TRACE-FREE CHARACTERS AND ABELIAN KNOT CONTACT HOMOLOGY II

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ABSTRACT. We calculate ghost characters for the $(5, 6)$-torus knot, and using them we show that the $(5, 6)$-torus knot gives a counter-example of Ng’s conjecture in [11] concerned with the relationship between degree 0 abelian knot contact homology and the character variety of the 2-fold branched covering of the 3-sphere branched along the knot.

1. Overview

In [10], L. Ng discovered a framework to construct a homology via a conormal unit bundle associated to a knot in the 3-space $\mathbb{R}^3$ using an idea of Legendrian contact homology. In general, the framework seems to be less manageable, however, he found its combinatorial and topological descriptions of the degree 0 part of the homology. Using them, he constructed a knot invariant, called the knot contact homology. One of the most interesting properties of the homology is the correspondence (Conjecture 1.1) of the abelianization of the knot contact homology, called degree 0 abelian knot contact homology, and the coordinate ring of the character variety of the 2-fold branched covering of the 3-sphere $\mathbb{S}^3$. The correspondence is still unclear in general, however as observed in [7], we have seen that the mechanism of the correspondence can be understood exactly by using the trace-free slice $S_0(K)$ and the associated algebraic set $F_2(K)$ of a knot $K$. Then we found an obstruction, called a ghost character of a knot [7, Definition 4.7], to the correspondence to be true [7, Theorem 4.9 (3)].

In this paper, using the ghost characters, we give a counter-example to the correspondence for the $(5, 6)$-torus knot $T_{5,6}$. Before giving the strategy, here we review the general setting for the correspondence (for details, refer to [7]).

Let $A_n^{ab}$ denote the polynomial ring over $\mathbb{Z}$ generated by indeterminates $a_{ij}$’s ($1 \leq i < j \leq n$) with $a_{ii} = -2$ for all $i$. For a knot $K$ in 3-space $\mathbb{R}^3$ given by a knot diagram $D_K$ with $n$ crossings, we define $I_{D_K} \subset A_n^{ab}$ to be the ideal generated by the elements $a_{lj} + a_{lk} + a_{li}a_{ij}$, where $i$ is the overcrossing strand and $j$, $k$ are the undercrossing strands of $D_K$ (we call such a triple $(i, j, k)$ with $i < k$ a Wirtinger triple), $l = 1, \cdots, n$ and $(i, j, k)$ ranges over all $n$ crossings of the knot diagram $D_K$. Namely, we have

$$I_{D_K} = \langle a_{lj} + a_{lk} + a_{li}a_{ij} \mid (i, j, k): \text{any Wirtinger triple}, \ l \in \{1, \cdots, n\}\rangle.$$
With this setting, the degree 0 abelian knot contact homology \( HC_0^{ab}(K) \) of a knot \( K \) is defined to be the quotient polynomial ring \( A_n^{ab}/I_{D_K} \):

\[
HC_0^{ab}(K) = \langle a_{12}, \ldots, a_{nn-1} \rangle_{\mathbb{Z}}[a_{ij} + a_{lk} + a_{li}a_{lj} \ (i,j,k): \text{any Wirtinger triple}, \ l \in \{1, \cdots, n\}] .
\]

Let \( K \) be a knot in the 3-sphere \( S^3 \) and \( p : C_2 K \to E_K \) the 2-fold cyclic covering of the knot exterior \( E_K \) such that the image \( p(\mu_2) \) of a meridian \( \mu_2 \) of \( C_2 K \) is the square \( \mu_2^2 \) of a meridian \( \mu_K \) of \( K \). The 2-fold branched covering \( \Sigma_2 K \) of \( S^3 \) branched along \( K \) is constructed from \( C_2 K \) by filling a solid torus trivially (i.e., attaching the standard meridian of the boundary of the solid torus to \( \mu_2 \)). Since the covering map \( p \) induces an injection \( p_* : \pi_1(C_2 K) \to G(K) \), by the above construction we have

\[
\pi_1(\Sigma_2 K) \cong \text{Im}(p_*)/\langle \langle \mu_2^2 \rangle \rangle,
\]

where \( \langle \langle \ast \rangle \rangle \) denotes the normal closure of the group \( \langle \ast \rangle \). In particular, by the Fox’s theorem in [2] (cf. [4], [7 Theorem 4.5]), for the knot group \( G(K) = \langle \langle m_1, \cdots, m_n \ | \ r_1, \cdots, r_{n-1} \rangle \rangle \) generated by \( n \) meridians \( m_1, \cdots, m_n \), if we take \( m_1 \) as \( \mu_K \), then we obtain

\[
\pi_1(\Sigma_2 K) \cong \langle m_1m_i \ (2 \leq i \leq n) \ | \ w(r_j), w(m_1r_jm_1^{-1}) \ (1 \leq j \leq n-1) \rangle,
\]

where \( w(r_j) \) (resp. \( w(m_1r_jm_1^{-1}) \)) is the word given by interpreting \( r_j \) (resp. \( m_1r_jm_1^{-1} \)) with the generators \( m_1, m_i \)'s. By [3], the set \( \mathfrak{X}(\Sigma_2 K) \) of characters of \( \text{SL}_2(\mathbb{C}) \)-representations of \( \pi_1(\Sigma_2 K) \) is parametrized by

\[
\begin{align*}
  z_{ab} &= t_{m_am_b}(\chi_\rho) = \text{tr}(\rho(m_am_b)) \quad (1 \leq a < b \leq n), \\
  z_{1cde} &= t_{m_1m_cm_dm_e}(\chi_\rho) = \text{tr}(\rho(m_1m_cm_dm_e)) \quad (2 \leq c < d < e \leq n).
\end{align*}
\]

The character variety \( X(\Sigma_2 K) \), which is a closed algebraic set, is constructed as the image of the map \( t : \mathfrak{X}(\Sigma_2 K) \to \mathbb{C}^{(n\choose 2)} \times \mathbb{C}^{n-1}, \ t(\chi_\rho) = (z_{ab}; z_{1cde}) \).

Then the coordinate ring \( \mathbb{C}[X(\Sigma_2 K)] \) of the algebraic set \( X(\Sigma_2 K) \) is defined as the set of regular functions on \( X(\Sigma_2 K) \), which is isomorphic to the quotient ring

\[
\mathbb{C}[X(\Sigma_2 K)] = \frac{\mathbb{C}[z_{ab}; z_{1cde}]}{\langle \text{polynomials in } \mathbb{C}[z_{ab}; z_{1cde}] \text{ vanishing on } X(\Sigma_2 K) \rangle}.
\]

where \( \sqrt{\langle \ast \rangle} \) denotes the radical of \( \langle \ast \rangle \). Through this algebraic setting, we can state Ng’s conjecture in detail.

**Conjecture 1.1** (Conjecture 5.7 in [11]). For any knot \( K \), the homomorphism \( g : HC_0^{ab}(K) \otimes \mathbb{C} \to \mathbb{C}[X(\Sigma_2 K)] \) defined by \( g(a_{ij}) = -z_{ij} \ (1 \leq i < j \leq n) \), \( g(1) = 1 \) gives an isomorphism. In particular, \( HC_0^{ab}(K) \otimes \mathbb{C} \) and \( \mathbb{C}[X(\Sigma_2 K)] \) are isomorphic.

Since the coordinate ring always has the trivial nilradical, it is reasonable to consider the nilradical quotient of \( HC_0^{ab}(K) \otimes \mathbb{C} \) as a conjecture. In [7, Theorem 4.8], we have shown that this modified conjecture is true for any 2-bridge and 3-bridge knots.

Meanwhile, in [7, Theorem 4.9 (3)] we give an obstruction to the modified conjecture to be true using a special kind of ghost characters. To be more precise, for a knot \( K \) we consider the set \( \mathfrak{G}_0(K) \) of characters \( \chi_\rho \) of the trace-free (traceless) representations \( \rho : G(K) \to \text{SL}_2(\mathbb{C}) \) of the knot group \( G(K) \), each of which satisfies
\( \text{tr}(\rho(\mu_K)) = 0 \) for a meridian \( \mu_K \) of \( K \). Again, let \( G(K) = \langle m_1, \cdots, m_n | r_1, \cdots, r_n \rangle \) be a Wirtinger presentation. Then it follows from [3] that \( \mathcal{S}_0(K) \) is parametrized by

\[
\begin{align*}
x_{ij} &= -t_{m_i m_j}(\chi_\rho) \quad (1 \leq i < j \leq n), \\
x_{ijk} &= -t_{m_i m_j m_k}(\chi_\rho) \quad (1 \leq i < j < k \leq n),
\end{align*}
\]

and has one-to-one correspondence under the map \( t \)

\[
t : \mathcal{S}_0(K) \rightarrow \mathbb{C}^\left(\begin{array}{c}
\binom{n}{2} + \binom{n}{3}
\end{array}\right), \quad t(\chi_\rho) = (x_{ij}; x_{ijk})
\]
to its image. In fact, that is a closed algebraic set called the \textit{trace-free slice} of the character variety of the knot group \( G(K) \) and denoted by \( S_0(K) \), whose defining equations consists of the following 3 types:

\textbf{(F2):} the fundamental relations

\[
x_{ak} = x_{ij} x_{ai} - x_{aj}
\]

\((1 \leq a \leq n, (i, j, k) : \text{any Wirtinger triple}), \)

\textbf{(H):} the hexagon relations

\[
x_{ii} x_{i3} x_{j1} x_{j2} x_{j3} = \frac{1}{2} \begin{vmatrix}
  x_{i1j1} & x_{i1j2} & x_{i1j3} \\
  x_{i2j1} & x_{i2j2} & x_{i2j3} \\
  x_{i3j1} & x_{i3j2} & x_{i3j3}
\end{vmatrix},
\]

\((1 \leq i_1 < i_2 < i_3 \leq n, 1 \leq j_1 < j_2 < j_3 \leq n), \)

\textbf{(R):} the rectangle relations

\[
\begin{vmatrix}
  2 & x_{12} & x_{1a} & x_{1b} \\
  x_{21} & 2 & x_{2a} & x_{2b} \\
  x_{a1} & x_{a2} & 2 & x_{ab} \\
  x_{b1} & x_{b2} & x_{ba} & 2
\end{vmatrix} = 0 \quad (3 \leq a < b \leq n).
\]

(For more details, refer to [7, Theorem 1.1].) By dropping the \( \binom{n}{3} \) coordinates \( (x_{ijk}) \) of the trace-free slice \( S_0(K) \), we have its projection to the algebraic set \( F_2(K) \) in \( \mathbb{C}^N, N = \binom{n}{2} \), defined by the fundamental relations (F2):

\[
F_2(K) = \left\{ (x_{12}, \cdots, x_{n-1,n}) \in \mathbb{C}^\left(\begin{array}{c}
\binom{n}{2}
\end{array}\right) \left|\begin{array}{c}
x_{ak} = x_{ij} x_{ai} - x_{aj} \text{ for any } 1 \leq a \leq n \\
\text{and any Wirtinger triple } (i, j, k)
\end{array}\right. \right\},
\]

If a point \( (x_{ij}) \in F_2(K) \) does not satisfy one of (H) and (R), which \( (x_{ij}) \) does not lift to \( S_0(K) \), then we call the point \( (x_{ij}) \) a ghost character of \( K \). Since the coordinate ring \( \mathbb{C}[F_2(K)] \) of the algebraic set \( F_2(K) \) has the description

\[
\mathbb{C}[F_2(K)] = \frac{\mathbb{C}[x_{12}, \cdots, x_{nn-1}]}{\sqrt{\langle x_{aj} + x_{ak} - x_{ai} x_{ij} | (i, j, k) : \text{any Wirtinger triple, } a \in \{1, \cdots, n\} \rangle}}
\]

it is clear that the map \( f : HC^{ab}_0(K) \otimes \mathbb{C} \rightarrow \mathbb{C}[F_2(K)], \) defined by \( f(a_{ij}) = -x_{ij} \) \((1 \leq i < j \leq n), f(1) = 1 \) gives a homomorphism as ring. Then the kernel of \( f \) is obviously the nilradical \( \sqrt{0} \) (refer to [7, Proposition 4.2]). So \( HC^{ab}_0(K) \otimes \mathbb{C}/\sqrt{0} \cong \mathbb{C}[F_2(K)] \) holds for any knot \( K \). Hence Conjecture [1.1] with the nilradical quotient claims that for any knot \( K \), the homomorphism

\[
h : \mathbb{C}[F_2(K)] \rightarrow \mathbb{C}[X(\Sigma_2K)]
\]
defined by \( h(x_{ij}) = t_{m_i m_j} \) (1 \( \leq i < j \leq n \)), \( h(1) = 1 \) gives an isomorphism. By basic arguments in algebraic geometry, this is true if and only if the following holds.

**Conjecture 1.2** (Conjecture 4.4 in [7]). For any knot \( K \), the map \( h^* : X(\Sigma_2 K) \to \mathbb{F}_2(K) \), which satisfies

\[
h(p)(z) = p(h^*(z))
\]

for any \( p \in \mathbb{C}[\mathbb{F}_2(K)] \) and \( z \in X(\Sigma_2 K) \), gives an isomorphism.

Note that it follows from [7, Section 4] that \( h^*((z_{ab}; z_{1cde})) = (z_{ab}) \). In this paper, we will give a counter example to Conjecture 1.2 for the \((5, 6)\)-torus knot \( T_{5,6} \).

**Theorem 1.3** (cf. Theorem 3.1). The map \( h^* \) is not surjective for the \((5, 6)\)-torus knot \( T_{5,6} \). In particular, Conjecture 1.2 (i.e., Conjecture 1.1 with the nilradical quotient) does not hold for \( T_{5,6} \).

The strategy for Theorem 1.3 is to find a point which breaks the surjectivity of \( h^* \). In general, the candidates which can break the surjectivity of \( h^* \) are only the ghost characters of a knot because the image \( \text{Im}(h^*) \) contains all non-ghost characters in \( \mathbb{F}_2(K) \). Indeed, for any non-ghost character \( x \in \mathbb{F}_2(K) \) we can always find a character \( \chi_\rho \in X(\Sigma_2 K) \) with \( h^*(\chi_\rho) = x \) by using the map \( \hat{\Phi} : S_0(K) \to X(\Sigma_2 K) \) constructed in [3]. (For more details, refer to [7, Section 4].)

In the following sections, we first calculate ghost characters for \( T_{5,6} \). Then we will show that there exists a ghost character whose preimage under the map \( h^* \) is empty. The fact proves that \( h^* \) is not surjective and thus not isomorphic. Moreover, this indicates that \( \mathbb{F}_2(T_{5,6}) \) and \( X(\Sigma_2 T_{5,6}) \) themselves are never isomorphic (refer to [7, Theorem 4.9 (3)]).

### 2. Ghost characters of a knot and Conjecture 1.2

#### 2.1. Ghost characters of a knot

As shown in [7, Theorem 4.8], a knot with bridge index less than 4 does not have ghost characters. A computer experiment shows that there exist 4-bridge and 5-bridge knots which have ghost characters. For example, the \((4, 5)\)-torus knot \( T_{4,5} \) as shown in [8], whose bridge index is 4, has a ghost character. Here we demonstrate the calculations of them for the \((5, 6)\)-torus knot \( T_{5,6} \), whose bridge index is 5. Let \( D \) be the diagram of the \((5, 6)\)-torus knot shown in Figure 2.1. Put the meridians \( m_1, \cdots, m_{24} \) as in Figure 2.1 for a Wirtinger presentation of \( G(T_{5,6}) \).

In this setting, the algebraic set \( \mathbb{F}_2(T_{5,6}) \) is given by

\[
\mathbb{F}_2(T_{5,6}) = \left\{ (x_{12}, \cdots, x_{23,24}) \in \mathbb{C}^{24} \left| \begin{array}{c} x_{ak} = x_{ij}x_{ai} - x_{aj} \\
\text{and any Wirtinger triple } (i, j, k) \end{array} \right. \right\}.
\]

We notice that a knot \( K \) in braid position has a nice reduction of the fundamental relations (F2) in general, which is similar to the case of a knot in bridge position (refer to [7, Theorem 4.8]). We review this fact for the current case \( T_{5,6} \). At first, we

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1The map \( h \) is the pull-back by \( h^* \).
have the following fundamental relations (F2) for $1 \leq a \leq 24$:

\[
x_{a24} = x_{14,18} x_{a18} - x_{a14}, \quad x_{a23} = x_{18,21} x_{a18} - x_{a21}, \\
x_{a22} = x_{18,20} x_{a18} - x_{a20}, \quad x_{a21} = x_{10,14} x_{a14} - x_{a10}, \\
x_{a20} = x_{14,17} x_{a14} - x_{a17}, \quad x_{a19} = x_{14,16} x_{a14} - x_{a16}, \\
x_{a18} = x_{14,15} x_{a14} - x_{a15}, \quad x_{a17} = x_{6,10} x_{a10} - x_{a6}, \\
x_{a16} = x_{10,13} x_{a10} - x_{a13}, \quad x_{a15} = x_{10,12} x_{a10} - x_{a12}, \\
x_{a14} = x_{10,11} x_{a10} - x_{a11}, \quad x_{a13} = x_{16} x_{a6} - x_{a1}, \\
x_{a12} = x_{69} x_{a6} - x_{a9}, \quad x_{a11} = x_{68} x_{a6} - x_{a8}, \\
x_{a10} = x_{67} x_{a6} - x_{a7}, \quad x_{a9} = x_{15} x_{a1} - x_{a5}, \\
x_{a8} = x_{14} x_{a1} - x_{a4}, \quad x_{a7} = x_{13} x_{a1} - x_{a3}, \\
x_{a6} = x_{12} x_{a1} - x_{a2}, \quad x_{a5} = x_{18,19} x_{a18} - x_{a19}, \\
x_{a4} = x_{5,18} x_{a5} - x_{a18}, \quad x_{a3} = x_{5,24} x_{a5} - x_{a24}, \\
x_{a2} = x_{5,23} x_{a5} - x_{a23}, \quad x_{a1} = x_{5,22} x_{a5} - x_{a22}.
\]

Note that the last 5 types of (F2) are described for the triple $(i, j, k)$ with $j > k$ (using the symmetry on $j$ and $k$) for a technical reason on the elimination process of (F2). Moreover, we add the following equations given by the fundamental relations for an efficient elimination:

\[
\begin{align*}
  x_{12} &= x_{5,22} x_{25} - x_{2,22} = x_{5,22} x_{5,23} - x_{5,23} x_{5,22} + x_{2,22} = x_{22,23}, \\
  x_{13} &= x_{5,24} x_{15} - x_{1,24} = x_{5,24} x_{5,22} - x_{5,22} x_{5,24} + x_{22,24}, \\
  x_{14} &= x_{5,18} x_{15} - x_{1,18} = x_{5,18} x_{5,22} - x_{5,22} x_{5,18} + x_{18,22}, \\
  x_{15} &= x_{5,22} = x_{18,19} x_{18,22} - x_{19,22}, \\
  x_{23} &= x_{5,23} x_{35} - x_{3,23} = x_{5,23} x_{5,24} - x_{5,24} x_{5,23} + x_{23,24}, \\
  x_{24} &= x_{5,24} x_{45} - x_{4,24} = x_{5,24} x_{5,18} - x_{5,18} x_{5,24} + x_{18,24}, \\
  x_{25} &= x_{5,23} = x_{18,19} x_{18,23} - x_{19,23}, \\
  x_{34} &= x_{5,24} x_{45} - x_{4,24} = x_{5,24} x_{5,18} - x_{5,18} x_{5,24} + x_{18,24}, \\
  x_{35} &= x_{5,24} = x_{18,19} x_{18,24} - x_{19,24}, \\
  x_{45} &= x_{5,18} = x_{18,19}.
\end{align*}
\]

We start with the elimination of $x_{a24}$ by applying $x_{a24} = x_{14,18} x_{a18} - x_{a14}$ to the right hand sides of the other fundamental relations and the added relations. We continue this elimination from $x_{a23}$ to $x_{a6}$. Then $x_{a24}, \ldots, x_{a6}$ ($1 \leq a \leq 24$) are described by
polynomials in

\[ R = \mathbb{C}[x_{12}, x_{13}, x_{14}, x_{15}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45}] . \]

This elimination process can be understood topologically as observed in the proof of \cite[Theorem 4.8]{7}. For example, a fundamental relation can be seen as sliding the corresponding loop at the corresponding crossing of the diagram \( D \) (this time from the left to the right) and resolving “the winding part” by the trace-free Kauffman bracket skein relation:

\[
\begin{align*}
&x_{ak}^k \text{ sliding} & \quad = \quad &x_{ai}^k - x_{aj}^k
\end{align*}
\]

We do this topological operation again and again until the loop under consideration is described by a polynomial in \( R \).

By the argument in the proof of \cite[Theorem 4.8]{7}, the resulting polynomial can be also obtained by sliding the arc to the right side of \( D \) first and resolving the winding parts along the way associated with the fundamental relations second. This understanding is also applied below. We summarize the topological elimination process case by case as follows:

1. for \( x_{ij} \) (1 ≤ \( i \leq j \leq 5 \)) in the original fundamental relations: slide the arc \( c_i \) (resp. \( c_j \)) from the left side to the right side of \( D \), meanwhile we fix the arc \( c_j \) (resp. \( c_i \)), and resolve the winding parts along the way associated with the fundamental relations (the resulting polynomial in \( R \) is denoted by \( g_i(x_{ij}) \) (resp. \( g_j(x_{ij}) \)),
2. for \( x_{ij} \) (1 ≤ \( i < j \leq 5 \)) in the added relations: just slide every loop from the left side to the right side of \( D \) (the resulting loop does not have winding parts for the current diagram \( D \)),
3. for the others: slide every loop all the way to the right side of \( D \) and resolve the winding parts along the way associated with the fundamental relations.

Process (1) gives \( x_{ij} = g_i(x_{ij}) \) and \( x_{ij} = g_j(x_{ij}) \) (1 ≤ \( i \leq j \leq 5 \)). Process (2) shows that \( x_{12} = x_{23} = x_{34} = x_{45} = x_{15} \) and \( x_{13} = x_{24} = x_{35} = x_{14} = x_{25} \) hold. Process (3) gives the description of \( x_{ij} \) (6 ≤ \( i \) or 6 ≤ \( j \)) by a polynomial in \( R \). Note that in Processes (2) and (3) \( x_{ii} \) (1 ≤ \( i \leq 24 \)) gives only the trivial relation \( x_{ii} = 2 \). So we omit them for the process. (The above idea is originally from \cite{5}. A similar idea can be seen in \cite{10}.)

By the above argument, we can define a biregular map (an isomorphic projection) \( i : F_2(T_5,6) \to \operatorname{Im}(i) \subset \mathbb{C}^{24} \),

\[
(x_{12}, \ldots, x_{23,24}) \mapsto (x_{12}, x_{13}, x_{14}, x_{15}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45}).
\]
Then the resulting equations in Process (1) turn out to be the defining polynomials of \(\text{Im}(i)\). Hence we obtain

\[
F_2(T_{5,6}) \cong \left\{ (x_{12}, \cdots, x_{45}) \in \mathbb{C}^5 \left| \begin{array}{l}
   x_{ij} = g_i(x_{ij}), x_{ij} = g_j(x_{ij}) \quad (1 \leq i \leq 5) \\
   x_{12} = x_{23} = x_{34} = x_{45} = x_{15}, \\
   x_{13} = x_{24} = x_{35} = x_{14} = x_{25}
\end{array} \right. \right\}.
\]

We notice that the above presentation of \(F_2(T_{5,6})\) can be naturally generalized for a knot \(K\) in \(m\)-braided position \((m \geq 2)\):

\[
F_2(K) \cong \left\{ (x_{12}, \cdots, x_{m-1,m}) \in \mathbb{C}^5 \left| x_{ij} = g_i(x_{ij}), x_{ij} = g_j(x_{ij}) \quad (1 \leq i \leq j \leq m) \right. \right\}.
\]

Now, we can eliminate \(x_{14}, x_{15}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45}\) by

\[
x_{12} = x_{23} = x_{34} = x_{45} = x_{15}, \quad x_{13} = x_{24} = x_{35} = x_{14} = x_{25}.
\]

Note that substituting

\[
\begin{align*}
   x_{12} &= x_{23}, \quad x_{24} = x_{34}, \quad x_{34} = x_{45}, \quad x_{45} = x_{15}, \quad x_{15} = x_{12}, \\
   x_{13} &= x_{24}, \quad x_{24} = x_{35}, \quad x_{35} = x_{14}, \quad x_{14} = x_{25}, \quad x_{25} = x_{13},
\end{align*}
\]

in \(x_{ij} = g_i(x_{ij})\) and \(x_{ij} = g_j(x_{ij})\) give \(x_{i+1,j+1} = g_{i+1}(x_{i+1,j+1})\) and \(x_{i+1,j+1} = g_{j+1}(x_{i+1,j+1})\), where every index shifts cyclically from 1 to 5, that is, if \(i\) (resp. \(j\)) is 5, then \(i + 1\) (resp. \(j + 1\)) means 1. Hence the relations \(x_{ij} = g_i(x_{ij})\), \(x_{ij} = g_j(x_{ij})\) for \((i, j) = (1, 5), (2, 3), (3, 4), (4, 5)\) are reduced to \(x_{12} = g_1(x_{12}), x_{12} = g_2(x_{12}), x_{13} = g_3(x_{13}), x_{13} = g_4(x_{13})\) by this substitution. Moreover, \(x_{ii} = g_i(x_{ii})\) is reduced to \(x_{11} = g_1(x_{11})\). Therefore \(F_2(T_{5,6})\) is eventually isomorphic to

\[
F_2(T_{5,6}) \cong \left\{ (x_{12}, x_{13}) \in \mathbb{C}^5 \left| x_{ij} = \tilde{g}_i(x_{ij}), \quad x_{ij} = \tilde{g}_j(x_{ij}) \quad (2 \leq j \leq 3) \right. \right\},
\]

where \(\tilde{g}_i(x_{ij})\) (resp. \(\tilde{g}_j(x_{ij})\)) denotes the polynomial given by substituting \(x_{14} = x_{15}, x_{15} = x_{12}, x_{23} = x_{12}, x_{24} = x_{13}, x_{25} = x_{13}, x_{34} = x_{12}, x_{35} = x_{13}, x_{45} = x_{12}\) in \(g_i(x_{ij})\) (resp. \(g_j(x_{ij})\)). By computer, we can calculate the following descriptions of the defining polynomials of \(F_2(T_{5,6})\). Let \(a = x_{12}, b = x_{13}\).

\[
\begin{align*}
  2 &= \tilde{g}_1(x_{11}) = a^6 - 5a^4b + 4a^3b^2 + 6a^2b^2 - 3a^3b - 6ab^2 - b^3 + 2ab + b^2 + 3a - 2, \\
  a &= \tilde{g}_1(x_{12}) = a^7 - 5a^5b - a^5 + 4a^4b + 6a^3b^2 - 3a^4 + 4a^3b - 6a^2b^2 - ab^3 - a^2b \\
       &\quad - 2ab^2 + 6a^2 + 2b^2 - a - 2, \\
  a &= \tilde{g}_2(x_{12}) = a^5b - a^4b - 4a^3b^2 + a^4 + 6a^2b^2 + 3ab^3 - 4a^2b - 2ab^2 - 3b^3 - a^2 \\
       &\quad + 2ab - a + 3b, \\
  b &= \tilde{g}_1(x_{13}) = a^6b - a^6 - 5a^4b^2 + 4a^4b + 4a^3b^2 + 6a^2b^2 + a^4 - 6a^3b - 3a^2b^2 - 6ab^3 \\
       &\quad - b^4 + 2a^3 - 3a^2b + 3ab^2 + b^3 + 5ab + b^2 - 3a - b, \\
  b &= \tilde{g}_3(x_{13}) = a^5b - a^5 - 4a^3b^2 + 3a^3b + 3a^2b^2 + 3ab^3 + a^3 - 4a^2b - ab^2 - 2b^3 \\
       &\quad + a^2 - 2ab + 2b.
\end{align*}
\]

Solving these equations, we obtain 10 points \((x_{12}, x_{13}) = (2, 2), (0, -1), (1, 1), (-2, 1), (\text{Root}(z^2 - 5z + 5), -1 + 2\text{Root}(z^2 - 5z + 5), (\text{Root}(z^2 - z - 1), 1), (\text{Root}(z^2 + z - 1), -1 - \text{Root}(z^2 + z - 1))\), giving the algebraic set \(F_2(T_{5,6})\). Here we focus on
(x_{12}, x_{13}) = (0, -1), (1, 1), (-2, 1). One can easily check that these do not satisfy at least one of the following rectangle relations (R):

\[
\begin{array}{c|c|c|c}
2 & x_{12} & x_{13} & x_{14} \\
\hline
x_{21} & 2 & x_{23} & x_{24} \\
x_{31} & x_{32} & 2 & x_{34} \\
x_{41} & x_{42} & x_{43} & 2
\end{array}
= 0,
\begin{array}{c|c|c|c}
2 & x_{12} & x_{13} & x_{15} \\
\hline
x_{21} & 2 & x_{23} & x_{25} \\
x_{31} & x_{32} & 2 & x_{35} \\
x_{51} & x_{52} & x_{53} & 2
\end{array}
= 0,
\begin{array}{c|c|c|c}
2 & x_{12} & x_{14} & x_{15} \\
\hline
x_{21} & 2 & x_{24} & x_{25} \\
x_{41} & x_{42} & 2 & x_{45} \\
x_{51} & x_{52} & x_{54} & 2
\end{array}
= 0.
\]

So we obtain the following.

**Proposition 2.1.** With the above setting, the points \((x_{12}, x_{13}) = (0, -1), (1, 1), (-2, 1)\) in \(F_2(T_{5,6})\) are ghost characters of \(T_{5,6}\).

Moreover, we can check that the other 7 points in \(F_2(T_{5,6})\) satisfy all the rectangle relations (R) and the hexagon relations (H), that is, the points shown in Proposition 2.1 are all ghost characters of \(T_{5,6}\). This fact will not be necessary here for the proof of Theorem 1.3 and so will be discussed in another paper in a general setting.

### 3. Character variety \(X(\Sigma_2 K)\), the algebraic set \(F_2(K)\) and Ng’s conjecture

Along the strategy stated at the end of Section 1, we try to find a ghost character which breaks the surjectivity of \(h^\ast\). Since, as shown in Proposition 2.1, the \((5,6)\)-torus knot \(T_{5,6}\) has ghost characters \((x_{12}, x_{13}) = (0, -1), (1, 1), (-2, 1) \in F_2(T_{5,6})\), the candidate which breaks the surjectivity is there. In fact, we can show that the ghost character \((x_{12}, x_{13}) = (1, 1)\) has the empty preimage under the map \(h^\ast\). Therefore we obtain the following.

**Theorem 3.1.** The ghost character \((x_{12}, x_{13}) = (1, 1)\) has the empty preimage under the map \(h^\ast\). Therefore, Conjecture 1.2 does not hold for the \((5, 6)\)-torus knot \(T_{5,6}\).

**Proof.** First, we need to calculate \(\pi_1(\Sigma_2 T_{5,6})\) using the presentation shown in Section 4 (cf. Theorem 4.5 in [7]).

**Lemma 3.2** (cf. Theorem 4.5 in [7]). For the Wirtinger presentation \(G(T_{5,6}) = \langle m_1, \ldots, m_{24} \mid r_1, \ldots, r_{23} \rangle\) associated with the diagram \(D\) in Figure 2.1, let \(x = m_1m_2, y = m_1m_3, z = m_1m_4, w = m_1m_5\). Then we obtain

\[\pi_1(\Sigma_2 T_{5,6}) = \langle x, y, z, w \mid w_i (1 \leq i \leq 8) \rangle,\]

where \(w_i (1 \leq i \leq 8)\) denote the following relators:

- \(w_1 = w^{-1}x^{-1}yz^{-1}wx^{-1}wz^{-1}y^{-1}x^{-1}w^{-1},\)
- \(w_2 = w^{-1}x^{-1}yz^{-1}wy^{-1}wz^{-1}y^{-1}x^{-1}w^{-1},\)
- \(w_3 = w^{-1}x^{-1}yz^{-1}wz^{-1}wz^{-1}y^{-1}x^{-1}w^{-1},\)
- \(w_4 = w^{-1}x^{-1}yz^{-1}w^{-1}y^{-1}x^{-1}w^{-1},\)
- \(w_5 = wxy^{-1}zw^{-1}xw^{-1}zy^{-1}xw,\)
- \(w_6 = wxy^{-1}zw^{-1}yw^{-1}zy^{-1}xw^{-1},\)
- \(w_7 = wxy^{-1}zw^{-1}zy^{-1}xw^{-1},\)
- \(w_8 = wxy^{-1}zw^{-1}zy^{-1}xw^{-1}.\)
We demonstrate how to calculate \( \pi_1(\Sigma_2T_{5,6}) \) in Lemma 3.2 below. First, by the relators \( r_1, \ldots, r_{23} \) of \( G(T_{5,6}) \), we have

\[
\begin{align*}
m_6 &= m_1m_2m_1^{-1}, \\
m_7 &= m_1m_3m_1^{-1}, \\
m_8 &= m_1m_4m_1^{-1}, \\
m_9 &= m_1m_5m_1^{-1}, \\
m_{10} &= m_6m_7m_6^{-1} = m_1m_2m_3m_2^{-1}m_2^{-1}m_1^{-1}, \\
m_{11} &= m_6m_8m_6^{-1} = m_1m_2m_4m_2^{-1}m_1^{-1}, \\
m_{12} &= m_6m_9m_6^{-1} = m_1m_2m_5m_2^{-1}m_1^{-1}, \\
m_{13} &= m_6m_1m_6^{-1} = m_1m_2m_1^{-1}m_1^{-1}, \\
m_{14} &= m_{10}m_{11}m_{10}^{-1} = m_1m_2m_3m_4m_3^{-1}m_2^{-1}m_1^{-1}, \\
m_{15} &= m_{10}m_{12}m_{10}^{-1} = m_1m_2m_5m_3^{-1}m_2^{-1}m_1^{-1}, \\
m_{16} &= m_{10}m_{13}m_{10}^{-1} = m_1m_2m_3m_1^{-1}m_2^{-1}m_1^{-1}, \\
m_{17} &= m_{10}m_6m_{10}^{-1} = m_1m_2m_3m_2^{-1}m_2^{-1}m_1^{-1}, \\
m_{18} &= m_{14}m_{15}m_{14}^{-1} = m_1m_2m_4m_5m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}, \\
m_{19} &= m_{14}m_{16}m_{14}^{-1} = m_1m_2m_4m_4m_3^{-1}m_3^{-1}m_2^{-1}m_1^{-1}, \\
m_{20} &= m_{14}m_{17}m_{14}^{-1} = m_1m_2m_4m_2^{-1}m_3^{-1}m_2^{-1}m_1^{-1}, \\
m_{21} &= m_{14}m_{10}m_{14}^{-1} = m_1m_2m_4m_3^{-1}m_3^{-1}m_2^{-1}m_1^{-1}, \\
m_{22} &= m_{18}m_{20}m_{18}^{-1} = m_1m_2m_4m_5m_5^{-1}m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}, \\
m_{23} &= m_{18}m_{21}m_{18}^{-1} = m_1m_2m_4m_5m_3^{-1}m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}, \\
m_{24} &= m_{18}m_{14}m_{18}^{-1} = m_1m_2m_4m_5m_5^{-1}m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}, \\
m_1 &= m_5m_2m_3^{-1}m_5m_2m_5^{-1}m_4^{-1}m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}m_5^{-1}, \\
m_2 &= m_5m_2m_3^{-1}m_5m_3m_5^{-1}m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}m_5^{-1}, \\
m_3 &= m_5m_2m_5^{-1} = m_5m_1m_2m_3m_4m_5^{-1}m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}m_5^{-1}, \\
m_4 &= m_5m_1m_5^{-1} = m_5m_1m_2m_3m_4m_5^{-1}m_3^{-1}m_2^{-1}m_1^{-1}m_5^{-1}.
\end{align*}
\]

By the Tietze transformations, the first 19 relations show that \( G(T_{5,6}) \) is generated by \( m_1, m_2, m_3, m_4, m_5 \) and then the set of relators of \( G(T_{5,6}) \) is generated normally by the last 4 relations. So we obtain

\[
G(T_{5,6}) = \langle m_1, m_2, m_3, m_4, m_5 \mid w_1, w_2, w_3, w_4 \rangle,
\]

where \( w_1, w_2, w_3 \) and \( w_4 \) are the following:

\[
\begin{align*}
w_1 &= m_5m_1m_2m_3m_4m_5m_2m_5^{-1}m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}m_5^{-1}m_1^{-1}, \\
w_2 &= m_5m_1m_2m_3m_4m_5m_3m_5^{-1}m_4^{-1}m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}m_5^{-1}m_2^{-1}, \\
w_3 &= m_5m_1m_2m_3m_4m_5m_4m_5^{-1}m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}m_5^{-1}m_1^{-1}m_3^{-1}, \\
w_4 &= m_5m_1m_2m_3m_4m_5m_4^{-1}m_3^{-1}m_2^{-1}m_1^{-1}m_5^{-1}m_4^{-1}.
\end{align*}
\]

Now, it follows from the presentation shown in Section 1 that

\[
\pi_1(\Sigma_2T_{5,6}) = \langle m_1m_2, m_1m_3, m_1m_4, m_1m_5 \mid w_i \ (1 \leq i \leq 8) \rangle,
\]
where \( w_5 = m_1 w_1 m_1^{-1}, w_6 = m_1 w_2 m_1^{-1}, w_7 = m_1 w_3 m_1^{-1} \) and \( w_8 = m_1 w_4 m_1^{-1} \). Note that the relators \( w_j \) (\( 1 \leq j \leq 8 \)) should be words in \( m_1 m_i \) (\( 1 \leq i \leq 5 \)). For simplicity, let \( x = m_1 m_2, y = m_1 m_3, z = m_1 m_4, w = m_1 m_5 \). Then we have

\[
\begin{align*}
  w_1 &= w^{-1} x^{-1} y z^{-1} w x^{-1} w z^{-1} y x^{-1} w^{-1}, \\
  w_2 &= w^{-1} x^{-1} y z^{-1} w y z^{-1} y x^{-1} w^{-1} x, \\
  w_3 &= w^{-1} x^{-1} y z^{-1} w z^{-1} y x^{-1} w^{-1} y, \\
  w_4 &= w^{-1} x^{-1} y z^{-1} w z^{-1} y x^{-1} w^{-1} z, \\
  w_5 &= w x y^{-1} z w^{-1} x w^{-1} z y^{-1} x w, \\
  w_6 &= w x y^{-1} z w^{-1} y w^{-1} y z^{-1} x w x^{-1}, \\
  w_7 &= w x y^{-1} z w^{-1} z w^{-1} y z^{-1} x w y^{-1}, \\
  w_8 &= w x y^{-1} z w^{-1} y z^{-1} x w z^{-1}.
\end{align*}
\]

This shows Lemma 3.2

To complete the proof, we show that the preimage \((h^*)^{-1}(g)\) of the ghost character \( g = (x_{12}, x_{13}) = (1, 1) \) is empty. (Note that \( g \) gives \( x_{ij} = 1 \) for \( 1 \leq i < j \leq 5 \).) To do this, we consider an \( SL_2(\mathbb{C})\)-representation \( \rho \) of \( \pi_1(\Sigma_2 T_{5,6}) \) satisfying \( t_{m_1 m_2}(\rho) = 1 \). Since \( t_{m_1 m_2}(\rho) \neq 2 \), up to conjugation, we can assume that

\[
(\rho(m_1 m_2), \rho(m_1 m_3), \rho(m_1 m_4), \rho(m_1 m_5))
= \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} b & c \\ d & e \end{pmatrix}, \begin{pmatrix} f & g \\ h & i \end{pmatrix}, \begin{pmatrix} j & k \\ l & m \end{pmatrix} \right) \in SL_2(\mathbb{C})^4.
\]

Then we can check by computer that there does not exist an \( SL_2(\mathbb{C})\)-representation \( \rho \) satisfying \( (t_{m_1 m_2}(\rho), t_{m_1 m_3}(\rho), t_{m_1 m_4}(\rho), t_{m_1 m_5}(\rho)) = (1, 1, 1, 1) \).

To ensure this, we can make use of the required condition \( t_{m_1 m_1}(\rho) = 1 \) (\( 1 \leq i < j \leq 5 \)). For example, \( t_{m_1 m_2}(\rho) = 1 \) gives \( a = (1 + \sqrt{3}i)/2, ((1 + \sqrt{3}i)/2)^{-1} \). Similarly, \( t_{m_1 m_3}(\rho) = t_{m_1 m_4}(\rho) = t_{m_1 m_5}(\rho) = 1 \) give \( e = 1 - b, i = 1 - f, m = 1 - j \). Since \( m_2 m_3 = (m_1 m_2)^{-1}(m_1 m_3) \) holds in \( \pi_1(\Sigma_2 T_{5,6}) \), we have

\[
\text{tr}(\rho(m_2 m_3)) = \text{tr}\left( \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & c \\ d & 1 - b \end{pmatrix} \right) = -3ib + (1 + \sqrt{3}i)/2 = 1
\]

So \( b = (3 + \sqrt{3}i)/6 \) holds. In the same way, using \( t_{m_2 m_4}(\rho) = t_{m_2 m_5}(\rho) = 1 \) we obtain \( f = j = (3 + \sqrt{3}i)/6 \). Moreover, \( t_{m_3 m_4}(\rho) = t_{m_3 m_5}(\rho) = t_{m_4 m_5}(\rho) = 1 \) show that \( ch + dg = -1/3, cl + dk = -1/3, gl + hk = -1/3 \) are required. On the other hand, \( cd = -2/3, gh = -2/3, kl = -2/3 \) are requested from the determinants. Combining these equations, we have \( 2g^2 - cg + 2c^2 = 0, 2k^2 - ck + 2c^2 = 0, 2k^2 - gk + 2g^2 = 0 \). The first two equations show that \( g \) and \( k \) are \( c(1 \pm \sqrt{15}i)/4 \). Substituting these to the third one, we obtain \( c^2 = 0 \), i.e., \( c = 0 \). This contradicts \( cd = -2/3 \neq 0 \), completing the proof.

We can also check by computer that the preimage of the ghost character \((x_{12}, x_{13}) = (0, -1)\) under \( h^* \) is empty. Additionally, for the ghost character \( g = (x_{12}, x_{13}) = (0, -1) \) we found the following representation \( \rho : \pi_1(\Sigma_2 T_{5,6}) \to SL_2(\mathbb{C}) \) satisfying
\( h^*(\chi_\rho) = g \). Let \( i = \sqrt{-1} \).

\[
\begin{align*}
\rho(m_1m_2) &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \Rightarrow \text{tr}(\rho(m_1m_2)) = 0, \\
\rho(m_1m_3) &= \begin{pmatrix} -1/2 & -(\sqrt{5} + 2i)/4 \\ (\sqrt{5} - 2i)/3 & -1/2 \end{pmatrix} \Rightarrow \text{tr}(\rho(m_1m_3)) = -1, \\
\rho(m_1m_4) &= \begin{pmatrix} -(1 + i)/2 & (\sqrt{5} - i)/4 \\ -(\sqrt{5} + i)/3 & -1/2 \end{pmatrix} \Rightarrow \text{tr}(\rho(m_1m_4)) = -1, \\
\rho(m_1m_5) &= \begin{pmatrix} -i/2 & 3i/4 \\ i & i/2 \end{pmatrix} \Rightarrow \text{tr}(\rho(m_1m_5)) = 0.
\end{align*}
\]

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