Jordanian Deformation of the Open XXX Spin Chain

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Outline

1 Introduction
   - Quantum Inverse Scattering Method
   - Twist of a Quantum Group
   - Jordanian Twist

2 Solutions of the Reflection Equation
   - Twisted Yang R-matrix
   - Reflection Equation
   - Scaling Limit of the XXZ Chain

3 Open spin chain in the framework of QISM
   - Transfer Matrix
   - Hamiltonian with the boundary terms

4 Conclusions
   - Summary
   - Outlook
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P. P. Kulish, N. Manojlović and Z. Nagy

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The quantum inverse scattering method (QISM) as an approach for constructing and solving quantum integrable systems lead to creating the theory of quantum groups.

A particularly interesting feature of quantum groups is a transformation that is called twist and allows constructing new quantum groups starting from already known ones.

Although the twist transformations generate equivalence relation among quantum groups, they produce different R-matrices.

These new R-matrices can in turn lead to new integrable systems.
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The twist of a quantum group or, more generally, a Hopf algebra \( \mathcal{A} \) is a similarity transformation of the coproduct \( \Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \) by an invertible twist element \( \mathcal{F} = \sum_j f_j^{(1)} \otimes f_j^{(2)} \in \mathcal{A} \otimes \mathcal{A}, \)

\[
\Delta(a) \rightarrow \Delta_t(a) = \mathcal{F} \Delta(a) \mathcal{F}^{-1}, \quad a \in \mathcal{A}.
\]

To guarantee the coassociativity property of the coproduct, the element \( \mathcal{F} \) has to satisfy certain compatibility condition, the so-called twist equation

\[
\mathcal{F}_{12} (\Delta \otimes \text{id}) \mathcal{F} = \mathcal{F}_{23} (\text{id} \otimes \Delta) \mathcal{F},
\]

where

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(\Delta \otimes \text{id}) \sum_j f_j^{(1)} \otimes f_j^{(2)} = \sum_j \Delta \left( f_j^{(1)} \right) \otimes f_j^{(2)} \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}.
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Twist of a Quantum Group

The **twist** of a quantum group or, more generally, a Hopf algebra $\mathcal{A}$ is a similarity transformation of the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ by an invertible twist element $\mathcal{F} = \sum_j f_j^{(1)} \otimes f_j^{(2)} \in \mathcal{A} \otimes \mathcal{A}$,

$$\Delta(a) \to \Delta_t(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \quad a \in \mathcal{A}.$$  

To guarantee the **coassociativity property of the coproduct**, the element $\mathcal{F}$ has to satisfy certain compatibility condition, the so-called **twist equation**

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Moreover, the transformation law of the coproduct also determines how the corresponding universal $R$-matrix changes

$$
R \rightarrow R^{(t)} = F_{21} R F^{-1}, \quad F_{21} = \sum_j f_j^{(2)} \otimes f_j^{(1)}.
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This new $R$-matrix allows building and studying new integrable models.
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A particular solution of the twist equation is provided by the Jordanian twist element for the $\mathcal{U}(sl(2))$ (M. Gerstenhaber, A. Giaquinto and S. D. Schack and O. V. Ogievetsky in ’92).

Because the twist preserves the regularity of the $R$-matrix ($R(0) = \mathcal{P}$), we can write the deformed Hamiltonian of the Heisenberg XXX spin chain with periodic boundary conditions

$$H = \sum_{j=1}^{N} \left( \frac{1}{2} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z \right) + \theta \left( \sigma_j^+ - \sigma_{j+1}^+ \right) + \theta^2 \sigma_j^+ \sigma_{j+1}^+ \right).$$

This operator is non-Hermitian, which results in additional difficulties in the application of the algebraic Bethe ansatz to this model. Although it can be seen that the extra terms added to the XXX Hamiltonian do not change the spectrum of the model, the explicit form of the Bethe states is not obvious (Kulish and Stolin ’97).
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We study the deformation by the Jordanian twist of the \textit{XXX}-model with nonperiodic boundary conditions, which are described by the reflection matrices $K^\pm(\lambda)$. The result is a classification of reflection matrices compatible with the twisted Jordanian $R$-matrix. We obtain the general solution of the reflection equation by a direct calculation; it is also confirmed by the singular scaling limit from the known reflection matrix of the XXZ-model. Using the general solution for $K(\lambda)$ and following Sklyanin approach, we construct the Hamiltonian with the general nonperiodic boundary conditions.
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P. P. Kulish, N. Manojlović and Z. Nagy

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Yangian $\mathcal{Y}(sl(2))$

The main relation of the quantum inverse scattering method

$$R_{12}(\lambda - \mu) T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) R_{12}(\lambda - \mu)$$

can be used to define a special infinite dimensional quantum algebra - the Yangian. The Yang R-matrix

$$R_{12}(\lambda) = \lambda 1 + \eta P \in \text{End} (\mathbb{C}^n \otimes \mathbb{C}^n)$$

gives rise to the Yangian $\mathcal{Y}(sl(n))$ with the entries of the $n \times n$ matrix $T(\lambda)$ as generating functions of the $\mathcal{Y}(sl(n))$ generators. The Heisenberg XXX-spin chain is related to the $\mathcal{Y}(sl(2))$, and the universal enveloping algebra of $sl(2)$ is a Hopf subalgebra of the Yangian: $U(sl(2)) \subset \mathcal{Y}(sl(2))$. 
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The two generators $h$ and $X^\pm$ of $sl(2)$

$$[[h, X^\pm], X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = h,$$

yield the Jordanian twist element

$$\mathcal{F} = \exp\left(\frac{h}{2} \otimes \ln(1 + 2\theta X^+)\right) \in U(sl(2)) \otimes U(sl(2))$$

which satisfies the Drinfeld twist equation. The matrix form of $\mathcal{F}$ in the spin-1/2 representation $\rho$ is $F_{12} \in \text{End} \left( \mathbb{C}^2 \otimes \mathbb{C}^2 \right)$

$$F_{12} = (\rho \otimes \rho) \mathcal{F} = \exp \left( \sigma^z \otimes \theta \sigma^+ \right) = 1 + \theta \sigma^z \otimes \sigma^+ = \begin{pmatrix} 1 & \theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\theta \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\sigma^z, \sigma^\pm = (\sigma^x \pm i\sigma^y)/2$ are Pauli matrices.
Twisted Yang R-matrix

Hence, the $R$-matrix of the twisted Yangian $\mathcal{Y}_\theta(sl(2))$ has the following form

$$R^{(i)}(\lambda) = F_{21} R_{12}(\lambda) F_{12}^{-1} = \lambda R^{(i)} + \eta \mathcal{P} = \begin{pmatrix} \lambda + \eta & -\lambda \theta & \lambda \theta & \lambda \theta^2 \\ 0 & \lambda & \eta & -\lambda \theta \\ 0 & \eta & \lambda & \lambda \theta \\ 0 & 0 & 0 & \lambda + \eta \end{pmatrix},$$

where $F_{21} = \mathcal{P} F_{12} \mathcal{P}$. This R-matrix is also a solution of the Yang-Baxter equation

$$R_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda - \mu).$$
Twisted Yang R-matrix

- The unitarity property of the $R$-matrix is unaffected by the twist, $R_{12}(\lambda)R_{21}(-\lambda) = g(\lambda)$ with $g(\lambda) = (-\lambda^2 + \eta^2)$.
- The PT symmetry is broken, $R_{21}(\lambda) \neq R_{12}(\lambda)^{t_1 t_2}$.
- The $R$-matrix also does not have the crossing symmetry, but it does satisfy the weaker condition

$$\left\{\{R_{12}(\lambda)^{t_2}\}^{-1}\right\}^{t_2} = \frac{g(\lambda + \eta)}{g(\lambda + 2\eta)} M_2 R_{12}(\lambda + 2\eta) M_2^{-1},$$

with a matrix $M = \begin{pmatrix} 1 & -2\theta \\ 0 & 1 \end{pmatrix}$. We note that the more general matrix

$$\tilde{M} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

commutes with the $R$-matrix, $[\tilde{M} \otimes \tilde{M}, R(\lambda)] = 0.$
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   - Summary
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A way to introduce nonperiodic boundary conditions which are compatible with the integrability of the bulk model, was developed by Sklyanin ’88. The boundary conditions at the left and right sites of the system are expressed in the left and right reflection matrices $K^-$ and $K^+$. The compatibility condition between the bulk and the boundary of the system takes the form of the so-called reflection equation (RE). For the left reflection matrix, it is written in the form

$$R_{12}(\lambda - \mu) K^-_1(\lambda) R_{21}(\lambda + \mu) K^-_2(\mu) = K^-_2(\mu) R_{12}(\lambda + \mu) K^-_1(\lambda) R_{21}(\lambda - \mu).$$

The compatibility at the right site of the model is expressed in the dual reflection equation

$$A_{12}(\lambda - \mu) K^+_1(\lambda) B_{12}(\lambda + \mu) K^+_2(\mu) = K^+_2(\mu) C_{12}(\lambda + \mu) K^+_1(\lambda) D_{12}(\lambda - \mu).$$
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$$ A_{12}(\lambda - \mu)K^+_1(t(\lambda))B_{12}(\lambda + \mu)K^+_2(t(\mu)) = K^+_2(t(\mu))C_{12}(\lambda + \mu)K^+_1(t(\lambda))D_{12}(\lambda - \mu), $$
A way to introduce nonperiodic boundary conditions which are compatible with the integrability of the bulk model, was developed by Sklyanin ’88. The boundary conditions at the left and right sites of the system are expressed in the left and right reflection matrices $K^-$ and $K^+$. The compatibility condition between the bulk and the boundary of the system takes the form of the so-called reflection equation (RE). For the left reflection matrix, it is written in the form

$$R_{12}^{}(\lambda - \mu)K_{1}^{}(\lambda)R_{21}^{}(\lambda + \mu)K_{2}^{}(\mu) = K_{2}^{}(\mu)R_{12}^{}(\lambda + \mu)K_{1}^{}(\lambda)R_{21}^{}(\lambda - \mu).$$

The compatibility at the right site of the model is expressed in the dual reflection equation

$$A_{12}^{}(\lambda - \mu)K_{1}^{+ t}(\lambda)B_{12}^{}(\lambda + \mu)K_{2}^{+ t}(\mu) = K_{2}^{+ t}(\mu)C_{12}^{}(\lambda + \mu)K_{1}^{+ t}(\lambda)D_{12}^{}(\lambda - \mu).$$
where the matrices $A, B, C, D$ are obtained from the $R$-matrix as

\[ A_{12}(\lambda) = (R_{12}(\lambda)^{t_{12}})^{-1} = D_{21}(\lambda), \]
\[ B_{12}(\lambda) = \left(\left(R_{21}(\lambda)^{t_1}\right)^{-1}\right)^{t_2} = C_{21}(\lambda), \]

or, explicitly,

\[ A(\lambda) = \frac{1}{\lambda^2 - \eta^2} \left( \begin{array}{cccc}
\lambda - \eta & 0 & 0 & 0 \\
\lambda \theta & \lambda & -\eta & 0 \\
-\lambda \theta & -\eta & \lambda & 0 \\
\lambda \theta^2 & \lambda \theta & -\lambda \theta & \lambda - \eta 
\end{array} \right), \]
\[ B(\lambda) = \frac{1}{\lambda(\lambda + 2\eta)} \left( \begin{array}{cccc}
\lambda + \eta & 0 & 0 & 0 \\
-\lambda \theta & \lambda + 2\eta & -\eta & 0 \\
\lambda \theta & -\eta & \lambda + 2\eta & 0 \\
-(3\lambda + 2\eta)\theta^2 & -\lambda \theta & \lambda \theta & \lambda + \eta 
\end{array} \right). \]
where the matrices $A, B, C, D$ are obtained from the $R$-matrix as

$$A_{12}(\lambda) = (R_{12}(\lambda)^{t_{12}})^{-1} = D_{21}(\lambda),$$

$$B_{12}(\lambda) = \left(\left(R_{21}^{t_1}(\lambda)^{-1}\right)^{t_2}\right) = C_{21}(\lambda),$$

or, explicitly,

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\lambda \theta^2 & \lambda \theta & -\lambda \theta & \lambda - \eta
\end{pmatrix},$$

$$B(\lambda) = \frac{1}{\lambda(\lambda + 2\eta)} \begin{pmatrix}
\lambda + \eta & 0 & 0 & 0 \\
-\lambda \theta & \lambda + 2\eta & -\eta & 0 \\
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-(3\lambda + 2\eta)\theta^2 & -\lambda \theta & \lambda \theta & \lambda + \eta
\end{pmatrix}.$$
But using the crossing property we can write the dual reflection equation in the equivalent form

\[ R_{12}(-\lambda + \mu)K_1^+(\lambda)M_2R_{21}(-\lambda - \mu - 2\eta)M_2^{-1}K_2^+(\mu) = \]
\[ K_2^+(\mu)M_1R_{12}(-\lambda - \mu - 2\eta)M_1^{-1}K_1^+(\lambda)R_{21}(-\lambda + \mu). \]

It can then be verified that the map

\[ K^+(\lambda) = K^-(-\lambda - \eta) M \]

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is a bijection between solutions of the reflection equation and the dual reflection equation.
Solving the RE

We first note that if $K(\lambda)$ is a solution of this equation, then obviously $f(\lambda)K(\lambda)$ is also. We use this freedom to fix $k_{11}(\lambda) = 1$. We seek the general solution in the form

$$K^{-}(\lambda) = \begin{pmatrix} 1 & k_{12}(\lambda) \\ k_{21}(\lambda) & k_{22}(\lambda) \end{pmatrix}.$$

Replacing this $K$-matrix into the reflection equation, we see that by adding elements 21 and 31 of the resulting matrix equation, we obtain

$$k_{21}(\lambda) (k_{22}(\mu) - 1) = k_{21}(\mu) (k_{22}(\lambda) - 1).$$

This is a functional equation of the form

$$f(\lambda)g(\mu) = f(\mu)g(\lambda).$$

We recall its general solution. The equation is obviously satisfied if either one of the functions is identically zero. If one of them is not identically zero, they are proportional to each other.
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According to this, the solution of the equation above splits in two cases:

1. $k_{22}(\lambda) = \phi k_{21}(\lambda) + 1$, $\phi \in \mathbb{C}$;

Then the entry 21 of the reflection equation yields

$$k_{21}(\lambda) \left(2\eta\mu + \eta\phi\mu k_{21}(\mu) + 2\theta\mu^2 k_{21}(\mu)\right) =$$
$$k_{21}(\mu) \left(2\eta\lambda + \eta\phi\lambda k_{21}(\lambda) + 2\theta\lambda^2 k_{21}(\lambda)\right),$$

which is an algebraic equation for $k_{21}(\lambda)$,

$$k_{21}(\lambda) = \frac{2\eta\lambda}{\xi - \eta\phi\lambda - 2\theta\lambda^2},$$

where $\xi \in \mathbb{C}$ is an arbitrary constant.
Solving the RE

We have now the expression of two elements,

\[ k_{22}(\lambda) = 1 + \frac{2\phi\eta\lambda}{\xi - \eta\phi\lambda - 2\theta\lambda^2}. \]

Substituting these expressions in the reflection equation we obtain an equation for \( k_{12}(\lambda) \)

\[ \lambda \, k_{12}(\mu) \left( \xi - (\eta\phi + 2\theta\mu)\mu \right) = \mu \, k_{12}(\lambda) \left( \xi - (\eta\phi + 2\theta\lambda)\lambda \right), \]

which has the solution

\[ k_{12}(\lambda) = \frac{\psi\lambda}{\xi - \eta\phi\lambda - 2\theta\lambda^2}, \]

with arbitrary constant \( \psi \in \mathbb{C} \).
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\[ \lambda \ k_{12}(\mu) (\xi - (\eta\phi + 2\theta\mu)\mu) = \mu \ k_{12}(\lambda) (\xi - (\eta\phi + 2\theta\lambda)\lambda), \]

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\[ \lambda \ k_{12}(\mu) (\xi - (\eta \phi + 2\theta \mu) \mu) = \mu \ k_{12}(\lambda) (\xi - (\eta \phi + 2\theta \lambda) \lambda), \]

which has the solution

\[ k_{12}(\lambda) = \frac{\psi \lambda}{\xi - \eta \phi \lambda - 2\theta \lambda^2}, \]

with arbitrary constant \( \psi \in \mathbb{C} \).
\[ k_{21}(\lambda) = 0; \]

Plugging this assumption into the RE leads to

\[
\lambda \left(1 + k_{22}(\lambda)\right) \left(k_{22}(\mu) - 1\right) = \mu \left(1 + k_{22}(\mu)\right) \left(k_{22}(\lambda) - 1\right). 
\]

Here we can assume without loss of generality that \( k_{22} \neq 1 \).

Then the general solution depends on an arbitrary parameter \( \xi \)

\[
k_{22}(\lambda) = \frac{\xi + \lambda}{\xi - \lambda}.
\]

Replacing this in the reflection equation leads to

\[
\lambda k_{12}(\mu)(\mu - \xi) - \mu k_{12}(\lambda)(\lambda - \xi) = 0,
\]

which has the solution depending on an arbitrary constant \( \psi \)

\[
k_{12}(\lambda) = \frac{\psi \lambda}{\xi - \lambda}.
\]
Solutions of the RE

We can thus identify two families of reflection matrices. The first family depends on three arbitrary parameters,

\[
K^-(\lambda, \psi, \phi, \xi) = \begin{pmatrix}
\xi - \phi \eta \lambda - 2\theta \lambda^2 & \psi \lambda \\
2\eta \lambda & \xi + \phi \eta \lambda - 2\theta \lambda^2
\end{pmatrix},
\]

and the second family depends on two,

\[
K^-(\lambda, \psi, \xi) = \begin{pmatrix}
\xi - \lambda & \psi \lambda \\
0 & \xi + \lambda
\end{pmatrix}.
\]
The obtained form of the solutions after rescaling and redefining the parameters can be transformed into one family with a more familiar form, reminiscent of the general XXX solution,

\[ K^-(\lambda, \xi_-, \phi_-, \psi_-) = K^-_{\text{XXX}}(\lambda, \xi_-, \phi_-, \psi_-) - \phi_- \theta \lambda^2 \mathbb{1} \]

\[ = \begin{pmatrix} \xi_- - \lambda - \phi_- \theta \lambda^2 & \psi_- \lambda \\ \eta \phi_- \lambda & \xi_- + \lambda - \phi_- \theta \lambda^2 \end{pmatrix} . \]

As previously mentioned, because of the weak crossing relation, the general solution of the dual equation is

\[ K^+(\lambda, \xi_+, \phi_+, \psi_+) = K^-(\lambda + \eta, \xi_+, \phi_+, \psi_+) M. \]
The obtained form of the solutions after rescaling and redefining the parameters can be transformed into one family with a more familiar form, reminiscent of the general XXX solution,

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K^- (\lambda, \xi_-, \phi_-, \psi_-) = K^-_{XXX} (\lambda, \xi_-, \phi_-, \psi_-) - \phi_- \theta \lambda^2 \mathbb{1} \\
= \begin{pmatrix}
\xi_- - \lambda - \phi_- \theta \lambda^2 & \psi_- \lambda \\
\eta \phi_- \lambda & \xi_- + \lambda - \phi_- \theta \lambda^2
\end{pmatrix}.
\]

As previously mentioned, because of the weak crossing relation, the general solution of the dual equation is

\[
K^+ (\lambda, \xi_+, \phi_+, \psi_+) = K^- (-\lambda - \eta, \xi_+, \phi_+, \psi_+) M.
\]
Outline

1. Introduction
   - Quantum Inverse Scattering Method
   - Twist of a Quantum Group
   - Jordanian Twist

2. Solutions of the Reflection Equation
   - Twisted Yang R-matrix
   - Reflection Equation
   - Scaling Limit of the XXZ Chain

3. Open spin chain in the framework of QISM
   - Transfer Matrix
   - Hamiltonian with the boundary terms

4. Conclusions
   - Summary
   - Outlook
Many relations of the **XXX spin chain** can be obtained from the **XXZ model** by simple **scaling** i.e., degeneration of the trigonometric functions to the rational ones. It is known that the Jordanian deformation of the XXX chain can also be obtained by a scaling limit with an additional (singular) similarity transformation of the XXZ model. For this, we start from the R-matrix related to the quantum algebra $U_q(sl(2))$

$$
\tilde{R}(u, q) = u \tilde{R}(q) - \frac{1}{u} \tilde{R}^{-1}(q),
\tilde{R}(q) = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & \omega(q) & 0 \\
0 & 0 & 0 & q
\end{pmatrix},
$$

where $\omega(q) = q - q^{-1}$ and we use the multiplicative parameter $u = e^\lambda$. 
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Many relations of the **XXX spin chain** can be obtained from the **XXZ model** by simple **scaling** i.e., degeneration of the trigonometric functions to the rational ones. It is known that the **Jordanian deformation of the XXX chain** can also be obtained by a **scaling limit** with an additional (singular) **similarity transformation of the XXZ model**. For this, we start from the R-matrix related to the quantum algebra $U_q(sl(2))$

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where $\omega(q) = q - q^{-1}$ and we use the multiplicative parameter $u = e^\lambda$. 

P. P. Kulish, N. Manojlović and Z. Nagy

**Jordanian Deformation of the Open XXX Chain**
After the transformation

$$\tilde{R}(u, q) \rightarrow \text{Ad}J(x) \otimes J(x) \tilde{R}(u, q),$$

with a 2x2 matrix

$$J(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

we consider the scaling limit

$$u = \exp(\epsilon \lambda), \quad q = \exp(\epsilon \eta), \quad x = \frac{\theta}{\eta \epsilon}, \quad \epsilon \to 0.$$ 

In the limit, the transformed R-matrix is proportional to the R-matrix of the twisted Yangian $\mathcal{Y}(sl(2))$

$$\tilde{R}(\lambda, \eta, \theta) = \lambda \tilde{R}^{(j)}(\theta) + \eta \mathbb{1},$$

and hence yields the deformed XXX model.
Scaling Limit

We apply the scaling also to the K-matrix

\[
K(u) = \begin{pmatrix}
  f + u^2 a & (u^2 - u^{-2})b \\
  (u^2 - u^{-2})c & f + u^{-2} a
\end{pmatrix},
\]

where the parameters \(a, b, c,\) and \(f\) are arbitrary. To obtain a finite solution after the similarity transformation

\[K(u) \rightarrow \text{Adj}(x)K(u),\]

we must use the scaling

\[
\begin{align*}
  f &= -a + \epsilon \zeta, \\
  a &= a_0 - \frac{2\theta c}{\eta \epsilon}, \\
  b &= b_0 + \frac{\theta}{\eta \epsilon}(a_0 - \frac{\theta}{\eta \epsilon}c),
\end{align*}
\]

and also to consider the first three terms in the expansion of \(u = e^{(\epsilon \lambda)}\).
We apply the scaling also to the K-matrix

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and also to consider the first three terms in the expansion of \( u = e^{(\epsilon \lambda)}. \)
We then obtain

\[
K(\lambda) = \begin{pmatrix}
\zeta + 2a_0\lambda - \frac{4\theta c}{\eta} \lambda^2 & 4b_0\lambda \\
4c\lambda & \zeta - 2a_0\lambda - \frac{4\theta c}{\eta} \lambda^2
\end{pmatrix}
\]

as the limiting K-matrix. Obviously if we set \( \zeta = \xi_-, 2a_0 = -1, 4c = \eta\phi_-, 4b_0 = \psi_- \) this K-matrix coincide the general solution of the RE obtained perviously.
Outline

1. Introduction
   - Quantum Inverse Scattering Method
   - Twist of a Quantum Group
   - Jordanian Twist

2. Solutions of the Reflection Equation
   - Twisted Yang R-matrix
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   - Summary
   - Outlook

P. P. Kulish, N. Manojlović and Z. Nagy

Jordanian Deformation of the Open XXX Chain
To construct deformed integrable open spin chains, we follow the method proposed by Sklyanin in '88. Using two arbitrary solutions $K^{-}(\lambda)$, $K^{+}(\lambda)$ of the reflection equations we obtain the open chain transfer matrix

$$t(\lambda) = tr_0 K_0^{+}(\lambda) T_0(\lambda) K_0^{-}(\lambda) \hat{T}_0(\lambda),$$

where the monodromy matrices are given by

$$T_0(\lambda) = R_{0N}(u) \cdots R_{01}(\lambda), \quad \hat{T}_0(\lambda) = R_{10}(\lambda) \cdots R_{N0}(\lambda).$$

The index 0 refers to the auxiliary space $\mathbb{C}^2$, while the indices $j = 1, 2, \ldots, N$ refer to the spin 1/2 spaces at the sites of the chain.
The transfer matrices at different values of the spectral parameter commute,

\[ [t(\lambda), t(\mu)] = 0. \]

The open spin chain Hamiltonian is obtained from \( t'(0) = \frac{d}{d\lambda} t(\lambda) |_{\lambda=0} \).

Normalizing the matrix \( K^- (0) = 1 \), we write the derivative of the transfer matrix in the form (all arguments are zero)

\[ t'(0) \propto \left( tr_0 K^+_0' \right) + \left( tr_0 K^+_0 \right) K^-_1 + \frac{2}{\eta} tr_0 \left( K^+_0 \tilde{R}'_{N0} \right) + \frac{2}{\eta} tr_0 K^+_0 \sum_{j=1}^{N-1} \tilde{R}'_{j,j+1}. \]
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\[
t'(0) \propto \left( \text{tr}_0 K_0^+ \right) + \left( \text{tr}_0 K_0^+ \right) K_1^- + \frac{2}{\eta} \text{tr}_0 \left( K_0^+ \tilde{R}_N^0 \right) + \frac{2}{\eta} \text{tr}_0 K_0^+ \sum_{j=1}^{N-1} \tilde{R}_{j,j+1}.'
\]
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   - Hamiltonian with the boundary terms

4 Conclusions
   - Summary
   - Outlook
From the expression above, we extract the Hamiltonian (Sklyanin ’88)

\[ H = \sum_{j=1}^{N-1} \hat{R}_{j,j+1} + \frac{tr_0 K_0^+ \hat{R}_{N0}'}{tr_0 K_0^+} + \frac{\eta}{2} K_1' \].

Substituting the general boundary matrices, we obtain the open chain Hamiltonian

\[ H = \sum_{j=1}^{N-1} \frac{1}{2} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z \right) + \theta \left( \sigma_j^+ - \sigma_{j+1}^+ \right) + \theta^2 \sigma_j^+ \sigma_{j+1}^-

+ \frac{\eta}{2} \left( -\sigma_1^z + \psi - \sigma_1^+ + \phi - \sigma_1^- \right)

+ \frac{\eta}{2 \xi_+} \left( (1 - \theta \psi) \sigma_N^z - (\theta (2 - \theta \phi) + \psi) \sigma_N^+ - \phi + \sigma_N^- \right). \]

This Hamiltonian has a general set of open boundary parameters compatible with the integrability of the $XXX_\theta$ model in the bulk.
From the expression above, we extract the Hamiltonian (Sklyanin ’88)

\[ H = \sum_{j=1}^{N-1} \mathcal{R}'_{j,j+1} + \frac{tr_0 K^+_0 \mathcal{R}'_{N0}}{tr_0 K^+_0} + \frac{\eta}{2} K^{-}_1. \]

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\[ + \frac{\eta}{2} \left( -\sigma^z_1 + \psi - \sigma^+_1 + \phi - \sigma^-_1 \right) \]
\[ + \frac{\eta}{2 \zeta_+} \left( (1 - \theta \psi_+) \sigma^z_N - (\theta (2 - \theta \phi_+) + \psi_+) \sigma^+_N - \phi + \sigma^-_N \right). \]

This Hamiltonian has a general set of open boundary parameters compatible with the integrability of the XXX\_\theta model in the bulk.
Like in the periodic case, the spectrum of the free end deformed model with constant reflection matrices \( K^{-}(\lambda) = \mathbb{1} \) and \( K^{+}(\lambda) = M \) coincides with the spectrum of the free end XXX-spin chain due to the connection of the corresponding Hamiltonians by the similarity transformation.
In order to study the spectrum for the general boundary condition we notice that the obtained reflection matrix $K(\lambda)$ can be put into diagonal form. Its eigenvalues are

$$\epsilon_{1,2}(\lambda) = \xi - \phi_+ \theta \lambda^2 \pm \lambda \sqrt{1 + \gamma^2}, \quad \gamma^2 = \eta \psi_- \phi_-$$

and the matrix of the corresponding eigenvectors $U$ does not depend on $\lambda$

$$K(\lambda) U = U \text{ diag } (\epsilon_1(\lambda), \epsilon_2(\lambda)),$$

$$U = \begin{pmatrix}
\frac{1}{(x + 1)} & \frac{-1}{(x - 1)} \\
\psi_- & \psi_-
\end{pmatrix},$$

with $x = \sqrt{1 + \gamma^2}$. 
However, the approach by J. Avan and collaborators ’04 to get the Bethe equations defining the parameters of the spin Hamiltonian eigenvectors is not valid in the case of the deformed model since the R-matrix is not $SL(2)$ invariant.
Outline

1. Introduction
   - Quantum Inverse Scattering Method
   - Twist of a Quantum Group
   - Jordanian Twist

2. Solutions of the Reflection Equation
   - Twisted Yang R-matrix
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   - Scaling Limit of the XXZ Chain

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   - Transfer Matrix
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4. Conclusions
   - Summary
   - Outlook
We have considered the XXX spin chain with nonperiodic boundary conditions deformed by the Jordanian twist.

The twisted $R$-matrix has weaker symmetry properties than usual. Despite this, the left and right reflection equations and their solutions are not independent, but are related by a simple bijection.

The obtained solutions $K^\pm(\lambda)$ of the reflection equation and its dual depend explicitly on the deformation parameter $\theta$.

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