A note on the sum-rate-distortion function of some lossy source coding problems involving infinite-valued distortion functions

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Abstract
For a number of lossy source coding problems it is shown that even if the usual single-letter sum-rate-distortion expressions may become invalid for non-infinite distortion functions, they can be approached, to any desired accuracy, via the usual valid expressions for appropriately truncated finite versions of the distortion functions.

Index Terms
Source coding, rate-distortion function, sum-rate, non-finite distortion function.

I. INTRODUCTION
In a number lossy source coding problems, the minimum sum-rate needed to attain a target expected distortion of $D$ or less — the operational sum-rate-distortion function — is characterized by an information-theoretic rate-distortion function $R(D)$ which is expressed in terms of an optimization problem of the following form:

$$R(D) := \inf_{p_{X,Y,U,\hat{X}} \in A \text{ s.t. } E[d(X,Y,U,\hat{X})] \leq D} f(p_{X,Y,U,\hat{X}}),$$

where $X \in \mathcal{X}, Y \in \mathcal{Y}, U \in \mathcal{U}$, and $\hat{X} \in \hat{\mathcal{X}}$ are, respectively, the source, side-information, auxiliary, and reconstruction random variables taking values in finite alphabets, $p_{X,Y,U,\hat{X}}$ is their joint pmf, $f$ is a finite linear combination of conditional mutual informations involving some or all the random variables, $A$ is a finite set of marginal consistency and Markov-chain constraints that the random variables need to satisfy, and $d$ is a real-valued, nonnegative distortion function which does not depend on $U$. Examples include Shannon rate-distortion [1] and Gray-Leiner conditional rate-distortion [2]–[4] where auxiliary random variables are not needed, Wyner-Ziv rate-distortion [5], [6] which uses a single auxiliary random variable, and Kaspi’s two-way rate-distortion [7] which uses multiple auxiliary random variables (collectively denoted by $U$ here for convenience).

For Shannon’s lossy source coding problem, the fact that the usual information-theoretic expression of the form (1) will continue to coincide with the operational rate-distortion function when the distortion function $d$ can take the value $\infty$, was established by Pinkston in [8] (also see [9][Ch 9,Historical Notes and References]). That this should also be true for the Gray-Leiner conditional rate-distortion function should be expected, since the encoder can group source samples that have the same side-information value into multiple conditional sources and code them separately like in Shannon’s lossy source coding problem. We have, however, been unable to locate a reference which discusses the extension of the Gray-Leiner conditional rate-distortion function to non-finite distortion functions.

An example of an infinite-valued distortion function is the so-called erasure distortion function [10][Chapter 13, Problem 7]. Here, the distortion function equals zero if the source and reconstruction symbols agree, it

1Specifically, $\sum_{u,\hat{u}} p_{X,Y,U,\hat{X}}(x, y, u, \hat{x}) = p_{X,Y}(x, y)$ for all $x, y$. 
equals one if the reconstruction is a special erasure symbol (irrespective of the source symbol), and it equals infinity otherwise. The erasure distortion function has appeared in a number of recent works: in [11][Sec. III-B] in the context of the CEO problem, in [13] in the context of the multiple descriptions coding problem, and in [12] in the context of some examples.

The erasure distortion function was used in [14] and [15][Secs. VII.B,C] to construct the first example which shows that the usual information-theoretic two-way rate-distortion function with two messages can be strictly smaller than the usual one-message Wyner-Ziv information-theoretic rate-distortion function. Specifically, it was shown that for sufficiently correlated doubly-symmetric binary sources [5][Sec.II,Eqn.(20)] and the erasure distortion function, it is possible to make the ratio of the one-message rate to the two-message sum-rate arbitrarily large while simultaneously making the ratio of the backward rate to the forward rate in the two-message sum-rate arbitrarily small.

However, in [16] it was shown that, contrary to the claim made in footnote 8 of [15][Appendix E], the usual information-theoretic rate-distortion functions for the Wyner-Ziv and two-way source coding problems, which are used in [14] and [15][Secs. VII.B,C], are in fact strictly smaller than their operational counterparts for the erasure distortion function and doubly-symmetric binary sources which satisfy a positivity condition. This implies that the usual information-theoretic rate-distortion expressions are not always valid for non-finite distortion functions, such as the erasure distortion function, in a distributed source coding setting even though they are for the Shannon and Gray-Leiner settings.

For the Wyner-Ziv and two-way source coding problems, the correct information-theoretic rate-distortion expressions that coincide with their operational counterparts were characterized in [16] for the erasure distortion function when the source and side information satisfy a positivity condition. It was shown, in particular, that the two-message two-way rate-distortion function for this problem exactly coincides with the one-message Wyner-Ziv rate-distortion function casting into doubt some of the conclusions reached in [14] and [15][Secs. VII.B,C] about the benefit of interaction (two-way coding) for lossy source reproduction.

The aim of this article is to demonstrate that all the conclusions reached in [14] and [15][Secs. VII.B,C] regarding the benefit of interaction for lossy source reproduction are correct if one amends footnote 8 of [15][Appendix E] to the following: “Although the usual information-theoretic expressions for the Wyner-Ziv and two-message rate-distortion functions are not operationally attainable for the non-finite erasure distortion function, they can be operationally approached, as closely as desired, by replacing the ∞ value in the erasure distortion function with a sufficiently large, but finite, positive real number.” The possibility of making such an approach work is contained in a suggestion of anonymous reviewers reported in Remark 1.1 of [16]. Such an approach is also taken in [11][Sec. III-B] where it is remarked that an infinite-valued distortion measure is unforgiving of decoding errors that have nonzero probability even if they are negligible. Decoding errors with vanishingly small but nonzero probability are unavoidable in most distributed source coding problems.

Formally speaking, our main result is that if $d_n$ is a sequence of bounded distortion functions that monotonically increases to a non-finite distortion function $d_∞$, then the corresponding sequence of information-theoretic rate-distortion functions $R_n(D)$ associated with $d_n$ also monotonically increases to the usual information-theoretic rate-distortion function $R_∞(D)$ associated with $d_∞$. Thus by making $d_n$ approach $d_∞$, it is possible to make the operational sum-rate-distortion function for $d_n$ as close as desired to $R_∞(D)$ even though $R_∞(D)$ itself may not be operationally attainable for $d_∞$. Thus intuitions, examples, and broad qualitative conclusions (such as the benefit of interaction) that can be formed on the basis of examining $R_∞(D)$ will be essentially correct even from an operational perspective in the sense that while they may not hold true operationally for $d_∞$ itself, they will hold for $d_n$, for all sufficiently large $n$.

II. MAIN RESULT

We first establish a general result and then discuss its application to rate-distortion functions.
A. Statement

Consider the following finite-dimensional optimization problem

$$\psi_n(D) := \inf_{p \in C_n(D) := A \cap B_n(D)} f(p)$$  \hspace{1cm} (2)

where $p$ is a probability vector (prob.vec.) in $\mathbb{R}^k$, $k$ a finite positive integer, $f$ is a real-valued continuous function of $p$, $A$ is a fixed, nonempty, compact subset of probability vectors in $\mathbb{R}^k$, $D$ is a finite nonnegative real number, and $B_n(D) := \{p \text{ prob.vec.} : \langle p, d_n \rangle \leq D \}$, where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean-space inner product and $d_n, n = 1, 2, \ldots$, is a sequence of vectors in $\mathbb{R}^k$ with finite, nonnegative, and nondecreasing components some of which (but not all) increase to $\infty$ while the rest monotonically increase to some finite nonnegative real numbers. We denote the limit of the $d_n$’s by $d_\infty$. Thus, $B_n(D)$ (and therefore also $C_n(D)$) is a nested nonincreasing sequence of compact subsets of probability vectors in $\mathbb{R}^k$. We define

$$B_\infty(D) := \bigcap_{n=1}^{\infty} B_n(D) = \{p \text{ prob.vec.} : \langle p, d_\infty \rangle \leq D \}$$  \hspace{1cm} (3)

with the convention $0 \cdot \infty := 0$ in the inner product and observe that the support of any $p$ in $B_\infty(D)$ excludes components where $d_\infty$ equals $\infty$ (since $D$ is finite) and that $B_\infty$ is a compact subset of probability vectors in $\mathbb{R}^k$. Let

$$\psi_\infty(D) := \inf_{p \in C_\infty(D) := A \cap B_\infty(D)} f(p).$$  \hspace{1cm} (4)

The main result is that $\psi_n(D) \uparrow \psi_\infty(D)$ for all finite nonnegative $D$ for which $C_\infty(D)$ is nonempty.

B. Proof

To prove this result, first observe that the constraint sets $C_n(D)$ are nested, nonincreasing, and contain $C_\infty(D)$. This implies that

$$\psi_n(D) \uparrow \lim_{n \to \infty} \psi_n(D) \leq \psi_\infty(D).$$

The main result is proved by establishing the reverse inequality. Towards this end, we note that the minimands in (2) and (4) are continuous functions and that the constraint sets are compact. This implies that there is a sequence of probability vectors $p_n^{(D)} \in C_n(D)$ and a probability vector $p_\infty^{(D)} \in C_\infty(D)$ such that $\psi_n(D) = f(p_n^{(D)})$ for all $n$ and $\psi_\infty(D) = f(p_\infty^{(D)})$. Since all these probability vectors belong to the compact set $C_\infty(D)$, there is a subsequence $p_{n_j}^{(D)}, j = 1, 2, \ldots$, converging to a probability vector $q_\infty^{(D)}$. We will shortly show that $q_\infty^{(D)}$ is in $C_\infty(D)$. Then, since $f$ is continuous, $\psi_{n_j}(D) = f(p_{n_j}^{(D)}) \uparrow f(q_\infty^{(D)}) \geq \psi_\infty(D)$ establishing the reverse inequality. Finally, to see why $q_\infty^{(D)}$ is in $C_\infty(D)$, note that for components of $d_{n_j}$ that increase to infinity, the corresponding components of $p_{n_j}^{(D)}$ must converge to zero since $D$ is finite. The remaining components of both $d_{n_j}$ and $p_{n_j}^{(D)}$ converge to finite nonnegative real numbers. Thus the subsequence of their (nonnegative) inner products, which is no more than $D$ (finite and nonnegative), converges to the inner product of their limits.

C. Application to rate-distortion functions

Comparing (1) and (2) it is apparent that they have the same form. To demonstrate that (1) is, in fact, a special case of (2), we only need to verify that the minimand, minimizing variable, and constraint sets of (1) satisfy all the assumptions in Sec. IIA-B that (2) is required to satisfy. First note that $p$ in (2) corresponds to $p_{X,Y,U,\hat{X}}$ in (1) with $k = |X \times Y \times U \times \hat{X}|$. The $f$ in (1) is a finite linear combination of conditional mutual informations involving some or all the variables $X, Y, U, \hat{X}$. This is a real-valued (in fact also nonnegative and bounded) continuous function of $p_{X,Y,U,\hat{X}}$ since conditional mutual informations are continuous functions of the joint pmf of all the variables that appear in them [17][Ch.2,Sec.2.3], marginal pmfs are linear (therefore continuous) functions of the joint pmf, and the composition of a finite number of continuous functions is continuous. The set $A$ in (1) is the set of pmfs which satisfy certain marginal
consistency and Markov-chain constraints associated with the random variables $X,Y,U,\hat{X}$. This set is compact because a marginal consistency constraint is a linear equality constraint on the joint pmf (hence it defines a closed hyperplane within the bounded simplex of joint pmfs) and Markov-chain constraints can be expressed as the zero-level sets of appropriate conditional mutual information functions which, as we just discussed, are continuous. The distortion function $d$ in (1) corresponds to $d_n$ in (2). They are both real-valued (finite) and nonnegative. The set $\mathcal{B}_n(D)$ in (2) then corresponds to the expected distortion constraint $E[d(X,Y,U,\hat{X})] \leq D$ in (1). Finally, in typical scenarios of the source coding problems discussed in Sec. I (including [14] and [15][Secs. VII.B,C]), $D$ is finite, and the feasible set $\{p_{X,Y,U,\hat{X}} \in \mathcal{A} \text{ s.t. } E[d(X,Y,U,\hat{X})] \leq D\}$ is nonempty.

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