RECONSTRUCTION OF CENTRALLY SYMMETRICAL CONVEX BODIES
BY PROJECTION CURVATURE RADII

R. H. Aramyan

The article considers the problem of existence and uniqueness of centrally symmetrical convex body
for which the projection curvature radius function coincides with a given flag function. A necessary
and sufficient condition is found that ensures a positive answer. An algorithm for construction the
body in question is proposed.

§1. INTRODUCTION

Let $F(\omega)$ be a function defined on the sphere $S^2$. The existence and uniqueness of convex body $B \subset \mathbb{R}^3$ for which
the mean curvature radius at a point on $\partial B$ with normal direction $\omega$ coincides with given $F(\omega)$ was posed by
Christoffel (see [4]). Let $R_1(\omega)$ and $R_2(\omega)$ be the principle curvature radii of the surface of the body at the point
with normal $\omega \in S^2$. The Christoffel problem asked about the existence of $B$ for which $R_1(\omega) + R_2(\omega) = F(\omega)$.
The corresponding problem for Gauss curvature $R_1(\omega)R_2(\omega) = F(\omega)$ was posed and solved by Minkovski. Blashke
reduced the Christoffel problem to a partial differential equation of second order for support function (see [4]).
Aleksandrov and Pogorelov generalized these problems, and proved the existence and uniqueness of a convex body
for which

$$G(R_1(\omega), R_2(\omega)) = F(\omega),$$

for a class of symmetric functions $G$ (see [4], [6]).

In this paper we consider a similar problem posed for the projection curvature radii of centrally symmetrical convex
bodies (see [2]). By $\mathcal{B}_o$ we denote the class of convex bodies $B \subset \mathbb{R}^3$ that have a center of symmetry at the origin
$O \in \mathbb{R}^3$. We use the notation:

- $S^2$ – the unit sphere in $\mathbb{R}^3$ (the space of spatial directions),
- $S_\omega \subset S^2$ – the great circle with pole at $\omega \in S^2$,
- $e(\omega, \psi)$ – the plane containing the origin of $\mathbb{R}^3$ and the directions $\omega \in S^2$ and $\psi \in S_\omega$,
- $B(\omega, \psi)$ – projection of $B \in \mathcal{B}_o$ onto $e(\omega, \psi),$
Let $F(\omega, \psi)$ be a symmetric function defined on the space of "flags" $\{(\omega, \psi) : \omega \in S^2, \psi \in S_{\omega}\}$ (see [1]). We pose the problem of existence and uniqueness of a convex body for which

$$R(\omega, \psi) = F(\omega, \psi),$$

and find a necessary and sufficient condition on $F(\omega, \psi)$ that ensures a positive answer. Note, that uniqueness (up to parallel shifts) follows from the classical uniqueness result on Christoffel problem.

Now we describe the main result. Let $F(\omega, \psi)$ be a function defined on the $\{(\omega, \psi) : \omega \in S^2, \psi \in S_{\omega}\}$. We say that the function $F(\omega, \psi)$ is symmetric if $F(\omega, \psi) = F(\omega, \pi + \psi) = F(-\omega, \psi)$. We define

$$\overline{F}(\omega) = \frac{1}{\pi} \int_0^{2\pi} F(\omega, \psi) d\psi.$$  \hfill (1.3)

Below we use the usual spherical coordinates $\nu, \phi$ on $S^2$ based on a choice $\omega$ for the North Pole and a choice of a reference point $\phi = 0$ on the equator $S_{\omega}$ (so points $(0, \phi)$ lie on the equator $S_{\omega}$). The point with coordinates $\nu, \phi$ in that coordinates system we will denote by $(\nu, \phi)_{\omega}$.

**Theorem 1.1.** A nonnegative symmetric continuously differentiable function $F(\omega, \psi)$ defined on the space $\{(\omega, \psi) : \omega \in S^2, \psi \in S_{\omega}\}$ represents the projection curvature radius of some convex body if and only if

$$F(\omega, \psi) = \frac{\overline{F}(\omega)}{2} - \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\pi} \frac{\overline{F}(\nu, \phi)_{\omega} \cos 2(\phi - \psi)}{\cos \nu} d\phi d\nu$$  \hfill (1.4)

for all $\omega \in S^2$ and all $\psi \in S_{\omega}$ (integration order is important).

Our proof of Theorem 1.1 suggests an algorithm of construction of the convex body $B$ for given $F(\omega, \psi)$. We will need the following facts from the convexity theory.

§2. **PRELIMINARIES**

It is well known (see [8]) that the support function of every sufficiently smooth $B \in B_o$ has the unique representation

$$H(\xi) = \int_{S^2} |(\xi, \Omega)| h(\Omega) d\Omega,$$  \hfill (2.1)

where $d\Omega$ is the usual area measure on $S^2$, $h(\Omega)$ is an even continuous function (not necessarily nonnegative) called the generating density of $B$. 


Below we will use the following result by N. F. Lindquist, see [8].

An even continuous function $h(\Omega)$ defined on $S^2$ is the generating density of a body $B \in B_o$ if and only if

$$\int_{S_\omega} \cos^2(\psi, \varphi) h_\omega(\varphi) \, d\varphi \geq 0 \quad (2.2)$$

for all $\omega \in S^2$ and $\psi \in S_\omega$.

Remarkably, the integral (2.2) has a clear geometrical interpretation.

In [2] it was proved that for any sufficiently smooth $B \in B_o$ and $\psi \in S_\omega$ the projection curvature radius can be calculated as

$$R(\omega, \psi) = 2 \int_{S_\omega} \cos^2(\psi, \varphi) h_\omega(\varphi) \, d\varphi, \quad (2.3)$$

where $(\psi, \varphi)$ is the angle between $\psi$ and $\varphi$, while $h_\omega(\varphi)$ is the restriction of $h(\Omega)$ of $B$ on $S_\omega$.

In Blaschke’s book [5] one can find the following representation of the generating density. For any $\Omega \in S^2$

$$2\pi h(\Omega) = \frac{1}{4\pi} \int_{S^2} (R_1 + R_2) \, d\tau - \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi (R_1((u, \tau)_{\Omega}) + R_2((u, \tau)_{\Omega}))^\prime \frac{1 - \sin u}{\sin u} \, du \, d\tau, \quad (2.4)$$

where $R_i((u, \tau)_{\Omega}), i = 1, 2$ are the principal curvature radii of $B$ at the point with normal $(u, \tau)_{\Omega}$ (which has the spherical coordinates $u, \tau$ with respect $\Omega$)

§3. PROOF OF THEOREM 1.1 AND CONSTRUCTION OF CONVEX BODY

Proof. Necessity: let $R(\omega, \psi)$ be the projection curvature radius of some convex body $B \in B_o$. We have to prove that $R(\omega, \psi)$ satisfies the condition (1.4). It follows from [5] that

$$\overline{R}(\omega) = \frac{1}{\pi} \int_0^{2\pi} R(\omega, \psi) \, d\psi = R_1(\omega) + R_2(\omega) \quad (3.1)$$

and using Fubini Theorem one can rewrite the expression (2.4) in the form

$$8\pi^2 h(\Omega) = \lim_{a \to 0} \left[ \frac{1}{\sin a} \int_0^{2\pi} \overline{R}((a, \tau)_{\Omega}) \, d\tau - \int_a^\pi \int_0^{2\pi} \overline{R}((u, \tau)_{\Omega}) \cos u \frac{\cos u}{\sin^2 u} \, du \, d\tau \right], \quad (3.2)$$

where $(u, \tau)_{\Omega}$ -is the point on $S^2$ with usual spherical coordinates $u, \tau$ with respect to $\Omega$. Taking some $\psi \in S_\omega$ for the reference point on $S_\omega$ and substituting the new expression for the generating density from (3.2) into (2.3) we get

$$R(\omega, \psi) = 2 \int_{S_\omega} \cos^2 \varphi h_\omega(\varphi) \, d\varphi =$$
\[
\lim_{a \to 0} \frac{1}{4\pi^2} \int_0^{2\pi} \cos^2 \varphi \left[ \frac{1}{\sin a} \int_0^{2\pi} R((a, \tau)_{\Omega}) \, d\tau - \int_0^{\frac{\pi}{2}} \int_a^{\frac{\pi}{2}} R((u, \tau)_{\Omega}) \frac{\cos u}{\sin^2 u} \, du \, d\varphi \right] \, d\varphi, \tag{3.3}
\]

where \( \Omega = (0, \varphi)_{\omega} \).

For \( u \in (0, \frac{\pi}{2}) \) and \( \varphi \in S_{\omega} \) we denote by \( S(u, \varphi) \) the circle with center at \( \varphi \in S_{\omega} \) and of radius \( \sin u \). The meridian passing through the point \( \omega \) and \( (0, \varphi)_{\omega} \) divides \( S(u, \varphi) \) into two halfcircles. When \( \varphi \) changes on \( S_{\omega} \), each halfcircle of \( S(u, \varphi) \) subtend the set \([0, 2\pi) \times [-u, u] \subset S^2 \). So on the latter set we have two parametrizations \((\nu, \phi)\) and \((\tau, \phi)\). One can prove the Jacobian relation

\[
d\tau \, d\varphi = \frac{\cos \nu}{\sqrt{\cos^2 u - \sin^2 \nu}} \, d\nu \, d\phi. \tag{3.4}
\]

Writing (3.3) as a sum of terms corresponding to the halfcircles of \( S(u, \varphi) \) and applying a change of variables (3.4), we obtain

\[
R(\omega, \psi) = \lim_{a \to 0} \frac{1}{4\pi^2} \left[ \int_0^{2\pi} \int_{-\frac{\pi}{2} + a}^{\frac{\pi}{2} - a} R((\nu, \phi)_{\omega}) \left[ \int_0^{\nu} \cos^2 (\phi - \beta) + \cos^2 (\phi + \beta) \right] \frac{\cos \nu}{\sqrt{\cos^2 a - \sin^2 \nu}} \, d\nu \, d\phi - \int_0^{2\pi} \int_{-\frac{\pi}{2} + a}^{\frac{\pi}{2} - a} R((\nu, \phi)_{\omega}) \left[ \int_0^{\nu} \cos^2 (\phi - \beta) + \cos^2 (\phi + \beta) \right] \frac{\cos \nu}{\sqrt{\cos^2 u - \sin^2 \nu}} \, d\nu \, d\phi \right], \tag{3.5}
\]

where \( \beta = \phi - \varphi \). We have \( \cos \beta = \frac{\sin u}{\cos \nu} \) and \( \sin \beta = \sqrt{\frac{\cos^2 \nu - \sin^2 u}{\cos \nu}} \). After some standard integral calculation (see [7]) and using symmetry we find

\[
R(\omega, \psi) = \lim_{a \to 0} \frac{1}{\pi^2} \left[ \int_0^{2\pi} \int_{-\frac{\pi}{2} + a}^{\frac{\pi}{2} - a} R((\nu, \phi)_{\omega}) \frac{\cos^2 \phi}{\cos \nu \sqrt{\cos^2 a - \sin^2 \nu}} \, d\nu \, d\phi - \int_0^{2\pi} \int_{-\frac{\pi}{2} + a}^{\frac{\pi}{2} - a} R((\nu, \phi)_{\omega}) \cos 2\phi \frac{\pi}{2} \arcsin \frac{\sin a}{\cos \nu} \, d\nu \, d\phi \right]. \tag{3.6}
\]

Calculating the limit requires decomposition of the integrals in the powers \( a \). The negative powers annihilate and we get (1.4).

Sufficiently: let \( F(\omega, \psi) \) be a nonnegative symmetric continuous differentable function satisfying the condition (1.4).

We consider \( F(\omega) \) (see (1.3)) and by means of (2.4) construct the function \( f(\Omega) \) defined on \( S^2 \):

\[
f(\Omega) = \frac{1}{8\pi^2} \int_{S^2} F((0, \tau)_{\Omega}) \, d\tau - \frac{1}{8\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} F((u, \tau)_{\Omega}) \frac{1 - \sin u}{\sin u} \, du \, d\tau. \tag{3.7}
\]

According to N. F. Lindquist, \( f(\Omega) \) has to satisfy the condition (2.2) to be the generating density of a body. Substituting (3.7) into (2.2) and applying the same procedure as above, by (1.4) we obtain \( F(\omega, \psi) \) which
is nonnegative by assumption. Hence $h(\Omega)$ is the generating function of $B$ and according (2.3), $F(\omega, \psi)$ is the projection curvature radius of $B$.

The author hopes to present soon the results on similar problem that do not depend on the assumption of central symmetry of the convex body $B$.

I would like to express my gratitude to Professor R. V. Ambartzumian for helpful discussions.

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Institute of Mathematics
Armenian Academy of Sciences
e.mail: rafik@instmath.sci.am