ON THE DIMENSION DISTORTIONS OF QUASI-SYMMETRIC HOMEOMORPHISMS

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Abstract. In this paper, we first generalize a result of Bishop and Steger [Representation theoretic rigidity in PSL(2, R). Acta Math., 170, (1993), 121-149] by proving that for a Fuchsian group $G$ of divergence type and non-lattice, if $h$ is a quasi-symmetric homeomorphism of the real axis $\mathbb{R}$ corresponding to a quasi-conformal compact deformation of $G$. Then for any $E \subset \mathbb{R}$, we have $\max(\dim E, \dim (\mathbb{R} \setminus E)) = 1$. Furthermore, we showed that Bishop and Steger’s result does not hold for the covering groups of all $d$-dimensional jungle gym (d is any positive integer) which generalizes Gönye’s results [Differentiability of quasi-conformal maps on the jungle gym. Trans. Amer. Math. Soc. Vol 359 (2007), 9-32] where the author discussed the case of '1-dimensional jungle gym'.

1. Introduction

Let $G$ be a non-elementary torsion free discrete Möbius transformations group acting on $\mathbb{R}^n = \mathbb{R}^n \cup \infty$ or $S^n = \partial \mathbb{B}^n$; the action of $G$ can extend to the $(n + 1)$-dimension hyperbolic upper half hyperplane $\mathbb{H}^{n+1} = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ or the hyperbolic unit ball $\mathbb{B}^{n+1}$. A discrete group $G$ is called a Kleinian group if $n = 2$ and Fuchsian group if $n = 1$. In this paper, we mainly focus our attention on Fuchsian groups.

Let $\Lambda(G)$ be the accumulation set of any orbit. A Fuchsian group $G$ is said to be of the first kind if the limit set $\Lambda(G)$ is the entire circle. Otherwise, it is of the second kind. A point $x \in \mathbb{R}$ is a conical point or radial point of $G$ if there is a sequence of elements $g_i \in G$ such that for any $z \in \mathbb{H}$, there exists a constant $C$ and a hyperbolic line $L$ with endpoint $x$ such that the hyperbolic distance between $g_i(z)$ and $L$ are bounded by $C$. Denote by $\Lambda_c(G)$ the set of all the conical limit points and $\Lambda_e(G)$ the set of all the escaping limit points. Let $S = \mathbb{H}/G$ be the corresponding surface of $G$. The points in $\Lambda_c(G)$ are just corresponding to the geodesics in $S$ which return to some compact set infinitely often and the points in $\Lambda_e(G)$ are corresponding to the geodesics in $S$ which eventually leave every compact subset of $S$.

For $g$ in $G$, we denote by $\mathcal{D}_z(g)$ the closed hyperbolic half-plane containing $z$, bounded by the perpendicular bisector of the segment $[z, g(z)]_h$. The Dirichlet fundamental domain $\mathcal{F}_z(G)$ of $G$ centered at $z$ is the intersection of all the sets $\mathcal{D}_z(g)$ with $g$ in $G – \{id\}$. For simplicity, in this paper we use the notation $\mathcal{F}$ for
the Dirichlet fundamental domain \( F_z(G) \) of \( G \) centered at \( z = 0 \). A Fuchsian group \( \Gamma \) is called a lattice if the area of its one Dirichlet fundamental domain is finite. Moreover, a lattice is said to be uniform if each of its Dirichlet domain is compact, for more details, see [8].

A Fuchsian group \( G \) is said to be of divergence type if \( \sum_{g \in G} (1 - \|g(0)\|) = \infty \). Otherwise, we say it is of convergence type. All the second kind groups are of convergence type but the converse is not true.

We call \( F \) a quasi-conformal deformation of \( G \) if it is a quasi-conformal homeomorphism of the upper half plane \( \mathbb{H} \) such that

\[
G' = \{ g' : g' = F \circ g \circ F^{-1} \text{ for every } g \in G \}
\]

is also a Fuchsian group and a compact quasi-conformal deformation of \( G \) if it is just a lifted mapping of a quasi-conformal mapping \( f \) defined on the surface \( \mathbb{H}/G \) whose Beltrami coefficients is supported on a compact subset of \( \mathbb{H}/G \). Such a \( F \) will extend unique to a homeomorphism of the real axis \( \mathbb{R} \), denoted by \( h \). The homeomorphism \( h \) is a quasi-symmetric mapping of \( \mathbb{R} \).

The quasi-symmetric mappings can be very singular in the measure theoretic sense. It is known that quasi-conformal mapping preserve the null-sets. However, the quasi-symmetric mappings may be very singular, which will not preserve null-sets, see [4].

In [13], Tukia showed that, for the unit interval \( I = [0, 1] \), there are a quasi-symmetric self mapping of \( I \) and a set \( E \subset I \) such that the Hausdorff dimensions of both \( I \setminus E \) and \( f(E) \) are less than 1. In [5], Bishop and Steger got the following result: for a lattice group \( G \) (i.e. \( G \) is finitely generated of first kind), there is a set \( E \subset \mathbb{R} \) such that the hausdorff dimensions of both \( E \) and \( h(\mathbb{R} \setminus E) \) are less than 1, where \( h \) is a quasi-symmetric conjugating homeomorphism of the real axis \( \mathbb{R} \).

Concerning the negative results we first give the definition of the ‘\( d \)-dimensional jungle gym’. Let \( S_0 \) be a compact surface of genus \( d \) and \( G_0 \) its covering group. Let \( N_0 \) be a normal subgroup of \( G_0 \) such that \( G_0/N_0 \) is isomorphic to \( \mathbb{Z}^d \). The surface \( S^* = \mathbb{H}/N_0 \) is the so called infinite ‘\( d \)-dimensional jungle gym’, that is, \( S^* = \mathbb{H}/N_0 \) can be quasi-isometrically embedded into \( \mathbb{R}^d \) as a surface \( S \) which is invariant under translations \( t_j, 1 \leq j \leq d \), in \( d \) orthogonal directions. Moreover \( S_0 \) is conformal equivalent to \( S/(< t_1, \ldots, t_d >) \).

In [10], Gönye showed that Tukia-Bishop-Steger’s results do not hold for the covering group of ‘1-dimensional jungle gym.’ Gönye constructed a conjugating map \( f \) between covering groups of two ‘1-dimensional jungle gym’ with the Beltrami coefficient being compactly supported, for which

\[
\max(dim(E), dim(f(\mathbb{R} \setminus E))) = 1
\]

for all \( E \subset \mathbb{R} \).
In this paper we continue to investigate the range of validity of Tukia-Bishop-Steger results. We first show the following result which is essentially due to Fernandez and Melian [9].

**Theorem 1.1.** Suppose $G$ is a non-lattice divergence type Fuchsian group. Then $\Lambda_e(G)$ has zero 1-dimensional Hausdorff measure, but its Hausdorff dimension is 1.

In [4], Bishop showed that the divergence Fuchsian groups have Mostow rigidity property, so if $h$ is any quasi-symmetric homeomorphism which conjugates a divergence Fuchsian group to another one, then $h$ is singular, i.e. $h$ is continuous but the derivation of $h$ vanishes almost everywhere in the real axis $\mathbb{R}$. For the quasi-symmetric homeomorphisms corresponding to a compact deformation of a divergence Fuchsian group, we have

**Theorem 1.2.** Let $G$ be a Fuchsian group of divergence type and non-lattice, and $h$ be a homeomorphism of the real axis $\mathbb{R}$ corresponding to a compact deformation of $G$. Then for any $E \subset \mathbb{R}$, we have

$$\max(dim(E), dim h(\mathbb{R} \setminus E)) = 1.$$  

Combine with Bishop and Steger’s result [5], we have

**Theorem 1.3.** Let $G$ be a Fuchsian group and $h$ a quasi-symmetric homeomorphism of the real axis $\mathbb{R}$ corresponding to a compact deformation of $G$. Then there exists a subset $E \subset \mathbb{R}$, such that

$$\max(dim(E), dim h(\mathbb{R} \setminus E)) < 1 \quad (1.1)$$

if and only if $G$ is a lattice.

Concerning the 'jungle gym', by Theorem 1.3, we can generalize Gönyle’s result to 'd-dimensional jungle gym', where $d$ is any positive integer number.

**Corollary 1.4.** For any positive integer number $d$, suppose $G$ be a covering group of a 'd-dimensional jungle gym’ and $h$ a homeomorphism of the real axis $\mathbb{R}$ corresponding to a compact deformation of $G$. Then for any $E \subset \mathbb{R}$, we have

$$\max(dim(E), dim h(\mathbb{R} \setminus E)) = 1. \quad (1.2)$$

**Remark:** By [2] we know that when $d = 1$ or 2; the covering groups of 'd-dimensional jungle gyms' are of the divergence type and when $d \geq 3$, the covering group of 'd-dimensional jungle gyms' are of the convergence type.

The remainder of the paper is organized as follows: In section 2 we recall some definitions. In section 3, we give some results about differentiability of Quasi-conformal mappings at escaping limit points. In section 4, we prove Theorem 1.1. In section 5, we prove Theorem 1.2 and in section 6, we prove Theorem 1.3.
2. Preliminaries

Before giving the proofs of the above results, we first recall some definitions.

2.1 Quasi-conformal mapping. Let $\mathbb{H}$ be the upper half-plane in the complex plane $\mathbb{C}$. We denote by $M(\mathbb{H})$ the unit sphere of the space $L^\infty(\mathbb{H})$ of all essentially bounded Lebesgue measurable functions in $\mathbb{H}$. For a given $\mu \in M(\mathbb{H})$, there exists a unique quasiconformal self-mapping $f^\mu$ of $\mathbb{H}$ fixing 0, 1 and $\infty$, and satisfying the following equation

$$\frac{\partial}{\partial \bar{z}} f^\mu(z) = \mu(z) \frac{\partial}{\partial z} f^\mu(z), \text{ a.e. } z \in \mathbb{H}.$$  

We call $\mu$ the Beltrami coefficient of $f^\mu$. It is well known that $f^\mu$ can be extended continuously to the real axis $\mathbb{R}$ such that $f^\mu$ restricted to $\mathbb{R}$ is a quasisymmetric homeomorphism.

Similarly, there exists a unique quasiconformal homeomorphism $f^\mu$ of the plane $\mathbb{C}$ which is holomorphic in the lower half-plane, fixing 0, 1 and $\infty$ and satisfying

$$\frac{\partial}{\partial \bar{z}} f^\mu(z) = \mu(z) \frac{\partial}{\partial z} f^\mu(z), \text{ a.e. } z \in \mathbb{H}.$$  

2.2 Poincaré exponent. The critical exponent (or Poincaré exponent) of a Fuchsian group $G$ is defined as

$$\delta(G) = \inf \{ t \in [0,\infty) : \sum_{g \in G} \exp(-t \rho(0, g(0))) < \infty \}$$

(2.1)

$$= \inf \{ t \in [0,\infty) : \sum_{g \in G} (1 - |g(0)|)^t < +\infty \},$$

(2.2)

where $\rho$ denotes the hyperbolic metric. It has been proven in [6] that for any non-elementary group $G$, $\delta(G)$ is equals to $\dim(\Lambda_c(G))$, the Hausdorff dimension of the conical limit set.

2.2 Hausdorff dimension. Let $E$ be a subset of the complex plane $\mathbb{C}$. Suppose $\varphi$ is an nonnegative increasing homeomorphism of $[0,\infty)$. For $\varphi$ and $0 < \delta \leq \infty$, we define

$$\mathcal{H}^\delta_\varphi(E) = \inf \{ \sum_{i=1}^\infty \varphi(|B_i|) : E \subset \bigcup_{i=1}^\infty B_i, |B_i| \leq \delta \},$$

where $B_i \subset \mathbb{C}$ is a set and $|B_i|$ denotes its diameter, the infimum is taken over all open coverings of $E$. Then the Hausdorff measure of $E$ to be

$$\mathcal{H}^\delta(E) = \lim_{\delta \to 0} \mathcal{H}^\delta_\varphi(E) = \sup_{\delta > 0} \mathcal{H}^\delta_\varphi(E)$$

and the Hausdorff content of $E$ is $\mathcal{H}_\infty^\delta(E)$.

If $\varphi(t) = t^\alpha$, $\alpha \in [0,2]$, we denote $\mathcal{H}^\delta(E)$ by $\mathcal{H}^\alpha(E)$. Then one defines the $\alpha$-dimensional Hausdorff measure of $E$ to be

$$\mathcal{H}^\alpha(E) = \lim_{\delta \to 0} \mathcal{H}^\alpha_\varphi(E) = \sup_{\delta > 0} \mathcal{H}^\alpha_\varphi(E).$$

One defines the Hausdorff dimension of $E$ to be
\[ \dim E = \inf \{ \alpha : \mathcal{H}^\alpha (E) = 0 \} . \]

3. Differentiability of Quasi-conformal mappings at escaping limit points revisited

It is well known that a quasi-conformal mapping of a domain \( \Omega \) is differentiable almost everywhere in \( \Omega \). In this paper we need the following criterion for pointwise conformality due to Lehto, see [11] or ([12], Theorem 6.1).

Lemma 3.1. Let \( \Omega \) and \( \Omega' \) be two domains in the complex plane \( \mathbb{C} \), and let \( f \) be a quasi-conformal mapping from \( \Omega \) to \( \Omega' \) with Beltrami coefficient \( \mu(z) \), where \( |\mu(z)| \leq k < 1 \) almost everywhere in \( \Omega \). If \( f \) satisfies

\[
\frac{1}{2\pi} \iint_{|z|<r} \frac{\mu(z)}{|z|^2} \, dx \, dy < \infty
\]

for some \( r > 0 \), then \( f \) is conformal at \( z = 0 \).

Suppose \( G \) be a non-lattice Fuchsian group of divergence type. Let \( f \) be a quasi-conformal mapping on the surface \( S = \mathbb{H}/G \) whose Beltrami coefficient \( \mu \) is supported on a compact subset of \( S \). Thus we can choose a point \( z_0 \in S \) and a sufficiently large \( r_0 \) such that the support set of \( \mu \) is contained in the disk \( B(z_0, r_0) \). Let \( S_{r_0} = S \setminus B(z_0, r_0) \) and let \( \Omega_{r_0} \) be the lift of \( S_{r_0} \) to the upper plane \( \mathbb{H} \). By the definition of escaping limit points we know that an escaping geodesic eventually stays inside in the region \( \Omega_{r_0} \) and far from the support of \( \mu \). We can lift \( f \) to the upper half plane \( \mathbb{H} \) and get a quasi-conformal homeomorphism \( F^\mu : \mathbb{H} \to \mathbb{H} \). Induced by the Beltrami coefficient of \( F^\mu \) we can get a quasi-conformal homeomorphism of the complex plane \( \mathbb{C} \) such that the Beltrami coefficient of \( F^\mu \) is almost everywhere equal to the one of \( F^\mu \) on the upper half plane \( \mathbb{H} \) and vanishes almost everywhere on the lower half plane \( \mathbb{L} \).

For the quasi-conformal mapping \( F^\mu \), we have following results which is similar to ([10], Theorem 1.1) where Gönye discussed the differentiability of quasi-symmetric homeomorphism conjugating the covering groups of 1-dimensional 'Jungle Gym'.

Theorem 3.2. Suppose \( G \) be a Fuchsian group of divergence type and non-lattice, and let \( f \) be a quasi-conformal mapping on the surface \( S = \mathbb{H}/G \) so that the Beltrami coefficient \( \mu \) of \( f \) is compactly supported on \( S \). Let \( F^\mu \) be the lifted mapping of \( f \) to the upper half plane \( \mathbb{H} \) extended to the real axis \( \mathbb{R} \). Then \( F^\mu \) is differentiable at the escaping points \( x \in \Lambda_e(G) \) with the Jacobian \( J(F^\mu) = |(F^\mu)'(x)|^2 \). Furthermore, if \( F^\mu \) be a quasi-conformal homeomorphism of the complex \( \mathbb{C} \) whose Beltrami coefficient is equal to the one of \( F^\mu \) almost everywhere on the upper half plane \( \mathbb{H} \) and vanishes on the lower half plane \( \mathbb{L} \). Then \( F^\mu \) is conformal at the escaping limit points \( x \in \Lambda_e(G) \).
Proof. As the statements of the theorem, we can choose a point \( p_0 \in S \) and a sufficiently large \( R_0 \) such that the support set of \( f \) is contained in the disk \( B(p_0, R_0) \). Let \( S_{R_0} = S \setminus B(p_0, R_0) \) and let \( \Omega_{R_0} \) be the lift of \( S_{R_0} \) to the upper half plane \( \mathbb{H} \).

By the definition of the escaping limit points, we know that an escaping geodesic eventually stays inside the region \( \Omega_{R_0} \) and far from the support of \( \mu \).

Since the Möbius transformations which keep the upper half plane invariant do not change the hyperbolic geometry properties (such as hyperbolic area of subset of \( \mathbb{H} \) and hyperbolic distance between two points) of the upper half plane, with the conjugation of such Möbius transformations, we suppose \( x = 0 \) and the initial point of the geodesic ray is \( i \), denote the geodesic by \( \gamma(t) \), where \( t \) is the arc-length parametrization with \( \gamma(0) = i \) and \( \lim_{t \to \infty} \gamma(t) = 0 \).

By the definition of escaping geodesic, there is a region such that none of the lifted pre-images of \( B(p_0, R_0) \) will hit the escaping geodesics eventually. Hence there is a sufficiently large \( t_0 (t_0 > 1) \) and a \( \delta \in (0, 1) \), for \( t > t_0 \), dist\((\gamma(t), \mathbb{H} \setminus \Omega_{R_0}) > \delta t > R_0 \), where dist\((\cdot, \cdot)\) denote the hyperbolic distance between two points.

Let \( r_0 = e^{-t_0} \) and \( \mu_F \) be the Beltrami coefficient of \( F_\mu \). In the following, we will show that the integral

\[
\frac{1}{2\pi} \int_{|z| < r_0} \frac{|\mu_F(z)|}{|z|^2} \, dx \, dy \tag{3.2}
\]

is finite.

Since the Beltrami coefficient \( \mu_F \) vanishes in the regions \( \Omega_{R_0} \) and the lower half plane \( \mathbb{L} \), we need to show that the integral (3.2) is finite in a neighborhood of 0 outside the regions \( \Omega_{R_0} \) and \( \mathbb{L} \). We will use polar coordinateto estimate the integral of (3.2). We have

\[
\frac{1}{2\pi} \int_{|z| < r_0} \frac{|\mu_F(z)|}{|z|^2} \, dx \, dy \leq \int_0^{r_0} dr \int_0^{\theta_1(r)} \frac{1}{r} \, d\theta + \int_0^{r_0} dr \int_0^{\theta_2(r)} \frac{1}{r} \, d\theta, \tag{3.3}
\]

where \( \theta_1(r) \) and \( \theta_2(r) \) are the arguments of the points which are the intersection of the hyperbolic circle \( \text{dist}(ir, z) = -\delta \ln r \) with the Euclidean circle \( |z| = r \), where \( r < 1 \). Since the region is relative \( \gamma(t) \) symmetry, we only need to show that the integral

\[
\int_0^{r_0} dr \int_0^{\theta_1(r)} \frac{1}{r} \, d\theta \tag{3.4}
\]

is finite. By some easy calculation or see ([3], Page 131), we know that the hyperbolic circle \( \text{dist}(ir, z) = -\delta \ln r \) is just the Euclidean circle

\[
|z - ir\frac{r^\delta + r^{-\delta}}{2}| = r\left(\frac{r^{-\delta} - r^\delta}{2}\right). \tag{3.4}
\]

Let \( Q = x + iy \) be the intersection points of the hyperbolic circle \( \text{dist}(ir, z) = -\delta \ln r \) with the Euclidean circle \( |z| = r \) in the first quadrant. Combine (3.5), we have the imaginary part of \( Q \) satisfies the equation

\[
y = \frac{2r}{r^\delta + r^{-\delta}}.
\]
Therefore

\[ \sin \theta_1(r) = \frac{y}{r} = \frac{2}{r^\delta} \leq r^\delta. \]

Since for \( \theta \in (0, \frac{\pi}{2}) \), \( \frac{2}{\pi} \theta \leq \sin \theta \), we have

\[ \theta_1(r) \leq \frac{2}{\pi} r^\delta. \quad (3.5) \]

Hence the integral

\[ \int_0^r dr \int_0^{\theta_1(r)} \frac{1}{r} d\theta \leq \int_0^r r^{1-\delta} dr \]

is finite. Further more we have

\[ \frac{1}{2\pi} \int \int |z| < r_0 \left| \frac{\mu F(z)}{|z|^2} \right| dxdy < \infty. \quad (3.6) \]

By Lemma 3.1 we know that \( F_\mu \) is conformal at 0.

Let

\[ F(z) = \begin{cases} F_\mu(z), & z \in \mathbb{H}, \\ F_\mu(\bar{z}), & z \in \mathbb{C} \setminus \mathbb{H}. \end{cases} \]

By the symmetry of \( F \) and (3.6), we have that the quasi-symmetric homeomorphism \( F_\mu| \mathbb{R} \) are differentiable at the points \( x \in \Lambda_c(G) \). □

For a compact quasi-conformal deformation as the statement of Theorem 3.1, in [6], Bishop and Jones use the properties of Schwarzian derivative of \( F_\mu \) to estimate the Hausdorff dimension of \( \Lambda_e(G) \) and \( F_\mu(\Lambda_e(G)) \), they showed

**Theorem 3.3.** [6] As the statements of Theorem 3.2, \( \dim \Lambda_e(G) \) is equal to \( F_\mu(\Lambda_e(G)) \), where \( \dim(\cdot) \) denotes the Hausdorff dimension of the set.

As an application of Theorem 3.2, we give a new proof of Bishop and Jones result.

**Proof.** By Theorem 3.2 we know

\[ \Lambda_e(G) = \{ x : F_\mu'(x) \text{ exists and non-zero, } x \in \Lambda_e(G) \}. \]

Define the set

\[ \Lambda_n = \{ x : \frac{1}{n} \leq |F_\mu'(x)|, x \in \Lambda_e(G) \}, \]

it is easy to see \( \Lambda_e(G) = \bigcup_{n=1}^{\infty} \Lambda_n \) and \( \Lambda_n \subset \Lambda_{n+1} \).

Hence \( \Lambda_e(G) = \lim_{n \to \infty} \Lambda_n \).

For \( x \in \Lambda_n \), we can choose a \( \delta_x \) such that, for \( |z - x| < \delta_x \),

\[ \frac{1}{2n} \leq \frac{|F_\mu(z) - F_\mu(x)|}{|z - x|} \leq 2n. \]

This means that for each \( x \in \Lambda_n \), there exists a constant \( \delta_x \), such that for all neighborhood \( B_x \) of \( x \) with \( |B_x| < \delta_x \),

\[ \frac{1}{2n}|B_x| \leq F_\mu(|B_x|) \leq 2n|B_x|. \quad (3.7) \]
Note that for fixed number $n$, the choice of constant $\delta_x$ depends on the points $x$. To get rid of the dependence on $x$, define the set

$$\Lambda_{n,k} = \{x : x \in \Lambda_n, \forall |B_x| < \frac{1}{k}, \frac{1}{2n}|B_x| \leq F_{\mu}(|B_x|) \leq 2n|B_x|.\}$$

(3.8)

It is easy to see that $\Lambda_{n,k} \subset \Lambda_{n,k+1}$ and $\Lambda_n = \lim_{k \to \infty} \Lambda_{n,k}$.

In the following, we will show that for $\alpha \in (0,2)$, the Hausdorff measures of $H_{\alpha}(F_{\mu}(\Lambda_{n,k}))$ and $H_{\alpha}(\Lambda_{n,k})$ satisfy

$$(\frac{1}{2n})^{\alpha}H_{\alpha}(\Lambda_{n,k}) \leq H_{\alpha}(F_{\mu}(\Lambda_{n,k})) \leq (2n)^{\alpha}H_{\alpha}(\Lambda_{n,k}).$$

(3.9)

We first show that the second inequality of (3.9) holds.

For fixed $n$ and $k$, suppose $\{B_i\}$ is a cover of $\Lambda_{n,k}$ with $|B_i| < \frac{1}{2nj}$, where $j \geq k$.

Then by the definition of $\Lambda_{n,k}$ we know that the sequence $\{F_{\mu}(B_i)\}$ is a cover of $F_{\mu}(\Lambda_{n,k})$ with

$$|F_{\mu}(B_i)| < \frac{1}{j}.$$  

For any $\alpha \in (0,2)$, we have

$$H_{\frac{1}{j}}^{\alpha}(F_{\mu}(\Lambda_{n,k})) \leq \sum_{i=1}^{\infty} |F_{\mu}(B_i)|^{\alpha} \leq \sum_{i=1}^{\infty} (2n|B_i|)^{\alpha}.$$  

(3.10)

Take the infimum of the right of (3.9), we obtain

$$H_{\frac{1}{j}}^{\alpha}(F_{\mu}(\Lambda_{n,k})) \leq (2n)^{\alpha}H_{\frac{1}{2nj}}^{\alpha}(\Lambda_{n,k}).$$

Let $j$ tend to infinity, the $\alpha$- dimensional Hausdorff measures of $F_{\mu}(\Lambda_{n,k})$ and $\Lambda_{n,k}$ satisfies

$$H_{\alpha}(F_{\mu}(\Lambda_{n,k})) \leq (2n)^{\alpha}H_{\alpha}(\Lambda_{n,k}).$$

(3.11)

By (3.8) and the similar discussion as above we can get that the first inequality of (3.9) also holds. By the definition of Hausdorff dimension, the inequalities (3.9) show that, for fixed $n,k$, the Hausdorff dimension of $\Lambda_{n,k}$ is the same as its image under the map $F_{\mu}$. Since the dimension is preserve for every $n$ and $k$, hence we have

$$\dim F_{\mu}(\Lambda_e) = \dim(\Lambda_e).$$

This completes the proof of this theorem. $\square$

Now it is time to give the proof of Theorem 1.1.

4. PROOF OF THEOREM 1.1

Let $G$ be a non-lattice divergence Fuchsian group and $f$ be a quasi-conformal mapping of the surface $S = \mathbb{H}/G$ whose Beltrami coefficients is compactly supported on $S$. As the statements of Theorem 3.2, let $F_{\mu}$ be a quasi-conformal of the complex plane $\mathbb{C}$ which has the same Beltrami coefficient with the lifted mapping $F^\mu$ of $f$ to the upper half plane $\mathbb{H}$ and is conformal on the lower half plane $\mathbb{L}$. By (7, Theorem 1.3), we know that the 1-dimensional Hausdorff measure of $F_{\mu}(\Lambda_{e}(G))$ is zero. Hence, as the notations in the proof of Theorem 3.2 for fixed numbers $n$ and
k, the Hausdorff measure of the subset \( F_n(\Lambda_{n,k}) \) is zero. By (3.9) in the proof of Theorem 3.2, we know, for fixed \( n \) and \( k \), the 1-dimensional Hausdorff measure of \( \Lambda_{n,k} \) is zero. Furthermore the 1-dimensional Hausdorff measure of \( \Lambda_e(G) \) is zero.

In the following of this section we will show that the Hausdorff dimension of \( \Lambda_e(G) \) is 1.

Since \( G \) is non-lattice, the area of the surface \( S \) and the generators of \( G \) are both infinity. The method we used here is from [9]. For the reader to better understand the distribution of the geodesics corresponding to \( \Lambda_e(G) \) on the surface \( S \), we give the detail of the proof here.

We first recall the definition of geodesic domain. A domain \( D \subset S \) is called a geodesic domain if its relative boundary consists of finitely many non-intersecting closed simple geodesics and its area is finite. Fix a point \( P_0 \in S \), by \([9]\), Theorem 4.1, we know that there exists a family \( \{D_i\}_{i=0}^{\infty} \) of pairwise disjoint (except the boundary) geodesic domains in \( S \) satisfying that the boundary of \( D_i \) and \( D_{i+1} \) have at least a simple closed geodesic in common and \( \lim_{i \to \infty} \text{dist}(P_0, D_i) = \infty \), where \( \text{dist}(\cdot, \cdot) \) denotes the hyperbolic distance of the surface \( S \).

Let \( \{D_i\}_{i=0}^{\infty} \) be the family of geodesic domains of \( S \) constructed as above. For any \( i \), let \( S_i \) be the Riemann surface obtained from \( D_i \) by gluing a funnel along each one of the simple closed geodesics of its boundary. For each \( i \), we choose a simple closed geodesic \( \gamma_i \) from the common boundary \( D_i \cap D_{i+1} \) and a point \( P_i \in \gamma_i \). By \([9]\), Theorem 4.1, we have \( \delta_i \to 1 \) when \( i \) tends to infinity, where \( \delta_i \) is the Poincare exponent of the surface \( S_i \).

For \( \theta \in (0, \frac{1}{2}\pi) \), by \([9]\), Theorem 5.1, we can choose a collection \( B_i \) of geodesics in \( S_i \) with initial and final endpoint \( P_i \) such that

\[
L_i \leq \text{length}(\gamma) \leq L_i + C(P_i), \quad \gamma \in B_i,
\]

where \( L_i \) is a constant such that \( L_i \to \infty \) as \( i \to \infty \), \( C(P_i) \) is a constant depending only on the length of the geodesic \( \gamma_i \), and \( \sigma_i < \delta(S_i), \sigma_i \to 1 \) as \( i \to \infty \).

The number of geodesic arcs in \( B_i \) is at least \( e^{L_i} \sigma_i \), and both the absolute value of the angles between \( \gamma_i \) and the closed geodesic \( \gamma_i^* \) are less than or equal to \( \theta \).

Note that for each \( i \), \( D_i \) is the convex core of \( S_i \), implying that every geodesic arc \( \gamma \in B_i \) is contained in the convex core \( D_i \).

Furthermore, for each \( i \), we may choose a geodesic arcs \( \gamma_i^* \) with initial point \( P_i \) and final endpoint \( P_{i+1} \) such that

\[
L_i \leq \text{length}(\gamma_i^*) \leq L_i + C(P_{i+1}),
\]

and both the absolute value of the angles between \( \gamma_i, \gamma_i^* \), and \( \gamma_i^*, \gamma_{i+1} \) are less than or equal to \( \theta \).

In order to show the distribution of geodesics on \( S \), we are going to construct a tree \( \mathcal{T} \) consisting of oriented geodesic arcs in the unit disk \( \Delta \).

Let us first lift \( \gamma_0^* \) to the unit disk starting at 0 (without loss of generality we may suppose that 0 projects onto \( P_0 \)). From the endpoint of the lifted \( \gamma_0^* \) (which
project onto $P_1$), lift the family $\mathcal{B}_1$; from each of the end points of these liftings (which still project onto $P_1$), lift again $\mathcal{B}_1$. Keep lifting $\mathcal{B}_1$ in this way a total of $M_1$ times.

Next, from each one of the endpoints obtained in the process above, we lift $\gamma_1^*$, and from each one of the endpoints of the liftings of $\gamma_1^*$ (which project onto $P_2$), we lift the collection $\mathcal{B}_2$ successively $M_2$ times as above. Continuously this process indefinitely we obtain a tree $\mathcal{T}$.

It is easy to see that $\mathcal{T}$ contains uncountably many branches. The tips of the branches of $\mathcal{T}$ are contained in the escaping limit set $\Lambda_e(G)$ of the covering group of $S$. For suitably choosing the sequence $\{M_i\}$ of repetitions, the dimension of the rims of tree $\mathcal{T}$ is 1. By the construction of the tree $\mathcal{T}$, we see that the tree $\mathcal{T}$ is a unilaterally connected graph. Hence the geodesic corresponding to any branch of $\mathcal{T}$ does not tend to the funnel with boundary $\gamma$. Hence the dimension of the escaping limit set $\Lambda_e(G)$ of the covering group $G$ is 1.

5. Proof of Theorem 1.2

To prove this theorem, we need the following lemma which is essentially due to Gönyle, see (10, P29.)

**Lemma 5.1.** Let $F$ be a quasi-symmetric homeomorphism of the real axis $\mathbb{R}$ and $A$ be a subset of $\mathbb{R}$ with Hausdorff dimension equal to 1. If for any $x \in A$, $F'(x)$ exists and is non-zero, then the Hausdorff dimension of $F(A)$ is also 1.

Now we give the proof of Theorem 1.2

**Proof.** Let $G$ be a Fuchsian group of divergence type and not a lattice. Let $f$ be a quasi-conformal mapping on the surface $H/G$. The lifting mapping $F^\mu$ of $f$ to the upper half plane $\mathbb{H}$ can extend to the real axis $\mathbb{R}$ naturally. We denote by $h = F^\mu|\mathbb{R}$. The mapping $h$ is a quasi-symmetric homeomorphism of $\mathbb{R}$. By Theorem 3.2, the homeomorphism $h$ is differentiable at $x$ in $\Lambda_e(G)$ with $|F^\mu(x)| \neq 0$. By Theorem 1.3 and Lemma 5.1 we know, for any $E \subset \mathbb{R}$,

$$\max(dim(E), dimf(\mathbb{R} \setminus E)) = 1.$$  

Hence the theorem holds.

6. Proof of Theorem 1.3

**Proof.** The necessity of the equivalence is from (5, Theorem 4).

For the sufficient condition, by Theorem 1.2, we only need to show the case when $G$ is a Fuchsian group of the convergence type.

If $G$ is a Fuchsian group of the second, the boundary of Dirichlet fundamental domain contains at least an arc (denoted by $\alpha^*$) in $\mathbb{R}$. It is easy to see that the homeomorphism is smooth on $\alpha^*$. Hence the sufficient condition holds. If $G$ is a Fuchsian group of convergence type and of the first kind, we need to show that the
Hausdorff dimension of the escaping limit set $\Lambda_{\omega}(G)$ is 1, actually it has positive 1-dimensional Hausdorff measure.

Suppose $\gamma$ be a closed geodesic on the surface $S = \mathbb{H}/G$. Consider the liftings of the closed geodesic $\gamma$ in the upper half plane $\mathbb{H}$. It consists of a nested set $\Sigma$ of hyperbolic lines: the one intersecting the Dirichlet fundamental domain cuts it in two parts and we may assume that the point $i$ belongs to a part that has infinite (hyperbolic) area. The hyperbolic lines in $\Sigma$ of the first generation define a two-by-two disjoint family $(I_j)$ of intervals of the real axis $\mathbb{R}$. Suppose $\bigcup_{j=1}^{\infty} I_j$ is equal to $\mathbb{R}$ except a zero Lebesgue measure set, then almost every geodesic issued from $i$ would visit $\gamma$ infinitely often, contradicting of (9, Theorem 1). Thus the set of geodesics from $i$ that never visit $\gamma$ has positive measure. It follows that the escaping limit set of $S$ has positive Lebesgue measure.

Hence if $F^\mu$ corresponding to a compact quasi-conformal deformation of $G$, we always have, for any $E \subset \mathbb{R}$,

$$\max(\dim(E), \dimf(\mathbb{R} \setminus E)) = 1.$$