A unified approach to degenerate problems in the half-space

G. Metafune ∗  L. Negro †  C. Spina ‡

Abstract
We study elliptic and parabolic problems governed by the singular elliptic operators

\[ \mathcal{L} = y^{\alpha_1} \Delta_x + y^{\alpha_2} \left( D_{yy} + \frac{c}{y} D_y - \frac{b}{y^2} \right), \quad \alpha_1, \alpha_2 \in \mathbb{R} \]

in the half-space \( \mathbb{R}^{N+1}_+ = \{(x, y) : x \in \mathbb{R}^N, y > 0\} \).

Mathematics subject classification (2020): 35K67, 35B45, 47D07, 35J70, 35J75.
Keywords: degenerate elliptic operators, boundary degeneracy, vector-valued harmonic analysis, maximal regularity.

1 Introduction
In this paper we study solvability and regularity of elliptic and parabolic problems associated to the degenerate operators

\[ \mathcal{L} = y^{\alpha_1} \Delta_x + y^{\alpha_2} \left( D_{yy} + \frac{c}{y} D_y - \frac{b}{y^2} \right) \quad \text{and} \quad D_t - \mathcal{L} \]

in the half-space \( \mathbb{R}^{N+1}_+ = \{(x, y) : x \in \mathbb{R}^N, y > 0\} \) or in \((0, \infty) \times \mathbb{R}^{N+1}_+ \).

Here \( b, c \) are constant real coefficients and we use \( L_y = D_{yy} + \frac{c}{y} D_y - \frac{b}{y^2} \). Note that singularities in the lower order terms appear when either \( b \) or \( c \) is different from 0. When \( b = 0 \), then \( L_y \) is a Bessel operator and we shall denote it by \( B_y \).

The real numbers \( \alpha_1, \alpha_2 \) satisfy \( \alpha_2 < 2 \) and \( \alpha_2 - \alpha_1 < 2 \) but are not assumed to be nonnegative.

The reasons for these restrictions will be explained later in this introduction.

\( \mathcal{L} \) is the sum of a degenerate diffusion \( y^{\alpha_1} \Delta_x \), tangential to \( \partial \mathbb{R}^{N+1}_+ \), and of a 1d degenerate normal diffusion \( y^{\alpha_2} L_y \) which commute only when \( \alpha_1 = 0 \). It satisfies the scaling property

\[ I_s^{-1} \mathcal{L} I_s = s^{2-\alpha_2} \mathcal{L}, \quad I_s \theta(x, y) = \theta(s^{\frac{(\alpha_2-\alpha_1)}{2}} x, sy). \]

When \( \alpha_1 = \alpha_2 = 0 \), \( \mathcal{L} = \Delta_x + \left( D_{yy} + \frac{c}{y} D_y - \frac{b}{y^2} \right) \) reduces to the so-called Caffarelli-Silvestre extension operators, studied in detail in [21] under Dirichlet and Neumann boundary conditions. We refer the reader also to [4], [5] for the case \( b = 0 \) and with variable coefficients.

∗Dipartimento di Matematica e Fisica “Ennio De Giorgi”, Università del Salento, C.P.193, 73100, Lecce, Italy.
- mail: giorgio.metafune@unisalento.it
†Dipartimento di Matematica e Fisica “Ennio De Giorgi”, Università del Salento, C.P.193, 73100, Lecce, Italy.
email: luigi.negro@unisalento.it
‡Dipartimento di Matematica e Fisica “Ennio De Giorgi”, Università del Salento, C.P.193, 73100, Lecce, Italy.
e-mail: chiara.spina@unisalento.it
The case $\alpha_1 = 0$ and $\alpha_2$ arbitrary (even without the restriction $\alpha_2 < 2$) can be easily reduced to that above, using the results of this paper and, in particular, the transformation of Section 3.

The case $\alpha_1 = \alpha_2 = 1$ and $b = 0$, namely $\mathcal{L} = y (\Delta_x + D_y y) + c D_y$, is also widely treated in the literature on degenerate problems. A comparison of our results with those already known is done in Section 7.4.

When $\alpha_2 = 0$ our operators generalize the class of Baouendi-Grushin operators $\mathcal{L} = y^\alpha \Delta_x + D_y y$, to which they reduce when $c = b = 0$. A comparison with known results is done in Section 7.3 and here we only point out that we allow also negative $\alpha$.

Finally, we mention that kernel estimates for operators in divergence form and with normal and tangential degeneracy on the hyperplane \{y = 0\} have been obtained in [30, 31].

The aim of this paper is to provide a unified approach which allows to prove elliptic and parabolic $L^p$ estimates and solvability of the associated problems. In the language of semigroup theory, we prove that $\mathcal{L}$ generates an analytic semigroup, characterize its domain as a weighted Sobolev space and show that it has maximal regularity, which means that both $D_t v$ and $\mathcal{L} v$ have the same regularity as $(D_t - \mathcal{L}) v$.

Surprisingly enough, the case $\alpha_1 = \alpha_2$ implies all other cases by a change of variables, as described in Section 3. However this modifies the underlying measure and the procedure works if one is able to deal with the simpler case described in Section 3. However this modifies the underlying measure and the procedure works if one is able to deal with the simpler case described in Section 3.

Let us explain how to obtain (1). Assuming that $y^\alpha (\Delta_x u + B_y u) = f$ and taking the Fourier transform with respect to $x$ (with covariable $\xi$) we get $-|\xi|^2 \hat{u}(\xi, y) + B_y \hat{u}(\xi, y) = y^{-\alpha} \hat{f}(\xi, y)$ and then $y^\alpha |\xi|^2 \hat{u}(\xi, y) = -y^\alpha |\xi|^2 (|\xi|^2 - B_y)^{-1} y^{-\alpha} \hat{f}(\xi, y)$. Denoting by $\mathcal{F}$ the Fourier transform with respect to $x$ we get

$$y^\alpha \Delta_x \mathcal{L}^{-1} = -\mathcal{F}^{-1} \left( y^\alpha |\xi|^2 (|\xi|^2 - B_y)^{-1} y^{-\alpha} \right) \mathcal{F}$$

and the boundedness of $y^\alpha \Delta_x \mathcal{L}^{-1}$ is equivalent to that of the multiplier $\xi \in \mathbb{R}^N \rightarrow M(\xi) = y^\alpha |\xi|^2 (|\xi|^2 - B_y)^{-1} y^{-\alpha} \in L_p(\mathbb{R}^N; L_p^{\alpha}(0, \infty)) = L_p^{\alpha}(\mathbb{R}^{N+1})$.

We prove this by a vector valued Mikhlin multiplier theorem which rests on square function estimates for the family $M(\xi)$ and its derivatives. The strategy for proving (2) is similar after taking the Fourier transform with respect to $t$.

Both the elliptic and parabolic estimates above share the name “maximal regularity” even though this term is often restricted to the parabolic case. We refer to [15] and the new books.
in $R$ alone can be treated for any normal and tangential directions. To treat operators which degenerate near the boundary of a domain with (possibly) different rates also added by perturbation. We shall deal with these consequences in a subsequent paper in order to Section 9.2, see in particular Example 9.10, without demanding for completeness.

Most of the result of this paper can be extended to operators with variable coefficients

$$\mathcal{L} = y^{\alpha_1} \sum_{i,j=1}^{N} a_{ij}(t,x,y)D_{x_ix_j} + y^{\alpha_2} \left( D_{yy} + \frac{c}{y^2} D_y - \frac{b}{y^2} \right)$$

assuming uniform ellipticity and appropriate continuity of the matrix $(a_{ij})$. In fact, the case of constant coefficients $(a_{ij})$ follows by a linear change of the $x$-variables and allows to use perturbation methods. The situation is easier in a finite strip $R^N \times [0,1]$ and for positive $\alpha_1, \alpha_2$ since the powers $y^{\alpha_1}, y^{\alpha_2}$ are bounded. First order terms like $y^{\alpha_1} b(t,x,y) \cdot \nabla_x$ with $b$ bounded can be also added by perturbation. We shall deal with these consequences in a subsequent paper in order to treat operators which degenerate near the boundary of a domain with (possibly) different rates along normal and tangential directions.

The paper is organized as follows. In Section 2 we briefly recall the harmonic analysis background needed in the paper, as square function estimates, $\mathcal{R}$-boundedness and a vector valued multiplier theorem.

In Section 3, we exploit an elementary change of variables, in a functional analytic setting, to reduce our operators to the simpler case where $\alpha_1 = \alpha_2$. 

[12], [13] for the functional analytic approach to maximal regularity we use. The whole theory relies on a deep interplay between harmonic analysis and structure theory of Banach spaces but largely simplifies when the underlying Banach spaces are $L^p$ spaces, by the use of classical square function estimates. This last approach has been employed extensively in [3], showing that uniformly parabolic operators have maximal regularity, under very general boundary conditions.

However the a-priori estimates (1) and (2) are not sufficient for the solvability of the equation $\lambda u - Lu = f$. In fact, $\mathcal{L}$ is not dissipative unless additional restrictions on the parameters and on the underlying measure are assumed, see Section 9.1, and approximation methods with uniformly parabolic operators do not need to converge.

In order to prove existence results, or generation results in the language of semigroups, we use that the operator valued map $\xi \in R^N \rightarrow N(\xi) = (\lambda + y^\alpha |\xi|^2 - y^\beta B_y)^{-1}, \lambda \in C_+$, is a Fourier multiplier in $L^p(\mathbb{R}^N; L^p_m(0,\infty)) = L^p(\mathbb{R}^N_+)$, see [23, Section 8] where the relevant one dimensional degenerate operators are studied in detail.

Before describing the content of the sections, let us explain the meaning of the restrictions $\alpha_2 < 2$, $\alpha_2 - \alpha_1 < 2$.

Let us first consider the case where $\alpha_1 = \alpha_2 = \alpha$, so that the unique requirement is $\alpha < 2$. It turns out that when $\alpha \geq 2$ the problem is easily treated in the strip $R^N \times [0,1]$ in the case of the Lebesgue measure, see [7], and all problems are due to the strong diffusion at infinity. The case $\alpha \geq 2$ in the strip $R^N \times [1,\infty]$ requires new investigation even though the operator $y^\alpha L_y$ alone can be treated for any $\alpha \in R$, by the similarity transformation of Section 3.

When $\alpha_1 \neq \alpha_2$, the change of variables of Section 3, namely $T_{\alpha_1,\alpha_2,\alpha}$, transforms $y^{\alpha_1} \Delta_x + y^{\alpha_2} B_y$ into $y^{\alpha}(\Delta_x + \hat{B}_y), \alpha = \frac{2\alpha_1}{\alpha_2 - \alpha_1 + 2}$. However, the strip $R^N \times [0,1]$ is mapped into itself only when $\alpha_2 - \alpha_1 < 2$. Under this condition it is possible, though not treated in this paper, that the restriction $\alpha_2 < 2$ can be removed, at least when the operator is studied in $R^N \times [0,1]$ rather than in $R^N_+$. But dealing with the case $\alpha_2 - \alpha_1 \geq 2$ requires further investigation, as explained above.

Assuming, in addition, that $\alpha_1 \geq 0$, the range of parameters for which we prove solvability is optimal, since it coincides with that of $L_y$. However, when $\alpha_1 < 0$ it can happen that $\mathcal{L}$ generates in a range of parameters for which the domain is less regular. We discuss these phenomena in Section 9.2, see in particular Example 9.10, without demanding for completeness.
In Sections 4 and 5 we recall some preliminary results concerning anisotropic weighted Sobolev spaces and one-dimensional Bessel operators.

In Section 6, which is the core of the paper, we prove generation results, maximal regularity and domain characterization for the operator \( y^{α_1} Δ_x + y^{α_2} L_y \) where \( B^0_y \) is the Bessel operator with Neumann boundary conditions. The general case both with Neumann and Dirichlet boundary conditions will be deduced in Sections 7 and 8 using the isometry of Section 3.

In Section 9, we complement our results by characterizing the contractivity range and investigating uniqueness. We show that many examples of degenerate operators, like the Baouendi-Grushin operators, are special cases of ours and that our results improve those already existing in the literature.

**Notation.** For \( N ≥ 0, \mathbb{R}^{N+1}_+ = \{(x, y) : x ∈ \mathbb{R}^N, y > 0\} \). For \( m ∈ \mathbb{R} \) we consider the measure \( y^m dx dy \) in \( \mathbb{R}^N_+ \) and write \( L^p_m(\mathbb{R}^{N+1}_+) \) for \( L^p(\mathbb{R}^{N+1}_+: y^m dx dy) \) and often only \( L^p_m \) when \( \mathbb{R}^{N+1}_+ \) is understood. Similarly \( W^k,p(\mathbb{R}^{N+1}_+) = \{u ∈ L^p_m(\mathbb{R}^{N+1}_+) : \partial^α u ∈ L^p_m(\mathbb{R}^{N+1}_+) \mid |α| ≤ k\} \).

\( C^+ = \{λ ∈ \mathbb{C} : Re λ > 0\} \) and, for \( |θ| ≤ π \), we denote by \( Σ_θ \) the open sector \( \{λ ∈ \mathbb{C} : λ ≠ 0, |Arg(λ)| < θ\} \).

## 2 Vector-valued harmonic analysis

Regularity properties for \( \mathcal{L} = y^{α_1} Δ_x + y^{α_2} L_y \) follow once we prove the estimate
\[
\|y^{α_1} Δ_x u\|_p + \|y^{α_2} L_y u\|_p ≤ C\|Lu\|_p
\] (3)
where the \( L^p \) norms are taken over \( \mathbb{R}^{N+1}_+ \) on a sufficiently large set of functions \( u \). This is equivalent to saying that the domain of \( \mathcal{L} \) is the intersection of the domain of \( y^{α_1} Δ_x \) and \( y^{α_2} L_y \) (after appropriate tensorization) or that the operator \( y^{α_1} Δ_x \mathcal{L}^{-1} \) is bounded. This strategy arose first in the study of maximal regularity of parabolic problems, that is for the equation \( u_t = Au + f, u(0) = 0 \) where \( A \) is the generator of an analytic semigroup on a Banach space \( X \). Estimates like
\[
\|u_t\|_p + \|Au\|_p ≤ \|f\|_p
\]
where now the \( L^p \) norm is that of \( L^p([0, T]; X) \) can be interpreted as closedness of \( D_t - A \) on the intersection of the respective domains or, equivalently, boundedness of the operator \( A(D_t - A)^{-1} \) in \( L^p([0, T]; X) \).

Nowadays this strategy is well established and relies on Mikhlin vector-valued multiplier theorems. Let us state the relevant definitions and main results we need, referring the reader to [3], [29] or [15].

Let \( S \) be a subset of \( B(X) \), the space of all bounded linear operators on a Banach space \( X \). \( S \) is \( R \)-bounded if there is a constant \( C \) such that
\[
\| \sum_i ε_i S_i x_i \|_{L^p(Ω; X)} ≤ C \| \sum_i ε_i x_i \|_{L^p(Ω; X)}
\]
for every finite sum as above, where \( (x_i) ∈ X, (S_i) ∈ S \) and \( ε_i : Ω → \{-1, 1\} \) are independent and symmetric random variables on a probability space \( Ω \). The smallest constant \( C \) for which the above definition holds is the \( R \)-bound of \( S \), denoted by \( R(S) \). It is well-known that this definition does not depend on \( 1 ≤ p < ∞ \) (however, the constant \( R(S) \) does) and that \( R \)-boundedness is equivalent to boundedness when \( X \) is an Hilbert space. When \( X \) is an \( L^p \) space (with respect to any \( σ \)-finite measure), testing \( R \)-boundedness is equivalent to proving square functions estimates, see [15, Remark 2.9].
Proposition 2.1 Let $S \subset B(L^p(\Sigma))$, $1 < p < \infty$. Then $S$ is $\mathcal{R}$-bounded if and only if there is a constant $C > 0$ such that for every finite family $(f_i) \in L^p(\Sigma), (S_i) \in S$

$$\left\| \sum_i |S_i f_i|^2 \right\|_{L^p(\Sigma)}^{\frac{1}{2}} \leq C \left\| \sum_i |f_i|^2 \right\|_{L^p(\Sigma)}^{\frac{1}{2}}.$$ 

The best constant $C$ for which the above square functions estimates hold satisfies $\kappa^{-1} C \leq \mathcal{R}(S) \leq \kappa C$ for a suitable $\kappa > 0$ (depending only on $p$). The proposition above $\mathcal{R}$-boundedness follows from domination.

Corollary 2.2 Let $S, T \subset B(L^p(\Sigma)), 1 < p < \infty$ and assume that $T$ is $\mathcal{R}$ bounded and that for every $S \in S$ there exists $T \in T$ such that $|S f| \leq |T f|$ pointwise, for every $f \in L^p(\Sigma)$. Then $S$ is $\mathcal{R}$-bounded.

Let $(A, D(A))$ be a sectorial operator in a Banach space $X$; this means that $\rho(-A) \supset \Sigma_{\pi-\phi}$ for some $\phi < \pi$ and that $\lambda(\lambda + A)^{-1}$ is bounded in $\Sigma_{\pi-\phi}$. The infimum of all such $\phi$ is called the spectral angle of $A$ and denoted by $\phi_A$. Note that $-A$ generates an analytic semigroup if and only if $\phi_A < \pi/2$. The definition of $\mathcal{R}$-sectorial operator is similar, substituting boundedness of $\lambda(\lambda + A)^{-1}$ with $\mathcal{R}$-boundedness in $\Sigma_{\pi-\phi}$. As above one denotes by $\phi_A^R$ the infimum of all $\phi$ for which this happens; since $\mathcal{R}$-boundedness implies boundedness, we have $\phi_A \leq \phi_A^R$.

The $\mathcal{R}$-boundedness of the resolvent characterizes the regularity of the associated inhomogeneous parabolic problem, as we explain now. An analytic semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$ with generator $-A$ has maximal regularity of type $L^q$ ($1 < q < \infty$) if for each $f \in L^q([0, T]; X)$ the function $t \mapsto u(t) = \int_0^t e^{-(t-s)A} f(s) \, ds$ belongs to $W^{1,q}([0, T]; X) \cap L^q([0, T]; D(B))$. This means that the mild solution of the evolution equation

$$u'(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = 0,$$

is in fact a strong solution and has the best regularity one can expect. It is known that this property does not depend on $1 < q < \infty$ and $T > 0$. A characterization of maximal regularity is available in UMD Banach spaces, through the $\mathcal{R}$-boundedness of the resolvent in a suitable sector $\omega + \Sigma_\phi$, with $\omega \in \mathbb{R}$ and $\phi > \pi/2$ or, equivalently, of the scaled semigroup $e^{-(A+\omega) t}$ in a sector around the positive axis. In the case of $L^p$ spaces it can be restated in the following form, see [15, Theorem 1.11]

Theorem 2.3 Let $(e^{-tA})_{t \geq 0}$ be a bounded analytic semigroup in $L^p(\Sigma), 1 < p < \infty$, with generator $-A$. Then $T(\cdot)$ has maximal regularity of type $L^q$ if and only if the set $\{\lambda(\lambda + A)^{-1}, \lambda \in \Sigma_{\pi/2+\phi}\}$ is $\mathcal{R}$-bounded for some $\phi > 0$. In an equivalent way, if and only if there are constants $0 < \phi < \pi/2$, $C > 0$ such that for every finite sequence $(\lambda_i) \subset \Sigma_{\pi/2+\phi}, (f_i) \subset L^p$

$$\left\| \sum_i |\lambda_i(\lambda_i + A)^{-1} f_i|^2 \right\|_{L^p(\Sigma)}^{\frac{1}{2}} \leq C \left\| \sum_i |f_i|^2 \right\|_{L^p(\Sigma)}^{\frac{1}{2}}$$

or, equivalently, there are constants $0 < \phi' < \pi/2, C' > 0$ such that for every finite sequence $(\xi_i) \subset \Sigma_{\phi'}, (f_i) \subset L^p$

$$\left\| \sum_i |e^{-\xi_i A} f_i|^2 \right\|_{L^p(\Sigma)}^{\frac{1}{2}} \leq C' \left\| \sum_i |f_i|^2 \right\|_{L^p(\Sigma)}^{\frac{1}{2}}.$$
Finally we state a version of the operator-valued Mikhlin multiplier theorem in the N-dimensional case, see [3, Theorem 3.25] or [15, Theorem 4.6].

**Theorem 2.4** Let $1 < p < \infty$, $M \in C^N(\mathbb{R}^N \setminus \{0\}; B(L^p(\Sigma))$ be such that the set

$$\left\{ |\xi|^{\alpha}|D_\xi^\alpha M(\xi) : \xi \in \mathbb{R}^N \setminus \{0\}, |\alpha| \leq N \right\}$$

is $\mathcal{R}$-bounded. Then the operator $T_M = \mathcal{F}^{-1}M\mathcal{F}$ is bounded in $L^p(\mathbb{R}^N, L^p(\Sigma))$, where $\mathcal{F}$ denotes the Fourier transform.

### 3 Degenerate operators and similarity transformations

We consider first the 1d operators

$$L = D_{yy} + \frac{c}{y}D_y - \frac{b}{y^2}, \quad B = D_{yy} + \frac{c}{y}D_y$$

on the half line $\mathbb{R}_+ = [0, \infty]$. Note that $B$ (which stands for Bessel) is nothing but $L$ when $b = 0$. Often we write $L_y, B_y$ to indicate that they act with respect to the $y$ variable.

The equation $Lu = 0$ has solutions $y^{-s_1}, y^{-s_2}$ where $s_1, s_2$ are the roots of the indicial equation

$$f(s) = -s^2 + (c - 1)s + b = 0$$

where

$$s_1 := \frac{c - 1}{2} - \sqrt{D}, \quad s_2 := \frac{c - 1}{2} + \sqrt{D} \quad (4)$$

and

$$D := b + \left(\frac{c - 1}{2}\right)^2. \quad (5)$$

The above numbers are real if and only if $D \geq 0$. When $D < 0$ the equation $u - Lu = f$ cannot have positive distributional solutions for certain positive $f$, see [26]. When $b = 0$, then $\sqrt{D} = |c - 1|/2$ and $s_1 = 0, s_2 = c - 1$ for $c \geq 1$ and $s_1 = c - 1, s_2 = 0$ for $c < 1$.

Next we consider, for $\alpha_1, \alpha_2 \in \mathbb{R}$, the operators

$$\mathcal{L} = y^{\alpha_1}\Delta_x + y^{\alpha_2}L_y$$

(keeping the assumption $D \geq 0$ on $L_y$) in the space $L^p_m = L^p_m(\mathbb{R}_+^{N+1})$.

We investigate when these operators can be transformed one into the other by means of change of variables and multiplications.

For $k, \beta \in \mathbb{R}, \beta \neq -1$ let

$$T_{k,\beta} u(x, y) := |\beta + 1|^{1/2} y^k u(x, y^{\beta+1}), \quad (x, y) \in \mathbb{R}_+^{N+1}. \quad (6)$$

Observe that

$$T_{k,\beta}^{-1} = T_{\frac{k}{\beta + 1}, -\frac{1}{\beta + 1}}.$$

**Proposition 3.1** Let $1 \leq p \leq \infty$, $k, \beta \in \mathbb{R}, \beta \neq -1$. The following properties hold.

(i) For every $m \in \mathbb{R}$, $T_{k,\beta}$ maps isometrically $L^p_m$ onto $L^p_m$ where

$$\bar{m} = \frac{m + kp - \beta}{\beta + 1}.$$
(ii) For every \( u \in \W_{\text{loc}}^{2,1}(\mathbb{R}^{n+1}_+) \) one has

1. \( y^\alpha T_{k,\beta} u = T_{k,\beta}(y^{\frac{\alpha}{\alpha+1}}u) \), for any \( \alpha \in \mathbb{R} \);
2. \( D_{x,x}(T_{k,\beta} u) = T_{k,\beta}(D_{x,x} u), \quad D_{x}(T_{k,\beta} u) = T_{k,\beta}(D_{x} u) \);
3. \( D_{y} T_{k,\beta} u = T_{k,\beta}\left(ky^{-\frac{\alpha}{\alpha+1}}u + (\beta + 1)y^{\frac{\beta}{\alpha+1}}D_{y} u\right) \),
   \( D_{yy}(T_{k,\beta} u) = T_{k,\beta}\left((\beta + 1)^2 y^{\frac{2\beta}{\alpha+1}}D_{yy} u + (\beta + 1)(2k + \beta)y^{\frac{\beta+1}{\alpha+1}}D_{y} u + k(k - 1)y^{-\frac{\alpha}{\alpha+1}}u\right) \).
4. \( D_{xy} T_{k,\beta} u = T_{k,\beta}\left(ky^{-\frac{\alpha}{\alpha+1}}D_{x} u + (\beta + 1)y^{\frac{\beta}{\alpha+1}}D_{xy} u\right) \)

Proof. The proof of (i) follows after observing the Jacobian of \((x, y) \mapsto (x, y^{\beta+1})\) is \( |1 + \beta|y^{\beta} \). To prove (ii) one can easily observe that any \( x \)-derivatives commutes with \( T_{k,\beta} \). Then we compute

\[
D_{y} T_{k,\beta} u(x, y) = |\beta + 1|^{\frac{1}{\alpha}}y^{\beta} \left(k \frac{u(x, y^{\beta+1})}{y} + (\beta + 1)y^{\beta} D_{y} u(x, y^{\beta+1})\right) = T_{k,\beta}\left(ky^{-\frac{\alpha}{\alpha+1}}u + (\beta + 1)y^{\frac{\beta}{\alpha+1}}D_{y} u\right)
\]

and similarly

\[
D_{yy} T_{k,\beta} u(x, y) = T_{k,\beta}\left((\beta + 1)^2 y^{\frac{2\beta}{\alpha+1}}D_{yy} u + (\beta + 1)(2k + \beta)y^{\frac{\beta+1}{\alpha+1}}D_{y} u + k(k - 1)y^{-\frac{\alpha}{\alpha+1}}u\right).
\]

\[\square\]

Proposition 3.2 Let \( T_{k,\beta} \) be the isometry above defined. The following properties hold.

(i) For every \( u \in \W_{\text{loc}}^{2,1}(\mathbb{R}^{n+1}_+) \) one has

\[
T_{k,\beta}^{-1}\left(y^{\alpha_1} \Delta_x + y^{\alpha_2} \tilde{L}_y\right) T_{k,\beta} u = \left(y^{\frac{\alpha_1}{\alpha+1}} \Delta_x + (\beta + 1)^2 y^{\frac{\alpha_2+2\beta}{\alpha+1}} \tilde{L}_y\right) u
\]

where \( \tilde{L} \) is the operator defined as in (1) with parameters \( b, c \) replaced, respectively, by

\[
\tilde{b} = \frac{b - k(c - 1 + k)}{(\beta + 1)^2}, \quad \tilde{c} = \frac{c + 2k + \beta(c + 1 + 2k + \beta)}{(\beta + 1)^2}.
\]

(ii) The discriminant \( \tilde{D} \) and the parameters \( \tilde{s}_{1,2} \) of \( \tilde{L} \) defined as in (11), (4) are given by

\[
\tilde{D} = \frac{D}{(\beta + 1)^2},
\]

and

\[
\tilde{s}_{1,2} = \frac{s_{1,2} + k}{\beta + 1} \quad (\beta + 1 > 0), \quad \tilde{s}_{1,2} = \frac{s_{2,1} + k}{\beta + 1} \quad (\beta + 1 < 0).
\]

7
Proof. Using Proposition 3.1 we can compute
\[
L_y T_{k,\beta} u(x, y) = T_{k,\beta} \left[ (\beta + 1)^2 y^{\frac{2\beta}{\beta + 1}} D_{yy} u + (\beta + 1)(2k + \beta)y^{\frac{2\beta}{\beta + 1}} D_y u + k(k - 1)y^{-\frac{4\beta}{\beta + 1}} u \\
+ cky^{-\frac{4\beta}{\beta + 1}} u + c(\beta + 1) y^{\frac{2\beta}{\beta + 1}} D_y u - by^{-\frac{4\beta}{\beta + 1}} u \right]
\]
\[
= T_{k,\beta} \left[ y^{\frac{2\beta}{\beta + 1}} (\beta + 1)^2 D_{yy} u + \frac{(\beta + 1)(2k + \beta + c)}{y} D_y u \\
- \left( b - k(c + k - 1) \right) u \right] = T_{k,\beta} \left( y^{\frac{2\beta}{\beta + 1}} L_y u \right)
\]
which implies
\[
T_{k,\beta}^{-1} (y^{\alpha_2} L_y) T_{k,\beta} u = y^{\frac{\alpha_2 + 2\beta}{\beta + 1}} L_y u.
\]
Similarly one has \( y^{\alpha_1} \Delta_x T_{k,\beta} u = T_{k,\beta} \left( y^{\frac{\alpha_1}{\beta + 1}} \Delta_x u \right) \). Adding the last equalities yields (i). The remaining properties follow directly from the definitions (4), (11).

4 Weighted Sobolev spaces

Let \( p > 1, m, \alpha_1 \in \mathbb{R}, \alpha_2 < 2 \). In order to describe the domain of the operator \( y^{\alpha_1} \Delta_x + y^{\alpha_2} B_y \), we collect in this section the main results concerning anisotropic weighted Sobolev spaces, referring to [22] for further details and all the relative proofs. We define the Sobolev space
\[
W^{2,p}(\alpha_1, \alpha_2, m) = \left\{ u \in W^{2,p}_{loc}(\mathbb{R}^n_+^+) : u, y^{\alpha_1} D_{x,i,j} u, y^{\frac{\alpha_1}{p}} D_x u, y^{\alpha_2} D_{yy} u, y^{\frac{\alpha_2}{p}} D_y u \in L^p_m \right\}
\]
which is a Banach space equipped with the norm
\[
\| u \|_{W^{2,p}(\alpha_1, \alpha_2, m)} = \| u \|_{L^p_m} + \sum_{i,j=1}^n \| y^{\alpha_1} D_{x,i,j} u \|_{L^p_m} + \sum_{i=1}^n \| y^{\frac{\alpha_1}{p}} D_x u \|_{L^p_m} + \| y^{\alpha_2} D_{yy} u \|_{L^p_m} + \| y^{\frac{\alpha_2}{p}} D_y u \|_{L^p_m}.
\]
Next we add a Neumann boundary condition for \( y = 0 \) in the form \( y^{\alpha_2-1} D_y u \in L^p_m \) and set
\[
W^{2,p}_N(\alpha_1, \alpha_2, m) = \left\{ u \in W^{2,p}(\alpha_1, \alpha_2, m) : y^{\alpha_2-1} D_y u \in L^p_m \right\}
\]
with the norm
\[
\| u \|_{W^{2,p}_N(\alpha_1, \alpha_2, m)} = \| u \|_{W^{2,p}(\alpha_1, \alpha_2, m)} + \| y^{\alpha_2-1} D_y u \|_{L^p_m}.
\]
We consider also an integral version of the Dirichlet boundary condition, namely a weighted summability requirement for \( y^{-2} u \) and introduce
\[
W^{2,p}_R(\alpha_1, \alpha_2, m) = \left\{ u \in W^{2,p}(\alpha_1, \alpha_2, m) : y^{\alpha_2-2} u \in L^p_m \right\}
\]
with the norm
\[
\| u \|_{W^{2,p}_R(\alpha_1, \alpha_2, m)} = \| u \|_{W^{2,p}(\alpha_1, \alpha_2, m)} + \| y^{\alpha_2-2} u \|_{L^p_m}.
\]
The symbol $\mathcal{R}$ stands for "Rellich", since Rellich inequalities concern with the summability of $y^{-2} u$.

We consider only the case $\alpha_2 < 2$. Analogous results can be recovered for $\alpha_2 > 2$ via the similarity transformation of Lemma 4.3.

We have made the choice not to include the mixed derivatives in the definition of $W^{2,p}_N(\alpha_1, \alpha_2, m)$ to simplify some arguments. However the following result follows from Theorem 7.1.

**Proposition 4.1** If $\alpha_2 - \alpha_1 < 2$ and $\alpha_1^- \leq \frac{m+1}{p}$ then for every $u \in W^{2,p}_N(\alpha_1, \alpha_2, m)$

$$
\| y^{\alpha_1^- + \alpha_2} \nabla_x u \|_{L^p_m} \leq C \| u \|_{W^{2,p}_N(\alpha_1, \alpha_2, m)}.
$$

**Remark 4.2** With obvious changes we consider also the analogous Sobolev spaces $W^{2,p}(\alpha_2, m)$ and $W^{2,p}_N(\alpha_2, m)$ on $\mathbb{R}^+$. For example we have

$$
W^{2,p}_N(\alpha, m) = \left\{ u \in W^{2,p}_{loc}(\mathbb{R}^+): u, y^{\alpha} D_y u, y^{\alpha+1} D_y u \in L^p_m \right\}.
$$

All the results of this section will be valid also in $\mathbb{R}^+$ changing (when it appears) the condition $\alpha_1^- \leq \frac{m+1}{p}$ to $0 < \frac{m+1}{p}$.

The next proposition shows how these spaces transform under the map of Section 3.

**Proposition 4.3** Let $p > 1$, $m, \alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_2 < 2$. Then one has

$$
W^{2,p}_N(\alpha_1, \alpha_2, m) = T_{0, \beta} \left( W^{2,p}_N(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{m}) \right), \quad \tilde{\alpha}_1 = \frac{\alpha_1}{\beta + 1}, \quad \tilde{\alpha}_2 = \frac{\alpha_2 + 2 \beta}{\beta + 1}.
$$

In particular, by choosing $\beta = -\frac{m+1}{p}$ one has

$$
W^{2,p}_N(\alpha_1, \alpha_2, m) = T_{0, -\frac{m+1}{p}} \left( W^{2,p}_N(\tilde{\alpha}_1, 0, \tilde{m}) \right), \quad \tilde{\alpha}_1 = \frac{2 \alpha_1}{2 - \alpha_2}, \quad \tilde{m} = \frac{m + \frac{m+1}{p}}{1 - \frac{m+1}{p}}.
$$

**Remark 4.4** It is essential to deal with $W^{2,p}_N(\alpha_1, \alpha_2, m)$: in general the map $T_{0, \beta}$ does not transform $W^{2,p}_N(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{m})$ into $W^{2,p}(\alpha_1, \alpha_2, m)$.

The next result clarifies in which sense the condition $y^{\alpha_2+1} D_y u \in L^p_m$ is a Neumann boundary condition.

**Proposition 4.5** The following assertions hold.

(i) If $\frac{m+1}{p} > 1 - \alpha_2$, then $W^{2,p}_N(\alpha_1, \alpha_2, m) = W^{2,p}(\alpha_1, \alpha_2, m)$.

(ii) If $\frac{m+1}{p} < 1 - \alpha_2$, then

$$
W^{2,p}_N(\alpha_1, \alpha_2, m) = \left\{ u \in W^{2,p}(\alpha_1, \alpha_2, m): \lim_{y \to 0} D_y u(x, y) = 0 \text{ for a.e. } x \in \mathbb{R}^N \right\}.
$$

In both cases (i) and (ii), the norm of $W^{2,p}_N(\alpha_1, \alpha_2, m)$ is equivalent to that of $W^{2,p}(\alpha_1, \alpha_2, m)$.

We provide an equivalent description of $W^{2,p}_N(\alpha_1, \alpha_2, m)$, adapted to the operator $D_{yy} + cy^{-1} D_y$.
Proposition 4.6 Let \( c \in \mathbb{R} \) and \( \frac{m+1}{p} < c + 1 - \alpha_2 \). Then

\[
W^{2,p}_N(\alpha_1, \alpha_2, m) = \left\{ u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N_{+}^{N+1}) : u, \ y^{\alpha_1} \Delta_x u \in L^p_m, \ y^{\alpha_2} \left( D_{yy} u + c \frac{D_y u}{y} \right) \in L^p_m \text{ and } \lim_{y \to 0} y^c D_y u = 0 \right\}
\]

and the norms \( \| u \|_{W^{2,p}_N(\alpha_1, \alpha_2, m)} \) and

\[
\| u \|_{L^p_m} + \| y^{\alpha_1} \Delta_x u \|_{L^p_m} + \| y^{\alpha_2} (D_{yy} u + cy^{-1} D_y u) \|_{L^p_m}
\]

are equivalent on \( W^{2,p}_N(\alpha_1, \alpha_2, m) \). Finally, when \( 0 < \frac{m+1}{p} \leq c - 1 \) then

\[
W^{2,p}_N(\alpha_1, \alpha_2, m) = \left\{ u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N_{+}^{N+1}) : u, \ y^{\alpha_1} \Delta_x u, y^{\alpha_2} \left( D_{yy} u + c \frac{D_y u}{y} \right) \in L^p_m \right\}.
\]

The following equivalent description of \( W^{2,p}_N(\alpha_1, \alpha_2, m) \) involves a Dirichlet, rather than Neumann, boundary condition, in a certain range of parameters.

Proposition 4.7 Let \( c \geq 1 \) and \( \frac{m+1}{p} < c + 1 - \alpha_2 \). The following properties hold.

(i) If \( c > 1 \) then

\[
W^{2,p}_N(\alpha_1, \alpha_2, m) = \left\{ u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N_{+}^{N+1}) : u, \ y^{\alpha_1} \Delta_x u \in L^p_m, \ y^{\alpha_2} \left( D_{yy} u + c \frac{D_y u}{y} \right) \in L^p_m \text{ and } \lim_{y \to 0} y^{-1} u = 0 \right\}.
\]

(ii) If \( c = 1 \) then

\[
W^{2,p}_N(\alpha_1, \alpha_2, m) = \left\{ u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N_{+}^{N+1}) : u, \ y^{\alpha_1} \Delta_x u \in L^p_m, \ y^{\alpha_2} \left( D_{yy} u + c \frac{D_y u}{y} \right) \in L^p_m \text{ and } \lim_{y \to 0} u(x, y) \in \mathbb{C} \right\}.
\]

The next results show the density of smooth functions in \( W^{2,p}_N(\alpha_1, \alpha_2, m) \). Let

\[
\mathcal{C} := \left\{ u \in C_c^{\infty} \left( \mathbb{R}^N \times [0, \infty) \right), \ D_y u(x, y) = 0 \text{ for } y \leq \delta \text{ and some } \delta > 0 \right\}, \quad (10)
\]

its one dimensional version

\[
\mathcal{D} = \left\{ u \in C_c^{\infty}([0, \infty)), \ D_y u(y) = 0 \text{ for } y \leq \delta \text{ and some } \delta > 0 \right\} \quad (11)
\]

and finally (finite sums below)

\[
C_c^{\infty}(\mathbb{R}^N) \otimes \mathcal{D} = \left\{ u(x, y) = \sum_i u_i(x) v_i(y), \ u_i \in C_c^{\infty}(\mathbb{R}^N), \ v_i \in \mathcal{D} \right\} \subset \mathcal{C}.
\]

Theorem 4.8 If \( \frac{m+1}{p} > \alpha_1^- \) then \( C_c^{\infty}(\mathbb{R}^N) \otimes \mathcal{D} \) is dense in \( W^{2,p}_N(\alpha_1, \alpha_2, m) \).
Proposition 4.9 Let \( \frac{m+1}{p} > \alpha_1^- \). The following properties hold for any \( u \in W^{2,p}_N(\alpha_1, \alpha_2, m) \).

(i) If \( \frac{m+1}{p} > 1 - \frac{\alpha}{2} \) then
\[
\| y^{\alpha_2^-} u \|_{L^p_m} \leq C \| y^{\alpha_2^-} D_y u \|_{L^p_m}.
\]

(ii) If \( \alpha_2 - \alpha_1 < 2 \) and \( \frac{m+1}{p} > 1 - \frac{\alpha_1 + \alpha_2}{2} \), then
\[
\| y^{\alpha_1 + \alpha_2 - 1} \nabla_x u \|_{L^p_m} \leq C \| y^{\alpha_1 + \alpha_2} D_y \nabla_x u \|_{L^p_m}.
\]

Finally, we investigate some relationships between \( W^{2,p}_R(\alpha_1, \alpha_2, m) \), \( W^{2,p}_N(\alpha_1, \alpha_2, m) \) and \( W^{2,p}_N(\alpha_1, \alpha_2, m) \).

Proposition 4.10 The following properties hold.

(i) If \( u \in W^{2,p}_R(\alpha_1, \alpha_2, m) \) then \( y^{\alpha_1 - 1} D_y u \in L^p_m \).

(ii) If \( \alpha_2 - \alpha_1 < 2 \) and \( \frac{m+1}{p} > 2 - \alpha_2 \), then
\[
W^{2,p}_R(\alpha_1, \alpha_2, m) = W^{2,p}_N(\alpha_1, \alpha_2, m) = W^{2,p}(\alpha_1, \alpha_2, m),
\]
with equivalence of the corresponding norms. In particular, \( C_\infty^\infty(\mathbb{R}^{N+1}_+) \) is dense in \( W^{2,p}_R(\alpha_1, \alpha_2, m) \).

We clarify the action of the multiplication operator \( T_{k,0} : u \mapsto y^k u \). The following lemma is the companion of Lemma 4.3 which deals with the transformation \( T_{0,\beta} \).

Lemma 4.11 Let \( \alpha_2 - \alpha_1 < 2 \) and \( \frac{m+1}{p} > 2 - \alpha_2 \). For every \( k \in \mathbb{R} \)
\[
T_{k,0} : W^{2,p}_N(\alpha_1, \alpha_2, m) \to W^{2,p}_R(\alpha_1, \alpha_2, m - kp)
\]
is an isomorphism (we shall write \( y^k W^{2,p}_N(\alpha_1, \alpha_2, m) = W^{2,p}_R(\alpha_1, \alpha_2, m - kp) \)).

5 One dimensional degenerate operators

In this section we summarize the main results proved in [23] for the one dimensional operator \( y^\alpha B_y - \mu y^\alpha = y^\alpha \left( D_{yy} + \frac{\mu}{2} D_y \right) - \mu y^\alpha, \mu \geq 0, \) in \( L^p_m \). To characterize the domain for \( \mu > 0 \), we denote by
\[
D(y^\alpha) = \{ u \in L^p_m : y^\alpha u \in L^p_m \}
\]
the domain of the potential \( V(y) = y^\alpha \) in \( L^p_m \).

Theorem 5.1 Let \( \alpha < 2 \), \( c \in \mathbb{R} \) and \( 1 < p < \infty \).

(i) If \( 0 < \frac{m+1}{p} < c + 1 - \alpha \), then the operator \( y^\alpha B \) endowed with domain \( W^{2,p}_N(\alpha, m) \) generates a bounded positive analytic semigroup of angle \( \frac{\pi}{2} \) on \( L^p(\mathbb{R}_+, y^m dy) \). 

11
In both cases the set \( \mathcal{D} \) defined in (11) is a core.

We shall use \( y^\alpha B^n \), \( n \) stands for Neumann, for \( y^\alpha B \) with domain \( W^{2,p}_N(\alpha,m) \cap D(y^\alpha) \) and similarly for \( y^\alpha B^n - \mu y^\alpha \). Note that the condition \( \alpha^- < \frac{m+1}{p} < c+1 - \alpha \) is equivalent to \( 0 < \frac{m+1}{p} < c+1 - \alpha \) and \(-\alpha < \frac{m+1}{p} < c+1 - \alpha \). The first guarantees that \( y^\alpha B^n \) is a generator in \( L^p_m \) and the second that \( B^n \) is a generator in \( L^p_{m+\alpha p} \).

In the next proposition we show that the multipliers
\[
\xi \in \mathbb{R}^N \rightarrow N_\lambda(\xi) = \lambda(\lambda - y^\alpha B_y + y^\alpha |\xi|^2)^{-1},
\]
\[
\xi \in \mathbb{R}^N \rightarrow M_\lambda(\xi) = |\xi|^2 y^\alpha(\lambda - y^\alpha B_y + y^\alpha |\xi|^2)^{-1}
\]
satisfy the hypothesis of Theorem 2.4. \( M_\lambda \) is used in Section 6 to characterize the domain of \( \mathcal{L} = y^\alpha(\Delta_x + B_y) \) whereas \( N_\lambda \) to prove that \( \mathcal{L} = y^\alpha(\Delta_x + B_y) \) generates an analytic semigroup.

**Proposition 5.2** Assume that \( \alpha^- < \frac{m+1}{p} < c+1 - \alpha \). Then the families
\[
\left\{ |\xi|^{\beta} D_\xi^\beta (M_\lambda)(\xi) : \xi \in \mathbb{R}^N \setminus \{0\}, |\beta| \leq N, \lambda \in \mathbb{C}^+ \right\},
\]
\[
\left\{ |\xi|^{\beta} D_\xi^\beta (N_\lambda)(\xi) : \xi \in \mathbb{R}^N \setminus \{0\}, |\beta| \leq N, \lambda \in \mathbb{C}^+ \right\}
\]
are \( \mathcal{R} \)-bounded in \( L^p_m \).

## 6 Domain and maximal regularity for \( y^\alpha \Delta_x + y^\alpha B^n_y \)

Let \( c,m \in \mathbb{R} \) and \( p > 1 \). In this section we prove generation results, maximal regularity and domain characterization for the degenerate operators
\[
\mathcal{L} := y^\alpha \Delta_x + y^\alpha B^n_y, \quad \alpha < 2
\]
in \( L^p_m(\mathbb{R}^{N+1}_+) \), where \( y^\alpha B^n_y = y^\alpha \left( D_{yy} + \frac{1}{y} D_y \right) \). We start with the \( L^2 \) theory.

### 6.1 The operator \( \mathcal{L} = y^\alpha \Delta_x + y^\alpha B^n_y \) in \( L^2_{c-\alpha} \)

We use the Sobolev spaces of Section 4 and also \( H^{1}_\alpha := \{ u \in L^2_{c-\alpha} : y^\frac{\alpha}{2} \nabla u \in L^2_{c-\alpha} \} \) equipped with the inner product
\[
\langle u, v \rangle_{H^{1}_\alpha} := \langle u, v \rangle_{L^2_{c-\alpha}} + \langle y^\frac{\alpha}{2} \nabla u, y^\frac{\alpha}{2} \nabla v \rangle_{L^2_{c-\alpha}}.
\]

Let \( \mathcal{L} \) be the operator defined on \( C^\infty_c(\mathbb{R}^{N+1}_+) \) by
\[
\mathcal{L} = y^\alpha \Delta + cy^{\alpha-1}D_y = y^{-c+\alpha} \text{div}(y^c \nabla u).
\]
Note that \( c = \alpha \) if and only if \( \mathcal{L} \) is formally self-adjoint with respect to the Lebesgue measure. \( \mathcal{L} \) is associated to the non-negative, symmetric and closed form in \( L^2_{x-\alpha}(\mathbb{R}^{N+1}) \)

\[
\mathcal{L} = \int_{\mathbb{R}^{N+1}} \langle y^\alpha \nabla u, \nabla \varphi \rangle y^{-\alpha} dx dy = \int_{\mathbb{R}^{N+1}} (\mathcal{L}u) \varphi y^{-\alpha} dx dy,
\]

\[
D(\alpha) = H^1_{x,c}.
\]

Accordingly we define the operator with Neumann boundary conditions by

\[
D(\mathcal{L}) = \{ u \in H^1_{x,c} : \exists f \in L^2_{x-\alpha} \text{ such that } \mathcal{L}u = f \text{ for every } v \in H^1_{x,c} \},
\]

\[
\mathcal{L}u = -f.
\]

By construction \( \mathcal{L} \) is a non-positive self-adjoint operator and, if \( u \in D(\mathcal{L}) \), then \( u \in H^2_{\text{loc}}(\mathbb{R}^{N+1}) \) and \( \mathcal{L}u = y^\alpha \Delta u + cy^{-\alpha} D_y u \) by standard arguments. \( \mathcal{L} \) generates a contractive analytic semigroup \( \{ e^{\tau y} \mathcal{L} : \tau \in \mathbb{C} \} \) in \( L^2_{x-\alpha}(\mathbb{R}^{N+1}) \) and our aim is to characterize its domain.

**Proposition 6.1** If \( c+1 > |\alpha| \) then the set \( C^\infty_c(\mathbb{R}^N) \otimes \mathcal{D} \), see (11), is a core for \( \mathcal{L} \) in \( L^2_{x-\alpha}(\mathbb{R}^{N+1}) \).

**Proof.** We observe, preliminarily, that under the given assumptions on \( \alpha, c \), the set \( C^\infty_c(\mathbb{R}^N) \otimes \mathcal{D} \) is contained in \( H^1_{x,c} \). Moreover, integrating by parts one sees that any \( u \in C^\infty_c(\mathbb{R}^N) \otimes \mathcal{D} \) satisfies

\[
(I - \mathcal{L}) (C^\infty_c(\mathbb{R}^N) \otimes \mathcal{D}) \subset D(\mathcal{L}).
\]

Since \( I - \mathcal{L} \) is invertible we have to show that \( (I - \mathcal{L}) (C^\infty_c(\mathbb{R}^N) \otimes \mathcal{D}) \) is dense in \( L^2_{x-\alpha} \) or, equivalently, that \( ((I - \mathcal{L}) (C^\infty_c(\mathbb{R}^N) \otimes \mathcal{D}))^\perp = \{0\} \). Let \( v \in L^2_{x-\alpha}(\mathbb{R}^{N+1}) \) be such that

\[
\int_{\mathbb{R}^{N+1}} (I - \mathcal{L}) f \varphi dx y^{-\alpha} dy = 0, \quad \forall f \in C^\infty_c(\mathbb{R}^N) \otimes \mathcal{D}.
\]

Let us choose \( f = a(x) u(y) \in \mathcal{D} \) with \( a \in C^\infty_c(\mathbb{R}^N) \) and \( u \in \mathcal{D} \). Taking the Fourier transform with respect to \( x \) we get \( \hat{f}(\xi, y) = \hat{a}(\xi) u(y) \) and

\[
\int_{\mathbb{R}^{N+1}} \left[ u(y) + y^\alpha |\xi|^2 u(y) - y^\alpha B_y u(y) \right] \hat{a}(\xi) \tilde{v}(\xi, y) d\xi y^{-\alpha} dy = 0.
\]

Fix \( \xi_0 \in \mathbb{R}^N, \ r > 0 \) and let \( u(\xi) = \frac{1}{|B(\xi_0, r)|} \chi_{B(\xi_0, r)} \in L^2(\mathbb{R}^N) \). Let \( (a_n)_n \in C^\infty_c(\mathbb{R}^N) \) a sequence of test functions such that \( a_n \to \bar{w} \) in \( L^2(\mathbb{R}^N) \); then \( \bar{a}_n \to w \) in \( L^2(\mathbb{R}^N) \) and writing (13) with \( \bar{a} \) replaced by \( a_n \) and letting \( n \to \infty \) we obtain

\[
\frac{1}{|B(\xi_0, r)|} \int_{B(\xi_0, r)} d\xi \int_0^\infty \left[ u(y) + y^\alpha |\xi_0|^2 u(y) - y^\alpha B_y u(y) \right] \tilde{v}(\xi, y) y^{-\alpha} dy = 0.
\]

Letting \( r \to 0 \) and using the Lebesgue Differentiation theorem, we have for a.e. \( \xi_0 \in \mathbb{R}^N \)

\[
\int_0^\infty \left[ u(y) + y^\alpha |\xi_0|^2 u(y) - y^\alpha B_y u(y) \right] \tilde{v}(\xi_0, y) y^{-\alpha} dy = 0,
\]

which is valid for every \( u \in \mathcal{D} \). Under the given hypotheses on \( c \) and \( \alpha \), Theorem 5.1 implies that \( \mathcal{D} \) is a core for the operator \( y^\alpha B^n_y - y^\alpha |\xi_0|^2 \) in \( L^2_{x-\alpha}(\mathbb{R}^+) \). The last equation then implies \( \tilde{v}(\xi_0, \cdot) = 0 \) for a.e. \( \xi_0 \in \mathbb{R}^N \) and the proof is complete. 

\[ \square \]
Lemma 6.3 Let which the operator acts.

\[ m \]

in \( L \) initially defined on \( W^{2,2}_{\alpha}(\alpha, c - \alpha) \) and that it is core for \( L \) by Proposition 6.1 and is dense in \( W^{2,2}_{\alpha}(\alpha, c - \alpha) \) by 4.8.

We have to show that the graph norm and that of \( W^{2,2}_{\alpha}(\alpha, c - \alpha) \) are equivalent on \( C^{\infty}_{c}(\mathbb{R}^{N}) \otimes D \). Since the second is obviously stronger, we have to show the converse.

We use Proposition 4.6 and endow \( W^{2,2}_{\alpha}(\alpha, c - \alpha) \) with the equivalent norm

\[
\| u \|_{W} = \| u \|_{L^{2}_{\alpha, 2}} + \| \alpha^{\alpha} \Delta u \|_{L^{2}_{\alpha, 2}} + \| \alpha^{\alpha} y_{y} u \|_{L^{2}_{\alpha, 2}}.
\]

Let \( u \in C^{\infty}_{c}(\mathbb{R}^{N}) \otimes D \) and \( f = u - L u \), so that \( \| u \|_{L^{2}_{\alpha, 2}} \leq \| f \|_{L^{2}_{\alpha, 2}} \). By taking the Fourier transform with respect to \( x \) (with co-variable \( \xi \)) we obtain

\[
(1 + |\xi|^{2} y_{y}^{\alpha} - \alpha^{\alpha} B_{y}^{n} u) = f(\xi, \cdot) - \alpha^{\alpha}|\xi|^{2} \hat{u}(\xi, \cdot) = y_{y}^{\alpha}|\xi|^{2} (1 + |\xi|^{2} y_{y}^{\alpha} - \alpha^{\alpha} B_{y}^{n})^{-1} f(\xi, \cdot). \tag{14}
\]

This means \( \alpha^{\alpha} \Delta u = - F^{-1} M(\xi) F f \), where \( F \) denotes the Fourier transform and \( M(\xi) = y_{y}^{\alpha}|\xi|^{2} (1 + |\xi|^{2} y_{y}^{\alpha} - \alpha^{\alpha} B_{y}^{n})^{-1} \).

The estimate \( \| \alpha^{\alpha} \Delta u \|_{L^{2}_{\alpha, 2}} \leq C \| f \|_{L^{2}_{\alpha, 2}} \) then follows from the boundedness of the multiplier \( M \) in \( L^{2}(\mathbb{R}^{N}, L^{2}_{\alpha, 2}(\mathbb{R})) \) which follows from Proposition 5.2 and Theorem 2.4 and yields \( \| \alpha^{\alpha} y_{y} u \|_{L^{2}_{\alpha, 2}} \leq C \| f \|_{L^{2}_{\alpha, 2}} \) by difference.

This gives the equivalence of the graph norm and of the norm of \( W^{2,2}_{\alpha}(\alpha, c - \alpha) \) on \( C^{\infty}_{c}(\mathbb{R}^{N}) \otimes D \) and concludes the proof. \( \square \)

6.2 The operator \( L = y_{y}^{\alpha} \Delta + \alpha^{\alpha} B_{y}^{n} \) in \( L^{p}_{m} \)

In this section we prove domain characterization and maximal regularity for the degenerate operator

\[
L = y_{y}^{\alpha} \Delta + \alpha^{\alpha} B_{y}^{n}, \quad \alpha < 2
\]

in \( L^{p}_{m} \). To avoid any misinterpretation, we often write \( L_{m, p} \) to emphasize the underlying space on which the operator acts.

We shall use extensively the set \( D \) defined in (11). In particular \( L \) is well defined on \( C^{\infty}_{c}(\mathbb{R}^{N}) \otimes D \) when \( (m + 1)/p > \alpha^{-} \).

Lemma 6.3 Let \( \alpha^{-} < \frac{m + 1}{p} < c + 1 - \alpha \). Then for any \( \lambda \in \mathbb{C}^{+} \) the operators

\[
(\lambda - L_{c, 2}^{-1}, \quad \alpha^{\alpha} \Delta x (\lambda - L_{c, 2}^{-1}, \quad \alpha^{\alpha} B_{y}^{-1}(\lambda - L_{c, 2}^{-1} \alpha^{\alpha})^{-1})
\]

initially defined on \( L^{p} \cap L^{2}_{\alpha, 2} \) by Theorem 6.2, extend to bounded operators on \( L^{p}_{m} \) which we denote respectively by \( R(\lambda), y_{y}^{\alpha} \Delta x R(\lambda), y_{y}^{\alpha} B_{y}^{-1} R(\lambda) \). Moreover the family \( \{ \lambda R(\lambda) : \lambda \in \mathbb{C}^{+} \} \) is \( R \)-bounded on \( L^{p}_{m} \).

Proof. Let \( u \in C^{\infty}_{c}(\mathbb{R}^{N}) \otimes D \) and \( f = \lambda u - \mathcal{L} u \). By taking the Fourier transform with respect to \( x \) we obtain

\[
(\lambda + |\xi|^{2} y_{y}^{\alpha} - \alpha^{\alpha} B_{y}^{n}) \hat{u}(\xi, \cdot) = \hat{f}(\xi, \cdot), \quad \hat{u}(\xi, \cdot) = (\lambda - y_{y}^{\alpha} B_{y}^{n} + |\xi|^{2} y_{y}^{\alpha})^{-1} \hat{f}(\xi, \cdot).
\]

14
This means \( u = \mathcal{F}^{-1} N_\lambda(\xi) \mathcal{F} f \), where
\[
N_\lambda(\xi) = (\lambda - y^\alpha B^n_y + |\xi|^2 y^\alpha)^{-1}.
\]
Since \( C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D} \) is a core for \( L_{c-a,2} \) we have proved the equality
\[
(\lambda - L_{c-a,2})^{-1} = \mathcal{F}^{-1} N_\lambda(\xi) \mathcal{F}.
\]
Proposition 5.2 and Theorem 2.4 yield the boundedness of the Fourier multiplier \( N_\lambda \) in the space \( L^p(\mathbb{R}^N, L_m^p(\mathbb{R}_+)) \) and the existence of a bounded operator \( R(\lambda) \in L_m^p \) which extends \( (\lambda - L_{c-a,2})^{-1} \). Furthermore [29, Theorem 4.3.9] and the \( R \)-boundedness with respect to \( \lambda \) of \( N_\lambda(\xi) \) and its \( \xi \)-derivatives, see again Proposition 5.2, imply that the family \( \{\lambda R(\lambda) : \lambda \in \mathbb{C}^+\} \) is \( R \)-bounded.

The proof for \( y^\alpha \Delta_y R(\lambda) \) is similar. As before we show that, see (14) in Theorem 6.2,
\[
y^\alpha \Delta_y (\lambda - L_{c-a,2})^{-1} = -\mathcal{F}^{-1} M_\lambda(\xi) \mathcal{F}
\]
where \( M_\lambda(\xi) = y^\alpha |\xi|^2 (\lambda + |\xi|^2 y^\alpha - y^\alpha B^n_y)^{-1} \), and use Proposition 5.2 for the boundedness of the multiplier \( M_\lambda \) in \( L^p(\mathbb{R}^N, L_m^p(\mathbb{R}_+)) \).

The boundedness of \( y^\alpha B^n_y R(\lambda) \) follows then by difference, since \( y^\alpha \Delta_y R(\lambda) + y^\alpha B^n_y R(\lambda) = \lambda R(\lambda) - I \).

**Proposition 6.4** If \( \alpha^- < \frac{m+1}{p} < c+1-\alpha \), an extension \( L_{m,p} \) of the operator \( L \), initially defined on \( C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D} \), generates a bounded analytic semigroup in \( L_m^p(\mathbb{R}_+^{N+1}) \) which has maximal regularity and it is consistent with the semigroup generated by \( L_{c-a,2} \) in \( L_2^{c-a}(\mathbb{R}_+^{N+1}) \).

**Proof.** Let us consider the \( R \)-bounded family of operators \( \{\lambda R(\lambda) : \lambda \in \mathbb{C}^+\} \) defined by Lemma 6.3. In particular it satisfies
\[
\|\lambda R(\lambda)\|_{\mathcal{B}(L_m^p(\mathbb{R}_+^{N+1}))} \leq C, \quad \forall \lambda \in \mathbb{C}^+.
\]
By construction \( R(\lambda) \) coincides with \( (\lambda - L_{c-a,2})^{-1} \) when restricted to \( L_m^p \cap L_2^{c-a} \). Hence, by density, the family \( \{\lambda R(\lambda) : \lambda \in \mathbb{C}^+\} \) satisfies the resolvent equation
\[
R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu), \quad \forall \lambda, \mu \in \mathbb{C}^+
\]
in \( L_m^p \) and therefore it is a pseudoresolvent, see [6, Section 4.a]. Furthermore \( \text{rg}(R(\lambda)) \) is dense in \( L_m^p \) for every \( \lambda \in \mathbb{C}^+ \), since it contains \( C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D} \).

Let us prove that \( R(\lambda) \) is injective for every \( \lambda \in \mathbb{C}^+ \). Let \( f \in L_m^p \) s.t. \( R(\lambda)f = 0 \) for some \( \lambda \in \mathbb{C}^+ \). Since \( \text{Ker}(R(\lambda)) = \text{Ker}(R(\mu)) \) for any \( \lambda, \mu \in \mathbb{C}^+ \), see [6, Lemma 4.5], we have \( R(\lambda)f = 0 \) for every \( \lambda > 0 \). Given \( \epsilon > 0 \), let us choose \( g \in L_m^p \cap L_2^{c-a} \) s.t. \( \|f - g\|_{L_m^p} < \epsilon \). Then
\[
\lambda R(\lambda) g = \lambda R(\lambda) (g - f), \quad \|\lambda R(\lambda) g\|_{L_m^p} \leq C\epsilon, \quad \forall \lambda > 0.
\]
Since \( \lambda R(\lambda) g = (\lambda - L_{c-a,2})^{-1} g \to g \) as \( \lambda \to \infty \) we may suppose, up to a subsequence, that \( \lambda R(\lambda) g \to g \) a.e.. Then Fatou’s Lemma yields
\[
\|g\|_{L_m^p} \leq \lim inf_{\lambda \to \infty} \|\lambda R(\lambda) g\|_{L_m^p} \leq C\epsilon
\]
which implies \( \|f\|_{L_m^p} \leq \|f - g\|_{L_m^p} + |g|_{L_m^p} \leq (1 + C) \epsilon \), hence \( f = 0 \) which proves the injectivity of \( R(\lambda) \).
At this point, [6, Proposition 4.6] yields the existence of a densely defined closed operator \(L_{m,p}\) such that \(C^+ \subseteq \rho(L_{m,p})\) and \(R(\lambda) = (\lambda - L_{m,p})^{-1}\) for any \(\lambda \in C^+\). By construction, \((L_{m,p}; D(L_{m,p}))\) extends \((L, C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D})\) and one has
\[
\|\lambda (\lambda - L_{m,p})^{-1}\|_{B(L_m^p)} \leq C, \quad \lambda \in C^+.
\]
Then from standard results on semigroup theory, see for example [1, Section AII, Theorem 1.14], \((L_{m,p}, D(L_{m,p}))\) generates a bounded analytic semigroup \((e^{z L_{m,p}})_{z \in \Sigma}^+\) for some \(\theta > 0\), in \(L_m^p\).

The maximal regularity of the semigroup follows, using Theorem 2.3, from the \(R\)-boundedness of the resolvent family \(\{\lambda (\lambda - L_{m,p})^{-1}, \lambda \in C^+\}\). Finally, the semigroup is consistent with that in \(L_{c,\alpha}^2\), since the resolvents are consistent.

Finally we characterize the domain of \(L_{m,p}\).

**Theorem 6.5** If \(\alpha^- < \frac{m+1}{p} < c + 1 - \alpha\), then
\[
D(L_{m,p}) = W_{N}^{2,p}(\alpha, \alpha, m)
\]
and in particular \(C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}\) is a core for \(L_{m,p}\).

**Proof.** With the notation of the above proposition, \(D(L_{m,p}) = R(1)(L_m^p)\). Let \(u = R(1)f = (I - L_{c,\alpha,2})^{-1}f\) with \(f \in L_{c,\alpha}^2 \cap L_m^p\). Then Lemma 6.3 yields
\[
\|y^\alpha D_x u\|_{L_m^p} + \|y^\alpha B_y u\|_{L_m^p} \leq C (\|L u\|_{L_m^p} + \|u\|_{L_m^p}).
\]

Using Theorem 5.1 and Theorem 6.2, we deduce that \(u(x, \cdot) \in D(y^\alpha B_{c,\alpha,2}^{-1})\) for a.e. \(x \in \mathbb{R}^n\). Moreover, \(u(x, \cdot), y^\alpha B_y u(x, \cdot) \in L_m^p(\mathbb{R}^n)\), for a.e. \(x \in \mathbb{R}^n\).

Let us show that \(u(x, \cdot) \in D(y^\alpha B_{m,p}^n)\). In fact, setting \(f := u(x, \cdot) - B_y u(x, \cdot) \in L_m^p(\mathbb{R}^n)\) \(\cap L_{c,\alpha}^2(\mathbb{R}^n)\) we have \(u = (I - y^\alpha B_\alpha^{-1}) f \in D(y^\alpha B_{m,p}^n) \cap D(y^\alpha B_{c,\alpha,2}^{-1})\) by the consistency of the resolvent \((I - y^\alpha B_\alpha^{-1})\) in \(L_m^p(\mathbb{R}^n)\) and in \(L_{c,\alpha}^2(\mathbb{R}^n)\).

Theorem 5.1 then implies
\[
\|y^\alpha D_y u\|_{L_m^p} + \|y^\alpha B_y u\|_{L_m^p} \leq C (\|D_y u\|_{L_m^p} + \|u\|_{L_m^p}).
\]

Then, raising to the power \(p\), integrating over \(\mathbb{R}^n\) and using Lemma 6.3 for the last inequality
\[
\|y^\alpha D_y u\|_{L_m^p} + \|y^\alpha B_y u\|_{L_m^p} \leq C (\|u\|_{L_m^p} + \|y^\alpha B_y u\|_{L_m^p}) \leq C (\|u\|_{L_m^p} + \|L u\|_{L_m^p}).
\]

By the density of \(L_{c,\alpha}^2 \cap L_m^p\) in \(L_m^p\), (15), (16) hold for every \(u \in D(L_{m,p})\) and this last is contained in \(W^2_{N}^{2,p}(\alpha, \alpha, m)\), by 4.6.

Moreover, since the graph norm is clearly weaker than the norm of \(W^2_{N}^{2,p}(\alpha, \alpha, m)\), (15), (16) again show that they are equivalent on \(D(L_{m,p})\), in particular on \(C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}\) which is dense in \(W^2_{N}^{2,p}(\alpha, \alpha, m)\), by 4.8.

Therefore \(D(L_{m,p}) = W^2_{N}^{2,p}(\alpha, \alpha, m)\) and in particular \(C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}\) is a core.

**Corollary 6.6** Under the hypotheses of Theorem 6.5 we have for every \(u \in W^2_{N}^{2,p}(\alpha, \alpha, m)\)
\[
\|y^\alpha D_{x,i,j} u\|_{L_m^p} + \|y^\alpha D_{y,i} u\|_{L_m^p} + \|y^\alpha-1 D_y u\|_{L_m^p} \leq C \|L u\|_{L_m^p}.
\]

**Proof.** By Theorem 6.5 the above inequality holds if \(\|u\|_{L_m^p(\mathbb{R}^n)}\) is added to the right hand side. Applying it to \(u(x, y) = u(x, \lambda y), \lambda > 0\) we obtain
\[
\|y^\alpha D_{x,i,j} u\|_{L_m^p} + \|y^\alpha D_{y,i} u\|_{L_m^p} + \|y^\alpha-1 D_y u\|_{L_m^p} \leq C (\|L u\|_{L_m^p} + \lambda^{\alpha-2} \|u\|_{L_m^p})
\]
and the proof follows letting \(\lambda \rightarrow \infty\).
6.3 Mixed derivatives

By using classical covering results, Rellich inequalities and Theorem 6.5, we obtain $L^p$ estimates for the mixed second order derivatives.

**Theorem 6.7** Let $\alpha^\prime < \frac{m+1}{p} < c+1-\alpha$. Then there exists $C > 0$ such that for every $u \in D(L_{m,p})$

$$\|y^{\alpha}D_y \nabla_x u\|_{L^p_m} \leq C\|Lu\|_{L^p_m}.$$  

We need a Rellich type inequality for smooth functions vanishing near $\{ y = 0 \}$.

**Lemma 6.8** Let $\alpha^\prime < \frac{m+1}{p} < c+1-\alpha$. Assume, in addition, $\alpha \neq 1 - \frac{m+1}{p}$, $\alpha \neq 2 - \frac{m+1}{p}$. Then there exists a positive constant $C$ such that for $u \in C^\infty_c(\mathbb{R}^N \times ]0, \infty[)$ we have

$$\|y^{\alpha^\prime-2}u\|_{L^p_m} \leq C\|Lu\|_{L^p_m}.$$  

**Proof.** Let $u \in C^\infty_c(\mathbb{R}^N \times ]0, \infty[)$. Let $\alpha \neq 1 - \frac{m+1}{p}$, $\alpha \neq 2 - \frac{m+1}{p}$. Then by [25, Proposition 3.10] (see also [16])

$$\int_{\mathbb{R}^+} |y^{\alpha^\prime-2}u|^p y^m dy \leq C \int_{\mathbb{R}^+} |y^{\alpha}D_y y u|^p y^m dy.$$  

Integrating the previous inequality over $\mathbb{R}^N$ and using Corollary 6.6 we get

$$\|y^{\alpha^\prime-2}u\|_{L^p_m} \leq C\|y^{\alpha}D_y y u\|_{L^p_m} \leq C\|Lu\|_{L^p_m}.$$  

We first prove mixed derivatives estimates for functions with support far away from $\{ y = 0 \}$.

**Lemma 6.9** Let $\alpha^\prime < \frac{m+1}{p} < c+1-\alpha$. Assume, in addition, $\alpha \neq 1 - \frac{m+1}{p}$, $\alpha \neq 2 - \frac{m+1}{p}$. Then for every $u \in C^\infty_c(\mathbb{R}^N \times ]0, \infty[)$

$$\|y^{\alpha}D_y \nabla_x u\|_{L^p_m} \leq C\|Lu\|_{L^p_m}.$$  

**Proof.** For every $n \in \mathbb{Z}$ let

$$I_n = [2^n, 2^{n+1}[,$$  

$$J_n = [2^{n-1}, 2^{n+2}[.$$  

We fix $\vartheta \in C^\infty_c(\mathbb{R})$ such that $0 \leq \vartheta \leq 1$, $\vartheta(y) = 1$ for $y \in [1, 2]$ and $\vartheta(y) = 0$ for $y \notin [\frac{1}{2}, 3]$ and set $\vartheta_n(y) = \vartheta \left( \frac{y}{\rho_n} \right)$, where $\rho_n = 2^n$.

We apply the classical $L^p$ estimates for elliptic operators with constant coefficients to the function $\vartheta_n u$ and obtain

$$\|\rho_n^\alpha D_y \nabla_x (\vartheta_n u)\|_{L^p(R^{N+1})} \leq C\|\rho_n^\alpha D_y y u\|_{L^p(R^N \times I_n)} + \rho_n^\alpha \Delta_x (\vartheta_n u)\|_{L^p(R^{N+1})}.$$  

Then we get

$$\|\rho_n^\alpha D_y \nabla_x u\|_{L^p(R^N \times I_n)} \leq C \left( \|\rho_n^\alpha D_y y u\|_{L^p(R^N \times J_n)} + \frac{1}{\rho_n} \|\rho_n^\alpha D_y y u\|_{L^p(R^N \times J_n)} \right) + \frac{1}{\rho_n} \|\rho_n^\alpha u\|_{L^p(R^N \times J_n)}$$.
Since \( \frac{m}{p} \leq y \leq 4\rho_m \) if \( y \in J_m \) then we get

\[
\| y^{\alpha + \frac{m}{p}} D_y \nabla_x u \|_{L^p(\mathbb{R}^N \times J_m)} \leq C \left( \| y^{\alpha + \frac{m}{p}} D_y y u + y^{\alpha + \frac{m}{p}} \Delta_x u \|_{L^p(\mathbb{R}^N \times J_m)} + \| y^{\alpha - 1 + \frac{m}{p}} D_y u \|_{L^p(\mathbb{R}^N \times J_m)} + \| y^{\alpha - 2 + \frac{m}{p}} u \|_{L^p(\mathbb{R}^N \times J_m)} \right).
\]

Summing over \( n \), since at most three among the intervals \( J_m \) overlap, it follows that

\[
\| y^{\alpha} D_y \nabla_x u \|_{L^p_m} \leq C \left( \| L_{m,p} u \|_{L^p_m} + \| y^{\alpha - 1} D_y u \|_{L^p_m} + \| y^{\alpha - 2} u \|_{L^p_m} \right).
\]

Using Corollary 6.6 and Lemma 6.8 we conclude the proof.

Next we remove the assumption on the supports and work in \( C^\infty_c(\mathbb{R}^N) \otimes D \) which is a core for \( L_{m,p} \).

**Lemma 6.10** Let \( \alpha^- < \frac{m+1}{p} < c + 1 - \alpha \) and assume also that \( \alpha \neq 1 - \frac{m+1}{p}, \alpha \neq 2 - \frac{m+1}{p} \). Then

\[
\| y^{\alpha} D_y \nabla_x u \|_{L^p_m} \leq C \| \mathcal{L} u \|_{L^p_m}
\]

for every \( u \in C^\infty_c(\mathbb{R}^N) \otimes D \).

**Proof.** Given \( u \in C^\infty_c(\mathbb{R}^N) \otimes D \), let \( v(x,y) = u(x, \lambda y) \). Then \( v \in C^\infty_c(\mathbb{R}^N) \otimes D \) and \( u(x,0) = v(x,0) \). It follows that \( w = u - v \in C^\infty_c(\mathbb{R}^N \times [0, \infty[) \). Moreover

\[
\| y^{\alpha} D_y \nabla_x v \|_{L^p_m} = \lambda^{1 - \alpha - \frac{m+1}{p}} \| y^{\alpha} D_y \nabla_x u \|_{L^p_m}
\]

and, by Corollary 6.6,

\[
\| \mathcal{L} v \|_{L^p_m} \leq \lambda^{-\alpha - \frac{m+1}{p}} \| y^{\alpha} \Delta_x u \|_{L^p_m} + \lambda^{2 - \alpha - \frac{m+1}{p}} \| y^{\alpha} B_y u \|_{L^p_m} \leq C(\lambda) \| \mathcal{L} u \|_{L^p_m}.
\]

Hence by applying Lemma 6.9 to \( w \), we have

\[
\| y^{\alpha} D_y \nabla_x u \|_{L^p_m} \leq C \left( \| y^{\alpha} D_y \nabla_x w \|_{L^p_m} + \| y^{\alpha} D_y \nabla_x v \|_{L^p_m} \right) \leq C \left( \| \mathcal{L} u \|_{L^p_m} + \| y^{\alpha} D_y \nabla_x v \|_{L^p_m} \right)
\]

\[
\leq C \left( \| \mathcal{L} u \|_{L^p_m} + \| \mathcal{L} v \|_{L^p_m} + \| y^{\alpha} D_y \nabla_x v \|_{L^p_m} \right) \leq C(\lambda) \| \mathcal{L} u \|_{L^p_m} + C \lambda^{1 - \alpha - \frac{m+1}{p}} \| y^{\alpha} D_y \nabla_x u \|_{L^p_m}.
\]

Choosing \( \lambda \) large enough or small enough accordingly to \( 1 - \alpha - \frac{m+1}{p} > 0 \) or \( 1 - \alpha - \frac{m+1}{p} < 0 \) we conclude the proof.

**Proof.** (Theorem 6.7). Since \( C^\infty_c(\mathbb{R}^N) \otimes D \) is a core for \( L_{m,p} \), by Lemma 6.10 the claim holds for \( \alpha \neq 1 - \frac{m+1}{p}, \alpha \neq 2 - \frac{m+1}{p} \).

Suppose now \( p = \frac{m+1}{1-\alpha} \) (in particular \( \alpha < 1 \) and \( m + \alpha > 0 \)). Observe that, by the previous part of the proof, the operator \( y^{\alpha} D_y \nabla_x (I - L_{m,q})^{-1} \) is bounded in \( L^q \) for \( q < \frac{m+1}{1-\alpha} \) and for \( q > \frac{m+1}{1-\alpha} \). Q.e.d.
7 The operator $y^{\alpha_1} \Delta_x + y^{\alpha_2} B_y^n$

In this section we consider for $\alpha_1 \in \mathbb{R}, \alpha_2 < 2$ the operator
\[ L^{\alpha_1, \alpha_2}_{m, p} = y^{\alpha_1} \Delta_x + y^{\alpha_2} B_y^n \]
in the space $L^p_m$. The generation and domain properties for $L^{\alpha_1, \alpha_2}$ are deduced from the case $\alpha_1 = \alpha_2$ by using the isometry
\[ T_{k, \beta} u(x, y) := |\beta + 1|^\frac{p}{2} y^k u(x, y^{\beta+1}), \quad (x, y) \in \mathbb{R}_+^{\mathbb{N}+1} \]
introduced in Section 3.

**Theorem 7.1** Let $\alpha_2 - \alpha_1 < 2$ and
\[ \alpha_1^- < \frac{m + 1}{p} < c + 1 - \alpha_2. \]
Then $L^{\alpha_1, \alpha_2}$ with domain $D(L^{\alpha_1, \alpha_2}) = W^{2, p}_N (\alpha_1, \alpha_2, m)$ generates a bounded analytic semigroup in $L^p_m$ which has maximal regularity. Moreover for every $u \in W^{2, p}_N (\alpha_1, \alpha_2, m)$
\[ \|y^{\alpha_1 + \alpha_2} D_y \nabla_x u\|_{L^p_m} \leq C \|Lu\|_{L^p_m}. \]

**Proof.** We use the isometry
\[ T_{0, \alpha_1 - \alpha_2} : L^p_m \rightarrow L^p_m, \quad \hat{m} = \frac{2m - \alpha_1 + \alpha_2}{\alpha_1 - \alpha_2 + 2} \]
which, according to Proposition 3.2, transforms $L^{\alpha_1, \alpha_2}$ into
\[ T^{-1}_{0, \alpha_1 - \alpha_2} L^{\alpha_1, \alpha_2} T_{0, \alpha_1 - \alpha_2} = y^\alpha \Delta_x + \left(\frac{\alpha_1 - \alpha_2 + 2}{2}\right)^2 y^\alpha \tilde{B}_y^n \]
where
\[ \alpha = \frac{2\alpha_1}{\alpha_1 - \alpha_2 + 2}, \quad \tilde{B}_y^n = D_{yy} + \frac{\tilde{c}}{y} D_y, \quad \tilde{c} = \frac{4c + (\alpha_1 - \alpha_2)(2c + 2 + \alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_2 + 2)^2}. \]
Observe that $\alpha < 2$ by assumption as well as $\alpha^- < \frac{\hat{m} + 1}{p} < \tilde{c} + 1 - \alpha$. Generation properties and maximal regularity for $L^{\alpha_1, \alpha_2}$ in $L^p_m$ are then immediate consequence of the same properties of
\[ y^\alpha \Delta_x + \left(\frac{\alpha_1 - \alpha_2 + 2}{2}\right)^2 y^\alpha \tilde{B}_y^n \]
in $L^p_{\hat{m}}$ proved in Proposition 6.4 and Theorem 6.5. Concerning the domain, we have
\[ D(L^{\alpha_1, \alpha_2}) = T_{0, \alpha_1 - \alpha_2} \left( W^{2, p}_N (\alpha, \alpha, \hat{m}) \right) \]
which, by 4.3, coincides with $W^{2, p}_N (\alpha_1, \alpha_2, m)$. The estimates for the mixed derivatives follow from the equality
\[ y^{\alpha_1 + \alpha_2} D_{xy} u = \frac{2 + \alpha_1 - \alpha_2}{2} T_{0, \alpha_1 - \alpha_2} \left( y^{\alpha_1 - \alpha_2} D_{xy} \tilde{u} \right). \]
and Theorem 6.7. \( \square \)

**Remark 7.2** The operator $y^\alpha \Delta_x + ay^\alpha B_y^n$, $a > 0$, has the same domain and properties of $y^\alpha \Delta_x + y^\alpha B_y^n$. This follows by using the map $Tu(x, y) = u(x, a^{-\frac{p}{2}} y)$ since $T^{-1} (y^\alpha \Delta_x + ay^\alpha B_y^n) T = a^\frac{p}{2} (y^\alpha \Delta_x + y^\alpha B_y^n)$. We used this in the above proof.
8 Degenerate operators with Dirichlet boundary conditions

In this section we add a potential term to $B$ and study the operator

$$L = L^{\alpha_1, \alpha_2} = y^{\alpha_1} \Delta_x + y^{\alpha_2} L_y = y^{\alpha_1} \Delta_x + y^{\alpha_2} \left( D_{yy} + \frac{c}{y} D_y - \frac{b}{y^2} \right), \quad \alpha_2 < 2$$

in $L^p_m$, under Dirichlet boundary conditions, in the sense specified below. We recall that the equation $L_y u = 0$ has solutions $y^{-s_1}, y^{-s_2}$ where $s_1, s_2$ are the roots of the indicial equation $f(s) = -s^2 + (c - 1)s + b = 0$ given by

$$s_1 := \frac{c - 1}{2} - \sqrt{D}, \quad s_2 := \frac{c - 1}{2} + \sqrt{D}$$

where

$$D := b + \left( \frac{c - 1}{2} \right)^2$$

is supposed to be nonnegative. When $b = 0$, then $\sqrt{D} = |c - 1|/2$ and $s_1 = 0, s_2 = c - 1$ for $c \geq 1$ and $s_1 = c - 1, s_2 = 0$ for $c < 1$.

Remark 8.1 All the results of this section will be valid, with obvious changes, also in $\mathbb{R}^+_{+}$ for the 1d operators $y^{\alpha_2} L_y$ changing (when it appears in the various conditions on the parameters) $\alpha_1$ to 0 (see also Remark 4.2). We also refer to [2, 17, 18, 20, 27] for the analogous results concerning the Nd version of $L_y$.

A multiplication operator transforms $L$ into an operator of the form $y^{\alpha_1} \Delta_x + y^{\alpha_2} B_y^m$ and allows to transfer the results of the previous sections to this situation. Indeed, we use the map defined in Section 3

$$T_{k,0} u(x, y) := y^k u(x, y), \quad (x, y) \in \mathbb{R}^{N+1}_+$$

for a suitable choice of $k$ and with $\beta = 0$. We recall that $T_{k,0}$ maps isometrically $L^p_{\tilde{m}}$ onto $L^p_m$ where $\tilde{m} = m + kp$ and for every $u \in W^{2,1}_{loc} (\mathbb{R}^{N+1}_+)$ one has

$$T_{k,0}^{-1} \left( y^{\alpha_1} \Delta_x + y^{\alpha_2} L_y \right) T_{k,0} u = \left( y^{\alpha_1} \Delta_x + y^{\alpha_2} \tilde{L}_y \right) u$$

where $\tilde{L}$ is the operator defined as above with parameters $b, c$ replaced, respectively, by

$$\tilde{b} = b - k (c - 1 + k), \quad \tilde{c} = c + 2k.$$  \hspace{1cm} (18)

Moreover the discriminant $\tilde{D}$ and the parameters $\tilde{s}_{1,2}$ of $\tilde{L}$ are given by

$$\tilde{D} = D, \quad \tilde{s}_{1,2} = s_{1,2} + k.$$  \hspace{1cm} (19)

Choosing $k = -s_i, i = 1, 2$, we get $\tilde{b} = 0, \tilde{c}_i = c - 2s_i$ and therefore

$$T_{-s_i,0}^{-1} \left( y^{\alpha_1} \Delta_x + y^{\alpha_2} \tilde{B}_y \right) T_{-s_i,0} u = y^{\alpha_1} \Delta_x + y^{\alpha_2} \left( D_{yy} + \frac{c - 2s_i}{y} D_y \right).$$
Theorem 8.2 Let \( \alpha_2 - \alpha_1 < 2 \) and
\[
s_1 + \alpha_1 - \frac{m+1}{p} < s_2 + 2 - \alpha_2.
\]
Then \( \mathcal{L}^{\alpha_1,\alpha_2} \) generates a bounded analytic semigroup in \( L^p_m \) which has maximal regularity. Moreover,
\[
D(\mathcal{L}^{\alpha_1,\alpha_2}) = y^{-s_1}W^2_{N,p}(\alpha_1, \alpha_2, m - s_1p).
\]
(20)

Finally, the estimate
\[
\|y^{s_1}D_{x,i}u\|_{L^p_m} + \|y^{s_2}L_yu\|_{L^p_m} \leq C\|\mathcal{L}^{\alpha_1,\alpha_2}u\|_{L^p_m}
\]
holds for every \( u \in D(\mathcal{L}^{\alpha_1,\alpha_2}) \).

PROOF. According to the discussion above the map \( T_{-s_1,0}: L^p_{m-s_1p} \to L^p_m \) transforms \( \mathcal{L}^{\alpha_1,\alpha_2} \) into
\[
y^{s_1}\Delta_x + y^{s_2}\tilde{B}_y^n \text{ where } \tilde{B}_y^n = D_{yy} + \frac{\hat{c}}{\hat{y}}D_y, \quad \hat{c} = c - 2s_1.
\]
Since \( s_1 + \alpha_1 - \frac{m+1}{p} < s_2 + 2 - \alpha_2 \) is equivalent to \( \alpha_1 - \frac{m-s_1p+1}{p} < 1 + \alpha_2 \), the statement on generation and maximal regularity is therefore a translation to \( \mathcal{L}^{\alpha_1,\alpha_2} \) and in \( L^p_m \) of the results of Section 7 for \( y^{s_1}\Delta_x + y^{s_2}\tilde{B}_y^n \) in \( L^p_{m-s_1p} \).

Also \( D(\mathcal{L}^{\alpha_1,\alpha_2}) = T_{s_1,0}(W^2_{N,p}(\alpha_1, \alpha_2, m - s_1p)) \). Finally, (21) holds since the similar statement holds for \( y^{s_1}\Delta_x + y^{s_2}\tilde{B}_y^n \) in \( L^p_{m-s_1p} \) and
\[
T_{-s_1,0}^{-1}(y^{s_1}D_{x,i})T_{s_1,0} = y^{s_1}D_{x,i}, \quad T_{-s_1,0}^{-1}(y^{s_2}L_y)T_{s_1,0} = y^{s_2}\tilde{B}_y.
\]

The following corollary explains why we use the term Dirichlet boundary conditions.

Corollary 8.3 Let \( \alpha_2 - \alpha_1 < 2 \) and \( s_1 + \alpha_1 - \frac{m+1}{p} < s_2 + 2 - \alpha_2 \).

(i) If \( D > 0 \) then
\[
D(\mathcal{L}^{\alpha_1,\alpha_2}) = \left\{ u \in W^2_{loc}(R_+^N) : u, y^{s_1}\Delta_x u, y^{s_2}L_y u \in L^p_m \text{ and } \lim_{y \to 0} y^{s_2}u(x,y) = 0 \right\}.
\]

(ii) If \( D = 0 \) then \( s_1 = s_2 \) and
\[
D(\mathcal{L}^{\alpha_1,\alpha_2}) = \left\{ u \in W^2_{loc}(R_+^N) : u, y^{s_1}\Delta_x u, y^{s_2}L_y u \in L^p_m \text{ and } \lim_{y \to 0} y^{s_2}u(x,y) \in \mathbb{C} \right\}.
\]

PROOF. Since \( \hat{c} = c - 2s_1 = 1 + 2\sqrt{-D} \geq 1 \), both points follow by the previous theorem and 4.7.

Remark 8.4 Equality (20) says that \( u \in D(\mathcal{L}^{\alpha_1,\alpha_2}) \) if and only for every \( i, j = 1, \ldots, N \) all functions
\[
u, y^{s_1}D_{x,i}, y^{s_1}D_{x,x,i}u, y^{s_1-1}(D_yu + s_1 \frac{u}{y}), y^{s_2}L_y u
\]
belong to \( L^p_m \) but one cannot deduce, in general, that \( y^{s_2-1}D_yu \) and \( y^{s_2}D_{yy}u \) belong to \( L^p_m \), as one can check on functions like \( y^{-s_1}u(x), u \in C_c^\infty(R^N) \), near \( y = 0 \). This is however possible in the special case below.
Corollary 8.5 Let \( \alpha_2 - \alpha_1 < 2 \) and \( s_1 + 2 - \alpha_2 < \frac{m+1}{p} < s_2 + 2 - \alpha_2 \). Then \( D(\mathcal{L}_{m,p}^{\alpha_1,\alpha_2}) = W^{2,p}_{\mathcal{R}}(\alpha_1, \alpha_2, m) \).

Proof. Observe that \( s_1 + 2 - \alpha_2 > s_1 + \alpha_1 \), since \( \alpha_2 < 2 \), \( \alpha_2 - \alpha_1 < 2 \). By Theorem 8.2 and Proposition 4.10

\[
D(\mathcal{L}_{m,p}^{\alpha_1,\alpha_2}) = y^{-s_1} \left( W^{2,p}_{\mathcal{N}}(\alpha_1, \alpha_2, m - s_1 p) \right) = W^{2,p}_{\mathcal{R}}(\alpha_1, \alpha_2, m)
\]
under the assumption \( \frac{m - p \alpha_1 + 1}{p} > 2 - \alpha_2 \) which is equivalent to \( s_1 + 2 - \alpha_2 < \frac{m+1}{p} \).

Concerning the mixed derivatives, we have the following result.

Corollary 8.6 Let \( \alpha_2 - \alpha_1 < 2 \) and

\[
s_1 + \alpha_1 < \frac{m+1}{p} < s_2 + 2 - \alpha_2, \quad \frac{m+1}{p} > s_1 + 1 - \frac{\alpha_1 + \alpha_2}{2}.
\]

Then

\[
\| y^{\frac{\alpha_1 + \alpha_2}{2} - 1} D_x u \|_{L_m^p} + \| y^{\frac{\alpha_1 + \alpha_2}{2}} D_{x,y} u \|_{L_m^p} \leq C \| \mathcal{L}_{m,p}^{\alpha_1,\alpha_2} u \|_{L_m^p}
\]
for every \( u \in D(\mathcal{L}_{m,p}^{\alpha_1,\alpha_2}) \).

Proof. Let us write \( u = y^{-s_1} v \) with \( v \in W^{2,p}_{\mathcal{N}}(\alpha_1, \alpha_2, m - s_1 p) \). Then

\[
y^{\frac{\alpha_1 + \alpha_2}{2}} D_{x,y} u = y^{\frac{\alpha_1 + \alpha_2}{2}} (y^{-s_1} D_{x,y} v - s_1 y^{-s_1 - 1} D_x v).
\]
The first term on the right hand side belongs to \( L_m^p(\mathbb{R}^{N+1}) \) by Theorem 7.1 and the second by Proposition 4.9, provided \( \frac{m+1}{p} > s_1 + 1 - \frac{\alpha_1 + \alpha_2}{2} \). This gives the estimate for \( y^{\frac{\alpha_1 + \alpha_2}{2}} D_{x,y} u \). That for \( y^{\frac{\alpha_1 + \alpha_2}{2} - 1} D_x u \) follows similarly, using Proposition 4.9 again.

Observe that the condition \( \frac{m+1}{p} > s_1 + 1 - \frac{\alpha_1 + \alpha_2}{2} \) in the previous corollary is necessary for the integrability of the mixed derivatives of functions like \( y^{-s_1} u(x) \), \( u \in C_c^\infty(\mathbb{R}^N) \), near \( y = 0 \).

Corollary 8.7 Let \( \alpha_2 - \alpha_1 < 2 \) and

\[
s_1 + \alpha_1 < \frac{m+1}{p} < s_2 + 2 - \alpha_2, \quad \frac{m+1}{p} > s_1 + 1 - \frac{\alpha_2}{2}.
\]

Then

\[
\| y^{\frac{\alpha_2}{2}} D_y u \|_{L_m^p} \leq C \left( \| u \|_{L_m^p} + \| \mathcal{L}_{m,p}^{\alpha_1,\alpha_2} u \|_{L_m^p} \right).
\]

Proof. Let us write \( u = y^{-s_1} v \) with \( v \in W^{2,p}_{\mathcal{N}}(\alpha_1, \alpha_2, m - s_1 p) \). Then

\[
y^{\frac{\alpha_2}{2}} D_y u = y^{\frac{\alpha_2}{2}} (y^{-s_1} D_y v - s_1 y^{-s_1 - 1} v)
\]
and the thesis follows from Proposition 4.9 (i).

The above results apply also to the operator \( \mathcal{L} = y^{\alpha_1} \Delta_x + y^{\alpha_2} B_y \), \( B_y = D_{yy} + \frac{c}{2} D_y \), when \( c < 1 \), so that \( s_1 = c - 1 \neq 0 \), and allow to construct a realization of \( \mathcal{L} \) different from that of Theorem 7.1.
Corollary 8.8 Let \( \alpha_2 - \alpha_1 < 2 \), \( c < 1 \) and and \( c - 1 + \alpha_1^- \frac{m+1}{p} < 2 - \alpha_2 \). Then \( \mathcal{L} = y^{\alpha_1} \Delta_x + y^{\alpha_2} B_y \) with domain

\[
D(\mathcal{L}^{\alpha_1, \alpha_2}) = \left\{ u \in W^{2,p}_0(\mathbb{R}^{N+1}) : u, y^{\alpha_1} \Delta_x u, y^{\alpha_2} B_y u \in L^p_m \text{ and } \lim_{y \to 0} u(x, y) = 0 \right\},
\]
generates a bounded analytic semigroup in \( L^p_m \) which has maximal regularity.

Proof. This follows from Corollary 8.3 (i), since \( s_1 = c - 1 \) and \( s_2 = 0 \).

Note that the generation interval \( c - 1 + \alpha_1^- < \frac{m+1}{p} < 2 - \alpha_2 \) under Dirichlet boundary conditions, is larger than \( \alpha_1^- < \frac{m+1}{p} < c + 1 - \alpha_2 \) given by Theorem 7.1 for Neumann boundary conditions.

Let us explain what happens in Theorem 8.2 if we choose the second root \( s_2 \) instead of \( s_1 \). Proceeding similarly, one proves an identical result under the condition

\[
s_2 + \alpha_1^- < \frac{m+1}{p} < s_1 + 2 - \alpha_2. \tag{22}
\]

However this requires the assumption \( s_2 < s_1 + 2 - \alpha_2 \) which is not always satisfied. When (22) holds this procedure leads to a different operator, as we explain in more detail in Section 9.2.

9 Further results, examples and applications

9.1 The range of contractivity

Here we investigate when the semigroups generated by our operators are contractive on the positive real axis.

Let \( \mathcal{L} = y^{\alpha_1} \Delta_x + y^{\alpha_2} B_y \) with \( \alpha_2 < 2 \), \( \alpha_2 - \alpha_1 < 2 \) and \( (\alpha_1^-) < \frac{m+1}{p} < c + 1 - \alpha_2 \) so that the generation conditions are satisfied and \( C^\infty_\omega (\mathbb{R}^N) \otimes D \) is a core.

If \( I_s u(x, y) = u(s^{1 - \frac{(m+1)^-}{m}} x, sy) \), then \( I_s^{-1} \mathcal{L} I_s = s^{2 - \alpha_2} \mathcal{L} \) and an estimate \( \| e^{t \mathcal{L}} \| \leq e^{ct} \) implies \( \| e^{t \mathcal{L}} \| \leq 1 \) (operator norms in \( L^p_m \)). Therefore quasi-contractivity is equivalent to contractivity.

Lemma 9.1 The operator \( \mathcal{L} = y^{\alpha_1} \Delta_x + y^{\alpha_2} B_y \) is dissipative on \( C^\infty_\omega (\mathbb{R}^N) \otimes D \subset L^p_m \) if and only if \( y^{\alpha_2} B \) is dissipative on \( D \subset L^p_m (\mathbb{R}^N) \).

Proof. For \( u \in C^\infty_\omega (\mathbb{R}^N) \otimes D \)

\[
- \int_{\mathbb{R}^{N+1}_+} (\mathcal{L}u) u |u|^{p-2} y^m \, dx \, dy = (p - 1) \int_{\mathbb{R}^{N+1}_+} |\nabla_x u|^2 |u|^{p-2} y^{\alpha_1 + m} \, dx \, dy
\]

\[
- \int_{\mathbb{R}^{N+1}_+} (y^{\alpha_2} B_y u) |u|^{p-2} y^m \, dx \, dy
\]

and the dissipativity of \( \mathcal{L} \) follows from that of \( y^{\alpha_2} B_m \). Conversely, assuming the dissipativity of \( \mathcal{L} \), we fix \( v \in D, 0 \neq \phi \in C^\infty_\omega (\mathbb{R}^N) \) and consider \( u_n(x, y) = \phi(x/n)v(y) \). Inserting in the above identity and letting \( n \to \infty \) it follows that \( -\int_0^\infty (y^{\alpha_2} B_y u) |u|^{p-2} y^m \, dy \geq 0 \).

The dissipativity of \( y^{\alpha_2} B \) will be deduced from the case \( \alpha = 0 \), via a change of variable.
Lemma 9.2 The best constant in the inequality
\[ \int_0^\infty u_0^2 |u|^{p-2} y^m \, dy \geq C \int_0^\infty |u|^p y^{m-2} \, dy, \quad u \in C_c^\infty (\mathbb{R}_+) \] (23)
is \( C = \left( \frac{m-1}{p} \right)^2 \). When \( m > 1 \) the inequality above holds also for every \( u \in D \).

Proof. A proof that the best constant is that indicated above can be found in [25, Proposition 8.3].

When \( m > 1 \) and \( u \in D \), let \( \phi \) be a smooth cut-off functions which is equal to 0 in \([0,1]\) and to 1 in \([2,\infty]\). We apply the inequality above to \( u_n(y) = u(y)\phi(ny) \) and get
\[ C \int_0^\infty |u_n|^p y^m \, dy \leq \int_0^\infty u_0^2 \phi(ny)u(y)|\phi(ny)u(y)|^{p-2} y^m \, dy + \int_0^\infty |u|^p |\phi(y)|^{p-2} y^m \, dy \]
and the last term tends to 0 as \( n \to \infty \), since \( m > 1 \) and \( u, \phi, \phi_y \) are bounded. One concludes by dominate convergence.

Proposition 9.3 Assume \( 0 < \frac{m+1}{p} < c+1 \). The operator \( B_p^m \) is dissipative in \( L_p^m(\mathbb{R}_+) \) if and only if

(i) \( m = c \) or

(ii) \( m \geq 1 \) and \( \frac{m-1}{p} < c-1 \).

Proof. For \( u \in D \), \( u \) constant in \([0,a]\) we have integrating by parts
\[ -\int_0^\infty (Bu)u|u|^{p-2} y^m \, dy = (p-1) \int_0^\infty u_0^2 |u|^{p-2} y^m \, dy + \frac{(1-m)(m-c)}{p} \int_a^\infty |u|^p y^{m-2} \, dy \] (24)
and (i) is immediate.

The inequality
\[ -\int_0^\infty (Bu)u|u|^{p-2} y^m \, dy = (p-1) \int_0^\infty u_0^2 |u|^{p-2} y^m \, dy + \frac{(1-m)(m-c)}{p} \int_a^\infty |u|^p y^{m-2} \, dy \geq 0 \]
holds in \( C_c^\infty (\mathbb{R}_+) \) if and only if \( \frac{(m-1)(m-c)}{p(p-1)} \leq \left( \frac{m-1}{p} \right)^2 \) by the above lemma, which means
\[ \frac{m-1}{p} \left( c - 1 - \frac{m-1}{p} \right) \geq 0. \] (25)
Therefore, dissipativity can hold when \( m \geq 1 \) only if (ii) holds. On the other hand, if \( m > 1 \) and (ii) holds, then letting \( a \to 0 \) in (24) we obtain
\[ -\int_0^\infty (Bu)u|u|^{p-2} y^m \, dy = (p-1) \int_0^\infty u_0^2 |u|^{p-2} y^m \, dy + \frac{(1-m)(m-c)}{p} \int_0^\infty |u|^p y^{m-2} \, dy \]
which is nonnegative since (23) holds in \( D \), by Lemma 9.2. Therefore (ii) is proved for \( m > 1 \). If \( m = 1 \) let us observe that (24) trivially holds when \( c \geq 1 \).
Finally we consider the case \( m \leq 1 \) and show that \( B^n \) is never dissipative for \( m < 1 \) and \( c \neq m \) or for \( m = 1 \) and \( c \leq 1 \), even though (23) can hold on \( C^\infty_0(\mathbb{R}_+) \).

Let assume that (25) holds, or \( c - 1 \leq (m - 1)/p \), otherwise dissipativity fails already on \( C^\infty_0(\mathbb{R}_+) \), and let \( u(y) = y^{-\beta} \) for \( y \geq 1 \) and constant in \([0,1]\). The function \( u \) is not properly in \( \mathcal{D} \) but smoothing and cutting at infinity do not make any problem.

Assuming \( (m - 1)/p < \beta \) all integrals in the right hand side of (24) converge and a straightforward computation shows that positivity is equivalent to

\[
\beta ((p - 1)\beta - (m - c)) \geq 0
\]

for every \( \beta > (m - 1)/p \). However this is false for \( m < 1 \) since the expression above is negative between 0 and \((m - c)/(p - 1)\) and \((m - 1)/p \leq (m - c)/(p - 1)\).

When \( m = 1 \) the inequality (25) is always verified and the positivity of (24) on \( y^{-\beta} \) is equivalent to

\[
\beta ((p - 1)\beta - (1 - c)) \geq 0
\]

for \( \beta > 0 \) which is false for small \( \beta > 0 \) when \( c < 1 \).

We can now state the final contractivity result.

**Theorem 9.4**

(i) Assume that \( \alpha_2 - \alpha_1 < 2 \) and

\[
\alpha_1^- < \frac{m + 1}{p} < c + 1 - \alpha_2.
\]

Then the semigroup generated by \( y^{\alpha_1} \Delta_x + y^{\alpha_2} B^n_y \) is contractive in \( L^p_m \) if and only if

\[
m = c - \alpha_2 \quad \text{or} \quad \frac{2 - \alpha_2}{p} \leq \frac{m + 1}{p} \leq c - 1 + \frac{2 - \alpha_2}{p}.
\]

(ii) Assume that \( \alpha_2 - \alpha_1 < 2 \) and

\[
s_1 + \alpha_1^- < \frac{m + 1}{p} < s_2 + 2 - \alpha_2.
\]

Then the semigroup generated by \( y^{\alpha_1} \Delta_x + y^{\alpha_2} L_y \), under Dirichlet boundary conditions, is contractive in \( L^p_m \) if and only if

\[
s_1 + \frac{2 - \alpha_2}{p} \leq \frac{m + 1}{p} \leq s_2 + \frac{2 - \alpha_2}{p}.
\]

**Proof.** Concerning (i), observe that by Lemma 9.1 it is enough to consider \( y^{\alpha_2} B^n_y \). According to Proposition 3.2, we use the isometry

\[
T_{0,-\alpha_n} : L^p_m(\mathbb{R}_+) \to L^p_{\hat{m}}(\mathbb{R}_+) \, , \, T_{0,-\alpha_n} u(y) = \left| 1 - \frac{\alpha_2}{2} \right|^\frac{p}{2} u(y^{1 + \frac{\alpha_2}{2}}),
\]

\( \hat{m} = \frac{m + \alpha_2}{1 - \frac{\alpha_2}{2}} \), under whose action \( y^{\alpha_2} B^n_y \) becomes isometrically equivalent to \( (1 - \frac{\alpha_2}{2})^2 \hat{B} \) where

\( \hat{B} = D_{yy} + \hat{c} D_y \) and \( \hat{c} = \frac{c - \frac{\alpha_2}{2}}{2} \).

The dissipativity for \( y^{\alpha_2} \hat{B} \) in \( L^p_m \) is then immediate consequence of that of \( \hat{B} \) in \( L^p_{\hat{m}} \) already proved in Proposition 9.3.
Concerning (ii), observe that, as in the previous Section, the map $T_{-s_1,0} : L_p^{m-s_1,p} \to L_p^{m}$ transforms $\mathcal{L}^{\alpha_1,\alpha_2}$ into $y^{\alpha_1}\Delta_x + y^{\alpha_2}B_y$ where $B_y^n = D_{yy} + \frac{\tilde{c}}{2}D_y$, $\tilde{c} = c - 2s_1$. Therefore the dissipativity of $\mathcal{L}^{\alpha_1,\alpha_2}$ in $L_p^{m}$ follows from that of $y^{\alpha_1}\Delta_x + y^{\alpha_2}B_y$ in $L_p^{m-s_1,p}$ proved in (i). We have that $\mathcal{L}^{\alpha_1,\alpha_2}$ is dissipative in $L_p^{m}$ if and only if $m - ps_1 = c - 2s_1 - \alpha_2$ or $m - ps_1 \geq 1 - \alpha_2$ and $\frac{m - ps_1 - 1 + \alpha_2}{p} \leq c - 2s_1 - 1$. The claim follows since $m - ps_1 \geq 1 - \alpha_2$ and $\frac{m - ps_1 - 1 + \alpha_2}{p} \leq c - 2s_1 - 1$ are equivalent respectively to $\frac{m+1}{p} \geq s_1 + \frac{2-\alpha_2}{p}$ and $\frac{m+1}{p} \leq s_2 + \frac{2-\alpha_2}{p}$ and after observing that $m - ps_1 = c - 2s_1 - \alpha_2$ is equivalent to $\frac{m+1}{p} = s_1 + \frac{2-\alpha_2}{p}$ and obviously $s_1 + \frac{2-\alpha_2}{p} < s_1 + \frac{2-\alpha_2}{p}$. 

9.2 Further generation results and uniqueness

Let $\mathcal{L} = y^{\alpha_1}\Delta_x + y^{\alpha_2}L_y$, $\alpha_2 < 2$, and keep the notation of Section 8, in particular $\mathcal{L}^{\alpha_1,\alpha_2}$ is the operator constructed therein. Let us define the maximal operator $\mathcal{L}^{\max}_{m,p}$ as $\mathcal{L}$ on the maximal domain

$$D(\mathcal{L}^{\max}_{m,p}) = \{ u \in L_p^{m} \cap W^{2,p}_{loc}(\mathbb{R}^{N+1}_+) : \mathcal{L}u \in L_p^{m}\}$$

and the minimal operator $\mathcal{L}^{\min}_{m,p}$ as the closure of $\mathcal{L}$ initially defined on $C_c^\infty (\mathbb{R}^{N+1}_+)$. By local elliptic regularity $\mathcal{L}^{\max}_{m,p}$ is closed and then, since $(\mathcal{L}, C_c^\infty (\mathbb{R}^{N+1}_+))$ admits the closed extension $\mathcal{L}^{\max}_{m,p}$, its closure is well defined. Clearly $\mathcal{L}^{\min}_{m,p} \subset \mathcal{L}^{\alpha_1,\alpha_2} \subset \mathcal{L}^{\max}_{m,p}$.

Integrating by parts one sees that the formal adjoint of $\mathcal{L}$ is the operator $\mathcal{L}^* = y^{\alpha_1}\Delta_x + y^{\alpha_2}L_y^*$ in $L_p^{m}$ where

$$L_y^* = D_{yy} + \frac{\tilde{c}}{y}D_y - \frac{\tilde{b}}{y^2}, \quad \tilde{c} = 2\alpha_2 + 2m - c, \quad \tilde{b} = b - (\alpha_2 + m - c)(\alpha_2 + m - 1).$$

Moreover, the characteristic numbers of $\mathcal{L}^*$ are given by

$$D^* = D, \quad s_1^* = \alpha_2 + m - 1 - s_2, \quad s_2^* = \alpha_2 + m - 1 - s_1.$$

Lemma 9.5 The dual of $\mathcal{L}^{\min}_{m,p}$ is $\mathcal{L}^{*,\max}_{m,p'}$ and the dual of $\mathcal{L}^{\max}_{m,p}$ is $\mathcal{L}^{*,\min}_{m,p'}$.

Proof. Since $L_p^{m}$ is reflexive, it is sufficient to prove the first equality. The second follows by duality from the first, changing $p$ with $p'$. If $u \in C_c^\infty (\mathbb{R}^{N+1}_+)$ and $v \in D(\mathcal{L}^{*,\max}_{m,p'})$ one can integrate by parts and get

$$\int_{\mathbb{R}^{N+1}_+} v(\mathcal{L}u) y^m \, dx \, dy = \int_{\mathbb{R}^{N+1}_+} u(\mathcal{L}^* v) y^m \, dx \, dy$$

and hence $\mathcal{L}^{*,\max}_{m,p'}$ is a restriction of the dual of $\mathcal{L}^{\min}_{m,p}$. Conversely, if $v \in L_p^{m'}$ and

$$\int_{\mathbb{R}^{N+1}_+} v(\mathcal{L}u) y^m \, dx \, dy = \int_{\mathbb{R}^{N+1}_+} uf y^m \, dx \, dy$$

for some $f \in L_p^{m}$, then by local elliptic regularity (the coefficients of $\mathcal{L}$ are smooth in the interior of $\mathbb{R}^{N+1}_+$), $v \in W^{2,p'}_{loc}(\mathbb{R}^{N+1}_+)$ and $\mathcal{L}^* v = f \in L_p^{m'}$. 

Proposition 9.6 Let $\alpha_2 - \alpha_1 < 2$ and $s_1 + 2 - \alpha_2 \leq \frac{m+1}{p} < s_2 + 2 - \alpha_2$. Then $\mathcal{L}^{\alpha_1,\alpha_2} = \mathcal{L}^{\min}_{m,p}$. 


PROOF. Observe that \( s_1 + 2 - \alpha_2 > s_1 + \alpha_1^- \), since \( \alpha_2 < 2, \alpha_2 - \alpha_1 < 2 \). By Theorem 8.2

\[
D(\mathcal{L}^{\alpha_1, \alpha_2}_{m,p}) = y^{-s_1} \left( W^2_{\infty} (\alpha_1, \alpha_2, m - s_1 p) \right).
\]

The assumption \( s_1 + 2 - \alpha_2 \leq \frac{m+1}{p} \) is equivalent to \( \frac{m-p\alpha+1}{p} \geq 2 - \alpha_2 \) and one concludes by ??.

Note that when \( s_1 + 2 - \alpha_2 \leq \frac{m+1}{p} < s_2 + 2 - \alpha_2 \), then we have also \( D(\mathcal{L}^{\alpha_1, \alpha_2}_{m,p}) = W^2_{\infty} (\alpha_1, \alpha_2, m) \), by Corollary 8.5.

**Proposition 9.7** Let \( \alpha_2 - \alpha_1 < 2 \) and \( s_1 < \frac{m+1}{p} \leq s_2 \). Then \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \) generates an analytic semigroup.

**Proof.** Let us consider the adjoint \( \mathcal{L}^* \). Then \( s_1^* + 2 - \alpha_2 \leq \frac{m+1}{p} < s_2^* + 2 - \alpha_2 \) and then, by the proposition above, \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \) generates a semigroup in \( L^p_m \). By standard semigroup duality in reflexive spaces, the dual operator \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \), see Lemma 9.5, generates a semigroup in \( L^p_m \).

Observe that when \( \alpha_1 < 0 \), \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \) generates a semigroup also when the condition \( s_1 + \alpha_1^- < \frac{m+1}{p} \) is violated. However, if this last holds, then \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} = \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \).

**Proposition 9.8** Let \( \alpha_2 - \alpha_1 < 2 \) and \( s_1 + \alpha_1^- < \frac{m+1}{p} \leq s_2 \). Then \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \) generates a semigroup.

**Proof.** In fact \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \) is well defined and generates a semigroup. Since \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \) extends \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \) and both are generators, they coincide.

By duality, we can extend the generation interval.

**Proposition 9.9** If \( \alpha_2 - \alpha_1 < 2 \) and \( \frac{m+1}{p} \in (s_1, s_2 + 2 - \alpha_2 - \alpha_1^-) \cup (s_1 + \alpha_1^-, s_2 + 2 - \alpha_2) \) a realization \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \subset \mathcal{L}^D \subset \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \) generates a semigroup.

**Proof.** In fact, if \( s_1 + \alpha_1^- < \frac{m+1}{p} < s_2 + 2 - \alpha_2 \) we can take \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \) and if \( s_1 < \frac{m+1}{p} \leq s_2 + 2 - \alpha_2 - \alpha_1^- \), we can take the adjoint of \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \), since the condition is equivalent to \( s_1^* + \alpha_1^- < \frac{m+1}{p} < s_2^* + 2 - \alpha_2 \).

For the 1d operator \( y^{\alpha_2} L_y \) it is known that a such a realization exists if and only if \( s_1 < \frac{m+1}{p} < s_2 + 2 - \alpha_2 \), see [24, Theorem 1.1, Theorem 1.2] for the case \( m = 0 \). For general \( m \) and \( \alpha_2 = 0 \) see [20, Propositions 2.4, 2.5] from which, by the transformation \( T_{\alpha_1^-} \), it is possible deduce the general case. However the above proposition yields a semigroup in this interval only when \( s_1 + \alpha_1^- < s_2 + 2 - \alpha_2 - \alpha_1^- \), for example, when \( \alpha_1 = \alpha_2 = \alpha < 0 \) this requires \( |\alpha| < s_2 + s_1 + 2 \).

Let us show that when the condition \( s_1 + \alpha_1^- < \frac{m+1}{p} \) is violated, the regularity estimate

\[
\| y^{\alpha_1} D_{x,y} u \|_{L^p_{m}} + \| y^{\alpha_2} L_y u \|_{L^p_{m}} \leq C \| u \|_{L^p_{m}}
\]

may fail for \( u \) in the domain of the operator.

**Example 9.10** Let \( \mathcal{L} = y^{-\beta} (\Delta_x + D_{yy}) \), \( \beta > 0 \). Then \( s_1 = -1, s_2 = 0, \mathcal{L} \) generates under Neumann boundary conditions when \( \beta < \frac{m+1}{p} < 1 + \beta \) and under Dirichlet boundary conditions when \( -1 + \beta < \frac{m+1}{p} < 2 + \beta \) and both operators satisfy (27).

However, when \( -1 < \frac{m+1}{p} \leq (-1 + \beta) \wedge 0 \), \( \mathcal{L}^{\alpha_1, \alpha_2}_{m,p} \) is a generator for which (27) fails. Indeed, let \( \eta \) be a smooth function equal to 1 in \([0, \frac{1}{2}]\) and to 0 in \([1, \infty]\) and

\[
u(x,y) = \eta(y) \left( \frac{y + 1}{(|x|^2 + (y + 1)^2)^{\frac{N+1}{2}}} + \frac{y - 1}{(|x|^2 + (y - 1)^2)^{\frac{N+1}{2}}} \right).
\]

27
Note that $u$ is, for small $y$, the difference of the Poisson kernels on the hyperplanes $y = \pm 1$. Then $u \in L^p_m$, since $(m+1)/p+1 > 0$, and $Lu \in L^p_m$, since $\Delta u = 0$ for $0 \leq y \leq \frac{1}{2}$, so that $u \in D(L^{m,p}_{m,p})$. However, $y^{-\beta} \Delta_x u$ and $y^{-\beta} D_{yy} u$ do not belong to $L^p_m$, since $(m+1)/p \leq -1 + \beta$.

A natural question arises if different boundary conditions can be imposed to produce different semigroups in $L^p_m$. This is the case, for example, for the operator $L = y^{\alpha_1} \Delta_x + y^{\alpha_2} B_y$ in Theorem 7.1 and Corollary 8.8, in the range $c < 1$, $\alpha_1 < \frac{m+1}{p} < c+1-\alpha_2$, where both boundary conditions $\lim_{y \to 0} u = 0$ and $\lim_{y \to 0} y^\beta D_y u = 0$ can be imposed and produce different semigroups.

As in [21, Section 5] we look for realizations $\mathcal{L}_D$ such that $L^{\text{min}}_{m,p} \subset \mathcal{L}_D \subset L^{\text{min}}_{m,p}$. From Propositions 9.6 and 9.7 it follows that $\mathcal{L}_D$ is unique in $L^p_m$ if $s_1 < \frac{m+1}{p} \leq s_2$ or $s_1 + 2 - \alpha - 2 \leq \frac{m+1}{p} < s_2 + 2 - \alpha_2$. Uniqueness then holds in the generation range of $L_y$, namely $(s_1, s_2 + 2 - \alpha_2)$, when these two intervals overlap, that is when $s_1 + 2 - \alpha_2 \leq s_2$ or equivalently $D \geq (1 - \frac{m}{p})^2$. In this case, uniqueness does not depend on $p$ and $m$.

Uniqueness may fail if $s_2 < s_1 + 2 - \alpha_2$ and $(m+1)/p \in (s_2, s_1 + 2 - \alpha_2)$, as we show under the stronger assumptions $s_1 \neq s_2$ and $s_2 + \alpha_1 < \frac{m+1}{p} < s_1 + 2 - \alpha_2$.

**Proposition 9.11** If $0 < D < (1 - \frac{m}{p})^2$ and $s_2 + \alpha_1 < \frac{m+1}{p} < s_1 + 2 - \alpha_2$, the operator $\mathcal{L} = y^{\alpha_1} \Delta_x + y^{\alpha_2} L_y$ with domain

$$D(\mathcal{L}) = y^{-s_2} W^2_{N,p}(\alpha_1, \alpha_2, m - s_2 p)$$

$$(28) \quad = \left\{ u \in W^2_{loc}(\mathbb{R}_+^{N+1}) : u, y^\alpha \Delta_x u, y^{\alpha_2} L_y u \in L^p_m \text{ and } \lim_{y \to 0} y^{s_1+1} \left( D_y u + s_2 \frac{u_y}{y} \right) = 0 \right\}$$

generates a semigroup in $L^p_m$.

**Proof.** We proceed as in the proof of Theorem 8.2 but in place of the isometry $T_{-s_1,0}$ we use $T_{-s_2,0} : L^p_{m-s_2 p} \to L^p_m$ which transform $\mathcal{L}$ into $y^{\alpha_1} \Delta_x + y^{\alpha_2} \tilde{B}_y$ where $\tilde{B}_y = D_y + \tilde{c} D_{yy}$, $\tilde{c} = c - 2 s_2$.

Observe that, under the given hypotheses, $\tilde{c} = 1 - 2 \sqrt{D} > -1 + \alpha_2$ and the claim follows by Theorem 7.1.

We point out that in the range $s_2 + \alpha_1 < \frac{m+1}{p} < s_1 + 2 - \alpha_2$ the operators $L^{\alpha_1,\alpha_2}_{m,p}$ of Theorem 8.2 and $(\mathcal{L}, D(\mathcal{L}))$ just constructed are different. In fact let $f = a(x)b(y) \in C^\infty_c(\mathbb{R}^N) \times \mathcal{D}$ a function in the core defined in (11). Then $u = y^{-s_1} f$ belongs to $D(L^{\alpha_1,\alpha_2}_{m,p})$ but not to $D(\mathcal{L})$ since $\lim_{y \to 0} y^{s_1+1} \left( D_y u + s_2 \frac{u_y}{y} \right) = s_2 - s_1 > 0$.

### 9.3 Baouendi-Grushin operator

Our results apply to generalized Baouendi-Grushin operators

$$\mathcal{L} = y^{\alpha} \Delta_x + L_y, \quad \alpha > -2$$

in the half space $\mathbb{R}^{N+1}$ both with Neumann and Dirichlet boundary conditions, but we restrict ourselves to the classical case $\mathcal{L} = y^{\alpha} \Delta_x + D_{yy}$ in the whole space $\mathbb{R}^{N+1}$ with the Lebesgue measure. Our results improve those from [19], allowing negative $\alpha$ and showing maximal regularity, besides domain characterization.
Proposition 9.12 Let $\alpha > -\frac{1}{p}$. Then $\mathcal{L} = |y|^\alpha \Delta_x + D_{yy}$ with domain

$$D(\mathcal{L}) = \{ u \in W^{2,p}_{\text{loc}}(\mathbb{R}^{N+1}) : u, y^\alpha D_x u, y^2 D_{xx} u, \ldots \in L^p(\mathbb{R}^{N+1}) \}$$

generates a bounded analytic semigroup in $L^p(\mathbb{R}^{N+1})$ which has maximal regularity.

PROOF. By Theorems 7.1, 8.2 the operator $\mathcal{L}$ generates an analytic semigroup in $L^p(\mathbb{R}^{N+1})$ both with Dirichlet and Neumann boundary conditions. We can therefore consider the operators $\mathcal{L}_i$, $i = 1, 2$, that is $\mathcal{L}$ with domains

$$D_1 = \{ u \in W^{2,p}_{\text{loc}}(\mathbb{R}^{N+1}) : u, y^\alpha D_x u, y^2 D_{xx} u, \ldots \in L^p(\mathbb{R}^{N+1}), \lim_{y \to 0} u(x, y) = 0 \}$$

$$D_2 = W^{2,p}_N(\alpha, 0, 0)
= \{ u \in W^{2,p}_{\text{loc}}(\mathbb{R}^{N+1}) : u, y^\alpha D_x u, y^2 D_{xx} u, \ldots \in L^p(\mathbb{R}^{N+1}), \lim_{y \to 0} D_y u(x, y) = 0 \}.$$  

The mixed derivatives estimates follows from Theorem 7.1 and Corollary 8.6. Let $P_1, P_2 : L^p(\mathbb{R}^{N+1}) \to L^p(\mathbb{R}^{N+1})$ be the even and odd projections

$$(P_1 f)(x, y) = \frac{f(x, y) + f(x, -y)}{2}, \quad (P_2 f)(x, y) = \frac{f(x, y) - f(x, -y)}{2}$$

and $E_1, E_2 : L^p(\mathbb{R}^{N+1}) \to L^p(\mathbb{R}^{N+1})$ the even and odd extensions

$$E_1 u(x, y) = \begin{cases} u(x, y), & \text{if } y > 0; \\ u(x, -y), & \text{if } y < 0; \end{cases}$$

$$E_2 u(x, y) = \begin{cases} u(x, y), & \text{if } y > 0; \\ -u(x, -y), & \text{if } y < 0. \end{cases}$$

Note that $E_1 P_1 + E_2 P_2 = I_{L^p(\mathbb{R}^{N+1})}$, $P_i(D(\mathcal{L})) \subset D_i$, $E_i(D_i) \subset D(\mathcal{L})$, $i = 1, 2$ and that $D(\mathcal{L}) = D_1 \oplus D_2$ algebraically and topologically with respect to the Sobolev norm. Then $\mathcal{L} = E_1 \mathcal{L}_1 P_1 + E_2 \mathcal{L}_2 P_2$ and everything follows from the properties of $\mathcal{L}_1, \mathcal{L}_2$. \qed

9.4 The operator $y \Delta_x + y B_y$

We specialize and comment here the results obtained in the special case $\alpha = 1$, that is for $\mathcal{L} = y \Delta_x + y B_y = y \Delta_x + y D_{yy} + c D_y$ where $B_y = D_{yy} + \frac{c}{y} D_y$.

Theorem 7.1 applies when $0 < \frac{m+1}{p} < c$ and yields generation and all other properties listed therein for $y \Delta_x + y B_y$ with domain

$$W^{2,p}_N(1, 1, m) = \{ u \in W^{2,p}_{\text{loc}}(\mathbb{R}^{N+1}) : u, \nabla_x u, D_y u, y D_x u, y D_{xx} u, \ldots \in L^p_m \},$$

29
using also Proposition 4.9 for $\nabla_x u$.

This result has been already proved in [14] but also in [28] when $c \geq 1$ and $m = 0$ and in [10] when $p = 2$, $m = 0$ (and $c > 4$).

Note that, when $m = 0$, then

$$W^2_p(1, 1, 0) = \{ u \in W^{1, p}(\mathbb{R}_+^N) : yD_{x,y}u, yD_{x,y}u \in L^p(\mathbb{R}_+^N) \}$$

and the associated elliptic and parabolic problems seem to have no boundary condition. In our approach, the Neumann boundary condition is indeed imposed to $yD_y u$, by requiring that

$$\frac{1}{p}(yD_y u) \in L^p(\mathbb{R}_+^N).$$

Theorem 8.2 says nothing new when $c \geq 1$, since then $s_1 = 0$ and the transformation $T_{-s_1,0}$ is the identity. However, when $c < 1$ then $s_1 = c - 1, s_2 = 0$ and Theorem 8.2 yields a different operator $L^1_m$ in the range $c - 1 < \frac{m+1}{p} < 1$. Its domain is

$$\{ u \in W^{2,p}(\mathbb{R}_+^N) : u, yD_{x,y}u, yD_{x,y}u + cD_y u \in L^p_m \text{ and } \lim_{y \to 0} u(x, y) = 0 \}$$

by Corollary 8.3(i) and Corollary 8.6 for the mixed derivative. However, it is not true that $yD_{x,y}u$ and $D_y u$ belong to $L^p_m$ separately, even when $m = 0$, see also [11]. On the other hand, when $c < \frac{m+1}{p} < 1$, then Corollary 8.5 applies and gives

$$D(L^1_{m,p}) = W^2_p(1, 1, m).$$

In particular, if $m = 0$, it follows that for $c < \frac{1}{p}$

$$D(L^1_{0,p}) = \{ u \in W^{1, p}(\mathbb{R}_+^N) : yD_{x,y}u, yD_{x,y}u \in L^p(\mathbb{R}_+^N) \},$$

a result already proved in [8].

Finally, let us specialize the results of Section 9.2, see also [9]. If $c < 1$

(i) $L^1_{m,p} = L^{\min}_{m,p}$ when $c \leq (m + 1)/p < 1$; $L^1_{m,p} = L^{\max}_{m,p}$ when $c - 1 < (m + 1)/p \leq 0$;

(ii) uniqueness fails if and only if $0 < (m + 1)/p < c$.

Instead, if $c \geq 1$

(i) $L^1_{m,p} = L^{\min}_{m,p}$ when $1 \leq (m + 1)/p < c$; $L^1_{m,p} = L^{\max}_{m,p}$ when $0 < (m + 1)/p \leq c - 1$;

(ii) uniqueness fails if and only if $c - 1 < (m + 1)/p < 1$.

References

[1] ARENDT, W. The abstract cauchy problem, special semigroups and perturbation. In One-parameter Semigroups of Linear Operators, R. Nagel, Ed., vol. 1184 of Lecture Notes in Mathematics. Springer, 1980.

[2] CALVARUSO, G., METAFUNE, G., NEGRO, L., AND SPINA, C. Optimal kernel estimates for elliptic operators with second order discontinuous coefficients. Journal of Mathematical Analysis and Applications 485, 1 (2020), 123763.
[3] Denk, R., Hieber, M., and Prüss, J. \textit{R-boundedness, Fourier multipliers and problems of elliptic and parabolic type}, vol. 166 (n.788) of Memoirs of the American Mathematical Society. Amer. Math. Soc., 2003.

[4] Dong, H., and Phan, T. On parabolic and elliptic equations with singular or degenerate coefficients, 2020. arxiv: 2007.04385.

[5] Dong, H., and Phan, T. Weighted mixed-norm $L_p$ estimates for equations in non-divergence form with singular coefficients: the dirichlet problem, 2021. arxiv: 2103.08033.

[6] Engel, K. J., and Nagel, R. \textit{One parameter semigroups for linear evolution equations}. Springer-Verlag, Berlin, 2000.

[7] Fornaro, S., Metafune, G., and Pallara, D. Analytic semigroups generated in lp by elliptic operators with high order degeneracy at the boundary. \textit{Note Mat.} 31, 1 (2011), 103–115.

[8] Fornaro, S., Metafune, G., Pallara, D., and Prüss, J. Lp-theory for some elliptic and parabolic problems with first order degeneracy at the boundary. \textit{Journal de Mathématiques Pures et Appliquées} 87, 4 (2007), 367–393.

[9] Fornaro, S., Metafune, G., Pallara, D., and Schnaubelt, R. One-dimensional degenerate operators in lp-spaces. \textit{Journal of Mathematical Analysis and Applications} 402, 1 (2013), 308–318.

[10] Fornaro, S., Metafune, G., Pallara, D., and Schnaubelt, R. Second order elliptic operators in $L^2$ with first order degeneration at the boundary and outward pointing drift. \textit{Communications on Pure & Applied Analysis} 14, 2 (2015), 407–419.

[11] Fornaro, S., Metafune, G., Pallara, D., and Schnaubelt, R. Multi-dimensional degenerate operators in $L^p$–spaces, 2021. preprint.

[12] Hytönen, T., Van Neerven, J., Veraar, M., and Weis, L. \textit{Analysis in Banach Spaces, Vol. I: Martingales and Littlewood-Paley Theory}. Springer, 2016.

[13] Hytönen, T., Van Neerven, J., Veraar, M., and Weis, L. \textit{Analysis in Banach Spaces, Vol. II: Probabilistic Methods and Operator Theory}. Springer, 2017.

[14] Koch, H. Non-euclidean singular integrals and the porous medium equation, 1999. Habilitation thesis: www.math.uni-bonn.de/~koch/public.html.

[15] Kunstmann, P. C., and Weis, L. Maximal $L^p$-regularity for parabolic equations, fourier multiplier theorems and $H^\infty$-functional calculus. In Iannelli M., Nagel R., Piazzera S. (eds) \textit{Functional Analytic Methods for Evolution Equations}, vol. 1855 of Lecture Notes in Mathematics. Springer, Berlin, 2004.

[16] Metafune, G., Negro, L., Sobajima, M., and Spina, C. Rellich inequalities in bounded domains. \textit{Mathematische Annalen} 379 (2021), 765–824.

[17] Metafune, G., Negro, L., and Spina, C. Sharp kernel estimates for elliptic operators with second-order discontinuous coefficients. \textit{Journal of Evolution Equations} 18 (2018), 467–514.

[18] Metafune, G., Negro, L., and Spina, C. Gradient estimates for elliptic operators with second-order discontinuous coefficients. \textit{Mediterranean Journal of Mathematics} 16, 138 (2019).
[19] Metafune, G., Negro, L., and Spina, C. $L^p$ estimates for Baouendi–Grushin operators. *Pure and Applied Analysis* 2, 3 (2020), 603–625.

[20] Metafune, G., Negro, L., and Spina, C. Maximal regularity for elliptic operators with second-order discontinuous coefficients. *Journal of Evolution Equations* (2020).

[21] Metafune, G., Negro, L., and Spina, C. $L^p$ estimates for the Caffarelli-Silvestre extension operators. *Submitted* (2021). Online preprint: https://arxiv.org/abs/2103.10314v1.

[22] Metafune, G., Negro, L., and Spina, C. Anisotropic sobolev spaces with weights. *Submitted* (2022). Online preprint on arxiv.

[23] Metafune, G., Negro, L., and Spina, C. Degenerate operators on the half-line. *Submitted* (2022). Online preprint on arxiv.

[24] Metafune, G., Okazawa, N., Sobajima, M., and Spina, C. Scale invariant elliptic operators with singular coefficients. *Journal of Evolution Equations* 16, 2 (Jun 2016), 391–439.

[25] Metafune, G., Sobajima, M., and Spina, C. Weighted Calderón–Zygmund and Rellich inequalities in $L^p$. *Mathematische Annalen* 361, 1 (Feb 2015), 313–366.

[26] Metafune, G., Sobajima, M., and Spina, C. Elliptic and parabolic problems for a class of operators with discontinuous coefficients. *Annali SNS XIX* (2019), 601–654.

[27] Negro, L., and Spina, C. Asymptotic behaviour for elliptic operators with second-order discontinuous coefficients. *Forum Mathematicum* 32, 2 (2020), 399–415.

[28] Prüss, J. On second-order elliptic operators with complete first-order boundary degeneration and strong outward drift. *Archiv der Mathematik* 108, 3 (Mar 2017), 301–311.

[29] Prüss, J., and Simonett, G. *Moving Interfaces and Quasilinear Parabolic Evolution Equations*, vol. 105. Springer-Verlag, 2016.

[30] Robinson, D. W., and Sikora, A. Analysis of degenerate elliptic operators of Grushin type. *Mathematische Zeitschrift* 260, 3 (Nov 2008), 475–508.

[31] Robinson, D. W., and Sikora, A. The limitations of the Poincaré inequality for Grušin type operators. *Journal of Evolution Equations* 14, 3 (Sep 2014), 535–563.