How short can the hair of a black hole be?

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Abstract

We show that in all theories in which black hole hair has been discovered, the region with non-trivial structure of the non-linear matter fields must extend beyond $3/2$ the horizon radius, independently of all other parameters present in the theory. We argue that this is a universal lower bound that applies in every theory where hair is present. This no short hair conjecture is then put forward as a more modest alternative to the now debunked no hair conjecture.

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The no–hair conjecture (NHC) proposed in the early 70’s, (for a detailed review, see [1]), was explicitly proven for the Einstein–Maxwell theory [2] and for the Einstein scalar theory [3,4], but the idea of its universal validity was based on a physical argument which suggested that all matter fields present in a black hole space time would eventually be either radiated to infinity or “sucked” into the black hole, except when those fields were associated with conserved charges defined at asymptotic infinity.

At this point, it seems convenient to define what we call “hair”: We will say that in a given theory there is black hole hair when the space time metric and the configuration of the other fields of a stationary black hole solution are not completely specified by the conserved charges defined at asymptotic infinity. Thus, in the language of Ref. [5], we do not consider secondary hair.

Recently, we have witness the discovery of precisely this type of black hole hair in multiple theories in which Einstein’s gravity is coupled with self interacting matter fields. This has been achieved using numerical analysis in theories like Einstein–Yang–Mills (EYM) [6], Einstein–Skyrme (ES) [7], Einstein–Yang–Mills–Dilaton (EYMD) [8] and Einstein–Yang–Mills–Higgs (EYMH) [9].

These results, which demonstrate the falsehood of the NHC, naturally give raise the question: What happened to the physical arguments put forward to support it? It seems clear now that the non-linear character of the matter content of the examples discussed plays an essential role: The interaction between the part of the field that would be radiated away and that which would be sucked in is responsible for the failure of the argument and, thus, for the existence of black hole hair.

On the other hand, this suggests that the non-linear behavior of the matter fields must be present both, in a region very close to the horizon (a region from which presumably the fields would tend to be sucked in) and in a region relatively distant from the horizon (a region from which presumably the fields would tend to be radiated away), with the self interaction being responsible for binding together the fields in these two regions. In this letter, we will present analytic evidence for this heuristic argument by showing the existence of a lower
bound for the size of the region where the non-linear behavior of the field is present. This lower bound value is valid for all theories in which black hole hair has been found, and it is independent of all parameters present in the theory; more specifically we will show that in all such cases the region of interest must extend beyond $3/2$ the horizon radius.

The theories under consideration are described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M,$$

where $\mathcal{L}_G = (16\pi)^{-1} \sqrt{-g} R$ is the standard Hilbert Lagrangian for Einstein’s gravity, and $\mathcal{L}_M$ is the Lagrangian for some set of non-linear matter fields minimally coupled to gravity. We will use units in which $c = G = 1$. We will focus on static spherically symmetric gravitational fields and so we write the line element of the black hole space-time as

$$ds^2 = -e^{-2\delta} \mu dt^2 + \mu^{-1} dr^2 + r^2 d\Omega^2,$$

where $\delta$ and $\mu = 1 - 2m(r)/r$ are functions of $r$ only, and we assume that there is a regular event horizon at $r_H$, so $m(r_H) = r_H/2$. The matter fields will also respect the symmetries of the space-time and in particular we will look at the specific ansatz employed in each of the cases where black hole hair has been discovered.

In all cases considered here, Einstein’s equations for the metric (2) and for the corresponding ansatz for the matter fields can be written as:

$$\delta' = \beta K, \quad \mu' = \frac{1}{r}[1 - \mu + \alpha (K + U)],$$

where $\alpha, \beta, K$ and $U$ take particular values in each case.

The crucial step in showing the validity of our conjecture in different cases lies in the analysis of the matter field equation applying the Lyapunov method in the fashion recently employed by one of us to prove a no-hair theorem in Einstein–Higgs theory [4]. That way of using the Lyapunov method allows us to obtain from the matter field equations and Einstein’s equations a generic relationship of the form $\mathcal{E}' = B(r)$ where $\mathcal{E} \propto e^{-\delta(K - U)}$ is a function of $r$ which is negative on the horizon and must tend to zero (or a positive value)
at infinity if we demand the existence of a black hole solution. Hence the function $B(r)$ must be positive in some region. Furthermore we show that in all such cases this region corresponds to values of $r$ that are larger than $3/2$ of the horizon radius.

We now proceed to prove the conjecture stated above for all cases in which black hole hair has been found.

i) In the pure EYM case, $(SU(2))$, the matter Lagrangian has the form

$$\mathcal{L}_M = -\frac{\sqrt{-g}}{16\pi f^2} F_{\mu\nu}^a F^{\mu\nu a},$$  \hfill (4)

where $F_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + \epsilon_{abc} A_{\mu}^b A_{\nu}^c$, is the field strength for the gauge field $A_{\mu}^a$, and $f$ represents the gauge coupling constant. Furthermore, we use the static spherically symmetric ansatz for the potential

$$A = \sigma_a A_{\mu}^a dx^\mu = \sigma_1 w d\theta + (\sigma_3 \cot \theta + \sigma_2 w) \sin \theta d\phi,$$  \hfill (5)

where $w$ is a function of $r$ only. Einstein’s equations for this case may be written as in Eq.(3), with $K = \mu w^2$, $U = (1 - w^2)/(2r^2)$, $\alpha = -2/f^2$, and $\beta = -2/(f^2\mu r)$. Applying the Lyapunov method, as in [4], we find

$$\mathcal{E}' \equiv [r^2 e^{-\delta}(K - U)]' = re^{-\delta}(3\mu - 1)w^2.$$  \hfill (6)

From the expressions for $K$ and $U$ we see that $\mathcal{E}$ is negative at the horizon because $K(r_H) = 0$, (since $\mu(r_H) = 0$), and $U(r_H) > 0$. On the other hand, it can be shown that asymptotic flatness implies that $\mathcal{E} \to 0$ as $r \to \infty$. Accordingly, $\mathcal{E}$ must be an increasing function of $r$ in some intermediate region. It follows then that the right hand side of Eq. (3) must become positive at some point, i.e. we must have $K$ substantially different from zero while $3\mu > 1$, a condition which is equivalent to $r > 3m(r)$, and since $m(r)$ is an increasing function, then we find that this occurs at some point with $r > 3m(r_H) = 3r_H/2$. This proves our conjecture.

ii) In the ES case, the matter Lagrangian is [4]

$$\mathcal{L}_M = \sqrt{-g} f^2 4 Tr(\nabla_\mu W \nabla^\mu W^{-1}) + \frac{\sqrt{-g}}{32 e^2} Tr[(\nabla_\mu W) W^{-1}, (\nabla_\nu) W^{-1}]^2,$$  \hfill (7)
where $\nabla_\mu$ is the covariant derivative, $W$ is the $SU(2)$ chiral field, and $f^2$ and $e^2$ are the coupling constants. For the $SU(2)$ chiral field we use the hedgehog ansatz $W(r) = \exp(\sigma \cdot \mathbf{r} F(r))$ where $\sigma$ are the Pauli matrices and $\mathbf{r}$ is a unit radial vector.

In writing the Einstein equations as in Eq.(3), we follow Ref. [7] and use the variables $\tilde{r} = e f r$, and $\tilde{m}(\tilde{r}) = e f m(r)$ so that the function $\mu$ defined above remains invariant. Dropping the tilde, the resulting equations are equivalent to Eq.(3) with $K = \mu (r^2/2 + \sin^2 F(r)) F' - 1$, $U = \sin^2 F [1 + \sin^2 F / (2 r^2)]$, $\alpha = -8 \pi f^2$, and $\beta = -8 \pi f^2 / (\mu r)$. Using the Lyapunov method as before, we obtain

$$\mathcal{E}' \equiv [e^{-\delta}(\mathcal{K} - U)]' = -e^{-\delta} \left[ \mu F'^2 + \frac{1 - \mu}{\mu r^2} \mathcal{K} - 2r \left( 1 + \frac{U}{r^2} - \sqrt{1 + 2 \frac{U}{r^2}} \right) \right]. \tag{8}$$

As before, the r.h.s. of this equation must be positive in some region. Moreover, $\mathcal{E}(r_H) < 0$ and, since the asymptotic behavior of the field equations implies $F(r) \approx 1/r^2$ at infinity, it follows that $\mathcal{E} \to 0$ from above. Therefore there must be a point where $\mathcal{E} = 0$, i.e. $\mathcal{K} = U$, and $\mathcal{E}' > 0$. Considering the r.h.s. of Eq.(8) at this point, we find

$$- \mu F'^2 - 2r \left( \sqrt{1 + 2 \frac{K}{r^2}} - 1 \right) + \frac{K}{r^2} (3\mu - 1) > 0. \tag{9}$$

Since the first and second terms of the last equation are negative, we conclude that $3\mu > 1$ at this point. This is the same condition as in case $i$).

iii) We will generalize the EYMD case by adding an arbitrary (positive semi-definite) potential term $V(\phi)$ (which is expected to arise in super-strings inspired models [10]). The corresponding matter Lagrangian is given by [8]

$$\mathcal{L}_M = \frac{\sqrt{-g}}{4\pi} \left( \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{4 f^2} e^{2\gamma \phi} F_{\mu \nu} a F_{\mu \nu} a - V(\phi) \right), \tag{10}$$

where $f$ is the gauge coupling constant, $\gamma$ is the dimensionless dilatonic coupling constant, and $F_{\mu \nu} a$ is the SU(2) Yang–Mills field strength. The ansatz for the gauge field configuration is the same as that given in case $i$), and $\phi = \phi(r)$.

The corresponding Einstein equations can be written in the generic form [3] with $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$, where $\mathcal{K}_1 = \mu \exp(2\gamma \phi) w^2 / f^2$, $\mathcal{K}_2 = \mu r^2 \phi'^2 / 2$ and $U = r^2 V(\phi) + \exp(2\gamma \phi)(1 -
Following the same procedure, which in this case involves the two matter field equations, we find

\[ E' \equiv \left[ r^2 e^{-\delta (K - U)} \right]' = r e^{-\delta} \left[ -2 K_2 - 4 r^2 V(\phi) + (3 \mu - 1) \frac{K}{\mu} \right]. \tag{11} \]

Since the first and second terms of the r.h.s. of Eq. (11) are negative, we again find the condition \( 3\mu > 1 \) in order to obtain asymptotically flat solutions.

iv) In the EYMH case, the matter Lagrangian is given by [9]

\[ L_M = -\frac{\sqrt{-g}}{4\pi} \left[ \frac{1}{4 f^2} F_{\mu \nu}^a F^{\mu \nu}_a + (D_\mu \Phi)^\dagger (D^\mu \Phi) + V(\Phi) \right], \tag{12} \]

where \( D_\mu \) is the usual gauge–covariant derivative, \( \Phi \) is a complex doublet Higgs field, and \( F_{\mu \nu}^a \) is the \textit{SU}(2) Yang–Mills field given above. In this case, the ansatz for the Yang–Mills field is the same as before, and for the Higgs field we have

\[ \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \varphi(r) \end{pmatrix}. \tag{13} \]

The Einstein equations are equivalent to Eq. (3) with \( K = K_1 + K_2 \), where \( K_1 = \mu r^2 \varphi^2 / 2 \), and \( K_2 = \mu w^2 / f^2 \), \( U = r^2 V(\varphi) + (1 - w^2)^2 / (2 f^2 r^2) + (1 + w)^2 \varphi^2 / 4 \), \( \alpha = -2 \), and \( \beta = -2 / (\mu r) \).

Finally, our procedure leads to

\[ E' \equiv \left[ r^2 e^{-\delta (K - U)} \right]' = r e^{-\delta} \left[ -2 K_2 - 4 r^2 V(\varphi) - \frac{1}{2} (1 + w)^2 \varphi^2 + (3 \mu - 1) \frac{K}{\mu} \right]. \tag{14} \]

As in the previous cases, the required behavior of the function \( E \) leads to the condition \( 3\mu > 1 \) for the region of interest.

This last case is particularly interesting because, in view of the fact that both, the scalar field and the Yang–Mills field become massive through spontaneous symmetry breaking and the Higgs mechanism respectively, one could have naively expected that the fields would fall rapidly (in a distance of the order of the inverse masses) to their asymptotic behavior (exponentially decreasing behavior) and therefore, that by choosing the parameters of the theory appropriately one could obtain a black hole with hair that is as short as one desires.

The above calculation shows explicitly that this is not what happens and thus we view this case as a strong piece of evidence in support of our conjecture.
This set of results which have been obtained under the assumption of spherical symmetry, in part because all the cases in which hair has been discovered also involve this simplifying assumption, should, we believe, generalize to the stationary black hole cases where we expect that the region of non-linear behavior of the matter fields, which we have dubbed the "Hairosphere" should also be characterized by the length $r_{Hair} = 3/2\sqrt{\frac{A}{4\pi}}$ where $A$ is the horizon area. To be more specific we call the "Hairosphere" the region of non-linear behavior of the matter fields, (as oppose to the region where the behavior of the fields is that which is found asymptotically). In all cases presented here, there is a change in the behavior of $\mathcal{E}$ (which starts always as a negative and decreasing function and needs to increase towards its asymptotic value) and we have shown that this change always occurs beyond the point characterized by $r = 3 m(r) > 3r_H/2$ and this changes implies that the fields have not achieved their asymptotic behavior.

In view of the evidence shown here and based on the physical arguments described at the beginning of this letter, that suggest the existence of such a universal lower bound, we are lead to conjecture that for all stationary black holes in theories in which the matter content is described via a Lagrangian in terms of self interacting fields, the "Hairosphere", if it exists, must extend beyond the above mentioned distance. In short: If a black hole has hair, then it can not be shorter that $3/2$ the horizon radius.
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