Berry phase of the Tavis-Cummings model with three modes of oscillation

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Abstract

In this paper we develop a general method to obtain the Berry phase of time-dependent Hamiltonians with a linear structure given in terms of the $SU(1,1)$ and $SU(2)$ groups. This method is based on the similarity transformations of the displacement operator performed to the generators of each group, and let us diagonalize these Hamiltonians. Then, we introduce a trilinear form of the Tavis-Cummings model to compute the $SU(1,1)$ and $SU(2)$ Berry phases of this model.

1 Introduction

The Jaynes-Cummings model is the simplest and completely soluble quantum-mechanical model which describes the interaction between radiation and matter [1]. The exact solution of this theoretical model has been found in the rotating wave approximation [2]. However, despite the simplicity of the Jaynes-Cummings model, it presents interesting quantum phenomena [3–10], all of them being experimentally corroborated [11–13].

The Tavis-Cummings model is another important model in quantum optics, which emerged from the study of $N$ identical two-level molecules interacting through a dipole coupling with a single-mode quantized radiation field at resonance [14,15]. This model has been studied in terms of the Holstein-Primakoff transformation [16], quantum inverse methods [17, 18], and polynomially deformed $su(2)$ algebras [19]. In general, the Jaynes-Cummings model and the Tavis-Cummings model are still under study as can be seen in the references [20–29].

On the other hand, since its introduction in 1984 the Berry phase [30] has been extensively studied in several quantum systems [31–34]. This is a phase factor gained by the wavefunction after the system is transported through a closed path via adiabatic variation of parameters. The aim of the present work is to compute the Berry phase of the Tavis-Cummings model with three modes of oscillation in terms of the $SU(1,1)$ and $SU(2)$ group theory.

This work is organized as follows. In Section 2, we introduce a general method to diagonalize a Hamiltonian with a linear $su(1,1)$ and $su(2)$ linear structure by means of the tilting transformation of each group. In Section 3, we compute the Berry phase for time-dependent Hamiltonians with a linear structure given in terms of the $SU(1,1)$ and $SU(2)$ groups. With all these previous results, Section 4 is dedicated to obtain the $SU(1,1)$ and $SU(2)$ Berry phases of the Tavis-Cummings model with three modes of oscillation.

2 Algebraic diagonalization method of a system with an $SU(1,1)$ or $SU(2)$ symmetry

As it is well known, the group algebraic methods are very powerful tools in the description, diagonalization as well as understanding of the nature of the physical structure of many problems with certain dynamical symmetry. In this Section, we outline an algebraic procedure for diagonalizing certain Hamiltonians of physical systems that can
be described through a linear structure of the Lie algebra, that is, Hamiltonians that can be written as a linear combination of the generators \( \{ N_i, A_q, A_q^\dagger \} \) of the \( G \) Lie algebra, i.e.

\[
H = \sum_i a_i N_i + \sum_q (b_q A_q + c_q A_q^\dagger),
\]

(1)

where \( a_i, b_q \) and \( c_q \) can be real or complex constants. The generators \( \{ N_i, A_q, A_q^\dagger \} \) form the standard Cartan-Weyl basis of semi-simple \( G \) Lie algebra satisfying the commutation relations \([N_i, N_j] = 0, [N_i, A_q] = q_i A_q, [A_q, A_q^\dagger] = q_i^\prime N_i, [A_p, A_q] = C_{p,q} A_{p+q} \) \((p \neq q)\). Some examples of this algebra can be constructed by combining the bosonic realizations of the generators \( su(1,1) \) and \( su(2) \) Lie algebras or by the bosonic generators of the \( Sp(4,R) \) algebra.

In general, based on a \( G \) Lie algebra with its respective generators \( \{ N_i, A_q, A_q^\dagger \} \) which satisfy certain commutation relations, we can introduce a unitary operator

\[
D(\xi_q) = \exp \left[ \sum_q \left( \xi_q A_q - \xi_q^* A_q^\dagger \right) \right].
\]

(2)

Here, \( D(\xi_q) \) is a generalized displacement operator and \( \xi_q \) are complex parameters to be determined. Now, by using the Baker-Campbell-Hausdorff identity

\[
e^{-A} Be^A = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!} [[[B, A], A], A] + ...,\]

and a suitable choice of the complex parameters \( \xi_q \), the operator \( D(\xi_q) \) allows us to compute the transformation of the Lie algebra generators \( \{ N_i, A_q, A_q^\dagger \} \) as a linear combination of them. Hence, a generator \( A_j \) of the Lie algebra can be transformed as

\[
D^\dagger(\xi_q) A_j D(\xi_q) = \sum_i \lambda_{ij}(\xi_q) A_i,
\]

(3)

where each coefficient \( \lambda_{ij} \) is required to be an analytical function of the complex parameters \( \xi_q \). From this, we have that a Hamiltonian with a semi-simple Lie algebra structure as in equation (1) can be transformed by \( D(\xi_q) \) to

\[
D^\dagger(\xi_q) H D(\xi_q) = \sum_i \Lambda_i(\xi, a, b, c) N_i + \sum_q \left( F_q(\xi, a, b, c) A_q + M_q(\xi, a, b, c) A_q^\dagger \right).
\]

(4)

Since the operators \( N_i \) commute, the Hamiltonian of equation (1) can be diagonalized by this transformation if we impose

\[
F_q(\xi, a, b, c) = 0, \quad M_q(\xi, a, b, c) = 0.
\]

(5)

The above expressions are a set of equations of the parameters \( \xi_i \) and the physical constants \( a_i, b_q \) and \( c_q \) of the Hamiltonian. The solution of the system of equations reduces the expression (4) to

\[
D^\dagger(\xi) HD(\xi) = \sum_i \Lambda_i(\xi, a, b, c) N_i.
\]

(6)

Therefore, if the eigenfunctions \( \Phi_n \) of the operator \( N_i \) \((N_i \Phi_n = \alpha_i \Phi_n)\) are known we have from equation (6) that

\[
HD(\xi_q) \Phi_n = \sum_i \Lambda_i(\xi, a, b, c) \alpha_i D(\xi_q) \Phi_n,
\]

(7)

\[
H \Phi_n^{(q)} = \Omega(\xi, a, b, c) \alpha_i \Phi_n^{(q)},
\]

(8)

where \( \Phi_n^{(q)} = D(\xi_q) \Phi_n \) and \( \Omega(\xi, a, b, c) = \sum_i \Lambda_i(\xi, a, b, c) \alpha_i \) are respectively the eigenfunctions and eigenvalues of the Hamiltonian of equation (1). It is important to point out that the diagonalization of the Hamiltonian \( H \) depends on whether the system of equations (6) have a solution.
2.1 Diagonalization of Hamiltonians with a linear $su(1,1)$ and $su(2)$ algebraic structure

During the last decade much attention has been given to studying different quantum optical models. As a result, it has been found that a wide variety of these models are expressed in terms of bosons and fermions or matrix-differential equations. By choosing an appropriate realization, many of these models can be put in the context of the $su(1,1)$ and $su(2)$ Lie algebras, as it is shown in the references [38, 39]. In this Section, as an simple but useful application of the theory developed earlier, we are going to diagonalize two Hamiltonians with a simple structure given in terms of the $su(1,1)$ and $su(2)$ Lie algebras. Thus, we introduce two Hamiltonians written as a linear combination of the generators $\{K_\pm, K_0\}$ of $SU(1,1)$ group and the generators $\{J_\pm, K_0\}$ of $SU(2)$ group as

$$H_{su(1,1)} = a_0 K_0 + a_1 K_+ + a_2 K_-,$$

$$H_{su(2)} = b_0 J_0 + b_1 J_+ + b_2 J_-.$$  \hspace{1cm} (9)

The generators of each group satisfy the commutation relations \[38\]

$$[K_0, K_\pm] = \pm K_\pm,$$

$$[K_-, K_+] = 2K_0,$$

$$[J_0, J_\pm] = \pm J_\pm,$$

$$[J_+, J_-] = 2J_0.$$ \hspace{1cm} (10)

The displacement operators $D(\xi_1)$ and $D(\xi_2)$ for these algebras are defined in terms of the creation and annihilation operators $\{K_+, K_\pm\}$ and $\{J_+, J_-\}$ as

$$D(\xi_1)_{su(1,1)} = \exp(\xi_1 K_+ - \xi_1^* K_-), \hspace{1cm} D(\xi_2)_{su(2)} = \exp(\xi_2 J_+ - \xi_2^* J_-),$$ \hspace{1cm} (12)

where $\xi_i = -\frac{1}{2} \tau_i e^{-i\varphi_i}$, $-\infty < \tau_i < \infty$ and $0 \leq \varphi_i \leq 2\pi$. With these operators we can transform the generators of the $su(1,1)$ and $su(2)$ Lie algebras as $K'_i = D(\xi_i) K_i D(\xi_i)$ and $J'_i = D(\xi_i) J_i D(\xi_i)$. Explicitly we obtain

$$K'_+ = \frac{\xi_1}{|\xi_1|} \alpha K_0 + (\beta + 1) K_+ + \frac{\xi_1^*}{|\xi_1|} \beta K_-,$$

$$K'_- = \frac{\xi_1}{|\xi_1|} \alpha K_0 + (\beta + 1) K_- + \frac{\xi_1^*}{|\xi_1|} \beta K_+,$$

$$K'_0 = (2\beta + 1) K_0 + \frac{\alpha \xi_1}{2|\xi_1|} K_+ + \frac{\alpha \xi_1^*}{2|\xi_1|} K_-,$$ \hspace{1cm} (13)

$$J'_+ = -\frac{\xi_2}{|\xi_2|} \delta J_0 + (\epsilon + 1) J_+ + \frac{\xi_2^*}{|\xi_2|} J_-,$$

$$J'_- = -\frac{\xi_2}{|\xi_2|} \delta J_0 + (\epsilon + 1) J_- + \frac{\xi_2^*}{|\xi_2|} J_+,$$

$$J'_0 = (2\epsilon + 1) J_0 + \frac{\delta \xi_2}{2|\xi_2|} J_+ + \frac{\delta \xi_2^*}{2|\xi_2|} J_-.$$ \hspace{1cm} (14)

where $\alpha = \sinh(2|\xi_1|)$, $\beta = \frac{1}{2} [\cosh(2|\xi_1|) - 1]$, $\delta = \sin(2|\xi_2|)$ and $\epsilon = \frac{1}{2} [\cos(2|\xi_2|) - 1]$.

Therefore, the Hamiltonians of equation \[38\] are transformed in terms of the displacement operators $D(\xi_1)$ and $D(\xi_2)$ to

$$H'_{su(1,1)} = A_0(\xi_1) K_0 + A_1(\xi_1) K_+ + A_2(\xi_1) K_-,$$

$$H'_{su(2)} = B_0(\xi_2) J_0 + B_1(\xi_2) J_+ + B_2(\xi_2) J_-.$$ \hspace{1cm} (16)

where new $A$’s and $B$’s constants are given by

$$A_0 = (2\beta + 1) a_0 + \frac{\xi_1 a_1}{|\xi_1|} + \frac{\xi_1^* a_2}{|\xi_1|}, \hspace{1cm} B_0 = (2\epsilon + 1) b_0 + \frac{\xi_2 b_1}{|\xi_2|} - \frac{\xi_2^* b_2}{|\xi_2|},$$ \hspace{1cm} (17)

$$A_1 = 2\alpha a_0 + (\beta + 1) a_1 + \frac{\xi_1^* \beta a_2}{|\xi_1|}, \hspace{1cm} B_1 = \frac{\xi_2^* \beta b_0}{|\xi_2|} - (\epsilon + 1) b_1 + \frac{\xi_2 b_2}{|\xi_2|},$$ \hspace{1cm} (18)

$$A_2 = \frac{\xi_1}{2|\xi_1|} a_0 + \frac{\xi_1^* \beta a_1}{|\xi_1|} + (\beta + 1) a_2, \hspace{1cm} B_2 = \frac{\xi_2^* \beta b_1}{|\xi_2|} - (\epsilon + 1) b_1 + \frac{\xi_2 b_2}{|\xi_2|}.$$ \hspace{1cm} (19)

The generators $K_\pm$ and $J_\pm$ can be removed from the Hamiltonians $H'_{su(1,1)}$ and $H'_{su(2)}$ if we impose that coefficients $A_1 = 0$, $A_2 = 0$, $B_1 = 0$, and $B_2 = 0$. From this condition we need to solve the following system of equations

$$\frac{\xi_1}{2|\xi_1|} a_0 + (\beta + 1) a_1 + \frac{\xi_1^* \beta a_2}{|\xi_1|} = 0,$$ \hspace{1cm} (20)

$$\frac{\xi_1^*}{2|\xi_1|} a_0 + \frac{\xi_1^* \beta a_1}{|\xi_1|} + (\beta + 1) a_2 = 0.$$ \hspace{1cm} (21)
and
\[\frac{\xi}{2|\xi|} \delta b_0 + (\epsilon + 1)b_1 + \frac{\xi}{\xi_2} c b_2 = 0,\]
(22)
\[\frac{\xi_2^2}{2|\xi|} \delta b_0 + \frac{\xi_2^2}{\xi} c b_1 + (\epsilon + 1)b_2 = 0.\]
(23)

Therefore, by choosing the coherent parameters \(\tau_1\) and \(\varphi_1\) of the complex numbers \(\xi_i = -\frac{i}{2}e^{-i\varphi_i}\) as
\[\tau_1 = \tanh^{-1} \left( \frac{2\sqrt{a_2}}{a_0} \right), \quad \varphi_1 = \frac{i}{2} \ln \left( \frac{a_1}{a_2} \right),\]
(24)
and
\[\tau_2 = \arctan \left( \frac{2\sqrt{b_2}}{b_0} \right), \quad \varphi_2 = \frac{i}{2} \ln \left( \frac{b_1}{b_2} \right),\]
(25)
we diagonalize the Hamiltonians of equation (16) to obtain
\[H'_{su(1,1)} = \sqrt{a_0^2 - 4a_1a_2K_0}, \quad H'_{su(2)} = \sqrt{a_0^2 + 4a_1a_2J_0}\]
(26)
Finally, if the eigenfunctions \(\Phi_n^{(1)}\) and \(\Phi_n^{(2)}\) of \(K_0\) and \(J_0\) are known, we have found the eigenvalues of the Hamiltonians of equation (16). The eigenfunctions are obtained by applying the operators \(D(\xi)\) on the functions \(\Phi_n^{(i)}\). Hence, the eigenfunctions of the Hamiltonian \(H_{su(1,1)}\) are given by the \(SU(1, 1)\) Perelomov number coherent states.

Similarly, the eigenfunctions of the Hamiltonian \(H_{su(2)}\) are given by the \(SU(2)\) Perelomov number coherent states.

3 Time-dependent Hamiltonians with a linear \(su(1, 1)\) or \(su(2)\) Lie algebraic structure

In this Section, we shall consider systems whose Hamiltonian \(H(t)\) is an explicit function of time and has a linear \(su(1, 1)\) or \(su(2)\) Lie algebraic structure, i.e.,
\[H(t)_{su(1,1)} = a_0(t)K_0 + a_1(t)K_1 + a_2(t)K_2, \quad H(t)_{su(2)} = b_0(t)J_0 + b_1(t)J_1 + b_2(t)J_2.\]
(29)
Now, because the Hamiltonians \(H(t)_{su(1,1)}\) and \(H(t)_{su(2)}\) are Hermitian operators, we have \(a_2(t) = a_2^*(t)\), \(b_2(t) = b_2^*(t)\). Moreover, we can write the coefficients \(a_1(t)\) and \(b_1(t)\) as
\[a_1(t) = \lambda_1(t)e^{i\varphi_1(t)}, \quad b_1(t) = \lambda_2(t)e^{i\varphi_2(t)}\]
(30)
where \( \lambda_i(t) \) and \( \varphi_i(t) \) are arbitrary real functions of time.

Since the Hamiltonians are time-dependent, in describing quantum dynamics we shall use the Schrödinger picture in which state vectors depend explicitly on time, but operators do not

\[
\frac{\text{i} \hbar}{\text{d}t} |\psi(t)\rangle = H(t)|\psi(t)\rangle.
\]  

(31)

Thus, in order to study the time evolution of the states of Hamiltonians \( H(t)_{\text{su}(1,1)} \) and \( H(t)_{\text{su}(2)} \), we will use the time-dependent nontrivial invariant Hermitian operator \( I(t) \) \[40\][41], which satisfies the conditions

\[
\text{i} \frac{\partial}{\partial t} I(t) + [I(t), H(t)] = 0.
\]

(32)

By using the \( SU(1,1) \) and \( SU(2) \) time-dependent displacement operators \( D(\xi_i(t)) \) given by the expressions \[12\], with \( \xi_i(t) = -\frac{1}{2}\theta_i(t)e^{-\gamma_i(t)} \) and where now \( \theta_i(t) \) and \( \gamma_i(t) \) are arbitrary real functions of time, we can define the invariant operators \( I(t)_{\text{su}(1,1)} \) and \( I(t)_{\text{su}(2)} \) as (see reference \[42\])

\[
I(t)_{\text{su}(1,1)} = D(\xi_1(t))K_0D^\dagger(\xi_1(t)), \quad I(t)_{\text{su}(2)} = D(\xi_2(t))J_0D^\dagger(\xi_2(t)),
\]

(33)

or explicitly as

\[
I(t)_{\text{su}(1,1)} = \cosh(\theta_1)K_0 + \frac{\sinh(\theta_1)}{2} e^{-\gamma_1} K_+ + \frac{\sinh(\theta_1)}{2} e^{\gamma_1} K_-,
\]

(34)

and

\[
I(t)_{\text{su}(2)} = \cos(\theta_2)J_0 + \frac{\sin(\theta_2)}{2} e^{-\gamma_2} J_+ + \frac{\sin(\theta_2)}{2} e^{\gamma_2} J_-.
\]

(35)

From the condition of equation \[52\] and the invariant operators \( I(t)_{\text{su}(1,1)} \) and \( I(t)_{\text{su}(2)} \), the time-dependent physical parameters \( b_0(t) \), \( a_0(t) \), \( \lambda_i(t) \) and \( \varphi_i(t) \) are related to parameters \( \theta_i(t) \) and \( \gamma_i(t) \) as follows

\[
\dot{\theta}_1 = -2\lambda_1 \sin(\varphi_1 + \gamma_1), \quad \gamma_1 - a_0 \sinh(\theta_1) = -2\lambda_1 \cosh(\theta_1) \cos(\varphi_1 + \gamma_1),
\]

(36)

\[
\dot{\theta}_2 = -2\lambda_2 \sin(\varphi_2 + \gamma_2), \quad \gamma_2 - b_0 \sin(\theta_2) = -2\lambda_2 \cosh(\theta_2) \cos(\varphi_2 + \gamma_2).
\]

(37)

On the other hand, the transformations of the generators \( \{K_0, K_\pm\} \) and \( \{J_0, J_\pm\} \) under its respective time-dependent displacement operators \( D(\xi(t)) \) remain unchanged and are given by the expressions \[15\]. In addition, by using of the BCH identity (equation \[2\]) the operator \( \text{i} \frac{\partial}{\partial t} \) is transformed under the \( SU(1,1) \) time-dependent displacement operators \( D(\xi_1) \) as

\[
D^\dagger_1(t) \left( \text{i} \frac{\partial}{\partial t} \right) D_1(t) = \frac{\gamma_1}{2} \left( \gamma_1 \sinh(\theta_1) + i\dot{\theta}_1 \right) K_+ + \frac{\gamma_1}{2} \left( \gamma_1 \sinh(\theta_1) - i\dot{\theta}_1 \right) K_-.
\]

(38)

Analogously, the transformation of \( \text{i} \frac{\partial}{\partial t} \) under the \( SU(2) \) time-dependent displacement operator \( D(\xi_2) \) is given by

\[
D^\dagger_2(t) \left( \text{i} \frac{\partial}{\partial t} \right) D_2(t) = \frac{\gamma_2}{2} \left( \gamma_2 \sin(\theta_2) + i\dot{\theta}_2 \right) J_+ - \frac{\gamma_2}{2} \left( \gamma_2 \sin(\theta_2) - i\dot{\theta}_2 \right) J_-.
\]

(39)

As it is shown in reference \[41\], if the eigenstates of the invariant operator satisfy the Schrödinger equation its eigenvalues are real. Therefore, given that \( K_0|k, n\rangle = |k + n\rangle|k, n\rangle \) and \( J_0|j, \mu\rangle = |j, \mu\rangle \) we have

\[
D(\xi_1)K_0|k, n\rangle = (k + n)D(\xi_1)|k, n\rangle, \quad D(\xi_2)J_0|j, \mu\rangle = \mu D(\xi_2)|j, \mu\rangle,
\]

\[
I(t)_{\text{su}(1,1)}D(\xi_1)|k, n\rangle = (k + n)D(\xi_1)|k, n\rangle, \quad I(t)_{\text{su}(2)}D(\xi_2)|j, \mu\rangle = \mu D(\xi_2)|j, \mu\rangle.
\]

Thus, the invariant operators \( I(t)_{\text{su}(1,1)} \) and \( I(t)_{\text{su}(2)} \) have as eigenstates, the states \( D(\xi_1)|k, n\rangle = |\zeta_1(t), k, n\rangle \) and \( D(\xi_2)|j, \mu\rangle = |\zeta_2(t), j, \mu\rangle \) respectively, which are the \( SU(1,1) \) and \( SU(2) \) Perelomov number coherent states. These states are given by the expressions \[27\] and \[28\] but now these are functions of time through the parameters

\[
\zeta_1(t) = -\text{tanh} \left( \frac{\theta_1(t)}{2} \right) e^{-\gamma_1(t)}, \quad \eta_1 = \ln(1 - |\zeta_1(t)|^2),
\]

(40)
Finally, the eigenvalues of $I(t)_{su(1,1)}$ are $(k + n)$ and the eigenvalues of $I(t)_{su(2)}$ are $\mu$.

Moreover, if the states $|\psi(t)\rangle_{su(1,1)}$ and $|\psi(t)\rangle_{su(2)}$ satisfy the relationship for the Hamiltonians $H(t)_{su(1,1)}$ and $H(t)_{su(2)}$, these states can be expanded through the states $|\zeta_1(t), k, n\rangle$ and $|\zeta_2(t), j, \mu\rangle$ in the form

$$|\psi(t)\rangle_{su(1,1)} = \sum_n a_n e^{i\alpha_n^{(1)}} |\zeta_1(t), k, n\rangle, \quad |\psi(t)\rangle_{su(2)} = \sum_j a_j e^{i\alpha_j^{(2)}} |\zeta_2(t), j, \mu\rangle,$$  \hspace{1cm} (42)

where according to reference [41] the phase $\alpha$ is given as

$$\alpha = \int_0^t dt' \langle \lambda, \kappa | \frac{\partial}{\partial t'} - H(t') | \lambda, \kappa \rangle.$$  \hspace{1cm} (43)

Here, $|\lambda, \kappa\rangle$ are the eigenstates and $\lambda$ are the eigenvalues of the invariant operator $I(t)$. Therefore, the phase of the eigenstate $|\zeta_1(t), k, n\rangle$ and $|\zeta_2(t), j, \mu\rangle$ in a non-adiabatic process is given by

$$\alpha_n^{(1)} = (n + k) \int_0^t [(\gamma_1 + a_0)(\cosh(\theta_1) - 1) + 2\lambda_1 \cos(\gamma_1 + \varphi_1) \sin(\theta_1) - a_0] dt', \quad (44)$$

$$\alpha_n^{(2)} = \mu \int_0^t [\gamma_2 - b_0)(\cos(\theta_2) - 1) - 2\lambda_2 \cos(\gamma_2 + \varphi_2) \sin(\theta_2) - a_0] dt'. \quad (45)$$

Unlike in a non-adiabatic process, in an adiabatic process we have that $\dot{\theta}_1 = \gamma_1 = 0$ and so the expressions (46) and (47) become

$$\varphi_1 + \gamma_1 = n\pi, \quad \tanh(\theta_1) = \frac{2\lambda_1}{a_0} \cos(n\pi), \quad (46)$$

$$\varphi_2 + \gamma_2 = n\pi, \quad \tan(\theta_2) = \frac{2\lambda_2}{b_0} \cos(n\pi). \quad (47)$$

If we set $n = 1$, the above conditions are reduced to the time-dependent versions of the expressions (48) and (49). Therefore, in an adiabatic process the phase of the states $|\zeta_1(t), k, n\rangle$ and $|\zeta_2(t), j, \mu\rangle$ are reduced to

$$\alpha_n^{(1)} = -(n + k) \int_0^t \sqrt{a_0(t')^2 - 4\lambda_1^2(t')} dt', \quad (48)$$

$$\alpha_n^{(2)} = -\mu \int_0^t \sqrt{a_0^2(t') + 4\lambda_2^2(t')} dt'. \quad (49)$$

These are known as the dynamical phases and are defined as

$$\dot{\epsilon}_n = \langle \lambda, \kappa | H(t') | \lambda, \kappa \rangle,$$  \hspace{1cm} (50)

while the Berry phase is written as

$$\dot{\gamma}_n = i \langle \lambda, \kappa | \frac{\partial}{\partial t} | \lambda, \kappa \rangle.$$  \hspace{1cm} (51)

Thus, the Berry phases of the states $|\zeta_1(t), k, n\rangle$ and $|\zeta_2(t), j, \mu\rangle$ are obtained in the adiabatic limit as follows

$$\gamma_n^{(1)}(T) = (n + k) \oint (\cosh(\theta_1) - 1)d\varphi_1, \quad (52)$$

$$\gamma_n^{(2)}(T) = \mu \oint (\cos(\theta_2) - 1)d\varphi_2, \quad (53)$$

where $T$ denotes the period. It is obvious that in these cases the Berry phases do not depend on an explicit form of the functions $\varphi_1(t)$ or $\varphi_2(t)$.
In this Section, we shall consider a trilinear time-dependent Hamiltonian which vary slowly in the time. This Hamiltonian can be considered as the model of Tucker and Walls [43,44] where only three modes interact with each other [45,47]. This Hamiltonian has the form (with $\hbar = 1$)

$$
\hat{H}(t) = \omega_1(t)\hat{a}^\dagger \hat{a} + \omega_2(t)\hat{b}^\dagger \hat{b} + \omega_3(t)\hat{c}^\dagger \hat{c} + \lambda(t)(\hat{a}^\dagger \hat{b} \hat{c} e^{-i\varphi(t)} + \hat{a} \hat{b}^\dagger \hat{c} e^{i\varphi(t)}) ,
$$

(54)

where the set of operators $\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger$ and $\hat{c}, \hat{c}^\dagger$ satisfy the bosonic algebra $[a, a^\dagger] = [b, b^\dagger] = [c, c^\dagger] = 1$. Moreover, $\omega_j(t)$ with $j = 1, 2, 3, \lambda(t)$ and $\varphi(t)$ are physical time-dependent constants that vary slowly.

In what follows it will be convenient to use the bosonic $su(1,1)$ and $su(2)$ Lie algebras realizations

$$
\begin{align*}
K_0 &= \frac{1}{2} \left( \hat{b}^\dagger \hat{b} + \hat{c}^\dagger \hat{c} + 1 \right) , & K_+ &= \hat{b}^\dagger \hat{c}^\dagger & K_- &= \hat{b} \hat{c} , & N_d &= \hat{b}^\dagger \hat{b} - \hat{c}^\dagger \hat{c} , \\
\hat{J}_0 &= \frac{1}{2} \left( \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \right) , & \hat{J}_+ &= \hat{a}^\dagger \hat{b} , & \hat{J}_- &= \hat{b}^\dagger \hat{a} & \hat{N}_s &= \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} ,
\end{align*}
$$

(55, 56)

such that these operators satisfy the commutation relations $[10]$ and $[11]$. Thus, in terms of these operators the Hamiltonian $[54]$ can be rewritten in the following forms

$$
\begin{align*}
\hat{H}(t)_{su(1,1)} &= \omega_1(t)\hat{a}^\dagger \hat{a} + (\omega_2(t) + \omega_3(t))K_0 + \lambda(t)(\hat{a}^\dagger K_- e^{-i\varphi(t)} + \hat{a}K_+ e^{i\varphi(t)}) + \frac{\omega_2(t) - \omega_3(t)}{2}N_d - \frac{\omega_2(t) + \omega_3(t)}{2} , \\
\hat{H}(t)_{su(2)} &= \omega_3(t)\hat{c}^\dagger \hat{c} + (\omega_2(t) - \omega_1(t))\hat{J}_0 + \lambda(t)(\hat{c} \hat{J}_+ e^{-i\varphi(t)} + \hat{c}^\dagger \hat{J}_- e^{i\varphi(t)}) + \frac{\omega_1(t) + \omega_2(t)}{2}\hat{N}_s .
\end{align*}
$$

(57, 58)

Therefore, the Hamiltonian $[54]$ possesses the $SU(1,1)$ and $SU(2)$ symmetries. By considering the $SU(1,1)$ Hamiltonian of $[57]$ and using the theory developed in Section 3, we have that the exact solution of the Schrödinger equation is obtained in terms of the eigenstates of the invariant operator $\hat{I}(t)_{su(1,1)} = D(\xi)K_0D^\dagger(\xi)$

$$
|\psi(t)\rangle_{su(1,1)} = \sum_n a_n e^{i\alpha_n^{(1)}(t)}|\zeta_1(t), k, n\rangle \otimes |n_a\rangle ,
$$

(59)

In this expression, $|n_a\rangle$ are the states of the one-dimensional harmonic oscillator and the states $|\zeta_1(t), k, n\rangle = D(\xi)|k, n\rangle$ are given in terms of the quantum numbers $n_l$ and $m_n$ as

$$
\begin{align*}
\psi_{\zeta_1, n_l, m_n} &= \sqrt{\frac{2\Gamma(n_l+1)}{\Gamma(n_l + n_l + 1)}} (-1)^n \sqrt{\frac{n_l!}{\pi}} e^{in_\phi} \left( \frac{(\zeta_1)^n}{(1 - |\zeta_1|^2)^{\frac{n_l + 1}{2}}} \right) \times e^{-\frac{\sigma^2}{2}} \rho^{m_n} L_{m_n}^{n|_{\|m=1}} \left( \frac{\rho^2 \sigma}{(1 - |\zeta_1|^2)} \right) ,
\end{align*}
$$

(60)

where $\zeta_1(t) = -\tanh(\frac{\theta(t)}{2})e^{-i\gamma_1(t)}$ and $\sigma(t)$ is defined as

$$
\sigma(t) = \frac{1 - |\zeta_1(t)|^2}{(1 - |\zeta_1(t)|^2)(-\zeta_1(t))} .
$$

On the other hand, the total phase $\alpha_n^{(1)}(t)$ is given by the expression $[43]$, which consist of the usual dynamical phase $\epsilon_n^{(1)}(t)$ and the Berry phase $\gamma^{(1)}(T)_n$. Thus, the dynamical phase of the $SU(1,1)$ Tavis-Cummings model is given by

$$
\epsilon_n^{(1)}(t) = -\int_0^t \langle n_a| \otimes \langle \zeta_1(t), k, n| H_{su(1,1)} |\zeta_1(t), k, n\rangle \otimes |n_a\rangle dt' ,
$$

(61)

$$
= \int_0^t \langle n_a| \otimes \langle k, n| H'_{su(1,1)} |k, n\rangle \otimes |n_a\rangle dt' ,
$$

(62)
where \( H'_{su(1,1)} = D^\dagger H_{su(1,1)} D \). Taking into account the adiabatic limit and by using the relationship \( \hat{a}^\dagger \frac{1}{\hat{a}} = 1 - |0\rangle \langle 0| \), the equations become

\[
\tanh(\theta_1(t)) = \frac{2\lambda(t)\sqrt{\hat{a}^\dagger \hat{a}}}{\omega_2(t) + \omega_3(t)}, \quad \gamma_1(t) = \frac{i}{2} \ln \left[ \frac{\hat{a}}{\hat{a}^\dagger} \right] - \varphi(t),
\]

which are the expression of the coherent state parameters in the reference [28]. Now, if we impose that \( n_a \gg 1 \) and use the following relationship

\[
\frac{1}{\hat{a}^\dagger} |n_a \rangle = (1 - \delta_{n_a,0}) \frac{1}{\sqrt{n_a}} |n_a + 1 \rangle,
\]

we have that

\[
(\tanh(\theta_1(t)))_a = \frac{2\lambda(t)\sqrt{n_a}}{\omega_2(t) + \omega_3(t)}, \quad (\gamma_1(t))_a = -\varphi(t).
\]

Therefore, the dynamical phase of the \( SU(1,1) \) Tavis-Cummings model in the adiabatic limit is

\[
\epsilon_a^{(1)}(t) = - \int_0^t E(t')_{su(1,1)} dt',
\]

where explicitly the term \( E(t)_{su(1,1)} \) is given by

\[
E(t)_{su(1,1)} = \sqrt{(\omega_2(t) + \omega_3(t))^2 - 4\lambda(t)^2 n_a} \left( n_l + \frac{m_n}{2} + \frac{1}{2} \right) + \omega_1(t)n_a + \frac{\omega_3(t) - \omega_2(t)}{2} m_n - \frac{\omega_2(t) + \omega_3(t)}{2}.
\]

In the same way, from expression (52) we obtain for the Berry phase of the \( SU(1,1) \) Tavis-Cummings model in the adiabatic limit the following result

\[
\gamma_{n_l}^{(1)}(T) = \left( n_l + \frac{m_n}{2} + \frac{1}{2} \right) (n_a |(\cos(\theta_1) - 1)|n_a) \int d\varphi,
\]

\[
\gamma_{n_l, n_n, m_n}^{(1)}(T) = (2n_l + m_n + 1) \frac{(\omega_2(t) + \omega_3(t)) - \sqrt{(\omega_2(t) + \omega_3(t))^2 - 4\lambda(t)^2 n_a}}{\sqrt{(\omega_2(t) + \omega_3(t))^2 - 4\lambda(t)^2 n_a}}.
\]

Now, form the \( SU(2) \) Tavis-Cummings model the invariant operator of the Hamiltonian \( H(t)_{su(2)} \) is given by

\[
\hat{I}(t)_{su(2)} = D(\xi_2)J_0 D^\dagger(\xi_2).
\]

The general solution of the Schrödinger equation is given by the eigenstates of the invariant operator \( \hat{I}_{su(2)} \)

\[
|\psi(t)\rangle_{su(2)} = \sum_j a_j e^{i\zeta_j^{(2)}} |\zeta_2(t), j, \mu \rangle \otimes |n_c \rangle,
\]

where \( |n_c \rangle \) are the states of the one-dimensional harmonic oscillator for the oscillation mode \( \hat{c} \) and \( |\zeta_2(t), j, \mu \rangle \) are the \( SU(2) \) Perelomov number coherent states for the two-dimensional harmonic oscillator. The states \( |\zeta_2(t), j, \mu \rangle \) in the coordinate space are given in terms of the quantum numbers \( n_l \) and \( m_n \) by

\[
|\zeta_2(t)\rangle_{n_l, m_n} = \frac{e^{-\frac{1}{4} (n_l + n_n)^2}}{\sqrt{\pi}} \sum_{s=0}^{n_l+n_n} \sum_{n=0}^{\frac{(n_l+n_n)}{2}} \frac{\left(-\zeta_2^{(2)}\right)^n}{n!} e^{\frac{1}{2} (m_n-2n) e^{i(m_n-2n+2s) \varphi} (-1)^n} \frac{\Gamma(n_l + n_n + 1)}{\Gamma(n_l + n_n + s + 1)} \rho^{(m_n-2n+2s)} L_{n_l+n+n-s}^{(m_n-2n+2s)} (\rho^2),
\]

where \( \zeta_2(t) = -\tan(\frac{\varphi_2(t)}{2}) e^{-i\gamma_2(t)} \).

On the other hand, the total phase \( \alpha_j^{(2)} \) is split into the dynamical phase \( \epsilon_j^{(2)} \) and the Berry phase \( \gamma_j^{(2)}(T) \). From the relationship \( \hat{c}^\dagger \frac{1}{\hat{c}} = 1 - |0\rangle \langle 0| \), we obtain that in the adiabatic limit the equations become

\[
\tan(\theta_2(t)) = \frac{2\lambda(t)\sqrt{\hat{c}^\dagger \hat{c}}}{\omega_2(t) - \omega_1(t)}, \quad \gamma_2(t) = \frac{i}{2} \ln \left[ \frac{\hat{c}}{\hat{c}^\dagger} \right] + \varphi.
\]
If we consider that \( n_a \gg 1 \) and use the relationship

\[
\frac{1}{c t} |n_c\rangle = (1 - \delta_{n_c,0}) \frac{1}{\sqrt{n}} |n_c + 1\rangle,
\]

we obtain that \( \langle \gamma_2(t) \rangle_c = \varphi(t) \). Therefore, the dynamical phase for the \( SU(2) \) Tavis-Cummings model is given by

\[
\epsilon^{(2)}_j(t) = -\int_0^t E(t')_{su(2)} dt',
\]

where explicitly the term \( E(t)_{su(2)} \) is

\[
E_{su(2)} = \frac{1}{2} \sqrt{\left(\omega_2(t) - \omega_1(t)\right)^2 + 4\lambda^2(t)n_c m_n} + \left(\omega_2(t) + \omega_1(t)\right)(n_l + \frac{m_n}{2}) + \omega_3(t)n_c.
\]

Finally, the Berry phase \( \gamma^{(2)}(T)_j \) of the \( SU(2) \) Tavis-Cummings model in the adiabatic limit is given by

\[
\gamma^{(2)}_j(T) = \frac{(m_n)}{2} \langle n_c | (\cos(\theta_2) - 1) | n_c \rangle \oint d\varphi,
\]

\[
\gamma^{(2)}_{m_n,n_c}(T) = \left(\pi m_n\right) \frac{\left(\omega_2(t) - \omega_1(t)\right) - \sqrt{\left(\omega_2(t) - \omega_1(t)\right)^2 + 4\lambda(t)^2n_c}}{\sqrt{\left(\omega_2(t) - \omega_3(t)\right)^2 + 4\lambda(t)^2n_c}}.
\]

5 Concluding remarks

In this paper we first introduced a method to diagonalize Hamiltonians with an \( SU(1,1) \) and \( SU(2) \) linear algebraic structure. For this method, we applied the similarity transformations of the displacement operator to the generators of each group. With these transformations we were able to obtain in a general way the Berry phase of time-dependent Hamiltonians with an \( SU(1,1) \) and \( SU(2) \) linear structure. Then, we introduced a trilinear time-dependent form of the Tavis-Cummings model, which possesses the \( SU(1,1) \) and \( SU(2) \) symmetry. Therefore, all the previous results allowed us to compute the \( SU(1,1) \) and \( SU(2) \) Berry phases for the Tavis-Cummings model with three modes of oscillation.

It is important to note that the method developed in this work not only allowed us to calculate the Berry phase exactly, but also allowed us to obtain the dynamical phase of the Tavis-Cummings model for each symmetry. Moreover, the general method developed to obtain the Berry phase and the dynamical phase can be applied to more Hamiltonians who have these symmetries.

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References

[1] E.T. Jaynes, and F.W. Cummings, Proc. IEEE 51, 89 (1963).
[2] S. Haroche, and J.M. Raimond, Exploring the Quantum: Atoms, Cavities and Photons, Oxford University Press, Oxford, 2007.
[3] J.H. Eberly, N.B. Narozhny, and J.J. Sanchez-Mondragon, Phys. Rev. Lett. 44, 1323 (1980).
[4] J.R. Kukliński, and J. Madajczyk, Phys. Rev. A 37, 3175 (1988).
[5] R. Short, and L. Mandel, Phys. Rev. Lett. 51, 384 (1983).
[6] F. Diedrich, and H. Walther, Phys. Rev. Lett. 58, 203 (1987).
[7] P.W. Milonni, J.R. Ackerhalt, and H.W. Galbraith, Phys. Rev. Lett. 50, 966 (1983).
[8] J.J. Slosser, P. Meystre, and S.L. Braunstein, Phys. Rev. Lett. 63, 934 (1989).
[9] J. Gea-Banacloche, Phys. Rev. Lett. 65, 3385 (1990).
[10] S.J.D. Phoenix, and P.L. Knight, Phys. Rev. Lett. 66, 2833 (1991).
[11] P. Goy, J.M. Raimond, M. Gross, and S. Haroche, Phys. Rev. Lett. 50, 1903 (1983).
[12] M. Brune et. al., Phys. Rev. Lett. 76, 1800 (1996).
[13] C. Guerlin et. al., Nature 448, 889 (2007).
[14] R.H. Dicke, Phys. Rev. 93, 99 (1954).
[15] M. Tavis, and F.W. Cummings, Phys. Rev. 170, 379 (1968).
[16] M.A. Bashir, and M.S. Abdalla, Phys. Lett. A 204, 21 (1995).
[17] N.M. Bogoliubov, R.K. Bullough, and J. Timonen, J. Phys. A: Math. Gen. 29, 6305 (1996).
[18] A. Rybin, G. Kastelewicz, J. Timonen, and N.M. Bogoliubov, J. Phys. A: Math. Gen. 31, 4705 (1998).
[19] I.P. Vadeiko, G.P. Miroshnichenko, A.V. Rybin, and J. Timonen, Phys. Rev. A 67, 053808 (2003).
[20] L. Lamata, J. León, T. Schätz, E. Solano, Phys. Rev. Lett. 98, 253005 (2007).
[21] R. Gerritsma et al., Nature (London) 463, 68 (2010).
[22] L. Lamata et al., New J. Phys. 13, 095003 (2011).
[23] A. Retzker, E. Solano, and B. Reznik, Phys. Rev. A 75, 022312 (2007).
[24] W. Kopylov et al., Phys. Rev. A 92, 063832 (2015).
[25] C. Sun, and N. Sinitsyn, Phys. Rev. A 94, 033808 (2016).
[26] D. Ojeda-Guillén, R.D. Mota, V.D. Granados, J. Math. Phys. 57, 062104 (2016).
[27] E. Choreño, D. Ojeda-Guillén, M. Salazar-Ramírez, and V.D. Granados, Ann. Phys. 387, 121 (2017).
[28] E. Choreño, D. Ojeda-Guillén, and V.D. Granados, J. Math. Phys. 59, 073506 (2018).
[29] E. Choreño, D. Ojeda-Guillén, and V.D. Granados, Eur. Phys. J. D 72, 142 (2018).
[30] M.V. Berry, Proc. R. Soc. Lond. A 392, 45 (1984).
[31] C.C. Gerry, Phys. Rev. A 39, 3204 (1989).
[32] B.-H. Xie, S. Jin, W.-X. Yan, S.-Q. Duan, and X.-G. Zhao, Eur. Phys. J. D 30, 411 (2004).
[33] S.-P. Bu, G.-F. Zhang, J. Liu, and Z.-Y. Chen, Phys. Scr. 78, 065008 (2008).
[34] A. Thilagam, J. Phys. A: Math. Theor. 43, 354004 (2010).
[35] R. Koços, M. Koca, and H. Tütüncüiler, J. Phys. A: Math. Gen 35, 9425 (2002).
[36] R. Koços, O. Özer, H. Tütüncüiler, and R.G. Yıldırım, Eur. Phys. J. B 59, 375 (2007).
[37] R. Koços, H. Tütüncüiler, M. Koca, and E. Olğar, Ann. Phys. 319, 333 (2005).
[38] A. Vourdas, Phys. Rev. A 41, 1653 (1990).
[39] D. Ojeda-Guillén, M. Salazar-Ramírez, R.D. Mota, and V.D. Granados, J. Nonlinear Math. Phys. 23, 607 (2016).
[40] H.R. Lewis Jr, J. Math. Phys. 9, 1976 (1968).

[41] H.R. Lewis Jr, and W.B. Riesenfeld, J. Math. Phys. 10, 1458 (1969).

[42] Y.-Z. Lai, J.-Q. Liang, H.J.W. Müller-Kirsten, and J.-G. Zhou, Phys. Rev. A 53, 3691 (1996).

[43] J. Tucker, and D.F. Walls, Phys. Rev. 178, 2036 (1969).

[44] J. Tucker, and D.F. Walls, Ann. Phys. NY 52, 1 (1969).

[45] E.A. Mishkin, and D.F. Walls, Phys. Rev. 185, 1618 (1969).

[46] G.P. Agrawal, and C.L. Mehta, J. Phys. A: Math. Gen. 7, 607 (1974).

[47] M.S. Abdalla, E.M. Khalil, A.S.-F. Obada, J. Peřina, and J. Křepelka, AIP Advances 7, 015013 (2017).