Revenue Maximization for Selling Multiple Correlated Items

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Abstract

We study the problem of selling \( n \) items to a number of buyers with additive valuation functions. We consider the items to be correlated, i.e., desirabilities of buyers for the items are not drawn independently. Ideally, the goal is to design a mechanism to maximize the revenue. However, it has been shown that the optimum-revenue mechanism might be very complicated and as a result inapplicable to real-world auctions. Therefore, our focus is on designing a simple mechanism that gets a constant fraction of the optimal revenue. This problem was posed by Babaioff et al. in paper “A Simple and Approximately Optimal Mechanism for an Additive Buyer” (FOCS 2014) as an open question. In their paper they show a constant approximation of the optimal revenue can be achieved by either selling the items separately or as a whole bundle in the independent setting. We show a similar result for the correlated setting when the desirabilities of buyers are drawn from a common-base correlation. It is worth mentioning that the core decomposition lemma which is mainly the heart of the proofs for efficiency of the mechanisms does not hold for correlated settings. Therefore we proposed a modified version of this lemma which plays a key role in proving bounds on the approximation of the mechanism. In addition, we introduce a generalized form of correlation for items and show the same mechanism can achieve an approximation of the optimal revenue in that setting.

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1 Introduction

Suppose an auctioneer wants to sell \( n \) items to a number of buyers. Each buyer’s value for a particular item comes from a known distribution, and the buyers’ values are assumed to be additive (i.e., value of a set of items for a buyer is equal to the summation of the values of the items in the set). Buyers are considered to be strategic, that is, each buyer is trying maximize \( v(S) - p(S) \) where \( S \) is the set of purchased items, \( v(S) \) is the value of these items to the buyer and \( p(S) \) is the price of the set. Knowing that the valuation of buyer \( i \) for item \( j \) is drawn from a given distribution \( D_{i,j} \), what is the revenue-optimal mechanism for the auctioneer to sell the items? Myerson [18] solves the problem for a very simple case where we only have a single item and one buyer. He shows that in this special case the optimal mechanism is to set a fixed reserved price for the item. Despite simplicity of the optimal mechanism for selling a single item, this problem becomes quite complicated when it comes to selling two items even when we have only one buyer in particular. Hart and Reny [13] show an optimal mechanism for selling two independent items is much more subtle, and may involve randomization.

Though there are several attempts to characterize the properties for an optimal mechanism of an auction, most approaches seem to be too complex and as a result impractical for real-world auctions [2, 1, 4, 5, 6, 7, 8, 11, 9, 12, 14]. Therefore, a new line of investigation is to design simple mechanisms that are approximately optimal. In a recent work of Babaioff, Immorlica, Lucier, and Weinberg [3] (FOCS 2014), it is shown that we can achieve a constant-factor approximation of the optimal revenue by selling items either separately or as a whole bundle in the independent setting. However, they leave the following important open problem in their paper:

- “Open Problem 3. Is there a simple, approximately optimal mechanism for a single additive buyer whose value for \( n \) items is sampled from a common base-value distribution? What about other models of limited correlation?”

Note that the idea of having a common base-value correlation has been used by Chawla, Malec, and Sivan [10] when they study mechanism design for selling multiple items in a unit-demand setting and prove to be fruitful. In this work we study the problem for the case of correlated valuation functions and answer the above open problem. In addition we also introduce a generalized form of correlation between items. Suppose you have a set of items and want to sell them to a single buyer. The buyer has a set of features in his mind and considers a value for each feature which is randomly drawn from a known distribution. Furthermore, he formulates the desirability for each item as a linear combination of the values of the features. More precisely, each buyer has \( l \) distributions \( F_1, F_2, \ldots, F_l \) and an \( l \times n \) matrix \( M \) (which is used for determining the valuation of items based on the valuation of features) such that the value of feature \( i \) (which is denoted by \( f_i \)) is drawn from \( F_i \) and the value of item \( j \) is calculated by \( V_f \cdot M_j \) where \( V_f = \langle f_1, f_2, \ldots, f_i \rangle \) and \( M_j \) is the \( j \)-th row of matrix \( M \).

This framework is a good model for studying correlated valuation functions especially when each item has different features and the buyer is setting values for each item based on the quality of that items in each feature. Note that every common base-value distribution is a special case of this general correlation model where we have \( n + 1 \) features \( F_1, F_2, \ldots, F_n, B \) and the value of item \( i \) is computed as \( v_i + b \) where \( v_i \) is drawn from \( F_i \) and \( b \) which is equal for all items is drawn from the constant distribution \( B \).

2 Related Work

As mentioned earlier, the problem originates from the seminal work of Myerson [18] in 1981 which characterizes a revenue-optimal mechanism for selling a single item to a single buyer. This result was important in the sense that it was simple and practical while promising the maximum possible revenue. In contrast to
this result, it is known that designing an optimal mechanism is much harder for the case of multiple items. There has been some efforts to find a revenue-optimal mechanism for selling two heterogeneous items [17] but, unfortunately so far too little is known about the problem even for this case, with more items.

Hardness of this problem is even more highlighted when Hart and Reny [13] observed randomization is necessary for the case of multiple items. This reveals the fact that even if we knew how to design an optimal mechanism for selling multiple items, it would be almost impossible to implement the optimal strategy in a real-world auction. Therefore, so far studies are focused on finding simple and approximately optimal mechanisms.

Speaking of simple mechanisms, it is very natural to think of selling items separately or as a whole bundle. The former mechanism is denoted by $SRev$ and the latter is referred to by $BRev$. Hart and Nissan [12] show $SRev$ mechanism achieves at least $\Omega(1/\log^2 n)$ of the optimal revenue in independent setting and $BRev$ mechanism yields at least an $\Omega(1/\log n)$ approximation for the case of identically independent distributions. Later on, this result was improved by the work of Li and Yao, who obtained $\Omega(1/\log n)$ for $SRev$ and a constant approximation for $BRev$ for identically independent distributions [15]. These bounds are tight up to a constant factor. Moreover, it is shown $BRev$ can be $\theta(n)$ times worse than the revenue of an optimal mechanism for independent setting. Therefore in order to achieve a constant-factor approximation mechanism we should think about more non-trivial strategies.

The seminal work of Babaioff et al. [3] shows despite the fact that both strategies $SRev$ and $BRev$ may separately result in a bad approximation factor, $\max\{SRev, BRev\}$ always has a revenue at least $\frac{1}{15}$ of an optimal mechanism. They also show we can find out which of these strategies has more revenue in polynomial time which yields a deterministic simple mechanism that can be implemented in polynomial time. However, there has been no significant progress in designing simple and approximate mechanisms for the case of correlated items, as [3] leave it as an open problem.

Another line of research investigated optimal mechanism for selling $n$ items to a single unit-demand buyer. Chawla et al. [10] show how complex the optimal strategies can become by proving that the gap between the revenue of deterministic mechanisms and revenue of non-determinstic mechanisms can be unbounded even when we have a constant number of correlated items. This highlights the fact that when it comes to general correlations, there is not much that can be achieved by deterministic mechanisms. However, Chawla et al. [10] study the problem with a mild correlation known as the common base-value correlation and present positive results for deterministic mechanisms in this case.

3 Results and Techniques

We study the mechanism design for selling $n$ items to buyers with additive valuation function when desirabilities of each buyer for items are correlated. The main result of the paper is that $\max\{SRev, BRev\}$ achieves a constant approximation of the optimal revenue when we have only one buyer and the distribution of valuations for this buyer is a common base-value distribution. This problem was left open in [3].

**Theorem 3.1** For an auction with one seller, one buyer, and common base-value distribution of valuations we have

$$\max\{SRev(D), BRev(D)\} \geq \frac{1}{15} \times \text{Rev}(D).$$

Furthermore, we generalize this result by considering linear correlations and proving that the same mechanism guarantees at least $\frac{1}{13k}$ of the optimal revenue when we have only one buyer and the correlation between items is a linear correlation in which value of each item depends on the value of at most $k$ features.
Theorem 3.2 Let $D$ be a distribution of valuation for one buyer in an auction such that the correlation between items is linear. If each row of $M_1$ has at most $k$ non-zero entries, then we have

$$\max\{S\text{Rev}(D), B\text{Rev}(D)\} \geq \frac{1}{7.5k} \times \text{Rev}(D).$$

Our approach for both problems is as follows. First we show $\max\{S\text{Rev}(D), B\text{Rev}(D)\} \geq \frac{\text{Rev}(D)}{7.5}$ for a semi-independent distribution $D$. This is achieved by applying the core decomposition lemma and setting proper values for $t_i$'s. Next, we analyze the behaviour of $\max\{S\text{Rev}, B\text{Rev}\}$ in each of the settings by creating another auction in which each item of the original auction is split into several items and the distributions are semi-independent. We show that the optimal revenue in the second auction is no less than the optimal revenue of the first auction and also selling all items together gets the same revenue in both auctions. Finally, we bound the revenue of $S\text{Rev}$ in the original auction by a fraction of $S\text{Rev}$ of the new auction and putting all inequalities together we show $\max\{S\text{Rev}, B\text{Rev}\}$ is at least a fraction of $\text{Rev}$.

We also introduce a new mechanism for pricing items which we call point mechanism. In this mechanism we set a price for each item. We only add one more constraint which does not allow the buyers to pay less than a set price $l$. In other words, if a buyer wants to buy some items, the price that he is paying for all items should be at least $l$, otherwise the seller does not sell anything to him. Unlike the other mechanism that we consider in this paper, in this mechanism pricing of items is not based on Myerson’s optimal mechanism. We show in the case of many buyers with i.i.d. distribution of desirabilities, $\text{PPRev}$ (the optimal revenue that we can achieve using point mechanism) is at least a constant fraction of the optimal revenue we can achieve by partition mechanism. Next, we give an example in which point mechanism achieves a revenue $\Omega(\log n)$ times more than what partition mechanism can possible obtain.

Theorem 3.3 Let $\text{PPRev}(D)$ be the revenue of the optimal partition mechanism for $n$ items with i.i.d. distributions and $m$ buyers. There is a point mechanism with revenue $\text{PPRev}(D) \geq \frac{1}{32} \text{PPRev}(D)$.

Proposition 3.1 There exists a setting with $n$ items with i.i.d. distributions and $m$ buyers such that $\text{PPRev} \leq \text{PPRev}/\Omega(\log n)$.

4 Preliminaries

Throughout this paper we study the optimal mechanisms for selling $n$ items to $m$ risk-neutral, quasilinear buyers. The items are considered to be indivisible and not necessarily identical i.e. buyers can have different distributions of desirabilities for different items. In our setting, distributions are denoted by $D = \langle D_1, D_2, \ldots, D_m \rangle$ where $D_i$ is an $n$-dimensional distribution for buyer $i$. Moreover, buyer $i$ has a valuation vector $V_i = \langle v_{i,1}, v_{i,2}, \ldots, v_{i,n} \rangle$ which is randomly drawn from $D_i$ specifying the values he has for the items. Note that, vectors of desirabilities are drawn independently for buyers, but values within each vector may be correlated.

Once a mechanism is set for selling items, each buyer purchases a set $S_{V_i}$ of the items that maximizes $v_i(S_{V_i}) - p_i(S_{V_i})$, where $v_i(S_{V_i})$ is the desirability of $S_{V_i}$ for that buyer and $p_i(S_{V_i})$ is the price that he pays. The revenue achieved by a mechanism is equal to $\sum E[p_i(S_{V_i})]$ where $V_i$ is randomly drawn from $D_i$. The following terminology is used in [3] in order to compare the performance of different mechanisms.

In this paper we use similar notations and introduce a new mechanism which we call point mechanism with maximal revenue $\text{PPRev}(D)$ as follows.

- $\text{Rev}(D)$: Maximum possible revenue that can be achieved by any truthful mechanism.
- $S\text{Rev}(D)$: The revenue that we get when selling items separately using Myerson’s optimal mechanism for selling a single item.
• $\text{BRev}(D)$: The revenue that we get when selling all items as a whole package using Myerson’s optimal mechanism for selling a single item.

• $\text{PRev}(D)$: The maximum possible revenue that we get by partitioning items into some packages and selling each package as a single item using Myerson’s optimal mechanism.

• $\text{PPRev}(D)$: The maximum possible revenue that we get by pricing items separately (not necessarily with Myerson’s mechanism), and setting a minimum purchase value $l$ which specifies that a buyer can buy a set $S$ of items if he pays $\max\{l, p(S)\}$ where $p(S)$ is the total price of items in the set.

We say an $n$-dimensional distribution $D_i$ of the desirabilities of a buyer is independent over the items if for every $a \neq b$, $v_{i,a}$ and $v_{i,b}$ are independent variables when $V_i = \langle v_{i,1}, v_{i,2}, \ldots, v_{i,n} \rangle$ is drawn from $D_i$. Furthermore, we define the semi-independent distributions as follows.

**Definition 4.1** Let $D_i$ be a distribution of valuations of a buyer over a set of items. We say $D$ is semi-independent iff the valuations of every two different items are either always equal or completely independent. Moreover, we say two items $a$ and $b$ are similar in a semi-independent distribution $D_i$ if for every $V_i \sim D_i$ we have $v_{i,a} = v_{i,b}$.

Moreover, we define the common-base value distributions as follows.

**Definition 4.2** We say a distribution $D_i$ is common-base value, if there exist independent distributions $F_1, F_2, \ldots, F_n, B$ such that for $V_i = \langle v_{i,1}, v_{i,2}, \ldots, v_{i,n} \rangle \sim D_i$ and every $1 \leq j \leq n$ we know $v_{i,j}$ comes from distribution $F_j + B$.

A natural generalization of common-base distributions are distributions in which the valuation of each item is determined by a linear combination of $k$ independent variables which are the same for all items. More precisely, we define the linear distribution in the following way.

**Definition 4.3** Let $D_i$ be a distribution of valuations of a buyer for $n$ items. We say $D_i$ is a linear distribution if there exist independent desirability distributions $F_1, F_2, \ldots, F_k$ and a $k \times n$ matrix $M$ with non-negative values such that $V_i = \langle v_{i,1}, v_{i,2}, \ldots, v_{i,n} \rangle \sim D_i$, can be written as $W \times M$ where $W = \langle w_1, w_2, \ldots, w_k \rangle$ is a vector such that $w_i$ is drawn from $F_i$.

5 The core decomposition technique

Most of the results in this area are mainly achieved by the core decomposition technique which was first introduced in [15]. Using this technique we can bound the revenue of an optimal mechanism without having any information about its behaviour. The underlying idea is to split distributions into two parts: the core and the tail. If we were to know in advance for which items the valuations in the core part will be and for which items the valuations in the tail part will be, we would achieve the optimal revenue without any additional information. This gives us an intuition which we can bound the optimal revenue by bounding the total sum of $2^n$ mechanisms where in each mechanism we know each valuation is in which part. Then the tricky part would be to separate the items whose valuations are in the core from the items whose valuations are in the tail in each mechanism and then sum them up separately. We use the same notation which was used in [3] for formalizing our arguments as follows.

• $r_i$: The revenue that we get by selling item $i$ using Mayerson’ auction.

• $r$: The revenue we get by using strategy $\text{SRev}$ which is equal to $\sum r_i$. 

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• $t_i$: A real number specifying the core and the tail for the distribution of item $i$. We say a valuation $v_i$ is in the core if $0 \leq v_i \leq r_i t_i$ and is in the tail otherwise.

• $p_i$: This is a real number equal to the probability that $\max\{v_{1,i}, v_{2,i}, \ldots, v_{m,i}\} > r_i t_i$.

• $p_A$ ($A$ is a subset of items): A real number equal to the probability that $\forall i \notin A, \max\{v_{1,i}, v_{2,i}, \ldots, v_{m,i}\} \leq r_i t_i$ and $\forall i \in A, \max\{v_{1,i}, v_{2,i}, \ldots, v_{m,i}\} > r_i t_i$.

• $D^{(i)}$: The distribution of desirabilities of the buyers for item $i$.

• $D^{(A)}$ ($A$ is a subset of items): The distribution of desirabilities of the buyers for items in $A$.

• $D^C_i$: A distribution of valuations of the $i$-th item for all buyers that is equal to $D^{(i)}$ conditioned on $\max\{v_{1,i}, v_{2,i}, \ldots, v_{m,i}\} \leq r_i t_i$.

• $D^T_i$: A distribution of valuations of the $i$-th item for all buyers that is equal to $D^{(i)}$ conditioned on $\max\{v_{1,i}, v_{2,i}, \ldots, v_{m,i}\} > r_i t_i$.

• $D^C_A$ ($A$ is a subset of items): A distribution of valuations of the items in $[N] - A$ for all buyers that is equal to $D([N] - A)$ conditioned on $\forall i \notin A, \max\{v_{1,i}, v_{2,i}, \ldots, v_{m,i}\} \leq r_i t_i$.

• $D^T_A$ ($A$ is a subset of items): A distribution of valuations of the items in $A$ for all buyers that is equal to $D^{(A)}$ conditioned on $\forall i \in A, \max\{v_{1,i}, v_{2,i}, \ldots, v_{m,i}\} > r_i t_i$.

• $D_A$: A distribution of valuations for all items and all buyers which is equal to $D$ conditioned on both $\forall i \notin A, \max\{v_{1,i}, v_{2,i}, \ldots, v_{m,i}\} \leq r_i t_i$ and $\forall i \in A, \max\{v_{1,i}, v_{2,i}, \ldots, v_{m,i}\} > r_i t_i$.

In Lemma 5.2 we provide an upper bound for $p_i$. Next we bound $\text{Rev}(D^C_i)$ and $\text{Rev}(D)$ in Lemmas 5.3 and 5.4 and at last in Lemma 5.6 which is known as Core Decomposition Lemma we prove an upper bound for $\text{Rev}(D)$. All these lemmas are proved in [3] for the case of independent setting. For all these lemmas we give similar proofs that work in the correlated setting as well. For brevity we omit the proofs and include them in the appendix.

**Lemma 5.1** For every $A \subset [N]$, if the valuation of items in $A$ are independent of items in $[N] - A$ then we have

$$\text{Rev}(D) \leq \text{Rev}(D^{(A)}) + \text{Val}(D([N] - A)).$$

**Lemma 5.2** $p_i \leq \frac{1}{t_i}$.

**Lemma 5.3** $\text{Rev}(D^C_i) \leq r_i$.

**Lemma 5.4** $\text{Rev}(D^T_i) \leq r_i/p_i$.

**Lemma 5.5** $\text{Rev}(D) \leq \sum_A p_A \text{Rev}(D_A)$.

For independent setting we can apply Lemma 5.1 to Lemma 5.5 and finally with help of some algebraic inequalities come up with the following inequality

$$\text{Rev}(D) \leq \text{Val}(D^C_i) + \sum_A p_A \text{Rev}(D^T_A).$$

Unfortunately this does not hold for correlated setting since in Lemma 5.1 we assume valuation of items of $A$ are independent of the items of $[N] - A$. Therefore, we need to slightly modify this lemma such that it becomes applicable to the correlated settings as well. Thus, we add the following restriction to the valuation of items: For each $A$ such that $p_A$ is non-zero, the valuation of items in $A$ are independent of items of $[N] - A$.  

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Lemma 5.6 If for every $A$ with $p_A > 0$ the values of items in $A$ are drawn independent of the items in $[N] - A$ we have $\text{Rev}(D) \leq \text{Val}(D_C^A) + \sum_A p_A \text{Rev}(D_A^T)$.

Proof: According to Lemma 5.5 we have

$$\text{Rev}(D) \leq \sum_A p_A \text{Rev}(D_A). \quad (5.1)$$

Since for every $A$ such that $p_A > 0$ we know the values of items in $A$ are drawn independent of items in $[N] - A$, we can apply Lemma 5.1 to Inequality (5.1) to come up with the following inequality.

$$\text{Rev}(D) \leq \sum_A p_A [\text{Val}(D_C^A) + \text{Rev}(D_A^T)].$$

Note that, $D_C^A$ is an upper bound for $\text{Val}(D_C^A)$ for all $A$. Therefore

$$\text{Rev}(D) \leq \sum_A p_A [\text{Val}(D_C^A) + \text{Rev}(D_A^T)].$$

We rewrite the inequality to separate $\text{Val}(D_C^A)$ from $\text{Rev}(D_A^T)$.

$$\text{Rev}(D) \leq \sum_A p_A \text{Rev}(D_A^T) + \sum_A p_A \text{Val}(D_C^A).$$

Since $\sum p_A = 1$

$$\text{Rev}(D) \leq \text{Val}(D_C^A) + \sum_A p_A \text{Rev}(D_A^T).$$

6 Semi-Independent distributions

In this section we give a deterministic mechanism for selling $n$ items to one buyer in the semi-independent setting. To do so, we first show $k \cdot \text{SRev}(D) \geq \text{Rev}(D)$ where we have $n$ items divided into $k$ types such that items of each type are similar. Next we use this lemma in order to prove $\max \{ \text{SRev}(D), \text{BRev}(D) \}$ achieves a constant-factor approximation of the revenue of an optimal mechanism. To prove this, we use the following lemma which is proved in [16].

Lemma 6.1 In an auction with one seller, one buyer, and multiple similar items we have $\text{Rev}(D) = \text{SRev}(D)$.

We also need Lemma 6.2 proved in [12] and [3] which bounds the revenue when we have a sub-domain $S$ two independent value distributions $D$ and $D'$ over disjoint sets of items.

Lemma 6.2 (“Marginal Mechanism on Sub-Domain [12, 3]”) Let $D$ and $D'$ be two independent distributions over disjoint sets of items. Let $S$ be a set of values of $D$ and $D'$ and $s$ be the probability that a sample of $D$ and $D'$ lies in $S$, i.e. $s = \text{Pr}(v, v') \sim D \times D' \in S$. $\text{sRev}(D \times D'|(v, v') \in S) \leq s \text{Val}(D|(v, v') \in S) + \text{Rev}(D')$.

Lemma 6.3 In a single-seller mechanism with $m$ buyers and $n$ items in which there are $k$ types of items we have $\text{Rev}(D) \leq mk \cdot \text{SRev}(D)$. 
Proof: First we prove the case \( m = 1 \). The proof is by induction on \( k \). For \( k = 1 \), all items are identical and by Lemma 6.1 \( \text{Rev}(D) = \text{SRev}(D) \). Now we prove the case in which we have \( k \) non-similar types assuming the theorem holds for \( k - 1 \). Consider a partition of \( D \) into two parts \( S_1 \) and \( S_2 \) where in \( S_1 \), \( v_1c_1 \geq c_iv_i \) for each \( i \) and in \( S_2 \) there is at least one type \( i \) such that \( c_iv_i > c_1v_1 \). Let \( D^1 \) and \( D^2 \) denote the valuations conditioned on \( S_1 \) and \( S_2 \), respectively, and let \( p_1 \) and \( p_2 \) denote the probability that the valuations lie in \( D^1 \) and \( D^2 \). Since we do not lose revenue due to having extra information about the domain

\[
\text{Rev}(D) \leq p_1 \text{Rev}(D^1) + p_2 \text{Rev}(D^2).
\]

(6.1)

Thus we need to bound \( p_1 \text{Rev}(D^1) \) and \( p_2 \text{Rev}(D^2) \). Let \( D_{-i} \) denote the distribution of valuations excluding the items of type \( i \). Using Lemma 6.2, \( p_1 \text{Rev}(D^1) \leq p_1 \text{Val}(D^1_{-1}) + \text{Rev}(D_1) \) and \( p_2 \text{Rev}(D^2) \leq p_2 \text{Val}(D^2_{-1}) + \text{Rev}(D_{-1}) \). Hence by Inequality (6.1),

\[
\text{Rev}(D) \leq p_1 \text{Val}(D^1_{-1}) + \text{Rev}(D_1) + p_2 \text{Val}(D^2_{-1}) + \text{Rev}(D_{-1}).
\]

(6.2)

Now the goal is to bound four terms in Inequality (6.2). For the first term consider the following truthful mechanism. Assume we only want to sell the items of type one. We take a sample \( v \sim D \) and then sell all \( c_1 \) items of type one in a bundle with price \( \max_{2 \leq i \leq k} \{c_iv_i\} \). With probability \( p_1 \), \( c_1v_1 \geq \max_{2 \leq i \leq k} \{c_iv_i\} \) and hence the bundle would be sold. Thus with probability \( p_1 \), valuations lie in \( D^1 \) which means for each \( i \)

\[
p_1 \text{Val}(D^1_{-1}) \leq k\text{Rev}(D_1).
\]

(6.3)

For the third term we provide another truthful mechanism which can sell all items except the items of type one. Take a sample \( v \sim D \), put all items of the same type in the same bundles, except the items of type one. Hence we have \( k - 1 \) bundles. Price all bundles equal to \( c_1v_1 \). With probability \( p_2 \) at least one bundle has valuation more than \( c_1v_1 \) and as a result would be sold and the revenue is more than \( \text{Val}(D^2_{-1}) \). Moreover by Lemma 6.1 in each bundle the maximum revenue is achieved by selling the items separately, thus

\[
p_2 \text{Val}(D^2_{-1}) \leq \text{SRev}(D_{-1}).
\]

(6.4)

Moreover by induction hypothesis,

\[
\text{Rev}(D_{-1}) \leq k\text{SRev}(D_{-1}).
\]

(6.5)

Summing up inequalities (6.3), (6.4), and (6.5), \( p_1 \text{Val}(D^1_{-1}) + \text{Rev}(D_1) + p_2 \text{Val}(D^2_{-1}) + \text{Rev}(D_{-1}) \leq k\text{SRev}(D_1) + \text{Rev}(D_1) + \text{SRev}(D_{-1}) + k\text{SRev}(D_{-1}) \). Therefore, \( \text{Rev}(D) \leq (k + 1)\text{SRev}(D) \), as desired.

Now we prove that for any \( m \geq 1 \), \( \text{Rev}(D) \leq mk\text{SRev}(D) \). Note that any mechanism for \( m \) buyers provides \( m \) single buyer mechanisms and \( \text{Rev}(D) = \sum_{i=1}^{m} \text{Rev}_i(D) \), where \( \text{Rev}_i(D) \) is the revenue for \( i \)-th buyer. Thus \( \max_i \text{Rev}_i(D) \geq \frac{1}{m} \text{Rev}(D) \) and as a result \( \text{Rev}(D) \leq m\text{SRev}(D) \).

Next, we show \( \max\{\text{SRev}(D), \text{BRev}(D)\} \geq \frac{1}{7.5} \cdot \text{Rev}(D) \). The proof is very similar in sprit to the proof of Babayoff et. al for showing \( \max\{\text{SRev}(D), \text{BRev}(D)\} \) achieves a constant approximation factor of the optimal revenue mechanism in independent setting. [3]. In this proof, we first apply the core decomposition lemma with \( t_i = 2r/(ri\cdot ni) \) and break down the problem into two sub-problems. In the first sub-problem we show \( \sum_A p_A \text{Rev}(D^1_{-1}) \leq \frac{3\sqrt{2}}{\sqrt{3}} \text{SRev}(D) \) and in the second sub-problem we prove \( 5 \max\{\text{SRev}(D), \text{BRev}(D)\} \geq \text{Val}(D^C_{-1}) \). Having these two bounds together, we can apply the core decomposition lemma to imply \( \max\{\text{SRev}(D), \text{BRev}(D)\} \geq \frac{1}{7.5} \cdot \text{Rev}(D) \).

Lemma 6.4 Let \( D \) be a semi-independent distribution of valuations for \( n \) items in single buyer setting. In this problem we have

\[
\max\{\text{SRev}(D), \text{BRev}(D)\} \geq \frac{1}{7.5} \cdot \text{Rev}(D).
\]
Proof: We use the core decomposition technique to prove this lemma. Let \( n_i \) be the number of items that are equal to item \( i \). We set \( t_i = 2r/(r_i n_i) \) and then apply the Core Decomposition Lemma to prove a lower bound for \( \max\{\text{SRev}(D), \text{BRev}(D)\} \). According to this lemma we have

\[
\text{Rev}(D) \leq \left[ \sum_A p_A \text{Rev}(D_A^T) \right] + \left[ \text{Val}(D_0^C) \right].
\]

To prove the theorem, we first show \( \sum_A p_A \text{Rev}(D_A^T) \leq \frac{3\sqrt{e}}{2}\text{SRev}(D) \) and next prove \( \text{Val}(D_0^C) \leq 5 \cdot \max\{\text{SRev}(D), \text{BRev}(D)\} \) which together imply

\[
\text{Rev}(D) \leq \left[ \sum_A p_A \text{Rev}(D_A^T) \right] + \left[ \text{Val}(D_0^C) \right] 
\leq \frac{3\sqrt{e}}{2}\text{SRev}(D) + 5 \max\{\text{SRev}(D), \text{BRev}(D)\} 
\leq (\frac{3\sqrt{e}}{2} + 5) \max\{\text{SRev}(D), \text{BRev}(D)\} \leq 7.5 \max\{\text{SRev}(D), \text{BRev}(D)\}.
\]

**Proposition 6.1** If we set \( t_i = 2r/(r_i n_i) \) the following inequality holds in the single buyer setting.

\[
\sum_A p_A \text{Rev}(D_A^T) \leq \frac{3\sqrt{e}}{2}\text{SRev}(D)
\]

(6.6) where \( D \) is a semi-independent valuation function for \( n \) items.

Proof: According to Lemma 6.4 we have

\[
\text{Rev}(D_A^T) \leq d_A \text{SRev}(D_A^T) \leq d_A \left( \sum_{i \in A} \text{Rev}(D_i^T) \right) \leq d_A \left( \sum_{i \in A} r_it_i \right).
\]

(6.7) Therefore, the following inequality holds.

\[
\sum_A p_A \text{Rev}(D_A^T) \leq \sum_A p_A d_A \left( \sum_{i \in A} r_it_i \right)
\]

(6.8) where \( d_A \) is the number of non-similar items in \( A \). By rewriting Equation (6.8) we get

\[
\sum_A p_A \text{Rev}(D_A^T) \leq \sum_{i=1}^n r_it_i \left( \sum_{A \ni i} p_A d_A \right) = \sum_{i=1}^n r_it_i \sum_{j=1}^n \left( \sum_{A \ni i \cap d_A = j} p_A \right).
\]

(6.9) In the next step we will show

\[
t_i \left( \sum_{A \ni i \cap d_A = j} p_A \right) \leq \frac{2^{1-j}}{(j-1)!}.
\]

(6.10) Since for every two equal items \( i \) and \( j \), \( r_i \) and \( n_i \) are equal to \( r_j \) and \( n_j \), \( t_i \) is also equal to \( t_j \) and thus \( p_A = 0 \) if \( i \in A \) but \( j \not\in A \). Therefore, we can assume that for every \( A \) such that \( p_A > 0 \), every set of equal items is either a subset of \( A \) or completely disjoint from \( A \). Therefore, we can consider each set of equal items as a package containing all the items in the set. Let \( S = \{ s_1, s_2, \ldots, s_{|S|} \} \) be a maximal set of equal items and \( p_S \) be the probability that all items of \( S \) lie in the tail (which is equal to \( r/r_{s_a} n_{s_a} \) for all \( 1 \leq a \leq |S| \)). Now we can formulate \( p_A \) as

\[
p_A = \prod_{S_j \subset A} p_{S_j}
\]

(6.11)
and thus
\begin{align}
\sum_{A \ni i, d_A = j} p_A &= \sum_{A = S_i \cup S_j} p_A = \sum_{A = S_i \cup S_j} p_i p_S p_S \cdots p_S, \\
\text{where } S_i &\text{ is the package containing item } i \text{ and all } S_a \text{'s are different packages of items other than } S_i.
\end{align}

By rewriting Equation (6.12) we have
\begin{align}
\sum_{A \ni i, d_A = j} p_A &= p_i \sum_{A = S_i \cup S_j} p_S p_S \cdots p_S. \\
&\leq \frac{1}{2^{j-1}(j-1)!} p_i (\sum_{j=1}^n p_S)^{j-1} = \frac{1}{2^{j-1}(j-1)!} p_i (\sum_{j=1}^n p_S)^{j-1}.
\end{align}

Since \( t_i = 1/p_i \) we have:
\begin{align}
t_i \left( \sum_{A \ni i, d_A = j} p_A \right) \leq \frac{2^{1-j}}{(j-1)!}.
\end{align}

By applying Inequality (6.14) to Inequality (6.9) we have
\begin{align}
\sum_{A} p_A \text{Rev}(D_A^T) &\leq \sum_{i=1}^n r_i \sum_{j=1}^n j t_i \left( \sum_{A \ni i, d_A = j} p_A \right) \\
&\leq \sum_{i=1}^n r_i \left( \sum_{j=1}^n j \frac{2^{1-j}}{(j-1)!} \right) = \text{SRev}(D) \left( \sum_{j=1}^n j \frac{2^{1-j}}{(j-1)!} \right).
\end{align}

Next, we use the Proposition 6.2 to bound \( \sum_{j=1}^n j \frac{2^{1-j}}{(j-1)!} \). For simplicity we give the proof of this proposition in the appendix.

**Proposition 6.2** For every \( n > 1 \) we have
\begin{align}
\sum_{j=1}^n j \frac{2^{1-j}}{(j-1)!} \leq \frac{3\sqrt{e}}{2}.
\end{align}

Proposition 6.2 states that \( \sum_{j=1}^n j \frac{2^{1-j}}{(j-1)!} \) is at most \( \frac{3\sqrt{e}}{2} \) and hence
\begin{align}
\sum_{A} p_A \text{Rev}(D_A^T) &\leq \sum_{i=1}^n r_i \left( \sum_{j=1}^n j \frac{2^{1-j}}{(j-1)!} \right) \leq \sum_{i=1}^n r_i \frac{3\sqrt{e}}{2} \leq \frac{3\text{SRev}(D)\sqrt{e}}{2}.
\end{align}

Next, we show that \( \max\{\text{SRev}(D), \text{BRev}(D)\} \) is at least \( \text{Val}(D_\emptyset^C) \) which completes the proof. In the proof of this proposition, we use the following Lemma which has been proved by Li and Yao in [15].

**Lemma 6.5** Let \( F \) be a one-dimensional distribution with optimal revenue at most \( c \) supported on \([0, t_*] \). Then \( \text{Var}(F) \leq (2t - 1)c^2 \).

**Proposition 6.3** For a single buyer in semi-independent setting we have
\begin{align}
5 \max\{\text{SRev}(D), \text{BRev}(D)\} &\geq \text{Val}(D_\emptyset^C),
\end{align}

where \( t_i = 2r_i/(r_i n_i) \).
Proof: Since $SRev(D) = r$, the proof is trivial when $Val(D^C_0) \leq 5r$. Therefore, we assume $Val(D^C_0) > 5r$ from now on. Next, we show that $Var(D^C_0) \leq 4r^2$ and then use this fact in order to show $BRev(D)$ is a constant approximation of $Rev(D^C_0)$. To do so, we formulate the variance of $D^C_0$ as follows:

$$Var(D^C_0) = Var(D^C_1 + D^C_2 + \ldots + D^C_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} Covar(D^C_i, D^C_j)$$

(6.16)

Note that $Covar(D^C_i, D^C_j) = Var(D^C_i)$ if items $i$ and $j$ are equal and 0 otherwise. Therefore,

$$Var(D^C_0) = \sum_{i=1}^{n} Var(D^C_i) \times n_i$$

(6.17)

Recall that Lemma 6.5 states that $Var(D^C_i) \leq 4rr_i/n_i$, and thus

$$Var(D^C_0) \leq \sum_{i=1}^{n} Var(D^C_i) \times n_i \leq \sum_{i=1}^{n} 4rr_i \leq 4r^2$$

(6.18)

Since $Val(D^C_0) \geq 5r$ and $Var(D^C_0) \leq 4r^2$, we can apply the Chebyshev’s Inequality to show

$$Pr\left[\sum v_i \leq \frac{2}{5} Val(D^C_0)\right] \leq \frac{4r^2}{(1 - \frac{2}{5})^2 Val(D^C_0)} \leq \frac{4}{9}$$

(6.19)

which implies that $BRev(D) \geq (\frac{5}{9} \cdot \frac{2}{5}) Val(D^C_0) \geq \frac{Val(D^C_0)}{5}$.

7 Common Base-Value Distributions

In this section we study the same problem with common base-value distributions. Recall that in these distributions disariblities of buyers are of the form $v_{i,j} = f_{i,j} + b_i$ where $f_{i,j}$ is drawn from a known distribution $F_{i,j}$ and $b_i$ is the same for all items and is drawn from a known distribution $B_i$. Again, we show $\max SRev$, $BRev$ achieves a constant approximation of the revenue optimal mechanism when we have only one buyer. Note that, this result answers an open question raised by Babaioff et al. in [3].

Theorem 7.1 For an auction with one seller, one buyer, and common base-value distribution of valuations we have

$$\max\{SRev(D), BRev(D)\} \geq \frac{1}{15} \times Rev(D).$$

Proof: Let $I$ be an instance of the problem. We create an instance $Cor(I)$ of the problem with $2n$ items such that the distribution of valuations is a semi-independent distribution $D'$ as where $D'_i = F_i$ for $1 \leq i \leq n$ and $D'_{i+n} = B$ for $n + 1 \leq i \leq 2n$. Moreover, the valuations of the items $n + 1, n + 2, \ldots, 2n$ are always equal and all other valuations are independent. Thus, by the definition, $D'$ is a semi-independent distribution of valuations and by Lemma 6.4 we have

$$\max\{SRev(D'), BRev(D')\} \geq \frac{1}{7.5} \times Rev(D').$$

(7.1)

Since every mechanism for selling the items of $D$ can be mapped to a mechanism for selling the items of $D'$ where items $i$ and $n + i$ are considered as a single package containing both items, we have

$$Rev(D) \leq Rev(D').$$

(7.2)
Moreover, since in the bundle mechanism we sell all of the items as a whole bundle, the revenue achieved by mechanism $\text{BRev}$ is the same in both auctions. Hence,

$$\text{BRev}(D) = \text{BRev}(D').$$ \hfill (7.3)

Note that, we can consider $\text{SRev}(D)$ as a mechanism for selling items of $\text{Cor}(I)$ such that items are packed into partitions of size 2 (items $i$ is packed with item $n+i$) and each partition is priced with Myerson’s optimal mechanism. Since for every two independent distributions $F_i, F_{i+n}$ we have $\text{SRev}(F_i \times F_{n+i}) \leq 2 \cdot \text{BRev}(F_i \times F_{n+i})$ we can imply

$$\text{SRev}(D) = \sum_{i=1}^{n} \text{BRev}(F_i \times F_{n+i}) \geq \sum_{i=1}^{n} \frac{\text{SRev}(F_i \times F_{i+n})}{2} = \frac{\text{SRev}(D')}{2}.$$ \hfill (7.4)

According to Inequalities 7.1, 7.2, and 7.3 we have

$$\max\{\text{SRev}(D), \text{BRev}(D)\} \geq \max\{\text{SRev}(D')/2, \text{BRev}(D')\} \geq \max\{\text{SRev}(D'), \text{BRev}(D')\}/2 \geq \text{Rev}(D')/15 \geq \text{Rev}(D)/15.$$

\[\square\]

### 8 Linear Correlation

A natural generalization of common base-value distributions is an extended correlation such that the valuation of each item for a buyer can be a linear combination of his desirabilities for some features where the distribution of desirabilities for features are independent and known. More precisely, let $F_{i,1}, F_{i,2}, \ldots, F_{i,l}$ be $l$ independent distributions of disarabilities of features for buyer $i$ and once each value $f_{i,j}$ is drawn from $F_{i,j}$, desirability of the $i$-buyer for $j$-th item is determined by $V_{f_i} \cdot M_{i,j}$ where $V_{f_i} = \langle f_{i,1}, f_{i,2}, \ldots, f_{i,l} \rangle$ and $M_i$ is an $n \times l$ matrix containing non-negative values.

Note that, a semi-independent distribution of valuations is a special case of linear correlation where we have $n+1$ features $F_{i,1}, F_{i,2}, \ldots, F_{i,n+1}$ and $M_i$ is a matrix such that $(M_i)_{a,b} = 1$ if either $a = b$ or $b = n+1$ and $(M_i)_{a,b} = 0$ otherwise. In this case, $F_{i,n+1}$ is the base value which is shared between all items and each of the other distributions is dedicated to a single item.

In this section we show that if we have only one buyer and the distribution of his desirabilities for items has a linear correlation and $M_1$ has at most $k$ none-zero elements in each row then $\max\{\text{SRev}, \text{BRev}\}$ achieves at least $\frac{1}{7.5k}$ of the revenue of the revenue optimal mechanism. This result generalizes the approximation result for common-base value distributions since each common-base value distribution is a special case of linear correlation where $M_i$ has at most two non-zero entries in each row.

**Theorem 8.1** Let $D$ be a distribution of valuation for one buyer in an auction such that the correlation between items is linear. If each row of $M_1$ has at most $k$ non-zero entries, then we have

$$\max\{\text{SRev}(D), \text{BRev}(D)\} \geq \frac{1}{7.5k} \times \text{Rev}(D).$$

**Proof:** The proof is similar to the proof of Theorem 7.1. Since we can multiply the entries of the matrix by any integer number and divide the values of distribution by that number without any impact on the setting of the problem, for the sake of simplicity, we assume all values of $M_1$ are integers. Let $J$ be an instance of our auction. We create an instance $\text{Cor}(I)$ of an auction with semi-independent distributions as follows. Let $n_i$ be the total sum of numbers in $i$-th column of $M_1$. For each feature we put a set of items in $\text{Cor}(I)$
containing \( n_i \) similar elements. Moreover, we consider every two items of different types to be independent. We refer to the distribution of items in \( Cor(I) \) with \( D' \). Each strategy of auction \( I \) can be mapped to a strategy of auction \( Cor(I) \) by just partitioning items of \( Cor(I) \) into some packages, such that package \( i \) has \( (M_1)_i,a \) items from \( a \)-th type, and then treating each package as a single item. Therefore we have

\[
\text{Rev}(D) \leq \text{Rev}(D').
\]

Moreover, bundle strategy has the same revenue in both auction since it sells all items as a whole package. Therefore the following equality holds.

\[
\text{BRev}(D) = \text{BRev}(D').
\]

Now, all that remains is to find a lower bound in terms of \( S\text{Rev}(D') \) for \( S\text{Rev}(D) \). Recall that each row of \( M_1 \) has at most \( k \) non-zero entries which implies that the corresponding strategy of \( S\text{Rev}(D) \) in \( Cor(I) \) has at most \( k \) non-similar items and hence

\[
S\text{Rev}(D) \geq \frac{S\text{Rev}(D')}{k}.
\]

By Lemma 6.4, we have

\[
\max\{S\text{Rev}(D'), \text{BRev}(D')\} \geq \frac{\text{Rev}(D')}{7.5}
\]

By applying Inequalities (8.1),(8.2), and (8.3) we get

\[
\max\{S\text{Rev}(D)/k, \text{BRev}(D)\} \geq \frac{\text{Rev}(D)}{7.5}
\]

which implies

\[
\max\{S\text{Rev}(D), \text{BRev}(D)\} \geq \frac{\text{Rev}(D)}{k \cdot 7.5}.
\]

### 9 Point mechanism

In this section we introduce a new simple mechanism which we call point mechanism. We show that in i.i.d. distributions the revenue obtained by point mechanism \( P\text{Rev} \in O(P\text{Rev}) \) and provide a lower bound \( P\text{Rev} \leq P\text{Rev}/\Omega(\log n) \).

In point mechanism we use the simple item pricing mechanism where each item has a separate price. The only difference is that buyers enter the mechanism sequentially and each buyer needs to pay at least a minimum payment amount \( M \) in order to buy her desired items. More precisely if the buyer is interested in buying a set of items \( S \), then she will be able to buy the set \( S \) iff she pays \( \max\{M, \sum_{i \in S} \pi_i\} \), where \( \pi_i \) is the price of \( i \)-th item. Note that point mechanism is more general than both item pricing and bundle pricing. By \( M = 0 \), we have a mechanism equivalent to item pricing, and by \( M = \sum \pi_i \), we have a mechanism equivalent to bundle pricing. Babaioff et al. [3] prove in a mechanism with a single additive buyer \( \max\{S\text{Rev}(D), \text{BRev}(D)\} \geq \frac{1}{6}\text{Rev}(D) \), thus by the aforementioned reductions for single additive buyer we have

\[
P\text{Rev}(D) \geq \frac{1}{6}\text{Rev}(D).
\]

However in multi-buyer mechanisms both \( \max\{S\text{Rev}(D), \text{BRev}(D)\} \) might be less than the optimal revenue by a factor of \( \log n \). Moreover [3] proves the revenue of any partition pricing might be less than the optimal revenue by a factor of \( \log n \). In the following we prove the revenue by point mechanism \( P\text{Rev}(D) \) is no less than a constant fraction of the revenue by partition pricing and also \( P\text{Rev}(D) \) might be greater than \( P\text{Rev}(D) \) by a \( \log n \) factor. First we should prove some characteristics of the partition mechanisms.
Lemma 9.1 Let $P_{\text{Rev}}(D)$ be the revenue of the optimal partition mechanism for $n$ items with i.i.d. distributions and $m$ buyers. There is a partition mechanism with revenue $P_{\text{Rev}}^*(D)$ such that the size and price of each part is the same and $P_{\text{Rev}}^*(D) \geq \frac{1}{2} P_{\text{Rev}}(D)$.

Proof: Since the distributions are assumed i.i.d. each partition mechanism is identified by the size and prices of the parts. Let the optimal partition mechanism partition the items into $k$ parts. Let $s_i$ and $\pi_i$ denote the size and price of the $i$-th part respectively. Let $e_i$ be the expected revenue from the $i$-th part, i.e., $e_i = \Pr[\text{$i$-th part is sold}] \times \pi_i$. We can write the revenue as

$$P_{\text{Rev}}(D) = \sum_{i=1}^{k} e_i$$

(9.1)

Now let $\rho_i$ denote the ratio of the expected revenue of the $i$-th part to the size of the $i$-th part, i.e., $\rho_i = \frac{e_i}{s_i}$. Thus one can rewrite Equality (9.1) as

$$P_{\text{Rev}}(D) = \sum_{i=1}^{k} e_i = \sum_{i=1}^{k} \rho_i \cdot s_i$$

(9.2)

Let $j$-th part be the part with the maximum ratio. More formally $j = \arg \max_{1 \leq i \leq k} \{\rho_i\}$. We claim that if we design a new partition mechanism which has only parts of size $s_j$ and for all of them the price is $\pi_j$, the revenue would be at least half of the optimal partition revenue. Note that $n$ is not necessarily dividable by $s_j$ but at least half of the items would be in the parts. Let $P_{\text{Rev}}^*$ denote the revenue for the new partition mechanism. We write Inequality (9.2) for $P_{\text{Rev}}^*$,

$$P_{\text{Rev}}^*(D) = \sum_{j=1}^{k'} \rho_j \cdot s_j$$

(9.3)

where $k' = \left\lfloor \frac{n}{s_j} \right\rfloor$ is the number of parts in the new mechanism. Since $\sum_{j=1}^{k'} s_j \geq \frac{1}{2} \sum_{i=1}^{k} s_i = n$ and for each $i$, $\rho_j \geq \rho_i$,

$$P_{\text{Rev}}^*(D) = \sum_{j=1}^{k'} \rho_j \cdot s_j \geq \frac{1}{2} \rho_j \sum_{i=1}^{k} s_i$$

(9.4)

$$\geq \frac{1}{2} \sum_{i=1}^{k} \rho_i \cdot s_i$$

(9.5)

$$\geq \frac{1}{2} \sum_{i=1}^{k} \rho_i \cdot s_i$$

(9.6)

$$= \frac{1}{2} P_{\text{Rev}}(D)$$

(9.7)

Now we can prove there is a point mechanism that obtains a constant fraction of the partition revenue.

Theorem 9.1 Let $P_{\text{Rev}}(D)$ be the revenue of the optimal partition mechanism for $n$ items with i.i.d. distributions and $m$ buyers. There is a point mechanism with revenue $P_{\text{PPrev}}(D) \geq \frac{1}{e} P_{\text{Rev}}(D)$, where $c$ is a constant.
First using Lemma 9.1 we consider a partition mechanism with \( k \) parts of same size \( s \) and same price \( \pi \) and revenue \( \text{PRev}^*(D) \). By Lemma 9.1

\[
\text{PRev}^*(D) \geq \frac{1}{2} \text{PRev}(D).
\] (9.8)

Now in the new partition mechanism each part is sold independently from the other parts and all parts have the same size \( s \) and price \( \pi \). Let \( k \) denote the number of parts in the partition. For each part and each buyer, let \( q \) be the probability that the buyer buys the part. Thus the probability that each buyer buys \( i \) number of parts is a binomial function \( (\frac{k}{i}) q^i (1-q)^{k-i} \). Let \( r \) denote the expected number of parts a buyer is going to buy from \( k \) parts. The revenue of the partition mechanism is bounded by

\[
\text{PRev}^*(D) \leq mr \pi.
\] (9.9)

Now we provide a point mechanism which obtains a revenue at least \( \frac{2}{3} \text{PRev}^*(D) \). Let the price for each item be \( \frac{\pi}{2} \) and the minimum payment amount be \( \frac{\pi}{4} \). Now to analyze \( \text{PPRev}(D) \) we consider the case that buyers enter the mechanism sequentially but each of them only sees half of the items, unless there exists less than \( \frac{n}{2} \) of items. If there is less than half of the items, then \( \text{PPRev}(D) \geq \frac{\pi}{2} \times \frac{n}{2} \geq \frac{1}{4} \text{PRev}^*(D) \). Thus we assume each buyer sees exactly half of the items. Since the number of parts a buyer is going to buy in partition mechanism is a binomial function, with probability \( \frac{1}{2} \) the buyer is going to buy at least \( r \) parts, where \( c' \) is a constant. Therefore if a buyer is going to buy at least \( r \) parts in the partition, then in point mechanism with probability \( \frac{1}{2} \) half of the desired parts are in the \( \frac{n}{2} \) items he sees. Therefore for each buyer, with probability \( \frac{1}{2c'} \) there is a set of items \( S \) such that \( \sum_{i \in S} v_i \geq \frac{\pi}{4} \). Thus

\[
\text{PPRev}(D) \geq \frac{1}{2c'} \times \frac{mvr \pi}{4}.
\] (9.10)

Thus by Inequalities (9.9) and (9.8)

\[
\text{PPRev}(D) \geq \frac{1}{8c'} \text{PRev}^*(D) \geq \frac{1}{16c'} \text{PRev}(D).
\] (9.11)

9.1 A Lower Bound: \( \text{PRev} \leq \text{PPRev} / \Omega(\log n) \)

Babaioff et al. [3] show a setting with \( n \) items with i.i.d. distributions and \( m \) buyers in which \( \text{PRev} \leq \text{Rev} / \Omega(\log n) \). We show that in the same setting the revenue obtained by partition is a poor approximation to the revenue obtained by point mechanism.

**Proposition 9.1** There exists a setting with \( n \) items with i.i.d. distributions and \( m \) buyers such that \( \text{PRev} \leq \text{PPRev} / \Omega(\log n) \).

**Proof:** Let the number of buyers \( m = \sqrt{n} \). Let the valuation distributions for each buyer and each item be 0 with probability \( 1 - \frac{1}{\sqrt{n}} \) and with probability \( \frac{1}{\sqrt{n}} \) be a distribution \( F \) with CDF

\[
F(x) = \begin{cases} 
1 - \frac{1}{x} & \text{for } x \in [1, n^{\frac{1}{3}}] \\
1 & \text{for } x > n^{\frac{1}{3}} 
\end{cases}
\]

Babaioff et al. show \( \text{PRev}(D) \in \Theta(n) \). We design a point mechanism which obtains a revenue \( \text{PPRev}(D) \in \Omega(n \log n) \). Let the minimum payment amount \( M = c \sqrt{n} \log n \) and the price for each item be \( 2c \log n \), where \( c \) is a constant. [3] considers a similar mechanism and proves for some constant \( c \), in expectation the \( \frac{c}{\sqrt{n}} \) first buyers will pay at least \( \frac{M}{2} \). Thus the overall revenue \( \text{PPRev}(D) \geq \frac{\sqrt{n}}{2} M = \frac{c}{4} n \log n \). Thus \( \text{PRev}(D) \leq \frac{\text{PPRev}(D)}{\Omega(\log n)} \).
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Appendix A: Omitted proofs of Section 5

Proof of Lemma 5.1: Suppose for the sake of contradiction that \(\text{Rev}(D) > \text{Rev}(D^{(A)}) + \text{Val}(D^{([1]-A)})\), we show one can sell items of \(A\) to obtain an expected revenue more than \(\text{Rev}(D^{(A)})\) which is impossible. To do so, we add items of \([N - A]\) (which are of no value to the buyers) and do the following:

- We draw a valuation for items in \([N] - A\) based on \(D\).
- We sell all items with the optimal mechanism for selling items of \(D\).
- Finally, each buyer can return every item that has bought from the set \([N] - A\) and get a money equal to the value that we have drawn for those items. Note that, since buyers have no desirability for these items, they will definitely return them.

Note that, we are faking the desirabilities of buyers for items in \([N] - A\) with the money that we return in the last step. Therefore, the bahaviour of buyers is as if they had value for those items as well. Since the money that we return to buyers is at most \(\text{Val}([N] - A)\) (in expectation), and we achieve \(\text{Rev}(D)\) (in expectation) at first, the expected revenue that we obtain is at least \(\text{Rev}(D) - \text{Val}([N] - A)\) which is greater than \(\text{Rev}(D^{(A)})\) which contradicts with maximality of \(\text{Rev}(D^{(A)})\).

Proof of Lemma 5.2: Suppose we run a second price auction with reserved price \(t_i r_i\). Since the revenue achieved by this auction is equal to \(p_i t_i r_i\) and is at most \(\text{Rev}(D) = r_i\) we have \(p_i \leq \frac{1}{t_i}\).

Proof of Lemma 5.3: This lemmas follows by the fact that \(D_i^{C}\) is stochastically dominated by \(D_i\). Therefore \(\text{Rev}(D_i) \geq \text{Rev}(D_i^{C})\) and thus \(\text{Rev}(D_i^{C}) \leq r_i\).

Proof of Lemma 5.4: By definition, the probability that a random variable drawn from \(D_i\) lies in the tail is equal to \(p_i\), therefore \(\text{Rev}(D_i^{T})\) cannot be more than \(p_i r_i\), since otherwise \(\text{Rev}(D_i)\) would be more than \(r_i\) which is a contradiction.

Proof of Lemma 5.5: Suppose the seller has a magical oracle that after the realization of desirabilities, it informs him for which items the highest valuation of buyers lies in the tail and for which items it lies in the core. Suppose \(A\) be the set of items with highest desirability in the tail. By definition, the maximum possible revenue (in expectation) that the seller can achieve in this situation is \(\text{Rev}(D_A)\) and this happens with probability \(p_A\), therefore having the magical oracle, the maximum expected revenue of the seller is \(\sum_A p_A \text{Rev}(D_A)\). Since this oracle gives the seller some additional information, the optimal revenue of the seller cannot be decreased and hence

\[
\text{Rev}(D) \leq \sum_A p_A \text{Rev}(D_A).
\]

Proof of Proposition 6.1: Suppose we have \(n\) similar items with valuation function \(D\) for buyers. By definition \(\text{Rev}(D) \geq \text{SRev}(D)\) since \(\text{Rev}(D)\) is the maximal possible revenue that we can achieve. Therefore we need to show \(\text{Rev}(D)\) cannot be more than \(\text{SRev}(D)\). Since all the items are similar, \(\text{SRev}(D) = n \text{Rev}(D^{(i)})\) for all \(1 \leq i \leq n\). In the rest we show \(\text{Rev}(D)\) cannot be more than \(n\) times of \(\text{Rev}(D^{(i)})\). We design the followiwig mechanism for selling just one of the items:
• Pick an integer number \( g \) between 1 and \( n \) uniformly at random and keep it private.

• Use the optimal mechanism for selling \( n \) similar items, except that the prices are divided over \( n \).

• At the end, give the item to the buyer that has bought item number \( g \) (if any), and take back all other sold items.

Note that, in buyers’ perspective both the prices and the expectation of the number of items they buy are divided over \( n \), therefore they’ll have the same behaviour as before. Since prices are divided over \( n \) the revenue we get by the above mechanism is exactly \( \frac{\text{Rev}(D)}{n} \) which implies \( \text{Rev}(D) \leq n \text{Rev}(D(i)) \) and completes the proof.

**Proof of Proposition 6.2:** Recall that in this proposition our aim is to prove the following inequality

\[
\sum_{j=1}^{n} j \frac{2^{1-j}}{(j-1)!} \leq \frac{3\sqrt{e}}{2}.
\]  

(B.1)

To do so, we show \( \sum_{j=1}^{\infty} j \frac{2^{1-j}}{(j-1)!} = \frac{3\sqrt{e}}{2} \) and since \( \sum_{j=1}^{\infty} j \frac{2^{1-j}}{(j-1)!} \geq \sum_{j=1}^{n} j \frac{2^{1-j}}{(j-1)!} \) we imply Inequality (B.1). We first write the Maclaurin series for \( \sqrt{e} \) and next prove Equation (B.1). By Maclaurin series we have:

\[
f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^n =
\]

\[
f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots + \frac{f^{(j)}(0)}{j!} x^j + \cdots.
\]

By setting \( f(t) = e^t \) and \( x = \frac{1}{2} \) we come up with the following equation.

\[
f\left(\frac{1}{2}\right) = e^{\frac{1}{2}} = \sqrt{e} = \sum_{j=0}^{\infty} \frac{1}{2j j!}
\]

(B.2)

Now we rewrite Equation B.2 to obtain

\[
\sqrt{e} = \sum_{j=1}^{\infty} \frac{1}{2j-1(j-1)!} = \sum_{j=1}^{\infty} \frac{2^{1-j}}{(j-1)!}
\]

(B.3)

We can write the coefficient \( j \) in the series as \((j - 1) + 1\), hence

\[
\sum_{j=1}^{n} j \frac{2^{1-j}}{(j-1)!} = \sum_{j=2}^{n} (j - 1) \frac{2^{1-j}}{(j-1)!} + \left[ \sum_{j=1}^{n} \frac{2^{1-j}}{(j-1)!} \right] =
\]

\[
\sum_{j=2}^{n} (j - 1) \frac{2^{1-j}}{(j-1)!} + \sqrt{e} = \sum_{j=2}^{n} \frac{2^{1-j}}{(j-2)!} + \sqrt{e} =
\]

\[
\sum_{j=1}^{n} \frac{2^{-j}}{(j-1)!} + \sqrt{e} = \left[ \frac{\sum_{j=1}^{n} \frac{2^{1-j}}{(j-1)!}}{2} \right] + \sqrt{e} = \frac{\sqrt{e}}{2} + \sqrt{e} = \frac{3\sqrt{e}}{2}.
\]

Note that, by Equation (B.2) the total sum of the series inside each bracket is equal to \( \sqrt{e} \).