A Heuristic Description of Fast Fourier Transform

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Abstract
Fast Fourier Transform (FFT) is an efficient algorithm to compute the Discrete Fourier Transform (DFT) and its inverse. In this paper, we pay special attention to the description of complex-data FFT. We analyze two common descriptions of FFT and propose a new presentation. Our heuristic description is helpful for students and programmers to grasp the algorithm entirely and deeply.

Keywords: Fast Fourier transform; Discrete Fourier transform

1 Introduction
Denote the set of all complex numbers by \( \mathbb{C} \). Let \( \omega = e^{2\pi i/n} \) be a primitive \( n \)th root of unity. Let
\[
f(x) = \sum_{0 \leq i < n} f_i x^i \in \mathbb{C}[x]
\]
(1)
be a polynomial of degree less than \( n \) with its coefficient vector \((f_0, \ldots, f_{n-1}) \in \mathbb{C}^n\). The \( \mathbb{C} \)-linear map
\[
\text{DFT}_\omega : \begin{cases} 
\mathbb{C}^n & \rightarrow \mathbb{C}^n \\
(f_0, \ldots, f_{n-1}) & \mapsto (f(1), f(\omega), f(\omega^2), \ldots, f(\omega^{n-1}))
\end{cases}
\]
which evaluates a polynomial at the powers of \( \omega \) is called the Discrete Fourier Transform. Apparently, the Discrete Fourier Transform is a special multipoint evaluation at the powers 1, \( \omega, \omega^2, \ldots, \omega^{n-1} \).

A beginner will often evaluate the polynomial (1) in a manner that: first \( f_{n-1}x^{n-1} \) is calculated, then \( f_{n-2}x^{n-2}, \ldots, f_1x \), and finally all of the terms of (1) are added together. Such a process involves lots of multiplications and additions. There is an elegant way to evaluate a polynomial, which is called Horner’s rule. It evaluates \( f(x) \) as follows: Rearrange
this computation as
\[ f(x) = (\cdots (f_{n-1}x + f_{n-2})x + \cdots)x + f_0 \]
(2)
Then start with \( f_{n-1} \), multiply by \( x \), add \( f_{n-2} \), multiply by \( x \), \( \cdots \), multiply by \( x \), add \( f_0 \). If we apply the Horner’s rule to compute \( \text{DFT}_\omega \), it needs to run the evaluation algorithm \( n \) times. Can we find more efficient algorithm to compute Discrete Fourier transform?

Fast Fourier Transform (FFT) is an efficient algorithm to compute the Discrete Fourier Transform (DFT) and its inverse. It was popularized by a publication of J. Cooley and J. Tukey [1] in 1965. It has been called the most important numerical algorithm of our lifetime. In this paper, we will pay special attention to the description of complex-data FFT. We will analyze two common descriptions of FFT [3, 4] and propose an explicit and heuristic presentation of FFT.

2 Analysis of two common descriptions of FFT

2.1 Description-1

We refer to Ref.[3] for the first description of FFT.

Let \( n = 2^k \in \mathcal{N} \) with \( k \in \mathcal{N} \), \( \omega \in \mathbb{C} \) be a primitive \( n \)th root of unity, and \( f(x) = \sum_{0 \leq i < n} f_i x^i \in \mathbb{C}[x] \) of degree less than \( n \). To evaluate \( f(x) \) at the powers 1, \( \omega, \omega^2, \cdots, \omega^{n-1} \), we divide \( f(x) \) by \( x^{n/2} - 1 \) and \( x^{n/2} + 1 \) with remainder:

\[ f(x) = q_0(x)(x^{n/2} - 1) + r_0(x) = q_1(x)(x^{n/2} + 1) + r_1(x) \]
(3)
for some \( q_0(x), r_0(x), q_1(x), r_1(x) \in \mathbb{C}[x] \) of degree less than \( n/2 \).

Plugging in a power of \( \omega \) for \( x \) in (3), we find

\[ f(\omega^{2\ell}) = q_0(\omega^{2\ell})(\omega^{n\ell} - 1) + r_0(\omega^{2\ell}) = r_0(\omega^{2\ell}) \]
\[ f(\omega^{2\ell+1}) = q_1(\omega^{2\ell+1})(\omega^{n\ell}\omega^{n/2} + 1) + r_1(\omega^{2\ell+1}) = r_1(\omega^{2\ell+1}) \]
for all \( 0 \leq \ell < n/2 \). We have used the facts that \( \omega^n = 1 \) and \( \omega^{n/2} = -1 \). It remains to evaluate \( r_0(x) \) at the even powers of \( \omega \) and \( r_1(x) \) at the odd powers. Now \( \omega^2 \) is a primitive \((n/2)\)th root of unity, and hence the first task is a DFT of order \( n/2 \). But also the evaluation of \( r_1(x) \) can be reduced to a DFT of order \( n/2 \) by noting that \( r_1(\omega^{2\ell+1}) = r_1^*(\omega^{2\ell}) \) for \( r_1^*(x) = r_1(\omega x) \). Since \( n \) is a power of 2, we can proceed recursively to evaluate \( r_0(x) \) and \( r_1^*(x) \) at the powers 1, \( \omega^2, \cdots, \omega^{2^{n-2}} \) of \( \omega^2 \), and obtain the following algorithm.
Input: \( n = 2^k \in \mathbb{N} \) with \( k \in \mathbb{N}, f(x) = \sum_{0 \leq i < n} f_i x^i \in \mathbb{C}[x], \) and
the powers \( \omega, \omega^2, \ldots, \omega^{n-1} \) of a primitive \( n \)th root of unity \( \omega \in \mathbb{C}. \)

Output: \( \text{DFT}_\omega(f) = (f(1), f(\omega), \cdots, f(\omega^{n-1})) \subseteq \mathbb{C}^n. \)

1. if \( n = 1 \) then return \( f_0 \)
2. \( r_0(x) \leftarrow \sum_{0 \leq j < n/2} (f_j + f_{j+n/2}) x^j, \quad r_1^*(x) \leftarrow \sum_{0 \leq j < n/2} (f_j - f_{j+n/2}) \omega^j x^j \)
3. call the algorithm recursively to evaluate \( r_0(x) \) and \( r_1^*(x) \) at the powers of \( \omega^2 \)
4. return \( (r_0(1), r_1^*(1), r_0(\omega^2), r_1^*(\omega^2), \cdots, r_0(\omega^{n-2}), r_1^*(\omega^{n-2})) \)

### 2.2 Analysis of Description-1

To investigate the working flow in description-1, we take the following polynomial as an example:

\[
f(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_7 x^7 \in \mathbb{C}[x], \quad \omega = e^{2\pi i/8} \tag{4}
\]

At first, we have the following two remainders:

\[
r_0(x) = (f_0 + f_1) + (f_1 + f_3) x + (f_2 + f_6) x^2 + (f_3 + f_7) x^3 \tag{5}
\]

\[
r_1^*(x) = (f_0 - f_1) + (f_1 - f_3) \omega x + (f_2 - f_6) \omega^2 x^2 + (f_3 - f_7) \omega^3 x^3 \tag{6}
\]

If we evaluate them at \( (\omega^2)^0 \), then \( r_0(1) = f(1), r_1^*(1) = f(\omega) \).

We now proceed to the step 3. By (5) we have the following two remainders:

\[
[(f_0 + f_1) + (f_2 + f_6)] + [(f_1 + f_3) + (f_3 + f_7)] x \tag{7}
\]

\[
[(f_0 + f_1) - (f_2 + f_6)] + [(f_1 + f_3) - (f_3 + f_7)] \omega x \tag{8}
\]

Likewise, by (6) we have the following two remainders:

\[
[(f_0 - f_1) + (f_2 - f_6) \omega^2] + [(f_1 - f_3) \omega + (f_3 - f_7) \omega^3] x \tag{9}
\]

\[
[(f_0 - f_1) - (f_2 - f_6) \omega^2] + [(f_1 - f_3) \omega - (f_3 - f_7) \omega^3] \omega x \tag{10}
\]

Should we evaluate them at \( (\omega^2)^1 \)? If that we can not find correct answers.

**Drawback:** The sentence that, “call the algorithm recursively to evaluate \( r_0(x) \) and \( r_1^*(x) \)
at the powers of \( \omega^{2n} \), is really too vague to specify the working flow.
2.3 Description-2

We refer to Ref.[4] for the second description of FFT.

Let \( n = 2^k \in \mathcal{N} \) with \( k \in \mathcal{N} \), \( \omega \in \mathcal{C} \) be a primitive \( n \)th root of unity, and \( f(x) = \sum_{0 \leq i < n} f_i x^i \in \mathcal{C}[x] \) of degree less than \( n \). To evaluate \( f(x) \) at the powers \( 1, \omega, \omega^2, \ldots, \omega^{n-1} \), we carry out the following scheme. (In these formulas the parameters \( s_j \) and \( t_j \) are either 0 or 1, so that each “pass” represents \( 2^k \) elementary computations.)

Table 2: Description-2

| Pass | Description |
|------|-------------|
| Pass 0. Let \( A^{[0]}(t_{k-1}, \ldots, t_0) = f_t \), where \( t = (t_{k-1}, \ldots, t_0)_2 \). |
| Pass 1. Set \( A^{[1]}(s_k-1, t_{k-2}, \ldots, t_0) \leftarrow \) \( A^{[0]}(0, t_{k-2}, \ldots, t_0) + \omega^{2^{k-1}s_k-1} A^{[0]}(1, t_{k-2}, \ldots, t_0) \). |
| Pass 2. Set \( A^{[2]}(s_k-1, s_k-2, t_{k-3}, \ldots, t_0) \leftarrow \) \( A^{[1]}(s_k-1, 0, t_{k-3}, \ldots, t_0) + \omega^{2^{k-2}(s_k-2)^2} A^{[1]}(s_k-1, 1, t_{k-3}, \ldots, t_0) \). |
| \ldots |
| Pass \( k \). Set \( A^{[k]}(s_k-1, \ldots, s_1, s_0) \leftarrow \) \( A^{[k-1]}(s_k-1, \ldots, s_1, 0) + \omega^{(s_0s_1\cdots s_{k-1})2} A^{[k-1]}(s_k-1, \ldots, s_1, 1) \). |

It is easy to find that

\[
A^{[k]}(s_k-1, \ldots, s_1, s_0) = f(\omega^s), \quad \text{where} \quad s = (s_0, s_1, \ldots, s_{k-1})_2
\]  

(11)

Notice that the binary digits of \( s \) are reversed in the final result (11).

2.4 Analysis of Description-2

To investigate the working flow in description-2, we also take the polynomial (4) as an example. Clearly, we have

\[
A^{[0]}(000) = f_0, \quad A^{[0]}(001) = f_1, \quad A^{[0]}(010) = f_2, \quad A^{[0]}(011) = f_3, \\
A^{[0]}(100) = f_4, \quad A^{[0]}(101) = f_5, \quad A^{[0]}(110) = f_6, \quad A^{[0]}(111) = f_7
\]

In pass 1,

\[
A^{[1]}(000) = A^{[0]}(000) + A^{[0]}(100) = f_0 + f_4, \quad A^{[1]}(100) = A^{[0]}(000) + \omega^2 A^{[0]}(100) = f_0 - f_4, \\
A^{[1]}(001) = A^{[0]}(001) + A^{[0]}(101) = f_1 + f_5, \quad A^{[1]}(101) = A^{[0]}(001) + \omega^2 A^{[0]}(101) = f_1 - f_5, \\
A^{[1]}(010) = A^{[0]}(010) + A^{[0]}(110) = f_2 + f_6, \quad A^{[1]}(110) = A^{[0]}(010) + \omega^2 A^{[0]}(110) = f_2 - f_6, \\
A^{[1]}(011) = A^{[0]}(011) + A^{[0]}(111) = f_3 + f_7, \quad A^{[1]}(111) = A^{[0]}(011) + \omega^2 A^{[0]}(111) = f_3 - f_7
\]
In pass 2,

\[ A^{[2]}(000) = A^{[1]}(000) + \omega^{2 \times (00)^2} A^{[1]}(010) = (f_0 + f_4) + (f_2 + f_6) \]
\[ A^{[2]}(001) = A^{[1]}(001) + \omega^{2 \times (00)^2} A^{[1]}(011) = (f_1 + f_3) + (f_3 + f_7) \]
\[ A^{[2]}(010) = A^{[1]}(000) + \omega^{2 \times (10)^2} A^{[1]}(010) = (f_0 + f_4) - (f_2 + f_6) \]
\[ A^{[2]}(011) = A^{[1]}(001) + \omega^{2 \times (10)^2} A^{[1]}(011) = (f_1 + f_3) - (f_3 + f_7) \]
\[ A^{[2]}(100) = A^{[1]}(100) + \omega^{2 \times (01)^2} A^{[1]}(110) = (f_0 - f_4) + \omega^2(f_2 - f_6) \]
\[ A^{[2]}(101) = A^{[1]}(101) + \omega^{2 \times (01)^2} A^{[1]}(111) = (f_1 - f_3) + \omega^2(f_3 - f_7) \]
\[ A^{[2]}(110) = A^{[1]}(100) + \omega^{2 \times (11)^2} A^{[1]}(110) = (f_0 - f_4) - \omega^2(f_2 - f_6) \]
\[ A^{[2]}(111) = A^{[1]}(101) + \omega^{2 \times (11)^2} A^{[1]}(111) = (f_1 - f_3) - \omega^2(f_3 - f_7) \]

In pass 3,

\[ A^{[3]}(000) = A^{[2]}(000) + \omega^{(000)^2} A^{[2]}(001) = [(f_0 + f_4) + (f_2 + f_6)] + [(f_1 + f_3) + (f_3 + f_7)] = f(1) \]
\[ A^{[3]}(001) = A^{[2]}(001) + \omega^{(100)^2} A^{[2]}(001) = [(f_0 + f_4) + (f_2 + f_6)] - [(f_1 + f_3) + (f_3 + f_7)] = f(4) \]
\[ A^{[3]}(010) = A^{[2]}(010) + \omega^{(010)^2} A^{[2]}(011) = [(f_0 + f_4) - (f_2 + f_6)] + \omega^2(f_1 + f_3) - (f_3 + f_7) = f(5) \]
\[ A^{[3]}(011) = A^{[2]}(010) + \omega^{(110)^2} A^{[2]}(011) = [(f_0 + f_4) - (f_2 + f_6)] - \omega^2(f_1 + f_3) - (f_3 + f_7) = f(6) \]
\[ A^{[3]}(100) = A^{[2]}(100) + \omega^{(001)^2} A^{[2]}(101) = [(f_0 - f_4) + \omega^2(f_2 - f_6)] + \omega^2(f_1 - f_3) + \omega^2(f_3 - f_7) = f(7) \]
\[ A^{[3]}(101) = A^{[2]}(100) + \omega^{(101)^2} A^{[2]}(101) = [(f_0 - f_4) + \omega^2(f_2 - f_6)] - \omega^2(f_1 - f_3) + \omega^2(f_3 - f_7) = f(8) \]
\[ A^{[3]}(110) = A^{[2]}(110) + \omega^{(011)^2} A^{[2]}(111) = [(f_0 - f_4) - \omega^2(f_2 - f_6)] + \omega^3(f_1 - f_3) - \omega^2(f_3 - f_7) = f(9) \]
\[ A^{[3]}(111) = A^{[2]}(110) + \omega^{(111)^2} A^{[2]}(111) = [(f_0 - f_4) - \omega^2(f_2 - f_6)] - \omega^3(f_1 - f_3) - \omega^2(f_3 - f_7) = f(10) \]

**Drawback:** The description-2 almost veil the elegant idea behind FFT.

### 3 A heuristic description of FFT

We now present an explicit and heuristic description of FFT, which can be regarded as the combination of the above two descriptions.

The basic idea of the new description is to recursively split polynomials and shrink coefficients. Concretely, given a polynomial \( f(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_7 x^7 \in \mathbb{C}[x] \) and a primitive 8th root of unity \( \omega = e^{2\pi i/8} \), we split it into two polynomials of degree less than 4. Their coefficients can be individually obtained by shrinking the symmetric two coefficients of original polynomial. See the following working flow for details.
Table 3: Splitting and shrinking method for FFT

In pass $i$: $f(x)$

\[ f(x) = \sum_{0 \leq j \leq n/2^i} (f_j + f_{j+n/2^i}) x^j \]

\[ f(x) = \sum_{0 \leq j \leq n/2^i} (f_j - f_{j+n/2^i}) (\omega^{2^{i-j}} x)^j \]

\[ f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + f_5 x^5 + f_6 x^6 + f_7 x^7 \]

pass 1:

\[ f^{[0]}(x) = (f_0 + f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7)x \]

\[ f^{[1]}(x) = (f_0 - f_1 + f_2 - f_3 - f_4 + f_5 - f_6 + f_7)\omega x \]

pass 2:

\[ f^{[00]}(x) = (f_0 + f_3 + f_6 + f_9 + f_4 + f_7 + f_2 + f_5) \]

\[ f^{[01]}(x) = (f_0 + f_3 - f_6 - f_9 + f_4 + f_7 - f_2 - f_5) \omega^2 x \]

pass 3:

\[ f^{[000]}(x) = f_0 + f_3 + f_6 + f_9 + f_4 + f_7 + f_2 + f_5 \]

\[ f^{[001]}(x) = f_0 + f_3 - f_6 - f_9 + f_4 + f_7 - f_2 - f_5 \]

Bit reversal permutation: $f(\omega^{[b]}) = f^{[b]}(x)$
Refining the method, we have the following heuristic description for FFT:

**Table 3: Description-3**

| Input: \( n = 2^k \in \mathcal{N} \) with \( k \in \mathcal{N} \), \( f(x) = \sum_{0 \leq i < n} f_i x^i \in \mathcal{C}[x] \), and the powers \( \omega, \omega^2, \ldots, \omega^{n-1} \) of a primitive \( n \)th root of unity \( \omega \in \mathcal{C} \). |
| Output: DFT \( \omega(f) = (f(1), f(\omega), \ldots, f(\omega^{n-1})) \in \mathcal{C}^n \). |

**Pass 1.** Set \( f_{b_1}(x) = \begin{cases} \sum_{0 \leq j < n/2} (f_j + f_{j+n/2}) x^j, & b_1 = 0 \\ \sum_{0 \leq j < n/2} (f_j - f_{j+n/2}) (\omega x)^j, & b_1 = 1 \end{cases} \)

**Pass 2.** For each \( b_i \in \{0, 1\}, 1 \leq i \leq 2 \), compute
\[
f_{b_1b_2}(x) = \begin{cases} \sum_{0 \leq j < n/2^2} (f_{b_1}^j + f_{b_1}^{j+n/2^2}) x^j, & b_2 = 0 \\ \sum_{0 \leq j < n/2^2} (f_{b_1}^j - f_{b_1}^{j+n/2^2}) (\omega^2 x)^j, & b_2 = 1 \end{cases}
\]

**Pass k.** For each \( b_i \in \{0, 1\}, 1 \leq i \leq k \), compute
\[
f_{b_1b_2\cdots b_k}(x) = \begin{cases} \sum_{0 \leq j < n/2^k} (f_{b_1,b_2\cdots b_k-1}^j + f_{b_1,b_2\cdots b_k-1}^{j+n/2^k}) x^j, & b_k = 0 \\ \sum_{0 \leq j < n/2^k} (f_{b_1,b_2\cdots b_k-1}^j - f_{b_1,b_2\cdots b_k-1}^{j+n/2^k}) (\omega^{2^{k-1}} x)^j, & b_k = 1 \end{cases}
\]

**Reversal permutation.** For each \( b_i \in \{0, 1\}, 1 \leq i \leq k \), set
\[
f(\omega^{(b_kb_{k-1}\cdots b_1)}x) = f_{b_1\cdots b_k}(x)
\]

It is easy to prove the correctness of above method by induction and show that the algorithm requires \( kn \) complex-number additions. Incidentally, we refer to Ref.[2] for an explicit pseudocode of FFT (ITERATIVE-FFT)

### 4 Conclusion

In this paper, we analyze two common descriptions of FFT and propose a heuristic presentation of complex-data FFT. We think the new presentation is helpful for students and programmers to grasp the method entirely and deeply.

*Acknowledgements* We thank the National Natural Science Foundation of China (Project 60873227), and the Key Disciplines of Shanghai Municipality (S30104).
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