EFFECTIVE POTENTIAL FOR SCALAR FIELD IN THREE DIMENSIONS: ISING MODEL IN THE FERROMAGNETIC PHASE

M.M. Tsypin

Department of Theoretical Physics,
P. N. Lebedev Physical Institute, Leninsky pr. 53, 117924 Moscow, Russia

Abstract

We compute the effective potential $V_{\text{eff}}(\varphi)$ for one-component real scalar field $\varphi$ in three Euclidean dimensions (3D) in the case of spontaneously broken symmetry, from the Monte Carlo simulation of the 3D Ising model in external field at temperatures approaching the phase transition from below. We study probability distributions of the order parameter on the lattices from $30^3$ to $74^3$, at $L/\xi \approx 10$. We find that, in close analogy with the symmetric case, $\varphi^6$ plays an important role: $V_{\text{eff}}(\varphi)$ is very well approximated by the sum of $\varphi^2$, $\varphi^4$ and $\varphi^6$ terms. An unexpected feature is the negative sign of the $\varphi^4$ term. As close to the continuum limit as we can get ($\xi \approx 7.2$), we obtain

$$\mathcal{L}_{\text{eff}} \approx \frac{1}{2} \partial_\mu \varphi \partial_\mu \varphi + 1.7(\varphi^2 - \eta^2)^2(\varphi^2 + \eta^2).$$

We also compute several universal coupling constants and ratios, including the combination of critical amplitudes $C^- (f^-)^{-3} B^{-2}$.
Introduction

This work continues our previous Monte Carlo study [1], which was devoted to the effective potential $V_{\text{eff}}(\varphi)$ for the theory of one-component real scalar field in three Euclidean dimensions in the symmetric (paramagnetic, PM) phase. There we have found that $V_{\text{eff}}$ is very well approximated by the sum of $\varphi^2$, $\varphi^4$ and $\varphi^6$ terms, and computed universal 4-point and 6-point couplings. Here we turn to the spontaneously broken (ferromagnetic, FM) phase of the same theory. To compute the effective potential in the FM case we use largely the same approach we have developed for the PM case, with some modifications. As the detailed discussion, with all necessary references, was given in [1], we describe it here only briefly, concentrating mostly on the points specific for the broken phase.

Monte Carlo computation

We consider the Ising model in external field,

$$ Z = \sum_{\{\phi_i\}} \exp \left\{ \beta \sum_{\langle ij \rangle} \phi_i \phi_j + J \sum_i \phi_i \right\}, \quad \phi_i = \pm 1, \quad (1) $$

on a simple cubic lattice of the size $L^3$ (from $30^3$ to $74^3$) with periodic boundary conditions. The critical coupling is $\beta_c \approx 0.221655$ [2]. We use Swendsen-Wang cluster Monte Carlo algorithm [3] to generate the Boltzmann ensemble of configurations, for given coupling $\beta$ and external field $J$. For every configuration we measure the order parameter (magnetization per site)

$$ \varphi = \frac{1}{N} \sum_{i=1}^{N} \phi_i, \quad (2) $$

where $N = L^3 = \Omega$ is the total number of sites. Thus we obtain probability distributions $P(\varphi)$, in form of histograms (Fig. 1). The relation between $P(\varphi)$ and $V_{\text{eff}}(\varphi)$, for sufficiently large volume $\Omega$, is [1]

$$ P(\varphi) \propto \sqrt{V''_{\text{eff}}(\varphi)} \exp \left\{ -\Omega V_{\text{eff}}(\varphi) + \Omega J \varphi \right\}. \quad (3) $$

Now one can check whether it is possible to fit the set of probability distributions for given $\beta$ and several values of $J$ with this formula, using this or that ansatz for $V_{\text{eff}}(\varphi)$.

We see that the situation is analogous to that in the PM phase [1]: the ansatz $V_{\text{eff}}(\varphi) = r\varphi^2 + u\varphi^4$ works poorly, while

$$ V_{\text{eff}}(\varphi) = r\varphi^2 + u\varphi^4 + w\varphi^6 \quad (4) $$

provides a perfect fit (Figs. [1] [2]). In the remaining part of the paper we discuss extraction of universal quantities from $r$, $u$, $w$ obtained from such fits.
Figure 1: Probability distributions $P(\phi)$ for magnetization per lattice site, for the Ising model (1). The points are from Monte Carlo simulation; the solid line is the fit with (3), $V_{\text{eff}}(\phi) = r\phi^2 + u\phi^4 + w\phi^6$ (all 5 histograms are fitted simultaneously with one $V_{\text{eff}}$).

Figure 2: The Monte Carlo points are the same as at Fig. 1. The solid line is given by (3), $V_{\text{eff}}(\phi) = r\phi^2 + u\phi^4$, where $r$ and $u$ are fixed by the requirement that spontaneous magnetization and zero field susceptibility are correctly reproduced.
Problems with fitting procedures

In the FM phase several complications arise, that make determination of parameters of the effective potential much more tricky than in the PM phase.

First, it turns out that the ratio of lattice size to correlation length $L/\xi$ has to be much larger in the FM phase than in the PM phase, to get finite size effects under control. The formula (3) is valid in the limit of large $L/\xi$. It turns out that to get finite volume corrections to $V_{\text{eff}}$ as small as they are in the PM phase at $L/\xi = 4$, one has to go to $L/\xi$ as large as 10 in the FM phase.

Thus it is difficult to go to large $\xi$, as the lattice size becomes prohibitively large. This, in its turn, makes it very difficult to study the finite cutoff effects and to extrapolate to $\xi \to \infty$. While it was possible to do such an extrapolation, with a reasonable degree of confidence, in the PM case, where we could study $\xi$ as large as 14.1, in the FM case we can only get a qualitative estimate of finite cutoff corrections, and have no choice other than to take the values from our largest lattice, without attempting any extrapolation.

Secondly, the applicability of eq. (3) for various values of $\varphi$ depends on the ratio $L/\xi(\varphi)$, where $\xi(\varphi)$ is determined by $V''_{\text{eff}}(\varphi)$, and deteriorates quickly at $|\varphi| < M$, where $M$ is spontaneous magnetization. Thus we have to use for the fitting only the subset of data corresponding to $|\varphi| > M$, losing considerable amount of information (practically, a half of the $J = 0$ histogram).

So one has to take into account that

1) Including larger values of $J$ into the fit reduces statistical errors, but increases systematic errors connected with the finite cutoff (i.e. the smallness of $\xi$), as larger $J$ mean smaller $\xi$.

2) It would be nice to use the information contained in the whole $J = 0$ histogram, including $|\varphi| < M$.

One can see how this works, from Table 1, where we have compiled some observables that can be measured directly, as well as position $M$ of the minimum of $V_{\text{eff}}$ (that corresponds to spontaneous magnetization $\langle |\varphi| \rangle$) and its second derivative in the minimum $V''_{\text{eff}}(M)$ (that corresponds to inverse susceptibility $\chi^{-1}$), obtained by fitting either 2 or 3 histograms.

From Table 1 one observes that

1) The best results for both $M$ and $V''_{\text{eff}}(M)$, as far as statistical errors are concerned, come from the direct measurement (apparently because the region $|\varphi| < M$ is lost for the fitting).

2) Fitting 2 histograms leads to results compatible with direct measurement, for all lattices, but with unsatisfactorily large statistical errors.

3) Fitting 3 histograms leads to considerable reduction of statistical errors. However, with the exception of the two largest lattices, there is a serious deviation of $M$ and $V''_{\text{eff}}(M)$ from the direct measurement. This is a manifestation of a systematic error associated with the finite cutoff, due to the smallness of the correlation length at $J = J_2$ (corresponding to the third histogram) on our smaller lattices.
The improved fitting procedure

Thus both fitting procedures in the Table 1 are far from satisfactory. So we have designed the following procedure that uses the $J = 0$ data to significantly reduce statistical errors of the 2-histogram fit: one should fix the minimum of $V_{\text{eff}}$ at the value of $M$ obtained from the direct measurement. This leads to 2-parameter fit instead of the 3-parameter one, with corresponding reduction of statistical errors. The results are collected in Table 2.

One observes that: 1) the quality of fits, as indicated by $\chi^2$, is very high: there is no discrepancy between Monte Carlo histograms and the fit, other than statistical noise; 2) $V''_{\text{eff}}(M)$ is completely consistent with the direct measurement; 3) Fitting 5 histograms for our largest lattice without fixing the position of the minimum of $V_{\text{eff}}$ (Fig. 1 and the last column in Table 2) provides both $M$ and $V''_{\text{eff}}(M)$ completely consistent with the direct measurement. All parameters are consistent with the 2-histogram fit.

Extraction of universal parameters

In the FM case the extraction of universal, i.e. dimensionless, parameters of $V_{\text{eff}}$ from the data becomes less trivial than in the PM case.

Having measured $r$, $w$, $u$ from the fits and the field renormalization constant $Z$ from the small-momentum behavior of the propagator, we obtain the effective Lagrangian in the form

$$L_{\text{eff}} = \frac{1}{2} Z^{-1} \partial_\mu \varphi \partial_\mu \varphi + r \varphi^2 + u \varphi^4 + w \varphi^6.$$  \hspace{1cm} (5)

Then we change the scale of $\varphi$, introducing the renormalized field $\varphi_R = Z^{-1/2} \varphi$:

$$L_{\text{eff}} = \frac{1}{2} \partial_\mu \varphi_R \partial_\mu \varphi_R + Zr \varphi_R^2 + Z^2 u \varphi_R^4 + Z^3 w \varphi_R^6.$$  \hspace{1cm} (6)

The coefficients in front of $\varphi_R^2$, $\varphi_R^4$ and $\varphi_R^6$ have, correspondingly, dimensionalities $m^2$, $m$ and 1. In the PM case it was natural to choose $\sqrt{2Zr}$ as a scale factor, and to use it to render the 4-point coupling dimensionless, obtaining two dimensionless parameters: $g_4 = Z^2 u / \sqrt{2Zr}$ and $g_6 = Z^3 w$.

In the FM case the coefficient in front of $\varphi_R^2$ does not determine the correlation length any more. Moreover, $r$ is determined with very large statistical errors. So we have to find something different that could serve as a scale factor. Spontaneous magnetization seems to be the best choice: $\langle |\varphi_R|^2 \rangle^2 \equiv M_R^2 = Z^{-1} M^2$ has the dimensionality of mass and very small statistical error.

Thus the coefficient in front of $\varphi_R^4$ provides a dimensionless parameter $Z^3 u / M^2$ (included in Table 2). Another interesting dimensionless parameter is obtained from the mass:

$$G \equiv m / M_R^2 = \frac{Z^{3/2} \sqrt{V''}}{M^2}.$$  \hspace{1cm} (7)

Its remarkable property is that, being a special universal combination of critical amplitudes, it can be measured, to high precision, without any fitting (Fig. 3):

$$G = \chi \xi^{-3} M^{-2},$$  \hspace{1cm} (8)
Figure 3: Dimensionless parameter \( G = Z^{3/2} \sqrt{V''} M^{-2} = \chi \xi^{-3} M^{-2} \), as a function of inverse correlation length. The diamonds are our Monte Carlo results, the square is from [4], and the solid line is from approximants given by Liu and Fisher [5] (see Appendix). The line has an uncertainty of order 3% in the overall factor. If one combines our points with \( m \)-dependence from Liu and Fisher’s approximants, one gets in the scaling limit \( G \to C^-(f^{-1}_{\xi})^{-3} B^{-2} \approx 5.0 \).

\[
\lim_{\xi \to \infty} G = C^-(f^{-1}_{\xi})^{-3} B^{-2}. \tag{9}
\]

(See Appendix for notation). More dimensionless parameters are obtained from the third and fourth derivatives of \( V_{\text{eff}} \) in the minimum (Table 2), such as

\[
\kappa = \frac{V'''M}{6V''}. \tag{10}
\]

**Expansion around the minimum**

Consider the situation in the vicinity of the minimum of \( V_{\text{eff}} \). One can write \( \varphi(x) = M + \tilde{\varphi}(x) \), where \( \tilde{\varphi} \) is a small deviation. Then for \( \tilde{\varphi} \) we get

\[
\mathcal{L}_{\text{eff}} = \frac{1}{2} Z^{-1} \partial_{\mu} \tilde{\varphi} \partial_{\mu} \tilde{\varphi} + \frac{1}{2} V'' \tilde{\varphi}^2 + \frac{1}{3!} V''' \tilde{\varphi}^3 + \frac{1}{4!} V'''' \tilde{\varphi}^4 + \ldots, \tag{11}
\]

where all derivatives of \( V_{\text{eff}} \) are taken in its minimum. After changing the scale, \( \tilde{\varphi} = \sqrt{Z} \tilde{\varphi}_R \),

\[
\mathcal{L}_{\text{eff}} = \frac{1}{2} \partial_{\mu} \tilde{\varphi}_R \partial_{\mu} \tilde{\varphi}_R + \frac{1}{2} Z V'' \tilde{\varphi}_R^2 + \frac{1}{3!} Z^{3/2} V''' \tilde{\varphi}_R^3 + \frac{1}{4!} Z^2 V'''' \tilde{\varphi}_R^4 + \ldots, \tag{12}
\]
Now we obtain the mass,

\[ m = \xi^{-1} = \sqrt{ZV''} \] (13)

and dimensionless 3-point and 4-point couplings:

\[ \tilde{g}_3 = \frac{1}{3!} \frac{Z^{3/2}V''''}{m^{3/2}} = \frac{1}{6} (Z^{-1}V'')^{-3/4} V''' \] (14)

\[ \tilde{g}_4 = \frac{1}{4!} \frac{Z^2V'''''}{m} = \frac{1}{24} \frac{Z^{3/2}V''''}{\sqrt{V''}}. \] (15)

They are in the following correspondence with \( \tilde{g}_R^{(3)} \) and \( \tilde{g}_R^{(4)} \) computed by J.-K. Kim and A. Patrascioiu [6]:

\[ \tilde{g}_3 = \frac{1}{6} \tilde{g}_R^{(3)}, \] (16)

\[ \tilde{g}_4 = \frac{1}{24} \left[ \tilde{g}_R^{(4)} + 3(\tilde{g}_R^{(3)})^2 \right]. \] (17)

As can be seen from Fig. 4, our values of \( \tilde{g}_3 \) are somewhat lower than those obtained in [4] by direct measurement of the 3-point correlation function. As a consistency check, we have computed \( \tilde{g}_3 \) and \( \tilde{g}_4 \) for two largest lattices directly from 1-particle irreducible 3- and 4-point functions, and obtained

\[ 58^3, \quad \beta = 0.2232 : \quad \tilde{g}_3 = 2.41(7), \quad \tilde{g}_4 = 5.4(7); \]

\[ 74^3, \quad \beta = 0.2227 : \quad \tilde{g}_3 = 2.35(8), \quad \tilde{g}_4 = 5.1 \pm 2.1, \] (18)
in complete agreement with Table 2.

**Compact formula for $V_{\text{eff}}$**

We see that quite a few dimensionless parameters of $V_{\text{eff}}$ can be constructed. However, one would like to have $V_{\text{eff}}$ written down in a concise form, so that its shape can be better understood. For the PM phase it could be written \[1\] as

$$V_{\text{eff}} = \frac{1}{2} m^2 \varphi_R^2 + mg_4 \varphi_R^4 + g_6 \varphi_R^6, \quad (19)$$

with the scale parameter $m$ and two dimensionless (and thus universal) parameters $g_4$ and $g_6$.

In the broken phase one can take as a starting point the following observation. If the constant term in $V_{\text{eff}}$ is chosen in such a way that $V_{\text{eff}} = 0$ in the minimum, it has double zeros at $\varphi_R = \pm M_R$, and must be proportional to $(\varphi_R^2 - M_R^2)^2$. Thus it must take the form

$$V_{\text{eff}}(\varphi_R) = (\varphi_R^2 - M_R^2)^2(a \varphi_R^2 + b M_R^2), \quad (20)$$

with two dimensionless parameters $a$ and $b$, that can be expressed via $G$ and $\kappa$ \[8,10\], leading to

$$V_{\text{eff}}(\varphi_R) = \frac{1}{16} G^2 (\varphi_R^2 - M_R^2)^2 \left[2 M_R^2 + (2\kappa - 1)(\varphi_R^2 - M_R^2)\right]. \quad (21)$$

Unlike the symmetric phase, where we had the range of $\xi$ sufficient to discuss extrapolation to $\xi \to \infty$, our results for the broken phase indicate the existence of the finite cutoff effects, but are clearly not sufficient to extrapolate to $\xi \to \infty$. So we can only write down the result for our largest $\xi$ (the last columns of Tables \[1\] and \[2\]):

$$G = 5.27(9), \quad \kappa = 0.998(14), \quad V_{\text{eff}} = 1.73(6) \cdot (\varphi_R^2 - M_R^2)^2 \left[2 M_R^2 + 1.00(3) \cdot (\varphi_R^2 - M_R^2)\right]. \quad (22)$$

The hope is that these numbers are already close to the continuum limit. It is interesting to observe that $V_{\text{eff}}$ turns out to be proportional to $(\varphi^2 - M^2)^2(\varphi^2 + M^2)$. This sheds some light on the negative sign of $\varphi^4$, as $(\varphi^2 - 1)^2(\varphi^2 + 1) = 1 - \varphi^2 - \varphi^4 + \varphi^6$.

**Comparison with analytical results**

The only analytical results on the effective potential in the broken phase we know of are from the $\varepsilon$-expansion of the scaling equation of state (see \[7\]), and those obtained recently by Berges, Tetradis and Wetterich \[4\] using a method based on an exact flow equation for a coarse grained free energy. In their notation,

$$U_R(\rho_R) \equiv V_{\text{eff}}(\varphi_R), \quad \rho_R \equiv \frac{1}{2} \varphi_R^2, \quad (24)$$
Figure 5: The plots of the “Ising equation of state”, \( J(\varphi) = dV_{\text{eff}}(\varphi)/d\varphi \), with effective potentials computed by different means, all of them normalized in such a way that the minimum of \( V_{\text{eff}} \) is at \( M = 1 \), and \( V''_{\text{eff}} = 1 \). The curves 0–3 are from the \( \varepsilon \)-expansion of the parametric representation of the equation of state \([7]\), in orders \( \varepsilon^0 \ldots \varepsilon^3 \), respectively. The curve \( W \) is from \([4]\), and the curve \( V \) represents our result \([23]\), which reduces in this normalization to \( J = \frac{1}{8}(-\varphi - 2\varphi^3 + 3\varphi^5) \).

They obtain

\[
\hat{\lambda}_R \equiv \frac{U''_R(\rho_0 R)}{\rho_0 R} = 61.6, \tag{25}
\]

\[
\hat{\nu}_R \equiv U'''_R(\rho_0 R) = 107. \tag{26}
\]

This translates, in our notation, to

\[
G = \left( \frac{1}{2} \frac{\hat{\lambda}_R}{\rho_0 R} \right)^{1/2} \approx 5.55, \tag{27}
\]

\[
\kappa \approx 1.08, \tag{28}
\]

\[
\tilde{g}_3 \approx 2.54. \tag{29}
\]

These values are included in Figs. 3, 4 and are 5–8% higher than our Monte Carlo results. Berges, Tetradis and Wetterich give also a complicated formula that serves as an approximant for \( J(\varphi) = V'_{\text{eff}}(\varphi) \). The corresponding curve is shown in Fig. 5, and goes quite close to ours. Also included are three known terms of the \( \varepsilon \)-expansion of the parametric representation of equation of state \([4]\).
Conclusions

Here are our main conclusions about the universal properties of the effective potential for the broken phase of the 3D Ising model in the scaling region:

1. The form \( r \varphi^2 + u \varphi^4 + w \varphi^6 \) provides a very good approximation for \( V_{\text{eff}}(\varphi) \) (Fig. 1), while without the \( \varphi^6 \) term this is not achievable (Fig. 2) — the same result as in the symmetric phase \([1]\).

2. Not only \( r \), but also \( u \) turns out to be negative (Table 2).

3. Quantitative results can be summarized by eq. (23).

4. The combination of critical amplitudes \( G = C - (f^-_1)^{-3}B^{-2} \) (Fig. 3) seems to be a quantity very suitable for a precise Monte Carlo computation, especially using improved estimators, as it does not suffer from uncertainties in \( T_c \) and critical exponents.

It should be noted that, thinking about \( V_{\text{eff}} \) in terms of Taylor expansion around the minimum, one would need at least 4 parameters to describe our data \((M, V''_{\text{eff}}(M), V'''_{\text{eff}}(M) \text{ and } V''''_{\text{eff}}(M))\), and the symmetry \( \varphi \to -\varphi \) would be lost. Thus the possibility to approximate \( V_{\text{eff}} \) with \( r \varphi^2 + u \varphi^4 + w \varphi^6 \), that has only 3 parameters and respects the symmetry, is by no means trivial, and demonstrates, once more, a special role of \( \varphi^6 \) in three dimensions.

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Appendix

In Table 1 and Fig. 3 we have included, for comparison, the values obtained from approximants given by Liu and Fisher [5]:

\[
\begin{align*}
t &= (T - T_c)/T_c = (\beta_c - \beta)/\beta, \\
\xi^2(t) &= (f^-_1)^2|t|^{-2\nu} - 0.430, \quad f^-_1 = 0.2502(8), \quad 2\nu = 1.267, \\
\chi(t) &= C^-|t|^{-\gamma} - 2.71, \quad C^- = 0.220(4), \quad \gamma = 1.2395, \\
M(t) &= B|t|^{\beta}(1 - 0.256\sqrt{|t|}), \quad B = 1.71(2), \quad \beta = 0.3305.
\end{align*}
\]

We use the best available estimate of the critical coupling, \( \beta_c = 0.221655 \) [3], rather than the value 0.22163 originally associated with these approximants.
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| $L^3$ | $30^3$ | $38^3$ | $46^3$ | $58^3$ | $74^3$ |
|-------|-------|-------|-------|-------|-------|
| $\beta$ | 0.226 | 0.2246 | 0.2239 | 0.2232 | 0.2227 |
| $J_0$ | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| $J_1$ | 0.004 | 0.0022 | 0.0014 | 0.00078 | 0.00043 |
| $J_2$ | 0.010 | 0.0056 | 0.0035 | 0.0020 | 0.0011 |
| $J_3$ | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| $J_4$ | 0.0017 | 0.0017 | 0.0017 | 0.0017 | 0.0017 |
| $N_{\text{config}}$ | $3 \times 360000$ | $3 \times 360000$ | $3 \times 360000$ | $3 \times 360000$ | $5 \times 100000$ |
| $Z_{(J=0)}^{-1}$ | 0.3167(11) | 0.3166(13) | 0.3149(15) | 0.3174(14) | 0.3133(32) |
| $M = \langle |\varphi| \rangle_{J=0}$ | 0.44975(17) | 0.39972(19) | 0.36743(18) | 0.32700(16) | 0.2892(3) |
| $M$ from $\beta$ | 0.4468 | 0.3962 | 0.3640 | 0.3235 | 0.2856 |
| $V''$ from the propagator | 0.0365(1) | 0.02207(8) | 0.01558(7) | 0.00958(4) | 0.00597(6) |
| $\chi^{-1}$ from $\beta$ | 0.0374 | 0.0224 | 0.0158 | 0.00982 | 0.00600 |
| $\xi = (Z^{-1}/V'')^{1/2}$ | 2.946(6) | 3.788(10) | 4.496(14) | 5.756(17) | 7.24(5) |
| $G \equiv Z^{3/2}/\sqrt{V''}M^{-2}$ | 5.30(3) | 5.22(3) | 5.23(4) | 5.12(4) | 5.27(9) |

Fitting 2 histograms ($J = J_0, J_1$) at $|\varphi| > M$

| $\sum \chi^2$ | 102 | 106 | 111 | 92 | 83 |
| $N_{\text{bins}}$ | 104 | 98 | 108 | 109 | 106 |
| $M$ | 0.4491(8) | 0.3993(13) | 0.3672(7) | 0.3271(7) | 0.2899(7) |
| $V''$ | 0.0353(13) | 0.0217(12) | 0.0154(5) | 0.0096(3) | 0.0063(3) |

Fitting 3 histograms ($J = J_0, J_1, J_2$) at $|\varphi| > M$

| $\sum \chi^2$ | 188 | 187 | 194 | 209 | 109 |
| $N_{\text{bins}}$ | 178 | 175 | 187 | 188 | 112 |
| $M$ | 0.4470(4) | 0.3979(4) | 0.3666(9) | 0.3281(5) | 0.2894(9) |
| $V''$ | 0.0320(5) | 0.0204(6) | 0.0149(3) | 0.0102(3) | 0.0060(3) |

Table 1: Monte Carlo numerical results, computed by direct measurement of some observables, as well as by two different fitting procedures. Values of spontaneous magnetization and inverse susceptibility from approximants proposed by Liu and Fisher (see Appendix) are listed for comparison. $N_{\text{config}}$ is the number of configurations used. $V'' \equiv V''_{\text{eff}}(M)$. $\sum \chi^2$ is the sum of $\chi^2$ for all histograms used in the fit (it is minimized by the fit). $N_{\text{bins}}$ is the total number of bins in these histograms. The numbers in parentheses are standard deviations of the last decimal digits.
Fitting 2 histograms \((J = J_0, J_1)\) at \(|\varphi| > M\), keeping minimum of \(V_{\text{eff}}(\varphi)\) fixed at \(\varphi = M\).

| \(L^3\) | 30\(^3\) | 38\(^3\) | 46\(^3\) | 58\(^3\) | 74\(^3\) | 74\(^3\) |
|---|---|---|---|---|---|---|
| \(\chi^2\) | 103 | 106 | 111 | 92 | 85 | 200 |
| \(N_{\text{bins}}\) | 104 | 98 | 108 | 109 | 106 | 204 |
| \(r\) | -0.00077(21) | -0.00084(13) | -0.00071(8) | -0.00036(8) | -0.00040(10) | -0.00038(4) |
| \(u\) | -0.0186(9) | -0.0120(7) | -0.0091(5) | -0.0078(6) | -0.0042(10) | -0.0044(3) |
| \(w\) | 0.0676(12) | 0.0612(13) | 0.0581(11) | 0.059(2) | 0.052(3) | 0.0532(7) |
| \(M\) | 0.44975 | 0.3997 | 0.3674 | 0.3270 | 0.2892 | 0.2893(6) |
| \(V''\) | 0.03625(26) | 0.02211(15) | 0.01555(9) | 0.00955(7) | 0.00599(12) | 0.00599(13) |
| \(m = \sqrt{ZV'''}\) | 0.3383(12) | 0.2643(9) | 0.2222(7) | 0.1735(7) | 0.1382(14) | 0.1382(15) |
| \(V'''\) | 0.537(4) | 0.353(3) | 0.265(2) | 0.1870(23) | 0.123(3) | 0.1239(10) |
| \(V''''\) | 4.48(7) | 3.23(6) | 2.60(4) | 2.09(5) | 1.47(8) | 1.497(10) |

The following parameters are dimensionless:

| \(Z^2u/M^2\) | -2.90(14) | -2.37(14) | -2.17(12) | -2.29(19) | -1.6(4) | -1.72(11) |
| \(g_6 = Z^3w\) | 2.13(4) | 1.93(4) | 1.86(4) | 1.85(5) | 1.70(10) | 1.73(2) |
| \(\tilde{g}_3\) | 2.55(3) | 2.43(3) | 2.39(3) | 2.41(4) | 2.27(9) | 2.29(2) |
| \(\tilde{g}_4\) | 5.50(10) | 5.08(11) | 4.92(9) | 4.99(14) | 4.5(3) | 4.60(8) |

The following parameters are dimensionless and do not contain \(Z\):

| \(\kappa = V''''M/6V''\) | 1.110(15) | 1.065(16) | 1.045(13) | 1.067(20) | 0.99(4) | 0.998(14) |
| \(V''''M^2/24V''\) | 1.041(23) | 0.972(24) | 0.942(20) | 0.975(30) | 0.86(7) | 0.87(2) |
| \(\tilde{g}_4/\tilde{g}_3^2\) | 0.844(5) | 0.857(5) | 0.863(4) | 0.857(6) | 0.878(11) | 0.875(4) |

Table 2: Numerical results obtained by the fitting procedure that we find most successful. For the largest lattice, fitting all 5 histograms gives the same results (the last column), providing an additional consistency check. All derivatives of \(V_{\text{eff}}\) are taken in the minimum.