Cube-Split: A Structured Grassmannian Constellation for Non-Coherent SIMO Communications

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Abstract

In this paper, we propose a practical structured constellation for non-coherent communication with single transmit antenna over flat and block fading channel without instantaneous channel state information. The constellation symbols belong to the Grassmannian of lines and are defined up to a complex scaling. The constellation is generated by partitioning the Grassmannian of lines into a collection of bent grids and defining a mapping onto each of these bent grids such that the resulting symbols are approximately uniformly distributed on the Grassmannian. With a reasonable choice of parameters, this so-called cube-split constellation has higher packing efficiency, represented by the minimum distance, than other structured constellations in the literature. Furthermore, exploiting the constellation structure, we design low-complexity greedy symbol decoder and low-complexity log-likelihood ratio computation. Numerical results show that the performance of the cube-split constellation is close to that of a numerically optimized constellation, and better than other structured constellations. It also outperforms a coherent pilot-based scheme in terms of error probability and achievable data rate.

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I. INTRODUCTION

In communication over fading channels, the knowledge of instantaneous channel state information (CSI) enables to adapt the transmission and reception to current channel conditions. The capacity of communication with a priori CSI at the receiver, a.k.a. coherent communication, is well known to increase linearly with the minimum number of transmit and receive antennas [2], [3]. In practice, however, the channel coefficients are not granted for free prior to communication. They need to be estimated, typically by periodic transmission of reference (or pilot) symbols known to the receiver [4]. If the channel state is not stable (in, e.g., time or frequency domain), accurate channel estimation requires regular pilot transmissions which can occupy a disproportionate fraction of communication resources. In this case, the cost of channel estimation is significant, and it might be beneficial to use a communication scheme that does not rely on the knowledge of instantaneous CSI. Communication without a priori CSI is said to be non-coherent [5].

We consider non-coherent single-input multiple-output (SIMO) communication in which a single-antenna transmitter transmits to an $N$-antenna receiver. We assume flat and block fading channel, i.e., the $N$-dimensional channel coefficient vector remains constant within each coherence interval of $T \geq 2$ symbols and changes independently between blocks. The non-coherent capacity of this channel with Rayleigh fading was calculated for the high signal-to-noise ratio (SNR) regime in [5], [6], [7] as

$$C(\text{SNR}, N, T) = \left(1 - \frac{1}{T}\right) \log_2 \text{SNR} + c(N, T) + o(1) \text{ bits/channel use,}$$

where $c(N, T)$ (given in [5]) is a constant independent of the SNR and $o(1)$ indicates a vanishing function of the SNR as $\text{SNR} \to \infty$. The pre-log factor $1 - \frac{1}{T}$ can be achieved by a pilot-based scheme: the transmitter sends a pilot symbol in one of the channel uses and data symbols in the remaining $T - 1$ channel uses of a coherence block; the receiver estimates the channel based on the known pilot symbol and performs coherent detection on the received data symbols based on the channel estimate [4], [8]. This approach, however, is at a constant performance gap below the full capacity $C(\text{SNR}, N, T)$ since it is short of achieving the constant term $c(N, T)$, which can be significant when the number of receive antennas $N$ is large [5].
In [5], it was shown that the optimal strategy achieving the high-SNR capacity \( C(\text{SNR}, N, T) \) is to transmit isotropically distributed vectors on \( \mathbb{C}^T \) belonging to the Grassmannian of lines, which is the space of one-dimensional subspaces in \( \mathbb{C}^T \) [9], and use the direction of these vectors to carry information.\(^1\) The intuition behind that result is that the random channel coefficients only scale the transmitted signal vector without changing its direction. In other words, the transmitted vector \( \mathbf{x} \) and the noise-free observation \( h_i \mathbf{x} \) at receive antenna \( i \in \{1, \ldots, N\} \) are identical in the Grassmannian of lines. Thus, the constellation design for non-coherent communication can be formulated as a sphere-packing problem on the Grassmannian manifold. The ultimate packing criteria is to minimize the detection error under noisy observation. This typically amounts to maximizing the distance between the constellation points, for which the packing efficiency limits are characterized in, e.g., [10], [11], [12]. A number of Grassmannian constellation designs have been proposed (mostly for the multiple-transmit-antenna case at large) with different criteria, constellation generation, and decoding methods. They follow two main approaches.

The first approach uses numerical optimization tools to solve the sphere-packing problem on the Grassman manifold by maximizing the minimum pairwise distance between constellation points [13], [14], [15] or directly minimizing the error probability upper bound [16]. This results in constellations with a good distance spectrum but no particular structure. Due to the lack of structure, this kind of constellation needs to be stored at both the transmitter and receiver, and decoded with the high-complexity maximum-likelihood (ML) decoder, which limits practical use to only small constellations.

The second approach imposes particular structure on the constellation using, e.g., algebraic construction [17], [18], [19], or parameterized mappings of unitary matrices [20], [21]. Specifically, the Fourier constellation [17] contains the rows of the discrete Fourier transform (DFT) matrix with optimized frequencies, while the exponential-mapped constellation [20] is obtained by mapping coherent quadrature amplitude modulation (QAM) vectors into non-coherent symbols via exponential maps. In [22], a semi-structured constellation is proposed with multi-layer construction: the points in a layer are generated by moving the points in the previous layer along a set of geodesics with numerically optimized direction and moving steps. The pilot-data structured input of a pilot-based scheme can also be seen as a non-coherent code [8]. In general,

\(^1\)When \( T < N + 1 \), a further condition for achieving the capacity is that the input norm square is beta distributed; the rate achieved with constant-norm isotropically distributed input approaches the capacity within a constant factor of \( O(\log N) \) [7].
the constellation structure facilitates low complexity constellation mapping and demapping, while it needs to be carefully designed so as to preserve good distance properties.

In this work, we introduce a structured Grassmannian constellation for non-coherent SIMO communications over block fading channel. This constellation is structurally generated by partitioning the Grassmannian of lines with a collection of bent grids and mapping the symbols’s coordinates in the Euclidean space onto one of these bent grids. The main advantages of our so-called cube-split constellation are as follows:

- It has a good packing efficiency: its minimum distance is larger than that of existing structured constellation and compares well with the fundamental limits.
- It allows for a systematic decoder which has low complexity, hence can be easily implemented in practice, yet achieves near-ML performance.
- It admits a very simple yet effective binary labeling which leads to a low bit error rate.
- It allows for an accurate approximation of the log-likelihood ratio (LLR) which can be efficiently computed.

We verify by simulation that under block-fading channel, in terms of error probability (with or without channel codes) and achievable data rate, our cube-split constellation achieves performance nearly as good as the numerically optimized constellation, and outperforms a (coherent) pilot-based scheme and existing structured constellations in the literature.

The remainder of this paper is organized as follows. The system model and an overview of Grassmannian constellations are presented in Section II. We describe the construction and labeling of our cube-split constellation in Section III. We next propose low-complexity symbol decoder and LLR computation in Section IV. Numerical results on the error rates and achievable data rate are provided in Section V. Section VI concludes the paper and discusses the extension to the MIMO case. The proofs are provided in the appendices.

Notations: Random quantities are denoted with non-italic letters: normal fonts, e.g., \( x \), for scalars; bold fonts, e.g., \( \mathbf{v} \), for vectors; and bold and sans serif fonts, e.g., \( \mathbf{M} \), for matrices. Deterministic quantities are denoted with italic letters, e.g., a scalar \( x \), a vector \( \mathbf{v} \), and a matrix \( \mathbf{M} \). The Euclidean norm is denoted by \( \| \mathbf{v} \| \) and the Frobenius norm \( \| \mathbf{M} \|_F \). The trace, conjugate, transpose, and conjugated transpose of \( \mathbf{M} \) are denoted \( \text{tr}\{ \mathbf{M} \} \), \( \mathbf{M}^* \), \( \mathbf{M}^T \) and \( \mathbf{M}^{\dagger} \), respectively.

\(^2\) Our constellation was used in [23] as a quantization codebook on the Grassmannian of lines. Although the constellation structure is similar in both problems, the labeling and LLR computation presented here do not appear in the quantization problem.
$e_i$ is the $T \times 1$ canonical basis vector with 1 at position $i$ and 0 elsewhere. The Grassmann manifold $G(\mathbb{K}^T, M)$ is defined as the space of $M$-dimensional subspaces in $\mathbb{K}^T$ with $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ \cite{9}. In particular, $G(\mathbb{K}^T, 1)$ is the Grassmannian of lines. We use a vector $x \in \mathbb{C}^T$ of unit Euclidean norm ($\|x\| = 1$) to represent the set $\{\lambda x, \lambda \in \mathbb{C}\}$, which is a point in $G(\mathbb{C}^T, 1)$. The chordal distance between two lines represented by $x_1$ and $x_2$ is $d(x_1, x_2) = \sqrt{1 - |x_1^* x_2|^2}$.

II. SYSTEM MODEL AND GRASSMANNIAN CONSTELLATIONS

A. System model

We consider a SIMO non-coherent channel in which a single-antenna transmitter transmits to a receiver equipped with $N$ antennas. The channel between the transmitter and the receiver is assumed to be flat and block fading with coherence time $T \geq 2$ symbol periods. That is, the channel vector $h \in \mathbb{C}^N$ remains constant during each coherence block of $T$ symbols, and changes to an independent realization in the next block, and so on. The inter-block independence is relevant, for example, in the context of sporadic transmission in which the time gap between successive transmissions is indefinite. We assume that the distribution\cite{3} of $h$ is known, but its realizations are unknown to both ends of the communication link. Within a coherence block, the transmitter sends a signal $x \in \mathbb{C}^T$, and the receiver receives the $T \times N$ signal matrix

$$Y = \sqrt{\rho T} x h^T + Z,$$

(2)

where $Z \in \mathbb{C}^{T \times N}$ is the additive white Gaussian noise with independent and identically distributed (i.i.d.) $\mathcal{CN}(0, 1)$ entries independent of $h$, and the block index is omitted for simplicity. We consider the power constraint $\mathbb{E}[\|x\|^2] = 1$, so that the transmit power $\rho$ is identified with the signal-to-noise ratio (SNR) at each receive antenna. The high-SNR channel capacity $C(\rho, N, T)$ of this channel is given in \cite{1} with $\text{SNR} = \rho$ and

$$c(N, T) = \frac{1}{T} \log_2 \left( \frac{(L - 1)!}{(N - 1)! (T - 1)!} \right) + \left(1 - \frac{1}{T}\right) \log_2 T + \frac{L}{T} \log_2 \frac{N}{L} + \frac{T}{T} (\psi(N) - 1),$$

(3)

where $L := \min\{N, T-1\}$, $\bar{L} := \max\{N, T-1\}$, and $\psi(\cdot)$ is Euler’s digamma function \cite{5, 7}.

We assume that the input vector $x$ is taken from a finite constellation $C$ of size $|C|$. The maximum-likelihood (ML) decoder is

$$\hat{x}^{\text{ML}} = \arg\max_{x \in C} p(Y|X).$$

(4)

\textsuperscript{3}The constellation design is valid for any block-fading channel. We will mainly consider Rayleigh fading in the analysis of the decoder and LLR computation.
In the case of i.i.d. Rayleigh fading, i.e., \( h \sim \mathcal{CN}(0, I_N) \), conditioned on \( x \), \( Y \) is a Gaussian matrix with independent columns having the same covariance matrix \( I_T + \rho T x x^H \), hence

\[
p(Y|x) = \frac{\exp\left(-\text{tr}\{Y^H (I_T + \rho T x x^H)^{-1} Y\}\right)}{\pi^T \det(I_T + \rho T x x^H)}
\]

(5)

\[
= \frac{\exp\left(-\|Y\|^2_F + \frac{\rho \|x\|^2}{1 + \rho T \|x\|^2} \|Y^H x\|^2\right)}{\pi^T \left(1 + \rho T \|x\|^2\right)}.
\]

(6)

Thus, for unit-norm input, the ML decoder is simply

\[
\hat{x}_{ML} = \max_{x \in \mathbb{C}} \|Y^H x\|^2.
\]

(7)

Assuming that all constellation symbols are equally likely to be transmitted, i.e., the input law is \( p_x(x) = \frac{1}{|C|} \mathbb{1}\{x \in C\} \) where \( \mathbb{1}\{.\} \) denotes the indicator function, the achievable (data) rate is given as

\[
R = \frac{1}{T} I(x; Y) = \frac{1}{T} \mathbb{E}\left[ \log_2 \frac{\sum_{c \in C} p(Y|x = c)}{\mathbb{1}\{|C|\} \sum_{c \in C} p(Y|x = c)} \right]
\]

(8)

\[
= \frac{\log_2 |C|}{T} - \frac{1}{T} \mathbb{E}\left[ \log_2 \frac{\sum_{c \in C} p(Y|x = c)}{p(Y|x)} \right] \text{ bits/channel use.}
\]

(9)

Here, \( \frac{\log_2 |C|}{T} \) is the rate that can be achieved in the absence of noise, and \( \frac{1}{T} \mathbb{E}\left[ \log_2 \frac{\sum_{c \in C} p(Y|x = c)}{p(Y|x)} \right] \) is the rate loss due to noise. The expectation does not have a closed form in general, and the achievable rate can be computed numerically via the Monte Carlo method.

**B. Grassmannian Constellations**

It was shown that the high-SNR capacity \( [1] \) is achieved with isotropically distributed input \( x \) such that its distribution is invariant under rotation, i.e., \( p(x) = p(Qx) \) for any deterministic unitary matrix \( Q \in \mathbb{C}^{T \times T} \) \([5], [7]\). Such \( x \) is uniformly distributed on Grassmannian of lines \( G(\mathbb{C}^T, 1) \) \[9\]. Motivated by this, the constellation \( C \) can be designed by choosing \( |C| \) elements of \( G(\mathbb{C}^T, 1) \), represented by \( |C| \) unit-norm vectors \( \{ e_1, \ldots, e_{|C|} \} \). By definition, \( x \) and the noise-free observation \( h_i x \) at receive antenna \( i \in \{1, \ldots, N\} \) are identical in \( G(\mathbb{C}^T, 1) \). Therefore, Grassmannian signaling guarantees error-free detection without CSI in the absence of the additive noise. When the noise \( Z \) is present, since its columns are almost surely not aligned with the transmitted signal \( x \), the signal direction is perturbed, and the observation at each receive antenna can be dragged away from \( x \) with respect to (w.r.t.) a distance measure leading to a detection.
error if $Y$ falls out of the decision region of the transmitted symbol. Under i.i.d. Rayleigh fading, the decision regions of the optimal ML decoder (7) correspond to the collection of Voronoi regions defined for symbol $c_i$ by

$$V_i = \{x \in G(\mathbb{C}^T, 1) : d(x, c_i) \leq d(x, c_j), \forall j \neq i\}, \quad i \in \{1, \ldots, |C|\},$$

(10)

where $d(x, y) = \sqrt{1 - |x^H y|^2}$ is the chordal distance between the symbols $x$ and $y$. The constellation $C$ should be designed so as to minimize the probability of decoding error.

Following the footsteps of [6], we can derive the pairwise error probability (PEP) of mistaking a symbol $c_i$ for another symbol $c_j$ of the ML decoder as

$$P_{i,j}^{\text{ML}} = \Pr \{|Y^i e_j|^2 > |Y^i e_i|^2 | x = c_i\} = \frac{1}{2} \left[ 1 - \left( 1 + \frac{4(1 + \rho T)}{(d(c_i, c_j) \rho T)^2} \right)^{-\frac{1}{2}} \right].$$

(11)

We can verify that the PEP is decreasing with the chordal distance. The error probability $P_e^{\text{ML}}$ of ML decoder can be upper bounded in terms of the PEP using the union bound as

$$P_e^{\text{ML}} = \frac{1}{|C|} \sum_{i=1}^{|C|} P \{\hat{x} \neq x | x = c_i\} \leq \frac{1}{|C|} \sum_{i=1}^{|C|} \sum_{j \neq i} P_{i,j}^{\text{ML}} \leq \frac{|C| - 1}{2} \left[ 1 - \left( 1 + \frac{4(1 + \rho T)}{(d_{\text{min}} \rho T)^2} \right)^{-\frac{1}{2}} \right],$$

(12)

(13)

where $d_{\text{min}} := \min_{1 \leq i < j \leq |C|} d(c_i, c_j)$ is the minimum pairwise chordal distance of the constellation. Therefore, maximizing the minimum pairwise distance minimizes the union bound. This leads to a commonly used constellation design criteria

$$\max_{c = \{c_1, \ldots, c_{|C|}\}} \min_{1 \leq i < j \leq |C|} d(c_i, c_j).$$

(14)

This optimization problem can be solved numerically. The resulting constellation, however, is hard to exploit in practice due to its lack of structure. In particular, the unstructured constellations are normally used with the high-complexity ML decoder, do not admit a straightforward binary labeling, and need to be stored at both ends of the channel. In our design, we would rather relax (slightly) the optimality requirement (14) to have a structured constellation while preserving good packing properties, as described in the next section.

III. CUBE-SPLIT CONSTELLATION

A. Constellation Construction

The building blocks of our constellation are as follows.

...
1) Partitioning of the Grassmannian: First, the Grassmannian of lines $G(\mathbb{C}^T, 1)$ is partitioned into $T$ cells in which cell $i$ is defined as $S_i := \{ \mathbf{x} \in G(\mathbb{C}^T, 1) : |\mathbf{x}^*\mathbf{e}_i| > |\mathbf{x}^*\mathbf{e}_j|, \forall j \neq i \}$. We ignore the cell boundaries for which $|\mathbf{x}^*\mathbf{e}_i| = |\mathbf{x}^*\mathbf{e}_j| \geq |\mathbf{x}^*\mathbf{e}_k|$ for some $i \neq j$ and any $k \notin \{i, j\}$ since this is a set of measure zero. Note that these cells correspond to the Voronoi regions associated to the constellation $\{\mathbf{e}_1, \ldots, \mathbf{e}_T\}$. In this way, a symbol $\mathbf{x}$ belongs to cell $S_{i^*}$ if

$$\arg \min_{i=1,\ldots,T} d(\mathbf{x}, \mathbf{e}_i) = \arg \max_{i=1,\ldots,T} |x_i| = i^*,$$

that is, $\mathbf{e}_i$ is the closest (w.r.t. the chordal distance) canonical basis vector to $\mathbf{x}$. Note that since the symbols are defined up to a complex scaling factor, the Grassmannian of lines has $T-1$ complex dimensions, i.e. $2(T-1)$ real dimensions, and so are the cells.

2) Mapping from the Euclidean space onto a cell: Since each cell has $2(T-1)$ real dimensions, any point on a cell can be parameterized by $2(T-1)$ real coefficients. We choose to define these coefficients in the Euclidean space of $2(T-1)$ real dimensions, and let the coefficients determine the symbol through a mapping $g_{i^*}(.)$ from this Euclidean space onto the cell $S_{i^*}$. When $|C|$ goes to infinity, the constellation can be regarded as having a continuous distribution, in which regime an optimal constellation has symbols uniformly distributed on the Grassmannian, as mentioned in Section II-B. Since we want to support large constellations, we focus on this asymptotic regime and design the mapping $g_{i^*}(.)$ such that the resulting symbols are (approximately) uniformly distributed on $G(\mathbb{C}^T, 1)$. To this end, let us consider the mapping $g_{i^*}(\mathbf{a}) : (0,1)^{2(T-1)} \rightarrow S_{i^*}$ defined as

$$g_{i^*}(\mathbf{a}) = \frac{1}{\sqrt{1 + \sum_{i=1}^{T-1} |v_i|^2}} [v_1 \ldots v_{i^*-1} 1 v_{i^*} \ldots v_{T-1}]^T,$$

(16)

where for $i = 1, \ldots, T-1$,

$$v_i = \frac{1 - \exp \left( -\frac{|w_i|^2}{2} \right) w_i}{\sqrt{1 + \exp \left( -\frac{|w_i|^2}{2} \right)|w_i|}}, \quad \text{with} \quad w_i = \mathcal{N}^{-1}(a_{2i-1}) + i\mathcal{N}^{-1}(a_{2i}),$$

(17)

and $\mathcal{N}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$ is the cumulative distribution function of the standard real Gaussian. Note that the image of a point in $(0,1)^{2(T-1)}$ by $g_{i^*}(.)$ is a unit vector representative of a point in the cell $S_{i^*}$, thus the codomain of $g_{i^*}(.)$ is $S_{i^*}$. This mapping has the following properties.

**Property 1.** The mapping $g_{i^*}(.)$ is bijective, thus invertible. The inverse mapping

$$g_{i^*}^{-1}(\mathbf{x}) : S_{i^*} \rightarrow (0,1)^{2(T-1)}$$

$$\mathbf{x} = (x_1, \ldots, x_T)^T \mapsto \mathbf{a} = (a_1, \ldots, a_{2(T-1)})^T$$

(18)
can be written explicitly as

\[ a_{2i-1} = \mathcal{N}(\text{Re}(w_i)), \quad \text{and} \quad a_{2i} = \mathcal{N}(\text{Im}(w_i)), \]  

for \( i = 1, \ldots, T - 1 \), where

\[ w_i = \sqrt{2\log \frac{1 + |t_i|^2}{1 - |t_i|^2}} t_i \quad \text{and} \quad t = \left( \frac{x_1}{x_i^*}, \ldots, \frac{x_{i-1}}{x_i^*}, \frac{x_{i+1}}{x_i^*}, \ldots, \frac{x_T}{x_i^*} \right)^T. \]  

(19)

Since the \( \mathbf{g}_{i^*}(\cdot) \) are bijective and have non-overlapping codomains for different \( i^* \), a symbol \( \mathbf{x} = \mathbf{g}_{i^*}(\mathbf{a}) \) on the Grassmannian is uniquely identified with the parameters \( \{i^*, \mathbf{a}\} \). We refer to \( i^* \) as the cell index of \( \mathbf{x} \) since it identifies the cell in which \( \mathbf{x} \) belongs to, and to the components of \( \mathbf{a} \) as the local coordinates of \( \mathbf{x} \) since they identify the location of \( \mathbf{x} \) within the cell (i.e. the relative position of \( \mathbf{x} \) w.r.t. the center \( \mathbf{e}_{i^*} \) of the cell \( S_{i^*} \)).

**Property 2.** Let \( \mathbf{x} \) be a random vector uniformly distributed on the cell \( S_{i^*} \) of \( G(\mathbb{C}^T, 1) \), then the output of the inverse mapping \( \mathbf{g}_{i^*}^{-1}(\mathbf{x}) \) is a random vector on \((0, 1)^{2(T-1)} \) with uniform marginals.

**Proof.** The proof is given in Appendix B.

Property 2 gives a necessary condition on \( \mathbf{a} \) for the resulting symbol \( \mathbf{g}_{i^*}(\mathbf{a}) \) to be uniformly distributed on the cell \( S_{i^*} \). More generally, if \( i^* \) is drawn uniformly from \( \{1, \ldots, T\} \), Property 2 gives a necessary condition for the resulting symbol to be uniformly distributed on \( G(\mathbb{C}^T, 1) \).

3) **Constructing the constellation:** The constellation is then defined as

\[ \mathcal{C} = \left\{ \mathbf{c} = \mathbf{g}_{i^*}(\mathbf{a}) : i^* \in \{1, \ldots, T\}, \mathbf{a} \in \bigotimes_{i=1}^{2(T-1)} A_i \right\}, \]  

(21)

where \( A_i \) is a set of points in \((0, 1)\) and \( \bigotimes \) denotes the Cartesian product. Motivated by Property 2, we let each \( A_i \) contain evenly spread points in \((0, 1)\) as

\[ A_i = \left\{ \frac{1}{2B_i+1}, \frac{3}{2B_i+1}, \ldots, \frac{2B_i+1}{2B_i+1} \right\}, \]  

(22)

so \( |A_i| = 2B_i \), \( i \in \{1, \ldots, 2(T-1)\} \). Note that assigning values for the components of \( \mathbf{a} \) independently enables simple component-wise decoding as will be shown in Section IV. The grid of symbols in each cell is analogous to a bent hypercube, hence the name *cube-split constellation*.

The constellation contains \( T \times 2^{B_1} \times 2^{B_2} \times \cdots \times 2^{B_{2(T-1)}} \) symbols. If \( T \) is a power of 2, the number of bits required to represent a symbol is

\[ B = \log_2(T) + \sum_{i=1}^{2(T-1)} B_i. \]  

(23)
For a given constellation size \( |C| = 2^B \), we can initially let \( B_1 = \cdots = B_{2(T-1)} = \left\lfloor \frac{B-\log_2(T)}{2(T-1)} \right\rfloor \), then add one more bit to each of \( B - \log_2(T) - 2(T-1) \left\lfloor \frac{B-\log_2(T)}{2(T-1)} \right\rfloor \) randomly selected dimensions. An example of the cube-split constellation is illustrated in Fig. 1. For the sake of representation, we plot the constellation built on the real Grassmannian \( G(\mathbb{R}^T, 1) \) following the same principle.

Fig. 1. Illustration of the cube-split constellation on \( G(\mathbb{R}^3, 1) \) for \( B_1 = B_2 = 3 \) bits. The cells are denoted by different gray levels. Note that each symbol is depicted twice due to the sign indeterminacy associated to the Grassmannian. The symbols are represented by the dots. The constellation defines \( T \times 2^{B_1+B_2} = 192 \) lines in \( \mathbb{R}^3 \), each intersecting twice with the unit sphere. Two symbols \( e_1 \) and \( e_2 \) with minimum distance are in the middle of an edge of a cell.

B. Minimum Distance

The following lemma provides theoretical benchmarks for the minimum distance of an optimal constellation with given size.

Lemma 1. The minimum distance \( \delta \) of an optimal constellation \( C_{\text{opt}} \) of cardinality \( 2^B \) on the complex Grassmannian of lines \( G(\mathbb{C}^T, 1) \) is bounded by

\[
2^{1-\frac{B}{2(T-1)}} \geq \delta \geq 2^{-\frac{B}{2(T-1)}}. \tag{24}
\]

Proof. According to \cite{24} (also stated in \cite{12, Corollary 1}), the volume of a metric ball \( B(\delta) \) of radius \( \delta \) on \( G(\mathbb{C}^T, 1) \) with normalized invariant measure \( \mu(.) \) is given by \( \mu(B(\delta)) = \delta^{2(T-1)} \). Let \( C_{\text{opt}} \) denotes the optimal constellation on \( G(\mathbb{C}^T, 1) \) with minimum distance \( \delta \), we have the Gilbert-Varshamov lower bound and Hamming upper bound on the size of the code as

\[
\frac{1}{\mu(B(\delta/2))} \geq |C_{\text{opt}}| = 2^B \geq \frac{1}{\mu(B(\delta))}. \]

Next, by substituting the volumes \( \mu(B(\delta/2)) \) and \( \mu(B(\delta)) \) into this, \cite{24} follows readily. \( \square \)
In Fig. 2 we compare the minimum distance of the cube-split constellation for \( T = 2 \) and \( T = 4 \) with these fundamental limits. We also plot the minimum distance of

- the numerically optimized constellation generated by approximating the optimization (14) by
  \[
  \max_\mathcal{C} \log \sum_{1 \leq i < j \leq |\mathcal{C}|} \exp \left( \frac{|c_i c_j^*|}{\epsilon} \right)
  \]
  with a small \( \epsilon \) for smoothness, then solving it by conjugate gradient descent on the Grassmann manifold using the Manopt toolbox [25];
- the Fourier constellation [17], which coincides with the algebraically constructed constellation in [18, Sec. III-A] when \( T = 2 \);
- the exponential-mapped constellation [20] obtained by mapping each coherent symbol \( q \) containing \( T - 1 \) QAM symbols to a non-coherent symbol
  \[
  x = \begin{bmatrix}
    \cos(\gamma \|q\|) \\
    -\frac{\sin(\gamma \|q\|)}{\|q\|} q^T
  \end{bmatrix}
  \]
  with the homothetic factor \( \gamma \) given in [20, Eq.(19)].

We observe that the cube-split constellation has the largest advantage over the other structured constellations whenever

\[
B = \log_2(T) + 2(T - 1)B_0 \quad \text{with} \quad B_0 := B_1 = \cdots = B_{2(T-1)}.
\]

In this symmetric case, all the real dimensions of a cell accommodate the same number of bits, thus the symbols are more evenly spread. While a proof of the minimum distance remains elusive, we conjecture, and have verified with all the cases depicted in Fig. 2, that a pair of symbols \( c_1 \) and \( c_2 \) has minimum distance if they are in the same cell and their respective local coordinates \( a^{(1)} \)
and \( a^{(2)} \) differ in only one component such that
\[
\alpha = \begin{cases} 
\frac{1}{2} - \frac{1}{2B_0+1}, & \text{if } i = 0 \\
\frac{1}{2} + \frac{1}{2B_0+1}, & \text{if } i \neq 0,
\end{cases}
\]
\[
a_i^{(1)} = a_i^{(2)} \in \left\{ \frac{1}{2B_0+1}, 1 - \frac{1}{2B_0+1} \right\}, \forall i \neq i_0,
\]
for some \( i_0 \in \{1, \ldots, 2(T-1)\} \). One such pair of symbols are illustrated in Fig. [1] The two symbols are in the middle of an edge of a cell. This conjectured minimum distance is given by
\[
\tilde{d}_{\text{min}}(T, B_0) := \sqrt{1 - \left| 1 + \frac{\alpha}{\alpha + \beta} (e^{2\varphi} - 1) \right|^2},
\]
where \( \alpha := \frac{1 - \exp\left(-\frac{m_0^2 + m_1^2}{2}\right)}{1 + \exp\left(-\frac{m_0^2 + m_1^2}{2}\right)} \), \( \beta := 1 + (T - 2) \frac{1 - e^{-m_0^2}}{1 + e^{-m_0^2}} \), and \( \varphi := \arctan\left(\frac{m_1}{m_0}\right) \), with \( m_0 := \mathcal{N}^{-1}\left(2^{-B_0 - 1}\right) \) and \( m_1 := \mathcal{N}^{-1}\left(\frac{1}{2} + 2^{-B_0 - 1}\right) \). Under this conjecture, one can show that the constellation is asymptotically optimal w.r.t. the bounds given in Lemma [1] up to a log factor and a constant.

**Lemma 2.** When \( B \) is large, it holds that
\[
\log_2 \left( \tilde{d}_{\text{min}}(T, B_0) \right) = -\frac{B}{2(T-1)} - \frac{1}{2} \log_2(B) + O_B(1).
\]

*Proof.* First, it is straightforward to see that, when \( B \) is large, \( m_0 \) goes to \(-\infty\) and \( m_1 \) goes to 0 and it follows that \( \alpha \) goes to 1, \( \beta \) goes to \( T - 1 \), and \( \varphi = \frac{m_1}{m_0} + o\left(\frac{m_1}{m_0}\right) \). Then
\[
\tilde{d}_{\text{min}}(T, B_0) = \sqrt{\frac{2\alpha}{\alpha + \beta} \left(1 - \cos(2\varphi) - \sin(2\varphi)\right) - \left(\frac{\alpha}{\alpha + \beta}\right)^2 |e^{2\varphi} - 1|^2}
\]
\[
= \sqrt{\frac{4}{T^2} \varphi^2 - \frac{4}{T^2} \varphi^2 + o(\varphi^2)}
\]
\[
= 2\sqrt{T - 1} \left| \varphi \right| + o(\left| \varphi \right|).
\]
On the other hand, using [26, Sec.26.2], it follows that \( m_1 = \sqrt{2\pi 2^{-B_0 - 1} + o(2^{-B_0})} \) and \( \left|m_0\right| = \sqrt{2 \log(2B_0 + 1) + o(\sqrt{2 \log(2B_0 + 1)})} \). Inserting this and \( \varphi = \frac{m_1}{m_0} + o\left(\frac{m_1}{m_0}\right) \) into (31) gives
\[
\log_2(\tilde{d}_{\text{min}}(T, B_0)) = \log_2(m_1) - \log_2(\left|m_0\right|) + O(1)
\]
\[
= -B_0 - \frac{1}{2} \log_2(B_0) + O(1),
\]
which yields the result. 

Hereafter, we denote by \( CS(T, B_0) \) the symmetric cube-split constellation with \( B_1 = \cdots = B_2(T-1) =: B_0 \). In Fig. [3] we plot the spectrum of the symbol-wise minimum distance, i.e., the
distance from each symbol to its nearest neighbor, of the $CS(2, 4)$ and $CS(4, 1)$ constellations. The symbol-wise minimum distances are concentrated and compare well to the fundamental bounds. Every symbol in the $CS(4, 1)$ constellation has the same distance to its nearest neighbor, which is $\tilde{d}_{\text{min}}(4, 1)$. This property holds for any $CS(T, 1)$ constellations due to symmetry.

![Graph showing symbol-wise minimum distance spectrum for $CS(2, 4)$ and $CS(4, 1)$ constellations.](image)

(a) $CS(2, 4)$ constellation (512 symbols)  
(b) $CS(4, 1)$ constellation (256 symbols)

Fig. 3. The symbol-wise minimum distance spectrum of $CS(2, 4)$ and $CS(4, 1)$ constellations. The dashed and dash-dotted lines are respectively the upper and lower bounds \cite{24} of the minimum distance of an optimal constellation of the same size.

C. Binary Labeling

We now consider another important aspect of designing a constellation which is to label each symbol with a binary vectors. These binary labels should be assigned such that a symbol error does not cause many bit errors. This requires that symbols which are likely to be mistaken for each other should differ by a minimal number of bits in their labels. In other words, symbols with small distance, w.r.t. the chordal distance in our case, are given labels with small Hamming distance. This is the principle of Gray labeling which was shown to be optimal in terms of average bit error probability for structured scalar constellations such as phase-shift keying (PSK), pulse amplitude modulation (PAM), and QAM \cite{27}, and has been widely used. Ideally, a Gray labeling scheme gives the neighboring symbols labels that differ by exactly one bit. It was shown in \cite[Thm. 1]{28} that this is possible for a special case of the Grassmannian constellation in \cite{19}. Nevertheless, this is rarely the case in general due to the irregular neighboring properties of the constellation. When a true Gray labeling is not possible, finding a quasi-Gray one requires
an exhaustive search over $|C|!$ candidate labelings. Therefore, one often resorts to sub-optimal labeling schemes.

An iterative labeling scheme consisting in propagating the labels along the edges of the neighboring graph was proposed in [29]. Unfortunately, building and storing such a graph is possible only for constellations of small dimension and small size. In [30], two matching-based methods to label a Grassmannian constellation, say $C$, are proposed. The first, so-called match-and-label algorithm, matches $C$ to an auxiliary constellation which can be Gray labeled. The second, so-called successive matching algorithm, matches the distance spectrum of $C$ with the Hamming distance spectrum of an auxiliary Gray label. Although these three schemes do not require exhaustive search, their complexity is still at least cubic in the constellation size. Furthermore, they require storage and do not offer any optimality guarantee.

For our cube-split constellation, we introduce a simple yet effective and efficient Gray-like labeling scheme by exploiting the constellation structure. Recall that the number of bits per symbol is $B = \log_2(T) + \sum_{i=1}^{2(T-1)} B_i$, and a symbol is entirely determined by the cell index $i^*$ and the set of local coordinates $\{a_1, \ldots, a_{2(T-1)}\}$. Our labeling scheme works as follows.

- We let the first $\log_2(T)$ bits represent the cell index $i^*$ and denote them by cell bits. These bits are defined simply as the binary representation of $i^* - 1$. Note that no optimization of the labels of the cell index is possible since each cell have common boundaries with all other cells, as can be seen in Fig. 1.

- We let each of the next groups of $B_i$ bits represent the local coordinate $a_i \in A_i$ and denote them by coordinate bits. These bits are mapped using the Gray code explicitly defined for $A_i$. For example, if $B_i = 3$ (as in Fig. 1), $A_i = \left\{ \frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \ldots, \frac{15}{16} \right\}$ and the corresponding coordinate bits are $\{000, 001, 011, 010, 110, 111, 101, 100\}$.

Thus, the complexity of assigning label to a symbol is only linear in $T$. In addition, the label assignment is independent between the symbols. Therefore, our Gray-like labeling can be done on-the-fly and requires no storage.

In Fig. 4, we compare the performance of this Gray-like labeling scheme with random labeling, graph propagation labeling [29], match-and-label labeling and successive matching labeling [30]. For the match-and-label scheme, we use the exponential-mapped constellation [20] as the auxiliary constellation. This constellation is mapped from coherent symbols $q \in \mathbb{C}^{T-1}$ containing $T - 1$ QAM symbols, and thus can be quasi-Gray labeled by taking the Gray label of $q$. It can be seen that for the considered $CS(2, 4)$ and $CS(4, 1)$ constellations, our Gray-like labeling scheme,
albeit being simpler, outperforms the other considered schemes.

Fig. 4. The bit error rate of the cube-split constellation with ML decoder and different labeling schemes in a single-receive-antenna system. The proposed Gray-like labeling outperforms the other schemes.

IV. LOW-COMPLEXITY RECEIVER DESIGN

In this section, leveraging the constellation structure, we design efficient symbol decoder and log-likelihood ratio computation from the observation $Y$.

A. Low-complexity greedy decoder

In order to avoid the high-complexity ML decoder, we propose to decode the symbol in a greedy manner by estimating sequentially the cell index $i^*$ and the local coordinates $a$.

1) Step 1 - Denoising: We first use the fact that, by construction, the signal of interest is supported by a rank-1 component of $Y$ (see (2)). We compute the left singular vector $u = [u_1, u_2, \ldots, u_T]^T$ corresponding to the largest singular value of $Y$, which is also the solution of

$$\arg\max_{u \in \mathbb{C}^T : \|u\|_2^2 = 1} \|Y^u u\|^2.$$  

Under i.i.d. Rayleigh fading, $u$ is the solution of a relaxed version of the ML decoder (7) if we disregard the discrete nature of the constellation. Thus, $u$ serves as a rough estimate of the transmitted symbol $x$ on the unit sphere.
2) **Step 2 - Estimating the cell index and the local coordinates:** We then find the closest symbol to \( \mathbf{u} \) by localizing \( \mathbf{u} \) on the system of bent grids defined for the constellation. To do so, we estimate the cell index and the local coordinates. The cell index estimate is obtained as

\[
\hat{i}^* = \arg\min_{i=1,\ldots,T} d(\mathbf{u}, \mathbf{e}_i) = \arg\max_{i=1,\ldots,T} |u_i|.
\]

Note that in the noise-free case, \( \mathbf{Y} = \sqrt{\rho T} \mathbf{x} \) is a rank-1 matrix and \( \mathbf{u} = e^{j\phi} \mathbf{x} \) for some \( \phi \in [0, 2\pi] \), thus \( \hat{i}^* = i^* \) since \( x_i^* \) is the strongest component in \( \mathbf{x} \) by construction (see (16)). The local coordinates are estimated by applying the inverse mapping \( g_{i^*}^{-1} \) to \( \mathbf{u} \). That is,

\[
\hat{\mathbf{a}} = g_{i^*}^{-1}(\mathbf{u}) = [\hat{a}_1 \ldots \hat{a}_{2T-2}] \text{ with }
\]

\[
\hat{a}_{2i-1} = \arg\min_{a \in A_{2i-1}} |N(\text{Re}(\hat{w}_i)) - a|, \quad \text{and} \quad \hat{a}_{2i} = \arg\min_{a \in A_{2i}} |N(\text{Im}(\hat{w}_i)) - a|
\]

for \( i = 1, \ldots, T - 1 \), where

\[
\hat{w}_i = \sqrt{2 \log \frac{1 + |t_i|^2}{1 - |t_i|^2}} t_i \quad \text{and} \quad \mathbf{t} = \left( \frac{u_1}{u_{i^*}}, \ldots, \frac{u_{i^*-1}}{u_{i^*}}, \frac{u_{i^*+1}}{u_{i^*}}, \ldots, \frac{u_T}{u_{i^*}} \right)^T.
\]

Again, in the absence of noise, \( \mathbf{u} = e^{j\phi} \mathbf{x}, \hat{i}^* = i^* \), and thus \( \hat{\mathbf{a}} = \mathbf{a} = g_{i^*}^{-1}(\mathbf{x}) \).

The decoded symbol \( \hat{\mathbf{x}} \) is then identified from the parameters \( \{\hat{i}^*, \hat{\mathbf{a}}\} \) as \( \hat{\mathbf{x}} = g_{\hat{i}^*}(\hat{\mathbf{a}}) \). The cell bits are decoded by taking the binary representation of \( \hat{i}^* - 1 \). The coordinate bits are demapped from \( \hat{a}_i \) using the Gray code defined for \( A_i \), independently for each real component \( i = 1, \ldots, 2(T - 1) \).

In Step 1, if a full singular value decomposition of \( \mathbf{Y} \) is carried out to find \( \mathbf{u} \), the complexity is \( O(N^2T) \). The complexity of Step 2 is \( O(T) \). The total complexity of the greedy decoder is thus linear in \( T \) for a fixed number of receive antenna, and independent of the constellation size. The decision regions of the proposed greedy decoder are close to the Voronoi regions, which are the optimal decision regions, as depicted in Fig. 5 for the constellation shown in Fig. 1.

**B. Demapping Error Analysis**

With the above greedy decoder, two types of error can occur. First, a cell error can occur due to false detection of the cell index \( i^* \). The probability of cell error is

\[
\Pr\{\hat{i}^* \neq i^*\} = \Pr\left\{ \arg\max_i |u_i| \neq \arg\max_i |x_i| \right\}.
\]

This is equivalent to the problem of quantizing \( \mathbf{u} \) using \( C \) as a quantization codebook on \( G(\mathbb{C}^T, 1) \), see [23].
Fig. 5. Illustration of the decision regions of the greedy decoder for the cube-split constellation on \( G(\mathbb{R}^3, 1) \) with \( B_1 = B_2 = 3 \) bits. These decision regions well match the Voronoi regions [10], which are optimal decision regions under i.i.d. Rayleigh fading.

Second, a coordinate error occurs when one of the local coordinates in \( a \) is wrongly detected. The probability of a coordinate error given correct cell detection is \( \Pr \{ \hat{a} \neq a | \hat{i}^* = i^* \} \). Then, the symbol error probability of the greedy decoder is

\[
P_e = \Pr \{ \hat{i}^* \neq i^* \} + (1 - \Pr \{ \hat{i}^* \neq i^* \}) \Pr \{ \hat{a} \neq a | \hat{i}^* = i^* \}. \tag{39}
\]

In the following, we derive analytically the error rate in the case of Rayleigh fading, single receive antenna, and \( B_1 = \cdots = B_2(T-1) = 1 \). In this case, the received signal is \( y = \sqrt{\rho T} h x + z \), where \( h \) is a scalar fading coefficient, and the symbols can be written simply as

\[
x = \left( \frac{1 + e^{-m^2}}{1 - e^{-m^2}} + T - 1 \right)^{-1/2} \left[ q_1 \cdots q_{i^*-1} \sqrt{\frac{1 + e^{-m^2}}{1 - e^{-m^2}}} q_{i^*+1} \cdots q_{T-1} \right]^T, \tag{40}
\]

where \( m := N^{-1}(3/4) \) and \( q_i := \pm \frac{1}{\sqrt{2}} \pm j \frac{1}{\sqrt{2}} \) are 4-QAM symbols with unit power.

**Proposition 1.** When \( N = 1, \) \( B_1 = \cdots = B_2(T-1) = 1, \) under Rayleigh block fading, the cell error probability is

\[
\Pr \{ \hat{i}^* \neq i^* \} = 1 - \int_{0}^{\infty} \int_{0}^{\infty} \left( 1 - Q_1(\sqrt{2c\rho_0 x}, \sqrt{2y}) \right)^{T-1} I_0(2\sqrt{\rho_0 xy}) e^{-y-(\rho_0+1)x} \, dy \, dx, \tag{41}
\]

where \( m := N^{-1}(3/4), \) \( c := \frac{1-e^{-m^2}}{1+e^{-m^2}}, \) \( \rho_0 := \frac{\rho T}{1+(T-1)c}, \) \( I_0(x) := \frac{1}{\pi} \int_{0}^{\pi} \exp(x \cos(\theta)) \, d\theta \) is the modified Bessel function of the first kind at order 0, and \( Q_1(a, b) := \int_{b}^{\infty} x \exp\left(-\frac{x^2+a^2}{2}\right) I_0(ax) \, dx \)
is the Marcum Q-function with parameter 1. Given correct cell detection, the error probability of one pair of local coordinates is given by

\[
\Pr\left\{ \hat{a}_{2i-1} \neq a_{2i-1}, \hat{a}_{2i} \neq a_{2i} \mid \hat{i}^* = i^* \right\} = \Pr\left\{ \hat{q}_i \neq q_i \mid \hat{i}^* = i^* \right\} = \frac{1}{2} \left( 1 - \frac{(1 - c) \rho_0}{\sqrt{(2 + (1 + c) \rho_0)^2 - 4c \rho_0^2}} \right)^{-1} \times \frac{1}{4} + \frac{\sqrt{2c \rho_0 \arccot \frac{1 + (c - \sqrt{c}) \rho_0}{\sqrt{1 + (c + 1) \rho_0} + \frac{c}{2} \rho_0}}}{\pi \sqrt{1 + (c + 1) \rho_0 + \frac{c}{2} \rho_0^2}} + \frac{(1 - c) \rho_0 \arccot \frac{2 + (1 - 2\sqrt{c} + c) \rho_0}{(2 + (1 + c) \rho_0)^2 - 4c \rho_0^2}}{\pi \sqrt{(2 + (1 + c) \rho_0)^2 - 4c \rho_0^2}} \right),
\]

for all \( i \in \{1, \ldots, T - 1\} \). The symbol error probability is then bounded by the union bound as

\[
P_e \leq \Pr\{\hat{i}^* \neq i^*\} + (T - 1)(1 - \Pr\{\hat{i}^* \neq i^*\}) \Pr\left\{ \hat{q}_i \neq q_i \mid \hat{i}^* = i^* \right\}.
\]

**Proof.** The proof is given in Appendix \[ \Box \]

In particular, if \( T = 2 \), the symbol error probability can be computed in closed form as follows.

**Corollary 1.** When \( N = 1 \), \( T = 2 \) and \( B_1 = \cdots = B_{2(T - 1)} = 1 \), under Rayleigh block fading, the cell error probability can be written explicitly as

\[
\Pr\{\hat{i}^* \neq i^*\} = \frac{1}{2} \left( 1 - \frac{(1 - c) \rho_0}{\sqrt{(2 + (1 + c) \rho_0)^2 - 4c \rho_0^2}} \right),
\]

Whereas, the conditional coordinate error probability \( \Pr\{\hat{a} \neq a \mid \hat{i}^* = i^*\} \) is exactly the right-hand side (RHS) of (43). Accordingly, the symbol error probability \( P_e \) is

\[
P_e = \frac{7}{8} - \frac{\sqrt{c \rho_0 \arccot \frac{1 + (c - \sqrt{c}) \rho_0}{\sqrt{1 + (c + 1) \rho_0} + \frac{c}{2} \rho_0}}}{\pi \sqrt{2 + 2(1 + c) \rho_0 + c \rho_0^2}} - \frac{(1 - c) \rho_0 \arccot \frac{2 + (1 - 2\sqrt{c} + c) \rho_0}{(2 + (1 + c) \rho_0)^2 - 4c \rho_0^2}}{2 \pi \sqrt{(2 + (1 + c) \rho_0)^2 - 4c \rho_0^2}}.
\]

**Proof.** The proof is given in Appendix \[ \Box \]

**C. Log-Likelihood Ratio**

When a channel code is employed, most channel decoders require the log-likelihood ratios (LLRs) of the coded bits as an input. LLR computation is performed independently from the code structure, assuming uniform input probabilities, i.e., all the symbols are equally likely to be
transmitted, and so are the bits. Denote the binary label of symbol \( x \) as \( b = [b_1(x) \ b_2(x) \ldots b_B(x)] \).

The log-likelihood ratio (LLR) of bit \( b_i \) given the observation \( Y \) under i.i.d. Rayleigh fading is

\[
\text{LLR}_i(Y) = \log \frac{p(Y|b_i=1)}{p(Y|b_i=0)} = \log \frac{\sum_{\alpha \in C_i^{(1)}} p(Y|x=\alpha)}{\sum_{\beta \in C_i^{(0)}} p(Y|x=\beta)}
\]

(47)

\[
= \log \frac{\sum_{\alpha \in C_i^{(1)}} \exp \left( \frac{\rho T}{1+\rho T} \|Y^H \alpha\|^2 \right)}{\sum_{\beta \in C_i^{(0)}} \exp \left( \frac{\rho T}{1+\rho T} \|Y^H \beta\|^2 \right)},
\]

(48)

where \( C_i^{(b)} \) denotes the set of all possible symbols in the constellation such that \( b_i = b \), i.e.,

\[
C_i^{(b)} := \{ c \in C : b_i(c) = b \}, \text{ for } i \in \{1, \ldots, B\} \text{ and } b \in \{0, 1\}, \text{ and } (48) \text{ follows from (6).}
\]

In general, the LLR distribution differs between the bit positions, thus have different error protection properties. This can be seen in Fig. 6, where we depict the LLR histogram of the cell bit and the first two coordinate bits (the remaining two coordinate bits have the same LLR distribution as the first two due to symmetry), given that 0 was sent in that bit, for the \( CS(2, 2) \) constellation and single receive antenna. It can also be observed that the LLR distribution truncated on \([0, \infty)\) (or \((-\infty, 0]\)) closely fits the exponential distribution (or the flipped exponential distribution, respectively). The distribution fitting of the LLR can be useful, e.g., to calculate the mutual information \( I(b_i(x); \text{LLR}_i(Y)) \), which reveals how much information is carried in different bit positions, as well as in the framework of an extrinsic information transfer (EXIT) chart [31] analysis.

![LLR histograms](image)

Fig. 6. Histograms of the LLR of the first 3 bits, given that 0 was sent, of \( CS(2, 2) \) constellation (5 bits/symbol), single receive antenna, and SNR = 10 dB. The red solid lines show the fitted exponential distribution of the LLR truncated on \([0, \infty)\) and the fitted flipped exponential distribution of the LLR truncated on \((-\infty, 0]\) obtained by matching the first moment (mean).

**Low-complexity LLR computation:** Computing the LLR according to (48) requires enumerating the whole constellation. To avoid this, we propose a low-complexity LLR computation as follows.
First, for any real-valued array \( x_1, \ldots, x_n \), denote by \( M_m \) the set of \( m \) largest values, i.e., \( x_j \leq x_i \) for all \( x_i \in M_m \) and \( x_j \not\in M_m \). We have that
\[
\log \sum_{i=1}^{n} e^{x_i} = \log \sum_{x_i \in M_m} e^{x_i} + \log \left( 1 + \frac{\sum_{x_j \in M_m} e^{x_j}}{\sum_{x_i \in M_m} e^{x_i}} \right) \\
\leq \log \sum_{x_i \in M_m} e^{x_i} + \log \left( 1 + (n-m) \exp \left( \max_{x_j \notin M_m} x_j - \max_{i \in \{1, \ldots, n\}} x_i \right) \right).
\]
(49)

For sufficiently large value of \( \max_{i \in \{1, \ldots, n\}} x_i - \max_{x_j \notin M_m} x_j \), the second term in the RHS vanishes and \( \log \sum_{i=1}^{n} e^{x_i} \) can be well approximated\(^6\) by \( \log \sum_{x_i \in M_m} e^{x_i} \). Applying this approximation to (48) yields
\[
\text{LLR}_i(Y) \approx \log \sum_{\alpha \in M^{(1)}_{\eta,i}} \exp \left( \frac{\rho T}{1 + \rho T} \| Y^\alpha \|^2 \right) - \log \sum_{\beta \in M^{(0)}_{\eta,i}} \exp \left( \frac{\rho T}{1 + \rho T} \| Y^\beta \|^2 \right),
\]
(51)
where \( M^{(b)}_{\eta,i} \) stores the \( \eta \) symbols corresponding to the \( \eta \) largest terms in \( \{ \| Y^\alpha \| \} \) for \( b \in \{0, 1\} \). Observe that one symbol in either \( M^{(1)}_{\eta,i} \) or \( M^{(0)}_{\eta,i} \) is exactly the output \( \hat{x}^{\text{ML}} \) of the ML decoder (7). The remaining symbols in \( M^{(1)}_{\eta,i} \) and \( M^{(0)}_{\eta,i} \) are expected to be close to \( \hat{x}^{\text{ML}} \) since they also lead to high likelihood of \( Y \). Furthermore, \( \hat{x}^{\text{ML}} \) can be approximated by the output \( \hat{x} \) of the greedy decoder\(^7\). Thus, the symbols in \( M^{(1)}_{\eta,i} \) and \( M^{(0)}_{\eta,i} \) are expected to be in the neighborhood of the greedy decoder’s output \( \hat{x} \). Therefore, the LLR can be further approximated by replacing \( M^{(1)}_{\eta,i} \) and \( M^{(0)}_{\eta,i} \) in (51) by the sets of the greedy decoder’s output \( \hat{x} \) and its neighbors as
\[
\text{LLR}_i(Y) \approx \log \sum_{\alpha \in B_i(\hat{x}, 1)} \exp \left( \frac{\rho T}{1 + \rho T} \| Y^\alpha \|^2 \right) - \log \sum_{\beta \in B_i(\hat{x}, 0)} \exp \left( \frac{\rho T}{1 + \rho T} \| Y^\beta \|^2 \right),
\]
(52)
where the set \( B_i(c, b) \) contains the symbol \( c \in C \) and its \( \eta \) nearest neighbors \( \hat{c}_1, \ldots, \hat{c}_\eta \) such that \( b_i(\hat{c}_1) = \cdots = b_i(\hat{c}_\eta) = b \), for \( i = 1, \ldots, B \).

The sets \( B_i(c, b) \) can be precomputed for each symbols \( c \in C \) prior to communication (with negligible complexity) and stored at the receiver. In this way, the complexity of computing the RHS of (52) is only \( O(N^2T + NT\eta) \) (\( O(N^2T) \) for the hard detection to find \( \hat{x} \) and \( O(NT\eta) \) for the computation of the RHS of (52)). Alternatively, one can look for an approximation of \( B_i(\hat{x}, b) \) (possibly constructed on-the-fly upon detecting \( \hat{x} \)) when the constellation size is too large. The latter option does not require storage but increases the complexity.

\(^6\)When \( m = 1 \), this approximation boils down to the well-known one \( \log \sum_i e^{x_i} \approx \max_i x_i \).

\(^7\)We will see in Section V that the greedy decoder achieves near-ML performance.
V. PERFORMANCE EVALUATION

In this section, we evaluate numerically the performance of our cube-split constellation in comparison with other constellations and a baseline (coherent) pilot-based scheme.

A. A baseline pilot-based scheme

We consider a baseline scheme based on channel training [4]. The transmitted signal is
\[
x = (\rho T)^{-1/2} \left[ \sqrt{\rho_r} \, \sqrt{\rho_d} \mathbf{x}_d^T \right]^T
\]
where the data \( \mathbf{x}_d = [x_2 \ldots x_T]^T \) is normalized such that \( \mathbb{E} [\mathbf{x}_d \mathbf{x}_d^H] = \mathbf{I}_{T-1} \). The power factors \( \rho_r \) and \( \rho_d \) satisfy \( \rho_r + (T-1)\rho_d = \rho T \) and can be optimized. The received signal can be written as
\[
\mathbf{y} = [\mathbf{y}_\tau \mathbf{y}_d^T]^T \quad \text{where} \quad \mathbf{y}_\tau = \sqrt{\rho_r} \mathbf{h} + \mathbf{z}_\tau \quad \text{and} \quad \mathbf{y}_d = \sqrt{\rho_d} \mathbf{x}_d \mathbf{h}^T + \mathbf{Z}_d
\]
are the received signals in the training phase and data transmission phase, respectively. The receiver uses minimum-mean-square-error (MMSE) channel estimation \( \hat{\mathbf{h}} = \frac{\sqrt{\rho_r}}{1+\rho_r} \mathbf{y}_\tau \).

According to [4, Thm. 3], a lower bound on the achievable rate of this pilot-based scheme is
\[
R_{\text{pilot}}(\rho, N, T) := \left( 1 - \frac{1}{T} \right) \mathbb{E} \left[ \log_2 \left( 1 + \rho_{\text{eff}} \| \hat{\mathbf{h}} \|^2 \right) \right],
\]
where \( \hat{\mathbf{h}} := \frac{1}{N} \mathbb{E} \| \hat{\mathbf{h}} \|^2 \) is the normalized version of \( \mathbf{h} \), and \( \rho_{\text{eff}} \) equals \( \frac{\rho^2}{1+2\rho} \) if \( T = 2 \) and \( \left( \frac{\sqrt{(T-1)(1+\rho T)}-\sqrt{T-1+\rho T}}{T-2} \right)^2 \) if \( T > 2 \). With i.i.d. Rayleigh fading \( \mathbf{h} \sim \mathcal{CN}(0, \mathbf{I}_N) \), we have \( \hat{\mathbf{h}} \sim \mathcal{CN}(0, \frac{\rho_r}{1+\rho_r} \mathbf{I}_N) \), \( \mathbf{h} \sim \mathcal{CN}(0, \mathbf{I}_N) \), and the lower bound can be derived using [52] 4.337.5 as
\[
R_{\text{pilot}}(\rho, N, T)
\]
\[
= \frac{T-1}{T} \log_2(e) \sum_{n=1}^{N} \frac{(N-1)!}{(N-n)!} \left[ e^{1/\rho_{\text{eff}}} E_1 \left( \frac{1}{\rho_{\text{eff}}} \right) + \sum_{m=1}^{N-n} (m-1)! (-\rho_{\text{eff}})^m \right],
\]
where \( E_1(x) := \int_x^{\infty} \frac{e^{-t}}{t} \, dt \) is the exponential integral function. This lower bound is achieved with i.i.d. Gaussian input \( \mathbf{x}_d \sim \mathcal{CN}(0, \frac{1}{T-1} \mathbf{I}_{T-1}) \) and the optimal power allocation
\[
\rho_r = \begin{cases} 
\rho, & \text{if } T = 2, \\
\sqrt{T-1+\rho T} \left( \frac{\sqrt{(T-1)(1+\rho T)}-\sqrt{T-1+\rho T}}{T-2} \right), & \text{if } T > 2.
\end{cases}
\]

Let \( \mathbf{h} = \mathbf{h} - \hat{\mathbf{h}} \) be the channel estimation error, then \( \mathbf{h} \sim \mathcal{CN}(0, \frac{1}{1+\rho_r} \mathbf{I}_N) \) and \( \hat{\mathbf{h}} \) and \( \mathbf{h} \) are uncorrelated. The output can be written as \( \mathbf{y}_d = \sqrt{\rho_d} \mathbf{x}_d \hat{\mathbf{h}}^T + \mathbf{Z}_d \), where \( \mathbf{Z}_d = \sqrt{\rho_d} \mathbf{x}_d \mathbf{h}^T + \mathbf{Z}_d \). Given
the input, the rows of $\tilde{Z}_d$ are independent and follows $CN\left(0, (1 + \frac{\rho_d|x_i|^2}{1+\rho_r})I_N\right)$, $i \in \{2, \ldots, T\}$. Thus, the likelihood function of the output at time slot $i \in \{2, \ldots, T\}$ is

$$p(Y_{[i]}|x_i, h) = \pi^{-N} \left(1 + \frac{\rho_d|x_i|^2}{1+\rho_r}\right)^{-N} \exp \left(-\frac{||Y_{[i]} - \sqrt{\rho_d x_i^* h^T}||^2}{1 + \frac{\rho_d|x_i|^2}{1+\rho_r}}\right),$$

where $Y_{[i]}$ denotes the $i$-th row of $Y$.

In practice, the data symbols in $x_d$ are normally taken from finite scalar constellations such as QAM or PSK in order to reduce the complexity of the ML decoder based on (57). A sub-optimal method consists in linear equalization followed by component-wise scalar demapping. With zero-forcing (ZF) or MMSE equalizer, the equalized symbols are respectively

$$\hat{x}_{zf}^d = \frac{Y_d}{\sqrt{\rho_d \|h\|^2}} h^*, \quad \text{or} \quad \hat{x}_{mmse}^d = \frac{Y_d}{\sqrt{\rho_d \|h\|^2 + 1/\rho_d}} h^*.$$  

In the remainder of this section, we consider i.i.d. Rayleigh fading $h \sim CN(0, I_N)$ and compare different schemes with the same transmission rate in terms of bits per symbol. We remark that, although we show the results for $T = 2$ and $T = 4$, our constellation is available for large $T$ and large $|C|$ since the symbol mapping, demapping, and labeling can all be done on-the-fly.

**B. Achievable data rate**

In Fig. 7, we compare the achievable rate (computed as in (9)) of cube-split constellation with the rate of the numerically optimized constellation and the high-SNR capacity $C(\rho, N, T)$ given in (1) for $T = 2$ and single receive antenna. We also include the rate lower bound $R_{pilot}(\rho, N, T)$ of a pilot-based scheme with Gaussian input given in (55), and the achievable rate of the pilot-based scheme with QAM input. The cube-split constellation can achieve almost the same rate as the numerically optimized constellation and a higher rate than the pilot-based scheme with QAM input at a given SNR. For example, at 25 dB, the cube-split constellation can achieve about 0.3 bits/channel use higher than the rate achieved with the pilot-based scheme. Furthermore, the achievable rate of a large cube-split constellation approaches the high-SNR capacity $C(\rho, N, T)$.

Next, in Fig. 8, we plot the achievable rate of the cube-split constellation, the numerically optimized constellation, the Fourier constellation [17], and the exponential-mapped constellation [20], and the pilot-based scheme with QAM input for $T = 4$ and $N = 2$. Again, the rate achieved with cube-split constellation is close to the rate achieved with the numerically optimized constellation and higher than that of other structured constellations and the pilot-based scheme.
Fig. 7. The achievable rate of the cube-split constellation in comparison with the channel capacity given in [1], and the rate achieved with the numerically optimized constellation and the pilot-based scheme with Gaussian input or QAM input for $T = 2, N = 1$ and $B = 3, 5, 7, 9, 11, 13$ bits/symbol.

Fig. 8. The achievable rate of cube-split constellation in comparison with the numerically optimized constellation, other structured constellations, and the pilot-based scheme with QAM input for $T = 4, N = 2$, and $B \in \{8, 14\}$ bits/symbol.

C. Error rates of uncoded constellations

In Fig. 9, we plot the symbol error rate of the cube-split constellation, the pilot-based scheme, the numerically optimized constellation, the Fourier constellation [17], the exponential-mapped
constellation [20]. For the three latter constellations, we use ML decoder (7). We observe that
the greedy decoder for cube-split constellation achieves near-ML performance. The cube-split
constellation outperforms the other structured constellation and the pilot-based scheme. The
cube-split constellation (with Gray-like labeling as in Sec. [III-C] is also better than the pilot-based
scheme (with Gray labeling for the QAM symbols) in terms of bit error rate, as shown in Fig. [10]

(a) $T = 2, N = 1$

(b) $T = 4, N = 2$

Fig. 9. Symbol error rate of cube-split constellation in comparison with the numerically optimized constellation, other structured
constellations, and the pilot-based scheme with QAM input.

Fig. 10. Bit error rate of cube-split constellation vs. pilot-based scheme for $T = 2, N = 1$, and $B = 3, 5, 7, 9, 11$ bits/symbol.
D. Performance with channel coding

Next, we integrate a systematic parallel concatenated rate-$1/3$ standard turbo code \[^{33}\]. The turbo encoder is applied in each packet of 640 bits. The coded bits are mapped into symbols using the Gray-like labeling scheme described in Section \[^{III-C}\] The turbo decoder calculates the LLR of the received coded bits and performs 10 decoding iterations for each bit packet. For the pilot-based scheme, the LLR of bit $b_i$ given $Y$ and the channel estimate $\hat{h}$ is calculated as

$$ \text{LLR}^{\text{pilot}}_{i}(Y, \hat{h}) = \log \frac{\sum_{\alpha \in Q_i^{(1)}} p(Y|\hat{h}, x_{\{i\}} = \alpha)}{\sum_{\beta \in Q_i^{(0)}} p(Y|\hat{h}, x_{\{i\}} = \beta)} $$

where $Q_i^{(b)}$, $b \in \{0, 1\}$, denotes a subset of the chosen QAM constellation such that $b_i = b$, $x_{\{i\}}$ denotes the data symbol accounting for bit $b_i$, and $p(Y|\hat{h}, x_{\{i\}})$ is given in (57).\[^{5}\]

Fig. \[^{11}\] presents the bit error rate of the coded cube-split constellation in comparison with the coded pilot-based scheme for the same number of uncoded bit per symbol. As can be seen, the performance with the approximate LLR (computed as in (52) with $\eta = 5$) is close to the performance with exact LLR computation. In spite of the uneven LLR distribution across the bits, turbo code works well for the cube-split constellation and enhances its advantage over the pilot-based scheme. For the same transmission rate of 9 bits per symbol, the power gain of the cube-split constellation for the same bit error rate is about 2.5 dB.

VI. CONCLUSION AND SOME DISCUSSIONS

We proposed a novel Grassmannian constellation for non-coherent SIMO communications. The structure of this constellation allows for on-the-fly symbol generation, a simple yet effective binary labeling, and low-complexity symbol decoding and bit-wise LLR computation. Analytical and numerical results show that this constellation is close to optimality in terms of packing properties, has larger minimum distance than other structured constellations in the literature and outperforms the coherent pilot-based approach in terms of error rates with/without channel codes and achievable data rate under flat and block fading channel.

It is natural to extend the proposed scheme to the MIMO case. If the transmitter has $M \leq \frac{T}{2}$ antennas, we may consider constellation symbols belonging to the Grassmannian $G(\mathbb{C}^T, M)$

\[^{5}\]One can also compute the LLR based on the equalized symbols $\hat{x}_i$. When $N = 1$, the likelihood function $p(\hat{x}_i|x_i)$ for ZF equalized symbols can be derived explicitly using Lemma 3 as

$$ p(\hat{x}^f_i|x_i) = \frac{1}{\pi \left( \frac{1}{\mu \rho d (T-1)} + \frac{|\hat{x}^f_i|^2}{\rho} \right)^2} = \frac{1}{\pi \left( \frac{1}{\mu d (T-1)^2} + \frac{|x_i|^2}{\rho} \right)^2} = \left( \frac{1}{\pi \left( \frac{1}{\mu d (T-1)^2} + \frac{|x_i|^2}{\rho} \right)^2} \right)^{1/2}, i \in \{2, \ldots, T\}. $$
reproduced by $T \times M$ truncated unitary matrices. To extend the cube-split design, we would follow two essential steps: partitioning the Grassmannian into cells and defining a mapping from an Euclidean space onto one of the cells. To partition $G(\mathbb{C}^T, M)$, we consider a set of reference subspaces $E_1, \ldots, E_{N_s}$, for some $N_s$, defining an initial constellation in $G(\mathbb{C}^T, M)$ and its associated Voronoi cells

$$S_i := \{X \in G(\mathbb{C}^T, M) : d(E_i, X) \leq d(E_j, X), \forall j \neq i\}$$

$$= \{X \in G(\mathbb{C}^T, M) : \|E_i^t X\|_F \geq \|E_j^t X\|_F, \forall j \neq i\},$$

where $d(Q_1, Q_2) := \sqrt{M - \text{tr}\{Q_1^t Q_2 Q_2^t Q_1\}}$ is the chordal distance between the subspaces spanned by the columns of $T \times M$ truncated unitary matrices $Q_1$ and $Q_2$. The problem of choosing the initial constellation $E_1, \ldots, E_{N_s}$ and designing a mapping from an Euclidean space onto each cell $S_i$ preserving a property similar to Property 2 is not evident and left as perspective for future work. In particular, it seems difficult to describe each Voronoi region $S_i$ as it is the case for the regions of Grassmannian of lines where the condition $\|E_i^t X\|_F \geq \|E_j^t X\|_F$ expressed on the canonical basis simply translates into a coordinate-wise condition $|\frac{x_j}{x_i}| < 1$.  

---

**Fig. 11.** Bit error rate of cube-split constellation vs. pilot-based scheme with turbo codes for $T = 2, N = 1$, and $B = 3, 5, 7, 9$ bits/symbol.
Appendix

A. Complex Gaussian ratio distribution

Lemma 3. Let \( U_1 \sim \mathcal{CN}(0, \sigma_1^2) \) and \( U_2 \sim \mathcal{CN}(0, \sigma_2^2) \) be two complex Gaussian random variables with correlation coefficient \( \frac{\mathbb{E}[U_1U_2]}{\sigma_1\sigma_2} = \beta \). The random variable \( X = \frac{U_1}{U_2} \) follows the complex Gaussian ratio distribution, denoted by \( X \sim \mathcal{CR}(\sigma_1^2, \sigma_2^2, \beta\sigma_1\sigma_2) \), with the probability density function (PDF) [34]

\[
f_X(x) = \frac{\frac{\sigma_1^2}{\sigma_2^2}(1 - |\beta|^2)}{\pi\left(\frac{\sigma_1^2}{\sigma_2^2}(1 - |\beta|^2) + |x - \frac{\sigma_1^2}{\sigma_2^2}\beta|^2\right)^{\frac{3}{2}}}. \tag{62}
\]

B. Proof of Property 2

It is enough to show that with \( x \) uniformly distributed on the cell \( S_i^* \), the variable \( w_i \) defined in (20) follows a circularly symmetric Gaussian distribution. According to [35, Thm.5], \( t_i \) follows a complex Cauchy distribution, thus \( \frac{t_i}{|t_i|} \) and \( |t_i| \) are independent. It follows that \( \frac{w_i}{|w_i|} \) is independent of \( |w_i| \). Therefore, it suffices to prove that \( |w_i| = \sqrt{2\log\frac{1+|t_i|^2}{1-|t_i|^2}} \) is Rayleigh distributed.

Since \( |t_i|^2 \) may be seen as the quotient of two independent chi-squared random variables with two degrees of freedom, its distribution is a Fisher(2, 2) truncated on \( (0, 1) \) whose CDF is \( F : x \mapsto \frac{2x}{x+1} \). Therefore, denoting the quantile of the Rayleigh distribution as \( Q : t \mapsto \sqrt{2\log\frac{1}{1-t}} \), it holds that \( |w_i| = Q(F(|t_i|^2)) \). So \( |w_i| \) is indeed Rayleigh distributed.

C. Proof of Proposition 1

Recall that when \( B_0 = 1 \), the constellation symbols can be written simply as

\[
x = (c^{-1} + T - 1)^{-1/2} [q_1 \ldots q_{i^*} -1 c^{-1/2} q_{i^*+1} \ldots q_{T-1}]^T,
\]

where \( c = \frac{1-e^{-m^2}}{1+e^{-m^2}} \), \( m = \mathcal{N}^{-1}(3/4) \), and \( q_i = \pm \frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}} \). The received symbols are \( y_i^* = \sqrt{\rho_0} h + z_{i^*} \) and \( y_i = \sqrt{\rho_0} q_i h + z_i \) if \( i \neq i^* \), where \( h \sim \mathcal{CN}(0, 1) \), \( z_i \sim \mathcal{CN}(0, 1) \), \( i \in \{1, \ldots, T\} \), and \( \rho_0 = \frac{\rho T}{1+(T-1)c} \).
1) Cell error probability: A cell error occurs if there exists \( i \neq i^* \) such that \(|y_i| > |y_{i^*}|\). Given \( h \) and \( y_{i^*} \),

\[
\Pr\{\hat{i} \neq i^* \mid h, y_{i^*}\} = \Pr\{\exists i \neq i^*: |y_i|^2 > |y_{i^*}|^2 \mid h, y_{i^*}\} = 1 - \prod_{i \neq i^*} \Pr(|y_i|^2 \leq |y_{i^*}|^2 \mid h, y_{i^*})
\]

where \( Q_1(., .) \) is the Marcum Q-function with parameter 1. Here, (65) holds because conditioned on \( h \) and \( y_{i^*} \), the events \(|y_i| \leq |y_{i^*}|\) are mutually independent for all \( i \neq i^* \); (66) is because given \( h \), the variables \( 2|y_i|^2 \) for \( i \neq i^* \) are independently non-central chi-squared distributed with two degrees of freedom and non-centrality parameters \( 2c\rho_0|h|^2 \), denoted by \( \chi_2^2(2c\rho_0|h|^2) \).

Next, by averaging \( \Pr\{\hat{i} \neq i^* \mid h, y_{i^*}\} \) over \(|y_{i^*}|^2\) and \(|h|^2\), taking into account that \(|h|^2\) is exponentially distributed with mean 1, and given \( h \), \( 2|y_{i^*}|^2 \sim \chi_2^2(2\rho_0|h|^2) \), we get

\[
\Pr\{\hat{i} \neq i^*\} = 1 - \mathbb{E}_{|h|^2}\mathbb{E}_{|y_{i^*}|^2|h}\left[1 - (1 - Q_1(\sqrt{2c\rho_0|h|}, \sqrt{2|y_{i^*}|}))^T^{-1}\right]
= 1 - \int_0^\infty \int_0^\infty \left[1 - (1 - Q_1(\sqrt{2c\rho_0|h|}, \sqrt{2|y_{i^*}|}))^T^{-1}\right] \exp(-|y_{i^*}|^2 - (1 + \rho_0)|h|^2) \times I_0(2\sqrt{\rho_0}|y_{i^*}|^2| |h||d|y_{i^*}|^2 |d|h|^2),
\]

where \( I_0(.) \) is the modified Bessel function of the first kind of order 0. From this, a simple change of variables gives (41).

2) Coordinate error probability given correct cell detection: We assume that the cell index \( i^* \) has been correctly decoded and, without loss of generality, that \( i^* = T \). The decoding strategy for the coordinate bits is similar to a 4-QAM demapper on \( t = [t_1 \ldots t_{T-1}] = [\frac{y_1}{y_T} \ldots \frac{y_{T-1}}{y_T}]^T \).

Given \( q_i \), we have \( y_i = \sqrt{c\rho_0} q_i h + z_i \sim \mathcal{C}\mathcal{N}(0, 1 + c\rho_0) \) for \( i < T \), \( y_T = \sqrt{\rho_0} h + z_T \sim \mathcal{C}\mathcal{N}(0, 1 + \rho_0) \), and \( \mathbb{E}[y_i y_T^*] = \sqrt{c\rho_0}q_i \). Then, conditioned on \( q_i \), \( t_i \) follows the complex Gaussian ratio distribution \( \mathcal{C}\mathcal{R}(1 + c\rho_0, 1 + \rho_0, \sqrt{c\rho_0}q_i) \) truncated on the unit circle. It follows from Lemma 3 that the conditional pdf of \( t_i \) is

\[
f_{t_i|q_i}(t) = \frac{\hat{f}_{t_i|q_i}(t)dt}{\int_{|t|\leq 1} \hat{f}_{t_i|q_i}(t) dt},
\]

\(^9\)Without the additive noise \( z \), \(|y_{i^*}|^2 = \frac{1 + m^2}{1 + c - m^2} |y_i|^2 > |y_i|^2 \) for all \( i \neq i^* \), and therefore, there is no cell error.

\(^{10}\) \( \frac{y_T}{\sqrt{\rho_0}} \) can be seen as an estimate of \( q_i \) using imperfect channel estimate \( \hat{h} = \frac{y_T}{\sqrt{\rho_0}} \). Here, \( x_i \) plays the role of a pilot symbol whose position is exploited to carry \( \log_2(T) \) more bits.
where
\[
\hat{f}_{t_i|q_i}(t)dt := \frac{(1 + \rho_0)^2(1 + (c + 1)\rho_0)}{\pi \left(1 + (c + 1)\rho_0 + |(1 + \rho_0)t - \sqrt{c\rho_0}q_i|^2\right)^2}.
\] (70)

An error happens at \( t_i \) if \( \text{Re}(t_i) \text{Re}(q_i) < 0 \) or \( \text{Im}(t_i) \text{Im}(q_i) < 0 \)\(^{11}\). Therefore,
\[
\Pr\{\tilde{q}_i \neq q_i|q_i, \hat{i}^* = i^*\} = 1 - \frac{\int_{\mathcal{R}_i} \hat{f}_{t_i|q_i}(t)dt}{\int_{|t| \leq 1} \hat{f}_{t_i|q_i}(t)dt},
\] (71)
where \( \mathcal{R}_i := \{t \in \mathbb{C} : |t| \leq 1, \text{Re}(t) \text{Re}(q_i) > 0 \) and \( \text{Im}(t) \text{Im}(q_i) > 0 \}. \) Using the polar coordinate \( \{r, \theta\} \), we have that \( \hat{f}_{t_i|q_i}(t)dt = \tilde{f}(r, \theta, q_i)r \, dr \, d\theta \), where \( \tilde{f}(r, \theta, q_i) \) is obtained by replacing \( t \) by \( re^{\theta} \) in \( \hat{f}_{t_i|q_i}(t) \). Then
\[
\int_{\mathcal{R}_i} \hat{f}_{t_i|q_i}(t)dt = \int_0^1 \int_0^{2\pi} \tilde{f}(r, \theta, q_i)r \, dr \, d\theta,
\] (72)
where \( \Theta_i \) is \([0, \pi/2]\) if \( q_i = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \), \([\pi/2, \pi]\) if \( q_i = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \), \([\pi, 3\pi/2]\) if \( q_i = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \), and \([3\pi/2, 2\pi]\) if \( q_i = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \). After some manipulations, we get
\[
\int_0^1 \int_0^{2\pi} \tilde{f}(r, \theta, q_i)r \, dr \, d\theta = \frac{1}{2} + \frac{(1 - c)\rho_0}{2\sqrt{(2 + (1 + c)\rho_0)^2 - 4c\rho_0}},
\] (73)
and for all \( q_i \in \left\{\pm \frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}\right\},
\[
\int_0^1 \int_{\Theta_i} \tilde{f}(r, \theta, q_i)r \, dr \, d\theta = \frac{1}{8} + \frac{\sqrt{2c\rho_0} \arccot \frac{1 + (\sqrt{c} - \sqrt{3})\rho_0}{\sqrt{1 + (1 + c)\rho_0} + \sqrt{c\rho_0}^2}}{2\pi \sqrt{1 + (1 + c)\rho_0} + \sqrt{c\rho_0}^2} + \frac{(1 - c)\rho_0 \arccot \frac{2 + (1 - 2\sqrt{c} + c)\rho_0}{\sqrt{(2 + (1 + c)\rho_0)^2 - 4c\rho_0}}}{2\pi \sqrt{(2 + (1 + c)\rho_0)^2 - 4c\rho_0}}.
\] (74)

Plugging these into (72), we obtain (43). The union bound for \( P_e \) follows readily.

D. Proof of Corollary 7

The pdf \( f_{t_i|q_i}(t) \) of \( t = \frac{y_i - n}{y_i} \) conditioned on \( q_i \) is given in (69). The conditional cell error is simply \( \Pr\{\hat{i}^* \neq i^*|q_i\} = 1 - \Pr\{|t| < 1|q_i\} = 1 - \int_{|t| \leq 1} f_{t_i|q_i}(t)dt \). By calculating the integral using polar coordinate (which results in the same value for any \( q_i \in \left\{\pm \frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}\right\} \) and averaging over \( q_i \), we get (45). Furthermore, when \( T = 2 \), the union bound (44) of \( P_e \) is tight.

\(^{11}\)As for the cell error, there is no coordinate error in the absence of additive noise since in this case, \( t_i = \sqrt{c}q_i \).
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