Parameter extraction by \textit{Planck} for a CDM model with broken scale invariance and cosmological constant

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ABSTRACT
We consider a class of spatially flat cold dark matter (CDM) models, with a cosmological constant and a broken-scale-invariant (BSI) steplike primordial spectrum of adiabatic perturbations, previously found to be in very good agreement with observations. Performing a Fisher matrix analysis, we show that in case of a large gravitational waves (GW) contribution some free parameters (defining the step) of our BSI model can be extracted with remarkable accuracy by the \textit{Planck} satellite, thanks to the polarisation anisotropy measurements. Further, cosmological parameters can still be found with very good precision, despite a larger number of free parameters than in the simplest inflationary models.

Key words: cosmology:theory - early Universe - cosmic microwave background.

1 INTRODUCTION

Current observations of the large-scale structure in the Universe, and of cosmic microwave background (CMB) anisotropies in particular, already allow a discrimination among different cosmological scenarios with increasing precision. Nevertheless, many possibilities are still viable, with different assumptions concerning e.g. the matter content of the Universe, and the primordial (initial) fluctuation power spectrum. However, most scenarios should be excluded by cross-correlating the forthcoming experiments, like, for instance, balloon and satellite measurements of small scale CMB anisotropies, and new redshift surveys (Wang, Spergel & Strauss 1999). The most precise scheduled experiment for the measurement of the CMB anisotropies is the \textit{Planck} satellite, the data from which will very likely favour a restricted family of cosmological scenarios, hopefully with a small number of free parameters.

As the simplest CDM model with a flat primordial power spectrum is already excluded, it is necessary to introduce some refinements either in the content of the Universe (i.e., in the transfer functions of matter and radiation), or in the generation of initial fluctuations (i.e., in the case of inflationary models, in the primordial power spectrum). By now, the simplest variant favoured by experimental data seems to be that of a flat universe with a cosmological constant, $\Omega_m + \Omega_\Lambda = 1$, and a scale-invariant primordial (or slightly tilted) power spectrum (Kofman, Gnedin & Bahcall 1993; Bagla, Padmanabhan & Narlikar 1996; Ostriker & Steinhardt 1995; Lineweaver 1998). In two recent papers (Lesgourgues, Polarski & Starobinsky 1998a, 1998b, further referred as LPS1, LPS2), the combination of this ΛCDM scenario with an inflationary model introduced by Starobinsky (1992), predicting a broken-scale-invariant (BSI) step-like primordial power spectrum, was investigated. In LPS1, the case of adiabatic primordial fluctuations was considered when the contribution of the tensorial fluctuations to the CMB anisotropies is negligible. In LPS2, the possible contribution of gravitational waves to the CMB anisotropy was taken into account: it is a most interesting peculiarity of these models that these distinct cases are possible and were shown to be viable regarding observations. Using polarization measurements on the precision level scheduled for \textit{Planck}, this large GW contribution will allow accurate parameter extraction. Briefly, the motivations for considering steplike models are the following. First, an even better agreement with the data can be found than in the case of a flat or tilted spectrum, inside a wider region of the cosmological parameter space. Second, a few authors point out the pos-

* For the instrumental specifications of the mission, see [http://astro.estec.esa.nl/SA-general/Projects/Planck/]
sible observational evidence for a spike in the matter power spectrum at \( k \simeq 0.05 \hmpc \) (Einasto et al. 1997a, 1997b, 1997c; Retzlaff et al. 1997; Gaztañaga & Baugh 1998). This is still a point of debate; however, as our BSI model predicts a similar feature, it would clearly be an excellent remaining candidate if the spike were to be confirmed by future redshift surveys.

One could argue that BSI inflationary models, by introducing additional free parameters in the primordial power spectrum, just increase the degeneracy among different scenarios; that instead of making real predictions, like the simplest slow-roll models, they are just introduced \textit{ad hoc}, in order to fit any observations; and finally, that in the case of BSI primordial spectra, the prospect of extracting the cosmological parameters at the per cent level with \textit{Planck} would collapse. However, we recall that our model is based on simple assumptions concerning the inflaton potential, and cannot be tailored at will in order to fit any given observational data. On the contrary, it predicts a very peculiar observable feature in the matter power spectrum at intermediate scales \((\simeq 125 \hmpc)\) while it makes, of course, predictions on all scales both for the matter and the radiation power spectrum. Even when only the radiation power spectrum is considered, we show in this work the following points:

- the future \textit{Planck} results should easily discriminate between our BSI model and other scale-free models;
- assuming that this model is indeed realized in nature, in spite of four additional degrees of freedom in the theory compared to the simplest versions of inflation, \textit{Planck} should still be able to measure accurately both the cosmological and the inflationary parameters. Furthermore, it turns out that one of the inflationary parameters, \( h \), which defines both the height and the shape of the step, should be constrained with remarkably high precision, a fact which could be of significant interest for building particle-physics-motivated inflationary models.

2 THE MODEL

We assume for simplicity that our Universe is known to be spatially flat, and that neutrino mass and reionisation can both be neglected (relaxing these assumptions would of course increase the uncertainties on all parameters). Then, our model contains three cosmological parameters \(( h, \Omega_{\Lambda}, \Omega_b)\), and five inflationary parameters, which can be understood as follows:

- the power spectrum of adiabatic perturbations has a scale-invariant tilt \( n_s \) on large scales, \( k < k_0 \), undergoes a break (the shape of which is defined by one single parameter \( p \)) at \( k > k_0 \), and is finally flat on small scale \( k > k_0 \). The ratio between the power spectrum on the small-scale plateau and at \( k_0 \) is given by \( p^{-7} \).
- the spectrum of GW has no break at \( k_0 \), while the tensor tilt on small scale \( k > k_0 \) is irrelevant for our purpose, because the corresponding contribution to the \( C_{l} \)'s is negligible. Using the slow-roll conditions valid on large scales, the scale-dependent tilt \( n_T(k) \) for \( k \leq k_0 \) can be found as a function of \( n_s \) and \( n_T(k_0) \).
- as the slow-roll approximation is still valid for large-scale perturbations, at \( k = k_0 \) one can relate the amplitude of the GW power spectrum to the dimensionless parameter \( H_0^2 \), and the scalar power spectrum amplitude to \( H_0^2 \).

In summary, the five free inflationary parameters are:

1. \( H_0^2 \), the overall dimensionless normalization factor. Varying \( H_0^2 \) (all other parameters being fixed) is exactly similar to varying the commonly used \( Q_{10} \), the 10th multipole of the temperature anisotropy power spectrum (Lineweaver & Barbosa 1998). Hence, we will further use this parameter instead of \( H_0^2 \).
2. \( k_0 \), the scale of the break.
3. \( p \), which defines the break’s amplitude and shape.
4. \( n_s \), the scale-invariant scalar tilt on scales \( k < k_0 \).
5. \( n_T(k_0) \), the (effective) tensor tilt at \( k_0 \).

The usual tensor-to-scalar ratio \( C_{10}^T/C_{10}^S \) does not appear in a natural way in this description. For fixed values of the cosmological parameters, there is a non-trivial dependence of \( C_{10}^T/C_{10}^S \) on \( n_T(k_0) \) and \( n_s \). Therefore, fixing the parameters \( n_T(k_0) \) and \( n_s \) fixes the ratio \( C_{10}^T/C_{10}^S \) as well.

In previous studies (LPS1, LPS2), \( \Omega_b h^2 = 0.015 \) was assumed while \( k_0 \) was fixed by the Einasto et al. cluster data (the spike in the matter power spectrum at \( k = 0.05 \hmpc \) requires \( k_0 = 0.016 \hmpc \)). Two possibilities for the scalar tilt were investigated:

A. \( n_s \approx 1 \), which implies \( d \ln n \approx n_s^2 \).
B. \( n_s \approx 1 + n_T \), which implies \( n_T(k) = n_T(k_0) \).

Further, a double normalization was performed to both \( Q_{10} = 18 \mu K \) (Bennett et al. 1996) and \( \sigma_8 = 0.60 \Omega^{-0.56} \) (White, Efstathiou & Frenk 1993).

3 THE FISHER MATRIX

Using the CMB Boltzmann code CMBFAST (Seljak & Zaldarriaga 1996), we compute the derivative of the \( C_{l} \)'s with respect to each parameter \( \theta_i \) \( i=1..8 \). The Fisher matrix (Jungman et al. 1996a, 1996b; Tegmark, Taylor & Heavens 1997; see also Bond, Efstathiou & Tegmark 1997; Copeland, Grivell & Liddle 1998; Stompor & Efstathiou 1998; Eisenstein, Hu & Tegmark 1998) is then obtained by adding the derivatives, weighted by the inverse of the covariance matrix of the estimators of the polarized and unpolarized CMB power spectra for the \textit{Planck} satellite mission, \( \text{Cov}(C_{l}^X, C_{l}^Y) \):

\[
F_{ij} = \sum_{\ell=2}^{\infty} \sum_{X,Y} \frac{\partial C_{\ell}^X}{\partial \theta_i} \text{Cov}^{-1}(C_{\ell}^X, C_{\ell}^Y) \frac{\partial C_{\ell}^Y}{\partial \theta_j},
\]

(1)
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where $\{X, Y\} \in \{T, E, TE\}$ (Kamionkowski, Kosowsky, & Stebbins 1997, Zaldarriaga, Spergel & Seljak 1997; Prunet, Sethi & Bouchet, 1998a, 1998b). The meaning of $F_{ij}$ is the following. Assuming that a fit to the Planck data yields a maximum likelihood for the model under consideration (for which the derivatives were computed), the 1-σ confidence region in the eight-dimensional parameter space would be inside the ellipsoid (Press et al. 1989):

$$\sum_{i,j} \Delta \theta_i \Delta \theta_j F_{ij} = 9.3.$$  \hfill (2)

Using $F_{ij}$ (or the dimensionless Fisher matrix $\tilde{F}_{ij} \equiv \theta_i \theta_j F_{ij}$), one can also compute the allowed region in lower dimensional cuts of the parameter space, making no assumptions on other parameters. In particular, the 1-σ uncertainty on a single parameter is just the square root of a diagonal coefficient of the inverse Fisher matrix:

$$\Delta \theta_i = \sqrt{(F^{-1})_{ii}}.$$ \hfill (3)

Each multipole will be measured by Planck with a precision of the order of 1%. As there are many more independent measurements than free parameters, one naively expects the parameter extraction to be much more precise. However, in general, the parameters are degenerate, i.e., some combinations of parameters produce a very weak change in the $C_l$ curve. Hence, even when some other combinations can be measured with very high precision, each parameter separately is constrained only at the percent level (unless its effect is “orthogonal” to the other ones).

A useful way to express the results of the Fisher matrix analysis, which does not depend on a particular choice of basis in the parameter space, and contains the most refined constraints that can be deduced from the experiment, is to diagonalize $\tilde{F}_{ij}$. The eigenvectors correspond to the axes of the likelihood ellipsoid, and the inverse square root of the eigenvalues to the 1-σ relative uncertainties on each eigenvector. Eigenvectors with the smallest uncertainties are the best constrained parameter combinations (they generate maximal changes in the anisotropy curve). Eigenvectors with the largest uncertainties are the worst-constrained combinations (they generate minimal changes), and are generally called degenerate directions in parameter space.

In computing the covariance matrix of the CMB power spectra, we accounted for the presence of foregrounds (both polarized and unpolarized) in the measurement of the CMB power spectra, using the method described in Bouchet, Prunet & Sethi 1998 (see also Prunet, Sethi & Bouchet 1998a, 1998b). The derivatives of the spectra have been computed with an error-minimizing routine derived from Numerical Recipes (Press et al. 1989).

4 RESULTS

The uncertainty on each parameter is presented in Table 1. A and B stand for the two models previously mentioned. T is a tilted model, with only three inflationary parameters instead of five, namely $(Q_{10}, n_S, n_T(k_0))$, with the same values as in model B. Of course, one should keep in mind that all the uncertainties quoted in Table 1 would increase if the space of free cosmological parameters were enlarged. In the lines without a ×-sign and with italic numbers, the polarisation measurement is not taken into account. The main conclusions to be drawn from the table are the following.

First, the three cosmological parameters are constrained with almost the same precision for the tilted and for the BSI models; this means that the step parameters $(k_0, p)$ do not “conspire” with the parameters $(h, \Omega_A, \Omega_b)$ in order to create directions of degeneracy. Hence, in general, the one percent parameter extraction proposed by Planck is not affected in the case of BSI steplike models.

The situation is somewhat different for the inflationary parameters. The normalization and tilts, $(Q_{10}, n_S, n_T)$, appear less constrained; on the other hand, the step parameters, $(k_0, p)$, can be predicted with excellent accuracy, up to a 0.09% 1-σ errorbar for $p$! These results can be easily understood, especially if one keeps in mind that the best constraints come from high $l$ multipoles, for which the cosmic variance can be neglected. For tilted models, the scalar tilt enters in all multipoles, and can be accurately determined from high $l$’s; the two remaining inflationary parameters $(Q_{10}, n_T)$ have a similar effect on high $l$’s (since $n_T$ is proportional to the tensor to scalar ratio), but measurements of the $C_l$’s for small $l$’s and polarisation measurements reduce the degeneracy. For our BSI models, the scalar tilt cannot be deduced from high $l$’s (it is defined at $k < k_0$, i.e., mainly $l < 100$); the three parameters $(Q_{10}, n_S, n_T(k_0))$ combine into several degeneracy directions that can be resolved only by small $l$ measurements, so the precision remains poor. The situation is exactly opposite for the step parameters $(k_0, p)$, which have the crucial property of playing a role only at $l > 150$. Hence, they are only marginally affected by cosmic variance. Further, $p$ is orthogonal to the degeneracy directions, and can be extracted with great precision.

All these features can be deduced with more accuracy from the Fisher matrix diagonalization, given in Table 2. The first lines give parameter combinations that are constrained with great precision; the last lines indicate the directions of degeneracy in parameter space. It is straightforward to see that the inflationary parameter $p$ contributes only to the first four lines. Therefore, it doesn’t suffer from any degeneracy, and is the best constrained parameter. The 5th and 6th lines show that a change in $k_0$ (resulting in a slight change in the location of the first acoustic peak, through the change in the primordial power spectrum), can be cancelled by a change in the cosmological parameters (i.e., in the sound horizon scale). The 6th eigenvector has a 1.5% uncertainty: this is already a small degeneracy, and $h$, $\Omega_A$, $\Omega_b$, $k_0$ are not as well constrained as $p$. The last two lines show the large degeneracy between $Q_{10}$ and $n_S$ and $n_T(k_0)$.

Let us compare the uncertainties when polarisation measurements are taken into account and when they are not. Usually, the addition of polarized spectra leads to a small precision increase (by a factor < 2) for all parameters which were not part of a specific degeneracy, as can be seen e.g. for tilted models, by comparing the (T) and (T×) data in Table 1. In our BSI model, the precision on $(n_S, n_T(k_0), p)$ increases by a much larger factor, and even by one order of magnitude for the parameter $p$! So, measuring the polarisation is even more important when one considers primordial spectra with additional free parameters (i.e., additional potential degeneracies to remove). One could be...
Table 1. In the upper part of the table, we give the parameter values for the chosen models: two BSI models A and B with 8 free parameters, and one tilted model T, with 6 free parameters. We also indicate the related value of $C_{10}^{\ell}/C_{10}^{S\ell}$. The corresponding relative 1-$\sigma$ uncertainties, $\Delta \theta_i/\theta_i$, are given in the lower part, in percent. In the lines without a $\times$-sign and with italic numbers, the polarisation measurement is not taken into account. The uncertainty on $C_{10}^{\ell}/C_{10}^{S\ell}$ was not calculated, but it is of the same order as the one on $n_T(k_0)$ since in a first order description, $C_{10}^{\ell}/C_{10}^{S\ell}$ is approximately proportional to $n_T(k_0)$.

| model | $h$ | $\Omega_\Lambda$ | $\Omega_b$ | $Q_{10}$ | $k_0$ | $p$ | $n_s$ | $n_T(k_0)$ | related $C_{10}^{\ell}/C_{10}^{S\ell}$ |
|-------|-----|------------------|------------|----------|-------|-----|-------|-----------|------------------|
| A     | 0.7 | 0.7              | 0.03       | 18 $\mu K$ | 0.016 $h$Mpc$^{-1}$ | 0.615 | 1    | -0.12     | 0.8               |
| B     | 0.7 | 0.7              | 0.03       | 18 $\mu K$ | 0.016 $h$Mpc$^{-1}$ | 0.51  | 0.825 | -0.175    | 0.8               |
| T     | 0.7 | 0.65             | 0.03       | 18 $\mu K$ | /      | /    | 0.825 | -0.175    | 0.8               |

Table 2. Orthonormal eigenvectors of the dimensionless Fisher matrix $\tilde{F}_{ij}$, with their 1-$\sigma$ uncertainty (in percent). The first lines show some combinations of the parameters that can be recovered with a precision much smaller than 1%. The last lines correspond to the directions of degeneracy in parameter space.

| eigenvector | uncertainty (%) |
|-------------|-----------------|
| $\Delta h$ | 0.3             |
| $\Delta \Omega_\Lambda$ | 0.3             |
| $\Delta \Omega_b$ | 0.3             |
| $\Delta Q_{10}$ | 0.3             |
| $\Delta k_0$ | 0.3             |
| $\Delta p$ | 0.3             |
| $\Delta n_s$ | 0.3             |
| $\Delta n_T(k_0)$ | 0.3             |

Concluded by the factor 10 found for $p$ in model B. In fact, when polarisation is not taken into account, $p$ enters into a single combination of parameters leading to a degeneracy. When polarisation is added, this degeneracy is supressed and, as we saw, $p$ doesn’t enter into any degeneracy at all. This mechanism is illustrated in figure 1.

5 CONCLUSION

In this paper, we considered an inflationary model with BSI primordial spectrum and we investigated the precision with which the cosmological parameters and the free inflationary parameters could be extracted by the Planck satellite. We first conclude that in the framework of the BSI steplike models considered here, the extraction of cosmological parameters can be as precise as in the case of tilted models. The step parameters $p$ and $k_0$ can be constrained with excellent accuracy, especially $p$, the effect of which on the $C_{10}\ell$'s can be easily distinguished from the effect of any parameter combinations. There is no degeneracy with tilted models, which are special cases of our model with respect to the CMB anisotropies whenever $k_0 \geq 0.25 h$ Mpc$^{-1}$. Further, if this class of models (or some other BSI model) were ever confirmed by future observations, it would be reasonnable to expect constraints on some of the inflaton Lagragian param-
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Figure 1. 1-$\sigma$ likelihood regions in the $(\Delta p/p, \Delta \Omega_\Lambda/\Omega_\Lambda)$ plane, with and without including polarisation measurement. The only degeneracy involving $p$ is removed by the introduction of polarisation measurement. Therefore, the ellipse appears vertical in all $(\Delta p/p, \Delta \theta_i/\theta_i)$ plots.

eters up to the 0.1% precision level. This is most interesting for building particle physics inspired inflationary models. On the other hand, precision is lost for the determination of the scalar and tensor tilts on large scales, as well as on the quantity $C_{10}^T/C_{10}^S$ related by the slow-roll equations to $n_T(k)$. Finally, in usual inflationary models, the inclusion of polarization measurements is known to increase the precision for the parameter extraction. In our model, polarization measurements by Planck are even shown to render the extraction of the inflationary parameters up to about 10 times more accurate.

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