Observation of the spatial emission spectrum in the experiments of Abbe-Porter and Talbot

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Abstract. The observation results of the spatial harmonic height are given. It was experimentally shown that the limitation associated with the final aperture and the parabolic approximation in the Fresnel integral does not interfere with the confident observation of the high harmonics reaching the fortieth.

1. Introduction
In the scientific literature, much attention is paid to the study and application of the Talbot effect [1-21]. This paper presents the results of observing high harmonics of the spatial frequency spectrum in the Talbot effect. This is important for the development and use of high harmonics in various applications - visualization of complex phase objects, for creating wavefront sensors, hiding information [7, 8, 21]. In the student experimental laboratory of physics in the Department of Physics of the BMSTU, conducted studies of the Talbot effect for waves of different nature on diffraction gratings [17-21]. The proposed paper presents a comparative analysis of the spectrum of spatial harmonics in the schemes of the Abbe-Porter experiment and the Talbot effect on the same transparency. A good contrast of the picture of harmonics up to the fortieth is demonstrated in the Abbe-Porter experiment and in the observing of the Talbot effect. The spectrum management of two-dimensional transparency and its effect on image synthesis is illustrated.

2. The spatial frequency spectrum of transparency in the Fourier plane and in the Talbot carpet.
According to the theory developed by Abbe, the phenomenon of the formation of an optical image by a collecting lens can be divided into a stage of Fourier analysis of the object wavefield and Fourier analysis of image synthesis. An important role is played by the Fourier - plane (focal plane of the lens), in which the spatial frequency spectrum of the object is formed.

It is of interest to compare the possibilities of studying the high harmonics of the spatial pattern of the frequency spectrum of a periodic structure in the Fourier plane with the picture of high harmonics in the Talbot carpet of the same periodic structure.

It is known that any periodic function can be decomposed into a Fourier series in an orthonormal trigonometric system:

\[ f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} [a_m \cos(mx) + b_m \sin(mx)], \]  

(1)
where the coefficients $a_m, b_m (\forall m \in \mathbb{N} \cup \{0\})$ are real numbers depending on the form of the given function $f(x)$ satisfying the conditions of the Dirichlet theorem [19]. Using a simple trigonometric transformation, the transition to the formula:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} A_m \cos(mx - \varphi_m),$$

the expression under the sum sign is a particular solution of the wave equation for a plane harmonic wave.

This means that the wavefront described by the law $f(x)$ can be represented with any degree of accuracy as a sum of harmonic plane waves. This result was obtained without imposing restrictions on the physical nature of the phenomena under consideration. Therefore, it can be electromagnetic, ultrasonic waves, waves of matter [17-20].

Moving from a trigonometric record to a complex one, we will present the expression in the form:

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{-i\gamma(\xi - x)} d\xi,$$

where $\gamma$ and $\zeta$ are some real parameters, their physical meaning depends on the nature of the wavefront decomposition, be it a temporal or spatial transformation.

In this paper, the analysis of the superposition of light waves is based on the principle outlined above. Consider how Fourier is formed - the spectrum of the object in the focal plane of the lens. By presenting the wavefront as a sum of harmonic plane waves directed at different angles to the normal of its propagation plane, one can use the principles of Fourier optics to describe the observed image (Figure 1) [13, 15]. Then, each elementary harmonic (flat harmonic wave) will correspond to a point in the Fourier-lens plane, i.e. in the focal plane of the lens, the spatial frequency spectrum of the flat image is observed, which is described using the Fourier transform.

![Figure 1. Formation of the image spatial spectrum](image_url)

From a mathematical point of view, the phenomenon is described as follows: we introduce the concept of spatial frequency as the projection of the wave vector onto the orthogonal axes Ox, Oy, Oz belonging to the plane of propagation of the wavefront. Then the spectral amplitude at an arbitrary point in space will be determined using an integral transform as a function of three variables [15]:

$$\xi(x; y; z) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} \xi_0(k_x; k_y) e^{-i(k_x x + k_y y + k_z z)} dk_y$$

However, this law is valid for the case if the wave propagates in a homogeneous medium. When a thin collecting lens is on its way, then introducing the concept of its frequency transfer coefficient, one can strictly prove that it is a material object that performs the spatial Fourier transform [22]:

$$\xi(x; y; f) = \frac{i}{2\pi} \xi_0(k_x; k_y) e^{-i(k_x^2 + k_y^2)/2f}.$$
As you can see, this dependence characterizes the distribution of the spectral amplitude in the focal plane. By itself, the quantity $\xi$ is the coefficient under the sign of the Fourier integral, but its physical meaning can be written using the expression $I \sim |\xi|^2$, where $I$ is the radiation intensity.

The use of the mathematical apparatus of Fourier analysis to solve the problem of diffraction of a cylindrical wave on a one-dimensional periodic grating with period $d$ allows us to obtain the distribution of the spectral amplitude in its immediate vicinity in the form of:

$$\xi_0(x) = \xi_0(x + d) = \sum_{m=-\infty}^{+\infty} \xi_m \cos \left(\frac{2\pi}{d} mx\right) = \sum_{m=-\infty}^{+\infty} \xi_m e^{2\pi imx/d}$$  \hspace{1cm} (6)

where $\xi_0(x + d)$ is a periodic function of the $x$ coordinate, containing the number of harmonics $m$, defined by the ratio $\frac{m\lambda}{d} \ll 1$.

It was shown by Lord Rayleigh that it is possible to solve the diffraction problem by decomposition into plane waves, which, in turn, is nothing more than proof of the principles of Fourier optics [2]:

$$\xi(x; z) = \sum_{m=-\infty}^{+\infty} \xi_m e^{-2\pi im^2 z z_T}$$ \hspace{1cm} (7)

Here, the quantity $z_T = \frac{2d^2}{\lambda}$ is called the Talbot period, and $\lambda$ is the radiation wavelength. If the condition $\frac{z}{z_T} \in \mathbb{N}$ holds, then the value of the spectral amplitude does not change, i.e. there is a distance, called the Talbot distance (period), on which the lattice image is self-reproducing. This phenomenon is called the Talbot effect.

The fractional effect of Talbot is the phenomenon that is observed under the condition $z = \frac{a}{b} z_T \ (a \in \mathbb{Z}, \ b \in \mathbb{N})$. \hspace{1cm} (8)

In this case, the original image is decomposed in harmonics, but not in the Fourier plane, but in space, along the normal to the wave surface.

3. Experimental setup

Experiments on the observation of the spatial spectra of one-dimensional and two-dimensional objects were carried out on the installations below.

The setup diagram is as follows (Figure 3): a coherent laser beam (1) passes through a telescopic system (2), consisting of an objective, a collecting focal lens and a pin-hall. The telescope expands the Gaussian beam to 20 mm. This beam illuminates the object (3), behind which the lens (4) is located with
a focal length of $f = 150$ mm. The spatial frequency spectrum is observed on screen 5 or with the help of a camera on a computer screen.

![Diagram](image)

**Figure 3.** Installation diagram for observing the spatial spectrum of one-dimensional and two-dimensional objects

Figure 4. Photo of installation

To demonstrate the possibilities of spatial harmonic filtering, Abbe-Porter experiment was reproduced (Figure 5). In the focal plane of the lens there was a filtering mask 5, which diaphragmed part of the spatial spectrum. Then, on screen 6, the source image is not reproduced but transformed as a result of the filtering.

![Diagram](image)

**Figure 5.** Scheme of the Abbe-Porter experiment

The scheme of the experiment to observe the Talbot effect is shown in Figure 6, where 3 is the amplitude diffraction grating, and 4 is the photo detector or screen.
4. Results and discussion
Let us compare the results of the observation of spatial spectra obtained from the same diffraction grating in experiments using Abbe-Porter and Talbot observation schemes.

The photographs above show a transparency - amplitude diffraction grating. The left column contains photographs of spectra in the Fourier plane. The top shot shows the total spectrum with zero harmonics.
in the center of the photo. The middle picture shows the fifteen first harmonics, where the fifteenth harmonic is in the upper right corner. The bottom shot shows the forty-first harmonics, where the fortieth harmonic is in the upper right corner. The right column shows the corresponding harmonics in the Talbot planes. The upper right image corresponds to zero harmonics (spectral images are located along the diagonal so that the full period fits on the screen). The middle picture corresponds to the fifteenth harmonic. The bottom picture corresponds to the fortieth harmonic. As can be seen from the comparison of these photographs, the spatial frequency spectrum is confidently observed up to the fortieth harmonic.

Below are photos of the first forty harmonics obtained by observing the Talbot effect. They consistently increase the number of lines from 2 to 41, which correspond to harmonic oscillations from zero to fortieth orders:

![Spatial harmonics from zero to twentieth orders in the Talbot experiment](image1)

**Figure 8.** Spatial harmonics from zero to twentieth orders in the Talbot experiment

![Spatial harmonics from twentieth to fortieth orders in the Talbot experiment](image2)

**Figure 9.** Spatial harmonics from twentieth to fortieth orders in the Talbot experiment
The presented photos show that the noise generated by higher harmonics of the spatial frequency spectrum does not significantly affect the implementation of harmonics up to \( m = 40 \), as well as aberrations.

In order to demonstrate the effect of filtering the spatial frequency spectrum of an object on its image in experiments with the Fourier plane, we will select a photograph of the main building of the BMSTU size 30 \( \times \) 40 mm (Figure 10). We selected the central part of the slide.

![Figure 10. Photograph of the BMSTU](image)

The spatial frequency spectrum was recorded using a Moticam1SP camera with a physical size of 8 \( \times \) 8 mm and the number of active pixels of \( 1.3 \times 10^6 \) through a microscope objective.

![Figure 11. The spatial spectrum of the central part of the building (left) and its reconstructed image (right)](image)
The spectrum consists of a set of main maxima (the brightest lines along the vertical and horizontal) and higher-order maxima that can be observed outside two orthogonal lines. The image restored on the screen is shown in Figure 11 on the right.

The object in the Fourier plane of the lens is transformed into a spatial frequency spectrum, which in turn is restored to the original image on the screen.

Figure 12 shows a view of the spectrum when placing a circular diaphragm with a diameter of 1 mm in the Fourier plane.

![Figure 12](image1.png)

**Figure 12** The spatial spectrum of the object, bounded by a circular aperture (left) and the image of the object reconstructed from the central part of the spectrum (right)

It differs from the previous lack of higher order harmonics. Then we should expect that the image would lose subtle details. It can be seen that the outline of the building is more noticeable, the minor details (elements on the windows, part of the building reflected from the lake, etc.) have disappeared.

Let us demonstrate the effect of a slit horizontal filtering mask, emitting harmonics along the horizontal axis, on the resulting image of a building. (Figure 13)

![Figure 13](image2.png)

**Figure 13.** The spatial spectrum of the object, limited by the aperture in the form of a horizontal slit (left) and the image of the object reconstructed from the horizontal part of the spectrum (right)

The distribution law of the spectral amplitude is written in a simpler form (here, we direct the axis Ox along the horizontal and the axis Oy along the vertical):
\[ \xi(x; y; f) = \frac{i}{k f} \xi_0(k x; 0) e^{-ik(f + \frac{x^2}{2f})}. \] (9)

Since there are no terms containing the component y, it is expected to see on the screen instead of the reconstructed image only a part of it with pronounced elements located along the Ox axis (Figure 12). The resulting image demonstrates the pronounced vertical columns of the main building, and the horizontal components of the image are blurred or absent.

5. Conclusion
In the experiments considered, spatial frequency spectra of a periodic and non-periodic structure were observed. The presented results allow us to conclude that the noise generated by higher harmonics of the spatial frequency spectrum does not significantly affect the ability to observe and use harmonics up to m = 40, both in the Talbot effect and in the Fourier plane.

Experimental implementation of sufficiently clear images of high harmonics will allow applying the results of experiments in interferometry [8,16], spectrography, cryptography, optical metrology and other areas of science and technology.

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