An improved threshold for the number of distinct intersections of intersecting families

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Abstract

A family $F$ of subsets of $\{1, 2, \ldots, n\}$ is called a $t$-intersecting family if $|F \cap G| \geq t$ for any two members $F, G \in F$ and for some positive integer $t$. If $t = 1$, then we call the family $F$ to be intersecting. Define the set $I(F) = \{F \cap G: F, G \in F$ and $F \neq G\}$ to be the collection of all distinct intersections of $F$. Frankl et al. proved an upper bound for the size of $I(F)$ of intersecting families $F$ of $k$-subsets of $\{1, 2, \ldots, n\}$. Their theorem holds for integers $n \geq 50k^2$. In this article, we prove an upper bound for the size of $I(F)$ of $t$-intersecting families $F$, provided that $n$ exceeds a certain number $f(k,t)$. Along the way we also improve the threshold $k^2$ to $k^{3/2+o(1)}$ for the intersecting families.

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1. Introduction

We denote the standard $n$-element set $\{1, 2, \ldots, n\}$ by $[n]$, the set of all subsets of $[n]$ by $2^n$, and for $0 \leq k \leq n$ the collection of all $k$-element subsets of $[n]$ by $\binom{[n]}{k}$. We use the standard notation $|S|$ for the cardinality of a set $S$. We also use the following standard analytical notations. For non-negative functions $f, g$, we write $g = \mathcal{O}(h)$ to mean that $g(x) \leq c \cdot h(x)$ for some positive constant $c$; $g = o(h)$ to mean that $\frac{g(x)}{h(x)} \to 0$; $g = \Theta(h)$ to mean that $g = \mathcal{O}(h)$ and $h = \mathcal{O}(g)$.

We call a family $F \subset 2^n$ to be intersecting if the intersection of any two members of $F$ is non-empty, and we call $F$ to be $t$-intersecting if $|F \cap G| \geq t$ for any two members $F, G \in F$ and for some positive integer $t$. We call a family to be the complete sunflower if every subset which contain a fix $t$-set (say) $X$ is a member of that family; we denote such a family by $S_X$. Any subset of $S_X$ is called Sunflower.
The Erdős-Ko-Rado theorem is a pioneer result in extremal combinatorics. 

**Theorem 1** (Erdős-Ko-Rado theorem). There exists some $n_0(k, t)$ such that if $n \geq n_0(k, t)$ and $\mathcal{F} \subset \binom{[n]}{k}$ is $t$-intersecting, then

$$|\mathcal{F}| \leq \binom{n - t}{k - t}.$$ 

Frankl proved that the Erdős-Ko-Rado theorem holds for $n_0(k, t) = (t + 1)(k - t + 1)$ and $t \geq 15$. Wilson proved the theorem holds with same $n_0(k, t)$ and for all $t$. The bound in the Erdős-Ko-Rado theorem is achieved for the complete sunflower $\mathcal{F}$.

Let $\mathcal{I}(\mathcal{F}) = \{F \cap G : F, G \in \mathcal{F} \text{ and } F \neq G\}$ be the collection of all distinct intersections of $\mathcal{F}$. In Frankl, Kiselev, and Kupavskii proved the following theorem.

**Theorem 2** (Frankl, Kiselev, and Kupavskii). Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting with $k \geq 2$ and $n \geq 50k^2$. Then

$$|\mathcal{I}(\mathcal{F})| \leq |\mathcal{I}(\mathcal{A})|,$$

where

$$\mathcal{A} = \left\{ A \in \binom{[n]}{k} : |A \cap \{1, 2, 3\}| \geq 2 \right\}.$$

In this paper, we prove that a similar result also holds for $t$-intersecting families $\mathcal{F}$ of $\binom{[n]}{k}$. More precisely, we prove the following theorem.

**Theorem 3.** There exists some number $f(k, t)$ depending on $k, t$ such that if $n \geq f(k, t)$ and $\mathcal{F} \subset \binom{[n]}{k}$ is $t$-intersecting, then

$$|\mathcal{I}(\mathcal{F})| \leq |\mathcal{I}(\mathcal{A}_t)|,$$

where

$$\mathcal{A}_t = \left\{ A \in \binom{[n]}{k} : |A \cap \{1, 2, \ldots, t + 2\}| \geq t + 1 \right\},$$

and $f(k, t)$ satisfies the following property:

1. if $t = \mathcal{O}(1)$, then $f(k, t) = \Theta(k^{3/2 + \epsilon})$, where $\epsilon = \frac{10 + t}{2(k - t - 2)}$,

2. if $t = o(k)$, then
   
   (a) if $t = \Theta(k^{\epsilon'})$ with $\epsilon' \leq 1/4$, then $f(k, t) = \Theta(k^{3/2 + \epsilon'})$,
   (b) if $t = \Theta(k^{\epsilon'})$ with $\epsilon' > 1/4$, then $f(k, t) = \Theta(k^{1 + 2\epsilon'})$,

3. if $t = \Theta(k)$, then $f(k, t) = \Theta(k^3)$. 

2. Preliminaries

We need certain notions which were introduced in [3]; apart from that we also define some analogous notions for $t$-intersecting families.

For a family $F$, we call

$$\mathcal{T}(F) := \{ T \in 2^{[n]} : |T| \leq k, |T \cap F| \geq t \text{ for all } F \in \mathcal{F} \}$$

to be the family of $t$-transversals. Then $F \subset \mathcal{T}(F)$ if and only if $F$ is $t$-intersecting. For a non-negative integer $\ell \leq n$ we call $\mathcal{F}^{(\ell)} = \{ F \in \mathcal{F} : |F| = \ell \}$ the $\ell$-th level of the family $\mathcal{F}$, and set $\mathcal{F}^{(\leq \ell)} = \bigcup_{i=1}^{\ell} \mathcal{F}^{(i)}$. We call a $t$-intersecting family $F$ to be saturated if the property that $|F \cap G| \geq t$ ceases to hold for addition of any new member to $F$. It is easy to observe that, an $t$-intersecting family $F \subset \binom{[n]}{k}$ is saturated if and only if $F = \mathcal{T}(F)^{(k)}$. So, in the rest of this paper we assume that $F$ is $t$-intersecting and saturated. A family $B$ is an antichain if for any $B, B' \in B$ with $B \subset B'$, then $B = B'$.

To prove Theorem 3 we need certain lemmas, which are just the $t$-intersecting analogue of the lemmas proved in [3] (see Lemma 1.3, Lemma 2.2, and Lemma 2.3 of [3]). All the lemmas in this section can be proved by exploiting the following fact: in a $t$-intersecting family $|F \cap G| < t$ works as disjoint sets $F, G$ of an intersecting family. So, we leave the details of the proof to the reader.

**Lemma 1.** Let $F \subset \binom{[n]}{k}$ be a saturated $t$-intersecting family. Let $B = B(F)$ be the family of minimal sets in $\mathcal{T}(F)$. Then

1. $B$ is a $t$-intersecting antichain,
2. $F = \left\{ D \in \binom{[n]}{k} : \exists B \in B, B \subset D \right\}$,
3. $B$ contains no sunflower of size $k+1$.

**Lemma 2.** Let $F \subset \binom{[n]}{2}$ be a $t$-intersecting family. Then $F$ is either a sunflower or a $(t+2)$-triangle of the form $\{1, 2, \ldots, t+1\}, \{2, 3, \ldots, t+2\}, \{1, t+2\} \cup D : D \subset \{2, 3, \ldots, t+1\}, |D| = t-1\}$.

To state the following lemma we need some further notions. We denote $s = s(B)$ for $\min\{|B| : B \in B\}$ and the $t$-covering number $\tau(B)$ for $\min\{|T| : |T \cap B| \geq t \text{ for all } B \in B\}$.

**Lemma 3.** Let $\ell$ be an integer such that $t + 1 \leq \ell \leq k$. Suppose that $F \subset \binom{[n]}{k}$ is a saturated $t$-intersecting family. Assume that $B = B(F), s \geq t + 1$, and $\tau(B^{(\leq \ell)}) \geq t + 1$. Then

$$|B^{(\ell)}| \leq s \cdot \ell \cdot (k-t+1)^{\ell-t-1}. \quad (1)$$
3. The proof of Theorem \[3\]

We begin with estimating the exact size of \( \mathcal{I}(A_t) \).

**Proposition 1.** We have

\[
|\mathcal{I}(A_t)| = \binom{t+2}{t} k^{t-1} \left(\binom{n-t-2}{j}\right) + \binom{t+2}{t+1} k^{t-2} \left(\binom{n-t-2}{j}\right) + \sum_{j=0}^{k-t-3} \left(\binom{n-t-2}{j}\right).
\]

**Proof.** Let \( A, A' \in \mathcal{A}_t \). As \( |A \cap A' \cap \{1, 2, \ldots, t+2\}| \geq t \), one has total \( \binom{t+2}{t} + \binom{t+2}{t+1} \) possibilities for \( |A \cap A' \cap \{1, 2, \ldots, t+2\}|. \) Consider one of such possibilities \( A \cap A' \cap \{1, 2, \ldots, t+2\} = \{1, 2, \ldots, t\} \). Then both \( A \cap \{1, 2, \ldots, t+2\} \) and \( A' \cap \{1, 2, \ldots, t+2\} \) are of the form \( \{1, 2, \ldots, t+2\} \setminus \{x\} \) for some \( x \in [t+2] \). As \( n \) large enough \( A \cap A' = \{1, 2, \ldots, t\} \cup B \) for some \( B \subset \{t+3, t+4, \ldots, n\} \). Further, as \( A \neq A' \) we must have \( |B| \leq k - t - 1 \). Thus, in this particular case there are total \( \sum_{i=0}^{k-t-1} \left(\binom{n-t-2}{i}\right) \) possible values for \( |A \cap A'| \). Considering all the possibilities for \( A \cap A' \cap \{1, 2, \ldots, t+2\} \) we obtain

\[
|\mathcal{I}(A_t)| = \binom{t+2}{t} k^{t-1} \left(\binom{n-t-2}{j}\right) + \binom{t+2}{t+1} k^{t-2} \left(\binom{n-t-2}{j}\right) + \sum_{j=0}^{k-t-3} \left(\binom{n-t-2}{j}\right).
\]

\[\square\]

**Proof of Theorem \[3\]** Let \( k = t + 1 \). Then by Lemma \[2\] the family \( \mathcal{F} \) is either a \((t+2)\)-triangle or a sunflower. In both the cases the theorem is trivial. Thus, we may assume that \( k \geq t + 2 \).

For a \( t \)-element subset \( X \) of \([n]\), let \( S_X \) be the complete sunflower. Then

\[
|\mathcal{I}(S_X)| = \sum_{j=0}^{k-t-1} \left(\binom{n-t}{j}\right).
\]

\[
= 2 \sum_{j=0}^{k-t-1} \left(\binom{n-t-1}{j}\right) - \left(\binom{n-t-1}{k-t-1}\right)
\]

\[
= 4 \sum_{j=0}^{k-t-1} \left(\binom{n-t-2}{j}\right) - 2 \left(\binom{n-t-2}{k-t-1}\right) - \left(\binom{n-t-1}{k-t-1}\right)
\]

\[
= 2 \left(\binom{n-t-2}{k-t-1}\right) + 4 \sum_{j=0}^{k-t-2} \left(\binom{n-t-2}{j}\right) - \left(\binom{n-t-1}{k-t-1}\right). \tag{3}
\]
Comparing Eq. (2) and Eq. (3) one can see that $|\mathcal{I}(S_X)| < |\mathcal{I}(A_t)|$. Thus, we may suppose that $\mathcal{F}$ is not a complete sunflower. This implies $B^{(t)} = \emptyset$.

Now, we partition $\mathcal{F}$ as $\mathcal{F} = \mathcal{F}^{(s)} \cup \cdots \cup \mathcal{F}^{(k)}$, where $s = s(\mathcal{B}(\mathcal{F}))$ and the subfamilies $\mathcal{F}^{(i)}$’s are defined as follows: $F \in \mathcal{F}^{(i)}$ if $\ell = \max\{|B| : B \in \mathcal{B}, B \subset F\}$. Set $\mathcal{I}^{(i)} = \{F \cap G : F \in \mathcal{F}^{(i)}, G \in \mathcal{F}^{(s)} \cup \cdots \cup \mathcal{F}^{(k)}\}$. Then

$$|\mathcal{I}(\mathcal{F})| \leq \sum_{\ell=s}^{k} |\mathcal{I}^{(i)}|$$

To calculate $|\mathcal{I}^{(i)}|$ we recall that, for every $F \in \mathcal{F}^{(i)}$ there exist $B \in \mathcal{B}^{(i)}$ such that $B \subset F$. Then for any $F' \in \mathcal{F}$, we have

$$F \cap F' = (B \cap F') \cup ((F \setminus B) \cap F').$$

Here there are at most $\sum_{i=t}^{\ell} \binom{\ell}{i}$ possibilities for $B \cap F'$ and $(F \setminus B) \cap F'$ can be any subset of $[n]$ of size $k - \ell$. Thus, using Eq. (1) we get

$$|\mathcal{I}^{(i)}| \leq |\mathcal{B}^{(i)}| \sum_{i=t}^{\ell} \binom{\ell}{i} \sum_{j=0}^{\ell} \binom{n}{j} < s\ell(k - t + 1)^{\ell-t-1} \sum_{i=t}^{\ell} \binom{\ell}{i} \sum_{j=0}^{\ell} \binom{n}{j}. \quad (4)$$

If $\tau(\mathcal{B}^{(t+1)}) = t + 1$, then by Lemma 2 we obtain that $\mathcal{F}$ is a triangle. In this case, Theorem 3 is trivial. Let $\alpha$ be the smallest integer such that $\tau(\mathcal{B}^{(\leq \alpha)}) \geq t + 1$. Thus $\alpha \geq t + 2$. This implies that the family $\bigcup_{i=1}^{\alpha-1} \mathcal{F}^{(i)}$ is a sunflower and hence

$$|\bigcup_{i=1}^{\alpha-1} \mathcal{I}^{(i)}| \leq |\mathcal{I}(S_X)| = 2 \binom{n-t-2}{k-t-1} + 4 \sum_{j=0}^{k-t-2} \binom{n-t-2}{j} - \binom{n-t-1}{k-t-1}. \quad (5)$$

On the other hand, by Eq. (4) we have

$$|\bigcup_{i=1}^{\alpha-1} \mathcal{I}^{(i)}| \leq \sum_{\ell=\alpha}^{k} |\mathcal{I}^{(i)}| < \sum_{\ell=\alpha}^{k} \ell^2(k - t + 1)^{\ell-t-1} \sum_{i=t}^{\ell} \binom{\ell}{i} \sum_{j=0}^{\ell} \binom{n}{j} \leq \sum_{\ell=t+2}^{k} \ell^2(k - t + 1)^{\ell-t-1} \sum_{i=t}^{\ell} \binom{\ell}{i} \sum_{j=0}^{\ell} \binom{n}{j}. \quad (6)$$
Adding Eq. (5) and Eq. (6) we obtain
\[ |T(F)| < 2 \binom{n-t-2}{k-t-1} + 4 \sum_{j=0}^{k-t-2} \binom{n-t-2}{j} - \binom{n-t-1}{k-t-1} + \sum_{\ell=t+2}^{k} \ell^2 (k-t+1)^{\ell-t-1} \sum_{i=t}^{\ell} \binom{\ell}{i} \sum_{j=0}^{\ell-\ell} \binom{n}{j}. \tag{7} \]

We claim that there exist some \( f(k, t) \) such that for \( n \geq f(k, t) \) we have
\[ 2 \binom{n-t-2}{k-t-1} + 4 \sum_{j=0}^{k-t-2} \binom{n-t-2}{j} - \binom{n-t-1}{k-t-1} \]
\[ + \sum_{\ell=t+2}^{k} \ell^2 (k-t+1)^{\ell-t-1} \sum_{i=t}^{\ell} \binom{\ell}{i} \sum_{j=0}^{\ell-\ell} \binom{n}{j} \]
\[ \leq \binom{t+2}{t} \binom{n-t-2}{k-t-1} + \binom{t+3}{t+1} \sum_{j=0}^{k-t-2} \binom{n-t-2}{j} + \sum_{j=0}^{k-t-3} \binom{n-t-2}{j} \]
\[ = \binom{t+2}{t} \sum_{j=0}^{k-t-1} \binom{n-t-2}{j} + \binom{t+2}{t+1} \sum_{j=0}^{k-t-2} \binom{n-t-2}{j} + \sum_{j=0}^{k-t-3} \binom{n-t-2}{j} \]
\[ \leq \binom{n-t-1}{k-t-1} + \frac{(t+2)(t+1)-4}{2} \cdot \binom{n-t-2}{k-t-1} + \frac{(t+3)(t+2)-8}{2} \sum_{j=0}^{k-t-2} \binom{n-t-2}{j} + \sum_{j=0}^{k-t-3} \binom{n-t-2}{j}. \tag{8} \]

Adjusting similar terms equivalently we claim that, for sufficiently large \( n \) we have
\[ \sum_{\ell=t+2}^{k} \ell^2 (k-t+1)^{\ell-t-1} \sum_{i=t}^{\ell} \binom{\ell}{i} \sum_{j=0}^{\ell-\ell} \binom{n}{j} \]
\[ \leq \frac{(t+2)(t+1)-4}{2} \cdot \binom{n-t-2}{k-t-1} + \frac{(t+3)(t+2)-8}{2} \sum_{j=0}^{k-t-2} \binom{n-t-2}{j} + \sum_{j=0}^{k-t-3} \binom{n-t-2}{j}. \tag{8} \]

We calculate the order of \( n \) (if written in the form of power of \( k \)) for which this inequality holds.

Suppose that \( n = \Theta(k^s) \) for some real number \( s \). Then the growth of \( \sum_{j=0}^{k-\ell} \binom{n}{j} \) is \( k^{s(k-t-2)} \) which we obtain by putting \( \ell = t+2 \) in \( \sum_{j=0}^{k-\ell} \binom{n}{j} \). On the other hand, the growth of \( \sum_{i=t}^{\ell} \binom{\ell}{i} \sum_{j=0}^{\ell-\ell} \binom{n}{j} \) is \( k^{3k/2-t+2} \) which we obtain by putting \( \ell = k \) in \( \sum_{i=t}^{\ell} \binom{\ell}{i} \sum_{j=0}^{\ell-\ell} \binom{n}{j} \). If \( \epsilon \geq \frac{10+t}{2(k-t-2)} \), then \( \frac{3}{2} + \epsilon \geq \frac{3k/2-t+2}{k-t-2} \).

Therefore, for any real \( s \geq \frac{3}{2} + \epsilon \) with \( \epsilon \geq \frac{10+t}{2(k-t-2)} \), the growth of \( \sum_{j=0}^{k-\ell} \binom{n}{j} \) is higher than \( \sum_{i=t}^{\ell} \binom{\ell}{i} \sum_{j=0}^{\ell-\ell} \binom{n}{j} \).

Now, comparing the growths of both sides of Eq. (8) we obtain
\[ O(t^4) \cdot k \cdot \Theta(n^{k-t-2}) \leq O(t^2) \cdot \Theta(n^{k-t-1}). \tag{9} \]
Case 1. If \( t = O(1) \), then the LHS of Eq. (9) is \( k \cdot \Theta(n^{k-t-2}) \), where as the RHS of Eq. (9) is \( \Theta(n^{k-t-1}) \). Thus by setting \( n = \Theta(k^{s'}) \) for some \( s' \), we see that LHS has growth \( k^{s'(k-t-2)+1} \) and RHS has growth \( k^{s'(k-t-1)} \). Thus, Eq. (9) holds for \( n \geq k^{\max\{3/2+\epsilon,1\}} = k^{3/2+\epsilon} \).

Case 2. If \( t = \Theta(k) \), then the LHS of Eq. (9) has growth \( \Theta(k^5) \cdot \Theta(n^{k-t-2}) \), where as the RHS of Eq. (9) has growth \( \Theta(k^3) \cdot \Theta(n^{k-t-1}) \). Thus by setting \( n = \Theta(k^{s'}) \) for some \( s' \), we see that LHS has growth \( k^{s'(k-t-2)+5} \) and RHS has growth \( k^{s'(k-t-1)+2} \). Thus, Eq. (9) holds for \( n \geq k^{\max\{3/2+\epsilon,3\}} \). As \( \epsilon \geq \frac{10+t}{2(k-t-2)} \), for \( t < \frac{3}{4}k - 4 \) we have \( \epsilon < \frac{3}{2} \), which further implies \( \frac{3}{2} + \epsilon < 3 \). Hence, Eq. (9) holds for \( n \geq c \cdot k^3 \).

Case 3. If \( t = o(k) \), then we have two cases.

Subcase 1. Let \( t = \Theta(k^{\epsilon'}) \) for some \( \epsilon' \) with \( \epsilon' \leq \frac{1}{4} \). Then \( 1 + 2\epsilon' \leq \frac{3}{2} + \epsilon \) as \( \epsilon > 0 \). In that case Eq. (9) is equivalent to \( O(t^2) \cdot O(k) = O(k^{1+2\epsilon'}) \leq \Theta(n) \), which holds when \( n = \Theta(k^{s'}) \) with \( s' \geq \max\{\frac{3}{2} + \epsilon, 1 + 2\epsilon'\} = \frac{3}{2} + \epsilon \).

Subcase 2. Let \( t = \Theta(k^{\epsilon'}) \) for some \( \epsilon' \) with \( \epsilon' = \frac{1}{4} + o(1) \). Then for sufficiently large \( k \) we have \( 1 + 2\epsilon' \geq \frac{3}{2} + \epsilon \) as \( \epsilon' \geq \frac{1}{4} + \frac{1}{2} \cdot \frac{10+\Theta(k^{\epsilon'})}{2(k-\Theta(k^{\epsilon'}))} \). In that case Eq. (9) is equivalent to \( O(t^2) \cdot O(k) = O(k^{1+2\epsilon'}) \leq \Theta(n) \), which holds when \( n = \Theta(k^{s'}) \) with \( s' \geq \max\{\frac{3}{2} + \epsilon, 1 + 2\epsilon'\} = 1 + 2\epsilon' \).

This completes the proof of the theorem.

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