0 Introduction

Four-dimensional webs $W(3, 2, 2)$ have been considered in many books and papers (see, for example, the books [AS 92], [G 88] and the papers [B 35], [C 36], [G 85, 86, 87, 99], [K 81, 83, 84, 96]). They are of special interest since

a) They are the first webs generalizing the notion of two-dimensional three-web introduced by Blaschke [Bl 28] to higher codimension (see [B 35]).

b) They provide examples illustrating different properties of webs (see [AS 92], [B 35], [C 36], [G 88], [G 99]).

c) Their torsion tensor has a simple structure: $a^i_{jk} = a_i [j_k]$, where $a_i$ is a covector (see [AS 92], [G 88]).

d) They are connected with the pseudoconformal structures $CO(2, 2)$ of signature $(2, 2)$ (see [AG 96], [AG 99], [K 81, 83, 96]).

If the covector $a = \{a_1, a_2\}$ of a web $W(3, 2, 2)$ does not vanish, then it defines a transversal $a$-distribution invariantly and intrinsically connected with a web. In general, this $a$-distribution is not integrable.

In Section 1 we find necessary and sufficient conditions of its integrability and prove the existence theorem for webs $W(3, 2, 2)$ with integrable transversal $a$-distributions (see Theorems 1 and 3).

In Section 2 we prove that for a web $W(3, 2, 2)$ with the integrable distribution $\Delta$, its integral surfaces $V^2$ are geodesically parallel in an affine connection of a certain bundle of affine connections (Theorem 4 (i)) and study three-webs for which the surfaces $V^2$ are geodesically parallel with respect to affine connections of this bundle (Theorem 4 (ii)).

In Section 3, we find conditions for webs $W(3, 2, 1)$ cut by the foliations of $W(3, 2, 2)$ on $V^2$ to be hexagonal (Theorem 6) and prove the existence theorem for such webs $W(3, 2, 2)$ (Theorem 7). We also prove the existence theorem for webs $W(3, 2, 2)$ of the subclass which is the intersection of subclasses considered in Sections 2 and 3 (Theorem 8), and establish some properties of webs $W(3, 2, 2)$ implied by a relationship existing between four-dimensional three-webs and pseudoconformal structures $CO(2, 2)$ of signature $(2, 2)$ (Theorem 9).

In addition, in Sections 2 and 3 we find an analytic characterization of three-webs considered in these sections not only in a specialized frame but also in the general frame.

Note that webs $W(3, 2, 2)$ with integrable transversal $a$-distributions as well as webs $W(3, 2, 2)$, for which integral surfaces $V^2$ of $\Delta$ are geodesically parallel in an affine connection of a certain bundle of affine connections, and webs $W(3, 2, 2)$, for which the three-subwebs $W(3, 2, 1)$ cut by the foliations of $W(3, 2, 2)$ on $V^2$ are hexagonal, are considered in this paper for the first time.
1 The transversal distribution of a web \( W(3, 2, 2) \)

1. The leaves of the foliation \( \lambda_u, \ u = 1, 2, 3, \) of a web \( W(3, 2, 2) \) are determined by the equations \( \omega^i = 0, \ i = 1, 2, \) where

\[
\omega^i + \frac{1}{2} \omega^i + \frac{1}{3} \omega^i = 0
\]  

(1)

(see, for example, [G 88], Section 8.1 or [AS 92], Section 1.3). The forms \( \omega^1 \) and \( \omega^2 \) are basis forms on a manifold \( M^4 \) carrying the web \( W(3, 2, 2) \).

The structure equations of such a web can be written in the form

\[
\begin{cases}
d\omega^i = \omega^j \wedge \omega^i + a^i_{jk} \omega^j \wedge \omega^k, \\
d\omega^i = \omega^j \wedge \omega^i - a^i_{jk} \omega^j \wedge \omega^k.
\end{cases}
\]  

(2)

The differential prolongations of equations (2) are (see [G 88], Sections 8.1 and 8.4 or [AS 92], Section 3.2):

\[
d\omega^i - \omega^k \wedge \omega^i = b^i_{jkl} \omega^k \wedge \omega^l, \\
d\omega^i = p^i_{jk} \omega^j + q^i_{jk} \omega^j,
\]  

(3)

(4)

where

\[
b^i_{j[l]k]l} = \delta^i_{j[k]} \delta^l_{j[l]}, \quad b^i_{[j[k]l]} = \delta^i_{j[l]} \delta^l_{j[k]},
\]  

(5)

The quantities

\[
a^i_{jk} = a^i_{[j} \delta^i_{k]}
\]  

(6)

and \( b^i_{jkl} \) are the torsion and curvature tensors of a three-web \( W(3, 2, 2) \). Note that for webs \( W(3, 2, 2) \) the torsion tensor \( a^i_{jk} \) always has structure (6), where \( a = \{a_1, a_2\} \) is its transversal covector. If \( a = 0 \), then a web \( W(3, 2, 2) \) is isoclinically geodesic. Such webs were studied in [A 69]. In what follows, we will assume that \( a \neq 0 \), i.e., a web \( W(3, 2, 2) \) is nonisoclinically geodesic.

The covector \( a_i \) is defined in a second-order differential neighborhood of a point \( x \in M^4 \), and the curvature tensor \( b^i_{jkl} \) as well as the tensors \( p^i_{jk} \) and \( q^i_{jk} \) are defined in a third-order neighborhood of \( x \in M^4 \). By conditions (5), the tensor \( b^i_{jkl} \) can be represented in the form

\[
b^i_{jkl} = s^i_{jkl} + \frac{2}{3} p^i_{jk} \delta^i_j - \frac{1}{3} p^i_{kl} \delta^i_j - \frac{1}{3} p^i_{lj} \delta^i_k - \frac{1}{3} q^i_{jk} \delta^i_l - \frac{1}{3} q^i_{kl} \delta^i_j + \frac{2}{3} q^i_{lj} \delta^i_k,
\]  

where \( s^i_{jkl} = b^i_{[jkl]} \) is the symmetric part of the tensor \( b^i_{jkl} \) (see [AS 92], p. 113). The last formula implies that in a third-order neighborhood of \( x \in M^4 \), there are 8 independent components of the tensors \( p^i_{jk} \) and \( q^i_{jk} \) and also 8 independent components of the tensor \( b^i_{jkl} \).
In this paper we will need the differential prolongations of equations (3), (4), and (5). They have the form

\[
[\nabla b^i_{jk} + b^i_{jk}a_m(\omega^m - \omega^m_0)] \wedge \omega^k \wedge \omega^l = 0,
\]

(7)

\[
(\nabla p_{jk} + p_{jk}a_1\omega^l) \wedge \omega^k + (\nabla q_{jk} - q_{jk}a_1\omega^l) \wedge \omega^k + a_m b^m_{jk} \omega^k \wedge \omega^l = 0,
\]

(8)

where

\[
\nabla b^i_{jk} = db^i_{jkl} - b^i_{bmklm} - b^i_{jmk\omega^m} - b^i_{jm\omega^m l} + b^m_{jm\omega^m} b^i_{jk},
\]

\[
\nabla p_{jk} = dp_{jk} - p_{mk\omega^m} - p_{jm\omega^m l},
\]

\[
\nabla q_{jk} = dq_{jk} - q_{mk\omega^m} - q_{jm\omega^m l}.
\]

Equations (7) and (8) prove that the forms \(\nabla b^i_{jk}, \nabla p_{jk}, \) and \(\nabla q_{jk}\) are linear combinations of the basis forms \(\omega^k_1\) and \(\omega^k_2\):

\[
\begin{align*}
\nabla b^i_{jk} &= b^i_{jklm} \omega^m + \bar{b}^i_{jklm} \omega^m_0, \\
\nabla p_{jk} &= p^i_{jk} \omega^l + \bar{p}^i_{jk} \omega^l, \\
\nabla q_{jk} &= q^i_{jk} \omega^l + \bar{q}^i_{jk} \omega^l.
\end{align*}
\]

(9)

Substituting decompositions (9) into equations (7) and (8) and using the linear independence of the forms \(\omega^i\), we find that the coefficients in (9) satisfy the following conditions:

\[
\begin{align*}
\bar{b}^i_{jklm[l]} + a_{m} b^i_{jkl} &= 0, \\
\bar{b}^i_{jkl[m]} - a_{m} b^i_{jkl[l]} &= 0, \\
\bar{p}^i_{jkl} + q_{j[l]k} &= 0, \\
\bar{q}^i_{j[l]k} - q_{j[k]l} &= 0, \\
a_m b^m_{jk} - \bar{p}^i_{jk} + \bar{p}^i_{jk} &= 0.
\end{align*}
\]

(10)

In addition, upon differentiating conditions (5) and applying equations (9), we find other conditions for the coefficients in (9):

\[
\begin{align*}
\bar{b}^i_{j[l]kl[m]} &= \delta^i_{[k} \bar{p}^i_{j]lm}, \\
\bar{b}^i_{j[l]kl} &= \delta^i_{[k} \bar{p}^i_{j]l}, \\
\bar{b}^i_{j[l]k} &= \delta^i_{[k} \bar{q}^i_{j]l}.
\end{align*}
\]

(11)

It follows from conditions (5) and (10) that in a fourth-order neighborhood of \(x \in M^4\), there are 6 independent components of the tensors \(\bar{p}_{jkl}, \bar{p}_{ijkl}, \bar{q}_{ijkl}\) and also 20 independent components of the tensors \(\bar{b}_{jkl}, \bar{b}_{ijkl}\).

2. For a web \(W(3,2,2),\) a transversally geodesic distribution is defined (cf. [AS 92], Section 3.1) by the equations

\[
\xi^2 \omega^1 - \xi^1 \omega^2 = 0, \quad \xi^2 \omega^1 - \xi^1 \omega^2 = 0.
\]
If we take $\xi^1 = \frac{a_2}{a_1}$, we obtain the invariant transversal distribution $\Delta$ defined by the equations
\[
a_1\omega^1 + a_2\omega^2 = 0, \quad a_1\omega^1 + a_2\omega^2 = 0. \tag{12}
\]
This distribution is defined by the 1-forms
\[
\omega = a_1\omega^1 + a_2\omega^2, \quad \alpha = 1, 2. \tag{13}
\]
It is connected with a web invariantly and intrinsically since it is defined by the torsion tensor of a web. We will call the distribution $\Delta$ the transversal a-distribution of a web $W(3, 2, 2)$. Note that for isoclinically geodesic webs $W(3, 2, 2)$, for which $a_1 = a_2 = 0$, the distribution $\Delta$ is not defined.

The following theorem gives the conditions of integrability of the distribution $\Delta$.

**Theorem 1** The transversal a-distribution $\Delta$ defined by the equations (12) is integrable if and only if
\[
\begin{align*}
    a_2^2p_{11} - 2a_1a_2p_{(12)} + a_1^2p_{22} &= 0, \\
    a_2^2q_{11} - 2a_1a_2q_{(12)} + a_1^2q_{22} &= 0.
\end{align*} \tag{14}
\]

**Proof.** A transversal distribution $\Delta$ defined by equations (12) is integrable if and only if
\[
d\omega \wedge \omega = 0, \quad \alpha = 1, 2. \tag{15}
\]
By (2) and (13), equations (15) take the forms:
\[
\begin{align*}
    (a_2\nabla a_1 - a_1\nabla a_2) \wedge \omega^1 \wedge \omega^2 &+ (a_1\omega^1 + a_2\omega^2) = 0, \\
    (a_2\nabla a_1 - a_1\nabla a_2) \wedge \omega^1 \wedge \omega^2 &+ (a_1\omega^1 + a_2\omega^2) = 0,
\end{align*}
\]
where $\nabla a_i = da_i - a_i\omega^j_i$. By (4) and the linear independence of the forms $\omega^j_\alpha$, the last equations imply conditions (14).

Note that for an arbitrary web $W(3, 2, 2)$, it is always possible to take a specialized frame in which there is a relation between the components $a_1$ and $a_2$ of the covector $a$. For example, if the transversal distribution $\Delta$ coincides with the distribution $\omega^1 = 0$ or $\omega^2 = 0$ or $\omega^1 + \omega^2 = 0$ or $\omega^1 - \omega^2 = 0$, then we have $a_2 = 0$ or $a_1 = 0$ or $a_1 = a_2$ or $a_1 = -a_2$, respectively. Note that in these cases the forms $\omega^1_2$, $\omega^1_1$, $\omega^1_1 + \omega^1_2 - \omega^2_1 - \omega^2_2$, and $-\omega^1_1 + \omega^1_2 - \omega^2_1 + \omega^2_2$, respectively, are expressed in terms of the basis forms $\omega^j_\alpha$, i.e., in these cases we have
\[
\pi^1_2 = 0, \quad \pi^2_2 = 0, \quad \pi^1_1 + \pi^2_1 - \pi^2_2 = 0, \quad -\pi^1_1 + \pi^2_1 - \pi^1_2 + \pi^2_2 = 0,
\]
respectively, where $\pi^j_i = \omega^j_\alpha|_{\omega^i_\alpha = 0}$.

In proving the existence theorems it is convenient to use one of these specializations. Let us reformulate Theorem 1 for the first specialization indicated above.
Corollary 2 If the frame bundle associated with a three-web \( W(3,2,2) \) is specialized in such a way that
\[
a_2 = 0, 
\]
then the \( a \)-distribution \( \Delta \) coincides with the coordinate distribution \( \omega^1 \) and the condition \( \pi^1_2 = 0 \) holds. In such a frame bundle the \( a \)-distribution is integrable if and only if
\[
p_{22} = 0, \quad q_{22} = 0. 
\]

Proof. This follows from equations (14) and (16). ■

Each of the relations (14) and (17) gives two conditions which Pfaffian derivatives \( p_{ij} \) and \( q_{ij} \) of the covector \( a \) must satisfy in order for the transversal distribution \( \Delta \) of a web \( W(3,2,2) \) to be integrable.

3. We will now prove an existence theorem for webs with integrable transversal \( a \)-distributions \( \Delta \).

Theorem 3 The webs with integrable transversal \( a \)-distributions \( \Delta \) exist, and a solution of a system of differential equations defining such webs depends on five arbitrary functions of three variables.

Proof. Suppose that specialization (16) has been made. Since our web \( W(3,2,2) \) has the integrable \( a \)-distribution \( \Delta \) defined by the equations \( \omega^1 = 0 \), we have conditions (16) and (17), and equations (4) take the form
\[
\begin{cases}
da_1 - a_1 \omega^1_1 = p_{1j} \omega^j_1 + q_{1j} \omega^j_2, \\
-a_1 \omega^1_2 = p_{21} \omega^1_1 + q_{21} \omega^1_2.
\end{cases}
\]

The exterior cubic and quadratic equations (7) and (8) become
\[
\begin{align*}
[\nabla b^i_{jkl} + b^i_{jkl} a_1 (\omega^1_1 - \omega^1_2) \wedge \omega^k_1 \wedge \omega^2_2, \\
(\nabla p_{1k} + p_{1k} a_1 \omega^1_1) \wedge \omega^k_1 + (\nabla q_{1k} - q_{1k} a_1 \omega^1_1) \wedge \omega^k_2 + a_1 b^1_{1k} \omega^k_1 \wedge \omega^2_2 = 0,
\end{align*}
\]

\[
\begin{align*}
\nabla p_{21} \wedge \omega^1_1 + \nabla q_{21} \wedge \omega^1_2 + a_1 b^1_{2kl} \omega^k_1 \wedge \omega^2_2 = 0.
\end{align*}
\]

First note that the last equation of (20) implies that
\[
b^1_{222} = 0.
\]

By (17), (21), and (5), the number of unknown 1-forms (6 forms \( \nabla p_{1i}, \nabla q_{1i}, \nabla p_{21}, \nabla q_{21} \) and 7 forms among the forms \( \nabla b^i_{jkl} \) is 13, \( q = 13 \) (see [BCGGG 91]).

Since we have 2 exterior quadratic equations and 4 exterior cubic equations (see (19) and (20)), the Cartan’s characters are: \( s_1 = 2, s_2 = 6, \) and \( s_3 = 13 - 8 = 5 \). As a result, we have \( Q = s_1 + 2s_2 + 3s_3 = 29 \).
By (10), 13 Pfaffian derivatives of the functions \( p_{1i}, q_{1i}, p_{21} \) and \( q_{21} \) are independent: 3 functions \( \mathfrak{p}_{jk} \), 4 functions \( \mathfrak{q}_{jk} \), 3 functions \( \mathfrak{q}_{1jk} \), and 3 functions \( \bar{\mathfrak{p}}_{211}, \bar{\mathfrak{q}}_{211} \). In addition, by (5), (10), and (11), there are 16 independent functions \( \mathfrak{p}_{ijk} \), \( \tilde{\mathfrak{p}}_{ijk} \), \( \tilde{\mathfrak{q}}_{ijk} \), and \( \tilde{\mathfrak{q}}_{211} \). This implies that the general third-order integral element depends on \( N = 13 + 16 = 29 \) parameters.

Thus, we have \( Q = N \). As a result, the system defining three-webs \( W(3, 2, 2) \) is in involution, and its solution depends on five arbitrary functions of three variables (see [BCGGG 91]).

2 Geodesicity of integral surfaces

1. Suppose that the specialization of frames indicated in Section 1 has been made, i.e., we have condition (16): \( a_2 = 0 \). Then by (12), the distribution \( \Delta \) is determined by the system of equations

\[
\omega^1 = 0, \quad \omega^2 = 0.
\]

In \( T_x(M) \), consider a vectorial frame \( \{ e^\alpha_i \} \) that is conjugate to the coframe \( \{ \omega^\alpha_i \} \). Thus for \( x \in M \), we obtain

\[
dx = e^\alpha_i \omega^i.
\]

Then on integral surfaces \( V^2 \) of the \( \alpha \)-distribution \( \Delta \), we have

\[
dx = e^1_2 \omega^2 + e^3_2 \omega^2.
\]

The 1-forms \( \omega^2 \) and \( \omega^2 \) are basis forms on surfaces \( V^2 \), and the vectors \( e^1_2 \) and \( e^3_2 \) are tangent to these surfaces.

Consider the affine connections \( \Gamma \) defined on the manifold \( M^4 \) by 1-forms

\[
\theta^u_v = \begin{pmatrix} \theta^i_j & 0 \\ 0 & \theta^j_i \end{pmatrix}, \quad i, j = 1, 2; \quad u, v = 1, 2, 3, 4,
\]

where

\[
\theta^j_i = \omega^j_i + a^j_k (p \omega^k_1 + q \omega^k_2)
\]

(see [AS 92], p. 35). For the three-web \( W(3, 2, 2) \) in question, by (6) and (16), formulas (25) take the form:

\[
\begin{cases}
\theta^1_1 = \omega^1_1, & \theta^2_1 = \omega^2_1 + \frac{1}{2} a_1 (p \omega^2_1 + q \omega^2_2), \\
\theta^1_2 = \omega^1_2, & \theta^2_2 = \omega^2_2 + \frac{1}{2} a_1 (p \omega^2_1 + q \omega^2_2).
\end{cases}
\]
When the vectorial frame \( \{ e^\alpha_i \} \) moves along the manifold \( M^4 \) endowed with a connection \( \Gamma \), we obtain

\[
\begin{align*}
d e_1^1 &= \theta_1^1 e_1^1 + \theta_2^1 e_2^1, \\
d e_2^2 &= \theta_2^2 e_1^2 + \theta_2^2 e_2^2.
\end{align*}
\] (27)

We will now prove the following result.

**Theorem 4** (i) *If the a-distribution \( \Delta \) is integrable on the web \( W(3, 2, 2) \), then its integral surfaces \( V^2 \) are totally geodesic on \( M^4 \) in any affine connection of the bundle (24)–(25).*

(ii) *If on a web \( W(3, 2, 2) \) the condition

\[
\omega_2^1 = 0
\] (28)

holds, then the integral surfaces \( V^2 \) of the a-distribution \( \Delta \) are geodesically parallel in any affine connection of the bundle (24)–(25).*

**Proof.**

(i) By the second of relations (18) and (22), on surfaces \( V^2 \) we have

\[
\omega_2^1|_{V^2} = 0.
\] (29)

This and equations (22) and (26) imply that on \( V^2 \) equations (27) take the form

\[
\begin{align*}
d e_1^1 &= \theta_2^1 e_1^1, \\
d e_2^2 &= \theta_2^2 e_2^2.
\end{align*}
\] (30)

It follows that the bivectors \( \Delta = e_1^1 \wedge e_2^2 \) are geodesically parallel on \( V^2 \) in any affine connection of the bundle (24)–(25). As a result, the integral surfaces \( V^2 \) are totally geodesic on \( M^4 \) in any of these connections.

(ii) If equations (28) hold on the entire manifold \( M^4 \), then equations (30) are identically satisfied on \( M^4 \). Therefore, the bivectors \( \Delta = e_1^1 \wedge e_2^2 \) are geodesically parallel on the entire manifold \( M^4 \) in any affine connection of the bundle (24)–(25). As a result, the integral surfaces \( V^2 \) of the a-distribution \( \Delta \) are not only totally geodesic but also geodesically parallel on the entire manifold \( M^4 \) in all these connections.

2. Preliminary considerations show that three-webs \( W(3, 2, 2) \), for which integral surfaces \( V^2 \) of the transversal a-distribution \( \Delta \) are geodesically parallel, exist, and a solution of a system defining such webs depends on four arbitrary functions of three variables. However, we were not able to check the Cartan test in detail.

3. Conditions (28) for integral surfaces \( V^2 \) of the transversal a-distribution \( \Delta \) to be geodesically parallel were obtained in a specialized frame, i.e., for \( a_2 = 0 \). To find these conditions in the general frame, we first note that by (18), equations (28) are equivalent to equations

\[
p_{21} = 0, \quad q_{21} = 0
\] (31)
Of course, conditions (17) of integrability of the $a$-distribution $\Delta$ in a specialized frame must be added to conditions (31).

In order to write equations (17) and (31) in the general frame, we write equations (13) in the form

$$\omega^{1'}_\alpha = a_1 \omega^1_\alpha + a_2 \omega^2_\alpha,$$

(32)

consider a relation

$$\omega^{2'}_\alpha = c_1 \omega^1_\alpha + c_2 \omega^2_\alpha,$$

(33)

along with equation (32), and assume that

$$D = \det \begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix} \neq 0.$$  

(34)

The 1-forms $\omega^{1'}_\alpha$ and $\omega^{2'}_\alpha$ form a basis on a manifold $M^4$ carrying a three-web $W(3, 2, 2)$ whose coordinate bivectors determined by the equations $\omega^{1'}_\alpha = 0$ and $\omega^{2'}_\alpha = 0$ are transversal. The first of these bivectors is defined by the torsion tensor of the three-web $W(3, 2, 2)$, and the second one is chosen arbitrarily.

Let us write equations (32) and (33) in the form

$$\omega^{i'}_\alpha = a^{i'}_j \omega^j_\alpha,$$

(35)

where the matrix

$$A = \begin{pmatrix} a^{i'}_j \\ \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix}$$

(36)

is nondegenerate. Its inverse matrix can be written in the form

$$A^{-1} = \begin{pmatrix} a^{i'}_j \\ \end{pmatrix} = \frac{1}{D} \begin{pmatrix} c_2 & -a_2 \\ -c_1 & a_1 \end{pmatrix}.$$ 

(37)

Under the coframe transformation (35), the tensors $p_{ij}$ and $q_{ij}$ of the web $W(3, 2, 2)$ undergo the regular tensor transformation:

$$p_{i'j'} = a^{i'}_i a^{j'}_j p_{ij}, \quad q_{i'j'} = a^{i'}_i a^{j'}_j q_{ij}.$$ 

(38)

Taking into account (37), we write formulas (38) for the components $p_{21}, q_{21}, p_{22}$ and $q_{22}$ of these tensors:

$$\begin{cases} p_{2'1'} = c_1 (a_2 p_{12} - a_1 p_{22}) + c_2 (a_1 p_{21} - a_2 p_{11}), \\
q_{2'1'} = c_1 (a_2 q_{12} - a_1 q_{22}) + c_2 (a_1 q_{21} - a_2 q_{11}), \\
p_{2'2'} = -a_1 (a_2 p_{12} - a_1 p_{22}) - a_2 (a_1 p_{21} - a_2 p_{11}), \\
q_{2'2'} = -a_1 (a_2 q_{12} - a_1 q_{22}) - a_2 (a_1 q_{21} - a_2 q_{11}). \end{cases}$$ 

(39)

Note that the right-hand sides of the last two expressions differ from the left-hand sides of equations (14) only by sign.

Conditions (39) imply the following result.
Theorem 5 The integral surfaces $V^2$ of the $a$-distribution $\Delta$ are geodesically parallel with respect to any affine connection of the bundle (24)–(25) if and only if the components of the covector $a = \{a_i\}$ and of the tensors $p_{ij}$ and $q_{ij}$ satisfy the following conditions:

$$\begin{cases} a_2 p_{12} - a_1 p_{22} = 0, & a_1 p_{21} - a_2 p_{11} = 0, \\ a_2 q_{12} - a_1 q_{22} = 0, & a_1 q_{21} - a_2 q_{11} = 0. \end{cases}$$  

Proof. In fact, by (17) and (31), necessary and sufficient conditions for integral surfaces $V^2$ of the transversal $a$-distribution $\Delta$ to be geodesically parallel in the general frame with respect to any affine connection of the bundle (24)–(25) have the form

$$\begin{align*}
p_{2'1'} &= 0, \\
q_{2'1'} &= 0, \\
p_{2'2'} &= 0, \\
q_{2'2'} &= 0.
\end{align*}$$

(41)

But by conditions (34) and (39), equations (41) are equivalent to conditions (40).

3 Hexagonality of two-dimensional three-subwebs

1. On integral surfaces of the $a$-distribution $\Delta$ defined on $M^4$ by the torsion tensor of a web $W(3,2,2)$, the leaves of this web cut two-dimensional three-subwebs $W(3,2,1)$. Let us find the structure equations of these subwebs.

In a specialized frame in which condition (16) holds, the integral surfaces $V^2$ are defined by the system of equations (22). In addition, the complete integrability of this system on a surface $V^2$ and equations (31) imply that equation (28) holds. Thus, on a surface $V^2$, we have

$$\omega_1^1 = 0, \omega_2^1 = 0, \omega_2^2 = 0,$$

(42)

and the forms $\omega_2^1$ and $\omega_2^2$ are basis forms on $V^2$. One-dimensional foliations of a web $W(3,2,1)$ are defined on $V^2$ by the equations

$$\omega_1^1 = 0, \omega_2^1 = 0, \omega_1^2 + \omega_2^2 = 0.$$

(43)

To find the structure equations of webs $W(3,2,1)$ on surfaces $V^2$, we substitute the values (31) of the forms $\omega_1^1$, $\omega_2^1$, and $\omega_2^2$ into equations (2) and (3). As a result, we obtain the following structure equations:

$$\begin{align*}
d\omega_1^2 &= \omega_1^2 \wedge \omega_2^1, \\
d\omega_2^2 &= \omega_2^2 \wedge \omega_2^1, \\
d\omega_2^2 &= b_{i22}^2 \omega_i^1 \wedge \omega_2^1.
\end{align*}$$

(44)

Comparing these equations with the structure equations of a two-dimensional three-web (see [AS 92], p. 18), we see that the form $\omega_2^2$ is the connection form
of the web $W(3, 2, 1)$, and the component $b_{222}^2$ of the curvature tensor of the web $W(3, 2, 2)$ is the curvature of the web $W(3, 2, 1)$:

$$K = b_{222}^2.$$  

Since the vanishing of the curvature of the web $W(3, 2, 1)$ is a necessary and sufficient condition for its hexagonality, we arrive at the following result.

**Theorem 6** Two-dimensional three-webs $W(3, 2, 1)$ cut on integral surfaces of the $a$-distribution $\Delta$ by the foliations of $W(3, 2, 2)$ are hexagonal if and only if in the specialized frame bundle defined by condition (16), the component $b_{222}^2$ of the curvature tensor of the web $W(3, 2, 2)$ vanishes.

2. We will now prove an existence theorem for webs, for which two-dimensional three-webs $W(3, 2, 1)$ cut on integral surfaces $V^2$ by the foliations of the web $W(3, 2, 2)$ are hexagonal.

**Theorem 7** The webs $W(3, 2, 2)$, for which two-dimensional three-webs $W(3, 2, 1)$ cut on integral surfaces $V^2$ by the foliations of the web $W(3, 2, 2)$ are hexagonal, exist, and a solution of a system of differential equations defining such webs depends on four arbitrary functions of three variables.

**Proof.** Suppose that specialization (16) has been made. Then $a_2 = 0$. Since the $a$-distribution $\Delta$ is integrable, we have conditions (17). As a result, equations (4) take the form (18). As we showed in the proof of Theorem 3, the second of equations (18) implies (21).

Finally, since two-dimensional three-webs $W(3, 2, 1)$ cut on integral surfaces $V^2$ by the foliations of the web $W(3, 2, 2)$ are hexagonal, we have

$$b_{222}^2 = 0.$$  

(45)

By (17), (21), (45), and (5), there are 4 exterior cubic equations (7) and two exterior quadratic equation (8).

By (17), (21), and (5), the number of unknown 1-forms (6 forms $\nabla p_{1i}$, $\nabla q_{1i}$, $\nabla p_{2i}$, $\nabla q_{2i}$ and 6 forms $\nabla b^i_{jkl}$, namely, the forms $\nabla b^i_{111}$, $\nabla b^i_{112}$, $\nabla b^i_{122}$, $\nabla b^i_{211}$, $\nabla b^i_{112}$, $\nabla b^i_{222}$) is 12, $q = 18$ (see [BCGGG 91]).

Thus, the Cartan’s characters are: $s_1 = 2, s_2 = 6$, and $s_3 = 12 - 8 = 4$. As a result, we have $Q = s_1 + 2s_2 + 3s_3 = 26$.

By (10) and (45), 14 Pfaffian derivatives of the functions $p_{1i}$ and $q_{1i}$ are independent: 4 functions $\overline{p}_{111}, \overline{p}_{112}, \overline{p}_{122}, \overline{p}_{211}$, 4 functions $\overline{p}_{1jk}$, and 4 functions $\overline{q}_{111}, \overline{q}_{112}, \overline{q}_{122}, \overline{q}_{211}$. In addition, by (5), (10), (11), and (45), there are 12 independent functions among $\overline{b}^i_{jklm}$ and $\tilde{b}^i_{jklm}$: $\overline{b}^i_{1111}, \overline{b}^i_{1112}, \overline{b}^i_{1122}, \overline{b}^i_{1111}, \overline{b}^i_{1112}, \overline{b}^i_{1122}$. This implies that the general third-order integral element depends on $N = 14 + 12 = 26$ parameters.

Thus, we have $Q = N$. As a result, the system defining three-webs, for which and two-dimensional three-webs $W(3, 2, 1)$ cut on integral surfaces $V^2$ by the foliations of the web $W(3, 2, 2)$ are hexagonal, is in involution, and its solution depends on four arbitrary functions of three variables (see [BCGGG 91]).
3. We will now prove an existence theorem for webs, for which integral surfaces $V^2$ of the transversal distribution $\Delta$ are geodesically parallel and two-dimensional three-webs $W(3, 2, 1)$ cut on $V^2$ by the foliations of the web $W(3, 2, 2)$, are hexagonal.

**Theorem 8** The webs $W(3, 2, 2)$, for which integral surfaces $V^2$ of the transversal distribution $\Delta$ are geodesically parallel, and two-dimensional three-webs $W(3, 2, 1)$ cut on $V^2$ by the foliations of the web $W(3, 2, 2)$, are hexagonal, exist, and a solution of a system of differential equations defining such webs depends on three arbitrary functions of three variables.

**Proof.** Suppose that specialization (16) has been made. Then $\alpha_2 = 0$. Since the surfaces $V^2$ are geodesically parallel, we have conditions (28), i.e., we have $\omega_1^2 = 0$. As a result, equations (4) take the form

$$\begin{cases} 
  da_1 - a_1\omega_1^1 = p_1j\omega_1^j + q_1j\omega_2^j, \\
  \omega_2^1 = 0.
\end{cases} \tag{46}$$

By (4), the second of equations (46) implies that

$$p_{2i} = 0, \quad q_{2i} = 0, \tag{47}$$

and by (3), the same equation implies that

$$b_{2kl}^1 = 0. \tag{48}$$

Since two-dimensional three-webs $W(3, 2, 1)$ cut on $V^2$ by the foliations of the web $W(3, 2, 2)$, are hexagonal, we have condition (45):

$$b_{222}^2 = 0.$$ 

Note that conditions (47) imply conditions (17) of integrability of the distribution $\Delta$ defined by the equations $\omega_\alpha^i = 0$.

By (47), (48), (45), and (5), there are 3 exterior cubic equations (7) and only one exterior quadratic equation (8).

By (47), (48), and (5), the number of unknown 1-forms (4 forms $\nabla p_{1i}$, $\nabla q_{1i}$, and 4 forms $\nabla b_{jkl}^i$, namely, the forms $\nabla b_{111}^1$, $\nabla b_{111}^2$, $\nabla b_{122}^2$, $\nabla b_{122}^2$) is 8, $q = 8$ (see [BCGGG 91]).

Thus, the Cartan’s characters are: $s_1 = 1$, $s_2 = 4$, and $s_3 = 8 - 5 = 3$. As a result, we have $Q = s_1 + 2s_2 + 3s_3 = 18$.

By (10) and (45), 10 Pfaffian derivatives of the functions $p_{1i}$ and $q_{1i}$ are independent: 3 functions $\bar{p}_{111}^1$, $\bar{p}_{112}^1$, $\bar{p}_{122}^1$, 4 functions $\bar{p}_{1jk}$, and 3 functions $\bar{q}_{111}$, $\bar{q}_{112}$, $\bar{q}_{122}$. In addition, by (5), (10), (11), (45), and (48), there are 8 independent functions among $\bar{b}_{jkl}^i$ and $\bar{b}_{jkl}^l$: $\bar{b}_{1111}^1$, $\bar{b}_{1111}^2$, $\bar{b}_{1112}^1$, $\bar{b}_{1112}^2$, $\bar{b}_{1222}^2$, $\bar{b}_{1111}^1$, $\bar{b}_{1111}^2$. This implies that the general third-order integral element depends on $N = 10 + 8 = 18$ parameters.
Thus, we have \( Q = N \). As a result, the system defining three-webs, for which integral surfaces \( V^2 \) of the transversal \( a \)-distribution \( \Delta \) are geodesically parallel, and two-dimensional three-webs \( W(3, 2, 1) \) cut on \( V^2 \) by the foliations of the web \( W(3, 2, 2) \) are hexagonal, is in involution, and its solution depends on three arbitrary functions of three variables (see [BCGGG 91]).

4. Theorem 6 does not give a condition for two-dimensional three-webs \( W(3, 2, 1) \) cut on integral surfaces \( V^2 \) of the transversal \( a \)-distribution \( \Delta \) by the foliations of the web \( W(3, 2, 2) \) to be hexagonal in the general frame. To find such a condition in the general frame, we note that under the coframe transformation (34), the curvature tensor of the web \( W(3, 2, 2) \) undergoes the regular tensor transformation:

\[
b_{ijkl}^{\prime} = a_i^l a_j^k a_k^l a_i^j b_{ijkl}.
\]

We write formulas (49) for the components \( b_{1222}^{\prime}, b_{2222}^{\prime} \) of the curvature tensor:

\[
b_{1222}^{\prime} = a_{i}^{l} b_{i}^{l}, \quad b_{2222}^{\prime} = a_{i}^{l} b_{i}^{l},
\]

where we denote by \( b^{i} \) the following contraction:

\[
b^{i} = b_{ijkl}^{j} a_{k}^{l} a_{i}^{j} a_{2}^{l}.
\]

By (37), this contraction can be written as

\[
b^{i} = \frac{1}{D^{3}}(-b_{111}^{i} a_{2}^{3} + 3b_{1(12)}^{i} a_{2}^{2} a_{1}^{1} - 3b_{(12)}^{i} a_{2}^{2} a_{1}^{1} + b_{222}^{i} a_{1}^{1}).
\]

Equation (51) shows that the contraction \( b^{i} \) is expressed only in terms of components of the torsion and curvature tensors of the web \( W(3, 2, 2) \); that is, \( b^{i} \) is completely determined by this web.

We will now prove the following result.

**Theorem 9** Let \( W(3, 2, 2) \) be a four-dimensional three-web with a nonvanishing covector \( a \) and with the integrable transversal \( a \)-distributions \( \Delta \). Two-dimensional three-webs \( W(3, 2, 1) \) cut on integral surfaces \( V^2 \) of the transversal \( a \)-distribution \( \Delta \) by the foliations of the web \( W(3, 2, 2) \) are hexagonal if and only if the torsion and curvature tensors of this web are connected by the relations

\[
b^{1} = 0, \quad b^{2} = 0.
\]

**Proof.** Since the \( a \)-distribution \( \Delta \) is integrable, then in the specialized frame condition (21) holds. The hexagonality of the webs \( W(3, 2, 1) \) implies that in the specialized frame \( b_{2222}^{2} = 0 \). In the general frame these two conditions have the form

\[
b_{2222}^{1} = 0, \quad b_{2222}^{2} = 0.
\]

By (36), (37), and (50), equations (53) can be written as follows:

\[
a_{1} b^{1} + a_{2} b^{2} = 0, \quad c_{1} b^{1} + c_{2} b^{2} = 0.
\]

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Since by (34) $D \neq 0$, equations (54) imply conditions (53).

5. A three-web $W(3, 2, 2)$ defines on a manifold $M^4$ a conformal structure $CO(2, 2)$ whose isotropic cones $C_x$ in the tangent space $T_x(M^4)$ are determined by the equation

$$\omega^1_1 \omega^2_2 - \omega^1_2 \omega^2_1 = 0$$

(see [AG 96], p. 196). Transversal bivectors of the three-web $W(3, 2, 2)$ form one of two families of planar generators of the cones $C_x$. These bivectors are defined by equations (12). They can be written in the form

$$\omega^1_1 + t' \omega^2_2 = 0, \quad \omega^1_2 + t' \omega^2_1 = 0,$$

where $t = \frac{a_2}{a_1}$. On the manifold $M^4$, these bivectors form a fiber bundle $E_\alpha$ whose base is $M^4$ and whose one-dimensional fibers are defined by the fiber parameter $t$.

The relative conformal curvature of these bivectors is defined by the formula

$$C(t) = s^2_{1111} t^4 - (3s^2_{112} - s^2_{111}) t^3 + 3 (s^2_{122} - 3s^1_{11}) t^2 - (3s^2_{222} - 3s^1_{122}) t - s^1_{222},$$

where $s^i_{jkl} = b^i_{(jkl)}$ is the symmetrized curvature tensor of the web in question ([AG 96], Ch. 5; see also [K 83, 84, 96]). The vanishing of the quantity $C(t)$ singles out four transversal bivectors on the cone $C_x$. These bivectors are called principal.

Next, on a web $W(3, 2, 2)$ we consider the following contraction:

$$b = b' a_i.$$ 

(56)

This quantity is an absolute invariant of a web $W(3, 2, 2)$. Substituting the values (51) of the quantities $b'$ into equations (56), we find that

$$b = -\frac{1}{D} [b^2_{111} a_2^4 - (3b^2_{112} - b^1_{111}) a_2^3 a_1 + 3(b^2_{122} - b^1_{112}) a_2^2 a_1^2$$

$$- (b^2_{222} - 3b^1_{122}) a_2 a_1^3 + b^1_{222} a_1^4].$$

(57)

Comparing equations (55) and (57), we easily find that

$$b = -\frac{a^4_1}{D^4} C(\frac{a_2}{a_1}).$$

(58)

This means that the invariant $b$ of a web $W(3, 2, 2)$ differs from the relative conformal curvature of the transversal bivector $\Delta$ defined by the torsion tensor of $W(3, 2, 2)$ only by a factor.

Relations (50) and (56) allow us to prove the following result.

**Theorem 10** Let $W(3, 2, 2)$ be a four-dimensional three-web with a nonvanishing covector $a$ and with the integrable transversal $a$-distribution $\Delta$ defined by
Two-dimensional three-webs $W(3, 2, 1)$ cut on integral surfaces $V^2$ of the transversal $a$-distribution $\Delta$ by the foliations of the web $W(3, 2, 2)$ are hexagonal if and only if the $a$-distribution $\Delta$ is one of four principal transversal distributions of the pseudoconformal structure $CO(2, 2)$ associated with the web $W(3, 2, 2)$.

Proof. Sufficiency. Using the same considerations which we used in the proof of Theorem 8, we find that the integrability of $\Delta$ and the hexagonality of $W(3, 2, 1)$ lead to conditions (52). Equations (52) and (56) give $b = 0$. By (57), the last condition means that the transversal bivectors of the $a$-distribution $\Delta$ are principal.

Necessity. If the $a$-distribution $\Delta$ is integrable and all bivectors $\Delta$ of the pseudoconformal structure $CO(2, 2)$ defined by the web $W(3, 2, 2)$ on $M^4$ are principal, then we have the first equation of (53), $b_{2'}^i 2'_{2'2'} = 0$, and $b = b^i a_i = 0$. These two conditions imply that

$$K = b_{2'2'2'} = 0,$$

i.e., the three-webs $W(3, 2, 1)$ cut on integral surfaces $V^2$ of the transversal $a$-distribution $\Delta$ by the foliations of the web $W(3, 2, 2)$ are hexagonal.

References

[A 69] Akivis, M. A., *Three-webs of multidimensional surfaces*, Trudy Geometr. Sem. 2 (1969), 7–31 (Russian).

[AG 96] Akivis, M. A. and V. V. Goldberg, *Conformal differential geometry*, John Wiley & Sons, 1996, xiv+383 pp.

[AG 99] Akivis, M. A. and V. V. Goldberg *Differential geometry of webs*, Chapter 1 in *Handbook of Differential Geometry*, Elsevier, 1999, 1–103.

[AS 92] Akivis, M. A. and A. M. Shelekhov, *Geometry and Algebra of Multidimensional Three-Webs*, Translated from Russian by V. V. Goldberg, Kluwer Academic Publishers, Dordrecht, 1992, xvii+358 pp.

[Bl 28] Blaschke, W., *Thomsens Sechseckgewebe. Zueinander diagonale Netze*, Math. Z. 28 (1928), 150–157.

[B 35] Bol, G., *Über 3-Gewebe in vierdimensionalen Raum*, Math. Ann. 110 (1935), 431–463.

[BCGGG 91] Bryant, R. L., S. S. Chern, R. B. Gardner, H. L. Goldsmith, and P. A. Griffiths, *Exterior differential systems*, Springer-Verlag, New York, 1991, vii+475 pp.
[C 36] Chern, S. S., Eine Invariantentheorie der Dreigewebe aus r-
dimensionalen Mannigfaltigkeiten in $\mathbb{R}_r$, Abh. Math. Sem. Univ.
Hamburg 11 (1936), no. 1–2, 333–358.

[G 85] Goldberg, V. V., 4-tissus isoclines exceptionnels de codimension
deux et de 2-rang maximal, C. R. Acad. Sci. Paris Sér. I Math. 301 (1985),
o. 11, 593–596.

[G 86] Goldberg, V. V., Isoclinic webs $W(4,2,2)$ of maximum 2-rank,
Differential Geometry, Peninsula 1985, Lecture Notes in Math., 1209,
Springer, Berlin-New York, 1986, 168–183.

[G 87] Goldberg, V. V., Nonisoclinic 2-codimensional 4-webs of maxi-
mum 2-rank, Proc. Amer. Math. Soc. 100 (1987), no. 4, 701–708.

[G 88] Goldberg, V. V., Theory of Multicodimensional $(n + 1)$-Webs,
Kluwer Academic Publishers, Dordrecht-Boston-Tokyo, 1988,
xxii+466 pp.

[G 99] Goldberg, V. V., A classification and examples of four-
dimensional isoclinic three-webs, Webs and Quasigroups, Tver St.
Univ., Tver’, 1998/1999, 32–66.

[K 81] Klekovkin, G. A., A pencil of Weyl connections associated with
a four-dimensional three-web, Geometry of Imbedded Manifolds,
Moskov. Gos. Ped. Inst., Moscow, 1981, 59–62 (Russian).

[K 83] Klekovkin, G. A., Weyl geometries generated by a four-
dimensional three-web, Ukraïn. Geom. Sb. 26 (1983), 56–63 (Rus-
sian).

[K 84] Klekovkin, G. A., Four-dimensional three-webs with a covariantly
constant curvature tensor, Webs and Quasigroups, Kalinin. Gos.
Univ., Kalinin, 1984, 56–63 (Russian).

[K 96] Klekovkin, G. A., Certain problems of the geometry of four-
dimensional three-webs, Proc. Annual Scientific Confer., Faculty
of Physics & Mathematics, Samara State Pedag. Univ., Samara,
1996, 9–11 (Russian).

Authors’ addresses:

M. A. Akivis
Department of Mathematics
Jerusalem College of Technology—Mahon Lev
Havaad Haleumi St., P. O. B. 16031
Jerusalem 91160, Israel
E-mail address: akivis@avoda.jct.ac.il

V. V. Goldberg
Department of Mathematical Sciences
New Jersey Institute of Technology
University Heights
Newark, N.J. 07102, U.S.A.
E-mail address: vlgold@m.njit.edu