A RIGIDITY THEOREM ON THE SECOND FUNDAMENTAL FORM
FOR SELF-SHRINKERS

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Abstract. In Theorem 3.1 of [12], we proved a rigidity result for self-shrinkers under
the integral condition on the norm of the second fundamental form. In this paper, we
relax the such bound to any finite constant (see Theorem 4.4 for details).

1. Introduction

Self-similar solutions for mean curvature flow play a key role in the understanding
the possible singularities that the flow goes through. Self-shrinkers are type I singularity
models of the flow. Huisken made a pioneer work on self-shrinking solutions of the flow [22,
23]. Colding and Minicozzi [8] gave a comprehensive study for self-shrinking hypersurfaces
and solve a long-standing conjecture raised by Huisken.

Colding-Ilmanen-Minicozzi [9] showed that cylindrical self-shrinkers are rigid in a very
strong sense. Namely, any other shrinker that is sufficiently close to one of them on
a large, but compact set must itself be a round cylinder. See [25] by Guang-Zhu for
further results. Lu Wang in [37, 38] proved strong uniqueness theorems for self-shrinkers
asymptotic to regular cones or generalized cylinders of infinite order.

For Bernstein type theorems, Ecker-Huisken [17] and Wang [36] showed the nonexis-
tence of nontrivial graphic self-shrinking hypersurfaces in Euclidean space. For $2 \leq n \leq 6$,
Guang-Zhu showed that any smooth complete self-shrinker in $\mathbb{R}^{n+1}$ which is graphical in-
side a large, but compact, set must be a hyperplane. Ding-Xin-Yang [14] studied the sharp
rigidity theorems with the condition on Gauss map of self-shrinkers. In high codimensions,
see [2, 3, 10, 13, 26] for more Bernstein type theorems.

Le-Sesum [30] showed that any complete embedded self-shrinking hypersurface with
polynomial volume growth must be a hyperplane provided the squared norm of the second
fundamental form $|B|^2 < \frac{1}{2}$. Cao-Li [1] showed that any complete self-shrinker (with high
codimension) with polynomial volume growth must be a generalized cylinder provided
$|B|^2 \leq \frac{1}{2}$. Later, Cheng-Peng [5] removed the condition of polynomial volume growth in
the case of $|B|^2 < \frac{1}{2}$ (See [4, 6, 12, 42] for more results on the gap theorems of the norm of
the second fundamental form). In [12], Ding-Xin proved a rigidity result for self-shrinkers
if the integration of $|B|^n$ is small. In this paper, we improve the small constant to any
finite constant.

For a complete properly immersed self-shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$, Ilmanen showed that there
exists a cone $C \subset \mathbb{R}^{n+1}$ with the cross section being a compact set in $\mathbb{S}^n$ such that $\lambda \Sigma^n \to C$
as $\lambda \to 0_+$ locally in the Hausdorff metric on closed sets (see [28] Lecture 2, B, remark

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on p.8). In [35], Song gave a simple proof by "maximum principle for self-shrinkers". For high codimensions, with backward heat kernel (see [8]) we show the uniqueness of tangent cones at infinity for self-shrinkers with Euclidean volume growth in the current sense with the condition on mean curvature(see Theorem 3.3).

$\epsilon$-regularity theorems for the mean curvature flow have been studied by Ecker [15, 16], Han-Sun [19], Ilmanen [27], Le-Sesum [29]. Now we use the one showed by Ecker [16] starting from self-similar solutions, and obtain the curvature estimates for self-shrinkers, see Theorem 4.2. Combining Theorem 3.3, Theorem 4.2 and backward uniqueness for parabolic operators [21], we can show that self-shrinkers with finite integration on $|B|^n$ must be planes, which improves a previous rigidity theorem in [12]. A little more, we obtain the following Theorem.

**Theorem 1.1.** Let $M$ be an $n$-dimensional properly non-compact self-shrinker with compact boundary in $\mathbb{R}^{n+m}$, $B$ denote the second fundamental form of $M$. If

$$\lim_{r \to \infty} \int_{M \cap B_2 \setminus B_r} |B|^n d\mu = 0,$$

then $M$ must be an $n$-plane through the origin.

2. Preliminary

Let $M$ be an $n$-dimensional $C^2$-submanifold in $\mathbb{R}^{n+m}$ with the induced metric. Let $\nabla$ and $\nabla$ be the Levi-Civita connections on $M$ and $\mathbb{R}^{n+m}$, respectively. We define the second fundamental form $B$ of $M$ by

$$B(V, W) = (\nabla_Y W)^N = \nabla_Y W - \nabla_Y W$$

for any $V, W \in \Gamma(TM)$, where the mean curvature vector $H$ of $M$ is given by $H = \text{trace}(B) = \sum_{i=1}^n B(e_i, e_i)$, where $\{e_i\}$ is a local orthonormal frame field of $M$.

In this paper, $M^n$ is said to be a self-shrinker in $\mathbb{R}^{n+m}$ if its mean curvature vector satisfies

$$H = -\frac{X^N}{2},$$

where $X = (x_1, \cdots, x_{n+m}) \in \mathbb{R}^{n+m}$ is the position vector of $M$ in $\mathbb{R}^{n+m}$, and $(\cdots)^N$ stands for the orthogonal projection into the normal bundle $NM$. Let $(\cdots)^T$ denote the orthogonal projection into the tangent bundle $TM$.

We define a second order differential operator $L$ as in [8] by

$$L f = e^{\frac{|X|^2}{4}} \text{div} \left( e^{-\frac{|X|^2}{4}} \nabla f \right) = \Delta f - \frac{1}{2} \langle X, \nabla f \rangle$$

for any $f \in C^2(M)$. Let $\Delta$ be the Laplacian of $M$, then for self-shrinkers,

$$\Delta |X|^2 = 2 \langle X, \Delta X \rangle + 2 |\nabla X|^2 = 2 \langle X, H \rangle + 2n = -|X^N|^2 + 2n.$$

In [8], Colding and Minicozzi defined a function $F_{x_0, t_0}$ for self-shrinking hypersurfaces in Euclidean space. Obviously, hypersurfaces can be generalized to submanifolds naturally in this definition. Set $\Phi_t \in C^\infty(\mathbb{R}^{n+m})$ for any $t > 0$ by

$$\Phi_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.$$
For an \( n \)-complete submanifold \( M \) in \( \mathbb{R}^{n+m} \), we define a functional \( F_t \) on \( M \) by
\[
F_t(M) = \int_M \Phi_t d\mu = \frac{1}{(4\pi t)^{n/2}} \int_M e^{-\frac{|x|^2}{4t}} d\mu \quad \text{for} \quad t > 0,
\]
where \( d\mu \) is the volume element of \( M \). Sometimes, we write \( F_t \) for simplicity if no ambiguous in the text. If a self-shrinker is proper, then it is equivalent to that it has Euclidean volume growth at most by [7] and [11]. We shall only consider proper self-shrinkers in the following text.

Now we use the backward heat kernel to give a monotonicity formula for self-shrinkers with arbitrary codimensions, which is essentially same as self-shrinking hypersurfaces established by Colding-Minicozzi in [8].

**Lemma 2.1.** For any \( 0 < t_1 \leq t_2 \leq \infty \), each complete immersed self-shrinker \( M^n \) with boundary \( \partial M \) (may be empty) in \( \mathbb{R}^{n+m} \) satisfies
\[
F_{t_2}(M) - F_{t_1}(M) = -\int_{t_1}^{t_2} \left( \int_{\partial M} \langle X^T, \nu_{\partial M} \rangle \frac{\Phi_s(X)}{2s} \right) ds
+ \int_{t_1}^{t_2} \frac{1}{4s} \left( 1 - \frac{1}{s} \right) \left( \int_M |X^N|^2 \Phi_s(X) d\mu \right) ds.
\]

**Proof.** We differential \( F_t(M) \) with respect to \( t \),
\[
F'_t = (4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+1)} \int_M \left( -\frac{n}{2} + \frac{|X|^2}{4t} \right) e^{-\frac{|x|^2}{4t}} d\mu.
\]
A straightforward calculation shows (see also [11])
\[
- e^{-\frac{|x|^2}{4t}} \text{div} \left( e^{-\frac{|x|^2}{4t}} \nabla |X|^2 \right) = - \Delta |X|^2 + \frac{1}{4t} \nabla |X|^2 \cdot \nabla |X|^2
= - 2\langle H, X \rangle - 2n + \frac{1}{t} |X^T|^2
= |X^N|^2 + \frac{|X^T|^2}{t} - 2n
= \left( 1 - \frac{1}{t} \right) |X^N|^2 + \frac{|X|^2}{t} - 2n,
\]
where the third equality above uses the self-shrinkers’ equation (2.1). Then
\[
F'_t = (4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+1)} \int_M \left( -\frac{1}{4} \text{div} \left( e^{-\frac{|x|^2}{4t}} \nabla |X|^2 \right) - \frac{1}{4} \left( 1 - \frac{1}{t} \right) |X^N|^2 e^{-\frac{|x|^2}{4t}} \right) d\mu
\]
\[
= \frac{1}{4} (4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+1)} \left( -2 \int_{\partial M} \langle X^T, \nu_{\partial M} \rangle e^{-\frac{|x|^2}{4t}} - \left( 1 - \frac{1}{t} \right) \int_M |X^N|^2 e^{-\frac{|x|^2}{4t}} d\mu \right)
= \frac{-1}{2t} \int_{\partial M} \langle X^T, \nu_{\partial M} \rangle \Phi_t(X) - \frac{1}{4t} \left( 1 - \frac{1}{t} \right) \int_M |X^N|^2 \Phi_t(X) d\mu,
\]
where \( \nu_{\partial M} \) is the normal vector of \( \partial M \) in \( \Gamma(TM) \). Then we complete the proof by integration from \( t_1 \) to \( t_2 \).

Denote
\[
G_t(M) \triangleq F'_t(M) + \frac{1}{2t} \int_{\partial M} \langle X^T, \nu_{\partial M} \rangle \Phi_t(X) = - \frac{1}{4t} \left( 1 - \frac{1}{t} \right) \int_M |X^N|^2 \Phi_t(X) d\mu.
\]
The above Lemma implies \( G_t(M) \leq 0 \) for each self-shrinker and \( t \geq 1 \). If \( \partial M \) is bounded and has finite \((n-1)\)-dimensional Hausdorff measure, then the limit
\[
\lim_{t \to \infty} \left( \int_1^t G_s(M) ds \right)
\]
always exists, and is a finite negative number. Hence, it’s clear that \( \lim_{t \to \infty} F_t(M) \) exists.

3. **Uniqueness of tangent cones at infinity for self-shrinkers**

For any \( n \)-rectifiable varifold \( V \subset \mathbb{R}^{n+m} \) with multiplicity one, we define a functional \( \Xi_t \) by
\[
\Xi_t(V, f) = \frac{1}{(4\pi t)^{n/2}} \int_{spt V} f e^{-\frac{|x|^2}{4t}} d\mu
\]
for any \( t > 0 \), where \( \mu \) is a measure on \( \mathbb{R}^{n+m} \) associated with the Radon measure of \( V \) in \( \mathbb{R}^{n+m} \times G(n, n+m) \).

We suppose that \( M \) is a self-shrinker in \( \mathbb{R}^{n+m} \setminus B_R \) with boundary \( \partial M \subset \partial B_R \) for some \( R \geq 1 \) and \( \mathcal{H}^{n-1}(\partial M) < \infty \). Let \( \phi \in C^1(\mathbb{R}^{n+m} \setminus \{0\}) \) be a homogeneous function of degree zero. Namely, for any \( 0 \neq X \in \mathbb{R}^{n+m} \),
\[
\phi(X) = \phi(|X| \xi) = \phi(\xi)
\]
with \( \xi = \frac{X}{|X|} \). Then
\[
\partial_x \phi = \sum_j \left( \frac{\delta_{ij}}{|X|} - \frac{x_i x_j}{|X|^2} \right) \partial_{\xi_j} \phi,
\]
and
\[
|h|^2 = \sum_{i,j,k} \left( \frac{\delta_{jk}}{|X|^2} - \frac{x_j x_k}{|X|^2} \right) \partial_{\xi_j} \phi \partial_{\xi_k} \phi \leq |X|^{-2} \sum_j \left( \partial_{\xi_j} \phi \right)^2 \triangleq |X|^{-2} |\phi|^2.
\]
Taking the derivative of \( \Xi_t(M, \phi) \) on \( t \) gets
\[
\partial_t \Xi_t(M, \phi) = (4\pi)^{-\frac{n}{2}} t^{-\left(\frac{n}{2}+1\right)} \int_M \left( -\frac{n}{2} + \frac{|X|^2}{4t} \right) \phi e^{-\frac{|X|^2}{4t}} d\mu
\]
\[
= (4\pi)^{-\frac{n}{2}} t^{-\left(\frac{n}{2}+1\right)} \int_M \left( -\frac{\phi}{4} \text{div} \left( e^{-\frac{|X|^2}{4t}} \nabla |X|^2 \right) - \frac{\phi}{4} \left( 1 - \frac{1}{t} \right) |X|^2 e^{-\frac{|X|^2}{4t}} \right) d\mu.
\]
Combining \( X \cdot \nabla \phi = 0 \), we have
\[
\int_M \frac{\phi}{4} \text{div} \left( e^{-\frac{|X|^2}{4t}} \nabla |X|^2 \right) d\mu
\]
\[
= \int_M \frac{1}{4} \text{div} \left( \phi e^{-\frac{|X|^2}{4t}} \nabla |X|^2 \right) d\mu + \int_M \frac{1}{4} \nabla \phi \cdot \nabla |X|^2 e^{-\frac{|X|^2}{4t}} d\mu
\]
\[
= -\frac{1}{2} \int_{\partial M} \phi \langle X^T, \nu_{\partial M} \rangle e^{-\frac{|X|^2}{4t}} + \int_M \frac{1}{2} X \cdot \nabla \phi e^{-\frac{|X|^2}{4t}} d\mu
\]
\[
= -\frac{1}{2} \int_{\partial M} \phi \langle X^T, \nu_{\partial M} \rangle e^{-\frac{R^2}{4t}} - \frac{1}{2} \int_M X^N \cdot \nabla \phi e^{-\frac{|X|^2}{4t}} d\mu.
\]
Set $c_R = 2^{-1}(4\pi)^{-\frac{n}{2}}R \cdot \mathcal{H}^{n-1}(\partial M)$. Substituting (3.2) and (3.4) into (3.3) gets

$$
\partial_t \Xi_t(M, \phi) \leq 2^{-1}(4\pi)^{-\frac{n}{2}}t^{-(\frac{n}{2}+1)} \left( \int_M |X^N| \cdot |\nabla \phi| e^{-\frac{|X|^2}{4t}} d\mu \
+ |\phi|_0 Re^{-\frac{|X|^2}{4t}} \mathcal{H}^{n-1}(\partial M) \right) + |\phi|_0 |G_t(M)|
$$

$$
\leq 2^{-1}(4\pi)^{-\frac{n}{2}}t^{-(\frac{n}{2}+1)} \int_M \frac{|X^N|}{|X|} |\phi|_1 e^{-\frac{|X|^2}{4t}} d\mu + |\phi|_0 \left( |G_t(M)| + c_R t^{-(\frac{n}{2}+1)} \right)
$$

$$
(3.5) \leq |\phi|_0 \left( |G_t(M)| + c_R t^{-(\frac{n}{2}+1)} \right)
$$

$$
+ 2^{-1}(4\pi)^{-\frac{n}{2}}t^{-(\frac{n}{2}+1)} |\phi|_1 \left( \int_M |X^N|^2 e^{-\frac{|X|^2}{4t}} d\mu \right)^{1/2} \left( \int_M |X|^{-2} e^{-\frac{|X|^2}{4t}} d\mu \right)^{1/2}
$$

$$
\leq |\phi|_0 \left( |G_t(M)| + c_R t^{-(\frac{n}{2}+1)} \right)
$$

$$
+ |\phi|_1 (G_t(M))^{1/2} \sqrt{\frac{t}{t-1}} \left( (4\pi)^{-\frac{n}{2}}t^{-(\frac{n}{2}+2)} \int_M |X|^{-2} e^{-\frac{|X|^2}{4t}} d\mu \right)^{1/2}.
$$

Put $D_r = M \cap B_r$ for every $r > 0$. There is a constant $c_0 > 0$ depending only on $M$ such that for all $r > 0$

$$
\int_{D_r} 1 d\mu < c_0 r^n.
$$

Note $M \subset \mathbb{R}^{n+m} \setminus B_R$. Then for $n \geq 2$, $t \geq R^2$, one has

$$
t^{-\frac{n}{2}} \int_M \frac{t}{|X|^2} e^{-\frac{|X|^2}{4t}} d\mu \leq t^{-\frac{n}{2}} \sum_{k=-1}^{\infty} \int_{D^{2k+1}} \sqrt{t} |D^{2k+1} \sqrt{t} | e^{-\frac{|X|^2}{4t}} d\mu
$$

$$
\leq t^{-\frac{n}{2}} \sum_{k=-1}^{\infty} \frac{1}{4^k} e^{-4k} \int_{D^{2k+1}} \sqrt{t} |D^{2k+1} \sqrt{t} | 1 d\mu
$$

$$
(3.6) \leq c_0 \sum_{k=0}^{\infty} 4^{-k} e^{-4k} 2^{(k+1)n} + c_0 \sum_{k=-1}^{\infty} 4^{-k} 2^{(k+1)n}
$$

$$
\leq c_0 \sum_{k=0}^{\infty} 2^k (n-2) e^{-4k} + c_0 \sum_{k=1}^{\infty} 2^{-k} (n-2) +
$$

$$
(4\pi)^{\frac{n}{2}} c_1 (1 + \log t - 2 \log R),
$$

where $c_1$ is a constant depending only on $n, c_0$. Therefore

$$
|\partial_t \Xi_t(M, \phi)| \leq c_1 t^{-\frac{n}{2}} \left( \int_M |G_t(M)|^2 \right)^{1/2} + |\phi|_0 \left( |G_t(M)| + c_R t^{-(\frac{n}{2}+1)} \right)
$$

$$
(3.7) \leq c_1 t^{-\frac{n}{2}} \left( |\phi|_1 + c_R t^{-(\frac{n}{2}+1)} |\phi|_0 + (|\phi|_0 + |\phi|_1) |G_t(M)| \right).
$$

**Theorem 3.1.** Let $M$ be an $n$-dimensional self-shrinker in $\mathbb{R}^{n+m}$ with Euclidean volume growth and boundary $\partial M \subset \partial B_R$. If

$$
\limsup_{r \to \infty} \left( r^{1-n} \int_{M \cap B_r} |H| \right) < \infty,
$$

(3.8)
then there is a sequence \( t_i \to \infty \) such that
\[
M_{t_i} \triangleq t_i^{-1} M = \{ X \in \mathbb{R}^{n+m} \mid t_i X \in M \}
\]
converges to a cone \( C \) in \( \mathbb{R}^{n+m} \).

**Proof.** By co-area formula, we can choose \( R' > 0 \) so that \( \mathcal{H}^{n-1}(\partial M) < \infty \) with \( \partial M \subset \partial B_{R'} \). Denote \( R' \) by \( R \) for convenience. Let \( M_t \triangleq t^{-1} M = \{ X \in \mathbb{R}^{n+m} \mid tX \in M \} \) for any \( t > 0 \). Since \( M \) has Euclidean volume growth and (3.8) holds, then by compactness of varifolds, there exists an \( n \)-rectifiable varifold \( T \) in \( \mathbb{R}^{n+m} \) with integer multiplicity and a sequence of \( t_i \) such that \( M_{t_i} = t_i^{-1} M \to T \) in the sense of Radon measure (See 42.7 Theorem of [34] for example).

Denote \( \phi \) and \( \Xi_t(M, \phi) \) as above. Set \( \mu_t \) be the volume element of \( M_t \). Since
\[
(3.9) \quad \Xi_{t_1^2 R^2}(M, \phi) = \frac{1}{(4\pi t_1^2)^{n/2}} \int_M \phi e^{-\frac{|\mathbf{x}|^2}{4t_1^2}} d\mu = \frac{1}{(4\pi)^{n/2}} \int_{M_t} \phi e^{-\frac{|\mathbf{x}|^2}{4}} d\mu_t = \Xi_1(M_t, \phi),
\]
then for all \( R > 0 \)
\[
(3.10) \quad \lim_{t \to \infty} \Xi_1(M_{t R}, \phi) = \lim_{t \to \infty} \Xi_{R^2}(M_{t^2 R}, \phi) = \frac{1}{(4\pi R^2)^{n/2}} \int_T \phi e^{-\frac{|\mathbf{x}|^2}{4R^2}} d\mu_T = \Xi_{R^2}(T, \phi).
\]

Note that \( G_t(M) \) does not change sign for \( t > 1 \). Fixing \( 0 < r < R < \infty \), from (3.7) we have
\[
\left| \Xi_{t_1^2 R^2}(M, \phi) - \Xi_{t_2^2 R^2}(M, \phi) \right| \leq \int_{t_1^2 r^2}^{t_2^2 r^2} |\partial_s \Xi_s(M, \phi)| ds
\]
\[
\leq \int_{t_1^2 r^2}^{t_2^2 r^2} \left( c_1 \frac{1 + \log s}{4s(s-1)} |\phi|_1 + c_R |\phi|_0 s^{-(\frac{n}{2}+1)} + (|\phi|_0 + |\phi|_1) |G_s(M)| \right) ds
\]
\[
\leq \frac{c_1}{4} \int_{t_1^2 r^2}^{t_2^2 r^2} \frac{1 + \log s}{s(s-1)} ds + \frac{2}{n} (t_i r)^{-n-2} c_R |\phi|_0 + (|\phi|_0 + |\phi|_1) \int_{t_1^2 r^2}^{t_2^2 r^2} G_s(M) ds
\]
for all \( t_i \) with \( r t_i \geq 2 \). Since
\[
\left| \int_{t_1^2 r^2}^{t_2^2 r^2} G_s(M) ds \right| \leq \int_{t_1^2 r^2}^{t_2^2 r^2} F'_t(M) ds + \int_{t_1^2 r^2}^{t_2^2 r^2} \left( \frac{1}{2s} \int_{\partial M} (X_t, \nu_{\partial M}) \Phi_s(X) \right) ds
\]
\[
\leq \int_{t_1^2 r^2}^{t_2^2 r^2} \left( \frac{R}{2s} \mathcal{H}^{n-1}(\partial M)(4\pi s)^{-n/2} \right) ds
\]
\[
= \int_{t_1^2 r^2}^{t_2^2 r^2} \left( \frac{R}{(4\pi)^{n/2}} \mathcal{H}^{n-1}(\partial M)(t_i r)^{-n} \right) ds
\]
and \( \lim_{t \to \infty} F_t \) exists, we obtain
\[
(3.13) \quad \lim_{t \to \infty} \Xi_1(M_{t r}, \phi) = \lim_{t \to \infty} \Xi_1(M_{t R}, \phi) = \Xi_{R^2}(T, \phi).
\]
Hence
\[
(3.14) \quad \Xi_t(T, \phi) = \frac{1}{(4\pi t)^{n/2}} \int_T \phi e^{-\frac{|\mathbf{x}|^2}{4t}} d\mu_T
\]
is independent of \( t \in (0, \infty) \).

Clearly,
\[
0 < \mathcal{H}^n(T \cap B_r) \leq c_2 r^n
\]
for some constant $c_2 > 0$ and all $r > 0$. By the following lemma for $V(r) = \int_{T \cap B_r} \phi \, d\mu_T$, we conclude that
\begin{equation}
(3.15) \quad r^{-n} \int_{T \cap B_r} \phi \, d\mu_T
\end{equation}
is a constant independent of $r$. An analog argument as the proof of 19.3 in [34] implies that $T$ is a cone.

**Lemma 3.2.** Let $V(r)$ be a monotone nondecreasing continuous function on $[0, \infty)$ with $V(0) = 0$ and $V(r) \leq c_3 r^n$ for some constant $c_3 > 0$. If the quantity
\begin{equation}
(3.16) \quad \frac{1}{(4\pi t)^{n/2}} \int_0^\infty e^{-\frac{r^2}{4t}} dV(r)
\end{equation}
is a constant for any $t > 0$, then $r^{-n}V(r)$ is a constant.

**Proof.** There are constants $\kappa_0, \kappa_1 > 0$ such that for all $t > 0$
\begin{equation}
(3.17) \quad \int_0^\infty e^{-\frac{r^2}{4t}} dV(r) = \kappa_0 t^{n/2} = \kappa_1 \int_0^\infty e^{-\frac{r^2}{4t}} dr^n,
\end{equation}
namely,
\begin{equation}
(3.18) \quad \int_0^\infty e^{-\frac{r^2}{4t}} d(V(r) - \kappa_1 r^n) = 0.
\end{equation}
Integrating by parts implies
\begin{equation}
(3.19) \quad \int_0^\infty (V(r) - \kappa_1 r^n) re^{-\frac{r^2}{4t}} dr = 0.
\end{equation}
Suppose that there is a constant $r_0 > 0$ such that $V(r_0) - \kappa_1 r_0^n > 0$ (Or else we complete the proof by (3.19)). Then there is a $0 < \delta < \frac{r_0}{2}$ and $\epsilon > 0$ such that $V(r) - \kappa_1 r^n \geq \epsilon$ for all $r \in (r_0 - \delta, r_0 + \delta)$. Set $t_p = \frac{2}{p} r_0^2$, then in $(0, \infty)$ the function
\[ r^p e^{-\frac{r^2}{4t}} \]
attains its maximal value at $r = r_0$.

Now we claim
\begin{equation}
(3.20) \quad \lim_{p \to \infty} \frac{p^{\frac{n}{2}} e^{\frac{x^2}{2}}}{r_0^{p+1}} \int_{r_0 - \delta}^{r_0 + \delta} r^p e^{-\frac{r^2}{4t}} dr = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.
\end{equation}
In fact,
\begin{equation}
(3.21) \quad I(p) \equiv \frac{p^{\frac{n}{2}} e^{\frac{x^2}{2}}}{r_0^{p+1}} \int_{r_0 - \delta}^{r_0 + \delta} r^p e^{-\frac{r^2}{4t}} dr = \frac{p^{\frac{n}{2}} e^{\frac{x^2}{2}}}{r_0^{p+1}} \int_{-\frac{\delta}{r_0}}^{\frac{\delta}{r_0}} (1 + s)^p e^{-\frac{s^2}{2t}(1+s)^2} ds
\end{equation}
When $-\frac{1}{2} \leq s < \infty$, a simple calculation implies
\[ \min \left\{ 0, \frac{8}{3} s^3 \right\} \leq \log(1 + s) - s + \frac{s^2}{2} \leq \frac{s^3}{3}. \]
Combining the above inequality, we get
\[
\limsup_{p \to \infty} I(p) \leq \limsup_{p \to \infty} \int_{-\frac{\delta}{\sqrt{p}}}^{\frac{\delta}{\sqrt{p}}} e^{-t^2 + \frac{3}{\sqrt{p}}} dt
\]
(3.22)
and
\[
\liminf_{p \to \infty} I(p) \geq \lim_{p \to \infty} \int_{0}^{\delta/\sqrt{p}} e^{-t^2} dt + \lim_{p \to \infty} \int_{-\delta/\sqrt{p}}^{0} e^{-t^2 + \frac{3}{\sqrt{p}}} dt
\]
(3.23)
\[
= \int_{0}^{\infty} e^{-t^2} dt + \lim_{p \to \infty} \int_{-\delta/\sqrt{p}}^{0} e^{-t^2 + \frac{3}{\sqrt{p}}} dt = \int_{-\infty}^{\infty} e^{-t^2} dt.
\]
Hence we have shown (3.20).

For \( p > 1 \),
\[
\frac{p^2 e^{\frac{p}{2}}}{r_0^{p+1}} \int_{r_0+\delta}^{\infty} r^{n+p} e^{-\frac{r^2}{2}} dr = \int_{0}^{\delta/\sqrt{p}} e^{(n+p) \log \left(1 + \frac{t^2}{\sqrt{p}}\right)} e^{-\sqrt{\pi} t^2} dt
\]
(3.24)
\[
\leq r_0^{n} \int_{0}^{\delta/\sqrt{p}} e^{(n+p) \frac{t^2}{\sqrt{p}}} e^{-\sqrt{\pi} t^2} dt \leq r_0^{n} \int_{0}^{\delta/\sqrt{p}} e^{\frac{t^2}{\sqrt{p}}} dt.
\]
Then
\[
\liminf_{p \to \infty} \frac{p^2 e^{\frac{p}{2}}}{r_0^{p+1}} \int_{0}^{\infty} (V(r) - \kappa_1 r^n) r^p e^{-\frac{r^2}{2}} dr
\]
\[
\geq \liminf_{p \to \infty} \frac{p^2 e^{\frac{p}{2}}}{r_0^{p+1}} \left( \int_{r_0-\delta}^{r_0+\delta} e^{p} e^{\frac{r^2}{2}} dr - \kappa_1 \int_{r_0-\delta}^{r_0+\delta} r^n e^{-\frac{r^2}{2}} dr - \kappa_1 \int_{r_0+\delta}^{\infty} r^n e^{-\frac{r^2}{2}} dr \right)
\]
\[
\geq \epsilon \sqrt{\pi} - \kappa_1 r_0^n \limsup_{p \to \infty} \left( \frac{p^2 e^{\frac{p}{2}}}{r_0^{p+1}} \int_{r_0-\delta}^{r_0+\delta} r^p e^{\frac{r^2}{2}} dr + \int_{-\frac{\delta}{\sqrt{p}}}^{\delta/\sqrt{p}} e^{\frac{t^2}{\sqrt{p}}} dt \right)
\]
\[
= \epsilon \sqrt{\pi} - \kappa_1 r_0^n \limsup_{p \to \infty} \left( \int_{-\frac{\delta}{\sqrt{p}}}^{\frac{\delta}{\sqrt{p}}} e^{p \log \left(1 + \frac{t^2}{\sqrt{p}}\right)} e^{-\sqrt{\pi} t^2} dt + \int_{-\frac{\delta}{\sqrt{p}}}^{\delta/\sqrt{p}} e^{-t^2 \left(1 - \frac{1}{2} \sqrt{\pi}\right)} dt \right)
\]
\[
\geq \epsilon \sqrt{\pi} - \kappa_1 r_0^n \limsup_{p \to \infty} \left( \int_{-\frac{\delta}{\sqrt{p}}}^{\frac{\delta}{\sqrt{p}}} e^{\frac{t^2}{\sqrt{p}}} e^{-\sqrt{\pi} t^2} dt \right) = \epsilon \sqrt{\pi}.
\]
Taking the derivative of \( t \) in (3.19) yields
\[
\int_{0}^{\infty} (V(r) - \kappa_1 r^n) r^{2k+1} e^{-\frac{r^2}{2}} dr = 0
\]
for any \( t > 0 \) and \( k = 0, 1, 2 \cdots \). If we choose \( p = 2k + 1, r_0^2 > e, t_p = \frac{2}{p} r_0^2 \) in (3.25), then we get the contradiction provided \( k \) is sufficiently large. Hence \( V(r) - \kappa_1 r^n \equiv 0 \).}

**Theorem 3.3.** Let \( M \) be an \( n \) dimensional smooth self-shrinker with Euclidean volume growth and boundary \( \partial M \subset \partial B_R \) in \( \mathbb{R}^{n+m} \). If (3.8) holds, then the limit \( \lim_{r \to \infty} r^{-1} M \) exists and is cone, namely, the tangent cone at infinity of \( M \) is a unique cone.
Proof. We claim

\[ \lim_{r \to \infty} \left( r^{-n} \int_{M \cap B_r} \phi \, d\mu \right) \]

exists for every homogeneous function \( \phi \in C^1(\mathbb{R}^{n+m} \setminus \{0\}) \) with degree zero. Suppose

\[ \limsup_{r \to \infty} r^{-n} \int_{M \cap B_r} \phi \, d\mu > \liminf_{r \to \infty} r^{-n} \int_{M \cap B_r} \phi \, d\mu \]

for some homogeneous function \( \phi \in C^1(\mathbb{R}^{n+m} \setminus \{0\}) \) with degree zero. Then there exist two sequences \( p_i \to \infty \) and \( q_i \to \infty \) such that

\[ \lim_{i \to \infty} p_i^{-n} \int_{M \cap B_{p_i}} \phi \, d\mu > \lim_{i \to \infty} q_i^{-n} \int_{M \cap B_{q_i}} \phi \, d\mu. \]

By compactness of varifolds and Theorem 3.1, there exist two cones \( C_1, C_2 \) in \( \mathbb{R}^{n+m} \) with integer multiplicities and subsequences \( p_{k_i} \) of \( p_i \) and \( q_{k_i} \) of \( q_i \) such that

\[ M_{p_{k_i}} \rightharpoonup C_1 \text{ and } M_{q_{k_i}} \rightharpoonup C_2 \]

in the sense of Radon measure. So we have

\[ \int_{C_1 \cap B_1} \phi \, d\mu_{C_1} = \lim_{i \to \infty} \int_{M_{p_{k_i}} \cap B_1} \phi \, d\mu_{p_{k_i}} = \lim_{i \to \infty} p_{k_i}^{-n} \int_{M \cap B_{p_{k_i}}} \phi \, d\mu \]

\[ > \lim_{i \to \infty} q_{k_i}^{-n} \int_{M \cap B_{q_{k_i}}} \phi \, d\mu = \lim_{i \to \infty} \int_{M \cap B_{q_{k_i}}} \phi \, d\mu_{q_{k_i}} \]

\[ = \int_{C_2 \cap B_1} \phi \, d\mu_{C_2}, \]

which implies

\[ \int_{C_1} \phi e^{-\frac{|X|^2}{4}} \, d\mu_{C_1} > \int_{C_2} \phi e^{-\frac{|X|^2}{4}} \, d\mu_{C_2} \]

by co-area formula.

From the previous argument, the limit

\[ \lim_{t \to \infty} \Xi_t(M, \phi) = \lim_{t \to \infty} \frac{1}{(4\pi t)^{n/2}} \int_M \phi e^{-\frac{|X|^2}{4t}} \, d\mu \]

exists. It infers that

\[ \int_{C_1} \phi e^{-\frac{|X|^2}{4}} \, d\mu_{C_1} = \lim_{i \to \infty} \int_{M_{p_{k_i}}} \phi e^{-\frac{|X|^2}{4}} = \lim_{t \to \infty} \frac{1}{(4\pi t)^{n/2}} \int_M \phi e^{-\frac{|X|^2}{4t}} \, d\mu \]

\[ = \lim_{i \to \infty} \int_{M_{q_{k_i}}} \phi e^{-\frac{|X|^2}{4}} = \int_{C_2} \phi e^{-\frac{|X|^2}{4}} \, d\mu_{C_2}. \]

However, (3.33) contradicts (3.31). Hence, the claim (3.27) holds.

If \( \lim_{t \to \infty} r_t^{-1} M \to C^+ \) and \( \lim_{t \to \infty} s_t^{-1} M \to C^- \) are cones, then from (3.33) one has

\[ \int_{C^+} \phi e^{-\frac{|X|^2}{4}} \, d\mu_{C^+} = \int_{C^-} \phi e^{-\frac{|X|^2}{4}} \, d\mu_{C^-} \]

for every homogeneous function \( \phi \in C^1(\mathbb{R}^{n+m} \setminus \{0\}) \) with degree zero. It’s clear that

\[ \int_{C^+ \cap \partial B_1} \phi = \int_{C^- \cap \partial B_1} \phi. \]
Arbitrariness of \( \phi \) implies \( C^+ = C^- \). Therefore, the tangent cone at infinity of \( M \) is a unique cone. \(\square\)

4. A Rigidity Theorem for Self-Shrinkers

Let us recall an \( \epsilon \)-regularity theorem for mean curvature flow showed by Ecker (A litter different from Theorem 1.8 in [16]).

**Theorem 4.1.** For \( p \in [n, n + 2] \), there exists a constant \( \epsilon_0 > 0 \) such that for any smooth properly immersed solution \( M = (\mathcal{M}_t)_{t\in(-4,0)} \) of mean curvature flow in \( \mathbb{R}^{n+m} \), every \( X_0 \) which the solution reaches at time \( t_0 \in [-1,0) \), the assumption

\[
(4.1) \quad I_{X_0,t_0} \triangleq \sup_{\sqrt{-t_0} \leq \rho \leq \sqrt{-t_0}} \frac{1}{(\rho^2 - \rho^2)^{n+\frac{2-p}{2}}} \int_{-\rho^2}^{-\rho^2} \int_{\mathcal{M}_t \cap \mathbb{R}^2} |B|^p \leq \epsilon_0
\]

implies

\[
(4.2) \quad \sup_{\sigma \in [0,1]} \left( \sigma^2 \sup_{t \in \left[0 - (1-\sigma)^2,t_0\right]} \sup_{\mathcal{M}_t \cap \mathbb{R}^{1-\sigma}(X_0)} |B|^2 \right) \leq \left( \epsilon_0^{-1} I_{X_0,t_0} \right)^{\frac{2}{p}}.
\]

For completeness, we give the proof in appendix which is based on Ecker’s proof. Let us consider the mean curvature flow in Theorem 4.1 which starts from a self-shrinker. Let \( M \) be a self-shrinker, then the one-parameter family \( \mathcal{M}_t = \sqrt{-t}M \) is a mean curvature flow for \(-4 \leq t < 0\). In this case,

\[
(4.3) \quad I_{X_0,t_0} = \sup_{\sqrt{-t_0} \leq \rho \leq \sqrt{-t_0}} \frac{1}{(\rho^2 - \rho^2)^{n+\frac{2-p}{2}}} \int_{-\rho^2}^{-\rho^2} \left( \int_{\mathcal{M}_t \cap \mathbb{R}^2} |B|^p \right) dt
\]

\[
= \sup_{\sqrt{-t_0} \leq \rho \leq \sqrt{-t_0}} \left( \rho^2 - \rho^2 \right)^{\frac{n+2-p}{2}} \int_{-\rho^2}^{0} \left( \int_{\mathcal{M}_t \cap \mathbb{R}^2} |B|^p \right) \frac{2}{\rho^p} dr
\]

\[
= \sup_{\sqrt{-t_0} \leq \rho \leq \sqrt{-t_0}} \frac{2}{\rho^p} \left( \rho^2 - \rho^2 \right)^{\frac{n+2-p}{2}} \int_{0}^{1} \left( \int_{\mathcal{M}_t \cap \mathbb{R}^2} |B|^p \right) \frac{2}{\rho^p} dr.
\]

For any \(-\frac{1}{4} < t_0 < 0 \) and \( X_0 \in \sqrt{-t_0}M \), \( I_{X_0,t_0} \leq \epsilon_0 \) implies

\[
(4.4) \quad \frac{1}{4} \sup_{t \in \left(0 - \frac{1}{4},t_0\right)} \sup_{\sqrt{-t}M \cap \mathbb{R}^2} \left( |B|^2 \right) \leq \left( \epsilon_0^{-1} I_{X_0,t_0} \right)^{\frac{2}{p}}.
\]

Hence

\[
(4.5) \quad \sup_{t \in \left(2,(-t_0)^{-1/2}\right)} \sup_{\frac{t}{2}M \cap \mathbb{R}^2} \left( |B|^2 \right) \leq 4 \left( \epsilon_0^{-1} I_{X_0,t_0} \right)^{\frac{2}{p}}.
\]

Now we have the following curvature estimates for self-shrinkers.

**Theorem 4.2.** Let \( M \) be an \( n \) dimensional proper self-shrinker in \( \mathbb{R}^{n+m} \). If for some \( p \in [n, n + 2] \) there is

\[
(4.6) \quad \lim_{R \to \infty} \int_{M \cap \mathbb{R}^2 \setminus \mathbb{R}^2} |B|^p d\mu = 0,
\]
then there exist constants \( c, r_0 > 0 \) such that for all \( r \geq r_0 \) and \( t > 4 \) we have

\[
(4.7) \quad \sup_{M \cap \partial B_{(r+1)t}} \frac{|B|}{|B|} \leq c \left( \frac{1}{t} \right) \left( \sup_{s \geq r} \int_{M \cap B_{2s} \setminus B_s} |B|^p \, d\mu \right)^{\frac{1}{p}}.
\]

**Proof.** For any \( \epsilon > 0 \), there exists a constant \( r_0 \geq 2 \) such that for any \( r_1 \geq r_0 \) we have

\[
\sup_{r \geq r_1} \int_{M \cap B_{2r} \setminus B_r} |B|^p \, d\mu < \epsilon.
\]

For any vector \( X_0 \in \mathbb{R}^{n+m} \) with \( |X_0| \geq 2r_1 + 2 \), it’s clear that

\[
B_{2r}(rX_0) \subset (B_{|X_0|+2r} \setminus B_{|X_0|-2r}) \subset (B_{2r} \setminus B_{|X_0|-2r}).
\]

Let \( X \in \sqrt{-1} M \) with \( |X| \geq 2r_1 + 2 \) and \( t < 0 \), then

\[
(4.8) \quad \int_{M \cap B_{2r} \setminus (rX)} |B|^p \, d\mu \leq \int_{M \cap (B_{2r} \setminus B_{|X|+2r})} |B|^p \, d\mu \leq \sup_{s \geq r_1} \int_{M \cap B_{2s} \setminus B_s} |B|^p \, d\mu < \epsilon.
\]

In view of (4.3), one has

\[
(4.9) \quad I_{X,t} \leq \sup_{0 \leq \rho \leq \rho' \leq 2} \left( \rho'^2 - \rho^2 \right)^{\frac{n+2-p}{2}} \int_{\mathbb{R}^n} 2 \rho^{p-n-3} d\rho \sup_{r \geq r_1} \int_{M \cap B_{2r} \setminus B_r} |B|^p \, d\mu \leq \frac{2}{2 + n - p} \sup_{0 \leq \rho \leq \rho' \leq 2} \left( \rho'^2 - \rho^2 \right)^{\frac{n+2-p}{2}} \left( \rho^2 - \rho^2 \right)^{\frac{n+2-p}{2}} \sup_{r \geq r_1} \int_{M \cap B_{2r} \setminus B_r} |B|^p \, d\mu.
\]

Since for each fixed \( \alpha \in (0, 1] \) and each \( s \geq 1 \),

\[
(4.10) \quad \frac{\partial}{\partial s} \left( \frac{s^{2\alpha} - 1}{(s^2 - 1)^{\alpha}} \right) = 2\alpha \frac{s - s^{2\alpha - 1}}{(s^2 - 1)^{\alpha}} \geq 0,
\]

then

\[
\sup_{s \geq 1} \frac{s^{2\alpha} - 1}{(s^2 - 1)^{\alpha}} = \lim_{s \to \infty} \frac{s^{2\alpha} - 1}{(s^2 - 1)^{\alpha}} = 1.
\]

So we obtain

\[
(4.11) \quad I_{X,t} \leq \frac{2}{2 + n - p} \sup_{r \geq r_1} \int_{M \cap B_{2r} \setminus B_r} |B|^p \, d\mu < \frac{2\epsilon}{2 + n - p}.
\]

Let \( \epsilon = \frac{2 + n - p}{2} \epsilon_0 \), \( |X| \geq 2r_1 + 2 \) and \( -\frac{1}{4} < t < 0 \), then combining (4.5) we have

\[
(4.12) \quad \sup_{s \in (2, (-t)^{-1/2})} \left( \sup_{B_{2s} \setminus B_s} |B| \right)^{\frac{1}{p}} \leq 2 \left( \epsilon^{-1} \sup_{r \geq r_1} \int_{M \cap B_{2r} \setminus B_r} |B|^p \, d\mu \right)^{\frac{1}{p}},
\]

which implies

\[
2 \left( \epsilon^{-1} \sup_{r \geq r_1} \int_{M \cap B_{2r} \setminus B_r} |B|^p \, d\mu \right)^{\frac{1}{p}} \geq \sup_{X \in M \cap \partial B_{2r_1 + 2}} \left( \sup_{s \geq r_1} \int_{M \cap B_{2s} \setminus B_s} |B| \right)^{\frac{1}{p}} \geq t \sup_{|X|=2r_1+2,t \in M \cap \partial B_{2t} \setminus B_t} |B|,
\]

for any \( r \geq r_1 \) and \( t > 2 \). This suffices to complete the proof. \( \square \)
Lemma 4.3. Let $M$ be an $n$ dimensional proper noncompact self-shrinker in $\mathbb{R}^{n+m}$ with

$$
\limsup_{r \to \infty} \int_{M \cap B_2 \setminus B_r} |H|^p \, d\mu < \infty
$$

for some $p \geq 2$. Then every end of $M$ has Euclidean volume growth at least.

Proof. For any end $E$ of $M$, there is a constant $r_0 > 0$ such that $\partial E \subset B_{r_0}$. Replacing $E$ by $E \setminus B_{r_0}$ if necessary, we have $\partial E \subset \partial B_{r_0}$. Set $E_r = E \cap B_r$. For $0 \leq s < 1$ and $r \geq r_0$, we have

$$
\frac{\partial}{\partial r} \left( r^{-n+s} \int_{E_r} 1 \, d\mu \right) = -(n-s) r^{-n+s-1} \int_{E_r} 1 \, d\mu + r^{-n+s} \int_{E_r \cap \partial B_r} \frac{|X|}{|X^T|} \left| \partial r \right| \geq (n-s) r^{-n+s-1} \int_{E_r} 1 \, d\mu + r^{-n+s-1} \int_{E_r \cap \partial B_r} |X|^T
$$

$$
\geq -(n-s) r^{-n+s-1} \int_{E_r} 1 \, d\mu + \frac{1}{2} r^{-n+s-1} \int_{E_r} \Delta |X|^2 + r^{-n+s-1} \int_{\partial E} |X|^T
$$

$$
\geq sr^{-n+s-1} \int_{E_r} 1 \, d\mu - 2r^{-n+s-1} \int_{E_r} |H|^2 \, d\mu
$$

$$
\geq sr^{-n+s-1} \int_{E_r} 1 \, d\mu - 2r^{-n+s-1} \left( \int_{E_r} |H|^p \, d\mu \right)^{2/p} \left( \int_{E_r} 1 \, d\mu \right)^{1-2/p}.
$$

Set

$$
\tilde{V}_s(r) = r^{-n+s} \int_{E_r} 1 \, d\mu,
$$

then

$$
\partial_r \tilde{V}_s \geq \frac{s}{r} \tilde{V}_s - 2r^{-2(n-s)-1} \tilde{V}_s^{1-2/p} \left( \int_{E_r} |H|^p \, d\mu \right)^{2/p}
$$

$$
= \frac{\tilde{V}_s}{r} \left( s - 2 \left( \int_{E_r} |H|^p \, d\mu \right)^{2/p} \left( \int_{E_r} 1 \, d\mu \right)^{-2/p} \right).
$$

For any $r > 0$, let $q \in \mathbb{N}$ with $2^q \leq r < 2^{q+1}$. By (4.14), there is a constant $c > 0$ such that

$$
\int_{E_r} |H|^p \, d\mu \leq \sum_{k=0}^{q} \int_{E_{2k+1} \setminus E_{2k}} |H|^p \, d\mu + \int_{E_{1}} |H|^p \, d\mu \leq c(q+2) \leq c \left( \frac{\log r}{\log 2} + 2 \right).
$$

From [31, 33], every end of any self-shrinker has linear growth at least. For any $\delta > 0$, there exists a constant $r_\delta > 0$ such that for all $r \geq r_\delta$

$$
\left( \int_{E_r} |H|^p \, d\mu \right)^{2/p} \left( \int_{E_r} 1 \, d\mu \right)^{-2/p} \leq \frac{\delta}{4},
$$

then (4.16) implies

$$
\partial_r \tilde{V}_\delta \geq \frac{\delta \tilde{V}_\delta}{2r}.
$$

By Newton-Leibniz formula,

$$
\log \tilde{V}_\delta(r) \geq \log \tilde{V}_\delta(r_\delta) + \int_{r_\delta}^{r} \frac{\partial_s \tilde{V}_\delta(s)}{\tilde{V}_\delta(s)} \, ds \geq \log \tilde{V}_\delta(r_\delta) + \frac{\delta}{2} \log \frac{r}{r_\delta}.
$$
Denote $\widetilde{V}(r) = \widetilde{V}_0(r)$. By (4.16),
\begin{equation}
\partial_r \widetilde{V}_r^2 \geq -\frac{4}{p} \left( \int_{E_r} |H|^p d\mu \right)^{\frac{2}{p}} r^{-\frac{2n}{p}-1}.
\end{equation}
There is a constant $s_0 > e$ such that for all $s \geq s_0$ the inequality $\log s < s^\frac{2}{p}$ holds. Hence combining (4.14) and (4.20), for any $r_2 \geq r_1 \geq \max \{s_0, r_0\}$ we have
\begin{equation}
\widetilde{V}_r^2(r_2) - \widetilde{V}_r^2(r_1) \geq -\frac{nc'}{p} \int_{r_1}^{r_2} r^{-\frac{2n}{p}-1} \log r dr \geq -\frac{nc'}{p} \int_{r_1}^{r_2} r^{-\frac{n}{p}-1} dr \geq -c'r_1^{-\frac{2}{p}}
\end{equation}
for some constant $c' > 0$. (4.19) infers
\[ \lim_{r \to \infty} r^\delta \widetilde{V}(r) = \infty \]
for any $\delta > 0$. Combining (4.21), we obtain
\begin{equation}
\widetilde{V}_r^2(r_2) \geq \frac{1}{2} \widetilde{V}_r^2(r_1) > 0
\end{equation}
for some fixed sufficiently large $r_1 \geq \max \{s_0, r_0\}$. This suffices to complete the proof. \qed

Now let us prove the following rigidity theorem.

**Theorem 4.4.** Let $M$ be an $n$-dimensional properly non-compact self-shrinker with compact boundary in $\mathbb{R}^{n+m}$. If
\begin{equation}
\lim_{r \to \infty} \int_{M \cap B_{2r} \setminus B_r} |B|^n d\mu = 0,
\end{equation}
then $M$ must be an $n$-plane through the origin.

**Proof.** From Theorem 4.2, we obtain
\begin{equation}
\lim_{r \to \infty} \left( r \sup_{B_{5r}} |B| \right) = 0.
\end{equation}
Let $M_r = r^{-1}M$ for any $r > 0$, then $M_r \cap \left( B_K \setminus B_{\frac{1}{K}} \right)$ for any $K > 0$ has bounded sectional curvature. On the one hand, $M_r \cap \left( B_K \setminus B_{\frac{1}{K}} \right)$ converges to a smooth manifold with $C^{1,\alpha}$ metric in the Gromov-Hausdorff sense. On the other hand, Theorem 3.3 implies that $M_r$ converges to a unique cone $C$ in $\mathbb{R}^{n+1}$ in the current sense. Hence there is a neighborhood $\Omega_x$ of $x$ such that $\Omega_x \cap C$ can be represented as a graph with $C^{1,\alpha}$ graphic function. Hence by Fatou lemma, $\Omega_x \cap C$ is flat by (4.24). So we conclude that $M_r$ converges to a union of finite $n$-planes through origin as $r \to \infty$. Note that every end of $M$ converges to a union of finite $n$-planes through origin by Lemma 4.3. Therefore, up to rotation there are a constant $R > 0$ and a smooth graph $\text{graph}_u \subset M$ over $\mathbb{R}^n \setminus B_R$ with the graphic function $u = (u^1, \cdots, u^m)$. Moreover, there is a constant $c_M$ such that
\begin{equation}
|D^j u^\alpha(x)| \leq c_M |x|^{-j+1}
\end{equation}
on $\mathbb{R}^n \setminus B_R$ for any $j = 0, 1, 2$ and $1 \leq \alpha \leq m$. Here, $c_M$ is a general constant, which may change from line to line.

Let $g_{ij} = \delta_{ij} + \sum_{1 \leq \alpha \leq m} u^\alpha u^\alpha_i$ and $(g^ij)$ be the inverse matrix of $(g_{ij})$. From the equation of self-shrinkers (see [10]) for instance
\begin{equation}
\sum_{1 \leq i, j \leq n} g^ij u^\alpha_{ij} = \frac{-u^\alpha + x \cdot Du^\alpha}{2},
\end{equation}
we have
\[
\Delta_M u^\alpha = \frac{1}{\sqrt{\det g_{ij}} \partial_{x^i}} \left( g^{kl} \sqrt{\det g_{ij}} u^\alpha \right)
\]
(4.27)
\[
= \frac{1}{\sqrt{\det g_{ij}}} \partial_{x^i} \left( g^{ij} \sqrt{\det g_{kl}} \right) u^\alpha_j + \frac{1}{2} x \cdot Du^\alpha - \frac{u^\alpha}{2}.
\]

Denote \( g^{ij}_t(x) = g^{ij}(x,t) = g^{ij} \left( \frac{x}{\sqrt{t}} \right) \), then
\[
\left| \delta_{ij} - g^{ij}_t \right| \leq c_1 \sum_{\beta} |\nabla_{\mathbb{R}^n} u^\beta|,
\]
(4.28)
where \( c_1 \) is a constant. Let \( Q(x,t,Du^\alpha,D^2u^\gamma) = \frac{1}{\sqrt{t}} \left( \delta_{ij} - g^{ij}_t \right) u^0_{ij} \), then on \((\mathbb{R}^n \setminus B_R) \times \mathbb{R}^+\), from (4.25) one has
\[
|Q(x,t,Du^\alpha,D^2u^\gamma)| \leq \frac{c_2}{|x|} \sum_{\beta} |\nabla_{\mathbb{R}^n} u^\beta|,
\]
(4.29)
where \( c_2 \) is a constant.

Denote \( a^{ij}(x,t) = a^{0ij} \left( \frac{x}{\sqrt{t}} \right) \) and \( U^\alpha(x,t) = \sqrt{t} u^\alpha \left( \frac{x}{\sqrt{t}} \right) \). Then
\[
\frac{\partial}{\partial t} U^\alpha + \Delta_{\mathbb{R}^n} U^\alpha = \frac{1}{2\sqrt{t}} u^\alpha \left( \frac{x}{\sqrt{t}} \right) - \frac{1}{2} Du^\alpha \left( \frac{x}{\sqrt{t}} \right) \cdot \frac{x}{t} + \frac{1}{\sqrt{t}} \Delta_{\mathbb{R}^n} u^\alpha \left( \frac{x}{\sqrt{t}} \right)
\]
(4.30)
\[
= - \frac{1}{\sqrt{t}} g^{ij}_t u^\alpha_{ij} + \frac{1}{\sqrt{t}} \Delta_{\mathbb{R}^n} u^\alpha \left( \frac{x}{\sqrt{t}} \right) = Q(x,t,Du^\alpha,D^2u^\gamma).
\]
Hence for any \((x,t) \in (\mathbb{R}^n \setminus B_R) \times \mathbb{R}^+\), combining (4.29) we have
\[
\left| \frac{\partial}{\partial t} U^\alpha + \Delta_{\mathbb{R}^n} U^\alpha \right| \leq \frac{c_2}{|x|} \sum_{\beta} |\nabla_{\mathbb{R}^n} U^\beta|.
\]
(4.31)
Due to Theorem 1 (with the version of vector-valued functions) showed by Escauriaza-Seregin-Sverák in [21] (see the following content in Theorem 1 of [21]), we obtain

\[
U^\alpha \equiv 0 \quad \text{on} \quad \mathbb{R}^n \setminus B_R,
\]
and then \( \text{graph}_u \) is an \( n \)-plane through the origin. Hence \( M \) is an \( n \)-plane through the origin by the rigidity of elliptic equations, and then we complete the proof. \( \Box \)

5. Appendix

Let us prove Theorem 4.1. There exist \( \sigma_1 \in (0,1) \), \( t_1 \in [t_0 - (1-\sigma_1)^2,t_0] \) and \( X_1 \in \mathcal{M}_{t_1} \cap B_{1-\sigma_1}(X_0) \) such that
\[
\sigma_1^2 |B| \bigg|_{(X_1,t_1)} = \sup_{\sigma \in [0,1]} \left( \frac{\sigma^2}{t_1} \sup_{t \in (t_0 - (1-\sigma)^2,t_0)} \sup_{\mathcal{M}_t \cap B_{1-\sigma}(X_0)} |B| \right).
\]

Denote \( \lambda_1 = |B|^{-1} \bigg|_{(X_1,t_1)} \). Then
\[
\sup_{t \in (t_0 - (1-\sigma_1)^2,t_0)} \sup_{\mathcal{M}_t \cap B_{1-\sigma_1}(X_0)} |B| \leq \frac{4}{\lambda_1^2}.
\]
Since
\[ B_{\sigma_1^2}(X_1) \times \left( t_1 - \frac{\sigma_1^2}{4}, t_1 \right) \subset B_{1 - \sigma_1^2}(X_0) \times \left( t_0 - \left( 1 - \sigma_1^2 \right)^{1/2}, t_0 \right), \]
then
\[ \sup_{t \in \left( t_1 - \frac{\sigma_1^2}{4}, t_1 \right)} \sup_{\mathcal{M}_t \cap B_{\sigma_1^2}(X_1)} |B|^2 \leq \frac{4}{\lambda_1^2}. \]

Let \( I_{X_0,t_0} \) be as in (4.1). It is sufficient to prove
\[ \sigma_1 \lambda_1^{-1} \leq \left( \epsilon_0^{-1} I_{X_0,t_0} \right)^{1/\beta} \]
for a certain uniform constant \( \epsilon_0 > 0 \) depending only on \( n \) provided \( I_{X_0,t_0} \leq \epsilon_0 \). By contradiction, we assume
\[ \sigma_1 \lambda_1^{-1} > \left( \epsilon_0^{-1} I_{X_0,t_0} \right)^{1/\beta}. \]

Denote \( \lambda \triangleq \lambda_1 \left( \epsilon_0^{-1} I_{X_0,t_0} \right)^{1/\beta} < \sigma_1 \).

Define
\[ \tilde{M}_s = \lambda^{-1} \left( M_{\lambda^2 s + t_1} - X_1 \right) \]
for \( s \in \left( -\frac{4+t_1}{\lambda^2}, \frac{t_0-t_1}{\lambda^2} \right) \), where we have changed variables by setting \( X = \lambda Y + X_1 \) and \( t = \lambda^2 s + t_1 \). Then \( \tilde{M}_s \) is a smooth solution of mean curvature flow satisfying
\[ 0 \in \tilde{M}_0, \quad |B| \big|_{(0,0)} = \left( \epsilon_0^{-1} I_{X_0,t_0} \right)^{1/\beta} \leq 1 \]
and
\[ \sup_{s \in \left( -\frac{4+t_1}{\lambda^2}, 0 \right)} \sup_{\tilde{M}_s \cap B_{\sigma_1^2}} |B|^2 \leq 4 \left( \epsilon_0^{-1} I_{X_0,t_0} \right)^{2/\beta}. \]

Since \( \sigma_1 > \lambda \), then
\[ \sup_{s \in \left( -\frac{1}{\lambda}, 0 \right)} \sup_{\tilde{M}_s \cap B_{\sigma_1^2}} |B|^2 \leq 4 \left( \epsilon_0^{-1} I_{X_0,t_0} \right)^{2/\beta}. \]

By scaling, it follows that
\[ (5.1) \quad I_{X_0,t_0} = \sup_{\sqrt{-\epsilon_0} \leq \rho < \epsilon_0} \left( \frac{\lambda^2}{\rho^2 - \rho^2} \right)^{\frac{n+2-p}{2}} \int_{\tilde{M}_s \cap B_{\epsilon_0}(X_1)} |B|^p. \]
Since \(-1 < \epsilon_0 < 0\) and \( t_0 - (1 - \sigma_1)^2 \leq t_1 \leq t_0 \), we choose \( \rho^2 = -t_1, \rho^2 - \rho^2 = \rho^2 + t_1 = 2\lambda^2 > 0 \). Noting \( X_1 \in \mathcal{M}_t \cap B_{1-\sigma_1}(X_0) \), so we have
\[ (5.2) \quad I_{X_0,t_0} \geq 2^{-\frac{n+2-p}{2}} \int_{-2 \epsilon_0^{-1/2}}^0 \int_{\mathcal{M}_s \cap B_{\epsilon_0}(X_1)} |B|^p \geq 2^{-\frac{n+2-p}{2}} \int_{-2 \epsilon_0^{-1/2}}^0 \int_{\mathcal{M}_s \cap B_{\epsilon_0}} |B|^p. \]

Now let’s recall the evolution equation for the norm of second fundamental form in [41]:
\[ (5.3) \quad \left( \frac{d}{ds} - \Delta_{\tilde{M}_s} \right) |B|^2 = -2|\nabla B|^2 + 2|R^N| + 2 \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leq 3|B|^4. \]

Since
\[ \sup_{s \in \left( -\frac{1}{\lambda}, 0 \right)} \sup_{\tilde{M}_s \cap B_{\sigma_1^2}} |B|^2 \leq 4 \left( \epsilon_0^{-1} I_{X_0,t_0} \right)^{2/\beta} \leq 4, \]
then
\[ (5.4) \quad \left( \frac{d}{ds} - \Delta_{\tilde{M}_s} \right) |B|^p \leq \frac{3p}{2} |B|^{p+2} \leq 6p |B|^p. \]
By the mean value inequality for mean curvature flow in [15][16] (where the case of sub-manifolds is similar to the case of hypersurfaces), there exists a constant $c(n)$ such that
\begin{equation}
|B|^p|_{(0,0)} \leq c(n) \int_{-\frac{1}{4}}^{0} \int_{\tilde{M} \cap B_{\frac{1}{4}}} |B|^p,
\end{equation}
which implies
\begin{equation}
\epsilon_0^{-1} I_{X_0,t_0} \leq c(n) 2^{\frac{n+2-p}{2}} I_{X_0,t_0}.
\end{equation}
This is impossible for the sufficiently small $\epsilon_0$. Hence we complete the proof of Theorem 4.1.

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