Unified Description of Multiplicity Distributions and Bose-Einstein Correlations at the LHC Based on the Three-Negative Binomial Distribution

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Abstract

Using the Monte Carlo data at 7 TeV collected by the ATLAS collaboration (PYTHIA 6), we examine the necessity of applying the three-negative binomial distribution (T-NBD). By making use of the T-NBD formulation, we analyze the multiplicity distribution (MD) and the Bose-Einstein correlation (BEC) at the Large Hadron Collider (LHC). In the T-NBD framework, the BEC is expressed by two degrees of coherence $\lambda_1$ and $\lambda_2$ and two kinds of exchange functions $E_1$ and $E_2$ that act over interaction ranges $R_1$ and $R_2$, respectively. Using the calculated $\lambda_1$ and $\lambda_2$ based on the T-NBD, along with free $\lambda_1$ and $\lambda_2$, we analyze the BEC data at 0.9 and 7 TeV. The estimated parameters $R_1$ and $R_2$ are almost coincident and seem to be consistent with $pp$ collisions. We also present an enlarged Koba-Nielsen-Olesen (KNO) scaling function based on the T-NBD, and apply it to KNO scaling at LHC energies. The enlarged scaling function describes the observed violation of the KNO scaling.

1 Introduction

1.1 Negative Binomial Distribution (NBD)

Approximately three decades ago, the UA5 collaboration \cite{UA5} discovered a violation of Koba-Nielsen-Olesen (KNO) scaling \cite{KNO} at CERN $\sqrt{s}$ppS collider. To explain those data, the UA5 collaboration assumed a double-negative binomial distribution (D-NBD) of the data. The NBD is given as

$$P_{\text{NBD}}(n, k; \langle n \rangle) = \frac{\Gamma(n + k)}{\Gamma(n + 1) \Gamma(k)} \left(\frac{\langle n \rangle}{k}\right)^n \left(1 + \frac{\langle n \rangle}{k}\right)^{n+k},$$

where $\langle n \rangle$ and $k$ are the averaged multiplicity and the intrinsic parameter of the NBD, respectively. The D-NBD is expressed as

$$P_{\text{(D-N)}}(n, \langle n \rangle, \alpha_i, k_i) = \sum_{i=1}^{2} \alpha_i P_{\text{NBD}_i}(n, \langle n_i \rangle, k_i),$$

where...
where $\alpha_1 + \alpha_2 = 1$. Using Eqs. (1) and (2), several authors [3–6] have analyzed the multiplicity distribution (MD) of data at high energies, such as the high-energy data of the Large Hadron Collider (LHC) [7, 8]. Among those analyses, the three-NBD (T-NBD) was applied by Zborovsky [9]:

$$P_{\text{T-NBD}}(n, \langle n \rangle, \alpha_i, k_i) = \sum_{i=1}^{3} \alpha_i P_{\text{NBD}_i}(n, \langle n_i \rangle, k_i).$$

(3)

where $\alpha_1 + \alpha_2 + \alpha_3 = 1.0$ (see also [6]).

Here, we address the question “Why must the MD of LHC data be analyzed in the T-NBD framework?” To find a plausible answer, we must consider the constraints adopted by the ATLAS and CMS collaborations (for example, $|\eta| < 2.5$ and $|\eta| < 2.4$). Even in restricted $\eta$ regions, Monte Carlo (MC) estimations have revealed that three processes, i.e., the non-diffractive dissociation (ND), the single-diffractive dissociation (SD), and the double-diffractive dissociation (DD) contribute to the MD at LHC energies [10–12].

For the reader's convenience, our analysis results of MC data at 7 TeV collected by the ATLAS collaboration are presented in Tables 11 and 12 and Figs. 5 and 6 of Appendix A. The individual data ensembles of ND, SD, and DD, defined by the partial probability distributions $P_{\text{ND}}(n)$, $P_{\text{SD}}(n)$, and $P_{\text{DD}}(n)$ respectively, are reasonably described by the D-NBD (see Table 1). Accordingly, the total probability $P_{\text{tot}} = P_{\text{ND}} + P_{\text{SD}} + P_{\text{DD}}$ is also described by the T-NBD. This behavior implies an important role for the T-NBD in MD analyses at LHC energies.

Table 1: Stochastic properties of the three ensembles computed by PYTHIA 8 in the ATLAS experiment at 7 TeV (see Figs. 5 and 6 in Appendix A).

| QCD acceptance and correction by ATLAS coll. | stochastic property of individual ensemble | stochastic description of sum of three ensembles |
|---------------------------------------------|-------------------------------------------|-----------------------------------------------|
| ND $P_{\text{ND}}(n)$ = $\sum P_{\text{NBD}}(n) = 0.787$ | (sum of) NBD$_1$ and NBD$_2$ | T-NBD Eq. (3) $P_{\text{T-NBD}}(n, \langle n \rangle, \alpha_i, k_i) = \sum_{i=1}^{3} \alpha_i P_{\text{NBD}_i}(n, \langle n_i \rangle, k_i)$ |
| SD $P_{\text{SD}}(n)$ = $0.121$ | NBD$_2$ and NBD$_3$ | |
| DD $P_{\text{DD}}(n)$ = $0.092$ | NBD$_3$ and NBD$_2$ | |
| sum of fractions: $f_{\text{ND}} + f_{\text{SD}} + f_{\text{DD}} = 1.0$ | NBD$_i$'s are specified by $(\langle n \rangle, k)$ | Eq. (3) is a possible candidate |

Very recently, Zborovsky [13] revealed the stochastic structure of the T-NBD studying the oscillations in combinations of T-NBDs. As pointed out by Wilk and Wlodarczyk [14], Zborovsky’s work supports the theoretical plausibility of the T-NBD. See also recent study on this subject [15]. Regarding recent T-NBD investigations, we approach the T-NBD from a different perspective, namely, the identical particle effect [16–18] observed at the LHC (see Refs. [19–21] and [22–24] for related theoretical and empirical studies, respectively).

1 In Ref. [29], for $N^{\text{BG}}$, an identical separation between two ensembles with $\alpha_1$ and $\alpha_2$ is assumed. For
1.2 Bose-Einstein correlation at the LHC

The moments of a charged-particle distributions are calculated as

\[ \langle n \rangle = \sum_{i=1}^{3} \alpha_i \langle n_i \rangle, \]

\[ \langle n(n-1) \rangle = \sum_{i=1}^{3} \alpha_i \langle n_i(n_i - 1) \rangle = \sum_{i=1}^{3} \alpha_i \langle n_i \rangle^2 \left( 1 + \frac{1}{k_i} \right), \]

\[ \langle n(n-1)(n-2) \rangle = \sum_{i=1}^{3} \alpha_i \langle n_i(n_i - 1)(n_i - 2) \rangle = \sum_{i=1}^{3} \alpha_i \langle n_i \rangle^3 \left( 1 + \frac{3}{k_i} + \frac{2}{k_i^2} \right). \]  (4)

When the particles are identical, we obtain the following relation (where the charge sign \( a \) is + or -):

\[ \langle n^a(n^a - 1) \rangle = \sum_{i=1}^{3} \alpha_i \langle n_i^a \rangle^2 \left( 1 + \frac{2}{k_i} \right), \]  (5)

Eq. (5) can be interpreted as

Eq. (5) : \[ \sum_{i=1}^{3} (\text{The number of pairs of identical charged particle in MD}(P(n)) \text{ with } \alpha_i) \times (\text{identical particle effect in MD}). \]

Meanwhile, the authors of [20] recently studied the interrelation between the MD and the Bose-Einstein correlation (BEC) under the D-NBD assumption. To extend the framework of T-NBD, we compute \( N^{BG} \) as a function of \( \{\alpha_i, \langle n_i \rangle : i = 1 \sim 3\} \) (see Ref. [21]). The BEC in the extended framework is given by

\[ \frac{N^{(2+: 2-)}}{N^{BG}} = \frac{\sum_{i=1}^{3} \alpha_i (\langle n_i^+ \rangle^2 + \langle n_i^- \rangle^2) \left( 1 + \frac{2}{k_i} \right)}{\sum_{i=0}^{2} 2 \times (\text{The number of pairs of different charged particles (+-) in MD with } \alpha_i)}. \]  (6)

To simplify the calculations, we denote \( \langle n_i^+ \rangle = \langle n_i^- \rangle = \langle n_i \rangle/2 \). In the denominator \( N^{BG} \) of Eq. (6), the three coefficients \( \alpha_i \langle n_i \rangle^2 \) are given by

\[ N^{BG} = \sum_{i=1}^{3} \alpha_i \langle n_i \rangle^2 = a_1 + a_2 + a_3 = s, \]

\[ \sum_{i=1}^{3} \left( \frac{a_i}{s} \right) = 1. \]  (7)

no-separation between them, the following formula is obtained:

\[ \frac{N^{(2+: 2-)}}{N^{BG}} = 1 + (a_1/s)E_1^2 + (a_2/s)E_2^2, \]

where \( s = a_1 + a_2 = a_1 \langle n_1 \rangle^2 + a_2 \langle n_2 \rangle^2 \) (see succeeding Ref. [21]).
In the framework of the T-NBD assumption, we have
\[
\frac{N^{(2+, 2-)}_{\text{BG}}}{N^{(2+, 2-)}} = \sum_{i=1}^{3} \left( \frac{a_i}{s} \right) \left( 1.0 + \frac{2}{k_i} E_{\text{BEC}}^2 \right)
\]
(8)

To describe the BEC in the \(0 \leq Q \leq 2\) GeV region, we assume the following exchange function \(E_{\text{BEC}}^2\)
\[
E_{\text{BEC}}^2 = \begin{cases} 
\exp(-RQ) \text{ (Exponential function) (E)}, \\
\exp(-(RQ)^2) \text{ (Gaussian distribution) (G)},
\end{cases}
\]
(9)

where \(R\) and \(Q\) are the interaction range and the momentum-transfer squared function, respectively. The latter is calculated as \(Q = \sqrt{-(p_1 - p_2)^2}\), where \(p_1\) and \(p_2\) are momenta of identical particles.

By making use of those calculations mentioned above, the BEC is then formulated as
\[
\text{BEC}_{(T-N)} = 1.0 + \sum_{i=1}^{3} \left( \frac{a_i}{s} \right) \left( \frac{2}{k_i} \right) E_{\text{BEC}}^2_i
\]
(10)

where \(\lambda_{(T-N)} = (a_i/s)(2/k_i) (i = 1, 2)\). It should be noted that the third component (with coefficient \(\alpha_3\)) exhibits a Poisson property. Because the \(k_3\) values are large, the third term \((i = 3)\) does not numerically contribute to \(\text{BEC}_{(T-N)}\).

In our BEC analysis of LHC data, we note that all three collaborations (ATLAS, CMS, and LHCb \[22-24\]) applied the well-known conventional formula
\[
\text{CF}_I = 1.0 + \lambda E_{\text{BEC}}^2,
\]
(11)

Regarding Eq. (10) as another conventional formula, we would like to propose that
\[
\text{CF}_II = 1.0 + \lambda_{(II)} E_{\text{BEC}}^2_{1} + \lambda_{(II)} E_{\text{BEC}}^2_{2},
\]
(12)

where \(\lambda_{(II)}^1\) and \(\lambda_{(II)}^2\) are free parameters. By analyzing the BEC data, we can compare \(\lambda_{(II)}^1\) and \(\lambda_{(II)}^2\) in Eq. (12) with the terms \(\lambda_{(T-N)} = (a_i/s)(2/k_i) (i = 1, 2)\) in Eq. (10), and the terms \(R_{(T-N)}^i\) and \(R_{(II)}^i\) in Eqs. (11), (11) and (12).

The second section of this paper analyzes the MD at 0.9 and 7 TeV through Eq. (3). The second and third moments, and \(a_i\)’s and \((a_i/s)(2/k_i)\)’s are displayed in this section. Section 3 analyzes the BEC through Eqs. (10), (11) and (12). This paper concludes with remarks and discussions in Section 4. The appendices analyze the MC data at 7 TeV collected by the ATLAS collaboration (Appendix A) and the KNO scaling data by an enlarged KNO scaling function based on the T-NBD assumption (Appendix B).

2 Multiplicity Distribution \( (P(n)) \) Analysis

We begin by analyzing the MD at 0.9 TeV and 7 TeV obtained by the ATLAS \[7\] and CMS \[8\] collaborations under the T-NBD assumption. The MINUIT program is initialized by assigning
random variables to the physical quantities. The estimated parameters are displayed in Fig. 1 and Table 2. Note that both collaborations obtained similar minimum \( \chi^2 \) values at 0.9 TeV.

To compare our results with those of Zborovsky [9], we adopt the same treatments to the probability distributions. Specifically, we renormalize Eq. (3) without the \( P(0) \) and \( P(1) \) as the MD obtained by the ATLAS collaboration, and also renormalize the MD obtained by the CMS collaboration after excluding \( P(0) \).

Our results in Table 2 almost match those of Zborovsky [9]. Moreover, the empirical values of the second and third moments almost equal those of the D-NBD and T-NBD (Table 3). From the values in Table 2 we obtain \( \lambda_{(T-N)}^i = (\alpha_i \langle n_i \rangle^2/s)(2/k_i) \) \( (i = 1, 2) \). The results are shown in Table 4.

Figure 1: Analysis of MD \( P(n) \) data collected by the ATLAS and CMS collaborations. The \( P(n) \) data were computed by Eq. (3). As done in Ref. [9], we exclude \( P(0) \) obtained by the CMS in our analysis. All data are renormalized.
Table 2: Analysis (by Eq. (3)) of the MD ($P(n)$) data at 0.9 and 7.0 TeV collected by the ATLAS and CMS collaborations.

|        | $i$ | $\alpha_i$  | $\langle n_i \rangle$ | $\alpha_i$  | $k_i$  |
|--------|-----|-------------|---------------------|-------------|--------|
| ATLAS  | 1   | 0.645±0.199 | 13.493±2.546       | 116.519±56.975 | 1.78±0.20 |
|        | 0.9 TeV | 2   | 0.250±0.164 | 28.488±3.588       | 202.892±142.572 | 5.01±1.37 |
| $\chi^2$ = 5.317 | 3   | 0.111±0.047 | 10.998±0.237       | 13.426±5.714       | 28.1±24.4 |
| ATLAS  | 1   | 0.737±0.053 | 21.934±2.392       | 354.571±81.430 | 1.50±0.08 |
|        | 7.0 TeV | 2   | 0.183±0.061 | 57.214±2.597       | 599.040±206.953 | 5.67±0.76 |
| $\chi^2$ = 5.964 | 3   | 0.080±0.010 | 11.164±0.169       | 9.971±1.282       | 23.4±8.4 |
| ATLAS  | 1   | 0.754±0.063 | 22.625±2.549       | 385.966±92.755 | 1.48±0.08 |
|        | 7.0 TeV | 2   | 0.164±0.063 | 57.936±2.618       | 550.479±217.238 | 5.94±0.98 |
| $\chi^2$ = 6.160 | 3   | 0.082±0.010 | 11.177±0.178       | 10.244±1.291       | 23.4±8.7 |
| CMS    | 1   | 0.743±0.179 | 15.852±2.454       | 186.705±73.245 | 2.08±0.20 |
|        | 0.9 TeV | 2   | 0.189±0.170 | 32.160±4.567       | 195.476±184.382 | 6.56±2.85 |
| $\chi^2$ = 4.289 | 3   | 0.068±0.032 | 11.624±0.814       | 9.188±4.511       | 896±817 |
| CMS    | 1   | 0.739±0.200 | 15.830±2.841       | 185.092±83.177 | 2.10±0.21 |
|        | 0.9 TeV | 2   | 0.193±0.179 | 32.093±5.002       | 199.236±194.669 | 6.49±3.03 |
| $\chi^2$ = 4.948 | 3   | 0.068±0.031 | 11.618±0.810       | 9.177±4.384       | $\infty$ |
| CMS    | 1   | 0.826±0.091 | 28.613±4.126       | 676.108±208.928 | 1.66±0.12 |
|        | 7.0 TeV | 2   | 0.103±0.098 | 67.206±6.727       | 465.270±452.384 | 6.67±2.92 |
| $\chi^2$ = 2.275 | 3   | 0.071±0.028 | 13.018±0.870       | 12.036±5.012       | 38.1±73.6 |

Table 3: Second and third moments of MD at 0.9 TeV and 7 TeV, measured (data) and calculated by Eqs. (2) (D-NBD) and (3) (T-NBD).

|        | $\langle n(n-1) \rangle$ ($\times 10^3$) | $\langle n(n-1)(n-2) \rangle$ ($\times 10^4$) |
|--------|----------------------------------------|-----------------------------------------------|
|        | data D-NBD T-NBD data D-NBD T-NBD        |
| ATLAS  | 0.454±0.026 0.450 0.439                | 1.55±0.13 1.54 1.53                           |
| 0.9 TeV | 1.35±0.08 1.35 1.30                     | 8.75±0.79 9.09 8.43                           |
| ATLAS  | 0.488±0.050 0.525 0.511                 | 1.73±0.22 1.86 1.82                           |
| 7.0 TeV | 1.57±0.13 1.67 1.63                     | 10.94±1.10 11.89 11.52                         |
| CMS    | 0.488±0.050 0.525 0.511                 | 1.73±0.22 1.86 1.82                           |
| 0.9 TeV | 1.57±0.13 1.67 1.63                     | 10.94±1.10 11.89 11.52                         |
| CMS    | 0.488±0.050 0.525 0.511                 | 1.73±0.22 1.86 1.82                           |
| 7.0 TeV | 1.57±0.13 1.67 1.63                     | 10.94±1.10 11.89 11.52                         |

Table 4: Values of $\lambda^{(T-N)}_i = (\alpha_i \langle n_i \rangle^2 / s)(2/k_i)$.

|        | $\lambda^{(T-N)}_1$ | $\lambda^{(T-N)}_2$ | $\lambda^{(T-N)}_3$ |
|--------|--------------------|--------------------|--------------------|
| ATLAS  | 0.9 TeV, $\chi^2 = 5.317$ | 0.393±0.214 0.244±0.103 |                   |
| ATLAS  | 7.0 TeV, $\chi^2 = 5.964$ | 0.490±0.130 0.219±0.045 |                   |
| ATLAS  | 7.0 TeV, $\chi^2 = 6.160$ | 0.552±0.152 0.196±0.050 | $\mathcal{O}(10^{-3} \sim 10^{-4})$ |
| CMS    | 0.9 TeV, $\chi^2 = 4.289$ | 0.460±0.241 0.152±0.102 |                   |
| CMS    | 0.9 TeV, $\chi^2 = 4.941$ | 0.448±0.250 0.156±0.110 |                   |
| CMS    | 7.0 TeV, $\chi^2 = 2.275$ | 0.706±0.296 0.121±0.091 |                   |

6
3 Analyses of BEC data by Eqs. (10), (11) and (12)

Our BEC results are displayed in Fig. 2 and Tables 5 and 6. In Tables 5 and 6, the combinations exhibiting high coincidence are indicated by (*1 and *2). In the BEC(T-N) analysis, we apply the calculated $\lambda^{(T-N)}_i (i = 1, 2)$ values in Table 4, which were fixed in the MINUIT computations. Contrarily, the four parameters of the CFII, calculations $\{ R^{(II)}_i, \lambda^{(II)}_i, i = 1, 2\}$ are free; however, the four parameters in Eq. (12) and the set of two parameters ($R^{(T-N)}_i, i = 1, 2$) and two fixed parameters ($\lambda^{(T-N)}_i, i = 1, 2$) give very similar results. We emphasize that the geometrical combinations $G+G$ at 0.9 TeV and $E+G$ at 7 TeV are identical in the CFII and BEC(T-N) formulations. This coincidence is likely attributable to the common stochastic properties of the MD and BEC ensembles (see Tables 5 and 6).

Table 5: Analysis results of the BEC data from ATLAS [22] using Eqs. (10), (11), and (12), where the BEC formulas are normalized with a consistent factor and the long-range correlation is assumed as $(1 + \varepsilon Q)$ (the labels (*1 and *2) indicate equivalence between the tabulated results by Eq. (10) and Eq. (12), respectively).

|            | $R$ [fm] | $\lambda$ (free) | $\chi^2$/ndf |
|------------|----------|------------------|--------------|
| ATLAS 0.9 TeV |          |                  |              |
| CFI        | 1.84±0.07 (E) | 0.74±0.03 | —           | 86.0/75 |
|            | 1.00±0.03 (G) | 0.34±0.01 | —           | 148/75  |
| CFII       | 4.52±1.02 (E) | 0.98±0.21 | 0.81±0.05 (G) | 0.21±0.04 | 78.2 |
|            | 2.82±0.28 (G) | 0.47±0.07 | 0.87±0.03 (G) | 0.26±0.02 | 79.8 (*1) |
| BEC(T-N)   | 2.55±0.10 (G) | 0.39 | 0.85±0.02 (G) | 0.25 | 81.1 (*1) |
| MD $\chi^2 = 5.32$ | 3.37±0.22 (E) | 0.39 | 0.89±0.02 (G) | 0.24 | 101 |
| ATLAS 7.0 TeV |          |                  |              |
| CFI        | 2.06±0.01 (E) | 0.72±0.01 | —           | 919/75  |
|            | 1.13±0.01 (G) | 0.33±0.00 | —           | 4578/75 |
| CFII       | 6.54±0.40 (E) | 0.73±0.05 | 1.80±0.02 (E) | 0.54±0.02 | 465 |
|            | 1.85±0.02 (E) | 0.59±0.01 | 3.51±0.12 (G) | 0.28±0.01 | 466 (*2) |
| BEC(T-N)   | 1.70±0.01 (E) | 0.49 | 2.52±0.03 (G) | 0.22 | 609 |
| MD $\chi^2 = 5.96$ | 2.40±0.02 (E) | 0.49 | 1.52±0.02 (E) | 0.22 | 836 |
|            | 1.80±0.01 (E) | 0.55 | 2.85±0.04 (G) | 0.20 | 531 (*2) |

4 Concluding remarks and discussions

Observing the results of Table II and Appendix A, the T-NBD appears to adequately describe the MD at LHC energies. The MD data are contributed by three processes, ND, SD, and DD. The total probability distribution $P_{tot}$ is expressed by the T-NBD (see Table II and Appendix A).
Figure 2: BEC data at 0.9 and 7 TeV, collected by the ATLAS and CMS collaborations and analyzed by Eqs. (10) and (11). Values in parentheses are the $\chi^2$ values in different geometrical combinations of the exponential function (E) and Gaussian distribution (G).
Table 6: Results of the BEC data collected by the CMS [23] and analyzed by Eqs. (10), (11), and (12), where the BEC formulas are normalized with a constant factor and the long-range correlation is assumed as \((1 + \varepsilon Q)\) (the labels \((\ast 1\) and \((\ast 2))\) indicate equivalence between the tabulated results by Eq. (10) and Eq. (12), respectively).

| CMS 0.9 TeV | \( R \) [fm] | \( \lambda \) (free) | \( \chi^2/\text{ndf} \) |
|-------------|---------------|---------------------|------------------|
| CF\(_I\)    | 1.56±0.02 (E) | 0.62±0.01           | 487/194          |
|             | 0.87±0.01 (G) | 0.30±0.00           |                  |
| CF\(_II\)   | 3.37±0.19 (E) | 0.62±0.01           | 356              |
|             | 2.06±0.07 (G) | 0.38±0.02           |                  |

| CMS 7.0 TeV | \( R \) [fm] | \( \lambda \) (free) | \( \chi^2/\text{ndf} \) |
|-------------|---------------|---------------------|------------------|
| CF\(_I\)    | 1.89±0.02 (E) | 0.62±0.01           | 738/194          |
|             | 1.03±0.01 (G) | 0.29±0.00           |                  |
| CF\(_II\)   | 3.88±0.18 (E) | 0.84±0.03           | 540 \((\ast 2))\) |
|             | 2.39±0.07 (G) | 0.40±0.01           |                  |

| BEC\(_{\text{T-N}}\) | \( R \) \((1-N)\) [fm] | \( \lambda \) \((1-N)\) (calcu.) | \( \chi^2/\text{ndf} \) |
|----------------------|---------------------------|-------------------------------------|------------------|
| MD \( \chi^2 = 4.29 \) | 2.02±0.02 (G) | 0.46 | 429 \((\ast 1))\) |
| MD \( \chi^2 = 4.94 \) | 2.06±0.02 (G) | 0.45 | 422 \((\ast 1))\) |
| cf.                  | 1.29±0.01 (E) | 0.45 | 454 |

| BEC\(_{\text{T-N}}\) | \( R \) \((1-N)\) [fm] | \( \lambda \) \((1-N)\) (calcu.) | \( \chi^2/\text{ndf} \) |
|----------------------|---------------------------|-------------------------------------|------------------|
| MD \( \chi^2 = 2.27 \) | 3.41±0.03 (E) | 0.71 | 559 \((\ast 2))\) |
|                     | 2.07±0.01 (E) | 0.71 | 817 |
Figure 3 compares the workflows of the T-NBD and CF II computations. The estimated interaction ranges and geometrical combinations are comparable between the two approaches.

**Figure 3**: Workflows of the T-NBD and CF II computations. The interaction ranges of both computations are comparable.

The main achievements of the study are summarized below.

**C1)** The values in Table 2 are estimated after initializing the MD($P(n)$) values with random variables in the MINUIT application. Our calculations consider the lack of $P(0)$ and $P(1)$ in the ATLAS collaboration and the exclusion problem on $P(0)$ in the CMS collaboration [9]. Renormalization is the necessary step in the application of Eq. (3).

**C2)** Utilizing the $\lambda_{i}^{(T-N)}$ in Table 4, we obtain the BEC data by Eq. (10). The $\chi^2$ values at 7 TeV are higher in CF II than those in BEC (T-N) and CF II (see also point 3), probably because the MD($P(n)$) at the LHC is governed by stochastic effects [25, 26].

**C3)** Using the CF II values calculated by Eq. (12), we estimate the numerical values of the four-parameter set $\{R_{i}^{(II)}, \lambda_{1}^{(II)}, i = 1, 2\}$. The results are presented in Table 7. The $R_1$ and $R_2$ values estimated by CF II and BEC (T-N) are satisfactorily similar. Table 8 summarizes the two degrees of coherence for the results indicated by ($\ast$1 and $\ast$2) in Tables 5 and 6. Despite the large error bars in $\lambda_{i}^{(T-N)}$'s, the $\lambda_{i}^{(II)}$ and $\lambda_{i}^{(T-N)}$ values are reasonably coincident, possibly reflecting the common stochastic properties of the MD and BEC ensembles, which are both described by the T-NBD.

### Table 7: Comparison of $R_1$ and $R_2$ values marked with ($\ast$1 and $\ast$2) in Tables 5 and 6 and $\chi^2$ of the comparison.

| $\sqrt{s}$ [TeV] | formula | $R_1$ [fm] | $R_2$ [fm] | $\chi^2$ |
|------------------|---------|------------|------------|----------|
| ATLAS 0.9        | CF II   | 2.82±0.28 (G) | 0.87±0.03 (G) | 79.8     |
|                  | BEC (T-N) | 2.55±0.10 (G) | 0.85±0.02 (G) | 81.1     |
| 7.0              | CF II   | 1.85±0.02 (E) | 3.51±0.12 (G) | 466      |
|                  | BEC (T-N) | 1.80±0.01 (E) | 2.85±0.04 (G) | 531      |
| CMS 0.9          | CF II   | 2.06±0.07 (G) | 0.65±0.01 (G) | 384      |
|                  | BEC (T-N) | 2.06±0.02 (G) | 0.62±0.01 (G) | 422      |
| 7.0              | CF II   | 3.88±0.18 (E) | 0.71±0.01 (G) | 540      |
|                  | BEC (T-N) | 3.41±0.03 (E) | 0.70±0.01 (G) | 559      |
Table 8: Comparisons of the two degrees of coherence for the results marked with (*1 and *2) in Tables 5 and 6.

| √s [TeV] | formulas   | λ_1       | λ_2       |
|----------|------------|-----------|-----------|
|          | ATLAS      |           |           |
| 0.9      | CF_{II}    | 0.47±0.07 | 0.26±0.02 |
|          | BEC_{(T-N)}| 0.39±0.21 | 0.24±0.10 |
| 7.0      | CF_{II}    | 0.59±0.01 | 0.28±0.01 |
|          | BEC_{(T-N)}| 0.55±0.15 | 0.20±0.05 |
|          | CMS        |           |           |
| 0.9      | CF_{II}    | 0.38±0.02 | 0.17±0.01 |
|          | BEC_{(T-N)}| 0.45±0.25 | 0.16±0.11 |
| 7.0      | CF_{II}    | 0.84±0.03 | 0.12±0.01 |
|          | BEC_{(T-N)}| 0.71±0.30 | 0.12±0.09 |

C4) Interesting interrelations are found between the results of BEC_{(T-N)} and CF_{II} marked with (*1 and *2) in Tables 5 and 6. The BEC_{(T-N)} and CF_{II} may both reasonably describe the BEC at the LHC. Moreover, the combination of $E_{BEC}^2$’s at 0.9 TeV satisfies the double-Gaussian distribution (G+G), whereas those at 7 TeV are combined exponential and Gaussian distribution (E+G). This finding implies different production mechanisms at 0.9 TeV and 7 TeV.

C5) Possible correspondences are found among the KNO scaling, the MD, and the BEC (see Table 9). These correspondences might be attributed to the violation of KNO scaling discovered in 1989 by the UA5 collaboration, that first proposed the D-NBD. The KNO scaling based on the T-NBD at LHC energies is calculated in Appendix B.

C6) Taking into account the $λ_i^{(II)}$’s as weight factors, the effective interaction ranges can be estimated as,

$$R_E = R_1 \times λ_1 + R_2 \times λ_2.$$  (13)

The estimated effective interaction ranges are displayed in Table 10 and Fig. 4. The $R_E$ values appear reasonable because they are larger at the higher colliding energy (7.0 TeV) than at the lower energy (0.9 TeV).

D1) We must also elucidate the physical meanings of the three intrinsic parameters $k_i$ and weight factor $α_i$. From the MC data in [10][12] and the results of Table 1 we infer the following correspondences:

- 1) The first NBD weighted with $α_1$ ↔ the main part of $σ_{ND}$.
- 2) The second NBD weighted with $α_2$ ↔ the main part of $σ_{SD}$, and parts of $σ_{ND}$ and $σ_{DD}$.
- 3) The third NBD weighted with $α_3$ ↔ the main part of $σ_{DD}$, and a part of $σ_{SD}$.

Here, $σ_{ND}$, $σ_{SD}$, and $σ_{DD}$ are the cross sections of the ND, SD, and double-diffractive dissociation (DD), respectively.

D2) In future work, we are planning the following improvements:

- The large error bars of $λ_i^{(T-N)}$ (i = 1, 2) must be reduced in future work. For this purpose we must improve the framework of the MD analysis.

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Table 9: Correspondences among KNO scaling, MD, and BEC [20, 21] (see also [17, 19]).

| KNO scaling | MD | BEC |
|-------------|----|-----|
| existence.  | Single NBD | CF \(_1^I\) |
| \(\psi_k(z) = \frac{k(kz)(k-1)e^{-kz}}{\Gamma(k)},\) where \(z = n/\langle n\rangle\). | Eq. (2) | Eq. (11) |

violation I:

\[\psi(z) = \sum_{i=1}^{2} \frac{\alpha_i}{r_i} \psi_{k_i}(z_i)\]
\[= \frac{\alpha_1}{r_1} \psi_{k_1}(z_1) + \frac{\alpha_2}{r_2} \psi_{k_2}(z_2),\]
where \(z_i = z/r_i\) [20].
\[\sum_{i=1}^{2} \alpha_i = 1.0\]

violation II:

\[\psi(z) = \sum_{i=1}^{3} \frac{\alpha_i}{r_i} \psi_{k_i}(z_i)\]
\[= \sum_{i=1}^{3} \alpha_i \langle n_i \rangle^2 = \sum_{i=1}^{3} \alpha_i,\]
\[\sum_{i=1}^{3} \alpha_i r_i = 1.0\]

The third term shows the contribution of the Poisson-like distribution.

\[\psi = \psi_{k_1}(z_1)\]
\[= Eq. (12)\]
\[BEC_{(T-N)} = \lambda_i^{(T-N)} E_{BEC_i}^2 + \lambda_i^{(D-N)} E_{BEC_i}^2,\]
where \(\lambda_i^{(T-N)} = (a_i/s)(2/k_i)\) (\(i = 1, 2\)). See Ref. [21].

Table 10: Effective ranges calculated by Eq. (13).

| formulas | \(R_E\) [fm] | 0.9 TeV | 7 TeV |
|----------|---------------|---------|-------|
| ATLAS    | CF\(_{II}\) | 1.55±0.24 | 2.07±0.05 |
|          | BEC\(_{(T-N)}\) | 1.21±0.55 | 1.56±0.31 |
| CMS      | CF\(_{II}\) | 0.89±0.05 | 3.34±0.19 |
|          | BEC\(_{(T-N)}\) | 1.03±0.34 | 2.51±1.02 |
D3) To properly validate the present theoretical formulation [27], we will analyze the MD and BEC (2.0 < \eta < 4.5) obtained by the LHCb collaboration.

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A Monte Carlo data at 7 TeV collected by the ATLAS collaboration and analyzed by Eqs. (1)–(3)

The MC data collected at 0.9 TeV and 7 TeV are presented in [10], which reported the MD results. The total MD is decomposed into three processes: ND, SD, and double-diffractive dissociation (DD), with probability distributions defined by \( P_{\text{ND}} \), \( P_{\text{SD}} \), and \( P_{\text{DD}} \), respectively. The total probability distribution is expressed as \( P_{\text{tot}} = P_{\text{ND}} + P_{\text{SD}} + P_{\text{DD}} \). The partial and total probability distributions are plotted in Figs. 5 and 6, respectively.

The NBD and D-NBD are calculated by Eqs. (1) and (2), respectively, and the results are shown in Table 11. Obviously, the single-NBD cannot describe the MD. The D-NBD probably constitutes three ensembles with different \( \langle n \rangle \) and \( k \) values: the first set with \( \langle n \rangle \approx 33, \, k=2–4 \), the second set with \( \langle n \rangle =4–13, \, k=4–16 \), and the third set with \( \langle n \rangle \approx 10 \) and \( k=200–1000 \). These ensembles appear to reasonably validate the T-NBD framework in the analysis of MD at the LHC. The T-NBD is computed from the total MC data at 7 TeV by Eq. (3). The analysis results are shown in Table 12.

We also analyze the MC data at 0.9 TeV by PYTHIA 6, and at 7 TeV by PHOJET and PYTHIA 8. The results are similar to those in Tables 11 and 12.
Figure 5: Partial probability distributions $P_{\text{ND}}$, $P_{\text{SD}}$, and $P_{\text{DD}}$ obtained by Eq. (2) (see Table 11).

Table 11: PYTHIA 6 analysis of MD at 7 TeV by the ATLAS collaboration calculated by Eqs. (1) and (2). The magnitude of the error bars is assumed as 10% of the data points. The $\chi^2$ values are markedly improved by D-NBD.

| PYTHIA 6 | ATLAS | single-NBD | | | D-NBD |
| --- | --- | --- | --- | --- | --- |
| 7 TeV | ratio | $\langle n \rangle$, $k$ | $\chi^2$ | $i$ | $\alpha_i$ | $\langle n_i \rangle$, $k_i$ | $\chi^2$ |
| --- | --- | --- | --- | --- | --- | --- | --- |
| ND | 0.787 | $\langle n \rangle = 30.17 \pm 0.36$; $k = 2.53 \pm 0.04$ | 150.4 | 1 | 0.814 | 32.5; 2.56 | 6.84 |
| SD | 0.121 | $\langle n \rangle = 7.08 \pm 0.08$; $k = 15.87 \pm 0.80$ | 257.3 | 1 | 0.716 | 4.63; 7.56 | 11.75 |
| DD | 0.092 | $\langle n \rangle = 6.15 \pm 0.08$; $k = 10.46 \pm 0.50$ | 240.7 | 1 | 0.798 | 4.14; 4.34 | 34.00 |

Table 12: Estimated parameters of T-NBD in the $P_{\text{tot}} = P_{\text{ND}} + P_{\text{SD}} + P_{\text{DD}}$ (calculated by Eq. (3)). The magnitude of the error bars is assumed as 10% of the data points.

| $i$ | $\alpha_i$ | $\langle n_i \rangle$, $k_i$ | $\chi^2$ |
| --- | --- | --- | --- |
| 1 | 0.711±0.131 | 15.37±3.62; 1.56±0.31 |
| 2 | 0.255±0.129 | 47.86±5.69; 7.40±2.88 | 0.168 |
| 3 | 0.035±0.023 | 12.99±2.05; 1000.0±551.0 |
B An Enlarged KNO scaling function for LHC energies

In this Appendix, we analyze the KNO scaling at LHC energies. Recall that the D-NBD was proposed to explain the KNO scaling violation found at the S\bar{p}pS energy (√s = 546 GeV) [1]. The KNO scaling function in the framework of the T-NBD [20] is given by

$$\psi_{(T-N)}(z = n/\langle n \rangle, \alpha_i, k_i, r_i; i = 1 \sim 3) = \sum_{i=1}^{3} \alpha_i \frac{1}{r_i} k_i^{k_i} (z/r_i)^{k_i-1} e^{-k_i z/r_i}. \quad (14)$$

After integrating the KNO scaling variables z, the KNO scaling function becomes

$$\int_0^{\infty} dz \psi_{(T-N)}(z, \alpha_i, k_i, r_i; i = 1 \sim 3) = \sum_{i=1}^{3} \alpha_i = 1.0. \quad (15)$$

The violation of the KNO scaling can be understood studying the energy dependences of the parameters $\alpha_i$, $k_i$, and $r_i$.

The results of Eq. (14) are shown in Fig. 7 and Table 13. The ratio $r_i = \langle n_i \rangle/\langle n \rangle$ should be large, because the average multiplicities $\langle n_i \rangle$ are required to be large as $\langle n \rangle$ itself; specifically, $r_i \geq 0.33 \,(\langle n_i \rangle \geq \langle n \rangle/3)$. The KNO scaling functions at 0.9 TeV obtained by the ATLAS and CMS collaborations differed from those at 7, 8, and 13 TeV (Fig 7). The 0.9 TeV data collected by the CMS collaboration must be constrained by $k_2 > 5$ and $k_3 > 7$ because $z_3$ is large (800 or infinity) in the MD analysis. Our KNO scaling analysis also includes the $P(0)$ data at 0.9 and 7 TeV obtained by the CMS collaboration. As shown in Table 13, the violation of KNO scaling occurs through the parameters $\alpha_i$ and $r_i = \langle n_i \rangle/\langle n \rangle \,(i = 1 \sim 3)$. 
Figure 7: Analysis of KNO data collected by the ATLAS and CMS collaborations. The KNO scaling functions were computed by Eq. (14).
Table 13: Analysis (by Eq. (14)) of KNO data at 0.9 and 7.0 TeV collected by the ATLAS and CMS collaboration.

|     | $i$ | $\alpha_i$  | $r_i$  | $k_i$     |
|-----|-----|-------------|--------|-----------|
| ATLAS | 1   | 0.761±0.134 | 0.84±0.10 | 1.79±0.08 |
| 0.9 TeV | 2   | 0.166±0.006 | 1.87±0.03 | 5.75±1.62 |
| $\chi^2 = 25.6$ | 3   | 0.074±0.031 | 0.67±0.01 | 11.0±3.2  |
| ATLAS | 1   | 0.664±0.067 | 0.68±0.09 | 1.54±0.10 |
| 7.0 TeV | 2   | 0.275±0.011 | 1.91±0.02 | 4.37±0.54 |
| $\chi^2 = 27.4$ | 3   | 0.061±0.011 | 0.41±0.01 | 10.0±1.8  |
| ATLAS | 1   | 0.692±0.069 | 0.73±0.11 | 1.50±0.10 |
| 8.0 TeV | 2   | 0.235±0.003 | 2.00±0.01 | 4.61±0.61 |
| $\chi^2 = 39.28$ | 3   | 0.073±0.015 | 0.38±0.01 | 7.77±1.34 |
| ATLAS | 1   | 0.751±0.037 | 0.81±0.06 | 1.27±0.04 |
| 13 TeV | 2   | 0.168±0.050 | 2.18±0.09 | 4.88±0.49 |
| $\chi^2 = 43.26$ | 3   | 0.081±0.007 | 0.35±0.01 | 7.33±0.66 |
| CMS  | 1   | 0.575±0.079 | 0.75±0.09 | 1.51±0.17 |
| 0.9 TeV | 2   | 0.286±0.050 | 1.66±0.04 | 5.0 (lower limit) |
| $\chi^2 = 18.80$ | 3   | 0.139±0.043 | 0.67±0.04 | 7.0 (lower limit) |
| CMS  | 1   | 0.809±0.019 | 0.83±0.02 | 1.42±0.05 |
| 7.0 TeV | 2   | 0.153±0.009 | 2.06±0.05 | 5.18±0.33 |
| $\chi^2 = 7.83$ | 2   | 0.038±0.020 | 0.45±0.04 | 15.2±10.9 |

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