Noise conversion in Kerr comb RF photonic oscillators

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I. INTRODUCTION

In a radio frequency (RF) photonic oscillator that utilizes an optical frequency comb the signal is produced by demodulating the equally spaced frequency harmonics of the comb on a fast photodiode. The relative coherence, or phase locking of the optical frequency harmonics, is a necessary condition for generation of spectrally pure RF signals. The performance of the best photonic generators is far superior to electronic devices. For instance, generation of 10 GHz signals with phase noise at the level of -100 dBc/Hz at 1 Hz offset and -179 dBc/Hz at 1 MHz and higher offsets were demonstrated with a femtosecond mode locked laser [1, 2]. This phase noise level is 40 dB lower compared with phase noise of the best room-temperature electronic oscillator. The achieved spectral purity of the photonic device could be spoiled by back-end electronic amplifiers because of their inherent noise. However, another attribute of the photonic oscillator is that amplifiers are usually not required, since the output power of a 10 GHz RF source can reach 25 mW when a uni-traveling carrier photodiode is utilized [3].

Phase noise is the main characteristic determining the performance of an oscillator and its practical usefulness. Optical monolith microresonator-based RF photonic oscillators are advantageous for providing low noise RF signals on a chip. Development of RF generation with hyper-parametric oscillators and mode-locked (Kerr) frequency combs has recently become the major direction in this area [4, 5]. For example, phase noise of a whispering gallery mode (WGM) resonator-based hyper-parametric RF photonic oscillator is much lower as compared with the noise of existing electronic oscillators such as dielectric resonator oscillators. Packaged 35 GHz Kerr comb RF photonic oscillators with phase noise of -115 dBc/Hz at 10 kHz offset and -130 dBc/Hz at 100 kHz offset were previously demonstrated and have became commercially available [6]. A miniature 10 GHz RF photonic oscillator characterized with phase noise better than -60 dBc/Hz at 10 Hz, -90 dBc/Hz at 100 Hz, -120 dBc/Hz at 1 kHz, and -150 dBc/Hz at 10 MHz was demonstrated recently [7]. The frequency stability of this device is at the level of $10^{-10}$ at 1-100 s integration time [8], which is better than existing RF photonic devices of similar size, weight, and power consumption.

Quantum noise limited phase diffusion of the microresonator-based hyper-parametric RF photonic oscillator can be described by a Schawlow-Townes-like formula [12]. A Leson model can be used far enough from the RF carrier to understand the behavior of the phase noise [6]. The phase noise of the Kerr comb oscillator operating in the mode locked regime has been recently analyzed as well [13]. In accordance with the obtained theoretical formula, the phase noise of the mode locked Kerr frequency comb oscillator made with a 7 mm magnesium fluoride WGM resonator, having a quality factor of $10^9$ and pumped with 5 mW of optical power, is expected to be -90 dBc/Hz at 10 Hz, -110 dBc/Hz at 100 Hz, -130 dBc/Hz at 1 kHz, and -154 dBc/Hz at 10 MHz. The predicted noise floor, determined by the shot noise of the oscillator, matches the measured value [10] well. The measured close-in noise, however, is significantly worse than the predicted number. Evidently, there exists a mechanism that increases the oscillator noise well above the quantum limit.

In this paper we analyze classical transfer functions describing the impact of amplitude and frequency noise of the pump laser on the repetition rate of the Kerr frequency comb generated in a continuously pumped monolithic microresonator (see Fig. 1 as an example of the frequency comb generator). Using this result, we analyze the noise of the Kerr comb RF photonic oscillator and show that classical noise of the pump laser can be an important limiting factor on the performance of the oscillator.

We start with a simple analytical model of the resonant hyper-parametric oscillation that involves interaction of light confined in three resonator modes. The central mode is pumped optically. The oscillation can be described as a nondegenerate parametric process in which two pump photons are transformed to two different photons of lower and higher frequencies. This occurs when the pump power exceeds a certain threshold determined by the loss of the system, and by nonlinear coupling efficiency between the resonator modes. We show that the beat frequency between the pump laser and generated sidebands does not depend on the laser parameters, if all the modes have the same quality (Q-) factor. The de-
dependence on the parameters of the pump light emerges as soon as the degeneracy of the Q-factor is lifted. Both amplitude and phase modulation of the pump can be transferred to the beat frequency of the optical harmonics of the oscillator. The phase and amplitude noise of the pump can impact the beat frequency as well.

A large variety of nonlinear processes takes place in an optically pumped nonlinear microresonator. We here focus at mode-locked Kerr frequency comb and mode-locked hyper-parametric oscillation that behave in a similar way. Both processes are characterized with hard excitation regime and the fundamental noise of both processes can be described by similar analytical expressions [13]. Generation of the mode-locked Kerr frequency comb is different from the hyper-parametric oscillation only in a sense that a large number of nearly equal power frequency harmonics are generated from the cw optical pump, in contrast with a pair of optical harmonics considered in the hyper-parametric oscillator model.

Mode-locked hyper-parametric oscillation occurs in resonators with relatively large group velocity dispersion, while Kerr combs are generated in resonators with a small dispersion value. To modify the dispersion value one needs to change loading of the resonator modes. It is possible to transfer a narrow frequency comb to a broad frequency comb adiabatically by changing the loading accompanied by proper change of the pump power and frequency. This fact proves physical identity of the combs.

Even though the narrow and broad combs are identical, an analytic description of noise transfer for the broad frequency comb is not straightforward. We perform a numerical simulation of the system and solve a set of ordinary differential equations describing the comb [14, 15] to find the impact of noise transfer process on a realistic Kerr comb oscillator. We evaluate the corresponding modulation of the comb power and repetition rate by adding modulation to the amplitude and phase of the pump, and subsequently infer the corresponding transfer functions that aid estimating the impact of the pump noise on those parameters. There is no transfer if the resonator spectrum is characterized with second order group velocity dispersion (GVD) only and all the modes have the same Q-factor. By substituting realistic asymmetry parameters to the model we find that the impact of a noisy pump light can be rather significant. A natural way to reduce this influence is to utilize a low noise pump laser.

This paper is organized as follows. In Section I we provide an analytical model of hyper-parametric oscillation and study the dependence of its frequency on parameters of the pump light. Numerical simulations involving Kerr frequency comb are described in Section II. We discuss how the approximations considered in the paper are related to realistic resonators in Section III. Section IV concludes the paper.

II. PUMP-DEPENDENT FREQUENCY OF A HYPER-PARAMETRIC OSCILLATOR

The theory of hyper parametric oscillation presented here is related to the theory of additive modulational instability in ring lasers [16, 17] and in fiber-based optical parametric oscillators [18, 22]. In the case of high-Q WGM hyper-parametric oscillator the theory was developed in [12]. We adopt the model [23] to analyze the dependence of the frequency of a hyper-parametric oscillator on the frequency and power of the pump laser. We show that an ideal symmetry in the oscillation process results in complete suppression of the influence of the pump parameters on the frequency interval between the pump light and generated optical frequency harmonics. This frequency interval coincides with the frequency of the RF signal generated by demodulation of the frequency comb on a fast photodiode. We argue that this is not the case in practice, since modes of an optical resonator have slightly different Q-factors. For instance, this difference is fundamental in the case of an evanescently coupled whispering gallery mode resonator.

A. Basic equations and their steady state solution

We consider a hyper-parametric oscillation process in which two monochromatic optical pump photons transform into two sideband photons mediated by the cubic nonlinearity of the ring resonator’s host material. The oscillation is described by the Kerr Hamiltonian

\[ V = -\hbar(g/2) : (a + b_+ + b_- + h.c.)^4 : , \]

where \( " : \cdots : " \) stand for normal ordering, \( a, b_+ \), and \( b_- \) are the annihilation operators for the optically pumped and sideband modes, respectively; \( g \) is the coupling constant.

The coupling constant \( g \) can be derived using definition of the nonlinear refractive index:

\[ n = n_0 + n_2 I, \]

where \( I \) denotes the time-averaged intensity of the optical field within the resonator mode, and \( n_0 \) and \( n_2 \) are the
linear and nonlinear refractive indices of the resonator host material. According to (1), the power-dependent frequency shift of the optically pumped mode depends on the number of photons in the mode \((a^\dagger a)\) as
\[
\Delta \omega_0 = -g(a^\dagger a).
\]
On the other hand, in accordance with (2),
\[
\Delta \omega_0 = -\omega_0 n_2 I/n_0 \quad (\omega_0 \text{ is the frequency of the mode}).
\]
Since \(h\omega_0(a^\dagger a) = V n_2 I/c \) (\(V \) is the mode volume), we get
\[
g = \omega_0 \frac{h\omega_0 c n_2}{V n_0 T n_0}.
\]
This simple derivation is valid for the case of completely spatially overlapped resonator modes. The mode volume parameter has to be modified appropriately in a more general case.

Using Eq. (1) and applying the rotating wave approximation, we derive equations describing the amplitude of the electric field within the optical modes
\[
\dot{a} = -(i\omega_0(T) + \gamma_0 + \gamma_{\text{co}})a + ig[a^* a + 2b^*_+ b_+] + (4)
\]
\[
2b^*_+ b_- a + 2iga^* b_+ b_- + \sqrt{\frac{2g^2 T_0}{\hbar \omega_0}} e^{-i\omega t},
\]
\[
\dot{b}_+ = -(i\omega_+(T) + \gamma_+ + \gamma_{\text{co}})b_+ + ig[2a^* a + b_+^* b_- + 2b^*_+ b_-]b_+ + igb^*_+ a^2, \quad (5)
\]
\[
\dot{b}_- = -(i\omega_-(T) + \gamma_- + \gamma_{\text{co}})b_- + ig[2a^* a + b_+^* b_- + b_+^* b_-]b_- + igb^*_- a^2,
\]
where \(\omega_0(T), \omega_+(T), \) and \(\omega_-(T)\) are the temperature (T)-dependent eigenfrequencies of the WGMs; \(\gamma_0, \gamma_+, \) and \(\gamma_-\) are the internal decay rates of the modes; \(\gamma_{\text{co}}, \gamma_{\text{ce}}, \) and \(\gamma_{\text{ce}}\) are the decay rates due to external coupling; \(P_0\) is the power of the external pump.

We are interested in a classical analysis of the oscillator, so we transfer the operators to the expectation values of field amplitudes. Additionally, to simplify the equations we introduce slowly varying amplitudes:
\[
a = A e^{-i\omega t}, b_+ = B_+ e^{-i\omega_+ t}, b_- = B_- e^{-i\omega_- t}, \quad (7)
\]
where \(\omega\) is the carrier frequency of the external pump, and \(\omega_+\) and \(\omega_-\) are the carrier frequencies of generated optical harmonics. Values of the frequencies of the harmonics are determined by the oscillation process and cannot be controlled externally. From energy conservation principle we find a relationship between them:
\[
2\omega = \omega_+ + \omega_-.
\]

We substitute (7) into (4) and derive a set of algebraic equations neglecting time derivatives of slowly varying amplitudes \(A\) and \(B_\pm\)
\[
\Gamma_0 A = ig\left[|A|^2 + 2|B_+|^2 + 2|B_-|^2\right] A + 2iga^* B_+ B_- + F_{\text{co}},
\]
\[
\Gamma_+ B_+ = ig(2|A|^2 + 2|B_-|^2 + |B_+|^2) B_+ + igB_+^* A^2,
\]
\[
\Gamma_- B_- = ig(2|A|^2 + |B_-|^2 + 2|B_+|^2) B_- + igB_-^* A^2.
\]
where
\[
\Gamma_0 = i(\omega_0(T) - \omega) + \gamma_0 + \gamma_{\text{co}},
\]
\[
\Gamma_+ = i(\omega_+(T) - \omega_+) + \gamma_+ + \gamma_{\text{ce}},
\]
\[
\Gamma_- = i(\omega_-(T) - \omega_-) + \gamma_- + \gamma_{\text{ce}},
\]
are the complex parameters determining detuning and attenuation in the system.

To solve the equations it is convenient to introduce dimensionless parameters:
\[
\bar{\gamma}_j = \gamma_j + \gamma_{\text{co}},
\]
\[
\xi = \frac{g|A|^2}{\gamma_0},
\]
\[
f = \left(\frac{g}{\gamma_0}\right)^{1/2} \frac{|F_{\text{co}}|}{\gamma_0},
\]
\[
A = |A|e^{i\phi_0},
\]
\[
B_\pm = |B_\pm|e^{i\phi_{\pm}},
\]
\[
\phi = 2\phi_0 - \phi_+ - \phi_-,
\]
\[
\Delta_+ = \frac{\omega_+(T) - \omega_+}{\gamma_0},
\]
\[
\Delta_- = \frac{\omega_-(T) - \omega_-}{\gamma_0},
\]
\[
D = \frac{2\omega_0(T) - \omega(T) - \omega(T)}{\gamma_0} \approx \frac{\beta_{2e} \omega_{FSR}^2}{\gamma_0 n_0}.
\]

It is easy now to transform the set of three equations \((9-11)\) to the set of six algebraic equations using this parametrization
\[
\sqrt{\xi}(1 - 2\xi B_+ B_- \sin \phi) = f \cos \psi, \quad (13)
\]
\[
\sqrt{\xi} \{\Delta_0 - \xi \left[1 + 2(B_+^2 + B_-^2 + B_+ B_- \cos \phi)\right]\} = f \sin \psi, \quad (14)
\]
\[
\gamma_+ B_+ + \gamma_0 \xi B_- \sin \phi = 0, \quad (15)
\]
\[
[\Delta_+ - \xi (2 + B_+^2 + 2B_-^2)] B_+ - \xi B_- \cos \phi = 0, \quad (16)
\]
\[
\gamma_- B_- + \gamma_0 \xi B_+ \sin \phi = 0, \quad (17)
\]
\[
[\Delta_- - \xi (2 + 2B_+^2 + B_-^2)] B_- - \xi B_+ \cos \phi = 0, \quad (18)
\]
\[
\Delta_+ + \Delta_- = 2\Delta_0 - D \quad (\omega_+ + \omega_- = 2\omega). \quad (19)
\]
Parameters \(\xi, \phi, \Delta_+, \Delta_-\) as well as ratio between \(B_+\)
and $B_-$ can be inferred from Eqs. (15-19). They are

$$\xi^2 = \frac{\bar{\gamma} + \bar{\gamma} - \bar{\gamma}_0}{\bar{\gamma}_0} + \frac{\bar{\gamma} + \bar{\gamma} - \bar{\gamma}_0}{(\bar{\gamma}_+ + \bar{\gamma}_-)^2}$$  \hspace{1cm} (20)$$

$$[2\Delta_0 - D - \xi(4 + 3(\bar{B}_+^2 + \bar{B}_-^2))]^2,$$

$$\sin \phi = -\sqrt{\bar{\gamma}_+ - \gamma},$$  \hspace{1cm} (21)

$$\cos \phi = \frac{2\Delta_0 - D - \xi(4 + 3(\bar{B}_+^2 + \bar{B}_-^2))}{(\bar{\gamma}_+ + \bar{\gamma}_-)^2} \sqrt{\bar{\gamma}_+ - \gamma},$$  \hspace{1cm} (22)

$$\Delta_+ = (2\Delta_0 - D) \frac{\bar{\gamma}_+}{\bar{\gamma}_- + \bar{\gamma}_+} + \xi(2 + \bar{B}_+^2 + \bar{B}_-^2) \times$$  \hspace{1cm} (23)

$$\Delta_- = (2\Delta_0 - D) \frac{\bar{\gamma}_-}{\bar{\gamma}_- + \bar{\gamma}_+} - \xi(2 + \bar{B}_+^2 + \bar{B}_-^2) \times$$  \hspace{1cm} (24)

$$\frac{\bar{\gamma}_- - \bar{\gamma}_+}{\bar{\gamma}_- + \bar{\gamma}_+} - \xi(\bar{B}_-^2 - \bar{B}_+^2),$$

$$\frac{\bar{\gamma}_- + \bar{\gamma}_+}{\bar{\gamma}_- + \bar{\gamma}_+} - \xi(\bar{B}_-^2 + \bar{B}_+^2),$$

$$\frac{B_+}{B_-} = \sqrt{\frac{\bar{\gamma}_-}{\bar{\gamma}_+}}$$  \hspace{1cm} (25)

Now, from Eq. (14) and Eq. (15) we find equations for $\psi$ and $B_-:

$$\left[1 + 2\frac{\bar{\gamma}_-}{\bar{\gamma}_0}B_+^2\right]^2 +$$  \hspace{1cm} (26)

$$\left[\Delta_0 - \xi - 2\xi B_+^2 \left(\frac{\bar{\gamma}_- - \bar{\gamma}_+}{\bar{\gamma}_- + \bar{\gamma}_+}\right)^2 +$$

$$\frac{\bar{\gamma}_- (2\Delta_0 - D)}{\xi(\bar{\gamma}_+ + \bar{\gamma}_-)} - \frac{3}{2} \frac{\bar{\gamma}_- B_+^2}{\bar{\gamma}_+}\right]^2 = \frac{f^2}{\xi},$$

$$\cos \psi = \sqrt{\frac{\xi}{f}} \left[1 + 2\frac{\bar{\gamma}_-}{\bar{\gamma}_0}B_+^2\right],$$  \hspace{1cm} (27)

$$\sin \psi = \sqrt{\frac{\xi}{f}} \left[\Delta_0 - \xi - 2\xi B_+^2 \left(\frac{\bar{\gamma}_- - \bar{\gamma}_+}{\bar{\gamma}_- + \bar{\gamma}_+}\right)^2 +$$

$$\frac{\bar{\gamma}_- (2\Delta_0 - D)}{\xi(\bar{\gamma}_+ + \bar{\gamma}_-)} - \frac{3}{2} \frac{\bar{\gamma}_- B_+^2}{\bar{\gamma}_+}\right].$$  \hspace{1cm} (28)

**B. Oscillation frequency**

The complex amplitude of an RF signal generated by the hyper-parametric oscillator output on a fast photodiode is proportional to $A^* B_+ + A B_+^*$, or

$$E_{RF} \sim |A|^2 e^{i \phi / 2}(B_+ e^{-i \phi / 2} + B_- e^{i \phi / 2}),$$  \hspace{1cm} (29)

where $\phi_- = \phi_+ - \phi_-$. Equation (29) shows that the phase of the RF signal does not depend on the phase of the pump if $B_+ = B_-$. In accordance with Eq. (25) this is not true if $\bar{\gamma}_- \neq \bar{\gamma}_+$. It means that the phase noise of the pump light will influence the RF phase noise if the modes of the resonator are not identical.

The frequency difference between the pump light and oscillation harmonics can be found from Eqs. (25) and (29) in a straightforward way

$$\frac{\bar{\omega}_+ - \bar{\omega}_-}{\bar{\gamma}_0} = \frac{\bar{\omega} - \bar{\omega}_-}{\bar{\gamma}_0} =$$  \hspace{1cm} (30)

$$\left[-\frac{D}{2} + \Delta_0 - \xi(2 + \bar{B}_+^2 + \bar{B}_-^2) \right] \frac{\bar{\gamma}_- - \bar{\gamma}_+}{\bar{\gamma}_+ + \bar{\gamma}_-} + \frac{\bar{\omega}_+ - \bar{\omega}_-}{2\bar{\gamma}_0}.$$

These frequency differences determine the RF frequency $\omega_{RF}$ produced by the optical signal. It is possible to rewrite Eq. (30) in conventional terms as

$$\omega_{RF} = \omega_{FSR}(T) + \frac{\Delta Q}{2Q} \frac{\omega_+ (T) + \omega_-(T) - 2\omega}{2} -$$  \hspace{1cm} (31)

$$g[2 |A|^2 + |B_+|^2 + |B_-|^2],$$

where $\omega_{FSR}(T)$ is the free spectral range of the resonator, $\Delta Q = Q_+ - Q_-$ is the difference of loaded quality factors of the two sideband modes, $Q = (Q_+ + Q_-)/2$ is the averaged Q-factor of the modes. Equation (31) means that the frequency of the RF signal generated at the output of optical hyper-parametric oscillator depends both on the detuning of the cw optical pump from the corresponding resonator mode and the power fluctuations of the pump light. Naturally, the noise of these parameters will be transferred to the frequency noise of the RF signal. The transfer coefficient is equal to $\Delta Q/(2Q)$ and is dependent on the size of the resonator.

If we assume that the loss is given by Rayleigh scattering, it inversely increases with wavelength as $\lambda^{-4}$ in the vicinity of the carrier frequency of the pump, or

$$\gamma = \gamma_0 \frac{\lambda^4}{\lambda_0^4},$$  \hspace{1cm} (32)

In accordance with Eq. (32) $\Delta Q/(2Q) \approx -3\omega_{FSR}/\omega_0$.

The coupling loss also has a dispersion. For instance, in the case of an evanescently coupled WGM resonator, if there is no gap between the coupling prism and the resonator, the frequency behavior of the resonator loading can be approximated as

$$\gamma_c = \gamma_0 \frac{\omega_{FSR}^3}{\omega_0^3},$$  \hspace{1cm} (33)

In accordance with Eq. (33) we get $\Delta Q_c/(2Q_c) \approx 5\omega_{FSR}/(2\omega_0)$. If a nonzero spatial gap between the coupler and the resonator surface is maintained, the frequency dependence becomes exponential and the coupling efficiency can change by several orders of magnitude if the frequency changes by an octave.

The ratio $\omega_0/\omega_{FSR}$ is equal to the mode order $l$ and is a large number. For instance, $l \approx 2 \times 10^5$ for a resonator with $\omega_{FSR} = 2\pi \times 10^{10}$ rad/s pumped at $\lambda_0 = 1,550$ nm. Therefore, the detuning and laser power noise is suppressed by $4 \times 10^8$ times at small frequencies. This value can be reduced if an additional process introducing asymmetry takes place, for example interaction among the modes of the resonator of frequency dependent attenuation resulting from impurities or contaminants (e.g. water).
III. NUMERICAL SIMULATION OF NOISE TRANSFER FROM OPTICAL PUMP TO FREQUENCY COMB REPETITION RATE

The analytical theory of hyper-parametric oscillation presented in the previous section suggests that there is a channel for transfer of noise from the optical pump to the phase noise of the RF signal generated at the output of the oscillator. This prediction is not accurate enough if we consider generation of the Kerr frequency comb instead of the hyper-parametric oscillation. The frequency comb has many interacting harmonics involved, which can result in modification of the noise transfer efficiency.

To study this particular case we perform a numerical simulation in which a set of coupled ordinary differential equations is solved to simulate Kerr frequency comb formation. We write the set as

\[
\dot{a}_j = -i(\gamma_j + \omega_j)a_j + \frac{i}{\hbar}[V_c, a_j] + F_{j0}(t)e^{-i\omega t}\delta_{j0,j},
\]

where \(V_c = -\hbar g(e^e)^2e^2/2\) describes nonlinear interaction among modes of the resonator (c.f. Eq. 1), \(e = \sum_{j=1}^N a_j\) is the operator describing amplitude of light confined in the resonator; \(a_j\) is an operator standing for complex amplitude of the field in the mode (annihilation operator); \(\omega_j\) is the frequency of the pumped mode. Practically the combs can be generated by proper nonadiabatic changing of the pump frequency on the cw pump per Eq. (36). The pulse trains are generated trains of freely propagating optical pulses by interacting with the input/output couplers during each round trip. The loss associated with the train generation is compensated by interaction of the intracavity pulse with the continuous wave background powered by the external pump.

The pulse duration, \(\tau\), and corresponding spectral width of the frequency comb, \(\Delta f_{\text{comb}}\), can be approximated analytically

\[
\tau \approx \sqrt{\frac{2b_2\lambda_0\Delta f_{\text{th}}}{\pi^2F_{\text{th}}^2n_2}},
\]

\[
\Delta f_{\text{comb}} \approx (\pi^2\tau)^{-1},
\]

where \(\lambda_0\) is the wavelength of the pump light \([13, 24, 25]\). As a rule, the smaller the GVD (\(\beta_2\)), the broader is the comb for the same set of parameters. We selected two cases of comparably large and comparably small GVD to study optical-to-RF noise transfer for the signal generated by the light produced by the hyper-parametric oscillation, and mode locked Kerr frequency comb.

The envelopes of the optical frequency combs we used in the simulation are shown in Fig. (2b). While the Kerr frequency comb corresponds to generation of short optical pulses, the hyper-parametric oscillator naturally forms pulses that rather resemble a beat note of several mutually coherent harmonics (Fig. 2b). Generation of the combs in both cases have hard excitation regime \([13\). To simulate them numerically we select an arbitrary nonzero initial conditions for the field localized in the resonator modes. Practically the combs can be generated by proper nonadiabatic changing of the pump frequency and/or power \([24\).

B. Simulation technique

Once the frequency and the power are selected to observe the frequency comb (Fig. 2b), we introduce modulation on the cw pump per Eq. (36). The pulse trains are
and pump frequency detuned from the resonance by $\Delta F_0$ by pump light to the frequency comb parameters. The transfer function we are looking for is represented by the frequency dependence of the modulation harmonic. We define the transfer functions as

$$H_{PM}(f) = \frac{4}{\kappa_{a}^2} S_{PM \ max}(f), \quad (42)$$

$$H_{AM}(f) = \frac{4}{P_{comb} \kappa_{a}^2} S_{AM \ max}(f). \quad (43)$$

where $S_{PM \ max}(f)$ and $S_{AM \ max}(f)$ are the maxima of the spectral power densities of the RF phase, Eq. (40), and comb power, Eq. (41), signals, respectively.

We found that the transfer functions is zero (their values are smaller than the digital accuracy of the simulation) when an ideal resonator with identical modes and only the second order GVD is considered. However, when higher order dispersive terms and/or frequency dependence of the resonator Q-factor are introduced, the transfer functions start to deviate from zero. This means that, in accordance with the simplified analytical treatment presented above, the modulation of the pump light can be transferred to the frequency comb parameters. Since small additive noise can be considered as modulation of a regular signal with a small noise signal, the transfer functions can be used to evaluate the noise transfer from the cw pump light to the frequency comb parameters.

FIG. 3. Examples of simulated power spectra $S(f)$ of the RF signal. Curve (1) stands for a narrow frequency comb, curve (2) stands for the broad frequency comb (see Fig. 2a). Top panel stands for the amplitude modulation, bottom one – for phase modulation of the pump. The modulation frequency is shown by the arrows. The resonator has $D = -0.05$ and $D_3 = -10^{-4}$ to achieve the noise transfer. The AM and PM modulation coefficients ($\kappa_a$ and $\kappa_o$) are selected to be the same, $10^{-3}$.
C. Conversion of optical amplitude noise to frequency comb noise due to frequency dependent Q-factor

Quality factors of modes of any realistic resonator are not identical. In WGM resonators, for instance, there are fundamental reasons that enforce the inequality of Q-factors of any two adjacent optical modes. This inequality results in fundamental mechanism for noise transfer from the pump light to the generated pulse train and to the corresponding optical frequency comb.

We introduce the frequency dependence of the quality factors and evaluate the transfer functions using Eqs. (12) and (13). The result of the simulation is shown in Fig. 4. The amplitude modulation impacts both the phase and the amplitude of generated frequency combs. There are two well pronounced resonances in the transfer functions. The resonances occur at frequency $f_1 \approx 9\text{HWHM}$ and $f_2 \simeq \Delta_0$. The low frequency, $f_1$, resonance corresponds to the relaxation oscillation. It depends on the Q-factor of the resonator and less on the other parameters of the system. The high frequency, $f_2$ resonance corresponds to the detuning of the pump frequency from the frequency of the cold resonator mode. This resonance has an analogy with Feshbach resonances in atomic systems [27].

The numerical simulation shows that the hyper-parametric oscillator has higher efficiency for optical-to-RF modulation transfer. It also predicts that the amplitude modulation of the pump light can be transferred with amplification to the power of the frequency comb if the modulation frequency corresponds with the relaxation oscillation resonance. It means that the amplitude noise of the pump laser can be efficiently imprinted on the amplitude noise of the RF signal generated by the optical frequency comb on a fast photodiode.

D. Conversion of optical phase noise to frequency comb noise due to frequency dependent Q-factor

We repeated the simulation for phase modulated pump light using identical parameters of the system. The results of the simulation are shown in Fig. 5. They are similar to those obtained for the amplitude modulated pump. Phase modulation (phase noise) of the pump light impacts both amplitude and phase of the generated frequency combs (power and timing of the corresponding pulse train).

![FIG. 4](image1.png)

**FIG. 4.** Transfer functions characterizing the impact of amplitude modulation (amplitude noise) of the pump light on the frequency comb, simulated for the case of frequency dependent loss of the resonator modes: $\gamma_{1}/\gamma_{0} = 1 + 10^{-2} (j - j_{0})$. The rest of the parameters are the same as in Figs. 2 and 3. Panel (a) describes AM to PM conversion in narrow and broad frequency combs, while panel (b) stands for AM to AM conversion. Red lines, (1), stand for hyper-parametric (narrow) oscillation comb, blue lines, (2), stand for the (broad) Kerr frequency comb.

![FIG. 5](image2.png)

**FIG. 5.** Transfer functions characterizing the impact of phase modulation (phase noise) of the pump light on the frequency comb simulated for the case of frequency dependent loss of the resonator modes. The parameters of the system are selected in the same way as in Figs. 2 and 3. Panels (a) and (b) correspond to PM to PM conversion and PM to AM conversion, respectively. Red lines, (1), stand for the hyper-parametric oscillation comb, blue lines, (2), stand for the broad Kerr frequency comb.

E. Conversion of optical noise to frequency comb noise due to the presence of third order dispersion

We found that frequency dependent Q-factor of the spectrum of a nonlinear resonator initiates transfer of modulation (noise) of the pump light to the parameters of generated optical frequency comb. There is another parameter of the spectrum that results in the noise transfer. This parameter is related to high-order dispersion of the resonator modes. The high order dispersion is always present in the resonator due to dispersion of the
resonator host material, which has nonlinear dispersion due to limited transparency bandwidth. The wavelength-dependent quality factor of the resonator spectrum also has to have certain correspondence with the dispersion, in accordance with the Kramers-Kronig relationship.

For the sake of clarity we neglect the wavelength dependence of the absorption and introduce a small third order dispersion (TOD) to the resonator spectrum. The numerical simulations performed with this kind of resonator clearly show that the presence of a small TOD results in noise transfer similar to what we observed with frequency dependent Q-factor. We present results of the simulation for the (broad) Kerr frequency comb only (Fig. 6), since the results for hyper-parametric oscillation have similar behavior, as shown for the frequency dependent Q-factor.

FIG. 6. Transfer functions describing impact of the pump modulation on the phase (panel (a)) and amplitude (panel (b)) of the Kerr frequency comb for $D = -0.05$ and $D_3 = -0.0001$. The other parameters of the system are selected in the same way as in Figs. 2 and 3.

F. Impact of limited mode number on simulation result

Similarly to the case of wavelength dependent Q-factor we observed two resonances at the transfer functions shown in Fig. 6. One of the resonances corresponds to the relaxation oscillation and the other – to a Feshbach resonance. We also observed a third resonance in the $H_{AM2PM}$ curve (Eq. 4). To understand the nature of the resonance we repeated the simulation with different second-order GVD parameters. We found that the resonance shifts beyond the dynamic range of the simulation accuracy (practically disappears) for larger GVD values (when we change $D = -0.05$ to $D = -0.2$); the relaxation oscillation resonance shifts to smaller frequencies, and the Feshbach resonance changes its contrast, not the frequency, by a small amount (Fig. 7). The third resonance also disappears if we change the number of modes in the simulation. These observations show that the resonance is a consequence of the limited mode number we have considered. Physically, the resonance can be interpreted as a feature occurring due to the presence of an additional filter function in the system, because the limited number of modes under consideration is equivalent to the presence of a square band-pass filter inserted in the nonlinear ring resonator.

FIG. 7. The result of numerical simulation showing that the resonance (3) depends on GVD of the resonator spectrum. The dashed blue line corresponds to $D = -0.05$, while the old red line stands for $D = -0.005$. The other conditions are the same for both curves. Resonance (1) corresponds to relaxation oscillation of the system and resonance (2) is the Feshbach resonance.

IV. DISCUSSION

Our simulations show that while transfer of the optical pump amplitude noise to the noise of the fundamentally mode locked frequency comb is present in realistic nonlinear resonators, it is rather small. For instance, if a pump laser has RIN of $-140$ dBc/Hz at 10 kHz and the value of the AM-to-PM transfer function is $-75$ dB, the noise transfer can be safely neglected, since the best RF phase noise realized with a Kerr frequency comb oscillator approaches $-130$ dBc/Hz at 10 kHz [11]; that is much larger than $-225$ dBc/Hz value resulting from the noise transfer. The same conclusion applies to higher offset frequencies.

The impact of the laser phase noise, however, can be more significant. For instance, if the phase noise of the pump laser is $-50$ dBc/Hz at 10 kHz, which is a realistic value for a stabilized diode laser, and the PM-to-PM transfer function value is $-80$ dB at 10 kHz, the associated phase noise of the RF oscillator is limited by $-130$ dBc/Hz at 10 kHz. This number is already significant for high-quality RF photonic oscillators.

The situation can become even worse if modes of the resonator interact. It was shown that mode interaction changes the frequency spacing (GVD) of the resonator modes rather significantly [28-32]. The change can be so large that fundamental hyper-parametric oscillations
with soft excitation take place in a resonator with small normal GVD. The mode interaction also destroys the symmetry of the mode Q-factors, leading to several percent differences among Q-factors of adjacent modes. It may result in five to six orders of magnitude increase in the noise transfer efficiency for both amplitude and phase noise.

The highest effect of the noise transfer is expected in the vicinity of the relaxation oscillation peak (the phase noise at smaller frequency is usually determined by thermal fluctuations of the resonator). If the phase noise arising from the detuning frequency noise of the laser and the resonator mode is -50 dBc/Hz at 10 kHz and the PM-PM transfer function flat part reaches -40 dB, the resultant RF phase noise will be at the level of -90 dBc/Hz, which is a large number for, e.g., an RF Kerr comb oscillator.

Another possibility for increased efficiency of the noise transfer occurs in multi-soliton combs, which have certain analogy with harmonic mode locking. Those combs correspond to multiple optical pulses confined in the nonlinear resonator. In such a regime the power of the RF frequency signal generated by the comb on a fast photodiode is higher than in the case of the fundamentally mode locked comb. On the other hand, it was found that the sensitivity of the multi soliton combs to the modulation of the pump light is several orders of magnitude stronger than the sensitivity of the fundamentally mode locked comb discussed in this paper [27].

Therefore, to achieve highly spectrally pure RF signals generated in a photonic oscillator based on a Kerr frequency comb, one needs to operate the device in the fundamentally mode locked regime and also use a pump laser with as low amplitude and phase noise as possible.

V. CONCLUSION

We have analyzed the impact of the noise of continuous wave pump light on the noise of the RF signal generated by a Kerr frequency comb based photonic oscillator. Using both analytical and numerical simulations we have shown that the optical noise can be transferred to the comb noise in the case where the resonator spectrum has a certain asymmetry, such as nonzero third- and higher order dispersion and frequency-dependent quality factor. The noise transfer can already be significant at the current level of the technology and future experimental research is needed to further explore the impact of the optical pump noise in Kerr frequency comb oscillators.

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[1] T. M. Fortier M. S. Kirchner, F. Quinlan, J. Taylor, J. C. Bergquist, T. Rosenband, N. Lemke, A. Ludlow, Y. Jiang, C. W. Oates, and S. A. Diddams, “Generation of ultrastable microwaves via optical frequency division,” Nature Photonics 5, 425-429 (2011).
[2] F. Quinlan, T. M. Fortier, H. Jiang, A. Hati, C. Nelson, Y. Fu, J. C. Campbell, and S. A. Diddams, “Exploiting shot noise correlations in the photodetection of ultrashort optical pulse trains,” Nature Photonics 7, 290–293 (2013).
[3] T. M. Fortier, F. Quinlan, A. Hati, C. Nelson, J. A. Taylor, Y. Fu, J. Campbell, and S. A. Diddams, “Photonic microwave generation with high-power photodiodes,” Opt. Lett. 38, 1712-1714 (2013).
[4] A. A. Savchenkov, A. B. Matsko, D. Strekalov, M. Mohageg, V. S. Ilchenko, and L. Maleki, "Low threshold optical oscillations in a whispering gallery mode CaF2 resonator," Phys. Rev. Lett. 93, 243905 (2004).
[5] A. A. Savchenkov, A. B. Matsko, V. S. Ilchenko, I. Solomatine, D. Seidel, and L. Maleki, "Tunable optical frequency comb with a crystalline whispering gallery mode resonator," Phys. Rev. Lett. 101, 093902 (2008).
[6] A. A. Savchenkov, E. Rubiola, A. B. Matsko, V. S. Ilchenko, and L. Maleki, “Phase noise of whispering gallery photonic hyper-parametric microwave oscillators,” Opt. Express 16, 4130-4144 (2008).
[7] S. B. Papp and S. A. Diddams, "Spectral and temporal characterization of a fused quartz microresonator optical frequency comb," Phys. Rev. A 84, 053833 (2011).
[8] P. DelHaye, T. Herr, E. Gavartin, M. L. Gorodetsky, R. Holzwarth, and T. J. Kippenberg, "Octave spanning tunable frequency comb from a microresonator," Phys. Rev. Lett. 107, 063901 (2011).
[9] A. A. Savchenkov, D. Eliyahu, W. Liang, V. S. Ilchenko, J. Byrd, A. B. Matsko, D. Seidel, and L. Maleki, "Stabilization of a Kerr frequency comb oscillator," Opt. Lett. 38, 2636-2639 (2013).
[10] W. Liang, D. Eliyahu, A. B. Matsko, V. S. Ilchenko, D. Seidel, L. Maleki, “Spectrally pure RF photonic source based on a resonant optical hyper-parametric oscillator,” Proc. SPIE 8960, 896010 (2014).
[11] W. Liang, D. Eliyahu, V. S. Ilchenko, A. A. Savchenkov, A. B. Matsko, D. Seidel, and L. Maleki, “Record high spectral purity with a Kerr frequency comb RF photonic oscillator,” To be published (2015).
[12] A. B. Matsko, A. A. Savchenkov, D. Strekalov, V. S. Ilchenko, and L. Maleki, “Optical hyperparametric oscillations in a whispering-gallery-mode resonator: Threshold and phase diffusion,” Phys. Rev. A 71, 033804 (2005).
[13] A. B. Matsko and L. Maleki, “On timing jitter of mode locked Kerr frequency combs,” Opt. Express 21, 28862–28876 (2013).
[14] Y. K. Chembo and N. Yu, “Modal expansion approach to optical-frequency-comb generation with monolithic whispering-gallery-mode resonators,” Phys. Rev. A 82, 033801 (2010).

[15] A. B. Matsko, A. A. Savchenkov, V. S. Ilchenko, D. Seidel, and L. Maleki, “Hard and soft excitation regimes of Kerr frequency combs,” Phys. Rev. A 85, 023830 (2012).

[16] M. Haelterman, S. Trillo, and S. Wabnitz, “Additive-modulation-instability ring laser in the normal dispersion regime of a fiber,” Opt. Lett. 17, 745-747 (1992).

[17] S. Coen and M. Haelterman, “Modulational instability induced by cavity boundary conditions in a normally dispersive optical fiber,” Phys. Rev. Lett. 79, 4139-4142 (1997).

[18] S. Coen and M. Haelterman, “Continuous-wave ultrahigh-repetition-rate pulse-train generation through modulational instability in a passive fiber cavity,” Opt. Lett. 26, 39-41 (2001).

[19] D. K. Serkland and P. Kumar, “Tunable fiber-optic parametric oscillator,” Opt. Lett. 24, 92-94 (1999).

[20] J. E. Sharping, M. Fiorentino, P. Kumar, and R. S. Windeler, “Optical parametric oscillator based on four wave mixing in microstructure fiber,” Opt. Lett. 27, 1675-1677 (2002).

[21] C. J. S. De Matos, J. R. Taylor, K. P. Hansen, “Continuous-wave, totally fiber integrated optical parametric oscillator using holey fiber,” Opt. Lett. 29, 983-985 (2004).

[22] Y. Deng, Q. Lin, F. Liu, G. P. Agrawal, and W. H. Know, “Broadly tunable femtosecond parametric oscillator using a photonic crystal fiber,” Opt. Lett. 30, 1234-1236 (2005).

[23] L. Maleki, V. S. Ilchenko, M. Mohageg, A. B. Matsko, A. A. Savchenkov, D. Seidel, N. P. Wells, J. C. Campanaro, and B. Jaduszliwer, “All-optical integrated atomic clock,” 2010 IEEE International Frequency Control Symposium (FCS), pp.119-124, 1-4 June 2010.

[24] S. Coen and M. Erkintalo, “Universal scaling laws of Kerr frequency combs,” Opt. Lett. 38, 1790-1792 (2013).

[25] T. Herr, V. Brusch, J. D. Jost, C. Y. Wang, N. M. Kondratiev, M. L. Gorodetsky, and T. J. Kippenberg, “Temporal solitons in optical microresonators,” Nature Photonics 8, 145-152 (2014).

[26] A. B. Matsko, W. Liang, A. A. Savchenkov, and L. Maleki, “Chaotic dynamics of frequency combs generated with continuously pumped nonlinear microresonators,” Opt. Lett. 38, 525-527 (2013).

[27] A. B. Matsko and L. Maleki, “Feshbach resonances in Kerr frequency combs,” To be published (2015).

[28] A. A. Savchenkov, A. B. Matsko, W. Liang, V. S. Ilchenko, D. Seidel, and L. Maleki, “Kerr frequency comb generation in overmoded resonators,” Opt. Express 20, 27290-27298 (2012).

[29] T. Herr, V. Brusch, J. D. Jost, I. Mirgorodskiy, G. Lilachev, M. L. Gorodetsky, and T. J. Kippenberg, “Mode spectrum and temporal soliton formation in optical microresonators,” arXiv.org>physics>arXiv:1311.1716 (2013).

[30] Y. Liu, Y. Xuan, X. Xue, P.-H. Wang, S. Chen, A. J. Metcalf, J. Wang, D. E. Leaird, M. Qi, and A. M. Weiner, “Investigation of Mode Coupling in Normal Dispersion Silicon Nitride Microresonators for Kerr Frequency Comb Generation,” lanl.arXiv.org>physics>arXiv:1405.6225 (2014).

[31] X. Xue, Y. Xuan, Y. Liu, P.-H. Wang, S. Chen, J. Wang, D. E. Leaird, M. Qi, and A. M. Weiner, “Mode interaction aided soft excitation of dark solitons in normal dispersion microresonators and offset-frequency tunable Kerr combs,” lanl.arXiv.org>physics>arXiv:1404.2865 (2014).

[32] S. Ramelow, A. Farsi, S. Clemmen, J. S. Levy, A. R. Johnson, Y. Okawachi, M. R. E. Lamont, M. Lipson, and A. L. Gaeta, “Strong polarization mode coupling in microresonators,” Opt. Lett. 39, 5134-5137 (2014).