SOLUTIONS OF NONLINEAR POLYHARMONIC EQUATION
WITH PERIODIC POTENTIAL

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Abstract. Quasi-periodic solutions of a nonlinear periodic polyharmonic equa-
tion in $\mathbb{R}^n$, $n > 1$, are studied. It is proven that there is an extensive "non-resonant"
set $G \subset \mathbb{R}^n$ such that for every $\vec{k} \in G$ there is a solution asymptotically close to a
plane wave $A e^{i\langle \vec{k}, \vec{x} \rangle}$ as $|\vec{k}| \to \infty$.

1. Introduction

Let us consider a nonlinear polyharmonic equation with quasi-periodic boundary
conditions:

$$(-\Delta)^l u(\vec{x}) + V(\vec{x}) u(\vec{x}) + \sigma |u(\vec{x})|^2 u(\vec{x}) = \lambda u(\vec{x}), \quad \vec{x} \in \mathbb{R}^n,$$

$$(1)$$

$$u(x_1, \ldots, 2\pi, \ldots, x_n) = e^{2\pi i t_s} u(x_1, \ldots, 0, \ldots, x_n),$$

$$\frac{\partial}{\partial x_s} u(x_1, \ldots, 2\pi, \ldots, x_n) = e^{2\pi i t_s} \frac{\partial}{\partial x_s} u(x_1, \ldots, 0, \ldots, x_n),$$

$$(2)$$

$$\ldots$$

$$\frac{\partial^{2l-1}}{\partial x_s^{2l-1}} u(x_1, \ldots, 2\pi, \ldots, x_n) = e^{2\pi i t_s} \frac{\partial^{2l-1}}{\partial x_s^{2l-1}} u(x_1, \ldots, 0, \ldots, x_n),$$

$$s = 1, \ldots, n.$$  

where $l$ is an integer, $n \geq 2$, $\vec{t} = (t_1, \ldots, t_n)$ is a parameter (quasimomentum),
$\vec{t} \in K := [0,1]^n$ and $V(\vec{x})$ is a periodic potential with an elementary cell $Q := [0,2\pi]^n$.
Restriction on smoothness of $V(\vec{x})$ is given by the inequality:

$$\sum_{q \in \mathbb{Z}^n} |v_q| < \infty,$$

$v_q$ being Fourier coefficients. Without the loss of generality, we assume $v_0 = 0$.

When $l = 1, n = 1, 2, 3$, equation (1) is a famous Gross-Pitaevskii equation for Bose-Einstein condensate, see e.g. [4]. In physics papers, e.g. [2], [3], [5], [6], numerical computations for Gross-Pitaevskii equation are made. However, they are
restricted to the one dimensional case. There is a lack of theoretical considerations
even for the case $n = 1$. In this paper we study the case $2l > n$, $n \geq 2$, preparing the
ground for a physically interesting case $l = 1, n = 2, 3$.

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The goal of the paper is to construct asymptotic formulas for \( u(\vec{x}) \) as \( \lambda \to \infty \). We show that there is an extensive "non-resonant" set \( G \subset \mathbb{R}^n \) such that for every \( \vec{k} \in G \) there is a quasiperiodic solution of (1) close to a plane wave \( Ae^{i(\vec{k}, \vec{x})} \) with \( \lambda = \lambda(\vec{k}, A) \) close to \( |\vec{k}|^2 + \sigma|A|^2 \) as \( |\vec{k}| \to \infty \) (Theorem 3.8). We assume \( A \in \mathbb{C} \),

\[
\sigma|A|^2 < \lambda^\gamma, \quad 0 < \gamma < (2l - n)/2l,
\]

the quasimomentum \( \vec{t} \) in (1) being defined by the formula: \( \vec{k} = \vec{t} + 2\pi j, \ j \in \mathbb{Z}^n \).

We show that the non-resonant set \( G \) has an asymptotically full measure in \( \mathbb{R}^n \):

\[
\lim_{R \to \infty} \frac{|G \cap B_R|}{|B_R|} = 1,
\]

where \( B_R \) is a ball of radius \( R \) in \( \mathbb{R}^n \) and \( | \cdot |_n \) is Lebesgue measure in \( \mathbb{R}^n \).

Moreover, we investigate a set \( D(\lambda, A) \) of vectors \( \vec{k} \in G \), corresponding to a fixed sufficiently large \( \lambda \) and a fixed \( A \). The set \( D(\lambda, A) \), defined as a level (isoenergetic) set for \( \lambda(\vec{k}, A), \)

\[
D(\lambda, A) = \left\{ \vec{k} \in G : \lambda(\vec{k}, A) = \lambda \right\},
\]

is proven to be a slightly distorted \( n \)-dimensional sphere with with a finite number of holes (Theorem 4.5). For any sufficiently large \( \lambda \), it can be described by the formula:

\[
D(\lambda, A) = \{ \vec{k} : \vec{k} = \varsigma(\lambda, A, \vec{\nu}) \vec{\nu}, \ \vec{\nu} \in B(\lambda) \},
\]

where \( B(\lambda) \) is a subset of the unit sphere \( S_{n-1} \). The set \( B(\lambda) \) can be interpreted as a set of possible directions of propagation for the almost plane waves. The set \( B(\lambda) \) has an asymptotically full measure on \( S_{n-1} \) as \( \lambda \to \infty \):

\[
|B(\lambda)| =_{\lambda \to \infty} \omega_{n-1} + O\left(\lambda^{-\delta}\right), \quad \delta > 0,
\]

here \( \omega_{n-1} \) is the standard surface measure of \( S_{n-1} \). The value \( \varsigma(\lambda, A, \vec{\nu}) \) in (7) is the “radius” of \( D(\lambda, A) \) in a direction \( \vec{\nu} \). The function \( \varsigma(\lambda, A, \vec{\nu}) = (\lambda - \sigma|A|^2)^{1/2l} \) describes the deviation of \( D(\lambda, A) \) from the perfect circle of the radius \( (\lambda - \sigma|A|^2)^{1/2l} \).

It is proven that the deviation is asymptotically small:

\[
\varsigma(\lambda, A, \vec{\nu}) =_{\lambda \to \infty} (\lambda - \sigma|A|^2)^{1/2l} + O\left(\left(1 + |\sigma||A|^2\right)\lambda^{-\gamma}\right), \quad 0 < \gamma < (2l - n)/2l.
\]

To prove the results above, we consider the term \( V + \sigma|u|^2 \) in equation (1) as a periodic potential and formally change the nonlinear equation to a linear equation with an unknown potential \( V(\vec{x}) + \sigma|u(\vec{x})|^2 \):

\[
(-\Delta)^lu(\vec{x}) + (V(\vec{x}) + \sigma|u(\vec{x})|^2)u(\vec{x}) = \lambda u(\vec{x}).
\]

Further, we use results obtained in [1] for linear polyharmonic equations. To start with, we consider a linear operator in \( L^2(Q) \) described by the formula

\[
H(\vec{t}) = (-\Delta)^l + V,
\]

and quasi-periodic boundary condition (2). The free operator \( H_0(\vec{t}) \), corresponding to \( V = 0 \), has eigenfunctions given by:

\[
\psi_j(\vec{x}) = e^{i(\vec{p}_j(\vec{t}), \vec{x})}, \quad \vec{p}_j(\vec{t}) := \vec{t} + 2\pi j, \ j \in \mathbb{Z}^n, \ \vec{t} \in K, \quad (11)
\]
and the corresponding eigenvalue is \( p^2_l(\vec{t}) := |\vec{p}_j(\vec{t})|^2 \). Perturbation theory for operator \( H(\vec{t}) \) is developed in [1]. It is shown that at high energies, there is an extensive set of generalized eigenfunctions being close to plane waves. Below (See Theorem 2.1), we describe this result in details.

Next, we define a map \( \mathcal{M} : L^\infty(Q) \to L^\infty(Q) \) by
\[
\mathcal{M}W(\vec{x}) = V(\vec{x}) + \sigma|u_{\tilde{W}}(\vec{x})|^2.
\]
(12)

Here, \( \tilde{W} \) is a shift of \( W \) by a constant such that
\[
\int_Q \tilde{W}(\vec{x})d\vec{x} = 0,
\]
and \( u_{\tilde{W}} \) is an eigenfunction of the linear operator, \((-\Delta)^l + \tilde{W}\) with (2). We consider a sequence \( \{W_m\}_{m=0}^\infty \):
\[
W_0 = V + \sigma|A|^2, \quad \mathcal{M}W_m = W_{m+1}.
\]
(13)

Note that the sequence is well defined, since for each \( m = 1, 2, 3, \cdots \) and \( \vec{t} \) in a neighborhood of a non-resonant set described in Section 2, there is an eigenfunction \( u_m(\vec{x}) \) corresponding to the potential \( \tilde{W}_m \):
\[
H_m(\vec{t})u_m = \lambda_m u_m,
\]
\[
H_m(\vec{t})u_m := (-\Delta)^l u_m + \tilde{W}_m u_m.
\]
where \( \lambda_m, u_m \) are as described in Theorem 2.1. Next, we prove that the sequence \( \{W_m\}_{m=0}^\infty \) is a Cauchy sequence of periodic functions in \( Q \) with respect to a norm
\[
||W||_* = \sum_{q \in \mathbb{Z}^n} |w_q|,
\]
(14)
w_\vec{q} being Fourier coefficients of \( W \). This implies that there is a periodic function \( W \) such that
\[
W_m \to W, \text{ with respect to the norm } ||\cdot||_*.
\]

Further, we show that
\[
u_m \to u_{\tilde{W}}, \text{ in } L^\infty(Q),
\]
\[
\lambda_m \to \lambda_{\tilde{W}}, \text{ in } \mathbb{R},
\]
where \( u_{\tilde{W}}, \lambda_{\tilde{W}} \) correspond to the potential \( \tilde{W} \) as described in Theorem 2.1. It follows from (12) and (13) that \( \mathcal{M}W = W \) and hence \( u := u_{\tilde{W}} \) solves the nonlinear equation with quasi-periodic boundary condition, (1) and (2).

The paper is organized as follows. In Section 2, we describe some known results for the linear operator. They include perturbation formulas for Bloch eigenvalues and the corresponding spectral projections at high energies. In Section 3, we prove the main lemma (Lemma 3.1), using the perturbation formulas for the linear operator. Based on the main lemma, we show the existence of a solution of (1) and (2) close to plane wave (Theorem 3.8). Section 4 is devoted to isoenergetic surface \( D(\lambda, A) \). We prove formulas (7) – (9) (Theorem 4.5).
Let us consider an operator
\[
H = (-\Delta)^l + V,
\]
in \(L^2(\mathbb{R}^n)\), \(2l > n\), and \(n \geq 2\), where \(l\) is an integer and \(V(x)\) satisfies \((3)\). Since potential \(V(x)\) is periodic with the elementary cell \(Q\), we can reduce spectral study of \((15)\) to that of a family of Bloch operators \(H(t)\) in \(L^2(Q)\), \(t \in K\), see formula \((10)\) and quasi-periodic conditions \((2)\).

The free operator \(H_0(t)\), corresponding to \(V = 0\), has eigenfunctions given by \((11)\) and the corresponding eigenvalue is \(\bar{p}_j^2(t) := |\bar{\varphi}_j(t)|^2\). Next, we describe an isoenergetic surface of \(H_0\) in \(Q\). To start with, we consider the sphere \(S(k)\) of radius \(k\) centered at the origin. We break it into pieces by the lattice \(\{\bar{p}_q(0)\}_{q \in \mathbb{Z}^n}\) and translate all the pieces into the elementary cell of the reciprocal lattice \(K := [0,1]^n\) in the parallel manner. By using the process, we obtain a sphere of radius \(k\) "packed" into \(K\). We denote it by \(S_0(k)\). Namely,
\[
S_0(k) = \{t \in K : \text{there is a } j \in \mathbb{Z}^n \text{ such that } p_j^2(t) = 2l\}.
\]
Obviously, operator \(H_0(t)\) has an eigenvalue equal to \(2l\) if and only if \(t \in S_0(k)\). For this reason, \(S_0(k)\) is called an isoenergetic surface of \(H_0(t)\). Note that, when \(t\) is a point of self-intersection of \(S_0(k)\), there exists \(q \neq j\) such that
\[
P_q^2(t) = p_j^2(t).
\]
In other words, there is a degenerated eigenvalue of \(H_0(t)\). We remove the \(k^{n+1-\delta}\)-neighborhoods of all self-intersections \((14)\) from \(S_0(k)\). We call the remaining set a non-resonant set and denote it by \(\chi_0(k,\delta)\). The removed neighborhood of self-intersections is shown to be relatively small \((1)\), and, therefore, \(\chi_0(k,\delta)\) has asymptotically full measure with respect to \(S_0(k)\):
\[
\frac{|\chi_0(k,\delta)|}{|S_0(k)|} = 1 + O(k^{-\delta/8}),
\]
here and below \(|\cdot|\) is Lebesgue measure of a surface in \(R^n\). It can be easily shown that for any \(t \in \chi_0(k,\delta)\) there is a unique \(j \in \mathbb{Z}^n\) such that \(p_j^2(t) = 2l\) and
\[
\min_{q \neq j} \left| p_q^2(t) - 2l \right| > 2k^{2l-n-\delta}.
\]

The following perturbative result is proven in \([1]\) for the linear operator \(H(t)\), \(2l > n\).

**Theorem 2.1.** Suppose \(t\) belongs to the \((k^{-n+1-2\delta})\)-neighborhood in \(K\) of the non-resonant set \(\chi_0(k,\delta)\), \(0 < 2\delta < 2l - n\), and \(V\) is a periodic potential satisfying \((3)\). Then, for sufficiently large \(k\), \(k > k_0(||V||_*,\delta)\), there exists a unique simple eigenvalue of the operator \(H(t)\) in the interval \(\varepsilon(k,\delta) \equiv (k^{2l} - k^{2l-n-\delta}, k^{2l} + k^{2l-n-\delta})\). It is given by the series:
\[
\lambda(t) = p_j^2(t) + \sum_{r=2}^{\infty} g_r(k,t),
\]
Theorem 2.4. Under the conditions of Theorem 2.1 the series (18), (20), can be differentiated with respect to $g$ character. Coefficients $(p_j)_{j \in \mathbb{Z}}$ converging absolutely, where the index $j$ is uniquely determined from the relation $p_j^2(\vec{t}) \in \varepsilon(k, \delta)$ and

$$g_r(k, \vec{t}) = \frac{(-1)^r}{2\pi i^r} \text{Tr} \int_{C_0} ((H_0(\vec{t}) - z)^{-1}V)^r dz,$$  \hspace{1cm} (19)

$C_0$ being the circle of the radius $k^{2l-n-\delta}$ centered at $k^{2l}$. The spectral projection, corresponding to $\lambda(\vec{t})$ is given by the series:

$$E(\vec{t}) = E_0 + \sum_{r=1}^{\infty} G_r(k, \vec{t}),$$ \hspace{1cm} (20)

which converges in the trace class $S_1$, $E_0$ being the spectral projection for $V = 0$, $(E_0)_{sq} = \delta_{s_3}\delta_{q_2}$,

$$G_r(k, \vec{t}) = \frac{(-1)^{r+1}}{2\pi i^r} \int_{C_0} ((H_0(\vec{t}) - z)^{-1}V)^r (H_0(\vec{t}) - z)^{-1} dz.$$ \hspace{1cm} (21)

Moreover, coefficients $g_r(k, \vec{t}), G_r(k, \vec{t})$ satisfy the following estimates:

$$|g_r(k, \vec{t})| < k^{2l-n-\delta} k^{-(2l-n-2\delta)r},$$ \hspace{1cm} (22)

$$\|G_r(k, \vec{t})\|_{S_1} \leq k^{-(2l-n-2\delta)r}.$$ \hspace{1cm} (23)

Corollary 2.2. For the perturbed eigenvalue and its spectral projection, the following estimates are valid:

$$|\lambda(\vec{t}) - p_j^2(\vec{t})| \leq k^{2l-n-\delta} \gamma_0,$$

$$\|E(\vec{t}) - E_0\|_{S_1} \leq k^{-\gamma_0}, \hspace{0.5cm} \gamma_0 = 2l - n - 2\delta.$$ \hspace{1cm} (24)

Remark 2.3. Further we use the following norm $\|T\|_1$ of an operator $T$ in $l_2(\mathbb{Z}^2)$:

$$\|T\|_1 = \max_{i} \sum_{p} |T_{pi}|.$$ \hspace{1cm} (25)

It can be easily seen from construction in [1] that estimates (23), (25) hold with respect to this norm too.

Let us introduce the notations:

$$T(m) \equiv \frac{\partial^{|m|}}{\partial t_1^{m_1} \partial t_2^{m_2} ... \partial t_n^{m_n}},$$ \hspace{1cm} (26)

$$0 \leq |m| < \infty, \hspace{0.5cm} T(0)f \equiv f.$$ \hspace{1cm} (27)

Theorem 2.4. Under the conditions of Theorem 2.1 the series (18), (27), can be differentiated with respect to $\vec{t}$ any number of times, and they retain their asymptotic character. Coefficients $g_r(k, \vec{t})$ and $G_r(k, \vec{t})$ satisfy the following estimates in the $(k^{-n+1-2\delta})$-neighborhood in $C^\alpha$ of the nonsingular set $\chi_0(k, \delta)$:

$$|T(m)g_r(k, \vec{t})| < m! k^{2l-n-\delta - \gamma_0 r + |m|(n-1+\delta)}$$ \hspace{1cm} (28)

$$\|T(m)G_r(k, \vec{t})\|_1 < m! k^{-\gamma_0 r + |m|(n-1+\delta)}$$ \hspace{1cm} (29)
Corollary 2.5. There are the estimates for the perturbed eigenvalue and its spectral projection:

\[ |T(m)(\lambda(I) - p_{j2}(I))| < 2m!k^{2l-n-\delta-2\gamma_0+|m|(n-1+\delta)}, \quad (29) \]
\[ \|T(m)(E(I) - E_j)\|_1 < 2m!k^{-\gamma_0+|m|(n-1+\delta)}. \quad (30) \]

Definition 2.6. Bloch eigenfunctions \( u_0(\vec{x}) \) corresponding to the one-dimensional projection operator \( E(I) \) are given by the formula:

\[ u_0(\vec{x}) = AE(\vec{I})e^{i(p_j(\vec{I}),\vec{x})} = A \sum_{m \in \mathbb{Z}^n} E(\vec{I})_{mj}e^{i(p_m(\vec{I}),\vec{x})} \]
\[ = Ae^{i(p_j(\vec{I}),\vec{x})}(1 + \sum_{q \neq 0} p_{j2}(\vec{I}) - p_{j2}(\vec{I})^2 e^{i(p_q(0),\vec{x})} + \cdots), \quad j, q \in \mathbb{Z}^n, \quad A \in \mathbb{C}. \quad (31) \]

Let \( B(\lambda) \subset S_{n-1} \) be the set of directions corresponding to a nonsingular set \( \chi_0(k, \delta) \) or, more precisely, to its image on the sphere \( S(k) \):

\[ B(\lambda) = \{ \vec{\nu} \in S_{n-1} : k\vec{\nu} = \vec{p}_j(\vec{I}), \ t \in \chi_0(k, \delta) \}, \quad k^{2l} = \lambda. \quad (32) \]

The set \( B(\lambda) \) can be interpreted as a set of possible directions of propagation for almost plane waves \([51]\). We define the non-resonance set \( G \subset \mathbb{R}^n \) as the union of all \( \chi_0(k, \delta) \):

\[ G = \cup_{k > k_0(||V||_*, \delta)} \chi_0(k, \delta) = \{ k\vec{\nu}, \vec{\nu} \in B(k^{2l}), k > k_0(||V||_*, \delta) \}. \quad (33) \]

Next, we describe isoenergetic surfaces for (15). Let \( D(\lambda) \) be the set of vectors \( \vec{k} \in G \), corresponding to a fixed sufficiently large \( \lambda \). The set \( D(\lambda) \), defined as a level (isoenergetic) set for \( \lambda(\vec{k}) \),

\[ D(\lambda) = \{ \vec{k} \in G : \lambda(\vec{k}) = \lambda \}, \quad (34) \]

Lemma 2.7. For any sufficiently large \( \lambda, \lambda > k_0(||V||_*, \delta)^{2l} \), and for every \( \vec{\nu} \in B(\lambda) \), there is a unique \( \vec{\nu} = \vec{\nu}(\lambda, \vec{\nu}) \) in the interval

\[ I := [k - k^{-2l+1+\gamma_0}, k + k^{-2l+1+\gamma_0}], \quad k^{2l} = \lambda, \]

such that

\[ \lambda(\vec{\nu}) = \lambda. \quad (35) \]

Furthermore, \( |\vec{\nu} - \vec{\nu}| \leq ck^{-2l+1-\gamma_0+\delta} \).

The lemma easily follows from \([29]\) for \( |m| = 1 \). For details see Lemma 2.10 in \([1]\).

Lemma 2.8. (1) For any sufficiently large \( \lambda, \lambda > k_0(||V||_*, \delta)^{2l} \), the set \( D(\lambda) \), defined by \([31]\) is a distorted circle with holes; it can be described by the formula

\[ D(\lambda) = \{ \vec{x} \in \mathbb{R}^n : \vec{x} = \vec{x}(\lambda, \vec{\nu}), \ \vec{\nu} \in B(\lambda) \}, \quad (36) \]

where \( \vec{x}(\lambda, \vec{\nu}) = k + h(\lambda, \vec{\nu}) \) and \( h(\lambda, \vec{\nu}) \) obeys the inequalities

\[ |h| < ck^{-2l+1-\gamma_0+\delta}, \quad |\nabla \nu h| < ck^{-2l+1-\gamma_0+n-1+2\delta}. \quad (37) \]

(2) The measure of \( B(\lambda) \subset S_{n-1} \) satisfies the estimate \([8]\).
where $\gamma$. Further we need the following obvious properties of norm

$$
\|f\|_* = \|f\|_*, \quad \|R(f)\|_* \leq \|f\|_*, \quad \|\Im(f)\|_* \leq \|f\|_*, \quad \|fg\|_* \leq \|f\|_* \|g\|_*.
$$

We define the value $k_1(k_1(\|V\|_*, \delta)$ as

$$
k_1(\|V\|_*, \delta) = \max \left\{ (8l)^{1/\delta}, (4\|V\|_*)^{1/\delta}, (2 + 2\|V\|_*)^{1/(2l-n-\delta)}, k_0(\|V\|_*, \delta) \right\},
$$

where $k_1(\|V\|_*, \delta)$ being as in Theorem 2.1.

**Lemma 3.1.** The following inequality holds for any $m = 1, 2, \cdots$:

$$
\|W_m - W_{m-1}\|_* \leq (|\sigma|A|^2k^{-\gamma_0})^m,
$$

where $\gamma_0 = 2l - n - 2\delta > 0$, $\delta > 0$ and $|\sigma|A|^2 < k^{\gamma_1}$, $\gamma_1 < \gamma_0$, $k$ being sufficiently large $k > k_1(\|V\|_*, \delta)$.

**Corollary 3.2.** There is a periodic function $W$ such that $W_m$ converges to $W$ with respect to the norm $\|\cdot\|_*$:

$$
\|W - W_m\|_* \leq 2(|\sigma|A|^2k^{-\gamma_0})^{m+1}.
$$

**Proof.** Clearly,

$$
\|W_0\|_* = \|V\|_* + |\sigma|A|^2,
$$

and the function $u_0(x)$ can be written in the form

$$
u_0(x) = \psi_0(x)e^{i\psi_0(x)}, \quad (44)
$$

where

$$
u_0(x) = A \sum_{q \in \mathbb{Z}^n} E(t_0 + q)\psi_0(x).
$$

is called the periodic part of $u_0$.

Let us prove (11) for $m = 1$. Indeed, it follows from (12), (13) and (32) that

$$
\|W_1 - W_0\|_* = |\sigma|||u_0|^2 - |A|^2|_* = |\sigma||\psi_0|^2 - |A|^2|_*
$$

$$
\leq |\sigma||\psi_0|^2 - |A|^2 + 2|\Im(\psi_0)|_* = |\sigma||\psi_0 - A||\psi_0 + A||_*
$$

$$
\leq |\sigma||\psi_0 - A|_* \|\psi_0 + A\|_*.
$$

Next, we estimate $\|\psi_0 - A\|_*$. Let

$$
B_0(z) = (H_0(t) - z)^{-1}V.
$$

(47)
It follows from (17) that
\[ \| (H_0(t) - z)^{-1} \|_1 < k^{-2l+n+\delta}, \text{ when } z \in C_0, \] (48)
and, hence,
\[ \| B_0(z) \|_1 < \| V \|_s k^{-(2l-n-\delta)}, \text{ } z \in C_0. \] (49)

By (21) and (17),
\[ G_r(k, t) = \frac{(-1)^{r+1}}{2\pi i} \int_{C_0} B_0(z)^r (H_0(t) - z)^{-1} dz. \] (50)

It is easy to see that
\[ \| G_r(k, t) \|_1 < \| V \|_s k^{-(2l-n-\delta)r}. \] (51)

Next, by (45) and (20):
\[ \| \psi_0 - A \|_s \leq |A| E(\bar{t})_{jj} - A | + |A| \sum_{q \in \mathbb{Z}^n \setminus \{0\}} |E(\bar{t})_{j+q,j}| \]
\[ \leq |A| \sum_{r=1}^{\infty} \| G_r(k, t) \|_1 \leq \| V \|_s |A| k^{-(2l-n-\delta)}(1 + o(1)). \] (52)

It follows:
\[ \| \psi_0 \|_s = \| \tilde{\psi}_0 \|_s \leq |A| + O(|A| k^{-(2l-n-\delta)}). \] (53)

Using (16), (52) and (53), we get
\[ \| W_1 - W_0 \|_s \leq 4|\sigma| |A|^2 \| V \|_s k^{-(2l-n-\delta)} \leq |\sigma| |A|^2 k^{-\gamma_0}, \text{ when } k > (4\| V \|_s)^{1/\delta}. \]

Next, we use mathematical induction. Suppose that for all 1 \leq s \leq m - 1,
\[ \| W_s - W_{s-1} \|_s \leq (|\sigma| |A|^2 k^{-\gamma_0})^s. \] (54)
The relation (43) gives that for all 1 \leq s \leq m - 1,
\[ \| W_s \|_s \leq 1 + \| V \|_s + |\sigma| |A|^2, \] (55)
\[ \| \tilde{W}_s \|_s \leq 1 + \| V \|_s. \] (56)

Let, by analogy with (31),
\[ u_s(\bar{x}) := A \sum_{m \in \mathbb{Z}^n} E_s(\bar{t})_{m,j} e^{i(\bar{p}_m(\bar{t}), \bar{x})}, \] (57)
where \( E_s(\bar{t}) \) is the spectral projection (20), corresponding to the potential \( \tilde{W}_s \). Obviously,
\[ u_s(\bar{x}) = \psi_s(\bar{x}) e^{i(\bar{p}_s(t), \bar{x})}, \] (58)
where the function,
\[ \psi_s(\bar{x}) = A \sum_{q \in \mathbb{Z}^n} E_s(\bar{t})_{j+q,j} e^{i(\bar{p}_s(0), \bar{x})}, \] (59)
is the periodic part of \( u_s \). Clearly,
\[ \| \psi_s \|_s \leq |A| \| E_s(\bar{t}) \|_1. \] (60)
Let
\[ B_s(z) = (H_0(t) - z)^{-1} W_s. \]

By (61),
\[ \|B_s(z)\|_1 < \|W_s\|_* k^{-2l-n-\delta}, \quad z \in C_0, \quad 1 \leq s \leq m - 1. \] (62)

It is easy to see that
\[ \|G_{s,r}(k,t)\|_1 < \|W_s\|_* k^{-2l-n-\delta} r, \quad 1 \leq s \leq m - 1, \] (63)

here \( G_{s,r}(k,t) \) is given by (21) with \( \tilde{W} \) instead of \( V \). It follows:
\[ \|E_s(t)\|_1 \leq 1 + \sum_{r=1}^{\infty} \|G_{s,r}(k,t)\|_1 \leq 1 + 2k^{-2l-n-\delta} (1 + \|V\|_*), \quad 1 \leq s \leq m - 1. \] (64)

Next, we note that
\[ \|G_{m-1,r}(k,t) - G_{m-2,r}(k,t)\|_1 \leq \max_{z \in C_0} \|B_{m-1}^r(z) - B_{m-2}^r(z)\|_1 \]
\[ \leq \max_{z \in C_0} \|B_{m-1}^r(z) - B_{m-2}^r(z)\|_1 (\|B_{m-1}(z)\|_1 + \|B_{m-2}(z)\|_1)^{r-1} \]
\[ \leq k^{-r(2l-n-\delta)} \|\tilde{W}_{m-1} - \tilde{W}_{m-2}\|_* (2 + 2\|V\|_*)^{r-1}. \] (65)

Estimate (65) yields that for sufficiently large \( k, k > (2 + 2\|V\|_*)^{1/(2l-n-\delta)} \):
\[ \|E_{m-1}(t) - E_{m-2}(t)\|_1 \leq \sum_{r=1}^{\infty} \|G_{m-1,r}(k,t) - G_{m-2,r}(k,t)\|_1 \]
\[ \leq 2k^{-2(l-n-\delta)} \|\tilde{W}_{m-1} - \tilde{W}_{m-2}\|_* \] (66)

Next, we obtain:
\[ \|W_m - W_{m-1}\|_* = \|M W_{m-1} - M W_{m-2}\|_* \]
\[ = \|\psi_{m-1} - \psi_{m-2}\|_* \]
\[ \leq \|\psi_{m-1} - \psi_{m-2}\|_* \]
\[ \leq \|\tilde{\psi}_{m-1} - \tilde{\psi}_{m-2}\|_* \]
\[ \leq \|\tilde{\psi}_{m-1} - \tilde{\psi}_{m-2}\|_* \]
\[ \leq \|\psi_{m-1} - \psi_{m-2}\|_* \]
\[ \leq \|\tilde{\psi}_{m-1} - \tilde{\psi}_{m-2}\|_* \]
\[ \leq \|\tilde{\psi}_{m-1} - \tilde{\psi}_{m-2}\|_* \]
\[ \leq \|\tilde{\psi}_{m-1} - \tilde{\psi}_{m-2}\|_* \] (67)

and, hence, by (65),
\[ \|W_m - W_{m-1}\|_* \leq |\sigma| |A|^2 \|E_{m-1}(t) - E_{m-2}(t)\|_1 (\|E_{m-1}(t)\|_1 + \|E_{m-2}(t)\|_1). \] (68)

Using (64) and (66), we obtain
\[ \|W_m - W_{m-1}\|_* \leq 6|\sigma| |A|^2 k^{-2l-n-\delta} \|\tilde{W}_{m-1} - \tilde{W}_{m-2}\|_* \] (69)

Considering that \( \|\tilde{W}_{m-1} - \tilde{W}_{m-2}\|_* \leq \|W_{m-1} - W_{m-2}\|_* \), we arrive at the estimate:
\[ \|W_m - W_{m-1}\|_* \leq 8|\sigma| |A|^2 k^{-2l-n-\delta} (|\sigma| |A|^2 k^{-n})^{m-1} \leq (|\sigma| |A|^2 k^{-n})^m, \]
when \( k > k_1(|V|_*), \delta) \).

**Definition 3.3.** Let \( u(x) \) be defined by Definition 2.6 for the potential \( W(x) \).
Let $\psi(x)$ be the periodic part of $u(x)$. Now, for a sufficiently large $k > k_1(\|V\|_*, \delta)$, we have the following two results.

**Lemma 3.4.** Suppose $\tilde{t}$ belongs to the $(k^{-n+1-2\delta})$-neighborhood in $K$ of the non-resonant set $\chi_0(k, \delta)$, $0 < 2\delta < 2l - n$. Then for every sufficiently large $k$, $k > k_1(\|V\|_*, \delta)$, the sequence $\psi_m(x)$ converges to the function $\psi(x)$ with respect to $\|\cdot\|_*$: 
\[
\|\psi_m - \psi\|_* < 4A^2 k^{-(2\delta-n)} (\|\sigma\| A^2 k^{-\gamma_0})^{m+1}.
\] (70)

**Corollary 3.5.** The sequence $u_m$ converges to $u$ in $L^\infty(Q)$.

**Corollary 3.6.**
\[\mathcal{M}W = W.\]

**Proof of Corollary 3.6.** Considering as in (64), we obtain:
\[
\|\mathcal{M}W_m - \mathcal{M}W\|_* \leq |\sigma| \|\psi_m - \psi\|_* \|\psi_m + \psi\|_*,
\] (71)
It immediately follows that $\mathcal{M}W_m \to \mathcal{M}W$ with respect to $\|\cdot\|_*$. Now, by (13) and Corollary 3.2, we have $\mathcal{M}W = W.$

**Proof of Lemma 3.4.** By the definition of the functions $u_m$ and $u$:
\[
\|\psi_m - \psi\|_* \leq |A| \|E_m(\tilde{t}) - E_{\tilde{W}}(\tilde{t})\|_1,
\] (72)
where $E_{\tilde{W}}(\tilde{t})$ is the spectral projection (20), corresponding to the potential $\tilde{W}$. By (55),
\[
\|W\|_* \leq 1 + \|V\|_* + |\sigma| A^2. \quad \|\tilde{W}\|_* \leq 1 + \|V\|_*.
\] (73)
Considering as in (68) and using the estimates (73), we obtain
\[
\|E_m(\tilde{t}) - E_{\tilde{W}}(\tilde{t})\|_1 \leq \sum_{r=1}^\infty \|G_{m,r}(k, \tilde{t}) - G_{\tilde{W},r}(k, \tilde{t})\|_1
\leq 2k^{-(2\delta-n)} \|W_m - \tilde{W}\|_*,
\] (74)
here $G_{\tilde{W},r}(k, \tilde{t})$ are given by (21) for $V := \tilde{W}$. Using (72) and (74), we arrive at:
\[
\|\psi_m - \psi\|_* \leq 2|A| k^{-(2\delta-n)} \|W_m - W\|_*.
\]
Taking into account (72), we arrive at (70).

Let $\lambda_m(\tilde{t})$, $\lambda_{\tilde{W}}(\tilde{t})$ be the eigenvalues (18) corresponding to $W_m$ and $\tilde{W}$, respectively.

**Lemma 3.7.** Suppose $\tilde{t}$ belongs to the $(k^{-n+1-2\delta})$-neighborhood in $K$ of the non-resonant set $\chi_0(k, \delta)$, $0 < 2\delta < 2l - n$. Then, for every sufficiently large $k$, $k > k_1(\|V\|_*, \delta)$, the sequence $\lambda_m(\tilde{t})$ converges to $\lambda_{\tilde{W}}(\tilde{t})$ in $\mathbb{R}$.

**Proof.** Considering (18) and taking into account that $2l > n$, we easily show that the trace class norm of $(H_0(\tilde{t}) - z)^{-1}$, is bounded uniformly for $z \in C_0$:
\[
\max_{z \in C_0} \|(H_0(\tilde{t}) - z)^{-1}\|_{S_1} < k^{2n-2l}.
\] (75)
It follows
\[
\|B_m(z) - B(z)\|_{S_1} < k^{2n-2l} (|\sigma| A^2 k^{-\gamma_0})^m,
\] (76)
here and below $B(z)$ is given by \( \| B_m(z) - B(z) \|_{S_1} (\| B_m(z) \| + \| B(z) \|)^{r-1} \).

Using (56), (62) and (76), we obtain that
\[
\left| g_{m,r}(k, t) - \tilde{g}_{W,m}(k, t) \right| < k^n \left( |\sigma| A^2 k^{-\gamma_0} \right)^m (2 + 2 \| V \|_\sigma)^{r-1} k^{-(2l-n-\delta)(r-1)}.
\]

Summarizing this estimate over $r \geq 2$, we obtain that for $k > k_1(\| V \|_\sigma, \delta)$:
\[
\left| \lambda_m(\tilde{t}) - \lambda_{\tilde{W}}(\tilde{t}) \right| \leq k^{n-\gamma_0} (|\sigma| A^2 k^{-\gamma_0})^m.
\]

Therefore, $\lambda_m(\tilde{t})$ converges to $\lambda_{\tilde{W}}(\tilde{t})$ in $\mathbb{R}$. 

We have the following main result for the nonlinear polyharmonic equation with quasi-periodic condition.

**Theorem 3.8.** Suppose $\tilde{t}$ belongs to the $(k^{-n+1-2\delta})$-neighborhood in $K$ of the nonresonant set $\chi_0(k, \delta)$, $0 < 2\delta < 2l - n$. Then for every sufficiently large $k$, $k > k_1(\| V \|_\sigma, \delta)$, there is a value $\lambda$ and a corresponding solution $u(\tilde{x})$ that satisfy the equation
\[
(-\Delta)^l u(\tilde{x}) + V(\tilde{x}) u(\tilde{x}) + |u(\tilde{x})|^2 u(\tilde{x}) = \lambda u(\tilde{x}), \quad \tilde{x} \in Q,
\]
and the quasi-periodic boundary condition (2). The following formulas hold:
\[
u(\tilde{x}) = Ae^{i(\tilde{p}_j(\tilde{t}), \tilde{x})} (1 + \tilde{u}(\tilde{x})�), \quad \lambda = k^{2l} + |A|^2 + O \left( (1 + |A|^2) k^{-\gamma_0 + \delta} \right), \quad |\tilde{p}_j(\tilde{t})| = k, \quad |\tilde{u}(\tilde{x})| \text{ is periodic and}
\]
\[
\| \tilde{u} \|_\sigma < k^{-\gamma_0}, \quad \gamma_0 = 2l - n - 2\delta > 0.
\]

**Proof.** Let us consider the function $u$ given by Definition 3.3 and the value $\lambda_{\tilde{W}}(\tilde{t})$. They solve the equation
\[
(-\Delta)^l u(\tilde{x}) + \tilde{W}(\tilde{x}) u(\tilde{x}) = \lambda_{\tilde{W}}(\tilde{t}) u(\tilde{x}), \quad \tilde{x} \in Q,
\]
and $u$ satisfies the quasi-boundary condition (2). By Corollary 3.3, we have
\[
\tilde{W}(\tilde{x}) = MW(\tilde{x}) = V(\tilde{x}) + |u_{\tilde{W}}(\tilde{x})|^2.
\]

Hence,
\[
\tilde{W}(\tilde{x}) = W(\tilde{x}) - \frac{1}{(2\pi)^n} \int_Q W(\tilde{x}) d\tilde{x} = V(\tilde{x}) + |u_{\tilde{W}}(\tilde{x})|^2 - \sigma \| u_{\tilde{W}} \|^2_{L_2(Q)}.
\]

Substituting the last expression into (81), we obtain that $u(\tilde{x})$ satisfies (77) with
\[
\lambda = \lambda_{\tilde{W}}(\tilde{t}) + \sigma \| u \|^2_{L_2(Q)} = \lambda_{\tilde{W}}(\tilde{t}) + \sigma |A|^2 \sum_{q \in \mathbb{Z}^2} \| (E_{\tilde{W}})_{qj} \|^2 = \lambda_{\tilde{W}}(\tilde{t}) + \sigma |A|^2 (E_{\tilde{W}})_{jj}
\]
Moreover, by the definition of $u(\vec{x})$, we have
\[ u(\vec{x}) := A e^{i(\vec{p}_j(t),\vec{x})} \left( 1 + \sum_{q \in \mathbb{Z}^N} \sum_{r=1}^{\infty} G_{\tilde{W},r}(k, t) q_{j} e_{r}(p_j(0), \vec{x}) \right). \] (83)

Using Theorem 2.1 and Corollary 2.2, we obtain (78), (80). □

4. Isoenergetic Surface

**Lemma 4.1.** The following inequalities hold for any $m = 0, 1, 2, \ldots$:
\[ \|\nabla_{\vec{r}}(W_m - W_{m-1})\|_s \leq 2k^{n-1+\delta}(8|A|^2k^{-\gamma_0})^m, \] (84)
\[ \|\nabla_{\vec{r}}(E_m - E_{m-1})\|_1 \leq 2k^{n-1-\gamma_0}(8|A|^2k^{-\gamma_0})^m, \] (85)
\[ |\nabla_{\vec{r}}(\lambda_m - \lambda_{m-1})| \leq k^{2l-1-\gamma_0}(8|A|^2k^{-\gamma_0})^m, \] (86)
where $W_1 := 0$, $E_1$, $\lambda_1$ correspond to the zero potential, $\nabla\lambda_{-1} = p_{j}^{2l-2}(t)\tilde{p}_j(t)$; $\gamma_0 = 2l - n - 2\delta > 0$, $\delta > 0$ and $|\sigma||A|^2 < k^{\gamma_1}$, $\gamma_1 < \gamma_0$, $k$ being sufficiently large $k > k_1(||V||_s, \delta)$.

**Corollary 4.2.**
\[ \|\nabla_{\vec{r}}(W - W_m)\|_s \leq 4k^{n-1+\delta}(8|A|^2k^{-\gamma_0})^{m+1}, \] (87)
\[ \|\nabla_{\vec{r}}(E_{\tilde{W}} - E_m)\|_1 \leq 4k^{n-1+\gamma_0}(8|A|^2k^{-\gamma_0})^{m+1}, \] (88)
\[ |\nabla_{\vec{r}}(\lambda_{\tilde{W}} - \lambda_m) \leq 4k^{2l+1-\gamma_0}(8|A|^2k^{-\gamma_0})^{m+1}. \] (89)

**Corollary 4.3.**
\[ \|\nabla_{\vec{r}}W\|_s \leq 32|\sigma||A|^2k^{n-1+\delta-\gamma_0}, \] (90)
\[ \|\nabla_{\vec{r}}E_{\tilde{W}}\|_1 \leq 4k^{n-1+\delta-\gamma_0}. \] (91)
\[ |\nabla_{\vec{r}}\lambda_{\tilde{W}} - p_{j}^{2l-2}(t)\tilde{p}_j(t)\| < 2k^{2l+1+\delta-\gamma_0}. \] (92)

**Proof of Corollary 4.3.** Setting $m = 0$ in (87) and $m = -1$ in (88) and (89) and taking into account that $\nabla_{\vec{r}}W_0 = 0$, $\nabla_{\vec{r}}E_{-1} = 0$ and $\nabla_{\vec{r}}\lambda_{-1} = p_{j}^{2l-2}(t)\tilde{p}_j(t)$, we obtain (87)–(89). □

**Proof of Lemma 4.1.** First, we establish a recurrent relation (102) for the left-hand part of (83). Indeed considering as in the proof of (68), we obtain:
\[ \|\nabla_{\vec{r}}(W_m - W_{m-1})\|_s \leq |\sigma||A|^2\|\nabla_{\vec{r}}(E_{m-1} - E_{m-2})\|_1 (||E_{m-1}||_1 + ||E_{m-2}||_1) \]
\[ + |\sigma||A|^2||E_{m-1} - E_{m-2}||_1 (||\nabla_{\vec{r}}E_{m-1}||_1 + ||\nabla_{\vec{r}}E_{m-2}||_1), \] (93)
where $m \geq 1$. Considering (25) and (30), we obtain:
\[ \|\nabla_{\vec{r}}(W_1 - W_0)\|_s \leq |\sigma||A|^2\|\nabla_{\vec{r}}E_0\|_1 (||E_0||_1 + ||E_0 - E_{-1}||_1) \leq 2|\sigma||A|^2k^{n-1+\delta-\gamma_0}. \] (94)

Obviously $\nabla_{\vec{r}}W_0 = 0$. Let $m \geq 2$. Using the estimates (66) and (11), we get
\[ \|\nabla_{\vec{r}}(W_m - W_{m-1})\|_s \leq 2|\sigma||A|^2\|\nabla_{\vec{r}}(E_{m-1} - E_{m-2})\|_1 \]
It easily follows:

\[
+ (\| \sigma \| A^2 k^{-\gamma_0})^m \left( \| \nabla_\mathbb{T} E_{m-1} \|_1 + \| \nabla_\mathbb{T} E_{m-2} \|_1 \right).
\]  

(95)

Now, we estimate \( \| \nabla_\mathbb{T} (E_{m-1} - E_{m-2}) \|_1 \). Obviously,

\[
\| \nabla_\mathbb{T} (E_{m-1} - E_{m-2}) \|_1 \leq \sum_{r \geq 1} \| \nabla_\mathbb{T} (G_{m-1,r} - G_{m-2,r}) \|_1.
\]

Next,

\[
\| \nabla_\mathbb{T} (G_{m-1,r} - G_{m-2,r}) \|_1
\]

\[
\leq \max_{z \in C_0} \| \nabla_\mathbb{T} (B_{m-1}^r - B_{m-2}^r) \|_1 + \max_{z \in C_0} \| (B_{m-1}^r - B_{m-2}^r) \|_1 \| \nabla_\mathbb{T} (H_0(t) - z)^{-1} \|_1
\]

\[
\leq \max_{z \in C_0} \| \nabla_\mathbb{T} (B_{m-1} - B_{m-2}) \|_1 \| (B_{m-1}^r - B_{m-2}^r) \|_1 \| \nabla_\mathbb{T} (H_0(t) - z)^{-1} \|_1
\]

\[
+ \max_{z \in C_0} \| (B_{m-1} - B_{m-2}) \|_1 \| \nabla_\mathbb{T} (B_{m-1} - B_{m-2}) \|_1 \| \nabla_\mathbb{T} (H_0(t) - z)^{-1} \|_1
\]

(96)

Obviously,

\[
\nabla_\mathbb{T} B_m = (H_0(t) - z)^{-1} \nabla_\mathbb{T} \tilde{W}_m + (\nabla_\mathbb{T} (H_0(t) - z)^{-1}) \tilde{W}_m,
\]

\[
\| \nabla_\mathbb{T} (H_0(t) - z)^{-1} \| < 2lk^{-2l+2n-1+2\delta}.
\]

It easily follows:

\[
\| \nabla_\mathbb{T} B_m \|_1 = \left\| \nabla_\mathbb{T} \tilde{W}_m \right\|_1 k^{-2l+n+\delta} + 2l(1 + \| V \|_*) k^{-2l+2n-1+2\delta},
\]

(97)

Using (13), we arrive at

\[
\| \nabla_\mathbb{T} (B_{m-1} - B_{m-2}) \|_1 = \left\| \nabla_\mathbb{T} (\tilde{W}_{m-1} - \tilde{W}_{m-2}) \right\|_1 k^{-2l+n+\delta} + 2lk^{-2l+2n-1+\delta} (|\sigma| A^2 k^{-\gamma_0})^m - 1.
\]

(98)

Substituting the estimates (97) and (98) into (96), and using (12), we obtain:

\[
\| \nabla_\mathbb{T} (G_{m-1,r} - G_{m-2,r}) \|_1 \leq \| \nabla_\mathbb{T} (W_{m-1} - W_{m-2}) \|_1 k^{-\gamma_0 r} + \left( \| \nabla_\mathbb{T} W_{m-1} \|_1 + \| \nabla_\mathbb{T} W_{m-2} \|_1 \right) \| \nabla_\mathbb{T} (W_{m-1} - W_{m-2}) \|_1 \| \sigma \| A^2 k^{-\gamma_0} \|^m - 1 k^{-\gamma_0 r}.
\]

(99)

Summarizing the last estimate over \( r \geq 1 \), we obtain:

\[
\| \nabla_\mathbb{T} (E_{m-1} - E_{m-2}) \|_1
\]

\[
< 2 \| \nabla_\mathbb{T} (W_{m-1} - W_{m-2}) \|_1 k^{-\gamma_0} + 2 \left( \| \nabla_\mathbb{T} W_{m-1} \|_1 + \| \nabla_\mathbb{T} W_{m-2} \|_1 \right) \| \nabla_\mathbb{T} (W_{m-1} - W_{m-2}) \|_1 \| \sigma \| A^2 k^{-\gamma_0} \|^m - 1 k^{-\gamma_0} + 6k^{n-1+\delta} (|\sigma| A^2 k^{-\gamma_0})^m - 1 k^{-\gamma_0}.
\]

Similarly,

\[
\| \nabla_\mathbb{T} E_{m-1} \|_1 < 2 \| \nabla_\mathbb{T} W_{m-1} \|_1 k^{-\gamma_0} + 2k^{n-1+\delta} k^{-\gamma_0}.
\]

(100)

Substituting the last two estimates into (95), we get:

\[
\| \nabla_\mathbb{T} (W_{m-1}) \|_s \leq 4 \| \nabla_\mathbb{T} (W_{m-1} - W_{m-2}) \|_1 \| \sigma \| A^2 k^{-\gamma_0} +
\]

(101)
Using (94) and mathematical induction, we easily get (84). Now, estimate (85) follows from (100). Next we apply a well known formula to for the first derivative of $\lambda_m$: $\nabla_i \lambda_m = \text{Tr} \left( E_m(t) \nabla_i (H_0(\vec{t}) + W_m) \right)$. Next, we use (84). Taking into account that $\|E_m\|_{S_1} = 1, \|E_m - E_{m-1}\|_{S_1} \leq 2\|E_m - E_{m-1}\|_1 \leq 2\|E_m - E_{m-1}\|_1$, since $E_m$ is a one dimensional projector, we arrive at (86).

\textbf{Lemma 4.4.} For any sufficiently large $\lambda$, every $A \in C : |A|^2 \sigma < k^{\gamma_0 - \delta}, k^{2l} = \lambda$, and for every $\vec{v} \in B(\lambda)$, there is a unique $\vec{x} = \vec{x}(\lambda, A, \vec{v})$ in the interval

$I := [k - k^{-2l+1+\gamma_0}, k + k^{-2l+1+\gamma_0}]$

such that

$$\lambda(\vec{x}, A) = \lambda.$$  \hfill (103)

Furthermore,

$$|\vec{x}(\lambda, A, \vec{v}) - \vec{k}| \leq c \left( 1 + |\sigma| |A|^2 \right) k^{-2l+1-\gamma_0+\delta}, \quad \vec{k} = (\lambda - |A|^2)^{1/2l}. \hfill (104)$$

\textbf{Proof.} Formulas (82), (91) and (92) yield

$$\frac{\partial \lambda(\vec{x}, A)}{\partial \vec{x}} = 2\vec{x}^{2l-1} + O(\vec{x}^{2l-1-\gamma_0}), \quad \hfill (105)$$

when $|\vec{x} - \vec{k}| < k^{-n+1-2\delta}$. The lemma easily follows from Corollary 2.2 and (105). For details see Lemma 2.10 in [1]. \hfill \blacksquare

\textbf{Theorem 4.5.} (1) For sufficiently large $\lambda$, the set $D(\lambda, A)$, defined by (6) is a distorted circle with holes; it can be described by the formula

$$D(\lambda, A) = \{ \vec{x} \in \mathbb{R}^n : \vec{x} = \vec{x}(\lambda, A, \vec{v}), \vec{v} \in B \}, \quad \hfill (106)$$

where $\vec{x}(\lambda, A, \vec{v}) = \vec{k} + h(\lambda, A, \vec{v})$ and $h(\lambda, A, \vec{v})$ obeys the inequalities

$$|h| < c \left( 1 + |\sigma| |A|^2 \right) k^{-2l+1-\gamma_0+\delta}, \quad |\nabla_{\vec{v}} h| < c \left( 1 + |\sigma| |A|^2 \right) k^{-2l+n-\gamma_0+2\delta}. \quad \hfill (107)$$

(2) The measure of $B(\lambda) \subset S_{n-1}$ satisfies the estimate

$$L(B) = \omega_{n-1} (1 + O(k^{-\delta})). \quad \hfill (108)$$

(3) The surface $D(\lambda, A)$ has the measure that is asymptotically close to that of the whole sphere of the radius $k$ in the sense that

$$|D(\lambda, A)| = \omega_{n-1} k^{n-1} (1 + O(k^{-\delta})), \quad \lambda = k^{2l}. \quad \hfill (109)$$

\textbf{Proof.} The proof is based on Implicit Function Theorem. It is completely analogous to Lemma 2.11 in [1]. \hfill \blacksquare
References

[1] Yulia E. Karpeshina, *Perturbation Theory for the Schrödinger Operator with a Periodic Potential*, Springer 1997.

[2] V. V. Konotop and M. Salerno, *Modulational instability in Bose-Einstein condensates in optical lattices*, 65, 021602, 2002.

[3] Pearl J.Y. Louis, Elena A. Ostrovskaya, Craig M. Savage and Yuri S. Kivshar, *Bose-Einstein Condensates in Optical Lattices: Band-Gap Structure and Solitons*, Phys. Rev. A 67, 013602, 2003.

[4] C. J. Pethick, H. Smith, *Bose-Einstein Condensation in Dilute Gases*, Cambridge, 2008.

[5] A. V. Yulin, Yu. V. Bludov, V. V. Konotop, V. Kuzmiak, and M. Salerno, *Superfluidity breakdown of periodic matter waves in quasi-one-dimensional annular traps via resonant scattering with moving defects*, Phys. Rev. A 87, 033625 Published 25 March 2013.

[6] Alexey V. Yulin and Dmitry V. Skryabin, *Out-of-gap Bose-Einstein solitons in optical lattices*, physical review A 67, 023611, 2003.

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