A New Model for Fermion Masses in Supersymmetric Grand Unified Theories

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Abstract

We present a simple model for fermion mass matrices and quark mixing in the context of supersymmetric grand unified theories and show its agreement with experiment. Our model realizes the GUT mass relations $m_d = 3m_e$, $m_s = m_\mu/3$, $m_b = m_\tau$ in a new way and is easily consistent with values of $m_t$ suggested by MSSM fits to LEP data.

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Understanding the values of fermion masses and quark mixing remains one of the most important unsolved problems in particle physics. The magnitude of the challenge is shown by the fact that at present it is not definitely clear which approach one should take to the related matter of electroweak symmetry breaking (EWSB). Intensive efforts continue in at least two quite different directions: (i) supersymmetry, and, in particular, the minimal supersymmetric standard model (MSSM), in which EWSB is perturbative [1]; (ii) dynamical (and nonperturbative) EWSB, e.g. slowly running technicolor [2], as well as others such as compositeness. Here we shall adopt the first approach, specifically the MSSM, which may provide the simplest way to understand the precision electroweak measurements from LEP [3] while stabilizing the Higgs sector in the standard model. A further motivation for the MSSM is that it achieves unification of gauge couplings at a single scale $m_G$, a prerequisite for a supersymmetric grand unified theory (SGUT).

In GUT’s such as SU(5) or SO(10), by well-known restrictions on (renormalizable) Higgs couplings, the elements of the lepton Yukawa matrix can be made equal to 1 or $-3$ times corresponding elements of down quark Yukawa matrix. In this approach, one can get a simple relation between the resultant (running) lepton and down quark masses, evaluated at the GUT scale. One interesting proposal [4] is that at the GUT scale

$$m_d = 3m_e, \quad m_s = \frac{m_\mu}{3}, \quad m_b = m_\tau$$  (1)

It is important to determine which (experimentally acceptable) forms for fermion mass matrices yield this relation. Since 1979, essentially only one such form (up to trivial equivalence transformations) has been found. With $-\mathcal{L}_m = v_u \bar{\psi}_{u,L} Y_u \psi_{u,R} + v_{de} (\bar{\psi}_{d,L} Y_d \psi_{d,R} + \bar{\psi}_{e,L} Y_e \psi_{e,R}) + h.c.$, this form has GUT scale Yukawa matrices

$$Y_u = \begin{pmatrix} 0 & A_u & 0 \\ A_u & 0 & B_u \\ 0 & B_u & C_u \end{pmatrix}$$  (2)

$$Y_d = \begin{pmatrix} 0 & A_{de} e^{i\phi} & 0 \\ A_{de} e^{-i\phi} & 0 & B_d \\ 0 & 0 & C_d \end{pmatrix}$$  (3)
Interesting studies of SGUT Yukawa matrices, and this ansatz in particular, have been carried out recently using renormalization group equations (RGE’s) in the MSSM [5, 6, 7] (where $v_u = 2^{-1/2}v \sin \beta$, $v_{de} = 2^{-1/2}v \cos \beta$, with $v = 2^{-1/4}G_F^{-1/2} = 246$ GeV). A general feature of this ansatz is that it tends to require a rather large top quark mass, $m_t$. Further recent works have considered more complicated mass matrices [8] and higher-dimension operators [9] which do not, in general, yield (1) (see also Ref. [10]).

In this paper we exhibit a new and quite different model for fermion mass matrices which yields the GJ mass relation (1). We show that this model agrees with experiment, in particular with values of $m_t$ around 135 GeV, in the range suggested by MSSM fits to the LEP data [3, 11]. Our model is defined at the SGUT scale by

$$Y_e = \begin{pmatrix} 0 & A_d & 0 \\ A_d & -3B_d & 0 \\ 0 & 0 & C_d \end{pmatrix}$$ (4)

A natural setting for this ansatz would be an SO(10) SGUT, where one can obtain symmetric (complex) Yukawa matrices by using 10 and 126 Higgs representations to couple to the fermion $16 \times 16$ bilinear. If one had a fundamental theory of fermion masses, one would presumably be able to derive the forms of the $Y_f$ and also the values of the parameters from first principles. In the absence of such a theory, we believe that models such as ours can give valuable hints about the underlying physics [12]. Our mass matrix model has seven real parameters $A_u, B_u, C_u, A_d, B_d, C_d,$ and $\phi$ (plus $\tan \beta$) to describe the nine fermion masses in the $u, d, e$ sectors and the four angles parametrizing the quark mixing matrix $V$ (plus $\tan \beta$);
hence it makes six predictions. (Because of the uncertainties regarding neutrino masses and
mixing, we do not consider these here.)

We first show that this ansatz yields the GUT scale GJ mass relation (1). The $Y_f, f = u, d, e$
are diagonalized by unitary transformations $U_f$: $U_f Y_f U_f^\dagger = \text{diag}\{\lambda_{f,1}, \lambda_{f,2}, \lambda_{f,3}\}$, where
$\lambda_{f,j} = \pm m_{f,j}/v_f$ (with the quantities defined at $\mu = m_G$, and $v_d = v_e \equiv v_{de}$). Without loss of
generality, one may pick $\lambda_{f,3} > 0$. For $Y_d$ one can take either of the choices
$(\lambda_{d,1}, \lambda_{d,2}, \lambda_{d,3}) = (+, -, +), (-, +, +)$; correspondingly, $(\lambda_{e,1}, \lambda_{e,2}, \lambda_{e,3}) = (-, +, +), (+, -, +)$. To leading
order in small fermion mass ratios, we find that $C_d = C_e \Rightarrow m_b = m_t$. Next, $B_d = \lambda_{d,2} = \mp m_s/v_{de}$ and
$\lambda_{e,2} = -3\lambda_{d,2}$, which together imply $m_s = m_\mu/3$. Finally, $A_d = \sqrt{-\lambda_{d,1}\lambda_{d,2}} = \sqrt{m_d m_s}/v_{de}$ and $(Y_e)_{12} = (Y_e)_{21} = -\lambda_{e,1}\lambda_{e,2} = \sqrt{m_e m_\mu}/v_{de}$ so that the
property $(Y_e)_{12} = A_d$ implies $\sqrt{m_e m_\mu} = \sqrt{m_d m_s}$ and hence $m_d = 3m_e$. (These relations
hold for either set of sign choices.) This proves our assertion. We will use the choice
$(-, +, +)$ for $Y_d$ since it gives a slightly better fit to the data on quark mixing. For $Y_u$ one
may also take $(\lambda_{u,1}, \lambda_{u,2}, \lambda_{u,3}) = (+, -, +)$ or $(-, +, +)$, and one finds $C_u = \lambda_{u,3} = m_t/v_u$, 
$B_u = \lambda_{u,2} = \mp m_c/v_u$, and $A_u = \sqrt{-\lambda_{u,1}\lambda_{u,2}} = \sqrt{m_u m_c}/v_u$. The different sign choices for the
$\lambda_{u,j}$ will imply picking $\phi$ values in the fit which differ by $\pi$; for definiteness, we use $(+, -, +)$
for the $\lambda_{u,j}$.

To test the model, we evolve the Yukawa matrices down from the SGUT scale to the
electroweak scale, $m_{EW}$ and diagonalize them there. We shall take $m_{EW} = m_t$, but our
results are not very sensitive to this choice. We use the illustrative value $m_t(m_t) = 135$ GeV
(i.e., pole mass $M_t = m_t(m_t)[1 + 4\alpha_3(M_t)/(3\pi) + O(\alpha_3(M_t)^2) \approx 142$ GeV), consistent with
MSSM fits to LEP data [3, 11]. To avoid complications involving further model-dependence,
we follow a common simplification [3, 7] of approximating the full SUSY mass spectrum by
a single mass scale, $m_{SUSY}$ [13] with $m_{SUSY} = m_t$. Given the theoretical uncertainties
associated with the SUSY spectrum and threshold corrections, 1-loop RGE’s are sufficiently
accurate in the range from $m_G$ to $m_{SUSY}$. The inputs $\alpha_{em}(m_Z)^{-1} = 127.9 \pm 0.2$ and $\sin^2 \theta_W = 
0.2325 \pm 0.008$ [14], which give $\alpha_1(m_Z)^{-1} = 58.90 \pm 0.11$ and $\alpha_2(m_Z)^{-1} = 29.74 \pm 0.11,$
together with the assumptions that the lightest Higgs has $m_h \simeq m_Z$ and $m_{SUSY} \simeq m_t = 135$
GeV lead to 1-loop unification in the MSSM at $m_G = 1.3 \times 10^{16}$, with $\alpha_G^{-1} = 24.8$. We
take \( \alpha_3(m_Z) = 0.125 \), consistent with current measurements \[3, 15\] and with MSSM gauge coupling unification. (Using the 3-loop QCD RGE’s \[16\], this gives \( \Lambda^{(5)}_{MS} = 302 \) MeV and \( \alpha_3(m_t) = 0.118 \).)

An interesting property of the model which we have established by analytic and numerical solutions of the RGE’s is that although the values of the parameters change, the forms of \( Y_u \) and \( Y_e \) are preserved to high accuracy under evolution from \( m_G \) to \( m_{EW} \). For \( Y_d \), the relation \((Y_d)_{21} = (Y_d)^*_{12}\) and the equality \((Y_d)_{22} = (Y_d)_{23}\) are maintained quite accurately, while \(|(Y_d)_{32}|\) decreases (by about 35 %) relative to \(|(Y_d)_{23}|\). Keeping dominant terms (and with \( \dot{f}(t) \equiv (16\pi^2)^{-1}df/dt, t = \ln(\mu/m_G) \)), the RGE’s for the Yukawa matrices thus reduce to RGE’s for the nonzero elements, which are

\[
\dot{X}_u = X_u(-\sum_j c_{u,j}g_j^2) \quad (8)
\]

for \( X_u = A_u, B_u; \)

\[
\dot{C}_u = C_u(-\sum_j c_{u,j}g_j^2 + 6C_u^2) \quad (9)
\]

\[
\dot{X}_d = X_d(-\sum_j c_{d,j}g_j^2) \quad (10)
\]

for \( X_d = A_d, (Y_d)_{22}, (Y_d)_{23}; \)

\[
\dot{Z}_d = Z_d(-\sum_j c_{d,j}g_j^2 + C_u^2) \quad (11)
\]

for \( Z_d = (Y_d)_{32}, (Y_d)_{33}; \) and

\[
\dot{X}_e = X_e(-\sum_j c_{e,j}g_j^2) \quad (12)
\]

for \( X_e = A_e, B_e, C_e, \) where for the MSSM in the relevant energy range \( \mu > m_{SUSY}, \)

\((b_1, b_2, b_3) = (\frac{33}{5}, 1, -3), \quad (c_{u,1}, c_{u,2}, c_{u,3}) = (\frac{13}{15}, 3, \frac{16}{3}), \quad (c_{d,1}, c_{d,2}, c_{d,3}) = (\frac{7}{15}, 3, \frac{16}{3}), \quad (c_{e,1}, c_{e,2}, c_{e,3}) = (\frac{9}{5}, 3, 0) \) \[17\]. To this order, the running of \( \phi \) is negligibly small.

Manipulating these RGE’s, we obtain the solutions

\[
\frac{m_b(m_{EW})}{m_{\tau}(m_{EW})} = \frac{C_d(t_{EW})}{C_e(t_{EW})} = \left(\frac{C_u(t_{EW})}{C_u(t_G)}\right)^{1/6} \rho_C(t_{EW}) \quad (13)
\]

where

\[
\rho_C(t) = \prod_{j=1}^3 \left(\frac{\alpha_G}{\alpha_j(t)}\right)^{\frac{c_{d,j}-c_{e,j}-(c_{u,j}/6)}{2\beta_j}} \quad (14)
\]
and $t_{EW} = \ln(m_{\ell}/m_G)$. We calculate $\rho_C(t_{EW}) = 1.92$. Further, we define $\eta_f = m_f(m_f)/m_f(m_{EW})$ for leptons and heavy quarks, and $\eta_{uds} = m_f(1\text{ GeV})/m_f(m_{EW})$ for $u, d, s$ and take the value $m_b(m_b) = 4.19\text{ GeV}$, consistent with Ref. [20]. Using 1-loop QED and 3-loop QCD RGE's, we obtain $\eta_r = 1.02$ and $\eta_b = 1.55$, whence $m_b(m_{EW})/m_r(m_{EW}) = 1.54$. We successfully fit this ratio by choosing an appropriate value of $C_u(t_G)$. This is a nontrivial test of the ansatz, since it is not, $a\ priori$, guaranteed that a perturbatively acceptable value of $C_u(t_G)$ would enable one to fit the observed $m_b/m_r$ mass ratio. We find $C_u(t_{EW})/C_u(t_G) = 0.269$ from (13); substituting this into the solution to the RGE for $C_u$ [18], we get $C_u(t_G) = 4.48$. (This is consistent with the validity of perturbation theory since loop corrections are small: $C^2_u/(16\pi^2) = 0.13$.) We then get $C_u(t_{EW}) = 1.21$ and, from the condition $m_t(m_t) = C_u(t_{EW})v\sin(\beta)/\sqrt{2}$, determine $\tan\beta = 0.84$.

From the RGE's, we obtain the solutions

$$\frac{A_d(t)}{A_e(t)} = \frac{B_d(t)}{B_e(t)} = \rho_{AB}(t)$$

(15)

where $(Y_d)_{22}(t) = (Y_d)_{23} \equiv B_d(t)$ and

$$\rho_{AB}(t) = \prod_{j=1}^{3}\left(\frac{\alpha_G}{\alpha_j(t)}\right)^{e_{d,j} - e_{e,j}}\frac{e^{e_{d,j} - e_{e,j}}}{2\eta_j}$$

(16)

We calculate $\rho_{AB}(t = t_{EW}) = 2.38$. Now

$$\frac{m_s(m_{EW})}{m_{\mu}(m_{EW})} = \frac{B_d(t_{EW})}{3B_e(t_{EW})} = 0.792$$

(17)

Using $m_{\mu}(m_{EW}) = 102.8\text{ MeV}$, the model predicts $m_s(m_{EW}) = 81.4\text{ MeV}$, i.e., with our 3-loop $\eta_{uds} = 2.94$, $m_s(1\text{ GeV}) = 239\text{ MeV}$. This is consistent with the (upper range of the) determination $m_s(1\text{ GeV}) = 175 \pm 55\text{ MeV}$ [20]. Further we find

$$\frac{(Y_d)_{32}(t)}{(Y_d)_{23}(t)} = \left(\frac{C_u(t)}{C_u(t_G)}\right)^{\frac{1}{6}} \prod_{j=1}^{3}\left(\frac{\alpha_G}{\alpha_j(t)}\right)^{-\frac{e_{d,j}}{2\eta_j}}$$

(18)

We use the results $A_d(t)v_{de} = \sqrt{m_d(t)m_s(t)}$, $A_e(t)v_{de} = \sqrt{m_e(t)m_{\mu}(t)}$ and eq. (15) to find $m_d(m_{EW})/m_e(m_{EW}) = 7.13$. Using $m_e(m_{EW}) = 0.4876\text{ MeV}$, this yields $m_d(m_{EW}) = 3.48\text{ MeV}$, and hence $m_d(1\text{ GeV}) = 10.2\text{ MeV}$, in agreement with the value $m_d(1\text{ GeV}) = 8.9 \pm 2.6\text{ MeV}$ [20].
Given that $Y_u$ retains its form to high accuracy under the evolution from $m_G$ to $m_{EW}$, the two remaining parameters in $Y_u$ are determined via the relations $B_u(t_{ew})v_u = -m_c(t)$ and $A_u(t_{ew})v_u = \sqrt{m_u(t)m_c(t)}$. From $m_c(m_c) = 1.27 \pm 0.05$ GeV \[\cite{21}\] we calculate $\eta_c = 2.32$, whence $m_c(m_{EW}) = 0.547 \pm 0.022$ GeV. With $m_u(1 \text{ GeV}) = 5.1 \pm 1.5$ MeV we get $m_u(m_{EW}) = 1.7 \pm 0.5$ MeV. Diagonalizing at $m_{EW}$, we have $v_u U_{u,L} Y_{u} U_{u,L}^\dagger = M_u$, $v_d^2 U_{d,L} Y_{d} Y_{d}^\dagger U_{d,R} = M_d^2$, whence $V = U_{u,L} U_{d,L}^\dagger$ for the Cabibbo-Kobayashi-Maskawa (CKM) matrix $V$ \[\cite{21}\]. To leading order $|V_{us}| \approx |\sqrt{m_u/m_c} - e^{i\phi} \sqrt{m_d/m_s}|$ (with quantities evaluated at $\mu = m_{EW}$ \[\cite{21}\]). We determine $\phi = 75.5^\circ$ by fitting $|V_{us}|$. Quoting as usual the rephasing-invariant $|V_{jk}|$ values, we get

$$\{|V_{ij}|\} = \begin{pmatrix} 0.9753 & 0.221 & 0.002 \\ 0.221 & 0.9748 & 0.030 \\ 0.006 & 0.030 & 0.9995 \end{pmatrix} \quad (19)$$

in agreement with current data \[\cite{14, 13}\] (with $|V_{cb}|$ in the lower range of acceptable values). For the reparametrization-invariant CP violation parameter $J$ \[\cite{22}\], we get $J = 0.95 \times 10^{-5}$, again in agreement with data. Further details will be given elsewhere \[\cite{23}\].

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[12] It has often been speculated that a given ansatz may arise in a technically natural way due to global symmetries. However, global symmetries are likely to be broken by quantum gravity (see, e.g., the review by M. Dine, in Proc. XXVI Conf. H. E. Physics, 1992 op. cit., p. 1386). Discrete symmetries do arise naturally in string theory, where they are typically local. Perhaps the zeroes in mass matrices should really be regarded as only good approximations to nonzero quantities dynamically determined by a more fundamental theory (e.g. R. Shrock, Phys. Rev. D 45 R10 (1992)). Our model specifically assumes that the underlying theory yields (i) zeroes as indicated; (ii) the relation $|Y_{d22}| = |Y_{d23}| = |Y_{d32}|$ and (iii) $Im((Y_{d})_{jk}) = 0$, $jk = 23, 32$. We do not assume that the properties which lead to (5)-(7) hold below the SGUT level. Interestingly, the zeroes, the reality of $(Y_{d})_{jk}$, $jk = 23, 32$, and the relation $(Y_{d})_{22} = (Y_{d})_{23}$ still hold to very good accuracy down to $t_{EW}$. We thank L. Lavoura for a helpful comment.

[13] To do significantly better than the assumption of a single $m_{SUSY}$ would require a self-consistent evolution of the full theory down from the SGUT level, which would predict
both the supersymmetric particle mass spectrum and the usual Yukawa mass matrices, while addressing issues such as the \( \mu \) problem, doublet-triplet splitting, flavor-changing neutral operators, etc; inevitably, this would involve more parameters and model-dependence. The effect of using \( m_{SUSY} = 1 \text{ TeV} \) as well as \( m_{SUSY} = m_t = 150 \) GeV has been studied in Ref. [7].

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