RATIONAL PROOFS FOR QUANTUM COMPUTING

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It is an open problem whether a classical client can delegate quantum computing to an
efficient remote quantum server in such a way that the correctness of quantum computing
is somehow guaranteed. Several protocols for verifiable delegated quantum computing
have been proposed, but the client is not completely free from any quantum technology:
the client has to generate or measure single-qubit states. In this paper, we show that the
client can be completely classical if the server is rational (i.e., economically motivated),
following the “rational proofs” framework of Azar and Micali. More precisely, we consider
the following protocol. The server first sends the client a message allegedly equal to the
solution of the problem that the client wants to solve. The client then gives the server
a monetary reward whose amount is calculated in classical probabilistic polynomial-
time by using the server’s message as an input. The reward function is constructed in
such a way that the expectation value of the reward (the expectation over the client’s
probabilistic computing) is maximum when the server’s message is the correct solution
to the problem. The rational server who wants to maximize his/her profit therefore has
to send the correct solution to the client.

Keywords: rational proofs, delegated quantum computing, verification of quantum
computing
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1 Introduction
One of the most important open problems in quantum physics and quantum computing is
the possibility of classically verifying quantum computing [1, 2, 3]. As is shown in Fig. 1, the
client, a classical computer, is connected to a remote quantum server via a classical channel.
The server does quantum computing for the client, and sends the result to the client. The
client, who does not trust the server, needs some guarantee that the result is correct. How can
the correctness of server’s quantum computing be guaranteed? There is an ironical dilemma
here: quantum computing is useful because it cannot be classically efficiently simulated, but
exactly because of the fact, it is impossible for the client to verify the correctness of server’s
quantum computing via the direct classical simulation.

181
So far, five different types of approaches have been taken to the open problem. First, if the client is allowed to be “slightly quantum”, verifiable delegated quantum computing is possible. For example, verification protocols of Refs. [4, 5] and verifiable blind quantum computing protocols [3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] assume some minimum quantum technologies for the client, such as small quantum memories, single-qubit state generations, or single-qubit measurements.

Second, if multiple entangling quantum servers who are not communicating with each other are allowed, a completely classical client can verify the correctness of servers’ quantum computing [17, 18, 19].

Third, several specific problems solvable with quantum computing have been shown to be classically verifiable. For example, Simon’s problem [20] and factoring [21] are trivially classically verifiable. Furthermore, the recursive Fourier sampling [22] has a poly-round-message-exchange verification protocol between a single quantum server and a completely classical client [23]. Certain problems regarding the output probability distributions of quantum circuits in the second level of the Fourier hierarchy [24] have single-message verification protocols [25, 26]. Calculating the order of solvable groups has two or three-message verification protocols [27].

Fourth, it is known that quantum computing is verifiable by using a technique so called the sum check protocol. However, if we use the sum check protocol, the server needs much stronger computational power than usual quantum computing. (More precisely, the servers needs to be #P [28], which is believed to be much stronger than NP.) In Ref. [29], authors constructed a “quantum version” of the sum check protocol so that the computational power of the server becomes weaker (but still stronger than usual quantum computing).

Finally, a recent innovative work has shown that a classical verification of quantum computing is indeed possible with the assumption that the learning with errors problem is hard for quantum computing [30].

In this paper, we take a new approach different from these previous works. We consider a delegated quantum computing with a rational server. As is shown in Fig. 2, the server first sends the client a message $b$ allegedly equal to the solution of the problem that the client wants to solve. The client then does a classical probabilistic polynomial-time computing to calculate a reward $\$ (b)$, and pays $\$ (b)$ to the server. The reward function $\$ is constructed in such a way that the expectation value of $\$ (over the client’s probabilistic computing) is maximum when $b$ is the correct solution. Therefore, the rational server who wants to maximize his/her
profit has to choose \( b \) as the correct solution.

\[
\text{Quantum server} \quad \xrightarrow{\text{solution } b} \quad \text{Classical probabilistic computer (Client)} \quad \xleftarrow{\text{reward $(b)$}}
\]

Fig. 2. The quantum rational proof system.

We propose two protocols. The first protocol is for decision problems solvable with polynomial-time quantum computing (i.e., BQP). (Actually, the same construction works also for other classes in PP.) The second protocol is for estimating output probability distributions of quantum circuits. Finally, some discussions are given.

The idea of the rational server was first introduced by Azar and Micali in the context of (classical) interactive proof systems [31], which is called “rational proof systems”. (In Sec. 4.1, we provide a brief summary of their results. Although understanding their results is not necessary to understand our results, we provide them because they are insightful, and therefore should be useful for readers.) Among several results, Azar and Micali constructed rational proof systems for \#P and PP. We bring the idea of the rational proof systems to the verification of quantum computing. To our knowledge, it is the first time that the concept of the rational proof systems is applied to quantum information. We believe that the rational proof systems will also be useful in many other areas of quantum information than the verification of quantum computing.

2 Results

2.1 First protocol

In this subsection, we propose our first protocol for BQP. Let us assume that the client wants to solve a decision problem \( L \) in BQP. The client asks the server to solve the problem, and the server sends a single bit \( b \in \{0,1\} \) to the client. If the server is honest, the server sends \( b = 1 \) when the answer is yes \((x \in L)\), and \( b = 0 \) when the answer is no \((x \notin L)\).

Since BQP is in PP, there exists a classical probabilistic polynomial-time algorithm \( A \) that outputs \( a \in \{0,1\} \) such that

- If \( x \in L \) then \( \Pr[a = 1] > \frac{1}{2} \).
- If \( x \notin L \) then \( \Pr[a = 1] < \frac{1}{2} \).

The client runs \( A \). If \( a = b \), the client gives the server the reward \( $ = 1 \). If \( a \neq b \), \$ = 0. The expectation value \( \langle $ \rangle_b \) of server’s reward when the server sends \( b \in \{0,1\} \) to the client is

\[
\langle $ \rangle_b \equiv 1 \times \Pr[a = b] + 0 \times \Pr[a \neq b] = \Pr[a = b].
\]
Therefore, if \( x \in L \),

\[
\langle b \rangle_{b=1} = \Pr[a = 1] > \frac{1}{2},
\]

\[
\langle b \rangle_{b=0} = \Pr[a = 0] < \frac{1}{2}.
\]

If \( x \notin L \),

\[
\langle b \rangle_{b=0} = \Pr[a = 0] > \frac{1}{2},
\]

\[
\langle b \rangle_{b=1} = \Pr[a = 1] < \frac{1}{2}.
\]

This means that the rational server wants to send the correct solution to the client.

Although we use the class PP, the important point here is that the server’s computational ability is enough to be BQP in our protocol. It is clear that the same proof holds for other classes in PP, such as AWPP, QCMA, QMA, SBQP, and \( C_{=P} \), etc.

2.2 Second protocol

Let us explain our second protocol, which is for estimating output probability distributions of quantum computing. Consider an \( n \)-qubit quantum circuit \( V \). Without loss of generality, we can assume that \( V \) consists of only classical gates (such as \( X \), CNOT, Toffoli, etc.) and Hadamard gates [32, 33]. (Generalizations to other gate sets, such as Clifford plus \( T \), are given in Sec. 4.2.) Let

\[
p_z \equiv \langle 0^n | V^\dagger (|z\rangle \otimes I^{\otimes n-k}) V | 0^n \rangle
\]

be the probability of obtaining \( z \in \{0, 1\}^k \) when the first \( k \) qubits of \( V | 0^n \rangle \) is measured in the computational basis. We assume that \( k = O(\log(n)) \). The client wants to know a probability distribution \( \tilde{p} \equiv \{\tilde{p}_z\}_{z \in \{0, 1\}^k} \), which is close to \( p \equiv \{p_z\}_{z \in \{0, 1\}^k} \) in the sense that \( |p_z - \tilde{p}_z| \leq \frac{1}{\text{poly}(n)} \) for all \( z \in \{0, 1\}^k \). Such an estimation \( \tilde{p} \) can be obtained in quantum \( \text{poly}(n) \) time (see Sec 4.3), but the client who is completely classical cannot do it by him/herself. The client therefore delegates the task to the server. We here provide a protocol where the client can receive such an estimation from the rational server.

From \( V \), we construct the \((n + 1)\)-qubit quantum circuit

\[
W_z = (I \otimes V^\dagger) \left( (I \otimes |z\rangle \langle z| + X \otimes (I^{\otimes k} - |z\rangle \langle z|)) \otimes I^{\otimes n-k} \right) (I \otimes V)
\]

for each \( z \in \{0, 1\}^k \). It is easy to see

\[
\langle 0^{n+1} | W_z | 0^{n+1} \rangle = p_z.
\]  \hspace{1cm} (1)

We can construct a classical probabilistic computing \( M_z \) that “simulates” \( W_z \) such that

\[
\langle 0^{n+1} | W_z | 0^{n+1} \rangle = 2^h (D_z(1) - D_z(2)),
\]  \hspace{1cm} (2)

where \( h \) is the number of Hadamard gates in \( V \), and \( D_z(w) \) is the probability that \( M_z \) outputs \( w \in \{1, 2, 3\} \). In fact, Let \( t \) be the number of elementary gates in \( W_z \). In other words, \( W_z = u_t \cdots u_1 \), where \( u_i \) (\( i = 1, 2, \ldots, t \)) is a classical gate or the Hadamard gate. We consider the following \( t \)-step classical probabilistic computing \( M_z \):
1. The state of the register is represented by the pair \((z, c)\) of an \((n + 1)\)-bit string \(z \equiv (z_1, \ldots, z_{n+1}) \in \{0, 1\}^{n+1}\) and a single bit \(c \in \{0, 1\}\). The initial state of the register is \((z = 0^{n+1}, c = 0)\).

2. For \(i = 1, 2, \ldots, t\), do the following:

2-a. If \(u_i\) is a classical gate, update the register as \((z, c) \rightarrow (u_i(z), c)\).

2-b. If \(u_i\) is the Hadamard gate acting on \(j\)th qubit, flip a fair coin. If heads, update the register as
\[
(z, c) \rightarrow (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_{n+1}, c).
\]
If tails, update the register as
\[
(z, c) \rightarrow (z_1, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_{n+1}, c \oplus z_j).
\]

3. If the state of the register is \((0^{n+1}, 0)\), output 1. If the state of the register is \((0^{n+1}, 1)\), output 2. Otherwise, output 3.

An example of the computational tree for \(n = 2\), \(t = 4\), and
\[
\begin{align*}
u_1 &= X \otimes I \otimes I, \\
u_2 &= H \otimes I \otimes I, \\
u_3 &= (|0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes X) \otimes I, \\
u_4 &= I \otimes H \otimes I,
\end{align*}
\]
is given in Fig. 3.

Now it is easy to check Eq. (2). (In the example of Fig. 3, \(D_z(1) = \frac{1}{4}\), \(D_z(2) = 0\), and \(D_z(3) = \frac{3}{4}\).) Therefore, from Eqs. (1) and (2), we obtain
\[
p_z = 2^h(D_z(1) - D_z(2)). \tag{3}
\]

Our protocol runs as follows.
1. The server sends a classical description of a probability distribution \( p' \equiv \{ p'_z \}_{z \in \{0,1\}^k} \) to the client. If the server is honest, \(|p'_z - p_z| \leq \frac{1}{\text{poly}(n)}\) for all \( z \).

2. The client chooses \( z \in \{0,1\}^k \) uniformly at random.

3. The client runs \( M_z \) and obtains the output \( w \in \{1,2,3\} \).

4. The client pays the reward to the server whose amount \( \$ \) is determined according to the following rule:
   - If \( w = 1 \), then \( \$ = S(z,p') + 2 \).
   - If \( w = 2 \), then \( \$ = -S(z,p') + 2 \).
   - Otherwise, \( \$ = 2 \).

Here,

\[
S(z,p') \equiv 2p'_z - \sum_{\alpha \in \{0,1\}^k} (p'_{\alpha})^2 - 1
\]

is called Brier’s scoring rule [34].

The expectation value \( \langle \$ \rangle \) of the prover’s reward is

\[
\langle \$ \rangle = \frac{1}{2^{k+h}} \sum_{z \in \{0,1\}^k} p_z S(z,p') + 2,
\]

where we have used Eq. (3).

Note that

\[
\sum_{z \in \{0,1\}^k} p_z S(z,p) - \sum_{z \in \{0,1\}^k} p_z S(z,p') = \sum_{z \in \{0,1\}^k} (p_z - p'_z)^2,
\]

which means that \( \langle \$ \rangle \) is larger if \( p' \) is closer to \( p \). As is explained in Sec. 4.3, the quantum polynomial-time prover can send \( p' \) such that \(|p'_z - p_z| \leq \frac{1}{\text{poly}(n)}\) for all \( z \). If the server sends another \( p' \) such that \(|p'_z - p_z| = \text{const.}\) for a certain \( z \), on the other hand, his/her expected profit becomes smaller. Therefore the rational prover will not do that.

3 Discussion

In this paper, we have constructed delegated quantum computing protocols with a classical client and a rational quantum server. Let us here mention three advantages of our protocols.

First, our protocols are zero-knowledge, which means that no information other than the solution of the problem itself is leaked from the server to the client.

Second, our protocols do not require any extra computational overhead for the server. For example, in the verification protocols of Refs. [7, 8], some extra trap qubits are needed, and in the verification protocols of Refs. [4, 5, 6], the server has to generate the Feynman-Kitaev history state

\[
\frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} (v_t \cdots v_1 |0^n\rangle) \otimes |t\rangle,
\]
where $V = v_T \cdots v_1$, which is more complicated than the mere output state, $V|0^n\rangle$, of the quantum computation. On the other hand, in our protocols, what the server has to do is only the original quantum computing that the client would do if the client had his/her own quantum computer.

Finally, our protocols neither generate any extra communication overhead between the server and the client. In the verification protocols of Refs. [7, 8], polynomially many bits have to be exchanged between the server and the client in order to verify that the server did the correct measurements on trap qubits. In the verification protocols of Refs. [4, 5, 6], the server has to send the client a Feynman-Kitaev history state, which consists of polynomially many qubits. On the other hand, in our protocols, what the server has to send to the client is only the solution of the problem that the client wants to solve.

In our protocols, reward gaps are exponentially small. It is an open problem whether the constant (or at least polynomial-inverse) reward gap is possible. Unfortunately, we can show that as long as we consider a single-round protocol with the server sending a single bit, it is not possible unless $\text{BQP} = \text{BPP}$. It is shown by using Theorem 16 of Ref. [35], but for readers’ convenience, we give a proof here in our notation. The expectation value of the server’s reward when the server sends $b \in \{0, 1\}$ to the client is

$$\langle \$ \rangle_b = \sum_w p_w \langle w, b \rangle,$$

where the classical probabilistic polynomial-time computing outputs $w$ with probability $p_w$. Since $|\langle w, b \rangle| \leq \text{const.}$ for any $w$ and $b$, the client can estimate the value of $\langle \$ \rangle_b$ within a $1/\text{poly}$ precision in classical probabilistic polynomial time by using the standard Chernoff-Hoeffding bound argument. In fact, let $w_1, w_2, \ldots, w_T$ be the random numbers sampled from the probability distribution $\{p_w\}_w$. The quantity

$$\eta = \frac{1}{T} \sum_{i=1}^T \langle w_i, b \rangle$$

is an $\epsilon$ precision estimator of $\langle \$ \rangle_b$ due to the Chernoff-Hoeffding bound:

$$\Pr \left[ |\eta - \langle \$ \rangle_b| \geq \epsilon \right] \leq 2 \exp \left[ - \frac{T\epsilon^2}{2M^2} \right],$$

where $M \equiv \max_{w, b} |\langle w, b \rangle|$. If $M \leq \text{poly}$, $T = \text{poly}$ is enough to get the $\epsilon = 1/\text{poly}$ precision. If $|\langle \$ \rangle_{b=1} - \langle \$ \rangle_{b=0}| \geq 1/\text{poly}$, the client can learn which $b$ gives larger $\langle \$ \rangle_b$ by itself in classical probabilistic polynomial time, which means $\text{BQP} = \text{BPP}$.

One might notice that the above argument does not work if the server sends the client not a single bit $b \in \{0, 1\}$ but a polynomial-length bit string $b \in \{0, 1\}^{\text{poly}(|x|)}$. In this case, it is no longer possible to calculate $\langle \$ \rangle_b$ for all exponentially many $b$ in classical polynomial time. However, such a generalization does not help, because, as is shown in Sec. 4.4, the power of such a rational proof system is in the third level of the polynomial-time hierarchy.

We also remark effects of errors. For the first protocol, errors in the server do not cause any problem as long as the bit $b$ is correct. For the second protocol, again, errors do not cause any problem as long as the final estimated probabilities are $1/\text{poly}$-close to the true values.
To conclude this paper, let us also mention security of our protocols. In our first protocol, if the client’s result $a$ is leaked to the server before the server sends $b$ to the client, the server can cheat. Therefore, the client’s result $a$ should be hidden from the server. On the other hand, a malicious client can cheat the server. For example, if the server sends $b = 1$ to the client, the malicious client will claim that he/she has generated $a = 0$ thus avoiding the payment. One way of preventing it would be that the client first commits $a$ to the server by using the bit commitment protocol. In this case, however, the security becomes a computational one.

4 Appendix

4.1 Brief summary of Ref. [31]

Here we briefly summarize some of results in Ref. [31]. To understand the essence, let us consider the following protocol:

1. The client samples $w$ from a probability distribution $D$.
2. The server sends the client the description of a probability distribution $D'$.
3. The client gives the server the reward $S(D', w)$.

Here,

$$S(D', w) \equiv 2D'(w) - \sum_{\alpha} (D'(\alpha))^2 - 1$$

is called Brier’s scoring rule [34]. In the above protocol, server’s expected profit is $\sum_w D(w)S(D', w)$. By the straightforward calculation,

$$\sum_w D(w)S(D, w) - \sum_w D(w)S(D', w) = \sum_w (D(w) - D'(w))^2.$$ 

Therefore, server’s expected profit is maximum when $D' = D$. In other words, if the server wants to maximize the expected profit, he/she has to send $D' = D$. The point is that this protocol enables the client, who can sample from $D$ but does not know the description of $D$, to learn the description of $D$ from the rational server.

In Ref. [31], this idea was used to construct a single-message rational protocol for $\#P$ problems. Let

$$\phi : \{0, 1\}^n \ni x \mapsto \phi(x) \in \{0, 1\}$$

be a Boolean function that can be calculated in classical polynomial time. The client first samples an $n$-bit string $x \in \{0, 1\}^n$ uniformly at random. He/She then outputs $\phi(x)$. The probability that the client outputs 0 is $\frac{\#\phi}{2^n}$, where $\#\phi$ is the number of $x \in \{0, 1\}^n$ such that $\phi(x) = 0$. In other words, the client can sample from the probability distribution

$$D : \{0, 1\} \ni w \mapsto D(w) \in [0, 1]$$

such that $D(0) = \frac{\#\phi}{2^n}$ and $D(1) = 1 - \frac{\#\phi}{2^n}$. The ability of sampling from $D$ is not enough for the BPP client to learn $\#\phi$, since the estimation of $D(0)$ with an exponential precision is required. However, if the client uses the above protocol, the client can learn $\#\phi$, since the rational server sends the client the description of $D'$ such that $D'(0) = \frac{\#\phi}{2^n}$ and $D'(1) = 1 - \frac{\#\phi}{2^n}$. 
4.2 Another gate set

Let us assume that a circuit $V$ consists of only Clifford and $T \equiv Z^\frac{1}{4}$ gates. In other words, $V = u_t \cdots u_1$, where $u_i \ (i = 1, 2, \ldots, t)$ is $H, CZ, S = \sqrt{Z}$, or $T$. Let us consider the following $t$-step non-deterministic computing:

1. The state of the register is represented by $(p, c, k)$, where $p$ represents the tensor product of $n$ Pauli operators, $c \in \{+1, -1\}$ represents the sign, and $k$ is an integer that counts the number of non-deterministic transitions experienced. The initial state of the register is $(p = Z \otimes I^{\otimes n-1}, c = +1, k = 0)$.

2. For $i = 1, 2, \ldots, t$, do the following:
   2-a. If $u_i$ is a Clifford gate $g$, update the register as $(p, c, k) \rightarrow (g^\dagger pg, c', k)$.
   2-b. If $u_i$ is $T$ gate acting on $j$th qubit, and if $j$th Pauli operator of $p$ is $Z$, do nothing on the register.
   2-c. If $u_i$ is $T$ gate acting on $j$th qubit, and if $j$th Pauli operator of $p$ is $X$, do the following non-deterministic transition:
   $$(p, c, k) \rightarrow \begin{cases} (p_1 \otimes \ldots \otimes p_{j-1} \otimes X \otimes p_{j+1} \otimes \ldots \otimes p_n, c, k + 1) \\ (p_1 \otimes \ldots \otimes p_{j-1} \otimes Y \otimes p_{j+1} \otimes \ldots \otimes p_n, c, k + 1). \end{cases}$$
   2-d. If $u_i$ is $T$ gate acting on $j$th qubit, and if $j$th Pauli operator of $p$ is $Y$, do the following non-deterministic transition:
   $$(p, c, k) \rightarrow \begin{cases} (p_1 \otimes \ldots \otimes p_{j-1} \otimes X \otimes p_{j+1} \otimes \ldots \otimes p_n, -c, k + 1) \\ (p_1 \otimes \ldots \otimes p_{j-1} \otimes Y \otimes p_{j+1} \otimes \ldots \otimes p_n, c, k + 1). \end{cases}$$

An example for $n = 3, t = 6, and$

\[
\begin{align*}
  u_1 &= H \otimes I \otimes I, \\
  u_2 &= T \otimes I \otimes I, \\
  u_3 &= CZ \otimes I, \\
  u_4 &= H \otimes I \otimes I, \\
  u_5 &= T \otimes I \otimes I, \\
  u_6 &= H \otimes I \otimes I,
\end{align*}
\]

is given in Fig. 4.

It is easy to check that

$$\langle 0^n | V^\dagger (Z \otimes I^{\otimes n-1}) V | 0^n \rangle = \sum_{i:\text{path}} \frac{c_i}{\sqrt{2^{k_i}}} f_i,$$

where the summation is taken over all paths, $(p_i, c_i, k_i)$ is the final state of the register corresponding to the path $i$, and

$$f_i = \begin{cases} 1 & \text{if } p_i \text{ consists of only } Z \text{ and } I, \\ 0 & \text{otherwise}. \end{cases}$$
Rational proofs for quantum computing

\[
\begin{align*}
(ZII,+1,0) & \quad |u_1 \rangle \\
(XII,+1,0) & \quad |u_2 \rangle \\
\quad |u_3 \rangle & \quad (XII,+1,1) (YII,+1,1) \\
\quad |u_4 \rangle & \quad (ZZI,+1,1) \\
\quad |u_5 \rangle & \quad (ZZI,+1,1) \\
\quad |u_6 \rangle & \quad (XZI,+1,1) (YZI,+1,2) (YZI,-1,2) (ZZI,+1,1) \\
\end{align*}
\]

Fig. 4. An example of the non-deterministic computation. For simplicity, the symbol $\otimes$ is omitted, i.e., $Z \otimes I \otimes I$ is written as $ZII$, for example.

Hence

\[
p_{acc} = \frac{1}{2} + \frac{1}{2} \langle 0^n | V^\dagger (Z \otimes I^{\otimes n-1}) V | 0^n \rangle
= \frac{1}{2} + \frac{1}{2} \sum_{i: \text{path}} c_i \frac{1}{\sqrt{2^k}} f_i.
\]

4.3 Estimation

We generate $V|0^n\rangle$ and measure the first $k$ qubits in the computational basis. Output $X = 1$ if the result is $z$. Otherwise, output $X = 0$. We repeat it for $T$ times to correct $X_1, \ldots, X_T \in \{0, 1\}$. If we define

\[
\eta_z \equiv \frac{1}{T} \sum_{i=1}^T X_i,
\]

it satisfies

\[
\Pr[|\eta_z - p_z| \geq \epsilon] \leq 2e^{-2T\epsilon^2}
\]
due to the Chernoff-Hoeffding bound. If we take $\epsilon = \frac{1}{2^k n}$, $T = poly(n)$ is enough to guarantee that $|\eta_z - p_z| \leq \frac{1}{2^k n}$ except for an exponentially small probability. We do this procedure for all $z \in \{0, 1\}^k$ to obtain $\{\eta_z\}_{z \in \{0, 1\}^k}$. Except for an exponentially small probability, $|\eta_z - p_z| \leq \frac{1}{2^k n}$ for all $z$. Let us define

\[
\tilde{p}_z \equiv \frac{\eta_z}{\sum_{z \in \{0, 1\}^k} \eta_z}
\]

for each $z$. Then,

\[
\tilde{p}_z \leq \frac{p_z + \epsilon}{1 - 2^k \epsilon} \leq p_z + 5 \times 2^k \epsilon,
\]
\[ p_z \geq \frac{p_z - \epsilon}{1 + 2^k \epsilon} \geq p_z - 5 \times 2^k \epsilon. \]

Therefore
\[ |\hat{p}_z - p_z| \leq \frac{1}{\text{poly}(n)} \]
for all \( z \) except for an exponentially small probability.

### 4.4 Longer message

Let \( L \) be a language and \( x \) be its instance. Assume that \( L \) has the following rational proof system:

1. The server sends \( b \in \{0, 1\}^m \) to the client, where \( m = \text{poly}(|x|) \).

2. The client samples a polynomial-length bit string \( w \) from a probability distribution \( D \), and sends the reward \((b, w)\) to the server.

3. The client calculates a predicate \( \pi(x, b) \in \{0, 1\} \) and accepts/rejects if \( \pi(x, b) = 1/0 \), where \( \pi \) is a polynomial-time computable Boolean function.

The expectation value of the server’s reward when he/she sends \( b \) to the client is
\[ \langle \$ \rangle_b = \sum_w D(w) \$$(b, w). \]

We require that the rational proof system satisfies the following:

- When \( x \in L \) then there exists \( b^* \in \{0, 1\}^m \) such that \( \pi(x, b^*) = 1 \), and \( \langle \$ \rangle_{b^*} - \langle \$ \rangle_b \geq \frac{1}{n} \) for all \( b \) that satisfies \( \pi(x, b) = 0 \), where \( n = \text{poly}(|x|) \).

- When \( x \notin L \) then there exists \( b^* \in \{0, 1\}^m \) such that \( \pi(x, b^*) = 0 \), and \( \langle \$ \rangle_{b^*} - \langle \$ \rangle_b \geq \frac{1}{n} \) for all \( b \) that satisfies \( \pi(x, b) = 1 \).

We can show that if \( \max_{b, w} |\$$(b, w)\| \leq \text{poly}(|x|) \), then \( L \) is in \( \text{NP}^{\text{MA}[1]} \), which is in \( \text{NP}^{\text{MA}} \subseteq \Sigma_3^P \) [28]. It means that the above rational proof system will not contain BQP, because BQP is not believed to be in the third level of the polynomial-time hierarchy.

In fact, let us consider the following probabilistic polynomial-time algorithm \( M \) on input \((x, b, a) \in \{0, 1\}^* \times \{0, 1\}^m \times \{0, 1\}^m:\)

1. Calculate \( \pi(x, b), \pi(x, a) \in \{0, 1\} \).

2. By using the Chernoff bound, calculate \( \frac{1}{n^2} \)-precision estimates, \( \eta_b \) and \( \eta_a \), of \( \langle \$ \rangle_b \) and \( \langle \$ \rangle_a \), respectively. Except for an exponentially small failure probability \( e^{-\text{poly}(|x|)} \), \( |\eta_b - \langle \$ \rangle_b| \leq \frac{1}{n^2} \) and \( |\eta_a - \langle \$ \rangle_a| \leq \frac{1}{n^2} \).

3. If \( \pi(x, b) = \pi(x, a) = 1 \), accept. If \( \pi(x, b) = 1, \pi(x, a) = 0 \), and \( \eta_b - \eta_a \geq \frac{1}{n} - \frac{2}{n^2} \), accept. Otherwise, reject.

Then \( L \) satisfies the following:
• If $x \in L$ then there exists $b$ such that for all $a$, $M(x, b, a)$ accepts with probability at least $1 - 2^{-r}$, where $r = \text{poly}(|x|)$.
• If $x \notin L$ then for all $b$ there exists $a$ such that $M(x, b, a)$ accepts with probability at most $2^{-r}$.

Therefore, $L$ is in $\text{NP}^{\text{MA}[1]}$.

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