Nonnegative measures belonging to $H^{-1}(\mathbb{R}^2)$.

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Abstract

Radon measures belonging to the negative Sobolev space $H^{-1}(\mathbb{R}^2)$ are important from the point of view of fluid mechanics as they model vorticity of vortex-sheet solutions of incompressible Euler equations. In this note we discuss regularity conditions sufficient for nonnegative Radon measures supported on a line to be in $H^{-1}(\mathbb{R}^2)$. Applying the obtained results, we derive consequences for measures on $\mathbb{R}^2$ with arbitrary support and prove elementarily, among other things, that measures belonging to $H^{-1}(\mathbb{R}^2)$ may be supported on a set of Hausdorff dimension 0. We comment on possible numerical applications.

Keywords: embeddings of measures, vorticity, Hausdorff dimension

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1 Introduction

Let $\mathcal{M}_+(\mathbb{R}^2)$ denote the space of nonnegative bounded Radon measures on $\mathbb{R}^2$ (see [7]) and let $H^{-1}(\mathbb{R}^2)$ be the space of all tempered distributions $f$ on $\mathbb{R}^2$ such that

$$
\int_{\mathbb{R}^2} (1 + |y|^2)^{-1} |\hat{f}(y)|^2 dy < \infty.
$$

Alternatively, $H^{-1}(\mathbb{R}^2)$ can be viewed as the space of all continuous functionals on the Sobolev space $W^{1,2}(\mathbb{R}^2)$ (see e.g. [1]). The following basic problem can be posed:

Problem A. Characterize the space $\mathcal{M}_+(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$.

Our motivation to study this problem originates in fluid mechanics. Namely, let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be the velocity field of a fluid in two-dimensional space and let

$$
\omega = \text{curl}(u) := \partial_{x_1} u_2 - \partial_{x_2} u_1
$$

be its vorticity field. Then $\omega \in \mathcal{M}_+(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ for compactly supported $\omega$ means that
• vorticity of the flow is everywhere nonnegative (condition $\omega \in \mathcal{M}_+(\mathbb{R}^2)$),

• kinetic energy of the fluid is locally finite, i.e. $\int_{\Omega} u^2(x) dx < \infty$ for every bounded $\Omega \subset \mathbb{R}^2$ (condition $\omega \in H^{-1}(\mathbb{R}^2)$).

The latter condition follows from the fact that the Biot-Savart operator mapping $\omega$ to $u$ by the convolution formula

$$u = K * \omega$$

for $K(x) = \frac{x^+}{2\pi|x|^2}$ is bounded from $H^{-1}$ to $L^2_{loc}$, see below.

Solutions of the incompressible Euler equations,

$$\partial_t u + u \nabla u + \nabla p = 0,$$

$$\text{div}(u) = 0.$$

with vorticity belonging to $\mathcal{M}_+(\mathbb{R}^2)$ were defined and studied in [5]. In [4] Delort proved a basic existence theorem, which states that for initial data $u(t = 0, x)$ such that $\omega(0, x) := \text{curl}(u(0, x))$ is a bounded nonnegative Radon measure belonging to $H^{-1}(\mathbb{R}^2)$ there exists a global solution $u(t, x)$ of the Euler equations such that $\omega(t, x) := \text{curl}(u(t, x))$ is a bounded nonnegative Radon measure belonging to $H^{-1}(\mathbb{R}^2)$ for every $t > 0$. Uniqueness of such solutions is still an outstanding open problem. To approach it, it seems reasonable to study Problem A see also the introduction in [3] for a more comprehensive physical background and motivations.

In the case of compactly supported measures Problem A can be solved as follows. Define the positive logarithmic energy of a measure $\omega \in \mathcal{M}_+(\mathbb{R}^2)$ by

$$\mathcal{H}^+(\omega) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log^+ \frac{1}{|x-y|} \omega(dx) \omega(dy), \quad (1)$$

where $\log^+(x) = \max(\log(x), 0)$. In [11], which builds upon previous ideas of Delort [4] the following crucial characterization was demonstrated.

**Lemma 1.1** (Lemma 3.1 in [11]). Let $\omega$ be a nonnegative measure of finite mass and compact support, and let $u = K * \omega$ be the velocity corresponding to the vorticity $\omega$. Then the following are equivalent:

1. $\omega$ is in $H^{-1}$.
2. $u$ is in $L^2_{loc}$.
3. $\mathcal{H}^+(\omega) < \infty$.

As a simple corollary, we obtain that measures belonging to $H^{-1}$ have no discrete part. Indeed, $\mathcal{H}^+(\delta_x) = +\infty$ for every $x \in \mathbb{R}^2$, where $\delta_x$ is the Dirac mass in $x$. For general measures, however, Formula (1) is not very convenient to use and we would like
to have more ‘tangible’ local conditions characterizing measures belonging to $H^{-1}$.

The study of Problem A in relation to spirals of vorticity was initiated in [3], where the authors proved that the so-called Prandtl and Kaden spirals belong locally to $H^{-1}(\mathbb{R}^2)$. The crucial tool in [3] was the following theorem.

**Theorem 1.2** (Theorem 1.1 from [3]). Let $\mu$ be a positive Radon measure supported in a ball $B(0, R_0) \subset \mathbb{R}^2$. Assume that there exists a positive constant $c_1$ such that for any $r \leq R_0$

$$\mu(B(0,r)) = c_1 r^\alpha,$$

where $\alpha > 0$.

Then $\mu \in H^{-1}(\mathbb{R}^2)$.

In this note, motivated by studies in [3], we go beyond Theorem 1.2. We investigate, namely, singular continuous measures belonging to $\mathcal{M}_+(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ and derive, using formula (1), simple analytical and geometric conditions characterizing such measures. We begin with measures supported on a line $\{(x_1, 0) : x_1 \in \mathbb{R}\}$ and then generalize the results to measures with more general support. In particular, we recover Theorem 1.2 as a special case. Let us note that our methods are based on transformation of formula (1), which, in contrast to t-energy methods (see [10]) used in [3] allow us to extract more detailed information on measures.

Measure supported on a line can be written in the form

$$\omega = \eta(dx_1)\delta_0(dx_2),$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $\eta$ is a compactly supported nonnegative Radon measure on $\mathbb{R}$ with no discrete part. Measure $\omega$ can be equivalently represented as

$$\omega = dF(x_1)\delta_0(dx_2),$$

where $F : \mathbb{R} \rightarrow [0, \infty)$ is the continuous, nondecreasing cumulative distribution function of $\eta$, given by

$$F(x) := \eta((-\infty, x)).$$

If $\eta$ is absolutely continuous with respect to the one-dimensional Lebesgue measure or, equivalently, $F \in W^{1,1}_{loc}(\mathbb{R})$, then we can represent $\omega$ as

$$\omega = f(x_1)dx_1\delta_0(dx_2),$$

where $f := F'$ is a nonnegative compactly supported function belonging to $L^1(\mathbb{R})$. In the following, we study, under which conditions on $F$ and $f$ does $\omega$ belong to $H^{-1}(\mathbb{R}^2)$. We consider the following cases:

- $f \in L^1$,
- $f \in L^\infty$ or equivalently $F$ – Lipschitz continuous,
• $f \in L^p$ for $1 < p < \infty$,
• $f \in L(\log L)^\gamma$, where $L(\log L)^\gamma$ is the Calderón-Zygmund class, see Section 3
• $F$ – continuous,
• $F$ – Hölder continuous with exponent $\alpha \in (0,1)$.

We prove that any of the conditions $f \in L^\infty$, $f \in L^p$, $F$ - Hölder continuous, $F$-Lipschitz continuous is sufficient (Section 2). On the other hand, we show that conditions $f \in L^1$, $f \in L(\log L)^\gamma$ for $\gamma < 1/2$ or $F$ being absolutely continuous are not sufficient (Section 3). Finally (Section 4) we apply these results to more general nonnegative measures $\omega$ and discuss the Hausdorff dimension of support of $\omega$. We comment also on possible numerical applications.

2 Classes of measures belonging to $H^{-1}$

For measures $\omega$ of the form (2) formula (1) reduces to

$$
\mathcal{H}^+(\omega) = \mathcal{H}^+(dF) := \int_R \int_R \log^+ \frac{1}{|x-y|} dF(x)dF(y), \tag{5}
$$

where integrals are understood in the Lebesgue-Stieltjes sense (i.e. $dF \equiv \eta$ is the Lebesgue-Stieltjes measure generated by equality (3), see [2]). Similarly, for measures $\omega$ of the form (4), we obtain

$$
\mathcal{H}^+(\omega) = \mathcal{H}^+(f) := \int_R \int_R \log^+ \frac{1}{|x-y|} f(x)f(y)dx\,dy. \tag{6}
$$

So prepared, we are ready to study particular cases of Problem A. By Lemma 1.1 it suffices to determine whether $\mathcal{H}^+(dF)$ or $\mathcal{H}^+(f)$ are finite, using formulas (5) and (6), respectively. We begin with the simple cases of $f \in L^\infty$ and $f \in L^p$, $p > 1$.

**Proposition 2.1.** If $f$ is bounded and compactly supported then $\mathcal{H}^+(f) < \infty$.

**Proof.**

$$
\mathcal{H}^+(f) \leq \|f\|_{L^\infty}^2 \int_{\supp(f)} \int_{\supp(f)} \log^+ \frac{1}{|x-y|} dx\,dy < \infty,
$$

where $\supp(f)$ denotes the support of function $f$. \qed

**Corollary 2.2.** For $F$ Lipschitz continuous $\mathcal{H}^+(dF) < \infty$.

**Proposition 2.3.** If $f \in L^p$, $1 < p \leq \infty$ and $f$ is compactly supported then $\mathcal{H}^+(f) < \infty$.
Proof. Let $f \in L^p$ have a compact support such that $\text{supp}(f) \subset B(0, R)$, where $B(0, R)$ is the closed ball centered at 0 and with radius $R$. Then, setting $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and using the H"older and Young inequalities we obtain

$$
\int_{\mathbb{R}} \int_{\mathbb{R}} \log^+ \frac{1}{|x-y|} f(x)f(y) dy dx = \int_{B(0,R+1)} \int_{B(0,R+1)} \log^+ \frac{1}{|x-y|} f(x)f(y) dy dx
$$

$$
\leq \left\| \int_{B(0,R+1)} \log^+ \frac{1}{|y|} f(y) dy \right\|_q \|f\|_p
$$

$$
\leq \left\| \int_{B(0,R+1)} \log^+ \frac{1}{|y|} f(y) dy \right\|_\infty [2(R+1)]^{\frac{1}{q}} \|f\|_p
$$

$$
\leq \left\| \log^+ \frac{1}{|\cdot|} 1_{B(0,R+1)}(\cdot) \right\|_q [2(R+1)]^{\frac{1}{q}} \|f\|_p^2 < +\infty.
$$

\qed

Next, we consider the more demanding case of $F$ being Hölder continuous. Recall that $F \in C^{0,\alpha}(\mathbb{R})$, $0 < \alpha \leq 1$, if there exists a constant $K > 0$ such that $|F(x+y) - F(x)| \leq K|y|^\alpha$ for every $x, y \in \mathbb{R}$.

**Proposition 2.4.** If $F \in C^{0,\alpha}$, $0 < \alpha \leq 1$ then $\mathcal{H}^+(dF) < \infty$.

Proposition 2.4 is a consequence of the following lemma.

**Lemma 2.5.** Suppose a bounded continuous nondecreasing $F : \mathbb{R} \to [0, \infty)$ satisfies:

1) $(F(x+\varepsilon) - F(x)) \log \varepsilon \to 0$ as $\varepsilon \to 0$ uniformly in $x$,

2) $(F(x-\varepsilon) - F(x)) \log \varepsilon \to 0$ as $\varepsilon \to 0$ uniformly in $x$,

3) $\int_0^1 \frac{F(x+y) - F(x)}{y} dy \leq C$ uniformly in $x$,

4) $\int_0^1 \frac{F(x-y) - F(x)}{y} dy \leq C$ uniformly in $x$.

Then

$$
\mathcal{H}^+(dF) = \int_{\mathbb{R}} \left( \int_0^1 \frac{1}{y} (F(x+y) - F(x-y)) dy \right) dF(x)
$$

and in particular, $\mathcal{H}^+(dF) < +\infty$. 

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Proof. Using the properties of Lebesgue-Stieltjes integrals (see [2]) we obtain:

\[
\mathcal{H}^+(dF) = \int_\mathbb{R} \int_\mathbb{R} \log \frac{1}{|x-y|} dF(x)dF(y)
\]

\[
= \int_\mathbb{R} \left[ \int_{x=1}^{x+1} \log \frac{1}{|x-y|} dF(y) \right] dF(x)
\]

\[
= \int_\mathbb{R} \left[ \int_{y=1}^{y+1} \log \frac{1}{|y|} dF(x+y) \right] dF(x)
\]

\[
= \int_\mathbb{R} \left[ \int_{y=0}^{1} \log \left( \frac{1}{y} \right) d(F(x+y) - F(x-y)) \right] dF(x)
\]

\[
= \int_\mathbb{R} \int_0^1 \log \left( \frac{1}{y} \right) d(F(x+y) - F(x)) dF(x)
\]

\[
+ \int_\mathbb{R} \int_0^1 \log \left( \frac{1}{y} \right) d(F(x) - F(x-y)) dF(x)
\]

\[
= \int_\mathbb{R} \lim_{\varepsilon \to 0} \left[ \int_{\varepsilon}^1 \log \left( \frac{1}{y} \right) d(F(x+y) - F(x)) \right] dF(x)
\]

\[
+ \int_\mathbb{R} \lim_{\varepsilon \to 0} \left[ \int_{\varepsilon}^1 \log \left( \frac{1}{y} \right) d(F(x) - F(x-y)) \right] dF(x)
\]

\[
= \int_\mathbb{R} \lim_{\varepsilon \to 0} \left[ \log \left( \frac{1}{x+y} \right) (F(x+y) - F(x)) \right]_{\varepsilon}^{1} + \int_{\varepsilon}^1 \frac{1}{y} (F(x+y) - F(x)) dy \] dF(x)

\[
+ \int_\mathbb{R} \lim_{\varepsilon \to 0} \left[ \log \left( \frac{1}{x+y} \right) (F(x) - F(x-y)) \right]_{\varepsilon}^{1} + \int_{\varepsilon}^1 \frac{1}{y} (F(x) - F(x-y)) dy \] dF(x)

\[
= \int_\mathbb{R} \left( \int_{0}^{1} \frac{1}{y} (F(x+y) - F(x) + F(x) - F(x-y)) dy \right) dF(x) \leq 2C \int_\mathbb{R} dF(x),
\]

where in the last equality we used the Lebesgue dominated convergence theorem and the fact that measure \(dF\) is bounded. \(\Box\)

Proof of Proposition 2.4. For \(F \in C^{0,\alpha}\), where \(0 < \alpha \leq 1\), we obtain

\[
|F(x \pm \varepsilon) - F(x)| \log(\varepsilon) \leq K \varepsilon^\alpha \log(\varepsilon) \to 0
\]

as \(\varepsilon \to 0\) and

\[
\int_0^1 \frac{|F(x+y) - F(x)|}{y} dy \leq K \int_0^1 y^{\alpha-1} dy = K/\alpha.
\]

Using Lemma 2.5 we conclude. \(\Box\)

Remark 2.6. Proofs of Lemma 2.5 and Proposition 2.4 show that if \(F\) satisfies

\[
|F(x+y) - F(x)| \leq K|y|^\alpha
\]

then

\[
\mathcal{H}^+(dF) \leq 2(K/\alpha) \omega(\mathbb{R}^2).
\]
Remark 2.7. Conditions i)-iv) from Lemma 2.5 encompass a larger class of functions than functions which are Hölder continuous. For instance, it suffices to assume that \(|F(x + y) - F(x)| \leq 1/|\log(|y|)|^\beta\) for \(|y| \leq \varepsilon, x \in \mathbb{R}\) and fixed \(\beta > 1\) and \(\varepsilon > 0\).

Remark 2.8. Due to embedding \(W^{1,p}(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})\) for \(p > 1\) (see e.g. [1]), using Proposition 2.4 we recover the result from Proposition 2.3.

Remark 2.9. Results of this section allow us to obtain embeddings of various spaces into the fractional Sobolev space \(H^{1/2}\) (see [12]) as follows. Distributions belonging to \(H^{-1}(\mathbb{R}^2)\), which are supported on the line \(\{(x_1,0) : x_1 \in \mathbb{R}\}\) may be identified with the space of \(H^{-1/2}(\mathbb{R})\) due to the fact that the trace operator \(T : W^{1,2}(\mathbb{R}^2) \rightarrow H^{1/2}(\mathbb{R})\) is bounded and has a bounded right inverse, see [12, Section 16]. Hence, if \(\omega \in H^{-1}(\mathbb{R}^2)\) is of the form (2) then \(dF\) belongs to \(H^{-1/2}(\mathbb{R})\) and consequently \(F\) belongs locally to \(H^{1/2}\).

It is not possible to extend the results of this section to arbitrary absolutely continuous \(F\). In the next section we show counterexamples.

3 Counterexamples

We begin by describing a class of functions, which we will use for construction of counterexamples for \(f \in L^1\) and \(f \in L(\log L)^\gamma\). Let, namely,

\[ f(x) = \sum_{n=1}^{\infty} h_n 1_{[a_n, a_n + d_n]}(x), \]

where for every \(n = 1, 2, \ldots\) we have \(a_n \in \mathbb{R}, h_n > 1, 0 < d_n \leq 1\) and \(a_n + d_n \leq a_{n+1}\). Observe that

\[ H^+(h 1_{[a_n, a_n + d_n]}) \geq \int_a^{a+d} \int_a^{a+d} \log^+ \left( \frac{1}{|x - y|} \right) h^2 dx dy \geq h^2 d^2 \log(1/d) \]

and hence

\[ H^+(f) \geq \sum_{n=1}^{\infty} h_n^2 d_n^2 \log(1/d_n). \] (7)

Proposition 3.1. There exists a nonnegative compactly supported \(f \in L^1\) such that \(H^+(f) = +\infty\).

Proof. Take \(d_n = \exp(-2^n)\) and \(h_n = 1/(2^nd_n)\). Then on the one hand

\[ \|f\|_{L^1} = \sum_{n=1}^{\infty} h_n d_n = 1. \]
On the other hand, however, by (7)
\[ H^+(f) \geq \sum_{n=1}^{\infty} 2^{-2n} \log(1/d_n) = +\infty. \]

\[ \square \]

**Corollary 3.2.** There exists an absolutely continuous \( F \) such that \( H^+(dF) = +\infty \).

Using the same construction we can generalize the result to the Calderón-Zygmund class \( L(\log L)^{\gamma} \), for \( \gamma < 1/2 \). Recall that \( f \in L(\log L)^{\gamma}(\mathbb{R}) \) if
\[ \int_{\mathbb{R}} |f(x)| (\log(1 + |f(x)|)^{\gamma} dx < \infty. \]

**Proposition 3.3.** For every \( \gamma < 1/2 \) there exists a nonnegative compactly supported \( f \in L(\log L)^{\gamma} \) such that \( H^+(f) = +\infty \).

**Proof.** A direct calculation shows that function \( f \) constructed in Proposition 3.1 belongs in fact to \( L(\log L)^{\gamma} \) for every \( \gamma < 1/2 \).

\[ \square \]

## 4 Applications

To apply the results of the previous sections it is useful to generalize them to the two-dimensional setting. We begin by defining the radial cumulative distribution function of a measure \( \omega \in \mathcal{M}_+(\mathbb{R}^2) \).

\[ G(r) := \begin{cases} \omega(B(0, r)) & \text{for } r > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{8} \]

where \( B(0, r) \) is the closed ball centered at 0 and with radius \( r \). Using \( G(r) \) we estimate \( H^+(\omega) \) by \( H^+(dG) \) as follows.

**Lemma 4.1.** Let \( \omega \) be a compactly supported nonnegative Radon measure on \( \mathbb{R}^2 \). Let \( G \) be its radial cumulative distribution function defined by (8). Then

i) for every Borel function \( h : [0, \infty) \rightarrow [0, \infty) \)
\[ \int_{\mathbb{R}^2} h(|x|) \omega(dx) = \int_{[0, \infty)} h(r) dG(r), \tag{9} \]

ii) \( H^+(\omega) \leq H^+(dG) \).

**Remark 4.2.** The reverse inequality in Lemma 4.1ii is false even up to a constant. For instance, both \( \nu_1 = \delta_{(1,0)} \) and \( \nu_2 \) – a probability measure distributed uniformly on the circle \( \{(x_1, x_2) : x_1^2 + x_2^2 = 1\} \) have the same radial cumulative distribution function
\[ G(r) = 1_{[1, \infty)}(r). \]

Nevertheless, \( H^+(\nu_1) = H^+(dG) = \infty \) yet \( H^+(\nu_2) < \infty \), see Remark 4.3.
Remark 4.3. Inequality in Lemma 4.1 holds for $G$ centered at any $x_0 \in \mathbb{R}^2$, i.e. $\mathcal{H}^+(\omega) \leq \mathcal{H}^+(dG_{x_0})$ for

\[ G_{x_0}(r) := \begin{cases} \omega(B(x_0, r)) & \text{for } r > 0, \\ 0 & \text{otherwise.} \end{cases} \]

The choice of $x_0$ is important in order to obtain a useful estimate. Taking, for instance, $x_0 = (1, 0)$ we obtain for measure $\nu_2$ from Remark 4.2 that

\[ G_{x_0}(r) = \begin{cases} 0 & \text{for } r < 0, \\ (2/\pi) \arcsin(r/2) & \text{for } 0 \leq r \leq 2, \\ 1 & \text{for } 2 \leq r, \end{cases} \]

which is Hölder continuous with exponent $1/2$. Thus, $\mathcal{H}^+(\nu_2) \leq \mathcal{H}^+(dG_{x_0}) < \infty$. On the other hand, the choice $x_0 = (0, 0)$ leads to $\mathcal{H}^+(\nu_2) \leq \mathcal{H}^+(dG_{x_0}) = \mathcal{H}^+(\delta_1) = \infty$, which does not allow us to conclude about finiteness of $\mathcal{H}^+(\nu_2)$.

Proof of Lemma 4.1. i) By definition of $G$, equality (9) holds for $h(r) = 1_{[r_1, r_2]}(r)$ with any $0 \leq r_1 < r_2 \leq \infty$. Standard approximation arguments for Radon measures and the Lebesgue monotone convergence theorem allow us to prove the case of general $h$.

ii) We observe that $\log^+ \frac{1}{|x-y|} \leq \log^+ \frac{1}{||x|-|y||}$, use repeatedly representation from i) as well as the Fubini theorem and calculate:

\[
\mathcal{H}^+(\omega) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log^+ \frac{1}{|x-y|} \omega(dx) \omega(dy) \\
\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log^+ \frac{1}{||x|-|y||} \omega(dx) \omega(dy) \\
= \int_{\mathbb{R}^2} \left[ \int_{[0,\infty)} \log^+ \frac{1}{|r_x-y|} dG(r_x) \right] \omega(dy) \\
= \int_{[0,\infty)} \int_{[0,\infty)} \log^+ \frac{1}{|r_x-r_y|} dG(r_x) dG(r_y) \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} \log^+ \frac{1}{|r_x-r_y|} dG(r_x) dG(r_y) = \mathcal{H}^+(dG).
\]

Corollary 4.4. Fix $\alpha > 0$ and let $\omega$ be a Radon measure such that $\omega(B(0,r)) = G(r)$ for

\[ G(r) = \begin{cases} cr^\alpha & \text{for } 0 \leq r \leq R, \\ cR^\alpha & \text{for } r > R, \\ 0 & \text{otherwise.} \end{cases} \quad (10) \]

Then $\omega \in H^{-1}(\mathbb{R}^2)$. Thus, we recover Theorem 1.2.
Proof. $\mathcal{H}^+(dG) < +\infty$, which follows by the fact that $G'(r) = r^\alpha 1_{[0, R]}(r)$ belongs to $L^p$ for some $p > 1$. Using Proposition 2.3 and Lemmas 4.1, we conclude. Alternatively, we can use Proposition 2.4 observing that $G(r) \in C^{\alpha, \alpha}$.

Next, let us investigate the Hausdorff dimension of the support of measures belonging to $H^{-1}(\mathbb{R}^2)$. As we will use Cantor sets and Cantor functions, we recall the definitions and basic properties of them.

**Definition 4.5.**  i) The standard Cantor set is the set $C \subset [0, 1]$ constructed inductively as follows.

- $Z_0 = [0, 1]$.
- $Z_1$ is obtained from $Z_0$ by removing the middle third of the interval, i.e. $Z_1 = [0, 1/3] \cup [2/3, 1]$.
- $Z_2$ is obtained from $Z_1$ by removing the middle third of every remaining interval in $Z_1$, i.e. $Z_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$.
- $Z_n$ is, in general, obtained by removing the middle third of every remaining interval in $Z_{n-1}$.

Finally, $C := \bigcap_{n=1}^\infty Z_n$.

ii) The standard Cantor function $\Gamma : [0, 1] \to [0, 1]$ can be constructed inductively as follows.

- $\gamma_0(x) = x$
- $\gamma_n(x) = \begin{cases} 1/2 \gamma_{n-1}(3x) & \text{for } 0 \leq x < 1/3, \\ 1/2 & \text{for } 1/3 \leq x \leq 2/3, \\ 1/2 + 1/2 \gamma_{n-1}(3x - 2) & \text{for } 2/3 < x \leq 1. \end{cases}$

We define $\Gamma := \lim_{n \to \infty} \gamma_n$, where the convergence is uniform on $[0, 1]$. If we prolong $\Gamma$ by $0$ for $x \leq 0$ and $1$ for $x \geq 1$ then we obtain a nondecreasing continuous function mapping $\mathbb{R}$ onto $[0, 1]$.

Let us summarize the basic properties of the standard Cantor set and Cantor function useful later on. For the proofs, we refer the reader to the survey paper [6].

**Proposition 4.6.**  i) The standard Cantor set is closed.

ii) The dimension of the standard Cantor set equals $\log(2)/\log(3)$.

iii) The standard Cantor function is Hölder continuous with exponent $\log(2)/\log(3)$.

iv) Measure $d\Gamma$ is supported on $C$. 

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Example 4.7. Let \( \omega \) satisfy
\[
\omega(B(0, r)) = \Gamma(r),
\]
where \( \Gamma(r) \) is the standard Cantor function. Then \( \omega \in H^{-1}(\mathbb{R}^2) \).

Proof. \( \Gamma(r) \) is Hölder continuous with exponent \( \alpha = \log(2)/\log(3) \). The assertion follows by Proposition 2.4 and Lemmas 1.1 and 4.1. \( \square \)

Now, we are ready to construct examples of measures belonging to \( H^{-1}(\mathbb{R}^2) \) supported on very small sets.

Proposition 4.8. A nonnegative Radon measure belonging to \( H^{-1}(\mathbb{R}^2) \) may be supported on a set of arbitrary small positive Hausdorff dimension.

Proof. Consider a modified Cantor set \( C_K \) obtained by removing in every step of the construction, described in Definition 4.5, the middle \( (K - 2)/K \) portion of every interval (note that for \( K = 3 \) we obtain the standard Cantor set). Let \( \Gamma_K(r) \) be the corresponding Cantor function, constructed similarly as in Definition 4.5, and consider the measure
\[
\omega_K = d\Gamma_K(x_1)\delta_0(dx_2).
\]
Then measure \( \omega_K \) is supported on the closed set \( C_K \) of dimension \( \alpha = \log(2)/\log(K) \). Moreover, \( \Gamma_K(r) \) is Hölder continuous with the same exponent \( \alpha = \log(2)/\log(K) \), see e.g. \( \square \), and hence \( \omega_K \in H^{-1}(\mathbb{R}^2) \).

Adapting the above construction, we can prove that a measure belonging to \( H^{-1}(\mathbb{R}^2) \) may be supported on a set of Hausdorff dimension 0.

Proposition 4.9. There exists a nonnegative bounded Radon measure belonging to \( H^{-1}(\mathbb{R}^2) \) which is supported on a bounded set of Hausdorff dimension 0.

Sketch of the proof. We construct a general Cantor set \( C_\infty \) by removing in step \( n \) of the construction the central \( 1 - 2c_n \) portion of every interval remaining from step \( n - 1 \). We obtain
\[
\begin{align*}
Z^0_\infty &= [0, 1], \\
Z^1_\infty &= [0, c_1] \cup [1 - c_1, 1], \\
Z^2_\infty &= [0, c_1 c_2] \cup [c_1 - c_1 c_2, c_1] \cup [1 - c_1, 1 - c_1 + c_1 c_2] \cup [1 - c_1 c_2, 1], \\
&\quad \ldots
\end{align*}
\]
(note that \( c_n \equiv 1/3 \) would lead to the standard Cantor set). Observe that the length of every of the \( 2^n \) intervals constituting \( Z^\infty_\infty \) is equal
\[
d_n = c_1 c_2 \ldots c_n.
\]
Fix $\beta > 1$ and set

$$d_n = e^{-2^n/\beta}.$$  

Then $c_n = d_n/d_{n-1}$ is decreasing and tends to 0 as $n \to \infty$. Define

$$C_\infty := \bigcap_{n=0}^{\infty} Z^n_\infty.$$  

Observe that $Z^n_\infty$ is a union of $2^n$ intervals of length $d_n$ and hence $C_\infty$ can be covered by $2^n$ balls of diameter $d_n$ for $n = 1, 2, \ldots$. Since for every fixed $\varepsilon > 0$ we have $2^n(d_n)^\varepsilon \to 0$ as $n \to \infty$, we conclude that the Hausdorff dimension of $C_\infty$ is equal 0.

Define

$$\omega_\infty := d\Gamma_\infty(x_1)\delta(x_2),$$

where $\Gamma_\infty$ is the corresponding Cantor function constructed as in Definition 4.5. More precisely, let

- $\gamma_\infty^0(x) = x$
- $\gamma_\infty^n(x) = \begin{cases} 1/2\gamma_\infty^{n-1}(x/c_n) & \text{for } 0 \leq x < c_n, \\
1/2 & \text{for } c_n \leq x \leq 1 - c_n, \\
1/2 + 1/2\gamma_\infty^{n-1}((x-1+c_n)/c_n) & \text{for } 1 - c_n < x \leq 1. \end{cases}$

and define $\Gamma_\infty := \lim_{n \to \infty} \gamma_n$, prolonging it by 0 for $x \leq 0$ and 1 for $x \geq 1$. We claim that

$$\Gamma_\infty(y) \leq 1/|\log(|y|)|^\beta$$

for $y \leq \exp(-(\beta + 1))$. Indeed,

- function $y \mapsto 1/|\log(|y|)|^\beta$ is increasing on the interval $[0, 1]$,
- function $y \mapsto 1/|\log(|y|)|^\beta$ is concave on the interval $[0, \exp(-(\beta + 1))]$,
- $\Gamma_\infty(d_n) = 2^{-n} = 1/|\log(|d_n|)|^\beta$ for $n = 0, 1, \ldots$,
- the graph of $\Gamma_\infty$ restricted to $[d_{n+1}, d_n]$ lies below the segment connecting points $(d_{n+1}, \Gamma_\infty(d_{n+1}))$ and $(d_n, \Gamma_\infty(d_n))$, i.e.

$$\Gamma_\infty(y) \leq \Gamma_\infty(d_{n+1}) + \frac{y - d_{n+1}}{d_n - d_{n+1}}(\Gamma_\infty(d_n) - \Gamma_\infty(d_{n+1}))$$

for every $y \in [d_{n+1}, d_n]$,
- the segment connecting points $(d_{n+1}, \Gamma_\infty(d_{n+1}))$ and $(d_n, \Gamma_\infty(d_n))$ lies, for $n$ satisfying $d_n \leq \exp(-(\beta + 1))$, below the graph of $y \mapsto 1/|\log(|y|)|^\beta$ due to concavity of the latter function.
Consequently, $\Gamma_\infty(y) \leq 1/|\log(|y|)|^\beta$ for $0 \leq y \leq \exp(-(\beta+1))$. Self-similarity of $\Gamma_\infty$ allows us to conclude that

$$|\Gamma_\infty(x+y) - \Gamma_\infty(x)| \leq 1/|\log(|y|)|^\beta$$

for $|y| \leq \exp(-(\beta+1))$ and arbitrary $x \in \mathbb{R}$. Using Remark 2.7 and Lemma 2.5 we obtain $\mathcal{H}^+(d\Gamma_\infty) < +\infty$ and hence $\omega_\infty \in H^{-1}(\mathbb{R}^2)$.

Finally, let us briefly comment on possible numerical applications of our results.

**Remark 4.10.** From the point of view of proving the convergence of numerical schemes it is important to know that $\omega^n$, a sequence of approximations of a compactly supported measure

$$\omega \in M_+(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2),$$

is such that $\mathcal{H}^+(\omega^n)$ remains bounded uniformly in $n$ (see e.g. [11] or [9]). Let, for instance, $\omega$ be the positive branch of the Kaden spiral (see [3]) at some point in time. Then function $r \mapsto \omega(B(0, r))$ is Hölder continuous with exponent $\alpha = 1/2$ (see [3]) and hence belongs locally to $H^{-1}(\mathbb{R}^2)$. Let $\omega_n$ be a smooth approximation of $\omega$, e.g. a vortex blob approximation, see [9]. To prove that $\mathcal{H}^+(\omega^n)$ is bounded uniformly with respect to $n$ it suffices, by Remark 2.6, to show that functions

$$r \mapsto \omega^n(B(0, r))$$

are uniformly Hölder continuous with constant $K$ and exponent $\alpha$ independent of $n$. Whether this is the case, depends on a particular form of vortex blob approximation. The goal is then to construct an approximation which satisfies the uniform Hölder condition. This, however, is relatively simple, since $r \mapsto \omega(B(0, r))$ is Hölder continuous.

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