ON NEW FAMILIES OF WAVELETS AND GABOR ANALYSIS

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Abstract. We construct two new families of wavelets: One family of frames which is well suited for frequency localized signals and interpolates between the standard wavelet frames and a version of a Gabor type frame. The second family is well suited for time localized signals and interpolated between a version of a wavelet frame and a standard Gabor frame. In particular we approximate Gabor analysis by wavelets. Our construction is based on certain realizations of the unitary representations of the Heisenberg group and of the affine group on \( L^2(\mathbb{R}) \). The main technical tool that we use for the interpolation procedures is contraction of Lie groups representations.

1. Introduction

Time-frequency analysis is one of the most important branches of signal analysis, and probably the most important one excluding Fourier analysis. The basic idea is to take a signal (some function of time) and to store all the relevant information in a way that for each time interval encodes those frequencies that appear. Usually one also wants an effective way of making changes in the stored data and a way of reconstructing the signal.

The two prominent types of time-frequency analyses are those which are known as wavelets analysis and Gabor analysis. Both are special cases of analysis which is based on a certain family of vectors that are called coherent states\(^{[1,2]}\). We shall review the needed parts of the theory of coherent states, wavelets analysis and Gabor analysis in the next sections.

In \(^{[3]}\) the following question was considered: Let \( V \) be the Hilbert space \( L^2(\mathbb{R}) \) with the standard inner product \( \langle f,g \rangle = \int_{\mathbb{R}} f(x)g(x)dx \); how can one find a sequence of functions \( \{g_n\}_{n \in \mathbb{N}} \subseteq V \) such that for any \( f \in V \)

\[
    f = \sum_{n \in \mathbb{N}} \langle g_n, f \rangle g_n
\]

(1.1)

Obviously any orthonormal basis of \( V \) satisfies (1.1). but as shown in \(^{[3]}\), there are many more solutions that do not constitute an orthonormal basis. These other discrete collections of vectors constitute what is known in the physics literature as an overcomplete set, i.e., a collection of vectors in Hilbert space that has a dense proper subset. In some cases these new collections of functions are better suited in practice than an orthonormal basis (see the introduction in \(^{[3]}\)). Schematically, these collections are constructed in \(^{[3]}\) as follows: they start with a unitary irreducible representation \( \pi \) of a group \( G \), which is either the Weyl-Heisenberg group or the affine group (also known as the "ax + b" group) on \( V \). They pick a certain discrete subset \( \{x_{n,m}\}_{n,m \in \mathbb{Z}} \subseteq G \) and an admissible function \( g \in V \) (we define admissibility in section \(^{[3]}\)). They consider admissible functions \( g \) that satisfy some
regularity conditions, with the most significant one that is (essentially) given by

$$\sum_{n \in \mathbb{N}} |\pi(x_{n,m})g(x)|^2 = C$$

where $C$ is a positive constant that is not dependent on $x \in \mathbb{R}$ and $m \in \mathbb{Z}$. Under these conditions the collection of functions $\{g_{n,m} = \pi(x_{n,m})g\}_{n,m \in \mathbb{Z}^2}$ constitutes a tight frame (to be defined below) and as such also satisfies equation (1.1). In the case of the Weyl-Heisenberg group the analysis that arose is of Gabor type, while for the affine group the analysis is of wavelets type. For the Weyl-Heisenberg group they show how one can find an admissible $g \in V$ that is compactly supported and generates a solution for (1.1). Such admissible functions are better suited for analysis of signals that are spread over a finite time interval. For the two-dimensional affine group they show how one can find an admissible $g \in V$ such that its Fourier transform is compactly supported and generates solution for (1.1). Such admissible functions are better suited for analysis of signals that are band limited, i.e., signals with bounded frequencies. In some sense these two analyses are complementary to each other but it is not clear how are they related.

In this work we show how Gabor analysis can be interpreted as a limit of wavelets analysis. This means that in practice Gabor analysis can be approximated by wavelets analysis. The groups that we consider here are a three dimensional central extension of the affine group, that we refer to as the extended affine group, and the Heisenberg group. Coherent states of the Heisenberg group give rise to the same analysis that is related to the Weyl-Heisenberg group and coherent states of the extended affine group give rise to the same analysis that is related to the affine group. The Heisenberg group can be approximated by a one parameter family of groups which are isomorphic to the extended affine group. The existence of such a family of groups is the fundamental fact that explains why, in principle, Gabor analysis can be approximated by wavelets analysis.

Some relations between Gabor analysis and wavelets analysis were established in [4, 5] by exhibiting both the Weyl-Heisenberg group and the affine group as subgroups of the four dimensional affine Weyl-Heisenberg group.

The main technical tool that we use is contraction of representations of Lie groups [6, 7, 8, 9]. Specifically we use the contraction of the unitary irreducible representations of the extended affine group on $L^2(\mathbb{R})$ to the unitary irreducible representations of the Heisenberg group on the same space. In [10], an approximation of the narrowband cross-ambiguity function by wideband cross-ambiguity functions is interpreted in terms of contraction of the extended affine group to the Heisenberg group.

This paper is organized as follows: Our main results are given in section 2. In section 3 we summarize briefly the theory of coherent states and frames. Afterwards we describe the unitary irreducible representations and coherent states of the Heisenberg group and of the extended affine group. In section 6 we show how to contract the extended affine group to the Heisenberg group. We then show how to contract the unitary irreducible representations, coherent states families, resolutions of the identity and tight frames. In section 7 we show how one can take a family of tight frames that is based upon a compactly supported function and to transform it to a family of tight frames that is based upon a function with compactly supported Fourier transform.
2. Main results

2.1. Frames for time localized signals.

Theorem 1. [Frames that are associated with the Heisenberg group (Daubechies et al. [3])] For any $A \in \mathbb{R}^*$, $B \in \mathbb{R}$, $L > 0$ and $0 \neq \psi \in L^2(\mathbb{R}, dx)$ that is supported in the interval $[-L, L]$, let $p_0 = \frac{2}{\pi}$ and fix $q_0 \in \mathbb{R}^*$ such that $|q_0 p_0| < 2\pi$. Suppose that there exists a positive constant $\chi$ such that for all $x \in \mathbb{R}$

\begin{equation}
\sum_{n \in \mathbb{Z}} |\psi(x + n q_0)|^2 = \chi.
\end{equation}

Then the sequence $\left\{ |\psi^A_n(x)| \right\}_{n \in \mathbb{Z}}$, where

\begin{equation}
|\psi^A_n(x)| = e^{-A n q_0} e^{-B n p_0} e^{-i A x} \psi(x + n q_0),
\end{equation}

constitutes a tight frame. Moreover, for any $f \in L^2(\mathbb{R})$

\begin{equation}
f = \frac{1}{2\chi L} \left( \sum_{n, m \in \mathbb{Z}} \langle \psi^A_n(x) | f \rangle \right) |\psi^A_n(x)|^2
\end{equation}

Theorem 2. [Frames that are associated with the extended affine group]
Under the assumptions of theorem 1 let $a(\epsilon) = a_0 + \frac{A}{\epsilon}$, $b(\epsilon) = b_0 - \frac{B}{\epsilon}$ such that $a_0 + b_0 = B$. Then, for any $\epsilon \in (0, 1]$, the sequence $\left\{ |\psi^A_{\epsilon}(n, m)| \right\}_{n, m \in \mathbb{Z}}$, where

\begin{equation}
|\psi^A_{\epsilon}(n, m)| = e^{i a(\epsilon) \gamma_m \epsilon n q_0} e^{i b(\epsilon) \beta_m \epsilon n q_0} e^{-i \epsilon n q_0} \psi(x + n q_0)
\end{equation}

with

\begin{align}
\beta_m &= -e^{-n q_0} \frac{\pi m}{b(\epsilon) \sinh(\epsilon L)} \\
\gamma_m &= \beta_m \epsilon n q_0 e^{-n q_0} - 1
\end{align}

constitutes a tight frame. Moreover, for any $f \in L^2(\mathbb{R})$ and any $\epsilon \in (0, 1]$

\begin{equation}
f = \frac{\epsilon}{2 \chi \sinh(\epsilon L)} \left( \sum_{n, m \in \mathbb{Z}} \langle \psi^A_{\epsilon}(n, m) | f \rangle \right) |\psi^A_{\epsilon}(n, m)|^2
\end{equation}

Theorem 3. [Contraction of time localized frames] Under the assumptions of theorem 1 and theorem 2 for any $f \in L^2(\mathbb{R})$ and $n, m \in \mathbb{Z}$,

\begin{equation}
\lim_{\epsilon \to 0^+} |\psi^A_{\epsilon}(n, m)| = |\psi^A(n, m)| \quad \text{in} \quad L^2(\mathbb{R}, dx)
\end{equation}

\begin{equation}
\lim_{\epsilon \to 0^+} \langle \psi^A_{\epsilon}(n, m) | f \rangle = \langle \psi^A(n, m) | f \rangle
\end{equation}

In addition for any $\epsilon \in (0, 1]$

\begin{equation}
f = \frac{1}{2 \chi L} \left( \sum_{n, m \in \mathbb{Z}} |\psi^A_{\epsilon}(n, m)|^2 \right) |\psi^A_{\epsilon}(n, m)|^2 =
\end{equation}

\begin{equation}
\sum_{n, m \in \mathbb{Z}} \frac{1}{2 \chi L} \left( \langle \psi^A_{\epsilon}(n, m) | f \rangle \right) |\psi^A_{\epsilon}(n, m)|^2
\end{equation}
2.2. Frames for frequency localized signals.

Theorem 4. [Frames that are associated with the Heisenberg group] Under the assumptions of theorem 4, let
\[ \psi_0(x) = \begin{cases} \frac{1}{\sqrt{2}} \psi(-\ln(x)), & x > 0 \\ 0, & x \leq 0 \end{cases} \]

Let \( \tilde{\psi}(x) \) be the inverse Fourier transform of \( \psi_0 \), i.e., \( \tilde{\psi}(x) = \mathcal{F}^{-1}(\psi_0)(x) \). Then the sequence \( \left\{ \hat{\psi}_{\psi}^{A,B}(n,m) \right\}_{n,m \in \mathbb{Z}} \), where
\[ \hat{\psi}_{\psi}^{A,B}(n,m) = e^{i(Amp_0 + B \frac{\nu_0}{2})} \tilde{\psi}(x) \left( \frac{\lambda_0 + A^* x}{x} \right) \]
constitutes a tight frame. Moreover, for any \( f \in L^2(\mathbb{R}) \)
\[ f = \sum_{n,m \in \mathbb{Z}} \frac{1}{2\chi}(\hat{\psi}_{\psi}^{A,B}(n,m)|f)(\hat{\psi}_{\psi}^{A,B}(n,m)) \]

Theorem 5. [Frames that are associated with the extended affine group] Under the assumptions of theorem 4 and theorem 2, for any \( \epsilon \in (0,1] \), let \( \psi_\epsilon(x) \) be given by
\[ \psi_\epsilon(x) = \frac{1}{\sqrt{2}} \psi(-\ln(x)), \quad x > 0, \]
\[ 0, \quad x \leq 0. \]

Then the sequence \( \left\{ \hat{\psi}_{\psi}^{\epsilon(a),\epsilon(b)}(n,m) \right\}_{n,m \in \mathbb{Z}} \), where
\[ \hat{\psi}_{\psi}^{\epsilon(a),\epsilon(b)}(n,m) = e^{i\epsilon(a)\gamma_0 + i\epsilon(b)\beta_0} \mathcal{F}^{-1}(e^{i\epsilon(b)\beta_0} e^{-nq_0 + \epsilon A^*w} \tilde{\psi}_{\psi}(x))(x) \]
constitutes a tight frame. Moreover, for any \( f \in L^2(\mathbb{R}) \) and any \( \epsilon \in (0,1] \)
\[ f = \sum_{n,m \in \mathbb{Z}} \frac{\epsilon}{2\chi \sinh(\epsilon L)} (\hat{\psi}_{\psi}^{\epsilon(a),\epsilon(b)}(n,m)|f)(\hat{\psi}_{\psi}^{\epsilon(a),\epsilon(b)}(n,m)) \]

Remark 1. The case of \( \epsilon = 1 \) of theorem 4 is proved in [3].

Theorem 6. [Contraction of frequency localized frames] Under the assumptions of theorem 4 and theorem 2, for any \( f \in L^2(\mathbb{R}) \) and \( n,m \in \mathbb{Z} \),
\[ \lim_{\epsilon \to 0^+} \hat{\psi}_{\psi}^{\epsilon(a),\epsilon(b)}(n,m) = \hat{\psi}_{\psi}^{A,B}(n,m) \quad \text{in} \quad L^2(\mathbb{R},dx) \]
\[ \lim_{\epsilon \to 0^+} (\hat{\psi}_{\psi}^{\epsilon(a),\epsilon(b)}|f) = (\hat{\psi}_{\psi}^{A,B}|f) \]
In addition, for any \( \epsilon \in (0,1] \)
\[ f = \sum_{n,m \in \mathbb{Z}} \frac{\epsilon}{2\chi \sinh(\epsilon L)} (\hat{\psi}_{\psi}^{\epsilon(a),\epsilon(b)}(n,m)|f)(\hat{\psi}_{\psi}^{\epsilon(a),\epsilon(b)}(n,m)) = \sum_{n,m \in \mathbb{Z}} \frac{1}{2\chi L} (\hat{\psi}_{\psi}^{A,B}(n,m)|f)(\hat{\psi}_{\psi}^{A,B}(n,m)) \]
3. Perelomov Coherent States and Frames

In this section we recall the construction of Perelomov coherent states \[1\] and review briefly the notion of frame. For further details see e.g., \[1, 2\].

Let \(\rho : G \to U(H)\) be a unitary irreducible representation of a Lie group \(G\) on a complex Hilbert space \(H\). For any nonzero \(\psi \in H\) let \(G_{\psi,\rho}\) denote the isotropy subgroup of \(\psi\), i.e., the subgroup that stabilizes the state that is determined by \(\psi\). Explicitly \(G_{\psi,\rho} = \{ g \in G | \rho(g)\psi \in \text{span}(\psi) \}\). Let \(X_{\psi,\rho} = G/G_{\psi,\rho}\) and let \(dX_{\psi,\rho}\) be a positive invariant Borel measure on \(X_{\psi,\rho}\) (we assume it exists). Let \(\sigma : X_{\psi,\rho} \to G\) be a global Borel section i.e., \(\sigma\) is Borel measurable and \(\sigma(x)G_{\psi,\rho} = x\). Then \(X_{\psi,\rho}\) parameterizes a family of vectors in \(H\) in the following way

\[
X_{\psi,\rho} \ni x \mapsto \rho(\sigma(x))\psi
\]

The vectors that constitute such a family are called generalized coherent states or Perelomov coherent states. For simplicity we will refer to these vectors as coherent states (CS). We will adopt the bra-ket notation and denote any vector \(v \in H\) by \(|v\rangle\). Similarly \(\langle u|v\rangle\) denotes the inner product of \(u, v \in H\). Our convention is that \(\langle u|v\rangle\) is linear in \(v\) and skew-linear in \(u\). The projection operator \(|v\rangle\langle v|\) is defined by \(|v\rangle\langle v|\langle u| = \langle v|u\rangle\langle v|\). By abuse of notation we will denote the coherent state that corresponds to \(x \in X_{\psi,\rho}\) by the following symbol

\[
|x\rangle = |\rho(\sigma(x))\psi\rangle = |\rho(x)\psi\rangle
\]

\(\psi\) is called admissible if

\[
C_{\psi,\rho} = \int_{X_{\psi,\rho}} |\langle \psi|x\rangle|^2 dX_{\psi,\rho}(x) < \infty
\]

Assuming admissibility, the coherent states family \(\{|x\rangle\}_{x \in X_{\psi,\rho}}\) induces the resolution of the identity

\[
Id = \frac{\|\psi\|^2}{C_{\psi,\rho}} \int_{X_{\psi,\rho}} |x\rangle\langle x|dX_{\psi,\rho}(x)
\]

i.e., for any \(u, v \in H\)

\[
\langle u|v\rangle = \frac{\|\psi\|^2}{C_{\psi,\rho}} \int_{X_{\psi,\rho}} \langle u|x\rangle\langle x|v\rangle dX_{\psi,\rho}(x)
\]

and

\[
|v\rangle = \frac{\|\psi\|^2}{C_{\psi,\rho}} \int_{X_{\psi,\rho}} \langle x|v\rangle|x\rangle dX_{\psi,\rho}(x)
\]

We note that the coherent states family depends on the section but the resolution of the identity does not. The discrete analogue for the resolution of the identity is given by the notion of tight frame, see, e.g., \[2, 3\].

**Definition 1.** Let \(H\) be a Hilbert space. A sequence of vectors in \(H\), \(\{\psi_n\}_{n=0}^{\infty}\), is called a frame if there exist positive numbers \(A, B\) such that for all \(f \in H\)

\[
A\|f\|^2 \leq \sum_{n=0}^{\infty} |\langle \psi_n|f\rangle|^2 \leq B\|f\|^2.
\]
If in addition \( A = B \) then the frame is called tight, and in that case condition (3.7) is equivalent to

\[
\frac{1}{A} \sum_{n=0}^{\infty} |\psi_n\rangle\langle\psi_n| = Id.
\]

In that case, for any \( v \in H \) we have the decomposition

\[
|v\rangle = \frac{1}{A} \sum_{n=0}^{\infty} (\langle\psi_n|v\rangle) |\psi_n\rangle.
\]

In the following, for any infinite dimensional unitary irreducible representations of the groups that we consider, we construct a tight frame as an "orbit" of an admissible fiducial vector under the action of a discrete subset of the group. This method was demonstrated in [3].

4. The Heisenberg Group: Unitary Irreducible Representations, Coherent States and Frames

We recall that the Heisenberg group is the semidirect product \( H \equiv \mathbb{R}^2 \rtimes_{\varphi} \mathbb{R} \) where \( \varphi : \mathbb{R} \to \text{Aut}(\mathbb{R}^2) \) is defined by \( \varphi_\alpha \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} x_1 + \alpha x_2 \\ x_2 \end{array} \right) \) for every \( \alpha \in \mathbb{R} \) and \( \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \in \mathbb{R}^2 \). And the product of \( (\alpha, \vec{v}) \), \( (\beta, \vec{u}) \in H \) is given by:

\[
(\alpha, \vec{v})(\beta, \vec{u}) = (\alpha + \beta, \varphi_\alpha(\vec{u}) + \vec{v}).
\]

4.1. Representation Theory of the Heisenberg group. Using "the Mackey machine" [11], one can show that, for every \( A, B \in \mathbb{R}^* \), we have a unitary irreducible representation \( \eta^{A,B} : H \to \mathcal{U}(L^2(\mathbb{R}, dx)) \)

\[
(\eta^{A,B}((c, \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right))))f(x) = e^{i[A(v_1 + x v_2) + B v_2]} f(c + x)
\]

and up to equivalence of representations, any infinite dimensional unitary irreducible representation of \( H \) is of that form.

Remark 2. \( \eta^{A_1,B_1} \cong \eta^{A_2,B_2} \) if and only if \( A_1 = A_2 \). Hence the family \( \{\eta^{A,0}\}_{A \in \mathbb{R}^*} \) exhausts all the \( \rho^{A,B} \).

4.2. Coherent states of the Heisenberg group. We recall briefly the construction of Perelomov coherent states for the unitary irreducible representations \( \eta^{A,B} \). For further details see [3]. Any nonzero \( \psi \in L^2(\mathbb{R}, dx) \) is admissible and for a generic \( \psi \in L^2(\mathbb{R}, dx) \) the isotropy subgroup relative to the representation \( \eta^{A,B} \) is given by \( Z(H) \equiv \{0, \left( \begin{array}{c} v_1 \\ 0 \end{array} \right) \}_{v_1 \in \mathbb{R}} \), the center of \( H \). Let \( X_H = X_{\psi, \eta^{A,B}} \) be the homogeneous space \( H/Z(H) \) and let \( \sigma_H : X_H \to H \) denote the natural Borel section \( \sigma_H((q, \left( \begin{array}{c} z \\ p \end{array} \right)), Z(H)) = (q, \left( \begin{array}{c} \frac{np}{z} \\ p \end{array} \right)) \). The plane \( \mathbb{R}^2 \) is naturally identified with \( X_H \) by the following rule

\[
(q, p) \mapsto \sigma_H((q, \left( \begin{array}{c} z \\ p \end{array} \right)), Z(H)) = (q, \left( \begin{array}{c} \frac{np}{z} \\ p \end{array} \right))
\]
and parameterizes the family of coherent states by the rule

\[(q, p) \mapsto \psi_{\sigma_H(q, p)} = \eta^{A,B}(q, \left( \frac{2p}{p} \right)) \psi\]

The Lebesgue measure on \(\mathbb{R}^2\) is an invariant measure for the action of \(H\). In cartesian coordinates of \(\mathbb{R}\), the interval \(4.3\).

For the canonical choice of the fiducial state, i.e., \((4.6)\)

\[
\begin{align*}
\int_{\mathbb{R}^2} |\psi_{\sigma_H(q, p)}(x)|^2 dq dp.
\end{align*}
\]

A straightforward calculation shows that

\[
C_{\psi, \eta^{A,B}} = \frac{2\pi \|\psi\| L^2(\mathbb{R}, dx)}{|A|}
\]

For the canonical choice of the fiducial state, i.e., \(\psi(x) = \pi^{-\frac{1}{2}} e^{-\frac{x^2}{2}}\), and the natural representation \(\eta^{A=1,B=0}\) we have

\[
|\psi_{\sigma_H(-q, p)}(x)| = \pi^{-\frac{1}{2}} e^{-i(\beta x^2)} e^{-\frac{(x-\beta)^2}{2}}
\]

4.3. Frames for the Heisenberg group. Let \(q_0\) and \(p_0\) be real numbers such that \(|q_0p_0| < 2\pi\). Consider the discrete subset of \(H\) which is given by \(H_{q_0,p_0} = \left\{ \left( nq_0, \left( \frac{mmp_0}{2} \right) \right) \mid n, m \in \mathbb{Z} \right\}\). It is well known (e.g., [1]) that for any \(0 \neq \psi \in L^2(\mathbb{R}, dx)\) the sequence

\[
\psi_{n,m}^{A,B} = \psi_{\sigma_H(nq_0, mp_0)} = \eta^{A,B}(nq_0, \left( \frac{nq_0mp_0}{2} \right)) \psi
\]

where \(n, m \in \mathbb{Z}\) is dense in \(L^2(\mathbb{R}, dx)\). Note that for \(q_0p_0 = 2\pi\) the sequence constitutes a Von Neumann lattice which was considerably studied and used in quantum mechanics, see e.g., [12, 13, 14, 15, 16].

**Proposition 1.** Let \(0 \neq \psi \in L^2(\mathbb{R}, dx)\) be such that its support is contained in the interval \([-L, L]\). Let \(p_0 = \frac{2\pi}{L}\) and fix \(q_0\) such that \(|q_0p_0| < 2\pi\) i.e., such that \(|q_0| < 2|A|L\). Suppose that there exists a positive constant \(\chi\) such that for all \(x \in \mathbb{R}\)

\[
\sum_{n \in \mathbb{Z}} |\psi(x + nq_0)|^2 = \chi.
\]

Then the sequence \(\psi^{A,B}_{(n,m)}\) constitutes a tight frame with frame constant that is equal to \(2L\chi\) i.e.,

\[
\sum_{n, m \in \mathbb{Z}} \left| \psi^{A,B}_{(n,m)} \right|^2 \left( \psi^{A,B}_{(n,m)} \right) = 2L\chi
\]

(For construction of such \(\psi\) see [10]).

**Proof.** Let \(f \in L^2(\mathbb{R}, dx)\). Recall that Parseval’s theorem implies that for \(f \in L^2([-L, L], dx) \subset L^2(\mathbb{R}, dx)\) we have

\[
2L \sum_{n \in \mathbb{Z}} |C_n|^2 \]
\[ \int_{\mathbb{R}} f(x) e^{-i\frac{2\pi}{n} x} \, dx. \] Using this we observe that

\[ (4.9) \quad \sum_{n,m \in \mathbb{Z}} \left| f|\psi_{(n,m)}^{A,B}\rangle \right|^2 = \sum_{n,m \in \mathbb{Z}} \left| \int_{\mathbb{R}} f(x) e^{i\frac{2\pi}{n} x} \psi(x + nq_0) \, dx \right|^2 = \]

\[ \sum_{n,m \in \mathbb{Z}} \left| \int_{\mathbb{R}} f(y - nq_0) e^{iy \frac{2\pi}{n} y} \psi(y) \, dy \right|^2 = 2L \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |f(y - nq_0)\psi(y)|^2 \, dy = \]

\[ 2L \chi \int_{\mathbb{R}} |f(x)|^2 \, dx = 2L \chi \| f \|^2_{L^2(\mathbb{R}, dx)} \]

(we used the change of variables \( y = x + nq_0 \) and we interchanged the order of the sum and the integral which is justified by the compactness of the support of \( \psi \).) \( \square \)

5. The Extended Affine Group; Unitary Irreducible Representations and Coherent States

5.1. The affine group. Let \( A \) be the affine group realized as the matrix group \( \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{R}^+, \beta \in \mathbb{R} \right\} \).

Using ”the Mackey machine” [11], one can show that for any \( b \in \mathbb{R}^* \) the formula

\[ (5.1) \quad \left( U^b \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \right) f(x) = \frac{1}{\sqrt{\alpha}} f\left( x + \frac{b\beta}{\alpha} \right) \]

defines a unitary irreducible representation of the affine group on the subspace of \( L^2(\mathbb{R}, dx) \) which consists of functions that their fourier transforms are supported in the positive half of the real line.

It is well known (e.g., [2]) that \( \psi \in L^2(\mathbb{R}, dx) \) is admissible if and only if

\[ (5.2) \quad \int_{0}^{\infty} |\mathcal{F}(\psi)(x)|^2 \, dx < \infty \]

where \( \mathcal{F}(\psi) \) is the Fourier transform of \( \psi \). Let \( \psi \) be admissible, then

\[ (5.3) \quad C_{\psi,U^b} = \frac{2\pi}{|b|} \| \psi \|^2_{L^2(\mathbb{R}, dx)} \int_{0}^{\infty} |\mathcal{F}(\psi)(x)|^2 \, dx < \infty \]

and \( G_{\psi,U^b} \) is the trivial group. The invariant measure on \( X_{\psi,U^b} = A \) in terms of the natural coordinates is given by \( \frac{d\alpha d\beta}{\alpha^2} \). The coherent states family is parameterized by the affine group, \( A \), according to:

\[ (5.4) \quad |\psi_{\alpha,\beta}^{b}(x) = \left( U^b \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \right) \psi(x) = \frac{1}{\sqrt{\alpha}} \psi\left( x + \frac{b\beta}{\alpha} \right) \]

and satisfies

\[ (5.5) \quad Id = \frac{\| \psi \|^2_{L^2(\mathbb{R}, dx)}}{C_{\psi,U^b}} \int_{\mathbb{R}^+ \times \mathbb{R}} |\psi_{\alpha,\beta}^{b}\rangle \langle \psi_{\alpha,\beta}^{b}| \frac{d\alpha d\beta}{\alpha^2} \]
There are two inequivalent infinite dimensional unitary irreducible representations of the affine group, one for each sign of \( b \). A frequent choice in the literature is \( U^{b = -1} \). In this representation the CS are given by

\[
|\psi_{\alpha, \beta}(x)\rangle = |\psi_{\alpha, \beta}^{b = -1}(x)\rangle = \frac{1}{\sqrt{\alpha}} \psi\left(\frac{x - \beta}{\alpha}\right)
\]

Any admissible function \( \psi \) is called a wavelet and for any \( f \in L^2(\mathbb{R}, dx) \) its wavelet transform with respect to \( \psi \) is given by:

\[
\langle \psi_{\alpha, \beta} | f \rangle = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} \psi\left(\frac{x - \beta}{\alpha}\right) f(x) dx
\]

### 5.2. Representation theory of the extended affine group

We define the extended affine group to be the direct product of the affine group, \( A \), with the additive group of the real numbers, \( \mathbb{R} \). We denote this group by \( EA \equiv A \oplus \mathbb{R} \). The product in \( EA \) is given by \((\alpha, \beta, \gamma) (x, y, z) = (ax, \alpha y + \beta, \gamma + z)\). For \( a \in \mathbb{R} \) let \( \chi_a(x) = e^{i\alpha x} \) be a unitary character of \( \mathbb{R} \). Let \( \mathcal{H} \) denote the subspace of \( L^2(\mathbb{R}, dx) \) which consists of all functions whose Fourier transforms vanish on the negative half of \( \mathbb{R} \). For every \( b \in \mathbb{R}^* \), \( a \in \mathbb{R} \) we have the unitary irreducible representation \( U^{b} \otimes \chi_a : EA \rightarrow \mathcal{U}(\mathcal{H}) \) which is given by:

\[
(U^{b} \otimes \chi_a)((\alpha, \beta, \gamma)) f(x) = e^{i\alpha \gamma} \frac{1}{\sqrt{\alpha}} f\left(\frac{x + b \beta}{\alpha}\right)
\]

Up to equivalence, any infinite dimensional unitary irreducible representation of \( EA \) is of that form. Applying Fourier transform to \( \mathcal{H} \) we obtain a unitary equivalent representation on the Hilbert space of functions in \( L^2(\mathbb{R}, dx) \) that are supported on the positive reals which is given by:

\[
(\mathcal{F} U^{b} \otimes \chi_a)((\alpha, \beta, \gamma)) f(x) = e^{i\alpha \gamma} e^{ibx} f(ax)
\]

The representation space is naturally identified with \( L^2(\mathbb{R}^+, dx) \). Applying the map \( f(x) \mapsto \sqrt{x} f(x) \) to \( L^2(\mathbb{R}^+, dx) \) we obtain a unitary equivalent representation on the Hilbert space \( L^2(\mathbb{R}^+, \frac{dx}{x}) \) which is given by:

\[
(\rho^{a,b}(\alpha, \beta, \gamma)) f(x) = e^{i\alpha \gamma} e^{ibx} f(ax)
\]

Our convention for the Fourier transform of \( f \) is \( \mathcal{F}(f)(w) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-iwx} dx \).

**Remark 3.** For \( b, c \in \mathbb{R}^* \), \( a \in \mathbb{R} \) we have \( \rho^{a,b} \simeq \rho^{a,c} \) and these are the only equivalences in the above parametrization.

For contraction of the unitary irreducible representations of \( EA \) we need another equivalent realization which is better suited for the contraction procedure. We obtain this realization as follows. For any \( \epsilon \in (0, 1] \) let \( V_\epsilon = L^2(\mathbb{R}, dx) \) and let \( T_\epsilon : L^2(\mathbb{R}^+, \frac{dx}{x}) \rightarrow V_\epsilon \) be given by \( T_\epsilon(f) = f \circ \psi_\epsilon \) where \( \psi_\epsilon(x) = e^{-\epsilon x} \). We note that \( T_\epsilon : L^2(\mathbb{R}^+, \frac{dx}{x}) \rightarrow V_\epsilon \) which is given by \( T_\epsilon(f) = f \circ \psi_\epsilon \) is a linear isomorphism. Its inverse \( T_\epsilon^{-1} \) is given by \( T_\epsilon^{-1}(f) = f \circ \psi_\epsilon^{-1} \). We also have

\[
\int_{-\infty}^{\infty} |(T_\epsilon(f))(z)|^2 dz = \int_{-\infty}^{\infty} |f(e^{-\epsilon x})|^2 dz = e^{-\epsilon} \int_{0}^{\infty} |f(x)|^2 \frac{dx}{x}.
\]
Hence $\sqrt{T_{\epsilon}}$ is an isometry. For each $\epsilon \in (0,1]$, $b \in \mathbb{R}^+$ and $a \in \mathbb{R}$, we intertwine $\rho^{a,b}$ with $T_{\epsilon}$ to get the equivalent representation

$$
(5.12) \quad \rho^{a,b}_{\epsilon} : EA \rightarrow \mathcal{U}(V_{\epsilon}) \quad \rho^{a,b}_{\epsilon}(f) = e^{a\gamma_1 - b\beta_2 x} f(-\frac{\log \alpha}{\epsilon} + x)
$$

where for every $f \in V_{\epsilon}$,

$$
(5.13) \quad T_{\epsilon} \circ \rho^{a,b}_{\epsilon}((\alpha, \beta, \gamma)) \circ T_{\epsilon}^{-1}(f)|_{x} = e^{i\alpha \gamma_1 e^{ib\beta_2 x - \epsilon \epsilon}} f(-\frac{\log \alpha}{\epsilon} + x)
$$

5.3. Coherent states of the extended affine group. In this section we describe the Perelomov coherent states for the unitary irreducible representations of $EA$.

One can show that for a generic normalized function $\psi \in L^2(\mathbb{R}^+, \frac{dx}{x})$ the isotropy subgroup relative to the representation $\rho^{a,b}$ is given by $Z(EA) = \{(1,0, \gamma) | \gamma \in \mathbb{R}\}$, the center of $EA$. Let $\psi \in L^2(\mathbb{R}^+, \frac{dx}{x})$ be a nonzero admissible fiducial state. Let $X_{EA}$ denote the homogeneous space $EA/Z(EA)$ and let $d_{X_{EA}}$ be a positive invariant Borel measure on $X_{EA}$. Let $\sigma_{EA} : X_{EA} \rightarrow EA$ be a global Borel section. The set $\mathbb{R}^+ \times \mathbb{R}$ naturally parameterizes $X_{EA}$ by the following rule

$$
(5.14) \quad (\alpha, \beta) \mapsto \sigma_{EA}(\alpha, \beta, 0)Z(H).
$$

In these coordinates (3.3) becomes

$$
(5.15) \quad Id = \frac{\|\psi\|^2}{C_{\psi, \rho^{a,b}}} \int_{\mathbb{R}^+ \times \mathbb{R}} |\psi^{a,b}_{\sigma_{EA}(\alpha, \beta, 0)}\rangle \langle \psi^{a,b}_{\sigma_{EA}(\alpha, \beta, 0)}| \frac{d\alpha d\beta}{\alpha^2}
$$

where

$$
(5.16) \quad |\psi^{a,b}_{\sigma_{EA}(\alpha, \beta, 0)}\rangle = \rho^{a,b}_{\epsilon}((\alpha, \beta, \gamma) Z(EA)) \psi
$$

For admissible $\psi \in L^2(\mathbb{R}^+, \frac{dx}{x})$, a straightforward calculation shows that

$$
(5.17) \quad C_{\psi, \rho^{a,b}} = \frac{2\pi}{|b|} \|\psi\|^2_{L^2(\mathbb{R}^+, \frac{dx}{x})} \|\psi\|^2_{L^2(\mathbb{R}^+, \frac{dx}{x})}
$$

We construct frames for the unitary irreducible representations of $EA$ in section 6.

6. Contracting the Extended Affine Group and Its Coherent States

In this section we define contraction of Lie groups and their representations and show how the extended affine group along with its unitary irreducible representations contract to the Heisenberg group and its unitary irreducible representations. Moreover we contruct the corresponding coherent state families, resolutions of the identity, and frames.

6.1. The contraction of $EA$ to $H$. Contraction of Lie algebras was first introduced by Segal [6] and Inönü and Wigner [7], as a natural notion for limit of Lie algebras. If one wants to define contraction for Lie groups, there are several approaches, see for example [17] [18] [19]. In our case (the contraction of $EA$ to $H$), since both groups are exponential, lifting the Lie algebras contraction using the exponential map is straightforward and we obtain the following definition:

**Definition 2.** Let $G_1, G_0$ be two Lie groups. Suppose that for every $\epsilon \in (0,1]$ we have a diffeomorphism $P_{\epsilon} : G_0 \rightarrow G_1$ such that the following conditions hold:

1. $P_{\epsilon}(e_0) = e_1$ where $e_0, e_1$ are the identity elements of $G_0, G_1$, respectively,
(2) For every \( x, y \in G_0 \)
\[
(6.1) \quad x \cdot_0 y = \lim_{\epsilon \to 0^+} P_\epsilon^{-1}(P_\epsilon(x) \cdot P_\epsilon(y)),
\]
then we say that \( G_0 \) is the contraction of \( G \) by \( P_\epsilon \) and we denote it by \( G_1 \xrightarrow{P(\epsilon)} G_0 \).

**Proposition 2.** For \( \epsilon \in (0, 1] \) let \( P_\epsilon : H \to EA \) be defined by
\[
(6.2) \quad P_\epsilon \left( x, \begin{pmatrix} y \\ z \end{pmatrix} \right) = \left( e^{-\epsilon x} z \left( \frac{e^{-\epsilon x} - 1}{-\epsilon x} \right) + \epsilon (y - \frac{z^2}{2}) \right)
\]
for \( x \neq 0 \) and
\[
(6.3) \quad P_\epsilon \left( 0, \begin{pmatrix} y \\ z \end{pmatrix} \right) = \lim_{\epsilon \to 0^+} \left( e^{-\epsilon x} z \left( \frac{e^{-\epsilon x} - 1}{-\epsilon x} \right) + \epsilon (y - \frac{z^2}{2}) \right) = \left( 1, z + \epsilon y \right)
\]
Then \( EA \xrightarrow{P(\epsilon)} H \) i.e., \( P_\epsilon \) realizes the contraction of \( EA \) to \( H \).

**Proof.** We first note that \( P(\epsilon) \) is smooth and note that
\[
(6.4) \quad P_\epsilon^{-1}(\alpha, \beta, \gamma) = \left( -\frac{1}{\epsilon} \ln \alpha, \left( \frac{1}{\epsilon} \left( \gamma - \beta \ln \frac{\alpha}{\alpha - 1} - \beta \frac{\ln^2 \alpha}{2(\alpha - 1)} \right) \right) \right)
\]
for \( x \neq 0 \) and
\[
(6.5) \quad P_\epsilon^{-1}(\alpha, \beta, \gamma) = \left( 0, \left( \frac{1}{\epsilon} (\gamma - \beta) \right) \right)
\]
Obviously \( P_\epsilon \) is a diffeomorphism that preserves the identity element. A straightforward calculation shows that for any \( x, y \in H \)
\[
(6.6) \quad x \cdot y = \lim_{\epsilon \to 0^+} P_\epsilon^{-1}(P_\epsilon(x) \cdot P_\epsilon(y))
\]
where the product on the left hand side is the one in \( H \) and on the right hand side the one in \( EA \). \( \square \)

6.2. The contraction of \( \rho^{a,b} \) to \( \eta^{A,B} \). In [9], for any representation of the Lie algebra of the Heisenberg group that arises from a unitary irreducible representation of the Heisenberg group, a family of representations of the Lie algebra of the extended affine that contracts to the given representation was constructed. In the following we explain the analogous picture at the level of the representations of the groups themselves.

**Proposition 3.** [Contraction of representations] Fix a representation \( \eta^{A,B} \) of the Heisenberg group on \( L^2(\mathbb{R}) \) and let \( a(\epsilon) = a_0 + \frac{\epsilon}{4}, b(\epsilon) = b_0 - \frac{\epsilon}{2} \) such that \( a_0 + b_0 = B \). Consider the family of representations of \( EA \) on \( L^2(\mathbb{R}) \) that is given by \( \left\{ \rho^{a(\epsilon), b(\epsilon)}_\epsilon \right\}_{\epsilon \in (0,1)} \). Then \( \eta^{A,B} \) is the strong contraction of the family of representations \( \left\{ \rho^{a(\epsilon), b(\epsilon)}_\epsilon \right\}_{\epsilon \in I} \) in the following sense:

1. **(Pointwise convergence)** For any \( f \in L^2(\mathbb{R}) \), any \( x \in \mathbb{R} \), and any \( g \in H \)
\[
\lim_{\epsilon \to 0^+} \left( \rho^{a(\epsilon), b(\epsilon)}_\epsilon(P_\epsilon(g)f) \right)(x) = \left( \eta^{A,B}(g)f \right)(x).
\]
(2) **(Norm convergence)** For any \( f \in L^2(\mathbb{R}) \) and any \( g \in H \)

\[
\lim_{\epsilon \to 0^+} \left( \rho_\epsilon^{a(\epsilon),b(\epsilon)}(P_\epsilon(g))f \right) = (\eta^{A,B}(g)f).
\]

**Proof.** We observe that for any \( f \in L^2(\mathbb{R}) \), any \( x \in \mathbb{R} \), and any \((c, \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right)) \in H\)

\[
\text{(6.7) } \lim_{\epsilon \to 0^+} \left( \rho_\epsilon^{a(\epsilon),b(\epsilon)} \left( P_\epsilon \left( c, \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) \right) \right) f \right)(x) =
\]

\[
\lim_{\epsilon \to 0^+} \left( \rho_\epsilon^{a(\epsilon),b(\epsilon)} \left( e^{-\epsilon c} v_2 \left( \frac{e}{-\epsilon} \right) v_2 + c(v_1 - \frac{e}{\epsilon}) \right) f \right)(x) =
\]

\[
\lim_{\epsilon \to 0^+} e^{iA[2(v_2 + c(v_1 + v_2)]}f(c + x) = \eta^{A,B} \left( c, \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) \right) (f)(x)
\]

For condition (2) we observe that

\[
\text{(6.8) } \left\| \rho_\epsilon^{a(\epsilon),b(\epsilon)} \left( P_\epsilon \left( c, \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) \right) \right) f - \eta^{A,B} \left( c, \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) \right) f \right\|^2 =
\]

\[
\int_{\mathbb{R}} |f(x + c)|^2 |e^{iA[2(v_2 + c(v_1 + v_2)]} - e^{iA[2(v_2 + A(v_1 + v_2)]}|^2 dx \leq
\]

\[
\int_{\mathbb{R}} 2|f(x + c)|^2 dx < \infty
\]

Hence by Lebesgue’s dominated convergence theorem

\[
\text{(6.9) } \lim_{\epsilon \to 0^+} \left\| \rho_\epsilon^{a(\epsilon),b(\epsilon)} \left( P_\epsilon \left( c, \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) \right) \right) f - \eta^{A,B} \left( c, \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) \right) f \right\|^2 =
\]

\[
\int_{\mathbb{R}} \lim_{\epsilon \to 0^+} |f(x + c)|^2 |e^{iA[2(v_2 + c(v_1 + v_2)]} - e^{iA[2(v_2 + A(v_1 + v_2)]}|^2 dx
\]

= 0

\[ \square \]

### 6.3. Contraction of coherent states

We can reinterpret the contraction \( EA \overset{P_\epsilon}{\to} H \) as a continuous family of Lie groups \( \{G_\epsilon\}_{\epsilon \in [0,1]} \), such that for \( \epsilon \in (0,1] \) \( G_\epsilon \) is isomorphic to \( EA \) and and \( \tilde{G}_0 = H \). More specifically, for \( \epsilon \in (0,1] \) let \( G_\epsilon \) be the group with \( \mathbb{R} \times \mathbb{R}^2 \) as the underlying smooth manifold and a product that for any \( x, y \in \mathbb{R} \times \mathbb{R}^2 \) is given by

\[
(x \cdot \epsilon y = P_{\epsilon^{-1}}(P_\epsilon(x) \cdot P_\epsilon(y))
\]

where the product on the right hand side is taking place in \( EA \). As we already saw,

\[
\text{(6.10) } \lim_{\epsilon \to 0^+} x \cdot \epsilon y = x \circ y
\]

where the product on the right hand side is in \( H \). Hence, we have a family of Lie products over the smooth manifold \( \mathbb{R} \times \mathbb{R}^2 \) that vary continuously in \( \epsilon \). In the following we use this continuous family of groups in order to contract coherent states families and tight frames.

The map \( \tilde{P}_\epsilon : G_\epsilon \to EA \) is an isomorphism of Lie groups and hence the map \( \tilde{\rho}_\epsilon^{a,b} = \rho_\epsilon^{a,b} \circ \tilde{P}_\epsilon \) defines a representation of \( G_\epsilon \) on \( V_\epsilon \). For generic \( \psi \in V_\epsilon = L^2(\mathbb{R}, dx) \), the isotropy subgroup of \( G_\epsilon \) for the representation \( \tilde{\rho}_\epsilon^{a,b} \) is \( Z(G_\epsilon) = \)
The center of $G_{\epsilon}$, i.e., the convergence is pointwise and also in $L^2$. Let $X_\epsilon$ denote the homogeneous space $G/G_{\epsilon}$ and let $\sigma_\epsilon : X_\epsilon \rightarrow G_{\epsilon}$ be the global Borel section that is given by

\[
\sigma_\epsilon \left( \left( q, \left( \begin{array}{c} z \\ p \end{array} \right) \right) \right) Z(G_{\epsilon}) = \left( q, \left( \frac{qp}{2} \right) \right)
\]

As before, $\mathbb{R}^2$ parameterizes $X_\epsilon$ and hence also the CS according to

\[
(q, p) \mapsto |\psi^{a,b}_\epsilon(q, p)| = \left( \widetilde{\rho}_\epsilon^{a,b} \left( q, \left( \frac{qp}{2} \right) \right) \right) \psi
\]

The invariant measure on $X_\epsilon$ with respect to the action of $G_{\epsilon}$ in the coordinates $(q, p)$ is easily calculated to be

\[
eq \frac{e^{eq} - 1}{eq} \, dqd\rho
\]

And the corresponding resolution of the identity is given by

\[
Id = \frac{||\psi||^2_{L^2(\mathbb{R}, dx)}}{C_{\psi, \widetilde{\rho}_\epsilon^{a,b}}} \int_{\mathbb{R}^2} |\psi^{a,b}_\epsilon(q, p)| \langle \psi^{a,b}_\epsilon(q, p), \psi \rangle \frac{e^{eq} - 1}{eq} \, dqd\rho
\]

**Proposition 4.** For admissible nonzero $\psi \in \mathcal{V}_\epsilon$

\[
C_{\psi, \widetilde{\rho}_\epsilon^{a(\epsilon), b(\epsilon)}} = \frac{2\pi}{|A - b_0\epsilon|} \frac{||\psi||^2_{L^2(\mathbb{R}, dx)}}{A - b_0\epsilon} \int_{\mathbb{R}} |\psi(x)|^2 \, e^{eq} \, dx = \frac{2\pi}{|A - b_0\epsilon|} \frac{||\psi||^2_{L^2(\mathbb{R}, dx)}}{A - b_0\epsilon} \frac{||\psi||^2_{L^2(\mathbb{R}, e^{eq} \, dx)}}{A - b_0\epsilon}
\]

The proof is given in the appendix.

**Proposition 5.** \textbf{[Contraction of coherent states]} Let $a(\epsilon), b(\epsilon)$ be as in proposition 4, then for any $x \in \mathbb{R}$, $(q, p) \in \mathbb{R}^2$

\[
\lim_{\epsilon \to 0^+} |\psi^{a(\epsilon), b(\epsilon)}_\epsilon(q, p)| = |\psi^{a,b}_{\sigma_H(q,p)}(x)
\]

and

\[
\lim_{\epsilon \to 0^+} \langle \psi^{a(\epsilon), b(\epsilon)}_\epsilon(q, p), \psi \rangle = \langle \psi^{a,b}_{\sigma_H(q,p)}, \psi \rangle
\]

i.e., the convergence is pointwise and also in $L^2(\mathbb{R}, dx)$.

**Proof.** For any $x \in \mathbb{R}$, $(q, p) \in \mathbb{R}^2$ we have

\[
\lim_{\epsilon \to 0^+} |\psi^{a(\epsilon), b(\epsilon)}_\epsilon(q, p)| = \lim_{\epsilon \to 0^+} \left( \widetilde{\rho}_\epsilon^{a(\epsilon), b(\epsilon)} \circ \sigma_\epsilon \left( q, \left( \begin{array}{c} 0 \\ p \end{array} \right) \right) \right) \psi(x) = \lim_{\epsilon \to 0^+} \left( \rho^{a(\epsilon), b(\epsilon)}_\epsilon \circ P_\epsilon \left( q, \left( \begin{array}{c} \frac{qp}{2} \\ p \end{array} \right) \right) \right) \psi(x) = \eta^{A,B}_{\sigma_H(q,p)}(x)
\]

The convergence in norm follows by the same argument that was used in the proof of proposition 4.
6.4. Contraction of the resolution of the identity. For any $\epsilon \in (0, 1]$ we have the resolution of the identity that is associated with a representation of $G_\epsilon(z \equiv E^A)$:

$$
(6.20) \quad Id = \frac{\|\psi\|^2_{L^2(\mathbb{R}, dx)}}{C_{\psi, \psi_{\,A,B}^H}} \int_{\mathbb{R}^2} |\psi_{\,A,B}^{\sigma}(\epsilon)\langle \psi_{\,A,B}^{\sigma}, (q,p)\rangle|^{\epsilon q - 1} \, dq dp = \frac{|b(\epsilon)|}{2\pi \|\psi\|^2_{L^2(\mathbb{R}, dx)}} \int_{\mathbb{R}^2} |\psi_{\,A,B}^{\sigma}(\epsilon)\langle \psi_{\,A,B}^{\sigma}, (q,p)\rangle|^{\epsilon q - 1} \, dq dp
$$

We would like to interpret its limit when $\epsilon$ goes to zero as

$$
(6.21) \quad Id = \frac{\|\psi\|^2_{L^2(\mathbb{R}, dx)}}{C_{\psi, \psi_{\,A,B}^H}} \int_{\mathbb{R}^2} |\psi_{\,A,B}^H(q,p)| \langle \psi_{\,A,B}^H(q,p) \rangle dq dp = \frac{|A|}{2\pi \|\psi\|^2_{L^2(\mathbb{R}, dx)}} \int_{\mathbb{R}^2} |\psi_{\,A,B}^H(q,p)| \langle \psi_{\,A,B}^H(q,p) \rangle dq dp
$$

which is a resolution of the identity that is associated with a representation of $H$. We formulate it in the following proposition:

**Proposition 6.** [Contraction of resolutions of the identity] Let $a(\epsilon), b(\epsilon)$ be as in proposition 5 then for any $f, g \in L^2(\mathbb{R}, dx)$

$$
(6.22) \quad \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^2} \langle f, \psi_{\,A,B}^{\sigma}(\epsilon)\langle \psi_{\,A,B}^{\sigma}, (q,p)\rangle g \rangle^{\epsilon q - 1} \, dq dp = \int_{\mathbb{R}^2} \langle f, \psi_{\,A,B}^H \rangle \langle \psi_{\,A,B}^H, g \rangle dq dp
$$

**Proof.** Since

$$
(6.23) \quad \lim_{\epsilon \to 0^+} \frac{e^{\epsilon q} - 1}{\epsilon q} = 1,
$$

then proposition 5 implies proposition 6 provided that we can interchange the limit and the integral in (6.22). We prove proposition 6 using a more direct approach. We note that proposition 5 holds if and only if

$$
(6.24) \quad \lim_{\epsilon \to 0^+} C_{\psi, \psi_{\,A,B}^{\sigma}(\epsilon)}^{\psi_{\,A,B}^{\sigma}(\epsilon)} = C_{\psi, \psi_{\,A,B}^H}
$$

By proposition 4 we have

$$
(6.25) \quad C_{\psi, \psi_{\,A,B}^{\sigma}(\epsilon)}^{\psi_{\,A,B}^{\sigma}(\epsilon)} = 2\pi \|\psi\|^2_{L^2(\mathbb{R}, dx)} \int_{\mathbb{R}} |\psi(x)|^2 \frac{1}{|A - b_0\epsilon|} e^{\epsilon x} dx
$$

and by the monotone convergence theorem we obtain

$$
(6.26) \quad \lim_{\epsilon \to 0^+} C_{\psi, \psi_{\,A,B}^{\sigma}(\epsilon)}^{\psi_{\,A,B}^{\sigma}(\epsilon)} = 2\pi \|\psi\|^2_{L^2(\mathbb{R}, dx)} \int_{\mathbb{R}} \lim_{\epsilon \to 0^+} |\psi(x)|^2 \frac{1}{|A - b_0\epsilon|} e^{\epsilon x} dx = 2\pi \frac{\|\psi\|^2_{L^2(\mathbb{R}, dx)}}{|A|} = C_{\psi, \psi_{\,A,B}^H}
$$

□
6.5. Frames with compactly supported fiducial states and their contractions. In this section, we construct tight frames for the unitary irreducible representations of the extended affine group. We show how the tight frames of the extended affine group contract (in a sense that is explained below) to tight frames of the Heisenberg group.

**Proposition 7.** [Tight frames for EA] Let \( a(\epsilon), b(\epsilon) \) be as in proposition 3 Under the assumptions of proposition 4 for any \( \epsilon \in (0, 1) \) let \( \text{EA}_{q_0,p_0}(\epsilon) \) be the discrete subset of \( \text{EA} \) that is defined by \( \{(\alpha_n(\epsilon), \beta_{mn}(\epsilon), \gamma_{mn}(\epsilon)) | m, n \in \mathbb{Z} \} \) where

\[
\alpha_n(\epsilon) = e^{-\epsilon q_0}
\]

\[
\beta_{mn}(\epsilon) = -\frac{2\pi \alpha_n(\epsilon)m}{b(\epsilon)(e^{\epsilon L} - e^{-\epsilon L})} = -e^{-\epsilon q_0} \frac{2\pi m}{b(\epsilon)(e^{\epsilon L} - e^{-\epsilon L})}
\]

\[
\gamma_{mn}(\epsilon) = \beta_{mn}(\epsilon) \ln \alpha_n(\epsilon) \frac{\alpha_n(\epsilon)}{\alpha_n(\epsilon) - 1} = e^{-\epsilon q_0} \frac{2\pi m}{b(\epsilon)(e^{\epsilon L} - e^{-\epsilon L})} \frac{\epsilon q_0}{e^{-\epsilon q_0} - 1}.
\]

The sequence \( \left\{ \psi_{(n,m)}^{\epsilon,a(\epsilon),b(\epsilon)} \right\}_{m,n \in \mathbb{Z}} \), where

\[
\psi_{(n,m)}^{\epsilon,a(\epsilon),b(\epsilon)} = \rho_{\epsilon}(\epsilon, \cdot) (\alpha_n(\epsilon), \beta_{mn}(\epsilon), \gamma_{mn}(\epsilon)) (Q_\epsilon \psi)
\]

with \( Q_\epsilon(x) = e^{-\frac{\epsilon}{2} x} \), constitutes a tight frame with frame constant that is equal to \( 2 \sinh(\epsilon L) / \epsilon \).

**Remark 4.** We refer to \( Q_\epsilon(x) \psi(x) \) as the new mother wavelet (fiducial state). We emphasize that if one uses \( \psi(x) \) as a mother wavelet instead of \( Q_\epsilon(x) \psi(x) \), then \( \psi(x) \) does not generate a tight frame for \( \epsilon \neq 0 \).

**Proof.** Let \( f \in L^2(\mathbb{R}, dx) \). Using the change of variables \( y = e^{-\epsilon x} \) and then \( z = y \alpha_n(\epsilon) \) we obtain

\[
\sum_{n,m \in \mathbb{Z}} \left| \langle f, \psi_{(n,m)}^{\epsilon,a(\epsilon),b(\epsilon)} \rangle \right|^2 =
\]

\[
\sum_{n,m \in \mathbb{Z}} \left| \int_{\mathbb{R}} f(x) e^{ib(\epsilon)\beta_{mn}(\epsilon)e^{-\epsilon x}} e^{-\frac{\epsilon}{2}(x - \ln \alpha_n(\epsilon))} \psi(x - \frac{\ln \alpha_n(\epsilon)}{\epsilon}) \, dx \right|^2 =
\]

\[
\sum_{n,m \in \mathbb{Z}} \left| \int_{\mathbb{R}} f(x) e^{ib(\epsilon)\beta_{mn}(\epsilon)e^{-\epsilon x}} e^{-\frac{\epsilon}{2}(x + nq_0)} \psi(x + nq_0) \, dx \right|^2 =
\]

\[
\sum_{n,m \in \mathbb{Z}} \left| \int_{\mathbb{R}^+} f(-\frac{\ln y}{\epsilon}) e^{ib(\epsilon)\beta_{mn}(\epsilon)y} e^{-\frac{\epsilon}{2}(-\frac{\ln y}{\epsilon} + nq_0)} \psi(-\frac{\ln y}{\epsilon} + nq_0) \, dy \right|^2 =
\]

\[
\sum_{n,m \in \mathbb{Z}} \left| \int_{\mathbb{R}^+} f(-\frac{\ln y}{\epsilon}) e^{ib(\epsilon)\beta_{mn}(\epsilon)y} e^{-\frac{\epsilon}{2}(nq_0)} \psi(-\frac{\ln y}{\epsilon} + nq_0) \, dy \right|^2 =
\]

\[
\sum_{n,m \in \mathbb{Z}} \left| \int_{e^{-\epsilon L}}^{e^{\epsilon L}} f(-\frac{\ln(z \alpha_n(\epsilon)^{-1}(\epsilon))}{\epsilon}) e^{ib(\epsilon)\beta_{mn}(\epsilon)z} e^{-\frac{\epsilon}{2}(nq_0)} \psi(-\frac{\ln z}{\epsilon} + nq_0) \, dz \right|^2 =
\]

\[
\sum_{n,m \in \mathbb{Z}} \left| \int_{e^{-\epsilon L}}^{e^{\epsilon L}} f(-\frac{\ln(z \alpha_n(\epsilon)^{-1}(\epsilon))}{\epsilon}) e^{-\frac{2\pi m}{e^{\epsilon L} - e^{-\epsilon L}} z} e^{-\frac{\epsilon}{2}(nq_0)} \psi(-\frac{\ln z}{\epsilon} + nq_0) \, dz \right|^2 =
\]
Using Parseval’s theorem we obtain that the above expression is equal to

\[
2 \sinh(\epsilon L) \sum_{n \in \mathbb{Z}} \int_{e^{-\epsilon L}}^{e^{\epsilon L}} \left| f\left(-\frac{\ln(z \alpha_n^{-1}(\epsilon))}{\epsilon}, e^{-\frac{1}{2}(nq_0)} \psi\left(-\frac{\ln z}{\epsilon}\right) \right| \frac{1}{\epsilon z} \right|^2 dz =
\]

\[
2 \sinh(\epsilon L) \epsilon \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}^+} \left| f\left(-\frac{\ln y}{\epsilon}, \psi\left(-\frac{\ln \alpha_n(\epsilon)}{\epsilon} + nq_0\right) \right| \frac{1}{\epsilon y} \right|^2 dy =
\]

\[
2 \sinh(\epsilon L) \epsilon \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}^+} \left| f(x) \psi(x + nq_0) \right|^2 dx =
\]

\[
2 \sinh(\epsilon L) \epsilon \int_{\mathbb{R}} \left| f(x) \right|^2 \sum_{n \in \mathbb{Z}} \left| \psi(x + nq_0) \right|^2 dx =
\]

\[
2 \epsilon \sinh(\epsilon L) \int_{\mathbb{R}} \left| f(x) \right|^2 dx = 2 \epsilon \sinh(\epsilon L) \left\| f \right\|^2_{L^2(\mathbb{R}, dx)}
\]

The interchange of the order of the sum and the integral is justified by the compactness of the support of \( \psi \).

\[
\text{Corollary 1. [Contraction of tight frames]} \quad \text{Let} \quad a(\epsilon), b(\epsilon) \quad \text{be as in proposition} \quad 3 \quad \text{then for any} \quad f \in L^2(\mathbb{R}, dx)
\]

\[
(6.30) \quad \lim_{\epsilon \to 0^+} \sum_{n,m \in \mathbb{Z}} \left| \langle f, \psi_{(n,m)}^{A,B} \rangle \right|^2 = \sum_{n,m \in \mathbb{Z}} \left| \langle f, \psi_{(n,m)}^{A,B} \rangle \right|^2
\]

where the right hand side is given in proposition 7.

Proof.

\[
(6.31) \quad \lim_{\epsilon \to 0^+} \sum_{n,m \in \mathbb{Z}} \left| \langle f, \psi_{(n,m)}^{A,B} \rangle \right|^2 = \lim_{\epsilon \to 0^+} 2 \epsilon \sinh(\epsilon L) \left\| f \right\|^2_{L^2(\mathbb{R}, dx)} =
\]

\[
2 \epsilon L \left\| f \right\|^2_{L^2(\mathbb{R}, dx)} = \sum_{n,m \in \mathbb{Z}} \left| \langle f, \psi_{(n,m)}^{A,B} \rangle \right|^2
\]

\[
\text{Corollary 2. [Contraction of frame expansion]} \quad \text{Let} \quad a(\epsilon), b(\epsilon) \quad \text{be as in proposition} \quad 3 \quad \text{then for any} \quad f \in L^2(\mathbb{R}, dx) \quad \text{and any} \quad \epsilon \in (0, 1]
\]

\[
(6.32) \quad f = \sum_{n,m \in \mathbb{Z}} \frac{1}{2 \epsilon \sinh(\epsilon L)} \left( \langle \psi_{(n,m)}^{A,B}, f \rangle \right) \psi_{(n,m)}^{A,B} =
\]

\[
\sum_{n,m \in \mathbb{Z}} \frac{1}{2 \epsilon L} \left( \langle \psi_{(n,m)}^{A,B}, f \rangle \right) \psi_{(n,m)}^{A,B}
\]

\[
\text{□}
\]
Proof. We observe that

\[ \lim_{\epsilon \to 0^+} |\psi_{(n,m)}^{\epsilon,a(x),b(x)}(x)| = |\psi_{(n,m)}^{A,B}(x)| \]

and

\[ \lim_{\epsilon \to 0^+} |\psi_{(n,m)}^{\epsilon,a(x),b(x)}| = |\psi_{(n,m)}^{A,B}| \]

i.e., the convergence is pointwise and in the norm of \( L^2(\mathbb{R}, dx) \).

**Proposition 8. Contraction of overcomplete frame bases** Let \( a(\epsilon), b(\epsilon) \) be as in proposition 8 then for any \( x \in \mathbb{R}, (q, p) \in \mathbb{R}^2 \)

\[ \lim_{\epsilon \to 0^+} |\psi_{(n,m)}^{\epsilon,a(x),b(x)}(x)| = |\psi_{(n,m)}^{A,B}(x)| \]

Equation 6.34 follows from the dominated convergence theorem.
Corollary 3. \textbf{Contraction of truncated frame expansions} Let \( a(\epsilon), b(\epsilon) \) be as in proposition 3 then for any \( f \in L^2(\mathbb{R}, dx) \), \( \epsilon \in (0, 1] \) and \( N_1, N_2, M_1, M_2 \in \mathbb{Z} \)

\[
\lim_{\epsilon \to 0^+} \sum_{n=N_1}^{n=N_2} \sum_{m=M_1}^{m=M_2} \frac{\epsilon}{2\chi \sinh(\epsilon L)} \left( \left| \psi_{(n,m)}^{(a(\epsilon),b(\epsilon))} \right| f \right) \left| \psi_{(n,m)}^{(a(\epsilon),b(\epsilon))} \right| =
\]

where the equality is in \( L^2(\mathbb{R}, dx) \) and pointwise.

7. Frames with band limited fiducial states and their contractions

In this section we construct another realization of the family of representations \( \{ \rho_{a(\epsilon),b(\epsilon)} \}_{\epsilon \in (0,1]} \) which also contracts to the desired representation of the Heisenberg group. This realization enables us to construct families of tight frames based on functions with their Fourier transform compactly supported.

Let us recall how we constructed the representation \( \rho_{a,b}^\epsilon : \mathcal{E}A \to U(L^2(\mathbb{R}, dx)) \). First we list some useful notations

- \( \mathcal{H} = \{ f \in L^2(\mathbb{R}, dx) | f(x) = 0, \forall x < 0 \} \)
- \( S_1 : \mathcal{H} \to L^2(\mathbb{R}^+, dx) \), \( S_1(f) = F(f)|_{\mathbb{R}^+} \) where \( F(f)(w) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} f(x)e^{-iwx} dx \).
- \( S_2 : L^2(\mathbb{R}^+, dx) \to L^2(\mathbb{R}^+, \frac{1}{2} dx) \) where \( S_2(f)(x) = \sqrt{2} f(x) \).
- For \( \epsilon > 0 \), \( S_3(\epsilon) : L^2(\mathbb{R}^+, \frac{1}{2} dx) \to L^2(\mathbb{R}, dx) \), \( S_3(\epsilon)(f) = f \circ \psi_\epsilon \) where \( \psi_\epsilon(x) = e^{-\epsilon x} \).

We started with the unitary irreducible representation \( U^b \otimes \chi_a : \mathcal{E}A \to U(\mathcal{H}) \) which is given by:

\[
(U^b \otimes \chi_a((\alpha, \beta, \gamma)) f)(x) = e^{ia\gamma} \frac{1}{\sqrt{\alpha}} f(x + b/\alpha)
\]

Then we conjugated \( U^b \otimes \chi_a \) by \( S_1 \) to obtain \( \overline{\rho}_{a,b} : \mathcal{E}A \to U(L^2(\mathbb{R}^+, dx)) \) which is given by:

\[
(\overline{\rho}_{a,b}((\alpha, \beta, \gamma)) f)(x) = (S_1 U^b \otimes \chi_a((\alpha, \beta, \gamma))) S_1^{-1} f(x) = \sqrt{\alpha} e^{ia\gamma} e^{ib\alpha} f(\alpha x)
\]

Next we conjugated \( \overline{\rho}_{a,b} \) by \( S_2 \) to obtain \( \rho_{a,b} : \mathcal{E}A \to U(L^2(\mathbb{R}^+, dx)) \) which is given by:

\[
(\rho_{a,b}((\alpha, \beta, \gamma)) f)(x) = (S_2 \rho_{a,b}((\alpha, \beta, \gamma))) S_2^{-1} f(x) = e^{ia\gamma} e^{ib\alpha} f(\alpha x)
\]

And finally we conjugated \( \rho_{a,b} \) by \( S_3(\epsilon) \) to obtain \( \rho_{a,b}^\epsilon : \mathcal{E}A \to U(L^2(\mathbb{R}, dx)) \) which is given by:

\[
(\rho_{a,b}^\epsilon((\alpha, \beta, \gamma)) f)(x) = S_3(\epsilon)(\rho_{a,b}((\alpha, \beta, \gamma))) S_3^{-1}(\epsilon)(f)\big|_{\epsilon} = e^{ia\gamma} e^{ib\alpha} e^{-\epsilon x} f(-\log \frac{\alpha}{\epsilon} + x)
\]

Now we are ready to define another realization for the above representations.

Let \( I \) denote the unitary operator of \( L^2(\mathbb{R}) \) that is given by the composition \( S_3^{-1} \circ S_2^{-1} \circ S_1^{-1}(1) \). Let \( \pi_{a,b} : \mathcal{E}A \to GL(\mathcal{H}) \) be the representation that is given
by \( I \circ \rho_{\epsilon}^{a,b} \circ I^{-1} \); explicitly we have

\[
\pi^{a,b}_{\epsilon}((\alpha,\beta,\gamma))(f)(x) = e^{\epsilon i\alpha \gamma} \mathcal{F}^{-1} \left( e^{ib\beta w^\gamma} \mathcal{F}(f)(\alpha \hat{w}) \right)(x)
\]

\[
= \frac{e^{\epsilon i\alpha \gamma}}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t)e^{ib\beta w^\gamma - i\alpha \epsilon \omega t + iw \epsilon x} dt dw
\]

Using the known representation of the delta distribution, \( \frac{1}{2\pi} \int_{\mathbb{R}} e^{ik(x-y)} dk = \delta(x-y) \) we verify that \( \pi^{a,b}_{\epsilon} = U^{a,b} \otimes \chi_a \) as follows

\[
\pi^{a,b}_{\epsilon}((\alpha,\beta,\gamma))(f)(x) = \frac{e^{\epsilon i\alpha \gamma}}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t)e^{ib\beta w^\gamma - i\alpha \epsilon \omega t + iw \epsilon x} dt dw = \frac{e^{\epsilon i\alpha \gamma}}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t)e^{iw(b\beta - \alpha \epsilon \omega t + x)} dt dw = e^{\epsilon i\alpha \gamma} \frac{1}{\sqrt{\alpha}} f\left(\frac{x + b\beta}{\alpha}\right) = (U^{a,b} \otimes \chi_a((\alpha,\beta,\gamma))f)(x)
\]

Let \( \bar{\eta}^{A,B} \) denote the representation of \( \tilde{H} \) that is given by \( I \circ \eta^{A,B} \circ I^{-1} \). A direct calculation shows that for any \( \left(c, \left(\begin{array}{c} v_1 \\ v_2 \end{array}\right)\right) \in \tilde{H} \)

\[
\left(\begin{array}{c}
\hat{\eta}^{A,B} \left(c, \left(\begin{array}{c} v_1 \\ v_2 \end{array}\right)\right) \\
\end{array}\right) f(x) = e^{i(Av_2 + Bv_1)} f \left(-\log(c + e^{Av_2-x})\right)
\]

The unitarity of \( I \) allows us to translate statements regarding the family \( \{\rho_{\epsilon}^{a,\gamma}(\cdot,\cdot)\}_{\epsilon \in [0,1]} \) and its contraction \( \eta^{A,B} \) to the family \( \{\pi^{a,\gamma}(\cdot,\cdot)\}_{\epsilon \in [0,1]} \) and its contraction \( \bar{\eta}^{A,B} \). Here we only translate the statement with regards to the tight frame:

**Proposition 9. [Band limited tight frames for \( \tilde{H} \)]** Let \( 0 \neq \psi \in L^2(\mathbb{R}, dx) \) satisfy the assumptions of proposition 7. Then the sequence \( \left\{\hat{\psi}^{A,B}_{(n,m)}\right\}_{n,m \in \mathbb{Z}} \) where

\[
\hat{\psi}^{A,B}_{(n,m)} = \bar{\eta}^{A,B} \left(nq_0, \left(\begin{array}{c} \frac{mq_p}{m^2} \\
\end{array}\right)\right)(I\hat{\psi})
\]

constitutes a tight frame with frame constant that is equal to \( 2L\chi \).

**Proposition 10. [Band limited tight frames for \( EA \)]** Let \( a(\cdot), b(\cdot) \) be as in proposition 3. Under the assumptions of proposition 7 for any \( \epsilon \in (0,1] \) let \( EA_{\psi_q,\psi_p}(\epsilon) \) be the discrete subset of \( EA \) that was defined in proposition 7 by \( \{(\alpha_{\epsilon}(\cdot),\beta_{mn}(\cdot),\gamma_{mn}(\cdot)) \mid m,n \in \mathbb{Z}\} \). Let \( \psi_{\epsilon} = IQ_{\epsilon} \psi \). The sequence \( \left\{\hat{\psi}_{(n,m)}^{a,\gamma}(\cdot,\cdot)\right\}_{m,n \in \mathbb{Z}} \) where

\[
\hat{\psi}_{(n,m)}^{a,\gamma}(\cdot,\cdot) = \pi^{a,\gamma}(\cdot,\cdot)_{\epsilon}((\alpha_{\epsilon}(\cdot),\beta_{mn}(\cdot),\gamma_{mn}(\cdot))(\psi_{\epsilon})
\]

constitutes a tight frame with frame constant that is equal to \( 2^{\sup_{\epsilon}L^{\epsilon}H} \chi \).

**Remark 5.** The tight frames given in propositions 9 and 10 are composed of functions whose Fourier transform is compactly supported, and hence suitable for analyzing band limited signals.
Corollary 4. \textbf{[Contraction of tight frames]} Let $a(\epsilon), b(\epsilon)$ be as in proposition \ref{prop:contraction} then for any $f \in L^2(\mathbb{R}, dx)$

\begin{equation}
\lim_{\epsilon \to 0^+} \sum_{n,m \in \mathbb{Z}} \left| \langle f | \hat{\psi}_{A,B}^{\epsilon}(n,m) \rangle \right|^2 = \sum_{n,m \in \mathbb{Z}} \left| \langle f | \hat{\psi}_{A,B}^{\epsilon}(n,m) \rangle \right|^2
\end{equation}

8. Concluding remarks

It should be noted that one can not directly relate the affine group to the Heisenberg group by contraction, since these groups are not of the same dimension. An essential step in our approach was to consider the extended affine group which gives rise to the same analysis as the affine group and to use the contraction of the extended affine group to the Heisenberg group.

In practice, when one analyzes signals by the standard wavelets or Gabor frames there is a hard problem of choosing the most suitable frame. The frames that are given in this paper come in natural families that vary continuously in the parameter $\epsilon$. This extra parameter, which is absent in the usual theory, gives rise to a setting in which one can potentially choose a more suitable frame.

For band limited signals our family of frames coincides for $\epsilon = 1$ with the standard wavelet frame. For $\epsilon = 0$ we obtain a new frame of Gabor type. This gives us a continuous family of frames, which can be regarded as a continuous analog of a discrete library (for an example of a discrete library see \cite{20}). Given a signal, it is a basic question to find an $\epsilon$ in the range $[0, 1]$ that gives the best approximating frame. An open question is, for which signals the new Gabor type frame ($\epsilon = 0$) is more suitable. Similar questions hold for the time localized frames.

9. Appendix

9.1. Proof of proposition \ref{prop:contraction} Using the repeated changes of variable

\begin{enumerate}
\item $(\alpha, \beta) = (e^{-\epsilon x}, p(e^{-\epsilon q} - 1))$
\item $(X, Y) = (e^{-\epsilon x}, e^{-\epsilon y})$
\item $x = -\ln X$
\item $t = -\ln \frac{1}{\epsilon}$
\item $s = x + t$
\end{enumerate}

we obtain

\begin{equation}
C_{\psi,\hat{\psi}^{\epsilon}(\cdot,\cdot)} = \int_{\mathbb{R}^2} |\langle \psi | \hat{\psi}_{\epsilon}(\cdot,\cdot) \rangle|_{L^2(\mathbb{R}, dx)}^2 e^{\epsilon q} - \frac{1}{\epsilon q} dq dp =
\end{equation}

\begin{align*}
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} & \frac{\psi(x) e^{ib(\epsilon)p} e^{\frac{-\epsilon q}{\epsilon q + 1}} e^{-\epsilon q}}{\psi(y + q) e^{\epsilon q} - \frac{1}{\epsilon q}} dxdydp dq = \\
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} & \frac{\psi(x) e^{ib(\epsilon)\beta e^{-\epsilon q}} e^{\frac{-\epsilon q}{\epsilon q + 1}} \psi(x - \frac{\ln \alpha}{\epsilon}) e^{-ib(\epsilon)\beta e^{-\epsilon q}}}{\psi(y - \frac{\ln \alpha}{\epsilon}) \frac{1 - \alpha^{-1}}{\ln \alpha} e^{\alpha(\alpha - 1)} dxdy\alpha =
\end{align*}
\[
\int_{\mathbb{R} \times \mathbb{R}} \int_{\mathbb{R}} \frac{(x)e^{ib(x)\beta - x}}{\alpha \epsilon} \frac{(y)}{\epsilon} e^{-ib(x)\epsilon - y} \frac{(y)}{\epsilon} e^{-ib(y)\beta - y} \frac{1}{\epsilon} e^{\alpha x} dx dy d\alpha d\beta = \\
\int_{\mathbb{R} \times \mathbb{R}^+} \int_{\mathbb{R}^+} \frac{\psi(\ln \frac{X}{\epsilon})}{\epsilon} e^{ib(x)\beta X} \frac{\psi(\ln \frac{X}{\epsilon} - \ln \frac{\alpha}{\epsilon})}{\epsilon} \frac{\psi(\ln \frac{Y}{\epsilon})}{\epsilon} e^{-ib(y)\beta Y} = \\
\psi(\ln \frac{Y}{\epsilon} - \ln \frac{\alpha}{\epsilon}) \frac{1}{\epsilon \alpha^2} \frac{dX}{\epsilon^X} \frac{dY}{\epsilon^Y} d\alpha d\beta = \\
\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{\psi(\ln \frac{X}{\epsilon})}{\epsilon} e^{ib(x)\beta (x - l(x)Y)} \frac{\psi(\ln \frac{X}{\epsilon} - \ln \frac{\alpha}{\epsilon})}{\epsilon} \frac{\psi(\ln \frac{Y}{\epsilon})}{\epsilon} e^{-ib(y)\beta Y} = \\
\psi(\ln \frac{Y}{\epsilon} - \ln \frac{\alpha}{\epsilon}) \frac{1}{\epsilon \alpha^2} \frac{dX}{\epsilon^X} \frac{dY}{\epsilon^Y} \frac{d\alpha}{\epsilon} = \\
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\psi(x)}{b(\epsilon)} \left[ \psi \left( \ln \frac{X}{\epsilon} - \ln \frac{\alpha}{\epsilon} \right) \right]^2 \frac{1}{\epsilon \alpha^2} \frac{dx}{\epsilon} e^{\alpha x} = \\
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\psi(x)}{b(\epsilon)} \left[ \psi \left( \ln \frac{X}{\epsilon} - \ln \frac{\alpha}{\epsilon} \right) \right]^2 \frac{1}{\epsilon \alpha^2} \frac{dx}{\epsilon} e^{\alpha x} = \\
2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\psi(x)|^2}{\epsilon} \frac{|\psi(x + t)|^2}{\epsilon} e^{\alpha x} dt = 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\psi(x)|^2}{\epsilon} \frac{|\psi(s)|^2}{\epsilon} \frac{1}{|A - b(\epsilon)|} e^{\alpha x} dx ds
\]

REFERENCES

[1] A. Perelomov  Generalized Coherent States and Their Applications, Springer-Verlag, New York, (1986).
[2] S. T. Ali, J.P. Antoine  and J.P. Gazeau, Coherent States, Wavelets and their Generalizations, Springer-Verlag, New York (1999).
[3] I. Daubechies, A. Grossmann and Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys. 27 1275 (1986).
[4] B. Torresani, Time-frequency representations: wavelet packets and optimal decomposition, Annales de l'I.H.P., section A 56 (2) 215 (1992).
[5] B. Torresani, Wavelets associated with representations of the affline Weyl-Heisenberg group, J. Math. Phys. 32 1273 (1991).
[6] I. E. Segal A class of operator algebras which are determined by groups Duke Math. J. 18 221 (1951).
[7] E. Inönü and E. P. Wigner, On the contraction of groups and their representations, Proc. Nat. Acad. Sci. U.S 39, 510 (1953).
[8] E. J. Saletan, Contraction of Lie groups J. Math. Phys. 2 1 (1961).
[9] E. M. Subag, E. M. Baruch, J. L. Birman and A. Mann, Strong contraction of the representations of the three dimensional Lie algebras, J. Phys. A: Math. Theor. 45, 265206 (2012).
[10] E. G. Kalnins and W. Miller Jr., A note on group contractions and radar ambiguity functions, Radar and sonar, Part II, 71, IMA Vol. Math. Appl. 39, Springer, New York, (1992).
[11] G. W. Mackey, Imprimitivity for representations of locally compact groups, Proc. Nat. Acad. Sci. U.S 35, 537 (1949).
[12] J. von Neumann, Mathematical Foundations of Quantum Mechanics, Princeton U. P., Princeton, NJ, (1955).
[13] V. Bargmann, P. Butera, L. Girardello, and J. R. Klauder, On the completeness of the coherent states Rep. Math. Phys. 2, 221 (1971).
[14] A. M. Perelomov, On the completeness of a system of coherent states, Teor. Mat. Fiz. 6, 213 (1971).
[15] H. Bacry, A. Grossmann, and J. Zak, Proof of completeness of lattice states in the kq representation, *Phys. Rev. B* **12**, 1118 (1975).
[16] R. Balian, Un principe d’incertitude fort en théorie du signal ou en mécanique quantique, *C. R. Acad. Sci. Paris, Ser. 2* **292**, 1357 (1981).
[17] J. Mickelsson and J. Niederle, Contractions of representations of de Sitter groups, *Commun. math. Phys.* **27**, 167 (1972).
[18] A. H. Dooley and J. W. Rice, On contractions of semisimple Lie groups, *Trans. Amer. Math. Soc.* **289**, 185 (1985).
[19] R. Fulvio, A contraction of $SU(2)$ to the Heisenberg, *Mh. Math.* **101**, 211 (1986).
[20] R. R. Coifman and M. V. Wickerhauser, Entropy-based algorithms for best basis selection, *IEEE Trans. Inform. Theory* **38**, 713 (1992).

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