The Berezin inequality on domains of infinite measure

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Abstract The Berezin inequality gives an upper bound on the Riesz means of the magnetic Schrödinger operator on a set of finite volume. We find an analogous inequality for the magnetic operator with homogeneous magnetic field on sets whose complement in \( \mathbb{R}^2 \) has finite measure. Similar bounds are obtained for the Heisenberg sub-Laplacian.

Keywords Berezin inequality · Magnetic field · Heisenberg sub-Laplacian

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1 Introduction

Let \( \Omega \) be an open subset of \( \mathbb{R}^d \) such that \( \Omega^c = \mathbb{R}^d \setminus \Omega \) is of finite measure. In [2] Berezin proved that the Dirichlet Laplacian operator \( -\Delta_{\Omega^c}^\gamma \) on \( \Omega^c \) satisfies the inequality

\[
\text{tr}( -\Delta_{\Omega^c}^\gamma - \lambda )_+^\gamma \leq (2\pi)^{-d} |\Omega^c| \int_{\mathbb{R}^d} (|p|^2 - \lambda)_+^\gamma \, dp = L_{\gamma,d}^{\text{cl}} |\Omega^c| \lambda^{\gamma + \frac{1}{2}}
\]

for all \( \lambda \geq 0, \gamma \geq 1 \). Here and below, the measure of a set \( S \subset \mathbb{R}^d \) is denoted by \( |S| \) and \( x_+ = \frac{1}{2}(x) - x \) is the negative part of a variable, a function or a self-adjoint operator. The so-called Lieb–Thirring constant \( L_{\gamma,d}^{\text{cl}} \) can be computed to be

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and is sharp, which follows from an asymptotic result by Weyl [17]. For the inequality to hold it is essential that the Laplace operator is considered on the set $\Omega^c$ of finite volume. This guarantees that $H$ only has discrete spectrum consisting of eigenvalues converging to infinity, showing that the left-hand-side of (1) exists.

In [5] a similar result to the Berezin inequality (1) has been established for the Dirichlet Laplace operator on the set $\Omega$ of infinite measure. To this end one introduces the orthogonal projection $P_\Omega : L^2(\mathbb{R}^d) \rightarrow L^2(\Omega)$, i.e. the multiplication with the characteristic function $\chi_\Omega$. The operator $P_\Omega(-\Delta)P_\Omega$ corresponds to the Laplacian on the set $\Omega$ with Dirichlet boundary conditions. Since the continuous spectrum of $P_\Omega(-\Delta)P_\Omega$ contains the positive real axis, the operator $(P_\Omega(-\Delta)P_\Omega - \lambda)_-$ is not trace-class on $L^2(\mathbb{R}^d)$. However, it can be compared to a suitable operator to achieve similar results to (1). The authors of [5] considered the difference $(-\Delta - \lambda)_- - (P_\Omega(-\Delta)P_\Omega - \lambda)_-$ and proved that

$$\text{tr} \left( (-\Delta - \lambda)_- - (P_\Omega(-\Delta)P_\Omega - \lambda)_- \right) \geq L_{1,d}^{\text{cl}} |\mathbb{R}^d \setminus \Omega| \lambda^{1 + \frac{d}{2}},$$

which can be seen as an analogue of the Berezin inequality for perturbations of the continuous spectrum of the Laplace operator. Bounds on traces for these types of problems are a fairly recent research area and we point to [5] for a generalisation of Lieb–Thirring inequalities to this setting.

In our paper we aim to find an analogous inequality to (2) for the magnetic operator $H_B = (-i \nabla + A(x))^2$. Similar to the case of the Laplacian, problems stem from the fact that $(P_\Omega H_B P_\Omega - \lambda)_-$ is not trace-class. Thus we consider the difference $(H_B - \lambda)_- - (P_\Omega H_B P_\Omega - \lambda)_-$ and establish lower bounds on the trace of this operator. We also prove a similar inequality for the sub-Laplacian $L$ on the first Heisenberg group $\mathbb{H}^1$. A key observation for our results is that, for any self-adjoint operator $H$, a formal computation involving the Berezin–Lieb inequality for convex functions (see [1] and [14]) yields the result

$$\text{tr} \left( ((H - \lambda)_- - (P_\Omega H P_\Omega - \lambda)_- \right) \geq \text{tr} \left( (H - \lambda)_- - P_\Omega (H - \lambda)_- P_\Omega \right).$$

It is the object of this work to give correct mathematical meaning to this observation and to explicitly calculate the right-hand-side for the two special choices of $H$.

The Berezin inequality (1) on domains of finite measure has inspired a number of authors and is related to the Li–Yau inequality [13]. In their paper the authors showed that the sum over the first $k$ eigenvalues $\lambda_1, \ldots, \lambda_k$ of $-\Delta_{\Omega^c}$ can be bounded from below as

$$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d + 2} \left( L_{0,d}^{\text{cl}} |\Omega^c| \right)^{-\frac{2}{d}} k^{-\frac{2}{d}}.$$
The Berezin inequality on domains of infinite measure

This was later proven to be a corollary of (1) via the Legendre transformation, see [11]. In [9] comparable inequalities were established for various classes of differential and pseudo-differential operators including \((-\Delta)^\alpha\frac{\partial}{\partial \Omega}\) with \(\alpha > 0\). A similar inequality to (1) can be found for Schrödinger operators with magnetic fields in the case \(d = 2\).

The operator \(H^{B,\Omega}_{B,\Omega} := (-i\nabla + A(x))^2\) on \(L^2(\Omega^c)\) with Dirichlet boundary conditions and arbitrary vector field \(A\) satisfies

\[
\text{tr}(H^{B,\Omega}_{B,\Omega} - \lambda)^\gamma \leq L_{\gamma,2}^{cl} \lambda^{\gamma + 1}|\Omega^c| \quad (4)
\]

for all \(\gamma \geq \frac{3}{2}\), which follows from a result by Laptev and Weidl in [12] (see also [6]). In [4] this was generalised to \(\gamma \geq 1\) under the restriction that the magnetic field \(B = dA\) is constant. In this case the upper bound in (4) can be improved by allowing it to depend on \(B\)

\[
\text{tr}(H^{B,\Omega}_{B,\Omega} - \lambda)^\gamma \leq |\Omega^c| \frac{B}{2\pi} \sum_{k=0}^{\infty} ((2k + 1)B - \lambda)^\gamma \quad (5)
\]

as shown in [6]. In their paper the authors also proved that, under the assumption that \(\Omega^c\) is a tiling domain, this inequality also holds if \(0 \leq \gamma < 1\), where it is sharp. For \(\gamma = 1\) the right-hand-side of (5) can be adapted to magnetic operators with additional external potentials \(V\), see [7]. The Berezin inequality was furthermore extended to the sub-Laplacian \(L\) on the Heisenberg group \(\mathbb{H}^1\). In [8] (see also [16]), it was proven that the Dirichlet realisation \(L^{\Omega}_{\Omega^c}\) of \(L\) on a domain \(\Omega^c \subset \mathbb{H}^1\) of finite measure satisfies

\[
\text{tr}(L^{\Omega}_{\Omega^c} - \lambda)^\gamma \leq |\Omega^c| \frac{1}{16 (\gamma + 1)(\gamma + 2)} \lambda^{\gamma + 2}. \quad (6)
\]

In our paper we obtain lower bounds on the traces of the differences \((H_B - \lambda)^\gamma \) and \((L - \lambda)^\gamma\) which are of the same form as the upper bounds in (5) and (6), respectively.

The paper is organised as follows. In Sect. 2 we discuss (3) in the general setting of \(H\) being a self-adjoint operator on \(L^2(\mathbb{R}^d)\). We then state our main results for the magnetic operator \(H_B\) with constant magnetic field and the sub-Laplacian \(L\) in Theorems 2 and 3, respectively. The complete proofs of these results are given in the subsequent sections.

2 Statement of the main results

Let \(H\) be a self-adjoint operator on the space \(L^2(\mathbb{R}^d)\) and let \(\varphi : \mathbb{R} \to \mathbb{R}\) be a convex function such that \(\varphi(H) - \varphi(P_\Omega H P_\Omega)\) and \(\varphi(H) - P_\Omega \varphi(H) P_\Omega\) are both trace-class. Under these assumptions a generalisation of the Berezin–Lieb inequality as derived in [10] states that

\[
\text{tr} (P_\Omega \varphi(H) P_\Omega - \varphi(P_\Omega H P_\Omega)) \geq 0.
\]
As a consequence, we obtain the inequality
\[ \text{tr} \left( \varphi(H) - \varphi(P_H P \varphi(H)) \right) \geq \text{tr} \left( \varphi(H) - P \varphi(H) P \right) \tag{7} \]
by making use of the additivity of the trace. We now simplify the right-hand-side of (7) as follows. To shorten notation denote the trace-class operator \( Q := \varphi(H) - P \varphi(H) P \) and let \( P \varphi(H) = I - P \) be the complementary projection of \( P \varphi(H) \). Clearly \( Q \) can be written as the sum of four operators corresponding to the decomposition of \( L^2(\mathbb{R}^d) \) into ran \( P \) and ran \( P \varphi(H) \), i.e.
\[
Q = P \varphi(H) P + P \varphi(H) P + P \varphi(H) P + P \varphi(H) P.
\]

In [15][Theorem VI.25] it is shown that, if \( T \) is trace-class and \( S \) is bounded, then \( \text{tr}(ST) = tr(TS) \). As a result \( \text{tr}(P \varphi(H) P) = 0 \) as well as \( \text{tr}(P \varphi(H) P) = 0 \). Thus the trace of \( Q \) consists only of the diagonal terms
\[
\text{tr} \left( \varphi(H) - P \varphi(H) P \right) = \text{tr} \left( P \varphi(H) P + P \varphi(H) P \right) = \text{tr} \left( P \varphi(H) P \right).
\]

These results are summarised in the following theorem.

**Theorem 1** Let \( H \) be a self-adjoint operator on \( L^2(\mathbb{R}^d) \) and \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) a convex function such that \( \varphi(H) - \varphi(P \varphi(H) P) \) and \( \varphi(H) - P \varphi(H) P \) are both trace-class. Then the Berezin–Lieb type inequality
\[
\text{tr} \left( \varphi(H) - P \varphi(H) P \right) \geq \text{tr} \left( \varphi(H) - P \varphi(H) P \right) = \text{tr} \left( P \varphi(H) P \right)
\]
holds.

While this result is true for arbitrary self-adjoint operators \( H \), we shall now apply it to two special choices of \( H \) to obtain the main results of this work. First, consider Schrödinger operators with magnetic fields. Let the magnetic field \( B(x) \) be a two-form on \( \mathbb{R}^d \) and the magnetic vector potential \( A(x) \) a one-form satisfying \( B(x) = dA(x) \). We shall restrict ourselves to the case \( d = 2 \) and in the remainder of this work, we furthermore assume that \( B \) is constant and positive. Consider the magnetic operator \( H_B = (-i\nabla + A(x))^2 \), which is defined as the closure of the form
\[
\langle \psi, H_B \psi \rangle := \int_{\mathbb{R}^2} \left| (-i\nabla + A(x)) \psi(x) \right|^2 \, dx
\]
on \( C_c^\infty(\mathbb{R}^2) \), the set of smooth functions with compact support. The obtained operator is found to be self-adjoint and we can state the first main result.

**Theorem 2** Assume \( d = 2, \lambda \geq 0, B > 0, \gamma \geq 1 \) and let \( \Omega \) be an open subset of \( \mathbb{R}^2 \) such that \( \mathbb{R}^2 \setminus \Omega \) has finite measure. Then the inequality
\[
\text{tr} \left( (H_B - \lambda)^\gamma - (P \varphi(H_B) P \varphi(H_B - \lambda)^\gamma P \right) \geq \text{tr} \left( (H_B - \lambda)^\gamma - P \varphi(H_B - \lambda)^\gamma P \right)
\]
\[
= \text{tr} \left( P \varphi(H_B - \lambda)^\gamma P \right) \tag{8}
\]
holds and the right-hand-side can be calculated explicitly as

$$\text{tr} \left( P_{\Omega^c} (H_B - \lambda)^\gamma P_{\Omega^c} \right) = |\mathbb{R}^2 \setminus \Omega| \frac{B}{2\pi} \sum_{k=0}^{\infty} \left( (2k + 1) B - \lambda \right)^\gamma.$$  \hfill (9)

The proof of Theorem 2 is provided in Sect. 3. The lower bound (9) coincides with the upper bound (5) for the magnetic operator on the set $\Omega^c$ of finite volume. In essence the proof is the same.

Similar results can also be obtained on the first Heisenberg group $\mathbb{H}^1$. Here, $\mathbb{H}^1$ is considered to be the three-dimensional space $\mathbb{R}^3$ equipped with the non-commutative multiplication

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = \left( x_1 + y_1, x_2 + y_2, x_3 + y_3 - \frac{1}{2} (x_1 y_2 - x_2 y_1) \right)$$

for $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$. On the Heisenberg group, we introduce the two left-invariant vector fields

$$X_1 = \frac{\partial}{\partial x_1} + \frac{1}{2} x_2 \frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_2} - \frac{1}{2} x_1 \frac{\partial}{\partial x_3}.$$  

Using these definitions, we consider the quadratic form

$$\ell(\psi) = \int_{\mathbb{R}^3} \left( |X_1 \psi|^2 + |X_2 \psi|^2 \right) \, dx_1 \, dx_2 \, dx_3$$

on $C^\infty_c(\mathbb{R}^3)$ and note that the closure of this form gives the self-adjoint sub-Laplacian $L = -X_1^2 - X_2^2$ on $\mathbb{H}^1$. For a detailed background we refer to the literature, e.g. [3]. The sub-Laplacian $L$ is found to satisfy the following analogue of the Berezin inequality.

**Theorem 3** Assume $\lambda \geq 0, \gamma \geq 1$ and let $\Omega$ be an open subset of $\mathbb{R}^3$ such that $\mathbb{R}^3 \setminus \Omega$ has finite measure. Then the inequality

$$\text{tr} \left( (L - \lambda)^\gamma - (P_{\Omega} L P_{\Omega} - \lambda)^\gamma \right) \geq \text{tr} \left( (L - \lambda)^\gamma - P_{\Omega}(L - \lambda)^\gamma P_{\Omega} \right)$$

$$= \text{tr} \left( P_{\Omega^c} (L - \lambda)^\gamma P_{\Omega^c} \right)$$  \hfill (10)

holds for the sub-Laplacian $L$ on $\mathbb{H}^1$ and the right-hand-side can be calculated explicitly as

$$\text{tr} \left( P_{\Omega^c} (L - \lambda)^\gamma P_{\Omega^c} \right) = |\mathbb{R}^3 \setminus \Omega| \frac{1}{16} \frac{1}{(\gamma + 1)(\gamma + 2)} \lambda^{\gamma + 2}. \hfill (11)$$

Similarly to the previous application, the lower bound (11) coincides with the upper bound in the case of the Heisenberg sub-Laplacian being defined on the domain $\Omega^c$.  

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of finite measure with Dirichlet boundary conditions, see (6). The proof of Theorem 3 is basically the same as in the case of finite measure [8] and can be found in Sect. 4. Note that this result can easily be generalised to the \( N \)-th Heisenberg group \( \mathbb{H}^N \).

**Remark 1** Using Theorem 1 we can also reproduce the results of Frank, Lewin, Lieb and Seiringer [5] and show that

\[
\text{tr} \left( (-\Delta - \lambda)^\gamma - (P_{\Omega}(-\Delta) P_{\Omega} - \lambda)^\gamma \right) \geq L_{\gamma,d}^{cl} |\mathbb{R}^d \setminus \Omega|^\gamma \frac{d}{2}
\]

for the Laplacian on a set \( \Omega \subset \mathbb{R}^d \) with complement of finite measure.

### 3 The proof of Theorem 2

Let \( \varphi_{\lambda,\gamma} : \mathbb{R} \to \mathbb{R} \) be the convex function defined as

\[
\varphi_{\lambda,\gamma}(t) = (t - \lambda)^\gamma = \begin{cases} 
(\lambda - t)^\gamma, & t \leq \lambda \\
0, & t > \lambda.
\end{cases}
\]

Applying Theorem 1 to this function and the operator \( H_B \) yields (8) and it only remains to prove (9). This can be done in complete analogy to calculations by Frank, Loss and Weidl [6]. The spectrum of \( H_B \) is entirely discrete and can be calculated to be \((2k + 1)B\) for \( k \in \mathbb{N} \cup \{0\} \). The projection onto the \( k \)-th Landau level is denoted by \( \Pi_{B,k} \). The spectral theorem implies that the operator \( \varphi_{\lambda,\gamma}(H_B) \) can then be written as

\[
\varphi_{\lambda,\gamma}(H_B) = \sum_{k=0}^{\infty} \varphi_{\lambda,\gamma}((2k + 1)B) \Pi_{B,k}.
\]

We multiply this identity from both sides with the projection \( P_{\Omega^c} \) and consider the trace of the obtained expression, that is

\[
\text{tr} \left( P_{\Omega^c} \varphi_{\lambda,\gamma}(H_B) P_{\Omega^c} \right) = \sum_{k=0}^{\infty} \varphi_{\lambda,\gamma}((2k + 1)B) \text{tr}(P_{\Omega^c} \Pi_{B,k}).
\]

To explicitly calculate the summands on the right-hand-side of (13), we observe that by the cyclicity of the trace

\[
\text{tr}(P_{\Omega^c} \Pi_{B,k}) = \text{tr}(P_{\Omega^c} \Pi_{B,k} \Pi_{B,k} P_{\Omega^c}) = \| P_{\Omega^c} \Pi_{B,k} \|_{\sigma_2}^2,
\]

where \( \| \cdot \|_{\sigma_2} \) denotes the Hilbert–Schmidt norm. This norm can be calculated explicitly by using the integral kernel of the operator \( P_{\Omega^c} \Pi_{B,k} \). Let \( \Pi_{B,k}(x,y) \) be the integral kernel of \( \Pi_{B,k} \) such that \( \Pi_{B,k}\psi(x) = \int_{\mathbb{R}^2} \Pi_{B,k}(x,y)\psi(y) \, dy \). The integral kernel of the composition \( P_{\Omega^c} \Pi_{B,k} \) is then given by

\[
(P_{\Omega^c} \Pi_{B,k})(x,y) = \chi_{\Omega^c}(x)\Pi_{B,k}(x,y).
\]
We can calculate the Hilbert-Schmidt norm on the right-hand-side of (14) by double integration of the square of the modulus of this integral kernel, that is

\[ \text{tr} (P_{\Omega^2} \Pi_{B,k}) = \| P_{\Omega^2} \Pi_{B,k} \|_{\sigma_2}^2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\Pi_{B,k}(x, y)|^2 \chi_{\Omega^2}(x) \, dy \, dx. \]  

To explicitly solve this integral, we point out some important properties of the function \( \Pi_{B,k}(x, y) \). As the orthogonal projection \( \Pi_{B,k} \) is self-adjoint it must hold that \( \Pi_{B,k}(x, y) = \Pi_{B,k}(y, x) \). By evaluating \( \Pi_{B,k} \) at the delta distribution \( \delta(x - x_0) \) and using the defining property of projections, \( \Pi_{B,k} = \Pi_{B,k} \Pi_{B,k} \), it can easily be concluded that

\[ \int_{\mathbb{R}^2} |\Pi_{B,k}(x_0, y)|^2 \, dy = \Pi_{B,k}(x_0, x_0). \]

It is furthermore a remarkable fact that the diagonal of the integral kernel of \( \Pi_{B,k} \) is given by the constant \( \Pi_{B,k}(x, x) = \frac{B}{2\pi} \) for all \( k \in \mathbb{N} \cup \{0\} \). Using these properties, identity (15) can be continued as

\[ \text{tr} (P_{\Omega^2} \Pi_{B,k}) = \int_{\mathbb{R}^2} \Pi_{B,k}(x, x) \chi_{\Omega^2}(x) \, dx = \frac{B}{2\pi} |\mathbb{R}^2 \setminus \Omega|. \]

Inserting this equation back into (13) yields the final result

\[ \text{tr} \left( P_{\Omega^2} \varphi_{\lambda, \gamma} (H_B) P_{\Omega^2} \right) = |\mathbb{R}^2 \setminus \Omega| \frac{B}{2\pi} \sum_{k=0}^{\infty} \varphi_{\lambda, \gamma} ((2k + 1) B) \]

which finishes the proof.

4 The proof of Theorem 3

Let the convex function \( \varphi_{\lambda, \gamma} : \mathbb{R} \to \mathbb{R} \) be defined as in (12). Theorem 1 applied to \( L \) yields (10) and it only remains to show (11) which can be proven following calculations by Hansson and Laptev [8]. Firstly, we introduce the Fourier transformation \( \mathcal{F}_{x_3} \) with respect to the variable \( x_3 \),

\[ \mathcal{F}_{x_3} \psi (x_1, x_2, x_3) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-ix_3 y_3} \psi (x_1, x_2, y_3) \, dy_3. \]

A simple calculation shows that the Heisenberg sub-Laplacian \( L \) satisfies the identity

\[ \mathcal{F}_{x_3} L \mathcal{F}^*_{x_3} = \left( i \frac{\partial}{\partial x_1} - \frac{1}{2} x_2 x_3 \right)^2 + \left( i \frac{\partial}{\partial x_2} + \frac{1}{2} x_1 x_3 \right)^2. \]
For fixed $x_3$, the right-hand-side of (17) can be identified with a two-dimensional Schrödinger operator with vector field $A = \frac{1}{2} (-x_2, x_1)$. The corresponding magnetic field $B = \text{d}A$ is constant and can be calculated to be $B = x_3$. The eigenvalues of this magnetic operator $H_{x_3}$ are $(2k + 1)|x_3|$ for $k \in \mathbb{N} \cup \{0\}$. Note that they depend on the variable $x_3$. Similar to the previous section we shall use the Landau projections $\Pi_{3,k}$ to prove (11). With respect to the variable $x_3$, the operator $\mathcal{F}_{x_3} L \mathcal{F}^*_{x_3}$ simply acts as a multiplication operator and consequently the spectral theorem allows us to write

$$\varphi_{\lambda,\gamma}(\mathcal{F}_{x_3} L \mathcal{F}^*_{x_3}) = \sum_{k=0}^{\infty} \varphi_{\lambda,\gamma}((2k + 1)|x_3|) \hat{\Pi}_{3,k}$$

(18)

where we have used the tensor product $\hat{\Pi}_{3,k} = \Pi_{3,k} \otimes \mathbb{I}_{L^2(\mathbb{R})}$ with $\mathbb{I}_{L^2(\mathbb{R})}$ denoting the identity on $L^2(\mathbb{R})$. For convenience we introduce the notation $\mu_k(x_3) = \varphi_{\lambda,\gamma}((2k + 1)|x_3|)$ swallowing the dependence on $\lambda$ and $\gamma$ for the moment. As a consequence of (18) we obtain the identity

$$\text{tr} \left( P_{\Omega} \varphi_{\lambda,\gamma}(L) P_{\Omega} \right) = \sum_{k=0}^{\infty} \text{tr} \left( P_{\Omega} \mathcal{F}^*_{x_3} \mu_k(x_3) \hat{\Pi}_{3,k} \mathcal{F}_{x_3} P_{\Omega} \right).$$

(19)

This result can be compared to the analogous equation in the case of a magnetic operator (13). While in this setting the magnetic field was a given constant, we now consider a magnetic field that changes with the variable $x_3$. In addition, the Fourier transformation $\mathcal{F}_{x_3}$ has to be dealt with. The summands on the right-hand-side of (19) can be written as Hilbert–Schmidt norms of certain operators, that is

$$\text{tr} \left( P_{\Omega} \mathcal{F}^*_{x_3} \mu_k(x_3) \hat{\Pi}_{3,k} \mathcal{F}_{x_3} P_{\Omega} \right) = \left\| P_{\Omega} \mathcal{F}^*_{x_3} \mu_k(x_3) \frac{1}{2} \hat{\Pi}_{3,k} \right\|_{\sigma_2}^2$$

for every $k \in \mathbb{N} \cup \{0\}$. Here we have used that the multiplication operator $\mu_k(x_3)$ and the projection $\Pi_{3,k}$ commute. The investigation of these Hilbert–Schmidt norms requires us to calculate the integral kernels of the operators involved. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. The integral kernel of $P_{\Omega} \mathcal{F}^*_{x_3} \mu_k(x_3) \frac{1}{2} \hat{\Pi}_{3,k}$ can then be computed to be $\chi_{\Omega}(x) \frac{1}{\sqrt{2\pi}} e^{i x_3 y_3} \Pi_{y_3,k}(x_1, x_2, y_1, y_2) \mu_k(y_3) \frac{1}{2}$. As a consequence we obtain the identity

$$\left\| P_{\Omega} \mathcal{F}^*_{x_3} \mu_k(x_3) \frac{1}{2} \hat{\Pi}_{3,k} \right\|_{\sigma_2}^2 = \frac{1}{2\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \chi_{\Omega}(x) |\Pi_{y_3,k}(x_1, x_2, y_1, y_2)|^2 \mu_k(y_3) \, dy \, dx$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \chi_{\Omega}(x) \int_{\mathbb{R}^2} |\Pi_{y_3,k}(x_1, x_2, y_1, y_2)|^2 \, dy_1 \, dy_2 \, dx \, dy_3.$$
To calculate this integral, we recall (16) and stress again that the diagonal of the integral kernel \( k(x_1, x_2, y_1, y_2) \) is known to be the constant \( \frac{|y_3|}{2\pi} \). This results in

\[
\left\| P_{\Omega^c} \mathcal{F}_{x_3} \mu_k(y_3) \right\|_{\mathcal{S}_2}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \mu_k(y_3) \int_{\mathbb{R}^3} \chi_{\Omega^c}(x) \frac{|y_3|}{2\pi} \, dx \, dy_3 
\]

\[
= \frac{1}{2\pi^2} \left| \mathbb{R}^3 \setminus \Omega \right| \int_0^{+\infty} (2k + 1) y_3 - \lambda \, (\gamma - 1) \, y_3 \, dy_3 
\]

where we have used the definition of \( \mu_k(y_3) \) to obtain the last equality. We insert this identity back into (19) and substitute \( p = (2k + 1) y_3 \) to conclude that

\[
\text{tr} \left( P_{\Omega^c} \varphi_{\lambda, \gamma}(L) P_{\Omega^c} \right) = \frac{1}{2\pi^2} \left| \mathbb{R}^3 \setminus \Omega \right| \sum_{k=0}^{+\infty} \frac{1}{(2k + 1)^2} \int_0^{+\infty} (p - \lambda)_-^\gamma p \, dp 
\]

\[
= \frac{1}{16} \left| \mathbb{R}^3 \setminus \Omega \right| \int_0^{+\infty} (p - \lambda)_-^\gamma p \, dp. 
\]

Here, the last equality follows from the well-known fact that \( \sum_{k=0}^{+\infty} \frac{1}{(2k + 1)^2} = \frac{\pi^2}{8} \). The remaining integral can be easily calculated using partial integration

\[
\int_0^{+\infty} (p - \lambda)_-^\gamma p \, dp = \frac{\lambda^{\gamma+2}}{(\gamma + 1)(\gamma + 2)} 
\]

and this yields the desired result.

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