EQUIVARIANT COHERENT SHEAVES, SOERGEL BIMODULES, AND CATEGORIFICATION OF AFFINE HECKE ALGEBRAS

CHRISTOPHER DODD

Abstract. We give a description of certain categories of equivariant coherent sheaves on Grothendieck’s resolution in terms of the categorical affine Hecke algebra of Soergel. As an application, we deduce a relationship of these coherent sheaf categories to the categories of perverse sheaves considered in [BY], generalizing results of [AB]. In addition, we deduce that the weak braid group action on sheaves of [BR] can be upgraded to a strict braid group action.

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1. Introduction.

1.1. Kazhdan-Lusztig Equivalence. Categories of equivariant coherent sheaves play a crucial role in geometric representation theory dating back at least to the work of Kazhdan and Lusztig in the 1980’s. The subject of their seminal work [KL] was a description of irreducible representations of the affine Hecke algebra associated to a given root datum. In order to accomplish this, they first gave a geometric construction of the affine Hecke algebra, which we shall now describe.

Let \( G \) be a complex reductive algebraic group with simply connected derived group, and let \( \tilde{G} \) be its (complex reductive) Langlands dual group. In particular, let us fix a pinning of \( G \), consisting of the choice of a Borel subgroup \( B \), and a maximal torus \( T \). To this pinning there is associated the root datum \((X, Y, \Phi, \tilde{\Phi})\), and the Weyl group \( W \), with \( S \) its set of simple reflections. Then, the root datum \((\tilde{X}, \tilde{Y}, \tilde{\Phi}, \Phi)\) also comes from an algebraic group, denoted \( \tilde{G} \), with its pinning \( \tilde{B} \) and \( \tilde{T} \). We let \( g \) and \( \tilde{g} \) denote the lie algebras of \( G \) and \( \tilde{G} \), respectively, and \( b, \tilde{b}, h, \tilde{h} \) the lie algebras of our fixed Borel and Cartan subalgebras.

Now, we let \( H_X \) denote the affine Hecke algebra associated to \( X \). Let us recall Bernstein’s presentation of this algebra:

**Definition 1.** \( H_X \) is the free \( \mathbb{Z}[q, q^{-1}] \)-module with basis \( \{e^\lambda T_w | \lambda \in X, w \in W\} \) satisfying

1) The \( T_w \) span a \( \mathbb{Z}[q, q^{-1}] \)-subalgebra isomorphic to the finite Hecke algebra \( H_W \).

2) The \( \{e^\lambda\} \) (by which we mean \( \{T_e e^\lambda\} \)) satisfy \( e^{\lambda_1} e^{\lambda_2} = e^{\lambda_1 + \lambda_2} \).

3) Let \( \alpha \) be a simple root and \( s_\alpha \) its simple reflection. Then for \( \lambda \in X \)

\[
T_{s_\alpha} e^{s_\alpha(\lambda)} - e^\lambda T_{s_\alpha} = (1 - q) \frac{e^\lambda - e^{s_\alpha(\lambda)}}{1 - e^{-\alpha}}
\]

Kazhdan and Lusztig have constructed \( H_X \) entirely in terms of the geometry of the group \( G \). We first recall the flag variety of \( G \), denoted \( \mathcal{F} \), which we shall regard as the variety of all Borel subalgebras of \( g \). Of course, given the pinning chosen in the previous paragraph, we can identify \( \mathcal{F} = G/B \) as homogeneous spaces. We next recall that there is associated to \( G \) a morphism \( \tilde{\mathcal{N}} \to \mathcal{N} \) called the springer resolution. Here \( \mathcal{N} \) denotes the nilpotent cone associated to \( G \), which is defined as follows: we first define

\( \mathcal{N}^* = \{x \in g | ad(x) \text{dim } g = 0\} \)

and then we define \( \mathcal{N} \subset g^* \) as the transport of \( \mathcal{N}^* \) under the natural isomorphism (given by the killing form) \( g^* \cong g \).

Next, \( \tilde{\mathcal{N}} \) denotes the incidence variety defined by

\( \tilde{\mathcal{N}} = \{(x, b) \in g^* \times B | x|_b = 0\} \)

Then the morphism \( \tilde{\mathcal{N}} \to \mathcal{N} \) is given by the first projection.

Finally, we can construct from here the Steinberg variety, defined as

\( St_G = \mathcal{N} \times_X \tilde{\mathcal{N}} \)

We see immediately from the definition that \( St_G \) has an action by \( G \times \mathbb{C}^* \), where \( G \) acts via its obvious action on \( \mathcal{N} \) and \( \mathcal{N} \), and the action of \( \mathbb{C}^* \) is given by dilation of the first coordinate in \( \tilde{\mathcal{N}} \). Therefore we can consider the (complexified) \( K \) group \( R^{G \times \mathbb{C}^*} (St_G) \), which is naturally a \( \mathbb{C}[q, q^{-1}] \)-module, where the parameter \( q \) acts by shifting the grading induced by the \( \mathbb{C}^* \)-action.
Further, this module also has the structure of an algebra, where the product is
given via the exterior product of sheaves: let \( \mathcal{F} \) and \( \mathcal{G} \) be two \( G \times \mathbb{C}^* \)-equivariant
sheaves on \( St_G \). Consider the variety \( \tilde{N} \times_N \tilde{N} \times_N \tilde{N} \), which has three projections
to \( St_G \), which we shall denote \( p_i \) where \( i = \{ 1, 2, 3 \} \) is the omitted factor. Then we
can consider the complex in \( D^{G \times \mathbb{C}^*}(\text{Coh}(St_G)) \) (the bounded derived category of
equivariant coherent sheaves; full details about such objects are recalled in section
2 below)
\[
p_{2*}(p_3^*(\mathcal{F}) \otimes p_1^*(\mathcal{G}))
\]
where all the functors are taken to be derived. This complex defines a class in \( K \)
theory, which is then the product of \([ \mathcal{F} ]\) and \([ \mathcal{G} ]\).

With all of this is hand, we can state the theorem of Kazhdan and Lusztig
(actually the slightly stronger version proved in \([ CG ]\)), which says that there is an
isomorphism:
\[
H_x \cong K^{G \times \mathbb{C}^*}(St_G)
\]
Let us explain how this isomorphism works, in terms of the presentation of \( H_x \)
given above. We shall follow the explanation of Riche \([ Ri ]\). First, given \( \lambda \in \mathbb{X} \), we
associate the bundle \( O(\lambda) \) on the flag variety \( B \), and then via pullback, a bundle
\( O_{\Delta N}(\lambda) \). Now, we have the diagonal embedding \( \Delta N \to St_G \). So we can consider
the equivariant coherent sheaf \( O_{\Delta N}(\lambda) \) on \( St_G \); the map will send \( e^\lambda \) to the class
\( [O_{\Delta N}(\lambda)] \in K^{G \times \mathbb{C}^*}(St_G) \).

Next, consider \( s \in S \). We associate to \( s \) a partial flag variety of \( G \), denoted
\( P_s = G/P_s \), where \( P_s \) is the standard parabolic in \( G \) of type \( s \) containing \( B \). Then
we can form the variety \( B \times P_s B \), which is naturally a closed subscheme of \( B \times B \).
From here we define the variety
\[
S'_s = \left\{ (X, g_1B, g_2B) \in \mathfrak{g}^* \times B \times P_s B | X|_{g_1b+g_2b=0} = 0 \right\}
\]
It is easy to see that this is a subvariety of \( St_G \), which is \( G \times \mathbb{C}^* \) equivariant. Then
we have our map send \( T_s \) to \(-q^{-1}[O_{S'_s}]\). The fact that this really is an isomorphism
of algebras is checked in \([ CG ]\), c.f. also the main result of Riche in \([ Ri ]\) (shown in
full generality in \([ BR ]\)). We shall discuss the results of these papers in more detail
later.

For our purposes in this work, the interest in this result lies in the fact that
the Steinberg variety can now be viewed as a “categorification” of the affine Hecke
algebra.

1.2. Soergel Bimodules. In this section, we discuss a different categorification of
the affine Hecke algebra, which can be found in the works of Wolfgang Soergel
\([ S1, S2, S3 ]\) and Rafael Rouquier \([ R ]\). For this categorification, it is appropriate
to consider the general case where \( (W, S) \) is any Coxeter system for which the
generating set of involutions \( S \) is finite.

Let us recall that in this generality, the Hecke algebra is defined as the \( \mathbb{Z}[q, q^{-1}] \)-
algebra generated by symbols \( \{ T_s \}_{s \in S} \), which satisfy the braid relations for \( S \), and
also the additional relation \( (T_s + 1)(T_s - q) = 0 \).

Then, over an algebraically closed field of sufficiently large characteristic (including
zero), \( k \), we have the geometric representation of \( W \), on a finite dimensional
vector space \( V \). Then there is a categorification which can be constructed from this
purely combinatorial set-up. We shall follow closely the notation and constructions
in \([ R ]\).
So, we let \( \{ e_s \} \) be the natural basis of \( V \), and we let \( \{ \alpha_s \} \) be its dual basis. Therefore we have that
\[
\ker(\alpha_s) = \ker(s - id)
\]
for all \( s \in S \). We define \( A \) to be \( \text{Sym}(V^*) \), i.e., the algebra of polynomial functions on \( V \), and we let \( A^{en} = A \otimes_k A \). We consider both of these as graded rings by putting \( V^* \) in degree 2. We shall work with certain graded \( A^{en} \)-modules which are known as Soergel bimodules (they first appeared in the paper [S2]).

To define these bimodules, we first note that for any \( s \in S \), we have the subalgebra \( A_s \) of elements fixed under the action of \( s \). We have a decomposition
\[
A = A^s \oplus A^s \alpha_s
\]
as \( A^s \)-modules (this follows immediately from the fact that \( s \) has order two).

Now, let \( w \in W \) be any element. We can write \( w = \prod_{i=1}^{m} s_i \), a minimal length decomposition (where \( s_i \in S \)). We associate to any such decomposition the graded \( A^{en} \)-module
\[
A \otimes_{A^{s_1}} A \otimes_{A^{s_2}} A \otimes_{A^{s_3}} \cdots \otimes_{A^{s_m}} A
\]
We note that the element \( e \in W \) gets the “diagonal” bimodule \( A \) by definition.

Then, as shown in the works of Soergel, each of these bimodules admits a decomposition as a direct sum of indecomposable graded \( A \)-\( A \) bimodules, called the Soergel bimodules. In fact, it is even shown that to each such \( w \in W \), one can associate a unique \( B_w \) which occurs as a summand (with multiplicity one) in \( A \otimes_{A^{s_1}} A \otimes_{A^{s_2}} A \otimes_{A^{s_3}} \cdots \otimes_{A^{s_m}} A \) and which occurs in no bimodule \( A \otimes_{A^{t_1}} A \otimes_{A^{t_2}} \otimes_{A^{t_3}} \cdots \otimes_{A^{t_n}} A \) which corresponds to a lower element in the Bruhat ordering.

Therefore, one makes the following

**Definition 2.** [S3] The category \( \mathcal{H}(W) \) is the smallest category of \( A^{en} \) bimodules containing the Soergel bimodules and closed under direct sums, summands, and tensor product. This is an additive category, which has a monoidal structure via
\[
(M, N) \to M \otimes_A N
\]

Next, we consider the (complexified) split Grothendieck group of this category, which is a \( \mathbb{C}[q, q^{-1}] \)-algebra, where the parameter \( q \) acts by shifting the grading, and the multiplication is the image in \( K \)-theory of the above monoidal structure. We state the main result of [S3] as
\[
K(\mathcal{H}(W)) \cong H(W)
\]
as \( \mathbb{C}[q, q^{-1}] \) algebras. The map \( H(W) \rightarrow K(\mathcal{H}(W)) \) is given by \( T_s \rightarrow [A \otimes_{A^s} A] - 1 \), and \( q \rightarrow A[1] \).

Thus the category \( \mathcal{H}(W) \) provides another categorification of the Hecke algebra of a Coxeter group, of which the affine Hecke algebra is a particular example (where \( W \) is the affine Weyl group of a given root datum).

### 1.3. Perverse Sheaves.

We now discuss a third categorical realization of the affine Hecke algebra- the one that, in fact, is closest to the actual definition of \( H_{aff} \). This realization follows from applying Grothendieck’s “sheaves-functions” correspondence to the affine flag variety.

In particular, for our given reductive group \( G \), we consider the formal loop group \( G((t)) \), and the Iwahori subgroup \( I \) (see section 5).
Since the affine Hecke algebra, by definition, is the algebra of $I$-equivariant functions on $\mathcal{F} l := G((t))/I$, the sheaves-functions philosophy would predict that it can be categorified as the category of $I$-equivariant mixed perverse sheaves on $\mathcal{F} l$, and this is indeed a theorem of Kazhdan and Lusztig [KL2]. So in particular we have an isomorphism of $\mathbb{C}[q, q^{-1}]$-algebras

$$K(\text{Perv}^\text{mix}_l(G((t))/I)) \cong H_Y$$

where the group on the left is the complexified $K$-group, and the graded element $q$ acts by a the Tate twist of mixed sheaves (c.f., e.g., [BGS]). Note that the Hecke algebra on the right is the one associated to the dual group of $G$.

We explain how the map works, in the case that $Y$ is the root lattice $Z\Phi$ (in the general case the variety $\mathcal{F} l$ is a finite cover of this one). By the standard Bruhat decomposition, the $I$-orbits on $G((t))/I$ are parametrized by the group $W_{aff}$. As is well known, each of these orbits is isomorphic to an affine space. For $w \in W_{aff}$, let $Z_w$ denote its orbit. Then we can consider the trivial sheaf $\mathcal{Q}_l, Z_w$. If we let $\bar{Z}_w$ denote the closure of $Z_w$ in $G((t))/I$, and $j_w : Z_w \to \bar{Z}_w$ the natural inclusion, then we can also consider $j_w!(\mathcal{Q}_l, Z_w[\dim Z_w])$. These classes of these objects in $K(\text{Perv}^\text{mix}_l(G((t))/I))$ form a natural $\mathbb{C}[q, q^{-1}]$-basis, and the map is then given by $\mathcal{T}_w \to q^{-l(w)}[j_w!(\mathcal{Q}_l, Z_w[\dim Z_w])]$ (here $l(w)$ denotes the length).

1.4. Description of the Main Theorems. In order to explain our goal, we first explain a variant of the above results, which is the categorification of the standard (or aspherical) module for a given affine Hecke algebra, $H_{\mathcal{X}}$. This module, denoted $M_{\text{asp}}$, is defined by the induction

$$M_{\text{asp}} = H \otimes_{H_{\text{fin}}} sgn$$

where $H_{\text{fin}} \subseteq H$ denotes the inclusion of the finite Hecke algebra, and $sgn$ is the one-dimensional sign representation of $H_{\text{fin}}$. The main result of [AB] is a categorification of this module (and the action of $H$ on it) as follows (c.f. [AB] section 1 for all definitions):

We work with the Langlands dual reductive group $\check{G}$. Let $I^-$ denote the opposite Iwahori subgroup, and let $\psi$ be a generic character on it. Then we define $D_{IW}$ to be the $(I^-, \psi)$-equivariant derived category of constructible sheaves on $\mathcal{F} l$; it admits a mixed version $D_{IW,m}$. There is an action of the category $D^b(\text{Perv}_l(\mathcal{F} l))$ on $D_{IW}$, defined by convolution\(^1\). This works as follows: by the definition of $\mathcal{F} l$ as a quotient, there is a map $a : \mathcal{F} l \times_I \mathcal{F} l \to \mathcal{F} l$ (the image of group multiplication). So, for $\mathcal{G} \in D_{IW}$ and $\mathcal{K} \in D^b(\text{Perv}_l(\mathcal{F} l))$, we take $a_*(\mathcal{G} \boxtimes \mathcal{K}) \in D_{IW}$. We note that, by using the same formulae, this action extends to the mixed versions of these categories.

Then the claim is that (the mixed version of) this action is a categorification of the action of $H_{\mathcal{X}}$ on $M_{\text{asp}}$, in the sense of the above sections (i.e., after taking graded $K$-groups one recovers this action).

Next, there is another categorification of $M_{\text{asp}}$ in terms of coherent sheaves. We work with the group $G$, over the field $\mathcal{Q}_l$. There is a convolution action of the Steinberg variety $St = \mathcal{N} \times_\mathcal{X} \mathcal{N}$ on the variety $\mathcal{N}$. This induces an action (in the same sense as above) of $D^b(G \times_{G_m}(\text{Coh}(St)))$ on $D^b(G \times_{G_m}(\text{Coh}(\mathcal{N})))$ which

\(^1\)In this case, by an action we simply mean that to each object of $D^b(\text{Perv}_l(\mathcal{F} l))$, there is a assigned an endofunctor of $D_{IW}$, and this assignment is natural.
categorifies the action of $H_S$ on $M_{asp}$. Then the main result of [AB] states that there is an equivalence of triangulated categories
\[ D^{b,G \times G_m}(\text{Coh}(\tilde{N})) \rightarrow D_{IW,m} \]
such that the shift in grading on the left corresponds to the shift in mixed structure on the right. The proof there relies on the main result of [G2] and the geometric Satake equivalence.

Below, we shall present a different approach to this result and several "deformed" versions of it, by relating both sides to relevant categories of Soergel bimodules. In particular, the category on the left admits two natural deformations, explained in detail in section 2 below, denoted $D^{b,G \times G_m}(\text{Coh}(\tilde{g}))$ and $D^{b,G \times G_m}(\text{Mod}(\tilde{D}_h))$. The first is a deformation purely in the world of commutative algebraic geometry, while the second is a noncommutative deformation quantization. We shall study these categories in detail, and give a bimodule description of them in section 4 below.

On the perverse sheaves side, several deformations of the category $D_{IW}$ have been defined and studied in the paper [BY]. There, they develop a Koszul duality formalism for these categories, the major technical part of which is to relate these categories to appropriate Soergel bimodules. Combining this with our description, we arrive at equivalences of categories generalizing the one above (see section 5 for details).

1.5. Summary of the Major Argument. Our strategy is to relate the three different species of categorification of the affine Hecke algebra (or rather its standard module). We shall indicate how the argument works. The first task is to define a tilting collection for this triangulated category.

Following [Ke], we recall that an algebraic triangulated category is one which can be constructed as the stable triangulated category of a Frobenius exact category. We shall not recall the details of these definitions, but the important thing for us is that all triangulated categories arising in algebra and algebraic geometry (e.g., derived and homotopy categories of abelian and dg categories) are algebraic triangulated categories.

Then we begin with the:

**Lemma 3.** [Ke] Let $\mathcal{C}$ be an algebraic triangulated category. Let $T \in \mathcal{C}$ be an object such that

1) $T$ generates $\mathcal{C}$ (in the sense that the full subcategory of $\mathcal{C}$ containing $T$ and closed under extensions, shifts, and direct summands is $\mathcal{C}$ itself).

2) We have $\text{Hom}_\mathcal{C}(T, T[n]) = 0$ for all $n \neq 0$.

Then the functor $M \rightarrow \text{Hom}_\mathcal{C}(T, M)$ is an equivalence of categories $\mathcal{C} \rightarrow \text{Perf}(\text{End}_\mathcal{C}(T))$ where the latter denotes the homotopy category of perfect complexes over $\text{End}_\mathcal{C}(T)$.

In this case, the object $T$ is called a tilting object of $\mathcal{C}$.

Unfortunately, our categories will not admit such a simple description. However, they will be generated by infinite collections of such objects. Therefore we make the

**Definition 4.** Let $\mathcal{C}$ be an algebraic triangulated category. Let $\mathcal{T}$ be a full subcategory such that

1) $\mathcal{T}$ generates $\mathcal{C}$ in the following sense: for each finite collection $I$ of objects of $\mathcal{T}$, let $P_I = \bigoplus_{i \in I} T_i$, and let $\mathcal{C}_I$ be the full subcategory generated by $\mathcal{P}_I$ in the
sense of the lemma above; then \( \lim I \rightarrow C \) (i.e. the natural inclusion is essentially surjective).

2) We have \( \text{Hom}_C(\mathcal{T}, \mathcal{T}[n]) = 0 \) for all \( n \neq 0 \).

Then \( \mathcal{T} \) is called a tilting subcategory of \( C \).

In the cases relevant to us, we can assume that \( \text{Ob}(\mathcal{T}) \) is a countable, totally ordered set. Then for each \( i \in \mathbb{N} \), we define \( P_i = \bigoplus_{j \leq i} T_j \). Then we can define the categories \( D_i = \text{Perf}(\text{End}(P_i)) \) with the natural inclusion functors \( D_i \rightarrow D_j \) for \( i \leq j \). Then the assumptions on \( \mathcal{T} \) and the previous lemma imply the

Claim 5. There is an equivalence of categories \( C \rightarrow \lim_{i}(D_i) \) which is given by the stable image of the functors \( F_i : C \rightarrow D_i, F_i(M) = \text{Hom}_C(P_i, M) \). We shall denote this limit by \( \text{Perf}(\mathcal{B}) \) where \( \mathcal{B} = \bigoplus_{i,j}(\text{Hom}(T_i, T_j)) \).

Remark 6. Let us note that the objects of \( \mathcal{T} \) correspond to (direct sums of) the summands of \( \mathcal{B} \). Thus we also have an equivalence \( \text{Perf}(\mathcal{B}) \rightarrow K^b(\mathcal{T}) \) where the category on the right denotes the homotopy category of complexes of objects in \( \mathcal{T} \).

In the main body of the paper, we shall explicitly identify tilting collections for each of the categories \( C \) of interest to us. We shall then define a functor \( \kappa : \mathcal{T} \rightarrow (\text{Mod}^{\text{PT}}(O(\mathfrak{h}^* \times \mathbb{A}^1 \times \mathfrak{h}^*/W))) \)

(for notation for algebraic groups as above) which is called the Kostant-Whittaker reduction. This functor is based on computing the action of the center of each of the above categories; in this sense, it is a generalization of the fundamental work of Soergel [S1]. A prototype also appears in the paper [BF].

We next show that when restricted to tilting modules, \( \kappa \) is fully faithful. We shall describe explicitly the image of \( \kappa \), and show that the sheaves that appear are a version of Soergel bimodules (for the module \( M_{asp} \)); thus obtaining, by the remarks above, a description of the entire category \( C \). 3

On the other hand, there is already a description, contained in the paper [BY], of the constructible categories under consideration in terms of sheaves on the space \( \mathfrak{h}^* \times \mathbb{A}^1 \times \mathfrak{h}^*/W \). This description also goes by finding a tilting collection, and also explicitly describes the image in terms of Soergel bimodules. From these compatible descriptions, we deduce the equivalence of categories above.

1.6. Further Results and Future Work. Given the results outlined above, it is natural to ask about the categorifications of the other standard modules. In particular, we know that the regular representation of \( H_{aff} \) on itself is categorified by certain \( G \)-equivariant sheaves on the Steinberg variety (as discussed in section 1.1), and also by the usual category of Soergel bimodules. In a future work, we shall extend the results of this paper to show that there is a fully-faithful functor \( \kappa \) which takes a certain subcategory of \( D^b(\text{Coh}^{G \times G_{\alpha}}(\bar{D}_h \boxtimes \bar{D}_h)) \) onto the category of Soergel bimodules for \( W_{aff} \) (see section 2 below for definitions). In addition, we shall show the comparable results for the categories of equivariant coherent sheaves on the deformations of partial flag varieties \( \mathfrak{g}_F \) (see section 2 below). The method will be to describe these categories in terms of the ones already considered here via the Barr-Beck theorem applied to the appropriate push-pull functors. This will allow us to upgrade the functor \( \kappa \) of this paper to functors on these other categories; and similar methods to the ones here can then be used to deduce the full-faithfulness on the appropriate objects. The details will appear in a forthcoming work.
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2. Main Players—Coherent Side

In this section, we shall give detailed explanations of the “coherent” categories that we shall use. The constructions in this section make sense over an arbitrary algebraically closed field $k$ of characteristic greater than the Coxeter number of $G$.

2.1. Varieties. We start with the varieties $\tilde{g}$ and $\tilde{g}_P$—details on these can be found in many places, e.g. [BMRI,II] and [BK].

2.1.1. Full Flag Varieties. The variety $\tilde{g}$ is the total space of a bundle over $\tilde{N}$, and is defined as

$$\tilde{g} = \{(x, b) \in g^* \times B|x|_{[b,b]} = 0\}$$

the first projection defines a morphism $\tilde{g} \to g^*$, and so we have a map $O(\tilde{g}) \to O(g^*)$.

Further, we can explicitly identify the global sections $\Gamma(O(\tilde{g}))$ as follows. Recall the Harish-Chandra isomorphism

$$O(g^*)/G \cong O(h^*/W)$$

This makes $O(h^*/W)$ into a subalgebra of $O(g^*)$. Then we have

$$\Gamma(O(\tilde{g})) \cong O(g^*) \otimes O(h^*/W)$$

Thus the map $\mu : \tilde{g} \to g^*$ factors through another map, which by abuse of notation we shall also call $\mu : \tilde{g} \to g^* \times h^*/W$. We shall state an important property of this map: we consider the locus of regular elements $g^*_{reg}$, defined as follows: one first defines $g^*_{reg}$ to be the set $\{x \in g|\dim(c_g(x)) = \text{rank}(g)\}$, and then transfers this open set to $g^*$ via the killing isomorphism.

Now, define $\tilde{g}_{reg} = \mu^{-1}(g^*_{reg})$. Then we have the following

**Lemma 7.** The map $\mu : (\tilde{g})_{reg} \to h^*/W; \tilde{g}^{*}_{reg}$ is an isomorphism.

The proof can be found in [G1], section 7. We shall see below that the regular elements play a major role in capturing the $G$-equivariant geometry of $\tilde{g}$.

2.1.2. Partial Flag Varieties. Now we wish to extend this definition by having parabolic subalgebras of different types play the role of the Borel subalgebra. Thus we let $P$ be a given partial flag variety. For a parabolic $P \in P$, we let $u(P)$ denote its nilpotent radical. Then we define

$$\tilde{g}_P = \{(x, p) \in g^* \times P|x|_{u(P)} = 0\}$$

In the case $P = B$, this recovers $\tilde{g}$. In case $P = pt$ (i.e., $p = g$), this is simply $g^*$.

As before, we have a map $\tilde{g}_P \to g^*$, and we now have the isomorphism

$$\Gamma(O(\tilde{g}_P)) \cong O(g^*) \otimes O(h^*/W) \otimes O(W(P))$$

Where $W(P)$ is the Weyl group of parabolic type associated to this partial flag variety.
Let us note that there are certain natural projection maps between these varieties. In particular, let \( p \subset q \) be two parabolic subalgebras. Then there is a natural projection map of partial flag varieties \( \pi : P \to Q \). This then induces a map

\[
\pi_{PQ} : \tilde{g}_P \to \tilde{g}_Q
\]

defined by sending \((x, p)\) to \((x, \pi(p))\).

Some special cases will be of interest to us. First, let us note that for any \( p \), taking \( q = g \) yields the natural map \( \tilde{g}_P \to g^* \) considered above.

Next, we want to consider the case of the projection \( B \to P_s \), where \( s \) is a simple reflection. We shall record two important properties of the map \( \pi_{BP_s} = \pi_s \). First, we have an isomorphism

\[
\pi_{s*}(O(\tilde{g})) = O(\tilde{g}_s) \otimes O(h^*/W(P)) O(h^*)
\]

thus this pushforward is a locally free sheaf of rank 2. The second, related fact, is that one can consider the restriction of this map to the regular locus as follows: the map \( \mu = \pi_{BP} : \tilde{g} \to g^* \) factors as

\[
\tilde{g} \to \tilde{g}_s \to g^*
\]

and so, taking the inverse image of the regular locus, we have a map

\[
(\tilde{g})^{reg} \to (\tilde{g}_s)^{reg}
\]

Then this map is a two sheeted covering map, with fibres are naturally isomorphic to \( W/W(P) \).

In fact, we shall state the somewhat more general

**Lemma 8.** For all parabolic subgroups, the projection map \( \tilde{g}_P^{reg} \to g_s^{reg} \) induces an isomorphism

\[
\tilde{g}_P^{reg} = h^*/W(\tilde{P}) \times h^*/W g^{reg}
\]

The proof of this is similar to that of the case \( P = B \) discussed above.

**2.1.3. Equivariance.** Here we would like to note that there is a natural action of the reductive group \( G \times \mathbb{G}_m \) on all the varieties we have considered. The \( G \) action comes essentially from the construction of the varieties- for any \( \tilde{g}_P \), we can set

\[
g \cdot (x, p) = (ad^*(g)(x), ad(g)(p))
\]

where \( ad^* \) and \( ad \) are the coadjoint and adjoint action, respectively.

The \( \mathbb{G}_m \)-action comes from the natural dilation action on the lie algebra, i.e.,

\[
c \cdot (x, p) = (cx, p)
\]

which obviously commutes with the action of \( G \).

Let us note that

\[
\Gamma(O(\tilde{g}_P))^G = O(h^*/W(\tilde{P}))
\]

simply by taking \( G \)-invariants on both sides of equation 2 above.

With this action defined, there are now abelian categories of equivariant coherent sheaves \( Coh^G(\tilde{g}_P) \) and \( Coh^{G \times \mathbb{G}_m}(\tilde{g}_P) \) (c.f. [CG], chapter 5 for general information about equivariant coherent sheaves), and their derived categories, which will be some of the main players in the paper. As it turns out, these categories admit a very nice “affine” description, as in the following:
Lemma 9. The line bundles $O_{\hat{g}}(\lambda)$ generate\(^2\) the category $\text{DbCoh}^G(\hat{g})$. The analogous statement holds for the graded version.

Proof. We first recall that by definition $\hat{g} = G \times_B b^*$. Therefore there are equivalences of categories

\[ i^* : \text{Coh}^G(\hat{g}) \sim \text{Coh}^B(b^*) \]

and

\[ i^* : \text{Coh}^{G \times G_m}(\hat{g}) \sim \text{Coh}^{B \times G_m}(b^*) \]

given by restriction to the fibre over the base point of $B$. The inverse of this functor is given by taking the associated sheaf of a $B$-module $M$ (c.f. [J], chapter 5), which yields a quasicoherent sheaf on $B$, and then noting that the additional compatible structure of a $\text{Sym}(b)$-module on $M$ is equivalent to an action of $p_*(\hat{g})$ on the associated sheaf.

So, to prove the lemma, we consider any finitely generated $B$-equivariant module $M$ over $O(b^*)$, and show that $M$ is in the triangulated category generated by $i^*O_{\hat{g}}(\lambda)$. We choose a finite dimensional $B$-stable generating space for $M$. We reduce to the case that $N$ acts trivially on $V$ by considering a filtration of $V$ such that $N$ acts trivially on the subquotients. But then the proof comes down to the statement is just that if we have a multigraded polynomial ring, then any module has a finite resolution by graded projective modules. The result for the entire bounded derived category follows by induction on the length of complex. □

Finally, we would like to end by describing one crucial property possessed by equivariant coherent sheaves (in a general context), as explained in [Kas]. In particular, any equivariant coherent sheaf $M$ comes with a morphism of Lie algebras

\[ L_\nu : \mathfrak{g} \to \text{End}_k(M) \]

obtained, essentially, by “differentiating the $G$-action” (c.f. [Kas] pg. 23 for details on the algebraic definition); thus $M$ can be considered a sheaf of $\mathfrak{g}$-modules, and hence a sheaf of $U(\mathfrak{g})$-modules. Further, for $A \in \mathfrak{g}$, the operator $L_\nu(A)$ is a derivation on $M$.

2.2. Deformations. Now we shall consider certain non-commutative deformations of the various varieties and maps considered above. Again these objects are more or less well known, c.f. [BK]; in addition [BMRI,II] consider the version in positive characteristic, and [Mil] has much of the material of the first subsection.

2.2.1. Full Flag Varieties. We shall start with $\hat{g}$. By definition, $\hat{g}$ is a vector bundle over $B$. At a given point $b \in B$, the fibre of this bundle, $\hat{g}_b$, is equal to \{ $x \in \mathfrak{g}^* | [x, b] = 0$ \}. This is a vector space that can naturally be considered the dual of $\mathfrak{b}$.

We can quantize this situation, following [Mil]. We start with the sheaf $U^0 = U(\hat{g}) \otimes O(B)$ - a trivial sheaf on $B$. Let us note that the multiplication in this sheaf is not the obvious one, but is instead given by the formula

\[ (f \otimes \xi)(g \otimes \eta) = f(\xi \cdot g) \otimes \eta + fg \otimes \xi \eta \]

where $\xi \cdot g$ denotes the action of a vector field on a function.

---

\(^2\)In the sense that the smallest subcategory containing the line bundles and closed under shifts, extensions, and direct factors is the entire category.
The PBW filtration on $U(g)$ gives a filtration on this sheaf. It is clear that with respect to this filtration we have
\[ \text{gr}(U(g) \otimes O(B)) \approx O(g^*) \otimes O(B) \]
Further, the sheaf on the right is equal to $p_* (O(g^* \times B))$, where $p : g^* \times B \to B$ is the obvious projection. Thus we can consider $U(g) \otimes O(B)$ as a quantization of $g^* \times B$.

Now, for a given point $b \in B$, we can consider $\mathfrak{n}(b) = [b, b]$, the nilpotent radical of $b$. We can define $\mathfrak{n}^0$ to be the ideal sheaf generated at each point $b$ by the subalgebra $\mathfrak{n}(b)$. Then we can form the quotient sheaf $U_0 / \mathfrak{n}^0$. This sheaf inherits the PBW filtration from $U_0$, and it is immediate from the definitions that
\[ \text{gr}(U_0 / \mathfrak{n}^0) \approx p_* O(\tilde{g}) \]
where we have here used $p : \tilde{g} \to B$ to denote the projection.

To give the quantization in its final form, let us recall that to any filtered sheaf of $k$-algebras $A$ on a space, we can associate the Rees algebra, as was done, e.g., in [BFG]. In particular, $\text{Rees}(A)$ is a graded sheaf of $k[h]$-algebras, such that $\text{Rees}(A)/h \approx \text{gr}(A)$ (the associated graded algebra of $A$).

So, we finally make the

**Definition 10.** The sheaf $\tilde{D}_h$ on $B$ is $\text{Rees}(U_0 / \mathfrak{n}^0)$.

Thus we have that $\tilde{D}_h / h \approx p_* (O(\tilde{g}))$ by construction.

We wish to consider the global sections of this object. To that end, we note that the algebra
\[ U_h(g) := \text{Rees}(U(g)) \]
(where the PBW filtration is used) maps naturally to $\Gamma(\tilde{D}_h)$, simply by following the chain of filtration preserving maps:
\[ U(g) \to \Gamma(U^0) \to \Gamma(U_0 / \mathfrak{n}^0) \]
Then, we also have a natural map
\[ O(\mathbb{A}^1 \times \mathfrak{h}^*) \otimes k[h] \otimes U(\mathfrak{h}) \to \Gamma(\tilde{D}_h) \]

simply by the fact that $\mathfrak{h} \subset g$.

Further, there are embeddings $O(\mathbb{A}^1 \times \mathfrak{h}^*/W) \to U_h(g)$ and $O(\mathbb{A}^1 \times \mathfrak{h}^*/W) \to O(\mathbb{A}^1 \times \mathfrak{h}^*)$; the first as the inclusion of the center, the second as the natural inclusion. Then, from [Mil], page 21, we have the

**Claim 11.** The natural maps $U_h(g) \to \Gamma(\tilde{D}_h)$ and $O(\mathbb{A}^1 \times \mathfrak{h}^*) \to \Gamma(\tilde{D}_h)$ agree upon restriction to $O(\mathbb{A}^1 \times \mathfrak{h}^*/W)$.

Thus, we in fact have a morphism
\[ O(\mathbb{A}^1 \times \mathfrak{h}^*) \otimes O(\mathbb{A}^1 \times \mathfrak{h}^*/W) U_h(g) \to \Gamma(\tilde{D}_h) \]
which is actually an isomorphism—this is proved in [Mil], Theorem 5, page 37. Upon taking $h \to 0$, we get isomorphism 1 (c.f. section 2.2.4 below for details about “taking $h \to 0$”).
2.2.2. Partial Flag Varieties. Now we wish to quantize the varieties $\tilde{g}_P$, by a similar explicit strategy. So we start with the sheaf

$$U^0_P := U(g^*) \otimes O(P)$$

with the multiplication as for $U^0$ above. For each point $p \in P$, we have the sub-lie algebra $u(p) \subset \tilde{g}$. We can now define the sheaf of ideals $u^0$ to be the sheaf generated at each point $p$ by $u(p)$. Then we have the sheaf $U^0_P/u^0$, and it is immediate from the definition that

$$gr(U^0_P/u^0) \cong p_*O(\tilde{g}_P)$$

where $p : \tilde{g}_P \to P$ is the natural projection. So we define

$$\tilde{D}_h.P := \text{Rees}(U^0_P/u^0)$$

Of course, we have that $\tilde{D}_h = \tilde{D}_h.B$. It also follows from the definition that in the case $P = pt$, $\tilde{D}_h.P = U_h(g)$.

Next, we can explain the behavior of these sheaves under the natural pushforward maps. In particular, let $\pi_s : B \to P_s$ be the natural projection morphism (this is a slight abuse of notation from the previous section). We wish to calculate $\pi_s^*(\tilde{D}_h.P)$, following [BMRI,II], [BK] (the answer will be a deformation of equation 2).

To proceed, let $p \in P_s$, and let $p^- = $ be the opposite parabolic, with Levi decomposition $p^\perp = u(p^-) \oplus j^-$. Under our assumptions, we have that $j^- = sl_2 \oplus h$. Then we have the open subset $J^- \cdot p \subset P$ (and $P$ is covered by such subsets). Further we have that

$$\pi_s^{-1}(J^- \cdot p) \cong P_s/B \times (J^- \cdot p) = P^1 \times (J^- \cdot p)$$

and the map $\pi_s$ becomes the projection to the second factor.

So, the above decompositions imply that we see that

$$\pi_{ss}(\tilde{D}_h)((J^- \cdot p)) \cong \Gamma(\tilde{D}_h(P/B)) \otimes_C O(P) \otimes_C U_h[(h^*)^s \oplus u^-(p)]$$

where $\tilde{D}_h(P/B)$ denotes $\tilde{D}_h$ in the case of the reductive group $SL_2$, with flag variety $P^1$. But we already know the global sections of this sheaf:

$$\Gamma(\tilde{D}_h(P/B)) \cong U_h(sl_2) \otimes_{O(t^<s>)} O(t^*)$$

where $t$ denotes the Cartan subalgebra for this $sl_2$, whose Weyl group is $< s >$. So we see that

$$\pi_{ss}(\tilde{D}_h) \cong \tilde{D}_h.P \otimes_{O(t^<s>)} O(h^*)$$

which becomes equation 2 after letting $h \to 0$ (c.f. section 2.2.4 below).

2.2.3. Equivariance. We would like to now explain how the $G \times \mathbb{G}_m$ action discussed above can be quantized. We start with the action of $G$ on $P$, which is of course a map

$$a : G \times P \to P$$

such that for each $g \in G$, $a(g) : P \to P$ is an isomorphism, yielding an isomorphism of sheaves $a(g)^* : O(P) \to O(P)$. This collection of isomorphisms satisfies the unit, associativity, and inverse properties, as with any group action.
Speaking in loose terms, we would like a $G$-equivariant $\tilde{D}_{h,p}$-module to be a $\tilde{D}_{h,p}$-module $M$ equipped with isomorphisms

$$a(g)^* M \rightarrow M$$

(where this is the quasicoherent pullback), which satisfy these compatibilities, and which “depend algebraically” on $g \in G$.

Formally speaking, we shall give the definition of \[Kas\]. Firstly, we define

$$O_G \boxtimes \tilde{D}_{h,p} := O_{G \times P} \otimes_{\text{pr}^{-1} O_P} \text{pr}^{-1} \tilde{D}_{h,p}$$

where $\text{pr} : G \times P \rightarrow P$ is the second projection. This is naturally a subsheaf of

$$D_G \boxtimes \tilde{D}_{h,p} := \text{pr}^{-1}_2 (D_G) \otimes \text{pr}^{-1}_1 (\tilde{D}_{h,p})$$

Next, let us recall that the maps $a$ and $\text{pr}$ induce pullback functors

$$a^* , \text{pr}^* : \text{Mod}(\tilde{D}_{h,p}) \rightarrow \text{Mod}(D_G \boxtimes \tilde{D}_{h,p})$$

These functors are simply the quasicoherent pullback of sheaves, but one endows them with the action of vector fields on $G$ by pushforward of vector fields as usual (c.f. \[HTT\], chapter 1). Given this, we make the

**Definition 12.** The category of quasi-$G$ equivariant-coherent $\tilde{D}_{h,p}$-modules, $\text{Mod}^G(\tilde{D}_{h,p})$ has consists of finitely generated $\tilde{D}_{h,p}$-modules $M$ equipped with an isomorphism

$$a^*(M) \rightarrow \text{pr}^*(M)$$

Further, we demand the usual cocycle compatibility spelled out, e.g., in \[Kas\].

The morphisms in this category are those which respect all structures.

We note that $\tilde{D}_{h,p}$ has the structure of a quasi-equivariant coherent module by the simple computation

$$a^*(\tilde{D}_{h,p}) \rightarrow \tilde{D}_{h,p} \rightarrow a^*(\tilde{D}_{h,p})$$

In addition to the formal definition, it will be extremely useful for us to use one of the basic properties of equivariant coherent $D$-modules, following the discussion in \[Kas\]. Since any $M \in \text{Mod}^G(\tilde{D}_{h,p})$ is a quasi-coherent equivariant $P$-module, we have the natural map

$$L_v : g \rightarrow \text{End}_k(M)$$

as described above. However, we also have another map

$$\alpha : g \rightarrow \text{End}_k(M)$$

given by using the natural map $g \rightarrow \Gamma(\tilde{D}_{h,p})$. These can be considered as the “adjoint action” and “left action” of $g$. Thus we can define a third action

$$\gamma = h \cdot L - \alpha : g \rightarrow \text{End}_k(M)$$

which will in fact commute with the action of $\tilde{D}_{h,p}$, i.e.,

$$\gamma : g \rightarrow \text{End}_{\tilde{D}_{h,p}}(M)$$

can be considered a “right action” of $g$ (and hence of $U(g)$).

We can extend this definition to define the category of $G \times \mathbb{G}_m$-equivariant coherent modules $\text{Mod}^{G \times \mathbb{G}_m}(\tilde{D}_{h,p})$ simply by demanding that the modules be graded, and the action respect the grading (we note that $\tilde{D}_{h,p}$ is graded by virtue of being a Rees algebra, but that we put $h$ in degree 2). These categories and their derived
versions will be the other major players in our story. We note that the subcategory of modules in 
\(\text{Mod}^G(D_h,p)\) (resp. \(\text{Mod}^G \times \mathbb{G}_m(D_h,p)\)) where \(h\) acts as zero is precisely the category \(\text{Mod}^G(\mathfrak{g}^p)\) (resp. \(\text{Mod}^G \times \mathbb{G}_m(\mathfrak{g}^p)\)).

To end this subsection, we shall state a result analogous to Lemma 9 at the end of the previous subsection:

**Lemma 13.** The functor \(i^*_e\) gives an equivalence of categories

\[\text{Mod}^G(D_h) \rightarrow \text{Mod}^B(U_h(b))\]

where the category on the right consists of \(U_h(b)\)-modules equipped with an algebraic action of \(B\) satisfying the natural compatibilities. The same is true of the graded versions of these categories.

The proof is exactly the same as that of the coherent case; the inverse is the induction functor. There is also a result concerning generation of this category, but it shall have to wait until the next section where we define the natural deformations of the sheaves \(O^\mathfrak{g}(\lambda)\).

2.2.4. Cohomology results. In this subsection we shall gather several results that we need concerning cohomology of modules in \(\text{Mod}(D_h,p)\). First of all, there is the following base change result:

**Lemma 14.** Let \(M \in \text{Mod}^{G_m}(D_h,p)\). Then we have an isomorphism in the derived category

\[R\Gamma(M) \otimes_{k[h]} L k_0 \cong R\Gamma(M \otimes_{k[h]} L k_0)\]

where \(k_0\) denotes the trivial \(k[h]\)-module.

**Proof.** First, note that there is a natural base change map

\[R\Gamma(M) \otimes_{k[h]} L k_0 \rightarrow R\Gamma(M \otimes_{k[h]} L k_0)\]

which comes from the map of sheaves \(M \rightarrow M/hM\). We shall split the problem of showing this is an isomorphism into two cases.

Let \(M_{\text{tors}}\) denote the subsheaf of \(h\)-torsion sections of \(M\). Then we have an exact sequence

\[0 \rightarrow M_{\text{tors}} \rightarrow M \rightarrow M/M_{\text{tors}} \rightarrow 0\]

we claim that \(M/M_{\text{tors}}\) is actually a flat \(k[h]\)-sheaf. To see this, consider \(M/M_{\text{tors}}(U)\) where \(U\) is affine. Choose any finite \(k[h]\)-submodule, \(V\). Then by the usual classification of modules over a PID, \(V\) is the direct sum of free and finite-dimensional components. The existence of a component of the form \(k[h]/(h - \lambda)^m\) implies that \(h\) has eigenvalue \(\lambda\) somewhere in \(V\). Since \(V\) is \(h\)-torsion free, we have \(\lambda \neq 0\). However, since \(h\) acts as a graded operator of degree 2 on \(M\), one sees that there are no nonzero eigenvalues for \(h\) either. So \(V\) is a free \(k[h]\) module, and we get that \(M/M_{\text{tors}}(U)\) is a direct limit of flat \(k[h]\)-modules, and hence flat.

So we must consider two cases- \(M\) is \(h\)-torsion, and \(M\) is flat. Suppose \(M\) is flat. Then \(R\Gamma(M)\) is equivalent to the Cech complex for \(M\) for a given covering \(\{U_i\}\), which is thus a complex of \(h\)-flat modules. So we have

\[R\Gamma(M) \otimes_{k[h]} L k_0 \cong C^*(\{U_i\}, M) \otimes_{k[h]} k \rightarrow C^*(\{U_i\}, M/hM)\]

where the last isomorphism is from the definition of the Cech complex. This takes care of the flat case.
For the torsion case, we note that there is a natural finite filtration of any torsion sheaf by $O(\mathfrak{h})$-modules (i.e. modules $M$ where $h$ acts trivially). In this case, we have an isomorphism

$$M \otimes^L_{k[h]} k_0 \cong M \oplus M[1]$$

and the same for the global sections $\Gamma(M)$, since $h$ also acts trivially on them. The result for such sheaves follows immediately, as does the general result by walking up the filtration. 

Let us note that the same result holds if $M$ is any bounded complex, since we can reduce to the case where $M$ is concentrated in a single degree by using cutoff functors and exact triangles. Now we can give the important

**Corollary 15.** Let $M \in D^{b,G_m}(\tilde{\mathcal{D}}_h)$. Suppose that $R\Gamma(M \otimes^L_{k[h]} k_0)$ is concentrated in degree zero. Then the same is true of $R\Gamma(M)$.

**Proof.** $R\Gamma(M)$ is a complex of graded, finitely generated $O(h^* \times \mathbb{A}^1)$-modules, whose reduction modulo an ideal of positively graded elements is concentrated in a single degree. Given this, the result follows from the graded Nakayama lemma for complexes- an appropriate version of which is the next proposition. 

**Proposition 16.** Let $M$ be a bounded complex in the category of graded $k[h]$ modules $(\text{deg}(h) > 0)$ whose cohomology sheaves all have grading bounded below. Let $k_0$ denote the trivial graded $k[h]$-module. Then we have

1) If $M \otimes^L_{k[h]} k_0 = 0$, then $M = 0$.

2) If $M \otimes^L_{k[h]} k_0$ is concentrated in a single degree, then the same is true of $M$.

**Proof.** Since $k[h]$ has global dimension one, any bounded complex $M$ is quasi-isomorphic as a complex of $k[h]$-modules to the direct sum of its appropriately shifted cohomology sheaves. Since our complex consists of graded modules and graded morphisms, the cohomology sheaves are graded as well. We write

$$M \cong \oplus H^i(M)[-i]$$

Then the complex $M \otimes^L_{k[h]} k_0$ is quasi-isomorphic to

$$\oplus(H^i(M)/h)[-i] \oplus \text{Tor}^1_{k[h]}(k_0, H^i(M))[-i + 1]$$

simply because, for any object $N \in k[h] - \text{mod}

$$N \otimes^L_{k[h]} k_0 \cong (N/h) \oplus \text{Tor}^1_{k[h]}(k_0, N)[1]$$

(a quasi-isomorphism of complexes). Now, if we are in the situation of 1, the assumption implies $H^i(M)/h = 0$ for all $i$, and so the graded Nakayama lemma for modules yields $H^i(M) = 0$ for all $i$, hence $M$ is equivalent to the trivial complex.

Next, suppose we are in the situation of 2, and shift so that $M \otimes^L_{k[h]} k_0$ is concentrated in degree zero. Then $H^i(M)/h$ must be trivial for all $i \neq 0$, so the same is true for $H^i(M)$ as required. 

We shall also have occasion to consider the functor $R\Gamma^G$ of $G$-invariant cohomology. The results above go through unchanged in this setting, as is easy to see by the fact that the functor $V \to V^G$ is exact on the category of algebraic $G$-modules.

We should like to end the section with some general remarks about the significance of these results for us. The three main “coherent” categories that appear in this work are $D^{b,G \times G_m}(\mathcal{N})$, $D^{b,G \times G_m}(\mathfrak{g})$, and $D^{b}(\text{Mod}^{G \times G_m}(\tilde{\mathcal{D}}_h))$. The above
results will be used to show that, for objects $M, N$ in a certain tilting subcategory $T$ of $D^b(\text{Mod}^G \times \mathbb{G}_m(\hat{D}_h))$, we have

$$\text{Hom}_{D^b(\text{Mod}^G \times \mathbb{G}_m(\hat{D}_h))}(M, N) \otimes_{\mathbb{K}[h]} k_0 \cong \text{Hom}_{D^b(\text{Mod}^G \times \mathbb{G}_m(\tilde{\mathbb{G}))}}(M \otimes_{\mathbb{K}[h]} k_0, N \otimes_{\mathbb{K}[h]} k_0)$$

making precise the notion that $D^b(\text{Mod}^G \times \mathbb{G}_m(\hat{D}_h))$ can be considered a one-parameter deformation of $D^b(\text{Mod}^G \times \mathbb{G}_m(\tilde{\mathbb{G}))}$.

On the other hand, we have the identification $\tilde{\mathcal{N}} \cong \hat{\mathfrak{g}} \times \mathfrak{h} \{0\}$; where $\hat{\mathfrak{g}} \to \mathfrak{h}$ is Grothendieck’s morphism described above, coming from the maps

$$O(\mathfrak{h}^*) \to O(\hat{\mathfrak{g}})^G \to O(\hat{\mathfrak{g}})$$

which follows from the well known fact that $\tilde{\mathcal{N}} \cong \mathfrak{g}^* \times_{\mathfrak{h}^*/\mathfrak{W}} \{0\}$ (c.f. [CG] chapter 3). As is also well known, the morphism $\hat{\mathfrak{g}} \to \mathfrak{h}^*$ is flat (c.f. [Mil]). Therefore the flat base change theorem implies that for $M, N \in D^b(\text{Coh}(\hat{\mathfrak{g}}))$ there is an isomorphism

$$\text{Hom}_{D^b(\text{Coh}(\hat{\mathfrak{g}}))}(M, N) \otimes_{\text{O}(\mathfrak{h}^*)} k_0 \cong \text{Hom}_{D^b(\text{Coh}(\mathfrak{N}))}(\text{Li}^* M, \text{Li}^* N)$$

where $i : \tilde{\mathcal{N}} \to \hat{\mathfrak{g}}$ denotes the inclusion. Thus $D^b(\text{Coh}(\hat{\mathfrak{g}}))$ (and its equivariant and graded versions) can be considered a $\dim(\mathfrak{h})$-parameter deformation of $D^b(\text{Coh}(\mathfrak{N}))$. The exact same set-up holds for the three main categories of perverse sheaves under consideration, and this will turn out to be a key point in proving the main equivalences.

3. Structure of Coherent Categories

In this chapter we discuss two interrelated and crucial pieces of structure: the braid group action and tilting generation of the categories defined in the previous chapter.

3.1. Braid Group Action. In this subsection we recall the main results of the papers [Ri] and [BR].

First of all, let us recall that for any Coxeter system $(W, S)$, there is associated the braid group $B(W, S)$, which is the group on generators $S$ satisfying only the braid relations

$$s_is_js_i \cdots = s_js_is_j \cdots$$

where the number of factors on each side is the $(i, j)$ entry in the associated Coxeter matrix.

The case of interest to us, as usual, is the case of the affine Weyl group $W_{aff}$. Of course, there is also the isomorphism $W_{aff} = W \rtimes \mathbb{Z}\Phi$, and, as in the case of the affine Hecke algebra, there is a presentation of the affine braid group based upon this isomorphism, which is actually slightly more general (c.f. the appendix to [Ri]). So we make the

**Definition 17.** Let $(W, S_{fin})$ be a finite Weyl group, with its root lattice $\mathbb{Z}\Phi$ and its weight lattice $\chi$. The extended affine braid group $B_{aff}$ is the group with generators $\{T_s\}_{s \in S_{fin}}$, and $\{\theta_x\}_{x \in \chi}$, and relations:

- $T_sT_{s'} = T_{s'}T_s$ whenever $<s, s'> = 0$.
- $\theta_x\theta_y = \theta_{x+y}$ for all $\chi, y \in \chi$.
- $T_s\theta_x = \theta_{x}T_s$ whenever $<x, \alpha_s > = 0$.
- $\theta_x = T_s\theta_{sx}T_s$ whenever $<x, \alpha_s > = 1$. 

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If we replace the lattice $X$ with the root lattice $\mathbb{Z}^\Theta$ in this presentation, then the resulting group is just isomorphic to the usual affine braid group $B_{aff}$; in particular $B_{aff}$ is a subgroup of finite index in $B_{aff}'$. We also note that the extended affine Weyl group $W_{aff}'$ is then the quotient of $B_{aff}$ by the relations
\[ s_i^2 = 1 \]
for all $s_i$. The group $W_{aff}$ is the analogous quotient for $B_{aff}$.

This presentation is useful for explaining how $B_{aff}'$ will act on categories of coherent sheaves. To set this up, we shall briefly recall the notion of a Fourier-Mukai functor.

Let $X$ and $Y$ be algebraic varieties, and let $F \in D^b\text{Coh}(Y \times X)$ be a complex of coherent sheaves whose support is proper over both $X$ and $Y$. Then we have a well defined functor on $D^b\text{Coh}(X) \to D^b\text{Coh}(Y)$
\[ F_F(M) = Rp_{2*}(F \otimes_{O_{Y \times X}} Lp_1^*(M)) \]
The sheaf $F$ is called the kernel of this functor. For example, the diagonal sheaf $O_{\Delta X}$ corresponds to the identity functor, while for a proper morphism $f$ the standard (derived) functors $f^*$ and $f_*$ can be realized via the sheaf of functions on the graph of $f$ (c.f. [Huy], page 114).

Further, we shall need the fact that the composition of functors corresponding to two kernels $F$ and $G$ can be realized as the “composition” of the kernels, as follows: suppose $F \in D^b(\text{Coh}(X \times Y))$ and $G \in D^b(\text{Coh}(Y \times Z))$. We define
\[ F \circ G := p_{XZ,*}(p_{XY}^*(F) \otimes p_{YZ}^*(G)) \in D^b(\text{Coh}(X \times Z)) \]
(all supports assumed proper, all functors derived). Then we have the

**Proposition 18.** There is an isomorphism $F_G \circ F_F \cong F_{F \circ G}$ of functors $D^b(\text{Coh}(X)) \to D^b(\text{Coh}(Z))$.

This is proved in [Huy], page 114.

The braid group action we shall present is given by Fourier-Mukai kernels. To explain the sorts of varieties we shall need, let us recall that $\tilde{G}$ has an open subvariety $\tilde{\alpha}^{rs}$ (defined above!) which has a natural action of the finite Weyl group $W$. For $s_\alpha \in S_{fin}$, we define the variety $S_\alpha \subset \tilde{G} \times \tilde{G}$ to be the closure of the graph of $s_\alpha$ in $\tilde{\alpha}^{rs} \times \tilde{\alpha}^{rs}$. We further define $S_{\alpha}^{\prime}$ to be the variety $S_\alpha \cap (\tilde{N} \times \tilde{N})$ (c.f. [Ri], section 4).

Then, we have

**Theorem 19.** There is an action$^3$ of the group $B_{aff}'$ on the category $D^b\text{Coh}(\tilde{G})$ which is specified by

- The action of $\theta_{\lambda}$ is given by the kernel $\Delta_*(O_{\tilde{G}}(\lambda))$ (where $\Delta$ is the diagonal inclusion)
- The action of $s_\alpha \in S_{fin}$ is given by the kernel $O_{S_\alpha}$.

This action restricts to an action on the category $D^b\text{Coh}(\tilde{N})$, in the sense that if we define kernels $\Delta_*(O_{\tilde{S}}(\lambda))$, and $O_{S_{\alpha}^{\prime}}$, we get an action of $B_{aff}$ on $D^b\text{Coh}(\tilde{N})$ which agrees with the previous action under the inclusion functor $i_*$. Further, these same kernels also define braid group actions on the equivariant categories $D^{b,G}(\tilde{G})$, $D^{b,G \times G_m}(\tilde{G})$; and the same for $\tilde{N}$.

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$^3$By an action in this case we mean a weak action. We shall discuss an extension of this to a stronger structure later in the paper.
We shall show later on that this action also extends to an action on the category $\mathcal{D}^b(\text{Mod}(\hat{D}_h))$.

We note, for later use, the following:

**Claim 20.** For each finite root $s_\alpha$, we have $s_\alpha \cdot O_{\hat{\theta}} = O_{\hat{\theta}}$.

This is proved in [BR, BM].

### 3.2. Highest Weight Structure.

In this section, we discuss a crucial piece of structure on the categories $\mathcal{D}^b\text{Coh}^G(\mathcal{N})$ and $\mathcal{D}^b\text{Coh}^{G \times \mathbb{G}_m}(\mathcal{N})$, which appears in the paper [B1]. In that paper the author defines a t-structure, known as the perversely exotic t-structure, which corresponds under the equivalence of [AB] to the perverse t-structure on the category $\mathcal{D}^b_{I_u,\chi}(G((t))/I)$ (see [BM, B1] for a proof of this fact).

A nice feature of this t-structure is that it can be defined in relatively elementary terms, using the braid group action. To give the statement, we shall have to recall some general facts, starting with the

**Definition 21.** Let $\mathcal{D}$ be a triangulated category, linear over a field $k$. Suppose that $\mathcal{D}$ is of finite type, so that $\text{Hom}(N,M)$ is always finite dimensional over $k$. Let $\nabla = \{\nabla^i | i \in I\}$ be an ordered set of objects in $\mathcal{D}$, which generate $\mathcal{D}$ as a triangulated category. This set is called exceptional if $\text{Hom}(\nabla^i, \nabla^j) = 0$ whenever $i < j$, and if $\text{Hom}(\nabla^i) = k$ for all $i$.

The classic example of such a set is the collection of Verma modules in the principal block of the BGG category $\mathcal{O}$.

Of course, part of the advantage of Verma modules is that there are dual Verma modules, and natural maps $M(\lambda) \to M^*(\lambda)$, whose image is the irreducible module $L(\lambda)$. It turns out that there is a general version of this fact as well. So let us make the

**Definition 22.** Let $\mathcal{D}$ be a triangulated category with a given exceptional set $\nabla$. We let $\mathcal{D}_{<i}$ denote the triangulated subcategory generated by $\{\nabla^j | j < i\}$. Then another set of objects $\{\Delta_i | i \in I\}$ is called a dual exceptional set if it satisfies $\text{Hom}(\Delta_n, \nabla^i) = 0$ for $n > i$, and if we have isomorphisms

$$\Delta_i \cong \nabla^i \mod \mathcal{D}_{<i}$$

for all $i$.

Whenever the dual exceptional set exists, it is unique (c.f. [B1]).

### 3.2.1. Existence of a t-structure.

Now we would like to recall, from the paper [B2], the existence of a t-structure which is compatible with the standard and costandard objects. In particular, under the above assumptions, with the additional assumption that the order set $I$ is either finite or (a finite union of copies of) $\mathbb{Z}_{>0}$, we have the following

**Theorem 23.** a) There exists a unique t-structure, $(\mathcal{D}^{\geq 0}, \mathcal{D}^{< 0})$, which satisfies $\nabla^i \in \mathcal{D}^{\geq 0}$ and $\Delta_i \in \mathcal{D}^{\leq 0}$ for all $i$.

b) This t-structure is bounded.

c) For any $X \in \text{Ob}(\mathcal{D})$, we have that $X \in \mathcal{D}^{\geq 0}$ iff $\text{Hom}^{< 0}(\Delta_i, X) = 0$ for all $i$, and similarly, $X \in \mathcal{D}^{< 0}$ iff $\text{Hom}^{\leq 0}(X, \nabla^i) = 0$.

d) We let $\mathcal{A} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ denote the heart of this t-structure. Every object of $\mathcal{A}$ has finite length. For each $i$ there is a canonical arrow

$$\tau_{\geq 0}(\Delta_i) \to \tau_{\geq 0}(\nabla^i)$$
whose image is an irreducible object of $A$, called $L_i$. The set \{L_i\}_{i \in I}$ is a complete, pairwise non-isomorphic set of irreducibles in $A$.

Let us remark that by part c, if our exceptional collection satisfies $\text{Hom}^{<0}(\nabla^i, \nabla^j) = 0$ and $\text{Hom}^{<0}(\Delta_i, \Delta_j) = 0$ for all $i$ and $j$, then in fact the collections $\nabla$ and $\Delta$ are actually in the heart $Z$. This will be the case in all of the examples we consider.

We should also note that the theorem holds in the slightly modified situation of a graded triangulated category. In particular, we suppose that $D$ is equipped with a triangulated autoequivalence $M \to M(1)$, which is the “shift in grading.” In this instance, we can define the graded hom $\text{Hom}_M^g(X, Y) = \oplus_n \text{Hom}(X, Y(n))$.

In this case, we say that a collection of objects \{X_i\} generates $D$ if $\{X_i(n)\}_{n \in \mathbb{Z}}$ generates $D$. Then a graded exceptional set is defined as above but using $\text{Hom}_M^g$ instead of $\text{Hom}$, and using this looser sense of “generate”. Then there is a graded analogue of theorem 9 where one replaces all instances of $\text{Hom}$ with $\text{Hom}_M^g$. See [B1] for details.

3.2.2. Perversely Exotic t-structure. Now we are ready to describe the perversely exotic t-structures on $D^b(\text{Coh}^G(\mathcal{N}))$ and on $D^b(\text{Coh}^G \times \mathbb{G}_m(\mathcal{N}))$. In fact, we shall, following [B1, BM], write down the exceptional and coexceptional sets explicitly.

Our indexing set $I$ will be the character lattice $X$. We first consider $X$ with the Bruhat partial ordering- this is the partial ordering induced from considering $X$ as a subset of the affine extended Weyl group $W_{\text{aff}}$.

More explicitly, we can define the order as follows: let $\lambda$ and $\nu$ be two elements of $X$. Choose $w(\lambda)$ and $w(\nu)$ in the group $W_{\text{fin}}$ so that $w(\lambda) \cdot \lambda$ sits in the dominant cone, and the same for $\nu$. Then $\lambda \leq \nu$ iff $w(\lambda) \cdot \lambda$ is below $w(\nu) \cdot \nu$ in the usual dominance ordering.

We note from this description that there are finitely many elements which are absolute minima under this ordering; these are precisely the set of minimal representatives in $X$ of the finite group $X/\mathbb{Z} \Phi = \Omega$. So, we complete $\leq$ to a complete ordering on $X$, which we choose to be isomorphic to a finite union of copies of $\mathbb{Z}_{>0}$.

Now we can define our exceptional and coexceptional sets as follows: we let $B^+_{\text{aff}}$ denote the subsemigroup of the affine braid group generated by \{s_a\}_{a \in I_{\text{aff}}}$, and $B^-_{\text{aff}}$ the subsemigroup generated by the inverses. Then our exceptional set is the collection of $\{b^- \cdot \omega O_{\mathcal{N}}\}_{\omega \in \Omega}$ and our coexceptional set is the collection of $\{b^+ \cdot \omega O_{\mathcal{N}}\}_{\omega \in \Omega}$. Then, one can in fact show c.f. [B1, BM], that these sets are indexed by $X$, by sending an element $b^\pm \omega$ its action on $0 \in X$; and thus we can also label them \{\Delta_\lambda\} and \{\nabla^\lambda\} for $\lambda \in X$.

These indexing sets have several nice properties- at the bottom of the ordering, the objects \{\omega O_{\mathcal{N}}\} are both standard and costandard. In addition, for any $\lambda$ which is in the dominant cone of $X$, we can choose the representative $\theta_\lambda \in B^-_{\text{aff}}$ as an element of the form $b^\pm \omega$. We know from the explicit presentation of the braid group action given above that the action of this element is given by tensoring by the line bundle $O_{\mathcal{N}}(\lambda)$. Thus we have that the set of dominant coexceptional objects is \{O_{\mathcal{N}}(\lambda)\}_{\lambda \in \mathbb{Y}^+}$, and the exceptional are \{O_{\mathcal{N}}(-\lambda)\}_{\lambda \in \mathbb{Y}^+}.

So, we now can define the perversely exotic t-structure to be the t-structure provided by the above theorem on $D^b(\text{Coh}^G(\mathcal{N}))$ and $D^b(\text{Coh}^G \times \mathbb{G}_m(\mathcal{N}))$ (using the graded version for the latter).

\[\text{In [B1], he works with the adjoint group, and so assumes that the ordering is isomorphic to a single copy of } \mathbb{Z}_{>0}. \text{ However, the results we need go over to our case without any difficulty.} \]
We would like to record one very important feature of this $t$-structure right now. We recall from [BM] the

**Definition 24.** A $t$-structure on one of the categories $D^b(Coh(\mathfrak{g}))$, $D^b(Coh(\tilde{N}))$ (or one of the equivariant versions) is said to be braid positive if for any affine root $\alpha$, the functor $s^{-1}_\alpha$ is left exact with respect to this $t$-structure. Of course, by adjointness, this implies immediately that $s_\alpha$ is right exact.

Then we have the very easy

**Lemma 25.** The perversely exotic $t$-structure is braid positive.

**Proof.** By the definition of the $t$-structure and part c of the theorem, we have that $X \in D^{\geq 0}$ iff $\text{Hom}^< b^+ \omega \cdot O_{\tilde{N}}, X) = 0$ for all $b^+ \in B^+_{aff}$. But then we have by adjointness

$$\text{Hom}^< (b^+ \omega \cdot O_{\tilde{N}}, s^{-1}_\alpha \cdot X) = \text{Hom}^< (s^{-1}_\alpha b^+ \omega \cdot O_{\tilde{N}}, X)$$

and the term on the right vanishes because $s_\alpha$ is positive, so $s_\alpha b^+$ is a positive element of the braid group also. □

In fact, a similar argument shows something a bit stronger:

**Lemma 26.** Suppose that $X \in \mathcal{A}$ is filtered (in $\mathcal{A}$) by standard objects $\nabla^\lambda$. Then for all simple affine roots, $s^{-1}_\alpha X$ is in $\mathcal{A}$. Similarly, if $X$ is filtered by costandard objects, then $s_\alpha X$ is in $\mathcal{A}$.

**Proof.** By definition, $X \in \mathcal{A}$ iff $X \in D^{\geq 0}$ and $X \in D^{\leq 0}$. So, to show the first claim, we must show that, under the assumptions, we have

$$\text{Hom}^< (b^+ \omega \cdot O_{\tilde{N}}, s^{-1}_\alpha \cdot X) = \text{Hom}^< (\alpha^{-1} X, b^+ \omega \cdot O_{\tilde{N}}) = 0$$

for all $b^+ \in B^+_{aff}$. The first equality holds simply because $X \in \mathcal{A}$. For the second, we shall walk up a standard filtration of $X$: if $X$ is itself standard, then by definition $s^{-1}_\alpha X$ is also standard, and hence in $\mathcal{A}$.

So suppose that we have the exact sequence

$$0 \to Y \to X \to b_1^+ \omega \cdot O_{\tilde{N}} \to 0$$

in $\mathcal{A}$, where $Y$ has a filtration by standard objects of length $n - 1$. Hitting this sequence with $s^{-1}_\alpha$ gives the triangle

$$s^{-1}_\alpha Y \to s^{-1}_\alpha X \to s^{-1}_\alpha b_1^+ \omega \cdot O_{\tilde{N}}$$

whose left and right terms are in $\mathcal{A}$, by induction. Then, by the long exact sequence for $\text{Hom}$, we have for all $b^- \omega$ the sequence

$$\text{Hom}^i (s^{-1}_\alpha b_1^- \omega O_{\tilde{N}}, b^- \omega \cdot O_{\tilde{N}}) \to \text{Hom}^i (s^{-1}_\alpha X, b^- \omega \cdot O_{\tilde{N}}) \to \text{Hom}^i (s^{-1}_\alpha Y, b^- \omega \cdot O_{\tilde{N}})$$

and the left and right terms are zero for $i > 0$; so the middle one is as well, proving the first claim. The second claim follows in exactly the same way. □

**Remark 27.** The proof actually shows that the object $s^{-1}_\alpha X$ is filtered by standard objects in $\mathcal{A}$: since the exact triangles

$$s^{-1}_\alpha Y \to s^{-1}_\alpha X \to s^{-1}_\alpha b_1^+ \omega \cdot O_{\tilde{N}}$$

are actually exact sequences in $\mathcal{A}$, this is shown by the same inductive argument. Clearly the analogous fact is true for $s_\alpha X$ if $X$ is filtered by costandard objects.
3.3. Reflection Functors. In this subsection we define the reflection functors-they will come naturally out of the braid group action, and will allow us to construct explicitly the tilting objects of our category. We shall also see in the next section that they are the key to lifting the braid group action from coherent sheaves to $\tilde{D}_h$-modules.

To motivate the definition, we need to recall a bit of geometry from the paper [Ri]. Recall that we have defined above the kernel $O_{S_\alpha}$ to be the structure sheaf of the variety $S_\alpha$, which in turn is defined as the closure of the graph of the action of the Weyl group element $s_\alpha$ acting on $\tilde{\mathfrak{g}}^{rs}$. Let us recall also that we have defined varieties $\tilde{\mathfrak{g}}_P$ associated to any partial flag variety $P$. In the case $P = G/P_\alpha$, we shall denote this variety $\tilde{\mathfrak{g}}_\alpha$, and the natural map $\pi_\alpha : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_\alpha$.

Now, we let us consider the algebraic variety $\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}_\alpha \tilde{\mathfrak{g}}$. This is not an irreducible variety, but instead has two components: the first is the diagonal $\Delta \tilde{\mathfrak{g}}$, and the second is $S_\alpha$. The natural restriction morphism leads to a short exact sequence of kernels:

$$K \to O_{\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}_\alpha \tilde{\mathfrak{g}}} \to O_{\Delta \tilde{\mathfrak{g}}}$$

and it is checked in [Ri] that $K$ is the kernel of functor inverse to $s_\alpha$. Further, when we consider the action on $D^b(G_m)(\text{Coh}(\tilde{\mathfrak{g}}))$, we get that $K(2)$ is inverse to $s_\alpha$.

So we should like to understand the kernel $O_{\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}_\alpha \tilde{\mathfrak{g}}}$. Fortunately, it is easy to describe, following [Ri]:

Lemma 28. There is an isomorphism of functors

$$F_{O_{\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}_\alpha \tilde{\mathfrak{g}}}} \cong \pi_s^* \pi_{ss}$$

where the functor on the right is taken in the derived sense. Further, the natural adjunction $\pi_s^* \pi_{ss} \to Id$ comes from the natural map of sheaves $O_{\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}_\alpha \tilde{\mathfrak{g}}} \to O_{\Delta \tilde{\mathfrak{g}}}$ (the restriction to a subvariety quotient map).

These facts lead us to the following

Definition 29. For a finite root $\alpha$, we define the reflection functor $R_\alpha$ to be the functor of the kernel $O_{\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}_\alpha \tilde{\mathfrak{g}}}$.

These functors have many nice properties. As already noted, there is a natural complex of functors $s_\alpha^{-1}(-2) \to R_\alpha \to Id$. In fact, there is also a natural adjunction morphism $Id \to R_\alpha(2)$ defined in [Ri], section 5, and an exact sequence $Id \to R_\alpha(2) \to s_\alpha$. Thus it is possible to describe completely the finite braid actions via the reflection functors.

We should note that the adjunction morphism $Id \to \pi_s^* \pi_{ss}(2)$ has a natural algebro-geometric explanation. We have an isomorphism $\pi_s^* \cong \pi_{ss}^*$ (noted in [BMRI,II]), and in fact $\pi_s^*$ is the right adjoint to $\pi_{ss}$ (c.f. [Ha]). In this instance it has the advantage of having been constructed by hand in terms of Fourier-Mukai kernels.

Next we would like to define the reflection functor corresponding to the affine root. Of course, there is no “affine root” partial flag variety, so we have to use a trick to get around it. The trick relies on the following

Claim 30. In the extended affine braid group $\mathbb{B}_aff'$, there exists a finite root element $s_\alpha$ which is conjugate to the affine root $s_{\alpha_0}$.

This claim is proved in [BM], lemma 2.1.1. It is interesting to note that in every type except $C$, the claim is true in the non-extended affine braid group $\mathbb{B}_aff$. 
This is helpful for the following reason. If we take the exact sequence of functors

\[ Id \to R_\alpha \to s_\alpha \]

and conjugate by an appropriate element in \( B_{aff} \), \( b \), we then arrive at a new sequence

\[ Id \to b^{-1}R_\alpha b \to s_{\alpha_0} \]

and the same holds for the exact sequence for \( s^{-1}_\alpha \). Thus if we define the affine reflection functor

\[ R_{\alpha_0} := b^{-1}R_\alpha b \]

then this is a functor which satisfies the same exact sequences for \( \alpha_0 \) as the other reflection functors for their roots. This functor will then do everything we need. Further, it will turn out that the action of this functor is unique up to a unique isomorphism.

3.4. Tilting Generators For Coherent Sheaves. In this section, we shall describe collections of tilting modules for the categories of our interest. We gave a general definition of a tilting subcategory above, which we shall use for \( D^b\text{Coh}^{G}(\tilde{g}) \) and \( D^b\text{Mod}^{G}(\tilde{D}_h) \). However, in the case of \( D^b\text{Coh}^{G}(\tilde{N}) \), there is a way of constructing tilting modules just using the highest weight structure. We shall take this to be our base case.

3.4.1. Tilting in Highest Weight Categories. In this section, we shall recall the general constructions and definitions of [S4, BBM, B1]. Let us suppose we are in the situation of section 3.2, where we have a triangulated category with a \( t\)–structure, whose heart \( \mathcal{A} \) contains a given set of exceptional and coexceptional objects. In this very special situation, we make the

**Definition 31.** A tilting object in \( \mathcal{A} \) is one which possesses a filtration (in \( \mathcal{A} \)) by standard objects, and a filtration (in \( \mathcal{A} \)) by costandard objects.

This, as it turns out, is a very strong condition. We recall from [S4, BBM, B1, B2] some properties that these objects satisfy:

**Lemma 32.** In the above situation, we have that:

1) To each \( i \in I \), there is a unique indecomposable tilting module \( T_i \), which has a unique (up to scalar) surjection \( T_i \to \nabla^i \) and a unique (up to scalar) injection \( \Delta_i \to T_i \).

2) Every tilting module is a direct sum of the \( T_i \).

3) For any two tilting modules \( \text{Hom}(T_i, T_j) = \text{Hom}^0(T_i, T_j) \).

4) The tilting modules generate the standard and costandard objects; thus they generate the entire triangulated category. Combining with the above observation, we obtain an equivalence of categories

\[ K^b(\mathcal{T}) \to \mathcal{D} \]

where the left hand side is the homotopy category of complexes of tilting modules (c.f. section 1.5).

5) Let \( X \) be any object of \( \mathcal{D} \) such that \( \text{Hom}^>0(\Delta_i, X) = \text{Hom}^>0(X, \nabla^i) \) for all \( i \). Then \( X \) lies in \( \mathcal{A} \) and is a tilting object therein.
Thus, our goal is now to compare this abstract characterization of tilting (which holds for the perversely exotic t-structure on $D^b\text{Coh}^G(N)$ and its graded version) with the concrete information listed above about our categories. The main tool in this will be the reflection functors.

3.4.2. Tilting via Reflection Functors. In this subsection, we will use the reflection functors to construct tilting modules in our various categories. There is a minor problem: the reflection functors only act on $\text{Coh}(\mathfrak{g})$, and do not restrict to functors on $\text{Coh}(\mathcal{N})$- so we have to construct our objects over $\mathfrak{g}$, and then restrict.

To start with, we should define natural lifts of our standard and costandard objects to $\mathfrak{g}$. This is easy- since the braid group action on the variety $\mathfrak{g}$ is consistent via the restriction functor with the braid group action on $\mathcal{N}$, we define

$$\Delta_\lambda(\mathfrak{g}) = b^+\omega \cdot O_\mathfrak{g}$$

and

$$\nabla^\lambda(\mathfrak{g}) = b^-\omega \cdot O_\mathfrak{g}$$

where we have that $\lambda = b^+\omega \cdot 0$.

Claim 33. These are well defined objects which restrict to our given standard and costandard objects on $\mathcal{N}$.

Proof. Evidently these objects restrict to our given standard and costandard objects on $\mathcal{N}$; we shall now argue that they are well defined. Suppose $\lambda \in \mathbb{Y}$ has two decompositions $b_1^+\omega_1 \cdot 0 = \lambda = b_2^+\omega_2 \cdot 0$, we wish to show $b_1^+\omega_1 \cdot O_\mathfrak{g} \cong b_2^+\omega_2 \cdot O_\mathfrak{g}$ (the argument for the standard objects will work the same way). Then, by the well-definedness on $\mathcal{N}$, we have a $G \times \mathbb{G}_m$-isomorphism

$$O_{\mathcal{N}} \cong \omega_1^{-1}b_1^-b_2^+\omega_2 \cdot O_{\mathcal{N}}$$

Thus $R^G(\omega_1^{-1}b_1^-b_2^+\omega_2 \cdot O_{\mathcal{N}}) \cong k$, and so $R^G(\omega_1^{-1}b_1^-b_2^+\omega_2 \cdot O_\mathfrak{g}) \otimes_{O(\mathfrak{g})}^L k \cong k$; implying the existence of a $G \times \mathbb{G}_m$-global section of $\omega_1^{-1}b_1^-b_2^+\omega_2 \cdot O_\mathfrak{g}$ and thus a morphism

(3.1)$$O_\mathfrak{g} \to \omega_1^{-1}b_1^-b_2^+\omega_2 \cdot O_\mathfrak{g}$$

lifting the one above. By the graded Nakayama lemma (applied locally), this is a surjective morphism of sheaves.

Now, if we play the same game with the inverse $\omega_2^{-1}b_2^-b_1^+\omega_1 \cdot O_{\mathcal{N}}$, we get a map the other way, such that the composition with the map 3.1 is a $G \times \mathbb{G}_m$-endomorphism of $O_{\mathfrak{g}}$ lifting the identity of $O_{\mathcal{N}}$, which therefore is the identity of $O_{\mathfrak{g}}$ since $R^G \times \mathbb{G}_m(O_{\mathfrak{g}}) = k$. Thus the map 3.1 is injective, and hence an isomorphism.

□

Before we proceed, let us make one notational convention. Given a collection of objects $\{D_i\}$ in a triangulated category $C$, we shall say that an object $X$ is filtered by the $\{D_i\}$ if there is a finite sequence of objects $\{X_j\}_{j=1}^n$ with $X_1 \in \{D_i\}$, $X_n = X$, and for all $j$ there are exact triangles:

$$X_{j-1} \to X_j \to Q$$

where $Q$ is an object in the set $\{D_i\}$.

Now, the very definition of reflection functors implies the following

Claim 34. Suppose that $X$ is an object in $D^b\text{Coh}^G(\mathfrak{g})$ (or $D^b\text{Coh}^{G \times \mathbb{G}_m}(\mathfrak{g})$) which is filtered by the $\Delta_\lambda$ (the $\Delta_\lambda(i)$, respectively). Then $R_\alpha X$ is also so filtered.

The same holds if the $\Delta_\lambda$ are replaced by the $\nabla^\lambda$.
Proof. The proof of the first part comes from the exact sequence
\[ X \to R_\alpha X \to s_\alpha X \]
since by definition, if \( X \) is filtered by the \( \Delta_\lambda \), so is \( s_\alpha X \), and thus so is \( R_\alpha X \). The second part follows by using the exact sequence for \( s_\alpha^{-1} \).

Now, we already know a finite collection of objects of \( D^b\text{Coh}^G(\tilde{g}) \) (and also \( D^b\text{Coh}^{G \times G_m}(\tilde{g}) \)) which are filtered by both the \( \{ \Delta_\lambda(\tilde{g}) \} \) and the \( \{ \nabla^\lambda(\tilde{g}) \} \): namely, the set \( \{ \omega \cdot O_{\tilde{g}} \}_{\omega \in \Omega} \), which are both standard and costandard. So it follows from the claim that all objects of the form
\[ R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{g}} \]
for all collections of affine simple roots \( \{ \alpha_i \}_{i=1}^n \) are filtered by both the \( \{ \Delta_\lambda(\tilde{g}) \} \) and the \( \{ \nabla^\lambda(\tilde{g}) \} \).

Now, we can consider the restriction of such an object to \( \tilde{N} \). We have the

Claim 35. All objects of the form
\[ R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{g}}|_\tilde{N} \]
are tilting objects in the heart of the perversely exotic \( t \)-structure. All indecomposable tilting objects for this \( t \)-structure are summands of such objects.

Proof. We shall show that these are tilting objects in \( A \) by from the definition of tilting. By the construction, these objects are filtered (in the triangulated sense) by both standard and costandard objects. Now, consider the exact triangles of the form
\[ R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{g}}|_\tilde{N} \to R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{g}}|_\tilde{N} \to s_\alpha R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{g}}|_\tilde{N} \]
We assume by induction that the leftmost object is in \( A \) and is even filtered, in \( A \), by standard and costandard objects. Then the right hand object is in \( A \) by Lemma 26. Thus the middle object is in \( A \) and its filtration by standard objects is actually a filtration in \( A \), by remark 27. Using the other exact sequence, we obtain that the same is true for its filtration by costandard objects, and it is a tilting object by definition.

To see that we obtain all indecomposable tilting objects as summands, we use the same exact sequences as in the first part- by going through all reduced words in \( W_{aff} \), we eventually can obtain objects which have any given \( b^{-1} \omega \cdot O_{\tilde{N}} \) at the top of the filtration. Thus we obtain all tilting modules. \( \square \)

Having thus obtained tilting modules explicitly as a restriction of certain objects on \( \tilde{g} \), we have obvious candidates for the deformation of these modules to \( \tilde{g} \), which we shall use. In particular, we can immediately deduce the following

Lemma 36. The objects \( R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{g}} \) satisfy
\[ \text{End}_{D^b\text{Coh}^G(\tilde{g})}(R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{g}}) = \text{End}_{D^b\text{Coh}^{G \times G_m}(\tilde{g})}(R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{g}}) \]
The same is true in \( D^b\text{Coh}^{G \times G_m}(\tilde{g}) \) for the objects \( R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} O_{\tilde{g}}(i) \).

Proof. By the deformation arguments of section 2, (3.2)
\[ \text{End}_{D^b\text{Coh}^G(\tilde{g})}(R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{g}}) \otimes_{O(\tilde{g})}^L \otimes_{O(\tilde{g})} \text{End}_{D^b\text{Coh}^{G \times G_m}(\tilde{g})}(R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{g}}|_\tilde{N}) \]
as complexes of graded modules.

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Now, the complex on the right of equation 3.2 has no terms outside of degree zero, because the objects there are tilting modules for a $t$-structure. But then the same must be true for the complex

$$\text{End}_{D^b Coh \overset{\eta}{\to} \tilde{\mathcal{O}}}(R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}})$$

by the graded Nakayama lemma (c.f. 16- to make the arguments given there apply, one should filter the bundle $\tilde{\mathcal{O}} \to \tilde{N}$ by line bundles indexed by a basis of $\mathfrak{h}^*$. Hence the first claim, and the second immediately follows since the complexes in question are summands of the first ones. \hfill $\square$

**Remark 37.** We should note that a much stronger version of this lemma (without any $G$-equivariance) is proved in [BM]. The proof there uses reduction to characteristic $p$, where the relation with modular representations of lie algebras is exploited.

Now, we would like to show the key property of tilting modules, namely, that $T$, the full subcategory on the objects $R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}}$, generates (in the sense of section 1.5) the whole category. We recall that all objects of the form $O_{\tilde{\mathcal{O}}}(\lambda)$ generate (both the graded and ungraded versions of) our category. It follows immediate that the collection $\{ b \cdot O_{\tilde{\mathcal{O}}} \}$ (where $b$ is an element of the extended affine braid group) also generates our category.

With this in hand, we can state and prove the

**Corollary 38.** The tilting objects generate (both the graded and ungraded versions of) our category.

**Proof.** Let $\mathcal{C}$ denote the full triangulated subcategory containing $T$ and closed under extensions, shifts, and direct summands. Then for any sequence of reflections and any $\omega \in \Omega$, we have the exact sequences

$$R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}} \to R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}} \to s_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}}$$

and

$$s_{\alpha_1}^{-1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}} \to R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}} \to R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}}$$

which imply that all objects of the form $s_{\alpha_1}^{\pm} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}}$ are in $\mathcal{C}$. But now consider the sequence

$$s_{\alpha_1}^{\pm} R_{\alpha_3} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}} \to s_{\alpha_1}^{\pm} R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}} \to s_{\alpha_1}^{\pm} s_{\alpha_2} R_{\alpha_3} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}}$$

By the previous implication, we obtain that all objects of the form $s_{\alpha_1}^{\pm} s_{\alpha_2} R_{\alpha_3} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}}$ are in $\mathcal{C}$; and considering the other exact sequence for $R_{\alpha_2} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}}$ yields in fact that all objects of the form $s_{\alpha_1}^{\pm} s_{\alpha_2} R_{\alpha_3} \cdots R_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}}$ are in $\mathcal{C}$.

Continuing in this way, we eventually see that all objects of the form $s_{\alpha_1}^{\pm} s_{\alpha_2} \cdots s_{\alpha_n} \omega \cdot O_{\tilde{\mathcal{O}}}$ are in $\mathcal{C}$. But the previous comments then imply that $\mathcal{C}$ is the entire category. \hfill $\square$

**3.5. $\tilde{D}_k$-Modules.** In this section we would like to show that the above considerations extend to the category $D^b(\text{Mod}^G(\tilde{D}_k))$ and its graded version. The point is to show that all of the definitions lift from the coherent case.
3.5.1. Line Bundles. In this section, we shall show that the notion of twisting by a line bundle lifts in a canonical way. This material follows easily from standard knowledge of twisted differential operators.

Let us start by recalling that for each \( \lambda \in \mathfrak{h}^* \), we define the sheaf of \( \lambda \)-twisted quantized differential operators, \( D^\lambda_h \), to be the quotient of \( \mathcal{D}_h \) by the ideal sheaf generated by \( \{ v - h \lambda(v) | v \in \mathfrak{h} \} \).

The sheaf \( D^\lambda_h/(h-1) = D^\lambda \) is the well known sheaf of twisted differential operators considered, e.g., in [BB] (c.f. also [Mil], Chapter 1). When \( \lambda \) is in the weight lattice, we have an isomorphism

\[
D^\lambda \cong D(O(\lambda))
\]

where the object on the right is the sheaf of differential operators of the equivariant line bundle \( O(\lambda) \) on \( B \). This isomorphism is even an isomorphism of filtered algebras, where the algebra on the left has the filtration induced from \( \mathcal{D}_h \), and the algebra on the right has the filtration by order of differential operators. The associated graded of these algebras is clearly \( O(\mathcal{N}) \) (considered as a sheaf on \( B \)). Further, the Rees algebra of \( D^\lambda \) is then isomorphic to \( D^\lambda_h \) (more or less by definition).

Now, the algebra \( D(O(\lambda)) \) comes with a natural action on the line bundle \( O(\lambda) \) - since by definition an element of \( D(O(\lambda)) \) is an operator on \( O(\lambda) \). Let us equip \( O(\lambda) \) with the trivial filtration- all terms in degree zero. Then the natural action of \( D^\lambda \) on \( O(\lambda) \) becomes a filtration respecting action, and we can therefore deform it to an action of \( D^\lambda_h \) on \( \text{Rees}(O(\lambda)) \cong O(\lambda)[h] \).

Under the natural morphism \( \mathcal{D}_h \to D^\lambda_h \), we obtain that \( O(\lambda)[h] \) is a graded \( \mathcal{D}_h \)-module. The natural \( G \)-equivariant structure on \( O(\lambda) \) lifts to \( O(\lambda)[h] \) (by letting \( G \) act trivially on \( h \)) to make it an element of \( \text{Mod}^{G \times \mathbb{G}_m}(\mathcal{D}_h) \).

Finally, this allows us to define the \( \lambda \)-twist of any \( \mathcal{D}_h \) module as follows: there is an isomorphism of sheaves

\[
M \otimes_{O(B)} O(\lambda) \cong M \otimes_{O(B \times \mathbb{G}^1)} O(\lambda)[h]
\]

and we can define the action of any \( \xi \in \mathcal{D}_h \) on the right hand side by the usual formula:

\[
\xi \cdot (m \otimes v) = \xi m \otimes v + m \otimes \xi v
\]

We shall denote by \( \mathcal{D}_h(\lambda) \) the module \( \mathcal{D}_h \otimes_{O(B)} O(\lambda) \) with this action. By the projection formula, \( \mathcal{D}_h(\lambda)/h \) is the sheaf \( O(\mathfrak{g})(\lambda) \) considered as a sheaf on \( B \).

3.5.2. Deforming Reflection Functors. Now we wish to define the deformation of our reflection functors, starting with the finite roots. Recall that for each finite reflection \( s \) we have a map

\[
\pi_s : B \to \mathcal{P}_s
\]

and also an extension of this map (which we also called \( \pi_s \)) \( \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_s \).

We can realize this latter map as follows: define the variety \( \tilde{\mathfrak{g}}(s) \) as the incidence variety

\[
\{(x, b) \in \mathfrak{g}^* \times B | x_{\mid \text{w}(\pi_s(b))} = 0 \}
\]

Equivalently, we have

\[
\tilde{\mathfrak{g}}(s) = \tilde{\mathfrak{g}}_s \times_{\mathcal{P}_s} B
\]

where \( \tilde{\mathfrak{g}}_s \to \mathcal{P}_s \) is the projection, and \( B \to \mathcal{P}_s \) is \( \pi_s \). Thus we see there are natural maps

\[
\tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}(s) \to \tilde{\mathfrak{g}}_s
\]
(inclusion and projection, respectively), and the composition is nothing but our standard map $\tilde{g} \to \tilde{g}_s$.

The benefit of writing things this way is that it makes deformation easy. In particular, we can define a sheaf (on $B$), called $\tilde{D}_h(s)$, quantizing $\tilde{g}_s$, as follows: recall the sheaf of algebras $U^0$ on $B$ from section 2, and define $u^0(s)$ as the ideal sheaf generated by $u(\pi_s(b))$ at each point. Then the quotient

$$\tilde{D}(s) = U^0 / u^0(s)$$

is naturally a filtered sheaf of algebras which deforms $\tilde{g}(s)$; and we set $\tilde{D}_h(s) = \text{Rees}(\tilde{D}(s))$.

Then there is the obvious quotient map $\tilde{D}_h(s) \to \tilde{D}_h$ which deforms the inclusion $\tilde{g} \to \tilde{g}_s$. The pushforward under this inclusion is then deformed by the functor which regards a $\tilde{D}_h$-module as a $\tilde{D}_h(s)$ module.

Further, if $M$ is a $\tilde{D}_h(s)$-module, then clearly $(\pi_s)_*(M)$ is a $\tilde{D}_h, p_s$-module. Thus we have defined a natural functor (which on the level of sheaves is simply $(\pi_s)_*$) from $\text{Mod}(\tilde{D}_h)$ to $\text{Mod}(\tilde{D}_h, p_s)$. Thus functor clearly respects $G$-equivariance and grading. Further, this is a deformation of the pushforward map on coherent sheaves, in the sense of the following

**Proposition 39.** We have an isomorphism

$$(\pi_s)_*(M) \otimes_{k[h]}^L k_0 \cong (\pi_s)_*(M \otimes_{k[h]}^L k_0)$$

where $M \in \text{Mod}^{G_m}(\tilde{D}_h)$ and the $(\pi_s)_*$ on the right is that of coherent sheaves.

The proof follows immediately from Lemma 14 and 15.

Next, we would like to define a functor

$$\pi_s^* : \tilde{D}_h, p_s \to \tilde{D}_h, B$$

This we can also do, following the definition for coherent sheaves. So, we first deform the pullback along the map

$$\tilde{g}(s) \to \tilde{g}_s$$

by defining a functor on $\text{Mod}(\tilde{D}_h, p_s)$ as

$$M \to O(B) \otimes_{\pi^{-1}(O(p))}^{\pi_s^{-1}}(M)$$

which is clearly an object of $\text{Mod}(\tilde{D}_h(s))$ (by the definition of $\tilde{D}_h(s)$).

Next we deform the pullback along the map $\tilde{g} \to \tilde{g}(s)$ by defining a functor on $\text{Mod}(\tilde{D}_h(s))$ as

$$M \to \tilde{D}_h \otimes_{\tilde{D}_h(s)} M$$

where $\tilde{D}_h$ is a module over $\tilde{D}_h(s)$ by the natural surjection of algebras; so in fact this functor is just

$$M \to M / \mathcal{I}$$

where $\mathcal{I}$ is the ideal sheaf kernel of $\tilde{D}_h(s) \to \tilde{D}_h$.

Taking the composition of these two functors yields a functor $\pi_s^* : \text{Mod}(\tilde{D}_h, p_s) \to \text{Mod}(\tilde{D}_h)$. As above, one checks immediately that this functor preserves graded and Equivariant versions of the category, and it is very easy to see that this functor deforms the pullback of coherent sheaves (as in the proposition above).
Let us note that there is a natural adjunction $\pi^*_s \pi_* \rightarrow Id$ for the usual reasons. Further, we note that these functors extend naturally to derived functors, and this adjunction continues to hold at that level.

Thus we can now make the

**Definition 40.** To any finite simple root $\alpha$, we associate the functor $R_{\alpha^*} = L\pi^*_s R(\pi_*)$ on $D^b(Mod(\tilde{D}_h))$. Thus functor also preserves the equivariant and graded versions of this category.

This is a crucial step in deforming tilting modules. We shall need to gain some further insight into the behavior of these functors. To do so, we shall reformulate them in terms of a “Fourier-Mukai” type set up.

3.6. Fourier-Mukai functors for Noncommutative rings. In this section, we would like to set up a version of Fourier-Mukai theory which applies to certain sheaves of non-commutative rings on nice spaces.

We start with the “affine” case of noncommutative rings themselves; more precisely, let $A$ and $B$ be two noncommutative flat noetherian $k$-algebras, and let $F$ be a complex in $D^b(A^{opp} \otimes_k B - mod)$. To $F$ we shall associate a functor $F^*: D^b(A - mod) \rightarrow D^b(B - mod)$ as follows:

For any $M \in D^b(A - mod)$, we can form the tensor product $M \otimes^L_k B$. Since $B$ is right $B$-module, we can consider this tensor product as an element of $D^b(A \otimes_k B^{opp} - mod)$; in addition, it carries a left action of $B$ via $b \rightarrow 1 \otimes b$.

So the complex

$$(M \otimes^L_k B) \otimes^L_{A \otimes_k B^{opp}} F$$

which is a priori just a complex of $k$-vector spaces, is in fact a complex of left $B$-modules also; this defines the functor $F^*$.

There are two main cases of interest. First up, we have the

**Example 41.** Let us suppose that $A = B$.

Then, the identity functor on $D^b(A - mod)$ can be realized via the kernel $A$ considered as an $A^{opp} \otimes_k A$-module. To see this, we note that for any $M \in D^b(A - mod)$, there is a morphism of complexes $M \rightarrow (M \otimes^L_k A) \otimes^L_{A \otimes_k A^{opp}} A$ given at the level of objects by $m \rightarrow m \otimes 1 \otimes 1$. However, the complex on the right is equivalent to another where $M$ is replaced by a finite complex of projective $A$ modules. Hence, to show that this morphism is an isomorphism, we can in fact assume that $M$ is a complex of free $A$-modules. But in that case the complex on the right is evidently a complex of free $A$-modules of the same rank, and the map just becomes the identity.

We shall also need to consider the cases provided by the following

**Example 42.** Now let us suppose that there is an algebra map $f: A \rightarrow B$ between two noetherian flat $k$-algebras. We shall express the Fourier-Mukai kernels for the functors $M \rightarrow M \otimes^L_A B$ (here $B$ is considered as a right $A$-module via $f$, and the resulting complex is a left $B$-module via the left action of $B$ on itself) and $N \rightarrow Res_A^B(N)$ (where the restriction is over the map $f$).
For the first functor, which is an arrow $D^b(A-\text{mod}) \rightarrow D^b(B-\text{mod})$, we consider the object $B \in D^b(A^{opp} \otimes_k B)$- here letting $A$ act on the right via $f$ and $B$ act on the left. Then we have a morphism of complexes of $B$-modules

$$M' \otimes_A^L B \rightarrow (M' \otimes_k^L B) \otimes_{A \otimes_k B^{opp}}^L B$$

which comes from the map on objects sending $m \otimes b \mapsto m \otimes 1 \otimes b$. As above, we can actually replace $M'$ by a complex of free $A$-modules to see that this is an isomorphism.

For the second functor, we use the object $B \in D^b(B^{opp} \otimes_k A)$ considered as a left $A$-module via $f$ and a right $B$-module. A similar argument says that this gives the functor $\text{Res}$.

3.6.1. Fourier Mukai Kernels for deformed reflection functors. Now we shall give the sheaf-theoretic versions of the above functors in the cases which are relevant to us. The general set-up simply copies the affine case: let $X$ and $Y$ be $k$-varieties, and let $A$ and $B$ be quasi-coherent sheaves of non-commutative rings on $X$ and $Y$, respectively. We assume $A$ and $B$ are noetherian and flat over $k$.

Then, the product variety $X \times Y$ carries the sheaf of rings $A \boxtimes B$. We consider a complex of sheaves $\mathcal{F}$ belonging to $D^b((A^{opp} \boxtimes B) - \text{mod})$. Then $\mathcal{F}$ defines a functor

$$M' \rightarrow R\pi_*(-(M' \boxtimes B) \otimes_{A^{opp}B^{opp}}^L \mathcal{F})$$

which goes from $D^b(A - \text{mod})$ to $D^b(B - \text{mod})$; here $p$ denotes the projection $X \times Y \rightarrow Y$. As above, the left action of $B$ comes from the additional action of $O_X \boxtimes B$ on the sheaf $(M' \boxtimes B) \otimes_{A^{opp}B^{opp}}^L \mathcal{F}$.

In the case of interest to us, the categories under consideration are $\text{Mod}(\hat{D}_h, \mathcal{P})$ for the various flag varieties $\mathcal{P}$, which will mainly be $B$ or $\mathcal{P}_s$. We shall use the principles of the above section to construct the functors $Id$, $(\pi_s)_*$, and $\pi^*_s$ (from now on we only consider the derived version of these functors; every functor in sight is taken to be derived).

**Example 43.** We start with the functor $Id$. Consider the sheaf of algebras $\hat{D}_h \boxtimes \hat{D}_h^{opp}$ on the scheme $B \times B$, and let $M'$ be a complex in $D^b(\text{Mod}(\hat{D}_h))$. We shall form the Fourier-Mukai functor associated to the diagonal bimodule $\hat{D}_h \in \text{Mod}(\hat{D}_h^{opp} \boxtimes \hat{D}_h)$. This is the functor

$$M' \rightarrow (p_2)_*(((M' \boxtimes \hat{D}_h) \otimes_{\hat{D}_h \boxtimes \hat{D}_h^{opp}}^L \hat{D}_h))$$

(where $p_2 : B \times B \rightarrow B$ is the second projection), which will therefore be an endofunctor of the category $D^b(\text{Mod}(\hat{D}_h))$.

To see that this is just the identity functor, we can reduce to the affine case discussed above by noting that the $(\hat{D}_h \boxtimes \hat{D}_h^{opp})$-module $\hat{D}_h$ is concentrated on the diagonal of $B \times B$, which easily implies that we can reduce to the affine case by working with a covering of the form $\{U \times U : U \subseteq B$ is affine\}.

Next, we take care of the push and pull functors.

**Example 44.** We have the sheaves of algebras $\hat{D}_h^{opp} \boxtimes \hat{D}_h, \mathcal{P}$ on the scheme $B \times \mathcal{P}$, and $\hat{D}_h, \mathcal{P} \boxtimes \hat{D}_h$ on $\mathcal{P} \times B$. Via the map $\pi_s : B \rightarrow \mathcal{P}_s$, we get the graph subschemes $\Gamma_{\pi_s} \subseteq B \times \mathcal{P}_s$ and $\Gamma'_{\pi_s} \subseteq \mathcal{P}_s \times B$ (the latter is the flip of the former).
We shall now construct certain modules supported on these graph subschemes: let \( i : B \to \Gamma_{\pi_s} \) be the natural isomorphism. We define \( (\tilde{D}_h)_{\pi_s} \) as the \( \tilde{D}_h^{opp} \otimes \tilde{D}_h, \pi_{\pi_s} \)-module which is simply \( i_*\tilde{D}_h \) as a sheaf, with the obvious right action of \( \tilde{D}_h \). The action by \( \tilde{D}_h, \pi_{\pi_s} \) then comes via the natural algebra morphism \( \pi_{\pi_s} : (\tilde{D}_h, \pi_{\pi_s}) \to \tilde{D}_h \).

Similarly, we can define \( \pi_s(\tilde{D}_h) \) over \( \Gamma_{\pi_s} \) as \( \tilde{D}_h^{opp} \otimes \tilde{D}_h \) as follows: we have the subscheme \( \Gamma_{\pi_s}' \subseteq \mathcal{P} \times B \), and a corresponding isomorphism \( i : B \to \Gamma_{\pi_s}' \). Then we can consider the module \( i'_*(\tilde{D}_h) \) as a left \( \tilde{D}_h \)-module, which also inherits a right action of \( \tilde{D}_h, \pi_{\pi_s} \) via the morphism \( \pi_{\pi_s} \); this will be our \( \pi_s(\tilde{D}_h) \).

Now, let \( M' \in D^b(\text{Mod}(\tilde{D}_h, \pi_{\pi_s})) \). Then we have the complex \( M' \otimes \tilde{D}_h \in D^b(\text{Mod}(\tilde{D}_h, \pi_{\pi_s})) \), and so we can form the tensor product

\[
(M' \otimes \tilde{D}_h) \otimes_{\tilde{D}_h, \pi_{\pi_s}}^{L} \tilde{D}_h^{opp} \pi_s(\tilde{D}_h)
\]

which carries an additional action of the sheaf \( O_{\pi_{\pi_s}} \otimes \tilde{D}_h \) (via the left action of \( \tilde{D}_h \) on itself in the first factor). Thus the complex

\[
(M' \otimes \tilde{D}_h) \otimes_{\tilde{D}_h, \pi_{\pi_s}}^{L} \tilde{D}_h^{opp} \pi_s(\tilde{D}_h)
\]

is naturally in \( D^b(\text{Mod}(\tilde{D}_h)) \). We claim that this complex is functorially isomorphic to our complex \( L\pi_{\pi_s}^*(M') \) defined above. The morphism of complexes is the same one as was used in the affine case, and in fact if we locally replace \( M' \) by a complex of free \( \tilde{D}_h, \pi_{\pi_s} \)-modules, we can use the same argument to prove that this map is an isomorphism (noting that \( \pi_{\pi_s}(\tilde{D}_h) \) is acyclic for \( (p_2)_s \) by the results of section 2).

In a very similar way, one shows that for \( M' \in D^b(\text{Mod}(\tilde{D}_h)) \), the functor

\[
M' \to (p_2)_s((M' \otimes \tilde{D}_h, \pi_{\pi_s}) \otimes_{\tilde{D}_h, \pi_{\pi_s}}^{L} \tilde{D}_h^{opp} \pi_s(\tilde{D}_h))
\]

is isomorphic to \( (R\pi_{\pi_s})_s \).

3.6.2. Convolution. With these preliminaries out of the way, we shall develop several important properties of our reflection functors. To reach our aims, we shall first explain how convolution works in a general setting setting.

We suppose varieties \( X, Y, \) and \( Z \), with flat noetherian sheaves of algebras \( \mathcal{A}, \mathcal{B}, \mathcal{C} \), respectively. We let \( M' \in D^b(\mathcal{A} \boxtimes \mathcal{B} - \text{mod}) \) and \( N' \in D^b(\mathcal{B}^{opp} \boxtimes \mathcal{C} - \text{mod}) \). We shall construct an object \( M' \star N' \in D^b(\mathcal{A} \boxtimes \mathcal{C} - \text{mod}) \), as follows:

We have the object \( M' \otimes \mathcal{C} \in D^b(\mathcal{A} \boxtimes \mathcal{B} \boxtimes \mathcal{C}^{opp} - \text{mod}) \) (by looking at \( \mathcal{C} \) as a right module over itself); this object admits an additional action of \( O_X \boxtimes O_Y \boxtimes \mathcal{C} \) via the left action of \( \mathcal{C} \) on itself. In the same vein, we can consider the object \( \mathcal{A}^{opp} \boxtimes N' \in D^b(\mathcal{A}^{opp} \boxtimes \mathcal{B}^{opp} \boxtimes \mathcal{C} - \text{mod}) \), with the additional action of \( \mathcal{A} \boxtimes O_Y \boxtimes O_Z \). Since \( M' \otimes \mathcal{C} \) and \( \mathcal{A}^{opp} \boxtimes N' \) are modules over opposed algebras, we can consider the tensor product

\[
(M' \otimes \mathcal{C}) \otimes_{\mathcal{A}^{opp} \boxtimes \mathcal{C}^{opp}} (\mathcal{A}^{opp} \boxtimes N')
\]

which is then naturally a module over \( \mathcal{A} \boxtimes O_Y \boxtimes \mathcal{C} \) via the additional actions discussed above. Thus the complex

\[
(Rp_{13})_s((M' \otimes \mathcal{C}) \otimes_{\mathcal{A}^{opp} \boxtimes \mathcal{C}^{opp}} (\mathcal{A}^{opp} \boxtimes N')) := M' \star N' \in D^b(\mathcal{A} \boxtimes \mathcal{C} - \text{mod})
\]

is an element of \( D^b(\mathcal{A} \boxtimes \mathcal{C} - \text{mod}) \).

On the other hand, by the general discussion of the previous sections, the object \( M' \) defines a functor \( F_M : D^b(\mathcal{A}^{opp} - \text{mod}) \to D^b(\mathcal{B} - \text{mod}) \), while \( N' \) defines a functor \( F_N : D^b(\mathcal{B} - \text{mod}) \to D^b(\mathcal{C} - \text{mod}) \), and \( M' \star N' \) defines a functor \( F_M \star N : D^b(\mathcal{A}^{opp} - \text{mod}) \to D^b(\mathcal{C} - \text{mod}) \). Then we have the
Lemma 45. There is an isomorphism of functors
\[ F_N \circ F_M \cong F_{M \ast N}. \]

This lemma is simply the statement that “convolution becomes composition” for Fourier-Mukai kernels (see section 3 above). The proof in the classical case (spelled out in great detail in [Huy]) works perfectly well in our situation.

Let us spell out exactly how this works in the case of interest to us: consider \((\tilde{D}_h)_{\pi_s} \boxtimes \tilde{D}_h\) as a \(D_h^{opp} \boxtimes \tilde{D}_h \boxtimes \tilde{D}^{opp}_h\)-module. This module admits an additional action by \(O_B \boxtimes O_B \boxtimes \tilde{D}_h\). Similarly, consider \(\tilde{D}_h \boxtimes \pi_s(\tilde{D}_h)\) as a \(\tilde{D}_h \boxtimes D^{opp}_h \boxtimes \tilde{D}_h\)-module; this module admits an additional action by \(\tilde{D}^{opp}_h \boxtimes O_B \boxtimes O_B\). Then we have:

Lemma 46. For a finite root \(\alpha\), the sheaf
\[ G_\alpha := (p_{13})_\ast (((\tilde{D}_h)_{\pi_s} \boxtimes \tilde{D}_h) \otimes L_{D_h^{opp} \boxtimes \tilde{D}_h}^{\text{opp}} \boxtimes \tilde{D}^{opp}_h) \otimes (\tilde{D}_h \boxtimes \pi_s(\tilde{D}_h))) \]
on \(B \times B\), which is naturally a \(\tilde{D}_h^{opp} \boxtimes \tilde{D}_h\)-module, is a kernel which gives the functor \(R_\alpha\).

Now we can compare formally this functor with its classical version. Firstly, let us note that, a priori, \(G_\alpha\) has three actions of \(h\) coming from the fact that is is the pushforward of a sheaf defined on a three-fold product. However, these actions of \(h\) must all agree, as can be seen from the fact that the left and right actions of \(h\) on \((\tilde{D}_h)_{\pi_s}\) and \(\pi_s(\tilde{D}_h)\) agree. Then we have the

Proposition 47. We have an isomorphism (of sheaves on \(B \times B\))
\[ (G_\alpha) \otimes L_{k[h]} k_0 \cong O_{\tilde{g} \times \tilde{g}} \]

Proof. The results of [Ri], section 5, show that we have an isomorphism
\[ (RP_{13})_\ast (O_{\tilde{g} \times \tilde{g}} \boxtimes O_{\tilde{g} \times \tilde{g}}) \cong O_{\tilde{g} \times \tilde{g}} \]
where all the products are taken over the natural map \(\pi_s\). But the sheaf on the left, considered as a quasicoherent sheaf on \(B \times B\), is isomorphic to \((G_\alpha) \otimes L_{k[h]} k_0\) by Lemma 14,15 and the fact that the functor \(\otimes L_{k[h]} k_0\) commutes with all tensor products.

From this proposition and the graded Nakayama lemma, it follows that \(G_\alpha\) is a sheaf (i.e., concentrated in a single cohomological degree).

Now we can deduce from these facts the following crucial:

Corollary 48. The adjunction morphism of kernels \(O_{\Delta \tilde{g}} \to O_{\tilde{g} \times \tilde{g}}(2)\) lifts to a morphism of \(D_h^{opp} \boxtimes D_h\)-modules \(\tilde{D}_h \to G_\alpha(2)\).

Proof. By the basic results of section 2 for the flag variety \(B \times B\), we have that the space \(R\Gamma^G(G_\alpha)\) admits an action of the algebra \(O(\mathfrak{h}^* \times \mathbb{A}^1) \otimes_k O(\mathfrak{h}^* \times \mathbb{A}^1)\). Further, there is a global section \(1 \in R\Gamma^G(G_\alpha)\) (obtained by looking at the image of \((1 \boxtimes 1) \otimes (1 \boxtimes 1) \in G_\alpha\)); and so the action on 1 produces a map
\[ O(\mathfrak{h}^* \times \mathbb{A}^1) \otimes_k O(\mathfrak{h}^* \times \mathbb{A}^1) \to R\Gamma^G(G_\alpha) \]
and applying \(\otimes L_{k[h]} k_0\) yields a map
\[ O(\mathfrak{h}^*) \otimes_k O(\mathfrak{h}^*) \to R\Gamma^G(O_{\tilde{g} \times \tilde{g}}) \]

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It is not difficult to check that this map is a surjection, and produces an isomorphism
\[ O(h^*) \otimes_{O(h^*/s)} O(h^*) \to R\Gamma^G (O_{\tilde{g} \times \emptyset, \tilde{g}}) \]
using the Leray spectral sequence for either of the two pushforwards \( p : \hat{g} \times \hat{g} \to \hat{g} \)
(and the fact that \( p_*(O_{\tilde{g} \times \emptyset, \tilde{g}}) = R\alpha_{\emptyset} O_{\tilde{g}} = O_{\tilde{g}} \oplus O_{\tilde{g}} \)).

Now, the morphism \( O_{\tilde{g} \times \emptyset} \to O_{\Delta \tilde{g}} \to O_{\tilde{g} \times \emptyset, \tilde{g}}(2) \) yields a non-trivial element \( v \) in
\[ R\Gamma^{G \times G_m}(O_{\tilde{g} \times \emptyset, \tilde{g}}(2)) \cong h^* \oplus h^* \]

Now, one can regard this element as a degree one element of \( O(h^* \times A^1) \otimes_k O(h^* \times A^1) \). Then its image in \( R\Gamma^G (G_\alpha) \) under the map \( 3.3 \) produces a nontrivial map \( \phi : D_{h'}^{opp} \boxtimes \hat{D}_h \to G_\alpha \).

Let \( \hat{D}_h \cong D_{h'}^{opp} \boxtimes \hat{D}_h / J \) as \( D_{h'}^{opp} \boxtimes \hat{D}_h \)-modules. We wish to show that the map \( \phi \) dies on the submodule \( J \). We know this is true upon setting both copies of \( h = 0 \).

However, \( J \subseteq ker(\phi) \) can then be seen from the fact that \( J \) is generated locally by elements in grade degree 2 which survive after killing \( h \); as well as the fact that the two actions of \( h \) on \( G_\alpha \) (coming from \( D_{h'}^{opp} \boxtimes \hat{D}_h \)) agree by definition of \( G_\alpha \). Thus \( \phi \) is a lift of the original map.

**Remark 49.** a) The choice of this lift depended only on the choice of an isomorphism \( O(h^* \times A^1) \cong R\Gamma^G (\hat{D}_h) \), which in turn depends only on the general data fixed at the beginning of the paper (c.f. section 2).

b) We can compute explicitly the element \( v \). By construction, it is a degree 2 element of \( O(h^*) \otimes_{O(h^*/s)} O(h^*) \) which satisfies \( (\alpha_s \otimes 1)v = (1 \otimes \alpha_s)v \). It is easy to see that, up to scalar, the only such is \( 1 \otimes \alpha_s + \alpha_s \otimes 1 \). We shall make this choice of \( v \) from now on.

**Proposition 50.** The morphism \( Id \to R_\alpha(2) \) is an adjointness of functors on \( D^{b,G_m}(\text{Mod}(\hat{D}_h)) \).

**Proof.** To see this, recall that we have morphisms of the sort
\[ (\pi_s)_* \to (\pi_s)_* R_\alpha \to (\pi_s)_* \]
(on \( D^b(\text{Mod}(\hat{D}_h,F)) \)) which we want to show are the identity. At the level of Fourier-Mukai kernels, we have a morphism
\[ (\hat{D}_h)_\pi_s \to (\hat{D}_h)_{\pi_s} \]
whose reduction mod \( h \) is the identity. Since this morphism is \( G \times G_m \)-equivariant, we conclude from \( R\Gamma (\hat{D}_h) = O(h^* \times A^1) \) that it is the identity as well; the same is true for the adjunction morphism in the other order.

From this proposition, we conclude that \( R_\alpha \) is a self-adjoint functor on \( D^{b,G}(\text{Mod}(\hat{D}_h)) \) and its graded version.

As a next step, we shall also need to deform the functor associated to the affine root. At the level of coherent sheaves, this functor can be obtained from a finite-root functor by a certain conjugation. Thus we need to lift the functors by which our finite root was conjugated. That is the goal of the next subsection.
3.7. **Deforming braid generators.** Our goal here is to deform the functors attached to braid generators. The previous two sections have told us exactly how to do this: the functor of action by a line bundle is given by

\[ M \rightarrow M \otimes_{O(B)} O(\lambda) \]

made into a \( \tilde{D}_h \)-module as explained above.

The functor associated to a finite root \( \alpha \) can then be defined using the reflection functors. By the natural adjointness property and 50, we have natural morphisms

\[ \text{Id} \rightarrow R_\alpha(2) \]

and

\[ R_\alpha \rightarrow \text{Id} \]

and therefore we can simply define functors (on \( D^b(Mod^G(\tilde{D}_h)) \)) to be triangles

\[ \text{Id} \rightarrow R_\alpha(2) \rightarrow s_\alpha \]

and

\[ s_\alpha^{-1}(-2) \rightarrow R_\alpha \rightarrow \text{Id} \]

(to get the ungraded version we simply ignore the shifts).

Our first order of business is to check that these functors are indeed inverse. For this, we shall use a very general lemma, which appears in [R]. The set-up is as follows:

Suppose that \( C \) and \( D \) are algebraic triangulated categories and let \( F : C \rightarrow D \) and \( G : D \rightarrow C \) be triangulated functors. Let \( \Phi \) be a triangulated self-equivalence of \( C \). Suppose we are given two adjoint pairs \((F,G)\) and \((G,F)\). Then we have the data of four morphisms of the adjunctions

\[ \eta : 1_D \rightarrow F\Phi G \quad \epsilon : G\Phi F \rightarrow 1_C \]

\[ \eta' : 1_C \rightarrow GF \quad \epsilon' : FG \rightarrow 1_D \]

Let \( \Psi \) be the cocone of \( \epsilon' \) and \( \Psi' \) be the cone of \( \eta \). Assume that

\[ 1_C \rightarrow GF \rightarrow \Phi^{-1} \]

is a split exact triangle. Then one concludes

**Proposition 51.** The functors \( \Psi \) and \( \Psi^{-1} \) are inverse self equivalences of \( D \).

We shall apply this proposition in the case where \( D = D^b(Mod(\tilde{D}_h)), C = D^b(Mod(\tilde{D}_h, P)) \) \( F = \pi_s^*, \ G = (\pi_s)_*, \) and \( \Phi = (2) \). The remaining issue is to show that the triangle

\[ 1_{D^b(Mod(\tilde{D}_h,P))} \rightarrow (\pi_s)_* \pi_s^* \rightarrow (2) \]

is split exact- but in fact the adjunction formula implies immediately that for any sheaf \( M \in D^b(Mod(\tilde{D}_h, P)) \),

\[ (\pi_s)_* \pi_s^* M \cong M \oplus M(2) \]

and so we can conclude that this the case. Thus our functors are indeed inverse as required.

Next, we would like to show that these functors satisfy the (weak) braid relations. We begin with the
Lemma 52. Consider any braid relation satisfied by the elements \( \{ T_{s_{\alpha}}^{\pm 1} \}_{\alpha \in W_{f,n}}, \{ \theta_{\lambda} \} \). For notational convenience, we consider \( T_{s_{\alpha}} \theta_{\lambda} = \theta_{\lambda} T_{s_{\alpha}} \) (for \( \lambda, \alpha > 0 \)). Then there is an isomorphism in \( D^b(\text{Mod}^G \times \mathbb{G}_m(\tilde{D}_h)) \)

\[
s_{\alpha} \cdot \theta_{\lambda} \cdot \tilde{D}_h \cong \theta_{\lambda} \cdot s_{\alpha} \cdot \tilde{D}_h
\]

Remark 53. In other words, the lemma says that the functors satisfy braid relations upon application to \( \tilde{D}_h \).

Proof. As all functors considered are invertible (c.f. the remarks right above the lemma), we see that this comes down to showing

\[
\tilde{D}_h \cong s_{\alpha}^{-1} \cdot \theta_{-\lambda} \cdot s_{\alpha} \cdot \theta_{\lambda} \cdot \tilde{D}_h
\]

We know that, upon restriction to \( h = 0 \), there is an isomorphism

\[
O_{\tilde{g}} \cong s_{\alpha}^{-1} \cdot \theta_{-\lambda} \cdot s_{\alpha} \cdot \theta_{\lambda} \cdot O_{\tilde{g}}
\]

Thus the complex \( s_{\alpha}^{-1} \cdot \theta_{-\lambda} \cdot s_{\alpha} \cdot \theta_{\lambda} \cdot \tilde{D}_h \) is concentrated in degree zero by the graded Nakayama lemma. Now, since \( RT^G(O_{\tilde{g}}) \cong O(\tilde{h}^*) \) as graded modules, we deduce that

\[
RT^G(s_{\alpha}^{-1} \cdot \theta_{-\lambda} \cdot s_{\alpha} \cdot \theta_{\lambda} \cdot \tilde{D}_h)|_{h=0} \cong O(\tilde{h}^*)
\]

as graded modules, and that

\[
RT^G(s_{\alpha}^{-1} \cdot \theta_{-\lambda} \cdot s_{\alpha} \cdot \theta_{\lambda} \cdot \tilde{D}_h) = 0
\]

for \( i > 0 \) (by Lemma 14 and 15). So we see that there is a nontrivial element of

\[
\text{Hom}_{(\text{Mod}^G \times \mathbb{G}_m(\tilde{D}_h))}(\tilde{D}_h, s_{\alpha}^{-1} \cdot \theta_{-\lambda} \cdot s_{\alpha} \cdot \theta_{\lambda} \cdot \tilde{D}_h)
\]

but the restriction of any such map to \( h = 0 \) is a nontrivial element of

\[
\text{Hom}_{\text{coh}^G \times \mathbb{G}_m(\tilde{g})}(O_{\tilde{g}}, s_{\alpha}^{-1} \cdot \theta_{-\lambda} \cdot s_{\alpha} \cdot \theta_{\lambda} \cdot O_{\tilde{g}}) = k
\]

and hence is an isomorphism. Thus we see that our map is surjective by the graded Nakayama lemma (applied locally).

To produce a morphism in the other direction, we simply run the same argument for the “inverse” complex \( \theta_{-\lambda} \cdot s_{\alpha}^{-1} \cdot \theta_{\lambda} \cdot s_{\alpha} \cdot \tilde{D}_h \); thus we can get a map

\[
s_{\alpha}^{-1} \cdot \theta_{-\lambda} \cdot s_{\alpha} \cdot \theta_{\lambda} \cdot \tilde{D}_h \to \tilde{D}_h
\]

such that the composition of the two is an endomorphism of \( \tilde{D}_h \) lifting the identity on \( O_{\tilde{g}} \); since any such is the identity, we conclude that our original map is injective. \( \square \)

Now we proceed to the full statement:

Corollary 54. The collection of functors \( \{ s_{\alpha}^{\pm 1} \}_{\alpha \in W_{f,n}}, \{ \theta_{\lambda} \} \) defined above satisfy the (weak) braid relations.

Proof. By definition, \( s_{\alpha} \) is the Fourier-Mukai kernel for the complex of bimodules \( \text{Id} \to G_{\alpha}(2) \), and \( s_{\alpha}^{-1} \) is the Fourier-Mukai kernel for \( G_{\alpha}(2) \to \text{Id}(2) \). Further, it is easy to verify that the functor \( M \to M \otimes_{O(\tilde{g})} O(\lambda) \) is represented by the bimodule \( \tilde{D}_h \otimes O(\Delta_B) \) \( O(\lambda) \); here \( \tilde{D}_h \) is taken to be the diagonal bimodule, and the action of \( \tilde{D}_h \otimes \tilde{D}_h^{\text{opp}} \) is inherited from the action on \( \tilde{D}_h \).

So, for \( ? = s_{\alpha}^{\pm 1} \) or \( \lambda \), we let \( M_? \) denote the associated bimodule. We consider any braid relation; again we shall take \( T_{s_{\alpha}} \theta_{\lambda} = \theta_{\lambda} T_{s_{\alpha}} \) for notational convenience. We wish to show the existence of an isomorphism inside

\[
\text{Hom}_{D^b(\text{Mod}^G \times \mathbb{G}_m(\tilde{D}_h \otimes \tilde{D}_h^{\text{opp}}))}(M_s \star M_\lambda, M_\lambda \star M_s) \cong \frac{34}{384}
\]
\[ \text{Hom}_{D^b(\text{Mod}^G \times \text{Gm}(\mathcal{D}_h \boxtimes \mathcal{D}_h^\text{aff})))}(\mathcal{D}_h, M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s) \]

However, the object \( M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s \) is known to satisfy \( (M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s) \otimes O(\Delta \mathfrak{g}) \) by [BR]; this is precisely the fact that the braid relations are known for \( D^b(\mathfrak{g} \times \mathfrak{g} = (\mathfrak{g}) \). This implies that \( M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s \) is a sheaf (i.e., concentrated in a single degree).

We shall use this to argue that \( M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s \), as a quasi-coherent sheaf on \( \mathcal{B} \times \mathcal{B} \), is scheme-theoretically supported on the diagonal \( \Delta \mathcal{B} \). The fact that is set-theoretically supported there is immediate from the above remark. To see the scheme-theoretic support, let \( \mathcal{I} \) denote the ideal sheaf of \( \Delta \mathcal{B} \) in \( \mathcal{B} \times \mathcal{B} \). Then \( (M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s) / \mathcal{I} \) is a graded sheaf whose grading is bounded below (because \( M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s \) is a finitely generated, graded \( \mathcal{D}_h \boxtimes \mathcal{D}_h^\text{aff} \) module and \( \mathcal{I} \) is an ideal of degree zero elements). Further, we have that
\[ (M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s / \mathcal{I}) \otimes O(\Delta \mathfrak{g}) = 0 \]

since \( (M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s) / \mathcal{I} \) is scheme-theoretically supported on \( \Delta \mathcal{B} \). Thus the graded Nakayama lemma for sheaves implies that \( (M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s) / \mathcal{I} = 0 \), which is what we wanted.

Given this, we have an isomorphism
\[ \text{Hom}_{D^b(\text{Mod}^G \times \text{Gm}(\mathcal{D}_h \boxtimes \mathcal{D}_h^\text{aff})))}(\mathcal{D}_h, M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s) \cong \]
\[ \text{Hom}_{D^b(\text{Mod}^G \times \text{Gm}(\mathcal{D}_h)))}(\mathcal{D}_h, R \mathcal{P}_s(M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s)) \]

since the projection \( p \) induces an equivalence of categories between quasi-coherent sheaves on \( \Delta \mathcal{B} \) and those on \( \mathcal{B} \) (we need scheme-theoretic support for this, not just set theoretic; hence the above discussion).

But now, \( R \mathcal{P}_s(M_{-\lambda} \star M_{s-1} \star M_{\lambda} \star M_s) \cong \theta_{-\lambda} \cdot s^{-1} \cdot \theta_{\lambda} \cdot s \cdot \mathcal{D}_h \) (by “convolution becomes composition” above), and so we finally conclude
\[ \text{Hom}_{D^b(\text{Mod}^G \times \text{Gm}(\mathcal{D}_h \boxtimes \mathcal{D}_h^\text{aff})))}(\mathcal{D}_h, \mathcal{M}_s \star \mathcal{M}_\lambda \star \mathcal{M}_s) \cong \]
\[ \text{Hom}_{D^b(\text{Mod}^G \times \text{Gm}(\mathcal{D}_h)))}(\mathcal{D}_h, \theta_{-\lambda} \cdot s^{-1} \cdot \theta_{\lambda} \cdot s \cdot \mathcal{D}_h) \]
and so the result follows from the lemma above.

Now let us note that it is possible to define functors associated to \( b \) and \( b^{-1} \), where these were elements of \( \mathbb{B}_\text{aff} \) such that
\[ \mathcal{R}_\alpha b = b^{-1} \mathcal{R}_\alpha \]

for a finite root \( \alpha \); this then defines an affine root functor for \( D^b(\text{Mod}(\mathcal{D}_h)) \). From the braid relations it follows that any two such choices are isomorphic; we shall see an even stronger uniqueness statement later.

3.7.1. Tilting Objects. Now it is straightforward to define the deformation of our tilting objects. Indeed, for any sequence of finite roots \( (\alpha_1, ..., \alpha_n) \), and any element \( \omega \in \Omega \) we define an object
\[ \mathcal{R}_{\alpha_1} \mathcal{R}_{\alpha_2} \cdots \mathcal{R}_{\alpha_n} \mathcal{O}_\omega \cdot \mathcal{D}_h \]
which lives in \( D^b(\text{Mod}^G \times \text{Gm}(\mathcal{D}_h)) \). From the definitions and the cohomological lemmas 14,15 it is clear that
\[ (\mathcal{R}_{\alpha_1} \mathcal{R}_{\alpha_2} \cdots \mathcal{R}_{\alpha_n} \mathcal{O}_\omega \cdot \mathcal{D}_h) \mid_{h=0} = \mathcal{R}_{\alpha_1} \mathcal{R}_{\alpha_2} \cdots \mathcal{R}_{\alpha_n} \mathcal{O}_\omega \]

Further, the \( G \)-equivariant version of these cohomological lemmas gives:
Lemma 55. Let $T_h$ denote any tilting object in $D^b(\text{Mod}^G \times G_m(\tilde{D}_h))$, and $T$ its reduction mod $h$ as above. Then:

a) $H^i(\text{RT}^G(T_h)) = 0$ for all $i \neq 0$.
b) $H^0(\text{RT}^G(T_h))|_{h=0} = H^0(\text{RT}^G(T))$

Then, by applying the self adjointness of the $\mathcal{R}_\alpha$ and the fact that $\text{Hom}_{D^b\text{Mod}^G(\tilde{D}_h)}(\tilde{D}_h, \cdot) = \text{RT}^G(\cdot)$, we deduce immediately (from the graded Nakayama lemma) the

Corollary 56. The objects $\mathcal{R}_{\alpha_1} \mathcal{R}_{\alpha_2} \cdots \mathcal{R}_{\alpha_n} \omega \cdot \tilde{D}_h$ satisfy $\text{End}_{D^b\text{Coh}^G(\tilde{D}_h)}(\mathcal{R}_{\alpha_1} \mathcal{R}_{\alpha_2} \cdots \mathcal{R}_{\alpha_n} \omega \cdot \tilde{D}_h) = \text{End}_{D^b\text{Coh}^G(\tilde{D}_h)}(\mathcal{R}_{\alpha_1} \mathcal{R}_{\alpha_2} \cdots \mathcal{R}_{\alpha_n} \omega \cdot \tilde{D}_h)$

The same is true in $D^b\text{Coh}^G(\tilde{D}_h)$ for the objects $\mathcal{R}_{\alpha_1} \mathcal{R}_{\alpha_2} \cdots \mathcal{R}_{\alpha_n} \omega \cdot \tilde{D}_h(i)$.

Therefore, to see that these are tilting objects in the sense that we need, we should show that they generate the category. As in the coherent case, one first shows that the objects $\tilde{D}_h(\lambda)$ generate our category, and then show that the tilting objects generate these objects.

To prove the generation by $\tilde{D}_h(\lambda)$'s we only have to prove the equivalent for $\text{Mod}^B(U_h(b))$. But in this case the proof is identical to the coherent version.

Next, we can copy the coherent argument to show that the full triangulated subcategory generated by the tilting objects contains all objects of the form $s_{\alpha_1} \cdots s_{\alpha_n} \cdot \tilde{D}_h$. Since the weak braid relations are satisfied for objects acting on $\tilde{D}_h$, we deduce right away that this collection contains all objects $\tilde{D}_h(\lambda)$, which is what we needed.

4. Kostant-Whittaker Reduction and Soergel Bimodules

The aim of this section is to prove our “combinatorial” description of the coherent categories via the Kostant-Whittaker reduction functor. By the results of the above sections, all we have to do is to completely encode these categories is to give a description of the Hom’s between tilting generators. We shall show that this can be done entirely in terms of the action of the (affine) Weyl group on its geometric representation- hence the use of the adjective “combinatorial.”

4.1. Kostant Reduction for $\mathfrak{g}$. In this section, we shall define our “functor into combinatorics.” This definition is a generalization of the main idea of [BF]. We shall start by making a few general remarks about the Kostant reduction- first found in the classic paper [K] (c.f. also [GG]). Let $\mathfrak{g}$ be our reductive lie algebra (over $k$) with its fixed pinning, such that $\mathfrak{n}^-$ is the “opposite” maximal nilpotent subalgebra. We choose $\chi$ a generic character for $\mathfrak{n}^-;$. in other words, we choose a linear functional on the space $\mathfrak{n}^-/\mathfrak{n}^-$ which takes a nonzero value on each simple root element $\mathfrak{e}_\alpha$. Our $\chi$ is the pullback of this functional to $\mathfrak{n}^-$, which is then a character by definition.

Next, we define a left ideal of the enveloping algebra $U(\mathfrak{g})$, called $I_\chi$, to be the left ideal generated by

$$\{n - \chi(n) | n \in \mathfrak{n}^-\}$$

and we can form the quotient $U(\mathfrak{g})/I_\chi$ - naturally a $U(\mathfrak{g})$-module. It is easy to check that this module retains the adjoint action of the lie algebra $\mathfrak{n}^-$, and hence we can further define the subspace

$$(U(\mathfrak{g})/I_\chi)^{ad(\mathfrak{n}^-)}$$
of $n^-$-invariant vectors. (This is equal, in characteristic zero, to the $N^-$-invariant vectors). In (large enough) positive characteristic, everything still works, but we should work with the group instead of the algebra.

As it turns out, this space has the structure of an algebra under the residue of the multiplication in $U(g)$ (c.f. [GG] section 2 for a more general result). Further, we see that since the center of $U(g)$, $Z(g)$, consists of all $G$-invariant vectors in $U(g)$, the natural quotient map yields a morphism

$$Z(g) \to (U(g)/I_\chi)^{ad(n^-)}$$

Kostant’s theorem assets that in fact this is an algebra isomorphism.

We can perform the same procedure with $U(g)$ replaced by its associated graded version, $S(g) = O(g^*)$. Then the quotient by the ideal $gr(I_\chi)$ corresponds to the restriction to affine subspace $n^+ + \chi \subseteq g^*$. Taking invariant vectors then corresponds to taking the quotient of this affine space by the action of the group $N^-$. This quotient exists (in the sense of GIT), and is isomorphic to an explicitly constructed affine space, as follows.

We choose a principal nilpotent element in $n^-$, called $F$, which can be taken to be the sum of all the $F_n$ associated to simple roots. Then by the well known Jacobson-Moroovz theorem, we can complete $F$ to an $\mathfrak{sl}_2$-triple- called $\{E,F,H\}$. Then we can define the subspace $\ker(ad(F)) \subset g$, and we can then transfer this space to $g^*$ via the isomorphism $g = g^*$, and we shall denote the resulting space $\ker(ad(F))^*$. Finally, we define the Kostant-Slodowy slice to be the affine space

$$S_\chi := \chi + \ker(ad(F))^*$$

This space lives naturally inside $n^+ + \chi$. What’s more, we have:

**Lemma 57.** The action map

$$a: N^- \times S_\chi \to n^- + \chi$$

is an isomorphism of varieties.

Therefore, Kostant’s theorem states that the space $S_\chi$ is naturally isomorphic to $h^*/W$, and that in fact this isomorphism is realized as the restriction of the natural adjoint quotient map $g^* \to h^*/W$. This is a deep result, and along the way he proves many interesting facts about $S_\chi$. One which we shall record for later use is:

**Proposition 58.** Every point of $S_\chi$ is contained in the regular locus of $g$.

Let us note one more nice property of the Kostant map. The action of the principal semisimple element $H$ equips the space $(U(g)/I_\chi)^{ad(n^-)}$ with a grading; which for convenience, we shift up by 2 (c.f. [GG], this is called the Kazhdan grading). In addition, $Z(g)$ is graded by considering the algebra $S(h)^W$ as a subalgebra of $S(h)$-which, of course, is graded by putting $h$ in degree 2. Then, with these conventions the Kostant map is actually an isomorphism of graded algebras.

4.2. **Kostant Reduction for $\hat{g}_P$.** Now we would like to extend the definition of the Kostant reduction to the varieties $\hat{g}_P$. In fact, we shall work with the sheaves of algebras $\hat{D}_{h,P}$. The Kostant reduction of these sheaves is easy to define: by using the natural map $n^- \to \Gamma(\hat{D}_{h,P})$, we define the sheaf of left ideals $I_\chi$ to be the left ideal sheaf generated by the image of $\{n - \chi(n)|n \in n^-\}$.

Then we form the sheaf of $\hat{D}_{h,P}$-modules $\hat{D}_{h,P}/I_\chi$, and, using the residual adjoint action of the group $N^-$, we take $\Gamma(\hat{D}_{h,P}/I_\chi)^{N^-}$. It is easy to check that this object
inherits a multiplication from the algebra structure of $\tilde{D}_{h, P}$ (c.f. [GG] section 2 for a more general result).

Further, we can consider the action of the principal semisimple element $H$ (chosen via the Jacobson-Morozov theorem above), which makes this into a graded algebra (the element $h$ is in degree 2). As above, we shift this grading by 2. Then, the natural map from the center

$$O(h^*/WP \times A^1) = \Gamma(D_{h, P})^G \to \Gamma(D_{h, P}/I_X)^{N^-}$$

(where $h^*$ is in degree 2 as well) becomes a morphism of graded algebras.

Claim 59. This map is a graded algebra isomorphism.

Proof. Since this is clearly a morphism of flat graded algebras, one reduces immediately to the coherent case where $h = 0$. By the construction of the spaces involved, we have the isomorphisms

$$O(\tilde{g}P/I_X) \cong O(\pi^{-1}(n^+ + \chi))$$

and

$$O(\tilde{g}P/I_X)^{N^-} \cong O(\pi^{-1}(S_\chi))$$

So, we really only have to show that the natural map

$$O(h^*/WP) \to O(\pi^{-1}(S_\chi))$$

is an isomorphism.

But we also have the isomorphism

$$\tilde{g}^{reg}_{P} \cong h^*/WP \times h^*/W\ h^{*, reg}$$

(c.f. Lemma 7). Since we already know that $S_\chi$ is a closed subscheme of $g^{reg}$ which is a section of the map $g^* \to h^*/W$ (by 58), the result follows. □

Remark 60. From the proof it follows that $\pi^{-1}(S_\chi)$ is an affine variety (indeed, it is a copy of affine space). Therefore the space $\pi^{-1}(n^- + \chi)$ is a copy of an affine space as well. So the use of the global sections functor in the definition is superfluous (see Lemma 64 below for a more detailed result in this direction).

4.3. The Functor $\kappa'$. We shall now proceed to define the first, naive version of our functor. This shall be a functor

$$\kappa'_P : Mod^{G \times S_m}(\tilde{D}_{h, P}) \to Mod^{\mathfrak{h}^*/A^1}(O(h^* \times A^1))$$

defined as

$$\kappa'_P(M) = \Gamma(M/I_X)^{N^-}$$

where the taking of $N^-$-invariants is via the adjoint action of the group $N^- \subset G$.

The fact that this functor lands in the category $Mod^{\mathfrak{h}^*/A^1}(O(h^*/WP \times A^1))$ follows immediately from the discussion in the previous section.

We should like to consider some general properties of this functor. First of all, let us note that the sheaf $M/I_X$ retains an $N^-$-Equivariance and an action of $H$, the principal semisimple element (we shall consider the $H$-grading shifted by 2, as above). This sheaf has the property that $(M/I_X)|_{h=0}$ is supported on the variety $\pi^{-1}(n^- + \chi)$. It has the further property that if we define a left action of $n^-$ on it by

$$n \cdot x = (n - \chi(n))x$$
then this is a nilpotent lie algebra action. This follows, essentially, from the commutation relations in the enveloping algebra $U_h(g)$.

To study $\kappa'$, we shall need to develop, briefly, some of the properties of the functor $\Gamma$, as applied to objects of the form $M/\mathcal{I}_\chi$. To that end, we make the

**Definition 61.** We let $\mathcal{C}_\chi$ be the category of modules $M \in \text{Mod}(\hat{D}_h)$ such that $\mathcal{I}_\chi \cdot M \subseteq hM$, such that the $\chi$-twisted left action of $n$ is nilpotent, and such that $M$ is graded by a semisimple action of the principal semisimple element $H$. We further demand that $h$ act by degree 2 with respect to this grading.

The morphisms in this category are those which respect all structures.

We note that for any equivariant module $M$, the object $M/\mathcal{I}_\chi$ is in $\mathcal{C}_\chi$ (the grading is the Kazhdan grading). Then, we have the

**Proposition 62.** The functor $\Gamma$ is exact on the subcategory $\mathcal{C}_\chi$.

**Proof.** This shall follow from the cohomological lemmas 14, 15. In particular, we need that for $M \in \mathcal{C}_\chi$, $R\Gamma(M)$ has grading bounded below, and that $M/hM$ has no higher cohomology.

To see that the grading is bounded below, note that $R\Gamma(M)$ satisfies $\mathcal{I}_\chi R\Gamma(M) \subseteq hR\Gamma(M)$, and that the space $S_\chi \times N^-$ is positively graded (c.f. [GG], section 2). So, we choose a PBW basis for $U_h(g)$ from a basis of $g$ as a graded module, but with elements $n - \chi(n)$ instead of $n$ for all $n$ with negative grading. Let elements of the form $n - \chi(n)$ be on the right. Then since they act nilpotently, and $R\Gamma(M)$ has cohomology consisting of finitely generated modules, we see that $R\Gamma(M)$ is indeed bounded below.

Finally, note that $M/hM$ is now supported on the affine variety $\tilde{S}_\chi \times N^-$, and hence has no higher cohomology. The lemma follows. □

From this proposition, one can go a bit further. Let $A_\chi$ be the subcategory of $U_h(g) \otimes_{O(h^* \cap W)} O(h^*)$-modules $M$ such that $\mathcal{I}_\chi \cdot M \subseteq hM$, such that the $\chi$-twisted left action of $n$ is nilpotent, and such that $M$ admits a semisimple action of the principal semisimple element $H$, with $h$ acting by degree 2 elements. Then we have the

**Corollary 63.** The functor $\Gamma$ is an equivalence of categories between $\mathcal{C}_\chi$ and $A_\chi$.

**Proof.** To see this, we only need show that $\Gamma$ is conservative; then the result will follow from the previous proposition and standard arguments (e.g. [HTT] chapter 1.4). So, let $V \in \mathcal{C}_\chi$ be nonzero. Choose $W$ a nonzero coherent subsheaf of $V$ (on $B$). Then there exists a line bundle $O(\lambda)$ such that

$$\Gamma(W \otimes O(\lambda)) \neq 0$$

and so the same is true of $V$. Next, the exact sequence

$$0 \to V \otimes O(\lambda) \to V \otimes O(\lambda) \to (V/hV) \otimes O(\lambda) \to 0$$

(where the first map is multiplication by $h$) gives a surjection $\Gamma(V \otimes O(\lambda)) \to \Gamma((V/hV) \otimes O(\lambda))$ with kernel equal to $h\Gamma(V \otimes O(\lambda))$, by the exactness of $\Gamma$. Thus the graded Nakayama lemma implies that $\Gamma((V/hV) \otimes O(\lambda)) \neq 0$. But now the sheaf $(V/hV)$ lives on a copy of affine space. Thus any tensor by a line bundle is an isomorphism of sheaves. So we deduce $\Gamma(V/hV) \neq 0$, which by the exact sequence

$$0 \to V \to V \to (V/hV) \to 0$$

implies $\Gamma(V) \neq 0$, as required. □
We wish to see what happens after taking $N^+$-invariants. To that end, we state the

**Lemma 64.** The functor $M \to M^{N^+}$ is exact on the category $A_{\chi}$. In fact, this functor gives an equivalence from $A_{\chi}$ to the category of graded $O(h^* \times A^1)$-modules.

This lemma is really just a restatement of Skryabin’s equivalence in our context. See [GG], section six, for a proof (the same one applies here).

Thus, we see that by taking flat resolutions, we can consider the derived functor

$$L_{\kappa'}: D^b Mod^{G \times G_m}(\tilde{D}_h, P) \to D^b(\text{Mod}^{gr}(O(h^*/W P \times A^1)))$$

which is obtained by taking the derived functor of the restriction $M \to M/I_{\chi}$ and then composing with the invariants functor (we shall omit the functor $\Gamma$ from now on, which we can do by the above propositions).

To see that this is a functor is the appropriate one, we should first show the

**Proposition 65.** We have $L_{\kappa'}(\tilde{D}_h, P) = O(h^*/W P \times A^1)$

**Proof.** The claim is simply that there are no higher derived terms. We note that by the construction

$$L^i_{\kappa'}(\tilde{D}_h, P)|_{h=0} = \text{Tor}^i_{O(G)}(S_{\chi}, O_{gP})$$

and the term on the right vanishes for nonzero $i$ (c.f. [BM], chapter 1, and [BR]). So the result follows from the graded Nakayama lemma.

Below, we shall denote $\kappa'_g$ simply by $\kappa'$, and for $P = P_s$, we denote $\kappa'_{P_s}$ by $\kappa'_s$.

Now we can state the two main results of this section:

**Lemma 66.** If $\alpha_s$ is a finite simple root, then we have a functorial isomorphism

$$L_{\kappa'}(R_{\alpha_s} M) = O(h^* \otimes O(h^*/\mathbb{A}^1)) L_{\kappa'_s}(M)$$

for any complex $M \in D^b Mod^{G \times G_m}(\tilde{D}_h)$.

and also

**Lemma 67.** For any integral weight $\lambda$, we have a functorial isomorphism

$$L_{\kappa'}(M \otimes O(\lambda)) = O(h^* \otimes \mathbb{A}^1) \otimes O(h^* \otimes \mathbb{A}^1) L_{\kappa'}(M)$$

where $O(h^* \otimes \mathbb{A}^1)$ is the $O(h^* \otimes \mathbb{A}^1) = \text{Sym}(h \oplus k \cdot e)$-module with the action defined as

$$(h, a) \cdot m = (h + a\lambda, a)m$$

abstractly, this is a one dimensional free $O(h^* \otimes \mathbb{A}^1)$-module, and so ultimately we get an isomorphism

$$L_{\kappa'}(M \otimes O(\lambda)) = L_{\kappa'}(M)$$

We shall prove these lemmas momentarily. Let us note right away, however, the crucial consequence that $L_{\kappa'}$ takes a tilting module to a complex concentrated in degree zero—simply because the above lemmas show that the action of the image of the reflection functors is exact on $\text{Mod}(O(h^* \otimes \mathbb{A}^1))$, and the tilting modules are built by applying the reflection functors to the basic objects $\omega \cdot \tilde{D}_h$.

In order to prove Lemma 66, we shall break the reflection functor into its two pieces. The proof of the lemma is immediately reducible to the following claim:
Claim 68. a) We have a functorial isomorphism

\[ \mathcal{L}_{\kappa}' \pi_*(M) \cong (pr_{\mathfrak{s}})_*(L_{\kappa}'(M)) \]

(where \( pr_{\mathfrak{s}} : \mathfrak{h}^* / s \) is the natural quotient map) for all \( M \in D^b Mod^{G \times G_m}(\hat{D}_h) \).

b) We have a functorial isomorphism

\[ \mathcal{L}_{\kappa}' \pi_*(N) \cong O(h^*) \otimes O(h^*/ s) L_{\kappa}')_{\alpha}(N) \]

for all \( N \in D^b Mod^{G \times G_m}(\hat{D}_h, \mathcal{P}) \).

Proof. We start with a). There is the obvious natural map of sheaves

\[ R\pi_*(M) / I_{\chi} \rightarrow R\pi_*(M / I_{\chi}) \]

which upon taking \( N^- \) invariants becomes a map

\[ L_{\kappa}'(R\pi_*(M)) \rightarrow pr_{\mathfrak{s}}(L_{\kappa}'(M)) \]

We shall show that the map 4.1 is an isomorphism. First of all, let us recall that we have the sheaf of algebras \( \hat{D}_h(s) \), which by definition is the coherent pullback \( \pi_*(\hat{D}_h, \mathcal{P}) \); we recall the natural surjection \( \hat{D}_h(s) \rightarrow \hat{D}_h \), which allows us to regard any \( \hat{D}_h \)-module as a \( \hat{D}_h(s) \)-module.

Now, we can define the ideal sheaf \( I_{\chi}(s) \) to be the ideal sheaf of \( \hat{D}_h(s) \) generated by \( \{ n - \chi(n) | n \in \mathfrak{n}^- \} \), and then we have an isomorphism

\[ M/I_{\chi} \cong M/I_{\chi}(s) \]

(as \( \hat{D}_h(s) \)-modules) following immediately from the fact that \( I_{\chi} \) is defined by global generators. So, we have to compute

\[ R\pi_*(M / I_{\chi}(s)) \]

over a given affine subset of \( \mathcal{P} \), denoted \( U \). To do that, we should first replace \( M \) by a complex of flat \( \hat{D}_h(s) \)-modules, \( F^* \), and then quotient each term of this complex by \( I_{\chi}(s) \). We compute cohomology by taking the Cech complex of this complex.

But now \( \hat{D}_h(s) \) is flat over \( \hat{D}_h, \mathcal{P} \) since it is obtained by (the quantization of) base change from the \( \mathbb{P}^1 \)-bundle \( \mathcal{B} \rightarrow \mathcal{P} \), which implies that \( (\pi_*)_{\mathcal{P}}(N^-) \) is a complex of \( \hat{D}_h(s) \)-flat modules if \( N^- \) is a complex of \( \hat{D}_h(s) \)-flat modules. So we can compute \( R\pi_*(M / I_{\chi}) \) by taking the Cech complex of \( F^- \) and then moding out by \( I_{\chi} \) (no further replacement necessary). These two procedures evidently yield isomorphic complexes.

So this shows that 4.1 is an isomorphism, and the result we want follows upon taking \( N^- \) invariants.

Part b) is simpler- in fact it follows easily from the statement that composition of pullback functors is the pullback of the composed map.

Now we proceed to Lemma 67.

Proof. (of Lemma 67). As a first step we note that since tensoring by a line bundle is exact, we have

\[ (M \otimes O(\lambda)) / I_{\chi} \cong (M / I_{\chi}) \otimes O(\lambda) \]

(where the quotient is taken in the derived sense). Then, as a module over

\[ \Gamma(\hat{D}_h)^G = O(\mathfrak{h}^*/ W \times A^1) \]
we have that \((M/\mathcal{I}_\chi) \otimes O(\lambda)\) is simply
\[
O(h^* \times \mathbb{A}^1)_{\lambda} \otimes_{O(h^* \times \mathbb{A}^1)} M/\mathcal{I}_\chi
\]
by the definition of the \(\hat{D}_h\)-action on the tensor product (we use the equivalence of the category of sheaves \(\mathcal{C}_\chi\) with the “affine” category \(\mathcal{A}_\chi\) above). Now the result follows from taking \(N^-\) invariants. □

**Remark 69.** A natural question is to ask where the morphisms of the adjunctions \(Id \rightarrow R\alpha(2)\) and \(R\alpha \rightarrow Id\) go under \(\kappa'\).

We claim that the former goes to the natural transformation
\[
M \rightarrow M \oplus (\alpha_s \otimes M)
\]
which sends \(m \rightarrow \alpha_s m + (\alpha_s \otimes m)\), while the latter goes to the multiplication map
\[
M \oplus (\alpha_s \otimes M) \rightarrow M
\]
which sends \(m_1 + (\alpha_s \otimes m_2) \rightarrow m_1 + \alpha_s m_2\).

These claims shall follow from the explicit description of the adjunction morphisms on Fourier-Mukai kernels given above. We saw that the morphism
\[
\hat{D}_h \rightarrow G_\alpha(2)
\]
was defined by sending 1 to the global section \(\alpha_s \otimes 1 + 1 \otimes \alpha_s\). Thus the morphism
\[
\hat{D}_h \rightarrow G_\alpha(2)
\]
must send a local section \(m \otimes 1 \otimes 1\) to the section \(m \otimes 1 \otimes (1 \otimes \alpha_s + \alpha_s \otimes 1)\). After restriction to the regular elements (i.e., moding out by \(\mathcal{I}_\chi\)) and taking \(N^-\) invariants this is evidently the same map as written above. The argument for the other adjunction is the same.

### 4.4. The functor \(\kappa\)

Now we shall extend our functor \(\kappa'\) to a functor into categories of bimodules. Let us first recall that there is an equivalence of categories
\[
K^b(\mathcal{T}) \rightarrow D^b Mod^{G \times G_\mathbb{C}}(\hat{D}_h)
\]
where as above \(K^b(\mathcal{T})\) denotes the homotopy category of graded tilting complexes (c.f. section 1.5). So our task shall be to extend the functors \(\kappa'\) to functors \(\kappa\) on the category \(\mathcal{T}\). To that end, we note that any \(T \in \mathcal{T}\) carries an action of \(Z(\mathfrak{g}) = O(h^*/W)\) which is inherited from the right \(U(\mathfrak{g})\)-module structure (c.f. the definition of an equivariant \(D\)-module above). This action is functorial, and hence the functor
\[
\kappa' : \mathcal{T} \rightarrow Mod^{\mathfrak{g}}(O(h^* \times \mathbb{A}^1))
\]
naturally carries a \(Z(\mathfrak{g})\)-action. Even better, since the right \(U(\mathfrak{g})\)-module structure respects the grading, we can in fact upgrade to a functor
\[
\kappa : \mathcal{T} \rightarrow Mod^{\mathfrak{g}}(O(h^* \times \mathbb{A}^1 \times h^*/W))
\]
by identifying the category on the right with the category of (graded) \(Z(\mathfrak{g})\)-module objects in the category \(Mod^{\mathfrak{g}}(O(h^* \times \mathbb{A}^1))\).

We then extend this to a functor
\[
\kappa : K^b(\mathcal{T}) \rightarrow K^b(\mathcal{Mod}^{\mathfrak{g}}(O(h^* \times \mathbb{A}^1 \times h^*/W)))
\]
in the canonical way.
We note right away that these bimodules yield exact functors on $\text{Mod}_{\text{aff}}$ and Soergel. The first observation here is that the geometric action of the extended affine action on the graph $\text{gr}_{\text{aff}}$ gives some context for the final result, extending the remarks in section 1.2.

4.5. The Categorical Affine Hecke Algebra. The goal of the next few sections is to extend our key lemmas 66 and 67 to the functor $\kappa$ itself. First, we wish to give some context for the final result, extending the remarks in section 1.2.

So, in the rest of this section, we adopt the language and notation of section 1.2, in the specific case $W = W_{\text{aff}}$. We have $V = \mathfrak{h}^* \times \mathbb{A}^1$ and $A = O(\mathfrak{h}^* \times \mathbb{A}^1)$. Let us re-state the constructions of that section explicitly in this language.

First we define, for a finite root $\alpha_s$

$$R_{\alpha_s} := O(\mathfrak{h}^* \times \mathbb{A}^1) \otimes_{O(\mathfrak{h}^* \times \mathbb{A}^1)^{\mathfrak{h}^* / W}} O(\mathfrak{h}^* \times \mathbb{A}^1)$$

and then we define, for a weight $\lambda$, the bimodule $J_\lambda$ to be the module of functions on the graph

$$\{(h_1, h_2, a) | h_2 = h_1 + a\lambda\}$$

We note right away that these bimodules yield exact functors on $\text{Mod}^{\mathfrak{h}^* / W}(O(\mathfrak{h}^* \times \mathbb{A}^1 \times \mathfrak{h}^* / W))$.

We wish to relate these functors to the categorical constructions of Rouquier and Soergel. The first observation here is that the geometric action of the extended affine Weyl group $W_{\text{aff}}$ on the space $\mathfrak{h}^* \times \mathbb{A}^1$ is given by the formulas

$$w \cdot (v, a) = (w \cdot v, a)$$

for $w \in W$, and

$$\lambda \cdot (v, a) = (v + a\lambda, a)$$

for $\lambda$ in the weight lattice. This action induces an action of $W'_{\text{aff}}$ on the category $\text{Mod}^{\mathfrak{h}^* / W}(O(\mathfrak{h}^* \times \mathbb{A}^1 \times \mathfrak{h}^* / W))$, and this action can be expressed via tensoring by bimodules. In particular, the action of $\lambda$ is given by tensoring by the $J_\lambda$, while the action of $w \in W_{\text{fin}}$ is given by the module $J_w$ defined as the module of functions on the graph of $w$, i.e., $\{(v, a), (w \cdot v, a) | v \in \mathfrak{h}^*, a \in \mathbb{A}^1\}$.

Then it follows directly from the definitions that for finite Coxeter generators $s_\alpha$ we have the exact sequence of bimodules

$$J_{s_\alpha}(-2) \rightarrow R_{\alpha} \rightarrow J_{Id}$$
where $J_{ld}$ is the diagonal bimodule (c.f. [R] section 3); in particular, the bimodule $\mathcal{R}_\alpha$ is just the module of functions on the union of the diagonal $\Delta$ and the graph of $s_\alpha$.

Further, there is a naturally defined affine reflection functor $R_{a_0}$ which is given by the bimodule
\[ O(h^* \times A^1) \otimes_{O(h^* \times A^1)^{a_0}} O(h^* \times A^1) \]
and which fits into the analogous exact sequence
\[ J_{s_{a_0}}(-2) \to R_{a_0} \to J_{ld} \]
Then we have $H_{aff} = H(W)$, the smallest category of $A \otimes A$ bimodules containing $J_{ld}$ and $\{R_{\alpha}\}_{\alpha \in S_{aff}}$ and closed under direct sums, summands, and tensor product.

Soergel’s results show that the objects of this category (the Soergel bimodules) are precisely the summands of the objects of the form $R_{\alpha_1}R_{\alpha_2} \cdots R_{\alpha_n}(A)$ for all sequences of simple affine roots.

It follows from this that the same isomorphism is true for the homotopy category of complexes $K^b(H_{aff})$; and we can even express the element $q^{-2}T_s$ as the image in $K$-theory of the complex $R_{a_0} \to J_{ld}$

Now, if we consider $b \in W_{aff}$ and $\alpha \in S_{fin}$ as in the definition of the affine reflection functor as given above (i.e., chosen so that $b^{-1}s_\alpha b = s_{a_0}$), then we have the

Claim 72. There is an isomorphism
\[ J_{b^{-1}}R_{a_0}J_b \cong R_{a_0} \]
where the term on the left indicates convolution of bimodules.

Proof. We note first that there is a multiplication isomorphism $\phi : J_{b^{-1}}(A \otimes A)J_b \to A \otimes A$ given by
\[ \phi(a_1 \otimes (a_2 \otimes a_3) \otimes a_4) = a_1a_2 \otimes a_3a_4 \]
Further, the group relations give us isomorphisms
\[ J_{b^{-1}}AJ_b \to A \]
\[ J_{b^{-1}}A_{s_{a_0}}J_b \to A_{s_{a_0}} \]
(both given by multiplication) from which we conclude the following: if we let $I_w$ is the kernel of the defining map $A \otimes A \to A_w$; then $\phi(I_e) = I_e$ and $\phi(I_{s_{a_0}}) = I_{s_{a_0}}$. Since $R_{a_0} = A \otimes A/(I_e \cap I_{s_{a_0}})$ (and the same for $R_{a_0}$), the result follows.

From here, one checks right away that the natural map $R_{a_0} \to Id$ corresponds under conjugation to the map $R_{a_0} \to Id$. Thus we deduce that the category of bimodules containing $\{R_{\alpha}\}_{\alpha \in S_{fin}}$ and $\{J_{b^{-1}}R_{a_0}J_b\}$ and closed under direct sums, summands, and tensor product is equivalent to the category $H_{aff}$.

With this in mind, we shall extend $H_{aff}$ to a categorification of the extended affine Hecke algebra $H'_{aff}$ corresponding to any given finite root system, with the finite group $\Omega = \mathbb{Y}/\mathbb{Z}\Phi$. In particular, we make the
Definition 73. We define $\mathcal{H}'_{aff}$ to be the smallest additive category of bimodules containing $\{J_{Id}\}$ and closed under the action of $\mathcal{H}$ and the modules $\{J_\omega\}_{\omega \in \Omega}$. So in particular this category consists of summands of objects of the form

$$R_{\alpha_1} \cdots R_{\alpha_n}(J_\omega)$$

where $\{\alpha_1, ..., \alpha_n\}$ is any sequence of affine roots.

We shall now argue that in fact $K(\mathcal{H}'_{aff}) \approx H'_{aff} \approx \Omega \rtimes H_{aff}$.

First, let us recall [Hai] that the action of $\Omega$ induces an action on the set of affine simple roots, which we denote $\alpha_i \rightarrow \omega(\alpha_i)$.

Therefore, for any $\omega \in \Omega$ and any affine simple root $\alpha_i$, we have the isomorphism of complexes

$$J_\omega(R_{\alpha_i} \rightarrow J_{Id})J_{\omega^{-1}} \approx R_{\omega(\alpha_i)} \rightarrow J_{Id}$$

by the same reasoning as the claim above. Therefore the map $H'_{aff} \rightarrow K(\mathcal{H}'_{aff})$ which takes $\omega T_s$ to $[J_\omega ([A \otimes A^* A] - 1)]$ satisfies the correct relations. The surjectivity of this map is evident from the definition of the category $\mathcal{H}'_{aff}$. In addition, this relation also demonstrates the above claim about how to index the objects in $\mathcal{H}'_{aff}$.

The injectivity of our map will be clear if we know that for any $\omega_1 \neq \omega_2$ and any two strings of affine roots $\{\alpha_1, ..., \alpha_n\}$ and $\{\beta_1, ..., \beta_m\}$, we have

$$\text{Hom}(R_{\alpha_1} \cdots R_{\alpha_n}(J_{\omega_1}), R_{\beta_1} \cdots R_{\beta_m}(J_{\omega_2})) = 0$$

this will be proved in section 83 below.

Finally, we should consider the category of interest to us:

Definition 74. Let $\mathcal{M}_{asp}$ be the smallest additive category of modules in $\text{Mod}^{gr}(O(h^* \times \mathbb{A}^1 \times h^*/W))$ which contains the “identity” module $O(h^* \times \mathbb{A}^1)$ and which is closed under the action of $\mathcal{H}'_{aff}$.

This is then a category of “singular Soergel bimodules” as defined in [W]. From the results of that paper (and an argument just like the one above for $\mathcal{H}_{aff}$) it follows that $K(\mathcal{M}_{asp}) \approx \mathcal{M}_{asp}$, the polynomial representation of $\mathcal{H}'_{aff}$ (c.f. section 1 above); and that one can describe the objects as summands of actions of the reflection functors on $A$. The same sorts of descriptions then hold for the homotopy category of complexes $K^b(\mathcal{M}_{asp})$.

Remark 75. The computation of $K(\mathcal{M}_{aff})$ which follows the arguments of [W] is purely algebraic- indeed, that paper follows the lines of argument of Sorgel’s paper [S3]. In fact, this computation is not needed to prove the main results of this paper in section 4.6 below; the equivalences there follow directly from the definition of the category $\mathcal{M}_{asp}$. Since the computation of this $K$-group on the geometric side is a well known result (c.f. e.g., [AB]), we can obtain from this result on the algebraic side.

4.6. The Key Properties of $\kappa$. We have the following corollary of the previous section:

Corollary 76. Let $\mathcal{T}$ be the full subcategory of $D^b \text{Mod}^{G \times \mathbb{G}_m}(\check{D}_h)$ on all objects obtained from $\check{D}_h$ via repeated application of the reflection functors or tensoring by a line bundle. Then for any $M \in \mathcal{T}$, we have functorial isomorphisms:

$$\kappa(R_\alpha M) \approx R_\alpha(M)$$
and
\[ \kappa(M \otimes O(\lambda)) \cong J_{\lambda}(M) \]

**Proof.** The second isomorphism follows from the observation that the right \( U(\mathfrak{g}) \)-module structure is unaffected by tensoring by a line bundle. This is because the left \( \mathfrak{g} \)-structure is defined by the tensor product rule, as is the adjoint action; so the action of their difference simply comes from the right \( \mathfrak{g} \)-action on \( M \). Combining this with the calculation of the left action in the previous section, we see immediately the isomorphism.

The first isomorphism is proved by showing that the action of \( R\pi_s \) and \( \pi_s^* \) correspond to the pushforward and pullback of bimodules. But this is an easy generalization of the proofs of 66 and 67. \( \square \)

**Remark 77.** It also follows right away that the adjunctions \( \text{Id} \to R\alpha \) and \( R\alpha \to \text{Id} \) are sent under \( \kappa \) to the morphisms described in 69, now considered as morphisms of bimodules. These maps make the \( R\alpha \) into self adjoint functors on \( \text{Mod}(A) \).

Therefore, we arrive at the

**Corollary 78.** For any tilting object \( \mathcal{R}_{\alpha_1} \cdots \mathcal{R}_{\alpha_n}(\tilde{D}_h) \), we have that
\[ \kappa(\mathcal{R}_{\alpha_1} \cdots \mathcal{R}_{\alpha_n}(\tilde{D}_h)) \cong R_{\alpha_1} \cdots R_{\alpha_n}(O(\mathfrak{g}^* \times A^1)) \]
as \( O(\mathfrak{h}^* \times A^1 \times \mathfrak{h}^*/W) \)-modules.

This result will now allow us to give a complete description of the category \( K^b(T) \), in particular the following

**Theorem 79.** The functor \( \kappa \) is fully faithful on tilting modules, and thus induces an equivalence of categories
\[ D^b(Mod^{G \times \mathbb{G}_m}(\tilde{D}_h)) \to K^b(T) \to K^b(M_{asp}) \]

Let us note right away the following

**Corollary 80.** Let
\[ \mathcal{B}_h = \bigoplus_i \text{End}_{D^b(Mod(\tilde{D}_h))}(T_i) \]
where the sum runs over all tilting modules. Then we have equivalences of categories
\[ D^b,G \times \mathbb{G}_m(Mod(\tilde{D}_h)) \to \text{Perf}_{gr}(\mathcal{B}_h) \]
\[ D^b,G \times \mathbb{G}_m(Coh(\tilde{N})) \to \text{Perf}_{gr}(\mathcal{B}_h \otimes_{k[h]} k_0) \]
and
\[ D^b,G \times \mathbb{G}_m(Coh(\tilde{G})) \to \text{Perf}_{gr}(\mathcal{B}_h \otimes_{O(h^* \times A^1)} k_0) \]
Where the categories on the right stand for (direct limits of) graded perfect complexes (c.f. section 1.5). The same statement also holds true if we remove the \( \mathbb{G}_m \) from the left and the \( \text{gr} \) from the right of all these equivalences.

This is an immediate consequence of the theorem and the descriptions of all three categories via tilting modules.

In order to approach the proof of the theorem, we start with an immediate reduction:
Lemma 81. The theorem follows from the statement that

\[ \text{Hom}_{D^b(\text{Mod}^G(\hat{D}_h))}(\hat{D}_h, T) \cong \text{Hom}_{O(h^* \times A^1 \times h^*/W)}(O(h^* \times A^1), \kappa T) \]

(as graded modules) for any tilting module \( T \).

Proof. The proof follows from the self-adjointness property of the reflection functors, and the obvious adjointness for the action of the line bundles. On the left hand side, this is discussed above, while on the right hand side the fact that the \( R_\alpha \) are self adjoint is explained [R]. Further, we wish to see that the diagram

\[ \begin{array}{ccc}
\text{Hom}_{D^b(\text{Mod}^G(\hat{D}_h))}(\mathcal{R}_\alpha T_1, T_2) & \xrightarrow{\alpha} & \text{Hom}_{D^b(\text{Mod}^G(\hat{D}_h))}(T_1, \mathcal{R}_\alpha T_2) \\
\kappa & & \kappa \\
\text{Hom}_{O(h^* \times A^1 \times h^*/W)}(\mathcal{R}_\alpha \kappa T_1, \kappa T_2) & \xrightarrow{=} & \text{Hom}_{O(h^* \times A^1 \times h^*/W)}(\kappa T_1, \mathcal{R}_\alpha \kappa T_2)
\end{array} \]

coming from functoriality commutes. This follows from the various remarks 69 and 77 above.

So the proof comes down to computing equivariant global sections. To approach this, let us note that the exact sequences

\[ s_\alpha(-2) \to \mathcal{R}_\alpha \to \text{Id} \]

imply that the object \( T \) admits a filtration in \( D^b(\text{Mod}^G(\hat{D}_h)) \) by objects of the form \( b \cdot \hat{D}_h(i) \) for \( b \in \mathbb{B}_{af}^+ \) (i.e., \( b \) is a product of positive elements). Further, let us note that there are isomorphisms of graded modules

\[ \text{RHom}_{D^b(\text{Mod}^G(\hat{D}_h))}(\hat{D}_h, b \cdot \hat{D}_h) \cong \text{RHom}_{D^b(\text{Coh}(G))}(O_{\bar{g}}, b \cdot O_{\bar{g}}) \]

as follows from the cohomological lemmas 14 and 15. Now, the term on the right is zero for any \( b \) such that \( b \cdot 0 \neq 0 \) in the representation \( M_{af} \) (this follows from the description of standard and costandard objects in section 3). Let us note that this is equivalent to \( b \notin W \). Thus the term on the left is zero for \( b \notin W \) also. Further, when \( b \in W_{fin} \), we have \( b \cdot \hat{D}_h = \hat{D}_h \) (c.f. proof of Lemma 52, and 20), and so

\[ \text{Hom}_{D^b(\text{Mod}^G(\hat{D}_h))}(\hat{D}_h, b \cdot \hat{D}_h) \cong O(h^* \times A^1) \]

as graded modules.

Now, on the bimodule side, one makes exactly the same type of argument: the exact sequences for reflection functors imply that

\[ \text{Hom}_{O(h^* \times A^1 \times h^*/W)}(O(h^* \times A^1), \mathcal{R}_\alpha, \cdots \mathcal{R}_\alpha O(h^* \times A^1)) \]

is filtered by terms of the form

\[ \text{Hom}_{O(h^* \times A^1 \times h^*/W)}(O(h^* \times A^1), J_b \cdot O(h^* \times A^1)) \]

where \( \bar{b} \in W_{af} \) is a positive element. Now, when \( \bar{b} \notin W \), the module \( J_b O(h^* \times A^1) \) is the module of functions on an affine subspace of \( h^* \times A^1 \times h^*/W \) which intersects \( O(h^* \times A^1) \) in a proper subspace. Thus the \( \text{Hom}'s \) between them are zero. When \( \bar{b} \in W \), then of course we have \( J_b O(h^* \times A^1) = O(h^* \times A^1) \) (since we are in \( h^* \times A^1 \times h^*/W \)) and so

\[ \text{Hom}_{O(h^* \times A^1 \times h^*/W)}(O(h^* \times A^1), J_b O(h^* \times A^1) \) \cong O(h^* \times A^1) \]

as graded modules. Now the proof of the lemma follows by walking up a standard filtration, and the following easy
Claim 82. For any object $b^+\omega \cdot \tilde{D}_h$, we have $\kappa(b^+\omega \cdot \tilde{D}_h) = J_{b^+\omega}(O(\h^* \times \mathbb{A}^1))$. The analogous result holds for objects $b^-\omega \cdot \tilde{D}_h$.

Proof. We already know the compatibility with the reflection functors, so the proof follows by applying the usual exact sequences defining the action of the braid group and the fact that the adjunction maps $\mathcal{R}_\alpha \rightarrow Id$ and $Id \rightarrow \mathcal{R}_\alpha(2)$ go to the corresponding maps for $R_{\alpha}$ on $D^{b-gr}(A - mod)$. □

Remark 83. Let us note that the argument given above also proves the unproved claim (of the previous section) that in $\mathcal{H}_{aff}$ we have

$$Hom(R_{\alpha_1} \cdots R_{\alpha_n}(J_{\omega_1}), R_{\beta_1} \cdots R_{\beta_m}(J_{\omega_2})) = 0$$

for $\omega_1 \neq \omega_2$ and any strings of affine roots. By adjointness and the definition of the action of $\Omega$ on $W_{aff}$ this comes down to showing that

$$Hom(O(\h^* \times \mathbb{A}^1), R_{\alpha_1} \cdots R_{\alpha_k}(J_{\omega})) = 0$$

for any $\omega \neq 0$ and any sequence $\{\alpha_1, \ldots, \alpha_k\}$ of affine roots. But now the object $R_{\alpha_1} \cdots R_{\alpha_k}(J_{\omega})$ will have a filtration by objects of the form $J_s(i)$ where no $s$ is $Id$. The claim follows. A similar argument works for the category $\mathcal{M}_{asp}$.

5. Applications

In this section, we shall give our two main applications of the above description. The first is the connection with perverse sheaves on affine flag manifolds, and the second is the strictification of the braid group action.

5.1. Connection with perverse sheaves. Let us fulfill our promise from the introduction of the paper. Given the results of the paper [BY] this is an easy consequence of the results of the previous section. Let us recall some generalities from [AB, BY] (c.f. also the references therein).

We consider the dual reductive group $\tilde{G}$ over a field $F = \mathbb{F}_p$. As is well known, we can associate to $\tilde{G}$ an ind-scheme (called the affine flag variety) as follows: we let $F([t])$ be the field of Laurent series in $F$, and $F[[t]]$ its ring of integers. Then $\tilde{G}(F[[t]])$ is a maximal compact subgroup of the topological group $\tilde{G}(F((t)))$, and there is a subgroup $I \subseteq \tilde{G}(F[[t]])$ called the standard Iwahori (it is the image of our standard Borel in $\tilde{G}$ under the evaluation map taking $\tilde{G}(F[[t]]) \rightarrow \tilde{G}$). Then we define $\mathcal{F}I := \tilde{G}(F((t)))/I$ with its natural ind-scheme structure.

Next, we recall that the action of $I$ on the left of $\mathcal{F}I$ induces a decomposition of $\mathcal{F}I$ into orbits, each of which is isomorphic to a copy of the affine space $\mathbb{A}^n$. Further, the orbits are indexed by the standard basis of the algebra $H_S$ which we associated to $G$ above (this is a combinatorial manifestation of Langlands duality). In fact, we can even say that the basis element $T_u$ gives an orbit of length $\mathbb{A}^{l(w)}$. Thus we have a stratification of the variety $\mathcal{F}I$ which is given by closures of $I$ orbits, and $I$ acts on each orbit through a finite quotient; as in section 1 we let $j_w$ denote the inclusion of an orbit into its closure.

Thus we are in a perfect setting to consider categories of equivariant constructible sheaves. We let $I^-$ be the opposite Iwahori subgroup, and $I^-$ its associated group scheme. We also consider their “unipotent radicals” $I_u^-$ and $I_u^-$. Let $\psi : I^- \rightarrow \mathbb{G}_a$ be a generic character (this is the affine lie algebra analogue of the situation of section 4.1, in which Kostant-Whittaker reduction was defined). Then we shall let $D_{I^W}$ denote the triangulated category of bounded complexes of $(I^-, \psi)$-equivariant
\(\mathbb{Q}_l\)-constructible sheaves on \(\mathcal{F}l\). This category is a main player of [AB], along with its mixed version obtained by taking into account the action of the Frobenius by weights, and denoted \(D_{IW,m}\). We note that these categories are linear over \(k = \mathbb{Q}_l\).

As explained in [BY], this category admits several natural deformations. We first explain how to deform the spaces involved.

We can define the extended affine flag manifold to be \(\tilde{\mathcal{F}}l = \tilde{G}(F((t)))/I_u\) which is a natural \(\tilde{T}\)-torsor over \(\mathcal{F}l\) (here \(\tilde{T}\) is the maximal torus of \(\tilde{G}\)), where the morphism is just the quotient morphism.

In addition, we recall that the group \(\tilde{G}(F((t)))\) admits a one dimensional central extension \(\tilde{G}\) (this is an example of a Kac-Moody group), and therefore we can define the quotient \(\tilde{\mathcal{F}}l = \tilde{G}/I_u\). This is naturally a \(\tilde{T} \times \mathbb{G}_m\)-torsor over \(\mathcal{F}l\).

By the general yoga of sheaves on torsors, we see that the category \(D_{IW}\) is equivalent to a certain category of \(\tilde{T}\)-equivariant sheaves on \(\tilde{\mathcal{F}}l\), and a certain category of \(\tilde{T} \times \mathbb{G}_m\)-equivariant sheaves on \(\mathcal{F}l\). Because these are tori, Equivariance for a constructible sheaf is equivalent to demanding that the associated monodromy action be trivial. We can therefore loosen this condition by demanding that a torus act unipotently with monodromy. In this way we obtain categories \(D^b((I_u^-\psi)\backslash\tilde{\mathcal{F}}l,\tilde{T})\) and \(D^b((I_u^-\psi)\backslash\mathcal{F}l,\mathcal{F}l,\tilde{T} \times \mathbb{G}_m)\), and their mixed versions. We note that taking the logarithm of monodromy then gives us actions by the polynomial rings \(\text{Sym}(t)\) and \(\text{Sym}(t \times \mathbb{A}_1)\), such that the augmentation ideal acts nilpotently.

As explained in the appendices to [BY], these categories alone do not have enough objects to be suitable for the Koszul duality formalism. This is remedied there by defining certain completions with respect to the action of the rings \(\text{Sym}(t)\) and \(\text{Sym}(t \times \mathbb{A}_1)\), roughly analogous to replacing modules such that the augmentation ideal acts nilpotently with modules over the completed algebra.

Therefore we obtain categories denoted \(\hat{D}^b((I_u^-\psi)\backslash\tilde{\mathcal{F}}l,\tilde{T} \times \mathbb{G}_m)\) (and similarly for \(\mathcal{F}l\)), and mixed versions \(\hat{D}^b_m((I_u^-\psi)\backslash\tilde{\mathcal{F}}l,\tilde{T} \times \mathbb{G}_m)\). Let us say a few words about their structure. First, the category \(D_{IW}\) inherits the \(t\)-structure of the middle perversity from \(D^b(\mathcal{F}l,\mathbb{Q}_l)\), the bounded derived category of constructible sheaves on \(\mathcal{F}l\). The heart of this \(t\)-structure is a highest weight category in the sense of section three. In particular, the standard and costandard objects are given by the collections \(\{\mathcal{J}w!(\mathbb{Q}_l,Z_{w,\text{dim}Z_{w}})\}_{W_{aff}/W}\) and \(\mathcal{J}w_*(\mathbb{Q}_l,Z_{w,\text{dim}Z_{w}})\}_{W_{aff}/W}\), henceforth simply denoted \(\{\mathcal{J}w!\}\) and \(\{\mathcal{J}w_*\}\).

We can rephrase this in terms of the convolution action (as defined in section 1.4); in particular, if we consider \(\mathcal{J}w!\) and \(\mathcal{J}w_*\) as objects in \(D^b(\mathcal{F}l,\mathbb{Q}_l)\), then we see that our standard and costandard collection is given by \(\{\mathcal{J}w!*\mathcal{J}e\}\) and \(\{\mathcal{J}w_*\mathcal{J}e\}\). This is exactly the same type of formula used to define the perversely exotic \(t\)-structure on coherent sheaves above; it also works for finite dimensional flag varieties and category \(\mathcal{O}\). By the general theory of highest weight categories, this category has a tilting collection which generates it.

In [BY] it is shown that the perverse \(t\)-structure and the standard and costandard objects admit deformations to both of the lifted categories. The deformations of \(\mathcal{J}w_*\) are denoted \(\mathcal{J}w_{u!}\), and \(\mathcal{J}w!\) deforms to \(\Delta_{w!}\). In addition, it is proved there that the deformed categories are generated by tilting collections, which deform the tilting objects in \(D_{IW}\).

Then the main result is the following...
**Theorem 84.** We have equivalences of triangulated categories
\[
D^b_{\hat{\mathcal{G}}\times \mathbb{C}}(\text{Mod}(\hat{\mathcal{D}}_h)) \to \hat{\mathcal{D}}^b_m((I^+_{\hat{\mathcal{G}}}, \psi)\backslash \hat{\mathcal{F}}l, \hat{T} \times \mathbb{G}_m)
\]
\[
D^b_{\hat{\mathcal{G}}\times \mathbb{C}}(\text{Coh}(\hat{\mathcal{G}})) \cong \hat{\mathcal{D}}^b_m((I^+_{\hat{\mathcal{G}}}, \psi)\backslash \hat{\mathcal{F}}l, \hat{T})
\]
\[
D^b_{\hat{\mathcal{G}}\times \mathbb{C}}(\text{Coh}(\hat{\mathcal{N}})) \to D_{IW,m}
\]
Where the varieties on the left are taken over the field \(k = \mathbb{Q}_l\). These equivalences take tilting modules to tilting modules and they respect the grading in the sense that the \(\mathbb{G}_m\)-shift on the left corresponds to shifting the mixed structure on the right.

**Proof.** The proof of this theorem is almost immediate from the results of [BY] and the previous section.

The first statement follows from the equivalence \(D^b_{\hat{\mathcal{G}}\times \mathbb{C}}(\text{Mod}(\hat{\mathcal{D}}_h)) \to \text{Perf}^g(W_h)\). This equivalence takes tilting modules to summands of \(\mathcal{B}_h\) and the \(\mathbb{G}_m\)-shift to the shift of grading on the right. On the other hand, there is a description, proved in [BY], chapter 4, of the category on the right of the first statement as \(\text{Perf}^g(W_h)\) where \(W_h\) is an explicitly defined ind-algebra. This equivalence takes tilting modules to finitely generated summands of \(W_h\), and it takes the shift of mixed structure to the shift of grading.

Thus it remains to identify \(W_h\) and \(\mathcal{B}_h\). This is done via the functor \(\mathcal{V}\) of [BY]; this functor takes sheaves to modules over the ring \(\text{Sym}(\mathbb{A}^1 \times \mathcal{M})\), and is fully faithful on tilting modules. Further, the result of [BY], appendix C, says that the image of the tilting modules in \(\text{Sym}(\mathbb{A}^1 \times \mathcal{M})\) is exactly the category \(\mathcal{M}_{\text{asp}}\) described above. The first statement follows.

We can deduce the other two statements from the first if we know that
\[
\hat{\mathcal{D}}^b_m((I^+_{\hat{\mathcal{G}}}, \psi)\backslash \hat{\mathcal{F}}l, \hat{T}) \to \text{Perf}^g(W_h \otimes \mathbb{Q}_l)_{k_0}
\]
and that
\[
D_{IW,m} \to \text{Perf}^g(W_h \otimes \mathbb{Q}_l)_{k_0}
\]
This is indeed the case, and follows from the description of the deformation categories in [BY], appendices A and B. \(\square\)

As a corollary, we can give a coherent sheaf interpretation of the non-completed versions of the categories in [BY] as well. These will be the full subcategories of the completed categories on objects such that \(\text{Sym}(\mathbb{A}^1)\) and \(\text{Sym}(\mathbb{A}^1 \times \mathbb{A}^1)\) act nilpotently. On the coherent side, one sees from the definitions that the corresponding full subcategories are those on objects which are set theoretically supported on \(\hat{\mathcal{N}}\). Thus we see

**Corollary 85.** We have equivalences of categories
\[
D^b_{\hat{\mathcal{N}}} \times \mathbb{C}(\text{Mod}(\hat{\mathcal{D}}_h)) \to D^b_{\hat{\mathcal{N}}}(\text{Mod}(I^+_{\hat{\mathcal{N}}}, \psi)\backslash \hat{\mathcal{F}}l, \hat{\mathcal{T}} \times \mathbb{G}_m)
\]
\[
D^b_{\hat{\mathcal{N}}} \times \mathbb{C}(\text{Coh}(\hat{\mathcal{G}})) \to D^b_{\hat{\mathcal{N}}}(\text{Coh}(\hat{\mathcal{N}})\backslash \hat{\mathcal{F}}l, \hat{\mathcal{T}})
\]
where the subscript \(\hat{\mathcal{N}}\) denotes the full subcategories on objects set-theoretically supported on \(\hat{\mathcal{N}}\).

In addition, by looking at the ungraded versions of the underlying DG-algebras, we conclude that are equivalences
\[
D^b_{\hat{\mathcal{N}}}(\text{Mod}(\hat{\mathcal{D}}_h)) \to D^b((I^+_{\hat{\mathcal{N}}}, \psi)\backslash \hat{\mathcal{F}}l, \hat{T} \times \mathbb{G}_m)
\]
\[ D^{b,G}(\text{Coh}(\mathfrak{g})) \rightarrow D^{b}((I^-_\mu, \psi)\backslash \mathcal{F}l, \cdot, \overline{T}) \]

5.2. Strict Braid Group Action. We shall now discuss how our results fit in with the main result of the paper [R], mentioned several times above. Let us recall some results from that paper. As discussed in section 4, the action of \( W'_{\text{aff}} \) on \( A \) induces a strict action of \( W'_{\text{aff}} \) on the category \( D^{b}(A - \text{mod}) \). This can be written in terms of bimodules as follows: for any collection of generators \( \{s_1, \ldots, s_n, \omega\} \), the multiplication map induces an isomorphism

\[ A_{s_1} \otimes \cdots \otimes A_{s_n} \otimes A_{\omega} \rightarrow A_{\omega} \]

where \( w = s_1 \cdots s_n \cdot \omega \) in \( W'_{\text{aff}} \). This map has a unique inverse. Thus any relation \( s_1 \cdots s_n \cdot \omega = s'_1 \cdots s'_m \cdot \omega' \) yields an isomorphism

\[ A_{s_1} \otimes \cdots \otimes A_{s_n} \otimes A_{\omega} \rightarrow A_{s'_1} \otimes \cdots \otimes A_{s'_m} \otimes A_{\omega'} \]

and this collection of isomorphisms is compatible with the multiplication of bimodules. Thus this collection yields a strict action of \( W'_{\text{aff}} \) on \( D^{b}(A - \text{mod}) \).

Rouquier’s observation is that, if we consider the complexes \( R_{\alpha} \rightarrow J_{Id} \) (denoted \( F_{\alpha, n}^{-1} \)), which satisfy weak braid relations (proved in [R], section 3), then for any braid relation \( s_1 \cdots s_n = s'_1 \cdots s'_m \) we have an isomorphism

\[ \text{Hom}_{K^b(A \otimes A - \text{mod}^\text{op})}(F_{s_1}^{-1} \cdots F_{s_n}^{-1}, F_{s'_1}^{-1} \cdots F_{s'_m}^{-1}) \rightarrow \text{End}_{K^b(A \otimes A - \text{mod}^\text{op})}(A) = k \]

\[ \rightarrow \text{Hom}_{D^b(A \otimes A - \text{mod}^\text{op})}(F_{s_1}^{-1} \cdots F_{s_n}^{-1}, F_{s'_1}^{-1} \cdots F_{s'_m}^{-1}) \]

where the second line is by the quasi-isomorphism \( F_{\alpha, n}^{-1} \rightarrow (R_{\alpha} \rightarrow J_{Id})(-2) \). Therefore we can lift the collection of isomorphisms given for \( W'_{\text{aff}} \), and obtain strict braid relations.

Our presentation of the group \( \mathbb{B}'_{\text{aff}} \) was slightly different from Rouquier’s, but let us note that the isomorphism

\[ J_{b^{-1}}(R_{\alpha} \rightarrow Id) J_{b} \cong R_{\alpha 0} \rightarrow Id \]

comes from the multiplication map itself; therefore the same proof shows that this presentation gives strict braid relations as well. We further deduce the independence of the affine reflection functor from the choice of \( b \). Given this, we can deduce right away the

**Theorem 86.** There are strict actions of the group \( \mathbb{B}'_{\text{aff}} \) on the categories \( D^{b,G \times G_m}(\text{Mod}(\hat{D}_h)), D^{b,G \times G_m}(\text{Coh}(\mathfrak{g})), D^{b,G \times G_m}(\text{Coh}(\hat{N})) \), as well as their ungraded versions. Further, there is a strict action on the categories \( D^{b}(\text{Mod}(\hat{D}_h)), D^{b}(\text{Coh}(\mathfrak{g})) \) and \( D^{b}(\text{Coh}(\hat{N})) \). The actions by braid generators are given by the functors of section 3.

**Proof.** Given the main theorem, the only thing that remains to be done is to see that the action extends to the non-equivariant categories; it clearly suffices to consider \( D^{b}(\text{Mod}(\hat{D}_h)) \); the other two follow by restriction.

As above we denote, for \( ? = s \in S \) or \( \lambda \in \mathcal{X} \), the bimodule \( M_{?} \in D^{b}(\hat{D}_h \otimes \hat{D}_h^{\text{op}}) \) which represents the functor associated to \( ? \). For any element of the braid group, \( b \), we wish to give a preferred isomorphism between any two bimodules given by convolutions of different decompositions of \( b \). As in section 3, we choose, for
notational convenience, the element \( b = s \cdot \lambda \) for \( s, \lambda \geq 0 \). So, we wish to give a preferred isomorphism between \( M_s \ast M_\lambda \) and \( M_\lambda \ast M_s \).

We have

\[
\text{Hom}_{D^b(M_{\text{mod}} \times G \times_m (D_h \boxtimes D_h^{\text{opp}}))}(M_s \ast M_\lambda, M_\lambda \ast M_s) = \ast
\]

\[
\text{Hom}_{D^b(M_{\text{mod}} \times G \times_m (D_h \boxtimes D_h^{\text{opp}}))}((\tilde{D}_h, R_\bullet (M_\lambda \ast M_s \ast M_\lambda)) = \ast
\]

\[
\text{Hom}_{D^b(M_{\text{mod}} \times G \times_m (\tilde{D}_h))}(\tilde{D}_h, \theta_{s^{-1}} \cdot \theta_\lambda \cdot s \cdot \tilde{D}_h) = \ast
\]

\[
\text{Hom}_{D^b(A \boxtimes \mathcal{A} \mod^{opp})}(A, F_{s^{-1}} \cdot F_s \cdot F_s \cdot A)
\]

The strict braid group action discussed above then produces a preferred isomorphism in the last \( \text{Hom} \) space, which we transfer to the first. It follows easily that the collection of isomorphisms obtained this way produces a strict braid group action on \( D^b(\text{Mod}(\tilde{D}_h)) \).

By the results of section 5.1, we obtain a compatible action on all the categories of perverse sheaves considered. In fact, this action is the same as the usual one, constructed, e.g., in [BB] (c.f. also [R], section 6). Let us recall this action: for each affine simple root \( s_\alpha \), we let \( j_{s_\alpha}^* \) and \( j_{s_\alpha}^! \) denote the standard and costandard objects, respectively, of the \( I \)-orbit associated to \( s_\alpha \). As noted above, these objects have deformations to the category \( D^\text{b}_{\infty}(I_u, \mathcal{F}_l, \mathcal{T} \times G_m) \), which we denote \( \tilde{\Delta}_{s_\alpha} \) and \( \tilde{\nabla}_{s_\alpha} \). Then the action of the braid generators is given by convolution with respect to \( \tilde{\nabla}_{s_\alpha} \), with the inverse being given by convolution with respect to \( \Delta_{s_\alpha} \); this is of course consistent with the description of the affine Hecke algebra in terms of perverse sheaves.

Let us denote by \( \tilde{\delta} \) the deformation of the constant sheaf on the trivial orbit. Then, according to [BY], appendix C, there are exact sequences

\[
0 \to \tilde{\Delta}_{s_\alpha} \to \tilde{T}_{s_\alpha} \to \tilde{\delta}(1/2) \to 0
\]

\[
0 \to \tilde{\delta}(1/2) \to \tilde{T}_{s_\alpha} \to \tilde{\nabla}_{s_\alpha} \to 0
\]

where \( \tilde{T}_{s_\alpha} \) is the (free-monodromic) tilting sheaf associated to \( s_\alpha \), and the \( (1/2) \) denotes Tate twist. The explicit calculation done there confirms that the action of functor \( \mathbb{V} \) transforms these exact sequences into the ones which define the braid generators on \( K^b(A - \text{mod}) \). Further, we recall that \( \mathbb{V} \) respects convolution. Thus we conclude that the actions coincide.

An immediate corollary of this is that the equivalence constructed here corresponds, at least on objects, with the one constructed in [AB], and therefore that the perversely exotic \( t \)-structure corresponds to the heart of the perverse \( t \)-structure of \( D_{IW} \) (as was also proved in [BM]).
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Department of Mathematics, Massachusetts Institute of Technology
cdodd@math.mit.edu