Strict Gr-categories and applications on classification of extensions of groups of the type of a crossed module

Nguyen Tien Quang
Hanoi National University of Education, Department of Mathematics,
E-mail: cn.nguyenquang@gmail.com

Pham Thi Cuc
Hongduc University, Natural Science Department,
E-mail: cucphamhd@gmail.com

Nguyen Thu Thuy
Academy of Finance, Science Faculty,
E-mail: nttthuy11@gmail.com

Abstract
In this paper we state some applications of Gr-category theory on the classification of crossed modules and on the classification of extensions of groups of the type of a crossed module.

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1 Introduction

Theory of Gr-categories, or 2-groups, and its generality are getting more and more applications. The relationship among Gr-categories, cohomology of groups, and extensions of groups is stated in [9]. The results on group extensions of the type of a crossed module are also presented by cohomology of groups in [2, 3, 4]. These motivate our studies on the representation of the concepts related to crossed modules by Gr-categories, and then one can apply the results of Gr-categories for crossed modules.

A Gr-category [10] is a monoidal category in which every morphism is invertible and every object has a weak inverse. (Here, a weak inverse of an object $x$ is an object $y$ such that $x \otimes y$ and $y \otimes x$ are both isomorphic to the unit object.)

A strict Gr-category is a strict monoidal category in which every morphism is invertible and every object has a strict inverse (so that $x \otimes y$ and $y \otimes x$ are actually equal to the unit object).

Strict Gr-categories can be identified with crossed modules, thus Gr-categories in general can be seen as a weakening of crossed modules.

A strict Gr-category is a group object in Cat. R. Brown and C. Spencer [5] studied group objects in the category of groupoids under the name $G$-groupoid, or group-groupoid by R. Brown and O. Mucuk in [3]. In [5] the
authors stated the construction of a crossed module from a $G$-groupoid and that of a $G$-groupoid from a crossed module. They are repeated by Forrester - Barker [7] as a $G$-groupoid is replaced by a strict Gr-category (see also J. Baez and A. Lauda [1]) thanks to the notion of internal categories.

In present paper, we show directly the above constructions based on strict Gr-categories, and prove that the strong homotopy category $HoGrstr$ of strict Gr-categories and single Gr-functors is isomorphic to the category $Cross$ of crossed modules (Theorem 6).

One obtains the first applications of the Gr-category theory as follows. In the classical method, the factor set of a homomorphism $h : G \to H$ is only defined if $h$ is a surjection. For the general case, we can overcome this limitation by building a Gr-functor between two appropriate Gr-categories. Then the results of Gr-category theory become the useful device to study crossed modules.

For any crossed module $B \to D$ and a group homomorphism $\psi : Q \to \text{Coker}d$ of an extension problem of type $B \to D$ (see [3, 4]), we construct a strict Gr-category $P$. Then, for a Gr-functor $F : \text{Dis}Q \to P$ we can determine an associated extension of the type of a crossed module. Therefore, there is a bijection (Theorem 9)

$$\text{Hom}_{\text{Dis}P}(Q,F) \leftrightarrow \text{Ext}_{B \to D}(Q,B,\psi),$$

where $\text{Ext}_{B \to D}(Q,B,\psi)$ is the set of equivalence classes of extensions of $B$ by $Q$ of type $B \to D$ inducing $\psi$. This result contains The classification Theorem of R. Brown and O. Mucuk (Theorem 5.2 [3]).

2 Preliminaries

For convenience, we recall here some well-known results on Gr-categories and Gr-functors (see [9]).

We often denote by $(G, \otimes, I, a, l, r)$ a Gr-category. If $(F, \tilde{F}, F_\ast)$ is a monoidal functor between Gr-categories, it is called a Gr-functor. Then the isomorphism $F_\ast : I' \to FI$ can be deduced from $F$ and $\tilde{F}$. Hereafter, we refer to $(F, \tilde{F})$ as a Gr-functor.

Two Gr-functors $(F, \tilde{F})$ and $(F', \tilde{F}')$ from $G$ to $G'$ are homotopic if there is a natural monoidal equivalence, or a homotopy $\alpha : (F, \tilde{F}, F_\ast) \to (F', \tilde{F}', F'_\ast)$ which is a natural isomorphism such that

$$F'_\ast = \alpha_I \circ F_\ast.$$

A Gr-category is equivalent to one of type $(\Pi, A)$, which can be described as follows. The set $\pi_0G$ of isomorphism classes of the objects in $G$ is a group with the operation induced by the tensor product in $G$, and the set $\pi_1G$ of automorphisms of the unit object $I$ is an abelian group with the operation,
denoted by $+$, induced by the composition of morphisms. Moreover, $\pi_1 G$ is a $\pi_0 G$-module under the action

$$su = \gamma_X^{-1} \delta_X(u), \quad X \in s, \ s \in \pi_0 G, \ u \in \pi_1 G,$$

where $\delta_X, \gamma_X$ are defined by the following commutative diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{\gamma_X(u)} & X \\
\downarrow{I_X} & & \downarrow{I_X} \\
I \otimes X & \xrightarrow{u \otimes \text{id}} & I \otimes X
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{\delta_X(u)} & X \\
\downarrow{r_X} & & \downarrow{r_X} \\
X \otimes I & \xrightarrow{\text{id} \otimes u} & X \otimes I
\end{array}
$$

The reduced Gr-category $S_G$ of $G$ is a category whose objects are the elements of $\pi_0 G$ and whose morphisms are automorphisms $(s,u) : s \rightarrow s$, where $s \in \pi_0 G, \ u \in \pi_1 G$. The composition of two morphisms is induced by the addition in $\pi_1 G$

$$(s,u).(s,v) = (s,u + v).$$

The category $S_G$ is equivalent to $G$ by canonical equivalences constructed as follows. For each $s = [X] \in \pi_0 G$, choose a representative $X_s \in G$ such that $X_1 = I$, and for each $X \in s$ choose an isomorphism $i_X : X_s \rightarrow X$ such that $i_{X_1} = \text{id}$. The family $(X_s, i_X)$ is called a stick of the Gr-category $G$ whenever

$$i_{I \otimes X_s} = I_{X_s}, \ i_{X_s \otimes I} = r_{X_s}.$$

For given stick $(X_s, i_X)$, we obtain two functors

$$
\begin{aligned}
G : G & \rightarrow S_G \\
G(X) & = [X] = s \\
G(X \xrightarrow{f} Y) & = (s, \gamma_X^{-1} (i_Y^{-1} f i_X))
\end{aligned}
\quad
\begin{aligned}
H : S_G & \rightarrow G \\
H(s) & = X_s \\
H(s,u) & = \gamma_X(u)
\end{aligned}
$$

Two functors $G$ and $H$ are categorical equivalences by natural transformations

$$\alpha = (i_X) : HG \cong \text{id}_{S_G}; \quad \beta = \text{id} : GH \cong \text{id}_{S_G}.$$  

They are canonical equivalences.

With the structure transport by the quadruple $(G, H, \alpha, \beta)$, $S_G$ becomes a Gr-category together with the following operations

$$s \otimes t = s.t, \quad s, t \in \pi_0 G,$$

$$(s,u) \otimes (t,v) = (st, u + sv), \quad u, v \in \pi_1 G.$$

The unit constraints of the Gr-category $S_G$ are therefore strict, and its associativity constraint is $a_{s,r,t} = (srt, k(s,r,t))$, where

$$k \in Z^3(\pi_0 G, \pi_1 G).$$
Moreover, the equivalences $G$ and $H$ become Gr-equivalences together with natural isomorphisms

\[ G_{X,Y} = G(i_X \otimes i_Y), \quad H_{s,t} = i_{X_s \otimes X_t}^{-1} : X_sX_t \to X_{st}. \]  

(1)

The Gr-category $S_{\mathbb{G}}$ is called a reduction of the Gr-category $\mathbb{G}$. $S_{\mathbb{G}}$ is said to be of type $(\Pi, A, k)$, or simply type $(\Pi, A)$ if $\pi_0 \mathbb{G}, \pi_1 \mathbb{G}$ are replaced with the group $\Pi$ and the $\Pi$-module $A$, respectively.

Let $S = (\Pi, A, k)$, $S' = (\Pi', A', k')$ be Gr-categories. A functor $F : S \to S'$ is of type $(\varphi, f)$ if

\[ F(x) = \varphi(x), \quad F(x, u) = (\varphi(x), f(u)), \]

where $\varphi : \Pi \to \Pi'$, $f : A \to A'$ are group homomorphisms satisfying $f(xa) = \varphi(x)f(a)$ for $x \in \Pi, a \in A$.

**Lemma 1** ([9]). Each Gr-functor $(F, \widetilde{F}) : G \to G'$ induces a Gr-functor $S_F : S_{\mathbb{G}} \to S_{\mathbb{G}'}$ of type $(\varphi, f)$, where $\varphi = F_0, f = F_1$ which are determined by

\[
F_0 : \pi_0 \mathbb{G} \to \pi_0 \mathbb{G}', \quad [X] \mapsto [FX],
\]

\[
F_1 : \pi_1 \mathbb{G} \to \pi_1 \mathbb{G}', \quad u \mapsto \gamma_{F_1}^{-1}(Fu)
\]

Moreover,

\[ S_F = G'FH, \]

where $H, G'$ are canonical equivalences.

Note that if $\Pi'$-module $A'$ is considered as a $\Pi$-module under the action $xa' = \varphi(x)a'$, then $f : A \to A'$ is a homomorphism of $\Pi$-modules. The compatibility of $(F, \widetilde{F})$ with the associativity constraint gives

\[ \varphi^*k' - f_*k = \partial(g_F), \]

where

\[
(f_*k)(x, y, z) = f(k(x, y, z)),
\]

\[
(\varphi^*k')(x, y, z) = k'(\varphi x, \varphi y, \varphi z),
\]

and $g_F : \Pi^2 \to A'$ is a function associated to $\widetilde{F}$.

If $F : S \to S'$ is a functor of type $(\varphi, f)$, then the function

\[ \xi = \varphi^*k' - f_*k \]

(2)

is called an obstruction of the functor $F$. We have

**Proposition 2** ([9]). The functor $F : S \to S'$ of type $(\varphi, f)$ induces a Gr-functor if and only if its obstruction $\xi$ vanishes in $H^3(\Pi, A')$. Then, there is a bijection

\[ \text{Hom}_{(\varphi, f)}[S, S'] \leftrightarrow H^2(\Pi, A'), \]

where $\text{Hom}_{(\varphi, f)}[S, S']$ is the set of homotopy classes of the Gr-functors of type $(\varphi, f)$ from $S$ to $S'$. 

4
3 Gr-categories associated to a crossed module

**Definition.** A crossed module is a quadruple \((B, D, d, \theta)\) where \(d : B \rightarrow D, \ \theta : D \rightarrow \text{Aut}B\) are group homomorphisms making the following diagram commute

\[
\begin{array}{ccc}
B & \xrightarrow{d} & D \\
\downarrow{\mu} & & \downarrow{\theta} \\
\text{Aut}B
\end{array}
\]

and satisfying the relation

\[
d(\theta_x(b)) = \mu_x(d(b)), \ x \in D, b \in B, \tag{3}
\]

where \(\mu_x\) is an inner automorphism given by conjugation of \(x\).

In present paper, the crossed module \((B, D, d, \theta)\) is sometimes denoted \(B \xrightarrow{d} D\), or \(B \rightarrow D\).

For convenience, we denote by the addition for the operation in \(B\) and by the multiplication for that in \(D\).

The following properties follow from the definition of a crossed module.

**Proposition 3.** Let \((B, D, d, \theta)\) be a crossed module.

i) \(\text{Kerd} \subset Z(B)\).

ii) \(\text{Imd}\) is a normal subgroup in \(D\).

iii) The homomorphism \(\theta\) induces a homomorphism \(\varphi : D \rightarrow \text{Aut(Kerd)}\) given by

\[
\varphi_x = \theta_x|_{\text{Kerd}}.
\]

iv) \(\text{Kerd}\) is a left Cokerd-module under the action

\[
sa = \varphi_x(a), \ a \in \text{Kerd}, \ x \in s \in \text{Cokerd}.
\]

For any crossed module \((B, D, d, \theta)\) we can construct a strict Gr-category \(P_{B \rightarrow D} = \mathbb{P}\) called the Gr-category associated to the crossed module \(B \rightarrow D\), as follows.

\[
\text{Ob} (\mathbb{P}) = D, \text{Hom}(x, y) = \{b \in B/x = d(b)y\},
\]

where \(x, y\) are objects of \(\mathbb{P}\). The composition of two morphisms is given by

\[
(x \xrightarrow{b} y \xrightarrow{c} z) = (x \xrightarrow{b+c} z).
\]

The tensor operation on objects is given by the multiplication in the group \(D\), and for two morphisms \((x \xrightarrow{b} y), (x' \xrightarrow{b'} y')\) then

\[
(x \xrightarrow{b} y) \otimes (x' \xrightarrow{b'} y') = (xx' \xrightarrow{b+b'+\theta_{x'b'}b'} yy'). \tag{4}
\]

By the definition of a crossed module, we can easily check that \(\mathbb{P}\) is a Gr-category with the identity constraints.
Conversely, for a strict Gr-category \((\mathbb{P}, \otimes)\) we define an associated crossed module \(C_{\mathbb{P}} = (B, D, d, \theta)\) as follows. Set

\[ D = \text{Ob}(\mathbb{P}), \]
\[ B = \{ x \xrightarrow{b} 1 / x \in D \}. \]

The operations of \(D\) and of \(B\) are given by

\[ xy = x \otimes y, \quad b + c = b \otimes c, \]

respectively. Then \(D\) becomes a group in which the unit is 1, the inverse of \(x\) is \(x^{-1}\) (\(x \otimes x^{-1} = 1\)). \(B\) is a group in which the unit is \((1 \xrightarrow{id} 1)\) and the inverse of \((x \xrightarrow{b} 1)\) is the morphism \((x^{-1} \xrightarrow{\overline{b}} 1)\) (\(b \otimes \overline{b} = \text{id}_1\)).

The homomorphisms \(d : B \rightarrow D\) and \(\theta : D \rightarrow \text{Aut}B\) are given by

\[ d(x \xrightarrow{b} 1) = x, \]
\[ \theta_y(x \xrightarrow{b} 1) = (xy^{-1} \xrightarrow{id_y + b + id_y^{-1}} 1), \]

respectively. It is easy to see that \((B, D, d, \theta)\) is a crossed module.

**Definition.** A morphism \((f_1, f_0) : (B, D, d, \theta) \rightarrow (B', D', d', \theta')\) of crossed modules consists of homomorphisms of groups \(f_1 : B \rightarrow B', f_0 : D \rightarrow D'\) such that the following diagram commutes

\[ \begin{array}{ccc}
B & \xrightarrow{d} & D \\
\downarrow{f_1} & & \downarrow{f_0} \\
B' & \xrightarrow{d'} & D'
\end{array} \]

and \(f_1\) is an operator homomorphism, i.e. for all \(x \in D, b \in B\),

\[ f_1(\theta_x b) = \theta'_{f_0(x)} f_1(b). \]

**Proposition 4.** Let \((f_1, f_0) : (B, D, d, \theta) \rightarrow (B', D', d', \theta')\) be a morphism of crossed modules. Let \(\mathbb{P}\) and \(\mathbb{P}'\) be two Gr-categories associated to crossed modules \((B, D, d, \theta)\) and \((B', D', d', \theta')\), respectively.

i) There exists a functor \(F : \mathbb{P} \rightarrow \mathbb{P}'\) defined by \(F(x) = f_0(x), \quad F(b) = f_1(b), \quad \) for \(x \in \text{Ob}(\mathbb{P}), b \in \text{Mor}(\mathbb{P}).\)

ii) A natural isomorphism \(\tilde{F}_{x,y} : F(x)F(y) \rightarrow F(xy)\) together with \(F\) is a Gr-functor if and only if \(\tilde{F}_{x,y}\) is a constant \(\tilde{c} \in \text{Ker}d'\) and

\[ \theta'_{f_0(x)}(\tilde{c}) = \tilde{c}. \]

We say that \(F\) is a Gr-functor of form \((f_1, f_0)\).
The natural isomorphisms

\[ \tilde{F}_{x,y} : F(x)F(y) \to F(xy) \]

such that \( F = (F, \tilde{F}) \) becomes a Gr-functor. We first see that if \( (x \xrightarrow{b} y) \) and \( (x' \xrightarrow{b'} y') \) are morphisms in \( \mathbb{P} \), then

\[
F(b \otimes b') = F(xx' \xrightarrow{b+θy' Butler} yy') = (f_0(x)x') \xrightarrow{f_1(b)+θf_0(y') Butler} f_0(y')).
\]

Besides, \( F(x)F(y) = F(xy) \) and \( F(x)F(y) = d'(\tilde{F}_{x,y})F(xy) \) lead to \( d'(\tilde{F}_{x,y}) = 1 \). Therefore,

\[ \tilde{F}_{x,y} \in \text{Ker} d' \subset Z(B'). \]

The natural isomorphisms \( \tilde{F}_{x,y} \) (if they exist) must satisfy the diagram

\[
\begin{array}{ccc}
F(x)F(x') & \xrightarrow{\tilde{F}_{x,x'}} & F(x'x') \\
F(b \otimes F(y')) & \xrightarrow{F(b \otimes b')} & F(b \otimes b') \\
F(y)F(y') & \xrightarrow{\tilde{F}_{y,y'}} & F(yy')
\end{array}
\]

From (8) and \( \tilde{F}_{x,y} \in Z(B') \), this commutative diagram implies \( \tilde{F}_{x,x'} = \tilde{F}_{y,y'} \). Write \( \tilde{F}_{x,y} = \tilde{c} \). The compatibility of \( (F, \tilde{F}) \) with the associativity constraints implies \( θ'_{F(x)}(\tilde{c}) = \tilde{c}, \) for \( x \in D. \)

A Gr-functor \((F, \tilde{F}) : \mathbb{P} \to \mathbb{P}'\) is single if \( \tilde{F}_{x,y} = \tilde{c} \in \text{Aut}(1') \) for all \( x, y \in \text{Ob}(\mathbb{P}). \)

We state the converse proposition of Proposition 4

**Proposition 5.** Let \( \mathbb{P}, \mathbb{P}' \) be corresponding Gr-categories associated to the crossed modules \((B, D, d, θ), (B', D', d', θ')\), and \( F : \mathbb{P} \to \mathbb{P}' \) be a single Gr-functor. Then, \( F \) induces a morphism of crossed modules \((f_1, f_0) : (B \to D) \to (B' \to D')\), where

\[
f_1(b) = F(b), \quad f_0(x) = F(x),
\]

for all \( b \in B, x \in D. \)

7
Proof. Any \( b \in B \) can be considered as a morphism \((db \to 1)\) in \( \mathbb{P} \), hence \((F(db) \overset{F(b)}{\to} 1')\) is a morphism in \( \mathbb{P}' \).

For each \( x,y \in D \), since \((FxFy \overset{\tilde{F}_{x,y}}{\to} F(xy))\) is a morphism in \( \mathbb{P}' \), and since \( \tilde{F}_{x,y} \) is in \( \text{Aut}(1') = \text{Ker}d' \), we obtain
\[
FxFy = d'(\tilde{F}_{x,y})F(xy) = F(xy)
\]
or \( f_0(xy) = f_0(x)f_0(y) \).

For any morphism \((x \to y)\) in \( \mathbb{P} \), we have \( x = d(b)y \). It follows that
\[
f_0(x) = f_0(d(b)y) = f_0(d(b))f_0(y).
\]
Besides, \((f_0(x) \overset{f_1(b)}{\to} f_0(y))\) is a morphism in \( \mathbb{P}' \), so
\[
f_0(x) = d'(f_1(b))f_0(y).
\]
Therefore, \( f_0(d(b)) = d'(f_1(b)) \), for all \( b \in B \). That means the diagram \([5]\) commutes.

Now, since \( \tilde{F}_{x,y} \in \text{Ker}d' \), the commutative \([8]\) implies \([5]\). According to the determination of \( F(b) \otimes F(b') \) and \( F(b \otimes b') \) in the proof of Proposition \([4]\) we have
\[
f_1(b) + f_1(\theta_y b') = f_1(b) + \theta'_{f_0(y)}f_1(b').
\]
Hence \( f_1(\theta_y b') = \theta'_{f_0(y)}f_1(b') \), for all \( b \in B, y \in D \), or \([6]\) holds.

We write \( \text{Grstr} \) for the category of strict Gr-categories and simple Gr-functors. Two Gr-functors \((F, \tilde{F})\) and \((G, \tilde{G})\) are strong homotopic if they are homotopic and \( F = G \). We can define the strong homotopy category \( \text{HoGrstr} \) of \( \text{Grstr} \) to be the quotient category with the same objects, but morphisms are homotopy classes of monoidal functors. We write \( \text{Hom}_{\text{Grstr}}[\mathbb{P}, \mathbb{P}'] \) for the homsets of the homotopy category, that is,
\[
\text{Hom}_{\text{Grstr}}[\mathbb{P}, \mathbb{P}'] = \frac{\text{Hom}_{\text{Grstr}}(\mathbb{P}, \mathbb{P}')}{}{\text{strong homotopies}}
\]
Let \( \text{Cross} \) for the category of crossed modules, we obtain the similar result to Theorem 1 \([5]\)

**Theorem 6** (Classification Theorem). *There exists an isomorphism*

\[
\Phi : \text{Cross} \to \text{HoGrstr}
\]
\[
(B \to D) \mapsto \mathbb{P}_{B \to D}
\]
\[
(f_1, f_0) \mapsto [F]
\]
where \( F(x) = f_0(x), F(b) = f_1(b) \), for \( x \in \text{Ob}(\mathbb{P}), b \in \text{Mor}(\mathbb{P}) \).
Proof. Let \( P, P' \) be corresponding Gr-categories associated to the crossed modules \( B \rightarrow D, B' \rightarrow D' \), and \( (F, \tilde{F}), (G, \tilde{G}) : P \rightarrow P' \) be two Gr-functors of the form \( (f_1, f_0) \). Then \( F = G \) and by Proposition 4 \( \tilde{F} = c \) and \( \tilde{G} = c' \) are constants satisfying the rule (7)

\[
\theta'_{f_0(x)}(c) = c, \quad \theta'_{f_0(x)}(c') = c'.
\]

We show that \( a = c - c' \) is a homotopy of \((F, \tilde{F})\) and \((G, \tilde{G})\), i.e., \([F] = [G]\) and hence, \( \Phi \) is a map on the homsets

\[
\Phi : \text{Hom}_{\text{Cross}}(B \rightarrow D, B' \rightarrow D') \rightarrow \text{Hom}_{\text{Grstr}}[P_B \rightarrow D, P'_B \rightarrow D'].
\]

Indeed, the naturality of \( a \) is trivial. The commutativity of the diagram

\[
\begin{array}{ccc}
F(x)F(y) & \xrightarrow{c} & F(xy) \\
\downarrow{a \otimes a} & & \downarrow{a} \\
G(x)G(y) & \xrightarrow{c'} & G(xy)
\end{array}
\]

means that

\[
c + a = a + \theta'_{G(x)}(a) + c'.
\]

Since \( c, c' \) satisfy the rule (7), so does \( a \). Now, since \( c, c' \in \text{Ker}d' \), so the above relation holds.

Since the homotopy between Gr-functors is strong, \( \Phi \) is an injection. By Proposition 5 every single Gr-functor \( F : P_B \rightarrow D \rightarrow P'_B \rightarrow D' \) determines a morphism \((f_1, f_0)\) of crossed modules, and clearly \( \Phi(f_1, f_0) = [F] \). Thus, \( \Phi \) is a surjection on morphisms.

By the construction of the Gr-category associated to a crossed module, \( \Phi \) is a bijection on objects. Thus, \( \Phi \) is an isomorphism. \( \square \)

4 Extensions of groups of the type of a crossed module

We now recall the notion of an extension of groups of the type of a crossed module due to Taylor [11] and Dedeker [6] (see also [4]).

Note that if \( B \) is a normal subgroup in \( D \), then the system \((B, D, d, \theta)\) is a crossed module in which \( d : B \rightarrow D \) is an inclusion, \( \theta : D \rightarrow \text{Aut}B \) given by conjugation.

Definition. Let \( d : B \rightarrow D \) be a crossed module. A group extension of \( B \) by \( Q \), of type \( d : B \rightarrow D \) is a diagram of homomorphisms of groups

\[
\mathcal{E} : \begin{array}{c}
0 \longrightarrow B \xrightarrow{j} E \xrightarrow{p} Q \longrightarrow 1, \\
\downarrow & & \downarrow & & \\
B \xrightarrow{d} & & D \end{array}
\]
where the top row is exact, and the family \((B, E, j, \theta')\) is a crossed module where \(\theta'\) is given by conjugation, and \((id, \varepsilon)\) is a morphism of crossed modules.

Two extensions of \(B\) by \(Q\) of type \(B \xrightarrow{d} D\) are said to be equivalent if there is a morphism of exact sequences

\[
\begin{array}{ccccccccc}
0 & \rightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q & \rightarrow & 1, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & B & \xrightarrow{j'} & E' & \xrightarrow{p'} & Q & \rightarrow & 1, \\
\end{array}
\begin{array}{cccc}
E & \xrightarrow{\varepsilon} & D \\
\downarrow & & \downarrow & & \\
E' & \xrightarrow{\varepsilon'} & D \\
\end{array}
\]

such that \(\varepsilon'\eta = \varepsilon\). Obviously, \(\eta\) is an isomorphism.

In the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q & \rightarrow & 1, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & B & \xrightarrow{d} & D & \xrightarrow{q} & \text{Coker} d & \rightarrow & q \\
\end{array}
\begin{array}{cccc}
E & \xrightarrow{\psi} & \text{Coker} d \\
\downarrow & & \downarrow & & \\
E' & \xrightarrow{\varepsilon'} & \text{Coker} d \\
\end{array}
\]

where \(q\) is a canonical homomorphism, since the top row is exact and \(q \circ \varepsilon \circ j = q \circ d = 0\), there is a homomorphism \(\psi : Q \rightarrow \text{Coker} d\) such that the right hand side square commutes. Moreover, \(\psi\) depends only on the equivalence class of the extension. Our objective is to study the set

\[
\text{Ext}_{B \rightarrow D}(Q, B, \psi)
\]

of equivalence classes of extensions of \(B\) by \(Q\) of type \(B \rightarrow D\) inducing \(\psi : Q \rightarrow \text{Coker} d\). Such group extensions have been classified by R. Brown and O. Mucuk (Theorem 5.2 [3]).

In the present paper, we use the obstruction theory of Gr-functors to prove Theorem 5.2 [3]. Further, the second assertion of this theorem can be seen as a consequence of Schreier Theory (Theorem [4] by Gr-functors between strict Gr-categories \(\mathbb{P}_{B \rightarrow D}\) and \(\text{Dis} Q\), where \(\text{Dis} Q\) is a Gr-category of type \((Q, 0, 0)\).

Let \(\mathbb{P} = \mathbb{P}_{B \rightarrow D}\) be the Gr-category associated to crossed module \(B \xrightarrow{d} D\). Since \(\pi_0\mathbb{P} = \text{Coker} d\), \(\pi_1\mathbb{P} = \text{Kerd}\), then

\[
S_{\mathbb{P}} = (\text{Coker} d, \text{Kerd}, k), \overline{k} \in H^3(\text{Coker} d, \text{Kerd}).
\]

The homomorphism \(\psi : Q \rightarrow \text{Coker} d\) induces an obstruction

\[
\psi^* k \in Z^3(Q, \text{Kerd}).
\]

Under this notion of obstruction, we state and prove the following theorem.
Theorem 7. Let \((B, D, d, \theta)\) be a crossed module and \(\psi : Q \to \text{Coker} d\) be a group homomorphism. Then, the vanishing of \(\psi^*k\) in \(H^3_\psi(Q, \text{Ker} d)\) is necessary and sufficient for there to exist an extension of \(B\) by \(Q\) of type \(B \to D\) inducing \(\psi\). Further, if \(\psi^*k\) vanishes, then the equivalence classes of such extensions are bijective with \(H^2_\psi(Q, \text{Ker} d)\).

The first assertion of Theorem 7 follows from the following lemma.

Lemma 8. For each Gr-functor \((F, \tilde{F}) : \text{Dis} Q \to \mathbb{P}\), there exists an extension \(E_F\) of \(B\) by \(Q\) of type \(B \to D\) inducing \(\psi : Q \to \text{Coker} d\). Such extension \(E_F\) is called an extension associated to the Gr-functor \(F\).

Proof. By Lemma 1, \((F, \tilde{F})\) induces a Gr-functor \(S_F : \text{Dis} Q \to \mathbb{P}\) of type \((\psi, 0)\). Let \((H, \tilde{H}) : \mathbb{P} \to \mathbb{P}\) be a canonical Gr-functor defined by the stick \((x_s, i_x)\). By (1), \(H(s) = x_s, H(s, b) = b, \tilde{H}_{s,r} = -i_x s \cdot x_r\).

Also by Lemma 1, \((F, \tilde{F})\) is homotopic to the composition \(\text{Dis} Q \xrightarrow{SF} \mathbb{P} \xrightarrow{H} \mathbb{P}\).

So we can choose \((F, \tilde{F}) = (H, \tilde{H}) \circ (S_F, \tilde{S_F})\). Then, by the determination of \(\tilde{HS_F}\),

\[ F(u) = x_s, \quad \tilde{F}_{u,v} = f(u, v) = h(u, v) - i_x s \cdot x_r \in B, \]

for \(u, v \in Q, s = \psi(u), r = \psi(v), h(u, v) = (\tilde{S_F})_{u,v}\).

By the compatibility of \((F, \tilde{F})\) with the strict constraints of \(\text{Dis} Q\) and \(\mathbb{P}\), \(f\) is a “normal” function satisfying

\[ \theta_{Fu}[f(v, t)] + f(u, vt) = f(u, v) + f(uv, t). \]

The function \(\varphi : Q \to \text{Aut} B\) given by

\[ \varphi(u) = \theta_{Fu} = \theta_{x_s} (s = \psi(u)) \]

satisfies the rule

\[ \varphi(u) \varphi(v) = \mu[f(u, v)] \varphi(uv). \]

Indeed, by (12) and \(h(u, v) \in \text{Ker} d\), we have

\[ \varphi(u) \varphi(v) = \theta_{x_s} \cdot \theta_{x_r} = \theta_{x_s x_r} = \theta_{d[\psi(u)v]} = \theta_{d(-i_x s x_r)} = \theta_{d(-i_x s x_r)} \theta_{x_r} = \mu(-i_x s x_r) \varphi(uv) = \mu[f(u, v)] \varphi(uv). \]
The pair \((\varphi, f)\) satisfies (13) and (14), so it is a factor set, and therefore one can define the crossed product \(E_0 = [B, \varphi, f, Q]\) (see [8]), i.e., \(E_0 = B \times Q\) with the operation
\[
(b, u) + (b', u') = (b + \varphi(u)b' + f(u, u'), uu').
\]
The unit is \((0, 1)\) and \(-(b, u) = (c, u^{-1})\), where \(\varphi(u)c = -b - f(u, u^{-1})\).

Then, we have an exact sequence
\[
E_F: 0 \to B \xrightarrow{j_0} E_0 \xrightarrow{p_0} Q \to 1,
\]
where
\[
j_0(b) = (b, 1); \quad p_0(b, u) = u, \quad b \in B, u \in Q.
\]
Clearly, \(j_0(B)\) is a normal subgroup in \(E_0\), so \(j_0: B \to E_0\) is a crossed module in which the action \(\theta': E_0 \to \text{Aut}B\) is given by conjugation.

We define a group homomorphism \(\varepsilon: E_0 \to D\) by
\[
\varepsilon(b, u) = d(b)x_{\psi(u)}, \quad (b, u) \in E_0,
\]
where \(x_{\psi(u)}\) is a representative of \(\psi(u)\) in \(D\). Then, \((id, \varepsilon)\) is a morphism of crossed modules. Indeed, \(\varepsilon \circ j_0 = d\). Further, for all \((b, u) \in E_0, c \in B\)
\[
\theta'(b, u)(c) = j_0^{-1}[\mu_{(b, u)}(c, 0)] = \mu_b c + \varphi(u)c,
\]
\[
\theta_{\varepsilon(b, u)}c = \theta_{d\varepsilon(b, u)}c = \mu_b c + \varphi(u)c.
\]
Hence, \(\theta'(b, u)(c) = \theta_{\varepsilon(b, u)}c\).

Therefore, we have an extension
\[
E_F: 0 \to B \xrightarrow{j_0} E_0 \xrightarrow{p_0} Q \xrightarrow{\psi} 1,
\]
where \(j_0(b) = (b, 1); \quad p_0(b, u) = u, \quad b \in B, u \in Q\).

For all \(u \in Q, q \varepsilon(0, u) = q(x_{\psi(u)}) = \psi(u), \text{ i.e., } E_F \text{ induces } \psi: Q \to \text{Coker}d\).

The proof of Theorem 7

Proof. Let’s recall that \(P\) is the Gr-category associated to the crossed module \(B \to D\). Then, its reduced Gr-category is \(S_P = (\text{Coker}d, \text{Ker}d, k)\), where \(k \in Z^3(\text{Coker}d, \text{Ker}d)\). The pair
\[
(\psi, 0): (Q, 0, 0) \to (\text{Coker}d, \text{Ker}d, 0)
\]
has \(-\psi^*k\) as an obstruction. By the assumption, \(\overline{\psi^*k} = 0\), hence by Proposition 2 it determines a Gr-functor \((\Psi, \Psi): \text{Dis}Q \to S_P\). Then the composition of \((\Psi, \Psi')\) and \((H, \tilde{H}): S_P \to P\) is a Gr-functor \((F, \tilde{F}): \text{Dis}Q \to P\), and by Lemma 8 we obtain an associated extension \(E_F\).
Conversely, suppose that there is an extension as in the diagram (10). Let \( \mathbb{P}' \) be the category associated to the crossed module \( B \to E \). By Proposition 4 there is a Gr-functor \( F : \mathbb{P}' \to \mathbb{P} \). Since the reduced Gr-category of \( \mathbb{P}' \) is \( \text{Dis} \mathbb{Q} \), by Lemma 1 \( F \) induces a Gr-functor of type \( (\psi, 0) \) from \( \text{Dis} \mathbb{Q} \) to \( (\text{Coker} d, \text{Ker} d, k) \). Now, by Proposition 2, the obstruction of the pair \( (\psi, 0) \) must vanish in \( H^3(Q, \text{Ker} d) \), i.e. \( \psi^* k = 0 \).

The final assertion of the theorem is obtained from Proposition 4.

**Theorem 9** (Schreier Theory for extensions of the type of a crossed module). There is a bijection

\[
\Omega : \text{Hom}_{(\psi, 0)[\text{Dis} \mathbb{Q}, \mathbb{P}]} \to \text{Ext}_{B \to D}(Q, B, \psi).
\]

**Proof.**

Step 1: Gr-functors \( (F, \tilde{F}) \) and \( (F', \tilde{F}') \) are homotopic if and only if the corresponding associated extensions \( \mathcal{E}_F, \mathcal{E}_{F'} \) are equivalent.

Let \( F, F' : \text{Dis} \mathbb{Q} \to \mathbb{P} \) be homotopic by a homotopy \( \alpha : F \to F' \). By Lemma 8, there exist the extensions \( \mathcal{E}_F, \mathcal{E}_{F'} \) associated to \( F \) and \( F' \), respectively. By the definition of Gr-morphisms, the following diagram commutes

\[
\begin{array}{ccc}
F_{uv} & \xrightarrow{\alpha_{uv}} & F'(uv) \\
\tilde{F}_{uv} & \xrightarrow{\alpha_{uv}} & \tilde{F}'_{uv} \\
F'_{uv} & \xrightarrow{\alpha_{uv}} & F'(uv)
\end{array}
\]

That means

\[
\tilde{F}_{uv} + \alpha_{uv} = \alpha_u \otimes \alpha_v + \tilde{F}'_{uv}
\]

(the sum in \( B \)). By relation (15),

\[
f(u, v) + \alpha_{uv} = \alpha_u + \theta_{F_{uv}}(\alpha_v) + f'(u, v),
\]

where \( f(u, v) = \tilde{F}_{u,v}, f'(u, v) = \tilde{F}'_{u,v} \). Now we set

\[
\alpha^* : \mathcal{E}_F \to \mathcal{E}_{F'}, \quad (b, u) \mapsto (b + \alpha_u, u).
\]

Then \( \alpha^* \) is a homomorphism thanks to the relation (15). Further, the following diagram commutes

\[
\begin{array}{ccccccc}
0 & \xrightarrow{j} & B & \xrightarrow{p} & E_F & \xrightarrow{\alpha^*} & Q & \xrightarrow{\epsilon} & 1, \\
& & \downarrow{\alpha^*} & & \downarrow{\alpha^*} & & \downarrow{\alpha^*} & & \\
0 & \xrightarrow{j'} & B & \xrightarrow{p'} & E_{F'} & \xrightarrow{\alpha^*} & Q & \xrightarrow{\epsilon'} & 1
\end{array}
\]

\[
E_F & \xrightarrow{\epsilon} & D \\
E_{F'} & \xrightarrow{\epsilon'} & D
\]

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It remains to show that $\varepsilon'\alpha^* = \varepsilon$.

Since $\alpha : F \to F'$ is a homotopy, and by (11), $Fu = x_{\psi(u)} = F'u$. Therefore $x_{\psi(u)} = d(\alpha_u)x_{\psi(u)}$, or $d(\alpha_u) = 1$. Then,

$$\varepsilon'\alpha^*(b,u) = \varepsilon'(b + \alpha_u, u) = d(b + \alpha_u)x_{\psi(u)} = d(b)d(\alpha_u)x_{\psi(u)} = d(b)x_{\psi(u)} = \varepsilon(b, u).$$

That means $\mathcal{E}_F$ and $\mathcal{E}_{F'}$ are equivalent.

Conversely, if $\alpha^* : E \to E'$ is an isomorphism then

$$\alpha^*(b,u) = (b + \alpha_u, u),$$

where $\alpha : Q \to B$ is a function satisfying $\alpha_1 = 0$. By retracing our steps, $\alpha$ is a homotopy of $F$ and $F'$.

**Step 2:** $\Omega$ is a surjection.

Assume that $\mathcal{E}$ is an extension $E$ of $B$ by $Q$ of type $B \to D$ inducing $\psi : Q \to \text{Coker } d$ as in the commutative diagram (10). We prove that $\mathcal{E}$ is equivalent to an extension $\mathcal{E}_F$ which is associated to a Gr-functor $(F, \tilde{F})$ : $\text{Dis}Q \to \mathcal{P}$.

Let $\mathcal{P}'$ be the Gr-category associated to the crossed module $(B, E, j, \theta')$. Then, by Proposition 4, the pair $(id_B, \varepsilon)$ determines a single Gr-functor $(K, \tilde{K}) : \mathcal{P}' \to \mathcal{P}$. Since $\pi_0\mathcal{P}' = Q, \pi_1\mathcal{P}' = 0$, then $S_{\mathcal{P}'} = \text{Dis}Q$.

Choose the stick $(e_u, i_v), e \in E, u \in Q$ in $\mathcal{P}'$, i.e., $\{e_u\}$ is a representative of $Q$ in $E$. By (11), the canonical Gr-functor $(H', \tilde{H}') : \text{Dis}Q \to \mathcal{P}'$ is given by

$$H'(u) = e_u, \tilde{H}'_{u,v} = -i_{e_u + e_v} = h'(u, v).$$

Then the composition $F = KH'$ determines a Gr-functor $\text{Dis}Q \to \mathcal{P}$, where

$$F(u) = \varepsilon(e_u), \tilde{F}_{u,v} = K(\tilde{H}'_{u,v}) = h'(u, v).$$

By Lemma 8, we determine an extension $\mathcal{E}_F$ of the crossed product $E_0 = [B, \varphi, h', Q]$, which is associated to $(F, \tilde{F})$. We now prove that $\mathcal{E}$ and $\mathcal{E}_F$ is equivalent, i.e., the following diagram commutes

$$\begin{array}{c}
\mathcal{E}_F : & 0 & \xrightarrow{j_0} & B & \xrightarrow{\eta} & E_0 & \xrightarrow{\rho_0} & Q & \xrightarrow{\eta} & 1, & \xrightarrow{\varepsilon_0} & E_0 & \xrightarrow{\rho_0} & D \\
\mathcal{E} : & 0 & \xrightarrow{j} & B & \xrightarrow{\eta} & E & \xrightarrow{\rho} & Q & \xrightarrow{\eta} & 1, & \xrightarrow{\varepsilon} & E & \xrightarrow{\rho} & D
\end{array}$$

The representative $e_u$ satisfies:

$$\varphi(u)c = \mu_{e_u}(c), \ c \in B. \quad (16)$$

$$e_u + e_v = -i_{e_u + e_v} + e_{uv} = h'(u, v) + e_{uv}. \quad (17)$$
The relation (16) holds since \((id_B, \varepsilon)\) is a morphism of crossed modules, the relation (17) holds thanks to the definition of morphism in \(P'\).

Each element of \(E\) is presented uniquely as \(b + e_u, b \in B\), so one can define a map

\[ \eta : E_0 \to E, \quad (b, u) \mapsto b + e_u \]

By the relations (16) and (17), \(\eta\) is an isomorphism.

Finally, choose the representative \(e_u\) such that

\[ \varepsilon(e_u) = x_{\psi(u)}. \]

Indeed, from (10) we have \(q(\varepsilon(e_u)) = \psi p(e_u) = \psi(u)\). Then

\[ \varepsilon \eta(b, u) = \varepsilon(b + e_u) = \varepsilon(b) \varepsilon(e_u) = d(b)x_{\psi(u)} = \varepsilon_0(b, u). \]

Thus, \(E\) and \(E_F\) are equivalent. This completes the proof. \(\square\)

Now, the second assertion of Theorem 5.2 [3] is obtained as follows. Firstly, there is a natural bijection

\[ \text{Hom}[\text{Dis} Q, P] \leftrightarrow \text{Hom}[\text{Dis} Q, S P] \]

Since \(\pi_0(\text{Dis} Q) = Q, \pi_1(S P) = \text{Ker} d\), then it follows from Theorem [9] and Proposition [2] that

\[ \text{Ext}_{B \to D}(Q, B, \psi) \leftrightarrow H^2(Q, \text{Kerd}). \]

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