Reality conditions for Ashtekar gravity from Lorentz-covariant formulation

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Abstract

We study the limit of the Lorentz-covariant canonical formulation where the Immirzi parameter approaches $\beta = i$. We show that, formulated in terms of a shifted spacetime connection, which also plays a crucial role in the covariant quantization, the limit is smooth and reproduces the canonical structure of the self-dual Ashtekar gravity. The reality conditions of Ashtekar gravity can be incorporated by means of the Dirac brackets derived from the covariant formulation and defined on an extended phase space which involves, besides the self-dual variables, also their anti-self-dual counterparts.

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1. Introduction

The complex Ashtekar variables [1–3] were at the origin of the canonical quantization program of general relativity, which grew up later into the loop approach to quantum gravity [4–6]. However, modern loop quantization is based on the use of a real version of the Ashtekar variables [7, 8]. The initial complex variables were given up due to a problem associated with the implementation of some reality conditions needed to ensure that one describes the real Einstein gravity [9, 10]. Despite many efforts to understand these conditions properly, their status in quantum theory remained obscure. Therefore, when it was realized that the real Barbero variables preserve the main advantages of the Ashtekar variables and still allow a quantization in the manner of loops, the interest moved in this direction.

However, there are at least two big differences between the complex real variables and the real variables. First, the former describe a theory with the Lorentz gauge group, whereas in the latter case the gauge group is reduced to $SU(2)$ [11, 12]. Second, the Ashtekar connection is a pull-back of a spacetime connection, whereas the real Barbero connection is not [13]. These differences warn us that the passage to the real variables may not be so harmless as it seems at first sight.
In fact, in some of our previous work [14, 15] it was argued that the loop quantization based on the real Barbero variables is very likely anomalous. The main reason for this is just the second fact mentioned above, from which it follows that the Barbero connection does not transform as a true connection under the time diffeomorphisms. Therefore, one expects an anomaly in the diffeomorphism symmetry at the quantum level. In particular, it was argued that one of the manifestations of such an anomaly is the appearance of the Immirzi parameter in physical results, such as spectra of geometric quantities [16–18]1.

All these conclusions were obtained in the framework of the so-called covariant loop quantization originating from a canonical formulation explicitly covariant under the full Lorentz gauge group [19]. Using this formulation, it was shown that there is only one connection which transforms properly under all classical symmetries (four diffeomorphisms and six local Lorentz transformations) and simultaneously diagonalizes the area operator [14]. As expected, it does not coincide with the Barbero connection and leads to results different from those found in loop quantum gravity with the $SU(2)$ gauge group. In particular, the area spectrum does not depend on the Immirzi parameter, is given by the Casimir operator of $SO(3,1)$ and is, therefore, continuous [14, 20].

We see that in many respects the Lorentz-covariant canonical formulation is quite similar to the original complex formulation of Ashtekar. Indeed, they both preserve the full Lorentz gauge symmetry and are based on connections which are pull-backs of true spacetime connections. Moreover, one of the key ingredients of the covariant formulation is the presence of second class constraints. As will be shown in this paper, but also expected on general grounds, they coincide with the reality conditions of Ashtekar gravity and, as a result, the two formulations are completely equivalent.

The reality conditions were the main obstacle to quantizing Ashtekar gravity. In the covariant approach the second class constraints, which are equivalent to the reality conditions, are taken into account via the Dirac bracket. Although some of the resulting expressions are quite complicated, in principle, this is enough to implement the constraints at the quantum level. Thus, one can ask: can one learn something useful about the reality conditions for Ashtekar gravity starting from the covariant formulation?

One of the problems in understanding the reality conditions was that they cannot be written as constraints on the phase space variables of Ashtekar gravity because they involve complex conjugate fields. The complex conjugation brings out of the phase space since there is no symplectic structure defined on the conjugate fields. Therefore, although it is clear that the reality conditions are a kind of second class constraint, it is difficult to make this statement precise2. As we will see, the correct statement is that the reality conditions can be viewed as true second class constraints when they are formulated on some extended phase space.

The simplest idea to construct such an extended phase space is to add the conjugate fields to the original ones with some symplectic structure defined on them. The original fields of Ashtekar gravity are the self-dual parts of the triad and the spacetime connection. The complex conjugate fields are their anti-self-dual counterparts. Thus, it is natural to expect that the symplectic structure we are looking for should be induced from a formulation which involves both self-dual and anti-self-dual fields. For example, it can be the covariant

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1 The issue however is not yet settled. The usual objection is that the results, which the statements mentioned above are based on, were obtained without having a rigorously derived Hilbert space structure and therefore cannot be trusted. Nevertheless, we believe that these results give strong evidence in favour of our conclusions.

2 On different points of view on this problem in the literature see [9, 21, 22]. In fact, for the reality conditions in the triad form [23], which are obtained by fixing the time gauge [22], one can define a symplectic structure on the conjugate fields since they are expressed in terms of the original fields. But this is impossible to do for the reality conditions in the metric form, which preserve the full Lorentz symmetry.
formulation taken for any value of the Immirzi parameter $\beta \neq \pm i$. Then Ashtekar gravity should be recovered in the limit $\beta \to i$.

However, one encounters an immediate problem that, considered for canonical variables, the latter limit is not well defined: the covariant formulation exists for an arbitrary Immirzi parameter except for just these two special values where various expressions become singular. This can be traced back, of course, to the disappearance of the (anti-)self-dual variables from the action.

A way to overcome this problem comes from the observation that the algebra of the Dirac brackets written for the spacetime connection diagonalizing the area operator, which played a crucial role in the covariant quantization, does not depend on the Immirzi parameter [24]. Hence, after we take the second class constraints (the reality conditions) into account and shift the connection properly, the limit $\beta \to i$ becomes smooth.

However, this is not the end of the story yet. First, one should show that the self-dual and anti-self-dual parts of the shifted connection can be associated with the Ashtekar connection and its complex conjugate, respectively. Second, since the limit involves some not well-defined intermediate steps, it is necessary to check that the resulting Dirac brackets define a consistent symplectic structure. And finally, one should explain how this helps to solve the problem of the reality conditions in quantum theory. This is what we are going to accomplish in this paper.

We start by reviewing some necessary elements of the Lorentz-covariant canonical formulation. In section 3 we rewrite it in terms of self-dual and anti-self-dual variables. Then in section 4 we take the limit $\beta \to i$ and obtain the complex Ashtekar gravity with an extended phase space and the reality conditions taken into account by means of the Dirac brackets. In section 5 we comment on the quantization of the resulting theory. In the appendices one can find some details of calculations.

2. Lorentz-covariant canonical formulation

The starting point to construct the covariant canonical formulation is the generalized Hilbert–Palatini action [25], which allows us to include the arbitrary Immirzi parameter $\beta$:

$$S(\beta) = \frac{1}{2} \int \varepsilon_{\alpha\beta\gamma} e^\alpha \wedge e^\beta \wedge \left( \Omega^{\gamma \delta} + \frac{1}{\beta} \star \Omega^{\gamma \delta} \right),$$

(1)

Here $\Omega^{\gamma \delta} = d\omega^{\gamma \delta} + \omega^{\gamma}{}_\alpha \wedge \omega^\alpha{}_{\delta}$ is the curvature of the spin connection $\omega^{\alpha\beta}$ and $\star$ is the Hodge operator acting on the tangent indices $\alpha, \beta, \ldots$. The notation we use for other indices is the following. The indices $i, j, \ldots$ from the middle of the alphabet label the space coordinates, $a, b, \ldots$ from the beginning are $so(3)$ indices in the tangent space and the capitalized letters $X, Y, \ldots$ take six values and are used to label the components of the adjoint representation of $sl(2, \mathbb{C})$.

The canonical formulation arises after the $3 + 1$ decomposition

$$e^0 = N \, dt + \chi_a E^a_i \, dx^i, \quad e^a = E^a_i \, dx^i + E^a_i N^i \, dt,$$

(2)

The field $\chi$ appearing in (2) describes the deviation from the time gauge $\chi = 0$, which is used to obtain the real Barbero formulation. To write the decomposed action, it is convenient to introduce the fields $A^X \_i$ and $\tilde{P}_{(\beta) X}$, which belong to the adjoint representation of the Lorentz group and are defined as follows [19]:

$$A^X \_i = \left( \omega^a_{\alpha i}, \frac{1}{2} \varepsilon^a_{bc} \omega^b_{\alpha i} \right), \quad \tilde{P}_{(\beta) X} = \tilde{P}^i_X - \frac{1}{\beta} \tilde{Q}^i_X.$$

(3)
where
\[ \tilde{P}^i_X = \left( \tilde{E}_a^i, e_a^{bc} \tilde{E}_b^c \right), \quad \tilde{Q}^i_X = \left( -e_a^{bc} \tilde{E}_b^c, \tilde{E}_a^i \right) \]
and \( \tilde{E}_a^i = h^{1/2} E_a^i \left( \sqrt{\hbar} = \det E_a^i \right) \) is the inverse densitized triad. The first field is just the space components of the spin connection \( \omega_{ab} \), the field \( \tilde{P}^i_X \) can be obtained from the bivector \( e_a^\alpha \wedge e_b^\beta \), and \( \tilde{Q}^i_X \) comes from its Hodge dual. This fact is encoded in the relation
\[ \tilde{P}^i_X = \Pi^i_Y \tilde{Q}^j_Y, \]
where the matrix \( \Pi^i_Y \) can be considered as a representation of the \( \star \) operator and is defined in appendix A. There one can also find the definition of the Killing form \( g^{XY} \) of the \( sl(2, \mathbb{C}) \) algebra, its structure constants \( f_{Z}^{XY} \) and various properties satisfied by these matrices and fields.

In terms of the introduced fields and after some redefinition of the lapse and shift, the decomposed action takes the following form [19]:
\[ S(\beta) = \int dt d^3x \left( \tilde{P}(\beta)_X X A^X + A^X \mathcal{G}_X + \mathcal{N}_p H_i + \mathcal{N}_n H \right), \]
\[ \mathcal{G}_X = \partial_t \tilde{P}(\beta)_X X + f_{YZ} A^Y \tilde{P}(\beta)_Z, \]
\[ H_i = -\tilde{P}(\beta)_X F^X_{ij}, \]
\[ H = -\frac{1}{2(1 + \frac{1}{\beta})} \tilde{P}(\beta)_X \tilde{P}(\beta)_Y f_{YZ} R^Z R^W, \]
where
\[ F^X_{ij} = \partial_i A^X_j - \partial_j A^X_i + f_{YZ} A^Y_i A^Z_j, \quad R^{XY} = g^{XY} - \frac{1}{\beta} \Pi^{XY}. \]

It is clear that \( A^X_i \) and \( \tilde{P}(\beta)_X X \) form the canonical pair, and \( \mathcal{G}_X, H_i \) and \( H \) are first class constraints generating the symmetry transformations. However, there are additional constraints coming from the fact that not all components of \( \tilde{P}(\beta)_X X \) are independent. It is easier to write them in terms of \( \tilde{Q}^i_X \):
\[ \phi^{ij} = \Pi^{XY} \tilde{Q}^i_X \tilde{Q}^j_Y = 0. \]

This constraint is very well known in the BF formulations of gravity and spin foam models by the name ‘simplicity constraint’ [26–28]. Requiring that \( \phi^{ij} = 0 \) is preserved by evolution, one obtains an additional constraint
\[ \psi^{ij} = 2 f^{XY} \tilde{Q}^i_X \tilde{Q}^j_Y \partial_i \tilde{Q}^j_Z - 2(\tilde{Q} \tilde{Q})^{ij} \tilde{Q}^j_Z A^X_j + 2(\tilde{Q} \tilde{Q})^{ij} \tilde{Q}^j_W A^X_j = 0. \]

Here \( (\tilde{Q} \tilde{Q})^{ij} = g^{XY} \tilde{Q}^i_X \tilde{Q}^j_Y \) and symmetrization \([\cdot]\) is taken with the weight 1/2. Together \( \phi^{ij} \) and \( \psi^{ij} \) form a set of second class constraints and require a modification of the symplectic structure to that of the Dirac brackets [29]. As a result, the canonical variables acquire the following non-trivial commutation relations:
\[ \{ \tilde{P}_X, \tilde{P}_Y \}_{D} = 0, \]
\[ \{ A^X_i, \tilde{P}_Y \}_{D} = \delta^i_Y \delta^X_A - \frac{1}{2} R^{XY} (\tilde{Q}^W_Z A^Y_W + \delta^W_Y \mathcal{I}^{W}_{Y/Z} \mathcal{G}_{WY}), \]
\[ \{ A^X_i, A^Y_j \}_{D} = \text{complicated}. \]

To write the result, we introduced the so-called inverse fields \( \tilde{E}^X_i \) and \( \tilde{Q}^X_i \) and the projectors
\[ \mathcal{I}^Y_P(\beta)_X = \tilde{P}^i_X P^i_Y, \quad \mathcal{I}^Y_{(Q)_X} = \tilde{Q}^i_X Q^i_Y. \]
We refer to appendix A for their definitions in terms of the triad $\tilde{E}_a$ and the field $\chi^a$ as well as for their properties. The commutator of two connections has not been specified since it will not be necessary here.

The connection $A^X_i$ is not well suited for the loop quantization. The reason is that its commutator with the triad multiplet $\tilde{P}_X^i$ given in (8) is not proportional to $\delta^i_j$ and therefore the area operator is not diagonal on holonomies of this connection [20]. However, as was mentioned in the introduction, there is a unique spacetime connection which does this job. It can be obtained from $A^X_i$ by shifting it by a term proportional to the Gauss constraint:

$$A^X_i = A^X_i + \frac{1}{2(N + 1)} R^X_i(t_{(Q)}^X T^X_k f_{(Q)}^k W F^W_e G^e_y = t_{(P)}^X \left( \delta^X_j + \frac{1}{2} \Pi^X_{j} \right) A^X_j + R^X_j \Gamma^X_j,$$

(10)

where

$$\Gamma^X_j = \frac{1}{2} f_{(Q)}^X t_{(Q)}^X Q^X_j \partial_l \tilde{Q}^l W + \frac{1}{2} f_{(Q)}^X (Q Q)_{i j} I_X^{i j} + Q_X^{i j} Q^X_j - Q_X^{i j} Q^X_j) \tilde{O}^l \partial_l \tilde{Q}^l W$$

(11)

and we used (7) to obtain the second equality. The quantity $\Gamma^X_i$ is nothing else but the $SL(2, \mathbb{C})$ connection compatible with the metric induced on the three-dimensional hypersurface [24]. At $\chi = 0$ it reduces to the connection $\Gamma^X_i(\tilde{E})$ appearing in the definition of the Barbero connection.

In terms of the new connection the Dirac brackets take a simpler form and do not depend at all on the Immirzi parameter:

$$\{ A^X_i(x), \tilde{P}_Y^j(y) \}_D = \delta^i_j I_{(P)} \delta(x, y),$$

$$\{ A^X_i(x), A^Y_j(y) \}_D = \frac{1}{2} (\Pi^X_{i j} M^{X Y}_{i j} - M^{X Y}_{i j} \Pi^X_{i j}) \delta(x, y).$$

(12)

(13)

Here $M^{X Y}_{i j}$ is a linear differential operator whose exact expression can be found in appendix A. An important consequence of (12) is that the field $\chi$ and, therefore, also the projectors $I_{(P)}$ and $I_{(Q)}$ commute with both $\tilde{P}$ and $A$.

3. Separation of chiral variables

This section is purely technical. Our aim here is to split all variables into the self-dual and anti-self-dual parts. For this let us introduce the corresponding projectors

$$R^{X Y}_{(\pm)} = \frac{1}{2} (g^{X Y} \mp i \Pi^{X Y}) = \left( \frac{1}{\mp i} \frac{\mp i}{-1} \right) \delta^{a b},$$

(14)

which satisfy the following properties:

$$R_{(\pm)} \cdot R_{(-)} = 0, \quad (R_{(\pm)})^2 = R_{(\pm)}, \quad \Pi \cdot R_{(\pm)} = \pm i R_{(\pm)}.$$

(15)

Applying these projectors to the canonical variables, one obtains

$$R^{X Y}_{(\pm)} \tilde{P}_Y^j = \frac{1}{2} (\tilde{P}^{(\pm)}_{(\pm)} + \mp i \tilde{P}^{(\pm)}_{(\pm)}), \quad \tilde{P}^{(\pm)}_{(\pm)} = \tilde{E}^{(\pm)}_{a \pm} \tilde{E}^{(\pm)}_{b c} \tilde{Q}^{(\pm)}_{(\pm)},$$

$$R^{X Y}_{(\pm)} A^X_i = \frac{1}{2} (A^{(\pm)}_{(\pm)} + \mp i A^{(\pm)}_{(\pm)}), \quad A^{(\pm)}_{(\pm)} = a^{(\pm)}_{a b \pm a_{b c}} \pm \frac{1}{2} e^{a b \pm a_{b c}},$$

(16)

(17)

and $R^{(\pm)} \cdot \tilde{Q} = \mp i R^{(\pm)} \cdot \tilde{P}$. Thus, each of the projected fields has only half of the independent components, so one can take $(\tilde{P}^{(\pm)}, A^{(\pm)})$ to be the basic variables. It is useful
to note also the following relations:

\[ R \cdot R(\pm) = \left(1 \mp \frac{i}{\beta}\right) R(\pm), \quad R(\pm) \cdot I(Q) \cdot R(\pm) = \frac{1}{2} R(\pm), \]

\[ R(\pm) \cdot I(P) \cdot R(\pm) = \frac{1}{2} R(\pm), \quad (R(\pm) \cdot I(P) \cdot R(\mp))^X_Y = \left(\frac{1}{\mp i} \pm i\right) \left(\pm X_\mp\right)^{ab}_{xy}. \tag{18} \]

where we introduced

\[ (\pm X_\mp)^{ab}_{xy} = \frac{\delta^{ab}(1 + \chi^2) - 2\chi^a \chi^b \mp 2i\epsilon^{abc}\chi^c}{1 - \chi^2}. \tag{19} \]

It is easy to check that the matrices \((\pm X_\mp)^{ab}_{xy}\) and \((-X_{\mp})^{ab}_{xy}\) are mutually inverse.

After the splitting of the variables into self-dual and anti-self-dual parts, the action (5) can be written as a sum of two actions. One of them depends only on the self-dual variables and the other one is a similar action for the anti-self-dual fields:

\[ S(\beta) = \frac{1 + i/\beta}{2} S(-) + \frac{1 - i/\beta}{2} S(+), \tag{20} \]

where

\[ S(\pm) = \int dt \, d^3x \left(\dot{P}(\pm)_{a\mu}^i A^{(\pm)a}_{i\mu} + A^{(\pm)a}_{i\mu} \frac{\partial}{\partial A^{(\pm)a}_{i\mu}} + \mathcal{N}_D H^{(\pm)} + \mathcal{N}_N H^{(\pm)}\right). \]

\[ G_a^{(\pm)} = \partial_\mu \dot{P}(\pm)_{a\mu}^i \pm i\epsilon_{abc} A^{(\pm)b}_{i} \dot{P}(\pm)_{c}, \]

\[ H^{(\pm)} = -\dot{P}(\pm)_{a\mu}^i F^{(\pm)a}_{ij}, \]

\[ H^{(\pm)} = \mp \frac{i}{2} \dot{P}(\pm)_{a\mu}^i \dot{P}(\pm)_{b\nu}^j \epsilon^{abc} F^{(\pm)c}_{ij}, \]

\[ \mathcal{F}^{(\pm)a}_{ij} = \partial_i A^{(\pm)a}_{j} - \partial_j A^{(\pm)a}_{i} \pm i\epsilon_{abc} A^{(\pm)b}_{i} A^{(\pm)c}_{j}. \]

Thus, the two chiral sectors do not interact with each other and the Immirzi parameter measures the ‘weight’ of each sector. The only non-vanishing Poisson brackets of the chiral variables are

\[ \left\{ A^{(\pm)a}_{i}, \dot{P}(\pm)_{b \mu}^j \right\} = \frac{2\delta_{j}^{b}}{1 \pm \frac{i}{\beta}}. \tag{22} \]

The two sectors become mixed when one takes into account the second class constraints (6) and (7). Let us also rewrite them in terms of the chiral variables. For the first class constraint one has

\[ 2i\phi^{ij} = \dot{P}(\pm)_{a}^{i} \dot{P}(\pm)_{b}^{j} - \dot{P}(\mp)_{a}^{i} \dot{P}(\mp)_{b}^{j} = 0. \tag{23} \]

Since \(\dot{P}(\pm)\) is the complex conjugate of \(\dot{P}(\pm)\), the meaning of this constraint is just that the spatial metric defined by the self-dual triad, \(g^{ij}_{(+)a} = \dot{P}(\pm)_{a}^{i} \dot{P}(\pm)_{b}^{j}\), is real. Thus, the first of the second class constraints is nothing else but the first reality condition in the metric form.

The second constraint can be written (with the use of (23)) as

\[ -i\psi^{ij} = i\epsilon_{abc} \dot{P}(\pm)_{a}^{i} \dot{P}(\pm)_{b}^{j} \partial_i \dot{P}(\pm)_{c} - g^{ij}_{(+)a} \dot{P}(\pm)_{a}^{i} A^{(+)a}_{i} + g^{j}^{(+)a} \dot{P}(\pm)_{a}^{i} A^{(+)a}_{i}. \]

\[ = -i \rightarrow i \]

\[ \quad \rightarrow -i \] \tag{24}

Since this constraint was obtained by commuting \(\phi^{ij}\) with the Hamiltonian, it coincides with the second reality condition which requires the reality of the spatial metric to be preserved under the time evolution. Thus, as was expected, the reality conditions are identical to the second class constraints of the covariant formulation.
In this formulation the constraints were taken into account by means of the Dirac bracket. Making projection to the two chiral sectors in the commutation relation (8), one finds the following results for the Dirac brackets of the chiral variables:

\[
\{ A^{(\pm)a}_i, \tilde{P}^{(\pm)b}_i \}_D = \frac{2\delta_i^j \delta^a_b}{1 \pm \frac{1}{\beta}} - \frac{1 + \frac{1}{\beta}}{2} (\tilde{P}^{(\pm)b}_i A^{(\pm)a}_i + \delta_i^j \delta^a_b),
\]

\[
\{ A^{(\pm)a}_i, \tilde{P}^{(-)b}_i \}_D = 0
\]

\[
\{ A^{(-)a}_i, \tilde{P}^{(+)b}_i \}_D = \frac{\delta^a_b}{2} (\tilde{P}^{(+)}_i A^{(-)} + \delta_i^j X^{(a)}_b),
\]

\[
\{ A^{(-)a}_i, \tilde{P}^{(-)b}_i \}_D = \frac{\delta^a_b}{2} (\tilde{P}^{(-)}_i A^{(-)} + \delta_i^j X^{(a)}_b),
\]

where we had to introduce

\[
P^{(\pm)a}_i = \frac{\delta^a_b - \chi^a X_b}{1 - \chi^2} E^b_j + i \epsilon^{abc} \tilde{E}^b_j X^c
\]

such that \( P^{(\pm)a}_i \tilde{P}^{(\pm)b}_i = \delta^a_b \) and \( P^{(\pm)a}_i \tilde{P}^{(-)b}_i = (\pm X^a)_b \).

Finally, we should find the chiral components of the shifted connection (10). A simple calculation gives

\[
A^{(\pm)a}_i = 2 \left(1 \pm \frac{1}{\beta} \right) \left(1 \mp \frac{1}{\beta} \right) \left(\frac{1}{2} (\pm X^a)_b A^{(\mp)} + \Gamma^{(\pm)a}_i \right),
\]

where \( \Gamma^{(\pm)a}_i \) are the chiral components of (11). Note that they cannot be written entirely in terms of the fields of one chirality. Instead, one has the following property:

\[
(\pm X^a)_b \Gamma^{(-)b}_i = -\Gamma^{(+)i}.
\]

For the variables (27) the Dirac brackets become

\[
\{ A^{(\pm)a}_i, \tilde{P}^{(\pm)b}_i \}_D = \delta_i^j \delta^a_b, \quad \{ A^{(\mp)a}_i, \tilde{P}^{(-)b}_i \}_D = \delta_i^j (\pm X^a)_b,
\]

\[
\{ A^{(-)a}_i, \tilde{P}^{(+)b}_i \}_D = \delta_i^j (-X^a)_b, \quad \{ A^{(\pm)a}_i, A^{(\mp)b}_j \}_D = 0,
\]

\[
\{ A^{(\pm)a}_i, A^{(-)b}_j \}_D = i (R^{(\pm)}_a \cdot M_{ij} \cdot R^{(-)} b).
\]

This finishes the preparation for taking the limit corresponding to the complex Ashtekar gravity, which will be investigated in the following section.

4. Ashtekar gravity with extended phase space

The complex Ashtekar gravity corresponds to the special case where the Immirzi parameter is chosen to be \( \beta = i \). Setting this value of \( \beta \) in (20), one finds that

\[
S_{(i)} = S_{(+)}.
\]

Of course, \( S_{(+)} \) coincides with the usual Ashtekar action [1–3]. As a result, only the self-dual variables contribute to the action and all anti-self-dual variables disappear. An immediate consequence of this is that the Poisson brackets (22) of the anti-self-dual variables become divergent.

The situation does not become better when one takes into account the second class constraints relating the self-dual and anti-self-dual fields and considers the corresponding Dirac bracket. Indeed, the results (25) show that the Dirac bracket of the anti-self-dual parts of the canonical connection and the triad still diverges at \( \beta = i \). Besides, we did not consider the commutator of two connections which can also contain some divergences. Thus, there is no consistent symplectic structure which can be defined on the space of \( (\tilde{P}^{(+)a}, A^{(+)a}) \) and \( (\tilde{P}^{(-)}a, A^{(-)a}) \).
Nevertheless, let us consider instead the phase space spanned by \((\tilde{P}_{+}, \mathcal{A}^{(\star)})\) and \((\tilde{P}_{-}, \mathcal{A}^{(-)})\). Remarkably, the Dirac brackets of these variables, given in (29), do not depend on \(\beta\) and therefore are well defined even at the point corresponding to the complex Ashtekar gravity. Thus, they represent a good candidate for the symplectic structure we are looking for, which will allow us to implement the reality conditions at the quantum level.

But how could it happen that all divergences disappeared? It is clear that this cannot be achieved by a simple invertible change of variables. To clarify the situation, let us consider the expressions for the chiral components of the shifted connection (27) in terms of the original variables at \(\beta = i\). One finds

\[
\mathcal{A}^{(+)} = \mathcal{A}^{(-)} + 2\Gamma^{(-)}A.
\]

We observe that none of the chiral components depend on \(A\). This means that the shifted connection \(\mathcal{A}\) contains nine components less than the original connection \(A\). The missing components, which are precisely \(\mathcal{A}^{(-)}\) at \(\beta = i\), were removed by means of the second class constraint \(\psi^{ij}\) and some part of the Gauss constraint (at \(\beta = i\) this is \(\tilde{G}^{\mu}_{\nu}\)). Thus, working at the surface of these constraints, we simply exclude the corresponding variables from the phase space.

However, now there is another problem. On one hand, \(\mathcal{A}^{(-)}\) can be expressed through other variables by means of (32) as

\[
\mathcal{A}^{(-)} = (\cdots X_{\nu})_{\bar{a}} A^{(+)} + 2\Gamma^{(-)}A.
\]

On the other hand, its Dirac brackets are already defined in (29). Thus, there is a non-trivial consistency condition which requires that, using expression (33) for \(\mathcal{A}^{(-)}\) to calculate the Dirac brackets, one obtains the same results as in (29). Equivalently, this means that relation (33) can be considered as a strong equality or a second class constraint on the phase space of \((\tilde{P}_{+}, \mathcal{A}^{(+)}, \mathcal{A}^{(-)})\) endowed with the symplectic structure (29). We check that this is indeed true in appendix B.

Note that this consistency condition is not ensured by the construction for generic \(\beta\). The problem is that the shifted connection contains terms proportional to \((1 - i/\beta)A^{-}\). At \(\beta = i\) such terms do not contribute to the expression for the connection, but they do contribute to the Dirac brackets since the vanishing factor \((1 - i/\beta)\) can be cancelled by the same factor from the denominator in (25). The simplest example of such a situation is the Dirac bracket of \(\mathcal{A}^{(+)}\) with \(\tilde{P}_{+}\); although \(\mathcal{A}^{(+)}\) coincides with \(A^{(+)}\) according to (31), the Dirac brackets (25) and (29) are different. The remarkable fact is that despite all these problems, which seem to appear at intermediate steps, the final result, namely the Dirac brackets (29), together with the constraint (33), is consistent.

As a result, one gets the following picture. The phase space of Ashtekar gravity can be extended to include, besides the original self-dual variables \(\mathcal{A}^{(+)}\) and \(\tilde{P}_{+}\), also \(\tilde{P}_{-}\) and \(\mathcal{A}^{(-)}\) with the constraint (33) imposed on it (so, in fact, \(\mathcal{A}^{(-)}\) can be excluded from the phase space). The symplectic structure on this extended phase space is defined by the Dirac brackets (29). Finally, on the extended phase space one can define the operation of complex conjugation, which acts according to

\[
(\tilde{P}_{+})^* = \tilde{P}_{-}, \quad (\mathcal{A}^{(+)})^* = \mathcal{A}^{(-)}.
\]

It is easy to see that the two structures, the symplectic structure and the complex conjugation (34), are mutually consistent, which means that

\[
\{ F^*, G^* \}^{D} = \{ F, G \}^{D}.
\]

\(^3\) Note that for \(\chi = 0\), relation (33) reduces to the well-known second reality condition in the triad form: \(\text{Im} A^\alpha = \Gamma^\alpha\). Our approach provides its generalization to the case of the full Lorentz gauge group.
Before using this construction for quantization of general relativity, one should check two additional conditions. First, it should ensure the reality of the metric. Second, the complex conjugation in (34) should agree with the usual one, which acts in the evident way being written in the original variables (we denote it by a bar):
\[
\tilde{P}(\gamma) = \tilde{P}(-\gamma), \quad \tilde{A}(\gamma) = A(-). 
\] (36)

The first requirement is fulfilled due to the fact that the symplectic structure is induced by the Dirac brackets, which take into account the second class constraints. As we saw above, these constraints are nothing else but the reality conditions for the metric. The second condition becomes satisfied if one also allows the use of the Gauss constraint, since in that case one has (see (10))
\[
\Delta^{(\gamma)} = A(-) \approx G = 0
\] (37)

Thus, using the Gauss constraint and shifting the canonical connection by a term proportional to it, allows us to achieve two things: the Dirac brackets become well defined and it becomes possible to endow the extended phase space with a complex conjugation consistent with the usual one. The resulting structure will be the starting point in discussing the quantization of Ashtekar gravity in the following section.

5. Quantization

By quantizing gravity in the loop approach, one chooses the space of connections as configuration space and the wavefunctions to be the loop, or the so-called spin network functionals of the connection [5, 30, 31]. In our case it is natural to take them to be the functionals of the self-dual connection \( A^{(\gamma)} \). Then the variables \( \tilde{P}(\gamma) \) are going to be the operators which are the usual functional derivatives with respect to \( A^{(\gamma)} \). In this way one obtains the standard loop quantization of the self-dual sector [3].

The main problem, which has existed for a long time, was how to implement the reality conditions of Ashtekar gravity in this framework. The idea was that it can be done by a clever choice of the scalar product on the space of loop functionals. But no such scalar product has been found.

The picture presented in the previous section suggests a new look at the problem of the reality conditions. These conditions will be automatically satisfied as soon as we find an appropriate representation of the algebra of the Dirac brackets (29) such that the fields, which are complex conjugate according to the rule (34), become Hermitian conjugate operators. In other words, the anti-self-dual fields should be realized as operators Hermitian conjugate to the self-dual ones:\footnote{It might be that the connection itself is not a well-defined operator on the Hilbert space as happens in the standard loop approach. In this case, the requirement (38) should be understood for appropriate functions of the connection.}
\[
(\tilde{P}(\gamma))^\dagger = \tilde{P}(-\gamma), \quad (\tilde{A}(\gamma))^\dagger = A(-). 
\] (38)

For example, it is trivial to check that in this case the operator of the spatial metric and, consequently, the area operator would be Hermitian operators. Thus, it is not necessary to deal explicitly with the constraint expressing the complex conjugate connection in terms of the original variables. Rather, the problem is moving in the direction of the representation theory of some complicated algebra.

In fact, the problem of finding the appropriate representation is still quite non-trivial, especially taking into account the very non-trivial form of the commutation relation between...
the self-dual and anti-self-dual connections (29). Indeed, the simplest solution to (38) would be that in the connection representation, which is extensively used in the loop approach, the two chiral connections are realized as multiplication operators by complex conjugate variables. But this contradicts their non-vanishing commutator. Nevertheless, the form of the Dirac algebra suggests that maybe it is possible to realize the self-dual and anti-self-dual connections as such multiplication operators when they act on the functions of only $A^+(\cdot)$ or $A^-(\cdot)$, respectively.

Although we do not know a representation of the algebra (29) where the self-dual connection is chosen as a configuration variable, it is easy to construct a representation with $\tilde{P}^+_a$ and $\tilde{P}^-_a$ being the configuration variables. In appendix C we show that the following operators:

$$\tilde{P}^+_a = \hat{P}^+_a, \quad \tilde{A}^+_a = i \frac{\delta}{\delta \tilde{P}^+_a} + i (\tilde{X}_+ \tilde{P}^+_a) \frac{\delta}{\delta \tilde{P}^+_b} + \Gamma^+_a \tilde{P}_b,$$

$$\tilde{P}^-_a = \hat{P}^-_a, \quad \tilde{A}^-_a = i \frac{\delta}{\delta \tilde{P}^-_a} + i (\tilde{X}_- \tilde{P}^-_a) \frac{\delta}{\delta \tilde{P}^-_b} + \Gamma^-_a \tilde{P}_b,$$

which act on the space of functions of $\tilde{P}$ and $\tilde{P}$ endowed with the usual scalar product, form the algebra isomorphic to (29) and satisfy (38). This shows that the search for representations of (29) is not hopeless. Also it may indicate that the so called triad representation, rather than the connection representation, might be more natural in quantum gravity.

In fact, a similar problem exists in the covariant approach to the loop quantization where the non-commutativity of the connection (see (13)) prevents it from being chosen as a configuration variable. This problem was either ignored or some tricks were made to achieve the commutativity for its holonomies [15]. In this respect the situation in Ashtekar gravity is more promising. It allows us to consider holonomies of the self-dual or anti-self-dual connection only and these chiral quantities are commutative. Therefore, the problem arises only when one considers their mutual commutators.

We conclude that the results of this paper show the similarity of the approaches based on the Lorentz-covariant formulation and on the complex Ashtekar formulation, both in the resulting structures and in the arising problems. We hope that they can help each other in solving these problems and finding the correct way to quantize gravity.

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Appendix A. Definitions and properties

Structure constants of the Lorentz algebra:

$$f^{A_1}_{A_2 A_3} = 0, \quad f^{A_1}_{B_2 A_3} = -\epsilon^{A_1 B_2 A_3}, \quad f^{A_1}_{A_2 B_3} = 0, \quad f^{B_1}_{B_2 B_3} = -\epsilon^{B_1 B_2 B_3}, \quad f^{B_1}_{A_2 A_3} = 0, \quad f^{B_1}_{A_2 B_3} = \epsilon^{A_1 A_2 B_3}. \quad (A.1)$$

Here we split the six-dimensional index $X$ into a pair of three-dimensional indices, $X = (A, B)$, so that $A, B = 1, 2, 3$. The indices $A$ correspond to the Lorentz boosts, whereas the indices $B$ label the $SO(3)$ subgroup.
Killing form:
\[ g_{XY} = \frac{1}{4} f_{XYZ} f_{Zi} \quad g^{XY} = (g^{-1})^{XY}, \quad g_{XY} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & -\delta_{ab} \end{pmatrix}. \quad (A.2) \]

Matrix algebra:
\[ \Pi^{XY} = (\Pi^{-1})^{XY} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \delta_a^b, \quad R^{XY} = \begin{pmatrix} 1 & -\frac{1}{\beta} \\ -\frac{1}{\beta} & -1 \end{pmatrix} \delta_a^b. \quad (A.3) \]

The matrices \( \Pi_X^X \), \( R_X^X \) and their inverses commute with each other. Furthermore, they commute with the structure constants in the following sense:
\[ f^{XYZ} \Pi_Z^X = f^{XYZ} \Pi_Y^X. \quad (A.4) \]

The contraction of two structure constants can be decomposed as follows:
\[ f_{XY} f_{YZ} = -g_{XZ} \delta^W_Y + \Pi_{XZ} \Pi_Y^W - \Pi_{YZ} \Pi_X^W. \quad (A.5) \]

Inverse fields:
\[ P^X_Y = \left( \frac{\delta^X_b - X^b X^X}{1 - \chi^2} E^b_i, -\frac{\epsilon^{abc} X^b X^c}{1 - \chi^2} \right), \quad Q^X_i = \left( \frac{\epsilon^{abc} E^b_i X^c}{1 - \chi^2}, \delta^X_b - X^b X^X E^b_i \right). \quad (A.6) \]

Projectors:
\[ I^Y_{(P)X} = \left( \frac{\delta^Y_b - X^b X^X}{1 - \chi^2} E^b_i, -\frac{\epsilon^{abc} X^b X^c}{1 - \chi^2} \right), \quad I^Y_{(Q)X} = \left( \frac{\epsilon^{abc} E^b_i X^c}{1 - \chi^2}, -\delta^Y_b + X^b X^X E^b_i \right). \quad (A.7) \]

Properties of the inverse fields and the projectors:
\[ P^X_Y = -\Pi^X_Y Q^Y_Y, \quad I^X_{(P)} = -\Pi^X_W I^W_{(Q)} \Pi^Y_W, \quad (A.8) \]

\[ \tilde{Q}^Y_X = \delta^Y_j, \quad \tilde{P}^X_j = \delta^X_j, \quad \tilde{Q}^X_j = \tilde{P}^X_j = 0 \quad (A.9) \]

\[ I^Y_{(P)Z} I^X_{(P)X} = I^X_{(P)X}, \quad I^Y_{(Q)Z} I^X_{(Q)X} = I^X_{(Q)X}, \quad I^Y_{(P)X} + I^Y_{(Q)X} = \delta^Y_X. \quad (A.10) \]

The projector \( I_{(P)} \) projects on \( \tilde{P} \) and \( P \), whereas \( I_{(Q)} \) projects on \( \tilde{Q} \) and \( Q \). For example, one has \( I^Y_{(P)X} P^Y_X = P^Y_X, I^Y_{(Q)X} Q^Y_X = 0 \) and other similar relations. Other two useful identities are
\[ f^{WYZ} I^X_{(P)W} \tilde{Q}^Y_{(Q)Z} = 0, \quad f^{WYZ} I^X_{(Q)W} \tilde{Q}^Y_{(Q)Z} = f^{XYZ} \tilde{Q}^Y_{(Q)Z}. \quad (A.10) \]

Gauge transformations (\( \tilde{G}(\xi) = \int d^3 x \, \tilde{G}^X G_X \)):
\[ \{ \tilde{G}(\xi), \tilde{Q}^X_\ell \}_{(D)} = f^X_{\ell} \tilde{G}^X \tilde{Q}^X_\ell, \quad [ \tilde{G}(\xi), \tilde{Q}^X_\ell \]_{(D)} = -f^X_{\ell} \tilde{G}^X \tilde{Q}^X_\ell. \quad (A.11) \]

Commutator of two shifted connection:
\[ \{ A^Y_i(x), A^Y_j(y) \} = \frac{1}{2} (\Pi^X_{i} M_{ij}^{XY} - M_{ij}^{XY} \Pi^X_{j}) \delta(x, y). \quad (A.12) \]

where\(^5\)
\[ M_{ij}^{XY} (x, y) = -\frac{1}{2} (V_{ij}(x) \delta^X_i + V_{ij}(y) \delta^Y_j) + W_{ij}^{XY}. \quad (A.13) \]

\(^5\) There is a sign mistake in the first term in (A.13) in the printed version of [24].
and
\[ \mathcal{V}_{ij}^{XY} = f_Q^{iP} \left[ \tilde{Q}_P \left( \{ Q \tilde{Q} \}_{ij} \right) \tilde{Q}_{ij}^X + \tilde{Q}_i^X \tilde{Q}_j^Y - \tilde{Q}_j^X \tilde{Q}_i^Y + \delta_i^j \tilde{Q}_Q^Y \tilde{P}_Q^P \right] \]  
(A.14)
\[ \mathcal{W}_{ij}^{XY} = \frac{1}{2} ( \mathcal{L}_{ij}^{XY} + \mathcal{L}_{ji}^{XY} ) + \frac{g^{SS}}{2} \left( \tilde{I}_{i(\tilde{P} \tilde{Q})}^S \mathcal{V}_{ij}^{SY} + \tilde{I}_{i(\tilde{Q} \tilde{Q})}^S \mathcal{L}_{ij}^{SY} \right) \tilde{Q}_{ij}^S \partial_j \tilde{Q}_Q^P, \]  
(A.15)
\[ \mathcal{L}_{ij}^{XY} = f_P^Q \left[ \tilde{Q}_Q^X \tilde{Q}_Q^Y, (Q \tilde{Q})_{ij} \tilde{I}_{i(\tilde{Q} \tilde{Q})}^Y - \tilde{Q}_j^X \tilde{Q}_j^Y \right] \tilde{Q}_P^i \partial_i \tilde{Q}_Q^P + f_Q^P \tilde{Q}_P^i \tilde{Q}_i^X \tilde{Q}_i^Y \partial_i \tilde{Q}_Q^P. \]  
(A.16)

Since this operator is implied to act on \( \delta(x, y) \), the argument of the last term in (A.13) is not important. The antisymmetry of the bracket is ensured by the antisymmetry property of the matrix (A.14)
\[ \mathcal{V}_{ij}^{XY} = -\mathcal{V}_{ji}^{XY}, \]  
(A.17)
which can be checked by straightforward calculations. As another consistency check of the commutator (A.12), one can check with the help of (A.11) that the gauge transformations of the two sides agree.

Appendix B. Consistency conditions

In this appendix we are going to check that relation (33) can be considered as a strong equality on the extended phase space spanned by \( (\tilde{P}_{(\alpha)}, \tilde{P}_{(-\alpha)}, A^{(+)}, A^{(-)}) \). In other words, one should prove that the Dirac brackets (29) remain true if one substitutes \( A^{(-)} \) by the rhs of (33). This can easily be done for the chiral components of the triad. Indeed, one obtains
\[ \{ \ldots \}^{a}_{\ldots} = \{ A^{(-)} \}^{a}_{\ldots} \left( \ldots, \tilde{P}_{(\alpha)} \right) = \left\{ A^{(-)} \right\}^{a}_{\ldots} \left( \ldots, \tilde{P}_{(\alpha)} \right). \]  
(B.1)
\[ \{ \ldots \}^{a}_{\ldots} = \{ A^{(+)} \}^{a}_{\ldots} \left( \ldots, \tilde{P}_{(-\alpha)} \right) = \left\{ A^{(+)} \right\}^{a}_{\ldots} \left( \ldots, \tilde{P}_{(-\alpha)} \right). \]  
(B.2)
For the two anti-self-dual connections one finds
\[ \{ \ldots \}^{c}_{\ldots} = \{ A^{(-)} \}^{c}_{\ldots} \left( \ldots, \tilde{P}_{(-\alpha)} \right) = \left\{ A^{(-)} \right\}^{c}_{\ldots} \left( \ldots, \tilde{P}_{(-\alpha)} \right). \]  
(B.3)
To reproduce the commutator \( \{ A^{(-)}, A^{(-)} \} \) one should prove that this expression vanishes. This is natural to expect since \( \tilde{\Gamma} \) is a connection compatible with the three-dimensional metric. Therefore, for \( \chi = 0 \) the vanishing of (B.3) reduces to the well-known statement that the Barbero connection is commutative. To perform the calculations, it might be easier to work in the explicitly Lorentz-covariant formulation where the statement we need to prove becomes
\[ R_{(-\alpha)}^X \left( \left\{ A^X, \Gamma^Y_{\alpha} \right\} + \left\{ \Gamma^Z_{\alpha}, A^Y_{\alpha} \right\} \right) R_{(-\alpha)} = 0. \]  
(B.4)
Using the explicit expression for \( \tilde{\Gamma}_j^X \) (11), the commutation relation (12) and various properties from appendix A, relation (B.4) can be checked by tedious and lengthy calculations.
Finally, it remains to prove that
\[ \left\{ A^{(-)}, \ldots, \tilde{P}_{(-\alpha)} \right\} = \left\{ \tilde{P}_{(\alpha)} \right\} = i \left( R_{(+)} \cdot M_{ij} \cdot R_{(-)} \right) \]  
(B.5)
where $ \mathcal{M}_{ij} $ is given in (A.13). Again, it is more convenient to work in the covariant notations. Then the statement to be proved reads

$$
2R_{(\hat{X})} X \{ A^X_i, \Gamma^X_{ij} \} \delta R_{(-Y)} Y = iR_{(\hat{X})} X \mathcal{M}^{XY}_{ij} R_{(-Y)} Y \quad .
$$

(B.6)

Note that the connection $ \Gamma^X_i $ is related to the quantity (A.14) as

$$
\Gamma^X_i = -\frac{1}{2} \nu^{WX,i} \partial_i \tilde{Q}^W_x \quad .
$$

(B.7)

Substituting this relation into (B.6) and using the antisymmetry property (A.17), one immediately reproduces the first term in (A.13) with derivatives. The remaining terms lead to the following statement:

$$
-iR_{(\hat{X})} X \{ A^X_i, \nu^{WY,i}_j \} \delta \tilde{Q}^W_x R_{(-Y)} Y
$$

$$
= \frac{1}{2} R_{(\hat{X})} X \left( \mathcal{L}^{WY}_{ij} \nu^{YX,i}_j + \nu^{WY,j}_i \mathcal{L}^{YX}_i - g_{WS} I^{WY}_j \nu^{XY}_j \tilde{Q}^X_x \delta \tilde{Q}^W_x + I^{XY}_j \nu^{WY}_j \right) R_{(-Y)} Y \quad .
$$

(B.8)

The Dirac bracket on the lhs can be easily evaluated and one obtains

$$
-iR_{(\hat{X})} X \{ A^X_i, \nu^{WY,i}_j \} \delta \tilde{Q}^W_x R_{(-Y)} Y
$$

$$
= R_{(+)} X \left( \nu^{WY}_j \mathcal{L}^{YX}_i + \nu^{XY}_i \mathcal{L}^{WY}_j - g_{WS} I^{WY}_j \nu^{XY}_j \tilde{Q}^X_x \tilde{Q}^W_x + I^{XY}_j \nu^{WY}_j \right) \partial \tilde{Q}^W_x R_{(-Y)} Y \quad .
$$

(B.9)

Using the explicit expressions for $ \nu^{WY}_j $ and $ \mathcal{L}^{YX}_i $ from (A.14) and (A.16), one can show that the rhs of (B.8) and (B.9) indeed coincide.

**Appendix C. Triad representation**

We want to check that the operators (39) and (40) give a representation of the algebra (29) satisfying condition (38). The latter fact is completely trivial as soon as the scalar product is defined with the trivial measure $ D \tilde{P} D \tilde{P} $. The commutation relations with the triad operators $ \tilde{P}_{(+)} $ and $ \tilde{P}_{(-)} $ also trivially give the necessary results. The only non-trivial check must be done for the commutation relations involving the connections. For the two self-dual connections one finds

$$
\left[ \tilde{\Gamma}^{(a)\alpha}_{ij}, \tilde{\Gamma}^{(b)\beta}_{ij} \right] = i \left( \frac{\delta \Gamma^{(b)\beta}_{ij}}{\delta P^a_i} + \left( X_{-} \right)^a \frac{\delta \Gamma^{(b)\beta}_{ij}}{\partial X_{-}^a} - \left( X_{+} \right)^a \frac{\delta \Gamma^{(a)\alpha}_{ij}}{\partial X_{+}^a} \right) \quad .
$$

(C.1)

Then the property (28) and the constraint (33) allow us to rewrite this as

$$
\left[ \tilde{\Gamma}^{(a)\alpha}_{ij}, \tilde{\Gamma}^{(a)\alpha}_{ij} \right] = \frac{i}{2} \left( \left( X_{-} \right)^a \tilde{A}^{(a)\alpha}_{ij} \right) D \left( -X_{+} \right)^a = 0 \quad .
$$

(C.2)

It is clear that the same result is valid for the two anti-self-dual connections. In a similar way the commutator of the self-dual and the anti-self-dual connections is found as

$$
\left[ \tilde{\Gamma}^{(a)\alpha}_{ij}, \tilde{\Gamma}^{(b)\beta}_{ij} \right] = i \left( \frac{\delta \Gamma^{(b)\beta}_{ij}}{\delta P^a_i} + \left( X_{-} \right)^a \frac{\delta \Gamma^{(b)\beta}_{ij}}{\partial X_{-}^a} - \left( X_{+} \right)^a \frac{\delta \Gamma^{(a)\alpha}_{ij}}{\partial X_{+}^a} \right) \quad .
$$

(C.3)

This completes the proof that the operators (39) and (40) give a representation of the Dirac algebra.
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