Tangent cones of Hermitian Yang–Mills connections with isolated singularities

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2018-01-26

Abstract

We give a simple direct proof of uniqueness of tangent cones for singular projectively Hermitian Yang–Mills connections on reflexive sheaves at isolated singularities modelled on a sum of $\mu$–stable holomorphic bundles over $\mathbb{P}^{n-1}$.

1 Introduction

A projectively Hermitian Yang–Mills (PHYM) connection $A$ over a Kähler manifold $X$ is a unitary connection $A$ on a Hermitian vector bundle $(E, H)$ over $X$ satisfying

$$F^{0,2}_{A} = 0 \quad \text{and} \quad i\Lambda F_{A} - \frac{\text{tr}(i\Lambda F_{A})}{\text{rk}E} \cdot \text{id}_{E} = 0.$$  

Since $F^{0,2}_{A} = 0$, $\mathcal{E} := (E, \tilde{\partial}_{A})$ is a holomorphic vector bundle, and $A$ is the Chern connection of $H$. A Hermitian metric $H$ on a holomorphic vector bundle is called PHYM if its Chern connection $A_{H}$ is PHYM. The celebrated Donaldson–Uhlenbeck–Yau Theorem [Don85; Don87; UY86] asserts that a holomorphic vector bundle $\mathcal{E}$ on a compact Kähler manifold admits a PHYM metric if and only if it is $\mu$–polystable; moreover, any two PHYM metrics are related by an automorphism of $\mathcal{E}$ and by multiplication with a conformal factor. If $H$ is a PHYM metric, then the connection $A^{0}$ on $PU(E, H)$, the principal $PU(r)$–bundle associated with $(E, H)$, induced by $A_{H}$ is Hermitian Yang–Mills (HYM), that is, it satisfies $F^{0,2}_{A^{0}} = 0$ and $i\Lambda F_{A^{0}} = 0$; it depends only on the conformal class of $H$. Conversely, any HYM connection $A^{0}$ on $PU(E, H)$ can be lifted to a PHYM connection $A$; any two choices of lifts lead to isomorphic holomorphic vector bundles $\mathcal{E}$ and conformal metrics $H$.

An admissible PHYM connection is a PHYM connection $A$ on a Hermitian vector bundle $(E, H)$ over $X \backslash \text{sing}(A)$ with $\text{sing}(A)$ a closed subset with locally finite $(2n - 4)$–dimensional Hausdorff measure and $F_{A} \in L_{\text{loc}}^{2}(X)$.\footnote{It should be pointed out that our notion of admissible PHYM connection follows Bando and Siu [BS94] and not Tian [Tia00]. The notion of admissible Yang–Mills connection introduced by Tian is stronger: it assumes that the Hermitian vector bundle extends to all of $X$.} Bando [Ban91] proved that if $A$ is an admissible PHYM connection,
then \((E, \tilde{\partial}_A)\) extends to \(X\) as a reflexive sheaf \(\mathcal{E}\) with \(\text{sing}(\mathcal{E}) \subset \text{sing}(A)\). Bando and Siu [BS94] proved that a reflexive sheaf on a compact Kähler manifold admits an admissible PHYM metric if and only if it is \(\mu\)-polystable.

The technique used by Bando and Siu does not yield any information on the behaviour of the admissible PHYM connection \(A_H\) near the singularities of the reflexive sheaf \(\mathcal{E}\) — not even at isolated singularities. The simplest example of a reflexive sheaf on \(\mathbb{C}^n\) with an isolated singularity at 0 is \(i_* \sigma^* \mathcal{F}\) with \(\mathcal{F}\) a holomorphic vector bundle over \(\mathbb{P}^{n-1}\); cf. Hartshorne [Har80, Example 1.9.1]. Here we use the obvious maps summarised in the following diagram:

\[
\begin{array}{cccc}
\mathbb{C}^n & \xrightarrow{i} & \mathbb{C}^n \setminus \{0\} & \xrightarrow{\pi} & S^{2n-1} & \xrightarrow{\rho} & \mathbb{P}^{n-1}.
\end{array}
\]

The main result of this article gives a description of PHYM connections near singularities modelled on \(i_* \sigma^* \mathcal{F}\) with \(\mathcal{F}\) a sum of \(\mu\)-stable holomorphic vector bundles.

**Theorem 1.2.** Let \(\omega = \frac{1}{2i} \partial \bar{\partial} |z|^2 + O(|z|^2)\) be a Kähler form on \(B_R(0) \subset \mathbb{C}^n\). Let \(A\) be an admissible PHYM connection on a Hermitian vector bundle \((E, H)\) over \(B_R(0) \setminus \{0\}\) with \(\text{sing}(A) = \{0\}\) and \((E, \tilde{\partial}_A) \equiv \sigma^* \mathcal{F}\) for some holomorphic vector bundle \(\mathcal{F}\) over \(\mathbb{P}^{n-1}\). Denote by \(F\) the complex vector bundle underlying \(\mathcal{F}\).

If \(\mathcal{F}\) is a sum of \(\mu\)-stable holomorphic vector bundles, then there exist a Hermitian metric \(K\) on \(F\), a connection \(A_*\) on \(\sigma^*(F, K)\) which is the pullback of a connection on \(\rho^*(F, K)\), and an isometry \((E, H) \equiv \sigma^*(F, K)\) such that with respect to this isometry we have

\[
|z|^{k+1} |\nabla^k_{A_*} (A^2 - A_*^2)| \leq C_k |z|^\alpha \quad \text{for each } k \geq 0.
\]

The constants \(C_k, \alpha > 0\) depend on \(\omega, \mathcal{F}, A|_{B_R(0) \setminus B_{R/2}(0)}\), and \(||F_A||_{L^2(B_R(0))}\).

**Remark 1.3.** Using a gauge theoretic Łojasiewicz–Simon gradient inequality, Yang [Yan03, Theorem 1] proved that the tangent cone to a stationary Yang–Mills connection—in particular, a PU\((r)\) HYM connection—with an isolated singularity at \(x\) is unique provided

\[
|F_A| \leq d(x, \cdot)^{-2}.
\]

In our situation, such a curvature bound can be obtained from Theorem 1.2. Our proof of this result, however, proceeds more directly—without making use of Yang’s theorem.

The hypothesis that \(\mathcal{F}\) be a sum of \(\mu\)-stable holomorphic vector bundles is optimal. This is a consequence of the following observation, which will be proved in Section 6.

**Proposition 1.4.** Let \((F, K)\) be a Hermitian vector bundle over \(\mathbb{P}^{n-1}\). If \(B\) is a unitary connection on \(\rho^*(F, K)\) such that \(A_* := \pi^* B\) is HYM with respect to \(\omega_0 := \frac{1}{2i} \partial \bar{\partial} |z|^2\), then there is a \(k \in \mathbb{N}\) and, for each \(j \in \{1, \ldots, k\}\), there are \(\mu_j \in \mathbb{R}\), a Hermitian vector bundle \((F_j, K_j)\) on \(\mathbb{P}^{n-1}\), and an irreducible unitary connection \(B_j\) on \(F_j\) satisfying

\[
F_{B_j}^{0,2} = 0 \quad \text{and} \quad i \Lambda F_{B_j} = (2n - 2) \pi \mu_j \cdot \text{id}_{F_j}.
\]
such that
\[
F = \bigoplus_{j=1}^{k} F_j \quad \text{and} \quad B = \bigoplus_{j=1}^{k} \rho^* B_j + i \mu_j \text{id}_{\rho^* F_j} \cdot \theta.
\]
Here \( \theta \) denotes the standard contact structure\(^2\) on \( S^{2n-1} \). In particular,
\[
\mathcal{E} = (\sigma^* F, \hat{\partial}_A) \equiv \bigoplus_{j=1}^{k} \sigma^* \mathcal{F}_j
\]
with \( \mathcal{F}_j = (F_j, \hat{\partial}_{B_j}) \) \( \mu \)-stable.

To conclude the introduction we discuss two concrete examples in which Theorem 1.2 can be applied.

**Example 1.5** (Okonek, Schneider, and Spindler \[OSS 11, Example 1.1.13\]). It follows from the Euler sequence that \( H^0(\mathbb{P}^n(-1)) \cong \mathbb{C}^4 \). Denote by \( s_v \in H^0(\mathbb{P}^n(-1)) \) the section corresponding to \( v \in \mathbb{C}^4 \). If \( v \neq 0 \), then the rank two sheaf \( \mathcal{E} = \mathcal{E}_v \) defined by
\[
0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{s_v} \mathbb{P}^n(-1) \to \mathcal{E}_v \to 0
\]
is reflexive and \( \text{sing}(\mathcal{E}) = \{[v]\} \).

\( \mathcal{E} \) is \( \mu \)-stable. To see this, because \( \mu(\mathcal{E}) = 1/2 \), it suffices to show that
\[
\text{Hom}(\mathcal{O}_{\mathbb{P}^n}(k), \mathcal{E}) = H^0(\mathcal{E}(-k)) = 0 \quad \text{for each} \quad k \geq 1.
\]
However, by inspection of the Euler sequence, \( H^0(\mathcal{E}(-k)) \cong H^0(\mathbb{P}^n(-k - 1)) = 0 \). It follows that \( \mathcal{E} \) admits a PHYM metric \( H \) with \( F_H \in L^2 \) and a unique singular point at \([v] \in \mathbb{P}^3\). To see that Theorem 1.2 applies, pick a standard affine neighborhood \( U \cong \mathbb{C}^3 \) in which \([v]\) corresponds to 0. In \( U \), the Euler sequence becomes
\[
0 \to \mathcal{O}_{\mathbb{C}^3} \xrightarrow{(1, z_1, z_2, z_3)} \mathcal{O}_{\mathbb{C}^3} \oplus \mathcal{O}_{\mathbb{C}^3} \to \mathbb{P}^n(-1)|_U \to 0,
\]
and \( s_v = [(1, 0, 0, 0)] \); hence,
\[
0 \to \mathcal{O}_{\mathbb{C}^3} \xrightarrow{(z_1, z_2, z_3)} \mathcal{O}_{\mathbb{C}^3} \oplus \mathcal{O}_{\mathbb{C}^3} \to \mathcal{E}_v|_U \to 0.
\]
On \( \mathbb{C}^3 \setminus \{0\} \), this is the pullback of the Euler sequence on \( \mathbb{P}^2 \); therefore, \( \mathcal{E}_v|_U \cong i_* \sigma^* \mathcal{T}_{\mathbb{P}^2} \).

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\(^2\)With respect to standard coordinates on \( \mathbb{C}^n \), the standard contact structure \( \theta \) on \( S^{2n-1} \) is such that \( \pi^* \theta = \sum_{j=1}^{n} (z_j dz_j - z_j d\bar{z}_j)/2|z|^2 \).
Example 1.6. For \( t \in \mathbb{C} \), define \( f_t : \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 5} \) by

\[
 f_t := \begin{pmatrix}
 z_0 & 0 \\
 z_1 & z_0 \\
 z_2 & z_1 \\
 t \cdot z_3 & z_2 \\
 0 & z_3 
\end{pmatrix},
\]

and denote by \( \mathcal{E}_t \) the cokernel of \( f_t \), i.e.,

\[
 (1.7) \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \xrightarrow{f_t} \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 5} \to \mathcal{E}_t \to 0.
\]

If \( t \neq 0 \), then \( \mathcal{E}_t \) is locally free; \( \mathcal{E}_0 \) is reflexive with \( \text{sing}(\mathcal{E}_0) = \{[0 : 0 : 0 : 1]\} \). The proof of this is analogous to that of the reflexivity of \( \mathcal{E}_0 \) from Example 1.5 given in [OSS, Example 1.1.1].

For each \( t \), \( H^0(\mathcal{E}_t) = H^1(\mathcal{O}_{\mathbb{P}^3}) = 0 \); hence, \( \mathcal{E}_t \) is \( \mu \)-stable according to the criterion of Okonek, Schneider, and Spindler [OSS, Remark 1.2.6(b)]. The former vanishing is obvious since \( H^0(\mathcal{O}_{\mathbb{P}^3}) = H^1(\mathcal{O}_{\mathbb{P}^3}(-2)) = 0 \). The latter follows by dualising (1.7), twisting by \( \mathcal{O}_{\mathbb{P}^3}(-1) \) and observing that the induced map \( H^0(\mathcal{F}^t) : H^0(\mathcal{O}_{\mathbb{P}^3})^{\oplus 5} \to H^0(\mathcal{O}_{\mathbb{P}^3}(-1))^{\oplus 2} \), which is given by

\[
 \begin{pmatrix}
 z_0 & z_1 & z_2 & t \cdot z_3 & 0 \\
 0 & z_0 & z_1 & z_2 & z_3 
\end{pmatrix},
\]

is injective.

In a standard affine neighborhood \( U \cong \mathbb{C}^3 \) of \( [0 : 0 : 0 : 1] \), we have \( \mathcal{E}_0|_U \cong i_* \sigma^*(\mathcal{F}(\mathbb{P}^2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \). To see this, note that the cokernel of the map \( g : \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^2} \) defined by

\[
 g := \begin{pmatrix}
 z_0 & 0 \\
 z_1 & z_0 \\
 z_2 & z_1 \\
 0 & z_2 \\
 0 & 1 
\end{pmatrix}
\]

is \( \mathcal{F}(\mathbb{P}^2) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \).

Conventions and notation. Set \( B_r := B_r(0) \) and \( \tilde{B}_r := B_r(0) \setminus \{0\} \). We denote by \( c > 0 \) a generic constant, which depends only on \( \mathfrak{F} \), \( \omega, s|_{B_1 \setminus B_1/2}, H_0 \), and \( \|F_H\|_{L^2(B_1(0))} \) (which will be introduced in the next section). Its value might change from one occurrence to the next. Should \( c \) depend on further data we indicate this by a subscript. We write \( x \lesssim y \) for \( x \leq cy \). The expression \( O(x) \) denotes a quantity \( y \) with \( |y| \lesssim x \). Since reflexive sheaves are locally free away from a closed subset of complex codimension three, without loss of generality, we will assume throughout that \( n \geq 3 \).
Acknowledgements. HSE and TW were partially supported by São Paulo State Research Council (FAPESP) grant 2015/50368-0 and the MIT–Brazil Lemann Seed Fund for Collaborative Projects. HSE is also funded by FAPESP grant 2014/24727-0 and Brazilian National Research Council (CNPq) grant PQ2 - 312390/2014-9.

2 Reduction to the metric setting

In the situation of Theorem 1.2, the Hermitian metric $H$ on $\mathcal{E}$ corresponds to a PHYM metric on $\sigma^* \mathcal{F}$ via the isomorphism $(E, \overline{\partial}_A) \cong \sigma^* \mathcal{F}$. By slight abuse of notation, we will denote this metric by $H$ as well.

Denote by $\mathcal{F}_1, \ldots, \mathcal{F}_k$ the $\mu$–stable summands of $\mathcal{F}$. Denote by $K_j$ the PHYM metric on $\mathcal{F}_j$ with

$$i\Lambda_{\omega_{FS}} F_{K_j} = \frac{2\pi}{(n-2)! \text{vol}(\mathbb{P}^{n-1})} \mu_j \cdot \text{id}_{F_j} = (2n-2) \pi \mu_j \cdot \text{id}_{F_j}$$

with $\omega_{FS}$ denoting the integral Fubini study form and for $\mu_j := \mu(\mathcal{F}_j)$. The Kähler form $\omega_0$ associated with the standard Kähler metric on $\mathbb{C}^n$ can be written as

$$(2.1) \quad \omega_0 = \frac{1}{2i} \bar{\partial} \partial |z|^2 = \pi r^2 \sigma^* \omega_{FS} + r dr \wedge \pi^* \theta$$

with $\theta$ as in Proposition 1.4. Therefore, we have

$$i\Lambda_{\omega_0} F_{\sigma^* K_j} = (2n-2) \mu_j r^{-2} \cdot \text{id}_{\sigma^* F_j},$$

and $H_{\sigma, j} := r^{2\mu_j} \cdot \sigma^* K_j$ satisfies

$$i\Lambda_{\omega_0} F_{H_{\sigma, j}} = i\Lambda_{\omega_0} F_{\sigma^* K_j} + i\Lambda_{\omega_0} \bar{\partial} \partial \log r^{2\mu_j} \cdot \text{id}_{\sigma^* F_j}$$

$$= i\Lambda_{\omega_0} F_{\sigma^* K_j} + \frac{1}{2} \Delta \log r^{2\mu_j} \cdot \text{id}_{\sigma^* F_j} = 0.$$ 

Denote by $A_{\sigma, j}$ the Chern connection associated with $H_{\sigma, j}$ and by $B_j$ the Chern connection associated with $K_j$. The isometry $r^{\mu_j} : (\sigma^* F_j, H_{\sigma, j}) \rightarrow (\sigma^* (F_j, K_j))$ transforms $A_{\sigma, j}$ into

$$A_{\sigma, j} := (r^{\mu_j})_* A_{\sigma, j} = \sigma^* B_j + i \mu_j \text{id}_{\sigma^* F_j} \cdot \pi^* \theta.$$ 

In particular,

$$A_* := \bigoplus_{j=1}^k A_{\sigma, j}$$

is the pullback of a connection $B$ on $S^{2n-1}$. Moreover, $A_*$ is unitary with respect to

$$H_* := \bigoplus_{j=1}^k \sigma^* K_j.$$
Proposition 2.2. Assume the above situation. Set $H_0 := \bigoplus_{j=1}^k H_{0,j}$ and fix $R > 0$. We have

\[(2.3) \quad \| |z|^{2+\ell} \nabla_{H_0}^\ell F_{H_0} \|_{L^\infty(B_R)} < \infty \quad \text{for each } \ell > 0.\]

Proof. Using the isometry $g := \bigoplus_{j=1}^k r_{H_j}$ both assertions can be translated to corresponding statements for $A_0$. The first assertion then follows since $A_0$ is the pullback of a connection $B$ on $S^{2n-1}$. \hfill \Box

In the situation of Theorem 1.2, after a conformal change, which does not affect $A^0$, we can assume that $\det H = \det H_0$. Setting

$$ s := \log(H_0^{-1} H) \in C^\infty(\tilde{B}_R, \mathfrak{su}(\sigma^* F, H_0))^3 $$

and

$$ \Upsilon(s) := \frac{e^{ads} - 1}{adj}, $$

we have

$$ e_s^{s/2} H = H_0 \quad \text{and} \quad e_s^{s/2} A = A_0 + a $$

with

$$ a := \frac{1}{2} \Upsilon(-s/2) \partial A_s - \frac{1}{2} \Upsilon(s/2) \bar{\partial} A_s; $$

see, e.g., [JW18, Appendix A]. Moreover, with $g := \bigoplus_{j=1}^k r_{H_j}$ we have

$$ g_e^{s/2} A = A_s + gag^{-1}. $$

Since

$$ |\nabla_{A_s}^k gag^{-1}|_{H_s} = |\nabla_{H_0}^k a|_{H_0} \quad \text{for each } k \geq 0, $$

Theorem 1.2 will be a consequence of Proposition 2.2 and the following result.

Theorem 2.4. Suppose $\omega = \frac{1}{2} \bar{\partial} \partial |z|^2 + O(|z|^2)$ is a Kähler form on $\tilde{B}_R \subset \mathbb{C}^n$, $\mathcal{E}$ is a holomorphic vector bundle over $\tilde{B}_R$, and $H_0$ is a Hermitian metric on $\mathcal{E}$ which is HYM with respect to $\omega_0$ and satisfies (2.3). If $H$ is an admissible HYM metric on $\mathcal{E}$ with $\text{sing}(A_H) = \{0\}$ and $\det H = \det H_0$, then

$$ s := \log(H_0^{-1} H) \in C^\infty(\tilde{B}_R, \mathfrak{su}(\pi^* F, H_0)) $$

satisfies

$$ |s| \leq C_0 \quad \text{and} \quad |z|^k |\nabla_{H_0}^k \omega| \leq C_k |z|^{\alpha} \quad \text{for each } k \geq 1. $$

The constants $C_k, \alpha > 0$ depend on $\omega, H_0, s|_{\tilde{B}_R \setminus B_R^{1/2}}$, and $\|F_H\|_{L^2(B_R)}$.

The next three sections of this paper are devoted to proving Theorem 2.4. Without loss of generality, we will assume that the radius $R$ is one. We set $B := B_1$ and $\tilde{B} := \tilde{B}_1$. 

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3If $H, K$ are two Hermitian inner products on a complex vector space $V$, then there is a unique endomorphism $T \in \text{End}(V)$ which is self-adjoint with respect to $H$ and $K$, has positive spectrum, and satisfies $H(Tv, w) = K(v, w)$. It is customary to denote $T$ by $H^{-1}K$, and thus $\log(H^{-1}K) = \log(T)$.
3 A priori $C^0$ estimate

As a first step towards proving Theorem 2.4 we bound $|s|$, using an argument which is essentially contained in Bando and Siu [BS94, Theorem 2(a) and (b)].

**Proposition 3.1.** We have $|s| \in L^\infty(B)$ and $\|s\|_{L^\infty(B)} \leq c$.

**Proof.** The proof relies on the differential inequality

\[(3.2) \quad \Delta \log \det H_0^{-1} H_1 \leq |K_{H_1} - K_{H_0}|\]

for Hermitian metrics $H_0$ and $H_1$ with $\det H_0 = \det H_1$, and with

\[K_H := i\Lambda_H - \frac{\text{tr}(i\Lambda_H)}{\text{rk} E} \cdot \text{id}_E;\]

see [Siu87, p. 13] for a proof.

**Step 1.** We have $\log \text{tr} e^s \in W^{1,2}(B)$ and $\|\log \text{tr} e^s\|_{W^{1,2}(B)} \leq c$.

Choose $1 \leq i < j \leq n$ and define the projection $\pi : B \to \mathbb{C}^{n-2}$ by

\[\pi(z) := (z_1, \ldots, \hat{z_i}, \ldots, \hat{z_j}, \ldots, z_n).\]

For $\xi \in \mathbb{C}^{n-2}$, denote by $\nabla_\xi$ and $\Delta_\xi$ the derivative and the Laplacian on the slice $\pi^{-1}(\xi)$ respectively. Set $f_\xi := \log \text{tr} e^s|\pi^{-1}(\xi)$. Applying (3.2) to $H|\pi^{-1}(\xi)$ and $H_0|\pi^{-1}(\xi)$ we obtain

\[\Delta_\xi f_\xi \leq |F_H| + |F_{H_0}|.\]

Fix $\chi \in C^\infty(\mathbb{C}^2; [0, 1])$ such that $\chi(\eta) = 1$ for $|\eta| \leq 1/2$ and $\chi(\eta) = 0$ for $|\eta| \geq 1/\sqrt{2}$. For $0 < |\xi| \leq 1/\sqrt{2}$ and $\epsilon > 0$, we have

\[
\int_{\pi^{-1}(\xi)} |\nabla_\xi (\chi f_\xi)|^2 \leq \int_{\pi^{-1}(\xi)} \chi^2 f_\xi (|F_H| + |F_{H_0}|) + 1 \\
\quad \leq \epsilon \int_{\pi^{-1}(\xi)} |\chi f_\xi|^2 + e^{-1} \int_{\pi^{-1}(\xi)} |F_H|^2 + |F_{H_0}|^2 + 1.
\]

Using the Dirichlet–Poincaré inequality and rearranging, we obtain

\[
\int_{\pi^{-1}(\xi)} |\chi f_\xi|^2 + |\nabla_\xi (\chi f_\xi)|^2 \leq \int_{\pi^{-1}(\xi)} |F_H|^2 + |F_{H_0}|^2 + 1.
\]

Integrating over $0 < |\xi| \leq 1/\sqrt{2}$ yields

\[
\int_B |\log \text{tr} e^s|^2 + |\nabla' \log \text{tr} e^s|^2 \leq \int_B |F_H|^2 + |F_{H_0}|^2 + 1
\]

with $\nabla'$ denoting the derivative along the fibres of $\pi$. Using (2.3) and $n \geq 3$, $F_{H_0} \in L^2(B)$. Since the choice of $i, j$ defining $\pi$ was arbitrary, the asserted inequality follows.
Step 2. The differential inequality
\[ \Delta \log tr e^s \lesssim |K_{H_s}| \]
holds on \( B \) in the sense of distributions.

Fix a smooth function \( \chi : [0, \infty) \to [0, 1] \) which vanishes on \([0, 1] \) and is equal to one on \([2, \infty) \). Set \( \chi_{\varepsilon} := \chi(|\cdot|/\varepsilon) \). By (3.2), for \( \phi \in C^0_0(B) \), we have
\[
\int_B \Delta \phi \cdot \log tr e^s = \lim_{\varepsilon \to 0} \int_B \chi_{\varepsilon} \cdot \Delta \phi \cdot \log tr e^s \leq \int_B \phi \cdot |K_{H_s}| + \lim_{\varepsilon \to 0} \int_B \phi \cdot (\Delta \chi_{\varepsilon} \cdot \log tr e^s - 2(\nabla \chi_{\varepsilon}, \nabla \log tr e^s)) .
\]
Since \( n \geq 3 \), we have \( \|\chi_{\varepsilon}\|_{W_2^2(B)} \leq \varepsilon^2 \). Because \( \log tr e^s \in W_1^2(B) \) this shows that the limit vanishes.

Step 3. We have \( \log tr e^s \in L^\infty(B) \) and \( \|\log tr e^s\|_{L^\infty(B)} \leq c \).

Since \( \text{tr } s = 0 \), we have \(|s| \leq \text{rk}(E) \cdot \log tr e^s \); in particular, \( \log tr e^s \) is non-negative. By hypothesis \( K_{H_s} = 0 \). Since \( H_s \) is HYM with respect to \( \omega_0 \) and \(|F_{H_s}| \lesssim |z|^{-2} \) by hypothesis (2.3), we have \(|K_{H_s}| \leq c \). The asserted inequality thus follows from Step 2 via Moser iteration; see [GT01, Theorem 8.1].

4 A priori Morrey estimates

The following decay estimate is the crucial ingredient of the proof of Theorem 2.4.

Proposition 4.1. There is a constant \( \alpha > 0 \), such that for \( r \in [0, 1] \) we have
\[
\int_{B_r} |\nabla H_s|^2 \lesssim r^{2n-2+2\alpha} .
\]

The proof of this proposition relies on a Neumann–Poincaré type inequality, which we describe in what follows. Denote by \( \nabla_{T,r} \) the connection on \( isu(E, H_s)|_{\partial B_r} \) induced by \( \nabla_{H_s} \). The linear operator \( \nabla_{T,r} : \Gamma(\partial B_r, isu(E, H_s)) \to \Omega^1(\partial B_r, isu(E, H_s)) \) has a finite dimensional kernel. Since \( \nabla_{H_s} \) is conical, we can identify
\[
\ker \nabla_{T,r} = \ker \nabla_{T,1} := K.
\]
Moreover, we can regard \( K \) as a subset of constant sections: \( K \subset \Gamma(\hat{B}_r, isu(E, H_s)) \). Denote by \( \pi_r : \Gamma(\partial B_r, isu(E, H_s)) \to K \) the \( L^2 \)-orthogonal projection onto \( K \) and define \( \Pi_r : \Gamma(\hat{B}_{2r}, isu(E, H_s)) \to K \) by
\[
\Pi_r s := \frac{1}{r} \int_r^{2r} \pi_t(s|_{\partial B_t}) \, dt .
\]

\(^4K\) can be determined explicitly from the from the decomposition of \( \mathcal{F} \) into \( \mu \)-stable summands, but we will not need a precise description of \( K \).
**Proposition 4.2.** We have

\[
\int_{B_{2r}\setminus B_r} |s - \Pi_r s|^2 \lesssim r^2 \int_{B_{2r}\setminus B_r} |\nabla_H s|^2.
\]

**Proof.** The asserted estimate is scale-invariant; hence, we may assume \( r = 1/2 \). To prove the estimate in this case it suffices to prove the cylindrical estimate

\[
\int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \Pi s(t, \cdot)|^2 \, d\hat{x}dt \lesssim \int_{1/2}^1 \int_{\partial B} |\partial_t s(t, \hat{x})|^2 + |\nabla T s(t, \hat{x})|^2 \, d\hat{x}dt
\]

with \( s \) denoting a section over \([1/2, 1] \times \partial B\), \( \pi := \pi_1 \), \( \Pi s := 2 \int_{1/2}^1 \pi s(t, \cdot) \, dt \), and \( \nabla_T := \nabla_{T, 1} \).

We have

\[
\int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \Pi s(t, \cdot)|^2 \, d\hat{x}dt = 4 \int_{1/2}^1 \int_{\partial B} \left| \int_{1/2}^1 s(t, \hat{x}) - \pi s(u, \cdot) \, du \right|^2 d\hat{x}dt
\]

\[
\lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \pi s(u, \cdot)|^2 \, d\hat{x}dudt
\]

\[
\lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \pi s(t, \cdot)|^2 + |\pi s(t, \cdot) - \pi s(u, \cdot)|^2 \, d\hat{x}dudt.
\]

The first summand can be bounded as follows

\[
\int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |s(t, \hat{x}) - \pi s(t, \cdot)|^2 \, d\hat{x}dudt \lesssim \int_{1/2}^1 \int_{1/2}^1 \int_{\partial B} |\nabla T s(t, \hat{x})|^2 \, d\hat{x}dudt
\]

\[
\lesssim \int_{1/2}^1 \int_{\partial B} |\nabla_T s(t, \hat{x})|^2 \, d\hat{x}dt.
\]

The second summand can be controlled as in the usual proof of the Neumann–Poincaré inequality. We have

\[
|\pi s(t, \cdot) - \pi s(u, \cdot)| = \left| \int_0^1 \partial_v \pi s(t + v(t - u), \cdot) \, dv \right|
\]

\[
\lesssim \left| \int_0^1 \pi(\partial_v s)(t + v(t - u), \cdot) \, dv \right|
\]

\[
\lesssim \left( \int_0^1 \int_{\partial B} |(\partial_v s)(t + v(t - u), x)|^2 \, d\hat{x}dv \right)^{1/2}.
\]
Plugging this into the second summand and symmetry considerations yield

$$
\int_{1/2}^{1} \int_{1/2}^{1} \int_{\partial B} \int_{0}^{1} \frac{1}{2} \left| \Pi s(t, \cdot) - \Pi s(u, \cdot) \right|^2 d\hat{x} dudt
\leq \int_{1/2}^{1} \int_{1/2}^{1} \int_{0}^{1} \left| (\partial_t s)(t + \nu(t - u), \hat{x}) \right|^2 d\hat{x} dudt
\leq \int_{1/2}^{1} \int_{\partial B} \left| \partial_t s(t, \hat{x}) \right|^2 d\hat{x} dt.
$$

This finishes the proof. □

The proof of Proposition 4.1 also uses the following observation about

$$
\hat{s}_r := \log(e^{-\Pi_r s} e^s).
$$

By construction, the section \(\hat{s}_r\) is self-adjoint with respect to \(H e^s\) as well as \(H e^{\Pi_r s}\), and

$$
H e^s = (H e^{\Pi_r s}) e^{\hat{s}_r}.
$$

**Proposition 4.3.** The section \(\hat{s}_r\) satisfies

$$
|\nabla H e^s| \leq |\nabla H e^{\hat{s}_r}|, \quad |\hat{s}_r| \leq |s - \Pi_r s|, \quad \text{and} \quad |\nabla H e^{\hat{s}_r}|^2 \leq 1 - \Delta |\hat{s}_r|^2.
$$

**Proof.** The first two inequalities follow by elementary considerations.

Since \(s\) is bounded in \(L^\infty(B)\), \(\Pi_r s\) is uniformly bounded and, consequently, so is \(\hat{s}_r\). By [\text{JW}18, Proposition A.9], we have

$$
\Delta |\hat{s}_r|^2 + 2 |\nabla H e^{\Pi_r s} \hat{s}_r|^2 \leq |K_{H e^s}| + |K_{H e^{\Pi_r s}}|
$$

with

$$
u(-\hat{s}_r) = \sqrt{\frac{1 - e^{-\Phi_{\Pi_r s}}}{\Phi_{\Pi_r s}}} \in \text{End}(\mathfrak{g}(E)).
$$

\(H e^s\) is HYM; that is: \(K_{H e^s} = 0\) Since \(\Pi_r s\) is constant with respect to \(\nabla H e\), we have

$$
K_{H e^{\Pi_r s}} = i\Lambda(e^{\Pi_r s} \partial H e^{-\Pi_r s}) = \text{Ad}(e^{\Pi_r s}) K_{H e}.
$$

which is bounded. Moreover, \(\nabla H e\) and \(\nabla H e^{\Pi_r s}\) differ by a bounded algebraic operator. Given this, the third inequality follows using

$$
\sqrt{\frac{1 - e^{-x}}{x}} \geq \frac{1}{\sqrt{1 + |x|}},
$$

\(\|K_{H e}\|_{L^\infty} \leq c\), which is a consequence of (2.3), and the fact that \(H e\) is HYM with respect to \(\omega_0\), and the bound on \(|s|\) established in Proposition 3.1. □
Proof of Proposition 4.1. Given the above discussion, the proof is very similar to that of [JW18, Proposition C.2]. Nevertheless, for the reader’s convenience we provide the necessary details.

Define \( g : [0, 1/2] \to [0, \infty) \) by

\[
g(r) := \int_{B_r} |z|^{2-2n}|\nabla H \hat{s}|^2.
\]

We will show that

\[
g(r) \leq cr^{2\alpha},
\]

which implies the asserted inequality.

Step 1. We have \( g \leq c \).

Fix a smooth function \( \chi : [0, \infty) \to [0, 1] \) which is equal to one on \([0, 1] \) and vanishes outside \( [0, 2] \). Set \( \chi_r(\cdot) := \chi(\cdot/|r|) \). For \( r > \varepsilon > 0 \), using Proposition 4.3 and Proposition 3.1, and with \( G \) denoting Green’s function on \( B \) centered at \( 0 \), we have

\[
\hat{B}_r \setminus B_{r/2} \leq \int_{B_r \setminus B_{r/2}} \chi_r(1 - \chi_{r/2})G(1 - \Delta |\hat{s}|^2) \\
\leq \int_{B_r \setminus B_{r/2}} |\nabla H \hat{s}|^2 \\
\leq \int_{B_r \setminus B_{r/2}} |z|^{2-2n}|s - \Pi_r s|^2 + r^2 \int_{B_r \setminus B_{r/2}} |s - \Pi_r s|^2 \\
\leq c.
\]

Step 2. There are constants \( \gamma \in [0, 1) \) and \( A > 0 \) such that

\[
g(r) \leq \gamma g(2r) + Ar^2.
\]

Continuing the inequality from Step 1 using Proposition 4.2, we have

\[
\int_{B_r \setminus B_{r/2}} |z|^{2-2n}|\nabla H \hat{s}|^2 \leq \int_{B_r \setminus B_{r/2}} |z|^{2-2n}|\nabla H \hat{s}|^2 + r^2 \int_{B_r \setminus B_{r/2}} |\nabla H \hat{s}|^2 + \varepsilon^{2-2n} \int_{B_r \setminus B_{r/2}} |s - \Pi_r s|^2 \\
\leq g(2r) - g(r) + r^2 + g(\varepsilon).
\]

By Lebesgue’s monotone convergence theorem, the last term vanishes as \( \varepsilon \) tends to zero; hence, the asserted inequality follows with \( \gamma = \frac{c}{c+1} \) and \( A = c \).

Step 3. We have \( g \leq cr^{2\alpha} \) for some \( \alpha \in (0, 1) \).

This follows from Step 1 and Step 2 and as in [JW18, Step 3 in the proof of Proposition C.2]. \( \square \)
5 Proof of Theorem 2.4

For $r > 0$, define $m_r : C^n \to C^n$ by $m_r(z) := rz$. Set

$$s_r := m_r(s|_{B_1 \setminus B_{i/2}}) \in C^\infty(B_1 \setminus B_{i/2}, isu(E, H)) \quad \text{and} \quad H_{s,r} := m_r^* H_0.$$

The metric $H_{s,r} e^{s_r}$ is HYM with respect to $\omega_r := r^{-2} m_r^* \omega$ and $\|F_{H_{s,r}}\|_{C^k(B_1 \setminus B_{i/2})} \leq c_k$. Proposition 3.1, (2.3) and interior estimates for HYM metrics [W18, Theorem C.1] imply that

$$\|s_r\|_{C^k(B_1 \setminus B_{i/2})} \leq c_k.$$

By Proposition 4.1, we have

$$\|\nabla_{H_{s,r}} s_r\|_{L^2(B_1 \setminus B_{i/2})} \leq r^\alpha.$$

Schematically, $K_{H_{s,r}} e^{s_r} = 0$ can be written as

$$\nabla_{H_{s,r}}^* \nabla_{H_{s,r}} s_r + B(\nabla_{H_{s,r}} s \otimes \nabla_{H_{s,r}} s_r) = C(K_{H_{s,r}}),$$

where $B$ and $C$ are linear with coefficients depending on $s$, but not on its derivatives; see, e.g., [W18, Proposition A.1]. Since $\|K_{H_{s,r}}\|_{C^k(B_1 \setminus B_{i/2})} \leq c_k r^2$, as in [W18, Step 3 in the proof of Proposition 5.1], standard interior estimates imply that

$$\|\nabla_{H_{s,r}}^k s_r\|_{L^2(B_1 \setminus B_i)} \leq c_k r^\alpha$$

and, hence, the asserted inequalities, for each $k \geq 1$. (The asserted inequality for $k = 0$ has already been proven in Proposition 3.1.)

6 Proof of Proposition 1.4

We will make use of the following general fact about connections over manifolds with free $S^1$–actions.

**Proposition 6.1.** Let $M$ be a manifold with a free $S^1$–action. Denote the associated Killing field by $\xi \in \text{Vect}(M)$ and let $q : M \to M/\mathbb{S}^1$ be the canonical projection. Suppose $\theta \in \Omega^1(M)$ is such that $\theta(\xi) = 1$ and $\mathcal{L}_\xi \theta = 0$. Let $A$ be a unitary connection on a Hermitian vector bundle $(E, H)$ over $M$. If $i(\xi) F_A = 0$, then there is a $k \in \mathbb{N}$ and, for each $j \in \{1, \ldots, k\}$, a Hermitian vector bundles $(F_j, K_j)$ over $M/\mathbb{S}^1$ such that

$$E = \bigoplus_{j=1}^k E_j \quad \text{and} \quad H = \bigoplus_{j=1}^k H_j,$$

with $E_j := q^* F_j$ and $H_j := q^* K_j$; moreover, the bundles $E_j$ are parallel and, for each $j \in \{1, \ldots, k\}$, there are a unitary connection $B_j$ on $F_j$ and $\mu_j \in \mathbb{R}$ such that

$$A = \bigoplus_{j=1}^k q^* B_j + i \mu_j \text{id}_{E_j} \cdot \theta.$$
Proof. Denote by $\xi \in \text{Vect}(U(E))$ the $A$–horizontal lift of $\xi$. This vector field integrates to an $R$–action on $U(E)$. Thinking of $A$ as an $\mathfrak{u}(r)$–valued 1–form on $U(E)$ and $F_A$ as an $\mathfrak{u}(r)$–valued 2–form on $U(E)$, we have
\[ \mathcal{L}_\xi A = i(\xi)F_A = 0; \]
hence, $A$ is invariant with respect to the $R$–action on $U(E)$.

The obstruction to the $R$–action on $U(E)$ inducing an $S^1$–action is the action of $1 \in R$ and corresponds to a gauge transformation $g_A \in \mathcal{G}(U(E))$ fixing $A$. If this obstruction vanishes, i.e., $g_A = \text{id}_{U(E)}$, then $E \cong q^*F$ with $F = E/S^1$ and there is a connection $A_0$ on $F$ such that $A = q^*A_0$.

If the obstruction does not vanish, we can decompose $E$ into pairwise orthogonal parallel subbundles $E_j$ such that $g_A$ acts on $E_j$ as multiplication with $e^{i\mu_j}$ for some $\mu_j \in R$. Set $A := A - \bigoplus_{j=1}^k i\mu_j \text{id}_{E_j} \cdot \theta$. This connection also satisfies $i(\xi)F_A = 0 \in \Omega^1(M, g_E)$ and the subbundles $E_j$ are also parallel with respect to $E_j$. Since $g_A = \text{id}_E$, the assertion follows. \qed

In the situation of Proposition 1.4, with $\xi \in \mathfrak{o}(2n-1)$ denoting the Killing field for the $S^1$–action we have $i(\xi)F_{A_0} = 0$; c.f., Tian [Tiao, discussion after Conjecture 2]. Therefore, we can write
\[ A_* = \bigoplus_{j=1}^k \sigma^*B_j + i\mu_j \text{id}_{E_j} \cdot \pi^*\theta. \]

Since $d\theta = 2\pi \rho^*\omega_{FS}$, we have
\[ F_{A_*} = \bigoplus_{j=1}^k \sigma^*g_{B_j} + 2\pi i\mu_j \text{id}_{E_j} \cdot \sigma^*\omega_{FS}. \]

Using (2.1), $A_*$ being HYM with respect to $\omega_0$ can be seen to be equivalent to
\[ F_{B_j}^{\mathfrak{o},2} = 0 \quad \text{and} \quad i\Delta g_{B_j} = (2n-2)\pi \mu_j \cdot \text{id}_{E_j}. \]

The isomorphism $\mathcal{E} = (E, \bar{\partial}_{A_*}) \cong \bigoplus_{j=1}^k \rho^*\mathcal{F}_j$ with $\mathcal{F}_j = (F_j, \bar{\partial}_{B_j})$ is given by $g^{-1}$ with $g := \bigoplus_{j=1}^k \rho_{B_j}$. \qed

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