In bounding the homology of a manifold, Forman’s Discrete Morse theory recovers the full precision of classical Morse theory: Given a PL triangulation of a manifold that admits a Morse function with \(c_i\) critical points of index \(i\), we show that some subdivision of the triangulation admits a boundary-critical discrete Morse function with \(c_i\) interior critical cells of dimension \(d - i\). This dualizes and extends a recent result by Gallais. Further consequences of our work are:

1. Every simply connected smooth \(d\)-manifold \((d \neq 4)\) admits a locally constructible triangulation. (This solves a problem by ˇZivaljević.)
2. Up to refining the subdivision, the classical notion of geometric connectivity can be translated combinatorially via the notion of collapse depth.

\[\text{Abstract}\]

In bounding the homology of a manifold, Forman’s Discrete Morse theory recovers the full precision of classical Morse theory: Given a PL triangulation of a manifold that admits a Morse function with \(c_i\) critical points of index \(i\), we show that some subdivision of the triangulation admits a boundary-critical discrete Morse function with \(c_i\) interior critical cells of dimension \(d - i\). This dualizes and extends a recent result by Gallais. Further consequences of our work are:

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2. Up to refining the subdivision, the classical notion of geometric connectivity can be translated combinatorially via the notion of collapse depth.

\[\text{1 Introduction}\]

Morse Theory, introduced by Marston Morse in the Twenties [43], has been a reservoir for breakthrough results ever since. It analyzes a smooth manifold \(M\) without boundary by looking at generic smooth functions \(f: M \to \mathbb{R}\). Via Morse theory, one can bound the homology of a manifold: The number of critical points of \(f\) of index \(i\) is not less than the \(i\)-th Betti number of \(M\). When these two numbers coincide, the Morse function is called “perfect”.

Plenty of manifolds do not admit perfect Morse functions. Yet sometimes non-perfect Morse functions may be “sharpened”: Smale’s cancellation theorem provides sufficient conditions for canceling critical points in pairs [49, 50]. For many interesting examples of manifolds, including spheres and complex manifolds, the sharpening process goes on until one eventually reaches a perfect Morse function. This is at the core of Smale’s proof of the higher-dimensional Poincaré conjecture [50].

In the last decade, Forman’s Discrete Morse Theory [19] has provided important contributions to computational geometry and to combinatorial topology. Discrete Morse Theory uses regular cell complexes in place of manifolds. It studies a complex \(C\) by looking at certain weakly-increasing maps \(f: (C, \subseteq) \to (\mathbb{R}, \leq)\), where \((C, \subseteq)\) is the poset of all faces of \(C\), ordered by inclusion. The “critical cells” in the discrete setting are simply the faces of \(C\) at which the function \(f\) is strictly increasing. As for smooth Morse theory, the critical cells of \(f\) of dimension \(i\) are not fewer than the \(i\)-th Betti number of \(C\). When equality is attained, \(f\) is called “perfect”.

There is also a discrete analogous of Smale’s cancellation theorem: A sufficient condition for pairwise canceling critical cells is the existence of a unique “gradient path” (see [20, Section 9] for the
definition) from the boundary of one cell to the other cell. Both cells are no longer critical if we reverse the gradient path [19].

Imagine a situation in which the two theories — the smooth one and the discrete one — can be applied at the same time. In fact, any smooth manifold will automatically admit PL triangulations [14].

1 Which theory is going to give the better bounds?

In a previous work [6] we showed that if the triangulation is fixed, smooth Morse theory typically wins at large. For each \( k \geq 0 \) and each \( d \geq 3 \), we constructed a PL triangulation of \( S^d \) on which any discrete Morse function must have more than \( k \) critical edges [6, Thms. 3.10 & 4.18] [7, Thm. 2.19]. Since the first Betti number of \( S^d \) is zero, this demonstrates that for a fixed triangulation of \( S^d \) the bounds given by discrete Morse theory may be arbitrarily bad — independently of the function chosen. In contrast, by the smooth Poincaré conjecture (proven in all dimension \( d \neq 4 \)), every closed \( d \)-manifold homotopy equivalent to \( S^d \) admits a (perfect) smooth Morse function with only two critical points, one of index 0 and the other one of index \( d \).

The ‘nasty’ triangulations of \( S^d \) are not rare. Typically, the problem is the presence in the \((d-2)\)-skeleton of a complicated \((d-2)\)-knot with relatively few facets. This is a local defect, which intuitively explains why knotted spheres are at least as numerous as unknotted ones. (This statement can be made precise by counting triangulations asymptotically with respect to the number of facets, cf. [6, Section 3.2].)

Yet in discretizing any continuous theory, one usually leaves the door open for successive refinements of the discrete structure chosen. What if we do not fix a triangulation? Or better, what if we initially choose a triangulation, but later allow for subdivisions? In this more natural setting, it turns out that we can recover the full precision of smooth Morse theory already at the discrete level. The first claim of this fact is contained in the recent work by Gallais [24, Thm. 3.1]:

\[
\text{Let } M \text{ be a closed } d \text{-dimensional smooth manifold that admits a smooth Morse function with } c_i \text{ critical points of index } i. \text{ (In particular, } M \text{ admits a PL handle decomposition into } c_i \text{ } i\text{-handles.) Then a suitable PL triangulation } M \text{ of } M \text{ admits a discrete Morse function with } c_i \text{ critical cells of dimension } i.
\]

The proof in [24] contains a minor gap, which we explain in Remark 3.9.

Here we present a “dual result”, obtained with a slightly simpler (and independent) combinatorial approach:

**Main Theorem 1** (Theorem 4.4). Let \( M \) be any PL triangulated \( d \)-manifold, with or without boundary, that admits a handle decomposition into \( c_i \) PL \( i \)-handles. Then a suitable subdivision of \( M \) admits a boundary-critical discrete Morse function with \( c_i \) critical interior cells of dimension \( d - i \).

Both results show that if one can sharpen “smoothly” (via Smale’s cancellation theorem) then, up to subdividing, one can sharpen also “discretely”. Moreover, the conclusion of Main Theorem 4.4 is exactly what we need to answer a question by Živaljević, which we will now explain.

Locally constructible triangulations of manifolds (or, shortly, LC manifolds) were introduced by Durhuus and Jonsson in 1994 [17], in connection with the discretization of quantum gravity. They are the manifolds obtainable from some tree of \( d \)-simplices by repeatedly identifying two adjacent \((d-1)\)-simplices in the boundary [7]. (A “tree of \( d \)-simplices” is a \( d \)-ball whose dual graph is a tree.) Durhuus and Jonsson showed that all LC manifolds are simply connected. Moreover, they showed with elementary methods that all LC closed 3-manifolds are spheres [17]. (This could also be derived via the

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1Kervaire constructed topological 10-manifolds that admit PL triangulations, without admitting any smooth structure [34].
3-dimensional Poincaré conjecture, since all LC manifolds are simply connected. However, Durhuus–Jonsson’s combinatorial proof appeared about ten years before Perelman’s proof of the Poincaré conjecture.

Even if for \( d \leq 3 \) all closed LC \( d \)-manifolds are spheres, it recently turned out that for \( d \geq 4 \) other topological types such as products of spheres are possible [5]. Meanwhile, LC closed manifolds have been characterized by the author and Ziegler as the manifolds admitting a discrete Morse function without critical faces of dimension \((d-1)\) [6, 7]. In 2009, Živaljević has conjectured that the class of topological types of LC triangulations consists of all simply connected manifolds. The intuition behind this conjecture is that the topological notion of simply connectedness is ‘captured’ by the combinatorial notion of local constructibility. Via Main Theorem 4.4, we are now able to prove Živaljević’s conjecture in all dimensions higher than four.

**Main Theorem 2** (Theorem 5.2). Every simply connected PL \( d \)-manifold (and thus every smooth \( d \)-manifold) admits an LC triangulation, except possibly when \( d = 4 \).

When \( d = 3 \), Main Theorem 2 relies on Perelman’s proof of the Poincaré conjecture, cf. [42]. In fact, modulo the elementary combinatorial proof that all closed LC 3-manifolds are spheres, Main Theorem 2 is indeed equivalent to the Poincaré conjecture when \( d = 3 \).

When \( d > 5 \), the proof of Main Theorem 2 is obtained by combining our Main Theorem 1 with Smale’s proof of the Poincaré conjecture. That said, our proof of Main Theorem 1 is relatively easy, by induction on the dimension \( d \) of \( M \), and can be sketched as follows:

1. We are given a handle decomposition of \( M \) with \( c \) PL \( i \)-handles, which topologically are \( d \)-balls. If the handles and their intersections are “nicely” triangulated, so is their union \( M \) (cf. Theorem 4.1).
2. Since the intersection of each handle \( H \) with the previous handles is a lower-dimensional submanifold of its boundary \( \partial H \), we may assume by induction that this intersection has been nicely triangulated already.
3. Thus, we are left with the problem of how to extend the given, nice triangulations of submanifolds of \( \partial H \) into a nice subdivision of the whole ball \( H \). We achieve this by adapting a result of classical PL topology by Zeeman, cf. Proposition 3.7.

Intuitively, after Main Theorem 2, LC triangulations should be regarded as the “nicest” triangulations of simply connected manifolds. More generally, what are the “nicest” triangulations of \( k \)-connected manifolds? To answer this question, we put to good use the notion of collapse depth, which we recently introduced in [6]. This collapse depth turns out to be a combinatorial analogous of the classical notion of geometrical connectivity, studied among others by Stallings [51] and Wall [52]. Building on their formidable work, we can prove the following:

**Main Theorem 3** (Corollary 4.6). Every \( k \)-connected smooth or PL \( d \)-manifold, if \( k \leq d - 4 \), admits a triangulation with collapse depth \( k + 1 \).

Thus, the collapse depth of a triangulation of a given \( d \)-manifold equals the geometric connectivity of the manifold, plus one, plus some “combinatorial noise” which depends only on the triangulation chosen. Intuitively, this noise can be progressively reduced by taking suitable subdivisions.

## 2 Preliminaries

Here we review the basic definitions from the world of triangulated manifolds and PL topology. We refer the reader to one of the books [13, 25, 31, 47, 54] for a more detailed introduction. Our notation differs from the standard one only in the following aspect: In order to avoid a possible linguistic ambiguity (namely, the fact that if the smooth Poincaré conjecture is false, some 4-ball might be “PL” as
manifold but “non-PL” as ball), we adopt a slightly stronger definition of “PL manifold”. In all practical examples, this new definition coincides with the old one. In fact, we do not even know whether the two definitions are really different: A concrete example on which the two definitions would disagree, would also disprove the smooth Poincaré conjecture, which is a long-standing open problem in topology. See Subsection 2.3 for details.

2.1 Manifolds and handle decompositions

By a \(d\)-dimensional TOP-manifold we mean a topological space \(\mathcal{M}\), Hausdorff and compact, in which every point has an open neighborhood that is either homeomorphic to \(\mathbb{R}^d\) or homeomorphic to the Euclidean half-space \(\{x \in \mathbb{R}^d \mid x_d \geq 0\}\). The boundary of a TOP-manifold is the set of points with neighborhood homeomorphic to the Euclidean half-space. By TOP-manifold with boundary (resp. without boundary) we mean that the boundary is non-empty (resp. empty). Closed is synonymous of “without boundary”. Thus the boundary of any \((d + 1)\)-manifold is either empty, or a disjoint union of closed \(d\)-manifolds. All the TOP-manifolds we consider here are connected and orientable. A \(d\)-TOP-ball (resp. a \(d\)-TOP-sphere) is a TOP-manifold homeomorphic to the \(d\)-simplex (resp. to the boundary of the \((d + 1)\)-simplex).

A smooth manifold is a TOP-manifold that admits a smooth structure. Some 4-dimensional TOP-manifolds do not admit any smooth structure [25, p. 105]. In contrast, some TOP-manifolds admit even more than one smooth structure: Using Morse theory, Milnor constructed a 7-dimensional smooth manifold that is homeomorphic, but not diffeomorphic, to the boundary of the unit ball in \(\mathbb{R}^8\) [41]. \(S^2 \times S^2\) has even infinitely many different smooth structures [4]. In contrast, any TOP-manifold of dimension different than four admits only a finite number of non-diffeomorphic smooth structures.

Let \(I = [0, 1]\) be the unit segment in \(\mathbb{R}\). Let \(\mathcal{M}\) be a \(d\)-dimensional TOP-manifold with boundary and let \(\mathcal{S}\) be a \(d\)-dimensional TOP-ball, so that \(\mathcal{S} \cap \mathcal{M} \subset \partial \mathcal{M}\). We say that \((\mathcal{S}, h)\) is a \(d\)-dimensional handle of index \(p\) on \(\mathcal{M}\), or simply a \(p\)-handle, if \(h : I^p \times I^{d-p} \to \mathcal{S}\) is a homeomorphism such that \(h(I^p \times \{0\}) = \mathcal{M} \cap \mathcal{S}\). We denote a \(p\)-handle by \(\mathcal{S}(p)\), carrying the index (and not the dimension!) in the notation. The TOP-manifold \(\mathcal{M} = \mathcal{M} \cup \mathcal{S}(p)\) is obtained from \(\mathcal{M}\) by “attaching a \(p\)-handle”. We refer to \(\mathcal{M} \cap \mathcal{S}(p)\) as the intersection of the \(p\)-handle \(\mathcal{S}(p)\). The notation \(\mathcal{M} = \mathcal{M} \cup \mathcal{S}(r) \cup \mathcal{S}(s)\) means that \(\mathcal{S}(r)\) is an \(r\)-handle on \(\mathcal{M}\) and \(\mathcal{S}(s)\) is an \(s\)-handle on \(\mathcal{M} \cup \mathcal{S}(r)\).

A handle decomposition of a TOP-manifold \(\mathcal{M}\) is an expression of the form

\[
\mathcal{M} = S_0^{(0)} \cup \ldots \cup S_m^{(r)} \cup S_m^{(s)},
\]

where \(S_0^{(0)}\) is a 0-handle and all other handles are \(p\)-handles with \(p > 0\). (This setting corresponds to the particular case \(V_0 = \emptyset\) and \(V_1 = \partial M\) of the more general notion of “handle decomposition for a cobordism \((M, V_0, V_1)\)” described in [47, 49].) We can assume that the handles are attached in order of increasing index [25, p. 107]. If \(\mathcal{B}\) is a TOP-ball, with slight abuse of notation we view \(\mathcal{B}\) as a 0-handle and regard the tautology \(\mathcal{B} = S_0^{(0)}\) as a handle decomposition. Only \(d\)-balls admit handle decompositions with only one handle.

The core of a \(d\)-dimensional \(p\)-handle \(\mathcal{S}\) is the image under the homeomorphism \(h : I^p \times I^{d-p} \to \mathcal{S}\) of the \(p\)-dimensional TOP-ball \(I^p \times \{0, \ldots, \frac{1}{2}\}\) in \(\mathbb{R}^d\). (We refer to [25, p. 100] or to [47, p. 74] for nice illustrations.) By definition, the core of a \(p\)-handle is a \(p\)-cell. By shrinking each handle onto its core, from a handle decomposition we obtain a CW-complex homotopy equivalent to \(\mathcal{M}\) [47, p. 83]. In particular, if a TOP-manifold admits a handle decomposition without 1-handles, then the TOP-manifold is simply connected. The converse is not true: Mazur constructed a contractible smooth 4-manifold all of whose handle decompositions contain 1-handles [23, 38]. More generally, let \(k\) be an integer in \(\{1, \ldots, d\}\). A \(d\)-dimensional TOP-manifold \(M\) is \(k\)-connected if all the homotopy groups \(\pi_0(M), \ldots, \pi_k(M)\) are zero; it is geometrically \(k\)-connected if it admits a handle decomposition with one 0-handle and no further handles of dimension \(\leq k\) [52]. Since every handle can be shrunk onto its core, every geometrically \(p\)-connected
TOP-manifold is also $p$-connected. The converse is false: Mazur’s smooth manifold is 1-connected, but not geometrically.

Some 4-dimensional TOP-manifolds do not admit any handle decomposition [25, p. 105]. However, every TOP-manifold that admits a smooth structure admits also some smooth Morse function; and any smooth Morse function induces in fact a handle decomposition, cf. Milnor [40].

2.2 Triangulations, joins and subdivisions

The underlying (topological) space $|C|$ of a simplicial complex $C$ is the union of all of its faces. Conversely, the simplicial complex $C$ is called a triangulation of $|C|$ (and of any topological space homeomorphic to $|C|$). If $C$ and $D$ are two simplicial complexes with the same underlying space, $C$ is called a subdivision of $D$ if every cell of $C$ is contained in a cell of $D$.

By a $d$-manifold we mean a simplicial complex whose underlying space is homeomorphic to a $d$-dimensional TOP-manifold. For example, by a $d$-ball or a $d$-sphere we mean a simplicial complex homeomorphic to a TOP-ball (resp. a TOP-sphere). In other words, all the manifolds we consider from now on are actually triangulations of TOP-manifolds. We point out that not all TOP-manifolds can be triangulated: There are counterexamples in each dimension $d \geq 4$. In fact, a 4-dimensional TOP-manifold admits a handle decomposition if and only if it admits a smooth structure, if and only if it is triangulable [25, p. 105].

Since TOP-balls can be triangulated, it makes sense to study handle decompositions in the triangulated category: Each $p$-handle should be a simplicial complex, and it should intersect the previous handles at a subcomplex of its boundary, homeomorphic to $\partial I^p \times I^{d-p}$. Every manifold, possibly after a suitable subdivision, admits a handle decomposition in the triangulated sense. We will use latin characters for handle decompositions in the triangulated category, writing

$$M' = H_0^{(0)} \cup \ldots \cup H_{m-1}^{(r)} \cup H_m^{(s)},$$

where $M'$ is either the manifold $M$ itself, or (possibly) a suitable subdivision of $M$.

Given two disjoint simplices $\alpha$ and $\beta$, the join $\alpha \ast \beta$ is a simplex whose vertices are the vertices of $\alpha$ plus the vertices of $\beta$. By convention, $\emptyset \ast \beta = \beta$ itself. The join of two simplicial complexes $A$ and $B$ is defined as $A \ast B := \{\alpha \ast \beta : \alpha \in A, \beta \in B\}$. If $\sigma$ is a face of a simplicial complex $C$, and $\hat{\sigma}$ is an arbitrary point in the interior of $\sigma$, we define

$$C' = (C - \text{star}(\sigma,C)) \cup \hat{\sigma} \ast \text{link}(\sigma,C).$$

This $C'$ is a subdivision of $C$. We say that $C'$ is obtained from $C$ by starring the face $\sigma$. A stellar subdivision is a subdivision obtained starring one or more faces, in some order. A first derived subdivision of $C$ is obtained by starring all the simplices of $C$, in order of (weakly) decreasing dimension. Recursively, an $r$-th derived subdivision is the first derived of an $(r-1)$-st derived. The barycentric subdivision is a first derived subdivision obtained by starring at the barycenters. With abuse of notation, we will denote any first derived subdivision of $C$ (including the barycentric) by $sdC$. Stellar subdivisions are particularly nice from a combinatorial perspective: For example, if $C$ is a shellable complex, any stellar subdivision of $C$ is shellable, while an arbitrary subdivision of $C$ might not be shellable.

2.3 PL topology

If a $k$-simplex $\Delta$ is the join of two disjoint faces $\sigma$ and $\tau$, then $\dim \sigma + \dim \tau = k - 1$ and

$$\partial \Delta = \partial(\sigma \ast \tau) = (\partial \sigma \ast \tau) \cup (\sigma \ast \partial \tau).$$

Assuming the pair $(\sigma, \tau)$ is ordered, the expression to the right hand side gives (up to isomorphism) $k$ different ways of expressing the boundary of a $k$-simplex. If $\sigma$ is a $k$-simplex inside a PL triangulated
$d$-manifold $M$, and $\text{link}(M, \sigma) = \partial \tau$ for some $(d-k)$-simplex $\tau$ not in $M$, the bistellar flip $\chi(\sigma, \tau)$ consists of changing $M$ to

$$\tilde{M} := (M - \sigma \ast \partial \tau) \cup (\partial \sigma \ast \tau).$$

A $d$-ball $B$ is called PL if one of the following equivalent [45] conditions holds:

(i) $B$ is piecewise-linearly homeomorphic to a $d$-simplex; this means that there exist subdivisions $B'$ of $B$ and $\Delta'$ of the $d$-simplex, and a bijective map $h : B' \to \Delta'$, such that $h$ maps vertices to vertices and simplices linearly to simplices;

(ii) $B$ is obtainable from the $d$-simplex via a finite sequence of bistellar flips.

Similarly, a $d$-sphere $S$ is called PL if one of the following equivalent [45] conditions holds:

(i) $S$ is piecewise-linearly homeomorphic to the boundary of the $(d+1)$-simplex;

(ii) $S$ is obtainable from the boundary of the $(d+1)$-simplex via a finite sequence of bistellar flips.

Given a $d$-sphere $S$, the fact that $S$ is PL implies that every vertex link inside $S$ is a PL $(d-1)$-sphere. Surprisingly, the converse also holds, except (possibly) when $d = 4$. The conjecture that this holds when $d = 4$ as well goes under the name of PL Poincaré conjecture. It is equivalent to the smooth Poincaré conjecture, which claims that every TOP-manifold with a smooth structure and the same homotopy type of a $4$-sphere is diffeomorphic to $S^4$ [35, Problem 4.89] [13, 39]. It is also equivalent to the conjecture that every $4$-sphere is PL. These three equivalent conjectures are typically believed to be false [22], but whenever interesting classes of $4$-spheres have been analyzed, they have always turn out to be PL; see Akbulut [1, 2]. That said, in each dimension $d \geq 5$ we already know that non-PL $d$-spheres exist by the work of Edwards [18].

A $d$-manifold is linkwise-PL if the link of any vertex on its boundary (resp. in its interior) is a PL $(d-1)$-ball (resp. a PL $(d-1)$-sphere). All PL balls and spheres are linkwise-PL. The three-dimensional Poincaré conjecture, recently proven by Perelman [42], implies that for $d \leq 4$ all $d$-manifolds are linkwise-PL. Linkwise-PL $d$-manifolds are usually called “PL manifolds” in the literature. We will refrain from this simplification, since it could potentially create some embarrassment when $d = 4$: In fact, all $4$-spheres are linkwise-PL (manifolds), but unless the smooth Poincaré conjecture is true, we would expect to find some non-PL $4$-sphere someday.

Let us define “PL handle decompositions” as handle decompositions inside the PL category. Formally, the definition is by induction on the dimension: All (triangulated) handle decompositions of a $d$-manifold are PL handle decompositions, if $d \leq 2$. Recursively, a handle decomposition of a $d$-manifold $M$ $(d > 2)$ is called a PL handle decomposition if and only if:

- all handles are PL balls;
- all intersections admit a PL handle decomposition.

For example, every PL $d$-sphere admits a PL handle decomposition with one PL 0-handle and one PL $d$-handle, whose intersection is a PL $(d-1)$-sphere. In the present paper, by PL manifold we will denote a manifold that admits a PL handle decomposition. This notion is consistent: A PL manifold homeomorphic to a sphere is a PL sphere, even if the smooth Poincaré conjecture turns out to be false. (The same cannot be said of linkwise-PL manifolds.) Clearly, if the smooth Poincaré conjecture is true, PL manifolds and linkwise-PL manifolds coincide.

Every smooth manifold admits a PL handle decomposition [14]. In fact, any smooth Morse function on the manifold induces one possible PL handle decomposition, cf. [47, Chapter 6] or [25, Chapter 4]. Neither the (linkwise) PL property nor the smooth structure are preserved under homeomorphisms: for example, two manifolds homeomorphic to $S^4$ need not be diffeomorphic [41]; one could be PL and the other one non-PL [18]. Interestingly, not all PL manifolds are smooth: Kervaire found examples of closed PL 10-manifolds that do not admit any smooth structure [34].
2.4 Discrete Morse functions, collapses and local constructions

The face poset \((C, \subset)\) of a simplicial complex \(C\) is the set of all the faces of \(C\), ordered with respect to inclusion. By \((\mathbb{R}, \leq)\) we denote the poset of the real numbers with the usual ordering. A discrete Morse function is an order-preserving map \(f\) from \((C, \subset)\) to \((\mathbb{R}, \leq)\), such that:

- the preimage \(f^{-1}(r)\) of any real number \(r\) consists of at most two elements;
- if \(f(\sigma) = f(\tau)\), then either \(\sigma \subseteq \tau\) or \(\tau \subseteq \sigma\).

A critical cell of \(C\) is a face at which \(f\) is injective.

The function \(f\) induces a perfect matching (called Morse matching) on the non-critical cells: Two cells are matched if and only if they have identical image under \(f\). The Morse matching can be represented by a system of arrows: Whenever \(\sigma \subset \tau\) and \(f(\sigma) = f(\tau)\), one draws an arrow from the barycenter of \(\sigma\) to the barycenter of \(\tau\). We consider two discrete Morse functions equivalent if they induce the same Morse matching. For example, up to replacing a discrete Morse function \(f\) with an equivalent one, we can assume that \(f(\sigma)\) is a positive integer for all \(\sigma\). Forman’s original definition of a discrete Morse function is weaker than the one presented here; but one can easily see that each Morse function in the sense of Forman is equivalent to a discrete Morse function in our sense.

An elementary collapse is the simultaneous removal from a simplicial complex \(C\) of a pair of faces \((\sigma, \Sigma)\), such that \(\Sigma\) is the only face of \(C\) that properly contains \(\sigma\). If \(C' = C - \sigma - \Sigma\), we say that \(C\) collapses onto \(C'\). We also say that the complex \(C\) collapses onto the complex \(D\) if \(C\) can be reduced to \(D\) by a finite sequence of elementary collapses. A collapsible complex is a complex that collapses onto a single vertex. Equivalently, a simplicial complex is collapsible if and only if it admits a discrete Morse function with one critical vertex and no critical cells of higher dimension. Collapsible complexes are contractible; collapsible PL manifolds are necessarily balls [53]. However, some PL 3-balls are not collapsible [8] and some collapsible 6-balls (for example, the cones over non-PL 5-balls) are not PL.

A \(d\)-manifold without boundary is endo-collapsible if it admits a discrete Morse function with only two critical faces, which have to be a vertex and a \(d\)-simplex. A \(d\)-manifold with boundary is endo-collapsible if it admits a discrete Morse function whose critical cells are all boundary faces plus exactly one interior face, which has to be \(d\)-dimensional. Both collapsibility and endo-collapsibility are weaker properties than shellability, a classical notion in combinatorial topology, cf. [6, 9]. Shellable manifolds are either balls or spheres [9]. In contrast, the topology of collapsible manifolds is not completely understood (or better, it is understood only in the PL case [53]). However, endo-collapsible manifolds are either balls or spheres [6, Theorem 3.12].

A discrete Morse function on a manifold \(M\) is boundary-critical if all of the boundary faces of \(M\) are critical cells. The collapse depth \(\text{cdepth} M\) of a \(d\)-manifold \(M\) is the maximal integer \(k\) for which there exists a boundary-critical discrete Morse function on \(M\) with one critical \(d\)-cell and no critical interior \((d-i)\)-cells, for each \(i \in \{1, \ldots, k-1\}\). In general \(1 \leq \text{cdepth} M \leq \dim M\). A manifold \(M\) is endo-collapsible if and only if \(\text{cdepth} M = \dim M\).

A tree of \(d\)-simplices is a \(d\)-ball whose dual graph is a tree. The locally constructible manifolds are the manifolds obtainable from some tree of \(d\)-simplices by repeatedly identifying two adjacent \((d-1)\)-simplices in the boundary [7]. Equivalently, the locally constructible manifolds are those with collapse depth \(\geq 2\) [6, 7]. From now on, we will shorten “locally constructible” into “LC”. Topologically, every LC 3-manifold is homeomorphic to a 3-sphere with a finite number of “cacti” of 3-balls removed [7, Theorem 1.2][17]. All LC \(d\)-manifolds are simply connected [6, 17]. Any stellar subdivision of an LC (resp. endo-collapsible) manifold is also LC (resp. endo-collapsible). Also, the stellar subdivision of a collapsible complex is always collapsible. Compare Lemma 4.3 below.

In contrast, an arbitrary subdivision might destroy some combinatorial properties. For example, although the 3-simplex is shellable, there exists subdivisions of the 3-simplex that are neither shellable, nor collapsible, nor endo-collapsible. Also, if \(S\) is the double suspension of the Poincaré homology sphere and \(\Delta\) is a 5-simplex of \(S\), the \(d'\)-ball \(S - \Delta\) is a non-PL subdivision of the 5-simplex.
3 The combinatorics of handles

Here we show that an arbitrary PL triangulation of any handle has a convenient subdivision, that preserves some of the combinatorial properties of the original boundary.

We start by recalling two classical results from the lecture notes by Zeeman [54]:

**Lemma 3.1** (Zeeman [54, Lemma 13]). If \( B \) is a PL \((d-1)\)-ball in the boundary of a PL \(d\)-ball \( A \), there exists an integer \( r \) and a subdivision of \( A \) that collapses onto the \( r \)-th derived subdivision of \( B \). This subdivision of \( A \) need not be stellar.

**Proof.** Let \( \Delta \) be the \( d \)-simplex and \( \Gamma \) a \((d-1)\)-face of it. By the definition of PL, we can find subdivisions \( \Delta', \Delta'' \) and a simplicial isomorphism \( h : \Delta' \to \Delta'' \) whose restriction to \( \Delta'' \) yields a simplicial isomorphism between \( \Delta' \) and \( \Gamma' \).

Let \( \pi_{\text{down}} : \Delta \to \Gamma \) be the linear ("vertical") projection, mapping the vertex opposite to \( \Gamma \) to the barycenter of \( \Gamma \). Choose subdivisions \( \Delta'', \Gamma'' \) of \( \Delta', \Gamma' \) such that \( \pi_{\text{down}} : \Delta'' \to \Gamma'' \) is simplicial. Let \( \Delta'', \Gamma'' \) be the isomorphic subdivisions of \( \Delta' \), \( \Gamma' \). For a sufficiently large integer \( r \), an \( r \)-th derived subdivision \( B''' := \text{sd}^r B \) of \( B \) will subdivide also \( \Gamma'' \). Let \( \Gamma''' \) be the subdivision of \( \Gamma'' \) corresponding to \( B''' \). We can extend \( \Gamma''' \) to a subdivision \( \Delta''' \) of \( \Delta'' \), such that the projection \( \pi_{\text{down}} : \Delta''' \to \Gamma''' \) is simplicial. Finally, let \( A''' \) be the subdivision of \( A'' \) corresponding to \( \Delta''' \). By construction, \( \Delta''' \) collapses vertically to \( \Gamma''' \), in decreasing order of dimension. Hence, \( A''' \) collapses onto \( B''' \).

**Proposition 3.2** (Zeeman). Every PL ball admits a collapsible subdivision.

**Proof.** Choose a \((d-1)\)-simplex \( B \) in the boundary of the \( d \)-ball \( A \), and apply Lemma 3.1: Some subdivision \( A' \) of \( A \) will collapse onto an \( r \)-th derived subdivision \( \text{sd}^r B \) of \( B \). Such a subdivision is collapsible, because simplices are collapsible and stellar subdivisions preserve collapsibility.

**Remark 3.3.** The previous results were used by Zeeman to show that the two notions of PL-collapsibility and simplicial collapsibility are equivalent up to subdividing [54, Theorem 4] [31, p. 12]. In fact, if a polyhedron \( C \) PL-collapses onto a polyhedron \( D \), then using Lemma 3.1 one can prove the existence of an integer \( r \) and of a subdivision \( C' \) of \( C \) such that \( C' \) is a simplicial complex that collapses simplicially onto the \( r \)-th derived subdivision of \( D \).

Recall that shellable manifolds are collapsible and endo-collapsible at the same time [6]. In the Seventies, Proposition 3.2 was strengthened by Bruggesser and Mani as follows:

**Proposition 3.4** (Bruggesser–Mani). Every \( d \)-dimensional PL ball admits a shellable subdivision with shellable boundary.

**Proof.** If \( A \) is a \( d \)-dimensional PL ball, there exists an integer \( r \) for which the \( r \)-th derived subdivision of the \( d \)-simplex is also a subdivision of \( A \). Since the simplex and its boundary are shellable, so are the \( r \)-th derived subdivision of the simplex and its boundary.

Both Proposition 3.2 and Proposition 3.4 claim that some nice subdivision exists, but do not specify how to get it. It is natural to ask whether a collapsible or shellable subdivision can always be reached just by performing barycentric subdivisions. Unfortunately, this is an open problem.

**Conjecture 3.5.** For every PL ball \( B \) there is an integer \( r \) such that the \( r \)-th derived subdivision of \( B \) is shellable.

The conjecture seems crucial for the topological application we have in mind, namely, to triangulate any handle decomposition ‘one handle at the time’. The plan we have in mind is to triangulate each handle \( H_i \) starting from a triangulation \( T_i \) of the intersection of \( H_i \) with the previous handles. Topologically,
We solve the dilemma with a hybrid approach. We show the existence of a subdivision we fix the triangulation of $C$ with a derived subdivision. The vertical projection simplicial there is no need to subdivide the top faces of $C$. Subdivision is pushed forward via $h$ from $G$ to $G'$, but we can make it simplicial by refining the triangulation, that is, by passing to a suitable subdivision of $C$. The boundary of $C'$ consists of two connected components $C'_{\text{bottom}}$ and $C'_{\text{top}}$, where $C'_{\text{bottom}}$ is the $r$-th derived subdivision of $C_{\text{bottom}}$ and $C'_{\text{top}}$ is the $s$-th derived subdivision of $\partial \Delta^d$.

**Proof.** Let $G$ be $(\partial \Delta^d) \times \mathbb{I}$. The boundary of $G$ consists of two connected components, $G_{\text{bottom}}$ and $G_{\text{top}}$, both homeomorphic to $S^{d-1}$. We choose subdivisions $C'$ and $G'$ so that there is a simplicial isomorphism $h$ from $C'$ to $G'$, which restricts to an isomorphism between $C'_{\text{top}}$ and $G_{\text{top}}$ (and also to an isomorphism between $C'_{\text{bottom}}$ and $G_{\text{bottom}}$). Without loss of generality, we can assume that $C'_{\text{top}}$ is the $s$-th derived subdivision of $\partial \Delta^d$, for some $s$. (If not, we choose $s$ large enough so that $sd' \partial \Delta^d$ is a subdivision of $C'_{\text{top}}$ and we replace $C'$ with some finer triangulation whose restriction to the top face is $sd' \partial \Delta^d$.)

Let us denote by $\pi_{\text{down}}$ the vertical projection from $G$ to $G_{\text{bottom}}$. In general $\pi_{\text{down}}$ is not a simplicial map, but we can make it simplicial by refining the triangulation, that is, by passing to a suitable subdivision $G''$ of $G'$. The refinement can be done without subdividing the top faces of $G'$, so we can assume that $G''_{\text{top}} = G_{\text{top}}$. Using the isomorphism $h$, we can pull-back $G''$ to a subdivision $C''$ of $C'$. Clearly $C''_{\text{top}} = C'_{\text{top}} = sd' \partial \Delta^d$.

Finally, for $r$ large enough the $r$-th derived subdivision of $C_{\text{bottom}}$ subdivides $C''_{\text{bottom}}$. This derived subdivision is pushed forward via $h$ to a subdivision of $G''_{\text{bottom}}$, which can be extended to a triangulation $G'''$ of $G$, so that the projection $\pi_{\text{down}}$ is simplicial. We can assume $C'''_{\text{top}} = G_{\text{top}}$, because in order to make the vertical projection simplicial there is no need to subdivide the top faces of $G''$. If we pull-back $G'''$ to a subdivision $C'''_C$ of $C$, we have that:

(i) The boundary of $C'''$ consists of two connected components $C'''_{\text{bottom}}$ and $C'''_{\text{top}}$, where $C'''_{\text{bottom}}$ is the $r$-th derived subdivision of $C_{\text{bottom}}$ and $C'''_{\text{top}} = C'_{\text{top}} = sd' \partial \Delta^d$.

(ii) $C'''$ simplicially collapses onto $C'''_{\text{bottom}}$, because $G'''$ collapses vertically onto $G'''_{\text{bottom}}$.

**Proposition 3.7.** Every PL $d$-ball $B$ admits some subdivision $B'$ with the following two properties:

(i) $B'$ is endo-collapsible, and

(ii) $\partial B'$ is the $r$-th derived subdivision of $\partial B$, for a suitable $r$.

**Proof.** Up to replacing $B$ with its second barycentric subdivision, we can assume that some facet $\Delta$ of $B$ is disjoint from $\partial B$. Applying Lemma 3.6 to $C := B - \Delta$, we can find a subdivision $C'$ of $C$ such that:

(i) The boundary of $C'$ consists of two connected components $C'_{\text{bottom}}$ and $C'_{\text{top}}$, where $C'_{\text{bottom}}$ is the $r$-th derived subdivision of $\partial B$ and $C'_{\text{top}}$ is the $s$-th derived subdivision of $\partial \Delta^d$. 


(ii) $C'$ simplicially collapses onto $C'_{\text{bottom}}$.

Let $D'$ be the $d$-ball obtained by taking a cone over $C'_{\text{top}} = \text{sd}^t \partial \Delta^d$. By gluing $D'$ onto $C'$ along $C'_{\text{top}}$, we can complete $C'$ to a triangulation $B' := C' \cup D'$ of $B$. Let $\Sigma$ be a facet of $D'$. Being a cone over an endo-collapsible sphere, $D'$ is endo-collapsible \cite[Theorem 4.9]{6}, which means that $D' - \Sigma$ collapses onto $C'_{\text{top}}$. The latter collapsing sequence can also be viewed as a collapse of $B' - \Sigma$ onto $C'$. By Lemma 3.6, $C'$ collapses onto $C'_{\text{bottom}} = \text{sd}^t \partial B$, which is also the boundary of $B'$. Therefore, $B' - \Sigma$ collapses onto $\partial B'$, which means that $B'$ is endo-collapsible.

**Remark 3.8.** Many triangulations of 3-balls are neither collapsible nor endo-collapsible \cite{6}. These “bad” triangulations usually contain complicated knots as subcomplexes with few edges. In \cite{7} we introduced a measure of \textit{complicatedness} for knots, which is the minimal number of generators for the knot group, minus one. For example, the connected sum of $m$ trefoil knots is $m$-complicated \cite{6.26}. In an arbitrary 3-ball, any $m$-complicated knot can be realized with only 3 edges. In contrast, in a collapsible or endo-collapsible 3-ball, there is no $m$-complicated knot that uses less than $m$ edges \cite{6}.

When we perform a subdivision, the complicatedness of the knot stays the same, while the edge number might increase. Intuitively, a sufficiently fine subdivision will make the knot-theoretical obstruction disappear. However, there is no universal upper bound on how fine this subdivision should be. In fact, for each positive integer $t$, consider a PL 3-sphere $S$ with a 3-edge knotted subcomplex isotopic to the sum of $3 \cdot 2^t$ trefoils. Since it contains an $(3 \cdot 2^t)$-complicated knot on $3 \cdot 2^t$ edges, any $t$-th derived subdivision of $S$ cannot be endo-collapsible.

**Remark 3.9.** A similar statement to Lemma 3.6 appears also in the work by Gallais \cite[Lemma 3.9]{24}:

\begin{quote}
Given a $d$-simplex $\Delta^d$ and an arbitrary subdivision $X_0$ of $\Delta_d$, one can construct a triangulation $X$ of the full cylinder $\Delta^d \times \mathbb{I}$ such that the ‘bottom’ of the cylinder is combinatorially equivalent to $X_0$, the ‘top’ is combinatorially equivalent to $\Delta_d$, and $X$ collapses to the top and to the bottom.
\end{quote}

Unfortunately, the proof presented in \cite[p. 240]{24} is incorrect in its “Step 3”: It is not true that any simplicial subdivision $X_0$ of the $d$-simplex is collapsible. There are explicit counterexamples already when $d = 3$: For example, take $X_0 = S - \Delta$, where $S$ is the non-shellable 3-sphere constructed by Lickorish \cite{37}. In higher dimensions, the situation gets even more complicated, since the 5-simplex admits subdivisions that are not collapsible and not even PL. In particular, the triangulation $X$ constructed in \cite[p. 240]{24}, which is a cone over the CW complex $X_0 \cup \Delta_d \cup (\mathbb{I} \times \partial \Delta_d)$, might not collapse down to its bottom $X_0$.

At present, we do not know whether the claim \cite[Lemma 3.9]{24} is still true (with a different construction for $X_0$, say). However, the main results in \cite{24} can be savaged, (for example) using stellar subdivisions and the ideas explained in the present paper.

### 4 Main Results

We start by recalling how to compose two discrete Morse functions together \cite[Theorem 3.18]{6}. Given a discrete Morse function $f$ on a manifold with boundary $M$, let $c^\text{int}_i(f)$ denote the number of critical $i$-faces of $f$ in the interior of $M$.

**Theorem 4.1** \cite[Theorem 3.18]{6}. \textit{Let $M = M_1 \cup M_2$ be three $d$-manifolds, $d \geq 2$, such that the $M_i$'s have non-empty boundary and $M_1 \cap M_2$ is a full-dimensional submanifold of $\partial M_i$ ($i = 1, 2$). Let $f$ and $g$ be boundary-critical discrete Morse functions on $M_1$ and $M_2$, respectively, with $c^\text{int}_d(f) = c^\text{int}_d(g) = 1$.}
Let $h$ be a boundary-critical discrete Morse functions on $M_1 \cap M_2$, with $c_{d-1}^\text{int}(h) = 1$. There exists a boundary-critical discrete Morse function $u$ on $M$ with

$$c_k^\text{int}(u) = \begin{cases} c_k^\text{int}(f) + c_k^\text{int}(g) + c_k^\text{int}(h) & \text{if } 0 \leq k \leq d - 2 \\ c_k^\text{int}(f) + c_k^\text{int}(g) + c_k^\text{int}(h) - 1 & \text{if } d - 1 \leq k \leq d. \end{cases}$$

Next, we show that stellar subdivisions behave nicely with respect to discrete Morse functions.

**Lemma 4.2** (Jojic [33]). Any stellar subdivision of the $d$-simplex is extendably shellable.

**Lemma 4.3.** Let $C$ be an arbitrary simplicial complex. Let $C'$ be a stellar subdivision of $C$. If there is a discrete Morse function on $C$ with $c_i$ critical $i$-cells, then there is also a discrete Morse function on $C'$ with exactly $c_i$ critical $i$-cells.

**Proof.** It suffices to prove the claim for some complex $C'$ obtained from $C$ by starring a single face $\tau$. Let us fix a Morse matching on $C$. Let $\Delta$ be a critical cell for such matching. There are 2 cases:

(a) $\Delta$ does not contain $\tau$. So $\Delta$ is not subdivided: It is also a face of $C'$. In this case we leave $\Delta$ unmatched also in $C'$. In other words, $\Delta$ will be a critical cell also in $C'$.

(b) $\Delta$ contains $\tau$. In passing from $C$ to $C'$, $\Delta$ is subdivided into several faces $\Delta_1, \ldots, \Delta_k$. Choose one of these faces, say, $\Delta_1$. If $T'$ denotes the subcomplex of $C'$ determined by the facets $\Delta_1, \ldots, \Delta_k$, then $T'$ is a stellar subdivision of $\Delta$, and therefore shellable. In particular, $T'$ is endo-collapsible, so $T'$ minus the interior of $\Delta_1$ collapses onto the boundary of $T'$. This sequence of elementary collapses shows how to match the faces in which $\Delta$ is subdivided, so that in the end they are all matched, except $\Delta_1$. In other words, $\Delta_1$ will be a critical face of $C'$. The other faces coming from the subdivision of $\Delta$, will not be critical.

Next, consider any two faces $\sigma \subset \Sigma$ that are paired in the Morse matching of $C$. There are 3 cases:

(a) None of $\sigma, \Sigma$ contains $\tau$. So $\sigma$ and $\Sigma$ are both faces of $C'$. We match $\sigma$ and $\Sigma$ also in $C'$.

(b) Both $\sigma, \Sigma$ contain $\tau$. Let $v$ be the vertex of $\Sigma$ opposite to $\sigma$. Passing from $C$ to $C'$, $\Sigma$ is re-triangulated as a cone with vertex $v$. For any face $F'$ of $C'$ with $|F'| \subset \sigma$, we match $F'$ with $v + F'$. After all these elementary collapses, $C'$ is reduced to a subcomplex with the same underlying space of $C - \sigma - \Sigma$. The claim boils down to matching a stellar subdivision of a simplex and thus endo-collapsible; so $T$ minus the interior of $\Sigma_1$ collapses onto the boundary of $T$. This list of elementary collapses explains how to match $\Sigma_2, \ldots, \Sigma_k$ with subfaces. It is easy to see that with the rules above we immediately obtain a Morse matching for $C'$. Also, for each critical face of $C$ we produce exactly one critical face in $C'$. For each non-critical face of $C$, we do not produce any new critical face in $C'$. Thus, there is a $1 - 1$ correspondence between the critical faces in the original Morse matching of $C$, and the critical faces in the output Morse matching of $C'$.

**Theorem 4.4.** Let $M$ be a PL $d$-manifold (with or without boundary) with a PL handle decomposition into $c_i$ $i$-handles. Then, a suitable subdivision of $M$ admits a boundary-critical discrete Morse function with $c_i$ critical interior cells of dimension $d - i$.

**Proof.** We proceed by induction on the dimension $d$. (Since $M$ is connected, without loss of generality we could assume $c_0 = 1$, cf. [19, p. 110].)

If $d = 1$, $M$ is either a 1-sphere (in which case $c_1 = 1$), or a 1-ball ($c_1 = 0$). The claim boils down to the two well-known facts that every polygon becomes a collapsible path after the removal of an edge, and every path becomes collapsible onto its endpoints after the removal of an edge.

Let $M$ be a $d$-manifold with a PL handle decomposition with $c_i$ $i$-handles, for $i \in \{0, \ldots, d\}$. Each $i$-handle $H_k^{(i)}$ is attached to the union of the previous handles $H_j^{(\alpha)}$ ($j < k$, $\alpha \leq i$) along a PL triangulation.
$T_k$ of $S^{i-1} \times I^{d-i}$. If $i > 1$, $S^{i-1} \times I^{d-i}$ has a PL handle decomposition into one 0-handle and one $(i-1)$-handle. (In case $i = 1$, $S^0 \times I^d$ is the disjoint union of two PL $d$-balls; this case is completely analogous to the one we describe here and left to the reader.) Moreover, $S^{i-1} \times I^{d-i}$ has dimension $d - 1$. By the inductive assumption, a certain subdivision $T'_k$ of $T_k$ will admit a boundary-critical discrete Morse function $u_k$ with one critical interior $(d - 1)$-cell and one critical interior $(d - i)$-cell.

We subdivide the handle $H_k^{(i)}$ according to the method explained in Proposition 3.7. The resulting subdivision is endo-collapsible; moreover, the effect of the subdivision to the boundary of $H_k^{(i)}$ is the same as that of an $r_k$-th derived subdivision, for some $r_k$. Since stellar subdivisions preserve the number of critical faces (Lemma 4.3), $sd^{r_k} T_k$ still admits a discrete Morse function $u'_k$ with one critical interior $(d - 1)$-cell and one critical interior $(d - i)$-cell. Also, up to replacing the triangulation of each $H_j$ ($j < k$) adjacent to $H_k$ with its $r_k$-th derived subdivision, we can assume that the triangulation of $H_k^{(i)}$ is compatible with the way we triangulated the previous handles.

Once we have subdivided all handles, by Theorem 4.1 all of the boundary-critical discrete Morse functions on the handles can be composed together, using the boundary-critical discrete Morse functions on the intersections. This way we obtain a boundary-critical discrete Morse function $u$ on the whole of $M$. All the critical cells of the handles and intersections “add up”: The starting 0-handle contributes a critical interior $0$-cell, every additional $i$-handle contributes no critical cell. (This is the effect of the “minus one” in the second line of the formula of Theorem 4.1, for $i = d$.) The intersection of each $i$-handle with the previous handles contributes $0$ critical interior $(d - 1)$-cells (which is also the effect of the “minus one” in the second line of the formula of Theorem 4.1, for $i = d - 1$), and exactly one critical interior $(d - i)$-cell. Therefore, the number of critical interior $(d - i)$-cells of $u$ is equal to the number $c_i$ of $i$-handles.

Theorem 4.4 is of particular interest when paired with the classical notion of geometrical connectivity. Recall that, given an integer $k$ in $\{1, \ldots, d\}$, a geometrically $k$-connected manifold is a manifold that admits a handle decomposition with one 0-handle and no further handles of dimension $\leq k$ [52]. Every geometrically $p$-connected manifold is also $p$-connected. This statement admits a surprising converse:

**Theorem 4.5** (Wall [52]). Let $M$ be a $p$-connected (linkwise) PL (or smooth) $d$-manifold. If $p \leq d - 4$, then $M$ is also geometrically $p$-connected.

The condition $p \leq d - 4$ in Theorem 4.5 is best possible. Mazur and Casson described a contractible 4-manifold all of whose handle decompositions contain 1-handles [38, 23]. One possible decomposition has one $i$-handle for each $i \in \{0, 1, 2\}$. This gives an example of a $(d - 3)$-connected $d$-manifold which is not geometrically $(d - 3)$-connected. However, every $(d - 3)$-connected $d$-manifold becomes geometrically $(d - 3)$-connected after sufficiently many “stabilizations”, cf. Quinn [46, Theorem 1.2].

**Corollary 4.6.** Let $M$ be a $p$-connected PL $d$-manifold.

1. If $M$ is not $(p + 1)$-connected, then for any subdivision $M'$ of $M$ one has $cdepth M' \leq p + 1$.
2. If $p \leq d - 4$, there exists a subdivision $M'$ of $M$ such that $cdepth M' \geq p + 1$.

**Proof.** If $cdepth M' \geq p + 2$, by the definition of collapse depth we can find on $M'$ a boundary-critical discrete Morse function without interior critical $i$-cells, for $1 \leq i \leq p + 1$. By [6, Theorem 3.3], the manifold $M'$ (and thus $M$ as well) is homotopy equivalent to a connected cell complex without $i$-cells, for $1 \leq i \leq p + 1$. This proves that $M$ is $(p + 1)$-connected.

As for the second item: If $M$ is $p$-connected, by Theorem 4.5 $M$ has a handle decomposition without handles in dimensions $1, \ldots, p$. By Theorem 4.4, there is a boundary-critical discrete Morse function $u$ (on some subdivision $M'$ of $M$), such that $u$ has no critical interior $(d - i)$-cells, for $i = 1, \ldots, p$. By definition, then, the collapse depth of $M'$ is at least $p + 1$.

In general, given any simply connected smooth manifold, if it has dimension $\geq 6$ we can predict the number of critical points of any ‘minimal’ Morse function on it [49, pp. 27–28]. Results of this type can be translated into a combinatorial language via Theorem 4.4. Here is an example:
**Theorem 4.7** (Sharko [49, pp. 27–28]). Every contractible smooth $d$-manifold $\mathcal{M}$, with $d \geq 6$, admits a handle decomposition with exactly one 0-handle, $m$ $(d - 3)$-handles and $m$ $(d - 2)$-handles, where $m$ is the minimal number of generators of the relative homotopy group $\pi_2(\mathcal{M}, \partial \mathcal{M})$. (When $\mathcal{M}$ is a ball, $m = 0$.)

**Corollary 4.8.** Let $M$ be any contractible PL $d$-manifold, with $d \geq 6$. Then some subdivision $M'$ of $M$ admits a boundary-critical discrete Morse function with one critical (interior) $d$-simplex, $m$ critical interior tetrahedra, and $m$ critical interior triangles, where $m$ is the minimal number of generators of the relative homotopy group $\pi_2(\mathcal{M}, \partial \mathcal{M})$.

**Proof.** Take the handle decomposition given by Theorem 4.7, and apply our Theorem 4.4. □

Understanding minimal Morse functions for smooth manifolds that are not simply connected, instead, seems to be a much more difficult problem. For a survey of what has been achieved so far, we refer the reader to Sharko [49, Chapter 7].

### 5 Local constructibility of simply-connected manifolds

In this section we describe an application of the previous ideas to combinatorial topology, proving Main Theorems 2 and 3.

Locally constructible (LC) manifolds are manifolds with collapse depth at least two. The LC notion was originally introduced by Durhuus and Jonsson in [17] and later studied by the author and Ziegler [7]. All LC closed 2- and 3-manifolds are spheres [17]. However, some LC 4-manifolds are not spheres [5]: For example, they may be homeomorphic to $S^2 \times S^2$, or $\mathbb{C}P^2$. Why so?

To explain this gap between dimensions 3 and 4, let us first recall a few properties of shellability. All shellable closed manifolds (of any dimension) are spheres. The converse is true only in dimension two: All 2-spheres are shellable, but some 3-spheres are not shellable. However, every PL $d$-sphere has a shellable subdivision (Proposition 3.4). Similarly, all LC closed manifolds (of any dimension) are simply connected manifolds. The converse is true only in dimension two: All simply connected closed 2-manifolds are LC, but some simply connected closed 3-manifolds are not LC [7]. In 2009, at the author’s dissertation defense, Rade˘Zivaljevi´c made the following insightful conjecture:

**Conjecture 5.1** (˘Zivaljevi´c, 2009). Every simply connected smooth manifold admits an LC triangulation.

˘Zivaljevi´c’s conjecture explains why all LC closed $d$-manifolds are spheres only for $d \leq 3$. In fact, by the Poincaré conjecture, all simply connected closed $d$-manifolds are all spheres only for $d \leq 3$. Theorem 4.4 enables us to answer ˘Zivaljevi´c’s conjecture positively, at least for all $d \neq 4$:

**Theorem 5.2.** Every simply connected PL $d$-manifold $(d \geq 2)$ admits an LC subdivision, except possibly when $d = 4$.

**Proof.** A simply connected 2-manifold $M$ is either a 2-sphere or a 2-ball, so $M$ is shellable and thus LC. A simply connected closed $3$-manifold $M$ is a 3-sphere by the Poincaré conjecture [42]. Every 3-sphere is PL and admits a shellable (hence LC) subdivision.

Let $M$ be a simply connected 3-manifold with boundary. Let $k + 1$ be the number of connected components of $\partial M$. A priori, each one of these connected components is a closed 2-manifold, or in other words, a genus-$g$ surface for some $g$; but it is easy to see that for all of the components the genus $g$ must be zero, otherwise some non-trivial loop in $\partial M$ would yield a non-trivial loop inside $M$ (which is simply connected, a contradiction). Using the Poincaré conjecture, $M$ can thus be viewed as the result of removing $k + 1$ disjoint 3-balls from a 3-sphere. So intuitively the picture of $M$ resembles a piece of Swiss cheese with $k$ holes.
To prove that $M$ has an LC subdivision, we proceed by induction on $k$. If $k = 0$, $M$ is a PL 3-ball, thus it has a shellable subdivision and we are done. If $k > 0$, up to refining the triangulation of $M$ we can find an embedded annulus $A$ inside $M$, around one of the holes, such that $M$ splits as $B \cup M_1$, where $B \cap M_1 = A$, $B$ is a 3-ball and $M_1$ is a piece of Swiss cheese with $k − 1$ holes. By the inductive assumption, $M_1$ will have an LC subdivision $M_1'$. By Proposition 3.7, we can choose an endo-collapsible subdivision $B'$ of $B$ that agrees with the $r$-th derived subdivision of $M_1'$ on $|A|$. In particular, $B'$ is LC. Since the LC property is maintained through stellar subdivisions, also the $r$-th derived subdivision of $M_1'$ is LC. Since any triangulation of $A$ is strongly connected, using [7, Lemma 2.23] we conclude that the triangulation $M' := B' \cup sd'M_1'$ of $M$ is LC.

Finally, suppose $d \geq 5$. Let $M$ be a 1-connected $d$-manifold. Since $1 = 5 − 4 \leq \dim M − 4$, we can apply Corollary 4.6 and conclude that a suitable subdivision $M'$ of $M$ will have cdepth $M' \geq 1 + 1 = 2$. By definition, $M'$ is LC.

6 Open questions

Due to the beautiful mysteriousness of 4-manifolds, Živaljević’s conjecture remains open for $d = 4$. The easiest examples of 4-manifolds, like $S^4$, $S^2 \times S^2$ or $\mathbb{C}P^2$, satisfy the conjecture. Our guess is that Živaljević’s conjecture should hold at least for closed 4-manifolds. A related open problem is the following:

**Conjecture 6.1** (cf. Kirby [35, Problem 4.18]). Every simply connected closed smooth 4-manifold is geometrically 1-connected.

Via Theorem 4.5, a proof of Conjecture 6.1 would imply the validity of Živaljević’s conjecture for closed manifolds. It is plausible that Conjecture 6.1 is true; compare the recent results by Akbulut [3]. But even if Conjecture 6.1 turned out to be false, a priori it could still be possible to prove Živaljević’s conjecture, basically because there is no theorem telling how to pass from a discrete Morse function to a smooth Morse function with the same number of critical cells. Some interesting progress on this particular problem is contained in the work by Jerše and Mramor [32]. Obviously, the examples by Kervaire [34] suggest that a smooth structure might not always be created from a triangulated structure. In conclusion, all these results leave the door open to the possibility that Živaljević’s conjecture for closed manifold may be easier to prove, perhaps via a direct combinatorial approach.

On the other hand, we are less optimistic on the validity of Živaljević’s conjecture for 4-manifolds with boundary (or in other words, on the validity of Theorem 5.2 in the case $d = 4$). The fact that a 4-manifold is simply connected provides essentially no information about its boundary, as explained in the work of Hirsch [29, Lemma 1]. By Theorem 4.4, every geometrically 1-connected manifold becomes LC after suitably many subdivisions; however, some simply connected 4-manifolds with boundary, like Mazur’s manifold, are simply connected without being geometrically 1-connected. Therefore, a first step towards a possible extension of Theorem 5.2 to the case $d = 4$ would consist in proving the following Conjecture:

**Conjecture 6.2.** Mazur’s 4-manifold admits an LC triangulation.

Conjecture 6.2 is interesting in view of the “fight for perfection” between smooth Morse and discrete Morse theory. We already know that smooth Morse functions on Mazur’s manifold do not give sharp bounds for the first Betti number. Can a discrete Morse function beat them all? Conjecture 6.2 envisions a positive answer.

A similar question arises from the studies on hyperbolic 3-manifolds by Li [36] and on graph manifolds by Schultens and Weidmann [48]. Given a closed 3-manifold $M$, the rank of $M$ is the minimal number of generators of the fundamental group $\pi_1(M)$, while the Heegaard genus $g(M)$ is the smallest $g$
for which $M$ admits a smooth Morse function with $g$ critical points of index one. One has $r(M) \leq g(M)$, but the inequality is sometimes strict, as first discovered by Boileau and Zieschang [10]. Very recently, Li and Schultens–Weidmann constructed closed 3-manifolds with Heegaard genus arbitrarily larger than the rank [36] [48]. It would be interesting to triangulate these examples to test the following conjecture:

**Conjecture 6.3.** Some closed 3-manifold with Heegaard genus $g$ admits discrete Morse functions with less than $g$ critical edges.

Conjectures 6.2 and 6.3 would be solved in the negative, had we discovered a recipe to construct a PL handle decomposition into $c_i$ $i$-handles of any PL manifold that admits discrete Morse functions with $c_i$ critical $i$-faces. But as we pointed out before, no such recipe (currently) exists.

Finally, we point out that the main proofs in the present paper could be significantly simplified by proving Conjecture 3.5, on the possibility of making all PL balls shellable via sufficiently many barycentric subdivisions. Along the same lines of the proof of Theorem 4.4, one could derive the following: If Conjecture 3.5 is true, then for every PL $d$-manifold $M$ there is an integer $r$ such that the $r$-th derived subdivision of $M$ has collapse depth equal to the geometric connectivity of $M$, plus one.

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