Fast thermalization and Helmholtz oscillations of an ultracold Bose gas

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Transport without friction is a signature property of superfluidity, spectacularly illustrated in the fountain effect of liquid Helium \[1\]. Its observation relies on the use of a superleak, which lets the superfluid through while blocking the normal part. Superleaks are familiar elements in the context of experiments on liquid helium \[2\], but their design in the context of ultracold gases remains an open question. Their implementation would allow, for instance, the implementation of new adiabatic cooling schemes \[3\], the efficient excitation of second sound \[2\], and, more generally, an advanced control over transport phenomena.

Recent experiments have initiated the exploration of the transport properties of ultracold atomic gases \[4–10\] in geometries comprising two reservoirs separated by a potential barrier or by a constriction (see Fig. 1). The constriction–based geometry is related to those investigated in mesoscopic physics \[11\]. It has already been used to demonstrate the concept of contact resistance \[7\], to investigate superflow \[8\], and to observe thermoelectric effects \[9, 12\] in ultracold Fermi gases. Superfluids trapped within two connected reservoirs are expected to undergo plasma oscillations, which are analogous to the oscillations of a gas in between two connected Helmholtz resonators \[13\; \text{[69]}\]. These oscillations have been extensively studied in the context of liquid helium \[14\]. Similar oscillations have also been observed with ultracold Bose gases in double–well potentials \[15, 10\].

In ultracold Fermi gases, the occurrence of BCS–type superfluidity occurs at reasonably high temperatures only in the presence of strong interactions \[17\]. In this case, both the superfluid and normal parts of the quantum fluid are deep in the hydrodynamic regime, which affords a strong analogy with superfluid Helium \[18\]. However, it also makes it more difficult to tell the behavior of the superfluid fraction apart from that of the normal fraction. Hence, in the present Letter, we focus on weakly–interacting bosonic gases, where the parameters can be chosen such that superfluid transport is hydrodynamic whereas normal transport through the channel is ballistic.

We develop a theory describing the transport properties of weakly–interacting uniform Bose gases under these conditions, reflecting the different transport regimes for the superfluid and normal parts. We use it to show that plasma oscillations are observable even at non–zero temperatures below \(T_c\), and provide a first description of the damping mechanism due to the coupling between the superfluid and normal parts. We also show that the large compressibility of the Bose gas leads to surprisingly fast thermalization compared to the damping time of the transport phenomena. Below \(T_c\), this causes an efficient transport of the normal part at short times; above \(T_c\), it yields a key difference compared to ideal fermionic gases.

We describe the ballistic transport of the normal part of the fluid using the Landauer–Büttiker formalism for quantum transport \[19\; \text{chap. 2}\]. To our knowledge, the present work is the first application of this formalism to massive bosons. It had previously been applied to (massless) phonons to determine the quantum properties of heat conductance \[20\].

We assume that the two compartments of Fig. 1 are box traps with the same volume \(V^L = V^R\), each enclosing a uniform superfluid. We model the constriction by an isotropic radial harmonic trap of frequency \(\omega_\perp/2\pi\). The hydrodynamic assumption for superfluid
transport through the constriction [21, chap. 5] is valid if \( \hbar \omega_\perp \ll gn \), where \( n \) is the mean gas density inside the constriction, \( g = 4 \pi \hbar^2 a / m \) is the interaction constant, \( a \) is the scattering length and \( m \) is the atomic mass.

We call \( \delta N_s = N_s^R - N_s^L \) and \( \delta N_n = N_n^R - N_n^L \) the difference in superfluid and normal atom numbers between the right and left compartments of Fig. 1, and \( \delta \mathcal{S} = S^R - S^L \) the analogous entropy difference. We focus on small deviations from the homogeneous situation. In this linear–response regime, the superfluid current \( I_{N_s} \), the normal current \( I_{N_n} \), and the entropy current \( I_S \), corresponding to the three differences above, are linear functions of the small differences in phase \( \delta \phi \), chemical potential \( \delta \mu \), and temperature \( \delta T \) between the two reservoirs, which we write in matrix form as:

\[
\begin{pmatrix}
I_{N_s} \\
I_{N_n} \\
I_S / k_B
\end{pmatrix} =
\begin{pmatrix}
2I_J & 0 & 0 \\
0 & L_{11} & L_{12} \\
0 & L_{12} & L_{22}
\end{pmatrix}
\begin{pmatrix}
\hbar \delta \phi \\
\delta \mu \\
k_B \delta T
\end{pmatrix}.
\tag{1}
\]

The first line in the matrix reflects the definition of the superfluid current, \( j_s = n_s \mathbf{v}_s \), where \( n_s \) is the mean superfluid density in the reservoirs and \( \mathbf{v}_s \) is the superfluid velocity. The latter satisfies \( m \mathbf{v}_s = \hbar \nabla \phi \), with \( m \) being the atomic mass. For the geometry of Fig. 1 we find \( I_J = n_s A / ml \), where \( l \) is the constriction length and \( A = \pi g n_s / m \omega_\perp^2 \) is its effective Thomas–Fermi section.

The two zeroes in the first column reflect the fact that the normal–part quantities \( \delta N_n \) and \( \delta \mathcal{S} \) do not explicitly depend on the superfluid phase difference \( \delta \phi \). The coefficients \( (L_{ij}) \) describe the ballistic transport of the normal part and the entropy. Assuming that \( k_B T \gg gn \), the excitations exchanged by the reservoirs are particles, and an analysis of the role of interactions using Hartree–Fock theory reveals that the ideal–gas expressions for the \( L_{ij} \)’s are applicable. This assumption on \( T \) rules out low–temperature collective phenomena, such as anomalous phonon transmission [22] or Andreev reflection [23].

For uniform Bose gases, this condition is easy to satisfy while maintaining the presence of superfluid \( (T < T_c) \), because the ratio \( gn / k_B T_c \) is of the order of 0.04.

We calculate the \( L_{ij} \)’s using the Landauer–Büttiker formalism [13, chap. 2]. We describe the excitations in each reservoir using Bose distribution functions \( \eta^B \), whose difference \( \delta \eta^B = \eta^B - \eta^B_L \) satisfies \( \delta \eta^B = \partial \eta^B / \partial \mu |_T \delta \mu + \partial \eta^B / \partial T |_\mu \delta T \). For temperatures below \( T_c \), the \( L_{ij} \)’s are given by:

\[
\begin{align*}
\hbar L_{11} &= -\frac{\pi^2}{6} \left( \frac{k_B T}{\hbar \omega_\perp} \right)^2, \\
L_{12} &= L_{21} = \frac{18}{10} \pi \chi(3) L_{11}, \\
L_{22} &= \frac{4\pi^2}{5} L_{11}.
\end{align*}
\tag{2}
\]

The \( L_{ij} \)’s do not depend on the constriction length \( (\sim 5 \mu m) \), because it is much shorter than the thermal mean free path inside the reservoirs \( (\sim 100 \mu m) \). Furthermore, the \( L_{ij} \)’s all share the same dependence on \( T \) and \( \omega_\perp \). This second property is an important difference with respect to the fermionic case [3], where the chemical potential is of the order of the Fermi energy and, hence, enters in the expression for the transport coefficients.

The coupling between the superfluid and normal parts arises from the equation of state, which involves the total gas density in each reservoir, e.g. \( n^R = (N^R + N^L) / V^R \), and from the equation dictating the evolution of the superfluid velocity, \( \hbar \partial_t \delta \phi = -\delta \mu \) [21]. Combining these equations with the currents given by Eq. (1), we obtain a differential system describing the evolution of \( \delta \phi, \delta N \), and \( \delta T \) [24]:

\[
\tau_1 \frac{d}{dt} \begin{pmatrix}
\frac{\hbar \delta \phi}{\tau_1} \\
\frac{\delta N}{\kappa_T} \\
\frac{\hbar \delta T}{\kappa_T}
\end{pmatrix} = \begin{pmatrix}
0 & -1 & 0 \\
\omega_{pl} \tau_1 & -1 & +S \\
0 & S / \ell & -\tau_1 / \tau_T
\end{pmatrix} \begin{pmatrix}
\frac{\hbar \delta \phi}{\tau_1} \\
\frac{\delta N}{\kappa_T} \\
\frac{\hbar \delta T}{\kappa_T}
\end{pmatrix}.
\tag{3}
\]

In Eq. (3), \( \kappa_T = \partial \mu / \partial T |_N \) is the isothermal compressibility, \( C_N = T \partial S / \partial T |_N \) is the heat capacity, \( \ell = C_N / \kappa_T \) is their ratio, and the Seebeck coefficient \( S = -\partial \mu / \partial T |_N - \tau_R / \tau_{11} \) encodes the thermoelectric properties of the gas. Equation (3) introduces three timescales:

\[
\tau_1 = \frac{\kappa}{L_{11}}, \quad \tau_{pl} = 2\pi \sqrt{\frac{\kappa}{2I_J}}, \quad \tau_T = \frac{C_N / T}{L_{22}},
\tag{4}
\]

where \( \tau_1 \) is a damping time associated with normal transport, the bare plasma period \( \tau_{pl} = 2\pi / \omega_{pl} \) is associated to superfluid transport, and \( \tau_T \) is the thermalization time. The time \( \tau_1 \) determines the damping of plasma oscillations and thermoelectric effects.

Weakly–interacting Bose gases are characterized by a very large compressibility \( (\kappa_T = N / gn \text{ for } T < T_c) \), whereas \( C_N / N k_B \) remains finite (see Fig. 3 left). Hence, the ratio \( \ell \) is very small, of the order of a few \( 10^{-2} \), which is a key difference with respect to both ideal Fermi gases \( (\ell \sim 1, \text{ Fig. 3 center}) \), and liquid Helium 4 \( (\ell \sim 10 \text{ close to the superfluid transition, Fig. 3 right}) \). This specific property of Bose gases leads to \( \tau_T \ll \tau_1 \), i.e. thermalization is much faster than the damping due to normal transport,
as illustrated on Fig. 2. Furthermore, for temperatures high enough for Hartree–Fock theory to hold \((k_B T > 10\) a few \(g_n\)), but low enough for the superfluid fraction \(\delta N/N = 1 - (T/T_c)^{3/2}\) to be substantial \((T/T_c > 0.5)\), the three timescales satisfy \(\tau_T \ll \tau_{pl} < \tau_1\). On the other hand, if \(T > T_c\), the superfluid is absent and our description reduces to the dynamical system on \(\delta N\) and \(\delta T\) introduced in Eq. 3, which corresponds to the lower right \(2 \times 2\) block of the matrix in Eq. 3.

**Plasma oscillations.** We now turn to the analysis of plasma oscillations in the geometry of Fig. 1. These oscillations can be excited by introducing an initial atom number mismatch \(\delta N\) between the two reservoirs. We have predicted their occurrence at \(T = 0\) by numerically solving the Gross–Pitaevskii equation, using a Crank–Nicolson scheme [26, 27]. We have investigated a 2D geometry inspired by [28], as well as the cylindrically symmetric three–dimensional geometry corresponding to Fig. 1. Our results validate the hydrodynamic approach for superfluid transport and yield promising orders of magnitude for the plasma oscillation frequency \(\omega_{pl}^{(0)}\), of the order of a few Hz.

Our model allows us to investigate plasma oscillations at non–zero temperatures. First, our Hartree–Fock description shows that the bare plasma frequency \(\omega_{pl}\) scales with \(a/\sqrt{T}\), whereas the damping factor \(\omega_{pl} \tau_1\) is proportional to \((T_c/T)^{3/2}\) and does not depend on \(a\). Therefore, the observation of oscillations will be favored by using smaller constriction lengths, lower temperatures \(T/T_c\), and larger scattering lengths \(a\). Plasma oscillations occur if the matrix entering Eq. 3 has two complex–conjugate eigenvalues with negative real parts, \((-\tau_{damp} \pm i \omega_{osc})\). In this case, the plasma oscillation frequency is \(\omega_{osc}/2\pi\) and the damping time is \(\tau_{damp}\).

**Thermalization at temperatures below \(T_c\).** In order to reveal the key role played by fast thermalization in ultracold Bose gases, we now consider the response of the system to an initial temperature mismatch \(\delta T_0\). We consider \(T/T_c < 0.5\). In this case, the dynamics of the system at small times of the order of \(\tau_T\) is driven by the relaxation of temperature towards \(\delta T = 0\). This fast process quickly converts the initial temperature mismatch \(\delta T_0\) into a number imbalance \(\delta N_{max}\):

\[
\frac{\delta N_{max}}{N} = \frac{15}{4} \frac{\zeta}{\zeta} \frac{S}{S^2 + L} \left(\frac{T}{T_c}\right)^{3/2} \frac{\delta T_0}{T}. \tag{6}
\]

The sign of \(\delta N_{max}\) is dictated by the Seebeck coefficient \(S\), which is negative, just like for fermions [3].
Furthermore, according to Eq. (1), temperature variations do not directly couple to the motion of the superfluid part. Hence, this fast relaxation process almost exclusively drives the transport of normal atoms. On a longer timescale, the oscillation then proceeds quasi-isothermally as before, with the frequency \( \omega_{osc} \) and the damping time \( \tau_{damp} \). This process is illustrated on Fig. 4 right, for \( \delta T_0/T = 0.1 \) and the parameters used in Fig. 2.

**Thermalization at temperatures above \( T_c \).** In Bose gases, the ratio \( \ell \) remains small for temperatures \( T > T_c \), where the physics is captured by the ideal-gas model and a direct comparison with fermions is possible (see Fig. 3 left and center). The gas contains no superfluid part, and the dynamics of \( \delta N \) and \( \delta T \) are described by the lower right 2 \( \times \) 2 block of Eq. (3), which coincides with the model of Ref. [3]. Equation (4) shows that the thermalization time \( \tau_T \) is determined by the specific heat, which is of the same order of magnitude for Bose and Fermi gases. However, the damping time \( \tau_1 \) involves the compressibility, which is much larger for bosons than for fermions. Therefore, damping is much slower in Bose gases than in Fermi gases. The variation of \( \delta N \) reflects the two timescales \( \tau_T \) and \( \tau_1 \). In both cases, the Seebeck coefficient \( S \) is negative, therefore \( \delta N \) first decreases towards negative values. It reaches a minimum for short times \( t_m \approx \tau_1 \ln(\tau_1/\tau_T) \), whose value \( \delta N_m = \delta T_0/T C_N S/(S^2 + \ell) \) does not depend critically on the statistics. However, the difference between bosons and fermions is apparent during the long-time relaxation towards \( \delta N = 0 \). Figure 5 compares the cases of bosonic \( ^{41} \text{K} \) at the temperature \( T/T_c = 1.1 \) and fermionic \( ^{40} \text{K} \) at the temperature \( T/T_F = 1.1 \), with \( T_F \) being the Fermi energy. These two isotopes differ only by the statistics which they obey, and the relaxation is 50 times longer for bosons (\( \tau_1^B \sim 1.5 \text{ s} \)) than for fermions (\( \tau_1^F \sim 30 \text{ ms} \)).

FIG. 4. Superfluid oscillations in an ultracold \(^{87} \text{Rb} \) Bose gas, for the parameters used in Fig. 2 Left: frequency \( \nu_{osc} = \omega_{osc}/2\pi \) (top) and damping time \( \tau_{damp} \) (bottom) for plasma oscillations at non-zero temperatures below \( T_c \), calculated using Eq. (3) (red “full”) and its isothermal limit (dashed green “isothermal”). Center: the initial imbalance in atom numbers \( \delta N_0/N = 0.1 \) causes quasi-isothermal oscillations. Right: the initial temperature mismatch \( \delta T_0/T = 0.1 \) yields fast thermalization accompanied by an efficient transport of the thermal part at short times, followed by quasi-isothermal oscillations. In both cases, \( T/T_c = 0.4 \), and we plot the time evolution of the differences in atom numbers \( \delta N/N \) (solid green), temperature \( \delta T/T \) (dotted red, multiplied by 10), and transported thermal part atoms \( \delta N^\text{th}/N \) (dashed brown).

**FIG. 5.** Evolution of \( \delta N/N \) following an initial temperature imbalance \( \delta T_0/T = 0.1 \) for bosonic \( ^{41} \text{K} \) (solid blue, \( T = 1.1T_c \)) and fermionic \( ^{40} \text{K} \) (dashed green, \( T = 1.1T_F \)). No superfluid is present, and the constriction is more stringent (\( \omega_{osc}/2\pi = 500 \text{ Hz} \)) to achieve bosonic decay times of the order of 1 s.
anomalously fast thermalization will disappear if $L_{22}$ is chosen sufficiently small to achieve $\tau_T > \tau_{pl}$.

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