end{bmatrix} \) is non degenerate and \( u \) is invertible, we get \( x = 0 \). Similarly \( y = 0 \). Therefore, \( B \) is non degenerate.

Now, if \( u, u' \in A^z \cap A^z_{\sim} \) then \( uJ^v_2n \) and \( u'J^v_2n \) are invertible skew symmetric matrices. In fact if \( u \in A^z_{\sim} \) then \( u^\sim = J^v_2uJ^v_2^{-1} = u \). Also \( J^v_2x = -J^v_2u \), so we get that \((uJ^v_2)^o = J^v_2u^o = uJ^v_2 = -uJ^v_2 \). Thus \( uJ^v_2n \) and \( u'J^v_2n \) represent a non degenerate skew symmetric bilinear form, therefore they are equivalent. So, there exists \( j \in A^z \) such that \( uJ^v_2n = ju'J^v_2n \). Thus,

\[ Q_u((xj, yj)) = [xj u', yj] \]
\[ = Q_u((x, y)) \]

If we choose \( u = I_{2n} \), the quadratic form \( Q_u \) is represented by the matrix \( \begin{bmatrix} 0 & -J_{2n} \\ J_{2n} & 0 \end{bmatrix} \). Thus, \( Q_u \) is split.

**Theorem 4.5.** The data \((M, \alpha, \gamma, \chi)\) describe a Generalized Weil Representation for \( G = O_q(2n, 2n) \). Furthermore, this representation is independent of the choice of the character \( \psi \).

**Proof 4.6.** We will check that \( \chi \) satisfies the corresponding conditions. Let \((x, y), (v, z) \in M, a \in A.\)

(a) \( \chi((x, y)a, (v, z)) = \psi([xa, z] - [ya, v]) \)
\[ = \psi([x, za^\sim] - [y, va^\sim]) \quad \text{by (3)} \]
\[ = \chi((x, y), (v, z)a^\sim). \]

(b) \( \chi((v, z), (x, y)) = \psi([v, y] - [z, x]) \)
\[ = \psi([x, z] - [y, v]) \]
\[ = \chi((x, y), (v, z)). \]

(c) Let us suppose that \( \chi((x, y), (v, z)) = 1 \) for all \((v, z) \in M. \) If \( v = 0 \), then \( \psi([x, z]) = 1 \) for all \( z \in V. \) If \( x \neq 0, \) then \([x, \cdot]: V \longrightarrow k \) is a non trivial linear functional. Therefore it is surjective. Let \( \lambda \in k \) such that \( \psi(\lambda) \neq 1, \) and \( t = t(\lambda) \in V \) such that \( \lambda = [x, t] \), then we get the following contradiction:
\[ 1 = \psi([x, t]) = \psi(\lambda). \]

Therefore \( x = 0, \) and similarly \( y = 0. \)
Now, we will prove that \( \gamma \) satisfies the corresponding properties. Let \( u, u' \in A^*_\sim, a \in A^\times, (x, y), (v, z) \in M; \)

\[
(a) \quad \gamma(u + u', (x, y)) = \psi([xu + xu', y]) \\
= \psi([xu, y])\psi([xu', y]) \\
= \gamma(u, (x, y))\gamma(u', (x, y)).
\]

\[
(b) \quad \gamma(u, (x, y)a) = \psi([xau, ya]) \\
= \psi([xaua^-, y]) \\
= \gamma(aua^-, (x, y)).
\]

\[
(c) \quad \gamma(u, (x, y) + (v, z)) = \psi([x + v)u, y + z]) \\
= \psi([xu, y])\psi([xu, z])\psi([vu, y])\psi([vu, z]) \\
= \gamma(u, (x, y))\gamma(u, (v, z))\chi((x, y), (v, z)u).
\]

Now, we must choose \( c \in \mathbb{C}^\times \) satisfying \( c^2|M| = 1 \) and show that for \( u \in A^\times \cap A^*_\sim \) the following equality holds:

\[
\sum_{(x, y) \in M} \gamma(u, (x, y)) = \sum_{x, y \in V} \psi([xu, y]) = \frac{1}{c}.
\]

According to the lemma \[4.3\] we know that \( \sum_{(x, y) \in M} \gamma(u, (x, y)) \) is a Gauss sum associated to a split quadratic form in a vector space of even dimension \( 4n \). This sum is calculated, for instance, in \[11\]. In fact,

\[
\sum_{(x, y) \in M} \gamma(u, (x, y)) = q^{2n},
\]

We choose \( c = \frac{1}{q^{2n}} \). Thus,

\[
\sum_{(x, y) \in M} \gamma(u, (x, y)) = \frac{1}{c}.
\]

Now, let \( \psi_1 \) and \( \psi_2 \) be two non trivial characters of \( k^\times \). Let us prove that the corresponding representations are isomorphic.

Let \( \lambda \in k^\times \) such that \( \psi_2(r) = \psi_1(\lambda r) \) for all \( r \in k \). Let \( (L^2(M), \rho_1) \) and \( (L^2(M), \rho_2) \) the Weil representations obtained from \( \psi_1 \) and \( \psi_2 \) respectively. Then, the linear automorphism \( \Psi : L^2(M) \rightarrow L^2(M) \) given by \( (\Psi f)(x, y) = f(x, \lambda y) \) is a isomorphism between the representations \( (L^2(M), \rho_1) \) and \( (L^2(M), \rho_2) \).

5 An initial decomposition.

**Definition 5.1.** The group \( U(\gamma, \chi) \) is the group of all \( A \)-linear automorphisms \( \beta \) of \( M \) such that:

1. \( \gamma(u, \beta(x, y)) = \gamma(u, (x, y)) \) for all \( u \in A^*_{\sim}, (x, y) \in M. \)
2. \( \chi(\beta(x, y), \beta(v, z)) = \chi((x, y), (v, z)) \) for all \( (x, y), (v, z) \in M. \)

In what follows we will denote \( U(\gamma, \chi) \) simply by \( U \).
Following the idea of [3], if we know the structure of the group $U$ and the set of its irreducible representations, we can find an initial decomposition of the Weil Representation in the sense that we do not know if the components obtained are irreducible. In what follows, we make this decomposition explicit.

For $\beta \in U$ and $x \in M$ we put $\beta.x = \beta(x)$. The group $U$ acts naturally on $L^2(M)$. That is to say the action is given by:

$$
\sigma : U \rightarrow Aut_\mathbb{C}(L^2(M)), \\
\sigma_\beta(f)(x) = f(\beta^{-1}.x)
$$

In [3] it is shown that the natural action of $U$ on $L^2(M)$ commutes with the action of the Weil Representation.

Let $\hat{U}$ be the set of the irreducible representations of $U$. We consider the isotypic decomposition of $L^2(M)$ with respect to $U$:

$$
L^2(M) \cong \bigoplus_{(V_\pi,\pi) \in \hat{U}} n_\pi V_\pi.
$$

Since $n_\pi = \text{dim}_\mathbb{C}(\text{Hom}_U(V_\pi, L^2(M))) = \text{dim}_\mathbb{C}(\text{Hom}_U(L^2(M), V_\pi))$, we can write this decomposition in the following way:

$$
L^2(M) \cong \bigoplus_{(V_\pi,\pi) \in \hat{U}} (\text{Hom}_U(L^2(M), V_\pi) \otimes_\mathbb{C} V_\pi).
$$

If we put $m_\pi = \text{dim}_\mathbb{C}(V_\pi)$, we get;

$$
L^2(M) \cong \bigoplus_{(V_\pi,\pi) \in \hat{U}} m_\pi \text{Hom}_U(L^2(M), V_\pi).
$$

If $(V_\pi,\pi) \in \hat{U}$ and $\beta \in U$, we denote by $\pi_\beta$ the map $\pi(\beta) : V_\pi \rightarrow V_\pi$. The space $\text{Hom}_U(L^2(M), V_\pi)$ is formed by linear functions $\Theta : L^2(M) \rightarrow V_\pi$ such that for any $\beta \in U$

$$
\Theta \circ \sigma_\beta = \pi_\beta \circ \Theta. \tag{4}
$$

Let us consider the Delta functions $\{e_x \mid x \in M\}$ and the map $\theta : M \rightarrow V_\pi$ such that $\theta(x) = \Theta(e_x)$ for all $x \in M$. Since $\sigma_\beta(e_x) = e_{\beta.x}$, condition (4) becomes:

$$
\Theta(e_x) = \pi_\beta \circ \Theta(e_x). \tag{5}
$$

Conversely, let $\theta : M \rightarrow V_\pi$ satisfying (5). We extend linearly and we get a map $\Theta : L^2(M) \rightarrow V_\pi$ such that (4) holds.

Thus, we can see the space $\text{Hom}_U(L^2(M), V_\pi)$ as the function space formed by maps $\theta : M \rightarrow V_\pi$ such that $\theta(\beta.x) = \pi_\beta \circ \theta(x)$ for all $\beta \in U, x \in M$. The group $G = SL_\mathbb{C}(2,A)$ acts on this space via the Weil representation, using the same formulas as defined in Theorem (4.1). Similarly, it is possible to define the natural action of the group $U$ in this space, because- like $L^2(M)$- it is formed for functions with domain $M$. 
Let $\rho$ denote the Weil action of $G$ on $L^2(M)$ and $\tilde{\rho}$ the Weil action of $G$ on $\bigoplus_{(V_\sigma, \pi) \in \mathcal{O}} m_\pi \text{Hom}_U(L^2(M), V_\sigma)$. Because of how we define the Weil representation, there exist scalars $K_\gamma(x, y) \in \mathbb{C}$ depending only on $g \in G$ and $x, y \in M$ such that for all $f \in L^2(M)$, $\Lambda \in \bigoplus_{(V_\sigma, \pi) \in \mathcal{O}} m_\pi \text{Hom}_U(L^2(M), V_\sigma)$ the following statements hold:

\[
\rho_\gamma(f) = \sum_{y \in M} K_\gamma(\cdot, y)f(y);
\]
\[
\tilde{\rho}_\gamma(\Lambda) = \sum_{y \in M} K_\gamma(\cdot, y)\Lambda(y).
\]

In this way, we get:

**Lemma 5.2.** $(L^2(M), \rho)$ and $(\bigoplus_{(V_\sigma, \pi) \in \mathcal{O}} m_\pi \text{Hom}_U(L^2(M), V_\sigma), \tilde{\rho})$ are isomorphic representations of $G$.

**Proof 5.3.** The linear isomorphism between $L^2(M)$ and $\bigoplus_{(V_\sigma, \pi) \in \mathcal{O}} m_\pi \text{Hom}_U(L^2(M), V_\sigma)$ is an isomorphism between representations.

Finally, we have:

**Proposition 5.4.** The space $\text{Hom}_U(L^2(M), V_\sigma)$ is invariant under the Weil action of $G$.

**Proof 5.5.** Let $g \in G$, $\theta \in \text{Hom}_U(L^2(M), V_\sigma)$, $\beta \in U$, $x \in M$:

\[
(\tilde{\rho}_\gamma \theta)(\beta \cdot x) = \sigma_{\beta^{-1}}(\tilde{\rho}_\gamma \theta)(x), \quad \text{(definition of } \sigma_\beta) \\
= \tilde{\rho}_\gamma(\sigma_{\beta^{-1}} \theta)(x) \\
= \tilde{\rho}_\gamma(\pi_\beta \circ \theta)(x), \quad (\sigma_{\beta^{-1}} \theta = \pi_\beta \circ \theta) \\
= \pi_\beta(\tilde{\rho}_\gamma \theta(x)).
\]

The last equality holds because (7).

Now, having made the decomposition above explicit, our purpose is to obtain an initial decomposition for our particular case $G = O_q(2n, 2n) \cong \text{SL}_2(2, M_{2n}(k))$. For this it is enough to know the structure of the group $U$ and the set of irreducible representations.

**Remark 5.6.** We note that since in our case $A = M_{2n}(k)$ and $A^*_{\varphi, \sim} \cap A^* \neq \emptyset$, the first condition in definition 5.1 implies the second one (see [3]).

**Theorem 5.7.** Let $\gamma$ and $\chi$ be the functions defined above. Then,

\[
U(\gamma, \chi) \cong \text{SL}_2(k)
\].

**Proof.** Let $\beta \in U(\gamma, \chi)$. In particular $\beta$ is $k$–linear, therefore we can suppose that $\beta \in M_{2n}(k)$. We can write the action of $A$ on $M$ in matrix language as follows:

\[
(x \quad y) \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = (xa \quad ya) \quad x, y \in V, a \in A.
\]

Since $\beta$ is $A$–linear we have that $\beta(x, y)a = \beta(xa, ya)$. In matrix language:

\[
\beta \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \beta.
\]

(8)
Let $\beta_1, \beta_2, \beta_3, \beta_4 \in A$ such that $\beta = \left( \begin{array}{cc} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{array} \right)$. Then, using (3) we get that each of these blocks must be scalar. Thus, there are $b_1, b_2, b_3, b_4 \in k$ such that $\beta = \left( \begin{array}{cc} b_1I_{2n} & b_2I_{2n} \\ b_3I_{2n} & b_4I_{2n} \end{array} \right)$ and hence

$$\gamma(u, \beta(x, y)) = \psi([(b_1x + b_3y)u, b_2x + b_4y]).$$

Let us note that the bilinear form $(x, y) \mapsto [xu, y]$ is skew symmetric for all $u \in A^*_{-\infty}$, hence $[xu, x] = 0$ for all $x \in V, u \in A^{\text{sym}}$. Thus for all $x, y \in V, u \in A^*_{-\infty}$

$$\gamma(u, \beta(x, y)) = \psi((b_1b_4 - b_2b_3)|xu, y|)$$

$$= \psi([xu, y])$$

$$= \gamma(u, (x, y)).$$

Consequently $\psi((b_1b_4 - b_2b_3 - 1)|xu, y|) = 1$ for all $x, y \in V, u \in A^*_{-\infty}$.

From this last equality it follows that $b_1b_4 - b_2b_3 = 1$. In fact, let $u \in A^*_{-\infty} \cap A^x, x \neq 0$ and let us suppose $b_1b_4 - b_2b_3 - 1 \neq 0$.

The map $F_{x,u} : V \rightarrow k$ given by $F_{x,u}(z) = [(b_1b_4 - b_2b_3 - 1)xu, z]$ is a non trivial linear functional and therefore is surjective. Let $\lambda \in k$ such that $\psi(\lambda) \neq 1$ and $z = z(\lambda) \in V$ such that $\lambda = [(b_1b_4 - b_2b_3 - 1)xu, z]$. Then $\psi(\lambda) = \psi([(b_1b_4 - b_2b_3 - 1)xu, z]) = 1$. This contradicts our assumption and therefore our result follows. \( \square \)

Thus, for our case, we get an initial decomposition of the Weil Representation $(L^2(M), \rho)$. We expect to address the question about irreducibility elsewhere.

6 Dual Pairs.

In this section we will prove that the representation $(L^2(M), \rho)$ of $O_q(2n, 2n)$ constructed in section 4 is equal to the restriction of the Weil representation to $O_q(2n, 2n)$ for the dual pair $(\text{Sp}(2, k), O_q(2n, 2n))$. Also, we will prove that the initial decomposition described above is the same as decomposition with respect to the action of $\text{Sp}(2, k)$ via the Weil representation.

Let $J_{2n} = \left( \begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array} \right) \in M_{2n}(k)$ and $F = \left( \begin{array}{cc} 0 & J_{2n} \\ -J_{2n} & 0 \end{array} \right) \in M_{4n}(k)$. The matrix $F$ defines the following non-degenerate split symmetric bilinear form in $V_1 = k^{4n}$

$$(u, v) = v^t Fu \quad u, v \in V_1$$

The group $G$ of isometries of this form is isomorphic to the split orthogonal group $O_q(2n, 2n)$. As before, set

$$a^\sim = J_{2n} a^t J_{2n} \quad a \in M_{2n}(k).$$

A direct calculation shows that the following matrices belong to the group $G$:

$$h_a = \left( \begin{array}{cc} a & 0 \\ 0 & (a^{-1})^t \end{array} \right), \quad w = \left( \begin{array}{cc} 0 & I_{2n} \\ -I_{2n} & 0 \end{array} \right), \quad u_s = \left( \begin{array}{cc} I_{2n} & s \\ 0 & I_{2n} \end{array} \right)$$

$(a \in M_{2n}(k)^*, s = s^\sim \in M_{2n}(k))$.

Therefore $G = \text{SL}_{2n}(2, M_{2n}(k))$. 

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Let \( V_2 = k^2 \) and \( W = \text{Hom}(V_1, V_2) \). The following formula defines a non-degenerate symplectic form on \( W \).

\[
\langle w_1, w_2 \rangle = \text{tr}(w_1 F w_2^t J_2) \quad (w_1, w_2 \in W)
\]

The group \( G \) acts on \( W \) by

\[
g(w_1) = w_1 g^{-1} \quad (g \in G, w_1 \in W).
\]

This action preserves the symplectic form \( \langle \cdot, \cdot \rangle \). In fact, since \( g \in G \),

\[
\langle w_1 g^{-1}, w_2 g^{-1} \rangle = \text{tr}(w_1 g^{-1} F(g^{-1})^t w_2^t J_2) = \text{tr}(w_1 F w_2^t J_2) = \langle w_1, w_2 \rangle.
\]

Let \( X = \{ (x, 0) \mid x \in M_{2,2n}(k) \} \); \( Y = \{ (0, y) \mid y \in M_{2,2n}(k) \} \).

Then \( W = X \oplus Y \) is a complete polarization. We will consider the Schrödinger model of the Weil representation of \( \text{Sp}(W) \) attached to the above complete polarization realized on \( L^2(X) \) as in [11]. Let \( (L^2(X), \omega) \) such representation.

We identify \( X \) with \( M_{2,2n}(k) \) in the canonical way

\[
X \ni (x, 0) \sim x \in M_{2,2n}(k).
\]

**Remark 6.1.** Let us note that the module \( M \) in section 4 is canonically isomorphic to \( X \). Consequently the spaces \( L^2(M) \) and \( L^2(X) \) are also isomorphic.

Let \( \psi \) a non-trivial character of the additive group \( k^+ \). For all \( x \in X, y \in Y \) it is clear that \( h_a(x) = x h_a^{-1} \in X \) and \( h_a(y) = y h_a^{-1} \in Y \), then the matrix \( h_a \) preserves \( X \) and \( Y \). Also, \( \det(h_a|_X) \in k^\times \).

Thus, proposition 34 in [11] shows that:

\[
\omega(h_a)f(x) = f(xa) \quad (f \in L^2(X))
\]

Thus, \( \omega(h_a) = \rho(h_a) \).

Now, let us see the action of \( \omega \) on \( u_a \). The matrix \( u_a \) acts trivially on \( Y \) and on \( W/Y \). Therefore, proposition 35 in [11] shows that:

\[
\omega(u_a)f(x) = \psi(\langle xc(-u_a), x \rangle) f(x),
\]

where \( c(-u_a) = \begin{pmatrix} 0 & -s/2 \\ 0 & 0 \end{pmatrix} \in M_{4n}(k) \) is the Cayley transform for \(-u_a\).

Let \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X, x_1, x_2 \in k^{2n} \). Then,

\[
\langle xc(-u_a), x \rangle = x_2 s J_{2n} x_1^t.
\]

In order to prove that \( \omega(u_a) = \rho(u_a) \) we have to check that

\[
[x_1 s, x_2] = \langle xc(-u_a), x \rangle,
\]

where \( [\ , \ ] \) is the symplectic form defined in section 4. In fact,

\[
[x_1 s, x_2] = [x_1, x_2 s] = -[x_2 s, x_1] = x_2 s J_{2n} x_1^t.
\]
It is clear that the matrix \( w \) maps \( X \) bijectively onto \( Y \) and \( Y \) onto \( X \), and \( w^2 = -1 \). Then, using proposition 36 of [1] we get:

\[
\omega(w)f(x) = \frac{1}{\sqrt{|X|}} \sum_{x' \in X} \psi(\llw(x), x'\gg) f(x')
\]

Thus, in order to prove that \( \omega(w) = \rho(w) \) we have to check that

\[
\chi(x, x') = \psi(\ll xw^{-1}, x'\gg).
\]

Let \( x = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix} \) and \( x' = \begin{pmatrix} x'_1 & 0 \\ x'_2 & 0 \end{pmatrix} \), \( x_1, x_2, x'_1, x'_2 \in k^{2n} \). So,

\[
\psi(\ll xw^{-1}, x'\gg) = \psi(x_2J_{2n}(x'_1)^t - x_1J_{2n}(x'_2)^t).
\]

On the other hand,

\[
\chi(x, x') = \psi([x_1, x'_2] + [x'_1, x_2])
\]

\[
= \psi((x_1, x_2) \begin{pmatrix} 0 & -J_{2n} \\ J_{2n} & 0 \end{pmatrix} (x'_1)^t (x'_2)^t)
\]

\[
= \psi(x_2J_{2n}(x'_1)^t - x_1J_{2n}(x'_2)^t)).
\]

Thus, we have showed that the representation constructed in section 4 is equal to the restriction of the Weil representation to \( O_q(2n, 2n) \) for the dual pair \( (\text{Sp}(2,k), O_q(2n, 2n)) \).

Furthermore, since an element \( g \in \text{Sp}(2,k) = \text{SL}_2(k) \) preserves \( X \) and \( Y \) and \( \det(g|_X) \in k^{\times 2} \), using proposition 34 in [1] we get that the group \( \text{SL}_2(k) \) acts on \( L^2(X) \) as follows:

\[
\omega(g)f(x) = f(g^{-1}x) \quad g \in \text{SL}_2(k), f \in L^2(X), x \in X.
\]

Therefore, the initial decomposition in section 5 is the same as the decomposition with respect to the action of \( \text{SL}_2(k) \) via the Weil representation.

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