Scattering of solitons on resonance

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Abstract

We investigate a propagation of solitons for nonlinear Schrodinger equation under small driving force. The driving force passes the resonance. The process of scattering on the resonance leads to changing of number of solitons. After the resonance the number of solitons depends on the amplitude of the driving force.

Nonlinear Schrodinger equation (NLSE) is a mathematical model for wide class of wave phenomenons from signal propagation into optical fibre [1, 2] to surface wave propagation [3]. This equation is integrable by inverse scattering transform method [4] and can be considered as an ideal model equation. The perturbations of this ideal model lead to nonintegrable equations. Here we consider such nonintegrable example which is NLSE perturbed driving force.

The most known class of the solutions of NLSE is solitons [4]. The structure of this kind of solutions is not changed in a case of nonperturbed NLSE. The perturbations usually lead to modulation of parameters of solitons [5, 6]. Number of solitons does not change.

In this work we investigate a new effect called scattering of solitons on resonance. We consider the process of scattering in detail and obtain the connection formula between pre-resonance and post-resonance solutions. In general case the passage through resonance leads to changing of the number of solitons. This effect is based on the soliton generation due to passage through resonance by external driving force [7].

We found that the scattering of solitary waves on resonance is a general effect for nonlinear equations described the wave propagation. In this work we investigate this effect for the simplest model. It allows to show the essence of this effect without unnecessary details.

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1 STATEMENT OF THE PROBLEM AND RESULT

This paper has the following structure. The first section contains the statement of the problem and the main result. The second section contains the asymptotic construction in the pre-resonance domain. In the third section we construct the asymptotic solution in the neighborhood of the resonance curve. The fourth section of the paper is devoted to construction of the post-resonance asymptotics. Asymptotics are constructed by multiple scale method [8] and matched [9].

1 Statement of the problem and result

Let us consider the perturbed NLSE

\[ i\partial_t \Psi + \partial^2_x \Psi + |\Psi|^2 \Psi = \varepsilon^2 f e^{iS/\varepsilon^2}, \quad 0 < \varepsilon \ll 1. \]  

(1)

The phase of the driving force is \( S/\varepsilon^2 = \omega t \). The amplitude \( f = f(\varepsilon x) \) is a smooth and rapidly vanished function.

In the simplest case the phase is linear function with respect to \( t \) \( S/\varepsilon^2 = \omega t, \omega = \text{const} \). In general situation the constant frequency of the driving force does not lead to scattering of solitons. Let us investigate the driving force with slowly varying frequency. The most simplest dependence on \( t \) for \( \omega \) has a form \( \omega = \varepsilon^2 t/2 \). The amplitude \( f \) of the driving force admit an additional dependency on \( \varepsilon^2 t \) but it leads to complicated formulas and no more.

Let us formulate the result of this work. Below we use the following variables \( x_j = \varepsilon^j x, \; t_j = \varepsilon^j t, \; j = 1, 2 \).

Let the asymptotic solution of (1) be

\[ \Psi(x, t, \varepsilon) = \varepsilon^1 u(x_1, t_2) + O(\varepsilon^2) \quad \text{as} \quad t_2 < 0, \]

where \( u(x_1, t_2) \) satisfies

\[ \partial_{t_2} \frac{1}{\varepsilon} u + \partial^2_{x_1} \frac{1}{\varepsilon} u + \left| \frac{1}{\varepsilon} u \right|^2 \frac{1}{\varepsilon} u = 0 \]

and initial condition

\[ \frac{1}{\varepsilon} u \big|_{t_2 = t_0} = h_1(x_1), \quad t_0 = \text{const} < 0. \]

Then in the domain \( t_2 > 0 \) the asymptotic solution of (1) has a form

\[ \Psi(x, t, \varepsilon) = \varepsilon^1 v(x_1, t_2) + O(\varepsilon^2), \]  

(2)
where \( \frac{1}{2} v(x_1, t_2) \) is a solution of NLSE with initial condition
\[
\left. \frac{1}{2} v \right|_{t_2=0} = \frac{1}{2} u(x_1, 0) + (1 - i) \sqrt{\pi f(x_1)}.
\]

Let us explain the result for soliton solution. If in the domain \( t_2 < 0 \) the solution has \( N \)-soliton form then in the domain \( t_2 > 0 \) the number of solitons is defined by initial condition for \( \frac{1}{2} v \).

2 Incident wave

In this section we construct the asymptotic solution of equation (1) in pre-resonance domain. This solution contains two parts. The first part is a specific solution of the nonhomogeneous equation. This solution oscillates with the frequency of the driving force. The amplitude is determined by an algebraic equation. The second part of the solution is a solution of the homogeneous equation. The solution contains undefined function due to integration. This undefined function usually determines by initial condition for Cauchy problem.

We construct the formal asymptotic solution of the form
\[
\Psi(x, t, \varepsilon) = \varepsilon \frac{1}{2} u(x_1, t_2) + \varepsilon^3 \frac{3}{2} u(x_1, t_2) + \varepsilon^2 \left( \frac{2}{4} B(x_1, t_2) \exp(iS/\varepsilon^2) + \right.
\]
\[
\left. + \varepsilon^4 \left( \frac{4}{4} B_1(x_1, t_2) \exp(iS/\varepsilon^2) + \frac{4}{4} B_{-1}(x_1, t_2) \exp(-iS/\varepsilon^2) \right) \right)
\]
\[
+ \varepsilon^5 \frac{5}{5} B_2(x_1, t_2) \exp(2iS/\varepsilon^2).
\]

To determine the coefficients of the asymptotics substitute (3) into equation (1). It yields
\[
\varepsilon^2 \left( - S^t \frac{2}{2} B - f \right) \exp(iS/\varepsilon^2) + \varepsilon^3 \left( i \frac{1}{2} u_{t_2} + \frac{1}{2} u_{x_1 x_1} + \frac{1}{2} u^2 \frac{1}{2} u \right)
\]
\[
+ \varepsilon^4 \left( - S^t \frac{4}{4} B_1 + i \frac{2}{2} B_{t_2} + \frac{2}{2} B_{x_1 x_1} + 2 \frac{1}{2} u^2 \frac{2}{2} B \right) \exp(iS/\varepsilon^2) + \right.
\]
\[
\left. + \left( S^t \frac{4}{4} B_{-1} + \frac{1}{2} u^2 \frac{2}{2} B^* \right) \exp(-iS/\varepsilon^2) \right)
\]
\[
+ \varepsilon^5 \left( i \frac{3}{3} u_{t_2} + \frac{3}{3} u_{x_1 x_1} + 2 \frac{1}{2} u^2 \frac{3}{3} u + \frac{1}{2} u^2 \frac{3}{3} u^* + 2 \frac{1}{2} u \frac{2}{2} B \right)
\]
\[
+ \left( - 2S^t \frac{5}{5} B_2 + \frac{1}{2} u^2 \frac{2}{2} B^2 \right) \exp(2iS/\varepsilon^2) \right) = O(\varepsilon^6 R(t_2, x_1)).
\]
The residue part of the asymptotics has a form

\[ R(t_2, x_1) = O\left( |\frac{2}{B}|^3 + \varepsilon^3 |\hat{u}|^3 + \varepsilon^6 |\frac{4}{B_1}|^3 + \varepsilon^6 |\frac{4}{B_{-1}}|\right. \]

Collect the terms with the same order of \( \varepsilon \) up to the order of \( \varepsilon^5 \) and reduce similar terms. It yields differential equations for \( \frac{1}{2}u, \frac{3}{4}u \) and algebraic equations for \( 2B, \frac{4}{B_1} \) and \( \frac{5}{B_2} \).

\[
i \frac{1}{2}u_t + \frac{1}{2}u_{x_1x_1} + \frac{1}{2}u |^2 \frac{1}{2}u = 0, \]
\[
i \frac{3}{4}u_t + \frac{3}{4}u_{x_1x_1} + 2\frac{1}{2}u |^2 \frac{3}{4}u + \frac{1}{2}u^2 \frac{3}{4}u^* = -2\frac{2}{2}B |^2 \frac{1}{2}u, \]
\[
-S' \frac{2}{B} = f, \]
\[
-S' \frac{4}{B_1} = i \frac{2}{B_t} + \frac{2}{B_{x_1x_1}} + \frac{1}{2}u |^2 \frac{2}{B}, \]
\[
S' \frac{4}{B_{-1}} = -\frac{1}{2}u^2 \frac{2}{B^*}, \]
\[
-2S' \frac{5}{B_2} = \frac{1}{2}u^2 \frac{2}{B^2}. \]

The coefficients \( \frac{1}{2}u, \frac{3}{4}u \) are uniquely determined by initial conditions at the moment \( t_2 = t^0 \). We suppose that \( t^0 = \text{const} < 0 \) and

\[ \frac{1}{2}u |_{t_2=t^0} = h_1(x_1); \quad \frac{3}{4}u |_{t_2=t^0} = h_3(x_1); \]

where functions \( h_1, h_3 \) are smooth and rapidly vanish as \( |x_1| \to \pm \infty \).

The coefficients of the representation (3) have a singularity as \( S' \to 0 \). The order of singularity of \( \frac{j}{k}B \) is easy calculated.

\[ \frac{2}{B} = O(t^{-1}), \quad \frac{4}{B_1} = O(t^{-3}). \]

To determine the asymptotics of \( \frac{3}{4}u \) as \( t_2 \to -0 \) we construct the solution of the form

\[ \frac{3}{4}u = t_2^{-1} \frac{3}{4}u (-1, 0)(x_1, t_2) + \ln |t_2| t_2 \frac{3}{4}u (0, 1)(x_1, t_2) + t_2 \ln |t_2| t_2 \frac{3}{4}u (1, 1)(x_1, t_2) + \frac{3}{4}u(x_1, t_2). \]

Substitute this representation into equation for \( \frac{3}{4}u \) and collect the terms of the same order with respect to \( t_2 \). It yields equations for coefficients of the
asymptotics
\[ 
\begin{align*}
\hat{3}u^{(-1,0)} &= i2|f|^2 \frac{1}{u}, \\
\hat{3}u^{(0,1)} &= -iL(\hat{3}u^{(-1,0)}), \\
\hat{3}u^{(1,1)} &= -iL(\hat{3}u^{(0,1)}), \\
L(\hat{3}u) &= it_2 \ln |t_2|L(\hat{3}u^{(1,1)}) + i \hat{3}u^{(1,1)}. 
\end{align*}
\]

Here \( L(u) \) is a linear operator of the form
\[ L(u) = i\partial_{t_2}u + \partial_{x_1}^2 u + 2|\frac{1}{u}|^2u + \frac{1}{u}u^* \cdot \]

Functions \( \hat{3}u^{(-1,0)}, \hat{3}u^{(0,1)} \), and \( \hat{3}u^{(1,1)} \) are determined from algebraic equations. These functions are bounded as \(-\text{const} < t_2 \leq 0, \text{const} > 0\).

The function \( \hat{3}u \) is a solution of nonhomogeneous linearized Schrödinger equation. The right hand side of the equation is a smooth function as \(-\text{const} < t_2 \leq 0, \text{const} > 0\). The solution of this equation can be obtained using results of [10]. In particularly if \( \hat{u} \) is N-solitons solution of NLSE then exists the bounded solution of nonhomogeneous linearized Schrödinger equation (4) as \(-\text{const} < t_2 \leq 0, \text{const} > 0\).

Coefficients of (3) have singularity at \( t_2 = 0 \). After substitution (3) into equation (1) we obtain a residue part. This residue part increases as \( t_2 \to 0 \). The domain of validity of (3) is determined by
\[ \varepsilon^6 R(t_2, x_1) = o(\varepsilon), \quad \varepsilon \to 0. \]

It yields
\[ -t_2 \gg \varepsilon \quad \text{or} \quad -t \gg \varepsilon^{-1}. \]

3 Scattering

In the neighborhood of the point \( t_2 = 0 \) the frequency of the driving force becomes resonant. It leads to changing of behaviour of the system described by (1). The forced mode of oscillations changes by a resonant mode. Formally it means representation (3) is not valid.

In this part of the work we construct another representation for the solution of equation (1). This representation is valid in the neighborhood of the
resonance line \( t_2 = 0 \).

\[
\Psi(x, t, \varepsilon) = \varepsilon \frac{1}{\varepsilon} w(x_1, t_1) + \varepsilon^2 \frac{2}{\varepsilon^2} w(x_1, t_1) + \\
\varepsilon^3 \ln \varepsilon \frac{3}{\varepsilon} w(x_1, t_1) + \varepsilon^3 \frac{3}{\varepsilon} w(x_1, t_1) \quad \varepsilon \to 0.
\]  

(5)

Here we use a new scaled variable \( t_1 = t_2/\varepsilon \). Representation (5) is matched with (3). It means these formulas are equivalent up to value \( o(\varepsilon^5) \) as \( t_2 \to -0 \).

The coefficients \( m \) of (5) are determined by ordinary differential equations (6), (8), (10) and matching conditions.

To obtain the behaviour of the coefficients of (5) as \( t_1 \to -\infty \) match (5) with (3). Write (3) in terms of \( t_1 \)

\[
\Psi(x, t, \varepsilon) = \varepsilon \left( \frac{1}{\varepsilon} u(x_1, 0) - (t_1^{-1} f + i t_1^{-3} f) \exp(i S/\varepsilon^2) \right) + \\
\varepsilon^2 \left( \partial_{t_2} \frac{1}{\varepsilon} u(x_1, t_2)|_{t_2=0} t_1 + t_1^{-1} i |f|^2 \frac{1}{\varepsilon} u(x_1, 0) + O(t_1^{-2}) \right) + \\
\varepsilon^3 \ln \varepsilon \left( -i L(2 |f|^2 \frac{1}{\varepsilon})|_{t_2=0} + o(1) \right) + \\
+ \varepsilon^3 \left( \frac{1}{2} \partial_{t_2}^2 \frac{1}{\varepsilon} u(x_1, t_2)|_{t_2=0} t_1^2 + \frac{3}{\varepsilon} u(x_1, 0) + o(1) \right), \quad 1 \ll -t_1 \ll \varepsilon^{-1}, \varepsilon \to 0.
\]

To obtain equations for coefficients of (5) substitute (5) into equation (11). It yields

\[
\varepsilon^2 \left( (\partial_{t_1} \frac{1}{\varepsilon} w - f \exp(i S/\varepsilon^2)) \right) + \varepsilon^3 \left( \partial_{t_1} \frac{2}{\varepsilon} w + \partial_{x_1}^2 \frac{1}{\varepsilon} w + \gamma | \frac{1}{\varepsilon} w |^2 \right) + \\
\varepsilon^4 \left( \partial_{t_1} \frac{3}{\varepsilon} w + \partial_{x_1}^2 \frac{2}{\varepsilon} w + \frac{1}{\varepsilon} w^2 w^* + 2 \gamma | \frac{1}{\varepsilon} w |^2 \frac{2}{\varepsilon} w \right) = O(\varepsilon^5 \rho(t_1, x_1, \varepsilon)).
\]

The function \( \rho(t_1, x_1, \varepsilon) \) can be represented in the form

\[
\rho(t_1, x_1, \varepsilon) = O(| \frac{1}{\varepsilon} w |^2 \frac{3}{\varepsilon} w + \partial_{x_1}^2 \frac{3}{\varepsilon} w + \varepsilon | \frac{2}{\varepsilon} w |^3 + \varepsilon^4 | \frac{3}{\varepsilon} w |^3).
\]

Collect the terms of the same order with respect to \( \varepsilon \). As result we obtain the equation for \( \frac{1}{\varepsilon} w \)

\[
i \partial_{t_1} \frac{1}{\varepsilon} w = f \exp(it_1^2/2).
\]  

(6)
The matching conditions give \( \psi = \frac{1}{t_1} u(x_1, 0) \) \( t_1 \to -\infty \). The solution of this problem is represented in terms of Fresnel integral

\[
\psi = \frac{1}{t_1} u(x_1, 0) - i f(x_1) \int_{-\infty}^{t_1} \exp(i \theta^2/2) d\theta.
\]

(7)

Equations for higher-order terms are

\[
i \partial_{t_1} \psi_2 = -\partial_{x_1}^2 \psi_1 + |\psi_1|^2 \psi_1,
\]

(8)

\[
i \partial_{t_1} \psi_3 = 0,
\]

(9)

\[
i \partial_{t_1} \psi_3 = -\partial_{x_1}^2 \psi_2 - 2|\psi_1|^2 \psi_2 - \psi_1^2 \psi_1^\ast.
\]

(10)

The higher-order terms satisfy first order ordinary differential equations with respect to \( t_1 \). The spatial variable \( x_1 \) is a parameter in these equations. The solutions of these equation are uniquely defined by terms of the order of \( 1 \) in asymptotics as \( t_1 \to -\infty \). The asymptotics as \( t_1 \to -\infty \) is obtained by matching

\[
\psi_2 = \partial_{t_2} \frac{1}{t_1} u(x_1, t_2)|_{t_2=0} + o(1);
\]

\[
\psi_3 = -i L \left( 2i |f|^2 \frac{1}{t_1} u \right)|_{t_2=0} + o(1),
\]

\[
\psi_3 = \frac{1}{2} \partial_{t_2}^2 \frac{1}{t_1} u(x_1, t_2)|_{t_2=0} + \frac{3}{2} \hat{u}(x_1, 0) + o(1).
\]

To determine the behaviour of the solution after resonance we need to calculate the asymptotics as \( \tau \to +\infty \) of the coefficients for representation (5).

Calculations give

\[
\psi(x_1, t_1) = \frac{1}{t_1} u(x_1, 0) - i f(x_1) \left[ ic_1 + \frac{\exp(it_1^2/2)}{it_1} + O(t_1^{-3}) \right],
\]

where \( c_1 = (1 - i)\sqrt{\pi} \).

Denote by

\[
\bar{\psi}(x_1, t_1)|_{t_1=0} = \psi_0(x_1)
\]

The function \( \bar{\psi}(x_1, t_1) \) has the asymptotics of the form

\[
\bar{\psi}(x_1, t_1) = t_1 \bar{\psi}_1(x_1) + \bar{\psi}_0(x_1) + g_1(x_1) \frac{\exp(it_1^2/2)}{it_1^2} + O(t_1^{-4})
\]

\]
where
\[ w_1^2 = -\partial_{x_1}^2 w_0 + |w_0|^2 w; \]
\[ w_0(x_1) = \lim_{t_1 \to \infty} \left( \int_{-\infty}^{t_1} |\partial_{x_1}^2 w(x_1, \theta) + |w(x_1, \theta)|^2 w(x_1, \theta)|d\theta - \frac{w_1 t_1}{2} \right), \]
\[ g_1(x_1) = k_1 \partial_{x_1} f + k_2 |f|^2 f, \]
\[ k_1 \text{ and } k_2 \text{ are constants.} \]
\[ 3 \left( w(x_1, t_1) = t_1^2 3 w_2(x_1) + o(t_1^2). \right. \]
\[ \left. \right] \]
where
\[ 3 w_2(x_1) = i \left( \partial_{x_1}^2 w_1 + 2 |w_0|^2 w_1 + \frac{1}{2} |w_0|^2 w_1 \right), \]

Representation (3) of the solution for (1) is valid as
\[ \varepsilon^5 \rho(t_1, x_1, \varepsilon) = o(\varepsilon). \]
The determined above behaviour of coefficients of asymptotics (4) give the domain of validity with respect to \( t_1 \)
\[ |t_1| \ll \varepsilon^{-1} \text{ or } |t| \ll \varepsilon^{-2}. \]

4 Scattered wave

In this section we construct the asymptotic solution of equation (1) after the resonance. The leading-order term of the solution satisfies NLSE and depends on \( x_1, t_2 \) as well as before resonance. But this leading-order term is determined by another solution of NLSE which contains generally speaking another number of solitons. This number depends on a condition on the resonance curve \( t_2 = 0. \)

After resonance we construct the asymptotic solution of the form
\[ \Psi(x, t, \varepsilon) = \varepsilon^1 v(x_1, t_2) + \varepsilon^2 2 v(x_1, t_2) + \]
\[ \varepsilon^2 A(t_2, x_1) \exp(iS/\varepsilon^2) + \varepsilon^4 (A_1(t_2, x_1) \exp(iS/\varepsilon^2) + \]
\[ A_{-1}(t_2, x_1) \exp(-iS/\varepsilon^2)) \right). \]

Substitute this representation into (1):
\[ \varepsilon^2 (-S' A - f) \exp(iS/\varepsilon^2) + \varepsilon^3 (\partial_{x_2} v + \partial_{x_1} v + |v|^2 v) + \]
Here $r(t_2, x_1, \varepsilon)$ depends on coefficients of the asymptotics (11). This dependence is easy calculated. The coefficients $\hat{A}$, $\hat{A}_1$, $\hat{A}_{-1}$ have singularity on the resonance curve. To determine the domain of validity of (11) we need to derive the explicit formula for $r$

$$r(t_2, x_1, \varepsilon) = O(1 + |A|) = \varepsilon \ln |\varepsilon| + \varepsilon^7 (|A_1| + |A_{-1}|)$$

Collect the terms of the same order of small parameter and the same exponents. It yields the equations for coefficients of representation (11).

$$\partial_t \hat{v} + \partial^2_{x_1} \hat{v} + |\hat{v}|^2 \hat{v} = 0; \quad (12)$$

$$\partial_t \hat{v} + \partial^2_{x_1} \hat{v} + |\hat{v}|^2 \hat{v} + \hat{v}^* = 0 \quad (13)$$

$$-S' \hat{A} = f;$$

$$-S' \hat{A}_1 = -\partial_t \hat{A} - \partial^2_{x_1} \hat{A} - 2|\hat{v}|^2 \hat{A};$$

$$S' \hat{A}_{-1} = -\frac{1}{2} \hat{v}^* \hat{A}.$$
References

[1] Kelley P.L. Self-focusing of optical beams. Phys.Rev.Lett., 1965, v.15, 1005-1008.

[2] Talanov V.I. O samofokusirovke malykh puchkov v nelineinykh sredakh, Pis’ma v ZhETF, 1965, n2, 218-222.

[3] Zakharov V.E. Ustoichivost’ periodicheskikh voln s konechnoi amplitudoi na poverkhnosti glubokoi zhidkosti. Zhurnal prikladnoi mekhaniki i tekhnicheskoi fiziki, 1968, n2, 86-94.

[4] Zakharov V.E., Manakov S.V., Novikov S.P., Pitaevskii L.P. Teoriya solitonov: metod obratnoi zadachi. M.:Nauka, 1980.

[5] Kaup D.J. A perturbation expansion for the Zakharov-Shabat inverse scattering transform. SIAM J.on Appl.Math., 1976, v. 31, 121–133.

[6] Karpman V.I., Maslov E.I. Teoriia vozmusheniy dlia solitonov ZhETPh, 1977, t.73, 537–559.

[7] Glebov S.G., Kiselev O.M., Lazarev V.A. Birth of soliton during passage through local resonance. Proceedings of Steklov Mathematical Institute. Suppl.1, 2003, S84-S90.

[8] Jeffrey A. and Kawahara T. Asymptotic methods in nonlinear wave theory. Pitman Publishing INC, 1982.

[9] Il’in A.M. Matching of Asymptotic Expansions of Solutions of Boundary Value Problem, AMS, 1992.

[10] Keener J.P., McLaughlin D.W., Solitons under perturbation. Phys. Rev. A. 1977, v.16, N2. 777-790.