Type II DFT solutions from Poisson–Lie $T$-duality/plurality

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Abstract

String theory has the $T$-duality symmetry when the target space has Abelian isometries. A generalization of the $T$-duality, where the isometry group is non-Abelian, is known as the non-Abelian $T$-duality, and it works well as a solution generating technique in supergravity. In this paper, we describe the non-Abelian $T$-duality as a kind of $O(D,D)$ transformation when the isometry group acts without isotropy. We then provide a duality transformation rule for the Ramond–Ramond fields by using the technique of double field theory (DFT). We also study a more general class of solution generating technique, the Poisson–Lie (PL) $T$-duality or $T$-plurality. We describe the PL $T$-plurality as an $SO(D,D)$ transformation and clearly show the covariance of the DFT equations of motion by using the gauged DFT. We further discuss the PL $T$-plurality with spectator fields, and study an application to the $\text{AdS}_5 \times S^5$ solution. The dilaton puzzle known in the context of the PL $T$-plurality is resolved with the help of DFT.
1 Introduction

The $T$-duality was discovered in [1] as a symmetry of string theory compactified on a circle. The mass spectrum or the partition function of string theory on a $D$-dimensional torus was studied for example in [2–6] and the $T$-duality was identified as an $O(D,D;\mathbb{Z})$ symmetry. The $T$-duality was further studied from a different approach [7,8], and the transformation rules for the background fields (i.e. metric, the Kalb–Ramond $B$-field, and the dilaton) under the $T$-duality were determined. In [9,10], the $T$-duality was understood as an $O(D,D)$ symmetry of the classical equations of motion of string theory. The classical symmetry was clarified in [11] by using the gauged sigma model and this approach works efficiently for example when we discuss the global structure of the $T$-dualized background [12]. The transformation rules for the Ramond–Ramond (R–R) fields and spacetime fermions were determined in [13–16]. This well-established symmetry of string theory is called the Abelian $T$-duality since it is relying on the existence of Killing vectors which commute with each other (see [17,18] for reviews).

An extension of the $T$-duality to the case of non-commuting Killing vectors was explored in [19] (see also [20,21] for earlier works) and it is called the non-Abelian $T$-duality (NATD). Various aspects were studied in [12,22–35]. Unlike the Abelian $T$-duality, there are still many things to be clarified. For example, the partition function in the dual model is not the same as that of the original model (see for example [36] for a recent study), and NATD may be rather regarded as a map between two string theories. The global structure of the dual geometry is also not clearly understood [12]. However, at least, NATD generates many new solutions of supergravity, and it can be utilized as a useful solution generating technique.

Under a NATD, the isometries are generally broken, and naively we cannot recover the original model from the dual model. However, this issue was resolved by relaxing the condition for the dualizability [37]. The generalized duality is called the Poisson–Lie (PL) $T$-duality [38], and it can be performed even in the absence of the usual Killing vectors. The PL $T$-duality is based on a pair of two groups with the same dimension, $G$ and $\tilde{G}$, that form a larger Lie group, known as the Drinfel’d double $\mathcal{D}$. The PL $T$-duality is a symmetry that exchanges the role of the subgroup $G$ and $\tilde{G}$. The standard NATD can be reproduced as a special case where one of the two groups is an Abelian group. Aspects of the PL $T$-duality and its generalization were studied in [39–47], and concrete applications are given for example in [38,48–51].

Low-dimensional Drinfel’d doubles were classified in [52–54], and it was stressed that some Drinfel’d double $\mathfrak{d}$ can be decomposed into several different pairs of subalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}}) \cong (\mathfrak{d}, \mathfrak{g}', \tilde{\mathfrak{g}}') \cong \cdots$. The decomposition is called the Manin triple, and each Manin triple corresponds to a sigma model. The existence of several decompositions suggests that many sigma models are related through a Drinfel’d double. This was explicitly realized in [55] and the classical equivalence of the sigma models was called the PL $T$-plurality (see [56,60] for reviews).
for more examples). Various aspects of the PL $T$-plurality are discussed in [61, 62] and in particular the quantum aspects (including the PL $T$-duality) are studied in [55, 63–71].

Recent developments in NATD were triggered by [72], which provided the transformation rule for the $\mathbb{R}$–$\mathbb{R}$ fields under NATD. Although the analysis was limited to the case where the isometry group acts freely, that restriction was relaxed in [73]. By exploiting the techniques, NATD for an SU(2) isometry was extensively studied in [74–100] (mainly in the context of AdS/CFT correspondence) and many novel solutions were constructed.

More recently, NATD is paid much attention in the context of integrable deformations of string theory, since a class of integrable deformation, called the homogeneous Yang–Baxter deformation, was shown to be a particular type of NATD [101–103] (see also [104]). In addition, other integrable deformations, such as the $\lambda$-deformation and the $\eta$-deformations, also can be understood as a subclass of the so-called $\mathcal{E}$-model [105], which was developed in the PL $T$-duality [37, 39]. Moreover, it is also discussed in [105–108] that the $\lambda$-deformation and the $\eta$-deformations are related by a PL $T$-duality and an analytic continuation. In this way, there is a close relationship between the PL $T$-duality and integrable deformations (see [109–112] for recent studies on the $\mathcal{E}$-model).

Another approach to the $T$-duality was developed in [113–129], which is called the double field theory (DFT). This manifest the Abelian $O(D, D)$ $T$-duality symmetry at the level of supergravity by formally doubling the dimensions of the spacetime. Several formulations of DFT have been proposed, such as the flux-formulation (or the gauged DFT) [130–133] and DFT on group manifolds (or DFT$_{WZW}$) [134–136]. Recently, by applying the idea of DFT$_{WZW}$, a formulation of DFT which manifests the Poisson-Lie $T$-duality was proposed in [137] and the transformation of the $\mathbb{R}$–$\mathbb{R}$ fields under the PL $T$-duality was discussed for the first time. The idea was developed in [138] and applications to various integrable deformations were studied (see also [139] for discussion on the PL $T$-duality, $O(D, D)$ symmetry, and integrable deformations). The covariance of the supergravity equations of motion under the PL $T$-duality was also shown in [140, 141] from mathematical approaches.

In this paper, we revisit the traditional NATD in a general setup where the non-vanishing $B$-field and the $\mathbb{R}$–$\mathbb{R}$ fields are included. By assuming that the isometry group acts freely on the target space, we describe the NATD as a kind of $O(D, D)$ rotation of the supergravity fields. Once the $O(D, D)$ matrix is explicitly identified, we can easily find the transformation rule for the $\mathbb{R}$–$\mathbb{R}$ fields by using the technique of DFT. By using the information of given (generalized) Killing vectors, we provide simple duality transformation rules for bosonic fields.

We then demonstrate the efficiency of the formula by studying some concrete examples. Since many examples are already studied in the literature, in this paper, we will basically consider the cases where the isometry group is non-unimodular $f_{a b}^{a} \neq 0$. This type of NATD is not studied well because the resulting dual geometry does not satisfy the supergravity
equations of motion \[23,25,28\]. However, as pointed out in \[142,143\], the dual geometry in fact satisfies the generalized supergravity equations of motion (GSE) \[144,145\]. When the target space satisfies the GSE, string theory has the scale invariance \[144,146\] and the \(\kappa\)-symmetry \[145\]. The conformal symmetry may be broken, but recently, a local counterterm that cancels out the Weyl anomaly was constructed in \[147\] (see also \[148\]), and string theory may be consistently defined even in the generalized background. Even if it is not the case, NATD for a non-unimodular algebra still works as a solution generating technique in supergravity, because an arbitrary GSE solution can be mapped to a solution of the usual supergravity \[144,148–150\] by performing a (formal) \(T\)-duality. Then, combining the NATD with \(f_{ab} \neq 0\) and the formal \(T\)-duality, we can generate a new supergravity solution.

We also study the PL \(T\)-plurality with the R–R fields. In fact, the PL \(T\)-plurality can be regarded as a constant \(\mathrm{SO}(D,D)\) transformation acting on “untwisted fields” \(\{\tilde{H}_{AB}, \tilde{d}, \tilde{F}\}\). By requiring the untwisted fields to satisfy the dualizability condition or the \(E\)-model condition of \[147\], we show that the DFT equations of motion in the original and the transformed background are covariantly related by the \(\mathrm{SO}(D,D)\) transformation. This shows that the if the original background satisfies the DFT equations of motion, the transformed background also is a solution of DFT. We also discuss the PL \(T\)-plurality with spectator fields. Again, requiring a certain \(T\)-dualizability condition, we show the dilaton equation of motion is satisfied in the dual background, although the full equations of motion are not checked. By using the duality rules, we study an example of the PL \(T\)-plurality with the R–R fields.

In the study of the PL \(T\)-plurality, the so-called dilaton puzzle has been discussed in \[55–58\]. Under a PL \(T\)-plurality transformation, a dual-coordinate dependence (i.e. dependence on the coordinates of the dual group \(\tilde{G}\)) can appear in the dilaton. When such coordinate dependence appears, the background does not have the usual supergravity interpretation, and we are forced to disallow such transformation. However, in DFT, we can treat the dual coordinates and the usual coordinates on an equal footing and we do not need to worry about the dilaton puzzle. As discussed in \[148,150\], a DFT solution with a dual-coordinate dependent dilaton can be regarded as a solution of GSE, and by performing a further formal \(T\)-duality, we can obtain a linear dilaton solution of the usual supergravity. In this way, the issue of the dilaton puzzle is totally resolved and we can consider an arbitrary \(T\)-plurality transformation.

This paper is organized as follows. In section 2 we briefly review DFT and GSE. In section 3 we begin with a review of the traditional NATD, and translate the results into the language of DFT. We then provide a general transformation rule for the R–R fields. Examples of NATD without and with the R–R fields are studied in section 4 and 5. In section 6 we study the PL \(T\)-plurality in terms of DFT and determine the transformation rules from the DFT equations of motion. As an example of the PL \(T\)-plurality, in section 7 we study the PL \(T\)-plurality transformation of \(\text{AdS}_5 \times S^5\) solution. Section 8 is devoted to conclusions and discussions.
2 A review of DFT and GSE

Generalized-metric formulation of DFT

There are several equivalent formulations of DFT, but the generalized-metric formulation \cite{119,120,122,126} may be most accessible one, and in this paper, we utilize this formulation as much as possible. In this formulation, the fundamental fields are a symmetric tensor, called the generalized metric $\mathcal{H}_{MN}(x)$, and a scalar density $e^{-2d(x)}$ called the DFT dilaton. The Lagrangian is given by

$$\mathcal{L}_{\text{DFT}} = e^{-2d} S,$$

$$S \equiv \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{PQ} \partial_N \mathcal{H}_{PQ} - \frac{1}{2} \mathcal{H}^{PQ} \partial_Q \mathcal{H}^{MN} \partial_N \mathcal{H}_{PM} + 4 \partial_M d \partial_N \mathcal{H}^{MN}$$

$$- 4 \mathcal{H}^{MN} \partial_M d \partial_N d - \partial_M \partial_N \mathcal{H}^{MN} + 4 \mathcal{H}^{MN} \partial_M \partial_N d. \quad (2.1)$$

Here, the fields are supposed to depend on the generalized coordinates $(x^M) = (x^m, \tilde{x}_m)$ ($M = 1, \ldots, 2D$, $m = 1, \ldots, D$), and we raise or lower the indices $M, N$ by using the $O(D, D)$-invariant metric $\eta_{MN}$ and its inverse $\eta^{MN}$,

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^m_n \\ \delta^m_n & 0 \end{pmatrix}, \quad \eta^{MN} = \begin{pmatrix} 0 & \delta^n_m \\ \delta^n_m & 0 \end{pmatrix}. \quad (2.2)$$

The generalized metric $\mathcal{H}_{MN}$ is defined to be an $O(D, D)$ matrix,

$$\mathcal{H}_M^P \mathcal{H}_N^Q \eta_{PQ} = \eta_{MN}, \quad (2.3)$$

and this property allows us to define projection operators satisfying $P_M^N + \bar{P}_M^N = \delta^N_M$ as

$$P^{MN} \equiv \frac{1}{2} \left( \eta^{MN} + \mathcal{H}^{MN} \right), \quad \bar{P}^{MN} \equiv \frac{1}{2} \left( \eta^{MN} - \mathcal{H}^{MN} \right). \quad (2.4)$$

In the generalized-metric formulation, for consistency, we assume that arbitrary fields or gauge parameters $A(x)$ and $B(x)$ satisfy the so-called section condition,

$$\eta^{MN} \partial_M \partial_N A = 0, \quad \eta^{MN} \partial_M A \partial_N B = 0. \quad (2.5)$$

According to this requirement, all of the fields cannot depend on more than $D$ coordinates. Under the section condition, the DFT action is invariant under the generalized Lie derivative

$$\hat{\mathcal{L}}_V \mathcal{H}_{MN} \equiv V^P \partial_P \mathcal{H}_{MN} + (\partial_M V^P - \partial^P \mathcal{H}_M) \mathcal{H}_{PN} + (\partial_N V^P - \partial^P \mathcal{H}_N) \mathcal{H}_{MP},$$

$$\hat{\mathcal{L}}_V d \equiv V^M \partial_M d - \frac{1}{2} \partial_M V^M, \quad (2.6)$$
and this generates the gauge symmetry of DFT, known as the generalized diffeomorphisms. Under the section condition, we can also check that the generalized Lie derivative is closed by means of the C-bracket,

\[
[V_1, V_2]_C^M = \frac{1}{2} \left( \hat{\mathcal{L}}_{V_1} V_2^M - \hat{\mathcal{L}}_{V_2} V_1^M \right)
= V_1^N \partial_N V_2^M - V_2^N \partial_N V_1^M - V_1^N \partial^M V_2^N. 
\] (2.7)

In particular when the inner product of \( V_a^M \) is constant, \( \eta_{MN} V_a^M V_b^N = 2 \epsilon_{ab} \) (\( \epsilon_{ab} \) : constant), we can easily show that the C-bracket coincides with the generalized Lie derivative

\[
[V_a, V_b]_C = \hat{\mathcal{L}}_{V_a} V_b^M = -\hat{\mathcal{L}}_{V_b} V_a^M,
\] (2.8)

similar to the case of the usual Lie derivative.

In fact, the scalar \( S \) in (2.11) can be understood as the generalized Ricci scalar curvature

\[
S = \frac{1}{2} (P^{MK} P^{NL} - \bar{P}^{MK} \bar{P}^{NL}) S_{MNL},
\] (2.9)

where the (semi-covariant) curvature tensor \( S_{MNPQ} \) is defined by

\[
S_{MNPQ} \equiv R_{MNPQ} + R_{PQMN} - \Gamma_{RMN} \Gamma_{RPQ},
\]

\[
R_{PQMN} \equiv \partial_M \Gamma_{NPQ} - \partial_N \Gamma_{MPQ} + \Gamma_{MPR} \Gamma_{NRQ} - \Gamma_{NPR} \Gamma_{MRQ}.
\] (2.10)

If we use the curvature tensor, the invariance of the DFT action under the generalized diffeomorphism is manifest. In other words, the DFT action can be understood as a natural generalization of the Einstein–Hilbert action.

The equations of motion are also summarized in a covariant form as

\[
S = 0, \quad S_{MN} = 0,
\] (2.11)

where we the generalized Ricci tensor is defined by

\[
S_{MN} \equiv (P_M^P \bar{P}_N^Q + \bar{P}_M^P P_N^Q) S_{RPQ}^R.
\] (2.12)

For concrete computation, the following expression may be more useful:

\[
S_{MN} = -2 (P_M^P \bar{P}_N^Q + \bar{P}_M^P P_N^Q) K_{PQ},
\]

\[
K_{MN} \equiv \frac{1}{8} \partial_M \mathcal{H}^{PQ} \partial_N \mathcal{H}_{PQ} - \frac{1}{2} \partial_{(M]} \mathcal{H}^{PQ} \partial_P \mathcal{H}_{(N)Q} + 2 \partial_M \partial_N d
\]

\[
1\text{They are summarized as } \mathcal{G}_{MN} \equiv S_{MN} - \frac{1}{2} S H_{MN} = 0. \text{ Here, the generalized Einstein tensor } \mathcal{G}_{MN} \text{ satisfies the Bianchi identity } \nabla^M \mathcal{G}_{MN} = 0 \text{ [15], where } \nabla_M \text{ is the covariant derivative for the connection } \Gamma_{MNP}.\]
\[ + (\partial_P - 2 \partial_P d) \left( \frac{1}{2} \mathcal{H}^{PQ} \partial_{[M} \mathcal{H}_{N]Q} + \frac{1}{2} \mathcal{H}^{Q}_{\langle M} \partial_Q \mathcal{H}^{P}_{\rangle N} - \frac{1}{4} \mathcal{H}^{PQ} \partial_Q \mathcal{H}_{MN} \right) . \]  

(2.14)

When we make the connection to the conventional supergravity, we remove the dependence on the dual coordinates \( \tilde{\partial}^m = 0 \) and parameterize the generalized metric and the dilaton as

\[ \mathcal{H}_{MN} = \begin{pmatrix} g_{mn} - B_{mp} g^{pq} B_{qn} & B_{mp} g^{mn} \\ -g^{np} B_{pn} & g_{mn} \end{pmatrix} , \quad e^{-2d} = e^{-2\Phi} \sqrt{|g|} , \]  

(2.15)

by using the standard NS–NS fields \((g_{mn}, B_{mn}, \Phi)\). Then, \(S\) and \(S_{MN}\) reduce to

\[ S = R + 4 D^m \partial_m \Phi - 4 D^m \Phi D_m \Phi - \frac{1}{12} H_{mnp} H^{mnp} , \]

\[ (S_{MN}) = \begin{pmatrix} 2 g_{[mk} s_{kl]} & s_{(mn)} - B_{mk} s_{(kl)} B_{ln} & B_{mk} s_{(kn)} - g_{mk} s_{[kn]} \\ s_{[mk]} & s_{mn} - s_{[mk]} B_{km} & s_{mn} \\ & s_{mn} & 2 D_m \partial_n \Phi - \frac{1}{2} D^k H_{kmn} + \partial_k \Phi H^k_{mn} \end{pmatrix} , \]  

(2.16)

and the standard supergravity Lagrangian

\[ \mathcal{L} = \sqrt{|g|} e^{-2\Phi} (R + 4 D^m \partial_m \Phi - 4 D^m \Phi D_m \Phi - \frac{1}{12} H_{mnp} H^{mnp}) , \]  

(2.17)

and the equations of motion are reproduced:

\[ R + 4 D^m \partial_m \Phi - 4 D^m \Phi D_m \Phi - \frac{1}{12} H_{mnp} H^{mnp} = 0 , \quad s_{(mn)} = 0 , \quad s_{[mn]} = 0 . \]  

(2.18)

We can also introduce the R–R fields in a manifestly \(O(D, D)\) covariant manner. However, the treatment of the R–R fields is slightly involved, and we will not write down the covariant expression explicitly here (for the detail, see for example [148, 152], which is consistent with our conventions). In the following, aimed at readers who are not familiar with DFT, we will try to describe the R–R fields as the usual \(p\)-form fields as much as possible.

**Gauged DFT**

When we manifest the covariance under the PL T-plurality, it is convenient to rewrite the DFT equations of motion (2.11) by using a technique of the gauged DFT [130–133].

Suppose that the generalized metric \(\mathcal{H}_{MN}\) has the form,

\[ \mathcal{H}_{MN}(x) = \left[ U(x) \hat{\mathcal{H}} U^T(x) \right]_{MN} , \quad U \equiv (U^A_M) , \]  

(2.19)

where the \(\hat{\mathcal{H}}_{AB}\) is a constant matrix, which we call the untwisted metric. In this case, it is
useful to define \( F_{ABC} \) and \( F_A \), which are called the gaugings or the generalized fluxes, as

\[
F_{ABC} \equiv -3 \Omega_{[ABC]}, \quad F_A \equiv \Omega^B_{BA} + 2 \mathcal{D}_A d, \quad \Omega_{ABC} \equiv \mathcal{D}_A U_B^M U_C^M = \Omega_{[ABC]}, \quad \mathcal{D}_A \equiv U_A^M \partial_M, \quad U_A^M = (U^{-1})_A^M, \tag{2.20}
\]

which behave as scalars under generalized diffeomorphisms.

By using the generalized fluxes, we can show that the DFT equations of motion (2.11), under the section condition, are equivalent to

\[
\mathcal{R} = 0, \quad G^{AB} = 0, \tag{2.21}
\]

where

\[
\mathcal{R} \equiv -2 \tilde{P}^{AB} (2 \mathcal{D}_A \mathcal{F}_B - \mathcal{F}_A \mathcal{F}_B) - \frac{1}{3} \tilde{P}^{ABCDEF} F_{ABC} F_{DEF}, \\
G^{AB} \equiv -4 \tilde{P}^{D[A} D^{B]} F_D + 2 (F_D - \mathcal{D}_D) \mathcal{F}_D^{[AB]} - 2 \mathcal{F}_D^{CD[A} F_{CD}^{B]}. \tag{2.22}
\]

Here, we have defined

\[
(\eta_{AB}) = \begin{pmatrix} 0 & \delta_a^b \\ \delta^b_a & 0 \end{pmatrix}, \quad (\eta^{AB}) = \begin{pmatrix} 0 & \delta^a_b \\ \delta^b_a & 0 \end{pmatrix}, \quad \mathcal{F}^{ABC} \equiv \tilde{P}^{ABCDEF} \mathcal{F}_{DEF}, \tag{2.23}
\]

\[
P_{AB} \equiv \frac{1}{2} (\eta_{AB} + \hat{\mathcal{H}}_{AB}), \quad \tilde{P}_{AB} \equiv \frac{1}{2} (\eta_{AB} - \hat{\mathcal{H}}_{AB}), \tag{2.24}
\]

\[
\tilde{P}^{ABCDEF} \equiv \tilde{P}^{AD} \tilde{P}^{BE} \tilde{P}^{CF} + \tilde{P}^{AD} \tilde{P}^{BE} \tilde{P}^{CF} + \tilde{P}^{AD} \tilde{P}^{BE} \tilde{P}^{CF} + \tilde{P}^{AD} \tilde{P}^{BE} \tilde{P}^{CF} \\
= \frac{1}{4} (\hat{\mathcal{H}}^{AD} \eta^{BE} \eta^{CF} - \hat{\mathcal{H}}^{AD} \eta^{BE} \eta^{CF} - \eta^{AD} \hat{\mathcal{H}}^{BE} \eta^{CF} - \eta^{AD} \hat{\mathcal{H}}^{BE} \eta^{CF}) + \frac{1}{2} \eta^{AD} \eta^{BE} \eta^{CF}, \tag{2.25}
\]

and the indices \( A, B \) are raised or lowered with \( \eta_{AB} \) and \( \eta^{AB} \). Under the section condition, we can check that \( \mathcal{R} = \mathcal{S} \). The equivalence between \( \mathcal{S}_{MN} = 0 \) and \( G^{AB} = 0 \) is slightly more non-trivial but it is concisely explained in [133].

In the flux formulation of DFT [133], we take the untwisted metric \( \hat{\mathcal{H}}_{AB} \) as a diagonal Minkowski metric, and then \( E_A^M \equiv U_A^M \) is regarded as the generalized vielbein. The fundamental fields are \( E_M^A \) and \( d \), and the equations of motion (2.21) can be derived from

\[
\mathcal{L} = e^{-2d} \mathcal{R}. \tag{2.26}
\]

On the other hand, in this paper, we rather interpret (2.19) as a reduction ansatz and the equations of motion (2.21) are just rewritings of (2.14), similar to the gauged DFT [130–132]. For our purpose, it is enough to consider the cases where the generalized fluxes are constant.

\footnote{The convention for \( F_{ABC} \) is opposite to the standard one.}
In that case, the equations of motion are simple algebraic equations

\[ R = \frac{1}{12} \mathcal{F}_{ABC} \mathcal{F}_{DEF} (3 \hat{H}^{AD} \eta^{BE} \eta^{CF} - \hat{H}^{AD} \hat{H}^{BE} \hat{H}^{CF}) - \hat{H}^{AB} \mathcal{F}_A \mathcal{F}_B = 0, \quad (2.27) \]

\[ G^{AB} = \frac{1}{2} (\eta^{CE} \eta^{DF} - \hat{H}^{CE} \hat{H}^{DF}) \mathcal{H}^{[A} \mathcal{F}_{CD}B] \mathcal{F}_{EFG} + 2 \mathcal{F}_D \mathcal{F}^{D[AB]} = 0, \quad (2.28) \]

where we have again used the section condition.

In general, the untwisted metric and the DFT dilaton may depend on the coordinates \( y^\mu \) on the uncompactified external spacetime. In this case, we denote the extended coordinates as \( (x^M) = (y^\mu, x^i, \tilde{y}_\mu, \tilde{x}_i) \) and consider

\[ \mathcal{H}_{MN} = \left[ U(x) \hat{H}(y^\mu) U^T(x) \right]_{MN}, \quad d = \hat{d}(y^\mu) + d(x). \quad (2.29) \]

By following [132], we assume that \( \hat{H}_{AB}(y^\mu) \) and \( \hat{d} \) satisfies

\[ \mathcal{D}_A \hat{H}_{BC}(y^\mu) = \partial_A \hat{H}_{BC}(y^\mu), \quad \mathcal{D}_A \hat{d}(y^\mu) = \partial_A \hat{d}(y^\mu), \quad (2.30) \]

and then the generalized Ricci scalar becomes [132]

\[ S = \hat{S} + \frac{1}{12} \mathcal{F}_{ABC} \mathcal{F}_{DEF} (3 \hat{H}^{AD} \eta^{BE} \eta^{CF} - \hat{H}^{AD} \hat{H}^{BE} \hat{H}^{CF}) - \hat{H}^{AB} \mathcal{F}_A \mathcal{F}_B 
+ \frac{1}{2} \mathcal{F}_{ABC} \hat{H}^{BD} \hat{H}^{CE} \mathcal{D}_D \hat{H}_{AE} - 2 \mathcal{F}_A \mathcal{D}_B \hat{H}^{AB} + 4 \mathcal{F}_A \hat{H}^{AB} \mathcal{D}_B \hat{d}, \quad (2.31) \]

where \( \hat{S} \) denotes the generalized Ricci scalar associated with \( (\hat{H}_{AB}, \hat{d}) \). It is important to note that the equation of motion \( S = 0 \) is covariant under a constant \( O(D,D) \) rotation

\[ \hat{H}_{AB} \rightarrow (C \hat{H}_{AB} C^T)_{AB}, \quad U_A^M \rightarrow C_A^B U_B^M, \quad (2.32) \]

under which the generalized fluxes are also transformed covariantly. As we discuss later, the PL \( T \)-plurality is similar to this type of a constant \( O(D,D) \) rotation, although it is totally different from the \( O(D,D) \) rotation [2.32] that does not change the generalized metric \( \mathcal{H}_{MN} \).

**GSE from DFT**

As we already explained, if we choose a section \( \tilde{\partial}^m = 0 \), the DFT equations of motion reproduce the usual supergravity equations of motion. On the other hand, we can derive the GSE by choosing another solution of the section condition [148][150],

\[ \mathcal{H}_{MN} = \mathcal{H}_{MN}(x^m), \quad d = d_0(x^m) + \Gamma^m \tilde{x}_m \quad (\Gamma^m : \text{constant}), \quad (2.33) \]
where the DFT dilaton has a linear dependence on the dual coordinates. In order to satisfy the section condition, we require the $I^m$ to satisfy

$$\hat{\mathcal{L}}_{X} \mathcal{H}_{MN} = \hat{\mathcal{L}}_{X} d = 0, \quad (X^M) \equiv \begin{pmatrix} I^m \\ 0 \end{pmatrix}. \quad (2.34)$$

These are equivalent to

$$X^P \partial_P \mathcal{H}_{MN} = X^P \partial_P d = X^P \partial_P d_0 = 0, \quad (2.35)$$

and indeed ensure the section condition,

$$\eta^{MN} \partial_M \eta_{PQ} \partial_N d = X^P \partial_P \mathcal{H}_{MN} = 0, \quad \eta^{MN} \partial_M d \partial_N d = 2 X^P \partial_P d_0 = 0. \quad (2.36)$$

If we take this section and parameterize $\mathcal{H}_{MN}$ as usual in terms of $(g_{mn}, B_{mn})$ and $d_0$ as $e^{-2d_0} = e^{-2\Phi} \sqrt{|g|}$, the DFT equations of motion (without R–R fields) become

$$R + 4 D^m \partial_m \Phi - 4 |\partial \Phi|^2 - \frac{1}{2} |H_3|^2 - 4 (I^m I_m + U^m U_m + 2 U^m \partial_m \Phi - D_m U^m) = 0, \quad (2.37)$$

$$R_{mn} - \frac{1}{4} H_{mpq} H_{n}^{pq} + 2 D_m \partial_n \Phi + D_m U_n + D_n U_m = 0, \quad - \frac{1}{2} D^k H_{kmn} + \partial_k \Phi H^k_{mn} + U^k H_{kmn} + D_m I_n - D_n I_m = 0,$$

where $U_m \equiv I^n B_{nm}$. They are precisely the GSE studied in [144–146]. When $I^m = 0$ (where the Killing equations are trivial), they reduce to the usual supergravity equations of motion.

Another way to derive the GSE is to make a modification

$$\partial_M d \rightarrow \partial_M d + X_M \quad (X_M : a \text{ generalized vector}), \quad (2.38)$$

everywhere in the DFT equations of motion [150]. As long as the $X^M$ satisfies

$$\hat{\mathcal{L}}_{X} \mathcal{H}_{MN} = \hat{\mathcal{L}}_{X} d = 0, \quad \eta_{MN} X^M X^N = 0, \quad (2.39)$$

we can choose a gauge such that $X^M$ takes the form (2.34) [148]. In terms of the generalized flux, obviously, this modification corresponds to

$$\mathcal{F}_A \rightarrow \mathcal{F}_A + 2 X_A, \quad X_A \equiv U_A^M X_M. \quad (2.40)$$

Even in the presence of the R–R fields, this replacement is enough to derive the type II GSE,
although we additionally need to require the isometry condition for the R–R fields,

\[ \mathcal{L}_IF = 0. \quad (2.41) \]

**A formal T-duality**

In generalized backgrounds, where the supergravity fields satisfy the GSE, the string theory may not have the conformal symmetry. Accordingly, when we obtain a generalized background as a result of NATD, it is usually regarded as a problematic example, and such background is not considered seriously. However, as discussed in [144, 148–150], by performing a formal T-duality, we can always transform a generalized background to a linear-dilaton solution of the usual supergravity. Here, we review what is the formal T-duality.

The DFT equations of motion are covariant under a constant O($D,D$) transformation,

\[ x^M \to \Lambda^M_N x^N, \quad \mathcal{H}_{MN} \to (\Lambda \mathcal{H} \Lambda^T)_{MN}, \quad \partial_M d \to \partial_M d. \quad (2.42) \]

In particular, if we consider an O($D,D$) matrix,

\[ \Lambda = \begin{pmatrix} 1 - ez & ez \\ ez & 1 - ez \end{pmatrix}, \quad ez \equiv \text{diag}(0, \ldots, 0, 1_{x^z}, 0, \ldots, 0), \quad (2.43) \]

it corresponds to the (factorized) T-duality along the $x^z$-direction. For a given GSE solution with $d = d_0 + I^z \tilde{x}_z$, an O($D,D$) rotation (2.42) with (2.43) exchanges the coordinates $x^z$ and $\tilde{x}_z$, and the dilaton becomes $d = d_0 + I^z x^z$. According to the Killing equation, the generalized metric is independent of $x^z$, and the dual coordinate $\tilde{x}_z$ does not appear in the resulting background. This means that the GSE background is transformed to a solution of the usual supergravity with a linear dilaton $d = d_0 + I^z x^z$.

The reason why we call this O($D,D$) transformation a “formal” T-duality is as follows. The usual Abelian T-duality is a O($D,D$) transformation,

\[ \mathcal{H}_{MN} \to \Lambda_M^P \Lambda_N^Q \mathcal{H}_{PQ}, \quad \partial_M d \to \partial_M d, \quad (2.44) \]

in the presence of $D$ Abelian isometries. The difference from (2.42) is whether the coordinates are transformed or not. If we transform the coordinates, (2.42) is always a symmetry of the DFT equations of motion even without isometries. In the presence of Abelian isometries, due to the coordinate independence, the transformation $x^M \to \Lambda^M_N x^N$ is trivial and the formal T-duality reduces to the usual T-duality (2.44). To stress the difference, when we perform a transformation (2.42) along a non-isometric direction, we call it a formal T-duality.
3 Non-Abelian T-duality

In this section, we study the traditional NATD in general curved backgrounds. We begin with a review of NATD for the NS–NS sector. We then describe the duality as a kind of $O(D,D)$ rotation and provide the general transformation rule for the R–R fields by employing the results of DFT. To provide a closed-form expression for the duality rule, we restrict our discussion to the case where we can take a simple gauge choice, $x^i(\sigma) = \text{const.}$

3.1 NS–NS sector

In the case of the Abelian T-duality, the dual action can be obtained by following the procedure of [8, 11]. When a target space has a set of Killing vectors $v_a^m$ that commute with each other $[v_a, v_b] = 0$, the sigma model has a global symmetry generated by $x^m(\sigma) \to x^m(\sigma) + \epsilon v_a^m(\sigma)$. This global symmetry can be made a local symmetry by introducing gauge fields $A^a(\sigma)$ and replacing $dx^m \to Dx^m = dx^m - v_a^m A^a$. We also introduce the Lagrange multipliers $\tilde{x}_a(\sigma)$, which constrain the field strengths to vanish. Then, by integrating out the gauge fields $A^a$, we obtain the dual action, where the Lagrange multipliers $\tilde{x}_a$ becomes the embedding function in the dual geometry. In [19], this procedure was generalized to the case of non-commuting Killing vectors. It was further developed later, and in the following, we review NATD in a general setup discussed in [25, 35].

We consider a target space with $n$ generalized Killing vectors $V_a$ $(a = 1, \ldots, n)$ satisfying

$$\hat{L}_{V_a} H_{MN} = 0, \quad [V_a, V_b]_C = f_{ab}^c V_c, \quad \eta_{MN} V_a^M V_b^N = 2 c_{ab}, \quad f^{ab}_c c_{dc} = 0. \quad (3.1)$$

Here, $c_{ab}$ is a constant symmetric matrix. If we choose a section $\hat{\partial}^m = 0$ and parameterize the generalized Killing vectors as

$$(V_a^M) \equiv \begin{pmatrix} v_a^m \\ \hat{v}_{am} \end{pmatrix} \equiv \begin{pmatrix} v_a^m \\ v_a^m + B_{mn} v_n^m \end{pmatrix}, \quad (3.2)$$

these conditions reduce to

$$\mathcal{L}_{v_a} g_{mn} = 0, \quad \iota_{v_a} H_3 + \hat{v}_a = 0, \quad v_a \cdot \hat{v}_b = c_{ab}, \quad (3.3)$$

where the dot denotes a contraction of the index $m$. They are precisely the requirements to perform NATD [25, 35] (see [153, 154] for the origin of the conditions).

Under the setup, we consider the gauged action by following the standard procedure [8, 11].

---

3 We can easily show $f^{ab}_c c_{da} + f^{ab}_d c_{ba} = 0$ and then the last condition can be expressed as $f^{ab}_c c_{ba} = 0$. We can further rewrite the same condition as $\frac{1}{4} \iota_{v_a} \iota_{v_b} \iota_{v_c} H_3 + \iota_{v_a} f_{bc}^d \hat{v}_d = 0$ that was used in [33].
Ignoring the dilaton term, the gauged action takes the form

\[ S = \frac{1}{4\pi\alpha'} \int_{\Sigma} (g_{mn} \, D x^m \wedge * D x^n - 2 A^a \wedge \tilde{v}_a + B_{ab} A^a \wedge A^b) + \frac{1}{2\pi\alpha'} \int_{B} H_3 + \frac{1}{4\pi\alpha'} \int_{\Sigma} (2 A^a \wedge d\tilde{x}_a + f_{abc} \, \tilde{x}_c \, A^a \wedge A^b) \] (3.4)

where we have introduced gauge fields \( A^a(\sigma) \equiv A^a_\sigma(\sigma) \, d\sigma^a \) \((a = 0, 1)\) and have defined

\[ D_a x^m = \partial_a x^m - A^a_m, \quad F^a = dA^a + \frac{1}{2} f_{bca} A^b \wedge A^c, \quad B_{ab} = \tilde{v}_{[a} \cdot v_{b]} . \] (3.5)

Under the conditions (3.1), this action is invariant under the local symmetry,

\[ \delta \epsilon x^m(\sigma) = \epsilon_a(\sigma) \, v^m_a(x), \quad \delta A^a(\sigma) = d\epsilon^a(\sigma) + f_{bca} A^b(\sigma) \, \epsilon^c(\sigma), \]
\[ \delta \epsilon \tilde{x}_a(\sigma) = c_{ab} \, \epsilon^b(\sigma) - f_{ab} \, \epsilon^c(\sigma) \, \tilde{x}_c(\sigma). \] (3.6)

If we first use the equations of motion for the Lagrange multipliers \( \tilde{x}_a \), the field strengths \( F^a \) are constrained to vanish and the gauge fields will become a pure gauge. Then, at least locally, we can choose a gauge \( A^a = 0 \) and the original theory will be recovered,

\[ S_0 = \frac{1}{4\pi\alpha'} \int_{\Sigma} g_{mn} \, d x^m \wedge * d x^n + \frac{1}{2\pi\alpha'} \int_{B} H_3 . \] (3.7)

On the other hand, by using the equations of motion for \( A^a \) first, we obtain the dual model. For this purpose, it is convenient to rewrite the action as

\[ S = \frac{1}{4\pi\alpha'} \int_{\Sigma} g_{mn} \, d x^m \wedge * d x^n + \frac{1}{2\pi\alpha'} \int_{\Sigma} B_2 + \frac{1}{4\pi\alpha'} \int_{\Sigma} [2 A^a \wedge \nu_a + g_{ab} A^a \wedge * A^b + (B_{ab} + f_{abc} \, \tilde{x}_c) \, A^a \wedge A^b] , \] (3.8)

where

\[ \nu_a \equiv d\tilde{x}_a - v^m_a \, g_{mn} \, * d x^n - \tilde{v}_a, \quad g_{ab} \equiv g_{mn} \, v^m_a \, v^n_b . \] (3.9)

Then, the equations of motion for \( A^a \) become

\[ \nu_a = - g_{ab} \, * A^b - (B_{ab} + f_{abc} \, \tilde{x}_c) \, A^b , \] (3.10)

\[ d\tilde{x}_a - (c_{ab} - f_{abc} \, \tilde{x}_c) \, A^b = v^m_a \, (g_{mn} \, * D x^n + B_{mn} \, D x^n) + \tilde{v}_a \, D x^m , \]

and reduce to the standard self-duality relation when \( \tilde{v}_a = 0 \) and \( f_{abc} = 0 \).
and this can be solved for $A^a$ as

$$A^a = -N^{(ab)} * \nu_b - N^{[ab]} \nu_{\bar{b}} , \quad (3.11)$$

where we have defined

$$(N^{ab}) \equiv (E^{ab} + f_{ab}^{c} \tilde{x}_{c})^{-1} . \quad (3.12)$$

After eliminating the gauge fields, the action becomes

$$S = \frac{1}{4\pi \alpha'} \int_{\Sigma} \left( g_{mn} \ dx^m \land \ * dx^n + B_{mn} \ dx^m \land dx^n + N^{(ab)} \nu_a \land \ * \nu_b + N^{[ab]} \nu_a \land \nu_{\bar{b}} \right)$$

$$= -\frac{1}{4\pi \alpha'} \int_{\Sigma} d^2 \sigma \sqrt{-\gamma} (\gamma^{ab} - \varepsilon^{ab}) \left( E_{mn} \partial_a x^m \partial_b x^n + N^{ab} \nu_a \nu_b \right) , \quad (3.13)$$

where $E_{mn} \equiv g_{mn} + B_{mn}$. In the above computation, we have assumed that the matrix $(E^{ab} + f_{ab}^{c} \tilde{x}_{c})$ is invertible\footnote{Note that the invertibility is not ensured even in the Abelian case $f_{ab}^{c} = 0$.}, but other than that the computation is general.

Now, a major difference from the Abelian case appears. In the Abelian case, by choosing the adapted coordinates $v_{m}^{a} = \delta_{m}^{a}$, we can always realize a gauge $x^{a}(\sigma) = 0$. However, in the non-Abelian case, such gauge choice is not always possible since we cannot realize $v_{m}^{a} = \delta_{m}^{a}$. In order to provide a closed-form expression for the duality transformation rule, in this paper, we assume that the gauge symmetries can be fixed as $x^{i}(\sigma) = c^{i}$ ($c^{i} : \text{constant}$) under a suitable decomposition of spacetime coordinates $(x^{m}) = (y^{\mu}, x^{i})$. This gauge choice removes $n$ coordinates and instead $n$ dual coordinates $\tilde{x}_{a}$ are introduced. Then, the situation is the same as the Abelian case.

Under the gauge choice $x^{i}(\sigma) = c^{i}$, the action \footnote{Corrections and clarifications are made to ensure consistency.} reproduces the dual action for the dual coordinates $x^{m} = (y^{\mu}, \tilde{x}_{a})$,

$$S = \frac{1}{4\pi \alpha'} \int_{\Sigma} d^2 \sigma \sqrt{-\gamma} (\gamma^{ab} - \varepsilon^{ab}) E'_{mn} \partial_a x^m \partial_b x^n , \quad (3.14)$$

$$E'_{mn} \equiv \left( E_{\mu\nu} - \left( v_{\mu a} - \hat{v}_{\mu a} \right) N^{ab} \left( v_{\nu b} + \hat{v}_{\nu b} \right) \left( v_{\nu b} - \hat{v}_{\nu b} \right) N^{cb} \right)|_{x^{i}=c^{i}} , \quad (3.15)$$

Then, the NATD can be understood as a transformation of the target space geometry,

$$E_{mn} \rightarrow E'_{mn} . \quad (3.16)$$

Regarding the transformation rule for the dilaton, we employ the result of \cite{19},

$$e^{-2\Phi'} = \frac{1}{|\det(N^{ab})|} e^{-2\Phi} . \quad (3.17)$$
3.2 NATD as O(D, D) transformation

In order to show a general transformation rule for the R–R fields, it is convenient to describe NATD as O(D, D) rotations. Starting with the original background,

\[
(E_{mn}) = \begin{pmatrix} E_{\mu\nu} & E_{\mu\tilde{\nu}} \\ E_{\nu\mu} & E_{ij} \end{pmatrix}
\]  

we construct the dual background (3.15) through the following three steps.

1. We first perform a GL(D) transformation,

\[
E \rightarrow E^{(1)} = \Lambda_v E \Lambda_v^T, \quad \Lambda_v \equiv \begin{pmatrix} \delta^\mu_\nu & 0 \\ v^\mu_a & v^\nu_b \end{pmatrix}.
\]  

As we assumed, we can fix the gauge symmetry \( \delta x^i = \epsilon^a v^i_a \) such that \( x^i(\sigma) = c^i \) is realized. For this to be possible, \( \det(v^i_a) \neq 0 \) should be satisfied and the matrix \( \Lambda_v \) is invertible. We then obtain

\[
E^{(1)} = \begin{pmatrix} E_{\mu\nu} & E_{\mu m} v^m_b \\ v^m_a E_{mn} v^m_b E_{mn} \end{pmatrix} = \begin{pmatrix} E_{\mu\nu} & (v_{b\mu} - \hat{v}_{b\mu}) + \tilde{v}_{b\mu} \\ (v_{a\nu} + \hat{v}_{a\nu}) - \tilde{v}_{a\nu} & E_{ab} + v^a_c v^b_d \end{pmatrix},
\]  

where we have used

\[
v^m_a B_{mn} = \hat{v}_{am} - \tilde{v}_{am}, \quad B_{ab} = \hat{v}^c_a v^b_c.
\]

2. We next perform a B-transformation

\[
E^{(1)} \rightarrow E^{(2)} = E^{(1)} + \Lambda_f, \quad \Lambda_f \equiv \begin{pmatrix} 0 & -\tilde{v}_{b\mu} \\ \tilde{v}_{a\nu} & f^c_{ab} \tilde{x}_c - v^c_{[a} \tilde{v}^d_{b]} \end{pmatrix},
\]

and obtain

\[
E^{(2)} = \begin{pmatrix} E_{\mu\nu} & (v_{b\mu} - \hat{v}_{b\mu}) \\ (v_{a\nu} + \hat{v}_{a\nu}) & E_{ab} + v^c_{[a} \tilde{x}^d_{b]} \end{pmatrix}.
\]

3. Finally, we perform a T-duality transformation,

\[
E^{(2)} \rightarrow E^{(3)} = (\Lambda_T + \Lambda_T E^{(2)}) (\Lambda_T + \tilde{\Lambda}_T E^{(2)})^{-1},
\]

\[
\Lambda_T \equiv \begin{pmatrix} 1_{d-n} & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\Lambda}_T \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix},
\]

15
and obtain

\[
(E_{mn}^{(3)}) = \begin{pmatrix}
E_{\mu\rho} & (v_{c\mu} - \hat{v}_{c\mu}) \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
(v_{c\nu} + \hat{v}_{c\nu}) & E_{cb} + f_{cb}^d \hat{x}_d
\end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix}
E_{\mu\nu} - (v_{a\mu} - \hat{v}_{a\mu}) N^{ab} (v_{b\nu} + \hat{v}_{b\nu}) (v_{c\mu} - \hat{v}_{c\mu}) N^{cb} \\
- N^{ac} (v_{c\nu} + \hat{v}_{c\nu})
\end{pmatrix}.
\tag{3.25}
\]

By choosing the gauge \( x^i = c^i \), this precisely reproduces the dual background (3.15).

Of course, each step is not a symmetry of supergravity, but this decomposition is useful when we determine the transformation rule of the R–R fields. In terms of the generalized metric \( H_{MN} \), NATD is expressed as an \( O(D, D) \) transformation,

\[
H_{MN} \rightarrow H'_{MN} = (h H h^T)_{MN} |_{x^i = c^i},
\]

\[
(h_M^N) = \begin{pmatrix}
\Lambda_T & \tilde{\Lambda}_T \\
\tilde{\Lambda}_T & \Lambda_T
\end{pmatrix}
\begin{pmatrix}
1 & \Lambda_f \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\Lambda_v & 0 \\
0 & (\Lambda_v)^{-1}
\end{pmatrix},
\tag{3.26}
\]

and the \( O(D, D) \) matrix \( H_M^N \) can be straightforwardly constructed from the given set of the generalized Killing vectors \( V_a = (v_m^a, \tilde{v}_m^a) \).

Under a general \( O(D, D) \) rotation,

\[
H_{MN} \rightarrow H'_{MN} = (h H h^T)_{MN},
\]

\[
E_{mn} \rightarrow E'_{mn} = [(q + p E)(s + r E)^{-1}]_{mn} = [(s^T - E^T r^T)^{-1} (-q^T + E p^T)]_{mn},
\tag{3.27}
\]

the determinant of the metric transforms as (see for example [155])

\[
\sqrt{|g|} \rightarrow \sqrt{|g'|} = |\det(s + r E)|^{-1} \sqrt{|g|}.
\tag{3.28}
\]

Therefore, under the NATD (3.26), we obtain

\[
\sqrt{|g'|} = |\det(\Lambda_T + \tilde{\Lambda}_T E^{(2)})|^{-1} |\det(\Lambda_v)| \sqrt{|g|} \bigg|_{x^i = c^i}
= |\det(N^{ab})||\det(v_a^i)| \sqrt{|g|} \bigg|_{x^i = c^i}.
\tag{3.29}
\]

Combining this with (3.17), we obtain the transformation rule for the DFT dilaton

\[
e^{-2d'} \bigg|_{x^i = c^i} = |\det(v_a^i)| e^{-2d} \bigg|_{x^i = c^i}.
\tag{3.30}
\]

This shows that the DFT dilaton \( e^{-2d} \) transforms covariantly under the \( O(D, D) \) rotation.
3.3 R–R sector

Since the NS–NS fields are transformed covariantly under NATD, it is natural to expect that the R–R fields are also transformed covariantly under the same $O(D,D)$ rotation. Indeed, as we see from the examples, under NATD $\tilde{H}_{MN} = (h \, h^\dagger)_{MN}$, the generalized Ricci tensors are always transformed covariantly,

$$S'_{MN} = (h \, S \, h^\dagger)_{MN}, \quad S' = S.$$  (3.31)

This shows that the R–R fields also should transform covariantly, in order to satisfy the equations of motion of type II DFT,

$$S_{MN} = \mathcal{E}_{MN}, \quad S = 0,$$  (3.32)

where $\mathcal{E}_{MN}$ is an $O(D,D)$-covariant energy-momentum tensor made of the R–R fields.

In DFT, there are basically two approaches to describe the R–R fields. One treats the R–R fields as an $O(D,D)$ spinor [124], which is based on the earlier work [156], and the other treats them as an $O(D) \times O(D)$ bi-spinor [127], which is based on the approach of [157, 158].

R–R fields as a polyform

We first explain the former because it is simpler. Since the treatment of $O(D,D)$ spinor can be rephrased in terms of the differential form, here we treat the R–R field strength as the usual polyform (we follow the convention of [148]),

$$F = \sum_{p: even/odd} \frac{1}{p!} F_{m_1 \cdots m_p} \, dx^{m_1} \wedge \cdots \wedge dx^{m_p} \quad \text{(type IIA/IIB)}.$$  (3.33)

Let us summarize the behavior of an $O(D,D)$ spinor in terms of the polyform.

1. Under a $GL(D)$ subgroup of $O(D,D)$ transformation,

$$(h_M^N) = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad A \in GL(D)$$  (3.34)

a polyform $F$ (that corresponds to an $O(D,D)$ spinor) transforms as a $GL(D)$ tensor,

$$F' = F^{(A)} = \sum_p \frac{1}{p!} F^{(A)}_{m_1 \cdots m_p} \, dx^{m_1} \wedge \cdots \wedge dx^{m_p},$$

$$F^{(A)}_{m_1 \cdots m_p} \equiv A_{m_1}^{n_1} \cdots A_{m_p}^{n_p} F_{n_1 \cdots n_p}.$$  (3.35)
2. Under the $B$-transformation,

\[
(h_M^N) = \begin{pmatrix} 1_d & \omega \\ 0 & 1_d \end{pmatrix},
\]  

(3.36)

a polyform $F$ transforms as

\[
F' = e^{\omega \wedge} F \equiv F + \omega \wedge F + \frac{1}{2!} \omega \wedge \omega \wedge F + \cdots.
\]  

(3.37)

3. Under the (factorized) $T$-duality along the $x^m$-direction, it transforms as

\[
F' = F \cdot T_{x^m}, \quad F \cdot T_{x^m} \equiv F \wedge d\tilde{x}_m + F \vee dx^m,
\]  

(3.38)

where $\tilde{x}_m$ is the coordinate dual to $x^m$, and $\vee dx^m$ denotes the interior product acting from the right.

4. An arbitrary $O(D,D)$ transformation can be decomposed into the above three types of transformations, but for later convenience, we also show that under the $\beta$-transformation,

\[
(h_M^N) = \begin{pmatrix} 1_d & 0 \\ \chi & 1_d \end{pmatrix},
\]  

(3.39)

the transformation rule is given by

\[
F' = e^{\chi \vee} F \equiv F + \chi \vee F + \frac{1}{2!} \chi \vee \chi \vee F + \cdots, \quad \chi \vee F \equiv \frac{1}{2} \chi^{mn} \iota_m \iota_n.
\]  

(3.40)

By using the rules, the general formula for R–R fields under the NATD \((3.26)\) becomes

\[
F' = [e^{A_f \wedge} F^{(A_f')}]. T_{y^1} \cdots T_{y^n} |_{x^i = c^i}, \quad A_f \equiv \frac{1}{2} (A_f)_{mn} dx^m \wedge dx^n,
\]  

(3.41)

where the order of $T_{y^1} \cdots T_{y^n}$ is not important since the overall sign flip is a trivial symmetry. Note that the field strength $F = dA$ is known as the field strength in the A-basis [159] (which is sometimes called the Page form). Another definition, $G \equiv dC + H_3 \wedge C$, is known as the C-basis (see Appendix [A]). In the dual background, $G$ can be obtained as

\[
G' = e^{-B_2 \wedge} F'.
\]  

(3.42)

We note that the approach of [80] based on the Fourier–Mukai transformation (see also [96] for an application) will be closely related to the procedure explained here.
**R–R fields as a bi-spinor**

Next, let us also explain the treatment of the R–R fields as a bi-spinor $\mathcal{G}^{\alpha \beta}$. Starting with a polyform $G$, by using a vielbein $e^m_a$ associated with $g_{mn}$, we define the flat components

$$G_{a_1 \ldots a_p} = e^{m_1}_a \cdots e^{m_p}_a G_{m_1 \cdots m_p}$$

and then define the bi-spinor $\mathcal{G}$ as

$$\mathcal{G} = \sum_p e^p \Gamma^{a_1 \cdots a_p},$$

(3.43)

where $\Gamma^{a_1 \cdots a_p} \equiv \Gamma^{a_1} \cdots \Gamma^{a_p}$ and $(\Gamma^a)_\beta^\alpha$ is the usual gamma matrix satisfying $\{\Gamma^a, \Gamma^b\} = 2 \eta^{ab}$. According to \[127, 157, 158\] (see also \[152\]), under a general $O(D,D)$ rotation

$$\mathcal{H}_{MN} \rightarrow (h \mathcal{H} h^\dagger)_{MN}, \quad h = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

(3.44)

the bi-spinor transforms as

$$\mathcal{G} \rightarrow \mathcal{G} \Omega^{-1},$$

(3.45)

where $\Omega$ is a spinorial representation of the Lorentz transformation $\Lambda^a_b$,

$$\Omega^{-1} \Gamma^a \Omega = \Lambda^a_b \bar{\Gamma}^b,$$

$$\Lambda^a_b = \left[ e^T (s + r E)^{-1} (s - r E^T) e^{-T} \right]^a_b,$$

(3.46)

and $\bar{\Gamma}^a \equiv \Gamma^{11} \Gamma^a$. In particular, under a $T$-duality along a (spatial) $x^i$-direction, we have

$$\Omega = \Omega^{-1} = e^s \frac{\Gamma^a}{\sqrt{g_{zz}}},$$

(3.47)

When the vielbein has a diagonal form $e^m_a \propto \delta^a_m$, $\Omega$ is just the gamma matrix $\Omega = \Gamma^a_z$. The $\Omega$ corresponding to the $\beta$-transformation

$$h = \begin{pmatrix} 1 & 0 \\ \chi & 1 \end{pmatrix},$$

(3.48)

was obtained in \[152\] as

$$\Omega = [\det(\mathcal{E}^I \mathcal{E}^J)]^{-\frac{1}{2}} \mathcal{E}(\frac{1}{2} \beta^{ab} \Gamma_{ab}) \mathcal{E}(\frac{1}{2} \beta^{ab} \Gamma_{ab}),$$

(3.49)

where $\mathcal{E}$ is similar to an exponential function defined in \[157\]

$$\mathcal{E}(\frac{1}{2} \beta^{ab} \Gamma_{ab}) = \sum_{p=0}^5 \frac{1}{2p!} \beta^{a_1 a_2} \cdots \beta^{a_{2p-1} a_{2p}} \Gamma_{a_1 \cdots a_{2p}},$$

(3.50)
where the position of the indices \(a, b\) are changed with the flat metric \(\eta_{ab}\), and we also defined

\[
\mathcal{E}^{ab} \equiv e^{am} e^{bn} E_{mn}^T, \quad \beta^{ab} \equiv -\mathcal{E}^{[ab]}, \quad e_m^a \equiv e_m^b \left(\mathcal{E}^T\right)_b^a,
\]

\[
\mathcal{E}^{\prime ab} \equiv e_m^a e_n^b \left(\mathcal{E}^\dagger + \chi_{mn}\right), \quad \beta^{\prime ab} \equiv -\mathcal{E}^{\prime [ab]}.
\]

(3.51)

Now, let us consider the NATD (3.26). Since it is not easy to find a general expression for \(\Omega\), let us truncate the \(B\)-field and restrict to a simple background,

\[
(E_{mn}) = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & e_i^a e_j^b \eta_{ab} \end{pmatrix}.
\]

(3.52)

We also suppose the generalized Killing vectors are given by \(V_a = v_a^i \partial_i (v_a^b e_j^b = \delta^b_a)\). Then, the vielbein \(e_m^a\) has the block-diagonal form

\[
(e_m^a) = \begin{pmatrix} \hat{e}_\mu^a & 0 \\ 0 & e_i^a \end{pmatrix},
\]

(3.53)

and using this, we define the flat components and the bi-spinor as

\[
G_{a_1 \cdots a_p} \equiv e_{m_1}^{a_1} \cdots e_{m_p}^{a_p} G_{m_1 \cdots m_p}, \quad \mathcal{G} = \sum_p \frac{e_\Phi}{p!} G_{a_1 \cdots a_p} \Gamma_{a_1 \cdots a_p}.
\]

(3.54)

Under the first \(\text{GL}(D)\) transformation, \(\mathcal{G}\) is invariant while the internal part of the vielbein becomes an identity matrix \(e_i^a = \delta_i^a\). We next perform the \(B\)-transformation and \(T\)-dualities, but it is useful to perform the \(T\)-dualities first because the vielbein is now just an identity matrix. Namely, we rewrite the \(B\)-transformation and \(T\)-dualities as \(T\)-dualities and the \(\beta\)-transformation with parameter \(\chi^{ab} \equiv f_{abc} \bar{x}_c\),

\[
\begin{pmatrix} \Lambda_\tau & \bar{\Lambda}_\tau \\ \bar{\Lambda}_\tau & \Lambda_\tau \end{pmatrix} \begin{pmatrix} 1 & \Lambda_f \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \Lambda_f & 1 \end{pmatrix}, \quad \Lambda_f = \begin{pmatrix} 0 & 0 \\ 0 & \chi^{ab} \end{pmatrix}.
\]

(3.55)

Under this transformation, the bispinor is transformed as

\[
\mathcal{G} \rightarrow \mathcal{G} \Omega^{-1}, \quad \Omega^{-1} = \left[\det(\delta_c^d + \chi_c^d)\right]^{-\frac{1}{2}} \mathcal{A} \left(\frac{1}{2} \chi^{ab} \Gamma_{ab}\right) \prod_{a=1}^n \Gamma^a.
\]

(3.56)

This appears to be consistent with the formula given in eq. (3.8) of [73] up to convention.

If we need to consider the spacetime fermions such as the gravitino and the dilatino, they are also transformed by this \(\Omega\), and this approach will be important. However, in order to determine the transformation rule for the \(R-R\) fields, the first approach will be more useful.
4 Examples without R–R fields

In this section, we study examples of NATD without the R–R fields. In the absence of the R–R fields, our setup is basically the same as the standard one. In order to find new solutions, we consider NATD for non-unimodular algebras \( f_{ba}^b \neq 0 \).

As found in [23], in non-unimodular cases, the dual geometry does not solve the supergravity equations of motion. However, as recently found in [142], the dual geometry is a solution of GSE. Additional examples were discussed in [143], and there, by using the result of [28], it was shown that the Killing vector \( I \) in GSE is given by a simple formula,

\[
I = f_{ba}^b \tilde{\partial}^a .
\]  

(4.1)

As we reviewed in section 2, an arbitrary solution of GSE can be regarded as a solution of DFT with linear dual-coordinate dependence. Through a formal \( T \)-duality, this can be mapped to a solution of the conventional supergravity. In this section, we generate new solutions of supergravity by combining the NATD for a non-unimodular algebra and the formal \( T \)-duality.

In fact, by allowing for non-unimodular algebras, we can perform a rich variety of NATD. In order to demonstrate that, we consider several non-Abelian \( T \) dualities of a single solution, the \( \text{AdS}_3 \times S^3 \times T^4 \) background with the \( H \)-flux.

4.1 \( \text{AdS}_3 \times S^3 \times T^4 \): Example 1

In the first example, we introduce the coordinates as

\[
\begin{align*}
\text{d}s^2 & = 2 \frac{\text{d}x^+ \text{d}x^- + \text{d}z^2}{z^2} + \text{d}^2_{S^3 \times T^4} + \text{d}^2_{T^4}, & B_2 & = \frac{\text{d}x^+ \wedge \text{d}x^-}{z^2} + \omega_2, \\
\text{d}^2_{S^3} & \equiv \frac{1}{4} \left[ \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 + (\text{d}\psi + \cos \theta \text{d}\phi)^2 \right], & \omega_2 & \equiv -\frac{1}{4} \cos \theta \text{d}\phi \wedge \text{d}\psi.
\end{align*}
\]  

(4.2)

We then consider the generalized isometries generated by

\[
\begin{align*}
V_1 \equiv (v_1, \tilde{v}_1) & \equiv (-(x^+)^2 \partial_+ + \frac{x^2}{2} \partial_- - x^+ z \partial_z, \ \text{d}x^+ - \frac{x^+}{2} \text{d}z), \\
V_2 \equiv (v_2, \tilde{v}_2) & \equiv (-x^+ \partial_+ - \frac{z}{2} \partial_z, -\frac{1}{2z} \text{d}z),
\end{align*}
\]

(4.3)

which satisfy the algebra \([V_1, V_2]_C = V_1\). The structure constant has the non-vanishing trace \( f_{12}^1 = 1 \) and the dual background will be a solution of GSE.

The \( B \)-field is not isometric along the \( v_1 \) direction \( \mathcal{L}_{v_1} B_2 \neq 0 \), and the dual component \( \tilde{v}_1 \) is necessary to satisfy the generalized Killing equations \( \mathcal{L}_{v_1} B_2 + \text{d}\tilde{v}_1 = 0 \). Moreover, in order
to realize \([V_1, V_2]_C = V_1\), the dual component of \(V_2\) is also necessary. In this case, we find

\[
(c_{ab}) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \neq 0,
\]

(4.4)

but the requirement \(f_{ab}^d c_{dc} = 0\) in (3.1) is not violated and we can perform the NATD. The gauge symmetry associated with the generalized Killing vector \(V_2\)

\[
\delta_\epsilon x^+(\sigma) = \epsilon^2 v_2^+(x) = -\epsilon^2(\sigma) x^+(\sigma),
\]

(4.5)

can be fixed by realizing \(x^+(\sigma) = 1\). Similarly, the gauge symmetry associated with \(V_1\)

\[
\delta_\epsilon z(\sigma) = e^1 v_1^z(x)_{x^+=1} = -e^1(\sigma) z(\sigma),
\]

(4.6)

can be also fixed as \(z(\sigma) = 1\).

The AdS parts of the matrices in the formula (3.26) (before the gauge fixing) become

\[
(\Lambda_v) = \begin{pmatrix} - (x^+)^2 & \frac{1}{x^+} x^- & -x^+ z \\ 0 & 1 & 0 \\ -x^+ & 0 & -\frac{1}{x^+} \end{pmatrix}, \quad (\Lambda_f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tilde{x}_+ - \frac{x^+}{2} \\ -\tilde{x}_+ + \frac{x^+}{2} & 0 & 0 \end{pmatrix},
\]

(4.7)

and under the gauge \(x^+ = 1\) and \(z = 1\), the dual background becomes

\[
ds^2 = \frac{d\tilde{x}_+^2}{4} + 2(1 - 4 \tilde{x}_+) \frac{d\tilde{x}_- dx^-}{\tilde{x}_+} + 2 \frac{dx^- d\tilde{z}}{\tilde{x}_+} + ds^2_{S^3}, \quad e^{-2\Psi'} = \tilde{x}_+^2,
\]

\[
B'_2 = \frac{(1 - 4 \tilde{x}_+) \frac{d\tilde{x}_+ dx^-}{\tilde{x}_+} - (d\tilde{x}_+ dx^-) \frac{d\tilde{z}}{\tilde{x}_+}}{4 \tilde{x}_+^2} + \omega_2.
\]

(4.8)

As expected, this background does not solve the conventional supergravity equations of motion, but instead satisfies the GSE with the Killing vector

\[
I' = f_{ab}^a \tilde{\partial}^b = \tilde{\partial}^z.
\]

(4.9)

Interestingly, this geometry is locally the same as the original \(\text{AdS}_3 \times S^3\) spacetime. Indeed, by changing coordinates as

\[
x'^+ \equiv \tilde{z} - \tilde{x}_+ + \frac{1}{4} \ln \tilde{x}_+, \quad x'^- \equiv x^-, \quad z' \equiv \sqrt{\tilde{x}_+},
\]

(4.10)
we obtain the expression

\[d s^2 = 2 d x'^+ d x'^- + d z'^2 + d s^2_{S^3 \times T^4}, \quad e^{-2 \Phi} = z'^4,\]

\[B_2 = \frac{d x'^+ \wedge d x'^-}{z'^2} + \frac{2 d x'^+ \wedge d z'}{z'} + \omega_2, \quad I = \partial'_x.\]

In fact, we can find a 2-parameter family of solutions

\[d s^2 = 2 d x'^+ d x'^- + d x'^- + d z'^2 + d s^2_{S^3 \times T^4}, \quad e^{-2 \Phi} = z'^4,\]

\[B_2 = \frac{d x'^+ \wedge d x'^-}{z'^2} + 2 c_1 d x'^+ \wedge d z' + \omega_2, \quad I = c_0 \partial'_x,\]

and the NATD connects the original background \((c_0, c_1) = (0, 0)\) and the dual \((c_0, c_1) = (1, 1)\).

The metric in (4.11) is the same as the original one (4.2) and the \(B\)-field is also just shifted by a closed form \(B_2 \rightarrow B_2 + 2 d x'^+ \wedge d \ln z\). The only difference from the original background is in the dilaton. We note that, unlike the case of “trivial solutions” [152], we cannot remove the Killing vector \(I^m\) in the dual geometry [111].

It will be natural to consider performing a \(B\)-field gauge transformation in order to undo the shift in the \(B\)-field. However, in the conventional GSE, where the only modification is given by the Killing vector \(I^m\), the gauge symmetry for the \(B\)-field is already fixed and we cannot perform a \(B\)-field gauge transformation. Indeed, if we truncate the closed-form in the \(B\)-field by hand, we find another solution

\[d s^2 = \frac{2 d x'^+ d x'^- + d z'^2}{z'^2} + d s^2_{S^3}, \quad e^{-2 \Phi} = z'^4 c_1,\]

\[B_2 = \frac{d x'^+ \wedge d x'^-}{z'^2} + 2 c_1 d x'^+ \wedge d z' + \omega_2, \quad I = c_0 \partial'_x,\]

where \(c_0\) is a free parameter and \(c_1\) can take two values, \(c_1 = 0\) or \(c_1 = 1\). This is an example of the trivial solution and \(c_0\) can be chosen as \(c_0 = 0\). Then, we get two \(AdS_3 \times S^3 \times T^4\) solutions of the supergravity, with a different dilaton \(c_1 = 0\) or \(c_1 = 1\).

For an arbitrary GSE solution, by taking a coordinate system with \(I = I^z \partial_z\), it can be regarded as a DFT solution with the DFT dilaton \(d = d_0 + I^z \tilde{x}_z (e^{-2 d_0} \equiv e^{-2 \Phi} \sqrt{|g|})\). Then, if we perform a formal T-duality that exchanges \(\tilde{x}_z\) with the physical coordinate \(x^z\), we can get a solution of the conventional supergravity where the DFT dilaton is \(d = d_0 + I^z x^z\). In the present example (4.12), we perform a formal T-duality along the \(x^+\)-direction, and then

\[^6\text{According to [160], a solution of GSE is a trivial solution (namely also satisfies the supergravity equations of motion with } I = 0) \text{ only when } \tilde{K}^m \equiv I^n B_{np} \Theta^m \text{ satisfies } L_{\tilde{K}} g_{mn} = 0, L_{\tilde{K}} \Phi + (I + \tilde{K})^2 = 0, \text{ and } d I_1 + i_{\tilde{K}} H_3 = 0 \left( I_1 \equiv I^m g_{mn} d x^n \right) \text{ but they are not satisfied here.}\]
the DFT dilaton becomes a function of the physical coordinates

\[ e^{-2d} = e^{-2c_0 x^+} z^4 c_0 c_1 \sqrt{\frac{\sin^2 \theta}{64 z^6}}. \]  

(4.14)

Then, the dual-coordinate dependence disappears from the background fields. However, in this case, the AdS part of the dualized generalized metric becomes

\[ (\mathcal{H}_{MN}) = \begin{pmatrix} g_{mn} - B_{mp} g^{pq} B_{qn} & B_{mp} g^{pn} \\ -g^{mp} B_{pn} & g^{mn} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2 c_1} & \frac{1}{2 c_1} \\ 0 & 0 & \frac{1}{2 c_1} & 0 \end{pmatrix}, \]  

(4.15)

and we cannot extract the supergravity fields \((g_{mn}, B_{mn}, \Phi)\) from \(\mathcal{H}_{MN}\) due to \(\det(g^{mn}) = 0\). This type of (genuinely) DFT solution is called the non-Riemannian background \([161]\), which is studied in detail in \([162–165]\). Using a parameterization given in \([163]\), we find

\[ (\mathcal{H}_{MN}) = \begin{pmatrix} \delta^p_m & B_{mp} \\ 0 & \delta^p_p \end{pmatrix} \begin{pmatrix} K_{pq} & X^p Y^q - \tilde{X}^p \tilde{Y}^q \\ Y^p X_q - \tilde{Y}^p \tilde{X}_q & H^{pq} \end{pmatrix} \begin{pmatrix} \delta^q_n & 0 \\ -B_{qn} & \delta^q_q \end{pmatrix}, \]  

(4.16)

\[ H = \begin{pmatrix} 4 c_1^2 & 0 & 2 c_1 z \\ 0 & 0 & 0 \\ 2 c_1 z & 0 & z^2 \end{pmatrix}, \quad K = \begin{pmatrix} \frac{1}{4 c_1^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -\frac{1}{2 c_1 z} \\ 0 & 0 & 0 \\ \frac{1}{2 c_1 z} & 0 & 0 \end{pmatrix}. \]

\[ X^1 = \begin{pmatrix} \frac{z}{2} \\ 0 \\ c_1 \end{pmatrix}, \quad \tilde{X}^1 = \begin{pmatrix} \frac{z}{2} \\ \frac{1}{c_1} \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 \\ -z \\ \frac{1}{c_1} \end{pmatrix}, \quad \tilde{Y}_1 = \begin{pmatrix} 0 \\ z \\ 0 \end{pmatrix}. \]

In the parameterization of \([163]\), in general, there are \(n\) pairs of vectors \((X^i, Y^i)\) and \(\tilde{n}\) pairs of vectors \((\tilde{X}_i, \tilde{Y}_i)\), and such a non-Riemannian background is called a \((n, \tilde{n})\) solution. In this notation, this background is a \((1,1)\) solution.

In this way, in the first example of NATD, the formal \(T\)-duality does not produce the usual supergravity solution, and we instead obtained a \((1,1)\) non-Riemannian background.

### 4.2 AdS\(_3\) × S\(_3\) × T\(_4\): Example 2

In the second example, we take the coordinates,

\[ ds^2 = -\frac{dt^2 + dx^2 + dz^2}{z^2} + ds^2_{S^3 \times T^4} + ds^2_{T^4}, \quad B_2 = \frac{dt \wedge dx}{z^2} + \omega_2, \]  

(4.17)
and consider the translation and the dilatation generators as the generalized Killing vectors,

\[ V_1 \equiv (v_1, \tilde{v}_1) \equiv (\partial_x, 0), \quad V_2 \equiv (v_2, \tilde{v}_2) \equiv \left( t \partial_t + x \partial_x + z \partial_z, 0 \right), \]  

(4.18)

which satisfy \([V_1, V_2]_C = V_1\) and \(c_{ab} = 0\). Here, we fix the gauge as \(x(\sigma) = 0\) and \(z(\sigma) = 1\).

The AdS\(_3\) parts of the transformation matrices are

\[
(\Lambda_v) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
t & x & z \\
\end{pmatrix}, \quad
(\Lambda_f) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \tilde{x} \\
-\tilde{x} & 0 & 0 \\
\end{pmatrix},
\]  

(4.19)

and the NATD gives

\[
ds^2 = -\tilde{x}^2 dt^2 + 2(1 - t \tilde{x}) dt dx + (1 - t^2) d\tilde{x}^2 + d\tilde{z}^2,
\]

\[
B'_2 = \frac{[(t - \tilde{x}) d\tilde{x} - \tilde{x} dt] \wedge d\tilde{z}}{1 - 2t \tilde{x} + \tilde{x}^2} + \omega_2, \quad e^{-2\Phi'} = 1 - 2t \tilde{x} + \tilde{x}^2.
\]

(4.20)

This satisfies the GSE by introducing the Killing vector as \(I' = f_{ab}^a \tilde{\partial}^b = \tilde{\partial}^z\).

Again, in order to remove the Killing vector \(I\), let us perform a formal \(T\)-duality along the \(\tilde{z}\)-direction. This yields a simple linear-dilaton solution of the supergravity,

\[
ds^2 = 2 dt d\tilde{x} + d\tilde{x}^2 - 2 \tilde{x} dt dz + 2(t - \tilde{x}) d\tilde{x} dz + (1 - 2t \tilde{x} + \tilde{x}^2) dz^2 + d\tilde{s}_{S^3 \times T^4}^2, \quad B_2 = \omega_2, \quad \Phi = z,
\]

(4.21)

where the AdS part of the \(B\)-field has disappeared.

### 4.3 \(\text{AdS}_3 \times S^3 \times T^4\): Example 3

We next use the Rindler-type coordinates,

\[
ds^2 = -\frac{x^2 dt^2 + dx^2 + dz^2}{z^2} + d\tilde{s}_{S^3 \times T^4}^2, \quad B_2 = \frac{x dt \wedge dx}{z^2} + \omega_2,
\]

(4.22)

and consider the generalized Killing vectors

\[ V_1 \equiv (v_1, \tilde{v}_1) \equiv (\partial_t, 0), \quad V_2 \equiv (v_2, \tilde{v}_2) \equiv (e^{-t}(x^{-1} \partial_t + \partial_x), 0), \]

(4.23)

which satisfy \([V_1, V_2]_C = -V_2\) and \(c_{ab} = 0\). Here, we take a gauge \(t(\sigma) = 0\) and \(x(\sigma) = 1\).
The AdS parts of the transformation matrices are
\[
(\Lambda_v) = \begin{pmatrix}
1 & 0 & 0 \\
e^{-t} & e^{-t} & 0 \\
0 & 0 & 1 
\end{pmatrix}, \quad (\Lambda_f) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & - \tilde{x} \\
\tilde{x} & 0 & 0
\end{pmatrix},
\]
and the dual background, which satisfies the GSE, becomes
\[
d\bar{s}^2 = \frac{dz^2 - 2 d\tilde{t} d\tilde{x}}{\tilde{x} (2 - \tilde{x} z^2)} + \frac{dz^2}{z^2} + ds^2_{S^3 \times T^4}, \quad e^{-2\Phi} = \frac{\tilde{x} (z^2 - 2)}{z^2},
\]
\[
B'_2 = \frac{1 - \tilde{x} z^2}{\tilde{x} (2 - \tilde{x} z^2)} \, d\tilde{t} \wedge d\tilde{x} + \omega_2, \quad t' = \tilde{\partial}^t.
\]

In order to obtain a solution of the supergravity, if we again perform a formal $T$-duality along the time direction $\tilde{t}$. This time again we find a non-Riemannian background,
\[
(H_{MN}) = \begin{pmatrix}
\tilde{x} (\tilde{x} z^2 - 2) & 1 - \tilde{x} z^2 & 0 & \tilde{x} z^2 - 1 & \tilde{x} (\tilde{x} z^2 - 2) & 0 \\
1 - \tilde{x} z^2 & z^2 & 0 & -z^2 & 1 - \tilde{x} z^2 & 0 \\
0 & 0 & \tilde{x} z^2 - 1 & -z^2 & 0 & 0 \\
\tilde{x} (\tilde{x} z^2 - 2) & 1 - \tilde{x} z^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z^2 & 0
\end{pmatrix},
\]
where the $S^3 \times T^4$ part of the generalized metric is not displayed. This is also a $(1,1)$ solution,
\[
(H_{MN}) = \begin{pmatrix}
\delta_m^n & B_{mp} \\
0 & \delta_p^m
\end{pmatrix} \begin{pmatrix}
K_{pq} & X_1^1 Y_1^q - \bar{X}_1^1 \bar{Y}_1^q \\
Y_1^p X_1^q - \bar{Y}_1^p \bar{X}_1^q & H^{pq}
\end{pmatrix} \begin{pmatrix}
\delta_n^q & 0 \\
-B_{qn} & \delta_q^n
\end{pmatrix},
\]
\[
H = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & z^2
\end{pmatrix}, \quad K = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{z^2}
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & -\frac{1}{z} & 0 \\
\frac{1}{z} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
\[
X_1 = \begin{pmatrix}
\tilde{x} \frac{z^2}{\tilde{x}^2 - 2} \\
-\frac{z^2}{\tilde{x}^2 - 2} \\
0
\end{pmatrix}, \quad \bar{X}_1^1 = \begin{pmatrix}
\frac{1}{\tilde{x} - \bar{x}} - \tilde{x} \\
1 \\
0
\end{pmatrix}, \quad Y_1 = \begin{pmatrix}
1 \\
\tilde{x} - \frac{z^2}{\tilde{x}^2 - 2} \\
0
\end{pmatrix}, \quad \bar{Y}_1 = \begin{pmatrix}
\frac{\tilde{x} z^2}{\tilde{x}^2 - 2} \\
\tilde{x} \frac{z^2}{\tilde{x}^2 - 2} \\
0
\end{pmatrix}.
\]

To summarize shortly, NATD works well a solution generating technique of DFT even if the isometry algebra is non-unimodular. If we additionally perform a formal $T$-duality, we usually obtain the usual supergravity solution, but sometimes, the parameterization of the generalized metric becomes singular, and we obtain a non-Riemannian background. The non-Riemannian backgrounds do not have the usual supergravity interpretation, but they have interesting applications [162,165] and they are interesting backgrounds by their selves. Therefore, it will be important to study NATD for non-unimodular algebras more seriously.
5 Examples with R–R fields

In this section, we consider NATD with the R–R fields. After reproducing a known example, again we consider examples for non-unimodular algebras.

For convenience, let us display the summary of the duality rules. Under the setup,

\[ \hat{\mathcal{L}}_{V} \mathcal{H}_{MN} = 0, \quad [V_{a}, V_{b}]_{C} = f_{ab}^{c} V_{c}, \quad \eta_{MN} V_{a}^{M} V_{b}^{N} = 2 \epsilon_{ab}, \quad f_{ab} \epsilon_{dc} = 0, \quad (5.1) \]

where \((V_{a}^{M}) = (v_{a}^{m}, \tilde{v}_{am})\), the dual background is given by

\[ \mathcal{H}'_{MN} = (h \mathcal{H} h^{\dagger})_{MN} \big|_{x' = c}, \quad e^{-2d'} = |\text{det}(v_{a}^{d})| e^{-2d} \big|_{x = e}, \quad F' = [e^{\Lambda} \wedge F(\Lambda_{\dagger})] \cdot \mathcal{T} y^{i} \cdots \mathcal{T} y^{n} \big|_{x' = e'}, \quad I = f_{ba} \hat{\partial}^{a}, \quad (5.2) \]

where

\[ (h_{MN}^{J}) \equiv \begin{pmatrix} \Lambda_{T} & \Lambda_{T} \Lambda_{f} \Lambda_{v} & 0 \\ \Lambda_{T} & 0 & \Lambda_{v} & 0 \\ \Lambda_{T} & \Lambda_{f} & 0 & \Lambda_{v}^{-1} \end{pmatrix}, \quad \Lambda_{f} \equiv \frac{1}{2} (\Lambda_{f})_{mn} dx^{m} \wedge dx^{n}, \quad (5.3) \]

\[ \Lambda_{v} \equiv \begin{pmatrix} \delta^{a}_{\mu} & 0 \\ v_{a}^{
u} & v_{a}^{\dagger} \end{pmatrix}, \quad \Lambda_{f} \equiv \begin{pmatrix} 0 & -v_{ab} \mu \\ \tilde{v}_{ab} & f_{ab}^{c} \tilde{x}_{c} - v_{[a} \cdot \tilde{v}_{b]} \end{pmatrix}, \quad \Lambda_{T} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda_{v} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \]

and the coordinates are transformed as \((x^{m}) = (y^{\mu}, x^{i}) \rightarrow (x'^{m}) = (y^{\mu}, \tilde{x}_{a})\).

5.1 AdS\(_{3} \times S^{3} \times T^{4}\)

As the first example of NATD with the R–R fields, let us review the example of [72] and demonstrate that our formula gives the same result. The original background is

\[ ds^{2} = -\frac{dt^{2} + dz^{2} + dz^{2}}{\ell^{2} z^{2}} + \frac{1}{4 \ell^{2}} [d \theta^{2} + \sin^{2} \theta d \phi^{2} + (d \psi + \cos \theta d \phi)^{2}] + ds_{T^{4}}^{2}, \quad (5.4) \]

\[ G_{3} = \frac{2 dt \wedge dx \wedge dz}{\ell^{2} z^{3}} - \frac{\sin \theta}{4 \ell^{2}} d \theta \wedge d \phi \wedge d \psi, \]

where AdS\(_{3}\) and S\(_{3}\) part have the curvature \(R = \mp 6 \ell^{2}\).

We perform the non-Abelian T-dualities associated with three generalized Killing vectors

\[ V_{1} = (\cos \psi \partial_{\theta} + \frac{\sin \psi}{\sin \theta} \partial_{\phi} - \frac{\sin \psi}{\sin \theta} \partial_{\psi}, 0), \]

\[ V_{2} = (- \sin \psi \partial_{\theta} + \frac{\cos \psi}{\sin \theta} \partial_{\phi} - \frac{\cos \psi}{\sin \theta} \partial_{\psi}, 0), \quad V_{3} = (\partial_{\psi}, 0), \quad (5.5) \]

on the S\(_{3}\), which satisfy

\[ [V_{1}, V_{2}]_{C} = V_{3}, \quad [V_{2}, V_{3}]_{C} = V_{1}, \quad [V_{3}, V_{1}]_{C} = V_{2}. \quad (5.6) \]
As clear from the explicit form of the Killing vectors, we can choose a gauge

$$\theta(\sigma) = \frac{\pi}{2}, \quad \phi(\sigma) = 0, \quad \psi(\sigma) = 0. \quad (5.7)$$

The \((\theta, \phi, \psi)\) parts of the transformation matrices are

$$(\Lambda_v) = \begin{pmatrix}
\cos \psi & \sin \psi & -\sin \psi \\
-\sin \psi & \cos \psi & -\sin \psi \\
0 & 0 & 1
\end{pmatrix}, \quad (\Lambda_f) = \begin{pmatrix}
0 & \tilde{\psi} & -\tilde{\phi} \\
-\tilde{\psi} & 0 & \tilde{\theta} \\
\tilde{\phi} & -\tilde{\theta} & 0
\end{pmatrix}, \quad (5.8)$$

and the NS–NS fields in the dual background are

$$d s^2 = \frac{-d t^2 + d x^2 + d z^2}{\ell^2 z^2} + \frac{4 \ell^2 (\delta_{ij} + 16 \ell^4 u_i u_j)}{1 + 16 \ell^4 u_k u^k} d u^i d u^j + d s^2_{T^4}, \quad (5.9)$$

$$B'_2 = -8 \ell^4 \epsilon_{ijk} u^i d u^j \wedge d u^k, \quad e^{-2\Phi} = \frac{1 + 16 \ell^4 u_k u^k}{64 \ell^6},$$

where we denoted \((u^i) \equiv (\tilde{\theta}, \tilde{\phi}, \tilde{\psi})\), \(u_i \equiv u^i\), and \(\epsilon_{123} = 1\).

Now, let us consider the R–R fields. Under the gauge \((5.7)\), the Page form becomes

$$F = \left( \frac{2 d t \wedge d x \wedge d z}{\ell^2 z^3} - \frac{d \theta \wedge d \phi \wedge d \psi}{4 \ell^2} \right) \wedge [1 - \text{vol}(T^4)]. \quad (5.10)$$

The first GL\((D)\) transformation is trivial \(\Lambda_v = 1\) under the gauge \((5.7)\). We next perform the \(B\)-transformation \(F \rightarrow e^{\Lambda_f} \wedge F\) where

$$\Lambda_f = u^1 d \phi \wedge d \psi + u^2 d \psi \wedge d \theta + u^3 d \theta \wedge d \phi. \quad (5.11)$$

Finally, by performing \(T\)-dualities along \((\theta, \phi, \psi)\)-directions, we obtain

$$F' = (F + \Lambda_f \wedge F) \wedge (\wedge d u^1 + \wedge d \theta) \wedge (\wedge d u^2 + \wedge d \phi) \wedge (\wedge d u^3) + \wedge \left[1 - \text{vol}(T^4)\right]. \quad (5.12)$$

From this Page form, we get the R–R field strengths in the C-basis as

$$G'_0 = \frac{1}{4 \ell^2}, \quad G'_2 = \frac{2 \ell^2 \epsilon_{ijk} u^i d u^j \wedge d u^k}{1 + 16 \ell^4 u_k u^k},$$

$$G'_4 = -\frac{2 d t \wedge d x \wedge d z \wedge u_i d u^i}{\ell^2 z^3} - \frac{\text{vol}(T^4)}{4 \ell^2}. \quad (5.13)$$

They are precisely the same as the results of [72] (where \(\ell = 1/2\)).

Since the R–R potential also behaves an \(O(D,D)\) spinor in DFT, let us also explain how to determine the R–R potential in the dual background. Due to the gauge fixing \((5.7)\), the
Page form takes the form (5.10). Then the R–R potential in the A-basis is
\[ A = -\left( \frac{dt \wedge dx}{\ell^2 z^2} + \frac{\theta d\phi \wedge d\psi}{4 \ell^2} \right) \wedge \left[ 1 - \text{vol}(T^4) \right], \]
where \( \theta \) should not be set to \( \theta = \pi/2 \) in order to realize \( F = dA \). Similar to the field strength, GL(\( D \)) transformation is trivial, and the B-transformation \( A \rightarrow e^{A_i \tau^i} A \) and \( T \)-dualities along \( (\theta, \phi, \psi) \)-directions give
\[ A' = \left[ \frac{\omega_1}{4 \ell^2} + \frac{dt \wedge dx \wedge (u_1 du^1 - du^1 \wedge du^2 \wedge du^3)}{\ell^2 z^2} \right] \wedge [1 - \text{vol}(T^4)], \]
where we denoted \( \omega_1 \equiv \theta \) as it is dual to \( u^1 = \tilde{\theta} \). As \( A \) depends on the dual coordinate explicitly, the relation between \( F \) and \( A \) is generalized as\(^7\)
\[ F = dA, \quad d \equiv dx^m \wedge \partial_m + t_m \tilde{\theta}^m, \]
and the \( A' \) in (5.15) correctly reproduces the \( F' \) obtained in (5.12). This result is consistent with \[125\] where the massive IIA supergravity was reproduced from DFT by introducing a linear dual-coordinate dependence into the R–R 1-form potential. The potential in the C-basis also can be obtained by computing \( C' = e^{-B_i \tau^i} A' \).

5.2 \( \text{AdS}_5 \times S^5 \)

As the second example, let us consider a NATD of the \( \text{AdS}_5 \times S^5 \) background associated with a non-unimodular algebra. The original background is
\[ ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dz^2 + ds^2_{S^5} \quad (\eta_{\mu\nu}) \equiv \text{diag}(-1, 1, 1, 1), \]
\[ G = 4 \left( -dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dz + \omega_5 \right) = F, \]
where
\[ ds^2_{S^5} \equiv dr^2 + \sin^2 r d\xi^2 + \sin^2 r \cos^2 \xi d\phi_1^2 + \sin^2 r \sin^2 \xi d\phi_2^2 + \cos^2 r d\phi_3^2, \]
\[ \omega_5 \equiv \sin^3 r \cos r \sin \xi \cos \xi dr \wedge d\xi \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3. \]

We consider a NATD associated with two Killing vectors,
\[ V_1^M = (z \partial_z + x^\mu \partial_\mu, 0), \quad V_2^M = (\partial_1, 0), \]
which satisfy \([V_1, V_2]_C = -V_2 \). The gauge symmetry can be fixed as \( z(\sigma) = 1 \) and \( x^1(\sigma) = 0 \),

\(^7\)The operator \( d \) is useful also in GSE. In GSE, the R–R fields have the dual-coordinate dependence as \( A = e^{-I_m \tau^m} A(x^m) \) and \( F = e^{-I_m \tau^m} F(x^m) \), and the relation \( F = dA \) reproduces \( F = e^{I_m \tau^m} dA = d\tilde{A} - \iota_1 \tilde{A} \). By considering \((\tilde{A}, \tilde{F})\) as the dynamical fields, we obtain the relation in GSE (A.10). See \[118\] for more detail.
and the AdS parts of the transformation matrices are

\[
(A_{\nu}) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
x^0 & x^1 & x^2 & x^3 & z
\end{pmatrix}, \quad (A_{f}) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{x}_1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -\tilde{x}_1 & 0 & 0 & 0
\end{pmatrix}.
\] (5.20)

For simplicity, we denote \((u^\mu) \equiv (x^0, \tilde{x}_1, x^2, x^3)\), and then the dual background becomes

\[
d s^2 = \frac{d\tilde{z}^2 + \alpha_{\mu\nu} u^\mu u^\nu}{1 + \eta_{\rho\sigma} u^\rho u^\sigma} + \eta_{\mu\nu} d u^\mu d u^\nu + d s^2_{S^5}, \quad e^{-2\Phi} = 1 + \eta_{\mu\nu} u^\mu u^\nu,
\]

\[
B_2' = \frac{(-u^0 d u^0 - u^1 d u^1 + u^2 d u^2 + u^3 d u^3)}{1 + \eta_{\rho\sigma} u^\rho u^\sigma} \wedge d \tilde{z},
\] (5.21)

where

\[
(a_{\mu\nu}) = \begin{pmatrix}
-u^0 u^0 & -u^0 u^1 & u^0 u^2 & u^0 u^3 \\
-u^1 u^0 & -u^1 u^1 & u^1 u^2 & u^1 u^3 \\
u^2 u^0 & u^2 u^1 & -u^2 u^2 & -u^2 u^3 \\
u^3 u^0 & u^3 u^1 & -u^3 u^2 & -u^3 u^3
\end{pmatrix}.
\] (5.22)

Regarding the R–R fields, the first GL\((D)\) transformation does not change the Page form and the next \(B\)-transformation gives

\[
F = 4 \left( -d u^0 \wedge d u^1 \wedge d u^2 \wedge d u^3 \wedge d z + \omega_5 \right) + 4 u^1 \omega_5 \wedge d u^1 \wedge d z.
\] (5.23)

The Abelian \(T\)-dualities along \(z\) and \(x^1\) directions give

\[
F' = -4 d u^0 \wedge d u^2 \wedge d u^3 + 4 \omega_5 \wedge d \tilde{z} \wedge d u^1 + 4 u^1 \omega_5.
\] (5.24)

From this Page form, we find

\[
G_3' = -4 d u^0 \wedge d u^2 \wedge d u^3, \quad G_5' = -4 u^1 d u^0 \wedge d u^1 \wedge d u^2 \wedge d u^3 \wedge d \tilde{z} + u^1 d \omega_4.
\] (5.25)

Then, by introducing \(I = f_{ba}^{\ d} \tilde{\partial}^a = \tilde{\partial}^z\), they satisfy the type IIB GSE.

In order to obtain a solution of the usual supergravity, we perform a formal \(T\)-duality along the \(z\)-direction. By using the \(T\)-duality rule \([A.13]\), we obtain a simple type IIA solution,

\[
\begin{align*}
d s^2 &= (1 + \eta_{\mu\nu} u^\mu u^\nu) d z^2 + 2 \left( -u^0 d u^0 - u^1 d u^1 + u^2 d u^2 + u^3 d u^3 \right) d z \\
&\quad + \eta_{\mu\nu} d u^\mu d u^\nu + d s^2_{S^5}, \quad \Phi = z, \quad G_4 = 4 e^{-z} d z \wedge d u^0 \wedge d u^2 \wedge d u^3.
\end{align*}
\] (5.26)
5.3 AdS\(_3 \times S^3 \times T^4\) with NS–NS and R–R fluxes

In order to demonstrate the effectiveness of our formula, let us consider a more non-trivial example. We consider the AdS\(_3 \times S^3 \times T^4\) with the NS–NS and the R–R fluxes,

\[
ds^2 = \frac{-dt^2 + dz^2 + dz^2}{z^2} + \frac{1}{4} [d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi + \cos \theta d\phi)^2] + ds_{T^4}^2,
\]

\[
B_2 = p \left( \frac{dt \wedge dz}{z^2} - \frac{\cos \theta \, d\phi \wedge d\psi}{4} \right), \quad G_3 = q \left( \frac{2 dt \wedge dx \wedge dz}{z^3} - \frac{\sin \theta \, d\theta \wedge d\phi \wedge d\psi}{4} \right),
\]

where \(p\) and \(q\) are constants satisfying \(p^2 + q^2 = 1\). The Page form has the following form:

\[
F = G_3 + F_3 - (G_3 + F_3) \wedge \text{vol}_{T^4}, \quad F_3 \equiv d \left( \frac{pq \cos \theta}{4z^2} \right) \wedge dt \wedge dx \wedge d\phi \wedge d\psi.
\]

Then, we consider the generalized Killing vectors,

\[
V_1 \equiv (v_1, \tilde{v}_1) \equiv (t \partial_t + x \partial_x + z \partial_z, 0), \quad V_2 \equiv (v_2, \tilde{v}_2) \equiv (-2t \partial_t + (-t^2 - x^2 + z^2) \partial_x - 2xz \partial_z, 2p \cos \theta \wedge 2p \cos \theta \wedge 2p \cos \theta \wedge 2p \cos \theta \wedge 2p \cos \theta),
\]

which satisfy \([V_1, V_2] = 0\). The \(B\)-field is isometric along the dilatation generator \(L_{v_1}B_2 = 0\), but it is not isometric along the special conformal generator \(L_{v_2}B_2 \neq 0\) and the dual component \(\tilde{v}_2\) is important. Here, we choose the gauge as \(t(\sigma) = 1\) and \(x(\sigma) = 1\).

The AdS parts of the transformation matrices are

\[
(\Lambda_v) = \begin{pmatrix} t & x & z \\ -2tx & -t^2 - x^2 + z^2 & -2xz \\ 0 & 0 & 1 \end{pmatrix}, \quad (\Lambda_f) = \begin{pmatrix} 0 & \tilde{x} & 0 \\ -\tilde{x} & 0 & -2p \tilde{\phi} \\ 2p \tilde{\phi} & 0 & 0 \end{pmatrix},
\]

and the NS–NS fields and the Killing vector take the form

\[
ds^2 = \frac{z^2 dt^2 + 2 \tilde{d} \tilde{d} \tilde{d} + d\tilde{d}^2 + \frac{(x-p)^2-1}{z^2} dz^2 - \frac{2z}{z^2} \left[ z \tilde{d} \tilde{d} + (\tilde{x} - p) d\tilde{d} \right] dz}{z^2 + (\tilde{x} + p)^2 - 1} + ds_{S^3}^2 + ds_{T^4}^2,
\]

\[
B'_2 = -z \frac{(x + p) d\tilde{d} \wedge d\tilde{x} - \left[ z^2 + 2p (\tilde{x} + p) - 2 \right] d\tilde{d} \wedge dz + d\tilde{x} \wedge dz}{z \left[ z^2 + (\tilde{x} + p)^2 - 1 \right]} - p \cos \theta d\phi \wedge d\psi,
\]

\[
e^{-2\Phi'} = z^2 + (\tilde{x} + p)^2 - 1, \quad I' = -\tilde{\phi}'.
\]

For the R–R fields, the first \(GL(D)\) transformation makes the replacement

\[
dt \wedge dx \wedge dz \rightarrow z^2 dt \wedge dx \wedge dz,
\]

in the Page form (5.28), and by further acting \(e^{\Lambda_f} r^\wedge\) and \(T_x \cdot T_z\), we obtain the Page form in
the dual background,

\[ F'_1 = -\frac{2q}{z} dz, \quad F'_3 = \frac{2q}{z} \left[ p dz \wedge \frac{\cos \theta \ d\phi \wedge d\psi}{4} + z (\bar{x} + p) \omega_{S3} \right], \]

\[ F'_5 = \frac{2q}{z} \left[ dz \wedge \text{vol}_{T^4} - (z \ d\bar{t} \wedge d\bar{x} + 2p \ d\bar{t} \wedge dz) \wedge \omega_{S3} \right], \]

\[ F'_7 = -\frac{2q}{z} \left[ p dz \wedge \frac{\cos \theta \ d\phi \wedge d\psi}{4} + z (\bar{x} + p) \omega_{S3} \right] \wedge \text{vol}_{T^4}, \]

\[ F'_9 = \frac{2q}{z} (z \ d\bar{t} \wedge d\bar{x} + 2p \ d\bar{t} \wedge dz) \wedge \omega_{S3} \wedge \text{vol}_{T^4}, \]

where \( \omega_{S3} \equiv \frac{1}{8} \sin \theta \ d\theta \wedge d\phi \wedge d\psi \). Finally, the field strength \( G' = e^{-B_3} F' \) becomes

\[ G'_1 = -\frac{2q}{z} dz, \quad G'_3 = 2q (\bar{x} + p) \left[ -\frac{z^{-1} d\bar{t} \wedge d\bar{x} \wedge dz}{(\bar{x} + p)^2 + z^2 - 1} + \omega_{S3} \right], \]

\[ G'_5 = 2q \left[ \frac{\bar{x} (z^2 - 2) - p z^2}{z} d\bar{t} \wedge d\bar{x} - (\bar{x} + p) d\bar{t} \wedge dz - z (z^2 - 1) d\bar{t} \wedge d\bar{x} \right] \wedge \omega_{S3} \]

\[ + \frac{2q}{z} dz \wedge \text{vol}_{T^4}. \]

They satisfy the GSE under the original constraint \( p^2 + q^2 = 1 \).

By performing a formal \( T \)-duality along \( t \)-direction, we obtain

\[ ds^2 = \frac{(z^2 + 4p^2 - 4) dz^2}{z^3} + \frac{2}{z^3} \left[ \frac{(z^2 + 2p \bar{x} + 2p^2 - 2) dt + 2p dz}{z^3} \right] dz + \frac{(z^2 + \bar{x}^2 + 2p \bar{x} + p^2 - 1) dt^2 + 2(\bar{x} + p) dt d\bar{x} + d\bar{x}^2}{z^2} + ds_{S^3}^2 + ds_{T^4}^2, \]

\[ B_2 = \frac{2}{z^2} \left[ x dt + d\bar{x} \right] \wedge dz - \frac{p \cos \theta \ d\phi \wedge d\psi}{4}, \quad e^{-2\Phi} = z^2, e^{2\iota}, \]

\[ G_2 = \frac{2q z}{z} e^{\iota} \left[ x (\bar{x} + p) dt + z d\bar{x} + 2p dz \right] \wedge \omega_{S3}, \quad G_4 = -\frac{2q z}{z} e^{\iota} \left[ x (\bar{x} + p) dt + z d\bar{x} + 2p dz \right] \wedge \omega_{S3}, \]

which is a solution of type IIA supergravity.

### 5.4 Extremal black D3-brane background

In order to show that the AdS factor is not important, let us consider an extremal black D3-brane background. In order to manifest the Bianchi type V symmetry, we employ a non-standard coordinate system,

\[ ds^2 = H^{\frac{1}{2}}(r) \left\{ -dt^2 + t^2 \left[ dx_1^2 + e^{2x_1} (dx_2^2 + dx_3^2) \right] \right\} + \frac{dr^2}{H^2(r)} \]

\[ + r^2 (d\theta^2 + \sin^2 \theta \ d\xi^2 + \sin^2 \theta \ cos^2 \xi \ d\phi_1^2 + \sin^2 \theta \ sin^2 \xi \ d\phi_2^2 + cos^2 \theta \ d\phi_3^2) , \]

\[ G_5 = -\frac{4 r^4 + 3 e^{2x_1} dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dr}{r^3} \]

\[ + 4 r^4 \sin^3 \theta \cos \theta \sin \xi \cos \xi \ d\theta \wedge d\xi \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3, \]

32
where \( H(r) \equiv 1 - (r_+/r)^4 \) and the four-dimensional metric inside the brackets \( \{\cdots\} \) is flat. We consider the following three Killing vectors,

\[
V_1 \equiv (\partial_1 + x^2 \partial_2 + x^3 \partial_3, 0), \quad V_2 \equiv (\partial_2, 0), \quad V_3 \equiv (\partial_3, 0),
\]

that satisfy the algebra

\[
[V_1, V_2]_C = -V_2, \quad [V_1, V_3]_C = -V_3, \quad [V_2, V_3]_C = 0.
\]

The \((x^1, x^2, x^3)\) parts of the matrices are

\[
(A_v) = \begin{pmatrix} 1 & x^2 & x^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (A_f) = \begin{pmatrix} 0 & -\tilde{x}_2 & -\tilde{x}_3 \\ \tilde{x}_2 & 0 & 0 \\ \tilde{x}_3 & 0 & 0 \end{pmatrix},
\]

and the gauge symmetry is fixed as \( x^i(\sigma) = 0 \) \((i = 1, 2, 3)\). The dual background becomes

\[
d\sigma^2 = -H^{\frac{2}{t}} dt^2 + \frac{t^4 H (d\tilde{x}_1^2 + d\tilde{x}_2^2 + d\tilde{x}_3^2) + \tilde{x}_2^2 d\tilde{x}_2 d\tilde{x}_3 + \tilde{x}_3^2 d\tilde{x}_3^2 - 2 \tilde{x}_2 \tilde{x}_3 d\tilde{x}_2 d\tilde{x}_3 + \tilde{x}_2^2 d\tilde{x}_2^2 + \tilde{x}_3^2 d\tilde{x}_3^2}{t^2 H^\frac{2}{t} (H t^4 + \tilde{x}_2^2 + \tilde{x}_3^2)} + \frac{dr^2}{H^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\xi^2 + \sin^2 \theta \cos^2 \theta d\phi_1^2 + \sin^2 \theta \sin^2 \xi d\phi_2^2 + \cos^2 \theta d\phi_3^2\right),
\]

\[
B'_2 = \frac{d\tilde{x}_1 \wedge (\tilde{x}_2 d\tilde{x}_2 + \tilde{x}_3 d\tilde{x}_3)}{H t^4 + \tilde{x}_2^2 + \tilde{x}_3^2}, \quad e^{-2\Phi} = t^2 H^\frac{2}{t} \left(H t^4 + \tilde{x}_2^2 + \tilde{x}_3^2\right), \quad t' = 2 \tilde{\theta}^l,
\]

\[
G'_2 = -\frac{4 r_+^4 t^3 dt \wedge dr}{r^5}, \quad G'_4 = \frac{-4 r_+^4 t^3 dt \wedge d\tilde{x}_1 \wedge (\tilde{x}_2 d\tilde{x}_2 + \tilde{x}_3 d\tilde{x}_3) \wedge dr}{r^5 (H t^4 + \tilde{x}_2^2 + \tilde{x}_3^2)},
\]

and this is a solution of type IIA GSE.

Again, by performing a formal \( T \)-duality along the \( \tilde{x}_1 \)-direction we obtain a solution of type IIB supergravity,

\[
d\sigma^2 = H^{\frac{2}{t}} \left(-dt^2 + t^2 dt_1^2\right) + \frac{(d\tilde{x}_2 - \tilde{x}_2 d\tilde{x}_1)^2 + (d\tilde{x}_3 - \tilde{x}_3 d\tilde{x}_1)^2}{H^\frac{2}{t} t^2} + \frac{dr^2}{H^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\xi^2 + \sin^2 \theta \cos^2 \xi d\phi_1^2 + \sin^2 \theta \sin^2 \xi d\phi_2^2 + \cos^2 \theta d\phi_3^2\right),
\]

\[
e^{-2\Phi} = t^4 e^{-4x^1} H(r), \quad G_3 = e^{-2x^1} \frac{4 r_+^4 t^3 dt \wedge dx_1 \wedge dr}{r^5}.
\]

We note that, as discussed in [147], some supergravity solutions, obtained by a combination of NATD and a formal \( T \)-duality, can be also obtained from another route, a combination of diffeomorphisms and the Abelian \( T \)-dualities. Similarly, solutions obtained in this section may also be realized from such procedure.
6 Poisson–Lie $T$-duality/plurality

Here we study a more general class of $T$-duality, known as the Poisson–Lie $T$-duality \cite{37,38} or $T$-plurality \cite{55}. We can perform the PL $T$-duality/plurality if the target space has a set of vectors $v_a$ satisfying the dualizability conditions \cite{37}

$$[v_a, v_b] = f_{ab}^c v_c, \quad \mathcal{L}_{v_a} E_{mn} = - \hat{f}^{bc} a E_{mp} v_b^p v_c^q E_{qn}. \quad (6.1)$$

The non-Abelian $T$-duality (with $\hat{v}_a = 0$) can be regarded a special case $\hat{f}^{bc} a = 0$. We begin with a brief review of the idea and techniques, and show the covariance of the DFT equations of motion under the PL $T$-plurality. Namely, if we start with a DFT solution, the PL $T$-dualized background is also a DFT solution. In some examples, the Killing vector $I^m$ appears, and they are regarded as solutions of GSE, but by performing a formal $T$-duality, we can transform the GSE solutions to linear-dilaton solutions of the conventional supergravity. As an example with the R–R fields, we consider the PL $T$-plurality for the AdS$_5 \times$ S$^5$ solution.

6.1 Review of PL $T$-duality

We review the PL $T$-duality as a symmetry of the classical equations of motion of the string sigma model. To make the discussion transparent, we first ignore spectator fields $y^\mu(\sigma)$, which are invariant under the PL $T$-duality. As studied in \cite{37,38}, it is straightforward to introduce the spectators, and their treatments are discussed in section 6.2.4.

Let us consider a sigma model with a target space $M$, on which a group $G$ acts transitively and freely (i.e. $M$ itself can be regarded as a group manifold),

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma E_{mn}(x) \left( dx^m \wedge * dx^n + dx^m \wedge dx^n \right). \quad (6.2)$$

Under an infinitesimal right-action of a group $G$, the coordinates $x^m$ are shifted as

$$g(x) \rightarrow g(x) (1 + \epsilon^a T_a) \equiv g(x + \delta x), \quad \delta x^m = \epsilon^a(\sigma) v^m_a(x), \quad (6.3)$$

where $T_a$ ($a = 1, \ldots, n$) are the generators of the algebra $\mathfrak{g}$ satisfying

$$[T_a, T_b] = f_{ab}^c T_c, \quad (6.4)$$

and $v^m_a$ are the left-invariant vector fields satisfying

$$[v_a, v_b] = f_{ab}^c v_c, \quad v^m_a \ell^b_m = \delta^b_a, \quad \ell \equiv \ell^a T_a \equiv g^{-1} dg. \quad (6.5)$$
In general, the variation of the action becomes

$$\delta S = \frac{1}{2\pi\alpha'} \int_{\Sigma} \left\{ -\epsilon^a \left[ dJ_a - \frac{1}{2} \mathcal{L}_{v_a} E_{mn} \left( dx^m \wedge * dx^n + dx^m \wedge dx^n \right) \right] + d(\epsilon^a J_a) \right\},$$

(6.6)

where

$$J_a \equiv v^m_a \left( g_{mn} * dx^n + B_{mn} dx^n \right).$$

(6.7)

If $v^m_a$ satisfy the Killing equation $\mathcal{L}_{v_a} E_{mn} = 0$, equations of motion for $x^m$ can be written as

$$dJ_a = 0.$$  

(6.8)

In particular, if $v^m_a$ further satisfy $[v_a, v_b] = 0$, we can find a coordinate system where $v^m_a = \delta^m_a$ is realized. Then, the Abelian $T$-duality can be realized as the exchange of $x^m(\sigma)$ with the dual coordinates $\tilde{x}_a(\sigma)$, which are defined as

$$d\tilde{x}_a \equiv J_a.$$  

(6.9)

The Bianchi identity $d^2 \tilde{x}_a = 0$ corresponds to the equations of motion in the original theory.

The PL $T$-duality is a generalization of this duality when the vector fields $v_a$ satisfy

$$\mathcal{L}_{v_a} E_{mn} = -\tilde{f}^{bc}_a E_{mp} v^p_b v^q_c E_{qn}.$$  

(6.10)

In this case, the variation becomes

$$\delta S = \frac{1}{2\pi\alpha'} \int_{\Sigma} \left[ -\epsilon^a \left( dJ_a - \frac{1}{2} \tilde{f}^{bc}_a J_b \wedge J_c \right) + d(\epsilon^a J_a) \right],$$

(6.11)

and the equations of motion for $x^m$ become the Maurer–Cartan equation,

$$dJ_a - \frac{1}{2} \tilde{f}^{bc}_a J_b \wedge J_c = 0.$$  

(6.12)

This suggests to introduce the dual coordinates $\tilde{x}_m(\sigma)$ through a non-Abelian generalization of (6.9), namely,

$$\tilde{r}_a \equiv J_a \quad \left( \tilde{r} \equiv \tilde{r}_a \tilde{T}^a \equiv d\tilde{g} \tilde{g}^{-1}, \quad \tilde{g} \equiv \tilde{g}(\tilde{x}) \in \tilde{G} \right),$$

(6.13)

where $\tilde{T}^a$ are the generators of the dual algebra $\tilde{g}$ (associated with a dual group $\tilde{G}$) satisfying

$$[\tilde{T}^a, \tilde{T}^b] = \tilde{f}^{abc} \tilde{T}^c.$$  

(6.14)
Then, under the equations of motion, the physical coordinates \( x^m(\sigma) \) describe a motion of the string on the group \( G \) while the dual coordinates \( \tilde{x}_m(\sigma) \) describe a motion of the string on the dual group \( \tilde{G} \).

It is important to note that the condition (6.10) and the identity
\[
\{\xi^a, \xi^b\} E_{mn} = \{\xi^a, \xi^b\} E_{mn},
\]
show a relation
\[
f_{ae}^{\ c} \tilde{f}_{ed}^{\ b} + f_{ae}^{\ d} \tilde{f}_{ce}^{\ b} - f_{be}^{\ c} \tilde{f}_{ed}^{\ a} - f_{be}^{\ d} \tilde{f}_{ce}^{\ a} = f_{ab}^{\ e} \tilde{f}_{cd}^{\ e}.
\]
By considering the vector space \( \tilde{g} \) as the dual space of \( g \),
\[
\langle T_a, \tilde{T}_b \rangle = \delta^b_a,
\]
the relation gives the structure of the Lie bialgebra. By defining an \( ad \)-invariant bilinear form as
\[
\langle T_A, T_B \rangle = \eta_{AB}, \quad (\eta_{AB}) = \begin{pmatrix} 0 & \delta^b_a \\ \delta^a_b & 0 \end{pmatrix},
\]
the commutation relations on a direct sum \( \mathfrak{d} \equiv \mathfrak{g} \oplus \mathfrak{\tilde{g}} \) are determined as
\[
[T_a, T_b] = f_{ab}^{\ c} T_c, \quad [T_a, \tilde{T}_b] = \tilde{f}_{bc}^{\ a} T_c - f_{ac}^{\ b} \tilde{T}_c, \quad [\tilde{T}_a, \tilde{T}_b] = \tilde{f}_{ab}^{\ c} \tilde{T}_c,
\]
and the pair of two algebras can be regarded as that of the Drinfel’d double \( \mathfrak{D} \). Given the structure of the Drinfel’d double, the differential equation (6.10) can be integrated as [37, 38]
\[
E_{ab} = v^m_a v^n_b E_{mn} = [a^{-1} \hat{E} (a^\top + b^\top \hat{E})^{-1}]_{ab},
\]
where the matrices \( a \) and \( b \) are defined by
\[
g^{-1} T_A g = (\text{Ad}_{g^{-1}})_A T_B, \quad \text{Ad}_{g^{-1}} = \begin{pmatrix} a_{b}^{\ a} & 0 \\ b_{ab} & (a^{-1})_{a}^{\ b} \end{pmatrix},
\]
and \( \hat{E}_{ab} \) is a constant matrix (that corresponds to \( E_{ab}(x) \) at \( g = 1 \)). We can check that the \( E_{mn} \) given by (6.19) indeed satisfies (6.10).\(^8\)

Now, we rewrite the relation (6.13)
\[
\tilde{r}_a = J_a = g_{ab} \ast \tilde{\ell}_b + B_{ab} \ell_b \quad (g_{ab} \equiv E_{(ab)}, \ B_{ab} \equiv E_{[ab]}),
\]
\^[8\] For example when \( E_{mn} \) is invertible, we can check an equivalent expression \( \mathcal{L}_{\xi_m} E_{mn} = \tilde{f}_{bc}^{\ a} \tilde{\xi}_m^a v^m_c \) by using the rewriting (6.32) and \( v^m_{\partial m} \Pi^{ab} = -(a^{-1})_{a}^{\ b} \tilde{f}_{bc}^{\ e} \hat{E}^{ec} \) that can be derived from (6.20) (see [44]).
into two equivalent expressions (by following the standard trick \[10\] in the Abelian case),

\[
\ell^a = - (g^{-1} B)^a_b \ast \ell^b + g^{ab} \ast \tilde{r}_b , \\
\tilde{r}_a = (g - B g^{-1} B)_{ab} \ast \ell^b + (B g)_{ab} \ast \tilde{r}_b .
\]

(6.22)

They can be neatly expressed as a self-duality relation,

\[
P^A = H^A_B(x) \ast P^B , \quad (P^A) \equiv \begin{pmatrix} \ell^a \\ \tilde{r}_a \end{pmatrix} , \\
(H_{AB}) \equiv \begin{pmatrix} (g - B g^{-1} B)_{ab} B_{ac} g^{cb} \\ -g^{ac} B_{cb} g^{ab} \end{pmatrix} ,
\]

(6.23)

where indices \(A, B, \cdots\) are raised or lowered with \(\eta_{AB}\) and its inverse \(\eta^{AB}\). In terms of the metric \(H_{AB}\), the relation \(6.19\) can be expressed as

\[
H_{AB}(x) = (\text{Ad}_g) A^C (\text{Ad}_g) B^D \hat{\mathcal{H}}_{CD} , \quad \hat{\mathcal{H}}_{AB} \equiv \begin{pmatrix} \hat{g} - \hat{B} \hat{g}^{-1} \hat{B} \hat{B} \hat{g}^{-1} \\ -\hat{g}^{-1} \hat{B} \hat{B} \hat{g}^{-1} \end{pmatrix} ,
\]

(6.24)

where \(\hat{g}_{ab} \equiv \hat{E}_{(ab)}\), \(\hat{B}_{ab} \equiv \hat{E}_{[ab]}\), and we can easily obtain

\[
\hat{P}^A = \hat{H}^A_B \ast \hat{P}^B , \quad \hat{P}(\sigma) \equiv \hat{P}^A T_A \equiv d l^{-1} , \quad l \equiv g \tilde{g} ,
\]

(6.25)

where we have used

\[
\hat{P}(\sigma) \equiv d l^{-1} = g \left( \ell^a T_a + \tilde{r}_a \tilde{T}_a \right) g^{-1} = P^B (\text{Ad}_g) B^A T_A .
\]

(6.26)

Expressed in this form, the equations of motion are given in terms of the Drinfel’d double \(\mathfrak{D}\); the decomposition \(l = g \tilde{g}\) is no longer important.

Similar to the Abelian \(T\)-duality, we can recover the same equations of motion from the dual model, by exchanging the role of \(g\) and \(\tilde{g}\). Starting with the dual background \(\hat{E}_{mn}\), which has a set of vector fields \(\tilde{v}^a\) satisfying

\[
[\tilde{v}^a, \tilde{v}^b] = \tilde{f}^{ab}_c \tilde{v}^c , \quad \mathcal{L}_{\tilde{v}^a} E_{mn} = - f_{bc}^a \hat{E}_{mp} \tilde{v}^{bp} \tilde{v}^{cq} \hat{E}_{qn} ,
\]

(6.27)

\footnote{If we expand the right-invariant form as \(\hat{P} = P^A_M d x^M T_A\), we find that \(P^A_M\) is not an \(O(D, D)\) matrix,

\[
(P^A_M) = \begin{pmatrix} \lambda^a_m & \Pi^{ab} a^c_{mb} \tilde{r}^c_m \\
0 & a^b_m \tilde{r}^m_b \end{pmatrix} .
\]
the equations of motion can be expressed as
\[
\mathbf{\hat{P}}_A = \mathbf{\hat{H}}_{AB}^{\mathbf{H}} \mathbf{\hat{P}}_B, \quad \mathbf{\hat{P}}_A T^A = d\tilde{l} \tilde{l}^{-1}, \quad \tilde{l} \equiv \tilde{h} h \quad (h \in \mathfrak{g}, \quad \tilde{h} \in \tilde{\mathfrak{g}}), \tag{6.28}
\]
by using a constant matrix \(\mathbf{\hat{H}}_{AB}^{\mathbf{H}}\). For the duality, we demand that (6.25) and (6.28) are equivalent. This leads to the identifications
\[
\mathbf{\hat{H}}_{AB} = \mathbf{\hat{H}}_{AB}^{\mathbf{H}}, \quad g \tilde{g} = l = \tilde{l} \equiv \tilde{h} h. \tag{6.29}
\]
Under this identification, string theory defined on the original background \(E_{mn}\) and the dual background \(E'_{mn}\) give the same equations of motion, and are classically equivalent.

In summary, in PL T-dualizable backgrounds, the generalized metric \(H_{MN}(x)\) is always related to a constant matrix \(\mathbf{\hat{H}}_{AB}\) as
\[
H_{MN} = (U \mathbf{\hat{H}} U^\dagger)_{MN}, \tag{6.30}
\]
and the matrix \(U\) is defined as
\[
U_M^A \equiv L_M^B (\text{Ad}_g)^B_A, \quad (L_M^A) \equiv \begin{pmatrix} \ell^a_n & 0 \\ 0 & \eta^{m}_n \end{pmatrix}. \tag{6.31}
\]
By comparing this with (2.19), we call the matrix \(U\) the twist matrix and call the constant matrix \(\mathbf{\hat{H}}_{AB}\) the untwisted metric. The dual geometry also has the same structure, where the twist matrix is \(\tilde{U}_{MA} \equiv \tilde{L}_{MB} (\text{Ad}_{\tilde{g}})^B_A\). The relation between the original and the dual background becomes
\[
\tilde{H}_{MN} = (h \mathbf{\hat{H}} h^\dagger)_{MN}, \quad h_{MN} \equiv \tilde{U}_{MA} \eta^{AB} U_B^M. \tag{6.32}
\]
For later convenience, we rewrite the twist matrix as
\[
U = \text{Ad}_g = R \Pi, \tag{6.33}
\]
where we have defined
\[
(R_M^A) \equiv \begin{pmatrix} r^a_m & 0 \\ 0 & e^m_a \end{pmatrix}, \quad (\Pi_A B) \equiv \begin{pmatrix} \delta^a_b & 0 \\ -\Pi^{ab} & \delta^a_b \end{pmatrix}, \tag{6.34}
\]
\[
r \equiv r^a T_a = dg g^{-1}, \quad r^a e^m_b = \delta^a_b, \quad \Pi^{ab} \equiv (ba^{-1})^{ab} = -(a^{-1} b^T)^{ab},
\]
\[
r^a c^m_b = \delta^a_b, \tag{6.34}
\]

38
and used \( r^a = (a^{-1})^a_b \ell^b \). Then, in terms of \( E_{mn}(x) \), (6.30) can be expressed as

\[
E_{mn}(x) = [\left(\hat{E}^{-1} - \Pi^{-1}\right)_{ab} r^a_m r^b_n],
\]

(6.35)

and similarly, the dual background is

\[
\tilde{E}_{mn}(\tilde{x}) = \left[\left(\hat{\Pi}^{-1} - \tilde{\Pi}^{-1}\right)_{ab} \tilde{r}^a_m \tilde{r}^b_n\right],
\]

(6.36)

In a special case, where \( \tilde{f}^a_{bc} = 0 \), by parameterizing \( \tilde{g} = e^{\tilde{x}_a \tilde{T}^a} \), we obtain \( \tilde{r} = d\tilde{x}_a \tilde{T}^a \), \( \Pi^{ab} = 0 \), and \( \tilde{\Pi}_{ab} = -\tilde{f}^c_{ab} \tilde{x}_c \). This is precisely the dual background for NATD. In the dualized background, in general, the isometries are broken, and in the traditional NATD, we cannot recover the original model. However, the background has the form

\[
\tilde{E}_{mn} = (\hat{E} - \tilde{\Pi})_{ab} \tilde{e}^a m \tilde{e}^b n = (\hat{E}_a + f^c_{ab} \tilde{x}_c) \tilde{v}^a m \tilde{v}^b n,
\]

(6.37)

where \( \tilde{e}^a = \tilde{v}^a = \tilde{\partial}^a \), and we find that the dual background is \( T \)-dualizable,

\[
\mathcal{L}_{\tilde{\mathcal{E}}} E_{mn} = \tilde{\mathcal{E}}^a E_{mn} = f_{bc}^a \tilde{v}^m \tilde{v}^b n.
\]

(6.38)

Thus, through the PL \( T \)-duality, we can recover the original background \( E_{mn} = E_{ab} r^a_m r^b_n \).

As a side remark, we note that, in the case of the Abelian \( O(D, D) \) \( T \)-duality, the covariant equations of motion of string \( dx^M = \hat{H}^M_N \ast dx^N \) \[10\], can be derived from the double sigma model (DSM) \[161, 166–170\]. The correspondent of the DSM for the PL \( T \)-duality is studied in \[40, 41, 64, 171–173\], and that approach will be useful to manifest the PL \( T \)-duality.

### 6.2 PL \( T \)-plurality

The Lie algebra \( \mathfrak{d} \) of the Drinfel’d double \( \mathcal{D} \) can be constructed as a direct sum of two algebras \( \mathfrak{g} \) and \( \tilde{\mathfrak{g}} \), which are maximally isotropic with respect to the bilinear form \( \langle \cdot, \cdot \rangle \), and the pair \( (\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}}) \) is called the Manin triple. In general, a Drinfel’d double has several decompositions into the Manin triples, and this leads to a notion of the PL \( T \)-plurality \[55\]. More concretely, let us consider a redefinition of the generators \( T_A \) of \( \mathfrak{d} \),

\[
T_A' = C_A^B T_B,
\]

(6.39)

such that the new generators also satisfy the algebra of the Drinfel’d double,

\[
[T_a', T^c_b] = f_{ab}^c T_c', \quad [T^c_a, \tilde{T}^b_d] = \tilde{f}^{bce} T^e_a - f_{ac}^b \tilde{T}^e_c, \quad [\tilde{T}^a_d, \tilde{T}^b_c] = \tilde{f}^{dab}_c \tilde{T}^c_d,
\]

(6.40)
and the bilinear form is preserved,
\begin{equation}
(T'_A, T'_B) = \eta_{AB}.
\end{equation}

The latter condition shows that the matrix $C_{AB}$ should be a certain $O(n,n)$ matrix. Since the rescaling of the generators is trivial, we choose $C_{AB}$ as an $SO(n,n)$ matrix.

The transformation of the background fields under the $SO(n,n)$ transformation can be found in the same manner as the PL $T$-duality. Starting with a background $E'_{mn}$ satisfying
\begin{equation}
[v_{a}', v_{b}'], = f_{a'b'c'} v_{c}', \quad \mathcal{L} v_a' E_{mn}' = - f^{b'c'}_{a'} E_{mp}' v_{b'}' v_{c'}' E_{qn}', \quad (6.42)
\end{equation}
again we obtain the same equations of motion
\begin{equation}
\mathcal{P}^{\mu A}_B = \hat{\mathcal{H}}^{\mu A}_B \ast \mathcal{P}^B, \quad \mathcal{P}^{\mu A}_B T'_A \equiv d l' l'^{-1}, \quad l' \equiv g' \tilde{g}'. \quad (6.43)
\end{equation}

From the identification, $l = l'$, we obtain
\begin{equation}
\hat{\mathcal{P}}^A T_A = dl l'^{-1} = dl' l'^{-1} = \hat{\mathcal{P}}^{\mu A} T'_A = \hat{\mathcal{P}}^{\mu A} C_{A} B T_B, \quad (6.44)
\end{equation}
and the relation between the untwisted fields becomes
\begin{equation}
\mathcal{H}'_{AB} = (C \hat{\mathcal{H}} C^T)_{AB}. \quad (6.45)
\end{equation}

The generalized metric in the transformed frame also has the form,
\begin{equation}
\mathcal{H}'_{MN} = (U' \hat{\mathcal{H}} U^T)_{MN}, \quad (6.46)
\end{equation}
the relation between the original and the dual generalized metric is
\begin{equation}
\mathcal{H}'_{MN} = (h \mathcal{H} h^T)_{MN}, \quad (h_{M}^{N}) \equiv U' C U^{-1}. \quad (6.47)
\end{equation}

In terms of $E_{mn}(x)$, the original background is
\begin{equation}
E_{mn}(x) = \left[ (\hat{E}^{-1} - \Pi) \right]^{-1}_{ab} r^a_m r^b_n, \quad (6.48)
\end{equation}
while the dual background is
\begin{equation}
E'_{mn}(x') = \left[ (\hat{E}'^{-1} - \Pi') \right]^{-1}_{ab} r^a'_m r^b'_n, \quad E'_{mn} = \left[ (q + p \hat{E})(s + r \hat{E})^{-1} \right]_{mn}, \quad (6.49)
\end{equation}
where we parameterized the SO($D, D$) matrix $C$ as

$$C = \begin{pmatrix} p_{mn} & q_{mn} \\ r_{mn} & s_{mn} \end{pmatrix}. \tag{6.50}$$

Note that the PL $T$-duality is a special case of the $T$-plurality where

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{6.51}$$

and the original background corresponds to the trivial choice $C = 1$.

### 6.2.1 Duality rule for dilaton

The transformation rule for the dilaton was studied in [64] in the context of the PL $T$-duality. This was improved in [55] in the study of the PL $T$-plurality. In our convention, the result is

$$e^{-2 \Phi'} = e^{-2 \Phi} \frac{|\text{det}(q + p \hat{E})|}{|\text{det}(E'_{ab}) \det a'^{-1}|} \left( E'_{ab} = e^m_a e^m_b E_{mn} \right), \tag{6.52}$$

where $\Phi(x)$ is an arbitrary function. By using the formula (3.28), we obtain

$$\sqrt{|g'|} = |\text{det}(r_m^a)| |\text{det}(1 - \Pi' \hat{E}')|^{-1} |\text{det}(s + r \hat{E})|^{-1} \sqrt{|g|},$$

$$= |\text{det}(r_m^a)| |\text{det}(E'_{ab})| |\text{det}(q + p \hat{E})|^{-1} \sqrt{|g|}, \tag{6.53}$$

and the DFT dilaton becomes

$$e^{-2 d'} = e^{-2 \tilde{d}} |\text{det}(r_m^a)| |\text{det} a'^{-1}| = e^{-2 \tilde{d}} |\text{det}(\ell_m^a)|, \quad e^{-2 \tilde{d}} \equiv e^{-2 \Phi} \sqrt{|g|}. \tag{6.54}$$

Namely, the duality rule for the DFT dilaton is

$$|\text{det}(v^m_n)| e^{-2 (d' - \tilde{d})} = |\text{det}(v^m_n)| e^{-2 (d - \tilde{d})}. \tag{6.55}$$

If $\tilde{d}$ (or equivalently $\tilde{\Phi}$) is constant, this duality rule coincides with the recent proposal [138], where the PL $T$-duality was studied by utilizing “the DFT on a Drinfel’d double” proposed in [137]. There, it was shown that the dilaton transformation rule is consistent also with [140]. Moreover, when the dual algebra is Abelian $\hat{f}_{ab}^c = 0$, we have $|\text{det}(v^m_n)| = 1$ and the result (3.30) known in NATD is also reproduced as a particular case.

In fact, as demonstrated in [55], the PL $T$-plurality works even if $\tilde{d}$ has a coordinate dependence. A problem is that when $e^{-2 \tilde{d}}$ depends on the original coordinates $x^m$, it is not clear how to understand the $x^m$-dependence in the dual model. A prescription proposed in [55]...
is as follows. We first identify the relation between coordinates \(x^I = (x^i, \tilde{x}_i)\) and \(x'^I = (x'^i, \tilde{x}'_i)\) through the identification,
\[g'(x') \tilde{g}'(\tilde{x}') = l = g(x) \tilde{g}(\tilde{x}).\]  
(6.56)

We next plug the relation \(x^I = x^I(x')\) into the expression of \(e^{-2 \tilde{d}(x)} = e^{-2 \tilde{d}(x')}) = e^{-2 \tilde{d}(x')}\). Then, the relation (6.55) should be understood as
\[|\det(v^m_a)| e^{-2 [d' - \tilde{d}(x')]} = |\det(v^m_a)| e^{-2 [d - \tilde{d}(x)]}.\]  
(6.57)

In general, the \(\tilde{d}(x')\) depends on the dual coordinates \(\tilde{x}'_m\) and the background does not have the usual supergravity description. However, in our examples, the DFT dilaton has at most a linear dependence on the dual coordinates, and it can be absorbed into the Killing vector \(I^m\) in the GSE.

6.2.2 Covariance of equations of motion

In the approach of [137,138], the PL T-duality was realized as a manifest symmetry of DFT. We here discuss the covariance under a more general PL T-plurality by using the gauged DFT. The approach may be slightly different from [137,138] but essence will be the same.

In PL T-dualizable backgrounds, the generalized metric always has a simple form
\[\mathcal{H}_{MN} = [U(x) \tilde{\mathcal{H}} U^T(x)]_{MN},\]  
(6.58)

Since the twist matrix \(U\) is explicitly determined, we can compute the generalized fluxes \(F_{ABC}\) and \(F_A\) defined in (2.20). In fact, as shown in [138], in PL T-dualizable backgrounds, the three-index flux is precisely the structure constant of the Drinfel’d double,
\[F_{abc} = 0, \quad F_{ab}^c = f_{ab}^c, \quad F^{ab}_c = \tilde{f}^{ab}_c, \quad F^{abc} = 0.\]  
(6.59)

We can check this by using the explicit form of the twist matrix and its inverse
\[(U_M^A) = \begin{pmatrix} r^a_m & 0 \\ -\epsilon^m_b \Pi^{ba} & e^m_a \end{pmatrix}, \quad (U_A^M) = \begin{pmatrix} e^m_a & 0 \\ \Pi^{ab} \epsilon^m_b & r^m_a \end{pmatrix},\]  
(6.60)

where we have used \(\mathcal{L}_{e_a} e_b = -f_{ab}^c e_c, \mathcal{L}_{e_a} e^b = f_{ab}^c e_c, \partial_m \Pi^{ab} = -(a^{-1})^a_d (a^{-1})^b_e f_{def} a^f_r e^c_m,\) and \(\tilde{f}^{ab}_c = (a^{-1})^a_d (a^{-1})^b_e a^f_c f_{def} - 2 f_{ce}[a \Pi^{be} (see \[44\] for useful identities).
We can also compute the single-index flux as

\[
\mathcal{F}_A = \begin{pmatrix}
2 e^m_a \partial_m \bar{d} + e^c_e \partial_c r^m_e e^m_a \\
-(a^{-1})^a_{\alpha} f^{ca}_{\beta} + \Pi^{ab} (2 e^m_e \partial_m \bar{d} + e^n_e \partial_n r^m_e e^m_a) + 2 r^a_m \tilde{\partial}^m \bar{d}
\end{pmatrix}.
\] (6.61)

By using the expression for the DFT dilaton (6.54), \(e^{-2 \bar{d}} = e^{-2 \bar{d}} |\det(r_m^a)| |\det a|\), we find

\[
\mathcal{F}_A = U_A^M \mathcal{F}_M, \quad \mathcal{F}_M \equiv 2 \partial_M \bar{d} + \begin{pmatrix} 0 \\ -f^{ha}_b v^m_a \end{pmatrix},
\] (6.62)

where we have used \(a^e_b a^f_e f^{ha}_b a^m_e \) and \(\partial_m a^b_a = a^c_e f^{cd}_b r^d_m\).

As we discuss below, for the covariance of the equation of motion under the PL \(T\)-plurality, \(\mathcal{F}_A\) needs to transform covariantly. However, for example, in a particular case \(\bar{d} = 0\), we find that \(\mathcal{F}_A\) does not transform covariantly. Indeed, we have \(\mathcal{F}_A = 0\) in the duality frame where \(\tilde{f}^{ab}_a = 0\) while \(\mathcal{F}_A\) appears in the frame where \(\tilde{f}^{ab}_a \neq 0\). Therefore, in order to transform \(\mathcal{F}_A\) covariantly, we eliminate the non-covariant term by adding a vector field \(X_M\) as

\[
\partial_M \bar{d} \rightarrow \partial_M \bar{d} + X_M, \quad (X_M) \equiv \begin{pmatrix} 0 \\ I^m \end{pmatrix}, \quad I^m = \frac{1}{2} f^{ha}_b v^m_a,
\] (6.63)

which was suggested in [138]. This shift is a bit artificial, but without this procedure, we need to abandon all Manin triples with non-unimodular dual algebra. In fact, this shift is precisely the modification of DFT equations of motion (2.38), that reproduces the GSE after removing the dual-coordinate dependence. After this prescription, we obtain a simple flux

\[
\mathcal{F}_A = 2 D_A \bar{d}.
\] (6.64)

In fact, as we see in our examples, \(\mathcal{F}_A = 2 D_A \bar{d}\) are covariantly transformed under the PL \(T\)-plurality \(\mathcal{F}_A' = C_A^B \mathcal{F}_B\) [41] and the prescription (6.63) works well in our examples.

Now, let us discuss the covariance of the equations of motion. Since the derivative \(D_A\) generally does not transform covariantly, we assume that \(\mathcal{F}_A = 2 D_A \bar{d}\) is constant. Since \(\mathcal{F}_{ABC}\) is also constant in PL \(T\)-dualizable backgrounds, the DFT equations of motion become simple algebraic equations (2.27) and (2.28).

Under the PL \(T\)-plurality \(T'_A = C_A^B T_B\), the generalized fluxes are mapped as

\[
\mathcal{F}'_{ABC} = C_A^D C_B^E C_C^F \mathcal{F}_{DEF}, \quad \mathcal{F}'_A = C_A^B \mathcal{F}_B,
\] (6.65)

\[\text{This is non-trivial, because in general the derivative } D_A \text{ does not transform covariantly } D'_A \neq C_A^B D_B, \text{ which can check by performing a coordinate transformation } x'^M = x'^M(x) \text{ through (6.54). Therefore, at the present time, the covariance of } \mathcal{F}_A \text{ needs to be checked on a case-by-case basis. Of course, when } \bar{d} \text{ is constant, the covariance is manifest because } \mathcal{F}_A = 0 \text{ and } \mathcal{F}'_A = 0.\]
by introducing $X_M$ when the dual algebra is non-unimodular. According to (6.45), the untwisted metric $\hat{\mathcal{H}}_{AB}$ is also related covariantly,

$$\hat{\mathcal{H}}'_{AB} = (C \hat{\mathcal{H}} C^\dagger)_{AB}. \quad (6.66)$$

Then, we find that the equations of motion in the original and the dual background are covariantly related by the SO$(D, D)$ transformation $C$. Thus, as long as the original configuration is a DFT solution, the dual background also satisfies the DFT equations of motion.

We note that, this SO$(D, D)$ transformation is totally different from the transformation (2.32). The transformation (2.32) is simply a redefinition of $U$, and the generalized metric $\mathcal{H}_{MN}$ is invariant. On the other hand, in the case of PL $T$-plurality, $U(x)$ in the original model and $U'(x')$ in the dual model are defined on a different manifold and there is no clear connection between $U(x)$ and $U'(x')$. Only the constant fluxes made out of $U(x)$ and $U'(x')$ are related by a constant SO$(D, D)$ transformation, and this non-trivial relation connects the two equations of motion in a covariant manner.

Before moving on to the R–R sector, we make a brief comment on the vector field $I^m$. In order to reproduce the (generalized) supergravity from (modified) DFT, we need to choose the standard section $\tilde{\partial}^m = 0$. Therefore, if $\bar{d}$ has a dual-coordinate dependence, we should make a field redefinition. Supposing that $\bar{d}$ only has a linear dual-coordinate dependence $\bar{d} = \bar{d}_0(x^m) + d^m \bar{x}_m$, we make a field redefinition.

$$\bar{d} \rightarrow \bar{d}' = \bar{d}_0(x^m), \quad I^m \rightarrow I'^m = \frac{1}{2} f_{bab} v^m_a + d^m. \quad (6.67)$$

Then, the dual-coordinate dependence disappears from the background. Note that this is different from the shift (6.65) but just a field redefinition. In the following, when display a (generalized) supergravity solution, we always make this redefinition.

Before studying the R–R fields, let us make a comment on the Killing vector $I^m$. In the case of NATD, the Killing vector $I^m$ is given by (4.1), but apparently, (4.1) is different from the formula (6.63) by the factor 2. Here, we will roughly sketch how to resolve the discrepancy by using the redefinition (6.67). In the case of NATD, $\partial_m |\det(v^m_a)| = 0$ and $\partial_m \Phi = 0$ are usually satisfied in the original background (under the gauge fixing $x^m = c^m$). Then, we have

$$\partial_m \bar{d} = \partial_m d = -\frac{1}{2} \partial_m \ln \sqrt{|g|} = -\frac{1}{2} \partial_m \ln |\det(r^m_a)|$$

$$= \frac{1}{2} \partial_m \ln |\det a| = \frac{1}{2} f_{ba}^b \ell^m_a. \quad (6.68)$$

Namely, $\bar{d}$ has a linear coordinate dependence along the $v^m_a$ direction,

$$v^m_a \partial_m \bar{d} = \frac{1}{2} f_{ba}^b. \quad (6.69)$$
After performing NATD, this gives a dual-coordinate dependence of $\tilde{d}$ in the dual theory,

$$\tilde{d} = \frac{1}{2} \tilde{f}^{\alpha b}_c \tilde{x}_a,$$

(6.70)

where the dual structure constants $\tilde{f}^{\alpha b}_c$ corresponds to $f_{\alpha b}^c$ in the original frame. Then, the modified $I^m$ in (6.67) recovers the formula (4.1)

$$I^m = \frac{1}{2} \tilde{f}^{a b}_c v^m_a + d^m = \tilde{f}^{a b}_c,$$

(6.71)

where we used $v^m_a = \delta^m_a$ in the dual theory. In a general setup, (4.1) does not work correctly, and we use the results discussed in this section.

### 6.2.3 Duality rule for R–R fields

Now, let us determine the duality rule for the R–R fields. We will first find the duality rule from a heuristic approach, and then clarify the result in terms of the gauged DFT.

In the presence of the R–R fields, the equations of motion for $H_{MN}$ and $d$ are

$$S_{MN} = E_{MN}, \quad S = 0,$$

(6.72)

and since $S_{MN}$ is transformed covariantly under the PL $T$-duality, the energy-momentum tensor $E_{MN}$ also should transform covariantly,

$$E'_{MN} = (h E h^T)_{MN}.$$

(6.73)

The energy-momentum tensor $E_{MN}$ is a bilinear form of a combination $e^d F$ and it does not contain a derivative of $F$. Therefore, we can covariantly transform $E_{MN}$ simply by rotating the combination $e^d F$ covariantly, and this gives the transformation rule for the R–R fields.

Under a PL $T$-plurality, $H'_{MN} = (h H h^T)_{MN}$ with $(h_M^N) = U' CU^{-1}$, the O($D,D$)-covariant transformation rule for the DFT dilaton should be

$$e^{-2 d(b)} = \frac{|\det(e^m_a)|}{|\det(e^m_a)|} e^{-2 d}.$$

(6.74)

Indeed, the twist matrix has the form $U = R \Pi$, and the DFT dilaton is invariant under the $\beta$-transformation $\Pi$ while it is multiplied by $|\det(e^m_a)|^{-1}$ under the twist $R$. Moreover, the DFT dilaton is invariant under the SO($D,D$) transformation $C$. Thus, $e^{-2 d(b)}$ in (6.74) is the covariantly transformed DFT dilaton.

On the other hand, let us denote the covariantly transformed R–R polyform as $\mathcal{F}^{(h)}$. By
denoting the action of an $O(D, D)$ transformation $M$ on the polyform as $F \rightarrow \mathbb{M} F$ \footnote{Explicit form of the operation $\mathbb{M}$ is given in section 3.3.} we have

$$F^{(h)} = \mathcal{U}' C \mathcal{U}^{-1} F. \quad (6.75)$$

Then, the energy-momentum tensor made from the combination $e^{d(h)} F^{(h)}$ is as expected $\mathcal{E}'_{MN}$. However, an important thing is that, the actual DFT dilaton is given by

$$|\text{det}(a'^{-1})||\text{det}(e^m_a)| e^{-2|\bar{d}(x')|} = |\text{det}(a^{-1})||\text{det}(e^m_a)| e^{-2|\bar{d}(x)|}, \quad (6.76)$$

and $e^{d(h)}$ is related to $e^{d'}$ as

$$e^{d'} = \frac{\sqrt{|\text{det} a|} e^{-\bar{d}(x)}}{\sqrt{|\text{det} a'|} e^{-\bar{d}(x')}} e^{d(h)}. \quad (6.77)$$

Therefore, if we identify the dual R–R polyform as

$$F' \equiv \frac{\sqrt{|\text{det} a'|} e^{-\bar{d}(x')}}{\sqrt{|\text{det} a|} e^{-\bar{d}(x)}} F^{(h)}, \quad (6.78)$$

the energy-momentum tensor made from $e^{d'} F' = e^{d(h)} F^{(h)}$ is $\mathcal{E}'_{MN}$. Namely, (6.78) is the rule for the R–R fields.

Now, as $\mathcal{E}_{MN}$ is transformed covariantly, it is already clear that the equations of motion for $\mathcal{H}_{MN}$ and $d$ are satisfied in the dual background. However, the equations of motion for the R–R fields are still not clear. To clarify the covariance, let us rewrite (6.78) as

$$\hat{\mathcal{F}}' = C \hat{\mathcal{F}}, \quad (6.79)$$

where we defined

$$\hat{\mathcal{F}} \equiv \frac{e^d}{\sqrt{|\text{det} a|}} \mathcal{U}^{-1} F = \frac{e^d}{\sqrt{|\text{det}(e^m_a)|}} \mathcal{U}^{-1} F. \quad (6.80)$$

Then, we find that the $\hat{\mathcal{F}}$ is precisely the R–R field strength appearing in the gauged DFT or the flux-formulation of DFT (see for example footnote 1 of \cite{131})

$$|\mathcal{F}'\rangle = \sum_p \frac{1}{p!} \hat{\mathcal{F}}_{a_1 \cdots a_p} \Gamma^{a_1 \cdots a_p} |0\rangle. \quad (6.81)$$

Here, $\Gamma^{a_1 \cdots a_p} \equiv \Gamma^{(a_1} \cdots \Gamma^{a_p]}$ and $(\Gamma^A) \equiv (\Gamma^a, \Gamma_\alpha)$ satisfies the algebra,

$$\{\Gamma^A, \Gamma^B\} = \eta^{AB}. \quad (6.82)$$

\footnote{Explicit form of the operation $\mathbb{M}$ is given in section 3.3.}
The so-called Clifford vacuum \( |0\rangle \) is defined by \( \Gamma_A |0\rangle = 0 \). By using a nilpotent operator

\[
\nabla = \vartheta - \frac{1}{2} \Gamma^A F_A - \frac{1}{3!} \Gamma^{ABC} F_{ABC} \quad (\vartheta \equiv \Gamma^A D_A),
\]

the Bianchi identity can be expressed as

\[
\nabla |F\rangle = \vartheta |F\rangle + \frac{1}{3!} \Gamma^{ABC} F_{ABC} |F\rangle - \frac{1}{2} \Gamma^A F_A |F\rangle = 0.
\]

As it is well-known in the democratic formulation \cite{156,159}, the Bianchi identity is equivalent to the equations of motion when the self-duality relation \( G_p = (-1)^{\frac{p(p-1)}{2}} * G_{10-p} \) is satisfied.

Now, we require the dualizability condition for the R–R fields as

\[
\vartheta |F\rangle = 0,
\]

which will be the same as the proposal of \cite{138}. Then, the Bianchi identity or the equation of motion for the R–R fields becomes an algebraic equation

\[
\frac{1}{3!} \Gamma^{ABC} F_{ABC} |F\rangle = \frac{1}{2} \Gamma^A F_A |F\rangle.
\]

Note that when the dual algebra is non-unimodular, \( F_A \) should be modified as \( F_A + 2 U_A^M X_M \) as we explained in the discussion of the NS–NS fields. By denoting the spinor representative of the SO\((D, D)\) transformation by \( S_C \), the duality relation \eqref{eq:6.79} simply becomes

\[
|F'\rangle = S_C |F\rangle.
\]

Then, we find that the equation of motion for the R–R fields \eqref{eq:6.86} in the original and the dual background is covariantly related by the SO\((D, D)\) PL T-plurality transformation.

We call the object \( \hat{F} \), the untwisted R–R fields, and once the untwisted R–R fields \( \hat{F}_{a_1 \cdots a_p} \) in the dual background is obtained from \eqref{eq:6.79}, the Page form in the dual background can be constructed as

\[
F' = e^{-d(x')} \sqrt{|\text{det} a'|} \ U \hat{F} = e^{-\tilde{d}(x')} \sqrt{|\text{det} a'|} \ e^{-\Pi'\vee} \left( \sum_p \frac{1}{p!} \hat{F}'_{a_1 \cdots a_p} \ r^{a_1} \wedge \cdots \wedge r^{a_p} \right),
\]

where \( \Pi'\vee \equiv \frac{1}{2} \Pi^{ab} \ t_{e_a'} \ t_{e_b'} \).
6.2.4 Spectator fields

In the following, we consider more general cases where spectator fields are also included. Namely, we suppose that the original model takes the form,

\[
S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2 \sigma \sqrt{-\gamma} (\gamma^{ab} - \varepsilon^{ab}) \left( \partial_\alpha y^\mu \ r^a \right) \begin{pmatrix} E_{\mu\nu} & E_{\mu b} \\ E_{\nu a} & E_{ab} \end{pmatrix} \left( \partial_b y^\mu \right) .
\]

(6.89)

In the following, we consider the coordinates as \((x^m) = (y^\mu, x^i) \ (i = 1, \ldots, n)\). By assuming that the background field \((E_{mn}) = (E_{\mu\nu}, E_{\mu b}, E_{a\nu}, E_{ab})\) satisfies the condition,

\[
\mathcal{L}_a E_{mn} = -\hat{f}_a E_{mp} v_p^b v_q^c E_{qn} ,
\]

(6.90)

we can again determine \(E_{mn}\) as \([37,38]\)

\[
E_{\mu\nu} = \hat{E}_{\mu\nu} + \hat{E}_{\mu c} \hat{E}^{cd} N_{cd} \hat{E}_{c\nu} , \quad E_{\mu b} = \hat{E}_{\mu c} \hat{E}^{cd} N_{db} ,
\]

\[
E_{a\nu} = N_{ad} \hat{E}^{de} \hat{E}_{e\nu} , \quad E_{ab} = N_{ab} ,
\]

(6.91)

where

\[
(N_{ab}) = (\hat{E}^{ab} - \Pi^{ab})^{-1} .
\]

(6.92)

This reduces to (6.48) when there is no spectator field. An important difference is that \(\hat{E}_{mn}\) is not necessarily constant, but can depend on the spectator fields \(y^\mu, \hat{E}_{mn} = \hat{E}_{mn}(y^\mu)\). The dependence should be determined from the DFT equations of motion and it is independent of the structure of the Drinfel’d double.

In terms of the generalized metric, we can clearly see that the above relation (6.91) is a straightforward generalization of (6.30),

\[
\mathcal{H}_{MN} = \left[ U(x) \hat{\mathcal{H}}(y^\mu) U^\top(x) \right]_{MN} , \quad U(x) \equiv R \Pi ,
\]

\[
(R_M^B) \equiv \begin{pmatrix} \delta_\beta^\mu & 0 & 0 \\ 0 & r_i^b & 0 \\ 0 & 0 & \delta_\beta^a \end{pmatrix} , \quad (\Pi_A^B) \equiv \begin{pmatrix} \delta_\alpha^\beta & 0 & 0 \\ 0 & \delta_a^b & 0 \\ 0 & 0 & \delta_\beta^a \end{pmatrix} ,
\]

(6.93)

48
where \((x^M) = (y^\mu, x^i, \tilde{y}_\mu, \tilde{x}_i)\). The \(T\)-plurality transformation \((6.45)\) is also generalized as

\[
\mathcal{H}'_{AB} = (C \mathcal{H} C^T)_{AB}, \quad (C^B_A) = \begin{pmatrix}
\delta^b_a & 0 & 0 & 0 \\
0 & p^{ab} & q_{ab} & 0 \\
0 & 0 & \delta^\alpha_\beta & 0 \\
0 & r^{ab} & 0 & s^{a_b}
\end{pmatrix}.
\]  

(6.94)

The dilaton will also have an additional dependence on the spectators similar to \((2.29)\),

\[
e^{-2\hat{d}'} = e^{-2\hat{d}'(y^\mu)} e^{-2\hat{d}'(x)}, \quad e^{-2\hat{d}'(x)} \equiv e^{-2\hat{d}(x)} |\det(\ell_m)|. \tag{6.95}
\]

We also suppose that the untwisted R–R fields are functions of the spectators \(\hat{F} = \hat{F}(y^\mu)\).

Then, by defining the fluxes \(F_{ABC}\) and \(F_A\) from \(U_M A(x)\) and \(d(x)\), we again obtain

\[
F_{ab}^c = f_{ab}^c, \quad F^{ab} = \tilde{f}^{ab}, \quad F_{abc} = F^{abc} = F_{aBC} = F^{aBC} = 0, \quad (F_A) = (F_\alpha, F_a, F^{\alpha}, F^a) = (0, 2\, D_a d, 0, 2\, D^a d).
\]  

(6.96)

Here, again we need to perform a shift \(\partial^M d \to \partial^M d + X^M\) \((6.63)\) when the dual algebra is non-unimodular.

The requirement \((2.30)\) is automatically satisfied with our twist matrix, and by using \((2.31)\), the dilaton equation of motion becomes

\[
\hat{S} + \frac{1}{12} F_{ABC} F_{DEF} \left( 3 \hat{H}^{AD} \eta^{BE} \eta^{CF} - \hat{H}^{AD} \hat{H}^{BE} \hat{H}^{CF} \right) - \hat{H}^{AB} F_A F_B + \frac{1}{2} F^{A}_{BC} \hat{H}^{BD} \hat{H}^{CE} D_D \hat{H}_{AE} - 2 F_A D_B \hat{H}^{AB} + 4 F_A \hat{H}^{AB} D_B \hat{d} = 0.
\]  

(6.97)

By requiring that \(\hat{H}_{AB}(y^\mu)\) and \(\hat{d}(y^\mu)\) in the original and the dual background are covariantly related by the \(\text{SO}(D, D)\) transformation,

\[
\hat{H}_{AB} = (C \hat{H} C^T)_{AB}, \quad \hat{d}' = \hat{d},
\]  

(6.98)

we can easily see that \(D_C \hat{H}_{AB} = \partial_C \hat{H}_{AB}\) and \(D_A \hat{d} = \partial_A \hat{d}\) are also transformed covariantly,

\[
D'_C \hat{H}_{AB}(y) = C_A^D C_A^E \partial_C \hat{H}_{DE}(y) = C_C^F C_A^D C_A^E D_F \hat{H}_{DE}(y),
\]

\[
D'_A \hat{d}(y) = \partial_A \hat{d}(y) = C_A^B D_B d(y).
\]  

(6.99)

Then, the dilaton equation of motion is satisfied in the dualized background if it is satisfied in the original background. The covariance of the equations of motion for the generalized metric and the R–R fields will be also shown in a similar manner, but here we just assume the covariance and move on to the explicit construction of solutions.
7 PL $T$-plurality for AdS$_5 \times S^5$

In this section, we show an example of the Poisson–Lie $T$-plurality. As already mentioned, the Lie algebra $\mathfrak{d}$ of the Drinfel’d doubles can be realized as a direct sum of two maximally isotropic algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ and the pair $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ is called the Manin triple. By following [54], we denote the pair simply as $(\mathfrak{g}|\tilde{\mathfrak{g}})$. The classification of six-dimensional real Drinfel’d doubles was worked out in [54], and there, the following series of Manin triples, which corresponds to a single Drinfel’d double $\mathfrak{d}$, has been found:

\[
(5|1) \cong (6_0|1) \cong (5|2.i) \cong (6_0|5.ii) \\
\cong (1|5) \cong (1|6_0) \cong (2.i|5) \cong (5.ii|6_0).
\]

(7.1)

Here, the characters in each slot denote the Bianchi type of the three-dimensional Lie algebra,

1: $[X_1, X_2] = 0, [X_2, X_3] = 0, [X_3, X_1] = 0$,

2.i: $[X_1, X_2] = 0, [X_2, X_3] = X_1, [X_3, X_1] = 0$,

5: $[X_1, X_2] = -X_2, [X_2, X_3] = 0, [X_3, X_1] = X_3$,

5.ii: $[X_1, X_2] = -X_1 + X_2, [X_2, X_3] = X_3, [X_3, X_1] = -X_3$,

6: $[X_1, X_2] = 0, [X_2, X_3] = X_1, [X_3, X_1] = -X_2$.

(7.2)

By using an SO(3,3) transformation $T'_A = C_A^B T_B$\footnote{As pointed out in [62], the matrix $C$ which connects two Manin triples may not be unique, and different choice of $C$ may give a different background. We will use the matrices $C$ that are given in [54].}, the PL $T$-plurality for this chain of Manin triples was studied in [55]. In [55], since the initial background is the flat space (or the Bianchi type V universe) the R–R fields were absent in any of the dual backgrounds. Moreover, due to a problem in the treatment of the dual-coordinate dependence of the dilaton, only the three backgrounds were discussed (see also [56–58] for the dilaton puzzle),

\[
(5|1) \cong (6_0|1) \cong (5|2.i).
\]

(7.3)

In this section, we identify the AdS$_5 \times S^5$ solution as a background with $(5|1)$ symmetry, and write down all of the eight backgrounds with the Manin triples given in (7.1).
For convenience, we summarize the procedure of the PL T-plurality.

Untwisted fields
\{\hat{H}_{AB}(y), \hat{d}(y), \hat{F}(y)\} \quad \{\hat{H}'_{AB}(y), \hat{d}'(y), \hat{F}'(y)\} \quad \{\hat{H}''_{AB}(y), \hat{d}''(y), \hat{F}''(y)\} \quad \ldots

twist \quad T_A \quad \text{twist} \quad T_A'=C_A^B T_B \quad \text{twist} \quad T_A''=C_A^B T_B
\{\mathcal{H}_{MN}, d, F\} \quad \{\mathcal{H}'_{MN}, d', F'\} \quad \{\mathcal{H}''_{MN}, d'', F''\} \quad \ldots

We first prepare the untwisted fields \{\hat{H}_{AB}(y^\mu), \hat{d}(y^\mu), \hat{F}(y^\mu)\} that satisfy

\begin{align}
\mathcal{D}_A\hat{H}_{BC}(y^\mu) &= \partial_A\hat{H}_{BC}(y^\mu), \\
\mathcal{D}_A\hat{d}(y^\mu) &= \partial_A\hat{d}(y^\mu), \\
\mathcal{D}_A\hat{F}(y^\mu) &= \partial_A\hat{F}(y^\mu).
\end{align}

(7.4)

They are independent of the structure of the Drinfel’d double and can be chosen freely. Under the SO\((D, D)\) PL T-plurality, they are transformed covariantly,

\begin{align}
\hat{H}_{AB} &\rightarrow (C \hat{H} C^T)_{AB}, \\
\hat{d} &\rightarrow \hat{d}, \\
\hat{F} &\rightarrow C \hat{F}.
\end{align}

(7.5)

By using the generators \(T_A\) in each frame, we construct the twist matrix \(U\) as

\begin{align}
U(x) &\equiv R \Pi, \\
\Pi^\text{\textit{v}} &\equiv \frac{1}{2} \Pi^{ab} \epsilon_{ca} \epsilon_{da}, \\
\text{Ad}_{g^{-1}} &\equiv \begin{pmatrix}
\delta_a^c & 0 \\
\Pi^{ac} & \delta_a^c
\end{pmatrix} \begin{pmatrix}
a_c^b & 0 \\
0 & (a^{-1})^{\text{\textit{v}}}_b
\end{pmatrix}, \\
(R_M^B) &\equiv \begin{pmatrix}
\delta^\beta_\mu & 0 & 0 \\
0 & r^b_i & 0 \\
0 & 0 & \delta^\mu_\beta \\
0 & 0 & e^i_b
\end{pmatrix}, \\
(\Pi_A^B) &\equiv \begin{pmatrix}
\delta_\beta^\beta & 0 & 0 \\
0 & \delta^b_a & 0 \\
0 & 0 & \delta^\alpha_\beta \\
0 & -\Pi^{ab} & 0
\end{pmatrix}.
\end{align}

(7.6)

Then, by twisting the untwisted fields, we construct the DFT fields as

\begin{align}
\mathcal{H}_{MN} &= [U(x) \hat{H}(y^\mu) U^T(x)]_{MN}, \\
e^{-2d} &= e^{-2\tilde{d}(y^\mu)} e^{-2\tilde{d}(x)} |\text{det}(\ell_m^a)|, \\
X^M &= \begin{pmatrix}
\frac{1}{2} \hat{f}^{ba}_m v^m_a \\
0
\end{pmatrix}, \\
F &= e^{-\hat{d}(x)/\sqrt{|\text{det} \, a|}} e^{-\Pi(x)/\sqrt{p} \sum_p \frac{1}{p!} \hat{f}^{a_1 \ldots a_p}_m (y^\mu) r^{a_1} \wedge \cdots \wedge r^{a_p}}.
\end{align}

(7.7)

The function \(\tilde{d}(x)\) is given in the initial configuration, and after the PL T-plurality, it is rewritten in the new coordinates determined by the invariance of \(l = g(x^i) \tilde{g}(\tilde{x}_i)\). When \(\tilde{d}(x)\) has a linear dual-coordinate dependence \(d^i \tilde{x}_i\), we make a redefinition and absorb the dependence into the Killing vector, \(I^i = \frac{1}{2} \hat{f}^{ba}_m v^m_a + d^i\).
7.1 \((5|1)\): \(\text{AdS}_5 \times S^5\)

We start with the \(\text{AdS}_5 \times S^5\) background, in a non-standard coordinate system,

\[
\begin{align*}
\text{d} s^2 &= -\text{d} t^2 + \frac{\text{d} x_1^2 + e^{-2x_1}(\text{d} x_2^2 + \text{d} x_3^2)}{z^2} + \text{d} z^2 + \text{d} s_{S^5}^2, \\
G_5 &= -4 e^{-2x_1} \ell^3 \text{d} t \wedge \text{d} x_1 \wedge \text{d} x_2 \wedge \text{d} x_3 \wedge \text{d} z + 4 \omega_5,
\end{align*}
\]

(7.8)

where

\[
\begin{align*}
\text{d} s_{S^5}^2 &\equiv \text{d} r^2 + \sin^2 r \text{d} \xi^2 + \cos^2 \xi \sin^2 r \text{d} \phi_1^2 + \sin^2 r \sin^2 \xi \text{d} \phi_2^2 + \cos^2 r \text{d} \phi_3^2, \\
\omega_5 &\equiv \sin^3 r \cos r \sin \xi \text{d} r \wedge \text{d} \xi \wedge \text{d} \phi_1 \wedge \text{d} \phi_2 \wedge \text{d} \phi_3.
\end{align*}
\]

(7.9)

Since this background has Killing vectors,

\[
v_1 \equiv \partial_1 + x^2 \partial_2 + x^3 \partial_3, \quad v_2 \equiv \partial_2, \quad v_3 \equiv \partial_3,
\]

(7.10)

satisfying

\[
[v_a, v_b] = f_{ab}^\ c v_c, \quad f_{12}^\ 2 = f_{13}^\ 3 = -1, \quad \mathcal{L}_{v_a} E_{mn} = 0,
\]

(7.11)

this background has the \((5|1)\) symmetry.

We can construct this background, by providing a parameterization

\[
l = g \tilde{g}, \quad g = e^{x_1 T_1} e^{x_2 T_2} e^{x_3 T_3}, \quad \tilde{g} = e^{\tilde{x}_1 \tilde{T}_1} e^{\tilde{x}_2 \tilde{T}_2} e^{\tilde{x}_3 \tilde{T}_3},
\]

(7.12)

where \((T_A) = (T_a, \tilde{T}^a)\) are generators of the Manin triple \((5|1)\). We obtain

\[
\begin{align*}
\ell &= \text{d} x_1 T_1 + (\text{d} x_2 - x^2 \text{d} x_1) T_2 + (\text{d} x_3 - x^3 \text{d} x_1) T_3, \\
r &= \text{d} x_1 (T_1 + e^{-x_1} (\text{d} x_2 + \text{d} x_3) T_3), \\
a &= \begin{pmatrix}
1 & -x^2 & -x^3 \\
0 & e^{x_1} & 0 \\
0 & 0 & e^{x_1}
\end{pmatrix}, \quad \Pi^{ab} = 0,
\end{align*}
\]

(7.13)

and they give the twist matrix \(U^A_M\). We can easily determine the untwisted metric from the relation \(\hat{\mathcal{H}}_{MN} = (U^{-1} \mathcal{H} U^{-1})_{MN}\) and the result is

\[
(\hat{E}_{mn}) = \text{diag} \left( \frac{1}{z^2}, \frac{t^2}{z^2}, \frac{t^2}{z^2}, \frac{1}{z^2}, 1, \sin^2 r, \sin^2 r \cos^2 \xi, \sin^2 r \sin^2 \xi, \cos^2 r \right),
\]

(7.14)

(7.15)

in a coordinate system \((x^n) = (t, x_1, x_2, x_3, z, r, \xi, \phi_1, \phi_2, \phi_3)\). Since the dilaton is absent
\( \Phi = 0 \), the DFT dilaton becomes

\[
e^{-2d} = \sqrt{|g|} = \frac{t^3 e^{-2x_1} \sin^3 r \cos r \sin \xi \cos \xi}{z^5}.
\]

(7.16)

We also have \(|\det(\ell_m^\alpha)| = 1\) and we obtain

\[
e^{-2d} = e^{-2d(y^\mu)} e^{-2\hat{d}(x)}, \quad e^{-2\hat{d}(y^\mu)} \equiv \frac{t^3 \sin^3 r \cos r \sin \xi \cos \xi}{z^5}, \quad e^{-2\hat{d}(x)} \equiv e^{-2x_1}.
\]

(7.17)

In addition, from \(e^{-\hat{d}(x)} \sqrt{|\det a|} = 1\), the untwisted R–R fields become

\[
\hat{F} \equiv \sum_p \frac{1}{p!} \hat{F}_{a_1 \cdots a_p} dx^{a_1} \cdots dx^{a_p} = -4 t^3 dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dz + 4 \omega_5,
\]

(7.18)

and it is a function of spectator fields \((y^\mu) = (t, z, \xi, \phi_1, \phi_2, \phi_3)\) as expected.

Note that if we choose the untwisted fields as

\[
(\hat{E}_{mn}) = \text{diag}(-1, t^2, t^2, t^2, 1, 1, 1, 1, 1, 1), \quad e^{-2d} = t^3, \quad \hat{F} = 0,
\]

(7.19)

the purely NS–NS solutions studied in [55] can be recovered.

7.2 \((1|5)\): type IIA GSE

In order to consider the NATD background, we perform a redefinition of generators,

\[
T_A' = C A^{\alpha} T_B^{(5|1)}, \quad C = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix},
\]

(7.20)

and give a parameterization,

\[
l = g' \tilde{g}', \quad \tilde{g}' = e^{x_1 T_1} e^{x_2 T_2} e^{x_3 T_3}, \quad \tilde{g}' = e^{x_1 \tilde{T}^1} e^{x_2 \tilde{T}^2} e^{x_3 \tilde{T}^3}.
\]

(7.21)

Then, from the identification with the original background,

\[
g(x) \tilde{g}(\tilde{x}) = l = g'(x') \tilde{g}'(\tilde{x'}),
\]

(7.22)
we find the following relation between the coordinates:

\[
\begin{align*}
x^1 &= \tilde{x}'_1, \quad x^2 = \tilde{x}'_2, \quad x^3 = \tilde{x}'_3, \\
\tilde{x}_1 &= x^1 + x^2 e^{-\tilde{x}'_1} \tilde{x}'_2 + x^3 e^{-\tilde{x}'_1} \tilde{x}'_3, \quad \tilde{x}_2 = e^{-\tilde{x}'_1} x^2, \quad \tilde{x}_3 = e^{-\tilde{x}'_1} x^3.
\end{align*}
\] (7.23)

From this relation, we can identify \(\tilde{d}\) as

\[
\begin{align*}
e^{-2\tilde{d}} &= e^{-2x^1} = e^{-2\tilde{x}'_1}.
\end{align*}
\] (7.24)

For notational simplicity, in the following we drop the prime.

The untwisted fields in this frame become

\[
\begin{align*}
(\hat{E}_{mn}) &= \text{diag} \left( -\frac{1}{z^2}, \frac{z^2}{z^2}, \frac{z^2}{z^2}, \frac{z^2}{z^2}, 1, \sin^2 r, \sin^2 r \cos^2 \xi, \sin^2 r \cos^2 e, \cos^2 r \right), \\
\frac{e^{-2\tilde{d}}}{z^3} &= \frac{4 t^3 dt \wedge dz}{z^5} + 4 \omega_5 \wedge dx^1 \wedge dx^2 \wedge dx^3, \\
\hat{F} &= \frac{4 t^3 dt \wedge dz}{z^5} + 4 \omega_5 \wedge dx^1 \wedge dx^2 \wedge dx^3, \\
\end{align*}
\] (7.25)

and we twist them by using the quantities,

\[
\begin{align*}
\ell &= dx^1 T_1 + dx^2 T_2 + dx^3 T_3, \quad r = dx^1 T_1 + dx^2 T_2 + dx^3 T_3, \\
v_1 &= \partial_1, \quad v_2 = \partial_2, \quad v_3 = \partial_3, \\
a &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Pi^{ab} = -\tilde{f}^{ab}_c x^c. \\
\end{align*}
\] (7.26)

The resulting metric and the \(B\)-field are

\[
\begin{align*}
ds^2 &= -dt^2 + dz^2 + \frac{z^2 \left[ t^4 \left( dx_1^2 + dx_2^2 + dx_3^2 \right) + z^4 \left( x^3 dx_2 - x^2 dx_3 \right)^2 \right]}{t^2 \left[ t^4 + (x_2^2 + x_3^2) z^4 \right]} + ds_{5}^2, \\
B_2 &= \frac{z^4 dx^1 \wedge \left( x^2 dx_2 + x^3 dx_3 \right)}{t^4 + (x_2^2 + x_3^2) z^4}. \\
\end{align*}
\] (7.27)

Since the dual algebra \(5\) is non-unimodular, we need to introduce the Killing vector

\[
I = \frac{1}{2} \tilde{f}^{ab}_c v^m_a \partial_m = \partial_1. \\
\] (7.28)

We can check that the flux \(\mathcal{F}_A\) is transformed covariantly from the original one \(\mathcal{F}_B^{(5|1)}\),

\[
\mathcal{F}_A = (0, 0, 0, 2, 0, 0) = C_A^B \mathcal{F}_B^{(5|1)}, \\
\] (7.29)

which shows that the equations of motion are transformed covariantly. In order to make the
background as a solution of GSE, we make the redefinition (6.67) which gives

\[ \bar{d} = 0, \quad I = \left( \frac{1}{2} f^{\alpha \beta} v_\alpha^m + \hat{\theta}^m \hat{d} \right) \partial_m = 2 \partial_1. \] (7.31)

After this redefinition, the dual geometry becomes

\[
\begin{align*}
ds^2 &= -dt^2 + dz^2 + \frac{z^2}{t^2} \left[ t^4 (dx^1_2 + dx^1_3 + dx^1_4) + z^4 (x^3 dx^2 - x^2 dx^3)^2 \right] + ds_{S^5}^2, \\
e^{-2\Phi} &= \frac{t^2 [t^4 + (x^2_2 + x^2_3) z^4]}{z^6}, \\
B_2 &= \frac{z^4 dx^1 \wedge (x^2 dx^2 + x^3 dx^3)}{t^4 + (x^2_2 + x^2_3) z^4}, \\
G_2 &= -\frac{4 t^3 dt \wedge dz}{z^5}, \\
G_4 &= -\frac{4 t^3 dt \wedge dx^1 \wedge (x^2 dx^2 + x^3 dx^3) \wedge dz}{[t^4 + (x^2_2 + x^2_3) z^4] z}, \\
I &= 2 \partial_1,
\end{align*}
\] (7.32)

which is a solution of type IIA GSE. We can explicitly check that this background has the \( (1|5) \) symmetry,

\[
[v_a, v_b] = f_{abc} v_c = 0, \quad \mathcal{L}_{v_a} E^{mn} = \tilde{f}^{abc} v_b^m v_c^n.
\] (7.33)

A formal \( T \)-duality along the \( x^1 \)-direction gives a simple solution of type IIB supergravity

\[
\begin{align*}
ds^2 &= -dt^2 + dz^2 + \frac{z^2}{t^2} \left[ (dx^2 - x^2 dx^1)^2 + (dx^3 - x^3 dx^1)^2 \right] + ds_{S^5}^2, \\
\Phi &= \ln \left( \frac{z^2}{t^2} \right) + 2 x^1, \\
G_3 &= \frac{4 t^3 e^{-2x^1} dt \wedge dx^1 \wedge dz}{z^5}.
\end{align*}
\] (7.34)

### 7.3 \( (6_0|1) \): type IIA SUGRA

We next perform the following redefinition from the original \( (5|1) \) generators:

\[
T'_A = C_{A B} T^{(5|1)}_B, \quad C = \begin{pmatrix}
0 & 0 & -\frac{1}{2} & 0 & 1 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & -1 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}.
\] (7.35)

This time, we provide a parameterization

\[
l = g' \tilde{g}', \quad g' = e^{-x^3 T^3_1} e^{x^2 T^2_1} e^{x^1 T^1_1}, \quad \tilde{g}' = e^{x^1 \tilde{T}^1_1} e^{x^2 \tilde{T}^2_1} e^{-x^3 \tilde{T}^3_1},
\] (7.36)
and the coordinates are related to the original ones as

\[ x^1 = x'^3, \quad x^2 = \frac{x'_1 + x'_2}{2}, \quad x^3 = \frac{x'^2 - x'^1}{2}, \]

\[ \tilde{x}_1 = \tilde{x}'_3 + \frac{(x'^1 + x'^2)(\tilde{x}'_1 + \tilde{x}'_2)}{2}, \quad \tilde{x}_2 = x'^1 + x'^2, \quad \tilde{x}_3 = \tilde{x}'_2 - \tilde{x}'_1. \]  

(7.37)

Then, in this frame, \( \tilde{d} \) becomes

\[ e^{-2\tilde{d}} = e^{-2x^1} = e^{-2x'^3}. \]  

(7.38)

Again we remove the prime, and then the \((t, x^1, x^2, x^3, z)^\text{-part of the untwisted metric becomes}\)

\[ (\hat{E}_{mn}) = \begin{pmatrix}
-\frac{1}{z^2} & 0 & 0 & 0 & 0 \\
0 & \frac{t^2}{4z^2} + \frac{z^2}{t^2} & \frac{t^2}{4z^2} & 0 & 0 \\
0 & \frac{z^2}{t^2} - \frac{t^2}{4z^2} & \frac{t^2}{4z^2} + \frac{z^2}{t^2} & 0 & 0 \\
0 & 0 & 0 & \frac{z^2}{t^2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{z^2}
\end{pmatrix}. \]  

(7.39)

In order to obtain the untwisted R–R fields, it may be useful to decompose the matrix \( C \) into products of GL(D) transformation, B-transformation, T-dualities, and \( \beta \)-transformation. In this case, for example we can use a decomposition

\[ C = \begin{pmatrix}
0 & 1 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 1 & \frac{1}{2} & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & -1 \\
0 & 0 & 0 & \frac{1}{2} & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \]  

(7.40)

Then, the \( T \)-duality along the \( x^2 \)-direction and the GL(3) transformation give

\[ \hat{F} = \frac{2t^3 \mathrm{d}t \wedge (\mathrm{d}x^1 - \mathrm{d}x^2) \wedge \mathrm{d}x^3 \wedge \mathrm{d}z}{z^5} - 4(\mathrm{d}x^1 + \mathrm{d}x^2) \wedge \omega_5. \]  

(7.41)

In order to obtain the twist matrix, we compute

\[ \ell = (\mathrm{d}x^1 + x^2 \mathrm{d}x^3) T_1 + (\mathrm{d}x^2 + x^2 \mathrm{d}x^3) T_2 - \mathrm{d}x^3 T_3, \]

\[ r = (\cosh x^3 \mathrm{d}x^1 + \sinh x^3 \mathrm{d}x^2) T_1 + (\sinh x^3 \mathrm{d}x^1 + \cosh x^3 \mathrm{d}x^2) T_2 - \mathrm{d}x^3 T_3, \]

\[ v_1 = \partial_1, \quad v_2 = \partial_2, \quad v_3 = x^2 \partial_1 + x^1 \partial_2 - \partial_3, \]

\[ a = \begin{pmatrix}
\cosh x^3 & -\sinh x^3 & 0 \\
-\sinh x^3 & \cosh x^3 & 0 \\
-x^2 & -x^1 & 1
\end{pmatrix}, \quad \Pi^{ab} = 0. \]  

(7.42)

(7.43)
Again, the flux $F_A$ is transformed covariantly,

$$(F_A) = (0, 0, -2, 0, 0, 0) = C_A^B F_B^{(5|1)}.$$  \hfill (7.44)

The background fields are determined as

\[
\begin{align*}
\text{ds}^2 &= \frac{-dt^2 + t^2 dx_3^2 + dz^2}{z^2} + \frac{e^{-2x^3} t^2 (dx_1^2 - dx_2^2)}{4z^2} + \frac{e^{2x_3} z^2 (dx_1^2 + dx_2^2)}{t^2} + ds_5^2 , \\
e^{-2\Phi} &= \frac{e^{-2x^3} t^2}{z^2} , \\
G_4 &= \frac{2e^{-2x^3} t^3 (dx_1 - dx_2) \wedge dt \wedge dx_3 \wedge dz}{z^5} ,
\end{align*}
\]

and this is a solution of type IIA supergravity.

7.4 $(1|6_0)$: type IIB GSE

The NATD of the $(6_0|1)$ background, namely $(1|6_0)$ can be realized by

$$T_A' = C_A^B T_B^{(5|1)}, \quad C = \begin{pmatrix} 0 & \frac{1}{t} & 0 & 0 & 0 & -1 \\ 0 & \frac{1}{t} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{t} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{t} & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \hfill (7.46)$$

We give a parameterization

$$l = g' \tilde{g}' , \quad g' = e^{x^1 T_1'} e^{x^2 T_2'} e^{-x^3 T_3'} , \quad \tilde{g}' = e^{-\tilde{x}_3' T_3'^c} e^{x_2' T_2'^c} e^{x_1' T_1'^c} . \hfill (7.47)$$

In order to determine $\tilde{d}$, it is enough to identify the coordinate $x^1$, and we find

$$e^{-2\tilde{d}} = e^{-2x^1} = e^{-2\tilde{x}_3'} . \hfill (7.48)$$

Note that the appearance of the dual-coordinate dependence was discussed in [55], but at that time, DFT had not been developed and the interpretation was not clear.

We can construct the twist matrix $U$ from

$$\begin{align*}
\ell &= dx^1 T_1 + dx^2 T_2 - dx^3 T_3 , \\
v_1 &= \partial_1 , \\
 v_2 &= \partial_2 , \\
v_3 &= -\partial_3 . \\
a &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\Pi^{ab}) = \begin{pmatrix} 0 & 0 & -x^3 \\ 0 & 0 & -x^1 \\ x^2 & x^1 & 0 \end{pmatrix} . \hfill (7.50)
\end{align*}$$

57
and the flux $F_A$ becomes

$$F_A = (0, 0, 0, 0, -2) = C^{AB}_A F^{(5|1)}_B.$$  \hspace{1cm} (7.51)

Thus, the DFT equations of motion are covariantly transformed.

Although the dual algebra is unimodular, in order to absorb the dual coordinate dependence in $\bar{d}$, we make a field redefinition (6.67) and obtain

$$e^{-2\bar{d}} = 1, \quad I = \partial_X.$$  \hspace{1cm} (7.52)

After the redefinition, we obtain a solution of type IIB GSE,

$$ds^2 = \frac{-dt^2 + dz^2}{z^2} + ds^2_{S5},$$

$$+ t^6(dx^1 + dx^2)^2 + 4 t^2 z^4 \left[ (dx^1 - dx^2)^2 + (x^1 dx^1 - x^2 dx^2)^2 + dx_3^2 \right],$$

$$B_2 = \frac{t^4 (x^1 + x^2) (dx^1 + dx^2) - 4 z^4 (x^1 - x^2) (dx^1 - dx^2)}{t^4 ((x^1 + x^2)^2 + 4 z^4 (x^1 - x^2)^2)} \wedge dx^3, \quad I = \partial_3,$$  \hspace{1cm} (7.53)

$$e^{-2\Phi} = \frac{t^4 ((x^1 + x^2)^2 + 4 z^4 (x^1 - x^2)^2)}{4 z^4}, \quad G_3 = \frac{2 t^3 (dx^1 + dx^2) \wedge dt \wedge dz}{z^5},$$

$$G_5 = 2 (x^1 - x^2) \left[ \frac{8 t^4 z^{-1} dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dz}{t^4 ((x^1 + x^2)^2 + 4 z^4 (x^1 - x^2)^2)} - 2 \omega_{S5} \right].$$

It is important to note that the duality (60|1) $\rightarrow$ (1|60) is a NATD for traceless structure constants. In the literature, it has been discussed that if the structure constants are traceless, the NATD background satisfies the supergravity equations of motion, and it is important to clarify the consistency with the example discussed here. Of course, the existence of the R–R fields is not important here. As already mentioned, we can obtain a purely NS–NS solution, by starting with the untwisted fields (7.19). The (60|1) background is

$$ds^2 = -dt^2 + t^2 dz_3^2 + \frac{1}{4} e^{-2x^3} t^2 (dx^1 - dx^2)^2 + e^{2x^3} t^{-2} (dx^1 + dx^2)^2 + s^2_{T6},$$  \hspace{1cm} (7.54)

while its NATD, namely the (1|60) background, is a GSE solution,

$$ds^2 = -dt^2 + ds^2_{T6},$$

$$+ \frac{t^6(dx^1 + dx^2)^2 + 4 t^2 \left[ (dx^1 - dx^2)^2 + (x^1 dx^1 - x^2 dx^2)^2 + dx^3_3 \right]}{t^4 ((x^1 + x^2)^2 + 4 (x^1 - x^2)^2)},$$

$$B_2 = \frac{t^4 (x^1 + x^2) (dx^1 + dx^2) - 4 (x^1 - x^2) (dx^1 - dx^2)}{t^4 ((x^1 + x^2)^2 + 4 (x^1 - x^2)^2)} \wedge dx^3, \quad I = \partial_3,$$  \hspace{1cm} (7.55)

$$e^{-2\Phi} = \frac{t^4 ((x^1 + x^2)^2 + 4 (x^1 - x^2)^2)}{4}. $$
It will be interesting to study string theory on these backgrounds in detail.

We also note that, in the $\mathbf{(1|6_0)}$ background (7.53), if we perform a formal $T$-duality along the $x^3$-direction, we obtain a solution of type IIA supergravity,

$$
\begin{align*}
\mathcal{F} &= \frac{e^{-2x^3 t^2}}{z^2}, \\
G_4 &= \frac{2e^{-x^3 t^2} (dx^1 + dx^2) \wedge dt \wedge dx^3 \wedge dz}{z^5}. 
\end{align*}
$$

7.5 $\mathbf{(5|2.i)}$: type IIB SUGRA

In order to obtain the Manin triple $\mathbf{(5|2.i)}$, we perform a redefinition

$$
T_A' = C_{AB} T_B^{(5|1)}, \quad C = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 1 & 0 & \frac{1}{2} & 0
\end{pmatrix}.
$$

Again we consider a parameterization,

$$
l = g' \tilde{g}', \quad g' = e^{x^1 T_1} e^{x^2 T_2} e^{-x^3 T_3}, \quad \tilde{g}' = e^{x_1' \tilde{T}_1} e^{x_2' \tilde{T}_2} e^{-x_3' \tilde{T}_3},
$$

and from the coordinate transformation, we obtain

$$
e^{-2 \tilde{d}} = e^{-2 x^1} = e^{2 x^1}.
$$

The necessary quantities are obtained as

$$
\begin{align*}
\ell &= dx^1 T_1 + (dx^2 - x^2 dx^1) T_2 - (dx^3 - x^3 dx^1) T_3, \\
r &= dx^1 T_1 + e^{-x^1} (dx^2 T_2 - dx^3 T_3), \\
v_1 &= \partial_1 + x^2 \partial_2 + x^3 \partial_3, \quad v_2 = \partial_2, \quad v_3 = -\partial_3, \\
a &= \begin{pmatrix}
1 & -x^2 & x^3 \\
0 & e^{x^1} & 0 \\
0 & 0 & e^{x^1}
\end{pmatrix}, \\
(P^{ab}) &= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -e^{-x^1} \sinh x^1 \\
e^{-x^1} \sinh x^1 & 0
\end{pmatrix},
\end{align*}
$$

and again the flux $\mathcal{F}_A$ is covariantly transformed

$$
(\mathcal{F}_A) = (-2, 0, 0, 0, 0) = C_{AB} \mathcal{F}_B^{(5|1)}.
$$
A straightforward computation gives

\[
\begin{align*}
\text{ds}^2 &= -dt^2 + t^2 \, dx_1^2 + dz^2 + \frac{4 e^{2x_1} t^2 \, z^2 \, (dx_2^2 + dx_3^2)}{4 e^{4x_1} t^4 + z^4} + dx_3^2, \\
B_2 &= -\frac{2 \, z^4 \, dx^2 \wedge dx^3}{4 e^{4x_1} t^4 + z^4}, \\
G_3 &= -e^{2x_1} \, \frac{4 \, t^3 \, dt \wedge dx^1 \wedge dz}{z^3}, \\
G_5 &= -\frac{8 \, e^{2x_1} t^3 \, dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dz}{4 e^{4x_1} t^4 + z^4} + 2 \omega_{5},
\end{align*}
\]

and this is a solution of type IIB supergravity.

7.6 (2.i|5): type IIA SUGRA

We next consider a transformation,

\[
T'_A = C_A^B T_B^{(5|1)}, \quad C = \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

and provide a parameterization,

\[
l = g' \tilde{g}', \quad g' = e^{x_1 T_1} e^{x_2 T_2} e^{-x_3 T_3}, \quad \tilde{g}' = e^{\tilde{x}_1 \tilde{T}_1} e^{\tilde{x}_2 \tilde{T}_2} e^{-\tilde{x}_3 \tilde{T}_3}.
\]

The coordinate transformation gives

\[
e^{-2 \tilde{d}} = e^{-2 \, x_1} = e^{2 \tilde{x}_1}.
\]

Again, we compute

\[
\begin{align*}
\ell &= (dx_1^3 - x^3 \, dx_2^2) \, T_1 + dx_2^3 \, T_2 - dx_3^3 \, T_3, \\
r &= (dx_1^3 - x^3 \, dx_2^2) \, T_1 + dx_2^3 \, T_2 - dx_3^3 \, T_3, \\
v_1 &= \partial_1, \quad v_2 = x^3 \, \partial_1 + \partial_2, \quad v_3 = -\partial_3, \\
a &= \begin{pmatrix}
1 & 0 & 0 \\
-x^3 & 1 & 0 \\
-x^2 & 0 & 1
\end{pmatrix}, \quad (\Pi^{ab}) = \begin{pmatrix}
0 & x^2 - x^3 & -x^2 \\
x^2 - x^3 & 0 & 0 \\
x^3 & 0 & 0
\end{pmatrix}, \quad (F_A) = (0, 0, 0, -2, 0, 0) = C_A^B \, F_B^{(5|1)}.
\end{align*}
\]
Since the dual algebra 5 is non-unimodular, we have

\[ I = \frac{1}{2} \tilde{f}_{a b} v^m_a \partial_m = \partial_1. \]  

(7.70)

We thus expect that this background is a solution of the GSE. However, according to the field redefinition (6.67), we obtain

\[ e^{-2 \tilde{d}} = 1, \quad I = \partial_1 - \partial_1 = 0. \]  

(7.71)

As the result, we obtain a solution of the conventional type IIA supergravity

\[
\begin{align*}
    ds^2 &= \frac{-dt^2 + dz^2}{z^2} + \frac{z^2}{4 t^2 (1 + x_2^2 + x_3^2)} \left[ 4 dx^1 (dx^1 - x^3 dx^2 - x^2 dx^3) + (x^3 dx^2 + x^2 dx^3)^2 \right] \\
    B_2 &= \frac{dx^1 \wedge (x^2 dx^2 + x^3 dx^3)}{1 + x_2^2 + x_3^2} + \frac{(1 + 2 x_2^2) dx^2 \wedge dx^3}{2 (1 + x_2^2 + x_3^2)}, \\
    e^{-2 \Phi} &= \frac{t^2 (1 + x_2^2 + x_3^2)}{z^2}, \quad G_4 = -\frac{4 t^3 dt \wedge dx^2 \wedge dx^3 \wedge dz}{z^5}.
\end{align*}
\]  

(7.72)

Namely, even if the dual algebra is non-unimodular, the background can satisfy the usual supergravity equations of motion. This is a remarkable example of such unusual cases.

### 7.7 (5.ii|60): type IIB GSE

We next consider

\[
    T'_A = C_A B T_B^{(5|1)}, \quad C = \begin{pmatrix}
    1 & 0 & 0 & 0 & -1 & 0 \\
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 \\
    0 & -1 & 0 & 0 & 0 & \frac{1}{2} \\
    0 & 1 & 0 & 1 & 0 & \frac{1}{2} \\
    -1 & 0 & 1 & 0 & \frac{1}{2} & 0
    \end{pmatrix},
\]  

(7.73)

and give a parameterization,

\[
    l = g' \tilde{g}', \quad g' = e^{x^1 T'_1} e^{(x^2 - x^1) T'_2} e^{x^3 T'_3}, \quad \tilde{g}' = e^{\tilde{x}_1^T \tilde{T}_3} e^{\tilde{x}_2^T \tilde{T}_2} e^{(\tilde{x}_1 + \tilde{x}_2)^T \tilde{T}_1}. \]  

(7.74)

We then obtain \( \tilde{d} \) as

\[
    e^{-2 \tilde{d}} = e^{-2 x^1} = e^{-2 (x^2 - \tilde{x}_3^2)}. \]  

(7.75)
From a straightforward computation,
\[
\ell = e^{x_1-x^2} \, dx^1 \, T_1 + (dx^2 - e^{x_1-x^2} \, dx^1) \, T_2 + (dx^3 + x^3 \, dx^2) \, T_3,
\]
\[
r = [e^{x_1} \, dx^1 + (1 - e^{x_1}) \, dx^2] \, T_1 + e^{x_1} (dx^2 - dx^1) \, T_2 + e^x \, dx^3 \, T_3,
\]
\[
v_1 = e^{x_2} \, \partial_1 + \partial_2 - x^3 \, \partial_3, \quad v_2 = \partial_2 - x^3 \, \partial_3, \quad v_3 = \partial_3,
\]
\[
a = \begin{pmatrix}
  e^{x_1-x^2} & 1-e^{x_1-x^2} & x^3 \\
  0 & 1 & 0 \\
  0 & 0 & e^{-x^2}
\end{pmatrix},
\]
we obtain the twist matrix \( U \), and the flux is covariantly transformed
\[
(F_A) = (2, 2, 0, 0, 0, -2) = C_{AB} \, f_B^{(5|1)}.
\]

Since the dual algebra is unimodular, originally we have \( I^m = 0 \). However, due to the dual-coordinate dependence of \( \bar{d} \), we make the field redefinition (6.67) and obtain
\[
e^{-2 \bar{d}} = e^{-2 \bar{x}^2}, \quad I = -\partial_3.
\]

After the redefinition, we obtain a solution of type IIB GSE,
\[
ds^2 = \frac{-dt^2 + dz^2}{z^2} + dx_5^2
\]
\[
+ t^2 \left\{ 4 e^{x_1} z^4 (e^{x_1} \, dx_1^2 + dx_3^2) + [4 (t^4 + z^4) + e^{4x_2} z^4] dx_2^2 \right\} \frac{\Delta^2}{\Delta^2}
\]
\[
+ 4 e^{x_1} t^4 z^4 \left[ e^{x_1} (dx_1 - dx_2)^2 - e^{3x_2} \, dx_1 \, dx_2 + 2 (dx_1 - dx_2) \, dx_2 \right] \frac{\Delta^2}{\Delta^2},
\]
\[
B_2 = -2 e^{2x_2} z^2 \left\{ 2 t^4 dx_2 - z^4 (2 e^{x_1} - e^{x_2} - 2) \left[ e^{x_1} \, dx_1 - (e^{x_1} - 1) \, dx_2 \right] \right\} \wedge dx^3 \frac{\Delta^2}{\Delta^2},
\]
\[
e^{-2 \Phi} = \frac{-4 e^{2x_2} \Delta^2}{4 z^4}, \quad I = -\partial_3, \quad G_3 = 4 e^{-2x_2} t^3 \, dt \wedge dx_2 \wedge dz \frac{\Delta^2}{z^5},
\]
\[
G_5 = (2 e^{x_1} - e^{x_2} - 2) \left[ 8 t^3 e^{x_1} z \, dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dz - 2 e^{-x_2} \omega_5 \right] \frac{\Delta^2}{\Delta^2},
\]
which is defined on the region
\[
\Delta^2 \equiv 4 t^4 (e^{2x_2} + 1) \, z^2 + e^{2x_2} z^6 (2 - 2 e^{x_1} + e^{x_2})^2 \geq 0.
\]
A formal $T$-duality along the $x^3$-direction gives a solution of type IIA supergravity,
\[
\begin{align*}
\notag ds^2 &= \frac{-dt^2 + dz^2}{z^2} + (t^4 + z^4) \left( dx^2 - dx^3 \right)^2 + z^4 \left( dx^1 - dx^2 + dx^3 \right)^2 \\


&\quad + z^2 e^{x^1} \frac{2 \left( dx^2 - dx^3 \right) \left( dx^1 - dx^2 + dx^3 \right) - e^{x^2} \left( dx^1 - dx^2 + dx^3 \right) dx^3}{t^2} \\


&\quad + z^2 e^{x^2} \frac{\left( dx^3 - dx^2 \right) dx^3}{t^2} + e^{-2x^2} \left( 4 t^4 + e^{4x^2} z^4 \right) \left( dx^2 \right)^2 + d\hat{s}_5^2, \\


e^{-2\Phi} &= \frac{l^2 e^{2(x^3-x^1)}}{z^2}, \quad G_4 = -4 e^{x^3-2x^2} t^3 \left( dx^2 \wedge dx^3 \wedge dz \right) / z^5. 
\end{align*}
\]

### 7.8 $(6|5.\text{ii})$: type IIA SUGRA

Finally, we consider a redefinition,
\[
T'_A = C_A^B T_B^{(5|1)}, \quad C = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & \frac{1}{2} \\ -1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.82}
\]

This time, we consider a parameterization\(^{13}\)
\[
l = g' g', \quad g' = e^{x^3 T^3} e^{x^2 T^2} e^{(x^1+x^2)} T^1, \quad \tilde{g}' = e^{\tilde{x}^3 T^3} e^{(\tilde{x}^2-\tilde{x}^1)} T^2 e^{\tilde{x}^1 T^1}, \tag{7.83}
\]

which leads to
\[
e^{-2\delta} = e^{-2x^1} = e^{-2(\tilde{x}^2-\tilde{x}^3)}. \tag{7.84}
\]

By using
\[
\ell = (dx^1 + dx^2 - x^2 dx^3) T_1 + [dx^2 - (x^1 + x^2) dx^3] T_2 + dx^3 T_3, \\
r = (\cosh x^3 dx^1 + e^{-x^3} dx^2) T_1 + (\sinh x^3 dx^1 + e^{-x^3} dx^2) T_2 + dx^3 T_3, \tag{7.85}
\]

\[

\notag v_1 = \partial_1, \quad v_2 = \partial_2 - \partial_1, \quad v_3 = \partial_3 - x^1 \partial_1 + (x^1 + x^2) \partial_2, \\

\notag a = \begin{pmatrix} \cosh x^3 & \sinh x^3 & 0 \\ \sinh x^3 & \cosh x^3 & 0 \\ -x^2 & -x^1 - x^2 & 1 \end{pmatrix}, \quad (\Pi^{ab}) = \begin{pmatrix} 0 & x^1 & e^{-x^3} - 1 \\ -x^3 & 0 & e^{-x^3} - 1 \\ 1 - e^{-x^3} & 1 - e^{-x^3} & 0 \end{pmatrix}, \tag{7.86}
\]

we can check the covariance of the flux,
\[
(\mathcal{F}_A) = (0, 0, -2, 2, 0) = C_A^B F_B^{(5|1)}. \tag{7.87}
\]

\(^{13}\text{We note that, in general, the parameterization should be carefully chosen such that the twist matrix does not break the section condition.}\)
Since the dual algebra \( 5.\text{ii} \) is non-unimodular, the Killing vector becomes
\[
I = \frac{1}{2} \tilde{f}^{ab}_b \tilde{v}_a = -(v_1 + v_2) = -\partial_2 \quad (\tilde{f}^{b1}_1 = -2, \quad \tilde{f}^{b2}_2 = -2),
\]
but by absorbing the dual-coordinate dependence of \( \tilde{d} \), we obtain
\[
e^{-2 \tilde{d}} = e^{2x^3}, \quad I^m = 0.
\]

Then, after the redefinition, we obtain a solution of type IIA supergravity,
\[
\begin{align*}
\text{ds}^2 &= -\frac{dt^2 + dz^2}{z^2} + t^2 \frac{e^{4x^3} dx_1^2}{z^2 [2 - 2e^{x^3} + (x_1^2 + 1)e^{x_3}]} - 2e^{3x^3}(dx_1 + x^1 dx_3) dx_1^1 \\
&\quad + z^2 \frac{(1 - e^{x_3}) dx_1 + 2 dx_2 - e^{x_3} x^1 dx_3 |^2}{4t^2 [2 - 2e^{x^3} + (x_1^2 + 1)e^{x_3}]}, \\
&\quad + \frac{e^{2x^3} t^2 [2 dx_1^2 + 4x_1 dx_1 dx_3 + (2x_1^2 + 1) dx_3^2]}{z^2 [2 - 2e^{x^3} + (x_1^2 + 1)e^{x_3}]} + \text{ds}_5^2, \\
B_2 &= e^{2x^3} x^1 dx_1 \wedge dx_2 + [1 + e^{2x^3} (\sinh x^3 - \frac{1}{2})] dx_1 \wedge dx_3 + (2 - e^{x^3}) dx_2 \wedge dx_3, \\
e^{-2\Phi} &= \frac{t^2 [2 - 2e^{x^3} + e^{2x^3}(x_1^2 + 1)]}{z^2}, \quad G_4 = -\frac{4t^3 e^{2x^3} dt \wedge dx_1 \wedge dx_3 \wedge dz}{z^5}.
\end{align*}
\]

8 Conclusion and Outlook

Summary of results

We discussed two approaches to the non-Abelian T-duality. One is the traditional NATD, obtained by integrating out the gauge fields associated with non-Abelian isometries, and the other is the PL T-duality/plurality, which is based on the Drinfel’d double.

In NATD, a closed-form expression for the duality rules including the R–R fields was known only for a certain isometry group SU(2), but we proposed a general formula by assuming that the isometry group freely acts on the target space. The duality rules, under the setup (3.1), are summarized in (5.2) and (5.3). In order to check the formula, we studied many examples, especially, the NATD for non-unimodular isometry groups.

For the PL T-duality, the treatments of the R–R fields were discussed in recent papers [137, 138, 141], but concrete examples were not studied well. We first considered the case without spectator fields, and translated the known transformation rules for \( (g_{mn}, B_{mn}, \Phi) \) into the rules for the generalized metric \( H_{MN} \) and the DFT dilaton \( d \). Then, using a result of the gauged DFT, we showed that the equations of motion are transformed covariantly under the PL T-plurality (by introducing a Killing vector \( I^m \) appropriately). We also introduced the R–R fields, and determined their transformation rule under the SO\((D,D)\) PL T-plurality transformation such that the equations of motion are covariantly transformed. We further
considered the case with spectator fields, and proposed the duality rules. We showed that the dilaton equation of motion is indeed satisfied even in the presence of spectators, but the covariance of other equations of motion is not completely analyzed. Finally, we studied a concrete example of PL T-plurality. Starting with the AdS$_5 \times$ S$^5$ solution, we obtained the following family of solutions.

| Solution | Type          | Type          | Type          |
|----------|---------------|---------------|---------------|
| $\text{AdS}_5 \times S^5$ | $\text{type IIA SUGRA}$ | $\text{type IIB SUGRA}$ | $\text{type IIB GSE}$ |
| $(\tilde{d} = x^1, I = 0)$ | $(\tilde{d} = x^3, I = 0)$ | $(\tilde{d} = -x^1, I = 0)$ | $(\tilde{d} = x^2 - \tilde{x}_3, I = 0)$ |
| (5|1) | (6|1) | (5|2.1) | (5.|60) |
| NATD | NATD | PL T-dual | PL T-dual |

Three of them are solutions of GSE. There are two origins of GSE; one is the Killing vector $I^m = \frac{1}{2} \tilde{f}^{aba} \eta_a^m$ that appears when the dual algebra is non-unimodular, and the other is the dual-coordinate dependence in $\tilde{d}$. In the last two examples, the two contributions are canceled with each other, and they are solutions of the usual supergravity even though their dual algebras are non-unimodular. In the literature, when $\tilde{d}$ has a dual-coordinate dependence, since its interpretation is not clear in string theory or supergravity, such Manin triple was ignored. However, in DFT, we can treat the dual coordinates in the same ways as the physical coordinates, and we can lift the restriction. In this way, the PL T-plurality is a solution generating technique of the DFT, rather than the usual supergravity.

Discussion and outlook

As we discussed, if we consider a supergravity solution that contains a four-dimensional Minkowski spacetime, $ds^2 = f^2(y) \eta_{\mu\nu} dx^\mu dx^\nu + \cdots$, we can choose the coordinates such that the (5|1) symmetry manifest. Then, as long as the $B$-field isometric along the three Killing vectors, we will obtain a family of eight solutions similar to the case of AdS$_5 \times$ S$^5$. Moreover, low-dimensional Drinfel’d doubles are already classified in [52–54] and a useful list is given in section 3 of [54]. If we have a DFT solution with an isometry algebra $\mathfrak{g}$, we may find a series of Manin triples,

$$\mathfrak{b}|\mathfrak{g} \cong \cdots \cong \cdots$$

and obtain a chain of DFT solutions. We may also start from a background with a $(\mathfrak{g}|\tilde{\mathfrak{g}})$-symmetry. For example, as discussed in [50], the Yang–Baxter deformed backgrounds are also
Indeed, a Yang–Baxter deformed background has the form

\[ E^{mn} = \tilde{g}^{mn} - \beta^{mn}, \quad \beta^{mn} \equiv 2 \eta r^{ab} v^m_a v^n_b \quad (r^{ab} = -r^{ba}), \quad (8.2) \]

where \( \eta \) and \( r^{ab} \) are constant, and \( \mathcal{L}_{v_a} \tilde{g}_{mn} = 0 \) and \([v_a, v_b] = f_{abc} v^c \) are satisfied. Then, we can easily check \( \mathcal{L}_{v_a} E^{mn} = \tilde{f}^{bc} a v^m_a v^n_c \) with

\[ \tilde{f}^{bc} a = 2 \eta (r^{bd} f_{da} c - r^{cd} f_{db} a), \quad (8.3) \]

and we can see that this background has the \( (g|\tilde{g}) \)-symmetry. Then, by finding a parameterization of the group element \( g(x) \), which realizes the set of Killing vectors \( v^m_a \) as the left-invariant vector fields and \( \beta^{mn} \) as \( \beta^{mn} = e^m_a e^n_b \Pi^{ab} \) [i.e. \( (a^\dagger b)^{ab} = 2 \eta r^{ab} \)], we can perform the PL T-plurality transformations of the Yang–Baxter deformed background. In this way, from a given solution, we can find new solutions one after another, and the PL T-plurality is a useful solution generating technique.

In the traditional approach to NATD, we introduced the generalized Killing vector \( (V^M_a) = (v^m_a, \tilde{v}^m_a) \). When the dual components \( \tilde{v}^m_a \) are present, we cannot regard the NATD as a particular case of the PL T-plurality. Also when the generalized Killing vectors depend on the spectator fields \( y^\mu \), we cannot realize them as the left-invariant vector fields. In this sense, the traditional NATD is not completely contained in the PL T-plurality discussed here. It is interesting to study whether it is possible to generalize the PL T-plurality such that the traditional NATD can be realized as a particular case. In the realm of NATD that is going beyond the PL T-plurality, it is not ensured that the dual background is a solution of DFT. By the definition of the NATD, the duality rules for the metric and \( B \)-field should not be modified, but the transformation rule for the dilaton and \( I^m \) may be modified from (5.2). Indeed, a modification was required in the example \((6_0, 1) \rightarrow (1, 6_0)\) of the PL T-plurality. It will be an important task to determine the general rule for the dilaton and \( I^m \) that is consistent with the DFT equations of motion. Once the modification of the rule for the dilaton is determined, the modification of the rule for the \( R-R \) fields (by an overall factor) can be also determined, and then we can check the equations of motion for the \( R-R \) fields.

In this paper, in both approaches, we have assumed that the isometry group acts on the target space freely, or without isotropy. If the assumption is not satisfied, we cannot take a gauge \( x^i = c^i \) and we need to consider a more non-trivial gauge fixing. Treatments in such cases are discussed for example in [19, 73, 77, 174] for the NATD, and in [41, 45, 47] for the PL T-duality. It is an interesting future direction to check whether the DFT equations of motion are covariantly rotated even in such general cases.
Toward non-Abelian U-duality

Another important future direction is an investigation of the non-Abelian U-duality. As an attempt toward the non-Abelian U-duality, let us first consider an extension of the traditional NATD. As a natural extension of (3.3), let us consider the following setup [154],

\[ \epsilon_{a} g_{ij} = 0, \quad \epsilon_{a} F_{4} + d \hat{v}_{a}^{(2)} = 0, \quad \epsilon_{a} v_{b} = f_{abc} v_{c}, \quad \epsilon_{a} \hat{v}_{b}^{(2)} = f_{abc} \hat{v}_{c}^{(2)}, \]  

(8.4)

where \(F_{4} \equiv dC_{3}\) is the four-form field strength in the eleven-dimensional supergravity. We define \(\hat{v}_{(1)}^{(1)} \equiv \epsilon_{a} \hat{v}_{a}^{(2)}\), and for simplicity, let us assume

\[ \epsilon_{a} \hat{v}_{(1)}^{(1)} (ab) = 0, \quad \epsilon_{a} \hat{v}_{(1)}^{(1)} [bc] = \epsilon_{a} \hat{v}_{(1)}^{(1)} [bc]. \]  

(8.5)

We also assume the existence of 1-forms \(\ell^{a} \equiv \ell^{a}_{i} dx^{i}\), that are dual to \(v_{a} (\epsilon_{a} \ell^{b} = \delta_{a}^{b})\), and then we find that the action

\[ S = \int_{\Sigma} \left[ \frac{1}{2} (g_{ij} Dx^{i} \wedge * Dx^{j} + * 1) + C_{3} + 2 y_{ab} F^{a} \wedge (\ell^{b} - A^{b}) \right] \]

\[ + \int_{\Sigma} \left[ - A^{a} \wedge \hat{v}_{a}^{(2)} + \frac{1}{2} A^{a} \wedge A^{b} \wedge \hat{v}_{ab}^{(1)} - \frac{1}{3!} A^{a} \wedge A^{b} \wedge A^{c} \epsilon_{a} \hat{v}_{(1)}^{(1)} \right]. \]  

(8.6)

is invariant under

\[ \delta_{x} x^{i}(\sigma) = \epsilon^{i}(\sigma) v_{a}^{i}(x), \quad \delta_{x} A^{a}(\sigma) = d \epsilon^{a}(\sigma) + f_{bc}^{a} A^{b}(\sigma) \epsilon^{c}(\sigma), \]

\[ \delta_{x} y_{ab} = \epsilon^{e} (f_{ca}^{d} y_{db} + f_{cb}^{d} y_{ad}). \]  

(8.7)

Here, by following the approach of [175] (see also [176]), we have introduced antisymmetric Lagrange multipliers \(y_{ab} = -y_{ba}\) that will ensure \(F^{a} = 0\).

In the Abelian limit, we can realize \(v_{a}^{i} = \delta_{a}^{i}\) and \(\ell^{a} = \delta_{a}^{i} dx^{i}\), and then we can always choose a gauge \(x^{i} = 0\). By further assuming \(\hat{v}_{a}^{(2)} = -\epsilon_{a} C_{3}\), the action reduces to

\[ S = \int_{\Sigma} \left[ \frac{1}{2} (g_{ij} A^{i} \wedge * A^{j} + * 1) + \frac{1}{3!} C_{abc} A^{a} \wedge A^{b} \wedge A^{c} + dy_{ab} \wedge A^{a} \wedge A^{b} \right]. \]  

(8.8)

This is precisely the action discussed in [175] and (8.6) can be regarded as a natural extension. However, unlike the case of the string action, it is not clear how to eliminate the gauge fields \(A^{a}\), and at the present time, we do not know how to obtain the dual action.

A more promising approach may be the following approach based on a generalization of DFT. The U-dual version of DFT is known as the exceptional field theory (EFT) [176] and it is actively studied. In DFT, the generalized coordinates are \((x^{M}) = (x^{m}, \tilde{x}_{m})\) and the

---

14In the string action [3.3], by adding a total-derivative term, the Lagrangian multiplier was introduced with derivative \(d \delta a\) (see [22] for the Abelian case), but here we only discuss the classical equations of motion without investigating such a total-derivative term.

67
dual coordinates \( \tilde{x}_m \) are associated with the string winding number. On the other hand, in EFT, we introduce the dual coordinates for all of the wrapped branes that are connected by \( U \)-duality transformations. For example, in M-theory on a \( n \)-torus, we have the M2-brane, the M5-brane, and the Kaluza–Klein monopole, and more exotic branes in general, and correspondingly, we introduce the generalized coordinates as

\[
(x^I) = (x^i, y_{i1}^{i2}, y_{i1}^{i2}...y_{i1}^{i7}, i; i, \cdots) \quad (i = 1, \ldots, n).
\]  

(8.9)

By understanding that the multiple indices separated by commas are totally antisymmetrized, we can easily see that the number of dimensions of the extended space \( x^I \) is the same as the dimension \( D \) of a fundamental representation of the \( E_{n(n)} \) \( U \)-duality group:

| \( n \) | \( U \)-duality group \( E_{n(n)} \) | \( E_{6(6)} \) | \( E_{7(7)} \) | \( E_{8(8)} \) |
|-------|----------------|---------|---------|---------|
| \( n \) | SL(5) | SO(5, 5) | \( E_{6(6)} \) | \( E_{7(7)} \) | \( E_{8(8)} \) |
| dimension \( D \) | 10 | 16 | 27 | 56 | 248 |

(8.10)

In such extended space, the generalized metric \( \mathcal{M}_{IJ} \) has been constructed in [176, 180], and it contains the bosonic fields, such as the metric \( g_{ij} \), the 3-form and 6-form potentials, \( C_{i1i2i3} \) and \( C_{i1...i6} \). It is a natural generalization of the generalized metric \( \mathcal{H}_{MN} \) in DFT.

In DFT, the section condition \( \eta^{IJ} \partial_I \partial_J = 0 \) reduces the doubled space to the physical subspace. The section condition in EFT (for \( n \leq 6 \)) also has a similar form \( \eta^{I; \hat{K}} \partial_I \partial_J = 0 \), where \( \eta^{I; \hat{K}} \) is known as the \( \eta \)-symbol and it has additional index \( \hat{K} \) transforming in another representation (see [184] for the explicit form of the \( \eta \)-symbol). When all of the fields depend only on the coordinates \( x^i \) of (8.9), we find

\[
\eta^{I; \hat{K}} \partial_I \partial_J = \eta^{i; \hat{K}} \partial_i \partial_j = 0 \quad (\because \eta^{ij; \hat{K}} = 0),
\]  

(8.11)

and the section condition is satisfied. This \( n \)-dimensional solution is called the M-theory section. Another solution, called the type IIB section, was found in [185], and in order to discuss the type IIB section, it is convenient to reparameterize the coordinates as

\[
(x^M) = (x^m, y_{m1m2m3}, y_{m1...m6}, y_{m1...m6, m}, \cdots) \quad (m = 1, \ldots, n - 1, \alpha = 1, 2),
\]  

(8.12)

where the dual coordinates are associated with the type IIB branes. If the fields depend only on the \( x^m \), the section condition is again satisfied because \( \eta^{mm; \hat{P}} = 0 \). Since we cannot introduce any more coordinate-dependence, the subspace spanned by \( x^m \) is also a maximally isotropic subspace, although it is \( (n - 1) \)-dimensional unlike the M-theory section. In this way, a single EFT can be reproduced from the two viewpoints, M-theory and type IIB theory.

\[\text{The explicit relation between } x^I \text{ and } x^M \text{ was determined in [186].}\]
One of the key relations in the PL T-duality is the self-duality relation,

\[ \eta_{AB} \hat{P}^B = \hat{H}_{AB} * \hat{P}^B, \quad \hat{P}(\sigma) = dl l^{-1}. \]  

(8.13)

This is a covariant rewriting of the string equations of motion, but a similar equation for the M2- or M5-brane theory has been discussed in [187,188] for the SL(5) and SO(5,5) case, and in [189] for higher exceptional groups. For the Mp-brane (p = 2,5), it has a similar form

\[ \eta_{IJ} \wedge \mathcal{P}^I = M_{IJ} * \mathcal{P}^J, \]  

(8.14)

where \( \eta_{IJ} \) is some \((p - 1)\)-form that contains \( dx^i \) and the field strengths of the worldvolume gauge fields. In the case of the flat torus, the equations of motion give \( d\mathcal{P}^I = 0 \) and we find the on-shell expression \( \mathcal{P}^I = dx^I \). On the other hand, by requiring a certain “dualizability condition” on \( M_{IJ} \) appropriately, the equations of motion may lead to \( \mathcal{P} = dl l^{-1} \), where \( l \) is an element of a certain large group \( \mathcal{E} \) with dimension \( D \). The corresponding algebra \( \mathfrak{e} \) will be endowed with a bilinear form, corresponding to the \( \eta \)-symbol. Then, the U-dual version of the PL T-plurality may be the equivalence between sigma models with \( n \)- or \( (n - 1) \)-dimensional target spaces that have an isometry algebra \([T_a, T_b] = f_{abc} T_c\) satisfying \( \eta^{ab} A^c = 0 \). The identification of the detailed structure of the group \( \mathcal{E} \) and the systematic construction of the twist matrix \( U \), whose flux gives the structure constant of \( \mathfrak{e} \), are interesting future directions.

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A Conventions

The symmerization and antisymmeterization are normalized as
\[
A_{(m_1 \cdots m_n)} = \frac{1}{n!} (A_{m_1 \cdots m_n} + \cdots), \quad A_{[m_1 \cdots m_n]} = \frac{1}{n!} (A_{m_1 \cdots m_n} \pm \cdots). \tag{A.1}
\]

For conventions of differential forms are as follows both for the spacetime and the worldsheet:
\[
(*q)_{m_1 \cdots m_{p+1-q}} = \frac{1}{q!} \varepsilon^{m_{1-n} q_{m_1 \cdots m_{p+1-q}}} \alpha_{n_1 \cdots n_q}, \quad d^D x = dx^1 \wedge \cdots \wedge dx^D, \\
* (dx^{m_1} \wedge \cdots \wedge dx^{m_q}) = \frac{1}{(p+1-q)!} \varepsilon^{m_1 \cdots m_q} dx^{m_1} \wedge \cdots \wedge dx^{n_{p+1-q}}, \tag{A.2}
\]
\[
(\iota \alpha_n) = \frac{1}{(n-1)!} \mu^n \alpha_{nm_1 \cdots m_{n-1}} dx^{m_1} \wedge \cdots \wedge dx^{m_{n-1}}.
\]

The epsilon tensors on the spacetime and the worldsheet are defined as follows:
\[
\varepsilon^{01} = \frac{1}{\sqrt{|\gamma|}}, \quad \varepsilon_{01} = -\sqrt{|\gamma|}, \quad \varepsilon^{1\cdots D} = -\frac{1}{\sqrt{|g|}}, \quad \varepsilon_{1\cdots D} = \sqrt{|g|}. \tag{A.3}
\]

For the R–R fields, we have the R–R potential in the A-basis \( A_{m_1 \cdots m_p} \) and the C-basis \( C_{m_1 \cdots m_p} \) \cite{159}. In terms of the polyform,
\[
A \equiv \sum_p \frac{1}{p!} A_{m_1 \cdots m_p} dx^{m_1} \wedge \cdots \wedge dx^{m_p}, \quad C \equiv \sum_p \frac{1}{p!} C_{m_1 \cdots m_p} dx^{m_1} \wedge \cdots \wedge dx^{m_p}, \tag{A.4}
\]
they are related as
\[
A = e^{B_2 \wedge} C, \quad C = e^{-B_2 \wedge} A. \tag{A.5}
\]

Their field strengths are defined as
\[
F = dA, \quad G = dC + H_3 \wedge C, \tag{A.6}
\]
and they are also related as
\[
F = e^{B_2 \wedge} G, \quad G = e^{-B_2 \wedge} F. \tag{A.7}
\]

In this paper, for simplicity we call the field strength \( F \) the Page form. In our convention, the \( G \) satisfies the self-duality relation
\[
*G_p = (-1)^{\frac{(p+1)}{2}} + 1 G_{10-p}, \quad G_p = (-1)^{\frac{p-1}{2}} * G_{10-p}. \tag{A.8}
\]
In the presence of the Killing vector \( I^m \) in the GSE, which satisfies
\[
\mathcal{L}_1 g_{mn} = \mathcal{L}_1 B_2 = \mathcal{L}_1 \Phi = \mathcal{L}_1 F = \mathcal{L}_1 G = 0 ,
\]  
the relations (A.6) are modified as
\[
F = dA - \iota I A , \quad G = dC + H_3 \wedge C - \iota I B_2 \wedge C - \iota I C .
\]  
and the Bianchi identities, which are equivalent to the equations of motion under (A.8), become
\[
dF - \iota I F = 0 , \quad dG + H_3 \wedge G - \iota I B_2 \wedge G - \iota I G = 0 .
\]

The GSE for the fields in the NS–NS sector can be summarized as
\[
R + 4 D^m \partial_m \Phi - 4 |\partial \Phi|^2 - \frac{1}{2} |H_3|^2 - 4 \left( I^m I_m + U^m U_m + 2 U^m \partial_m \Phi - D_n U^m \right) = 0 ,
\]  
\[
R_{mn} - \frac{1}{4} H_{mpq} H_{n}^{pq} + 2 D_m \partial_n \Phi + D_m U_n + D_n U_m = T_{mn} ,
\]  
\[
- \frac{1}{2} D^k H_{k mn} + \partial_k \Phi H^k_{mn} + U^k H_{k mn} + D_m I_n - D_n I_m = K_{mn} ,
\]
where \( U_1 \equiv U_m \ dx^m \) is defined as \( U_1 \equiv \iota I B_2 \), and \( T_{mn} \) and \( K_{mn} \) are
\[
T_{mn} \equiv \frac{e^{2 \Phi}}{4} \sum_p \left[ \frac{1}{(p-1)!} G_m^{q_1 \cdots q_{p-1}} G_n^{q_1 \cdots q_{p-1}} - \frac{1}{2} g_{mn} |G_p|^2 \right] ,
\]  
\[
K_{mn} \equiv \frac{e^{2 \Phi}}{4} \sum_p \frac{1}{(p-2)!} G_m^{q_1 \cdots q_{p-2}} G_n^{q_1 \cdots q_{p-2}} .
\]

In the presence of the Killing vector \( (I^m) = (I^i, I^z) \), if we perform a formal \( T \)-duality along the \( x^z \)-direction, the supergravity fields are transformed as follows (148):
\[
g'_{ij} = g_{ij} - \frac{g_{iz} g_{jz} - B_{iz} B_{jz}}{g_{zz}} , \quad g'_{iz} = g_{iz} , \quad g'_{zz} = \frac{1}{g_{zz}} ,
\]
\[
B'_{ij} = B_{ij} - \frac{g_{iz} g_{jz} - g_{iz} B_{jz}}{g_{zz}} , \quad B'_{iz} = g_{iz} , \quad B'_{zz} = \frac{1}{g_{zz}} ,
\]
\[
\Phi' = \Phi + \frac{1}{4} \ln \left( \frac{\det(g'_{mn})}{\det(g_{mn})} \right) + I^z z , \quad I^t = I^i , \quad I^{t'} = 0 ,
\]  
\[
A'_{i_1 \cdots i_{p-1} z} = e^{-I^z z} A_{i_1 \cdots i_{p-1} z} , \quad A'_{i_1 \cdots i_p z} = e^{-I^z z} [ A_{i_1 \cdots i_{p-2} z} ] ,
\]
\[
C'_{i_1 \cdots i_{p-1} z} = e^{-I^z z} \left( C_{i_1 \cdots i_{p-1} z} - (p-1) \frac{C_{i_1 \cdots i_{p-2} z} |g_{i_{p-1} z}|}{g_{zz}} \right) ,
\]
\[
C'_{i_1 \cdots i_p z} = e^{-I^z z} \left( C_{i_1 \cdots i_{p-1} z} + p C_{i_1 \cdots i_{p-1} z} B_{i_p z} + p (p-1) \frac{C_{i_1 \cdots i_{p-2} z} B_{i_{p-1} z} |g_{i_p z}|}{g_{zz}} \right) .
\]
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