Covering a Sphere with Four Random Circular Caps

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February 24, 2015

Abstract. Let \( p(\omega) \) denote the probability that four random circular caps of angular radius \( 70^\circ < \omega < 90^\circ \) cover the unit sphere \( S^2 \). An exact expression for \( p(\omega) \) is unknown. We give nontrivial lower bounds for \( p(\omega) \) when \( \omega > 84^\circ \); no improvement on the inequality \( p(\omega) \geq 0 \) for \( \omega < 84^\circ \) is yet feasible. A dual problem involving randomly inscribed well-centered tetrahedra is also examined.

Let \( S^2 \) denote the two-dimensional sphere in \( \mathbb{R}^3 \) of unit radius. Let \( X_1, X_2, X_3, X_4 \) be four points that are independent and uniformly distributed over the surface of \( S^2 \). Each \( X_j \) is taken to be the center of a circular cap of angular radius \( \omega \):

\[
\{ Y \in S^2 : \arccos(X_j \cdot Y) \leq \omega \}, \quad 0 \leq \omega \leq \pi
\]

where \( X \cdot Y \) is the usual inner product between vectors in \( \mathbb{R}^3 \). A point, hemisphere and \( S^2 \) itself are the outcomes if \( \omega = 0, \pi/2 \) and \( \pi \). The sphere is said to be covered if each point of \( S^2 \) belongs to at least one cap. It is known that the coverage probability \( p(\omega) \) satisfies

\[
p(\omega) = \begin{cases} 
0 & \text{if } 0 \leq \omega \leq \omega_0, \\
1 - 2 \cos \left( \frac{\omega}{2} \right) \left[ 8 - 9 \cos^2 \left( \frac{\omega}{2} \right) \right] & \text{if } \frac{\pi}{2} \leq \omega \leq \pi 
\end{cases}
\]

where

\[
\omega_0 = \arccos \left( \frac{1}{3} \right) = 1.23095941... \approx 70.53^\circ.
\]

Finding an exact expression for \( p(\omega) \) in the interval \( \omega_0 < \omega < \pi/2 \) remains an open problem. Early works in this area include \[14,5\]; computer simulations appear in \[18,22\] and historical overviews in \[17,6,3\]. All such papers deal with \( n \) circular caps, where \( n \) is often large. Our restriction to the case \( n = 4 \) is somewhat unusual, as far as is known.

Gilbert [5] gave the following bounds on \( p(\omega) \):

\[
0 \leq p(\omega) \leq \sum_{k=0}^{2} \binom{2}{k} (-1)^k \left( 1 - k \sin^2 \left( \frac{\omega}{2} \right) \right)^4 = 1 - 2 \cos^8 \left( \frac{\omega}{2} \right) + \cos^4(\omega)
\]
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(his left-hand side is negative for \( n = 4 \) and thus replaced by zero). The upper bound is weak but not relevant to us. We will focus on the lower bound. Our improvement, based on elaborate formulation, occurs only when \( \omega > 84.25^\circ \). For example, if \( \omega = 88^\circ \), then \( 0.0765 < p(\omega) < 0.8567 \) via our work and \( p(\omega) \approx 0.08 \) via computer simulation.

Dual to the coverage problem is the following. Given \( A, B, C, D \) on \( S^2 \), there is a unique circular cap of minimal angular radius \( \theta_{\text{min}} \) containing all four points. If \( A, B, C, D \) are independent and uniform on \( S^2 \), then the cumulative distribution function \( \Phi \) for the random variable \( \theta_{\text{min}} \) satisfies \[ 1 - \Phi(\pi - \omega) = p(\omega), \quad 0 \leq \omega \leq \pi. \]

Define a distribution
\[
F(\xi) = 8 \left( \Phi(\xi) - \frac{1}{8} \right) = 8 \mathbb{P} \left\{ \frac{\pi}{2} < \theta_{\text{min}} \leq \xi \right\}, \quad \frac{\pi}{2} < \xi < \pi.
\]
Calculating \( F(\xi) \) is hopeless; therefore we study a different variable \( \theta_{abc} \) with distribution \( G \) that is computationally feasible and satisfies \( cG(\xi) \geq F(\xi) \) for some constant \( c > 0 \).

Any three points from \( \{A, B, C, D\} \) almost surely form both a chordal triangle (with sides as straight lines through the interior of \( S^2 \)) and a spherical triangle (with sides as great circles on the surface of \( S^2 \)).

The chordal triangle \( ABC \) determines a plane that partitions \( S^2 \) into two complementary caps. Let \( ABCD \) denote the tetrahedron inscribed in \( S^2 \) with base \( ABC \) and apex \( D \). Such a tetrahedron is called well-centered if it contains the origin \[19\].

The hypothesis \( \pi/2 < \theta_{\text{min}} \) implies that \( ABCD \) is well-centered and, without loss of generality, \( ABC \) is acute. Further, \( D \) is in the larger of the two complementary caps and the angular radius \( \theta_{abc} \) of the cap satisfies \( \pi/2 < \theta_{abc} \). The converse is not true. Our main objective is to find the conditional probability density function \( g \) for \( \theta_{abc} \) and the associated distribution \( G \). Since \( ABC \) is one of four possible choices of base, we have a crude estimate \( 4G(\xi) \geq F(\xi) \).

While finding \( g \), we compute the probability
\[
\kappa = \mathbb{P} \{ABCD \text{ is well-centered} \& \ ABC \text{ is acute} \} = 0.10191818...
\]
which extends a well-known result \( \mathbb{P} \{ABCD \text{ is well-centered} \} = 1/8 \) from \[21, 20, 9 \] \[7 \] \[11 \] and another result \( \mathbb{P} \{ABC \text{ is acute} \} = 1/2 \) from \[13 \]. Details underlying the acuteness probability are given in the next section. The expected number of acute faces of \( ABCD \), given well-centeredness, is \( 32 \kappa \); a refined estimate is hence \( (3.26138193...)G(\xi) \geq F(\xi) \).
1. Random Triangles of Circumradius $r$

For comparison’s sake, consider triangles formed by selecting three independent points on the circle $rS^1$, where the radius $r$ is fixed. The bivariate density for two arbitrary angles $\alpha$, $\beta$ is

$$
\begin{cases}
\frac{2}{\pi^2} & \text{if } 0 < \alpha < \pi, 0 < \beta < \pi \text{ and } \alpha + \beta < \pi, \\
0 & \text{otherwise}
\end{cases}
$$

thus the acuteness probability for such triangles is $1/4$. The bivariate density for two arbitrary sides $a, b$ is

$$
\begin{cases}
\frac{4}{\pi^2 r^2} \frac{1}{\sqrt{4 - \left(\frac{a}{r}\right)^2} \sqrt{4 - \left(\frac{b}{r}\right)^2}} & \text{if } 0 < \frac{a}{r} < 2 \text{ and } 0 < \frac{b}{r} < 2, \\
0 & \text{otherwise};
\end{cases}
$$

the sides $a, b$ are independent despite the fact that angles $\alpha, \beta$ are dependent and $a = 2r \sin(\alpha), b = 2r \sin(\beta)$.

Consider now chordal triangles formed by selecting three independent points on the sphere $S^2$. Given triangle $ABC$, let $r$ denote the radius of the unique circle passing through vertices $A, B, C$. Unlike before, $r$ is a random variable. Intuition might suggest that the density of $\alpha, \beta$ and of $a/r, b/r$ should be similar to the preceding. In fact, however, the bivariate density of $\alpha, \beta$ is

$$
\begin{cases}
\frac{8}{3\pi} \sin(\alpha) \sin(\beta) \sin(\alpha + \beta) & \text{if } 0 < \alpha < \pi, 0 < \beta < \pi \text{ and } \alpha + \beta < \pi, \\
0 & \text{otherwise}
\end{cases}
$$

thus the acuteness probability for such triangles is $1/2$. The trivariate density of $a, b, r$ is

$$
\frac{a b}{6\pi r^4} \left\{ \frac{\frac{a}{r} \sqrt{4 - \left(\frac{b}{r}\right)^2} + \frac{b}{r} \sqrt{4 - \left(\frac{a}{r}\right)^2}}{\sqrt{4 - \left(\frac{a}{r}\right)^2} \sqrt{4 - \left(\frac{b}{r}\right)^2}} \right\}
$$

for $0 < a < 2r$ and $0 < b < 2r$. This is a $(2/3, 1/3)$-weighted mixture of densities. The portion coming from acute chordal triangles is

$$
\frac{a b}{6\pi r^4} \frac{\frac{a}{r} \sqrt{4 - \left(\frac{b}{r}\right)^2} + \frac{b}{r} \sqrt{4 - \left(\frac{a}{r}\right)^2}}{\sqrt{4 - \left(\frac{a}{r}\right)^2} \sqrt{4 - \left(\frac{b}{r}\right)^2}} = \frac{a b}{6\pi r^4} \left\{ \frac{a}{\sqrt{4r^2 - a^2}} + \frac{b}{\sqrt{4r^2 - b^2}} \right\}
$$
for \( 0 < a < 2r, \ 0 < b < 2r \) and \( a^2 + b^2 > 4r^2 \). We call this portion \( \delta(a, b, \theta) \) for future reference, where \( r = \sin(\theta) \).

Proof of the above follows via a Jacobian determinant calculation based on formula (4.18) of [12]; see also Theorem 3.2 of [13]. This is one of several crucial contributions by Miles to our study.

2. Conditional Probabilities and Angular Radius

Let \( \mathcal{E} \) denote the event \( \{ABCD \text{ is well-centered \& } ABC \text{ is acute}\} \). Following the proof of Theorem 3.3 of [13], we have

\[
P\{\theta < \theta_{abc} < \theta + d\theta\} \cdot P\{\mathcal{E} | \theta\} = P\{\theta < \theta_{abc} < \theta + d\theta | \mathcal{E}\} \cdot P\{\mathcal{E}\}
\]

where \( \pi/2 < \theta < \pi \). Formula (3.3) of [13] yields

\[
P\{\theta < \theta_{abc} < \theta + d\theta\} = \frac{3}{2} \sin^3(\theta) \ d\theta
\]

and finding

\[
P\{\theta < \theta_{abc} < \theta + d\theta | \mathcal{E}\} = g(\theta) \ d\theta
\]

is our main objective. We turn attention to \( P\{\mathcal{E} | \theta\} \).

Fix a chordal triangle \( ABC \) for consideration. Let \( \Delta \) denote the spherical triangle with vertices \( A, B, C \). The set of all points \( D \) for which the tetrahedron \( ABCD \) is well-centered is \( \Delta' \), the spherical triangle with vertices \( A', B', C' \) antipodal to \( A, B, C \). This pictured in Figure 2 of [19] and is stated in [1]. Note that the area of \( \Delta' \) is equal to the area of \( \Delta \).

Letting \( ABC \) vary over all acute chordal triangles of fixed angular radius \( \pi/2 < \theta < \pi \), the random quantity \( \text{area}(\Delta) \) can be as large as \(-6 \arccot(\sqrt{3} \cos(\theta)) - \pi \). This is the area of an equilateral spherical triangle with vertices on a circle of radius \( \sin(\theta) \) [2]. A formula for the mean area, as opposed to the maximum area, turns out to be quite complicated. We derive this in the next section.

3. Kolmogorov-Robbins Theorem

The desired probability \( P\{\mathcal{E} | \theta\} \) is the expected value of \( \text{area}(\Delta)/(4\pi) \), where random spherical triangles \( \Delta \) correspond to acute chordal triangles \( ABC \) of fixed angular radius \( \pi/2 < \theta < \pi \) [10, 16, 8, 11].

Let \( r = \sin(\theta) \) for convenience. If angles \( \alpha, \beta \) of an acute chordal triangle are known, then the remaining angle \( \gamma \) satisfies

\[
\cos(\gamma) = \cos\left(\pi - \arcsin\left(\frac{a}{2r}\right) - \arcsin\left(\frac{b}{2r}\right)\right)
= \frac{a b - \sqrt{4r^2 - a^2} \sqrt{4r^2 - b^2}}{4r^2}.
\]
The corresponding spherical triangle has angle $\tilde{\gamma}$ given by

$$\tilde{\gamma} = \arccos \left( \frac{4 \cos(\gamma) - a \ b}{\sqrt{4 - a^2 \sqrt{4 - b^2}}} \right)$$

$$= \arccos \left( \frac{(1 - r^2) a \ b - \sqrt{4r^2 - a^2 \sqrt{4r^2 - b^2}}}{r^2 \sqrt{4 - a^2 \sqrt{4 - b^2}}} \right).$$

Call this expression $\lambda(a, b, \theta)$. While area($\Delta$) = $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} - \pi$ and analogous expressions for $\tilde{\alpha}, \tilde{\beta}$ are possible, it is simpler to employ $\mathbb{E}(\text{area}(\Delta)) = 3 \mathbb{E}(\tilde{\gamma}) - \pi$. This yields

$$\mathbb{P}\{\mathcal{E} | \theta\} = \int_0^{2r} \int_0^{2r} \frac{1}{4\pi} \left( 3\lambda(a, b, \theta) - \pi \right) \delta(a, b, \theta) \ da \ db$$

and

$$\mathbb{P}\{\mathcal{E}\} = \int_{\pi/2}^{\pi} \frac{3}{2} \sin^3(\theta) \mathbb{P}\{\mathcal{E} | \theta\} \ d\theta = 0.10191818... = \kappa;$$

therefore we conclude that

$$G(\theta) = \int_{\pi/2}^{\theta} g(t) \ dt = \int_{\pi/2}^{\theta} \frac{3}{2\kappa} \sin^3(t) \mathbb{P}\{\mathcal{E} | t\} \ dt.$$

In Figure 1, a plot of the density $g(\theta)$ is superimposed on a histogram of simulated $\theta_{abc}$-values, $\theta \in [\pi/2, \pi]$, based on $10^6$ random well-centered tetrahedra $ABCD$ with acute faces $ABC$. The fit is excellent. First and second moments are also indicated. All required integrals were evaluated via numerical methods because no simplification via computer algebra seems likely.

Of course, our real interest is in $\theta_{\text{min}}$, not $\theta_{abc}$. Define a function

$$\psi(\theta) = \begin{cases} 
24 \sin^5 \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right) \left[ 2 - 3 \sin^2 \left( \frac{\theta}{2} \right) \right] & \text{if } 0 \leq \theta \leq \frac{\pi}{2}, \\
\frac{1}{8} (32\kappa) g(\theta) & \text{if } \frac{\pi}{2} < \theta \leq \pi.
\end{cases}$$

In Figure 2, a plot of $\psi(\theta)$ is superimposed on a histogram of simulated $\theta_{\text{min}}$-values, $\theta \in [0, \pi]$, based on $10^6$ random (arbitrary) tetrahedra $ABCD$. The curve $\psi$ is represented by the solid line and the fit is excellent on $[0, \pi/2]$. The area under $\psi$ over $[0, \pi/2]$ is $7/8$; the area under $\psi$ over $[\pi/2, \pi]$ is $4\kappa = 0.40767274...$. It visibly dominates the (unknown) density function $\varphi$ for $\theta_{\text{min}}$. 
Figure 1: Experimental histogram of $\theta_{abc}$-values and theoretical fit.
Figure 2: Experimental histogram of $\theta_{\min}$-values and theoretical fit/upper bounds.
4. Bounds on Coverage Probability

Define a function

\[ q(\omega) = \frac{1}{8} (1 - 32\kappa G(\pi - \omega)), \quad 0 < \omega < \frac{\pi}{2} \]

then \( q(\omega) \) is a nontrivial lower bound for \( p(\omega) \) when \( q(\omega) > 0 \), that is, when \( \omega > 84.25^\circ \). As discussed earlier, \( q(88^\circ) = 0.0765 \).

Our method fails to improve upon the inequality \( p(\omega) \geq 0 \) for \( \omega < 84.25^\circ \). We assess the effect of more closely approximating the density \( \varphi \). Let

\[ \theta_0 = \pi - \omega_0 = \arccos \left( -\frac{1}{3} \right) = 1.91063323... \approx 109.47^\circ. \]

Assuming the function \( 1 + \varphi \) is logarithmically convex on \([\pi/2, \theta_0]\), it follows that

\[ \psi_{\text{lcv}}(\theta) = \left( \frac{5}{2} \right)^{\frac{\theta_0 - \theta}{\pi_0 - \pi/2}} - 1 \]

dominates \( \varphi \). This is suggested in Figure 2, where the curve \( \psi_{\text{lcv}} \) is represented by the dashed line. Integrating \( \psi_{\text{lcv}} \) from \( \pi/2 \) to \( \theta \), we obtain \( \Psi_{\text{lcv}} \), which in turn is used to define

\[ q_{\text{lcv}}(\omega) = \frac{1}{8} - \Psi_{\text{lcv}}(\pi - \omega), \quad 0 < \omega < \frac{\pi}{2}. \]

This is a tighter bound than the preceding; for example, \( q_{\text{lcv}}(88^\circ) = 0.0766 \). It is nontrivial when \( q_{\text{lcv}}(\omega) > 0 \), that is, when \( \omega > 83.90^\circ \). Thus the effect is only slight. A rigorous proof that log convexity holds, however, is not known.

A sharper upper bound for \( p(\omega) \) than that provided in [5] is also desired.

5. Addendum

The preceding text was written in 2011. A question, “Does \( \kappa \) possess a closed-form expression?” appeared in [23]; an affirmative answer

\[ \kappa = \frac{11}{96} - \frac{1}{8\pi^2} \]

appeared in [24] with a different line of reasoning. We acknowledge that the preceding text is heuristic and non-rigorous. Sample open issues include: Must a well-centered tetrahedron \( ABCD \) possess at least one acute-angled triangular face? (Simulation suggests, in fact, that there are at least two.) If all faces \( ABC, ABD, ACD, BCD \) are acute, then

\[ \theta_{\text{min}} = \min \{ \theta_{abc}, \theta_{abd}, \theta_{acd}, \theta_{bcd} \} \]

and the inequality \( 4G(\xi) \geq F(\xi) \) becomes clear. It is possible, however, that only two or three faces are acute. Let \( N \in \{2, 3, 4\} \) be the number of acute faces, given a random well-centered tetrahedron \( ABCD \). What precisely is the distribution of \( N \)? All we know is \( E(N) = 32\kappa \). How is the inequality \((32\kappa)G(\xi) \geq F(\xi) \) proved? (The apparent continuity of \( \psi(\theta) \) at \( \theta = \pi/2 \) gives only an impression that this may be correct.) Discussion would be appreciated.
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