Intertwining of Complementary Thue-Morse Factors

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March 8, 2022

Abstract

We consider the positions of occurrences of a factor $x$ and its binary complement $\overline{x}$ in the Thue-Morse word $t = 01101001 \cdots$, and show that these occurrences are “intertwined” in essentially two different ways. Our proof method consists of stating the needed properties as a first-order logic formula $\varphi$, and then using a theorem-prover to prove $\varphi$.

1 Introduction

The Thue-Morse sequence $t = t_0 t_1 t_2 \cdots = 01101001 \cdots$ is a famous binary sequence with many interesting properties [1]. In this short note we prove yet another in a long list of such properties, this time concerning complementary factors.

A factor of an infinite word $w$ is a contiguous block sitting inside $w$. In this paper we will only be concerned with factors of finite length. Define $\overline{1} = 0$ and $\overline{0} = 1$, and extend this notion to words in the obvious way, so that if $w = a_1 a_2 \cdots a_n$, then $\overline{w} = \overline{a_1} \overline{a_2} \cdots \overline{a_n}$. We say two binary words $x, y$ are complementary if $x = \overline{y}$. Thus, for example, 0110 and 1001 are complementary.

It is well known that the Thue-Morse word $t$ is recurrent, that is, every factor that occurs, occurs infinitely often (first observed by Morse [3]). Further, it is complement-invariant: if a factor $x$ occurs in $t$, then so does its binary complement $\overline{x}$. This suggests looking at the positions of the occurrences of $x$ and $\overline{x}$ in $t$.

For example, let us do this for the complementary factors 00 and 11, marking the occurrences of 00 in red and 11 in blue:

$$01101001100101101001 \cdots$$
Seeing this, it is natural to conjecture that occurrences of 11 and 00 strictly alternate in \( t \), a conjecture that is not hard to prove and appears in [2]. Spiegelhofer [8, Lemma 2.10] handled the case of 01 and 10.

However, strict alternation, as in this example, is not the only possibility for other factors. If we consider the complementary factors 01101 and 10010 instead, then the occurrences behave differently:

\[
01101001100101101001011001101001\cdots.
\]

If we write \( A \) for an occurrence of 01101 and \( B \) for an occurrence of its complementary factor, then experiments quickly lead to the conjecture that these factors occur in the repeating pattern \((ABBA)^\omega = ABBAABBAABBA\cdots\).

In this paper we prove that the two patterns \((AB)^\omega\) and \((ABBA)^\omega\) are essentially the only nontrivial possibilities for intertwining of complementary factors.

For a deep study of the gaps between successive occurrences of factors in \( t \), see the recent paper of Spiegelhofer [8].

\section{The main theorem}

Let \( x \) be a finite, nonempty factor of the Thue-Morse word \( t \). Consider all occurrences of \( x \) and \( \overline{x} \) in \( t \) and identify their starting positions, writing \( A \) for an occurrence of \( x \) and \( B \) for an occurrence of \( \overline{x} \). (An occurrence of \( x \) may overlap that of \( \overline{x} \).) Call the resulting infinite sequence of \( A \)'s and \( B \)'s the \emph{intertwining sequence} of \( x \), and write it as \( I(x) \).

The following is our main result.

\textbf{Theorem 1.} The only possibilities for \( I(x) \) are as follows:

1. \( ABBAABBAABBA\cdots \), which is the Thue-Morse word itself under the coding \( 0 \to A, 1 \to B \);
2. \( BAABABBAABBA\cdots \), which is the Thue-Morse word itself under the coding \( 0 \to B, 1 \to A \);
3. \( (AB)^\omega \);
4. \( (BA)^\omega \);
5. \( (ABBA)^\omega \);
6. \( (BAAB)^\omega \).

Furthermore, possibility 1 only occurs if \( x = 0 \) and possibility 2 only occurs if \( x = 1 \).

\textbf{Proof.} It is trivial to see the claim for \( x = 0 \) and \( x = 1 \). So in what follows, we assume \(|x| \geq 2\).
The idea of our proof is to write first-order logic formulas for assertions that imply our desired results, and then use the theorem-prover Walnut to prove the results. This is a strategy that has been used many times now (see, e.g., [5]). For more about Walnut, see [4].

We describe the formulas in detail for the cases $\omega$ and $\omega$, leaving the other cases to the reader.

To assert that the pattern $(AB)\omega$ describes the occurrences of $x$ and $\overline{x}$ in $t$, we create first-order logic formulas asserting each of the following:

(a) one of the two words $x$ and $\overline{x}$ occurs at positions $j, k$ for $j < k$, and furthermore that neither of the two words occurs at any position between $j$ and $k$. This ensures that $j$ and $k$ mark the starting position of two consecutive factors chosen from $\{x, \overline{x}\}$.

(b) if $j, k$ are two positions as in (a), then one must be the position of $x$, while the other is the position of $\overline{x}$. This forces the consecutive occurrences of the factors to alternate, and hence form either the pattern $(AB)\omega$ or $(BA)\omega$.

(c) the first occurrence of either $x$ or $\overline{x}$ in $t$ is actually an occurrence of $x$. This, together with (b), forces the pattern to be of the form $(AB)\omega$.

We specify the word $x$ by giving one of its occurrences, that is, two integers $i, n$ such that $x = t[i..i + n - 1]$.

Here is the meaning of each logical formula we now define.

- $\text{feq}(i, j, n)$ asserts that $t[i..i + n - 1] = t[j..j + n - 1]$;
- $\text{feqc}(i, j, n)$ asserts that $t[i..i + n - 1] = \overline{t}[j..j + n - 1]$;
- $\text{either}(i, j, n)$ asserts that either $t[i..i + n - 1] = t[j..j + n - 1]$ or $t[i..i + n - 1] = \overline{t}[j..j + n - 1]$;
- $\text{consec}(i, j, k, n)$ asserts that $j < k$ and $t[j..j + n - 1] \in \{x, \overline{x}\}$ and $t[k..k + n - 1] \in \{x, \overline{x}\}$, where $x = t[i..i + n - 1]$, but no factor starting in between these two equals either $x$ or $\overline{x}$.
- $\text{ab}(i, j, k, n)$ asserts $t[j..j + n - 1] = x$ and $t[k..k + n - 1] = \overline{x}$, for $x = t[i..i + n - 1]$.
- $\text{first}(i, j, n)$ asserts that $t[j..j + n - 1]$ is the first occurrence of the factor $t[i..i + n - 1]$ in $t$;
- $\text{afirst}(i, n)$ asserts that the first occurrence of the factor $x = t[i..i + n - 1]$ precedes the first occurrence of $\overline{x}$ in $t$;
- $\text{abpat}(i, n)$ asserts that the intertwining sequence of $x = t[i..i + n - 1]$ and $\overline{x}$ is $(AB)\omega$.
- $\text{bapat}(i, n)$ asserts that the intertwining sequence of $x = t[i..i + n - 1]$ and $\overline{x}$ is $(BA)\omega$. 

3
\[
\begin{align*}
\text{feq}(i, j, n) &:= \forall k \ (k < n) \implies t[i + k] = t[j + k] \\
\text{feqc}(i, j, n) &:= \forall k \ (k < n) \implies t[i + k] \neq t[j + k] \\
\text{either}(i, j, n) &:= \text{feq}(i, j, n) \lor \text{feqc}(i, j, n) \\
\text{consec}(i, j, k, n) &:= (j < k) \land \text{either}(i, j, n) \land \text{either}(i, k, n) \land \forall l \ (j < l \land l < k) \\
\implies \neg \text{either}(i, l, n) \\
\text{ab}(i, j, k, n) &:= \text{feq}(i, j, n) \land \text{feqc}(i, k, n) \\
\text{first}(i, j, n) &:= \text{feq}(i, j, n) \land \forall k \ (k < j) \implies \neg \text{feq}(i, k, n) \\
\text{afirst}(i, n) &:= \forall j, k \ (\text{first}(i, j, n) \land \text{feqc}(i, k, n)) \implies j < k \\
\text{abpat}(i, n) &:= (n > 0) \land \text{afirst}(i, n) \land \forall j, k \ \text{consec}(i, j, k, n) \\
&\implies (\text{ab}(i, j, k, n) \lor \text{ab}(i, k, j, n)) \\
\end{align*}
\]

The translation into Walnut is

```
def feq "Ak (k<n) => T[i+k]=T[j+k]":
def feqc "Ak (k<n) => T[i+k]!=T[j+k]":
def either "$feq(i,j,n)|$feqc(i,j,n)":
def consec "j<k & $either(i,j,n) & $either(i,k,n) & Al (j<l & l<k) \\
implies \neg$either(i,l,n)":
def ab "$feq(i,j,n) & $feqc(i,k,n)":
def first "$feq(i,j,n) & Ak (k<j) => \neg$feq(i,k,n)":
def afirst "Aj,k ($first(i,j,n) & $feqc(i,k,n)) => (j<k & Al (j<l & l<k) \\
($ab(i,j,k,n)|$ab(i,k,j,n))":
def abpat "(n>0) & $afirst(i,n) & Aj,k $consec(i,j,k,n) \\
implies (ab(i,j,k,n) \lor ab(i,k,j,n))":
```

We now do the same thing for the patterns \((ABBA)^\omega\) and \((BAAB)^\omega\). The one complication is that to assert that the intertwining sequence is \((ABBA)^\omega\), for example, then one must assert that

(a) the first two occurrences of either \(x\) or \(\overline{x}\) form the pattern \(AB\);

(b) three consecutive occurrences of either \(x\) or \(\overline{x}\) in \(t\) must form the pattern \(ABB\) or \(BBA\) or \(BAA\) or \(AAB\).

An easy induction now shows that the overall pattern can only be \((ABBA)^\omega\).

We give only the Walnut commands for checking this.

```
def firstc "$feqc(i,j,n) & Ak (k<j) => \neg$feqc(i,k,n)":
# j is the first occurrence of the complement of t[i..i+n-1]
def abfirst "Aj,k ($first(i,j,n) & $firstc(i,k,n)) => (j<k & Al (j<l & l<k) \\
($ab(i,j,k,n)|$ab(i,k,j,n))":
```

4
Now we are ready to finish the proof of the theorem. First we check that
\[
I(11) = I(t[1..2]) = (AB)\omega
\]
\[
I(00) = I(t[5..6]) = (BA)\omega
\]
\[
I(101) = I(t[2..4]) = (ABBA)\omega
\]
\[
I(010) = I(t[3..5]) = (BAAB)\omega,
\]
as follows:

\[
\text{eval alloccur } abpat(1,2) \& bapat(5,2) \& abbapat(2,3) \& baabpat(3,3)\ :
\]
and \text{Walnut} returns \text{TRUE}.

Next, we check that for all \(i\) and all \(n \geq 2\), the intertwining sequence of \(t[i..i+n-1]\) is either \((AB)\omega\), \((BA)\omega\), \((ABBA)\omega\), or \((BAAB)\omega\).

\[
\text{eval checkeach } Ai,n \ (n \geq 2) \Rightarrow (abpat(i,n) \& bapat(i,n) \& abbapat(i,n) \& baabpat(i,n))\ :
\]
and \text{Walnut} returns \text{TRUE}.

This completes the proof. \(\square\)

For factors of length \(n = 2\), the only intertwining patterns that occur are \((AB)\omega\) and \((BA)\omega\). However, for each \(n \geq 3\), we can prove that each of the four patterns actually occurs.

\textbf{Theorem 2.} \textit{For every }\(n \geq 3\), \textit{and each of the four patterns }\(p \in \{AB, BA, ABBA, BAAB\}\), there is a length-\(n\) factor \(x\) of \(t\) whose occurrence pattern is \(p^\omega\).

\textit{Proof.} We use \text{Walnut} with the command

\[
\text{eval checklen } An \ (n \geq 3) \Rightarrow E_i,j,k,l \ abpat(i,n) \& bapat(j,n) \& abbapat(k,n) \& baabpat(l,n)\ :
\]
and \text{Walnut} returns \text{TRUE}. \(\square\)
3 Automata

We now give the four automata for the four cases of intertwining sequence. Each automaton takes, as input, the base-2 expansions of $i$ and $n$ in parallel, starting with the most significant bit, and accepts if and only if $t[i..i + n - 1]$ has the specified intertwining sequence.

Figure 1: Automaton for $(i, n)$ such that $I(t[i..i + n - 1]) = (AB)^\omega$. 
Figure 2: Automaton for \((i, n)\) such that \(I(t[i..i+n-1]) = (BA)\omega\).

Figure 3: Automaton for \((i, n)\) such that \(I(t[i..i+n-1]) = (ABBA)\omega\).
Figure 4: Automaton for \((i,n)\) such that \(I(t[i..i+n-1]) = (BAAB)^\omega\).

Remark 3. We can combine these automata, as in [6], to get a single DFAO that, on input \((i,n)\), computes which of the six possibilities in Theorem 1 occurs. However, the resulting automaton has 30 states and is rather complicated in appearance, so we don’t give it here.

4 Number of factors of each type

We now determine the number of length-\(n\) factors of each of the four types. It is easy to see that there is a 1-1 correspondence between length-\(n\) factors where the intertwining sequence is \((AB)^\omega\) and those where the intertwining sequence is \((BA)^\omega\), and similarly for those with intertwining sequence \((ABBA)^\omega\) and \((BAAB)^\omega\). Thus it suffices to just handle \((AB)^\omega\) and \((ABBA)^\omega\).

Let \(f(n)\) be the number of length-\(n\) factors \(x\) of \(t\) where \(I(x) = (AB)^\omega\), and let \(g(n)\) be the number of length-\(n\) factors \(x\) of \(t\) where \(I(x) = (ABBA)^\omega\). Here is a table of the first few values of these functions:

| \(n\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| \(f(n)\) | 0 | 2 | 4 | 4 | 6 | 8 | 8 | 8 | 10 | 12 | 14 | 16 | 16 | 16 | 16 |
| \(g(n)\) | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 4 | 4 | 4 | 5 | 6 | 6 |

It turns out that both of these are expressible in terms of known sequences.
Theorem 4. We have

\[ f(n+1) = 2 \cdot A006165(n) \quad \text{for } n \geq 1; \]
\[ g(n+1) = A060973(n) \quad \text{for } n \geq 0, \]

where the sequence numbers refer to sequences in the On-Line Encyclopedia of Integer Sequences (OEIS) \[7\].

Proof. Let us start with \((AB)^\omega\). Using the Walnut commands

def firstocc "Aj (j<i) => ~$feq(i,j,n)"

eval mab n "$firstocc(i,n+1) & $abpat(i,n+1)"

we can construct the linear representation for \(f(n+1)\). It is

\[ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \gamma_1(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \gamma_1(1) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}. \]

On the other hand, from the known relations for \(A006165(n)\), namely

\[ A006165(2n) = 2A006165(n) - [n = 0] - [n = 1] \]
\[ A006165(2n + 1) = A006165(n + 1) + A006165(n) - [n = 0] \]

from which we can compute its linear representation:

\[ v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \gamma_2(0) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}, \quad \gamma_2(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \]

Here we are using the Iverson bracket, where (for example) the expression \([n = 0]\) evaluates to 1 if \(n = 0\) and 0 otherwise.

From these two linear representations, we can easily compute the linear representation for \(f(n+1) - 2 \cdot A006165(n)\) and then minimize it. When we do so, we get a linear representation of rank 1 that evaluates to the function \(-2[n = 0]\), so indeed \(f(n+1) = 2 \cdot A006165(n)\) for all \(n \geq 1\).

We can do the same thing for \(g(n+1)\), using the Walnut command:

eval mabba n "$firstocc(i,n+1) & $abbapat(i,n+1)"

\[ v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \gamma_3(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad \gamma_3(1) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \]

On the other hand, from the known relations for \(A060973(n)\), namely

\[ A060973(2n) = 2A060973(n) + [n = 1] \]
\[ A060973(2n + 1) = A060973(n + 1) + A060973(n) \]
from which we can compute its linear representation:

\[ v_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \quad \gamma_4(0) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \gamma_4(1) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad w_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \]

Once again we can compute the linear representation for \( g(n+1) - \text{A060973}(n) \) and minimize it. When we do so, we get a linear representation of rank 0, computing the constant function 0.

Finally, using the known expressions for the two sequences \text{A006165} and \text{A060973}, we arrive at the following result:

**Corollary 5.** For \( n \geq 2 \) we have

\[
\begin{align*}
  f(n) &= \begin{cases} 
  2^k, & \text{if } 3 \cdot 2^{k-2} < n \leq 2^k + 1; \\
  2n - 2^k - 2, & \text{if } 2^k + 1 < n \leq 3 \cdot 2^{k-1}. 
  \end{cases}
\end{align*}
\]

For \( n \geq 3 \) we have

\[
\begin{align*}
  g(n) &= \begin{cases} 
  2^{k-1}, & \text{if } 2^k + 1 < n \leq 3 \cdot 2^{k-1} + 1; \\
  n - 2^{k-1} - 1, & \text{if } 3 \cdot 2^{k-1} + 1 < n \leq 2^k + 1.
  \end{cases}
\end{align*}
\]

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