CONTROLLABILITY AND QUALITATIVE PROPERTIES OF THE SOLUTIONS TO SPDES DRIVEN BY BOUNDARY LÉVY NOISE

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ABSTRACT. Let \( u \) be the solution to the following stochastic evolution equation
\[
\begin{aligned}
\left\{ \begin{array}{l}
\mathrm{d}u(t,x) = Au(t,x) \, dt + B \sigma(u(t,x)) \, dL(t), \quad t > 0; \\
u(0,x) = x
\end{array} \right.
\end{aligned}
\]

taking values in an Hilbert space \( H \), where \( L \) is a \( \mathbb{R} \) valued Lévy process, \( A : H \to H \) an infinitesimal generator of a strongly continuous semigroup, \( \sigma : H \to \mathbb{R} \) bounded from below and Lipschitz continuous, and \( B : \mathbb{R} \to H \) a possible unbounded operator. A typical example of such an equation is a stochastic Partial differential equation with boundary Lévy noise. Let \( \mathcal{P} = (\mathcal{P}_t)_{t \geq 0} \) the corresponding Markovian semigroup.

We show that, if the system
\[
\begin{aligned}
\left\{ \begin{array}{l}
\mathrm{d}u(t) = Au(t) \, dt + B v(t), \quad t > 0; \\
u(0) = x
\end{array} \right.
\end{aligned}
\]

is approximate controllable in time \( T > 0 \), then under some additional conditions on \( B \) and \( A \), for any \( x \in H \) the probability measure \( \mathcal{P}_t \delta_x \) is positive on open sets of \( H \). Secondly, as an application, we investigate under which condition on the Lévy process \( L \) and on the operator \( A \) and \( B \) the solution of Equation (1) is asymptotically strong Feller, respective, has a unique invariant measure. We apply these results to the damped wave equation driven by Lévy boundary noise.

1. INTRODUCTION

To present the aim of this paper, let \( H \) be a Hilbert space. Let \( u \) be the unique solution of the infinite dimensional system with Poissonian noise, formally written as
\[
\begin{aligned}
\left\{ \begin{array}{l}
\mathrm{d}u(t,x) = Au(t,x) \, dt + \int_{\mathbb{R}} B \sigma(u(t)) z \, \tilde{\eta}(dz,dt), \quad t > 0, \\
u(0,x) = x
\end{array} \right.
\end{aligned}
\]

In this equation, \( A : H \to H \) is a linear operator generating a strongly continuous semigroup on \( H \), \( B : \mathbb{R} \to H \) is a certain mappings specified later, \( \sigma : H \to \mathbb{R} \) bounded from below and Lipschitz continuous, and \( \eta : \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^+) \to \mathbb{N}_0 \cup \{ \infty \} \) is a compensated Poisson random measure over a probability space \( \Omega = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) and intensity measure \( \nu \). Let \( \mathcal{P} = (\mathcal{P}_t)_{t \geq 0} \) be the Markovian semigroup induced on \( H \), i.e.
\[
\mathcal{P}_t \phi(x) := \mathbb{E} \phi(u(t,x)), \quad x \in H, \quad t > 0, \quad \phi \in C(H).
\]

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A typical example of such an equation is a stochastic partial differential equation with boundary noise. The aim of this paper is to verify under which conditions on $A$, $B$ and $\eta$ the Markovian semigroup generated by the solution of (3) is irreducible and admits a unique invariant measure.

Regularity properties of the Markovian semigroups of stochastic processes play an important role in studying the long time behavior of the process. Concerning the uniqueness of the invariant measure of SPDEs driven by Lévy processes some results exist. One of the first results in this direction were established in the articles of Chojnowska-Michalik [7, 8]. Next, Fournier [14] investigated SPDEs driven by space time Poissonian noise. Applebaum analysed in [3] the analytic property of the generalised Mehler semigroup induced by Lévy noise and in [2] the self-decomposability of a Lévy noise in Hilbert space. Further works are the two articles of Priola and Zabczyk [27, 28]. We also refer to [18], [29], [30] for some recent results and review of progress for the study of the ergodicity of the Markovian semigroup associated to the solution of a Lévy driven SPDEs. The proofs of the results in [28, 29, 30] rely on the cylindrical and $\alpha$-stability of the noise, hence their approach does not cover the case we are treating in this paper.

In the present work we show that if the system (2) is null controllable, then the Markovian semigroup of solutions to (1) is irreducible. We applied our result to stochastic evolution equation with Lévy noise boundary conditions. For results related to SPDEs with white-noise boundary condition we refer to [11], [20], [6]. For stochastic evolution equation driven by Wiener noise a similar result was established long ago. Indeed the Markovian semigroup of an Ornstein-Uhlenbeck is irreducible and strong Feller if (2) is null controllable. For this result we refer to the books of Da Prato and Zabczyk [11] and [13] and references therein. We also note that in the present paper we prove the uniqueness of invariant measure for the Markovian semigroup of solution to (1) if a certain notion of null controllability is satisfied by (2). In fact if (2) is null controllable with vanishing energy (see Section 3 for the definition), then we are able to show that the Markovian semigroup of (1) satisfies the asymptotic strong Feller property. The irreducibility and the asymptotic strong Feller property which is introduced by Hairer and Mattingly in [15] will imply the uniqueness of invariant measure. For SPDEs driven by Lévy noise it is proved in [31] that the null controllability implies the strong Feller property of the solution to Ornstein-Uhlenbeck system driven by Lévy noise with non-zero Gaussian part (see [31, Corollary 1.2]). Unfortunately the result in [31] tells us nothing about the property of the Markovian semigroup when we consider an Ornstein-Uhlenbeck driven by pure jump noise. Hence our work is an extension of the results in [11], [13] and [31], in the sense that we can prove uniqueness of invariant measure for SPDEs driven by multiplicative and pure jump noise.

The structure of the paper is the following. In Section 2 we give the hypotheses used throughout the paper and prove an important relation between the irreducibility property and approximate null controllability. Roughly speaking we could prove in Section 2 that any ball centered at the origin (resp., at any $x \in H$) has positive measure if (2) is approximate (resp., exactly) null controllable. Section 3 is devoted to the proof of the uniqueness of the invariant measure of the Markovian semigroup associated to the solution of (3). In fact, we established that the Markovian semigroup satisfies the asymptotic strong Feller property if (2) is null controllable with vanishing energy. The asymptotic strong Feller and the irreducibility of the aforementioned semigroup implies the uniqueness of the invariant measure. We apply our results in Section 4 to a damped wave equations driven by boundary Lévy noise. The last part of the paper is some appendices collecting some technical results about the change of measure. The proofs of our results are a combination of the change of measure formula given
by Bismuth, Graveraux and Jacod \cite{Bismuth:1986} and Sato \cite{Sato:1999} (see also \cite{Jacod:2003}) and the method used by Maslowski and Seidler \cite{Maslowski:2002}.

**Notation 1.** Let $\mathbb{R}^+ := (0, \infty)$, $\mathbb{R}_0^+ := (0, \infty)$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\overline{\mathbb{N}} := \mathbb{N}_0 \cup \{\infty\}$. Let $(Z, \mathcal{Z})$ be a measurable space. By $M_+(Z)$ we denote the family of all positive measures on $Z$, by $M_+(Z)$ we denote the σ-field on $M_+(Z)$ generated by functions $i_B : M_+(Z) \ni \mu \mapsto \mu(B) \in \mathbb{R}_+, B \in \mathcal{Z}$. By $M^1(Z)$ we denote the family of all σ-finite integer valued measures on $Z$, by $M^1(Z)$ we denote the σ-field on $M^1(Z)$ generated by functions $i_B : M^1(Z) \ni \mu \mapsto \mu(B) \in \overline{\mathbb{N}}, B \in \mathcal{Z}$. By $M^+_i(Z)$ we denote the σ-field on $M^+_i(Z)$ generated by functions $i_B : M^+_i(Z) \ni \mu \mapsto \mu(B) \in \mathbb{R}, B \in \mathcal{Z}$. We denote by $B(\mathcal{Z})$ the set of all Borel measurable, real-valued, bounded functions.

For a Hilbert space $H$, by $C_0(H)$ the space of all uniformly continuous and bounded mappings $\phi : H \to \mathbb{R}$ endowed with the norm $|\phi|_\infty = \sup_{x \in H} |\phi(x)|$.

2. **Irreducibility of the Markovian semigroup associated to the equation (1)**

One way to handle Lévy processes is to work with the associated Poisson random measure. In this section we will define the setting in which the results can be formulated. We start with defining a time homogenous Poisson random measure.

**Definition 2.1.** Let $(Z, \mathcal{Z})$ be a measurable space and let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. A time homogeneous Poisson random measure $\eta$ on $(Z, \mathcal{Z})$ over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, is a measurable function $\eta : (\Omega, \mathcal{F}) \to (M^1(Z \times [0, \infty)), \mathcal{M}^1(Z \times [0, \infty)))$, such that

(i) $\eta(\emptyset \times I) = 0$ a.s. for $I \in \mathcal{B}(\{0, \infty\})$ and $\eta(A \times \emptyset) = 0$ a.s. for $A \in \mathcal{Z}$;
(ii) for each $B \times I \in \mathcal{Z} \times \mathcal{B}(\{0, \infty\})$, $\eta(B \times I) := i_B \circ \eta : \Omega \to \overline{\mathbb{N}}$ is a Poisson random variable with parameter $\nu(B)\lambda(I)$.
(iii) $\eta$ is independently scattered, i.e. if the sets $B_j \times I_j \in \mathcal{Z} \times \mathcal{B}(\{0, \infty\}), j = 1, \cdots, n$, are pairwise disjoint, then the random variables $\eta(B_j \times I_j), j = 1, \cdots, n$ are mutually independent.
(iv) for each $U \in \mathcal{Z}$, the $\overline{\mathbb{N}}$-valued process $(N(t, U))_{t \geq 0}$ defined by

\[ N(t, U) := \eta(U \times (0, t]], \ t > 0 \]

is $(\mathcal{F}_t)_{t \geq 0}$-adapted and its increments are independent of the past, i.e. if $t > s \geq 0$, then $N(t, U) - N(s, U) = \eta(\{s, t]\times U)$ is independent of $\mathcal{F}_s$.

The measure $\nu$ defined by

\[ \nu : \mathcal{Z} \ni A \mapsto \mathbb{E}\eta(A \times (0, 1]) \in \overline{\mathbb{N}} \]

is called the intensity of $\eta$.

If the intensity of a Poisson random measure is a Lévy measure, then one can construct from the Poisson random measure a Lévy process. Vice versa, tracing the jumps, one can find a Poisson random measure associated to each Lévy process. For more details on this relationship we refer to \cite{Jacod:2003}.

Let $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability measure with right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, $\eta$ be a time homogeneous Poisson random measure on $\mathbb{R}$ over $\mathfrak{A}$ with intensity

\[ \lambda(I) = \infty, \text{ then obviously } \eta(B \times I) = \infty \text{ a.s.} \]
ν being a Lévy measure and compensator γ defined by
\[ \gamma : \mathcal{B}(\mathbb{R}) \times \mathcal{B}([0, \infty)) \ni (A \times I) \mapsto \gamma(A \times I) := \nu(A) \lambda(I) \in \mathbb{R}_0^+. \]

**Hypothesis 1.** We assume that the Lévy measure has a density k and there exist an index \( \alpha \in (1, 2] \) and constants \( K_0 > 0 \) and \( r_0 > 0 \) such that
\[ k(r) = K_0 |r|^{-\alpha - 1}, \quad \text{for all} \quad |r| \geq r_0. \]

From here and throughout the rest of the paper, let us assume that \( H \) is a Hilbert space, \( A : H \to H \) a generator of a strongly continuous semigroup \((e^{-tA})_{t \geq 0}\) on \( H \) and \( B : \mathbb{R} \to D(A^{-\gamma}) \) is bounded for some \( \gamma < \frac{1}{2}. \) Also let \( \sigma : H \to \mathbb{R} \) be a Lipschitz mapping satisfying
\[ C_\sigma < |\sigma(u)| \leq \ell (1 + |u|), \]
for some positive constants \( C_\sigma, \ell \) and for any \( u \in H. \) Let \( u \) be the solution of the following stochastic evolution equation
\[
\begin{aligned}
\begin{cases}
\displaystyle du(t, x) &= Au(t, x) + \int_{\mathbb{R}} B\sigma(u(t, x)) z\tilde{\eta}(dz, ds), \\
u(t, x) &= x \in H.
\end{cases}
\end{aligned}
\]

Typical examples of such system are SPDEs with boundary noise and are presented in the following examples (for more details we refer to section II).

**Example 2.2.** We consider the vibration of a string of length \( 2\pi \) where one end is fixed and the other end is perturbed by a Lévy noise. To be more precise, let \( T > 0 \) and \( \alpha > 0. \) We consider the system
\[
\begin{aligned}
\begin{cases}
\displaystyle u_{tt}(t, \xi) - u_{\xi\xi}(t, \xi) + \alpha u(t, \xi) &= 0, \quad t \in (0, T), \quad \xi \in (0, 2\pi), \\
u(t, 0) &= 0, \quad t \in (0, T), \\
u(\xi, 2\pi) &= \log(2 + |u(t)|_{L^2(\Omega)}) \dot{L}_t, \quad t \in (0, T), \\
u(0, \xi) &= x_0(\xi), \quad \nu(0, \xi) = x_1(\xi), \quad \xi \in (0, 2\pi),
\end{cases}
\end{aligned}
\]

where \( \dot{L} \) is the Radon Nikodym derivative of a real valued Lévy process with intensity measure \( \nu, \ x_0 \in H^1_0(0, 2\pi) \) and \( x_1 \in L^2(0, 2\pi). \)

**Example 2.3.** We consider a one–dimensional rod \((0, 1). \) A Lévy noise is added at the boundary \( \xi = 1, \) while the boundary \( \xi = 0 \) is assumed to be perfectly isolated. To be more precise, let \( T > 0. \) We consider the system
\[
\begin{aligned}
\begin{cases}
\displaystyle u_t(t, \xi) - u_{\xi\xi}(t, \xi) &= 0, \quad t \in (0, T), \quad \xi \in (0, 1), \\
u(t, 0) &= 0, \quad t \in (0, T), \\
u(\xi, 1) &= \dot{L}_t, \quad t \in (0, T), \\
u(0, \xi) &= x_0(\xi), \quad \xi \in (0, 1).
\end{cases}
\end{aligned}
\]

Here \( \dot{L} \) is the Radon Nikodym derivative of a real valued Lévy process with intensity measure \( \nu, \ x_0 \in L^2(0, 1). \)

---

2 A Lévy measure on \( \mathbb{R} \) is a \( \sigma \)-finite measure such that \( \nu(\{0\}) = 0 \) and \( \int_\mathbb{R}(z^2 \wedge 1)\nu(dz) < \infty. \)
The existence of solution to the stochastic equations in these examples can be established by fixed point argument as used in [23] and [20]. If \( \int_{\mathbb{R}} |z|^2 \, \nu(dz) < \infty \), then the Markovian semigroup \( (\mathcal{P}_t)_{t \geq 0} \) defined by
\[
\mathcal{P}_t \phi(x) := \mathbb{E}\phi(u(x,t)), \quad \phi \in \mathcal{C}_b(H), \quad t \geq 0,
\]
is a stochastically continuous Feller semigroup on \( \mathcal{C}_b(H) \). That is \( (\mathcal{P}_t)_{t \geq 0} \) satisfies (see [10])

1. \( \mathcal{P}_t \mathcal{P}_s = \mathcal{P}_{t+s} \);

2. for all \( \phi \in \mathcal{C}_b(H) \) and for all \( x \in H \) we have \( \lim_{t \to 0} \mathcal{P}_t \phi(x) = \phi(x) \).

Item (1) is clear. In order to verify (2) let \( \phi \in \mathcal{C}_b(H) \) with \( |\phi|_\infty = 1 \). Item (2) follows by the fact that \( \lim_{t \to 0} \mathbb{E}\phi(u(t,x)) = \phi(x) \), or, for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that
\[
|\mathbb{E}\phi(u(t,x)) - \phi(x)| \leq \epsilon \quad \text{for all } 0 \leq t < \delta.
\]

Fix \( \epsilon > 0 \). Since \( \phi \) is uniformly continuous on \( H \), there exists a \( \delta_1 > 0 \) such that \( |\phi(x) - \phi(y)| \leq \frac{\epsilon}{2} \) for all \( x, y \in H \), \( |x - y|_H \leq \delta_1 \). Then for \( t \leq \delta := \frac{\epsilon}{6} \delta_1^2 \) we know by the Chebyshev inequality that
\[
\mathbb{P}(|u(t,x) - x|_H \geq \delta_1) \leq \frac{\epsilon}{2}.
\]

It follows that Markovian \( (\mathcal{P}_t)_{t \geq 0} \) on \( \mathcal{C}_b(H) \) is a stochastically continuous.

Before continuing we would like to introduce some definitions from control theory. Again, \( H \) denotes a Hilbert space, \( A : H \to H \) a generator of a strongly continuous semigroup \( (e^{-tA})_{t \geq 0} \) on \( H \) and \( B : \mathbb{R} \to H \). Fix \( T > 0 \). Then we say that the system
\[
\begin{cases}
\dot{u}^c(t,x,v) = Au^c(t,x,v) + Bv(t), & t \geq 0, \\
u^c(0,x,v) = x,
\end{cases}
\]
is \textit{null controllable in time} \( T \), iff for any \( x \in H \) there exists a \( v \in L^2([0,T];\mathbb{R}) \) such that \( u^c(T,x,v) = 0 \). We say that the system (3) is \textit{approximate null controllable in time} \( T \), iff for any \( x \in H \) and \( \epsilon > 0 \) there exists a \( v \in L^2([0,T];\mathbb{R}) \) such that \( |u^c(T,x,v)|_H \leq \epsilon \). We say that the system (3) is \textit{controllable in time} \( T \) in \( x \in H \) if for each \( y \in H \) and \( \epsilon > 0 \) there exists a control \( v \in L^2([0,T];\mathbb{R}) \) such that \( u^c(T,x,v) = y \). We say that the system (3) is \textit{approximate controllable in time} \( T \) in \( x \in H \) if for each \( y \in H \) and \( \epsilon > 0 \) there exists a control \( v \in L^2([0,T];\mathbb{R}) \) such that \( |u^c(T,x,v) - y| \leq \epsilon \).

**Remark 1.**

- The system (3) associated to the wave equation with boundary control described in Example (2.2) is exactly controllable in time \( T > 2\pi \) (Zuazua [34]);
- The system (3) associated to the heat equation with Neumann boundary control described in Example (2.3) is approximate controllable in time \( T > 0 \) (Laroche, Martin and Rouchon [19]).

For all \( C > 0 \) we set
\[
\mathcal{D}_H(C) := \{ z \in H : |z| \leq C \}.
\]

Let \( u \) be the solution of the stochastic evolution equation
\[
\begin{cases}
du(t,x) = Au(t,x) \, dt + \int_{\mathbb{Z}} B \, z\eta(dz,dt) \\
u(0,x) = x.
\end{cases}
\]

Then the following Theorem can be shown.
Theorem 2.4. Assume that the system (8) is approximate null controllable in time $T > 0$ and that Hypothesis 1 is satisfied. Let $u$ be a solution of Eq. (9). Fix $x \in H$. Then for any $\delta > 0$ there exists a $\kappa > 0$ such that

$$\mathbb{P}(u(T, x) \in \mathcal{D}_H(\delta)) \geq \kappa. \quad (10)$$

In case the system is exactly controllable the result of the above theorem can be strengthened as follows.

Theorem 2.5. Assume that the system (8) is exactly null controllable in time $T > 0$ and that Hypothesis 1 is satisfied. Let $u$ be a solution of Eq. (9). Then for all $C > 0$, for all $x \in \mathcal{D}_H(C)$ and all $\delta > 0$ there exists $\kappa > 0$ such that

$$\mathbb{P}(u(T, x) \in \mathcal{D}_H(\delta)) \geq \kappa. \quad (11)$$

Remark 2. If the system (8) is controllable for a time $T$, then one can replace the disk $\mathcal{D}_H(C)$ centered at the point 0 with a disk centered at any point $y \in H$.

In case the solution $u$ is càdlàg in $H$, the result can be strengthened as well. Let $u$ be the solution of the stochastic evolution equation

$$
\left\{
\begin{array}{l}
\frac{du(t, x)}{dt} = Au(t, x) dt + \int_Z B \sigma(u(t, x)) z \tilde{\eta}(dz, dt) \\
u(0, x) = x,
\end{array}
\right.
$$

where $\sigma : H \to \mathbb{R}$ is a Lipschitz mapping of linear growth and such that for certain $C_\sigma > 0$ we have $|\sigma(x)| \geq C_\sigma$, $\forall x \in H$.

Then the following two Theorems can be shown.

Theorem 2.6. Assume that the system (8) is approximate null controllable in time $T > 0$ and that Hypothesis 1 is satisfied. Let $u$ be a solution of Eq. (12). If $u$ is càdlàg in $H$, then for any $\delta > 0$ there exists a $\kappa > 0$ such that

$$\mathbb{P}(u(T, x) \in \mathcal{D}_H(\delta)) \geq \kappa. \quad (13)$$

Theorem 2.7. Assume that the system (8) is exactly null controllable in time $T > 0$ and that Hypothesis 1 is satisfied. Let $u$ be a solution of Eq. (12). If $u$ is càdlàg in $H$, then for all $C > 0$, for all $x \in \mathcal{D}_H(C)$ and all $\delta > 0$ there exists $\kappa > 0$ such that

$$\mathbb{P}(u(T, x) \in \mathcal{D}_H(\delta)) \geq \kappa. \quad (14)$$

Example 2.8. In Section 4 we will see that the linear problem (8) for Example 2.2 is exactly controllable and the solution is càdlàg in $L^2(O)$. That means, the solution of system (6) satisfies the assumptions of Theorem 2.7.

Example 2.9. The linear system (8) for Example 2.3 is only approximately controllable and the solution is not càdlàg in $L^2(O)$. Thus the solution of system (7) does not satisfy the assumptions of Theorem 2.5, Theorem 2.6 nor Theorem 2.7. However, the assumptions of Theorem 2.4 are satisfied.

Proof of Theorem 2.4. We will switch for technical reasons to another representation of the Poisson random measure. Let $\mathfrak{F} = (\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space with filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ and let $\mu$ be a Poisson random measure on $\mathbb{R}$ over $\mathfrak{F}$ having intensity measure $\lambda$ (Lebesgue measure). The compensator of $\mu$ is denoted by $\gamma$ and given by

$$\mathcal{B}(\mathbb{R}) \times \mathcal{B}([0, \infty)) \ni A \times I \mapsto \gamma(A \times I) := \lambda(A) \lambda(I).$$
Let
\[ c : \mathbb{R}^+ \ni r \mapsto \sup_{\rho > 0} \left\{ \int_{\rho}^{\infty} k(s) \, ds \geq r \right\} \quad \text{if } r > 0. \]  
\[ \text{(15)} \]

**Remark 3.** Observe that Hypothesis 4 implies that there exists a number \( r_1 > 0 \) and a constant \( \delta_0 \) such that for all \( r \geq r_1 \)

\[ c(r) = \delta_0 r^{-\frac{1}{\alpha}}, \quad \text{and} \quad c(-r) = -\delta_0 r^{-\frac{1}{\alpha}}. \]  
\[ \text{(16)} \]

A short calculation shows that the distributions of \( L = \{L(t) : 0 \leq t < \infty\} \) and \( L^c = \{L^c(t) : 0 \leq t < \infty\} \) are equal, where

\[ L(t) := \int_0^t \int_{\mathbb{R}} z \tilde{\eta}(dz, ds), \quad t \geq 0, \]

and

\[ L^c(t) := \int_0^t \int_{\mathbb{R}} c(z) \tilde{\mu}(dz, ds), \quad t \geq 0. \]

Now, the stochastic evolution equation given in (9) reads as follows
\[ \begin{cases} du(t, x) = Au(t, x) + \int_{\mathbb{R}} B c(z) \tilde{\mu}(dz, ds), \\ u(0, x) = x \in H. \end{cases} \]  
\[ \text{(17)} \]

Fix \( \delta > 0 \), \( T > 0 \) and \( x \in D_H(C) \). In order to prove Lemma 2.4 we need a result from control theory. Given \( v \in L^2([0, \infty); \mathbb{R}) \), let \( u^c \) be the solution to (see system (8))
\[ \begin{cases} du^c(t, x, v) = Au^c(t, x, v) dt + Bv(t) dt, \\ u^c(0, x, v) = x. \end{cases} \]  
\[ \text{(18)} \]

Since the system (15) is approximate null controllable, there exists \( v \in L^2([0, \infty); \mathbb{R}) \) such that
\[ |u^c(T, x, v)| \leq \frac{\delta}{3}. \]  
\[ \text{(19)} \]

Choose \( R \geq r_1 \) such that
\[ \left( \frac{3}{\delta} \right)^2 C T R^{1 - \frac{\alpha}{2}} \leq \frac{1}{2}, \]
(here, \( C \) is a generic constant, not depending on \( \delta, T \) and \( R \), see (25)) and put
\[ g_R = \int_{D_R(R)} c(z) \lambda(dz). \]  
\[ \text{(20)} \]

Let \( \theta : \Omega \times [0, T] \) be a predictable transformation of \( \mathbb{R} \) such that
\[ v(s) + g_R = \int_{\mathbb{R}\setminus B_R(\rho)} [c(z) - c(\theta(s, z))] \lambda(dz), \quad s \in [0, T]. \]

The existence of such a transformation is given by Lemma A.1. Let \( \mu_\theta \) the following random measure defined by
\[ \mu_\theta : \mathcal{B}(\mathbb{R}) \times \mathcal{B}([0, T]) \ni (A \times I) \mapsto \int_{\mathbb{R}^2} \int_{I} 1_A(\theta(s, z)) \mu(dz, ds). \]
Let $\mathcal{Q}$ be the probability measure on $\mathfrak{A}$ such that $\mu_\theta$ has compensator $\gamma$. Then, the process $u^\theta_\mu$ defined by

$$
\begin{align*}
\left\{ \begin{array}{l}
\quad \text{due to Lemma B.1 the density process} \quad G \quad \text{and} \quad N \\
\quad \text{have under} \quad \mathcal{Q} \quad \text{the same law as} \quad u, \quad \text{in particular}
\end{array} \right.
\end{align*}
$$

$$
E^\mathbb{P} \left[ |u(T, x_0)| \right] = E^\mathbb{Q} \left[ |u^\theta_\mu(T, x_0)| \right].
$$

Due to Lemma B.1 the density process $G_\theta(t) = \frac{dG^\theta_{t}}{dz}$ satisfy the following stochastic differential equation (see also page 21)

$$
\begin{align*}
\left\{ \begin{array}{l}
\quad dG_\theta(t) = G_\theta(t) (\rho_z(\kappa(|v(s)|), z)) \text{sign}(v(s))) \, (\mu - \gamma) \, (dz, ds) \\
\quad G_\theta(0) = 1.
\end{array} \right.
\end{align*}
$$

Here $\rho$ is defined in (63) and $\kappa$ is the inverse of $\theta$ and defined on page 28. By the choice of $\rho$, we know that $G_\theta$ of finite variation and we obtain for $0 \leq t \leq T$

$$
E^\mathbb{P} \sup_{0 \leq s \leq t} |G_\theta(s)| \leq 1 + E^\mathbb{P} \int_0^t \int_{\mathbb{R}^+} |G_\theta(s^-)| |\rho_z(\kappa(|v(s)|), z)| \, dz \, ds,
$$

respectively, we obtain for $0 \leq r_1 \leq T$

$$
E^\mathbb{P} \sup_{0 \leq s \leq r_1} |G_\theta(s)| \leq 1 + E^\mathbb{P} \int_0^{r_1} \int_{\mathbb{R}^+} |G_\theta(s^-)| |\rho_z(\kappa(|v(s)|), z)| \, dz \, ds
$$

$$
\leq 1 + E^\mathbb{P} \sup_{0 \leq s \leq r_1} |G_\theta(s)| \times \int_0^{r_1} \int_{\mathbb{R}^+} |\rho_z(\kappa(|v(s)|), z)| \, dz \, ds.
$$

By Corollary A.3 we have

$$
E^\mathbb{P} \sup_{0 \leq s \leq r_1} |G_\theta(s)| \leq 1 + E^\mathbb{P} \sup_{0 \leq s \leq r_1} |G_\theta(s)| \times \int_0^{r_1} |v(s)|^2 \, ds.
$$

Now, if $\int_0^{r_1} |v(s)|^2 \, ds \leq \frac{1}{2}$, we get

$$
E^\mathbb{P} \sup_{0 \leq s \leq r_1} |G_\theta(s)| \leq 1 \times \left( \frac{1}{2} \right)^{-1}.
$$

Since $v \in L^2([0, T]; \mathbb{R})$, there exists a partition $\{t_j : 1 \leq j \leq N\}$ of $[0, T]$ with

$$
\int_{t_{j-1}}^{t_j} |v(s)|^2 \, ds \leq \frac{1}{2}, \quad j = 2, \ldots, N,
$$

and $N \leq \lfloor 2|v|_{L^2([0, T]; \mathbb{R})} \rfloor + 1$. Therefore,

$$
E^\mathbb{P} \sup_{0 \leq s \leq t_j} |G_\theta(s)| \leq \left( \frac{1}{2} \right)^{-j},
$$

and we can conclude that there exists a constant $C > 0$

$$
E^\mathbb{P} \sup_{0 \leq s \leq T} |G_\theta(s)| \leq C \, 2^{|v|_{L^2([0, T]; \mathbb{R})}^2}.
$$
On the other hand we know that under $\bar{P}$ the process $u^\theta_{\mu}$ follows the following differential equation

\begin{equation}
\begin{cases}
  du^\theta_{\mu}(t, x) = Au^\theta_{\mu}(t, x)dt + \int_{R} B [c(\theta(t, z)) - c(z)] (\mu - \gamma)(dz, dt) \\
  \quad + \int_{R} B [c(\theta(t, z)) - c(z)] \gamma(dz, dt) + \int_{R} Bc(z)(\mu - \gamma)(dz, dt), \\
  u^\theta_{\mu}(0, x) = x.
\end{cases}
\end{equation}

Hence we can write

\[ \bar{P}(|u(T, x)| \leq \delta) = Q(|u^\theta_{\mu}(T, x)| \leq \delta) = \mathbb{E}^Q \left[ 1_{|u^\theta_{\mu}(T, x)| \leq \delta} \right] = \mathbb{E}^\bar{P} \left[ G_\theta(T)1_{|u^\theta_{\mu}(T, x)| \leq \delta} \right]. \]

By the inverse Hölder inequality we get

\[ \bar{P}(|u(T, x)| \leq \delta) \geq \frac{\mathbb{E}^\bar{P} \left[ 1_{u^\theta_{\mu}(T, x)} \leq \delta \right]}{\mathbb{E}^\bar{P} \left[ G_\theta(T) \right]}. \]

The denominator, i.e. $\mathbb{E}^\bar{P} \left[ |G_\theta(T)| \right]$, is bounded. In particular, we have by (30) and Hypothesis 1 that

\[ \mathbb{E}^\bar{P} \left[ |G_\theta(T)| \right] \leq C' \exp \left( r_1^{\frac{\alpha_1 + 1}{\alpha}} \int_0^T |v(s)|^2 ds \right). \]

Next, we handle the numerator. Observe that by (19)

\begin{equation}
\begin{align*}
  \mathbb{E}^\bar{P} \left[ 1_{u^\theta_{\mu}(T, x)} \leq \delta \right] & \geq \mathbb{E}^\bar{P} \left[ 1_{u_\mu^\theta(T, x, v) \leq \delta} 1_{u_\mu^\theta(T, x, v) - u_\mu^\theta(T, x) \leq \frac{\delta}{\alpha}} \right] = \mathbb{E}^\bar{P} \left[ 1_{u_\mu^\theta(T, x, v) - u_\mu^\theta(T, x) \leq \frac{\delta}{\alpha}} \right].
\end{align*}
\end{equation}

Rewriting the difference $\Delta(T) = u^\theta_{\mu}(T, x) - u^\theta_{\mu}(T, x, v)$ as follows

\[ \Delta(T) = \int_0^T \int_{R} e^{-(t-s)A} B [c(\theta(t, z)) - c(z)] (\mu - \gamma)(dz, ds) \]
\[ + \int_0^T \int_{R} e^{-(t-s)A} Bc(z)(\mu - \gamma)(dz, ds) + \int_0^T e^{-(t-s)A} BgR ds \]
\[ = \int_0^T \int_{R} e^{-(t-s)A} Bc(\theta(t, z))(\mu - \gamma)(dz, ds) \]
\[ + \int_0^T e^{-(t-s)A} BgR ds. \]

we have

\[ \mathbb{E}^\bar{P} \left[ 1_{u^\theta_{\mu}(T, x)} \leq \delta \right] \geq \mathbb{E}^\bar{P} \left[ 1_{|\Delta(T)| \leq \delta} \right]. \]
To give a lower estimate of $\mathbb{P} (|\Delta(T)| \leq \frac{\delta}{3})$ we apply the Bayes Theorem and get

\[
\mathbb{P} \left( |\Delta(T)| \leq \frac{\delta}{3} \right) = \mathbb{P} \left( \mu(B_\mathbb{R}(R) \times [0, T]) = 0 \right) \\
\times \mathbb{P} \left( \left\| \int_0^{T_\rho} \int_{\mathbb{R}\setminus \mathcal{D}_\mathbb{R}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right\| \leq \frac{\delta}{3} \left| \mu(\mathcal{D}_\mathbb{R}(R) \times [0, T]) \right| = 0 \right) \\
+ \mathbb{P} \left( \left( \left\| \int_0^{T_\rho} \int_{\mathbb{R}} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right\| \leq \frac{\delta}{3} \left| \mu(\mathcal{D}_\mathbb{R}(R) \times [0, T]) \right| > 0 \right) \\
\times \mathbb{P} \left( \left\| \int_0^{T_\rho} \int_{\mathbb{R}\setminus \mathcal{D}_\mathbb{R}(R)} e^{-(t-s)A} Bg_{R\rho} ds \right\| \leq \frac{\delta}{3} \left| \mu(\mathcal{D}_\mathbb{R}(R) \times [0, T]) \right| = 0 \right) \\
\geq \mathbb{P} \left( \left( \left\| \int_0^{T_\rho} \int_{\mathbb{R}\setminus \mathcal{D}_\mathbb{R}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right\| \leq \frac{\delta}{3} \left| \mu(\mathcal{D}_\mathbb{R}(R) \times [0, T]) \right| = 0 \right) \right)
\]

By the Chebyshev inequality we know that

\[
\mathbb{P} \left( \left\| \int_0^{T_\rho} \int_{\mathbb{R}\setminus \mathcal{D}_\mathbb{R}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right\| \geq \frac{\delta}{3} \left| \mu(\mathcal{D}_\mathbb{R}(R) \times [0, T]) \right| = 0 \right) \leq \left( \frac{3}{\delta} \right)^2
\]

\[
\times \mathbb{E}^\mathbb{P} \left[ \left\| \int_0^{T_\rho} \int_{\mathbb{R}\setminus \mathcal{D}_\mathbb{R}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right\|^2 \left| \mu(\mathcal{D}_\mathbb{R}(R) \times [0, T]) \right| = 0 \right].
\]

Note that due to the fact that the random variables $\mu(\mathcal{D}_\mathbb{R}(R) \times [0, T])$ and $\mu(A \times [0, T])$ for all $A \in \mathcal{B}(\mathbb{R} \setminus \mathcal{D}_\mathbb{R}(R))$ are independent, we get

\[
\mathbb{E}^\mathbb{P} \left[ \left\| \int_0^{T_\rho} \int_{\mathbb{R}\setminus \mathcal{D}_\mathbb{R}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right\|^2 \left| \mu(\mathcal{D}_\mathbb{R}(R) \times [0, T]) \right| = 0 \right] = \mathbb{E}^\mathbb{P} \left[ \left\| \int_0^{T_\rho} \int_{\mathbb{R}\setminus \mathcal{D}_\mathbb{R}(R)} e^{-(t-s)A} Bc(\theta(s, z))(\mu - \gamma)(dz, ds) \right\|^2 \right].
\]
The Burkholder inequality and the fact that \(c(\theta(t,z)) \leq c(z)\) give

\[
\mathbb{E}^\mathbb{P}\left[ \int_0^T \int_{\mathbb{R}\setminus D_H(R)} e^{-(t-s)A} Bc(\theta(s,z))(\mu - \gamma)(dz,ds) \right]^2 \\
\leq \mathbb{E}^\mathbb{P}\left[ \int_0^T \int_{\mathbb{R}\setminus D_H(R)} |e^{-(t-s)A} Bc(\theta(s,z))|^2 \lambda \, ds \right] \\
\leq \mathbb{E}^\mathbb{P}\left[ \int_0^T \int_{\mathbb{R}\setminus D_H(R)} |e^{-(t-s)A} Bc(z)|^2 \lambda \, ds \right] \leq C T R^{1-\frac{2}{\alpha}}.
\]

Therefore, collecting all together

\[
\mathbb{E}^\mathbb{P}\left[ 1_{|u^c(T,x,v)-u^c_0(T,x)| \leq \frac{\delta}{2}} \right] \geq \frac{1}{C(R)} \left( 1 - \left( \frac{3}{5} \right)^2 C T R^{1-\frac{4}{\alpha}} \right).
\]

Since \(R\) is chosen in such a way that

\[
\left( \frac{3}{5} \right)^2 C T R^{1-\frac{4}{\alpha}} \leq \frac{1}{2}
\]

and using the fact that

\[
\mathbb{P}(\mu(D_H(R) \times [0,T]) = 0) = e^{-\lambda(D_H(R))^T}
\]

we get

\[
\mathbb{E}^\mathbb{P}\left[ 1_{|u^c_0(T,x)| \leq \delta} \right] \geq \mathbb{E}^\mathbb{P}\left[ 1_{|\Delta(T)| \leq \frac{\delta}{2}} \right] \geq \mathbb{P}(\mu(D_H(R) \times [0,T]) = 0) \\
\times \left( 1 - \mathbb{P}\left( \int_0^T \int_{\mathbb{R}\setminus D_H(R)} e^{-(t-s)A} Bc(\theta(s,z))(\mu - \gamma)(dz,ds) \geq \frac{\delta}{3} \mid \mu(D_H(R) \times [0,T]) = 0 \right) \right) \\
\geq e^{-\lambda(D_H(R))^T} C(R) \left( 1 - \left( \frac{3}{5} \right)^2 C T R^{1-\frac{4}{\alpha}} \right) \geq \frac{1}{2} e^{-\lambda(D_H(R))^T} C(R).
\]

Hence, we have shown that

\[
\mathbb{P}(|u(T,x)| \leq \delta) \geq \frac{1}{2} e^{-\lambda(D_H(R))^T} C(R) \exp(-|v|^2_{L^2([0,T];\mathbb{R})}),
\]

which gives the assertion. \(\square\)

**Proof of Theorem 2.5.** Let \(x \in \mathcal{B}_H(C)\) and \(u^c\) the solution to

\[
\begin{cases}
\dot{u}^c(t,x,v) = Au^c(t,x,v) + Bv_x(t), & t \geq 0, \\
u^c(0,x,v) = x,
\end{cases}
\]

In order to show Theorem 2.5, we have to show that the RHS of Inequality (26) can be estimated from below for all \(x \in \mathcal{B}_H(\delta)\). That is, we have to show that for any \(x \in \mathcal{D}_H(C)\) there exists a control \(v_x \in L^2([0,T];\mathbb{R})\) and a constant \(K > 0\) such that \(u^c(T,x,v_x) = 0\) and

\[
\|v_x\|_{L^2([0,T];\mathbb{R})} \leq K.
\]
However, this constant is given by continuity properties of the system (27). To be more precise, since (27) is exactly controllable the mapping $\Phi_T : L^2(0,T;\mathbb{R}) \to H$ defined by

$$
\Phi_T(v) = \int_0^T e^{-(T-s)A} Bv(s)ds
$$

is invertible. Furthermore, it is bounded thanks to our assumptions on the semigroup generated by $A$ and on the operator $B$. Hence, its inverse $\Phi_T^{-1}$ is also a bounded operator. Let $x \in H$ and $y \in H$, we have

$$
u_c(T,x,v_x) = y = S(T)x + \Phi_T(v_x).
$$

From this identity we infer that

$$
\int_0^T |v_x(s)|^2 ds \leq \|\Phi_T^{-1}\| \cdot |u_c(T,x,v_x) - S(T)x|
$$

$$
\leq \|\Phi_T^{-1}\| |y - S(T)x|.
$$

\[\square\]

**Proof of Theorem 2.6** The proof starts by the same consideration as the proof of Theorem 2.5. We list here only the points where the proof differs.

First, choose $R \geq r_1$ such that

$$
\left(\frac{3}{\delta}\right)^2 CTR^{1 - \frac{2}{\sigma}} C_\sigma (1 + K) \leq \frac{1}{2},
$$

where $K := \int_0^T E|u(s)|^2_H ds$. By the assumptions on $B$ and $\sigma$, $K$ is finite. The next difference is that one has to find a predictable transformation $\theta : \Omega \times [0,T] \to \mathbb{R}$ such that

$$
\frac{v(s)}{\sigma(u(s-))} + g_R = \int_{\mathbb{R} \setminus B_a(\rho)} [c(z) - c(\theta(s,z))] \lambda(dz), \quad s \in [0,T].
$$

Again, the existence of such a transformation is given by Lemma A.1.

Let $\mu_\theta$ the following random measure defined by

$$
\mu_\theta : \mathcal{B}(\mathbb{R}) \times \mathcal{B}([0,T]) \ni (A \times I) \mapsto \int_{\mathbb{R}^2} \int_I 1_A(\theta(s,z)) \mu(dz,ds).
$$

Again, let $\mathbb{Q}$ be the probability measure on $\mathfrak{A}$ such that $\mu_\theta$ has compensator $\gamma$. Then, the process $u^\theta_{\mu}$ defined by

$$
\left\{\begin{array}{ll}
du^\theta_{\mu}(t,x) &= Au^\theta_{\mu}(t,x)dt + \int_{\mathbb{R}} Bc(z) \sigma(u(s-)) (\mu_\theta - \gamma)(dz,dt) \\
u^\theta_{\mu}(0,x) &= x.
\end{array}\right.
$$

has under $\mathbb{Q}$ the same law as $u$, in particular

$$
\mathbb{E}^p \left[ 1_{[0,\delta]}(|u(T,x_0)|) \right] = \mathbb{E}^Q \left[ 1_{[0,\delta]}(|u^\theta_{\mu}(T,x_0)|) \right].
$$

Setting

$$
v_\sigma(t) := \frac{v(s)}{\sigma(u(s-))}
$$
again, due to Lemma [3.1] the density process $G_{\theta}(t) = \frac{dG_{\theta}^t}{d\mathbb{P}}$ satisfy the following stochastic differential equation

$$
\begin{cases}
    dG_{\theta}(t) = G_{\theta}(t) \left( \rho_\gamma(k(|v_\sigma(s)|), z) \right) \text{sgn}(v_\sigma(s)) \ (\mu - \gamma)(dz, ds) \\
    G_{\theta}(0) = 1.
\end{cases}
$$

Similarly, by Corollary [A.3] we have

$$
\mathbb{E}^P \sup_{0 \leq s \leq r_1} |G_{\theta}(s)| \leq 1 + \mathbb{E}^P \sup_{0 \leq s \leq r_1} |G_{\theta}(s)| \times \int_0^{r_1} |v_\sigma(s)|^2 \, ds.
$$

Now, if $\int_0^{r_1} |v_\sigma(s)|^2 \, ds \leq \frac{1}{2}$, we get

$$
\mathbb{E}^P \sup_{0 \leq s \leq r_1} |G_{\theta}(s)| \leq 1 \times \left( \frac{1}{2} \right)^{-1}.
$$

Since $v \in L^2([0, T]; \mathbb{R})$ and $\sigma$ is bounded from below by $C_\sigma$ we know there exists a constant $C_v = |v|_{L^2([0, T]; \mathbb{R})}/C_\sigma > 0$ such that $\mathbb{P}$-a.s.

$$
|v_\sigma|_{L^2([0, T]; \mathbb{R})} \leq C_v.
$$

Arguing as before, we conclude that there exists a constant $C > 0$

$$
(30) \quad \mathbb{E}^P \sup_{0 \leq s \leq T} |G_{\theta}(s)| \leq C \cdot 2^{\frac{|v_\sigma|^2_{L^2([0, T]; \mathbb{R})}}{C_v}} \sim C \cdot 2^{C_v}.
$$

On the other hand we know that under $\mathbb{P}$ the process $u_\mu^\theta$ follows the following differential equation

$$
\begin{cases}
    du_\mu^\theta(t, x) = A u_\mu^\theta(t, x) \, dt + \int_\mathbb{R} B \sigma(u_\mu^\theta(t, x)) \left[ c(\theta(t, z)) - c(\theta(t, z) - \gamma)(dz, dt) \\
    + \int_\mathbb{R} B \sigma(u_\mu^\theta(t, x)) \left[ c(\theta(t, z) - \gamma)(dz, dt) + \int_\mathbb{R} B \sigma(u_\mu^\theta(t, x)) c(\theta(t, z) - \gamma)(dz, dt), \\
    u_\mu^\theta(0, x) = x.
\end{cases}
$$

Hence we can again write

$$
\mathbb{P} (|u(T, x)| \leq \delta) = \mathbb{Q} (|u_\mu^\theta(T, x)| \leq \delta) = \mathbb{E}^Q \left[ \mathbb{1}_{|u_\mu^\theta(T, x)| \leq \delta} \right] = \mathbb{E}^\mathbb{P} \left[ G_{\theta}(T) \mathbb{1}_{|u_\mu^\theta(T, x)| \leq \delta} \right].
$$

By the inverse Hölder inequality we get

$$
\mathbb{P} (|u(T, x)| \leq \delta) \geq \frac{\mathbb{E}^\mathbb{P} \left[ \mathbb{1}_{|u_\mu^\theta(T, x)| \leq \delta} \right]}{\mathbb{E}^\mathbb{P} \left[ G_{\theta}(T) \right]}.
$$

Again, the denominator, i.e. $\mathbb{E}^\mathbb{P} [G_{\theta}(T)]$, is bounded. In particular, we have by the assumption on $\sigma$, [30] and Hypothesis [4] that

$$
\mathbb{E}^\mathbb{P} [G_{\theta}(T)] \leq C' \frac{C}{C_\sigma} \exp \left( r_1^{\frac{\alpha_\beta + 1}{\alpha}} \int_0^T |v(s)|^2 \, ds \right).
$$

The next steps are similar to the lines of the proof of Theorem 2.5. To be more precise, we obtain first

$$
\mathbb{E}^\mathbb{P} \left[ \mathbb{1}_{|u_\mu^\theta(T, x)| \leq \delta} \right] \geq \mathbb{E}^\mathbb{P} \left[ \mathbb{1}_{|u^c(T, x, v)| \leq \delta} \ - \mathbb{1}_{|u^c(T, x, v)| \leq \delta} \ - \mathbb{1}_{|u^c(T, x, v)| \leq \delta} \right] = \mathbb{E}^\mathbb{P} \left[ \mathbb{1}_{|u^c(T, x, v)| \leq \delta} \right].
$$
Again, rewriting the difference $\Delta(T) = u^\mu(T, x) - u^c(T, x, v)$ as follows

$$
\Delta(T) = \int_0^T \int_{\mathbb{R}} e^{-(t-s)A} B\sigma(u(s-)) c(\theta(t, z)) (\mu - \gamma)(dz, ds)
+ \int_0^T e^{-(t-s)A} B\sigma(u(s-)) g_R ds,
$$

we have

$$
\mathbb{E}^\mathbb{P} \left[ 1_{|u^\mu(T, x)| \leq \delta} \right] \geq \mathbb{E}^\mathbb{P} \left[ 1_{|\Delta(T)| \leq \frac{\delta}{3}} \right].
$$

Continuing as before, we get

$$
\tilde{\mathbb{P}} \left( \left| \Delta(T) \right| \leq \frac{\delta}{3} \right) \geq \mathbb{P} \left( \mu(\mathcal{D}_R(R) \times [0, T]) = 0 \right)
\times \mathbb{P} \left( \left| \int_0^T \int_{\mathbb{R}\setminus\mathcal{D}_b(R)} e^{-(t-s)A} B\sigma(u(s-)) c(\theta(s, z)) (\mu - \gamma)(dz, ds) \right| \leq \frac{\delta}{3} \mid \mu(\mathcal{D}_R(R) \times [0, T]) = 0 \right)
\geq \tilde{\mathbb{P}} \left( \mu(\mathcal{D}_R(R) \times [0, T]) = 0 \right) \times \left[ 1 - \mathbb{P} \left( \left| \int_0^T \int_{\mathbb{R}\setminus\mathcal{D}_b(R)} e^{-(t-s)A} B\sigma(u(s-)) c(\theta(s, z)) (\mu - \gamma)(dz, ds) \right| > \frac{\delta}{3} \mid \mu(\mathcal{D}_R(R) \times [0, T]) = 0 \right) \right]
:= \tilde{\mathbb{P}} \left( \mu(\mathcal{D}_R(R) \times [0, T]) = 0 \right) \times \left[ 1 - I(T) \right].
$$

If we can show, that $I(T) \leq \frac{1}{2}$, we are done. Again, we apply the Chebyscheff inequality to estimate $I(T)$ and use the fact that the random variables $\mu(\mathcal{D}_R(R) \times [0, T])$ and $\mu(A \times [0, T])$ for all $A \in \mathcal{B}(\mathbb{R} \setminus \mathcal{D}_b(R))$ are independent. Doing so we obtain

$$
I(T) \leq \frac{3}{\delta} \times \mathbb{E}^\mathbb{P} \left[ \int_0^T \int_{\mathbb{R}\setminus\mathcal{D}_b(R)} e^{-(t-s)A} B\sigma(u(s-)) c(\theta(s, z)) (\mu - \gamma)(dz, ds) \right]^2.
$$

The Burkholder inequality, the fact that $c(\theta(t, z)) \leq c(z)$ and the linear grow condition on $\sigma$ give

$$
I(T) \leq \frac{3}{\delta} \times \mathbb{E}^\mathbb{P} \left[ \int_0^T \int_{\mathbb{R}\setminus\mathcal{D}_b(R)} e^{-(t-s)A} B\sigma(u(s-)) c(z) \right]^2 ds
\leq C \frac{3}{\delta} \times TR^{1-\frac{2}{\gamma}} C_\sigma \left( 1 + \mathbb{E}^\mathbb{P} \int_0^T |u(s)|^2 ds \right).
$$

Therefore, collecting all together

$$
\mathbb{E}^\mathbb{P} \left[ 1_{|u^c(T, x, v) - u^\mu(T, x)| \leq \frac{\delta}{4}} \right] \geq \frac{1}{C(R)} \left( 1 - \left( \frac{3}{\delta} \right)^2 CTR^{1-\frac{2}{\gamma}} C_\sigma (1 + K) \right).
$$

Since $R$ satisfies (28) we get

$$
\mathbb{E}^\mathbb{P} \left[ 1_{|u^\mu(T, x)| \leq \delta} \right] \geq \frac{1}{2} e^{-\lambda(\mathcal{D}_b(R))^T C(R)}.
$$
Hence, we have shown that
\[ \tilde{P}(|u(T, x)| \leq \delta) \geq \frac{1}{2} e^{-\lambda(D_K(R))^T C(R) \exp(-|v|^2_{L^2([0,T];\mathbb{R})})}, \]
which gives the assertion. 

**Proof of Theorem 2.7.** Let \( x \in \mathcal{B}_H(C) \) and \( u^c \) the solution to
\[ \begin{cases} \dot{u}^c(t, x, v) = Au^c(t, x, v_x) + Bv_t(t), & t \geq 0, \\ u^c(0, x, v_x) = x, \end{cases} \]
In order to show Theorem 2.7 we use the same arguments as we have used in the proof of Theorem 2.5. That means, first, \( R \) can be chosen in such a way, that (28) holds for all initial conditions \( x \). Since \( K \) depends linearly on \( x \), this will not be a problem. Secondly, we have to show that the RHS of Inequality (33) can be estimated from below for all \( x \in \mathcal{B}_H(\delta) \). That means, we have to show that there exists a constant \( C' > 0 \) such that for any \( x \in \mathcal{D}_H(C) \) there exists a control \( v_x \in L^2([0,T];\mathbb{R}) \) such that \( u^c(T, x, v_x) = 0 \) and
\[ \|v_x\|_{L^2([0,T];\mathbb{R})} \leq C'. \]
However, again, this constant is given by continuity properties of the system (34). 

3. **Uniqueness of the invariant measure and the asymptotic strong Feller property**

Let \( u \) be the solution of the following stochastic evolution equation
\[ \begin{cases} du(t, x) = Au(t, x) + \int_{\mathbb{R}} B\sigma(u(t, x)) z\tilde{\eta}(dz, ds), \\ u(0, x) = x \in H. \end{cases} \]
Let us assume in addition that \( \sigma \) is bounded. In particular, there exists a \( K_\sigma \) such that \( |\sigma(x)| \leq K_\sigma \) for all \( x \in H \). In case, \( u \) is not càdlàg in \( H \), \( \sigma \) is supposed to be a constant.

If the semigroup generated by \( A \) is of contractive type, i.e. there exists a \( \omega > 0 \) and \( M > 0 \) such that \( \|e^{-tA}\|_{L(H,H)} \leq Me^{-\omega t}, t \geq 0 \), then by direct calculations the following a priori estimate can be shown
\[ \mathbb{E}|u(t, x)|^2_H \leq e^{-\omega t} M \mathbb{E}|x|^2_H + MC(t, \gamma) \int_{\mathbb{R}} |z|^2 \nu(dz) \]
where \( C(t, \gamma) = K_\sigma^2 \int_0^t e^{-\omega u}s^{-2\gamma} ds \). Note that \( \lim_{t \to \infty} C(t, \gamma) < \infty \).

Here, the existence of the invariant measure can be shown by an application of the Krylov-Bogoliubov Theorem (see [13, Theorem 3.1.1]). First, we will define for \( T > 0 \) and \( x \in H \) the following probability measure on \((H, \mathcal{B}(H))\)
\[ \mathcal{B}(H) \ni \Gamma \mapsto R_T^x(x, \Gamma) := \frac{1}{T} \int_0^T P_t(x, \Gamma) dt. \]
In addition, for any \( \rho \in M_1(H) \), let \( R_T^x \rho \) be defined by
\[ \mathcal{B}(H) \ni \Gamma \mapsto \int_H R_T(x, \Gamma) \rho(dx). \]
Corollary 3.1.2 in [13] says, that if for some probability measure \( \rho \) on \((H, \mathcal{B}(H))\) and some sequence \( T_n \uparrow \infty \), the sequence \( \{R_T^x \rho : n \in \mathbb{N}\} \) is tight, then there exists an invariant measure for \((P_t)_{t \geq 0}\). That means, it is sufficient for the existence of an invariant measure to show that
for all $\epsilon > 0$ for all $x \in H$, there exists a compactly embedded subspace $E \hookrightarrow H$, a $R > 0$ such that we have for all $T > 0$

$$\frac{1}{T} \int_0^T \mathbb{P}(\|u(t, x)\|_E \geq R) \, dt < \epsilon.$$  \hfill (38)

However, if there exists a constant $C > 0$ and a number $p > 0$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E}\|u(t, x)\|^p_E \leq C, \quad T \geq 0,$$

then (38) holds.

Observe, if $A$ generates a strongly continuous semigroup $(e^{-tA})_{t \geq 0}$ of contraction on $H$, then

$$\int_0^\infty |e^{-tA}|^2_{L(H, D(A^{-\gamma}))} \, dt < \infty.$$

**Theorem 3.1.** Let $H_1$ be a Hilbert space such that $H_1 \hookrightarrow H$ compactly, Assume $A$ generates a strongly continuous semigroup $(e^{-tA})_{t \geq 0}$ of contraction on $H$ and $H_1$, and

$$\int_0^\infty |S(t-s)B|_{L(D(A^{-\gamma}, H_1))}^2 \, ds < \infty,$$

Then, if

$$\int_\mathbb{R} |z|^2 \nu(dz) < \infty,$$

and $\sigma$ is bounded, then the Markovian semigroup $(\mathcal{P}_t)_{t \geq 0}$ admits an invariant measure.

**Proof.** First, Equation (38) can be written as follows

$$\begin{cases}
    du(t, x) = Au(t, x) \, dt + \int_Z B \sigma(u(s-)) z \tilde{\eta}(dz, ds), \quad t > 0, \\
    u(0, x) = x.
\end{cases}$$

If $A$ is a semigroup of contractions then (see [17])

$$\int_0^\infty \int_Z |e^{-tA}Bz|^2 \nu(dz) \, dt < \infty$$

Now, we will show that there exists a constant $C > 0$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E}\|u(t, x)\|^2_{H_1} \leq C, \quad T \geq 0.$$

Due to standard arguments (see [17]) we get

$$\mathbb{E}\|u(t, x)\|^2_{H_1} \leq C_1 \|e^{-tA}x\|^2_{H_1} + C_2 \mathbb{E} \left\| \int_0^t e^{-(t-s)A} B \sigma(u(s-)) z \nu(dz) \, ds \right\|^2_{H_1}$$

$$\leq C_1 \|e^{-tA}x_0\|^2_{H_1} + C_2 K_\sigma \mathbb{E} \left( \int_0^t \int_\mathbb{R} |e^{-(t-s)A} Bz|^2 \nu(dz) \, ds \right).$$

Due to the assumption, the first and the second summand are bounded uniformly for all $t \geq 0$. Hence, there exists a constant $C > 0$ such that

$$\sup_{0 \leq t < \infty} \mathbb{E}\|u(t, x)\|^2_{H_1} \leq C.$$
Now, the Chebyshev inequality leads for any $R > 0$ to
\[
\frac{1}{T} \int_0^T \mathbb{P}(|u(t,x)|_{H_1} \geq R) \, dt \leq \frac{1}{T} \int_0^T \mathbb{E}|u(t,x)|_{H_1}^2 \, ds \frac{1}{R^2} \leq \frac{C}{R^2}.
\]
Given $\epsilon > 0$ and taking $R > \left(\frac{C}{\epsilon}\right)^{\frac{1}{2}}$, inequality (38) follows. \hfill \Box

By [15, Corollary 3.17] we know that if the semigroup is asymptotically strong Feller (for the definition of asymptotic strong Feller, we refer to [15]) and there exists a point $x \in H$ such that $x \in \text{supp}(\sigma)$, whenever $\sigma$ is an invariant measure of the Markovian semigroup $(\mathcal{P}_t)_{t \geq 0}$, then the semigroup $(\mathcal{P}_t)_{t \geq 0}$ admits at most one invariant measure.

To be more precise, assume for the time being that the semigroup $(\mathcal{P}_t)_{t \geq 0}$ is asymptotically strong Feller. Hence, it remains to prove that there exists a $x \in H$ such that $x \in \text{supp}(\sigma)$. Now, the two following properties imply that $0 \in \text{supp}(\sigma)$ whenever $\sigma$ is an invariant measure.

- There exists a constant $C > 0$ such that
  \[
  \inf_{\{\rho \text{ is an invariant measure}\}} \rho(\mathcal{D}_H(C)) > 0.
  \]

- For all $\delta > 0$ and for all $x \in \mathcal{D}_H(C)$ there exists a time $T_\delta > 0$ and some $\kappa > 0$ such that
  \[
  \mathbb{P}(u(T_\delta, x) \in \mathcal{D}_H(\delta)) \geq \kappa.
  \]

It follows that $0 \in \text{supp}(\sigma)$ by the following observations. Since $\sigma$ is invariant we have
\[
\sigma(\mathcal{D}_H(\delta)) \geq \kappa \cdot \inf_{\{\rho \text{ is an invariant measure}\}} \rho(\mathcal{D}_H(C)).
\]

Now, estimates (39) and (40) give the assertion.

Inequality (40) can be verified by Theorem 2.4. Estimate (39) follows by the fact that for any invariant measure $\rho$ of the semigroup $(\mathcal{P}_t)_{t \geq 0}$ there exists a constant $C > 0$ such that
\[
\int_H |u|_{H_1}^2 \rho(du) \leq C.
\]
Estimate (41) follows by the same lines as in [22] Lemma B.1 and the a priori estimate (36). Now, an application of the Chebyscheff inequality leads to (39).

It remains to investigate under which conditions the semigroup is asymptotically strong Feller. However, before continuing we introduce a second concept of controllability. Again, $H$ denotes a Hilbert space, $A : H \to H$ a generator of a strongly continuous semigroup $(e^{-tA})_{t \geq 0}$ on $H$ and $B : \mathbb{R} \to H$. Then we say that the system
\[
\begin{cases}
\dot{u}^c(t, x, v) = Au^c(t, x, v) + B v(t), & t \geq 0, \\
u^c(0, x, v) = x,
\end{cases}
\]
is \textit{null controllable with vanishing energy} (see [24] [26]), if it is null controllable and for any $x \in H$ there exists a sequence of times $\{t_n \geq 0 : n \in \mathbb{N}\}$ and a sequence of controls $\{v_n : n \in \mathbb{N}\}$

\footnote{Note, that $x \in \text{supp}(\sigma)$ iff for all $\delta > 0$, $\sigma(\mathcal{D}_H(\delta)) > 0$.}
Now we are ready to present our result about the asymptotically strong Feller property.

**Theorem 3.2.** If \( \alpha > 1 \), and if the system \( (5) \) is null controllable with vanishing energy, then the Markovian semigroup of system \( (55) \) is asymptotically strong Feller.

**Corollary 3.3.** Assume that \( \alpha > 1 \) and that the conditions of Theorem \( 3.1 \) and Theorem \( 3.2 \) are satisfied. Then, the Markovian semigroup \((P_t)_{t \geq 0}\) generated by the solution to \( (55) \) admits an invariant measure, and this invariant measure is unique.

**Proof of Theorem 3.2:** Again, we will switch for technical reasons to another representation of the Poisson random measure. Let \( \Omega = (\Omega, F, \mathbb{F}, \mathbb{P}) \) be a filtered probability space with filtration \( \mathbb{F} = (\mathbb{F}_t)_{t \geq 0} \) and let \( \mu \) be a Poisson random measure on \( \mathbb{R} \) over \( \Omega \) having intensity measure \( \lambda \) (Lebesgue measure). The compensator of \( \mu \) is denoted by \( \gamma \) and given by

\[
\gamma(A \times I) := \lambda(A) \lambda(I).
\]

Let

\[
(42) \quad c : \mathbb{R}^+ \ni r \mapsto \sup_{\rho > 0} \left\{ \int_0^\infty k(s) \, ds \geq r \right\} \quad \text{if } r > 0.
\]

Now, the stochastic evolution equation given in \( (5) \) reads as follows

\[
(43) \quad \begin{cases} du(t, x) = Au(t, x) + \int_{\mathbb{R}} B\sigma(u(s-)) \, c(z) \tilde{\mu}(dz, ds), \\ u(0, x) = x \quad \text{if } x \in H.
\end{cases}
\]

We split the proof in several steps.

**Step 1:** Fix \( x \in H \). In order to show the asymptotically strong Feller property of \((P_t)_{t \geq 0}\) in \( x \), we have to show that there exist an increasing sequence \( \{t_n : n \in \mathbb{N}\} \) and a totally separating sequence of pseudometrics \( \{d_n : n \in \mathbb{N}\} \) such that

\[
(44) \quad \lim_{\epsilon > 0} \limsup_{n \to \infty} \sup_{y \in B(x, \epsilon)} \|P_{t_n}(x, \cdot) - P_{t_n}(y, \cdot)\|_{d_n} = 0.
\]

Let \( \{a_n : n \in \mathbb{N}\} \) be a sequence of positive real numbers converging to zero. Let

\[
d_n(y, z) := 1 \wedge \|z - y\|_H / a_n, \quad z, y \in H, \quad n \in \mathbb{N}.
\]

\( ^4 \) An increasing sequence \( \{d_n : n \in \mathbb{N}\} \) of pseudometrics is called a **totally separating system** of pseudometrics for \( \mathcal{X} \), if \( \lim_{n \to \infty} d_n(z, y) = 1 \) for all \( z, y \in \mathcal{X}, z \neq y \).

\( ^5 \) Let \( d \) be a pseudo-metric on \( \mathcal{X} \), we denote by \( L(\mathcal{X}, d) \) the space of \( d \)-Lipschitz functions from \( \mathcal{X} \) into \( \mathbb{R} \). That is, the function \( \phi : \mathcal{X} \to \mathbb{R} \) is an element of \( L(\mathcal{X}, d) \) if

\[
\|\phi\|_d := \sup_{z, y \in \mathcal{X}} \frac{|\phi(z) - \phi(y)|}{d(z, y)} < \infty.
\]

\( ^6 \) For a pseudo-metric \( d \) on \( \mathcal{X} \) we define the distance between two probability measures \( P_1 \) and \( P_2 \) wrt to \( d \) by

\[
\|P_1 - P_2\|_d := \sup_{\phi \in L(\mathcal{X}, d)} \int_{\mathcal{X}} \phi(x) (P_1 - P_2)(dx).
\]
Fix $h \in H$ with $|h| = 1$. Since the system \((3)\) is null controllable with vanishing energy we can find a sequence of times $\{t_n : n \in \mathbb{N}\}$ and controls $\{v^n : n \in \mathbb{N}\}$ such that $\alpha_n^2 \geq \int_0^{t_n} |v^n(s)|^2 \, ds$, $n \in \mathbb{N}$, and the solution $u^n$ to
\[
\begin{align*}
du^n(t, x, v^n) &= Au^n(t, x, v^n) \, dt + Bv^n(t) \, dt, \quad 0 \leq t \leq t_n \\
u^n(0, x, v^n) &= x,
\end{align*}
\]
satisfy $u^n(t_n, x, v^n) = u^n(t_n, x + h, 0)$. For simplicity, put $y = x + \epsilon h$ and $v^n_\epsilon := \epsilon \cdot v^n$. Then, it follows by the linearity that $u^n(t_n, x, \epsilon v^n) = u^n(t_n, x + \epsilon h, 0) = u^n(t_n, y, 0)$. In order to give an estimate of
\[
\|\mathcal{P}_n(x, \cdot) - \mathcal{P}_n(y, \cdot)\|_{d_n}
\]
in terms of $\epsilon$ and $n$, we define the following sequence of continuous functions. Let $\phi \in C_b(H)$, there exists a sequence $\{\phi_n : n \in \mathbb{N}\}$, $\phi_n \in C_b^\infty(H)$, such that $\phi_n \to \phi$ pointwise, $\|\phi_n\|_\infty \leq \|\phi\|_\infty$ and $\|\phi_n\|_{d_n} \leq 1$ for all $n \in \mathbb{N}$. Now, the following identity holds
\[
\|\mathcal{P}_n(x, \cdot) - \mathcal{P}_n(y, \cdot)\|_{d_n} = \left| \mathbb{E}^\mathbb{P} [\phi_n(u(t_n, x + \epsilon h))] - \mathbb{E}^\mathbb{P} [\phi_n(u(t_n, x))] \right|.
\]
Hence, we have to show
\[
\lim_{\epsilon \to 0} \limsup_{n \to \infty} \sup_{y \in B(x, \epsilon)} \left| \mathbb{E}^\mathbb{P}[\phi_n(u(t_n, x + \epsilon h))] - \mathbb{E}^\mathbb{P}[\phi_n(u(t_n, x))] \right| = 0.
\]

**Step II:** Next, let us introduce a transformation $\theta^n_\epsilon : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, such that we have
\[
\int_{\mathbb{R}} (c(\theta^n_\epsilon(s, z)) - c(z)) \lambda(dz) = \frac{v^n_\epsilon(s)}{\sigma(u(s-))} \quad \text{for all } s \in [0, t_n].
\]
From now on we denote
\[
v^n_{\epsilon, \sigma}(s) = \frac{v^n_\epsilon(s)}{\sigma(u(s-))}.
\]
In fact, by Lemma \([A.1]\) we can suppose that such a transformation $\theta^n_\epsilon$ exists and is given by (see definition \((55)\))
\[
\theta^n_\epsilon : [0, \infty) \times \mathbb{R} \ni (s, z) \mapsto z + \rho(\kappa([v^n_{\epsilon, \sigma}(s)]), z) \operatorname{sgn}(v^n_{\epsilon, \sigma}(s)),
\]
Here $\kappa$ denotes the inverse of $\theta$ and is defined on page \([28]\). In addition, let $\mu^n_\epsilon$ be a random measure defined by
\[
\mathcal{B}(\mathcal{Z}) \times \mathcal{B}([0, t]) \ni A \times I \mapsto \mu^n_\epsilon(A \times I) = \int_I \int_A 1_A(\theta^n_\epsilon(s, z)) \mu(dz, ds)
\]
Let $\mathcal{Q}^{\epsilon, n}$ be that probability measure on $\mathbb{R}$ such that $\mu^n_\epsilon$ has compensator $\gamma = \lambda \cdot \lambda^\sharp$. Let $u^n_\epsilon$ be the solution to
\[
\begin{align*}
du^n_\epsilon(t, x) &= Au^n_\epsilon(t, x) \, dt + \int_{\mathbb{R}} B\sigma(u(t-)) [c(\theta^n_\epsilon(t, z)) - c(z)] \mu(dz, dt) \\
u^n_\epsilon(0, x) &= x,
\end{align*}
\]
and let $u^n_{\mu, n, \epsilon}$ be solution to
\[
\begin{align*}
du^n_{\mu, n, \epsilon}(t, x, v^n_{\epsilon, \sigma}) &= Au^n_{\mu, n, \epsilon}(t, x, v^n_{\epsilon, \sigma}) \, dt + Bv^n_{\epsilon}(t) \, dt + \int_{\mathbb{R}} B\sigma(u(t-)) c(z)(\mu - \gamma)(dz, dt), \\
u^n_{\mu, n, \epsilon}(0, x, v^n_{\epsilon}) &= x.
\end{align*}
\]
Observe that, first, by the choice of the transformation \( \theta^n \) under \( Q^{c,n} \) the random variable \( u^n(t, x) \) is identical in law to the process \( u(t_n, x) \). In particular, we have

\[
\mathbb{E}^{Q^{c,n}}[\phi_n(u^n(t_n, x))] = \mathbb{E}^\mathbb{P}[\phi_n(u(t_n, x))].
\]

Secondly, by the choice of the control and the linearity of \( \bar{A} \) we have \( u^n(t_n, x, v^n) = u(t_n, x + \epsilon h) \). For \( t \geq 0 \) let \( Q^{c,n}_t \) be the restriction of \( Q^{c,n} \) onto \( \bar{F}_t \) and \( \mathbb{P}_t \) be the restriction of \( \mathbb{P} \) onto \( \bar{F}_t \). Now we are ready to give an estimate of

\[
\|\mathcal{P}_{t_n}(x, \cdot) - \mathcal{P}_{t_n}(y, \cdot)\|\phi_n = \left| \mathbb{E}^\mathbb{P}[\phi_n(u(t_n, x + \epsilon h))] - \mathbb{E}^\mathbb{P}[\phi_n(u(t_n, x))] \right|
\]

and

\[
\mathbb{E}^\mathbb{P}[\phi_n(u(t_n, x + \epsilon h))] - \mathbb{E}^\mathbb{P}[\phi_n(u(t_n, x))]
\]

\[
= \mathbb{E}^\mathbb{P}\left[\phi_n(u^n(t_n, x, v^n)) - \phi_n(u^n(t_n, x)) + \phi_n(u^n(t_n, x))\right] + \mathbb{P}\left[1 - \frac{dQ^{c,n}_t}{d\mathbb{P}_{t_n}}\right]\phi_n(u^n(t_n, x)) + E^{Q^{c,n}}[\phi_n(u^n(t_n, x)) - \mathbb{P}[\phi_n(u(t_n, x))]
\]

Next,

\[
\left| \mathbb{E}^\mathbb{P}[\phi_n(u(t_n, x + \epsilon h))] - \mathbb{E}^\mathbb{P}[\phi_n(u(t_n, x))] \right|
\]

\[
\leq \frac{1}{a_n} \mathbb{E}^\mathbb{P}\left|u^n_{\mu,n,c}(t_n, x, v^n) - u^n_{\epsilon}(t_n, x)\right| + \|\phi_n\|_{\infty} \mathbb{P}\left|1 - \frac{dQ^{c,n}_t}{d\mathbb{P}_{t_n}}\right|
\]

Let us put

\[
I_1 := \mathbb{E}^\mathbb{P}\left|u^n_{\mu,n,c}(t_n, x, v^n) - u^n_{\epsilon}(t_n, x)\right|, \quad \text{and} \quad I_2 := \mathbb{E}^\mathbb{P}\left|1 - \frac{dQ^{c,n}_t}{d\mathbb{P}_{t_n}}\right|
\]

Next, by the construction of \( u^n(t, x) \) and \( u^n_{\mu,n,c}(t, x, v^n) \) we see that

\[
u^n_{\epsilon}(t_n, x) - u^n_{\mu,n,c}(t_n, x, v^n) = \int_0^{t_n} \int_{\mathbb{R}} e^{-(t_n-s)A} B\sigma(u(s-)) [c(z) - c(\theta^n_{\epsilon}(s, z))] (\gamma - \mu) \lambda(dz, ds)
\]

and therefore

\[
\mathbb{E}^\mathbb{P}\left|u^n_{\epsilon}(t_n, x) - u^n_{\mu,n,c}(t_n, x, v^n)\right|^2
\]

\[
\leq \mathbb{E}^\mathbb{P}\left|\int_0^{t_n} \int_{\mathbb{R}} e^{-(t_n-s)A} B\sigma(u(s-)) [c(z) - c(\theta^n_{\epsilon}(s, z))] \lambda(dz, ds)
\]

\[
\leq \frac{1}{C\bar{\sigma}} \mathbb{E}^\mathbb{P}\left|\int_0^{t_n} e^{-(t_n-s)A} B|u^n_{\epsilon}(s)| ds\right|. 
\]
Hence,

\[ |I_1| \leq \mathbb{E}^\mathbb{P} \left| u(t_n, x) - u^n_{\mu_n}(t_n, x, v^n_{\epsilon}) \right|^2 \leq \frac{C}{C_\sigma} \left( \int_0^{t_n} \left| e^{-(t_n-s)A}B \right|^2 ds \right)^{\frac{1}{2}} \mathbb{E}^\mathbb{P} \left( \int_0^{t_n} |v^n_{\epsilon}(s)|^2 ds \right)^{\frac{1}{2}} \]

\[ \leq \frac{C||B||^2}{C_\sigma} \left( \int_0^{t_n} (t_n - s)^{-2\gamma} e^{-2(t_n-s)^\rho} ds \right)^{\frac{1}{2}} \left( \int_0^{t_n} |v^n_{\epsilon}(s)|^2 ds \right)^{\frac{1}{2}} \]

\[ \leq \frac{C(\gamma, \rho, B)}{C_\sigma} \left( \int_0^{t_n} |v^n_{\epsilon}(s)|^2 \right)^{\frac{1}{2}}. \]

To give an estimate of the second term \( I_2 \) we apply \[16, \text{Theorem 1}\] to get an exact representation of the Radon Nikodym derivative. In particular, we obtain

\[ I_2 \leq \mathbb{E}^\mathbb{P} \left[ (1 - \frac{dG^n_\epsilon}{dt_n}) \phi(u^n_{\epsilon}(t_n, x)) \right] \]

\[ \leq \mathbb{E}^\mathbb{P} \left[ (1 - G^n_\epsilon(t_n)) \phi(u^n_{\epsilon}(t_n, x)) \right] \]

where \( G^n_\epsilon \) is defined by (see Lemma \[B.1\] and (47))

\[
\begin{cases}
  dG^n_\epsilon(t) = G^n_\epsilon(t) (\rho_z (\kappa(v^n_{\epsilon,\sigma}(s))) z) \operatorname{sgn}(v^n_{\epsilon,\sigma}(s))) (\mu - \gamma)(dz, ds) \\
  G^n_\epsilon(0) = 1.
\end{cases}
\]

Now, the Hölder inequality gives

\[ I_2 \leq \mathbb{E}^\mathbb{P} |1 - G^n_\epsilon(t_n)| |\phi|_\infty. \]

First we will give an estimate of \( \mathbb{E} \sup_{0 \leq s \leq t_n} |G^n_\epsilon(s)| \). An application of the Itô formula and the estimate (66) give for \( 0 \leq t \leq t_n \)

\[
\mathbb{E}^\mathbb{P} \sup_{0 \leq s \leq t} |G^n_\epsilon(s)| \leq 1 + \int_0^t \int_{\mathbb{R}^+} |G^n_\epsilon(s -) ||\kappa(v^n_{\epsilon,\sigma}(s))| |\rho_z (|\kappa(v^n_{\epsilon,\sigma}(s))|, z)| dz ds
\]

(51)

\[
\leq 1 + C \int_0^t |G^n_\epsilon(s -) ||\kappa(v^n_{\epsilon,\sigma}(s))| \int_{\mathbb{R}^+} |\rho_z (|\kappa(v^n_{\epsilon,\sigma}(s))|, z)| dz ds.
\]

By Corollary \([A.3]\) and assumption on \( \sigma(\cdot) \) it follows

\[
\int_{\mathbb{R}^+} |\rho_z (|\kappa(v^n_{\epsilon,\sigma}(s))|, z)| dz \leq \frac{C(r_1)}{C_\sigma} |v^n_{\epsilon}(s)|^2.
\]

Substituting this last estimate in (51) we obtain

\[
\mathbb{E}^\mathbb{P} \sup_{0 \leq s \leq t} |G^n_\epsilon(s)| \leq 1 + \frac{C(r_1)}{C_\sigma} \int_0^t |G^n_\epsilon(s -) ||v^n_{\epsilon}(s)||^2 ds
\]

(52)

\[
\leq 1 + \frac{C(r_1)}{C_\sigma} \mathbb{E} \sup_{0 \leq s \leq t} |G^n_\epsilon(s)| \int_0^t |v^n_{\epsilon}(s)||^2 ds.
\]

(53)

Since \( \int_0^{t_n} |v^n_{\epsilon}(s)|^2 ds \leq a_n^2 \) and \( a_n \to 0 \), there exists a \( n_0 \in \mathbb{N} \) such that \( C(r_1)a_n^2 < 1/2 \). Therefore, for \( n \geq n_0 \) we obtain

\[
\mathbb{E}^\mathbb{P} \sup_{0 \leq s \leq t} |G^n_\epsilon(s)| \leq 2.
\]
Again applying the Itô formula and the considerations above we obtain

\[ \mathbb{E}^F |G^n(t_n) - 1| \leq \frac{C(r_1)}{C_\sigma} \mathbb{E}^\pi \int_0^{t_n} |G^n(s-)||v^n_v(s)|^2 \, ds \]

\[ \leq \frac{C(r_1)}{C_\sigma} \mathbb{E}^\pi \sup_{0 \leq s \leq t_n} |G^n(s)| \int_0^{t_n} |v^n_v(s)|^2 \, ds, \]

\[ \leq \frac{C(r_1)}{C_\sigma} 2\epsilon a_n^2. \]

Going back to Ansatz (49) and taking the limit, it follows that there exists some constants \( C_1, C_2 > 0 \) and some \( n_0 \in \mathbb{N} \), such that for all \( n \geq n_0 \)

\[ \mathbb{E}^\pi \left[ \phi_n (u(t_n, x + \epsilon h)) - \mathbb{E}^\phi \left[ \phi_n (u(t_n, x)) \right] \right] \leq \left\{ \frac{C_1}{a_n} \left( \int_0^{t_n} |v^n_v(s)|^2 \, ds \right)^{\frac{1}{2}} + C_2 \|\phi\|_{\infty} 2\epsilon^2 a_n^2 \right\}. \]

Hence, we have

\[ \leq \left\{ \frac{C_1 \epsilon a_n}{a_n} + C_2 \epsilon a_n \right\}. \]

Taking the limit \( n \to \infty \) we get

\[ \limsup_{n \to \infty} \sup_{y \in B(x, \epsilon)} \|P_{t_n}(x, \cdot) - P_{t_n}(y, \cdot)\|_{d_n} \leq C\epsilon. \]

Taking the limit \( \epsilon \to 0 \), the assertion follows. \( \square \)

4. AN EXAMPLE - THE DAMPED WAVE EQUATION WITH BOUNDARY NOISE

As mentioned in the introduction, as example we consider an elastic string, fixed at one end and perturbed at the other end by a Lévy noise. Mathematically, the system can be formulated as damped wave equation with boundary Lévy noise.

Throughout this section suppose that we are given a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) such that the filtration \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) satisfies the usual condition. On this probability space we assume that we are given a real valued Lévy process \( L \). Let \( T > 0 \) and \( \alpha \in \mathbb{R} \). We consider the system

\[
\begin{cases}
  u_{tt}(t, \xi) - \Delta u(t, \xi) + \alpha u_t(t, \xi) = 0, & t \in (0, T), \xi \in (0, 2\pi), \\
  u(t, 0) = 0, & t \in (0, T), \\
  u(0, \xi) = u_0(\xi), & u_t(0, \xi) = u_1(\xi),
\end{cases}
\]

where \( \Delta = \Delta \) the Laplacian and \( \dot{L} \) is the Radon Nikodym derivative of a real valued Lévy process with characteristic measure \( \nu \), \( u_0 \in H_0^1(0, 2\pi) \) and \( u_1 \in L^2(0, 2\pi) \). Here we have set \( \sigma(u(t)) = \log(2 + |u(t)|_{L^2(0, 2\pi)}) \) for any \( t \in (0, T) \).

Equation (54) can be reformulated as a evolution equation of order one. Henceforth, let us introduce the Hilbert space \( \mathcal{H} = D(\Delta^{\frac{1}{2}}) \times L^2(0, 2\pi) \) equipped with the scalar product

\[ \langle w, z \rangle_{\mathcal{H}} = \langle \Lambda^{\frac{1}{2}} w_1, \Lambda^{\frac{1}{2}} z_1 \rangle + \langle w_1, w_2 \rangle, \quad w = \left( \begin{array}{c} w_1 \\ w_2 \end{array} \right) \in \mathcal{H} \text{ and } z = \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) \in \mathcal{H}, \]
where $\langle \cdot,\cdot \rangle$ denotes the scalar product on $L^2(O)$. Define an operator $A$ with domain $D(A) = D(\Lambda) \times D(\Lambda^2) \to H$ by

$$A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

and $B_\alpha : H \to H$ by

$$B_\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha z_1 \end{pmatrix}.$$ 

It is not difficult to prove that $A$ generates a $C_0$ semigroup $(S(t))_{t \geq 0}$ on $H$. To be more precise, if $\{\lambda_n =: n \in \mathbb{N}\}$ are the eigenvalues and $\{\phi_n : n \in \mathbb{N}\}$ the eigenfunction of $A$, then $\{\mu_n : n \in \mathbb{R}\}$ with $\mu_n = \sqrt{|\lambda_n|}$, $\mu_{-n} = \mu_n$, $n \in \mathbb{N}$, are the eigenvalues and

$$\left\{ \psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\mu_n}{\phi_n} \\ \phi_n \end{pmatrix} : n \in \mathbb{Z} \right\},$$

are the eigenfunction of $A$ (see [33, Proposition]). The semigroup $S$ can be written as

$$S(t) \begin{pmatrix} f \\ g \end{pmatrix} = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{R}} e^{\mu_n t} \left( \begin{array}{c} \mu_n \frac{df}{dx} + \frac{d\phi_n}{dx} \rho_{L^2([0,1])} + \langle g, \phi_n \rangle_{L^2([0,1])} \end{array} \right) \psi_n, \begin{pmatrix} f \\ g \end{pmatrix} \in \mathbb{R}. $$

To rewrite (54) as a stochastic evolution equations on the Hilbert space $H$ we need to find a way of transforming the nonhomogeneous boundary conditions in (54) to homogeneous one. Therefore we introduce the operator $D_{B,1}$. For every $a \in \mathbb{R}$, $v = D_{B,1}a$ is a solution to the problem

$$\begin{cases} Av(\xi) = \lambda v(\xi), & \xi \in O, \\ v(\xi)(2\pi) = a, & v(\xi)(0) = 0. \end{cases}$$

By a short calculation it follows that given $a \in \mathbb{R}$,

$$v = D_{B,1}(\xi) = \frac{a}{e^{e^{2\pi}-e^{-2\pi}}(e^{-\xi} + e^{\xi})}, \quad \xi \in [0,2\pi].$$

Following the approach in [20] and [25] we see that (54) can be transformed to the following

$$\begin{cases} dX = (A + B_\alpha) X(t)dt + (A - I) \left( \begin{array}{c} 0 \\ D_{B,1}(\sigma \circ \Pi_1(X(t))) \end{array} \right) dL, \\ X(0) = X_0, \end{cases} \tag{56}$$

where $X = (u, \dot{u})^T$, $X_0 = (u_0, u_1)^T$. Here $\Pi_1 = u$ denotes the projection from $H$ onto $D(\Lambda^2)$. From now on we will work with (56). First, note that by mimicking the proof of [25, Theorem 15.7.2] (see also [20, 11]) one can show that Problem (56) is well posed. Moreover, if $\int z^2 \nu(dz) < \infty$, then (56) has a unique mild solution which is a Markov-Feller process. In particular, the family of operators $(P_t)_{t \geq 0}$ defined by

$$P_t \phi(x) := E \phi(X), \quad \phi \in C_b(H), \quad t \geq 0. \tag{57}$$

is indeed a semigroup on $C_b(H)$. By means of Theorem [3,2] and Lemma [2,4] the following result can be obtained.
Theorem 4.1. There exists a time \( T > 0 \) such that for any \( C > 0 \) and \( \rho > 0 \), there exists a \( \kappa > 0 \) such that for any \( x, y \in B_{L^2(\mathcal{O})} \) we have
\[
\mathcal{P}_T \delta_x (\mathcal{D}_{L^2(\mathcal{O})}(\rho, y)) \geq \kappa,
\]
where \( \mathcal{D}_{\mathcal{H}}(\rho, y) = \{ z \in \mathcal{H} : |z - y|_{\mathcal{H}} \leq \rho \} \).

Corollary 4.2. If \( \alpha > 0 \) then the Markovian semigroup \( (\mathcal{P}_t)_{t \geq 0} \) defined by (57) has at most one invariant measure.

Remark 4. Since the system is approximate controllable with vanishing energy in case \( \alpha = 0 \) it is asymptotically strong Feller also for \( \alpha = 0 \).

We need to show some facts which are essential for the results in the previous sections to be applicable in for our example. First note, that the following system

\[
\begin{cases}
\partial u(t, \xi) - \Lambda u(t, \xi) + \alpha u(t, \xi) &= 0, \quad t \in (0, T), \quad \xi \in (0, 2\pi), \\
u(t, 0) &= 0, \quad t \in (0, T), \\
u(\xi, 2\pi) &= v(t), \quad t \in (0, T), \\
u(0, \xi) &= \alpha, \quad \nu(0, \xi) = u_1(\xi),
\end{cases}
\]

with control \( v \in L^2((0, \infty); \mathbb{R}) \) is approximate null controllable with vanishing energy. This statement is proved in the following Lemma.

Lemma 4.3. The system

\[
\begin{cases}
\frac{\partial X(t)}{\partial t} = (\mathcal{A} + \mathcal{B}_\alpha)X(t) + (\mathcal{A} - \mathcal{I})(0) \\
X(0) = X_0,
\end{cases}
\]

is approximate null controllable with vanishing energy.

Proof. It was proved in [9, Section 2.4] (see also e.g. [33, Example 11.2.6]) that (54) is exactly controllable at any time \( T \), hence it is null controllable. Thanks to [9, Theorem 2.45] it is approximately null controllable. Now it remains to prove that it is approximate null controllable with vanishing energy. For this purpose we mainly follow the idea in [26]. Let us write \( \mathcal{H} \) as the direct sum \( \mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_u \) where \( \mathcal{H}_u = \{ 0 \} \) and \( \mathcal{H}_u = \mathcal{H} \). Therefore we see that [26, Hypothesis 1.1] are satisfied in our case. Moreover, since \( \mathcal{A} \) is the infinitesimal generator of a strongly continuous semigroup of contractions we can deduce from [23, Chapter 1, Corollary 3.6] that the spectrum \( \sigma(\mathcal{A}) \) is contained in \( \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) \leq 0 \} \). This fact implies that \( S(\mathcal{A}) = \sup\{ \text{Re}(\lambda) : \lambda \in \sigma(\mathcal{A}) \} \leq 0 \). Therefore we can deduce from [26, Theorem 1.1] that (59) is null controllable with vanishing energy. \( \square \)

Now we are ready to prove the existence and uniqueness of the invariant measure.

Proof of Theorem 4.2. To show the existence of the invariant measure we can argue exactly as in Theorem 4.1.

It remains to show the uniqueness of the invariant measure. Owing to the Lemma 4.3 and Theorem 3.2 the semigroup \( \mathcal{P}_t \) is asymptotically strong Feller. By [15, Corollary 3.17] we know that if the semigroup is asymptotically strong Feller and there exists a point \( x \in \mathcal{H} \) such that \( x \in \text{supp}(\rho) \), whenever \( \rho \) is an invariant measure of \( \mathcal{P}_t \), then the Markovian \( \mathcal{P}_t \) semigroup admits almost one invariant measure. Therefore, we have to show that there exists a point \( x \in \mathcal{H} \) such that for any invariant measure \( \rho, x \in \text{supp}(\rho) \), i.e. for all \( \kappa > 0 \), \( \nu(\mathcal{D}_{\mathcal{H}}(\kappa)) > 0 \). 


Since null controllability implies approximate null controllability, Theorem 2.4 can be applied and there exists a time $T > 0$ such that for all $C > 0$ and $\gamma > 0$ there exists a $\kappa > 0$ with

$$P(u(T,x) \in \mathcal{D}_H(\gamma)), \quad x \in \mathcal{D}(C).$$

It remains to show (39). In particular, we should check that there exists a constant $C > 0$ such that

$$\inf_{\mu \text{ is an invariant measure}} \mu(\mathcal{D}_H(C)) > 0.$$ 

It follows that $0 \in \text{supp}(\mu)$ by the following observations. First, since $\mu$ is invariant we have

$$\inf_{\mu \text{ is invariant measure}} \mu(\bar{\mathcal{D}}(C)) > 0.$$ 

Now, the estimates (61) and (40) give the assertion from which we easily complete the proof of the Theorem 4.2.

□

Appendix A. Technical Preliminaries

In this section we will show that one can find a transformation $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that for any $v \in \mathbb{R},$

$$\int_{\mathbb{R}} z(\nu_\theta - \nu)(dz) = v,$$

and $\nu_\theta$ is a Lévy measure. Here

$$\nu_\theta : \mathcal{B}(\mathbb{R}) \ni B \mapsto \int_{\mathbb{R}} 1_B(\theta(v,z)) \nu(dz).$$

In order to find the transformation, it is convenient to switch the representation of the Poisson random measure as in the beginning of the proof of Theorem 3.2. Let $\nu$ be a Lévy measure satisfying Hypotheses 1. Let $c : \mathbb{R}^+ \ni r \mapsto \sup_{\rho > 0} \left\{ \int_\rho^\infty k(s) ds \geq r \right\}.$

To analyse the effect of the perturbation, we define a function $\theta$ by

$$[0, \infty) \ni K \mapsto \theta(K) := \int_{\mathbb{R}^+} (c(z) - c(z + \rho(K,z))) \, dz \in \mathbb{R},$$

where $\rho : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$\rho(K, z) := \begin{cases} K z^{-\beta_1} & z \in (K^{\gamma_3} r_1, K^{\gamma_2} 2r_1) \text{ and } K \geq 1, \\ 0 & z \notin (K^{\gamma_3} r_1 - \frac{1}{4}, K^{\gamma_2} 2r_1 + \frac{1}{4}) \text{ and } K \geq 1, \\ C z^{-\beta_2} & z \in (r_1, r_1(1 + K^{\gamma_2})) \text{ and } K < 1, \\ 0 & z \notin (r_1 - \frac{1}{4}, r_1(1 + K^{\gamma_2}) + \frac{1}{4}) \text{ and } K < 1, \\ \text{differentiable interpolated elsewhere} & \end{cases}$$

and

$$\beta_1 = \frac{3 - 2\alpha}{\alpha(3\alpha - 5)}, \quad \beta_2 = -1.$$
\[ \gamma_1 = 5\alpha - 3\alpha^2, \quad \gamma_2 = \frac{\alpha}{2} \]

The constant \( C > 0 \) has to be chosen in such a way, that \( K \mapsto \theta(K) \) is continuously in \( K \). Moreover, let

\[ \theta : \mathbb{R}_0^+ \ni x \mapsto \theta(x). \]

**Lemma A.1.** The function \( \theta : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) is invertible.

**Proof.** We will show that there exists a function \( \kappa : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \kappa(\theta(x)) = \theta(\kappa(x)) = x \) for all \( x \in \mathbb{R}_0^+ \) with \( \theta(x) > 0 \).

Hence, we have to show that for \( K \in \mathbb{R}_0^+ \) \( \theta(K) \) defined by

\[ \theta(K) = \int_0^\infty \left[ c(z) - c(z + \rho(K, z)) \right] \, dz \]

is invertible on \( \mathbb{R}_0^+ \). We start by verifying the following properties

1. \( \theta(K) \in \mathbb{R}_0^+ \);
2. the function \( \mathbb{R}_0^+ \ni K \mapsto \theta(K) \in \mathbb{R}_0^+ \) is continuous.
3. the function \( \mathbb{R}^+ \ni K \mapsto \theta(K) \in \mathbb{R}_0^+ \) is injective.
4. the function \( \mathbb{R}^+ \ni K \mapsto \theta(K) \in \mathbb{R}_0^+ \) is surjective.

It follows, in particular, from (2), (3) and (4), that the function \( \theta \) is invertible.

In fact, (1) is clear by the definition of \( c \). In order to show Item (2) we take into account that the function \( \mathbb{R}_0^+ \ni K \mapsto \theta(K) \in \mathbb{R}_0^+ \) is strictly decreasing and continuous. In order to show Item (3) we will show that \( \lim_{K \to \infty} \theta(K) = \infty \). Since \( \theta(0) = 0 \) and \( \theta \) is continuous on \( \mathbb{R}_0^+ \), the claim follows.

Firstly, we consider the case \( K > 1 \). Here we have

\[
\theta(K) = \int_0^\infty [c(z) - c(z + \rho(K, z))] \, dz = \int_{r_1 K^{-\gamma_1}}^{K^{\gamma_1} 2r_1} \int_{z+\rho(K, z)}^{z+\rho(K, z) + \rho(K, z)} \frac{d}{dy} c(y) \, dy \, dz \\
= \int_{K^{-\gamma_1} r_1}^{K^{\gamma_1} 2r_1} \int_{z}^{z+K \rho(K, z)} \frac{d}{dy} c(y) \, dy \, dz \geq \int_{K^{-\gamma_1} r_1}^{K^{\gamma_1} 2r_1} \rho(K, z) \frac{d}{dy} c(z + \rho(K, z)) \, dz.
\]
Hypotheses 1 give for \( \tilde{\gamma}_1 = \gamma_1(1 + \beta_1) - 1 = \frac{17(\frac{3\alpha-2}{\alpha})}{3\alpha-2} - 1 = 5 \)

\[ \ldots \geq \delta_0 K \int_{r_1 K^{-\alpha}}^{K^{\gamma_1 r_1}} \frac{z^{-\beta_1}}{(z + K^{-\beta_1})^{\frac{1}{\alpha}+1}} \, dz \geq \delta_0 K \int_{r_1 K^{-\alpha}}^{K^{\gamma_1 r_1}} \frac{z^{-\beta_1 + \gamma_1(\frac{3\alpha-2}{\alpha})+1}}{(z + K^{-\beta_1})^{\frac{1}{\alpha}+1}} \, dz \]

\[ = \delta_0 K \frac{K^{\frac{1}{\alpha}}}{K^{\gamma_1 r_1}} \int_{r_1 K^{-\alpha}}^{K^{\gamma_1 r_1}} \frac{z^{\frac{1}{\alpha}}}{(u + 1)^{\frac{1}{\alpha}+1}} \, du \]

First, note that we have \( \tilde{\gamma}_1 = -3\alpha^2 + 7\alpha - 4 < 0 \) for \( 1 < \alpha \leq 2 \) and \( (1 + \beta_1) = \frac{6(3\alpha-2)}{1\alpha} > 0 \).

That means, we have for all \( u \geq r_1^{1+\beta_1}K^{-\gamma_1} \)

\[ \frac{1}{(1 + u)^{\frac{1}{\alpha}+1}} \geq \frac{(r_1^{1+\beta_1}K^{-\gamma_1})^{\frac{1+\alpha}{\alpha}}}{(1 + r_1^{1+\beta_1}K^{-\gamma_1})^{\frac{1}{\alpha}+1}} u^{-(\frac{1}{\alpha}+1)} \geq u^{-(\frac{1}{\alpha}+1)}. \]

Integration and substituting of \( \beta_1 \) give for \( K > 1 \)

\[ \theta = \int_{0}^{\infty} [c(z) - c(z + K \rho(z))] \, dz \]

\[ \geq K^{-\frac{1}{\alpha}(1+\beta_1)} \frac{r_1^{1+\beta_1}}{(1 + r_1^{1+\beta_1})^{\frac{1}{\alpha}+1}} \int_{r_1^{1+\beta_1}}^{(2r_1)^{1+\beta_1} K^{-\gamma_1}} \frac{u^{(1-\omega)}(\beta_1(1-\omega))}{(u^{(\beta_1+1)(\alpha)+1})^{-1}} \, du \]

\[ = K^{-\frac{1}{\alpha}(1+\beta_1)} \frac{r_1^{1+\beta_1}}{(1 + r_1^{1+\beta_1})^{\frac{1}{\alpha}+1}} \int_{r_1^{1+\beta_1}}^{(2r_1)^{1+\beta_1} K^{-\gamma_1}} \frac{u^{-(\frac{1}{\alpha}+1)}}{(u^{(\beta_1+1)(\alpha)+1})^{-1}} \, du \]

\[ = C K^{-\frac{1}{\alpha}(1+\beta_1)} \frac{r_1^{1+\beta_1}}{(1 + r_1^{1+\beta_1})^{\frac{1}{\alpha}+1}} \int_{r_1^{1+\beta_1}}^{(2r_1)^{1+\beta_1} K^{-\gamma_1}} \frac{u^{-\frac{1}{\alpha}+1}}{(u^{(\beta_1+1)(\alpha)+1})^{-1}} \, du \]

\[ \geq C K^{-\frac{1}{\alpha}(1+\beta_1)} \frac{r_1^{1+\beta_1}}{(1 + r_1^{1+\beta_1})^{\frac{1}{\alpha}+1}} \frac{u^{-\frac{1}{\alpha}+1}}{(u^{(\beta_1+1)(\alpha)+1})^{-1}} \]

It follows that

\[ (66) \quad K \leq r_1^{1+\beta_1} \theta^{\frac{1}{\alpha-1}}, \]
and, therefore
\[
\lim_{K \to \infty} \int_{r_1}^{\infty} [c(z) - c(z + \rho(K, z))] \, dz = \infty.
\]

From (1), (2) and (3) it follows that the function \( \theta : \mathbb{R}^+ \to \mathbb{R}^+ \) defined by (65) is invertible.

For any \( z \in \mathbb{R}^+ \) let us write \( \kappa(z) = K \) iff \( \theta(K) = z \).

It remains to investigate the rate of grow for \( 0 < K \leq 1 \). Here, we have

\[
\int_{0}^{\infty} [c(z) - c(z + \rho(K, z))] \, dz \\
= \int_{r_1}^{r_1(K + 1)^{1/2}} \rho(K, z) \frac{d}{dz} c(z + \rho(K, z)) \, dz \, du,
\]

which implies the existence of a positive constant \( C(r_1, \alpha, \gamma_2, \beta_2) \) such that

\[
\int_{0}^{\infty} [c(z) - c(z + \rho(K, z))] \, dz \\
= C(r_1, \alpha, \gamma_2, \beta_2) \int_{r_1}^{r_1(K + 1)^{1/2}} \frac{z^{-\beta_2}}{(z + z^{-\beta_2})^{1 + \frac{\alpha}{\beta}}} \, dz
\]

By changing of variables we get that
\[
\int_{0}^{\infty} [c(z) - c(z + \rho(K, z))] \, dz \\
\geq C(r_1, \alpha, \gamma_2, \beta_2) \int_{r_1}^{r_1(K + 1)^{1/2}} \frac{z^{-\beta_2}}{(z + z^{-\beta_2})^{1 + \frac{\alpha}{\beta}}} \, du
\]

Since \( r_1^{1+\beta_2} \leq u \leq r_1^{1+\beta_2}(1 + K^{-2})(1 + \beta_2) \), we get

\[
\int_{0}^{\infty} [c(z) - c(z + \rho(K, z))] \, dz \\
\geq C(r_1, \alpha, \gamma_2, \beta_2) \frac{1}{(1 + [r_1(1 + K^{-2})]^{1+\beta_2})(1 + \frac{\alpha}{\beta})} \int_{r_1^{1+\beta_2}}^{[r_1(1 + K^{-2})]^{1+\beta_2}} \frac{r_1^{-\beta}}{u^{-\alpha/\beta + 1} + \frac{\alpha}{\beta}} \, du
\]

which implies that

\[
\int_{0}^{\infty} [c(z) - c(z + \rho(K, z))] \, dz \\
\geq C(r_1, \alpha, \gamma_2, \beta_2) \frac{r_1^{-\frac{1}{\beta}}}{(1 + [r_1(1 + K^{-2})]^{\beta_2+1})(1 + \frac{\alpha}{\beta})} K^{-\gamma_2} \frac{1+\alpha+\beta_2}{\alpha} \\
\geq C(r_1, \alpha, \gamma_2, \beta_2) K^{-\gamma_2} \frac{1+\alpha+\beta_2}{\alpha} \geq C(r_1, \alpha, \gamma_2, \beta_2) K^{\frac{1}{2}}.
\]
This proves Lemma A.1. □

The following two corollaries are following.

**Corollary A.2.** Under Hypothesis 1 for any $\tilde{r} > r_1$ and $v \in \mathbb{R}_0^+$ there exists a number $K > 0$ such that

$$\int_0^\infty [c(r) - c(K,r)] dr = v. \quad (67)$$

Moreover, there exists a constant $C(\tilde{r}) > 0$ such that $v \in \mathbb{R}_0^+$

$$\int_0^\infty |\rho_\lambda(\kappa(v),r)| dr \leq C(\tilde{r}) v^2.$$

Taking into account the negative jumps, we will define the following transformation.

**Corollary A.3.** Under Hypothesis 1 for any $\tilde{r} > r_1$ and $v \in \mathbb{R}$ there exists a transformation $\theta : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\theta(v,z) := \begin{cases} z + \rho(|v|,z) & \text{if } v \geq 0, \\ -z - \rho(|v|,z) & \text{if } v < 0, \end{cases}$$

such that

$$\int_0^\infty [c(z) - c(\theta(v,z))] dz = v.$$ 

Moreover, there exists a constant $C(\tilde{r}) > 0$ such that

$$\int_0^\infty |\rho_\lambda(\kappa(|v|),z)| dz \leq C(\tilde{r}) |v|^2.$$ 

**Appendix B. Change of measure formula**

Let $\mu$ be a Poisson random measure over $\mathfrak{A} = (\Omega, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F})$ with compensator $\gamma = \lambda \cdot \lambda$. Let $c : \mathbb{R} \rightarrow \mathbb{R}$ the transformation defined by (42). Let $g \in L^2([0,\infty); \mathbb{R})$ be a predictable process and let

$$\psi : [0,\infty) \times \mathbb{R} \ni (s,z) \mapsto z + g(s) \rho(z) \in \mathbb{R}. \quad (68)$$

Combining Corollary A.3 and Example 1.9 of [16] one can verify the following Lemma.

**Lemma B.1.** There exists a probability measure $\mathbb{Q}^\psi$ on $\mathfrak{A}$ such that the Poisson random measure $\mu_\psi$ defined by

$$\mathcal{B}(\mathbb{R}) \times \mathcal{B}([0,\infty)) \ni A \times I \mapsto \int_I \int_\mathbb{R} 1_A(\psi(s,z)) \mu(dz,ds)$$

has compensator $\gamma$. For $t \geq 0$ let $\mathbb{Q}_t^\psi$, respectively, $\mathbb{P}_t$, be the projection of $\mathbb{Q}^\psi$ onto $\mathcal{F}_t$, respectively, of $\mathbb{P}$ onto $\mathcal{F}_1$. Then the density process given by

$$[0,\infty) \ni t \mapsto \mathcal{G}(t) := \frac{d\mathbb{Q}_t^\psi}{d\mathbb{P}_t}, \quad t > 0,$$
satisfy

\[
\begin{aligned}
   \frac{dG(t)}{dt} &= G(t- \int_R (1 - \psi(z))(\mu - \gamma)(dz, dt), \\
   &= G(t- \int_R g(s)\rho_z(z) (\mu - \gamma)(dz, dt), \\
   G(0) &= 1,
\end{aligned}
\]

where \( \rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is defined by \( \text{(63)} \) and \( \rho_z \) denotes the derivative of \( \rho \).

**Proof.** The proof is done via the Laplace transform. Let \( \xi = \{\xi(t) : 0 \leq t < \infty\} \) be given by

\[
\begin{aligned}
   \frac{d\xi(t)}{dt} &= \int_R c(z)(\mu \psi - \gamma)(dz, dt), \\
   \xi(0) &= 0.
\end{aligned}
\]

Then under \( Q^\psi \) the Laplace transform is given by

\[
E_{Q^\psi} e^{-\lambda \xi(t)} = e^{\int_R \left[ e^{-\lambda c(z)} - 1 + \lambda c(z) \right] \lambda(dz)}
\]

Rewriting \( \xi \) gives

\[
\begin{aligned}
   \frac{d\xi(t)}{dt} &= \int_R c(\psi(t, z))(\mu - \gamma)(dz, dt) + \int_R \left[ c(\psi(s, z)) - c(z) \right] \gamma(dz, dt), \\
   \xi(0) &= x_0.
\end{aligned}
\]

Let \( M_\lambda = \{M_\lambda(t) : 0 \leq t < \infty\} \) be given by \( M_\lambda(t) = e^{-\lambda \xi(t)} \), \( 0 \leq t < \infty \). Now, we will show that \( E^s M_\lambda(t) \mathcal{G}(t) = E_{Q^\psi} e^{-\lambda \xi(t)} \). First \( M_\lambda(t) \) solves

\[
\begin{aligned}
   \frac{dM_\lambda(t)}{dt} &= -\lambda \int_R M_\lambda(t- \int_R \left[ c(\psi(t, z)) - c(z) \right] \gamma(dz, dt) \\
   &\quad + \int_R M_\lambda(t- \int_R \left[ e^{-\lambda c(\psi(t, z))} - 1 \right] (\mu - \gamma)(dz, dt) \\
   &\quad + \int_R M_\lambda(t- \int_R \left[ e^{-\lambda c(\psi(t, z))} - 1 + \lambda c(\psi(t, z)) \right] \gamma(dz, dt), \\
   M_\lambda(0) &= 1.
\end{aligned}
\]
Therefore, $\mathcal{Z}_\lambda(t) = M_\lambda(t) \mathcal{G}(t)$ is given by

$$\mathbb{E}^\mathcal{P} \mathcal{Z}_\lambda(t) = -\lambda \mathbb{E}^\mathcal{P} \int_0^t \int_\mathbb{R} \mathcal{Z}_\lambda(s) \left[ c(\psi(s, z)) - c(z) \right] \lambda(dz) ds$$

$$+ \mathbb{E}^\mathcal{P} \int_0^t \int_\mathbb{R} \mathcal{Z}_\lambda(s) \left[ e^{-\lambda c(\psi(s, z))} - 1 + \lambda c(\psi(s, z)) \right] \lambda(dz) ds$$

$$+ \mathbb{E}^\mathcal{P} \int_0^t \int_\mathbb{R} \mathcal{Z}_\lambda(s) \left[ e^{-\lambda c(\psi(s, z))} - 1 \right] \left[ \psi_z(s, z) - 1 \right] \lambda(dz) ds$$

$$= \mathbb{E}^\mathcal{P} \int_0^t \int_\mathbb{R} \mathcal{Z}_\lambda(s) \left[ \lambda c(z) - \lambda c(\psi(s, z)) + e^{-\lambda c(\psi(s, z))} - 1 \right.$$

$$+ \lambda c(\psi(s, z)) + e^{-\lambda c(\psi(s, z))} \psi_z(s, z) - \psi_z(s, z) + \lambda c(z) \left. \right] \gamma(dz, ds).$$

$$= \mathbb{E}^\mathcal{P} \int_0^t \int_\mathbb{R} \mathcal{Z}_\lambda(s) \left[ e^{-\lambda c(\psi(s, z))} \psi_z(s, z) - \psi_z(s, z) + \lambda c(z) \right] \gamma(dz, ds).$$

$$= \mathbb{E}^\mathcal{P} \int_0^t \int_\mathbb{R} \mathcal{Z}_\lambda(s) \left[ e^{-\lambda c(\psi(s, z))} - 1 \right] \psi_z(s, z) \lambda(dz) ds$$

$$+ \lambda \mathbb{E}^\mathcal{P} \int_0^t \int_\mathbb{R} \mathcal{Z}_\lambda(s) c(z) \gamma(dz, ds).$$

Substitution gives

$$\mathbb{E}^\mathcal{P} \mathcal{Z}_\lambda(t) = \mathbb{E}^\mathcal{P} \int_0^t \int_\mathbb{R} \mathcal{Z}_\lambda(s) \left[ e^{-\lambda c(z)} - 1 + \lambda c(z) \right] \gamma(dz, ds).$$

Since

$$\mathbb{E}^{\mathcal{Q}_\psi} \left[ e^{-\lambda \xi(t)} \right] = \mathbb{E}^\mathcal{P} \left[ \mathcal{G}(t) e^{-\lambda \xi(t)} \right] = \mathbb{E}^\mathcal{P} \left[ \mathcal{Z}_\lambda(t) \right]$$

$$= \exp \left( \int_0^t \int_\mathbb{R} \left[ e^{-\lambda c(z)} - 1 + \lambda c(z) \right] \gamma(dz, dt) \right),$$

the Proposition follows. \qed

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