New Constructions of Cross Z-Complementary Pairs

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Abstract—Spatial modulation (SM) is a new paradigm of multiple-input multiple-output (MIMO) systems, in which only one antenna at the transmitter is activated during each symbol period. Recently, it is observed that SM training sequences derived from cross Z-complementary pairs (CZCPs) lead to optimal channel estimation performance over frequency-selective channels. CZCPs are special form of sequence pairs which have zero aperiodic autocorrelation zones and cross-correlation zone at certain time-shifts. Recent paper by Liu et al. discussed only perfect CZCPs. In this paper, we focus on non-perfect CZCPs. We introduce the term cross Z-complementary ratio and re-categorise the CZCPs, both perfect and non-perfect, based on that. We propose a systematic construction of CZCPs based on generalised Boolean functions (GBFs). We further extend the lengths of the CZCPs by using the insertion method. The proposed CZCPs are all of new lengths of the form $2^{10}26^2+2$ $(\alpha \geq 1)$, $10^5+2$, $26^2+2$ and $10^526^2+2$. Finally we propose a construction of optimal binary CZCPs having parameters $(12,N)$ and $(24,N)$ from binary Barker sequences. These CZCPs are also extended to $(12N,5N)$- CZCPs and $(24N,11N)$- CZCPs, where $N$ is the length of a binary Golay complementary pair (GCP). During the proof, we also found a new structural property of binary CZCPs and concluded all binary GCPs are CZCPs too. Finally, we give some numerical simulations to confirm that depending on the number of multi-paths, the proposed CZCPs can be used to design SM training matrix which attains the minimum mean square error.

Index Terms—Barker sequences, cross Z-complementary pairs (CZCPs), generalised Boolean functions, insertion method, spatial modulation.

I. INTRODUCTION

In 1950, M. J. Golay introduced complementary pairs in his work on multislit spectrometry [1]. Golay complementary pairs (GCPs) are pair of sequences whose aperiodic autocorrelation sums (AACSs) are zero everywhere, except at the zero shift [2]. Binary GCPs are available only for limited lengths of the form $2^{10}26^2$ (where $\alpha$, $\beta$, and $\gamma$ are non-negative integers) [2], [3]. In 1972, Tseng and Liu [4] extended the idea of complementary pairs to complementary sets (CSs) of sequences. Since then, CSs found a number of applications in communication systems [5]–[9]. Due to the limited availability of binary GCPs Fan et al. [10] proposed binary Z-complementary pairs (ZCPs) in 2007. Since then, a lot of research has been done afterwards towards the systematic and structural analysis of ZCPs [11]–[17].

Spatial modulation (SM) is a special kind of multiple-input multiple-output (MIMO) technique, which optimizes multiplexing gain with complexity and performance [18]–[22]. The main difference of SM system with a traditional MIMO is that it is equipped with a single radio-frequency (RF) chain. In SM, only one transmit antenna is activated over every symbol duration. During each time-slot, an SM symbol can be divided into two parts, spatial symbol and constellation symbol. Spatial symbol is responsible for the transmit antenna elements and constellation symbol is selected from a conventional phase shift keying (PSK)/quadrature amplitude modulation (QAM) constellation and transmitted from the active transmit antenna element. Such unique transmission principle of SM allows it to have the salient advantages of zero inter-channel interference, low energy consumption [23], and low receiver complexity over traditional MIMO systems. Till date, however, little has been understood on channel estimation of SM in frequency-selective channels. Early literature on SM mostly assume that channel state information (CSI) is perfectly known at the receiver [24], [25]. Note that the one-RF-chain principle of SM prevents the transmitter from using simultaneous pilot transmission over all the transmit antennas. Consequently, it implies that dense training sequences proposed in [26]–[28] for traditional MIMO are not applicable in SM systems. Although an identity training matrix has been employed for joint channel estimation and data detection in SM systems [29], extension to frequency-selective channels is not straightforward. A naive scheme is to extend a perfect sequence (having zero autocorrelation sidelobes) with cyclic prefix (CP) and then send the extended sequence in turn over multiple transmit antennas. But this training scheme would be inefficient in highly dispersive channels.

To deal with this problem, recently Liu et al. [30] proposed a new class of sequence pairs called cross Z-complementary sequence pairs (CZCPs). The authors also proposed a generic training framework for SM training over frequency selective channels. Under the proposed framework in [30], the authors derived the lower bound on channel estimation mean square error (MSE) using least square (LS) estimator and conditions to meet the lower bound with equality. In [30], the authors show that CZCPs play an instrumental role in the design of optimal SM training sequences (which are equivalent to certain sparse matrices). The authors also show that the numerical simulations indicate that the proposed SM training sequences in [30] lead to minimum channel estimation MSE w.r.t. the aforementioned lower bound.

A. Concept of Cross Z-Complementarity

Let $a$ and $b$ be two sequences of length $N$. Also let $\rho_{a,b}(\tau)$ denotes the aperiodic cross-correlation function of $a$ and $b$ at
time-shift $\tau$ given by
\[
\rho_{a,b}(\tau) := \begin{cases} 
\sum_{k=0}^{N-1-\tau} \omega^{a_k b_{k+\tau}}, & 0 \leq \tau \leq N - 1; \\
\sum_{k=0}^{N-1+\tau} \omega^{a_k b_{k-\tau}} - (N - 1) \leq \tau \leq -1; \\
0, & |\tau| \geq N; 
\end{cases}
\]
where \(\omega = \exp(2\pi\sqrt{-1}/q)\) (\(q \geq 2\), is a positive integer). When the two sequences are identical, i.e., \(a = b\), \(\rho_{a,b}(\tau)\) is known as an aperiodic auto-correlation function (AACF) of \(a\) and it is denoted by \(\rho_a(\tau)\). Then \(a, b\) is said to be a \((N, Z + 1)\)-Z-complementary pair (ZCP) if
\[
\rho_a(\tau) + \rho_b(\tau) = 0, \text{ for } 1 \leq \tau \leq Z. \tag{2}
\]
ZCPs are related to the AACS only. To define CZCPs we need the following two sets. For an integer \(Z\), let \(T_1 = \{1, 2, \ldots, Z\}\) and \(T_2 = \{N - Z, N - Z + 1, \ldots, N - 1\}\). Then \((a, b)\) is called an \((N, Z)\)-CZCP if it possesses symmetric zero (out-of-phase) AACSs for time shifts over \(T_1 \cup T_2\) and zero aperiodic cross-correlation sums (ACCSs) for time shifts over \(T_2\) [30]. In short, it needs to satisfy the following two conditions:
\[
\begin{aligned}
C1: \rho_a(\tau) + \rho_b(\tau) = 0, & \text{ for all } |\tau| \in T_1 \cup T_2; \\
C2: \rho_{a,b}(\tau) + \rho_{b,a}(\tau) = 0, & \text{ for all } |\tau| \in T_2.
\end{aligned} \tag{3}
\]
From C1 it is clear that CZCPs have two zero auto-correlation zones (ZACZ) when the AACSs are considered. In this paper we will call them “front-end-ZACZ” and “tail-end-ZACZ” for the time shifts over \(T_1\) and \(T_2\), respectively. From C2 we get that each CZCP needs to have “tail-end zero cross-correlation zone (ZCCZ)” when ACCS are considered.

In [30], Liu et al. proposed the concept of cross Z-complementarity and constructed a class of GCPs of length \(N\) achieving the maximum ZACZ and ZCCZ width of \(N/2\). The authors in [30] termed them as “strengthened GCPs” and the \((N, Z)\)-CZCPs for which the value of \(Z\) is \(N/2\), as perfect CZCPs. On contrary, the \((N, Z)\)-CZCPs for which \(Z < N/2\), the authors termed them as non-perfect CZCPs. Since, \(Z = N/2\) can only be achievable when \(N\) is the length of a complementary pair, we re-categorise the CZCPs based on cross-Z-complementary ratio.

### B. Cross Z-Complementary Ratio

Let \((a, b)\) be a CZCP of length \(N\) having ZACZ and ZACZ of length \(Z\). If the maximum achievable length of \(Z\) is \(Z_{\text{max}}\), then we define the Cross Z-Complementary ratio (\(CZC_{\text{ratio}}\)) as
\[
CZC_{\text{ratio}} = \frac{Z}{Z_{\text{max}}}. \tag{4}
\]
When \(CZC_{\text{ratio}} = 1\), we call it optimal. When \(0.8 \leq CZC_{\text{ratio}} < 1\) we call it almost-optimal. Figure 1 clearly categorizes the CZCPs and also shows the relation of CZCPs with the existing GCPs and ZCPs. Figure 1 also corrects [30] Figure 3], since there are a lot of “non-strengthened GCPs”, where \(Z < N/2\), which were not shown in [30] Figure 3].

### C. Our Contributions

Based on the simulations given in [30], it can be observed that CZCPs can be used as an alternative to strengthened GCPs in the design of optimal SM training sequences over frequency selective channels. Since the availability of binary GCPs only for lengths of the form \(2^n 10^3 26^7\) is still a conjecture, from [30, Table I] it can be partially proved that the maximum ZACZ and ZCCZ for binary non-perfect CZCPs of length \(N\) is \(N/2 - 1\). In [30], the authors proposed two constructions of perfect \((N, N/2)\)-CZCPs, where for binary case \(N = 2^{n+1}10^3 26^7\), \(\alpha, \beta, \gamma \in Z^+, \alpha \geq 1\). Motivated by the work of Liu et al. [30], we propose several new constructions of \((N, Z)\)-CZCPs of new lengths. Specifically, the contribution of this paper are the following:

- We propose construction of \((2^{n-1} + 1, 2^{n-1} + 1)\)-CZCPs by using generalized Boolean functions (GBFs), where \(m \geq 4\) and \(\pi\) is a permutation over \(\{0, 1, \ldots, m - 3\}\).
- We further extend the construction by applying insertion method on GCPs, which are constructed via Turyn’s method. By exploiting the intrinsic structural properties of the GCPs found in [16], we propose systematic construction of CZCPs of new lengths of the form \(2^{n+1}10^3 26^7\) \(\alpha \geq 1\), \(10^2 + 26^7 + 10^2 26^7 + 2\) based on insertion method. The constructions which are based on insertion method, all the GCPs are constructed by applying Turyn’s method over kernel GCPs given in Table 1.
- We propose an optimal construction of binary \((12, 5)\)-CZCP and \((24, 11)\)-CZCP using binary Barker sequences. These two optimal CZCPs leads to two new sets of CZCPs with parameters \((12N, 5N)\) and \((24N, 11N)\), where \(N\) is the length of a GCP.
- We also found one interesting property of CZCPs that if \(e(d)\) is an \((N, Z)\)-CZCP, then \(c_i = d_i\), and \(c_{N-1-i} = -d_{N-1-i}\), for all \(i \in \{0, 1, \ldots, Z - 1\}\).
- Along with this, we also found that all the binary GCPs are also CZCPs. We clearly describe the relationships between binary CZCPs with ZCPs and GCPs in Figure 3.
- Through numerical simulations we show that our proposed optimal and almost-optimal CZCPs can be used to design optimal training sequences for SM systems, based on the framework given in [30].
- We analysed through numerical simulations that although some of the CZCPs may not be optimal or almost-optimal in sequence design point of view, however, depending on the number of multi-paths, those CZCPs can still be used to design optimal training sequences for SM systems.

### TABLE I: [3] GCP Kernels of Lengths 2, 10 and 26.

| \(N\) | \(K\) | Notation |
|---|---|---|
| 2 | \((\text{++})\) | \(K_2\) |
| 10 | \((\text{++}++++)\) | \(K_{10}\) |
| 26 | \((\text{++}++++)\) | \(K_{26}\) |
afterwards in [30], we can say that our proposed constructions will add flexibility in choosing the CZCPs of various lengths for designing training sequences in SM systems. Since, in practical scenarios, a longer training sequence will give rise to a higher training overhead. Therefore, selection of the training length is a trade-off between channel estimation performance and training overhead. For example, let us consider that in a practical scenario, we need to consider 22 multi-path. If we only consider CZCPs in [30], we have to use a (64, 32)-CZCP. However, by our proposed constructions, now we can use a (48, 22)-CZCP to design the training sequences in SM systems using the framework given in [30]. This will improve the system performance.

The rest of the paper is organised as follows. In Section II we discuss some previous results on binary GCPs and the system performance. In Section IV, we propose the construction of CZCPs of various lengths for optimal channel estimation performance (w.r.t. the performance of the proposed CZCPs to design sparse training sequences in SM systems. Since, in practical scenarios, a longer training sequence will give rise to a higher training overhead. Therefore, selection of the training length is a trade-off between channel estimation performance and training overhead. For example, let us consider that in a practical scenario, we need to consider 22 multi-path. If we only consider CZCPs in [30], we have to use a (64, 32)-CZCP. However, by our proposed constructions, now we can use a (48, 22)-CZCP to design the training sequences in SM systems using the framework given in [30]. This will improve the system performance.

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a sequence of all ones. In this paper, we concern about \((N, Z)\)-CZCPs, where \(N \neq 2^m\). Hence we define the truncated sequence \(\Psi_L(f)\) corresponding to GBF \(f\) by eliminating the first and last \(L\) elements of the sequence \(\Psi(f)\).

**Example 1:** Let us consider \(m = 3, q = 2\) and \(f = x_0x_1 + x_1x_2\), then

\[
x_0x_1 = (0, 0, 0, 1, 0, 0, 0, 1),
\]
\[
x_1x_2 = (0, 0, 0, 0, 0, 1, 0, 0),
\]
and therefore \(\Psi(f) = (1+1+1+1+1+1+1+1)\). If we assume \(L = 1\), then \(\Psi_1(f) = (1+1+1+1)\).

**Lemma 1 (Turyn’s Method \([31]\]):** Consider a GBF \(f : \mathbb{Z}_q^m \rightarrow \mathbb{Z}_q\), given by

\[
f = \frac{q}{2} \sum_{\alpha=0}^{m-2} x_\pi(\alpha)x_\pi(\alpha+1) + \sum_{i=0}^{m-1} g_i x_i + g',
\]

and

\[
\bar{f} = \frac{q}{2} \sum_{\alpha=0}^{m-2} \bar{x}_\pi(\alpha)x_\pi(\alpha+1) + \sum_{i=0}^{m-1} g_i \bar{x}_i + g',
\]

where \(g, g' \in \mathbb{Z}_q\). Then, \(\Psi(\bar{f} + \bar{x}_\pi(m-1)) = (1+1+1+1+1+1+1+1+1)\) is one of the complementary mates of \((\Psi(f), \Psi(f + x_\pi(m-1)))\).

**Lemma 2 (Turyn’s Method \([31]\)):** Let \(A = (a, b)\) and \(B = (c, d)\) be binary GCPs of lengths \(N\) and \(M\), respectively and denote \(A\) as the 1st pair and \(B\) as the 2nd pair. Then \((e, f) \triangleq Turyn(A, B)\) is a GCP of length \(MN\) where,

\[
e = c \otimes (a + b) / 2 - d \otimes (b - a) / 2,
\]
\[
f = d \otimes (a + b) / 2 + \bar{c} \otimes (b - a) / 2.
\]

**Result 1:** \([16]\) Let \(A = (a, b)\) be a binary GCP kernel \(K_N\) where \(N \in \{2, 10, 26\}\), \(B = (c, d)\) is a GCP of length \(M\) and \((e, f) = Turyn(A, B)\). If the \(i\)-th column of \(B\) has elements with the same sign, then the next \(N\) columns of \((e, f)\) will have elements with same sign, starting from \(N-i\)-th column. If the \(i\)-th column of \(B\) have elements with different signs, then the next \(N\) columns of \((e, f)\) will have elements with different sign, starting from \(N-i\)-th column.

**Result 2:** \([16]\) Let \((e, f)\) be a GCP of length \(2^a M\), constructed recursively using Turyn’s method on kernel GCPs as follows:

\[
(e_0, f_0) = K_2,
\]
\[
(e_i, f_i) = Turyn(A, (e_{i-1}, f_{i-1})), A = K_2, K_{10} \text{ or } K_{26},
\]

where \(M = 10^a 26^7\) and \(\alpha, \beta, \text{ and } \gamma\) be non-negative integers and \(\alpha \geq 1\). Then the first \(2^a - 1\) columns of \((e, f)\) will have elements with identical sign in each column.

**Result 3:** \([16]\) Let \((e, f)\) be a GCP of length \(10^3 26^7\), constructed iteratively using Turyn’s method on \(K_{10}\) or \(K_{26}\), respectively. Also suppose there are \(t\) consecutive columns of \(K_{10}\) or \(K_{26}\), having with identical signs (or different signs) in each column, starting from the \(i\)-th column index. Then, the \(t \times 10^3 - 1\) or \(t \times 26^7 - 3\) consecutive columns of \((e, f)\) will have elements with identical sign (or different sign) in each column, starting from the \(iN^{p-1}\)-th column, respectively.

**Result 4:** \([16]\) Let \((e, f)\) be a GCP of length \(10^3 26^7\), constructed iteratively by employing Turyn’s method on \(K_{10}\) and \(K_{26}\) as follows:

\[
(e_0, f_0) = K_{26},
\]
\[
(e_i, f_i) = Turyn(A, (e_{i-1}, f_{i-1})), A = K_{10} \text{ or } K_{26},
\]

where \(\beta\) and \(\gamma\) are non-negative integers. Then the first \(12 \times 26^7 - 10^3\) columns of \((e, f)\) will have elements with identical signs in each column.

In the following section we propose another construction of CZCPs with the help of GBFs.

### III. CONSTRUCTION OF CZCPs THROUGH GBFS

The proposed construction is discussed in this subsection. We need the following lemmas for the construction.

**Lemma 3:** Let \(r_{2^m - 2 + \tau, \pi(m-3)}\) and \(r_{3 \times 2^m - 2 + \tau, \pi(m-3)}\) be as defined above. Then,

\[
r_{2^m - 2 + \tau, \pi(m-3)} + r_{3 \times 2^m - 2 + \tau, \pi(m-3)} = 1, \forall \tau.
\]

**Proof:** Note that binary representation of \(x\) and \(2^m - x - 1\) for \(0 \leq x < 2^m\) are always complementary to each other. Let \(x = 2^m - 2 + \tau - 1\), then \(2^m - (2^m - 2 + \tau - 1) - 1 = 3 \times 2^m - 2 + \tau\). Hence, the binary representation of \(2^m - 2 + \tau\) is complementary with \(3 \times 2^m - 2\). Hence the proof follows.

**Lemma 4:** For any integer \(m \geq 4\), let \(\pi\) be a permutation of \(\{0, 1, 2, \ldots, m - 3\}\). Then, for \(2^m - 1 - 2^m(m-3) < \tau < 2^m - 1 + 1\), \(r_{2^m - 2 + \tau, \pi(m-3)}\) is always 1.

**Proof:** For \(2^m - 1 - 2^m(m-3) < \tau < 2^m - 1\), we need to consider the \(m\) bit binary representation of \(i\) where \(3 \times 2^m - 2^m(m-3) \leq i \leq 3 \times 2^m - 2\) and show that \(r_{i, \pi(m-3)} = 1\). Since \(3 \times 2^m - 2 = 2^m + 2^m\), therefore, the binary representation of \(3 \times 2^m - 2\) is \((11 \overbrace{00 \ldots 0}^{m-2}\overbrace{111}^{m-2})\). Hence the binary representation of \(3 \times 2^m - 2 - 1\) is \((10 \overbrace{111}^{m-2}\overbrace{00 \ldots 0}^{m-2})\). Let \(\pi(m-3) = v\). Also, let the binary representation of \(i\) be \((i_0, i_1, \ldots, i_{m-3}, 0, 1)\). If \(i_s = 1\) for \(s = m - 3, m - 4, \ldots, v\), \(v \leq m - 3\), then \(3 \times 2^m - 2 - 1 > 2^v < i\). Therefore for \(3 \times 2^m - 2 - 2^v \leq i \leq 3 \times 2^m - 2 - 1\), \(r_{i, v} = 1\). Hence the proof follows.

**Theorem 1:** For any integer \(m \geq 4\), let \(\pi\) be a permutation of \(\{0, 1, 2, \ldots, m - 3\}\). For \(d \in \mathbb{Z}_2\), let the GBF \(g^d : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q\) be given as follows:

\[
g^d = \frac{q}{2} x_{m-1}x_m - \zeta^d + x_m - \bar{x}_m - 2\eta^d + d\bar{x}_m - x_m - 2
\]

where \(\zeta^d : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2\) is

\[
\zeta^d = \sum_{\alpha=0}^{m-4} x_\pi(\alpha)x_\pi(\alpha+1) + dx_\pi(m-3),
\]

\(\eta^d : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2\) is

\[
\eta^d = \sum_{\alpha=0}^{m-4} \bar{x}_\pi(\alpha)x_\pi(\alpha+1) + d\bar{x}_\pi(m-3) + d1,
\]

where \(\zeta \in \mathbb{Z}_2^m\) and \(\eta \in \mathbb{Z}_2^m\).
and \(e_i, f_i \in \mathbb{Z}_q\) for \(0 \leq i \leq m - 3\). Then
\[
(a, b) = (\Psi_{2m-2-1}(g^i), \Psi_{2m-2-1}(g^j))
\] (19)
forms a \((2^{m-1} + 2, 2^{m-1})\) CZCP.

Proof: Let \(p = \{p_0, p_1, \ldots, p_{m-3}\}\) be the \(m - 2\) bit binary representation of \(\tau - 1 \mod 2^{m-2}\). Then \(s = \sum_{i \in p} |e_i - f_i|\) where \(p' = \{r|p_i \neq 0, 0 \leq r \leq m - 3\}\). For \(0 < \tau < 2^{m-2}\) we have
\[
\rho_a(\tau) + \rho_b(\tau) = \left[\omega^{s-\frac{3}{2}}c_0(r_{2m-2-\tau+1}) + \omega^{s-\frac{5}{2}}\eta_1(r_{3x2m-2-\tau})\right]
\] (20)
\[
\omega^{s-\frac{3}{2}}c_0(r_{2m-2-\tau}) + \omega^{s-\frac{5}{2}}\eta_1(r_{3x2m-2-\tau})
\]
\[
\omega^s \left[\omega^{2s}c_0(r_{2m-2-\tau}) + \omega^{2s+1}\eta_1(r_{3x2m-2-\tau})\right]
\] (21)
\[
\omega^{s-\frac{1}{2}}c_0(r_{3x2m-2-\tau}) + \omega^{s-\frac{3}{2}}\eta_1(r_{3x2m-2-\tau})
\]

Note that,
\[
\frac{q}{2} \zeta^i (r_{2m-2-\tau+1}) = \frac{q}{2} \left[\zeta^0(r_{2m-2-\tau+1}) + r_{2m-2-\tau+1}, \pi(m-3)\right]
\] (22)

and
\[
\frac{q}{2} \eta_1(r_{3x2m-2-\tau}) = \frac{q}{2} \left[\eta^0(r_{3x2m-2-\tau}) + r_{3x2m-2-\tau}, \pi(m-3)\right].
\] (23)

Applying (21) and (22) in (20) we get
\[
\rho_a(\tau) + \rho_b(\tau) = \omega^{s-\frac{3}{2}}c_0(r_{2m-2-\tau + 1}) \left[1 + \omega^{s-\frac{1}{2}}r_{2m-2-\tau + 1}, \pi(m-3)\right]
\]
\[
- \omega^{s-\frac{1}{2}}\eta_1(r_{3x2m-2-\tau}) \left[\omega^{2s}r_{3x2m-2-\tau}, \pi(m-3) - 1\right].
\] (24)

We have the following two sub-cases:

1) For \(r_{2m-2-\tau+1}, \pi(m-3) = 0\) we have \(r_{3x2m-2-\tau}, \pi(m-3) = 1\). In this case it follows from (24) that
\[
\rho_a(\tau) + \rho_b(\tau) = 0.
\] (25)

2) For \(r_{2m-2-\tau+1}, \pi(m-3) = 1\) we have \(r_{3x2m-2-\tau}, \pi(m-3) = 0\). In this case we have from (24)
\[
\rho_a(\tau) + \rho_b(\tau) = 0.
\] (26)

For \(2^{m-1} - 2\pi(m-3) < \tau < 2^{m-1} + 1\) we have
\[
\rho_a(\tau) + \rho_b(\tau) = \left[\omega^{s-\frac{3}{2}}c_0(r_{2m-2-\tau+1}) + \omega^{s-\frac{5}{2}}\eta_1(r_{3x2m-2-\tau})\right]
\]
\[
+ \left[\omega^{s-\frac{1}{2}}\eta_1(r_{2m-2-\tau+1}) + \omega^{s+\frac{1}{2}}c_0(r_{3x2m-2-\tau})\right]
\]
\[
= \omega^s \left[\omega^{2s}c_0(r_{2m-2-\tau}) + \omega^{2s+1}\eta_1(r_{3x2m-2-\tau})\right]
\]
\[
- \omega^s \left[\omega^{2s}c_0(r_{3x2m-2-\tau}) + \omega^{2s+1}\eta_1(r_{3x2m-2-\tau})\right]
\] (27)

Since from (17) and (18) we have
\[
\eta_1(r_{3x2m-2-\tau+1}) = \zeta^0(r_{3x2m-2-\tau+1}) + 1.
\] (28)

and
\[
\frac{q}{2} \zeta^i (r_{2m-2-\tau+1}) = \frac{q}{2} \left[\zeta^0(r_{2m-2-\tau+1}) + r_{2m-2-\tau+1}, \pi(m-3)\right];
\] (29)

Applying (28) and (29) in (27) we get
\[
\rho_a(\tau) + \rho_b(\tau) = \omega^{s-\frac{3}{2}}\eta_1(r_{2m-2-\tau+1}) \left[\omega^{s-\frac{3}{2}}r_{2m-2-\tau+1}, \pi(m-3) + 1\right]
\]
\[
- \omega^{s+\frac{1}{2}}c_0(r_{3x2m-2-\tau}) \left[1 - \omega^{s}r_{3x2m-2-\tau}, \pi(m-3)\right].
\] (30)

Again we have the following sub-case

1) For \(r_{2m-2-\tau+1}, \pi(m-3) = 1\) we have \(r_{3x2m-2-\tau}, \pi(m-3) = 0\). In this case we have from (30)
\[
\rho_a(\tau) + \rho_b(\tau) = 0.
\] (31)

For \(\tau = 2^{m-1} + 1\) we have
\[
\rho_a(\tau) + \rho_b(\tau) = \omega^{s-\frac{3}{2}} + \omega^s = 0.
\] (32)

Hence the ZACZ is \(2^{\pi(m-3)} + 1\). Now we will check the ZCCZ. Before that let us define \(k(i)\). For a given \(i\) and \(\tau\), consider the first \((m - 2)\) bits of the binary representation of \(3 \times 2^{m-2} - \tau + i\) and \(2^{m-2} + \tau + i - 1\) as \(q_0, q_1, \ldots, q_{m-2}\) and \((s_0, s_1, \ldots, s_{m-2})\), respectively. Let \(p' = \{|r|qr \neq sr\}\) and
\[
\sigma = \begin{cases} 0 & \text{if } qr = 0, \\ 1 & \text{if } qr = 1. \end{cases}
\] (33)

Then
\[
k(i) = \sum_{s \in p'} (-1)^\sigma (f_i - e_i),
\] (34)

For calculating ZCCZ we only consider \(2^{m-1} - 2\pi(m-3) < \tau < 2^{m-1} + 1\). When \(2^{m-1} - 2\pi(m-3) < \tau < 2^{m-1} + 1\), we have
\[
\rho_{a,b}(\tau) = \omega^{s-\frac{3}{2}}\eta_1(r_{2m-2-\tau+1})
\]
\[
+ \sum_{i=0}^{2^m-3-1} \omega^s c_0(r_{3x2m-2-\tau+i}) - \frac{1}{2} \zeta^0(r_{2m-2-\tau+i}) + k(i)
\]
\[
+ \omega^{s+\frac{3}{2}}c_0(r_{3x2m-2-\tau})
\] (35)

and
\[
\rho_{b,a}(\tau) = \omega^{s-\frac{3}{2}}\eta_1(r_{2m-2-\tau+1})
\]
\[
+ \sum_{i=0}^{2^m-3-1} \omega^s c_0(r_{3x2m-2-\tau+i}) - \frac{1}{2} \zeta^0(r_{2m-2-\tau+i}) + k(i)
\]
\[
+ \omega^{s+\frac{3}{2}}c_0(r_{3x2m-2-\tau})
\] (36)

Since from (17) and (18) we have
\[
\eta_1(r_{2m-2-\tau+i+1}) = \zeta^0(r_{3x2m-2-\tau+i}) + 1,
\] (37)

and
\[
\zeta^i(r_{3x2m-2-\tau+i}) = \eta_0(r_{2m-2-\tau+i+1}).
\] (38)
Using (37) and (38) in (35) and (36), respectively, we have
\[
\rho_{a,b}(\tau) + \rho_{b,a}(\tau)
= \omega^\tau \left[ -\frac{\omega}{2} \eta^i(r_{2m-2} + r_{1}) + \frac{\omega}{2} \eta^i(r_{2m-2} + r_{1}) \right]
+ \sum_{i=0}^{2m-3} \omega^k(i) \left[ \omega^{-\frac{i}{2} + 1} \right]
+ \omega^\tau \left[ \frac{\omega}{2} \xi^i(r_{3x3m-2} + r_{1}) + \frac{\omega}{2} \xi^i(r_{3x3m-2} + r_{1}) \right].
\]  
(39)

Further using (28) and (29) in (39) we get
\[
\rho_{a,b}(\tau) + \rho_{b,a}(\tau)
= \omega^\tau \omega^\tau \left[ 1 + \omega^{-\frac{i}{2}} r_{2m-2} + r_{1} \right]
+ \omega^\tau \left[ 1 - \omega^{-\frac{i}{2}} r_{3x3m-2} + r_{1} \right].
\]  
(40)

We have the following sub-case
1) For \( r_{2m-2} + r_{1} = 1 \) we have \( r_{3x3m-2} + r_{1} = 0 \). In this case we have from
\[
\rho_{a,b}(\tau) + \rho_{b,a}(\tau) = \omega^\tau + \omega^{-\frac{i}{2}} = 0.
\]  
(41)

Consequently, e and f are sequences of length 2N, having the following structural property
\[
f_i = \sigma((i/L)) e_i.
\]  
(50)

Also within sequence e, we have the following property
\[
e_i = \begin{cases} 
1 & \text{if } i < L, \\
0 & \text{if } i \geq L.
\end{cases}
\]  
(51)

Define \( \delta_{\tau,L} \) as
\[
\delta_{\tau,L} = \begin{cases} 
0 & \text{if } \tau \mod L = 0 \text{ or } \tau \geq 3N/2, \\
1 & \text{otherwise}.
\end{cases}
\]  
(52)

A. CZCPs of lengths of the form 2^\alpha 10^\beta 26^\gamma + 2 (\alpha \geq 1)

Theorem 2: Let \((a, b)\) be GCPs of length \( N = 2^\alpha 10^\beta 26^\gamma + 2 \) constructed via Result 2 and \((c, d)\) be one of its complementary mate. Also let \( e = (a||c)\), \( f = (b||d)\), \( g = I_a(e, 0, \{x_0, y_0\})\), and \( h = I_b(f, 0, \{x_1, y_1\}) \) where \( x_0, y_0, x_1, y_1 \in U_q \). Then \((g, h)\) is \((2N + 2, N/2 + 1)\) - CCP if the following conditions hold.
\[
\begin{align*}
x_0 - y_0^* & = 0, \quad x_1 + y_0^* = 0, \\
x_0 = x_1, \quad y_0^* = y_1^*.
\end{align*}
\]  
(47)

Proof: From (10) we know that \((a, b)\) is a GCP having the property
\[
a_i = b_{i-1} \text{ for } 0 \leq i \leq N/2, \\
\text{and } a_i = -b_{i-1} \text{ for } N/2 \leq i \leq N.
\]  
(48)

Let us define two sets \( A = \{0, 2\} \) and \( B = \{1, 3\} \). Also let \( L = N/2 \). Define \( \sigma \) as follows:
\[
\sigma(i) = \begin{cases} 
1 & \text{if } i \in A, \\
-1 & \text{if } i \in B.
\end{cases}
\]  
(49)

Consider \((c, d)\) as one of the complementary mates of \((a, b)\), then \((c, d)\) will also have the same property as in (48). Consequently, \( e \) and \( f \) are sequences of length \( 2N \), having the following structural property
\[
f_i = \sigma((i/L)) e_i.
\]  
(50)

Similarly, we calculate the autocorrelation of sequence h. For \( i \leq \tau < jL \), where \( 0 \leq i < 4, j = i + 1 \), we have
\[
\begin{align*}
\rho_h(\tau) &= x_0 e_{\tau-1} + \sum_{k=0}^{4-j} \rho e_{\tau-1}^{k+1}(\tau - iL) \\
&+ \sum_{k=0}^{3-j} \rho e_{\tau-1}^{k+1}(jL - \tau) + e_{2N-\tau} y_0^*.
\end{align*}
\]  
(53)

For \( \tau = 4L \), we have
\[
\begin{align*}
\rho_h(\tau) &= x_0 e_{2N-1} + e_0 y_0^* \\
&= x_0 y_0^*.
\end{align*}
\]  
(54)

Similarly, we calculate the autocorrelation of sequence h. For \( i \leq \tau < jL \), where \( 0 \leq i < 4, j = i + 1 \), we have
\[
\begin{align*}
\rho_h(\tau) &= x_1 f_{\tau-1} + \sum_{k=0}^{4-j} \rho f_{\tau-1}^{k+1}(\tau - iL) \\
&+ \sum_{k=0}^{3-j} \rho f_{\tau-1}^{k+1}(jL - \tau) + f_{2N-\tau} y_0^*.
\end{align*}
\]  
(56)
For $\tau = 4L$, we have
$$\rho_h(\tau) = x_1 f_{2N-1} + f_0 y_1^*.$$  \hfill (57)
For $\tau = 4L + 1$, we have
$$\rho_h(\tau) = x_1 y_1^*.$$  \hfill (58)
Recall that $(e, f)$ is a GCP, using [50] and [51] we get
$$\rho_g(\tau) + \rho_h(\tau) = \begin{cases} 
(x_0 + y_0^* + x_1 + y_1^*) e_{\tau-1} & \text{if } 0 < \tau \leq N/2, \\
(x_0 - y_0^* - x_1 - y_1^*) e_{\tau-1} & \text{if } N/2 < \tau \leq N, \\
(-x_0 + y_0^* + x_1 + y_1^*) e_{2N-\tau} & \text{if } 3N/2 < \tau < 2N, \\
(x_0 + y_0^* - x_1 + y_1^*) e_{2N-1} & \text{if } \tau = 2N, \\
x_0 y_0^* + x_1 y_1^* & \text{if } \tau = 2N + 1.
\end{cases} \hfill (59)
$$
Therefore we can get a ZACZ of $(N/2 + 1)$ for the sequence pair $(g, h)$ if the given conditions hold.

To check the condition $C2$, we calculate the cross-correlation of $g$ and $h$. For $iL \leq \tau < jL$, where $0 \leq i < 4$, $j = i + 1$, we have
$$\rho_{g,h}(\tau) = \sigma(i) \cdot x_0 e_{\tau-1} + \sum_{k=0}^{3-j} \sigma(k+i) \cdot \rho_{e_{iL}, e_{iL}}^\tau(\tau - iL) + \delta_{L,L} \sum_{k=0}^{3-j} \sigma(k+j) \cdot \rho_{e_{iL}, e_{iL}}^\tau(jL - \tau) + \sigma(i) \cdot e_{2N-\tau} y_1^*.$$  \hfill (60)
For $\tau = 4L$, we have
$$\rho_{g,h}(\tau) = x_0 f_{2N-1} + e_0 y_1^*$$
$$= -x_0 e_{2N-1} + e_0 y_1^*$$
$$= (-x_0 + y_1^*) e_0.$$  \hfill (61)
For $\tau = 4L + 1$, we have
$$\rho_{g,h}(\tau) = x_0 y_1^*.$$  \hfill (62)
Similarly, for $iL \leq \tau < jL$, where $0 \leq i < 4$, $j = i + 1$, we have
$$\rho_{h,g}(\tau) = x_1 e_{\tau} + \sum_{k=0}^{3-j} \sigma(k) \cdot \rho_{e_{iL}, e_{iL}}^\tau(\tau - iL) + \delta_{L,L} \sum_{k=0}^{3-j} \sigma(k+j) \cdot \rho_{e_{iL}, e_{iL}}^\tau(jL - \tau) + e_{2N-\tau} y_1^*.$$  \hfill (63)
For $\tau = 4L$, we have
$$\rho_{h,g}(\tau) = x_1 e_{2N-1} + e_0 y_0^*$$
$$= (x_1 + y_0^*) e_0.$$  \hfill (64)
For $\tau = 4L + 1$, we have
$$\rho_{h,g}(\tau) = x_1 y_0^*.$$  \hfill (65)
Then for $1 \leq \tau < 2N \rho_{g,h}(\tau) + \rho_{h,g}(\tau)$ is given in [66].

For rest of the cases, we have
$$\rho_{g,h}(\tau) + \rho_{h,g}(\tau) = \begin{cases} 
(-x_0 + y_1^* + x_1 + y_0^*) e_0 & \text{if } \tau = 2N, \\
x_0 y_1^* + x_1 y_0^* & \text{if } \tau = 2N + 1.
\end{cases} \hfill (66)
$$
Since according to the setup $\sigma(i) = -1$ when $3N/2 \leq \tau < 2N$, therefore $\rho_{g,h}(\tau) + \rho_{h,g}(\tau) = 0$ if $(-x_0 + x_1 - y_1^* + y_0^*) = 0$. Therefore, from (66) and the above explanation, we conclude that the ZCCZ of $(g, h)$ is $(N/2 + 2)$.

Hence, we conclude that $(g, h)$ is a $(2N + 2, N/2 + 1)$- CZCP.

In the following example we will illustrate the proposed construction step by step.

**Example 3:** Step 1: Let $(a, b)$ be a GCP of length 8, constructed via Result [2] as follows:
$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + \end{pmatrix} \hfill (68)
$$
Step 2: $(c, d)$ be a complementary mate of $(a, b)$, therefore,
$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} - & - & - & - & - & - & - & - \\ + & + & + & + & + & + & + & + \end{pmatrix} \hfill (69)
$$
Step 3: $e = (a | c)$ and $f = (b | d)$. Therefore,
$$\begin{pmatrix} c \\ f \end{pmatrix} = \begin{pmatrix} + & - & - & - & + & + & + & + & + & + \\ + & + & - & - & - & - & - & - & - & - \end{pmatrix} \hfill (70)
$$
Step 4: $g = I_s(e, 0, \{x_0, y_0\})$, and $h = I_s(f, 0, \{x_1, y_1\})$. Let $x_0 = 1$, $x_1 = 1$, $y_0 = -1$, and $y_1 = 1$. Therefore,
$$\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} + & + & + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + & + & + \end{pmatrix} \hfill (71)
$$
The pair $(g, h)$ is a length $(18, 5)$- CCP, because
$$|\rho_g(\tau) + \rho_h(\tau)|_{\tau = 0}^{17} = (30, 0, 8, 4, 0, 8, 4, 0, 0). \hfill (72)
$$
and
$$|\rho_{g,h}(\tau) + \rho_{h,g}(\tau)|_{\tau = 0}^{17} = (0, 4, 2, 8, 4, 0, 4, 8, 4, 0, 0). \hfill (73)
$$
We give the following theorems without proofs. Since the proofs are similar to Theorem [2].

**B. CZCPs of lengths of the form $10^\beta + 2$**

**Theorem 3:** Let $(a, b)$ be GCPs of length $N = 10^\beta$ constructed via Result [3] and $(e, d)$ be one of its complementary mate. Also let $e = (a | c)$, $f = (b | d)$, $g = I_s(e, 0, \{x_0, y_0\})$, and $h = I_s(f, 0, \{x_1, y_1\})$ where $x_0, y_0, x_1, y_1 \in U_q$. Then $(g, h)$ is $(2N + 2, 4N/10 + 1)$- CZCP if the following conditions hold.
$$x_0 - y_1^* = 0, \quad x_1 + y_0^* = 0 \hfill (74)
$$
$$x_0 = x_1, \quad y_0^* = y_1^*.$$  \hfill (75)

**C. CZCPs of lengths of the form $26^\gamma + 2$**

**Theorem 4:** Let $(a, b)$ be GCPs of length $N = 26^\gamma$ constructed via Result [3] and $(e, d)$ be one of its complementary mate. Also let $e = (a | c)$, $f = (b | d)$, $g = I_s(e, 0, \{x_0, y_0\})$, and $h = I_s(f, 0, \{x_1, y_1\})$ where $x_0, y_0, x_1, y_1 \in U_q$. Then $(g, h)$ is $(2N + 2, 12N/26 + 1)$- CZCP if the following conditions hold.
$$x_0 - y_1^* = 0, \quad x_1 + y_0^* = 0 \hfill (75)
$$
$$x_0 = x_1, \quad y_0^* = -y_1^*.$$  \hfill (76)
\[
\rho_{g,h}(\tau) + \rho_{h,g}(\tau) = (\sigma(i) \cdot x_0 + x_1)e_{\tau-1} + \sum_{k=0}^{4-j} (\sigma(k+i) + \sigma(k)) \cdot \rho_{e^k e^{i+1}}(\tau - iL)
+ \delta_{\tau,L} \sum_{k=0}^{3-j} (\sigma(k+i) + \sigma(k)) \cdot \rho_{e^k e^{i+1}}((-i) L - \tau) + e_{2N-\tau}(\sigma(i) \cdot y_1^* + y_0^*).\]

Using their correlation properties given in Table II, we have the following theorem.

**Theorem 6:** Let \(a\) and \(b\) be binary Barker sequences of lengths \(M\) and \(N\), respectively, and \(M \leq N\). Then the sequence pair \((c,d)\) given by

\[
\begin{align*}
c &= a \parallel b; \\
d &= a \parallel -b;
\end{align*}
\]
forms a \((M+N,M)\) CZCP, if \(\rho_a(\tau) = -\rho_b(\tau)\) for \(0 < \tau < M\) and \(\rho_a(M) = 0\) when \(M < N\).

**Proof:** As per the given in (77), we have for \(0 < \tau < M\),

\[
\rho_c(\tau) + \rho_d(\tau) = 0,
\]

since, \(\rho_a(\tau) = -\rho_b(\tau)\) for \(0 < \tau < M\).

For \(M \leq \tau < N\), we have

\[
\rho_c(\tau) + \rho_d(\tau) = 2\rho_b(\tau).
\]

For \(N \leq \tau < M + N\), we have

\[
\rho_c(\tau) + \rho_d(\tau) = 0.
\]

Now, to analyse the cross-correlation, we have for \(0 < \tau < M\),

\[
\begin{align*}
\rho_{c,d}(\tau) &= \rho_d(\tau) - 2\rho_b(\tau); \\
\rho_{d,c}(\tau) &= \rho_c(\tau) - 2\rho_b(\tau);
\end{align*}
\]

Therefore, from (78) we have for \(0 < \tau < M\),

\[
\rho_{c,d}(\tau) + \rho_{d,c}(\tau) = -4\rho_b(\tau).
\]

Similarly, we have for \(M \leq \tau < N\),

\[
\rho_{c,d}(\tau) + \rho_{d,c}(\tau) = -2\rho_b(\tau).
\]

And, for \(N \leq \tau < M + N\), we have

\[
\rho_{c,d}(\tau) + \rho_{d,c}(\tau) = 0.
\]

Hence the theorem is proved.

**Remark 1:** Based on the result of the Theorem 6 when \(M = 5\) and \(N = 7\), using Table II, we get an optimal binary \((12, 5)\) CZCP. When \(M = 11\) and \(N = 13\), using Table II we get an optimal binary \((24, 11)\) CZCP. Note that these CZCPs are different than that of the “best possible” CZCPs given in Table I, obtained through computer search.

In the next theorem we will enlarge the length of the CCPs generated by Theorem 6 with the help of GCPs and utilizing Turyn’s method. To prove the theorem, we need the following lemma.

**Lemma 5:** For a binary sequence pair \((c,d)\) of length \(N\),

\[
\rho_{c,d}(\tau) = \rho_{d,c}(\tau).
\]
This completes the proof.

Using (87) to reduce the above equations we get,
\[ c_i = d_i, \quad \text{and} \quad c_{N-1-i} = -d_{N-1-i}, \] (86)
for all \( i \in \{0, 1, \cdots, Z-1\} \).

Proof: If \( a, b \) can only take the values +1 or −1, then we know that
\[ ab \equiv a + b - 1 \pmod{4}. \] (87)

From C2 of (3), we get for \( Z \leq \tau < N \)
\[ c_0d_{N-1} + d_0c_{N-1} = 0, \]
\[ c_0d_{N-2} + c_1d_{N-1} + d_0c_{N-2} + d_1c_{N-1} = 0, \]
\[ \cdots \]
(88)
\[ c_0d_{N-Z} + c_1d_{N-Z-1} + c_2d_{N-Z-2} + \cdots + d_{N-1}c_{N-1} = 0. \]

Using (87) to reduce the above equations we get,
\[ c_i + d_i + c_{N-1-i} + d_{N-1-i} = 2 \pmod{4}. \] (89)

Which is equivalent to
\[ c_i c_{N-1-i} + d_i d_{N-1-i} = 0. \] (90)

Equivalently, we can conclude that \( c_i = d_i \) and \( c_{N-1-i} = -d_{N-1-i} \).

Theorem 7: Any binary GCP \((a, b)\) of length \(N\) is also a CZCP.

Proof: We just need to check C2 of (3), for \( \tau = N - 1 \).

Since we already know that
\[ a_0a_{N-1} + b_0b_{N-1} = 0, \] (91)
using (87), we can conclude that
\[ a_0b_{N-1} + b_0a_{N-1} = 0. \] (92)

This completes the proof.

Remark 2: Theorem 7 changes our understanding for the binary CZCPs, which we get from Figure 1. The modified figure, describing the relationships of binary CZCPs with ZCPs and GCPs, is given in Figure 3.

Lemma 6: Let \((c, d)\) be a binary \((N, Z)\)-CZCP. Then
\[ \rho_{c, \bar{d}}(\tau) - \rho_{a, \bar{a}}(\tau) = 0, \quad \text{for all} \quad |\tau| \geq Z. \] (93)

Theorem 8: Let \(A = (a, b)\) be a GCP of length \(M\) and \(B = (c, d)\) be a \((N, Z)\)-CZCP. Also, let
\[ (e, f) = \text{Turyn}(A, B). \] (94)
TABLE III: Parameters of CCPs and CZCPs.

| Ref. | Type | \(N\) | \(Z\) | \(CZC_{ratio}\) | Remarks |
|------|------|------|------|--------------|---------|
| [30] | CCP  | \(2^\alpha\) | \(2^\alpha-1\) | 1 | Optimal |
| [30] | CCP  | \(2^{\alpha+1}10^\beta26^\gamma\) \((\alpha \geq 1)\) | \(2^{\alpha-1}10^\beta26^\gamma\) | 1 | Optimal |
| Th. 1 | CZCP | \(2^{m-1}+2\) \((m \geq 4)\) | \(2^{m-3}+1\) \(\approx \frac{1}{2}\) | Not optimal |
| Th. 2 | CZCP | \(2^{\alpha+1}10^\beta26^\gamma\) \((\alpha \geq 1)\) | \(2^{\alpha-1}10^\beta26^\gamma\) | \(\approx \frac{1}{2}\) | Not optimal |
| Th. 3 | CZCP | \(2N+2\) \((N = 10^k)\) | \(4N/10+1\) \(\approx \frac{2}{5}\) | Not optimal |
| Th. 4 | CZCP | \(2N+2\) \((N = 10^k)\) | \(12N/26+1\) \(\approx \frac{6}{13}\) | Not optimal |
| Th. 5 | CZCP | \(2N+2\) \((N = 10^k)\) | \(12N/26+1\) | \(\approx \frac{6}{13}\) | Not optimal |
| Th. 6 | CZCP | 12 | 5 | 1 | Optimal |
| Th. 7 | CZCP | 24 | 11 | 1 | Optimal |
| Th. 8 | CZCP | 12N \((N = 10^k)\) | 5N | \(\approx \frac{5}{6}\) Almost-optimal |
| Th. 9 | CZCP | 12N \((N = 10^k)\) | 11N | \(\approx \frac{11}{12}\) Almost-optimal |

Fig. 4: MSE comparison, No. of multi-paths 5, (12, 5)- CZCP.

Fig. 5: MSE comparison, No. of multi-paths 5, (18, 5)- CZCP.

compare its channel estimation performance with SM training matrices from (16, 8)- CZCP given in [30], the length-16 GCP (which is not a CZCP) given below
\[
\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} + + + + + - - - - - + + + + + + - - - + + \\ + + + + + - - - - - - - - - - - + + + + + \end{pmatrix}.
\] (96)

and also with a randomly generated sequence, which is generated “on-the-fly”. Fig. 6 shows that when the number of multi-paths is 9 (or less) (16, 8)- CZCP of [30] can be used to design SM training matrix which achieves the minimum MSEs. (48, 22)- CZCP, constructed using Theorem 6 can be used to design SM training matrix which achieves the minimum MSEs when the number of multi-paths is less than or equal to 23.

Hence, in the given conditions, it can be used to design optimal training matrix for SM systems using the framework given in [30].

VII. CONCLUDING REMARKS AND OPEN PROBLEMS

In this paper, we have introduced the concept of \(CZC_{ratio}\) and re-categorised the CZCPs based on that. We analysed the non-perfect CZCPs and proposed three constructions of non-perfect CZCPs. The first construction is based on GBFs while the second one is based on applying insertion method on the GCPs, which are constructed via Turyn’s method. By using GBFs we have constructed CZCPs of lengths \(2^{m-1}+2\) \((m \geq 4)\). By applying insertion method we have constructed CZCPs
of lengths $2^210^326^7 + 2$ ($\alpha \geq 1$), $10^2 + 2$, $26^7 + 2$ and $10^226^7 + 2$. In the final construction, we proposed construction of binary optimal $(12,5)$- CZCPs and $(24,11)$- CZCPs using Barker sequences. These two optimal CZCPs leads to $(12N,5N)$- CZCPs and $(24N,11N)$- CZCPs, where $N$ is the length of a GCP. All these CZCPs can be used to construct cross Z-complementary sets by the method given in [30]. Also, during this work we found one beautiful property of binary cases, the “best possible” CZCPs for $\frac{N}{2}$ - CZCPs, stated in Property 1. Through numerical simulations we show that depending on the number of multi-paths our proposed CZCPs can be used to design optimal training sequences for SM systems, based on the framework proposed by Liu et al. in [30].

While calculating the $CZC_{ratio}$ for non-perfect $(N,Z)$- CZCPs, we always take the maximum value of $Z$, i.e., $Z_{max} = N/2 - 1$. However, by taking a closer look at [30] Table I, we can see that for binary cases, the “best possible” CZCPs for lengths 18 and 22, one can obtain from computer search, have $Z_{max}$ values 7 and 9, respectively. So, it is highly possible to tighten the upper-bound of $Z_{max}$ for certain lengths. Along with the above, systematic constructions of optimal CZCPs with new lengths can be considered.

**APPENDIX**

**PROOF OF THEOREM 8**

By the Euclidean division theorem, we have $\tau = k_1M + k_2$ where $0 \leq k_1 < N$ and $0 \leq k_2 < M$. By the definition of AACF, we have

$$
\rho_{a}(\tau) = \sum_{m=0}^{N-1-k_1} \left( \frac{d_m + d_{N-1-m}}{2} \right) \left( \frac{d_{m+k_1} + d_{N-1-m-k_1}}{2} \right) \rho_{a}(k_2) + \left( \frac{c_m - d_{N-1-m}}{2} \right) \left( \frac{c_{m+k_1} - d_{N-1-m-k_1}}{2} \right) \rho_{b}(k_2) + \frac{c_m + d_{N-1-m}}{2} \frac{c_{m+k_1} + d_{N-1-m-k_1}}{2} \rho_{a,b}(k_2)
$$

where $M - k_2 = k_3$. Similarly,

$$
\rho_{e}(\tau) = \sum_{m=0}^{N-1-k_1} \left( \frac{d_m - c_{N-1-m}}{2} \right) \left( \frac{d_{m+k_1} - c_{N-1-m-k_1}}{2} \right) \rho_{a}(k_2) + \left( \frac{d_m + c_{N-1-m}}{2} \right) \left( \frac{d_{m+k_1} + c_{N-1-m-k_1}}{2} \right) \rho_{b}(k_2) + \frac{d_m - c_{N-1-m}}{2} \frac{d_{m+k_1} - c_{N-1-m-k_1}}{2} \rho_{a,b}(k_2) + \frac{d_m + c_{N-1-m}}{2} \frac{d_{m+k_1} + c_{N-1-m-k_1}}{2} \rho_{b,a}(k_2) + \frac{d_m - c_{N-1-m}}{2} \frac{d_{m+k_1+1} - c_{N-1-m-k_1-1}}{2} \rho_{a}(k_3) + \frac{d_m + c_{N-1-m}}{2} \frac{d_{m+k_1+1} + c_{N-1-m-k_1-1}}{2} \rho_{b}(k_3) + \frac{d_m - c_{N-1-m}}{2} \frac{d_{m+k_1+1} + c_{N-1-m-k_1-1}}{2} \rho_{b,a}(k_3) + \frac{d_m + c_{N-1-m}}{2} \frac{d_{m+k_1+1} - c_{N-1-m-k_1-1}}{2} \rho_{a,b}(k_3)
$$

where $M - k_2 = k_3$. Therefore, by some elementary operations and Lemma 5 we have

$$
\rho_{a}(\tau) + \rho_{e}(\tau) = \frac{1}{4} \left( \rho_{a}(k_1) + \rho_{a}(k_1) + \rho_{e}(k_1) + \rho_{e}(k_1) \right) \left( \rho_{b}(k_2) + \rho_{a}(k_2) \right) + \frac{1}{4} \left( \rho_{e}(k_1 + 1) + \rho_{a}(k_1 + 1) + \rho_{e}(k_1 + 1) + \rho_{a}(k_1 + 1) \right) \left( \rho_{b}(M - k_2) + \rho_{a}(M - k_2) \right)
$$

(99)
When \( \tau = MZ + M \), then \( k_1 = (Z + 1) \) and \( k_2 = 0 \), therefore from (99), it is clear that

\[
\rho_e(MZ + M) + \rho_f(MZ + M) = \frac{1}{2} (\rho_e(Z + 1) + \rho_a(Z + 1)) (\rho_a(0) + \rho_b(0)).
\] (100)

otherwise, for all other values of \( \tau \),

\[
\rho_e(\tau) + \rho_f(\tau) = 0.
\] (101)

Now to calculate the cross-correlation, for \( M - k_2 = k_3 \), we have

\[
\rho_e(\tau) = \sum_{n=0}^{N-1-k_1} \left[ \left( \frac{c_m + d_{N-1-m}}{2} \right) \left( \frac{d_{m+k_1} - c_{N-1-m-k_1}}{2} \right) \right] \rho_a(k_2) + \left( \frac{c_m - d_{N-1-m}}{2} \right) \left( \frac{d_{m+k_1} + c_{N-1-m-k_1}}{2} \right) \rho_b(k_2) + \left( \frac{c_m + d_{N-1-m}}{2} \right) \left( \frac{d_{m+k_1} + c_{N-1-m-k_1}}{2} \right) \rho_a(k_2) + \left( \frac{c_m - d_{N-1-m}}{2} \right) \left( \frac{d_{m+k_1} + c_{N-1-m-k_1}}{2} \right) \rho_b(k_2).
\] (102)

Similarly,

\[
\rho_f(\tau) = \sum_{n=0}^{N-1-k_1} \left[ \left( \frac{d_m - c_{N-1-m}}{2} \right) \left( \frac{c_{m+k_1} + d_{N-1-m-k_1}}{2} \right) \right] \rho_a(k_2) + \left( \frac{d_m - c_{N-1-m}}{2} \right) \left( \frac{c_{m+k_1} - d_{N-1-m-k_1}}{2} \right) \rho_b(k_2) + \left( \frac{d_m + c_{N-1-m}}{2} \right) \left( \frac{c_{m+k_1} + d_{N-1-m-k_1}}{2} \right) \rho_a(k_2) + \left( \frac{d_m + c_{N-1-m}}{2} \right) \left( \frac{c_{m+k_1} - d_{N-1-m-k_1}}{2} \right) \rho_b(k_2) + \left( \frac{d_m - c_{N-1-m}}{2} \right) \left( \frac{c_{m+k_1} + d_{N-1-m-k_1}}{2} \right) \rho_a(k_2) + \left( \frac{d_m + c_{N-1-m}}{2} \right) \left( \frac{c_{m+k_1} - d_{N-1-m-k_1}}{2} \right) \rho_b(k_2) + \left( \frac{d_m - c_{N-1-m}}{2} \right) \left( \frac{c_{m+k_1} + d_{N-1-m-k_1}}{2} \right) \rho_a(k_2) + \left( \frac{d_m + c_{N-1-m}}{2} \right) \left( \frac{c_{m+k_1} - d_{N-1-m-k_1}}{2} \right) \rho_b(k_2) + \left( \frac{d_m - c_{N-1-m}}{2} \right) \left( \frac{c_{m+k_1} + d_{N-1-m-k_1}}{2} \right) \rho_a(k_2) + \left( \frac{d_m + c_{N-1-m}}{2} \right) \left( \frac{c_{m+k_1} - d_{N-1-m-k_1}}{2} \right) \rho_b(k_2).
\] (103)

Simplifying (102), (103) and using Lemma 6 we get for \( ZM \leq \tau < NM \),

\[
\rho_e(\tau) + \rho_f(\tau) = 0,
\] (104)
[24] J. Jeganathan, A. Ghrayeb, and L. Szczecinski. “Spatial modulation: Optimal detection and performance analysis,” IEEE Commun. Lett., vol. 12, no. 8, pp. 545-547, Aug. 2008.

[25] M. D. Renzo and H. Haas, “Bit error probability of SM-MIMO over generalized fading channels,” IEEE Trans. Veh. Technol., vol. 61, no. 3, pp. 1124-1144, Mar. 2012.

[26] S. A. Yang and J. Wu, “Optimal binary training sequence design for multiple-antenna systems over dispersive fading channels,” IEEE Trans. Veh. Technol., vol. 51, pp. 1271-1276, Sept. 2002.

[27] C. Fragouli, N. Al-Dhahir, and W. Turin, “Training-based channel estimation for multiple-antenna broadband transmissions,” IEEE Trans. Wireless Commun., vol. 2, no. 2, pp. 384-391, Mar. 2003.

[28] P. Fan and W. H. Mow, “On optimal training sequence design for multiple-antenna systems over dispersive fading channels and its extensions,” IEEE Trans. Veh. Technol., vol. 53, no. 5, pp. 1623-1626, Sep. 2004.

[29] S. Sugiura and L. Hanzo, “Effects of channel estimation on spatial modulation,” IEEE Signal Process. Lett., vol. 19, no. 12, pp. 805-808, Dec. 2012.

[30] Z. Liu, P. Yang, Y. L. Guan, P. Xiao, “Cross-complementary pairs for optimal training in spatial modulation over frequency selective channels,” IEEE Trans. Inf. Theory, Early Access, Feb. 2020.

[31] R. Turyn, “Hadamard matrices, Baumert-Hall units, four-symbol sequences, pulse compression and surface wave encodings,” J. Combin. Theory (A), vol. 16, pp. 313-333, 1974.

[32] A. Rathinakumar A. K. Chaturvedi, “Complete mutually orthogonal Golay complementary sets from Reed-Muller codes” IEEE Trans. Inf. Theory vol. 54 no. 3 pp. 1339-1346 Mar. 2008.

[33] P. Fan and M. Darnell, Sequence Design for Communications Applications. New York, NY, USA: Wiley, 1996.