SIMPLICIAL GEOMETRY OF UNITAL LATTICE-ORDERED
ABELIAN GROUPS

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Abstract. By an \( \ell \)-group \( G \) we mean a lattice-ordered abelian group. This paper is concerned with the category \( \text{U}_{fp} \) of finitely presented unital \( \ell \)-groups, those \( \ell \)-groups having a distinguished order-unit \( u \). Using the duality between \( \text{U}_{fp} \) the category of rational polyhedra, we will provide (i) a construction of finite limits and co-limits in \( \text{U}_{fp} \); (ii) a Cantor-Bernstein-Schröder theorem for finitely presented unital \( \ell \)-groups; (iii) a geometrical characterization of finitely generated subalgebras of free objects of \( \text{U}_{fp} \).

MSC2010 Primary: 06F20, 52B20, 18B30. Secondary: 05E45, 52B11, 18A35, 55U05, 55U10 57Q05.

1. Introduction

A unital \( \ell \)-group \((G, u)\) is an abelian group \( G \) equipped with a translation invariant lattice-order and with a distinguished order unit, i.e. an element \( 0 \leq u \in G \) whose positive integer multiples eventually dominate every element of \( G \). A unital \( \ell \)-homomorphism between unital \( \ell \)-groups is a group homomorphism that preserves the order unit and the lattice structure.

As a particular case of a general definition [4, p. 286], a unital \( \ell \)-group \((G, u)\) is said to be finitely presented if there exists a finite set \( \{g_1, \ldots, g_n\} \subseteq G \) along with a finite set of equations \( s_1 = t_1, \ldots, s_m = t_m \) in the language of unital \( \ell \)-groups with \( n \)-variables such that

(i) \( s_i(g_1, \ldots, g_n) = t_i(g_1, \ldots, g_n) \) for each \( i = 1, \ldots, m \) and

(ii) if \((H, v)\) is a unital \( \ell \)-group and \( h_1, \ldots, h_n \in H \) satisfy \( s_i(h_1, \ldots, h_n) = t_i(h_1, \ldots, h_n) \) for each \( i = 1, \ldots, m \), then there exists a unique unital \( \ell \)-homomorphism \( h: (G, u) \to (H, v) \) such that \( h(g_1) = h_1, \ldots, h(g_n) = h_n \).

We denote by \( \text{U}_{fp} \) the category of finitely presented unital \( \ell \)-groups with unital \( \ell \)-homomorphisms.

For \( n = 1, 2, \ldots \) we let \( \mathcal{M}([0, 1]^n) \) denote the unital \( \ell \)-group of all continuous functions \( f: [0, 1]^n \to \mathbb{R} \) having the following property: there are linear polynomials \( p_1, \ldots, p_m \) with integer coefficients such that for all \( x \in [0, 1]^n \) there is \( i \in \{1, \ldots, m\} \) with \( f(x) = p_i(x) \). \( \mathcal{M}([0, 1]^n) \) is equipped with the pointwise operations +, −, max, min of \( \mathbb{R} \), and with the constant function 1 as the distinguished order unit.

\( \mathcal{M}([0, 1]^n) \) is a free object in the category of unital \( \ell \)-groups, in the following sense:

Date: February 28, 2012. The main results of this paper have been obtained while the author was holding a postdoctoral position in the Mathematisches Institute of the University of Bern.
Proposition 1.1. ([21, Corollary 4.16]) The coordinate maps $\xi_i: [0,1]^n \to \mathbb{R}$ together with the order unit 1 form a generating set of $M([0,1]^n)$. For every unital $\ell$-group $(G,u)$ and $0 \leq g_1, \ldots, g_n \leq u$, if the set $\{g_1, \ldots, g_n, u\}$ generates $G$ then there is a unique unital $\ell$-homomorphism $\psi$ of $M([0,1]^n)$ onto $G$ such that $\psi(\xi_i) = g_i$ for each $i = 1, \ldots, n$.

An ideal $i$ of a unital $\ell$-group $(G,u)$ is the kernel of a unital $\ell$-homomorphism of $(G,u)$, ([13, p.8 and 1.14]). $i$ is principal if it is singly (= finitely) generated.

As a consequence of Proposition 1.1, a unital $\ell$-group $(G,u)$ is finitely presented iff for some $n = 1, 2, \ldots$, $(G,u)$ is isomorphic to the quotient of $M([0,1]^n)$ by some principal ideal $j$, in symbols, $(G,u) \cong M([0,1]^n)/j$.

The characterization of finitely presented unital $\ell$-groups presented in Proposition 1.1 relies on the free objects $M([0,1]^n)$ and their universal property. In [6] an intrinsic characterization of finitely presented unital $\ell$-groups is given in terms of special sets of generators, called bases. The notion of basis was introduced in [19] as a purely algebraic counterpart of Schauder bases. In [19, Theorem 4.5] it is proved that if a unital $\ell$-group $(G,u)$ is isomorphic to an $\ell$-group of real-valued functions defined on some set $X$ (that is, $G$ is Archimedean) then it is finitely presented iff it has a basis. In [6, Theorem 3.1] it is proved that the Archimedean assumption can be dropped: thus, $(G,u)$ is finitely presented iff it has a basis.

In Section 2 we give a detailed account of the main tools used in this paper, namely the categorical duality between finitely presented unital $\ell$-groups and rational polyhedra, and the combinatorial representation of rational polyhedra as weighted abstract simplicial complexes.

Section 3 is devoted to the construction of limits and co-limits in these two categories.

Finally, in Section 4, all the machinery of the earlier chapters will be combined with the algebraic-topological analysis of projective unital $\ell$-groups in [7] to give geometric and algebraic characterizations of finitely generated subalgebras of the free unital $\ell$-groups $M([0,1]^n)$.

2. Preliminaries

2.1. Regular triangulations. We refer to [10], [12] and [28] for background in elementary polyhedral topology and simplicial complexes.

For any simplex $S$ we denote by $\text{vert}(S)$ the set of its vertices. For any $F \subseteq \text{vert}(S)$, the convex hull $\text{conv}(F)$ is called a face of $S$. A polyhedron $P$ in $\mathbb{R}^n$ is a finite union of (always closed) simplexes $P = S_1 \cup \cdots \cup S_l$ in $\mathbb{R}^n$.

A simplex $S$ is said to be rational if the coordinates of each $v \in \text{vert}(S)$ are rational numbers. $P$ is said to be a rational polyhedron if there are rational simplexes $T_1, \ldots, T_l$ such that $P = T_1 \cup \cdots \cup T_l$.

For every simplicial complex $\Delta$, its support $|\Delta|$ is the pointset union of all simplexes of $\Delta$, and $\text{vert}(\Delta)$ is the set of its vertices, i.e. the set of the vertices of its simplexes. We say that the simplicial complex $\Delta$ is rational if all simplexes of $\Delta$ are rational.

Given a rational polyhedron $P$, a triangulation of $P$ is a rational simplicial complex $\Delta$ such that $P = |\Delta|$. In [1, Theorem 1] it is proved that $\Delta$ exists for every rational polyhedron $P$. 
In the rest of this paper every simplex, polyhedron, and simplicial complex will be rational. Accordingly, the adjective “rational” will be omitted unless it is strictly necessary.

For $v$ a rational point in $\mathbb{R}^n$ we let $\text{den}(v)$ denote the least common denominator of the coordinates of $v$. The vector $\tilde{v} = \text{den}(v)(v, 1) \in \mathbb{Z}^{n+1}$ is called the homogenous correspondent of $v$. A simplex $S$ is called regular if the set of homogenous correspondents of its vertices is part of a basis of the free abelian group $\mathbb{Z}^{n+1}$. By a regular triangulation of a polyhedron $P$ we understand a triangulation of $P$ consisting of regular simplexes.

The following proposition was proved in [21, Theorem 1.2] under the assumption that $X \subseteq [0, 1]^n$. However, it is easy to see that the proof is the same for all $X \subseteq \mathbb{R}^n$ (also see [24, Lemma 3.1]):

**Proposition 2.1.** For any set $X \subseteq \mathbb{R}^n$ the following statements are equivalent:

(i) $X$ coincides with the support of some regular complex $\Delta$;

(ii) $X$ is a rational polyhedron.

2.2. **Farey subdivisions.** Given a polyhedron $P$ and triangulations $\Delta$ and $\Sigma$ of $P$ we say that $\Delta$ is a subdivision of $\Sigma$ if every simplex of $\Delta$ is contained in a simplex of $\Sigma$. For any point $p \in P$, the blow-up $\Delta(p)$ of $\Delta$ at $p$ is the subdivision of $\Delta$ given by replacing every simplex $S \in \Delta$ that contains $p$ by the set of all simplexes of the form $\text{conv}(F \cup \{p\})$, where $F$ is any face of $S$ that does not contain $p$ (see [30, p. 376] or [10, III, Definition 2.1], where blow-ups are called stellar subdivisions).

For any regular $m$-simplex $S = \text{conv}(v_0, \ldots, v_m) \subseteq \mathbb{R}^n$, the Farey mediant of $S$ is the rational point $v$ of $S$ whose homogeneous correspondent $\tilde{v}$ equals $\tilde{v}_0 + \cdots + \tilde{v}_m$. If $S$ belongs to a triangulation $\Delta$ and $v$ is the Farey mediant of $S$ then the blow-up $\Delta(v)$ is a regular triangulation iff so is $\Delta$ ([10, V, 6.2]). $\Delta(v)$ will be called the Farey blow-up of $\Delta$ at $v$. By a (Farey) blow-down we understand the inverse of a (Farey) blow-up.

The proof of the “weak Oda conjecture” by Morelli [20] and Włodarczyk [30] yields:

**Lemma 2.2.** Let $P$ be a polyhedron. Then any two regular triangulations of $P$ are connected by a finite path of Farey blow-ups and Farey blow-downs.

For later use in this paper, we recall here some properties of regular triangulations.

**Lemma 2.3.** Let $P \subseteq Q \subseteq \mathbb{R}^n$ be rational polyhedra and $\Delta$ be a regular triangulation of $P$. Then there exists a regular triangulation $\Delta_Q$ of $Q$ such that the set $\Delta_P = \{S \in \Delta_Q \mid S \subseteq P\}$ is a subdivision of $\Delta$. Moreover, $\Delta_Q$ can be so chosen that $\Delta_P$ is a full subcomplex of $\Delta_Q$, in the sense that $\Delta_P = \{S \in \Delta_Q \mid \text{vert}(S) \subseteq P\}$.

**Proof.** Let $\nabla_0$ be a rational triangulation of $Q$. From [25, Addendum 2.12] we obtain a triangulation $\nabla_1$ of $Q$ which is a subdivision of $\nabla_0$ and also satisfies $S = \bigcup\{T \in \nabla_1 \mid T \subseteq S\}$, for each $S \in \Delta$. By [25, Lemma 3.4], there is a subdivision $\nabla_2$ of $\nabla_1$ such that $\{T \in \nabla_2 \mid T \subseteq P\}$ is a full subcomplex of $\nabla_2$. By [1, Corollary, p. 242], there is no loss of generality to assume that $\nabla_2$ is rational.

The desingularization process described in [8, Chapter 9] then yields a regular triangulation $\nabla_3$ which is a subdivision of $\nabla_2$. Since $\nabla_3$ is obtained form $\nabla_2$ by blow-ups, then $\{T \in \nabla_3 \mid T \subseteq P\}$ is a regular triangulation of $P$ which is also a subdivision of $\Delta$ and a full subcomplex of $\nabla_3$. \qed
Lemma 2.4. Let $P \subseteq Q \subseteq \mathbb{R}^n$ be rational polyhedra, and $\Delta_P$ and $\Delta_Q$ be regular triangulations of $P$ and $Q$ such that $\Delta_P \subseteq \Delta_Q$. If $\nabla_P$ is a regular subdivision of $\Delta_P$ then there exists a regular triangulation $\nabla_Q$ of $Q$ such that $\nabla_P \subseteq \nabla_Q$ and $\nabla_Q$ is a subdivision of $\Delta_Q$.

Proof. Let $K = \{ S \in \Delta_Q \mid \text{vert}(S) \subseteq P \text{ and } S \not\subseteq P \}$ and $S$ be a maximal element in $K$. Then the Farey blow up $(\Delta_Q)_v = \nabla_1$, where $v$ is the Farey median of $S$, is a regular triangulation of $Q$ such that $\Delta_P \subseteq \nabla_1$. By the maximality of $S$ in $K$, $K_1 = \{ S \in \nabla_1 \mid \text{vert}(S) \subseteq P \text{ and } S \not\subseteq P \} = K \setminus \{ S \}$. Repeating this process, we obtain a sequence of regular complexes $\nabla_1, \ldots, \nabla_r$ and a sequence of sets $K = K_0 \supseteq K_1 \supseteq \ldots \supseteq K_r$ where each $\nabla_{k+1}$ is obtained by blowing-up $\nabla_k$ at the Farey median of some maximal $S$ in $K_k$ and $K_{k+1} = K_k \setminus S$. Since $K$ is finite this process terminates at some $t$. By construction, $\Delta_P$ is a full subcomplex of $\nabla_t$ and $\nabla_t$ is a subdivision of $\Delta_Q$.

For each $S \in \nabla_t$ we define:

$$\text{vert}_P(S) = \text{vert}(S) \cap P, \quad \text{vert}_Q(S) = \text{vert}(S) \setminus \text{vert}_P(S), \quad \text{and}$$

$$U_S = \{ \text{conv}(\text{vert}_Q(S) \cup T) \mid T \in \nabla_P \text{ and } T \subseteq S \cap P \}.$$

Since $\Delta_P$ is a full subcomplex of $\nabla_t$ and $\nabla_Q$ is a subdivision of $\Delta_Q$, it follows that $\nabla_0 = \bigcup \{ U_S \mid S \in \nabla_1 \}$ is a simplicial complex and a triangulation of $Q$.

An application to $\nabla_0$ of the desingularization procedure of [22, 1.2] yields a regular triangulation $\nabla_Q$. $\nabla_Q$ is obtained from $\nabla_0$ by a suitable sequence of blow ups at non-regular simplexes. Since $\nabla_P \subseteq \nabla_0$ is regular, none of its simplexes is modified by the application of this procedure. Therefore, $\nabla_P \subseteq \nabla_Q$ and $\nabla_Q$ is a subdivision of $\Delta_Q$, as desired. \[\square\]

2.3. The category of rational polyhedra.

Definition 2.5. ([23, Definition 3.1]) Given a rational polyhedra $P \subseteq \mathbb{R}^n$ a map $\eta: P \to \mathbb{R}^m$ is called a $\mathbb{Z}$-map if there is a triangulation $\Delta$ of $P$ such that over every simplex $T$ of $\Delta$, $\eta$ coincides with an affine linear map $\eta_T$ with integer coefficients.

The following lemmas are easy consequences of the definition. For detailed proofs in the case of rational polyhedra contained in some $n$-cube $[0,1]^n$ see [23, §3].

Lemma 2.6. Suppose $P \subseteq \mathbb{R}^n$ and $\eta: P \to \mathbb{R}^m$ is a $\mathbb{Z}$-map. Then $\eta(P)$ is a polyhedron in $\mathbb{R}^m$.

Lemma 2.7. Let $\eta: P \to Q$ be a $\mathbb{Z}$-map and $\Delta$ a regular triangulation of $P$. Then there exists a regular triangulation $\nabla$ of $Q$ such that $\nabla$ is a subdivision of $\Delta$ and $\eta$ is linear over each simplex in $\nabla$.

Lemma 2.8. Given polyhedra $P \subseteq \mathbb{R}^n$, $Q \subseteq \mathbb{R}^m$ and a map $\eta: P \to Q$, let $\xi_1, \ldots, \xi_m: Q \to \mathbb{R}$ denote the coordinate maps. Then $\eta$ is a $\mathbb{Z}$-map iff $\xi_i \circ \eta: P \to \xi_i(P)$ is a $\mathbb{Z}$-map for each $i = 1, \ldots, m$.

Lemma 2.9. Given a rational polyhedron $P \subseteq \mathbb{R}^n$. A subset $P \subseteq Q$ is a rational polyhedron iff there exists a $\mathbb{Z}$-map $f: Q \to \mathbb{R}$ such that $P = f^{-1}(0)$.

Proof. In [19, Propositions 5.1 and 5.2] and [24, Lemma 3.2], the result was proven for the case $P \subseteq [0,1]^n$. The same argument replacing $[0,1]^n$ by an arbitrary rational polyhedron $Q$ proves the result of this lemma. \[\square\]
We denote by \( P_z \) the category whose objects are rational polyhedra in \( \mathbb{R}^n \) (for \( n = 1, 2, \ldots \)), and whose arrows are \( z \)-maps.

Following [24, Definition 4.4] and [23, Definition 3.9], a map \( \eta: P \to Q \) is a \( z \)-homeomorphism if it is a one-one \( z \)-map of \( P \) onto \( Q \) and its inverse \( \eta^{-1} \) is also a \( z \)-map. Thus \( z \)-homeomorphisms are the same as iso-arrows of the category \( P_z \).

A proof of the following result can be obtained from [23, §3]:

**Theorem 2.10 (Duality).** Let the functor \( M: P_z \to U_{fp} \) be defined by

**Objects:** For \( P \in P_z \) a polyhedron,
\[
M(P) = \text{the set of all } z \text{-maps from } P \text{ into } \mathbb{R}.
\]

**Arrows:** For \( \eta: P \to Q \) a \( z \)-map,
\[
M(\eta)(f) = f \circ \eta, \text{ for each } f \in M(Q).
\]

Then \( M \) yields a duality between the categories \( P_z \) and \( U_{fp} \). Stated otherwise, \( M \) is a categorical equivalence between \( P_z \) and the opposite category of \( U_{fp} \).

**Lemma 2.11.** Suppose \( P \subseteq \mathbb{R}^n \) and \( Q \subseteq \mathbb{R}^m \) are polyhedra and \( \eta: P \to Q \) is a \( z \)-map. We then have
\[
M(Q)/\ker(M(\eta)) \cong M(\eta(P)).
\]

**Proof.** Observe that \( f \in \ker(M(\eta)) \) iff \( f \circ \eta(P) = f(\eta(P)) = \{0\} \). Therefore the kernel of the onto map \( h: M(Q) \to M(\eta(P)) \) given by \( f \mapsto f \upharpoonright \eta(P) \), coincides with \( \ker(M(\eta)) \). In conclusion, \( M(Q)/\ker(M(\eta)) \cong M(\eta(P)) \).

2.4. **Combinatorics of rational polyhedra.** Building on [7] and [24], in this section we introduce a functor from the category of abstract simplicial complexes with weighted vertices into the category of rational polyhedra. This will be used to construct limits and co-limits of rational polyhedra. Using the dual equivalence of Theorem 2.10 we will then characterize finitely generated subalgebras of free unital \( \ell \)-groups.

Let us recall that a \( (\text{finite}) \) abstract simplicial complex is a pair \((V, \Sigma)\), where \( V \) is a finite set, whose elements are called the vertices of \((V, \Sigma)\), and \( \Sigma \) is a collection of subsets of \( V \) whose union is \( V \), having the property that every subset of an element of \( \Sigma \) is again an element of \( \Sigma \).

A **weighted abstract simplicial complex** is a triple \((V, \Sigma, \omega)\) where \((V, \Sigma)\) is an abstract simplicial complex and \( \omega \) is a map of \( V \) into the set \( \{1, 2, 3, \ldots\} \).

Given two weighted abstract simplicial complexes \( \mathcal{W} = (V, \Sigma, \omega) \) and \( \mathcal{W}' = (V', \Sigma', \omega') \) a simplicial map \( \gamma: V \to V' \), (that is, \( \gamma(S) \subseteq \Sigma' \) for each \( S \in \Sigma \)) is a morphism from \( \mathcal{W} \) into \( \mathcal{W}' \), if \( \omega'(\gamma(v)) \) divides \( \omega(v) \) for all \( v \in V \).

We denote \( \mathcal{W} \) the category of weighted abstract simplicial complexes.

It is easy to see that two weighted abstract simplicial complexes \( \mathcal{W} = (V, \Sigma, \omega) \) and \( \mathcal{W}' = (V', \Sigma', \omega') \) are isomorphic in \( \mathcal{W} \) iff there is a one-one map \( \gamma \) from \( V' \) onto \( V \) having the following properties:

- \( \gamma(v) = \gamma(v) \) for all \( v \in V \), and
- \( \{w_1, \ldots, w_k\} \subset \Sigma \) iff \( \gamma(w_1), \ldots, \gamma(w_k) \in \Sigma' \) for each \( \{w_1, \ldots, w_k\} \subset V \).

When this is the case we say that \( \mathcal{W} \) and \( \mathcal{W}' \) are combinatorially isomorphic.

Let \( \mathcal{W} = (V, \Sigma, \omega) \) be a weighted abstract simplicial complex with vertex set \( V = \{v_1, \ldots, v_n\} \). Let \( e_1, \ldots, e_n \) be the standard basis vectors of \( \mathbb{R}^n \). We then use
the notation $\Delta_\mathcal{W}$ for the complex whose vertices are the following points in $\mathbb{R}^n$

$$v'_1 = e_1/\omega(v_1), \ldots, v'_n = e_n/\omega(v_n),$$

and whose $k$-simplices $(k = 0, \ldots, n)$ are given by

$$\text{conv}(v'_i(0), \ldots, v'_i(k)) \in \Delta_\mathcal{W} \iff \{v_i(0), \ldots, v_i(k)\} \in \Sigma.$$ 

Trivially, $\Delta_\mathcal{W}$ is a regular triangulation of the polyhedron $|\Delta_\mathcal{W}| \subseteq [0,1]^n$. The polyhedron $|\Delta_\mathcal{W}|$ is called the geometric realization of $\mathcal{W}$ and will be denoted $\mathcal{P}(\mathcal{W})$.

In order to extend $\mathcal{P}$ to a functor from $\mathcal{W}$ into $\mathcal{P}_Z$ we prepare

**Lemma 2.12.** [23, Lemma 3.7] Let $S = \text{conv}(x_1, \ldots, x_k) \subseteq \mathbb{R}^n$ be a regular $(k-1)$-simplex, and $\{y_1, \ldots, y_k\}$ a set of rational points in $\mathbb{R}^n$. Then the following conditions are equivalent:

(i) For each $i = 1, \ldots, k$, $\text{den}(y_i)$ is a divisor of $\text{den}(x_i)$.

(ii) For some integer matrix $M \in \mathbb{Z}^{n \times n}$ and integer vector $b \in \mathbb{Z}^n$, $Mx_i + b = y_i$.

**Corollary 2.13.** Let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ be polyhedra and $\eta: P \to Q$ a $\mathbb{Z}$-map. Then for every rational point $x \in P$, $\text{den}(\eta(x))$ divides $\text{den}(x)$.

**Corollary 2.14.** Let $P \subseteq \mathbb{R}^n$ be a polyhedron, $\Delta$ a regular triangulation of $P$ and $f: \text{vert}(\Delta) \to \mathbb{Q}^m$ a map such that $\text{den}(f(v))$ divides $\text{den}(v)$ for each $v \in \text{vert}(\Delta)$. Then there exists a unique $\mathbb{Z}$-map $\eta: P \to \mathbb{R}^m$ satisfying the following two conditions:

(i) $\eta$ is linear on each simplex of $\Delta$;

(ii) $\eta\mid\text{vert}(\Delta) = f$.

Let $\mathcal{W} = (V, \Sigma, \omega)$ and $\mathcal{W}' = (V', \Sigma', \omega')$ be weighted abstract simplicial complexes, with $V = \{v_1, \ldots, v_n\}$ and $V' = \{v'_1, \ldots, v'_m\}$. Let $\gamma: V \to V'$ be a morphism of weighted abstract simplicial complexes. Corollary 2.14 yields a unique $\mathbb{Z}$-map $\mathcal{P}(h): \mathcal{P}(W) \to \mathcal{P}(W')$ with the following properties:

- $\mathcal{P}(h)$ is linear on each simplex $S$ of $\Delta_W$, and
- for each $i = 1, \ldots, n$, $\eta(e_i/\omega(v_i)) = e_j/\omega'(v'_j)$ whenever $\gamma(v_i) = v'_j$.

As a consequence, $\mathcal{P}$ is a faithful functor from $\mathcal{W}$ into $\mathcal{P}_Z$.

For every regular complex $\Delta$, the skeleton of $\Delta$ is the weighted abstract simplicial complex $\mathcal{W}(\Delta) = (V, \Sigma, \omega)$ given by the following stipulations:

(i) $V = \text{vert}(\Delta)$.

(ii) For every $v \in \text{vert}(\Delta)$, $\omega(v) = \text{den}(v)$.

(iii) For every subset $W = \{w_1, \ldots, w_k\}$ of $V$, $W \in \Sigma$ iff $\text{conv}(w_1, \ldots, w_k) \in \Delta$.

Let $\{v_1, \ldots, v_m\}$ be the vertices of a regular triangulation $\Delta$ of a polyhedron $P$. Let

$$\iota_{\Delta}: P \to \mathcal{P}(\mathcal{W}(\Delta))$$

be the unique $\mathbb{Z}$-map given by Corollary 2.14 which is linear on each simplex of $\Delta$ and also satisfies $\iota_{\Delta}(v_i) = e_i/\text{den}(v_i)$. Then $\iota_{\Delta}$ is a $\mathbb{Z}$-homeomorphism. Since $\mathcal{P}(\mathcal{W}(\Delta)) \subseteq [0,1]^m$, as a byproduct we obtain that each rational polyhedron is $\mathbb{Z}$-homeomorphic to a polyhedron contained in some $m$-cube.
We have just proved that for each object \( P \) of \( P_Z \) there exists an object \( \mathfrak{W} \) of \( W \) such that \( \mathcal{P}(\mathfrak{W}) \) is isomorphic to \( P \) in \( P_Z \). Yet, \( \mathcal{P} \) does not define an equivalence between the categories \( W \) and \( P_Z \), because \( \mathcal{P} \) is not full:

**Example 2.15.** Let the \( Z \)-map \( \eta: [1/4, 1/3] \to [0, 2/3] \) be defined by \( \eta(x) = 8x - 2 \).

Let the rational point \( a \in [1/4, 1/3] \) be such that \([1/4, a]\) is a regular 1-simplex.

Writing \( a = k/l \) for \( k, l \in \{1, 2, \ldots\} \) with \( \gcd(k, l) = 1 \), it follows that \( l = 4k - 1 \), whence \( \eta(a) = 8(k/(4k - 1)) - 2 = 2/(4k - 1) \). Thus \([0, 2/(4k - 1)] = \eta([1/4, a])\) is not regular. As a consequence, there is no regular \( \Delta \) triangulation of \([1/4, 1/3]\). Hence, \( \eta(\Delta) \) is a regular triangulation of the simplex \([0, 2/3]\). Since for each morphism \( \gamma: \mathfrak{W} \to \mathfrak{W}' \) of weighted abstract simplicial complexes, \( \mathcal{P}(\gamma) \) satisfies \( \mathcal{P}(\eta(\Delta)) = \mathcal{P}(\gamma) \), we conclude that \( \mathcal{P} \) is not full.

### 3. Properties of the Category of Rational Polyhedra

In this section we will study some properties of the category \( P_Z \) of rational polyhedra, and the dual properties of the category of finitely presented unital \( \ell \)-groups.

#### 3.1. \( Z \)-maps and limits. The category of unital \( \ell \)-groups is small complete and small co-complete. This is a consequence of the categorical equivalence between unital \( \ell \)-groups and the equational class of MV-algebras, [21, Theorem 3.9].

It follows that finitely presented unital \( \ell \)-groups are closed under finite co-limits, whence, by Theorem 2.10, \( P_Z \) is closed under finite limits.

We next construct finite limits in \( P_Z \). This will be the key tool to describe monic and epic arrows in the category \( P_Z \) in Theorem 3.2.

**Theorem 3.1 (Limits).** The category \( P_Z \) is closed under finite limits.

*Proof.* In view of [16, §V.2. Corollary 2], we only need to prove that \( P_Z \) has finite products and equalizers.

**Finite products.** It is easy to see that the set \( \{1\} \) is the terminal object \( P_Z \). Therefore, \( P_Z \) admits the empty product.

Suppose that \( P \subseteq \mathbb{R}^n \) and \( Q \subseteq \mathbb{R}^m \) are polyhedra. The product of \( P \) and \( Q \) in \( \text{SET} \),

\[
P \times Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in P \text{ and } y \in Q\}
\]

is a rational polyhedron. Using Lemma 2.8, it is easy to see that the projections \( \pi_P \) and \( \pi_Q \) are \( Z \)-maps.

Suppose that \( R \subseteq \mathbb{R}^l \) is a polyhedron and \( \eta: R \to P \) and \( \mu: R \to Q \) are \( Z \)-maps.

An application of Lemma 2.8 shows that the unique map \( (\eta, \mu): R \to P \times Q \) such that \( \pi_P \circ (\eta, \mu) = \eta \) and \( \pi_Q \circ (\eta, \mu) = \mu \) is a \( Z \)-map. Thus the cone

\[
P \xrightarrow{\pi_P} P \times Q \xrightarrow{\pi_Q} Q
\]

is universal in \( P_Z \), and \( P \times Q \) is the product of \( P \) and \( Q \) in the category \( P_Z \).

**Equalizers.** Let \( P \subseteq \mathbb{R}^n \) and \( Q \subseteq \mathbb{R}^m \) be polyhedra and \( \eta, \mu: P \to Q \) two \( Z \)-maps. Let

\[
E = \{x \in P \mid \mu(x) = \eta(x)\}
\]

1The small completeness (small co-completeness) for equational classes of algebras follows from Birkhoff Theorem (see [4, Theorem 11.9]) and the construction of limits (co-limits) by products and equalizers (co-products and co-equalizers) (see [16, §5.2. Theorem 1]).
be the equalizer of $\mu$ and $\eta$ in $\mathbf{SET}$. To see that $E$ is a polyhedron, for each $i = 1, \ldots, m$, let $f_i : P \to \mathbb{R}$ be defined by $f_i = |\xi_i \circ \mu - \xi_1 \circ \eta|$, where $\xi_i : Q \to \mathbb{R}$ are the coordinate maps. Writing $f = f_1 + \cdots + f_m$ it follows that $E = f^{-1}(0)$. By Lemma 2.9, $E$ is a rational polyhedron.

If $R \subseteq \mathbb{R}^l$ is a polyhedron and $\nu : R \to P$ is a $\mathbb{Z}$-map such that $\nu \circ \eta = \nu \circ \mu$ then $\nu(R) \subseteq E$. Since inclusions are $\mathbb{Z}$-maps, the proof is complete. \hfill $\Box$

**Theorem 3.2.** Let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ be polyhedra and $\eta : P \to Q$ a $\mathbb{Z}$-map. Then

(i) $\eta$ is monic in $P_Z$ iff it is one-one.

(ii) $\eta$ is epic in $P_Z$ iff it is onto.

**Proof.** (i) For the nontrivial direction, suppose $\eta$ is a monic arrow in $P_Z$. If $\eta$ is not one-one (absurdum hypothesis) let

$$E = \{(x, y) \in P \times P \mid \eta(x) = \eta(y)\}.$$ 

The proof of Theorem 3.1 shows that $E$ is the equalizer of $\eta \circ \pi_1$ and $\eta \circ \pi_2$, where $\pi_1$ and $\pi_2$ are the projections of $P \times P$ onto $P$. By our assumption, there exists a rational point $(x, y) \in E$ such that $x, y \in P$ and $x \neq y$. Let $d = \text{den}(x) \cdot \text{den}(y)$ and $S = \{\frac{1}{d}\} \subseteq \mathbb{R}$. By Lemma 2.12, the maps $\mu_1, \mu_2 : S \to P$ defined by $\mu_1(\frac{1}{d}) = x$ and $\mu_2(\frac{1}{d}) = y$ are $\mathbb{Z}$-maps. Clearly, $\eta \circ \mu_1 = \eta \circ \mu_2$ and $\mu_1 \neq \mu_2$, a contradiction with the assumption that $\eta$ is monic in $P_Z$.

(ii) For the nontrivial direction, suppose that $\eta$ is epic in $P_Z$ but is not onto $Q$ (absurdum hypothesis). By Lemma 2.11, $\eta(P) \subseteq Q$ is a polyhedron. By Lemma 2.3, there exists a regular triangulation $\Delta_Q$ of $Q$ such that $\{S \in \Delta_Q \mid \text{vert}(S) \subseteq \eta(P)\}$ is a regular triangulation of $\eta(P)$.

Let $\mu_1 : Q \to Q \times [0, 1]$ be defined by

$$\mu_1(v) = (v, 0).$$

Let $\mu_2$ be the unique $\mathbb{Z}$-map of Corollary 2.14, satisfying the following conditions for each $S \in \Delta_Q$ and every $v \in \text{vert}(S)$:

$$\mu_2(v) = \begin{cases} (v, 0) & \text{if } v \in \eta(P), \\
(v, 1) & \text{if } v \notin \eta(P), \end{cases}$$

with $\mu_2$ being linear on each simplex of $\Delta_Q$.

It is easy to see that $\mu_1 \circ \eta = \mu_2 \circ \eta$ but $\mu_1 \neq \mu_2$, a contradiction with the assumption that $\eta$ is epic. \hfill $\Box$

The foregoing result is well known to the specialist: in particular, (ii) can also be derived from Theorem 2.10 in combination with [23, Lemma 3.8]. We have given a proof for the sake of completeness.

### 3.2. Cantor-Bernstein-Schröder theorem

In the previous subsection we have characterized monic and epic arrows in the category $P_Z$. In [23, Proposition 3.15] a characterization of iso-arrows in $P_Z$ is given in terms of preservation of denominators of rational points. Using this result, in Corollary 3.10 we will prove a (dual) Cantor-Bernstein-Schröder theorem for finitely presented unital $\ell$-groups.

By Theorem 2.10 we immediately have
Lemma 3.3. Let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ be polyhedra and $\eta: P \to Q$ a Z-map. Then $\eta$ is a Z-homeomorphism in $P_\mathbb{Z}$ iff $M(\eta): M(Q) \to M(P)$ is an isomorphism in $U_\mathbb{Z}$.

Theorem 3.4. [23, Proposition 3.5] Let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ be polyhedra and $\eta: P \to Q$ a one-one Z-map of $P$ onto $Q$. Then the following conditions are equivalent:

(i) $\eta$ is a Z-homeomorphism.
(ii) $\text{den}(\eta(x)) = \text{den}(x)$ for each rational point $x \in P$.
(iii) For each regular simplex $S \subseteq P$, $\eta(S)$ is a regular simplex of $Q$ and $\text{den}(\eta(x)) = \text{den}(x)$ for each $x \in \text{vert}(S)$.
(iv) For some (equivalently, for every) regular triangulation $\Delta$ of $P$ such that $\eta$ is linear on each simplex of $\Delta$, $\eta(\Delta)$ is a regular triangulation of $Q$ and $\text{den}(\eta(x)) = \text{den}(x)$ for each $x \in \text{vert}(\Delta)$.

Definition 3.5. Let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ be polyhedra and $\eta: P \to Q$ a Z-map. Then $\eta$ is a strict Z-map if it is a Z-homeomorphism onto its range.

From Theorem 3.4 we obtain:

Corollary 3.6. Let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ be polyhedra and $\eta: P \to Q$ a one-one Z-map. Then the following conditions are equivalent:

(i) $\eta$ is a strict Z-map.
(ii) $\text{den}(\eta(x)) = \text{den}(x)$ for each rational point $x \in P$.
(iii) For each regular simplex $S \subseteq P$, $\eta(S)$ is a regular simplex of $Q$ and $\text{den}(\eta(x)) = \text{den}(x)$ for each $x \in \text{vert}(S)$.
(iv) For some (equivalently, for every) regular triangulation $\Delta$ of $P$ such that $\eta$ is linear on each simplex of $\Delta$, $\eta(\Delta)$ is a regular triangulation of $\eta(P)$ and $\text{den}(\eta(x)) = \text{den}(x)$ for each $x \in \text{vert}(\Delta)$.

By Theorem 3.2, every strict Z-map is a monic Z-map, but the converse does not hold in general. From Theorem 2.10, monic Z-maps correspond to epi unital $\ell$-homomorphisms. The following theorem shows that strict Z-maps correspond to onto (or equivalently regular epi) unital $\ell$-homomorphisms:

Theorem 3.7. Let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ be polyhedra and $\eta: P \to Q$ a Z-map. Then $\eta$ is a strict Z-map iff $M(\eta): M(Q) \to M(P)$ is an onto map.

Proof. ($\Rightarrow$) In order to prove that $M(\eta)$ is onto $M(P)$, let $f \in M(P)$. By Proposition 2.1 and Lemma 2.7, there exists a regular triangulation $\Delta_P$ of $P$ such that $\eta$ and $f$ are linear over each simplex $S \in \Delta$. Since $\eta$ is a strict Z-map, by Theorem 3.4, $\eta(\Delta_P)$ is a regular triangulation of $\eta(P)$. By Lemma 2.3 there exists a regular triangulation $\Delta_Q$ of $Q$ such that the set $\nabla = \{S \in \Delta_Q \mid \text{vert}(S) \subseteq \eta(P)\}$ is a subdivision of $\eta(\Delta_P)$. Then $\eta^{-1}: \eta(P) \to P$ is linear over each simplex of $\nabla$. By Theorem 3.4, $\eta^{-1}(\nabla)$ is a regular triangulation of $P$ and is a subdivision of $\Delta_P$.

Let $g \in M(Q)$ be uniquely determined by the following conditions:

(i) $g$ is linear over each simplex $S \in \Delta_Q$.
(ii) for every $v \in \text{vert}(\Delta_Q)$,

$$g(v) = \begin{cases} f(\eta^{-1}(v)) & \text{if } v \in \eta(P), \\ 0 & \text{if } v \notin \eta(P). \end{cases}$$
The existence and uniqueness of \( g \) is ensured by Lemma 2.12. It follows that \( g \circ \eta \) is linear over each \( S \in \eta^{-1}(\nabla) \). For every \( x \in \eta^{-1}(\nabla) \) we can write
\[
g \circ \eta(x) = g(\eta(x)) = f(\eta^{-1}(\eta(x))) = f(x).
\]
Therefore, \( g \circ \eta = h(g) = f \), and \( \mathcal{M}(\eta) \) is onto.

\((\Leftarrow)\) Assume \( \mathcal{M}(\eta) \) is an onto map. By Lemma 2.11(ii),
\[
\mathcal{M}(P) \cong \mathcal{M}(Q)/\ker(\mathcal{M}(\eta)) \cong \mathcal{M}(\eta(P)).
\]
By Lemma 3.3, \( \eta: P \to \eta(P) \) is a \( \mathbb{Z} \)-homeomorphism. □

**Theorem 3.8.** Let \( P \subseteq \mathbb{R}^n \) be a polyhedron and \( \eta: P \to P \) a one-one (equivalently, a monic) \( \mathbb{Z} \)-map. Then \( \eta \) is a \( \mathbb{Z} \)-homeomorphism.

**Proof.** By Theorem 3.4, it is enough to prove that \( \eta \) preserves denominators and is onto \( P \). For each \( k = 1, 2, \ldots \), let \( P_k = \{ x \in P \cap \mathbb{Q}^n \mid \text{den}(x) = k \} \). Since \( P \) is a bounded set, each \( P_k \) is a finite set.

**Claim:** \( \eta \) preserves denominators. Equivalently, \( \eta(P_k) = P_k \) for each \( k = 1, 2, \ldots \).

The proof is by induction on \( k \). For the basis case, let \( x \in P_1 \). Then \( \text{den}(\eta(x)) \) divides \( \text{den}(x) = 1 \), i.e. \( \eta(P_1) \subseteq P_1 \). Since \( \eta \) is one-one and \( P_1 \) is finite, \( \eta(P_1) = P_1 \).

For the induction step, suppose that for every \( j < k \), \( \eta(P_j) = P_j \). Let \( x \in P_k \). Since \( \text{den}(\eta(x)) \) divides \( k \), then \( \text{den}(\eta(x)) \leq k \). Assume \( \eta(x) \notin P_k \) (absurdum hypothesis). Then \( \eta(x) \in P_{k'} \) for some \( k' < k \). By hypothesis, there exists \( y \in P_{k'} \) such that \( \eta(y) = \eta(x) \). Since \( \text{den}(x) = k' \neq k = \text{den}(y) \), then \( x \neq y \), thus contradicting the fact that \( \eta \). Our claim is settled.

To see that \( \eta \) is onto \( P \), first let us observe that since \( P \) is a rational polyhedron,
\[
P = \text{cl}(\bigcup_{k \geq 1} P_k),
\]
where \( \text{cl} \) denotes topological closure. The continuity of \( \eta \) now yields
\[
\eta(P) = \eta(\text{cl}(\bigcup_{k \geq 1} P_k)) = \text{cl}(\eta(\bigcup_{k \geq 1} P_k)) = \text{cl}(\bigcup_{k \geq 1} \eta(P_k)) = \text{cl}(\bigcup_{k \geq 1} P_k) = P,
\]
i.e. \( \eta \) is onto \( P \). The proof is complete. □

**Theorem 3.9.** Let \( P \subseteq \mathbb{R}^n \) and \( Q \subseteq \mathbb{R}^m \) be polyhedra, and \( \eta: P \to Q \) and \( \mu: Q \to P \) be one-one \( \mathbb{Z} \)-maps. Then \( P \) is \( \mathbb{Z} \)-homeomorphic to \( Q \).

**Proof.** Since \( \nu = \mu \circ \eta \) is a one-one \( \mathbb{Z} \)-map from \( P \) into itself, by the previous theorem \( \nu \) is a \( \mathbb{Z} \)-homeomorphism.

We claim that \( \nu^{-1} \circ \mu: Q \to P \) is the inverse of \( \eta \). Denoting by \( \text{Id}_P \) and \( \text{Id}_Q \) the identity maps over \( P \) and \( Q \), we get
\[
(\nu^{-1} \circ \mu) \circ \eta = \nu^{-1} \circ (\mu \circ \eta) = \nu^{-1} \circ (\nu) = \text{Id}_P.
\]
Symmetrically, \( \mu \circ (\nu \circ \nu^{-1} \circ \mu) = \mu = \mu \circ \text{Id}_Q \). Since \( \mu \) is a one-one \( \mathbb{Z} \)-map, then \( \eta \circ (\nu^{-1} \circ \mu) = \text{Id}_Q \). □

As a corollary we obtain a (dual) Cantor-Bernstein-Schröder theorem for finitely presented unital \( \ell \)-groups:

**Corollary 3.10.** For any finitely presented unital \( \ell \)-groups \( (G_1, u_1) \) and \( (G_2, u_2) \) the following conditions are equivalent:
(i) \((G_1, u_1)\) and \((G_2, u_2)\) are isomorphic.
(ii) There are onto homomorphisms \(f: (G_1, u_1) \to (G_2, u_2)\) and \(g: (G_2, u_2) \to (G_1, u_1)\).
(iii) There are epic homomorphisms \(h: (G_1, u_1) \to (G_2, u_2)\) and \(l: (G_2, u_2) \to (G_1, u_1)\).

It is easy to check that if in item (ii) we replace “onto” by “one-one” the result is no longer valid.

3.3. Co-limits. Having constructed finite limits in the category of rational polyhedra, we devote this section to a special type of co-limits that will find use in the rest of the paper.

From Lemmas 2.3, 2.4, 2.7 and Theorem 3.4 we obtain

Lemma 3.11. Let \(P \subseteq \mathbb{R}^n\), \(Q \subseteq \mathbb{R}^m\) and \(D \subseteq \mathbb{R}^k\) be polyhedra and \(\eta: P \to Q\) and \(\mu: P \to D\) be strict \(\mathbb{Z}\)-maps. Then there exist regular triangulations \(\Delta_P, \Delta_Q, \Delta_R\) of \(P, Q, R\) satisfying the following conditions:

(i) \(\eta\) and \(\mu\) are linear over each simplex of \(\Delta_P\);
(ii) \(\eta(\Delta_P)\) is a full subcomplex of \(\Delta_Q\);
(iii) \(\mu(\Delta_P)\) is a full subcomplex of \(\Delta_R\).

Moreover, for any regular triangulations \(\nabla_P, \nabla_Q, \nabla_R\) of \(P, Q, R\) we may insist that \(\Delta_P, \Delta_Q, \Delta_R\) subdivide \(\nabla_P, \nabla_Q, \nabla_R\), respectively.

Theorem 3.12 (Co-limits). \(PZ\) has finite co-products and pushouts of strict \(\mathbb{Z}\)-maps.

Proof. Finite co-products: Considered as a rational polyhedron, the empty set is the initial object of \(PZ\), and is also the empty co-product. Next, for any polyhedra \(P \subseteq \mathbb{R}^m\) and \(Q \subseteq \mathbb{R}^n\) let \(k = \max\{m, n\}\) and

\[ C = \{(x, 0_{m+1}, \ldots, 0_{k+1}) | x \in P\} \cup \{(y, 1_{n+1}, \ldots, 1_{k+1}) | y \in Q\}. \]

Then \(C\) is the co-product of \(P\) and \(Q\) in \(\text{SET}\) and the inclusion maps \(i_P: P \to C\) and \(i_Q: Q \to C\)

\[ i_P(x) = (x, 0_{m+1}, \ldots, 0_{k+1}) \quad \text{and} \quad i_Q(y) = (y, 1_{n+1}, \ldots, 1_{k+1}) \]

are \(\mathbb{Z}\)-maps.

Let \(R \subseteq \mathbb{R}^l\) be a polyhedron and \(\eta: P \to R\) and \(\mu: Q \to R\) be \(\mathbb{Z}\)-maps. By Lemma 2.8, the unique map \((\eta, \mu): C \to R\) defined by \((\eta, \mu) \circ i_P = \eta\) and \((\eta, \mu) \circ i_Q = \mu\) is a \(\mathbb{Z}\)-map. Then \(C\) is the co-product of \(P\) and \(Q\) in the category \(PZ\).

Pushouts of strict \(\mathbb{Z}\)-maps: Let \(P \subseteq \mathbb{R}^n\), \(Q \subseteq \mathbb{R}^m\) and \(D \subseteq \mathbb{R}^k\) be polyhedra and \(\eta: P \to Q\) and \(\mu: P \to D\) be strict \(\mathbb{Z}\)-maps.

With reference to Lemma 3.11, let \(\Delta_P, \Delta_Q\) and \(\Delta_R\) be regular triangulations of \(P, Q\) and \(R\) such that \(\eta\) and \(\mu\) are linear over each simplex of \(\Delta_P\), \(\eta(\Delta_P)\) is a full subcomplex of \(\Delta_Q\), and \(\mu(\Delta_P)\) is a full subcomplex of \(\Delta_R\).
Let $f = \eta \circ \text{vert}(\Delta_P)$ and $g = \mu \circ \text{vert}(\Delta_P)$. By definition, $f : \mathcal{W}(\Delta_P) \to \mathcal{W}(\Delta_Q)$ and $g : \mathcal{W}(\Delta_P) \to \mathcal{W}(\Delta_R)$ are $d$-maps and the following diagram commutes:

\[
\begin{array}{ccc}
Q & \xrightarrow{\iota_{\Delta_Q}} & \mathcal{P}(\mathcal{W}_{\Delta_Q}) \\
\downarrow{\eta} & & \downarrow{\mathcal{P}(f)} \\
P & \xrightarrow{\iota_{\Delta_P}} & \mathcal{P}(\mathcal{W}_{\Delta_P}) \\
\downarrow{\mu} & & \downarrow{\mathcal{P}(g)} \\
R & \xrightarrow{\iota_{\Delta_R}} & \mathcal{P}(\mathcal{W}_{\Delta_R})
\end{array}
\]

Since $\eta$ and $\mu$ are one-one, then so are $f$ and $g$, and we can write
\[
\begin{align*}
\text{vert}(\Delta_P) &= \{v_1, \ldots, v_r\} \\
\text{vert}(\Delta_Q) &= \{f(v_1), \ldots, f(v_m), w_1, \ldots, w_s\} \\
\text{vert}(\Delta_R) &= \{g(v_1), \ldots, g(v_m), z_1, \ldots, z_t\}
\end{align*}
\]

Let $W = \text{vert}(\Delta_Q) \setminus f(\text{vert}(\Delta_P))$ and $Z = \text{vert}(\Delta_R) \setminus g(\text{vert}(\Delta_P))$. Without loss of generality, $W \cap Z = \emptyset$.

We now define the weighted abstract simplicial complex $\mathcal{W} = (V, \Sigma, \omega)$ by the following stipulation:

- $V = \text{vert}(\Delta_P) \cup W \cup Z$;
- $\omega(v) = \text{den}(v)$ for each $v \in V$;
- $X \in \Sigma$ if
  \begin{enumerate}
  \item either $X \subseteq \text{vert}(\Delta_P) \cup W$ and $f(X \cap \text{vert}(\Delta_P)) \cup (X \cap W) \in \Sigma_{\Delta_Q}$, or
  \item $X \subseteq \text{vert}(\Delta_P) \cup Z$ and $g(X \cap \text{vert}(\Delta_P)) \cup (X \cap Z) \in \Sigma_{\Delta_R}$.
  \end{enumerate}

Let $i_Q : \text{vert}(\Delta_Q) \to V$ and $i_R : \text{vert}(\Delta_R) \to V$ be defined as follows:
\[
\begin{align*}
i_Q(v) &= \begin{cases} v & \text{if } v \in W \\ w & \text{if } w \in \text{vert}(\Delta_P) \text{ and } v = f(w) \end{cases} \\
i_R(v) &= \begin{cases} v & \text{if } v \in Z \\ w & \text{if } w \in \text{vert}(\Delta_P) \text{ and } v = g(w) \end{cases}
\end{align*}
\]

By definition of $\Sigma$ and $\omega$, the maps $i_Q$ and $i_R$ preserve weights, whence they are morphisms in $\mathcal{W}$. By Corollary 3.6, $\mathcal{P}(i_Q)$ and $\mathcal{P}(i_R)$ are strict $\mathbb{Z}$-maps and we have a commutative diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{\iota_{\Delta_Q}} & \mathcal{P}(\mathcal{W}_{\Delta_Q}) \\
\downarrow{\eta} & & \downarrow{\mathcal{P}(f)} \\
P & \xrightarrow{\iota_{\Delta_P}} & \mathcal{P}(\mathcal{W}_{\Delta_P}) \\
\downarrow{\mu} & & \downarrow{\mathcal{P}(g)} \\
R & \xrightarrow{\iota_{\Delta_R}} & \mathcal{P}(\mathcal{W}_{\Delta_R})
\end{array}
\]

Observe that $\rho_Q = \mathcal{P}(i_Q) \circ \iota_{\Delta_Q}$ and $\rho_R = \mathcal{P}(i_R) \circ \iota_{\Delta_R}$ are strict $\mathbb{Z}$-maps. Since $i_Q(f(v)) = i_R(g(v)) = v$ for each $v \in \text{vert}(\Delta_P)$, we have $\rho_Q \circ \eta = \rho_R \circ \mu = \rho_P$.

We claim that for all $x \in Q$ and $y \in R$ with $\rho_Q(x) = \rho_R(y)$, there exists $z \in P$ such that $\eta(z) = x$ and $\mu(z) = y$. 

Thus \(\gamma\) is a well defined simplicial complex. Moreover, \(\Lambda\) is a regular triangulation of \(Q\) into a free Heyting algebra (see also Section 4.5).

The result follows from Theorems 2.10, 3.7 and 3.12, upon observing that for \(f\) as the distinguished order unit)

\[ \rho_Q(Q) \cap \rho_R(R) = \rho_Q \circ f(P) = \rho_R \circ g(P) = \rho_P(P). \]

In order to prove that \(\mathcal{P}(\mathcal{W})\) is the pushout \(Q \coprod_R P \in W\), let \(U\) be a rational polyhedron, together with \(\mathcal{Z}\)-maps \(\gamma_Q: Q \to U\) and \(\gamma_R: R \to U\) such that \(\gamma_Q \circ \eta = \gamma_R \circ \mu\). Repeated applications of Lemma 2.7 provide regular triangulations \(\nabla_Q\) of \(Q\) and \(\nabla_R\) of \(R\) such that \(\gamma_Q\) and \(\rho_Q\) are linear on each simplex of \(\nabla_Q\), and \(\gamma_R\) and \(\rho_R\) are linear on each simplex of \(\nabla_R\). Lemma 3.11 yields regular triangulations \(\Lambda_P, \Lambda_Q, \Lambda_R\) of \(P, Q, R\) such that

- (i) \(\eta\) and \(\mu\) are linear on each simplex of \(\Lambda_P\);
- (ii) \(\eta(\Lambda_P) \subseteq \Lambda_Q\) and \(\mu(\Lambda_P) \subseteq \Lambda_R\);
- (iii) \(\Lambda_Q, \Lambda_R\) are subdivisions of \(\nabla_Q, \nabla_R\).

Thus \(\gamma_Q\) and \(\rho_Q\) are linear on each simplex of \(\Lambda_Q\), and \(\gamma_R\) and \(\rho_R\) are linear on each simplex of \(\Lambda_R\).

As a consequence, \(\rho_Q(\Lambda_Q) \cap \rho(\Lambda_R) = \rho_Q \circ \eta(\Lambda_P) = \rho_R \circ \mu(\Lambda_P)\) is a regular triangulation of \(\rho_Q(Q) \cap \rho_R(R) = \rho_Q \circ \eta(P) = \rho_R \circ \mu(P)\). Thus, \(\rho_Q(\Lambda_Q) \cup \rho(\Lambda_R) = \Lambda\) is a well defined simplicial complex. Moreover, \(\Lambda\) is a regular triangulation of \(\mathcal{P}(\mathcal{W})\).

Finally, let \(\zeta: \mathcal{P}(\mathcal{W}) \to U\) be the unique \(\mathcal{Z}\)-map given by Corollary 2.14 which is linear over each simplex of \(\Lambda\) and also satisfies

\[ \zeta(v) = \begin{cases} \gamma_Q(x) & \text{if } v = \rho_Q(x) \\ \gamma_R(y) & \text{if } v = \rho_R(y), \end{cases} \]

on each vertex \(v\) of \(\Lambda\). Since \(\rho_Q(\nabla_Q) \cap \rho(\nabla_R) = \rho_Q \circ \eta(\nabla_P)\) then \(\zeta\) is well defined. For any vertex \(x\) of \(\nabla_Q\), \(\gamma_Q(x) = \zeta(\rho(x))\). Since \(\gamma\) is linear over each simplex of \(\nabla_Q\) then \(\gamma_Q = \zeta \circ \rho_Q\). Similarly, \(\gamma_R = \zeta \circ \rho_R\), which proves that \(\mathcal{P}(\mathcal{W})\) is the pushout \(Q \coprod_R P \in W\).

\(\square\)

**Corollary 3.13.** Let \((G_1, u_1), (G_2, u_2), (G_3, u_3)\) be finitely presented unital \(\ell\)-groups with onto homomorphisms \(f: G_1 \to G_3\), \(g: G_2 \to G_3\). Then the fiber product \(G = \{(a, b) \in G_1 \times G_2 \mid f(a) = g(b)\}\) (with \((u_1, u_2)\) as the distinguished order unit) is a finitely presented unital \(\ell\)-group.

**Proof.** The result follows from Theorems 2.10, 3.7 and 3.12, upon observing that for all rational polyhedra \(P, Q, R\) and strict \(\mathcal{Z}\)-maps \(\eta: P \to Q\), \(\mu: P \to R\) the unital \(\ell\)-group \(\mathcal{M}(Q \coprod_P R)\) is isomorphic to the fiber product \(\{(f, g) \in \mathcal{M}(Q) \times \mathcal{M}(R) \mid f \circ \eta = g \circ \mu\}\).

\(\square\)

In Corollary 4.12 we will prove that if in Corollary 3.13 we also assume that \((G_1, u_1), (G_2, u_2),\) and \((G_3, u_3)\) are projective, then so is \(G\).

**4. Exact Unital \(\ell\)-Groups**

Working in the framework of intuitionistic logic, in his paper [9], de Jongh calls “exact” a formula \(\varphi\) such that the Heyting algebra presented by \(\varphi\) is embeddable into a free Heyting algebra (see also Section 4.5).
Accordingly, in this paper we say that a unital \( \ell\)-group \((G,u)\) is exact if it is finitely presented and there exist a positive integer \(n\) and a one-one unital \(\ell\)-homomorphism \(g\) of \((G,u)\) into the free unital \(\ell\)-group \(M([0,1]^n)\). By Lemma 2.6 and Theorem 2.10, a unital \(\ell\)-group is exact iff it is isomorphic to a finitely generated unital \(\ell\)-subgroup of \(M([0,1]^n)\). This equivalent definition can also be obtained as an application of the equivalence between MV-algebras and unital \(\ell\)-groups and [23, Corollary 6.6].

In this section we will give a characterization of exact unital \(\ell\)-groups.

A unital \(\ell\)-group \((G,u)\) is projective if whenever \(\psi: (G_1,u_1) \to (G_2,u_2)\) is a unital \(\ell\)-homomorphism onto \((G_2,u_2)\) and \(\phi: (G,u) \to (G_2,u_2)\) is a unital \(\ell\)-homomorphism, there is a unital \(\ell\)-homomorphism \(\theta: (G,u) \to (G_1,u_1)\) such that \(\phi = \psi \circ \theta\).

A finitely generated unital \(\ell\)-group \((G,u)\) is projective iff it is a retraction of the free unital \(\ell\)-group \(M([0,1]^n)\) for some \(n \in \{1,2,\ldots\}\). In other words, there is a homomorphism \(i: (G,u) \to M([0,1]^n)\) and a homomorphism \(\sigma: M([0,1]^n) \to (G,u)\) such that \(\sigma \circ i\) is the identity map \(Id_G\) on \(G\).

Lemma 2.11 yields the following inclusions for unital \(\ell\)-groups:

\[(3) \text{ Finitely Presented } \supseteq \text{ Exact } \supseteq \text{ Finitely generated projective.}\]

As shown in [7], the class of finitely generated projective unital \(\ell\)-groups (includes but) does not coincide with the class of finitely presented unital \(\ell\)-groups.

An example of an exact unital \(\ell\)-group which is not projective is as follows:

**Example 4.1.** Let \(P = \{(x,y) \in [0,1]^2 \mid x \in \{0,1\} \text{ or } y \in \{0,1\}\}\). The map \(\eta: [0,1] \to P\) defined by

\[
\eta(a) = \begin{cases} 
(3x,0) & \text{if } 0 \leq a \leq 1/3, \\
(1,6x-2) & \text{if } 1/3 \leq a \leq 1/2, \\
(4-6x,1) & \text{if } 1/2 \leq a \leq 2/3, \\
(0,3-3x) & \text{if } 2/3 \leq a \leq 1,
\end{cases}
\]

is a \(Z\)-map onto \(P\). By Theorem 3.2, \(M(\eta): M(P) \to M([0,1])\) is one-one. Then \(M(P)\) is exact. Since \(P\) is not simply connected, by [7, Theorem 4.2] \(M(P)\) is not projective.

As is well known (see [2]), for \(\ell\)-groups the three notions in (3) coincide.

4.1. **Strongly regular triangulations.** The notion of strongly regular triangulation was introduced in [7, Definition 3.1] and it has a key role in our characterization of exact unital \(\ell\)-groups.

**Definition 4.2.** A simplex \(S\) is said to be strongly regular if it is regular and the greatest common divisor of the denominators of the vertices of \(S\) is equal to 1. A triangulation \(\Delta\) of a polyhedron \(P \subseteq [0,1]^n\) is said to be strongly regular if each maximal simplex of \(\Delta\) is strongly regular.

**Lemma 4.3.** Let \(S\) be a regular \(k\)-simplex. Then for every regular \(k\)-simplex \(T\) such that \(T \subseteq S\), the greatest common divisor of the denominators of the vertices of \(T\) is equal to the greatest common divisor of the denominators of the vertices of \(S\).

**Proof.** In view of Lemma 2.2 it is no loss of generality to assume that \(T\) is one of the maximal simplexes obtained by blowing up \(S\) at the Farey median \(v\) of its vertices.
Let \( v_1, \ldots, v_k \) be the vertices of \( S \). Since \( \text{den}(v) \) is equal to \( \sum_{i=0}^{k} \text{den}(v_i) \), then for each \( i = 1, \ldots, k \) the greatest common divisor of the integers \( \text{den}(v_1), \ldots, \text{den}(v_k) \) coincides with the greatest common divisor of the set of integers

\[
\{ \text{den}(v_0), \ldots, \text{den}(v_{i-1}), \text{den}(v_i), \text{den}(v_{i+1}), \ldots, \text{den}(v_k) \}
\]

Thus the greatest common divisor of the denominators of the vertices \( S \) is equal to the greatest common divisor of the denominators of the vertices \( S \). \( \square \)

The following result was first proved in [7, Lemma 3.2]:

**Corollary 4.4.** Let \( \Delta \) and \( \nabla \) be regular triangulations of a polyhedron \( P \subseteq [0,1]^n \). Then \( \Delta \) is strongly regular iff so is \( \nabla \).

Since strong regularity does not depend on the regular triangulation \( \Delta \) of \( P \), without fear of ambiguity we may say that a polyhedron \( P \) is strongly regular if some (equivalently, each) of its regular triangulations is strongly regular.

**Example 4.5.** The \( n \)-cube \( [0,1]^n \) is strongly regular. To see this, let us equip the set \( [0,1]^n \) with the following partial order \( (a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \) iff \( a_i \leq b_i \) for each \( i = 1, \ldots, n \). Let \( \Delta \) be the triangulation of \( [0,1]^n \) formed by the simplexes \( \text{conv}(C) \) whenever \( C \) is a chain in the poset \( ([0,1]^n, \leq) \). This is called the standard triangulation of the cube in [27, p. 60]. Since the denominator of every vertex of \( \Delta \) is 1, it follows that \( \Delta \) is strongly regular. The desired conclusion now follows from Corollary 4.4.

For all \( v, w \in \mathbb{R}^n \), we let \( \text{dist}(v, w) \) denote their Euclidean distance in \( \mathbb{R}^n \). For each \( 0 \leq \delta \in \mathbb{R} \) and \( v \in \mathbb{R}^n \), we use the notation \( B(\delta, v) = \{ w \in \mathbb{R}^n \mid \text{dist}(v, w) < \delta \} \). The dimension of the ambient space will always be clear from the context.

From Lemma 4.3 we obtain the following characterization of strong regularity:

**Corollary 4.6.** Let \( P \subseteq \mathbb{R}^n \) be a polyhedron. Then the following conditions are equivalent:

(i) \( P \) is strongly regular.

(ii) For each \( v \in P \) and each \( 0 < \delta \in \mathbb{R} \), there exists \( w \in P \) such that \( \text{dist}(v, w) < \delta \), and \( \text{den}(v) \) and \( \text{den}(w) \) are coprime, in the sense that \( \gcd(\text{den}(v), \text{den}(w)) = 1 \).

For any set \( T \subseteq \mathbb{R}^n \), we let \( \text{aff}(T) \) denote the affine hull of \( T \), i.e.

\[
\text{aff}(T) = \left\{ \sum_{i=0}^{m} \lambda_i v_i \mid \text{for some } v_i \subseteq T, \lambda_i \in \mathbb{R}, \sum_{i=0}^{m} \lambda_i = 1 \text{ and } m = 1, 2, \ldots \right\}.
\]

Further, \( \text{relint}(T) \) denotes the relative interior of \( T \), that is, the interior of \( T \) in the relative topology of \( \text{aff}(T) \).

For later use in the proofs of Theorems 4.8 and 4.22, we record here the following elementary characterization:

**Lemma 4.7.** Let \( \Delta \) be a triangulation of a polyhedron \( P \subseteq \mathbb{R}^n \) and \( T \in \Delta \). Then the following conditions are equivalent:

(i) \( T \) is maximal in \( \Delta \).

(ii) Whenever \( w \in \text{relint}(T) \) and \( v \in \mathbb{R}^n \) does not lie in the affine hull of \( T \), then \( \text{conv}(v, w) \) is not contained in \( P \).
(iii) For every \( v \in \text{relint}(T) \) there exists \( 0 < \delta \in \mathbb{R} \) such that \( B(\delta, v) \cap P \subseteq T \).

**Theorem 4.8.** Let \( P \) and \( Q \) be polyhedra and \( \eta: P \to Q \) be a \( \mathbb{Z} \)-map onto \( Q \). If \( P \) has a strongly regular triangulation then \( Q \) has a strongly regular triangulation.

**Proof.** Let \( \Delta \) be a regular triangulation of \( Q \) and \( S \) a maximal simplex of \( \Delta \). Let \( d \) denote the greatest common divisor of the denominators of the vertices of \( S \). Let \( v \in \text{relint}(S) \). By Lemma 4.7 there exists \( 0 < \epsilon \in \mathbb{R} \) such that \( B(\epsilon, v) \cap Q \subseteq S \). Since \( \eta \) is a continuous onto map, there exist \( w \in P \) and \( 0 < \delta \in \mathbb{R} \) such that \( \eta(B(\delta, w) \cap P) \subseteq B(\epsilon, v) \).

Since \( P \) is strongly regular, Corollary 4.6 yields \( z \in B(\delta, w) \cap P \) such that \( \text{den}(w) \) and \( \text{den}(z) \) are coprime. Then \( \eta(w) = v, \eta(z) \in S, \) and \( d \) is a common divisor of \( \text{den}(\eta(w)) \) and \( \text{den}(\eta(z)) \). By Corollary 2.13, \( d \) is a common divisor of \( \text{den}(w) \) and \( \text{den}(z) \). We conclude that \( d = 1\). \( \Box \)

### 4.2 Finitely generated projective unital \( \ell \)-groups

We now collect some definitions and results from [7] that will be necessary for the geometrical description of exact unital \( \ell \)-groups in Theorem 4.13. In Theorem 4.11 it is also proved that projectiveness is preserved under fiber products of onto homomorphisms of unital \( \ell \)-groups.

A \( \mathbb{Z} \)-map \( \sigma: P \to P \) is a \( \mathbb{Z} \)-retraction of \( P \) if \( \sigma \circ \sigma = \sigma \). The rational polyhedron \( R = \sigma(P) \) is said to be a \( \mathbb{Z} \)-retract of \( P \). A rational polyhedron \( Q \) is said to be a \( \mathbb{Z} \)-retract if it is a \( \mathbb{Z} \)-retract of \( [0, 1]^n \) for some \( n \in \{1, 2, \ldots\} \).

The following is a consequence of Theorem 2.10 (see [5, Theorem 1.2] for details):

**Theorem 4.9.** A unital \( \ell \)-group \( (G, u) \) is finitely generated projective iff it is isomorphic to \( M(P) \) for some \( \mathbb{Z} \)-retract \( P \).

A simplex \( T \in \Sigma \) of an abstract simplicial complex \( \langle V, \Sigma \rangle \) is said to have a free face \( F \) if

- \( \emptyset \neq F \subseteq T \) is a facet (maximal proper subset) of \( T \), and
- whenever \( F \subseteq S \in \Sigma \) then \( S = F \) or \( S = T \).

It follows that \( T \) is a maximal simplex of \( \Sigma \), and the removal from \( \Sigma \) of both \( T \) and \( F \) results in the subcomplex \( \langle V', \Sigma' = \Sigma \setminus \{T, F\} \rangle \) of \( \langle V, \Sigma \rangle \), where \( V' = V \setminus F \) if \( F \) is a singleton and otherwise \( V' = V \). The transition from \( \langle V, \Sigma \rangle \) to \( \langle V', \Sigma' \rangle \) is called an (abstract) elementary collapse. If a simplicial complex \( \langle W, \Gamma \rangle \) can be obtained from \( \langle V, \Sigma \rangle \) by a sequence of elementary collapses we say that \( \langle V, \Sigma \rangle \) collapses to \( \langle W, \Gamma \rangle \). We say that the simplicial complex \( \langle V, \Sigma \rangle \) is collapsible if it collapses to the abstract simplicial complex consisting of one of its vertices (equivalently, any of its vertices [29, p.248]).

See [10, §III, Definition 7.2],[29, p.247] for the geometrical counterpart of collapsibility. For the purposes of this paper it is enough to observe that a regular triangulation \( \Delta \) is collapsible iff its skeleton \( \mathfrak{M}(\Delta) \) is collapsible.

**Theorem 4.10.** [7, Theorem 6.1] Let \( P \subseteq [0, 1]^n \) be a polyhedron. Suppose

(i) \( P \) has a collapsible triangulation \( \nabla \);

(ii) \( P \) contains a vertex \( v \) of \( [0, 1]^n \);

(iii) \( P \) is strongly regular.

Then \( P \) is a \( \mathbb{Z} \)-retract of \( [0, 1]^n \).

The following result states that \( \mathbb{Z} \)-retracts are preserved under pushouts of strict \( \mathbb{Z} \)-maps.
Theorem 4.11. Let $P \subseteq [0,1]^n$, $Q \subseteq [0,1]^m$ and $R \subseteq [0,1]^k$ be $\mathbb{Z}$-retracts and $\eta: P \to Q$ and $\mu: P \to D$ be strict $\mathbb{Z}$-maps. Then the pushout $Q \coprod_P R$ (whose existence is ensured by Theorem 3.12) is a $\mathbb{Z}$-retract.

Proof. With the notation of the proof of Theorem 3.12, the pushout $Q \coprod_P R$ was realized therein as the rational polyhedron $P(\mathbb{M}) \subseteq [0,1]^{r+s+t}$ of a certain weighted abstract simplicial complex $\mathbb{M}$. The embeddings of $P$, $Q$, $R$ into $Q \coprod_P R$, were denoted $\rho_P$, $\rho_Q$, and $\rho_R$. Letting $A = [0,1]^{r+s} \times \{0\}$ and $B = [0,1]^{r} \times \{0\} \times [0,1]^{t}$, it was shown:

(i) $\rho_Q(Q) \subseteq A$;
(ii) $\rho_R(R) \subseteq A$;
(iii) $\rho_Q(Q) \cup \rho_R(R) = Q \coprod_P R$;
(iv) $\rho_Q(Q) \cap \rho_R(R) = \rho_P(P)$.

Since $P$, $Q$, and $R$ are $\mathbb{Z}$-retracts, by [7, Lemma 4.2] $\rho_P(P)$, $\rho_Q(Q)$, and $\rho_R(R)$ are $\mathbb{Z}$-retracts. We let $\gamma_P: [0,1]^{r+s+t} \to \rho_P(P)$, $\gamma_Q: [0,1]^{r+s+t} \to \rho_Q(Q)$ and $\gamma_R: [0,1]^{r+s+t} \to \rho_R(R)$ denote the corresponding $\mathbb{Z}$-retractions.

The $\mathbb{Z}$-retraction for $Q \coprod_P R$ will be constructed in three steps.

Step 1:
If $x \in A \cup B$, then $\mathrm{conv}(0,x) \subseteq A \cup B$, where 0 denotes the origin of $\mathbb{R}^{r+s+t}$. Therefore, by [5, Theorem 1.4], there exists a $\mathbb{Z}$-retraction $\gamma_1: [0,1]^{r+s+t} \to A \cup B$ onto $A \cup B$.

Step 2:
Combining Proposition 2.1 and Lemma 2.3, there is a regular triangulation $\Delta$ of $A \cup B$, such that $\bigcup_{T \subseteq \Delta} T$ is a full subcomplex of $\Delta$. Let $\Delta_Q \coprod_P R = \{T \subseteq \Delta : T \subseteq Q \coprod_P R\}$ be the corresponding $\mathbb{Z}$-map $\gamma_2: \Delta \to \Delta$ satisfying

$$\gamma_2(v) = \begin{cases} v & \text{if } v \in Q \coprod_P R, \\ \gamma_P(v) & \text{otherwise}, \end{cases}$$

with $\gamma_2$ linear over each simplex $S \subseteq \Delta$. By construction, $\gamma_2$ has the following properties:

(a) if $v \in Q \coprod_P R$, then $\gamma_2(v) = v$;
(b) if $\gamma_2(A) \cap \gamma_2(B) = \rho_P(P)$

Step 3:
By (a), we have $Q \coprod_P R \subseteq \gamma_2(A \cup B)$. Again by Proposition 2.1 and Lemma 2.3, there is a regular triangulation $\Lambda$ of $\gamma_2(A \cup B)$, such that $\bigcup_{T \subseteq \Delta} T$ is a full subcomplex of $\Lambda$. Let $\gamma_3: \gamma_2(A \cup B) \to Q \coprod_P R$ be defined by:

$$\gamma_3(v) = \begin{cases} \gamma_Q(v) & \text{if } v \in \gamma_2(A), \\ \gamma_R(v) & \text{if } v \in \gamma_2(B). \end{cases}$$

By (b) and (iv), if $v \in \gamma_2(A) \cap \gamma_2(B)$, $v \in \rho_P(P) = \rho_Q(Q) \cap \rho_R(R)$, which implies that $\gamma_Q(v) = \gamma_R(v) = v$. This shows that $\gamma_3$ is well defined.

We claim that the map $\gamma = \gamma_3 \circ \gamma_2 \circ \gamma_1: [0,1]^{r+s+t} \to Q \coprod_P R$ is a $\mathbb{Z}$-retraction. If $v \in Q \coprod_P R$, by definition of $\gamma_1$, $\gamma_2$ and $\gamma_3$ it follows that $\gamma_1(v) = \gamma_2(v) = \gamma_3(v) = v$. If $v \notin Q \coprod_P R$, then $\gamma_1(v) \notin A \cup B$. Assume first $\gamma_1(v) \in A$. Then $\gamma_2 \circ \gamma_1(v) \in \gamma_2(A)$. Further, $\gamma(v) = \gamma_3(\gamma_2 \circ \gamma_1(v)) = \gamma_3(v) = \gamma_Q(v) \subseteq Q \coprod_P R$. Similarly, if $\gamma_1(v) \in B$
then $\gamma(v) \in Q \coprod_p R$. Therefore, $\gamma[0,1]^{r+s+t} \to Q \coprod_p R$ is a $\mathbb{Z}$-retraction onto $Q \coprod_p R$, as claimed.

The proof is complete \qed

Combining Corollary 3.13 with the foregoing theorem we get:

**Corollary 4.12.** Let $(G_1, u_1), (G_2, u_2), (G_3, u_3)$ be finitely generated projective unital $\ell$-groups and $f: G_1 \to G_3$ and $g: G_2 \to G_3$ onto homomorphisms. Then the fiber product $G = \{(a, b) \in G_1 \times G_2 \mid f(a) = g(b)\}$ (with $(u_1, u_2)$ as the distinguished order unit) is a finitely generated projective unital $\ell$-group.

### 4.3. Geometric realization of exact unital $\ell$-groups

The rest of the section is devoted to proving

**Theorem 4.13.** A unital $\ell$-group $(G, u)$ is exact iff there exists a polyhedron $P \subseteq \mathbb{R}^n$ satisfying the following conditions:

(i) $(G, u) \cong M(P)$;

(ii) $P$ is connected;

(iii) $P \cap \mathbb{Z}^n \neq \emptyset$;

(iv) $P$ is strongly regular.

For the proof we prepare:

**Lemma 4.14.** A unital $\ell$-group $(G, u)$ is exact iff there exist integers $m, n \geq 0$, and a $\mathbb{Z}$-map $\eta: [0,1]^n \to \mathbb{R}^m$ such that $(G, u) \cong M(\eta([0,1]^n))$.

**Proof.** ($\Rightarrow$) For some $m, n \in \{1, 2, \ldots\}$ there exist unital $\ell$-homomorphisms

$$f: M([0,1]^m) \to (G, u)$$

and

$$g: (G, u) \to M([0,1]^n)$$

with $f$ onto $(G, u)$ and $g$ one-one. Then

$$(G, u) \cong M([0,1]^m)/\ker f = M([0,1]^m)/\ker(f \circ g).$$

Theorem 2.10 yields a $\mathbb{Z}$-map $\eta: [0,1]^n \to [0,1]^m$ such that $g \circ f = M(\eta)$. Therefore, $h = \ker(f \circ g)$ if $h \circ \eta = 0$ if $h(\eta([0,1]^n)) = \{0\}$, whence $M([0,1]^m)/\ker(f \circ g) = M([0,1]^m)/\ker(M(\eta)) = M(\eta([0,1]^n))$.

($\Leftarrow$) As observed in Section 2, (see (1) in particular) every polyhedron is $\mathbb{Z}$-homeomorphic to a polyhedron contained in some unit cube $[0,1]^m$. Thus without loss of generality we can assume $\eta([0,1]^n) \subseteq [0,1]^m$. Let $\eta: [0,1]^n \to [0,1]^m$ be a $\mathbb{Z}$-map such that $(G, u) \cong M(\eta([0,1]^n))$. Let further $\mu: \eta([0,1]^n) \to [0,1]^m$ and $\nu: [0,1]^n \to \eta([0,1]^n)$ respectively be a strict and an onto $\mathbb{Z}$-map such that $\eta = \mu \circ \nu$. By Theorems 3.2 and 3.7, $M(\mu): M([0,1]^m) \to M(\eta([0,1]^n))$ is an onto unital $\ell$-homomorphism and $M(\nu): M(\eta([0,1]^n)) \to M([0,1]^n)$ is a one-unit $\ell$-homomorphism. Since $M(\eta([0,1]^n)) \cong (G, u)$, $(G, u)$ is finitely generated and is isomorphic to a subalgebra of the free unital $\ell$-group $M([0,1]^n)$. \qed

**Lemma 4.15.** Let $P \subseteq \mathbb{R}^n$ be a polyhedron. Then for some $l = 1, 2, \ldots$ there is a $\mathbb{Z}$-map $\eta$ of $[0,1]^l$ onto $P$ iff $P$ satisfies the following three conditions:

(i) $P$ is connected;

(ii) $P \cap \mathbb{Z}^n \neq \emptyset$;

(iii) $P$ is strongly regular.
Proof: \((\Rightarrow)\) If \(\eta: [0,1]^l \to P\) is an onto \(\mathbb{Z}\)-map, then \(P\) is connected because \(\eta\) is continuous. Combining Example 4.5 and Theorem 4.8, it follows that \(P\) is strongly regular. By Corollary 2.13, \(\text{den}(\eta(0,\ldots,0))\) is a divisor of \(\text{den}(0,\ldots,0)\), that is, \(\text{den}(\eta(0,\ldots,0)) = 1\). Then \(\eta(0,\ldots,0) \in \mathbb{Z}^n\).

\((\Leftarrow)\) For some suitable strongly regular collapsible triangulation \(\nabla\), we will define onto \(\mathbb{Z}\)-maps \(\eta_1: [0,1]^n \to |\nabla|\), and \(\eta_2: |\nabla| \to P\) providing the required \(\eta\).

Construction of \(\nabla\):

Let \(\Delta\) be a regular triangulation of \(P\). \(\nabla\) will be defined as the geometric realization of a weighted abstract simplicial complex \(\mathcal{W}\) arising from \(\Delta\).

Since \(P\) is connected, the simple graph \(H\) given by the 1-simplexes of \(\Delta\) is connected. As is well known, a spanning tree of \(H\) is a tree \(T \subseteq \Delta\) such that \(\text{vert}(T) = \text{vert}(\Delta) = \{v_1,\ldots,v_n\}\). By (ii), there is no loss of generality to assume

\[(4) \quad \text{den}(v_1) = 1.\]

Vertices of \(\mathcal{W}\): Let us set

\[
J = \{(i,j) \in \{1,\ldots,n\}^2 \mid i \neq j \text{ and } \text{conv}(v_i,v_j) \in \Delta\}.
\]

For each \(i \neq j\) such that \(\text{conv}(v_i,v_j) \in T\) let \(S_{i,j}\) be a maximal simplex in \(\Delta\) such that \(\text{conv}(v_i,v_j) \subseteq S_{i,j}\). Let \(K = \{(i,j,k) \in \{1,\ldots,n\}^3 \mid i \neq j, j \neq k, i \neq k \text{ and } \text{conv}(v_i,v_j,v_k) \subseteq S_{i,j}\}\). Then

\[
V = \{i,\ldots,n\} \cup J \cup K.
\]

Simplexes of \(\mathcal{W}\): For each \(i \in \{1,\ldots,n\}\), let the set \(F_i \subseteq \mathcal{P}(V)\) be defined by: \(X \in F_i\) if there are \(j_1,\ldots,j_m \in \{1,\ldots,n\}\) such that \(\text{conv}(v_i,v_{j_1},\ldots,v_{j_m}) \in \Delta\) and \(X \subseteq \{i,(i,j_1),\ldots,(i,j_m)\}\). For each \(i,j \in \{1,\ldots,n\}\) such that \(i \neq j\) and \(\text{conv}(v_i,v_j) \in T\), we further define \(B_{i,j} \subseteq \mathcal{P}(V)\) as follows: \(X \in B_{i,j}\) if \(\text{conv}(v_i,v_j,v_{k_1},\ldots,v_{k_m}) = S_{i,j}\) and \(X \subseteq \{i,j,(i,j,k_1),\ldots,(i,j,k_m)\}\). We next let

\[
\Sigma = \bigcup F_i \cup \bigcup B_{i,j}
\]

By definition, if \(X \subseteq Y\) and \(Y \in \Sigma\) then \(X \in \Sigma\). Moreover, for all \(x \in V\) there exists \(X \in \Sigma\) such that \(x \in X\). Therefore, \((V,\Sigma)\) is an abstract simplicial complex.

Weights: Finally we define \(w: V \to \{1,2,\ldots\}\) as follows:

\[
(5) \quad w(i) = \text{den}(v_i),
\]

\[
\begin{align*}
    w(i,j) &= \text{den}(v_j), \\
    w(i,j,k) &= \text{den}(v_k).
\end{align*}
\]

Claim 1: For every maximal simplex \(X\) in \(\Sigma\), the greatest common divisor of the denominators of the vertices of \(X\) is 1.

By definition, we either have

\[
X = \{i\} \cup \{(i,j_1),\ldots,(i,j_m)\} \quad \text{and} \quad \text{conv}(v_i,v_{j_1},\ldots,v_{j_m}) \text{ is maximal in } \Delta,
\]

or

\[
X = \{i,j,(i,j,k_1),\ldots,(i,j,k_m)\} \quad \text{and} \quad \text{conv}(v_i,v_j,v_{k_1},\ldots,v_{k_m}) = S_{i,j}.
\]

In either case the claim follows by definition of \(w\), because \(\Delta\) is strongly regular.

Claim 2: \(\mathcal{W}\) is collapsible.
By definition of $\mathfrak{M}$, we have:

(a) $\mathcal{F}_i \cap \mathcal{F}_j = \{\emptyset\}$ whenever $i \neq j$;
(b) $\mathcal{F}_i \cap B_{j,k} = \{\emptyset\}$ whenever $i \neq j$ and $i \neq k$;
(c) $\mathcal{F}_i \cap B_{j,k} = \{\emptyset, \{i\}\}$ whenever $i = j$ or $i = k$;
(d) $B_{i,j} \cap B_{k,l} = \{\emptyset, \{i\}\}$ whenever $i = k$ or $i = l$;
(e) $B_{i,j} \cap B_{k,l} = \{\emptyset\}$ whenever $\{i, j\} \cap \{k, l\} = \emptyset$.

The claim is now proved in 3 steps as follows:

**Step 1** (\(\mathcal{F}_i\)): For each $i \in \{1, \ldots, n\}$, \(\langle\text{vert}(\mathcal{F}_i), \mathcal{F}_i\rangle\) is combinatorially isomorphic to the closed star of $v_i$ (see [10, §III Definition 1.11]), and therefore, it is a collapsible abstract simplicial complex. Then \(\langle\text{vert}(\mathcal{F}_i), \mathcal{F}_i\rangle\) collapses to the vertex $i$. By (a-e), $\mathfrak{M}$ collapses to $(V \cup B_{i,j})$.

**Step 2** (\(B_{i,j}\)): For each $i, j \in \{1, \ldots, 2\}$ such that $i \neq j$ and $\text{conv}(v_i, v_j) \subseteq T$, the abstract simplicial complex $\langle\text{vert}(B_{i,j}), B_{i,j}\rangle$ is combinatorially isomorphic to the skeleton of the complex given by the simplex $S_{i,j}$ and its faces. Therefore, $\langle\text{vert}(B_{i,j}), B_{i,j}\rangle$ can be collapsed to any of its faces. In particular, it can be collapsed to $\{\{i, j\}, \{\emptyset, \{i\}, \{j\}, \{i, j\}\}\}$. Using (d) we see that $(V \cup B_{i,j})$ collapses to the abstract simplicial complex $(V, \Sigma')$ where $X \in \Sigma'$ iff $X \subseteq \{i, j\}$ and $\text{conv}(v_i, v_j) \subseteq T$.

**Step 3**: The sequence of collapses defined in Steps 1 and 2 leads to an abstract simplicial complex $(V, \Sigma')$ which is combinatorially isomorphic to the skeleton of $T$. Since $T$ is a tree, it is collapsible and therefore $(V, \Sigma')$ is collapsible, too.

Thus $\mathfrak{M}$ is collapsible, and Claim 2 is settled.

By (5) and (4), $w(1) = \text{den}(v_1) = 1$. From Claims 1 and 2 it follows that $P(\mathfrak{M})$ satisfies the hypotheses of Theorem 4.10. Therefore, for some integer $l > 0$ there is an onto $\mathbb{Z}$-map $\eta: [0, 1]^l \to P(\mathfrak{M})$.

Finally let $f: V \to \text{vert}(\Delta)$ be defined as follows:

\[
\begin{align*}
    f(i) &= v_i, \\
    f(i, j) &= v_j, \\
    f(i, j, k) &= v_k.
\end{align*}
\]

By (5), $\text{den}(f(x)) = w(x)$ for each $x \in V$. By definition of $\mathcal{F}_i$ and $B_{i,j}$, $f$ is a morphism from $\mathfrak{M}$ into the skeleton $\mathfrak{M}(\Delta)$ of $\Delta$. Then $P(f): P(\mathfrak{M}) \to P(\mathfrak{M}(\Delta))$ is a $\mathbb{Z}$-map. Since for every $m$-simplex $S = \text{conv}(v_{i_0}, \ldots, v_{i_m}) \in \Delta$ there is $X \in \Sigma$ (specifically, $X \in \mathcal{F}_{i_0}$) such that $f(X) = \{v_{i_0}, \ldots, v_{i_m}\}$, it follows that $P(f)$ is onto $P(W_\Delta)$.

In conclusion, $\iota^{-1}_\Delta \circ P(f) \circ \eta: [0, 1]^m \to P$ is the desired $\mathbb{Z}$-map onto $P$. \(\Box\)

**Proof of Theorem 4.13.** This immediately follows from Lemmas 4.14 and 4.15.

4.4. **Intrinsic characterization of exact unital $\ell$-groups.** In [17, Definition 2.1], *abstract Schauder basis* were defined for abelian $\ell$-groups as isomorphic copies of Schauder basis. In [17, Theorem 3.1] a characterization of abstract Schauder bases is presented. Using this characterization, in [19, Definition 4.3], the notion of abstract Schauder basis was extended to unital $\ell$-groups and called *basis*. In [19, Theorem 4.5] it is proved that an *archimedean* unital $\ell$-group $(G, u)$ is finitely presented iff it has a basis. In [6, Theorem 3.1], the archimedean assumption was
shown to be unnecessary. Using this latter result in combination with Theorem 4.13, will provide in Theorem 4.19 an algebraic description of exact unital ℓ-groups.

We first need to recall some definitions. We denote by maxspec\((G,u)\) the set of maximal ideals of \((G,u)\) equipped with the spectral topology: a basis of closed sets for maxspec\((G,u)\) is given by sets of the form \(\{p \in \text{maxspec}(G,u) \mid g \in p\}\), where \(g\) ranges over all elements of \(G\) (see \([3, \S10]\)). As is well known, maxspec\((G,u)\) is a nonempty compact Hausdorff space, \([3, \text{Theorem 10.2.2}]\).

**Definition 4.16.** \([19, \text{Definition 4.3}]\) Let \((G,u)\) be a unital ℓ-group. A basis of \((G,u)\) is a finite set \(B = \{b_1, \ldots, b_n\}\) of elements \(\neq 0\) of the positive cone \(G^+ = \{g \in G \mid g \geq 0\}\) such that

(i) \(B\) generates \(G\) using the group and lattice operations;

(ii) for each \(k = 1, 2, \ldots\) and \(k\)-element subset \(C\) of \(B\) with \(0 \neq \bigwedge\{b \mid b \in C\}\), the set \(\{m \in \text{maxspec}(G,u) \mid m \supseteq B \setminus C\}\) is homeomorphic to a \((k - 1)\)-simplex;

(iii) there are integers \(1 \leq m_1, \ldots, m_n\) such that \(\sum_{i=1}^{n} m_i b_i = u\).

**Theorem 4.17.** \([6, \text{Theorem 3.1}]\) Let \((G,u)\) be unital ℓ-group. Then the following are equivalent:

(i) \((G,u)\) is finitely presented;

(ii) \((G,u)\) has a basis.

Given a unital ℓ-group and a basis \(B = \{b_1, \ldots, b_m\}\) of \((G,u)\), let \(\mathcal{W}_B = \{B, \Sigma_B, \omega_B\}\) be the weighted abstract simplicial complex given by the following stipulations:

- \(S \in \Sigma_B\) iff \(\bigwedge S \neq 0\)
- \(\omega_B(b_i) = m_i\).

**Proposition 4.18.** Let \((G,u)\) be unital ℓ-group and \(B\) be a basis for \((G,u)\). Then 
\[G,u\] \(\cong \mathcal{M}(\mathcal{P}(\mathcal{W}_B))\).

Recall that \(\mathcal{M}(\mathcal{P}(\mathcal{W}_B))\) is the unital ℓ-group of \(\mathbb{Z}\)-maps from the canonical realization of \(\mathcal{W}_B\) into \(\mathbb{R}\).

**Proof.** This is essentially the content of the proof of \([6, \text{Theorem 3.1}]\). \(\Box\)

**Theorem 4.19.** Let \((G,u)\) be a unital ℓ-group. Then \((G,u)\) is exact iff it has a basis \(B = \{b_1, \ldots, b_n\}\) satisfying the following conditions:

(i) There is an element \(b_i \in B\), such that \(m_i = 1\);

(ii) For each maximal \(S \in \mathcal{W}_B\) the greatest common divisor of \(\{m_j \mid b_j \in S\}\) is 1;

(iii) For each \(b_i, b_j \in B\) there exist a sequence \(b_i = b_{k_1}, b_{k_2}, \ldots, b_{k_m} = b_j\), such that \(b_{k_l} \land b_{k_{l+1}} \neq 0\) for each \(l \in \{1, \ldots, m - 1\}\).

**Proof.** Immediate from Theorem 4.13, Proposition 4.18, upon noting that

- Condition (i) is equivalent to \(\mathcal{P}(\mathcal{W}_B) \cap \mathbb{Z}^n \neq \emptyset\);
- Condition (ii) is equivalent to \(\mathcal{P}(\mathcal{W}_B)\) being strongly regular;
- Condition (iii) is equivalent to \(\mathcal{P}(\mathcal{W}_B)\) being connected. \(\Box\)
Remark 1. In [18, Definition p.3], working in the framework of Abelian $\ell$-groups, the author introduced the notion of a regular set of positive elements. This definition only depends on the algebraic/combinatorial notions of starread set and 1-regularity. In [18, Lemmas 2.1 and 2.6] it is proved that, for Abelian $\ell$-groups, regular set of positive generators coincide with abstract Schauder bases. Using this result one can prove that a subset $\mathcal{B}$ of a unital $\ell$-group is a basis iff it is a regular set of positive generators satisfying condition (iii) in Definition 4.16. This leads to a reformulation of Theorem 4.19 where the exactness of a unital $\ell$-group is characterized only in terms of algebraic-combinatorial notions.

4.5. Admissible rules in Łukasiewicz infinite-valued calculus. Throughout this paper we have been going back and forth from unital $\ell$-groups, rational polyhedra and weighted abstract simplicial complexes. Using the categorical equivalence $\Gamma$ between unital $\ell$-groups and MV-algebras, the span of our paper can be further extended the algebraic counterparts of Łukasiewicz infinite-valued calculus $L_\infty$. This gives us an opportunity to discuss the underlying algorithmic and proof-theoretic aspects of the theory developed so far. We refer to [8] and [23] for background on $L_\infty$ and MV-algebras, and to [26] for background on admissible rules.

For any set $X$, we will denote $\text{FORM}_X$ to the algebra of formulas in the language $\{\top, \neg, \oplus\}$ where $\top$ is a constant $\neg$ is a unary connective and $\oplus$ is a binary connective and whose variables are in $X$. By definition, a substitution $\sigma : \text{FORM}_X \rightarrow \text{FORM}_Y$ is a homomorphism of the algebra $\text{FORM}_X$ into $\text{FORM}_Y$.

Two formulas $\psi, \varphi$ are said to be equivalent in $L_\infty$ (in symbols, $\psi \equiv_{L_\infty} \varphi$) if the equation $\psi \approx \varphi$ is valid in every MV-algebra. The algebra $\text{Free}_MV = \text{FORM}_X / \equiv_L$ is the free algebra on $X$ generators in the variety of MV-algebras.

Let $\Gamma$ be the categorical equivalence of [21, §3] between MV-algebras and unital $\ell$-groups. Then for any $n$-element set $X$, $\text{Free}_MV = \Gamma(M([0, 1]^n))$.

A rule in $L_\infty$ is a pair $(\Theta, \Sigma)$ where $\Theta \cup \Sigma$ is a finite subset of $\text{FORM}_X$ for some $X$. A rule $(\Theta, \{\varphi\})$ is derivable in $L_\infty$ if the quasi-equation $\wedge \{\psi \approx \top \mid \psi \in \Theta\} \rightarrow \varphi \approx \top$ is valid in every MV-algebra. A formula $\varphi$ is a theorem of $L_\infty$ if $(\emptyset, \{\varphi\})$ is derivable in $L_\infty$. A rule $(\Theta, \Sigma)$ is said to be admissible in $L_\infty$ if for every substitution $\sigma$ such that $\sigma(\psi)$ is a theorem of $L_\infty$ for each $\psi \in \Theta$, then there is $\varphi \in \Sigma$ such that $\sigma(\varphi)$ is a theorem of $L_\infty$.

In [15], the author provides a basis for the admissible rules of $L_\infty$. To this purpose, he introduced the notion of admissibly saturated formula. An equivalent reformulation is as follows:

Definition 4.20. [15, Definition 3.1] A formula $\varphi$ is admissibly saturated in $L_\infty$ if for every finite set $\Sigma$ of formulas the following conditions are equivalent:

(i) the rule $(\{\varphi\}, \Sigma)$ is admissible in $L_\infty$;

(ii) there exists $\psi \in \Sigma$ such that $(\{\varphi\}, \{\psi\})$ is derivable in $L_\infty$.

A formula $\varphi$ whose set of variables is $X$ is said to be exact in $L_\infty$ if there exists a substitution $\sigma : \text{FORM}_X \rightarrow \text{FORM}_Y$ such that $\sigma(\psi)$ is a theorem of $L_\infty$ if $(\{\varphi\}, \{\psi\})$ is derivable in $L_\infty$. Equivalently, $\varphi$ is exact iff there exists $Y$ such that $\text{Free}_MV / \theta(\{\varphi\} \equiv_{L_\infty}, [\top] \equiv_{L_\infty})$ is isomorphic to a subalgebra of $\text{Free}_MV$, where $\theta(\{\varphi\} \equiv_{L_\infty}, [\top] \equiv_{L_\infty})$ denotes the principal congruence generated by $(\{\varphi\} \equiv_{L_\infty}, [\top] \equiv_{L_\infty})$. Using the categorical equivalence $\Gamma$ between MV-algebras and unital $\ell$-groups, it follows that $\varphi$ is exact iff $\text{Free}_MV / \theta(\{\varphi\} \equiv_{L_\infty}, [\top] \equiv_{L_\infty})$ is isomorphic to $\Gamma(G, u)$ for some exact unital $\ell$-group.
As is well known, exact formulas are admissibly saturated. To help the reader, we supply a short proof for the special case of $L_\infty$.

Let $\varphi$ be an exact formula whose set of variables is $X$. Let $(\{\varphi\}, \Sigma)$ be an admissible rule in $L_\infty$. By Definition 4.20 we need to prove that there exists $\psi \in \Sigma$ such that $(\{\varphi\}, \{\psi\})$ is derivable in $L_\infty$. Since $\varphi$ is exact, there exist a substitution $\sigma$: $\text{FORM}_X \rightarrow \text{FORM}_Y$ such that $\sigma(\psi)$ is a theorem of $L_\infty$ and $(\{\varphi\}, \{\psi\})$ is derivable in $L_\infty$ iff $\sigma(\psi)$ is a theorem of $L_\infty$.

Since $\sigma(\varphi)$ is a theorem of $L_\infty$ and $(\{\varphi\}, \Sigma)$ is an admissible rule, there exists $\psi \in \Sigma$ such that $\sigma(\psi)$ is a theorem of $L_\infty$, i.e. $(\{\varphi\}, \{\psi\})$ is derivable in $L_\infty$. This proves that $\varphi$ is admissibly saturated.

In Theorem 4.23 we will prove that exact and admissibly saturated formulas coincide in $L_\infty$.

The following notion was introduced in [14, Definition 4.5] to study the decidability of admissible rules in Łukasiewicz infinite-valued calculus, and used in [15] to characterize admissibly saturated formulas: A set $X \subseteq [0,1]^n$ is called a anchored if for some $v_1, \ldots, v_m \in [0,1]^n$, $X = \text{conv}(v_1, \ldots, v_m)$ and the affine hull of $X$ intersects $Z^n$.

**Lemma 4.21.** Let $S \subseteq \mathbb{R}^n$ be a regular $t$-simplex, $t = 0, 1, \ldots, n$. Then the following conditions are equivalent:

1. $S$ is strongly regular;
2. $S$ is anchored;
3. $S$ is union of finitely many anchored sets.

**Proof.** The equivalence (ii)$\iff$(iii) immediately follows by definition.

To prove (i)$\iff$(ii), let $\{v_0, \ldots, v_t\}$ be the vertices of $S$. ($\Rightarrow$) There exist integers $m_0, \ldots, m_t$ such that $\sum_{i=0}^t m_i \text{ den}(v_i) = 1$. Since $\text{den}(v_i)v_i \in \mathbb{Z}^n$, the affine linear combination $\sum_{i=0}^t m_i \text{ den}(v_i)v_i$ lies in $\mathbb{Z}^n$, whence $S$ is anchored.

($\Leftarrow$) By hypothesis, there are $\lambda_0, \ldots, \lambda_t \in \mathbb{R}$ such that $v = \sum_{i=0}^t \lambda_i v_i \in \mathbb{Z}^n$ and $\sum_{i=0}^t \lambda_i = 1$. Then
\[
\bar{v} = \left(\sum_{i=0}^t \lambda_i v_i, 1\right) = \left(\sum_{i=0}^t \lambda_i v_i, \sum_{i=0}^t \lambda_i\right) = \sum_{i=0}^t \lambda_i \text{ den}(v_i)(v_i, 1) = \sum_{i=0}^t \frac{\lambda_i}{\text{den}(v_i)} \bar{v}_i.
\]

Since $S$ regular, $\frac{\lambda_i}{\text{den}(v_i)} m_i \in \mathbb{Z}$. Therefore, $\sum_{i=0}^t m_i \text{ den}(v_i) = \sum_{i=0}^t \lambda_i = 1$, whence the greatest common divisor of $\text{den}(v_0), \ldots, \text{den}(v_t)$ is 1. $\square$

**Theorem 4.22.** A rational polyhedron $P \subseteq \mathbb{R}^n$ is strongly regular iff it is a finite union of anchored sets.

**Proof.** ($\Rightarrow$) Immediate from Lemma 4.21.

($\Leftarrow$) Suppose that $P$ is such that $P = S_1 \cup \cdots \cup S_m$ for some anchored sets $S_1, \ldots, S_m \subseteq \mathbb{R}^n$.

Let $\Delta$ be a regular triangulation of $P$, and $T$ a maximal simplex in $\Delta$, with the intent of proving that $T$ is strongly regular.

Let $v \in \text{relint}(T)$. Since $T \subseteq P \subseteq S_1 \cup \cdots \cup S_m$ there exists $S_i$ such that $v \in S_i$. Lemma 4.7 yields $0 < \delta_1 \in \mathbb{R}$ such that $B(\delta_1, v) \cap P \subseteq T$. Since $v \in S_i$, there exists a point $w$ in $B(\delta_1, v) \cap \text{relint}(S_i)$, whence $w \in \text{relint}(T) \cap \text{relint}(S_i)$. 

Again by Lemma 4.7, there exists $0 < \delta_2 \in \mathbb{R}$ such that $B(\delta_2, w) \cap P \subseteq T$. By definition of the relative interior of $S_i$, there exists $0 < \delta_3 \in \mathbb{R}$ such that $B(\delta_1, w) \cap \text{aff}(S_i) \subseteq S_i$. Letting $\delta = \min\{\delta_2, \delta_3\}$ we obtain

\[ B(\delta, w) \cap \text{aff}(S_i) \subseteq B(\delta, w) \cap S_i \subseteq B(\delta, w) \cap P \subseteq T. \]

Therefore, $\text{aff}(S_i) \subseteq \text{aff}(T)$, and $T$ is anchored. Since $T$ is a regular simplex, by Lemma 4.21, $T$ is strongly regular, whence so is $P$. \hfill \Box

**Theorem 4.23.** Let $\varphi$ be a formula in the language of MV-algebras. Then the following are equivalent

(i) $\varphi$ is admissibly saturated;

(ii) $\varphi$ is exact.

**Proof.** A combination of Theorems 4.13 and 4.22 with [15, Theorem 3.5], again using the categorical equivalence $\Gamma$ of [21, Theorem 3.9]. \hfill \Box

**Acknowledgements.** The author would like to thank Professor Daniele Mundici for his helpful comments and suggestions on previous drafts of this paper.

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