Four-Dimensional Lie Algebras Revisited

Laurent Manivel, Bernd Sturmfels and Svala Sverrisdóttir

Abstract

The projective variety of Lie algebra structures on a 4-dimensional vector space has four irreducible components of dimension 11. We compute their prime ideals in the polynomial ring in 24 variables. By listing their degrees and Hilbert polynomials, we correct an earlier publication and we answer a 1987 question by Kirillov and Neretin.

1 Introduction

We examine the projective variety \( \text{Lie}_4 \) whose points are the Lie algebra structures on the vector space \( \mathbb{C}^4 \), with standard basis \( \{e_1, e_2, e_3, e_4\} \). Each point in \( \text{Lie}_4 \) is given by a matrix

\[
A = \begin{bmatrix}
a_{121} & a_{131} & a_{141} & a_{241} & a_{341} \\
a_{122} & a_{132} & a_{142} & a_{242} & a_{342} \\
a_{123} & a_{133} & a_{143} & a_{243} & a_{343} \\
a_{124} & a_{134} & a_{144} & a_{244} & a_{344}
\end{bmatrix}.
\]

The 24 matrix entries are homogeneous coordinates on \( \mathbb{P}^{23} \). They represent the structure constants of a Lie algebra, by setting \( [e_i, e_j] = a_{ij1}e_1 + a_{ij2}e_2 + a_{ij3}e_3 + a_{ij4}e_4 \) for \( 1 \leq i \leq j \leq 4 \). Since \( [e_i, e_j] = -[e_j, e_i] \), we fix the conventions \( a_{ijk} = 0 \) and \( a_{jik} = -a_{ijk} \). The Jacobi identity

\[
[e_i, [e_j, e_k]] + [e_j, [e_k, e_i]] + [e_k, [e_i, e_j]] = 0
\]

yields 16 quadratic equations in the 24 unknowns \( a_{ijk} \), shown explicitly in Section 2. These equations cut out the variety \( \text{Lie}_4 \subset \mathbb{P}^{23} \). Our discussion revolves around the following result.

**Theorem 1.** The variety \( \text{Lie}_4 \) has four irreducible components \( C_1, C_2, C_3, C_4 \) in the ambient space \( \mathbb{P}^{23} \). Each of these components has dimension 11. Their degrees are 55, 361, 121, 295.

The article [7] had reported the degrees 660, 57, 121, 195, where the computation used methods from intersection theory. However, three of those numbers are incorrect. Thus the present paper also serves as an erratum for [7]. The corrections will be discussed in Section 5.

Our study grew out of a student summer project by the third author at MPI Leipzig. When applying the software \texttt{HomotopyContinuation.jl} [2] to the 16 quadrics defining \( \text{Lie}_4 \), she discovered numerically that the degree equals 832 and not \( 1033 = 660 + 57 + 121 + 195 \).

We here use computer algebra to determine the prime ideals for \( C_1, C_2, C_3, C_4 \). Their generators are presented in Sections 3 and 4. The latter incorporates the action of \( \text{GL}(4, \mathbb{C}) \). Theorem 2 answer a question raised at the end of [6] by presenting the Hilbert polynomials.
2 Sixteen Quadrics

Two Lie algebra structures on $\mathbb{C}^4$ are isomorphic if they are in the same orbit under the action of the general linear group $G = \text{GL}(4, \mathbb{C})$. In this paper we use representation theory of the group $G$ at the level of [8, §10.2]. The matrix entries in (1) generate the polynomial ring $\mathbb{C}[A] = \mathbb{C}[a_{121}, a_{122}, \ldots, a_{344}]$. We write the $G$-action by matrix multiplication as follows:

$$G \times \mathbb{C}[A] \to \mathbb{C}[A], \quad (g, A) \mapsto g^{-1} \cdot A \cdot \wedge_2(g).$$

(3)

The entries of the $6 \times 6$ matrix $\wedge_2(g)$ are the $2 \times 2$ minors of $g$, with rows and columns labeled to match (1). The $\mathbb{Z}^4$-grading of the polynomial ring $\mathbb{C}[A]$ induced by (3) equals

$$\deg(a_{ijk}) = e_i + e_j - e_k.$$

The space of linear forms in $\mathbb{C}[A]$ decomposes into two irreducible representations:

$$\mathbb{C}[A]_1 = S_{(1,0,0,0)}(\mathbb{C}^4) \oplus S_{(1,1,0,-1)}(\mathbb{C}^4) \simeq \mathbb{C}^4 \oplus \mathbb{C}^{20}. \quad (4)$$

Explicitly, with highest weight vectors underlined, the two $G$-invariant subspaces are

$$S_{(1,0,0,0)}(\mathbb{C}^4) = \mathbb{C}\{a_{122} + a_{133} + a_{144}, -a_{121} + a_{233} + a_{244}, -a_{131} - a_{232} + a_{344}, -a_{141} - a_{242} - a_{334}\},$$

$$S_{(1,1,0,-1)}(\mathbb{C}^4) = \mathbb{C}\{a_{124} + a_{123}, a_{132}, a_{134}, a_{142}, a_{143}, a_{231}, a_{234}, a_{241}, a_{243}, a_{341}, a_{342}, a_{122} - a_{133}, -a_{133} - a_{144}, a_{121} + a_{233} + a_{244}, -a_{131} - a_{232} + a_{344}, -a_{141} - a_{242} - a_{334}\}.$$

The 16 quadrics in $\mathbb{C}[A]_2$ that encode the Jacobi identity (2) are organized into a $4 \times 4$ matrix $\Theta = (\theta_{ij})$. The columns are labeled $e_4, e_3, e_2, e_1$, in this order. The rows, labeled $123, -124, 134, -234$, contain the coefficients of (2). For instance, in the second row of $\Theta$,

$$-[e_1, [e_2, e_4]] - [e_2, [e_4, e_1]] - [e_4, [e_1, e_2]] = \theta_{21}e_4 + \theta_{22}e_3 + \theta_{23}e_2 + \theta_{24}e_1.$$

The entries in the upper left of our $4 \times 4$ matrix $\Theta$ are

$$\theta_{11} = a_{124}a_{131} - a_{121}a_{134} + a_{124}a_{232} + a_{134}a_{233} - a_{122}a_{234} - a_{133}a_{234} + a_{144}a_{234} - a_{134}a_{244} - a_{124}a_{344},$$

$$\theta_{12} = a_{123}a_{131} - a_{121}a_{133} + a_{123}a_{232} - a_{122}a_{233} + a_{143}a_{234} - a_{134}a_{243} + a_{124}a_{343},$$

$$\theta_{21} = -a_{124}a_{141} + a_{121}a_{144} + a_{123}a_{243} - a_{124}a_{242} + a_{134}a_{243} + a_{122}a_{244} + a_{123}a_{344}, \text{ etc.}$$

Viewed invariantly, our matrix represents the linear map given by the following composition:

$$\Theta : \wedge_3 \mathbb{C}^4 \hookrightarrow \wedge_2 \mathbb{C}^4 \otimes \mathbb{C}^4 \to \mathbb{C}^4 \otimes \mathbb{C}^4 \to \wedge_2 \mathbb{C}^4 \to \mathbb{C}^4.$$

The second and fourth map are given by Lie algebra multiplication, i.e. by the matrix $A$. The Jacobi identity states $\Theta = 0$. Hence $\text{Lie}_4 \subset \mathbb{P}^{23}$ is defined by the 16 quadrics $\theta_{ij} \in \mathbb{C}[A]_2$.

The space $\mathbb{C}[A]_2 \simeq \mathbb{C}^{300}$ decomposes into eight irreducible representations. The quadrics $\theta_{ij}$ account for two of these Schur modules. After relabeling and matrix inversion, the group $G = \text{GL}(4, \mathbb{C})$ acts on the $4 \times 4$ matrix of quadrics by congruence. In symbols, $\Theta \mapsto g^T \Theta g$. This action is compatible with the decomposition of $4 \times 4$ matrices into their symmetric and skew-symmetric parts. Our space of quadrics $\mathbb{C}\{\text{entries of } \Theta\} \simeq \mathbb{C}^{16}$ is the direct sum of

$$S_{(1,1,1,-1)}(\mathbb{C}^4) = \mathbb{C}\{\text{entries of } \Theta + \Theta^T\} \simeq \mathbb{C}^{10} \quad \text{and}$$

$$S_{(1,1,0,0)}(\mathbb{C}^4) = \mathbb{C}\{\text{entries of } \Theta - \Theta^T\} \simeq \mathbb{C}^{6}.$$

(5)
Highest weight vectors for these two $G$-modules are $\theta_{11}$ and $\theta_{12} - \theta_{21}$ respectively. The ideal of $\mathbb{C}[A]$ generated by (5) is not radical. Its radical is the intersection of four prime ideals. These four prime ideals and their varieties in $\mathbb{P}^{23}$ will be described in the next section. Explicit lists of all ideal generators and Macaulay2 code verifying Theorems 1 and 2 are available at the repository website MathRepo [4] of MPI-MiS via the link https://mathrepo.mis.mpg.de/Lie4.

3 Four Components

The irreducible components of Lie$_n$ were described by Kirillov and Neretin [6] for $n \leq 6$ and by Carles and Diakité [3] for $n = 7$. Here we revisit the basic case $n = 4$. The variety Lie$_4$ has four irreducible components. We denote these by $C_1, C_2, C_3, C_4$ as in [7, Proposition 2.1]. This differs from the orderings used in [1, 6]. We begin by presenting a parametrization for each component of Lie$_4$, starting from a generic $G$-orbit. For this we use the matrices

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (6)$$

The variety $C_i$ is the closure of the $G$-orbit of $A_i$ where $x$ and $y$ range over $\mathbb{C}$. For instance, $C_2 = \text{closure of } \{ g^{-1} \cdot A_2(x) \cdot \wedge_2(g) : x \in \mathbb{C}, g \in G \} \subset \mathbb{P}^{23}. \quad (7)$

The number of free parameters is consistent with the table in [6, page 25]. We see there that generic $G$-orbits are dense in the varieties $C_1$ and $C_4$, and they have codimension 1 in $C_2$ and codimension 2 in $C_3$. Given our rational parametrizations, such as (7), we can try to find the prime ideals $I_{C_1}, \ldots, I_{C_4}$ by performing implicitization [8, §4.2] in a computer algebra system, such as Macaulay2 [5]. While this works in theory, it is not easy to do in practise.

We next recall the Lie-theoretic descriptions of the general point on each component:

$C_1$ : The Lie algebra $\mathfrak{gl}_2$ of the general linear group GL(2, $\mathbb{C}$). Its derived algebra is $\mathfrak{sl}_2$.

$C_2$ : Lie algebras whose derived algebra is the Heisenberg algebra $\mathfrak{h}_{3}$, of dimension 3.

$C_3$ : Lie algebras whose derived algebra is abelian and three-dimensional.

$C_4$ : The Lie algebra $\mathfrak{2aff}_2$. Its derived algebra is abelian and two-dimensional.

The derived algebra of a Lie algebra is generated by all the commutators. As a subspace of $\mathbb{C}^4$, the derived algebra is simply the image of our $4 \times 6$ matrix $A$. This means that the $4 \times 4$-minors of $A$ vanish on all components, while the $3 \times 3$-minors of $A$ vanish only on $C_4$. We now present the prime ideals $I_{C_1}, I_{C_2}, I_{C_3}$ and $I_{C_4}$ in $\mathbb{C}[A]$ that define the components.
Theorem 2. The ideal \( IC_1 \) is generated by 4 linear forms, 10 quadrics and 20 cubics. The linear forms and quadrics are \( S_{1,0,0,0}(\mathbb{C}^4) \) and \( S_{1,1,1,-1}(\mathbb{C}^4) \) in Section 2. The ideal \( IC_2 \) is generated by 16 quadrics and 44 cubics, \( IC_3 \) is generated by 26 quadrics and 40 cubics, and \( IC_4 \) is generated by 16 quadrics and 60 cubics. The Hilbert polynomials of the varieties are

\[
\begin{align*}
\text{Hilb}_{C_1} &= 55\, [P^{11}] - 120\, [P^{10}] + 86\, [P^9] - 20\, [P^8], \\
\text{Hilb}_{C_2} &= 361\, [P^{11}] - 1184\, [P^{10}] + 1526\, [P^9] - 964\, [P^8] + 298\, [P^7] - 36\, [P^6], \\
\text{Hilb}_{C_3} &= 121\, [P^{11}] - 284\, [P^{10}] + 220\, [P^9] - 56\, [P^8], \\
\text{Hilb}_{C_4} &= 295\, [P^{11}] - 920\, [P^{10}] + 1114\, [P^9] - 652\, [P^8] + 184\, [P^7] - 20\, [P^6].
\end{align*}
\]

Here we write \([P^d]\) for the Hilbert polynomial \( n \mapsto \binom{n+d}{d} \) of projective space \( \mathbb{P}^d \). The leading coefficients are the degrees of the varieties. Theorem 1 is a corollary to Theorem 2.

Proof. We found the quadrics, cubics and quartics that vanish on each variety by solving linear systems of equations over \( \mathbb{Q} \). Namely, we substituted the parametrizations presented in (6) into a polynomial with unknown coefficients, and we solved for these coefficients. In order for the linear systems to have small sizes, we organized the computations by \( \mathbb{Z}_4 \)-degrees.

In this manner, we identified the polynomials of degree at most four in each ideal \( IC_i \). We verified that the dimension of each projective variety defined by the equations we found equals 11. We also checked that the degree matches that predicted by the numerical approach with \texttt{homotopyContinuation.jl}. The Hilbert polynomials listed above were computed in \texttt{Macaulay2} by applying the command \texttt{hilbertPolynomial} to the proposed ideal generators.

What remains to be shown is that, in each of the four cases, our polynomials do indeed generate a prime ideal. For the ideals \( IC_1 \) and \( IC_3 \), this was done by simply running the command \texttt{isPrime} in \texttt{Macaulay2}. The computation was harder for \( IC_2 \) and \( IC_4 \). To show that \( IC_4 \) is generated by the 26 quadrics and 40 cubics, we ran the commands \# \texttt{minimalPrimes C4} and \texttt{radical C4 == C4}. Their outputs are 1 and \texttt{true}, respectively. This proves the claim.

It remains to prove the claim for the variety \( C_2 \). We perform implicitization as follows:

```plaintext
R = QQ[f1,f2,f3,f4,f5,k1,k2,k3,k4,k5,k6,m, a121,a122,a123,a124, a131,a132,a133,a141,a142,a143,a144,a231,a232,a233,a234, a241,a242,a243,a244,a341,a342,a343,a344, MonomialOrder => Eliminate 12]; I = ideal(a121-f1*f4*k5-f2*f5*k5+f1*k1+f2*k3+f3*k5, a122+f4*k5-k1, a132+f4*k6-k2, a142+f4^2*k5+f4*f5*k6+f4*k4-f5*k2, a232+f1*f4*k6-f2*f4*k5+f1*k2+f2*k1+f4*m, a242+f1*f4^2*k5+f1*f4*f5*k6+f1*f4*k4-f1*f5*k2-f3*f4*k5+f4*f5*m+f3*k1, a342+f2*f4^2*k5+f2*f4*f5*k6+f2*f4*k4-f2*f5*k2-f3*f4*k6-f4^2*m+f3*k2, a123+f5*k5-k3, a133+f5*k6-k4, a143+f4*f5*k5+f5^2*k6-f4*k3+f5*k1, a131-f1*f4*k6-f2*k6+f1*k1+f2*k4+f3*k6, a233+f1*f5*k6-f2*f5*k5-f1*k4+f2*k3+f5*m, a243+f1*f4*f5*k5+f1*f5^2*k6-f1*f4*k3+f1*f5*k1-f3*f5*k5+f5^2*m+f3*k3, a343+f2*f4*f5*k5+f2*f5^2*k6-f2*f4*k3+f2*f5*k1-f3*f5*k6-f4*f5*m+f3*k4, a144-f4*k5-f5*k6-k1-k4, a244+f1*f4*f5*k6-f1*k1+f1*k4+f3*k5-f5*m, a234-f1*f4*k6-f2*k6+f2*k4+f3*k6+f4*m, a344-f2*f4*k5-f2*f5*k6+f2*k1-f2*k4+f3*k6+f4*m, a141-f1*f4^2*k5-f1*f4*f5*k6-f2*f4*f5*k5-f2*f5^2*k6 -f1*f4*k4+f1*f5*k2+f2*f4*k3 -f2*f5*k1+f3*f4*k5+f3*f5*k6+f3*k1+f3*k4, a231-f1^2*f4*k6+f1*f2*f4*k5-f1*f2*f5*k6+f2^2*f5*k5+f1^2*k2-f1*k2*f2*k1 +f1*f2*k4+f1*f3*k6-f1*f4*m-f2^2*k3-f2*f3*k5-f2*f5*m+f3*m, a124-k5, a134-k6,
```

This Macaulay2 code represents the birational parametrization of $C_2$ introduced in [7, Section 3.2] and revisited in Section 5 below. The ideal $I$ defines the graph of that parametrization. It is prime because each of its 24 generators is equal to $a_{ijk}$ minus a polynomial in the 12 parameters. The ideal $C_2$ is obtained by eliminating the 12 parameters $f_1, f_2, \ldots, k_6, m$, so it is prime as well. The last line yields that it is generated by 16 quadrics and 44 cubics.

The parametrization is a modification of (7). Namely, our Macaulay2 code says that

$$A = \wedge^2 g \cdot B \cdot \det(g) \cdot g^{-1}$$

where $g = \begin{pmatrix} 1 & f_1 & f_2 & f_3 \\ 0 & 1 & 0 & f_4 \\ 0 & 0 & 1 & f_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_1 & k_2 & 0 & 0 & 0 & 0 \\ k_3 & k_4 & 0 & 0 & 0 \\ k_5 & k_6 & k_1 + k_4 & m & 0 & 0 \end{pmatrix}$.

The five parameters $f_i$ seen in $g$ are local coordinates on the flag variety $Fl(1,3,\mathbb{C}^4)$. The flag amounts to the inclusion of the second derived algebra, spanned by the last column of the matrix $g$, into the derived algebra, which is spanned by the last three columns of $g$.

The matrix $B$ depends linearly on the parameters $k_1, \ldots, k_6, m$. It defines a rank seven vector bundle $F$ over $Fl(1,3,\mathbb{C}^4)$. The structure of $B$ ensures that the derived algebra is the Heisenberg algebra. In symbols, $\text{image}(B) \simeq \mathfrak{h}_3$. The projective bundle $\mathbb{P}(F)$ is a nonsingular variety of dimension $11 = 6 + 5$ that maps birationally onto $C_2$. The map is given by $I$. 

We close this section by showing that the 16 quadrics in (5) do not give a radical ideal.

**Corollary 3.** The radical ideal of $\text{Lie}_4$ is minimally generated by 16 quadrics and 15 quartics. The quadrics are those in (5) and the quartics are the $4 \times 4$ minors of the matrix $A$ in (1).

**Proof.** The radical ideal of $\text{Lie}_4$ is the intersection of the prime ideals $I_{C_1}, I_{C_2}, I_{C_3}, I_{C_4}$. We computed this intersection using Macaulay2, and we verified that it has the asserted minimal generators. It can also be shown directly that the ideal generated by the 16 quadrics in (1) is not radical. Namely, the $4 \times 4$ minors of $A$ are not in this ideal, but their squares are. 

## 4 Polynomials: Explicit versus Invariant

We now examine our ideal generators through the lens of the $G$-action. We identify the Schur modules $S_\lambda(\mathbb{C}^4)$ that generate the ideals $I_{C_i}$, and we identify polynomials that serve as highest weight vectors. The space $\mathbb{C}[A)_2 \simeq \mathbb{C}^{300}$ decomposes into five isotypical components:

| $\lambda$  | $(3,1,-1,-1)$ | $(2,1,0,-1)$ | $(2,0,0,0)$ | $(1,1,1,-1)$ | $(1,1,0,0)$ |
|------------|---------------|---------------|---------------|---------------|---------------|
| dim        | 126           | 64            | 10            | 10            | 6            |
| mult       | 1             | 2             | 2             | 2             | 1             |
The last two columns were seen already in (5). The space of cubics $\mathbb{C}[A]_3 \simeq \mathbb{C}^{2600}$ decomposes into 11 isotypical components. We display the four components that are relevant for us:

\[
\begin{array}{cccccc}
\lambda & (2, 1, 1, -1) & (3, 0, 0, 0) & (2, 1, 0, 0) & (1, 1, 1, 0) & \cdots \\
\text{dim} & 36 & 20 & 20 & 4 & \cdots \\
\text{mult} & 5 & 3 & 4 & 3 & \cdots \\
\end{array}
\]

One ingredient in an invariant description of these $G$-modules is the adjoint $\text{ad}(u)$ of an element $u$ in our Lie algebra. This is the endomorphism $\mathbb{C}^4 \to \mathbb{C}^4$ given by $v \mapsto [u, v]$. For any index $i \in \{1, 2, 3, 4\}$, the adjoint of $e_i$ is represented by the $4 \times 4$ matrix $(a_{ijk})_{1 \leq j, k \leq 4}$. The traces of the matrices $\text{ad}(e_i)$ are the four linear forms that span $S_{(1,0,0,0)}(\mathbb{C}^4) \subset I_{C_1}$. To be explicit, $\text{ad}(e_1)$ is obtained by prepending a zero column to the left three columns of $A$.

We begin with the module $S_{(3,0,0,0)}(\mathbb{C}^4)$. This has dimension 20 and occurs with multiplicity three in the space of cubics $\mathbb{C}[A]_3$. A highest weight vector for one embedding is

\[
f_{3000} = a_{122}^3 - a_{122}^3 a_{133} - a_{122}^3 a_{144} + 4a_{122} a_{123} a_{132} + 4a_{122} a_{124} a_{142} - a_{122} a_{133}^2 + 2a_{122} a_{133} a_{144} - 4a_{122} a_{134} a_{143} - a_{122} a_{144}^2 + 4a_{123} a_{132} a_{144} + 4a_{123} a_{134} a_{142} + 8a_{123} a_{134} a_{142} + 8a_{124} a_{132} a_{143} - 4a_{124} a_{133} a_{142} + 4a_{124} a_{142} a_{144} + 4a_{133} a_{134} a_{143} - a_{133} a_{134} a_{144} + 4a_{134} a_{143} a_{144} + a_{144}^3.
\]

This module generates the ideal $I_{C_1}$, together with the linear forms and quadrics seen in Theorem 2. The module $Gf_{3000}$ is also contained in $I_{C_2}$. But this ideal has 24 additional cubic generators. These additional cubics for $C_2$ are given by two irreducible $G$-modules:

\[
S_{(2,1,0,0)}(\mathbb{C}^4) \oplus S_{(1,1,1,0)}(\mathbb{C}^4) \simeq \mathbb{C}^{20} \oplus \mathbb{C}^{4}.
\]

The highest weight vectors for these two irreducible $G$-modules in $\mathbb{C}[A]_3$ are

\[
\text{trace}(\text{ad}(e_1) \cdot \text{ad}(e_2) \cdot \text{ad}(e_3)) - \text{trace}(\text{ad}(e_2) \cdot \text{ad}(e_1) \cdot \text{ad}(e_3))
\]

and

\[
\text{trace}(\text{ad}(e_1)) \cdot \text{trace}(\text{ad}(e_1) \cdot \text{ad}(e_2)) - \text{trace}(\text{ad}(e_2)) \cdot \text{trace}(\text{ad}(e_1)^2).
\]

These are cubic polynomials in the 24 unknowns $a_{ijk}$, having 51 and 39 terms respectively.

We also consider the trace of the third power of the adjoint of $e_1$. This is the cubic

\[
g_{3000} = \text{trace}(\text{ad}(e_1)^3) = a_{122}^3 + a_{133}^3 + a_{144}^3 + 3(a_{122} a_{123} a_{132} + a_{122} a_{124} a_{142} + a_{123} a_{132} a_{133} + a_{123} a_{134} a_{142} + a_{124} a_{132} a_{143} + a_{124} a_{134} a_{142} + a_{133} a_{134} a_{143} + a_{134} a_{143} a_{144}).
\]

This is the highest weight vector of another embedding of $S_{(3,0,0,0)}(\mathbb{C}^4)$ into $I_{C_1}$. This $G$-module can also serve to generate $I_{C_1}$. We note that $g_{3000} - f_{3000}$ lies in the ideal generated by the four linear forms in $I_{C_1}$. However, unlike $f_{3000}$, the cubic $g_{3000}$ does not lie in $I_{C_2}$.

We now turn to the two components $C_3$ and $C_4$. Lie algebras in these components have the property that their derived algebra is abelian. Hence the second derived algebra is zero.

The ideal $I_{C_3}$ requires 10 additional quadrics and 40 cubics. These 10 quadrics span a module $S_{(1,1,1,-1)}(\mathbb{C}^4)$ inside $\mathbb{C}[A]_2$, with highest weight vector $a_{124} a_{124} - a_{134} a_{124} + a_{144} a_{124}$. Hence both embeddings of $S_{(1,1,1,-1)}(\mathbb{C}^4)$ into the space of quadrics $\mathbb{C}[A]_2$ vanish on $C_3$.

The second derived algebra of a Lie algebra is the column space of the matrix $A \cdot (\wedge_2 A)$. This is the product of a $4 \times 6$ matrix with a $6 \times 15$ matrix. The 60 entries of the $4 \times 15$
matrix $A \cdot (\wedge_2 A)$ are cubics. These polynomials lie in both $I_{C_3}$ and $I_{C_4}$ because the second derived algebra is zero. Twenty of the 60 matrix entries are already accounted for: they are in the ideal of (5). The remaining 40 cubics are generators of $I_{C_3}$ and they form a $G$-module

$$S_{(2,1,1,-1)}(C^4) \oplus S_{(1,1,1,0)}(C^4) \simeq \mathbb{C}^{36} \oplus \mathbb{C}^4.$$  \hfill (8)

This module also gives 40 of the minimal generators of $I_{C_4}$. The remaining 20 cubic generators of $I_{C_4}$ form a Schur module $S_{(3,0,0,0)}(C^4)$. They arise from the constraint that the matrix $A$ has rank two. A highest weight vector is the lower left $3 \times 3$-minor of the matrix $A$, which equals $a_{122}a_{133}a_{144} - a_{122}a_{134}a_{143} - a_{123}a_{132}a_{144} + a_{123}a_{134}a_{142} + a_{124}a_{132}a_{143} - a_{124}a_{133}a_{142}$.

In conclusion, the polynomials in a $G$-invariant ideal can be described either invariantly, using the language of representation theory, or explicitly as linear combinations of monomials. This section straddles between these two paradigms. We explored the four associated prime ideals $I_{C_i}$ of $\text{Lie}_4$. While some readers favor the invariant description, others will prefer to work with the explicit polynomials which we posted at https://mathrepo.mis.mpg.de/Lie4.

5 From Vector Bundles to Degrees

The approach in [7] is based on desingularization of each component $C_i$, following Basili [1]. The key idea is to linearize the Jacobi identity, i.e. to express (2) by linear equations in $A$ over a variety derived from $\text{GL}(4, \mathbb{C})$. This yields alternative parametrizations which allow for the computation of degrees using Chern classes. We now correctly derive the numbers in Theorem 1 by this method. In what follows we present this for $C_2$ and then for $C_1$. The derivation for $C_4$ is similar to that for $C_2$, and that for $C_3$ was correct in [7, Section 3.3].

The linearized parametrization for $C_2$ was shown in the Macaulay2 code in the proof of Theorem 2. It is given by a vector bundle $F$ of rank 7 over the 5-dimensional flag variety $\text{Fl}(1, 3, \mathbb{C}^4)$. The degree of $C_2$ in $\mathbb{P}^{23}$ is computed using Chern classes and Segre classes:

$$\deg(C_2) = \int_{C_2} c_1(O(1))^{11} = \int_{\text{Fl}(1, 3, \mathbb{C}^4)} c_1(O_F(1))^{11} = \int_{\text{Fl}(1, 3, \mathbb{C}^4)} s_5(F).$$ \hfill (9)

From now set $V = \mathbb{C}^4$. The pair $(L, U) \in \text{Fl}(1, 3, V)$ gives the second and first derived algebra. We compute the Segre class in (9), as in [7, Section 3.2]. This uses the exact sequence

$$0 \longrightarrow K \simeq \text{Hom}(V/U, \text{End}_L^0(U)) \longrightarrow F \longrightarrow M = \text{Hom}(\wedge^2(U/L), L) \longrightarrow 0.$$

Here, $M$ is a line bundle and $K$ is a rank six vector bundle. Note the matching parameters $m$ and $k_1, k_2, k_3, k_4, k_5, k_6$ in the proof of Theorem 2. The kernel $K$ fits into the exact sequence

$$0 \longrightarrow K \longrightarrow \text{Hom}(V/U, \text{End}(U)) \longrightarrow \text{Hom}(V/U, \text{Hom}(L, U)) \longrightarrow 0.$$

From the convention that the Segre class and Chern class are inverse to each other we get

$$s(F) = s(M) \cdot s(K) = s(M) \cdot s(\text{Hom}(V/U, \text{End}(U)) \cdot c(\text{Hom}(V/U, \text{Hom}(L, U))). \hfill (10)$$

This is a rational generating function $s(x, y)$ in $x = -c_1(L)$ and $y = -c_1(U) = c_1(V/U)$. These classes satisfy $x^4 = y^4 = 0$ because they are induced from $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ respectively.
Let $s_5(x, y)$ denote the component of degree 5 in $s(x, y)$. We need to integrate it on the flag variety. Since $\text{Fl}(1, 3, V)$ is a divisor of type $(1, 1)$ in $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$, we can write

$$\deg(C_2) = \int_{F(1, 3, V)} s_5(F) = \int_{\mathbb{P}(V) \times \mathbb{P}(V^\vee)} (x + y)s_5(x, y).$$

This says that the desired degree is the coefficient of $x^3y^3$ in the polynomial $(x + y)s_5(x, y)$.

To find $s(x, y)$ we compute the three factors on the right of (10). Since $M$ is a line bundle, we have $s(M) = 1/(1 + y - 2x)$. The second factor is pulled back from $\mathbb{P}(V^\vee)$. We obtain

$$s(\text{Hom}(V/U, \text{End}(U))) = \frac{(1 - 2y)^4}{(1 - y)^{11}}.$$

A twist of the tautological sequence from $\mathbb{P}(V^\vee)$ finally gives

$$c(\text{Hom}(V/U, \text{Hom}(L, U))) = \frac{(1 + x - y)^4}{1 + x}.$$

Multiplying all these rational functions and extracting the coefficient of $x^3y^3$ gives 361.

We next consider the first component $C_1$ and we show degree($C_1$) = 55. This corrects [7, Section 3.1]. The mistake was due to the fact that the model in [7, Proposition 3.1] was only birational, with non-empty indeterminacy locus which was not properly taken into account.

To correct this we will use a slightly different, better behaved model. We recall from [1, §2] or [7, §2.1] that, for $U \simeq \mathbb{C}^3$, a Lie bracket $\theta \in \text{Hom}(\wedge^2 U, U)$ defining a Lie algebra structure isomorphic to $\mathfrak{so}_2$ must belong to the space $\text{Hom}_s(\wedge^2 U, U)$ of symmetric matrices. This relies on the fact that $\wedge^2 U \simeq U^\vee \otimes \det(U)$, since $U$ is 3-dimensional, and therefore

$$\text{Hom}(\wedge^2 U, U) \simeq U \otimes U \otimes \det(U)^\vee = (S^2 U \oplus \wedge^2 U) \otimes \det(U)^\vee.$$

The first component $S^2 U \otimes \det(U)^\vee =: \text{Hom}_s(\wedge^2 U, U)$ is a 6-dimensional subspace of $\text{Hom}(\wedge^2 U, U) \simeq \mathbb{C}^6$. Explicitly, if $u$ is a vector in $U$ and $\Omega \in \det(U)^\vee$ is a volume form on $U$, then the Lie bracket defined by $u^2 \otimes \Omega$ is given by the formula

$$\theta(x, y) = \Omega(x, y, u) \cdot u \quad \text{for all } x, y \in U.$$

By linearity, this formula extends to any element $q = u_1^2 + u_2^2 + u_3^2$ of $S^2 U$, and the resulting Lie algebra structure on $U$ is isomorphic to $\mathfrak{so}_2$ exactly when $q$ is non-degenerate.

Now suppose that $V = \mathbb{C}^4$ has a Lie bracket $\theta$ such that the Lie algebra structure is isomorphic to $\mathfrak{gl}_2$. The center $L$ is 1-dimensional. We have $\theta(L \wedge V) = 0$ since elements of $L$ commute with all vectors in $V$. Therefore $\theta$ descends to an element of $\text{Hom}(\wedge^2 Q, V)$, where $Q = V/L$. Composing with the projection to $Q$, we get a Lie bracket $\overline{\theta} \in \text{Hom}(\wedge^2 Q, Q)$ which, by the previous paragraph, must be a non-degenerate element of $\text{Hom}_s(\wedge^2 Q, Q)$. In other words, we can describe $\theta$ as a generic matrix in the subspace of $\text{Hom}(\wedge^2 Q, V) \subset \text{Hom}(\wedge^2 V, V)$ consisting of elements whose projection to $\text{Hom}(\wedge^2 Q, Q)$ is symmetric.

When the line $L$ varies, the construction above defines a rank nine vector bundle $E$ over $\mathbb{P}(V) = \mathbb{P}^3$. This fits into the following exact sequence, where $L$ and $Q$, with a slight abuse of notation, now denote the tautological line bundle and the quotient vector bundle:

$$0 \to \text{Hom}(\wedge^2 Q, L) \to E \to \text{Hom}_s(\wedge^2 Q, Q) \to 0.$$  (11)
Since $E$ was constructed as a subbundle of the trivial vector bundle with fiber $\text{Hom}(\wedge^2 V, V)$ there is a natural morphism from the total space of $\mathbb{P}(E)$ to $\mathbb{P}((\text{Hom}(\wedge^2 V, V)) = \mathbb{P}^{23}$. This is a resolution of singularities of the irreducible component $C_1$. This allows us to compute the degree of $C_1$ in its embedding into $\mathbb{P}^{23}$ exactly as before:

$$\deg(C_1) = \int_{C_1} c_1(O(1))^{11} = \int_{\mathbb{P}(E)} c_1(O_E(1))^{11} = \int_{\mathbb{P}(V)} s_3(E).$$

(12)

The exact sequence (11) lets us compute the Segre class of $E$ in terms of $x = -c_1(L)$. We get

$$s(E) = \frac{1 - 3x}{(1 - x)^{10}} = 1 + 7x + 25x^2 + 55x^3.$$  

This establishes the desired equation $\deg(C_1) = 55$.

Acknowledgments. We thank Marc Härkönen and Leonid Monin for help with this project.

References

[1] R. Basili: Resolutions of singularities of varieties of Lie algebras of dimensions 3 and 4, J. Lie Theory 12 (2002) 397–407.
[2] P. Breiding and S. Timme: HomotopyContinuation.jl: A Package for Homotopy Continuation in Julia, Math. Software – ICMS 2018, 458–465, Springer, 2018.
[3] R. Carles and Y. Diakité: Sur les variétés d’algèbres de Lie de dimension ≤ 7, Journal of Algebra 91 (1984) 53–63.
[4] C. Fevola and C. Görgen: The mathematical research-data repository MathRepo, Computer-algebra Rundbrief 70 (2022) 16–20.
[5] D. Grayson and M. Stillman: Macaulay2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2/.
[6] A. Kirillov and Y. Neretin: The variety $A_n$ of structures of $n$-dimensional Lie algebras, American Mathematical Society Translations 137 (1987) 21–30.
[7] L. Manivel: On the variety of four dimensional Lie algebras, J. Lie Theory 26 (2016) 1–10.
[8] M. Michałek and B. Sturmfels: Invitation to Nonlinear Algebra, Graduate Studies in Mathematics, vol 211, American Mathematical Society, Providence, 2021.

Authors’ addresses:
Laurent Manivel, Paul Sabatier University, Toulouse laurent.manivel@math.cnrs.fr
Bernd Sturmfels, MPI-MiS Leipzig and UC Berkeley bernd@mis.mpg.de
Svala Sverrisdóttir, UC Berkeley svalasverris@berkeley.edu