Mapping of Brans-Dicke theory to chameleon gravity: Phase space analysis and observational constraints

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It is possible to reconstruct chameleon field equations by transformation of Brans-Dicke equations from Jordan frame (JF) to Einstein frame (EF) under conformal metric $g_{\mu\nu} = e^{-2\beta\phi} g_{\mu\nu}$, where $g_{\mu\nu}$ and $g_{\mu\nu}$ are metrics in Einstein and Jordan Frames respectively and $\beta$ is the chameleon- matter coupling parameter which would be related to Brans-Dicke parameter $\omega_{BD}$ by $\beta = (2\omega_{BD} + 3)^{\frac{1}{2}}$. In this paper, we study the phase space analysis of mapping of Brans-Dick theory to chameleon gravity and investigate the mathematical equivalency of the models in Jordan frame (JF) and Einstein frame (EF). The equivalency of the models in two different frames has twofold advantage for our cosmological studies. Firstly, for those features of the chameleon study which focus on observational measurements it is more appropriate to transform chameleon gravity to Brans-Dicke theory in (JF) where experimental data have their usual interpretation. Secondly, for those aspects of the study that focus on dynamical behavior of the Brans-Dicke theory and its stability in (JF), it provides us the possibility to find the corresponding equations in chameleon mechanism in (EF) which are free from singularity and more easier to discussed then transform them to the Brans-Dick theory in (JF).

I. INTRODUCTION

It is quite evident that the universe has undergone a smooth transition from a decelerated phase to its present accelerated phase of expansion [1,2]. Discovering the source of cosmic acceleration is one of the biggest challenges of modern cosmology. This remarkable discovery has led cosmologists to hypothesize the presence of unknown form of energy called dark energy (DE), which is an exotic matter with negative pressure [3]. This surprising finding has now been confirmed by more recent data coming from SNeIa surveys [6, 7, 9,13], large scale structure [15, 19] and cosmic microwave background (CMBR) anisotropy spectrum [20, 26]. All current observations are consistent with a cosmological constant (CC); while this is in some sense the most economical possibility, the CC has its own theoretical and naturalness problems [27–28], so it is worthwhile to consider alternatives. The above observational data properly complete each other and point out that the dark energy (DE) is the dominant component of the present universe which occupies about 73% of the energy of our universe, while dark matter (DM) occupies 23%, and the usual baryonic matter about 4%. There are prominent candidates for DE such as the cosmological constant [30, 31], a dynamically evolving scalar field (like quintessence) [32, 33] or phantom (field with negative energy) [34] that explain the cosmic accelerating expansion. Meanwhile, the accelerating expansion of universe can also be obtained through modified gravity [35], brane cosmology and so on [36–42]. The DE can track the evolution of the background matter in the early stage, and only recently, it has negative pressure, and becomes dominant. Thus, its current condition is nearly independent of the initial conditions [43–47].

On the other hand, to explain the early and late time acceleration of the universe, it is most often the case that such fields interact with matter; directly due to a matter Lagrangian coupling, indirectly through a coupling to the Ricci scalar or as the result of quantum loop corrections [48–53]. If the scalar field self-interactions are negligible, then the experimental bounds on such a field are very strong; requiring it to either couple to matter much more weakly than gravity does, or to be very heavy [54–57]. Unfortunately, such scalar field is usually very light and its coupling to matter should be tuned to extremely to small values in order not to be conflict with the Equivalence Principal [58].

An alternative attempt to overcome the problem with light scalar fields has been suggested in chameleon cosmology [59–61]. In the proposed model, a scalar field couples to matter with gravitational strength, in harmony with general expectations from string theory whilst at the same time remaining very light on cosmological scales. The scalar field which is very light on cosmological scales is permitted to couple to matter much more strongly than gravity does, and yet still satisfies the current experimental and observational constraints. The cosmological value of such a field evolves over Hubble time-scales and could potentially cause the late-time acceleration of our Universe [62–65]. The crucial feature that these models possess are that the mass of the scalar field depends on the local background matter density. While the idea of a density-dependent mass term is not new [66–70], the work presented in [61,62] is novel in that the scalar field can couple directly to matter with gravitational strength. We show that it is possible

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to derive chameleon mechanism by transformation of the structure of the scalar tensor theory of gravity (STTG) in Jordan frame (JF) to Einstein frame (EF) by use of appropriate conformal metric. In particular case, it is possible to reconstruct chameleon field equations by transformation of Brans-Dick from JF to EF under conformal metric $g_\mu^\nu = e^{-2\beta \phi} g_{\mu \nu}$ where $g_\mu^\nu$ and $g_{\mu \nu}$ are metrics in Einstein and Jordan Frames respectively. The Brans-Dicke theory of gravity is one of the most popular modified gravity theory which conducted by Brans and Dicke[71] and was related with some previous work of Jordan and Fierz [72] for developing an alternative to GR. It is widely used to describe a modification of Einstein’s original formulation of General Relativity to bring it into conformity with some form of Mach’s Principle. This theory, has been investigated on various aspects for over 50 years.

II. MAPPING BETWEEN BRANS-DICKE, CHAMELEON FIELD AND GENERAL SCALAR TENSOR THEORY

JFBD theories are included in a more general set of the so-called scalar-tensor theories. These theories allow for a constant $\omega_{BD}$ depending on the scalar field itself and may also include a potential to the scalar field Lagrangian density. Scalar-tensor theories are usually formulated in two different frameworks, the Jordan Frame (JF) and the Einstein Frame (EF). It is easier to work in the EF. This frame has the advantage of diagonalizing the kinetic terms for the spin-0 (the scalar field) and spin-2 (the graviton) degrees of freedom so that the mathematical consistency of the solutions of the theory are more easily discussed. In this case, the scalar field is coupled with matter [68, 69]. We start with the usual Scalar Tensor Theory (STT) action in (JF) [69]

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g} \left( F(\Phi) \left( R - Z(\Phi) g^{\mu \nu} \partial_\mu \Phi \partial_\nu \Phi \right) - 2U(\Phi) \right) + S_m[\psi_m; g_{\mu \nu}] .$$

Here, $G_*$ denotes the bare gravitational coupling constant , $R$ is the scalar curvature of $g_{\mu \nu}$, and $g$ its determinant. The variation of action (1) gives straightforwardly

$$F(\Phi) \left( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \right) = 8\pi G_* T_{\mu \nu}$$

$$+ Z(\Phi) \left( \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu \nu} (\partial_\beta \Phi)^2 \right)$$

$$+ \nabla_\mu \partial_\nu F(\Phi) - g_{\mu \nu} \Box F(\Phi) - g_{\mu \nu} U(\Phi) ,$$

$$2Z(\Phi) \Box \Phi = \frac{dF}{d\Phi} R - \frac{dZ}{d\Phi} (\partial_\beta \Phi)^2 + 2 \frac{dU}{d\Phi} ,$$

$$\nabla_\mu T^\mu_{\nu} = 0 ,$$

In the Brans-Dicke theory $F(\Phi) = \Phi$ and $Z(\Phi) = \omega_{BD}/\Phi$ and $2ZF + 3(dF/d\Phi)^2 = 2\omega_{BD} + 3$. Thus the field equations will be

$$3H^2 = \frac{8\pi G_* \rho}{\Phi} - 3H \frac{\dot{\Phi}}{\Phi} + \frac{\omega_{BD} \dot{\Phi}^2}{2 \Phi^2} + \frac{U(\Phi)}{\Phi} ;$$

$$H = - \frac{4\pi G_*(\rho + P)}{\Phi} + H \frac{\dot{\Phi}}{2\Phi} + \frac{\omega_{BD} \dot{\Phi}^2}{2 \Phi^2} - \frac{\Phi}{2\Phi} ;$$

$$\ddot{\Phi} + 3H \dot{\Phi} = \frac{3\Phi(\dot{H} + 2H^2)}{\omega_{BD}} + \frac{\Phi}{2\Phi} - \frac{\Phi}{\omega_{BD}} \frac{dU(\Phi)}{d\Phi} ;$$

$$\dot{\rho} + 3H (\rho + P) = 0$$

The above equations are written in the so-called Jordan frame (JF). However, it is usually much clearer to analyze the equations and the mathematical consistency of the solutions in the so-called Einstein frame (EF). This is achieved by
conformal transformation of the metric and a redefinition of the scalar field. Let us call $g^*_{\mu\nu}$ and $\varphi$ the new variables, and define

$$g^*_{\mu\nu} \equiv F(\Phi) g_{\mu\nu},$$  \hspace{1cm} (9)

$$\left(\frac{d\varphi}{d\Phi}\right)^2 = \frac{3}{4} \left(\frac{d\ln F(\Phi)}{d\Phi}\right)^2 + \frac{Z(\Phi)}{2F(\Phi)}$$  \hspace{1cm} (10)

$$A(\varphi) \equiv F^{-1/2}(\Phi),$$  \hspace{1cm} (11)

$$2V(\varphi) \equiv U(\Phi) F^{-2}(\Phi).$$  \hspace{1cm} (12)

Action (1) then takes the form

$$S = \frac{1}{4\pi G_s} \int d^4x \sqrt{-g_s} \left( \frac{R^s}{4} - \frac{1}{2} g^*_{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right)$$  \hspace{1cm} (13)

$$+ S_m[\psi_m; A^2(\varphi) g^*_{\mu\nu}],$$

where $g_s$ is the determinant of $g^*_{\mu\nu}$, $g^*_{\mu\nu}$ its inverse, and $R^s$ its scalar curvature. Note that the above action looks like the action of chameleon gravity [49] where originally proposed by [59]. Note that matter is explicitly coupled to the scalar field $\varphi$ through the conformal factor $A^2(\varphi)$. The field equations deriving from action (13) take the simple form

$$R^s_{\mu\nu} - \frac{1}{2} R^s g^*_{\mu\nu} = 8\pi G_s T^s_{\mu\nu} + 2\partial_\mu \varphi \partial_\nu \varphi$$  \hspace{1cm} (14)

$$- g^*_{\mu\nu}(g^s_{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi) - 2V(\varphi) g^*_{\mu\nu},$$

$$\Box \varphi = -4\pi G_s \beta(\varphi) T_s + dV(\varphi)/d\varphi,$$  \hspace{1cm} (15)

$$\nabla^s_{\mu} T^s_{\nu\mu} = \beta(\varphi) T_s \partial_\nu \varphi.$$  \hspace{1cm} (16)

Where, quantities referring to the Einstein frame will always have an asterisk and $\beta(\varphi) \equiv \frac{d\ln A}{d\varphi}$ is the coupling strength of the scalar field to matter sources. Where for Brans-Dicke theory, using equation (10)-(12) and for constant $\omega_{BD}$ it will be as

$$\beta = (2\omega_{BD} + 3)^{-\frac{1}{2}}$$  \hspace{1cm} (17)

This shows that the GR limit ($\omega_{BD} \rightarrow \infty$) corresponds to the vanishing coupling ($\beta \rightarrow 0$). In the absence of the potential $U(\Phi)$ the BD parameter $\omega_{BD}$ is constrained to be $\omega_{BD} > 4 \times 10^{3}$ from solar-system experiments [53]. Hence using equation (17), we can find the local gravity bound for chameleon- matter coupling constant $\beta$ as

$$|\beta| < 3.5 \times 10^{-3},$$  \hspace{1cm} (18)

Also $T_s \equiv g^s_{\mu\nu} T^s_{\mu\nu}$ is the trace of the matter energy-momentum tensor $T^s_{\mu\nu} \equiv (2/\sqrt{-g_s}) \delta S_m/\delta g^*_{\mu\nu}$ in Einstein-frame units. From its definition, one can deduce the relation $T^s_{\mu\nu} = e^{2\beta \varphi} T_{\mu\nu}$ with its Jordan-frame counterpart [69].

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu} = e^{-2\beta \varphi} T^s_{\mu\nu}$$  \hspace{1cm} (19)

Using equations (9), (11) and (19), it can be concluded that $p_s = e^{4\beta \varphi} \rho$ and $P_s = e^{4\beta \varphi} P$. Also from equations (9) and (11), $d\tilde{s}^2 = e^{2\beta \varphi} ds^2$, where $ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2$ and $d\tilde{s}^2 = -dt^2 + a(t_s)^2 d\mathbf{x}^2$ are the JF and EF metrics respectively. This relation indicates that $dt = A(\varphi) dt_s$ and $a(t) = e^{3\beta \varphi} a_s(t_s)$. Thus, the variables in Brans-Dick theory in (EF) can be related to their corresponding in (JF) as

$$H_s = \Phi^{-\frac{1}{2}} (H + \frac{\dot{\Phi}}{2\Phi})$$  \hspace{1cm} (20)

$$\rho_s = \frac{\rho}{\Phi^2}$$  \hspace{1cm} (21)

$$\frac{d\varphi}{dt_s} = -\frac{\dot{\Phi}}{2\Phi^2}$$  \hspace{1cm} (22)
Hence the field equations in (EF) would be

\[ 3H_*^2 = 8\pi G_* \rho_* + \dot{\varphi}^2 + 2V(\varphi) \]  
\[ \dot{H}_* = -4\pi G_* (\rho_* + P_*) - \dot{\varphi}^2 \]  
\[ \ddot{\varphi} + 3H_* \dot{\varphi} + \frac{dV(\varphi)}{d\varphi} = -4\pi G_* \beta (\rho_* - 3P_*) \]

(23) (24) (25)

Here, dot denotes derivative respect to \( t_* \). Note that the field equations (23) to (25) are similar to those obtained for chameleon gravity\(^{49}\). Here, dot denotes derivative respect to \( t_* \). Note that the field equations (23) to (25) are similar to those obtained for chameleon gravity\(^{49}\). We also can derive the equations by replacing the physical time \( t \) with the conformal time \( \eta \). Hence, equation (31) indicates that considering positive values of coupling constant \( \beta \) the power law potential with \( m > 2 \) in Brans-Dick theory in (EF) would be mapped to a runaway potential in chameleon mechanism as

\[ V(\varphi) \equiv \frac{1}{2} U(\Phi) F^{-2}(\Phi) = V_0 e^{-2\beta(m-2)\varphi}. \]

(31)

The potential \( V(\varphi) \) plays important role in chameleon theory. It should be a monotonically decreasing function of \( \varphi \). Hence, equation (31) indicates that considering positive values of coupling constant \( \beta \) the power law potential with \( m > 2 \) in Brans-Dick would be transformed to a runaway potential in chameleon gravity. By considering \( 8\pi G = 1 \) the effective potential will be

\[ V_{\text{eff}}(\varphi) \equiv \rho_* + 2V(\varphi) = \rho e^{4\beta \varphi} + V_0 e^{-2\beta(m-2)\varphi} \]

(32)

\( V_{\text{eff}}(\varphi) \) has a minimum at

\[ \varphi_{\text{min}} = \frac{1}{2\beta m} \ln \left( \frac{V_0(m-2)}{2\rho} \right) \]

(33)

III. STABILITY ANALYSIS OF BRANS-DICK THEORY AND CHAMELEON GRAVITY IN (EF)

One of the advantages of mathematical consistency of the equations of models in two deferent frames is that it provide us this possibility to study the dynamical stability of the models in EF which more easier for discussed. In this section we are going to investigate the stability of Brans-Dick theory or its equivalent model (chameleon gravity) in EF. We consider the power low potential \( U(\Phi) = U_0 \Phi^m \) in (JF) which would be mapped to the exponential potentials \( V = V_0 e^{\alpha \varphi} \) in (EF) where \( \alpha = 2\beta(2-m) \). The system of equations (27) to (30) can be transformed to an autonomous system of differential equations by means of the transformations

\[ \Omega_1^2 = \frac{\dot{\rho}_*}{3H_*^2}, \quad \Omega_2^2 = \frac{\dot{\varphi}^2}{3H_*^2}, \quad \Omega_3^2 = \frac{2V(\varphi)\rho_*^2}{3H_*^2} \]

(34)
Equation (27) gives the following constraint between the variables
\[ \Omega_3^2 = 1 - 2\Omega_1^2 - \Omega_2^2 \] (35)

Now, for the autonomous equations of motions, we obtain
\[
\frac{d\Omega_1}{dN_*} = -\frac{1}{2} \left( 1 + 3c_*^2 \right) \Omega_1 + \frac{1}{2} \sqrt{3}\beta \left( 1 - 3c_*^2 \right) \Omega_1 \Omega_2
- \Omega_1 \left( 1 - 3\Omega_2^2 - 3 \left( 1 + 3c_*^2 \right) \Omega_1^2 \right)
\] (36)
\[
\frac{d\Omega_2}{dN_*} = -3\Omega_2 - \sqrt{3}\beta \left( 1 - 3c_*^2 \right) \Omega_1^2 + 3\Omega_2^3 + 3\Omega_2 \Omega_1^2
+ 3c_*^2 \Omega_2 \Omega_1^2 - \frac{\alpha}{2} \left( 1 - 2\Omega_1^2 - \Omega_2^2 \right)
\] (37)

Where \( N_* = \ln a_* \). In order to investigate the evolution of the universe, we need the essential parameter, \( \frac{\mathcal{H}'}{\mathcal{H}_0^2} \). In term of the new variables it would be
\[ \frac{\mathcal{H}'}{\mathcal{H}_0^2} = 1 - 3\left( 1 + c_*^2 \right) \Omega_1^2 - 3\Omega_2^2 \] (38)

Where, one can obtain the deceleration parameter, \( q_* \), in (EF) as,
\[ q_* = -\left( 1 + \frac{\mathcal{H}'}{\mathcal{H}_0^2} \right) = -\frac{\mathcal{H}'}{\mathcal{H}_0^2} = -1 + 3\left( 1 + c_*^2 \right) \Omega_1^2 + 3\Omega_2^2 \] (39)

In the following discussions, we use the Jacobin stability of a dynamical system as the robustness of the system to small perturbations of the whole trajectory. Jacobin stability analysis offers a powerful and simple method for constraining the physical properties of different systems, described by second order differential equations\[80\]. It is especially important in oscillatory systems where the phase paths can spiral in towards zero, spiral out towards infinity, or reach neutrally stable situations called centers. The eigenvalues of jacobian matrix can be used to determine the stability of periodic orbits, or limit cycles and predict if the system oscillates near the critical point. In cosmology where there is the problem of initial conditions, phase space analysis gives us the possibility of studying all of the evolution paths admissible for all initial conditions [81]-[84]. It is useful in visualizing the behavior of the system.

In previous section the critical points of the system have been obtained in term of important parameters (\( \beta, \alpha \)). The nature of these points can be determined by the corresponding eigenvalues. Here the eigenvalues of the system are as follows

\[
Ev_1 = \begin{bmatrix}
-\frac{3}{2} - \frac{3}{2}c_*^2 + \frac{3}{4} + \frac{3}{2}c_*^2 \beta \alpha - \frac{3}{4}c_*^2 \beta \alpha \\
\end{bmatrix}
\]
investigate the properties of each of the fixed points for the barotropic equation of state from it along some other eigenvectors. The behavior of the system near a critical point is spiral if and only if its called saddle points, and those trajectories which approach to a saddle fixed point along some eigenvectors may recede values. This fixed point is called unstable point. The fixed points with both positive and negative eigenvalues are This fixed point is called stable point, also the trajectories recede from a fixed point if all eigenvalues have positive

Generally speaking, the trajectories of the phase space approach to a fixed point if all eigenvalues get negative values. This fixed point is called stable point, also the trajectories recede from a fixed point if all eigenvalues have positive values. This fixed point is called unstable point. The fixed points with both positive and negative eigenvalues are called saddle points, and those trajectories which approach to a saddle fixed point along some eigenvectors may recede from it along some other eigenvectors. The behavior of the system near a critical point is spiral if and only if its eigenvalue be complex as \( \lambda_{1,2} = \lambda_r \pm i\lambda_i \). Because of reality of parameters \( \beta \) and \( \alpha \), it is obvious that only the eigenvalues \( E_{V6} \) and \( E_{V7} \) can be complex. Thus we can expect the spiral behavior near the points \( p_6 \) and \( p_7 \). We investigate the properties of each of the fixed points for the barotropic equation of state \( c_s^2 = 0 \), i.e., dust.

\[ \textbf{A: Critical point} \ \textbf{P}_1 (\Omega_1 = 0, \Omega_2 = -\frac{\alpha \sqrt{3}}{6}). \] This critical point corresponds to a solution where the constraint Eqs. (35) and (27) is dominated by potential-kinetic-scaling solution. This solution exists for all potentials and only depends on slope of potential \( \alpha \). This scaling solution has two eigenvalues which depend on the slope of potential \( \alpha \) and coupling constant \( \beta \).

\[ E_{V1} = \begin{bmatrix} \frac{-3 + \frac{4}{9} \beta}{2} \end{bmatrix} \]

The eigenvalue shows that the critical point is stable under the condition

\[ \textbf{CI} : \begin{cases} \beta < -\frac{6 + \alpha^2}{\alpha}, & -2\sqrt{3} < \alpha < 0 \\ \beta > -\frac{6 + \alpha^2}{\alpha}, & 2\sqrt{3} > \alpha > 0 \end{cases} \]

Fig.1 shows the behavior of the dynamical system in the \( \Omega_1, \Omega_2 \) phase plane for \( \beta = 1 \) and \( \alpha = 1 \). As can bee seen under the condition \( \text{CI} \), \( p_1 \) is stable, \( p_2 \) and \( p_3 \) are unstable points, \( p_4 \) and \( p_5 \) are saddle points and \( p_6 \) and
Fig.1. The behavior of the dynamical system in the $\Omega_1, \Omega_2$ phase plane for $\beta = 1$ and $\alpha = 1$. As can be seen $p_1$ is stable, $p_2$ and $p_3$ are unstable points, $p_4$ and $p_5$ are saddle points and $p_6$ and $p_7$ don’t exist.

Fig.2. The behavior of the dynamical system in the $\Omega_1, \Omega_2$ phase plane for $\beta = -2$ and $\alpha = -5$. As can be seen $p_1$ and $p_3$ are unstable, $p_2$ is stable, $p_4, p_5, p_6$ and $p_7$ are saddle points.

$p_7$ don’t exist. The non complexity of the eigenvalues implies that the the system has no spiral behavior near this critical point.

**B: Critical point** $P_2(\Omega_1 = 0, \Omega_2 = 1)$, corresponds to a *kinetic-scaling solution*. This solution exists for all potentials and is independent of slope of potential $\alpha$ and coupling constant $\beta$. This scaling solution has two eigenvalues which depend on the slope of potential $\alpha$ and coupling constant $\beta$.

$$E_{v_2} = \left[ \begin{array}{c} 6 + \alpha \sqrt{3} \\ \frac{3}{2} + \frac{1}{2} \beta \sqrt{3} \end{array} \right]$$ (41)

The eigenvalues show that the critical point is stable for

**CII:** ($\beta < -\sqrt{3}, \alpha < -2\sqrt{3}$)

Fig.2 shows the behavior of the dynamical system in the $\Omega_1, \Omega_2$ phase plane for $\beta = -2$ and $\alpha = -5$. As can be seen $p_1$ and $p_3$ are unstable, $p_2$ is stable, $p_4, p_5, p_6$ and $p_7$ are saddle points.

**C: Critical point** $P_3(\Omega_1 = 0, \Omega_2 = -1)$, corresponds to a *kinetic-scaling solution*. This solution exists for all potentials and is independent of slope of potential $\alpha$ and coupling constant $\beta$ however its eigenvalues are depend on slope of potential $\alpha$ and coupling constant $\beta$.

$$E_{v_3} = \left[ \begin{array}{c} 6 - \alpha \sqrt{3} \\ \frac{3}{2} - \frac{1}{2} \beta \sqrt{3} \end{array} \right]$$ (42)

The eigenvalues show that the critical point is stable for

**CIII:** ($\beta > \sqrt{3}, \alpha > 2\sqrt{3}$)
Fig.3. The behavior of the dynamical system in the $\Omega_1, \Omega_2$ phase plane for $\beta = 2$ and $\alpha = 5$. As can be seen $p_1$ and $p_2$ are unstable, $p_3$ is stable, $p_4, p_5$ don’t exist and $p_6$ and $p_7$ are saddle points.

Fig.4. The behavior of the dynamical system in the $\Omega_1, \Omega_2$ phase plane for $\beta = -1$ and $\alpha = -6$. As can be seen $p_1$ and $p_3$ are unstable, $p_2, p_6$ and $p_7$ are saddle points and $p_4$ and $p_5$ are stable points.

Fig.3 shows the behavior of the dynamical system in the $\Omega_1, \Omega_2$ phase plane for $\beta = 2$ and $\alpha = 5$. As can be seen $p_1$ and $p_2$ are unstable, $p_3$ is stable, $p_4, p_5$ don’t exist and $p_6$ and $p_7$ are saddle points.

**D:** Critical points $P_4, P_5(\Omega_1 = \pm \sqrt{\frac{1}{\beta^2}}, \Omega_2 = -\sqrt{\frac{3}{\beta^2}})$. These critical points are mirror images of each other. These solution exists for $\beta^2 < 3$ and all potentials. The solution has two eigenvalues which depend on slope of potential $\alpha$ and coupling constant $\beta$.

$$E\nu_{4,5} = \begin{bmatrix} -\frac{\beta^2}{\beta} + \frac{\beta}{3} \\ 3 + \beta^2 - \beta \alpha \end{bmatrix}$$ (43)

The eigenvalues show that the critical point is stable for

CIV: $\begin{cases} \alpha < \frac{1 + \beta^2}{\beta}, -\sqrt{3} < \beta < 0 \\ \alpha > \frac{1 + \beta^2}{\beta}, \sqrt{3} > \beta > 0 \end{cases}$

Fig.4 shows the behavior of the dynamical system in the $\Omega_1, \Omega_2$ phase plane for $\beta = -1$ and $\alpha = -6$. As can be seen $p_1$ and $p_3$ are unstable, $p_2, p_6$ and $p_7$ are saddle points and $p_4$ and $p_5$ are stable points.

**E:** Critical point $P_6, P_7(\Omega_1 = \pm \frac{1}{2} \sqrt{-\frac{12 + 2\alpha^2 - 2\beta \alpha}{-\beta + \alpha}}, \Omega_2 = \frac{\sqrt{3}}{\beta - \alpha})$.

These critical points are mirror images of each other. The solution exists for

$$\begin{cases} \beta < -\frac{\alpha^2}{\alpha}, \alpha > 0 \\ \beta > -\frac{\alpha^2}{\alpha}, \alpha < 0 \end{cases}$$

The solution has two eigenvalues which depend on slope of potential $\alpha$ and coupling constant $\beta$. 
Fig. 5. The behavior of the dynamical system in the $\Omega_1, \Omega_2$ phase plane for $\beta = 6$ and $\alpha = -5$. As can be seen $p_1$ is unstable, $p_2$ and $p_3$ are saddle points, $p_4$ and $p_5$ do not exist, and $p_6$ and $p_7$ are stable focus.

Fig. 6. The region of stability for different critical points

\begin{equation}
E_{0,7} : \begin{cases}
\beta < \frac{\alpha^2 - 6}{\alpha} \\
-2\sqrt{3} < \alpha < 0
\end{cases}
\end{equation}

Fig. 5. shows the behavior of the dynamical system in the $\Omega_1, \Omega_2$ phase plane for $\beta = 6$ and $\alpha = -5$. As can be seen $p_1$ is unstable, $p_2$ and $p_3$ are saddle points, $p_4$ and $p_5$ do not exist, and $p_6$ and $p_7$ are stable focus.

IV. MAPPING OF STABILITY ANALYSIS TO (JF)

In this section using same procedure in (EF), the field equations (5) to (8) can be transformed to an autonomous system of differential equations by introducing the following dimensionless variables,

\begin{equation}
\Gamma_1^2 = \frac{4\pi G_\ast \rho}{3H^2 \Phi}, \Gamma_2 = \frac{\Phi}{3\Phi H}, \Gamma_3^3 = \frac{U(\Phi)}{3\Phi H^2}
\end{equation}

However, equations (20) to (22) are more complicated than equations (27) to (30). Hence, in order to derive the autonomous differential equations in (JF), it is more appropriate to implement equations (20) to (22), to make relation between

\begin{equation}
\begin{bmatrix}
-6\alpha + 3\alpha + \sqrt{180\beta^2 - 108\alpha - 63\alpha^2 - 48\beta^2 \alpha^2 + 24\beta \alpha + 24\beta \alpha^2 + 432} \\
-6\alpha - \sqrt{180\beta^2 - 108\alpha - 63\alpha^2 - 48\beta^2 \alpha^2 + 24\beta \alpha + 24\beta \alpha^2 + 432}
\end{bmatrix}
\end{equation}
TABLE II: critical points in Jordan Frame

| Points | $\Gamma_1$ | $\Gamma_2$ |
|--------|-----------|-----------|
| $P_1$  | 0         | $-\frac{2}{3} \alpha \beta + \frac{2}{3} \alpha^2$ |
| $P_2$  | 0         | $-\frac{2}{3} \alpha \beta + \frac{2}{3} \alpha^2$ |
| $P_3$  | 0         | $-\frac{2}{3} \alpha \beta + \frac{2}{3} \alpha^2$ |
| $P_4$  | $\sqrt{18-6 \beta^2}$ | $-\frac{2}{3} \beta^2$ |
| $P_5$  | $\sqrt{18-6 \beta^2}$ | $-\frac{2}{3} \beta^2$ |
| $P_6$  | $\sqrt{-2(\alpha + 2 \alpha^2 - 12)}$ | $-\frac{2}{3} \beta^2$ |
| $P_7$  | $\sqrt{-2(\alpha + 2 \alpha^2 - 12)}$ | $-\frac{2}{3} \beta^2$ |

the new variables (45) in (JF) and variables 34 in (EF) as

$$
\Gamma_2 = \frac{2 \beta \Omega_2}{\sqrt{3 - 3 \beta \Omega_2}} = \frac{2 \Omega_2}{\sqrt{3(2 \omega_{BD} + 3)^{\frac{1}{2}} - 3 \Omega_2}} \quad (46)
$$

$$
\Gamma_1 = \frac{\sqrt{3} \Omega_1}{\sqrt{3 - 3 \beta \Omega_2}} = \frac{\sqrt{3} \Omega_1(2 \omega_{BD} + 3)^{\frac{1}{2}}}{\sqrt{3(2 + 3)^{\frac{1}{2}} - 3 \Omega_2}} \quad (47)
$$

$$
\Gamma_3 = \frac{\sqrt{3} \Omega_3}{\sqrt{3 - 3 \beta \Omega_2}} = \frac{\sqrt{3} \Omega_3(2 \omega_{BD} + 3)^{\frac{1}{2}}}{\sqrt{3(2 \omega_{BD} + 3)^{\frac{1}{2}} - 3 \Omega_2}} \quad (48)
$$

Note that the equations (46) to (48) confirm that

$$
2 \Gamma_1^2 - 3 \Gamma_2 + \frac{3 \omega_{BD}}{2} \Gamma_2^2 + \Gamma_3^2 = 1 \quad (49)
$$

Which can be derived from equation (50) directly. Also

$$
\frac{d N_*}{d N} = \frac{\mathcal{H}_*}{\mathcal{H}} = \frac{2 + 3 \Gamma_2}{2} \quad (50)
$$

Now, for the autonomous equations of motions in (JF), we obtain

$$
\frac{d \Gamma_i}{d N} = \frac{d \Gamma_i}{d N_*} \frac{d N_*}{d N} = \frac{2 + 3 \Gamma_2}{2} \frac{d \Gamma_i}{d N_*} \quad (51)
$$

Hence using equation (51) and equations (46) to (48), the autonomous equations of motions in (JF) can be related to the corresponding equations in (EF) as

$$
\frac{d \Gamma_1}{d N} = \frac{3(2 + 3 \Gamma_2)}{2(\sqrt{3 - 3 \beta \Omega_2})^2} \left( \frac{d \Omega_1}{d N_*} - \sqrt{3} \beta (\Omega_2 \frac{d \Omega_1}{d N_*} - \Omega_1 \frac{d \Omega_2}{d N_*}) \right) \quad (52)
$$

$$
\frac{d \Gamma_2}{d N} = \frac{2 + 3 \Gamma_2}{2} \frac{2 \sqrt{3} \beta \Omega_2}{(\sqrt{3 - 3 \beta \Omega_2})^2} \frac{d \Omega_2}{d N_*} \quad (53)
$$

$$
\frac{d \Gamma_3}{d N} = \frac{3(2 + 3 \Gamma_2)}{2(\sqrt{3 - 3 \beta \Omega_2})^2} \left( \frac{d \Omega_3}{d N_*} - \sqrt{3} \beta (\Omega_2 \frac{d \Omega_3}{d N_*} - \Omega_3 \frac{d \Omega_2}{d N_*}) \right) \quad (54)
$$

Equations (52) to (54) indicate that when \( \frac{d \Omega_1}{d N_*} = \frac{d \Omega_2}{d N_*} = \frac{d \Omega_3}{d N_*} = 0 \) then their corresponding in (JF) would also be zero \( \frac{d \Gamma_1}{d N} = \frac{d \Gamma_2}{d N} = \frac{d \Gamma_3}{d N} = 0 \). This implies that critical points of dynamical system in (EF) would be mapped to their corresponding in (JF) by transformation relations (46) to (48).

Here the eigenvalues of the system are as follows

$$
E v_1 = \begin{bmatrix}
-\frac{2}{3} \alpha \beta + \frac{2}{3} \alpha^2 + 6 \\
\frac{2}{3} \alpha \beta + 2 \\
\frac{1}{2} \beta \alpha + 2
\end{bmatrix}
$$
Ev$_2$ = \[
\begin{bmatrix}
\frac{3}{2} \sqrt{3} - \beta \\
\frac{3}{2} \sqrt{3} + \sqrt{3} \\
\frac{3}{2} \sqrt{3} + \sqrt{3}
\end{bmatrix}
\]

Ev$_3$ = \[
\begin{bmatrix}
\frac{3}{2} \sqrt{3} - 17\beta - 12\alpha \beta^2 \\
(3\beta + \sqrt{3}) \\
(3\beta + \sqrt{3})
\end{bmatrix}
\]

Ev$_4$ = \[
\begin{bmatrix}
\frac{1}{2} \beta^2 - 3 \\
\frac{1}{2} \beta^2 - 3 \\
(\beta^2 + 1)
\end{bmatrix}
\]

Ev$_5$ = \[
\begin{bmatrix}
\frac{1}{2} \beta^2 - 3 \\
\frac{1}{2} \beta^2 - 3 \\
(\beta^2 + 1)
\end{bmatrix}
\]

Ev$_{6,7}$ = \[
\begin{bmatrix}
\frac{6\beta - 3\alpha + \sqrt{180}\beta^2 - 108\beta \alpha - 63\alpha^2 - 48\beta^2 \alpha^2 + 24\beta^3 \alpha + 24\beta^3 \alpha^3 + 432}{4(3\beta^2 + \alpha)} \\
\frac{6\beta - 3\alpha - \sqrt{180}\beta^2 - 108\beta \alpha - 63\alpha^2 - 48\beta^2 \alpha^2 + 24\beta^3 \alpha + 24\beta^3 \alpha^3 + 432}{4(3\beta^2 + \alpha)}
\end{bmatrix}
\]

As can be seen, the eigenvalues in (JF) are different from those obtained in (EF). This implies that while the critical points in (EF) will be mapped to their corresponding in (JF), however the nature of the critical points may be changed under the transformation. In Fig.7 the behavior of dynamical system in phase space have been shown in (EF) and its map in (JF) for the same values of ($\alpha$, $\beta$). As can be seen, the critical point $p_3$ is unstable in (EF) while its corresponding is stable in (JF). It is also interesting to note that dynamic of the deceleration parameters is different in two frames. From equation (26), $\dot{H} = \dot{\beta} \varphi''$, Hence, the deceleration parameter in (JF) can be derived as

\[ q = \frac{\dot{H}}{H^2} = \frac{\dot{H} + \beta \varphi''}{(\dot{H} + \beta \varphi')^2} = \frac{q_* - \beta \frac{\ddot{\varphi}}{\varphi}^2}{(1 + \beta \frac{\ddot{\varphi}}{\varphi})^2} \]  

(55)

Where using equations (29) and (31), it will be simplified as,

\[ q = \frac{q_* + 2\sqrt{3}\beta \Omega_2 + \frac{\alpha \beta}{2} \Omega_3^2 + 3\beta^2(1 - 3e^2)\Omega_1^2}{(1 + \sqrt{3}\beta e^2 \Omega_2)^2} \]  

(56)

This is an important point to remember: Although we are looking for cosmological FRW backgrounds whose expansion is accelerating, however, equations (55) and (30) indicate that, acceleration universe in (JF) may be correspond to deceleration universe in (EF). For example, vanishing potential in (EF) implies that $q_0 > 0$, while the deceleration parameter $q$ in (JF) may be negative (This can be proved from equations (33), (35) and (30)). As another straightforward example, at critical point $P_2$ in (EF) with ($\Omega_1 = 0$, $\Omega_2 = 1$, $\Omega_3 = 0$), the deceleration parameter in (EF) is $q_* = 2$, while from equation (56), at this critical point $q = \frac{2+2\sqrt{3}\beta}{(1+\sqrt{3}\beta e^2 \Omega_2)^2}$. This indicates that for $\beta < -\frac{\sqrt{3}}{3}$, the deceleration parameter $q < 0$.

V. CONCLUSION

In this paper we have reconstructed the chameleon gravity from transformation of of Brans-Dick from (JF) to (EF) under conformal metric $\gamma_{\mu\nu} = e^{-2\beta \varphi} g_{\mu\nu}$. Under this conformal transformation the monotonically decreasing potential of scalar field $\varphi$ which is essential for chameleon gravity can be achieved from power law potential $U = \Phi^m$ with $m > 2$ in Brans-Dick theory. The mathematical equivalency of the models in two different frames has twofold advantages for our cosmological studies. Firstly, for those features of the chameleon study which focus on observational measurements it is more appropriate to use the corresponding Brans-Dick theory in (JF) where experimental data have their usual interpretation. For example, the consistency between the two theory provides the possibility to derive confidence regions for the value of chameleon-matter coupling constant $\beta$ (which is still controversial) from
corresponding coupling constant \( \omega_{BD} \) which severely has been constrained by some observations in (JF) Brans-Dicke theory. Solar System data put very strong constraints on the \( \omega_{BD} \) parameter. The measurement of the Parameterized Post-Newtonian parameter \( \gamma \) (see [73],[74]) from the Cassini mission gives \( \omega_{BD} > 40000 \) at the 2\( \sigma \) confidence level [74],[55]. This enable us to find the confidence region for chameleon- matter coupling parameter as \( |\beta| < 5 \times 10^{-3} \) in solar system. On cosmological scales, a wide range of values \( \omega_{BD} > \{50,2000\} \) have been reported in different studies [75]-[78] which determine different confidence region for parameter \( \beta \) in cosmological scale. An improvement of pervious studies has been done by [79] using Cosmic Microwave Background data from Planck. They implemented two types of models. First, the initial condition of the scalar field is fixed to give the same effective gravitational strength today as the one measured on the Earth. In this case they find that \( \omega_{BD} > 692 \) at the 99 confidence level. In the second type by considering that the initial condition for the scalar is a free parameter they find \( \omega_{BD} > 890 \) at the same confidence level. These confidence regions for \( \omega_{BD} \) put new constraints on parameter \( \beta \) as \( \beta < 0.023 \) and \( \beta < 0.026 \) in cosmological scale.

The other advantage of consistency between Brans-Dicke theory and chameleon gravity is that for those aspects of the study that focus on investigation of dynamical behavior of the Brans-Dicke theory and its stability, for those aspects of the study that focus on dynamical behavior of the Brans-Dicke theory and its stability in (JF), it provides us the possibility to find the corresponding equations in chameleon mechanism in (EF) which are free from singularity and more easier to discussed then transform them to the Brans-Dick theory in (JF). There are also important points that we found in this analysis;

I) While the critical points of dynamical system in (EF) would be mapped to their corresponding in (JF) by using appropriate transformation (A critical point in (EF) is mapped to a critical point in (JF) under the conformal metric \( g_{\mu\nu} = e^{-2\beta\phi} g_{\mu\nu} \) ) the nature (stability) of the critical points may be changed and a stable point in one frame may be mapped to an unstable point in the other frame.

II) The deceleration parameter have different dynamic in two frames where a positive deceleration universe in (EF) may be correspond to an acceleration universe in (JF) and vise versa.

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