Symmetries of Coefficients of Three-Term Relations for the Hypergeometric Series

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April 14, 2022

Abstract

Any three hypergeometric series whose respective parameters, \( a, b \) and \( c \), differ by integers satisfy a linear relation with coefficients that are rational functions of \( a, b, c \) and the variable \( x \). These relations are called three-term relations. This paper shows that the coefficients of three-term relations have properties called symmetries, and gives explicit formulas describing the symmetries.

Key Words and Phrases: The hypergeometric series; Three-term relation; Contiguous relation; Symmetry.

2010 Mathematics Subject Classification Numbers: 33C05.

1 Introduction

The hypergeometric series is defined by

\[
F(a, b, c; x) = F\left(\begin{array}{c} a \\ b \\ c \end{array}; x \right) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{1}{(1)_n} x^n.
\]

Here, \((\alpha)_n\) denotes \(\Gamma(\alpha+n)/\Gamma(\alpha)\), which equals \(\alpha(\alpha+1)\cdots(\alpha+n-1)\) for any positive integer \(n\). It is assumed that \(c\) is such that the denominator factor \((c)_n\) is never zero.

As mentioned in [1, Section 2.5, p.94], it is known that for any triples of integers \((k, l, m)\) and \((k', l', m')\), the three hypergeometric series

\[
F\left(\begin{array}{c} a+k \\ b+l \\ c+m \end{array}; x \right), \quad F\left(\begin{array}{c} a+k' \\ b+l' \\ c+m' \end{array}; x \right), \quad F\left(\begin{array}{c} a \\ b \\ c \end{array}; x \right)
\]

satisfy a linear relation with coefficients that are rational functions of \(a, b, c\) and \(x\). We call such a relation a three-term relation. Gauss obtained the three-term relations in the cases

\[
(k, l, m), (k', l', m') \in \{(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1), (0, 0, -1)\},
\]

where \((k, l, m) \neq (k', l', m')\); thus, there are \(\binom{6}{2} = 15\) pairs of \((k, l, m)\) and \((k', l', m')\). See [5, Chapter 4, p.71] for the 15 three-term relations obtained by Gauss.
We consider the three-term relations of the following form:

\[ F\left(\frac{a + k, b + l}{c + m} ; x\right) = Q \cdot F\left(\frac{a + 1, b + 1}{c + 1} ; x\right) + R \cdot F\left(\frac{a, b}{c} ; x\right). \]  

(1.1)

Note that the pair \((Q, R)\) of rational functions of \(a, b, c\) and \(x\) is uniquely determined by \((k, l, m)\) (cf. [4, Chapter 6, Section 23]). Ebisu [3, Section 2.3] noticed that the coefficient \(Q\) in (1.1) has 48 symmetries, and using these symmetries, he gave many special values of the hypergeometric \(\text{F}{}_{3}\). On the other hand, Víðunas considered the three-term relations of the form

\[ F\left(\frac{a + k, b + l}{c + m} ; x\right) = \bar{Q} \cdot F\left(\frac{a + 1, b + 1}{c + 1} ; x\right) + \bar{R} \cdot F\left(\frac{a, b}{c} ; x\right), \]

and gave an explicit formula describing \(\bar{Q}\)'s symmetry [6, p. 509, (11)]. We will see that \(Q\) also has the same symmetry.

In this paper, combining the 48 symmetries noticed by Ebisu and another symmetry the \(Q\) has 96 symmetries. In addition, we give a relation between \(Q\) and \(R\), and using the relation, we derive 96 symmetries of \(R\) from the 96 symmetries of \(Q\).

To avoid ambiguity, we first define the notion of a symmetry of \(Q\) and \(R\). For the parameters \(a, b, c\) and the variable \(x\), let \(S_{abc}, S_{x}\) and \(S\) be the sets defined by

\[ S_{abc} := \{n_0 + n_1a + n_2b + n_3c \mid n_i \in \mathbb{Z}\}, \]
\[ S_{x} := \left\{x, \frac{x}{x - 1}, \frac{1 - x}{x}, \frac{x - 1}{x}, \frac{1}{x - 1}, \frac{1}{1 - x}\right\}, \]
\[ S := \left\{(k, l; \alpha_1, \alpha_2, \alpha_3 ; \beta) \mid k, l, m \in \mathbb{Z}, \alpha_i \in S_{abc}, \beta \in S_{x}\right\}, \]

and let \(T\) be the set of all rational functions of \(a, b, c\) and \(x\). Also, let \(\text{Map}(S, T)\) denote the set of all functions \(P : S \rightarrow T\). Then, \(Q\) and \(R\) can be regarded as elements of \(\text{Map}(S, T)\); namely,

\[ Q = Q\left(\frac{k, l}{m} ; \frac{a, b}{c} ; x\right), \quad R = R\left(\frac{k, l}{m} ; \frac{a, b}{c} ; x\right) \in \text{Map}(S, T). \]

We define the notion of a symmetry of elements of \(\text{Map}(S, T)\) as follows:

**Definition 1.** Let \(G\) be a group that acts on \(\text{Map}(S, T)\), and take any \(\varphi \in G\) and \(P \in \text{Map}(S, T)\). If for any \(k, l, m \in \mathbb{Z}\), there exist \(\alpha_1, \ldots, \alpha_n \in S_{abc}\) and \(i_1, \ldots, i_n, j_1, j_2, j_3 \in \mathbb{Z}\) satisfying

\[ P\left(\frac{k, l}{m} ; \frac{a, b}{c} ; x\right) = (\alpha_1)^{i_1}(\alpha_2)^{i_2}\cdots(\alpha_n)^{i_n}(-1)^{j_1}x^{j_2}(1 - x)^{j_3}(\varphi P)\left(\frac{k, l}{m} ; \frac{a, b}{c} ; x\right), \]

then we say that \(P\) has a symmetry under \(\varphi\). If \(P\) has a symmetry under an arbitrary \(\varphi \in G\), then we say that \(P\) has symmetries under the action of \(G\).

We introduce a transformation group \(G\) and define an action of \(G\) on \(\text{Map}(S, T)\). Let \(G\) be the group generated by the following four mappings so that \(G\) acts on \(S\):

\[ \sigma_0 : \left(\frac{k, l}{m} ; \frac{a, b}{c} ; x\right) \mapsto \left(\frac{-k, -l}{-m} ; \frac{a + k, b + l}{c + m} ; x\right). \]
where group operation is defined as the composition of elements in $G$. We will see that $G$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times (S_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3)$; thus, the order of $G$ equals 96 (see Lemma 5). We define an action of $G$ on $\text{Map}(S, T)$ by $(\sigma P)(z) := P(\sigma^{-1}z)$, where $\sigma \in G$, $P \in \text{Map}(S, T)$ and $z \in S$.

The following theorem provides $Q$’s symmetries.

**Theorem 2.** The coefficient $Q$ of (11) has symmetries under the action of $G$; thus, $Q$ has 96 symmetries. In fact, $Q$ has the following symmetries:

\[
Q^{(k,l:a,b)}_m(x) = \frac{(c+1)m(c-m(-1)^{m-l-1}x^{-m}(-1-x)^{m-l})}{(a+1)(b+1)(c-a)(c-b)}(Q_0^{(k,l:a,b)}_m(x)),
\]

(1.2)

\[
Q^{(k,l:a,b)}_m(x) = -\frac{a}{c-a}(1-x)^{2-l}(Q_1^{(k,l:a,b)}_m(x)),
\]

(1.3)

\[
Q^{(k,l:a,b)}_m(x) = \frac{(c+1)(c-a-b-1)x^{m-l}}{(c-a)(c-b)}(Q_2^{(k,l:a,b)}_m(x)),
\]

(1.4)

\[
Q^{(k,l:a,b)}_m(x) = (Q_3^{(k,l:a,b)}_m(x)).
\]

(1.5)

In addition, combining these formulas, we are able to obtain the other 92 explicit formulas describing $Q$’s symmetries.

The identity (1.2) is counterpart of the symmetry of $\tilde{Q}$ given in [6, p. 509, (11)]. The explicit formulas in [3, (2.7), (2.11)] describe the symmetries of $Q$ under $\sigma_1 \sigma_3 \sigma_1 \sigma_3$ and $\sigma_3 \sigma_1 \sigma_3$, respectively.

The following lemma is used to derive $R$’s symmetries from the $Q$’s symmetries.

**Lemma 3.** The coefficients of (11) satisfy the following relation:

\[
R^{(k,l:a,b)}_m(x) = \frac{c(c+1)}{(a+1)(b+1)x(1-x)}Q^{(k-1,l-1:a+1,b+1)}_m(x).
\]

We introduce a transformation group $G$ and define an action of $G$ on $\text{Map}(S, T)$. Let $G$ be the group generated by the following four mappings so that $G$ acts on $S$:

\[
\tilde{\sigma}_0 := \tau \sigma_0^{-1} : \begin{pmatrix} k, l : a, b \\ m : c \end{pmatrix} \mapsto \begin{pmatrix} 2-k, 2-l : a+k-1, b+l-1 \\ 2-m : c+m-1 \end{pmatrix},
\]

\[
\tilde{\sigma}_1 := \tau \sigma_1^{-1} : \begin{pmatrix} k, l : a, b \\ m : c \end{pmatrix} \mapsto \begin{pmatrix} m+1-k, l : c-a-1, b \\ m : c \end{pmatrix},
\]

\[
\tilde{\sigma}_2 := \tau \sigma_2^{-1} : \begin{pmatrix} k, l : a, b \\ m : c \end{pmatrix} \mapsto \begin{pmatrix} k, l : a, b \\ k+l-m : a+b+1-c \end{pmatrix},
\]

\[
\tilde{\sigma}_3 := \tau \sigma_3^{-1} : \begin{pmatrix} k, l : a, b \\ m : c \end{pmatrix} \mapsto \begin{pmatrix} l, k : b, a \\ m : c \end{pmatrix}.
\]
where \( \sigma_i \) (\( i = 0, 1, 2, 3 \)) are the mappings defined in the above, \( \tau \) is the mapping defined by

\[
\tau : \left( \frac{k, l, a, b}{m \ c} ; x \right) \mapsto \left( \frac{k + 1, l + 1, a - 1, b - 1}{m + 1 \ c - 1} ; x \right).
\]

and group operation is defined as the composition of elements in \( \tilde{G} \). It immediately follows from this definition that \( \tilde{G} \) is isomorphic to \( G \); thus, the order of \( \tilde{G} \) also equals 96. We define an action of \( \tilde{G} \) on \( \text{Map}(S, T) \) in the same way as the action of \( G \) on \( \text{Map}(S, T) \).

The following theorem provides \( R \)'s symmetries.

**Theorem 4.** The coefficient \( R \) of (1.1) has symmetries under the action of \( \tilde{G} \); thus, \( R \) has 96 symmetries. In fact, \( R \) has the following symmetries:

\[
R \left( \frac{k, l, a, b}{m \ c} ; x \right) = \frac{(c + 1)m-1(c)m-1}{(a + 1)k-1(b + 1)l-1(c - a)m-k(c - b)m-l} \times (-1)^{m-k-l}x^{l-m-1}x^{m+1-k-l}(\tilde{\sigma}_0 R) \left( \frac{k, l, a, b}{m \ c} ; x \right),
\]

In addition, combining these formulas, we are able to obtain the other 92 explicit formulas describing \( R \)'s symmetries.

## 2 Proof of Theorem

After characterizing structure of \( G \), we prove Theorem

Let \( \sigma_4 := \sigma_1 \sigma_3 \sigma_1 \sigma_3 \) and \( \sigma_5 := \sigma_2 \sigma_4 \sigma_2 \sigma_4 \sigma_3 \) to make them become

\[
\sigma_4 : \left( \frac{k, l, a, b}{m \ c} ; x \right) \mapsto \left( \frac{m-k, m-l, c-a, c-b}{m \ c} ; x \right),
\]

\[
\sigma_5 : \left( \frac{k, l, a, b}{m \ c} ; x \right) \mapsto \left( \frac{-k-l, 1-a, 1-b}{-m \ 2-c} ; x \right).
\]

Then, we obtain the following lemma.

**Lemma 5.** The structure of \( G \) is identified as

\[
G = \langle \sigma_0 \rangle \times (\langle \sigma_1, \sigma_2 \rangle \ltimes (\langle \sigma_3 \rangle \times \langle \sigma_4 \rangle \times \langle \sigma_5 \rangle)) \cong \mathbb{Z} / 2 \mathbb{Z} \times \left( \mathbb{Z} / 2 \mathbb{Z} \right)^3, \]

where \( S_3 \) is the symmetric group of degree 3; thus, the order of \( G \) equals \( 2 \cdot 3! \cdot 2^3 = 96 \).

**Proof.** First, \( G = \langle \sigma_0 \rangle \times (\langle \sigma_1, \sigma_2, \sigma_3 \rangle \) holds. Next, since \( \langle \sigma_3, \sigma_4, \sigma_5 \rangle \) is normal in \( \langle \sigma_1, \sigma_2, \sigma_3 \rangle \) and satisfies \( \langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_3, \sigma_4, \sigma_5 \rangle = \{ \text{Id}_G \} \), where \( \text{Id}_G \) denotes the identity element of \( G \), it holds that \( \langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong \langle \sigma_1, \sigma_2 \rangle \ltimes \langle \sigma_3, \sigma_4, \sigma_5 \rangle \). Finally, from \( \sigma_i^2 = \text{Id}_G \) (\( 0 \leq i \leq 5 \)), \( \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \sigma_2 \sigma_1 \) and \( \sigma_i \sigma_j = \sigma_j \sigma_i \) (\( 3 \leq i, j \leq 5 \)), the proof of the lemma is complete. \( \square \)
We prove Theorem\(^2\). From the uniqueness of analytic continuation, it is sufficient to show that (1.2)–(1.5) hold for \(|x| < 1/2\); thus, below, we assume \(|x| < 1/2\). Also, we assume that
\[a, b, c - a, c - b, c, c - a - b, a - b \notin \mathbb{Z}.
\]

First, we prove (1.2) and (1.4). For the purpose, we introduce two expressions for \(Q\) in [2]. Let \(f_i\) \((i = 1, 2, 5, 6)\) be the functions defined by
\[
\begin{align*}
\frac{a}{c} f_i(a, b; x) := & f(a, b; x), \\
\frac{a}{c} f_i(a, b; x) := & f(a, b; a + b + 1 - c; 1 - x), \\
\frac{a}{c} f_i(a, b; x) := & x^{a+c} f(a + 1 - c, b + 1 - c; 2 - c), \\
\frac{a}{c} f_i(a, b; x) := & (1 - x)^{a - b} f(c - a, c - b; c + 1 - a - b; 1 - x),
\end{align*}
\]
where \(f(a, b; x) := \frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} F(a, b; c; x)\). Then, \(Q\) can be expressed as
\[
Q = \frac{ab(c)_m}{c(a)_k(b)_l} q,
\]
where
\[
q := q(k, l; a, b; c; x)
\]
\[
= \frac{a + k, b + l, c + m}{a + k, b + 1, c + m} f_1(a, b; x) f_1(a + 1, b + 1; c + 1; x) - f_1(a, b; x) f_3(a + k, b + l; c + m; x) f_3(a + k, b + 1; c + 1; x).
\]
\[
(2.1)
\]
\[
= \frac{(-1)^{m+1-k-l}}{(c - a)m-k(c - b)m-l} f_6(a, b; x) f_6(a + 1, b + 1; c + 1; x) - f_6(a, b; x) f_6(a + k, b + l; c + m; x) f_6(a + 1, b + 1; c + 1; x).
\]
\[
(2.2)
\]
These expressions (2.1) and (2.2) follow immediately from [2] p.260, (3.5)] and [2] p.264, the expression above Theorem 3.8], respectively.

Using (2.1), we prove (1.2). Applying \(\sigma_0\) to (2.1), we have
\[
(\sigma_0 q(k, l; a, b; c; x) = \frac{W(a, b, c; x)}{W(a + k, b + 1, c + m; x)} q(k, l; a, b; c; x),
\]
where \(W(a, b, c; x)\) denotes the denominator of (2.1); namely,
\[
W(a, b, c; x) := f_6(a, b; x) f_1(a + 1, b + 1; c + 1; x) - f_1(a, b; x) f_3(a + 1, b + 1; c + 1; x).
\]
From the formula [2] p.262, Lemma 3.6]
\[
W(a, b, c; x) = -\frac{\Gamma(a) \Gamma(b) \Gamma(a + 1 - c) \Gamma(b + 1 - c)}{\Gamma(c) \Gamma(1 - c)} x^{c} (1 - x)^{c - a - b - 1},
\]
\[
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\]
we obtain

\[
\frac{W(a, b, c \mid x)}{W(a + k, b + l, c + m \mid x)} = (-1)^{k+l-m}(c - a)_{m-k}(c - b)_{m-l} x^m(1 - x)^{k+l-m}.
\]

Therefore, it follows that

\[
(\sigma_0 q)^{k, l \atop m \atop c} x = \frac{(-1)^{k+l-m}(c - a)_{m-k}(c - b)_{m-l} x^m(1 - x)^{k+l-m}}{q^{k, l \atop m \atop c} x}.
\]  \hspace{1cm} (2.3)

Multiplying both sides of (2.3) by \((a + k)(b + l)(c + m)/(a + k)\_p(b + l)\_q(c + m)\), we prove (1.4). When we apply \(\sigma_2\) to (2.1), the numerator becomes

\[
f(k, a, b \atop a + b + 1 - c \atop 1 - x) f_1(k, a + k, b + l \atop a + b + 1 - c + k + l - m \atop 1 - x) - f_1(k, a + b + 1 - c \atop 1 - x) f(k, a + k, b + l \atop a + b + 1 - c + k + l - m \atop 1 - x).
\]  \hspace{1cm} (2.4)

and the denominator becomes

\[
f(k, a, b \atop a + b + 1 - c \atop 1 - x) f_1(k, a + b + 1 - c + k + l \atop a + b + 1 - c + k + l - m \atop 1 - x) - f_1(k, a + b + 1 - c \atop 1 - x) f(k, a + b + 1 - c + k + l \atop a + b + 1 - c + k + l - m \atop 1 - x).
\]  \hspace{1cm} (2.5)

From the definitions of \(f_i (i = 1, 2, 5, 6)\), we can rewrite (2.4) and (2.5) as

\[
f(k, a, b \atop c \atop x) f(k, a + k, b + l \atop c + m \atop x) - f(k, a, b \atop c \atop x) f(k, a + k, b + l \atop c + m \atop x),
\]  \hspace{1cm} (2.6)

\[
f(k, a, b \atop c \atop x) f(k, a + 1, b + 1 \atop c + 1 \atop x) - f(k, a, b \atop c \atop x) f(k, a + 1, b + 1 \atop c + 1 \atop x),
\]  \hspace{1cm} (2.7)

respectively. Comparing (2.6)/(2.7) with (2.2), we obtain

\[
(\sigma_2 q)^{k, l \atop m \atop c} x = (-1)^{k+l-m-1}(c - a)_{m-k}(c - b)_{m-l} q^{k, l \atop m \atop c} x.
\]  \hspace{1cm} (2.8)

Multiplying both sides of (2.8) by \(ab(a + b + 1 - c)_{k+l-m}/(a + b + 1 - c)(a)_p(b)_q\) completes the proof of (1.4).

Next, we prove (1.3) by using the following two formulas:

\[
F(k, a, b \atop c \atop x) = (1 - x)^{-k} F(k - a, b \atop c \atop x),
\]  \hspace{1cm} (2.9)

\[
F(k, a, b \atop c + 1 \atop x) = \frac{a - x}{a(1 - x)} F(k, a + 1, b + 1 \atop c + 1 \atop x) + a F(k, a, b \atop c \atop x).
\]  \hspace{1cm} (2.10)

where (2.9) is the Pfaff’s identity, and (2.10) is an immediate consequence of [3, p.14, (2.5)].

Applying (2.9) to both sides of (1.1), we have

\[
F(k, a, b \atop c + m \atop x) = Q(k, l \atop m \atop c) (1 - x)^{-k} F(k - a, b \atop c + 1 \atop x) + R(k, l \atop m \atop c) (1 - x)^{-l} F(c - a, b \atop c \atop x).
\]  \hspace{1cm} (2.11)

\[
+ R(k, l \atop m \atop c) (1 - x)^{-l} F(c - a, b \atop c \atop x).
\]  \hspace{1cm} (2.12)

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Moreover, from \(2.10\), we have
\[
F\left(c - a + m - k, b + l; \frac{x}{c + m}\right) = \frac{a - c}{a}(1 - x)^{l-1}Q\left(k, l; a, b; \frac{x}{c + 1}; x\right)F\left(c - a + 1, b + 1; \frac{x}{c + 1}; x\right)
\]
\[
+ \frac{c}{a}(1 - x)^{l-1}Q\left(k, l; a, b; \frac{x}{c + 1}; x\right) + (1 - x)^{l}
\]
\[
Q\left(k, l; a, b; \frac{x}{c + 1}; x\right) = F\left(c - a + 1, b + 1; \frac{x}{c + 1}; x\right).
\]

On the other hand, replacing \((k, l, m, a, b, c, x)\) by \((m - k, l, c - a, b, c, x/(x - 1))\) in \(1.1\), we have
\[
F\left(c - a + m - k, b + l; \frac{x}{c + m}\right) = Q\left(m - k, l; c - a, b; \frac{x}{c + 1}; x\right)F\left(c - a + 1, b + 1; \frac{x}{c + 1}; x\right)
\]
\[
+ R\left(m - k, l; c - a, b; \frac{x}{c + 1}; x\right)F\left(c - a + 1, b + 1; \frac{x}{c + 1}; x\right).
\]

Equating the coefficients of \(F(c - a + 1, b + 1, c + 1); x/(x - 1)\) in \(2.11\) and \(2.12\) completes the proof of \(1.3\).

Finally, we can immediately obtain \(1.5\) from the fact that \(F(\alpha, \beta, \gamma; x)\) is symmetric with respect to the exchange of \(\alpha\) and \(\beta\).

\section{3 Proof of Lemma 3}

In this section, we prove Lemma 3.

Replacing \((k, l, m, a, b, c)\) by \((k - 1, l - 1, m - 1, a + 1, b + 1, c + 1)\) in \(1.1\), we have
\[
F\left(a + k, b + l; \frac{x}{c + m}\right) = Q^{'}\cdot F\left(a + 2, b + 2; \frac{x}{c + m}\right) + R^{'}\cdot F\left(a + 1, b + 1; \frac{x}{c + m}\right),
\]
where
\[
Q^{'} := Q\left(k - 1, l - 1; a + 1, b + 1; \frac{x}{c + 1}; x\right), \quad R^{'} := R\left(k - 1, l - 1; a + 1, b + 1; \frac{x}{c + 1}; x\right).
\]

As is well known, \(F(a, b, c; x)\) satisfies
\[
\partial F\left(a, b; \frac{x}{c}\right) = \frac{ab}{c}F\left(a + 1, b + 1; \frac{x}{c + 1}; x\right),
\]
where \(\partial := d/dx\), and is a solution of the hypergeometric differential equation \(L_{abc} y = 0\), where
\[
L_{abc} := \partial^2 + \frac{c - (a + b + 1)x}{x(1 - x)}\partial - \frac{ab}{x(1 - x)}.
\]

Using these facts, we have
\[
0 = \partial^2 F\left(a, b; \frac{x}{c}\right) + \frac{c - (a + b + 1)x}{x(1 - x)}\partial F\left(a, b; \frac{x}{c}\right) - \frac{ab}{x(1 - x)}F\left(a, b; \frac{x}{c}\right).
\]
Also, for any \( \sigma \), therefore we obtain
\[
F^{(a + 1, b + 1)}_{c + 1} = \frac{ab(a + 1)(b + 1)}{c(c + 1)} F^{(a + 2, b + 2)}_{c + 2} + \frac{ab(c - (a + b + 1)x)}{c x(1 - x)} F^{(a + 1, b + 1)}_{c + 1} - \frac{ab}{x(1 - x)} F^{(a, b)}_{c}.
\]

Using this, we rewrite (3.1) as
\[
F^{(a + k, b + l)}_{c + m} = \left\{ -\frac{(c + 1)\{c - (a + b + 1)x\}}{(a + 1)\{b + 1\}x(1 - x)} F^{(a + 1, b + 1)}_{c + 1} + \frac{c(c + 1)}{(a + 1)\{b + 1\}x(1 - x)} Q R^{(a, b)}_{c} \right\}.
\]

Equating the coefficients of \( F(a, b, c ; x) \) in (3.1) and (3.2) completes the proof of Lemma 3.

4 Proof of Theorem 4

Using Theorem 2 and Lemma 3, we prove Theorem 4.

For any \((k, l, m) \in \mathbb{Z}^3\), let \( \lambda_z \) be the rational function of \( a, b, c \) and \( x \) defined by
\[
\lambda_z(z) := \frac{c(c + 1)}{(a + 1)\{b + 1\}x(1 - x)},
\]
where \( z := (k, l, m ; a, b, c ; x) \). Then, the relation in Lemma 3 can be written as
\[
R(z) = \lambda_z(z) (\tau Q)(z).
\]

Also, for any \( \sigma \in G \) and \((k, l, m) \in \mathbb{Z}^3\), let \( \lambda_z \) be the rational function of \( a, b, c \) and \( x \) satisfying
\[
Q(z) = \lambda_z(z) (\sigma Q)(z).
\]

Then, we obtain
\[
R(z) = \lambda_z(z) (\tau Q)(z) = \lambda_z(z) (\tau \lambda_z)(z) (\tau \sigma Q)(z) = \lambda_z(z) (\tau \lambda_z)(z) (\tau \sigma \tau^{-1})(z) = \lambda_z(z) (\tau \lambda_z)(z) (\tau \sigma \tau^{-1})(z).
\]

This implies that \( R \) has a symmetry under \( \tau \sigma \tau^{-1} \) for each \( \sigma \in G \); namely, \( R \) has symmetries under the action of \( \tilde{G} \). In particular, letting \( \sigma = \sigma_i \) \((i = 0, 1, 2, 3) \) in (4.1), we can derive (1.6)–(1.9) from (1.2)–(1.5), respectively.

Acknowledgements We are deeply grateful to Prof. Hiroyuki Ochiai for helpful comments. Also, we would like to thank Akihito Ebisu for his comments and suggestions.
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