ZERO DISSIPATION LIMIT TO RAREFACTION WAVE WITH VACUUM FOR A ONE-DIMENSIONAL COMPRESSIBLE NON-NEWTONIAN FLUID

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Abstract. In this paper, we study the zero dissipation limit toward rarefaction waves for solutions to a one-dimensional compressible non-Newtonian fluid for general initial data, whose far fields are connected by a rarefaction wave to the corresponding Euler equations with one end state being vacuum. Given a rarefaction wave with one-side vacuum state to the compressible Euler equations, we construct a sequence of solutions to the one-dimensional compressible non-Newtonian fluid which converge to the above rarefaction wave with vacuum as the viscosity coefficient $\epsilon$ tends to zero. Moreover, the uniform convergence rate is obtained, based on one fact that the viscosity constant can control the degeneracies caused by the vacuum in rarefaction waves and another fact that the energy estimates are obtained under some a priori assumption.

1. Introduction. The present work is concerned with the following system about a one-dimensional compressible non-Newtonian fluid

$$\begin{cases}
\rho_t + (\rho u)_x = 0, \\
\rho u_t + \rho uu_x + p_x(\rho) = \epsilon((\mu_0 + u_x^2)\alpha/2) u_x,
\end{cases}$$

where $\mu_0 > 0$, $\epsilon > 0$ and $\alpha > 2$ are given constants. Here, $\rho(x, t) \geq 0$ and $u(x, t)$ represent the density and the velocity, respectively. Let the pressure $p$ be given by the $\gamma-$law

$$p(\rho) = \frac{\rho^\gamma}{\gamma}$$

with $\gamma > 1$ denoting the adiabatic exponent.

Formally, as $\epsilon$ tends to zero, the limit system of the system (1) is the following inviscid Euler equations

$$\begin{cases}
\rho_t + (\rho u)_x = 0, \\
\rho u_t + \rho uu_x + p_x(\rho) = 0.
\end{cases}$$

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The Euler system (2) is a strictly hyperbolic one for $\rho > 0$ and its two distinct eigenvalues are

$$
\lambda_1(\rho, u) = u - \sqrt{p'(\rho)}, \quad \lambda_2(\rho, u) = u + \sqrt{p'(\rho)},
$$

with the corresponding right eigenvectors are

$$
r_i(\rho, u) = (1, (-1)^i \frac{p'(\rho)}{\rho})^t, \quad i = 1, 2
$$

such that

$$
r_i(\rho, u) \cdot \nabla_{\rho, u} \lambda_i(\rho, u) = (-1)^i \frac{\rho p''(\rho) + 2 p'(\rho)}{2 \rho \sqrt{p'(\rho)}} \neq 0, \quad i = 1, 2.
$$

We define $i$–Riemann ($i = 1, 2$) invariants as follows

$$
\Sigma_i(\rho, u) = u + (-1)^{i+1} \int_{\rho}^{\rho(s)} \frac{p'(s)}{s} ds
$$

such that

$$
\nabla_{\rho, u} \Sigma_i(\rho, u) \cdot r_i(\rho, u) \equiv 0 \quad \forall \rho > 0, \quad u.
$$

There are several literatures about mathematical studies on compressible non-Newtonian fluid with various initial and boundary conditions. For example, Zhikov and Pastukhova [30] obtained the existence of weak solutions of initial boundary value problem for multidimensional non-Newtonian fluids under some restrictions. Mamontov [18] established the global existence of sufficiently regular solutions to two-dimensional and three-dimensional equations of compressible non-Newtonian fluids. Yuan and his cooperators obtained the existence result on the local strong solutions of initial-boundary-value problem in one dimensional non-Newtonian fluids see [28, 29] and the references therein. Later Fang and Guo [5] gave the blowup criterion for the local strong solutions obtained in [28], constructed an analytical solutions to a class of compressible non-Newtonian fluids in [4], and considered the initial boundary problem for a compressible non-Newtonian fluid with density-dependent viscosity [6]. For more related results, we refer the reader to [3, 27] and the references therein.

As pointed out by Liu and Smoller [12], among the two non-linear waves, i.e., shock and rarefaction waves, to the one-dimensional compressible isentropic Euler equation, only rarefaction waves can be connected to vacuum when vacuum appears. On the stability of the rarefaction waves to the one-dimensional compressible Navier-Stokes equations, there has been great interest and intensive studies completed in the development of the theory of viscous conservation laws from 1985; this started with studies on the nonlinear stability of viscous shock profiles by Matsumura and Nishihara [14]. Along with new phenomena of compressible fluids have been discovered, new techniques, such as weighted characteristic energy methods and uniform approximate Green’s functions, have been developed based on the intrinsic properties of the underlying nonlinear waves see [13, 14, 16, 20] and the references therein. In [26], Xin first studied the stability toward contact waves for solutions of systems of viscous conservation laws. Later, Liu and Xin [13] obtained pointwise asymptotic behavior towards viscous contact waves, which leads to the nonlinear stability of the viscous contact wave in $L^p$—norms for all $p \geq 1$.

However, the qualitative behaviors of the solutions to compressible non-Newtonian fluids were addressed only recently. In order to study the large time behavior of
compressible non-Newtonian fluid, Shi, Wang and Zhang [22] obtained the asymptotical stability of the rarefaction wave solution for a compressible non-Newtonian fluid, provided the initial disturbance is suitably small. But, the rarefaction wave itself is away from the vacuum in [22]. To our knowledge, there are no results on the zero dissipation limit of the system (1) in the case when the Euler system (2) contains the rarefaction wave connected to the vacuum. In this paper, our concern is this fundamental problem and we intend to obtain the decay rate with respect to the given viscosity constant $\epsilon$.

Now, we give a description of the rarefaction wave connected to the vacuum to the compressible Euler equations (2) see also [23] and [24]. If the compressible Euler system (2) is investigated with the Riemann initial data
\[
\begin{align*}
\rho(0,x) &= 0, & \quad x < 0, \\
(\rho, u)(0,x) &= (\rho_+, u_+), & \quad x > 0,
\end{align*}
\]
with the left side being the vacuum state and $\rho_+ > 0$, $u_+$ being prescribed constants on the right state, then the Riemann problem (2)-(3) admits a 2-rarefaction wave connected to the vacuum on the left side. By the fact that along the 2-rarefaction wave curve, 2-Riemann invariant $\Sigma_2(\rho, u)$ is constant in $(x,t)$.

So, the velocity $u_\ast = \Sigma_2(\rho_+, u_+)$ coming into the vacuum $\rho = 0$ to $(\rho_+, u_+)$ is the self-similar solution $(\rho^r, u^r)(\xi)$ of (2) defined by
\[
\lambda_2(\rho^r_\ast(\xi), u^r_\ast(\xi)) = \begin{cases} 
\rho^r_\ast(\xi) \equiv 0, & \xi < \lambda_2(0, u_\ast) = u_\ast, \\
\xi, & \lambda_2(0, u_\ast) \leq \xi \leq \lambda_2(\rho_+, u_+), \\
\lambda_2(\rho_+, u_+), & \xi > \lambda_2(\rho_+, u_+) 
\end{cases}
\]
and
\[
\Sigma_2(\rho^r_\ast(\xi), u^r_\ast(\xi)) = \Sigma_2(0, u_\ast) = \Sigma_2(\rho_+, u_+),
\]
where $\xi = \frac{x}{t}$. Thus, the momentum of 2-rarefaction wave is defined by
\[
m^r_\ast(\xi) = \begin{cases} 
\rho^r_\ast(\xi)u^r_\ast(\xi), & \rho^r_\ast > 0, \\
0, & \rho^r_\ast = 0.
\end{cases}
\]

In the present paper, we intend to construct a sequence of solutions $(\rho^\epsilon, m^\epsilon)(x,t)$ to the system (1) which converge to the 2-rarefaction wave $(\rho^r_\ast, m^r_\ast)(\frac{x}{t})$ defined over as $\epsilon$ tends to zero. The effects of initial layers will be ignored by choosing the well-prepared initial data depending on the viscosity for the compressible non-Newtonian fluids.

The main novelty and difficulty of the paper is determining how to control the strong non-linear term in $(1)_2$ and the degeneracies caused by the vacuum in the rarefaction wave. To overcome the difficulty coming from the vacuum, we first cut off the 2-rarefaction wave with vacuum along the rarefaction wave curve. More precisely, for any $\mu > 0$ to be determined, the cut-off rarefaction wave will connect the state $(\rho, u) = (\mu, u_\mu)$ and $(\rho_+, u_+)$, where $u_\mu$ can be obtained uniquely by the definition of the 2-rarefaction wave curve. To deal with the difficulty from the strong non-linear term in $(1)_2$, the classical iterative technology is taken. Then an approximate rarefaction wave to this cut-off rarefaction wave will be constructed through the Burgers equation. Finally, the desired solution sequences to the system (1) could be established around the approximate rarefaction wave.

**Theorem 1.1.** Let $\alpha > 2$ and $(\rho^r_\ast, m^r_\ast)(\frac{x}{t})$ be the 2-rarefaction wave defined by (4)-(6) with one-side vacuum state. Then there exists a small positive constant $\epsilon_0$
such that for any $\epsilon \in (0, \epsilon_0)$, a global smooth solution $(\rho^\epsilon, m^\epsilon = \rho^\epsilon u^\epsilon)(x, t)$ with initial values (28) to the system (1) can be constructed which satisfies

$$(\rho^\epsilon - \rho^\epsilon^\gamma, m^\epsilon - m^\epsilon^\gamma), (\rho^\epsilon - \rho^\epsilon^\gamma, m^\epsilon - m^\epsilon^\gamma)^\gamma \in C((0, +\infty); L^2(\mathbb{R}))$$

$m_c^\epsilon \in C((0, +\infty); L^n(\mathbb{R}))$, $m_{xc}^\epsilon \in L^2(0, +\infty; L^2(\mathbb{R}))$

and $(\rho^\epsilon, m^\epsilon)(x, t)$ converges to $(\rho^\epsilon^\gamma, m^\epsilon^\gamma)(\frac{\epsilon}{\gamma})$ point wise except in the origin $(0,0)$ as $\epsilon \to 0$. Furthermore, for any given positive constant $h$, there exists a constant $C_h$, independent of $\epsilon$ such that

$$\sup_{t \geq h} \|\rho(\cdot, t) - \rho^\epsilon(\cdot, t)\|_{L^\infty} \leq C_h \epsilon^a |\ln \epsilon|$$

and

$$\sup_{t \geq h} \|m(\cdot, t) - m^\epsilon(\cdot, t)\|_{L^\infty} \leq C_h \epsilon^b$$

holds for $\alpha > 2$, where the positive constants $a$ and $b$ are given as follows if $1 < \gamma \leq 2$

$$a = \begin{cases} 
\frac{\alpha - 2}{2\alpha - 3} \cdot (2 + (\alpha - 1) + \max \{\gamma + \alpha, \frac{2(\alpha + 1)}{\alpha - 2}, \frac{5\alpha - 4}{2(4 - \alpha)}\}^{-1}, & 2 < \alpha < 4, \\
\frac{\alpha - 2}{2\alpha - 3} \cdot (2 + (\alpha - 1) + \max \{\gamma + \alpha, \frac{2(\alpha + 1)}{\alpha - 2}, \frac{5\alpha - 4}{2(4 - \alpha)}\}^{-1}, & 4 \leq \alpha < 6, \\
\frac{\alpha - 2}{2\alpha - 3} \cdot (2 + (\alpha - 1) + \max \{\gamma + \alpha, \frac{2(\alpha + 1)}{\alpha - 2}, \frac{5\alpha - 4}{2(4 - \alpha)}\}^{-1}, & \alpha \geq 6 
\end{cases}$$

and if $\gamma \geq 2$

$$a = \begin{cases} 
\frac{\alpha - 2}{2\alpha - 3} \cdot (5\gamma + 8\alpha - 10 + \max \{\gamma + \alpha, \frac{2(\alpha + 1)}{\alpha - 2}, \frac{5\alpha - 4}{2(4 - \alpha)}\}^{-1}, & 2 < \alpha < 4, \\
\frac{\alpha - 2}{2\alpha - 3} \cdot (5\gamma + 8\alpha - 10 + \max \{\gamma + \alpha, \frac{2(\alpha + 1)}{\alpha - 2}, \frac{5\alpha - 4}{2(4 - \alpha)}\}^{-1}, & 4 \leq \alpha < 6, \\
\frac{\alpha - 2}{2\alpha - 3} \cdot (5\gamma + 8\alpha - 10 + \max \{\gamma + \alpha, \frac{2(\alpha + 1)}{\alpha - 2}, \frac{5\alpha - 4}{2(4 - \alpha)}\}^{-1}, & \alpha \geq 6 
\end{cases}$$

and

$$b = \frac{2\alpha}{3\alpha - a}.$$  

A few remarks follow.

**Remark 1.** It is also interesting to study the zero dissipation limit of compressible non-Newtonian system (1) in the case when the Euler system (2) has two rarefaction waves with vacuum in middle. However, there is a strong non-linear term in system (1) and it is nontrivial to cut off these rarefaction wave with vacuum along the corresponding rarefaction wave curves. In fact, wave structure containing two rarefaction waves with the medium vacuum is destroyed and some new wave may occur in the cut-off process, which is quite different from the single rarefaction wave case considered in the present paper.

**Remark 2.** It is noted that the estimates for $\phi^2$ from the potential energy hold with the weight $\rho^{\gamma - 2}$ in Lemma 3.1 below, which is degenerate at vacuum when $\gamma > 2$. Therefore, the convergence rate obtained in Proposition 1 and thus in Theorem 1.1 not only depends on the value $\alpha$ but also dependents on $\gamma$.

**Remark 3.** All the estimates obtained in Theorem 1.1 is strongly dependent on the the $\alpha$ and $\gamma$, which can be regarded as the extending results related to compressible Navier-Stokes equations in [8] as the case of $\alpha = 2$ in (1) for $\mu_0 > 0$. If $\mu_0 = 0$,
i.e., for the pseudo-plastic fluid ($\alpha < 2$) or dilatant fluid ($\alpha > 2$) (see [4]), the strong nonlinearity and highly degeneracy can be produced. This is a big challenge which will be left for future. It is worth to point out that, in order to deal with the strong non-linear term $\epsilon((\mu_0 + u_x)^{\frac{\alpha}{2}}u_x)_x$ in (1), new technique is taken besides the classical iterative method.

The rest of the paper is organized as follows. We construct a smooth 2-rarefaction wave which approximates the cut-off rarefaction wave based on the inviscid Burgers equation in Section 2, and then the proof of Theorem 1.1 is given in Section 3.

2. Rarefaction waves. Consider the Riemann problem for the typical Burgers equation

$$\begin{cases}
w_t + w w_x = 0, & t > 0, \quad x \in \mathbb{R}, \\
w(x, 0) = w_0(x) = \begin{cases} w_-, & x < 0, \\
w_+, & x > 0. \\ \end{cases}
\end{cases}$$ (12)

If $w_- < w_+$, then the system (12) has a rarefaction wave solution of the form $w^\tau(x, t) = w^\tau(\frac{x}{t})$ given by

$$w^\tau\left(\frac{x}{t}\right) = \begin{cases} w_-, & \frac{x}{t} < w_-, \\
w^\tau, & \frac{x}{t} \leq w_+ \leq w_+, \\
w_+, & \frac{x}{t} > w_+. \\
\end{cases}$$ (13)

As in [25], the approximate rarefaction wave $w(x, t)$ to the system (1) can be constructed by the solution of the Burgers equation

$$\begin{cases}
w_t + w w_x = 0, \\
w(x, 0) = w_0(x) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} \tanh \frac{x}{\delta},
\end{cases}$$ (14)

where $\delta > 0$ is a small parameter to be determined. In fact, we choose $\delta = \epsilon^\alpha$ in (46) with $a$ given by (9) and (10). Moreover, the solution $w_\delta^\tau(t, x)$ of the problem (14) is given by

$$w_\delta^\tau(t, x) = w_\delta(x_0(t, x)), \quad x = x_0(t, x) + w_\delta(x_0(t, x))t.$$ (15)

Then, $w_\delta^\tau(t, x)$ has properties stated in Lemma 2.1, which can be found in [25] and [8].

Lemma 2.1. The problem (14) has a unique smooth global solution $w_\delta^\tau(x, t)$ for each $\delta > 0$ such that the following hold:

1. $w_- < w_\delta^\tau(x, t) < w_+$, $\partial_x w_\delta^\tau(x, t) > 0$ for $x \in \mathbb{R}$, $t > 0$, $\delta > 0$.

2. The following estimates hold for all $t > 0$ and $p \in [1, +\infty] :$

$$\|\partial_x w_\delta^\tau(\cdot, t)\|_{L^p} \leq C_p(w_+ - w_\delta)^{\frac{p}{2}}(\delta + t)^{-\frac{1}{2}}(\delta + t)^{-\frac{1}{2}},$$ (16)

$$\|\partial_{xx} w_\delta^\tau(\cdot, t)\|_{L^p} \leq C_p(w_+ - w_\delta)^{\frac{p}{2}}(\delta + t)^{-1-\frac{1}{2}},$$ (17)

$$\|\partial_{xx} w_\delta^\tau(\cdot, t)\|_{L^p} \leq \frac{4}{\delta} \partial_x w_\delta^\tau(x, t).$$ (18)

3. There exists a constant $\delta_0 \in (0, 1)$ such that $\delta \in (0, \delta_0]$, $t > 0$

$$\|w_\delta^\tau(\cdot, t) - w_\delta^\tau(\cdot, \frac{t}{t})\|_{L^\infty} \leq C\delta t^{-1}[\ln(1 + t) + |\ln \delta|].$$ (19)
Now, we turn to cut off the 2-rarefaction wave with vacuum along the wave curve in order to overcome the difficulty caused by the vacuum. More precisely, for any \( \mu > 0 \) to be determined, we can get a state \( (\rho, u) = (\mu, u_\mu) \) belonging to the 2-rarefaction wave curve. From the fact that 2-Riemann invariant \( \Sigma_2(\rho, u) \) is constant along the 2-rarefaction wave curve, \( u_\mu \) can be computed explicitly by \( u_\mu = \Sigma_2(\rho_+, u_+) + \frac{2}{\gamma-1} \mu^{\frac{\gamma}{\gamma-2}} \). So, we get a new 2-rarefaction wave \( (\rho^*_{\mu}, u^*_{\mu})(\xi) (\xi = \frac{x}{t}) \) connecting the state \( (\mu, u_\mu) \) to the state \( (\rho_+, u_+) \) which can be expressed explicitly by

\[
\lambda_2(\rho^*_{\mu}, u^*_{\mu})(\xi) = \begin{cases} 
\lambda_2(\mu, u_\mu), & \xi < \lambda_2(\mu, u_\mu), \\
\lambda_2(\mu, u_\mu) \leq \xi \leq \lambda_2(\rho_+, u_+), \\
\lambda_2(\rho_+, u_+), & \xi > \lambda_2(\rho_+, u_+) 
\end{cases}
\]  

and

\[
\Sigma_2(\rho^*_{\mu}, u^*_{\mu}) = \Sigma_2(\mu, u_\mu)(\xi) = \Sigma_2(\rho_+, u_+). \tag{21}
\]

Correspondingly, it is reasonable to define the momentum function \( m^*_{\mu}(\xi) \) and it is easy to prove that the cut-off 2-rarefaction wave \( (\rho^*_{\mu}, m^*_{\mu})(\xi) \) converges to the original 2-rarefaction wave \( (\rho^*, m^*)(\xi) \) in sup-norm with the convergence rate \( \mu \) as \( \mu \) tends to zero, which is stated as follows and can be found in [8].

**Lemma 2.2.** There exists a constant \( \mu_0 \in (0, 1) \) such that for \( \mu \in (0, \mu_0), t > 0 \)

\[
\| (\rho^*_{\mu}, m^*_{\mu})(\cdot, t) - (\rho^*, m^*)(\frac{\cdot}{t}) \|_{L^\infty} \leq C\mu. \tag{22}
\]

From now on, we define the approximate rarefaction wave \( (\bar{\rho}_{\mu, \delta}, \bar{u}_{\mu, \delta})(x, t) \) of the cut-off 2-rarefaction \( (\bar{\rho}^*_{\mu}, m^*_{\mu})(\xi) \) to the system (1) as

\[
\begin{cases} 
\bar{w}_+ = \lambda_2(\rho_+, u_+), & w_{\bar{w}} = \lambda_2(\mu, u_\mu), \\
\bar{w}^*_{\delta}(t, x) = \lambda_2(\bar{\rho}_{\mu, \delta}, \bar{u}_{\mu, \delta})(x, t), \\
\Sigma_2(\bar{\rho}_{\mu, \delta}, \bar{u}_{\mu, \delta})(x, t) = \Sigma_2(\rho_+, u_+) = \Sigma_2(\mu, u_\mu),
\end{cases} \tag{23}
\]

where \( \bar{w}^*_{\delta} \) is the solution of Burgers equation (14) defined in (15). From then on, the subscription of \( (\bar{\rho}_{\mu, \delta}, \bar{u}_{\mu, \delta})(x, t) \) will be omitted as \((\bar{\rho}, \bar{u})(x, t)\) for simplicity. Then the approximate cut-off 2-rarefaction wave \( (\bar{\bar{\rho}}, \bar{\bar{u}}) \) defined above satisfies

\[
\begin{cases} 
\bar{\bar{\rho}}_t + (\bar{\bar{\rho}}\bar{u})_x = 0, \\
(\bar{\bar{\rho}}\bar{u})_t + (\bar{\bar{\rho}}\bar{u}_x^2 + p(\bar{\bar{\rho}}))_x = 0
\end{cases} \tag{24}
\]

and the properties of the approximate rarefaction wave \( (\bar{\bar{\rho}}, \bar{\bar{u}}) \) are listed without proof in the following lemma, which can be found in [8].

**Lemma 2.3.** The approximate cut-off 2-rarefaction wave \( (\bar{\bar{\rho}}, \bar{\bar{u}}) \) defined in (23) satisfies the following properties:

1. \( \bar{u}_x(x, t) = \frac{2}{\gamma+1}(w^*_{\delta} x > 0 \quad \text{for} \quad x \in \mathbb{R}, \ t \geq 0, \ \bar{\rho}_x = \bar{\rho}^{\frac{\gamma}{\gamma-2}} \bar{u}_x \quad \text{and} \ \bar{\rho}_{xx} = \bar{\rho}^{\frac{\gamma}{\gamma-2}} \bar{u}_{xx} + \frac{3-\gamma}{\gamma} \bar{\rho}^{\frac{3-2\gamma}{\gamma}} (\bar{u}_x)^2. \)

2. The following estimates hold for all \( t > 0 \) and \( p \in [1, +\infty] \),

\[
\| \bar{u}_x(\cdot, t) \|_{L^p} \leq C_p(w_+ - w_{\bar{w}})^{\frac{1}{p}}(\delta + t)^{-1+\frac{1}{p}},
\]

\[
\| \bar{u}_{xx}(\cdot, t) \|_{L^p} \leq C_p(\delta + t)^{-1}\delta^{1+\frac{1}{p}}. \tag{26}
\]

3. There exists a constant \( \delta_0 \in (0, 1) \) such that for \( \delta \in (0, \delta_0], t > 0, \)

\[
\| (\bar{\bar{\rho}} - \rho^*_{\mu}, \bar{\bar{u}} - u^*_{\mu})(\cdot, t) \|_{L^\infty} \leq C\delta t^{-1}[\ln(1+t) + |\ln \delta|]. \tag{27}
\]
The following lemma is a key to get the estimate stated in Proposition 1, which can be found in [17].

**Lemma 2.4.** Let \( k \) and \( l \) be strictly positive integers. Then
\[
\sup_{\mathbb{R}} |\varphi(x)| \leq C(\int |\varphi(x)|^{2k-2l+2}dx \cdot \int |\varphi(x)|^{2l-2}\varphi^2(x)dx)^{\frac{1}{2k-l}}
\]
holds for \( \varphi(x) \) when the right-hand side is finite.

3. **Proof of Theorem 1.1.** To prove Theorem 1.1, the solution \((\rho^*, u^*)\) is constructed as the perturbation around the approximate rarefaction wave \((\bar{\rho}, \bar{u})\). The Cauchy problem for (1) is considered with initial values
\[
(\rho^*, u^*)(x, 0) = (\bar{\rho}, \bar{u})(x, 0).
\]
Then we introduce the perturbation
\[
(\phi, \psi)(y, \tau) = (\rho^*, u^*)(x, t) - (\bar{\rho}, \bar{u})(x, t)
\]
where \( y, \tau \) are the scaled variables as
\[
y = \frac{x}{\epsilon}, \quad \tau = \frac{t}{\epsilon},
\]
and \((\rho^*, u^*)\) is assumed to be the solution to the problem (1). For simplicity of notation, the superscription of \((\rho^*, u^*)\) will be omitted as \((\rho, u)\) from now on if there is no confusion of notation. Substituting (29) and (30) into (1) and using the definition of \((\bar{\rho}, \bar{u})\), we obtain that
\[
\begin{align*}
\phi_\tau + \rho\psi_y + u\phi_y &= -(\bar{u}\phi + \bar{\rho}\psi), \\
\rho\psi_\tau + \rho\psi_y + p'(\rho)\phi_y &= -((\mu_0 + (\frac{uy}{\epsilon})^2)\frac{\gamma^2}{\gamma - 2}\psi_y)_y, \\
&= -(\rho u\bar{\psi}_y + \bar{\rho}(p' - \frac{p'x}{\bar{\rho}})) - ((\mu_0 + (\frac{uy}{\epsilon})^2)\frac{\gamma^2}{\gamma - 2}\bar{\psi}_y)_y, \\
(\phi, \psi)(y, 0) &= 0.
\end{align*}
\]
We seek a global-in-time solution \((\phi, \psi)\) to the reformulated problem (31)-(33). To this end, the solution space for (31)-(33) is defined by
\[
X(0, \tau_1) = \{(\phi, \psi) | (\phi, \psi) \in C([0, \tau_1]; H^1(\mathbb{R})), \phi_y \in L^2(0, \tau_1; L^2(\mathbb{R})), \psi_y \in L^2(0, \tau_1; H^1(\mathbb{R})), \psi_y \in L^\infty(0, \tau_1; L^\infty(\mathbb{R}))\}
\]
with \( 0 < \tau_1 \leq +\infty \).

**Proposition 1.** Let \( \alpha > 2 \). The problem (31)-(33) admits a unique global-in-time solution \((\phi, \psi) \in X(0, +\infty)\). Furthermore, there exist positive constants \( \epsilon_0 \) and \( C \) independent of \( \epsilon \) such that if \( 0 < \epsilon \leq \epsilon_0 \), then
\[
\begin{align*}
&\sup_{\tau \in [0, +\infty]} \int \left( \frac{\psi_y^2}{2} + \epsilon^{-(\alpha - 2)}|\psi_y|^\alpha \right) dy \\
&+ \epsilon^{(2\alpha + \gamma - 5)\alpha} \sup_{\tau \in [0, +\infty]} \int \left( \mu_0 \frac{\alpha - 2}{2\rho} \phi_y^2 + \bar{\rho}\psi^2 + \bar{\rho}^{(-\alpha - 2)} \right) dy \\
&+ \int_0^{+\infty} \int \left( \frac{1}{\mu_0 + (\frac{uy}{\epsilon})^2} \frac{\alpha - 2}{2\rho} \psi_{yy}^2 + \epsilon^{-(\alpha - 2)} \frac{1}{\mu_0 + (\frac{uy}{\epsilon})^2} \frac{\alpha - 2}{2}\psi_y |\alpha - 2| \psi_y^2 \right) dy ds \\
&+ \epsilon^{(2\alpha + \gamma - 5)\alpha} \int_0^{+\infty} \int |\psi_y|^2 + \epsilon^{-(\alpha - 2)}|\psi_y|^\alpha + \bar{\rho}\psi_y^2 + \bar{\rho}^{\gamma - 2} \bar{\psi}_y \phi^2 + \bar{\rho}^{\gamma - 3} \phi_{yy}^2 dy ds
\end{align*}
\]
\[
+ \int_0^{+\infty} \int \frac{\bar{u}_y \psi_y^2}{2} + \epsilon^{-(\alpha-2)} \frac{1}{\alpha} \bar{u}_y(\psi_y^2) \frac{3}{2} \, dy \, ds \\
\leq C \epsilon^{\frac{2(\alpha-2)}{m-a} - \max\{2(\gamma+\alpha), \frac{4(\alpha+1)}{m-a}, \frac{5\alpha-4}{m-a}\} a}
\]  \tag{34}
holds for \(2 < \alpha < 4\),

\[
+ \epsilon^{(2\alpha+\gamma-5) a} \sup_{\tau \in [0, +\infty)} \int \left( \frac{\bar{u}_y \psi_y^2}{2} + \epsilon^{-(\alpha-2)} \frac{1}{\alpha} \bar{u}_y(\psi_y^2) \frac{3}{2} \right) \, dy \\
+ \epsilon^{(2\alpha+\gamma-5) a} \int_0^{+\infty} \int \left( \frac{1}{\bar{u}_0 + \left( \frac{\bar{u}_y \psi_y^2}{2} + \epsilon^{-(\alpha-2)} \frac{1}{\alpha} \bar{u}_y(\psi_y^2) \frac{3}{2} \right)}{2} + \rho \psi^2 + \rho^{\gamma-2} \phi^2 \right) dy \\
+ \epsilon^{(2\alpha+\gamma-5) a} \int_0^{+\infty} \int \left( \int \left| \psi_y \right|^2 + \epsilon^{-(\alpha-2)} \frac{1}{\alpha} \bar{u}_y(\psi_y^2) \frac{3}{2} \right) \, dy \\
\leq C \epsilon^{\frac{2(\alpha-2)}{m-a} - \frac{(2\alpha+1)}{m-a} + 2\gamma) a}
\]  \tag{35}

holds for \(4 \leq \alpha \leq 6\) and

\[
+ \epsilon^{(2\alpha+\gamma-5) a} \sup_{\tau \in [0, +\infty)} \int \left( \frac{\bar{u}_y \psi_y^2}{2} + \epsilon^{-(\alpha-2)} \frac{1}{\alpha} \bar{u}_y(\psi_y^2) \frac{3}{2} \right) \, dy \\
+ \epsilon^{(2\alpha+\gamma-5) a} \int_0^{+\infty} \int \left( \frac{1}{\bar{u}_0 + \left( \frac{\bar{u}_y \psi_y^2}{2} + \epsilon^{-(\alpha-2)} \frac{1}{\alpha} \bar{u}_y(\psi_y^2) \frac{3}{2} \right)}{2} + \rho \psi^2 + \rho^{\gamma-2} \phi^2 \right) dy \\
+ \epsilon^{(2\alpha+\gamma-5) a} \int_0^{+\infty} \int \left( \int \left| \psi_y \right|^2 + \epsilon^{-(\alpha-2)} \frac{1}{\alpha} \bar{u}_y(\psi_y^2) \frac{3}{2} \right) \, dy \\
\leq C \epsilon^{\frac{2(\alpha-2)}{m-a} - \frac{(2\alpha+1)\gamma}{m-a} + \alpha}
\]  \tag{36}

holds for \(\alpha \geq 6\), where \(a\) is given by (9) and (10). Consequently, when \(1 < \gamma \leq 2\) it is held that

\[
\sup_{0 \leq \tau \leq +\infty} \| \phi \|_{L^\infty} \leq \begin{cases} 
C \epsilon^{\frac{2(\alpha-2)}{m-a} - (\gamma+2(\alpha-2)) \frac{2(\alpha+1)}{m-a} \frac{5\alpha-4}{m-a} a}, & 2 < \alpha < 4, \\
C \epsilon^{\frac{2(\alpha-2)}{m-a} - (2\gamma+2(\alpha-2)) \frac{4(\alpha+1)}{m-a} a}, & 4 \leq \alpha < 6, \\
C \epsilon^{\frac{2(\alpha-2)}{m-a} - (2\gamma+3\alpha-4) a}, & \alpha \geq 6
\end{cases}
\]  \tag{37}

and when \(\gamma > 2\) it is held that

\[
\sup_{0 \leq \tau \leq +\infty} \| \phi \|_{L^\infty} \leq \begin{cases} 
C \epsilon^{\frac{2(\alpha-2)}{m-a} - (2\gamma+2(\alpha-2)) \frac{2(\alpha+1)}{m-a} \frac{5\alpha-4}{m-a} a}, & 2 < \alpha < 4, \\
C \epsilon^{\frac{2(\alpha-2)}{m-a} - (2\gamma+2(\alpha-2)) \frac{4(\alpha+1)}{m-a} a}, & 4 \leq \alpha < 6, \\
C \epsilon^{\frac{2(\alpha-2)}{m-a} - (2\gamma+3\alpha-4) a}, & \alpha \geq 6
\end{cases}
\]  \tag{38}
By similar way, if $\gamma > 1$ then

$$
\sup_{0 \leq \tau \leq +\infty} \left\| \psi \right\|_{L^\infty} \\
\leq \begin{cases} 
Ce^{-\gamma/2} \left( \frac{\alpha-2}{\alpha-5} - (\gamma + 2(\alpha - 2) + \max\{\gamma + \alpha, \frac{2(\alpha + 1)}{\alpha - 2} \cdot \frac{5\alpha - 4}{2(\alpha - 2)} \}) |a|, \quad 2 < \alpha < 4, \\
Ce^{-\gamma/2} \left( \frac{\alpha-2}{\alpha-5} - (\frac{4(\alpha + 1)}{\alpha - 2} + 2(\alpha - 2) |a| \right), \quad 4 \leq \alpha < 6, \\
Ce^{-\gamma/2} \left( \frac{\alpha-2}{\alpha-5} - (\frac{2(\alpha + 1)}{\alpha - 2} - 3) |a| \right), \quad \alpha \geq 6.
\end{cases}
$$

(39)

Here we omit the proof for the local existence of the solution to (31)-(33) for brevity, since it is standard. Note that the local existence time interval, denoted by $\tau_0$, may depend on $\epsilon$, in order to get the convergence rate of the local solution with respect $\epsilon$. The next step is to extend the local solution to the global solution in $[0, +\infty)$ for small but fixed constant $\epsilon$. To do so, it is sufficient to obtain the following a priori estimates for fixed $\epsilon$.

**Lemma 3.1.** Let $\alpha > 2$ and $(\phi, \psi) \in X(0, \tau_1(\epsilon))$ be a solution to the problem (31)-(33), where $\tau_1(\epsilon)$ is the maximum time of the solution. Then there exist positive constants $\epsilon_0$ and $C$ independent of $\epsilon$ and $\tau_1(\epsilon)$ such that if $0 < \epsilon \leq \epsilon_0$, then

$$
\sup_{\tau \in [0, \tau_1(\epsilon)]} \left[ \left( \int \frac{\psi^2}{2} + \epsilon^{-(\alpha - 2)} \frac{\left| \psi_y \right|^\alpha}{\alpha(\alpha - 1)} \right) dy \right]
+ \epsilon^{(2\alpha + \gamma - 5)\alpha} \sup_{\tau \in [0, \tau_1(\epsilon)]} \left[ \int \left( \mu_0 + \left( \frac{u_y}{\epsilon} \right)^2 \right) \frac{\phi_y^2}{2\rho^3} \right. \\
\left. + \rho \psi^2 + \rho^{\gamma - 2} \phi^2 \right] dy ds
+ \int_0^{\tau_1(\epsilon)} \left[ \frac{\bar{\psi}_y \bar{y}^2}{2} + \epsilon^{-(\alpha - 2)} \frac{\left. \rho \bar{u} \psi^2 + \rho^{\gamma - 2} \bar{u} \phi^2 + \rho^{\gamma - 3} \phi_y^2 \right]} {dy ds} \right.
\leq Ce^{-\epsilon/2 - \max\{2(\gamma + \alpha), \frac{4(\alpha + 1)}{\alpha - 2}, \frac{5\alpha - 4}{2(\alpha - 2)} \}}
$$

(40)

holds for $2 < \alpha < 4$,

$$
\sup_{\tau \in [0, \tau_1(\epsilon)]} \left[ \left( \int \frac{\psi^2}{2} + \epsilon^{-(\alpha - 2)} \frac{\left| \psi_y \right|^\alpha}{\alpha(\alpha - 1)} \right) dy \right]
+ \epsilon^{(2\alpha + \gamma - 5)\alpha} \sup_{\tau \in [0, \tau_1(\epsilon)]} \left[ \int \left( \mu_0 + \left( \frac{u_y}{\epsilon} \right)^2 \right) \frac{\phi_y^2}{2\rho^3} \right. \\
\left. + \rho \psi^2 + \rho^{\gamma - 2} \phi^2 \right] dy ds
+ \int_0^{\tau_1(\epsilon)} \left[ \frac{\bar{\psi}_y \bar{y}^2}{2} + \epsilon^{-(\alpha - 2)} \frac{\left. \rho \bar{u} \psi^2 + \rho^{\gamma - 2} \bar{u} \phi^2 + \rho^{\gamma - 3} \phi_y^2 \right]} {dy ds} \right.
\leq Ce^{-\epsilon/2 - \max\{2(\gamma + \alpha), \frac{4(\alpha + 1)}{\alpha - 2}, \frac{5\alpha - 4}{2(\alpha - 2)} \}}
$$

(41)
holds for $4 \leq \alpha \leq 6$ and
\[
\sup_{\tau \in [0, \tau_1(\epsilon)]} \int \left( \frac{\psi_y^2}{2} + \epsilon^{-(\alpha - 2)} \frac{|\psi_y|^{\alpha}}{\alpha(\alpha - 1)} \right) dy \\
+ \epsilon^{(2\alpha+\gamma-5)a} \sup_{\tau \in [0, \tau_1(\epsilon)]} \int \left( \frac{\mu_0}{\epsilon} - \frac{\phi_0^2}{2\rho} + \bar{\rho} \psi^2 + \bar{\rho}^{\gamma-2} \phi^2 \right) dy \\
+ \int_0^{\tau_1(\epsilon)} \int 0 \left( \frac{1}{\rho} \left( \frac{u_y}{\epsilon} \right)^2 \frac{\alpha-2}{\alpha} \psi_y^2 + \epsilon^{-(\alpha - 2)} \frac{1}{\rho} (\mu_0 + \frac{u_y}{\epsilon}) \left( \frac{\alpha-2}{\alpha} |\psi|^{\alpha-2} \psi^2 \right) \right) dy ds \\
+ \epsilon^{(2\alpha+\gamma-5)a} \int_0^{\tau_1(\epsilon)} \int \left( \frac{\alpha-2}{\alpha} \psi_y^2 \left( \frac{\mu_0}{\epsilon} - \frac{\phi_0^2}{2\rho} + \bar{\rho} \psi^2 + \bar{\rho}^{\gamma-2} \phi^2 \right) \right) dy ds \\
\leq Ce^{\frac{2(\alpha-2) - (2\gamma+3a-4)}{2}\epsilon} - (2\gamma+3a-4) \alpha
\] (42)
holds for $\alpha \geq 6$, where $a$ is given by (9) and (10). Consequently, when $1 < \gamma \leq 2$ it is held that
\[
\sup_{0 \leq \tau \leq \tau_1(\epsilon)} \| \phi \|_{L^\infty} \leq \begin{cases} Ce^{\frac{2(\alpha-2) - (\gamma+2(\alpha-2)+\max(\gamma+\alpha, 2(\alpha+1) \frac{(\alpha-4)}{\alpha-1}))a}{2}} & , 2 < \alpha < 4, \\
Ce^{\frac{2(\alpha-2) - (2\gamma+2(\alpha-2)+4(\alpha+1) \frac{(\alpha-4)}{\alpha-1})a}{2}} & , 4 \leq \alpha < 6, \\
Ce^{\frac{2(\alpha-2) - (2\gamma+3a-4) \alpha}{2}} & , \alpha \geq 6
\end{cases}
\] (43)
and when $\gamma > 2$ it is held that
\[
\sup_{0 \leq \tau \leq \tau_1(\epsilon)} \| \phi \|_{L^\infty} \leq \begin{cases} Ce^{\frac{2(\alpha-2) - (\gamma+2(\alpha-2)+\max(\gamma+\alpha, 2(\alpha+1) \frac{(\alpha-4)}{\alpha-1}))a}{2}} & , 2 < \alpha < 4, \\
Ce^{\frac{2(\alpha-2) - (2\gamma+2(\alpha-2)+4(\alpha+1) \frac{(\alpha-4)}{\alpha-1})a}{2}} & , 4 \leq \alpha < 6, \\
Ce^{\frac{2(\alpha-2) - (2\gamma+3a-4) \alpha}{2}} & , \alpha \geq 6
\end{cases}
\] (44)
Moreover, if $\gamma > 1$ then
\[
\sup_{0 \leq \tau \leq \tau_1(\epsilon)} \| \psi \|_{L^\infty} \leq \begin{cases} Ce^{\frac{2(\alpha-2) - (\gamma+2(\alpha-2)+\max(\gamma+\alpha, 2(\alpha+1) \frac{(\alpha-4)}{\alpha-1}))a}{2}} & , 2 < \alpha < 4, \\
Ce^{\frac{2(\alpha-2) - (4(\alpha+1) + 2\gamma+2(\alpha-2))a}{2}} & , 4 \leq \alpha < 6, \\
Ce^{\frac{2(\alpha-2) - (2\gamma+3a-4) \alpha}{2}} & , \alpha \geq 6
\end{cases}
\] (45)
Before giving the proof of Lemma 3.1, we take
\[
\mu = \epsilon^a |\ln \epsilon|, \quad \delta = \epsilon^a.
\] (46)
Then, it is obvious that $\mu \geq 2\epsilon^a$ if $\epsilon \ll 1$.

Proof of Lemma 3.1. In order to prove Lemma 3.1, we assume that the solution to the problem (31)-(33) satisfies
\[
\sup_{\tau \in [0, \tau_1(\epsilon)]} \| \phi(\cdot, \tau) \|_{L^\infty} \leq \epsilon^a, \quad \sup_{\tau \in [0, \tau_1(\epsilon)]} \| \psi_y \|_{L^\infty} \leq 1 \quad \text{and} \quad \epsilon^{-(\alpha - 2)} \int_0^{\tau_1(\epsilon)} \| \psi_y \|_{L^\infty}^a ds \leq 1.
\] (47)
Under the a priori assumption (46), we can get that
\[
\frac{\bar{\rho}}{2} \leq \rho \leq \frac{3\bar{\rho}}{2},
\]
(48)
\[
C_1\bar{\rho}^{\gamma -2}\phi^2 \leq p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi \leq C_2\bar{\rho}^{\gamma -2}\phi^2,
\]
where \(C_1, C_2\) are positive constants independent of \(\epsilon\). Now, we divide the proof of Lemma 3.1 into the following four steps.

**Step 1.** First, we define
\[
E := \Phi(\rho, \bar{\rho}) + \frac{\psi^2}{2},
\]
where
\[
\Phi(\rho, \bar{\rho}) := \int_0^\rho \frac{p(\xi) - p(\bar{\rho})}{\xi^2}\,d\xi = \frac{1}{\gamma - 1}(p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi).
\]
(50)

Direct computations yield that
\[
(\rho E)_\tau + [\rho uE - (\mu_0 + \frac{u y}{\epsilon})^2 \psi_y\psi_y + (p(\rho) - p(\bar{\rho}))\psi]_y + (\mu_0 + \frac{u y}{\epsilon})^2 \psi_y^2
\]
\[
+ \bar{u}_y(p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi) + \bar{u}u_y\psi^2 = ((\mu_0 + \frac{u y}{\epsilon})^2 \bar{u}_y )_y \psi
\]
Then, integrating the above equation over \(\mathbb{R} \times [0, \tau]\) and using (46), (48) and (49) imply that
\[
\int (\bar{\rho}\psi^2 + \bar{\rho}^{\gamma -2}\phi^2)\,dy + \int_0^\tau \int ((\mu_0 + \frac{u y}{\epsilon})^2 \psi_y^2 + \rho\bar{u}_y\psi^2 + \bar{\rho}^{\gamma -2}\bar{u}_y\phi^2)\,dy\,ds
\]
\[
\leq |\int_0^\tau \int ((\mu_0 + \frac{u y}{\epsilon})^2 \bar{u}_y )_y \psi\,dy\,ds|
\]
(51)

By the Sobolev inequality and Lemma 2.3, one has that
\[
|\int_0^\tau \int ((\mu_0 + \frac{u y}{\epsilon})^2 \bar{u}_y )_y \psi\,dy\,ds| \leq |\int_0^\tau \int ((\mu_0 + \frac{u y}{\epsilon})^2 \bar{u}_y )_y \psi\,dy\,ds|
\]
\[
+ \epsilon^{-2}(\mu_0 + \frac{u y}{\epsilon})^2 \frac{\alpha - 1}{\alpha - 2} |\bar{u}^{\alpha - 2}_y \bar{u} \psi| + \epsilon^{-(\alpha - 2)} |\bar{u}^{\alpha - 2}_y \bar{u} \psi| + \epsilon^{-(\alpha - 2)} |\bar{u}_y \psi|)
\]
(52)

Now we estimate the terms on the right-hand side of (52) one by one. By Lemma 2.3 and Cauchy’s inequality, it holds that
\[
I_1 \leq C \int_0^\tau \int (\mu_0 + \frac{u y}{\epsilon})^2 \bar{u} \psi\,dy\,ds
\]
\[
\leq C \int_0^\tau \int (\bar{u} \psi) + \epsilon^{-(\alpha - 2)} |\bar{u}^{\alpha - 2}_y \bar{u} \psi| + \epsilon^{-(\alpha - 2)} |\bar{u}_y \psi|)
\]
\[
:= I_1 + I_2 + I_3.
\]
(53)

In fact,
\[
I_1 \leq C \int_0^\tau \int \bar{u} \psi\,dy\,ds \leq C \int_0^\tau \|\bar{u} \psi\|_{L^1} \|\psi\|_{L^2} \|\psi\|_{L^2} \,ds
\]
\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(e)]} \|\sqrt{\rho} \psi\|_{L^2}^2 + C\mu^{-\frac{1}{2}} \int_0^\tau \|\bar{u} \psi\|_{L^1} \|\psi\|_{L^2} \,ds
\]
\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(e)]} \|\sqrt{\rho} \psi\|_{L^2}^2 + C\mu^{-\frac{1}{2}} \left( \int_0^\tau \|\bar{u} \psi\|_{L^1} \|\psi\|_{L^2} \,ds \right)^{\frac{1}{2}}
\]
\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \left\| \sqrt{\theta} \psi \right\|_{L^2}^2 + \frac{\mu^2}{32} \int_0^\tau \left\| \psi_y \right\|_{L^2}^2 ds + C \mu^{-\frac{1}{2}} \left( \int_0^\tau \left\| \bar{u}_{yy} \right\|_{L^2}^\frac{4}{\alpha} ds \right)^\frac{3}{2}
\]
\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \left\| \sqrt{\theta} \psi \right\|_{L^2}^2 + \frac{\mu^2}{32} \int_0^\tau \left\| \psi_y \right\|_{L^2}^2 ds + C \epsilon^2 \delta^{-\frac{1}{2}} \mu^{-\frac{1}{2}}
\]
\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \left\| \sqrt{\theta} \psi \right\|_{L^2}^2 + \frac{\mu^2}{32} \int_0^\tau \left\| \psi_y \right\|_{L^2}^2 ds + C \epsilon^2 \delta^{-\frac{1}{2}} \mu^{-\frac{1}{2}}
\]
\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \left\| \sqrt{\theta} \psi \right\|_{L^2}^2 + C \epsilon^2 \delta^{-\frac{1}{2}} |\ln \epsilon|^{-\frac{1}{2}}, \tag{54}
\]

\[I_1^2 \leq C \epsilon^{-{(\alpha-2)}} \int_0^\tau \int |\bar{u}_y^{-\delta} \bar{u}_{yy} \psi| dy ds \leq C \epsilon^{-{(\alpha-2)}} \int_0^\tau \left\| \bar{u}_y \right\|_{L^\infty}^{\alpha-2} \left\| \bar{u}_{yy} \right\|_{L^2} \left\| \psi \right\|_{L^2} ds
\]
\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \left\| \sqrt{\theta} \psi \right\|_{L^2}^2 + C \epsilon^{-{(\alpha-2)}}(\mu^{-1}) \left( \int_0^\tau \left\| \bar{u}_y \right\|_{L^\infty}^{\alpha-2} \left\| \bar{u}_{yy} \right\|_{L^2} ds \right)^2
\]
\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \left\| \sqrt{\theta} \psi \right\|_{L^2}^2 + C \epsilon^{-{(\alpha-2)}}(\mu^{-1}) \left( \int_0^\tau \left\| \bar{u}_y \right\|_{L^\infty}^{\alpha-2} \left\| \bar{u}_{yy} \right\|_{L^2} ds \right)^2 \tag{55}
\]

and
\[
I_1^3 \leq C \epsilon^{-{(\alpha-2)}} \int_0^\tau \int |\psi \bar{u}_y^{-\delta} \bar{u}_{yy} \psi| dy ds \leq C \epsilon^{-{(\alpha-2)}} \int_0^\tau \left\| \psi_y \right\|_{L^\infty}^{\alpha-2} \left\| \bar{u}_y \right\|_{L^2} \left\| \psi \right\|_{L^2} ds
\]
\[
\leq C \epsilon^{-{(\alpha-2)}} \int_0^\tau \left\| \psi_y \right\|_{L^\infty}^{\alpha-2} \left\| \psi \right\|_{L^2} \left( \int_0^\tau \left\| \bar{u}_y \right\|_{L^\infty}^{\alpha-2} \left\| \bar{u}_{yy} \right\|_{L^2} ds \right)^2
\]
\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \left\| \sqrt{\theta} \psi \right\|_{L^2}^2 + C \epsilon^{-{(\alpha-2)}} \left( \int_0^\tau \left\| \bar{u}_y \right\|_{L^\infty}^{\alpha-2} \left\| \bar{u}_{yy} \right\|_{L^2} ds \right)^2
\]
\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \left\| \sqrt{\theta} \psi \right\|_{L^2}^2 + C \epsilon^{-{(\alpha-2)}} \left( \int_0^\tau \left\| \bar{u}_y \right\|_{L^\infty}^{\alpha-2} \left\| \bar{u}_{yy} \right\|_{L^2} ds \right)^2 + C \epsilon^{-{(\alpha-2)}}(\mu^{-1}) |\ln \epsilon|^{\frac{3}{2}}. \tag{56}
\]

Similarly, \(I_2\) can be estimated as
\[
I_2 \leq C \epsilon^{-2} \left| \int_0^\tau \left( \mu_0 + \left( \frac{u_y}{\bar{u}_y} \right)^2 \bar{u}_y^{-\delta} \bar{u}_{yy} \psi \right) ds \right|
\]
\[
\leq C \epsilon^{-2} \int_0^\tau \left[ \left\| \bar{u}_y \right\|_{L^\infty} \left\| \bar{u}_{yy} \right\|_{L^2} \left\| \psi \right\|_{L^2} \right] \left( \int_0^\tau \left\| \bar{u}_y \right\|_{L^\infty}^{\alpha-2} \left\| \bar{u}_{yy} \right\|_{L^2} ds \right)^2 \]
\[
+ C \epsilon^{-2} \left( \int_0^\tau \left\| \bar{u}_y \right\|_{L^\infty}^{\alpha-2} \left\| \bar{u}_{yy} \right\|_{L^2} ds \right)^2 \tag{57}
\]

and the right-hand side of the \(I_2\) is estimated one by one
\[
I_2^1 \leq C \epsilon^{-2} \int_0^\tau \left\| \bar{u}_y \right\|_{L^4} \left\| \bar{u}_{yy} \right\|_{L^2} \left\| \psi \right\|_{L^2} \left( \int_0^\tau \left\| \bar{u}_y \right\|_{L^\infty}^{\alpha-2} \left\| \bar{u}_{yy} \right\|_{L^2} ds \right)^\frac{2}{\alpha}
\]
\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \left\| \sqrt{\theta} \psi \right\|_{L^2}^2 + \delta^{-2(\alpha-2)} \int_0^\tau \left\| \psi \right\|_{L^2}^\frac{2}{\alpha} ds
\]
\[
+ C \mu^{-\frac{2}{\alpha}} \sqrt{\delta} (\alpha-2) \left( \int_0^\tau \left\| \bar{u}_y \right\|_{L^4}^{\frac{4}{\alpha}} \left\| \bar{u}_{yy} \right\|_{L^2}^{\frac{4}{\alpha}} ds \right)^{\frac{3}{2} \alpha^{-2}}
\]

\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \left\| \sqrt{\theta} \psi \right\|_{L^2}^2 + \delta^{-2(\alpha-2)} \int_0^\tau \left\| \psi \right\|_{L^2}^\frac{2}{\alpha} ds
\]
\[
+ C \mu^{-\frac{2}{\alpha}} \sqrt{\delta} (\alpha-2) \left( \int_0^\tau \left\| \bar{u}_y \right\|_{L^4}^{\frac{4}{\alpha}} \left\| \bar{u}_{yy} \right\|_{L^2}^{\frac{4}{\alpha}} ds \right)^{\frac{3}{2} \alpha^{-2}}
\]
\begin{align*}
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \| \sqrt{\rho} \psi \|_{L^2} + \delta^{-2(\alpha - 2)} \int_0^\tau \| \frac{\psi_{yy}}{\sqrt{\rho}} \|_{L^2} ds + C\epsilon \frac{\alpha - 2}{2} \mu^{-2} \delta \left( \frac{\alpha - 2}{\alpha} \right)^{-3} \\
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \| \sqrt{\rho} \psi \|_{L^2} + \delta^{-2(\alpha - 2)} \int_0^\tau \| \frac{\psi_{yy}}{\sqrt{\rho}} \|_{L^2} ds + C\mu^{-4} \int_0^\tau \| \sqrt{\rho} \psi \|_{L^2} ds
\end{align*}

(57)

with the help that \( C \epsilon \delta^{-2(\alpha - 2)} \mu^{-1} \leq \frac{1}{32} \) if \( \epsilon \ll 1 \),

\[
I_2^3 \leq C\epsilon^{-2} \int_0^\tau \int \| \frac{\psi_{yy}}{\sqrt{\rho}} \|_{L^2} ds
\]

\[
\leq C\epsilon^{-2} \mu^{-2} \frac{1}{2} \int_0^\tau \| \frac{\psi_{yy}}{\sqrt{\rho}} \|_{L^2} ds
\]

\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \| \sqrt{\rho} \psi \|_{L^2} \] if \( \epsilon \ll 1 \), \( \mu \geq 4 \),

(58)

Now, we estimate \( I_3 \) as follows

\[
I_3 \leq C\epsilon^{-2} \int_0^\tau \left( \mu_0 + \left( \frac{\psi_{yy}}{\sqrt{\rho}} \right)^{\alpha - 2} \mu^{-1} \right) dx dy ds
\]

\[
\leq C\epsilon^{-2} \mu^{-2} \frac{1}{2} \int_0^\tau \left( \| \frac{\psi_{yy}}{\sqrt{\rho}} \|_{L^2} + \| \psi_{yy} \|_{L^2} \right) dx dy ds
\]

\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \| \sqrt{\rho} \psi \|_{L^2} \] if \( \epsilon \ll 1 \), \( \alpha \geq 4 \).

(59)

where

\[
I_3^j \leq C\epsilon^{-2} \int_0^\tau \int \| \frac{\psi_{yy}}{\sqrt{\rho}} \|_{L^2} ds
\]

\[
\leq C\epsilon^{-2} \mu^{-2} \frac{1}{2} \int_0^\tau \| \sqrt{\rho} \psi \|_{L^2} \] ds

\[
\leq \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} \| \sqrt{\rho} \psi \|_{L^2} \]
\[
\begin{align*}
&\leq \frac{1}{32} \sup_{\tau \in [0, T(\epsilon)]} \left\| \sqrt{\rho} \psi \right\|_{L^2} + \frac{\epsilon^{-(a-2)}}{32} \int_0^T \left\| \psi_y \right\|_{L^\infty} ds \\
&\quad + C \epsilon^{-\frac{2(a+2)}{a-2}} \mu^{-\frac{a}{2a-2}} \left( \int_0^T \left( \left\| \bar{u}_{yy} \right\|_{L^{\frac{2a}{a-2}}} \left\| \bar{u}_y \right\|_{L^\infty} \right)^{\frac{a}{a-2}} ds \right)^{\frac{2(a-1)}{a-2}} \\
&\leq \frac{1}{32} \sup_{\tau \in [0, T(\epsilon)]} \left\| \sqrt{\rho} \psi \right\|_{L^2}^2 + \frac{\epsilon^{-(a-2)}}{32} \int_0^T \left\| \psi_y \right\|_{L^\infty}^2 ds + C \epsilon^{-\frac{2(a-1)}{a-2} - \frac{2(a+2)}{a-2} \sigma_1} \ln \epsilon^{-\frac{a}{a-2}}, \\
&\quad \text{(60)}
\end{align*}
\]

\[
\begin{align*}
I_3^2 &\leq C \epsilon^{-(a-2)} \int_0^T \int_0^T \left| \bar{u}_y \right|^3 \psi_y \psi_{yy} \psi ds \quad dyds \\
&\leq C \epsilon^{-(a-2)} \mu^{-\frac{1}{2}} \int_0^T \left\| \sqrt{\rho} \psi \right\|_{L^2} \left\| \psi_y \right\|_{L^\infty} \left\| \bar{u}_{yy} \right\|_{L^{\frac{2a}{a-2}}} \left\| \bar{u}_y \right\|_{L^\infty}^3 ds \\
&\leq \frac{1}{32} \sup_{\tau \in [0, T(\epsilon)]} \left\| \sqrt{\rho} \psi \right\|_{L^2}^2 + \frac{\epsilon^{-(a-2)}}{32} \int_0^T \left\| \psi_y \right\|_{L^\infty}^2 ds \\
&\quad + C \epsilon^{-2(a-1)} \mu^{-\frac{a}{2a-2}} \left( \int_0^T \left( \left\| \bar{u}_{yy} \right\|_{L^{\frac{2a}{a-2}}} \left\| \bar{u}_y \right\|_{L^\infty} \right)^{\frac{a}{a-2}} ds \right)^{\frac{2(a-1)}{a-2}} \\
&\leq \frac{1}{32} \sup_{\tau \in [0, T(\epsilon)]} \left\| \sqrt{\rho} \psi \right\|_{L^2}^2 + \frac{\epsilon^{-(a-2)}}{32} \int_0^T \left\| \psi_y \right\|_{L^\infty}^2 ds + C \epsilon^{\frac{2(a-1)}{a-2} - \frac{2(a+2)}{a-2} \sigma_1} \ln \epsilon^{-\frac{a}{a-2}}, \\
&\quad \text{(61)}
\end{align*}
\]

holds for \( \alpha \geq 4 \) and

\[
\begin{align*}
I_3^3 &\leq C \epsilon^{-(a-2)} \int_0^T \int_0^T \left| \psi_y \right|^3 \psi_y \bar{u}_{yy} \psi ds \quad dyds \\
&\leq C \epsilon^{-(a-2)} \mu^{-\frac{1}{2}} \int_0^T \left\| \sqrt{\rho} \psi \right\|_{L^2} \left\| \psi_y \right\|_{L^\infty} \left| \psi_{yy} \right|_{L^{\frac{2a}{a-2}}} \left\| \psi_y \right\|_{L^\infty}^{\alpha(a-3)} \left\| \bar{u}_{yy} \right\|_{L^2} \left\| \bar{u}_y \right\|_{L^\infty} ds \\
&\leq \frac{1}{32} \sup_{\tau \in [0, T(\epsilon)]} \left\| \sqrt{\rho} \psi \right\|_{L^2}^2 + \epsilon^{-2(a-2)} \int_0^T \frac{\left\| \psi_y \right\|_{L^\infty}^4 \left\| \psi_{yy} \right\|_{L^2}^2}{\rho} ds + C \epsilon^{4(a-5)} \ln \epsilon^{-\frac{2a-2}{a+1}} \\
&\quad \text{(62)}
\end{align*}
\]

holds for \( \alpha \geq 4 \). Then, \( I_4 \) is estimated similar to \( I_2 \) as follows:

\[
\begin{align*}
I_4 &\leq C \epsilon^{-2} \left\| \left( \mu_0 + \frac{u_y}{\epsilon} \right)^2 \frac{\alpha^2}{\epsilon} \bar{u}_y \psi_y \psi_{yy} \psi \right\|_{L^2} ds \\
&\leq \mu_0^{-\frac{\alpha^2}{2}} \epsilon^{-2} \int_0^T \int_0^T \left( \mu_0 + \frac{u_y}{\epsilon} \right)^2 \frac{\alpha^2}{\epsilon} \bar{u}_y \psi_{yy} ds \\
&\quad + C \delta^{2(a-2)} \epsilon^{-2} \int_0^T \int_0^T \left( \mu_0 + \frac{u_y}{\epsilon} \right)^2 \frac{\alpha^2}{\epsilon} \bar{u}_y \psi_{yy} ds \\
&\leq \mu_0^{-\frac{\alpha^2}{2}} \epsilon^{-2} \int_0^T \int_0^T \left( \mu_0 + \frac{u_y}{\epsilon} \right)^2 \frac{\alpha^2}{\epsilon} \bar{u}_y \psi_{yy} ds \\
&\quad + C \delta^{2(a-3)} \epsilon^{-2} \int_0^T \left\| \bar{u}_y \right\|_{L^\infty}^4 ds \\
&\leq \epsilon^{-2(a-2)} \int_0^T \bar{u}_y \psi_{yy} ds
\end{align*}
\]
\begin{align}
&+ \mu_0 \frac{\alpha^2}{2} \delta^{-(\alpha-2)} \epsilon^{-(\alpha-2)} \int_0^t \int \rho^{-1} \psi_y \epsilon^{-2} \psi_{yy}^2 \rho \psi y ds + \frac{1}{32} \sup_{\tau \in [0, \tau(e)]} \| \sqrt{\rho} \|_{L^2} \tag{63} \\
\end{align}

holds for $2 \leq \alpha < 4$ if $\epsilon \ll 1$ and when $\alpha \geq 4$

\begin{align}
I_4 \leq & C \epsilon^{-2} | \int_0^t \int (\mu_0 + \frac{(\mu_0^2)}{\epsilon^2}) \frac{\alpha^2}{2} \bar{u}_y \psi_y y \psi y ds | \\
\leq & C \epsilon^{-2} \int_0^t \int [\bar{u}_y \psi_y y \psi y] + \epsilon^{-(\alpha-4)} | \bar{u}_y \psi_y y \psi_y y | \\
& + \epsilon^{-(\alpha-4)} | \bar{u}_y \psi_y \psi_y y \psi | dyds := \Sigma_{j=1}^3 I_4
\end{align}

where

\begin{align}
I_4 \leq & C \epsilon^{-2} \int_0^t \int [\bar{u}_y \psi_y y \psi y \psi] dyds \\
\leq & C \epsilon^{-2} \mu^{-1} \int_0^t \int \left[ \frac{|\psi_y|^{\alpha-2} \psi_{yy}}{\sqrt{\rho}} \right] \left[ \frac{\bar{u}_y \psi_y y \psi y}{\sqrt{\rho}} \right] ds \\
\leq & (\epsilon^{-(\alpha-2)} \int_0^t \int \frac{|\psi_y|^{\alpha-2} \psi_{yy}}{\sqrt{\rho}} ds) + \int_0^t \frac{|\psi_{yy}}{\sqrt{\rho}} ds \\
& + C \mu^{-\delta} \sup_{\tau \in [0, \tau(e)]} \| \sqrt{\rho} \psi \|_{L^2}^2 \int_0^t \int \frac{|\bar{u}_y \psi_y y \psi y \psi y |}{\sqrt{\rho}} ds \\
\leq & (\epsilon^{-(\alpha-2)} \int_0^t \int \frac{|\psi_y|^{\alpha-2} \psi_{yy}}{\sqrt{\rho}} ds) + \int_0^t \frac{|\psi_{yy}}{\sqrt{\rho}} ds + \frac{1}{32} \sup_{\tau \in [0, \tau(e)]} \| \sqrt{\rho} \psi \|_{L^2}^2. \tag{64}
\end{align}

if $\epsilon \ll 1$,

\begin{align}
I_4 \leq & C \epsilon^{-(\alpha-2)} \int_0^t \int [\bar{u}_y \psi_y \psi_y y \psi y \psi y] dyds \\
\leq & C \epsilon^{-(\alpha-2)} \mu^{-\frac{1}{2}} \int_0^t \int \left[ \frac{|\psi_y|^{\alpha-2} \psi_{yy}}{\sqrt{\rho}} \right] \left[ \frac{\bar{u}_y \psi_y y \psi y}{\sqrt{\rho}} \right] ds \\
\leq & \mu^{-(\alpha-2)} (\epsilon^{-(\alpha-2)} \int_0^t \int \frac{|\psi_y|^{\alpha-2} \psi_{yy}}{\sqrt{\rho}} ds) + \int_0^t \frac{|\psi_{yy}}{\sqrt{\rho}} ds \\
& + \frac{1}{32} \sup_{\tau \in [0, \tau(e)]} \| \sqrt{\rho} \psi \|_{L^2}^2. \tag{65}
\end{align}

if $\epsilon \ll 1$ and

\begin{align}
I_3 \leq & C \epsilon^{-(\alpha-2)} \int_0^t \int [\bar{u}_y \psi_y \psi_y y \psi y \psi y] dyds \\
\leq & \mu^{-\frac{2(\alpha-1)}{\alpha-2}} (\epsilon^{-(\alpha-2)} \int_0^t \int \frac{|\psi_y|^{\alpha-2} \psi_{yy}}{\sqrt{\rho}} ds) + (\epsilon^{-(\alpha-2)} \int_0^t \int \frac{|\psi_y^{\alpha-2} \psi_{yy}}{\sqrt{\rho}} ds) \\
& + \frac{1}{32} \sup_{\tau \in [0, \tau(e)]} \| \sqrt{\rho} \psi \|_{L^2}^2. \tag{66}
\end{align}

Last, we estimate $I_5$ and

\begin{align}
I_5 \leq & C \epsilon^{-2} | \int_0^t \int (\mu_0 + \frac{(\mu_0^2)}{\epsilon^2}) \frac{\alpha^2}{2} \bar{u}_y y \psi y dyds |
\end{align}
\[ \leq Ce^{-2} \int_{0}^{T} |\bar{u}_{yy}\psi| + \epsilon^{-(\alpha-4)}|\bar{u}_{yy}^2\bar{u}_{yy}\psi| + \epsilon^{-(\alpha-4)}|\bar{u}_{y}^{\alpha-2}\bar{u}_{yy}\psi| \, dyds \]

\[ := \Sigma_{j=1}^{3} \mathcal{I}_{j}^{2}, \]

where

\[ \mathcal{I}_{1}^{2} \leq Ce^{-2} \int_{0}^{T} \int |\bar{u}_{y}^{\alpha-2}\bar{u}_{yy}\psi| \, dyds \]

\[ \leq Ce^{-2} \mu^{-\gamma} \int_{0}^{T} \|\bar{u}_{y}\|_{L^{\infty}}^{\frac{3}{2}} \|\bar{u}_{yy}\|_{L^{2}} \|\sqrt{\rho\bar{u}_{y}}\psi\|_{L^{2}} \, ds \]

\[ \leq \frac{1}{32} \int_{0}^{T} \int \rho\bar{u}_{y}\psi^{2} \, dyds + Ce^{-4} \mu^{-1} \int_{0}^{T} \|\bar{u}_{y}\|_{L^{\infty}}^{\frac{3}{2}} \|\bar{u}_{yy}\|_{L^{2}}^{2} \, ds \]

\[ \leq \frac{1}{32} \int_{0}^{T} \int \rho\bar{u}_{y}\psi^{2} \, dyds + Ce^{2-6\alpha} \ln |\epsilon|^{-1}, \quad (67) \]

\[ \mathcal{I}_{2}^{2} \leq Ce^{-(\alpha-2)} \int_{0}^{T} \int |\bar{u}_{y}^{\alpha-4}\bar{u}_{yy}\psi| \, dyds \]

\[ \leq Ce^{-(\alpha-2)} \mu^{-\gamma} \int_{0}^{T} \|\bar{u}_{y}\|_{L^{\infty}}^{\frac{3}{2}} \|\bar{u}_{yy}\|_{L^{2}} \|\sqrt{\rho\bar{u}_{y}}\psi\|_{L^{2}} \, ds \]

\[ \leq \frac{1}{32} \int_{0}^{T} \int \rho\bar{u}_{y}\psi^{2} \, dyds + Ce^{-(\alpha-2)} \mu^{-1} \int_{0}^{T} \|\bar{u}_{y}\|_{L^{\infty}}^{2\alpha-5} \|\bar{u}_{yy}\|_{L^{2}}^{2} \, ds \]

\[ \leq \frac{1}{32} \int_{0}^{T} \int \rho\bar{u}_{y}\psi^{2} \, dyds + Ce^{2-(\alpha-1)\alpha} \ln |\epsilon|^{-1}, \quad (\alpha \geq 4). \quad (68) \]

\[ \mathcal{I}_{3}^{2} \leq Ce^{-(\alpha-2)} \int_{0}^{T} \int |\bar{u}_{y}^{2}\alpha\psi y\bar{u}_{yy}\psi| \, dyds \]

\[ \leq \frac{1}{32} \sup_{\tau \in [0,\tau(\alpha)]} \|\sqrt{\rho}\psi\|_{L^{2}}^{2} + \epsilon^{2(\alpha-2)} \int_{0}^{T} \|\psi_{y}^{\alpha-2}\psi_{yy}^{3}\psi\|_{L^{2}}^{2} \, ds \]

\[ + \frac{1}{32} (\epsilon^{-(\alpha-2)} \int_{0}^{T} \|\psi_{y}\|_{L^{2}}^{2} \, ds) + Ce^{\frac{4\alpha-5}{3}} \delta \mu^{-\frac{\alpha-1}{2}} \]

\[ \leq \frac{1}{32} \sup_{\tau \in [0,\tau(\alpha)]} \|\sqrt{\rho}\psi\|_{L^{2}}^{2} + \epsilon^{2(\alpha-2)} \int_{0}^{T} \|\psi_{y}^{\alpha-2}\psi_{yy}^{3}\psi\|_{L^{2}}^{2} \, ds \]

\[ + \frac{1}{32} (\epsilon^{-(\alpha-2)} \int_{0}^{T} \|\psi_{y}\|_{L^{2}}^{2} \, ds) + Ce^{\frac{4\alpha-5}{3}} \delta \ln |\epsilon|^{-\frac{\alpha-1}{6}}, \quad (\alpha \geq 4). \quad (69) \]

Then, one substitute (54)-(69) into (51) to obtain that

\[ \int (\bar{\rho}\psi^{2} + \bar{\rho}^{\gamma-2}\phi^{2}) \, dy + \int_{0}^{T} \int |\psi_{y}|^{2} + \epsilon^{-(\alpha-2)} |\psi_{y}|^{\alpha} + \rho\bar{u}_{y}\psi^{2} + \rho^{\gamma-2}\bar{u}_{y}\phi^{2} \, dyds \]

\[ \leq 8\mu_{0}^{\frac{2}{3}} \delta^{-2(\alpha-2)} (\epsilon^{-(\alpha-2)} \int_{0}^{T} \|\psi_{y}^{\alpha-2}\psi_{yy}^{2}\|_{L^{2}}^{2} \, ds) \]

\[ + 8\delta^{-(\alpha-2)} \int_{0}^{T} \|\psi_{y}^{2}\psi_{yy}^{}\, dyds + 8(\epsilon^{-2(\alpha-2)} \int_{0}^{T} \|\psi_{y}^{\alpha-2}\psi_{yy}^{2}\|_{L^{2}}^{2} \, ds) \]

\[ + Ce^{\frac{2\alpha-5}{3}} \ln |\epsilon|^{-\frac{1}{2}} \]

holds for \( 2 < \alpha < 4 \) and

\[ \int (\bar{\rho}\psi^{2} + \bar{\rho}^{\gamma-2}\phi^{2}) \, dy + \int_{0}^{T} \int |\psi_{y}|^{2} + \epsilon^{-(\alpha-2)} |\psi_{y}|^{\alpha} + \rho\bar{u}_{y}\psi^{2} + \rho^{\gamma-2}\bar{u}_{y}\phi^{2} \, dyds \]
\[ \leq 8\mu_0^{-\frac{3}{2}} \frac{2^{2-(\alpha-2)}}{\delta-2(\alpha-2)} \int_0^\tau \left\| \frac{\psi_y}{\sqrt{\rho}} \right\|_{L^2}^2 ds \]

\[ + 8\epsilon^{2(\alpha-2)} \int_0^\tau \int_0^\rho \psi_y^2 \psi_y ds dy \]

\[ + 2(\epsilon^{-2(\alpha-2)} \sup_{s \in [0,\tau]} \left\| \psi_y \right\|_{L^2}^2) + C\epsilon^{\frac{\alpha-2}{\delta} - \max(2(\alpha+1),\frac{\alpha-2}{\delta})} \delta^{\frac{1}{2}} \ln \epsilon^{-\frac{1}{2}} \] (71)

holds for \( \alpha \geq 4 \).

Step 2. We make use of the idea in [22] with modifications to derive the estimation of \( \phi_y \). Differentiating (31) with respect to \( y \) and then multiplying the resulting equation by \( \frac{\phi_y}{\rho^2} \) yields that

\[ \left( \frac{\phi_y^2}{2\rho^3} \right)_y + \frac{u\phi_y^2}{2\rho^3} \left( \frac{\phi_y}{\rho} \right)_y + \psi_y \psi_y \phi_y \left( \frac{\phi_y}{\rho} \right)_y = -\frac{\phi_y}{\rho^3} \left( \bar{u}_{yy} \phi + \bar{r}_{yy} \psi + 2\bar{p}_y \psi_y \right). \] (72)

Then, multiplying (32) by \( \frac{\phi_y}{\rho^2} \), one has that

\[ \frac{\psi_y \phi_y}{\rho} - \left( \frac{\psi_y \phi_y}{\rho} + \frac{\bar{r}_{yy} \psi}{\rho^2} \right)_y - \psi_y \bar{u}_{yy} \phi - \psi_y \bar{r}_{yy} \phi - \psi_y \bar{p}_y \phi_y \left( \frac{\phi_y}{\rho^2} \right)_y + \frac{\bar{r}_{yy} \psi_y}{\rho^2} \]

\[ + \bar{p}_y \psi_y \psi_y \phi + \frac{\psi_y \bar{u}_{yy} \phi}{\rho^2} - \psi_y \bar{r}_{yy} \phi - \psi_y \bar{p}_y \phi_y \left( \frac{\phi_y}{\rho^2} \right)_y + \frac{\bar{r}_{yy} \psi_y}{\rho^2} \]

\[ + \left( \mu_0 + \frac{u_y}{\epsilon} \right) \frac{\bar{r}_{yy} \phi_y}{\rho^2} \] (73)

Adding (72) \( \times \mu_0^{\frac{1}{2}} \) and (73) together, and then integrating the result over \( \mathbb{R} \times [0,\tau] \), one gets that

\[ \int \left( \mu_0^{-\frac{1}{2}} \frac{\phi_y^2}{2\rho^3} + \frac{\psi_y \phi_y}{\rho} \right) dy + \int_0^\tau \int \frac{p'(\rho)}{\rho^2} \frac{\phi_y^2}{2\rho^3} dy ds = \int_0^\tau \int \psi_y^2 dy ds \]

\[ + \int_0^\tau \left\{ \mu_0^{-\frac{1}{2}} \psi_y \left( \bar{u}_{yy} \phi + \bar{r}_{yy} \psi + 2\bar{p}_y \psi_y \right) - \frac{\bar{r}_{yy} \psi_y^2}{\rho^2} - \frac{2\bar{p}_y \psi_y \psi_y}{\rho^2} + \frac{\psi_y \bar{u}_{yy} \phi}{\rho^2} \right\} dy ds \]

\[ + \left( \mu_0 + \frac{u_y}{\epsilon} \right) \frac{\bar{r}_{yy} \phi_y}{\rho^2} \] (74)

Now, one can estimate the terms on the right-hand side of (74) in the following. By Lemma 2.3 and Cauchy’s inequality, one obtains that

\[ \Pi_1 = -\mu_0^{\frac{1}{2}} \int_0^\tau \int_0^\rho \frac{\phi_y^2}{\rho^3} \bar{u}_{yy} \phi dy ds \]

\[ \leq \mu_0^{\frac{1}{2}} \int_0^\tau \int_0^\rho \left| \frac{\phi_y^2}{\rho^3} \bar{u}_{yy} \phi \right| dy ds \leq \mu_0^{\frac{1}{2}} \int_0^\tau \int_0^\rho \left| \bar{r}_{yy} \phi_y \right| \left| \bar{r}_{yy} \phi \right| dy ds \]

\[ \leq \frac{1}{32} \int_0^\tau \int_0^\rho \left| \bar{r}_{yy} \phi_y \right|^2 dy ds + C \int_0^\tau \int_0^\rho \left| \rho \left( \gamma + 1 \right) \phi_y \right|^2 dy ds \]

\[ \leq \frac{1}{32} \int_0^\tau \int_0^\rho \left| \bar{r}_{yy} \phi_y \right|^2 dy ds + \frac{1}{32} \sup_{\tau \in [0,\tau(\epsilon)]} \left\| \bar{r}_{yy} \phi \right\|_{L^2}^2 \text{ if } \epsilon \ll 1. \] (75)
Recalling (18) from Lemma 2.1 and fact (i) in Lemma 2.3, one can arrive at
\[ |\tilde{\rho}_{yy}| = \left| \frac{2}{\gamma + 1} \tilde{\rho}^{\frac{3-\gamma}{2}} \omega_{yy}^{\delta} + \frac{2(3-\gamma)}{(\gamma + 1)^2} \tilde{\rho}^{4-2\gamma}(\omega_{\delta y})^{2} \right| \leq C(\tilde{\rho}^{\frac{3-\gamma}{2}} \frac{\epsilon u_y}{\delta} + \tilde{\rho}^{2-\gamma} \bar{u}_y^2). \]
and one can obtain that
\[ \Pi_2 = -2\mu_0^{\frac{\alpha-2}{2}} \int_0^T \int \phi_y \tilde{\rho}_{yy} \psi dyds \leq C \int_0^T \int \phi_y \tilde{\rho}_{yy} \psi dyds \]
\[ \leq \frac{1}{32} \int_0^T \int \tilde{\rho}^{-\frac{\alpha-2}{2}} \phi_y^2 dyds + C \int_0^T \int \tilde{\rho}^{-2\gamma} \bar{u}_y^2 \psi_y^2 dyds \]
\[ \leq \frac{1}{32} \int_0^T \int \tilde{\rho}^{-\frac{\alpha-2}{2}} \phi_y^2 dyds + C \int_0^T \int \tilde{\rho}^{-2\gamma} \bar{u}_y^2 \psi_y^2 dyds \]
\[ \leq \frac{1}{32} \int_0^T \int \tilde{\rho}^{-\frac{\alpha-2}{2}} \phi_y^2 dyds + C \epsilon \int_0^T \int \bar{u}_y \psi_y^2 dyds. \]
From Lemma 2.3(i), one has that
\[ \bar{\rho}_y = \tilde{\rho}^{\frac{3-\gamma}{2}} \bar{u}_y \]
and then
\[ \Pi_3 = -2\mu_0^{\frac{\alpha-2}{2}} \int_0^T \int \phi_y \tilde{\rho}_y \psi dyds \leq C \int_0^T \int \phi_y \tilde{\rho}_y \psi dyds \]
\[ \leq \frac{1}{32} \int_0^T \int \tilde{\rho}^{-\frac{\alpha-2}{2}} \phi_y^2 dyds + C \int_0^T \int \tilde{\rho}^{-2\gamma} \bar{u}_y^2 \psi_y^2 dyds \]
\[ \leq \frac{1}{32} \int_0^T \int \tilde{\rho}^{-\frac{\alpha-2}{2}} \phi_y^2 dyds + C \int_0^T \int \tilde{\rho}^{-2\gamma} \bar{u}_y^2 \psi_y^2 dyds \]
\[ \leq \frac{1}{32} \int_0^T \int \tilde{\rho}^{-\frac{\alpha-2}{2}} \phi_y^2 dyds + C \epsilon \int_0^T \int \bar{u}_y \psi_y^2 dyds. \]
Similar to \( \Pi_2, \) one can obtain that
\[ \Pi_4 = -\int_0^T \int \frac{\tilde{\rho}_{yy} \psi_y^2}{\tilde{\rho}} dyds \]
\[ \leq C \delta \int_0^T \int |\bar{u}_y|^2 \tilde{\rho}^{-\frac{\alpha-1}{2}} \tilde{\rho}^{-(\gamma-1)} dyds + C \int_0^T \int |\tilde{\rho}_y|^2 \tilde{\rho}^{4-2\gamma} dyds \]
\[ \leq C \epsilon \int_0^T \int |\tilde{\rho}_y|^2 \tilde{\rho}^{4-2\gamma} dyds. \]
Similar to \( \Pi_3, \) using H"older inequality and Lemma 2.3 one can get that
\[ \Pi_5 = -\int_0^T \int \frac{2\bar{\rho}_y \psi_y u_y}{\tilde{\rho}} dyds \leq \int_0^T \int \psi_y^2 dyds + C \int_0^T \int |\bar{\rho}_{yy} \psi_y^2 \tilde{\rho}^{-2\gamma} dyds \]
\[
\Pi_0 = - \int_0^T \int \frac{\psi u_y}{\rho} \phi dyds \\
\leq C \int_0^T \int |\psi^{\frac{\alpha+2}{\alpha+1}} u_y| \phi dyds \\
\leq C \epsilon \int_0^T \int |\psi^{\frac{\alpha+2}{\alpha+1}} u_y| \phi dyds
\]
\[
\leq \frac{1}{32} \int_0^\tau \int \rho \gamma^{-3} \phi_y^2 \, dyds + C \varepsilon^3 \delta^{-2} \mu^{-(\gamma+1)}
\]
\[
\leq \frac{1}{32} \int_0^\tau \int \rho \gamma^{-3} \phi_y^2 \, dyds + C \varepsilon^{3-(\gamma+3)a} \ln \varepsilon^{-(\gamma+1)},
\]
(86)

\[
\Pi_{11}^2 = C \varepsilon^{-(\alpha-2)} \int_0^\tau \int \left\| \psi_y \right\|^{\alpha-2} \bar{u}_{yy} \frac{\phi_y}{\rho^2} \, dyds
\]
\[
\leq C \varepsilon^{-(\alpha-2)} \mu^{-\frac{3\alpha+1}{\alpha}} \int_0^\tau \left\| \psi_y \right\|_{L^\alpha} \left\| \frac{\phi_y}{\rho} \right\|_{L^2} \left\| \frac{\psi_{yy}}{\rho} \right\|_{L^{2\alpha}} \, dyds
\]
\[
\leq \frac{1}{32} \int_0^\tau \int \rho \gamma^{-3} \phi_y^2 \, dyds + \mu^{-(\gamma+1)} (\varepsilon^{-(\alpha-2)} \int_0^\tau \left\| \psi \right\|_{L^\alpha} \left\| \frac{\phi_y}{\rho} \right\|_{L^2} \left\| \frac{\psi_{yy}}{\rho} \right\|_{L^{2\alpha}} \, dyds)
\]
\[
+ C \varepsilon^{-(\alpha-2)(\alpha+1)} \mu^{-(\gamma+1)} \int_0^\tau \left\| \psi_y \right\|_{L^\alpha} \left\| \bar{u}_{yy} \right\|_{L^2} \, dyds
\]
\[
\leq \frac{1}{32} \int_0^\tau \int \rho \gamma^{-3} \phi_y^2 \, dyds + \mu^{-(\gamma+1)} (\varepsilon^{-(\alpha-2)} \int_0^\tau \left\| \psi \right\|_{L^\alpha} \left\| \frac{\phi_y}{\rho} \right\|_{L^2} \left\| \frac{\psi_{yy}}{\rho} \right\|_{L^{2\alpha}} \, dyds)
\]
\[
+ C \varepsilon^{\frac{3\alpha+1}{\alpha}-(2\alpha+1)} \mu^{-(\gamma+1)} \left( \varepsilon^{-(\alpha-2)} \sup_{\tau \in [0, \tau]} \left\| \psi_y \right\|_{L^\alpha} \right) + C \varepsilon^{\frac{8\alpha+1}{\alpha}-(\gamma+1)} \ln \varepsilon^{-(\gamma+1)}
\]
(87)

holds for $2 < \alpha < 4$ and

\[
\Pi_{11}^{*} = C \varepsilon^{-(\alpha-2)} \int_0^\tau \int \left\| \psi \right\|_{L^\alpha} \left\| \frac{\phi_y}{\rho} \right\|_{L^2} \left\| \frac{\psi_{yy}}{\rho} \right\|_{L^{2\alpha}} \, dyds
\]
\[
\leq C \varepsilon^{-(\alpha-2)} \mu^{-\frac{3\alpha+1}{\alpha}} \int_0^\tau \left\| \psi_y \right\|_{L^\alpha} \left\| \frac{\phi_y}{\rho} \right\|_{L^2} \left\| \frac{\psi_{yy}}{\rho} \right\|_{L^{2\alpha}} \, dyds
\]
\[
\leq \frac{1}{32} \int_0^\tau \int \rho \gamma^{-3} \phi_y^2 \, dyds
\]
\[
+ C \varepsilon^{-2(\alpha-2)} \mu^{-(\gamma+1)} \int_0^\tau \left\| \psi_y \right\|_{L^\alpha} \left\| \frac{\psi_{yy}}{\rho} \right\|_{L^{2\alpha}} \, dyds
\]
\[
\leq \frac{1}{32} \int_0^\tau \int \rho \gamma^{-3} \phi_y^2 \, dyds + \left( \varepsilon^{-2(\alpha-2)} \int_0^\tau \left\| \psi \right\|_{L^\alpha} \left\| \frac{\phi_y}{\rho} \right\|_{L^2} \left\| \frac{\psi_{yy}}{\rho} \right\|_{L^{2\alpha}} \, dyds \right)
\]
\[
+ C \varepsilon^{-2(\alpha-2)} \mu^{-(\gamma+1)} \int_0^\tau \left\| \psi_y \right\|_{L^\alpha} \left\| \bar{u}_{yy} \right\|_{L^2} \, dyds
\]
\[
\leq \frac{1}{32} \int_0^\tau \int \rho \gamma^{-3} \phi_y^2 \, dyds + \left( \varepsilon^{-2(\alpha-2)} \int_0^\tau \left\| \psi \right\|_{L^\alpha} \left\| \frac{\phi_y}{\rho} \right\|_{L^2} \left\| \frac{\psi_{yy}}{\rho} \right\|_{L^{2\alpha}} \, dyds \right)
\]
\[
+ \mu^{-(\gamma+1)} (\varepsilon^{-(\alpha-2)} \sup_{\tau \in [0, \tau]} \left\| \psi_y \right\|_{L^\alpha} \right) + C \varepsilon^{\frac{3\alpha+1}{\alpha}-(\gamma+1)} \ln \varepsilon^{-(\gamma+1)}
\]
(88)

holds for $4 \leq \alpha < 6$ and

\[
\Pi_{11}^{**} = C \varepsilon^{-(\alpha-2)} \int_0^\tau \int \left\| \psi \right\|_{L^\alpha} \left\| \frac{\phi_y}{\rho} \right\|_{L^2} \left\| \frac{\psi_{yy}}{\rho} \right\|_{L^{2\alpha}} \, dyds
\]
\[
\leq C \varepsilon^{-(\alpha-2)} \mu^{-\frac{3\alpha+1}{\alpha}} \int_0^\tau \left\| \psi_y \right\|_{L^\alpha} \left\| \frac{\phi_y}{\rho} \right\|_{L^2} \left\| \frac{\psi_{yy}}{\rho} \right\|_{L^{2\alpha}} \, dyds
\]
\[
\leq \frac{1}{32} \int_0^\tau \int \rho \gamma^{-3} \phi_y^2 \, dyds + \left( \varepsilon^{-2(\alpha-2)} \int_0^\tau \left\| \psi \right\|_{L^\alpha} \left\| \frac{\phi_y}{\rho} \right\|_{L^2} \left\| \frac{\psi_{yy}}{\rho} \right\|_{L^{2\alpha}} \, dyds \right)
\]
\[ + \mu^{-(\gamma+1)}(\varepsilon^{-\alpha-2}) \sup_{\tau \in (0, \tau(\sigma))} \left| \frac{\partial}{\partial y} \right|_{L^\infty} + C \varepsilon^{4-\alpha-2} \left(\gamma+1+\frac{3\alpha-2}{\gamma}\right) \right| \ln \varepsilon \right|^{-\gamma+1} \]

(89)

holds for \( \alpha \geq 6 \), and

\[
\Pi_{11}^3 = C \varepsilon^{-\alpha-2} \int_0^\tau \int \left| \tilde{u}_y \right|^{\alpha-2} \left| \tilde{u}_{yy} \right| \phi_y |dyds \\
\leq \frac{1}{32} \int_0^\tau \int \tilde{\rho}^{\gamma-3} \phi_y^2 dyds + C \varepsilon^{2-\alpha-2} \int_0^\tau \int \tilde{\rho}^{-(\gamma+1)} \left| \tilde{u}_y \right|^2 \tilde{u}_{yy}^2 dyds \\
\leq \frac{1}{32} \int_0^\tau \int \tilde{\rho}^{\gamma-3} \phi_y^2 dyds + C \varepsilon^{3-\alpha-2} \| \varepsilon \|^2 \ln \varepsilon \right|^{-\gamma+1}. \]

(90)

Now, we estimate the last term in the right-hand side of (85).

\[
\Pi_{11}^1 \leq C \int_0^\tau \int \left| \frac{u_y}{\varepsilon} \right|^{\alpha-2} \left| \frac{\psi_{yy}}{\rho^2} \right| |dyds \\
\leq \frac{1}{32} \int_0^\tau \int \tilde{\rho}^{\gamma-3} \phi_y^2 dyds \\
+ C \varepsilon^{-2(\alpha-2)} \mu_{-(\gamma+1)} \int_0^\tau \int \tilde{\rho}^{-1} \tilde{u}_y^2 (\alpha-2) \psi_y^2 + \tilde{\rho}^{-1} \psi_y^2 (\alpha-2) \tilde{u}_{yy}^2 |dyds \\
\leq \frac{1}{32} \int_0^\tau \int \tilde{\rho}^{\gamma-3} \phi_y^2 dyds + \delta^{-2(\alpha-2)} \mu_{-(\gamma+1)} \int_0^\tau \int \frac{|\psi_{yy}|}{\sqrt{\rho}} \| \frac{|\psi_{yy}|}{\sqrt{\rho}} \|_{L^2} ds \\
+ C \mu_{-(\gamma+1)} \varepsilon^{-2(\alpha-2)} \int_0^\tau \int \frac{\left| \frac{\psi_{yy}}{\rho^2} \right|^2}{\sqrt{\rho}} |dyds \]  \tag{91}

holds for \( 2 < \alpha < 4 \). When \( \alpha \geq 4 \), one can rewrite \( \Pi_{11}^1 \) as

\[
\Pi_{11}^1 \leq C \int_0^\tau \int \left| (u_0 + \frac{u_y}{\varepsilon})^2 \right|^{\alpha-2} \left| \frac{\psi_{yy}}{\rho^2} \right| |dyds \\
\leq C \int_0^\tau \int \left| \frac{\varepsilon^{-2}}{u_y^2} \left| \psi_{yy} \right| \phi_y^2 |dyds \\
+ \varepsilon^{-2(\alpha-2)} \left| \frac{\tilde{u}_y^2}{\psi_{yy}} \frac{\phi_y}{\rho^2} \right| + \varepsilon^{-(\alpha-2)} \left| \tilde{u}_y^2 \psi_{yy} \frac{\phi_y}{\rho^2} \right| + \varepsilon^{-(\alpha-2)} \left| \tilde{u}_y^2 \frac{\phi_y}{\rho^2} \right| \\
+ \varepsilon^{-2(\alpha-2)} \left| \frac{\psi_{yy}}{\rho^2} \right| |dyds := \Sigma_{j=1}^6 \Pi_{11}^1(j), \]  \tag{92}

where

\[
\Pi_{11}^1(1) \leq C \varepsilon^{-2} \int_0^\tau \int \left| \tilde{u}_y^2 \psi_{yy} \frac{\phi_y}{\rho^2} \right| |dyds \\
\leq \frac{1}{32} \int_0^\tau \int \tilde{\rho}^{\gamma-3} \phi_y^2 dyds + C \delta^{-4} \mu_{-(\gamma+1)} \int_0^\tau \int \frac{\left| \frac{\psi_{yy}}{\rho} \right|}{\sqrt{\rho}} |dyds, \tag{93}
\]

\[
\Pi_{11}^1(2) \leq C \varepsilon^{-2} \int_0^\tau \int \left| \frac{\psi_{yy}}{\rho^2} \right| |dyds \\
\leq \frac{1}{32} \int_0^\tau \int \tilde{\rho}^{\gamma-3} \phi_y^2 dyds \\
+ C \mu_{-(\gamma+1)} \varepsilon^{-4} \int_0^\tau \int \frac{\left| \frac{\psi_{yy}}{\rho^2} \right|}{\sqrt{\rho}} \| \frac{\psi_{yy}}{\rho} \|_{L^2}^{2(\alpha-4)} |dyds \\
\leq \frac{1}{32} \int_0^\tau \int \tilde{\rho}^{\gamma-3} \phi_y^2 dyds + \mu_{-(\gamma+1)} \int_0^\tau \int \frac{\left| \frac{\psi_{yy}}{\rho} \right|}{\sqrt{\rho}} |dyds. \]
Substituting (75)-(98) into (74), one can obtain that

\[
\Pi_{11}^4(3) \leq C \epsilon^{-\alpha(2)} \int_0^\tau \int \frac{\rho^{\gamma-3} \phi_y^2}{\rho^2} \, dyds \\
+ C \mu^{-\gamma(1)} \epsilon^{-2(\alpha-2)} \int_0^\tau \frac{\psi_y |\alpha-2\psi_{yy}|}{\sqrt{\rho}} \|_{L^2}^2 \, ds,
\]

(94)

\[
\Pi_{11}^4(4) \leq C \epsilon^{-\alpha(2)} \int_0^\tau \int \frac{\rho^{\gamma-3} \phi_y^2}{\rho^2} \, dyds \\
+ C \mu^{-\gamma(1)} \epsilon^{-2(\alpha-2)} \int_0^\tau \frac{\psi_y |\alpha-2\psi_{yy}|}{\sqrt{\rho}} \|_{L^2}^2 \, ds,
\]

(95)

\[
\Pi_{11}^4(5) \leq C \epsilon^{-\alpha(2)} \int_0^\tau \int \frac{\rho^{\gamma-3} \phi_y^2}{\rho^2} \, dyds \\
+ C \mu^{-\gamma(1)} \epsilon^{-2(\alpha-2)} \int_0^\tau \frac{\psi_y |\alpha-2\psi_{yy}|}{\sqrt{\rho}} \|_{L^2}^2 \, ds,
\]

(96)

and

\[
\Pi_{11}^4(6) \leq C \epsilon^{-\alpha(2)} \int_0^\tau \int \frac{\rho^{\gamma-3} \phi_y^2}{\rho^2} \, dyds \\
+ C \mu^{-\gamma(1)} \epsilon^{-2(\alpha-2)} \int_0^\tau \frac{\psi_y |\alpha-2\psi_{yy}|}{\sqrt{\rho}} \|_{L^2}^2 \, ds,
\]

(97)

Substituting (75)-(98) into (74), one can obtain that

\[
\int \frac{\psi_y |\alpha-2\psi_{yy}|}{\sqrt{\rho}} \|_{L^2}^2 \, ds \\
+ C \delta^{-2(\alpha-2)} \mu^{-\gamma(1)} \int_0^\tau \frac{\psi_y |\alpha-2\psi_{yy}|}{\sqrt{\rho}} \|_{L^2}^2 \, ds \\
+ C \mu^{-\gamma(1)} \int_0^\tau \frac{\psi_y |\alpha-2\psi_{yy}|}{\sqrt{\rho}} \|_{L^2}^2 \, ds \\
+ C \epsilon^{-\gamma(2\alpha)} |\ln \epsilon|^{-\gamma(1)} \int_0^\tau \frac{\rho \psi_y^2}{\rho^2} \, dyds \\
+ C \epsilon^{-\gamma(2\alpha)} \int_0^\tau \frac{\rho \psi_y^2}{\rho^2} \, dyds \\
+ C \epsilon^{3(\gamma+2\alpha)} |\ln \epsilon|^{-\gamma(1)}
\]

(98)
holds for $2 < \alpha < 4$,
\[
\int \left( \frac{\alpha^2}{2} \frac{\phi_y^2}{2} + \bar{\rho} \psi^2 \right) dy + \int_0^\tau \int \bar{\rho} \gamma^{-3} \phi_y^2 dy ds \leq \int_0^\tau \int |\psi_y|^2 dy ds \\
+ C\delta^{(\alpha-2)} \mu^{-(\gamma+1)} \left( \int_0^\tau \left\| \frac{\psi_{yy}}{\sqrt{\rho}} \right\|_{L^2}^2 ds + \epsilon^{-2(\alpha-2)} \int_0^\tau \left\| \frac{\psi_y^{\alpha-2} \psi_{yy}}{\sqrt{\rho}} \right\|_{L^2}^2 ds \right) \\
+ C\epsilon \delta^{-3} \mu^{-(\gamma+1)} \left( \int_0^\tau \int \bar{\rho} \psi^2 dy ds + \int_0^\tau \int \bar{\rho} \gamma^{-2} \bar{u}_y \phi^2 dy ds \right) \\
+ 2\mu^{-(\gamma+1)}(\epsilon^{-(\alpha-2)} \sup_{\tau \in [0,\tau]} |\psi_y|_{L^\alpha}) + C\epsilon^{3-2(\gamma - \frac{\alpha(\alpha+1)}{4\alpha})} |\ln \epsilon|^{-\gamma+1} (100)
\]

holds for $4 \leq \alpha < 6$ and
\[
\int \left( \frac{\alpha^2}{2} \frac{\phi_y^2}{2} + \bar{\rho} \psi^2 \right) dy + \int_0^\tau \int \bar{\rho} \gamma^{-3} \phi_y^2 dy ds \leq \int_0^\tau \int |\psi_y|^2 dy ds \\
+ C\delta^{2(\alpha-2)} \mu^{-(\gamma+1)} \left( \int_0^\tau \left\| \frac{\psi_{yy}}{\sqrt{\rho}} \right\|_{L^2}^2 ds + \epsilon^{-2(\alpha-2)} \int_0^\tau \left\| \frac{\psi_y^{\alpha-2} \psi_{yy}}{\sqrt{\rho}} \right\|_{L^2}^2 ds \right) \\
+ C\epsilon \delta^{-3} \mu^{-(\gamma+1)} \left( \int_0^\tau \int \bar{\rho} \psi^2 dy ds + \int_0^\tau \int \bar{\rho} \gamma^{-2} \bar{u}_y \phi^2 dy ds \right) \\
+ 2\mu^{-(\gamma+1)}(\epsilon^{-(\alpha-2)} \sup_{\tau \in [0,\tau]} |\psi_y|_{L^\alpha}) + C\epsilon^{3-2(\gamma - \frac{\alpha(\alpha+1)}{4\alpha})} |\ln \epsilon|^{-\gamma+1} (101)
\]

holds for $\alpha \geq 6$.

Multiplying (99) and (100) by $\frac{1}{12}$, and adding the results to (70) and (71) respectively, if $\epsilon \ll 1$ one can get that
\[
\int \left( \frac{\alpha^2}{2} \frac{\phi_y^2}{2} + \bar{\rho} \psi^2 + \bar{\rho}^{-2} \phi^2 \right) dy \\
+ \int_0^\tau \int \left( |\psi_y|^2 + \epsilon^{-2(\alpha-2)} |\psi_y|^\alpha + \bar{\rho} \psi^2 + \bar{\rho} \gamma^{-2} \bar{u}_y \phi^2 + \bar{\rho} \gamma^{-3} \phi_y^2 \right) dy ds \\
\leq C\delta^{2(\alpha-2)} \mu^{-(\gamma+1)} \left( \int_0^\tau \left\| \frac{\psi_{yy}}{\sqrt{\rho}} \right\|_{L^2}^2 ds + \epsilon^{-2(\alpha-2)} \int_0^\tau \left\| \frac{\psi_y^{\alpha-2} \psi_{yy}}{\sqrt{\rho}} \right\|_{L^2}^2 ds \right) \\
+ C(\epsilon^{-(\alpha-2)} \sup_{\tau \in [0,\tau]} |\psi_y|_{L^\alpha}) + C\epsilon^{\frac{\alpha^2}{2} - \max\{2(\gamma + \alpha), \frac{4(\alpha+1)}{3\alpha}\}} |\ln \epsilon|^{-\gamma} (102)
\]

holds for $2 < \alpha < 4$,
\[
\int \left( \frac{\alpha^2}{2} \frac{\phi_y^2}{2} + \bar{\rho} \psi^2 + \bar{\rho}^{-2} \phi^2 \right) dy \\
+ \int_0^\tau \int \left( |\psi_y|^2 + \epsilon^{-2(\alpha-2)} |\psi_y|^\alpha + \bar{\rho} \psi^2 + \bar{\rho} \gamma^{-2} \bar{u}_y \phi^2 + \bar{\rho} \gamma^{-3} \phi_y^2 \right) dy ds \\
\leq C\delta^{2(\alpha-2)} \mu^{-(\gamma+1)} \left( \int_0^\tau \left\| \frac{\psi_{yy}}{\sqrt{\rho}} \right\|_{L^2}^2 ds + \epsilon^{-2(\alpha-2)} \int_0^\tau \left\| \frac{\psi_y^{\alpha-2} \psi_{yy}}{\sqrt{\rho}} \right\|_{L^2}^2 ds \right) \\
+ C(\epsilon^{-(\alpha-2)} \sup_{\tau \in [0,\tau]} |\psi_y|_{L^\alpha}) + C\epsilon^{\frac{\alpha^2}{2} - \max\{2(\gamma + \alpha), \frac{4(\alpha+1)}{3\alpha}\}} |\ln \epsilon|^{-\gamma} (103)
\]
holds for $4 \leq \alpha < 6$ and

$$\int (\mu_0^2 + \tau \phi_y^2 + \rho \psi^2 + \rho^2 \phi^2)dy$$

$$+ \int_0^\tau \left[ |y|^2 + c^2 \alpha |y| + \rho u_y y^2 + \rho^2 \bar{u}_y y^2 + \rho^2 \phi_y^2 |dydsight.$$

$$\leq C\delta^{-2(\alpha - 2)} \mu^{-2(\gamma + 1)} \left( \int_0^\tau \frac{|\psi_{yy}|^2}{\sqrt{\rho}} \right)^{2/3} + \epsilon^{-2(\alpha - 2)} \int_0^\tau \left[ |\psi_y|^{\alpha - 2} |\psi_{yy}|^2 \right] ds$$

$$+ C(\epsilon^{-2(\alpha - 2)} \int_0^\tau \left[ |\psi_y|^{\alpha - 2} |\psi_{yy}|^2 \right] ds)$$

$$+ 2\mu^{-2(\alpha - 2)} \epsilon^{-2(\alpha - 2)} \sup_{\tau \in [0, \tau(\epsilon)]} ||\psi_y||_{L^\alpha} + C\epsilon^{-2(\alpha - 2)(\gamma + 1)} ||\psi||_{L^\alpha}.$$

(104)

holds for $\alpha \geq 6$

Step 3. We estimate $\sup_{\tau \in [0, \tau(\epsilon)]} ||\psi_y||_{L^2}$ in this step. For this, we multiply (32) by $-\frac{\psi_{yy}}{\rho}$ and get that

$$\left(\frac{\psi_y^2}{2}\right)_{\tau} - \frac{u_y \psi_y^2}{2} + \psi_y \psi_{y\tau} + u_y \psi_y^2 - \rho' \frac{\rho}{\rho^2} \psi_y \psi_{yy} + \psi \bar{u}_y \psi_{yy}$$

$$+ \bar{\rho}_y \left[ \frac{\rho' \rho}{\rho^2} \right] \psi_{yy} + \left( \mu_0 + \frac{u_y}{\epsilon} \right)^2 \bar{u}_y y \psi_{yy} = 0.$$

(105)

Integrating the above equation over $\mathbb{R} \times [0, \tau]$ yields that

$$\int \frac{\psi_y^2}{2} dy + \int_0^\tau \frac{\bar{u}_y \psi_y^2}{2} dyds$$

$$+ \int_0^\tau \int \frac{1}{\rho} \left( \mu_0 + \frac{u_y}{\epsilon} \right)^2 \frac{\psi_{yy}}{4} dyds$$

$$= \int_0^\tau \int \left[ \rho' \frac{\rho}{\rho^2} \psi_y \psi_{yy} - \frac{\psi_y^3}{2} - \psi \bar{u}_y \psi_{yy} \right] dyds + \int_0^\tau \int \bar{\rho}_y \left[ \frac{\rho' \rho}{\rho^2} \right] \psi_{yy} dyds$$

$$- \int_0^\tau \int \frac{1}{\rho} \left( \mu_0 + \frac{u_y}{\epsilon} \right)^2 \frac{\psi_{yy}}{4} dyds.$$

(106)

First, one has that

$$\int_0^\tau \int \rho' \frac{\rho}{\rho^2} \psi_y \psi_{yy} dyds$$

$$\leq \frac{H_0^2}{32} \int_0^\tau \int \bar{\rho}^{-1} \psi_{yy}^2 dyds + C\mu^{-2\alpha} \epsilon \int_0^\tau ||\bar{\rho}||_{L^\infty}^2 \int \bar{\rho}^{-3} \phi_y^2 dyds$$

$$\leq \frac{H_0^2}{32} \int_0^\tau \int \bar{\rho}^{-1} \psi_{yy}^2 dyds$$

$$+ C\epsilon^{\frac{2\alpha - 2}{2}} \mu^{-\frac{1 + 2\alpha}{2} \max(2(\gamma - 3), 4(\gamma - 2), 0)} \sup_{\tau \in [0, \tau(\epsilon)]} ||\bar{\rho}^{-\frac{2\alpha}{2}} \phi_y||_{L^2}.$$ 

(107)

Furthermore, one can compute that

$$\int_0^\tau \int \frac{\psi_y^3}{2} dyds \leq C \epsilon^{-2(\alpha - 2)} \int_0^\tau \left[ |\psi_y|^{\alpha - 2} |\psi_{yy}|^2 \right] ds$$

$$+ \frac{1}{32} \int_0^\tau \left[ |\psi_y|^{\alpha - 2} |\psi_{yy}|^2 \right] ds + \frac{1}{32} \sup_{\tau \in [0, \tau(\epsilon)]} ||\psi_y||_{L^2}.$$

(108)
\[ + \frac{1}{64}(\epsilon^{-1}) \sup_{s \in [0, \tau]} \| \psi_y \|_{L^\infty}^2 + C \epsilon^2. \]

Moreover, one has that
\[
- \int_0^\tau \int \bar{u}_y \psi_y \psi_y dyds \leq \frac{1}{32} \int_0^\tau \int \rho \psi_y^2 \psi_y dyds + C \int_0^\tau \int \bar{u}_y \psi_y \psi_y dyds
\]
\[
\leq \frac{\mu_0^{a - 2}}{32} \int_0^\tau \int \rho \psi_y^2 \psi_y dyds + C \epsilon \psi \int_0^\tau \int \bar{u}_y \psi_y \psi_y dyds
\]
\[
\leq \frac{\mu_0^{a - 2}}{32} \int_0^\tau \int \rho \psi_y^2 \psi_y dyds + C (\epsilon \psi) \int_0^\tau \int \bar{u}_y \psi_y \psi_y dyds
\]
\[
\leq \frac{\mu_0^{a - 2}}{32} \int_0^\tau \int \rho \psi_y^2 \psi_y dyds + C (\epsilon \psi) \int_0^\tau \int \bar{u}_y \psi_y \psi_y dyds.
\]

Now, we estimate the non-linear term in (106)
\[
- \int_0^\tau \int \frac{1}{\rho} (\mu_0 + (\frac{\bar{u}_y}{\epsilon})^2 (\mu_0 + (\alpha - 1)(\frac{\bar{u}_y}{\epsilon})^2) \bar{u}_y \psi_y \psi_y dyds
\]
\[
\leq C \int_0^\tau \int \frac{1}{\rho} \bar{u}_y \psi_y \psi_y dyds + C \epsilon^3 |\ln \epsilon|^{-1},
\]
where
\[
III_1 \leq C \int_0^\tau \int \frac{1}{\rho} \bar{u}_y \psi_y \psi_y dyds
\]
\[
\leq \frac{\mu_0^{a - 2}}{32} \int_0^\tau \int \bar{u}_y \psi_y \psi_y dyds + C \epsilon^3 |\ln \epsilon|^{-1},
\]
\[
III_2 \leq C \epsilon^{-1} \int_0^\tau \int \bar{u}_y \psi_y \psi_y dyds
\]
\[
\leq \frac{\mu_0^{a - 2}}{32} \int_0^\tau \int \bar{u}_y \psi_y \psi_y dyds + C \epsilon^3 |\ln \epsilon|^{-1}
\]
and
\[
III_3 \leq C \epsilon^{-1} \int_0^\tau \int \bar{u}_y \psi_y \psi_y dyds
\]
\[
\leq \frac{1}{64} (\epsilon^{-1} \sup_{s \in [0, \tau]}) \int_0^\tau \int \psi_y \psi_y dyds
\]
\[
\leq \frac{1}{64} (\epsilon^{-1} \sup_{s \in [0, \tau]}) \int_0^\tau \int \psi_y \psi_y dyds + C \epsilon^3 |\ln \epsilon|^{-1}.
\]
Recalling (106), it follows from (107)-(114) that
\[
\int \frac{|\psi_y|^2}{2} dy + \int_0^T \int \bar{u}_y \psi_y^2 dyds + \int_0^T \int \frac{1}{\rho} (\mu_0 + (\frac{u_y}{\epsilon})^2) |\psi|^2 dyds \\
\leq C \epsilon^\delta \frac{1}{2} \mu^{-\frac{3}{2}} \max(2(\gamma-3),4(\gamma-2),0) \sup_{s \in [0,\tau(\epsilon)]} \|\rho^{-\frac{1}{2}} \phi_y\|_{L^2}^2 \\
+ C \epsilon^{\delta-1} \left( \int_0^T \int \bar{u}_y \psi_y^2 dyds + \int_0^T \int \rho \bar{u}_y \phi_y^2 dyds \right) \\
+ \frac{1}{64} (\epsilon^{-(\alpha-2)} \sup_{s \in [0,\tau(\epsilon)]} \|\psi_y\|_{L^x}^2) + \frac{1}{4} (\epsilon^{-2(\alpha-2)} \int_0^T \int |\psi_y|^2 \psi_y^2 dyds) \\
+ C \epsilon^{2(\alpha+1)\alpha} \ln \epsilon^{-1}.
\]

Step 4. As the last step, we estimate \(\sup_{\tau \in [0,\tau(\epsilon)]} \|\psi_y\|_{L^\alpha}\). Multiplying (32) by
\[-\frac{|\psi_y|^2 \psi_{yy}}{\rho}\]
gives that
\[
\left( \frac{|\psi_y|^2}{\alpha(\alpha-1)} \right)_\tau - \left( \frac{u(\psi_y)^2}{\alpha} + 1 \right) |\psi_y|^2 \psi_y \psi_{\tau} + u_y \left( \frac{\psi_y^2}{\alpha} \right)_y \\
- \frac{p'(\rho)}{\rho} \phi_y |\psi_y|^2 \psi_y + \psi \bar{u}_y |\psi_y|^2 \psi_y + \bar{\rho}_y \left( \frac{p'(\rho)}{\rho} \right) |\psi_y|^2 \psi_y \\
+ \left( \mu_0 + \left( \frac{u_y}{\epsilon} \right)^2 \right) u_y |\psi_y|^2 \psi_y = 0.
\]

Integrating the above equation over \(\mathbb{R} \times [0, \tau]\) yields that
\[
\int \frac{|\psi_y|^2}{\alpha(\alpha-1)} dy + \int_0^T \int \frac{1}{\rho} \bar{u}_y (\psi_y^2) dyds \\
+ \int_0^T \int (\mu_0 + (\frac{u_y}{\epsilon})^2) |\psi_y|^2 \psi_y dyds \\
= \int_0^T \int \left( \frac{p'(\rho)}{\rho} \phi_y |\psi_y|^2 \psi_y - \psi \bar{u}_y |\psi_y|^2 \psi_y \right) dyds \\
+ \int_0^T \int \left( \bar{\rho}_y p'(\rho) \right) (\psi_y^2 \psi_y) dyds - \int_0^T \int \frac{1}{\alpha} \psi_y (\psi_y)^2 dyds \\
- \int_0^T \int \left( \mu_0 + \left( \frac{u_y}{\epsilon} \right)^2 \right) u_y |\psi_y|^2 \psi_y dyds.
\]

First, one has that
\[
\int_0^T \int \frac{p'(\rho)}{\rho} \phi_y |\psi_y|^2 \psi_y dyds \\
\leq \frac{1}{32} \epsilon^{-(\alpha-2)} \int_0^T \int \bar{\rho}_y |\psi_y|^2 (\alpha-2) \psi_y^2 dyds \\
+ C \epsilon^{\alpha-2} \frac{1}{2} \delta \mu^{-\frac{1}{2}} \sup_{s \in [0,\tau(\epsilon)]} \|\rho^{-\frac{1}{2}} \phi_y\|_{L^2}^2.
\]

Furthermore, one can calculate that
\[
- \int_0^T \int \psi \bar{u}_y |\psi_y|^2 \psi_y dyds \\
\leq \frac{1}{32} \epsilon^{-(\alpha-2)} \int_0^T \int \bar{\rho}_y |\psi_y|^2 (\alpha-2) \psi_y^2 dyds + C \epsilon^{\alpha-2} \int_0^T \int \rho \psi^2 \bar{u}_y^2 dyds.
\]
\[
\leq \frac{1}{32} \epsilon^{-(\alpha-2)} \int_0^\tau \int \bar{\rho}^{-1} |\psi|^{2(\alpha-2)} \psi_y^2 \, dy \, ds + C \epsilon^{\alpha-1} \delta^{-1} \int_0^\tau \int \bar{p} \bar{u}_y \psi^2 \, dy \, ds
\]  
and
\[
- \int_0^\tau \int \bar{\rho} \left[ \psi_y' \left( \frac{\psi_y}{\rho} \right) - \frac{\rho'}{\rho} \psi_y \right] \psi_y \, dy \, ds
\leq \frac{1}{32} \epsilon^{-(\alpha-2)} \int_0^\tau \int \bar{p}^{-1} |\psi|^{2(\alpha-2)} \psi_y^2 \, dy \, ds + C \epsilon^{\alpha-1} \delta^{-1} \int_0^\tau \int \bar{p} \bar{u}_y \phi^2 \, dy \, ds.
\]

Similar to (108), one can get that
\[
- \int_0^\tau \int \frac{1}{\alpha} \left( \frac{1}{\alpha} \right) \frac{\psi_y'^2}{\psi_y} \, dy \, ds \leq C \int_0^\tau \left\| \frac{\psi}{\sqrt{\rho}} \right\|_{L^2} \left\| \psi \right\|_{L^\infty} \left\| \psi \right\|_{L^2} \, ds
\leq \frac{\mu_{\alpha-2}}{32} \int_0^\tau \left\| \frac{\psi}{\sqrt{\rho}} \right\|_{L^2} \, ds + \frac{1}{64} \sup_{\tau \in [0, \tau(\epsilon)]} \left\| \psi \right\|_{L^\infty} + C \epsilon^{(2\alpha-3)(\alpha-2)}.
\]

Next, one has that
\[
- \int_0^\tau \int \frac{1}{\rho} \left( \mu + \frac{\mu_y}{\varepsilon} \right) \frac{\psi_y'^2}{\rho^2} \, dy \, ds \leq C \int_0^\tau \left\| \psi \right\|_{L^2} \left\| \psi \right\|_{L^\infty} \left\| \psi \right\|_{L^2} \, ds
\leq \frac{1}{32} \epsilon^{-(\alpha-2)} \int_0^\tau \int \bar{p}^{-1} |\psi|^{2(\alpha-2)} \psi_y^2 \, dy \, ds + C \epsilon^{\alpha-2} \int_0^\tau \int \bar{p} \bar{u}_y^2 \, dy \, ds
\leq \frac{1}{32} \epsilon^{-(\alpha-2)} \int_0^\tau \int \bar{p}^{-1} |\psi|^{2(\alpha-2)} \psi_y^2 \, dy \, ds + C \epsilon^{\alpha+1} \delta^{-2} \mu^{-1}
\leq \frac{1}{32} \epsilon^{-(\alpha-2)} \int_0^\tau \int \bar{p}^{-1} |\psi|^{2(\alpha-2)} \psi_y^2 \, dy \, ds + C \epsilon^{\alpha+1-3a} \ln \epsilon^{-1},
\]

where,
\[
IV_1 \leq C \int_0^\tau \int \frac{1}{\rho} \bar{u}_y \psi_y |\psi|^{\alpha-2} \psi_y \, dy \, ds
\leq \frac{1}{32} \epsilon^{-(\alpha-2)} \int_0^\tau \int \bar{p}^{-1} |\psi|^{2(\alpha-2)} \psi_y^2 \, dy \, ds + C \epsilon^{\alpha-2} \int_0^\tau \int \bar{p} \bar{u}_y^2 \, dy \, ds
\leq \frac{1}{32} \epsilon^{-(\alpha-2)} \int_0^\tau \int \bar{p}^{-1} |\psi|^{2(\alpha-2)} \psi_y^2 \, dy \, ds + C \epsilon^{\alpha+1} \delta^{-2} \mu^{-1}
\leq \frac{1}{32} \epsilon^{-(\alpha-2)} \int_0^\tau \int \bar{p}^{-1} |\psi|^{2(\alpha-2)} \psi_y^2 \, dy \, ds + C \epsilon^{\alpha+1-3a} \ln \epsilon^{-1},
\]

\[
IV_2 \leq C \epsilon^{-(\alpha-2)} \int_0^\tau \int \frac{1}{\rho} \bar{u}_y \psi_y |\psi|^{\alpha-2} \psi_y \, dy \, ds
\leq \frac{1}{32} \epsilon^{-(\alpha-2)} \int_0^\tau \int \bar{p}^{-1} |\psi|^{2(\alpha-2)} \psi_y^2 \, dy \, ds + C \epsilon^{\alpha+1} \delta^{-2} (\alpha-1) \mu^{-1}.
\]

\[
IV_3 \leq C \epsilon^{-(\alpha-2)} \int_0^\tau \int \left| \psi \right|_{L^2} \left| \psi \right|_{L^2} \, dy \, ds
\leq C \epsilon^{-(\alpha-2)} \int_0^\tau \int \left| \psi \right|_{L^2} \left| \psi \right|_{L^2} \, dy \, ds
\leq C \epsilon^{-(\alpha-2)} \int_0^\tau \int \left| \psi \right|_{L^2} \left| \psi \right|_{L^2} \, dy \, ds
\leq C \epsilon^{-(\alpha-2)} \int_0^\tau \int \left| \psi \right|_{L^2} \left| \psi \right|_{L^2} \, dy \, ds
holds for $2 < \alpha < 4$,

$$
IV_4' \leq C \varepsilon^{-(\alpha - 2)} \int_0^T \left\| \psi_y \right\|_{L^2}^2 dy ds \\
\leq C \varepsilon^{-(\alpha - 2)} \int_0^T \left( \frac{\left\| \psi_y \right\|^2_{L^2}}{\varepsilon} \right)^\frac{2(\alpha - 2)}{3} ds + C \varepsilon^{-(\alpha - 2)} \int_0^T \left\| \psi_y \right\|_{L^2}^2 dy ds \\
\leq \frac{1}{32} \varepsilon^{-(\alpha - 2)} \int_0^T \left\| \psi_y \right\|_{L^2}^2 dy ds + \frac{1}{32} \sup_{s \in [0, \tau(\varepsilon)]} \left\| \psi_y \right\|_{L^2}^\alpha + C \varepsilon^{\frac{2(\gamma - 2)(\alpha - 6)}{\varepsilon - \alpha} - \frac{5\alpha - 4}{6 - \alpha}}
$$

holds for $4 \leq \alpha < 6$ and

$$
IV_4'' \leq C \varepsilon^{-(\alpha - 2)} \int_0^T \left\| \psi_y \right\|_{L^2}^2 dy ds \\
\leq C \varepsilon^{-(\alpha - 2)} \int_0^T \left( \frac{\left\| \psi_y \right\|^2_{L^2}}{\varepsilon} \right)^\frac{2(\alpha - 2)}{3} ds + C \varepsilon^{-(\alpha - 2)} \int_0^T \left\| \psi_y \right\|_{L^2}^2 dy ds \\
\leq \frac{1}{32} \varepsilon^{-(\alpha - 2)} \int_0^T \left\| \psi_y \right\|_{L^2}^2 dy ds + \frac{1}{32} \sup_{s \in [0, \tau(\varepsilon)]} \left\| \psi_y \right\|_{L^2}^\alpha + C \varepsilon^{\frac{4\alpha - 2 - \frac{3(\gamma - 2)\alpha}{\varepsilon - \alpha}}}
$$

holds for $\alpha \geq 6$.

Recalling (117), one can deduce from (118)-(127) that

$$
\int \frac{\left| \psi_y \right|^\alpha}{\alpha(\alpha - 1)} dy + \int_0^T \left( \frac{1}{\alpha} \bar{u}_y \left( \psi_y^2 \right) \right)^\frac{\alpha}{2} dy ds \\
+ \int_0^T \left( \frac{1}{\bar{\rho}} (\mu_0 + \left( \frac{u_y}{\varepsilon} \right)^2) \right)^{\frac{\alpha - 2}{2}} \left| \psi_y \right|^{\alpha - 2} \psi_y^2 dy ds \\
\leq C \varepsilon^{(\alpha - 2)} \varepsilon^\frac{\gamma}{3} \delta^{-\frac{2}{3}} \mu^{-(1 + \frac{1}{4} max(2(\gamma - 3), 4(\gamma - 2), 0))} \sup_{s \in [0, \tau(\varepsilon)]} \left\| \bar{\rho}^{-\frac{3}{2}} \phi_y \right\|_{L^2}^2 \\
+ C \varepsilon^{1 - \delta^{-1}} \int \left\| \bar{\rho} \bar{\psi}^2 \right\|_{L^2}^2 dy ds + \int_0^T \left( \bar{\rho}^{\gamma - 2} \bar{u}_y \phi_y^2 \right) dy ds + C \varepsilon^{\frac{2\alpha - 6 - \alpha}{\varepsilon - \alpha} - \frac{5\alpha - 4}{6 - \alpha}}
$$

holds for $2 < \alpha < 4$,
+(129)

holds for $4 \leq \alpha < 6$ and

\[
\int \frac{|\psi_\gamma|^\alpha}{\alpha(\alpha-1)}dy + \int_0^\tau \int \frac{1}{\alpha} \bar{u}_y(\psi_y^2)^2 dyds
+ \int_0^\tau \int \frac{1}{\rho} (\mu_0 + \left(\frac{\mu y}{\epsilon}\right)^2) \frac{\alpha-2}{2} |\psi_\gamma|^\alpha \psi_{yy}^2 dyds
\leq C \epsilon^{\frac{5}{2}} \delta - \frac{5}{2} \mu^{-\frac{1}{2}} \max\{2(\gamma-3),4(\gamma-2),0\} \sup_{s \in [0,\tau(\epsilon)]} \|\bar{\rho}^{-\frac{1}{2}} \phi_y\|^2_{L^2}
\]

\[
+ C \epsilon^{\frac{1}{2}} \bar{\alpha}^{-1} \left(\int_0^\tau \int \bar{\rho} \bar{u}_y \psi^2 dyds + \int_0^\tau \int \rho^\gamma \bar{u}_y \phi^2 dyds\right) + C \epsilon \left(\frac{\alpha-1}{2} \delta - 5\frac{2}{\epsilon} \mu^{-\frac{1}{2}} \max\{2(\gamma-3),4(\gamma-2),0\}\right)
\]

(130)

holds for $\alpha \geq 6$.

Finally, based on (115) and (128)-(130), one can get that

\[
\int \left(\frac{\psi_y^2}{2} + \epsilon^{-\frac{(\alpha-2)}{2}} \frac{|\psi_\gamma|^\alpha}{\alpha(\alpha-1)}dy + \int_0^\tau \int \left(\frac{\bar{u}_y \psi_y^2}{2} + \epsilon^{-\frac{(\alpha-2)}{2}} \frac{1}{\alpha} \bar{u}_y(\psi_y^2)^2\right) dyds
+ \int_0^\tau \int \left(\frac{1}{\rho} (\mu_0 + \left(\frac{\mu y}{\epsilon}\right)^2) \frac{\alpha-2}{2} \psi_{yy}^2 \right) dyds
\leq C \epsilon^{\frac{5}{2}} \delta - \frac{5}{2} \mu^{-\frac{1}{2}} \max\{2(\gamma-3),4(\gamma-2),0\} \sup_{s \in [0,\tau(\epsilon)]} \|\bar{\rho}^{-\frac{1}{2}} \phi_y\|^2_{L^2}
\]

\[
+ C \epsilon \bar{\alpha}^{-1} \left(\int_0^\tau \int \bar{\rho} \bar{u}_y \psi^2 dyds + \int_0^\tau \int \rho^\gamma \bar{u}_y \phi^2 dyds\right) + C \epsilon \left(\frac{\alpha-1}{2} \delta - 5\frac{2}{\epsilon} \mu^{-\frac{1}{2}} \max\{2(\gamma-3),4(\gamma-2),0\}\right)
\]

(131)

holds for $2 < \alpha < 4$,

\[
\int \left(\frac{\psi_y^2}{2} + \epsilon^{-\frac{(\alpha-2)}{2}} \frac{|\psi_\gamma|^\alpha}{\alpha(\alpha-1)}dy + \int_0^\tau \int \left(\frac{\bar{u}_y \psi_y^2}{2} + \epsilon^{-\frac{(\alpha-2)}{2}} \frac{1}{\alpha} \bar{u}_y(\psi_y^2)^2\right) dyds
+ \int_0^\tau \int \left(\frac{1}{\rho} (\mu_0 + \left(\frac{\mu y}{\epsilon}\right)^2) \frac{\alpha-2}{2} \psi_{yy}^2 \right) dyds
\leq C \epsilon^{\frac{5}{2}} \delta - \frac{5}{2} \mu^{-\frac{1}{2}} \max\{2(\gamma-3),4(\gamma-2),0\} \sup_{s \in [0,\tau(\epsilon)]} \|\bar{\rho}^{-\frac{1}{2}} \phi_y\|^2_{L^2}
\]

\[
+ C \epsilon \bar{\alpha}^{-1} \left(\int_0^\tau \int \bar{\rho} \bar{u}_y \psi^2 dyds + \int_0^\tau \int \rho^\gamma \bar{u}_y \phi^2 dyds\right) + C \epsilon \left(\frac{\alpha-1}{2} \delta - 5\frac{2}{\epsilon} \mu^{-\frac{1}{2}} \max\{2(\gamma-3),4(\gamma-2),0\}\right)
\]

(132)

holds for $4 \leq \alpha < 6$ and

\[
\int \left(\frac{\psi_y^2}{2} + \epsilon^{-\frac{(\alpha-2)}{2}} \frac{|\psi_\gamma|^\alpha}{\alpha(\alpha-1)}dy + \int_0^\tau \int \left(\frac{\bar{u}_y \psi_y^2}{2} + \epsilon^{-\frac{(\alpha-2)}{2}} \frac{1}{\alpha} \bar{u}_y(\psi_y^2)^2\right) dyds
+ \int_0^\tau \int \left(\frac{1}{\rho} (\mu_0 + \left(\frac{\mu y}{\epsilon}\right)^2) \frac{\alpha-2}{2} \psi_{yy}^2 \right) dyds
\leq C \epsilon^{\frac{5}{2}} \delta - \frac{5}{2} \mu^{-\frac{1}{2}} \max\{2(\gamma-3),4(\gamma-2),0\} \sup_{s \in [0,\tau(\epsilon)]} \|\bar{\rho}^{-\frac{1}{2}} \phi_y\|^2_{L^2}
\]

\[
+ C \epsilon \bar{\alpha}^{-1} \left(\int_0^\tau \int \bar{\rho} \bar{u}_y \psi^2 dyds + \int_0^\tau \int \rho^\gamma \bar{u}_y \phi^2 dyds\right) + C \epsilon \left(\frac{\alpha-1}{2} \delta - 5\frac{2}{\epsilon} \mu^{-\frac{1}{2}} \max\{2(\gamma-3),4(\gamma-2),0\}\right)
\]

(133)

holds for $\alpha \geq 6$. 

\[
\int \rho \bar{u}_y \psi^2 dyds + \int \rho^\gamma \bar{u}_y \phi^2 dyds + C \epsilon \left(\frac{\alpha-1}{2} \delta - 5\frac{2}{\epsilon} \mu^{-\frac{1}{2}} \max\{2(\gamma-3),4(\gamma-2),0\}\right)
\]
Therefore, multiplying (102) and (103) by $\delta^{2(\alpha-2)} \mu^\gamma$, adding the results to (131)-(133) respectively, and selecting $\epsilon$ small enough, one can get that

$$
\int \left( \frac{\psi^2}{2} + \epsilon^{-(\alpha-2)} \frac{|\psi|^\alpha}{\alpha(\alpha-1)} \right) dy + \delta^{2(\alpha-2)} \mu^\gamma \int \left( \frac{\alpha-2}{2} \frac{\phi_y^2}{\rho^2} + \rho \psi^2 + \rho^{\gamma-2} \phi^2 \right) dy 
+ \int_0^\infty \int \left\{ \frac{1}{\rho} (\mu_0 + \left( \frac{\psi}{\epsilon} \right)^2) \alpha \frac{\alpha-2}{2} \psi^2 + \epsilon^{-(\alpha-2)} \frac{1}{\rho} (\mu_0 + \left( \frac{\psi}{\epsilon} \right)^2) \alpha \frac{\alpha-2}{2} |\psi|^\alpha \phi_{yy}^2 \right\} dy ds 
+ \delta^{2(\alpha-2)} \mu^\gamma \int_0^\infty \int \left[ |\psi|^2 + \epsilon^{-(\alpha-2)} |\psi|^\alpha \right] \phi_y^2 + \rho \psi \phi^2 + \rho \psi \phi_{yy} \phi^2 + \rho \psi \phi_{yy} \phi^2 \] dy ds 
+ \int_0^\infty \int \left[ \frac{\psi_y^2}{2} + \epsilon^{-(\alpha-2)} \frac{1}{\alpha} \psi_y (\psi^2)^2 \right] dy ds 
\leq C \epsilon^{\frac{2(\alpha-2)}{\alpha-\max(2(\gamma+\alpha), 4(\alpha+1) + \frac{5\alpha-4}{\alpha+1})}} \left( \int \phi_y^2 dy \right)^{\frac{1}{\alpha}}
\tag{134}
$$

holds for $2 < \alpha < 4$ and

$$
\int \left( \frac{\psi^2}{2} + \epsilon^{-(\alpha-2)} \frac{|\psi|^\alpha}{\alpha(\alpha-1)} \right) dy + \delta^{2(\alpha-2)} \mu^\gamma \int \left( \frac{\alpha-2}{2} \frac{\phi_y^2}{\rho^2} + \rho \psi^2 + \rho^{\gamma-2} \phi^2 \right) dy 
+ \int_0^\infty \int \left\{ \frac{1}{\rho} (\mu_0 + \left( \frac{\psi}{\epsilon} \right)^2) \alpha \frac{\alpha-2}{2} \psi^2 + \epsilon^{-(\alpha-2)} \frac{1}{\rho} (\mu_0 + \left( \frac{\psi}{\epsilon} \right)^2) \alpha \frac{\alpha-2}{2} |\psi|^\alpha \phi_{yy}^2 \right\} dy ds 
+ \delta^{2(\alpha-2)} \mu^\gamma \int_0^\infty \int \left[ |\psi|^2 + \epsilon^{-(\alpha-2)} |\psi|^\alpha \right] \phi_y^2 + \rho \psi \phi^2 + \rho \psi \phi_{yy} \phi^2 + \rho \psi \phi_{yy} \phi^2 \] dy ds 
+ \int_0^\infty \int \left[ \frac{\psi_y^2}{2} + \epsilon^{-(\alpha-2)} \frac{1}{\alpha} \psi_y (\psi^2)^2 \right] dy ds 
\leq C \epsilon^{\frac{2(\alpha-2)}{\alpha-\max(2(\gamma+\alpha), 4(\alpha+1) + 2\gamma)\alpha}} \left( \int \phi_y^2 dy \right)^{\frac{1}{\alpha}}
\tag{135}
$$

holds for $4 < \alpha \leq 6$, and

$$
\int \left( \frac{\psi^2}{2} + \epsilon^{-(\alpha-2)} \frac{|\psi|^\alpha}{\alpha(\alpha-1)} \right) dy + \delta^{2(\alpha-2)} \mu^\gamma \int \left( \frac{\alpha-2}{2} \frac{\phi_y^2}{\rho^2} + \rho \psi^2 + \rho^{\gamma-2} \phi^2 \right) dy 
+ \int_0^\infty \int \left\{ \frac{1}{\rho} (\mu_0 + \left( \frac{\psi}{\epsilon} \right)^2) \alpha \frac{\alpha-2}{2} \psi^2 + \epsilon^{-(\alpha-2)} \frac{1}{\rho} (\mu_0 + \left( \frac{\psi}{\epsilon} \right)^2) \alpha \frac{\alpha-2}{2} |\psi|^\alpha \phi_{yy}^2 \right\} dy ds 
+ \delta^{2(\alpha-2)} \mu^\gamma \int_0^\infty \int \left[ |\psi|^2 + \epsilon^{-(\alpha-2)} |\psi|^\alpha \right] \phi_y^2 + \rho \psi \phi^2 + \rho \psi \phi_{yy} \phi^2 + \rho \psi \phi_{yy} \phi^2 \] dy ds 
+ \int_0^\infty \int \left[ \frac{\psi_y^2}{2} + \epsilon^{-(\alpha-2)} \frac{1}{\alpha} \psi_y (\psi^2)^2 \right] dy ds 
\leq C \epsilon^{\frac{2(\alpha-2)}{\alpha-\max(2(\gamma+\alpha), 4(\alpha+1) + \gamma)\alpha}} \left( \int \phi_y^2 dy \right)^{\frac{1}{\alpha}}
\tag{136}
$$

holds for $\alpha \geq 6$.

So, it deduced from (134)-(136) directly that

$$
\sup_{0 \leq \tau \leq T(\epsilon)} \| \psi_y \|_{L^2} \leq \left\{ \begin{array}{ll}
C \epsilon^{\frac{2(\alpha-2)}{\alpha-\max(2(\gamma+\alpha), 4(\alpha+1) + \frac{5\alpha-4}{\alpha+1})(\alpha-1)+\alpha)}} & 2 < \alpha < 4, \\
C \epsilon^{\frac{2(\alpha-2)}{\alpha-\max(4(\alpha+1) + \gamma)\alpha}} & 4 \leq \alpha < 6, \\
C \epsilon^{\frac{2(\alpha-2)}{\alpha-\max(\gamma+\alpha)\alpha}} & \alpha \geq 6.
\end{array} \right.
\tag{137}
$$

Obviously, the a priori assumption (47) is verified if $\epsilon$ is suitably small. Moreover, it follows (40) and (41) that if $1 < \gamma \leq 2$ then

$$
\sup_{0 \leq \tau \leq T(\epsilon)} \| \phi \|_{L^\infty} \leq \sqrt{2} \sup_{0 \leq \tau \leq T(\epsilon)} \| \phi(\cdot, \tau) \|_{L^2} \| \phi_y(\cdot, \tau) \|_{L^2} \leq C \sup_{0 \leq \tau \leq T(\epsilon)} \left( \int \rho \psi \phi\phi_y^2 dy \right)^{\frac{1}{2}} \left( \int \phi_y^2 dy \right)^{\frac{1}{2}}
$$
and if \( \gamma > 2 \), then

\[
\sup_{0 \leq \tau \leq \tau(\epsilon)} \| \phi \|_{L^\infty} \leq C \sup_{0 \leq \tau \leq \tau(\epsilon)} \| \phi(\cdot, \tau) \|_{L^2} \| \phi_y(\cdot, \tau) \|_{L^2}^{1/2}
\]

\[
\leq C \mu \frac{\gamma}{\alpha} \left( \int \bar{\rho}^{\gamma-2} \phi^2 \, dy \right)^{1/2} \left( \int \phi^2 \, dy \right)^{1/2}
\]

\[
\leq \begin{cases} 
C \mu \frac{\gamma}{\alpha} \left( \frac{\alpha}{6} - (\gamma + 2)(\alpha - 2) + \max \{ \gamma + \alpha, \frac{2(\alpha + 1)}{\alpha}, \frac{5(\alpha - 4)}{4(\alpha - 1)} \} \right) \alpha, & 2 \leq \alpha < 4, \\
C \mu \frac{\gamma}{\alpha} \left( \frac{\alpha}{6} - (\frac{4(\alpha + 1)}{\alpha} + 2\gamma + 2(\alpha - 2)) \right) \alpha, & 4 \leq \alpha < 6, \\
C \mu \frac{\gamma}{\alpha} \left( \frac{\alpha}{6} - (\gamma + 3\alpha - 4) \right) \alpha, & \alpha \geq 6.
\end{cases}
\]

By similar way, by using the Sobolev inequality one can get that if \( \gamma > 1 \) then

\[
\sup_{0 \leq \tau \leq \tau(\epsilon)} \| \psi \|_{L^\infty} \leq C \| \psi \|_{L^\infty} \| \psi \|_{L^2}^{1/2}
\]

\[
\leq \begin{cases} 
C \mu \frac{\gamma}{\alpha} \left( \frac{\alpha}{6} - (\gamma + 2)(\alpha - 2) + \max \{ \gamma + \alpha, \frac{2(\alpha + 1)}{\alpha}, \frac{5(\alpha - 4)}{4(\alpha - 1)} \} \right) \alpha, & 2 \leq \alpha < 4, \\
C \mu \frac{\gamma}{\alpha} \left( \frac{\alpha}{6} - (\frac{4(\alpha + 1)}{\alpha} + 2\gamma + 2(\alpha - 2)) \right) \alpha, & 4 \leq \alpha < 6, \\
C \mu \frac{\gamma}{\alpha} \left( \frac{\alpha}{6} - (\gamma + 3\alpha - 4) \right) \alpha, & \alpha \geq 6.
\end{cases}
\]

Thus the convergence rate (47) is justified and the proof of Lemma 3.1 is completed.

Based on these a priori estimates, we can claim \( \tau(\epsilon) = \infty \). If \( \tau(\epsilon) \leq 0 \), then by again using the local existence at time \( \tau = \tau_1(\epsilon) \), we can find another time \( \tau_2(\epsilon) > \tau_1(\epsilon) \) so that the solution satisfies the assumption (47) in the time interval \([0, \tau_2(\epsilon)]\) which contradicts the assumption that \( \tau_1(\epsilon) \) is the maximum time. Therefore we extend the local solution to the global solution in \([0, \infty)\) for small but fixed \( \epsilon \).

Proof of Theorem 1.1 It remains to prove (7) and (8) with \( a, b \) given in (9) and (11), respectively. From Lemma 2.2, Lemma 2.3(iii), and Proposition 1 and recalling that \( \mu = \epsilon^a \ln \epsilon, \delta = \epsilon^b \), one can get that for any given positive constant \( h \), there exists a constant \( C_h > 0 \) which is independent of \( \epsilon \) such that if \( 1 < \gamma < 2 \)

\[
\sup_{t \geq h} \| \rho(\cdot, t) - \rho^\ast(\cdot, t) \|_{L^\infty}
\]

\[
\leq \sup_{0 \leq \tau \leq +\infty} \| \phi(\cdot, \tau) \|_{L^\infty} + \sup_{t \geq h} \| \bar{\rho}(\cdot, t) - \rho^\ast(\cdot, \tau) \|_{L^\infty} + \sup_{t \geq h} \| \rho_y(\cdot, t) - \rho^\ast_y(\cdot, \tau) \|_{L^\infty}
\]

\[
\leq \begin{cases} 
C \mu \frac{\gamma}{\alpha} \left( \frac{\alpha}{6} - (\gamma + 2)(\alpha - 2) + \max \{ \gamma + \alpha, \frac{2(\alpha + 1)}{\alpha}, \frac{5(\alpha - 4)}{4(\alpha - 1)} \} \right) \alpha + C_h \epsilon^a |\ln \epsilon|, & 2 \leq \alpha < 4, \\
C \mu \frac{\gamma}{\alpha} \left( \frac{\alpha}{6} - (\gamma + 3\alpha - 4) \right) \alpha + C_h \epsilon^a |\ln \epsilon|, & \alpha \geq 6.
\end{cases}
\]

and if \( \gamma > 2 \)

\[
\sup_{t \geq h} \| \rho(\cdot, t) - \rho^\ast(\cdot, t) \|_{L^\infty}
\]
The authors would like to thank the referees for the valuable

Therefore, the proof of Theorem 1.1 is completed.

\[ \sup_{0 \leq t \leq \tau + \infty} \| \phi(\cdot, \tau) \|_{L^\infty} + \sup_{t \geq h} \| \bar{\rho}^\mu(t) - \rho^\mu(t) \|_{L^\infty} + \sup_{t \geq h} \| \rho^\mu(t) - \rho^\tau(t) \|_{L^\infty} \leq C_0 \epsilon^a \| \ln \epsilon \| + \]

\[ C \epsilon^{\frac{2-2\gamma}{2\gamma}} \left( 2\alpha - 2\gamma + 2(\alpha - 2) + \max(\gamma + \alpha, 2\frac{(\alpha + 1)}{2\gamma - 4\alpha}) \right)^2 \alpha, \quad 2 < \alpha < 4, \]

\[ C \epsilon^{\frac{2-2\gamma}{2\gamma}} \left( 2\alpha - 2\gamma + 2(\alpha - 2) + \frac{2\alpha + 1}{2\gamma - 4\alpha} \right)^2 \alpha, \quad 4 \leq \alpha < 6, \]

\[ C \epsilon^{\frac{2-2\gamma}{2\gamma}} \left( 2\alpha - 2\gamma + 2(\alpha - 2) + \frac{2\alpha + 1}{2\gamma - 4\alpha} \right)^2 \alpha, \quad \alpha \geq 6 \]

\[ \leq C_h \epsilon^b \ln \epsilon. \]

Moreover, if \( 1 < \tau < 2 \)

\[ \sup_{t \geq h} \| m(\cdot, t) - m^\tau(\cdot) \|_{L^\infty} \leq \sup_{t \geq h} \left( \| m(\cdot, t) - \bar{m}(\cdot, t) \|_{L^\infty} + \| \bar{m}(\cdot, t) - m^\mu(t) \|_{L^\infty} + \| m^\mu(t) - m^\tau(t) \|_{L^\infty} \right) \]

\[ \leq \sup_{0 \leq \tau \leq \infty} \| \phi(\cdot, \tau) \|_{L^\infty} + \| \psi(\cdot, \tau) \|_{L^\infty} + C_h \epsilon^a \| \ln \epsilon \| \leq C_h \epsilon^b \]

\[ \leq C_h \epsilon^b \]

and if \( \tau \geq 2 \) one can get that

\[ \sup_{t \geq h} \| m(\cdot, t) - m^\tau(\cdot) \|_{L^\infty} \leq \sup_{t \geq h} \left( \| m^\mu(t) - m^\tau(t) \|_{L^\infty} + \| \bar{m}(\cdot, t) - m^\mu(t) \|_{L^\infty} + \| m(\cdot, t) - \bar{m}(\cdot, t) \|_{L^\infty} \right) \]

\[ \leq \sup_{0 \leq \tau \leq \infty} \| \phi(\cdot, \tau) \|_{L^\infty} + \| \psi(\cdot, \tau) \|_{L^\infty} + C_h \epsilon^a \| \ln \epsilon \| \leq C_h \epsilon^b \]

\[ \leq C_h \epsilon^b \]

Therefore, the proof of Theorem 1.1 is completed.

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