Research Article

Differential Calculus of Fermatean Fuzzy Functions: Continuities, Derivatives, and Differentials

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ABSTRACT

Fermatean fuzzy sets are an effective way to handle uncertainty and vagueness by expanding the spatial scope of membership and nonmembership of the intuitionistic fuzzy set and the Pythagorean fuzzy set. However, existing studies only analyzed the discrete information and neglected the continuous state of Fermatean fuzzy sets. In this paper, we investigated the properties of continuous Fermatean fuzzy information by firstly proposing Fermatean fuzzy functions, then defining the subtraction and division operations of Fermatean fuzzy functions and discussing their properties. Further, we examined the continuity, derivatives, and differentials of Fermatean fuzzy functions. Effective approximate calculations regarding nonlinear problems in the Fermatean fuzzy environment were provided, and some examples were presented to verify the feasibility and effectiveness of approximate calculations using the Fermatean fuzzy functions.

1. INTRODUCTION

Since Zadeh proposed the concept of fuzzy sets [1], many scholars have researched fuzzy set theory. For example, Atanassov [2] proposed the intuitionistic fuzzy sets (IFs) to characterize uncertainty information according to the degree of membership and nonmembership, providing a basis for other scholars to develop some new fuzzy states of interval IFs [3,4], intuitionistic 2-tuple linguistic sets [5], intuitionistic trapezoidal fuzzy sets [6,7], intuitionistic normal fuzzy sets [8], intuitionistic uncertain linguistic [9], triangular intuitionistic fuzzy numbers [10–12], linguistic interval-valued intuitionistic neutrosophic fuzzy sets [13], generalized intuitionistic fuzzy Einstein hybrid geometric fuzzy sets [14], interval type-2 fuzzy sets [15], among others. However, IFS and its extensions must meet the criterion that the sum of the degree of membership and nonmembership is less than 1, thereby restricting its application in some decision and information environments [16–18]. For instance, when decision-makers independently evaluate the degree of membership and nonmembership, the sum may be greater than 1 but their quadratic sum would not be greater than 1. To handle this problem, Yager proposed the Pythagorean fuzzy set (PFS) [19,20] to satisfy the quadratic sum of membership and nonmembership degree, i.e., not greater than 1, allowing us to easily infer that the PFS is more useful than IFS in depicting fuzzy information.

After this presentation, PFS-based multi-attribute decision-making (MADM) methods were conducted by many scholars. Peng and Yang [21] investigated the division and subtraction operations of PFS and developed some aggregation operators. Liang et al. [22] combined the TOPSIS and three-way decisions model to present a novel Pythagorean fuzzy decision-making method. Verma and Merigo [23] examined the similarity measures between PFSs. Liu et al. [24] proposed an interval-valued Pythagorean hesitant fuzzy best-worst MADM method to support the product selecting. Rahman and Abdullah [25] developed the family of induced generalized Einstein geometric aggregation operators under interval-valued Pythagorean fuzzy environment. Moreover, Russian and Yang [26] developed a method for measuring the distance between PFSs using the Hausdorff metric theory. Muhammad et al. [27] developed the ELECTRE I-based MADM method under the Pythagorean fuzzy information. Similarly, Ren et al. [28] presented the TOSIM-based MADM method based on the PFSs. Apart from above, Garg [29] established the generalized Einstein operations in PFSs. Yang et al. [30] developed some interval-valued Pythagorean fuzzy Frank power aggregation operators and analyzed its several limiting cases. Meanwhile, Garg [31] defined the neutrality operations in PFSs to aggregate the Pythagorean fuzzy information.

Although IFS and PFS facilitate the resolution of fuzzy decision problems, they still have obvious shortcomings, especially in extremely contradictory decision environments. PFS and IFS are unable to handle a situation where the sum of membership and nonmembership is greater than 1 and the sum of squares is still greater than 1, but the sum of three is less than 1 [32]. For these cases, Senapati and Yager [32] developed the novel concept of Fermatean fuzzy set (FFS), which satisfies the criterion that the sum of the third power of membership and nonmembership must be

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less than 1. Compared to IFS and PFS, FFS gains a stronger ability to describe uncertain information by expanding the spatial scope of membership and nonmembership. Based on FFS, Wang et al. [33] developed a hesitant Fermatean fuzzy multicriteria decision-making method using Archimedean Bonferroni mean operators, Senapati and Yager [34] proposed Fermatean fuzzy information weighted aggregation operators, and Liu et al. [35] developed a distance measure method for Fermatean fuzzy linguistic term sets. Furthermore, Liu et al. [36] defined a new concept of a Fermatean fuzzy linguistic set and Senapati and Yager [37] developed some new operations between Fermatean fuzzy numbers (FFNs).

However, previous research only focused on how to address the FFS-based decision-making problems with discrete information and did not consider the continuous states of FFS. In many decision-making situations, people need to make decisions in a continuous information environment, such as the diagnosis of the patient’s condition, predict the weather and traffic condition, etc. For these issues, some authors have carried out some studies on the continuous Fermatean fuzzy information. For instance, Lei and Xu [38], Lei et al. [39] defined the intuitionistic fuzzy function (IFFs) to depict the continuous fuzzy information. For function $X$, $\alpha$ and $\beta$ designate the degree of membership and nonmembership of element $x \in X$ to set $A$, respectively, thereby satisfying $0 \leq \alpha_A(x) \leq 1$ and $0 \leq \beta_A(x) \leq 1$, and meeting the condition $0 \leq \alpha_A(x) + \beta_A(x) \leq 1$.

Definition 2.1. [2] With a nonempty set, the form of IFS was defined by Atanassov as

$$ A = \{(x, \alpha_A(x), \beta_A(x)) | x \in X\}, $$

where $\alpha_A(x)$ and $\beta_A(x)$ designate the degree of membership and nonmembership of element $x \in X$ to set $A$, respectively, thereby satisfying $0 \leq \alpha_A(x) \leq 1$ and $0 \leq \beta_A(x) \leq 1$, and meeting the condition $0 \leq \alpha_A(x) + \beta_A(x) \leq 1$.

Definition 2.2. [19] Let $X$ be an ordinary set, with the form of PFS defined by Yager as

$$ P = \{(x, \alpha_P(x), \beta_P(x)) | x \in X\}, $$

where $\alpha_P(x)$ and $\beta_P(x)$ designate the degree of membership and nonmembership of element $x \in X$ to set $P$, respectively, thereby satisfying $0 \leq \alpha_P(x) \leq 1$ and $0 \leq \beta_P(x) \leq 1$, and meeting the condition $0 \leq \alpha_P^2(x) + \beta_P^2(x) \leq 1$. The degree of indeterminacy is given as $\pi_P(x) = \left(1 - (\alpha_P(x))^2 - (\beta_P(x))^2\right)^{1/2}$.

Definition 2.3. [32] Assuming that $X$ is a universe of discourse, the form of FFS was defined by Senapati and Yager as

$$ F = \{(x, \alpha_F(x), \beta_F(x)) | x \in X\}, $$

where $\alpha_F(x)$ and $\beta_F(x)$ designate the degree of membership and nonmembership of element $x \in X$ to set $F$, respectively, thereby satisfying $0 \leq \alpha_F(x) \leq 1$ and $0 \leq \beta_F(x) \leq 1$, and meeting the condition $0 \leq \alpha_F^2(x) + \beta_F^2(x) \leq 1$. For any FFS, $F \subseteq X$, $\pi_F(x) = \left(1 - (\alpha_F(x))^3 - (\beta_F(x))^3\right)^{1/3}$ is identified as the degree of indeterminacy of $x$ to $F$.

For convenience, we considered $\left(\alpha_F, \beta_F\right)$ to be a FFN, which was denoted as $F = (\alpha_F, \beta_F)$. For simplicity, we considered the FFNs to be the components of the FFS.

The membership grades related to FFSs are herein referred to as Fermatean membership grades (FMGs).

Definition 2.4. [32] Let $F_1 = (\alpha_{F_1}, \beta_{F_1})$, $F_2 = (\alpha_{F_2}, \beta_{F_2})$, and $F = (\alpha_F, \beta_F)$ be three FFNs, defined as

1. $F_1 \cap F_2 = (\min\{\alpha_{F_1}, \alpha_{F_2}\}, \max\{\beta_{F_1}, \beta_{F_2}\})$,
2. $F_1 \cup F_2 = (\max\{\alpha_{F_1}, \alpha_{F_2}\}, \min\{\beta_{F_1}, \beta_{F_2}\})$,
3. $F' = (\beta_F, \alpha_F)$.

Definition 2.5. [32] Let $F_1 = (\alpha_{F_1}, \beta_{F_1})$, $F_2 = (\alpha_{F_2}, \beta_{F_2})$, and $F = (\alpha_F, \beta_F)$ be three FFNs and $\lambda$ be a positive real number, defined as

1. $(F_1 \otimes F_2) = \left((\alpha_{F_1}^3 + \beta_{F_1}^3 - \alpha_{F_1}^3 \beta_{F_1}^3)^{1/3}, \beta_{F_1} \beta_{F_2}\right)$,
2. $(F_1 \odot F_2) = \left((\alpha_{F_1} \alpha_{F_2} + \beta_{F_1} \beta_{F_2} - \alpha_{F_1} \beta_{F_1} \beta_{F_2}^3)^{1/3}\right)$,
3. $\lambda F = \left((1 - (1 - \alpha_F^3)^{1/3}, \beta_F^3\right)$,
4. $P^\lambda = \left((\alpha_F^3, (1 - (1 - \beta_F^3)^{1/3}\right)$.

The remaining content of this paper is organized as follows. Section 2 recalls some basic concepts of IFS, PFS, and FFS. The subtraction and division operations on FFNs are defined in Section 3. The definition of FFS and its continuous properties are described in Section 4. The derivative operations between FFSs are defined in Section 5. The differentials of FFNs are introduced in Section 6. And some numerical examples are given to illustrate the properties of FFS in Section 7. Finally, the conclusions are summarized in Section 8.

2. PRELIMINARIES

In this section, we briefly recall some basic concepts about IFS, PFS, and FFS.
Theorem 2.1. [32] Let \( F_1 = (\alpha_{F_1}, \beta_{F_1}) \), \( F_2 = (\alpha_{F_2}, \beta_{F_2}) \), and \( F = (\alpha, \beta) \) be three FFNs, and \( \lambda, \lambda_1, \) and \( \lambda_2 \) be three positive real numbers, therefore, the following operations are valid:

1. \( F_1 \odot F_2 = F_2 \odot F_1 \),
2. \( F_1 \otimes F_2 = F_2 \otimes F_1 \),
3. \( \lambda (F_1 \oplus F_2) = \lambda F_1 \oplus \lambda F_2 \),
4. \( (\lambda_1 + \lambda_2) F = \lambda_1 F \oplus \lambda_2 F \),
5. \( (F_1 \otimes F_2)^\lambda = F_1^\lambda \otimes F_2^\lambda \),
6. \( F_1^{\lambda_1} \otimes F_2^{\lambda_2} = F^{(\lambda_1 + \lambda_2)} \).

Definition 2.6. [32] Let \( F = (\alpha, \beta) \) be a FFN, therefore, the score function, \( S \) of \( F \), is defined as

\[
S(F) = \alpha^3 - \beta^3, \tag{4}
\]

and the accuracy function, \( H \) of \( F \), is defined as

\[
H(F) = \alpha^3 + \beta^3. \tag{5}
\]

Theorem 2.2. For any FFN \( F = (\alpha, \beta) \), the suggested function score is \( S(F) \in [\alpha, 1] \), whereby the bigger the value of \( H(F) \), the bigger the accuracy of the FFS \( F \).

Theorem 2.3. For any FFN \( F = (\alpha, \beta) \), the accuracy function is \( H(F) \in [0, 1] \), whereby the bigger the value of \( H(F) \), the greater the accuracy of \( F \).

According to Definition 2.3 and 2.6, \( \pi^3 + H(F) = 1 \) was identified.

Definition 2.7. [32] Let \( F_1 = (\alpha_{F_1}, \beta_{F_1}) \) and \( F_2 = (\alpha_{F_2}, \beta_{F_2}) \) be any two FFNs, with \( S(F_1) \) and \( S(F_2) \) being the score functions of \( F_1 \) and \( F_2 \), respectively, and \( H(F_1) \) and \( H(F_2) \) being the accuracy functions of \( F_1 \) and \( F_2 \), respectively, then,

1. If \( S(F_1) < S(F_2) \), then \( F_1 < F_2 \).
2. If \( S(F_1) > S(F_2) \), then \( F_1 > F_2 \).
3. If \( S(F_1) = S(F_2) \), then:
   i. If \( H(F_1) < H(F_2) \), then \( F_1 < F_2 \).
   ii. If \( H(F_1) > H(F_2) \), then \( F_1 > F_2 \).
   iii. If \( H(F_1) = H(F_2) \), then \( F_1 = F_2 \).

Definition 2.8. Let \( F_1 = (\alpha_{F_1}, \beta_{F_1}), F_2 = (\alpha_{F_2}, \beta_{F_2}), \) and \( F_3 = (\alpha_{F_3}, \beta_{F_3}) \) be three FFNs, therefore, the natural quasi-ordering can be defined as

1. \( F_1 \geq F_2 \) when and when \( \alpha_{F_1} \geq \alpha_{F_2} \) and \( \beta_{F_1} \leq \beta_{F_2} \),
2. \( F_1 \leq F_2 \) when and when \( \alpha_{F_1} \leq \alpha_{F_2} \) and \( \beta_{F_1} \geq \beta_{F_2} \).

3. SUBTRACTION AND DIVISION OPERATIONS ON FFNs

To further derive the derivative and differential of FFNs, we define the subtraction and division operations based on Definition 2.5 and Gao et al’s study [43] as follows.

Definition 3.1. Let \( F_1 = (\alpha_{F_1}, \beta_{F_1}) \) and \( F_2 = (\alpha_{F_2}, \beta_{F_2}) \) be two FFNs, therefore, the subtraction operation of FFNs was defined as

\[
F_2 \ominus F_1 = \left( \frac{\alpha_{F_2} - \alpha_{F_1}}{1 - \alpha_{F_1}^3}, \frac{\beta_{F_2} - \beta_{F_1}^3}{1 - \beta_{F_1}^3} \right), \tag{6}
\]

which satisfies

\[
0 \leq \frac{\beta_{F_2} - \beta_{F_1}^3}{1 - \beta_{F_1}^3} \leq \left( \frac{1 - \alpha_{F_2}^3}{1 - \alpha_{F_1}^3} \right)^{1/3} \leq 1, \tag{7}
\]

and can be replaced by

\[
\alpha_{F_2} \geq \alpha_{F_1}, \beta_{F_2} \leq \beta_{F_1}, \beta_{F_2} \pi_{F_1} \leq \beta_{F_1} \pi_{F_2}.
\]

3.3. SUBTRACTION AND DIVISION OPERATIONS ON FFNs

To further derive the derivative and differential of FFNs, we define the subtraction and division operations based on Definition 2.5 and Gao et al’s study [43] as follows.

Definition 3.2. Let \( F_1 = (\alpha_{F_1}, \beta_{F_1}) \) and \( F_2 = (\alpha_{F_2}, \beta_{F_2}) \) be two FFNs, therefore, the division operation between FFNs was defined as follows:

\[
F_2 \oslash F_1 = \left( \frac{\alpha_{F_2}}{\alpha_{F_1}}, \frac{\beta_{F_2}^3 - \beta_{F_1}^3}{1 - \beta_{F_1}^3} \right)^{1/3}, \tag{8}
\]

which satisfies

\[
0 \leq \frac{\beta_{F_2}^3 - \beta_{F_1}^3}{1 - \beta_{F_1}^3} \leq \left( \frac{1 - \alpha_{F_2}^3}{1 - \alpha_{F_1}^3} \right)^{1/3} \leq 1. \tag{9}
\]

Theorem 3.1. Let \( F_1 = (\alpha_{F_1}, \beta_{F_1}) \) and \( F_2 = (\alpha_{F_2}, \beta_{F_2}) \) be two FFNs, then the result of the subtraction operation of the FFNs, i.e., \( F_2 \ominus F_1 \), is an FFN, and the result of the division operation of FFNs, i.e., \( F_2 \oslash F_1 \), is also an FFN.

The following two conditions required proof.

i. \( 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1 \),
ii. \( 0 \leq \alpha^3 + \beta^3 \leq 1 \).

Since \( 0 \leq \alpha_{F_1} \leq 1, 0 \leq \alpha_{F_2} \leq 1, 0 \leq \beta_{F_1} \leq 1, \) and \( 0 \leq \beta_{F_2} \leq 1, \) and \( \alpha_{F_2} \geq \alpha_{F_1}, \beta_{F_2} \leq \beta_{F_1}, \beta_{F_2} \pi_{F_1} \leq \beta_{F_1} \pi_{F_2} \),

\[
\alpha_{F_2}^3 - \alpha_{F_1}^3 \leq 1 - \alpha_{F_1}^3 \text{ was obtained.}
\]

Further, \( \alpha_{F_2}^3 - \alpha_{F_1}^3 \leq 1 - \alpha_{F_1}^3 \) then,

\[
0 \leq \frac{\alpha_{F_2}^3 - \alpha_{F_1}^3}{1 - \alpha_{F_1}^3} \leq 1,
\]

and \( 0 \leq \frac{\beta_{F_2}^3 - \beta_{F_1}^3}{1 - \beta_{F_1}^3} \leq 1, \)

therefore, \( \alpha \leq 1 \).

Similarly, \( 0 \leq \beta \leq 1 \) was obtained, satisfying condition (i).

Since \( \beta_{F_2} \pi_{F_1} \leq \beta_{F_1} \pi_{F_2} \),

\[
\beta_{F_2} \left( 1 - \alpha_{F_1}^3 - \beta_{F_1}^3 \right)^{1/3} \leq \beta_{F_1} \left( 1 - \alpha_{F_2}^3 - \beta_{F_2}^3 \right)^{1/3} \text{ was obtained.}
\]

Further, \( \frac{\beta_{F_2}}{\beta_{F_1}} \leq \frac{1 - \alpha_{F_2}^3}{1 - \alpha_{F_1}^3} \).
According to [43] and [44], a novel order relationship based on the multiplication and division regions were analyzed, as shown in Figures 1–3. According to Definition 2.5 and 3.1, the partition relationships between the addition and subtraction regions and between the multiplication and division regions were analyzed, as shown in Figures 1–3.

Theorem 3.2. If $F_2 \ominus F_1$ is an FFN, then $\alpha_{F_2} \geq \alpha_{F_1}$ and $\beta_{F_1} \geq \beta_{F_2}$ must exist. Similarly, if $F_2 \odot F_1$ is an FFN, $\alpha_{F_2} \leq \alpha_{F_1}$ and $\beta_{F_1} \leq \beta_{F_2}$ must exist.

According to [43] and [44], a novel order relationship based on the addition and subtraction of FFNs is presented as follows.

Definition 3.3. Let $F_1 = (\alpha_{F_1}, \beta_{F_1})$, $F_2 = (\alpha_{F_2}, \beta_{F_2})$, and $F_3 = (\alpha_{F_3}, \beta_{F_3})$ be three FFNs, thereby satisfying $F_2 = F_1 \oplus F_3$, therefore, $F_1$ is less than or equal to $F_2$ as denoted by $F_1 \preceq F_2$. Particularly, $F_1 \prec F_2$ if $F_3 \neq (0, 1)$.

Definition 3.4. Let $F_1 = (\alpha_{F_1}, \beta_{F_1})$ and $F_2 = (\alpha_{F_2}, \beta_{F_2})$ be two FFNs, with the set $F$ consisting of all FFNs, therefore,

\[
F_{\oplus}(F_1) = \{F_2 \otimes F_1 \in F | F_2 \in F\},
\]

\[
F_{\ominus}(F_1) = \{F_1 \ominus F_2 \in F | F_2 \in F\},
\]

\[
F_{\odot}(F_1) = \{F_2 \odot F_1 \in F | F_2 \in F\},
\]

\[
F_{\oslash}(F_1) = \{F_1 \oslash F_2 \in F | F_2 \in F\}.
\]

According to Definition 2.5 and 3.1, the partition relationships between the addition and subtraction regions and between the multiplication and division regions were analyzed, as shown in Figures 1–3.

Theorem 3.3. Let $F_1 = (\alpha_{F_1}, \beta_{F_1})$, $F_2 = (\alpha_{F_2}, \beta_{F_2})$, and $F_3 = (\alpha_{F_3}, \beta_{F_3})$ be three FFNs, with the set $F$ consisting of all FFNs. For any $F_3 \in F_{\ominus}(F_1)$, $F_3 = F_1 \ominus F_2$ implies $F_2 = F_3 \ominus F_1$.

\[
F_1 \leq F_3, \quad \frac{\beta_{F_3}^3}{\beta_{F_1}^3} \leq \frac{1 - \alpha_{F_3}^3}{1 - \alpha_{F_1}^3},
\]

therefore,

\[
\frac{\beta_{F_3}^3}{\beta_{F_1}^3} + \frac{\alpha_{F_3}^3}{1 - \alpha_{F_1}^3} \leq \frac{1}{1 - \alpha_{F_1}^3}.
\]
iii. \((F_1 \ominus F_2) \ominus (F_3 \ominus F_4) = (F_1 \ominus F_3) \oplus (F_2 \ominus F_4)\).

Proof for (ii).

\[
(F_1 \oplus F_2) \ominus (F_3 \ominus F_4) = \left( \frac{\alpha_1^3 + \alpha_2^3 - \alpha_3^3}{1 - \alpha_3^3 + \alpha_4^3} \right) \psi_{F_1} \psi_{F_2} \\
\oplus \left( \frac{\alpha_3^3 - \alpha_4^3}{1 - \alpha_4^3 + \alpha_5^3} \right) \psi_{F_3} \psi_{F_4}
\]

The others could be proven in a similar way, therefore, the procedures were omitted from explanation.

4. FFF AND ITS CONTINUITY

In this section, we first propose a definition of FFF based on the basic information regarding the FFNs. Besides, we introduce the continuous information of FFF, which plays a fundamental role in calculus. Then, we discuss the calculus properties of FFFs.

**Definition 4.1.** The continuous functions of the multivariable were defined as

\[
h(\alpha_F, \beta_F) : [0, 1] \times [0, 1] \mapsto [0, 1],
\]

and

\[
y(\alpha_F, \beta_F) : [0, 1] \times [0, 1] \mapsto [0, 1].
\]

In addition, we assumed that

\[
0 \leq h^3 + y^3 \leq 1.
\]

Let

\[
\varphi (F) = \left( h(\alpha_F, \beta_F), y(\alpha_F, \beta_F) \right) := \left( h_F, y_F \right),
\]

then the function \( \varphi (F) \) is considered as an FFF in terms of \( h \) and \( y \).

The function \( \varphi \) in Equation (13) was still observed to be an FFN. For the analysis below, we guaranteed that all basic operations could be closed in FFS, so that \( \varphi (F) \oplus \varphi (I), \varphi (F) \odot \varphi (I), \varphi (F) \otimes \varphi (I), \) and \( \varphi (F) \oslash \varphi (I) \) still made sense for any \( F, I \in F \). Therefore, we introduced some subsets of \( F \) into the theorem.

According to Definition 2.5, Definitions 3.1, and 3.2, we obtained the following:

**Theorem 4.1.** Let \( \varphi (F) \) be an FFF, as introduced in Definition 4.1.

1. For a fixed \( F \in F \), we defined

\[
F_{\psi}^\varphi (F) := \left\{ I \in F \mid \varphi (I) \oplus \varphi (I) \ominus \varphi (F) \text{ was also an FFN} \right\}.
\]

where \( F_{\psi}^\varphi (F) \) was taken from Equation (9). Therefore, for any \( \varphi (I) \in F_{\psi}^\varphi (F), \varphi (I) \ominus \varphi (F) \) was also an FFN.

2. For a fixed \( F \in F \), we defined

\[
F_{\psi}^\varphi (F) := \left\{ I \in F \mid 0 \leq \frac{y_I}{y_F} \leq \frac{1 - h_I^3}{1 - h_F^3} \right\},
\]

where \( F_{\psi}^\varphi (F) \) was taken from Equation (11). Therefore, for any \( \varphi (I) \in F_{\psi}^\varphi (F), \varphi (I) \ominus \varphi (F) \) was also an FFN.

Based on the basic properties introduced in Definition 4.1 and Theorem 4.1, the calculus properties of FFFs can be discussed, starting with the continuity of FFFs and considering whether the following limit was suitable for FFFs regarding their operations.

\[
\lim_{I \in F \rightarrow (0, 1)} \varphi (I) \ominus \varphi (F) = (0, 1), \varphi (I) \in F_{\psi}^\varphi (F)
\]
Equation (18) was described as follows:

**Definition 4.2.** (ε – δ rule) Let $\varphi (F)$ be an FFF, as defined in Definition 4.1. For any fixed $\varepsilon = (\alpha_{\varepsilon}, \beta_{\varepsilon}) \in F$, if $\delta = (\alpha_{\delta}, \beta_{\delta})$ exists, $\alpha$ and $\varepsilon$ should be depended on, such that

$$\varphi (I) \in I F_{\#} (F) \big| F_{\#} (F) \big( F \oplus \varepsilon \big)$$

holds true for all FFN $I$, satisfying

$$I \in I F_{\#} (F) \big| F_{\#} (F) \big( F \oplus \delta \big).$$

Therefore, $\varphi$ is continuous at $F$, and $\varphi (F)$ is continuous in $F$ on the condition that $\varphi (F)$ is continuous at every $F$.

**Definition 4.3.** If Equation (18) is unsuitable for FFFs in their operations, then $\varphi (F)$ is called a discontinuous FFF at $F \in F$.

**Theorem 4.2.** Assuming that $\varphi (F) \oplus \phi (F)$ are both continuous FFFs, $\varphi (F) \oplus \phi (F)$, $\varphi (F) \oplus \phi (F)$, $\varphi (F) \oplus \phi (F)$, as well as their compositions $\varphi (\varphi (F))$ and $\phi (\varphi (F))$, are all continuous, as long as their operations make sense.

### 5. THE DERIVATIVE OPERATOR OF FFFs

The derivative operation is an essential concept in calculus, and the essence of the derivative is the local linear approximation of the function through limit operation, which can indicate the rate of change of a function value relative to its variable. Therefore, it can be used as an effective method to calculate the instantaneous rate of change in many fields, such as economics, physics, and medicine.

For this section, we modified classical calculus theories and introduced a definition for the derivative operations of FFFs. A derive formula of FFF is proposed in this section, allowing us to further discuss some properties of the derivative, such as Chain’s law and other operations.

**Definition 5.1.** Let $\varphi (F) = (h_F, y_F)$ and $\varphi (I) = (h_I, y_I)$ exist, as given in Definition 4.1, and if the quantity

$$\lim_{I \in \mathbb{F} \to (0,1)} \frac{\varphi (I) \ominus \varphi (F)}{I \ominus F}$$

made sense, and it was still an FFN in terms of $F$, then $\varphi$ allowed for a derivative at $F$, as denoted by

$$\frac{d \varphi (F)}{d F} = \lim_{I \in \mathbb{F} \to (0,1)} \frac{\varphi (I) \ominus \varphi (F)}{I \ominus F}.$$  \hspace{1cm} (19)

**Theorem 5.1.** The uniqueness of the limit showed that if

$$\frac{d \varphi (F)}{d F} \big|_F = \lim_{I \in \mathbb{F} \to (0,1)} \frac{\varphi (I) \ominus \varphi (F)}{I \ominus F},$$

then we obtained

$$\frac{d \varphi (F)}{d F} \big|_F = \frac{d (h_F, y_F)}{d F} \big| = \left( \frac{(1 - \alpha_i^3) h_i^2}{(1 - h_i^3) \alpha_i^2} \frac{\partial h_i}{\partial \alpha_i} \right) \cdot \left( 1 - \frac{\beta_i y_i}{y_i \beta_i} \right)^{1/3},$$

which played an essential role in our research.

For proof, according to Equations (6) and (7), we obtained

$$\frac{d \varphi (F)}{d F} \big|_F = \lim_{I \in \mathbb{F} \to (0,1)} \frac{\varphi (I) \ominus \varphi (F)}{I \ominus F} = \lim_{I \in \mathbb{F} \to (0,1)} \frac{(h_I, y_I) \ominus (h_F, y_F)}{(\alpha_i, \beta_i) \ominus (\alpha_F, \beta_F)}$$

$$= \lim_{I \in \mathbb{F} \to (0,1)} \frac{h_i^3 - h_F^3 \alpha_i^3 - \alpha_F^3}{1 - \alpha_i^3 - \alpha_F^3} \cdot \left( \frac{y_I}{y_F} \right) \cdot \left( \frac{\beta_i}{\beta_F} \right)^{1/3}$$

$$= \left( \lim_{I \in \mathbb{F} \to (0,1)} I, \lim_{I \in \mathbb{F} \to (0,1)} II \right)$$

We first considered $I$ and utilized the binomial expression formula

$$a^q - b^q = (a - b) \left( \sum_{i=0}^{q-1} a^{q-1-i} b^i \right), q \in \mathbb{N}_+,$$

and the continuity assumption, thereby obtaining

$$\lim_{I \in \mathbb{F} \to (0,1)} \frac{h_i^3 - h_F^3}{\alpha_i^3 - \alpha_F^3}$$

$$= \lim_{I \in \mathbb{F} \to (0,1)} \frac{h_i^3 - h_F^3}{\alpha_i^3 - \alpha_F^3} \sum_{i=0}^{2} \frac{\alpha_i^3 - 1}{\alpha_i^3}$$

$$= \frac{h_i^3}{\alpha_i^3} \lim_{I \in \mathbb{F} \to (0,1)} \frac{h_i^3 - h_F^3}{\alpha_i - \alpha_F}$$

$$= \frac{h_i^3}{\alpha_i^3} \lim_{I \in \mathbb{F} \to (0,1)} \frac{h_i - h (\alpha_i, \beta_i)}{\alpha_i - \alpha_F} + \frac{h (\alpha_i, \beta_i) - h_F \beta_i - \beta_F}{\alpha_i - \alpha_F}$$

$$= \frac{h_i^3}{\alpha_i^3} \left( \frac{\partial h_F}{\partial \alpha_i} + \frac{\partial h_F}{\partial \beta_F} \cos (\Theta, F) \right)$$

where $\cos (\Theta, F) := \lim_{I \in \mathbb{F} \to (0,1)} \frac{\beta_i \beta_F}{\alpha_i \alpha_F}$ denoted the tangential derivative at the point $F = (\alpha_F, \beta_F)$.

According to Theorem 4.1, we obtained

$$h_F \leq h_I \leq \beta_i \leq \beta_F$$

when $\beta \in F_{\#} (F) \cap F_{\#} (F)$, therefore,

$$\frac{y_F}{\beta_F} = \lim_{I \in \mathbb{F} \to (0,1)} \frac{h_I - h_F}{\beta_i - \beta_F} \leq 0.$$  \hspace{1cm} (26)
Based on Equations (26) and (29), we derived
\[
y_F \cdot \frac{y_F}{\beta_F} \cdot \cos(\theta, F) \geq 0.
\]
Thus, the last equal sign in (25) was reasonable.

Hence,
\[
\lim_{\theta \to F \to (0, 1)} I = \left( \lim_{\theta \to F \to (0, 1)} \frac{1 - \alpha_F^3}{1 - h_F^3} \right) \frac{h_F^3 - h_F^3}{1 - \alpha_F^3 - \alpha_F^3}^{1/3}
\]
\[
= \left( \frac{1 - \alpha_F^3}{1 - h_F^3} \right) \frac{h_F^3 - h_F^3}{\alpha_F^3} \left( \frac{\delta h_F}{\delta \alpha_F} + \frac{\delta h_F}{\delta \beta_F} \cos(\theta, F) \right)^{1/3}.
\]

Next, we considered II.
\[
\lim_{\theta \to F \to (0, 1)} II = \left( \lim_{\theta \to F \to (0, 1)} II \right)^{1/3}
\]
\[
= \left( \lim_{\theta \to F \to (0, 1)} \left( \frac{\left( \frac{\delta}{\delta x} \right)_F^3 - \left( \frac{\delta}{\delta y} \right)_F^3}{\beta_F^3 - \beta_F^3} \right) \right)^{1/3}
\]
\[
= \left( \lim_{\theta \to F \to (0, 1)} \frac{\left( \frac{\delta x}{\delta y} \right)_F^3 - \left( \frac{\delta y}{\delta x} \right)_F^3}{\beta_F^3 - \beta_F^3} \right)^{1/3}
\]
\[
= \left( \lim_{\theta \to F \to (0, 1)} \frac{\frac{\beta_F^3}{y_F^3} \left( \frac{\delta x}{\delta y} \right)_F^3 - \frac{\beta_F^3}{y_F^3} \left( \frac{\delta y}{\delta x} \right)_F^3}{\beta_F^3 - \beta_F^3} \right)^{1/3}
\]
\[
= \left( 1 - \frac{\frac{\beta_F^3}{y_F^3} \left( \frac{\delta x}{\delta y} \right)_F^3 - \frac{\beta_F^3}{y_F^3} \left( \frac{\delta y}{\delta x} \right)_F^3}{\beta_F^3 - \beta_F^3} \right)^{1/3}.
\]

\[
\lim_{\theta \to F \to (0, 1)} I \lim_{\theta \to F \to (0, 1)} II = \left( \frac{1 - \alpha_F^3}{1 - h_F^3} \right) \frac{h_F^3 - h_F^3}{\alpha_F^3} \left( \frac{\delta h_F}{\delta \alpha_F} + \frac{\delta h_F}{\delta \beta_F} \cos(\theta, F) \right)^{1/3},
\]
which was equal to Equation (22).

**Theorem 5.4.** We found that functions \( h \) and \( y \) were both single-variable dependent.

**Theorem 5.5.** In terms of Theorem 3.2, \( I \Theta F \) implies that \( \alpha_F \leq \alpha_I \) and \( \beta_F \geq \beta_I \); therefore,
\[
\lim_{I \Theta F \to (0, 1)} \frac{\beta_I - \beta_F}{\alpha_I - \alpha_F} \leq 0.
\]

**Definition 5.2.** The FFF \( \varphi (F) = (\alpha_F, \beta_F) \) admits derivative for all \( F \in F \). Particularly, \( \frac{d \varphi (F)}{dF} = (1, 0) \).

For proof, according to Equation (17), we obtained
\[
h_F = \alpha_F \text{ and } y_F = \beta_F.
\]

For all \( F \in F \),
\[
\frac{1 - \alpha_F^3}{1 - h_F^3} \frac{\partial h_F}{\partial \alpha_F} = \frac{1 - \alpha_F^3}{1 - h_F^3} \frac{\partial h_F}{\partial \beta_F} = 1.
\]

Therefore, Equations (20) and (21) were guaranteed.

Furthermore, from Equation (22), we obtained
\[
\frac{d \varphi (F)}{dF} = (1, 0).
\]

The basic operations of the derivatives of FFFs are discussed in the following theorems.

**Theorem 5.6.** Assuming the existence of FFF derivatives, then,
\[
\varphi_1 (F) = (h_1 (\alpha_F, \beta_F), y_1 (\alpha_F, \beta_F)) = (h_1, y_1)
\]
and
\[
\varphi_2 (F) = (h_2 (\alpha_F, \beta_F), y_2 (\alpha_F, \beta_F)) = (h_2, y_2)
\]
are derivable.

Then,
\[
\frac{d (\varphi_1 (F) \Theta \varphi_2 (F))}{dF} = \left( \frac{1 - \alpha_F^3}{1 - h_F^3} \frac{\partial h_1}{\partial \alpha_F} + \frac{1 - \alpha_F^3}{1 - h_F^3} \frac{\partial h_2}{\partial \alpha_F} \right)^{1/3},
\]
\[
\left( \frac{1 - \alpha_F^3}{1 - h_F^3} \frac{\partial h_1}{\partial \beta_F} - \frac{1 - \alpha_F^3}{1 - h_F^3} \frac{\partial h_2}{\partial \beta_F} \right)^{1/3},
\]
and
\[
\frac{d (\varphi_1 (F) \Theta \varphi_2 (F))}{dF} = \left( \frac{1 - \alpha_F^3}{1 - h_F^3} \frac{\partial h_1}{\partial \alpha_F} + \frac{1 - \alpha_F^3}{1 - h_F^3} \frac{\partial h_2}{\partial \alpha_F} \right)^{1/3},
\]
\[
\left( \frac{1 - \alpha_F^3}{1 - h_F^3} \frac{\partial h_1}{\partial \beta_F} - \frac{1 - \alpha_F^3}{1 - h_F^3} \frac{\partial h_2}{\partial \beta_F} \right)^{1/3}.
\]
For proof, regarding Equation (31),

\[
\frac{d(\varphi_1(F) \oplus \varphi_2(F))}{dF} = \frac{d}{dF} \left( \frac{h_1^3 - h_2^3}{1 - h_2^3} \right)^{1/3}, \quad y_1 y_2,
\]

then we obtained

\[
\frac{d}{dF} \left( \frac{h_1^3 - h_2^3}{1 - h_2^3} \right)^{1/3} = \frac{1}{3} \left( \frac{h_1^3 - h_2^3}{1 - h_2^3} \right)^{1/2} \frac{d}{dF} \left( \frac{h_1^3 - h_2^3}{1 - h_2^3} \right),
\]

and

\[
1 - \left( \frac{h_1^3 - h_2^3}{1 - h_2^3} \right)^{1/3} = (1 - h_1^3)(1 - h_2^3).
\]

According to Equation (22), we derived

\[
\frac{d(\varphi_1(F) \otimes \varphi_2(F))}{dF} = \frac{d}{dF} \left( \frac{(1 - \alpha_h^3) h_2^3}{h_2^3 - h_1^3} \right)^{1/3}.
\]

This is Equation (32).

The proof for Equations (33) and (34) was similar to the proof for Equations (31) and (32), so was omitted from the explanation.

As a direct consequence, we obtained

\[
\frac{d(\varphi_1(F) \oplus C)}{dF} = \frac{d}{dF} (\varphi_1(F) \oplus C) = \frac{d}{dF} (\varphi_1(F)),
\]

where C is a constant FFF.

In terms of Chain's law of derivatives, we obtained the following derivative operations for compound FFFs \( \varphi (\phi (F)) \) and \( \phi (\varphi (F)) \).

**Theorem 5.7.** Let \( \varphi (\phi (F)) \) and \( \phi (\varphi (F)) \) be two FFF compounds. Assuming that both the FFFs \( \varphi (F) \) and \( \phi (F) \) exist as derivatives, we obtained

\[
\frac{d}{dF} \left( \varphi (\phi (F)) \right) = \frac{d}{dF} \left( \phi (\varphi (F)) \right) = \frac{d}{dF} \left( \varphi (\phi (F)) \right).
\]

As the first instance of proof,

\[
\frac{d}{dF} (\phi (F)) = \frac{d}{dF} \left( \frac{\Delta F}{\phi (F) \oplus \phi (F)} \right) \frac{\phi (\Delta F) \oplus \phi (\phi (F))}{\Delta F} = \frac{\phi (\phi (F)) \oplus \phi (\phi (F))}{\Delta F} = \frac{d}{dF} \left( \varphi (\phi (F)) \right).
\]

\[
\frac{d}{dF} (\phi (F)) = \frac{d}{dF} \left( \frac{\Delta F}{\phi (F) \oplus \phi (F)} \right) \frac{\phi (\Delta F) \oplus \phi (\phi (F))}{\Delta F} = \frac{d}{dF} \left( \varphi (\phi (F)) \right).
\]
As the second instance of proof, let \( \varphi(F) = \) and \((h_{\varphi} (h_{\varphi} (\alpha)),  \gamma_{\varphi} (y_{\varphi} (\beta))) = (H(\alpha), Y(\beta))\) be an FFF compound, therefore, based on Equation (17), we obtained

\[
\frac{d\varphi (F)}{dF} = \left( 1 - \frac{\beta}{Y(\beta)} \frac{dY}{d\beta} \right)^{1/3} \left( 1 - \frac{\beta}{Y(\beta)} \frac{dY}{d\beta} \right) + \left( 1 - \frac{\beta}{Y(\beta)} \frac{dY}{d\beta} \right)^{3/3} \left( 1 - \frac{\beta}{Y(\beta)} \frac{dY}{d\beta} \right)
\]

the proof of \( \frac{d\varphi (F)}{dF} = \frac{d\varphi (F)}{d\varphi} \otimes \frac{d\varphi (F)}{dF} \) was completed.

6. FFF DIFFERENTIALS

In mathematics, a differential operator is an operator defined as a function of the differentiation operation. Differential operators encompass an effective method to estimate function changes based on the proportion of variable variation, which can be regarded as an effective way to linearly approximate nonlinearity problems on the condition that the changes are appropriate. Besides, the FFF differential can be used to estimate the approximate value. For example, decision-makers want to change the value or add some new values after calculation, but the workload of recalculation is too large in some decision-making environments to obtain the accurate results. For this situation, we can use differentiation to estimate the approximate value.

In this section, the intrinsic properties of differential operators in Fermatean fuzzy environment are studied, and the theories of FFFs are discussed. The definition of an FFF differential operator is first discussed, followed by the necessary assumptions needed to ensure an FFF is differentiable. We further propose a differential formula usually used in a variety of applications and numerical calculations.

**Definition 6.1.** Let \( F \oplus \Delta F \in F_{\mathbb{R}}^F \) exist with \( F \in F \), and let \( \Delta \varphi = \varphi(F \oplus \Delta F) \oplus \varphi(F) \) be the difference in \( \varphi \) in light of \( \Delta F \). If there an FFN exists, i.e., \((F_1, F_2)\), which depends only on \( F \), as denoted by

\[
\Delta \varphi = \left( F_1 + o \left( \alpha_{F_{\mathbb{R}} \Delta F} - \alpha_{F} \right), F_2 + o \left( \alpha_{F_{\mathbb{R}} \Delta F} - \alpha_{F} \right) \right) \approx (F_1, F_2).
\]

(35)

with \( o \left( \alpha_{F_{\mathbb{R}} \Delta F} - \alpha_{F} \right) \) and \( o \left( \alpha_{F_{\mathbb{R}} \Delta F} - \alpha_{F} \right) \) satisfying

\[
\lim_{\Delta \alpha \to (0,1)} o \left( \alpha_{F_{\mathbb{R}} \Delta F} - \alpha_{F} \right) \alpha_{F_{\mathbb{R}} \Delta F} - \alpha_{F} = 0
\]

(36)
and

\[
\lim_{\Delta \alpha \to (0, 1)} \frac{o (\hat{\beta}_{F \Delta \Delta \Delta} - \hat{\beta}_F)}{\hat{\beta}_{F \Delta \Delta \Delta} - \hat{\beta}_F} = 0,
\]

then the FFF \( \varphi \) is considered to be differential at \( F \), as denoted by

\[
d\varphi = (F_1, F_2).
\]

**Theorem 6.1.** Let \( F \oplus \Delta F \in \mathbb{F}^F \) exist with \( F \in \mathbb{F} \), assuming the hypotheses in Theorem 5.6 hold true, then FFN \( \varphi \) is differentiable. In particular,

\[
d\varphi = (F_1, F_2) = \left. \frac{d \varphi}{dF} \right| \otimes \Delta F
\]

As proof, according to Equation (6), we obtained

\[
\Delta F = \left( \frac{\alpha_{F \Delta \Delta \Delta}^3 - \alpha_F^3}{1 - \hat{\beta}_F^3}, \frac{\hat{\beta}_{F \Delta \Delta \Delta}}{\hat{\beta}_F} \right)
\]

If the hypotheses in Theorem 5.6 hold true, namely, \( \frac{d \varphi}{dF} \mid F \) exists, then in terms of Theorem 5.6 and Equation (39), we obtained

\[
\frac{d \varphi}{dF} \mid F \otimes \Delta F
\]

Based on Definition 3.2, we deduced that

\[
\varphi (F \oplus \Delta F) \oplus \varphi (F) = \left( \frac{\alpha_{F \Delta \Delta \Delta}^3 - \alpha_F^3}{1 - \hat{\beta}_F^3}, \frac{\hat{\beta}_{F \Delta \Delta \Delta}}{\hat{\beta}_F} \right),
\]

Next, we compared Equations (40) with (41) to identify whether Definition 6.1 holds true. We explore Taylor’s expansion theorem to handle the membership degree, as follows:

\[
\frac{\alpha_{F \Delta \Delta \Delta}^3 - \alpha_F^3}{1 - \hat{\beta}_F^3} = \frac{\hat{\beta}_{F \Delta \Delta \Delta}}{\hat{\beta}_F} + o (\alpha_{F \Delta \Delta \Delta} - \alpha_F)
\]

In a similar way, we derived

\[
\frac{y_{F \Delta \Delta \Delta}}{\hat{\beta}_F} = 1 - \frac{\hat{\beta}_{F \Delta \Delta \Delta}^3 - \hat{\beta}_F^3}{\hat{\beta}_F} \frac{\partial y_F}{\partial \hat{\beta}_F} + o (\alpha_{F \Delta \Delta \Delta} - \alpha_F).
\]

Based on Equations (39–43), we obtained

\[
\varphi (F \oplus \Delta F) \oplus \varphi (F) = \left( \frac{\alpha_{F \Delta \Delta \Delta}^3 - \alpha_F^3}{1 - \hat{\beta}_F^3}, \frac{\hat{\beta}_{F \Delta \Delta \Delta}}{\hat{\beta}_F} \right),
\]

Therefore, Equation (41) was regarded as Equation (40), i.e.,

\[
\Delta \varphi = \varphi (F \oplus \Delta F) \oplus \varphi (F) \approx \left. \frac{d \varphi}{dF} \right| \otimes \Delta F := (F_1, F_2).
\]

By combining all of the above, including Equations (36) and (37) \( \varphi \) was deduced to be differentiable at \( \alpha \).

**Theorem 6.2.** If \( \varphi (F) \) is differentiable, then \( \frac{d \varphi}{dF} \) exists.

Based on [43], we proposed an approach approximating a nonlinear FFF in the following corollary:

**Corollary 6.1.** Let \( F \oplus \Delta F \in \mathbb{F}^F \) and \( F \in \mathbb{F} \). FFF \( \varphi \) was differentiable, then,

\[
\varphi (F \oplus \Delta F) = \varphi (F) \oplus \varphi (F \oplus \Delta F) \oplus \varphi (F) \approx \left. \frac{d \varphi}{dF} \right| \otimes \Delta F
\]

As proof, based on Definition 3.2, Equations (35), (38), and (40), we obtained

\[
\varphi (F \oplus \Delta F) = \varphi (F) \oplus \varphi (F \oplus \Delta F) \oplus \varphi (F) \approx \left. \frac{d \varphi}{dF} \right| \otimes \Delta F
\]

where \( o (\alpha_{F \Delta \Delta \Delta} - \alpha_F) \) differed from line to line and satisfied Equation (36).
which was equal to Equation (45). Therefore, the proof of Corollary 6.1 was completed.

**Corollary 6.2.** If the FFF \( \varphi (h, y) \) satisfied

\[
\frac{\partial^2 h}{\partial \alpha^2} = 0 = \frac{\partial^3 y}{\partial \beta^2},
\]

then

\[
\varphi (F \oplus \Delta F) \ominus \varphi (F) = \left( \varphi (F) \right) \left( \frac{\Delta \varphi}{\Delta F} \right) = 0 = \left( \frac{\partial \varphi}{\partial h} \right) \left( \frac{\partial \varphi}{\partial y} \right).
\]

The proof of Corollary 6.2 was similar to the proof of Theorem 6.1, so was omitted from explanation.

When comparing Equation (35) with Equation (47), the sign “\( \approx \)” in Equation (35) was replaced with “\( =\)” in Equation (47), on the condition that Equation (47) is true. Therefore, the differential \( d\varphi \) represents the value of the difference \( \Delta \varphi \) on the condition that FFF \( \varphi (F) \) is linear.

### 7. NUMERICAL EXAMPLES AND APPLICATIONS

In order to illustrate the validity and accuracy of the differential approximate calculation formula in this paper, some numerical examples of the approximation of nonlinear FFF with Fermatean fuzzy continuous information is provided as follows:

**Example 7.1.** Let \( \Delta F = (0.1, 0.9), \lambda = 0.2 \) and the form of FFF \( \varphi (F) = (h, y) \) be shown as

\[
\varphi (F) = \left( \left( 1 - \left( 1 - \alpha^2 \right)^{0.2} \right)^{1/3}, \beta/\alpha^2 \right).
\]

Based on Definition 2.5 and Theorem 3.4, we derived

\[
\Delta \varphi (F) = \varphi (F \oplus \Delta F) \ominus \varphi (F) = 0.2 (F \oplus \Delta F) \ominus 0.2 F = (\Delta h, \Delta y) = (0.0585, 0.9816).
\]

Based on Theorem 5.6 and Theorem 6.1, we obtained

\[
d\varphi = \frac{d\varphi}{dF} \otimes \Delta F
\]

\[
= \left( \left( 1 - \alpha^2 \right)^{0.2} \frac{d (1 - \alpha^2)}{d (\alpha^2)} \right)^{1/3} \left( 1 - \beta/\alpha^2 \right)^{1/3} \otimes \Delta F
\]

\[
= \left( 0.2/\alpha^2 - \beta/\alpha^2 \right)^{1/3} \otimes \Delta F
\]

\[
= (0.0585, 0.9791) \otimes \Delta F
\]

\[
= (\Delta h, \Delta y).
\]

According to the result of Equation (49), we can find the validity of the differential approximate calculation formula, then we can further conduct its application as follows:

**Example 7.2.** Five FFN decision values were derived using the Fermatean method from five experts, i.e.,

\[
F_1 = (0.6, 0.46), F_2 = (0.93, 0.52), F_3 = (0.92, 0.46)
\]

\[
F_4 = (0.65, 0.43), F_4 = (0.72, 0.82)
\]

where the weight of their evaluation is \( \omega = (0.25, 0.2, 0.15, 0.25, 0.15)^T \). According to Definition 2.5 and Fermatean fuzzy weighted average (FFWA) operator in [32], the aggregation value is

\[
\varphi (F) = FFWA (F_1, F_2, F_3, F_4, F_5)
\]

\[
= \left( 1 - \prod_{i=1}^{5} (1 - \alpha^2)^{\omega_i} \right)^{1/3}, \prod_{i=1}^{5} \beta_i^{\omega_i}
\]

\[
= (0.8113, 0.5055).
\]

In case some experts wanted to change their preference, we assumed that first expert wanted to change the value of \( F_1 \) and give a new assessment, i.e., \( F_1 = (0.79, 0.42) \), then we can obtained

\[
\Delta F_1 = (0.7070, 0.9130),
\]

and the aggregation value is

\[
\varphi (F) = FFWA (F_1, F_2, F_3, F_4, F_5)
\]

\[
= \left( 1 - \prod_{i=1}^{5} (1 - \alpha^2)^{\omega_i} \right)^{1/3}, \prod_{i=1}^{5} \beta_i^{\omega_i}
\]

\[
= (0.8350, 0.4942).
\]

For simplicity, based on Theorem 5.6, and Theorem 6.1, we assumed \( F_1 \in F_{\oplus} (F_1) \) and used Corollary 6.1, thereby obtaining the following approximated aggregation:

\[
\varphi (F \oplus \Delta F) = \varphi (F) \oplus \varphi (F \oplus \Delta F) \ominus \varphi (F)
\]

\[
\approx \varphi (F) \oplus \frac{d\varphi (F_1)}{dF_1} \otimes \Delta F_1
\]

\[
\varphi (F) \oplus (F_1, \hat{F}_1)
\]

\[
= (0.8113, 0.5055) \oplus (0.4454, 0.9797)
\]

\[
= (0.8316, 0.4952).
\]

In conclusion, there were tiny differences observed between the results of Equations (51) and (52). Hence, the differential approximate calculation formulae regarding FFFs were effective and feasible.

If the first expert changes the value of \( F_1 \) and gives an extreme value, i.e., \( F_1 = (1, 0) \), then the aggregation value is

\[
\varphi (F_1) = FFWA (F_1, F_2, F_3, F_4, F_5)
\]

\[
= \left( 1 - \prod_{i=1}^{5} (1 - \alpha^2)^{\omega_i} \right)^{1/3}, \prod_{i=1}^{5} \beta_i^{\omega_i}
\]

\[
= (1, 0).
\]
For simplicity, based on Theorem 5.6, and Theorem 6.1, we assumed \( F_1' \in F_\Theta(F_1) \) and used Corollary 6.1 to obtain the following approximated aggregation:

\[
\varphi(F \oplus \Delta F) = \varphi(F) \oplus (\varphi(F \oplus \Delta F) \ominus \varphi(F)) \\
\approx \varphi(F) \oplus \left[ \varphi(F) \oplus \frac{d\varphi(F_1)}{dF_1} \right] \ominus \Delta F_1 \\
= \varphi(F) \oplus \left[ \varphi(F) \oplus \frac{d\varphi(F_1)}{dF_1} \right] \ominus (F_1, F_1') \\
= (0.8113, 0.5055) \oplus (0.75) \\
= (0.8113, 0.3791) 
\]

We find that there is a big difference between Equations (53) and (54). The reason is the FFW A operator cannot handle extreme cases. So, we further discuss the extreme value situation through the q-ROFIWMSM operator in [45].

According to the q-ROFIWMSM operator, we obtain the following aggregation value:

\[
\varphi(F) = q - ROFIWMSM(F_1, F_2, F_3, F_4, F_5) \\
= (0.7387, 0.8829) 
\]

If the first expert changes the value of \( F_1 \) and gives an extreme value, i.e., \( F_1' = (1, 0) \), then the aggregation value is

\[
\varphi(F') = q - ROFIWMMSM(F_1', F_2, F_3, F_4, F_5) \\
= (0.8781, 0.7204) 
\]

For simplicity, based on Theorem 5.6, and Theorem 6.1, we assumed \( F_1' \in F_\Theta(F_1) \) and used Corollary 6.1 to get the following approximated aggregation:

\[
\varphi(F \oplus \Delta F) = \varphi(F) \oplus (\varphi(F \oplus \Delta F) \ominus \varphi(F)) \\
\approx \varphi(F) \oplus \left[ \varphi(F) \oplus \frac{d\varphi(F_1)}{dF_1} \right] \ominus \Delta F_1 \\
= \varphi(F) \oplus \left[ \varphi(F) \oplus \frac{d\varphi(F_1)}{dF_1} \right] \ominus (F_1, F_1') \\
= (0.8699, 0.7268) 
\]

Obviously, there are little differences between Equations (55) and (56). Therefore, the differential approximate calculation formulae regarding FFFs are effective and feasible.

8. CONCLUSION

Considering the continuous Fermatean fuzzy information, we discussed the continuities, derivatives, and differentials of FFF. In it, we firstly proposed the subtraction and division operations between FFNs and discussed their properties, which laid the foundation for further discuss the derivatives and differentials. Then, we defined the concept of FFF and investigated its continuity that plays an important role in calculus. Moreover, we defined the derivatives and differentials of FFFs and examined the algebraic and compound operations of the derivatives of FFFs. Finally, we conducted some examples to verify the feasibility and effectiveness of the approximate calculation on FFFs. Compared to the existing studies, our method is nonlinear with its application in a more expansive range. Besides, the proposed method can address both continuous and discrete information.

In further studies, the elastic coefficients of FFFs and their relationships with the derivatives can be investigated. The inverse operations of the derivatives of the FFFs, such as indefinite integral and definite integral derivatives, can be further examined. Apart from those, the FFF-based decision-making method can be applied to complex uncertainty information decision-making environments, such as the selection of business partners in the supply chain, the prediction of traffic conditions, etc [46–48].

CONFLICTS OF INTEREST

We declare that there are no conflicts of interest regarding the publication of this paper.

AUTHORS’ CONTRIBUTIONS

Zaoli Yang: Conceptualization, Funding acquisition, Supervision, Software, Resources, Data curation, Supervision, Writing - review & editing; Harish Garg: Conceptualization, Data curation, Supervision, Software, Resources, Writing - review & editing; Xin Li: Formal analysis, Investigation, Writing - review & editing.

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