Composability of partially entanglement breaking channels via entanglement assisted local operations and classical communication

Ryo Namiki
Department of Physics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan
(Dated: May 7, 2014)

We consider composability of quantum channels from a limited amount of entanglement via local operations and classical communication (LOCC). We show that any $k$-partially entanglement breaking channel can be composed from an entangled state with Schmidt number of $k$ via one-way LOCC. From the entanglement assisted construction we can reach an alternative definition of partially entanglement breaking channels.

Quantum entanglement is considered to be a resource of quantum information processing. In measurement-based one-way quantum computation any unitary-gate operation can be implemented by sequentially performing local measurements based on classical communication of measurement outcomes once a large entangled state is shared between the nodes \cite{1,2,3}. An outstanding example is a quantum teleportation gate that transmits the state of an input node to a possibly remote output node by consumption of a maximally entangled state. It establishes an identity quantum channel via local operations and classical communication (LOCC) with the help of non-local coherence due to entanglement. Beside such an LOCC implementation of unitary gates, an interesting research direction would be to identify the role of entanglement for non-unitary quantum gates or channels. It is always possible to simulate non-unitary gates by deliberately inducing errors on unitary gates. In turn, a direct construction of non-unitary gates naturally addresses the possibility to save the consumption of entangled resources.

An important class of non-unitary quantum gates in entanglement study is the class of entanglement breaking (EB) channels \cite{4}. Any EB channel has a measure-and-prepare form and can be decomposed into a local measurement and local state preparation based on the classical communication of the measurement outcome. Hence, the class of EB channels constitutes the class of non-unitary gates that can be implemented without entanglement. As a generalization of EB channels, partially entanglement breaking (PEB) channels have been introduced in Ref. \cite{5} (See also Ref. \cite{6}). The notion of PEB channels gives a classification of whole quantum channels based on Schmidt number of entanglement \cite{7}. Schmidt number of $k = 1$ identifies the class of separable states, and the corresponding class of EB channels (1-PEB channels) can be composed solely by LOCC. On the contrary, one has to consume an entangled resource in order to construct a PEB channel of Schmidt class $k$ (a $k$-PEB channel) for $k \geq 2$. It might be fascinating if the class of $k$-PEB channels constitutes the class of non-unitary gates that can be implemented by entanglement assisted LOCC protocols from entangled resources of Schmidt number $k$. However, the original definition of PEB channels is based on Schmidt number of channel’s isomorphic quantum states given by Choi-Jamiolkowski (CJ) correspondence, and has not been connected to the LOCC implementation of quantum channels.

In the context of quantum benchmark problems \cite{8,9,10}, the performance of the quantum channels can be estimated by an amount of entanglement of CJ states. It can be related to the amount of entanglement generated by a single use of the channel. To continue the study of quantum benchmarks in quantitative regime \cite{9,10}, it might be valuable to investigate the relation between this attainable amount of entanglement due to the channel action and the amount of the resource entanglement to compose the channel by using LOCC.

In this report we investigate how to implement non-unitary quantum channels via entanglement assisted LOCC. We show that any $k$-PEB channel can be constructed from a $k$-dimension maximally entangled state via LOCC. It gives an alternative definition of PEB channels. Throughout this report we assume quantum channels acting on $d$-level (qudit) quantum systems and $k \leq d$.

We begin by giving a physical meaning of CJ correspondence. Suppose that Alice prepares a maximally entangled pair $\hat{\Phi}_{AB}$ and sends system $B$ to Bob through a quantum channel $\mathcal{E}$. Then, the final state shared between Alice and Bob $J_{E} = \text{id}_{A} \otimes \mathcal{E}(\hat{\Phi}_{AB})$ is the CJ state isomorphic to the channel $\mathcal{E}$. The amount of possibly shared entanglement $J$ can be associated with the performance of the quantum channel $\mathcal{E}$ in generalization of the quantum benchmarks \cite{4,10}. In this sense, quantitative quantum benchmarks concern the obtainable amount of entanglement by a single use of the quantum channel $\mathcal{E}$.

Let us consider another situation that Alice and Bob implement the quantum channel by an entanglement assisted LOCC protocol as in Fig. 1. Suppose that an entangled pair $\psi_{AB}$ is shared between Alice and Bob (Alice and Bob possess systems $A$ and $B$, respectively). They try to compose a quantum channel $\mathcal{E}$ from the shared entanglement $\psi$ via LOCC provided that an input density operator $\rho$ is given at Alice’s station and the output $\mathcal{E}(\rho)$ has to be made at Bob’s station. If they can establish the channel, it is possible for them to share the entanglement $J$ by transmitting a half of a maximally entangled pair locally prepared by Alice through the channel. In this procedure, they start from an initial entanglement
be saturated by composing a one-way LOCC protocol.\footnote{Before to start, it might be worth to note that an inequality holds. Conversely, the existence of a representation \( \{ K_\alpha \} \) with rank(\( K_\alpha \)) \( \leq k \) directly implies that the “\( \Leftarrow \)” direction is fulfilled. By using Eq. \ref{eq:3} the set of EB channels is given by \( \mathcal{O}_1 \) and the set of whole quantum channels for qudit states is denoted by \( \mathcal{O}_d \). Unitary channels have Schmidt number of \( d \) and belong to \( \mathcal{O}_d \). They can preserve entanglement and maintain full-d-dimensional coherence. This is in sharp contrast to the class of EB channels which cannot maintain any coherence as they cannot preserve any entanglement. Hence, a higher value of Schmidt number \( k \) suggests the capability of maintaining a higher order coherence. Based on this concept Schmidt-number benchmark enables us to demonstrate the multi-dimensional coherence in quantum channels.} In this representation, the input state is transposed, and the output state \( \mathcal{E}(\rho) \) has to be prepared at Bob’s side.

\( \psi \) and end up with the final entanglement \( J \) via LOCC.\footnote{Here, we may ask whether this inequality can be saturated. Could it be possible to construct the channel from the entangled resource specified by the isomorphic states? This is true for the case of unitary channels where \( J \) is a maximally entangled state and a local unitary action after the quantum teleportation process enables us to implement any unitary channel from a maximally entangled state via LOCC. This is also true for the case of EB channels where \( J \) is separable and the channel can be implemented without entanglement. However, in general, it seems difficult to know how to compose non-unitary channels and to address the relation between \( E(\psi) \) and \( E(J) \) beyond the general inequality of Eq. \ref{eq:1}. In what follows we consider Schmidt number as a measure of entanglement and show that the inequality of Eq. \ref{eq:1} can be saturated by composing a one-way LOCC protocol.}

Before to start, it might be worth to note that an inverse map of the CJ correspondence is given by \( \mathcal{E}(\rho) = d \text{Tr}_A[(\rho^T \otimes I)J_\mathcal{E}] \) where \( \rho^T \) is transposition of \( \rho \).\footnote{In this representation, the input state is transposed, and it could not give us the physical implementation of the channel. Nevertheless, an entanglement cost for quantum channels is defined through an amount of entanglement to generate a type of CJ state for tensor product of the channel \( \mathcal{E} \otimes \alpha \) in Ref. \ref{eq:1}. Note also that there have been extensive works to quantify the amount of entanglement for an implementation of global unitary operation. }\footnote{Schmidt number of a bipartite density operator \( \sigma \) can be defined as.} \footnote{where \( \{ p_i, \varphi_i \} \) is a decomposition of \( \sigma = \sum_i p_i \varphi_i \) and Schmidt rank of a pure state \( \varphi \) is the rank of its marginal density operator, i.e., \( \text{SR}(\varphi) = \text{rank}[\text{Tr}_B(\varphi)] \). Schmidt number of a quantum channel \( \mathcal{E} \) is defined by Schmidt number of its CJ state: \( \text{SN}(\mathcal{E}) := \text{SN}(J_\mathcal{E}) \). We call a quantum channel \( \mathcal{E} \) is \( k \)-partially entanglement breaking (\( k \)-PEB) if \( \text{SN}(\mathcal{E}) \leq k \). Let us consider the Kraus representation \( \mathcal{E}(\rho) = \sum_{\alpha} K_\alpha \rho K_\alpha^\dagger \) where the set of Kraus operators \( \{ K_\alpha \} \) fulfills the closure relation \( \sum_\alpha K_\alpha^\dagger K_\alpha = I \). Then, from Eq. \ref{eq:2} Schmidt number of the quantum channel can be associated with the rank of the Kraus operators as

\begin{equation}
\text{SN}(\mathcal{E}) = \min_{\{ K_\alpha \}} \{ \text{max rank}(K_\alpha) \}. \quad \text{(3)}
\end{equation}

Therefore, if \( \text{SN}(\mathcal{E}) = k \), there exists a Kraus representation with \( \text{rank}(K_\alpha) \leq k \) for all \( \alpha \).

It might be instructive to define the set of \( k \)-PEB channels \( \mathcal{O}_k \) as follows

\begin{equation}
\mathcal{O}_k = \left\{ \mathcal{E} \mid \mathcal{E}(\rho) = \sum_{\alpha} K_\alpha \rho K_\alpha^\dagger \wedge \forall \alpha, \text{rank}(K_\alpha) \leq k \right\}. \quad \text{(4)}
\end{equation}

Note that this definition includes no explicit optimization process. We can write

\begin{equation}
\mathcal{E} \in \mathcal{O}_k \iff \exists \{ K_\alpha \} \text{s.t.}, \mathcal{E}(\rho) = \sum_{\alpha} K_\alpha \rho K_\alpha^\dagger \wedge \forall \alpha, \text{rank}(K_\alpha) \leq k. \quad \text{(5)}
\end{equation}

Now, we proceed our main result to construct PEB channels by an entanglement assisted LOCC protocol.\footnote{In sharp contrast to the class of EB channels which cannot maintain any coherence as they cannot preserve any entanglement. Hence, a higher value of Schmidt number \( k \) suggests the capability of maintaining a higher order coherence. Based on this concept Schmidt-number benchmark enables us to demonstrate the multi-dimensional coherence in quantum channels.}

\textbf{Theorem.} — Any quantum channel \( \text{SN}(\mathcal{E}) \leq k \) can be constructed by an entanglement assisted one-way LOCC protocol from an entangled state \( \psi \) with \( \text{SN}(\psi) = k \).

\textbf{Proof.} — Let us suppose \( \mathcal{E} \in \mathcal{O}_k \). Let \( \{ K_\alpha \} \) be a set of Kraus operators of \( \mathcal{E} \) with \( \text{rank}(K_\alpha) \leq k \) for any \( \alpha \). For any given Kraus representation one can associate an environment system prepared in a certain initial state \( |e\rangle_E \) and a global unitary operator \( U \) so that \( K_\alpha = \langle \alpha | U | e \rangle \) holds due to Stinespring dilution, namely, there is a decomposition with the environment and global operations.
by performing the POVM, and from the outcome tanglement by locally acting the channel on Alice's state of Schmidt number. This implies that the amount of entanglement of its CJ state as long as the equality in Eq. (1) is achieved for the case of Schmidt number. This contradicts the LOCC monotonicity of entanglement. 

Finally, we can identify the class of k-PEB channels via entanglement assisted LOCC protocols: k-PEB channels are quantum channels which can be constructed from an entanglement state of Schmidt class k via LOCC. We can observe the validity of this novel definition through a simple closed chain between three equivalent definitions in Table I (Proof of the converse statement constitutes the step iii = i). The original definition of PEB channels (Table I(i)) is given by the condition on the CJ state. The second definition (Table I(ii)) is given by the condition regarding the Kraus form. Our definition (Table I(iii)) is in a more abstract manner based on a limited amount of entanglement and LOCC. Thereby, we can enjoy three of fundamental aspects on quantum channels to define the class of PEB channels.

In conclusion we have investigated composability of quantum channels from a limited amount of entanglement via entanglement assisted LOCC. It has been shown that the class of k-PEB channels can be constructed from entangled resources of Schmidt class k via LOCC (Theorem). This gives an alternative definition of PEB channels together with an establishment of a closed chain between different definitions (Table I). We hope that the results offer a basic tool to analyze non-unitary quantum gates and play a key role to find out a general structure on quantum channels.

This work was supported by GCOE Program “The Next Generation of Physics, Spun from Universality and Emergence” from MEXT of Japan, and World-Leading Innovative R&D on Science and Technology (FIRST).

TABLE I: Three equivalent definitions of k-PEB channels. The equivalence is proven in the following chain: i ⇒ ii ⇒ iii ⇒ i. The original definition (i) implies (ii) [See Eq. (3)] when we concern the Kraus representation. Our Theorem of an entanglement assisted LOCC construction leads to ii ⇒ iii. The LOCC monotonicity leads to iii ⇒ i (See main text).

(unitary evolution $\mathcal{E}(\rho) = \text{Tr}_E[\rho \otimes \langle e \rangle \langle e \rangle_E U^{\dagger}] = \sum_{\alpha} \langle e \rangle \langle e \rangle_E U^{\dagger} \rho \otimes \langle e \rangle \langle e \rangle_E U^{\dagger} |\alpha\rangle$, where $\{|\alpha\rangle\}$ forms a positive operator valued measure (POVM) on system E and fulfills $\sum_{\alpha} |\alpha\rangle \langle \alpha| = I_E$. Then, it is possible for Alice to implement the channel action locally by preparing ancilla system with $| e \rangle \langle e \rangle$ and subsequent application of $U$ and the POVM $\{|\alpha\rangle\}$. In this scenario, Alice can conceive which one of the Kraus operators $K_{\alpha}$ is applied from the measurement outcome of $\alpha$. Since $\text{rank}(K_{\alpha}) \leq k$, Alice can also conceive which one of $k$-dimensional subspaces the conditional state belongs to. Hence, the channel output can be transported to Bob’s side faithfully by an ideal quantum teleportation of the state in the $k$-dimensional subspace. This can be executed by using a maximally entangled state with Schmidt number of $k$. Therefore, Alice and Bob can simulate the single action of any quantum channel with $\text{SN}(\mathcal{E}) \leq k$ by an entanglement assisted one-way LOCC protocol from an entangled state of Schmidt class $k$.

Note that Alice and Bob can save some amount of entanglement by locally acting the channel on Alice's station. Actually, Alice deliberately induces a decoherence by performing the POVM, and from the outcome $\alpha$ she can see that a full $d$-dimensional entanglement is not necessary. An important point is that the rank of Kraus operators determines the amount of entanglement to implement the channel. From the construction we can say that the equality in Eq. (11) is achieved for the case of Schmidt number. This implies that the amount of entanglement to implement a k-PEB channel is equivalent to the amount of entanglement of its CJ state as long as Schmidt number is concerned.

We can immediately prove the converse statement of Theorem: Any quantum channel composed from an entangled state of Schmidt number $k$ via LOCC has a Kraus representation in which the maximal rank of the Kraus operators is at most $k$. This statement is a direct result of Eq. (11). We can write a formal proof as follows.

Proof— If it is not the case, Schmidt number of the CJ state becomes more than $k$. This contradicts the LOCC monotonicity of entanglement. 

1. [D. Gottesman and I. L. Chuang, Nature (London) 402, 390 (1999).]
2. [R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 5188 (2001).]
3. [E. Knill, R. Laflamme, and G. J. Milburn, Nature (London) 409, 46 (2001).]
4. [M. Horodecki, P. Shor, and M. B. Ruskai, Rev. Math. Phys. 15, 629 (2002).]
5. [D. Chruscinski and A. Kossakowski, Open Sys. Information Dyn. 13, 17-26 (2006).]
6. [S. Huang, Phys. Rev. A 73, 052318 (2006).]
7. [B. M. Terhal and P. Horodecki, Phys. Rev. A 61 040301 (2000).]
8. [K. Hammerer, M. M. Wolf, E. S. Polzik, and J. I. Cirac, Phys. Rev. Lett. 94, 150503 (2005); J. Rigas, O. Gühne and N. Lütkenhaus, Phys. Rev. A 73, 012341 (2006); R. Namiki, M. Koashi, and N. Imoto, Phys. Rev. Lett. 101, 100502 (2008); R. Namiki, Phys. Rev. A 78, 032333 (2008); M. Owari et al., New J. Phys. 10, 113014 (2008); H. Häselter, T. Moroder, and N. Lütkenhaus, Phys. Rev.
A 77, 032303 (2008); J. Calsamiglia et al., Phys. Rev. A 79, 050301(R) (2009); H. Häseler and N. Lütkenhaus, Phys. Rev. A 80, 042304 (2009); Phys. Rev. A 81, 060306(R) (2010); R. Namiki, Phys. Rev. A 83, 042323 (2011).

[9] N. Killoran, H. Häseler, and N. Lütkenhaus, Phys. Rev. A 82 052331 (2010); N. Killoran and N. Lütkenhaus, Phys. Rev. A 83 052320 (2011).

[10] R. Namiki and Y. Tokunaga, Phys. Rev. A 85, 010305(R) (2012).

[11] G. Vidal, J. Mod. Opt. 47, 355 (2000).

[12] M.A. Nielsen and I.L. Chuang, Phys. Rev. Lett. 79, 321 (1997).

[13] S. Ishizaka and T. Hiroshima, Phys. Rev. Lett. 101, 240501 (2008).

[14] M. Berta, F.G.S.L. Brandao, M. Christandl, and S. Wehner, arXiv:1108.5357v2.

[15] J. I. Cirac, A. K. Ekert, S. F. Huelga, and C. Machiavello, Phys. Rev. A 59, 4249 (1999).

[16] J. Eisert, K. Jacobs, P. Papadopoulos, and M. B. Plenio, Phys. Rev. A 62, 052317 (2000); D. Stahlke and R. B. Griffiths, Phys. Rev. A 84, 032316 (2011); A. Soeda, P.S. Turner, and M. Murao, Phys. Rev. Lett. 107, 180501 (2011).