I. INTRODUCTION

Traditional views on Hall effect have undergone dramatic changes over the past several decades, most prominently thanks to the observation of quantized Hall effect in two-dimensional electronic systems and subsequent realization that it is the band topology, rather than the magnetic field itself, that determines the Hall response of electronic systems [1,2]. It became manifest over the years, both theoretically and experimentally, that even non-electronic systems support Hall-like transport of their elementary excitations such as photons [3], phonons [4–5], magnons [6–14], and triplons [15] due to the topological character in their respective band structures or the emergent magnetic field governing their dynamics. More recently, there is growing experimental evidence of Hall-like heat (thermal) transport in magnetic materials that remain in paramagnetic, spin-liquid-like phases [16–20]. The physical picture regarding the origin of Hall-like phenomena for such correlated paramagnetic insulators remains poorly understood, as the Berry curvature effect only pertains to the band picture of weakly interacting quasiparticles. Schwinger-boson mean-field approximation was introduced in Refs. [20,21] as a way to partly address the Hall effect in the paramagnetic phase. Magnetic materials exhibiting the thermal Hall effect are typically frustrated, with the pyrochlore or the kagome lattice structure [7,8,16–20] responsible for the geometric frustration, or possess significant amount of Kitaev-type interaction leading to the emergence of novel Majorana excitation [19].

With this background, the recent observation of significant thermal Hall signal in the family of cuprate compounds comes as a surprise [22]. A few salient features of the experiment may be summed up: first, the undoped antiferromagnetically ordered compound La$_2$CuO$_4$ exhibits large thermal Hall conductivity $\kappa_{xy}$ in the absence of electronic charge carriers; phonon effects are ruled out by the authors; secondly, $\kappa_{xy}$ is reduced in magnitude as doping increases, and even undergoes a sign change at some finite temperature, reflecting a mixed contribution of electronic and magnetic origins upon doping. For underdoped (and presumably undoped) La$_{2-x}$Sr$_x$CuO$_4$, the Hall effect is almost linear in the applied magnetic field $B$. Magnons, on the other hand, must have an energy gap increasing with $B$ and lead to the suppressed Hall effect at larger $B$ field. A general picture thus emerging is that the underdoped antiferromagnetic compound might have some non-trivial magnetic correlations, which are presumably gapless and revealed by the applied magnetic field through the transverse heat conduction.

What are the quasiparticles responsible for the observed transverse heat conductivity? First of all, the magnon in the experimental system has a sizable gap [22]. Second, even assuming this gap to be small, we expect the gap to grow with magnetic field, whereas the thermal Hall effect initially increases with applied field. There are other objections arising from purely theoretical consideration, such as the “no-go” theorem [6], disfavoring the formation of topological Hall effect in un-frustrated square-lattice magnets. A way round this “theorem” was invented recently [23], by adopting a model complicated enough to break spatial symmetries of the square lattice; such models do not seem to apply readily to cuprates, though. Despite these objections, we categorically look into the magnon-based scenario and add various tweaks to it, with the hope that one such model might capture...
the thermal Hall phenomenology. In conclusion, as we report in Secs. 2 and 3, the answer is negative; hardly any magnon-based scenario is likely to account for the thermal Hall effect in the square-lattice antiferromagnet. In Sec. 4 we outline a completely different scenario based on the spinon picture of magnetic excitation. Treating spin excitations in terms of fractionalized fermions known as spinons is an old idea, dating back to Anderson’s RVB (Resonating Valence Bond) proposal. The task of applying the spinon idea to work out magnetic excitations in the cuprates was taken up in the past, notably in Refs. [24] and [25]. We show that a small modification of their models explicitly breaks time reversal symmetry and has net spin chirality spontaneously generated. These models will not have thermal Hall effect that is linear in magnetic field and generally speaking hysteresis may be expected.

II. MAGNON THEORY OF THERMAL HALL EFFECT IN SQUARE-LATTICE ANTI FERROMAGNET

We begin by (re-)visiting the well-known microscopic spin Hamiltonian of the cuprates

\[ H = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \sum_{\langle ij \rangle} \mathbf{D}_{i,j} \cdot \mathbf{S}_i \times \mathbf{S}_j - \mathbf{B} \cdot \sum_i \mathbf{S}_i. \]  

(2.1)

In addition to the familiar spin exchange \( J \), we allow the Dzyaloshinskii-Moriya (DM) interaction, originating from the small buckling of the oxygen atom out of the CuO₂ plane [27] [28], and the Zeeman interaction. The DM vectors as dictated by symmetry consideration were first worked out by Coffey et al. [29]: \( \mathbf{D}_{i,j} = \sqrt{2D}(-1)^i(\cos \theta_j \sin \theta_i) \), and \( \mathbf{D}_{i,j} = -\sqrt{2D}(-1)^i(\sin \theta_j \cos \theta_i, 0) \). The ordered spins are forced to lie in the CuO₂ plane due to the DM interaction, with a small out-of-plane ferromagnetic component also dictated by the same interaction. The mean-field ansatz can be chosen as

\[ \langle \mathbf{S}_i \rangle = n_0 \hat{z} - n_1(-1)^i \hat{a}. \]  

(2.2)

It proves convenient to work with a new pair of orthonormal axes \( \hat{a} = (1, 1)/\sqrt{2} \), \( \hat{b} = (-1, 1)/\sqrt{2} \) instead of \( \hat{x}, \hat{y} \) axes which extend along Cu-Cu directions. An orthonormal triad is formed by \( \hat{a} \times \hat{b} = \hat{z} \). The mean-field energy comes out as

\[ E = 2J(n_0^2 - n_1^2) - 4D(\cos \theta_d - \sin \theta_d)n_0n_1 - Bn_0. \]  

(2.3)

The Zeeman energy scale at \( B = 10T \) is only a meV, whereas the DM energy may be several meV in the cuprates. As a result, the canting angle \( \theta_c \) defined as \( (n_0, n_1) = (\sin \theta_c, \cos \theta_c) \) is dictated by the ratio \( D/J \), and not so much by the Zeeman field. Minimizing the energy \( E \) with respect to the canting angle \( \theta_c \) gives

\[ \tan 2\theta_c = (D/J)(\cos \theta_d - \sin \theta_d) \]  

(2.4)

at \( B = 0 \). The sign of the DM energy \( D \) and the angle \( \theta_d \) are chosen in such a way that the canting angle is positive, \( \theta_c > 0 \).

Next we introduce a general formalism that allows one to convert the spin Hamiltonian (2.1) to a magnon Hamiltonian, defined around a mean-field ground state given in (2.2). In doing so, we aim to see if the magnon theory or some of its variant can account for the thermal Hall phenomena in the undoped square-lattice antiferromagnet. The method is based on parameterizing the spin operator \( \mathbf{S}_i \) as

\[ \mathbf{S}_i = a_i \mathbf{n}_i + \mathbf{t}_i, \]  

(2.5)

where \( a_i \) refers to the amplitude reduction along the direction of the classical ground state spin \( \mathbf{n}_i \), due to the transverse fluctuation \( \mathbf{t}_i \). The well-known Holstein-Primakoff (HP) substitution follows from the formula

\[ a_i = S - b_i^\dagger b_i, \quad \mathbf{t}_i = t_i^\theta \mathbf{\theta}_i + t_i^\phi \mathbf{\phi}_i, \]  

(2.6)

where

\[ t_i^\theta = \sqrt{\frac{S}{2}}(b_i^\dagger + b_i), \quad t_i^\phi = \frac{i}{\sqrt{2}}(b_i^\dagger - b_i) \]  

(2.7)

and \( \mathbf{\theta}_i \) and \( \mathbf{\phi}_i \) are a pair of orthonormal vectors forming the local triad \( \mathbf{\theta}_i \times \mathbf{\phi}_i = \mathbf{n}_i \). For this choice of triad we are guaranteed the transversality condition \( \mathbf{t}_i \cdot \mathbf{n}_i = 0 \).

Substituting (2.5) and the rest of the HP formulas into the spin Hamiltonian gives the magnon Hamiltonian,
The magnon Hamiltonian in real space becomes

$$
H = \sum_{ij} [J \theta_i \cdot \theta_j + D_{ij} \cdot \theta_i \times \theta_j] t^\theta_i t^\theta_j + \sum_{ij} [J \phi_i \cdot \phi_j + D_{ij} \cdot \phi_i \times \phi_j] t^\phi_i t^\phi_j
+ \sum_{ij} [J \phi_i \cdot \phi_j + D_{ij} \cdot \phi_i \times \phi_j] t^\phi_i t^\phi_j - \sum_i \mu_i b_i^\dagger b_i,
$$

(2.8)

where $\mu_i = J \sum_j \mathbf{n}_i \cdot \mathbf{n}_j + \sum_j D_{ij} \cdot \mathbf{n}_i \times \mathbf{n}_j - \mathbf{B} \cdot \mathbf{n}_i$.

The spin size $S = 1/2$ can be absorbed by various redefinitions of the physical constants and will not be shown from now. Our notation is such that $(ij)$ refers to the nearest-neighbor (NN) bond, and $j \in i$ refers to the summation over the (four) NN sites $j$ that surround the site $i$. The mean-field spin configuration was already laid out in (2.2), and we need to complete the orthonormal triad as

$$
\mathbf{n}_i = -n_1(-1)^i \hat{a} + n_0 \hat{z},
\theta_i = n_1(-1)^i \hat{z} + n_0 \hat{a},
\phi_i = \hat{b}.
$$

(2.9)

This choice of parametrizing the triad is convenient because several terms in the Hamiltonian (3.8) vanish automatically: $\theta_i \times \theta_j = 0$, $\theta_i \cdot \phi_j = \phi_i \cdot \theta_j = 0$. Remaining terms are $\theta_i \cdot \theta_j = -1$, $\phi_i \cdot \phi_j = \cos 2\theta_k = -\mathbf{n}_i \cdot \mathbf{n}_j$, $D_{ij} \cdot \phi_i \times \phi_j = J \sin 2\theta_k \tan 2\theta_k$ for both $i = j + x$ and $j = i + y$, $D_{ij} \cdot \theta_i \times \theta_j = D_{ij} \cdot \phi_i \times \theta_j = \pm n_1 D \cos \theta_k \sin \theta_k$.

The magnon Hamiltonian in real space becomes

$$
H = (4J'S + Bn_0) \sum_i b_i^\dagger b_i - J \sum_{ij} t^\theta_i t^\theta_j + J' \sum_{ij} t^\phi_i t^\phi_j
+ D' \sum_i (t^\theta_i t^\theta_j + t^\phi_i t^\phi_j) - D'' \sum_i (t^\theta_i t^\theta_j + t^\phi_i t^\phi_j),
$$

(2.10)

Using abbreviations $X = \cos k_x, Y = \cos k_y$, we obtain the magnon energy spectrum

$$
E_k = \sqrt{A_k^2 - |B_k|^2} = \sqrt{[4J' - 2J(X + Y) + Bn_0][2J'(2 + X + Y) + Bn_0] - (2D')^2(X - Y)^2}.
$$

(2.13)

The spectrum has two local minima, at $k = 0$ and $k = Q = (\pi, \pi)$, with the minimum energy at $k = 0$ given by

$$
E_0 = \sqrt{(4(J' - J) + Bn_0)[8J' + Bn_0]}.
$$

(2.14)

It is governed by the larger of the DM energy $J' - J \sim D^2/J$ and the Zeeman energy $Bn_0$. Spin-rotation invariance of the Hamiltonian is completely lost due to the DM vector, and one sees a magnon gap of order $D^2/J$ even in the absence of Zeeman field.

The magnon spectrum derived from the Hamiltonian (2.1) is well-known [29], but little attention has been paid to the magnon eigenstates and the associated Berry curvature. The magnon eigenstate is given in the spinor form

$$
|\psi_k⟩ = \begin{pmatrix} \cosh \theta_k / 2 \\ -e^{-i\phi_k} \sinh \theta_k / 2 \end{pmatrix}, \quad e^{i\phi_k} = \frac{B_k}{|B_k|},
\cosh \frac{\theta_k}{2} = \sqrt{\frac{A_k}{2E_k} + \frac{1}{2}} \sinh \frac{\theta_k}{2} = \sqrt{\frac{A_k - 1}{2}}.
$$

(2.15)

Transformation to the quasiparticle operator $γ_k$ is implemented by the formula

$$
b_k = \cosh \frac{\theta_k}{2} γ_k - e^{i\phi_k} \sinh \frac{\theta_k}{2} γ_−_k.
$$

(2.16)

The Berry curvature of the magnon band can be calcu-
lated exactly [9] as \((\partial_\mu = \partial/\partial k_\mu, \sigma_3=\text{Pauli matrix})\)

\[ B_k = i(\partial_x \psi_k | \sigma_3 | \partial_y \psi_k) - i(\partial_y \psi_k | \sigma_3 | \partial_x \psi_k) = \frac{1}{2} \sin \theta_k (\partial_x \psi_k \partial_y \theta_k - \partial_y \psi_k \partial_x \theta_k) = 2D'(J' + J)(4J' + Bn_0) \sin k_x \sin k_y. \] (2.17)

The proportionality \(B_k \propto D'\) implies that the Berry curvature is possible only by the DM interaction. In the vicinity of \(k = 0\), one can write approximately

\[ B_k \approx \frac{16D'J^2}{E_0^3} k_x k_y, \] (2.18)

which highlights the \(d_{xy}\) character in the curvature function.

The thermal Hall conductivity \(\kappa_{xy}\) is deduced from the Berry curvature through the formula developed by Murakami and collaborators [9]

\[ \frac{\kappa_{xy}}{T} = -\frac{k_B^2}{h} \int \frac{d^2k}{(2\pi)^2} \left( c_2(E_k) - \frac{\pi^2}{3} \right) B_k \] (2.19)

where \(c_2(E_k)\) is some generalized Bose-Einstein distribution function of magnons. We find \(\kappa_{xy} = 0\) by symmetry of the integral in (2.19). Specifically, \(B_k\) changes sign under either \(k_x \to -k_x\) or \(k_y \to -k_y\), but \(E_k\) does not.

### III. LOCAL DEFECT SCENARIO

#### A. Local spirals

In a series of papers, Sushkov and collaborators have argued that holes introduced by doping Sr atom at the La site, for instance, get localized and distort the local spin configuration into a spiral with the wavevector \(K = \sqrt{2}x(\pi, -\pi)\) for a given doping concentration \(0 < x \lesssim 0.055\) \([10, 22]\). For \(0.055 \lesssim x \lesssim 0.12\) the \(K\) vector is directed along the crystallographic axis in accordance with the stripe scenario: \(K = 2\pi(x, \pm \pi, 0)\).

Inspired by this proposal, we generalize the ground state triad [2.9] to incorporate the spiral structure by writing

\[ n_i = n_1(-1)^i \hat{a}_i - n_0 \hat{z}, \quad \theta_i = (-1)^i \hat{b}_i \]
\[ \phi_i = n_0(-1)^i \hat{a}_i + n_1 \hat{z}, \] (3.1)

where the local orthonormal vectors \(\hat{a}_i\) and \(\hat{b}_i\) are now position-dependent:

\[ \hat{a}_i = \hat{a} \cos \theta_i + \hat{b} \sin \theta_i \]
\[ \hat{b}_i = \hat{b} \cos \theta_i - \hat{a} \sin \theta_i. \] (3.2)

Having \(\theta_i = 0\) irrespective of the site \(i\) corresponds to the magnetic order considered previously. Having \(\theta_i = K \cdot r_i\) with \(|K| \ll 1\) corresponds to the uniform spiral of slow modulation. Sushkov’s scenario corresponds to having a finite rotation angle \(\theta_i\) only in the vicinity of the impurity site. We will first consider the uniform spiral and the effect it has on the magnon Hall effect. Local spiral scenario will be considered subsequently.

There is an immediate consequence of having a finite spiral rotation angle \(\theta_i\). The inner product \(\psi_i \cdot \phi_j\) and \(\phi_i \cdot \theta_j\), previously equal to zero in the general magnon Hamiltonian [6,8], is now finite:

\[ \psi_i \cdot \phi_j = n_0 \sin(\theta_i - \theta_j) = -\phi_i \cdot \theta_j. \] (3.3)

Note that this term is nonzero only if the uniform moment \(n_0\) is present simultaneously. As a consequence, the Hamiltonian matrix \(H^0_k\) in (2.11) and (2.12) is modified to \(H^0_k + H^1_k\), where

\[ H^1_k = -2Jn_0(\sin K_x \sin k_x + \sin K_y \sin k_y). \] (3.4)

This new piece of Hamiltonian creates a simple shift in the magnon spectrum \(E_k \to E_k + \delta E_k\)

\[ \delta E_k = -2Jn_0(\sin K_x \sin k_x + \sin K_y \sin k_y). \] (3.5)

This is reminiscent of the Doppler shift; magnons whose momentum is parallel (anti-parallel) to \(K = (K_x, K_y)\) experience a red-shift (blue-shift) in energy.

Meanwhile, the magnon wave function [2.15] and the Berry curvature [2.17] obtained earlier remain unchanged. In particular the various energy factors in the wave function and the Berry curvature are still those of the unperturbed Hamiltonian, maintaining the symmetries \(E(k_x, -k_y) = E(-k_x, k_y) = E(k_x, k_y)\). The new magnon energy \(E_k + \delta E_k\) enters solely through the distribution function \(c_2(E_k + \delta E_k)\) of the thermal Hall conductivity formula (2.19), which undergoes correction

\[ \delta \kappa_{xy} \propto \sum_k \frac{\partial c_2(E_k)}{\partial E_k} B_k \delta E_k \]
\[ \propto \sum_k \frac{\partial c_2(E_k)}{\partial E_k} B_k (\sin K_x \sin k_x + \sin K_y \sin k_y) = 0. \] (3.6)

The first two terms in the sum, \(\frac{\partial c_2}{\partial E_k}\) and \(B_k\), are even under the change \(k \to -k\), while \(\delta E_k\) is odd. As a result, the sum must be zero. The uniform spiral state fails to produce Hall effect.

Akin to the original Sushkov proposal, we now look into the influence of localized spirals on the thermal Hall transport of magnons. First of all, we lay down some general strategy for attacking such problem. The continuum language is more appropriate for dealing with problems that break translation symmetry, and we begin with a continuum form of the Hamiltonian \(H_1\) introduced in [3.4]:

\[ H_1 = iJn_0 \sum_{(ij)} \sin(\theta_i - \theta_j)(\hat{b}_j^\dagger \hat{b}_i - \hat{b}_j \hat{b}_i^\dagger) \]
\[ \to iJn_0 \int_r \nabla \theta \cdot (\hat{b} \nabla \hat{b} - \hat{b} \nabla \hat{b}^\dagger). \] (3.7)
Born scattering calculation based on this Hamiltonian

\[ H_{1} = -iJS\sum_{k,p} p \cdot (p + 2k)\theta_{p}b_{k+p}^{\dagger}b_{k}, \]

(3.9)

where \( \theta_{p} \) is the Fourier transform of the real-space \( \theta \). The Born scattering amplitude \( \langle k+p|H_{1}|k \rangle \) is proportional to the factor \( p \cdot (p + 2k) = (k+p)^{2} - k^{2} \). Under the continuum approximation, however, the quasiparticle energy \( E_{k} \) is a quadratic function of \( k \) (see Eq. 2.13 for the full energy dispersion). Elastic scattering process satisfies \( E_{k+p} = E_{k} \), hence \( (k+p)^{2} - k^{2} = 0 \). The Born scattering amplitude vanishes. Higher-order contributions from \( H_{1} \) involves higher powers of the uniform moment \( n_{0} \) and are expected to give negligible contribution.

Upon expanding to one higher order in the phase gradient, we do find an additional correction in the form

\[ H_{2} = \frac{1}{4}JS\sum_{r}(\nabla\theta)^{2}(b^{\dagger} - b)^{2}. \]

(3.10)

Born scattering calculation based on this Hamiltonian also gives negative results for the magnon Hall effect. Details are not illuminating and omitted from readership.

### B. Local skyrmions

Speculations of skyrmion formulation in the cuprates have been around for a long time \([33,35]\) and revived recently with the report of their sightings in a member of the cuprate family \( \text{La}_{2}\text{Cu}_{0.37}\text{Li}_{0.63}\text{O}_{4} \) \([36]\). It has been well-established in the recent skyrmion literature that a magnon sees a localized skyrmion as two units of flux quanta \([10,37,40]\), and will experience Aharonov-Bohm scattering. We examine whether such scenario can apply to the antiferromagnetic skyrmions, assuming they do form localized defects in the underdoped or undoped cuprates.

In a nutshell, an antiferromagnetic skyrmion \textit{per se} does not give rise to magnon Hall effect, although the ferromagnetic skyrmion does. The difference can be outlined most simply in the continuum field theory of magnons for each case. For ferromagnetic model we switch \( J \rightarrow -J \) in the magnon Hamiltonian and treat \( \mathbf{n}, \theta, \phi \) as smooth functions of the coordinates, as there is no staggered component in any of them. Continuum limit of the magnon Hamiltonian with \( D_{ij} = 0 \) and \( B = 0 \) is easily obtained as

\[ H_{\text{FM}} \sim -\frac{J}{2}\sum_{\mu} b^{\dagger}[\partial_{\mu} - ia_{\mu}]b + \cdots \]

(3.11)

where the curl of the vector potential \( \partial_{x}a_{y} - \partial_{y}a_{x} = (2\pi)^{-1}\mathbf{n} \cdot (\partial_{x}\mathbf{n} \times \partial_{y}\mathbf{n}) \) represents the local skyrmion density. Integral of the curl \( (\nabla \times \mathbf{n})_{z} \) is -2 for a skyrmion charge -1. This is the basis of the claim that the local skyrmion magnetic structure acts as two units of flux quanta for the magnons. The magnon Hall effect due to skyrmion has been observed experimentally in ferromagnetic thin films \([12]\).

A very different effective theory of magnons is found for antiferromagnetic ground states. The smooth texture is realized for the \textit{staggered magnetization}, so the ground state triad is parameterized as

\[ \mathbf{n}_{i} \rightarrow (-1)^{i}\mathbf{n}_{i}^{s}, \quad \theta_{i} \rightarrow (-1)^{i}\theta_{i}^{s}, \quad \phi_{i} \rightarrow \phi_{i}. \]

(3.12)

Both \( \mathbf{n} \) and \( \theta \) are staggered but not \( \phi \), which is defined as the cross product \( \phi_{i} = \mathbf{n}_{i} \times \mathbf{\theta}_{i} \). Now treating \( \mathbf{n}_{i}^{s} \) and \( \theta_{i}^{s} \) as smooth, we obtain the continuum magnon Hamiltonian

\[ H_{\text{AFM}} = \frac{J}{2}\sum_{\mu}[(\partial_{\mu}b)^{2} + (\partial_{\mu}b^{\dagger})^{2}] + \cdots . \]

(3.13)

Various other terms proportional to \( b^{2} \), \( (b^{\dagger})^{2} \) and \( bb^{\dagger} \) are not shown. Crucially, there is no analogue of the covariant derivative \( \partial_{\mu} - ia_{\mu} \) in this theory and no source of emergent magnetic field. Magnon Hall effect originating from skyrmion spin texture must be absent in the antiferromagnetic ground state.

\[ \theta_{0i} = -n_{0i}\theta_{1i}, \quad \phi_{i} = n_{1i} \times \theta_{1i}, \]

\[ \theta_{1i} = n_{0i}\theta_{1i}, \quad \phi_{i} = n_{1i} \times \theta_{1i}. \]

**FIG. 1.** Coordinate system used in developing the magnon dynamics of ferrimagnetic ground state

As we saw earlier, however, undoped cuprate is weakly ferrimagnetic, due to the DM interaction and the consequent canting of spins. Since ferrimagnet has characters of both ferromagnet and antiferromagnet, we find it worth exploring possible low-energy magnon dynamics for a ferrimagnetic spin-textured ground state. To this end one needs a more elaborate setup for treating
magnon dynamics by allowing the triad of orthonormal vectors \((\mathbf{n}_0, \mathbf{\theta}_0, \phi_0)\) to carry both staggered and uniform components locally:

\[
\begin{align*}
\mathbf{n}_i &= (-1)^i \mathbf{n}_{i1} + n_0 \mathbf{\theta}_{i1}, \\
\mathbf{\theta}_i &= (-1)^i \mathbf{\theta}_{i1} - n_0 \mathbf{n}_{i1}, \\
\phi_i &= \mathbf{n}_{i1} \times \mathbf{\theta}_{i1},
\end{align*}
\]

(3.14)

Words of explanation are in order for this choice of parametrization. The uniform moment \(n_0\) is by assumption orthogonal to the staggered moment \(\mathbf{n}_{i1}\). The staggered component of \(\mathbf{\theta}_i\), denoted \(\mathbf{\theta}_{i1}\), is also orthogonal to \(\mathbf{n}_{i1}\). Since both \(\mathbf{n}_0\) and \(\mathbf{\theta}_0\) are required to be orthogonal to \(\mathbf{n}_{i1}\), and there is a \text{U}(1) degree of freedom in choosing the orthonormal vector \(\mathbf{\theta}_{i1}\), we invoke this freedom to choose \(\mathbf{\theta}_{i1}\) to be parallel to \(\mathbf{n}_{i1}\), or write \(\mathbf{n}_{i1} = n_0 \mathbf{\theta}_{i1}\). This explains the parametrization of \(\mathbf{n}_i\) in the first line of (3.14). The second line for \(\mathbf{\theta}_i\) follows naturally from requiring \(\mathbf{n}_i \cdot \mathbf{\theta}_i = 0\). Orthogonality of all three vectors in (3.14) is ensured up to first order in the small moment \(n_0\). The parameterization we propose is summed up pictorially in Fig. 1.

Substituting (3.14) into the general magnon Hamiltonian \((3.8)\) yields terms, linear in \(n_0\),

\[
\begin{align*}
\mathbf{\theta}_i \cdot \mathbf{\phi}_j &\approx -n_0 \mathbf{n}_{i1} \cdot (\mathbf{n}_{j1} \times \mathbf{\theta}_{j1}) \rightarrow -n_0 \cdot (\mathbf{n}_1 \times \partial_\mu \mathbf{n}_1), \\
\mathbf{\phi}_i \cdot \mathbf{\theta}_j &\approx -n_0 \mathbf{n}_{i1} \times \mathbf{\theta}_{j1} \cdot \mathbf{n}_{j1} \rightarrow n_0 \cdot (\mathbf{n}_1 \times \partial_\mu \mathbf{n}_1).
\end{align*}
\]

(3.15)

In arriving at the expressions at the far right we assumed continuum approximation and introduced \(\partial_\mu \) for the spatial derivative in the direction \(j = i + \mu\). A new contribution to the magnon dynamics arises from

\[
\begin{align*}
H_1 &= iJ \sum_{\mu = x,y} \int dx dy \left( \mathbf{n}_0 \cdot \mathbf{n}_1 \times \partial_\mu \mathbf{n}_1 \right) (b^\dagger \partial_\mu b - b \partial_\mu b^\dagger) \\
&= J \sum_{\mu = x,y} \int dx dy \mathbf{a} \cdot \mathbf{j},
\end{align*}
\]

(3.16)

where the vector potential \(\mathbf{a}\) and the magnon current density \(\mathbf{j}\) are defined by \(a_\mu = -n_0 \mathbf{n}_1 \times \partial_\mu \mathbf{n}_1\), and \(j_\mu = -i(b^\dagger \partial_\mu b - b \partial_\mu b^\dagger)\), respectively.

For non-textured ground state, the uniform and staggered moments are related by \(\mathbf{n}_0 = n_0 \mathbf{\theta}_0 \mathbf{n}_1\) through the DM interaction. If we assume that this relation continues to hold even for the textured spin configuration such as that of a skyrmion, it turns out one can write the vector potential in a much simpler form: \(\mathbf{a} = -n_0 \nabla (b \cdot \mathbf{n}_1)\). In this case, the Hamiltonian \(H_1\) reduces exactly to the form \(H_1 \sim J n_0 \nabla \cdot \mathbf{j}\) we discussed in the earlier subsection. The Born scattering amplitude there was zero, and so is it here. To conclude, even the ferrimagnetic skyrmion scenario fails to produce skew scattering at the level of Born scattering. Again, more elaborate theories are likely to involve higher powers of \(n_0\) and very small.

All of the local defect scenarios considered in this section fail to show skew scattering, at least at the lowest order in the uniform moment \(n_0\). There is also a general issue how to reconcile the impurity-induced defects with the undoped cuprate, where the impurities are nominally absent. Finally, the magnon gap grows with the magnetic field and suppresses the response function in any magnon-based scenarios. The experiment on \(\kappa_{xy}\) does not show such activation behavior \cite{22}.

**IV. FERMIONIC SPINON THEORY OF THERMAL HALL EFFECT**

With the general inability of the magnon theory to account for the observed thermal Hall effect in the undoped to lightly doped cuprates, we turn to look for a theory whose first requirement is the absence of energy gap under the applied magnetic field. A very natural candidate is a fermion model with the Fermi surface structure, wherein the Zeeman field would only create a shift in the relative positions of the Fermi surfaces among spin-up and spin-down fermions. This is a well-known spinon model of the spin liquid phase of quantum magnets, of course, and most recently its applicability was confined to paramagnetic states devoid of magnetic order \cite{11}. The spinon idea had been adopted also to compute spin dynamics in the undoped cuprates \cite{24, 25}, even though long-range magnetic ordering in such compound was well established. The spinon-based theories were rationalized by the fact that some aspects of high-energy spin excitations are not captured by the spin-wave picture alone, and that a vestige of spinon excitations must remain in the physical spectrum to account for the spin dynamics fully. The spinon-based theories were not applied to low-energy transport properties such as the thermal Hall conduction. We proceed to first present a simple spinon-based model of magnetic dynamics, and use it to compute thermal Hall conductivity and spin chirality.

We outline general requirements in a candidate spinon model. Firstly, it will consist of spin-up and spin-down fermion bands with identical dispersions and opposite Berry curvatures. As such, the Hall effect of one species of fermions will be cancelled out by that of the other. The applied magnetic field will then split the energy degeneracy and lead to the non-cancellation of Berry curvatures, resulting in non-zero thermal Hall conductivity. In such picture, the predicted Hall signal will be naturally proportional to the field strength \(B\): \(\kappa_{xy} \propto B\) - a prominent feature in the observed thermal Hall effect in underdoped cuprates \cite{22}.

The model we present can be summed up as a \(2 \times 2\) fermion Hamiltonian

\[
H = \frac{1}{2} \sum_{k\sigma} \psi^\dagger_{k\sigma} H_{k\sigma} \psi_{k\sigma}.
\]

(4.1)

For each spin \(\sigma = \uparrow, \downarrow\) we have the spinor \(\psi_{k\sigma} = (\alpha_{k\sigma} \beta_{k\sigma})\), and the Hamiltonian matrix

\[
H_{k\sigma} = \begin{pmatrix}
4\sigma h_2 s_x s_y - \sigma B & 2h_1 (c_x + ic_y) \\
2h_1 (c_x - ic_y) & -4\sigma h_2 s_x s_y - \sigma B
\end{pmatrix}.
\]

(4.2)
We have used the abbreviations $c_{\xi(y)} = \cos k_{\xi(y)}$, $s_{\xi(y)} = \sin k_{\xi(y)}$. The hopping amplitudes in the nearest-neighbor and the diagonal directions are as displayed in Fig. 2. Without the diagonal hopping this is the $\pi$-flux Hamiltonian whose energy spectrum has Dirac nodes. The diagonal hopping term $h_2$ opens up a gap at the Dirac points and creates bands with Chern numbers. The spin-dependent diagonal hopping amplitude is designed to generate opposite signs of Berry curvature between the two spin orientations. The Zeeman energy $-\sigma B$ is included in the Hamiltonian.

\begin{equation}
E_{n\kappa\sigma} = 2n \left( h_1^2 (c_x^2 + c_y^2) + 4h_2^2 s_x s_y \right)^{1/2} - \sigma B,
\end{equation}

\begin{equation}
B_{n\kappa\sigma} = 2n\sigma \frac{h_1^2 h_2 (1 - c_x^2 c_y^2)}{(h_1^2 (c_x^2 + c_y^2) + 4h_2^2 s_x s_y)^{3/2}}. \tag{4.3}
\end{equation}

The band index $n = \pm 1$ refers to the upper and the lower band, respectively. The Berry curvature $B_{n\kappa\sigma}$ has opposite signs between the two bands, and between the two spins. For visualization of the band dispersion and the Berry curvature, see Fig. 5. The upper and lower bands are separated by a gap of magnitude $8|h_2|$ at $(k_x, k_y) = (\pm \pi/2, \pm \pi/2)$.

The zero-temperature Hall conductivity at the putative chemical potential $\epsilon$ for each spin species is derived from the Berry curvature through the TKNN formula

\begin{equation}
\sigma_{xy\sigma}(\epsilon) = \sum_{n\kappa} B_{n\kappa\sigma} \theta(\epsilon - E_{n\kappa\sigma}), \tag{4.4}
\end{equation}

which involves the sum over all states whose energies lie below $\epsilon$. In the quantized case we obtain $\sigma_{xy} = C/2\pi$, where $C$ is the Chern number. The lower band in our fermion model has the spin-dependent Chern number $C_\sigma = -\sigma$ for $\sigma = \pm 1$ (↑, ↓). For calculation of thermal conductivity in the fermionic model we use the formula derived in Ref. [42],

\begin{equation}
\frac{k_{\text{2D}}^{xy}}{T} = \frac{1}{4T^3} \int d\epsilon \frac{(\epsilon - \mu)^2}{\cosh^2(\beta (\epsilon - \mu)/2)} \sigma_{xy}^{\text{tot}}(\epsilon). \tag{4.5}
\end{equation}

This has the form of a well-known Mott formula relating the thermal conductivity to the electric conductivity. To restore physical units to the dimensionless form of $k_{\text{2D}}^{xy}/T$ given above, one has to multiply by $k_B^2/h$, the ratio of Boltzmann’s constant and the Planck’s constant. It is useful to note that $k_B^2/h = 1.81 \times 10^{-12}$ W/K². As an example, consider a bulk La$_2$CuO$_4$ sample whose $c$-axis constant is $d = 13.2\text{Å}$. Since there are two CuO$_2$ layers per unit cell, the effective interlayer distance is half that, $d_{\text{eff}} = 6.6\text{Å}$. If each CuO$_2$ layer carries a two-dimensional $k_{\text{2D}}^{xy}/T$ worth the universal value $k_B^2/h$, the three-dimensional thermal Hall conductivity of the bulk La$_2$CuO$_4$ would be given by $k_{\text{3D}}^{xy}/T = k_{\text{2D}}^{xy}/(T \cdot d_{\text{eff}}) = 2.76$ mW/K²m. The recently observed thermal Hall conductivity in cuprates reaches maximal $k_{xy}^{3D}$ values in the vicinity of 30-40 mW/Km at $T \approx 10\text{K}$, consistent with a per layer value of $k_{\text{2D}}^{xy}/T$ roughly equal to $k_B^2/h$ at that temperature. The thermal Hall conductivity formula (4.5) predicts values of $k_{\text{2D}}^{xy}/T$ in the range of $k_B^2/h$ for $\sigma_{xy} \approx 1$.

The Hall conductivity $\sigma_{xy}^{\text{tot}}$ itself is given as the sum of contributions from the two spin species: $\sigma_{xy}^{\text{tot}}(\epsilon) = \sigma_{xy,\uparrow}(\epsilon) + \sigma_{xy,\downarrow}(\epsilon)$. In the absence of Zeeman field we have the opposite signs of the Berry curvature and the degenerate energy bands, i.e. $B_{nk\uparrow} = -B_{nk\downarrow}$ and $E_{nk\uparrow} = E_{nk\downarrow}$, hence a vanishing Hall conductivity: $\sigma_{xy,\uparrow}(\epsilon) + \sigma_{xy,\downarrow}(\epsilon) = 0$. The energy degeneracy of $\uparrow$, $\downarrow$-spinons are split by the Berry curvature itself remains unaffected by it. The Hall conductivity formula in the presence of $B$ becomes

\begin{equation}
\sigma_{xy}^{\text{tot}}(\epsilon) = \sigma_{xy}(\epsilon + B) - \sigma_{xy}(\epsilon - B). \tag{4.6}
\end{equation}

Here $\sigma_{xy}(\epsilon) = \sigma_{xy,\uparrow}(\epsilon)$ is the Hall conductivity of $\uparrow$-spinons. There is more occupation of $\uparrow$-spinons than $\downarrow$-spinons, because the chemical potential for the former (latter) particle has been raised (lowered) by $B$. In the model Hamiltonian we chose, the $\uparrow$-spinon band carries the Chern number $-1$ and results in negative values of $k_{\text{2D}}^{xy}$.

Numerical calculation of the thermal Hall conductivity as a function of temperature and magnetic fields are shown in Fig. 5. The chemical potential was chosen in such a way that the average occupation number was $(f_{\epsilon,\pm}^{\uparrow,\downarrow}) = n$ at zero temperature and magnetic field. The linear-$B$ dependence of $k_{\text{2D}}^{xy}/T$ in the numerical plot is easy to understand, since $\sigma_{xy}(\epsilon + B) - \sigma_{xy}(\epsilon - B) \propto B$ at small values of $B$. Thermal smearing reduces the Hall conductivity.
signal at higher temperatures. The magnitude of \( \kappa_{xy}^{2D} / T \) values calculated within our model can reach values close to one \((k_B^2 / h \text{ in physical units})\) with suitable choices of \( h_2 \) and \( \mu \).

The spinon density \( n \) was chosen to be 0.98 in the calculation of thermal Hall conductivity, Fig. 3. In the slave fermion model the spinon density equals the electron density on average, and at the Mott insulator limit \( n \) should be unity. In our model for \( n = 1 \) the thermal Hall effect is zero at zero temperature because the chemical potential will lie in the gap. However, it will be finite for sufficiently large \( B \) and/or temperature. The value 0.98 may be considered slightly doped. Results for other values of \( n \) will be shown later.

The fermion model we study supports the spin chirality as well. In the mean-field theory, average of the spin-chirality operator \( S_i \cdot (S_j \times S_k) \) of the \( (ijk) \) triangle becomes, through the substitution \( S_i = (1/2) f_i^\dagger \sigma f_i \) with \( f_i = (f_i^\uparrow, f_i^\downarrow) \),

\[
\langle S_i \cdot (S_j \times S_k) \rangle = -\frac{i}{2} \left( \chi_{ij} \chi_{jk} \chi_{ki} - \chi_{ik} \chi_{kj} \chi_{ji} \right),
\]

where \( \chi_{ij} = \sum_\sigma \langle f_i^\dagger \sigma f_j^\sigma \rangle \). Calculations of \( \chi_{ij} \) in the mean-field theory is straightforward. The essential point, as it turns out, is that the triple product of hopping parameters \( \chi_{ijk} \equiv \chi_{ij} \chi_{jk} \chi_{ki} \) contains an imaginary term only at finite magnetic field and diagonal hopping, thus \( \langle S_i \cdot (S_j \times S_k) \rangle = \text{Im}[\chi_{ijk}] \propto h_2 \cdot B \).

FIG. 3. Two-dimensional thermal Hall conductivity \( \kappa_{xy}^{2D} / T \) (in physical units of \( k_B^2 / h \)) vs. (a) magnetic field \( B \) at several temperatures \( T \) and (b) vs. temperature \( T \) at several magnetic fields \( B \). Parameters chosen are \( h_1 = 1, h_2 = 0.1, \) and the chemical potential \( \mu = -0.6, \) corresponding to the filling factor \( n = 0.98 \) at \( T = 0, B = 0 \). Temperature and magnetic field scales are measured in units of \( h_1 \).

Explicit calculation shows all elementary triangles having the same spin chirality. In other words, finite magnetic field induces uniform spin chirality state within our model. Numerical evaluation of spin chiralities through the four triangles of the elementary square are shown in Fig. 4, displaying the expected linear growth with \( B \) at small fields. Our observation suggests that an interaction of the form \( \sim BS_i \cdot (S_j \times S_k) \) might be present and play a hitherto neglected role in the transport of underdoped cuprates. Such interaction Hamiltonian is well-known to derive from the large-\( U \) expansion of the Hubbard interaction, when an external magnetic field is present. Application of such spin chirality Hamiltonian to the understanding of the behavior of spin liquid phase under external magnetic field was taken up in Ref. [44], where the focus had been the orbital effects of the magnetic field such as the Landau level formation of spinons, without explicit consideration of the Zeeman splitting of the spinons as we do. The spinon hopping parameters in Ref. [44] pick up an imaginary part as a result of the Aharonov-Bohm effect, while our hopping parameters are deemed fixed and unchanged under the magnetic field. We also note that a spin chirality induced by a magnetic field was considered earlier by Katsura et al. [6] to generate a thermal Hall effect. However that effect is extrinsic, i.e. it depends on the scattering of the spinons by disorder, whereas the effect we consider in this paper is intrinsic.

FIG. 4. Magnetic field dependence of spin chirality \( \langle S_i \cdot (S_j \times S_k) \rangle \) for the triangles of the elementary square. It grows linearly with \( B \) at small fields. Parameters used are \( h_1 = 1.0, h_2 = 0.1, \) and \( \mu = -0.6 \) (\( n = 0.98 \)) as in Fig. 3 (inset) four corners of the elementary square are labeled by \( i,j,k,l \). Spin chirality is calculated for each of the four triangles by going in the counter-clockwise fashion. All four triangles carry the same value of spin chirality.
at \( n = 1 \), since the two orientations of spinons actually carry opposing sense of circulation, i.e. \( \chi_{ij} \uparrow \chi_{jk} \uparrow \chi_{ki} \uparrow \approx -\chi_{ij} \downarrow \chi_{jk} \downarrow \chi_{ki} \downarrow \), and it is the residual part of their sum which contribute to the spin chirality. At \( n = 1 \) the cancellation is almost complete, hence the spin chirality becomes very small. Additionally, one can check that spin-spin correlation \( \langle S_i \cdot S_j \rangle \) preserves the lattice symmetries as well, and the loss of translational and rotational symmetry in the hopping patterns of our ansatz is only an artifact of the spinon theory. The aspect of projective symmetry restoration was discussed in Ref. [26] also.

Having demonstrated a fair proximity of the computed thermal Hall conductivity to the actual experiment, we come to ask if the spinon model we propose has much foundation. On the theoretical side, the conventional view is that starting from a spinon model, the Néel state can emerge as a confinement transition, where the spinons become gapped and confined [45]. Thus we normally do not expect the co-existence of anti-ferromagnetic order and nearly free spinons. Such co-existence will be considered highly exotic. The problem is exacerbated by the requirement that the spinon gap must be small in the insulator, in order to give a thermal Hall effect at relatively low temperature and magnetic field. Furthermore, the particular spinon dynamics that we assume, with spin-dependent hopping, is highly unusual and at the moment we have no way of justifying it microscopically, except to say that spin-dependent hopping most likely is tied to spin-orbit interaction. The model on the whole is an attempt to fit the observation. On the experimental side, the renormalized spin-wave theory does a good job in accounting for the magnetic excitations in the square-lattice antiferromagnet. Even a recent measurement of the magnetic excitation in the ultrathin, 1- and 2-layer La\(_2\)CuO\(_4\) by RIXS (Resonant Inelastic X-ray Scattering) found good agreement with the conventional spin-wave theory [46]. On the other hand, some high-energy features in the magnetic excitation are not fully explained within the spin-wave theory alone - a fact that prompted speculations about alternative spinon excitations [24, 25]. It is fair to say that at this point, spinons as low-energy excitations in square-lattice quantum antiferromagnet has, at best, quite weak experimental support. On the other hand, the two theories - magnons and spinons - give contrasting predictions in regard to their behavior under the magnetic field. In the spin-wave scenario, a magnon gap inevitably opens and suppresses magnon contribution to transport. For the spinon-based scenario, as demonstrated here, linear growth of the response function \( \kappa_{xy}/T \) with the field is natural. The diagonal hopping term \( \sim h_2 \), necessary for the opening of the gap, existence of Berry curvature, and ultimately the thermal Hall transport, seems closely related to the spin chirality correlation, given that the latter scales with \( h_2 \) in our model. In turn, including the three-spin exchange interaction on top of the Heisenberg interaction might be a necessary ingredient for the complete understanding of magnetic dynamics in undoped cuprates.

**Note added:** Spinon theory of thermal Hall effect in magnets with Dzyaloshinskii-Moriya interaction was also advanced in a recent preprint [47] and applied to the Kagome lattice. We also mention a preprint by Chatterjee et al. [24] which also used the \( \pi \) flux spinon as a starting point. A key ingredient is the term \( J_4 \sum_{\Delta} S_i \cdot (S_j \times S_k) \) in their Eq. (2), where \( J_4 \) is proportional to the magnetic flux through a triangular plaquette. This term generates a net chirality which produces a thermal Hall effect. We had considered this term in the last section but did not discuss it further because of the very small magnitude. One can make an estimate of \( J_4 \) using the \( t/U \) expansion by Motrunich [44], to find \( J_4 = -48\pi (t_2t^2/U^2) (\phi/\phi_0) \) where \( \phi_0 = \hbar c/e = 2.07 \times 10^{-15} \text{Wb} \) is the flux quantum, and \( \phi = BA_0 \) is the magnetic flux through a triangular plaquette of area \( A_0 \approx (3.8A)^2/2 \) for the cuprate. At \( B = 10\text{T} \) we find \( \phi/\phi_0 \approx 3.5 \times 10^{-4} \). Further using commonly accepted values of \( t_2 = -0.3t \), \( U = 8t \) and \( J = 4t^2/U \), we find \( J_4 \approx 5.6 \times 10^{-4} J \) at \( B = 10\text{T} \). The use of a smaller effective \( U \) may increase this number a bit, but in any case a very small number is expected for \( J_4 \), due to the small ratio \( \phi/\phi_0 \). As we emphasized in this paper, the unexpected nature of the experimental data means that all avenues should be explored. Nevertheless, the small value of this term should be kept in mind. The assumed proximity to a quantum critical point also makes it challenging to explain the linear \( B \) dependence of \( \kappa_{xy} \) observed over a large range from 5T to 15T.
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