Energy spectrum of bilayer graphene with magnetic quantum structures studied using the Dirac equation

Daehan Park, Heesang Kim and Nammee Kim

Department of Physics, Soongsil University, Seoul 06978, Republic of Korea

E-mail: nammee@ssu.ac.kr

Received 22 June 2021, revised 21 September 2021
Accepted for publication 23 September 2021
Published 14 October 2021

Abstract
The electronic energy spectrum of bilayer graphene with a magnetic quantum dot (MQD) and a magnetic quantum ring (MQR) are investigated. The energy eigenvalues and wavefunctions of quasiparticle states are calculated analytically by solving decoupled fourth-order differential equations. For the MQD, in the case of a negative inner magnetic field, two peculiar characteristics of the eigenvalue evolution are found: (a) the energy eigenstates change in a stepwise manner owing to energy anticrossing and (b) the quantum states approach zero energy. For the MQR, there is an angular momentum transition of eigenvalue as the inner radius of the ring varies, and the Aharonov–Bohm effect is observed in the eigenvalue spectra for both positive and negative magnetic fields inside the inner radius.

Keywords: bilayer graphene, Dirac equation, Aharonov–Bohm effect, magnetic quantum dot, magnetic quantum ring, energy spectrum

(Some figures may appear in colour only in the online journal)

1. Introduction

Since the first demonstration of the fabrication of monolayer graphene [1], graphene has attracted considerable attention owing to its exotic properties, such as its long mean free path, high carrier mobility, and an anomalous quantum Hall effect [1–5]. These properties of graphene originate mainly from the existence of two independent Dirac points, which are conventionally called the \( K \) and \( K' \) valleys. In the vicinity of these valleys, graphene has a unique band structure, namely, a gapless and linear energy dispersion. As a result, electrons in graphene behave as if they were massless relativistic particles, governed by the Dirac equation.

However, these relativistic characteristics are a considerable hindrance to the realization of electronic devices based on graphene. Owing to the Klein effect [6], near-perfect transmission of Dirac electrons through an electrostatic barrier at normal incidence is possible. That is, unlike electrons in a two-dimensional electron gas (2DEG), Dirac electrons cannot be confined by an electrostatic potential. However, De Martino et al [7] showed that this problem could be overcome in graphene by the application of a magnetic field. In contrast to an electrostatic potential, by deflection of the trajectory of electrons through the Lorentz force, a magnetic field allows trapping of electrons within a restricted region. Various magnetic quantum structures such as magnetic quantum dots (MQDs), magnetic step barriers, and magnetic quantum rings (MQRs) with impurities have been investigated [7–13].

Graphene multilayers have also been widely investigated. In multilayer graphene, the layers are weakly coupled by van der Waals interaction, and its properties are very different from those of monolayer graphene. For instance, pristine bilayer graphene (BLG) exhibits gapless and parabolic energy dispersion near the \( K \) and \( K' \) valleys [14]. Also, in a uniform
magnetic field \( \vec{B} \), the Landau levels (LLs) of BLG in the lowest band have a linear dependence on \( \vec{B} \), in contrast to the \( \sqrt{\vec{B}} \) dependence of LLs in monolayer graphene [15].

In recent years, semiconductor-based quantum dots and quantum rings have become important subjects of study in mesoscopic physics [16]. These quantum structures have been investigated both experimentally and theoretically in BLG. There have been theoretical studies of quantum dots and quantum rings in AA-stacked [17–19], AB-stacked [20–25], and twisted [26] BLG. The properties of two or more coupled quantum dots in BLG have also been studied experimentally [27–29]. Recently, the electronic density within BLG quantum dots has been visualized using the tip of a scanning tunneling microscope [30, 31]. The valley properties of an electrostatically confined BLG quantum dot have been investigated experimentally [32, 33], and a valley filter device based on these properties has been proposed [34].

In this paper, based on a single-particle approach, we investigate the electronic properties of BLG with two kinds of magnetic structures: a MQD and a MQR. In most previous work on these quantum structures in a 2DEG and in monolayer graphene, there was no magnetic field inside the MQD, and the spatial area of the MQD was kept constant. Here, however, we consider the behavior of the energy eigenvalues of BLG as the magnetic field inside an MQD varies. We show that there are dramatic differences depending on the direction of the magnetic field inside the dot. The results for the eigenvalue spectra of BLG for an MQR with fixed spatial area reveal nothing unusual compared with the cases of a 2DEG and monolayer graphene. Therefore, we vary the inner radius of the ring and introduce antiparallel magnetic fields inside the inner circle and outside the outer circle of the ring to look for interesting phenomena that might arise.

The remainder of the paper is organized as follows. In section 2, we introduce a mathematical model of our magnetic quantum structures and present the mathematical solution procedure for the Hamiltonian of the BLG, including complicated boundary conditions for the fourth-order differential equations satisfied by the Dirac wavefunctions. In section 3.1, we present our results for BLG with an MQD and discuss its energy anticrossing by using effective potentials and the probability density at subatoms. We discuss the energy eigenvalues and the Aharonov–Bohm (AB) effect in BLG with an MQR in section 3.2. Finally, we summarize our results in section 4.

2. Model and formalism

We calculate the energy spectra of BLG with an MQD or an MQR, formed by a spatially nonuniform distribution of magnetic fields, by solving the Dirac equation. The geometries of the MQD and MQR are shown schematically in Figure 1. In figure 1(a), the MQD is modeled as \( \vec{B} = \alpha_1 \vec{B}_0 \hat{e}_z \) for \( r < r_0 \) and \( \vec{B} = \alpha_2 \vec{B}_0 \hat{e}_z \) for \( r > r_0 \), where \( \alpha = \vec{B}/\vec{B}_0 \) is the ratio of the applied magnetic field \( \vec{B} \) to the standard magnetic field \( \vec{B}_0 \). The MQDs studied previously correspond to \( \alpha_1 = 0 \). In figure 1(b), the MQR is modeled as \( \vec{B} = \alpha_1 \vec{B}_0 \hat{e}_z \) for \( r < r_1 \), \( \vec{B} = 0 \) for \( r_1 < r < r_2 \), and \( \vec{B} = \alpha_2 \vec{B}_0 \hat{e}_z \) for \( r > r_2 \). The model of the MQD can be obtained from that for the MQR in the limit as \( r_1 \to r_2 \). Therefore, we shall describe the calculations for the MQR only. Because the MQR has rotational symmetry, it is convenient to use plane polar coordinates in our system. We can express the vector potential \( \vec{A}(r) \) of the magnetic field \( \vec{B} \) in the symmetric gauge as:

\[
\vec{A}(r) = \begin{cases} 
\frac{1}{2} \alpha_1 B_0 \hat{\theta} & \text{for } r < r_1, \\
\frac{1}{2} \frac{\alpha_1 B_0 r^2}{r} \hat{\theta} & \text{for } r_1 < r < r_2, \\
\frac{1}{2} \alpha_2 B_0 \hat{\theta} - \frac{B_0}{2r} (\alpha_2 r^2 - \alpha_1 r_1^2) \hat{\theta} & \text{for } r > r_2.
\end{cases}
\]

(1)

The Hamiltonian for BLG with Bernal (AA) stacking near the \( K \) valley is given by [14]:

\[
H = \begin{pmatrix} 0 & \pi & t_L & 0 \\
\pi & 0 & 0 & 0 \\
t_L & 0 & 0 & \pi \\
0 & 0 & \pi & 0 \end{pmatrix},
\]

(2)

where \( \pi = \Pi_x + i\Pi_y \), \( \Pi_x = i\Pi_y \), \( \Pi_x = v_F |p_x + (e/c)A_1| \), the Fermi velocity \( v_F \approx c/300 \), \( t_L = 400 \text{ meV} \) is an interlayer hopping parameter, and \( c \) is the speed of light. The wavefunctions of the Hamiltonian in equation (2) can be expressed as a four-spinor wavefunction \( \vec{\psi} = (\psi_A, \psi_B, \psi_B^*, \psi_A^*)^T \). In our system, the vector potentials have spatial circular symmetry, and thus this four-spinor wavefunction can be written as [20]:

\[
\vec{\psi} = e^{i m \theta} \begin{pmatrix} \phi_A(r) \\
 ie^{-i \theta} \phi_B(r) \\
\phi_B^*(r) \\
 ie^{i \theta} \phi_A^*(r) \end{pmatrix}^T,
\]

(3)

where \( m \) is an angular momentum quantum number. To simplify our problem, we nondimensionalize quantities by taking the unit of length to be the magnetic length \( l_B = \sqrt{2\hbar/e\vec{B}_0} \) and the unit of energy to be \( \epsilon_0 = \sqrt{2\hbar v_F/l_B} \). For a standard magnetic field \( \vec{B}_0 = 10 \text{ T} \), the values of \( l_B \) and \( \epsilon_0 \) are 11.4 nm and 81.05 meV, respectively.

In dimensionless units, by solving the Dirac equation, \((H - E)\vec{\psi} = 0\), we obtain the following four coupled first-order differential equations:
\[
\frac{d}{dr} \left( \frac{m-1}{r} - v(r) \right) \phi_B = \sqrt{2} (\epsilon \phi_A - t \phi_B),
\]
(4)

\[
\frac{d}{dr} + \frac{m}{r} + v(r) \phi_A = -\sqrt{2} \epsilon \phi_B,
\]
(5)

\[
\frac{d}{dr} + \frac{m+1}{r} + v(r) \phi_{A^*} = \sqrt{2} \left[ \phi_B'(r) - t \phi_{A^*} \right],
\]
(6)

\[
\frac{d}{dr} - \frac{m}{r} - v(r) \phi_{B^*} = -\sqrt{2} \epsilon \phi_{A^*},
\]
(7)

where \( t = t_1 / \epsilon_0 = 4.96 \). The expressions for \( v(r) \) in each region are as follows:

\[
v(r) = \begin{cases} 
\alpha_1 r & \text{for } r < r_1, \\
\frac{\alpha_1 r^2}{r} & \text{for } r_1 < r < r_2, \\
\alpha_2 r - \frac{\alpha_2 r^2 - \alpha_1 r^2}{r} & \text{for } r_2 < r.
\end{cases}
\]
(8)

Therefore, we need to solve four coupled first-order differential equations. Fortunately, however, equations (4)–(7) can be decoupled to give a fourth-order differential equation for \( \phi_A(r) \) in each region. To decouple equations (4)–(7), eliminate \( \phi_B \) and \( \phi_{A^*} \) by substituting equations (5) and (7) into equations (4) and (6), respectively. Then, four coupled first-order differential equations become two coupled second-order differential equations. For instance, these differential equations in region I \((r < r_1)\) become as follows:

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - 2 \alpha_1 (m-1) - \alpha_1^2 r^2 + 2 \epsilon^2 \right] \phi_A^I(r) = 2te \phi_A^I(r),
\]
(9)

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - 2 \alpha_1 (m+1) - \alpha_1^2 r^2 + 2 \epsilon^2 \right] \phi_B^I(r) = 2te \phi_B^I(r).
\]
(10)

To eliminate \( \phi_{B^*} \), the equation (9) is substituted into equation (10). Then the equation becomes:

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - 2 \alpha_1 (m+1) - \alpha_1^2 r^2 + 2 \epsilon^2 \right] \times \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - 2 \alpha_1 (m-1) - \alpha_1^2 r^2 + 2 \epsilon^2 \right] \phi_A^I(r) = 4r^2 \epsilon^2 \phi_A^I(r).
\]
(11)

If the terms in above equation are rearranged, the equation for region I \((r < r_1)\) becomes as follows:

in region I \((r < r_1)\),

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - 2 \alpha_1 m - \alpha_1^2 r^2 + 2 \epsilon^2 \right] \phi_A^I \left( \frac{\alpha_1^2}{4} + r^2 \right) \phi_A^I = 4r^2 \epsilon^2 \phi_A^I.
\]
(12)

In the same way, we can derive decoupled differential equations corresponding to region II \((r_1 < r < r_2)\) and region III \((r_2 < r)\) as follows:

in region II \((r_1 < r < r_2)\),

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - 2 \alpha_2 m - \alpha_2^2 r^2 + 2 \epsilon^2 \right] \phi_A^II \left( \frac{\alpha_2^2}{4} + r^2 \right) \phi_A^II = 4r^2 \epsilon^2 \phi_A^II.
\]
(13)

in region III \((r_2 < r)\),

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - 2 \alpha_2 m - \alpha_2^2 r^2 + 2 \epsilon^2 \right] \phi_A^III \left( \frac{\alpha_3^2}{4} + r^2 \right) \phi_A^III = 4r^2 \epsilon^2 \phi_A^III.
\]
(14)

The solutions of equations (12)–(14) for \( \phi_A \) are:

\[
\phi_A(r) = \begin{cases}
\text{for } r < r_1, \\
\text{for } r_1 < r < r_2, \\
\text{for } r_2 < r,
\end{cases}
\]
(15)

where

\[
\delta_i^\pm = -\frac{\beta_i^2}{2 \alpha_i} + \frac{1}{2} \left[ |m| + \frac{\alpha_i}{|\alpha_i|} |m| + 1 \right],
\]
\[
\beta_i^\pm = \frac{\sqrt{\epsilon^2 \pm \sqrt{\alpha_i^2 + (te)^2}}}{},
\]
\[
\gamma_i = \sqrt{\epsilon^2 \pm te},
\]

\( M \) and \( U \) are confluent hypergeometric functions, and \( J_m \) and \( N_m \) are Bessel functions of the first and second kinds, respectively. The wavefunctions for the other subatoms, namely, \( \phi_B \), \( \phi_{A^*} \), and \( \phi_{B^*} \), can also be derived from equations (4)–(7).

Unlike the Schrödinger equation and the Dirac equation for monolayer graphene, the equations for \( \phi_A \) for BLG are fourth-order differential equations. Therefore, we need to derive appropriate boundary conditions for \( \phi_A \). These boundary
conditions can be obtained from the continuity of the wavefunctions for all subatoms, \( \phi_A, \phi_B, \phi'_A, \) and \( \phi'_B, \) at the boundaries as follows:

at \( r = r_1, \) which is the boundary between regions I and II,
\[
\phi_A(r_1) - \phi_B(r_1) = 0
\]  
(16)
\[
\frac{d}{dr} \phi_A(r_1) - \frac{d}{dr} \phi_B(r_1) = 0, 
\]  
(17)
\[
\frac{d^2}{dr^2} \phi_A(r_1) + 2\alpha_1 \phi_A(r_1) - \frac{d^2}{dr^2} \phi_B(r_1) = 0, 
\]  
(18)
\[
\frac{d^3}{dr^3} \phi_A(r_1) - \frac{d^3}{dr^3} \phi_B(r_1) + 2\alpha_1 \frac{d}{dr} \phi_A(r_1) - 2\alpha_1 \frac{d}{dr} \phi_B(r_1) = 0, 
\]  
(19)

and at \( r = r_2, \) which is the boundary between regions II and III,
\[
\phi'''_A(r_2) - \phi'''_B(r_2) = 0, 
\]  
(20)
\[
\frac{d}{dr} \phi'''_A(r_2) - \frac{d}{dr} \phi'''_B(r_2) = 0, 
\]  
(21)
\[
\frac{d^2}{dr^2} \phi'''_A(r_2) + 2\alpha_2 \phi'''_A(r_2) - \frac{d^2}{dr^2} \phi'''_B(r_2) = 0, 
\]  
(22)
\[
\frac{d^3}{dr^3} \phi'''_A(r_2) - \frac{d^3}{dr^3} \phi'''_B(r_2) + 2\alpha_2 \frac{d}{dr} \phi'''_A(r_2) - \left( \frac{2\alpha_2}{r_2} + \frac{m_1^2 - m_2^2}{r_2^2} + \frac{2\alpha_2}{r_2} \right) \phi'''_B(r_2) = 0. 
\]  
(23)

The corresponding energy eigenvalues of an MQR in BLG can be obtained from the above boundary conditions as follows:

Plugging the solution \( \phi_A(r) \) in equation (15) into the boundary conditions in equations (16)–(23) would produce eight coupled equations for eight coefficients in \( \phi_A(r), \) namely, \( A, B, C_1, C_2, D_1, D_2, E \) and \( F. \) The coupled equations can be written into an \( 8 \times 8 \) matrix equation. For the set of coefficients to have a non-trivial solution, the secular equation has to be satisfied, whose zeroes would give the corresponding energy eigenvalues.

3. Results and discussion

3.1 Magnetic quantum dot

In this subsection, we discuss the electronic properties of BLG with an MQD. In figures 2(a) and (b), the energy eigenvalues \( \epsilon_{nm} \) are plotted as functions of \( \alpha_2 \) with \( \alpha_1 = 0; \) (b) \( \epsilon_{nm} \) as functions of \( \alpha_1 \) with \( \alpha_2 = 1.0. \)

The magnetic field \( \alpha_2 \) of MQD is zero and the outside magnetic field \( \alpha_2 \) varies. The results are shown in figure 2(a). There is a noticeable difference in that we have zero-energy states for the MQD in BLG. These zero-energy states are also present for an MQD in monolayer graphene. The red dashed lines in the figure represent the dimensionless LLs for BLG [35], i.e.

\[
\sqrt{2}\alpha \sqrt{\frac{\epsilon}{\alpha+4N+2} - \left( \frac{\epsilon}{\alpha+4N+2} \right)^2} - N(N+1), 
\]

where \( N \) is the Landau index corresponding to \( n + (m + |m|)/2 \) in our calculation. For a magnetic field that is weak compared with \( r, \) the LLs are approximately proportional to \( B. \) Except for these features, the energy spectra of BLG with a typical MQD show the general characteristics of an MQD in the 2DEG case [7, 36, 37]. For a small outer magnetic field \( \alpha_2, \) the wavefunctions are mostly distributed outside the MQD, as a consequence of which the energies are close to the LLs of BLG. On the other hand, as \( \alpha_2 \) increases, the cyclotron radius...
of the electrons becomes smaller than the radius of the dot, and the energies start to deviate from the LLs. Because the Lorentz force leads to strong localization of the electrons within MQD, the energy evolution is very similar to that of an MQD in the 2DEG case. These energy changes are also the same as for a conventional circular dot that is electrostatically confined by hard walls without magnetic fields.

In figure 2(b), the inside magnetic field $\alpha_1$ varies, while the outside magnetic field is fixed at $\alpha_2 = 1$. The LLs inside the MQD (red dashed lines) are fan-shaped because $\alpha_1$ increases, and the LLs outside the MQD (black dashed lines) are flat because $\alpha_2$ remains constant. When $\alpha_1$ increases in the positive direction, the wavefunctions are localized within the MQD. Thus, the energy eigenvalues approach the LLs inside the MQD. However, when $\alpha_1$ becomes negative, the evolution of the eigenvalue exhibits two peculiar characteristics: (a) the energy eigenvalues converge to the LLs outside the dot in a stepwise manner through energy anticrossing; (b) the energy eigenvalues of some quantum states approach zero energy. We now consider these characteristics in greater detail.

To explain the stepped shape of the energy evolution for a negative inner magnetic field $\alpha_1$, in figure 3 we plot $\epsilon_n$ as functions of negative $\alpha_1$, the effective potential of MQD for $m = 0$, and the probability density of a given quantum state $(n, m)$.

In figure 3(a), the red dashed lines and black dashed lines indicate the LLs inside and outside the MQD, respectively. The $(0, 0)$ state has zero energy regardless of $|\alpha_1|$, and the $(1, 0)$ state tends to zero energy as $|\alpha_1|$ increases. The energy eigenvalues $(n \geq 2, m)$ are determined by one of the LLs inside or outside the MQD, depending on the value of $\alpha_1$. For small $\alpha_1$ near zero, the wavefunctions are located mainly at the boundary of the MQD, and the eigenvalues deviate from the LLs both inside and outside the MQD. However, as $|\alpha_1| = 0$ starts to increase, the wavefunctions move to the inside of the MQD and are affected by the inside magnetic field $\alpha_1$, and the energies start to follow the LLs inside the MQD. When an LL inside the MQD becomes equal to an LL outside the MQD at $\alpha_1 = -0.81$, the state experiences energy anticrossing once and moves out again at $\alpha_1 = 0.85$ (figure 3(c)). These anticrossing phenomena are also found in electrostatically confined quantum dots and rings in monolayer graphene and BLG [19, 22, 38].

Figure 3(b) is a qualitative representation of the effective potential for the $m = 0$ state of BLG with an MQD. We shall explain these anticrossing phenomena in terms of the effective potential. If whole wavefunctions can be localized either inside or outside the MQD, we can simply assume that the system energies are determined only by the LLs in the respective region. As the magnetic field $\alpha_1$ increases, the energy-level spacing inside the MQD becomes wider, and the energy levels start to become equal to the LLs outside the MQD. When the first LL inside the MQD becomes higher than the first LL outside the MQD, to maintain the energy level sequence for a fixed $m$, the state with the first LL inside the MQD moves to the first LL outside the MQD. Therefore, energy anticrossing occurs whenever LLs in the two different regions become the same. As the magnetic field increases, the wavefunction moves back and forth between the outside and inside of the MQD until the first inside LL becomes larger than the other fixed outside LLs, and, as a result, the wavefunctions of all states are located outside the MQD.

The $(2, 0)$ state experiences energy anticrossing once and the $(3, 0)$ state three times in the region $0 < |\alpha_1| < 3$, as shown in figure 3(a). To explain the alternating movement of wavefunctions as the magnetic field inside the MQD varies, we show the probabilities densities of the wavefunctions for the $(2, 0)$ and $(3, 0)$ states in figures 3(c) and (d), respectively. For the $(2, 0)$ state, the wavefunction moves from inside the MQD at $\alpha_1 = -0.81$ to outside it at $\alpha_1 = -1.45$ (figure 3(c)). For the $(3, 0)$ state, the wavefunction moves from outside the MQD at $\alpha_1 = -0.81$ to inside it at $\alpha_1 = -1.45$, and then moves out again at $\alpha_1 = -1.85$ (figure 3(d)). Before and after energy anticrossing occurs, the probability densities are located alternately inside and outside the MQD.
and (b), while the wavefunctions of the $B$ and $B'$ subatoms are more strongly localized within the MQD than those for $\alpha_1 = 0.00$, as shown in figures 4(c) and (d). As a result, $|\phi_A|^2$ and $|\phi_A'|^2$ are distributed in a ring shape along the MQD boundary, while $|\phi_B|^2$ and $|\phi_B'|^2$ are distributed inside the MQD. The $(n,m \geq 0)$ states, where $n = 0, 1$, tend toward zero energy, because the wavefunctions of each subatom start to be separated when $\alpha_1$ increases in the negative direction. From these results, we can infer that wavefunction separation also occurs for zero-eigenvalue states in BLG, even with a nonuniform magnetic field distribution.

3.2. Magnetic quantum ring

In this subsection, we discuss the electronic properties of BLG with an MQR. For a fixed ring area with parallel magnetic fields ($\alpha_1 = \alpha_2$), the results for the eigenvalue spectra show nothing unusual compared with the cases of 2DEG and monolayer graphene [36, 39–41]. Here, we investigate BLG with an MQR from a different perspective.

First, we investigate the energy eigenvalues by varying the inner radius $r_1$ of the MQR with parallel magnetic fields ($\alpha_1 = \alpha_2$). We fix the outer radius of the MQR as $r_2 = 5$. The results are presented in figure 5. In figure 5(a), the energy eigenvalues $\epsilon_{nm}$ are shown as functions of $r_1$. The eigenvalue of the $(1,0)$ state (blue line) increases until it converges to the LL ($n = 1$) inside the ring as $r_1$ increases. The eigenvalue of the $(2,0)$ state (orange line) exhibits a similar tendency, and it converge to the LL ($n = 2$) inside the ring through energy anticrossing near the LL ($n = 1$), as shown in figure 5(a). The average kinetic energy of an electron in a magnetic field is higher than in the absence of the field. Increasing the inner radius $r_1$ causes a decrease in the missing magnetic flux in the ring area, and so the energy eigenvalues of the BLG increase.

To explain the wavefunction movement near anticrossing, in figure 5(b), we plot the probability density distribution of the $(1,0)$ state as a function of $r$ for different values of the inner ring radius $r_1$. For $r_1 = 2.93$ (blue lines) before anticrossing, the wavefunction is distributed widely over the MQR. For $r_1 = 3.09$ (green lines) after anticrossing, the wavefunction is localized inside the inner circle of the MQR. The energy converges to the LL ($n = 1$) inside the ring when $r_1 > 3.04$. This anticrossing can be explained using the same arguments as in the MQD case. At anticrossing, the size of the inner magnetic field region is large enough to encompass the wavefunctions of all subatoms for a lower energy. The wavefunctions within the ring move to the inner magnetic field region ($r_1 < r$), while a part of the wavefunction for higher energy moves to the ring region $r_1 < r < r_2$. Before and after energy anticrossing occurs, the wavefunctions are alternately located in regions with $B = 0$ and $B \neq 0$ (inside the inner circle or outside the outer circle) of the MQR.

To study the angular momentum transition as the inner radius of the MQR varies, in figure 5(c), we plot $\epsilon_{nm}$ as functions of $r_1$ in the region $\epsilon_{nm} < 0.5$ for the $(1,m)$ states. For an MQR with a finite width and for a conventional quantum ring (CQR) confined by an electrostatic potential, an angular momentum transition is observed with increasing magnetic
Figure 5. (a) Energy eigenvalues $\epsilon_{nm}$ as functions of $r_1$ for an MQR in BLG. Colored lines are for the state with $m = 0$, and gray lines represent other states $(n, m)$, where $n \in [-2, 2]$ and $m \in [-3, 3]$. Red dashed lines show the LLs of BLG. (b) Probability densities $|\psi(r)|^2$ of the $(1,0)$ state for different values of $r_1$ near energy anticrossing. The red dashed line indicates the outer radius $r_2$ of the ring. (c) $\epsilon_{nm}$ as a function of $r_1$ in the region of $\epsilon_{nm} < 0.5$ for the $(1,m)$ states.

Figure 6. Energy spectra of BLG with an MQR. The inner radius $r_1$ and outer radius $r_2$ of the MQR are 3 and 5, respectively. (a) Energy eigenvalues $\epsilon_{nm}$ as functions of $\alpha_1$. (b) Energy eigenvalues $\epsilon_{nm}$ of the states $(2,m)$, with $m \in [16, 25]$ as functions of magnetic flux. (c) Energy eigenvalues $\epsilon_{nm}$ of the states $(1,m)$, with $m \in [-20, -30]$, as functions of magnetic flux.

When the size of the ring is fixed, this transition is due mainly to the missing flux quanta in the ring area. However, in the present study, an angular momentum transition is observed with increasing inner ring radius $r_1$ for a fixed magnetic field. As the inner ring radius increases, the number of missing flux quanta in the ring area ($r_1 < r < r_2$) decreases, but the number of flux quanta within the inner ring radius increases. The small value of $m$ leads to rapid recovery of the LL as the inner circle radius $r_1$ increases. From the existence of this phenomenon, we can infer that the angular momentum transition depends more on the inner circle flux quanta than on the missing flux quanta in the ring area. The argument based on missing flux quanta is only valid when the ring area remains constant.

In figure 6, we show the energy eigenvalues $\epsilon_{nm}$ of BLG with an MQR as functions of $\alpha_1$ for fixed $r_1 = 3$, $r_2 = 5$, and $\alpha_2 = 1$. Here, we change the magnetic field $\alpha_1$ of the inner circle. As we vary $\alpha_1$ from $-1.5$ to $+1.5$, we can see the energy spectra of the MQR in parallel magnetic fields as well as in antiparallel fields. The energies of the $(0,m \geq 0)$ and $(1,m \geq 0)$ states of the MQR approach zero, similarly to what happens for the MQD, although the states of the MQR reach zero energy at a larger $|\alpha_1|$ value than those of the MQD. Geometrically, the difference between an MQD and an MQR is the presence in the latter of the zero-magnetic-field region $r_1 < r < r_2$. This difference is responsible for the very different behavior of $\epsilon_{nm}$. 
For CQRs formed by an electrostatic potential or by an etched boundary and based on a 2DEG and monolayer graphene, the AB effect has been studied both theoretically and experimentally [16, 42–46]. On the other hand, little attention has been paid to the AB effect for an MQR in a 2DEG. In a CQR, the energy of an electron bound in the ring oscillates as a function of the magnetic flux \( \phi \) inside the ring. We find that the energy dispersion of BLG with an MQR also exhibits this effect for both large positive and large negative values of \( |\alpha_1| \). To investigate the AB effect for an MQR, we calculate the period of the magnetic flux \( \Delta \phi \) of the MQR numerically, and we find that this period is close to \( \phi_0 \): for \( \alpha_1 < 0 \), it is \( 1.01 \phi_0 \), and for \( \alpha_1 > 0 \), it is \( 0.97 \phi_0 \). For small \( \alpha_1 \), the wavefunctions are not well localized in the region ring. Thus, the AB effect is not clearly seen in the range between \( r_1 \) and \( r_2 \), and this transition depends more on the magnetic fields inside the inner ring and outside the outer ring. However, as \( |\alpha_1| \) increases, the quantum states start to be strongly localized in the ring region, and exhibit an obvious AB effect. The AB effect in MQR can be shown by changing the inner radius \( r_1 \) because the AB effect is coming from variation of internal magnetic flux. However, the angular momentum transition due to the AB effect is allowed in a certain range. The maximum value of the angular momentum quantum number \( n_{\text{max}} \) allowed for the transition is approximately \( (\alpha_1 r_1^2 \pi)/\phi_0 \). Unlike CQRs, whose energy has symmetry with regard to the direction of the magnetic flux inside the ring [16], the energy in our case does not possess such symmetry, because our MQR is formed by a nonuniform magnetic field.

4. Conclusion

In this paper, we have analytically studied the electronic properties of BLG with a MQD and an MQR. We have calculated energy eigenvalues and wavefunctions based on a continuum model of the BLG using the Dirac equation. In contrast to previous studies, we have considered as significant variables the nonzero magnetic field inside the quantum dot, the size of the dot, the radius of the inner ring, and the directions of the magnetic fields inside the inner ring and outside the outer ring.

For the MQD, as the inside magnetic field increases in the positive direction, the corresponding energy eigenvalues of the BLG approach the LLs inside the dot, and its states are localized inside the dot. However, when the magnetic field inside the dot has the opposite direction to that outside the dot, we see two characteristic features. The first is a stepwise evolution of states through energy anticrossing. The second is that there are quantum states approaching zero energy. In these zero-energy states, the wavefunctions at each subatom are separated to lie either inside or outside the MQD. We explain these characteristics by using an effective potential and a separation of the wavefunction probability density.

For the MQR, an angular momentum transition occurs when \( r_1 \) is varied, and this transition depends more on the magnetic flux inside the inner circle of the ring than on the missing flux quanta in the ring area. For a negative magnetic flux \( \alpha_1 \), the MQR behaves similarly to the MQD, with some quantum states approaching zero energy, although the critical value of \( \alpha_1 \) associated with zero energy is larger than in the case of an MQD. We have investigated the AB effect on energy spectra of the MQR. For a large magnetic flux inside the ring, the period of the AB effect in the ring is close to the value of the magnetic flux quantum \( \phi_0 \).

We believe that our results here could be helpful as a basis for further study of quantum transport in BLG with these magnetic quantum structures.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Acknowledgments

This work was supported by Basic Science Research Program through National Research Foundation of Korea (NRF) funded by the Ministry of Science and ICT (Grant No. NRF-2019R1A2C1088327) and the Ministry of Education (Grant No. NRF-2018R1D1A1B07046338).

Conflict of interest

The authors have no conflicts to disclose.

ORCID IDs

Daehan Park https://orcid.org/0000-0001-5181-4571
Heesang Kim https://orcid.org/0000-0003-4985-0594
Nammee Kim https://orcid.org/0000-0003-3015-4353

References

[1] Novoselov K S, Geim A K, Morozov S V, Jiang D, Zhang Y, Dubonos S V, Grigorieva I V and Firsov A A 2004 Science 306 666–9
[2] Novoselov K S, Geim A K, Morozov S V, Jiang D, Katsnelson M I, Grigorieva I V, Dubonos S V and Firsov A A 2005 Nature 438 197–200
[3] Zomer P J, Dash S P, Tombre N and van Wees B J 2011 Appl. Phys. Lett. 99 232104
[4] Gusynin V P and Sharapov S G 2005 Phys. Rev. Lett. 95 146801
[5] Zhang Y, Tan Y W, Stormer H L and Kim P 2005 Nature 438 201–4
[6] Klein O 1929 Z. Phys. 53 157–65
[7] De Martino A, Dell’Anna L and Egger R 2007 Phys. Rev. Lett. 98 066802
[8] De Martino A, Dell’Anna L and Egger R 2007 Solid State Commun. 144 547–50
[9] Ramezani Masir M, Vasilopoulos P and Peeters F M 2008 Appl. Phys. Lett. 93 242103
[10] Ramezani Masir M, Vasilopoulos P and Peeters F M 2009 Phys. Rev. B 79 035409
[11] Wang D and Jin G 2009 Phys. Lett. A 373 4082–5
[12] Ramezani Masir M, Matulis A and Peeters F M 2009 Phys. Rev. B 79 155451
[13] Lee C M, Chan K S and Ho J C Y 2014 J. Phys. Soc. Japan 83 034007
