Chain rules and inequalities for the BHT fractional calculus on arbitrary time scales

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Abstract

We develop the Benkhettou–Hassani–Torres fractional (noninteger order) calculus on time scales by proving two chain rules for the $\alpha$-fractional derivative and five inequalities for the $\alpha$-fractional integral. The results coincide with well-known classical results when the operators are of (integer) order $\alpha = 1$ and the time scale coincides with the set of real numbers.

Keywords: local fractional calculus; calculus on time scales; chain rules; integral inequalities.

MSC 2010: 26A33; 26D10; 26E70.

1 Introduction

The study of fractional (noninteger order) calculus on time scales is a subject of strong current interest \cite{1,2,3,4}. Recently, Benkhettou, Hassani and Torres introduced a (local) fractional calculus on arbitrary time scales $\mathbb{T}$ (called here the BHT fractional calculus) based on the $T_\alpha$ differentiation operator and the $\alpha$-fractional integral \cite{5}. The Hilger time-scale calculus \cite{6} is then obtained as a particular case, by choosing $\alpha = 1$. In this paper we develop the BHT time-scale fractional calculus initiated in \cite{5}. Precisely, we prove two different chain rules for the fractional derivative $T_\alpha$ (Theorems 4 and 6) and several inequalities for the $\alpha$-fractional integral: Hölder’s inequality (Theorem 7), Cauchy–Schwarz’s inequality (Theorem 8), Minkowski’s inequality (Theorem 10), generalized Jensen’s fractional inequality (Theorem 11) and a weighted fractional Hermite–Hadamard inequality on time scales (Theorem 12).

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The paper is organized as follows. In Section 2 we recall the basics of the BHT fractional calculus. Our results are then formulated and proved in Section 3.

2 Preliminaries

We briefly recall the necessary notions from the BHT fractional calculus [5]: fractional differentiation and fractional integration on time scales. For an introduction to the time-scale theory we refer the reader to the book [6].

**Definition 1** (See [5]). Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$, and $\alpha \in (0, 1]$. For $t > 0$, we define $T_\alpha(f)(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a $\delta$-neighbourhood $\mathcal{V}_\delta = (t - \delta, t + \delta) \cap \mathbb{T}$ of $t$, $\delta > 0$, such that $[f(\sigma(t)) - f(s)]t^{1-\alpha} - T_\alpha(f)(t) |\sigma(t) - s| \leq \epsilon |\sigma(t) - s|$ for all $s \in \mathcal{V}_\delta$. We call $T_\alpha(f)(t)$ the $\alpha$-fractional derivative of $f$ of order $\alpha$ at $t$, and we define the $\alpha$-fractional derivative at 0 as $T_\alpha(f)(0) := \lim_{t \to 0^+} T_\alpha(f)(t)$.

If $\alpha = 1$, then we obtain from Definition 1 the Hilger delta derivative of time scales [6]. The $\alpha$-fractional derivative of order zero is defined by the identity operator: $T_0(f) := f$. The basic properties of the $\alpha$-fractional derivative on time scales are given in [5], together with several illustrative examples. Here we just recall the item (iv) of Theorem 4 in [5], which is needed in the proof of our Theorem 4.

**Theorem 2** (See [5]). Let $\alpha \in (0, 1]$ and $\mathbb{T}$ be a time scale. Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. If $f$ is $\alpha$-fractional differentiable of order $\alpha$ at $t$, then

$$f(\sigma(t)) = f(t) + \mu(t)t^{\alpha-1}T_\alpha(f)(t).$$

The other main operator of [5] is the $\alpha$-fractional integral of $f : \mathbb{T} \to \mathbb{R}$, defined by

$$\int f(t)\Delta^\alpha t := \int f(t)t^{\alpha-1}\Delta t,$$

where the integral on the right-hand side is the usual Hilger delta-integral of time scales [5 Def. 26]. If $F_\alpha(t) := \int f(t)\Delta^\alpha t$, then one defines the Cauchy $\alpha$-fractional integral by $\int_a^b f(t)\Delta^\alpha t := F_\alpha(b) - F_\alpha(a)$, where $a, b \in \mathbb{T}$ [5 Def. 28]. The interested reader can find the basic properties of the Cauchy $\alpha$-fractional integral in [5]. Here we are interested to prove some fractional integral inequalities on time scales. For that, we use some of the properties of [5 Theorem 31].

**Theorem 3** (Cf. Theorem 31 of [5]). Let $\alpha \in (0, 1]$, $a, b, c \in \mathbb{T}$, $\gamma \in \mathbb{R}$, and $f, g$ be two rd-continuous functions. Then,

(i) $\int_a^b (f(t) + g(t))\Delta^\alpha t = \int_a^b f(t)\Delta^\alpha t + \int_a^b g(t)\Delta^\alpha t$;

(ii) $\int_a^b (\gamma f(t))\Delta^\alpha t = \gamma \int_a^b f(t)\Delta^\alpha t$. 

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(iii) $\int_a^b f(t)\Delta^\alpha t = -\int_b^a f(t)\Delta^\alpha t$;
(iv) $\int_a^b f(t)\Delta^\alpha t = \int_a^c f(t)\Delta^\alpha t + \int_c^b f(t)\Delta^\alpha t$;
(v) if there exist $g : T \to \mathbb{R}$ with $|f(t)| \leq g(t)$ for all $t \in [a, b]$, then $\left|\int_a^b f(t)\Delta^\alpha t\right| \leq \int_a^b g(t)\Delta^\alpha t$.

3 Main Results

The chain rule, as we know it from the classical differential calculus, does not hold for the BHT fractional calculus. A simple example of this fact has been given in [5, Example 20]. Moreover, it has been shown in [5, Theorem 21], using the mean value theorem, that if $g : T \to \mathbb{R}$ is continuous and fractional differentiable of order $\alpha \in (0, 1]$ at $t \in T$ and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then there exists $c \in [t, \sigma(t)]$ such that $T_\alpha(f \circ g)(t) = f'(g(c))T_\alpha(g)(t)$. In Section 3.1 we provide two other chain rules. Then, in Section 3.2 we prove some fractional integral inequalities on time scales.

3.1 Fractional chain rules on time scales

Theorem 4 (Chain Rule I). Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, $T$ be a given time scale and $g : T \to \mathbb{R}$ be $\alpha$-fractional differentiable. Then, $f \circ g : T \to \mathbb{R}$ is also $\alpha$-fractional differentiable with

$T_\alpha(f \circ g)(t) = \left[\int_0^1 f'(g(t) + h\mu(t)t^{\alpha-1}T_\alpha(g)(t))dh\right]T_\alpha(g)(t)$ \hspace{1cm} (1)

Proof. We begin by applying the ordinary substitution rule from calculus:

$f(g(\sigma(t))) - f(g(s)) = \int_{g(s)}^{g(\sigma(t))} f'(\tau)d\tau$

$= \left[|g(\sigma(t)) - g(s)| \int_0^1 f'(hg(\sigma(t))) + (1-h)g(s))dh\right].$

Let $t \in \mathbb{T}^\infty$ and $\epsilon > 0$. Since $g$ is $\alpha$-fractional differentiable at $t$, we know from Definition 11 that there exists a neighbourhood $U_1$ of $t$ such that

$\left|[g(\sigma(t)) - g(s)]t^{1-\alpha} - T_\alpha(g(t)(\sigma(t) - s)\right| \leq \epsilon^*|\sigma(t) - s| \quad \text{for all } s \in U_1,$

where

$\epsilon^* = \frac{\epsilon}{1 + 2 \int_0^1 |f'(hg(\sigma(t))) + (1-h)g(t)|dh}.$

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Moreover, $f'$ is continuous on $\mathbb{R}$ and, therefore, it is uniformly continuous on closed subsets of $\mathbb{R}$. Observing that $g$ is also continuous, because it is $\alpha$-fractional differentiable (see item (i) of Theorem 4 in [5]), there exists a neighbourhood $U_2$ of $t$ such that

$$|f'(hg(\sigma(t)) + (1 - h)g(s)) - f'(hg(\sigma(t)) + (1 - h)g(t))| \leq \frac{\epsilon}{2(\epsilon^* + |T_\alpha(g)(t)|)}$$

for all $s \in U_2$. To see this, note that

$$|hg(\sigma(t)) + (1 - h)g(s) - (hg(\sigma(t)) + (1 - h)g(t))| = (1 - h)|g(s) - g(t)|$$

holds for all $0 \leq h \leq 1$. We then define $U := U_1 \cap U_2$ and let $s \in U$. For convenience, we put

$$\gamma = hg(\sigma(t)) + (1 - h)g(s) \quad \text{and} \quad \beta = hg(\sigma(t)) + (1 - h)g(t).$$

Then we have

$$\left|[(f \circ g)(\sigma(t)) - (f \circ g)(s)]t^{1-\alpha} - T_\alpha(g)(t)(\sigma(t) - s) \int_0^1 f'(\beta)dh\right|$$

$$= \left|t^{1-\alpha}[g(\sigma(t)) - g(s)]\int_0^1 f'(\gamma)dh - T_\alpha(g)(t)(\sigma(t) - s)\int_0^1 f'(\beta)dh\right|$$

$$= \left|\left(t^{1-\alpha}[g(\sigma(t)) - g(s)] - (\sigma(t) - s)T_\alpha(g)(t)\right)\right|$$

$$\times \int_0^1 f'(\gamma)dh + T_\alpha(g)(t)(\sigma(t) - s)\int_0^1 (f'(\gamma) - f'(\beta))dh\right|$$

$$\leq \left|t^{1-\alpha}[g(\sigma(t)) - g(s)] - (\sigma(t) - s)T_\alpha(g)(t)\right|\int_0^1 |f'(\gamma)|dh$$

$$+ |T_\alpha(g)(t)||\sigma(t) - s|\int_0^1 |f'(\gamma) - f'(\beta)|dh$$

$$\leq \epsilon^*|\sigma(t) - s|\int_0^1 |f'(\gamma)|dh + [\epsilon^* + |T_\alpha(g)(t)|]|\sigma(t) - s|\int_0^1 |f'(\gamma) - f'(\beta)|dh$$

$$\leq \frac{\epsilon}{2}|\sigma(t) - s| + \frac{\epsilon}{2}|\sigma(t) - s|$$

$$= \epsilon|\sigma(t) - s|.$$ 

Therefore, $f \circ g$ is $\alpha$-fractional differentiable at $t$ and (1) holds. $\square$

Let us illustrate Theorem 4 with an example.

**Example 5.** Let $g : \mathbb{Z} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(t) = t^2 \quad \text{and} \quad f(t) = e^t.$$
Then, \( T_\alpha(g)(t) = (2t + 1)t^{1-\alpha} \) and \( f'(t) = e^t \). Hence, we have by Theorem 4 that
\[
T_\alpha(f \circ g)(t) = \left[ \int_0^1 f'(g(t) + h\mu(t)t^{\alpha-1}T_\alpha(g)(t))dh \right]T_\alpha(g)(t)
\]
\[
= (2t + 1)t^{1-\alpha}\int_0^1 e^{t^2 + h(2t+1)}dh
\]
\[
= (2t + 1)t^{1-\alpha}e^{t^2}\int_0^1 e^{h(2t+1)}dh
\]
\[
= (2t + 1)t^{1-\alpha}e^{t^2}\left[ \frac{1}{2t+1}e^{2t+1} - 1 \right]
\]
\[
= t^{1-\alpha}e^{t^2}\left[ e^{2t+1} - 1 \right].
\]

**Theorem 6** (Chain Rule II). Let \( \mathbb{T} \) be a time scale. Assume \( \nu : \mathbb{T} \to \mathbb{R} \) is strictly increasing and \( \tilde{\mathbb{T}} := \nu(\mathbb{T}) \) is also a time scale. Let \( w : \tilde{\mathbb{T}} \to \mathbb{R}, \alpha \in (0,1] \), and \( \tilde{T}_\alpha \) denote the \( \alpha \)-fractional derivative on \( \tilde{\mathbb{T}} \). If for each \( t \in \mathbb{T}^\kappa, \tilde{T}_\alpha(w)(\nu(t)) \) exists and for every \( \epsilon > 0 \), there is a neighbourhood \( V \) of \( t \) such that
\[
|\tilde{\sigma}(\nu(t)) - \nu(s) - \tilde{T}_\alpha(\nu(t))(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad \text{for all } s \in V,
\]
where \( \tilde{\sigma} \) denotes the forward jump operator on \( \tilde{\mathbb{T}} \), then
\[
T_\alpha(w \circ \nu)(t) = [\tilde{T}_\alpha(w) \circ \nu](t)T_\alpha(\nu)(t).
\]

**Proof.** Let \( 0 < \epsilon < 1 \) be given and define \( \epsilon^* := \epsilon \left[ 1 + |T_\alpha(\nu)(t)| + |\tilde{T}_\alpha(w)(\nu(t))| \right]^{-1} \).
Note that \( 0 < \epsilon^* < 1 \). According to the assumptions, there exist neighbourhoods \( U_1 \) of \( t \) and \( U_2 \) of \( \nu(t) \) such that
\[
|\tilde{\sigma}(\nu(t)) - \nu(s) - \tilde{T}_\alpha(\nu(t))(\sigma(t) - s)| \leq \epsilon^*|\sigma(t) - s|
\]
for all \( s \in U_1 \) and
\[
|\tilde{\sigma}(\nu(t)) - \nu(s) - \tilde{T}_\alpha(\nu(t))(\sigma(t) - s)| \leq \epsilon^*|\sigma(t) - s|
\]
for all \( r \in U_2 \). Let \( U := U_1 \cap \nu^{-1}(U_2) \). For any \( s \in U \), we have that \( s \in U_1 \) and \( \nu(s) \in U_2 \) with
\[
[|w(\nu(s)) - w(\nu(t))||t^{1-\alpha} - (\nu(t) - s)|\tilde{T}_\alpha(w)(\nu(t))|]T_\alpha(\nu)(t)
\]
\[
= [|w(\nu(s)) - w(\nu(t))||t^{1-\alpha} - (\tilde{\sigma}(\nu(t)) - \nu(s))\tilde{T}_\alpha(w)(\nu(t))]
\]
\[
+ [\tilde{\sigma}(\nu(t)) - \nu(s) - \tilde{T}_\alpha(\nu(t))(\sigma(t) - s)|\tilde{T}_\alpha(w)(\nu(t))|]
\]
\[
\leq \epsilon^*|\tilde{\sigma}(\nu(t)) - \nu(s)| + \epsilon^*|\sigma(t) - s||\tilde{T}_\alpha(w)(\nu(t))|]
\]
\[
\leq \epsilon^*\left[ |\tilde{\sigma}(\nu(t)) - \nu(s) - (\sigma(t) - s)\tilde{T}_\alpha(\nu)(t)|
\right.
\]
\[
+ |\sigma(t) - s||\tilde{T}_\alpha(\nu)(t)| + |\sigma(t) - s||\tilde{T}_\alpha(w)(\nu(t))|]}
\]
\[
= \epsilon^* \left[ |\tilde{\sigma}(\nu(t)) - \nu(s) - (\sigma(t) - s)\tilde{T}_\alpha(\nu)(t)|
\right.}
\[
+ |\sigma(t) - s||\tilde{T}_\alpha(\nu)(t)| + |\sigma(t) - s||\tilde{T}_\alpha(w)(\nu(t))|]
\]
\[ \leq \epsilon^* [\epsilon^* |\sigma(t) - s| + |\sigma(t) - s||T_\alpha(\nu(t))| + |\sigma(t) - s||T_\alpha(w)(\nu(t))|] \]
\[ = \epsilon^* |\sigma(t) - s| [\epsilon^* + |T_\alpha(\nu(t))| + |T_\alpha(w)(\nu(t))|] \]
\[ \leq \epsilon^* [1 + |T_\alpha(\nu(t))| + |T_\alpha(w)(\nu(t))|] |\sigma(t) - s| \]
\[ = \epsilon |\sigma(t) - s|. \]

This proves the claim. \qed

## 3.2 Fractional integral inequalities on time scales

The \( \alpha \)-fractional integral on time scales was introduced in \[5, \text{Section 3}\], where some basic properties were proved. Here we show that the \( \alpha \)-fractional integral satisfies appropriate fractional versions of the fundamental inequalities of Hölder, Cauchy–Schwarz, Minkowski, Jensen and Hermite–Hadamard.

**Theorem 7** (Hölder’s fractional inequality on time scales). Let \( \alpha \in (0,1] \) and \( a, b \in \mathbb{T} \). If \( f, g, h : [a, b] \to \mathbb{R} \) are rd-continuous, then

\[
\int_a^b |f(t)g(t)||h(t)|^{\Delta \alpha} t \leq \left[ \int_a^b |f(t)|^p |h(t)|^{\Delta \alpha} t \right]^{\frac{1}{p}} \left[ \int_a^b |g(t)|^q |h(t)|^{\Delta \alpha} t \right]^{\frac{1}{q}},
\]

where \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** For nonnegative real numbers \( A \) and \( B \), the basic inequality

\[ A^{\frac{1}{p}} B^{\frac{1}{q}} \leq \frac{A}{p} + \frac{B}{q} \]

holds. Now, suppose, without loss of generality, that

\[ \left[ \int_a^b |f(t)|^p |h(t)|^{\Delta \alpha} t \right] \left[ \int_a^b |g(t)|^q |h(t)|^{\Delta \alpha} t \right] \neq 0. \]

Applying Theorem 3 and the above inequality to

\[ A(t) = \frac{|f(t)||h(t)|}{\int_a^b |f(\tau)|^p |h(\tau)|^{\Delta \alpha} \tau} \]

and

\[ B(t) = \frac{|g(t)||h(t)|}{\int_a^b |g(\tau)|^q |h(\tau)|^{\Delta \alpha} \tau}, \]

and integrating the obtained inequality between \( a \) and \( b \), which is possible since
all occurring functions are rd-continuous, we find that

\[
\int_{a}^{b} |A(t)|^{1/p} |B(t)|^{1/q} \Delta^\alpha t
\]

\[
= \int_{a}^{b} \left[ \frac{|f(t)| |h(t)|^{1/p}}{\int_{a}^{b} |f(\tau)|^{p} |h(\tau)|^{\Delta^\alpha \tau}} \right]^{1/p} \left[ \frac{|g(t)| |h(t)|^{1/q}}{\int_{a}^{b} |g(\tau)|^{q} |h(\tau)|^{\Delta^\alpha \tau}} \right]^{1/q} \Delta^\alpha t
\]

\[
\leq \int_{a}^{b} \left[ \frac{A(t)}{p} + \frac{B(t)}{q} \right] \Delta^\alpha t
\]

\[
= \int_{a}^{b} \left[ \frac{1}{p} \int_{a}^{b} |f(t)|^{p} |h(t)| |h(\tau)|^{\Delta^\alpha \tau} + \frac{1}{q} \int_{a}^{b} |g(t)|^{q} |h(t)| |h(\tau)|^{\Delta^\alpha \tau} \right] \Delta^\alpha t
\]

\[
= \frac{1}{p} \int_{a}^{b} \left[ \frac{|f(t)|^{p} |h(t)|}{\int_{a}^{b} |f(\tau)|^{p} |h(\tau)|^{\Delta^\alpha \tau}} \right] \Delta^\alpha t + \frac{1}{q} \int_{a}^{b} \left[ \frac{|g(t)|^{q} |h(t)|}{\int_{a}^{b} |g(\tau)|^{q} |h(\tau)|^{\Delta^\alpha \tau}} \right] \Delta^\alpha t
\]

\[
\leq \frac{1}{p} + \frac{1}{q}
\]

\[
= 1.
\]

This directly yields the Hölder inequality \(2\). \(\square\)

As a particular case of Theorem \(7\), we obtain the following inequality.

**Theorem 8** (Cauchy–Schwarz’s fractional inequality on time scales). Let \(\alpha \in (0,1)\) and \(a, b \in \mathbb{T}\). If \(f, g, h : [a, b] \to \mathbb{R}\) are rd-continuous, then

\[
\int_{a}^{b} |f(t)g(t)||h(t)| \Delta^\alpha t \leq \left[ \int_{a}^{b} |f(t)|^{2} |h(t)|^{\Delta^\alpha \tau} \right]^{\frac{1}{2}} \left[ \int_{a}^{b} |g(t)|^{2} |h(t)|^{\Delta^\alpha \tau} \right]^{\frac{1}{2}}.
\]

**Proof.** Choose \(p = q = 2\) in Hölder’s inequality \(2\). \(\square\)

Using Hölder’s inequality \(2\), we can also prove the following result.

**Corollary 9.** Let \(\alpha \in (0,1)\) and \(a, b \in \mathbb{T}\). If \(f, g, h : [a, b] \to \mathbb{R}\) are rd-continuous, then

\[
\int_{a}^{b} |f(t)g(t)||h(t)| \Delta^\alpha t \geq \left[ \int_{a}^{b} |f(t)|^{p} |h(t)|^{\Delta^\alpha \tau} \right]^{\frac{1}{p}} \left[ \int_{a}^{b} |g(t)|^{q} |h(t)|^{\Delta^\alpha \tau} \right]^{\frac{1}{q}},
\]

where \(\frac{1}{p} + \frac{1}{q} = 1\) and \(p < 0\) or \(q > 0\).

**Proof.** Without loss of generality, we may assume that \(p < 0\) and \(q > 0\). Set \(P = -\frac{p}{q}\) and \(Q = \frac{1}{q}\). Then, \(\frac{1}{P} + \frac{1}{Q} = 1\) with \(P > 1\) and \(Q > 0\). From \(2\) we
can write that
\[
\int_a^b |F(t)G(t)||h(t)|\Delta^\alpha t
\leq \left[ \int_a^b |F(t)|^P|h(t)|\Delta^\alpha t \right]^{\frac{1}{p^*}} \left[ \int_a^b |G(t)|^Q|h(t)|\Delta^\alpha t \right]^{\frac{1}{q^*}} \tag{3}
\]

for any rd-continuous functions \(F, G : [a, b] \to \mathbb{R}\). The desired result is obtained by taking \(F(t) = [f(t)]^q\) and \(G(t) = [f(t)]^q[g(t)]^p\) in inequality (3).

Next, we use Hölder’s inequality (2) to deduce a fractional Minkowski’s inequality on time scales.

**Theorem 10** (Minkowski’s fractional inequality on time scales). Let \(\alpha \in (0, 1]\), \(a, b \in \mathbb{T}\) and \(p > 1\). If \(f, g, h : [a, b] \to \mathbb{R}\) are rd-continuous, then
\[
\int_a^b |(f + g)(t)||h(t)|\Delta^\alpha t
\leq \left[ \int_a^b |f(t)|^p|h(t)|\Delta^\alpha t \right]^{\frac{1}{p}} + \left[ \int_a^b |g(t)|^p|h(t)|\Delta^\alpha t \right]^{\frac{1}{p}} \tag{4}
\]

**Proof.** We apply Hölder’s inequality (2) with \(q = p/(p - 1)\) and items (i) and (v) of Theorem 3 to obtain
\[
\int_a^b |(f + g)(t)||h(t)|\Delta^\alpha t
= \int_a^b |(f + g)(t)|(f + g)(t)||h(t)|\Delta^\alpha t
\leq \int_a^b |f(t)||(f + g)(t)|^p|h(t)|\Delta^\alpha t + \int_a^b |g(t)||(f + g)(t)|^p|h(t)|\Delta^\alpha t
\leq \left[ \int_a^b |f(t)||h(t)|\Delta^\alpha t \right]^{\frac{1}{p^*}} \left[ \int_a^b |(f + g)(t)|^{p^*}|h(t)|\Delta^\alpha t \right]^{\frac{1}{q^*}}
\leq \left[ \int_a^b |f(t)||h(t)|\Delta^\alpha t \right]^{\frac{1}{p}} + \left[ \int_a^b |g(t)||h(t)|\Delta^\alpha t \right]^{\frac{1}{p}} \tag{4}
\]

\[\times \left( \left[ \int_a^b |f(t)||h(t)|\Delta^\alpha t \right]^{\frac{1}{p^*}} \right)^{\frac{1}{q^*}} \right).\]
Dividing both sides of the obtained inequality by $\left[ \int_a^b |(f + g)(t)|^p |h(t)|^{\Delta_{\alpha} t} \right]^{\frac{1}{p}}$, we arrive at the Minkowski inequality (1).

Jensen’s classical inequality relates the value of a convex/concave function of an integral to the integral of the convex/concave function. We prove a generalization of such relation for the BHT fractional calculus on time scales.

**Theorem 11** (Generalized Jensen’s fractional inequality on time scales). Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, $c, d \in \mathbb{R}$, $\alpha \in (0, 1]$, $g \in C([a, b] \cap \mathbb{T}; (c, d))$ and $h \in C([a, b] \cap \mathbb{T}; \mathbb{R})$ with

$$\int_a^b |h(s)|^{\Delta_{\alpha} s} > 0.$$

- If $f \in C((c, d); \mathbb{R})$ is convex, then

$$f \left( \frac{\int_a^b g(s)|h(s)|^{\Delta_{\alpha} s}}{\int_a^b |h(s)|^{\Delta_{\alpha} s}} \right) \leq \frac{\int_a^b f(g(s))|h(s)|^{\Delta_{\alpha} s}}{\int_a^b |h(s)|^{\Delta_{\alpha} s}}. \quad (5)$$

- If $f \in C((c, d); \mathbb{R})$ is concave, then

$$f \left( \frac{\int_a^b g(s)|h(s)|^{\Delta_{\alpha} s}}{\int_a^b |h(s)|^{\Delta_{\alpha} s}} \right) \geq \frac{\int_a^b f(g(s))|h(s)|^{\Delta_{\alpha} s}}{\int_a^b |h(s)|^{\Delta_{\alpha} s}}. \quad (6)$$

**Proof.** We start by proving (5). Since $f$ is convex, for any $t \in (c, d)$ there exists $a_t \in \mathbb{R}$ such that

$$a_t(x - t) \leq f(x) - f(t) \quad \text{for all } x \in (c, d). \quad (7)$$

Let

$$t = \frac{\int_a^b g(s)|h(s)|^{\Delta_{\alpha} s}}{\int_a^b |h(s)|^{\Delta_{\alpha} s}}.$$

It follows from (7) and item (v) of Theorem 3 that

$$\int_a^b f(g(s))|h(s)|^{\Delta_{\alpha} s} - \left( \int_a^b |h(s)|^{\Delta_{\alpha} s} \right) f \left( \frac{\int_a^b g(s)|h(s)|^{\Delta_{\alpha} s}}{\int_a^b |h(s)|^{\Delta_{\alpha} s}} \right)
\begin{align*}
&= \int_a^b f(g(s))|h(s)|^{\Delta_{\alpha} s} - \left( \int_a^b |h(s)|^{\Delta_{\alpha} s} \right) f \left( \frac{\int_a^b g(s)|h(s)|^{\Delta_{\alpha} s}}{\int_a^b |h(s)|^{\Delta_{\alpha} s}} \right) f(t) \\
&= \int_a^b (f(g(s)) - f(t))|h(s)|^{\Delta_{\alpha} s}
\end{align*}$$

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\[
\geq a_t \int_a^b (g(s) - t) |h(s)| \Delta^\alpha s \\
= a_t \left( \int_a^b g(s)h(s)|\Delta^\alpha s - t \int_a^b |h(s)| \Delta^\alpha s \right) \\
= a_t \left( \int_a^b g(s)h(s)|\Delta^\alpha s - \int_a^b g(s)h(s)|\Delta^\alpha s \right) \\
= 0.
\]

This proves (5). To prove (6), we simply observe that \( F(x) = -f(x) \) is convex (because we are now assuming \( f \) to be concave) and then we apply inequality (5) to function \( F \).

We end with an application of Theorem 11.

**Theorem 12** (A weighted fractional Hermite–Hadamard inequality on time scales). Let \( \mathbb{T} \) be a time scale, \( a, b \in \mathbb{T} \) and \( \alpha \in (0, 1] \). Let \( f : [a, b] \to \mathbb{R} \) be a continuous convex function and let \( w : \mathbb{T} \to \mathbb{R} \) be a continuous function such that \( w(t) \geq 0 \) for all \( t \in \mathbb{T} \) and \( \int_a^b w(t)\Delta^\alpha t > 0 \). Then,

\[
f(x_{w,\alpha}) \leq \frac{1}{\int_a^b w(t)\Delta^\alpha t} \int_a^b f(t)w(t)\Delta^\alpha t \leq \frac{b - x_{w,\alpha} - a}{b - a} f(a) + \frac{x_{w,\alpha} - a}{b - a} f(b), \quad (8)
\]

where \( x_{w,\alpha} = \frac{\int_a^b tw(t)\Delta^\alpha t}{\int_a^b w(t)\Delta^\alpha t} \).

**Proof.** For every convex function one has

\[
f(t) \leq f(a) + \frac{f(b) - f(a)}{b - a} (t - a).
\]

Multiplying this inequality with \( w(t) \), which is nonnegative, we get

\[
w(t)f(t) \leq f(a)w(t) + \frac{f(b) - f(a)}{b - a} (t - a)w(t).
\]

Taking the \( \alpha \)-fractional integral on both sides, we can write that

\[
\int_a^b w(t)f(t)\Delta^\alpha t \leq \int_a^b f(a)w(t)\Delta^\alpha t + \int_a^b \frac{f(b) - f(a)}{b - a} (t - a)w(t)\Delta^\alpha t,
\]

which implies

\[
\int_a^b w(t)f(t)\Delta^\alpha t \\
\leq f(a) \int_a^b w(t)\Delta^\alpha t + \frac{f(b) - f(a)}{b - a} \left( \int_a^b tw(t)\Delta^\alpha t - a \int_a^b w(t)\Delta^\alpha t \right),
\]

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that is,
\[
\frac{1}{\int_a^b w(t) \Delta^\alpha t} \int_a^b f(t) w(t) \Delta^\alpha t \leq \frac{b - x_{w,\alpha}}{b - a} f(a) + \frac{x_{w,\alpha} - a}{b - a} f(b).
\]

We have just proved the second inequality of (8). For the first inequality of (8), we use (5) of Theorem 11 by taking \( g : T \to T \) defined by \( g(s) = s \) for all \( s \in T \) and \( h : T \to \mathbb{R} \) given by \( h = w \).

Note that if in Theorem 12 we consider a concave function \( f \) instead of a convex one, then the inequalities of (8) are reversed.

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