F and M Theories as Gauge Theories of Area Preserving Algebra

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Abstract

F theory and M theory are formulated as gauge theories of area-preserving diffeomorphism algebra. Our M theory is shown to be 1-brane formulation rather than 0-brane formulation of M theory of Banks, Fischler, Shenker and Susskind and the F theory is shown to be 1-brane formulation rather than -1-brane formulation of type IIB matrix theory of Ishibashi, Kawai, Kitazawa and Tsuchiya.

Area preserving diffeomorphism algebra

Our starting point is the following 2 + 1 dimensional energy-momentum algebra of Dirac\(^{(1)}\), Schwinger\(^{(2)}\) and DeWitt\(^{(3)}\).

\[
\begin{align*}
[T_{00}(\sigma), T_{00}(\sigma')] &= -i(T_{i0}(\sigma) + T_{i0}(\sigma')) \partial_i \delta(\sigma - \sigma') \quad (1) \\
[T_{i0}(\sigma), T_{00}(\sigma')] &= -iT_{00}(\sigma) \partial_i \delta(\sigma - \sigma'), \quad (2) \\
[T_{i0}(\sigma), T_{j0}(\sigma')] &= -i(T_{i0}(\sigma') \partial_j \delta(\sigma - \sigma') + T_{j0}(\sigma) \partial_i \delta(\sigma - \sigma')), \quad (3)
\end{align*}
\]

where

\[\sigma = (\sigma^1, \sigma^2), (i, j = 1, 2)\]

and

\[
\partial_i \delta(\sigma - \sigma') = \frac{\partial}{\partial \sigma_i} [\delta(\sigma_1 - \sigma'_1) \delta(\sigma_2 - \sigma'_2)].
\]

These commutation relations are valid in an arbitrary Riemannian space where the indices 1 and 2 describe a time-like plane and the index 0 corresponds to the normal direction to the plane.

The area preserving diffeomorphism is defined as

\[
T(\sigma) = \frac{1}{\sqrt{g}}(\partial_1 T_{20}(\sigma) - \partial_2 T_{10}(\sigma)), \quad (4)
\]

where

\[g = \det g_{ij}(\sigma)\]
$g_{ij}(\sigma)$ is the metric tensor of the given time-like plane.

A straightforward calculation leads to the following commutation relation;

$$[T(\sigma), T'(\sigma')] = \frac{-i}{\sqrt{g}} (\partial_1 T(\sigma) \partial_2 \delta(\sigma - \sigma') - \partial_2 T(\sigma) \partial_1 \delta(\sigma - \sigma')).$$  \hspace{1cm} (5)

This commutation relation plays a fundamental role in our formulation of F and M theory which are basically membrane theories\(^{(4)}\).

Corresponding to each two dimensional field $A(\sigma)$ we can define the following operator $A$;

$$A = \int A(\sigma) T(\sigma) \sqrt{g} d\sigma_1 d\sigma_2$$  \hspace{1cm} (6)

Then for two operators $A$ and similar operator $B$ we get,

$$[A, B] = i \int \{A(\sigma), B(\sigma)\} T(\sigma) \sqrt{g} d\sigma_1 d\sigma_2$$  \hspace{1cm} (7)

with

$$\{A(\sigma), B(\sigma)\} \equiv \frac{1}{\sqrt{g}} \left( \frac{\partial A}{\partial \sigma_1} \frac{\partial B}{\partial \sigma_2} - \frac{\partial A}{\partial \sigma_2} \frac{\partial B}{\partial \sigma_1} \right).$$  \hspace{1cm} (8)

Equation (7) describes the most important property of our diffeomorphism algebra which relates directly to the gauge theory of the diffeomorphism to the theory of membranes.

It is easy to see that $T(\sigma)$ generates the area preserving diffeomorphism in the following way. Consider the transformation

$$U[f] = e^{\int f(\sigma) T(\sigma) \sqrt{g} d\sigma_1 d\sigma_2}$$  \hspace{1cm} (9)

Straightforward calculation gives:

if

$$A' = U^{-1} A U,$$  \hspace{1cm} (10)

then

$$A'(\sigma) = -\sqrt{g} \{f(\sigma), A(\sigma)\}$$  \hspace{1cm} (11)

for the infinitesimal function $f(\sigma)$.

The relation to the area preserving diffeomorphism studied previously\(^{(5)}\sim{(10)}\) is as follows.

Let $\{Y_{nm}(\sigma)\}$ be the complete set of functions on a given surface $S$. Then $T(\sigma)$ can be expanded in the following way;

$$T(\sigma) = \sum_{n,m} T_{nm} Y_{nm}(\sigma).$$  \hspace{1cm} (12)

With the following normalization and the completeness condition,

$$\int Y_{nm}(\sigma) Y_{nm'}^{*}(\sigma) \sqrt{g} d\sigma_1 d\sigma_2 = \delta_{nn'} \delta_{mm'},$$  \hspace{1cm} (13)

$$\sum_{n,m} Y_{nm}(\sigma) Y_{nm}^{*}(\sigma') = \frac{1}{\sqrt{g}} \delta(\sigma - \sigma'),$$  \hspace{1cm} (14)
we obtain
\[ [T_{nm}, T_{n'm'}] = i \int \{ Y_{nm}^*(\sigma), Y_{n'm'}^* \} T(\sigma) \sqrt{g} d\sigma_1 d\sigma_2. \]  

(15)

This commutation relation is equivalent to that of the area preserving diffeomorphism studied earlier. \( Y_{nm}(\sigma) \) is the ordinary spherical harmonics \( Y_{lm}(\theta, \phi) \) in the case of two dimensional sphere and the equation (15) reduces to
\[ [T_{lm}, T_{l'm'}] = ig'_{lm, l'm'} T_{l'' m''}, \]  

(16)

where \( g'_{lm, l'm'} \) is defined in reference (4).

As was shown by de Wit, Hoppe and Nicolai (4) this reduces to the SU(N) algebra when truncated to \( l \leq N - 1 \). This in turn means that \( \lim_{N \to \infty} SU(N) \) coincides with our area preserving algebra defined through the equation (5) in the special case of 2-dimensional sphere. We also get \( w^{(11)} \) algebra in the case of torus:
\[ [T_{\vec{m}}, T_{\vec{m}'}] = -\frac{i}{4\pi^2}(\vec{m} \times \vec{m}')T_{(\vec{m} + \vec{m}')}, \]  

(17)

where \( \vec{m} \times \vec{m}' = m_1 m_2' - m_1' m_2 \).

Gauge theory of area preserving diffeomorphism algebra

\( X^\mu (\mu = 0, \ldots, D - 1) \) stands for the D dimensional coordinate and \( X^\mu (\rho) \) describes a \( p \) dimensional space. A gauge field \( A_\mu (X(\rho)) \) is a function of \( \rho \) through \( X^\mu (\rho) \) (not a functional) and is an element of area preserving diffeomorphism algebra:
\[ A_\mu (X(\rho)) = \int A_\mu (X(\rho), \sigma) T(\sigma) \sqrt{g} d\sigma_1 d\sigma_2. \]  

(18)

D-dimensional Majorana spinor is also introduced in a similar manner:
\[ \psi(X(\rho)) = \int \psi(X(\rho), \sigma) T(\sigma) \sqrt{g} d\sigma_1 d\sigma_2. \]  

(19)

The Lagrangian for the gauge theory of area preserving diffeomorphism algebra is given by:
\[ L = -\frac{1}{4g^2}[T_r F_{\mu \nu} F^{\mu \nu} + 2i T_r \bar{\psi} \gamma^\mu [D_\mu, \psi]] \]  

(20)

with
\[ D_\mu = \frac{\delta}{\delta X^\mu (\rho)} + i A_\mu (X(\rho)), \]  

(21)

where the trace will be calculated using
\[ Tr T(\sigma) T(\sigma') = \frac{1}{\sqrt{g}} \delta(\sigma - \sigma'), \]  

(22)
and \( T_\nu T_\sigma = 0 \).
The Lagrangian is invariant under

\[
A'_\mu = U^{-1} A_\mu U - i U^{-1} \frac{\delta}{\delta X^\rho} U,
\]

and

\[
\psi' = U^{-1} \psi U
\]

where \( U \) is defined in the equation (9).
The action integral is given by:

\[
S = \int L d\Omega_\rho,
\]

where \( d\Omega_\rho \) is an appropriate measure in the \( \rho \)-space.

**Example 1. M theory**

In this case \( D=11 \) with the metric \((−+\cdots+)\). \( X^\rho(t) \) is chosen to be

\[
\begin{align*}
X^0 &= \tau, \\
X^{10} &= \rho, \\
X^1 &= X^2 = \cdots X^9 &= 0
\end{align*}
\]

This choice of \( X^\rho(t) \) can be interpreted to imply that the 10th space-like coordinate \( X^{10} \) is compactified.\(^{16}\)

This theory is now shown to coincide with the M-theory formulated by Banks, Fischler, Shenker and Susskind,\(^{17}\) and therefore, with the membrane theory of deWit, Hoppe and Nicolai\(^{4}\) in a certain limit.

First define \( X^\pm = X^{10} \pm X^0 \) and \( A_\pm = \frac{1}{\sqrt{2}} (A_0 \pm A_{10}) \).

The gauge condition \( (A_- = 0) \) will be imposed from here on.

This gauge condition will be supplemented as in the case of Green-Schwarz formulation of the superstring theory by the condition,

\[
\left( \frac{\delta X^\mu}{\delta \tau} - \frac{\delta X^\mu}{\delta \rho} \right) \alpha_\mu \psi = 0,
\]

where \( \alpha_\mu \) is defined as usual:

\[
\alpha_0 = \beta = \gamma_0, \quad i\alpha_a = \gamma_0 \gamma_a (a = 1, \ldots, 10).
\]

The meaning of this condition will become clear shortly. To eliminate the \( A_+ \) component entirely from the Lagrangian we go to the light cone system by the following Lorentz transformation:
\[
\begin{pmatrix} X_0 \\ X_{10} \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} X'_0 \\ X'_{10} \end{pmatrix}
\]
(29)

with \( \gamma = \frac{1}{\sqrt{1-v^2}} \) and take the limit \( v \to 1 \).

The bosonic part of the Lagrangian then becomes

\[
L_B = -\frac{T_{r}}{4g^2} \left[ 4(\partial_+ A_a \partial_- A_a) - ([A_a, A_b])^2 \right],
\]
(30)

with \( a, b = 1, \ldots, 9 \).

This is easily shown to be essentially equivalent to the bosonic part of the Lagrangian of deWit, Hoppe and Nicolai(4) with the help of the equations (7) and (22) in the limit \( R_{10} \to 0 \).

The vanishing compactification radius \( R_{10} \) can be taken to imply that our 1-brane formulation goes to the 0-brane formulation of Banks et al(17).

\( (\alpha_0, \ldots, \alpha_{10}) \) defined in equation (28) generate a Clifford algebra just as the original \( \gamma_{\mu} \). The spinorial condition (27) can be written as:

\[
\gamma_- \psi = 0,
\]
(31)

where

\[
\gamma_\pm = \frac{1}{\sqrt{2}} (\beta \pm \alpha_{10}) = \frac{\gamma_0}{\sqrt{2}} (1 \mp i\gamma_{10}) = \frac{\gamma_0}{\sqrt{2}} (1 \mp \gamma_{ch})
\]
(32)

and where \( \gamma_{ch} \) is the 10-dimensional chiral operator. This implies that the light-cone condition in the \( (\beta, \alpha) \) algebra is the chirality condition in the original \( \gamma_{\mu} \) algebra. This condition, therefore, guarantees the invariance of our Lagrangian under the following supersymmetric transformation:

\[
\delta A_\mu = \bar{\epsilon} \gamma_\mu \psi, \delta \psi_\alpha = \frac{1}{2} F_{\mu\nu} (\gamma^{\mu\nu} \epsilon)_\alpha + \epsilon'_\alpha
\]
(33)

The solution to equation (31) can be written as:

\[
\psi = \psi_+ = \frac{1}{\sqrt{2i}} \gamma_- \psi_-,
\]
(34)

@ with \( \gamma_+ \psi_- = 0 \).

The whole Lagrangian now takes the form

\[
L = -\frac{T_{r}}{4g^2} \left[ 4(\partial_+ A_a \partial_- A_a) - ([A_a, A_b])^2 + \sqrt{2i}\gamma_- \left\{ \gamma_- [D_0, \psi_-] + \gamma_- \alpha^a [A_a, \psi_-] \right\} \right].
\]
(35)

The \( R_{10} \to 0 \) limit of this Lagrangian is nothing but the M-theory Lagrangian proposed by Banks et al(17), based on the work of deWit et al.(4)
Example 2. F theory

The 12-dimensional spinor with the metric $(g_{0,0}, g_{1,1}, \cdots, g_{10,10}, g_{11,11}) = (-1, +1, \cdots, +1, -1)$ can have the Majorana-Weyl representation which is 32 dimensional. In addition to the 11-dimensional $\gamma$-matrices $\gamma_\mu (\mu = 0, \cdots, 10)$ we add $\gamma_{11} = 1$ to construct the 12-dimensional gauge theory. The Lagrangian is formally the same as in the equation (20)

$$L = -\frac{T_r}{4g^2} (F_{\mu\nu} F^{\mu\nu} + 2i \bar{\psi} \gamma^\mu [D_\mu, \psi]), \quad (36)$$

where

$$\mu, \nu = 0, 1, \cdots, 10, 11$$

and $\psi$ is a 32 component spinor.

$X^\mu (\sigma)$ in this case is defined by

$X^{11} = \tau, X^{10} = \rho$, and all the other $X^\mu = 0$,

and the spinor is assumed to satisfy

$$(i \frac{\delta X^\mu}{\delta \rho} - \frac{\delta X^\mu}{\delta \tau}) \gamma_\mu \psi = 0. \quad (37)$$

We define

$$A_\pm = \frac{1}{\sqrt{2}} (A_{11} - A_{10}) \quad (38)$$

and $X_\pm = X^{10} \pm X^{11}$. We take the $A_-=0$ gauge and go to the light cone gauge by the transformation defined in equation (29) expect that $X^0$ is replaced by $X^{11}$.

We then obtain

$$L = -\frac{T_r}{4g^2} \left[ 4g^{\mu\nu} \partial_+ A_\mu \partial_- A_\nu - g^{\mu\nu} g^{\rho\kappa} [A_\rho, A_\mu] [A_\kappa, A_\nu] - 2 \bar{\psi} \gamma^\mu [A_\mu, \psi] \right], \quad (39)$$

where $\mu, \nu = 0, \cdots, 9$.

This Lagrangian coincides with that of Ishibashi, Kawai, Kitazawa and Tsuchiya\(^{19}\) when the compactification radii in both 11 and 10 directions are taken to vanish.

*We are currently working on F-theory based on the volume preserving diffeomorphism. Let us call tentatively the 12-dimensional theory in this paper an F-theory simply because it is a 12-dimensional theory which leads to the type IIB string theory (to be published).
Example 3. $SU(\infty)$ QCD

In this case

$$L = -\frac{1}{4g^2} \left[ [T, F_{\mu\nu} F^{\mu\nu} - 2i T, \bar{\psi}\gamma^{\mu} [D_{\mu}, \psi] \right]$$

with $\mu, \nu = 0, 1, 2, 3$ and $X^\mu = \rho^\mu$. We use the notation $X^\mu$ rather than $\rho^\mu$ in this case.

The vacuum equation is $F_{\mu\nu} = 0$.

The solution to this equation will satisfy

$$\oint_C A_\mu dX^\mu = -i \int [A_\mu, A_\nu] dX^\mu dX^\nu$$

$$= \frac{1}{2} \int_S \varepsilon_{\alpha\beta\varepsilon_{ijk}} \int_{\Sigma} T(\sigma) A_i \partial_\alpha A_j \partial_\beta A_k d\sigma_1 d\sigma_2 dS,$$  \hspace{1cm} (40)

where $A_0 = 0$ gauge was taken. $dS$ is the surface element of $S$ which has the boundary $C$. $\Sigma$ is an internal sphere parametrized by $(\sigma_1, \sigma_2)$. $(n, A)$ is the component of $A$ in the normal direction to the surface $S$.

Following Arafune et al (20) we can rewrite this in the following way:

$$\oint_C A_\mu dX^\mu = d \int_S \int_{\Sigma} \frac{T(\sigma)}{(n, A)} \sqrt{G} d\sigma_1 d\sigma_2 dS,$$  \hspace{1cm} (41)

where $G = det(\partial A_i / \partial \sigma_1 \partial \sigma_2)$ and $d$ is the wrapping number.

In contrast to the case of Arafune et al (20) the internal space and the external space are interchanged in our case.

The vacuum of $SU(\infty)QCD$, therefore, is the state where the T’Hooft- Polyakov (21)(22) monopole in the internal space is condensed.

Super area preserving Algebra

As in the case of string theory where we have the light-cone formulation (Green-Schwartz formulation) on one hand and the Neveu-Schwartz-Ramond formulation with the world sheet supersymmetry on the other, our formulation of F and M theory should also be supplemented by another formulation which keeps the world sheet supersymmetry. For this purpose the two dimensional surface $(\sigma_1, \sigma_2)$ will be extended to the super surface parametrized by $(\sigma_1, \sigma_2, \theta_1, \theta_2)$ where $\theta_\alpha (\alpha = 1, 2)$ is the two component spinor with the Grassmannian property. The commutation relation (5) is now extended to:

$$[T(\sigma, \theta), T(\sigma', \theta')] = -\frac{i}{\sqrt{gg'}} \left\{ \frac{\partial T}{\partial \sigma_1} \partial_2 (\sigma - \sigma') \delta(\theta - \theta') \right\}$$
\[-\frac{\partial T}{\partial \sigma_2} \partial_1 \delta(\sigma - \sigma') \delta(\theta - \theta') - \frac{\partial T}{\partial \theta_\alpha} \frac{\partial}{\partial \theta_\alpha} \delta(\sigma - \sigma') \delta(\theta - \theta') \]  
\( (42) \)

It is easy to prove that for

\[ A_\mu = \int \sqrt{g} d\sigma_1 d\sigma_2 d\theta_1 d\theta_2 A_\mu(\sigma, \theta) T(\sigma, \theta), \]  
\( (43) \)

\[ [A_\mu, A_\nu] = i \int d\Omega \frac{T(\sigma, \theta)}{\sqrt{g}} \left( \frac{\partial A_\mu}{\partial \sigma_1} \frac{\partial A_\nu}{\partial \sigma_2} - \frac{\partial A_\mu}{\partial \sigma_2} \frac{\partial A_\nu}{\partial \sigma_1} - \frac{\partial A_\mu}{\partial \theta_\alpha} \frac{\partial A_\nu}{\partial \theta_\alpha} \right), \]  
\( (44) \)

where \( d\Omega = \sqrt{g} d\sigma_1 d\sigma_2 d\theta_1 d\theta_2 \).

This is consistent with the graded area preserving algebra studied earlier\(^9\). By expanding the \( T(\sigma, \theta) \):

\[ T(\sigma, \theta) = S(\sigma) + V_\alpha(\sigma) \theta_\alpha + T(\sigma) \theta_1 \theta_2, \]  
\( (45) \)

the commutation relation \((42)\) can be rewritten using these component fields,

\[ [S(\sigma), S(\sigma')] = 0, \]  
\( (46) \)

\[ [V_\alpha(\sigma), S(\sigma')] = -\frac{i}{\sqrt{gg'}} \varepsilon_{\alpha \beta} V_\beta(\sigma) \delta(\sigma - \sigma'), \]  
\( (47) \)

\[ [T(\sigma), S(\sigma')] = -\frac{i}{\sqrt{gg'}} \left( \frac{\partial S(\sigma)}{\partial \sigma_1} \partial_2 \delta(\sigma - \sigma') - \frac{\partial S(\sigma)}{\partial \sigma_2} \partial_1 \delta(\sigma - \sigma') \right), \]  
\( (48) \)

\[ [V_\alpha(\sigma), V_\beta(\sigma')] = -\frac{i}{\sqrt{gg'}} \left( \frac{\partial S(\sigma)}{\partial \sigma_1} \delta(\sigma - \sigma') - \frac{\partial S(\sigma)}{\partial \sigma_2} \partial_1 \delta(\sigma - \sigma') \right) \varepsilon_{\alpha \beta} \]  
\( + -\frac{i}{\sqrt{gg'}} T(\sigma) \delta(\sigma - \sigma') \delta_{\alpha \beta}, \)  
\( (49) \)

\[ [V_\alpha(\sigma), T(\sigma')] = -\frac{i}{\sqrt{gg'}} \left( \frac{\partial V_\alpha(\sigma)}{\partial \sigma_1} \partial_2 \delta(\sigma - \sigma') - \frac{\partial V_\alpha(\sigma)}{\partial \sigma_2} \partial_1 \delta(\sigma - \sigma') \right), \]  
\( (50) \)

and

\[ [T(\sigma), T(\sigma')] = -\frac{i}{\sqrt{gg'}} \left( \frac{\partial T(\sigma)}{\partial \sigma_1} \partial_2 \delta(\sigma - \sigma') - \frac{\partial T(\sigma)}{\partial \sigma_2} \partial_1 \delta(\sigma - \sigma') \right). \]  
\( (51) \)

\( T(\sigma) \) coincides with the original \( T(\sigma) \), and \( V_\alpha \) is expected to be written in terms of the 2 + 1 supercurrent. The gauge theory based on the graded algebra \((42)\) is expected to preserve the space-time supersymmetry after the GSO projection.

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References

[1] P.A.M. Dinac, Canad. J. Math. 3 (1951) 1
[2] J. Schwinger, Phys. Rev. 130 (1963) 406 ; 130 (1963) 800
[3] B.S. DeWitt, Phys. Rev. 160 (1967) 1113
[4] J. Hoppe, MIT Ph. D. Thesis (1982); B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. B305 (1988) 545; E. Bergshoeff, E. Sezgin, Y. Tanii and P. Townsend, preprint IC/88/186
[5] I. Bakas, preprint (April 1989)
[6] E. Bergshoeff, M. Blencowe and K. Stelle preprint, Imperial/TH/88-89/9
[7] E. Sezgin and E. Sokatchev, Phys. Lett. 227B (1989) 103
[8] C.N. Pope and P.K. Townsend, Phys. Lett. 225B (1989) 245
[9] E. Sezgin, in Proc. Strings '89 (Texas A & M) (1989)
[10] C.N. Pope and X. Shen, Phys. Lett. 236 (1990) 21
[11] For example, C.N. Pope, L.J. Romans and X. Shen, in Strings 90 (1991))287. This article contains extensive references.
[12] E.G. Floratos and J. Iliopoulos, Phys. Lett. B201 (1988) 237 ; E.G. Floratos, Phys. Lett. B228 (1989) 335
[13] E.G. Floratos, J. Iliopoulos and G. Tiktopoulos, Phys. Lett. B217 (1989) 285
[14] D.B. Fairlie, P. Fletcher and C.K. Zakos, Phys. Lett. B218 (1989) 203; D.B. Fairlie and C.K. Zakos, Phys. Lett. B224 (1989) 101; D.B. Fairlie, P. Fletcher and C.K. Zakos, J.Math.Phys. 31 (1990) 1088; C.K. Zakos, in Physics and Geometry, NATO ASI Series, Plenum (1990) 423
[15] An. Kavalov and B. Sakita, Ann, Phys. 255 (1997) 1
[16] W. Taylor, hep-th/9611042; O.J. Ganov, S. Ramgoolam and W. Taylor IV, hep-th/9602022
[17] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, hep-th/9610043
[18] C. Vafa, Nucl.Phys. B469 (1996) 403, hep-th/9602022
[19] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612113 (KEK-TH-503)
[20] J. Arafune, P.G.O. Freund and C.J. Goebal, J.Math.Phys. 16 (1975) 433
[21] G. 'tHooft, Nucl.Phys. B79 (1974) 276
[22] A.M. Polyakov, JETP Lett. 20 (1974) 194