A GENERALIZED INVERSE EIGENVALUE PROBLEM AND
\textit{m}-FUNCTIONS

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Abstract. In this manuscript, a generalized inverse eigenvalue problem is considered
that involves a linear pencil \((zJ_{[0,n]} - H_{[0,n]})\) of matrices arising in the theory of rational
interpolation and biorthogonal rational functions. In addition to the reconstruction of the
Hermitian matrix \(H_{[0,n]}\) with the entries \(b_j's\), characterizations of the rational functions
that are components of the prescribed eigenvectors are given. A condition concerning
the positive-definiteness of \(J_{[0,n]}\) and which is often an assumption in the direct problem
is also isolated. Further, the reconstruction of \(H_{[0,n]}\) is viewed through the inverse of the
pencil \((zJ_{[0,n]} - H_{[0,n]})\) which involves the concept of \(m\)-functions.

1. Introduction

A generalized inverse eigenvalue problem (GIEP) concerns the reconstruction of ma-
trices from a given set of spectral data. The spectral data may be completely or only
partially specified in terms of eigenvalues and eigenvectors. Precisely, a GIEP for a pair
\((H, J)\) of square matrices involves the generalized eigenvalue equation
\(H\Phi = zJ\Phi\). With
the prescribed spectral data, the solution to the problem consists in the reconstruction of
the matrices \(H\) and/or \(J\) \cite{10,14}. In general, it is often necessary both from the point of view of practical applications and
of mathematical interest that the matrices involved have a specified structure \cite{6}. This
introduces a structural constraint on the solution in addition to the spectral constraint.
Thus, one may require that both the matrices \(H\) and \(J\) or one of them to be, for instance,
banded or Hermitian or Hamiltonian \cite{10,15} and so on.

In the present manuscript, we consider, as an inverse problem, the generalized eigen-
value equation arising from the continued fraction

\[
\frac{1}{u_0(z) - \frac{v_0^L(z)v_0^R(z)}{u_1(z) - \frac{v_1^L(z)v_1^R(z)}{u_2(z) - \cdots}}, \tag{1.1}
\]

where \(u_j(z)\), \(v_j^L(z)\) and \(v_j^R(z)\) are non-vanishing polynomials of degree one \cite{2}. If we
terminate the above continued fraction at \(u_n(z)\), then it is a rational function denoted by
\(S_{n+1}(z) = Q_{n+1}(z)/P_{n+1}(z)\) where, the polynomials \(Q_n(z)\) of degree \(\leq n - 1\) and \(P_n(z)\)
of degree \(\leq n\) satisfy the three term recurrence relation \cite{13}

\[
X_{n+1}(z) = u_n(z)X_n(z) - v_{n-1}^L(z)v_{n-1}^R(z)X_{n-1}(z), \quad n = 0, 1, 2, \cdots, \tag{1.2}
\]

2010 Mathematics Subject Classification. Primary 15A29, 30C10.

Key words and phrases. Generalized inverse eigenvalue problem; Linear pencil of tridiagonal matrices;
\(m\)-functions.

This research is supported by the Dr. D. S. Kothari postdoctoral fellowship scheme of University
Grants Commission (UGC), India.
with the initial conditions
\[ Q_{-1}(z) = -1, \quad Q_0(z) = 0, \quad P_{-1}(z) = 0 \quad \text{and} \quad P_0(z) = 1. \] (1.3)

The linear pencil that is associated with \( S_{n+1}(z) \) and (1.2) is \( (zJ_{[0,n]} - H_{[0,n]}) \) where the matrices \( H_{[0,n]} \) and \( J_{[0,n]} \) are tridiagonal.

In the present manuscript, we consider the matrices

\[
H_{[0,n]} = \begin{pmatrix}
  a_0 & b_0 & 0 & \cdots & 0 & 0 \\
  b_0 & a_1 & b_1 & \cdots & 0 & 0 \\
  0 & b_1 & a_2 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\
  0 & 0 & 0 & \cdots & b_{n-1} & a_n
\end{pmatrix},
\]

\[
J_{[0,n]} = \begin{pmatrix}
  c_0 & d_0 & 0 & \cdots & 0 & 0 \\
  d_0 & c_1 & d_1 & \cdots & 0 & 0 \\
  0 & d_1 & c_2 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & c_{n-1} & d_{n-1} \\
  0 & 0 & 0 & \cdots & d_{n-1} & c_n
\end{pmatrix}, \quad d_j \neq 0, \quad 0 \leq j \leq n - 1,
\]

and the following generalized inverse eigenvalue problem.

**GIEP 1.** Given: the symmetric matrix \( J_{[0,n]} \), the hermitian matrix \( H_{[0,k]} \), real numbers \( \lambda \) and \( \mu \), and vectors \( p_k^R = (p_{k,0}^R, p_{k,1}^R, \cdots, p_{k,n}^R)^T \) and \( s_k^R = (s_{k,0}^R, s_{k,1}^R, \cdots, s_{k,n}^R)^T \), where \( 1 \leq k \leq n - 1 \). To find:

(i) hermitian matrix \( H_{[0,n]} \) with eigenvalues \( \lambda \) and \( \mu \) such that \( H_{[0,k]} \) is the leading principal sub-matrix of \( H_{[0,n]} \),

(ii) vectors \( p_{[0,k-1]}^R = (p_{0,0}^R, p_{1,0}^R, \cdots, p_{k-1,0}^R)^T \) and \( s_{[0,k-1]}^R = (s_{0,0}^R, s_{1,0}^R, \cdots, s_{k-1,0}^R)^T \) such that

\[
p_{[0,n]}^R = \begin{pmatrix}
  p_{[0,k-1]}^R \\
  p_{[k,n]}^R
\end{pmatrix} \quad \text{and} \quad s_{[0,n]}^R = \begin{pmatrix}
  s_{[0,k-1]}^R \\
  s_{[k,n]}^R
\end{pmatrix},
\]

are the right eigenvectors of the matrix pencil \( (zJ_{[0,n]} - H_{[0,n]}) \), corresponding to the eigenvalues \( \lambda \) and \( \mu \) respectively.

The pencil \( zJ_{[0,n]} - H_{[0,n]} \), which is a linear pencil of tridiagonal matrices arises in the theory of biorthogonal rational functions and rational interpolation \([4, 10]\). A particular case, in which the \( b \)'s appearing in \( H_{[0,n]} \) are purely imaginary and the \( c \)'s appearing in \( J_{[0,n]} \) are unity, has its origins in the continued fraction representation of Nevanlinna functions, which in turn are obtained via the Cayley transformation of the continued fraction representation of a Carathéodory function \([13]\) (see also \([7]\)). As further specific illustrations, the rational functions arising as components of eigenvectors in such cases have been related to a class of hypergeometric polynomials orthogonal on the unit circle \([8]\) as well as pseudo-Jacobi polynomials (or Routh-Romanovski polynomials) \([7]\).

In the direct problem, the components of the right eigenvector of the linear pencil \( (zJ_{[0,n]} - H_{[0,n]}) \) are rational functions with poles at \( z = b_j/d_j \) while that of the left eigenvector have poles at \( z = b_j/d_j \). In addition to poles, the entries of the matrices \( J_{[0,n]} \) and \( H_{[0,n]} \) also completely specify the numerators of such rational functions appearing as polynomial solutions of a three term recurrence relation.
It is also known that the zeros of these numerator polynomials are the eigenvalues of the linear pencil under consideration \[8, \text{Theorem 1.1}\]. These numerator polynomials are precisely normalized \(\mathcal{P}_j(z)\), the denominator of the convergents \(\mathcal{S}_j(z)\) of the continued fraction (1.1). Further, it can be verified with the recurrence relation (1.2) and the initial conditions (1.3) that the following expressions satisfy (1.1). Further, it can be verified with the recurrence relation (1.2) and the initial conditions (1.3) that the following expressions hold which lead to the formula

\[
\mathcal{P}_{n+1}(z) = \det \left( z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]} \right), \quad \mathcal{Q}_{n+1}(z) = \det \left( z\mathcal{J}_{[1,n]} - \mathcal{H}_{[1,n]} \right),
\]

hold which lead to the formula

\[
\mathcal{S}_{n+1}(z) = \frac{\mathcal{Q}_{n+1}(z)}{\mathcal{P}_{n+1}(z)} = \langle (z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})^{-1}e_0, e_0 \rangle,
\]

with the standard inner product

\[
\langle x, y \rangle = \sum_{j=0}^{\infty} x_j \overline{y}_j, \quad x = (x_0, x_1, \cdots) \in \ell^2, \quad y = (y_0, y_1, \cdots) \in \ell^2.
\]

on the space \(\ell^2\) of complex square summable sequences.

A fundamental object related to a pair \((\mathcal{H}, \mathcal{J})\) of matrices is the function

\[
m(z) = \langle (z\mathcal{J} - \mathcal{H})^{-1}e_0, e_0 \rangle, \quad z \in \rho(\mathcal{H}, \mathcal{J})
\]

called the \(m\)-function or the Weyl function of the linear pencil \((z\mathcal{J} - \mathcal{H})\) \[4\] (see also \[13\]). Here \(\sigma(\mathcal{H}, \mathcal{J})\) and \(\rho(\mathcal{H}, \mathcal{J}) := \mathbb{C} \setminus \sigma(\mathcal{H}, \mathcal{J})\) are, respectively, the spectrum and the resolvent set of the pencil \((z\mathcal{J} - \mathcal{H})\). We can denote similarly the \(m\)-function

\[
m(z, j + 1) = \frac{\mathcal{Q}_{j+1}(z)}{\mathcal{P}_{j+1}(z)} = \langle (z\mathcal{J}_{[0,j]} - \mathcal{H}_{[0,j]})^{-1}e_0, e_0 \rangle, \quad z \in \rho(\mathcal{H}_{[0,j]}, \mathcal{J}_{[0,j]}),
\]

of the finite pencil \((z\mathcal{J}_{[0,j]} - \mathcal{H}_{[0,j]})\).

Thus, a way to interpret the reconstruction of the matrix \(\mathcal{H}_{[0,n]}\) is to determine its entries in terms of rational functions with arbitrary poles. These rational functions enter into the problem as components of a prescribed eigenvector, while the structural constraint of the pencil being tridiagonal characterizes these poles.

Our primary goal in this manuscript is to find a representation of the entries \(b'_j\)'s of the matrix \(\mathcal{H}_{[0,n]}\) in terms of given spectral points and corresponding eigenvectors. We find characterizations of both the given poles and the entries \(b'_j\)'s which appear in special matrix pencils as mentioned earlier. A condition concerning the positive-definiteness of \(\mathcal{J}_{[0,n]}\) and which is often an assumption in the direct problem is also isolated. Further, we have a view at the entries \(b'_j\)'s through the \(m\)-functions (1.4) which, as is obvious, involve a point in the resolvent set and not in the spectrum of the pair \((\mathcal{H}_{[0,n]}, \mathcal{J}_{[0,n]}))\).

The manuscript is organized as follows. Section 2 includes preliminary results that illustrate the key role played by the entry \(b_k\) in the inverse approach to the linear pencil \((z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}))\). The matrix \(\mathcal{H}_{[0,n]}\) is reconstructed in Section 3 thereby solving GIEP \[1\]. In Section 4 we have a further look at the problem through \(m\)-functions that involves computing the inverse of the matrix \((z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}))\).

2. Preliminary results

In this section, we derive some results that will help in solving the GIEP. Though the entries are yet to be determined, we use them as symbols in the computation, with the final result depending only on \(b_k\) and given components of the eigenvector. In a way, these results exhibit the role played by the specific entry \(b_k\) in the solution.
First of all, it can be seen that if \( H_{[0,n]} \) were completely specified, the leading minors \( \mathcal{P}_m(z) \) of \((zJ_{[0,n]} - H_{[0,n]})\) satisfy the three term recurrence relation

\[
\mathcal{X}_{m+1}(z) = (zc_m - a_m)\mathcal{X}_m(z) - (zd_m - b_m)(zd_m - \bar{b}_m)\mathcal{X}_{m-1}(z), \tag{2.1}
\]

for \( 0 \leq m \leq n \), where we define \( \mathcal{P}_m(z) := 0 \) and \( \mathcal{P}_0(z) := 1 \). If \( \kappa_m \) is the leading coefficient of \( \mathcal{P}_m(z) \), then from (2.1), we have \( \kappa_{m+1} = c_m\kappa_m - d_m^2\kappa_{m-1} \), with \( \kappa_0 = 1 \) and \( \kappa_1 = c_0 \). Hence, if

\[
\frac{\kappa_m}{\kappa_{m-1}} \neq \frac{d_m^2}{c_m}, \quad m \geq 1,
\]

then \( \mathcal{P}_{m+1}(z) \) is a polynomial of degree \( m + 1 \). Further, \((zJ_{[0,n]} - H_{[0,n]})p^R_{[0,n]} = 0\) yields the following relations

\[
(zd_m - \bar{b}_m)p^R_{m-1}(z) + (zc_m - a_m)p^R_m(z) + (zd_m - b_m)p^R_{m+1}(z) = 0, \tag{2.2}
\]

for \( m = 0, 1, \ldots, n \), where \( p^R_{n+1}(z) = 0 \) and we define \( \mathcal{P}^R_{m}(z) := 0 \). The former equality occurs if \( z \notin \sigma(H_{[0,n]}, J_{[0,n]}) \). Moreover, with \( p^R_0(z) \) a non-vanishing function to be specified, the components of the eigenvector \( p^R_{[0,n]} \) can be obtained from (2.2), for instance by induction, in the form of the rational functions

\[
p^R_m(z) = \frac{\mathcal{P}_m(z)}{\prod_{j=0}^{m-1}(b_j - zd_j)}p^R_{m}(z), \quad m = 1, 2, \ldots, k, k + 1, \ldots, n - 1, \tag{2.3}
\]

and \( p^R_n(z) \) obtained from \((zc_n - a_n)p^R_n(z) = (\bar{b}_n - zd_{n-1})p^R_{n-1}(z)\). However, because of the prescribed data, we will assume that the components of the vector \( p^R_{[k,n]} \) are given in the form

\[
p^R_m(z) = \frac{\mathcal{T}_m(z)}{\eta_m(z)\prod_{j=0}^{m-1}(\alpha_j - z)}, \quad \eta_m(z) \in \mathbb{R} \setminus \{0\}, \quad m = k, k + 1, \ldots, n, \tag{2.4}
\]

where \( \mathcal{T}_m(z) \) is a polynomial of degree \( m \) with leading coefficient \( \kappa_m \). But, we note that once \( H_{[0,n]} \) is determined, a component of the right eigenvector of \((zJ_{[0,n]} - H_{[0,n]})\) must be of the form (2.3). In particular, if we look at \( p^R_k(z) \), this would imply that the set \( \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\} \) is necessarily a permutation of the set \( \{b_0/d_0, b_1/d_1, \ldots, b_{k-1}/d_{k-1}\} \), which is known. The inverse problem thus consists of \( \alpha_j, j = k, \ldots, n-1 \), being arbitrary, on which the determination of \( b_j, j = k, \ldots, n-1 \), depends.

**Lemma 2.1.** Given \( (\lambda, p^R_{[0,n]}) \) an eigen-pair for \((H_{[0,n]}, J_{[0,n]})\), let \( \lambda \notin \sigma(H_{[0,k]}, J_{[0,k]}) \), then the components of \( p^R_{[0,k-1]}(z) \) are given by

\[
p^R_m(z) = \frac{(b_k - zd_k)\prod_{j=m}^{k-1}(b_j - zd_j)\mathcal{P}_m(z)}{\mathcal{P}_{k+1}(z)}p^R_{k+1}(z), \quad m = 1, 2, \ldots, k - 1,
\]

at \( z = \lambda \) and \( p^R_0(\lambda) \) assumed to be a non-vanishing function of \( \lambda \).

**Proof.** Since \( \lambda \notin \sigma(H_{[0,k]}, J_{[0,k]}) \), \( \det(\lambda J_{[0,k]} - H_{[0,k]}) \neq 0 \), which implies that \( \mathcal{P}_{k+1}(\lambda) \neq 0 \). If we use the form of \( p^R_{k+1}(\lambda) \) as suggested by (2.3) with \( b_k \) unknown at the moment, we will have that \( p^R_{k+1}(\lambda) \neq 0 \) and

\[
p^R_0(\lambda) = \frac{\prod_{j=0}^{k}(b_j - \lambda d_j)p^R_{k+1}(\lambda)}{\mathcal{P}_{k+1}(\lambda)} = \frac{(b_k - \lambda d_k)p^R_{k+1}(\lambda)}{\mathcal{P}_{k+1}(\lambda)}\prod_{j=0}^{k-1}(b_j - \lambda d_j)p^R_0(\lambda).
\]
Then, $p^R_1(\lambda)$ can be obtained using (2.2) for $m = 0$ as $(\lambda c_0 - a_0)p^R_0(\lambda) + (\lambda d_0 - b_0)p^R_1(\lambda) = 0$ giving

$$p^R_1(\lambda) = \frac{(b_k - \lambda d_k)p^R_{k+1}(\lambda)}{P_{k+1}(\lambda)} \prod_{j=1}^{k-1} (b_j - \lambda d_j) P_1(\lambda).$$

The rest of the proof can be completed by induction using (2.2) and (2.3). $\square$

We note that $P_{k+1}(\lambda)$ is known since (2.1) involves $a_k$ and $b_{k-1}$. Thus, Lemma 2.1 shows that once $b_k$ is computed, the vector $p^R_{[0,k-1]}$ can be uniquely obtained in terms of $p^R_{k+1}(\lambda)$, which is now known in the form given by (2.4). This would determine $p^R_{[0,k-1]}$ at $z = \lambda$ completely.

To proceed further, we will make use of rational functions of the form

$$p^L_m(z) = \frac{P_m(z)}{\prod_{j=0}^{m-1} (b_j - zd_j)} p^L_0(z), \quad m = 1, 2 \cdot \cdot \cdot, k - 1, k, \cdot \cdot \cdot, n - 1, \quad (2.5)$$

and $p^R_0(z)$ obtained from the equation $(zc_n - a_n)p^L_n(z) = (b_{n-1} - zd_{n-1})p^L_{n-1}(z)$. These arise as components of the left eigenvector $p^L_{[0,n]}$ of $(zJ_{[0,n]} - H_{[0,n]})$ corresponding to the eigenvalue $z = \lambda$ and owing to the underlying hermitian character of the problem, satisfy $p^L_j(\lambda) = \overline{p^R_j(\lambda)}$, $0 \leq j \leq n$. Again, we reiterate that $p^L_j(z)$ is specified in the form (2.5) only for $j = 0, 1, \cdot \cdot \cdot, k - 1$, while $p^L_m(z)$ is obtained in the form suggested by (2.4) for $m = k, k + 1, \cdot \cdot \cdot, n$.

Similarly, we define the rational functions $s^L_m(\mu)$ and $s^R_m(\mu)$ corresponding to the eigenvalue $\mu$. Lemma 2.1 with $\lambda$ replaced by $\mu$ and the assumption $\mu \notin \sigma(H_{[0,k]}, J_{[0,k]})$ gives the corresponding expressions for $s^R_j(z)$ at $z = \mu$ for $j = 0, 1, \cdot \cdot \cdot, k - 1$.

For ease of notations, we use $p^R_j := p^R_j(z)$ and similarly for others. Let us also denote by $J_{[k+1,n]}$ and $H_{[k+1,n]}$, as is clear from the notations, the trailing matrices obtained by removing the first $k + 1$ rows and columns from $J_{[0,n]}$ and $H_{[0,n]}$ respectively.

**Lemma 2.2.** Suppose $\lambda, \mu \notin \sigma(H_{[0,i]}, J_{[0,i]})$ for $i = k - 1, k$. Then, the following identities

$$
(\lambda - \mu) p^L_{[k+1,n]} J_{[k+1,n]} s^R_{[k+1]} = (b_k - \lambda d_k) p^L_{k+1} s^R_{k+1} - (\overline{b_k} - \mu d_k) p^L_{k+1} s^R_{k+1}, \quad (2.6)
$$

$$
(\lambda - \mu) s^L_{[0,k]} J_{[0,k]} p^R_{[0,k]} = (\overline{b_k} - \mu d_k) s^L_{k+1} p^R_{k+1} - (b_k - \lambda d_k) s^L_{k+1} p^R_{k+1}, \quad (2.7)
$$

hold.

**Proof.** The relations (2.2) for $m = k + 1, \cdot \cdot \cdot, n$ can be written as

$$
\begin{pmatrix}
  c_{k+1} & d_{k+1} & 0 & \cdots & 0 \\
  d_{k+1} & c_{k+2} & d_{k+2} & \cdots & 0 \\
  0 & d_{k+2} & c_{k+3} & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & c_n
\end{pmatrix}
\begin{pmatrix}
  p^R_{k+1} \\
  p^R_{k+2} \\
  p^R_{k+3} \\
  \vdots \\
  p^R_n
\end{pmatrix}

= \begin{pmatrix}
  a_{k+1} & b_{k+1} & 0 & \cdots & 0 \\
  \bar{b}_{k+1} & a_{k+2} & d_{k+2} & \cdots & 0 \\
  0 & \bar{b}_{k+2} & a_{k+3} & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & a_n
\end{pmatrix}
\begin{pmatrix}
  p^R_{k+1} \\
  p^R_{k+2} \\
  p^R_{k+3} \\
  \vdots \\
  p^R_n
\end{pmatrix} + (\overline{b_k} - zd_k)
\begin{pmatrix}
  p^R_k \\
  0 \\
  \vdots \\
  0
\end{pmatrix},
$$

or in the compact form

$$
H_{[k+1,n]} p^R_{[k+1,n]} = zJ_{[k+1,n]} p^R_{[k+1,n]} - (\overline{b_k} - zd_k) p^R_k e_1, \quad (2.8)
$$
where \( \vec{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{n-k} \). Post-multiplying (2.8) at \( z = \lambda \) by \( s_{[k+1,n]}^L \), we obtain
\[
\mathcal{H}_{[k+1,n]} p_{[k+1,n]}^R | s_{[k+1,n]}^L = \lambda \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R | s_{[k+1,n]}^L - (\vec{b}_k - \lambda d_k)^T p_{[k+1,n]}^R | \vec{e}_1^T \cdot s_{[k+1,n]}^L,
\]
while pre-multiplying the conjugate transpose of (2.8) at \( z = \mu \) by \( p_{[k+1,n]}^R \), we obtain
\[
p_{[k+1,n]}^R | s_{[k+1,n]}^L \mathcal{H}_{[k+1,n]} = \mu p_{[k+1,n]}^R | s_{[k+1,n]}^L \mathcal{J}^{(k+1)}_{n+1} - (\vec{b}_k - \mu d_k)^T s_{[k+1,n]}^L p_{[k+1,n]}^R | \vec{e}_1^T.
\]
We proceed with the well-known technique of subtracting traces of the respective sides of (2.9) and (2.10). The left hand side upon subtraction is zero owing to the fact that \( \text{Tr}(A\mathcal{B}) = \text{Tr}(\mathcal{B}A) \) for any well-defined matrix product. Consequently, we have
\[
(\lambda - \mu) \text{Tr}[p_{[k+1,n]}^R | s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]}] = (\vec{b}_k - \lambda d_k)^T p_{[k+1,n]}^R | \vec{e}_1^T \text{Tr}[s_{[k+1,n]}^L] - (\vec{b}_k - \mu d_k)^T s_{[k+1,n]}^L \text{Tr}[p_{[k+1,n]}^R | \vec{e}_1^T].
\]
The left hand side above is equal to the matrix product \((\lambda - \mu) s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R \), while the right hand side can be simplified to \((\vec{b}_k - \lambda d_k)^T s_{k+1}^L p_{k+1}^R - (\vec{b}_k - \mu d_k)^T s_{k}^L p_{k+1}^R \) which gives (2.6). A similar computation starting from
\[
\mathcal{H}_{[0,k]} p_{[0,k]}^R | = z \mathcal{J}_{[0,k]} p_{[0,k]}^R - (\vec{b}_k - z d_k)^T p_{k+1}^R | \vec{e}_{k+1}^T, \quad \vec{e}_{k+1} = (0, 0, \ldots, 1) \in \mathbb{R}^{k+1},
\]
leads to (2.7). □

The assumptions in Lemma (2.2) are necessary for the right hand sides of (2.6) and (2.7) to be non-vanishing. Later, we will use (2.7) to make an observation regarding the positive-definiteness of \( \mathcal{J}_{[0,k]} \). Before that we solve the stated inverse problem GIEP

### 3. Solution to the GIEP

The given data in GIEP suggest that we write the equation \((\lambda \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}) p_{[0,n]}^R = 0 \) in the form
\[
\begin{pmatrix}
\lambda \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]} & \mathcal{O}_\lambda \\
\mathcal{O}_\lambda & \lambda \mathcal{J}_{[k+1,n]} - \mathcal{H}_{[k+1,n]}
\end{pmatrix}
\begin{pmatrix}
p_{[0,k]}^R \\
p_{[k+1,n]}^R
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]
where
\[
\mathcal{O}_\lambda = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda d_k - b_k & 0 & \cdots & 0
\end{pmatrix}.
\]
Pre-multiplying (3.1) by \( s_{[0,n]}^L \) gives the relation
\[
s_{[0,k]}^L \mathcal{H}_{[0,k]} p_{[0,k]}^R + s_{[k+1,n]}^L \mathcal{H}_{[k+1,n]} p_{[k+1,n]}^R = s_{[0,k]}^L \mathcal{O}_\lambda p_{[k+1,n]}^R - s_{[k+1,n]}^L \mathcal{O}_\lambda^* p_{[0,k]}^R
\]
\[
= \lambda (s_{[0,k]}^L \mathcal{J}_{[0,k]} p_{[0,k]}^R + s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R).
\]
Similarly, from \( s_{[0,n]}^L (\mu \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}) = 0 \), we obtain
\[
s_{[0,k]}^L \mathcal{H}_{[0,k]} p_{[0,k]}^R + s_{[k+1,n]}^L \mathcal{H}_{[k+1,n]} p_{[k+1,n]}^R = s_{[0,k]}^L \mathcal{O}_\mu p_{[k+1,n]}^R - s_{[k+1,n]}^L \mathcal{O}_\mu^* p_{[0,k]}^R
\]
\[
= \mu (s_{[0,k]}^L \mathcal{J}_{[0,k]} p_{[0,k]}^R + s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R),
\]
which used with (3.3) to eliminate \( \mathcal{H}_{[0,n]} \) and \( \mathcal{H}_{[k+1,n]} \) gives
\[
s_{[0,k]}^L [\mathcal{O}_\mu - \mathcal{O}_\lambda] p_{[k+1,n]}^R + s_{[k+1,n]}^L [\mathcal{O}_\mu^* - \mathcal{O}_\lambda^*] p_{[0,k]}^R
\]
\[
= (\lambda - \mu) (s_{[0,k]}^L \mathcal{J}_{[0,k]} p_{[0,k]}^R + s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R).
\]
The left hand side can be further simplified to finally obtain
\[
s_{[0,k]}^L \mathcal{J}_{[0,k]} p_{[0,k]}^R + s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R + d_k (s_{k+1}^L p_{k+1}^R + s_{k}^L p_{k+1}^R) = 0,
\]
Theorem 3.2. which simplifies further to give
\[ \frac{p_{k+1}^L - p_k^R}{p_{k+1}^L p_k^R} = - \left( (p_{[0,k]}^L)^* J_{[0,k]} p_{[0,k]}^R + (p_{[k+1,n]}^L)^* J_{[k+1,n]} p_{[k+1,n]}^R \right). \] (3.4)

The next lemma provides a crucial characterization of the pole \( \alpha_k \), that is, it should not be a real number if the entry \( b_k \) is to be determined uniquely.

Lemma 3.1. Suppose \( \alpha_k \notin \mathbb{R} \). Then
\[ \Delta_k = \frac{p_k^L}{p_{k+1}^L} \begin{vmatrix} s_k^L & s_k^L \cr s_{k+1}^L & s_{k+1}^L \end{vmatrix} - \frac{p_k^R}{p_{k+1}^R} \begin{vmatrix} s_k^R & s_k^R \cr s_{k+1}^R & s_{k+1}^R \end{vmatrix} \neq 0, \] (3.5)
if \( \lambda, \mu \notin \sigma(H_{[0,i]}, J_{[0,i]}) \) for \( i = k - 1, k \).

Proof. Using the forms as suggested by (2.4), we first note that
\[ \begin{vmatrix} p_k^L & p_k^R \cr p_{k+1}^L & p_{k+1}^R \end{vmatrix} = - \frac{T_k(\lambda) \overline{T}_{k+1}(\lambda) \prod_{i=0}^k |\alpha_i - \lambda|^2}{\eta_k^L \eta_k^R \eta_{k+1}^L \eta_{k+1}^R \prod_{i=0}^k |\alpha_i - \lambda|^2} \frac{2i \text{Im} \alpha_k}{\lambda - \mu}, \]
by given assumptions and where \( \text{Im} \alpha_k \) is the imaginary part of \( \alpha_k \). Then,
\[ \Delta_k = \frac{T_k(\lambda) \overline{T}_{k+1}(\lambda) \prod_{i=0}^k |\alpha_i - \lambda|^2}{\eta_k^L \eta_k^R \eta_{k+1}^L \eta_{k+1}^R \prod_{i=0}^k |\alpha_i - \lambda|^2} \frac{1}{|\alpha_k - \lambda|^2} \frac{1}{|\alpha_k - \mu|^2} \]
which simplifies further to give
\[ \Delta_k = \frac{T_k(\lambda) \overline{T}_{k+1}(\lambda) \prod_{i=0}^k |\alpha_i - \lambda|^2}{\eta_k^L \eta_k^R \eta_{k+1}^L \eta_{k+1}^R \prod_{i=0}^k |\alpha_i - \lambda|^2} \frac{2i(\lambda - \mu)}{\lambda - \mu} \text{Im} \alpha_k. \]
Since \( \alpha_k \notin \mathbb{R} \), we have that \( \Delta_k \) is not zero. \( \square \)

Now, with \( \Delta_k \) a non-zero purely imaginary number or equivalently, \( i \Delta_k \) a non-vanishing real number, we proceed to show that \( b_k \) can be determined uniquely.

Theorem 3.2. Suppose that the given spectral point points \( \lambda, \mu \notin \sigma(H_{[0,i]}, J_{[0,i]}) \) for \( i = k - 1, k \). If \( \alpha_k \notin \mathbb{R} \), then
\[ b_k = (\lambda + \mu) d_k + \frac{d_k}{\Delta_k} \begin{vmatrix} \mu s_k^L s_k^R \cr p_{k+1}^L p_{k+1}^R \end{vmatrix} - \lambda p_{k+1}^L p_k^R \begin{vmatrix} s_k^L & s_k^R \cr s_{k+1}^L & s_{k+1}^R \end{vmatrix}, \] (3.6)
where \( \Delta_k \) is given by (3.5).

Proof. We have from (2.2)
\[ (\lambda d_{m-1} - b_{m-1}) p_{m-1}^R + (\lambda c_m - a_m) p_m^R + (\lambda d_m - b_m) p_{m+1}^R = 0, \]
\[ (\lambda d_{n-1} - b_{n-1}) p_{n-1}^R + (\lambda c_n - a_n) p_n^R = 0, \] (3.7)
and the corresponding equations for the components of \( p_{[0,k]}^L \)
\[ (\lambda d_{m-1} - b_{m-1}) p_{m-1}^L + (\lambda c_m - a_m) p_m^L + (\lambda d_m - b_m) p_{m+1}^L = 0, \]
\[ (\lambda d_{n-1} - b_{n-1}) p_{n-1}^L + (\lambda c_n - a_n) p_n^L = 0. \] (3.8)
Eliminating \( p_m^R \) and \( p_m^L \) between the first equations of (3.7) and (3.8) we obtain
\[ (p_{m-1}^L p_m^R b_{m-1} - p_{m-1}^R p_m^R b_{m-1}) + (p_{m+1}^L p_m^R b_{m} - p_{m+1}^R p_m^R b_{m-1}) \]
\[ = \lambda d_{m-1}(p_{m-1}^L p_m^R - p_{m-1}^R p_m^R) + \lambda d_m(p_{m+1}^L p_m^R - p_{m+1}^R p_m^R). \]
which on summing respective sides from \( m = k + 1 \) to \( m = n - 1 \), gives
\[
(p_k^L p_{k+1}^R b_k - p_k^L p_{n-1}^R b_{n-1}) + (p_k^L p_{n-1}^R \tilde{b}_{n-1} - p_{k+1}^L p_k^R \tilde{b}_k) = \lambda d_k(p_k^L p_{k+1}^R - p_{k+1}^L p_k^R) + \lambda d_{n-1}(p_n^L p_{n-1}^R - p_{n-1}^L p_n^R).
\]
(3.9)

From the last two relations of (3.7) and (3.8), we get
\[
p_{n-1}^L p_{n-1}^R \tilde{b}_{n-1} = \lambda d_{n-1}(p_n^L p_{n-1}^R - p_{n-1}^L p_n^R),
\]
(3.10)
which when added to (3.9) yields
\[
p_k^L p_{k+1}^R b_k - p_{k+1}^L p_k^R \tilde{b}_k = \lambda d_k(p_k^L p_{k+1}^R - p_{k+1}^L p_k^R).
\]
(3.11)
A computation similar to the relations (3.7), (3.8) and (3.9) for the eigenpair \((\mu, s_{[o,n]}^R)\) gives
\[
s_k^L s_{k+1}^R b_k - s_{k+1}^L s_k^R \tilde{b}_k = \mu d_k(s_k^L s_{k+1}^R - s_{k+1}^L s_k^R).
\]
(3.12)

We solve the system of equations (3.11) and (3.12) for \( b_k \) and \( b_k \). First, the determinant of the system is
\[
\Delta_k = p_k^L p_{k+1}^R s_{k+1}^L s_k^R - p_k^L p_{k+1}^R s_{k+1}^R s_k^L = \left| \begin{array}{ll} s_k^L p_k^R & s_{k+1}^L p_{k+1}^R \\ p_k^R & p_{k+1}^R \end{array} \right|,
\]
which by Lemma 3.1 is non-zero. It is now a matter of computation to obtain
\[
\Delta_k b_k = \lambda d_k p_k^L p_{k+1}^R s_{k+1}^L s_k^R + \mu d_k s_k^L s_{k+1}^R p_k^L p_{k+1}^R
- \lambda d_k p_k^L p_{k+1}^R s_{k+1}^L s_k^R - \mu d_k s_{k+1}^L s_k^R p_k^L p_{k+1}^R,
\]
which can be further simplified to obtain
\[
\Delta_k b_k = (\lambda + \mu)d_k \Delta_k + d_k[\lambda p_k^L p_{k+1}^R (s_{k+1}^L s_k^R - s_k^L s_{k+1}^R) - \mu s_k^L s_{k+1}^R (p_k^L p_{k+1}^R - p_{k+1}^L p_k^R)],
\]
leading to (3.6) and specifying the entry \( b_k \) uniquely. \( \square \)

A similar computation for \( \tilde{b}_k \) gives
\[
\tilde{b}_k = (\lambda + \mu)d_k \Delta_k + \frac{d_k}{\Delta_k} \left[ \mu s_k^L s_{k+1}^R p_k^L p_{k+1}^R - \lambda p_k^L p_{k+1}^R s_{k+1}^L s_k^R \right].
\]

Theorem 3.2 finds the unique expression for \( b_k \). Summing (3.9) from \( m = j \) to \( m = n - 1 \) for each \( j = k + 1, \ldots, n - 1 \) yields an expression similar to (3.6) for each \( b_j \), \( j = k + 1, \ldots, n - 1 \). The entry \( b_{n-1} \) is found from the system of equations consisting of (3.11) and the equivalent equation in \( \mu \). The assumptions are \( \alpha_j \notin \mathbb{R} \) and \( \lambda, \mu \notin \sigma(\mathcal{H}_{[0,j]}, \mathcal{J}_{[0,j]}) \), \( j = k + 1, \ldots, n - 1 \). Thus, with \( b_j, j = k, k+1, \ldots, n-1 \), determined, the \( a_j \)'s are found using (2.2) as
\[
a_i = \begin{cases}
\lambda c_i + \frac{(\lambda d_{i-1} - b_{i-1}) p_i^R}{p_i^L}, & i = k + 1, k + 2 \cdots n - 1; \\
\lambda c_n + \frac{(\lambda d_{n-1} - b_{n-1}) p_n^R}{p_n^L}, & i = n.
\end{cases}
\]
(3.13)
This completes the reconstruction of the matrix \( \mathcal{H}_{[0,n]} \).

**Remark 3.3.** Since \( \lambda \) and \( \mu \) are zeros of \( \mathcal{P}_{n+1}(z) \), the assumptions for the determination of the matrix \( \mathcal{H}_{[0,n]} \) requires that \( \mathcal{P}_j(z), j = k - 1, k, \ldots, n, \) do not vanish at \( \lambda \) and \( \mu \). However, we emphasize that the determination of each entry \( b_j \) requires that \( \mathcal{P}_{j-1}(z) \) and \( \mathcal{P}_j(z) \) do not share a common zero at \( \lambda \) and \( \mu \). This condition is often implicit, both in the direct and inverse problems, in the form of the requirement that the zeros of \( \mathcal{P}_{j-1}(z) \)
and \( p_j(z) \) or equivalently, the eigenvalues of the corresponding pencil matrices satisfy a separation property known as interlacing.

**Corollary 3.4.** For \( j = k, k + 1, \ldots, n - 1 \), \( b_j \) is purely imaginary and equals \( \pm h_j \) if

\[
\begin{bmatrix}
p^L_j & p^R_j \\
p^L_{j+1} & p^R_{j+1}
\end{bmatrix}
\begin{bmatrix}
p^L_j & -p^R_j \\
p^L_{j+1} & -p^R_{j+1}
\end{bmatrix}
\begin{bmatrix}
s^L_j \\
s^R_j
\end{bmatrix}
= \begin{bmatrix}
\lambda(\lambda - \mu) d_j \\
\lambda(\lambda - \mu) d_j
\end{bmatrix},
\]

\[
\begin{bmatrix}
s^L_j & s^R_j \\
s^L_{j+1} & s^R_{j+1}
\end{bmatrix}
\begin{bmatrix}
s^L_j & -s^R_j \\
s^L_{j+1} & -s^R_{j+1}
\end{bmatrix}
\begin{bmatrix}
\mu(\lambda - \mu) d_j \\
\mu(\lambda - \mu) d_j
\end{bmatrix},
\]

\[(3.14)\]

**Proof.** First, let us find the real and imaginary parts of \( b_j \). Since \( \alpha_j \notin \mathbb{R} \), we write \( b_j = x_j + iy_j \) to obtain from (3.11) and (3.12) the system of equations

\[
x_j + \frac{i}{p^L_j p^R_{j+1} - p^L_{j+1} p^R_j} y_j = \lambda d_j; \quad x_j + \frac{s^L_j s^R_{j+1} + s^L_{j+1} s^R_j}{s^L_j s^L_{j+1} - s^R_j s^R_{j+1}} y_j = \mu d_j,
\]

which can be solved to yield

\[
x_j = \frac{d_j}{2\Delta_j} \begin{bmatrix}
\mu & p^L_j & -p^R_j \\
p^L_{j+1} & p^R_{j+1} & s^L_j \\
p^L_{j+1} & p^R_{j+1} & s^R_j
\end{bmatrix}
\begin{bmatrix}
s^L_{j+1} & s^R_{j+1} \\
\lambda & s^L_{j+1} & -s^R_{j+1} \\
\lambda & s^L_{j+1} & -s^R_{j+1}
\end{bmatrix},
\]

\[
y_j = \frac{(\lambda - \mu) d_j}{2i \Delta_j} \begin{bmatrix}
\mu & p^L_j & -p^R_j \\
p^L_{j+1} & p^R_{j+1} & s^L_j \\
p^L_{j+1} & p^R_{j+1} & s^R_j
\end{bmatrix}
\begin{bmatrix}
s^L_{j+1} & s^R_{j+1} \\
\lambda & s^L_{j+1} & -s^R_{j+1} \\
\lambda & s^L_{j+1} & -s^R_{j+1}
\end{bmatrix}.
\]

The above (which can be noted to be in the form \( x_j = A_j x_j - B_j y_j \) and \( y_j = C_j x_j y_j \)) solved further for \( x_j = 0 \) and \( y_j = h_j \), gives the required relations (3.14).

If \( \alpha_j \notin \mathbb{R} \), the proof of Lemma 3.1 implies that \( y_j \neq 0 \) and hence \( b_j \notin \mathbb{R} \), \( j = k, \ldots, n-1 \). However, if we assume \( x_j = 0 \), that is if \( \lambda, \mu \) satisfy

\[
\frac{\lambda}{\mu} = \begin{bmatrix}
p^L_j & -p^R_j \\
p^L_{j+1} & -p^R_{j+1}
\end{bmatrix}
\begin{bmatrix}
s^L_j & s^R_j \\
s^L_{j+1} & s^R_{j+1}
\end{bmatrix}, \quad j = k, k + 1, \ldots, n - 1,
\]

\[(3.15)\]

then \( b_j = i \epsilon_j d_j \), \( \epsilon_j \neq 0 \), that is \( b_j \) is a scalar multiple of \( id_j \).

**Remark 3.5.** The emphasis on \( b_j \) being purely imaginary arises from a particular form of the pencil \( (z I_{[0,n]} - H_{[0,n]}) \), where in fact \( b_j = id_j \) and \( c_j = 1, j = 0, 1, \ldots, n \). As mentioned in Section 4, such pencils appear in analytic function theory and have been studied extensively in the field of pencils as Wall pencils.

Further, as an inverse approach to these pencils, the relation (3.15) shows that the expression on the right hand side of (3.15) must be a constant and equal to the ratio of the given spectral points for \( b_j \) to be at least purely imaginary. Hence, if we begin with \( b_j = id_j, j = 0, \cdots, k - 1 \) and \( c_j = 1, j = 0, \cdots, k \), appropriate conditions can be added to Corollary 3.2 so that \( b_j \) is equal to \( d_j, j = k, \cdots, n - 1 \) and we obtain a Wall pencil.

The matrix \( J_{[0,n]} \) in Wall pencils is positive-definite, while no such assumption has been made in the present manuscript. However, since the matrix \( H_{[0,n]} \) has been reconstructed,
let us have a look in this direction. Suppose the assumptions of Theorem 3.2 hold. We put \( \lambda = \mu = h \) in (2.7) to get
\[
\lambda \left( \frac{R_{[0,k]} \mathcal{J}_{[0,k]} p^R_{[0,k],s}}{h} \right) = \frac{(b_k - \mu d_k) s^R_{k+1} p^R_k - (b_k - \lambda d_k) s^L_{k+1} p^L_k}{h},
\]
(3.16)
so that as \( \lambda \to \mu \) or \( h \to 0 \), we have the left hand side as \( (s^R_{[0,k]})^\ast \mathcal{J}_{[0,k]} s^R_{[0,k]} \). If \( \mu \neq \alpha_j \), \( j = 0, 1, \ldots, k \), \( s^R_{k+1} \) is finite at \( \mu \) so that by Lemma 2.1 \( s^R_{[0,k]} \) is a vector with finite component. By L'Hopital's rule, (3.16) yields
\[
(s^R_{[0,k]})^\ast \mathcal{J}_{[0,k]} s^R_{[0,k]} = (b_k - \mu d_k) s^R_{k+1} (s^R_k)' - (b_k - \mu d_k) s^L_{k+1} (s^R_k)' + d_k s^L_{k+1},
\]
(3.17)
so that if \( \mathcal{J}_{[0,k]} \) is a positive-definite matrix, the right hand side above is a positive quantity.

The essence of this observation is the following. For \( \Delta_k \neq 0 \) to hold, it is necessary that \( \alpha_k \notin \mathbb{R} \). Since \( \mu \in \mathbb{R} \), \( \mu \neq \alpha_k \) follows trivially. Recalling the set \( \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\} \) is just a permutation of \( \{b_0/d_0, b_1/d_1, \ldots, b_{k-1}/d_{k-1}\} \), it follows that \( \mu \neq b_j/d_j \), \( j = 0, \ldots, k-1 \). Hence, for (3.17) to hold as an identity with finite quantities on both sides, no zero of \( P_{n+1}(z) = \det (z \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}) \) should coincide with \( b_j/d_j \), \( j = 0, 1, \ldots, k-1 \).

### 4. A View with m-functions

In this section, we have a look at the reconstruction of the matrix \( \mathcal{H}_{[0,n]} \) through the concept of m-functions. For the general theory of m-functions arising in the context of orthogonal polynomials, we refer to [12] and for that in the context of linear pencil, we refer to [4]. But because of the problem under consideration, we will have use only of the representation (1.4) for a point outside the spectrum of the pencil.

In the present case, in addition to \( p_m^R(z) \) as defined in (2.3), we will use the rational functions
\[
q_0^R(z) = 0, \quad q_m^R(z) = \frac{Q_m(z)}{\prod_{j=0}^{m-1} (zd_j - b_j)}, \quad m = 1, \ldots, n,
\]
where \( Q_m(z) \) satisfy (2.1) with initial conditions (1.3). A key role will be played by the following relation
\[
P_{m+1}(z) Q_m(z) - P_m(z) Q_{m+1}(z) = \prod_{j=0}^{m-1} (zd_j - b_j)(zd_j - b_j),
\]
(4.1)
called the Liouville-Ostrogradsky formula and which follows by induction from the recurrence relation (2.1) along with the initial conditions (1.3). Using (4.1), the matrix representation of the bounded operator \( (\omega \mathcal{J} - \mathcal{H})^{-1}, \omega \in \rho(\mathcal{H}, \mathcal{J}) \) has been found in terms of \( p_m^{R(L)}(z) \) and \( q_m^{R(L)}(z) \) [4] Theorem 2.3. The inverse of banded matrices has been studied, for instance, in [9], but we follow [4] to obtain a finite version, that is the inverse of the pencil \( z \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]} \).

**Lemma 4.1.** Let us denote \( m_j := m(\omega, j) - m(\omega, n+1) \). Then the inverse \( \mathcal{R}_{[0,n]}(\omega) \) of \( (\omega \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})^{-1}, \omega \in \rho(\mathcal{H}_{[0,n]}, \mathcal{J}_{[0,n]}) \) is given by
\[
\mathcal{R}_{[0,n]}(\omega) = \begin{pmatrix}
p_1^L m_0 p_1^R & p_1^L m_0 p_0^R & p_2^L m_0 p_0^R & \cdots & p_{n-1}^L m_0 p_0^R & p_n^L m_0 p_0^R \\
p_1^L m_1 p_1^R & p_1^L m_1 p_0^R & p_2^L m_1 p_0^R & \cdots & p_{n-1}^L m_1 p_0^R & p_n^L m_1 p_0^R \\
p_1^L m_2 p_2^R & p_1^L m_2 p_1^R & p_2^L m_2 p_1^R & \cdots & p_{n-1}^L m_2 p_1^R & p_n^L m_2 p_1^R \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
p_1^L m_{n-1} p_{n-1}^R & p_1^L m_{n-1} p_{n-2}^R & p_2^L m_{n-1} p_{n-2}^R & \cdots & p_{n-1}^L m_{n-1} p_{n-2}^R & p_n^L m_{n-1} p_{n-2}^R \\
p_1^L m_n p_n^R & p_1^L m_n p_{n-1}^R & p_2^L m_n p_{n-1}^R & \cdots & p_{n-1}^L m_n p_{n-1}^R & p_n^L m_n p_{n-1}^R
\end{pmatrix}
\]

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Proof. Consider the $1 \times (n + 1)$ vector $p_{[0, n]}^F := (p_0^F, p_1^F, \ldots, p_n^F, 0, \ldots, 0)$ and similarly the vector $q_{[0, n]}^F$. Using \eqref{2.2} for the left eigenvector we obtain

$$
p_{[0, n]}^F (\omega J_{[0, n]} - \mathcal{H}_{[0, n]}) = -(\omega d_j - \bar{b}_j) p_{j+1}^F \bar{e}^T_j + (\omega d_j - b_j) p_j^F \bar{e}^T_j + 1,
$$

$$
q_{[0, n]}^F (\omega J_{[0, n]} - \mathcal{H}_{[0, n]}) = \bar{e}^T_0 - (\omega d_j - \bar{b}_j) q_{j+1}^F \bar{e}^T_j + (\omega d_j - b_j) q_j^F \bar{e}^T_j + 1,
$$

which in view of \eqref{1.1} leads to

$$
[q_j^F p_{[0, n]}^F] (\omega J_{[0, n]} - \mathcal{H}_{[0, n]}) = \bar{e}^T_j - p_j^F \bar{e}^T_0. \tag{4.2}
$$

The vectors $p_{[0, n]}^F$ and $q_{[0, n]}^F$ with the above computation yield

$$
[q_{n+1}^F p_{[0, n]}^F] (\omega J_{[0, n]} - \mathcal{H}_{[0, n]}) = -p_{n+1}^F \bar{e}^T_0,
$$

which in terms of $m$-functions can also be written as

$$
[q_{[0, n]}^F - m(\omega, n + 1) p_{[0, n]}^F] (\omega J_{[0, n]} - \mathcal{H}_{[0, n]}) = \bar{e}^T_0. \tag{4.3}
$$

Eliminating $\bar{e}^T_0$ between \eqref{4.2} and \eqref{4.3}, we obtain

$$
p_j^F [q_{[0, n]}^F - m(\omega, n + 1) p_{[0, n]}^F] - q_{[0, n]}^F + m(\omega, j) p_{[0, n]}^F] (\omega J_{[0, n]} - \mathcal{H}_{[0, n]}) = \bar{e}^T_j,
$$

which upon further simplification gives the matrix $R_{[0, n]}(\omega)$. \hfill \square

In compact form, the $(i, j)^{th}$ entry of $\mathcal{R}_{[0, n]}(\omega)$ is given by $p_j^F m_{\min(i, j)} p_i^R$. Next, for $\omega \in \rho(\mathcal{H}_{[0, n]}, J_{[0, n]})$, let us factorize

$$
\mathcal{R}_{[0, n]}(\omega) = (\omega J_{[0, n]} - \mathcal{H}_{[0, n]})^{-1} = L_{[0, n]}(\omega) D_{[0, n]}(\omega) U_{[0, n]}(\omega). \tag{4.4}
$$

Then, owing to the Hermitian nature of the eigenvalue equation involved, we choose

$$
U_{[0, n]}(\omega) = L^*_{[0, n]}(\omega), \quad \text{where} \quad L_{[0, n]}(\omega) = \left(\begin{array}{cccccc}
p_0^R & 0 & 0 & \ldots & 0 & 0 \\
p_1^R & p_1^R & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
p_n^R & p_{n-1}^R & p_{n-2}^R & \ldots & p_{n-1}^R & 0 \\
p_n^R & p_n^R & p_n^R & \ldots & p_n^R & p_n^R
\end{array}\right),
$$

and $D_{[0, n]}(\omega)$ is the diagonal matrix $\text{diag}(d_0(\omega), d_1(\omega), \ldots, d_n(\omega))$ where

$$
d_0(\omega) = m_0, \quad d_j(\omega) = (m_j - m_{j-1}) = m(\omega, j) - m(\omega, j - 1), \quad j = 1, 2, \ldots, n.
$$

As a matter of verification, with $m_{-1} := 0$, we have in the right hand side of \eqref{4.4}

$$
i^{th} \text{ row} \times j^{th} \text{ column} = p_j^R \sum_{k=0}^{\min(i, j)} (m_k - m_{k-1}) p_k^F = p_j^F m_{\min(i, j)} p_i^R.
$$

With this decomposition we can easily invert $\mathcal{R}_{[0, n]}(\omega)$ again, so that $[\mathcal{R}_{[0, n]}(\omega)]^{-1} = (\omega J_{[0, n]} - \mathcal{H}_{[0, n]})$ will be a matrix in which the entries are given in terms of the $m$-functions. We illustrate this for the trailing submatrix $[\Psi_{[k+1, n]}(\omega)]^{-1}$.

**Lemma 4.2.** Suppose $m(\omega, k + 1) \neq m(\omega, n + 1)$ and $m(\omega, i) \neq m(\omega, i - 1)$, $i = k + 2, \ldots, n$. The entries of the inverse of the trailing sub-matrix $\Psi_{[k+1, n]}(\omega)$ are given by

$$
[\Psi_{[k+1, n]}(\omega)]^{-1} = \begin{cases}
p_j^F [m(\omega, i) - m(\omega, i - 1)] p_i^R & \text{if } i = j; \\
p_j^F [m(\omega, j+1) - m(\omega, j)] p_i^R & \text{if } i < j; \\
0, & \text{if } |i - j| > 1,
\end{cases} \tag{4.5}
$$
for \(i, j = k + 2, k + 3, \ldots, n - 1\), while

\[
[\Psi_{[k+1,n]}(\omega)]^{-1}_{i,j} = \begin{cases} 
p^R_j [m(\omega, i) - m(\omega, n + 1)] p^R_i, & i = j = k + 1; \\
p^R_j [m(\omega, i) - m(\omega, j)] p^R_j, & i = j = n.
\end{cases} \tag{4.6}
\]

**Proof.** We start with the decomposition

\[
R_{[k+1,n]}(\omega) = L_{[k+1,n]}(\omega) D_{[k+1,n]}(\omega) U_{[k+1,n]}(\omega)
\]

where

\[
U_{[k+1,n]}(\omega) = L^*_{[k+1,n]}(\omega) \quad \text{with} \quad L_{[k+1,n]}(\omega) = \begin{pmatrix} 1/p^R_{k+1} & 0 & 0 & \ldots & 0 & 0 \\ -1/p^R_{k+1} & 1/p^R_{k+2} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1/p^R_{n-1} & 0 \\ 0 & 0 & 0 & \ldots & -1/p^R_{n-1} & 1/p^R_n \end{pmatrix},
\]

and \(D_{[k+1,n]} = \text{diag}\{d_{k+1}, d_{k+2}, \ldots, d_n\}\) given by

\[
d_{k+1}(\omega) = m_{k+1} = m(\omega, k + 1) - m(\omega, n + 1),
\]

\[
d_j(\omega) = (m_j - m_{j-1}) = m(\omega, j) - m(\omega, j - 1), \quad j = k + 2, k + 3, \ldots, n.
\]

It can be easily verified that

\[
[L_{[k+1,n]}(\omega)]^{-1} = \begin{pmatrix} 1/p^R_{k+1} & 0 & 0 & \ldots & 0 & 0 \\ -1/p^R_{k+1} & 1/p^R_{k+2} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1/p^R_{n-1} & 0 \\ 0 & 0 & 0 & \ldots & -1/p^R_{n-1} & 1/p^R_n \end{pmatrix},
\]

and \(U_{[k+1,n]}(\omega)^{-1} = [L^*_{[k+1,n]}(\omega)]^{-1}\). Then,

\[
[R_{[k+1,n]}(\omega)]^{-1} = [U_{[k+1,n]}(\omega)]^{-1} [D_{[k+1,n]}(\omega)]^{-1} [L_{[k+1,n]}(\omega)]^{-1},
\]

gives the required entries (4.5) and (4.6). \(\square\)

We are now ready to view the entries of \(\mathcal{H}_{[0,n]}\) in terms of \(m\)-functions. Let us denote

\[
m^j(\omega) = \frac{1}{p^R_j [m(\omega, j) - m(\omega, i)] p^R_i} \quad \text{and} \quad \tilde{m}^j(\omega) = \frac{1}{p^R_j [m(\omega, j) - m(\omega, i)] p^R_j}.
\]

**Theorem 4.3.** Suppose the \(m\)-functions \(m(\omega, j)\) of the pencil \((\omega J_{[0,j]} - \mathcal{H}_{[0,j]})\) are known and satisfy the assumptions of Lemma 4.2 for \(j = k + 1, k + 2, \ldots, n, n + 1\). If \(\omega \in \rho(\mathcal{H}_{0,k}, J_{[0,k]})\), then the matrix \(\mathcal{H}_{[0,n]}\) can be reconstructed with the entries given by

\[
b_j = \omega d_j + \tilde{m}^j_{j+1}(\omega), \quad j = k + 1, k + 2, \ldots, n - 1, \tag{4.7}
\]

and

\[
a_j = \begin{cases} 
\omega c_j - m^{j+1}(\omega) + \frac{[\omega d_k - b_k]^2}{m^2_{j-1}(\omega)} & j = k + 1; \\
\omega c_j + m^{-j+1}(\omega) & j = k + 2, \ldots, n - 1, \\
\omega c_j + m^{j-1}(\omega) & j = n.
\end{cases} \tag{4.8}
\]

**Proof.** We use the following representation of the inverse obtained through the concept of Schur’s complement [5]

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ [D - CA^{-1}B]^{-1} & I \end{pmatrix}, \tag{4.9}
\]
where $I$ is the identity matrix of appropriate order. Since $\Psi_{[k+1,n]}(\omega)$ is the trailing submatrix of the inverse of the pencil, we also have

$$
\begin{pmatrix}
\mathcal{J}_{[0,k]} - \mathcal{H}_{[0,k]} & \mathcal{O}_\omega \\
\mathcal{O}_\omega^* & \mathcal{J}_{[k+1,n]} - \mathcal{H}_{[k+1,n]}
\end{pmatrix}^{-1} = \begin{pmatrix}
\Psi_{[0,k]}(\omega) & \Psi_{[k+1,n]}(\omega)^*
\end{pmatrix}.
$$

(4.10)

Since $\omega \in \rho(\mathcal{H}_{[0,k]}, \mathcal{J}_{[0,k]} \mathcal{J}_{[0,k]}^* \mathcal{H}_{[0,k]} )$, we can substitute $A = \omega \mathcal{J}_{[0,k]} - \mathcal{H}_{[0,k]}$, $B = \mathcal{O}_\omega$, given by (3.2) and $D = \omega \mathcal{J}_{[k+1,n]} - \mathcal{H}_{[k+1,n]}$. Then comparing the respective blocks of (4.9) and (4.10), we get

$$(\omega \mathcal{J}_{[k+1,n]} - \mathcal{H}_{[k+1,n]}) = [\Psi_{[k+1,n]}(\omega)]^{-1} + \mathcal{O}_\omega^*(\omega \mathcal{J}_{[0,k]} - \mathcal{H}_{[0,k]})^{-1}\mathcal{O}_\omega.$$  

(4.11)

We use Lemma 4.1 to obtain the inverse $R_{[0,k]}$ of $\omega \mathcal{J}_{[0,k]} - \mathcal{H}_{[0,k]}$ so that

$$\mathcal{O}_\omega^*[\omega \mathcal{J}_{[0,k]} - \mathcal{H}_{[0,k]}]^{-1} \mathcal{O}_\omega = p_k^L[w(\omega, k) - m(\omega, k + 1)]p_k^L[w d_k - b_k^2 \vec{c}_0 \vec{e}^T],$$

where $\vec{c}_0 \in \mathbb{R}^{k+1}$. The entries of $[\Psi_{[k+1,n]}]^{-1}$ obtained from (4.5) and (4.6) and used in (4.11) yields the required expressions (4.8) and (4.7). 

It may be observed that the rational functions $q_j^{R(L)}(\omega)$ are only intermediary since the final expressions for the entries depend on the $\mathcal{m}$-functions defined by (1.4). As opposed to (3.13), the $a_j$'s in the present case, except for $a_{k+1}$, are determined independent of $b_j$'s while for $a_{k+1}$, in addition to $b_k$, we need prior information on $m(\omega,n + 1)$. Finally, the results of the present section can themselves be seen as a sort of inverse problem of reconstructing the matrix $\mathcal{H}_{[0,n]}$ from the knowledge of $\mathcal{m}$-functions, components of an eigenvector and a point in the resolvent set of the linear pencil.

**References**

[1] A. I. Aptekarev, V. Kaliaguine and W. Van Assche, Criterion for the resolvent set of nonsymmetric tridiagonal operators, Proc. Amer. Math. Soc. 123 (1995), no. 8, 2423–2430.

[2] G. A. Baker, Jr. and P. Graves-Morris, Padé approximants. Part I, Encyclopedia of Mathematics and its Applications, 13, Addison-Wesley Publishing Co., Reading, MA, 1981.

[3] B. Beckermann, V. Kaliaguine, The diagonal of the Pad table and the approximation of the Weyl function of second order difference operators, Constr. Approx. 13 (1997) 481–510.

[4] B. Beckermann, M. Derevyagin and A. Zhedanov, The linear pencil approach to rational interpolation, J. Approx. Theory 162 (2010), no. 6, 1322–1346.

[5] D. Carlson, What are Schur complements, anyway?, Linear Algebra Appl. 74 (1986), 257–275.

[6] M. T. Chu, Inverse eigenvalue problems, SIAM Rev. 40 (1998), no. 1, 1–39.

[7] M. Derevyagin, A note on Wall’s modification of the Schur algorithm and linear pencils of Jacobi matrices, J. Approx. Theory 221 (2017), 1–21.

[8] M. E. H. Ismail and A. Sri Ranga, $R_{II}$ type recurrence, generalized eigenvalue problem and orthogonal polynomials on the unit circle, Linear Algebra Appl. 562 (2019), 63–90.

[9] E. Kilic and P. Stanica, The inverse of banded matrices, J. Comput. Appl. Math. 237 (2013), no. 1, 126–135.

[10] P. Lancaster and Q. Ye, Inverse spectral problems for linear and quadratic matrix pencils, Linear Algebra Appl. 107 (1988), 293–309.

[11] M. Sen and D. Sharma, Generalized inverse eigenvalue problem for matrices whose graph is a path, Linear Algebra Appl. 446 (2014), 224–236.

[12] B. Simon, Orthogonal polynomials on the unit circle. Part 1, American Mathematical Society Colloquium Publications, 54, Part 1, American Mathematical Society, Providence, RI, 2005.

[13] H. S. Wall, Analytic Theory of Continued Fractions, D. Van Nostrand Company, Inc., New York, NY, 1948.

[14] Y.-X. Yuan and H. Dai, A generalized inverse eigenvalue problem in structural dynamic model updating, J. Comput. Appl. Math. 226 (2009), no. 1, 42–49.

[15] H. Zhang and Y. Yuan, Generalized inverse eigenvalue problems for Hermitian and J-Hamiltonian/skew-Hamiltonian matrices, Appl. Math. Comput. 361 (2019), 609–616.
[16] A. Zhedanov, Biorthogonal rational functions and the generalized eigenvalue problem, J. Approx. Theory 101 (1999), no. 2, 303–329.

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