IMPLICATIONAL COMPLETENESS

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ABSTRACT. For the implicational propositional calculus, we present a proof of completeness based on a variant of the Lindenbaum procedure.

0. INTRODUCTION

A standard formulation of the Implicational Propositional Calculus (IPC) has \( \supset \) as its only connective, has modus ponens as its only inference rule, and has the following axiom schemes:

\[
A \supset (B \supset A)
\]

\[
[A \supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)]
\]

\[
[(A \supset B) \supset A] \supset A
\]

the last of which is due to Peirce. A (Boolean) valuation is a map \( v \) from the set \( \text{wf} \) of all well-formed formulas to the set \( \{0, 1\} \) such that \( v(A \supset B) = 0 \) precisely when \( v(A) = 1 \) and \( v(B) = 0 \); a tautology is a well-formed formula that takes the value 1 in all such valuations.

The Completeness Theorem for IPC asserts that every tautology is a theorem: if a well-formed formula has value 1 in each valuation then there is a proof of it using modus ponens and the three axiom schemes listed above. One proof of completeness for IPC is indicated in Exercises 6.3, 6.4, 6.5 of Robbin [2]; that proof adapts the Kalmár approach for the classical propositional calculus. Our purpose in this brief paper is to present a proof of completeness for IPC that adapts the Lindenbaum approach for the classical propositional calculus.

1. THEOREM AND PROOF

We begin with some simple observations regarding IPC. To say that \( A \in \text{wf} \) may be deduced from \( \Gamma \subseteq \text{wf} \) we may write \( \Gamma \vdash A \) as usual; in particular (taking \( \Gamma \) to be empty) \( \vdash A \) asserts that \( A \) is a theorem. Modus ponens and the first two axiom schemes together ensure that if \( A \in \text{wf} \) is any well-formed formula then \( A \supset A \) is a theorem. Modus ponens and the first two axiom schemes further ensure that the Deduction Theorem (DT) holds: if \( \Gamma \cup \{A\} \vdash B \) then \( \Gamma \vdash A \supset B \); in particular, if \( \vdash A \supset B \) then \( \vdash A \supset B \).

The lack of negation in IPC is in part repaired by an elegant device. Fix an arbitrary well-formed formula \( Q \in \text{wf} \). When \( A \in \text{wf} \) write \( QA := Q(A) := A \supset Q \).

**Theorem 1.** Fix any well-formed formula \( Q \in \text{wf} \). If \( A, B, C \in \text{wf} \) then each of the following well-formed formulas is a theorem of IPC:

1. \( (A \supset B) \supset [(B \supset C) \supset (A \supset C)] \)
2. \( (A \supset B) \supset (QB \supset QA) \)
3. \( A \supset QQA \)
4. \( QQQA \supset QA \)
5. \( QQB \supset QQ(A \supset B) \)
6. \( QQQA \supset [QB \supset Q(A \supset B)] \)
7. \( QA \supset QQ(A \supset B) \)
8. \( (QA \supset B) \supset [(QQA \supset B) \supset QQB] \).
Proof. This is Exercise 6.3 in Chapter 1 of Robbin [2]. As noted by Robbin, part (7) requires the Peirce axiom scheme; the other parts need only the first two axiom schemes. □

The classical propositional calculus presented by Church [1] and followed by Robbin [2] incorporates a propositional symbol \( \uparrow \) (falsity) having value 0 under each valuation; it takes \( \uparrow A = A \uparrow \uparrow \) for the negation \( \sim A \) of \( A \); and it replaces the Peirce axiom scheme by the ‘double negation’ axiom scheme \( \sim \sim A \Rightarrow A \). The symbol \( \uparrow \) has no place in the present paper; however, significant aspects of its function will be served by a non-theorem \( Q \).

From this point on, we shall consider extensions of IPC obtained by enlarging the set of theorems, so we shall modify our notation accordingly. Let us write \( L \) for the system IPC as formulated above and write \( T(L) = \{ A \in \text{wf} : \vdash A \} \) for its set of theorems. An extension \( M \) of \( L \) is produced by adding an axiom (or axioms); thus \( T(L) \subseteq T(M) \) and the Deduction Theorem continues to hold for \( M \). To indicate that a well-formed formula \( A \) is a theorem of \( M \) we prefer to write \( A \in T(M) \) rather than the customary \( \vdash_M A \).

**Theorem 2.** Let \( Q \in \text{wf} \) and let \( M \) be an extension of \( L \). The following are equivalent:
1. \( Q \in T(M) \);
2. some \( A \in \text{wf} \) has \( A \in T(M) \) and \( QA \in T(M) \);
3. some \( A \in \text{wf} \) has \( QQA \in T(M) \) and \( QA \in T(M) \).

Proof. (2) \( \Rightarrow \) (3) Follows from Theorem 1 part (3) by modus ponens.

(3) \( \Rightarrow \) (1) Follows by modus ponens from \( T(M) \vdash QA \) and \( T(M) \vdash QQA (= QA \uparrow Q) \).

(1) \( \Rightarrow \) (2) Simply let \( A = Q \) and recall that \( QQ (= Q \uparrow Q) \in T(L) \subseteq T(M) \). □

We say that \( M \) is \( Q \)-inconsistent precisely when it satisfies one (hence each) of the equivalent conditions in this theorem; we say that \( M \) is \( Q \)-consistent otherwise.

Henceforth, we shall let \( Q \in \text{wf} \) be a well-formed formula that is not a theorem of \( L \); thus, \( L \) is \( Q \)-consistent.

**Theorem 3.** Let \( M \) be a \( Q \)-consistent extension of \( L \) and \( A \in \text{wf} \) a well-formed formula. If \( QQA \) is not a theorem of \( M \), then the extension \( N \) of \( M \) obtained by adding \( QA \) as an axiom is \( Q \)-consistent.

Proof. To prove the contrapositive, assume that \( Q \in T(N) \): a deduction of \( Q \) within the system \( N \) is a deduction of \( Q \) from \( QA \) within the system \( M \); by the Deduction Theorem, it follows that \( QQA = (QA \uparrow Q) \in T(M) \). □

We say that \( M \) is \( Q \)-complete precisely when each \( A \in \text{wf} \) satisfies either \( QQA \in T(M) \) or \( QA \in T(M) \); that is, either \( QQA \) or \( QA \) is a theorem of \( M \).

**Theorem 4.** Each \( Q \)-consistent extension \( M \) of \( L \) has a \( Q \)-complete \( Q \)-consistent extension.

Proof. List all the well-formed formulas: say \( \text{wf} = \{ A_n : n \geq 0 \} = \{ A_0, A_1, \ldots \} \).

Put \( N_0 = M \). If \( QQA_0 \in T(N_0) \) then let \( N_1 = N_0 \); if \( QQA_0 \notin T(N_0) \) then let \( N_1 \) be \( N_0 \) with \( QA_0 \) as an extra axiom. Repeat inductively: if \( QQA_n \in T(N_n) \) then let \( N_{n+1} = N_n \); if \( QQA_n \notin T(N_n) \) then let \( N_{n+1} \) be \( N_n \) with \( QA_n \) as an extra axiom. Finally, let \( N \) be the extension of \( M \) produced by adding as axioms all those \( \text{wf} \)s introduced at each stage of this inductive process.

Claim: \( N \) is \( Q \)-consistent. [Any proof of \( Q \) in \( N \) would involve only finitely many axioms and would therefore be a proof of \( Q \) in \( N_n \) for some \( n \geq 0 \); but Theorem 3 guarantees inductively that the extension \( N_n \) is \( Q \)-consistent for each \( n \geq 0 \).]

Claim: \( N \) is \( Q \)-complete. [Take any \( \text{wf} \): say \( A_n \). If \( QQA_n \in T(N_n) \) then \( QQA_n \in T(N) \); if \( QQA_n \notin T(N_n) \) then \( QA_n \in T(N_{n+1}) \) so that \( QA_n \in T(N) \). Thus, if \( A \in \text{wf} \) is arbitrary then either \( QQA \) or \( QA \) is a theorem of \( N \) as required.] □
Now, let $M$ be a $Q$-consistent extension of $L$ and let $N$ be a $Q$-complete $Q$-consistent extension of $M$. Let $A$ be a well-formed formula: when $QA$ is a theorem of $N$ we put $v_N(A) = 1$; when $QA$ is a theorem of $N$ we put $v_N(A) = 0$. As $N$ is both $Q$-complete and $Q$-consistent, this defines a function $v_N : \text{wf} \rightarrow \{0, 1\}$.

**Claim:** $v = v_N$ is a valuation: that is, $v(A \supset B) = 0$ precisely when $v(A) = 1$ and $v(B) = 0$.

**Proof:** Suppose that $v(A) = 0$: that is, suppose $QA \in \mathbb{T}(N)$; Theorem 1 part (7) tells us that

$$QA \supset QQ(A \supset B) \in \mathbb{T}(L) \subseteq \mathbb{T}(N)$$

whence modus ponens places $QQ(A \supset B)$ in $\mathbb{T}(N)$ and $v(A \supset B) = 1$. Suppose $v(B) = 1$: that is, suppose $QQB \in \mathbb{T}(N)$; Theorem 1 part (5) tells us that

$$QQB \supset QQ(A \supset B) \in \mathbb{T}(L) \subseteq \mathbb{T}(N)$$

whence modus ponens places $QQ(A \supset B)$ in $\mathbb{T}(N)$ and $v(A \supset B) = 1$. Thus

$$(v(A) = 0) \lor (v(B) = 1) \Rightarrow v(A \supset B) = 1$$

and so

$$v(A \supset B) = 0 \Rightarrow (v(A) = 1) \land (v(B) = 0).$$

Conversely, let $v(A) = 1$ and $v(B) = 0$: thus, $QA$ and $QB$ are theorems of $N$; part (6) of Theorem 1 tells us that

$$QA \supset [QB \supset Q(A \supset B)] \in \mathbb{T}(L) \subseteq \mathbb{T}(N)$$

whence two applications of modus ponens yield $Q(A \supset B) \in \mathbb{T}(N)$ and so $v(A \supset B) = 0$.

We are now able to prove the completeness of IPC.

**Theorem 5.** The Implicational Propositional Calculus is complete.

**Proof.** Suppose that $Q$ is not a theorem of $L$; thus, $L$ is $Q$-consistent. Theorem 1 fashions a $Q$-complete $Q$-consistent extension $N$ of $L$ by means of which we define the valuation $v_N$ as above. Before stating Theorem 1 we noted that $Q \supset Q$ is a theorem of $L$; thus $QQ (= Q \supset Q)$ is a theorem of $N$ and so $v_N(Q) = 0$. We have found a valuation under which $Q$ does not take the value 1; $Q$ is not a tautology.

**REFERENCES**

[1] Alonzo Church, *Introduction to Mathematical Logic*, Princeton University Press (1956).

[2] Joel W. Robbin, *Mathematical Logic - A First Course*, W.A. Benjamin (1969); Dover Publications (2006).

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