FUNCTIONAL STRONG LAWS OF LARGE NUMBERS
FOR EULER CHARACTERISTIC PROCESSES
OF EXTREME SAMPLE CLOUDS

ANDREW M. THOMAS AND TAKASHI OWADA

Abstract. This study demonstrates functional strong law large numbers for the Euler characteristic process of random geometric complexes formed by random points outside of an expanding ball in \( \mathbb{R}^d \), in two distinct extreme value theoretic scenarios. When the points are drawn from a heavy-tailed distribution with a regularly varying tail, the Euler characteristic process grows at a regularly varying rate, and the scaled process converges uniformly and almost surely to a smooth function. When the points are drawn from a distribution with an exponentially decaying tail, the Euler characteristic process grows logarithmically, and the scaled process converges to another smooth function in the same sense. All of the limit theorems take place when the points inside the expanding ball are densely distributed, so that the simplex counts outside of the ball of all dimensions contribute to the Euler characteristic process.

1. Introduction

To recover the topology of a manifold using point cloud data, one needs to have a strong understanding of how the points are perturbed from the manifold. In [9], given a “nice” manifold, it was shown that one can recover the topology of the manifold by a sufficiently dense random sampling of points if the noise is bounded. In [10] it was shown that the recovery is still possible by a sufficiently dense random sampling of points if the noise is standard multivariate Gaussian and the variance is bounded by a function of the reach and dimension of the manifold. However, if the noise distribution has a heavy tail, the recovery of topology from point cloud data will be severely impacted because of layers of extraneous homology elements generated by the heavy-tailed noise. This layered structure is referred to as topological crackle, which is visualized in Figure 1 as a layer of Betti numbers in \( \mathbb{R}^d \), \( d \geq 2 \). Loosely speaking, the \( k \)th Betti number, denoted as \( \beta_k \) in Figure 1, counts the number of \( k \)-dimensional topological cycles which can be interpreted as the boundary of a \((k+1)\)-dimensional body. For each layer in Figure 1 there is a particular dimension \( k \in \{0, \ldots, d-1\} \), such that the \( k \)th Betti number converges to a Poisson distribution as the sample size \( n \) increases, while all the other Betti numbers either vanish or diverge as \( n \to \infty \). The earliest results examining topological crackle from a probabilistic viewpoint are given in [1]. One of the main findings in [1] (see also [14]) is that the layered structure in Figure 1 appears only when the noise distribution has a tail at least as heavy as that of an exponential distribution. After this pioneering paper, the probabilistic features of crackle has repeatedly been investigated via the behavior of Betti numbers [12, 13, 14, 15].

More specifically, let \( X_n = \{X_1, \ldots, X_n\} \subset \mathbb{R}^d \) be a random sample in \( \mathbb{R}^d \). Given an integer \( k \in \{0, \ldots, d-1\} \) and a sequence of non-random radii \( R_n \to \infty \), the central object for the studies cited in the last paragraph has been the \( k \)th Betti number, defined by

\[
\beta_{k,n}(t) := \beta_k \left( \bigcup_{x \in X_n \cap B(0, R_n)^c} B(x, t) \right),
\]

2020 Mathematics Subject Classification. Primary 60F15, 60G70. Secondary 55U10, 60F17.

Key words and phrases. Functional strong law of large numbers, Euler characteristic, random geometric complex, topological crackle.

This research is partially supported by the NSF grant DMS-1811428.
ANDREW M. THOMAS AND TAKASHI OWADA

Figure 1. Topological crackle is a layered structure of topological invariants such as Betti numbers. The kth Betti number is denoted by $\beta_k$, and “Poi” represents a Poisson distribution. A core is a centered ball in which random points are densely scattered, so that the union of the unit balls around them is contractible—that is, the union is topologically equivalent to a single point [11, 14].

where $B(x,t) = \{ y \in \mathbb{R}^d : \| y - x \| < t \}$ (resp. $\bar{B}(x,t)$) is an open (resp. closed) ball of radius $t$ around $x \in \mathbb{R}^d$ (here $\| \cdot \|$ denotes the Euclidean norm). Then, $\beta_{k,n}(t)$ counts the number of $k$-dimensional topological cycles outside of the expanding ball $B(0,R_n)$. Defining the kth Betti number this way, [14] studied the case in which (1.1) weakly converges to a Poisson distribution as $n \to \infty$, while [12] established the central limit theorem for (1.1), in the case that infinitely many $k$-dimensional topological cycles appear outside of $B(0,R_n)$ as $n \to \infty$. Furthermore, [13] gave a rigorous description of the limiting Betti numbers when the random points are generated by a classical moving average process, and [15] discussed the weak convergence of a standard graphical representation of topological cycles.

In contrast to these previous papers, the primary objective of this paper is to examine the crackle phenomena from the viewpoint of the Euler characteristic. In the context of topological crackle, the Euler characteristic is constructed from a higher-dimensional analogue of a geometric graph, called the geometric complex. Among many varieties of geometric complexes (see [5]), the Vietoris-Rips complex and the Čech complex are specific examples that deserve our attention.

Definition 1.1. Given a point set $\mathcal{X} = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d$ and a positive number $t > 0$, the Vietoris-Rips complex $R(\mathcal{X},t)$ is defined as follows.

- The 0-simplices are the points in $\mathcal{X}$.
- A k-simplex $\sigma = [x_{i_0}, \ldots, x_{i_k}]$ is in $R(\mathcal{X},t)$ if $\bar{B}(x_{i_p}, t/2) \cap \bar{B}(x_{i_q}, t/2) \neq \emptyset$ for every $0 \leq p < q \leq k$.

Definition 1.2. Given the same $\mathcal{X}$ and $t > 0$, the Čech complex $\check{C}(\mathcal{X},t)$ is defined as follows.

- The 0-simplices are the points in $\mathcal{X}$.
- A k-simplex $\sigma = [x_{i_0}, \ldots, x_{i_k}]$ is in $\check{C}(\mathcal{X},t)$ if a family of balls $\{\bar{B}(x_{i_p}, t/2), p = 0, \ldots, k\}$ has a non-empty intersection.

In this paper, we first define a geometric complex that generalizes both $R(\mathcal{X},t)$ and $\check{C}(\mathcal{X},t)$ above, and then we establish the functional strong law of large numbers (FSLLN) for the corresponding Euler characteristic. In conjunction with the recent development of Topological Data Analysis, the literature dealing with the asymptotics of the Euler characteristic of random geometric complexes has flourished [3, 4, 7, 8, 18]. However, none of these studies have paid sufficient attention to the topology of the tail of a probability distribution. In fact, they have only investigated the topology of random geometric complexes in $\mathbb{R}^d$ without any geometric constraints, so that the resulting limit theorems are robust to the choice of probability density. In contrast, the current study considers geometric complexes outside of an expanding ball $B(0,R_n)$. As a consequence, the nature of the obtained limit theorems will depend heavily on the decay rate of the probability density.

The remainder of this paper is structured as follows. Section 2 provides a discussion of the background material necessary for this paper. The paper then proceeds to the heavy tailed setup...
and presents the FSLLN for the Euler characteristic in Section 3. The paper continues with a discussion of the intricacies of the exponentially decaying tail case along with the corresponding FSLLN in Section 4. The proofs of the main results for both setups are deferred to Section 5. From a technical point of view, the studies most relevant to this paper are [6, 18], in which the authors established strong laws of large numbers for topological invariants—such as Betti numbers and the Euler characteristic—in the non-extreme value theoretic setup. In particular, these studies revealed that if the topological invariants are scaled proportionally to the sample size, they converge almost surely to a finite and positive constant. Owing to this fact, the main machinery in their proofs is a direct application of the Borel-Cantelli lemma, together with the calculation of lower-order moments. On the contrary, the main challenge in this paper is that the scaling sequence of the Euler characteristic may grow very slowly (e.g., logarithmically), in which case, a direct application of the Borel-Cantelli lemma does not work. To overcome this difficulty, we need to identify suitable subsequential upper and lower bounds of the Euler characteristic to which one can apply the Borel-Cantelli lemma. This is a standard technique in the theory of random geometric graphs—see Chapter 3 of the monograph [16]. It is possible to extend these arguments to our higher-dimensional setup since the geometric complexes such as those in Definitions 1.1 and 1.2 are higher-dimensional analogues of a geometric graph.

As a final remark, we point out that other types of the limit theorems for the Euler characteristic still remain as a future topic. For example, it seems feasible to establish a (functional) central limit theorem for the Betti numbers and the Euler characteristic—in the non-extreme value theoretic setup. In particular, these studies revealed that if the topological invariants are scaled proportionally to the sample size, they converge almost surely to a finite and positive constant. Owing to this fact, the main machinery in their proofs is a direct application of the Borel-Cantelli lemma, together with the calculation of lower-order moments. On the contrary, the main challenge in this paper is that the scaling sequence of the Euler characteristic may grow very slowly (e.g., logarithmically), in which case, a direct application of the Borel-Cantelli lemma does not work. To overcome this difficulty, we need to identify suitable subsequential upper and lower bounds of the Euler characteristic to which one can apply the Borel-Cantelli lemma. This is a standard technique in the theory of random geometric graphs—see Chapter 3 of the monograph [16]. It is possible to extend these arguments to our higher-dimensional setup since the geometric complexes such as those in Definitions 1.1 and 1.2 are higher-dimensional analogues of a geometric graph.

2. Preliminaries

The point cloud of interest in this study is the sample \( X_n := \{X_1, \ldots, X_n\} \) of \( n \) i.i.d random points in \( \mathbb{R}^d \), \( d \geq 2 \) with spherically symmetric density \( f \). Spherical symmetry of \( f \) is far from necessary; the results in this paper could be extended to densities with ellipsoidal level sets fairly easily. Denote \( \lambda \) to be Lebesgue measure on \( \mathbb{R}^d \) and \( S^{d-1} = \{x \in \mathbb{R}^d : ||x|| = 1\} \). Let us here define the spherical measure
\[
\nu_{d-1}(A) := d \cdot \lambda\left(\{x \in B(0, 1) : ||x|| \in A\}\right)
\]
for Borel sets \( A \subset S^{d-1} \). We denote \( \omega_d := \lambda(B(0, 1)) = 2\pi^{d/2}/(d\Gamma(d/2)) \), and \( s_{d-1} := \nu_{d-1}(S^{d-1}) = d\omega_d \).

Let \( \mathcal{F}(\mathbb{R}^d) \) be the collection of all non-empty, finite subsets of \( \mathbb{R}^d \). For \( X \in \mathcal{F}(\mathbb{R}^d) \), a simplicial complex \( \mathcal{K}(X) \) is a collection of subsets of \( X \) such that if \( \sigma \in \mathcal{K}(X) \) and \( \tau \subset \sigma \) then \( \tau \in \mathcal{K}(X) \). Evidently, the Vietoris-Rips complex \( \mathcal{R}(X,t) \) and the Čech complex \( \mathcal{C}(X,t) \) satisfy this condition. We call \( \sigma \in \mathcal{K}(X) \) a \( k \)-simplex if \( |\sigma| = k + 1 \).

Subsequently, let \( h : \mathcal{F}(\mathbb{R}^d) \to \{0, 1\} \) be an indicator function satisfying the following conditions.

\begin{itemize}
  \item [(H1)] \( h(X) \leq h(Y) \) for all \( Y \subset X \).
  \item [(H2)] \( h \) is translation invariant—that is, for every \( X \in \mathcal{F}(\mathbb{R}^d) \) and \( y \in \mathbb{R}^d \), we have \( h(X+y) = h(X) \).
  \item [(H3)] \( h \) is locally determined—that is, there exists \( c > 0 \) so that \( h(X) = 0 \) whenever \( \text{diam}(X) > c \), where \( \text{diam}(X) := \max_{x,y \in X} ||x - y|| \).
\end{itemize}

By abusing notation slightly, for \( X = \{x_1, \ldots, x_m\} \in \mathcal{F}(\mathbb{R}^d) \), we write \( h(X) = h(x_1, \ldots, x_m) \). Moreover, for \( X = \{x_1, \ldots, x_m\} \) and \( a \in \mathbb{R} \), we write \( aX = \{ax_1, \ldots, ax_m\} \). We then define a scaled version of \( h \) by
\[
h_t(X) := h(t^{-1}X), \quad X \in \mathcal{F}(\mathbb{R}^d), \quad t > 0,
\]
with the additional assumption that
(H4) $h_4(\mathcal{X}) \leq h_t(\mathcal{X})$ for all $\mathcal{X} \in \mathcal{F}(\mathbb{R}^d)$ and $0 \leq s \leq t < \infty$.

Given such a scaled indicator $h_t$, we can construct the simplicial complex
\[
K(\mathcal{X}, t) := \{Y \subset \mathcal{X} : h_t(Y) = 1\}.
\] (2.1)
By virtue of (H4) above, (2.1) induces a filtration over a point set $\mathcal{X}$—that is, $K(\mathcal{X}, s) \subset K(\mathcal{X}, t)$ for all $0 \leq s \leq t$.

Note that if one takes
\[
h(\mathcal{X}) = 1\{\text{diam}(\mathcal{X}) \leq 1\}, \quad \mathcal{X} \in \mathcal{F}(\mathbb{R}^d),
\]
then, (2.1) induces a Vietoris-Rips filtration. Moreover, if we define
\[
h(\mathcal{X}) = 1\{\bigcap_{x \in \mathcal{X}} B(x, 1/2) \neq \emptyset\}, \quad \mathcal{X} \in \mathcal{F}(\mathbb{R}^d),
\]
then, (2.1) induces a Čech filtration.

As mentioned in the Introduction, the objective of this paper is to study “extreme-value” behavior of random geometric complexes via the Euler characteristic. More concretely, with a non-random sequence $R_n \to \infty$, we study the complex
\[
K(\mathcal{X}_n \cap B(0, R_n)^c, t), \quad t \geq 0,
\] (2.2)
which is distributed increasingly further from the origin as $n \to \infty$. We now define the Euler characteristic pertaining to (2.2) by
\[
\chi_n(t) := \sum_{k=0}^{\infty} (-1)^k S_{k,n}(t), \quad t \geq 0,
\] (2.3)
where $S_{k,n}(t)$ denotes the $k$-simplex counts in the complex (2.2)—namely,
\[
S_{k,n}(t) := \sum_{Y \subset \mathcal{X}_n, |Y| = k+1} h_t(Y) 1\{\min_{y \in Y} \|y\| \geq R_n\}.
\]
Note that for every $n \in \mathbb{N} = \{1, 2, \ldots\}$, (2.3) is almost surely a finite sum as $S_{k,n}(t) \equiv 0$ for all $k \geq n$. Furthermore, (2.3) can be seen as a stochastic process in parameter $t$, with right continuous sample paths and left limits. In the following, we establish the FSLLN for the Euler characteristic process $\chi_n(t)$, $t \geq 0$ in the space $D[0, \infty)$ of right continuous functions on $[0, \infty)$ with left limits. In particular, we equip $D[0, \infty)$ with the uniform topology.

3. Regularly varying tail case

In this section, we detail the large-sample behavior of (2.3) of an extreme sample cloud when the distribution of points has a heavy tail. Recall that $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ denotes a random sample in $\mathbb{R}^d$ with a spherically symmetric density $f$. We assume that there exists $\alpha > d$, such that
\[
\lim_{r \to \infty} \frac{f(r \theta)}{f(r \bar{\theta})} = t^{-\alpha}, \quad t > 0,
\] (3.1)
for every (equivalently, some) $\theta \in S^{d-1}$. Because of the spherical symmetry of the density, we can define $f(r) := f(r \theta)$ for all $r \geq 0$ and $\theta \in S^{d-1}$.

It is known from the literature of topological crackle (see [11]) that the behavior of topological invariants significantly depends on the limit value of $nf(R_n)$. The present study focuses exclusively on the case when the limit of $nf(R_n)$ is a positive and finite constant—that is,
\[
nf(R_n) \to \xi \quad \text{as } n \to \infty \quad \text{for some } \xi \in (0, \infty).
\] (3.2)
In this instance, the ball $B(0, R_n)$ coincides with the notion of a weak core (see [11]). The configuration of points between the outside and inside of a weak core is very different. Inside of the weak core, the random points are very densely distributed, and the topology of the union of balls around
those points (with a fixed radius) is nearly trivial. Outside of a weak core, however, the random points are distributed more sparsely, and the topology of the union of balls becomes much more non-trivial. In the FSLLN below, by an appropriate scaling, the simplex counts of all dimensions in (2.3) will contribute to the limit. In contrast, if $R_n$ grows faster, such that $nf(R_n) \to 0$ as $n \to \infty$, then even under an appropriate scaling, the Euler characteristic is dominated asymptotically by the 0-simplices, or the extremal points. In this case, the Euler characteristic simply counts points in $X_n$ outside $B(0, R_n)$ as $n \to \infty$.

Finally, (3.2) implies that one can typically take

$$R_n = \xi^{-1/\alpha} (1/f)^+ (n);$$

thus, $(R_n)_{n \geq 1}$ is a regularly varying sequence of exponent $1/\alpha$. We can now present the FSLLN for the Euler characteristic process in this setup.

**Theorem 3.1.** Suppose that $f$ is a spherically symmetric density satisfying (3.1). Assume that $nf(R_n) \to \xi$ as $n \to \infty$ for some $\xi \in (0, \infty)$. Then, the Euler characteristic process in (2.3) satisfies the following functional SLLN, i.e., as $n \to \infty$,

$$\left( \frac{\chi_n(t)}{R_n}, t \geq 0 \right) \to \left( \sum_{k=0}^{\infty} (-1)^k s_k(t), t \geq 0 \right), \text{ a.s. in } D[0, \infty).$$

where

$$s_k(t) := s_{d-1} \xi^{k+1} \frac{1}{(k+1)!((\alpha(k+1) - d)} \int_{(\mathbb{R}^d)^k} h_t(0, y_1, \ldots, y_k) \, dy, \quad t \geq 0, \quad k \geq 1,$$

and $s_0(t) \equiv s_{d-1} \xi / (\alpha - d)$.

The following example illustrates the uniform convergence that takes place in the above theorem.

**Example 3.2.** Consider the density defined by

$$f(x) = \frac{2}{\pi^2 (1 + \|x\|^4)}, \quad x \in \mathbb{R}^2.$$

Define $R_n := (2n/\pi^2)^{1/4}$, so that $nf(R_n) \to 1$. We consider the Vietoris-Rips complex induced by $h(X) = 1\{\text{diam}(X) \leq 1/\sqrt{2}\}$, where diam is calculated here with respect to the $\ell^\infty$ norm. Then, it follows from Theorem 3.1 that, as $n \to \infty$,

$$\left( \frac{\pi \chi_n(t)}{\sqrt{2n}}, t \geq 0 \right) \to \left( \sum_{k=0}^{\infty} (-1)^k s_k(t), t \geq 0 \right), \text{ a.s. in } D[0, \infty).$$

The limiting function above can be simplified as follows:

$$\sum_{k=0}^{\infty} (-1)^k s_k(t) = 2\pi \sum_{k=0}^{\infty} \frac{(-1)^k (t/\sqrt{2})^{2k}}{(k+1)!((4(k+1) - 2)} \int_{(\mathbb{R}^2)^k} h_{\sqrt{2}t}(0, y_1, \ldots, y_k) \, dy$$

$$= \pi \sum_{k=0}^{\infty} \frac{(-t^2/2)^k (k+1)^2}{(k+1)!((2k+1)}$$

$$= \frac{\pi}{2} e^{-t^2/2} + \left( \frac{\pi}{2} \right)^{3/2} t \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (t/\sqrt{2})^{2k+1}}{k!(2k+1)}$$

$$= \frac{\pi}{2} \left[ e^{-t^2/2} + \sqrt{\pi} \left( \frac{2\Phi(t) - 1}{t} \right) \right],$$

where $\Phi$ is the standard normal distribution function. This limiting function is plotted in red in Figure 2. This figure indicates that $\pi \chi_n(t) / \sqrt{2n}$ converges uniformly as the sample size increases.
the case \( \tau > 0 \) does not belong to the scope of our study. Note that (4.2) trivially holds for every limit value of the regular variation exponent of \( \tau \).

Since the main theme of this study is topological crackle, we do not treat the case \( \tau > 0 \).

One of the implications of Theorem 3.1 is that one can immediately obtain various limit theorems of functionals of the Euler characteristic process. For every continuous functional \( T \) on \( D[0, \infty) \), it indeed holds that, as \( n \to \infty \),

\[
T \left( \frac{\pi \chi_n}{\sqrt{2n}} \right) \to T \left( \sum_{k=0}^{\infty} (-1)^k s_k \right), \quad a.s.
\]

For example, if \( T : D[0, \infty) \to [0, \infty) \) is a continuous functional defined by \( T(x) = \sup_{a \leq x \leq b} |x(t)| \), \( 0 \leq a < b < \infty \), we have

\[
\frac{\pi}{\sqrt{2n}} \sup_{a \leq t \leq b} |\chi_n(t)| \to \sup_{a \leq t \leq b} \left| \sum_{k=0}^{\infty} (-1)^k s_k(t) \right| = \sum_{k=0}^{\infty} (-1)^k s_k(a), \quad a.s.
\]

4. Exponentially decaying tail case

In this section, we consider a density of an exponentially decaying tail. We assume that the density \( f \) is specified by

\[
f(x) = C \exp \left\{ -\psi(||x||) \right\}, \quad x \in \mathbb{R}^d,
\]

where \( C \) is a normalizing constant and \( \psi : [0, \infty) \to [0, \infty) \) is a regularly varying function (at infinity) of an exponent \( \tau \in (0, 1) \). Moreover, \( \psi \) is assumed to be twice differentiable, such that \( \psi'(x) > 0 \) for all \( x > 0 \), and \( \psi' \), \( \psi'' \) are both eventually non-increasing. Namely, there exists \( z_0 > 0 \) such that \( \psi' \) and \( \psi'' \) are non-increasing in \([z_0, \infty)\). By the spherical symmetry of (4.1), we often use the same abbreviation as in Section 3—that is, \( f(r) := f(r\theta) \) for all \( r \geq 0 \) and \( \theta \in S^{d-1} \). Under this setup, let \( a(z) := 1/\psi'(z) \); then, \( a \) is also regularly varying with index \( 1 - \tau \) (see, e.g., Proposition 2.5 in [17]).

Here, it is important to note that the occurrence of topological crackle depends on the limit value of \( a(z) \) as \( z \to \infty \) (see [14]). In particular, [14] showed that crackle occurs if and only if

\[
\zeta := \lim_{z \to \infty} a(z) \in (0, \infty).
\]

Since the main theme of this study is topological crackle, we do not treat the case \( \zeta = 0 \). In terms of the regular variation exponent of \( \psi \), we exclude the case \( \tau > 1 \). So, for instance, the multivariate Gaussian densities do not belong to the scope of our study. Note that (4.2) trivially holds for every \( \tau \in (0, 1) \).

As in the regularly varying tail case in Section 3, the behavior of (2.3) is crucially determined by the limit value of \( nf(R_n) \)—see [11, 14, 15]. By the same reasoning as in Section 3, we study only the case \( nf(R_n) \to \xi \) as \( n \to \infty \) for some \( \xi \in (0, \infty) \). Now, we are ready to state the FSLLN below.
Interestingly, if $\zeta = \infty$ in (4.2), the limiting function in (4.3) agrees with (3.4) up to multiplicative constants.

**Theorem 4.1.** Suppose that $f$ is a density specified by (4.1) with $\tau \in (0,1]$. If $d = 2$, we restrict the range of $\tau$ to $(0,1)$. Suppose further that $nf(R_n) \to \xi$ as $n \to \infty$ for some $\xi \in (0,\infty)$. Then, we have, as $n \to \infty$,

$$
\left( \frac{\chi_n(t)}{a(R_n)R_n^{-1}}, t \geq 0 \right) \to \left( \sum_{k=0}^{\infty} (-1)^k s_k(t), t \geq 0 \right), \text{ a.s. in } D[0,\infty),
$$

where

$$
s_k(t) := \frac{\zeta^{k+1}}{(k+1)!} \int_0^\infty \int_{S^{d-1}} \int_{(\mathbb{R}^d)^k} h_t(0,y_1,\ldots,y_k) e^{-(k+1)\rho - \zeta^{-1} \sum_{i=1}^k \langle \theta, y_i \rangle} \prod_{i=1}^k 1\{\rho + \zeta^{-1} \langle \theta, y_i \rangle \geq 0\} \, dy \, d\nu_{d-1}(\theta) \, d\rho, \quad t \geq 0, \quad k \geq 1,
$$

with $\langle \cdot, \cdot \rangle$ being the Euclidean inner product and $s_0(t) \equiv s_{d-1}\xi$.

**Example 4.2.** We consider a special case of the density in (4.1),

$$
f(x) = Ce^{-\|x\|^\tau/\tau}, \quad x \in \mathbb{R}^d, \quad \tau \in (0,1].
$$

Define $R_n = (\tau \log n + \tau \log C)^{1/\tau}$ so that $nf(R_n) = 1$. Then, $a(z) = z^{1-\tau}$, $z > 0$. According to Theorem 4.1

$$
\left( \frac{\chi_n(t)}{(\tau \log n + \tau \log C)^{(d-\tau)/\tau}}, t \geq 0 \right) \to \left( \sum_{k=0}^{\infty} (-1)^k s_k(t), t \geq 0 \right), \text{ a.s. in } D[0,\infty).
$$

where $s_k(t)$ is defined in (4.3). Moreover, applying the functional $T(x) = \sup_{a \leq t \leq b} |x(t)|$ for $x \in D[0,\infty)$, $0 \leq a < b < \infty$, we have that

$$
\sup_{a \leq t \leq b} \left| \frac{\chi_n(t)}{(\tau \log n + \tau \log C)^{(d-\tau)/\tau}} \right| \to \sup_{a \leq t \leq b} \left| \sum_{k=0}^{\infty} (-1)^k s_k(t) \right| \quad \text{a.s.}
$$

5. **Proofs of main theorems**

Throughout the proof, denote by $C^*$ a positive constant that is independent of $n$ and may vary between (and even within) the lines. Denote by $\text{RV}_\rho$ the collection of regularly varying sequences (or functions) at infinity with exponent $\rho \in \mathbb{R}$. For $a, b \in \mathbb{R}$, write $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. For two sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$, $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$.

First, we present a fundamental result which allows us to extend a pointwise SLLN to a functional SLLN in the space $D[0,\infty)$.

**Proposition 5.1** (Proposition 4.2 in [15]). Let $(X_n, n \in \mathbb{N})$ be a sequence of random elements in $D[0,\infty)$ with non-decreasing sample paths. Suppose $a : [0,\infty) \to \mathbb{R}$ is a deterministic, continuous, and non-decreasing function. If we have

$$
X_n(t) \to a(t), \quad n \to \infty, \quad \text{a.s.,}
$$

for every $t \geq 0$, then it follows that

$$
\sup_{t \in [0,T]} |X_n(t) - a(t)| \to 0, \quad n \to \infty, \quad \text{a.s.,}
$$

for every $0 \leq T < \infty$. Hence, it holds that $X_n(t) \to a(t)$ a.s. in $D[0,\infty)$ under the uniform topology.
By virtue of this proposition, for the proof of Theorem 3.1 it suffices to show that as \( n \to \infty \),
\[
\frac{\chi_n(t)}{R_n^d} \to \sum_{k=0}^{\infty} (-1)^k s_k(t), \quad \text{a.s.}
\]
for every \( t \geq 0 \). Subsequently, we divide the Euler characteristic process into two terms:
\[
\chi_n(t) = \sum_{k=0}^{\infty} S_{2k,n}(t) - \sum_{k=0}^{\infty} S_{2k+1,n}(t) =: \chi_n^{(1)}(t) - \chi_n^{(2)}(t). \tag{5.1}
\]
In addition, the limiting function can also be decomposed as
\[
\sum_{k=0}^{\infty} (-1)^k s_k(t) = \sum_{k=0}^{\infty} s_{2k}(t) - \sum_{k=0}^{\infty} s_{2k+1}(t) =: K_1(t) - K_2(t). \tag{5.2}
\]
From (5.1) and (5.2), it is now sufficient to prove that for every \( t \geq 0 \) and \( i = 1, 2 \),
\[
\frac{\chi_n^{(i)}(t)}{R_n^d} \to K_i(t), \quad n \to \infty, \quad \text{a.s.} \tag{5.3}
\]
In the case of Theorem 4.1 defining \( \chi_n^{(i)}(t) \) and \( K_i(t) \) analogously, it suffices to show that for each \( t \geq 0 \) and \( i = 1, 2 \),
\[
\frac{\chi_n^{(i)}(t)}{a(R_n) R_n^{d-1}} \to K_i(t), \quad n \to \infty, \quad \text{a.s.} \tag{5.4}
\]

5.1. Proof of Theorem 3.1. The goal of this subsection is to prove (5.3). We handle the case \( i = 1 \) only, as the proof is totally the same regardless of \( i \in \{1, 2\} \). Let
\[
u_m = \lfloor e^{m\gamma} \rfloor, \quad m = 0, 1, 2, \ldots, \tag{5.5}
\]
for some \( \gamma \in (0, 1) \). Then, for every \( n \in \mathbb{N} \), there exists a unique \( m = m(n) \) such that \( u_m \leq n < u_{m+1} \). Let us also define
\[
p_m = \arg\max \{ u_m \leq \ell \leq u_{m+1} : R_{\ell} \}, \tag{5.6}
\]
\[
q_m = \arg\min \{ u_m \leq \ell \leq u_{m+1} : R_{\ell} \}. \tag{5.7}
\]
Below, we offer a lemma on the asymptotic moments of certain variants of the process \( \chi_n^{(1)}(t) \), defined by
\[
T_m(t) := \sum_{k=0}^{\infty} \sum_{Y \subset X_{u_{m+1}}, |Y| = 2k+1} h_t(Y) 1 \{ \min_{y \in Y} \|y\| \geq R_{q_m} \}, \tag{5.8}
\]
\[
U_m(t) := \sum_{k=0}^{\infty} \sum_{Y \subset X_{u_m}, |Y| = 2k+1} h_t(Y) 1 \{ \min_{y \in Y} \|y\| \geq R_{p_m} \}. \tag{5.9}
\]

Lemma 5.2. Under the assumptions of Theorem 3.1, we have the following asymptotic results on the first and second moments of \( T_m(t) \) and \( U_m(t) \).
\[
\lim_{m \to \infty} R^{-d}_{q_m} \mathbb{E} [T_m(t)] = K_1(t), \tag{5.10}
\]
\[
\lim_{m \to \infty} R^{-d}_{p_m} \mathbb{E} [U_m(t)] = K_1(t), \tag{5.11}
\]
\[
\sup_{m \geq 1} R^{-d}_{q_m} \text{Var} (T_m(t)) < \infty, \tag{5.12}
\]
\[
\sup_{m \geq 1} R^{-d}_{p_m} \text{Var} (U_m(t)) < \infty. \tag{5.13}
\]
Proof. We begin by offering the proofs of (5.10) and (5.11) by extending the argument in the proof of Proposition 7.2 of [11]. As for (5.10), it is clear that

\[ R_{q_m}^{-d} \mathbb{E}[T_m(t)] = \sum_{k=0}^{\infty} R_{q_m}^{-d} \left( \frac{u_{m+1}}{2k+1} \right) \mathbb{E} \left[ h_t(X_1, \ldots, X_{2k+1}) \mathbf{1} \left\{ \min_{1 \leq i \leq 2k+1} \|X_i\| \geq R_{q_m} \right\} \right], \]

where \( X_1, \ldots, X_{2k+1} \) are i.i.d random variables with density \( f \). From this, we have

\[
\begin{align*}
R_{q_m}^{-d} \left( \frac{u_{m+1}}{2k+1} \right) & \mathbb{E} \left[ h_t(X_1, \ldots, X_{2k+1}) \mathbf{1} \left\{ \min_{1 \leq i \leq 2k+1} \|X_i\| \geq R_{q_m} \right\} \right] \\
& = R_{q_m}^{-d} \left( \frac{u_{m+1}}{2k+1} \right) \int_{[\mathbb{R}^d]^{2k+1}} h_t(x_1, \ldots, x_{2k+1}) \prod_{i=1}^{2k+1} f(x_i) \mathbf{1} \left\{ \|x_i\| \geq R_{q_m} \right\} \, dx \\
& = R_{q_m}^{-d} \left( \frac{u_{m+1}}{2k+1} \right) \int_{\mathbb{R}^d} \int_{[\mathbb{R}^d]^{2k}} h_t(0, y_1, \ldots, y_{2k}) f(x) \mathbf{1} \left\{ \|x\| \geq R_{q_m} \right\} \\
& \quad \times \prod_{i=1}^{2k} f(x + y_i) \mathbf{1} \left\{ \|x + y_i\| \geq R_{q_m} \right\} \, dy \, dx,
\end{align*}
\]

by the change of variables \( x_i = x + y_{i-1}, i = 1, \ldots, 2k + 1 \) (with \( y_0 \equiv 0 \)) and the translation invariance of \( h_t \). Furthermore, we make the change of variables by \( x = R_{q_m} \rho \theta \) with \( \rho \geq 0 \) and \( \theta \in S^{d-1} \), to get that

\[
R_{q_m}^{-d} \left( \frac{u_{m+1}}{2k+1} \right) \mathbb{E} \left[ h_t(X_1, \ldots, X_{2k+1}) \mathbf{1} \left\{ \min_{1 \leq i \leq 2k+1} \|X_i\| \geq R_{q_m} \right\} \right] \\
= \left( \frac{u_{m+1}}{2k+1} \right) f(R_{q_m})^{2k+1} \int_{1}^{\infty} \int_{S^{d-1}} \int_{[\mathbb{R}^d]^{2k}} h_t(0, \mathbf{y}) \rho^{d-1} f(R_{q_m} \rho) \frac{f(R_{q_m})}{f(R_{q_m})} \\
\times \prod_{i=1}^{2k} f(R_{q_m} \rho \theta + y_i/R_{q_m}) \mathbf{1} \left\{ \|\rho \theta + y_i/R_{q_m}\| \geq 1 \right\} \, dy \, d\nu_{d-1}(\theta) \, d\rho,
\]

where \( \mathbf{y} = (y_1, \ldots, y_{2k}) \in ([\mathbb{R}^d]^{2k} \). Next, for a fixed constant \( \eta \in (0, \alpha - d) \), Potter’s bounds (see Proposition 2.6 in [17]) yield that

\[ \frac{f(R_{q_m} \rho)}{f(R_{q_m})} \leq 2 \rho^{-\alpha + \eta}, \]

and

\[ \prod_{i=1}^{2k} f(R_{q_m} \|\rho \theta + y_i/R_{q_m}\|) \mathbf{1} \left\{ \|\rho \theta + y_i/R_{q_m}\| \geq 1 \right\} \leq 2^{2k} \]

for sufficiently large \( m \). Since \( \int_{1}^{\infty} \rho^{d-1-\alpha + \eta} \, d\rho < \infty \) and \( \int_{([\mathbb{R}^d]^{2k})} h_t(0, \mathbf{y}) \, dy < \infty \) by property \( (H3) \) of \( h \), we can see that the regular variation of \( f \), as well as the dominated convergence theorem, ensures that the triple integral in (5.15) converges to

\[ s_{d-1} \int_{1}^{\infty} \rho^{d-1-\alpha(2k+1)} \, d\rho \int_{([\mathbb{R}^d]^{2k})} h_t(0, \mathbf{y}) \, dy = \frac{s_{d-1}}{\alpha(2k+1) - d} \int_{([\mathbb{R}^d]^{2k})} h_t(0, \mathbf{y}) \, dy. \]

Furthermore, (3.2) ensures that as \( m \to \infty \),

\[ \left( \frac{u_{m+1}}{2k+1} \right) f(R_{q_m})^{2k+1} \sim \left( \frac{u_{m+1} f(R_{q_m})}{(2k+1)!} \right)^{2k+1} \sim \frac{\xi^{2k+1}}{(2k+1)!} \left( \frac{u_{m+1}}{q_m} \right)^{2k+1}, \]

so that

\[ 1 \leq \frac{u_{m+1}}{q_m} \leq \frac{u_{m+1}}{u_m} \leq \frac{e^{(m+1)\gamma - m\gamma}}{1 - e^{-m\gamma}} = \frac{e^{m\gamma - (\gamma + o(1))}}{1 - e^{-m\gamma}}. \]
As $0 < \gamma < 1$, the rightmost term above converges to $1$ as $m \to \infty$. Hence,

$$
\left( \frac{u_{m+1}}{2k+1} \right) f(R_{qm})^{2k+1} \to \frac{\xi^{2k+1}}{(2k+1)!}, \quad \text{as } m \to \infty.
$$

Combining all of these results, it follows that

$$
R_{qm}^{\ell} \left( \left( \frac{u_{m+1}}{2k+1} \right) \mathbb{E} \left[ h_t(X_1, \ldots, X_{2k+1}) \mathbf{1}\left\{ \min_{1 \leq i \leq 2k+1} \|X_i\| \geq R_{qm} \right\} \right] \right)_{s2k(t)}, \quad \text{as } m \to \infty,
$$

which yields (5.10) as desired. The proof of (5.11) is almost the same, so we omit it here.

Now we will prove (5.12). We see that

$$
\mathbb{E}[T_m(t)^2] = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{\ell=0}^{2(k_1+k_2)+1} \mathbb{E} \left[ \prod_{i=1}^{2} \left( \sum_{y \in Y_i \cup X_{\min,m+1}, |Y_i|=2k_i+1} h_t(Y_i) \mathbf{1}\left\{ \min_{y \in Y_i} \|y\| \geq R_{qm} \right\} \mathbf{1}\{|Y_1 \cap Y_2| = \ell\} \right]
$$

$$
= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{\ell=0}^{2(k_1+k_2)+1} \left( \frac{u_{m+1}}{2k_1+k_2+2-\ell} \right) \frac{(2(k_1+k_2)+2-\ell)!}{(2k_1+1-\ell)!(2k_2+1-\ell)! \ell!} \mathbb{E}[I_{k_1,k_2,\ell}],
$$

where $X_1, \ldots, X_{2(k_1+k_2)+2-\ell}$ are i.i.d random points with density $f$. In the above, if $\ell = 0$, we take that

$$
h_t(X_1, \ldots, X_\ell, X_{2k_1+2}, \ldots, X_{2(k_1+k_2)+2-\ell}) := h_t(X_{2k_1+2}, \ldots, X_{2(k_1+k_2)+2}).
$$

From this, we see that

$$
R_{qm}^{\ell} \text{Var}(T_m(t)) \quad (5.19)
$$

$$
\leq \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{\ell=1}^{2(k_1+k_2)+1} R^{\ell}_{qm} \left( \frac{u_{m+1}}{2k_1+k_2+2-\ell} \right) \frac{(2(k_1+k_2)+2-\ell)!}{(2k_1+1-\ell)!(2k_2+1-\ell)! \ell!} \mathbb{E}[I_{k_1,k_2,\ell}],
$$

where the last inequality comes from

$$
\left( \frac{u_{m+1}}{2k_1+k_2+2} \right) \left( \frac{2(k_1+k_2)+2}{2k_1+1} \right) - \left( \frac{u_{m+1}}{2k_1+1} \right) \left( \frac{u_{m+1}}{2k_2+1} \right) < 0.
$$
For our purposes, we must examine the appropriate upper bounds of $\mathbb{E}[I_{k_1,k_2,\ell}]$ for $\ell \geq 1$. For every $\ell \geq 1$, performing the change of variables, $x_i = x + y_{i-1}$, $i = 1, \ldots, 2(k_1 + k_2) + 2 - \ell$ (with $y_0 \equiv 0$), we have that

$$
\mathbb{E}[I_{k_1,k_2,\ell}] = \int_{(\mathbb{R}^d)^{2(k_1+k_2)+2-\ell}} h_t(x_1, \ldots, x_{2k_1+1})h_t(x_1, \ldots, x_{\ell}, x_{2k_1+2}, \ldots, x_{2(k_1+k_2)+2-\ell})
\times \prod_{i=1}^{2(k_1+k_2)+2-\ell} \frac{f(x_i)1\{\|x_i\| \geq R_{q_m}\}}{f(x)} \, dx
$$

$$
= \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^{2(k_1+k_2)+1-\ell}} h_t(0, y_1, \ldots, y_{2k_1})h_t(0, y_1, \ldots, y_{\ell-1}, y_{2k_1+1}, \ldots, y_{2(k_1+k_2)+1-\ell})
\times f(x)1\{\|x\| \geq R_{q_m}\} \prod_{i=1}^{2(k_1+k_2)+1-\ell} \frac{f(x+y_i)1\{\|x+y_i\| \geq R_{q_m}\}}{f(x)} \, dy \, dx.
$$

As in (5.15), we apply the polar coordinate transform $x = R_{q_m} \rho \theta$ with $\rho \geq 0$ and $\theta \in S^{d-1}$, to obtain that

$$
\mathbb{E}[I_{k_1,k_2,\ell}] = R_{q_m}^d f(R_{q_m})^{2(k_1+k_2)+2-\ell} \int_1^\infty \int_{S^{d-1}} \int_{(\mathbb{R}^d)^{2(k_1+k_2)+1-\ell}} h_t(0, y_0, y_1)h_t(0, y_0, y_2)\rho^{d-1}
\times \prod_{i=1}^{2(k_1+k_2)+1-\ell} \frac{f(R_{q_m})\|\rho\theta + y_i/R_{q_m}\|}{f(R_{q_m})} \, d(y_0 \cup y_1 \cup y_2) \, d\nu_{d-1}(\theta) \, d\rho,
$$

where $y_0 = (y_1, \ldots, y_{\ell-1})$, $y_1 = (y_\ell, \ldots, y_{2k_1})$ and $y_2 = (y_{2k_1+1}, \ldots, y_{2(k_1+k_2)+1-\ell})$. Appealing to Potter’s bounds as in (5.16) and (5.17), as well as (3.2), there exists $N \in \mathbb{N}$ such that for all $m \geq N$,

$$
R_{q_m}^{-d} \left(\frac{u_{m+1}}{2(k_1 + k_2) + 2 - \ell}\right) \mathbb{E}[I_{k_1,k_2,\ell}] 
\leq \left(\frac{u_{m+1}f(R_{q_m})}{2(k_1 + k_2) + 2 - \ell!}\right)^{2(k_1+k_2)+2-\ell} \int_1^\infty \int_{S^{d-1}} \int_{(\mathbb{R}^d)^{2(k_1+k_2)+1-\ell}} h_t(0, y_0, y_1)h_t(0, y_0, y_2)\rho^{d-1}
\times \prod_{i=1}^{2(k_1+k_2)+1-\ell} \frac{f(R_{q_m})\|\rho\theta + y_i/R_{q_m}\|}{f(R_{q_m})} \, d(y_0 \cup y_1 \cup y_2) \, d\nu_{d-1}(\theta) \, d\rho
\leq \left(\frac{4\xi}{2(k_1 + k_2) + 2 - \ell!}\right)^{2(k_1+k_2)+2-\ell} \prod_{i=1}^{2k_1} \frac{s_{d-1}}{\alpha - d - \eta} \int_{(\mathbb{R}^d)^{2(k_1+k_2)+1-\ell}} h_t(0, y_0, y_1)h_t(0, y_0, y_2) \, d(y_0 \cup y_1 \cup y_2).
$$

By virtue of property (H3) of $h$,

$$
\int_{(\mathbb{R}^d)^{2(k_1+k_2)+1-\ell}} h_t(0, y_0, y_1)h_t(0, y_0, y_2) \, d(y_0 \cup y_1 \cup y_2) \leq \int_{(\mathbb{R}^d)^{2(k_1+k_2)+1-\ell}} \prod_{i=1}^{2(k_1+k_2)+1-\ell} 1\{\|y_i\| \leq ct\} \, dy
\leq \left(\frac{ct}{d}\omega_d\right)^{2(k_1+k_2)+1-\ell}.
$$

Therefore, for all $m \geq N$,

$$
R_{q_m}^{-d} \left(\frac{u_{m+1}}{2(k_1 + k_2) + 2 - \ell}\right) \mathbb{E}[I_{k_1,k_2,\ell}] \leq \left(\frac{4\xi}{2(k_1 + k_2) + 2 - \ell!}\right)^{2(k_1+k_2)+2-\ell} \prod_{i=1}^{2k_1} \frac{s_{d-1}}{\alpha - d - \eta} \leq \left(\frac{C}{2(k_1 + k_2) + 2 - \ell!}\right).
$$

(5.20)
Note that the constant $C^*$ does not depend on $k_1$, $k_2$, and $\ell$. Returning to (5.19) and applying the bound in (5.20), we have that
\[
R_{q_m}^{-d} \text{Var}(T_m(t)) \leq \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{\ell=1}^{\infty} \frac{(C^*)^{2(k_1+k_2)+2-\ell}}{(2k_1+1-\ell)(2k_2+1-\ell)\ell!} \leq 2 \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{\ell=1}^{k_1+k_2+2-\ell} \frac{(C^*)^{k_1+k_2+2-\ell}}{(k_1+1-\ell)(k_2+1-\ell)\ell!} \leq 2 \sum_{\ell=1}^{\infty} \sum_{k_1=1-\ell}^{\infty} \sum_{k_2=1-\ell}^{\infty} \frac{(C^*)^{k_1+1-\ell}}{(k_1+1-\ell)(k_2+1-\ell)\ell!} \cdot (C^*)^{\ell} \leq 2e^{3C^*} < \infty.
\]

Since the proof of (5.13) is very similar to that of (5.12), we will omit it.

**Proof of Theorem 7.1.** By the definition of $u_m$, $p_m$, and $q_m$, we have, for every $n \geq 1$,
\[
\frac{U_m(t)}{R^d_{p_m}} \leq \frac{\chi_n^{(1)}(t)}{R^n} \leq \frac{T_m(t) - \mathbb{E}[T_m(t)]}{R^d_{q_m}}.
\]
Then, Lemma 5.2 gives that
\[
\limsup_{n \to \infty} \frac{\chi_n^{(1)}(t)}{R^n} \leq K_1(t) + \limsup_{m \to \infty} \frac{T_m(t) - \mathbb{E}[T_m(t)]}{R^d_{q_m}}, \text{ a.s.,} \tag{5.21}
\]
and
\[
\liminf_{n \to \infty} \frac{\chi_n^{(1)}(t)}{R^n} \geq K_1(t) + \liminf_{m \to \infty} \frac{U_m(t) - \mathbb{E}[U_m(t)]}{R^d_{p_m}}, \text{ a.s.} \tag{5.22}
\]

Let us continue by showing that
\[
R_{q_m}^{-d}(T_m(t) - \mathbb{E}[T_m(t)]) \to 0, \ m \to \infty, \text{ a.s.,} \tag{5.23}
\]
and
\[
R_{p_m}^{-d}(U_m(t) - \mathbb{E}[U_m(t)]) \to 0, \ m \to \infty, \text{ a.s.} \tag{5.24}
\]
For (5.23), it follows from (5.12) in Lemma 5.2 and Chebyshev’s inequality that, for every $\epsilon > 0$,
\[
\sum_{m=1}^{\infty} \mathbb{P}\left(\left|T_m(t) - \mathbb{E}[T_m(t)]\right| > \epsilon R^d_{q_m}\right) \leq \sum_{m=1}^{\infty} \frac{\text{Var}(T_m(t))}{\epsilon^2 (R^d_{q_m})^2} \leq C^* \sum_{m=1}^{\infty} \frac{1}{R^d_{q_m}}.
\]
As $R_n \in RV_{1/\alpha}$ (see (3.3)), we have that
\[
R_{q_m} \geq C^* q_m^{1/(2\alpha)} \geq C^* u_m^{1/(2\alpha)} \geq C^* e^{m^{\gamma}/(3\alpha)}
\]
for all $m \geq 1$. Now, we have $\sum_m R_{q_m}^{-d} \leq C^* \sum_m e^{-dm^{\gamma}/(3\alpha)} < \infty$, and the Borel-Cantelli lemma concludes (5.23). The proof of (5.24) is analogous by virtue of (5.13) in Lemma 5.2. Now, combining (5.21), (5.22), (5.23), and (5.24) can complete the proof.

**5.2. Proof of Theorem 4.1.** The goal here is to prove (5.4). Once again, we deal with the case $i = 1$ only. The proof is essentially similar in character to the proof of Theorem 3.1 but involves more complex machinery. First we take
\[
\gamma \in \left(\frac{\tau}{d - \tau}, 1\right) \tag{5.25}
\]
(recall that we have restricted the range of $\tau$ to $(0, 1)$ when $d = 2$), and define
\[
u_m = [e^{m^{\gamma}}], \ m = 0, 1, 2, \ldots
\]
as in \((5.5)\). Moreover, let \(p_m\) and \(q_m\) remain as before—see \((5.6)\) and \((5.7)\). Additionally, we also introduce
\[
\begin{align*}
v_m & := \text{argmax}\{u_m \leq \ell \leq u_{m+1} : a(R_\ell)R_{\ell}^{d-1}\}, \\
w_m & := \text{argmin}\{u_m \leq \ell \leq u_{m+1} : a(R_\ell)R_{\ell}^{d-1}\}.
\end{align*}
\]
Let \(T_m(t)\) and \(U_m(t)\) be defined in the same way as \((5.8)\) and \((5.9)\).

The lemma below is analogous to Lemma 5.2 (see also Proposition 7.4 in [11]) that provides the asymptotic moments of \(T_m(t)\) and \(U_m(t)\).

**Lemma 5.3.** Under the assumptions of Theorem 4.1, we have the following.
\[
\begin{align*}
limit_{m \to \infty} & \left[ a(R_{w_m})R_{w_m}^{d-1} \right]^{-1} \mathbb{E}[T_m(t)] = K_1(t), \tag{5.26} \\
limit_{m \to \infty} & \left[ a(R_{v_m})R_{v_m}^{d-1} \right]^{-1} \mathbb{E}[U_m(t)] = K_1(t), \tag{5.27} \\
\sup_{m \geq 1} & \left[ a(R_{w_m})R_{w_m}^{d-1} \right]^{-1} \text{Var}(T_m(t)) < \infty, \tag{5.28} \\
\sup_{m \geq 1} & \left[ a(R_{v_m})R_{v_m}^{d-1} \right]^{-1} \text{Var}(U_m(t)) < \infty.
\end{align*}
\]

**Proof.** We begin by proving \((5.26)\) and \((5.27)\). By the same change of variables as in \((5.14)\) and the translation invariance of \(h_t\),
\[
\begin{align*}
\left[ a(R_{w_m})R_{w_m}^{d-1} \right]^{-1} \mathbb{E}[T_m(t)] \\
= & \sum_{k=0}^{\infty} \left[ a(R_{w_m})R_{w_m}^{d-1} \right]^{-1} \left( \frac{u_{m+1}}{2k+1} \right) \int_{(\mathbb{R}^d)^{2k+1}} \int_{\mathbb{R}^d} \sum_{i=1}^{2k+1} f(x_i) 1\{\|x_i\| \geq R_{q_m}\} \ dx \\
= & \sum_{k=0}^{\infty} \left[ a(R_{w_m})R_{w_m}^{d-1} \right]^{-1} \left( \frac{u_{m+1}}{2k+1} \right) \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^{2k}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_t(0,y) f(x) 1\{\|x\| \geq R_{q_m}\} \\
& \times \prod_{i=1}^{2k} f(x + y_i) 1\{\|x + y_i\| \geq R_{q_m}\} \ dy \ dx,
\end{align*}
\]
where \(y = (y_1, \ldots, y_{2k}) \in (\mathbb{R}^d)^{2k}\). Here, we make the change of variable by \(x = (R_{q_m} + CR_{q_m} \rho) \theta\) with \(\rho \geq 0\) and \(\theta \in S^{d-1}\). Then, the integral above becomes
\[
\begin{align*}
& a(R_{q_m}) \int_0^{\infty} \int_{S^{d-1}} \int_{(\mathbb{R}^d)^{2k}} h_t(0,y) (R_{q_m} + CR_{q_m} \rho)^{d-1} f(R_{q_m} + CR_{q_m} \rho) \\
& \times \prod_{i=1}^{2k} f\left(\|(R_{q_m} + CR_{q_m} \rho) \theta + y_i\|\right) 1\{\|(R_{q_m} + CR_{q_m} \rho) \theta + y_i\| \geq R_{q_m}\} \ dy \ d\nu_{d-1}(\theta) \ d\rho,
\end{align*}
\]
which implies that
\[
\begin{align*}
\left[ a(R_{w_m})R_{w_m}^{d-1} \right]^{-1} \mathbb{E}[T_m(t)] \\
= & \sum_{k=0}^{\infty} \frac{a(R_{q_m})R_{q_m}^{d-1} \left( \frac{u_{m+1}}{2k+1} \right) f(R_{q_m})^{2k+1}}{a(R_{w_m})R_{w_m}^{d-1}} \\
& \times \int_0^{\infty} \int_{S^{d-1}} \int_{(\mathbb{R}^d)^{2k}} h_t(0,y) \left(1 + \frac{a(R_{q_m})}{R_{q_m}}\right)^{d-1} \frac{f(R_{q_m} + CR_{q_m} \rho)}{f(R_{q_m})} \\
& \times \prod_{i=1}^{2k} f\left(\|(R_{q_m} + CR_{q_m} \rho) \theta + y_i\|\right) 1\{\|(R_{q_m} + CR_{q_m} \rho) \theta + y_i\| \geq R_{q_m}\} \ dy \ d\nu_{d-1}(\theta) \ d\rho.
\end{align*}
\]
For the last expression, we claim that
\[
\frac{a(R_{q_m})R_{q_m}^{d-1}}{a(R_{w_m})R_{w_m}^{d-1}} \to 1, \quad m \to \infty, \quad (5.31)
\]
and
\[
\left(\frac{u_{m+1}}{2k+1}\right) f(R_{q_m})^{2k+1} \to \frac{\xi^{2k+1}}{(2k+1)!}, \quad m \to \infty. \quad (5.32)
\]
Because of (3.2), we have \(\psi(R_n) \sim \log(Cn/\xi)\) as \(n \to \infty\). Let \(\psi^{-}\) be the inverse function of \(\psi\). With the assumptions on the density (4.1), Proposition 2.6 in [17] gives that \(\psi^{-} \in RV_{1/\tau}\). It follows from the uniform convergence of regularly varying sequences (see Proposition 2.4 in [17]) that
\[
\frac{\psi^{-}(\psi(R_n))}{\psi^{-}(\log(Cn/\xi))} \sim \left(\frac{\psi(R_n)}{\log(Cn/\xi)}\right)^{1/\tau} \to 1, \quad n \to \infty.
\]
Since \(\psi^{-}(\psi(R_n)) \sim R_n\) as \(n \to \infty\), the above relation and \(\log(Cn/\xi) \sim \log n\), \(n \to \infty\), implies
\[
R_n \sim \psi^{-}(\log n), \quad n \to \infty. \quad (5.33)
\]
Now we are ready to prove (5.31). By the uniform convergence of regularly varying sequences,
\[
\frac{R_{q_m}}{R_{w_m}} \sim \frac{\psi^{-}(\log q_m)}{\psi^{-}(\log w_m)} \sim \left(\frac{\log q_m}{\log w_m}\right)^{1/\tau}, \quad m \to \infty. \quad (5.34)
\]
Notice that
\[
\frac{u_m}{u_{m+1}} \leq \frac{q_m}{w_m} \leq \frac{u_{m+1}}{u_m},
\]
so that \(u_{m+1}/u_m \to 1\) as \(m \to \infty\) (see (5.18)), and hence, \(q_m/w_m \to 1\), \(m \to \infty\). Now, (5.34) implies \(R_{q_m}/R_{w_m} \to 1\) as \(m \to \infty\). Recalling also \(a \in RV_{1-\tau}\) and using the uniform convergence of regularly varying sequences,
\[
\frac{a(R_{q_m})}{a(R_{w_m})} \sim \left(\frac{R_{q_m}}{R_{w_m}}\right)^{1-\tau} \to 1, \quad m \to \infty;
\]
hence, (5.31) is obtained.

Turning to (5.32), we note that by (3.2),
\[
\left(\frac{u_{m+1}}{2k+1}\right) f(R_{q_m})^{2k+1} \sim \frac{1}{(2k+1)!} \left(\frac{u_{m+1}}{q_m}\right)^{2k+1} f(R_{q_m})^{2k+1} \sim \frac{\xi^{2k+1}}{(2k+1)!} \left(\frac{u_{m+1}}{q_m}\right)^{2k+1} \to \frac{\xi^{2k+1}}{(2k+1)!}, \quad m \to \infty,
\]
where the last convergence is obtained as a result of (5.18).

Returning to (5.30), let us next calculate the limits for each term in the integral, while finding their appropriate upper bounds. Under our setup, it is straightforward to see that \(a' \in RV_{1-\tau}\)—see, e.g., Proposition 2.5 in [17]. Therefore, \(a(z)/z \to 0\) as \(z \to \infty\), and for all \(\rho > 0\),
\[
\left(1 + \frac{a(R_{q_m})}{R_{q_m}}\right)^{d-1} \to 1, \quad m \to \infty. \quad (5.35)
\]
Note also that (5.35) is bounded by \(2(1 \lor \rho)^{d-1}\) for sufficiently large \(m\).

Next we deal with \(f(R_{q_m} + a(R_{q_m})\rho)/f(R_{q_m})\). Write
\[
\frac{f(R_{q_m} + a(R_{q_m})\rho)}{f(R_{q_m})} = \exp \left\{ -\psi(R_{q_m} + a(R_{q_m})\rho) + \psi(R_{q_m}) \right\} \quad (5.36)
\]
\[
= \exp \left\{ -\int_0^\rho \frac{a(R_{q_m})}{a(R_{q_m} + a(R_{q_m})\rho)} \, dr \right\}.
\]
By the uniform convergence of regularly varying functions and \(a(z)/z \to 0\) as \(z \to \infty\), we have for every \(r \geq 0\) that
\[
\frac{a(R_{qm})}{a(R_{qm} + a(R_{qm})r)} \to 1, \quad m \to \infty.
\]

Therefore, for every \(\rho > 0\),
\[
\frac{f(R_{qm} + a(R_{qm})\rho)}{f(R_{qm})} \to e^{-\rho}, \quad m \to \infty.
\]

Additionally, we define a sequence \((s_\ell(m), \ell \geq 0, m \geq 0)\) by
\[
s_\ell(m) = \frac{\psi^{-}(\psi(R_{qm}) + \ell) - R_{dm}}{a(R_{qm})},
\]
equivalently, \(\psi(R_{qm} + a(R_{qm})s_\ell(m)) = \psi(R_{qm}) + \ell\). Then, Lemma 5.2 in [2] implies that for any \(\epsilon \in (0, d^{-1})\), there exists a positive integer \(N = N(\epsilon)\) such that \(s_\ell(m) \leq \epsilon^{-1} e^{\ell}\) for all \(m \geq N\) and \(\ell \geq 0\). Since \(\psi\) is increasing, we can establish the bound of (5.36) as follows: for \(m \geq N\),
\[
\begin{align*}
\exp \left\{ -\psi(R_{qm} + a(R_{qm})\rho) + \psi(R_{qm}) \right\} & 1\{\rho > 0\} \\
= \sum_{\ell=0}^{\infty} 1\{s_\ell(m) \leq \rho \leq s_{\ell+1}(m)\} & \exp \left\{ -\psi(R_{qm} + a(R_{qm})\rho) + \psi(R_{qm}) \right\} \\
\leq \sum_{\ell=0}^{\infty} 1\{0 < \rho \leq \epsilon^{-1} e^{(\ell+1)\epsilon}\} e^{-\ell}.
\end{align*}
\]

We now discuss the final untreated term from the integral in (5.30). Let us give a helpful fact about \(\|(R_{qm} + a(R_{qm})\rho + y_i\|\) for \(i \in \{1, \ldots, 2k\}\). We have that
\[
\frac{\left\|(R_{qm} + a(R_{qm})\rho + y_i\right) - \left(R_{qm} + a(R_{qm})\rho + \langle \theta, y_i\rangle\right)}{\|y_i\|^2 - \langle \theta, y_i\rangle^2} = \frac{\|\left(R_{qm} + a(R_{qm})\rho + \langle \theta, y_i\rangle\right) + R_{qm} + a(R_{qm})\rho + \langle \theta, y_i\rangle\}}{\|R_{qm} + a(R_{qm})\rho + \langle \theta, y_i\rangle\|^2} := \gamma_m(\rho, \theta, y_i).
\]

In particular, if \(\|(R_{qm} + a(R_{qm})\rho + y_i\| \geq R_{qm}\), then
\[
\gamma_m(\rho, \theta, y_i) \leq \frac{\|y_i\|^2 - \langle \theta, y_i\rangle^2}{2R_{qm}^2 + \langle \theta, y_i\rangle^2} \to 0, \quad m \to \infty. \quad (5.37)
\]

This convergence takes place uniformly for \(\rho > 0, \theta \in S^{d-1}\), and \(y_i \in \mathbb{R}^d\) with \(\|y_i\| \leq ct\), where \(c\) is determined by property \((H3)\) of \(h\)—see Section [2]. Continuing onward, let
\[
A_m = \left\{ y \in \mathbb{R}^d : \|(R_{qm} + a(R_{qm})\rho + y\| \geq R_{qm}\right\};
\]
then, for each \(i \in \{1, \ldots, 2k\}\),
\[
\begin{align*}
\frac{f\left(\|(R_{qm} + a(R_{qm})\rho + y_i\|\right)}{f(R_{qm})} & 1_{A_m}(y_i) \\
= \exp \left\{ -\psi(R_{qm} + a(R_{qm})\rho + \langle \theta, y_i\rangle + \gamma_m(\rho, \theta, y_i) + \psi(R_{qm}) \right\} 1_{A_m}(y_i) \\
= \exp \left\{ -\int_{0}^{\rho + \xi_m(\rho, \theta, y_i)} \frac{a(R_{qm})}{a(R_{qm} + a(R_{qm})r)} \, dr \right\} 1_{A_m}(y_i),
\end{align*}
\]
where $\xi_m(\rho, \theta, y_i) = a(R_{qm})^{-1}\left(\langle \theta, y_i \rangle + \gamma_m(\rho, \theta, y_i)\right)$. Note that the last term is bounded by 1, due to the fact that
\[
\left\| (R_{qm} + a(R_{qm})\rho)\theta + y_i \right\| \geq R_{qm} \iff \rho + \xi_m(\rho, \theta, y_i) \geq 0.
\]
Additionally, (4.2) and (5.37) yield that
\[
\exp\left\{ -\int_0^\rho + \xi_m(\rho, \theta, y_i) \right\} \frac{a(R_{qm})}{a(R_{qm} + a(R_{qm})\rho)} \, dr \rightarrow \exp\left\{ -\rho - \xi^{-1}(\theta, y_i) \right\},
\]
and
\[
1_{A_m}(y_i) \rightarrow 1\{\rho + \xi^{-1}(\theta, y_i) \geq 0\}.
\]
Combining all the bounds derived thus far, the integral in (5.30) is bounded above by
\[
2 \int_0^\infty \int_{S^{d-1}} \int_{|\mathbb{R}^d|2k} h_t(0, y) (1 \vee \rho)^{d-1} \sum_{\ell=0}^\infty 1\{0 < \rho \leq \epsilon^{-1}e^{(\ell+1)\epsilon}\} e^{-\ell} \, dy \, \nu_{d-1}(\theta) \, d\rho
\]
\[
= C^* \int_0^\infty \sum_{\ell=0}^\infty 1\{0 < \rho \leq \epsilon^{-1}e^{(\ell+1)\epsilon}\} e^{-\ell} (1 \vee \rho)^{d-1} \, d\rho
\]
\[
\leq C^* \left( \frac{\epsilon}{\epsilon} \right)^d \sum_{\ell=0}^\infty e^{-(1-cd)\ell} < \infty,
\]
as $\epsilon^{-1}e^{(\ell+1)\epsilon} > 1$ and $cd < 1$. Now, by the dominated convergence theorem, we can see that the integral in (5.30) converges to
\[
\int_0^\infty \int_{S^{d-1}} \int_{|\mathbb{R}^d|2k} h_t(0, y) e^{-(2k+1)\rho - \xi^{-1}\sum_{i=1}^k \rho_i} \prod_{i=1}^k \left( \rho + \xi^{-1}(\theta, y_i) \right) \, dy \, \nu_{d-1}(\theta) \, d\rho.
\]
Because of this convergence, as well as (5.31) and (5.32), we can get (5.26) as required.

The proof of (5.27) is almost the same as above, so we skip it. We can now conclude this lemma by showing (5.28) and (5.29). The proof is however omitted as it has essentially the same character as the proofs of (5.12) and (5.13) in Lemma 5.2, albeit with different upper bounds as discussed above, due to the differing nature of the tail of probability densities. \hfill \Box

Proof of Theorem 4.1. First, for every $n \geq 1$,
\[
\frac{U_m(t)}{a(R_{wm})R_{wm}^{d-1}} \leq \frac{\chi_n^{(1)}(t)}{a(R_n)R_n^{d-1}} \leq \frac{T_m(t)}{a(R_{wm})R_{wm}^{d-1}}.
\]
Lemma 5.3 yields that
\[
\limsup_{n \to \infty} \frac{\chi_n^{(1)}(t)}{a(R_n)R_n^{d-1}} \leq K_1(t) + \limsup_{m \to \infty} \frac{T_m(t) - \mathbb{E}[T_m(t)]}{a(R_{wm})R_{wm}^{d-1}},
\]
and
\[
\liminf_{n \to \infty} \frac{\chi_n^{(1)}(t)}{a(R_n)R_n^{d-1}} \geq K_1(t) + \liminf_{m \to \infty} \frac{U_m(t) - \mathbb{E}[U_m(t)]}{a(R_{wm})R_{wm}^{d-1}}.
\]
Now, the proof can be finished, if one can show that
\[
[a(R_{wm})R_{wm}^{d-1}]^{-1} (T_m(t) - \mathbb{E}[T_m(t)]) \to 0, \quad m \to \infty, \quad a.s., \tag{5.38}
\]
\[
[a(R_{wm})R_{wm}^{d-1}]^{-1} (U_m(t) - \mathbb{E}[U_m(t)]) \to 0, \quad m \to \infty, \quad a.s. \tag{5.39}
\]
By (5.28) in Lemma 5.3, for every $\epsilon > 0$,
$$
\sum_{m=1}^{\infty} \mathbb{P} \left( \left| T_m(t) - \mathbb{E}[T_m(t)] \right| > \epsilon a(R_{w_m}) R_{w_m}^{d-1} \right) \leq \frac{1}{c^2} \sum_{m=1}^{\infty} \frac{\text{Var}(T_m(t))}{(a(R_{w_m}) R_{w_m}^{d-1})^2} \leq C^* \sum_{m=1}^{\infty} \frac{1}{a(R_{w_m}) R_{w_m}^{d-1}}.
$$
Because of the constraint in (5.25), there exist $\delta_i > 0, i = 1, 2$, so that
$$
\gamma(d - \tau - \delta_1) \left( \frac{1}{\tau} - \delta_2 \right) > 1.
$$
Then, $a \in \text{RV}_{1-\tau}$ implies that
$$
a(R_{w_m}) R_{w_m}^{d-1} \geq C^* R_{w_m}^{d-\tau-\delta_1}
$$
for all $m \geq 1$. Note that by (5.33),
$$
R_{w_m} \geq C^* \psi^{-c} (\log w_m) \geq C^* \psi^{-c} (\log u_m) \geq C^* m^{\gamma(1/\tau - \delta_2)}
$$
again for all $m \geq 1$. Therefore,
$$
a(R_{w_m}) R_{w_m}^{d-1} \geq C^* m^{\gamma(d-\tau-\delta_1)(1/\tau-\delta_2)},
$$
and
$$
\sum_{m=1}^{\infty} \frac{1}{a(R_{w_m}) R_{w_m}^{d-1}} \leq C^* \sum_{m=1}^{\infty} \frac{1}{m^{\gamma(d-\tau-\delta_1)(1/\tau-\delta_2)}} < \infty.
$$
Now, the Borel-Cantelli lemma completes the proof of (5.38). The proof of (5.39) is the same by virtue of (5.29) in Lemma 5.3.

References

[1] R. J. Adler, O. Bobrowski, and S. Weinberger. Crackle: The homology of noise. *Discrete & Computational Geometry*, 52(4):680–704, Dec. 2014. ISSN 0179-5376, 1432-0444. doi: 10.1007/s00454-014-9621-6. URL [http://link.springer.com/10.1007/s00454-014-9621-6](http://link.springer.com/10.1007/s00454-014-9621-6).

[2] G. Balkema and P. Embrechts. Multivariate excess distributions. *ETHZ Preprint*, 2004.

[3] O. Bobrowski and R. J. Adler. Distance functions, critical points, and the topology of random Čech complexes. *Homology, Homotopy and Applications*, 16:311–344, 2014.

[4] O. Bobrowski and S. Mukherjee. The topology of probability distributions on manifolds. *Probability Theory and Related Fields*, 161:651–686, 2015.

[5] R. Ghrist. *Elementary Applied Topology*. Createspace, 2014.

[6] A. Goel, K. Trinh, and K. Tsunoda. Strong law of large numbers for Betti numbers in the thermodynamic regime. *Journal of Statistical Physics*, 174(4):865–892, Feb. 2019. ISSN 0022-4715. doi: 10.1007/s10955-018-2201-z.

[7] D. Hug, G. Last, and M. Schneider. Second-order properties and central limit theorems for geometric functionals of Boolean models. *The Annals of Probability*, 26:73–135, 2016.

[8] J. Krebs, B. Roycraft, and W. Polonik. On approximation theorems for the Euler characteristic with applications to the bootstrap. arXiv:2005.07557, 2020.

[9] P. Niyogi, S. Smale, and S. Weinberger. Finding the homology of submanifolds with high confidence from random samples. *Discrete & Computational Geometry*, 39(1-3):419–441, mar 2008. ISSN 0179-5376. doi: 10.1007/s00454-008-9053-2. URL [http://link.springer.com/10.1007/s00454-008-9053-2](http://link.springer.com/10.1007/s00454-008-9053-2).

[10] P. Niyogi, S. Smale, and S. Weinberger. A topological view of unsupervised learning from noisy data. *SIAM Journal on Computing*, 40(3):646–663, 2011. doi: 10.1137/090762932. URL [http://epubs.siam.org/doi/10.1137/090762932](http://epubs.siam.org/doi/10.1137/090762932).

[11] T. Owada. Functional central limit theorem for subgraph counting processes. *Electronic Journal of Probability*, 22:38 pp., 2017. doi: 10.1214/17-EJP30. URL [https://doi.org/10.1214/17-EJP30](https://doi.org/10.1214/17-EJP30).
[12] T. Owada. Limit theorems for Betti numbers of extreme sample clouds with application to persistence barcodes. *The Annals of Applied Probability*, 28(5):2814–2854, 2018.

[13] T. Owada. Topological crackle of heavy-tailed moving average processes. *Stochastic Processes and their Applications*, 129:4965–4997, 2019.

[14] T. Owada and R. J. Adler. Limit theorems for point processes under geometric constraints (and topological crackle). *The Annals of Probability*, 45(3):2004–2055, 2017. ISSN 00911798. doi: 10.1214/16-AOP1106.

[15] T. Owada and O. Bobrowski. Convergence of persistence diagrams for topological crackle. *Bernoulli*, 26(3):2275–2310, aug 2020. ISSN 1350-7265. doi: 10.3150/20-BEJ1193. URL https://projecteuclid.org/euclid.bj/1587974541

[16] M. Penrose. *Random Geometric Graphs*, volume 5. Oxford university press, 2003.

[17] S. I. Resnick. *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*. Springer Science & Business Media, 2007.

[18] A. Thomas and T. Owada. Functional limit theorems for the Euler characteristic process in the critical regime. To appear in *Advances in Applied Probability*, 2020.

Department of Statistics, Purdue University, IN, 47907, USA

Email address: thoma186@purdue.edu, owada@purdue.edu