Scalar normal modes of higher dimensional gravitating kinks

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Abstract

The scalar normal modes of higher dimensional gravitating kink solutions are derived. By perturbing to second order the gravity and matter parts of the action in the background of a five-dimensional kink, the effective Lagrangian of the scalar fluctuations is derived and diagonalized in terms of a single degree of freedom which invariant under infinitesimal diffeomorphisms. The spectrum of the normal modes is discussed and applied to the analysis of short distance corrections to Newton law.

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Higher dimensional kink solutions [1] have been introduced in order to discuss the localization properties of fields in the context of infinite extra dimensions [2]. Of particular interest is, in this framework, the problem of localization of gravitational interactions [6,7] and of gauge interactions [8–14].

It has been recently shown that five-dimensional gravitating kinks possess an interesting structure of zero modes [15] (see also [16] and [8]). In their simplest realization, gravitating kinks can appear in five-dimensional scalar-tensor theories of gravity when the five-dimensional bulk coordinate is infinite [17,19–21]. The gravity theory can be taken to be, for simplicity, of Einstein-Hilbert type (appropriately extended to higher dimensions). However, also quadratic gravity theories with Gauss-Bonnet self-interactions allow the same type of static solutions [22,23]. More generally, string inspired solutions with Gauss-Bonnet terms have been discussed from different points of view [24–29].

Gravitating kinks have scalar, vector and tensor normal modes [15] with respect to four-dimensional Poincaré transformations which are always unbroken. The tensor normal modes have been extensively analyzed in the context of brane solutions with AdS$_5$ geometries [17] where it has been shown that Poincaré invariance in four-dimensions implies the existence of a localized tensor zero mode. The tower of massive states, when resummed, leads to computable corrections to the Newton potential at short distances. The vector modes of the geometry (the so-called graviphoton fields) are not localized on five-dimensional gravitating kinks and they are always massless [15,30].

A different situation appears in the case of scalar degrees of freedom since they do have continuum modes. As it will be shown in detail, the scalar fluctuations of the action perturbed to second order combine in a single degree of freedom. This combination diagonalizes the full (second order) action and it is invariant under infinitesimal diffeomorphisms.

It is the purpose of the present paper to derive precisely the effective Lagrangian of the scalar normal modes of the gravitating five-dimensional kink solutions. There is no a priori
reason to expect the simplicity of the result from the complicated structure of the scalar action perturbed to second order. The strategy is, in short, the following. Consider the five-dimensional extension of the Einstein-Hilbert action minimally coupled to a scalar field $\varphi$:

$$ S = \int d^5 x \sqrt{|G|} \left[ -\frac{R}{2\kappa} + \frac{1}{2} G^{AB} \partial_A \varphi \partial_B \varphi - V(\varphi) \right]. \quad (1.1) $$

In this framework, kink solutions can be obtained in a geometry of the type

$$ ds^2 = a^2(w) [\eta_{\mu\nu} dx^\mu dx^\nu - dw^2], \quad (1.2) $$

where $w$ is the bulk coordinate and $\eta_{\mu\nu}$ is the Minkowski metric. The potential appearing in Eq. (1.1) is symmetric for $\varphi \to -\varphi$. The gravity and matter parts of the action (1.1) will then be perturbed to second order in the amplitude of the scalar fluctuations of the metric without fixing a specific gauge $^1$:

$$ \delta^{(2)} S = \delta^{(2)} S_{gr} + \delta^{(2)} S_m. \quad (1.3) $$

In this procedure various total derivatives appear. Some of them are expected. For instance the known total derivative coming from the surface term of the Einstein-Hilbert action. Other total derivatives are accidental in the sense that they appear only when the perturbed gravity and matter part of the action are combined and evaluated on-shell, i.e. on the background configuration.

A naive way of thinking would suggest that the perturbation of the action to second order should led exactly to the same results one would obtain by perturbing to first order in the amplitude of the scalar fluctuations the equations of motion derived from the action (1.1). This procedure has been already discussed and the present analysis shows that the naive expectation is only partially true. From the equations of motion various gauge-invariant quantities can be defined all leading to decoupled equations for the fluctuations.

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$^1$In the present paper, the first order fluctuations of a given quantity will be denoted by $\delta^{(1)}$ while the second order fluctuations will be denoted by $\delta^{(2)}$. 

These variables, even if invariant under infinitesimal diffeomorphisms, do not diagonalize the action and, hence, are not the correct normal modes of the scalar tensor action (1.1) in the background of a gravitating kink.

The canonical structure of the action is particularly important if the fluctuations of a given spin are quantized. It would not be correct to quantize a fluctuation whose action is not canonical. For instance, it could be easily shown that the scalar mode associated with the fluctuation of the field \( \varphi \) obeys a Schrödinger-like equation when the coupling to the metric fluctuations is ignored. The canonical normal modes are the correct classical quantities to be promoted to field operators.

The present paper is organized as follows. In Section II the basic properties of the formalism will be introduced. In Section III the second order fluctuation of the gravity and matter parts of the action will be derived. Section IV contains the diagonalization of the full action in terms of its canonical normal modes. In Section V the corrections to Newton potential coming from the tower of scalar fluctuations of a gravitating kink will be addressed. Finally Section VI contains a summary of the main findings and some concluding remarks. For purposes of presentation, various technical details needed for the derivations have been collected in the Appendix.

II. GRAVITATING KINKS AND THEIR FLUCTUATIONS

In the metric (1.2) the background equations of motion derived from the action (1.1) become

\[
\mathcal{H}' + \mathcal{H}^2 = -\frac{1}{6} \left[ \varphi'^2 + 2V \varphi^2 \right], \quad (2.1)
\]

\[
\mathcal{H}^2 = \frac{1}{12} \left[ \varphi'^2 - 2V \varphi^2 \right], \quad (2.2)
\]

\[
\varphi'' + 3 \mathcal{H} \varphi' - a^2 \frac{\partial V}{\partial \varphi} = 0, \quad (2.3)
\]

where \( \mathcal{H} = a'/a \) and the prime denotes derivation with respect to the bulk coordinate. Notice that natural gravitational units \( 2\kappa = 1 \) are used. Eqs. (2.1)–(2.3) admit gravitating
kink solutions. For instance, we will be interested in solutions of the type

$$\varphi(w) = \sqrt{6} \arctan (bw),$$  \hspace{1cm} (2.4)

$$a(w) = (b^2 w^2 + 1)^{-\frac{1}{2}},$$  \hspace{1cm} (2.5)

arising in sine-Gordon potentials

$$V(\varphi) = 3b^2 \left[ 5 \cos^2 (\varphi/\sqrt{6}) - 4 \right],$$ \hspace{1cm} (2.7)

or in other classes of symmetric potentials \cite{17-27}.

In the case of scalar fluctuations the total metric can be written, in its most general form, as

$$ds^2 = a^2(w) \left\{ \eta_{\mu\nu} + 2 \left( \eta_{\mu\nu} + \partial_{\mu}E \right) dx^\mu dx^\nu + 2\partial_\alpha C dx^\alpha dw - (1 - 2\xi) dw^2 \right\}$$ \hspace{1cm} (2.8)

where $\psi$, $\xi$, $C$ and $E$ are four functions characterizing the scalar (Poincaré-invariant) modes of the first order fluctuations of the metric $G_{AB}$, i.e., in our notations, $\delta^{(1)}G_{AB}$.

The infinitesimal diffeomorphisms preserving the scalar nature of the fluctuation are

$$x^\mu \to \tilde{x}^\mu = x^\mu + \partial^\mu \epsilon,$$

$$w \to \tilde{w} = w + \epsilon^w.$$ \hspace{1cm} (2.9)

Under the infinitesimal coordinate transformation (2.9) the scalar fluctuations of the metric change as

$$\tilde{E} = E - \epsilon,$$ \hspace{1cm} (2.10)

$$\tilde{\psi} = \psi - \mathcal{H}\epsilon_w,$$ \hspace{1cm} (2.11)

$$\tilde{C} = C - \epsilon^w + \epsilon_w,$$ \hspace{1cm} (2.12)

$$\tilde{\xi} = \xi + \mathcal{H}\epsilon_w + \epsilon_w.$$ \hspace{1cm} (2.13)

In spite of this, two gauge-invariant fluctuations can be constructed \cite{15}:
\[ \tilde{\Psi} = \tilde{\psi} - \mathcal{H}(\tilde{E}' - \tilde{C}'), \quad (2.14) \]
\[ \tilde{\Xi} = \tilde{\xi} - \frac{1}{a}[a(\tilde{C} - \tilde{E}')]'. \quad (2.15) \]

The fluctuations of the domain-wall itself (i.e. the fluctuations of \( \varphi \))
\[ \varphi(x^\mu, w) = \varphi(w) + \chi(x^\mu, w), \quad \delta^{(1)} \varphi = \chi, \quad (2.16) \]
are also non gauge-invariant
\[ \tilde{\chi} = \chi - \varphi' \epsilon_w. \quad (2.17) \]

The gauge-invariant scalar field fluctuation will be
\[ \tilde{\chi} = \tilde{\chi} - \varphi'(\tilde{E}' - \tilde{C}). \quad (2.18) \]

It is worth noticing that Eqs. (2.14) and (2.15) are reminiscent of the Bardeen potentials which are normally introduced in the gauge-invariant theory of gravitational fluctuations in four-dimensional cosmological backgrounds [31]. Of course the problem treated here is very different: we deal with static backgrounds, we are in five dimensions and four-dimensional Poincaré symmetry (unlike five-dimensional Poincaré symmetry) is unbroken.

Already from this analysis we can guess that there are three gauge-invariant scalar functions subjected to the Hamiltonian and momentum constraint. Hence only one degree of freedom describing the scalar fluctuations of the gravitating kink should be expected. This degree of freedom will emerge naturally from the structure of the effective action perturbed to second order in the amplitude of the fluctuations appearing in (2.8).

**III. GRAVITY AND MATTER ACTIONS TO SECOND ORDER**

The gravity part of the action (1.1) will now be perturbed to second order in the amplitude of the scalar fluctuations of the metric. First of all it should be noticed that the Einstein-Hilbert action can be written in a form where the known surface term is already absent, namely,
\[ S_{gr} = - \int d^5 x \sqrt{|G|} R = \int d^5 x \sqrt{|G|} G^{AB} \left[ \Gamma^M_{AB} \Gamma^N_{MN} - \Gamma^M_{AN} \Gamma^N_{MB} \right]. \] (3.1)

Hence the second order fluctuation of the gravity part of the action can be written as

\[
\delta^{(2)} S_{gr} = \int d^5 x \left\{ \sqrt{|G|} \left[ \delta^{(2)} G^{AB} \left( \Gamma^M_{AB} \Gamma^N_{MN} - \Gamma^M_{AN} \Gamma^N_{MB} \right) + \frac{G^{AB}}{G^{MN}} \left( \delta^{(2)} \Gamma^M_{AB} \Gamma^N_{MN} - \delta^{(2)} \Gamma^M_{AN} \Gamma^N_{MB} \right) \right] + \delta^{(1)} G^{AB} \left( \delta^{(1)} \Gamma^M_{AB} \Gamma^N_{MN} - \delta^{(1)} \Gamma^M_{AN} \Gamma^N_{MB} \right) \right\},
\] (3.2)

where \( G^{AB} \) and \( \Gamma^C_{AB} \) are, respectively, the background values of the metric and of the Christoffel connections. In (3.2) there are different kinds of contributions coming both from the second order fluctuations of the inverse metric (and of its determinant) and from the second order fluctuations of the Christoffel connections. All the results needed in order to obtain the explicit form of (3.2) in terms of the degrees of freedom appearing in Eq. (2.8) are separately reported in the Appendix. Thus, using the results of the Appendix, and, in particular, inserting Eqs. (A.2)-(A.3), (A.4)-(A.5) and (A.6)-(A.7) into Eq. (3.2) the second order form of the gravity part of the action is obtained:

\[
\delta^{(2)} S_{gr} = \int d^5 x \left\{ a^3 \left[ \mathcal{H}^2 \left( 48 \psi^2 + 18 \xi^2 + 12(\xi + 2\psi) \square E + 48\psi \xi + 6(\square E)^2 \right) - 6\partial_\alpha C \partial^\alpha C - 12\partial_\alpha \partial_\beta E \partial^\alpha \partial^\beta E \right] + \mathcal{H} \left( 48 \psi^2 + 12\psi \square E + 6\psi \square E + 6\square E \square E' - 12\partial_\alpha \partial_\beta E \partial^\alpha \partial^\beta E \right. \\
\left. - 12\partial^\alpha C \partial_\alpha \psi + 3\partial^\alpha C \partial_\alpha \xi + 24\psi \psi' + 6\xi \square E' + 6\square E' \right) - 6\partial_\alpha \psi (\partial^\alpha \psi - \partial^\alpha \xi) + 12\psi^2 + 6\psi \square E' - 2\psi \square C - 4\partial_\alpha C' \partial^\alpha \psi + \xi' \square C - \partial^\alpha C' \partial_\alpha \xi \right\} + D_1 + D_2 + D_3, \] (3.3)
where $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are the following total derivatives:

\[
\mathcal{D}_1 = \partial_\alpha \left\{ -a^3 \mathcal{H} \left[ 3 \partial^\alpha C (4\psi + \square E) + 3\xi \partial^\alpha C + 6 \partial^\alpha C \psi + \partial^\alpha \partial^3 E \partial_\beta C \right] \right\},
\]

\[
\mathcal{D}_2 = \partial_\alpha \left\{ a^3 \left[ \partial^\alpha \partial_\beta E \partial^\beta E - \partial^\alpha \partial^3 E \partial^\beta \partial_\beta E + 2 \partial_\beta C \partial^\alpha \partial^3 E' \right. \right.
\]

\[
\left. \quad - 2 \partial^\alpha C \square E' - 2 \partial^\alpha C' \square E + \partial^\alpha E' \square E' - \partial_\beta E' \partial^\alpha \partial^3 E' \right\},
\]

\[
\mathcal{D}_3 = \left\{ a^3 \square C \square E \right\}'.
\]  

(3.4)

The same procedure discussed in the case of the gravity part of the action, should be repeated for the matter part. Recalling the notation for the fluctuations of the domain-wall itself, i.e. Eq. (2.16), the perturbed matter part of the action can be written, in general terms, as

\[
\delta^{(2)} S_m = \int d^5 x \frac{1}{2} \left\{ \delta^{(2)} \sqrt{|G|} \left[ G^{AB} \partial_A \varphi \partial_B \varphi - V(\varphi) \right] \right.
\]

\[
\left. + \sqrt{|G|} \left[ \delta^{(2)} G^{AB} \partial_A \varphi \partial_B \varphi + G^{AB} \partial_A \chi \partial_B \chi \right. \right.
\]

\[
\left. \quad + \delta^{(1)} G^{AB} (\partial_A \varphi \partial_B \chi + \partial_A \chi \partial_B \varphi) - \frac{1}{2} \frac{\partial^2 V}{\partial \varphi^2} \chi^2 \right]\n
\]

\[
\left. \quad + \delta^{(1)} \sqrt{|G|} \left[ \delta^{(1)} G^{AB} \partial_A \varphi \partial_B \varphi + G^{AB} (\partial_A \chi \partial_B \varphi + \partial_A \varphi \partial_B \chi) - \frac{\partial V}{\partial \varphi} \chi \right] \right\}. \quad (3.5)
\]

Using now Eqs. (A.2)–(A.3) and Eqs. (A.4)–(A.5) into Eq. (3.5) we get the explicit form of the matter action perturbed to second order:

\[
\delta^{(2)} S_m = \int d^5 x \left\{ \frac{1}{2} \left[ \frac{V}{2} + Va^2 \right] \left[ \xi^2 - 8 \psi^2 - (\square E)^2 + 2 \partial_\alpha \partial_\beta E \partial^\alpha \partial^3 E - 4 \psi \square E \right. \right.
\]

\[
\left. \quad + 8 \psi \xi + 2 \xi \square E - \partial_\alpha C \partial^\alpha C \right]\n
\]

\[
\left. \quad + (\xi - 4 \psi - \square E) \left[ \chi' \varphi' + \varphi'^2 \xi + a^2 \frac{\partial V}{\partial \varphi} \chi \right] - \frac{1}{2} \chi'^2 + \partial_\alpha \chi \partial^\alpha \chi + \varphi' \partial_\alpha \chi \partial^\alpha C \right.
\]

\[
\left. \quad - 2 \varphi' \chi' \xi + \frac{\varphi'^2}{2} \left[ \partial_\alpha C \partial^\alpha C - 4 \xi^2 \right] - \frac{1}{2} \frac{\partial^2 V}{\partial \varphi^2} \chi^2 \right\}. \quad (3.6)
\]

The contributions to the second order action coming, respectively, from the gravity (3.3) and matter parts (3.6) should be combined. In this procedure various simplifications occur.

\[\text{\textsuperscript{2}}\text{Notice that, in order to shorten the notation, the convention } \square = \eta^{\alpha\beta} \partial_\alpha \partial_\beta \text{ has been adopted.}\]
First of all, since the action should be evaluated on the background of the gravitating kink, Eqs. (2.1)-(2.3) can be imposed. The resulting (total) action is, therefore,

\[ \delta^{(2)}S = \delta^{(2)}S_{gr} + \delta^{(2)}S_m = \int d^5x \left\{ a^3 \left[ 12 \psi' \psi + 3(\mathcal{H}' + 3\mathcal{H}^2)\xi^2 - 6\partial_\alpha \psi (\partial^\alpha \psi - \partial^\alpha \xi) + 24\mathcal{H}\xi \psi' + (\xi' + 4\psi')\varphi' \chi + 2\xi \frac{\partial V}{\partial \varphi} a^2 \chi - \frac{1}{2} \chi'^2 + \frac{1}{2} \partial_\alpha \chi \partial^\alpha \chi 
\right.
\]
\[\left. - \frac{1}{2} \frac{\partial^2 V}{\partial \varphi^2} a^2 \chi^2 + \Box (E' - C)(6\mathcal{H}\xi + 6\psi' + \varphi' \chi) \right\} + \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_5 \right\}, \quad (3.7)\]

where \( \mathcal{D}_4 \) and \( \mathcal{D}_5 \) are two further total derivatives which appear as a consequence of the use of the background equations (2.1)-(2.3) in the perturbed action

\[
\mathcal{D}_4 = \partial_\alpha \left\{ a^3 \left[ 12 \mathcal{H} \left( 2\partial^\alpha E'^\beta E - 2\partial_\beta E'^\alpha E \right) + 6(\mathcal{H}' + 3\mathcal{H}^2) \left( \partial^\alpha E \Box E - \partial_\beta E \partial^\alpha \partial^\beta E \right) \right]
\]
\[+ \mathcal{H} \left( (12\psi - 3\xi + \varphi' \chi) \partial^\alpha C - (4\psi + \xi) \partial^\alpha C' \right) \right\}, \]
\[
\mathcal{D}_5 = \left\{ a^3 \left[ -3\mathcal{H} (\Box E)^2 + 12\mathcal{H} \Box E + 24\mathcal{H} \psi^2 - \varphi' \chi \Box E + (4\psi + \xi) \Box C 
\right.
\]
\[\left. - (\xi + 4\psi') \varphi' \chi \right\} \right\}'. \quad (3.8)

The action obtained in Eq. (3.7) will now be diagonalized.

**IV. CANONICAL NORMAL MODES OF THE SECOND ORDER ACTION**

The variation of the action (3.7) with respect to \((E' - C)\) leads to the constraint

\[ 6\mathcal{H}\xi + 6\psi' + \varphi' \chi = 0. \quad (4.1) \]

From the gauge-invariant analysis of the evolution equations of the fluctuations we do know that there are variables obeying simple (Schrödinger-like) equations. However, we cannot say, only from the equations of motion, that these variables diagonalize the second order action.

In order to diagonalize the action (3.7) let us look first at a small portion of it, namely the kinetic terms of the various fluctuations. If a variable diagonalizing the kinetic terms
can be found, then it will be worth trying to see if also all the other terms of (3.7) will be
diagonal in the same variable. From Eq. (3.7) the kinetic part of the second order action is
\[ \delta^{(2)} S_{\text{kin}} = \int d^5 x \left\{ a^3 \left[ -6 \partial_\alpha \psi (\partial^\alpha \psi - \partial^\alpha \xi) + \frac{1}{2} \partial_\alpha \chi \partial^\alpha \chi + [\ldots] \right] + [\ldots] \right\}, \] (4.2)
where the ellipses stand both for the other terms of (3.7) and for the five total derivatives.
Eliminating now \( \xi \) through Eq. (4.1) we can see that Eq. (4.2) can be written as
\[ \delta^{(2)} S_{\text{kin}} = \int d^5 x \left\{ \frac{1}{2} \partial_\alpha \mathcal{G} \partial^\alpha \mathcal{G} + \mathcal{D}_6 \right\}, \] (4.3)
where \( \mathcal{D}_6 \) is a total derivative and \( \mathcal{G} \) is given by
\[ \mathcal{G} = \frac{a^3}{2} \frac{\chi}{\mathcal{H}} - z \psi, \quad z = \frac{a^{3/2} \varphi'}{\mathcal{H}}. \] (4.4)
Hence, \( \mathcal{G} \) diagonalizes the kinetic part of the action (3.7). It will now be shown that the
same variable \( \mathcal{G} \) diagonalizes the full action (3.7).

Notice, preliminary, that the variable \( \mathcal{G} \) is gauge-invariant. In fact, reading-off, from Eqs.
(2.11) and (2.17), the gauge-variations of \( \psi \) and \( \chi \) for infinitesimal diffeomorphisms we also
see that
\[ \tilde{\mathcal{G}} \equiv \mathcal{G} = a^{3/2} X - z \Psi, \] (4.5)
where \( \Psi \) and \( X \) are, respectively, the gauge-invariant longitudinal fluctuation and the wall
perturbation as defined in Eqs. (2.14) and (2.18).

From Eqs. (4.1) and (4.4), the wall fluctuation and the derivative of the longitudinal
fluctuation of the metric can be expressed as
\[ \chi = \frac{\mathcal{G}}{a^{3/2}} + \left( \frac{\varphi'}{\mathcal{H}} \right) \psi, \]
\[ \psi' = -\mathcal{H} \left[ \xi + \frac{\varphi'^2}{6 \mathcal{H}^2} \psi \right] - \frac{\varphi'}{6} \left( \frac{\mathcal{G}}{a^{3/2}} \right). \] (4.6)
Inserting Eqs. (4.3) into Eq. (3.7) we find, after a rather straightforward but algebraically
long procedure, that
\[ \delta^{(2)}S = \delta^{(2)}S_{gr} + \delta^{(2)}S_m \]
\[ = \int d^5x \left\{ \frac{1}{2} \left[ \partial_\alpha G \partial^\alpha G - G'' - \frac{z''}{z} G^2 \right] + \sum_{i=1}^{7} D_i \right\}, \tag{4.7} \]
where \(D_7\) is the last total derivative
\[ D_7 = \left\{ \left( \frac{\varphi'^2}{H} \right) a^3 \xi \psi - \frac{a^4}{2} \left( \frac{\varphi'}{H} \right)^2 \left( \frac{a}{a} \right)' \psi^2 \right. \]
\[ \left. - \varphi' a^{3/2} \xi G - \frac{a^{5/2}}{H} \left( \frac{\varphi'}{a} \right) G \psi - \frac{a^3}{6} \frac{\varphi'^3}{H^2} \left( \frac{\varphi'}{\varphi'} - H \right) \psi \left( \frac{G}{a^{3/2}} \right) \right\}'. \tag{4.8} \]

V. CORRECTIONS TO NEWTONIAN POTENTIAL

The correction to Newton’s law at short distances is well known in the case of the tensor modes. In the case of tensor fluctuations the second order action is well known and it is, for each polarization,
\[ \delta^{(2)}S_{(T)} = \int d^5x \left\{ \frac{1}{2} \left[ \partial_\alpha \mu \partial^\alpha \mu - \mu'' - \frac{(a^{3/2})''}{a^{3/2}} \mu^2 \right] \right\}, \tag{5.1} \]
where \(\mu = \sqrt{2} h a^{3/2}\) and \(h\) stands for each polarization of the tensor modes of the metric. In general terms, the equation for the mass eigenstates of the tensor normal modes can be written as
\[ \mu''_m + \left[ m^2 - \frac{(a^{3/2})''}{a^{3/2}} \right] \mu_m = 0, \tag{5.2} \]
and the related equation for \(h_m\) is
\[ h''_m + 3H h'_m + m^2 h_m = 0. \tag{5.3} \]

The tensor zero mode is always normalized since the integral
\[ \int_0^\infty |\mu_0|^2 dw = \int_0^\infty dw a^3(w), \tag{5.4} \]
is always convergent provided the four-dimensional Planck mass
\[ M^2_p = M^2 \int_0^\infty a^3(w) dw, \tag{5.5} \]
is finite. In the case of the higher-dimensional kink solution we can focus the attention on the case where, according to the example of Eqs. (2.4)-(2.7), \( a(w) = (b^2 w^2 + 1)^{-1/2} \). In this case the four-dimensional Planck mass is clearly finite. In Eq. (5.2)

\[
\left( \frac{a^{3/2}}{a^{3/2}} \right)'' \sim \frac{15}{4w^2}, \quad bw \geq 1.
\]

The solution for the continuum modes will be

\[
\mu_m = \sqrt{w} \left[ AJ_2(mw) + BY_2(mw) \right],
\]

where \( J_\nu(mw) \) and \( Y_\nu(mw) \) are Bessel functions of index \( \nu \) [32]. The solution given in Eq. (5.7) is the same one appearing in the case of Ref. [6,7] and determines the known correction to Newton’s law

\[
V(r) \sim G_N \frac{m_1}{r} \frac{m_2}{r^2} \left[ 1 + \frac{1}{(br)^2} \right],
\]

which arises from the contribution of the bulk continuum modes.

In the scalar case, the equation obeyed by the normal modes is given by

\[
\mathcal{G}_m'' + \left[ m^2 - \frac{z''}{z} \right] \mathcal{G}_m = 0.
\]

As an example, consider the solution given in Eqs. (2.4)-(2.7) where

\[
z(w) = \frac{a^{3/2} \varphi'}{\mathcal{H}} = -\left( \frac{\sqrt{6}}{bw (1 + b^2 w^2)^{3/2}} \right).
\]

As previously discussed [15], the scalar zero mode is not normalizable. The solution for the zero mode is

\[
\mathcal{G}_0(w) = c_1 z(w) + c_2 z(w) \int_w^w \frac{dw'}{z^2(w')},
\]

and the integrand of

\[
\int_0^\infty |\mathcal{G}_0|^2 dw,
\]

diverges for \( w \to 0 \) for both linearly independent solutions parametrized by the two arbitrary constants \( c_1 \) and \( c_2 \). Notice also, incidentally, that \( z''/z \) diverges when the zero-mode
diverges. This means that the zero mode is decoupled from the four dimensional dynamics. If we want to discuss the corrections coming from the continuum modes it is useful to work with the field \( g_m = (1/z) \mathcal{G}_m \) whose equation is

\[
g''_m + 2 \frac{z'}{z} g'_m + m^2 g_m = 0. \tag{5.13}
\]

The differential operator of Eq. (5.13) is self-adjoint provided

\[
\frac{dg_m}{dw} \bigg|_{1/b} = 0, \quad \frac{dg_m}{dw} \bigg|_{w_{\text{max}}} = 0 \tag{5.14}
\]

The effective cut-off \( w_{\text{max}} \) will be taken to \( \infty \) after having determined the spectrum of mass eigenstates which is discrete for finite \( w_{\text{max}} \) but becomes continuous for \( w_{\text{max}} \to \infty \).

The solution for the massive modes can be obtained by noticing that

\[
\frac{z''}{z} \sim \frac{35}{4w^2}, \quad w \geq 1/b. \tag{5.15}
\]

Consequently, from Eqs. (5.9) and (5.13)

\[
\mathcal{G}_m(w) = \sqrt{w} \left[ A_S J_3(mw) + B_S Y_3(mw) \right], \quad bw \geq 1 \tag{5.16}
\]

\[
g_m(w) = \sqrt{w(bw)^{5/2}} \left[ A_S J_3(mw) + B_S Y_3(mw) \right], \quad bw \geq 1 \tag{5.17}
\]

Imposing now the boundary conditions (5.14) we have that

\[
A_S = -B_S \frac{Y_2(m/b)}{J_2(m/b)}. \tag{5.18}
\]

At infinity the boundary conditions imply, instead,

\[
A_S = -B_S \frac{Y_2(mw_{\text{max}})}{J_2(mw_{\text{max}})}. \tag{5.19}
\]

Equating Eqs. (5.18) and (5.19) we find that

\[
\frac{Y_2(m/b)}{J_2(m/b)} = \frac{Y_2(mw_{\text{max}})}{J_2(mw_{\text{max}})} \tag{5.20}
\]

which allows to determine the spectrum of mass eigenstates by noticing that at the right hand side Eq. (5.20) the Bessel functions have a large argument (i.e. \( mw_{\text{max}} \)), whereas at the
left hand side the Bessel functions have a small argument, i.e. \( m/b \ll 1 \). Consequently, if the small and large argument limit is taken appropriately \[^{32,33}\] in Eq. (5.20), the resulting relation leads to the mass spectrum

\[
m_n \simeq \frac{\pi}{2} \left( \frac{3}{2} + 2n \right) \frac{1}{w_{\text{max}}}, \quad n = 1, 2, 3... \tag{5.21}
\]

which becomes continuous in the limit \( w_{\text{max}} \to \infty \). The normalization condition

\[
\int_{1/b}^{\infty} G_m(w) G_m'(w) dw \equiv \int_{1/b}^{\infty} z^2(w) g_m(w) g_m'(w) dw = \delta(m - m') \tag{5.22}
\]

can be used in order to determine \( B_S \). the correction to the Newton’s potential will be given by resumming

\[
V(r) \sim G_N \frac{m_1}{r} \frac{m_2}{r} \left[ 1 + \frac{\pi}{2bw_{\text{max}}} \sum_n \left( \frac{m}{b} \right)^3 e^{-m_r} \right]. \tag{5.23}
\]

Transforming now the sum in an integral\[^{3}\] and taking, consequently, the limit \( w_{\text{max}} \to \infty \) it is found that

\[
V(r) \sim G_N \frac{m_1}{r} \frac{m_2}{r} \left[ 1 + \frac{3}{(br)^4} \right]. \tag{5.24}
\]

Hence, in the case of the specific example discussed in the present section, the correction to the Newtonian potential coming from the bulk (scalar) continuum modes are more suppressed than the corrections coming from the tensor continuum modes. This situation is reminiscent of what happens in the case of six-dimensional solutions when a string-like defect is included in the matter sector \[^{34-38}\]. The difference is that in the case of \[^{38}\] the suppressed contribution comes from the tensor modes (in six dimensions), whereas, in the present five-dimensional context, it comes from the scalar modes.

\[^{3}\]Recall that \( \delta m_n = m_{n+1} - m_n = \pi/w_{\text{max}} \). For \( w_{\text{max}} \to \infty \) \( (\pi/w_{\text{max}}) \sum_n \to \int dm \).
VI. CONCLUDING REMARKS

In this paper the scalar effective action for the normal modes of five-dimensional kink solutions has been derived. By perturbing the full action to second order in the amplitude of the scalar fluctuations the action for the scalar modes is

$$\delta^{(2)}S = \int d^5 x \left\{ \frac{1}{2} \left[ \partial_\alpha G \partial^\alpha G - G'^2 - \frac{z''}{z} G^2 \right] \right\}, \quad (6.1)$$

The action is then expressed in terms of a single gauge-invariant fluctuation

$$G(x^\mu, w) = a^{3/2}(w) X(x^\mu, w) - z(w) \Psi(x^\mu, w),$$
$$z(w) = a^{3/2} \frac{\varphi'}{\mathcal{H}}, \quad (6.2)$$

where $X$ is the gauge-invariant wall fluctuation and $\Psi$ is the gauge-invariant longitudinal fluctuation of the metric. Furthermore $a(w)$ is the warp factor and $\varphi$ is the kink background.

It is interesting to notice that neither the longitudinal fluctuations of the metric nor the wall fluctuations are the correct normal modes. The correct normal mode is obtained through a combination (with background-dependent coefficients) of the wall fluctuation and of the longitudinal metric perturbation. The variable $(6.2)$ is independent on the specific choice of coordinate system since it is invariant under infinitesimal diffeomorphisms. Furthermore, it should be appreciated that the derivation of $(6.2)$ only assumes the validity of the background equations of motion and not of any specific background solution. The variable $(6.2)$ is the correct quantity to use in order to discuss the possible effects of scalar fluctuations in different frameworks. The zero mode associated with $G$ is not localized but, still, the massive modes can lead to corrections to Newton’s law at short distances which have been computed in the case of a specific kink configuration.

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APPENDIX A: SECOND ORDER FLUCTUATIONS OF THE GEOMETRY

In this Appendix the first and second order fluctuations of the inverse metric and of the Christoffel connections will be reported. They are needed in order to obtain the second order form of the gravity and matter part of the action.

The scalar fluctuations of the metric can be written as

$$\delta^{(1)} G_{AB} = a^2(w) \begin{pmatrix} 2(\eta_{\mu\nu} \psi + 2\partial_\mu \partial_\nu E) & \partial_\mu C \\ \partial_\mu C & 2\xi \end{pmatrix}. \quad (A.1)$$

The first order fluctuations of the inverse metric are

$$\delta^{(1)} G^{\mu\nu} = -\frac{2}{a^2}(\eta^{\mu\nu} \psi + \partial^{\mu} E), \quad (A.2)$$

$$\delta^{(1)} G^{\mu w} = \frac{\partial^\mu C}{a^2}, \quad$$

$$\delta^{(1)} G^{ww} = -\frac{2\xi}{a^2}.$$  

The first order fluctuation of square root of the determinant of the metric are

$$\delta^{(1)} \sqrt{|G|} = a^5 \left[ \Box E + 4\psi - \xi \right]. \quad (A.3)$$

In order to perturb consistently the action (1.1) the second order fluctuations of the inverse metric and of the square root of the determinant are needed. They are:

$$\delta^{(2)} G^{\mu\nu} = \frac{4}{a^2} \left[ \psi^2 \eta^2 + \partial^\mu \partial^\nu \right] E + 2\psi \partial^\mu \partial^\nu E \partial_\alpha \partial^\nu E] - \frac{\partial^\mu C \partial^\nu C}{a^2}, \quad (A.4)$$

$$\delta^{(2)} G^{\mu w} = \frac{2}{a^2} \xi \partial^\mu C - \frac{2}{a^2} [\eta^{\mu\alpha} \psi + \partial^{\mu} E] \partial_\alpha C, \quad$$

$$\delta^{(2)} G^{ww} = \frac{1}{a^2} \partial_\alpha C \partial^\alpha C - \frac{4}{a^2} \xi^2.$$ 

and

$$\delta^{(2)} \sqrt{|G|} = \frac{a^5}{2} \left[ 8\psi^2 + 4\Box E \psi - \xi^2 - 8\psi \xi + \partial_\alpha \partial^\alpha C - 2\xi \Box E + (\Box E)^2 + \partial_\alpha \partial_\beta E \partial^\alpha \partial^\beta E \right]. \quad (A.5)$$

The first order fluctuations of the Christoffel connections are
\( \delta^{(1)} \Gamma^w_{\mu \nu} = \eta_{\mu \nu} [\psi' + 2\mathcal{H}(\xi + \psi)] + \partial_\mu \partial_\nu [E' + 2\mathcal{H}E - C], \)
\( \delta^{(1)} \Gamma^w_{\mu w} = \partial_\mu [\mathcal{H}C - \xi], \)
\( \delta^{(1)} \Gamma^\mu_{ww} = \partial^\mu [C' + \mathcal{H}C - \xi], \)
\( \delta^{(1)} \Gamma^w_{ww} = -\xi', \)
\( \delta^{(1)} \Gamma^\mu_{\alpha \beta} = \partial_\alpha \psi \delta^\mu_\beta + 2\partial_\beta \psi \delta^\mu_\alpha - \partial^\mu \psi \delta_{\alpha \beta} - \mathcal{H} \partial^\mu \eta_{\alpha \beta} + \partial^\mu \partial_\alpha \partial_\beta E. \)
\( \delta^{(1)} \Gamma^\alpha_{\mu w} = \psi' \delta^\alpha_\mu + \frac{1}{2} \left( \partial_\mu \delta^\alpha C - \partial^\alpha \partial_\mu C \right) + \partial_\alpha \partial^\mu E. \)  

(A.6)

Finally, the second order fluctuations of the Christoffel connections are

\( \delta^{(2)} \Gamma^w_{\mu \nu} = \mathcal{H}(4\xi^2 - \partial_\alpha C \partial^\alpha C) \eta_{\mu \nu} + \partial^\alpha C \left[ \partial_\mu H_{\nu \alpha} + \partial_\nu H_{\mu \alpha} - \partial_\alpha H_{\mu \nu} \right] - 2\xi \partial_\mu \partial_\nu C + 2\xi \left[ H'_{\mu \nu} + 2\mathcal{H}H_{\mu \nu} \right], \)
\( \delta^{(2)} \Gamma^w_{ww} = \mathcal{H} \partial_\alpha C \partial^\alpha C + \partial_\alpha C \partial^\alpha C' - \partial_\alpha C \partial^\alpha \xi - 2\xi \xi', \)
\( \delta^{(2)} \Gamma^w_{\mu w} = 2\mathcal{H} \xi \partial_\mu C - 2\xi \partial_\mu \xi + \partial^\alpha CH'_{\alpha \mu}, \)
\( \delta^{(2)} \Gamma^w_{ww} = \partial^\mu C \xi' - 2\mathcal{H} H^\mu_\alpha \partial_\alpha C - 2H^\alpha_\mu \partial_\alpha C' + 2H^\mu_\alpha \partial_\alpha \xi, \)
\( \delta^{(2)} \Gamma^\mu_{\alpha \beta} = -\mathcal{H} \partial^\mu C \partial_\alpha C - 4\mathcal{H} H^\mu_\beta H_{\alpha \beta} - 2H^\mu_\beta H'_{\alpha \beta} + \partial^\mu C \partial_\alpha \xi, \)
\( \delta^{(2)} \Gamma^\mu_{\alpha w} = B^\mu \partial_\beta \partial_\alpha C + 2\mathcal{H} \left[ H^\mu_\lambda \partial_\alpha C - \xi \partial^\mu C \right] \eta_{\alpha \beta} - H'_{\alpha \beta} \partial^\mu C - 2\mathcal{H} \partial^\mu CH_{\alpha \beta} + 2H^\mu_\lambda \left( \partial_\lambda H_{\alpha \beta} - \partial_\beta H_{\lambda \alpha} - \partial_\alpha H_{\beta \lambda} \right). \)  

(A.7)

where, in order to reduce the already lengthy expressions, the following quantity

\( H_{\mu \nu} = \psi \eta_{\mu \nu} + \partial_\mu \partial_\nu E, \)  

(A.8)

has been defined.
REFERENCES

[1] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B 125, 136 (1983).

[2] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B 125, 139 (1983).

[3] K. Akama, in Proceedings of the Symposium on Gauge Theory and Gravitation, Nara, Japan, eds. K. Kikkawa, N. Nakanishi and H. Nariai, (Springer-Verlag, 1983), [hep-th/0001113].

[4] M. Visser, Phys. Lett. B 159 (1985) 22.

[5] S. Randjbar-Daemi and C. Wetterich, Phys. Lett. B 166, 65 1986.

[6] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 3370 (1999).

[7] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 4690 (1999).

[8] S. Randjbar-Daemi, and M. Shaposhnikov, [hep-th/0206010].

[9] V. Rubakov, Usp. Fiz. Nauk 171, 913 (2001) [Phys. Usp. 44, 871 (2001)].

[10] G. Dvali and M. Shifman, Phys. Lett. B 396, 64 [Erratum-ibid. B 407, 452 (1992)].

[11] I. Oda, Phys. Lett. B 496, 113 (2000).

[12] S.L. Dubovsky and V.A. Rubakov, Int. J. Mod. Phys. A 16, 4331 (2001).

[13] M. Shaposhnikov and P. Tinyakov, Phys. Lett. B 515, 442 (2001).

[14] M. Giovannini, J.-V. Le Bé, and S. Riederer Class. Quant. Grav. 19, 3357 (2002).

[15] M. Giovannini, Phys. Rev. D 64, 064023 (2001); Phys.Rev.D 65, 064008 (2002).

[16] J. E. Kim, B. Kyae, and H. Min Lee, [hep-th/0110103].

[17] M. Gremm, Phys. Lett. B 478, 434 (2000).

[18] M. Gremm, Phys. Rev. D 62, 044017 (2000).
[19] A. Kehagias and K. Tamvakis, Phys.Lett. B 504, 38 (2001).

[20] A. Kehagias and K. Tamvakis, [hep-th/0011006].

[21] O. DeWolfe, D.Z. Freedman, S.S. Gubser, A. Karch, Phys.Rev.D 62, 046008 (2000).

[22] O. Corradini and Z. Kakushadze, Phys.Lett. B 494, 302 (2000).

[23] M. Giovannini, Phys.Rev. D 64, 124004 (2001).

[24] N. Mavromatos and J. Rizos, Phys. Rev. D 62, 124004 (2000).

[25] N. Mavromatos and J. Rizos, [hep-th/0205299].

[26] E. Kiritsis, N. Tetradis, and T.N. Tomaras, JHEP 0108, 012 (2001).

[27] S. Nojiri and S. D Odintsov, Phys. Rev. D 65, 023521 (2002).

[28] I. P. Neupane, Class. Quant. Grav. 19, 1167 (2002).

[29] S. Nojiri, O. Obregon, S. Odintsov, and V. I. Tkach, Phys.Rev.D 64, 043505 (2001).

[30] M. Giovannini, Phys. Rev. D 65, 124019 (2002).

[31] J. M. Bardeen, Phys. Rev. D 22, 1882 (1980).

[32] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, (Dover, New York, 1972).

[33] A. Erdelyi, W. Magnus, F. Oberhettinger, and F.R. Tricomi, Higher Trascendental Functions, (Mc Graw-Hill, New York, 1953).

[34] A. G. Cohen and D. B. Kaplan, Phys. Lett. B 470, 52 (1999).

[35] A. Chodos and E. Poppitz, Phys. Lett. B 471, 119 (1999).

[36] I. Olasagasti and A. Vilenkin, Phys. Rev. D 62, 044014 (2000).

[37] R. Gregory, Phys. Rev. Lett. 84, 2564 (2000).
[38] T. Gherghetta and M. Shaposhnikov, Phys.Rev.Lett. 85, 240 (2000).