Perturbations of continuous-time Markov chains

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Abstract. The equivalence of regularity of a $Q$-matrix with its bounded perturbations is proved and a integration by parts formula is established for the associated Feller minimal transition functions.

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1 Introduction

One of the basic questions in studying continuous-time Markov chains is to find the regularity criterion, i.e., to investigate the conditions under which the given $Q$-matrix is regular, or, equivalently, the corresponding Feller minimal process is honest in the sense that the corresponding transition function $P(t) = \{P_{ij}(t); i, j \in \mathbb{N}\}$ satisfies $\sum_{j=0}^{\infty} P_{ij}(t) = 1$ for all $i \geq 0$ and $t \geq 0$. Here we assume the chain has state space $\mathbb{N} := \{0, 1, 2, \ldots\}$. We refer to Anderson (1991) and Chen (2004) for the general theory of continuous-time Markov chains. In this note we show that the regularity property is preserved under a bounded perturbation of the $Q$-matrix. We also establish a integration by parts formula for the corresponding Feller minimal processes without the regularity condition.

Given two $Q$-matrices $R = (r_{ij}; i, j \in \mathbb{N})$ and $A = (a_{ij}; i, j \in \mathbb{N})$, we call $Q = (q_{ij}; i, j \in \mathbb{N}) := R + A$ the perturbation of $R$ by $A$. Throughout this note, we assume all $Q$-matrices are stable and conservative.

The main purpose of this note is to prove the following theorems:

Theorem 1.1 Suppose that $A$ is a bounded $Q$-matrix. Then $Q = R + A$ is regular if and only if $R$ is regular.

Theorem 1.2 Let $Q(t) = \{Q_{ij}(t); i, j \in \mathbb{N}\}$ and $R(t) = \{R_{ij}(t); i, j \in \mathbb{N}\}$ be the Feller minimal transition functions of $Q$ and $R$, respectively. Then we have the following integration by parts formula

$$\sum_{k \in \mathbb{N}} \int_0^t R_{ik}(s)a_k Q_{kj}(t-s)ds = \sum_{l \in \mathbb{N}, m \neq l} \int_0^t R_{il}(t-v)a_{lm} Q_{mj}(v)dv + R_{ij}(t) - Q_{ij}(t). \quad (1.1)$$
In particular, when \( \sum_{k \in \mathbb{N}} \int_0^t R_{ik}(s) a_k Q_{kj}(t-s) ds < \infty \), we can rewrite (1.1) as

\[
Q(t) - R(t) = \int_0^t R(s) A Q(t-s) ds. \tag{1.2}
\]

The perturbation theory of infinitesimal generators has been a very useful tool in the hands of analysts and physicists. A considerable amount of research has been done on the perturbation of linear operators on a Banach space. The effect on a semigroup by adding a linear operator to its infinitesimal generator was studied by Phillips (1952) and Yan (1988). However, these authors did not show the equivalence of the regularity of a \( Q \)-matrix with its bounded perturbations. The integration by parts formula (1.1) was given by Chen (2004, p510) under a stronger condition. The \( Q \)-matrix of the branching processes with immigration and/or resurrection introduced in Li and Chen (2006) can be regarded as the perturbations of a given branching \( Q \)-matrix.

**Example 1.3** Let \( R = (r_{ij}; i, j \in \mathbb{N}) \) be a branching \( Q \)-matrix given by

\[
r_{ij} = \begin{cases} 
  ib_{j-i+1} & j \geq i - 1, i \geq 1; \\
  0 & \text{otherwise},
\end{cases}
\]

where

\[
b_j \geq 0 \quad (j \neq 0), \quad \sum_{n \neq 1}^{\infty} b_j = -b_1 \geq 0.
\]

Let \( A = (a_{ij}; i, j \in \mathbb{N}) \) be a bounded \( Q \)-matrix given by

\[
a_{ij} = \begin{cases} 
  c_{j-i+1} & j \geq i, i \geq 1; \\
  h_j & j \geq 0, i = 0; \\
  0 & \text{otherwise},
\end{cases}
\]

where

\[
\begin{align*}
  \{ h_j \geq 0 \quad (j \neq 0), & \quad \sum_{j=1}^{\infty} h_j = -h_0 \geq 0; \\
  c_j \geq 0 \quad (j \neq 0), & \quad \sum_{j=1}^{\infty} c_j = -c_0 \geq 0.
\end{align*}
\]

Then the \( Q \)-matrix \( Q = (q_{ij}; i, j \in \mathbb{Z}^+) := R + A \) is called a branching \( Q \)-matrix with immigration and resurrection. The corresponding continuous-time Markov chain is called a branching process with immigration and resurrection. Note that the regularity criterion of the branching \( Q \)-matrix \( R \) is given by Harris (1963). Since \( A \) is a bounded \( Q \)-matrix, by Theorem 1.1 we see \( Q \) is regular if and only \( R \) is regular. This simplifies considerably the proof of Theorem 2.1 in Li and Chen (2006).

### 2 Bounded perturbations

In this section, we assume \( A \) is a bounded \( Q \)-matrix. We shall prove that the regularity of \( R \) and \( Q \) are equivalent. Let \( \gamma = \sup_i a_i = - \inf_i a_{ii} \). Let \( q'_{ii} = \gamma - a_i > 0 \) and \( q'_{ij} = q_{ij} \) for \( i \neq j \). Let \( a'_{ij} = a_{ij} + \gamma \delta_{ij} > 0 \). Then we have \( q'_{ik} = a'_{ik} + (1 - \delta_{ik}) r_{ik} > 0 \),
Proposition 2.1  The backward Kolmogorov equation of \( Q \) is equivalent to the following equation:

\[
Q_{ij}(t) = \sum_{k \in \mathbb{N}} \int_0^t e^{-(r_i + \gamma)(t-s)} q'_{ik} Q_{kj}(s) ds + \delta_{ij} e^{-(r_i + \gamma)t}. \tag{2.3}
\]

Proof. Suppose that \( Q(t) = \{Q_{ij}(t); i, j \in \mathbb{N}\} \) is a solution of the backward Kolmogorov equation \( \partial_t Q(t) = QQ(t) \). Then

\[
\partial_t Q_{ij}(t) + (r_i + \gamma)Q_{ij}(t) = \sum_{k \in \mathbb{N}} q'_{ik} Q_{kj}(t).
\]

Multiplying both sides by the integrating factor \( e^{(r_i + \gamma)t} \), we find

\[
\partial_t (e^{(r_i + \gamma)t} Q_{ij}(t)) = e^{(r_i + \gamma)t} \sum_{k \in \mathbb{N}} q'_{ik} Q_{kj}(t).
\]

Integrating and dividing both sides by \( e^{(r_i + \gamma)t} \) give (2.3). Conversely, suppose \( Q_{ij}(t) \) is a solution of (2.3). By differentiating both sides of the equation we get the backward Kolmogorov equation \( \partial_t Q(t) = QQ(t) \). □

Let \( Q(t) = \{Q_{ij}(t); i, j \in \mathbb{N}\} \) and \( R(t) = \{R_{ij}(t); i, j \in \mathbb{N}\} \) be the minimal transition functions of \( Q \) and \( R \), respectively. By the second successive approximation scheme; see, e.g., Chen (2004, p64), we see

\[
Q_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t) \quad \text{and} \quad R_{ij}(t) = \sum_{n=0}^{\infty} R_{ij}^{(n)}(t), \tag{2.4}
\]

where

\[
R_{ij}^{(0)}(t) = \delta_{ij} e^{-r_i t}, \quad R_{ij}^{(n+1)}(t) = \sum_{k \neq i} \int_0^t e^{-r_i (t-s)} r_{ik} R_{kj}^{(n)}(s) ds \tag{2.5}
\]

and

\[
Q_{ij}^{(0)}(t) = \delta_{ij} e^{-(r_i + \gamma)t}, \quad Q_{ij}^{(n+1)}(t) = \sum_{k \in \mathbb{N}} \int_0^t e^{-(r_i + \gamma)(t-s)} q'_{ik} Q_{kj}^{(n)}(s) ds. \tag{2.6}
\]

Lemma 2.2  For any \( n \geq 0 \) we have

\[
Q_{ij}^{(n)}(t) = \sum_{p=0}^{n-1} \sum_{l,k \in \mathbb{N}} \int_0^t e^{-\gamma (t-s)} R_{il}^{(n-p-1)}(t-s) q'_{ik} Q_{kj}^{(p)}(s) ds + R_{ij}^{(n)}(t) e^{-\gamma t} \tag{2.7}
\]

with \( \sum_{p=0}^{n-1} = 0 \) by convention.
Proof. For $n = 0$, we have (2.7) trivially. Suppose that (2.7) holds for $n = 0, 1, \ldots, m$. Recall that $q'_{ik} = a'_{ik} + (1 - \delta_{ik})r_{ik}$. By the second equality in (2.6) we have
\[
Q^{(m+1)}_{ij}(t) = \sum_{k \neq i} \int_0^t e^{-(r_{ik}(t-s))} r_{ik}Q^{(m)}_{kj}(s)ds + \sum_{k \in \mathbb{N}} \int_0^t e^{-(r_{ik}(t-s))} a'_{ik}Q^{(m)}_{kj}(s)ds
=: I_1 + I_2.
\]
By (2.1) and (2.5) we have
\[
I_1 = \sum_{k \neq i} \int_0^t e^{-(r_{ik}(t-s))} r_{ik} \left[ \sum_{p=0}^{m-1} \sum_{l,r \in \mathbb{N}} \int_0^s e^{-(s-u)} R_{kl}^{(m-p-1)}(s-u) a'_{lr}Q^{(p)}_{rj}(u)du \right]ds
+ \sum_{k \neq i} \int_0^t e^{-(r_{ik}(t-s))} r_{ik} R_{kj}^{(m)}(s)e^{-\gamma s}ds
= \sum_{p=0}^{m-1} \sum_{l,r \in \mathbb{N}} \int_0^t e^{-(s-u)} \left[ \sum_{k \neq i} \int_0^s e^{-r_{ik}(t-s)} r_{ik} R_{kl}^{(m-p-1)}(s-u)du \right] a'_{lr}Q^{(p)}_{rj}(u)du
+ \sum_{k \neq i} \int_0^t e^{-(r_{ik}(t-s))} r_{ik} R_{kj}^{(m)}(s)e^{-\gamma s}ds
= \sum_{p=0}^{m-1} \sum_{l,r \in \mathbb{N}} \int_0^t e^{-(s-u)} R_{il}^{(m-p)}(t-u) a'_{lr}Q^{(p)}_{rj}(u)du + R_{ij}^{(m+1)}(t)e^{-\gamma t}.
\]
On the other hand, using the first equality in (2.5) we obtain
\[
I_2 = \sum_{k \in \mathbb{N}} \int_0^t e^{-(s-u)} R_{ii}^{(0)}(t-s) a'_{ik}Q^{(m)}_{kj}(s)ds
= \sum_{l,k \in \mathbb{N}} \int_0^t e^{-(s-u)} R_{il}^{(0)}(t-s) a'_{lk}Q^{(m)}_{kj}(s)ds.
\]
Summing up the above expressions of $I_1$ and $I_2$, we see (2.7) also holds when $n = m + 1$. That gives the desired result. \(\square\)

Proposition 2.3 Let $Q(t) = \{Q_{ij}(t); i,j \in \mathbb{N}\}$ and $R(t) = \{R_{ij}(t); i,j \in \mathbb{N}\}$ be the minimal transition functions of $Q$ and $R$, respectively. Then $Q_{ij}(t)$ is the unique solution of the following equation
\[
Q_{ij}(t) = \sum_{l,k \in \mathbb{N}} \int_0^t e^{-(s-u)} R_{il}(t-s) a'_{lk}Q^{(m)}_{kj}(s)ds + R_{ij}(t)e^{-\gamma t}. \quad (2.8)
\]
Proof. We first prove the uniqueness of (2.8). Let $\tilde{Q}_{ij}(t)$ be another solution of (2.8). Let $c_{ij}(t) = |Q_{ij}(t) - \tilde{Q}_{ij}(t)|$ and $c_j(t) = \sup_i c_{ij}(t)$. Then we have
\[
c_{ij}(t) \leq \sum_{l,k \in \mathbb{N}} \int_0^t R_{il}(t-s) a'_{lk}c_{kj}(s)ds.
\]
Taking the supremum we have

\[ c_j(t) \leq \sup_i \sum_{l,k \in \mathbb{N}} \int_0^t R_{il}(t-s) d_{lk}' c_j(s) ds = \gamma \int_0^t c_j(s) ds. \]

Using Gronwall’s inequality we have that \( c_j(t) = 0 \). Thus (2.8) has at most one solution.

Next we will show that \( Q_{ij}(t) \) satisfies (2.8). Using (2.4) and (2.7) we have

\[ Q_{ij}(t) = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \int_0^t e^{-\gamma(t-s)} R_{il}^{(n-p)}(t-s) d_{lk}' Q_{kj}^{(p)}(s) ds + \sum_{n=0}^{\infty} R_{ij}^{(n)}(t) e^{-\gamma t}. \]

Interchanging the order of summation and using (2.4) again we obtain

\[
Q_{ij}(t) = \sum_{l,k \in \mathbb{N}} \int_0^t e^{-\gamma(t-s)} \sum_{n=0}^{\infty} R_{il}^{(n)}(t-s) d_{lk}' \sum_{p=0}^{\infty} Q_{kj}^{(p)}(s) ds + \sum_{n=0}^{\infty} R_{ij}^{(n)}(t) e^{-\gamma t}.
\]

That completes the proof. \( \square \)

**Proof of Theorem 1.1.** Summing up both sides of (2.8) over \( j \), we see that \( x_i(t) := \sum_{j=0}^{\infty} Q_{ij}(t) \) is a solution to the following equation:

\[ x_i(t) = \sum_{l,k \in \mathbb{N}} \int_0^t e^{-\gamma(t-s)} R_{il}(t-s) d_{lk}' x_k(s) ds + e^{-\gamma t} \sum_{j=0}^{\infty} R_{ij}(t). \quad (2.9) \]

Suppose that \( R \) is regular. Then we have \( \sum_{j=0}^{\infty} R_{ij}(t) = 1 \), so \( x_i(t) \equiv 1 \) is a solution of (2.9). Let \( \bar{x}_i(t) \) be another solution of (2.9). Set \( c_i(t) = |x_i(t) - \bar{x}_i(t)| \) and \( c(t) = \sup_t c_i(t) \). By (2.9) we obtain

\[ c_i(t) \leq \sum_{l,k \in \mathbb{N}} \int_0^t R_{il}(t-s) d_{lk}' c_k(s) ds. \]

Taking the supremum we get

\[ c(t) \leq \sup_i \sum_{l,k \in \mathbb{N}} \int_0^t R_{il}(t-s) d_{lk}' c(s) ds = \gamma \int_0^t c(s) ds. \]

Using Gronwall’s inequality we have \( c(t) = 0 \). Then we see \( x_i(t) \equiv 1 \) is the unique solution to (2.9). Hence \( Q \) is regular.

Conversely, suppose that \( Q \) is regular. Then \( x_i(t) = \sum_{j=0}^{\infty} Q_{ij}(t) = 1 \). Let \( y_i(t) = \sum_{j=0}^{\infty} R_{ij}(t) \). From (2.9) we have

\[ 1 - e^{-\gamma t} \leq \int_0^t e^{-\gamma(t-s)} y_i(t-s) ds. \]

Then we must have \( y_i(t) \equiv 1 \), so \( R \) is regular. \( \square \)
3 Integration by parts formula

Recall that $R(t)$ and $Q(t)$ are the Feller minimal transition functions of $R$ and $Q$, respectively. By the second successive approximation scheme; see, e.g. Chen (2004, p64) we have

$$Q_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t), \quad \text{(3.10)}$$

where

$$Q_{ij}^{(0)}(t) = \delta_{ij} e^{-q_{ij}t}, \quad Q_{ij}^{(n+1)}(t) = \sum_{k \neq i} \int_{0}^{t} e^{-q_{ik}(t-s)} Q_{kj}^{(n)}(s)ds. \quad \text{(3.11)}$$

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \xi_t, P_t)$ be a realization of $(R_{ij}(t))_{t \geq 0}$.

**Lemma 3.1** Let $\sigma^t_s$ denote the number of jumps of the trajectory $t \mapsto \xi_t$ on the interval $(s, t]$. Then for $n \geq 0$ we have

$$Q_{ij}^{(n)}(t) = \sum_{p=0}^{n-1} \sum_{k \in \mathbb{N}, l \neq k} \int_{0}^{t} P_i(M_{t-s}^0; A_{n-p-1,k}(0, t-s)) a_{kl} Q_{ij}^{(p)}(s)ds$$

$$+ P_i(M_{t}^0; A_{n,j}(0, t)) \quad \text{(3.12)}$$

with $\sum_{p=0}^{n-1} = 0$ by convention, where $A_{n,j}(s, t) = \{\sigma^t_s = n, \xi_t = j\}$ and $M_s^t = e^{-\int_{0}^{t} a(\xi_s)ds}$.

**Proof.** For $n = 0$ we have (3.12) trivially. Suppose that (3.12) holds for $n = 0, 1, \ldots, m$. By (3.11) we have

$$Q_{ij}^{(m+1)}(t) = \sum_{k \neq i} \int_{0}^{t} e^{-q_{ij}(t-s)} r_{ik} Q_{kj}^{(m)}(s)ds$$

$$+ \sum_{k \neq i} \int_{0}^{t} e^{-q_{ij}(t-s)} a_{ik} Q_{kj}^{(m)}(s)ds =: I_1 + I_2.$$

Denote $\tau = \inf\{t \geq 0 : \xi_t \neq \xi_0\}$. By the Markov property we have

$$I_1 = \sum_{k \neq i} \int_{0}^{t} e^{-q_{ij}(t-s)} r_{ik} \sum_{p=0}^{m-1} \sum_{r \in \mathbb{N}, l \neq r} \int_{0}^{s} P_k(M_{s-v}^0; A_{m-1-p,r}(0, s-v)) a_{rl} Q_{ij}^{(p)}(v)dvds$$

$$+ \sum_{k \neq i} \int_{0}^{t} e^{-q_{ij}(t-s)} r_{ik} P_k(M_{t}^0; A_{m,j}(0, s))ds$$

$$= \sum_{p=0}^{m-1} \sum_{r \in \mathbb{N}, l \neq r} \int_{0}^{t} e^{-q_{ij}(t-s)} r_{ik} P_k(M_{s-v}^0; A_{m-1-p,r}(0, s-v)) a_{rl} Q_{ij}^{(p)}(v)dvds$$

$$+ \sum_{k \neq i} \int_{0}^{t} e^{-q_{ij}(t-s)} r_{ik} P_k(M_{t}^0; A_{m,j}(0, s))r_i e^{-r_i(t-s)}ds.$$
On the other hand, we have

\[
\sum_{i=0}^{m-1} \int_0^t \left[ \int_0^t r_i e^{-r_i s} a_{ij} Q^{(m)}_{ij}(v) dv + \sum_{k \neq i} \int_0^t r_i e^{-r_i s} a_{ik} P_k(M_{t-s}^0; A_{m-1-p,r}(0, t-s)) r_i e^{-r_i s} ds \right] dv
\]

\[
= \sum_{i=0}^{m-1} \int_0^t P_i \left[ e^{-a_i t} \sum_{r=0}^{\infty} r_i e^{-r_i s} a_{ij} Q^{(m)}_{ij}(v) dv + \sum_{k \neq i} e^{-a_i t} P_k(M_{t-s}^0; A_{m-1-p,r}(0, t-s)) r_i e^{-r_i s} ds \right] dv
\]

\[
= \sum_{i=0}^{m-1} \int_0^t P_i \left[ e^{-a_i t} P_{\xi_t}(M_{t-s}^0; A_{m-1-p,r}(0, t-s)) a_{ij} Q^{(m)}_{ij}(v) dv + \sum_{k \neq i} e^{-a_i t} P_k(M_{t-s}^0; A_{m-1-p,r}(0, t-s)) r_i e^{-r_i s} ds \right] dv
\]

On the other hand, we have

\[
I_2 = \sum_{i=0}^{m-1} \int_0^t e^{-a_i t} a_{ij} Q^{(m)}_{ij}(v) dv
\]

\[
= \sum_{i=0}^{m-1} \int_0^t P_i \left( e^{-a_i t} 1_{\{a_0 = 0\}} \right) a_{ij} Q^{(m)}_{ij}(v) dv
\]

\[
= \sum_{i=0}^{m-1} \int_0^t P_i(M_{t-s}^0; A_{0, r}(0, t-s)) a_{ij} Q^{(m)}_{ij}(v) dv.
\]

Summing up the above expressions of \(I_1\) and \(I_2\) we see (3.12) also holds when \(n = m + 1\). That gives the desired result. \(\square\)

**Theorem 3.2** The Feller minimal transition functions \(Q(t)\) and \(R(t)\) satisfy the following equation

\[
Q_{ij}(t) = \sum_{k \neq i} \int_0^t P_i(M_{t-s}^0; 1_{\xi_t = k}) a_{kl} Q_{lj}(s) ds + P_i(M_{t-s}^0 1_{\xi_t = j}).
\]

**Proof.** Using (3.11) and (3.12) we have

\[
Q_{ij}(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} \sum_{k \neq i} \int_0^t P_i(M_{t-s}^0; A_{k,n-m}(0, t-s)) a_{kl} Q^{(m)}_{lj}(s) ds + P_i(M_{t-s}^0 1_{\xi_t = j}).
\]

Interchanging the order of summation we see (3.14) holds. \(\square\)
Proof of Theorem 1.2. By the Markov property of \{\xi_t : t \geq 0\},
\[
\sum_{k \in \mathbb{N}} \int_0^t R_{ik}(s) a_k P_k(M_{t-s}^0 1_{\{\xi_{t-s} = j\}}) ds \\
= \int_0^t P_i \left[ a(\xi_s) P_{\xi_s}(M_{t-s}^0 1_{\{\xi_{t-s} = j\}}) \right] ds \\
= \int_0^t P_i \left[ a(\xi_s) P_i(M_t^s 1_{\{\xi_{t} = j\}} | \mathcal{F}_s) \right] ds \\
= \int_0^t P_i \left[ a(\xi_s) M_t^s 1_{\{\xi_{t} = j\}} \right] ds \\
= P_i \left[ 1_{\{\xi_{t} = j\}} \int_0^t a(\xi_s)e^{-\int_0^s f_{t-s} a(\xi_u) du} ds \right] \\
= P_i \left[ 1_{\{\xi_{t} = j\}} \left( 1 - e^{-\int_0^t a(\xi_u) du} \right) \right] \\
= R_{ij}(t) - P_i(M_t^0 1_{\{\xi_{t} = j\}}).
\]

On the other hand, by the Markov property, we have
\[
\sum_{l \in \mathbb{N}, m \neq l} \int_0^t R_{il}(t - v) a_{lm} Q_{mj}(v) dv \\
= \sum_{l \in \mathbb{N}, m \neq l} P_i \left[ \int_0^t 1_{\{\xi_{t-v} = l\}} a_{lm} Q_{mj}(v) \left( 1 - e^{-\int_0^t f_{t-v} a(\xi_u) du} \right) dv \right] \\
+ \sum_{l \in \mathbb{N}, m \neq l} P_i \left( M_{t-v}^0 1_{\{\xi_{t-v} = l\}} \right) a_{lm} Q_{mj}(v) dv \\
= \sum_{l \in \mathbb{N}, m \neq l} P_i \left[ \int_0^t 1_{\{\xi_{t-v} = l\}} a_{lm} Q_{mj}(v) dv \int_0^{t-v} a(\xi_s)e^{-\int_0^s f_{t-v} a(\xi_u) du} ds \right] \\
+ \sum_{l \in \mathbb{N}, m \neq l} P_i \left( M_{t-v}^0 1_{\{\xi_{t-v} = l\}} \right) a_{lm} Q_{mj}(v) dv \\
= \sum_{l \in \mathbb{N}, m \neq l} P_i \left[ \int_0^t a(\xi_s) \int_0^{t-s} M_{t-v}^s 1_{\{\xi_{t-v} = l\}} a_{lm} Q_{mj}(v) dv ds \right] \\
+ \sum_{l \in \mathbb{N}, m \neq l} P_i \left( M_{t-v}^0 1_{\{\xi_{t-v} = l\}} \right) a_{lm} Q_{mj}(v) dv \\
= \sum_{l \in \mathbb{N}, m \neq l} P_i \left[ \int_0^t a(\xi_s) P_i \left( \int_0^{t-s} M_{t-v}^s 1_{\{\xi_{t-v} = l\}} a_{lm} Q_{mj}(v) dv | \mathcal{F}_s \right) ds \right] \\
+ \sum_{l \in \mathbb{N}, m \neq l} P_i \left( M_{t-v}^0 1_{\{\xi_{t-v} = l\}} \right) a_{lm} Q_{mj}(v) dv \\
= P_i \left[ \int_0^t a(\xi_s) \sum_{l \in \mathbb{N}, m \neq l} P_{\xi_s} \left[ \int_0^{t-s} M_{t-s-v}^0 1_{\{\xi_{t-s-v} = l\}} a_{lm} Q_{mj}(v) dv \right] ds \right] \\
+ \sum_{l \in \mathbb{N}, m \neq l} P_i \left( M_{t-v}^0 1_{\{\xi_{t-v} = l\}} \right) a_{lm} Q_{mj}(v) dv \\
= \sum_{k \in \mathbb{N}} \int_0^t R_{ik}(s) a_k \sum_{l \in \mathbb{N}, m \neq l} \int_0^{t-s} P_k(M_{t-s-v}^0 1_{\{\xi_{t-s-v} = l\}}) a_{lm} Q_{mj}(v) dv ds.
\begin{equation}
+ \sum_{l \in \mathbb{N}, m \neq l} P_l \left( M_{l-v}^0 1_{\{\xi_{l-v} = l\}} \right) a_{lm} Q_{mj}(v) dv.
\end{equation}

By the above two equations and (3.14) we obtain (1.1). Suppose that

\[ \sum_{k \in \mathbb{N}} \int_0^t R_{ik}(s) a_k Q_{kj}(t-s) ds < \infty. \]

Then subtracting it from both sides of (1.1) yields (1.2).

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