Grassmann sheaves and the classification of vector sheaves

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Abstract

Given a sheaf of unital commutative and associative algebras \( \mathcal{A} \), first we construct the \( k \)-th Grassmann sheaf \( \mathcal{G}_\mathcal{A}(k, n) \) of \( \mathcal{A}^n \), whose sections induce vector subsheaves of \( \mathcal{A}^n \) of rank \( k \). Next we show that every vector sheaf over a paracompact space is a subsheaf of \( \mathcal{A}^\infty \). Finally, applying the preceding results to the universal Grassmann sheaf \( \mathcal{G}_\mathcal{A}(n) \), we prove that vector sheaves of rank \( n \) over a paracompact space are classified by the global sections of \( \mathcal{G}_\mathcal{A}(n) \).

Introduction

Let \( \mathcal{A} \) be a sheaf of unital commutative and associative algebras over the ring \( \mathbb{R} \) or \( \mathbb{C} \). A vector sheaf \( \mathcal{E} \) is a locally free \( \mathcal{A} \)-module. For instance, the sections of a vector bundle provide such a sheaf. However, a vector sheaf is not necessarily free, as is the case of the sections of a non trivial vector bundle.

Recently, vector sheaves gained a particular interest because they serve as the platform to abstract the classical geometry of vector bundles and their connections within a non smooth framework. This point of view has already been developed in [7] (see also [8] for applications to physics, and [10] for the reduction of the geometry of vector sheaves to the general setting of principal sheaves).

A fundamental result of the classical theory is the homotopy classification of vector bundles (of rank, say, \( n \)) over a fixed base. The construction of the classifying space, and the subsequent classification, are based on the

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Grassmann manifold (or variety) $G_k(\mathbb{R}^n)$ of $k$-dimensional subspaces of $\mathbb{R}^n$. In this respect we refer, e.g., to [5] and [6]. However, considering vector sheaves, we see that a homotopy classification is not possible, since the pull backs of a vector sheaf by homotopic maps need not be isomorphic, even in the trivial case of the free $A$-module $A$, as we prove in Section 1. Consequently, any attempt to classify vector sheaves (over a fixed space $X$) should not involve pull-backs and homotopy.

In this paper we develop a classification scheme based on a sort of universal Grassmann sheaf. More explicitly, for fixed $k \leq n \in \mathbb{N}$, in Section 2 we construct –in two equivalent ways– a sheaf $G_A(k, n)$, legitimately called the $k$-th Grassmann sheaf of $A^n$, whose sections coincide (up to isomorphism) with vector subsheaves of $A^n$ of rank $k$ (Proposition 2.3). Then, inducing in Section 3 the vector sheaf $A^\infty$, we show that every vector sheaf over a paracompact space is a subsheaf of $A^\infty$ (Theorem 3.1). A direct application of the previous ideas leads us to the construction of the universal Grassmann sheaf $G_A(n)$ of rank $n$. The main result here (Theorem 3.5) asserts that arbitrary vector sheaves of rank $n$, over a paracompact base space, coincide –up to isomorphism– with the sections of $G_A(n)$.

1. Vector sheaves and homotopy

For the general theory of sheaves we refer to standard sources such as [1], [2], [4], and [9]. In what follows we recall a few definitions in order to fix the notations and terminology of the present paper.

Throughout the paper $A$ denotes a fixed sheaf of unital commutative and associative $K$-algebras ($K = \mathbb{R}, \mathbb{C}$) over a topological space $X$. An $A$-module $E \equiv (E, \pi, X)$ is a sheaf whose stalks $E_x$ are $A_x$-modules so that the respective operations of addition and scalar multiplication

$$E \times_X E \longrightarrow E \quad \text{and} \quad A \times_X E \longrightarrow E$$

are continuous. In particular, a vector sheaf of rank $n$ is an $A$-module $E$, locally isomorphic to $A^n$. This means there is an open covering $U = \{U_\alpha\}, \alpha \in I,$ of $X$ and $A|_{U_\alpha}$-isomorphisms

$$\psi_\alpha: E|_{U_\alpha} \longrightarrow A^n|_{U_\alpha}, \quad \alpha \in I.$$  

The category of vector sheaves of rank $n$ over $X$ is denoted by $V^n(X)$. More details, examples and applications of vector sheaves can be found in [7], [8], and [10].
As already mentioned in the Introduction, we shall show, by a concrete counterexample, that homotopic maps do not yield isomorphic pull-backs, even in the simplest case of the free \( A \)-module \( A \). In fact, we consider two non-isomorphic algebras \( A_0 \) and \( A_1 \) and a morphism of algebras \( \rho: A_0 \to A_1 \).

Given now a topological space \( X \) and a fixed point \( x_0 \in X \), for every open \( U \subseteq X \) we set
\[
A(U) := \begin{cases} 
A_0, & x_0 \in U, \\
A_1, & x_0 \notin U,
\end{cases}
\]
while, for every open \( V \subseteq U \), \( \rho_U^V: A(U) \to A(V) \) denotes the corresponding (restriction) map, defined by
\[
\rho_U^V := \begin{cases} 
id: A_0 \to A_0, & \text{if } x_0 \in V, \\
\rho: A_0 \to A_1, & \text{if } U \supset x_0 \notin V, \\
id: A_1 \to A_1, & \text{if } x_0 \notin U.
\end{cases}
\]

It is not difficult to show that \((A(U), \rho_U^V)\) is a presheaf whose sheafification is a sheaf of algebras, denoted by \( \mathcal{A} \equiv (\mathcal{A}, \pi, X) \). It is clear that
\[
\mathcal{A}_{x_0} = \lim_{U \in \mathcal{N}(x_0)} A(U) = A_0.
\]

On the other hand, if there is an \( x_1 \in X \) admitting a neighborhood \( V \in \mathcal{N}(x_1) \) with \( x_0 \notin V \) (a fact always ensured if \( X \) is a \( T_1 \)-space), then
\[
\mathcal{A}_{x_1} = \lim_{x_1 \in W \subseteq V} A(W) = A_1,
\]
which is not isomorphic to \( \mathcal{A}_{x_0} \). Hence, a vector sheaf, even a free one, need not have locally isomorphic fibres.

Let now \( \alpha: [0, 1] \to X \) be a continuous path with \( \alpha(0) = x_0 \) and \( \alpha(1) = x_1 \). For any topological space \( Y \), we define the map
\[
f: [0, 1] \times Y \to X \quad \text{with} \quad f(t, y) := \alpha(t).
\]

Obviously, this is a homotopy between the constant maps \( f_0 = x_0: Y \to X \) and \( f_1 = x_1: Y \to X \). As a result, taking the pull-backs of \( \mathcal{A} \) by the latter, we see that, for every \( y \in Y \),
\[
f_0^*(\mathcal{A})_y = \{(y, a) \mid a \in \mathcal{A}_{x_0} \} = \{y\} \times A_0,
\]
\[
f_1^*(\mathcal{A})_y = \{(y, b) \mid b \in \mathcal{A}_{x_1} \} = \{y\} \times A_1;
\]
that is, we obtain two non-isomorphic stalks, thus proving the claim.
2. The Grassmann sheaf of rank $k$ in $A^n$

As in Section 1, $A$ is a sheaf of unital commutative and associative $K$-algebras over a given topological space $X$. We denote by $\Sigma_X$ the topology of $X$ and fix $n \in \mathbb{N}$. For $k \in \mathbb{N}$ with $k \leq n$ and any $U \in \Sigma_X$, we define the set

$$G_A(k, n)(U) := \{ \mathcal{S} \text{ subsheaf of } A^n|_U : \mathcal{S} \cong A^k|_U \},$$

that is, $G_A(k, n)(U)$ consists of the free submodules of $A^n|_U$ of rank $k$. If, for every $U, V \in \Sigma_X$ with $V \subseteq U$,

$$\rho^U_V : G_A(k, n)(U) \longrightarrow G_A(k, n)(V) : \mathcal{S} \mapsto \mathcal{S}|_V$$

denotes the natural restriction, it is clear that the collection

(2.1) $\quad G_A(k, n) := (G_A(k, n)(U), \rho^U_V)$

determines a presheaf. Moreover, it is a monopresheaf. Indeed, if $U = \bigcup_{i \in I} U_i$ and $\mathcal{E}_1, \mathcal{E}_2 \in G_A(n, k)(U)$ with $\mathcal{E}_1|_{U_i} = \mathcal{E}_2|_{U_i}$, for all $i \in I$, then $\mathcal{E}_1 = \mathcal{E}_2$. However, it is not complete: If $\mathcal{E}_i \in G_A(n, k)(U_i)$, with $\mathcal{E}_i|_{U_i \cap U_j} = \mathcal{E}_j|_{U_i \cap U_j}$, then $\mathcal{E} := \bigcup_{i \in I} \mathcal{E}_i$ is a vector sheaf over $U$, but not necessarily a free $A|_U$-module.

**Definition 2.1.** The $k$-th Grassmann sheaf of $A^n$, denoted by $G_A(k, n)$, is defined to be the sheaf generated by the presheaf $G_A(k, n)$.

Since $G_A(k, n)$ is not complete, it does not coincide with the complete presheaf

(2.2) $\quad (G_A(k, n)(U), r^U_V),$

of (continuous) sections of $G_A(k, n)$. We shall describe $G_A(k, n)$ via another complete presheaf. As a matter of fact, we consider the presheaf

(2.3) $\quad V_A(k, n) := (V_A(k, n)(U), \lambda^U_V),$

where now

$$V_A(k, n)(U) := \{ \mathcal{S} \text{ subsheaf of } A^n|_U : \mathcal{S} \in \mathcal{V}^k(U) \}$$

and

$$\lambda^U_V : V_A(k, n)(U) \longrightarrow V_A(k, n)(V) : \mathcal{S} \mapsto \mathcal{S}|_V$$

are the natural restrictions. In contrast to $G_A(k, n)$, $V_A(k, n)$ is obviously a complete presheaf.
Lemma 2.2. The sheaf generated by $V_A(k, n)$ is isomorphic to $G_A(k, n)$.

Proof. Clearly, for every $U \in \mathcal{T}_X$, $G_A(k, n)(U) \subseteq V_A(k, n)(U)$, that is, $G_A(k, n)$ is a sub-presheaf of $V_A(k, n)$. Besides, for every $E \in V_A(k, n)(U)$ and every $x \in U$, there is $V \in \mathcal{T}_X$ with $x \in V \subseteq U$, so that $E|_V$ is free of rank $k$, namely $E|_V \in G_A(k, n)(V)$. Thus, $G_A(k, n)(U)$ and $V_A(k, n)(U)$ define the same sheaf $G_A(k, n)$.

Since $V_A(k, n)$ is complete, it is isomorphic with the sheaf of sections of $G_A(k, n)$, thus we have the following interpretation of the elements of $G_A(k, n)(X)$.

Proposition 2.3. The global sections of the $k$-th Grassmann sheaf $G_A(k, n)$ coincide –up to isomorphism– with the vector subsheaves of $A^n$ of rank $k$.

3. The universal Grassmann sheaf

The preliminary results of the preceding section hold for every base space $X$. Here we prove that if $X$ is a paracompact space, then any vector sheaf can be interpreted as a section of an appropriate universal Grassmann sheaf.

First we prove a Whitney-type embedding theorem. To this end, for every sheaf of algebras $A$, we consider the presheaf

$$ U \mapsto \prod_{i \in \mathbb{N}} A_i(U), \quad U \in \mathcal{T}_X, $$

where $A_i = A$, for every $i \in \mathbb{N}$, with the obvious restrictions. This presheaf generates the infinite fibre product

$$ A^\infty := \prod_{i \in \mathbb{N}} A_i, $$

which is a free $A$-module. Then, we obtain:

Theorem 3.1. Let $X$ be a paracompact space. Then every vector sheaf $E$ of finite rank over $X$ is a subsheaf of $A^\infty$.

Proof. Let $E$ be a vector sheaf of rank, say, $k$. Since $X$ is paracompact, a reasoning similar to that of [5, Proposition 5.4] proves that $E$ is free over a countable open covering $\{U_i\}_{i \in \mathbb{N}}$ of $X$. Let $\psi_i : E|_{U_i} \to A^k|_{U_i}$, $i \in \mathbb{N}$, be the
respective family of \( \mathcal{A} \)-module isomorphisms. The same open covering has a countable locally finite open refinement (\[3\] Ch. VIII, Theorem 1.4), with a subordinate partition of unity \( \{\alpha_i : X \to \mathbb{R}\}_{i \in \mathbb{N}} \) (ibid., Ch. VIII, Theorem 4.2). For every \( i \in \mathbb{N} \), we define the map \( \alpha_i \psi_i : \mathcal{E} \to \mathcal{A}^k \) by
\[
\alpha_i \psi_i(u) := \begin{cases} 
\alpha_i(\pi(u)) \psi_i(u), & \pi(u) \in U_i, \\
0, & \pi(u) \notin U_i.
\end{cases}
\]
Therefore, \( \alpha_i \psi_i \) is an \( \mathcal{A} \)-module morphism, whose restriction to the interior of \( \text{supp} \alpha_i \subseteq U_i \) is an isomorphism.

We consider the fibre product \( \prod_{i \in \mathbb{N}} (\mathcal{A}^k)_i \), where \( (\mathcal{A}^k)_i \equiv \mathcal{A}^k \), for every \( i \in \mathbb{N} \), and we denote by \( p_i : \prod_{i \in \mathbb{N}} (\mathcal{A}^k)_i \longrightarrow (\mathcal{A}^k)_i \) the corresponding projections. The universal property of the product ensures the existence of a unique \( \mathcal{A} \)-morphism
\[
\psi : \mathcal{E} \longrightarrow \prod_{i \in \mathbb{N}} (\mathcal{A}^k)_i,
\]
such that
\[
p_i \circ \psi = \alpha_i \psi_i.
\]
Then \( \psi \) is a monomorphism. In fact, let \( 0 \neq u \in \mathcal{E}_x \) with \( \psi_x(u) = 0 \). There is \( i \in \mathbb{N} \), with \( \alpha_i(x) > 0 \), thus \( \alpha_i(x) \psi_{i,x}(u) \neq 0 \), a contradiction. Hence, \( \mathcal{E} \) is identified with its image \( \psi(\mathcal{E}) \leq \prod_{i \in \mathbb{N}} (\mathcal{A}^k)_i \). Since
\[
\prod_{i \in \mathbb{N}} (\mathcal{A}^k)_i \equiv \prod_{i \in \mathbb{N}} \mathcal{A}_i,
\]
where \( \mathcal{A}_i = \mathcal{A} \), for every \( i \in \mathbb{N} \), the assertion is proven.

We shall show that a further restriction on the topology of \( X \) leads to an embedding of \( \mathcal{E} \) into a smaller sheaf. To this end, assume that \( \mathcal{E} \) is a vector sheaf of rank \( k \), which is free over a finite open covering \( \{U_i\}_{1 \leq i \leq n} \) of \( X \). Let \( \psi_i : \mathcal{E}|_{U_i} \to \mathcal{A}^k|_{U_i} \) be the respective \( \mathcal{A} \)-module isomorphisms, and \( \{\alpha_i : X \to \mathbb{R}\}_{1 \leq i \leq n} \) a subordinate partition of unity. Considering the maps \( \alpha_i \psi_i : \mathcal{E} \to \mathcal{A}^k \), as before, we obtain the sheaf morphism
\[
f : \mathcal{E} \longrightarrow \mathcal{A}^{kn} : u \mapsto (\alpha_1(\pi(u)) \cdot \psi_1(u), \ldots, \alpha_n(\pi(u)) \cdot \psi_n(u))
\]
which embeds \( \mathcal{E} \) into the free \( \mathcal{A} \)-module \( \mathcal{A}^{kn} \). Therefore we have proved the following:
Proposition 3.2. Over a compact space $X$, every vector sheaf of finite rank is a subsheaf of a free $\mathcal{A}$-module of finite rank.

Clearly, every free $\mathcal{A}$-module $\mathcal{A}^n$ is a submodule of the free $\mathcal{A}$-module $\bigoplus_{i \in \mathbb{N}} \mathcal{A}_i$, with $\mathcal{A}_i = \mathcal{A}$, for every $i \in \mathbb{N}$. Thus we obtain:

Corollary 3.3. Over a compact space $X$, every vector sheaf of finite rank is a subsheaf of the free $\mathcal{A}$-module $\bigoplus_{i \in \mathbb{N}} \mathcal{A}_i$.

We are now in a position to repeat the constructions of Section 2 in a more general way. For every $n \in \mathbb{N}$, we define the set

$$G_{\mathcal{A}}(n)(U) := \{S \text{ subsheaf of } \mathcal{A}^\infty|_U : S \cong \mathcal{A}^n|_U\}.$$ 

Then the collection

$$G_{\mathcal{A}}(n) := (G_{\mathcal{A}}(n)(U), \rho^n_U), \quad U \in \mathcal{T}_X,$$

where $\rho^n_U$ denotes the obvious restriction, is a non-complete monopresheaf.

Definition 3.4. The sheaf $G_{\mathcal{A}}(n)$, generated by the presheaf $G_{\mathcal{A}}(n)$, is called the universal Grassmann sheaf of rank $n$.

The respective complete presheaf of the sections of $G_{\mathcal{A}}(n)$ is isomorphic to the presheaf

$$V_{\mathcal{A}}(n) := (V_{\mathcal{A}}(n)(U), \lambda^n_U), \quad U \in \mathcal{T}_X,$$

where now

$$V_{\mathcal{A}}(n)(U) := \{S \text{ subsheaf of } \mathcal{A}^\infty|_U : S \in V^n(U)\}$$

and $\lambda^n_U$ are the natural restrictions.

As a result, adapting the proof of Proposition 2.3 to the present situation, we obtain the main result of this work, namely the following classification of vector sheaves:

Theorem 3.5. If $X$ is a paracompact space, then the vector sheaves of rank $n$ (over $X$) coincide –up to isomorphism– with the global sections of the universal Grassmann sheaf $G_{\mathcal{A}}(n)$. 
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