CONFINED STEADY STATES OF THE RELATIVISTIC
VLASOV–MAXWELL SYSTEM IN
AN INFINITELY LONG CYLINDER

JÖRG WEBER*
University of Bayreuth
Universitätsstraße 30
95440 Bayreuth, Germany

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Abstract. The time evolution of a collisionless plasma is modeled by the relativistic Vlasov–Maxwell system which couples the Vlasov equation (the transport equation) with the Maxwell equations of electrodynamics. In this work, the setting is two and one-half dimensional, that is, the distribution functions of the particles species are independent of the third space dimension. We consider the case that the plasma is located in an infinitely long cylinder and is influenced by an external magnetic field. We prove existence of stationary solutions and give conditions on the external magnetic field under which the plasma is confined inside the cylinder, i.e., it stays away from the boundary of the cylinder.

1. Introduction. If a plasma is sufficiently rarefied or hot, collisions among the plasma particles can be neglected and the time evolution of this plasma can be modeled by the relativistic Vlasov–Maxwell system. We consider the case that the plasma is contained in some open set $\Omega \subset \mathbb{R}^3$ and that the particles and electromagnetic fields, respectively, are subject to purely reflecting and perfect conductor boundary conditions, respectively. In particular, the system reads

\[
\begin{align*}
\partial_t f + \tilde{v}_a \cdot \nabla_x f + q_a \left( E + \tilde{v}_a \times B^{\text{tot}} \right) \cdot \nabla_v f &= 0 \quad \text{on } [0, T] \times \Omega \times \mathbb{R}^3, \quad (1a) \\
\frac{\partial}{\partial t} f_a^+ - K f_a^- &= 0 \quad \text{on } \Gamma_T, \quad (1b) \\
f_a^+(0) &= f_a^0 \quad \text{on } \Omega \times \mathbb{R}^3, \quad (1c) \\
\partial_t E - \text{curl}_x B &= -4\pi j \quad \text{on } [0, T] \times \Omega, \quad (1d) \\
\partial_t B + \text{curl}_x E &= 0 \quad \text{on } [0, T] \times \Omega, \quad (1e) \\
\text{div}_x E &= 4\pi \rho \quad \text{on } [0, T] \times \Omega, \quad (1f) \\
\text{div}_x B &= 0 \quad \text{on } [0, T] \times \Omega, \quad (1g) \\
E \times n &= B^{\text{tot}} \cdot n = 0 \quad \text{on } [0, T] \times \partial \Omega, \quad (1h) \\
(E, B)(0) &= \left( \tilde{E}, \tilde{B} \right) \quad \text{on } \Omega. \quad (1i)
\end{align*}
\]
This set of equations, imposed on some time interval \([0, T]\), describes the time evolution of a collisionless plasma which consists of \(N\) particle species. Equations \((1a)\) to \((1c)\) are to hold for each \(\alpha = 1, \ldots, N\), where \((1a)\) is the Vlasov equation for the density \(f^\alpha = f^\alpha(t, x, v)\) of the \(\alpha\)-th particle species. These densities depend on time \(t \in [0, T]\), position \(x \in \Omega\) and momentum \(v \in \mathbb{R}^3\), from which the relativistic velocity is computed via

\[
\hat{v}_\alpha = \frac{v}{\sqrt{m^2_\alpha + |v|^2}}.
\]

Here and throughout this paper, \(|·|\) denotes the Euclidean norm. The quantities \(m_\alpha\) and \(q_\alpha\) are the rest mass and charge of a particle of the \(\alpha\)-th species.

Equation \((1c)\) is the initial condition for \(f^\alpha\) and \((1b)\) describes the boundary condition on \(\partial \Omega\). Here, \(f^\alpha_{\pm}\) are the restrictions of \(f^\alpha\) to \(\gamma_{\pm}^\Omega := \{(t, x, v) \in [0, T] \times \partial \Omega \times \mathbb{R}^3 \mid v \cdot n(x) \gtrless 0\}\),

The operator \(K\) describes pure reflection on \(\partial \Omega\) via

\[
(Kh)(t, x, v) = h(t, x, v - 2(v \cdot n(x))),
\]

Above, \(n(x)\) denotes the outer unit normal of \(\partial \Omega\) at \(x \in \partial \Omega\).

Equations \((1d)\) to \((1g)\) are the Maxwell equations for the electromagnetic fields \(E = E(t, x), B = B(t, x)\) with initial condition \((1i)\). The source terms are

\[
\begin{align*}
  j &:= \sum_{\alpha=1}^{N} q_\alpha \int_{\mathbb{R}^3} \hat{v}_\alpha f^\alpha \, dv, &
  \rho &:= \sum_{\alpha=1}^{N} q_\alpha \int_{\mathbb{R}^3} f^\alpha \, dv,
\end{align*}
\]

the current and charge density \(j\) and \(\rho\) induced by the plasma particles. Moreover, \((1h)\) is the perfect conductor boundary condition.

Furthermore, we consider the case that an external magnetic field \(B^{\text{ext}}\) influences the plasma particles. Accordingly, the total magnetic field \(B^{\text{tot}} = B + B^{\text{ext}}\) appears in the Lorentz force in \((1a)\).

The aim of this paper is to answer the following two questions: First, for given time-independent external magnetic field, is there a stationary solution of \((1)\)? Second, are there stationary solutions that are confined in \(\Omega\), i.e., the particles stay away from the boundary of their container, if the external magnetic field is adjusted suitably?

Before we analyze these problems, we first discuss the basic ideas for plasma confinement—more information on fusion plasma physics can be found in the classical book of Stacey [19]. The physical basis for confinement is the fact that charged particles spiral about magnetic field lines. The so-called gyroradii, that is, the radius of such a spiral, is inversely proportional to the strength of the magnetic field. This gives rise to the idea of linear confinement devices: The fusion reactor is a long cylinder and the external magnetic field points in the direction of the symmetry axis of this cylinder. If this external magnetic field is sufficiently strong, the gyroradii of the plasma particles will be smaller than the radius of the cylinder, whence the plasma is confined in the fusion device. However, this setting cannot prevent the plasma current from having a nonvanishing component in the direction of the symmetry axis. Thus, there will be losses at the ends of the long cylinder. In practice, one can try to overcome this problem by one of the two following modifications: First, so-called magnetic mirrors are added at these ends. Second, the
long cylinder is bent into a torus. This second idea is pursued typically in modern research. Toroidal geometry has the advantage of avoiding such losses but has the disadvantage that it gives rise to drifts of the plasma particles, which finally cause the particles moving radially outwards and thus make confinement impossible. Therefore, the external magnetic field needs to have a poloidal component additional to its toroidal one. This approach then leads to Tokamak devices.

However, analyzing the problem of existence of confined steady states from a mathematics point of view in toroidal geometry seems quite hard. As a first step towards this, we consider the set-up of a linear confinement device instead. For mathematical reasons, it will be convenient to assume that the cylinder is infinitely long (which is of course not conceivable from a practical point of view). Thus, we fix $R_0 > 0$ and let
\[ \Omega := \{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < R_0^2 \}. \]

Because of the axial symmetry of the set-up, it is natural to work with cylindrical coordinates $(r, \varphi, x_3)$. In these coordinates, we simply have $\Omega = \{ x \in \mathbb{R}^3 \mid r < R_0 \}$.

In the following, there often occur cylindrical coordinates and the corresponding local, orthonormal coordinate basis $(e_r, e_\varphi, e_3)$, where
\[ e_r = (\cos \varphi, \sin \varphi, 0), \quad e_\varphi = (-\sin \varphi, \cos \varphi, 0), \quad e_3 = (0, 0, 1). \]

For a vector $w \in \mathbb{R}^3$, we denote with $w_r, w_\varphi, w_3$ the coordinates of $w$ in this local coordinate system, i.e.,
\[ w_r = w \cdot e_r, \quad w_\varphi = w \cdot e_\varphi, \quad w_3 = w \cdot e_3. \]

Note that the perfect conductor boundary condition $E \times n = 0 = B_{\text{tot}} \cdot n$ now reduces to $E_\varphi = E_3 = B_{\text{tot}}^r = 0$ in the case of $\Omega$ being an infinitely long cylinder, since here $n = e_r$.

It is convenient to introduce electromagnetic potentials, which will be the functions we work with mostly, namely the electric scalar potential $\phi$ and the magnetic vector potential $A_{\text{tot}} = A + A_{\text{ext}}$, which splits into the internal and external potentials $A$ and $A_{\text{ext}}$. The electromagnetic fields and potentials are related via
\[ E = -\partial_x \phi - \partial_t A, \quad B = \text{curl}_x A, \quad B_{\text{ext}} = \text{curl}_x A_{\text{ext}}. \]

Then, Gauss’s law for magnetism ($\text{div}_x B = 0$) and Faraday’s law ($\partial_t B + \text{curl}_x E = 0$) are automatically satisfied. There is some freedom to demand a certain gauge condition on the potentials. We will consider Lorenz gauge for the internal potentials
\[ \partial_t \phi + \text{div}_x A = 0, \]
which of course is the same as Coulomb gauge
\[ \text{div}_x A = 0 \]
if the potentials are independent of time, and similarly $\text{div}_x A_{\text{ext}} = 0$ for the external potential.

Similar set-ups have already been studied earlier, for example, in [16, 17]. The basic strategy to obtain steady states was first mentioned in [7]. Closely related to our considerations is [1], where (among other set-ups) existence of steady states in an infinitely long cylinder without external magnetic field was proved. However, an important condition there is that there is only one particle species and thus only a fixed sign of particle charges appears. Therefore, $\rho$ has a fixed sign and $\phi$ is monotone, which is crucial for the considerations in [1]. As opposed to this, we allow positively and negatively charged particles.
The question about existence of confined steady states for a Vlasov–Poisson plasma (that is, $B = 0$) by means of an external magnetic field was considered in [18] and [12]. The approach of the latter work is similar to ours but needs some smallness assumption on the ansatz functions, which we can avoid, and is restricted to homogeneous external magnetic fields parallel to the symmetry axis. Also, we refer to [3, 4, 5, 6] for considerations about confinement of a Vlasov–Poisson plasma.

There are also some papers concerning Vlasov–Maxwell plasmas and the problem of their confinement as well as concerning their stability [10, 14, 15, 24, 25].

Another approach to control a plasma by means of external fields has been pursued by Knopf and the author in [11, 13, 22, 23, 21].

This work is organized as follows: In Section 2, we state some basic assumptions on the symmetry of the appearing functions and state the corresponding invariant quantities, which lead to the natural ansatz concerning the densities $f^\alpha$. This ansatz, together with a basic definition and some useful preliminary lemmas and tools, is the content of Section 3. In Sections 4 and 5, we answer the above-mentioned questions. In particular, we prove existence of a steady state for a given external magnetic field and give conditions on the external magnetic potential under which the steady state is confined; see Theorems 4.4 and 5.1.

2. Symmetries and invariants. Due to the symmetry properties of $\Omega$, it is natural to consider the case that the tuple $(f^\alpha, \phi, A, A^{\text{ext}})$ has some symmetry properties as well:

Firstly, as $\Omega$ is invariant under translations in the $e_3$-direction, we assume that the tuple $(f^\alpha, \phi, A, A^{\text{ext}})$ is independent of $x_3$, that is,

$$f^\alpha = f^\alpha(t, x_1, x_2, v_1, v_2, v_3), \quad \phi = \phi(t, x_1, x_2),$$

$$A = A(t, x_1, x_2), \quad A^{\text{ext}} = A^{\text{ext}}(t, x_1, x_2).$$

Then of course the same property also holds for $E$, $B$, and $B^{\text{ext}}$. With this assumption, the resulting system is also called the “two and one-half dimensional” relativistic Vlasov–Maxwell system, since an $f^\alpha$ as above only depends on two space and three momentum variables. Due to Glassey and Schaeffer [9], unique, classical solutions of the resulting system in case of $\Omega = \mathbb{R}^3$ and $B^{\text{ext}} = 0$ exist globally in time under suitable assumptions about the initial data.

Secondly, as $\Omega$ is invariant under rotations about the $x_3$-axis, we assume that the tuple $(f^\alpha, \phi, A, A^{\text{ext}})$ has the following property:

$$f^\alpha(t, Rx, Rv) = f^\alpha(t, x, v), \quad \phi(t, Rx) = \phi(t, x),$$

$$A(t, Rx) = RA(t, x), \quad A^{\text{ext}}(t, Rx) = RA^{\text{ext}}(t, x)$$

for any rotation $R \in \mathbb{R}^{3 \times 3}$ about the $x_3$-axis. With the use of cylindrical coordinates, this assumption about the potentials is equivalent to the assumption that

$$\phi = \phi(t, r, x_3)$$

and that the components of the vector potentials in the local coordinate basis $(e_r, e_\varphi, e_3)$ be independent of the angle $\varphi$, that is,

$$A_r = A_r(t, r, x_3), \quad A_\varphi = A_\varphi(t, r, x_3), \quad A_3 = A_3(t, r, x_3),$$

$$A^{\text{ext}}_r = A^{\text{ext}}_r(t, r, x_3), \quad A^{\text{ext}}_\varphi = A^{\text{ext}}_\varphi(t, r, x_3), \quad A^{\text{ext}}_3 = A^{\text{ext}}_3(t, r, x_3).$$

With this symmetry, we can also reduce the number of variables in $(x, v)$-space from six to five and can write $f = f(r, x_3, \theta, u, v_3)$ where $u = \sqrt{v_1^2 + v_2^2}$ and $\theta$ is the
angle between \((x_1, x_2)\) and \((v_1, v_2)\). However, we will not make use of the Vlasov equation written in these variables.

Additionally to these two space symmetries, we consider time symmetry, i.e., the tuple \(((f^\alpha)_\alpha, \phi, A, A^{\text{ext}})\) is assumed to be independent of \(t\), since we are interested in the existence of (confined) steady states.

In cylindrical coordinates, there holds (for any scalar function \(\phi\) and any vector-valued function \(A\))
\[
\partial_x \phi = e_r \partial_r \phi + \frac{1}{r} e_\phi \partial_\phi \phi + e_3 \partial_{x_3} \phi,
\]
\[
\text{curl}_x A = e_r \left( \frac{1}{r} \partial_\phi A_3 - \partial_{x_3} A_\phi \right) + e_\phi (\partial_{x_3} A_r - \partial_r A_3) + \frac{1}{r} e_3 (\partial_r (r A_\phi) - \partial_\phi A_r).
\]

Thus, assuming time symmetry and the two space symmetries, (2) becomes
\[
E_r = -\partial_r \phi, \quad E_\phi = E_3 = 0, \quad B_r = 0, \quad B_\phi = -\partial_r A_3, \quad B_3 = \frac{1}{r} \partial_r (r A_\phi),
\]
\[
B_r^{\text{ext}} = 0, \quad B_\phi^{\text{ext}} = -\partial_r A_3^{\text{ext}}, \quad B_3^{\text{ext}} = \frac{1}{r} \partial_r (r A_\phi^{\text{ext}}).
\]

Hence, perfect conductor boundary conditions on \(\partial \Omega\) are always satisfied in this case and we can let \(A_r = 0\) without loss of generality since \(A_r\) does not affect the electromagnetic fields.

Using the gauge (3), the remaining Maxwell’s equations, i.e., \(\partial_t E - \text{curl}_x B = -4\pi j\) and \(\text{div}_x E = 4\pi \rho\), become
\[
\partial^2_t \phi - \Delta_x \phi = 4\pi \rho, \quad \partial^2_t A - \Delta_x A = 4\pi j,
\]
where the latter equation is to be understood componentwise (in Cartesian coordinates). In cylindrical coordinates, we have (for any scalar function \(\phi\) and any vector-valued function \(A\))
\[
\Delta_x \phi = \frac{1}{r} \partial_r (r \partial_r \phi) + \frac{1}{r^2} \partial^2_\phi \phi + \partial^2_{x_3} \phi,
\]
\[
\Delta_x A = e_r \left( \Delta_x A_r - \frac{A_r}{r^2} - \frac{2}{r^2} \partial_\phi A_\phi \right) + e_\phi \left( \Delta_x A_\phi - \frac{A_\phi}{r^2} + \frac{2}{r^2} \partial_r A_r \right) + e_3 \Delta_x A_3.
\]

Thus, assuming time symmetry, the two space symmetries, and \(A_r = 0\), on the one hand the gauge (3) is automatically satisfied, as there holds
\[
\text{div}_x A = \frac{1}{r} \partial_r (r A_r) + \frac{1}{r} \partial_\phi A_\phi + \partial_{x_3} A_3
\]
(5) in general, and on the other hand (4) becomes
\[
-\frac{1}{r} (r \phi')' = 4\pi \rho, \quad -\frac{1}{r} (r A_\phi)' = 4\pi j_\phi, \quad -\frac{1}{r} (r A_3') = 4\pi j_3.
\]
(6)

As \(\phi, A_\phi,\) and \(A_3\) only depend on \(r\), we denote the \(r\)-derivative with simply \('\). Note that the choice \(A_r = 0\) launches the constraint
\[
j_r = 0,
\]
i.e., no radial currents are allowed to appear.
A basic physical principle is that to each symmetry there corresponds an invariant. For each of the two space symmetries, we can derive an invariant from the Lagrangian (without the use of any gauge)

$$L^\alpha = \mathcal{L}^\alpha(t, x, \dot{x}) = -\sqrt{1 - |\dot{x}|^2} - q_\alpha(\phi(t, x) - \dot{x} \cdot A^{\text{tot}}(t, x)).$$

In particular, the invariant

$$G^\alpha := \partial_{x_3} L^\alpha = v_3 + q_\alpha A_3^{\text{tot}}$$

corresponds to translation invariance and

$$F^\alpha := \partial_{\phi} L^\alpha = \alpha(v_\alpha + q_\alpha A_\alpha^{\text{tot}})$$

corresponds to rotational symmetry. Note that in the formulae for $F^\alpha$ (the “canonical angular momentum”) and $G^\alpha$, components of the so-called “canonical momentum”

$$p_\alpha = v + q_\alpha A^{\text{tot}}$$

appear. In the variables $(x, p_\alpha)$, the particle energy

$$E^\alpha := v_\alpha^0 + q_\alpha \phi := \sqrt{m_\alpha^2 + |v|^2 + q_\alpha \phi} = \sqrt{m_\alpha^2 + |p_\alpha - q_\alpha A^{\text{tot}}|^2 + q_\alpha \phi}$$

is the (in general time-dependent) Hamiltonian governing the motion of the particles of the $\alpha$-th species. Assuming that the electromagnetic potentials are independent of time, $E^\alpha$ is also independent of time and thus another invariant, the one corresponding to time symmetry.

3. Steady states—definition and ansatz. The preceding considerations about symmetry motivate the definition of what we call a (confined) steady state in our set-up. Before that we collect our symmetry assumptions.

Definition and Remark 3.1. (a) A function $f : \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}$ / a function $\phi : \overline{\Omega} \to \mathbb{R}$ / a vector field $A : \overline{\Omega} \to \mathbb{R}^3$ is called

(i) independent of $x_3$ if $\partial_{x_3} f = 0$ / $\partial_{x_3} \phi = 0$ / $\partial_{x_3} A = 0$;

(ii) axially symmetric if $f(Rx, Rv) = f(x, v)$ for any $x \in \overline{\Omega}$, $v \in \mathbb{R}^3$, and rotation $R \in \mathbb{R}^{3 \times 3}$ about the $x_3$-axis / $\phi(Rx) = \phi(x)$ for any $x \in \overline{\Omega}$ and rotation $R \in \mathbb{R}^{3 \times 3}$ about the $x_3$-axis / $A(Rx) = RA(x)$ for any $x \in \overline{\Omega}$ and rotation $R \in \mathbb{R}^{3 \times 3}$ about the $x_3$-axis. 

(b) With these two symmetries, the functions $\phi$, $A_r$, $A_\varphi$, and $A_3$ only depend on $r$. Accordingly, we will often view them as functions on $[0, R_0]$.

(c) An axially symmetric vector field $A$ automatically satisfies $A_1(x) = A_2(x) = 0$ if $x_1 = x_2 = 0$, i.e., if $x$ lies on the $x_3$-axis.

We proceed with an assumption about the external potential, which is supposed to hold henceforth.

Condition 3.2. The external potential $A^{\text{ext}} : \overline{\Omega} \to \mathbb{R}$ is independent of $x_3$ and axially symmetric such that $A_r^{\text{ext}} = 0$ and $A_\varphi^{\text{ext}}$, $A_3^{\text{ext}} \in C^1([0, R_0])$ (viewed as functions of $r$) with $A_\varphi^{\text{ext}}(0) = A_3^{\text{ext}}(0) = (A_3^{\text{ext}})'(0) = 0$.

Note that $A_3^{\text{ext}}(0) = 0$ can be assumed—for simplicity—without loss of generality since adding a constant to $A_3^{\text{ext}}$ does not affect $B^{\text{ext}}$ because of $\text{curl}_x e_3 = 0$ (as opposed to this, this invariance under adding constants does not hold for $A_\varphi^{\text{ext}}$, as $\text{curl}_x e_\varphi \neq 0$).

We first prove some technicalities.
Lemma 3.3. Let $\phi, A_\varphi, A_3 \in C^1([0, R_0])$ with

$$\phi'(0) = A_\varphi(0) = A_3'(0) = 0$$

and assume $A_r = 0$. Then there holds:

(i) The potentials $\phi = \phi(x)$ and $A = A(x)$ are continuously differentiable on $\overline{\Omega}$.

Thus, the electromagnetic fields

$$E = -\partial_x \phi = -\phi' e_r, \quad B = \text{curl}_x A = -A_r' e_\varphi + \frac{1}{r} (r A_\varphi)' e_3$$

are continuous on $\overline{\Omega}$. Moreover, $\text{div}_x A = 0$ on $\overline{\Omega}$.

(ii) If $\phi, A_3 \in C^2([0, R_0])$, they are even twice continuously differentiable on $\overline{\Omega}$ with respect to $x$. Accordingly, $E$ is of class $C^1$ on $\overline{\Omega}$. If moreover $A_\varphi \in C^2([0, R_0])$ such that

$$A_r'(r) - \frac{A_\varphi(r)}{r} = \mathcal{O}(r), \quad A_\varphi''(r) = \mathcal{O}(1) \quad \text{for } r \to 0,$$

then $A \in W^{2, \infty}(\Omega; \mathbb{R}^3) \cap C^1(\overline{\Omega} \setminus \mathbb{R} e_3; \mathbb{R}^3)$. Accordingly, $B$ is of class $W^{1, \infty}$ on $\Omega$ and of class $C^1$ on $\overline{\Omega} \setminus \mathbb{R} e_3$.

Proof. We easily see that the maps $x \mapsto \phi(x)$ and $x \mapsto A_3(x)e_3$ are (twice) continuously differentiable on $\overline{\Omega}$ if the maps $r \mapsto \phi(r)$ and $r \mapsto A_3(r)$ are (twice) continuously differentiable on $[0, R_0]$ since $\phi'(0) = A_3'(0) = 0$. There remains to take care of $x \mapsto A_\varphi(x)e_\varphi(x)$, in particular at $r = 0$. Indeed, this map can be continuously extended to whole $\overline{\Omega}$ because of $A_\varphi(0) = 0$ and is differentiable for $r > 0$ with

$$\partial_x (A_\varphi e_\varphi)(r, \varphi) = \begin{pmatrix} -\sin \varphi \cos \varphi \left( A_\varphi'(r) - \frac{A_\varphi(r)}{r} \right) & -\sin^2 \varphi \left( A_\varphi'(r) - \frac{A_\varphi(r)}{r} \right) & -\sin \varphi \cos \varphi \left( A_\varphi'(r) - \frac{A_\varphi(r)}{r} \right) \\ \cos^2 \varphi \left( A_\varphi'(r) - \frac{A_\varphi(r)}{r} \right) & \sin \varphi \cos \varphi \left( A_\varphi'(r) - \frac{A_\varphi(r)}{r} \right) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where all entries have a limit as $r \to 0$. Hence, also $A_\varphi e_\varphi$ is continuously differentiable on whole $\overline{\Omega}$. Furthermore, $A$ is divergence free with respect to $x$, as was already observed in Section 2 because of (5). Thus, 3.3.(i) is proved. If moreover the assumptions about $A_\varphi$ in 3.3.(ii) are satisfied, all second order derivatives (with respect to $x$) of $A_\varphi e_\varphi$ are bounded for $r \to 0$, since we see by differentiating the entries of (10) once more that these second order derivatives are expressions in $\sin \varphi$, $\cos \varphi$, $\frac{1}{r} \left( A_\varphi'(r) - \frac{A_\varphi(r)}{r} \right)$, and $A''_\varphi(r)$, and thus bounded by assumption. Therefore, all second order derivatives exist on $\Omega$ in the weak sense, coincide with the classical derivatives almost everywhere, and are bounded. This proves the remaining part of 3.3.(ii).

Note that this lemma yields that under Condition 3.2 the external potential $A^{\text{ext}}$ is continuously differentiable on $\overline{\Omega}$ and divergence free, and that the external magnetic field $B^{\text{ext}} = \text{curl}_x A^{\text{ext}}$ is continuous on $\overline{\Omega}$.

Remark 3.4. In Lemma 3.3.(ii), we cannot expect that $A \in C^2(\overline{\Omega}; \mathbb{R}^3)$ in general if $A_\varphi \in C^2([0, R_0])$ and (9) holds, as the example $A_\varphi(r) = r^2$ shows since

$$\Delta_x (A_\varphi e_\varphi)_1 = -\Delta_x (r^2 \sin \varphi) = -3 \sin \varphi$$
has no limit for \( r \to 0 \).

We proceed with a basic definition.

**Definition 3.5.** Let Condition 3.2 hold.

(a) A tuple \(((f^\alpha)_\alpha, \phi, A)\) is called an axially symmetric steady state of the two and one-half dimensional relativistic Vlasov–Maxwell system on \( \overline{\Omega} \) with external potential \( A^{\text{ext}} \) (hereafter abbreviated as steady state) if the following conditions are satisfied:

(i) For each \( \alpha = 1, \ldots, N \), the functions \( f^\alpha: \overline{\Omega} \times \mathbb{R}^3 \to [0, \infty[ \) are continuously differentiable satisfying \( f^\alpha(x, \cdot) \in L^1(\mathbb{R}^3) \) for each \( x \in \overline{\Omega} \).

(ii) The potentials satisfy

\[
\phi \in C^2(\overline{\Omega}), \quad A \in C^1(\Omega; \mathbb{R}^3) \cap C^2(\overline{\Omega} \setminus \{ e_3 \}; \mathbb{R}^3) \cap W^{2, \infty}(\Omega; \mathbb{R}^3).
\]

(This condition is motivated in view of Lemma 3.3.)

(iii) Any \( f^\alpha \) and \( \phi, A \) are independent of \( x_3 \) and axially symmetric.

(iv) The equations

\[
\hat{v}_\alpha \cdot \partial_x f^\alpha + q_\alpha (E + \hat{v}_\alpha \times B^{\text{tot}}) \cdot \partial_v f^\alpha = 0 \quad \text{on} \quad \overline{\Omega} \times \mathbb{R}^3, \quad (11a)
\]

\[
f^\alpha(x, v - 2v_r e_r) = f^\alpha(x, v), \quad x \in \partial \Omega, \quad v \in \mathbb{R}^3, \quad v_r < 0, \quad (11b)
\]

\[- \Delta_x \phi = 4\pi \rho, \quad - \Delta_x A = 4\pi j, \quad \text{div}_x A = 0 \quad \text{on} \quad \overline{\Omega}, \quad (11c)
\]

are satisfied. Here, \( e_r = e_r(x) \), \( v_r = v \cdot e_r \), and

\[
E = - \partial_x \phi, \quad B^{\text{tot}} = \text{curl}_x (A + A^{\text{ext}}),
\]

\[
\rho = \sum_{\alpha=1}^N q_\alpha \int_{\mathbb{R}^3} f^\alpha dv, \quad j = \sum_{\alpha=1}^N q_\alpha \int_{\mathbb{R}^3} \hat{v}_\alpha f^\alpha dv.
\]

(b) A steady state \(((f^\alpha)_\alpha, \phi, A)\) is said to

(i) have finite charge if

\[
\int_{B_{R_0}} \int_{\mathbb{R}^3} f^\alpha dv(x_1, x_2) < \infty
\]

for each \( \alpha = 1, \ldots, N \);

(ii) be compactly supported with respect to \( v \) if there is \( S > 0 \) such that \( f^\alpha(x, v) = 0 \) for each \( \alpha = 1, \ldots, N \), \( x \in \overline{\Omega} \), \( |v| \geq S \);

(iii) be nontrivial if \( f^\alpha \neq 0 \) for each \( \alpha = 1, \ldots, N \);

(iv) be confined with radius at most \( R \) if \( 0 < R < R_0 \) such that \( f^\alpha(x, v) = 0 \) for each \( \alpha = 1, \ldots, N \), \( x \in \overline{\Omega} \) with \( |(x_1, x_2)| \geq R \), and \( v \in \mathbb{R}^3 \).

Note that perfect conductor boundary conditions are automatically satisfied due to symmetry, as was already observed in Section 2, and are thus omitted in (11).

**Remark 3.6.** A physically reasonable steady state should have finite charge, which usually means \( f^\alpha \in L^1(\Omega \times \mathbb{R}^3) \) for each \( \alpha = 1, \ldots, N \). However, this is impossible in our setting (unless all \( f^\alpha \) vanish identically) by \( f^\alpha \) being independent of \( x_3 \). Thus, here we have to modify this definition suitably as above.

According to [7], the natural ansatz for \( f^\alpha \) is that

\[
f^\alpha = \eta^\alpha(\mathcal{E}_\alpha, \mathcal{F}_\alpha, \mathcal{G}_\alpha)
\]

is a function of the three invariants obtained in Section 2. We collect some basic assumptions about the ansatz functions \( \eta^\alpha \).
**Condition 3.7.** For each $\alpha = 1, \ldots, N$ there holds:

- $(i)$ $\eta^\alpha \in C^1(\mathbb{R}^3; [0, \infty))$;
- $(ii)$ there exists $\eta^\alpha_{\#} \in L^1(\mathbb{R}^2)$ such that
  \[
  \int_{\mathbb{R}^2} |\mathcal{E}\eta^\alpha (x, \mathcal{G})| \, d(\mathcal{E}, \mathcal{G}) < \infty
  \]
  and
  \[
  |\eta^\alpha (x, \mathcal{F}, \mathcal{G})| \leq \eta^\alpha_{\#} (\mathcal{E}, \mathcal{G})
  \]
  for all $(\mathcal{E}, \mathcal{F}, \mathcal{G}) \in \mathbb{R}^3$;
- $(iii)$ there exists $\eta^\alpha_{\#} : \mathbb{R}^2 \to \mathbb{R}$ such that
  \[
  \forall \mathcal{d} \in \mathbb{R} : \eta^\alpha_{\#} |E\eta^\alpha_{\#} (x) \in L^1([d, \infty) \times \mathbb{R})
  \]
  and
  \[
  |\nabla \eta^\alpha (x, \mathcal{F}, \mathcal{G})| \leq \eta^\alpha_{\#} (\mathcal{E}, \mathcal{G})
  \]
  for all $(\mathcal{E}, \mathcal{F}, \mathcal{G}) \in \mathbb{R}^3$.

We first prove that the ansatz (12) already ensures (11a) and (11b). Here and in the following, we will always write $A^\text{tot} = A + A^\text{ext}$.

**Lemma 3.8.** Let Conditions 3.2 and 3.7.$(i)$ hold and let $\phi, A_\varphi, A_3 \in C^1([0, R_0])$ with

\[
\phi'(0) = A_\varphi(0) = A_3(0) = 0.
\]
Then, for each $\alpha = 1, \ldots, N$,

\[
f^\alpha : \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}, \quad f^\alpha (x, v) = \eta^\alpha (\mathcal{E}^\alpha(x, v), \mathcal{F}^\alpha(x, v), \mathcal{G}^\alpha(x, v))
\]

\[
= \eta^\alpha (v_0 + q_\alpha \phi(r), r(v_\varphi + q_\alpha A^\text{tot}_\varphi(r)), v_3 + q_\alpha A^\text{tot}_3(r))
\]

(13)
is continuously differentiable, independent of $x_3$, axially symmetric, and satisfies (11a) and (11b).

**Proof.** We first note that $f^\alpha$ is continuously differentiable because of $r \nu = x_1 v_2 - x_2 v_1$ and $\phi'(0) = (r A^\text{tot}_\varphi)'(0) = (A^\text{tot}_3)'(0) = 0$. Clearly, $f^\alpha$ is independent of $x_3$ and axially symmetric. Furthermore, it is easy to see that (11b) holds since $\mathcal{E}^\alpha$ is even in $v$, and $\mathcal{F}^\alpha, \mathcal{G}^\alpha$ do not depend on $v_r$. To ensure (11a) for $f^\alpha$ it suffices to prove that $\mathcal{E}^\alpha, \mathcal{F}^\alpha$, and $\mathcal{G}^\alpha$ themselves satisfy (11a)—this clearly holds, as they are invariants of the motion; for the sake of completeness, we carry out the computation. Since they are of class $C^1$ on $\overline{\Omega} \times \mathbb{R}^3$, this only needs to be verified for $r > 0$. In the following, have (8) in mind. Firstly,

\[
\hat{v}_\alpha \cdot \partial_x \mathcal{E}^\alpha + q_\alpha (E + \hat{v}_\alpha \times B^\text{tot}) \cdot \partial_v \mathcal{E}^\alpha = -q_\alpha \hat{v}_\alpha \cdot E + q_\alpha (E + \hat{v}_\alpha \times B^\text{tot}) \cdot \hat{v}_\alpha = 0.
\]

Secondly,

\[
\hat{v}_\alpha \cdot \partial_x \mathcal{F}^\alpha + q_\alpha (E + \hat{v}_\alpha \times B^\text{tot}) \cdot \partial_v \mathcal{F}^\alpha
\]

\[
= \hat{v}_\alpha \cdot (v_\varphi + q_\alpha A^\text{tot}_\varphi) e_\varphi - \hat{v}_\alpha \cdot v_\varphi e_\varphi + q_\alpha \hat{v}_\alpha \cdot r(A^\text{tot}_\varphi)' e_r + q_\alpha (E + \hat{v}_\alpha \times B^\text{tot}) \cdot r e_\varphi
\]

\[
= q_\alpha \hat{v}_\alpha \cdot e_r (A^\text{tot}_\varphi + r(A^\text{tot}_\varphi)' - r \cdot \frac{1}{r} (r A^\text{tot}_\varphi)' ) = 0.
\]

Thirdly,

\[
\hat{v}_\alpha \cdot \partial_x \mathcal{G}^\alpha + q_\alpha (E + \hat{v}_\alpha \times B^\text{tot}) \cdot \partial_v \mathcal{G}^\alpha
\]
Lemma 3.9. Let \( \phi : [0, R_0] \to \mathbb{R} \), \( A : [0, R_0] \to \mathbb{R}^3 \), Condition 3.7.(ii) hold, and \( f^\alpha \) be defined as in (13) for each \( \alpha = 1, \ldots, N \). Then, \( f^\alpha(x, \cdot) \in L^1(\mathbb{R}^3) \) for each \( x \in \Omega \). Furthermore, \( \rho \) and \( j \) are independent of \( x_3 \) and axially symmetric, and we have

\[
\begin{align*}
4\pi \rho(r) &= g_1(r, \phi(r), A_{\phi}^{\mathrm{tot}}(r), A_3^{\mathrm{tot}}(r)), \quad j_r(r) = 0 \quad (14a) \\
4\pi j_\phi(r) &= g_2(r, \phi(r), A_{\phi}^{\mathrm{tot}}(r), A_3^{\mathrm{tot}}(r)), \quad 4\pi j_3(r) = g_3(r, \phi(r), A_{\phi}^{\mathrm{tot}}(r), A_3^{\mathrm{tot}}(r)) \quad (14b)
\end{align*}
\]

for \( r \in [0, R_0] \), where \( g_1, g_2, g_3 : [0, R_0] \times \mathbb{R}^3 \to \mathbb{R} \),

\[
\begin{pmatrix}
g_1 \\
g_2 \\
g_3
\end{pmatrix}(r, a, b, c)
= 4\pi \sum_{\alpha=1}^{N} q_\alpha \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{0}^{2\pi} \frac{\mathcal{E} - q_\alpha a}{\sqrt{(\mathcal{E} - q_\alpha a)^2 - (G - q_\alpha c)^2 - m_\alpha^2 \sin \theta}} \cdot \eta^\alpha \left( \mathcal{E}, r \sqrt{(\mathcal{E} - q_\alpha a)^2 - (G - q_\alpha c)^2 - m_\alpha^2 \sin \theta + rq_\alpha b, G} \right) \, d\theta \, d\mathcal{E} \, dG
\]

are continuous functions. Moreover,

\[
\left| (g_2^\alpha, g_3^\alpha) \right| \leq |g_1^\alpha| \quad (16)
\]
on \( [0, R_0] \times \mathbb{R}^3 \) for each \( \alpha = 1, \ldots, N \).

Proof. At least formally we have

\[
\begin{align*}
\int_{\mathbb{R}^3} \frac{1}{\varepsilon_3} \left( \tilde{\nu}_\alpha \cdot \epsilon_r \right) \eta^\alpha (\mathcal{E}^\alpha, \mathcal{F}^\alpha, G^\alpha) \, dv &= \int_{\mathbb{R}} \int_{0}^{\infty} \int_{0}^{2\pi} \frac{u}{\sqrt{m_\alpha^2 + u^2 + v_3^2}} \left( \frac{u \cos \theta}{u \sin \theta} \right) \eta^\alpha \left( \sqrt{m_\alpha^2 + u^2 + v_3^2} \right) \, d\theta \, du \, dv_3 \\
&= \int_{\mathbb{R}} \int_{0}^{\infty} \int_{0}^{2\pi} \frac{u}{\sqrt{m_\alpha^2 + u^2 + v_3^2} + q_\alpha \phi(r)} \, ru \sin \theta + rq_\alpha A_{\phi}^{\mathrm{tot}}(r), v_3 + q_\alpha A_3^{\mathrm{tot}}(r) \right) \, d\theta \, du \, dv_3
\end{align*}
\]
the problem of finding a steady state with the ansatz (12) reduces to finding $\phi$
is continuous.
is sufficient that $r \phi$
state obtained in the following sections has finite charge. Indeed, for this it is
Remark 3.10. The proof of preceding lemma additionally shows that any steady

where we introduced polar coordinates in the ($v_1, v_2$)-plane with basis ($e_r, e_\varphi$) and
then substituted firstly $E = \sqrt{m_0^2 + u^2 + v_3^2 + q_0 \phi(r)}$ and secondly $G = v_3 +
Note that the integral in the second line vanishes after substituting $y = \sin \theta$. Due to Condition 3.7.(ii), the modulus of the integrand in the first line
can be estimated by

and is hence integrable. Because of $|\tilde{\varepsilon}_a| < 1$ also the other integrals exist. Thus, the
above calculation is legitimated. Multiplying these identities with $q_a$ and summing
over $a$ yields the representation. The above estimate on the integrands also implies
that $g_i$ is continuous, $i = 1, 2, 3$. Finally, (16) is also a consequence of $|\tilde{\varepsilon}_a| < 1$. □

According to Lemma 3.9, after integrating (6) and using the representation (14)
the problem of finding a steady state with the ansatz (12) reduces to finding $\phi$, 
$A_3 \in C^2([0, R_0])$, $A_\varphi \in C^2([0, R_0]) \cap C^4([0, R_0])$ satisfying (7), (9), and

\begin{align}
\phi(r) &= -\int_0^r \frac{1}{s} \int_s^R g_1(\sigma, \phi(\sigma), A_{\varphi}^{\text{tot}}(\sigma), A_3^{\text{tot}}(\sigma)) \, d\sigma \, ds, \\
A_\varphi(r) &= -\frac{1}{r} \int_0^r s \int_0^s g_2(\sigma, \phi(\sigma), A_{\varphi}^{\text{tot}}(\sigma), A_3^{\text{tot}}(\sigma)) \, d\sigma \, ds, \\
A_3(r) &= -\int_0^r \frac{1}{s} \int_0^s g_3(\sigma, \phi(\sigma), A_{\varphi}^{\text{tot}}(\sigma), A_3^{\text{tot}}(\sigma)) \, d\sigma \, ds
\end{align}

for $r > 0$ in view of Lemmas 3.3 and 3.8; note that we could prescribe arbitrary values for $\phi$ and $A_3$ at $r = 0$—adding constants to $\phi$ or $A_3$ does not affect the
electromagnetic fields $E$ and $B$—and we choose both of these values to be zero.

Therefore, it is convenient to introduce the map

\[ M: C([0, R_0]; \mathbb{R}^3) \to C([0, R_0]; \mathbb{R}^3), \]

\[ M(\phi, A_\varphi, A_3) = \begin{pmatrix} 0, R_0 \end{pmatrix} \ni r \mapsto \begin{pmatrix} -\int_0^r \frac{1}{s} \int_s^R g_1(\sigma, \phi(\sigma), A_{\varphi}^{\text{tot}}(\sigma), A_3^{\text{tot}}(\sigma)) \, d\sigma \, ds \\ -\frac{1}{r} \int_0^r s \int_0^s g_2(\sigma, \phi(\sigma), A_{\varphi}^{\text{tot}}(\sigma), A_3^{\text{tot}}(\sigma)) \, d\sigma \, ds \\ -\int_0^r \frac{1}{s} \int_0^s g_3(\sigma, \phi(\sigma), A_{\varphi}^{\text{tot}}(\sigma), A_3^{\text{tot}}(\sigma)) \, d\sigma \, ds \end{pmatrix} \]

with the interpretation $M(\phi, A_\varphi, A_3)(0) = (0, 0, 0)$. The following lemma shows
that indeed $M$ is well-defined and that it suffices to search for fixed points of $M$.

Lemma 3.11. Assume Conditions 3.2, 3.7.(i), and 3.7.(ii).

(i) For any $(\phi, A_\varphi, A_3) \in C([0, R_0]; \mathbb{R}^3)$ we have

\[ \left( \tilde{\phi}, \tilde{A}_\varphi, \tilde{A}_3 \right) := M(\phi, A_\varphi, A_3) \in C^2([0, R_0]; \mathbb{R}^3). \]
Furthermore, \( \left( \tilde{\phi}, \tilde{A}_2, \tilde{A}_3 \right) \) satisfies (7) and (9).

(ii) If \((\phi, A_2, A_3) \in C([0, R_0]; \mathbb{R}^3)\) is a fixed point of \( \mathcal{M} \), then \((f^\alpha)_\alpha, \phi, A)\) is a steady state, where the \(f^\alpha\) are defined via the ansatz (12).

Proof. Due to Lemma 3.9, the functions

\[
\tilde{g}_i: [0, R_0] \to \mathbb{R}, \quad \tilde{g}_i(\sigma) = g_i(\sigma, \phi(\sigma), A_1^{\text{tot}}(\sigma), A_3^{\text{tot}}(\sigma))
\]

are continuous, \(i = 1, 2, 3\), and hence bounded by some constant \(C > 0\). Thus, there holds

\[
\left| \tilde{\phi}'(r) \right|, \left| \tilde{A}_3(r) \right| \leq C \int_0^r \frac{1}{s} \int_0^s \sigma d\sigma = \frac{C}{4} r^2,
\]

\[
\left| \tilde{A}_3(r) \right| \leq C \frac{r}{r} \int_0^r \int_0^s \sigma d\sigma = \frac{C}{3} r^2
\]

for \(r \in [0, R_0]\). Hence, \(\tilde{\phi}, \tilde{A}_2,\) and \(\tilde{A}_3\) are continuous also in \(r = 0\), and \(\tilde{A}_2(r) = O(r)\) for \(r \to 0\). Furthermore, the ‘tilde’-potentials are twice continuously differentiable on \([0, R_0]\) with

\[
\tilde{\phi}'(r) = -\frac{1}{r} \int_0^r \tilde{g}_1(s) ds, \quad \tilde{\phi}''(r) = \frac{1}{r^2} \int_0^r \tilde{g}_1(s) ds - \tilde{g}_1(r),
\]

\[
\tilde{A}_3'(r) = \frac{1}{r^2} \int_0^r \tilde{g}_2(s) ds - \int_0^r \tilde{g}_2(s) ds,
\]

\[
\tilde{A}_3''(r) = \frac{2}{r^3} \int_0^r \tilde{g}_2(s) ds + \frac{1}{r} \int_0^r \tilde{g}_2(s) ds - \tilde{g}_2(r).
\]

Because of

\[
\left| \tilde{\phi}'(r) \right|, \left| \tilde{A}_3(r) \right| \leq C \int_0^r \int_0^s \sigma d\sigma = \frac{C}{2} r,
\]

\[
\left| \tilde{A}_3'(r) \right| \leq C \frac{r}{r^2} \int_0^r \int_0^s \sigma d\sigma + C r = \frac{4C}{3} r
\]

they are continuously differentiable on whole \([0, R_0]\) with vanishing derivative at \(r = 0\), and moreover \(\tilde{A}_2'(r) = O(r)\) for \(r \to 0\). Furthermore, by l’Hôpital’s rule we have

\[
\lim_{r \to 0} \tilde{\phi}''(r) = \lim_{r \to 0} \frac{r \tilde{g}_1(r)}{2r} - \frac{\tilde{g}_1(0)}{2} = -\frac{\tilde{g}_1(0)}{2},
\]

\[
\lim_{r \to 0} \tilde{A}_3''(r) = -\lim_{r \to 0} \frac{2r \tilde{g}_2(s) ds}{3r^2} + \tilde{g}_2(0) - \tilde{g}_2(0) = -\frac{2\tilde{g}_2(0)}{3},
\]

\[
\lim_{r \to 0} \tilde{A}_3''(r) = \lim_{r \to 0} \frac{r \tilde{g}_3(r)}{2r} - \frac{\tilde{g}_3(0)}{2} = -\frac{\tilde{g}_3(0)}{2}.
\]

Therefore, \(\tilde{\phi}, \tilde{A}_2, \tilde{A}_3 \in C^2([0, R_0])\) and clearly \(\tilde{A}_2'(r) = O(1)\) for \(r \to 0\). Finally, from Lemmas 3.3, 3.8 and 3.9 it follows that \((f^\alpha)_\alpha, \phi, A)\) is a steady state if \((\phi, A_2, A_3)\) is a fixed point of \(\mathcal{M}\); note that (17) implies (6) and this yields \(-\Delta_x \phi = 4\pi \rho\) on \(\bar{\Omega}\) and \(-\Delta_x A = 4\pi j\) on \(\bar{\Omega} \setminus \mathbb{R}e_3\) in the classical sense, and \(-\Delta_x A = 4\pi j\) on \(\Omega\) in the weak sense. \(\square\)
4. Existence of steady states.

4.1. A priori estimates. Hence, there only remains to find a fixed point of $M$. For this, the most important tool is to derive a priori bounds for the potentials. Therefore, we assume that we already have a solution $(\phi, A_\phi, A_3) \in C([0, R_0]; \mathbb{R}^3)$ of (17) for the time being. Due to (15), we first have the following estimate on $g_1^\alpha$ for each $(r, a, b, c) \in [0, R_0] \times \mathbb{R}^3$:

$$|g_1^\alpha(r, a, b, c)| \leq 4\pi |q_\alpha| \cdot 2\pi \int_{\mathbb{R}^2} (|E| + |q_\alpha| |a|) \eta_\alpha^\phi(E, G) \, d(E, G).$$

Using (16) and summing over $\alpha$ yield

$$|g_i(r, a, b, c)| \leq c_1 + c_2 |a|, \quad i = 1, 2, 3,$$

(18)

where we introduced the abbreviations

$$c_1 := 8\pi^2 \sum_{\alpha=1}^{N} |q_\alpha| \int_{\mathbb{R}^2} |E| \eta_\alpha^\phi(E, G) \, d(E, G) < \infty,$$

$$c_2 := 8\pi^2 \sum_{\alpha=1}^{N} |q_\alpha|^2 \int_{\mathbb{R}^2} \eta_\alpha^\phi(E, G) \, d(E, G) < \infty.$$

Therefore, in view of (17a) an integral inequality for $\phi$ follows, in particular

$$|\phi(r)| \leq \int_{0}^{r} \frac{1}{s} \int_{0}^{s} \sigma(c_1 + c_2 |\phi(\sigma)|) \, d\sigma \, ds = \frac{c_1}{4} r^2 + c_2 \int_{0}^{r} \frac{1}{s} \int_{0}^{s} \sigma |\phi(\sigma)| \, d\sigma \, ds$$

(19)

for $r \in [0, R_0]$. We could thus easily derive the inequality

$$|\phi(r)| \leq \frac{c_1}{4} R_0^2 + c_2 R_0 \int_{0}^{r} |\phi(s)| \, ds$$

(20)

and therefore

$$|\phi(r)| \leq \frac{c_1}{4} R_0^2 e^{c_2 R_0 r}$$

(21)

via Gronwall’s lemma. However, (20) is way too crude and hence (21) is not very sharp. If we were to use this a priori estimate later to show confinement of a steady state, the needed assumption about the external potential would be quite strong. Consequently, in order to allow a wider class for external potentials ensuring confinement later, we now search for a sharper a priori estimate on $\phi$.

Thus, we search for a solution of the integral equation corresponding to (19), that is,

$$\xi(r) = \frac{c_1}{4} r^2 + c_2 \int_{0}^{r} \frac{1}{s} \int_{0}^{s} \sigma \xi(\sigma) \, d\sigma \, ds.$$
with nonnegative, square integrable Volterra kernel

\[ V : [0, R_0]^2 \to \mathbb{R}, \quad V(r, s) = \begin{cases} c_2 (\ln r - \ln s)s, & 0 < s \leq r \leq R_0, \\ 0, & \text{else.} \end{cases} \]

It is well known that Volterra integral equations such as (24) have a unique square integrable solution; see [20, Section 1.5]. To find this solution, we rather work with (22), which suggests a series ansatz

\[ \xi(r) = \sum_{k=0}^{\infty} a_k r^k \]

for \( \xi \). With this ansatz, at least formally we demand

\[
\sum_{k=0}^{\infty} a_k r^k = \frac{c_1}{4} r^2 + c_2 \int_{0}^{r} \frac{1}{s} \int_{0}^{s} \sigma \xi(\sigma) \, d\sigma \, ds = \frac{c_1}{4} r^2 + c_2 \int_{0}^{r} \frac{1}{s} \int_{0}^{s} \sum_{k=0}^{\infty} a_k \sigma^k \, d\sigma \, ds
\]

\[
= \frac{c_1}{4} r^2 + c_2 \int_{0}^{r} \sum_{k=0}^{\infty} a_k \frac{k+2}{k+1} s^{k+1} \, ds = \frac{c_1}{4} r^2 + c_2 \sum_{k=0}^{\infty} a_k \frac{k+2}{k+1} r^{k+2}
\]

\[
= \frac{c_1}{4} r^2 + \sum_{k=2}^{\infty} \frac{c_2 a_{k-2}}{k^2} r^k.
\]

Thus,

\[ a_0 = a_1 = 0, \quad a_2 = \frac{c_1}{4} + \frac{c_2 a_0}{2^2} = \frac{c_1}{4} . \]

Therefore, \( a_k = 0 \) if \( k \) is odd, and

\[ a_{2m} = \frac{c_2 a_{2(m-1)}}{4m^2} \]

for \( m \geq 2 \). Hence, we have

\[ a_{2m} = \frac{c_1 c_{2m-1}}{4m^2 (m!)^2} \]

for \( m \in \mathbb{N} \) by induction. Consequently, we define

\[ \xi : \mathbb{R} \to \mathbb{R}, \quad \xi(r) = \sum_{k=1}^{\infty} \frac{c_1 c_{2k-1}}{4^k (k!)^2} r^{2k}. \]

Obviously, this series is uniformly convergent on any bounded interval, whence the calculation (25) is legitimated and \( \xi \) indeed is the unique square integrable solution of (24) on \([0, R_0]\) by (23). Moreover, \( \phi \) satisfies the corresponding integral inequality

\[ |\phi(r)| \leq \frac{c_1}{4} r^2 + c_2 \int_{0}^{r} (\ln r - \ln s)|\phi(s)| \, ds. \]

Thus, there holds

\[ |\phi(r)| \leq \xi(r) \quad (26) \]

for all \( r \in [0, R_0] \) as a consequence of the positivity of Volterra operators in the case \( V \geq 0 \); see [2, Theorem 5]. Therefore, we have established a quite sharp a priori bound on \( \phi \).
In order to obtain similar estimates also for $A_\varphi$ and $A_3$, we insert (18) and (26) into (17b) and (17c). On the one hand, we conclude assume that Condition 3.7 holds and equip the space do exist via some fixed point argument. Throughout the rest of this section, we

\[ |A_\varphi(r)| \leq \frac{1}{r} \int_0^r s \int_0^s (c_1 + c_2 |\phi(\sigma)|) d\sigma ds \leq \frac{c_1}{3} r^2 + \frac{c_2}{r} \int_0^r s \int_0^s \xi(\sigma) d\sigma ds \]

\[ = \frac{c_1}{3} r^2 + \frac{c_2}{r} \int_0^r \sum_{k=1}^\infty \frac{c_1 c_2^{k-1}}{(2k+1)(2k+3)4^k(k!)^2} s^{2k+2} ds \]

\[ = \frac{c_1}{3} r^2 + \sum_{k=1}^\infty \frac{c_1 c_2^{k-1}}{(2k+1)(2k+3)4^k(k!)^2} r^{2k+2} = \sum_{k=1}^\infty \frac{c_1 c_2^{k-1}}{(1 - \frac{1}{4k^2})4^k(k!)^2} r^{2k} =: \zeta(r) \]

and on the other hand

\[ |A_3(r)| \leq \int_0^r \frac{1}{s} \int_0^s \sigma(c_1 + c_2 |\phi(\sigma)|) d\sigma ds \leq \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s \sigma(\xi(\sigma)) d\sigma ds = \xi(r) \]

(28)

for $r \in [0,R_0]$. Note that the a priori bound on $A_\varphi$ is slightly weaker than the bounds on $\phi$ and $A_3$ since obviously $\xi \leq \zeta$.

Thus, we have proved the following important a priori estimate.

**Lemma 4.1.** Let $(\phi, A_\varphi, A_3) \in C([0,R_0];\mathbb{R}^3)$ be a fixed point of $\mathcal{M}$. Then there holds

\[ |\phi(r)|, |A_3(r)| \leq \xi(r), \quad |A_\varphi(r)| \leq \zeta(r) \]

for $r \in [0,R_0]$.

For the sake of completeness, we remark that $\xi$ can be written in terms of a Bessel function, which corresponds to the fact that (22) implies

\[ r^2 \xi'' + r \xi' - c_2 r^2 \xi = c_1 r^2, \]

whence

\[ z(r) := \frac{c_2}{c_1} \xi \left( r \sqrt{\frac{r}{c_2}} \right) + 1 \]

solves the modified Bessel equation

\[ r^2 z'' + rz' - r^2 z = 0. \]

Endowed with the initial condition $\xi(0) = \xi'(0) = 0$, this yields $z = I_0$, where $I_0$ is the modified Bessel function of the first kind (with parameter 0). Consequently,

\[ \zeta(r) = \frac{c_1}{c_2} (I_0(\sqrt{c_2}r) - 1). \]

4.2. Fixed point argument. We proceed with proving that steady states really do exist via some fixed point argument. Throughout the rest of this section, we assume that Condition 3.7 holds and equip the space $C([0,R_0];\mathbb{R}^3)$ with the norm

\[ ||(\phi, A_\varphi, A_3)||_{C([0,R_0];\mathbb{R}^3)} = \sup_{r \in [0,R_0]} |(\phi(r), A_\varphi(r), A_3(r))|. \]

The a priori bounds obtained in the last section are an important tool to prove existence of solutions to (17). In view of Schaefer’s fixed point theorem—see [8, Section 9.2.2.], for example—we have to prove that $\mathcal{M}$ is continuous and compact, and we have to establish a priori bounds on possible fixed points of the operators.
\( \lambda M \) for \( 0 \leq \lambda \leq 1 \). The second task is easily carried out by using the results of Section 4.1.

**Lemma 4.2.** Let \((\phi, A_\varphi, A_3) \in C([0, R_0]; \mathbb{R}^3)\) such that there holds \((\phi, A_\varphi, A_3) = \lambda M(\phi, A_\varphi, A_3)\) for some \(0 \leq \lambda \leq 1\). Then we have

\[
|\phi(r)|, |A_\varphi(r)| \leq \xi(r), \quad |A_3(r)| \leq \zeta(r)
\]

for \(r \in [0, R_0]\). In particular, the set

\[
\{(\phi, A_\varphi, A_3) \in C([0, R_0]; \mathbb{R}^3) \mid (\phi, A_\varphi, A_3) = \lambda M(\phi, A_\varphi, A_3) \text{ for some } 0 \leq \lambda \leq 1\}
\]

is bounded.

**Proof.** By (18), we obtain

\[
|\phi(r)| \leq \lambda \int_0^r \frac{1}{s} \int_0^s \sigma(c_1 + c_2|\phi(\sigma)|) \, d\sigma \, ds \leq \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s |\phi(\sigma)| \, d\sigma \, ds
\]
similarly to (19). Hence, there holds \(|\phi(r)| \leq \xi(r)\) for \(r \in [0, R_0]\). Similarly to (27) and (28), we also have

\[
|A_\varphi(r)| \leq \frac{\lambda}{r} \int_0^r s \int_0^s (c_1 + c_2|\phi(\sigma)|) \, d\sigma \, ds \leq \frac{c_1}{3} r^2 + \frac{c_2}{r} \int_0^r s \int_0^s |\phi(\sigma)| \, d\sigma \, ds = \zeta(r),
\]

\[
|A_3(r)| \leq \lambda \int_0^r \frac{1}{s} \int_0^s \sigma(c_1 + c_2|\phi(\sigma)|) \, d\sigma \, ds \leq \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s |\sigma| \phi(\sigma) | \, d\sigma \, ds
\]

for \(r \in [0, R_0]\). \(\square\)

Thus, there remains to prove the following lemma.

**Lemma 4.3.** The map \( M \) is (even locally Lipschitz) continuous and compact.

**Proof.** Let \( S > 0 \) and \((\phi, A_\varphi, A_3), (\bar{\phi}, \bar{A}_\varphi, \bar{A}_3) \in \overline{B}_S \subset C([0, R_0]; \mathbb{R}^3)\). On the one hand, following the calculation in the proof of Lemma 3.9, we have for each \(r \in [0, R_0]\) for some \((a, b, c)\), possibly depending on the integration variables, in the line segment connecting \((\phi(r), A_\varphi(r), A_3(r))\) and \((\bar{\phi}(r), \bar{A}_\varphi(r), \bar{A}_3(r))\),

\[
\left|(g_1, g_2, g_3)(r, \phi(r), A_\varphi^\text{tot}(r), A_3^\text{tot}(r)) - (g_1, g_2, g_3)(r, \bar{\phi}(r), \bar{A}_\varphi^\text{tot}(r), \bar{A}_3^\text{tot}(r))\right|
\]

\[
= 4\pi \sum_{\alpha=1}^N q_\alpha^2 \int_{\mathbb{R}} \int_0^\infty \int_0^{2\pi} \frac{u}{\sqrt{m_\alpha^2 + u^2 + v_3^2}} \left( \frac{\sqrt{m_\alpha^2 + u^2 + v_3^2}}{u \sin \theta} \right) \nabla \eta^\alpha \left( \sqrt{m_\alpha^2 + u^2 + v_3^2} + q_\alpha a, ru \sin \theta + r q_\alpha b + r q_\alpha A_\varphi^\text{ext}(r), v_3 + q_\alpha c + q_\alpha A_3^\text{ext}(r) \right) \cdot \left( \begin{array}{c} \phi(r) - \bar{\phi}(r) \\ r (A_\varphi(r) - \bar{A}_\varphi(r)) \\ A_3(r) - \bar{A}_3(r) \end{array} \right) \, d\theta \, du \, dv_3
\]

\[
= 4\pi \sum_{\alpha=1}^N q_\alpha^2 \int_{\mathbb{R}} \int_0^{2\pi} \int_{-\infty}^\infty \frac{u}{\sqrt{m_\alpha^2 + (g_3 v_3 - q_\alpha c - q_\alpha A_3^\text{ext}(r))^2}} \, dv_3
\]
\[ \nabla \eta^\alpha \left( \mathcal{E}, r \sqrt{(\mathcal{E} - q_\alpha a)^2 - (G - q_\alpha c - q_\alpha A_3^{ext}(r))^2 - m^2 a sin \theta} \right) \]

\[ = C(S) \left| (\phi, A_\varphi, A_3)(r) \right| - (\overline{\phi}, \overline{A_\varphi}, \overline{A_3})(r) \right|, \]  

where the constant \( C(S) \) is finite due to Condition 3.7.(iii) (with \( d := -|q_\alpha| S \) there).

Integrating this estimate, we conclude

\[ |M(\phi, A_\varphi, A_3)(r) - M(\overline{\phi}, \overline{A_\varphi}, \overline{A_3})(r)| \]

\[ \leq C(S) \| (\phi, A_\varphi, A_3) - (\overline{\phi}, \overline{A_\varphi}, \overline{A_3}) \|_{C([0,R_0];\mathbb{R}^3)} \]

\[ \cdot \left( \int_0^r \int_0^s \overline{\sigma} d\sigma ds, \int_0^r \int_0^s \sigma d\sigma ds \right) \]

\[ = C(S) \cdot \frac{\sqrt{34}}{12} r^2 \| (\phi, A_\varphi, A_3) - (\overline{\phi}, \overline{A_\varphi}, \overline{A_3}) \|_{C([0,R_0];\mathbb{R}^3)}, \]

whence

\[ \| M(\phi, A_\varphi, A_3) - M(\overline{\phi}, \overline{A_\varphi}, \overline{A_3}) \|_{C([0,R_0];\mathbb{R}^3)} \]

\[ \leq C(S) \cdot \frac{\sqrt{34}}{12} R_0^2 \| (\phi, A_\varphi, A_3) - (\overline{\phi}, \overline{A_\varphi}, \overline{A_3}) \|_{C([0,R_0];\mathbb{R}^3)}. \]

Therefore, \( M \) is locally Lipschitz continuous.

On the other hand, by (18) we have

\[ |g_i(r, \phi(r), A_\varphi^{ext}(r), A_3^{ext}(r))| \leq c_1 + c_2|\phi(r)| \leq c_1 + c_2 S =: \tilde{C}(S) \]

for \( i = 1, 2, 3 \) and \( r \in [0, R_0] \). Furthermore, there holds

\( (M(\phi, A_\varphi, A_3))'(0) = (0, 0, 0) \)

by (the proof of) Lemma 3.11.(i) and for \( 0 < r \leq R_0 \)

\[ \left| (M_i(\phi, A_\varphi, A_3))'(r) \right| = \left| -\frac{1}{r} \int_0^r s g_i(s, \phi(s), A_\varphi^{ext}(s), A_3^{ext}(s)) ds \right| \leq \frac{\tilde{C}(S)r}{2} \]

for \( i = 1, 3 \) and

\[ \left| (M_2(\phi, A_\varphi, A_3))'(r) \right| = \left| \frac{1}{r^2} \int_0^r s \int_0^s g_2(\sigma, \phi(\sigma), A_\varphi^{ext}(\sigma), A_3^{ext}(\sigma)) d\sigma ds - \int_0^r g_2(s, \phi(s), A_\varphi^{ext}(s), A_3^{ext}(s)) ds \right| \]
\[
\leq \frac{\tilde{C}(S)r}{3} + \tilde{C}(S)r \leq \frac{4\tilde{C}(S)R_0}{3}.
\]

Therefore, for each \((\phi, A_\varphi, A_3) \in \overline{B_S}\), we have that \(\mathcal{M}(\phi, A_\varphi, A_3)\) is Lipschitz continuous with a uniform Lipschitz constant, i.e., a Lipschitz constant only depending on \(S\). By the theorem of Arzelà–Ascoli, \(\mathcal{M}\) thus maps bounded sets to precompact sets, that is, \(\mathcal{M}\) is compact. \(\Box\)

**Theorem 4.4.** Let Conditions 3.2 and 3.7 hold. Then \(\mathcal{M}\) has a unique fixed point. Thus, there exists an axially symmetric steady state \((f^a)_{a}, \phi, A\) of the two and one-half dimensional relativistic Vlasov–Maxwell system on \(\Omega\) with external potential \(A^{\text{ext}}\), where the \(f^a\) are written in terms of \(\phi\) and \(A\); cf. (13).

**Proof.** Combining Lemmas 4.2 and 4.3 and invoking Schaefer’s fixed point theorem we conclude that \(\mathcal{M}\) has a fixed point. Due to Lemma 3.11, we obtain a corresponding steady state.

There remains to prove that a fixed point of \(\mathcal{M}\) is unique. If we have two fixed points \((\phi, A_\varphi, A_3)\), \((\tilde{\phi}, \tilde{A}_\varphi, \tilde{A}_3)\) of \(\mathcal{M}\), let \(S > 0\) such that

\[
(\phi, A_\varphi, A_3), (\tilde{\phi}, \tilde{A}_\varphi, \tilde{A}_3) \in C([0, R_0]; \mathbb{R}^3).
\]

By (29) and \(0 \leq \sigma \leq s \leq r \leq R_0\) there holds

\[
|\phi(r) - \tilde{\phi}(r)| = |\mathcal{M}(\phi, A_\varphi, A_3)(r) - \mathcal{M}(\tilde{\phi}, \tilde{A}_\varphi, \tilde{A}_3)(r)|
\leq C(S)\left(\int_0^r \int_0^s |\phi, A_\varphi, A_3|(|\phi, A_\varphi, A_3| - |\tilde{\phi}, \tilde{A}_\varphi, \tilde{A}_3|)\, d\sigma\, ds, \right.
\left.\int_0^r \int_0^s \sigma |\phi, A_\varphi, A_3| - |\tilde{\phi}, \tilde{A}_\varphi, \tilde{A}_3|\, d\sigma\, ds, \right.
\left.\int_0^r \int_0^s \sigma |\phi, A_\varphi, A_3| - |\tilde{\phi}, \tilde{A}_\varphi, \tilde{A}_3|\, d\sigma\, ds\right)
\leq C(S) \cdot \sqrt{3R_0} \int_0^r |\phi - \tilde{\phi}, A_\varphi - \tilde{A}_\varphi, A_3 - \tilde{A}_3|\, ds
\]

for each \(r \in [0, R_0]\). Thus, the two fixed points coincide due to Gronwall’s lemma. \(\Box\)

**4.3. Direct construction.** Since the above proof of existence of steady states is not constructive, we now provide a method to obtain steady states which is constructive. To this end, we define an approximating sequence \((\phi^k, A^k_\varphi, A^k_3)\) \(k \in \mathbb{N}_0\) recursively via

\[
(\phi^0, A^0_\varphi, A^0_3) = (0, 0, 0), \quad (\phi^{k+1}, A^{k+1}_\varphi, A^{k+1}_3) = \mathcal{M}(\phi^k, A^k_\varphi, A^k_3).
\]

To show that this sequence indeed converges to a (and thus the) fixed point of \(\mathcal{M}\), we first prove that this sequence is bounded. In fact, the a priori estimates of Section 4.1 carry over.

**Lemma 4.5.** For each \(k \in \mathbb{N}_0\) and \(r \in [0, R_0]\) there holds

\[
|\phi^k(r)|, |A^k_\varphi(r)| \leq \xi(r), \quad |A^k_3(r)| \leq \zeta(r).
\]

In particular,

\[
\left\| (\phi^k, A^k_\varphi, A^k_3) \right\|_{C([0, R_0]; \mathbb{R}^3)} \leq \sqrt{2\xi(R_0)^2 + \zeta(R_0)^2} =: S.
\]
Proof. We prove

$$|\phi^k(r)|, |A_3^k(r)| \leq \sum_{j=1}^{k} \frac{c_1 c_2^{j-1}}{4^j(j!)^2} r^{2j}, \quad |A_3^k(r)| \leq \sum_{j=1}^{k} \left( \frac{c_1 c_2^{j-1}}{4^j(j!)^2} r^{2j} \right)$$

via induction, from which the assertion follows. Indeed, this obviously holds true for $k = 0$, and thanks to (18) we also have

$$|\phi^{k+1}(r)|, |A_3^{k+1}(r)| \leq \int_0^r \frac{1}{s} \int_0^s \sigma (c_1 + c_2 |\phi^k(\sigma)|) \, d\sigma \, ds$$

$$\leq \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s \sigma \sum_{j=1}^{k} \frac{c_1 c_2^{j-1}}{4^j(j!)^2} \sigma^{2j} \, d\sigma \, ds$$

$$= \frac{c_1}{4} r^2 + c_2 \int_0^r \frac{1}{s} \int_0^s \sum_{j=1}^{k} \frac{c_1 c_2^{j-1}}{4^j(j!)^2} (2j + 2) s^{2j+1} \, ds = \frac{c_1}{4} r^2 + \sum_{j=1}^{k} \frac{c_1 c_2^{j-1}}{4^j(j!)^2} r^{2j+2}$$

$$= \sum_{j=1}^{k+1} \frac{c_1 c_2^{j-1}}{4^j(j!)^2} r^{2j}$$

and

$$|A_3^{k+1}(r)| \leq \frac{1}{r} \int_0^r \frac{1}{s} \int_0^s (c_1 + c_2 |\phi^k(\sigma)|) \, d\sigma \, ds$$

$$\leq \frac{c_1}{3} r^2 + \frac{c_2}{r} \int_0^r \frac{1}{s} \int_0^s \sum_{j=1}^{k} \frac{c_1 c_2^{j-1}}{4^j(j!)^2} \sigma^{2j} \, d\sigma \, ds$$

$$= \frac{c_1}{3} r^2 + \frac{c_2}{r} \int_0^r \frac{1}{s} \int_0^s \sum_{j=1}^{k} \frac{c_1 c_2^{j-1}}{4^j(j!)^2} (2j + 1) s^{2j+1} \, ds$$

$$= \frac{c_1}{3} r^2 + \sum_{j=1}^{k} \frac{c_1 c_2^{j-1}}{4^j(j!)^2} (2j + 1) r^{2j+2} = \sum_{j=1}^{k+1} \frac{c_1 c_2^{j-1}}{4^j(j!)^2} r^{2j}.$$

We can now prove the following result.

Theorem 4.6. Let Conditions 3.2 and 3.7 hold. Then, $((\phi^k, A_3^k, A_3^k))_{k \in \mathbb{N}_0}$, where

$$(\phi^0, A_3^0, A_3^0) = (0, 0, 0), \quad (\phi^{k+1}, A_3^{k+1}, A_3^{k+1}) = M(\phi^k, A_3^k, A_3^k), \quad k \in \mathbb{N}_0,$$

is a Cauchy sequence in $C([0, R_0]; \mathbb{R}^3)$. The limit $(\phi, A_3, A_3)$ is the fixed point of $M$, whence $(f^n)_{\alpha, \phi, A}$ is an axially symmetric steady state of the two and one-half dimensional relativistic Vlasov–Maxwell system on $\bar{\Omega}$ with external potential $A^{\text{ext}}$, where the $f^n$ are written in terms of $\phi$ and $A$; cf. (13).

Proof. We abbreviate $P^k := (\phi^k, A_3^k, A_3^k)$ for $k \in \mathbb{N}_0$. By Lemma 4.5, (29), and $0 \leq \sigma \leq s \leq r$ we have

$$|\phi^{k+1}(r) - \phi^k(r)|, |A_3^{k+1}(r) - A_3^k(r)|, |A_3^{k+1}(r) - A_3^k(r)| \leq C(S) \int_0^r \int_0^s |P^k(\sigma) - P^{k-1}(\sigma)| \, d\sigma \, ds$$

We abbreviate $P^k := (\phi^k, A_3^k, A_3^k)$ for $k \in \mathbb{N}_0$. By Lemma 4.5, (29), and $0 \leq \sigma \leq s \leq r$ we have

$$|\phi^{k+1}(r) - \phi^k(r)|, |A_3^{k+1}(r) - A_3^k(r)|, |A_3^{k+1}(r) - A_3^k(r)| \leq C(S) \int_0^r \int_0^s |P^k(\sigma) - P^{k-1}(\sigma)| \, d\sigma \, ds$$
For each condition 4.7, corresponding theorem, under which a steady state indeed has these two properties and then prove the property which should hold is that the steady state is nontrivial—for example, we guarantee this property also in our setting, as is shown below. Another obvious further property of a steady state is that it is compactly supported with respect to $v$.

Further properties. A desirable property of a steady state is that it is compactly supported with respect to $v$. It is well known in similar settings that for this there should exist a cut-off energy. Indeed, the existence of such a cut-off energy for each $r \in [0, R_0]$, $k \in \mathbb{N}$ via induction: Indeed, this estimate obviously holds true for $k = 0$, and moreover we have

$$|P^{k+1}(r) - P^k(r)| \leq C \int_0^r \int_0^s |P^k(\sigma) - P^{k-1}(\sigma)| \, d\sigma \, ds$$

for each $r \in [0, R_0]$, $k \in \mathbb{N}_0$ via induction: Indeed, this estimate obviously holds true for $k = 0$, and moreover we have

$$|P^{k+1}(r) - P^k(r)| \leq C \int_0^r \int_0^s \frac{SC^{k-1}}{(2k-2)!} \sigma^{2k-2} \, d\sigma \, ds = \frac{SC^k}{(2k)!} \int_0^r s^{2k-1} \, ds = \frac{SC^k}{(2k)!} r^{2k}$$

for $k \geq 1$. Therefore, for each $m \geq k$ and $r \in [0, R_0]$ there holds

$$|P^m(r) - P^k(r)| \leq \sum_{j=k}^{m-1} |P^{j+1}(r) - P^j(r)| \leq \sum_{j=k}^{m-1} \frac{SC^j}{(2j)!} r^{2j} \leq \sum_{j=k}^{\infty} \frac{SC^j}{(2j)!} R_0^{2j}.$$

Since the series $\sum_{j=0}^{\infty} \frac{SC^j}{(2j)!} R_0^{2j}$ converges, it follows that $(P^k)$ is a Cauchy sequence in $C([0, R_0]; \mathbb{R}^3)$. Passing to the limit, we easily see that

$$(\phi, A_\varphi, A_3) = \lim_{k \to \infty} (\phi^{k+1}, A_\varphi^{k+1}, A_3^{k+1}) = \lim_{k \to \infty} \mathcal{M}(\phi^k, A_\varphi^k, A_3^k) = \mathcal{M}(\phi, A_\varphi, A_3)$$

since $\mathcal{M}$ is continuous due to Lemma 4.3. Hence, $(\phi, A_\varphi, A_3)$ is a (and by Theorem 4.4 the) fixed point of $\mathcal{M}$ and the corresponding tuple $((f^\alpha)_{\alpha}, \phi, A)$ is a steady state.

4.4. Further properties. A desirable property of a steady state is that it is compactly supported with respect to $v$. It is well known in similar settings that for this there should exist a cut-off energy. Indeed, the existence of such a cut-off energy guarantees this property also in our setting, as is shown below. Another obvious property which should hold is that the steady state is nontrivial—for example, we have not excluded the pointless possibility $\eta^\alpha = 0$ yet. We first state conditions under which a steady state indeed has these two properties and then prove the corresponding theorem.

Condition 4.7. For each $\alpha = 1, \ldots, N$ there holds:

(i) there exists $\mathcal{E}_0^\alpha > 0$ such that $\eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G}) = 0$ if $\mathcal{E} \geq \mathcal{E}_0^\alpha$;

(ii) there exist $\mathcal{E}_0^\alpha > m_\alpha$, $\mathcal{G}_0^\alpha < 0$, $\mathcal{G}_0^\alpha > 0$, and

$$(1) \quad \mathcal{F}_0^\alpha < 0, \mathcal{F}_0^\alpha \geq 0$$

(2) $\mathcal{F}_0^\alpha \leq 0, \mathcal{F}_0^\alpha > 0$

such that

$$\forall (\mathcal{E}, \mathcal{F}, \mathcal{G}) \in [m_\alpha, \mathcal{E}_0^\alpha] \times [\mathcal{F}_0^\alpha, \mathcal{F}_0^\alpha] \times [\mathcal{G}_0^\alpha, \mathcal{G}_0^\alpha] : \eta^\alpha(\mathcal{E}, \mathcal{F}, \mathcal{G}) > 0.$$

Theorem 4.8. Let Conditions 3.2 and 3.7 hold and let $((f^\alpha)_{\alpha}, \phi, A)$ be a steady state, where $(\phi, A_\varphi, A_3)$ is the fixed point of $\mathcal{M}$ and the $f^\alpha$ are given by (13). Then we have:
(i) If Condition 4.7.(i) is satisfied, then the steady state is compactly supported with respect to v.
(ii) If Condition 4.7.(ii) is satisfied, then the steady state is nontrivial.

Proof. As for 4.8.(i), we find that, if

\[ |v| \geq \max_{\alpha=1,\ldots,N} (\mathcal{E}^\alpha_0 + |q_\alpha|\xi(R_0)), \]

then for each \( \alpha = 1, \ldots, N \) and \( x \in \overline{\Omega} \) there holds

\[ \mathcal{E}^\alpha(x,v) = v_\alpha^0 + q_\alpha \phi(r) \geq |v| - |q_\alpha|\xi(R_0) \geq \mathcal{E}^\alpha_0 \]

due to Lemma 4.1 and hence \( f^\alpha(x,v) = 0 \).

As for 4.8.(ii), we follow the idea of [12]. For fixed \( \alpha \in \{1, \ldots, N\} \) choose \( 0 < r_\alpha \leq \frac{R_0}{2} \) small enough such that

\[ \sqrt{m_\alpha^2 + m_\alpha^2} + \frac{1}{2} |q_\alpha|\xi(2r_\alpha) > m_\alpha, \quad \sqrt{m_\alpha^2 + m_\alpha^2} + \frac{1}{2} |q_\alpha|\xi(2r_\alpha) < \mathcal{E}^\alpha_u, \]

\[ \sqrt{\alpha^2} + |q_\alpha|\xi(2r_\alpha) + \sup_{0 \leq r \leq 2r_\alpha} |A_{33}^\alpha(r)| < \min \{-G^\alpha_l, G^\alpha_u\} \]

and

\[ 4r_\alpha^2 + 2|q_\alpha|\xi(2r_\alpha) + 2|q_\alpha| \sup_{0 \leq r \leq 2r_\alpha} |A_{33}^\alpha(r)| \leq -F^\alpha_u, \]

\[ -\frac{1}{\sqrt{2}} r_\alpha^2 + 2|q_\alpha|\xi(2r_\alpha) + 2|q_\alpha| \sup_{0 \leq r \leq 2r_\alpha} |A_{33}^\alpha(r)| < 0 \]

in case 4.7.(ii).(1) and

\[ 4r_\alpha^2 + 2|q_\alpha|\xi(2r_\alpha) + 2|q_\alpha| \sup_{0 \leq r \leq 2r_\alpha} |A_{33}^\alpha(r)| \geq \mathcal{F}^\alpha_u, \]

\[ \frac{1}{\sqrt{2}} r_\alpha^2 - 2|q_\alpha|\xi(2r_\alpha) - 2|q_\alpha| \sup_{0 \leq r \leq 2r_\alpha} |A_{33}^\alpha(r)| > 0 \]

in case 4.7.(ii).(2), respectively. Indeed, this choice of \( r_\alpha \) is possible since there holds \( \xi(r), \xi(r), rA^\alpha_\varphi(r) = \mathcal{O}(r^2) \) for \( r \to 0 \), \( A^\alpha_3(0) = 0 \), and \( \frac{3}{2} \leq \mathcal{F} \leq 2 \). Next, let \( \theta_\alpha := \frac{3}{2} \) in case 4.7.(ii).(1) and \( \theta_\alpha := \frac{3}{2} \) in case 4.7.(ii).(2), respectively, and let

\[ S_\alpha := \left\{ (r, u, \theta, v_3) \in [0, R_0] \times [0, \infty) \times [0, 2\pi] \times \mathbb{R} \mid r_\alpha < r < 2r_\alpha, \right. \]

\[ \sqrt{r_\alpha} < u < 2\sqrt{r_\alpha}, \theta_\alpha - \frac{\pi}{4} < \theta < \theta_\alpha + \frac{\pi}{4}, -\sqrt{r_\alpha} < v_3 < \sqrt{r_\alpha} \}. \]

In \( (r, u, \theta, v_3) \)-coordinates, where \( u = \sqrt{v_1^2 + v_2^2} \) and \( \theta \) is the polar angle in the \((v_1, v_2)\)-plane with basis \((e_r, e_\varphi)\), there holds

\[ \mathcal{E}^\alpha(r, u, \theta, v_3) = \sqrt{m_\alpha^2 + u^2 + v_3^2} + q_\alpha \phi(r), \]

\[ \mathcal{F}^\alpha(r, u, \theta, v_3) = r(u \sin \theta + q_\alpha A_{33}^\alpha(r) + q_\alpha A_{33}^\alpha(r)), \]

\[ \mathcal{G}^\alpha(r, u, \theta, v_3) = v_3 + q_\alpha A_3^\alpha(r) + q_\alpha A_{33}^\alpha(r). \]

For each \( (r, u, \theta, v_3) \in S_\alpha \), we have by Lemma 4.1

\[ \mathcal{E}^\alpha(r, u, \theta, v_3) \geq \sqrt{m_\alpha^2 + r_\alpha} - |q_\alpha|\xi(2r_\alpha) > m_\alpha, \]

\[ \mathcal{E}^\alpha(r, u, \theta, v_3) \leq \sqrt{m_\alpha^2 + m_\alpha^2} + |q_\alpha|\xi(2r_\alpha) < \mathcal{E}^\alpha_u, \]

\[ \mathcal{G}^\alpha(r, u, \theta, v_3) \geq -\sqrt{r_\alpha} - |q_\alpha|\xi(2r_\alpha) - |q_\alpha| \sup_{0 \leq r \leq 2r_\alpha} |A_{33}^\alpha(r)| > \mathcal{G}^\alpha_l. \]
\[ G^\alpha(r, u, \theta, v_3) \leq \sqrt{r_{\alpha}} + |q_{\alpha}| \xi(2r_{\alpha}) + |q_{\alpha}| \sup_{0 \leq r \leq 2r_{\alpha}} |A^\text{ext}_3(r)| < G^\alpha_u \]

and
\[ F^\alpha(r, u, \theta, v_3) \geq -4r_{\alpha}^3 - 2|q_{\alpha}|r_{\alpha}\xi(2r_{\alpha}) - 2|q_{\alpha}|r_{\alpha} \sup_{0 \leq r \leq 2r_{\alpha}} |A^\text{ext}_\phi(r)| > F^\alpha_\phi, \]
\[ F^\alpha(r, u, \theta, v_3) \leq - \frac{1}{\sqrt{2}}r_{\alpha}^3 + 2|q_{\alpha}|r_{\alpha}\xi(2r_{\alpha}) + 2|q_{\alpha}|r_{\alpha} \sup_{0 \leq r \leq 2r_{\alpha}} |A^\text{ext}_\phi(r)| < 0 \leq F^\alpha_u \]

in case 4.7.(ii).(1) and
\[ F^\alpha(r, u, \theta, v_3) \leq 4r_{\alpha}^3 + 2|q_{\alpha}|r_{\alpha}\xi(2r_{\alpha}) + 2|q_{\alpha}|r_{\alpha} \sup_{0 \leq r \leq 2r_{\alpha}} |A^\text{ext}_\phi(r)| < F^\alpha_u, \]
\[ F^\alpha(r, u, \theta, v_3) \geq \frac{1}{\sqrt{2}}r_{\alpha}^3 - 2|q_{\alpha}|r_{\alpha}\xi(2r_{\alpha}) - 2|q_{\alpha}|r_{\alpha} \sup_{0 \leq r \leq 2r_{\alpha}} |A^\text{ext}_\phi(r)| > 0 \geq F^\alpha_i \]
in case 4.7.(ii).(2), respectively. Therefore,
\[ ru\eta^\alpha(\xi^\alpha(r, u, \theta, v_3), F^\alpha(r, u, \theta, v_3), G^\alpha(r, u, \theta, v_3)) > 0. \]

Thus, we have
\[
\int_{B_{R_0}} \int_\mathbb{R}^3 f^a \, d\nu(x_1, x_2) = 2\pi \int_0^{R_0} \int_\mathbb{R}^3 \eta^a(\xi^\alpha, F^\alpha, G^\alpha) \, dv \, dr = 2\pi \int_0^{R_0} \int_\mathbb{R} \int_0^\infty \int_0^{2\pi} ru\eta^a(\xi^\alpha, F^\alpha, G^\alpha) \, d\theta \, dv \, dr \geq \int_{S_\alpha} ru\eta^a(\xi^\alpha, F^\alpha, G^\alpha) \, d(r, u, \theta, v_3) > 0
\]
since \( S_\alpha \) has positive Lebesgue measure. In particular, \( f^a \neq 0 \).

\[ \square \]

\textbf{Remark 4.9.} Vividly, the proof of Theorem 4.8.(ii) shows that, for each species, there are some particles near the symmetry axis with small momentum. Moreover, it was proved that in case 4.7.(ii).(1) (or 4.7.(ii).(2), respectively) there are some particles with negative (or positive, respectively) canonical angular momentum.

5. **Confined steady states.** There remains to find conditions on the external potential \( A^\text{ext} \) and the ansatz functions \( \eta^a \) under which a corresponding steady state is confined. We consider two possibilities:

- A suitable \( A^\text{ext} \) (corresponding to an external magnetic field in the \( e_3 \)-direction) ensures confinement. This configuration is often called “\( \theta \)-pinch”.
- A suitable \( A^\text{ext}_3 \) (corresponding to an external magnetic field in the \( e_1 \)-direction) ensures confinement. This configuration is often called “\( z \)-pinch”.

A combination of these two—often called “screw-pinches”—would of course also be possible, whence the following options are not exhaustive.

\textbf{Theorem 5.1.} Let Conditions 3.2, 3.7 and 4.7 hold and let \( (f^a)_{\alpha}, \phi, A \) be a steady state, where \((\phi, A_\phi, A_3)\) is the fixed point of \( M \) and the \( f^a \) are given by (13).

We define
\[ \mathcal{N} := \{ \alpha \in \{1, \ldots, N\} \mid q_{\alpha} < 0 \}, \quad \mathcal{P} := \{ \alpha \in \{1, \ldots, N\} \mid q_{\alpha} > 0 \}. \]

Furthermore, let \( 0 < R < R_0 \) and one of the four following options hold:

\begin{enumerate}[(i)]
  \item \( \theta \)-pinch
  \item \( z \)-pinch
  \item \( \phi \)-pinch
  \item \( z \times \theta \)-pinch
\end{enumerate}
In the following, always let $r$ by Lemma 4.1. Thus, for each $\alpha \in \mathcal{N}$, case 4.7.(ii).(1) is satisfied and we have $\eta^\alpha(E,F,G) = 0$ if $F \geq 0$ (thus, necessarily $F^\alpha_u = 0$). Moreover, assume

$$A^\text{ext}_\varphi(r) \leq -a_\varphi(r), \quad R \leq r \leq R_0.$$  

(b) For each $\alpha \in \mathcal{N}$, case 4.7.(ii).(2) is satisfied and we have $\eta^\alpha(E,F,G) = 0$ if $F \leq 0$ (thus, necessarily $F^\alpha_l = 0$). For each $\alpha \in \mathcal{P}$, case 4.7.(ii).(1) is satisfied and we have $\eta^\alpha(E,F,G) = 0$ if $F \geq 0$ (thus, necessarily $F^\alpha_u = 0$). Moreover, assume

$$A^\text{ext}_\varphi(r) \geq a_\varphi(r), \quad R \leq r \leq R_0.$$  

Here,

$$a_\varphi(r) := \max_{\alpha = 1,\ldots,N} \left( \frac{\sqrt{(E^\alpha_0 + |q_\alpha|\xi(r))^2 - m_\alpha^2}}{|q_\alpha|} + \xi(r) \right).$$

(ii) (z-pinch)

(a) For each $\alpha \in \mathcal{N}$, there exists $G^\alpha_0 < 0$ such that $\eta^\alpha(E,F,G) = 0$ if $G \leq G^\alpha_0$. For each $\alpha \in \mathcal{P}$, there exists $G^\alpha_0 > 0$ such that $\eta^\alpha(E,F,G) = 0$ if $G \geq G^\alpha_0$. Moreover, assume

$$A^\text{ext}_\varphi(r) \geq a_3(r), \quad R \leq r \leq R_0.$$  

(b) For each $\alpha \in \mathcal{N}$, there exists $G^\alpha_0 > 0$ such that $\eta^\alpha(E,F,G) = 0$ if $G \geq G^\alpha_0$. For each $\alpha \in \mathcal{P}$, there exists $G^\alpha_0 < 0$ such that $\eta^\alpha(E,F,G) = 0$ if $G \leq G^\alpha_0$. Moreover, assume

$$A^\text{ext}_\varphi(r) \leq -a_3(r), \quad R \leq r \leq R_0.$$  

Here,

$$a_3(r) := \max_{\alpha = 1,\ldots,N} \left( \frac{|G^\alpha_0| + \sqrt{(E^\alpha_0 + |q_\alpha|\xi(r))^2 - m_\alpha^2}}{|q_\alpha|} + \xi(r) \right).$$

Then the steady state is confined with radius at most $R$, compactly supported with respect to $v$, and nontrivial.

Proof. First note that for each $(x,v) \in \mathcal{O} \times \mathbb{R}^3$ and $\alpha = 1,\ldots,N$ we have $f^\alpha(x,v) = 0$ if

$$|v| \geq \sqrt{(E^\alpha_0 + |q_\alpha|\xi(r))^2 - m_\alpha^2}$$

since then

$$E^\alpha(x,v) \geq \sqrt{m_\alpha^2 + |v|^2 - |q_\alpha|\xi(r)} \geq E^\alpha_0$$

by Lemma 4.1. Thus, for each $\alpha = 1,\ldots,N$ it suffices to consider $v \in \mathbb{R}^3$ with

$$|v| < \sqrt{(E^\alpha_0 + |q_\alpha|\xi(r))^2 - m_\alpha^2}.$$  

In the following, always let $r \in [R,R_0]$, $\alpha \in \mathcal{N}$, $\beta \in \mathcal{P}$, and $v$ as above.

If option 5.1.(i).(a) is satisfied, there holds

$$F^\alpha(x,v) \geq r(-|v| + q_\alpha \zeta(r) + q_\alpha A^\text{ext}_\varphi(r)) \geq r(-|v| + q_\alpha \zeta(r) - q_\alpha a_\varphi(r)) \geq$$
and thus $f^\alpha(x, v) = f^\beta(x, v) = 0$.

If option 5.1.(i),(b) is satisfied, there holds

$$
\mathcal{F}^\alpha(x, v) \leq r(|v| - q_\alpha \zeta(r) + q_\alpha A^{\text{ext}}_{\text{ext}}(r)) \leq r(|v| - q_\alpha \zeta(r) + q_\alpha a_\varphi(r)) \leq
$$

$$
\mathcal{F}^\beta(x, v) \geq r(-|v| - q_\beta \zeta(r) + q_\beta A^{\text{ext}}_{\text{ext}}(r)) \geq r(-|v| - q_\beta \zeta(r) + q_\beta a_\varphi(r)) \geq
$$

and thus $f^\alpha(x, v) = f^\beta(x, v) = 0$.

If option 5.1.(ii),(a) is satisfied, there holds

$$
\mathcal{G}^\alpha(x, v) \leq |v| - q_\alpha \xi(r) + q_\alpha A^{\text{ext}}_{\text{ext}}(r) \leq |v| - q_\alpha \zeta(r) + q_\alpha a_\varphi(r) \leq
$$

$$
\mathcal{G}^\beta(x, v) \geq -|v| - q_\beta \xi(r) + q_\beta A^{\text{ext}}_{\text{ext}}(r) \geq -|v| - q_\beta \zeta(r) + q_\beta a_\varphi(r) \geq
$$

and thus $f^\alpha(x, v) = f^\beta(x, v) = 0$.

If option 5.1.(ii),(b) is satisfied, there holds

$$
\mathcal{G}^\alpha(x, v) \geq -|v| + q_\alpha \xi(r) + q_\alpha A^{\text{ext}}_{\text{ext}}(r) \geq -|v| + q_\alpha \zeta(r) - q_\alpha a_\varphi(r) \geq
$$
Therefore, the steady state is confined for a sufficiently strong external magnetic field, that is to say, if $\mathbf{B}^\text{ext} = B_0 \mathbf{e}_3$ and $B_0$ is sufficiently large, then $A^\alpha_\varphi(0) = 0$ due to Condition 3.2 and $a_\varphi(0) \neq a_3(0)$ due to Condition 4.7. $|A^\varphi_\varphi|$ or $|A^\text{ext}_3|$, respectively, has to increase sufficiently fast on $[0, R]$ to satisfy the respective condition on $[R, R_0]$. Moreover, $a_\varphi$ and $a_3$ increase when the ansatz functions $\psi^\alpha$ (and hence $\xi$, $\zeta$) increase. Thus, a larger external magnetic field is necessary to confine a larger amount of particles (as one would expect).

To obtain a specific example for an external magnetic field ensuring confinement, we consider a $\theta$-pinch configuration and a homogeneous external magnetic field parallel to the symmetry axis, i.e., $B^\text{ext} = B_3^\text{ext} \mathbf{e}_3$ and $B_3^\text{ext} \equiv b$ for some constant $b \in \mathbb{R}$. As $B_3^\text{ext}(r) = \frac{1}{r} (r A^\text{ext}_3(r))^\prime$ and $A^\text{ext}_3(0) = 0$, there has to hold $A^\text{ext}_3(r) = \frac{b}{r}$. Therefore, the steady state is confined for a sufficiently strong external magnetic field, that is to say, if $|b| \geq 2 \sup_{r \in [R, R_0]} \frac{a_\varphi(r)}{r}$ and $b < 0$ (if option 5.1.(i).(a) is satisfied) or $b > 0$ (if option 5.1.(i).(b) is satisfied), respectively. As opposed to this, no configuration can exist where the $\varphi$-component of the external magnetic field is constant (and nontrivial) since in this case $A^\text{ext}_3$ would have to be a linear function of $r$ because of $B^\text{ext}_\varphi = -(A^\text{ext}_3)^\prime$, which contradicts the necessary condition $(A^\text{ext}_3)^\prime(0) = 0$.

We finish with an important remark:
Remark 5.2. Another interesting setting is that there is no confinement device and thus no boundary at \( r = R_0 \) in the first place. In this case, \( \Omega = \mathbb{R}^3 \) and no boundary conditions at \( r = R_0 \) have to be imposed. Moreover, Definition 3.5 can be suitably adapted to this new setting by abolishing (11b) and setting \( R_0 = \infty \). If we seek a steady state of this new setting that is confined with radius at most \( R > 0 \), we firstly choose a (slightly) larger \( R > R_0 \), secondly consider the confinement problem as before with boundary at \( r = R_0 \) and choose \( A^{\text{ext}}_x \) or \( A^{\text{ext}}_z \) suitably to ensure confinement of the obtained steady state with radius at most \( R \), and thirdly “glue” this steady state defined on \([0, R_0]\) and the vacuum solution on \([R_0, \infty]\) together, i.e., extend each \( f^n \) by zero and the potentials by their respective integral formula, that is,

\[
\phi(r) = -4\pi \int_0^r \frac{1}{s} \int_0^s \sigma \rho(\sigma) \, d\sigma \, ds
\]

\[
= -4\pi \int_0^R \frac{1}{s} \int_0^s \sigma \rho(\sigma) \, d\sigma \, ds - 4\pi \int_R^r \frac{1}{s} \int_0^s \sigma \rho(\sigma) \, d\sigma \, ds
\]

\[
= -4\pi \int_0^R \frac{1}{s} \int_0^s \sigma \rho(\sigma) \, d\sigma \, ds - 4\pi \int_0^R s \rho(s) \, ds \cdot (\ln r - \ln R),
\]

\[
A_\varphi(r) = -4\pi \int_0^r \frac{1}{s} \int_0^s j_\varphi(\sigma) \, d\sigma \, ds
\]

\[
= -4\pi \int_0^R \frac{1}{s} \int_0^s j_\varphi(\sigma) \, d\sigma \, ds - 4\pi \int_R^r \frac{1}{s} \int_0^s j_\varphi(\sigma) \, d\sigma \, ds
\]

\[
= -4\pi \int_0^R \frac{1}{s} \int_0^s j_\varphi(\sigma) \, d\sigma \, ds - 2\pi \int_0^R j_\varphi(s) \, ds \cdot (r - R^2/r),
\]

\[
A_3(r) = -4\pi \int_0^r \frac{1}{s} \int_0^s \sigma j_3(\sigma) \, d\sigma \, ds
\]

\[
= -4\pi \int_0^R \frac{1}{s} \int_0^s \sigma j_3(\sigma) \, d\sigma \, ds - 4\pi \int_R^r \frac{1}{s} \int_0^s \sigma j_3(\sigma) \, d\sigma \, ds
\]

\[
= -4\pi \int_0^R \frac{1}{s} \int_0^s \sigma j_3(\sigma) \, d\sigma \, ds - 4\pi \int_0^R s j_3(s) \, ds \cdot (\ln r - \ln R)
\]

for \( r \geq R \). Note that for this procedure it is important that the \( f^n \) already vanish on \([R, R_0]\) so that the composite \( f^n \) have no jumps at \( r = R_0 \). With the identities above we can furthermore determine the asymptotics of the potentials for \( r \to \infty \). In particular,

\[
\phi(r) = -2a \ln r + \text{const.}, \quad A_3(r) = -2b \ln r + \text{const.}, \quad r \geq R,
\]

\[
A_\varphi(r) + cr = O\left(r^{-1}\right) \quad \text{for} \ r \to \infty
\]

where

\[
a = 2\pi \int_0^R s \rho(s) \, ds, \quad b = 2\pi \int_0^R s j_3(s) \, ds, \quad c = 2\pi \int_0^R j_\varphi(s) \, ds.
\]

Here, \( a \) and \( b \) can be interpreted as the total charge and the third component of the total current on each slice perpendicular to the symmetry axis.

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E-mail address: joerg.weber@uni-bayreuth.de