GEOMETRIC THEORY OF EQUIAFFINE CURVATURE TENSORS

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Dedicated to the memory of Katsumi Nomizu

Abstract. From [4] we continue the algebraic investigation of generalized and equiaffine curvature tensors in a given pseudo-Euclidean vector space and study different orthogonal, irreducible decompositions in analogy to the known decomposition of algebraic curvature tensors. We apply the decomposition results to characterize geometric properties of Codazzi structures and relative hypersurfaces; particular emphasis is on projectively flat structures.

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1. Introduction

In their famous paper [30], I.M. Singer and J.A. Thorpe stated the orthogonal decomposition of the Riemannian curvature tensor on a 4-manifold into three components, described by their properties (Definition 6.3):

1. constant curvature type,
2. Ricci-traceless,
3. Ricci-flat.

This result led to a better understanding of the relations between algebraic and geometric properties of the Riemannian curvature tensor \( R \) and the associated Riemann curvature operator \( R \). The studies initiated a systematic investigation of algebraic curvature tensors; [11] contains a more complete bibliography than is possible in this paper.

Let \( V \) be a vector space of dimension \( n \geq 3 \). Let \( \mathfrak{a}(V) \) be the space of tensors of type \((0,4)\) with the same symmetries as those of the Riemann curvature tensor. Let \( O(V, g) \) be the orthogonal group associated to a non-degenerate scalar product \( g \) on \( V \). In Theorem 6.4, we will present the well known result that \( \mathfrak{a}(V) \) has an irreducible \( O(V, g) \) decomposition into the three subspaces described above.

It was Katsumi Nomizu [19] who initiated the study of so called generalized curvature tensors and generalized curvature operators, see Definition 2.1 below; later other authors, e.g. N. Bokan [5], extended his investigations. Our paper is devoted to this topic and its geometric applications.

We denote the real vector space of generalized curvature operators by \( \mathfrak{R}(V) \). These are the operators with the same symmetries as the curvature operator of a torsion free connection. Bokan proved that the representation of the orthogonal group \( O(V, g) \) on \( \mathfrak{R}(V) \) can be decomposed as the sum of eight irreducible subspaces; she dealt with the case that \( g \) is positive definite; we refer to [4] for the generalization to arbitrary signatures. This decomposition is not unique owing to the fact that two of the representations occur with multiplicity 2 (Lemma 6.1).

In relative hypersurface theory, in the theory of statistical manifolds, in the study of Codazzi structures, and in Weyl geometry there appear geometric structures...
relating to equiaffine connections, pseudo-Riemannian metrics and their induced conformal classes. In general their curvature tensors do not have the symmetries of the Riemann curvature tensor, but the connections involved are torsion free and admit parallel volume forms.

In [4] we extended known results about the decomposition of generalized curvature tensors, and we developed an algebraic theory of so called equiaffine curvature tensors (Definition 2.5). In particular we studied different orthogonal decompositions of generalized and also of equiaffine curvature tensors for a given pseudo-Euclidean vector space.

In this paper we apply our foregoing results and develop a geometric theory of generalized and of equiaffine curvature tensors. We study their geometric properties for Codazzi structures with conjugate connections and in relative hypersurface theory.

For a better understanding of the applications to geometry it was necessary to extend our algebraic investigations from [4] in more detail in Sections 2 through 6 below. For \((V, g)\) given, we introduce the following notation: the space \(C_0(V)\) is the space of \((1,3)\) curvature operators satisfying only the standard skew symmetry in the first two arguments; so called generalized curvature operators additionally satisfy the first Bianchi identity; this space is denoted by \(R(V)\). In \(C_0(V)\) we introduce the concept of \(g\)-conjugate curvature operators \(R\) and \(R^*\).

The space \(C_0(V)\) of generalized \((0,4)\) curvature tensors is \(g\)-associated to the space \(C_0(V)\), and the space \(r(V)\) of generalized \((0,4)\) curvature tensors is \(g\)-associated to the space \(R(V)\) of \((1,3)\) curvature operators. Taking traces with respect to \(g\), for \(R \in r(V)\) there appear only two essentially different Ricci type tensors, denoted by \(Ric\) and \(Ric^*\); their role is interchanged by conjugation. Both Ricci type tensors have the same trace (with respect to the scalar product considered). We study two different irreducible, orthogonal decompositions of the space \(r(V)\) under the action of the orthogonal group, each decomposition leads to eight subspaces:

\[ r(V) = W_1 \oplus ... \oplus W_8 = A_1 \oplus ... \oplus A_8. \]

The \(W\)-decomposition induces a decomposition of the space of projective curvature operators. We will use it subsequently to define additional projective invariants on manifolds. Similarly, the \(A\)-decomposition induces a decomposition of the space of algebraic curvature tensors. We point out that the concept of conjugation of generalized curvature tensors is a suitable instrument for investigations; this can be seen from the following statement:

1. We have an orthogonal \(W\)-decomposition into 3 subspaces

\[ r(V) = W_1 \oplus [\bigoplus_{2}^5 W_j] \oplus [\bigoplus_{6}^8 W_j]. \]

2. Any element of \(W_1\) is of constant curvature type.
3. Any element of \(\bigoplus_{2}^5 W_j\) is Ricci traceless and also \(Ric^*\) traceless.
4. Any element of \(\bigoplus_{6}^8 W_j\) is Ricci flat and also \(Ric^*\) flat.

A similar statement is true for the \(A\)-decomposition as we shall discuss presently. Let \(\mathcal{F}(V) \subset R(V)\) be the set of equiaffine curvature operators (Definition 2.5). In Observation 6.2, we discuss an irreducible, orthogonal decomposition of \(\mathcal{F}(V)\) into seven subspaces.

Our geometric investigations in the second part of the paper mainly concern the equiaffine setting. In many applications we show how the summands in the two different decompositions reflect geometric properties; in particular we find new projective invariants. In the final part we indicate relations to non-linear PDEs of fourth order that appear as Euler-Lagrange equations of variational problems in equiaffine hypersurface theory; it is very interesting, that some critical points
of the Euler-Lagrange equations can be characterized by the vanishing of some of the components in the decompositions that we study. Applications to geometric structures with non-symmetric Ricci tensors, in particular to Weyl geometries, shall follow in a subsequent paper.

Here is a brief guide to the paper. The first part of the paper is algebraic in nature. In Section 2, we introduce the algebraic theory of curvature tensors and operators, and we present geometric motivations (Theorem 2.8). We also discuss the conjugate tensor, generalized Ricci tensors, and generalized scalar curvatures. In Section 3, we touch briefly on the structure of these spaces as GL(V) modules. In Section 4, we introduce the W-decomposition, and in Section 5 we introduce the A-decomposition of r(V) as O(V, g) modules. Some geometric results are stated in these sections concerning these decompositions, and the decompositions are related to the Ricci and Ricci* tensors. It is of particular importance that, in the space r(V), the Ricci symmetry of R and R* is equivalent (Lemma 4.9 and Theorem 4.10), thus this property is purely algebraic; so far, a proof was only known in the context of Codazzi structures on manifolds in terms of analytic tools (Remark 7.4).

Let a(V) be the space of algebraic curvature tensors (Definition 2.6). In Section 6, these two decompositions are related and compared to the Singer-Thorpe decomposition of a(V).

The second part of the paper is more geometric in flavor. Section 7 deals with conjugate connections on manifolds. Section 8 examines Codazzi structures on manifolds. Section 9 studies projective and conformal changes of connections. Section 10 treats relative hypersurface theory. The paper concludes in Section 11 with an examination of the W-decomposition in the framework of relative hypersurfaces.

K. Nomizu did not only initiate the study of generalized curvature tensors, he significantly contributed to the geometry of conjugate connections and affine hypersurface theory. Our paper treats these topics. We dedicate our investigations to the memory of this great geometer of the 20-th century.

2. Spaces of curvature tensors and operators

In this section we establish notation and provide geometric motivations.

2.1. Basic Definitions. Let V be a real vector space of dimension n; to simplify the discussion, we shall assume that n ≥ 3 henceforth. Let g be a non-degenerate scalar product of signature (p, q) on V.

Definition 2.1. We say that R ∈ ⊗²V* ⊗ End(V) is a generalized curvature operator if it satisfies the following relations for all x, y, z ∈ V:

\[ R(x, y)z = -R(y, x)z, \]
\[ R(x, y)z + R(y, z)x + R(z, x)y = 0. \]

As already stated we denote the space of all R satisfying (2.a) by Co(V), and the space of generalized curvature operators, satisfying (2.a) and (2.b), by R(V). Equation (2.b) is called the first Bianchi identity. We use the scalar product to raise and lower indices. For R ∈ R(V) we define a corresponding (0,4)-tensor R ∈ r(V) by means of the identity:

\[ R(x, y, z, w) = g(R(x, y)z, w). \]

Such a tensor is called a generalized curvature tensor and is characterized by the identities:

\[ R(x, y, z, w) = -R(y, x, z, w), \]
\[ R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0. \]
Let \( \tau(V) \) be the space of all generalized curvature tensors. The spaces \( \mathcal{R}(V) \) and \( \tau(V) \) are invariant under the action of the general linear group \( \text{GL}(V) \). The isomorphism sending \( \mathcal{R} \) to \( R \) depends on the scalar product \( g \) or, equivalently, upon the identification of \( V \) with \( V^* \); \( \mathcal{R}(V) \) and \( \tau(V) \) are not isomorphic as \( \text{GL}(V) \) modules (Remark 3.2).

**Definition 2.2.** Let \( \mathcal{R} \in \mathcal{R}(V) \). There are several generalized Ricci tensors:

\[
\begin{align*}
\rho_{14}(\mathcal{R})(x, y) & := \operatorname{Tr}\{ z \rightarrow \mathcal{R}(z, x)y \}, \\
\rho_{24}(\mathcal{R})(x, y) & := \operatorname{Tr}\{ z \rightarrow \mathcal{R}(x, z)y \}, \\
\rho_{34}(\mathcal{R})(x, y) & := \operatorname{Tr}\{ z \rightarrow \mathcal{R}(x, y)z \};
\end{align*}
\]

where \( \operatorname{Tr} \) indicates the associated trace operation. These maps are equivariant with respect to the natural action of \( \text{GL}(V) \); there is no corresponding \( \text{GL}(V) \) equivariant map from \( \tau(V) \) to \( S^2(V^*) \). It follows from Equations (2.a) and (2.b) that:

\[
\begin{align*}
\rho_{24}(\mathcal{R})(x, y) &= -\rho_{14}(\mathcal{R})(x, y), \quad \text{and} \\
\rho_{34}(\mathcal{R})(x, y) &= -\rho_{14}(\mathcal{R})(x, y) + \rho_{14}(\mathcal{R})(y, x).
\end{align*}
\]

In particular we have that \( \rho_{34}(\mathcal{R}) = 0 \) if and only if \( \rho_{14}(\mathcal{R}) \) is symmetric.

We adopt the *Einstein convention* and sum over repeated indices. If \( \{e_i\} \) is a basis for \( V \), we expand \( \mathcal{R}(e_i, e_j)e_k = \mathcal{R}_{ijk}e_i \), \( x = x^ie_i \), and \( y = y^je_j \). We then have

\[
\rho_{14}(x, y) = x^iy^j\mathcal{R}_{ijk}^k, \quad \rho_{24}(x, y) = x^iy^j\mathcal{R}_{ijk}^k, \quad \rho_{34}(x, y) = x^iy^j\mathcal{R}_{ijk}^k.
\]

**Definition 2.3.** Given a scalar product \( g \), let \( g_{ij} := g(e_i, e_j) \) and let \( g^{ij} \) be the inverse matrix. We use \( g \) to define Ricci tensors associated to a generalized curvature \( R \in \tau(V) \) by setting:

\[
\begin{align*}
\rho_{13}(R)(x, y) & := g^{ij}R(e_i, x, e_j, x), \\
\rho_{14}(R)(x, y) & := g^{ij}R(e_i, x, e_j, y), \\
\rho_{23}(R)(x, y) & := g^{ij}R(e_i, e_j, e_j, y), \\
\rho_{24}(R)(x, y) & := g^{ij}R(e_i, e_j, e_j, y), \\
\rho_{34}(R)(x, y) & := g^{ij}R(x, y, e_j, e_j).
\end{align*}
\]

**Definition 2.4.** There is only one relevant scalar geometric invariant which we shall call the generalized scalar curvature

\[
\tau := g^{lk}\mathcal{R}_{ijk}^k = g^{il}g^{jk}\mathcal{R}_{ijkl}.
\]

**Definition 2.5.** We say that \( F \in \mathfrak{g}(V) \) is an *equiaffine curvature operator* (this notation is motivated by Definition 2.7) if, additionally to Equations (2.a) and (2.b), we have the Ricci symmetry:

\[
\rho_{14}(\mathcal{R})(x, y) = \rho_{14}(\mathcal{R})(y, x).
\]

Let \( \mathfrak{g}(V) \subset \mathcal{R}(V) \) be the subspace of all equiaffine curvature operators. Again, we use Equation (2.c) to raise indices to define \( f(V, g) \subset \tau(V) \); the scalar product \( g \) plays a crucial role. The space \( \mathfrak{g}(V) \) is a \( \text{GL}(V) \) module and the space \( f(V, g) \) is an \( O(V, g) \) module.

**Definition 2.6.** The space \( \mathfrak{a}(V) \subset \otimes^4 V \) of *algebraic curvature tensors* is defined by the following identities:

\[
\begin{align*}
(2.h) \quad A(x, y, z, w) &= A(z, w, x, y), \\
(2.i) \quad A(x, y, z, w) &= -A(y, x, z, w), \\
(2.j) \quad A(x, y, z, w) + A(y, z, x, w) + A(z, x, y, w) &= 0.
\end{align*}
\]

This space is invariant under the action of \( \text{GL}(V) \). If \( A \in \mathfrak{a}(V) \) is an algebraic curvature tensor, then we may use Equation (2.c) to define a corresponding *algebraic*
curvature operator \( A \in \otimes^2 V^* \otimes \text{End}(V) \); let \( \mathfrak{A}(V,g) \) be the space of all algebraic curvature operators; this is an \( O(V,g) \) module. It is then immediate that
\[
\mathfrak{a}(V) \subset \mathfrak{f}(V) \subset \mathfrak{r}(V) \subset \mathfrak{c}(V),
\]
\[
\mathfrak{A}(V,g) \subset \mathfrak{g}(V) \subset \mathfrak{R}(V) \subset \mathfrak{C}(V).
\]

We shall use capital Roman letters \( A, F, R \) for curvature tensors in \( \mathfrak{a}(V), \mathfrak{f}(V), \) and \( \mathfrak{r}(V) \), respectively. We shall use capital caligraphic letters \( \mathcal{A}, \mathcal{F}, \) and \( \mathcal{R} \) for the corresponding curvature operators in \( \mathfrak{A}(V,g), \mathfrak{g}(V), \) and \( \mathfrak{R}(V) \), respectively. Despite a tendency in the literature to confuse these objects, it is helpful to distinguish them notationally since the relevant structure groups and module actions differ.

2.2. Geometric representability I. We now present some representability results which provide geometric motivation for our study. We first establish notation in the geometric setting:

**Definition 2.7.** Let \( \nabla \) be a connection on the tangent bundle \( TM \) of a smooth \( n \)-dimensional manifold \( M \).

1. If \( p \in M \), and if \( v, w \in T_pM \), the associated curvature operator is given by
\[
\mathcal{R}^\nabla_p(v, w) := \nabla_v \nabla_w - \nabla_w \nabla_v - \nabla_{[v, w]}.
\]
2. If \( \nabla \) is torsion free, we say \( \nabla \) is **equiaffine** if locally there exists a \( \nabla \)-parallel volume element. This is equivalent to assuming that \( \rho_{14} \) is symmetric [23].
3. If \( g \) is a Riemannian metric on \( M \), let \( \nabla(g) \) be the associated Levi-Civita connection. This is an equiaffine connection.

Let \( 0 \) denote the origin of a finite dimensional vector space \( V \); if \( \nabla \) is a connection on \( TV \), we let \( \mathcal{R}^\nabla_0 \) denote the curvature on \( T_0V \). We have [12]:

**Theorem 2.8.**

1. If \( \nabla \) is a torsion free connection on \( M \), then \( \mathcal{R}^\nabla_p \in \mathfrak{R}(T_pM) \). Conversely, given \( \mathcal{R} \in \mathfrak{R}(V) \), there exists a torsion free connection \( \nabla \) on \( TV \) so that \( \mathcal{R}^\nabla_0 = \mathcal{R} \).
2. If \( \nabla \) is an equiaffine connection on \( M \), then \( \mathcal{R}^\nabla_p \in \mathfrak{g}(T_pM) \). Conversely, given \( \mathcal{F} \in \mathfrak{g}(V) \), there exists an equiaffine connection \( \nabla \) on \( TV \) so that \( \mathcal{R}^\nabla_0 = \mathcal{F} \).
3. We have \( \mathcal{R}^\nabla(g) \in \mathfrak{A}(T_pM, g_p) \). Conversely, given \( \mathcal{R} \in \mathfrak{A}(V, g_0) \), there exists a pseudo-Riemannian metric \( g \) on \( TV \) so that \( g|_{T_0V} = g_0 \) and \( \mathcal{R}^\nabla(g) = \mathcal{R} \).

We postpone until Section 3.1 additional questions of geometric realizability which arise naturally from the study of \( \mathfrak{R}(V) \) as a \( \text{GL}(V) \) module.

2.3. Conjugation. We return to the algebraic study in \( (V, g) \). The conjugate of a tensor of type \((0,4)\) is defined purely algebraically; to define the conjugate of an operator requires a scalar product. This is a central notion despite the fact that the conjugate of a generalized curvature tensor (or operator) need not be a generalized curvature tensor (or operator).

**Definition 2.9.** Let \( R \in \mathfrak{c}(V) \). We define the conjugate tensor
\[
R^*(x, y, z, w) := -R(x, y, w, z).
\]

Given a scalar product \( g \), let \( \mathcal{R} \) be the associated curvature operator. Then \( \mathcal{R}^* \) is characterized by the identity:
\[
g(\mathcal{R}(x, y) z, w) + g(z, \mathcal{R}^*(x, y)w) = 0.
\]

For this reason, we use the notation **conjugate tensor** and **conjugate operator** rather than dual tensor and dual operator.
Clearly $R^{**} = R$. We observe that, for $R \in \mathfrak{v}(V)$, $R^* \in \mathfrak{co}(V)$ need not belong to $\mathfrak{v}(V)$. In the presence of Equations (2.4) and (2.5), Equation (2.51) is equivalent to the identity $A(x, y, z, w) = -A(x, y, w, z)$ [4]. Thus we have

$$\mathfrak{a}(V) = \{ R \in \mathfrak{v}(V) \mid R(x, y, z, w) = -R(x, y, w, z) \} \subset \{ R \in \mathfrak{v}(V) \mid R = R^* \}.$$

Thus these tensors are alternating in the last two arguments. It is also useful to introduce the space of generalized curvature tensors which are symmetric in the last two arguments by setting:

$$\mathfrak{s}(V) = \{ R \in \mathfrak{v}(V) \mid R(x, y, z, w) = R(x, y, w, z) \} \subset \{ R \in \mathfrak{v}(V) \mid R = -R^* \}.$$

**Lemma 2.10.** Let $R \in \mathfrak{v}(V)$; then $R \in \mathfrak{a}(V) \oplus \mathfrak{s}(V)$ if and only if $R^* \in \mathfrak{v}(V)$.

**Proof.** If $R \in \mathfrak{a}(V)$, then $R^* = R$. Similarly, if $R \in \mathfrak{s}(V)$, then $R^* = -R$. Thus if $R \in \mathfrak{a}(V) \oplus \mathfrak{s}(V)$, one has that $R^* \in \mathfrak{a}(V) \oplus \mathfrak{s}(V) \subset \mathfrak{v}(V)$. This establishes one implication of the Lemma. Conversely, suppose $R \in \mathfrak{v}(V)$ and $R^* \in \mathfrak{v}(V)$. We average over the natural $\mathbb{Z}_2$ action interchanging the last two arguments to define

$$R^*_a := \frac{1}{2}(R + R^*) \in \mathfrak{a}(V) \quad \text{and} \quad R^*_s := \frac{1}{2}(R - R^*) \in \mathfrak{s}(V).$$

This shows that $R = R^*_a + R^*_s \in \mathfrak{a}(V) \oplus \mathfrak{s}(V)$ which establishes the other implication of the Lemma. \qed

We introduce the notation

$$\text{Ric}(\mathcal{R}) := \rho_{11}(\mathcal{R}) \quad \text{and} \quad \text{Ric}^*(\mathcal{R}) := -\rho_{13}(\mathcal{R}) = \rho_{23}(\mathcal{R}).$$

We then have that

$$\text{Ric}^*(\mathcal{R}) = \text{Ric}(\mathcal{R}^*).$$

3. THE STRUCTURE OF $\mathfrak{R}(V)$ AND $\mathfrak{F}(V)$ AS $GL(V)$ MODULES

We have a decomposition of $V^* \otimes V^*$ into irreducible $GL(V)$ modules of the form

$$V^* \otimes V^* = \Lambda^2(V^*) \oplus S^2(V^*).$$

Let $\mathfrak{B}(V) := \ker(\rho_{14})$. Note that

$$\dim \{ \mathfrak{B}(V) \} = \frac{1}{4}m^2(m^2 - 4), \quad \dim \{ \Lambda^2(V^*) \} = \frac{1}{4}m(m - 1),$$

$$\dim \{ S^2(V^*) \} = \frac{1}{2}m(m + 1), \quad \dim \{ \mathfrak{R}(V) \} = \frac{1}{4}m^2(m^2 - 1),$$

$$\dim \{ \mathfrak{F}(V) \} = \frac{m(m - 1)(2m^2 + 2m - 3)}{6}.$$

For $\omega \in \Lambda^2(V^*)$ and $\Theta \in S^2(V^*)$, define:

$$\sigma_1(\omega)(x, y)z := \frac{1}{1-m-z} \{ 2\omega(x, y)z + \omega(x, z)y - \omega(y, z)x \},$$

$$\sigma_2(\Theta)(x, y)z := \frac{1}{1-m-z} \{ \Theta(x, z)y - \Theta(y, z)x \}.$$

We refer to Strichartz [32] for the proof of the following theorem:

**Theorem 3.1.** The map $\rho_{14}$ defines two $GL(V)$ equivariant short exact sequences

$$0 \rightarrow \mathfrak{B}(V) \rightarrow \mathfrak{R}(V) \overset{\rho_{14}}{\rightarrow} \Lambda^2(V^*) \oplus S^2(V^*) \rightarrow 0,$$

$$0 \rightarrow \mathfrak{B}(V) \rightarrow \mathfrak{F}(V) \overset{\rho_{14}}{\rightarrow} S^2(V^*) \rightarrow 0,$$

which are equivariantly split by the maps $\sigma_1 + \sigma_2$ and $\sigma_2$, respectively. This gives a $GL(V)$ equivariant decomposition of

$$\mathfrak{R}(V) = \mathfrak{B}(V) \oplus \Lambda^2(V^*) \oplus S^2(V^*),$$

$$\mathfrak{F}(V) = \mathfrak{B}(V) \oplus S^2(V^*)$$

as the direct sum of irreducible $GL(V)$ modules.
Remark 3.2. Recall that \( \dim \mathfrak{a}(V) = m^2(m^2 - 1)/12 \) and that \( \mathfrak{a}(V) \) is an irreducible \( GL(V) \) module [32]. Suppose that \( \mathfrak{r}(V) \) and \( \mathfrak{R}(V) \) were isomorphic as \( GL(V) \) modules. We would then have \( \mathfrak{r}(V) \) as the direct sum of modules of dimension \( m^2(m^2 - 4)/3, m(m + 1)/2, \) and \( m(m - 1)/2 \) which is impossible. We conclude therefore that the natural representations of \( GL(V) \) on \( \mathfrak{R}(V) \) and on \( \mathfrak{r}(V) \) are not isomorphic. As our primary focus in this paper is on the \( O(V, g) \) module structure, we shall not continue our analysis further of the \( GL(V) \) module structure of these spaces and instead refer to [4, 32].

3.1. Geometrical representability II. There are 8 additional natural geometric realization questions which arise in this context and whose realizability may be summarized in the following table:

| \( \mathfrak{P}(V) \) | \( S^2(V^*) \) | \( \Lambda^2(V^*) \) | \( \mathfrak{P}(V) \) | \( S^2(V^*) \) | \( \Lambda^2(V^*) \) |
|----------------|-------------|-------------|----------------|-------------|-------------|
| *              | *           | yes         | 0              | *           | yes         |
| *              | *           | 0           | yes            | 0           | *           | 0           | yes         |
| *              | 0           | *           | yes            | 0           | 0           | *           | no          |
| *              | 0           | 0           | yes            | 0           | 0           | 0           | yes         |

Thus, for example, if

\[
\mathcal{R}(u, v)w = \frac{1}{n-1}[Ric(v, w)u - Ric(u, w)v]
\]

and if \( Ric(\mathcal{R}) \) is symmetric, then \( \mathcal{R} \) can be geometrically realized by a projectively flat, Ricci symmetric, torsion free connection. But if \( \mathcal{R} \neq 0 \) is projectively flat and if \( Ric(\mathcal{R}) \) is antisymmetric, then \( \mathcal{R} \) can not be geometrically realized by a projectively flat, Ricci antisymmetric, torsion free connection. We refer to [14] for further details.

3.2. Rescaling. The spaces \( \mathfrak{a}(V), \mathfrak{A}(V), \mathfrak{r}(V), \mathfrak{R}(V) \) are \( GL(V) \) modules. We have fixed a scalar product \( g \) on \( V \) to raise and lower indices and thereby identify \( \mathfrak{R}(V) \) with \( \mathfrak{r}(V) \), and \( \mathfrak{A}(V, g) \) with \( \mathfrak{a}(V) \). In terms of components, this isomorphism may be described by:

\[
(3.a) \quad \mathcal{R}_{hij}^k \mapsto \mathcal{R}_{hij}^k : = \mathcal{R}_{hij}^k g_{kt}.
\]

We can rescale the scalar product setting \( g_c := cg \) for \( c > 0 \). Thus the isomorphism of Equation (3.a) has trivial consequences and both, the \( A \)-decomposition and the \( W \)-decomposition, are unchanged. Such rescalings, however, play a crucial role in invariance theory. H. Weyl’s classical theory of invariance [37] shows that all \( O(V, g) \) scalar invariants of the curvature tensor (and of its covariant derivatives) arise by contractions of indices. The multiplication of a scalar product \( g \) on \( V \) by a non-zero factor is called a pseudo-conformal change; studying its effect induces a natural filtration on this space which is central in many applications. We refer to [15] for a detailed application of this theory in the context of heat trace and heat content asymptotics, for example. We also refer to [13] where this analysis is used to study the graded (or super) trace of the twisted de Rham complex.

4. The \( W \)-Decomposition of \( \mathfrak{r}(V) \) as an \( O(V, g) \) Module

Before stating the first \( O(V, g) \) decomposition results for \( \mathfrak{r}(V) \), we recall some standard notation.

Definition 4.1. Let \( h \) and \( k \) be bilinear forms.

1. Let \( S^2_0(V^*) \subset S^2(V^*) \) be the space of \( g \)-traceless symmetric bilinear forms.
2. Set \( h \cdot k(x, y, z, w) := h(x, y)k(z, w) \).
3. For \( r = 0, 1, 2, \ldots, \) define:

\[
(h \wedge_r k)(x, y, z, w) : = h(x, z)k(y, w) - h(y, z)k(x, w) - r[h(x, w)k(y, z) - h(y, w)k(x, z)].
\]
We set $\wedge := \wedge_0$ and note that $\wedge_1$ is the Kulkarni-Nomizu product:

\[(h \wedge k)(x, y, z, w) = h(x, z)k(y, w) - h(y, z)k(x, w),\]

\[(h \wedge_1 k)(x, y, z, w) = h(x, z)k(y, w) - h(y, z)k(x, w) - h(x, w)k(y, z) + h(y, w)k(x, z).\]

(4) Set $\Delta h(x, y) := \frac{1}{2}[h(x, y) - h(y, x)].$

(5) Set $Sh(x, y) := \frac{1}{2}[h(x, y) + h(y, x)].$

(6) Define mappings $\psi$ and $\mu$ from $\otimes^4 V^*$ to $\otimes^4 V^*$ by setting

\[4\psi(R)(x, y, z, w) := R(x, y, z, w) + R(y, x, w, z) + R(z, w, x, y) + R(w, z, y, x);\]

\[8\mu(R)(x, y, z, w) := 3R(x, y, z, w) + 3R(x, y, w, z) + 8R(x, w, z, y) + 8R(w, x, y, z) + 8R(z, y, w, x).\]

If we take $R \in \tau(V)$, then $\psi(R) \in a(V)$ and $\mu(R) \in s(V)$. Furthermore, $\psi(\psi(R)) = \psi(R)$ and $\mu(\mu(R)) = \mu(R)$, so these are idempotents [5].

4.1. Components of the $W$-decomposition. We summarize and extend results from [4, 5]. For fixed data $g$ and $R \in \tau(V)$, we simply write $Ric := Ric(R)$, $Ric^* := Ric(R^*)$, and $\tau := \tau(R)$. As our calculations are straight forward we shall omit proofs in the interests of brevity. We may define the $W$-components as follows:

**Definition 4.2.** Let $\pi_j : \tau(V) \to W_j$ be the following natural projections:

\[\pi_1(R) := \frac{-1}{(n-1)}g \wedge g,\]

\[\pi_2(R) := \frac{1}{(n-1)}g \wedge g,\]

\[\pi_3(R) := \frac{1}{n+1}[2\Lambda Ric \cdot g + \Lambda Ric \wedge g],\]

\[\pi_4(R) := \frac{-1}{(n-1)}[2\Lambda Ric^* \cdot g + \Lambda Ric^* \wedge_{n+1} g] - \frac{1}{(n-2)(n+1)}[2\Lambda Ric \cdot g + \Lambda Ric \wedge_{n+1} g],\]

\[\pi_5(R) := \frac{1}{(n-1)(n-2)}[\tau \cdot g \wedge g - \frac{1}{n}S(Ric + (n-1)Ric^*) \wedge_{n-1} g],\]

\[\pi_6(R) := \psi(R) + \frac{1}{2(n-2)}S(Ric + Ric^*) \wedge_1 g - \frac{\tau}{(n-1)(n-2)}g \wedge g,\]

\[\pi_7(R) := \mu(R) + \frac{1}{2n}S(Ric - Ric^*) \wedge_{-1} g + \frac{1}{2(n+2)}\Lambda(3Ric - Ric^*) \cdot g + \frac{1}{n(n+2)}\Lambda(3Ric - Ric^*) \wedge_{-1} g,\]

\[\pi_8(R) := R - \psi(R) - \mu(R) + \frac{1}{2n}S(Ric + Ric^*) \cdot g + \frac{1}{2(n+2)}\Lambda(Ric + Ric^*) \wedge_{-1} g.\]

The main result of this section is the following:

**Theorem 4.3.** [W-Decomposition Theorem] There is an $O(V, g)$ equivariant orthogonal decomposition of $\tau(V) = W_1 \oplus \ldots \oplus W_8$ as the direct sum of irreducible $O(V, g)$ modules.

We note that the isomorphism induced by $g$ identifies $\mathfrak{g}(V)$ with $\tau(V)$ as $O(V, g)$ modules. Consequently, Theorem 4.3 also gives the structure of $O(V, g)$ as an $O(V, g)$ module. Let

\[p(V) := \{ R \in \tau(V) : Ric(R) = 0 \},\]

\[t(V) := \{ R \in \tau(V) : Ric(R) = Ric^*(R) = 0 \} \subset p(V).\]

We have the following characterization of the subspaces $W_j$:

**Lemma 4.4.**

1. $R \in W_1$ if and only if $R = cg \wedge g$ for some $c \in \mathbb{R}$.
2. $R \in W_2$ if and only if $R \in p(V)^{1}$ and $Ric(R) \in S^2(V^*)$. 
Ricci and the Ricci representations corresponding to column contains the 2 components where the Ricci tensor vanishes but the Ricci I contains the three components where the Ricci tensor is non-zero, the second

One may summarize this information in a tabular form. Denote the projection of $R$ to $t(V)$ by $R_o := R - \sum_{1 \leq i \leq 5} \pi_i(R)$. Then:

$$R_o := R + \frac{2}{n-4} \Lambda[(n-1)\text{Ric} + \text{Ric}^*] \cdot g + \left(\frac{1}{n-1} - \frac{1}{(n-1)(n+1)}\right) \Lambda(3\text{Ric} + (n+1)\text{Ric}^*) \land g + \frac{1}{(n-1)(n-2)} S(\text{Ric} + (n-1)\text{Ric}^*) \land g - \frac{7}{(n-1)(n-2)} g \land g.$$  

| Ric $\neq 0$ | Ric $= 0$, Ric $^* \neq 0$ | Ric $= \text{Ric}^* = 0$ |
|----------------|-----------------------------|-----------------------------|
| $W_1$ $(\tau \neq 0)$ | $W_5$ $(\text{Ric}^* \in S_0)$ | $W_6 = t(V) \cap a(V)$ |
| $W_2$ $(\text{Ric} \in \Lambda)$ | $W_7 = t(V) \cap s(V)$ | $W_4$ $(\text{Ric}^* \in \Lambda)$ |
| $W_3$ $(\text{Ric} \in \Lambda)$ | $W_8 = t(V) \cap (a(V) \oplus s(V))^\perp$ |

Of course, in the table we ignore the element $0 \in \tau(V)$. The first column in Table I contains the three components where the Ricci tensor is non-zero, the second column contains the 2 components where the Ricci tensor vanishes but the Ricci* tensor is non-zero, and the third column contains the 3 components where both, the Ricci and the Ricci* tensors, vanish; thus the third column gives the decomposition of $t(V)$. The first two entries in the third row contain the 2 components where Ricci and Ricci* tensors are symmetric and traceless, and the first two entries in the fourth row contain the 2 components where the Ricci and Ricci* tensors are skew symmetric.

The $O(V, g)$ modules $W_i$ are discussed in [4, 5]. The representations defined by $W_1$, $W_6$, $W_7$, and $W_8$ appear with multiplicity 1 in the natural representation of $O(V, g)$ on $t(V)$. Thus these summands are unique. On the other hand, the representations corresponding to $W_2$ and $W_5$ are isomorphic as are the representations corresponding to $W_3$ and $W_4$. Thus these components in the decomposition of $t(V)$ as an $O(V, g)$ module are not unique. This gives rise to the fact that there can be different decompositions as we shall see when we discuss the $A$-decomposition in Section 5.

$W_6$ is the space of Weyl conformal curvature tensors. One then has that

$$\pi_6(R) = \psi(R_o), \quad \pi_7(R) = \mu(R_o), \quad \pi_8(R) = R_o - \psi(R_o) - \mu(R_o).$$

4.2. Properties of the $W$-decomposition. A straightforward calculation shows that the Ricci tensors and the Ricci* tensors for these components are given by:

**Lemma 4.5.** The Ricci tensors of the $W$-components are given by:

1. $\text{Ric}(\pi_1(R)) = \frac{2}{n} g$.
2. $\text{Ric}(\pi_2(R)) = -\frac{2}{(n-1)} g + SRic.$
3. $\text{Ric}(\pi_3(R)) = \Lambda \text{Ric}.$
4. $\text{Ric}(\pi_j(R)) = 0$ for $j = 4, ..., 8$.
5. $\text{Tr}_g(\text{Ric}(\pi_j(R))) = 0$ for $j = 2, ..., 8$.

**Lemma 4.6.** The Ricci* tensors of the $W$-components are given by:

1. $\text{Ric}^*(\pi_1(R)) = \frac{2}{n} g$.
2. $\text{Ric}^*(\pi_2(R)) = \frac{1}{n-1} [\frac{2}{n} g - SRic]$. 

\((3)\) \(\text{Ric}^*(\pi_3(R)) = \frac{8}{n+1} \Lambda \text{Ric}\).

\((4)\) \(\text{Ric}^*(\pi_4(R)) = \Lambda(\text{Ric}^* + \frac{4}{n+1} \text{Ric})\).

\((5)\) \(\text{Ric}^*(\pi_5(R)) = \frac{8}{n+1} g + S(\frac{1}{n-1} \text{Ric} + \text{Ric}^*)\).

\((6)\) \(\text{Ric}^*(\pi_j(R)) = 0\) for \(j = 6, 7, 8\).

\((7)\) \(\text{Tr}_g(\text{Ric}^*(\pi_j(R)) = 0\) for \(j = 2, \ldots, 8\).

**Lemma 4.7.** The following vanishing results hold:

1. \(\pi_1(R) = 0\) if and only if \(\tau = 0\).
2. \(\pi_2(R) = 0\) if and only if \(\text{SRic} = \frac{\tau}{n} g\).
3. \(\pi_3(R) = 0\) if and only if \(\text{Ric}\) is symmetric.
4. \(\pi_4(R) = 0\) if and only if \(\Lambda(\text{Ric}^* + \frac{4}{n+1} \text{Ric}) = 0\).
5. \(\pi_5(R) = 0\) if and only if \(S(\frac{1}{n-1} \text{Ric} + \text{Ric}^*) = \frac{\tau}{n-1} g\).

We recall the definition of \(R_o\) given in Equation (4.a). If \(\text{Ric}\) and \(\text{Ric}^*\) are symmetric then the \(W\)-components simplify.

**Lemma 4.8.** If \(\text{Ric}\) and \(\text{Ric}^*\) are symmetric then:

1. \(\pi_1(R) = \frac{1}{n(n-1)} g \land g\).
2. \(\pi_2(R) = \frac{1}{n(n-1)} [\frac{2g}{n} - \text{Ric} \land g]\).
3. \(\pi_3(R) = 0\).
4. \(\pi_4(R) = 0\).
5. \(\pi_5(R) = \frac{1}{n(n-1)(n-2)} [\tau \cdot g \land g - \frac{1}{n}(\text{Ric} + (n-1)\text{Ric}^*) \land_{n-1} g]\).
6. \(\text{R}_o = R + \frac{1}{n(n-2)} (g \land_{n-1} \text{Ric} + \text{Ric}^* \land_{n-1} g) - \frac{\tau}{(n-1)(n-2)} g \land g\).
7. \(\pi_6(R) = \psi(R) + \frac{1}{2(n-2)} (\text{Ric} + \text{Ric}^*) \land_{n-1} g - \frac{\tau}{(n-1)(n-2)} g \land g\).
8. \(\pi_7(R) = \mu(R) + \frac{1}{2(n-2)} (\text{Ric} - \text{Ric}^*) \land_{n-1} g\).
9. \(\pi_8(R)(x, y, z, w) = (R - \psi(R) - \mu(R))(x, y, z, w)\)

\[= \frac{1}{8} (3R(x, y, z, w) - R(x, y, w, z) + R(x, z, w, y)) \]

\[+ \frac{1}{8} (3R(x, w, z, y) - R(z, y, w, x) + 3R(y, w, z, x))\,.

**Lemma 4.9.** Let \(R \in \text{v}(V)\) and \(R^* \in \text{v}(V)\), then:

1. \(\pi_8(R) = 0\) if and only if \(\text{Ric}^*\) is symmetric.
2. \(\text{Ric}\) is symmetric if and only if \(\text{Ric}^*\) is symmetric.

**Proof.** The proof of (2) is elementary, but technical. From the assumptions we have \(R \in \text{a}(V) \oplus \text{s}(V, g)\) and \(R^* \in \text{a}(V) \oplus \text{s}(V)\), thus \(R = \mu(R) + \psi(R)\). We insert the definitions of the mappings \(\mu\) and \(\psi\) and get:

\[R(x, y, z, w) = \frac{1}{8} [R(x, y, z, w) + R(y, x, w, z) + R(z, w, x, y) + R(w, z, y, x)]\]

\[+ \frac{1}{8} [3R(x, y, z, w) + 3R(x, y, w, z) + R(x, w, z, y)]\]

\[+ \frac{1}{8} [R(x, z, w, y) + R(w, z, y, x) + R(z, w, x, y)]\,.

Using the skew symmetry and the Bianchi identity, this implies

\[8R(x, y, z, w) = 2 [R(x, y, z, w) + R(y, x, w, z) + R(z, w, x, y) + R(w, z, y, x)]\]

\[+ 3R(x, y, z, w) + 3R(x, y, w, z) - R(w, z, x, y) - R(z, x, w, y)\]

\[+ R(x, z, w, y) - R(y, z, w, x) - R(z, w, y, x) + R(z, y, w, x)\,.

We summarize:

\[3R(x, y, z, w) - R(x, y, w, z) = 3R(z, w, x, y) + 2R(x, z, w, y)\]

\[+ 3R(z, w, y, x) - 3R(y, w, z, x)\,.

For the last term use again the Bianchi identity:

\[3R(x, y, z, w) - R(x, y, w, z)\]

\[= 3R(z, w, x, y) + 2R(x, z, w, y) - 3R(z, w, y, x)\]

\[+ 2R(z, w, y, x) + 2R(w, y, z, x)\,.

Take the trace \( \text{Tr}\{w \rightarrow R(w, y)z\} \), that yields
\[
3\text{Ric}(y, z) + \text{Ric}^*(y, z) = 3\text{Ric}^*(z, y) - 2\text{Ric}^*(z, y) + 3\text{Ric}(z, y) - 2\text{Ric}(z, y) + 2\text{Ric}(y, z),
\]
and thus the identity \( \text{Ric}(y, z) - \text{Ric}(z, y) = \text{Ric}^*(z, y) - \text{Ric}^*(y, z) \), from which the desired result follows. \( \square \)

We get the following corollary which we state as a Theorem according to its importance (see section 10.1.4 below).

**Theorem 4.10.** Let \( R, R^* \in \mathfrak{v}(V) \). Then:

1. \( \mathcal{R} \) is equiaffine if and only if \( R^* \) is equiaffine.
2. \( R = \pi_1(R) + (\pi_2(R) + \pi_3(R)) + (\pi_6(R) + \pi_7(R)). \)

**Lemma 4.11.** Let \( R \in \mathfrak{j}(V, g) \) and \( R^* \in \mathfrak{j}(V, g) \); then

1. \( \pi_1(R) = \frac{1}{n(n-1)} g \wedge g = \pi_1(R^*). \)
2. \( \pi_2(R) = \frac{1}{n-1} [\frac{1}{n} \text{Ric} - \text{Ric}] \wedge g. \)
3. \( \pi_3(R) = 0 = \pi_3(R^*). \)
4. \( \pi_4(R) = 0 = \pi_4(R^*). \)
5. \( \pi_5(R) = \frac{1}{n-1} [\frac{n}{n-2}] [\text{Ric} + (n-1)\text{Ric}^*] \wedge (n-1)g].\)
6. \( \pi_6(R) = \mu(R) + \frac{1}{2(n-2)}(\text{Ric} + \text{Ric}^*) \wedge (n-1)g - \frac{7}{(n-1)(n-2)}g \wedge g = \pi_6(R^*). \)
7. \( \pi_7(R) = \mu(R) + \frac{1}{2(n-2)}(\text{Ric} - \text{Ric}^*) \wedge (n-1)g = -\pi_7(R^*). \)
8. \( \pi_8(R) = 0 = \pi_8(R^*). \)

**Remark 4.12.** Let \( R \in \mathfrak{v}(V) \) and \( R^* \in \mathfrak{v}(V) \), then:
\[
\mu(R) = \frac{1}{2}(R - R^*) \text{ and } \psi(R) = \frac{1}{2}(R + R^*).
\]

### 4.3. The Projective Curvature Tensor and Operator

**Definition 4.13.** Let \( g \) be fixed. The projective curvature tensor \( p(R) \) is the projection of \( R \) on \( p(V) \): this is the space of generalized curvature tensors with \( \text{Ric} = 0 \). Thus
\[
p(R) := \pi_4(R) \oplus ... \oplus \pi_8(R) = R - [\pi_1(R) \oplus \pi_2(R) \oplus \pi_3(R)].
\]
The \( g \)-associated (1,3) operator is called the *projective curvature operator* and is denoted by \( \mathcal{P}(R) \) [5]. Note that this definition yields the projective curvature tensor of a torsion free connection on a manifold.

Define
\[
(4.\text{b}) \quad B^* := S[\text{Ric}^* + (n-1)\text{Ric}] - \tau g.
\]

**Lemma 4.14.** Let \( R \in \mathfrak{j}(V, g) \). Then

1. \( p(R) = R + \frac{1}{n-1}(\text{Ric} \wedge g). \)
2. If additionally \( R^* \in \mathfrak{v}(V) \), then \( p(R) = \pi_5(R) + \pi_6(R) + \pi_7(R). \) Furthermore, the following conditions are equivalent:
   a. \( \mathcal{P}(R^*) = \mathcal{P}(R). \)
   b. \( R^* = R. \)
   c. \( R \) is algebraic.
3. If additionally \( R^* \in \mathfrak{v}(V) \) and \( p(R^*) = 0 \), then \( B^* = 0. \)

### 5. The \( A \)-Decomposition of \( \mathfrak{v}(V) \) as an \( O(V, g) \) Module

As already stated, the orthogonal decomposition of \( \mathfrak{v}(V) \) into irreducible subspaces is not unique. In this section we collect and extend results on a decomposition different from the \( W \)-decomposition. We call it the *\( A \)-decomposition*. The \( A \)-components are defined analogously to the \( W \)-components. In analogy to Definition 4.2, we set:
Definition 5.1. Let $\alpha_j : \mathfrak{g}(V) \to A_i$ be the following natural projections:

1. $\alpha_1(R) := \frac{-\tau}{n(n-1)} g \wedge g$.
2. $\alpha_2(R) := \frac{-1}{2n-2} S(Ric + Ric^*) \wedge_1 g + \frac{2\tau}{n(n-2)} g \wedge g$.
3. $\alpha_3(R) := \frac{2\tau}{2n} S(Ric - Ric^*) \wedge_1 g$.
4. $\alpha_4(R) := \frac{2\tau}{4n(n-2)} [2\Lambda(3Ric - Ric^*) g + \Lambda(3Ric - Ric^*) \wedge_1 g]$.
5. $\alpha_5(R) := \frac{2\tau}{4n(n-2)} [2\Lambda(Ric + Ric^*) g + \Lambda(Ric + Ric^*) \wedge_3 g]$.
6. $\alpha_6(R) := \psi(R) - \alpha_1(R) - \alpha_2(R)$.
7. $\alpha_7(R) := \mu(R) - \alpha_3(R) - \alpha_4(R)$.
8. $\alpha_8(R) := R - \mu(R) - \psi(R) - \alpha_5(R)$.

We have the following analogue of Theorem 4.3:

Theorem 5.2. [A-Decomposition Theorem] There is an $O(V, g)$ equivariant orthogonal decomposition of $\mathfrak{g}(V) = A_1 \oplus ... \oplus A_8$ as the direct sum of orthogonal, irreducible $O(V, g)$ modules.

We use Theorem 5.2 to establish the following useful fact.

Lemma 5.3. If $R \in (\mathfrak{a}(V) \oplus \mathfrak{s}(V))^\perp$ and if $Ric(R) \neq 0$, then $Ric$ is skew symmetric and $3Ric = Ric^*$.

Proof. Suppose that $R \in (\mathfrak{a}(V) \oplus \mathfrak{s}(V))^\perp$. It then follows that $R$ belongs to $\ker(\psi)$ and to $\ker(\mu)$. After some calculations (compare the proof of Lemma 4.9) this leads to the identity:

$$0 = R(x, y, z, w) + 2R(x, y, w, z) + R(x, z, w, y) + R(z, y, w, x).$$

Taking trace over $\rho_{14}$ then yields $Ric^*(y, w) = 3Ric(y, w)$, and taking trace over $\rho_{14}$ yields similarly $Ric(y, z) = -Ric(z, y)$.

In analogy to Lemma 4.4 we have:

Lemma 5.4.

1. $R \in A_1$ if and only if $R = cg \wedge g$ for some $c \in \mathbb{R}$.
2. $R \in A_2$ if and only if $R \in \mathfrak{a}(V)$, $\alpha_6(R) = 0$, and $\tau = 0$.
3. $R \in A_3$ if and only if $R \in \mathfrak{s}(V)$, $\alpha_7(R) = 0$, $Ric(R) \in S_3(V^*)$.
4. $R \in A_4$ if and only if $R \in \mathfrak{s}(V)$, $\alpha_7(R) = 0$, and $Ric(R) \in \Lambda^2(V^*)$.
5. $R \in A_5$ if and only if $R \in (\mathfrak{s}(V) \oplus \mathfrak{a}(V) \oplus A_8)^\perp$.
6. $R \in A_6$ if and only if $R \in \mathfrak{a}(V) \cap p(V)$.
7. $R \in A_7$ if and only if $R \in \mathfrak{s}(V) \cap p(V)$.
8. $R \in A_8$ if and only if $R \in p(V) \cap (\mathfrak{s}(V) \oplus \mathfrak{a}(V))^\perp$.

Again, it is useful to summarize this information in a tabular form:

Table II – the $A$-decomposition

| \( \mathfrak{a}(V) \) | \( \mathfrak{s}(V) \) | \((\mathfrak{a}(V) \oplus \mathfrak{s}(V))^\perp\) |
|---------------------|---------------------|----------------------------------|
| $A_1$ \( (\tau \neq 0) \) | $A_4$ \( Ric \in \Lambda \) | $A_5$ \( Ric \in \Lambda, 3Ric = Ric^* \) |
| $A_2$ \( Ric \in S_0 \) | $A_3$ \( Ric \in S_0 \) | |
| $Ric = Ric^* = 0$ | $A_6$ | $A_7$ |
| $A_8$ | |

The first column in Table II contains the components giving the decomposition of $\mathfrak{a}(V)$, the second column contains the components giving the decomposition of $\mathfrak{s}(V)$, and the third column contains the components giving the decomposition of \((\mathfrak{a}(V) \oplus \mathfrak{s}(V))^\perp\); such decompositions are not available from Table I. We also can read off the symmetry ($S_0$) and skew symmetry ($\Lambda$) of the Ricci tensor from this table.
5.1. Properties of the $A$-decomposition. Lemmas 4.5 and 4.6 extend to this setting to become:

**Lemma 5.5.** The Ricci tensors of the $A$-components are given by:

1. \( \text{Ric}(\alpha_1(R)) = \frac{1}{n}g. \)
2. \( \text{Ric}(\alpha_2(R)) = \frac{1}{n}g + \frac{1}{2}S(\text{Ric} + \text{Ric}^*). \)
3. \( \text{Ric}(\alpha_3(R)) = \frac{1}{4}S(\text{Ric} - \text{Ric}^*). \)
4. \( \text{Ric}(\alpha_4(R)) = \frac{1}{2}\Lambda(3\text{Ric} - \text{Ric}^*). \)
5. \( \text{Ric}(\alpha_5(R)) = \frac{1}{2}\Lambda(\text{Ric} + \text{Ric}^*). \)
6. \( \text{Ric}(\alpha_j(R)) = 0 \) for \( j = 6, 7, 8. \)
7. \( \text{Tr}_g(\text{Ric}(\alpha_j(R))) = 0 \) for \( j = 2, ..., 8. \)

**Lemma 5.6.** The Ricci* tensors of the $A$-components are given by:

1. \( \text{Ric}^*(\alpha_j(R)) = \text{Ric}(\alpha_j(R)) \) for \( j = 1, 2. \)
2. \( \text{Ric}^*(\alpha_j(R)) = -\text{Ric}(\alpha_j(R)) \) for \( j = 3, 4. \)
3. \( \text{Ric}^*(\alpha_5(R)) = 3\text{Ric}(\alpha_5(R)). \)
4. \( \text{Ric}^*(\alpha_j(R)) = 0 \) for \( j = 6, 7, 8. \)
5. \( \text{Tr}_g(\text{Ric}^*(\alpha_j(R))) = 0 \) for \( j = 2, ..., 8. \)

As for the $W$-components, there are important vanishing results:

**Lemma 5.7.** The following vanishing results hold:

1. \( \alpha_1(R) = 0 \) if and only if \( \tau = 0. \)
2. \( \alpha_2(R) = 0 \) if and only if \( \text{Ric} + \text{Ric}^* = \frac{2\tau}{n}g. \)
3. \( \alpha_3(R) = 0 \) if and only if \( S\text{Ric} = S\text{Ric}^*. \)
4. \( \alpha_4(R) = 0 \) if and only if \( 3\Lambda\text{Ric} = \Lambda\text{Ric}^*. \)
5. \( \alpha_5(R) = 0 \) if and only if \( 8\Lambda\text{Ric} = -\Lambda\text{Ric}^*. \)
6. The following conditions are equivalent:
   1. \( R \in \tau(V) \) and \( R^* \in \tau(V). \)
   2. \( R \in \mathfrak{a}(V) \oplus \mathfrak{s}(V). \)
   3. \( \alpha_5(R) = \alpha_8(R) = 0. \)

We have as an immediate consequence of Lemma 5.7(6) that:

**Lemma 5.8.** If \( R \in \tau(V) \) and if \( R^* \in \tau(V) \) then:

1. \( \alpha_j(R) = \alpha_j(R^*) = \alpha_j(R)^* \) for \( j = 1, 2, 6. \)
2. \( \alpha_2(R^*) = (\alpha_3(R))^* = -\alpha_3(R). \)
3. \( \alpha_j(R^*) = -\alpha_j(R) \) for \( j = 4, 7. \)
4. \( \alpha_2(R^*) = 0 = \alpha_3(R) \) for \( j = 5, 8. \)

**Lemma 5.9.** If \( R \in \mathfrak{f}(V, g) \) and if \( R^* \in \mathfrak{f}(V, g) \) then:

1. \( \alpha_1(R) = -\frac{\tau}{n(n-1)}g \wedge g. \)
2. \( \alpha_2(R) = -\frac{1}{2(n-2)}(\text{Ric} + \text{Ric}^*) \wedge_1 g + \frac{2\tau}{n(n-2)} g \wedge g. \)
3. \( \alpha_3(R) = \frac{1}{2n}(\text{Ric} - \text{Ric}^*) \wedge_1 g. \)
4. \( \alpha_4(R) = 0 = \alpha_5(R). \)
5. \( \alpha_6(R) = \frac{1}{2}(R + R^*) - \alpha_1(R) - \alpha_2(R). \)
6. \( \alpha_7(R) = \frac{1}{2}(R - R^*) - \alpha_1(R) - \alpha_2(R). \)
7. \( \alpha_8(R) = 0. \)

6. Comparing the $A$-decomposition and the $W$-decomposition

The two decomposition theorems show that, for given scalar product \( g \), there exist different decompositions of \( \tau(V) \) into irreducible and orthogonal subspaces; as noted above, this occurs because not all the representations which appear have multiplicity one. Referring to the two different decompositions above, we use the terminology $W$-decomposition and $A$-decomposition.
Lemma 6.1.

(1) We have the following relations:

\[ A_1 = W_1, \quad W_2 \oplus W_5 = A_2 \oplus A_3, \quad W_3 \oplus W_4 = A_4 \oplus A_5, \]
\[ A_6 = W_6, \quad A_7 = W_7, \quad A_8 = W_8. \]

(2) As representation spaces of \( O(V,g) \), we have isomorphisms

\[ W_2 \approx W_5 \approx A_2 \approx A_3, \quad \text{and} \quad W_3 \approx W_4 \approx A_4 \approx A_5. \]

For the convenience of the reader, we recall once again Tables I and II.

| Table I – the \( W \)-decomposition |
|--------------------------------------|
| Ric \( \neq 0 \) | Ric = 0, Ric* \( \neq 0 \) | Ric = Ric* = 0 |
| \( W_1 \) (\( \tau \neq 0 \)) | \( W_6 = \{ V \} \cap a(V) \) |
| \( W_2 \) (Ric \( \in S_0 \)) | \( W_5 \) (Ric* \( \in S_0 \)) | \( W_7 = \{ V \} \cap g(V) \) |
| \( W_3 \) (Ric \( \in A \)) | \( W_4 \) (Ric* \( \in A \)) | \( W_8 = \{ V \} \cap \{ a(V) \oplus g(V) \} \) |

| Table II – the \( A \)-decomposition |
|--------------------------------------|
| \( a(V) \) | \( g(V) \) | \( (a(V) \oplus g(V))^\perp \) |
| \( A_1 \) (\( \tau \neq 0 \)) | \( A_4 \) (Ric \( \in A \)) | \( A_5 \) (Ric \( \in A, 3 \text{Ric} = \text{Ric}^* \)) |
| \( A_2 \) (Ric \( \in S_0 \)) | \( A_3 \) (Ric \( \in S_0 \)) |
| Ric = Ric* = 0 | A_6 | A_7 | A_8 |

Table I contains the decomposition of the space where Ric = 0; this is the space of projective curvature tensors. This decomposition is not available from Table II. On the other hand, the components giving the decomposition of \( a(V) \), \( g(V) \), and \( (a(V) \oplus g(V))^\perp \) are available from Table II but not from Table I. The elements of the fourth row in Table II are the same as the elements of the third column in Table I and give the decomposition of \( t(V) \). We summarize this information as follows:

Observation 6.2.

(1) The \( W \)-decomposition allows to recover the subspace of projective curvature tensors as the direct sum \( \oplus_{j=4}^8 W_j \). This is not possible in the \( A \)-decomposition.

(2) The \( W \)-decomposition allows to recover separately the equiaffine character of \( R \) and \( R^* \), respectively, that means the symmetry of Ric and that of Ric*, respectively; that is not possible in the \( A \)-decomposition.

(3) The \( W \)-decomposition permits us to express

\[ f(V,g) = W_1 \oplus W_2 \oplus \bigoplus_{j=4}^8 W_j. \]

The \( A \)-decomposition permits us to express

\[ f(V,g) = \bigoplus_{j=1}^3 A_j \oplus \bigoplus_{j=6}^8 A_j. \]

There are two different decompositions, yet.

(4) In the decomposition of algebraic curvature tensors we use the notation constant curvature type, Ricci traceless, and Ricci flat; additionally we shall use the notation Ricci* traceless and Ricci* flat.

(5) The \( A \)-decomposition allows us to express \( a(V) = A_1 \oplus A_2 \oplus A_6 \). This is not possible in the \( W \)-decomposition.

(6) We have \( \oplus_{j=1}^3 A_j = \oplus_{j=6}^8 W_j \); this direct sum is completely traceless and corresponds to the Weyl part in the decomposition of algebraic curvature tensors.
6.1. **Algebraic curvature tensors.** We recall the well known decomposition of algebraic curvature tensors. To compare this decomposition with our decomposition results in Theorem 6.4 below, for the convenience of the reader we recall the notation that is often used in the literature (see e.g. [1], p. 46):

**Definition 6.3.**

1. Let \( u(V, g) \) be the space of all algebraic curvature tensors of constant curvature type. \( R \in u(V, g) \) if and only if there exists \( c \in \mathbb{R} \) so that
   \[
   R(x, y, z, w) = c \{ g(x, w)g(y, z) - g(x, z)g(y, w) \}.
   \]
2. Let \( \mathfrak{z}(V, g) \) be the space of all algebraic curvature tensors that are Ricci traceless. \( R \in \mathfrak{z}(V, g) \) if and only if there exists a symmetric trace free bilinear form \( \Xi \) so that
   \[
   R(x, y, z, w) = \Xi(x, w)g(y, z) + g(x, w)\Xi(y, z) - \Xi(x, z)g(y, w) - g(x, z)\Xi(y, w) = -\Xi \wedge_1 g.
   \]
3. Define the subspace \( \mathfrak{w}(V, g) \) as space of all algebraic curvature tensors that are Ricci flat; they are of the type of the Weyl conformal curvature tensor.

We have \( R \in \mathfrak{w}(V, g) \) if and only if \( R \in a(V) \) and \( \text{Ric} = 0 \).

For the comparison, now also recall our notation from Lemma 4.4. The following is the celebrated theorem of Singer and Thorpe [30].

**Theorem 6.4.** [Algebraic Curvature Decomposition Theorem] There is an \( O(V, g) \) equivariant orthogonal decomposition of

\[
\mathfrak{a}(V) = u(V, g) \oplus \mathfrak{z}(V, g) \oplus \mathfrak{w}(V, g)
\]

as the direct sum of irreducible and inequivalent \( O(V, g) \) modules.

The following result compares the decomposition of algebraic curvature tensors into 3 orthogonal subspaces with the \( W \)-decomposition and the \( A \)-decompositions.

**Proposition 6.5.** We have the orthogonal decomposition into 3 subspaces

\[
\mathfrak{r}(V) = W_1 \oplus \frac{\mathfrak{z}}{W_j} \oplus [\frac{\mathfrak{a}}{W_j}] = A_1 \oplus [\frac{\mathfrak{z}}{A_j}] \oplus [\frac{\mathfrak{a}}{A_j}].
\]

1. Any \( R \in \frac{\mathfrak{z}}{W_j} = \frac{\mathfrak{z}}{A_j} \) is Ricci traceless and Ricci* traceless.
2. Any \( R \in [\frac{\mathfrak{a}}{W_j}] = [\frac{\mathfrak{a}}{A_j}] \) is Ricci flat and Ricci* flat.
3. \( u(V, g) = W_1 = A_1 \), \( \mathfrak{z}(V, g) = A_2 \), and \( \mathfrak{w}(V, g) = A_6 = W_6 \).

7. **Conjugate Connections on Manifolds**

It is well known that the decomposition of algebraic curvature tensors reflects geometric properties. The following sections show that also our foregoing decomposition results reflect geometric properties. Let \( M \) be a differentiable manifold of dimension \( n \geq 3 \). We assume \( M \) to be equipped with a pseudo-Riemannian metric \( g \) of signature \((p, q)\). Let \( \nabla \) be a torsion free connection on \( M \); in general, \( \nabla \) will be different from the Levi-Civita connection \( \nabla(g) \). We denote vector fields by \( u, v, w, \ldots \). Our considerations have local character. We refer [29] for the proofs. As already stated in the Introduction, the structure \( (\nabla, g) \) appears in many situations.

We say that a smooth differential form \( \omega \) of maximal dimension \( n \) is a volume form if \( \omega \) is nowhere vanishing; we say that \( \omega \) is \( \nabla \)-parallel if \( \nabla \omega = 0 \). We summarize well known facts for later applications.

**Lemma 7.1.** Let \( \nabla \) be torsion free. The following conditions are equivalent:

1. Locally, \( \nabla \) admits a parallel volume form \( \omega \).
2. The Ricci tensor \( \text{Ric} := \text{Ric}(\nabla) := \text{Ric}(\mathcal{R}(\nabla)) \) is symmetric.
Note that the local volume form $\omega$ in question is unique modulo a non-zero constant factor.

**Definition 7.2.**

(1) A pair $(\nabla, g)$ is called a *Codazzi pair* if it satisfies Codazzi equations, that means the covariant derivative $\nabla g$ is totally symmetric.

(2) A triple $(\nabla, g, \nabla^\ast)$ of a non-degenerate metric and two affine connections $\nabla$ and $\nabla^\ast$ is called *conjugate* if it satisfies, for all tangent fields $u, v, w$, the relation $u g(v, w) = g(\nabla_u v, w) + g(v, \nabla^\ast_u w)$. Here we admit that the connections $\nabla$ and $\nabla^\ast$ have torsion. The relation generalizes the well known Ricci Lemma from Riemannian geometry.

We have the following:

**Theorem 7.3.** Let $\nabla$ be an affine connection on a pseudo-Riemannian manifold $(M, g)$. Then the following assertions hold:

(A) [21], [28]. Let $\nabla^2$ be an affine connection. Then the triple $(\nabla, g, \nabla^2)$ is conjugate if and only if the pair $(\nabla, g)$ is a Codazzi pair and the torsion tensors coincide: $T(\nabla) = T(\nabla^2)$.

(B) Assume that $\nabla$ is torsion free. Then:

1. The $(1,2)$ tensor field $C := \nabla - \nabla(g)$ is symmetric.
2. The triple $(\nabla, g, \nabla^\ast)$ with $\nabla^\ast := \nabla(g) - C$ is conjugate.
3. The connection $\nabla^\ast$ is torsion free if and only if the pair $(\nabla, g)$ is a Codazzi pair.
4. The connection $\nabla^\ast$ is Ricci symmetric if and only if the connection $\nabla$ is Ricci symmetric.
5. The curvature operator $\mathcal{R}$ satisfies the first Bianchi identity if and only if $\mathcal{R}^\ast$ does. Thus $\mathcal{R} \in \mathfrak{t}(T_p M)$ if and only if $\mathcal{R}^\ast \in \mathfrak{t}(T_p M)$.
6. Let $\nabla$ be Ricci symmetric. One has that the associated curvature operators $\mathcal{R} = \mathcal{R}(\nabla)$ and $\mathcal{R}^\ast = \mathcal{R}(\nabla^\ast)$ in a conjugate triple are $g-$conjugate.

**Proof.** (5) is the only statement without proof in the literature, so far. Its proof follows from 4.4.10.d in [29]; see also Lemma 7.6 and Observation 7.8 below.

**Remark 7.4.** (i) Note that (4) in Theorem 7.3 needed an analytic proof so far (see section 4.4.7 in [29]). The proof of Theorem 4.10 above is purely algebraic and pointwise.

(ii) The sections 4.4.1i and 4.4.10.d in [29] imply also the following: Assume that $\nabla$ is torsion free as above; then $\nabla^\ast$ is torsion free if and only if $C$ is symmetric. On the other hand, $\nabla^\ast$ is torsion free if and only if $(\nabla, g)$ is a Codazzi pair. The symmetry of $C$ implies that $\gamma_{jkl}^i := C_{j|i}^b C_{k|i}^l - C_{j|k}^b C_{i|l}^l$ is algebraic. Thus we can state: Let $\nabla$ be torsion free, satisfy the first Bianchi identity, and let $(\nabla, g)$ be a Codazzi pair. Then $\nabla^\ast$ satisfies the first Bianchi identity.

(iii) Consider the conformal class $\mathcal{C}$ of pseudo-Riemannian metrics generated by the metric $g$. While the conjugation of a connection $\nabla$ in general depends on the choice of a metric in $\mathcal{C}$, the conjugation of a generalized curvature tensor in $T_p M$ for $p \in M$ does not.

Below we summarize some facts we shall need concerning the *cubic form* $C$ and the *Tchebychev form $T^\ast$*:

**Observation 7.5.** Let $(\nabla, g, \nabla^\ast)$ be a conjugate triple with torsion free connections $\nabla, \nabla^\ast$. Then:

1. We have $C = \frac{1}{2}(\nabla - \nabla^\ast) = \nabla(g) - \nabla^\ast = \nabla - \nabla(g)$. The trace $T^\ast$ of $C$ is a 1-form, called the *Tchebychev form*, it is given by

$$n T^\ast(v) := \text{Tr}(u \mapsto C(u, v)).$$
Observation 8.1. This has the following consequences: information in both structures, called a gauge transformation.

8.1. Conformal and projective classes. Consider a pseudo-Riemannian metric $g$ and a connection $\nabla^*$ that is torsion free. The metric generates a conformal structure $\mathcal{C} = \{g\}$ and the connection a projective class $\mathfrak{P}^* = \{\nabla^*\}$ of torsion free connections. Any positive function $q \in C^\infty(M)$ induces a simultaneous transformation in both structures, called a gauge transformation with transition function $q$, by

(a) a conformal change $g^2 = q \cdot g$;
(b) a projective change

$$\nabla_v^* w - \nabla_v^* w = (d \ln q)(v)w + (d \ln q)(w)v,$$

This has the following consequences [6, 24, 25, 26, 27]:

Observation 8.1. Under the given assumptions for $\mathcal{C}$ and $\mathfrak{P}^*$ we have:
(1) Transform a given pair \((\nabla^*, g)\) simultaneously in the conformal and the projective class according to the above transformations, respectively; then \((\nabla^*, g)\) is a Codazzi pair if and only if \((\nabla^{*\sharp}, g^{\sharp})\) is a Codazzi pair \([24, 29]\).

(2) For a Ricci symmetric connection \(\nabla^*\), under a projective change with transition function \(g, \nabla^{*\sharp}\) is again Ricci symmetric \([24]\).

(3) Starting with a Codazzi pair \((\nabla^*, g)\), the foregoing extension to \((\nabla, g, \nabla^*)\) and transformation to \((\nabla^\sharp, g^\sharp, \nabla^{*\sharp})\) give a conjugate triple with torsion free connections s.t. \((\nabla^{*\sharp}, g^{\sharp})\) and \((\nabla^\sharp, g^\sharp)\) are Codazzi pairs. If additionally \(\nabla^*\) is Ricci symmetric then \(\nabla, \nabla^{\sharp}\) and \(\nabla^{*\sharp}\) are Ricci symmetric.

(4) In \([6]\) we proved that the foregoing transformation of conjugate triples with Codazzi pairs \((\nabla^*, g)\) and \((\nabla, g)\) is equivalent to a gauge transformation in an appropriate Weyl geometry. For this reason, for the simultaneous conformal and projective transformations with the same transition function within the classes \(\mathcal{C}\) and \(\mathfrak{P}^*\), we adopt the terminology \textit{gauge transformations}; invariants under gauge transformations are called \textit{gauge invariants} \([27]\).

(5) As above, consider the conformal structure \(\mathcal{C} = \{g\}\) and the \textit{Ricci symmetric projective structure} \(\mathfrak{P}^* = \{\nabla^*\}\), that means the generating connection \(\nabla^*\) is torsion free and Ricci symmetric. If \((\nabla^*, g)\) is a Codazzi pair it generates a conjugate triple \((\nabla, g, \nabla^*)\) with Ricci symmetric connections, and Ricci symmetric conjugate triples go to Ricci symmetric conjugate triples under gauge transformations. As in \([6]\) we call a structure, given by a conformal and such a projective class \(\mathfrak{P}^*\), both related by Codazzi equations, a \textit{Codazzi structure}. The Codazzi pairing induces a bijective mapping \(\mathfrak{P}^* \longleftrightarrow \mathcal{C}\).

8.2. The gauge invariant difference tensor. In \([27]\) we listed gauge invariants of conjugate triples. Above we defined the difference tensor \(C\) and its Tchebychev form \(T^\flat\); then the trace free part \(\tilde{C}\) of \(C\) is a gauge invariant:

\[
\tilde{C}_{jk}^i := C_{jk}^i - \frac{n}{n+2}(T_j \delta_k^i + T_k \delta_j^i + g_{jk} T^i).
\]

We use \(\tilde{C}\) in section 11.2 below.

8.3. Blaschke structures. Let a Codazzi structure be given by a conformal class \(\mathcal{C} = \{g\}\) and a projective, Ricci symmetric class \(\mathfrak{P}^* = \{\nabla^*\}\), related by Codazzi equations. Then there exists a unique Codazzi pair \((\nabla^*, g)\) \(\in \mathfrak{P} \times \mathcal{C}\), satisfying \(\omega(g) = \omega^*\) (apolarity), where the equality holds modulo a non-zero constant factor. We call the associated conjugate triple a \textit{Blaschke structure} or \textit{equiaffine structure} on \(M\) and use a notational mark “\(\ast\)” for \textit{equiaffine}. The existence of a Blaschke structure follows in analogy to Proposition 5.3.1.1 in \([29]\). As \(\omega(g) = c \cdot \omega^*\) with \(0 \neq c \in \mathbb{R}\) is equivalent to \(T^\flat = 0\) we conclude that \(\tilde{C} = C(e)\), that is: \(C(e)\) is a trace free, gauge invariant tensor field.

8.4. Codazzi structures and curvature operators. For a conjugate triple, the connections \(\nabla\) and \(\nabla^*\) induce curvature operators \(\mathcal{R} := \mathcal{R}(\nabla)\) and \(\mathcal{R}^* := \mathcal{R}(\nabla^*)\), respectively. \(\mathcal{R}\) is equiaffine if and only if \(\mathcal{R}^*\) is equiaffine, see Theorem 4.10. Assume that both operators satisfy this condition, then both curvature operators \(\mathcal{R}\) and \(\mathcal{R}^*\) are conjugate, and then a gauge transformation transforms a conjugate triple to a conjugate triple and thus induces a transformation of conjugate curvature operators, and if \(\mathcal{R}\) is equiaffine then it easily follows from the foregoing that equiaffine conjugate curvature tensors \(R\) and \(R^*\) give, via gauge transformations, equiaffine conjugate curvature tensors \(R^\sharp\) and \(R^{*\sharp}\).

8.5. Equiaffine Einstein spaces. Let \((M, g)\) be pseudo-Riemannian and \(\nabla\) be a torsion free and Ricci symmetric connection. \((M, g, \nabla)\) is called \textit{equiaffine Einstein} if \(\text{Ric} := \text{Ric}(\nabla)\) satisfies \(\text{Ric} = \lambda \cdot g\) for \(\lambda \in C^\infty(M)\); this relation then holds for
any other metric $g^\sharp$ in the conformal class of $g$. One has the following result which gives an equivalent condition in terms of the $W$-decomposition of Section 4:

**Lemma 8.2.** $(M, g, \nabla)$ is equiaffine Einstein if and only if $\pi_2(R) = 0 = \pi_3(R)$.

### 8.6. $W$-Decomposition, A-Decomposition, and Codazzi structures.

Let $\nabla$ be a torsion free connection on a pseudo-Riemannian manifold $(M, g)$. We extend $(\nabla, g)$ to a conjugate triple $(\nabla, g, \nabla^\ast)$; from above we know that $\nabla^\ast$ is torsion free if and only if $(\nabla, g)$ is a Codazzi pair. From now on we assume that both connections $\nabla, \nabla^\ast$ in the conjugate triple $(\nabla, g, \nabla^\ast)$ are torsion free.

Again, the metric $g$ generates a conformal structure $\mathcal{C} := \{g\}$, and the connection $\nabla^\ast$ a projective class $\mathfrak{P}^\ast := \{\nabla^\ast\}$ of torsion free connections.

At each point $p \in M$ the tangent space is a vector space with a scalar product $g_p$. The metric $g$ and both connections induce curvature operators $\mathcal{R}, \mathcal{R}^\ast$. As $\mathfrak{P}^\ast$ is a projective class the projective $(1,3)$ curvature operator $\mathcal{P}(\mathfrak{R}^\ast)$ is an invariant of the class, denoted by

$$\mathcal{P}(\mathfrak{P}^\ast) := \mathcal{P}(\mathfrak{R}^\ast) \quad \text{for} \quad \nabla^\ast \in \mathfrak{P}^\ast \quad \text{and} \quad \mathfrak{R}^\ast = \mathcal{R}(\nabla^\ast).$$

Now we study relations between the foregoing geometric structures on $M$ and the pointwise algebraic $W$-decomposition of generalized curvature tensors. Recall that a connection $\nabla$ with curvature operator $\mathcal{R}$ is called Ricci symmetric if its Ricci tensor $Ric$ is symmetric at any point $p \in M$; as already stated, this is equivalent to the fact that $\nabla$ locally admits a parallel volume form.

**Theorem 8.3.** A conjugate triple $(\nabla, g, \nabla^\ast)$, its induced curvature operators $\mathcal{R}$ and $\mathcal{R}^\ast$, and their decompositions satisfy the following equivalences:

1. $\nabla$ is Ricci symmetric.
2. $\nabla$ locally admits a parallel volume form $\omega$, thus $\nabla \omega = 0$.
3. $\mathcal{R}$ is an equiaffine curvature tensor.
4. $\nabla^\ast$ is Ricci symmetric.
5. $\nabla^\ast$ locally admits a parallel volume form $\omega^\ast$, thus $\nabla^\ast \omega^\ast = 0$.
6. $\mathcal{R}^\ast$ is an equiaffine curvature tensor.
7. $\pi_3(R) = 0 = \pi_8(R)$ at any point.
8. $\pi_3(R^\ast) = 0 = \pi_8(R^\ast)$ at any point.
9. $\alpha_4(R) = 0 = \alpha_5(R) = \alpha_6(R)$ at any point.
10. $\alpha_4(R^\ast) = 0 = \alpha_5(R^\ast) = \alpha_6(R^\ast)$ at any point.

**Proof.** It follows from Theorem 7.3 and Remark 7.4 that both curvature tensors $\mathcal{R}$ and $\mathcal{R}^\ast$ satisfy $\mathcal{R}, \mathcal{R}^\ast \in \mathfrak{r}(T_pM)$ on $M$. Theorem 4.10 implies that $\nabla$ is Ricci symmetric if and only if $\nabla^\ast$ is Ricci symmetric [29]. For the rest of the proof see the results given above. $\square$

**Remark 8.4.** The relation $\pi_3(R) = 0$ implies $\pi_4(R) = 0$; analogously $\pi_3(R^\ast) = 0$ implies $\pi_4(R^\ast) = 0$.

### 9. Projective and Conformal Changes

On a given pseudo-Riemannian manifold we consider a torsion free connection $\nabla^\ast$ and the projective class $\mathfrak{P}^\ast$ of torsion free connections generated by $\nabla^\ast$. As before we write $\mathfrak{R}^\ast := \mathcal{R}(\nabla^\ast)$ etc.

#### 9.1. $W$-Decomposition and projective classes.

From [5] we have:

**Theorem 9.1.** Consider a projective class $\mathfrak{P}^\ast$ of torsion free connections; for any $\nabla^\ast \in \mathfrak{P}^\ast$ the five components $\pi_j(R^\ast)$, for $j = 4, \ldots, 8$, in the $W$-decomposition of the $(0,4)$ curvature tensor

$$p(\mathfrak{P}^\ast) = p(R^\ast) = \sum_{j=4}^{8} \pi_j(R^\ast)$$
give rise to projective invariants, namely their $g$-associated $(1,3)$ curvature operators in the associated decomposition of the projective $(1,3)$ curvature operator in $\mathfrak{R}(V)$. Analogously, the $(1,3)$ curvature operator that is $g$-associated to 
\[
R^* - (\pi_1(R^*) \oplus \pi_2(R^*) \oplus \pi_3(R^*)) ,
\]
is a projective invariant.

Proof. For all $\nabla^* \in \mathfrak{B}^*$ the projective curvature operators coincide, and according to Section 2 one can characterize the subspaces in the $W$-decomposition, and from this the components $\pi_j(R^*)$ for $j = 4, \ldots, 8$ are uniquely determined, and we have
\[
\pi_j(R^*) = \pi_j(p(R^*)) .
\]
Thus the components in the associated decomposition of the projective $(1,3)$ curvature tensor are the same for any $\nabla^*$, that means that their $g$-associated $(1,3)$ curvature operators are projectively invariant. \qed

Remark. In case that we study equiaffine curvature tensors, we have
\[
p(\mathfrak{P}^*) = p(R^*) = \sum_{j=5}^7 \pi_j(R^*),
\]
and each component and also
\[
R^* - (\pi_1(R^*) \oplus \pi_2(R^*))
\]
lead to projective invariants taking the the associated decomposition of the projective $(1,3)$ curvature operator in $\mathfrak{E}(V)$.

The following Theorem is a corollary of the foregoing decomposition in the space $\mathfrak{E}(V)$, but because of its importance it is stated as a separate result; it generalizes results of [31]. Here we get a projectively invariant symmetric bilinear form of a Codazzi structure in a very general situation. See the material above in Section 4. We recall definition 4.13 and that of $B^*$ from Equation (4.b).

**Theorem 9.2.** The symmetric bilinear form $B^*$ is invariant under a projective change in the Ricci symmetric class $\mathfrak{P}^*$.

**Proof.** The $(1,3)$ curvature operator in $\mathfrak{E}(V)$ that is $g$-associated to $\pi_5(R^*) \subset \mathfrak{f}(V, g)$ is a projective invariant. Take the trace $\rho_{13}$. \qed

The following results are useful as well; as before, in this section we assume that the connections $\nabla, \nabla^*$ are torsion free:

**Lemma 9.3.** Let $(\nabla, g, \nabla^*)$ be a conjugate triple with Ricci symmetric $\nabla^*$. Then
\[
p(R^*) = \sum_{j=5}^7 \pi_j(R^*);
\]
if additionally $\mathcal{P}(\mathcal{R}) = \mathcal{P}(\mathcal{R}^*)$ then $\text{Ric} = \text{Ric}^*$.

The foregoing statements give the following result:

**Corollary 9.4.** Let $\nabla^*$ be Ricci symmetric and $(\nabla, g, \nabla^*)$ be a conjugate triple as above. The following equations are equivalent:

1. $\mathcal{P}(\mathcal{R}) = \mathcal{P}(\mathcal{R}^*)$.
2. $\mathcal{R} = \mathcal{R}^*$.

We now discuss the $W$-Decomposition and projective changes. According to Weyl two connections $\nabla^*$ and $\nabla^{*2}$ are projectively equivalent if and only if there exists a one-form $\theta$ s.t.
\[
\nabla^*_\theta - \nabla^{*2}_\theta w = \theta(v)w + \theta(w)v .
\]
In section 8.1 we considered the special case of $\theta = d\ln q$, where $q \in C^\infty$ is a transition function. Observation 8.1 gives the following:

**Proposition 9.5.** Let $(\nabla, g, \nabla^*)$ be a conjugate triple and $0 < q \in C^\infty$ a transition function so that $(\nabla, g, \nabla^*) \mapsto (\nabla^1, g^2, \nabla^{*2})$ is a Codazzi transformation. The following equations are equivalent:

1. $\pi_3(R) = 0$.
2. $\pi_3(R^*) = 0$.
3. $\pi_3(R^\flat) = 0$.
4. $\pi_3(R^*) = 0$.

We relate the $W$-Decomposition and projective flatness. Recall that a projective class $P^* := \{\nabla^*\}$ that is generated from $\nabla^*$ by gauge transformations with transition functions is said to be a *Ricci symmetric projective class* if $\nabla^*$ is torsion free and Ricci symmetric. From the foregoing then any connection in $\mathfrak{P}^*$ is torsion free and Ricci symmetric. It is well known that projective flatness can be characterized as follows [10]:

**Lemma 9.6.** Let $n \geq 2$ and $\nabla^*$ be torsion free and Ricci symmetric. Then $\nabla^*$ is projectively flat if and only if the projective curvature tensor $P(R^*)$ vanishes and the covariant derivative $\nabla^* \text{Ric}^*$ is totally symmetric.

It is also well known that the two conditions in Lemma 9.6 are dependent; see e.g. [10, 20, 23, 29]:

**Lemma 9.7.** (1) In dimension $n = 2$ the projective curvature tensor vanishes identically and projective flatness is equivalent to the total symmetry of $\nabla^* \text{Ric}^*$. (2) In dimension $n \geq 3$ projective flatness is equivalent to the vanishing of the projective curvature tensor $P(R^*)$; the total symmetry of $\nabla^* \text{Ric}^*$ is a consequence.

**Lemma 9.8.** Let $n \geq 3$ and $\nabla^*$ be Ricci symmetric. Then the following assertions are equivalent: (1) $\nabla^*$ is projectively flat; (2) $P(R^*) = 0$; (3) $R^* \in W_1 \oplus W_2$.

The following Corollary is now immediate; it is a technical use we shall need subsequently.

**Corollary 9.9.** Let $n \geq 3$ and $(\nabla, g, \nabla^*)$ be a conjugate triple; assume that $\nabla^*$ is Ricci symmetric and projectively flat. Then:

$$B^* := (n - 1)\text{Ric} + \text{Ric}^* - \tau g = 0.$$

We now have:

**Theorem 9.10. [Equivalence Theorem]** Let $n \geq 3$ and $(\nabla, g, \nabla^*)$ be a conjugate triple; assume that $\nabla^*$ is Ricci symmetric and projectively flat. Then the following assertions are equivalent:

1. $\nabla$ is projectively flat.
2. $\text{Ric} = \text{Ric}^*$.
3. $\pi_2(R) = 0$.
4. $\pi_2(R^*) = 0$.
5. $n \cdot \text{Ric} = \tau \cdot g$; that means: $(\nabla, g)$ is equiaffine Einstein.
6. $n \cdot \text{Ric}^* = \tau \cdot g$; that means: $(\nabla^*, g)$ is equiaffine Einstein.
7. $\mathcal{R} = R^*$.
8. $-n(n - 1) \cdot \mathcal{R} = \tau \cdot (g \wedge g)$.
9. The covariant derivative $\nabla(g)C^\flat$ is a totally symmetric $(0,4)$-tensor field.

We can draw the following consequence:
Corollary 9.11. Let \( n \geq 3 \) and \( (\nabla, g, \nabla^*) \) be a conjugate triple; assume that \( \nabla^* \) is torsion free, Ricci symmetric, projectively flat and equiaffine Einstein. Then \( \tau = \text{const.} \)

Proof. \( \nabla^* \) is torsion free, thus \( (\nabla, g) \) and also \( (\nabla^*, g) \) are Codazzi pairs. The projective flatness implies that also \( (\nabla^*, \text{Ric}^*) \) is a Codazzi pair [10]. \( \nabla^* \)-covariant differentiation of the equation \( \text{Ric}^* = \frac{1}{n} \cdot \tau g \), and the Codazzi properties together with a contraction finally give the assertion. \( \Box \)

The following result is another simple consequence of Theorem 9.10; because of its geometric importance we state it as a Theorem; namely, it is remarkable, that, under the given assumptions, the projective flatness of \( \nabla \) is equivalent to the vanishing of a symmetric bilinear form, and there is no need to calculate its \((1,3)\) projective curvature operator.

Theorem 9.12. Let \( n \geq 3 \) and \( (\nabla, g, \nabla^*) \) be a conjugate triple; assume that \( \nabla^* \) is torsion free, Ricci symmetric and projectively flat. Then the following conditions are equivalent:

(1) \( \nabla \) is projectively flat.
(2) \( B := (n - 1)\text{Ric}^* + \text{Ric} - \tau \cdot g = 0. \)

Proof. The projective flatness of \( \nabla^* \) implies \( B^* = 0. \) Then \( B = 0 \) is equivalent to the identity \( \text{Ric}^* = \text{Ric}. \) An easy computation now completes the proof. \( \Box \)

9.2. Projective flatness and PDEs. Theorem 9.10 (9) relates properties of equiaffine curvature tensors with that of PDEs. This admits important applications. We give the following example that generalizes the local classification of locally strongly convex equiaffine spheres with constant sectional curvature of the Blaschke metric. It is remarkable that, in the context of conjugate connections, the essential assumptions can be expressed in terms of the \( W \)-decomposition of the three curvature operators \( R, \ R(g), \) and \( R^*. \) The restriction to dimension \( n \geq 3 \) is due to the fact that in the following we characterize projective flatness by the vanishing of the projective curvature tensor.

Theorem 9.13. Let \( n \geq 3 \) and \( (\nabla, g, \nabla^*) \) be a Blaschke structure with a Riemannian metric \( g. \) Assume that \( R(g) = \pi_1(R(g)), \) that \( R^* = \pi_1(R^*) \oplus \pi_2(R^*), \) and that \( \pi_2(R) = 0. \) Then:

(1) If \( C = 0 \) then trivially all three connections coincide.
(2) If \( \|C\| \neq 0 \) then \( (M, g) \) is flat and \( \tau \) is a negative constant.

Proof. From the assumptions, from the results of Section 7 and from Theorem 9.10 the following conditions are satisfied:

(1) \( C^\circ \) is totally symmetric.
(2) \( \text{Tr}(C) = 0. \)
(3) \( g(C(u, v), C(w, z)) - g(C(w, v), C(u, z)) \)
\( = (\tau - \kappa)(g(u, v) g(w, z) - g(w, v) g(u, z)). \)
(4) \( \tau = \text{const.} \) and \( \kappa = \text{const} \) (see Remark 7.7).
(5) \( (\nabla(g)_u C)(v, w) = (\nabla(g)_v C)(u, w). \)

The proof follows now the lines of the proof of the Main Theorem in [35]. \( \Box \)

Concerning the local classification of equiaffine spheres with indefinite metric and constant Blaschke sectional curvature, there is the famous solution of the so called Magid-Ryan conjecture by Vrancken [33], [34]. This result and its proof can be generalized to conjugate connections as follows:

Theorem 9.14. Let \( n \geq 3 \) and \( (\nabla, g, \nabla^*) \) be a Blaschke structure with indefinite metric \( g. \) Assume that
The curvature operator $R$

Theorem 9.16. Let $\tau$ be a gauge transformation with a transition function as in Section 8.1. Then:

1. $\pi_2(R) = 0$.
2. $R(g) = \pi_1(R(g))$.
3. $R^* = \pi_1(R^*) \oplus \pi_2(R^*)$.

Moreover, assume that $\tau - \kappa \neq 0$. Then $(M, g)$ is flat.

Proof. As in the proof of Theorem 9.13, the conditions (1) - (5) are satisfied. Apply now Theorem 6 in [33].

Theorem 9.15. Let $n \geq 3$ and $(\nabla, g, \nabla^*)$ be a Blaschke structure with indefinite metric $g$. Assume that $\pi_5(R) = 0$, assume that $R(g) = \pi_1(R(g))$, and assume that $R^* = \pi_1(R^*) \oplus \pi_2(R^*)$. Then $(M, g)$ is flat and $\|C\|^2 = 0$.

Proof. Apply Theorem 11 in [33]. Analogously one can generalize Theorem 12 in Vrancken’s paper [33].

9.3. W-Decomposition and gauge transformations of conjugate triples.

We studied “pointwise” conformal changes in Section 3.2, while projective changes on a manifold where investigated above. It is well known that the Ricci symmetry is invariant under a projective change with a transition function [22]. Considering a conjugate triple $(\nabla, g, \nabla^*)$ and their $(1,3)$ curvature operators $R$ and $R^*$, and also the curvature operator $R(g)$ of the metric $g$, Section 8.1 and the foregoing results give the following Theorem.

Theorem 9.16. Let $\nabla$ and $\nabla^*$ be torsion free connections. Let

$$(\nabla, g, \nabla^*) \mapsto (\nabla^2, g^4, \nabla^{\ast 2})$$

be a gauge transformation with a transition function as in Section 8.1. Then:

1. If $\nabla$ or $\nabla^*$, respectively, is Ricci symmetric, then all connections appearing in the conjugate triple and under gauge transformations of this triple are Ricci symmetric.
2. We have that

$$\sum_{j=4}^7 \pi_j(R^*) = p(R^*) = p(R^*^{\ast 4}) = \sum_{j=4}^7 \pi_j(R^*^{\ast 4}).$$

In particular: each component of the decompositions of $(1,3)$ curvature operators that is $g$-associated to $\pi_j(R^*)$, for $j = 4, ..., 7$, is gauge invariant itself; if $\nabla$ or $\nabla^*$, respectively, is Ricci symmetric then $\pi_4(R) = 0 = \pi_4(R^*)$.
3. The conformal class satisfies:

$$\alpha_6(R(h)) = \pi_6(R(h)) = \pi_6(R(h^2)) = \alpha_6(R(h^2)).$$

Corollary 9.17. Let $\nabla$ and $\nabla^*$ be torsion free and Ricci symmetric connections. Let

$$(\nabla, g, \nabla^*) \mapsto (\nabla^2, g^4, \nabla^{\ast 2})$$

be a gauge transformation with a transition function. If for one of the curvature tensors, say $R$, we have $\pi_j(R) = 0$ for $j = 3$ ($j = 4$, resp.) then for any curvature tensor under conjugation or gauge transformation, the components $\pi_j$ vanish for $j = 3$ ($j = 4$, resp.).

We also have:

Theorem 9.18. Let $(\nabla, g, \nabla^*)$ be a conjugate triple with $\nabla^*$ torsion free and Ricci symmetric. Then the bilinear form $B^*$ in Corollary 9.9 is gauge invariant.

Proof. For fixed metric $g$ the $(1,3)$ curvature operator, that is $g$-associated to the component $\pi_5(R^*)$, is projectively invariant. Following (2) in the foregoing Theorem, a conformal change $g^* = \lambda \cdot g$ gives $\pi_5(R^*) = \pi_5(R^*)$, thus the associated $(1,3)$ curvature operators in $\mathfrak{R}(V)$ are independent of the conformal change. Taking traces $\rho_{13}$ on both sides gives $B^{\ast 2} = B^*$. □
9.4. The decomposition of equiaffine curvature tensor fields. According to its geometric importance, we summarize the following observations in the special case of equiaffine curvature tensor fields. Throughout we assume that \((\nabla, g, \nabla^*)\) be a conjugate triple with torsion free connections \(\nabla\) and \(\nabla^*\).

**Observation 9.19.** If \(\nabla^*\) is Ricci symmetric then \(R^*\) is an equiaffine curvature operator, and then \(R\) is an equiaffine curvature operator.

For the \(W\)-decomposition one has that:

**Observation 9.20.**

(1) \(R\) and \(R^*\), respectively, satisfy the orthogonal decompositions

\[
R = \pi_1(R) \oplus (\pi_2(R) \oplus \pi_3(R)) \oplus (\pi_6(R) \oplus \pi_7(R)),
\]

\[
R^* = \pi_1(R^*) \oplus (\pi_2(R^*) \oplus \pi_3(R^*)) \oplus (\pi_6(R^*) \oplus \pi_7(R^*)).
\]

(2) The sums \((\pi_2(R) \oplus \pi_3(R))\) and \((\pi_2(R^*) \oplus \pi_3(R^*))\) are Ricci traceless and also Ricci\(^*\) traceless.

(3) The sums \((\pi_6(R) \oplus \pi_7(R))\) and \((\pi_6(R^*) \oplus \pi_7(R^*))\) are Ricci flat and also Ricci\(^*\) flat.

For the \(A\)-decomposition, we have:

**Observation 9.21.** \(R\) and \(R^*\), respectively, satisfy the orthogonal decomposition

\[
R = \alpha_1(R) \oplus (\alpha_2(R) \oplus \alpha_3(R)) \oplus (\alpha_6(R) \oplus \alpha_7(R)),
\]

\[
R^* = \alpha_1(R^*) \oplus (\alpha_2(R^*) \oplus \alpha_3(R^*)) \oplus (\alpha_6(R^*) \oplus \alpha_7(R^*))
\]

where

(1) both of the two orthogonal sums \(\alpha_2(R) \oplus \alpha_3(R)\) and \(\alpha_2(R^*) \oplus \alpha_3(R^*)\) are Ricci traceless and Ricci\(^*\) traceless;

(2) both of the two orthogonal sums \(\alpha_6(R) \oplus \alpha_7(R)\) and \(\alpha_6(R^*) \oplus \alpha_7(R^*)\) are Ricci flat and Ricci\(^*\) flat;

(3) \(\pi_1(R^*) = \pi_1(R) = \alpha_1(R) = \alpha_1(R^*)\) and \(\pi_6(R) = \pi_6(R^*)\);

(4) \(\pi_2(R) \oplus \pi_5(R) = \alpha_2(R) \oplus \alpha_3(R)\)

\[
= \frac{1}{}\mbox{min}_{n-1}[2\tau g \wedge g + g \wedge_{n-1} \mbox{Ric} - \mbox{Ric}^* \wedge_{n-1} g];
\]

(5) \(\alpha_3(R) = \alpha_5(R^*)\).

Concerning projective invariants, Theorem 9.1 now reads:

**Observation 9.22.** The \((0,4)\) curvature tensor \(p(R^*)\) satisfies

\[
p(R^*) = \sum_{j=5}^{7} \pi_j(R^*).
\]

The \((1,3)\) tensor components \(g\)-associated to \(\pi_j(R^*)\) are projective invariants for \(j = 5, 6, 7\).

**Remark 9.23.** We would like to comment on the foregoing summary of decomposition results; for this, we recall the decomposition of algebraic curvature tensors in section 5.1 and that of generalized curvature tensors discussed in Section 6. Considering conjugate connections and their equiaffine curvature tensor fields on a manifold, we see how the decomposition reflects geometric properties of a triple \((\nabla, g, \nabla^*)\); moreover, we learn that the concepts of conjugate connections and conjugate curvature tensors, the latter induced from the first, are appropriate tools to understand how properties of algebraic curvature tensors generalize to equiaffine curvature tensors.
10. Relative Hypersurface Theory

We recall basics from relative hypersurface theory [18, 29].

10.1. Review of relative hypersurface theory. We describe the duality of the vector space $\mathbb{R}^{n+1}$ and its dual $\mathbb{R}^{(n+1)*}$ in terms of a non-degenerate scalar product $(\cdot, \cdot) : \mathbb{R}^{(n+1)*} \times \mathbb{R}^{n+1} \to \mathbb{R}$.

Associated to each of the vector spaces there is a one-dimensional vector space of determinant forms, fixing volumes modulo scaling. By $\det$ and $\det^*$ we denote an arbitrary pair of dual determinant forms on $\mathbb{R}^{n+1}$ and $\mathbb{R}^{(n+1)*}$. By the same symbol $\nabla$ we denote the canonical flat connections on $\mathbb{R}^{n+1}$ and $\mathbb{R}^{(n+1)*}$, respectively. For a hypersurface immersion $x : M \to \mathbb{R}^{n+1}$ we define a normalization: it is a pair $(Y, z)$ with $\langle Y, z \rangle = 1$ where $z : M \to \mathbb{R}^{n+1}$ is an arbitrary transversal field, and $Y : M \to \mathbb{R}^{(n+1)*}$, satisfying $\langle Y, dz(v) \rangle = 0$ for all tangent vectors $v$ on $M$, is a conormal field of $x$. While a transversal field $z$ extends a tangential basis to the ambient space, a conormal fixes the tangent plane. A normalized hypersurface is a triple $(x, Y, z)$.

10.1.1. Structure equations. The geometry of $(x, Y, z)$ can be described in terms of geometric invariants defined via the structure equations of Gauß and Weingarten, respectively:

$$\nabla_v dx(w) = dx(\nabla_v w) + h(v, w)z,$$
$$dz(v) = dx(-S(v)) + \sigma(v)z.$$

As before $u, v, w, \ldots$ denote tangent vectors and fields, respectively. The induced connection $\nabla$ is torsion free, $h$ is bilinear and symmetric, $S$ is the shape or Weingarten operator and $\sigma$ is a 1-form, the connection form; the sign in front of $S$ in the Weingarten equation is a convention corresponding to an appropriate choice of the orientation of $z$. All coefficients in the structure equations depend on the normalization, they are invariant under the affine group of transformations in $\mathbb{R}^{n+1}$.

10.1.2. Non-degenerate hypersurfaces. A hypersurface $x$ is non-degenerate if the bilinear form $h$ in the Gauß structure equation is non-degenerate; it is well known that this property is independent of the choice of the normalization as all such symmetric bilinear forms are conformally related, defining a conformal class $\mathcal{C}$. Thus, on a non-degenerate hypersurface, any transversal field induces a pseudo-Riemannian metric $h \in \mathcal{C}$ with Levi-Civita connection $\nabla(h)$ and Riemannian volume form $\omega(h)$; similarly we denote its curvature tensor by $R(h)$, its Ricci tensor by $\text{Ric}(h)$, its normalized scalar curvature by $\kappa(h)$, etc.

The non-degeneracy of $x$ is equivalent to the fact that any conormal field $Y$ itself is an immersion $Y : M \to \mathbb{R}^{(n+1)*}$ with transversal position vector $Y$. The associated Gauß structure equation reads

$$\nabla_v dY(w) = dY(\nabla^*_v w) + \frac{1}{n-1} \text{Ric}^*(v, w)(-Y)$$

where the conormal connection $\nabla^*$ is torsion free and Ricci symmetric. It is well known that all conormal connections are projectively related; we denote the projective class of all conormal connections by $\mathfrak{P}^*$.

10.1.3. Relative normalizations and curvature operators. Within the class of all normalizations of a non-degenerate hypersurface there is a distinguished large subclass, namely the class of all relative normalizations. This class can be characterized by the property that $\sigma = 0$ in the Weingarten structure equation. This is equivalent to the fact that the triple $(\nabla, h, \nabla^*)$ is conjugate. In the following we restrict to this class; this can be geometrically justified [27]. We denote a relative normalization by $(Y, y)$ and call such a triple $(x, Y, y)$, where $x$ is non-degenerate, a relative hypersurface.
We recall the notation

\[ \mathcal{R} := \mathcal{R}(\nabla), \quad \mathcal{R}^* := \mathcal{R}(\nabla^*), \quad \mathcal{R}(h) \]

for the curvature operators. Note that, if we have an arbitrary normalization that is not relative, then \( \nabla \) is not Ricci symmetric and thus \( \mathcal{R} \) is not equiaffine, while \( \nabla^* \) is always Ricci symmetric and thus \( \mathcal{R}^* \) equiaffine; \cite{27}. Using Theorem 4.10, we are able to characterize the important class of relative normalizations in terms of their \( W \)-decomposition as follows:

**Lemma 10.1.** A normalization of a non-degenerate hypersurface is relative if and only if \( \pi_3(R) = 0 \).

10.1.4. The cubic form and the Tchebychev form. We recall section 7.

\[ C(v, w) = \nabla(h)_{v} w - \nabla^{*}_{v} w \]

is a symmetric \((1,2)\)-tensor field (both connections are torsion free), and its trace

\[ nT^b(v) = \text{Tr}(w \mapsto C(v, w)) \]

is a closed 1-form as both connections locally admit parallel volume forms. \( T^b \) is called the Tchebychev form, the associated vector field \( T \), implicitly defined by \( h(T, v) := T^b(v) \), is called the Tchebychev field. Associated to \( C \) is the totally symmetric cubic form \( C^b \), defined by \( C^b(u, v, w) := h(C(u, v, w)) \).

10.1.5. Relative Gauß maps. In case of a relative normalization we know that the shape operator \( S \) is \( h \)-selfadjoint and satisfies

\[ \text{Ric}^c(v, w) = (n - 1)h(Sv, w) =: (n - 1)S^b(v, w). \]

The symmetric bilinear form \( S^b \) is called the Weingarten form. In the case that for a relative normalization \( \text{rank} \ S = n \), both, the relative normal and conormal fields

\[ y : M \to \mathbb{R}^{(n+1)} \quad \text{and} \quad Y : M \to \mathbb{R}^{(n+1)*}, \]

are immersions, called Gauß maps. For a relative hypersurface \((x, Y, y)\) the induced triple \((\nabla, h, \nabla^* )\) is a conjugate triple with torsion free and Ricci symmetric connections \( \nabla, \nabla^* \). Thus we can apply the results stated in sections 7 and 8. Recall that \( \text{Tr}(S) = nH \) where \( H \) denotes the relative mean curvature.

10.1.6. Integrability conditions. The integrability conditions for a relative hypersurface can be stated as follows (compare Lemma 9.6 - 9.7):

1. The conormal connection \( \nabla^* \) is projectively flat, that means the \((1,3)\) projective curvature operator \( P^* := \mathcal{P}(\mathcal{R}^*) \)

\[ P^*(u, v)w := \mathcal{R}^*(u, v)w - \frac{1}{n+1}[\text{Ric}^*(v, w)u - \text{Ric}^*(u, w)v] \]

vanishes identically, and \((\nabla^*, \text{Ric}^*)\) is a Codazzi pair, see Definition 7.2.

2. \((\nabla^*, h)\) is a Codazzi pair.

10.2. Examples of relative normalizations. We refer to \cite{29} for well known examples and further details for relative normalizations.

10.2.1. The equiaffine (Blaschke) normalization. There is a (modulo sign) unique normalization within all relative normalizations, characterized by the vanishing of its Tchebychev field: \( T(e) = 0 \) (apolarity condition); following section 8.3 here we use the notational mark "\( e \)" for the induced equiaffine geometry. The transversal field \( y = y(e) \) in this normalization historically is called affine normal field. Equivalent to the equation \( T(e) = 0 \) is the relation \( \omega^* = \omega(h) \) (modulo a positive constant factor). This relation characterizes a unique Codazzi pair \((\nabla^*, h)\) within the Cartesian product \( \mathbb{R}^+ \times \mathbb{C} \). Nowadays the unimodular geometry is often called Blaschke geometry. The geometry induced from the Blaschke normalization is invariant under the unimodular transformation group (including parallel translations).
10.2.2. The centroaffine normalization. For a non-degenerate hypersurface it is well known that the set \( \{ p \in M \mid x(p) \text{ tangential} \} \) is nowhere dense. Thus the position vector \( x \) is transversal almost everywhere. We call a non-degenerate hypersurface \( x \) with always transversal position vector centroaffine, and denote the position vector also by \( x \) [20]. For such a hypersurface one can choose \( y(c) := \varepsilon x \) as relative normal where \( \varepsilon = +1 \) or \( \varepsilon = -1 \) is chosen appropriately; in analogy to the foregoing we use “\( c \)” as a mark in case of a centroaffine normalization \((Y(c), y(c))\). \( Y(c) \) is oriented such that
\[
1 = \langle Y(c), y(c) \rangle.
\]

10.3. Gauge invariance. From the foregoing it is obvious that the conformal and the projective structure are of particular importance in relative hypersurface theory; both classes do not depend on a particular choice of a normalization, thus it is of interest how their invariants reflect the geometry of a given hypersurface [27]. Following the terminology of [27], gauge invariants are invariants that do not depend on a particular choice of a normalization. In relative hypersurface theory, the class \( \mathcal{P}^* \) is torsion free, Ricci symmetric and projectively flat; the last geometric property is equivalent to one of the integrability conditions of the structure equations, and this equivalence gives a geometric understanding of a version of the relative fundamental theorem that is an extension of the original result of Dillen, Nomizu and Vrancken in the theory of Blaschke hypersurfaces [29, 27]. From this the projective class \( \mathcal{P}^* \) and its geometry are well understood.

The situation is different with the conformal class \( \mathcal{C} \). One knows that \( \mathcal{C} \) is a class of Riemannian metrics if and only if \( x \) is locally strongly convex; this implies that a connected, closed (compact without boundary) hypersurface with definite class \( \mathcal{C} \) is a hyperovaloid. But e.g., so far there is no characterization of the class of hypersurfaces for which \( \mathcal{C} \) is locally conformally flat, even not under strong additional assumptions like locally strong convexity and compactness. One knows many local examples of hypersurfaces that are locally conformally flat, and besides the ellipsoid one knows that the following types of hypersurfaces are conformally flat:

1. hypersurfaces of rotation;
2. centroaffine Tchebychev hypersurfaces with complete centroaffine metric and non-constant unimodular support function [27];
3. decomposable hypersurfaces [2].

But one is far from a general understanding of the conformal properties in relative hypersurface theory, as there are only few results in special relative hypersurface theories. Concerning conformal properties, this motivates a particular interest in further investigations, and thus we consider special relative hypersurfaces in the following subsections, restricting to locally strongly convex relative hypersurfaces with appropriate orientation such that the class \( \mathcal{C} \) is (positive) definite.

From section 8.2 recall the definition of the trace free part \( \tilde{C} \) of \( C \).

**Proposition 10.2.** On a non-degenerate hypersurface, let \( Y \) be an arbitrary conormal field; from \( Y \) one can define the corresponding metric \( h \) and the projectively flat connection \( \nabla^* \), and from this \( C, T \) and finally \( \tilde{C} \); we have:

1. \( \tilde{C} \) is gauge invariant.
2. \( \tilde{C} = C(e) \).

10.3.1. *Gauge invariant relative geometries.* See [27]. We recall that the important relative hypersurface theories are in fact gauge invariant; more precisely:

**Lemma 10.3.** One has that:

1. The centroaffine metric, and thus its intrinsic geometry, is gauge invariant.
The class \( \{ c \cdot h \mid h \text{ Blaschke metric}, 0 \neq c \in \mathbb{R} \} \) is gauge invariant and thus also the intrinsic geometry of the Blaschke metric (modulo a non-zero constant factor).

10.4. Some special classes of relative hypersurfaces. We list some special classes of hypersurfaces that are well known in relative hypersurface theory.

10.4.1. Quadrics. We have the following characterization of quadrics in terms of the gauge invariant cubic form [29, 27].

**Theorem 10.4.** A relative hypersurface is a quadric if and only if \( \tilde{C} = 0 \).

10.4.2. Relative spheres. Let \( (x, Y, y) \) be a relative hypersurface; it is called a proper relative sphere if, for some \( x_0 \in \mathbb{R}^{n+1} \), we have \( y = \lambda(x - x_0) \) for an appropriate nowhere vanishing differentiable function \( \lambda \); it is called an improper relative sphere if \( y \) is a constant transversal field. Furthermore, \( (x, Y, y) \) is a relative sphere if and only if \( S^o = \lambda \cdot h = H \cdot h \), and the latter relation implies \( H = \text{const} \). A relative sphere is proper if \( H \neq 0 \), and it is improper if \( H = 0 \).

In the sense of the definition of relative spheres, any centro affine hypersurface with centro affine normalization is a proper relative sphere, thus in the centro affine geometry the notion of “relative spheres” is meaningless. In Blaschke’s geometry the relative spheres are called affine spheres. For proper affine spheres the Blaschke normalization and the centro affine normalization coincide (modulo a non-zero constant factor), thus the equation \( T(c) = 0 \) characterizes proper affine spheres within the centro affine geometry. The class of affine spheres is so large that one is far from any local classification. Under additional assumptions there are partial local and global classifications [18, 34].

10.4.3. Extremal hypersurfaces. For any non-degenerate hypersurface \( x \) which has a given conormal \( Y \) the area functional of its pseudo-Riemannian volume form \( \omega(h) \), on a domain \( D \) with compact support, is given by

\[
\mathfrak{B} := \int_D \omega(h).
\]

In case a hypersurface is a critical point of the functional it satisfies the Euler-Lagrange equation; then the hypersurface is called an extremal hypersurface.

10.4.4. Equiaffine extremal hypersurfaces. We use the notation from section 8.3. The Euler-Lagrange equation takes the form \( nH(e) := Tr(S(e)) = 0 \) in the Blaschke geometry; it is a PDE of fourth order. The expression for the second variation of the area functional is very complicated. E. Calabi [8] proved:

**Theorem 10.5.** On locally strongly convex, extremal hypersurfaces, any of the following conditions (1) and (2) implies that the second variation is negative; in this case the affine extremal hypersurfaces are called affine maximal:

(1) \( n = 2 \).

(2) \( n \geq 2 \) and \( x \) can be represented as a graph.

11. W-decomposition and relative hypersurfaces

11.1. W-decomposition and integrability conditions. One of the integrability conditions in relative hypersurface theory is given by the projective flatness of \( \nabla^* \); for \( n \geq 3 \), Lemma 9.3 and Lemma 10.1 imply:

\[
\pi_5(R^*) = \pi_6(R^*) = \pi_7(R^*) = 0.
\]
11.2. Characterization of hyperquadrics. Recall the definition of \( \tilde{C} \) from section 8.2 and the fact that a hypersurface is locally strongly convex if and only if the conformal class of relative metrics is (positive) definite.

**Theorem 11.1.** Consider a relative hypersurface. Then:

1. The expression \( \tilde{\gamma}^{i j k l} := \tilde{C}^{h i} \tilde{C}^{j k} - \tilde{C}^{h j} \tilde{C}^{i k} \) defines an algebraic, gauge invariant curvature operator.
2. The Blaschke geometry in the Codazzi structure has the following property:
   \[ R(e) + R^*(e) - 2R(h(e)) = \tilde{\gamma}. \]
3. In case the conformal class is (positive) definite, we have the equivalences:
   a. \( \tilde{\gamma} \) vanishes identically.
   b. \( \tilde{C} = 0 \).
   c. \( R(e) + R^*(e) = 2R(h(e)) \).
   d. The gauge invariant connections in the Blaschke geometry coincide:
   \[ \nabla(e) = \nabla(h(e)) = \nabla^*(e). \]
   e. \( x \) is a hyperquadric.
   f. \( \pi_1(R) = n(n-1) \pi_1(R(h)) \).

**Proof.** We restrict to some remarks, as the proof is routine. In (3), the equivalences of (b), (d), (e) are true for any relative hypersurface. In the Blaschke geometry, (f) yields if and only if the Pick invariant \( J \) satisfies \( n(n-1)J = n(n-1)(\kappa - H) = n(n-1)\kappa - \tau = 0 \). As \( g \) is positive definite, this is equivalent to \( \tilde{C} = C = 0 \) (see Theorem 10.4). \( \square \)

11.3. Characterization of relative and affine spheres. We recall the following result from [29], Theorem 6.3.5.2.

**Theorem 11.2.** Let \( x: M \to \mathbb{R}^{n+1} \) be a centroaffine hypersurface of dimension \( n > 2 \) with relative normalization \( (Y, y) \) and induced conjugate triple \( (\nabla, h, \nabla^*) \). Then the relative normalization coincides with the centroaffine normalization if and only if \( \nabla \) is projectively flat.

This together with Section 11.5 above gives:

**Theorem 11.3.** Let \( x: M \to \mathbb{R}^{n+1} \) be a centroaffine hypersurface of dimension \( n > 2 \) with relative normalization \( (Y, y) \) and induced conjugate triple \( (\nabla, h, \nabla^*) \). Then we have the equivalence of the following properties:

1. \( \nabla \) is projectively flat.
2. \( \text{Ric} = \text{Ric}^* \).
3. \( \pi_2(R) = 0 \).
4. \( \pi_2(R^*) = 0 \).
5. \( n \cdot \text{Ric} = \tau \cdot g \), that means \( (M, \nabla, h) \) is equiaffine Einstein.
6. \( n \cdot \text{Ric}^* = \tau \cdot g \).
7. \( \mathcal{R} = \mathcal{R}^* \).
8. \( -n(n-1) \cdot R = \tau \cdot (h \wedge h) \).
9. The relative normalization coincides with the centroaffine normalization.
10. \( (x, Y, y) \) is a relative sphere.

If we apply Theorem 6.3.5.2 in [29] and the foregoing Theorem we get:

**Theorem 11.4.** The following assertions are equivalent for a relative hypersurface \( (x, Y, y) \) in dimension \( n \geq 3 \).

1. \( (x, Y, y) \) is a relative sphere.
2. \( \nabla \) is projectively flat.
3. \( B := (n-1)\text{Ric}^* + \text{Ric} - \tau \cdot h = 0 \).
4. \( \pi_2(R) = 0 \).
Remark 11.5. The foregoing Theorems reflects the importance of the symmetric bilinear form $B = \rho_1(\pi_2(R))$; for a relative hypersurface one calculates that

$$B = n(n - 2)(S^b - H \cdot h).$$

In particular, one can characterize affine spheres by (3) above in the Blaschke geometry. Recall Section 10.4.2 and the fact that this class of hypersurfaces is very large.

Theorem 11.6. Let $(x, Y, y)$ be a non-degenerate relative hypersurface. Then one has $R \in W_5$ if and only if the hypersurface is an improper relative hypersphere.

Proof. $R \in W_5$ implies that

$$0 = \text{Ric}(R^\ast) = n(n - 1)S^b,$$

thus $(x, Y, y)$ is an improper relative hypersphere. The converse is trivial. □

The results in Section 11.5 specialize to affine spheres. Recall that for non-degenerate hypersurfaces the conormal connection $\nabla^\ast$ is always projectively flat, and the projective flatness is equivalent to the relation

$$R^\ast = \pi_1(R^\ast) \oplus \pi_2(R^\ast).$$

Moreover, we have the equivalences:

Observation 11.7. For a Blaschke hypersurface adopt the notation established above. Then:

(1) $\pi_2(R) = 0$ if and only if $x$ is an affine sphere; this is equivalent to the total symmetry of $\nabla(h)C$.

(2) $R(g) = \pi_1(R(g))$ is equivalent to the fact that the Blaschke metric has constant sectional curvature.

11.4. Characterization of affine maximal hypersurfaces. Let $x$ be a locally strongly convex Blaschke hypersurface.

Proposition 11.8. The following properties are equivalent:

(1) $x$ is affine maximal.

(2) $\pi_1(R) = 0$.

(3) $\pi_1(R^\ast) = 0$.

11.5. Characterization of classes of Blaschke hypersurfaces in terms of PDEs and curvature tensors. In the foregoing section we characterized some special classes of hypersurfaces in terms of their equiaffine curvature tensors. On the other hand it is well known that some of these classes can be also locally characterized in terms of PDEs for a graph representation. We combine characterizations in terms of decomposition results from section 10 with known characterizations in terms of PDEs from [18].

Let $x : M \to \mathbb{R}^{n+1}$ be a hypersurface with a local representation by a strongly convex graph function $f : \Omega \to \mathbb{R}$ with $f = f(x^1, ..., x^n)$, where $\Omega$ is a domain in $\mathbb{R}^n$ s.t. the origin $O \in \Omega \cap x(M)$ lies in the tangent plane $T_0x(M) = \mathbb{R}^n$.

Theorem 11.9. Let $f$ be the above graph function. Then

(1) $x$ is a proper affine sphere

(a) if and only if the Legendre transform function $u = u(\xi^1, ..., \xi^n)$ of $f$ satisfies the PDE

$$\det \left( \frac{\partial^2 u}{\partial \xi^i \partial \xi^j} \right) = (H u)^{-(n+2)},$$

(b) if and only if $\pi_2(R) = 0$. 

(2) \( x \) is an improper affine sphere with constant affine normal vector \((0,\ldots,0,1)\)
(a) if and only if the graph function satisfies the Monge-Ampère PDE
\[
\det \left( \frac{\partial^2 f}{\partial x^i \partial x^j} \right) = 1,
\]
(b) if and only if \( \pi_5(R) = 0 \).

(3) \( x \) is affine maximal
(a) if and only if the graph function \( f \) satisfies the PDE
\[
\Delta \left( \left[ \det \left( \frac{\partial^2 f}{\partial x^i \partial x^j} \right) \right]^{\frac{1}{n+2}} \right) = 0,
\]
(b) if and only if \( \pi_1(R) = 0 \).

Lemma 7.6(4) shows that, in a similar way, affine spheres in Blaschke’s geometry
can be characterized in terms of PDEs for the cubic form, using results from [7].

The following Corollary states modifications of two well known global results,
namely the Theorem of Blaschke and Deicke and the affine Bernstein problem in
the version of Calabi; see [18].

**Corollary 11.10.** Let \( x : M \to \mathbb{R}^{n+1} \) be a locally strongly convex Blaschke hypersurface.

1. If \( M \) is compact and if \( \pi_2(R) = 0 \) then \( x \) is a hyperellipsoid.
2. If \( n = 2 \), \((M,h)\) is complete, and \( \pi_1(R) = 0 \), then \( x \) is an elliptic paraboloid.

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