Effects of uncertainties and errors on a Lyapunov control

X X Yi, B Cui, Chunfeng Wu and C H Oh

1 School of Physics and Optoelectronic Technology, Dalian University of Technology, Dalian 116024, People’s Republic of China
2 Centre for Quantum Technologies and Department of Physics, National University of Singapore, 117543, Singapore

E-mail: yixx@dlut.edu.cn

Received 19 March 2011, in final form 11 June 2011
Published 20 July 2011
Online at stacks.iop.org/JPhysB/44/165503

Abstract

A Lyapunov (open-loop) control is often confronted with uncertainties and errors in practical applications. In this paper, we analyse the robustness of the Lyapunov control against the uncertainties and errors in quantum control systems. The analysis is carried out by examining the uncertainties and errors and calculating the control fidelity under the influence of the uncertainties and errors. Two examples, a closed control system and an open control system, are presented to illustrate the general formalism, discussions on the effect caused by the uncertainties and errors are given.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Quantum control is the manipulation of time evolution of a quantum system in order to obtain a desired target state or value of physical observable. The implementation of quantum control [1–3] is a fundamental challenge in many fields, including atomic physics [4], molecular chemistry [5] and quantum information [6]. Several strategies of quantum control have been introduced and developed from classical control theory. For example, optimal control theory has been used to assist in control design for molecular systems and spin systems [7, 8]. A learning control method has been presented for guiding the chemical reactions [5]. Quantum feedback control approaches including measurement-based feedback and coherent feedback have been used to improve the performance of several classes of tasks such as preparing quantum states, quantum error correction and controlling quantum entanglement [9, 10]. Robust control tools have been introduced to enhance the robustness of quantum feedback networks and linear quantum stochastic systems [11–13].

Control systems are broadly classified as either closed-loop or open-loop. An open-loop control system is controlled directly and only by an input signal, whereas a closed-loop control system uses an input signal that is determined in part by the system response. Among the open-loop controls, the Lyapunov control has been proven to be a sufficient control that was analysed rigourously. Moreover, this control can be shown to be highly effective for systems that satisfy certain sufficient conditions equivalent to the controllability of the corresponding linearized system.

The Lyapunov control for quantum systems in fact uses a feedback design to construct an open-loop control. In other words, the Lyapunov control is used to first design a feedback law which is then used to find the open-loop control by simulating the closed-loop system. Then the control is applied to the quantum system in an open-loop way. From the above description of the Lyapunov control, we find that it includes two steps: (1) for any initial state and a system Hamiltonian (assumed to be known exactly), design a control law, i.e. calculate the control field by simulating the dynamics of the closed-loop system, and (2) apply the control law to the control system in an open-loop way. Although some progress has been made, more research effort is necessary in the Lyapunov control; especially, the robustness of quantum control has been recognized as a key issue in developing practical quantum technology. In this paper, we study the effect of uncertainties and errors on the performance of the Lyapunov control. The uncertainties come from initial states and the system Hamiltonian, and errors may occur in applying the control field (control law). Through this study, we examine the robustness of the Lyapunov control against uncertainties and errors. In particular, the relation between the uncertainties...
and the fidelity is established for a closed two-level control system and an open four-level control system.

This paper is organized as follows. In section 2, we introduce the Lyapunov control and formulate the problem. A general formalism is given to examine the robustness of the Lyapunov control. In section 3, we exemplify the general formulation in section 2 through a closed and an open quantum control system. The stability of the control system is analysed in section 4. Concluding remarks are given in section 5.

2. Problem formulation

A control quantum system can be modelled in different ways, either as a closed system evolving unitarily governed by a Hamiltonian, or as an open system governed by a master equation. In this paper, we restrict our discussion to an N-dimensional open quantum system and consider its dynamics as Markovian. The discussion is applicable for closed systems, since a closed system is a special case of an open system with zero decoherence rates. Therefore, we here consider a system that obeys the Markovian master equation ($\hbar = 1$, throughout this paper),

$$\dot{\rho} = -i[H, \rho] + L(\rho) \quad (1)$$

with

$$L(\rho) = \frac{1}{2} \sum_{m=1}^{M} \lambda_m [(J_m, \rho J_m^\dagger) + (J_m^\dagger, \rho J_m)]$$

and

$$H = H_0 + \sum_{n=1}^{F} f_n(t)H_n,$$

where $\lambda_m$ ($m = 1, 2, \ldots, M$) are positive and time-independent parameters, which characterize the decoherence and are called decoherence rates. Furthermore, $J_m$ ($m = 1, 2, \ldots, M$) are the Lindblad operators, $H_0$ is the free Hamiltonian and $H_n$ ($n = 1, 2, \ldots, F$) are control Hamiltonians, while $f_n(t)$ ($n = 1, 2, \ldots, F$) are control fields. $F$ denotes the number of control fields. For a system without decoherence (closed system), the number of control fields can be determined by the condition of controllability. For open systems, however, the number of control fields is smaller than that for closed systems because we can only drive the system into the decoherence-free subspace (DFS) (as time tends to infinity). In this case, $F$ can be determined in such a way that the system is controllable in DFS. Equation (1) is of Lindblad form, which means that the solution to equation (1) has all of the required properties of a physical density matrix at any of the times. Since the free Hamiltonian can usually not be turned off, we take nonstationary states $\rho_D(t)$ as target states that satisfy

$$\dot{\rho}_D(t) = -i[H_0, \rho_D(t)]. \quad (2)$$

The control fields $[f_n(t), n = 1, 2, 3, \ldots]$ can be established by a Lyapunov function. Define $V(\rho_D, \rho)$,

$$V(\rho_D, \rho) = \text{Tr}(\rho_D^2) - \text{Tr}(\rho_D \rho_D). \quad (3)$$

We find $V \geq 0$ for the pure target state $\rho_D(t) = |\Psi_D(t)\rangle\langle\Psi_D(t)|$ and

$$\dot{V} = -\sum_{n} f_n(t) \text{Tr}[\rho_D[-iH_n, \rho]) - \text{Tr}[\rho_D \mathcal{L}(\rho)]. \quad (4)$$

For $V$ to be a Lyapunov function, it is required that $\dot{V} \leq 0$ and $V \geq 0$. If we choose a $n_0$ such that $f_{n_0}(t)\text{Tr}([\rho_D[-iH_{n_0}, \rho]) + Tr[\rho_D \mathcal{L}(\rho)] = 0$ and $f_{n}(t) = \text{Tr}([\rho_D[-iH_n, \rho])$ for $n \neq n_0$, then $\dot{V} = -\sum_{n \neq n_0} f_{n}^2(t) \leq 0$. With these choices, $V$ is a Lyapunov function. Therefore, the evolution of the open system with the Lyapunov control governed by the following nonlinear equations [15]:

$$\dot{\rho} = -i[H_0 + \sum_{n} f_{n}(t)H_n, \rho(t)] + L(\rho), \quad (5)$$

$$f_{n}(t) = \frac{\text{Tr}[\rho_{\mathcal{D}}\mathcal{L}(\rho)]}{\text{Tr}[\rho_{\mathcal{D}} \rho_{\mathcal{D}}]} \quad \text{and} \quad \rho_{\mathcal{D}}(t) = -i[H_0, \rho_{\mathcal{D}}(t)].$$

is stable in Lyapunov sense at least. In equations (2) and (3), we have identified $\rho_D(t)$ with target states, this means that if a quantum system is driven into the target states, it will be maintained in these states under the action of the free Hamiltonian. However, in practical applications, it is inevitable that errors and uncertainties exist in the free Hamiltonian, in the initial states and in the control fields. These uncertainties and errors would disturb the dynamics and steer the system away from the target state. In the following, we suppose that the uncertainties can be approximately described as perturbations $\delta H_0$ in the free Hamiltonian, and as deviations $\delta \rho_0$ in the initial state, as well as fluctuations $\delta f_n$ (n may take 1, 2, 3, . . .) in the control fields. Then, the actual final state $\rho_F(t)$ of the control system starting from $(\rho_0 + \delta \rho_0)$ governed by equation (5) with $(H_0 + \delta H_0)$ and $[f_n(t) + \delta f_n(t)]$ instead of $H_0$ and $f_n(t)$ would be different from $\rho_D(t)$. We quantify the difference between the target states $\rho_D(t)$ and the practical states $\rho_F(t)$ by using the fidelity defined by

$$F(\rho_D, \rho_F) = \text{Tr}\left[\left|\sqrt{\frac{1}{2}}\rho_D + \frac{1}{2}\rho_F\right|\sqrt{\frac{1}{2}}\rho_D + \frac{1}{2}\rho_F\right] \quad \text{and} \quad \mathcal{E}_n = \{\rho_{n,a} : \text{Tr}[\rho_D H_n \rho_{n,a} - H_n \rho_D \rho_{n,a}] = 0\}, \quad n \neq n_0. \quad (6)$$

For a Lyapunov control with negative gradient of the Lyapunov function in the neighbourhood of target states, the state of the controlled system will be almost attracted to and maintained in the target state, when there are no uncertainties and errors. With uncertainties and errors, the problem of robustness of the control system is not trivial because the Lyapunov-based feedback design for the control law would induce nonlinearity in the control system. By almost we mean almost all (but not all) initial states will converge to the target state by this control. We will discuss this in detail before the concluding section.

The equilibrium states are determined by the LaSalle invariant principle [14], which tells that the autonomous dynamical system (5) converges to an invariant set defined by $\mathcal{E} = \{\rho_{\mathcal{D}} : \dot{V} = 0\}$, which is equivalent to $f_{n}(t) = 0$, $n = 1, 2, 3, \ldots$, by equation (5). This set is in general not empty and the final state will be in it. From equations (4) and (5) we find that the invariant set is an intersection of all sets $\mathcal{E}_n$ ($n = 1, 2, 3, \ldots, n \neq n_0$), each satisfies

$$\mathcal{E}_n = \{\rho_{n,a} : \text{Tr}[\rho_D H_n \rho_{n,a} - H_n \rho_D \rho_{n,a}] = 0\}, \quad n \neq n_0.$$
Meanwhile, this invariant set must be in the DFS. Note that the last requirement is covered by the requirement on \( f_{\text{in}}(t) = 0 \). Since the control fields are proportional to \( V \), the errors in the control fields would change the invariant set. The uncertainties in the initial state affect the invariant set in the same way, and the uncertainties in the free Hamiltonian change the target sets \( \rho_0(t) \), leading to an invariant set different from that without uncertainties. In the next section, we will illustrate and exemplify the effect of errors and uncertainties on the fidelity through simple examples.

### 3. Illustration

In this section, we first introduce a Lyapunov control on a closed two-level quantum system, then we study the robustness of this Lyapunov control by examining the effects of uncertainties and errors on the fidelity of control. Next, we extend this study into open systems by considering a dissipative level system and steering it to a target state in its DFS.

We start with a closed two-level system described by the Hamiltonian

\[
H = \frac{\omega}{2} \sigma_z + f(t) \sigma_x = H_0 + H_1,
\]

where \( H_0 = \frac{\omega}{2} \sigma_z \) denotes the free Hamiltonian of the system and \( H_1 = f(t) \sigma_x \) is the control Hamiltonian with the control field \( f(t) \). We define one of the eigenstates of \( H_0 \), say the ground state \( |g\rangle \), as the target state; the Lyapunov function in equation (3) for this closed system is then

\[
V(|g\rangle, |\Phi(t)\rangle) = |\langle g|\Phi(t)\rangle|^2.
\]\n
For closed systems, the Liouvillian \( \mathcal{L}(\rho) \) vanishes; thus, we do not need to choose a control field \( f_{\text{in}}(t) \) in equation (5) to cancel the drift term. The only control field \( f(t) \) that can be derived from equation (5) is

\[
f(t) = 2\text{Im}(\langle g|\sigma_x|\Phi(t)\rangle|\Phi(t)\rangle\langle g\rangle).
\]\n
Here, \( |\Phi(t)\rangle \) represents states at time \( t \) starting from an initial state

\[
|\Phi(0)\rangle = \cos|\beta_0\rangle e^{i\phi_0} + \sin|\beta_0\rangle e^{i\phi_0}|g\rangle
\]

under the action of the Hamiltonian \( H \) without any uncertainties and errors. We further suppose that the uncertainties in the free Hamiltonian \( H_0 \) can be described as a perturbation,

\[
\delta H_0 = \delta_\sigma \sigma_z + \delta_\xi \sigma_x,
\]

and the uncertainties in the initial state \( |\Phi(0)\rangle \) can be characterized by replacing \( \beta_0 \) and \( \phi_0 \) with \( (\beta_0 + \delta \beta_0) \) and \( (\phi_0 + \delta \phi_0) \), respectively. We describe the errors in the control fields \( f(t) \) as fluctuations \( \delta(t) \cdot f(t) \) with random number \( \delta(t) \). With these descriptions, the practical control system can be described by

\[
\frac{\text{d}}{\text{d}t} |\psi(t)\rangle_R = [H_0 + \delta H_0 + f(t) H_1(1 + \delta(t))] |\psi(t)\rangle_R,
\]

with the initial condition \( |\Phi(0) + \delta |\Phi(0)\rangle = \cos(\beta_0 + \delta \beta_0)|\epsilon\rangle + \sin(\beta_0 + \delta \beta_0)e^{i(\phi_0 + \delta \phi_0)}|g\rangle \).

Figure 1. Fidelity of the Lyapunov control versus the uncertainties in the free Hamiltonian (left) and in the initial states (right). \( \omega = 4 \) (in arbitrary units) and \( \phi_0 = \beta_0 = \frac{\pi}{4} \) are chosen for this plot.

![Figure 1](image1.png)

Figure 2. Fidelity of the Lyapunov control as a function of time.

This plot shows the effects of fluctuations in the control field \( f(t) \) on the fidelity. Figures (a), (b) and (c) are for different types of fluctuations. (a) The fluctuation \( \delta(t) \) was taken from \((-1) \) to \(0\); (b) from \((-1) \) to \((+1)\); and (c) from \(0 \) to \((+1)\). All fluctuations are taken randomly. The other parameters chosen are the same as in figure 1. There are no uncertainties in the free Hamiltonian and in the initial states.

![Figure 2](image2.png)

We have performed numerical simulations for equation (11), and selected results are presented in figures 1 and 2. Figure 1 shows the control fidelity as a function of uncertainties \( \delta_\sigma, \delta_\xi \) in the free Hamiltonian and uncertainties \( \delta \beta_0, \delta \phi_0 \) in the initial state. Two observations can be made from the figures. (1) The control fidelity rapidly depends on the uncertainties \( \delta_\sigma, \delta_\xi \), whereas it is not sensitive to \( \delta \beta_0, \delta \phi_0 \). (2) The control fidelity is an oscillating function of \( \delta \beta_0 \) and \( \delta \phi_0 \) with different periods. These observations indicate that the Lyapunov control on closed systems is robust against the uncertainties that commute with the control Hamiltonian, while it is fragile with the other uncertainties in the free Hamiltonian. This claim is confirmed by figure 2, where the effect of fluctuations in the control field on the control fidelity is shown. One can clearly see from figure 2 that there are almost no effects on the fidelity from the fluctuations with zero mean. This can be understood as follows. Since the fluctuations are randomly chosen, the net effect intrinsically equals an average over all fluctuations, which must be zero for fluctuation with zero mean.

Now we turn to another example and explore the robustness of the Lyapunov control on open quantum systems. We borrow the model from [16] as shown in figure 3, where...
that the two degenerate eigenstates of a dissipative four-level system coupled to external lasers has been considered. The Hamiltonian of this system has the form

\[ H_0 = \sum_{j=0}^{2} \Delta_j |j\rangle\langle j| + \sum_{j=1}^{2} \Omega_j |0\rangle\langle j| + \text{h.c.}, \]

where \( \Omega_j (j = 1, 2) \) are coupling constants. Without loss of generality, in the following the coupling constants are parameterized as \( \Omega_1 = \Omega \cos \phi \) and \( \Omega_2 = \Omega \sin \phi \) with \( \Omega = \sqrt{\Omega_1^2 + \Omega_2^2} \). The excited state \( |0\rangle \) is not stable; it decays to the three stable states with rates \( \gamma_1, \gamma_2 \) and \( \gamma_3 \), respectively. We assume this process Markovian and can be described by the Liouvillian

\[ \mathcal{L}(\rho) = \sum_{j=1}^{3} \gamma_j \left( \sigma_j^{+} \rho \sigma_j^{-} - \frac{1}{2} \sigma_j^{+} \sigma_j^{-} \rho - \frac{1}{2} \rho \sigma_j^{+} \sigma_j^{-} \right), \]

with \( \sigma_j^{-} = |j\rangle\langle 0| \) and \( \sigma_j^{+} = (\sigma_j^{-})^{\dagger} \). It is not difficult to find that the two degenerate eigenstates \( |D_1\rangle = \cos \phi |2\rangle - \sin \phi |1\rangle \), \( |D_2\rangle = |3\rangle \), of the free Hamiltonian \( H_0 \) form a DFS. In fact, the scheme works for the case where \( |D_1\rangle \) and \( |D_2\rangle \) are not degenerate (but they must be eigenstates of \( H_0 \) and long lived). The scheme also does not depend on \( \Delta_j \), as long as \( \Delta_1 = \Delta_2 \) and \( \Delta_0 > \Delta_j (j \neq 0) \).

Now we show how to control the system to a desired target state (e.g. \( |D_1\rangle \)) in the DFS. For this purpose, we choose the control Hamiltonian

\[ H_c = \sum_{j=1}^{3} f_j(t) H_j, \]

with \( H_j \) is a 4 \times 4 matrix with all elements equal to 1. \( H_2 = |D_2\rangle\langle D_2| + |D_3\rangle\langle D_3| \), \( H_0 = |D_1\rangle\langle D_1| + |D_2\rangle\langle D_2| \). One may wonder how to choose the control Hamiltonians. Intuitively, when \( |D_1\rangle \) is chosen as the target state, we have to transfer the population on the state \( |D_2\rangle \) to the target state \( |D_1\rangle \), so \( H_2 \) is necessary. \( H_1 \) is designed to cancel the decoherence, so \( \text{iTr}[A, H_1 \rho] \) cannot be zero for any state, provided the actual state is not in DFS. \( H_3 \) is chosen to speed up the population transfer from \( |D_2\rangle \) to \( |D_1\rangle \).

We shall use equation (5) to determine the control fields \( f_n(t) \) and choose

\[ |\Psi(0)\rangle = \sin \beta_1 \cos \beta_3 |0\rangle + \cos \beta_1 \cos \beta_2 |1\rangle + \cos \beta_1 \sin \beta_2 |2\rangle + \sin \beta_1 \sin \beta_3 |3\rangle \]

as initial states for the numerical simulation, where \( \beta_1, \beta_2 \) and \( \beta_3 \) are allowed to change independently. The initial state written in equation (15) omits all (three) relative phases between the states \( |0\rangle \), \( |1\rangle \), \( |2\rangle \) and \( |3\rangle \) in the superposition and satisfies the normalization condition. \( f_1(t) \) here is specified to cancel the drift term \( \text{Tr}[L(\rho)A] \) in \( V \), which means that

\[ f_1(t) = -i \frac{\text{Tr}[L(\rho)A]}{\text{Tr}[A, H_1 \rho]} \]

and \( f_2(t) \) and \( f_3(t) \) are determined by equation (5).

We examine how the uncertainties in the free Hamiltonian and initial states as well as the errors in the control fields \( f_n(t) (n = 1, 2, 3, \ldots) \) affect the fidelity of the control. These effects can be illustrated by numerical simulations on equations (12)–(14) with the free Hamiltonian \( H_0 \), the initial state \( |\Phi(0)\rangle \) and the control fields \( f_n(t) \) replaced by \( (H_0 + \delta H_0) \), \( |\Phi(0) + \delta \Phi(0)\rangle \) and \( f_n(t) + \delta f_n \), respectively. Here,

\[ \delta H_0 = \Delta_x (|1\rangle\langle 1| + |2\rangle\langle 2|) + \Delta_z (|0\rangle\langle 0| + |2\rangle\langle 2|), \]

\[ |\Phi(0) + \delta \Phi(0)\rangle = \sin(\beta_1 + \delta \beta_1) \cos \beta_3 |0\rangle + \cos(\beta_1 + \delta \beta_1) \cos(\beta_2 + \delta \beta_2) |1\rangle + \cos(\beta_1 + \delta \beta_1) \sin(\beta_2 + \delta \beta_2) |2\rangle + \sin(\beta_1 + \delta \beta_1) \sin \beta_3 |3\rangle \]

and \( \delta f_n \) is random numbers ranging from \(-1\) to \(+1\). The fidelity versus the uncertainties (characterized by \( \Delta_x, \Delta_z, \delta \beta_1, \delta \beta_2 \)) and errors \( (\delta_n, n = 1, 2, 3, \ldots) \) is presented in figures 4 and 5. Figures 4 and 5 tell us that the Lyapunov control on an open system with the target state \( |D_1\rangle \) is robust against the uncertainties in the initial state, and the fidelity is above 95% when the uncertainties in the free Hamiltonian is bounded by 0.5 (in units of \( \gamma \)). The Lyapunov control is also robust against the fluctuations in the control fields \( f_n(t) \) as figure 5 shows. We note that the effects of fluctuations with zero mean are different from that with non-zero mean. This can be understood as an average of results taken over all fluctuations. In addition to the aforementioned uncertainties and errors, the uncertainties in the system–environment coupling characterized by \( \gamma_j \) in equation (13) are also important. The effects of these uncertainties on the control fidelity are similar to that in \( f_1(t) \) because \( f_1(t) = -i \frac{\text{Tr}[L(\rho)A]}{\text{Tr}[A, H_1 \rho]} \). In other words, uncertainty \( \delta \mathcal{L}(\rho) = \sum_{j=1}^{3} \delta \gamma_j (\sigma_j^{+} \rho \sigma_j^{-} - \frac{1}{2} \sigma_j^{+} \sigma_j^{-} \rho - \frac{1}{2} \rho \sigma_j^{+} \sigma_j^{-}) \) leads to an
In other words, we say that \( \rho_c \) in the state \(| H \rangle \) in the Hilbert space. We denote by critical points of normalized eigenvectors and eigenvalues of \( \hat{H} \) fluctuations without any perturbations.

By trajectory stability we mean that as the perturbations tend to zero, the trajectory of the control system is stable in figure 2. This means that as the uncertainties approach zero, the trajectories of the system into the eigenstate with the smallest eigenvalue of \( \hat{A} \) is the local minimum if \( A_j \) is the smallest eigenvalue \( \hat{A} \), and \( \rho_c \) is a saddle point otherwise. This observation suggests that the minimum of \( V \) is asymptotically attractive. In other words, the control field based on this Lyapunov function would drive the open system into the eigenstate with the smallest eigenvalue of \( \hat{A} \).

We next use the fidelity defined by \( S(t) = \text{Tr}(\rho_c(t)\rho(t))/\text{Tr}(\rho^2(t)) \) to quantify the trajectory stability of the system. Here, \( \rho(t) \) is the state at time \( t \) without uncertainties, while \( \rho_c(t) \) denotes the system state under the effect of uncertainties. We say that the system is stable if \( S(t) \rightarrow 1 \), when the uncertainties approach zero. This means that as the uncertainties approach zero, the trajectories of the system in Hilbert space return to that without uncertainties. The trajectory stability is a rigorous requirement on the control; it can be dropped for a quantum control with a specific target state as in this paper. In other words, for a quantum control aiming at a specific target state, its performance depends only on the fidelity between the target state and the resulting state of the control system, regardless of the system’s trajectory. On the other hand, trajectory stability in this paper can be found by examining the dependence of fidelity on time. For example, in figure 2 the trajectory stability \( S(t) \) is smaller than the largest difference among the fidelities at a fixed time. Clearly, the trajectory of the control system is stable in figure 2.

Finally, we analyse the convergence of the control system from the aspect of fixed points. The fixed points of the open system (1) must be in the DFS; thus, the open system eventually (after reaching equilibrium) has less than \( n! \) additional term in \( f_1(t) \), \( \delta f_1(t) = -i\frac{\text{Tr}(\rho(\hat{A}^\dagger))}{\text{Tr}(A^\dagger A)} \). This suggests that the uncertainties in the Liouvillean result in an effect on the control fidelity similar to the errors in \( f_1(t) \).

4. Stability and convergence of the control system

Before concluding the results, we discuss the stability and convergence of the control system treating the uncertainties as perturbations. Firstly, we analyse the stability of the control system around the fixed points. Next, we consider the trajectory stability of the control system, and finally we present a brief discussion on the convergence of the control system. By trajectory stability we mean that as the perturbations tend to zero, the trajectory of the system approaches the trajectory without any perturbations.

To analyse the stability of the system around the fixed points [15, 16], we rewrite the Lyapunov function as

\[
V(\rho) = \text{Tr}(\rho \hat{A})
\]

where \( \hat{A} \) is Hermitian and time independent. To be specific, for the Lyapunov function in equation (3), \( \hat{A} \) is \( \hat{A} = 1 - \rho \); \( \hat{A} = 1 - |g\rangle \langle g| \) for the Lyapunov function in equation (8); and \( \hat{A} = |D_2\rangle \langle D_2| - |D_1\rangle \langle D_1| \) in the control system equation (14). The state space is a compact set of all positive and trace-1 operators on the Hilbert space spanned by the eigenstates of the Hamiltonian \( H_0 \). The operator \( \hat{A} \) is also assumed in this Hilbert space. We denote by critical points and eigenvalues of \( \hat{A} \) the set of states with which \( V(\rho) \) arrives at its maximum or minimum. In other words, we say that \( \rho_c \) is a critical point of \( V(\rho) \) if the variation of \( V \) due to independent variations of \( \rho_c \) vanishes.

We first analyse the structure of critical points of \( V(\rho) \) with restriction \( \text{Tr}(\rho) = 1 \) and positivity of \( \rho \). To determine the structure of \( V(\rho) \) around one of its critical points, for example \( \rho_c = \sum_j p_j^c |A_j\rangle \langle A_j| \), we consider an infinitesimal variation \( \delta \rho \) in \( \rho_c \) such that \( \text{Tr}(\rho_c + \delta \rho) = 1 \). Here, we denote the normalized eigenvectors and eigenvalues of \( \hat{A} \) by \( |A_j\rangle \) and \( A_j \) [(i = 1, 2, 3, \ldots, N)], respectively. Expressing \( \rho_c + \delta \rho \) in the basis of the eigenvectors of \( \hat{A} \), we have

\[
\rho_c + \delta \rho = \sum_j p_j^c |A_j + \delta A_j\rangle \langle A_j + \delta A_j|,
\]

where \( |A_j + \delta A_j\rangle = |A_j\rangle + \sum_{a=1}^N \delta_j^a |A_a\rangle \).

At first sight \( \rho_c + \delta \rho \) represents a state which has the same spectrum with \( \rho_c \), but this is not the case and \( \rho_c + \delta \rho \) can be any state here. This can be understood by noting that \( |A_j + \delta A_j\rangle \) is unnormalized. The norm of \( |A_j + \delta A_j\rangle \) would change the spectrum of the density matrix and it makes \( \rho_c + \delta \rho \) and \( \rho_c \) different in spectrum. The diagonal elements of the density matrix is \( (\rho_c + \delta \rho)_{jj} = p_j^c (1 + |\delta_j^a|^2 + \delta_j^a + \delta_j^a) \). The off-diagonal elements of the density matrix is \( (\rho_c + \delta \rho)_{ij} = p_j^c p_i^c \delta_{ij} \). Since there are \( N^2 - 1 \) independent variations \( \delta_j^a \), \( \rho_c + \delta \rho \) is a general (state) variation. The normalization condition \( \text{Tr}(\rho_c + \delta \rho) = 1 \) follows

\[
\sum_j p_j^c (\delta_j^a + \delta_j^a) + \sum_j p_j^c \sum_a \delta_j^a \delta_j^a = 0.
\]

Then

\[
V(\rho_c + \delta \rho) - V(\rho_c) = \sum_j p_j^c (A_j - A_j) \delta_j^a \delta_j^a.
\]

Considering \( \delta_j^a \) as variation parameters and noting \( \delta_j^a \delta_j^a \geq 0 \), we find that the structure of \( V(\rho_c) \) around the critical point \( \rho_c \) depends on the ordering of the eigenvalues. \( \rho_c \) is a local maximum if \( A_j \) is the largest eigenvalue of \( \hat{A} \), \( \rho_c \) is a local minimum if \( A_j \) is the smallest eigenvalue \( \hat{A} \), and \( \rho_c \) is a saddle point otherwise. This observation suggests that the minimum of \( V \) is asymptotically attractive. In other words, the control field based on this Lyapunov function would drive the open system into the eigenstate with the smallest eigenvalue of \( \hat{A} \).

Figure 5. Fidelity of the Lyapunov control versus time \( t \). The fluctuations \( \delta_n \) \( (n = 1, 2, 3, \ldots) \) range from \((-1) \) to \((+1) \) for the upper panel, while from \((-1) \) to \(0 \) (or \(0 \) to \((+1) \)) for the lower panel. In both panels (a) denotes the probability in the state \(| D_1\rangle \), while (b) in the state \(| D_2\rangle \). The other parameters chosen are the same as in figure 4.
stationary states [17], where $n$ denotes the number of different eigenvalues in the initial state. The target state is of course a fixed point (stationary state) called a dynamical sink. The dynamical source may not be in the DFS, and the other fixed points (less than $n! - 2$) must be saddle. Therefore, for the open system, the stable fixed points are less than the corresponding closed system. In this sense, we conclude that almost all initial states converge to the target state [17], namely the target state can be considered almost globally asymptotically stable.

5. Concluding remarks

To summarize, we have examined the robustness of a Lyapunov control in quantum systems. The robustness is characterized by the fidelity of the quantum state to the target state. Uncertainties in the free Hamiltonian and in the initial states, as well as the errors in the control fields, diminish the fidelity of control. The relation between the uncertainties (errors) and the fidelity is established for a closed two-level control system and an open four-level control system. These results show that the Lyapunov control is robust against the type of uncertainties which commute with the control Hamiltonian, while it is fragile to the others. The fidelity is not sensitive to zero mean random fluctuations (white noise) in the control fields, but it really decreases due to the non-zero (positive or negative) mean fluctuations.

Acknowledgments

This work is supported by NSF of China under grant nos 61078011 and 10935010, as well as the National Research Foundation and Ministry of Education, Singapore under academic research grant no WBS: R-710-000-008-271.

References

[1] Dong D and Petersen I R 2009 arXiv:0910.2350
[2] Wiseman H M and Milburn G J 2010 Quantum Measurement and Control (Cambridge: Cambridge University Press)
[3] Rabitz H 2009 New J. Phys. 11 105030
[4] Chu S 2002 Nature 416 206
[5] Rabitz H, de Vivie-Riedle R, Motzkus M and Kompa K 2000 Science 288 824
[6] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[7] Khaneja N, Brockett R and Glaser S J 2001 Phys. Rev. A 63 032308
[8] D’Alessandro D and Dahleh M 2001 IEEE Trans. Autom. Control 46 866
[9] Wiseman H M and Milburn G J 1993 Phys. Rev. Lett. 70 548
[10] Doherty A C, Habib S, Jacobs K, Mabuchi H and Tan S M 2000 Phys. Rev. A 62 012105
[11] D’Helon C and James M R 2006 Phys. Rev. A 73 053803
[12] James M R, Nurdin H I and Petersen I R 2008 IEEE Trans. Autom. Control 53 1787
[13] Pravia M A, Boulan N, Emerson J, Fortunato E M, Havel T F, Cory D G and Farid A 2003 J. Chem. Phys. 119 9993
[14] LaSalle J and Lefschetz S 1961 Stability by Lyapunov’s Direct Method with Applications (New York: Academic)
[15] Yi X X, Huang X L, Wu C F and Oh C H 2009 Phys. Rev. A 80 052316
[16] Wang W, Wang L C and Yi X X 2010 Phys. Rev. A 82 034308
[17] Wang X and Schirmer S 2010 IEEE Trans. Autom. Control 55 1406