Symmetric Affine Theories and Nonlinear Einstein-Proca System

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May 27, 2021

Abstract
We review the correspondence between symmetric affine theories and the nonlinear Einstein-Proca system that was found by Einstein and Schrödinger. With the use of this correspondence, we investigate static spherically symmetric solutions in symmetric affine theory with no matter fields. Use of the correspondence leads to a significant simplification of the calculation. A special instance of “no-hair” theorem for symmetric affine theory is established.

We use reduced Planck units in which \( c = 1, \hbar = 1, 8\pi G = 1, \epsilon_0 = 1 \).

1 Introduction
The usual Einstein-Hilbert formulation of General Relativity has several disadvantages. The Hilbert Lagrangian contains second derivatives of \( g_{ab} \), yet it gives second order field equations (as opposed to 4-th order that one would expect). This is because of rather exceptional cancellations of higher order terms. It has been shown \([1, 2]\) that symmetric affine theories provide a natural remedy, that considerably clarifies the canonical structure of General Relativity. Symmetric affine theories introduce a nonmetric connection, which is desirable as recent developments suggest that Riemannian description is not valid on all scales \([6]\).

We will review symmetric affine theory and show its equivalence to the nonlinear Einstein-Proca system. Although the symmetric affine formalism is presented as more fundamental, certain calculations may be easier in the corresponding Einstein-Proca system. Exploiting the equivalence, we show a version of the “no-hair” theorem for symmetric affine theory in spherically symmetric spacetimes.

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2 Symmetric Affine Theories

Symmetric affine theories assume that Lagrangian density \( \mathcal{L} \) is a function of the connection \( \Gamma_{a}^{b} \) and its derivatives. It is further assumed that the connection is symmetric in its lower indices. This approach introduces more freedom as there are 40 independent components of connection, as opposed to 10 independent components of the metric.

It is reasonable to assume that the Lagrangian depends on the connection only through the Riemann tensor \( R_{abcd} \), or restrict to an even smaller class of Lagrangians that depend only on the Ricci tensor \( R_{ab} = R_{eaeb} \). For a general symmetric connection \( R_{ab} \) is not necessarily symmetric, but the first Bianchi identity \( R_{a[bcd]} = 0 \) implies \( R_{eab} = 2R_{[ab]} \) and so the only independent contraction of \( R_{abcd} \) is \( R_{ab} = R_{eaeb} \). (This gives a partial justification for why we consider only Lagrangians that depend only on \( R_{ab} \).) We write \( R_{ab} = S_{ab} + F_{ab} \), where \( S_{ab} \) is symmetric and \( F_{ab} \) is antisymmetric. The second Bianchi identity \( R_{ab[cde]} = 0 \) then implies \( F_{[abc]} = 0 \), i.e. \( F = dA \).

Symmetric affine theories consider the connection as a basic concept. The metric density is a derived concept defined as

\[
\mathcal{g}^{ab} = 2 \frac{\partial \mathcal{L}}{\partial S_{ab}} = \frac{\partial \mathcal{L}}{\partial R_{ab}} + \frac{\partial \mathcal{L}}{\partial R_{ba}}. \tag{1}
\]

We define also

\[
\mathcal{g}^{ab} = -2 \frac{\partial \mathcal{L}}{\partial F_{ab}} = -\frac{\partial \mathcal{L}}{\partial R_{ab}} + \frac{\partial \mathcal{L}}{\partial R_{ba}}, \tag{2}
\]

and current

\[
\mathcal{O}^{a} = -\mathcal{g}^{ab} ; b = -\mathcal{g}^{ab} , b. \tag{3}
\]

Clearly \( \mathcal{O}^{a} , a = 0 \), i.e. the current is conserved.

Variation of the action \( S = \int \mathcal{L}(R_{ab}) d^{n}x \) and the use of Palatini lemma leads to the field equations

\[
2\mathcal{g}^{ab} ; c - (\mathcal{O}^{a} + \mathcal{g}^{ad} ; d) \delta^{b}_{c} - (\mathcal{O}^{b} + \mathcal{g}^{bd} ; d) \delta^{a}_{c} = 0. \tag{4}
\]

This is equivalent to

\[
\mathcal{g}^{ab} ; c + \frac{1}{n-1} \mathcal{O}^{a} \delta^{b}_{c} + \frac{1}{n-1} \mathcal{O}^{b} \delta^{a}_{c} = 0, \tag{5}
\]

where semicolon denotes covariant derivative with connection \( \Gamma_{a}^{b} \).

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\(^{1}\)Using coordinates in which \( \Gamma_{a}^{b} \) is zero, it can be easily seen that \( R_{abcd} \) is the only independent tensor that depends on connection and its first derivatives, and is linear in the first derivatives.
Given the metric, there is a unique symmetric metric-compatible connection \( \{_{a}^{b}c \} \) (Levi-Civita connection). It is given by

\[
\{_{a}^{b}c \} = \frac{1}{2} g^{bc} (g_{ac,c} + g_{ec,a} - g_{ac,e}),
\]

where \( g_{ab} \) is the dedensitized metric and \( g^{ab} \) is the inverse metric.

It follows from the definition of covariant derivative and metric compatibility of \( \{_{a}^{b}c \} \), that the nonmetricity tensor \( Z_{a}^{b}c = \Gamma_{a}^{b}c - \{_{a}^{b}c \} \) satisfies

\[
\frac{1}{n-1} g^{bc} (g_{ae,c} + g_{ec,a} - g_{ac,e}) = 0,
\]

from which we obtain by contraction \( Z_{e}^{a}c = \frac{2}{n-1(n-2)} O_{a} \), where \( g_{ab} \) is used to lower indices.

The system of equation (8) can be solved for \( Z_{a}^{b}c \). This gives

\[
\Gamma_{a}^{b} = \{_{a}^{b}c \} - \frac{1}{n-2} g_{ac} O^{c} + \frac{1}{(n-1)(n-2)} (\delta_{b}^{c} O_{a} + \delta_{a}^{c} O_{b}),
\]

and its contraction

\[
\Gamma_{a}^{b} = (\ln \sqrt{g})_{a} + \frac{2}{(n-1)(n-2)} O_{a}.
\]

Working in a frame in which \( \{_{a}^{b}c \} = 0 \) and using \( O_{a} = O_{b} = 0 \), one can easily find an expression for \( R^{M}_{ab} \), the Ricci tensor arising from metric \( g_{ab} \)

\[
R_{ab} = R^{M}_{ab} - \frac{1}{(n-1)(n-2)} O_{a} O_{b} + \frac{1}{(n-1)(n-2)} (O_{b,a} - O_{a,b}).
\]

Comparing to \( R_{ab} = S_{ab} + F_{ab} \), we obtain

\[
S_{ab} = R^{M}_{ab} - \frac{1}{(n-1)(n-2)} O_{a} O_{b},
\]

\[
F_{ab} = \frac{1}{(n-1)(n-2)} (O_{b,a} - O_{a,b}).
\]

The interpretation of these equations is that the presence of matter fields gives rise to nonmetricity of the connection and nonmetricity of the Ricci tensor, according to equations (9), (11) and (12).
3 Equivalence of Symmetric Affine Theory and Nonlinear Einstein-Proca Theory

There is a correspondence between symmetric affine theories and nonlinear Einstein-Proca theories.

To see this we perform a Legendre transform of the original Lagrangian
\[ \bar{L}(g^{ab}, F_{ab}) = L(S_{ab}, F_{ab}) - \frac{1}{2} g^{ab} S_{ab}. \] (13)
\[ S_{ab} \]

is eliminated in favour of \( g^{ab} \) using the definition \( g^{ab} = 2 \frac{\partial \bar{L}}{\partial S_{ab}}(S_{ab}, F_{ab}). \)

Standard properties of the Legendre transform give
\[ \frac{\partial \bar{L}}{\partial F_{ab}} = \frac{\partial L}{\partial F_{ab}}, \] (14)
\[ \frac{\partial \bar{L}}{\partial g^{ab}} = -\frac{1}{2} S_{ab}. \] (15)

We have \( \mathcal{G}^{ab} = -2 \frac{\partial \bar{L}}{\partial g^{ab}} \) and thus we can identify \( \mathcal{G}^{ab} \) with the Ampere tensor. We introduce the rescaled current \( A^{a} = \frac{1}{(n-1)(n-2)} O^{a} \). Equations (11), (12) can then be rewritten as
\[ R^{M}_{\quad ab} = -2 \frac{\partial \bar{L}}{\partial g^{ab}} + (n-1)(n-2) A^{a} A_{b}, \] (16)
\[ G^{ab}_{\quad ;b} = -(n-1)(n-2) A^{a}, \] (17)
\[ F_{ab} = A_{b,a} - A_{a,b}. \] (18)

Equations (16), (17), are precisely the nonlinear Einstein-Proca equations, which can be derived from the Lagrangian
\[ L_{EP} = \frac{1}{2} R^{M} + \bar{L} - \frac{m^{2}_{V}}{2} A^{a} A_{a}, \] (19)

where the mass of the vector boson is \( m^{2}_{V} = (n-1)(n-2) \).

Note that if \( n > 2 \), the mass term always gives a negative contribution to the Raychaudhuri equation. Thus the mass term always has an attractive effect. From the Einstein-Proca Lagrangian we see that \( -\bar{L}(g^{ab}, 0) \) plays the role of the cosmological constant density.

Conversely, given a Lagrangian \( \bar{L} \) of Einstein-Proca theory, we can perform an inverse Legendre transform
\[ \bar{L}(S_{ab}, F_{ab}) = \bar{L}(g^{ab}, F_{ab}) + \frac{1}{2} g^{ab} S_{ab}, \] (20)
to obtain the Lagrangian of the corresponding symmetric affine theory.
We require that in the weak field limit \((A_a \rightarrow 0, F_{ab} \rightarrow 0)\), the Ampere tensor and the Faraday tensor are approximately equal, i.e. \(G_{ab} \approx F_{ab}\). This requirement guarantees that the theory gives Maxwell equations with a Proca term as the weak field limit. This imposes restrictions on possible Lagrangians and it fixes overall normalization of the Lagrangian density.

In the weak field limit, equation (17) can be rewritten as
\[
A_b^{\;;a;b} - \Box A_a = -m_V^2 A_a,
\]
which gives
\[
-\Box A_a + m_V^2 A_a - R_{ca} A^c = 0.
\] (21)
This is the expected equation governing a vector field of mass \(m_V^2\).

In general it is quite difficult to perform the Legendre transform explicitly. As an example, we consider the Lagrangian suggested by Eddington
\[
\mathcal{L} = -\sqrt{|\det R_{ab}|}.
\] (22)
(The sign and normalization are determined by the requirement on weak field limit mentioned above.)

For this Lagrangian we obtain metric density
\[
\gamma^{ab} = -\sqrt{|\det R_{ab}|} R^{(ab)},
\] (23)
and Ampere tensor
\[
\mathfrak{G}^{ab} = \sqrt{|\det R_{ab}|} R^{[ab]},
\] (24)
where \(R^{[ab]}\) is defined as the transposed inverse of \(R_{ab}\).

Although it does not seem to be possible to obtain an explicit formula for \(\mathcal{L}\) in terms of standard functions, we can eliminate \(R_{ab}\) to get the relation between the Faraday and the Ampere tensor (constitutive equation)
\[
F_{ad}(\delta_{d}^{\;e} + G_{d}^{\;e} G^{e} b) = G_{ab} \left[|\det(\delta_{q}^{\;p} + G_{q}^{\;p})|\right]^{1/2}.
\] (25)

In the case of \(\mathcal{L} = -\sqrt{|\det R_{ab}|}\), we get for the cosmological constant
\[
\Lambda = -\frac{n - 2}{2}.
\] (26)
4 Symmetric Affine Theory and Scalar Field

In the framework of symmetric affine theories, we can easily accommodate bosonic matter fields. We assume that the Lagrangian density depends on the connection and also on matter fields and their derivatives. By analogous derivation to that above, it can be shown that symmetric affine theory with matter fields is equivalent to Einstein-Proca theory with additional matter fields. In general however, the matter fields are not minimally coupled.

An interesting Lagrangian depending on the connection and scalar field was considered by Kijowski [1, 2]

\[ \mathcal{L} = \frac{1}{V(\varphi)} \sqrt{\left| \det S_{ab} - \varphi,^a \varphi,^b \right|}. \]  

(27)

We assume for simplicity that there is only one scalar field, but all results carry over straightforwardly to the case of \( N \) scalar fields. The Lagrangian is independent of \( F_{ab} \), which results in vector field \( A_a \) being identically zero.

Performing the Legendre transform of \( \mathcal{L} \) as before, we arrive at the Lagrangian

\[ \mathcal{\bar{L}}(g_{ab}, \varphi, \varphi,^a) = -\sqrt{g} \left( \frac{1}{2} g^{ab} \varphi,^a \varphi,^b + V(\varphi) \right). \]  

(28)

And the field equations can be rewritten as

\[ R^M_{ab} = -2 \frac{\partial \mathcal{\bar{L}}}{\partial g^{ab}}, \]  

(29)

\[ g^{ab} \varphi,^a \varphi,^b - V'(\varphi) = 0. \]  

(30)

Equations (29) and (30) are precisely the nonlinear Einstein-Klein-Gordon equations. In this case the scalar field turns out to be minimally coupled. A similar approach leads to Lagrangians for other matter fields (minimally coupled in the absence of \( F_{ab} \)). The Lagrangians that depend on \( \Gamma_{a}^{bc} \) only through \( R_{(ab)} \) generally result in \( \Gamma_{a}^{bc} \) being metric connection. If on the other hand the Lagrangian depends on \( \Gamma_{a}^{bc} \) in some other way than through \( R_{(ab)} \), the presence of matter field gives rise to the nonmetricity of \( \Gamma_{a}^{bc} \).

Interestingly, the Lagrangian (27) naturally arises as a brane action with scalar fields representing transverse displacement of the brane.
5 Spherically Symmetric Solutions

We use the equivalence of symmetric affine theories and the nonlinear Einstein-Proca system to investigate static spherically symmetric solutions in symmetric affine theories.

We assume that the metric and fields are independent of time and invariant under time reversal. As shown above, the equation of motion can be derived from the Lagrangian

$$\frac{1}{2} R - \frac{\bar{L}}{2} - \frac{m^2}{2} A^a A_a. \quad (31)$$

$\bar{L}$ in general depends on two invariants $F^{ab}F_{ab}$ and $*F^{ab}F_{ab}$ of the electromagnetic field. Since $A_a$ is a vector, the first invariant is a scalar, whereas the second is a pseudoscalar. The time reversal invariance then requires $*F^{ab}F_{ab} = 0$.

This allows us to restrict to the Lagrangian of the form

$$\frac{1}{2} R - \frac{1}{4} f(F^2) - \frac{m^2}{2} A^a A_a, \quad (32)$$

where $-\frac{1}{4} f(F^2)$ represents a nonlinear term which is assumed to depend only on $F^2 = F_{ab}F^{ab}$. Note that $\frac{1}{4} f(0)$ can be interpreted as the cosmological constant. Since symmetric affine theory gives $m^2 = (n - 1)(n - 2)$, we consider only the case $m^2 > 0$.

Spherical symmetry enables us to put the metric in the form

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (33)$$

The vector potential can be put in the form

$$A_a = (u(r), v(r), 0, 0). \quad (34)$$

The nonlinear Einstein-Proca equations now read

$$R_{ab} - \frac{1}{2} R g_{ab} = T_{ab}, \quad (35)$$

$$G^{ab}_{;b} + m^2 A^a = 0, \quad (36)$$

where

$$G_{ab} = f'(F^2)F_{ab}, \quad (37)$$

$$F_{ab} = A_{b,a} - A_{a,b}, \quad (38)$$

$$T_{ab} = f'(F^2)F_{ad}F_b^d - \frac{1}{4} f(F^2) g_{ab} + m^2 (A_a A_b - \frac{1}{2} A^2 g_{ab}). \quad (39)$$

\[2\text{We restrict to metric of signature } (- + ++). \text{ Result for general signature can be deduced by analytic continuation.}\]
The 01 component of the Einstein equation reads \( m_V^2 u v = 0 \). If \( u = 0 \) leads to \( F^2 = 0 \) and so \( v = 0 \). Therefore it suffices to consider only the case \( v = 0 \).

Then the Einstein equations and the field equations are equivalent to

\[
\left[ r^2 e^{-(\alpha + \beta)w} \right]' = m_V^2 r^2 u e^{\beta - \alpha} \quad \text{where} \quad w = f'(F^2)u',
\]

\[
\alpha' + \beta' = \frac{1}{2} m_V^2 u^2 re^{2(\beta - \alpha)},
\]

where \( F^2 = -2u'^2 e^{-2(\alpha + \beta)} \) (\( \prime \) denotes derivative with respect to \( r \)).

Equation (40) gives

\[
\left[ r^2 e^{-(\alpha + \beta)wu} \right]' = r^2 e^{\alpha + \beta} \left( -\frac{1}{4} F^2 f'(F^2) + \frac{1}{2} m_V^2 u^2 e^{-2\alpha} \right).
\]

We will now assume that the matter fields satisfy the weak energy condition. This imposes constraints on possible functions \( f(F^2) \). The weak energy condition requires that \( T_{ab} U^a U^b \geq 0 \) for all timelike and null vectors \( U^a \). This requires \( T^0_0 \leq 0, T^0_0 - T^i_i \leq 0 \) for \( i = 1, 2, 3 \). In the case we are considering, this translates to

\[
\frac{1}{4} f(F^2) - \frac{1}{2} F^2 f'(F^2) + \frac{1}{2} m_V^2 u^2 e^{-2\alpha} \geq 0,
\]

\[
-\frac{1}{2} F^2 f'(F^2) + m_V^2 u^2 e^{-2\alpha} \geq 0.
\]

Thus the weak energy condition implies that \( r^2 e^{-(\alpha + \beta)wu} \) is increasing with \( r \), and \( M(r) = (1 - e^{-2\beta})r \) is increasing with \( r \).

Suppose spacetime has an event horizon. By spherical symmetry, the horizon is a surface \( r = \text{const} \) given by \( \beta \to \infty \). The event horizon does not present any natural barrier and a freely falling observer would not be able to tell (based on local observations) that he has passed the horizon. Thus the physically observable quantities are bounded at the horizon.

\( g = r^2 e^{\alpha + \beta} \sin \theta \) is bounded at the horizon, which implies that \( \alpha \to -\infty \) at the horizon. \( -R = T^a_a = F^2 f'(F^2) - f(F^2) - m_V^2 u^2 e^{-2\alpha} \) and \( F^2 \) are bounded at the horizon. Thus we conclude that \( u = 0 \) at the horizon.
Suppose spacetime has two horizons. As $r^2 e^{-(\alpha + \beta)wu}$ is increasing by the weak energy condition and $u = 0$ at the horizons, this means that $r^2 e^{-(\alpha + \beta)wu} = 0$ between two horizons. Finally $r^2 e^{-(\alpha + \beta)wu} = 0$ implies $u = 0$. ($w = 0$ leads to $u = \text{const}$, but $u = 0$ at horizons.)

Suppose that spacetime has one horizon and we impose the following decay at infinity: We assume that $u \to 0$, $u' \to 0$, $r^2uu' \to 0$ as $r \to \infty$. Then we have that $r^2 e^{-(\alpha + \beta)wu} \to 0$ as $r \to \infty$, ($\alpha + \beta$ is increasing and $f(F^2)$ is bounded). As $r^2 e^{-(\alpha + \beta)wu}$ is an increasing function that vanishes on the horizon and at infinity, $r^2 e^{-(\alpha + \beta)wu} = 0$ and hence $u = 0$ between the horizon and infinity.

If there is no horizon and no singularity at $r = 0$ we have $r^2 e^{-(\alpha + \beta)wu} = 0$ at $r = 0$, and as before we may deduce that $r^2 e^{-(\alpha + \beta)wu} = 0$ and hence $u = 0$ between $r = 0$ and $\infty$.

We could imagine a solution with several horizons at $r_1 < r_2 \ldots < r_k$. If $r_i$, $r_{i+1}$ are two horizons such that $t$ is timelike and $r$ is spacelike between the horizons, then the preceding analysis shows that $u = 0$ between $r_i$ and $r_{i+1}$.

The analysis above can be extended even for the regions in which $r$ is timelike and $t$ is spacelike and we may conclude that $u = 0$ in $[r_1, \infty)$.

6 Conclusions

Symmetric affine formulation of General Relativity suggests that the connection is the fundamental concept whereas metric is only a derived notion. Presence of matter gives rise to nonintegrability and generally even nonmetricity of the connection.

Symmetric affine theory with no additional matter fields is equivalent to Einstein gravity with nonlinear electrodynamics including the Proca term. The mass of the vector boson in the Proca term is $m_V^2 = (n - 1)(n - 2)$ in reduced Planck units. This equivalence can be exploited to show that if the weak energy condition is imposed, there are no nontrivial (meaning including nonzero $F_{ab}$) static spherically symmetric solutions with horizons and there are no nontrivial static spherically symmetric solutions without a singularity. With additional constraints on $f(F^2)$ (for example $f'(F^2) > 0$) it could be shown that there are no static solutions with horizons even in general nonspherical case. This is a particular instance of the “no-hair” theorem in the framework of symmetric affine theories.

Symmetric affine theory can naturally include bosonic fields. Symmetric affine theory with scalar fields gives a natural action for branes.
Acknowledgements

This work has been done under supervision of Prof. G. W. Gibbons. I would like to thank him for his valuable explanations and very useful insights. I would also like to acknowledge Trinity College Heilbronn Fund from which I was funded under Trinity College Summer Research Scheme.

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