A Solution of the Word Problem for Braid Groups via the Complex Reflection Group $G_{12}$

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Abstract. It is known that if there exists a Gröbner-Shirshov basis for a group $G$, then we say that one of the decision problem, namely the word problem, is solvable for $G$ as well. Therefore, as the main target of this paper, we will present a (non-commutative) Gröbner-Shirshov basis for the braid group associated with the congruence classes of complex reflection group $G_{12}$ which will give us normal forms of the elements of $G_{12}$ and so will obtain a new algorithm to solve the word problem over it.

1. Introduction and Preliminaries

The Gröbner basis theory for commutative algebras was introduced by Buchberger [10] and provides a solution to the reduction problem for commutative algebras. In [4] Bergman generalized the Gröbner basis theory to associative algebras by proving the “Diamond Lemma”. On the other hand, the parallel theory of Gröbner bases was developed for Lie algebras by Shirshov [20]. In [6] Bokut noticed that Shirshov’s method works for also associative algebras. Hence, for this reason, Shirshov’s theory for Lie algebras and their universal enveloping algebras is called the Gröbner-Shirshov basis theory. There are some important studies on this subject related to the groups (see, for instance, [7, 11]). We may finally refer the papers [2, 3, 8, 14–17] for some other recent studies over Gröbner-Shirshov bases.

Algorithmic problems such as the word, conjugacy and isomorphism problems have played an important role in group theory since the work of M. Dehn in early 1900’s. These problems are called decision problems which ask for a yes or no answer to a specific question. Among these decision problems especially the word problem has been studied widely in groups (see [1]). It is well known that the word problem for finitely presented groups is not solvable in general; that is, given any two words obtained by generators of the group, there may be no algorithm to decide whether these words represent the same element in this group. Gröbner-Shirshov basis theory, which is the main theme of this paper, is one of the most effective

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and computational method to solve the word problem for a given algebraic structure not only for groups, monoids and semigroups but also other important algebras.

In the study [5], the authors first defined the presentations for the braid groups associated with the complex reflection groups $G_{24}$ and $G_{27}$ and they used VKCURVE that is a GAP package implementing Van Kampen’s method to obtain these presentations. Moreover, they added some conjectures for the cases of $G_{29}$, $G_{31}$, $G_{33}$ and $G_{34}$.

Shephard and Todd (1954) classified all finite complex reflection groups in [18]. Later, Cohen (1976) gave a more systematic description for these groups in terms of root systems, vector graphs and root graphs [12]. Recently, in [13], Howlett and Shi defined a simple root system $(B)$ and in [19], Shi studied the finite primitive complex reflection groups $G = [G_7, G_{11}, G_{13}, G_{19}, G_{27}]$ and in [19], Shi studied the finite primitive complex reflection groups $G = [G_{12}, G_{24}, G_{25}, G_{26}]$ (in the notations of Shephard and Todd, 1954). So by considering the presentation of $G_{12}$ given in [19], our aim in this paper is to find Gröbner-Shirshov bases of these important group.

The method of Gröbner-Shirshov bases which is the main theme of this paper gives a new algorithm to get normal forms of elements of groups, and so a new algorithm for solving the word problem in these groups. By considering this fact, our aim in this paper is to find Gröbner-Shirshov bases for the braid group associated with the congruence classes of complex reflection group $G_{12}$.

Throughout this paper, by considering the lengths of any two words, we will use the deg-lax ordering if the lengths of these words are different or lexicographically ordering if otherwise. Additionally the notations $(i) \land (j)$ and $(i) \lor (j)$ will denote the intersection and inclusion compositions of relations $(i)$ and $(j)$, respectively.

2. Gröbner-Shirshov Bases and Composition-Diamond Lemma

Let $K$ be a field and $K(X)$ be the free associative algebra over $K$ generated by $X$. Denote $X^*$ the free monoid generated by $X$, where the empty word is the identity denoted by 1. For a word $w \in X^*$, we denote the length of $w$ by $|w|$. Suppose that $X^*$ is a well ordered set. Then every nonzero polynomial $f \in K(X)$ has the leading word $\bar{f}$. If the coefficient of $\bar{f}$ in $f$ is equal to 1, then $f$ is called monic.

Let $f$ and $g$ be two monic polynomials in $K(X)$. We then have two compositions as follows:

- If $w$ is a word such that $w = \bar{f}b = a\bar{g}$ for some $a, b \in X^*$ with $|\bar{f}| + |\bar{g}| > |w|$, then the polynomial $(f, g)_w = fb - ag$ is called the intersection composition of $f$ and $g$ with respect to $w$. The word $w$ is called an ambiguity of intersection.

- If $w = \bar{f} = a\bar{g}b$ for some $a, b \in X^*$, then the polynomial $(f, g)_w = f - ab$ is called the inclusion composition of $f$ and $g$ with respect to $w$. The word $w$ is called an ambiguity of inclusion.

If $g$ is monic, $\bar{f} = a\bar{g}b$ and $\alpha$ is the coefficient of the leading term $\bar{f}$, then transformation $f \mapsto f - \alpha abg$ is called elimination (ELW) of the leading word of $g$ in $f$.

Let $S \subseteq K(X)$ with each $s \in S$ is monic. Then the composition $(f, g)_w$ is called trivial modulo $(S, w)$ if $(f, g)_w = \sum a_i a_i b_i$, where each $a_i \in K, a_i, b_i \in X^*, s_i \in S$ and $a_i \bar{a}b_i < w$. If this is the case, then we write $(f, g)_w \equiv 0 \mod (S, w)$.

We call the set $S$ endowed with the well ordering $\prec$ a Gröbner-Shirshov basis for $K(X \mid S)$ if any composition $(f, g)_w$ of polynomials in $S$ is trivial modulo $S$ and corresponding $w$.

The following lemma was proved by Shirshov [20] for free Lie algebras with deg-lax ordering,
Theorem 3.1 ([19]). The braid group associated with the congruence classes of complex reflection group $G_{12}$ admits the presentation
\[
\langle s, t, u; s^2 = u^2 = t^2 = 1, suts = utsu, tsut = suts \rangle.
\]

Lemma 2.1 (Composition-Diamond Lemma). Let $K$ be a field, $A = K \langle X \mid S \rangle = K(X)/Id(S)$ and $<$ a monomial ordering on $X^*$, where $Id(S)$ is the ideal of $K(X)$ generated by $S$. Then the following statements are equivalent:

1. $S$ is a Gröbner-Shirshov basis.
2. $f \in Id(S) \Rightarrow \overline{f} = \overline{a} \overline{b}$ for some $s \in S$ and $a, b \in X^*$.
3. $Irr(S) = \{u \in X^* \mid u \neq \overline{a} \overline{b}, s \in S, a, b \in X^*\}$ is a basis for the algebra $A = K \langle X \mid S \rangle$.

If a subset $S$ of $K(X)$ is not a Gröbner-Shirshov basis, then we can add to $S$ all nontrivial compositions of polynomials of $S$, and by continuing this process many times (maybe infinitely), we eventually obtain a Gröbner-Shirshov basis of polynomials of $S$. We should note that such a process is called the Shirshov algorithm.

3. Main Results

In this section, we would like to obtain normal form of elements of the braid group associated with the congruence classes of complex reflection group $G_{12}$ by using Gröbner-Shirshov basis theory. In [19], the author has just obtained a presentation for $G_{12}$ he did not give the structure of elements of the group $G_{12}$. On that respect the result given here is worth to study in Algebra. The presentation of the braid group associated with the congruence classes of complex reflection group $G_{12}$ is as follows:

Theorem 3.1 ([19]). The braid group associated with the congruence classes of complex reflection group $G_{12}$ admits the presentation
\[
\langle s, t, u; s^2 = u^2 = t^2 = 1, suts = utsu, tsut = suts \rangle.
\]

Now we consider the monoid presentation of $G_{12}$ given in (1). Since we have the relations $s^2 = u^2 = t^2 = 1$ for the generators $s, u, t$, it can be easily seen that the monoid presentation is the same with the group presentation of $G_{12}$ given in (1). To obtain Gröbner-Shirshov basis of the group $G_{12}$, let us consider an ordering among the generators as $t > u > s$.

The main result of this paper is the following.

Theorem 3.2. A Gröbner-Shirshov basis of the braid group associated with the congruence classes of the complex reflection group $G_{12}$ consists of the following polynomials:

\[
\begin{align*}
(1) & \quad s^2 - 1, & (2) & \quad u^2 - 1, & (3) & \quad t^2 - 1, & (4) & \quad utsu - suts, \\
(5) & \quad usts - stus, & (6) & \quad utus - stut, & (7) & \quad utsu - stus, & (8) & \quad usuts - tusu, \\
(9) & \quad ustut - stus, & (10) & \quad ut^2s - stu^2, & (11) & \quad (us)^2 - t - (ts)^2, & (12) & \quad (us)^2 - tu - (ts)^2, \\
(13) & \quad (us)^3 - su^3, & (14) & \quad ustt - su - (ut)^2, & (15) & \quad usut^2 - su^2t, & (16) & \quad tuts - suts, \\
(17) & \quad tuts - sutus, & (18) & \quad tsut - suts, & (19) & \quad tusu - stus, & (20) & \quad tuts - stut, \\
(21) & \quad tuts - stus, & (22) & \quad (ts)^2 - s - (ut)^2, & (23) & \quad (ts)^2 - t - (us)^2, & (24) & \quad (tu)^2 - (su)^2.
\end{align*}
\]

Proof. We need to prove that all compositions among relations (1) – (24) are trivial. We start by listing all compositions among relations. Actually we have the following ambiguities $w$:

\[
\begin{align*}
(1) & \land (1) : w = s^3, & (2) & \land (2) : w = u^3, & (2) & \land (4) : w = u^2st, \\
(2) & \land (5) : w = u^2tst, & (2) & \land (6) : w = u^3tus, & (2) & \land (7) : w = u^2stu, \\
(2) & \land (8) : w = u^2suts, & (2) & \land (9) : w = u^2sstu, & (2) & \land (10) : w = u^2(tu)^2, \\
(2) & \land (11) : w = u^2sysut, & (2) & \land (12) : w = u^2(su)^2t, & (2) & \land (13) : w = u^2(su)^2st, \\
(2) & \land (14) : w = u^2(st)^2s, & (2) & \land (15) : w = u^2(su)^2t, & (3) & \land (3) : w = t^3, \\
(3) & \land (16) : w = t^2sut, & (3) & \land (17) : w = t^2uts, & (3) & \land (18) : w = t^2stus, \\
(3) & \land (19) : w = t^2usu, & (3) & \land (20) : w = t^2ust, & (3) & \land (21) : w = t^2stu, \\
(3) & \land (22) : w = t^2stus, & (3) & \land (23) : w = t^2(st)u, & (3) & \land (24) : w = t^2(ut)^2, \\
(4) & \land (2) : w = ustu^2, & (4) & \land (4) : w = u^3tsut, & (4) & \land (5) : w = (uts)^2t, \\
(4) & \land (6) : w = utstu, & (4) & \land (7) : w = u^3stu, & (4) & \land (8) : w = u^3stu, \\
(4) & \land (9) : w = u^3stus, & (4) & \land (10) : w = u^3uts(ut)u, & (4) & \land (11) : w = u^3uts(ut)^2.
\end{align*}
\]
(4) \land (12) : w = ut(su)\^3t, (4) \land (13) : w = ut(su)\^3s, (4) \land (14) : w = utsu(t)s\^3,
(4) \land (15) : w = ut(su)\^3ut, (4) \land (16) : w = utsu, (4) \land (18) : w = utsus,
(5) \land (3) : w = ust\^2t, (5) \land (16) : w = utsusu, (5) \land (17) : w = utsust,
(5) \land (18) : w = utsusus, (5) \land (19) : w = utsusu, (5) \land (20) : w = utsust,
(5) \land (21) : w = utsusu, (5) \land (21) : w = utsu, (5) \land (22) : w = ut(s)^{2}su,
(5) \land (22) : w = ut(s)^{2}u, (5) \land (23) : w = ut(s)^{3}, (5) \land (23) : w = ut(s)^{2}t,
(5) \land (24) : w = ut(su)^{2}, (6) \land (1) : w = utu^2, (6) \land (7) : w = ustu,
(6) \land (8) : w = utusu, (6) \land (9) : w = utusu, (6) \land (11) : w = ut(us)^{2}t,
(6) \land (12) : w = ut(usu)^{2}ut, (6) \land (13) : w = ut(us)^{3}, (6) \land (14) : w = utus(us),
(6) \land (15) : w = utus(usu)^{2}, (6) \land (19) : w = ususu, (6) \land (20) : w = ustut,
(7) \land (2) : w = ust\^2u, (7) \land (4) : w = ustusu, (7) \land (5) : w = ustust,
(7) \land (6) : w = ustusus, (7) \land (7) : w = (ust)^{2}u, (7) \land (8) : w = ustusus,
(7) \land (9) : w = (ust)^{2}su, (7) \land (10) : w = us(tu)^{2}, (7) \land (11) : w = us(tu)^{2}t,
(7) \land (12) : w = us(tu)^{2}ut, (7) \land (13) : w = us(tu)^{3}, (7) \land (14) : w = us(tu)^{2}st,
(7) \land (15) : w = us(tu)^{2}ut, (7) \land (17) : w = ustus, (7) \land (19) : w = ustus,
(7) \land (20) : w = (ust)^{2}, (7) \land (24) : w = us(tu)^{2}, (8) \land (1) : w = usus\^2,
(8) \land (4) : w = ususu, (8) \land (5) : w = ususu, (8) \land (16) : w = utu\^2,
(8) \land (18) : w = ususu, (8) \land (21) : w = ususu, (8) \land (22) : w = usu(tu)^{2}u,
(8) \land (23) : w = ususu, (9) \land (2) : w = ususu, (9) \land (4) : w = ususu,
(9) \land (5) : w = ususu, (9) \land (6) : w = ususu, (9) \land (7) : w = ususu,
(9) \land (8) : w = ususu, (9) \land (9) : w = ususu, (9) \land (10) : w = ususu,
(9) \land (11) : w = ususu, (9) \land (12) : w = ususu, (9) \land (13) : w = ususu,
(9) \land (14) : w = ususu, (9) \land (15) : w = ususu, (9) \land (16) : w = ususu,
(9) \land (18) : w = ususu, (10) \land (2) : w = (ut)^{2}u^2, (10) \land (4) : w = (ut)^{2}su,
(10) \land (5) : w = (ut)^{2}st, (10) \land (6) : w = (ut)^{3}us, (10) \land (6) : w = (ut)^{2}us,
(10) \land (7) : w = (ut)^{2}ustu, (10) \land (8) : w = (ut)^{2}usu, (10) \land (9) : w = (ut)^{2}usu,
(10) \land (10) : w = (ut)^{2}u, (10) \land (10) : w = (ut)^{3}u, (10) \land (11) : w = (ut)^{2}u^2,
(10) \land (12) : w = (ut)^{2}(us)^{2}ut, (10) \land (13) : w = (ut)^{2}(us)^{3}, (10) \land (14) : w = (ut)^{2}us(tu)^{2},
(10) \land (15) : w = (ut)^{2}usus, (10) \land (17) : w = (ut)^{3}s, (10) \land (19) : w = (ut)^{2}usu,
(10) \land (20) : w = (ut)^{2}us, (10) \land (24) : w = (ut)^{2}, (10) \land (24) : w = (ut)^{2},
(11) \land (3) : w = (us)^{2}t^2, (11) \land (7) : w = (us)^{2}tu, (11) \land (9) : w = (us)^{2}usu,
(11) \land (14) : w = (us)^{2}(ts)^{2}, (11) \land (16) : w = (us)^{2}tsu, (11) \land (17) : w = (us)^{2}usu,
(11) \land (18) : w = (us)^{2}usu, (11) \land (19) : w = (us)^{2}usu, (11) \land (20) : w = (us)^{2}tsu,
(11) \land (21) : w = (us)^{2}usu, (11) \land (22) : w = (us)^{2}(ts)^{2}u, (11) \land (23) : w = (us)^{2}(ts)^{2}u,
(11) \land (24) : w = (us)^{2}(tu)^{2}t, (12) \land (3) : w = (us)^{2}ut^2, (12) \land (4) : w = (us)^{2}utu,
(12) \land (5) : w = (us)^{2}ust, (12) \land (6) : w = (us)^{2}utus, (12) \land (8) : w = (us)^{2}usu,
(12) \land (10) : w = (us)^{2}(ut)^{2}u, (12) \land (15) : w = (us)^{2}(ut)^{2}u, (12) \land (16) : w = (us)^{2}usu,
(12) \land (17) : w = (us)^{2}(ut)^{2}s, (12) \land (18) : w = (us)^{2}utsu, (12) \land (19) : w = (us)^{2}utsu,
(12) \land (20) : w = (us)^{2}ustu, (12) \land (21) : w = (us)^{2}usu, (12) \land (22) : w = (us)^{2}usu,
(12) \land (23) : w = (us)^{2}ust, (12) \land (24) : w = (us)^{2}(ut)^{2}, (13) \land (1) : w = (us)^{2}s,
(13) \land (7) : w = (us)^{2}tu, (13) \land (8) : w = (us)^{2}usu, (13) \land (9) : w = (us)^{2}tsu,
(13) \land (11) : w = (us)^{4}t, (13) \land (11) : w = (us)^{3}t, (13) \land (12) : w = (us)^{4}ut,
(13) \land (13) : w = (us)^{5}, (13) \land (13) : w = (us)^{4}, (13) \land (14) : w = (us)^{3}ts,
(13) \land (15) : w = (us)^{3}(ut)^{2}, (14) \land (1) : w = u(s)^{2}s^2, (14) \land (16) : w = u(s)^{2}sut,
(14) \land (18) : w = u(st)^2 s u_s,  
(14) \land (21) : w = u(st)^3 u,  
(14) \land (22) : w = u(st)^3 s u,  
(14) \land (23) : w = u(st)^4,  
(15) \land (3) : w = u s(u t)^2 t_s,  
(15) \land (4) : w = u s(u t)^2 s u,  
(15) \land (5) : w = u s(u t)^2 s t,  
(15) \land (6) : w = u s(u t)^2 u_s,  
(15) \land (10) : w = u s(u t)^3 u,  
(15) \land (17) : w = u s(u t)^3 s,  
(15) \land (18) : w = u s(u t)^2 s u_s,  
(15) \land (19) : w = u s(u t)^2 u_s u,  
(15) \land (20) : w = u s(u t)^3 u s,  
(15) \land (21) : w = u s(u t)^2 s t u_s,  
(15) \land (22) : w = u s(u t)^2 s t u u,  
(15) \land (23) : w = u s(u t)^2 (s t),  
(16) \land (4) : w = t s u t,  
(16) \land (5) : w = t s u t s,  
(16) \land (6) : w = t s u t,  
(16) \land (10) : w = t s(u t)^2 u,  
(16) \land (16) : w = (t s u t)^2,  
(16) \land (17) : w = t s(u t)^3 s,  
(16) \land (18) : w = t s u s u_s,  
(16) \land (19) : w = t s u s u,  
(16) \land (20) : w = t s u s t,  
(16) \land (21) : w = t s u s t u,  
(16) \land (22) : w = t s u s t u s,  
(16) \land (23) : w = t s u s t u s u,  
(16) \land (24) : w = t s(u t)^3,  
(17) \land (3) : w = t u t s t,  
(17) \land (5) : w = t u t s t,  
(17) \land (10) : w = t u(t)^2 s t,  
(17) \land (17) : w = t u t s u,  
(17) \land (18) : w = t u t s u,  
(17) \land (21) : w = t u t s u s,  
(17) \land (22) : w = t u(t)^2 u,  
(17) \land (23) : w = t u t(s)^2 t,  
(18) \land (1) : w = t s u s^2,  
(18) \land (7) : w = t s u t s,  
(18) \land (8) : w = t s(u t)^2 s,  
(18) \land (9) : w = t s(u t)^2 s t,  
(18) \land (11) : w = t(s u)^2 t,  
(18) \land (12) : w = t(s u)^2 u t,  
(18) \land (13) : w = t(s u)^2 u t,  
(18) \land (14) : w = t(s u)^2 (s t)^2,  
(18) \land (15) : w = t(s u)^3 u t,  
(18) \land (2) : w = t s u t^2,  
(19) \land (4) : w = t u s t u,  
(19) \land (6) : w = t u s t u s,  
(19) \land (7) : w = t u s t u s t,  
(19) \land (8) : w = t u s t u s u,  
(19) \land (9) : w = t u(s) t s u,  
(19) \land (10) : w = t u(s) t s t,  
(19) \land (11) : w = t(u s)^2 t,  
(19) \land (12) : w = t(u s)^2 u t,  
(19) \land (13) : w = t(u s)^4,  
(19) \land (14) : w = t(u s)^2 (u s t)^2,  
(19) \land (15) : w = t(u s)^2 (u s t)^2,  
(19) \land (2) : w = t(u s)^2 u t,  
(20) \land (3) : w = t u s^2 t,  
(20) \land (7) : w = t u s u,  
(20) \land (9) : w = t u s t,  
(20) \land (14) : w = t u(s)^2 s,  
(20) \land (16) : w = t u s t u,  
(20) \land (17) : w = t u s t u s,  
(20) \land (18) : w = t u(s)^2 u,  
(20) \land (19) : w = t u(s)^2 u s,  
(20) \land (20) : w = t u(s)^2 u s t,  
(20) \land (21) : w = t u(s)^2 u s t u,  
(20) \land (22) : w = t u(s)^2 u s t u s,  
(20) \land (23) : w = t u(s)^2 u s t u s u,  
(22) \land (4) : w = (t s)^2 u t s,  
(22) \land (5) : w = (t s)^2 u t s t,  
(22) \land (6) : w = (t s)^2 u t s u,  
(22) \land (7) : w = (t s)^2 u t s t u,  
(22) \land (8) : w = (t s)^2 u t s u s,  
(22) \land (9) : w = (t s)^2 u t s u t,  
(22) \land (10) : w = (t s)^2 (u t)^2 u,  
(22) \land (11) : w = (t s)^2 (u t)^2 u s,  
(22) \land (12) : w = (t s)^2 (u t)^2 u t,  
(22) \land (13) : w = (t s)^2 (u t)^3,  
(22) \land (14) : w = (t s)^2 u s(t s)^2,  
(22) \land (15) : w = (t s)^2 u s(u t)^2,  
(22) \land (16) : w = (t s)^2 u t,  
(22) \land (18) : w = (t s)^2 u s,  
(23) \land (3) : w = (t s)^2 u t,  
(23) \land (17) : w = (t s)^2 u t s,  
(23) \land (18) : w = (t s)^3 u s,  
(23) \land (19) : w = (t s)^2 u t s u,  
(23) \land (20) : w = (t s)^3 u t s,  
(23) \land (21) : w = (t s)^3 u t,  

A.S. Cevik, A.H. Albargi / Filomat 34:2 (2020), 461–467
Corollary 3.3. The word problem for the braid group associated with the congruence classes of complex reflection group $G_{12}$ is solvable.
Remark 3.5. The Gröbner-Shirshov basis is one of the best methods to obtain the solvability of the word problem since it can be adapted to some software based algorithms. Specially it has a good advantage in which the old algorithms cannot be applied to obtain solvable groups, monoids or some other algebraic structures.

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