\[ \alpha \text{-GAUSS CURVATURE FLOWS WITH FLAT SIDES} \]

LAMI KIM, KI-AHM LEE, AND EUNJAI RHEE

**Abstract.** In this paper, we study the deformation of the 2 dimensional convex surfaces in \( \mathbb{R}^3 \) whose speed at a point on the surface is proportional to \( \alpha \)-power of positive part of Gauss Curvature. First, for \( \frac{1}{2} < \alpha \leq 1 \), we show that there is smooth solution if the initial data is smooth and strictly convex and that there is a viscosity solution with \( C^{1,1} \)-estimate before the collapsing time if the initial surface is only convex. Moreover, we show that there is a waiting time effect which means the flat spot of the convex surface will persist for a while. We also show the interface between the flat side and the strictly convex side of the surface remains smooth on \( 0 < t < T_0 \) under certain necessary regularity and non-degeneracy initial conditions, where \( T_0 \) is the vanishing time of the flat side.

1. Introduction

We are concerned with the regularity of the \( \alpha \)-Gauss Curvature flow with flat sides, which is associated to the free boundary problem. This flow explains the deformation of a compact convex subjects moving with collision from any random angle. The probability of impact at any point \( P \) on the surface \( \Sigma \) is proportional to the \( \alpha \)-Gauss Curvature \( K^\alpha \). Then the deformation of the surface \( \Sigma \) can be described by the flow

\[
\frac{\partial X}{\partial t}(x, t) = -K^\alpha(x, t) \nu(x, t)
\]

(1.1)

where \( \nu \) denotes the unit outward normal and \( \alpha > 0 \). Now we are going to summarize the known results for the evolution of the strictly convex surfaces following (1.1), [C1, C2].

Various application of (1.1) has been discussed at [A2]: rolling stone on the hyperplanes (\( \alpha = 1, [2] \)), the affine normal flows (\( \alpha = \frac{1}{n+2}, [ST1, ST2] \)), the gradient flows of the mean width in \( L^p \)-norm (\( \alpha = \frac{1}{p-1}, [A2] \)), and image process (\( p = \frac{1}{4}, [ACLIM] \)).

The dynamics and degeneracy of the diffusion varies depending on \( \alpha \). If \( \alpha \) is smaller, it becomes more singular and the solution gets regular instantaneously. On the other hand, if \( \alpha \) is greater than \( \frac{1}{n} \), it becomes degenerate and has waiting time effect which means that the flat spot of the surface stays for a while, [A2]. Waiting time and finite speed of propagation caused by the degeneracy have been studied
at the other well-known degenerate equations: the Porous Medium Equation
\[ u_t = \Delta u^m \]
and Parabolic \( p \)-Laplace Equation
\[ u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u). \]

1.1. The known results. Let \( X(\cdot, \cdot) : S^n \times [0, T) \to \mathbb{R}^{n+1} \) be a embedding and set \( \Sigma_t = X(S^n, t) \). Since the volume is decreasing in time and vanishes at finite time, there is the first time, \( T_0 \), when \( \text{vol}(\Sigma_t) \) becomes zero.

Let us assume \( \Sigma_0 \) be strictly convex and smooth. Then \( \Sigma_t \) is also smooth and strictly convex for \( 0 < t < T_0 \) and \( \alpha = 1, [A1] \) or for \( \alpha = \frac{1}{n} \) and \( n \geq 2, [C1] \). It is also true if \( \alpha > \frac{1}{n} \) and if the initial surface is sufficiently close to the sphere, \( [C1] \). And \( \Sigma_t \) converges to a point if \( \alpha \in \left( \frac{1}{n} + \frac{2}{n+2}, \frac{1}{n} \right) \), or if \( \alpha \in (0, \frac{1}{n}] \) and \( \Sigma_t \) has bounded isoperimetric ratio which is the ratio between the radius of the inner sphere and that of outer sphere, \( [A2] \).

Now let \( \Sigma_0 \) be convex and smooth. For \( \alpha > 0 \), there is a viscosity solution, \( \Sigma_t \), for \( 0 < t < T_0 \) which has uniform Lipschitz bound, \( [A2] \). For \( \frac{1}{2} < \alpha \leq 1 \) and \( n = 2 \), the convex viscosity solution, \( \Sigma_t \), has a uniform \( C^{1,1} \)-estimate for \( 0 < t < T_0 \), \( [KL] \). For \( \alpha = 1 \) and \( n = 2 \), the \( C^\infty \)-regularity of the strictly convex part of the surface and the smoothness of the interface between the strictly convex part and flat spot have been proved at \( [DL3] \).

1.2. The balance of terms. In this paper, we are going to study the regularity of \( \Sigma_t \), when the initial surface, \( \Sigma_0 \), has a flat spot for \( n = 2 \).

We will assume for simplicity that the initial surface \( \Sigma_0 \) has only one flat spot, namely that at \( t \) we have \( \Sigma_t = \Sigma_t^1 \cup \Sigma_t^2 \) where \( \Sigma_t^1 \) is the flat spot and \( \Sigma_t^2 \) is strictly convex part of \( \Sigma_t \). The intersection between two regions is the free boundary \( \Gamma_t = \Sigma_t^1 \cap \Sigma_t^2 \). The lower part of the surface \( \Sigma_0 \) can be written as a graph \( z = f(x) \).

And similarly we can write the lower part of \( \Sigma_t \) as \( z = f(x, t) \) for \( x \in \Omega \subset \mathbb{R}^n \) where \( \Omega \) is an open subset of \( \mathbb{R}^n \).

The function \( f(x, t) \) satisfies \( \alpha \)-Gauss Curvature flow:
\[
(1.2) \quad f_t = \frac{[\det(D^2 f)]^a}{(1 + |\nabla f|^2)^{\alpha n + 2a - 1}}.
\]

Let’s consider rotationally symmetric case first to see the balance between terms for \( n = 2 \). If \( f = f(r) \) is rotationally symmetric, \( (1.2) \) can be written as
\[
(1.3) \quad f_t = \frac{f_r f_{rr}^a}{r^{a}(1 + f_r^2)^{\frac{2a-1}{2}}}
\]
Let \( r = \gamma(t) \) be the equation of the free boundary \( \Gamma(f) = \partial\{f = 0\} \). The speed of boundary is given by

\[
\gamma' = -f_t = -\frac{f_{r} \alpha - 1 \ f_{rr}}{r^{\alpha}(1 + f_{r}^2)^{\frac{4\alpha - 1}{2}}}. 
\]

The regularity comes from the nondegenerate finite speed of the free boundary before the flat spot converges to a lower dimensional singularity at a focusing time. When \( f = (r - 1)^{\beta} \) at a given time \( t \), for \( r \approx 1 \),

\[
|\gamma_t| = s^{(\alpha - 1)(\beta - 1)} s^{(\alpha - 2)} \approx 1
\]

for \( s = r - 1 \), which implies \( \beta = \frac{3\alpha - 1}{2\alpha - 1} \).

For the general \( f = f(x, y, t) \), let \( f = \frac{1}{B} g^{\beta} \) for \( \beta = \frac{3\alpha - 1}{2\alpha - 1} \). The equation for this pressure \( g \) will be

\[
g_t = \frac{[g \ \det(D^2 g) + \theta(\alpha)(g_{xx} g_{yy} + g_{xy}^2 g_{xx} - 2g_{x} g_{y} g_{xy})]^{\alpha}}{(1 + g^{2\beta - 2} |\nabla g|^2)^{\frac{4\alpha - 1}{2}}} 
\]

for \( \theta(\alpha) = \beta - 1 = \frac{\alpha}{2\alpha - 1} \).

Assuming \( g_{\tau} = 0 \) at the boundary, the speed of boundary will be

\[
\gamma_t = -\frac{g_t}{g_{\nu}} = -\theta(\alpha) \frac{\alpha}{2\alpha - 1} \frac{g_{2\alpha - 1} g_{\tau}}{g_{\nu}}
\]

for a tangential direction \( \tau \) and a normal direction \( \nu \) to \( \partial \Omega \).

1.3. Conditions for \( f \).

**Condition 1.1.** Set \( \Lambda(f) = \{f = 0\} \) and \( \Gamma(f) = \partial \Lambda(f) \).

(1) (Nondegeneracy Condition) Our basic assumption on the initial surface is that the function \( f \) vanishes of the order \( dist(X, \Lambda(f))^{\frac{3\alpha - 1}{2\alpha - 1}} \) and that the interface \( \Gamma(f) \) is strictly convex so that the interface moves with finite nondegenerate speed. Namely, setting \( g = (\beta f)^{\frac{1}{\beta}} \), we assume that at time \( t = 0 \) the function \( g \) satisfies the following nondegeneracy condition: at \( t = 0 \),

\[
0 < \lambda < |Dg(X)| < \frac{1}{\lambda} \quad \text{and} \quad 0 < \lambda^2 < D_{\tau \tau}^2 g(X) < \frac{1}{\lambda^2}
\]

for all \( X \in \Gamma_0 \) and some positive number \( \lambda > 0 \), where \( D_{\tau \tau}^2 \) denotes the second order tangential derivative at \( \Gamma \). Then the initial speed of free boundary has the speed, at \( t = 0 \),

\[
0 < \lambda^{\frac{\alpha - 1}{2}} < |\gamma| < \frac{1}{\lambda^{\frac{\alpha - 1}{2}}}
\]
(II) (Before Focusing of Flat Spot) Let \( T \) be any number on \( 0 < T < T_0 \), so that the flat side \( \Sigma_1^1 \) is non-zero. Since the area is non-zero, \( \Sigma_1^1 \) contains a disc \( D_{p_0} \) for some \( p_0 > 0 \). We may assume that

\[
\Phi(x, y, t) = 0
\]

implies that under (1.6) and initial regularity conditions, the linearized operator

\[
P \Phi(x, y, t) = 0
\]

by rotating the coordinates. Also by transforming the free-boundary to a fixed boundary 

\[
Q = (x, y) \in R^2
\]

for \( 0 \leq t \leq T_0 \).

(III) (Graph on a Neighborhood of the Flat spot \( \Sigma_1^1 \)) We will also assume, without loss of generality, throughout the paper that

\[
\max_{x \in \Omega(t)} f(x, t) \geq 2, \quad 0 \leq t \leq T_0
\]

where \( \Omega(t) = \{ (x, y) \in R^2 : |Df(x, t) < \infty \} \). Set

\[
\Omega_p(t) = \{ (x, y) : f(x, y, t) \leq f(P) \}
\]

1.4. The concept of regularity. Let \( s_0 = (x_0, y_0, t_0) \) is an interface point and \( t_0 \) is sufficiently small. Then condition (1.6) is satisfied at \( t_0 \) for small constant \( c \). We can assume

\[
g_s(s_0) \geq c > 0 \quad \text{for some} \quad c > 0
\]

by rotating the coordinates. Also by transforming the free-boundary to a fixed boundary near \( s_0 \), we can obtain the map \( x = h(z, y, t) \) where \( (z, y, t) \) is around \( Q_0 = (0, y_0, t_0) \) and then the free-boundary \( g = 0 \) is transformed into the fixed boundary \( z = 0 \). From the calculation on \( g(h(z, y, t), y, t) = z \), we have the fully nonlinear degenerate equation

\[
-\frac{h_y}{h_z} = \frac{(z(h_{yy} - h_{yy}) -\theta(\alpha)h_z h_{yy})^{\alpha}}{h_z^{\alpha}} \quad z > 0
\]

implying that under (1.6) and initial regularity conditions, the linearized operator

\[
\tilde{h} = z a_{11} \tilde{h}_{zz} + 2 \sqrt{z} a_{12} \tilde{h}_{zy} + a_{22} \tilde{h}_{yy} + b_1 \tilde{h}_z + b_2 \tilde{h}_y
\]

where \( (a_{ij}) \) is strictly positive and \( b_1 \geq v > 0 \) for some \( v > 0 \).

Definition 1.2. For the Riemannian metric \( ds^2 = \frac{dz^2}{z^2} + dz^2 \), let distance between \( Q_1 = (z_1, y_1) \) and \( Q_2 = (z_2, y_2) \) in the metric \( s \) be \( s(Q_1, Q_2) = |\sqrt{z_1} - \sqrt{z_2}| + |y_1 - y_2| \) and the parabolic distance between \( Q_1 = (z_1, y_1, t_1) \) and \( Q_2 = (z_2, y_2, t_2) \) be \( s(Q_1, Q_2) = |\sqrt{z_1} - \sqrt{z_2}| + |y_1 - y_2| + \sqrt{|t_1 - t_2|} \). Then we define \( C^\gamma_s \) as the space of \( \alpha \) continuous functions with respect to the metric \( s \) as the space of all functions \( h \) with

\[
h, h_z, h_y, h_t, z h_{zz}, z \sqrt{z} h_{zy}, h_{yy} \in C^\gamma_s.
\]
Remark 1.3. When we consider the equation
\[ h_t = z h_{zz} + h_{yy} + \nu h_z \]
on the half-space with $\nu > 0$, which doesn’t have the other condition of $h$ on $z = 0$, the Riemannian metric $ds$ decides the diffusion of the equation.

Remark 1.4. If the transformed function $h \in C^{2+\gamma}_x$, we say that $g \in C^{2+\gamma}_x$ around the interface $\Gamma$.

1.5. Main Theorems. Now we are going to state main theorems when the initial surface is just convex having a flat spot.

Theorem 1.5. Let us assume $\frac{1}{2} < \alpha \leq 1$. If $\Sigma_0$ is convex, any viscosity solution $\Sigma_t$ of (1.1) is $C^{1,1}$ for $0 < t < T_0$. Moreover the strictly convex part, $\Sigma^2_t$, is smooth for $0 < t < T_0$.

The following short time existence of $C^\infty_z$-solution with a flat spot has been essentially proved in [DH] since the linearized equation for $h$, (5.3), is in the same class of operators considered in [DH] because of the conditions, (1.6), as [DH]. Therefore their Schauder theory can be applied to (5.3) and then the application of implicit function theorem gives the short time existence as [DH].

Theorem 1.6. [Short Time Regularity] [DH] For $\frac{1}{2} < \alpha \leq 1$, assume that $g = (\beta f)^{\frac{1}{\beta}}$ is of class $C^{2+\gamma}_x$ up to the interface $z = 0$ at time $t = 0$, for some $0 < \gamma < 1$, and satisfies Conditions (1.1) for $f$. Then there exists a time $T > 0$ such that the $\alpha$--Gauss Curvature Flow (1.1) admits a solution $\Sigma(t)$ on $0 \leq t \leq T$. In addition the function $g = (\beta f)^{\frac{1}{\beta}}$ is smooth up to the interface $z = 0$ on $0 < t \leq T$. In particular the junction $\Gamma(t)$ between the strictly convex and the flat side will be a smooth curve for all $t$ in $0 < t \leq T$.

One of main results in this paper is the following long time regularity of the solution.

Theorem 1.7. [Long Time Regularity] Under the assumptions of Theorem 1.6 the function $g = (\beta f)^{\frac{1}{\beta}}$ remains smooth up to the interface $z = 0$ on $0 < t < T$ for all $T < T_0$. And the interface $\Gamma(t)$ between the strictly convex and the flat side will be smooth curve for all $t$ in $0 < t < T_0$.

To show Theorem 1.7 we follow the main steps at [DL3]. But the exponent $\alpha$ creates large number of nontrivial terms especially in the estimate of the second derivatives. New quantities have been considered to absorb the effect of terms depending on $(1 - \alpha)$ at Lemma 4.3. Optimal regularity and Aronson-Bénilan type estimate have been proved at Lemma 4.4 and 4.5.
2. Convex surface.

2.1. Evolution of the metric and curvature. The metric and second fundamental form can be defined by

\[ g_{ij} = \left( \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right) \quad \text{and} \quad h_{ij} = -\left( \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right) \]

with respect to a local coordinates \( \{x_1, \ldots, x_n\} \) of \( \Sigma_t \) and \( \nu \) is the outward unit normal to \( \Sigma_t \). Also the Weingarten map is given by

\[ h^j_i = g^{ik} h_{kj} \]

and then \( \sigma_k = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \cdots \lambda_k \), \( H = \text{trace}(h) = \sigma_1 = \sum_{1 \leq i \leq n} \lambda_i \), \( K = \det(h) = \sigma_n = \lambda_1 \lambda_2 \cdots \lambda_n \), and \( |A|^2 = h_{ij} h^{ij} = \lambda_1^2 + \cdots + \lambda_n^2 \) where \( \lambda_1, \cdots, \lambda_n \) are the eigenvalues of the Weingarten map.

The evolution of the metric, second fundamental form, and curvature are the following.

**Lemma 2.1.** Let \( X(x, t) \) be a smooth solution of (1.1). Then we have the following. The proof can be referred to Chapter 2, [Z]. Let \( \Box \) denote \( K^a(h^{-1})^{bi} \nabla_b \nabla_i \).

(i) \[ \frac{\partial g_{ij}}{\partial t} = -2K^a h_{ij} \]

(ii) \[ \frac{\partial K^a}{\partial t} = -g^{ij} \frac{\partial K^a}{\partial x^i} \frac{\partial X}{\partial x^j} = -\nabla^j K^a \frac{\partial X}{\partial x^j} \]

(iii) \[
\frac{\partial h_{ij}}{\partial t} = \alpha \Box h_{ij} + \alpha^2 K^a (h^{-1})^{kl} (h^{-1})^{mn} \nabla_i h_{kl} \nabla_j h_{mn} - \alpha K^a (h^{-1})^{kl} (h^{-1})^{mn} \nabla_i h_{mn} \nabla_j h_{kl} + \alpha K^a h_i h_j - (1 + 2\alpha) K^a h_i h_j
\]

(iv) \[ \frac{\partial K}{\partial t} = \alpha \Box K + \alpha (\alpha - 1) K^{a-1} (h^{-1})^{ij} \nabla_i \nabla_j K + K^{a+1} H \]

(v) \[ \frac{\partial K^a}{\partial t} = \alpha \Box K^a + \alpha K^a H \]

2.2. Curvature Estimates. Now we are going to show the regularity of \( \Sigma_t \). The following Lemma was proved in [A2, K1].

**Lemma 2.2.** [A2, K1] Let \( \Sigma_0 \) be convex and \( \alpha > 0 \). Then

(i) There is a constant \( C > 0 \) such that

\[ \sup_{x \in \Sigma_t} K^a(x, t) \leq C(\alpha) = \max \left( \sup_{x \in \Sigma} K^a(x, 0), \left( \frac{2\alpha + 1}{2\alpha \rho_0} \right)^{2\alpha} \right) \]

(ii) \[ \inf_{x \in \Sigma_t} K^a \geq e^{C(t_0)} \inf_{x \in \Sigma} K^a(x, 0) \] where \( C(t_0) \) is some constant for \( 0 < t_0 < T_0 \)

(iii) There is a unique viscosity solution \( \Sigma_t \).
Lemma 2.3. Set $\psi(x,t) = \langle x, \nu \rangle$ and let $B_{R_0}(0)$ be a ball of radius $R_0$ about the origin and $P = \frac{H}{\psi + 4R^2 - |x|^2}$, where $\Sigma_0$ is contained in $B_{R_0}(0)$ and $R^2 = \max(R_0^2, R_0)$. Then there exists a constant $C = C(\sup_{x \in \Sigma, 0 \leq t < T_0} K^\alpha, R) > 0$ for $\frac{1}{2} < \alpha \leq 1$ such that

$$\sup_{x \in \Sigma, 0 \leq t < T_0} H(x,t) \leq C.$$

Proof. Since $|x|$ is decreasing, $\psi + 4R^2 - |x|^2$ is positive and then we have

$$\frac{\partial}{\partial t} |x|^2 = \Box |x|^2 + 2K^\alpha \langle x, \nu \rangle - 2K^\alpha (h^{-1})^{kl} g_{kl}.$$

By using $\nabla_i P = 0$ at the maximum point, we can obtain

$$\alpha \Box P = \frac{\alpha \Box H}{\psi + 4R^2 - |x|^2} + \frac{\alpha H \Box |x|^2}{(\psi + 4R^2 - |x|^2)^2} - \frac{\alpha H \Box \psi}{(\psi + 4R^2 - |x|^2)^2}$$

and then since $\nabla_i \nabla_j P \leq 0$ at the maximum point, we get

(2.1)

$$\frac{\partial}{\partial t} P \leq \frac{(1 - \alpha)H \Box |x|^2}{\psi + 4R^2 - |x|^2} + \frac{H \left(2(\alpha + 1)K^\alpha + 2K^\alpha \psi \right)}{(\psi + 4R^2 - |x|^2)^2} - \frac{H^2 (\alpha K^\alpha \psi + 2K^\alpha - 1)}{(\psi + 4R^2 - |x|^2)^2}$$

$$+ \frac{\alpha K^\alpha}{\psi + 4R^2 - |x|^2} \left(\alpha g^{ij}(h^{-1})^{kl}(h^{-1})^{mn} \nabla_i h_{kl} \nabla_j h_{mn} - g^{ij}(h^{-1})^{km}(h^{-1})^{nl} \nabla_i h_{mn} \nabla_j h_{kl} \right)$$

$$+ \frac{1}{\psi + 4R^2 - |x|^2} \left(\alpha K^\alpha H^2 + (1 - 2\alpha)K^\alpha |A|^2 \right)$$

at the maximum point. Now, we can estimate the third term of (2.1) by the following inequality

$$\alpha g^{ij}(h^{-1})^{kl}(h^{-1})^{mn} \nabla_i h_{kl} \nabla_j h_{mn} - g^{ij}(h^{-1})^{km}(h^{-1})^{nl} \nabla_i h_{mn} \nabla_j h_{kl}$$

$$= (\alpha - 1) \left[ (h^{-1})^{11} \nabla_1 h_{11} + (h^{-1})^{22} \nabla_1 h_{22} \right]^2 + (h^{-1})^{11} \nabla_2 h_{11} + (h^{-1})^{22} \nabla_2 h_{22} \right]^2$$

$$+ 2(h^{-1})^{11} (h^{-1})^{22} \left[ \nabla_1 h_{11} \nabla_2 h_{22} + \nabla_2 h_{11} \nabla_2 h_{22} \right] - 2(h^{-1})^{11} (h^{-1})^{22} \left[ (\nabla_2 h_{11})^2 + (\nabla_1 h_{22})^2 \right]$$

$$\leq 2(h^{-1})^{11} (h^{-1})^{22} \left[ (\nabla_2 h_{12})^2 - PV_1 h_{22} (\nabla_1 |x|^2 - \nabla_1 \psi) - (\nabla_2 h_{11})^2 - PV_2 h_{11} (\nabla_2 |x|^2 - \nabla_2 \psi) \right]$$

$$\leq 2(h^{-1})^{11} (h^{-1})^{22} \left[ (1 - \frac{\tilde{h}}{2})^2 (|x|^2 - \langle x, \nu \rangle^2) \right]^2$$

where $\tilde{h} = \min \{h_{11}, h_{22}\}$. So

$$\frac{\partial}{\partial t} P \leq \frac{2\alpha K^\alpha - 1 - \frac{\tilde{h}}{2}}{\psi + 4R^2 - |x|^2} \left( |x|^2 - \psi^2 \right) + K^\alpha \left( 4(1 - \alpha)R^2 - (1 - \alpha)|x|^2 \right) + (1 - 2\alpha)K^\alpha \psi - 2K^\alpha - 1 \right] P^2$$

$$+ \frac{1}{\psi + 4R^2 - |x|^2} \left[ (1 - \alpha) \Box |x|^2 + \left( 2\psi + (2\alpha + 1) \right) K^\alpha \right] P + \frac{2(2\alpha - 1)K^\alpha}{\psi + 4R^2 - |x|^2}.$$
For $\frac{1}{2} < \alpha \leq 1$, we can make the coefficient of $P^2$ be negative, which can be achieved if we consider $\eta$ small enough. The reason is if we begin with $\eta \Sigma_0$ for any given $\Sigma_0$, we can make $K \geq \frac{C_0}{\eta^2}$ where $C_0$ is some constant depending on initial surface, which comes from Lemma 2.2 and $|\nu|^2 \leq \eta^2$, $R^2 \leq \eta^2$, and $\psi \leq \eta$ for sufficiently small $\eta$. Then the first term and second term of coefficient of $P^2$ are $O(\eta^{2-2\alpha})$ and the third term is negative with $K^\alpha \psi = O(\eta^{1-2\alpha})$ for $\eta$ small enough. This implies $\frac{\partial P}{\partial t} \leq -\frac{1}{2}P^2 + C$ where $C = C(\sup_{x \in \Sigma_0} \inf_{0 \leq t < T_0} K^\alpha, R)$ and then if $-\frac{1}{2}P^2 + C < 0$, it is contradiction. So $P$ is bounded and hence $H$ is bounded before $\Sigma$ shrinks a point. □

2.3. Strictly convexity away from the flat spot. To apply Harnack principle, let us introduce new coordinate defined on the sphere $S^n$. If $\Sigma_t$ is strictly convex, $\nu(x,t)$ is a one-to-one map from $\Sigma_t$ to $S^n$, which means for each $z \in S^n$, there is $X(x,t) = \nu^{-1}(z,t)$. $K(z,t)$ denotes Gauss Curvature $K$ at $\nu^{-1}(z,t)$. If $\Sigma_t$ is convex, we still use the same coordinate $(z,t)$ for strictly convex part $\Sigma^2_t$ by using approximates with strictly convex surfaces.

**Lemma 2.4.** Assume that the flat spot, $\Sigma^1_t$, is a part of the plane orthogonal to $e_{n+1}$. For any $\eta > 0$, there is a constant $c_\eta > 0$ such that

$$K^\alpha(z,t) \geq c_\eta$$

for $z \in S_\eta := \{z \in S^n \text{ and } ||z + e_{n+1}|| \geq \eta > 0\}$.

**Proof.** We can immediately obtain the result from the Harnack estimate in [C3]:

For any points $z_1, z_2 \in S_\eta$ and times $0 \leq t_1 < t_2$

$$\frac{K^\alpha(z_2,t_2)}{K^\alpha(z_1,t_1)} \geq e^{-\Theta/4} \left(\frac{t_2}{t_1}\right)^{(1+2\alpha)^{-1}}$$

where $\Theta = \Theta(z_1, z_2, t_1, t_2) = \inf_{\gamma} \int_{t_1}^{t_2} |\nu(t)|^2 |\nu(t)|^2_{\nu(t)} dt$ and the infimum is taken over all paths $\gamma$ in $\Sigma$ whose graph $(\gamma(t), t)$ joins $(z_1, t_1)$ to $(z_2, t_2)$. The short time existence of smooth surfaces implies that, for $z \in S_\eta$, $X(z,t)$ is the strictly convex part, $\Sigma^2_t$, for $0 \leq t \leq \delta_0$ for some $\delta_0 > 0$. Therefore we can take $0 < \delta_0 \leq t_1 < t_2 \leq T$, which implies $K^\alpha(z_2,t_2) \geq c_1 K^\alpha(z_1, \delta_0) \geq c_\eta$ for some $c_1, c_\eta > 0$ and then the conclusion. □

We finally know (1.2) is uniformly parabolic, which comes from Lemmas 2.2, 2.3

And then we can show that $\Sigma_t$ is $C^\alpha$ on the point being away from flat spot.

**Corollary 2.5.** Under the same condition of Lemma 2.4, $\tilde{\Sigma}^2_t := \{X(z,t) \in \Sigma^2_t : z \in S^2_\eta\}$ is smooth.
Proof. Let \( \lambda_i \) be the eigenvalues of \( (h'_i) \). From the convexity, \( \lambda_i \geq 0 \). And from the upper bound of Mean Curvature and the lower bound of Gauss Curvature, \( \lambda_1 + \cdots + \lambda_n < C_1 \) and \( K = \lambda_1 \cdots \lambda_n > c_2 \). Now we have

\[
C_1 \geq \lambda_i \geq \frac{c_2}{\prod_{j \neq i} \lambda_j} \geq \frac{c_2}{C_1^{n-1}} > 0.
\]

It implies there are \( 0 < \lambda \leq \Lambda < \infty \) such that

\[
\lambda |\xi|^2 \leq K \langle h^{-1} \rangle^j |\xi|^j \leq \Lambda |\xi|^2
\]

and the support function \( S(z, t) \) satisfies a uniformly parabolic equation in \( \Sigma_t^2 \). Therefore \( S(z, t) \) is \( C^2, \gamma \) and then \( C^\infty \) in \( \Sigma_t^2 \) through the standard bootstrap argument using the Schauder theory. \( \square \)

2.4. Proof of Theorem 1.5. Recall that \( |A|^2 \) is the square sum of principle curvatures of a given surface. First, we approximate the initial surface \( \Sigma_0 \) with strictly convex smooth functions, \( \Sigma_{0,\epsilon} \) whose \( |A_{0,\epsilon}|^2 \) is uniformly bounded by \( 2|A_0|^2 \) of \( \Sigma_0 \). Then there are smooth solutions \( \Sigma_{t,\epsilon} \) of \( (1.1), [KL], \) and \( |A_{0,\epsilon}|^2 \leq 2H_\epsilon^2 < 4|A_0|^2 < C \) uniformly. As \( \epsilon \to 0 \), \( \Sigma_{t,\epsilon} \) converges to a viscosity solution \( \Sigma_t \) as \( [A1] \). \( |A|^2 \) of \( \Sigma_t \) will be uniformly bounded, which implies that \( \Sigma_t \) is \( C^{1,1} \). And for any \( X \in \Sigma_t^2 \), there is a small \( \eta > 0 \) such that \( ||v_X + \epsilon_{n+1}|| \geq \eta > 0 \) and then \( X \in \Sigma_t^2 \). Since \( \Sigma_t^2 \) is smooth at \( X \), so \( \Sigma_t^2 \).

2.5. A Waiting Time Effect. We now are going to show the flat spot of the convex surface will persist for some time.

Lemma 2.6. Let \( \Sigma_0 \) be convex. For \( \frac{1}{2} < \alpha \leq 1 \), there is a waiting time of flat spot: if \( P_0 \in \text{int}_n(\Sigma_0 \cap \Pi) \) where \( \Pi \) is a \( n \)-dimensional plane and \( \text{int}_n(A) \) is the interior of \( A \) with respect to the topology in \( \Pi \), there is \( t_0 > 0 \) such that \( P_0 \in \text{int}_n(\Sigma_t \cap \Pi) \) for \( 0 < t < t_0 \).

Proof. Let \( h^* = C_+ \frac{|X-P_0|^\mu}{(r-r^*)^{\gamma}} \) for \( \mu = \frac{4\gamma}{2\gamma - 1}, \gamma = \frac{1}{2m-1}, \) and \( C_+ = \left( \frac{\gamma}{\mu^2(\mu-1)^2} \right)^{\frac{1}{2m-1}} \). Then \( h^* \) is a super-solution of \( (1.2) \). Now we are going to compare the solution \( f \) with \( h^* \). From \( C^{1,1} \)-estimate of \( f, f_1 \) is bounded and then there is a ball \( B_{P_0}(P_0) \subset \text{int}_n(\Sigma_0 \cap \Pi) \) and \( t_0 > 0 \) such that \( f(X, t) \leq h^*(X, t) \) on \( \partial B_{P_0}(P_0) \) for \( 0 \leq t \leq t_0 \) and \( f(X, 0) \leq h^*(X, 0) \). From the comparison principle, we have \( f(X, t) \leq h^*(X, t) \) for \( (X, t) \in B_{P_0}(P_0) \times [0, t_0) \), which implies \( f(P_0, t) = 0 \) and \( P_0 \in \Sigma_t \) for \( 0 \leq t \leq t_0 \). \( \square \)

3. Optimal Gradient Estimate near Free Interface

3.1. Finite and Non-Degenerate Speed of level sets. From using the differential Harnack inequalities, we can show that the free-boundary \( \Gamma(t) \) has finite and non-degenerate speed as \( [DL3] \). As Theorem 1.6, we assume that \( z = f(x, t) \) is a solution.
of (1.2) and $C^{1,1}$ on $\Omega(t)$ for all $0 < t \leq T$ and $g = (\beta f)^{\frac{1}{\beta}}$ is smooth up to the interface $\Gamma(t)$ on $0 < t \leq \tau$ for some $\tau < T$.

Let us consider the function

$$f_\varepsilon(x,t) = \frac{(1 - Ae)^{(4\alpha - 1)/2}(1 + \varepsilon)^{\frac{4\alpha}{2}}}{(1 + Be)^{2\alpha - 1}} f((1 + \varepsilon)x, (1 - Ae)t) (3.1)$$

and then the consequences of [DL3] can be applied to our equation by the similar ways.

We may assume condition (1.8) and let $r = \gamma(\theta, t)$ be the interface $\Gamma(t)$ and $r = \gamma_\varepsilon(\theta, t)$ be the $\varepsilon$-level set of the function $f$ with $0 \leq \theta < 2\pi$ by expressing in polar coordinates. Then

**Lemma 3.1.** There exist constants $A, B, C > 0$ and $\tilde{A}, \tilde{B}, \tilde{C} > 0$ such that

$$e^{-\frac{t - t_0}{\tilde{A} + \tilde{B} \theta}} \gamma(\theta, t_0) \geq \gamma(\theta, t) \geq e^{-\frac{t - t_0}{\tilde{C} \theta}} \gamma(\theta, t_0)$$

and

$$e^{-\frac{t - t_0}{A + B \theta}} \gamma_\varepsilon(\theta, t_0) \geq \gamma_\varepsilon(\theta, t) \geq e^{-\frac{t - t_0}{C \theta}} \gamma(\theta, t_0)$$

for all $0 < t_0 \leq t \leq T, 0 \leq \theta < 2\pi$. In particular, the free-boundary $r = \gamma(\theta, t)$ and the $\varepsilon$-level set $r = \gamma_\varepsilon(\theta, t)$ of $f$ for each $\varepsilon > 0$ move with finite and nondegenerate speed on $0 \leq t \leq T$.

### 3.2. Gradient Estimates

Throughout this section, we will assume that $g = (\beta f)^{\frac{1}{\beta}}$ is solution of (1.4) and smooth up to the interface on $0 \leq t \leq T$ and also is satisfied with

$$\max_{x \in \Omega(t)} g(x,t) \geq 2, \quad \text{for } 0 \leq t \leq T,$$

which comes from (1.9). We now will show that the gradient $|Dg|$ has the bound from above and below.

**Lemma 3.2** (Optimal Gradient estimates). With the same assumptions of Theorem 1.6 and (3.4), there is a positive constant $C_0$ such that

$$|Dg| \leq C_0, \quad \text{on } 0 \leq g(\cdot, t) \leq 1, \quad 0 \leq t \leq T.$$

Moreover if (1.8) is satisfied and if $g$ is smooth up to the interface on $0 \leq t \leq T$, then there is a positive constant $c_0$ such that

$$|Dg| \geq c_0, \quad \text{on } g(\cdot, t) > 0, \quad 0 \leq t \leq T.$$
Proof. (i) First, we are going to show the upper bound of $\nabla g$. Suppose that $f$ is approximated by $f_\varepsilon$ of (1.2) which is a decreasing sequence of solutions satisfying the positivity, strictly convexity and smoothness on $\{ x \in \mathbb{R}^2 : |Df_\varepsilon(x)| < \infty \}$ for $0 \leq t \leq T$. Set $g_\varepsilon = (\beta f_\varepsilon)^{1/\beta}$. We can choose the $f_\varepsilon$'s such that $|Dg_\varepsilon| \leq C_0$ at $t = 0$, on the set $\{ x : 0 \leq g_\varepsilon \leq 1 \}$ and $|Dg_\varepsilon| \leq C_0$ at $g_\varepsilon = 1$, $0 \leq t \leq T$, for some uniform constant $C_0$. The last estimate holds because of (1.9) and (3.4).

Let's denote $g_\varepsilon$ by $g$ for convenience of notation, where $g = (\beta f)^{1/\beta}$ is a strictly positive and a smooth solution of (1.4) with convex $f$. Let us apply the maximum principle to $X = |Dg|^2 = \frac{g_x^2 + g_y^2}{2}$ and assume $X$ has an interior maximum at the point $P_0 = (x_0, y_0, t_0)$. By rotating the coordinates, we can assume $g_x > 0$ and $g_y = 0$ at $P_0$. Then we have $X_t \leq 0$ by using the facts that $X_x = X_y = 0$, $X_{xx} \leq 0$ and $X_{yy} \leq 0$ are satisfied at $P_0$. On the other hand $|\nabla g|$ is bounded at $t = 0$ from the condition on the initial data and on $\{ g = 1 \}$, $|\nabla g| = \frac{|\nabla f|}{g^{\frac{\beta}{2}}} = |\nabla f|$ is bounded since $f$ is convex. Hence $X \leq C$, on $0 \leq g \leq 1$, $0 \leq t \leq T$, provided that $X \leq C$ at $t = 0$ and $g = 1$, $0 \leq t \leq T$ so that

$$|Dg| \leq C_0, \quad \text{on } 0 \leq g(t) \leq 1, \quad 0 \leq t \leq T.$$

(ii.) Now we are going to show the lower bound of the gradient. Consider

$$X = x g_x + y g_y.$$

Using the maximum principle as (i), we have that

$$X_t \geq -C X$$

where $C$ is a constant depending on $\rho_0$ and

$$\frac{d}{dt} X(\gamma(t), t) \geq -C X$$

at a interior or boundary minimum point $P_0$ of $X$. Then

$$\min_{\{g(\cdot, t) > 0\}} X(t) \geq \min_{\{g(\cdot, 0) > 0\}} X(0) e^{-Ct}$$

for all $0 \leq t \leq T$ by Gronwall's inequality, and it implies the desired estimate. □

**Theorem 3.3.** Under the same assumptions of Lemma [3.2] there exist positive constants $C_1$, $C_2$ and $\varepsilon_0$, depending only on $\rho_0$ and the initial data, for which

$$-C_2 \leq (\gamma_\varepsilon)_t(\theta, t) \leq -C_1 < 0, \quad \text{for } 0 \leq t \leq T \text{ and } 0 < \varepsilon < \varepsilon_0.$$
4. Second derivative estimate

4.1. Decay Rate of $\alpha$-Gauss Curvature. Under the same conditions with sections (3.1) and (3.2), we will show a priori bounds of the Gauss Curvature $K = \det(D^2f)/(1 + |Df|^2)$ and the second derivatives of $f$ and $g$.

**Lemma 4.1.** For the same hypothesis of Theorem 1.6 and (1.8), there exists a positive constant $c$ such that

\begin{equation}
(4.1) \quad c \leq \frac{K^\alpha}{g^{\frac{1}{\alpha-1}}} \leq c^{-1}, \quad \text{on } 0 \leq t \leq T
\end{equation}

for $K = \det D^2f/(1 + |Df|^2)$.  

**Proof.** We will only consider the bound of (4.1) around the interface. It suffices to show the bound of $g_t$ from $g_t = K^\alpha/(1 + |Df|^2)^{\frac{\alpha}{2}} \cdot g^{\frac{\alpha}{\alpha-1}}$ because $|Df|$ is bounded around $\{g = 0\}$. For $r = \gamma_{\varepsilon}(\theta, t)$ which is the $\varepsilon$-level set of $g$ in polar coordinates,

\[ g_t = -g_r \cdot \dot{\gamma}_{\varepsilon}(\theta, t) \]

since $g(\gamma_{\varepsilon}(\theta, t), \theta, t) = \varepsilon$ and the level sets of $g$ is convex. Then we know that $c < g_r < c^{-1}$ and $-C_2 \leq \dot{\gamma}_{\varepsilon}(\theta, t) \leq -C_1 < 0$ for $0 \leq t \leq T$ from Lemma 3.2 and Theorem 3.3 implying that $C_1 c < g_t < C_2 c^{-1}$, so proof is completed. \(\Box\)

**Corollary 4.2.** Under the assumptions of Lemma 4.1, the solution $g$ of (1.4) satisfies the bound

\begin{equation}
(4.2) \quad c \leq g_t \leq c^{-1}.
\end{equation}

4.2. Upper bound of the Curvature of Level Sets.

**Lemma 4.3.** With the assumptions of Theorem 1.6 and condition (1.8), there exists a constant $C > 0$ such that

\[ 0 < g_{\tau\tau} \leq C \]

with $\tau$ denoting the tangential direction to the level sets of $g$. 

**Proof.** Strictly convexity of the level sets of $g$ directly implies $g_{\tau\tau} > 0$. We will obtain the bound from above by using the maximum principle on

\begin{equation}
(4.3) \quad X = g_y^2 g_{xx} - 2g_x g_y g_{xy} + g_{xx}^2 g_{yy} + (g(g_{xx} + g_{yy}) + \theta|\nabla g|^2).
\end{equation}

Let $\nu$ and $\tau$ denote the outward normal and tangential direction to the level sets of $g$ respectively. Then we can write $X$ as

\begin{equation}
(4.4) \quad X = (g + g_{\tau}^2) g_{\tau\tau} + (g_{\nu\nu} + \theta g_{\tau}^2)
\end{equation}
since \( g_\tau = 0 \). We have also known that
\[
0 < c \leq g_\nu \leq c^{-1} \quad \text{on } g > 0, \quad 0 \leq t \leq T
\]
for some \( c > 0 \), depending on \( \rho_0 \) and the initial data. Also \( g(g_{xx} + g_{yy}) + \theta|\nabla g|^2 \) is bounded since \( f \in C^{1,1} \). Therefore, an upper bound on \( X \) will imply the desired upper bound on \( g_\tau \). We will apply the maximum principle on the evolution of \( X \).

The term \( (g(g_{xx} + g_{yy}) + \theta|\nabla g|^2) \) on \( X \) will control the sign of error terms. Corollary 4.2 implies
\[
X \leq C \quad \text{at } g = 0,
\]
since we know that \( X = \frac{1}{\theta} g_\mu \) at the free-boundary \( g = 0 \). Then we can assume that \( X \) has its space-time maximum at an interior point \( P_0 = (x_0, y_0, t_0) \).

Let’s assume that
\[
\text{(4.5) } g_\tau = g_y = 0 \quad \text{and } g_\nu = g_x > 0 \quad \text{at } P_0
\]
without loss of generality, since \( X \) is rotationally invariant. Also let’s consider the following transformation
\[
\tilde{g}(x, y) = g(\mu, \eta)
\]
where \( \mu = x \) and \( \eta = y - ax \) with \( a = \frac{g_{\mu\eta}(x_0, y_0, t_0)}{g_{\mu\mu}(x_0, y_0, t_0)} \). Then we can obtain \( \tilde{g}_x = g_\mu - \frac{g_\eta g_{\mu\eta}}{g_{\eta\eta}} = g_\mu > 0, \quad \tilde{g}_y = g_\eta = 0, \quad \text{and}
\]
\[
(\tilde{g}_{ij}) = \begin{bmatrix}
g_{\mu\mu} - \frac{g_{\mu\eta}^2}{g_{\eta\eta}} & 0 \\
0 & g_{\eta\eta}
\end{bmatrix}
\]
at \( P_0 \). Here \( \tilde{g}_{yy} = g_{\eta\eta} > 0 \) and \( \tilde{g}_{xx} < 0 \) at \( P_0 \). Hence equation is maintained with this change of coordinate. Also we can drop off the third derivative term of \( \tilde{g} \) because it is changed under the perfect square of the third derivative of \( g \). Hence we can assume
\[
\text{(4.6) } \tilde{g}_{xy} = 0
\]
at \( P_0 \) without loss of generality. Then we will proceed with the function \( g \) instead of \( \tilde{g} \) for convenient of notation. From (4.3), we get
\[
X = (g + g_x^2)g_{yy} + (g g_{xx} + \theta g_x^2), \quad \text{at } P_0.
\]
At the maximum point \( P_0 \), we also have \( X_x = 0 \) and \( X_y = 0 \) implying that
\[
\text{(4.7) } g_{xyy} = \frac{-g_{g_{xx}} + 2g_x \det D^2 g + (2\theta + 1)g_x g_{xx} + g_x g_{yy}}{g + g_x^2} \quad \text{and } g_{yyy} = \frac{-g_{g_{xyy}}}{g + g_x^2}.
\]
We next compute the evolution equation of $X$ from the evolution equation of $g$ to find a contradiction saying that

$$0 \leq X_t < 0 \quad \text{at} \quad P_0,$$

when $X > C > 0$ for some constant $C$. This implies that $X \leq C$, on $0 \leq t \leq T$.

First we will consider the following simpler case that $f$ satisfies the evolution

$$f_t = (\det D^2 f)^{\alpha}$$

for the convenience of the reader. Then $g = (\beta f)^{\frac{1}{\beta}}$ satisfies the equation

$$(4.8) \quad g_t = (g \det D^2 g + \theta(g^2 g_{xy} - 2g_x g_y + g^2 g_{yy}))^\alpha.$$  

To compute the evolution of $X$ we differentiate twice the equation (4.8). Set

$$K_g = g \det D^2 g + \theta(g^2 g_{xx} - 2g_x g_y g_{xy} + g^2 g_{yy}),$$

$$I = 1 + g^{2\beta - 2}\|\nabla g\|^2,$$  

and $J = g + \|\nabla g\|^2$.

Let $L$ denote the operator

$$LX := X_t - \alpha K_g^{-1} \left( (g g_{yy} + \theta g^2) X_{xx} - 2(g g_{xy} + \theta g_x g_y) X_{xy} + (g g_{xx} + \theta g^2) X_{yy} \right).$$

Then after many tedious calculations, we have that at the maximum point $P_0$,

$$(4.9) \quad LX = A + \frac{1}{(1 + 2\gamma)^2(g + \gamma^2)^2 K_g^2} B$$

where $\gamma = \theta - 1$ and

$$(4.10) \quad A = -4 g^2 g_{3xy} - \frac{4 g^3}{g^2 + g} \left( g_{xxx} + \frac{6 g_x g_{xx} + 3 g_x g_{xx} g_{yy}}{g} \right)^2$$

$$+ g \left\{ - (2 g^2 - g g_{xx})^2 + 3 g_{xx} (g^2 + g) (g^2 - g_{xx}) \right\}.$$

In addition, $B = 0$ if $\gamma = 0$, otherwise

$$(4.11) \quad B = -B_1 g^2 (g_x^2 + g) g_{3xy} - g^3 B_1 (g_{xxx} + B_{11})^2 + \left( (1 + 2\gamma)^2(g + \gamma^2)^2 K_g^2 \right) \frac{E_1}{E_2}.$$  

Here

$$B_1 = 4(1 + 2\gamma)^2 K_g^{-1} \left( \frac{1 + \gamma}{1 + 2\gamma} - K_g^2 \right) + (1 + \gamma) K_g^2 \left( (g_x^2 + g) g_{yy} - \left( g g_{xx} + (1 + \gamma) g_x^2 \right) \right)^2.$$
and set \( Z = g^2 g_{yy} \) so that

\[
E_2 = 4(1 + 2\gamma)^3 g X^2 + g^2 K^{\frac{6 + 12\gamma}{4}} Z^4 \left( \frac{(1 + \gamma)(1 + 2\gamma)}{(1 + 2\gamma)^2} - K^{\frac{\gamma}{4}} \right)
+ g\gamma(1 + \gamma)(1 + 2\gamma) g X^2 + g^2 K^{\frac{5 + 11\gamma}{4}} Z^6 + l.o.t.
\geq g(E_1 Z^4 + \gamma E_1 Z^6 + l.o.t.),
\]

where \( l.o.t. \) means lower order term. We may assume that \( P_0 \) lies close to the free-boundary and that \( K^{\frac{\gamma}{4}} < \frac{1}{2} \) by considering a scaled solution \( g_1(x,t) = \lambda^{1 + \frac{\gamma}{2}} \) \( \lambda \) \( g(\lambda x, \lambda^{\frac{4\gamma-5}{4\gamma-1}} t) \) as \( g \) at the beginning of proof with \( \lambda^{\frac{4\gamma-5}{4\gamma-1}} \leq \frac{1}{|K_0||g|^\infty} \left( \frac{1}{2} \right)^{\frac{1}{4\gamma-1}} \). Then on \( g \leq 1 \), we have \( \Lambda \) is negative in \((4.10)\) since \( g_{xx} \) is negative and \( E_{11}, E_{12} \geq \delta_0(g_{xx}, K_g) > 0 \) uniformly, which implies \( E_2 \) is positive. And we also have, in \((4.11)\),

\[
E_1 = -\gamma(1 + \gamma)(g X^2 + g^2 K^{\frac{\gamma}{4}} Z^6 C_8 + \gamma(Z^7 C_7 + l.o.t.)
\]

with \( C_8 = \left( (1 + \gamma)^2(g^2 + g)(2\gamma g + 3(5 + 4\gamma)g X^2) + \gamma(1 + 2\gamma) K^{\frac{\gamma}{4}} g \right)^2 - (1 + 2\gamma)(7 + 2\gamma) K^{\frac{\gamma}{4}} g X^2 - 3(1 + \gamma)(5 + 4\gamma) K^{\frac{\gamma}{4}} g X^2 \right) \). 

Now we can show \( C_8 \geq \delta_1(g_{xx}, K_g) > 0 \) uniformly and then \( E_1 < 0 \) for sufficiently large \( Z \). Therefore \( B \) is negative. Hence we can obtain desired result.

We now return to the case of the \( \alpha \)-Gauss Curvature Flow. Let us set

\[
I = 1 + g^2 |Dg|^2, \quad J = g + |Dg|^2 \quad \text{and} \quad Q = \left( g \det D^2 g + \theta(g^2 g_{xx} - 2 g_x g_y g_{xy} + g_x^2 g_{yy}) \right)^\alpha.
\]

Also let \( C = C(||g||_{C^1}, ||f||_{C^1}) \) denote various constants and \( \tilde{L}X \) denote the operator

\[
\tilde{L}X := X_t - \alpha K^{\alpha-1} I^{-\frac{4\alpha-1}{2}} \left( (g_{g_{yy}} + \theta g_{yy}^2) X_{xx} - 2(g_{g_{xy}} + \theta g_{xy}) X_{xy} + (g_{g_{xx}} + \theta g_{xx}^2) X_{yy} \right).
\]

We find, after several calculations, that at the maximum point \( P_0 \), where \((4.5)\) and \((4.6)\) hold, \( X \) satisfies the inequality

\[
\tilde{L}X = I^{-\frac{4\alpha-1}{2}} LX - \frac{4\alpha - 1}{2} (g + g^2) I^{-\frac{4\alpha-1}{2}} Q_{I_y y} - (4\alpha - 1) g X_t \theta + g_{yy} I^{-\frac{4\alpha-1}{2}} Q_{I_x}
+ g \left\{ - (4\alpha - 1) I^{-\frac{4\alpha-1}{2}} Q_{I_x} - \frac{4\alpha - 1}{2} I^{-\frac{4\alpha-1}{2}} Q_{I_{x} x} + \frac{16\alpha^2 - 1}{4} I^{-\frac{4\alpha+3}{2}} Q_{I_{x}}^2 \right\} + I^{-\frac{4\alpha-1}{2}} Q_{g_{xx}}.
\]
Lemma 4.3 at $P$ becomes minimum at the interior point $P$ with some positive constant $b$ and from (4.9) and tedious computation we obtain that

$$
\bar{L}X \leq I^{-\frac{4\alpha-1}{2}}A + \frac{I^{-\frac{4\alpha-1}{2}}}{(1 + 2\gamma)^2(g + g^2_\gamma)K^2_S} \left\{ - B_1 g^2(g^2_\gamma + g)g^2_{\bar{x}xy} - g^3 B_1(g_{xxx} + B_{11})^2 
+ \left( (1 + 2\gamma)^2(g + g^2_\gamma)K^2_S \frac{E_1}{E_2} \right) - 8\gamma(\gamma + 1) I^{-\frac{4\alpha-1}{2}} K^{\alpha-1}_S \frac{g^2}{g + g^2_\gamma} ((\gamma + 1)g^2_\gamma + gg_{xx})^2 - (g + g^2_\gamma)K^\gamma_S \right\} g_{xxx} 
+ g^{2\gamma+1} \cdot I.\!.t. + I^{-\frac{4\alpha-1}{2}} K^\alpha_S g_{xx}
\leq I^{-\frac{4\alpha-1}{2}}A + \frac{I^{-\frac{4\alpha-1}{2}}}{(1 + 2\gamma)^2(g + g^2_\gamma)K^2_S} \left\{ - B_1 g^2(g^2_\gamma + g)g^2_{\bar{x}xy} - g^3 B_1(g_{xxx} + B_{11} + O(g))^2 
+ \left( (1 + 2\gamma)^2(g + g^2_\gamma)K^2_S \frac{E_1}{E_2} + O(g) \right) \right\} + O(g) + I^{-\frac{4\alpha-1}{2}} K^\alpha_S g_{xx}.
$$

Here $O(g)$ denotes various terms satisfying $|O(g)| \leq Cg$ with constant $C$. We can know the first term and the second term are negative as in the case of $X$ provided that $X \geq C$ is sufficiently large. And then $\bar{L}X \leq C \bar{X}$ depending on $g \leq 1$, which implies that $X - Cl_1 \leq 0$. Applying the evolution of $\bar{X} = X - Ct$ with a simple trick implies $\bar{X} \leq C$ where $C$ is positive constant. This immediately gives the desired contradiction. $\square$

4.3. Aronson-Bénilan type Estimate.

**Lemma 4.4.** Under the assumptions of Theorem 1.7 and condition (1.8), there exists a constant $C > 0$ for which

$$
\det(D^2 g) \geq -C
$$

for a uniform constant $C > 0$.

**Proof.** To establish the bound of $\det(D^2 g)$ from below, we will use the maximum principle on the quantity

$$
Z = \frac{\det D^2 g}{g_x^2 g_{yy} + g_y^2 g_{xx} - 2g_x g_y g_{xy}} + b |\nabla g|^2
$$

with some positive constant $b$ on $\{ g(\cdot, t) > 0, 0 \leq t \leq T \}$. Let us assume that $Z$ becomes minimum at the interior point $P_0$. We can assume $g_y = 0$, $g_x > 0$ and $g_{xy} = 0$ at $P_0$ by using similar transformation and the change of coordinate in Lemma 4.3 at $P_0$. Then we have

$$
a_{ij}Z_{ij} \leq 0
$$
where $Z$ with a positive constant $a$ implies $su$ and $E$ on sufficiently large $b \geq t$.

(4.14)  
\[ Z_t \geq B_1(g_{xy} + B_2)^2 + A_0 + \frac{1}{(\alpha + 1)(2\alpha - 1)} O(g)Z + O(Z^2) \]

where

\[ B_1 = \frac{\alpha}{(1 - 2\alpha)^2 g_{xxx} g_{yy}} \left( 1 + g^{\frac{2\alpha}{3\alpha - 1}} g_x^2 \right)^{\frac{1}{2} - 2\alpha} K_g^{-2} \left[ \alpha g_x^2 + (2\alpha - 1) g g_{xx} \right] \]

\[ \cdot \left[ (4\alpha - 2)K_g + (\alpha - 1) \left( \alpha g_x^2 + (2\alpha - 1) g g_{xx} \right) g_{yy} \right] \]

and

\[ A_0 = A_{0,0} g^2 b^4 + E_0 + A_{0,5}(\alpha - 1) g^3 b^5 + (\alpha - 1) E_1 \]

with

\[ A_{0,0} = \frac{g_{xy}^{10} g_{yy} (24 + 12g^{2} g_{x}^{2} - 21g^{4} g_{y}^{4})}{2(1 + g^{2} g_{x}^{2})^{3/2}} \]

and

\[ A_{0,5} = -30g^{1 + \frac{4}{3\alpha - 1}}(1 - 2\alpha)g_{y}^2 (g^{\frac{2\alpha}{3\alpha - 1}} g_x^2 + g_x^2) \left( 1 + g^{\frac{2\alpha}{3\alpha - 1}} g_x^2 \right)^{\frac{1}{2} - 2\alpha} K_g^{-2} \frac{g_{xy}^{2} g_{yy}^2}{(4\alpha - 2)K_g + (\alpha - 1) \alpha g_x^2 g_{yy}} \]

where $E_0 = O(b^3, g^2)$ and $E_1 = O(b^4, g^3)$. Here we can also show $A_{0,0} \geq \delta_1 (g_x, g_{yy}) > 0$ uniformly and we have

\[ (4\alpha - 2)K_g + (\alpha - 1) \alpha g_x^2 g_{yy} = (4\alpha - 2) (g g_{xx} g_{yy} + \theta g_x^2 g_{yy}) + (\alpha - 1) \alpha g_x^2 g_{yy} \]

\[ = (4\alpha - 2) g g_{xx} g_{yy} + (\theta(4\alpha - 2) + \alpha^2 - \alpha) g_x^2 g_{yy} > 0 \]

on $g \leq 1$ since $\frac{1}{2} < \alpha \leq 1$. Then $(\alpha - 1) A_{0,5}$ is nonnegative so that $A_0$ is positive for sufficiently large $b \gg 1$. Also we can know $B_1$ is positive on $g \leq 1$ from (4.15). This implies

\[ Z_t > 0 > -aZ \]

with a positive constant $a$. By Gronwall’s inequality, we have

\[ Z \geq Z_0 e^{-\bar{a}t} \]

where $Z_0$ is initial data of $Z$ at $t = 0$ and $\bar{a}$ is constant, which concludes the proof. □
4.4. Global Optimal Regularity. Let's consider the quantity

\[
Z = \max_{\gamma} (gD_{\gamma\gamma}g + \theta|D_{\gamma}g|^2).
\]

Now we will show that \(Z\) is bounded from above through next lemma.

**Lemma 4.5.** With the same assumptions of Theorem 1.7 and condition (1.8), there exists a positive constant \(C = C(\theta, \rho, \lambda, \|g\|_{C^2(\partial\Omega)})\) with

\[
\max_{\Omega(g)} Z \leq C
\]

where \(\Omega(g) = \{x|g(x) > 0\}\).

**Proof.** First, we know that \(Z\) is nonnegative from \(Z = \beta f^2 - f \phi \phi\) and a convexity of \(f\). Also Lemma 3.2 implies \(Z \leq C(\theta, \rho, \lambda, \|g\|_{C^2(\partial\Omega)})\) at \(g = 0\), since \(Z = \theta|D_{\gamma\gamma}g|^2\) at the free-boundary \(g = 0\). Then we can assume that \(Z\) has its maximum at an interior point \(P_0 \in \Omega(g)\) and at a direction \(\gamma\). To show the bound of \(Z\), we consider \(\gamma = \lambda_1 \nu + \lambda_2 \tau\) with \(\lambda_1^2 + \lambda_2^2 = 1\), where \(\nu, \tau\) denote the outward normal and tangential directions to the level sets of \(g\) respectively. Then \(Z(P_0) = gD_{\gamma\gamma}g + \theta|D_{\gamma}g|^2\) and (1.4) can be rewritten as

\[
Z(P_0) = g[\lambda_1^2 g_{\nu\nu} + 2\lambda_1 \lambda_2 g_{\nu\tau} + \lambda_2^2 g_{\tau\tau}] + \lambda_1^2 g_{\nu}\nu
\]

(4.17)

and \((g g_{\nu\nu} + \theta g_{\nu}) g_{\tau\tau} = g_{\nu\nu}^2 + (g_t(1 + g_{\nu\nu}^2 - 2g_{\nu\nu} g_{\nu\tau})^\frac{d-1}{2})\). Here if \(g g_{\nu\nu}\) is not sufficiently large at \(P_0\), we have

\[
Z(P_0) \leq C(\theta, \rho, \lambda, \|g\|_{C^2(\partial\Omega)})
\]

from Lemmas 3.2, 4.2 and 4.3 implying the desired result immediately. On the other hand, if \(\theta g_{\nu}\nu \leq g g_{\nu\nu}\) at \(P_0\), then we get

\[
g_{\nu\nu}^2 \leq 2g g_{\nu\nu} g_{\tau\tau} \leq C(\theta, \rho, \lambda, \|g\|_{C^2(\partial\Omega)}) g g_{\nu\nu}
\]

implying \(g_{\nu\nu} \leq C(\theta, \rho, \lambda, \|g\|_{C^2(\partial\Omega)}) \sqrt{g g_{\nu\nu}}\). Then we can know that \(Z(P_0)\) is maximum when \(\lambda_2 = 0\) from Lemma 4.3 and (4.17) so that \(Z(P_0) = g g_{\nu\nu} + \theta g_{\nu}\nu\). Also we get \(g g_{\nu\nu} + \theta g_{\nu} g_{\tau} = 0\) at \(P_0\) implying that \(g_{\nu\nu} = 0\) at \(P_0\). Here by the similar transformation in Lemma 4.3, we can assume

\[
g_{\nu} = g y = 0, \quad g_{\nu} = g_{\tau} > 0 \quad \text{and} \quad g_{xy} = 0 \quad \text{at} \quad P_0.
\]

Then we have

\[
a_{ij} Z_{ij} \leq 0
\]
with (4.12) at the maximum point $P_0$. And since $g_{xxx} = \frac{-g_x g_{xx} - 2g_y g_{xy}}{g}$ and $g_{xyy} = 0$ at $P_0$, we obtain

$$
Z_t = g_t g_{xx} + g g_{xxt} + 2\theta g_x g_{xt}
$$

$$
\leq -(1 + g^{\frac{2\beta + 1}{\beta - 1}} g_y^2)^\frac{3\alpha + 4\beta}{3\alpha + 4\beta - 1} \left[ (1 - 2\alpha)^2 (\alpha - 1)(4\alpha - 1)(2\alpha(2\alpha - 3\theta + \alpha + 1) + 2\theta - 1)g_x^6 \\
+ (1 - 2\alpha)^2 (\alpha - 1)g_{xx}(1 \alpha - 1)(4\alpha - 1)g_x^2 g_{xx} \\
+ (2\alpha - 1)g_x^2 g_{xx} [(\alpha - 1)(16\alpha^2 - 5\alpha + 1) + 3\alpha(8\alpha^2 - 6\alpha + 1)g^{\frac{4\alpha - 1}{2\alpha - 1}} g_{xx}] \\
+ \alpha g_4 [(4\alpha^2 - 5\alpha + 1)(\theta(4\alpha - 2) + 1) + 6\alpha(10\alpha^2 - 9\alpha + 2)g^{\frac{4\alpha - 1}{2\alpha - 1}} g_{xx}]]
$$
at the point $P_0$. Also from $g_{xy} = \frac{Z - \theta g_y^2}{g}$, we have

$$
Z_t \leq \left(1 + g^{\frac{2\alpha - 1}{\alpha - 1}} g_y^2\right)^{\frac{3\alpha + 4\beta}{3\alpha + 4\beta - 1}} \left[ Z(2\alpha^2 - 3\alpha + 1)\left((4\alpha - 1)Z g^{\frac{2\alpha - 1}{\alpha - 1}} + \alpha - 1\right) \\
+ g^{\frac{4\alpha - 1}{2\alpha - 1}} g_y^2 (\alpha - 1)2\alpha - 3\alpha (8\alpha^2 - 6\alpha + 1)Z^2 - (\alpha - 1)(8\alpha^2 - 3\alpha + 1)Z g^{\frac{2\alpha - 1}{\alpha - 1}} \\
+ \alpha (\alpha - 1)g_x^2 (2\alpha - 1)g^{\frac{2\alpha - 1}{\alpha - 1}} - 6\alpha Z + (\alpha - 1)g_x^2 \right]
$$

$$
\leq (1 - 4\alpha)(1 + g^{\frac{2\alpha - 1}{\alpha - 1}} g_y^2)^{\frac{3\alpha + 4\beta}{3\alpha + 4\beta - 1}} \left[ g_y^2 Z^{2\alpha + \beta} g^{\frac{2\alpha - 1}{\alpha - 1}} + O(Z^{2\alpha + \beta})g^{\frac{2\alpha - 1}{\alpha - 1}} + O(Z^{1 + \alpha})g^{\frac{1}{\alpha - 1}} \right].
$$

Then on $g \leq 1$, $Z_t \leq 0$ at $P_0$ since $1 - 4\alpha < 0$. Hence we can obtain the desired result. □

### 4.5. Decay rates of Second Derivatives.

**Corollary 4.6.** With the same assumptions of Lemma 4.3, there exist a positive constant $c = c(\rho_0, f_0)$ such that

1. $c \leq g_{\tau\tau} \leq c^{-1}$
2. $c \leq f_{\nu\nu} \frac{f_{\tau\tau}}{g^{\frac{1}{\beta - 1}}} \leq c^{-1}$ and $\frac{|f_{\nu\tau}|}{g^{\frac{1}{\beta - 1}}} \leq c^{-1}$

with $\tau$ denoting the tangential direction to the level sets of $g$.

**Proof.** (i.) The upper bound of $g_{\tau\tau}$ comes from Lemma 4.3. Now we are going to show the lower bound. From Lemma 4.1, we have

$$
\det D^2 f \geq c g^{\frac{1}{\beta - 1}} = c g^{2\beta - 3}.
$$

which implies

$$
f_{\nu\nu} f_{\tau\tau} \geq c g^{2\beta - 3} + f_{\nu\tau}^2 \geq c g^{2\beta - 3}
$$

(4.18)
and then
\[ f_{tt} \geq \frac{c g^{2\beta-3}}{f_{vv}} \geq c g^{\beta-1} \]
since \( f_{vv} \leq C g^{\beta-2} \) from Lemma 4.5. Since \( f_{tt} = g^{\beta-1} g_{tt} + (\beta-1)g^{\beta-2} g^2_t \), we conclude that
\[ g_{tt} = \frac{f_{tt}}{g^{\beta-1}} \geq c, \]
for some positive constant \( c \) depending only on the initial data and \( \rho_0 \).

(ii.) \( f_{tt} = g^{\beta-1} g_{tt} \) and the bound on \( g_{tt} \) tell us
\[ c \leq \frac{f_{tt}}{g^{\beta-1}} \leq c^{-1}. \]

(iii.) Third, we are going to show
\[ f_{vv} + f_{tt} \geq c. \]

Let us denote by \( \lambda_1, \lambda_2 \) the two eigenvalues of the matrix \( \det D^2 f \) such that \( \lambda_1 \geq \lambda_2 \).

Then, from Lemma 4.1 we have
\[ c \leq \frac{\lambda_1 \lambda_2}{g^{2\beta-3}} \leq c^{-1} \]
and \( \lambda_2 \leq f_{tt} \leq c^{-1} g^{\beta-1} \), implying that \( c^{-1} g^{\beta-2} \geq \lambda_1 \geq C g^{\beta-2} \) for some positive constant \( c \) since \( 2 < \beta \). Hence, \( f_{vv} + f_{tt} \geq \lambda_1 + \lambda_2 \geq c > 0 \) as desired.

(iv.) From (i) and Lemma 4.5 we have
\[ c \leq f_{vv} + g^{\beta-1} g_{tt} \leq 2f_{vv} \leq \sup |D^2 f| < C \]
for a uniformly small \( g \), which is true in a uniformly small neighborhood of free boundary. The convexity of \( f \) says \( 0 \leq f_{tt}^2 - f_{vv} f_{tt} \). By using the bound \( f_{vv} \leq c^{-1} \), we obtain
\[ f_{tt}^2 \leq f_{vv} f_{tt} \leq c^{-1} g^{\beta-1} \]

\( \square \)

5. Higher Regularity

5.1. Local Change of Coordinates. For any point \( P_0 = P_0(x_0, y_0, t_0) \) at the interface \( \Gamma \) with \( 0 < t_0 \leq T \), let’s assume that \( n_0 \) is the unit vector in the direction of the vector \( P_0 - OP_0 \) and \( n_0 \) satisfies
\[ n_0 := \frac{P_0}{|P_0|} = e_1 \]
by rotating the coordinates. Then we will have the following Lemma as Lemma 4.6, [DL3].
Lemma 5.1. There exist positive constants \( c \) and \( \eta \), depending only on the initial data and the constant \( \rho_0 \) in (1.8), for which

\[
c \leq g_x(P) \leq c^{-1} \quad \text{and} \quad c \leq f_{xx}(P) \leq c^{-1}
\]

at all points \( P = (x, y, t) \) with \( f(P) > 0, |P - P_0| \leq \eta \) and \( t \leq t_0 \) under (5.1).

5.2. Class of Linearized Equation. In this subsection, we are going to show our transformed function \( h \) from \( g \) near the free boundary satisfies the same class of operators considered at [DL3] so that all the results in [DL3] can be applied to our equation by using the similar methods.

We will assume, throughout this section, that at time \( t = 0 \) the function \( g = (\beta f)^{1/\gamma} \) satisfies the hypotheses of Theorem 1.7 and that \( g \) is smooth up to the interface on \( 0 \leq t \leq T \), where \( T > 0 \) is such that condition (1.8) holds.

We will state the results of uniform \( C_1^{1,\gamma} \)-estimate in [DL3]. The readers can see detailed proofs with reference to [DL3].

Let \( P_0 = (x_0, y_0, t_0) \) be a point on the interface curve \( \Gamma(t_0) \) at time \( t = t_0, 0 < t_0 \leq T \). We may assume, without loss of generality, that \( \tau \leq t_0 \leq T \), for some \( \tau > 0 \). Indeed, the short time regularity result in Theorem 1.6 shows that solutions are smooth up to the interface on \( 0 \leq t \leq T \), where \( T > 0 \) is such that condition (5.1) holds. By Lemma 5.1, \( g_x(P) > 0 \) for all points \( P = (x, y, t) \) with \( t \leq t_0 \), sufficiently close to \( P_0 \) and then from (1.12) (see in [DH], Section II),

\[
(5.2) \quad h_t = -\left\{ \frac{z(h_{zz} h_{yy} - h_{zy}^2) - \theta(\alpha) h_z h_{zy}}{z^{2(\beta-1)} + h_z^2 + z z^{2(\beta-1)} h_y^2} \right\}^{1/\alpha}, \quad z > 0.
\]

Set \( K_h = z(h_{zz} h_{yy} - h_{zy}^2) - \theta(\alpha) h_z h_{zy} \) and \( J = z z^{(\beta-1)} + h_z^2 + z^{(\beta-1)} h_y^2 \). If we linearize this equation around \( h \), we obtain the equation

\[
(5.3) \quad \tilde{h}_t = \frac{\alpha K_h^{\alpha-1}}{J^{\alpha-1}} \left\{ -z h_{yy} \tilde{h}_{zz} + 2 z h_{zy} \tilde{h}_{zy} + (\theta(\alpha) h_z - z h_{zz}) \tilde{h}_{yy} \right\} + \frac{(4\alpha - 1) z^{2(\beta-1)} K_h^{\alpha} h_y}{J^{\alpha-1}} \tilde{h}_y
\]

\[
+ \frac{(4\alpha - 1) K_h^{\alpha} h_z + \alpha K_h^{\alpha-1} \theta(\alpha)(h_z^2 + z^{(\beta-1)}(1 + h_y^2)) h_{yy}}{J^{\alpha-1}} \tilde{h}_z.
\]

Let us denote by \( \mathcal{B}_\eta \) the box

\[
\mathcal{B}_\eta = \left\{ 0 \leq z \leq \eta^2, |y - y_0| \leq \eta, t_0 - \eta^2 \leq t \leq t_0 \right\}
\]
around the point \(Q_0 = (0, y_0, t_0)\). We can obtain a-priori bounds on the matrix

\[
A = (a_{ij}) = \begin{pmatrix}
-h_{yy} & \sqrt{z} h_{zy} \\
\sqrt{z} h_{zy} & (\theta(\alpha) h_z - z h_{zz})
\end{pmatrix}
\]

and the coefficient

\[
b = \frac{(4\alpha - 1) K_h^\alpha h_z + \alpha K_h^{\alpha - 1} \theta(\alpha)(h_z^2 + z^{2(\beta-1)}(1 + h_y^2))h_{yy}}{J^{\frac{\alpha+1}{2}}}.
\]

Then we obtain the following.

**Lemma 5.2.** There exist positive constants \(\eta, \lambda\) and \(\nu\), depending only on the initial data and the constant \(\rho_0\) in (1.8) such that

\[
\lambda |\xi|^2 \leq a_{ij} \xi_i \xi_j \leq \lambda^{-1} |\xi|^2, \quad \forall \xi \neq 0
\]

and

\[
|b| \leq \lambda^{-1} \quad \text{and} \quad b \geq \nu > 0 \quad \text{on the box } B_{\eta}.
\]

Notice that \(b \geq \nu > 0\) comes from the decay rates of second derivatives, Corollary 4.6 and Aronson-Bénilan type Estimate, Lemma 4.4. Similarly, we can get the bound of \(\tilde{A} := (\tilde{a}_{ij})\) and \(\tilde{b}_i, i = 1, 2\) to be the coefficients

\[
\tilde{a}_{ij} = \frac{a_{ij} \alpha K_h^{\alpha - 1}}{(z^{2(\beta-1)} + h_z^2 + z^{2(\beta-1)} h_y^2)^{\frac{\alpha-1}{2}}}
\]

and

\[
\tilde{b}_1 = b - \frac{\alpha \theta(\alpha) K_h^{\alpha - 1} h_{yy}}{J^{\frac{\alpha+1}{2}}} \quad \text{and} \quad \tilde{b}_2 = \frac{(4\alpha - 1) z^{2(\beta-1)} K_h^\alpha h_y}{J^{\frac{\alpha+1}{2}}}
\]

of equation (5.3).

**Lemma 5.3.** There exist constants \(\eta > 0, \lambda > 0\) and \(\nu > 0\), depending only on the initial data and the constant \(\rho_0\) in (1.8), for which

\[
\lambda |\xi|^2 \leq \tilde{a}_{ij} \xi_i \xi_j \leq \lambda^{-1} |\xi|^2, \quad \forall \xi \neq 0
\]

and

\[
|\tilde{b}_i| \leq \lambda^{-1} \quad \text{and} \quad \tilde{b}_1 \geq \nu > 0 \quad \text{on the box } B_{\eta}.
\]
5.3. Regularity theory. Recall the Definition 1.2. Then Lemma 5.3 tells us the linearized equation (5.3) is in the same class of operators considered at Lemma 5.2 of [DL3] and [DL2].

We are now in position to show the uniform Hölder bounds of the first order derivatives $h_t$, $h_y$ and $h_z$ of $h$ on $B_\eta$. In [DL3], the authors have obtained the $C^{2,\gamma}_s$ regularity of $h$ on the box.

**Lemma 5.4.** There exist numbers $\gamma$ and $\mu$ in $0 < \gamma, \mu < 1$, and positive constants $\eta$ and $C$, depending only on the initial data and $\rho_0$, such that

$$
\|h_y\|_{C^{2,\gamma}_s(B_{\frac{\eta}{2}})} \leq C, \quad \|h_t\|_{C^{2,\gamma}_s(B_{\frac{\eta}{2}})} \leq C, \quad \text{and} \quad \|h_z\|_{C^{\mu,\gamma}_s(B_{\frac{\eta}{2}})} \leq C.
$$

Following the proof Theorem 6.2 of [DL3], we will have the following theorem.

**Theorem 5.5.** With the same assumptions of Theorem 1.6 and condition (1.8) which satisfies at $T < T_c$, there exist constants $0 < \alpha_0 < 1$, $C < \infty$ and $\eta > 0$, depending only on the initial data and $\rho_0$, for which $x = h(x,y,t)$ fulfills

$$
\|h\|_{C^{2,\gamma}_s(B_{\frac{\eta}{2}})} \leq C
$$
on $B_\eta = \{0 \leq z \leq \eta^2, |y - y_0| \leq \eta, t_0 - \eta^2 \leq t \leq t_0\}$ for $P_0 = (x_0, y_0, t_0)$ with $0 < \tau < t_0 < T$, which is any free-boundary point holding condition (5.1).

**Proof of Theorem 1.7** By the short time existence Theorem 1.6, there exists a maximal time $T > 0$ for which $g$ is smooth up to the interface on $0 < t < T$. Assuming that $T < T_0$, we will show that at time $t = T$, the function $g(\cdot, T)$ is of class $C^{2,\gamma}_s$, up to the interface $z = 0$, for some $\gamma > 0$, and satisfies the non-degeneracy conditions (1.6). Therefore, by Theorem [DH], there exists a number $T' > 0$ for which $g$ is of class $C^{2,\gamma}_s$, for all $\tau < T + T'$, and hence $C^{\infty}$ up to the interface, according to Theorem 9.1 in [DH]. This will contradict the fact that $T$ is maximal, proving the Theorem. From Lemma 5.2 and Corollary 4.6 the functions $g(\cdot, t)$ satisfy conditions (1.6), for all $0 \leq t < T$, with constant $c$ independent of $t$. Hence, it will be enough to establish the uniform $C^{2,\gamma}_s$ regularity of $g$, on $0 \leq t \leq T$, up to the interface, whose proof follows the same line of argument at [DL3]. □

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Seoul National University, Seoul 151-747, Korea
E-mail address: lnkim@snu.ac.kr

Seoul National University, Seoul 151-747, Korea
E-mail address: kiahm@snu.ac.kr

Seoul National University, Seoul 151-747, Korea
E-mail address: erhee@math.snu.ac.kr