Dynamical precession of spin in the two-dimensional spin-orbit coupled systems

Tsung-Wei Chen, Zhi-Yang Huang, and Dah-Wei Chiou

1Department of Physics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan
2Center for Condensed Matter Sciences, National Taiwan University, Taipei 10617, Taiwan

(Dated: December 3, 2018)

We investigate the spin dynamics in the two-dimensional spin-orbit coupled system subject to an in-plane (x-y plane) constant electric field, which is assumed to be turned on at the moment \( t = 0 \). The equation of spin precession in linear response to the switch-on of the electric field is derived in terms of Heisenberg’s equation by the perturbation method up to the first order of the electric field. The dissipative effect, which is responsible for bringing the dynamical response to an asymptotic result, is phenomenologically implemented à la the Landau-Lifshitz-Gilbert equation by introducing damping terms upon the equation of spin dynamics. Mediated by the dissipative effect, the resulting spin dynamics asymptotes to a stationary state, where the spin and the momentum-dependent effective magnetic field are aligned again and have nonzero components in the out-of-plane (z) direction. In the linear response regime, the asymptotic response obtained by the dynamical treatment is in full agreement with the stationary response as calculated in the Kubo formula, which is a time-independent approach treating the applied electric field as completely time-independent. Our method provides a new perspective on the connection between the dynamical and stationary responses.

PACS numbers: 71.70.Ej, 72.25.Dc, 73.43.Cd, 75.47.-m

I. INTRODUCTION

The phenomenon of the spin-Hall effect is the appearance of lateral bulk spin current in the spin-orbit coupled systems driven solely by applying an electric field [1, 2]. The fact that the spin-Hall current, arising from the separation of opposite spin orientations without breaking the time-reversal symmetry, is dissipationless, has enormous advantages in the development of spintronics [3]. A lot of attention has been devoted to investigating the theoretical foundations [4, 5] of the spin-Hall effect and performing the experiments [6] that test the validity of theory [7] and advance the technology of spintronics.

In two-dimensional (2D) spin-orbit coupled systems [8–11], the spin-Hall effect becomes very important, not only for its relation to the topological Berry phase [12–15] but also for the development of the quantum spin-Hall effect [16–18] (2D topological insulators) and Chern insulators [19], where the definition of the bulk spin current [20, 21] plays the key role in the bulk-edge correspondence [10]. Recently, it was shown that the spin-Hall effect in the two-dimensional Weyl fermion system is caused by the spin torque current [22]. The phenomenon of the bulk spin current is usually understood as the stationary response of the system to the applied in-plane electric field, which, as well known, can be directly calculated by the Kubo formula. However, the dynamical origin of this response remains rather mysterious. That is, if the applied electric field is switched on at the moment \( t = 0 \), how does the (expectation value of) spin dynamically evolve from its original in-plane direction to yield an out-of-plane component and eventually asymptote to the stationary value?

The connection between the dynamical evolution and the stationary response is essential to understanding the underlying mechanism of the spin-Hall effect. This connection has been addressed for the Berry curvature induced spin dynamics in the 3D p-type semiconductor [4], the 2D k-linear Rashba system [5], the semiclassical Drude model [24], and the stationary response of the kinetic equation [25]. Recently, the spin dynamics in the honeycomb lattice [26] has also been investigated using the Landau-Lifshitz-Gilbert equation [27], by which the classical and quantum correspondence appears at low-energy spectra. Besides theoretical importance, understanding the dynamical evolution of spin is also crucial to the performance of spintronic devices, which largely depends on their response time to the applied field. In this work, we focus on the two-dimensional spin-orbit coupled system subject to a constant in-plane electric field switched on at \( t = 0 \).

For the two-dimensional spin-orbit coupled systems, the Hamiltonian before the electric field is turned on is in general written as
\[
H_0 = \epsilon_k + \sigma_x d_x(k) + \sigma_y d_y(k),
\]
where \( \epsilon_k \) is the kinetic energy, \( d \) is referred to as the effective magnetic field, and \( \sigma_i \) are Pauli matrices representing the real electron spin in the system. It follows from the algebra \( [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \) that the equation of motion of spin (i.e., spin precession) is described by the Larmor precession around the direction of \( d \). As the quantum average of spin is parallel to the effective magnetic field, its component in the out-of-plane (z) direction remains zero if the effective magnetic field is in the in-plane (x-y) direction. Therefore, in order to have a nonzero stationary response of z component of spin, the effective magnetic field has to be tilted from the in-plane direction by some...
mechanism.

It turns out that applying an in-plane electric field provides such a mechanism. The quantum average of spin is given by the stationary response to the applied electric field \( \mathbf{E} \), which exhibits a nonzero component in the \( z \) direction. That is, in the presence of an in-plane electric field, the total Hamiltonian \( H = H_0 + e \mathbf{E} \cdot \mathbf{x} \), when evaluated as expectation values with respect to the eigenstates of \( H_0 \), can be rendered into the form \( \sigma \cdot \mathbf{D}(k) \), where \( \mathbf{D}(k) \) represents the new effective magnetic field and has a nonzero \( z \) component. However, the dynamical origin of the spin-\( z \) component — especially, the question how the electric field tilts the effective magnetic field from the in-plane direction — remains obscure.

To address the dynamical issues, we treat the Hamiltonian as time-dependent by assuming that the electric field is turned off for \( t < 0 \) and turned on for \( t \geq 0 \). Heisenberg’s equation is then used to solve the dynamical evolution perturbatively up to the first order of the electric field. In this dynamical picture, the effective magnetic field becomes nonstatic due to the switch-on of the electric field and exhibits a time-dependent component on the \( x-y \) plane. The time-varying effective magnetic field is no longer aligned with the spin for \( t > 0 \) and therefore drives the spin to precess around it.

The dynamical evolution is expected to asymptote to a stationary result, where the spin and the effective magnetic field are aligned again. To obtain the asymptotic behavior, we have to take into account the dissipative process that attenuates and eventually ceases the spin precession. The fundamental mechanism for the dissipation remains unclear and complicated, but it can be phenomenologically implemented \( \text{à la} \) the Landau-Lifshitz-Gilbert equation by introducing damping terms upon the equation of spin dynamics obtained from the first-order perturbation. Via the dissipative process, the precession of spin gives rise to a backreaction that alters the effective magnetic field and tilts it from the \( x-y \) plane. The resulting dynamical evolution asymptotically approaches a stationary state, where the spin and the effective magnetic field are aligned again and both have nonzero components in the \( z \) direction.

Meanwhile, we also directly compute the stationary response in a time-independent approach where the electric field is treated as always turned on. The linear term of the stationary response is exactly equal to the asymptotic result obtained in the dynamical analysis. Our dynamical treatment not only reveals the dynamical origin of the spin-\( z \) component in terms of the dynamical response to the switch-on of the electric field but also establishes its connection to the stationary response. In particular, we uncover that the dissipative effect is crucial for connecting the dynamical and stationary responses.

This paper is organized as follows. In Sec. III, the spin dynamics for spin-orbit coupled systems subject to an constant electric field turned on at \( t = 0 \) is derived in terms of Heisenberg’s equation up to the first order of the electric field. In Sec. IV, the equation of spin dynamics is explicitly solved for a two-dimensional system. By phenomenologically implementing the dissipative effect, the spin dynamics is shown to approach an asymptotic result. In Sec. V we use the time-independent method to directly calculate the stationary response of spin. The linear response of the spin-\( z \) component is exactly the same as the asymptotic result obtained from the spin dynamics. The spin-Hall current in relation to spin-\( z \) component is also discussed in this section. Finally, our conclusion is summarized and discussed in Sec. V

II. EQUATION OF SPIN DYNAMICS IN 3D SYSTEMS

In the presence of a constant and in-plane electric field \( \mathbf{E} \), the full Hamiltonian is given by

\[
H = H_0 + e \mathbf{E} \cdot \mathbf{x},
\]

where \( H_0 \) is the unperturbed Hamiltonian, and in general can be written as the form

\[
H_0 = \epsilon_k + \sigma_x d_x(k) + \sigma_y d_y(k) + \sigma_z d_z(k),
\]

where \( \sigma_x, \sigma_y \) and \( \sigma_z \) are Pauli matrices and represents real electron spin. The Hamiltonian \( H_0 \) is time-reversal symmetric as it is invariant under the transformation of \( \sigma_i \rightarrow -\sigma_i, \ p_i \rightarrow -p_i \), and \( k \rightarrow -k \). To study the spin dynamics, we treat the full Hamiltonian \( H \) as time-dependent by assuming that \( \mathbf{E} \) is switched off for \( t < 0 \) and switched on for \( t \geq 0 \). More precisely, Eq. (1) is modified as

\[
H = H_0 + e \mathbf{E} \cdot \mathbf{x} \theta(t),
\]

where \( \theta(t) \) is a step function.

The corresponding equation of motion of momentum for \( t \geq 0 \) is given by

\[
\hbar \frac{d \mathbf{k}_t}{dt} = -e \mathbf{E}.
\]

The solution of Eq. (4) is \( \mathbf{k}_t = \mathbf{k} - e \mathbf{E} t / \hbar \), where \( \mathbf{k} \) is defined as the momentum at \( t = 0 \). Because the dynamics of momentum \( \mathbf{k}_t = \mathbf{k} - e \mathbf{E} t / \hbar \) shows that time and electric field couples in the form \( \mathbf{E} t \), this implies that the linear response is valid also for a very short time. The Heisenberg picture of an observable \( \mathcal{O} \) is defined as

\[
\mathcal{O}^H(t) = e^{iHt/\hbar} \mathcal{O} e^{-iHt/\hbar}.
\]

The dynamics of spin can always be cast in the following Heisenberg’s equation

\[
\frac{\partial}{\partial t} \sigma^H(t) = \Omega^H(t) \times \sigma^H(t)
\]
for some function $\Omega^H(t)$, which is referred to as the effective magnetic field.\footnote{Rigorously speaking, it is $-\Omega^H$, not $+\Omega^H$, that should be referred to as the effective magnetic field (in $k$ space). In this paper, we nevertheless call $+\Omega^H$ the effective magnetic field for convenience.} Eq. (6) can be exactly solved if $\Omega^H(t)$ is time-independent. For the time-dependent effective magnetic field $\Omega^H(t)$, as far as we know, Eq. (6) has no exact solution in general due to the complication that the unitary transformation for diagonalizing the effective magnetic field is time dependent.

We can solve Eq. (6) perturbatively by expanding the effective magnetic field $\Omega^H(t)$ in series of different orders of the applied electric field: $\Omega^H(t) = \Omega_0 + \Omega_1(t) + o(\lambda^2)$, where $\Omega_0$ is of $o(\lambda^3)$ and $\Omega_1$ is of $o(\lambda)$. The dimensionless perturbative parameter $\lambda$ with $|\lambda| < 1$ is given as a constant proportional to $E$ as

$$\lambda = \frac{e l}{\hbar \Omega_0},$$

where $\Omega_0 = 2|d|/\hbar$ is the interband gap of the unperturbed system $H_0$ and $l$ denotes a characteristic length, which is sensibly taken to be the mean free path of the electron. Since we must keep $|\lambda| < 1$, the result of the first-order perturbation is valid only if $t$ is short enough. More precisely, by taking $l = v_0 t$ with $v_0$ being the drift velocity, the short-time condition is given by

$$t < \frac{\hbar \Omega_0}{e v_0 |E|}. \quad (8)$$

The spin is also expanded in series accordingly: $\sigma^H(t) = \sigma^{H_0}(t) + \sigma^\lambda(t) + o(\lambda^2)$. Up to the linear order of $\lambda$, Eq. (11) can be written as

$$\frac{\partial}{\partial t} \sigma^\lambda(t) = \Omega_0 \times \sigma^\lambda(t) + \Omega_1(t) \times \sigma^{H_0}(t), \quad (9)$$

where the unperturbed equation

$$\frac{\partial}{\partial t} \sigma^{H_0}(t) = \Omega_0 \times \sigma^{H_0}(t) \quad (10)$$

was used. The right hand side of Eq. (9) exhibits two different kinds of torque. The first one $\Omega_0 \times \sigma^\lambda(t)$ gives the Larmor precession around the direction of the static magnetic field $\Omega_0$. The second one $\Omega_1(t) \times \sigma^{H_0}(t)$ gives a non-Larmor precession.

As $E$ is turned on at $t = 0$, the initial state at $t = 0$ is given by the eigenstate of the unperturbed Hamiltonian $H_0$ denoted as $|n k\rangle$, with $n$ being the band index. We now apply the expectation value $\langle n k | O^H(t) | n k \rangle$, which is also simply written as $\langle O^H(t) \rangle$. For the unperturbed system, we have $\langle n k | O^{H_0}(t) | n k \rangle = \langle n k | O | n k \rangle$. Consequently, Eq. (9) leads to

$$\frac{\partial}{\partial t} \langle \sigma^\lambda(t) \rangle = \Omega_0 \times \langle \sigma^\lambda(t) \rangle + \Omega_1(t) \times \langle \sigma^{H_0}(t) \rangle. \quad (11)$$

Eq. (11) is the equation of motion of spin subject to the extra torque $\Omega_1(t) \times \langle \sigma^{H_0}(t) \rangle$ in addition to the Larmor torque. Furthermore, $\langle \sigma^{H_0}(t) \rangle$ is parallel (or anti-parallel) to $\Omega_0$. This can be deduced from the fact that the left hand side of Eq. (10) vanishes because $\langle \sigma^{H_0}(t) \rangle$ is independent of time. Therefore, the second term of Eq. (11) becomes $\Omega_1 \times \langle \sigma^{H_0}(t) \rangle = \pm \Omega_1 \times \Omega_0 / \Omega_0$, i.e., perpendicular to $\Omega_0$. This implies that the linear response $\langle \sigma^\lambda(t) \rangle$ is always perpendicular to $\Omega_0$,

$$\Omega_0 \cdot \langle \sigma^\lambda(t) \rangle = 0. \quad (12)$$

Eq. (12) can also be obtained as follows. Because the first term and the second term on the right hand side of Eq. (11) is always perpendicular to $\Omega_0$ and $\Omega_0$ is time-independent, we have $\partial(\Omega_0 \cdot \langle \sigma^\lambda(t) \rangle) / \partial t = 0$. This implies $\Omega_0 \cdot \langle \sigma^\lambda(t) \rangle$ is a constant for all time. On the other hand, as the perturbation term $\langle \sigma^\lambda(t) \rangle$ vanishes at $t = 0$, the constant is zero and therefore we obtain Eq. (12). Furthermore, Eq. (12) also leads to the result that $|\langle \sigma^\lambda(t) \rangle|^2 = |\langle \sigma^{H_0}(t) \rangle|^2 + o(\lambda^2)$. This means that the magnitude of spin remains unchanged up to the first order. That is, even though the non-Larmor torque can alter the magnitude of spin, the change is of $o(\lambda^2)$.

Up to the linear order of the applied electric field, the second derivative of Eq. (11) can be written as

$$\frac{\partial^2}{\partial t^2} \langle \sigma^\lambda(t) \rangle = \Omega_0 \times \langle \Omega_0 \times \langle \sigma^\lambda(t) \rangle \rangle + \Omega_0 \times \langle \Omega_1 \times \langle \sigma^{H_0}(t) \rangle \rangle + \Omega_1 \times \langle \Omega_0 \times \langle \sigma^{H_0}(t) \rangle \rangle + \frac{\partial \Omega_1}{\partial t} \times \langle \sigma^{H_0}(t) \rangle. \quad (13)$$

For the first term, we have

$$\Omega_0 \times \langle \Omega_0 \times \langle \sigma^\lambda(t) \rangle \rangle = \Omega_0 \cdot \langle \sigma^\lambda(t) \rangle \Omega_0 - \Omega_0^2 \langle \sigma^\lambda(t) \rangle = -\Omega_0^2 \langle \sigma^\lambda(t) \rangle, \quad (14)$$

where $\Omega_0 \cdot \langle \sigma^\lambda(t) \rangle = 0$ [see Eq. (12)] is used. The third term vanishes, i.e., $\Omega_1 \times \langle \Omega_0 \times \langle \sigma^{H_0}(t) \rangle \rangle = 0$ because $\Omega_0$ is parallel to the unperturbed spin $\langle \sigma^{H_0}(t) \rangle = \langle \sigma \rangle$, which is independent of $t$. Therefore, the second derivative of $\langle \sigma^\lambda(t) \rangle$ is given by

$$\frac{\partial^2}{\partial t^2} \langle \sigma^\lambda(t) \rangle = -\Omega_0^2 \langle \sigma^\lambda(t) \rangle + \Omega_0 \times \langle \Omega_1 \times \langle \sigma^{H_0}(t) \rangle \rangle + \frac{\partial \Omega_1}{\partial t} \times \langle \sigma^{H_0}(t) \rangle. \quad (15)$$

Eq. (11), Eq. (12) and Eq. (15) play the key role in obtaining the spin dynamics, and can be further simplified in two-dimensional systems. In the next section, we will solve the spin dynamics in the intrinsic (impurity-free) system.
III. INTRINSIC 2D SYSTEMS

In two-dimensional systems, both $\Omega_0$ and $\Omega_1$ lie on the $x$-$y$ plane; we have $\Omega_0 = (\Omega_{0x}, \Omega_{0y}, 0)$ and $\Omega_1 = (\Omega_{1x}, \Omega_{1y}, 0)$. This implies that $\Omega_1 \times \Omega_0$ is always perpendicular to the plane. To the first order of the electric field, $\langle \sigma^x(t) \rangle$ precesses around $\Omega_0$ but also tends to be dragged out of plane at the same time. In this sense, we separate the solution of $\langle \sigma^x(t) \rangle$ into a harmonic term describing the Larmor motion and an anharmonic term describing the non-Larmor motion for the drag effect. Therefore, the first order perturbation of spin $\langle \sigma^x(t) \rangle$ can be separated into two parts,

$$\langle \sigma^x(t) \rangle = \Sigma^L(t) + \Sigma^N(t).$$

(16)

$\Sigma^L$ is referred to as the Larmor component of spin and is a harmonic function of time. $\Sigma^N$ is referred to as the non-Larmor component of spin and is anharmonic in time. Correspondingly, $\Sigma^L$ and $\Sigma^N$ respectively satisfy

$$\frac{\partial}{\partial t} \Sigma^L = \Omega_0 \times \Sigma^L,$$

(17)

and

$$\frac{\partial}{\partial t} \Sigma^N = \Omega_0 \times \Sigma^N + \Omega_1 \times \langle \sigma \rangle,$$

(18)

where

$$\langle \sigma^H_0(t) \rangle = \langle n_k|\sigma|n_k \rangle \equiv \langle \sigma \rangle$$

(19)

was used.

Eq. (17) shows that $\Sigma^L$ precesses around the static magnetic field $\Omega_0$ harmonically and as a consequence the magnitude $|\Sigma^L|$ is independent of time. On the other hand, as $\Omega_0$ and $\Omega_1$ spin the 2D plane, the third term at the right hand side of Eq. (15) is perpendicular to the 2D plane (the unperturbed spin $\langle \sigma \rangle$ is parallel to $\Omega_0$). Furthermore, the second term at the right hand side of Eq. (15) lies on the 2D plane. Therefore, the in-plane spin, denoted as $\langle \sigma^x_0(t) \rangle = \langle \sigma^x_0(t) \rangle$, and the out-of-plane spin, denoted as $\langle \sigma^z(t) \rangle$, satisfy the following equations, respectively,

$$\frac{\partial^2}{\partial t^2} \langle \sigma^x_0(t) \rangle + \Omega_0^2 \langle \sigma^x_0(t) \rangle = G_\| (t),$$

(20)

$$\frac{\partial^2}{\partial t^2} \langle \sigma^z(t) \rangle + \Omega_0^2 \langle \sigma^z(t) \rangle = G_z(t),$$

where

$$G_\| (t) = (G_x, G_y) = \Omega_0 \times [\Omega_1 \times \langle \sigma \rangle],$$

(21)

$$G_z(t)e_z = \frac{\partial}{\partial t} \times \langle \sigma \rangle.$$

Importantly, $G_z$ in Eq. (21) gives the dynamical origin of the nonzero spin-$z$ component. We note that $G_z$ is related to time derivative of $\Omega_1$, which is nonzero in general.

The unperturbed Hamiltonian of the spin-orbit coupled systems in two dimensions can be written as ($d_z = 0$)

$$H_0 = \epsilon_k + \sigma_x d_x (k) + \sigma_y d_y (k),$$

(22)

where $\epsilon_k$ is the kinetic energy, and $d_x$ and $d_y$ are functions of momentum $k = (k_x, k_y)$, which describe the spin-orbit coupling. The eigenenergy of Eq. (22) satisfying $H_0|n_k \rangle = E_{n_k}|n_k \rangle$ is given by $E_{n_k} = \epsilon_k - nd$, where $d = \sqrt{d_x^2 + d_y^2}$ and $n = \pm$ represents the band index. For generic k-linear systems, $d_i$ can be written as $d_i = \sum \beta_{ij} k_j$ [13]. For the k-cubic Rashba system [8], $d_x = \alpha_k k^3 \sin(3\phi)$ and $d_y = -\alpha_k k^3 \cos(3\phi)$. For the k-cubic Rashba-Dresselhaus system [9], $d_x = \alpha_k \sin(3\phi) + \beta_h \cos(3\phi) k^3$, and $d_y = -\alpha_k \sin(3\phi) + \beta_h \sin(3\phi) |k|^3$. The precession frequency is given by $\Omega_0 = 2d/h$. The expectation values of spin with respect to the unperturbed eigenstates are given by $\langle \sigma \rangle = \langle n_k|\sigma|n_k \rangle$, and we have (see Appendix [A])

$$\langle n_k|\sigma_x|n_k \rangle = -n \frac{d_x}{d} = -n \frac{\Omega_{0x}}{\Omega_0},$$

(23)

$$\langle n_k|\sigma_y|n_k \rangle = -n \frac{d_y}{d} = -i n \frac{\Omega_{0y}}{\Omega_0},$$

$$\langle n_k|\sigma_z|n_k \rangle = 0.$$

Since $\Omega_0 = (\Omega_{0x}, \Omega_{0y}, 0)$ lies on the plane, the quantum average of spin-$z$ component is zero, i.e., $\langle n_k|\sigma_z|n_k \rangle = 0$, as expected. To simplify the following calculations, we define $d_x = d \sin \theta$, $d_y = -d \cos \theta$, and we have

$$\frac{\partial \theta}{\partial k_a} = \frac{\partial}{\partial k_a} \tan^{-1} \left( \frac{d_x}{d - d_y} \right) = \frac{1}{d^2} \left( d_x \frac{\partial d_y}{\partial k_a} - d_y \frac{\partial d_x}{\partial k_a} \right).$$

(24)

The equation of motion of spin for the total Hamiltonian $H = H_0 + \epsilon E \cdot \mathbf{x}$ is then given by

$$\frac{\partial}{\partial t} \sigma^H(t) = \frac{2d^H(t)}{h} \times \sigma^H(t),$$

(25)

where $d^H(t) = (d^H_x(t), d^H_y(t), 0)$. The corresponding effective magnetic field is given by $\Omega^H(t) = 2d^H(t)/h$. Eq. (25) can be perturbatively expanded up to first order of $\epsilon E_a x_a$ [27], and the result is given by

$$\sigma^H(t) = \sigma^{H_0}(t) + \epsilon E_a \Gamma_a \sigma^{H_0}(t) + o(\lambda^2),$$

(26)

where $\sigma^{H_0}(t) = \exp(i H_0 t/h) \sigma \exp(-i H_0 t/h)$, and the operator $\Gamma_a$ is given by

$$\Gamma_a = i \frac{1}{\hbar} \int_0^t dt' e^{i H_0 t'/\hbar} x_a e^{-i H_0 t'/\hbar}.$$

(27)

By using Eq. (26) and Eq. (27), we have

$$d^H_i(t) = \epsilon \frac{E_a}{\hbar} \frac{d k_a}{\partial x_a} e^{-i H_0 t/h}$$

$$= d_i - \frac{\epsilon E_a}{\hbar} \frac{d k_a}{\partial x_a} + o(\lambda^2).$$

(28)
We note that the result Eq. (28) is valid in a short time as given in Eq. (8). The original (i.e., zeroth-order) effective magnetic field is given by

$$\mathbf{\Omega}_0 = \left(\frac{2d_x}{\hbar}, \frac{2d_y}{\hbar}, 0\right),$$

and the first-order effective magnetic field is given by

$$\mathbf{\Omega}_1 = \frac{2}{\hbar} (-\epsilon E_a \frac{\partial d_x}{\partial k}\hat{n}_x, -\epsilon E_a \frac{\partial d_y}{\partial k}\hat{n}_y, 0).$$

The magnitude of $\mathbf{\Omega}_0$ is $|\mathbf{\Omega}_0| = 2d/\hbar = \mathbf{\Omega}_0$. We first focus on the solution of $\langle \sigma^z(t) \rangle$. The solution of the Larmor component $\Sigma^L_z$ in Eq. (20) is obtained by setting $G_z = 0$, and we must have $\Sigma^L_z = A \sin(\Omega_0 t) + B \cos(\Omega_0 t)$. On the other hand, as the first-order magnetic field $\mathbf{\Omega}_1$ is linear in time, its time derivative is a constant in time. The solution of the non-Larmor component $\Sigma^N_z$ must be anharmonic, and thus, we have $\Omega_0^2 \Sigma^N_z = G_z$,

$$\Sigma^N_z = \frac{1}{\Omega_0^2} \left( \frac{\partial \mathbf{\Omega}_1}{\partial t} \times \langle n k | \sigma| n k \rangle \right)_z,$$

$$= \frac{1}{\Omega_0^2} \left( \frac{\partial \Omega_1}{\partial t} \times \langle n k | \sigma_y| n k \rangle - \frac{\partial \Omega_1}{\partial t} \times \langle n k | \sigma_x| n k \rangle \right),$$

$$= -\epsilon E_a \frac{2}{\hbar^2} \left[ \frac{\partial d_x}{\partial k} \left( -n \frac{d_y}{d} \right) - \frac{\partial d_y}{\partial k} \left( -n \frac{d_x}{d} \right) \right],$$

$$= -\frac{\epsilon E_a}{\hbar \Omega_0} \frac{\partial \theta}{\partial k_a}.$$

The general solution of $\Sigma^L_z$ and $\Sigma^N_z$ is given by $\Sigma^L_z(t) = \cos(\Omega_0 t) + \sin(\Omega_0 t)$ and $\Sigma^N_z(t) = \sin(\Omega_0 t) - \cos(\Omega_0 t) + \frac{\Omega_0}{\hbar} \frac{\partial \theta}{\partial k_a}$.

By requiring that $\Sigma^L_z + \Sigma^N_z = 0$ at the initial time $t = 0$, we have $C_\parallel = 0$. For the non-Larmor precession, we find that Eq. (18) is satisfied by substitution of Eqs. (31) and (22) into Eq. (18). Up to this step, the remaining coefficients are $K_x$, $K_y$, and $A$ for direct precession. By substituting $\Sigma^L_z$ and $\Sigma^N_z$ into Eq. (17), we can obtain $A = 0$, $K_x = -\Omega_0 y \Sigma^N_z / \Omega_0$, and $K_y = \Omega_0 x \Sigma^N_z / \Omega_0$. Furthermore, by using the condition $n k|\sigma^H(t)| = 0$, we have $\Omega_0 x K_x + \Omega_0 y K_y = 0$, and it is easy to check that the result is satisfied. Therefore, the solution of $\langle n k | \sigma^H(t) | n k \rangle$ can be written as

$$\langle n k | \sigma^H(t) | n k \rangle = \sigma + \Sigma^N(t) + \Sigma^L(t),$$

where the Larmor component of spin $\Sigma^L = (\Sigma^L_x, \Sigma^L_y, \Sigma^L_z)$ is given by $[d_x = d \sin \theta, d_y = -d \cos \theta]$ is used

$$\Sigma^L(t) = \Sigma^N \cos \theta \sin(\Omega_0 t) \hat{e}_x + \Sigma^N \sin \theta \sin(\Omega_0 t) \hat{e}_y - \Sigma^N \cos(\Omega_0 t) \hat{e}_z,$$

and the non-Larmor component of spin $\Sigma^N(t) = (\Sigma^D_x, \Sigma^D_y, \Sigma^D_z)$ is given by

$$\Sigma^N = (\mathbf{\Omega}_0 \times \Sigma^N \hat{e}_z) t + \Sigma^N \hat{e}_z.$$

For the Larmor precession [see Eq. (23)], the magnitude of $\Sigma^L$ is $|\Sigma^L| = |\Sigma^N|$, and $\Sigma^L$ precesses about the in-plane axis $\mathbf{\Omega}_0$ with frequency $\Omega_0$ as can be verified from Eq. (17). For the non-Larmor precession, the spin-$z$ component is not necessary to be zero. The spin-$z$ component behaves as $\Sigma^N_z (1 - \cos(\Omega_0 t))$, which has a maximum value $2 \Sigma^N_z$ and a minimum value 0.

The solution of the in-plane and out-of-plane components of spin obtained in Eqs. (33), (34) and Eq. (35) can be recast into the following form [see also Eq. (31a)]

$$\langle n k | \sigma^H(t) | n k \rangle = \frac{-\Omega_0}{\hbar} \mathbf{\Omega}_0 \times \hat{e}_z \left[ \frac{\sin(\Omega_0 t)}{\Omega_0} \right],$$

$$\langle n k | \sigma^H(t) | n k \rangle \hat{e}_z = \Sigma^N \left[ 1 - \cos(\Omega_0 t) \right] \hat{e}_z.$$

The term linear in $t$ for $\langle \sigma^H \rangle$ seems to grow arbitrarily large with $t$, but this is only an artifact due to the perturbation method, which is valid only for a short range of $t$ as delimited by Eq. (8). In fact, the time-dependent part of $\langle \sigma^H \rangle$ and the whole part of $\langle \sigma^H(t) \rangle$ are both of $o(\lambda)$, reassuring $|\langle \sigma^H(t) \rangle|^2 = 1 + o(\lambda^2)$. From Eq. (36), we see that, after $\mathbf{E}$ is turned on at $t = 0$, the out-of-plane component $\langle \sigma^H(t) \rangle$ arises from zero and begins a harmonic oscillation around $\Sigma^N_z$ with the frequency $\Omega_0$. The typical value of the frequency $\Omega_0$ on the Fermi surface is about $10^{18}$ per second.

In reality, after a while, the harmonic oscillation will decay and $\langle \sigma^H(t) \rangle$ will asymptote to a new value due to some dissipative process, which has not been taken into account.

\footnote{For pure Rashba system, $\Omega_0 \sim 2ak_F/h$ and $k_F \sim 10^{-2} \AA^{-1}$ and $\alpha \sim 10^{-3} eV \AA$ for GaAs [32], and we have $\Omega_0 \sim 10^{10}$.}
consideration. The switch-on of $\mathbf{E}$ bridges the two time-independent systems described by $H_0$ and $H_0 + e\mathbf{E} \cdot \mathbf{x}$, respectively. Measurement of the spin in the latter system does not exhibit any harmonic oscillation, because a time-independent system gives a stationary solution (which will be the topic of Sec. [LV]). However, on the other hand, if the latter system is transited from the former system, the transition will induce a harmonic oscillation as we just calculated. Without any dissipative process, the harmonic oscillation will remain persistent, which is unphysical because after long enough the system should become oblivious of when the transition has taken place.

The exact mechanism for the dissipation is unclear and complicated, but it can be implemented phenomenologically. In particular, the time-varying behavior of a magnetic dipole subject to a magnetic filed can be modeled by various forms of the Landau-Lifshitz-Gilbert equation \[27\], which phenomenologically includes a damping term to account for the dissipation. The Landau-Lifshitz-Gilbert equation can be obtained more fundamentally in the context of irreversible statistical mechanics \[28\]. Unfortunately, as the Landau-Lifshitz-Gilbert equation models the time-varying behavior only for the magnetic dipole but assumes the magnetic field as static, it does not serve our purpose. Instead, we phenomenologically model the dissipative process by introducing damping terms directly upon the precessional equation Eq. \[39\]:

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} \langle \sigma^H \rangle(t) + \Omega_0^2 \langle \sigma^H \rangle(t) + \beta_\parallel \frac{\partial}{\partial t} \langle \sigma^H \rangle(t) &= G_\parallel(t) e^{-\alpha t}, \\
\frac{\partial^2}{\partial t^2} \langle \sigma_z \rangle(t) + \Omega_0^2 \langle \sigma_z \rangle(t) + \beta_z \frac{\partial}{\partial t} \langle \sigma^H \rangle(t) &= G_z(t),
\end{align*}
\]

(37)

where the constants $\beta_\parallel$ and $\beta_z$ are two damping parameters responsible for damping the harmonic oscillation. For generality, we keep $\beta_\parallel$ and $\beta_z$ as two different coefficients. Additionally, we also include an exponential decay factor $e^{-\alpha t}$ for the “source” term $G_\parallel(t)$. This factor is not only prescribed to subdue the pathological trait of the linear growth in $t$ for $G_\parallel(t)$ but in fact is required to render the damping with $\beta_\parallel$ consistent (as will be seen shortly). Provided that the dissipative effects are strong enough, i.e., phenomenologically characterized by

\[
\frac{1}{\beta_\parallel}, \frac{1}{\beta_z}, \frac{1}{\alpha} \ll \frac{\hbar \Omega_0}{e\epsilon_{\parallel \parallel} |\mathbf{E}|},
\]

(38)

the spin precession will reach an asymptotic state within the valid range of time given by Eq. \[8\].

If we require that the solution of Eq. \[37\] agrees with Eq. \[36\] when $\beta_\parallel, \beta_z, \alpha \rightarrow 0$, we then have the new solution

\[
\langle n \mathbf{k} | \sigma^H \rangle(t) | n \mathbf{k} \rangle = -\frac{n \Omega_0}{\Omega_0} + \sum_N \left( \frac{\Omega_0}{\Omega_0^2 - \alpha \beta_\parallel + \alpha^2} \right)^2 \times \left[ e^{-\alpha t} \left( \frac{\Omega_0}{\Omega_0^2 - \alpha \beta_\parallel + \alpha^2} \right)^2 \right] \\
+ \left( 2\alpha - \beta_\parallel \right) \left( e^{-\alpha t} - e^{-\beta_\parallel + i\Omega_0 t} \right) \\
- \Omega_0 e^{-\beta_\parallel + i\Omega_0 t} \sin(\Omega_0 t),
\]

(39)

where the new oscillatory frequencies $\Omega_\parallel, \Omega_z$ are given by

\[
\Omega_\parallel^2 = \Omega_0^2 - \left( \frac{\beta_\parallel}{2} \right)^2.
\]

(40)

Prescribing $\beta_\parallel \neq 0$ gives rise to a pathological trait that $\langle \sigma^H \rangle(t)$ in general becomes complex, with an imaginary part of $o(\lambda)$. This problem can be avoided by choosing $2\alpha = \beta_\parallel$.

\[
\Omega_\parallel = \Omega_0 - \left( \frac{\beta_\parallel}{2} \right)^2.
\]

(41)

Therefore, to implement dissipation consistently, inclusion of the damping term with $\beta_\parallel$ entails the exponential decay term with $\alpha$.

By defining

\[
F_0(t) = \frac{\Omega_0^3}{\Omega_0^2 - \alpha \beta_\parallel + \alpha^2} e^{-\alpha t} t, \\
F_1(t) = \frac{\Omega_0^3 (2\alpha - \beta_\parallel)}{(\Omega_0^2 - \alpha \beta_\parallel + \alpha^2)^2} \left( e^{-\alpha t} - e^{-\beta_\parallel + i\Omega_0 t} \right), \\
F_2(t) = \frac{\Omega_0^3}{\Omega_0^2 - \alpha \beta_\parallel + \alpha^2} e^{-\beta_\parallel + i\Omega_0 t} t, \\
F_3(t) = e^{-\beta_\parallel + i\Omega_0 t} t,
\]

Eqs. \[39\] can be rewritten as

\[
\langle \sigma^H \rangle(t) = -\frac{n \Omega_0}{\Omega_0} + \sum_N \left( \frac{\Omega_0}{\Omega_0^2 - \alpha \beta_\parallel + \alpha^2} \right)^2 \times \left[ F_0(t) + F_1(t) - F_2(t) \right] \\
+ \left( 2\alpha - \beta_\parallel \right) \left( e^{-\alpha t} - e^{-\beta_\parallel + i\Omega_0 t} \right) - \Omega_0 e^{-\beta_\parallel + i\Omega_0 t} \sin(\Omega_0 t),
\]

(43)

which follows

\[
\frac{\partial}{\partial t} \langle \sigma^H \rangle(t) = \sum_N \left( \frac{\Omega_0}{\Omega_0^2 - \alpha \beta_\parallel + \alpha^2} \right)^2 \times \left[ F_0(t) + F_1(t) - F_2(t) \right],
\]

(44)

Eqs. \[43\] and \[44\] can be organized into the following form

\[
\frac{\partial}{\partial t} \langle \sigma^H \rangle(t) = \Omega_0^2 \langle \sigma^H \rangle(t) + o(\lambda^2),
\]

(45)
with the effective magnetic field given by

\[
\Omega^\lambda(t) = \Omega_0 \\
+ n\sum_N \left( \frac{\Omega_0}{\Omega_0} \times \hat{e}_z \right) \left[ -\Omega_0 F_0(t) - \Omega_0 F_1(t) \\
+ (\Omega_0 F_2(t) - F'_{2}(t)) \right] \\
+ n\sum_N \hat{e}_z \left[ -\Omega_0 + F_0'(t) + F'_1(t) \\
+ (\Omega_0 F_2(t) - F'_{2}(t)) \right] + o(\lambda^2).
\]

(46)

With \( \beta_1, \beta_2, \alpha > 0 \), we have \( F_0(t) = \Omega_0 t, F_1(t) = 0, \) \( \Omega_0 F_2(t) - F'_{2}(t) = 0 \) and \( \Omega_0 F_3(t) - F'_{3}(t) = 0 \), and consequently Eq. (46) leads to

\[
\Omega^\lambda(t) = \Omega_0 - n\sum_N (\Omega_0 \times \hat{e}_z)\Omega_0 t + o(\lambda).
\]

(47)

This affirms the consistency that the effective magnetic field \( \Omega^\lambda(t) \) obtained from the equation of spin precession is identical to \( \Omega^H(t) \) in Eq. (45), where our calculation begins.

On the other hand, with \( \beta_1, \beta_2, \alpha > 0 \), we have \( F_0(t) = \Omega_0 t, F_1(t) = 0, \) \( \Omega_0 F_2(t) - F'_{2}(t) = 0 \) and \( \Omega_0 F_3(t) - F'_{3}(t) = 0 \), and consequently Eq. (46) leads to

\[
\Omega^\lambda(t) = \Omega_0 - n\sum_N (\Omega_0 \times \hat{e}_z)\Omega_0 t + o(\lambda^2).
\]

(48)

This affirms the consistency that the effective magnetic field \( \Omega^\lambda(t) \) obtained from the equation of spin precession is identical to \( \Omega^H(t) \) in Eq. (45), where our calculation begins.

In the asymptotic limit, both the spin \( \langle \sigma^H(t) \rangle \) and the effective magnetic field \( \Omega^\lambda(t) \) are parallel to each other and have constant nonzero components in the out-of-plane direction. The asymptotic values of \( \langle \sigma^H(t) \rangle \) and \( \Omega^\lambda(t) \) obtained from the dynamical response to the switch-on of the applied electric field should be the same as those obtained as the stationary response to the electric field that is viewed as never-changing in time. In the next section, we will perform the time-independent analysis upon the time-independent Hamiltonian \( H_0 + eE \cdot x \). In terms of the matrix elements with respect to the eigenstates of \( H_0 \), the full Hamiltonian \( H_0 + eE \cdot x \) takes the form

\[
\hat{H} = \epsilon_k - d + V_E \langle +k|x_a|\cdot|+k\rangle eE_a
\]

and

\[
\Omega^\lambda(t) \rightarrow \Omega_0 - n\Omega_0 \sum_N \hat{e}_z + o(\lambda^2)
\]

(49)

In the asymptotic limit, both the spin \( \langle \sigma^H(t) \rangle \) and the effective magnetic field \( \Omega^\lambda(t) \) are parallel to each other and have constant nonzero components in the out-of-plane direction. The asymptotic values of \( \langle \sigma^H(t) \rangle \) and \( \Omega^\lambda(t) \) obtained from the dynamical response to the switch-on of the applied electric field should be the same as those obtained as the stationary response to the electric field that is viewed as never-changing in time. In the next section, we will perform the time-independent analysis upon the time-independent Hamiltonian \( H_0 + eE \cdot x \). In terms of the matrix elements with respect to the eigenstates of \( H_0 \), the full Hamiltonian \( H_0 + eE \cdot x \) takes the form

\[
\hat{H} = \epsilon_k + \sigma \cdot D.
\]

The direction of the effective magnetic field (i.e., \( D/D \)) as a stationary response is in full agreement with the asymptotic value of the dynamical response given in Eq. (48).

Therefore, we have arrived at a good understanding about the dynamical origin of the out-of-plane spin component. In the beginning, before the electric field \( E \) is turned on, the spin is aligned with the original effective magnetic field \( \Omega_0 \), lying on the \( x-y \) plane. When \( E \) is turned on at \( t = 0 \), it deflects \( \Omega_0 \) into \( \Omega_0 + \Omega_1 + o(\lambda^2) \), which remains on the \( x-y \) plane. The spin is no longer aligned with the new effective magnetic field and starts to precess around it, thereby giving rise to the spin-\( z \) component. The precession of spin alters the effective magnetic field and tilts it from the \( x-y \) plane as a back-reaction via the dissipative process. Eventually, the precession of spin and the evolution of the effective magnetic field asymptotically reach a stationary balance, where the spin is aligned again with the effective magnetic field and has a nonzero spin-\( z \) component. The dissipative effect plays a crucial role in establishing the stationary balance.

We close this section with a remark: The mathematical result obtained from solving Eq. (49) up to the first order of the electric field should be the same with that obtained directly from the Heisenberg picture. The later calculation is presented in Appendix B.

IV. TIME-INDEPENDENT ANALYSIS

As demonstrated in the previous section, an effective out-of-plane magnetic field is dynamically generated by an in-plane electric field and it asymptotes to an asymptotic result. In this section, complementary to the dynamical treatment, we conduct a time-independent analysis to directly derive the stationary response to the applied electric field, which is now treated as always turned on and completely time independent. It will shown that the linear term of the stationary response is exactly equal to the asymptotic response obtained in the dynamical treatment.

The unperturbed Hamiltonian Eq. (22) is represented in the spin space \( |\uparrow\rangle \) and \( |\downarrow\rangle \). In the basis \( |n\rangle \), \( H_0 \) is diagonalized, while the total Hamiltonian is not but reads as

\[
\hat{H} = \epsilon_k - d + V_E \langle +k|x_a|\cdot|+k\rangle eE_a
\]

where we have defined \( \langle +k|x_a|+k\rangle eE_a \equiv V_E \) and \( \langle -k|x_a|\cdot|+k\rangle eE_a \equiv -V_E \), as in general the vector potential \( \langle n|k|x_a|n\rangle \) depends on the band index \( n \). In general, the quantity \( V_E \equiv eE_a(\langle +k|x_a|+k\rangle - \langle -k|x_a|\cdot|+k\rangle) \) is not invariant under the gauge transformation \( |+k\rangle \rightarrow e^{i\phi_+(k)}|+k\rangle, |+k\rangle \rightarrow e^{-i\phi_-(k)}|+k\rangle \) unless we choose \( \phi_+(k) = \phi_-(k) \). This raises an issue of how to define a gauge-independent spin current in the spin-Hall effect. Following Ref. [20], one has to apply an intricate prescription to render the spin current gauge independent. Nevertheless, up to the first order of the electric field, the linear response is independent of \( V_E \) and thus free of this problem as will be seen shortly.

For the off-diagonal matrix elements of \( x_a \), we can write \( \langle \pm k|x_a|\mp k \rangle \) in terms of matrix elements of \( \sigma_z \),
which is valid for all choice of wave functions, as proved in Appendix A. Using Eq. (A7), we have

\[
\langle +k| x_a | -k \rangle = \frac{1}{2} \langle +k| \sigma_z | -k \rangle \frac{\partial}{\partial k_a} e^{i E_a k}, \\
\langle -k| x_a | +k \rangle = \frac{1}{2} \langle -k| \sigma_z | +k \rangle \frac{\partial}{\partial k_a} e^{i E_a k}.
\]  

(52)

Substituting Eq. (52) into Eq. (51), \( \mathcal{H} \) can be written as

\[
\mathcal{H} = \epsilon_k + a_x \tau_x + a_y \tau_y + a_z \tau_z,
\]

(53)

where

\[
a_x = \frac{1}{2} \text{Re} \langle -k| \sigma_z | +k \rangle \frac{\partial}{\partial k_a} e^{i E_a k}, \\
a_y = \frac{1}{2} \text{Im} \langle -k| \sigma_z | +k \rangle \frac{\partial}{\partial k_a} e^{i E_a k}, \\
a_z = V_E - d.
\]

The matrices \( \tau_i \), which are called pseudo-spin matrices, are mathematically Pauli matrices, but they are not real spin. This can also be seen as follows. The position operator \( x \) and the electric field \( E \) are even under the time-reversal transformation. The real spin is odd \( \sigma_i \to -\sigma_i \) and the momentum is also odd under the time-reversal operation \( k \to -k \). The effective magnetic field \( d \) must be odd under the time-reversal operation \( d_x \to -d_x \) and \( d_y \to -d_y \). This implies that the Hamiltonian \( H_0 \) is invariant under the time-reversal transformation. Since the Hamiltonian \( H_0 \) is invariant under time-reversal transformation, the Hamiltonian in the basis \( | \pm \rangle \) must not break the time reversal symmetry. Therefore, we find that \( a_x, a_y \) and \( a_z \) are even under time reversal operation, and thus \( \tau_i \) must be even, which means that \( \tau_i \) are not the real spin. In order to transform Eq. (53) back to the real spin, we have to transform the spin in basis \( | \pm \rangle \) to the spin space \( | \uparrow \rangle \) and \( | \downarrow \rangle \).

The original spin \( \sigma_i \) in basis \( | \pm \rangle \) can be written in terms of the new spin matrices denoted as \( \tilde{\sigma}_i \). For the spin-\( z \) component, we have

\[
\tilde{\sigma}_z = \begin{pmatrix} \langle +k| \sigma_z | +k \rangle & \langle +k| \sigma_z | -k \rangle \\ \langle -k| \sigma_z | +k \rangle & \langle -k| \sigma_z | -k \rangle \end{pmatrix} \\
= \frac{2}{\text{Re} \sigma_{a_e}} (a_x \tau_x + a_y \tau_y).
\]

(55)

For spin-\( x \) nd spin-\( y \) components, we have

\[
\tilde{\sigma}_x = \begin{pmatrix} \langle +k| \sigma_x | +k \rangle & \langle +k| \sigma_x | -k \rangle \\ \langle -k| \sigma_x | +k \rangle & \langle -k| \sigma_x | -k \rangle \end{pmatrix} \\
= -\frac{d_x}{d} \tau_x + \frac{d_y}{d} \frac{2}{\text{Re} \sigma_{a_e}} (a_y \tau_x - a_x \tau_y).
\]

(56)

and

\[
\tilde{\sigma}_y = \begin{pmatrix} \langle +k| \sigma_y | +k \rangle & \langle +k| \sigma_y | -k \rangle \\ \langle -k| \sigma_y | +k \rangle & \langle -k| \sigma_y | -k \rangle \end{pmatrix} \\
= -\frac{d_y}{d} \tau_y - \frac{d_x}{d} \frac{2}{\text{Re} \sigma_{a_e}} (a_y \tau_x - a_x \tau_y).
\]

(57)

By using Eq. (A8), it is easy to show that \( \tilde{\sigma}_i \) satisfies the algebra of the Pauli matrices, i.e., \{\( \tilde{\sigma}_i, \tilde{\sigma}_j \) = 2\( \delta_{ij} \) and \( [\tilde{\sigma}_i, \tilde{\sigma}_j] = 2i\epsilon_{ijk} \tilde{\sigma}_k \). By using Eq. (55), Eq. (56) and Eq. (57), the Hamiltonian Eq. (53) can be written in terms of \( \tilde{\sigma}_i \) and the result is given by

\[
\mathcal{H} = \epsilon_k + D_x \tilde{\sigma}_x + D_y \tilde{\sigma}_y + D_z \tilde{\sigma}_z,
\]

(58)

where

\[
D_x = -\frac{d_x}{d} a_z = d_x - \frac{d_x}{d} V_E, \\
D_y = -\frac{d_y}{d} a_z = d_y - \frac{d_y}{d} V_E, \\
D_z = \frac{1}{2} \frac{\partial}{\partial k_a} e^{i E_a k}.
\]

(59)

Therefore, the Hamiltonian in terms of the expectation values with respect to \( | \pm k \rangle \) can be again cast into the form \( \mathcal{H} = \epsilon_k + \mathbf{D} \cdot \mathbf{\tilde{\sigma}} \). That is, the spin is aligned with an effective magnetic field given by Eq. (59). We also find that \( \mathbf{D} = (D_x, D_y, D_z) \) are odd under time-reversal operation. Therefore, the Hamiltonian in terms of the expectation values with respect to \( | \pm k \rangle \) can be cast into the form

\[
\tilde{\mathcal{H}} = \epsilon_k + D_x \sigma_x + D_y \sigma_y + D_z \sigma_z,
\]

(60)

where \( \sigma_i \) are Pauli matrices. Importantly, Eq. (60) shows that the \( z \)-component of effective magnetic field is non-zero. The effective magnetic field is being tilted up in the presence of an electric field. In the absence of electric field, Eq. (60) goes back to the unperturbed Hamiltonian \( H_0 = \epsilon_k + \sigma_x d_x + \sigma_y d_y \). It should be noted that \( D_x \) and \( D_y \) in Eq. (59) contain unphysical gauge-dependent pieces involving \( V_E \). Nevertheless, the gauge-dependent terms are of \( o(\lambda^2) \) and are exactly cancelled out when we compute \( \mathbf{D}/D \) (i.e., the direction of \( \mathbf{D} \) in the linear response regime. Noting that \( D^2 = d^2 + 2V_E d + o(\lambda^2) \), we have

\[
\frac{\mathbf{D}}{D} = \frac{d_x}{d} \hat{\tau}_x + \frac{d_y}{d} \hat{\tau}_y + \frac{1}{2d} \frac{\partial}{\partial k_a} e^{i E_a k} \hat{\tau}_z + o(\lambda^2).
\]

(61)

Therefore, up to \( o(\lambda^2) \), the result of Eq. (61) is gauge independent. We find that the direction of the effective magnetic field given in Eq. (61) is exactly the same as that of Eq. (49). The eigenstates of \( \tilde{\mathcal{H}} \) are given by

\[
| \Psi_{-k} \rangle = \frac{1}{\sqrt{2(1 + D_z)}} \left( \hat{D}_x + i \hat{D}_y \right), \\
| \Psi_{+k} \rangle = \frac{1}{\sqrt{2(1 + D_z)}} \left( -\hat{D}_x + i \hat{D}_y \right),
\]

(62)

where \( \hat{D}_i = D_i / D \) and \( D = \sqrt{D_x^2 + D_y^2 + D_z^2} \). The corresponding eigenenergies are given by \( \tilde{\mathcal{H}} | \Psi_{\pm k} \rangle = \epsilon_{\pm k} | \Psi_{\pm k} \rangle \).
with $E_{\ell k} = \epsilon_k - \ell D$. In the absence of electric field, the eigenenergy is given by $E_{\ell k} = \epsilon_k - \ell d$, which is the eigenenergy of the unperturbed Hamiltonian $H_0$. As mentioned above, the spin is aligned in the direction $\mathbf{D}/D$, and by using Eq. (62), we have $\langle \Psi_{\ell k}|\sigma|\Psi_{\ell k}\rangle = -\ell \mathbf{D}/D$. In the presence of the electric field, up to the first order of $E$, we have

$$
\langle \Psi_{\ell k}|\sigma|\Psi_{\ell k}\rangle = -\ell \mathbf{D}/D
$$

On the other hand, consider the projection of spin on the $\mathbf{D}$ direction, which is given by

$$
\hat{\Sigma}_c = \frac{1}{D^2} \mathbf{D}(\sigma \cdot \mathbf{D}).
$$

By noting that $D^2 = d^2 - 2V_d d + o(\lambda^2)$, the spin $\hat{\Sigma}_c = (\hat{\Sigma}_{cx}, \hat{\Sigma}_{cy}, \hat{\Sigma}_{cz})$ in the unperturbed basis $|nk\rangle$ is given by

$$
\langle nk|\hat{\Sigma}_{cx}|nk\rangle = -n\frac{d x}{d} + o(\lambda^2),
\langle nk|\hat{\Sigma}_{cy}|nk\rangle = -n\frac{d y}{d} + o(\lambda^2),
\langle nk|\hat{\Sigma}_{cz}|nk\rangle = -\frac{n}{2d} \frac{\partial}{\partial k_a} e E_a + o(\lambda^2).
$$

It can be shown that $\langle \Psi_{nk}|\sigma|\Psi_{nk}\rangle = \langle nk|\hat{\Sigma}_c|nk\rangle + o(\lambda^2)$. Importantly, compare to Eq. (31), we have

$$
\langle nk|\hat{\Sigma}_{cz}|nk\rangle = \Sigma^N_z.
$$

Eq. (65) and Eq. (69) are in full agreement with the asymptotic response obtained in Eq. (49).

We close this section by showing the relation between the spin current and $\Sigma^N$ by using the Kubo formula [29]. The spin-$z$ component satisfies the following continuity equation [21, 31]

$$
\frac{\partial}{\partial t} \Psi^1 S_z \Psi + \nabla_i \text{Re} \left[ \Psi^1 J_i^z \Psi \right] = \text{Re} \left[ \Psi^1 \frac{d S_z}{dt} \Psi \right],
$$

where the conventional spin current $J_i^z$ is given by

$$
J_i^z = \frac{1}{2} \langle J_h \sigma_i, v_i \rangle
$$

where $v_i = \partial H/\hbar \partial k_i$ and $J = 1/2$ for spin 1/2 and so on. The Kubo formula for spin current [4] is given by

$$
J_i^z = \frac{2\hbar}{V} \sum_{n(\neq n')} f_{nk} \frac{\text{Im} \langle nk|J_i^z|n'k\rangle \langle n'k|v_j|nk\rangle}{(E_{nk} - E_{n'k})^2} E_j,
$$

which is purely imaginary. Substituting Eq. (70) into Eq. (69), we have

$$
J_i^z = \frac{J}{V} \sum_{n(\neq n')} f_{nk} \frac{\partial \epsilon_k}{\partial k_i} \left( -\frac{n}{2d} \right) \frac{\partial \theta}{\partial k_j} e E_j
$$

In the second equality of Eq. (70), the definition of $\Sigma^N_z$ [see Eq. (31)] was used. The third equality of Eq. (70) can be directly obtained from Eq. (69) in the linear response regime, and the result is in agreement with the Kubo formula. Furthermore, since the source term in Eq. (67) is $d\langle nk|\sigma_i^\mu(t)|nk\rangle/dt \to d\Sigma^N_i/dt = 0$, and thus we find that the spin current shown in Eq. (71) is the conserved spin current. Another definition of spin current from the source term $J_i^z = \frac{1}{2} \{ x_i, \sigma_i \}$ is in agreement with the present result. The spin-torque current is non-zero only when the electric field is nonhomogeneous in the space $\exp(\text{i} qx)$, and $J_i^z$ will be the rate of change of the torque spin density with respect to $q$ in the limit $q \to 0$ [20]. In addition, Eq. (66) shows that the linear response of the in-plane spin is zero. This can be seen as follows. By using Eq. (62) and Eq. (66), we have

$$
\langle nk|\sigma_y|n'k\rangle \langle n'k|v_j|nk\rangle
$$

which is purely real, and therefore the imaginary part is zero. For $\sigma_y$, similar to the derivation shown in Eq. (72) (by using Eq. (62) and Eq. (66)), it can be shown that $\langle nk|\sigma_y|n'k\rangle \langle n'k|v_j|nk\rangle$ is also purely real.

V. CONCLUSION

We obtain the dynamical equation of spin in two-dimensional spin-orbit coupled systems by solving Heisengerg’s equation perturbatively up to the linear order of the applied electric field, which is assumed to be turned on at $t = 0$. As shown in Eqs. (29) and (30), the switch-on of the electric field deflects the effective magnetic field from its original direction by giving a time-dependent component on the $x$-$y$ plane. As the spin is no longer aligned with the effective magnetic field, it starts to precess around the new direction.

Taking into account the dissipative effect that attenuates and eventually ceases the spin precession, we phenomenologically add damping terms upon the equation
of spin dynamics as in Eq. (37). The solution of the resulting dynamics is given in Eq. (39) for the spin and Eq. (10) for the effective magnetic field. When \( t \) is large enough, the dynamical solution asymptotes to an asymptotic state given by Eqs. (18) and (20), where the spin and the effective magnetic field are aligned again and exhibit nonzero components in the \( z \) direction.

On the other hand, treating the applied electric field as always turned on, we also directly compute the stationary response in the time-independent approximation by projecting the full Hamiltonian on the spin space as in Eq. (37). The stationary response is obtained in Eqs. (65) and (66), which is exactly equal to the asymptotic result (18) obtained from the dynamical treatment. The direction of effective magnetic field [Eq. (61)] is also in agreement with that of the asymptotic result [Eq. (19)]. Furthermore, the relation between the stationary response of the effective magnetic field and the spin current is derived, and the result is in agreement with the Kubo formula.

Our dynamical treatment reveals the dynamical origin of the spin-\( z \) component and provides a method to study the connection between the dynamical and stationary responses. The dissipative effect is found to be crucial for establishing the connection. However, our prescription of dissipative effect remains phenomenological and it should be derived more fundamentally by the methods of irreversible statistical mechanics following the lines of Ref. 28. In the dynamical treatment, we study the evolution of the expectation values of the kind \( \langle nk|\sigma_z|nk\rangle \) but disregard the off-diagonal terms of the kind \( \langle +k|\sigma_z|−k\rangle \) and \( \langle −k|\sigma_z|+k\rangle \), while in the time-independent approach, both are included [see Eq. (37)]. The fact that the dynamical treatment with dissipation asymptotically leads to the stationary result of the time-independent approach strongly suggests that the dissipative effect in the dynamical picture is closely related to the equilibrium of the interband transition in the stationary picture. This relation should become more transparent if the dissipation can be more fundamentally derived.

Acknowledgments

T.-W. Chen would like to thank Wang-Chuang Kuo for valuable discussions on the stationary response. This work was supported in part by the Ministry of Science and Technology, Taiwan under the Grant MOST 106-2112-M-110-010.

Appendix A: Matrix Elements

In this appendix, we calculate unperturbed matrix elements of the spin and velocity operators used in this article without specifying any form of the wave functions \( |nk\rangle \). By using \( \{\sigma_x, H_0\} = \{\sigma_x, \varepsilon_k + \sigma_x d_x + \sigma_y d_y\} = 2\varepsilon_k \sigma_x + 2d_x \), and \( \langle nk|\{\sigma_z, H_0\}|nk\rangle = 2\langle nk|\sigma_z|nk\rangle E_{nk} = 2\varepsilon_k \langle nk|\sigma_z|nk\rangle - 2nd\langle nk|\sigma_z|nk\rangle \). We have the diagonal matrix element of \( \sigma_x \) in the helicity basis,

\[
\langle nk|\sigma_x|nk\rangle = -n \frac{d_x}{d}.
\]  

(A1)

For the off-diagonal matrix elements, we note that \( \{\sigma_x, H_0\} = \{\sigma_x, \varepsilon_k + \sigma_x d_x + \sigma_y d_y\} = 2i\varepsilon_k d_y \). It follows

\[
\langle nk|\sigma_x|mk\rangle = \frac{2id_y}{E_{mk} - E_{nk}} \langle nk|\sigma_z|mk\rangle.
\]  

(A2)

Similar to the derivation, for the spin-\( y \)-component, we have

\[
\langle nk|\sigma_y|nk\rangle = -n \frac{d_y}{d},
\]

\[
\langle nk|\sigma_y|mk\rangle = -\frac{2id_x}{E_{mk} - E_{nk}} \langle nk|\sigma_z|mk\rangle.
\]  

(A3)

For the spin-\( z \) component, we have \( \langle nk|\sigma_z|nk\rangle = 2\varepsilon_k \sigma_z \), and this implies \(-2nd\langle nk|\sigma_z|nk\rangle = 0 \). If the splitting \( d \) is nonzero (the spin-orbit coupling does not vanish and \( k \neq 0 \)), we have

\[
\langle nk|\sigma_z|nk\rangle = 0.
\]  

(A4)

The off-diagonal matrix element of \( \sigma_z \) cannot be further determined, and in general \( \langle nk|\sigma_z|mk\rangle \) depends on the choice of wave functions \( |nk\rangle \). However, from \( \sigma_z^2 = 1 \), we have \( \langle +k|\sigma_z|+k\rangle = 1 \). By inserting \( \sum_n\langle nk|\sigma_z|nk\rangle = 1 \) into the result, we obtain \( \langle +k|\sigma_z|+k\rangle \langle +k|\sigma_z|+k\rangle + \langle +k|\sigma_z|−k\rangle \langle −k|\sigma_z|+k\rangle = 1 \). Because of \( \langle +k|\sigma_z|+k\rangle = 0 \) as shown above, we have in general

\[
|\langle nk|\sigma_z|mk\rangle|^2 = 1, \quad n \neq m.
\]  

(A5)

The velocity operator is defined as \( v_b = \partial H_0/\hbar \partial k_b = \partial \varepsilon_k/\hbar \partial k_b + \sigma_x (\partial d_x/\hbar \partial k_b) + \sigma_y (\partial d_y/\hbar \partial k_b) \). By using Eqs. (A2), (A3), the off-diagonal matrix element of velocity operator is given by

\[
\langle nk|v_b|mk\rangle = -\frac{2id^2}{\hbar(E_{mk} - E_{nk})} \langle nk|\sigma_z|mk\rangle \frac{\partial \theta}{\partial k_b},
\]  

(A6)

where Eq. (24) was used. The off-diagonal matrix element of \( v_b \) is related to the position operator \( x_b \) by \( v_b = dx_b/\partial t = [x_b, H_0]/i\hbar \). By using \( \langle nk|m_k\rangle = 0 \) for \( n \neq m \), we have

\[
\langle nk|x_b|mk\rangle = \frac{i\hbar}{E_{mk} - E_{nk}} \langle nk|v_b|mk\rangle.
\]  

(A7)

For the diagonal part of \( v_b \), by using Eqs. (A1) and (A3), we can obtain \( \langle nk|v_b|nk\rangle = \partial E_{nk}/\hbar \partial k_b \).

Appendix B: Heisenberg Operator

In this appendix, we directly obtain the solutions of spin dynamics in terms of the time-evolving spin operators in the Heisenberg picture. We will show that the result obtained from the method of solving equation of motion is the same with that obtained from the Heisenberg picture.
time evolution method. The Hamiltonian under consideration is given by

\[ H_0 = \epsilon_k + K, \]  

where \( K \equiv \sigma_z d_x + \sigma_y d_y \), and \( d_x \) and \( d_y \) depends on the momentum and the spin-orbit coupling. The band index \( n \) is defined as \( E_{nk} = \epsilon_k - nd \), and the diagonal matrix element of \( K \) is \( \langle nk | K | nk \rangle = -nd \). The Pauli matrices \( \sigma_x \), \( \sigma_y \) and \( \sigma_z \) satisfy \( \{ \sigma_i, \sigma_j \} = 2\delta_{ij} \) and \( [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \).

By using \( K^2 = d_x^2 + d_y^2 \equiv d^2 \), we have \( K^{2m+1} = d^{2m}K \).

The time evolution operator of the unperturbed Hamiltonian is given by

\[ e^{iH_0 t/\hbar} = e^{i\epsilon_k t/\hbar} \left[ \cos \left( \frac{d}{\hbar} t \right) + iK \frac{d}{\hbar} \sin \left( \frac{d}{\hbar} t \right) \right]. \]  

The time evolution of \( \sigma_x \), \( \sigma_y \) and \( \sigma_z \) under the unperturbed Hamiltonian can be written as

\[ \sigma^H(t) = e^{iH_0 t/\hbar} \sigma e^{-iH_0 t/\hbar} \]

\[ = \sigma \left[ \cos \left( \frac{d}{\hbar} t \right) + i \frac{d}{\hbar} \sin \left( \frac{d}{\hbar} t \right) \right] [K, \sigma] \]

\[ + 1 \frac{d}{\hbar} \sin^2 \left( \frac{d}{\hbar} t \right) (K \sigma K). \]  

The operator \([K, \sigma]\) is given by

\[ [K, \sigma_z] = -2i\sigma_z d_y, \quad [K, \sigma_y] = 2i\sigma_z d_z, \quad [K, \sigma_x] = -2i(d_x \sigma_y - d_y \sigma_x). \]  

The operator \( K \sigma K \) is given by

\[ K \sigma x K = 2d_y (d_x \sigma_y - d_y \sigma_x) + \sigma_x d^2, \quad K \sigma y K = -2d_x (d_x \sigma_y - d_y \sigma_x) + \sigma_y d^2, \quad K \sigma z K = -\sigma_z d^2. \]  

Therefore, we have

\[ \sigma^H_x(t) = \sigma_x + \frac{d_y}{d} \sin (\Omega_0 t) \sigma_z \]

\[ - \frac{d_y}{d} (d \times \sigma)_z \cos (\Omega_0 t) - 1 \]

\[ \sigma^H_y(t) = \sigma_y - \frac{d_x}{d} \sin (\Omega_0 t) \sigma_z \]

\[ + \frac{d_x}{d} (d \times \sigma)_z \cos (\Omega_0 t) - 1 \]

\[ \sigma^H_z(t) = \sigma_z \cos (\Omega_0 t) + \frac{(d \times \sigma)_z}{d} \sin (\Omega_0 t), \]  

where \( \Omega_0 = 2d/\hbar \) was used. In order to simplify the calculations, we define \( d_x = d \sin \theta \) and \( d_y = -d \cos \theta \), and we have Eq. \( [B2] \). For position operator \( x_a \), we have

\[ x^H_a(t) = e^{iH_0 t/\hbar} x_a e^{-iH_0 t/\hbar} \]

\[ = x_a + e^{iH_0 t/\hbar} \left( i \frac{\partial}{\partial k_a} e^{-iH_0 t/\hbar} \right), \]  

By substitution of Eq. \( [B2] \), we have

\[ x^H_a(t) = x_a + \left[ \frac{\partial \epsilon_k}{\partial k_a} + \frac{K}{2d} \frac{\partial \Omega_0}{\partial k_a} \right] t + \frac{1}{2} \frac{\sigma_z}{\partial k_a} [\cos(\Omega_0 t) - 1] \]

\[ + \frac{1}{2d} \frac{\partial \theta}{\partial k_a} (d \times \sigma)_z \sin(\Omega_0 t). \]  

In the presence of applied electric field, the operator \( \mathcal{O} \) can be perturbatively expanded up to the first order of the electric field \( [28] \), and the result is given by

\[ \mathcal{O}^H(t) = \mathcal{O}^{H_0}(t) + eE_a \Gamma_a \mathcal{O}^H(t) + o(\lambda^2), \]  

where the operator \( \Gamma_a \) is given by

\[ \Gamma_a = i \int_0^t dt' e^{iH_0 t'/\hbar} x_a e^{-iH_0 t'/\hbar}. \]  

By substituting Eq. \( [B6] \) and Eq. \( [B8] \) into Eq. \( [B9] \) and Eq. \( [B10] \), after straightforward calculations, we have

\[ \sigma^H(t) = \sigma^H(t) + i \frac{e}{\hbar} \left[ A_1 \sigma_x + A_2 \sigma_y + A_3 \sigma_z + B K + C (d \times \sigma)_z \right], \]  

where \( A_1, A_2 \), \( A_3 \), \( B_x \) and \( C_x \) are for the spin-x component, the spin-y component and so on. The separation is convenient for obtaining the diagonal matrix element \( \langle nk | \sigma_z | nk \rangle \) because \( \langle nk | \sigma_z | nk \rangle = 0 \) and \( \langle nk | (d \times \sigma)_z | nk \rangle = 0 \). The results of the spin-z component are given by

\[ A_1 = \frac{it}{2d^2} \left[ \cos(\Omega_0 t) - 1 \right] \frac{\partial d_x}{\partial k_a}, \]

\[ A_2 = -i \frac{d_y}{2d^2} \left[ \cos(\Omega_0 t) - 1 \right] \frac{\partial d_y}{\partial k_a} + i \frac{\theta}{2d} \left[ \frac{1}{\Omega_0} \sin(\Omega_0 t) - t \right], \]

\[ A_3 = i \frac{t}{2d^2} \left[ \frac{\partial d_x}{\partial k_a} \sin(\Omega_0 t) - it \frac{\partial d_y}{\partial k_a} \cos(\Omega_0 t) \right] \]

\[ + \frac{\theta}{2d^2} \left[ \frac{1}{\Omega_0} \sin(\Omega_0 t) - t \right] d_x, \]

\[ B_x = i \frac{\theta}{2d} \left[ \frac{1}{\Omega_0} \sin(\Omega_0 t) - t \right] d_y \left[ \cos(\Omega_0 t) - 1 \right] - \frac{\theta}{2d^2} \left[ \frac{1}{\Omega_0} \sin(\Omega_0 t) - t \right] d_x \sin(\Omega_0 t), \]

\[ C_x = -i \frac{t}{2d^2} \frac{\partial d_y}{\partial k_a} \sin(\Omega_0 t) - it \frac{\partial d_x}{\partial k_a} \left[ \frac{d_y}{d} \cos(\Omega_0 t) - 1 \right]. \]  

By using \( \langle nk | \sigma^H(t) | nk \rangle = \langle nk | \sigma_z | nk \rangle = -nd_x/d, \langle nk | \sigma_y | nk \rangle = -nd_y/d, \langle nk | K | nk \rangle = -nd \) and neglecting the irrelevant terms \( C_x \) and \( A_3 \), we obtain
\[ \langle \mathbf{n} | \sigma_z^H(t) | \mathbf{n} \rangle = -n \frac{d_x}{d} + \frac{i}{\hbar} \left\{ \frac{d_y}{d} \left[ \cos(\Omega t) - 1 \right] \right\} \frac{\int d}{\Omega} \left( \frac{\partial \mathbf{d}}{\partial \mathbf{k}_a} \times \mathbf{d} \right) \left( -\frac{\partial \theta}{\partial \mathbf{k}_a} \frac{\hbar}{2d} \sin(\Omega t) - t \right) \frac{d_y}{d} \cos(\Omega t) \right\} = -n \frac{d_x}{d} + \frac{neE_a}{\hbar} \frac{d_y}{d} \left[ \frac{1}{\Omega_0} \sin(\Omega t) - t \right]. \]  

(B13)

Similar to the derivation of \( \langle \mathbf{n} | \sigma_x^H(t) | \mathbf{n} \rangle \), the spin-\( y \) component is given by

\[ \langle \mathbf{n} | \sigma_y^H(t) | \mathbf{n} \rangle = -nd_y/d, \]

For the spin-\( z \) component, the coefficients are given by

\begin{align*}
A_{z1} &= -it\sin(\Omega t) \frac{1}{\hbar} \frac{d\theta}{d\mathbf{k}_a}, \\
A_{z2} &= it\sin(\Omega t) \frac{1}{\hbar} \frac{d\mathbf{k}_a}{d\theta}, \\
A_{z3} &= -\frac{it^2}{2} \frac{\partial \mathbf{k}_a}{d\theta} \sin(\Omega t), \\
B_z &= \frac{it}{d} \frac{\partial \mathbf{k}_a}{d\theta} \sin(\Omega t) + \frac{i\hbar}{2} \frac{\partial \mathbf{k}_a}{d\theta} \left[ \sin(\Omega t) - 1 \right], \\
C_z &= \frac{it^2}{2} \frac{\partial \mathbf{k}_a}{d\theta} \cos(\Omega t) - it\sin(\Omega t) \frac{1}{d^2} \frac{d\mathbf{k}_a}{d\theta}.
\end{align*}

(B15)

By using \( \langle \mathbf{n} | \sigma_x^H(t) | \mathbf{n} \rangle = -nd_x/d \), we can

\[ \langle \mathbf{n} | \sigma_y^H(t) | \mathbf{n} \rangle = -nd_y/d, \langle \mathbf{n} | K^H(t) | \mathbf{n} \rangle = -nd, \langle \mathbf{n} | \sigma_z^H(t) | \mathbf{n} \rangle = 0 \]

and neglecting the irrelevant terms \( C_z \) and \( A_{z3} \), we can obtain

\[ \langle \mathbf{n} | \sigma_z^H(t) | \mathbf{n} \rangle = -n \frac{d_x}{d} \frac{neE_a}{\hbar} \frac{d_y}{d} \left[ \frac{1}{\Omega_0} \sin(\Omega t) - 1 \right]. \]  

(B16)

By using the definition of \( \Sigma_z^N \), the components of spin can be written as

\[ \langle \mathbf{n} | \sigma_z^H(t) | \mathbf{n} \rangle \hat{e}_z = \Sigma_z^N \left[ 1 - \cos(\Omega t) \right] \hat{e}_z. \]  

(B17)

and we can see that Eq. (B17) is exactly the same with Eq. (63).

\[ \langle \mathbf{n} | \sigma_x^H(t) | \mathbf{n} \rangle = -nt \frac{\partial \mathbf{k}_a}{d\theta} \left[ \sin(\Omega t) - 1 \right], \]

We can see that Eq. (B17) is exactly the same with Eq. (63).
[15] T.-W Chen, J.-H. Li and C. D. Hu, Phys. Rev. B 90, 195202 (2014).
[16] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005).
[17] X.-L. Qi, Y.-S. Wu, and S.-C. Zhang, Phys. Rev. B 74, 085308 (2006).
[18] S. Murakami and N. Nagaosa, Comprehensive Semiconductor Science and Technology, 1st ed., Vol. 1, edited by P. Bhattacharya, R. Fornari, and H. Kamimura (Elsevier Science, New York, 2011).
[19] Haldane, Phys. Rev. Lett. 61, 2015 (1988).
[20] J. Shi, P. Zhang, D. Xiao, Q. Niu, Phys. Rev. Lett. 96, 076604 (2006).
[21] Q.-F. Sun, and X. C. Xie, Phys. Rev. B 72, 245305 (2005); Y. Wang, K. Xia, Z.-B. Su, and Z. Ma, Phys. Rev. Lett. 96, 066601 (2006); P.-Q. Jin, Y.-Q. Li, and F.-C. Zhang, J. Phys. A 39, 7115 (2006); J. Wang, B. Wang, W. Ren, and H. Guo, Phys. Rev. B 74, 155307 (2006); Q.-F. Sun, X. C. Xie, and J. Wang, Phys. Rev. B 77, 035327 (2008).
[22] S. G. Tan, M. B. A. Jalil, C. S. Ho, Z. Siu and S. Murakami, Sci. Rep. 5, 18409 (2015).
[23] T. T. Ong and N. Nagaosa, Phys. Rev. Lett. 121, 066603 (2018).
[24] S.-Q. Shen, Phys. Rev. Lett. 95, 187203 (2005); E. M. Chudnovsky, Phys. Rev. Lett. 99, 206601 (2007); S. G. Tan, and M. B. A. Jalil, J. Phys. Soc. Jpn. 82, 094714 (2013); C. Ho, S. G. Tan, and M. B. A. Jalil, Euro. Phys. Lett. 107, 37005 (2014).
[25] K. Morawetz, Euro. Phys. Lett. 104, 27005 (2013).
[26] A. M. Samarakoon, G. Wachtel, Y. Yamaji, D. A. Tennant, C. D. Batista, and Y. B. Kim, phys. Rev. B 98, 045121 (2018).
[27] A. Amikam, Introduction to the Theory of Ferromagnetism, 2nd Ed., (Oxford University Press, New York 2000); S. Chikazumi, Physics of Ferromagnetism ((Oxford University Press, New York 1997).
[28] T. Iwata, J. Magn. Magn. Mater. 31-34, 1013 (1983); T. Iwata, J. Magn. Magn. Mater. 59, 215 (1986); V. G. Baryakhtar, Zh. Eksp. Teor. Fiz. 87, 1501 (1984); W. M. Saslow, J. Appl. Phys. 105, 07D315 (2009).
[29] G. D. Mahan, Many-particle Physics (Kluwer Academic/Plenum, 2000)
[30] J. B. Miller, D. M. Zumbuhl, C. M. Marcus, Y. B. Lyanda-Geller, D. Goldhaber-Gordon, K. Campman, and A. C. Gossard Phys. Rev. Lett. 90, 076807 (2003)
[31] T.-W. Chen and G. Y. Guo, Phys. Rev. B 79, 125301 (2009)