Limit behavior of a class of Cantor-integers

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Abstract
In this paper, we study a class of Cantor-integers \( \{C_n\}_{n \geq 1} \) with the base conversion function 
\( f : \{0, \ldots, m\} \rightarrow \{0, \ldots, p\} \) being strictly increasing and satisfying 
\( f(0) = 0 \) and \( f(m) = p \). Firstly we provide an algorithm to compute the superior and inferior of the sequence \( \left\{ \frac{C_n}{n^\alpha} \right\}_{n \geq 1} \)
where \( \alpha = \log_{m+1} p \), and obtain the exact values of the superior and inferior when \( f \) is a class of quadratic function. Secondly we show that the sequence \( \left\{ \frac{C_n}{n^\alpha} \right\}_{n \geq 1} \) is dense in the close interval with the endpoints being its inferior and superior respectively. As a consequence, (i) we get the upper and lower pointwise density \( 1/\alpha \)-density of the self-similar measure supported on \( C \) at 0, where \( C \) is the Cantor set induced by Cantor-integers. (ii) the sequence \( \left\{ \frac{C_n}{n^\alpha} \right\}_{n \geq 1} \) does not have cumulative distribution function but have logarithmic distribution functions (given by a specific Lebesgue integral). Lastly we obtain the Mellin-Perron formula for the summation function of Cantor-integers. In addition, we investigate some analytic properties of the limit function induced by Cantor-integers.

Keywords: Cantor-integers, Pointwise density, Self-similar measure, Mellin-Perron formula

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1. Introduction

Given two positive integer \( m \) and \( p \) with \( m < p \). For any non negative integer \( n \), the 
\( (m+1) \)-ary expansion of \( n = \sum_{j=0}^\alpha \varepsilon_j (m+1) \) is denoted by \( [\varepsilon_k \cdots \varepsilon_0]_{m+1} \) for some \( k \geq 0 \), \( \varepsilon_j \in \{0,1,\ldots,m\} \) and \( \varepsilon_0 > 0 \). The integer sequence \( \{C_n\}_{n \geq 0} \) is called Cantor-integers if there 
exists a non-decreasing function \( f : \{0, \ldots, m\} \rightarrow \{0, \ldots, p\} \) such that the \( f(m) \leq p \) and \( (p+1) \)- 
aray expansion of \( C_n \) is \( [f(\varepsilon_k \cdots \varepsilon_0)]_{p+1} \) whenever \( n = [\varepsilon_k \cdots \varepsilon_0]_{m+1} \). Here \( f \) will be called base 
conversion function and note that \( f \) is not unique.

To avoid triviality we always assume \( f(x) = x \) and \( f(m) = p \) do not hold simultaneously. When \( f(x) = 2x, m = 1 \) and \( p = f(m) = 2 \), we have that \( C_{2n+1} = 3C_n + 2i \) for every \( n \geq 0 \) and 
\( i \in \{0,1\} \). Namely, \( \{C_n\}_{n \geq 0} \) is the only non-negative integers whose ternary expansion contains 
no 1’s (cf. [11], A005823], which is an analogue of Cantor’s triadic set. That’s why it is called 
Cantor-integers. Throughout the paper, set \( \alpha := \log_{m+1}^\alpha. \) It is easy to get that the order of 
growth for \( C_n \) is \( n^\alpha \) with \( \alpha > 1 \). Recently, Liu et al. in [9] introduced quasi-linear sequence 
g(\( n \)), in which the order of growth of quasi-linear sequence \( \{g(n)\} \) must be strictly larger than 
\( \{g(n) - g(n-1)\} \). They showed that

\[
\left\{ \frac{g(n)}{n^\beta} \right\}_{n \geq 1} \text{ is dense in } \left[ \inf_{n \geq 1} \frac{g(n)}{n^\beta}, \sup_{n \geq 1} \frac{g(n)}{n^\beta} \right],
\]

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where $\beta$ is the order of growth of $g(n)$. It is not hard to check that for every Cantor-integers sequence $\{C_n\}$, the order of growth of $\{C_n - C_{n-1}\}$ is also $\alpha$. This implies $C_n$ is not quasi-linear. In fact, as an arithmetic function, $C_n$ has been widely investigated:

- When $f(x) = x$, $m = 1$ and $p \geq f(m) + 1 = 2$, $\{C_n\}_{n \geq 0}$ is the lexicographically earliest increasing sequence of nonnegative numbers that contains no arithmetic progression of length 3 (cf. [1]. A005836). Flajolet et al. in [2, Section 5] considered the Mellin-Perron formulae of the summation function $S(n) := \sum_{k<n} C_k$.

- When $f(x) = 2x$, $m = 1$ and $p = f(m) + 1 = 3$, then $\alpha = 2$. Gawron and Ulas in [8, Section 3] investigated the density property of $\{C_n/n^2\}_{n \geq 1}$ (here they used the notation $\{b_k/k^2\}_{k \geq 1}$). They obtained that

$$\left\{ \frac{C_n}{n^2} \right\}_{n \geq 1} \text{ is contained and dense in } \left\lfloor \frac{2}{3}, 2 \right\rfloor.$$

- When $f(x) = 2x$ and $p \in \{f(m), f(m) + 1\}$, Cao and Li in [4] generalized Gawron and Ulas’s result in [8, Section 3] and reviewed the result about density property in the sense of self-similar measure. In details, they showed that

$$\left\{ \frac{C_n}{n^\alpha} \right\}_{n \geq 1} \text{ is contained and dense in } \left[ \frac{f(m)}{p}, f(1) \right],$$

and

$$\left\{ \frac{x}{\mu_\mathcal{E}([0, x])} : x \in \mathcal{C} \cap \left[ \frac{2}{p+1}, 1 \right] \right\} = \left[ \frac{f(m)}{p}, f(1) \right],$$

where $\mathcal{C}$ and $\mu_\mathcal{C}$ are defined as in (1.3) and (1.4) respectively.

- When $f(x) = qx + r$ with $q \geq 2$ and $r \geq 0$, and $p + 1 \leq q + f(m)$, Cao and Yu in [4] generalized Cao and Li’s results in [3] and they showed that

$$\left\{ \frac{C_n}{n^\alpha} \right\}_{n \geq 1} \text{ is contained and dense in } \left[ \frac{f(m)}{p}, 1 + \frac{f(0)}{p} \right],$$

and

$$\left\{ \frac{x}{\mu_\mathcal{E}([0, x])} : x \in \mathcal{C} \cap \left[ \frac{f(1)}{p+1}, 1 \right] \right\} = \left[ \frac{f(m)}{p}, 1 + \frac{f(0)}{p} \right].$$

Let the homogeneous Cantor set $\mathcal{C} := \mathcal{C}_{f,m,p}$ be the self-similar set generated by the iterated function system (IFS)

$$S_i(x) := \frac{x + f(i)}{1+p}, i = 0, \ldots, m,$$

and $\mu_\mathcal{C}$ be the unique self-similar measure supported on $\mathcal{C}$ satisfying

$$\mu_\mathcal{C} = \sum_{i=0}^{m} \frac{1}{m+1} \mu_\mathcal{C} \circ S_{i}^{-1},$$

Much research has focused on the Cantor set satisfying $S_0(0) = 1$ and $S_m(1) = 1$ (or equivalently $f(0) = 0$ and $f(m) = p$) such as [2] and [10]. In this paper, we are concerned with the case that $f$ is strictly increasing and satisfies $f(0) = 0$ and $f(m) = p$. The first term of $\{C_n\}_{n \geq 0}$ is always zero and will be omitted. For simplicity, write $\Sigma_m := \{0, 1, \ldots, m\}$ and $\Sigma_m^+ := \{1, \ldots, m\}$. First we give an algorithm to compute the superior and inferior of $\left\{ \frac{C_n}{n^\alpha} \right\}_{n \geq 1}$, and obtain the exact values when the base conversion function satisfies (2.6).
Theorem 1. The superior and inferior of \( \{ \frac{C_n}{n^a} \}_{n \geq 1} \) are
\[
\max_{\epsilon \in \Sigma_m^\ell} \frac{C_\ell}{\epsilon^a} \quad \text{and} \quad \min_{1 \leq n < (m+1)^\ell} \frac{C_n + 1}{(n+1)^a}
\]
respectively, where \( \ell_0 \) is the smallest positive integer satisfying
\[
\alpha \frac{(f(m) + 1)^{\ell_0}}{(m + 1)^{\ell_0 + 1}} \geq \max_{\epsilon \in \Sigma_m \setminus \{m\}} \frac{f(m) - f(\epsilon)}{m - \epsilon}.
\]
In particular, when the base conversion function satisfies (2.6), then we have
\[
\inf_{n \geq 1} \frac{C_n}{n^a} = \min_{1 \leq n < (m+1)^2} \frac{C_n + 1}{(n+1)^a} = \begin{cases} 
1, \\
\min \left\{1, \min_{\epsilon \in \Sigma_m} \frac{C_{m+1} + 1}{(m+2)^a} \right\}, \quad \text{if } a \leq 0,
\end{cases} \quad \text{if } a > 0.
\]

Remark 1.1. It should be noticed that \( \ell_0 \) as in Theorem 1 is not optimal. The experiment suggests that \( \ell_0 \leq 2 \) for every strictly increasing \( f \) satisfying \( f(0) = 0 \) and \( f(m) = p \). So far we can not prove this. However, it holds for quadratic \( f \) satisfying (2.6). For the exact values of the superior and inferior with respect to this kind of base conversion function, see Proposition 2.5 and Proposition 2.6 respectively.

Similarly with [4, Theorem 2] and [5, Theorem 1.2], we also have the following density property.

Theorem 2. The sequence \( \{ \frac{C_n}{n^a} \}_{n \geq 1} \) is dense in the close interval \([\inf_{n \geq 1} \frac{C_n}{n^a}, \sup_{n \geq 1} \frac{C_n}{n^a}]\).

Recall that the Hausdorff dimension of \( \mathcal{C} \) is equal to \( 1/\alpha \), and the density of intervals of the form \([0, x]\) on the interval \((0, 1]\) with respect to \( \mu_{\xi} \) is defined by \( d(x) := \frac{\mu_{\xi}[0, x]}{x^{1/\alpha}} \). Note that the lower and upper \( 1/\alpha \)-density of \( \mu_{\xi} \) at 0 are defined by
\[
\Theta^L_{1/\alpha}(\mu_{\xi}, 0) := \liminf_{x \to 0^+} d(x) \quad \text{and} \quad \Theta^U_{1/\alpha}(\mu_{\xi}, 0) := \limsup_{x \to 0^+} d(x)
\]
respectively. For more details of pointwise density of self-similar measure, see [6] and [10]. Let
\[
\Lambda_0 := \{ \ell : \text{there exists a positive and vanishing sequence } \{x_n\}_{n \geq 1} \text{ such that } \lim_{n \to \infty} d(x_n) = \ell \}.
\]
Now we have the following corollary about pointwise density of self-similar measure \( \mu_{\xi} \) at 0.

Corollary 1.1. We have that
\[
\Lambda_0 = \left[ \inf_{n \geq 1} \frac{n}{C_n^{1/\alpha}}, \sup_{n \geq 1} \frac{n}{C_n^{1/\alpha}} \right].
\]
As a consequence,
\[
\Theta^L(\mu_{\xi}, 0) = \left( \sup_{n \geq 1} \frac{C_n}{n^a} \right)^{-1/\alpha} \quad \text{and} \quad \Theta^U(\mu_{\xi}, 0) = \left( \inf_{n \geq 1} \frac{C_n}{n^a} \right)^{-1/\alpha}.
\]

Remark 1.2. Kong, Li and Yao in [10] considered the homogeneous Cantor set generated by the iterated function system
\[
\left\{ S_i(x) = \rho x + \frac{i(1 - \rho)}{m} : i = 0, \ldots, m \right\},
\]
where \( m \geq 1 \) and \( \rho \in (0, 1/(m+1)^2] \). If \( \alpha \geq 2 \), then we can take \( \rho := \frac{1}{f(m)+1} \). This implies that \( f(i) = \frac{f(m)}{m}i \), which is the linear case when \( m \mid f(m) \).
Arithmetic functions related to number representation systems exhibit various periodicity phenomenon. For example, let \( \{ r(n) \}_{n \geq 0} \) be the Rudin-Shapiro sequence. Brillhart, Erdős and Morton in [3] considered the properties of the partial sum \( s(x) := \sum_{k=1}^{[x]} r(k) \) and \( t(x) := \sum_{k=1}^{[x]} (-1)^k r(k) \), and introduced the limit functions
\[
\tau(x) := \lim_{k \to \infty} \frac{s([4^k x])}{\sqrt{4^k x}} \quad \text{and} \quad \mu(x) := \lim_{k \to \infty} \frac{t([4^k x])}{\sqrt{4^k x}},
\]
which are defined for \( x > 0 \). Note that \( \tau(4x) = \tau(x) \) and \( \mu(4x) = \mu(x) \). They showed that \( \tau(x) \) and \( \mu(x) \) are continuous at every \( x > 0 \), but are non-differentiable almost everywhere, and they also estimated the decay of coefficients of logarithmic Fourier series of \( \tau(x) \) and \( \mu(x) \).

For the Cantor-integers sequence, write \( C_x := C_{|x|} \) for every \( x \in (0, \infty) \). Define a function \( \lambda : (0, \infty) \to (0, \infty) \) defined by
\[
\lambda(x) := \lim_{k \to \infty} \frac{C(m+1)^k x}{(m+1)^k x^\alpha}.
\]
It follows from (1.2) that \( \lambda(x) \) has a logarithmic Fourier series expansion. For the other analytic properties of \( \lambda(x) \), see Proposition 1.1 Proposition 1.2 and Proposition 1.3.

Let the cumulative and logarithmic distribution function of the sequence \( \{ C_{n^a} \}_{n \geq 1} \) be defined as in Section 7 and Section 8 in [3] respectively. Using Theorem 2 and applying similar arguments of Theorem 1.4 and Theorem 1.5 in [3], we have the following two corollaries.

**Corollary 1.2.** The cumulative distribution function of the sequence \( \{ C_{n^a} \}_{n \geq 1} \) does not exists for any point of \( [\inf_{n \geq 1} C_{n^a}, \sup_{n \geq 1} C_{n^a}] \).

**Corollary 1.3.** If \( \gamma \in [\inf_{n \geq 1} C_{n^a}, \sup_{n \geq 1} C_{n^a}] \), then the logarithmic distribution function of the sequence \( \{ C_{n^a} \}_{n \geq 1} \) exists at \( \gamma \), and has the value
\[
L(\gamma) = \frac{1}{\log(m+1)} \int_{E_\gamma} \frac{1}{x} \, dx,
\]
where \( E_\gamma := \{ x \in [1, m+1] : \lambda(x) \leq \gamma \} \).

Similarly with [3, Section 5], we have the Mellin-Perron formula for the summation function of Cantor-integers. Let \( \zeta(s, a) \) and \( \zeta(s) \) be the well-known Hurwitz zeta function and Riemann zeta function respectively, i.e.,
\[
\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad \text{and} \quad \zeta(s) := \zeta(s, 1).
\]
For simplicity, write \( \Delta f(r) := f(r) - f(r-1) \) for \( r \in \mathbb{Z}^+ \).

**Theorem 3.** For the summation function \( S(n) := \sum_{1 \leq k < n} C_n \), we have
\[
S(n) = n^{a+1} F(\log_{m+1}^n) - n^2 \frac{f(m)}{f(m) - m} - \frac{\sum_{r=1}^{m} f(r)}{f(m)(m+1)} - \frac{f(m)(m+1) - \sum_{r=1}^{m} f(r)}{f(m)(m+1) + m},
\]
where \( F(u) \) is the Fourier series
\[
F(u) = \frac{1}{(f(m) + 1) \log(m+1)} \sum_{k \in \mathbb{Z}} \sum_{r=1}^{m} \Delta f(r) \zeta(\gamma_k, \frac{r}{m+1}) - \frac{f(m) \zeta(\gamma_k)}{\gamma_k} e^{2\pi i ku},
\]
with \( \gamma_k := \alpha + \frac{2\pi ki}{\log(m+1)} \).

This paper is organized as follows. In Section 2, we prove Theorem 1. In Section 3, we prove Theorem 2 and Corollary 1.1. In Section 4, we prove some some analytic properties of the limit function induced by Cantor-integers. In Section 5, we prove Theorem 3.
2. Proof of Theorem [1]

With a little abuse of notation, write \( m^\ell := m \cdots m \) for every \( m \geq 1 \). Given any finite word \( u \), let \( |u| \) denote the length of \( u \). We have the following observation for Cantor-integers. For simplicity, here we assume \( n = [\epsilon_\ell \epsilon_{\ell-1} \cdots \epsilon_0]_{m+1} \) with \( \epsilon_i \in \Sigma_m \) for some \( 0 \leq i \leq \ell \). For every integer \( n \geq 1 \) and \( r \in \Sigma_m \), we have

\[
C_{[\epsilon_\ell \cdots \epsilon_0]}_{m+1} = (f(m) + 1)C_{[\epsilon_\ell \cdots \epsilon_0]}_{m+1} + f(r).
\]

In particular, for every integer \( k \geq 1 \), we have

\[
C_{[\epsilon_\ell \cdots \epsilon_0]}_{m+1} = (f(m) + 1)^k C_{[\epsilon_\ell \cdots \epsilon_0]}_{m+1}.
\]

First we give some useful propositions about Cantor-integers for general base conversion function.

Proposition 2.1. For every \( k \geq 2 \) and \( \epsilon_\ell \cdots \epsilon_1 \in \Sigma_m^+ \times \Sigma_m^{k-1} \), we have

\[
\frac{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}}{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}} \geq \frac{f(m) + m + 1}{2m + 1}.
\]

Proof. Firstly we claim that for every \( k \geq 2 \) and \( \epsilon_\ell \cdots \epsilon_1 \in \Sigma_m^+ \times \Sigma_m^{k-1} \), we have

\[
\frac{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}}{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}} \geq \frac{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}}{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}}.
\]

In fact, if \( k \geq 3 \), we have

\[
\frac{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}}{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}} = \frac{(f(m) + 1)C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1} + f(\epsilon_1)}{(m + 1)C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1} + \epsilon_1} \geq \frac{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}}{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}}.
\]

The last inequality is deduced using the fact that \( f(m) > m \). Applying the above inequality \((k - 2)\) times, we have that the claim holds. Meanwhile, note that

\[
\frac{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}}{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}} = \frac{(f(m) + 1)f(\epsilon_k) + f(\epsilon_{k-1})}{(m + 1)\epsilon_k + \epsilon_{k-1}} \geq \frac{(f(m) + 1)\epsilon_k + \epsilon_{k-1}}{(m + 1)\epsilon_k + \epsilon_{k-1}} = \frac{f(m) + 1}{2m + 1}.
\]

This completes the proof.

Proposition 2.2. For every \( n \geq m + 1 \), we have

\[
\frac{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}}{C_{[\epsilon_\ell \cdots \epsilon_1]}_{m+1}} \leq \frac{C_n}{n^\alpha}.
\]
Proof. Note that
\[
\frac{C_{(m+1)n+m}}{((m+1)n+m)^\alpha} = \frac{(f(m)+1)C_n + f(m)}{((m+1)n+m)^\alpha} = \frac{C_n + \frac{f(m)}{f(m)+1}}{(n + \frac{m}{m+1})^\alpha}.
\]
By Bernoulli inequality, it suffices to show that
\[
\frac{\alpha C_n}{n} \geq \frac{f(m)(m+1)}{(f(m)+1)m}.
\]
Following from Proposition 2.1, we have
\[
\frac{C_n}{n} \geq \frac{f(m) + 1 + m^2 + 1}{2m+1}.
\]
By the fact that \(f(m) \geq m + 1\), we have
\[
\frac{f(m) + 1 + m^2 + 1}{2m+1} \geq \frac{f(m)(m+1)}{(f(m)+1)m}.
\]
This completes the proof.

Similarly, we have the following results.

**Proposition 2.3.** There exists \(k_0 \in \mathbb{N}\) such that for every \(n \geq (m+1)^{k_0}\) and every \(\epsilon \in \{1, \cdots, m\}\), we have
\[
\frac{C_{(m+1)n+\epsilon}}{((m+1)n+\epsilon)^\alpha} \leq \frac{C_n}{n^\alpha}.
\]

**Proof.** As in the proof of 2.3, it suffices to show that there exists \(n_0 \in \mathbb{N}\) such that for every \(n \geq n_0\) and every \(\epsilon \in \{1, \cdots, m\}\),
\[
\frac{\alpha C_n}{n} \geq \frac{f(\epsilon)(m+1)}{\epsilon(f(m)+1)},
\]
which follows from the fact that \(\frac{C_n}{n} \to \infty\) when \(n \to \infty\).

**Proposition 2.4.** There exists \(k_1 \in \mathbb{N}\) such that for every \(n \geq (m+1)^{k_1}\), we have
\[
\frac{C_{(m+1)n}}{((m+1)n)^\alpha} > \frac{C_{(m+1)n+1}}{((m+1)n+1)^\alpha} > \cdots > \frac{C_{(m+1)n+m}}{((m+1)n+m)^\alpha}.
\]

**Proof.** Given any \(n = [\epsilon\ell \cdots \epsilon_1]_{m+1}\) for some \(\ell \geq 1\), we consider the function \(T_\ell : [0, m] \to \mathbb{R}\) defined by
\[
T_\ell(x) := \frac{C_{[\epsilon\ell \cdots \epsilon_1]_{m+1}}}{([\epsilon\ell \cdots \epsilon_1]_{m+1})^\alpha} = \frac{(f(m) + 1)C_{[\epsilon\ell \cdots \epsilon_1]_{m+1}} + f(x)}{((m+1)[\epsilon\ell \cdots \epsilon_1]_{m+1} + x)^\alpha}.
\]
Note that \(\partial T_\ell(x)/\partial x < 0\) if and only if
\[
f'(x)((m+1)[\epsilon\ell \cdots \epsilon_1]_{m+1} + x) - \alpha ( (f(m) + 1)C_{[\epsilon\ell \cdots \epsilon_1]_{m+1}} + f(x)) < 0.
\]
In fact, following from that \(C_{[\epsilon\ell \cdots \epsilon_1]_{m+1}}/([\epsilon\ell \cdots \epsilon_1]_{m+1}) \to \infty(n \to \infty)\), if \(\ell\) is sufficiently large, then for every \(x \in [0, m]\), \(\partial T_\ell(x)/\partial x < 0\). This completes the proof.

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2.1. The superior of \( \{ \frac{C_n}{m^n} : n \geq 1 \} \)

For simplicity, set \( M := \max_{\epsilon \in \Sigma_m} \frac{C_n}{\epsilon^m} \). We only need to show that for every \( k \geq 0 \),

\[
\sup_{(m+1)^k \leq n < (m+1)^{k+1}} \frac{C_n}{n^k} = M.
\]

We will prove this by induction. If \( k = 0 \), it holds trivially. Assume the desired result holds for \( k \geq 0 \). By \( 2.1 \), it suffices to consider \( C_{(m+1)n+i}/((m+1)n+i)^\alpha \) for every \( 1 \leq i \leq m \) and \((m+1)^k \leq n < (m+1)^{k+1} \). In fact, note that

\[
C_{(m+1)n+i} - M((m+1)n+i)^\alpha = (f(m)+1)C_n + f(i) - M((m+1)n+i)^\alpha \\
\leq (m+1)^\alpha Mn^\alpha + Mi^\alpha - M((m+1)n+i)^\alpha \\
= M((m+1)n)^\alpha \left( 1 + \left( \frac{i}{(m+1)n} \right)^\alpha - \left( 1 + \frac{i}{(m+1)n} \right)^\alpha \right) \leq 0.
\]

The last inequality can be deduced from Bernoulli inequality. This completes the proof of the superior as in Theorem 1.

It should be noticed that when \( f \) satisfies \( 2.3 \), we have the following result.

**Proposition 2.5.** Define \( T(x) := (2 - \alpha)ax - b(\alpha - 1) \). Then we have that

\[
\max_{\epsilon \in \Sigma_m^+} \frac{C_\epsilon}{\epsilon^\alpha} = \begin{cases} 
(f(1), & \text{if } a \leq 0 \text{ or } a = 1, b = 1, \\
\frac{f(m)}{m^\alpha}, & \text{if } a = 2, b = -1, m = 2 \text{ or } a = 1, b = 0, \\
(f(1), & \text{if } a > 0, \alpha \geq 2 \text{ and } b \geq 0, \\
\frac{f(m)}{m^\alpha}, & \text{if } a > 0, \alpha \geq 2, b < 0 \text{ and } T(m) \geq 0, \\
(f(1), & \text{if } a > 0, \alpha \geq 2, b < 0 \text{ and } T(m) < 0 < T(1), \\
\max \left\{ \frac{f(\xi_m)}{\xi_m^\alpha}, \frac{f(\xi_m+1)}{(\xi_m+1)^\alpha} \right\}, & \text{if } a > 0, \alpha \geq 2, b < 0 \text{ and } T(m) < 0 < T(1).
\end{cases}
\]

where \( \xi_m := \left\lfloor -\frac{b(\alpha-1)}{a(\alpha-2)} \right\rfloor \).

**Proof.** If \( a > 0, \alpha \geq 2 \) and \( b < 0 \), the result follows from the fact that \( T(m) \leq T(1) \). The other cases can be verified directly. \( \square \)

2.2. The inferior of \( \{ \frac{C_n}{m^n} : n \geq 1 \} \) for general \( f \)

It follows from \( 2.3 \) that for every finite word \( u \in \Sigma_m^{|u|} \) with \( |u| \geq 2 \) and the first digit being positive, we have

\[
\frac{C_{|u|} m_{+1}}{|u| m_{+1}} \geq \frac{C_{|um|} m_{+1}}{|um| m_{+1}} \geq \frac{C_{|umm|} m_{+1}}{|umm| m_{+1}} \geq \cdots \geq \lim_{t \to \infty} \frac{C_{|um^t|} m_{+1}}{|um^t| m_{+1}} = \frac{C_{|u|} m_{+1} + 1}{(|u| m_{+1} + 1)^\alpha}.
\]

By Proposition 2.3 and Proposition 2.4, the inferior must be of the form \( \frac{C_{|u|} m_{+1} + 1}{(|u| m_{+1} + 1)^\alpha} \) for some \( u \). It suffices to show that if \( |u| \geq \ell_0 \) with \( \ell_0 \) defined as in Theorem 1, then we have

\[
\min_{\epsilon \in \Sigma_m} \frac{C_{|u\epsilon|} m_{+1} + 1}{(|u\epsilon| m_{+1} + 1)^\alpha} = \frac{C_{|um|} m_{+1} + 1}{(|um| m_{+1} + 1)^\alpha} = \frac{C_{|u|} m_{+1} + 1}{(|u| m_{+1} + 1)^\alpha}.
\]
Note that, following from the fact that \( u \in \varepsilon \), we only need to show that for every finite word \( u \) with \( |u| \geq \ell_0 \) and every \( \epsilon \in \Sigma_m \setminus \{m\} \),
\[
\frac{\alpha ((f(m) + 1)C_u|_{m+1} + f(\epsilon) + 1)}{(m + 1)|u|_{m+1} + \epsilon + 1} \geq \frac{f(m) - f(\epsilon)}{m - \epsilon}.
\]
In fact, if \( |u| = k \geq \ell_0 \), then we have
\[
\frac{\alpha ((f(m) + 1)C_u|_{m+1} + f(\epsilon) + 1)}{(m + 1)|u|_{m+1} + \epsilon + 1} \geq \frac{\alpha (f(m) + 1)C_u|_{m+1}}{(m + 1)(|u|_{m+1} + 1)} \geq \frac{\alpha (f(m) + 1)k}{(m + 1)k + 1} \geq \max_{\epsilon \in \Sigma_m \setminus \{m\}} \frac{f(m) - f(\epsilon)}{m - \epsilon} \geq \frac{f(m) - f(\epsilon)}{m - \epsilon}.
\]
This completes the proof of the inferior for general \( f \).

2.3. The inferior of \( \{\frac{C_n}{n^2} : n \geq 1\} \) for a class of quadratic \( f \)

In this subsection, we will investigate the sequence \( \{\frac{C_n}{n^2} : n \geq 1\} \) with the corresponding base conversion function \( f \) being quadratic with zero constant term. i.e., \( f(x) = ax^2 + bx \) with \( a, b \in \mathbb{Z} \) and \( f'(x) = 2ax + b > 0 \) for every \( x \in [1, m] \).

If \( a = 0 \) or \( m = 1 \), then \( f \) can be replaced with a linear function. Following from (1.1) or (2.3), we have \( \inf_{n \geq 1} \frac{C_n}{n^2} = 1 \). Without loss of generality we always assume \( a \neq 0 \) and \( m \geq 2 \). There are some useful facts for this kind of base conversion function:

- We have that \( f(1) = a + b \geq 1 \).
- When \( 1 < a < 2 \), we have that \( -am + 2 \leq b \leq (1 - a)m + 1 \) since \( m < f(m) < m^2 + 2m \).
- In details, if \( a < 0 \), following from (2.6), we have that \( a = -1 \) and \( b \in \{2m, 2m + 1\} \);
- Similarly, if \( a > 0 \), we have that either \( a = 2, b = -1, m = 2 \) or \( a = 1, b \in \{0, 1\} \).

- It is not hard to get that \( f(m) \geq m^2 \) and \( \alpha \geq \frac{\log 5}{\log 3} \).

Lemma 2.1. For every \( \epsilon_2 \epsilon_1 \in \Sigma_m^+ \times \Sigma_m \), we have
\[
\inf_{\ell \geq 0} \left\{ \frac{C_{\epsilon_2 \epsilon_1 \ell}C_{|u|_{m+1}}}{|\epsilon_2 \epsilon_1 \ell|_{m+1} + \alpha} : v \in \Sigma_m^\ell \right\} = \frac{C_{|\epsilon_2 \epsilon_1 \ell|_{m+1}}}{|\epsilon_2 \epsilon_1 \ell|_{m+1} + \alpha}.
\]

Proof. As in the proof in Section 2.2 we only need to show that for every finite word \( u \) with \( |u| \geq 2 \) and every \( \epsilon \in \Sigma_m \setminus \{m\} \),
\[
\frac{\alpha ((f(m) + 1)C_u|_{m+1} + f(\epsilon) + 1)}{(m + 1)|u|_{m+1} + \epsilon + 1} \geq \frac{f(m) - f(\epsilon)}{m - \epsilon} = a(m + \epsilon) + b. \tag{2.7}
\]
Note that, following from the fact that \( |u|_{m+1} \geq m + 1 \) and Proposition 2.1, we have
\[
\frac{\alpha ((f(m) + 1)C_u|_{m+1} + f(\epsilon) + 1)}{(m + 1)|u|_{m+1} + \epsilon + 1} \geq \frac{\alpha (m + 1)(f(m) + 1)C_u|_{m+1}}{m + 2 - (m + 1)|u|_{m+1}}.
\]
Hence (2.7) can be deduced from
\[
\frac{\alpha(f(m) + 1)(f(m) + m + 1)}{(m + 2)(2m + 1)} \geq a(m + \epsilon) + b.
\] (2.8)

In fact, (2.7) always holds for \( \epsilon = 0 \), which follows from (2.4) and that fact that \( \frac{\alpha(m+1)}{m+2} \geq 1 \). It remains to consider the case \( \epsilon \in \Sigma_m \setminus \{0, m\} \). We distinguish two cases according to the sign of \( a \).

Case 1: \( a < 0 \). Then \( a(m + \epsilon) + b \leq a(m + 1) + b \leq \frac{f(m)}{m} \). Therefore by (2.4), the desired inequality (2.8) holds.

Subcase 2(1): \( a < 0 \). Then \( a(m + \epsilon) + b \leq \frac{2f(m)}{m} - f(1) \).

Subcase 2(2): \( a \geq 2 \). Then \( \alpha(7C_{[u]3} + 2) \geq 5 \).

In fact, the above inequality follows from the fact that \( C_{[u]3} \geq \frac{2}{3}[u]3 \) by Proposition 2.1.

Now we investigate the case \( a = 1, b = 1 \). Then \( f(m) = m^2 + m \) and \( \alpha = \frac{\log(m^2 + m + 1)}{\log(m+1)} > 1.7 \). Hence that
\[
\frac{\alpha(f(m) + 1)(f(m) + m + 1)}{(m + 2)(2m + 1)} = \frac{\alpha(m^2 + m + 1)(m^2 + 2m + 1)}{(m + 2)(2m + 1)}
\]
\[
\geq \frac{\alpha(m^2 + m + 1)m}{2m + 1}
\]
\[
\geq 2m = \frac{2f(m)}{m} - f(1).
\]

At last, we focus on the case \( a = 1, b = 0 \). If \( m = 2 \), then \( \alpha = \frac{5}{\log 3} \), and it suffices to check that
\[
\frac{\alpha(5C_{[u]3} + 2)}{3[u]3 + 2} \geq 3.
\]

Following from Proposition 2.1 \( C_{[u]3} \geq \frac{2}{3}[u]3 \). This implies the above inequality holds.

Now we assume \( m \geq 3 \). Recall that \( \frac{2f(m)}{m} - f(1) = 2m - 1 \). Then we have
\[
\frac{\alpha(f(m) + 1)(f(m) + m + 1)}{(m + 2)(2m + 1)} = \frac{\alpha(m^2 + 1)(m^2 + m + 1)}{(m + 2)(2m + 1)} \geq 2m - 1,
\]
where the last inequality follows from the monotonicity of \( \frac{\alpha(m^2 + 1)(m^2 + m + 1)}{(m + 2)(2m + 1)} - 2m + 1 \) with respect to \( m \geq 3 \). This completes the proof.
Set $\delta_a := 1$ if $a > 0$ otherwise $\delta_a := 0$. We have the following lemma.

**Lemma 2.2.** If $\epsilon_2 \in \{1 + \delta_a, \cdots, m\}$, then we have

$$
\min_{\epsilon \in \Sigma_m} \frac{C_{|\epsilon_2\epsilon_{m+1}|+1}}{\epsilon_{m+1} + 1)\epsilon_{2} + \epsilon + 1} = \frac{f(\epsilon_2) + 1}{(\epsilon_2 + 1)^a}.
$$

**Proof.** Fixed $\epsilon_2 \in \{1 + \delta_a, \cdots, m\}$. Like the proof of Lemma 2.1 it suffices to show that for every $\epsilon \in \Sigma_m \setminus \{m\}$,

$$
\frac{\alpha ((f(m)+1)e_2 + f(\epsilon+1)}{(m+1)e_2 + \epsilon + 1} \geq \frac{f(m) - f(\epsilon)}{m - \epsilon} = a(m + \epsilon) + b. \tag{2.9}
$$

**Case 1:** $a < 0$. Then $a(m + \epsilon) + b \leq am + b = \frac{f(m)}{m}$. Note that $b \geq -2am$ if $a < -1$ otherwise $b \geq 2m$. Hence $f(\epsilon)/\epsilon_2 \geq ma + b \geq m$ for every $\epsilon_2 \geq 1$. Hence that

$$
\frac{\alpha ((f(m)+1)e_2 + f(\epsilon+1)}{(m+1)e_2 + \epsilon + 1} \geq \frac{\alpha ((f(m)+1)me_2 + \epsilon + 1)}{(m+1)e_2 + \epsilon + 1} = \frac{\alpha ((f(m)+1)m + (\epsilon + 1)/e_2)}{m + 1 + (\epsilon + 1)/e_2} \geq \frac{\alpha((f(m)+1)m + m)}{2m + 1} \geq \frac{f(m)}{m} \geq a(m + \epsilon) + b.
$$

**Case 2:** $a > 0$. Then $a(m + \epsilon) + b \leq a(2m - 1) + b = \frac{2f(m)}{m} - f(1)$ and $f(\epsilon)/\epsilon_2 \geq 2a + b$ for every $\epsilon_2 \geq 2$. To prove (2.9), we only need to show that

$$
\frac{\alpha((2a + b)(f(m)+1)m + m/2)}{3m/2 + 1} \geq \frac{2f(m)}{m} - f(1). \tag{2.10}
$$

This follows from that

$$
\frac{\alpha ((f(m)+1)e_2 + f(\epsilon+1)}{(m+1)e_2 + \epsilon + 1} \geq \frac{\alpha ((2a + b)(f(m)+1)e_2 + \epsilon + 1)}{(m+1)e_2 + \epsilon + 1} = \frac{\alpha ((2a + b)(f(m)+1) + (\epsilon + 1)/e_2)}{m + 1 + (\epsilon + 1)/e_2} \geq \frac{\alpha((2a + b)(f(m)+1)m + m/2)}{3m/2 + 1}.
$$

**Subcase 2(1):** $\alpha \geq 2$. Then (2.10) follows from the fact that $2a + b \geq 2$.

**Subcase 2(2):** $\alpha < 2$. First we consider the case $a = 2, b = -1, m = 2$ or $a = 1, b = 1$. Then $2a + b = 3$ and $\alpha \geq \frac{\log 7}{\log 3}$. This implies (2.10) holds. Now for the case $a = 1, b = 0$. Then (2.9) is equivalent to

$$
\alpha((m^2 + 1)e_2^2 + \epsilon^2 + 1) - (m + \epsilon)((m+1)e_2 + \epsilon + 1) \geq 0.
$$
Note that the left-hand side of the above inequality is decreasing with respect to \( \epsilon \in [0, m - 1] \). Hence that
\[
\alpha((m^2 + 1)\epsilon^2 + \epsilon^2 + 1) - (m + \epsilon)((m + 1)\epsilon_2 + \epsilon + 1) \\
\geq \alpha((m^2 + 1)\epsilon_2^2 + (m - 1)^2 + 1) - (2m - 1)((m + 1)\epsilon_2 + m) \\
\geq \alpha(4(m^2 + 1) + (m - 1)^2 + 1) - (2m - 1)(2(m + 1) + m) \geq 0.
\]
This completes the proof. \( \square \)

**Lemma 2.3.** We have
\[
\min_{x \in \{1, \ldots, m\}} \frac{f(x) + 1}{(x + 1)^\alpha} = \begin{cases} 
1, & \text{if } a < 0, \\
\min \left\{ 1, \frac{a + b + 1}{2^a} \right\}, & \text{if } a > 0.
\end{cases}
\]

**Proof.** Consider the function \( F(x) := \frac{f(x) + 1}{(x + 1)^\alpha} \) defined on \([1, m]\). It is easy to get that for each \( x \in [1, m] \), \( F'(x) \leq 0 \) if and only if \( f'(x)(x+1) - \alpha(f(x)+1) \leq 0 \). Set \( \tilde{F}(x) := f'(x)(x+1) - \alpha(f(x)+1) \). Note that \( \tilde{F}'(x) = 2a(x+1) - (\alpha - 1)f'(x) \).

**Case 1:** \( a < 0 \). Then \( \tilde{F}'(x) \leq 0 \) for each \( x \in [1, m] \). We will only need to show that \( \tilde{F}(1) \leq 0 \).

Since if this holds, then we have \( \tilde{F}(x) \leq 0 \) for each \( x \in [1, m] \), which implies that
\[
\min_{x \in \{1, \ldots, m\}} \frac{f(x) + 1}{(x + 1)^\alpha} = 1.
\]

In fact, we have
\[
\tilde{F}(1) = 4a + 2b - \alpha(a + b + 1) \leq \begin{cases} 
2a - 2 \leq 0, & \text{if } \alpha \geq 2, \\
\frac{4m^2}{(m^2 + 1)\log(m+1)} - 4 \leq 0, & \text{if } a = -1, b = 2m, \\
\frac{m(2m+1)}{(m^2 + 1)\log(m+1)} - 4 \leq 0, & \text{if } a = -1, b = 2m + 1.
\end{cases}
\]

**Case 2:** \( a > 0 \).

**Subcase 2(1):** \( 1 < \alpha < 2 \). In other words, \( a = 2, b = -1, m = 2 \) or \( a = 1, b \in \{0, 1\} \). We have that \( \tilde{F}'(x) \geq 0 \) for each \( x \in [1, m] \). Note that \( \tilde{F}(1) = 4a + 2b - \alpha(a + b + 1) \geq 0 \). This implies \( \tilde{F}(x) \geq 0 \) for each \( x \in [1, m] \), Hence that
\[
\min_{x \in \{1, \ldots, m\}} \frac{f(x) + 1}{(x + 1)^\alpha} = \frac{a + b + 1}{2^a}.
\]

**Subcase 2(2):** \( \alpha \geq 2 \). Note that
\[
\tilde{F}(x) = f'(x)(x+1) - \alpha(f(x)+1) = (2 - \alpha)ax^2 + (2a + b - \alpha b)x + b - \alpha.
\]

Hence if \( \tilde{F}(1) \geq 0 \), namely \( \alpha \leq \frac{4a + 2b}{a + b + 1} \), then we have
\[
\min_{x \in [1, m]} F(x) = \min\{F(1), F(m)\} = \min \left\{ 1, \frac{a + b + 1}{2^a} \right\}.
\]
Now it remains to consider the case $\alpha > \frac{4a+2b}{a+b+1}$ (or equivalently $\tilde{F}(1) < 0$). Note that $\tilde{F}'(x) \leq \tilde{F}'(1)$ for each $x \in [1, m]$. We only need to show that $\tilde{F}'(1) \leq 0$. Namely, $\alpha > \frac{6a+b}{2a+b}$. If this inequality holds, then we have $\tilde{F}(x) \leq 0$ for each $x \in [1, m]$. Hence that

$$\min_{x \in \{1, \ldots, m\}} \frac{f(x) + 1}{(x + 1)^\alpha} = 1.$$

In fact, we have $\frac{4a+2b}{a+b+1} < \frac{6a+b}{2a+b}$ if and only if $2a^2 + ab - 6a - b + b^2 < 0$ if and only if one of the following cases occurs: (i) $a = 1, b \in \{0, 1\}$; (ii) $a = 2, b \in \{-1, 0, 1\}$; (iii) $a = 3, b = -1$. For each case, it is not hard to get that $\alpha < \frac{4a+2b}{a+b+1}$ which contradicts the assumption. It follows that $\alpha > \frac{6a+b}{2a+b}$. This completes the proof. \hfill \Box

Proof of the inferior in Theorem 4 for $f$ satisfying (2.6). It follows from Lemma 2.1, Lemma 2.2 and Lemma 2.3 directly. \hfill \Box

When $f$ satisfies (2.6), it should be noticed that the exact values of the inferior in Theorem 4 depend on the values of $a, b$ and $m$. In details, we will give the following result.

**Proposition 2.6.** Given any $a > 0$ and define

$$T(x) := (2ax + b)(x + m + 2) - \alpha ((f(m) + 1)f(1) + f(x) + 1).$$

Then we have that

$$\min_{\epsilon \in \Sigma_m} \frac{C_{m+\epsilon+1} + 1}{(m + \epsilon + 2)\alpha} = \begin{cases} \frac{a+b+1}{2a}, & \text{if } T(m) \leq 0, \\ \min \left\{ \frac{C_{m+\xi_m+1} + 1}{(m + \xi_m + 2)\alpha}, \frac{C_{m+\xi_m+2} + 1}{(m + \xi_m + 3)\alpha} \right\}, & \text{otherwise.} \end{cases}$$

where $\xi_m := \min\{x \in \Sigma_m : T(x) \leq 0 \text{ but } T(x+1) > 0\}$.

Proof. Consider the function $S(x) := \frac{C_{m+\epsilon+1} + 1}{(m + \epsilon + 2)\alpha}$. Then we can get that $S'(x) \leq 0$ if and only if $T(x) \leq 0$. Rewriting the function $T$ by

$$T(x) := (2 - \alpha)ax^2 + (2a(m+2) - b(\alpha - 1))x + b(m+2) - \alpha((f(m) + 1)f(1) + 1).$$

Clearly $T(0) \leq 0$. It suffices to investigate the case that $T(m) \leq 0$ and $\alpha > 2$. Since if $T(m) > 0$ or $\alpha \leq 2$, then this proposition holds directly. Note that

$$T'(x) := 2ax(2 - \alpha) + 2a(m+2) - b(\alpha - 1).$$

Without loss of generality we can assume that $T'(0) \geq 0$ and $T'(m) \leq 0$ (otherwise this proposition holds directly). This implies that $\alpha \geq 3$ if $b < 0$. Now we claim that $T(x) \leq 0$ for every $x \in R$. We only need to show that the discriminant $\Delta$ of $T$ is non-positive. In fact,

$$\Delta := \frac{\Delta}{2a(\alpha - 2)} \geq \frac{(2a(m+2) - b(\alpha - 1))^2/(2a(\alpha - 2)) - 2\alpha((f(m) + 1)f(1) + 1) + 2b(m+2)}{m(2am + 4a - ba + b) - 2\alpha(f(m) + 2) + 2b(m+2) \geq 0 \text{ (By } T'(0) \geq 0 \text{ and } T'(m) \leq 0)} \leq \begin{cases} m(2am + 4a - b) - 4(am^2 + bm + 2) + 2b(m+2) \leq 0, & \text{if } b \geq 0, \\ 2a(3 - \alpha) + 10(1 - \alpha) \leq 0, & \text{if } b < 0. \end{cases}$$

The last inequalities for the case $b < 0$ can be deduced from the fact that $\alpha \geq 3$ and the monotonicity of $m(2am + 4a - ba + b) - 2\alpha(f(m) + 2) + 2b(m+2)$ with respect to $b(\geq 1 - \alpha)$ and $m(\geq 2)$. This completes the proof. \hfill \Box
3. Proof of Theorem 2 and Corollary 1.1

Proof of Theorem 2. For simplicity, set \( \tilde{C}_n := \frac{C_n}{m} \) for every \( n \geq 1 \). By (2.1) and the superior of \( \{\tilde{C}_n\}_{n \geq 1} \), we have

\[
\limsup_{n \to \infty} \tilde{C}_n = \sup_{n \geq 1} \tilde{C}_n.
\]

Following from (2.3), we have that

\[
\lim_{k \to \infty} \tilde{C}_n = \inf_{n \geq 1} \tilde{C}_n
\]

and the limit \( \lim_{k \to \infty} \tilde{C}_{[um^k]_{m+1}} \) exists for each \( u \in \cup_{\ell \geq 0} \Sigma^r_m \).

Set \( E := \{\lim_{k \to \infty} \tilde{C}_{[um^k]_{m+1}} : u \in \cup_{\ell \geq 0} \Sigma^r_m\} \). Fixed any \( \gamma \in (\inf_{n \geq 1} \tilde{C}_n, \sup_{n \geq 1} \tilde{C}_n) \setminus E \), we will construct a subsequence \( \{\tilde{C}_m\}_{k \geq 1} \) of \( \{\tilde{C}_n\}_{n \geq 1} \) converging to \( \gamma \). Let \( k_0 \) and \( k_1 \) be defined as Proposition 2.3 and Proposition 2.4 respectively. Set \( k_2 := \max\{k_0, k_1\} \) and \( L(\gamma) := \{\tilde{C}_\xi : \xi \in \{1, \cdots, m\}, \tilde{C}_\xi \geq \gamma\} \). Now we take

\[
n_1 := \min\{\epsilon \in \{1, \cdots, m\} : \tilde{C}_\epsilon = \min L(\gamma)\},
\]

and \( n_k = (m+1)^{k-1}n_1 \) for \( 2 \leq k \leq k_2 \). Assume that \( n_k(k \geq k_2) \) has been chosen. We will choose \( n_{k+1} \) recursively as follows:

\[
n_{k+1} = (m+1)n_k + \max\{\epsilon \in \{0,1,\cdots, m\} : \tilde{C}_{(m+1)n_k+\epsilon} \geq \gamma\}.
\]

Then we have that for every \( k \geq 1 \),

\[
\tilde{C}_{n_{k+1}} \leq \tilde{C}_{n_k}.
\]

This implies that \( \lim_{k \to \infty} \tilde{C}_{n_k} \) exists and is larger than \( \gamma \). Now it suffices to show that

\[
\lim_{k \to \infty} \tilde{C}_{n_k} \leq \gamma.
\]

Define the set

\[
S := \{k \in \mathbb{N} : n_k \neq m \mod (m+1)\}.
\]

Note that \( S \) is non-empty. We claim that \( S \) is infinite. Otherwise if it is finite, then there exists \( k^* \) such that for every \( k \geq k^* \), \( n_{k+1} = (m+1)n_k + m \). This contradicts with the value of \( \gamma \). By Proposition 2.4, we have that for every \( k \geq k_2 \),

\[
\tilde{C}_{(m+1)n_k} > \tilde{C}_{(m+1)n_k+1} > \cdots > \tilde{C}_{(m+1)n_k+m}.
\]

Hence that there exist infinite \( k \geq k_2 \) such that \( \tilde{C}_{(m+1)n_k+m} < \gamma \). i.e.,

\[
\frac{(f(m)+1)C_{n_k} + f(m)}{(m+1)n_k + m)^\alpha} < \gamma.
\]

Therefore, we have

\[
\lim_{k \to \infty} \tilde{C}_{n_k} = \lim_{k \to \infty} \frac{C_{n_k}}{n_k^\alpha} = \lim_{k \to \infty} \frac{(f(m)+1)C_{n_k} + f(m)}{(m+1)n_k + m)^\alpha} \leq \gamma.
\]

This completes the proof. \( \square \)
Now we can prove Corollary 1.1.

**Proof of Corollary 1.1.** Recall that \( d(x) := \frac{e(x)}{x^{1/\alpha}} \). Note that \( S_0(0) = 0 \) and \( S_m(1) = 1 \). By blow-up principle in [2, Lemma 2.1], we have

\[
\{ d(x) : x \in (0,1] \} = \left\{ d(x) : x \in \left[ \frac{1}{f(m) + 1}, 1 \right] \right\}.
\]

Since \( d(x) \) is continuous, the minimum and maximum of \( d(x) \) on the interval \( \left[ \frac{1}{f(m) + 1}, 1 \right] \) are attained, denoted by \( d(x_m) \) and \( d(x_M) \) respectively. Then we have that \( x_m \) and \( x_M \) are both contained in the set \( C \cap \left[ \frac{1}{f(m) + 1}, 1 \right] \). On the other hand, using the same argument of (1.2) or [5, Corollary 1.3], we have that

\[
\left\{ \frac{x}{\mu^2((0, x])} : x \in C \cap \left[ \frac{f(1)}{f(m) + 1}, 1 \right] \right\} = \sup_{n \geq 1} \frac{C_n}{n^\alpha} \sup_{n \geq 1} \frac{C_n}{n^\alpha}.
\]

In other words, we have

\[
\left\{ d(x) : x \in \left[ \frac{1}{f(m) + 1}, 1 \right] \cap C \right\} = \inf_{n \geq 1} \frac{n}{C_n^{1/\alpha}} \sup_{n \geq 1} \frac{n}{C_n^{1/\alpha}}.
\]

This implies that

\[
\{ d(x) : x \in (0,1] \} = \left\{ d(x) : x \in \left[ \frac{1}{f(m) + 1}, 1 \right] \cap C \right\}.
\]

Hence that

\[
\left\{ \liminf_{x \to 0^+} d(x), \limsup_{x \to 0^+} d(x) \right\} \subset \{ d(x) : x \in (0, 1) \}.
\]

Following from the fact that \( d \left( \frac{x}{(f(m) + 1)^\epsilon} \right) = d(x) \) for any \( k \geq 1 \) and \( x \in (0, 1] \), we have

\[
\{ d(x) : x \in (0, 1] \} \subset \left[ \liminf_{x \to 0^+} d(x), \limsup_{x \to 0^+} d(x) \right].
\]

This completes the proof. \( \square \)

4. The limit function induced by Cantor-integers

First we prove the existence of \( \lambda(x) := \lim_{k \to \infty} \frac{C_{(m+1)^kx}}{(m+1)^kx} \) for every \( x > 0 \). In fact, we claim that

\[
\lambda(x) = C_x + \sum_{r=1}^{\infty} f(\epsilon_r)(f(m) + 1)^{-r} \frac{x^{\alpha}}{x^{\alpha}},
\]

where \( x = [\epsilon_\ell, \epsilon_0, \epsilon_1 \epsilon_2 \cdots]_{m+1} := \sum_{r=\ell}^{\infty} \epsilon_r (m + 1)^{-r} \) with \( \epsilon_r \in \Sigma_m \) for each \( r \geq -\ell \). In fact, \( [(m+1)^k x] = [\epsilon_\ell, \epsilon_0, \epsilon_1 \cdots \epsilon_k]_{m+1} \). Hence

\[
\frac{C_{(m+1)^kx}}{(m+1)^kx} = \frac{(f(m) + 1)^k C_{[\epsilon_\ell, \cdots, \epsilon_0]_{m+1}} + \sum_{r=1}^{k} f(\epsilon_r)(f(m) + 1)^{k-r}}{(f(m) + 1)^k x^\alpha} = C_x + \sum_{r=1}^{k} f(\epsilon_r)(f(m) + 1)^{-r} \frac{x^{\alpha}}{x^{\alpha}}.
\]

This implies (1.1) holds. There are some observations for \( \lambda(x) \).
• For every positive integer $n$, we have $\lambda(n) = \frac{C_n}{m^n}$.

• For every real number $x > 0$, we have

$$\lambda((m + 1)x) = \lambda(x).$$

(4.2)

We know that the $(m + 1)$-expansion is unique by excluding the expansions with a tail of $(m)$s. Recall that a real number $x$ is called $(m + 1)$-rational if its $(m + 1)$-expansion is finite, otherwise $x$ is called $(m + 1)$-irrational. By (4.2), without loss of generality we can always assume that $x \in [1, m + 1)$. For simplicity, define a function

$$D(x) := \sum_{r=1}^{\infty} f(\epsilon_r)(f(m) + 1)^{-r} \text{ for } x = \sum_{r=1}^{\infty} \epsilon_r(m + 1)^{-r} \text{ with } \epsilon_r \in \Sigma_m.$$  

(4.3)

Let $C := C_{m,f}$ and $\mu_C$ be defined as in (1.3) and (1.4) respectively, define a function $g_C : [0, 1) \rightarrow [0, 1]$ by

$$g_C(t) := \mu_C([0, t]),$$

which is a generalization of the Cantor function since $g_C(t)$ is the Cantor function if $C$ is the middle-thirds Cantor set. Equivalently,

$$g_C(t) = \begin{cases} \sum_{r=1}^{\infty} \epsilon_r(m + 1)^{-r}, & t = \sum_{r=1}^{\infty} \epsilon_r(m + 1)^{-r} \in C \text{ for } \epsilon_r \in \Sigma_m, \\
\sup_{y \leq t, y \in C} g_C(y), & t \in [0, 1) \setminus C. \end{cases}$$

It is not hard to check that $D(x) = \sup\{t \in [0, 1) : g_C(t) \leq x\}$. Hence the function $D(x)$ is an inversion of the generalized Cantor function in some sense. Since $D(x)$ is strictly increasing, it is differentiable almost everywhere by the Lebesgue theorem about monotone function. This implies $\lambda(x)$ is also differentiable almost everywhere. Moreover, for the continuity of $\lambda(x)$, we have the following explicit result by [3, Remark 2].

**Proposition 4.1.** (i) The function $\lambda(x)$ is continuous at every positive $(m + 1)$-irrational number.

(ii) The function $\lambda(x)$ is right continuous at every $(m + 1)$-rational number, and $\lambda(x)$ is left continuous at $x_0 = [\epsilon_0, \epsilon_1 \epsilon_2 \cdots \epsilon_{n+1}]_{m+1}$ if and only if $\epsilon_n \in \{r \in \Sigma_m : \Delta f(r) = 1\}$.

At the same time, $\lambda(x)$ satisfies $(\alpha - \delta)$-Hölder condition almost everywhere for every sufficiently small $\delta > 0$. To prove this, first we give the definition of normality.

**Definition 4.1.** Let $\ell \geq 1$, and let $B_{\ell}$ be a block of $\ell$ digits to the base $(m+1)$ (recall that $m \geq 1$). Also let $x_0 = [x_0] + \sum_{r \geq 1} \epsilon_r(m + 1)^{-r}$, and denote by $N(k, B_{\ell})$ the number of occurrences of the block $B_{\ell}$ in the initial block $\epsilon_1 \epsilon_2 \cdots \epsilon_k$ of $x_0 - [x_0]$. Then $x_0$ is normal if and only if

$$\lim_{k \to \infty} \frac{N(k, B_{\ell})}{k} = \frac{1}{(m + 1)^{\ell}},$$

for all $\ell \geq 1$ and all blocks $B_{\ell}$ of length $\ell$.

It is well-known that almost all positive real numbers are normal. Now we have the following result.
**Proposition 4.2.** If $x_0$ is normal to the base $(m+1)$, then we have that for every sufficiently small $\delta > 0$,

$$|\lambda(x_0 + h) - \lambda(x_0)| = \mathcal{O}(|h|^\alpha-\delta), \text{ as } h \to 0,$$

where the implied constant depends only on $x_0$.

**Proof.** It suffices to show that $|D(x_0 + h) - D(x_0)| = \mathcal{O}(|h|^\alpha-\delta)$ as $h \to 0$. Let $x_0 = [x_0] + \sum_{r \geq n+1} \epsilon_r (m+1)^{-r}$, and assume $(m+1)^{-(n+1)} < h < (m+1)^{-n}$ for some $n \geq 1$. Then

$$h = \sum_{r \geq n+1} h_r (m+1)^{-r}, 0 \leq h_r \leq m, h_{n+1} \neq 0.$$

Then

$$x_0 + h = [x_0] + \sum_{r=1}^n \epsilon_r (m+1)^{-r} + \sum_{r \geq n+1} (\epsilon_r + h_r)(m+1)^{-r}.$$

There maybe a carry into the $n$-th place in the right of the above equation. We need to estimate how long the carrying continues. Fixed any $\delta \in (0,1)$, we may choose $n_0$ and $B_k := m\ell$ (i.e., the $\ell$ consecutive block of $m$’s) such that $\frac{2}{(m+1)\ell} + \frac{\ell}{n_0} < \frac{\delta}{\alpha}$ and

$$N(k, B_k) < \frac{2}{(m+1)^\ell} k, \text{ for } k \geq n_0.$$

Hence if $n \geq n_0$, then there exists $t \in (1 - \delta/\alpha, 1)$ such that the number of occurrences of the block $B_k$ for $\epsilon_r$ between $tn$ and $n$ is at most

$$\frac{2}{(m+1)^\ell} n < (1-t)n - \ell.$$

Hence there is $r_0 > tn$ for which $\epsilon_{r_0} \neq m$, and this implies that

$$x_0 + h = [x_0] + \sum_{r=1}^{r_0-1} \epsilon_r (m+1)^{-r} + \sum_{r \geq r_0} \epsilon_r^* (m+1)^{-r}$$

where $\epsilon_r^* \in \Sigma_m$ are digits. Therefore

$$|D(x_0 + h) - D(x_0)| = \left| \sum_{r \geq r_0} (f(\epsilon) - f(\epsilon^*)) (m+1)^{-r} \right| \leq (m+1)^{-tn} \leq (1+f(m))h^{\alpha-\delta},$$

for $n \geq n_0$. Thus

$$|D(x_0 + h) - D(x_0)| = \mathcal{O}(h^{\alpha-\delta}), \text{ as } h \to 0^+. $$

A similar discussion shows that for every $h > 0$,

$$|D(x_0 - h) - D(x_0)| = \mathcal{O}(h^{\alpha-\delta}), \text{ as } h \to 0^+. $$

This completes the proof. \qed
Proposition 4.3. The function \( \lambda(x) \) has the logarithmic Fourier series expansion

\[
\lambda(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi i \log x / \log(m+1)}
\]  

where

\[
c_n = \frac{1}{\log(m+1)} \int_1^{m+1} \frac{\lambda(x)}{x^{1-\alpha+\gamma_n}} \, dx \quad \text{with} \quad \gamma_n := \alpha + \frac{2\pi n i}{\log(m+1)},
\]

and where the infinite series converges to \( \lambda(x) \) in the sense of Cesàro sum at each continuity point as in Proposition 4.1. Moreover, \( c_n = O(1/|n|) \) as \( n \to \infty \) or \( n \to -\infty \).

Proof. Let \( g(x) := \lambda((m+1)^{x/(2\pi)}) \). Then \( g(x) \) has the period \( 2\pi \) and thus \( g(x) \) has a Fourier series

\[
g(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta}, \quad \text{with} \quad c_n = \frac{1}{2\pi} \int_0^{2\pi} g(\theta)e^{-in\theta} \, d\theta,
\]

Then by [12, Theorem 5.2 in p. 53], we have that the infinite series converges to \( g(x) \) in the sense of Cesàro sum at each point \( x_0 \) which makes \( (m+1)x_0/(2\pi) \) be a continuity point as in Proposition 4.1. Using \( \lambda(x) = g(2\pi \log x / \log(m+1)) \), this easily yields the desired result. Recall that \( D(x) \) is monotone, it follows from [13] that \( c_n = O(1/|n|) \).

5. Proof of Theorem 3

Following from [7, Theorem 2.1] with the case \( m = 1 \), we have the following lemma.

Lemma 5.1. Let \( c \neq 0 \) lie in the half-plane of absolute convergence of \( \sum_k v_k k^{-s} \). Then we have

\[
\sum_{1 \leq k < n} v_k \left( 1 - \frac{k}{n} \right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \sum_{k \geq 1} \frac{v_k}{k^s} \right) n^s \frac{ds}{s(s+1)}
\]

Consequently, we have the following identity (cf. [7, Equation (2.6)]).

\[
\int_{-1/4-i\infty}^{-1/4+i\infty} \zeta(s)n^s \frac{ds}{s(s+1)} = 0.
\]

Proof of Theorme 3. Recall that \( \Delta C_n := C_n - C_{n-1} \) with \( n \geq 1 \). It is not hard to get that for every \( n \geq 1 \),

\[
\Delta C_{(m+1)n} = (f(m) + 1)\Delta C_n - f(m),
\]

and for every \( n \geq 0 \) and every \( r \in \{1, \ldots, m\} \),

\[
\Delta C_{(m+1)n+r} = \Delta f(r).
\]

It follows that

\[
\sum_{n=1}^{\infty} \frac{\Delta C_n}{n^s} = \sum_{r=1}^{m} \sum_{n=0}^{\infty} \frac{\Delta f(r)}{(m+1)n+r)^s} + \sum_{n=1}^{\infty} \frac{(f(m) + 1)\Delta C_n - f(m)}{(m+1)n^s} \]

\[
= \sum_{r=1}^{m} \frac{\Delta f(r)}{(m+1)^s} \zeta\left( s, \frac{r}{m+1} \right) + \frac{f(m) + 1}{(m+1)^s} \sum_{n=1}^{\infty} \Delta C_n \frac{1}{n^s} \zeta(s).
\]
Hence that
\[ \sum_{n=1}^{\infty} \frac{\Delta C_n}{n^s} = \frac{\sum_{r=1}^{m} \Delta f(r)\zeta \left( s, \frac{r}{m+1} \right) - f(m)\zeta(s)}{(f(m) + 1)((m + 1)^{s-\alpha} - 1)}. \]

By Lemma 5.1 with \( v_k = \Delta C_k \), for any \( c > \alpha + 1 \), we have
\[ \frac{S(n)}{n} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\sum_{r=1}^{m} \Delta f(r)\zeta \left( s, \frac{r}{m+1} \right) - f(m)\zeta(s)}{(f(m) + 1)((m + 1)^{s-\alpha} - 1)} \frac{n^s}{s(s+1)} ds, \]
(5.2)
The integrand in (5.2) has simple poles at \( s = 0, s = 1 \) and \( s = \gamma_k \) for every \( k \in \mathbb{Z} \). Shifting the line of integration to \( \text{Re} \, s = -\frac{1}{4} \) and taking residue into account, we get
\[ \frac{S(n)}{n} = \frac{\sum_{r=1}^{m} f(r)}{(m + 1)f(m)} - 2 + F(\log^{n+1}) + R(n), \]
(5.3)
where the Fourier series of \( F(u) \) is
\[ \frac{n^\alpha}{(f(m) + 1)\log(m + 1)} \sum_{k \in \mathbb{Z}} \sum_{r=1}^{m} \frac{\Delta f(r)\zeta \left( \gamma_k, \frac{r}{m+1} \right) - f(m)\zeta(\gamma_k)}{\gamma_k(\gamma_k + 1)} e^{2\pi i ku}, \]
which is the sum of residues of the integrand at the poles \( s = \gamma_k \). We still have to analyse the remainder term
\[ R(n) := \frac{1}{2\pi i(f(m) + 1)} \int_{-1/4-i\infty}^{-1/4+i\infty} \frac{\sum_{r=1}^{m} \Delta f(r)\zeta \left( s, \frac{r}{m+1} \right) - f(m)\zeta(s)}{(m + 1)^{s-\alpha} - 1} \frac{n^s}{s(s+1)} ds. \]
The integral converges since \( |\zeta(-\frac{1}{4} + it, a) - a^{\frac{1}{4} + it}| \ll |t|^{3/4} \) where the constant does not depend on \( a \) and \( t \) (cf. [1, Theorem 12.23]). Using the expansion
\[ \frac{1}{(m + 1)^{s-\alpha} - 1} = -1 - (m + 1)^{s-\alpha} - (m + 1)^{2s-2\alpha} - (m + 1)^{3s-3\alpha} - \cdots, \]
in the integrand of \( R(n) \), which is legitimate since \( \text{Re}(s - \alpha) < 0 \). By (5.1), we have
\[ R(n) = - \frac{1}{2\pi i(f(m) + 1)} \int_{-1/4-i\infty}^{-1/4+i\infty} \frac{\sum_{r=1}^{m} \Delta f(r)R'((m + 1)^{k+1}n/r)}{(m + 1)^{k+1}n(r+1)} \frac{n^s}{s(s+1)} ds, \]
where
\[ R'(n, r) := \frac{1}{2\pi i} \int_{-1/4-i\infty}^{3/2+i\infty} \zeta \left( s, \frac{r}{m+1} \right) \frac{n^s}{s(s+1)} ds. \]
Applying [7, Equation (2.3)] by setting \( \lambda_k \equiv 1, \mu_k \equiv k + \frac{r}{m+1}, x \equiv n^{-1} \) and \( f(x) \equiv (1 - x)H_0(x) \), we have
\[ \sum_{1 \leq k \leq n} \left( 1 - \frac{k + \frac{r}{m+1}}{n} \right) = \frac{1}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} \zeta \left( s, \frac{r}{m+1} \right) \frac{n^s}{s(s+1)} ds. \]
Shifting the the contour of $R'(n, r)$ back to $\text{Re } s = \frac{3}{2}$, by taking into account the residue at $s= 0$ and $s= 1$,

$$R'(n, r) = \frac{n+1}{2} - \frac{r}{m+1} + \sum_{1 \leq k < n} \left( 1 - \frac{k + \frac{r}{m+1}}{n} \right) = n - \left( 2 - \frac{1}{n} \right) \frac{r}{m+1}.$$

Therefore, we have

$$R(n) = -n \sum_{k \geq 0} \sum_{m=1}^{n} \Delta f(r) \left( (m+1)^{k+1} - \frac{1}{(m+1)^k} \right) \frac{r}{(f(m) + 1)^{k+1}}$$

$$= -n \frac{f(m)}{f(m) - m} + 2 - \frac{2 \sum_{r=1}^{m} f(r)}{f(m)(m+1)} - \frac{1}{n} \frac{f(m)(m+1) - \sum_{r=1}^{m} f(r)}{f(m)(m+1) + m}.$$

Substituting $R(n)$ into (5.3), the proof is complete.

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