THE GEOMETRY OF RANDOM TOURNAMENTS

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ABSTRACT. A tournament is an orientation of a graph. Each edge represents a match, directed towards the winner. The score sequence lists the number of wins by each team. Landau (1953) characterized score sequences of the complete graph. Moon (1963) showed that the same conditions are necessary and sufficient for mean score sequences of random tournaments.

We present short and natural proofs of these results that work for any graph using zonotopes from convex geometry. A zonotope is a linear image of a cube. Moon’s Theorem follows by identifying elements of the cube with distributions and the linear map as the expectation operator. Our proof of Landau’s Theorem combines zonotopal tilings with the theory of mixed subdivisions. We also show that any mean score sequence can be realized by a tournament that is random within a subforest, and deterministic otherwise.

1. INTRODUCTION

Let $G = (V,E)$ be a graph on $V = [n]$. A tournament on $G$ is an orientation of $E$. Intuitively, for each edge $(i,j) \in E$, teams $i$ and $j$ play a match, and then this edge is directed towards the winner. The score sequence $s = (s_1, \ldots, s_n)$ lists the number of wins by each team.

Motivated by observations by Allee [2] on pecking orders in animal populations, Rapoport [17,19] and Landau [10,12] pioneered the mathematical study of dominance relations. Applications include paired comparisons, elections and sporting events, see e.g. [9,15,20]. Landau is well-known for his characterization [12] of score sequences in the case that $G = K_n$ is the complete graph. Note that, in such a tournament, a single match is held between each pair of teams.

Theorem 1 (Landau’s Theorem). Any $s \in \mathbb{Z}_n$ is the score sequence for a tournament on $K_n$ if and only if $\sum_i s_i = \binom{n}{2}$ and $\sum_{i \in A} s_i \geq \binom{k}{2}$ for all $A \subset [n]$ of size $k$.

The necessity of these conditions is clear, since the total wins by any $k$ teams is at least the number of matches held between them. Sufficiency is less obvious; however, many proofs have since appeared, see e.g. [3,6,22].

Ten years later, Moon [14] discovered that the same conditions are necessary and sufficient for random tournaments on $K_n$.

Definition 2. A random tournament on $G$ is a collection of real numbers $p_{ij} \in [0,1]$, $i < j$, one for each edge $(i,j) \in E$. The mean score sequence $x = (x_1, \ldots, x_n)$ of a
random tournament is given by
\[ x_i = \sum_{j>i \atop (i,j) \in E} p_{ij} + \sum_{j<i \atop (i,j) \in E} (1 - p_{ji}). \]

Intuitively, each edge in \( G \) corresponds to a match between two teams \( i < j \) and the value \( p_{ij} \) is the probability that team \( i \) wins and \( 1 - p_{ij} \) is the probability that team \( j \) wins. The entry \( x_i \) of the mean score sequence is the expected number of wins for team \( i \).

**Theorem 3** (Moon’s Theorem).  Any \( x \in \mathbb{R}^n \) is the mean score sequence for a random tournament on \( K_n \) if and only if \( \sum_i x_i = \binom{n}{2} \) and \( \sum_{i \in A} x_i \geq \binom{|A|}{2} \) for all \( A \subset [n] \) of size \( k \).

Generalizations of these results have been studied in e.g. [4,6,8].

Our purpose is to provide short and natural proofs of Landau’s and Moon’s Theorems for any graph \( G \) using zonotopes from convex geometry. We were led in this direction by realizing that the hyperplane description of the permutahedron, the graphic zonotope \( Z_G \) when \( G = K_n \) (see below for definitions), coincides with the conditions in Theorems 1 and 3. It appears that this connection has not been fully capitalized on in the literature. For instance, seen in this light, Theorem 3 is immediate by earlier work of Rado [16]. We think the zonotopal perspective will be useful for studying further properties of tournaments. Finally, we mention here that our arguments extend immediately to any multigraph \( M \), however, as this becomes notationally cumbersome, we leave this to the interested reader.

Throughout, we fix a graph \( G = (V,E) \) with \( V = [n] \) and \( |E| = m \).

### 2. A ZONOTOPAL PROOF OF MOON’S THEOREM

A zonotope [23] is an affine image of a cube. In particular, the graphic zonotope \( Z_G \) of the graph \( G = (V,E) \) is the polytope given by the Minkowski sum
\[ Z_G = \sum_{(i,j) \in E \atop i < j} [e_i, e_j]. \tag{1} \]

In this case, \( Z_G \) is the image of the cube \([0,1]^m\), where recall we let \( |E| = m \). To understand the projection map, let \( e_{ij} \), for \( i < j \) and \( (i,j) \in E \), denote a basis vector of \( \mathbb{R}^m \) and let \( e_i \) denote a basis vector of \( \mathbb{R}^n \). Then the projection map \( \pi : \mathbb{R}^m \to \mathbb{R}^n \) satisfies
\[ \pi(a_{ij} e_{ij}) = a_{ij} e_i + (1 - a_{ij}) e_j. \tag{2} \]

Hence, for any \( \{a_{ij} : i < j\} \in [0,1]^m \), the image under \( \pi \) of any \( \sum_{i<j} a_{ij} e_{ij} \) is the vector with \( i \)th coordinate \( \sum_{j<i} a_{ij} + \sum_{j<i} (1 - a_{ij}) \). Hence Moon’s Theorem essentially follows by identifying \( \{a_{ij} : i < j\} \) with a random tournament. Then the projection map \( \pi \) is simply the expectation operator.

**Theorem 4** (Generalized Moon’s Theorem).  For any \( A \subset [n] \) let \( \phi(A) \) be the number of edges in the subgraph \( G[A] \) of \( G \) induced by \( A \). Then any \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) is the mean score sequence of a random tournament on \( G \) if and only if \( \sum_i x_i = m \) and \( \sum_{i \in A} x_i \geq \phi(A) \) for all \( A \subset [n] \).
**Proof.** Identify the cube \([0,1]^m\) with the set of random tournaments on \(G\) by mapping \(\{a_{ij} : i < j\}\) to the random tournament \(X\) on \(G\) where \(p_{ij} = a_{ij}\). As discussed, it follows by (2) that the image of \(X\) under \(\pi\) is its mean score sequence. Therefore, \(x \in \mathbb{R}^n\) is a mean score sequence if and only if \(x \in \text{image}(\pi) = Z_G\).

To conclude, we use the following hyperplane description of the graphic zonotope \(Z_G\), which follows from [1]:

\[
Z_G = \{x \in \mathbb{R}^n : \sum_i x_i = m, \sum_{i \in A} x_i \geq \phi(A), \forall A \subset [n]\}.
\]  

(3)

This description is obtained from the one in [1] by realizing that \(\phi(A) = \mu([n]) - \mu(A)\), where \(\mu\) is the submodular function that defines \(Z_G\). ■

**Remark.** In the classical case, when \(G = K_n\), the conditions above coincide with \(x\) being majorized, as in [13], by \((0,1,\ldots,n-1)\), and (3) is a result of Rado [16].

3. A Refinement

Next, by combining zonotopal tilings with the theory of mixed subdivisions, we obtain a refinement of Theorem 4 that implies a generalization of Landau’s theorem. Informally, we find that any mean score sequence \(x\) can be realized by a tournament with at most a “forest’s worth of randomness.”

**Theorem 5.** For any \(x \in Z_G\) there exists a forest \(F \subset G\) and a random tournament \(X\) on \(G\) with mean score sequence \(x\) such that for every edge \((i, j) \not\in E(F)\), we have \(p_{ij} = 0\) or 1 in the tournament \(X\). Furthermore, if \(x \in \mathbb{Z}^n\), then the same is true for \((i, j) \in E(F)\). Hence, in this case, \(x\) is the score sequence of a (deterministic) tournament on \(G\).

We obtain the following immediately.

**Corollary 6** (Generalized Landau’s Theorem). Any \(s = (s_1,\ldots,s_n) \in \mathbb{Z}^n\) is the score sequence of a tournament on \(G\) if and only if \(\sum_i s_i = m\) and \(\sum_{i \in A} s_i \geq \phi(A)\) for all \(A \subset [n]\).

To prove these results, we will need to recall two different types of subdivisions of polytopes. First, a **zonotopal subdivision** of a zonotope \(P\) is a collection of zonotopes \(\{P_i\}\) such \(\bigcup P_i = P\) and any two zonotopes \(P_i\) and \(P_j\) intersect properly; i.e., \(P_i\) and \(P_j\) intersect at a face of both or not at all, and their intersection is also in the collection \(\{P_i\}\). We call the zonotopes \(P_i\) the **tiles** of the subdivision. The following lemma is implicit in the proof of Theorem 2.2 in Stanley [21].

**Lemma 7.** There are vectors \(v_F \in \mathbb{R}^n\) for each forest \(F \subset G\) such that \(\{v_F + Z_F\}\) is a zonotopal subdivision of \(Z_G\). The full-dimensional tiles are those corresponding to spanning forests. Furthermore, every lattice point in \(Z_G\) appears as a vertex of a zonotope \(v_F + Z_F\) for some spanning forest \(F\).

The last statement of this lemma is true since the number of lattice points contained in the half-open parallelopiped generated by a linearly independent set of vectors is given by the determinant of the matrix whose columns are vectors in the
set. In the case of graphic zonotopes, each half-open paralleloiped is generated by vectors corresponding to an edge in a forest and the corresponding matrix has determinant 1. See [21] for more details.

The second type of subdivision comes from the theory of mixed subdivisions. Let \( P = P_1 + \cdots + P_k \) be the Minkowski sum of polytopes. A mixed cell \( \sum B_i \) is a Minkowski sum of polytopes, where the vertices of \( B_i \) are contained in the vertices of \( P_i \). A mixed subdivision of \( P \) is a collection of mixed cells which cover \( P \) and intersect properly; i.e., for any two mixed cells \( \sum B_i \) and \( \sum B_i' \) the polytopes \( B_i \) and \( B'_i \) intersect at a face of both, or not at all.

In the case of graphic zonotopes, the subdivision given by Lemma 7 is a mixed subdivision, see Lemma 9.2.10 in De Loera et al. [7]. This means that every tile \( v_F + Z_F \) can be written as \( \sum_{(i,j) \in E} B_{ij} \), where \( B_{ij} \) is a face of the line segment \([e_i, e_j]\). Since \( Z_F = \sum_{(i,j) \in E(F)} [e_i, e_j] \) this means that \( E \) can be partitioned \( A \cup B \cup E(F) = E \) in such a way that

\[
v_F + Z_F = \sum_{(i,j) \in A} e_i + \sum_{(i,j) \in B} e_j + \sum_{(i,j) \in E(F)} [e_i, e_j]. \tag{4}
\]

With this at hand, we prove our main result.

**Proof of Theorem 5.** If \( x \in Z_G \), then it is contained in one of the full-dimensional tiles of the subdivision of \( Z_G \) given by Lemma 7. Let \( F \) be a forest corresponding to one of the tiles \( v_F + Z_F \) that contains \( x \). Then (4) tells us that \( x \) is of the form

\[
x = \sum_{(i,j) \in A} e_i + \sum_{(i,j) \in B} e_j + \sum_{(i,j) \in E(F)} a_{ij} e_i + (1 - a_{ij}) e_j.
\]

for some \( 0 \leq a_{ij} \leq 1 \).

Let \( X \) be the random tournament where, for \( i < j \),

\[
p_{ij} = \begin{cases} 
0 & \text{if } (i, j) \in A \\
1 & \text{if } (i, j) \in B \\
a_{ij} & \text{otherwise.}
\end{cases}
\]

Then the mean score sequence of \( X \) is \( x \). If \( x \in \mathbb{Z}^m \) is integer-valued, then in fact (also by Lemma 7) all \( a_{ij} \in \{0, 1\} \), in which case \( X \) is a deterministic tournament.

**Remark.** Geometrically, this theorem states that every lattice point of \( Z_G \) is the image of some vertex of the cube \([0, 1]^m\) under the map \( \pi \).

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