ON COMPLEX SURFACES DIFFEOMORPHIC TO RATIONAL SURFACES

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Introduction

The goal of this paper is to prove the following:

Theorem 0.1. Let $X$ be a complex surface of general type. Then $X$ is not diffeomorphic to a rational surface.

Using the results from [13], we obtain the following corollary, which settles a problem raised by Severi:

Corollary 0.2. If $X$ is a complex surface diffeomorphic to a rational surface, then $X$ is a rational surface. Thus, up to deformation equivalence, there is a unique complex structure on the smooth 4-manifolds $S^2 \times S^2$ and $\mathbb{P}^2 \# n\overline{\mathbb{P}^2}$.

In addition, as discussed in the book [15], Theorem 0.1 is the last step in the proof of the following, which was conjectured by Van de Ven [37] (see also [14,15]):

Corollary 0.3. If $X_1$ and $X_2$ are two diffeomorphic complex surfaces, then

$$\kappa(X_1) = \kappa(X_2),$$

where $\kappa(X_i)$ denotes the Kodaira dimension of $X_i$.

The first major step in proving that every complex surface diffeomorphic to a rational surface is rational was Yau’s theorem [40] that every complex surface of the same homotopy type as $\mathbb{P}^2$ is biholomorphic to $\mathbb{P}^2$. After this case, however, the problem takes on a different character: there do exist nonrational complex surfaces with the same oriented homotopy type as rational surfaces, and the issue is to show that they are not in fact diffeomorphic to rational surfaces. The only known techniques for dealing with this question involve gauge theory and date back to Donaldson’s seminal paper [9] on the failure of the $h$-cobordism theorem in dimension 4. In this paper, Donaldson introduced analogues of polynomial invariants for 4-manifolds $M$ with $b_2^+(M) = 1$ and special $SU(2)$-bundles. These invariants depend in an explicit way on a chamber structure in the positive cone in $H^2(M;\mathbb{R})$. Using these invariants, he showed that a certain elliptic surface (the Dolgachev surface with multiple fibers of multiplicities 2 and 3) was not diffeomorphic to a rational surface. In [13], this result was generalized to cover all Dolgachev surfaces and their blowups (the case of minimal Dolgachev surfaces was also treated in [28]).
and Donaldson's methods were also used to study self-diffeomorphisms of rational surfaces. The only remaining complex surfaces which are homotopy equivalent (and thus homeomorphic) to rational surfaces are then of general type, and a single example of such surfaces, the Barlow surface, is known to exist [2]. In 1989, Kotschick [18], as well as Okonek-Van de Ven [29], using Donaldson polynomials associated to $SO(3)$-bundles, showed that the Barlow surface was not diffeomorphic to a rational surface. Subsequently Pidstrigach [30] showed that no complex surface of general type which has the same homotopy type as the Barlow surface was diffeomorphic to a rational surface, and Kotschick [20] has outlined an approach to showing that no blowup of such a surface is diffeomorphic to a rational surface. All of these approaches use $SO(3)$-invariants or $SU(2)$-invariants for small values of the (absolute value of) the first Pontrjagin class $p_1$ of the $SO(3)$-bundle, so that the dependence on chamber structure can be controlled in a quite explicit way.

In [33], the second author showed that no surface $X$ of general type could be diffeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or to $F_1$, the blowup of $\mathbb{P}^2$ at one point. Here the main tool is the study of $SO(3)$-invariants for large values of $-p_1$, as defined and analyzed in [19] and [21]. These invariants also depend on a chamber structure, in a rather complicated and not very explicitly described fashion. In [34], these methods are used to analyze minimal surfaces $X$ of general type under certain assumptions concerning the nonexistence of rational curves, which are always satisfied if $X$ has the same homotopy type as $\mathbb{P}^1 \times \mathbb{P}^1$ or $F_1$, by a theorem of Miyaoka on the number of rational curves of negative self-intersection on a minimal surface of general type. The main idea of the proof is to show the following: Let $X$ be a minimal surface of general type, and suppose that $\{E_0, \ldots, E_n\}$ is an orthogonal basis for $H^2(X; \mathbb{Z})$ with $E_0^2 = 1$, $E_i^2 = -1$ for $i \geq 1$, and $[K_X] = 3E_0 - \sum_{i \geq 1} E_i$. Finally suppose that the divisor $E_0 - E_i$ is nef for some $i \geq 1$. Then the class $E_0 - E_i$ cannot be represented by a smoothly embedded 2-sphere. (Actually, in [34], the proof shows that an appropriate Donaldson polynomial is not zero whereas it must be zero if $X$ is diffeomorphic to a rational surface. However, using [26], one can also show that if $E_0 - E_i$ is represented by a smoothly embedded 2-sphere, then the Donaldson polynomial is zero.) At the same time, building on ideas of Donaldson, Pidstrigach and Tyurin [31], using Spin polynomial invariants, showed that no minimal surface of general type is diffeomorphic to a rational surface.

We now discuss the contents of this paper and the general strategy for the proof of Theorem 0.1. The bulk of this paper is devoted to giving a new proof of the results of Pidstrigach and Tyurin concerning minimal surfaces $X$. Here our methods apply as well to minimal simply connected algebraic surfaces of general type with $p_g$ arbitrary. Instead of looking at embedded 2-spheres of self-intersection 0 as in [34], we consider those of self-intersection $-1$. We show in fact the following in Theorem 1.10 (which includes a generalization for blowups):

**Theorem 0.4.** Let $X$ be a minimal simply connected algebraic surface of general type, and let $E \in H^2(X; \mathbb{Z})$ be a class satisfying $E^2 = -1$, $E \cdot [K_X] = 1$. Then the class $E$ cannot be represented by a smoothly embedded 2-sphere.

In particular, if $p_g(X) = 0$, then $X$ cannot be diffeomorphic to a rational surface. The method of proof of Theorem 0.4 is to show that a certain value of a Donaldson polynomial invariant for $X$ is nonzero (Theorem 1.5), while it is a result of Kotschick that if the class $E$ is represented by a smoothly embedded 2-sphere, then the value of the Donaldson polynomial must be zero (Proposition 1.1). In case $p_g(X) = 0$,
once we have have found a polynomial invariant which distinguishes $X$ from a rational surface, it follows in a straightforward way from the characterization of self-diffeomorphisms of rational surfaces given in [13] that no blowup of $X$ can be diffeomorphic to a rational surface either (see Theorem 1.7). This part of the argument could also be used with the result of Pidstrigach and Tyurin to give a proof of Theorem 0.1.

Let us now discuss how to show that certain Donaldson polynomials do not vanish on certain classes. The prototype for such results is the nonvanishing theorem of Donaldson [10]: if $S$ is an algebraic surface with $p_g(S) > 0$ and $H$ is an ample line bundle on $S$, then for all choices of $w$ and all $p \ll 0$, the $SO(3)$-invariant $\gamma_{w,p}(H, \ldots, H) \neq 0$. We give a generalization of this result in Theorem 1.4 to certain cases where $H$ is no longer ample, but satisfies: $H^k$ has no base points for $k \gg 0$ and defines a birational morphism from $X$ to a normal surface $\bar{X}$, and where $p_g(X)$ is also allowed to be zero (for an appropriate choice of chamber). Here we must assume that there is no exceptional curve $C$ such that $H \cdot C = 0$, as well as the following additional assumption concerning the singularities of $\bar{X}$: they should be rational or minimally elliptic in the terminology of [22]. The proof of Theorem 1.4 is a straightforward generalization of Donaldson’s original proof, together with methods developed by J. Li in [23, 24].

Given the generalized nonvanishing theorem, the problem becomes one of constructing divisors $M$ such that $M$ is orthogonal to a class $E$ of square $-1$ and moreover such that $M$ is eventually base point free. (Here we recall that a divisor $M$ is eventually base point free if the complete linear system $|kM|$ has no base points for all $k \gg 0$.) There are various methods for finding base point free linear systems on an algebraic surface. For example, the well-studied method of Reider [35] implies that, if $X$ is a minimal surface of general type and $D$ is a nef and big divisor on $X$, then $M = K_X + D$ is eventually base point free. There is also a technical generalization of this result due to Kawamata [16]. However, the methods which we shall need are essentially elementary. The general outline of the construction is as follows. Let $E$ be a class of square $-1$ with $K_X \cdot E = 1$. It is known that, if $E$ is the class of a smoothly embedded 2-sphere, then $E$ is of type $(1, 1)$ [6]. Thus $K_X + E$ is a divisor orthogonal to $E$. If $K_X + E$ is ample we are done. If $K_X + E$ is nef but not ample, then there exist curves $D$ with $(K_X + E) \cdot D = 0$, and the intersection matrix of the set of all such curves is negative definite. Thus we may contract the set of all such curves to obtain a normal surface $X'$. If $X'$ has only rational singularities, then the divisor $K_X + E$ induces a Cartier divisor on $X'$ which is ample, by the Nakai-Moishezon criterion, and so some multiple of $K_X + E$ is base point free. Next suppose that $X'$ has a nonrational singular point $p$ and let $D_1, \ldots, D_t$ be the irreducible curves on $X$ mapped to $p$. Then we give a dual form of Artin’s criterion [1] for a rational singularity, which says the following: the point $p$ is a nonrational singularity if and only if there exist nonnegative integers $n_i$, with at least one $n_i > 0$, such that $(K_X + \sum_i n_i D_i) \cdot D_j \geq 0$ for all $j$. Moreover there is a choice of the $n_i$ such that either the inequality is strict for every $j$ or the contraction of the $D_j$ with $n_j \neq 0$ is a minimally elliptic singularity. In this case, provided that $K_X$ is itself nef, it is easy to show that $K_X + \sum_i n_i D_i$ is nef and big and eventually base point free, and defines the desired contraction. The remaining case is when $K_X + E$ is not nef. In this case, by considering the curves $D$ with $(K_X + E) \cdot D < 0$, it is easy to find a $\mathbb{Q}$-divisor of the form $K_X + \lambda D$, where $D$ is an irreducible curve and $\lambda \in \mathbb{Q}^+$, which is nef and big and such that some multiple
is eventually base point free, and which is orthogonal to $E$. The details are given in Section 3. These methods can also handle the case of elliptic surfaces (the case where $\kappa(X) = 1$), but of course there are more elementary and direct arguments here which prove a more precise result.

We have included an appendix giving a proof, due to the first author, R. Miranda, and J.W. Morgan, of a result characterizing the canonical class of a rational surface up to isometry. This result seems to be well-known to specialists but we were unable to find an explicit statement in the literature. It follows from work of Eichler and Kneser on the number of isomorphism classes of indefinite quadratic forms of rank at least 3 within a given genus (see e.g. [17]) together with some calculation. However the proof in the appendix is an elementary argument.

The methods in this paper are able to rule out the possibility of embedded 2-spheres whose associated class $E$ satisfies $E^2 = -1$, $E \cdot [K_X] = 1$. However, in case $p_g(X) = 0$ and $b_2(X) \geq 3$, there are infinitely many classes $E$ of square $-1$ which satisfy $|E \cdot K_X| \geq 3$. It is natural to hope that these classes also cannot be represented by smoothly embedded 2-spheres. More generally we would like to show that the surface $X$ is strongly minimal in the sense of [15]. Likewise, in case $p_g(X) > 0$, we have only dealt with the first case of the “$(-1)$-curve conjecture” (see [6]).

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1. Statement of results and overview of the proof

1.1. Generalities on $SO(3)$-invariants

Let $X$ be a smooth simply connected 4-manifold with $b_2^+(X) = 1$, and fix an $SO(3)$-bundle $P$ over $X$ with $w_2(P) = w$ and $p_1(P) = p$. Recall that a wall of type $(w, p)$ for $X$ is a class $\zeta \in H^2(S; \mathbb{Z})$ such that $\zeta \equiv w \mod 2$ and $p \leq \zeta^2 < 0$. Let

$$\Omega_X = \{x \in H^2(X; \mathbb{R}) : x^2 > 0\}.$$ 

Let $W^\zeta = \Omega_X \cap (\zeta)^\perp$. A chamber of type $(w, p)$ for $X$ is a connected component of the set

$$\Omega_X - \bigcup \{W^\zeta : \zeta \text{ is a wall of type } (w, p)\}.$$ 

Let $C$ be a chamber of type $(w, p)$ for $X$ and let $\gamma_{w,p}(X; C)$ denote the associated Donaldson polynomial, defined via [19] and [21]. Here $\gamma_{w,p}(X; C)$ is only defined up to $\pm 1$, depending on the choice of an integral lift for $w$, corresponding to a choice of orientation for the moduli space. The actual choice of sign will not matter, since we shall only care if a certain value of $\gamma_{w,p}(X; C)$ is nonzero. In the complex case we shall always assume for convenience that the choice has been made so that the orientation of the moduli space agrees with the complex orientation. Via Poincaré duality, we shall view $\gamma_{w,p}(X; C)$ as a function on either homology or cohomology classes. Given a class $M$, we use the notation $\gamma_{w,p}(X; C)(M^d)$ for the evaluation $\gamma_{w,p}(X; C)(M, \ldots, M)$ on the class $M$ repeated $d$ times, where $d = -p - 3$ is the expected dimension of the moduli space. We then have the following vanishing result for $\gamma_{w,p}(C)$, due to Kotschick [19, (6.13)]:
Proposition 1.1. Let $E \in H^2(X; \mathbb{Z})$ be the cohomology class of a smoothly embedded $S^2$ in $X$ with $E^2 = -1$. Let $w$ be the second Stiefel-Whitney class of $X$, or more generally any class in $H^2(X, \mathbb{Z}/2\mathbb{Z})$ such that $w \cdot E \neq 0$. Suppose that $M \in H_2(X; \mathbb{Z})$ satisfies $M^2 > 0$ and $M \cdot E = 0$. Then, for every chamber $C$ of type $(w, p)$ such that the wall $W^E$ corresponding to $E$ passes through the interior of $C$,

$$\gamma_{w, p}(X; C)(M^d) = 0.$$

Note that if $w$ is the second Stiefel-Whitney class of $X$, then $W^E$ is a wall of type $(w, p)$ (and so does not pass through the interior of any chamber) if and only if $E^2$ is even. This case arises, for example, if $X$ has the homotopy type of $(S^2 \times S^2)^\# \mathbb{R}^2$ and $E$ is the standard generator of $H^2(\mathbb{R}^2; \mathbb{Z}) \subseteq H^2(X; \mathbb{Z})$.

For the proof of Theorem 0.1, the result of (1.1) is sufficient. However, for the slightly more general result of Theorem 1.10, we will also need the following variant of (1.1):

Theorem 1.2. Let $E \in H^2(X; \mathbb{Z})$ be the cohomology class of a smoothly embedded $S^2$ in $X$ with $E^2 = -1$. Let $w$ be a class in $H^2(X, \mathbb{Z}/2\mathbb{Z})$ such that $w \cdot E \neq 0$. Suppose that $M \in H_2(X; \mathbb{Z})$ satisfies $M^2 > 0$ and $M \cdot E = 0$. Then, for every chamber $C$ of type $(w, p)$ containing $M$ in its closure,

$$\gamma_{w, p}(X; C)(M^d) = 0,$$

unless $p = -5$ and $w$ is the mod 2 reduction of $E$. Thus, except in this last case, $\gamma_{w, p}(X; C)$ is divisible by $E$.

Proof. If $W^E$ is not a wall of type $(w, p)$ we are done by (1.1). Otherwise, $E$ defines a wall of type $(w, p)$ containing $M$. Next let us assume that $E^\perp \cap C$ is a codimension one face of the closure $\overline{C}$ of $C$. We have an induced decomposition of $X$:

$$X = X_0 \# \mathbb{R}^2.$$

Identify $H_2(X_0; \mathbb{Z})$ with the subspace $E^\perp$ of $H_2(X; \mathbb{Z})$, and let $\overline{C}_0 = E^\perp \cap \overline{C}$. Then $\overline{C}_0$ is the closure of some chamber $C_0$ of type $(w - e, p + 1)$ on $X_0$, where $e$ is the mod 2 reduction of $E$. Choose a generic Riemannian metric $g_0$ on $X_0$ such that the cohomology class $\omega_0$ of the self-dual harmonic 2-form associated to $g_0$ lies in the interior of $C_0$. By the results in [39], there is a family of metrics $h_t$ on the connected sum $X_0 \# \overline{C} \mathbb{R}^2$ which converge in an appropriate sense to $g_0 \parallel g_1$, where $g_1$ is the Fubini-Study metric on $\mathbb{C}^2$, and such that the cohomology classes of the self-dual harmonic 2-forms associated to $h_t$ lie in the interior of $\overline{C}$ and converge to $\omega_0$.

Standard gluing and compactness arguments (see for example [15], Appendix to Chapter 6) and dimension counts show that the restriction of the invariant $\gamma_{w, p}(X; C)$ to $H_2(X_0; \mathbb{Z})$ vanishes.

Consider now the general case where $W^E$ is a wall of type $(w, p)$ and the closure of $C$ contains $M$ but where $W^E \cap C$ is not necessarily a codimension one face of $\overline{C}$. Since $W^E$ is a wall of type $(w, p)$ and $M \in E^\perp$, there exists a chamber $C'$ of type $(w, p)$ whose closure contains $M$ such that $W^E$ is a codimension one face of $\overline{C}'$. By the previous argument, $\gamma_{w, p}(X; C')(M^d) = 0$ and so it will suffice to show that

$$\gamma_{w, p}(X; C)(M^d) = \gamma_{w, p}(X; C')(M^d).$$
Note that $\mathcal{C}$ and $\mathcal{C}'$ are separated by finitely many walls of type $(w, p)$ all of which contain the class $M$. Thus, we have a sequence of chambers of type $(w, p)$:

$$\mathcal{C} = C_1, C_2, \ldots, C_{k-1}, C_k = C'$$

such that for each $i$, $C_{i-1}$ and $C_i$ are separated by a single wall $W_i = W_{C_i}$ of type $(w, p)$ which contains $M$. Since $W_i$ contains $M$, $M \cdot \zeta_i = 0$. By \cite[(3.2)(3)]{39} (see also \cite{21}), the difference $\gamma_{w,p}(X; C_{i-1}) - \gamma_{w,p}(X; C_i)$ is divisible by the class $\zeta_i$ except in the case where $p = -5$ and $w$ is the mod 2 reduction of $E$. It follows that, except in this last case, for each $i$,

$$\gamma_{w,p}(X; C_{i-1})(M^d) = \gamma_{w,p}(X; C_i)(M^d).$$

Hence $\gamma_{w,p}(X; C)(M^d) = \gamma_{w,p}(X; C')(M^d) = 0$. □

We shall also need the following “easy” blowup formula:

**Lemma 1.3.** Let $\mathbb{P}^2$ be a blowup of $X$, and identify $H_2(X; \mathbb{Z})$ with a subspace of $H_2(\mathbb{P}^2; \mathbb{Z})$ in the natural way. Given $w \in H^2(X; \mathbb{Z}/2\mathbb{Z})$, let $\hat{C}$ be a chamber of type $(w, p)$ for $\mathbb{P}^2$ containing the chamber $C$ in its closure. Then

$$\gamma_{w,p}(X; \mathbb{P}^2; \hat{C})|H_2(X; \mathbb{Z}) = \pm \gamma_{w,p}(X; C).$$

**Proof.** Choose a generic Riemannian metric $g$ on $X$ such that the cohomology class $\omega$ of the self-dual harmonic 2-form associated to $g$ lies in the interior of $\mathcal{C}$. We again use the results in \cite{39} to choose a family of metrics $h_i$ on the connected sum $\mathbb{P}^2$ which converge in an appropriate sense to $g \Pi g'$, where $g'$ is the Fubini-Study metric on $\mathbb{P}^2$, and such that the cohomology classes of the self-dual harmonic 2-forms associated to $h_i$ lie in the interior of $\hat{C}$ and converge to $\omega$. Standard gluing and compactness arguments (see e.g. \cite{15}, Chapter 6, proof of Theorem 6.2(i)) now show that the restriction of $\gamma_{w,p}(X; \mathbb{P}^2; \hat{C})$ to $H_2(X; \mathbb{Z})$ (with the appropriate orientation conventions) is just $\gamma_{w,p}(X; C)$. □

### 1.2. The case of a minimal $X$

In this subsection we shall outline the results to be proved concerning minimal surfaces of general type. One basic tool is a nonvanishing theorem for certain values of the Donaldson polynomial:

**Theorem 1.4.** Let $X$ be a simply connected algebraic surface with $p_g(X) = 0$, and let $M$ be a nef and big divisor on $X$ which is eventually base point free. Denote by $\varphi: X \to \tilde{X}$ the birational morphism defined by $|kM|$ for $k \gg 0$, so that $\tilde{X}$ is a normal projective surface. Suppose that $\tilde{X}$ has only rational or minimally elliptic singularities, and that $\varphi$ does not contract any exceptional curves to points. Let $w \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ be the mod 2 reduction of the class $|K_X|$. Then there exists a constant $A$ depending only on $X$ and $M$ with the following property: For all integers $p < A$, let $\mathcal{C}$ be a chamber of type $(w, p)$ containing $M$ in its closure and suppose that $\mathcal{C}$ has nonempty intersection with the ample cone of $X$. Set $d = -p - 3$. Then

$$\gamma_{w,p}(X; \mathcal{C})(M^d) > 0.$$
We shall prove Theorem 1.4 in Section 2, where we shall also recall the salient properties of rational and minimally elliptic singularities. The proof also works in the case where \( p_g(X) > 0 \), in which case \( \gamma_{w,p}(X) \) does not depend on the choice of a chamber.

We can now state the main result concerning minimal surfaces, which we shall prove in Section 3:

**Theorem 1.5.** Let \( X \) be a minimal simply connected algebraic surface of general type, and let \( E \in H^2(X;\mathbb{Z}) \) be a \((1,1)\)-class satisfying \( E^2 = -1, E \cdot K_X = 1 \). Let \( w \) be the mod 2 reduction of \([K_X]\). Then there exist:

(i) an integer \( p \) and (in case \( p_g(X) = 0 \)) a chamber \( C \) of type \((w,p)\) and

(ii) a \((1,1)\)-class \( M \in H^2(X;\mathbb{Z}) \)

such that \( M \cdot E = 0 \) and \( \gamma_{w,p}(X)(M^d) \neq 0 \) (or, in case \( p_g(X) = 0 \), \( \gamma_{w,p}(X;C)(M^d) \neq 0 \)).

The method of proof of (1.5) will be the following: we will show that there exists an orientation preserving self-diffeomorphism \( \psi \) of \( X \) with \( \psi^*[K_X] = [K_X] \) and a nef and big divisor \( M \) on \( X \) such that:

(i) \( M \cdot \psi^*E = 0 \).

(ii) \( M \) is eventually base point free, and the corresponding contraction \( \varphi : X \to \bar{X} \) maps \( X \) birationally onto a normal surface \( \bar{X} \) whose only singularities are either rational or minimally elliptic.

Using the naturality of \( \gamma_{w,p}(X;C) \), it suffices to prove (1.5) after replacing \( E \) by \( \psi^*E \). In this case, by Theorem 1.4 with \( w = [K_X] \), \( \gamma_{w,p}(X;C)(M^d) \neq 0 \) for all \( p \ll 0 \).

**Corollary 1.6.** Let \( X \) be a simply connected minimal surface of general type with \( p_g(X) = 0 \). Then there exist

(i) a class \( w \in H^2(X;\mathbb{Z}/2\mathbb{Z}) \);

(ii) an integer \( p \in \mathbb{Z} \);

(iii) a chamber \( C \) for \( X \) of type \((w,p)\), and

(iv) a homotopy equivalence \( \alpha : X \to Y \), where \( Y \) is either the blowup of \( \mathbb{P}^2 \) at \( n \) distinct points or \( Y = \mathbb{P}^1 \times \mathbb{P}^1 \),

such that, for \( w' = (\alpha^*)^{-1}(w) \) and \( C' = (\alpha^*)^{-1}(C) \),

\[
\alpha^* \gamma_{w',p}(Y; C') \neq \pm \gamma_{w,p}(X; C).
\]

**Proof.** If \( X \) is homotopy equivalent to \( \mathbb{P}^1 \times \mathbb{P}^1 \) then the theorem follows from [33]. Otherwise \( X \) is oriented homotopy equivalent to \( \mathbb{P}^2 \# n \mathbb{P}^2 \), for \( 1 \leq n \leq 8 \), and we claim that there exists a homotopy equivalence \( \alpha : X \to Y \) such that \( \alpha^*[K_Y] = -[K_X] \). Indeed, every integral isometry \( H^2(Y;\mathbb{Z}) \to H^2(X;\mathbb{Z}) \) is realized by an oriented homotopy equivalence. Thus it suffices to show that every two characteristic elements of \( H^2(Y;\mathbb{Z}) \) of square \( 9 - n \) are conjugate under the isometry group, which follows from the appendix to this paper. Choosing such a homotopy equivalence \( \alpha \), let \( e \) be the class of an exceptional curve in \( Y \) and let \( E = \alpha^* e \). Then \( E^2 = -1 \) and \( E \cdot [K_X] = 1 \). We may now apply Theorem 1.5 to the class \( E \), noting that \( E \) is a \((1,1)\)-class since \( p_g(X) = 0 \). Let \( M \) and \( C \) be a divisor and a chamber which satisfy the conclusions of Theorem 1.5 and let \( m = (\alpha^*)^{-1} M \).

If \( w \) is the mod 2 reduction of \([K_X]\), then \( w' \) is the mod two reduction of \([K_Y]\), so
exists a diffeomorphism of $\tilde{r}$  

By Theorem 10A on p. 355 of [13], for an integral isometry $P$ defined by exceptional classes and define the super preserving the set of connected components. Thus if $r$ of $Y$ to together with $[H]$ convex subset of $H$ that $\psi$ choice of a diffeomorphism be $(\tilde{r})$. Moreover, for every choice of an isometry $Y$ that $\tilde{r}$ is realized by a diffeomorphism, the proof above shows that the conclusions of the corollary hold for every homotopy equivalence $\alpha : X \to Y$.

1.3. Reduction to the minimal case

We begin by recalling some terminology and results from [13]. A good generic rational surface $Y$ is a rational surface such that $K_Y = -C$ where $C$ is a smooth curve, and such that there does not exist a smooth rational curve on $Y$ with self-intersection $-2$. Every rational surface is diffeomorphic to a good generic rational surface.

**Theorem 1.7.** Let $X$ be a minimal surface of general type and let $\tilde{X} \to X$ be a blowup of $X$ at $r$ distinct points. Let $E_1', \ldots, E_r'$ be the homology classes of the exceptional curves on $\tilde{X}$. Let $\psi_0 : \tilde{X} \to \tilde{Y}$ be a diffeomorphism, where $\tilde{Y}$ is a good generic rational surface. Then there exists a diffeomorphism $\psi : \tilde{X} \to \tilde{Y}$ and a good generic rational surface $Y$ with the following properties:

(i) The surface $\tilde{Y}$ is the blowup of $Y$ at $r$ distinct points.

(ii) If $e_1, \ldots, e_r$ are the classes of the exceptional curves in $H^2(\tilde{Y}; \mathbb{Z})$ for the blowup $\tilde{Y} \to Y$, then possibly after renumbering $\psi^*(e_i) = E_i'$ for all $i$.

(iii) Identifying $H^2(X)$ with a subgroup of $H^2(\tilde{X})$ and $H^2(Y)$ with a subgroup of $H^2(\tilde{Y})$ in the obvious way, we have $\psi^*(H^2(Y)) = H^2(X)$.

Moreover, for every choice of an isometry $\tau$ from $H^2(Y)$ to $H^2(X)$, there exists a choice of a diffeomorphism $\psi$ satisfying (i)–(iii) above and such that $\psi^*|H^2(Y) = \tau$.

**Proof.** Let $e'_i \in H^2(\tilde{Y}; \mathbb{Z})$ satisfy $\psi_0^*(e'_i) = E'_i$. Thus the Poincaré dual of $e'_i$ is represented by a smoothly embedded 2-sphere in $\tilde{Y}$. It follows that reflection $r_{e'_i}$ in $e'_i$ is realized by an orientation-preserving self-diffeomorphism of $\tilde{Y}$. To see what this says about $e'_i$, we shall recall the following terminology from [13].

Let $H(\tilde{Y})$ be the set $\{ x \in H^2(\tilde{Y}; \mathbb{R}) \mid x^2 = 1 \}$ and let $K(\tilde{Y}) \subset H^2(\tilde{Y}; \mathbb{R})$ be intersection of the closure of the Kähler cone of $\tilde{Y}$ with $H(\tilde{Y})$. Then $K(\tilde{Y})$ is a convex subset of $H(\tilde{Y})$ whose walls consist of the classes of exceptional curves on $\tilde{Y}$ together with $[-K_Y]$ if $b_2(\tilde{Y}) \geq 10$, which is confusingly called the exceptional wall of $K(\tilde{Y})$. Let $R$ be the group generated by the reflections in the walls of $K(\tilde{Y})$ defined by exceptional classes and define the super $P$-cell $S = S(P)$ by

$$S = \bigcup_{\gamma \in R} \gamma \cdot K(\tilde{Y}).$$

By Theorem 10A on p. 355 of [13], for an integral isometry $\varphi$ of $H^2(\tilde{Y}; \mathbb{R})$, there exists a diffeomorphism of $\tilde{Y}$ inducing $\varphi$ if and only if $\varphi(S) = \pm S$. (Here, if $b_2(\tilde{Y}) \leq 9$, $S = H$ and the result reduces to a result of C.T.C. Wall [38].) Note that $H(\tilde{Y})$ has two connected components, and reflection $r_e$ in a class $e$ of square $-1$ preserves the set of connected components. Thus if $r_e(S) = \pm S$, then necessarily $r_e(S) = S$.

Next we have the following purely algebraic lemma:
Lemma 1.8. Let $e$ be a class of square $-1$ in $H^2(\hat{Y};\mathbb{Z})$ such that the reflection $r_e$ satisfies $r_e(S) = S$. Then there is an isometry $\varphi$ of $H^2(\hat{Y};\mathbb{Z})$ preserving $S$ which sends $e$ to the class of an exceptional curve.

Proof. We first claim that, if $W$ is the wall corresponding to $e$, then $W$ meets the interior of $S$. Indeed, the interior int $S$ of $S$ is connected, by Corollary 5.5 of [13] p. 340. If $W$ does not meet int $S$, then the sets

$$\{ x \in \text{int } S \mid e \cdot x > 0 \}$$

and

$$\{ x \in \text{int } S \mid e \cdot x < 0 \}$$

are disjoint open sets covering int $S$ which are exchanged under the reflection $r_e$. Since at least one is nonempty, they are both nonempty, contradicting the fact that int $S$ is connected. Thus $W$ must meet int $S$.

Now let $C$ be a chamber for the walls of square $-1$ which has $W$ as a wall. It follows from Lemma 5.3(b) on p. 339 of [13] that $C \cap S$ is a $P$-cell $P$ and that $W$ defines a wall of $P$ which is not the exceptional wall. By Lemma 5.3(e) of [13], $S$ is the unique super $P$-cell containing $P$, and the reflection group generated by the elements of square $-1$ defining the walls of $P$ acts simply transitively on the $P$-cells in $S$. There is thus an element $\varphi$ in this reflection group which preserves $S$ and sends $P$ to $K(\hat{Y})$ and $W$ to a wall of $K(\hat{Y})$ which is not an exceptional wall. It follows that $\varphi(e)$ is the class of an exceptional curve on $\hat{Y}$. □

Returning to the proof of Theorem 1.7, apply the previous lemma to the reflection in $e'_r$. There is thus an isometry $\varphi$ preserving $S$ such that $\varphi(e'_r) = e_r$, where $e_r$ is the class of an exceptional curve on $\hat{Y}$. Moreover $\varphi$ is realized by a diffeomorphism.

Thus after composing with the diffeomorphism inducing $\varphi$, we can assume that $e'_r = e_r$, or equivalently that $\psi_0 e_r = [E'_e]$. Let $\hat{Y} \to \hat{Y}_1$ be the blowing down of the exceptional curve whose class is $e_r$. Then $\hat{Y}_1$ is again a good generic surface by [13] p. 312 Lemma 2.3. Since $e'_1, \ldots, e'_{r-1}$ are orthogonal to $e_r$, they lie in the subset $H^2(\hat{Y}_1)$ of $H^2(\hat{Y})$. For $i \neq r$, the reflection in $e'_i$ preserves $W \cap S$, where $W = (e_r)\perp$. Now $W$ is just $H^2(\hat{Y}_1)$ and $K(\hat{Y}) \cap H^2(\hat{Y}_1) = K(\hat{Y}_1)$ by [13] p. 331 Proposition 3.5. The next lemma relates the corresponding super $P$-cells:

Lemma 1.9. $W \cap S$ is the super $P$-cell $S_1$ for $\hat{Y}_1$ containing $K(\hat{Y}_1)$.

Proof. Trivially $S_1 \subset W \cap S$, and both sets are convex subsets with nonempty interiors. If they are not equal, then there is a $P$-cell $P' \subset S_1$ and an exceptional wall of $P'$ which passes through the interior of $S \cap W$. If $\kappa(P')$ is the exceptional wall meeting $S \cap W$, then, by [13] p. 335 Lemma 4.6, $\kappa(P') - e_r$ is an exceptional wall of $P$ for a well-defined $P$-cell in $S$, and $\kappa(P') - e_r$ must pass through the interior of $S$. This is a contradiction. Hence $S \cap W = S_1$ is a super $P$-cell of $\hat{Y}_1$, and we have seen that it contains $K(\hat{Y}_1)$. □

Returning to the proof of Theorem 1.7, reflection in $e'_{r-1}$ preserves $S_1$. Applying Lemma 1.8, there is a diffeomorphism of $\hat{Y}_1$ which sends $e'_{r-1}$ to the class of an exceptional curve $e_{r-1}$. Of course, there is an induced diffeomorphism of $\hat{Y}$ which fixes $e_r$. Now we can clearly proceed by induction on $r$.

The above shows that after replacing $\psi_0$ by a diffeomorphism $\psi$ we can find $Y$ as above so that (i) and (ii) of the statement of Theorem 3 hold. Clearly $\psi^*(H^2(Y)) =$
$H^2(X)$. By the theorem of C.T.C. Wall mentioned above, there is a diffeomorphism of $Y$ realizing every integral isometry of $H^2(Y)$. So after further modifying by a diffeomorphism of $Y$, which extends to a diffeomorphism of $\tilde{Y}$ fixing the classes of the exceptional curves, we can assume that the diffeomorphism $\psi$ restricts to $\tau$ for any given isometry from $H^2(Y)$ to $H^2(X)$.

We can now give a proof of Theorem 0.1:

**Theorem 0.1.** No complex surface of general type is diffeomorphic to a rational surface.

**Proof.** Suppose that $X$ is a minimal surface of general type and that $\rho: \tilde{X} \to X$ is a blowup of $X$ diffeomorphic to a rational surface. We may assume that $\tilde{X}$ is diffeomorphic via $\psi$ to a good generic rational surface $\bar{Y}$, and that $\rho': \bar{Y} \to Y$ is a blow up of $Y$ such that $Y$ and $\psi$ satisfy (i)–(iii) of Theorem 1.7. Choose $w, p, \alpha, C$ for $X$ such that the conclusions of Corollary 1.6 hold, with $C'$ the corresponding chamber on $Y$, and let $\tilde{C}'$ be any chamber for $\bar{Y}$ containing $C'$ in its closure. Then $\psi^*\tilde{C}' = \tilde{C}$ is a chamber on $\tilde{X}$ containing $C$ in its closure. Using the last sentence of Theorem 1.7, we may assume that $\psi^*H^2(Y) = \alpha^*$. Thus $\psi^*(\rho')^* = \rho^*\alpha^*$. By the functorial properties of Donaldson polynomials, and viewing $H^2(X;\mathbb{Z}/2\mathbb{Z})$ as a subset of $H^2(\tilde{X};\mathbb{Z}/2\mathbb{Z})$, and similarly for $\bar{Y}$, we have

$$\psi^*\gamma_{w,p}(\bar{Y},\tilde{C}') = \pm \gamma_{w',p}(\bar{X},\tilde{C}) = \pm \gamma_{w,p}(\tilde{X},\tilde{C}).$$

Restricting each side to $\psi^*H_2(Y) = H_2(X)$, we obtain by repeated application of Lemma 1.3 that

$$\alpha^*\gamma_{w',p}(Y;\tilde{C}') = \pm \gamma_{w,p}(X;C).$$

But this contradicts Corollary 1.6. □

Using Theorem 1.5, we have the following generalization of Theorem 0.4 in the introduction to the case of nonminimal algebraic surfaces:

**Theorem 1.10.** Let $X$ be a minimal simply connected surface of general type, and let $E \in H^2(X;\mathbb{Z})$ satisfy $E^2 = -1$ and $E \cdot K_X = 1$. Let $\tilde{X}$ be a blowup of $X$. Then, viewing $H^2(X;\mathbb{Z})$ as a subset of $H^2(\tilde{X};\mathbb{Z})$, the class $E$ is not represented by a smoothly embedded 2-sphere in $\tilde{X}$.

**Proof.** Suppose instead that $E$ is represented by a smoothly embedded 2-sphere. If $p_g(X) > 0$, then it follows from the results of [6] that $E$ is a $(1,1)$-class, i.e. $E$ lies in the image of Pic $X$ inside $H^2(X;\mathbb{Z})$. Of course, this is automatically true if $p_g(X) = 0$. Next assume that $p_g(X) = 0$. By Theorem 1.5, there exists a $w \in H^2(X;\mathbb{Z}/2\mathbb{Z})$, an integer $p$, and a chamber $C$ of type $(w,p)$, such that $\gamma_{w,p}(X;C)(M^d) \neq 0$, where $M$ is a class in the closure of $C$ and $M \cdot E = 0$. Consider the Donaldson polynomial $\gamma_{w,p}(X;C)$, where we view $w$ as an element of $H^2(X;\mathbb{Z}/2\mathbb{Z})$ in the natural way and $C$ is a chamber of type $(w,p)$ on $\tilde{X}$ containing $C$ in its closure. Then $C$ also contains $M$ in its closure. Thus, by Theorem 1.2, $\gamma_{w,p}(\tilde{X};\tilde{C})(M^d) = 0$. On the other hand, by Lemma 1.3, $\gamma_{w,p}(\tilde{X};\tilde{C})(M^d) = \pm \gamma_{w,p}(X;C)(M^d) \neq 0$. This is a contradiction. The case where $p_g(X) > 0$ is similar. □

We also have the following corollary, which works under the assumptions of Theorem 1.10 for surfaces with $p_g > 0$:
Corollary 1.11. Let $X$ be a simply connected surface of general type with $p_g(X) > 0$, not necessarily minimal, and let $E \in H^2(X; \mathbb{Z})$ satisfy $E^2 = -1$ and $E \cdot K_X = -1$. Suppose that $E$ is represented by a smoothly embedded 2-sphere. Then $E$ is the cohomology class associated to an exceptional curve.

Proof. Using [15] and [6], we see that if $E$ is not the cohomology class associated to an exceptional curve, then $E \in H^2(X_{\text{min}}; \mathbb{Z})$, where $X_{\text{min}}$ is the minimal model of $X$ and we have the natural inclusion $H^2(X_{\text{min}}; \mathbb{Z}) \subseteq H^2(X; \mathbb{Z})$. We may then apply Theorem 1.10 to conclude that $-E$ cannot be represented by a smoothly embedded 2-sphere, and thus that $E$ cannot be so represented, a contradiction. □

2. A generalized nonvanishing theorem

2.1. Statement of the theorem and the first part of the proof

In this section, we shall prove Theorem 1.4. We first recall its statement:

Theorem 1.4. Let $X$ be a simply connected algebraic surface with $p_g(X) = 0$, and let $M$ be a nef and big divisor on $X$ which is eventually base point free. Denote by $\phi : X \to \bar{X}$ the birational morphism defined by $|kM|$ for $k \gg 0$, so that $\bar{X}$ is a normal projective surface. Suppose that $\bar{X}$ has only rational or minimally elliptic singularities, and that $\phi$ does not contract any exceptional curves to points. Let $w \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ be the mod 2 reduction of the class $[K_X]$. Then there exists a constant $A$ depending only on $X$ and $M$ with the following property: For all integers $p \leq A$, let $C$ be a chamber of type $(w, p)$ containing $M$ in its closure and suppose that $C$ has nonempty intersection with the ample cone of $X$. Set $d = -p - 3$. Then

$$
\gamma_{w,p}(X; C)(M^d) > 0.
$$

A similar conclusion holds if $p_g(X) > 0$.

Proof. We begin by fixing some notation. For $L$ an ample line bundle on $X$, given a divisor $D$ on $X$ and an integer $c$, let $\mathcal{M}_L(D, c)$ denote the moduli space of isomorphism classes of $L$-stable rank two holomorphic vector bundles on $X$ with $c_1(V) = D$ and $c_2(V) = c$. Let $w$ be the mod 2 reduction of $D$ and let $p = D^2 - 4c$. Then we also denote $\mathcal{M}_L(D, c)$ by $\mathcal{M}_L(w, p)$, the moduli space of equivalence classes of $L$-stable rank two holomorphic vector bundles on $X$ corresponding to the choice of $(w, p)$. Here we recall that two vector bundles $V$ and $V'$ are equivalent if there exists a holomorphic line bundle $F$ such that $V' = V \otimes F$. The invariants $w$ and $p$ only depend on the equivalence class of $V$. Let $\mathcal{M}_L(w, p)$ denote the Gieseker compactification of $\mathcal{M}_L(w, p)$, i.e. the Gieseker compactification $\overline{\mathcal{M}_L(D, c)}$ of $\mathcal{M}_L(D, c)$. Thus $\overline{\mathcal{M}_L(w, p)}$ is a projective variety.

We now fix a compact neighborhood $N$ of $M$ inside the positive cone $\Omega_X$ of $X$. Note that, since $M$ is nef, such a neighborhood has nontrivial intersection with the ample cone of $X$. Using a straightforward extension of the theorem of Donaldson [10] on the dimension of the moduli space (see e.g. [12] Chapter 8, [32], [24]), there exist constants $A$ and $A'$ such that, for all ample line bundles $L$ such that $c_1(L) \in N$, the following holds:

1. If $p \leq A$, then the moduli space $\overline{\mathcal{M}_L(w, p)}$ is good, in other words it is generically reduced of the correct dimension $-p - 3$;
2. $\mathcal{M}_L(w, p)$ is a dense open subset of $\overline{\mathcal{M}_L(w, p)}$ and the generic point of $\overline{\mathcal{M}_L(w, p)} - \mathcal{M}_L(w, p)$ correspond to a torsion free sheaf $V$ such that the
length of \( V^VV/V \) is one and such that the support of \( V^VV/V \) is a generic point of \( X \);

(3) For all \( p' \geq A \), the dimension \( \dim \mathfrak{M}_L(w, p') \leq A' \).

We shall need to make one more assumption on the integer \( p \). Let \( \varphi : X \to \bar{X} \) be the contraction morphism associated to \( M \). For each connected component \( E \) of the set of exceptional fibers of \( \varphi \), fix a (possibly nonreduced) curve \( Z \) on \( X \) whose support is exactly \( E \). In practice we shall always take \( Z \) to be the fundamental cycle of the singularity, to be defined in Subsection 2.3 below. A slight generalization ([12], Chapter 8) of Donaldson’s theorem on the dimension of the moduli space then shows the following: after possibly modifying the constant \( A \),

(4) The generic \( V \in \mathfrak{M}_L(w, p) \) satisfies: the natural map

\[
H^1(X; ad V) \to H^1(Z; ad V|Z)
\]

is surjective. In other words, the local universal deformation of \( V \) is versal when viewed as a deformation of \( V|Z \) (keeping the determinant fixed).

We now assume that \( p \leq A \). Let \( L \) be an ample line bundle which is not separated from \( M \) by any wall of type \((w, p)\) (or equivalently of type \((D, c)\)), and moreover does not lie on any wall of type \((w, p)\). Thus by assumption, none of the points of \( \mathfrak{M}_L(D, c) \) corresponds to a strictly semistable sheaf. Let \( C \subset X \) be a smooth curve of genus \( g \). Suppose that \( C \cdot D = 2a \) is even. Choosing a line bundle \( \theta \) of degree \( g - 1 - a \) on \( C \), we can form the determinant line bundle \( \mathcal{L}(C, \theta) \) on the moduli functor associated to torsion free sheaves corresponding to the values \( w \) and \( p \) ([15], Chapter 5). Using Proposition 1.7 in [23], this line bundle descends to a line bundle on \( \mathfrak{M}_L(w, p) \), which we shall continue to denote by \( \mathcal{L}(C, \theta) \). Moreover, by the method of proof of Theorem 2 of [23], the line bundle \( \mathcal{L}(C, \theta) \) depends only on the linear equivalence class of \( C \), in the sense that if \( C \) and \( C' \) are linearly equivalent and \( \theta' \) is a line bundle of degree \( g - 1 - a \) on \( C' \), then \( \mathcal{L}(C, \theta) \cong \mathcal{L}(C', \theta') \).

Next we shall use the following result, whose proof is deferred to the next subsection:

**Lemma 2.1.** In the above notation, if \( k \gg 0 \) and \( C \in |kM| \) is a smooth curve, then, for all \( N \gg 0 \), the linear system associated to \( \mathcal{L}(C, \theta)^N \) has no base points and defines a generically finite morphism from \( \mathfrak{M}_L(w, p) \) to its image. In particular, if \( d = \dim \mathfrak{M}_L(w, p) \), then

\[
c_1(\mathcal{L}(C, \theta))^d > 0.
\]

It follows by applying an easy adaptation of Theorem 6 in [23] or the results of [25] to the case \( p_g(X) = 0 \) that, since the spaces \( \mathfrak{M}_L(w, p') \) have the expected dimension for an appropriate range of \( p' \gg p \), \( c_1(\mathcal{L}(C, \theta))^d \) is exactly the value \( k^{d-g(X;C)}(M^d) \). Thus we have proved Theorem 1.4, modulo the proof of Lemma 2.1. This proof will be given below.

**2.2. A generalization of a result of Bogomolov**

We keep the notation of the preceding subsection. Thus \( M \) is a nef and big divisor such that the complete linear system \(|kM|\) is base point free whenever \( k \gg 0 \). Throughout, we shall further assume that \( M \) is divisible by 2 in \( \text{Pic} \ X \). Moreover \( w \) and \( p \) are now fixed and \( L \) is an ample line bundle such that \( c_1(L) \in N \) is not separated from \( M \) by a wall of type \((w, p)\) and moreover that \( c_1(L) \) does not
lie on a wall of type \((w, p)\). In particular the determinant line bundle \(L(C, \theta)\) is defined for all smooth \(C\) in \(|kM|\) for all \(k \gg 0\).

We then have the following generalization of a restriction theorem due to Bogomolov [4]:

**Lemma 2.2.** With the above notation, there exists a constant \(k_0\) depending only on \(w, p, M, \) and \(L\), such that for all \(k \geq k_0\) and all smooth curves \(C \in |kM|\), the following holds: for all \(c' \leq c\) and \(V \in \mathcal{M}_L(D, c')\), either \(V|C\) is semistable or there exists a divisor \(G\) on \(X\), a zero-dimensional subscheme \(Z\) and an exact sequence

\[
0 \to \mathcal{O}_X(G) \to V \to \mathcal{O}_X(D - G) \otimes I_Z \to 0,
\]

where \(2G - D\) defines a wall of type \((w, p)\) containing \(M\) and \(C \cap \text{Supp} \ Z \neq \emptyset\).

**Proof.** The proof follows closely the original proof of Bogomolov’s theorem [4] or [15] Section 5.2. Choose \(k_0 \geq -p\) and assume also that there exists a smooth curve \(C\) in \(|kM|\) for all \(k \geq k_0\). Suppose that \(V|C\) is not semistable. Then there exists a surjection \(V|C \to F\), where \(F\) is a line bundle on \(C\) with \(\deg F = f < (D \cdot C)/2\).

Let \(W\) be the kernel of the induced surjection \(V \to F\). Thus \(W\) is locally free and there is an exact sequence

\[
0 \to W \to V \to F \to 0.
\]

A calculation gives

\[
p_1(\text{ad } W) = p_1(\text{ad } V) + 2D \cdot C + (C)^2 - 4f
\]

\[
> p + k^2(M)^2 \geq p + p^2 \geq 0.
\]

By Bogomolov’s inequality, \(W\) is unstable with respect to every ample line bundle on \(X\). Thus there exists a divisor \(G_0\) and an injection \(\mathcal{O}_X(G_0) \to W\) (which we may assume to have torsion free cokernel) such that \(2(L \cdot G_0) > L \cdot (D - C), \) i.e. \(L \cdot (2G_0 - D + C) > 0\). By hypothesis there is an exact sequence

\[
0 \to \mathcal{O}_X(G_0) \to W \to \mathcal{O}_X(-G_0 + D - C) \otimes I_{Z_0} \to 0.
\]

Thus

\[
0 < p_1(\text{ad } W) = (2G_0 - D + C)^2 - 4\ell(Z_0) \leq (2G_0 - D + C)^2.
\]

It follows that \((2G_0 - D + C)^2 > 0\). As \(L \cdot (2G_0 - D + C) > 0\) and \((2G_0 - D + C)^2 > 0\), \(M \cdot (2G_0 - D + C) \geq 0\) as well, i.e. \(-(M \cdot (2G_0 - D)) \leq k(M)^2\). On the other hand, since \(V\) is \(L\)-stable, \(L \cdot (2G_0 - D) < 0\). Since \(L\) and \(M\) are not separated by any wall of type \((w, p)\), it follows that \(M \cdot (2G_0 - D) \leq 0\). Finally using

\[
p_1(\text{ad } W) = (2G_0 - D + C)^2 - 4\ell(Z_0)
\]

\[
= p_1(\text{ad } V) + 2D \cdot C + (C)^2 - 4f
\]

\[
> p + k^2(M)^2,
\]

we obtain

\[
(2G_0 - D)^2 + 2k(2G_0 - D) \cdot M > p.
\]
Let \( m = -(2G - D) \cdot M \). As we have seen above \( m \leq kM^2 \) and \( m \geq 0 \). The above inequality can be rewritten as
\[
2km < (2G - D)^2 - p.
\]
We claim that \( m = 0 \). Otherwise
\[
2k < \left(\frac{2G - D}{m}\right)^2 - \frac{p}{m}.
\]
By the Hodge index theorem \( (2G - D)^2 M^2 \leq [(2G - D) \cdot M]^2 = m^2 \), so that \( (2G - D)^2 \leq m^2/M^2 \). Plugging this into the inequality above, using \(-p \geq 0\), gives
\[
2k < \frac{m}{M^2} - \frac{p}{m} \leq k - p,
\]
i.e. \( k < -p \), contradicting our choice of \( k \). Thus \( m = -(2G - D) \cdot M = 0 \).

Now the inclusions \( \mathcal{O}_X(G_0) \subset W \subset V \) define an inclusion \( \mathcal{O}_X(G_0) \subset V \). Thus there is an effective divisor \( E \) and an inclusion \( \mathcal{O}_X(G_0 + E) \to V \) with torsion free cokernel. Let \( G = G_0 + E \). Thus there is an exact sequence
\[
0 \to \mathcal{O}_X(G) \to V \to \mathcal{O}_X(-G + D) \otimes I_Z \to 0.
\]
We claim that \( (2G - D) \cdot M = 0 \). Since \( V \) is L-stable, \( (2G - D) \cdot L < 0 \), and since \( L \) and \( M \) are not separated by a wall of type \((w, p)\), \( (2G - D) \cdot M \leq 0 \). On the other hand,
\[
(2G - D) \cdot M = (2G - D) \cdot M + 2(E \cdot M) = -m + 2(E \cdot M) = 2(E \cdot M).
\]
As \( E \) is effective and \( M \) is nef, \( 2(E \cdot M) \geq 0 \). Thus \( (2G - D) \cdot M = 0 \). As \( M^2 > 0 \), we must have \( (2G - D)^2 < 0 \). Using \( p = (2G - D)^2 - 4\ell(Z) \leq (2G - D)^2 \), we see that \( 2G - D \) is a wall of type \((w, p)\).

Finally note that \( \text{Supp } Z \cap C \neq \emptyset \), for otherwise we would have \( V|C \) semistable. This concludes the proof of Lemma 2.2. \( \square \)

Returning to the proof of Lemma 2.1, we claim first that, given \( k \gg 0 \) and \( C \in |kM| \), for all \( N \) sufficiently large the sections of \( \mathcal{L}(C, \theta)^N \) define a base point free linear series on \( \mathfrak{M}_L(w, p) \). To see this, we first claim that, for \( k \gg 0 \), and for a generic \( C \in |kM| \), the restriction map \( V \to V|C \) defines a rational map \( r_C: \mathfrak{M}_L(w, p) \to \mathfrak{M}(C) \), where \( \mathfrak{M}(C) \) is the moduli space of equivalence classes of semistable rank two bundles on \( C \) such that the parity of the determinant is even.
It suffices to prove that, for every component \( N \) of \( \mathfrak{M}_L(w, p) \) there is one \( V \in N \) and one \( C \in |kM| \) such that \( V|C \) is semistable, for then the same will hold for a Zariski open subset of \( |kM| \). Now given \( V \), choose a fixed \( C_0 \in |kM| \). If \( V|C_0 \) is not semistable, then by Lemma 2.2 there is an exact sequence
\[
0 \to \mathcal{O}_X(G) \to V \to \mathcal{O}_X(-G + D) \otimes I_Z \to 0,
\]
where \( Z \) is a zero-dimensional subscheme of \( X \) meeting \( C_0 \). Choosing \( C \) to be a curve in \( |kM| \) disjoint from \( Z \), which is possible since \( |kM| \) is base point free, it follows that the restriction \( V|C \) is semistable.
For \( C \) fixed, let

\[
B_C = \{ V \in \mathfrak{M}_L(w, p) : \text{either } V \text{ is not locally free over some point of } C \\
\text{or } V | C \text{ is not semistable} \}.
\]

By the openness of stability and local freeness, the set \( B_C \) is a closed subset of \( \mathfrak{M}_L(w, p) \) and \( r_C \) defines a morphism from \( \mathfrak{M}_L(w, p) - B_C \) to \( \mathfrak{M}(C) \). Standard estimates (cf. [10], [12], [32], [24], [27]) show that, possibly after modifying the constant \( A \) introduced at the beginning of the proof of Theorem 1.4, the codimension of \( B_C \) is at least two in \( \mathfrak{M}_L(w, p) \) provided that \( p \leq A \) (where as usual \( A \) is independent of \( k \) and depends only on \( X \) and \( M \)). Indeed the set of bundles \( V \) which fit into an exact sequence

\[
0 \to \mathcal{O}_X(G) \to V \to \mathcal{O}_X(D - G) \otimes I_Z \to 0,
\]

where \( G \) is a divisor such that \( (2G - D) \cdot M = 0 \), may be parametrized by a scheme of dimension \(-\frac{2}{3}p + O(\sqrt{p})\) by e.g. [12], Theorem 8.18. Moreover the constant implicit in the notation \( O(\sqrt{p}) \) can be chosen uniformly over \( N \). The case of nonlocally free \( V \) is taken care of by assumption (2) in the discussion of the constant \( A \): it follows from standard deformation theory (see again [12], [24]) that at a generic point of the locus of nonlocally free sheaves corresponding to the semistable torsion free sheaf \( V \) the deformations of \( V \) are versal for the local deformations of the singularities of \( V \). Thus for a general nonlocally free \( V, V \) has just one singular point which is at a general point of \( X \) and so does not lie on \( C \). Thus the set of \( V \) which are not locally free at some point of \( C \) has codimension at least two (in fact exactly two) in \( \mathfrak{M}_L(w, p) \).

Let \( L_C \) be the determinant line bundle on \( \mathfrak{M}(C) \) associated to the line bundle \( \theta \) (see for instance [15] Chapter 5 Section 2). Then by definition the pullback via \( r_C \) of \( L_C \) is the restriction of \( L(C, \theta) \) to \( \mathfrak{M}_L(w, p) - B_C \). Since \( B_C \) has codimension two, the sections of \( L_C^N \) pull back to sections of \( L(C, \theta)^N \) on \( \mathfrak{M}_L(w, p) \). Since \( L_C \) is ample, given \( V \in \mathfrak{M}_L(w, p) - B_C \), there exists an \( N \) and a section of \( L_C^N \) not vanishing at \( r_C(V) \), and thus there is a section of \( L(C, \theta)^N \) not vanishing at \( V \). Moreover by [23], for all smooth \( C' \in [kM] \) and choice of an appropriate line bundle \( \theta' \) on \( C' \), there is an isomorphism \( L(C, \theta)^N \cong L(C', \theta')^N \). Next we claim that, for every \( V \in \mathfrak{M}_L(w, p) \), there exists a \( C \) such that \( V \) is locally free above \( C \) and \( V | C \) is semistable. Given \( V \), it fails to be locally free at a finite set of points, and its double dual \( W \) is again semistable. Thus applying the above to \( W \), and again using the fact that \( [kM] \) has no base points, we can find \( C \) such that \( V \) is locally free over \( C \) and such that \( V | C = W | C \) is semistable. Thus, given \( V \), there exists an \( N \) and a section of \( L(C, \theta)^N \) which does not vanish at \( V \). Since \( \mathfrak{M}_L(w, p) \) is of finite type, there exists an \( N \) which works for all \( V \), so that the linear system corresponding to \( L(C, \theta)^N \) has no base points.

Finally we must show that, for \( k \gg 0 \), the morphism induced by \( L(C, \theta)^N \) is in fact generically finite for \( N \) large. We claim that it suffices to show that the restriction of the rational map \( r_C \) to \( \mathfrak{M}_L(w, p) - B_C \) is generically finite (it is here that we must use the condition on the singularities of \( X \) in the statement of Theorem 1.4). Supposing this to be the case, and fixing a \( V \in \mathfrak{M}_L(w, p) - B_C \) for which \( r_C^{-1}(r_C(V)) \) is finite, we consider the intersection of all the divisors in \( L(C, \theta)^N \).
containing \( V \), where \( N \) is chosen so that \( \mathcal{L}_C^N \) is very ample. This intersection always contains \( V \) and is a subset of \( r_C^{-1}(r_C(V)) \cup B_C \). In particular \( V \) is an isolated point of the fiber, and so the morphism defined by \( \mathcal{L}(C, \theta)^N \) cannot have all fibers of purely positive dimension. Thus it is generically finite.

To see that \( r_C \) is generically finite, we shall show that, for generic \( V \), the restriction map

\[
  r: H^1(X; \text{ad} V) \to H^1(C; \text{ad} V|C)
\]

is injective. The map \( r \) is just the differential of the map \( r_C \) from \( \mathfrak{M}_L(w, p) \) to \( \mathfrak{M}(C) \) at the point corresponding to \( V \), and so if \( V \) is generic then \( r_C \) is finite. Now the kernel of the map \( r \) is a quotient of \( H^1(X; \text{ad} V \otimes \mathcal{O}_X(-C)) \), and we need to find circumstances where this group is zero, at least if \( C \in |kM| \) for \( k \) sufficiently large. By Serre duality it suffices to show that \( H^1(X; \text{ad} V \otimes \mathcal{O}_X(C) \otimes K_X) = 0 \) for \( k \) sufficiently large. By applying the Leray spectral sequence to the morphism \( \varphi: X \to \bar{X} \), it suffices to show that

\[
  H^1(\bar{X}; R^0 \varphi_* (\text{ad} V \otimes \mathcal{O}_X(C) \otimes K_X)) = 0
\]

and that \( R^1 \varphi_* (\text{ad} V \otimes \mathcal{O}_X(C) \otimes K_X) = 0 \). Now \( \bar{M} \) is the pullback of an ample line bundle \( \bar{M} \) on \( \bar{X} \), and \( \mathcal{O}_X(C) \) is the pullback of \( (\bar{M})^\otimes k \). Thus for fixed \( V \) and \( k \gg 0 \),

\[
  H^1(\bar{X}; R^0 \varphi_* (\text{ad} V \otimes \mathcal{O}_X(C) \otimes K_X)) = H^1(\bar{X}; R^0 \varphi_* (\text{ad} V \otimes K_X) \otimes (\bar{M}^k)) = 0.
\]

Moreover \( R^1 \varphi_* (\text{ad} V \otimes \mathcal{O}_X(C) \otimes K_X) = R^1 \varphi_* (\text{ad} V \otimes K_X) \otimes (\bar{M}^k) \), so that it is enough to show that \( R^1 \varphi_* (\text{ad} V \otimes K_X) = 0 \). By the formal functions theorem,

\[
  R^1 \varphi_* (\text{ad} V \otimes K_X) = \lim_m H^1(mZ; \text{ad} V \otimes K_X|mZ),
\]

where \( Z = \bigcup Z_i \) is the union of the connected components \( Z_i \) of the one-dimensional fibers of \( \varphi \). Thus it suffices to show that, for all \( i \) and all positive integers \( m \),

\[
  H^1(mZ_i; \text{ad} V \otimes K_X|mZ_i) = 0.
\]

Now by the adjunction formula \( \omega_{mZ_i} = K_X \otimes \mathcal{O}_X(mZ_i)|mZ_i \), where \( \omega_{mZ_i} \) is the dualizing sheaf of the Gorenstein scheme \( mZ_i \). Thus \( K_X|mZ_i = \mathcal{O}_X(-mZ_i)|mZ_i \otimes \omega_{mZ_i} \) and we must show the vanishing of

\[
  H^1(mZ_i; \text{ad} V \otimes \mathcal{O}_X(-mZ_i)|mZ_i \otimes \omega_{mZ_i}).
\]

By Serre duality, it suffices to show that, for all \( m > 0 \),

\[
  H^0(mZ_i; \text{ad} V \otimes \mathcal{O}_X(mZ_i)|mZ_i) = 0.
\]

We shall deal with this problem in the next subsection.

Remark. (1) Instead of arguing that the restriction map \( r_C \) was generically finite, one could also check that it was generically one-to-one by showing that for generic \( V_1, V_2 \), the restriction map

\[
  H^0(X; \text{Hom}(V_1, V_2)) \to H^0(C; \text{Hom}(V_1, V_2)|C)
\]
is surjective (since then an isomorphism from $V_1|C$ to $V_2|C$ lifts to a nonzero map from $V_1$ to $V_2$, necessarily an isomorphism by stability). In turn this would have amounted to showing that $H^1(X; \mathrm{Hom}(V_1, V_2) \otimes \mathcal{O}_X(-C)) = 0$ for generic $V_1$ and $V_2$, and this would have been essentially the same argument.

(2) Suppose that $\varphi: X \to \bar{X}$ is the blowup of a smooth surface $\bar{X}$ at a point $x$, and that $M$ is the pullback of an ample divisor on $X$. Let $Z \cong \mathbb{P}^1$ be the exceptional curve. In this case, if $c_1(V) \cdot Z$ is odd, say $2a + 1$, then the generic behavior for $V|Z = V|Z \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a + 1)$ and the restriction map exhibits $\mathfrak{M}_L(w, p)$ (generically) as a $\mathbb{P}^1$-bundle over its image (see for instance [5]). Thus the hypothesis that $\varphi$ contracts no exceptional curve is essential.

\section{Restriction of stable bundles to certain curves}

Let us recall the basic properties of rational and minimally elliptic singularities. Let $x$ be a normal singular point on a complex surface $\bar{X}$, and let $\varphi: X \to \bar{X}$ be the minimal resolution of singularities of $\bar{X}$. Suppose that $\varphi^{-1}(x) = \bigcup_j D_j$. The singularity is a rational singularity if $(R^1\varphi_*\mathcal{O}_X)_x = 0$. Equivalently, by [1], $x$ is rational if and only if, for every choice of nonnegative integers $n_i$ such that at least one of the $n_i$ is strictly positive, if we set $Z = \bigoplus_i n_i D_i$, the arithmetic genus $p_a(Z)$ of the effective curve $Z$ satisfies $p_a(Z) \leq 0$. Here $p_a(Z) = 1 - \chi(\mathcal{O}_Z) = 1 - h^0(\mathcal{O}_Z) + h^1(\mathcal{O}_Z) \leq h^1(\mathcal{O}_Z)$; moreover we have the adjunction formula

$$p_a(Z) = 1 + \frac{1}{2}(K_X + Z) \cdot Z.$$ 

Now every minimal resolution of a normal surface singularity $x$ has a fundamental cycle $Z_0$, which is an effective cycle $Z_0$ supported in the set $\varphi^{-1}(x)$ and satisfying $Z_0 \cdot D_i \leq 0$ and $Z_0 \cdot D_i < 0$ for some $i$ which is minimal with respect to the above properties. We may find $Z_0$ as follows [22]: start with an arbitrary component $A_1$ of $\varphi^{-1}(x)$ and set $Z_1 = A_1$. Now either $Z_0 = A_1$ or there exists another component $A_2$ with $Z_1 \cdot A_2 > 0$. Set $Z_2 = Z_1 + A_2$ and continue this process. Eventually we reach $Z_k = Z_0$. Such a sequence $A_1, \ldots, A_k$ with $Z_i = \sum_{j \leq i} A_j$ and $Z_i \cdot A_{i+1} > 0$, $Z_k = Z_0$ is called a computation sequence. By a theorem of Artin [1], $x$ is rational if and only if $p_a(\overline{Z}_0) \leq 0$, where $\overline{Z}_0$ is the fundamental cycle, if and only if $p_a(\overline{Z}_0) = 0$. Moreover, if $x$ is a rational singularity, then every component $D_i$ of $\varphi^{-1}(x)$ is a smooth rational curve, the $D_i$ meet transversally at at most one point, and the dual graph of $\varphi^{-1}(x)$ is contractible.

Next we recall the properties of minimally elliptic singularities [22]. A singularity $x$ is minimally elliptic if and only if there exists a minimally elliptic cycle $Z$ for $x$, in other words a cycle $Z = \sum_i n_i D_i$ with all $n_i > 0$ such that $p_a(Z) = 1$ and $p_a(Z') \leq 0$ for all nonzero effective cycles $Z' < Z$ (i.e. such that $Z' = \sum_i n'_i D_i$ with $0 \leq n'_i \leq n_i$ and $Z' \neq Z$). In this case it follows that $Z = Z_0$ is the fundamental cycle for $x$, and $(K_X + Z_0) \cdot D_i = 0$ for every component $D_i$ of $\varphi^{-1}(x)$. If $Z_0$ is reduced, i.e. if $n_i = 1$ for all $i$, then the possibilities for $x$ are as follows:

1. $\varphi^{-1}(x)$ is an irreducible curve of arithmetic genus one, and thus is either a smooth elliptic curve or a singular rational curve with either a node or a cusp;
2. $\varphi^{-1}(x) = \bigcup_{i=1}^t D_i$ is a cycle of $t \geq 2$ smooth rational curves meeting transversally, i.e. $D_i \cdot D_{i+1} = 1$, $D_i \cdot D_j \neq 0$ if and only if $i \equiv j \pm 1 \pmod{t}$, except for $t = 2$ where $D_1 \cdot D_2 = 2$;
(3) \( \varphi^{-1}(x) = D_1 \cup D_2 \), where the \( D_i \) are smooth rational, \( D_1 \cdot D_2 = 2 \) and \( D_1 \cap D_2 \) is a single point (so that \( \varphi^{-1}(x) \) has a tacnode singularity) or \( \varphi^{-1}(x) = D_1 \cup D_3 \cup D_3 \) where the \( D_i \) are smooth rational, \( D_1 \cdot D_3 = 1 \) but \( D_1 \cap D_2 \cap D_3 \) is a single point (the three curves meet at a common point).

Here \( x \) is called a simple elliptic singularity in case \( \varphi^{-1}(x) \) is a smooth elliptic curve, a cusp singularity if \( \varphi^{-1}(x) \) is an irreducible rational curve with a node or a cycle as in (2), and a triangle singularity in the remaining cases. If \( Z_0 \) is not reduced, then all components \( D_i \) of \( \varphi^{-1}(x) \) are smooth rational curves meeting transversally and the dual graph of \( \varphi^{-1}(x) \) is contractible.

With this said, and using the discussion in the previous subsection, we will complete the proof of Theorem 1.4 by showing that \( H^0(mZ; \text{ad} V \otimes \mathcal{O}_X(mZ)|mZ) = 0 \) for all \( i \), where \( x_1, \ldots, x_k \) are the singular points of \( X \) and \( Z_i \) is an effective cycle with \( \text{Supp} \, Z_i = \varphi^{-1}(x_i) \). The precise statement is as follows:

**Theorem 2.3.** Let \( \varphi \colon X \to \tilde{X} \) be a birational morphism from \( X \) to a normal projective surface \( \tilde{X} \), corresponding to a nef, big, and eventually base point free divisor \( M \). Let \( w \) be the mod 2 reduction of \([K_X]\), and suppose that

(i) \( \varphi \) contracts no exceptional curve; in other words, if \( E \) is an exceptional curve of the first kind on \( X \), then \( M \cdot E > 0 \).

(ii) \( \tilde{X} \) has only rational and minimally elliptic singularities.

Then there exists a constant \( A \) depending only on \( p \) and \( \mathcal{N} \) with the following property: for every singular point \( x \) of \( \tilde{X} \), there exists an effective cycle \( Z \) with \( \text{Supp} \, Z = \varphi^{-1}(x) \) such that, for all ample line bundles \( L \) in \( \mathcal{N} \), all \( p \) with \( p \leq A \), and generic bundles \( V \) in \( \mathfrak{M}(w, p) \),

\[
H^0(mZ; \text{ad} V \otimes \mathcal{O}_X(mZ)|mZ) = 0
\]

for every positive integer \( m \).

The statement of (i) may be rephrased by saying that \( X \) is the minimal resolution of \( \tilde{X} \). As \( \text{ad} V \subset \text{Hom}(V, V) \), it suffices to prove that \( H^0(mZ; \text{Hom}(V, V) \otimes \mathcal{O}_X(mZ)|mZ) = 0 \). We will consider the case of rational singularities and minimally elliptic singularities separately. Let us begin with the proof for rational singularities. Let \( \varphi^{-1}(x) = \bigcup_i D_i \), where each \( D_i \) is a smooth rational curve. By the assumption (4) of the previous subsection, we can assume that the constant \( A \) has been chosen so that \( V|D_i \) is a generic bundle over \( D_i \cong \mathbb{P}^1 \) for every \( i \). Thus either there exists an integer \( a \) such that \( V|D_i \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a+1) \), if \( w \cdot D_i \neq 0 \), or there exists an \( a \) such that \( V|D_i \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a) \), if \( w \cdot D_i = 0 \). Next, we have the following claim:

**Claim 2.4.** Suppose that \( x \) is a rational singularity. Let \( \varphi \colon X \to \tilde{X} \) be a resolution of \( x \). There exist a sequence of curves \( B_0, \ldots, B_k \), such that \( B_i \subseteq \varphi^{-1}(x) \) for all \( i \), with the following property:

1. Let \( C_i = \sum_{j \leq i} B_j \). Then \( B_i \cdot C_i \leq B_i^2 + 1 \).
2. \( C_k = Z_0 \), the fundamental cycle of \( x \).

**Proof.** Since \( (K_X + Z_0) \cdot Z_0 < 0 \), there must exist a component \( B^{(0)} = D_i \) of \( \text{Supp} \, Z_0 = \varphi^{-1}(x) \) such that \( (K_X + Z_0) \cdot B^{(0)} < 0 \). Thus

\[
Z_0 \cdot B^{(0)} < -K_X \cdot B^{(0)} = (B^{(0)})^2 + 2.
\]
Set $Z_1 = Z_0 - B^{(0)}$. Suppose that $Z_1$ is nonzero. Then $Z_1$ is again effective, and by Artin’s criterion $p_a(Z_1) \leq 0$. Thus by repeating the above argument there is a $B^{(1)}$ contained in the support of $Z_1$ such that $Z_1 \cdot B^{(1)} < (B^{(1)})^2 + 2$. Continuing, we eventually find $B^{(2)}, \ldots, B^{(k)}$ with $B^{(i)}$ contained in the support of $Z_i$, $Z_{i+1} = Z_i - B^{(i)}$ and $k = B^{(k)}$, and such that $Z_i \cdot B^{(i)} < (B^{(i)})^2 + 2$. If we now relabel $B^{(i)} = B_{k-i}$, then $Z_i = \sum_{j \leq n-i} B_j$ and the curves $B_0, \ldots, B_k$ are as claimed. □

Returning to the proof of Theorem 2.3, we first prove that

$$H^0(Z_0; Hom(V, V) \otimes O_X(Z_0)|Z_0) = 0.$$ 

We have the exact sequence

$$0 \to O_{C_{i-1}}(C_{i-1}) \to O_{C_i}(C_i) \to O_{B_i}(C_i) \to 0.$$ 

Tensor this sequence by $Hom(V, V)$. We shall prove by induction that

$$H^0(Hom(V, V) \otimes O_{C_i}(C_i)) = 0$$

for all $i$. It suffices to show that $H^0(Hom(V, V) \otimes O_{B_i}(C_i)) = 0$ for all $i$. Now $O_{B_i}(C_i)$ is a line bundle on the smooth rational curve $B_i$. If $V|B_i \cong O_{p^1}(a) \oplus O_{p^1}(a+1)$, then $w \cdot B_i \neq 0$ and so $B_i^2$ is odd. Since $B_i$ is not an exceptional curve, $B_i^2 \leq -3$ and so $B_i \cdot C_i \leq -2$. Thus, as

$$Hom(V, V)|B_i = O_{p^1}(-1) \oplus O_{p^1} \oplus O_{p^1} \oplus O_{p^1}(1),$$

we see that $H^0(Hom(V, V) \otimes O_{B_i}(C_i)) = 0$. Likewise if $V|D_i \cong O_{p^1}(a) \oplus O_{p^1}(a)$, then using $B_i \cdot C_i \leq -1$ we again have $H^0(Hom(V, V) \otimes O_{B_i}(C_i)) = 0$. Thus by induction

$$H^0(Hom(V, V) \otimes O_{C_i}(C_k)) = H^0(Hom(V, V) \otimes O_{Z_0}(Z_0)) = 0.$$ 

The vanishing of $H^0(mZ_0; Hom(V, V) \otimes O_X(mZ_0)|mZ_0)$ is similar, using instead the exact sequence

$$0 \to O_{mZ_0+C_{i-1}}(mZ_0+C_{i-1}) \to O_{mZ_0+C_i}(mZ_0+C_i) \to O_{B_i}(mZ_0+C_i) \to 0.$$ 

This concludes the proof in the case of a rational singularity.

For minimally elliptic singularities, we shall deduce the theorem from the following more general result:

**Theorem 2.5.** Let $\varphi: X \to \bar{X}$ be a birational morphism from $X$ to a normal projective surface $\bar{X}$, corresponding to a nef, big, and eventually base point free divisor $M$. Let $w$ be an arbitrary element of $H^2(X; \mathbb{Z}/2\mathbb{Z})$, and suppose that

(i) $\varphi$ contracts no exceptional curve; in other words, if $E$ is an exceptional curve of the first kind on $X$, then $M \cdot E > 0$.

(ii) If $D$ is a component of $\varphi^{-1}(x)$ such that $w \cdot D \neq 0$, then $Z_0 \cdot D < 0$, where $Z_0$ is the fundamental cycle of $\varphi^{-1}(x)$. 


Then the conclusions of Theorem 2.3 hold for the moduli space $\mathcal{M}_k(w,p)$ for all $p \ll 0$. In particular the conclusions of Theorem 2.3 hold if $\varphi^{-1}(x)$ is irreducible.

Proof that (2.5) implies (2.3). We must show that every minimally elliptic singularity satisfies the hypotheses of Theorem 2.5(ii), provided that $w$ is the mod 2 reduction of $K_X$. Suppose that $x$ is minimally elliptic and that $w \cdot D \neq 0$. Thus $K_X \cdot D$ is odd. Moreover if $D$ is smooth rational then $D^2 \neq -1$ and $K_X \cdot D \geq 0$ so that $K_X \cdot D \geq 1$. Now $(K_X + Z_0) \cdot D = 0$. Thus $Z_0 \cdot D = -(K_X \cdot D) \leq -1$. Likewise if $p_a(D) \neq 0$, so that $D$ is not a smooth rational curve, then $\varphi^{-1}(x) = D$ is an irreducible curve and (2.3) again follows. \(\square\)

Proof of Theorem 2.6. We begin with a lemma on sections of line bundles over effective cycles supported in $\varphi^{-1}(x)$, which generalizes (2.6) of [22]:

Lemma 2.6. Let $Z_0$ be the fundamental cycle of $\varphi^{-1}(x)$ and let $\lambda$ be a line bundle on $Z_0$ such that $\deg(\lambda|D) \leq 0$ for each component $D$ of the support of $Z_0$. Then either $H^0(Z_0; \lambda) = 0$ or $\lambda = \mathcal{O}_{Z_0}$ and $H^0(Z_0; \lambda) \cong \mathbb{C}$.

Proof. Choose a computation sequence for $Z_0$, say $A_1, A_2, \ldots, A_k$. Thus, if we set $Z_i = \sum_{j \leq i} A_j$, then $Z_i \cdot A_{i+1} > 0$, and $Z_k = Z_0$. Now we have an exact sequence

$$0 \to \mathcal{O}_{A_{i+1}}(-Z_i) \to \mathcal{O}_{Z_{i+1}} \to \mathcal{O}_{Z_i} \to 0.$$ 

Thus $\deg(\mathcal{O}_{A_{i+1}}(-Z_i) \otimes \lambda|A_{i+1}) < 0$. It follows that $H^0(\mathcal{O}_{Z_{i+1}} \otimes \lambda) \subseteq H^0(\mathcal{O}_{Z_i} \otimes \lambda)$ for all $i$. By induction $\dim H^0(\mathcal{O}_{Z_i} \otimes \lambda) \leq 1$ for all $i$, $1 \leq i \leq k$. Thus $\dim H^0(Z_0; \lambda) \leq 1$. Moreover, if $\dim H^0(Z_0; \lambda) = 1$, then the natural map

$$H^0(\mathcal{O}_{Z_{i+1}} \otimes \lambda) \to H^0(\mathcal{O}_{Z_i} \otimes \lambda)$$

is an isomorphism for all $i$, and so the induced map $H^0(Z_0; \lambda) \to H^0(A_1; \lambda|A_1)$ is an isomorphism and $\dim H^0(A_1; \lambda|A_1) = 1$. Thus $\lambda|A_1$ is trivial and a nonzero section of $H^0(Z_0; \lambda)$ restricts to a generator of $\lambda|A_1$. Since we can begin a computation sequence with an arbitrary choice of $A_1$, we see that a nonzero section $s$ of $H^0(Z_0; \lambda)$ restricts to a nonvanishing section of $H^0(D; \lambda|D)$ for every $D$ in the support of $\varphi^{-1}(x)$. Thus the map $\mathcal{O}_{Z_0} \to \lambda$ defined by $s$ is an isomorphism. \(\square\)

Remark. The lemma is also true if $\lambda$ is allowed to have degree one on some components $D$ of $Z_0$ with $p_a(D) \geq 2$, provided that $\lambda|D$ is general for these components, and a slight variation holds if $\lambda$ is also allowed to have degree one on some components $D$ of $Z_0$ with $p_a(D) = 1$.

We next construct a bundle $W$ over $Z_0$ with certain vanishing properties:

Lemma 2.7. Suppose that $\varphi: X \to \bar{X}$ is the minimal resolution of the normal surface singularity $x$. Let $\mu$ be a line bundle over the scheme $Z_0$. Suppose further that, if $D$ is a component of $\varphi^{-1}(x)$ such that $\deg(\mu|D)$ is odd, then $Z_0 \cdot D < 0$, where $Z_0$ is the fundamental cycle of $\varphi^{-1}(x)$. Then there exists a rank two vector bundle $W$ over $Z_0$ with $\det W = \mu$ and such that

$$H^0(Z_0; Hom(W, W) \otimes \mathcal{O}_X(mZ_0)|Z_0) = 0$$

for every $m \geq 1$.

Proof. Let $\varphi^{-1}(x) = \bigcup D_i$. Then there exists an integer $a_i$ such that $\deg(\mu|D_i) = 2a_i$ or $2a_i + 1$, depending on whether $\deg(\mu|D_i)$ is odd or even. Since $\dim Z_0 = 1$, the
natural maps \( \text{Pic } Z_0 \to \text{Pic}(Z_0)_{\text{red}} \to \bigoplus_i \text{Pic } D_i \) are surjective. Thus we may choose a line bundle \( L_1 \) over \( Z_0 \) such that \( \deg(L_1|D_i) = a_i \). It follows that \( \mu \otimes L_1^{\otimes -2}|D_i \) is a line bundle over \( D_i \) of degree zero or 1, and if it is of degree 1, then \( Z_0 \cdot D_i < 0 \). Hence \( \mu \otimes L_1^{\otimes -2} \otimes O_{Z_0}(Z_0) \) has degree at most zero on \( D_i \) for every \( i \).

Set \( L_2 = \mu \otimes L_1^{-1} \). Thus \( L_1 \otimes L_2 = \mu \) and \( \deg(L_2|D_i) = a_i \) or \( a_i + 1 \) depending on whether \( \deg(\mu|D_i) \) is even or odd. The line bundle \( L_1^{-1} \otimes L_2 = \mu \otimes L_1^{\otimes -2} \) thus has degree zero on those components \( D_i \) such that \( \deg(\mu|D_i) \) is even and 1 on the components \( D_i \) such that \( \deg(\mu|D_i) \) is odd. Moreover \( \deg(L_1^{-1} \otimes L_2 \otimes O_{Z_0}(mZ_0)|D_i) \leq 0 \) for every \( i \).

**Claim 2.8.** Under the assumptions of (2.7), there exists a nonsplit extension \( W \) of \( L_2 \) by \( L_1 \) except in the case where \( x \) is rational, \( \deg(\mu|D_i) \) is odd for at most one \( i \), and the multiplicity of \( D_i \) in \( Z_0 \) is one for such \( i \), or \( \chi(O_{Z_0}) = 0 \) and \( \deg(\mu|D_i) \) is even for every \( i \).

**Proof.** A nonsplit extension exists if and only if \( h^1(L_2^{-1} \otimes L_1) \neq 0 \). Now by the Riemann-Roch theorem applied to \( Z_0 = \sum_i n_i D_i \), we have

\[
h^1(Z_0; L_2^{-1} \otimes L_1) = h^0(Z_0; L_2^{-1} \otimes L_1) - \sum_i n_i \deg(L_2^{-1} \otimes L_1|D_i) - \chi(O_{Z_0}).
\]

Here \( \deg(L_2^{-1} \otimes L_1|D_i) = 0 \) on those \( D_i \) with \( \deg(\mu|D_i) \) even and \( = -1 \) on the \( D_i \) with \( \deg(\mu|D_i) \) odd. Moreover \( h^0(O_{Z_0}) = 1 \) by Lemma 2.6 and so \( \chi(O_{Z_0}) \leq 1 \), with \( \chi(O_{Z_0}) = 1 \) if and only if \( x \) is rational. Thus

\[
h^1(Z_0; L_2^{-1} \otimes L_1) \geq \sum \{ n_i : \deg(\mu|D_i) \text{ is odd} \} - \chi(O_{Z_0}).
\]

Hence if \( h^1(Z_0; L_2^{-1} \otimes L_1) = 0 \), then either \( x \) is rational, \( \deg(\mu|D_i) \) is odd for at most one \( i \), and for such \( i \) the multiplicity of \( D_i \) in \( Z_0 \) is one, or \( \deg(\mu|D_i) \) is even for all \( i \) and \( \chi(O_{Z_0}) = 0 \). \( \square \)

Returning to the proof of (2.7), choose \( W \) to be a nonsplit extension of \( L_2 \) by \( L_1 \) if such exist, and set \( W = L_1 \oplus L_2 \) otherwise. To see that \( H^0(Z_0; \text{Hom}(W,W) \otimes O_X(mZ_0)|Z_0) = 0 \), we consider the two exact sequences

\[
0 \to L_1 \to W \to L_2 \to 0;
0 \to L_1 \otimes O_{Z_0}(mZ_0) \to W \otimes O_{Z_0}(mZ_0) \to L_2 \otimes O_{Z_0}(mZ_0) \to 0.
\]

Clearly \( H^0(Z_0; \text{Hom}(W,W) \otimes O_X(mZ_0)|Z_0) = 0 \) if

\[
H^0(L_1^{-1} \otimes L_2 \otimes O_{Z_0}(mZ_0)) = H^0(O_{Z_0}(mZ_0)) = H^0(L_2^{-1} \otimes L_1 \otimes O_{Z_0}(mZ_0)) = 0.
\]

The line bundles \( O_{Z_0}(mZ_0) \) and \( L_2^{-1} \otimes L_1 \otimes O_{Z_0}(mZ_0) \) have nonpositive degree on each \( D_i \) and (since \( Z_0 \cdot D_i < 0 \) for some \( i \)) have strictly negative degree on at least one component. Thus by Lemma 2.6 \( H^0(O_{Z_0}(mZ_0)) \) and \( H^0(L_2^{-1} \otimes L_1 \otimes O_{Z_0}(mZ_0)) \) are both zero. Let us now consider the group \( H^0(L_1^{-1} \otimes L_2 \otimes O_{Z_0}(mZ_0)) \). By the hypothesis that \( Z_0 \cdot D_i < 0 \) for each \( D_i \) such that \( \deg(\mu|D_i) \) is odd, the line bundle \( L_1^{-1} \otimes L_2 \otimes O_{Z_0}(mZ_0) \) has nonpositive degree on all components \( D_i \). Thus by Lemma 2.6 either \( H^0(L_1^{-1} \otimes L_2 \otimes O_{Z_0}(mZ_0)) = 0 \) or \( L_1^{-1} \otimes L_2 \otimes O_{Z_0}(mZ_0) \cong O_{Z_0} \). Clearly this last case is only possible if \( m = 1 \) and \( L_1 \cong L_2 \otimes O_{Z_0}(Z_0) \), and if
moreover $Z_0 \cdot D_i = 0$ if $\deg(\mu|D_i)$ is even and $Z_0 \cdot D_i = -1$ if $\deg(\mu|D_i)$ is odd. As $Z_0 \cdot D_i < 0$ for at least one $i$, $\deg(\mu|D_i)$ is odd for at least one $i$ as well. In this case, if the nonzero section of $L_1^{-1} \otimes L_2 \otimes \mathcal{O}_{Z_0}(Z_0)$ lifts to give a map $L_1 \to W \otimes \mathcal{O}_{Z_0}(Z_0)$, then the image of $L_1$ in $W \otimes \mathcal{O}_{Z_0}(Z_0)$ splits the exact sequence

$$0 \to L_1 \otimes \mathcal{O}_{Z_0}(Z_0) \to W \otimes \mathcal{O}_{Z_0}(Z_0) \to L_2 \otimes \mathcal{O}_{Z_0}(Z_0) \to 0.$$ 

Thus $W$ is also a split extension. By Claim 2.8, since $\deg(\mu|D_i)$ is odd for at least one $i$, it must therefore be the case that $x$ is rational, $\deg(\mu|D_i)$ is odd for exactly one $i$, and for such $i$ the multiplicity of $D_i$ in $Z_0$ is one. Moreover $Z_0 \cdot D_j \neq 0$ exactly when $j = i$ and in this case $Z_0 \cdot D_i = -1$. But as the multiplicity of $D_i$ in $Z_0$ is 1, we can write $Z_0 = D_i + \sum_{j \neq i} n_j D_j$, and thus

$$Z_0^2 = Z_0 \cdot D_i = -1.$$

By a theorem of Artin [1], however, $-Z_0^2$ is the multiplicity of the rational singularity $x$. It follows that $x$ is a smooth point and $\varphi$ is the contraction of a generalized exceptional curve, contrary to hypothesis. This concludes the proof of (2.7). □

We may now finish the proof of (2.5). Start with a generic vector bundle $V_0 \in \mathcal{M}_L(w, p)$ on $X$ satisfying the condition that $H^1(X; \text{ad } V_0) \to H^1(Z_0; \text{ad } V_0|Z_0)$ is surjective. If $\mu = \det V_0|Z_0$, note that, according to the assumptions of (2.5), $\mu$ satisfies the hypotheses of Lemma 2.7. For $V \in \mathcal{M}_L(w; p)$, let

$$H(mZ_0) = H^0(mZ_0; \text{ad } V \otimes \mathcal{O}_X(mZ_0)|mZ_0).$$

Using the exact sequence

$$0 \to H((m - 1)Z_0) \to H(mZ_0) \to H^0(Z_0; \text{ad } V_0 \otimes \mathcal{O}_X(mZ_0)|Z_0),$$

we see that it suffices to show that, for a generic $V$, $H^0(Z_0; \text{ad } V \otimes \mathcal{O}_X(mZ_0)|Z_0) = 0$ for all $m \geq 1$. For a fixed $m$, the condition that $H^0(Z_0; \text{ad } V \otimes \mathcal{O}_X(mZ_0)|Z_0) \neq 0$ is a closed condition. Thus since the moduli space cannot be a countable union of proper subvarieties, it will suffice to show that the set of $V$ for which $H^0(Z_0; \text{ad } V \otimes \mathcal{O}_X(mZ_0)|Z_0) = 0$ is nonempty for every $m$. Let $S$ be the germ of the versal deformation of $V_0|Z_0$ keeping $\det V_0|Z_0$ fixed. By the assumption that the map from the germ of the versal deformation of $V_0$ to that of $V_0|Z_0$ is submersive, it will suffice to show that, for each $m \geq 1$, the set of $W \in S$ such that $H^0(Z_0; \text{ad } W \otimes \mathcal{O}_{Z_0}(mZ_0)) = 0$ is nonempty. One natural method for doing so is to exhibit a deformation from $V_0|Z_0$ to the $W$ constructed in the course of Lemma 2.7; roughly speaking this amounts to the claim that the “moduli space” of vector bundles on the scheme $Z_0$ is connected. Although we shall proceed slightly differently, this is the main idea of the argument.

Choose an ample line bundle $\lambda$ on $Z_0$. After passing to some power, we may assume that both $(V_0|Z_0) \otimes \lambda$ and $W \otimes \lambda$ are generated by their global sections. A standard argument shows that, in this case, both $V_0|Z_0$ and $W$ can be written as an extension of $\mu \otimes \lambda$ by $\lambda^{-1}$: Working with $W$ for example, we must show that there is a map $\lambda^{-1} \to W$, corresponding to a section of $W \otimes \lambda$, such that the quotient is again a line bundle. It suffices to show that there exists a section $s \in H^0(Z_0; W \otimes \lambda)$ such that, for each $z \in Z_0$, $s(z) \neq 0$ in the fiber of $W \otimes \lambda$ over $z$. 
Now for $z$ fixed, the set of $s \in H^0(Z_0; W \otimes \lambda)$ such that $s(z) = 0$ has codimension two in $H^0(Z_0; W \otimes \lambda)$ since $W \otimes \lambda$ is generated by its global sections. Thus the set of $s \in H^0(Z_0; W \otimes \lambda)$ such that $s(z) = 0$ for some $z \in Z_0$ has codimension at least one, and so there exists an $s$ as claimed.

Now let $W_0 = \lambda^{-1} \oplus (\mu \otimes \lambda)$. Let $(S_0, s_0)$ be the germ of the versal deformation of $W_0$ (with fixed determinant $\mu$). As $Z_0$ has dimension one, $S_0$ is smooth. Both $V_0|Z_0$ and $W$ correspond to extension classes $\xi, \xi' \in \text{Ext}^1(\mu \otimes \lambda, \lambda^{-1})$. Replacing, say, $\xi$ by the class $t\xi, t \in \mathbb{C}^*$, gives an isomorphic bundle. In this way we obtain a family of bundles $\mathcal{V}$ over $Z_0 \times \mathbb{C}$, such that the restriction of $\mathcal{V}$ to $Z_0 \times t$ is $V_0|Z_0$ if $t \neq 0$ and is $W_0$ if $t = 0$. Hence in the germ $S_0$ there is a subvariety containing $s_0$ in its closure and consisting of bundles isomorphic to $V_0|Z_0$, and similarly for $W$. As $H^0(Z_0; \text{ad} W \otimes \mathcal{O}_{Z_0}(mZ_0)) = 0$, the locus of bundles $U$ in $S_0$ for which $H^0(Z_0; \text{ad} U \otimes \mathcal{O}_{Z_0}(mZ_0)) = 0$ is a dense open subset. Since $S_0$ is a smooth germ, it follows that there is a small deformation of $V_0|Z_0$ to such a bundle. Thus the generic small deformation $U$ of $V_0|Z_0$ satisfies $H^0(Z_0; \text{ad} U \otimes \mathcal{O}_{Z_0}(mZ_0)) = 0$, and so the generic $V \in \mathcal{M}_L(w, p)$ has the property that $H^0(Z_0; \text{ad} V \otimes \mathcal{O}_X(mZ_0)|Z_0) = 0$ for all $m \geq 1$ as well. As we saw above, this implies the vanishing of $H^0(mZ_0; \text{ad} V \otimes \mathcal{O}_X(mZ_0)|Z_0)$.

**Remark.** (1) Suppose that $\bar{X}$ is a singular surface, but that $\varphi: X \to \bar{X}$ is not the minimal resolution. We may still define the fundamental cycle $Z_0$ for the resolution $\varphi$. Moreover it is easy to see that $Z_0 \cdot E = 0$ for every component of a generalized exceptional curve contained in $\varphi^{-1}(x)$. Thus the hypothesis of (ii) of Theorem 2.5 implies that $w \cdot E = 0$ for such curves.

(2) We have only considered contractions of a very special type, and have primarily been interested in the case where $w$ is the mod two reduction of $[K_X]$. However it is natural to ask if the analogues of Theorem 2.3 and 2.5 (and thus Theorem 1.4) holds for more general contractions and choices of $w$, provided of course that no smooth rational curve of self-intersection $-1$ is contracted to a point. Clearly the proof of Theorem 2.5 applies to a much wider class of singularities. Indeed a little work shows that the proof goes over (with some modifications in case there are components of arithmetic genus one) to handle the case where we need only assume condition (ii) of (2.5) for those components $D$ which are smooth rational curves. Another case where it is easy to check that the conclusions of (2.5) hold is where $w$ is arbitrary and the dual graph of the singularity is of type $A_k$. We make the following rather natural conjecture:

**Conjecture 2.9.** The conclusions of Theorem 1.4 hold for arbitrary choices of $w$ and $\varphi$, provided that $\varphi$ does not contract any exceptional curves of the first kind.

3. Nonexistence of embedded 2-spheres

3.1. A base point free theorem

**Theorem 3.1.** Let $\pi: X \to X'$ be a birational morphism from the smooth surface $X$ to a normal surface $X'$, not necessarily projective. Suppose that $X$ is a minimal surface of general type, and that $p \in X'$ is an isolated singular point which is a nonrational singularity. Let $\pi^{-1}(p) = \bigcup_i D_i$. Then:

- (i) There exist nonnegative integers $n_i$ with $n_i > 0$ for at least one $i$ such that $K_X + \sum_i n_i D_i$ is nef and big.
(ii) Suppose that \( q(X) = 0 \). Then there further exists a choice of \( D = \sum_i n_i D_i \) satisfying (i) with \( D \) connected and such that there exists a section of \( K_X + D \) which is nowhere vanishing in a neighborhood of

\[
E = \bigcup \{ D_j : (K_X + D) \cdot D_j = 0 \}.
\]

In this case either \( E = \emptyset \) or \( E = \text{Supp} \, D \) and \( D \) is the fundamental cycle of the minimal resolution of a minimally elliptic singularity.

(iii) With \( D \) satisfying (i) and (ii), the linear system \( K_X + D \) is eventually base point free. Moreover, if \( \varphi : X \to \bar{X} \) is the associated contraction, then \( \bar{X} \) is a normal projective surface all of whose singular points are either rational or minimally elliptic.

**Proof.** To prove (i), consider the set of all effective cycles \( D = \sum a_i D_i \), where the \( a_i \) are nonnegative integers, not all zero, and such that \( h^1(\mathcal{O}_D) \neq 0 \). This set is not empty by the definition of a nonrational singularity, and is partially ordered by \( \leq \), where \( D' \leq D \) if \( D - D' \) is effective. Choose a minimal element \( D \) in the set. This means that \( D = \sum n_i D_i \) where either \( n_i = 1 \) for exactly one \( i \) and \( h^1(\mathcal{O}_{D_i}) \neq 0 \), or for every irreducible \( D_i \) contained in the support of \( D \), \( D - D_i = D' \) is effective and \( h^1(\mathcal{O}_{D'}) = 0 \). If \( D'' \) is then any nonzero effective cycle with \( D'' < D \), then there exists an \( i \) such that \( \mathcal{O}_{D_i - D} \to \mathcal{O}_{D''} \) is surjective. By a standard argument, \( H^1(\mathcal{O}_{D - D_i}) \to H^1(\mathcal{O}_{D''}) \) is surjective and thus \( h^1(\mathcal{O}_{D''}) = 0 \) for every nonzero effective \( D'' < D \). Finally note that \( D \) is connected, since otherwise we could replace \( D \) by some connected component \( D_0 \) with \( h^1(\mathcal{O}_{D_0}) \neq 0 \).

Next we claim that \( K_X + D \) is nef. Since \( K_X \) is nef, it is clear that \( (K_X + D) \cdot C \geq 0 \) for every irreducible curve \( C \) not contained in the support of \( D \), and moreover, for such curves \( C \), \( (K_X + D) \cdot C = 0 \) if and only if \( C \) is a smooth rational curve of self-intersection \(-2\) disjoint from the support of \( D \). Next suppose that \( D_i \) is a curve in the support of \( D \) and consider \( (K_X + D) \cdot D_i \). If \( D = D_i \) then \( K_X + D_i | D_i = \omega_{D_i} \), the dualizing sheaf of \( D_i \), and this has degree \( 2p_a(D_i) - 2 \geq 0 \) since \( p_a(D_i) = h^1(\mathcal{O}_{D_i}) > 0 \). Otherwise let \( D' = D - D_i \) consider the exact sequence

\[
0 \to \mathcal{O}_{D_i}(-D') \to \mathcal{O}_D \to \mathcal{O}_{D'} \to 0.
\]

Thus the natural map \( H^1(\mathcal{O}_{D_i}(-D')) \to H^1(\mathcal{O}_D) \) is surjective since \( H^1(\mathcal{O}_{D'}) = 0 \), and so \( H^1(\mathcal{O}_{D_i}(-D')) \neq 0 \) as \( H^1(\mathcal{O}_D) \neq 0 \). By duality \( H^0(D_i; \omega_{D_i} \otimes \mathcal{O}_{D'}(D')) \neq 0 \). On the other hand \( \omega_{D_i} = K_X + D_i | D_i \), and so \( \deg(K_X + D + D_i | D_i) = (K_X + D) \cdot D_i \geq 0 \); moreover \( (K_X + D) \cdot D_i \geq 0 \) only if the divisor class \( K_X + D | D_i \) is trivial.

Next, \( (K_X + D)^2 \geq K_X^2 > 0 \), so that \( K_X + D \) is big. In fact,

\[
(K_X + D)^2 \geq K_X^2 > 0.
\]

Thus \( K_X + D \) is big.

To see (iii), let \( E = \bigcup \{ D_j : (K_X + D) \cdot D_j = 0 \} \). We shall also view \( E \) as a reduced divisor. We claim that \( \mathcal{O}_E(K_X + D) = \mathcal{O}_E \). First assume that \( E = D \) (and thus in particular that \( D \) is reduced); in this case we need to show that \( \omega_D = \mathcal{O}_D \). By assumption \( D \) is connected. Then \( \omega_D \) has degree zero on every reduced irreducible component of \( D \), and by Serre duality \( \chi(\omega_D) = -\chi(\mathcal{O}_D) = \frac{1}{2} (K_X + D) \cdot D = 0 \). As \( h^1(\omega_D) = h^0(\mathcal{O}_D) = 1 \), \( h^0(\omega_D) = 1 \) as well. As \( \omega_D \) has
degree zero on every component of $D$, if $s$ is a section of $\omega_D$, then the restriction of $s$ to every component $D_i$ of $D$ is either identically zero or nowhere vanishing. Thus if $s$ is nonzero, since $D$ is connected, $s$ must be nowhere vanishing. It follows that the map $\mathcal{O}_D \to \omega_D$ is surjective and is thus an isomorphism.

If $D \neq E$, we apply the argument that showed above that $(K_X + D) \cdot D_i \geq 0$ to each connected component $E_0$ of the divisor $E$, with $D' = D - E_0$, to see that there is a section of $\mathcal{O}_{E_0}(K_X + D)$. Since $\mathcal{O}_{E_0}(K_X + D)$ has degree zero on each irreducible component of $E_0$, the argument that worked for the case $D = E$ also works in this case.

Now let us show that, provided $q(X) = 0$, a nowhere zero section of $\mathcal{O}_E(K_X + D) = \mathcal{O}_E$ lifts to a section of $K_X + D$. It suffices to show that, for every connected component $E_0$ of $E$, a nowhere vanishing section of $\mathcal{O}_{E_0}(K_X + D)$ lifts to a section of $K_X + D$. Let $D' = D - E_0$. If $D' = 0$ then $D = E = E_0$ and we ask if the map $H^0(\mathcal{O}_X(K_X + D)) \to H^0(\mathcal{O}_D(K_X + D))$ is surjective. The cokernel of this map lies in $H^1(K_X) = 0$ since $X$ is regular. Otherwise $D' \neq 0$. Beginning with the exact sequence

$$0 \to \mathcal{O}_{D'}(-E_0) \to \mathcal{O}_D \to \mathcal{O}_{E_0} \to 0,$$

and tensoring with $\mathcal{O}_X(K_X + D)$, we obtain the exact sequence

$$0 \to \mathcal{O}_{D'}(K_X + D - E_0) \to \mathcal{O}_D(K_X + D) \to \mathcal{O}_{E_0} \to 0.$$

Now $\mathcal{O}_{D'}(K_X + D - E_0) = \mathcal{O}_{D'}(K_X + D) = \omega_{D'}$ and by duality $h^0(\omega_{D'}) = h^1(\mathcal{O}_{D'}) = 0$. Thus $H^0(\mathcal{O}_D(K_X + D))$ includes into $H^0(\mathcal{O}_{E_0}) = \mathbb{C}$ and so it suffices to prove that $H^0(\mathcal{O}_D(K_X + D)) \neq 0$, in which case it has dimension one. On the other hand, using the exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + D) \to \mathcal{O}_D(K_X + D) \to 0,$$

we see that $h^0(\mathcal{O}_D(K_X + D)) \geq h^0(\mathcal{O}_X(K_X + D)) - p_g(X)$. Since $h^2(K_X + D) = h^0(-D) = 0$, the Riemann-Roch theorem implies that

$$h^0(\mathcal{O}_X(K_X + D)) = h^1(\mathcal{O}_X(K_X + D)) + \frac{1}{2}(K_X + D) \cdot D + 1 + p_g(X).$$

Since all the terms are positive, we see that indeed $h^0(\mathcal{O}_X(K_X + D)) - p_g(X) \geq 1$, and that $h^0(\mathcal{O}_X(K_X + D)) - p_g(X) = 1$ if and only if $h^1(\mathcal{O}_X(K_X + D)) = 0$ and $(K_X + D) \cdot D_i = 0$ for every component $D_i$ contained in the support of $D$. This last condition says exactly that $E = \text{Supp} D$, and thus, as $D$ is connected, that $E_0 = E$. We claim that in this last case $D$ is minimally elliptic. Indeed, for every effective divisor $D'$ with $0 < D' < D$, we have

$$p_g(D') = 1 - h^0(\mathcal{O}_{D'}) + h^1(\mathcal{O}_{D'}) = 1 - h^0(\mathcal{O}_{D'}).$$

Thus $D$ is the fundamental cycle for the resolution of a minimally elliptic singularity.

Finally we prove (iii). The irreducible curves $C$ such that $(K_X + D) \cdot C = 0$ are the components $D_i$ of the support of $D$ such that $(K_X + D) \cdot D_i = 0$, as well as smooth rational curves of self-intersection $-2$ disjoint from $\text{Supp} D$. These last contribute rational double points, so that we need only study the $D_i$ such that $(K_X + D) \cdot D_i = 0$. We have seen in (ii) that either there are no such $D_i$, or every
$D_i$ in the support of $D$ satisfies $(K_X + D) \cdot D_i = 0$ and the contraction of $D$ is a minimally elliptic singularity.

Let $\bar{X}$ be the normal surface obtained by contracting all the irreducible curves $C$ on $X$ such that $(K_X + D) \cdot C = 0$. The line bundle $\mathcal{O}_X(K_X + D)$ is trivial in a neighborhood of these curves, either because they correspond to a rational singularity or because we are in the minimally elliptic case and by (ii). So $\mathcal{O}_X(K_X + D)$ induces a line bundle on $\bar{X}$ which is ample, by the Nakai-Moishezon criterion. Thus $|k(K_X + D)|$ is base point free for all $k \gg 0$. □

3.2. Completion of the proof

We now prove Theorem 1.5:

**Theorem 1.5.** Let $X$ be a minimal simply connected algebraic surface of general type, and let $E \in H^2(X; \mathbb{Z})$ be a $(1, 1)$-class satisfying $E^2 = -1$, $E \cdot K_X = 1$. Let $w$ be the mod 2 reduction of $[K_X]$. Then there exist:

(i) an integer $p$ and (in case $p_g(X) = 0$) a chamber $C$ of type $(w, p)$ and

(ii) a $(1, 1)$-class $M \in H^2(X; \mathbb{Z})$ such that $M \cdot E = 0$ and $\gamma_{w, p}(X)(M^d) \neq 0$ (or, in case $p_g(X) = 0$, $\gamma_{w, p}(X; C)(M^d) \neq 0$).

**Proof.** We begin with the following lemma:

**Lemma 3.2.** With $X$ and $E$ as above, there exists an orientation preserving diffeomorphism $\psi: X \to X$ such that $\psi^*[K_X] = [K_X]$ and such that $\psi^*E \cdot [C] \geq 0$ for every smooth rational curve $C$ on $X$ with $C^2 = -2$.

**Proof.** Let $\Delta = \{[C_1], \ldots, [C_k]\}$ be the set of smooth rational curves on $X$ of self-intersection $-2$, and let $r_i: H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ be the reflection about the class $[C_i]$. Then $r_i$ is realized by an orientation-preserving self-diffeomorphism of $X$, $r_i^*[K_X] = [K_X]$, and $r_i$ preserves the image of $\text{Pic}X$ inside $H^2(X; \mathbb{Z})$. Let $\Gamma$ be the finite group generated by the $r_i$. Since the classes $[C_i]$ are linearly independent, the set

$$\{ x \in H^2(X; \mathbb{R}) : x \cdot [C_i] \geq 0 \}$$

has a nonempty interior. Moreover, if $\Delta' = \Gamma \cdot \Delta$, and we set $W^d = \delta^d$ for $\delta \in \Delta'$, then the connected components of the set $H^2(X; \mathbb{R}) \setminus \bigcup_{\delta \in \Delta'} W^d$ are the fundamental domains for the action of $\Gamma$ on $H^2(X; \mathbb{R})$. Clearly at least one of these connected components lies inside $\{ x \in H^2(X; \mathbb{R}) : x \cdot [C_i] \geq 0 \}$. Thus given $E$ (or indeed an arbitrary element of $H^2(X; \mathbb{R})$), there exists a $\gamma \in \Gamma$ such that $\gamma(E) \cdot [C_i] \geq 0$ for all $i$. As every $\gamma \in \Gamma$ is realized by an orientation preserving self-diffeomorphism $\psi$, this concludes the proof of (3.2). □

Thus, to prove Theorem 1.5, it is sufficient by the naturality of the Donaldson polynomials to prove it for every class $E$ satisfying $E^2 = -1$, $E \cdot K_X = 1$, and $E \cdot [C] \geq 0$ for every smooth rational curve $C$ on $X$ with $C^2 = -2$. We therefore make this assumption in what follows. Given Theorem 1.4, it therefore suffices to find a nef and big divisor $M$ orthogonal to $E$, which is eventually base point free, such that the contraction morphism defined by $|kM|$ has an image with at worst rational and minimally elliptic singularities (note that, since $X$ is assumed minimal, no exceptional curves can be contracted). Thus we will be done by the following lemma:
Lemma 3.3. There exists a nef and big divisor $M$ which is eventually base point free and such that

1. $M \cdot E = 0$.
2. The contraction $\bar{X}$ of $X$ defined by $|kM|$ for all $k \gg 0$ has only rational and minimally elliptic singularities.

Proof. To find $M$ we proceed as follows: consider the divisor $K_X + E = M$. As $K_X \cdot E = 1$ and $E^2 = -1$, $M$ is orthogonal to $E$. Moreover $M^2 = (K_X + E)^2 = K_X^2 + 1 > 0$. We now consider separately the cases where $M$ is nef and where $M$ is not nef.

Case I: $M = K_X + E$ is nef.

Consider the union of all the curves $D$ such that $M \cdot D = 0$. The intersection matrix of the $D$ is negative definite, and so we can contract all the $D$ on $X$ to obtain a normal surface $X'$. If $X'$ has only rational singularities, then $M$ induces an ample divisor on $X'$ and so $M$ itself is eventually base point free. In this case we are done. Otherwise we may apply Theorem 3.1 to find a subset $D_1, \ldots, D_t$ of the curves $D$ with $M \cdot D = 0$ and positive integers $a_i$ such that the divisor $K_X + \sum a_i D_i$ is nef, big, and eventually base point free, and such that the contraction $\bar{X}$ of $X$ has only rational and minimally elliptic singularities, with exactly one nonrational singularity. Note that $D_i \cdot K_X = 0$, or in other words if and only if $D_i \cdot K_X = 0$, or in other words if and only if $D_i$ is a smooth rational curve of self-intersection $-2$. Setting $e = -\sum a_i (D_i \cdot E)$, we have $e \geq 0$, and $e = 0$ if and only if $D_i \cdot E = 0$ for all $i$. But as $\bar{X}$ has a nonrational singularity, we cannot have $D_i \cdot E = 0$ for all $i$, for then all singularities would be rational double points. Thus $e > 0$. Now the $\mathbb{Q}$-divisor $M' = K_X + \sum a_i D_i$ is a rational convex combination of $K_X$ and $K_X + \sum a_i D_i$, and $M' \cdot E = 0$. Moreover either $M'$ is a strict convex combination of $K_X$ and $K_X + \sum a_i D_i$ (if $e > 1$) or $M' = K_X + \sum a_i D_i$ (if $e = 1$). In the second case, $M'$ satisfies (1) of Lemma 3.3, and it is eventually base point free by (iii) of Theorem 3.1. Thus $M'$ satisfies the conclusions of Lemma 3.3. In the first case, $M'$ is nef and big, and the only curves $C$ such that $M' \cdot C = 0$ are curves $C$ such that $K_X \cdot C = 0$ and $K_X + \sum a_i D_i \cdot C = 0$. The set of all such curves must therefore be a subset of the set of all smooth rational curves on $X$ with self-intersection $-2$. Hence, if $X''$ denotes the contraction of all the curves $C$ on $X$ such that $M' \cdot C = 0$, then $X''$ has only rational singularities and $M'$ induces an ample $\mathbb{Q}$-divisor on $X''$. Once again some multiple of $M'$ is eventually base point free and (1) and (2) of Lemma 3.3 are satisfied. Thus we have proved the lemma in case $K_X + E$ is nef.

Case II: $M = K_X + E$ is not nef.

Let $D$ be an irreducible curve with $M \cdot D < 0$. We claim first that in this case $D^2 < 0$. Indeed, suppose that $D^2 \geq 0$. As $\text{Pic} X \otimes_{\mathbb{Z}} \mathbb{R}$ has signature $(1, \rho - 1)$, the set

$$Q = \{ x \in \text{Pic} X \otimes_{\mathbb{Z}} \mathbb{R} : x^2 \geq 0, x \neq 0 \}$$

has two connected components, and two classes $x$ and $x'$ are in the same connected component of $Q$ if and only if $x \cdot x' \geq 0$ (cf. [13] p. 320 Lemma 1.1). Now $(K_X + E) \cdot K_X = (K_X + E)^2 = K_X^2 + 1 > 0$, so that $K_X + E$ and $K_X$ lie in the same connected component of $Q$. Likewise, if $D^2 \geq 0$, then since $K_X \cdot D \geq 0$, $K_X$ and $D$ lie in the same connected component of $Q$. Thus $D$ and $K_X + E$ lie in the same
connected component of $\mathcal{Q}$, so that $(K_X + E) \cdot D \geq 0$. Conversely, if $M \cdot D < 0$, then $D^2 < 0$.

Fix an irreducible curve $D$ with $M \cdot D < 0$, and let $d = -E \cdot D > K_X \cdot D \geq 0$. Recall that by assumption $E \cdot D \geq 0$ if $D$ is a smooth rational curve of self-intersection $-2$. If $p_a(D) \geq 1$, then set $M' = K_X + \frac{1}{d}D$. Then $M' \cdot E = 0$ by construction. Moreover we claim that $M'$ is nef and big. Indeed

$$(M')^2 = (K_X + \frac{1}{d}D)^2 = K_X \cdot (K_X + \frac{1}{d}D) + \frac{1}{d}(K_X + \frac{1}{d}D) \cdot D.$$ 

Thus $M'$ is big if it is nef and to see that $M'$ is nef it suffices to show that $M' \cdot D \geq 0$. But

$$M' \cdot D = K_X \cdot D + \frac{1}{d}D^2 = 2p_a(D) - 2 - \left(1 - \frac{1}{d}\right)D^2.$$ 

As $D^2 < 0$, we see that $M' \cdot D \geq 0$, and $M' \cdot D = 0$ if and only if $p_a(D) = 1$ and $d = 1$. Suppose that $p_a(D) = 1$ and $M' \cdot D = 0$. Using the exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + D) \to \omega_D \to 0,$$

and arguments as in the proof of Theorem 3.1, we see that the linear system $M'$ is eventually base point free and that the associated contraction has just rational double points and a minimally elliptic singular point which is the image of $D$. In all other cases, $M' \cdot D > 0$, so that the curves orthogonal to $M'$ are smooth rational curves of self-intersection $-2$. Again, some positive multiple of $M'$ is eventually base point free and the contraction has just rational singularities.

Thus we may assume that $p_a(D) = 0$ for every irreducible curve $D$ such that $M \cdot D < 0$. By assumption $D^2 \neq -1, -2$, so that $D^2 \leq -3$. Thus $d = -D \cdot E \geq 2$. If either $D^2 \leq -4$ or $D^2 = -3$ and $d \geq 3$, then again let $M' = K_X + \frac{1}{d}D$. Thus $M' \cdot E = 0$ and

$$M' \cdot D = K_X \cdot D + \frac{1}{d}D^2 = -2 - \left(1 - \frac{1}{d}\right)D^2 \geq 0.$$ 

Thus $M'$ is nef and big, and some multiple of $M'$ is eventually base point free, and the associated contraction has just rational singularities. The remaining case is where there is a smooth rational curve $D$ on $X$ with self-intersection $-3$ and such that $-D \cdot E = 2$. In this case $K_X \cdot D = 1$, and so $D - E$ is orthogonal to $K_X$. Note that $D - E$ is not numerically trivial since $D$ is not numerically equivalent to $E$. Thus, by the Hodge index theorem $(D - E)^2 < 0$. But

$$(D - E)^2 = -3 + 4 - 1 = 0,$$

a contradiction. Thus this last case does not arise. \hfill \Box

Appendix: On the canonical class of a rational surface

Let $\Lambda_n$ be a lattice of type $(1, n)$, i.e. a free $\mathbb{Z}$-module of rank $n + 1$, together with a quadratic form $q: \Lambda_n \to \mathbb{Z}$, such that there exists an orthogonal basis $\{e_0, e_1, \ldots, e_n\}$ of $\Lambda_n$ with $q(e_0) = 1$ and $q(e_i) = -1$ for all $i > 0$. Fix once
Then \( \kappa_n \) is characteristic, i.e. \( \kappa_n \cdot \alpha \equiv q(\alpha) \mod 2 \) for all \( \alpha \in \Lambda_n \).

The goal of this appendix is to give a proof, due to the first author, R. Miranda, and J.W. Morgan, of the following:

**Theorem A.1.** Suppose that \( n \leq 8 \) and that \( \kappa \in \Lambda_n \) is a characteristic vector satisfying \( q(\kappa) = 9 - n \). Then there exists an automorphism \( \varphi \) of \( \Lambda_n \) such that \( \varphi(\kappa) = \kappa_n \). A similar statement holds for \( n = 9 \) provided that \( \kappa \) is primitive.

**Proof.** We shall freely use the notation and results of Chapter II of [13] and shall quote the results there by number. For the purposes of the appendix, chamber shall mean a chamber in \( \{ x \in \Lambda_n \otimes \mathbb{R} \mid x^2 = 1 \} \) for the set of walls defined by the set \( \{ \alpha \in \Lambda_n \mid \alpha^2 = -1 \} \). Let \( C_n \) be the chamber associated to \( \kappa_n \) [13, p. 329, 2.7(a)]: the oriented walls of \( C_n \) are exactly the set

\[
\{ \alpha \in \Lambda_n \mid q(\alpha) = -1, \alpha \cdot \kappa_n = 1 \}.
\]

Then \( \kappa_n \) lies in the interior of \( \mathbb{R}^+ \cdot C_n \), by [13, p. 329, 2.7(a)]. Similarly \( \kappa \) lies in the interior of a set of the form \( \mathbb{R}^+ \cdot C \) for some chamber \( C \), since \( \kappa \) is not orthogonal to any wall (because it is characteristic) and \( q(\kappa) > 0 \). But the automorphism group of \( \Lambda_n \) acts transitively on the chambers, by [13 p. 324]. Hence we may assume that \( \kappa \in C_n \). In this case we shall prove that \( \kappa = \kappa_n \). We shall refer to \( C_n \) as the fundamental chamber of \( \Lambda_n \otimes \mathbb{Z} \). Let us record two lemmas about \( C_n \).

**Lemma A.2.** An automorphism \( \varphi \) of \( \Lambda_n \) fixes \( C_n \) if and only if it fixes \( \kappa_n \).

**Proof.** The oriented walls of \( C_n \) are precisely the \( \alpha \in \Lambda_n \) such that \( q(\alpha) = -1 \) and \( \kappa_n \cdot \alpha = 1 \). Thus, an automorphism fixing \( \kappa_n \) fixes \( C_n \). The converse follows from [13, p. 335, 4.4]. \( \square \)

**Lemma A.3.** Let \( \alpha = \sum_i \alpha_i e_i \) be an oriented wall of \( C_n \), where \( e_0, \ldots, e_n \) is the standard basis of \( \Lambda_n \). After reordering the elements \( e_1, \ldots, e_n \), let us assume that

\[
|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_n|.
\]

Then for \( n \leq 8 \), the possibilities for \( (\alpha_0, \ldots, \alpha_n) \) are as follows (where we omit the \( \alpha_i \) which are zero):

1. \( \alpha_0 = 0, \alpha_1 = 1; \)
2. \( \alpha_0 = 1, \alpha_1 = \alpha_2 = -1 (n \geq 2); \)
3. \( \alpha_0 = 2, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = -1 (n \geq 5); \)
4. \( \alpha_0 = 3, \alpha_1 = -2, \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = -1 (n \geq 7); \)
5. \( \alpha_0 = 4, \alpha_1 = \alpha_2 = \alpha_3 = 2, \alpha_4 = \cdots = \alpha_8 = -1 (n = 8); \)
6. \( \alpha_0 = 5, \alpha_1 = \cdots = \alpha_6 = -2, \alpha_7 = \alpha_8 = -1 (n = 8); \)
7. \( \alpha_0 = 6, \alpha_1 = -3, \alpha_2 = \cdots = \alpha_8 = -2 (n = 8). \)

**Proof.** This statement is extremely well-known as the characterization of the lines on a del Pezzo surface (see [7], Table 3). We can give a proof as follows. It clearly suffices to prove the result for \( n = 8 \). But for \( n = 8 \), there is a bijection between
the $\alpha$ defining an oriented wall of $C_8$ and the elements $\gamma \in \kappa^+_8$ with $q(\gamma) = -2$. This bijection is given as follows: $\alpha$ defines an oriented wall of $C_8$ if and only if $q(\alpha) = -1$ and $\kappa_8 \cdot \alpha = 1$. Map $\alpha$ to $\alpha - \kappa_8 = \gamma$. Thus, as $q(\kappa_8) = 1$, $q(\gamma) = -2$ and $\gamma \cdot \kappa_8 = 0$. Conversely, if $\gamma \in \kappa^+_8$ satisfies $q(\gamma) = -2$, then $\gamma + \kappa_8$ defines an oriented wall of $C_8$.

Now the number of $\alpha$ listed above, after we are allowed to reorder the $e_i$, is easily seen to be

$$8 + \binom{8}{2} + \binom{8}{5} + 8 \cdot 7 + \binom{8}{3} + \binom{8}{2} + 8 = 240.$$  

Since this is exactly the number of vectors of square $-2$ in $-E_8$, by e.g. [36], we must have enumerated all the possible $\alpha$.  

Write $\kappa = \sum_{i=0}^n a_i e_i$, where $e_i$ is the standard basis of $\Lambda_n$ given above. Since $\kappa \cdot e_i > 0$, $a_i < 0$. After reordering the elements $e_1, \ldots, e_n$, we may assume that

$$|a_1| \geq |a_2| \geq \cdots \geq |a_n|.$$

By inspecting the cases in Lemma A.3, for every $\alpha = \sum a_i e_i$ not of the form $e_i$, $a_i \leq 0$ for all $i \geq 1$. Given $\alpha = \sum a_i e_i$ with $\alpha \neq e_i$ for any $i$, let us call $\alpha$ well-ordered if

$$|a_1| \geq |a_2| \geq \cdots \geq |a_n|.$$

Quite generally, given $\alpha = a_0 e_0 + \sum_{i>0} a_i e_i$, we define the reordering $r(\alpha)$ of $\alpha$ to be

$$r(\alpha) = a_0 e_0 + \sum_{i>0} a_{\sigma(i)} e_i,$$

where $\sigma$ is a permutation of $\{1, \ldots, n\}$ such that $r(\alpha)$ is well-ordered. Clearly $r(\alpha)$ is independent of the choice of $\sigma$.

We then have the following:

**Claim A.4.** $\kappa \in C_n$ if and only if $\kappa \cdot \alpha > 0$ for every well-ordered wall $\alpha$.

**Proof.** Clearly if $\kappa \in C_n$, then $\kappa \cdot \alpha > 0$ for every $\alpha$, well-ordered or not. Conversely, suppose that $\kappa \cdot \alpha > 0$ for every well-ordered wall $\alpha$. We claim that

$$(*)\quad \alpha \cdot \kappa \geq r(\alpha) \cdot \kappa,$$

which clearly implies (A.4) since $r(\alpha)$ is well-ordered. Now

$$\alpha \cdot \kappa = a_0 a_0 - \sum_{i>0} a_i a_i.$$

Since $a_i < 0$ and $a_i < 0$, $(*)$ is easily reduced to the following statement about positive real numbers: if $c_1 \geq \cdots \geq c_n$ is a sequence of positive real numbers and $d_1, \ldots, d_n$ is any sequence of positive real numbers, then a permutation $\sigma$ of $\{1, \ldots, n\}$ is such that $\sum c_i d_{\sigma(i)}$ is maximal exactly when $d_{\sigma(1)} \geq \cdots \geq d_{\sigma(n)}$. We leave the proof of this elementary fact to the reader.  

Next, we claim the following:
Lemma A.5. View $\Lambda_n \subset \Lambda_{n+1}$. Defining $\kappa_{n+1}$ and $C_{n+1}$ in the natural way for $\Lambda_{n+1}$, suppose that $\kappa \in C_n$. Then $\kappa' = \kappa - e_{n+1} \in C_{n+1}$.

Proof. We have ordered our basis $\{e_0, \ldots, e_n\}$ so that

$$|a_1| \geq |a_2| \geq \cdots \geq |a_n|.$$  

Since $a_i < 0$ for all $i$, $|a_0| \geq 1$. Thus the coefficients of $\kappa'$ are also so ordered. Note also that all coefficients of $\kappa'$ are less than zero, so that the inequalities from (1) of Lemma A.3 are automatic. Given any other wall $\alpha'$ of $C_{n+1}$, to verify that $\kappa' \cdot \alpha' > 0$, it suffices to look at $\kappa' \cdot r(\alpha')$, where $r(\alpha')$ is the reordering of $\alpha'$. Expressing $\alpha'$ as a linear combination of the standard basis vectors, if some coefficient is zero, then $r(\alpha') \in \Lambda_n$. Clearly, in this case, viewing $r(\alpha')$ as an element of $\Lambda_n$, it is a wall of $C_n$. Since then $\kappa' \cdot r(\alpha') = \kappa \cdot r(\alpha')$, we have $\kappa' \cdot r(\alpha') > 0$ in this case.

In the remaining case, $r(\alpha')$ does not lie in $\Lambda_n$. This can only happen for $n = 1, 4, 6, 7$, with $\alpha'$ one of the new types of walls corresponding to the cases (2) — (7) of Lemma A.3. Thus, the only thing we need to check is that, every time we introduce a new type of wall, we still get the inequalities as needed. Since $r(\alpha')$ is well-ordered, we can assume that it is in fact one of the walls listed in Lemma A.3.

The $n = 1$ case simply says that $a_0 > -a_1 + 1$. However, we can easily solve the equations $a_0^2 - a_1^2 = 8, a_1 < 0$ to get $a_0 = 3, a_1 = -1$. Since $3 > 1 + 1$, we are done in this case.

Next assume that $n = 4$. We have $\kappa = a_0 e_0 + \sum_{i=1}^4 a_i e_i$. We must show that $2a_0 > -a_1 - a_2 - a_3 - a_4$. We know that $a_0 > -a_1 - a_2$, hence that $a_0 \geq a_1 - a_3 + 1$. Moreover $a_0 \geq -a_3 - a_4$ since $|a_1| \geq |a_2| \geq |a_3| \geq |a_4|$. Adding gives $2a_0 \geq -\sum_{i=1}^4 a_i + 2 = (\sum_{i=1}^4 a_i) + 1$ and therefore $2a_0 > -\sum_{i=1}^4 a_i + 1$. The case where $n = 6$ is similar: we must show that $3a_0 > -2a_1 - \sum_{i=2}^6 a_i + 1$. But we know that $2a_0 \geq -\sum_{i=1}^5 a_i + 1$ and that $a_0 \geq a_1 - a_6 + 1$. Adding gives the desired inequality. For $n = 7$, we have three new inequalities to check. The inequality

$$4a_0 > -2a_1 - 2a_2 - 2a_3 - \sum_{i=4}^7 a_i + 1$$

follows by adding the inequalities $3a_0 > -2a_1 - \sum_{i=2}^7 a_i$ and $a_0 > -a_2 - a_3$. The inequality

$$5a_0 > -2 \sum_{i=1}^6 a_i - a_7 + 1$$

follows from adding the inequalities $3a_0 > -2a_1 - \sum_{i=2}^7 a_i$ and $2a_0 > -\sum_{i=2}^5 a_i$. Likewise, the last inequality

$$6a_0 > -3a_1 - 2 \sum_{i=2}^7 a_i + 2$$

follows by adding up the three inequalities $3a_0 > -2a_1 - \sum_{i=2}^7 a_i, 2a_0 > -\sum_{i=1}^5 a_i$, and $a_0 > -a_6 - a_7$. Thus we have established the lemma.

Completion of the proof of Theorem A.1. Begin with $\kappa$. Applying Lemma A.5 and induction, if $n < 8$, then the vector $\eta = \kappa - \sum_{j=n+1}^8 e_j$ lies in the fundamental
chamber of $\Lambda_8$. Moreover $\eta$ is a characteristic vector of square 1. Thus $\eta^+ \equiv -E_8$. The same is true for $\kappa_8 = 3\eta_0 - \sum_{i=1}^8 e_i = \kappa_n - \sum_{j=n+1}^8 e_j$. Clearly, then, there is an automorphism $\varphi$ of $\Lambda_8$ such that $\eta = \varphi(\kappa_8)$. But both $\eta$ and $\kappa_8$ lie in the fundamental chamber for $\Lambda_8$. Since the automorphism group preserves the chamber structure, the automorphism $\varphi$ must stabilize the fundamental chamber. By Lemma A.2, $\varphi(\kappa_8) = \kappa_8$. Thus $\eta = \kappa_8$. Hence $\kappa - \sum_{j=n+1}^8 e_j = \kappa_n - \sum_{j=n+1}^8 e_j$. It follows that $\kappa = \kappa_n$. □

**Note.** To handle the case $n = 9$, we argue that every vector $\kappa \in \Lambda_9$ which is primitive of square zero and characteristic is conjugate to $\kappa_9$ as above. To do this, an easy argument shows that, if $\kappa$ is such a class, then there is an orthogonal splitting

$$\Lambda_9 \cong \langle \kappa, \delta \rangle \oplus (-E_8),$$

where $\delta$ is an element of $\Lambda_9$ satisfying $\delta \cdot \kappa = 1$ and $q(\delta) = 1$. Thus clearly every two such $\kappa$ are conjugate.

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