GENERIC TRANSFER FOR GENERAL SPIN GROUPS

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Abstract. We prove Langlands functoriality for the generic spectrum of general spin groups (both odd and even). Contrary to other recent instances of functoriality, our resulting automorphic representations on the general linear group will not be self-dual. Together with cases of classical groups, this completes the list of cases of split reductive groups whose $L$-groups have classical derived groups. The important transfer from $GSp_4$ to $GL_4$ follows from our result as a special case.

1. Introduction

Let $G$ be a connected reductive group over a number field $k$. Let $G = G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of $k$. Let $L^G$ denote the $L$-group of $G$ and fix an embedding

$$i : L^G \hookrightarrow GL_N(\mathbb{C}) \times W(\overline{k}/k),$$

where $W(\overline{k}/k)$ is the Weil group of $k$. Without loss of generality we may assume $N$ to be minimal. Let $\pi = \otimes'_v \pi_v$ be an automorphic representation of $G$. Then for almost all $v$, the local representation $\pi_v$ is an unramified representation and its class is determined by a semisimple conjugacy class $[t_v]$ in $L^G$. Here $v$ is a finite place of $k$. Let $\Pi_v$ be the unramified representation of $GL_N(k_v)$ determined by the conjugacy class $[i(t_v)]$ generated by $i(t_v)$. Langlands’ functoriality conjecture then demands the existence of an automorphic representation $\Pi' = \otimes'_v \Pi'_v$ of $GL_N(\mathbb{A})$ such that $\Pi'_v \simeq \Pi_v$ for all the unramified places $v$. In this paper we prove functoriality in the cases where $G$ is not classical but the derived group $LG^0$ of the connected component of its $L$-group is. (We follow the convention that a classical group is the stabilizer of a symplectic, orthogonal, or hermitian non-degenerate bilinear form. Hence, for example, spin groups would not be considered classical.)

As we explain later, a major difficulty in establishing this result is the absence of any useful matrix representation when the groups themselves are not classical, the subject matter of the present paper, forcing us to use rather complicated abstract structure theory in order to prove stability of the corresponding root numbers.

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We shall be mainly concerned with quasi-split groups and those automorphic representations which are induced from generic cuspidal ones. The problem clearly reduces to establishing functoriality for generic cuspidal representations of $G = G(\mathbb{A})$.

The cases when $G$ is a quasi-split classical group were addressed in [5, 6, 21], unless $G$ is a quasi-split special orthogonal group which should be taken up by the authors of [6].

In this paper we will prove the functorial transfer of generic cuspidal representations when $G = \text{GSpin}^*_{2n}$, the split general spin group of semisimple rank $n = [m/2]$. We describe their structure in detail in Section 2.1. In particular, these are split reductive linear algebraic groups of type $B_n$ or $D_n$ whose derived groups are double coverings of split special orthogonal groups. Moreover, the connected component of their Langlands dual groups are $L_{G^0} = \text{GSp}_{2n}(\mathbb{C})$ or $\text{GSO}_{2n}(\mathbb{C})$, respectively. Then, $L_{G^0} = \text{GSO}_{2n}(\mathbb{C}) \rtimes \text{W}(k/k)$ or $\text{GSp}_{2n}(\mathbb{C}) \rtimes \text{W}(k/k)$ according to whether $m$ is even or odd. The map $\iota$ is the natural embedding. Observe that $L_{G^0}$ is now a classical group and these groups are precisely the ones for which $G^0$ is not classical but $L_{G^0}$ is. The transfer is to the space of automorphic representations of $\text{GL}_{2n}(\mathbb{A})$.

It is predicted by the theory of (twisted) endoscopy of Kottwitz, Langlands, and Shelstad [25, 28] that the representations of $\text{GL}_{2n}(\mathbb{A})$ which are in the image of this transfer must be of the form

$$\Pi = \tilde{\Pi} \otimes \omega$$

for some grössencharacter $\omega$. If $\omega_{\Pi}$ is the central character of $\Pi$, this implies that $\omega_{\Pi}/\omega_{\pi}$ must be a quadratic character $\mu$ of $k^\times \backslash \mathbb{A}^\times$. Each $\mu$ then determines a quadratic extension of $k$ via class field theory and the group $G$ which has transfers to automorphic representations of the type just mentioned would be the quasi-split form $\text{GSpin}^*_{2n}$ of $\text{GSpin}_{2n}$ associated to the quadratic extension. The split case then corresponds to $\mu \equiv 1$ which is the content of the present paper.

If a representation $\pi$ of $\text{GSpin}^*_{2n}(\mathbb{A})$ transfers to $\Pi$ on $\text{GL}_{2n}(\mathbb{A})$ satisfying $\Pi \simeq \tilde{\Pi} \otimes \omega$ for some grössencharacter $\omega$, then $\omega = \omega_{\pi}$ and $\omega_{\Pi} = \omega_{\pi}^{\mu}$, where $\omega_{\pi}$ and $\omega_{\Pi}$ denote the central characters of $\pi$ and $\Pi$, respectively, and $\mu$ is a quadratic grössencharacter associated with the quasi-split $\text{GSpin}^*_{2n}$. While we are not able to show that every $\Pi$ satisfying (1) is transfer of an automorphic representation $\pi$, we show that our transfers satisfy (1). (In fact, we will prove that $\Pi$ is nearly equivalent to $\tilde{\Pi} \otimes \omega$ for now. See Theorem 1.1)

We should note here that if $\Pi$ is an automorphic transfer to $\text{GL}_{2n+1}(\mathbb{A})$ satisfying (1), then $\omega = \theta^2$ for some $\theta$ and $\Pi \otimes \theta^{-1}$ is then self-dual. Therefore, this is already subsumed in the self-dual case which is a case of standard twisted endoscopy. On the other hand, the case of $\text{GL}_{2n}(\mathbb{A})$ discussed above
is an example of the most general form of transfer that twisted endoscopy can handle.

As explained earlier, in this paper we prove Langlands’ functoriality conjecture in the form discussed, for all generic cuspidal representations of split $\text{GSpin}_m(\mathbb{A})$. In other words, we establish generic transfer from $\text{GSpin}_m(\mathbb{A})$ to $\text{GL}_{2n}(\mathbb{A})$. Extension of this transfer to the non-generic case would require either the use of models other than Whittaker models or of Arthur’s twisted trace formula. As far as we know new models for these groups have not been developed and the fact that these groups are not classical may make the matters complicated. On the other hand, the use of Arthur’s twisted trace formula is at present depending on the validity of the fundamental lemmas which are not available for these groups. We refer to [1] for information on the case of $\text{GSp}_4$.

To state our main theorem, fix a Borel subgroup $B$ in $G$ and a maximal (split) torus $T$ in $B$, and denote the unipotent radical of $B$ by $U$. Let $\psi$ be a non-trivial continuous additive character of $k \setminus \mathbb{A}$. As usual, we use $\psi$ and a fixed splitting (i.e., the choice of Borel above along with a collection of root vectors, one for each simple root of $T$, cf. page 13 of [25], for example) to define a non-degenerate additive character of $U(k) \setminus U(\mathbb{A})$, again denoted by $\psi$. (Also see Section 2 of [38]).

Let $(\pi, V_\pi)$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. The representation $\pi$ is said to be globally generic if there exists a cusp form $\phi \in V_\pi$ such that

$$\int_{U(k) \setminus U(\mathbb{A})} \phi(ng)\psi^{-1}(n)dn \neq 0.$$  

(2)

Note that cuspidal automorphic representations of general linear groups are always globally generic. Two irreducible automorphic representations $\Pi$ and $\Pi'$ of $\text{GL}_N(\mathbb{A})$ are said to be nearly equivalent if there is a finite set of places $T$ of $k$ such that $\Pi_v \simeq \Pi'_v$ for all $v \not\in T$. Our main result is the following:

**Theorem 1.1.** Let $k$ be a number field and let $\pi = \otimes' \pi_v$ be an irreducible globally generic cuspidal automorphic representation of either $\text{GSpin}_{2n+1}(\mathbb{A})$ or $\text{GSpin}_{2n}(\mathbb{A})$. Let $S$ be a non-empty finite set of non-archimedean places $v$ such that for $v \not\in S$ we have that $\pi_v$ and $\psi_v$ are unramified. Then, there exists an automorphic representation $\Pi = \otimes' \Pi_v$ of $\text{GL}_{2n}(\mathbb{A})$ such that for all archimedean $v$ and all non-archimedean $v$ with $v \not\in S$ the homomorphism parameterizing the local representation $\Pi_v$ is given by

$$\Phi_v = \iota \circ \phi_v : W_v \rightarrow \text{GL}_{2n}(\mathbb{C}),$$

where $W_v$ denotes the local Weil group of $k_v$ and $\phi_v : W_v \rightarrow L^G(0)$ is the homomorphism parameterizing $\pi_v$. Moreover, if $\omega_{\Pi}$ and $\omega_{\pi}$ denote the central characters of $\Pi$ and $\pi$, respectively, then $\omega_{\Pi} = \omega_{\pi}^n$. Furthermore, if $v$ is an
archimedean place or a non-archimedean place with \( v \notin S \), then \( \Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v} \).

In particular, the representations \( \Pi \) and \( \tilde{\Pi} \otimes \omega_{\pi} \) are nearly equivalent.

At the non-archimedean places \( v \) where \( \pi_v \) is unramified with the semisimple conjugacy class \([t_v]\) in \( LG^0 \) as its Frobenius-Hecke (or Satake) parameter, this amounts to the fact that the local representation \( \Pi_v \) is the unramified irreducible admissible representation determined by the conjugacy class generated by \( \iota(t_v) \) in \( GL_2n(C) \).

Our method of proof is that of applying an appropriate version of converse theorem \([7, 9]\) to a family of \( L \)-functions whose required properties, except for one, are proved in \([36, 13, 22, 18]\). The exception, i.e., the main stumbling block for applying the converse theorem, is that of stability of certain root numbers under highly ramified twists. In \([38]\) the root numbers, or more precisely the local coefficients, were expressed as a Mellin transform of certain Bessel functions. Applying this to our case requires a good amount of development and calculations. This is particularly important since the necessary Bruhat decomposition of these groups are more complicated than the classical groups. For that we have to resort to the use of abstract theory of roots which are harder since no reasonable matrix representation is available for these groups. Moreover, the main theorem in \([38]\) is based on certain assumptions whose verification requires our calculations.

The fact that \( GSpin_{2n} \) has a disconnected center makes matters even more complicated. This led us to use an extended group \( GSpin^\sim_{2n} \) of \( GSpin_{2n} \) so that our proof of stability proceeds smoothly.

There are two important transfers that are special cases of this theorem. The first is the generic transfer from \( GSp_4 = GSpin_5 \) to \( GL_4 \) whose proof, as far as we know, has never been published before. We should point out that even the unpublished proofs of this result are based on methods that are fairly disjoint from ours. We finally remark that our result in this case also gives an immediate proof of the holomorphy of spinor \( L \)-functions for generic cusp forms on \( GSp_4 \) (cf. Remark \([24]\)).

The second special case is when \( G = GSpin_6 \). In this case our transfer gives the exterior square transfer from \( GL_4 \) to \( GL_6 \) due to H. Kim \([20]\) which, when composed with symmetric cube of a cuspidal representation on \( GL_2(\mathbb{A}) \), leads to its symmetric fourth.

The issue of strong transfer, which has been successfully treated in the cases of classical groups thanks to existence of descent from \( GL_n \) to classical groups \([14, 39]\), still needs to wait until the descent or other techniques are established for representations of \( GL_{2n}(\mathbb{A}) \) which satisfy \([1]\).

Further applications such as non-local estimates towards the Ramanujan conjecture as well as some of the other applications established in \([5, 6]\) will
be addressed in future papers. As mentioned earlier, the cases of quasi-split GSpin groups is the subject matter of our next paper.

Here is an outline of the contents of each section. In Section 2 we review the structure theory of the groups involved in this paper. In particular, we give detailed description of the root data for GSpin groups and their extensions. We then prove the necessary analytic properties of local $L$-functions in Section 3. In particular, we discuss standard module conjecture which is another local ingredient. In Section 4 we go on to prove the most crucial local result, i.e., stability of $\gamma$-factors under twists by highly ramified characters. This is where we do the calculations with root data mentioned above and use the extended group. We then prove the necessary analytic properties of the global $L$-functions in Section 5 which will prepare us to apply the converse theorem in Section 6. In Section 7 we include the special cases mentioned above along with some other local and global consequences of the main theorem.

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2. Structure Theory

We review the structure theory for the families of algebraic groups relevant to the current work, namely, GSpin\(_{2n+1}^\sim\) and GSpin\(_{2n}^\sim\) as well as their duals GSp\(_{2n}\) and GSO\(_{2n}\). We will also introduce the group GSpin\(_{2n}^\sim\) which is closely related to GSpin\(_{2n}\). It shares the same derived group as that of GSpin\(_{2n}\). However, contrary to GSpin\(_{2n}\) which has disconnected center, the center of GSpin\(_{2n}^\sim\) is connected. We will need this group for our purposes as we will explain later.

2.1. Root data for GSpin groups. We first describe the algebraic group GSpin and its standard Levi subgroups in terms of their root data. We will heavily rely on these description in the computations of Section 4.

Let $G = G\text{Spin}_m$, where $m = 2n + 1$ or $m = 2n$. We now describe the root datum for $G$.

**Definition 2.1.** Let

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$$

and

$$X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*$$
and let \( \langle , \rangle \) be the standard \( \mathbb{Z} \)-pairing on \( X \times X^\vee \). Then \((X, R, X^\vee, R^\vee)\) is the root datum for \( \text{GSpin}_m \), with \( R \) and \( R^\vee \) generated, respectively, by

\[
\Delta = \{ \alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n \}, \\
\Delta^\vee = \{ \alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = 2e_n^* - e_0^* \},
\]

if \( m = 2n + 1 \) and

\[
\Delta = \{ \alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n + e_{n-1} \}, \\
\Delta^\vee = \{ \alpha_1^\vee = e_1^* - e_2^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_n^* + e_{n-1}^* - e_0^* \},
\]

if \( m = 2n \).

In the odd case, \( G \) has a Dynkin diagram of type \( B_n \):

\[
\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-2} & \alpha_{n-1} & \alpha_n \\
& & & & \downarrow \\
\end{array}
\]

In the even case, it has a Dynkin diagram of type \( D_n \):

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-3} & \alpha_{n-2} & \alpha_{n-1} & \alpha_n \\
\end{array}
\]

**Proposition 2.2.** The derived group of \( G \) is isomorphic to \( \text{Spin}_{2n+1} \) or \( \text{Spin}_{2n} \), the double coverings, as algebraic groups, of special orthogonal groups. In fact, \( G \) is isomorphic to

\[
\frac{\text{GL}_1 \times \text{Spin}_m}{\{(1,1),(-1,c)\}}
\]

where \( c = \alpha_n^\vee(-1) \) if \( m = 2n + 1 \) or \( c = \alpha_{n-1}^\vee(-1)\alpha_n^\vee(-1) \) if \( m = 2n \). The dual of \( G \) is \( \text{GSp}_{2n} \) if \( m = 2n + 1 \) and \( \text{GSO}_{2n} \) if \( m = 2n \).

Moreover, if \( M \) is the Levi component of a maximal standard parabolic subgroup of \( G \), then it is isomorphic to \( \text{GL}_k \times \text{GSpin}_{m-2k} \) with \( k = 1, 2, \ldots, n \) if \( m = 2n + 1 \) and \( k = 1, 2, \ldots, n - 2, n \) if \( m = 2n \).

**Proof.** See Section 2 of [2]. \( \square \)
We can also describe the Levi subgroup $M$ in terms of its root datum. Without loss of generality we may assume $M$ to be maximal. Obviously, $M$ will have the same character and cocharacter lattices as those of $G$. Denote the set of roots of $M$ by $R_M$ and its coroots by $R_M^\vee$. They are generated by $\Delta - \{\alpha\}$ and $\Delta^\vee - \{\alpha^\vee\}$, respectively, where $\alpha = \alpha_k$ unless $m = 2n$ and $k = n$ in which case $\alpha$ can be either of $\alpha_n$ or $\alpha_{n-1}$ (resulting in two non-conjugate isomorphic Levi components). In the sequel, the case of $m = 2n$ and $k = n - 1$ is therefore always ruled out and we will not repeat this again.

**Proposition 2.3.**

(a) The center of $G$ is given by
\[
Z_G = \begin{cases} 
A_0 & \text{if } m = 2n + 1, \\
A_0 \cup (\zeta_0 A_0) & \text{if } m = 2n,
\end{cases}
\]
where
\[A_0 = \{e_0^\ast(\lambda) : \lambda \in GL_1\}\]
and $\zeta_0 = e_1^\ast(-1)e_2^\ast(-1) \cdots e_n^\ast(-1)$.

(b) The center of $M$ is given by
\[
Z_M = \begin{cases} 
A_k & \text{if } m = 2n + 1, \\
A_k \cup (\zeta_k A_k) & \text{if } m = 2n,
\end{cases}
\]
where
\[A_k = \{e_0^\ast(\lambda)e_1^\ast(\mu)e_2^\ast(\mu) \cdots e_k^\ast(\mu) : \lambda, \mu \in GL_1\}\]
and $\zeta_k = e_{k+1}^\ast(-1)e_{k+2}^\ast(-1) \cdots e_n^\ast(-1)$.

**Proof.** The maximal torus $T$ of $G$ (or $M$) consists of elements of the form
\[t = \prod_{j=0}^{n} e_j^\ast(t_j)\]
with $t_j \in GL_1$. Now $t$ is in the center of $G$, respectively $M$, if and only if it belongs to the kernel of all simple roots of $G$, respectively $M$. For $G$, this leads to
\[t_1/t_2 = t_2/t_3 = \cdots = t_{n-1}/t_n = t_n = 1\]
if $m = 2n + 1$ and
\[t_1/t_2 = t_2/t_3 = \cdots = t_{n-1}/t_n = t_{n-1}t_n = 1\]
if $m = 2n$. For $M$ we get
\[t_1/t_2 = t_2/t_3 = \cdots = t_{k-1}/t_k, \quad t_{k+1}/t_{k+2} = \cdots = t_{n-1}/t_n = t_n = 1\]
if $m = 2n + 1$ and
\[t_1/t_2 = t_2/t_3 = \cdots = t_{k-1}/t_k, \quad t_{k+1}/t_{k+2} = \cdots = t_{n-1}/t_n = t_{n-1}t_n = 1\]
if $m = 2n$. These relations prove the proposition. □
Remark 2.4. When \( m = 2n \), the non-identity component of \( \mathbf{Z}_G \) can also be written as \( z' \mathbf{A}_0 \), where \( z' \) is a non-trivial element in the center of \( \text{Spin}_{2n} \), the derived group of \( G \). We now specify this element explicitly in terms of the central element \( z \) of Proposition 2.2 of [2]. There is a typographical error in the description of \( z \) in that article which we correct here:

\[
    z = \begin{cases} 
    \prod_{j=1}^{n-2} \alpha_j \cdot \alpha_{n-1} \cdot (-1) & \text{if } n \text{ is even}, \\
    \prod_{j=1}^{n-2} \alpha_j \cdot \alpha_{n-1} \cdot (-\sqrt{-1}) \alpha_n \cdot (\sqrt{-1}) & \text{if } n \text{ is odd}.
    \end{cases}
\]

To compute \( z' \) note that with \( m = 2n \) we have,

\[
    e_1^* + \cdots + e_{n-1}^* + e_n^* = \sum_{j=1}^{n-2} j\alpha_j + (\frac{n}{2} - 1)\alpha_{n-1} + \frac{n}{2} \alpha_n + \frac{n}{2} e_0^*
\]

which, when evaluated as a character at \((-1)\), yields

\[
    \zeta_0 = \begin{cases} 
    z & \text{if } n = 4p, \\
    ze_0^*(\sqrt{-1}) & \text{if } n = 4p + 1, \\
    ce_0^*(-1) & \text{if } n = 4p + 2, \\
    cze_0^*(-\sqrt{-1}) & \text{if } n = 4p + 3.
    \end{cases}
\]

Therefore, \( \zeta_0 \mathbf{A}_0 = z' \mathbf{A}_0 \), where \( z' \) is an element in the center of \( \text{Spin}_{2n} \) given by

\[
    z' = \begin{cases} 
    z & \text{if } n \equiv 0, 1 \pmod{4}, \\
    cz & \text{if } n \equiv 2, 3 \pmod{4}.
    \end{cases}
\]

2.2. Root data for \( \text{GSpin} \sim \) groups. We describe the structure theory for the group \( \text{GSpin}_{2n} \) as well as its standard Levi subgroups in this section. For our future discussion on stability of \( \gamma \)-factors in Section 4 we will need to work with a group with connected center. However, center of \( \text{GSpin}_{2n} \) is not connected as we saw in Proposition 2.3. To remedy this we define a new group which is just \( \text{GSpin}_{2n} \) with a one-dimensional torus attached to it. This group will have a connected center while its derived group is the same as that of \( \text{GSpin}_{2n} \), i.e., \( \text{Spin}_{2n} \). This will allow us to work with \( \text{GSpin} \sim \) as we will explain in Section 4.

Definition 2.5. Let \( \text{GSpin} \sim ) \) be the group

\[
    \frac{\text{GL}_1 \times \text{GSpin}_{2n}}{\{(1,1), (-1, \zeta_0)\}}
\]

where \( \zeta_0 \) is as in Proposition 2.3. Note that the derived group of \( \text{GSpin} \sim ) \) is clearly isomorphic to \( \text{Spin}_{2n} \).

We now describe the root datum of this group.
Proposition 2.6. Let
\[ X = ZE_{-1} \oplus ZE_0 \oplus ZE_1 \oplus \cdots \oplus ZE_n \]
and
\[ X^\vee = ZE_{-1}^* \oplus ZE_0^* \oplus ZE_1^* \oplus \cdots \oplus ZE_n^* \]
and let \( \langle , \rangle \) be the standard \( \mathbb{Z} \)-pairing on \( X \times X^\vee \). Then \( (X, R, X^\vee, R^\vee) \) is the root datum for \( G\text{Spin}_{2n} \), with \( R \) and \( R^\vee \) generated, respectively, by
\[ \Delta = \{ \alpha_1 = E_1 - E_2, \ldots, \alpha_{n-1} = E_{n-1} - E_n, \alpha_n = E_{n-1} + E_n - E_1 \}, \]
and
\[ \Delta^\vee = \{ \alpha_1^\vee = E_1^* - E_2^*, \ldots, \alpha_{n-1}^\vee = E_{n-1}^* - E_n^*, \alpha_n^\vee = E_{n-1}^* + E_n^* - E_0^* \}. \]

Proof. Our proof will be similar to the proof of Proposition 2.4 of [2]. We will compute the root datum of \( G\text{Spin}_{2n} \) using that of \( G\text{Spin}_{2n} \) described earlier and verify that it can be written as above.

Start with the character lattice of \( GL_1 \times G\text{Spin}_{2n} \) which can be written as the \( \mathbb{Z} \)-span of \( e_0, e_1, \ldots, e_n \) and \( e_{-1} \). Now, characters of \( G\text{Spin}_{2n} \) are those characters of \( GL_1 \times G\text{Spin}_{2n} \) which are trivial when evaluated at the element \((-1, \zeta_0)\). Note that \( e_i(\zeta_0) = -1 \) for \( 1 \leq i \leq n, \) \( e_0(\zeta_0) = 1, \) and \( e_{-1}(-1) = -1 \). This implies that the character lattice of \( G\text{Spin}_{2n} \) can be written as the \( \mathbb{Z} \)-span of \( 2e_{-1}, e_0, e_1 + e_{-1}, \ldots, e_n + e_{-1} \). Now, set \( E_{-1} = 2e_{-1}, E_0 = e_0 \) and \( E_i = e_{-1} + e_i \) for \( 1 \leq i \leq n \). Using the \( \mathbb{Z} \)-pairing of the root datum, we can compute a basis for the cocharacter lattice which turns out to consist of \( E_{-1}^* = e_{-1}^*/2 - (e_1^* + \cdots e_n^*)/2, E_0^* = e_0^*, \) and \( E_i^* = e_i^*, \) for \( 1 \leq i \leq n \). Writing the simple roots and coroots now in terms of the new bases finishes the proof. For example,
\[ \alpha_n = e_{n-1} + e_n = (e_{n-1} + e_{-1}) + (e_n + e_{-1}) - 2e_{-1} = E_{n-1} + E_n - E_1 \]
and
\[ \alpha_n^\vee = e_{n-1}^* + e_n^* - e_0^* = E_{n-1}^* + E_n^* - E_0^*. \]

We can also describe the root datum of any standard Levi subgroup \( M \) in \( G\text{Spin}_{2n} \). Again, without loss of generality, we may assume \( M \) to be maximal. Similar to the case of \( G\text{Spin}_{2n} \), the roots and coroots of \( M \) are, respectively, generated by \( \Delta - \{ \alpha_k \} \) and \( \Delta^\vee - \{ \alpha_k^\vee \} \) for some \( k \). The character and cocharacter lattices are the same as those of \( G\text{Spin}_{2n} \).

Proposition 2.7. (a) The center of \( G\text{Spin}_{2n} \) is given by
\[ \{ E_0^*(\mu)E_1^*(\lambda)E_2^*(\lambda) \cdots E_n^*(\lambda)E_{n-1}^*(\lambda^2) : \lambda, \mu \in GL_1 \}, \]
and is hence connected.
(b) The center of $\mathbf{M}$ is given by

$$\left\{ E_{*}^{*}(\mu)E_{1}^{*}(\nu)\cdots E_{k}^{*}(\nu)E_{k+1}^{*}(\lambda)\cdots E_{n}^{*}(\lambda)E_{n-1}^{*}(\lambda^{2}) : \lambda, \mu, \nu \in GL_{1}\right\} ,$$

and is hence connected.

**Proof.** The maximal torus of $GSpin_{2n}$ (or $\mathbf{M}$) consists of elements of the form

$$t = \prod_{j=-1}^{n} E_{j}^{*}(t_{j})$$

with $t_{j} \in GL_{1}$. Now $t$ is in the center of $\mathbf{G}$, respectively $\mathbf{M}$, if and only if it belongs to the kernel of all simple roots of $\mathbf{G}$, respectively $\mathbf{M}$. For $\mathbf{G}$, this leads to

$$t_{1}/t_{2} = t_{2}/t_{3} = \cdots = t_{n-1}/t_{n} = (t_{n-1}t_{n})/t_{-1} = 1.$$

For $\mathbf{M}$ we get

$$t_{1}/t_{2} = t_{2}/t_{3} = \cdots = t_{k-1}/t_{k}, \quad t_{k+1}/t_{k+2} = \cdots = t_{n-1}/t_{n} = (t_{n-1}t_{n})/t_{-1} = 1.$$

These relations prove the proposition. \qed

We also describe the structure of standard Levi subgroups in $GSpin_{2n}$.

**Proposition 2.8.** Standard Levi subgroups of $GSpin_{2n}$ are isomorphic to

$$GL_{k_{1}} \times \cdots \times GL_{k_{r}} \times GSpin_{2l},$$

where $k_{1} + \cdots + k_{r} + l = n$.

**Proof.** Without loss of generality we may assume $\mathbf{M}$ to be maximal. The character and cocharacter lattices of $\mathbf{M}$, which are the same as those of $\mathbf{G}$, were described in Proposition 2.6, and can be written as

$$\left( \mathbb{Z}E_{1} \oplus \cdots \oplus \mathbb{Z}E_{k} \right) \oplus \left( \mathbb{Z}E_{-1} \oplus \mathbb{Z}E_{0} \oplus \mathbb{Z}E_{k+1} \cdots \oplus \mathbb{Z}E_{n} \right)$$

and

$$\left( \mathbb{Z}E_{1}^{*} \oplus \cdots \oplus \mathbb{Z}E_{k}^{*} \right) \oplus \left( \mathbb{Z}E_{-1}^{*} \oplus \mathbb{Z}E_{0}^{*} \oplus \mathbb{Z}E_{k+1}^{*} \cdots \oplus \mathbb{Z}E_{n}^{*} \right),$$

respectively. This along with the description of roots and coroots of $\mathbf{M}$ given above implies that the root datum of $\mathbf{M}$ can be written as a direct sum of two root data. The first one is now the well-known root datum of $GL_{k}$, and the second is just our earlier description of root datum of $GSpin_{2(n-k)}$. Therefore, $\mathbf{M}$ is isomorphic to $GL_{k} \times GSpin_{2(n-k)}$. \qed
2.3. Root data for GSp\(_{2n}\) and GSO\(_{2n}\). We describe the root data for the
two groups GSp\(_{2n}\) and GSO\(_{2n}\) in detail. Since these two groups are usually
introduced as matrix groups, we will also describe the root data in terms of
their usual matrix representation. It will be evident from this description that
the two groups GSpin\(_{2n+1}\) and GSp\(_{2n}\) as well as GSpin\(_{2n}\) and GSO\(_{2n}\) are pairs
of connected reductive algebraic groups with dual root data.

**Definition 2.9.** Let

\[ X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n \]

and

\[ X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^* \]

and let \(\langle , \rangle\) be the standard \(\mathbb{Z}\)-pairing on \(X \times X^\vee\). Then \((X, R, X^\vee, R^\vee)\) is the
root datum for the connected reductive algebraic group GSp\(_{2n}\) or GSO\(_{2n}\), with
\(R\) and \(R^\vee\) generated, respectively, by

\[
\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n - e_0\},
\]

\[
\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_n^*\},
\]

for GSp\(_{2n}\) (cf. pages 133–136 of [42]) and

\[
\Delta = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n + e_0\},
\]

\[
\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_n^* + e_0^*\},
\]

for GSO\(_{2n}\).

The Dynkin diagrams are of type \(C_n\) and \(B_n\), respectively. A computation
similar to the proof of Proposition 2.3 proves the following:

**Proposition 2.10.** Let \(G\) be either GSp\(_{2n}\) or GSO\(_{2n}\). Then the center of \(G\)
is given by

\[
\mathbb{Z} = \{e_0^*(\lambda^2) e_1^*(\lambda) \cdots e_n^*(\lambda) : \lambda \in GL_1\}
\]

Alternatively, consider the group defined via

\[
\{g \in GL_{2n} : {}^t g J g = \mu(g) J\},
\]

where the \(2n \times 2n\) matrix \(J\) is defined via

\[
J = \begin{pmatrix}
1 & & & & \\
& 1 & & & \\
& & \ddots & & \\
& & & -1 & \\
& & & & 1
\end{pmatrix}
\]

or

\[
J = \begin{pmatrix}
1 & & & & \\
& 1 & & & \\
& & \ddots & & \\
& & & 1 & \\
& & & & 1
\end{pmatrix},
\]

respectively. The former is the connected reductive algebraic group GSp\(_{2n}\).
However, the latter is not connected as an algebraic group. This group is
sometimes denoted by GO\(_{2n}\) (Section 2 of [31]). Its connected component is
the group $\text{GSO}_{2n}$ (also denoted by $\text{SGO}_{2n}$). The maximal split torus in both of these groups can be described as

$$
\widehat{T} = \left\{ t(a_1, \cdots, a_n, b_n, \cdots, b_1) = \begin{pmatrix} a_1 & & \cdots & b_n \\
 & & & \ddots \\
 & & & & \ddots \\
 & & & & & b_1 \end{pmatrix} : a_i b_i = \mu \right\}
$$

We can now describe $e_i$ and $e_i^*$ in terms of matrices. In either case we have

$$
e_0(t) = \mu \quad , \quad e_0^*(\lambda) = t(1, \cdots, 1, \lambda, \cdots, \lambda)
$$

$$
e_i(t) = a_i \quad , \quad e_i^*(\lambda) = t(1, \cdots, 1, \lambda, 1, \cdots, 1, \lambda^{-1}, 1, \cdots, 1), \ 1 \leq i \leq n.
$$

3. Analytic Properties of Local $L$-functions

Let $F$ denote a local field of characteristic zero, either archimedean or non-archimedean. Let $G_n$ denote the algebraic group $\text{GSpin}_{2n+1}$ (respectively, $\text{GSpin}_{2n}$) and let $\sigma$ be an irreducible admissible generic representation of $M = \text{M}(F)$ in $G = \text{G}_{r+n}(F)$, where $\text{M} \simeq \text{GL}_r \times G_n$ is the Levi subgroup of a standard parabolic $P$ in $G_{r+n}$. Let $\widehat{M} \simeq \text{GL}_r(\mathbb{C}) \times \text{GSp}_{2n}(\mathbb{C})$ (respectively, $\widehat{M} \simeq \text{GL}_r(\mathbb{C}) \times \text{GSO}_{2n}(\mathbb{C})$) be the Levi component of the corresponding standard parabolic subgroup $\widehat{P}$ in the dual group $\widehat{G} = LG^0 = \text{GSp}_{2n}(\mathbb{C})$ (respectively, $\widehat{G} = \text{GSO}_{2n}(\mathbb{C})$). Let $r$ denote the adjoint action of $\widehat{M}$ on the Lie algebra of the unipotent radical of $\widehat{P}$. Then by Proposition 5.6 of [2] $r = r_1 \oplus r_2$ if $n \geq 1$ (respectively, $n \geq 2$) with $r_1 = \rho_r \otimes \widehat{R}$ and $r_2 = \text{Sym}^2 \rho_r \otimes \mu^{-1}$ (respectively, $r_2 = \lambda^2 \rho_r \otimes \mu^{-1}$). Here, $\rho_r$ denotes the standard representation of $\text{GL}_r(\mathbb{C})$, $\widehat{R}$ denotes the contragredient of the standard representation of $\text{GSp}_{2n}(\mathbb{C})$ (respectively, $\text{GSO}_{2n}(\mathbb{C})$), and $\mu$ denotes the multiplicative character defining $\text{GSp}_{2n}(\mathbb{C})$ (respectively, $\text{GSO}_{2n}(\mathbb{C})$). If $n = 0$, then $r = r_1$ with $r_1 = \text{Sym}^2 \rho_r \otimes \mu^{-1}$ (respectively, $r_1 = \wedge^2 \rho_r \otimes \mu^{-1}$). Recall that we have excluded $n = 1$ in the even case. The Langlands-Shahidi method defines the $L$-functions $L(s, \sigma, r_i)$ and $\epsilon$-factors $\epsilon(s, \sigma, r_i, \psi)$ for $1 \leq i \leq 2$, where $\psi$ is a non-trivial additive character of $F$. (In the global setting, it will be the local component of our fixed global additive character $\psi$ of Section 1[1].) If $\pi$ denotes a representation of $G_n(F)$ and $\tau$ denotes one of $\text{GL}_r(F)$, then we sometimes employ the following notations for these $L$-functions as well as their global analogues:
\[ L(s, \pi \times \tau) := L(s, \tau \otimes \tilde{\pi}, \rho_r \otimes \tilde{R}) = L(s, \tau \otimes \tilde{\pi}, r_1), \quad (5) \]

\[ \epsilon(s, \pi \times \tau, \psi) := \epsilon(s, \tau \otimes \tilde{\pi}, \rho_r \otimes \tilde{R}, \psi) = \epsilon(s, \tau \otimes \tilde{\pi}, r_1, \psi). \quad (6) \]

**Proposition 3.1.** Assume that \(\sigma\) is tempered. Then the local \(L\)-function \(L(s, \sigma, r_i)\) is holomorphic for \(\Re(s) > 0\) for \(1 \leq i \leq m\).

**Proof.** The result is well-known for archimedean \(F\). For non-archimedean \(F\) this is Theorem 5.7 of [2]. Here, \(i = 1, 2\) and the first \(L\)-function gives the Rankin-Selberg product while the second is twisted symmetric or exterior square. When, \(n = 0\), we only get the second \(L\)-function. (cf. Proposition 5.6 of [2]).

**Proposition 3.2.** (Standard Module conjecture for \(G_n\)) Let \(\sigma\) be an irreducible admissible generic representation of \(M(F)\) in \(G_n(F)\) and let \(\nu\) be an element in the positive Weyl chamber. Let \(I(\nu, \sigma)\) be the representation unitarily induced from \(\nu\) and \(\sigma\), called the standard module, and denote by \(J(\nu, \sigma)\) its unique Langlands quotient. Assume that \(J(\nu, \sigma)\) is generic. Then, \(J(\nu, \sigma) = I(\nu, \sigma)\). In particular, \(I(\nu, \sigma)\) is irreducible. (A similar result also holds for general linear groups [44].)

**Proof.** For archimedean \(F\), this is due to Vogan for general groups. When \(F\) is a non-archimedean field, the proof is the subject of W. Kim’s thesis [23], which we rely on.

However, for small values of \(n\) we need not rely on [23] and can obtain the result from published results as we now explain: the group \(GSpin_5\) is isomorphic to \(GSp_4\) whose derived group is \(Sp_4\). G. Muić has proved the Standard Module conjecture for (quasi-split) classical groups (Theorem 1.1 of [30]). The result for \(GSpin_5\) now follows from Corollary 3.4 below.

Similarly, note that the derived group of \(GSpin_6\) is isomorphic to \(Spin_6 \cong SL_4\), hence equal to the derived group of \(GL_4\). Therefore, again by Corollary 3.4, the result for \(GSpin_6\) follows from the Standard Module conjecture for \(GL_4\).

**Proposition 3.3.** Let \(G \subset \tilde{G}\) be two connected reductive groups whose derived groups are equal. Let \(\tilde{\mathbf{P}} = \tilde{\mathbf{M}}\mathbf{N}\) be a maximal standard Levi subgroup of \(\tilde{G}\) and let \(\mathbf{P} = \mathbf{M}\mathbf{N}\) be the corresponding one in \(G\) with \(\mathbf{M} = \tilde{\mathbf{M}} \cap G\). Also, let \(\tilde{T} \subset \tilde{\mathbf{M}}\) and \(\mathbf{T} = \mathbf{T} \cap G \subset \mathbf{M}\) be maximal tori in \(\tilde{G}\) and \(G\). Let \(\tilde{\sigma}\) be a quasi-tempered representation of \(\tilde{M} = \tilde{\mathbf{M}}(F)\) and denote by \(\sigma\) its restriction to \(M = \mathbf{M}(F)\). Write \(\sigma = \bigoplus_i \sigma_i\). Let \(I(\tilde{\sigma})\) denote the induced representation \(\text{Ind}_{\tilde{M}N\cap G}^{\tilde{G}} (\tilde{G}(F))\) and \(I(\sigma_i)\) denote \(\text{Ind}_{\mathbf{M}N\cap G}^{\mathbf{G}} (\mathbf{M}(F))\). Then, \(I(\tilde{\sigma})\) is irreducible and standard if and only if each \(I(\sigma_i)\) is standard and irreducible.
Proof. We only need to address the reducibility questions. Write $\tilde{\sigma}|M = \oplus_i \sigma_i$. By irreducibility of $\tilde{\sigma}$ and the fact that $\tilde{M} = \tilde{T}M$, choose
\[\left\{t_1 = 1, t_2, \ldots, t_k : t_i \in \tilde{T} = \tilde{T}(F)\right\}\]
such that $\sigma_i(m) = \sigma_1(t_i^{-1}mt_i)$. Observe that
\[I(\tilde{\sigma})|G = \oplus_i I(\sigma_i). \tag{7}\]
In fact, if $f_1 \in V(\sigma_1)$, define $f_i(g) = f_1(t_i^{-1}gt_i)$. Then $f_i \in V(\sigma_i)$, the space of $I(\sigma_i)$, and the representation $I(\sigma_i)(t_i^{-1}gt_i)$ on $V(\sigma_1)$ is isomorphic to $I(\sigma_i)$ since
\[(I(\sigma_i)(t_i^{-1}gt_i)f_1)_i = I(\sigma_i)(g)f_i, \tag{8}\]
for all $g \in G$. In particular, $I(\sigma_i)$ is irreducible if and only if $I(\sigma_i)$ is. Observe moreover that $\tilde{T}$ acts transitively on the set of $I(\sigma_i)$'s using $\tilde{G} = \tilde{T}G$.

If each $I(\sigma_i)$ is irreducible, then $I(\tilde{\sigma})$ has to be irreducible. In fact, if $(\tilde{I}_1, \tilde{V}_1)$ is an irreducible subrepresentation of $I(\tilde{\sigma})$, then
\[\tilde{I}_1|G = \oplus_j I_j, \quad I_j \neq \{0\}, \tag{9}\]
and given $j$, there exists $i$ such that $I_j \subset I(\sigma_i)$. Conversely, for each $i$ there exists $j$ such that $0 \neq I_j \subset I(\sigma_i)$. In fact, fix $i$ such that $V(\sigma_i) \cap \tilde{V}_1 \neq 0$. Since $\tilde{V}_1$ is invariant under $\tilde{T}$, applying $I(\tilde{\sigma})(\tilde{T})$ to this intersection, then implies that $V(\sigma_i) \cap \tilde{V}_j \neq \{0\}$ for all $i$. Consequently, $\{0\} \neq I_j \subset I(\sigma_i)$ which is a contradiction.

Conversely, suppose $I(\tilde{\sigma})$ is irreducible but $I(\sigma_i)$ are (all) reducible. Let $V_i$ be an irreducible $G$-subspace of $V(\sigma_i)$. Then
\[\oplus_i I(\tilde{\sigma})(t_i)V_i \tag{10}\]
is a $\tilde{G}$-invariant subspace of $V(\tilde{\sigma})$ which is strictly smaller than $V(\tilde{\sigma})$, a contradiction. $\square$

Corollary 3.4. Suppose $G$ and $G'$ are two connected reductive groups having the same derived group. Then the standard module conjecture is valid for $G$ if and only if it is valid for $G'$.

The Langlands-Shahidi method defines the local $L$-functions via the theory of intertwining operators. With notation as above, let the standard maximal Levi $M$ in $G$ correspond to the subset $\theta$ of the set of simple roots $\Delta$ of $G$. Then $\theta = \Delta - \{\alpha\}$ for a simple root $\alpha \in \Delta$. We denote by $w$ the longest element in the Weyl group of $G$ modulo that of $M$. Then $w$ is the unique element with $w(\theta) \subset \Delta$ and $w(\alpha) < 0$. Let $A(s, \sigma, w)$ denote the intertwining
operator as in (1.1) on page 278 of \textsuperscript{36} and let \( N(s, \sigma, w) \) be defined via
\[
A(s, \sigma, w) = r(s, \sigma, w)N(s, \sigma, w),
\] (11)
\[
r(s, \sigma, w) = \frac{L(s, \sigma, \tilde{r}_1)L(2s, \sigma, \tilde{r}_2)}{L(1 + s, \sigma, \tilde{r}_1)\epsilon(s, \sigma, \tilde{r}_1, \psi)L(1 + 2s, \sigma, \tilde{r}_2)\epsilon(2s, \sigma, \tilde{r}_2, \psi)}
\] (12)

In fact, the Langlands-Shahidi method inductively defines the \( \gamma \)-factors using the theory of local intertwining operators out of which the \( L \)- and \( \epsilon \)-factors are defined via the relation
\[
\gamma(s, \sigma, r_i, \psi) = \epsilon(s, \sigma, r_i, \psi)\frac{L(1 - s, \sigma, \tilde{r}_i)}{L(s, \sigma, r_i)}.
\] (13)

The following proposition is the main result of this section about analytic properties of local \( L \)-functions which we will use to prove the necessary global analytic properties.

**Proposition 3.5.** Let \( \sigma \) be a local component of a globally generic cuspidal automorphic representation of \( M(\mathbb{A}) \). Then the normalized local intertwining operator \( N(s, \sigma, w) \) is holomorphic and non-zero for \( \Re(s) \geq 1/2 \).

**Proof.** First assume \( \sigma \) to be tempered. Then by Harish-Chandra we know that \( A(s, \sigma, w) \) is holomorphic for \( \Re(s) > 0 \). Moreover, for \( \Re(s) > 0 \) we have that \( r(s, \sigma, w) \) is non-zero by definition and holomorphic by Proposition 3.1. This implies that \( N(s, \sigma, w) \) is also holomorphic for \( \Re(s) > 0 \).

Next, assume that \( \sigma \) is not tempered but still unitary. Write \( \sigma = \tau \otimes \tilde{\pi} \), where \( \tau \) is a representation of \( GL_r(F) \) and \( \tilde{\pi} \) is one of \( G_n(F) \). (We used \( \tilde{\pi} \) in order to get the usual Rankin-Selberg factors for pairs of general linear groups below.) By Proposition 3.2, we can write \( \tau \) and \( \tilde{\pi} \) as follows:
\[
\tau = \text{Ind}(\nu^{\alpha_1} \tau_1 \otimes \cdots \otimes \nu^{\alpha_p} \tau_p \otimes \tau_{p+1} \otimes \nu^{-\alpha_p} \tau_{p+1} \otimes \cdots \otimes \nu^{-\alpha_1} \tau_1)
\]
and
\[
\tilde{\pi} = \text{Ind}(\nu^{\beta_1} \pi_1 \otimes \cdots \otimes \nu^{\beta_q} \pi_q \otimes \pi_0),
\]
with \( 0 = \alpha_{p+1} < \alpha_p < \cdots < \alpha_1 < 1/2 \) and \( 0 < \beta_q < \cdots < \beta_1 \) where, \( \tau_1, \ldots, \tau_{p+1} \) and \( \pi_1, \ldots, \pi_q \) are tempered representations of the corresponding \( GL(F) \), and \( \pi_0 \) is a generic tempered representation of \( G_t(F) \) for some \( t \). Here, \( \nu(\cdot) \) denotes \( |\det(\cdot)|_F \). Therefore,
\[
N(s, \sigma, w) = \prod_{i=1, \ldots, p+1} N_1(s \pm \alpha_i \pm \beta_j) \cdot \prod_{i=1, \ldots, p+1} N_2(s \pm \alpha_i),
\]
where the terms \( N_1(s \pm \alpha_i \pm \beta_j) \) in the first product are products of four rank one operators for \( GL_k \times GL_l \subset GL_{k+l} \) with complex parameters \( s \pm \alpha_i \pm \beta_j \) and the terms \( N_2(s \pm \alpha_i) \) in the second product are products of two rank one operators for \( GL_k \times G_l \subset G_{k+l} \) with complex parameters \( s \pm \alpha_i \) respectively.
If \( \Re(s) \geq 1/2 \), then \( \Re(s \pm \alpha_i) > 0 \) for all \( i \) and the terms in the second product are holomorphic by the first part of this proof. The operators in the first product are those associated with Rankin-Selberg factors for pairs of general linear groups and, by Lemma 2.10 of [19], they are holomorphic if we show that \( \Re(s - \alpha_1 - \beta_1) > -1 \). This happens if \( \Re(s - \alpha_1 - \beta_1) > -1 \) or \( \beta_1 < 1 \). Therefore, Lemma 3.6 below completes the proof of holomorphy part of the proposition.

The fact that \( N(s, \sigma, w) \) is a non-vanishing operator now follows from applying a result of Y. Zhang (cf. pages 393–394 of [45]) to our case. Note that in view of Proposition 4.1 no assumptions are needed in applying [45]. □

**Lemma 3.6.** If \( \pi \) is a supercuspidal representation of \( G_n(F) \) written as above, then \( \beta_1 < 1 \).

**Proof.** This is Lemma 3.3 of [19]. However, note that one should use our \( \tilde{\pi} \) in the argument. □

4. **Stability of \( \gamma \)-factors**

We continue to denote by \( G_n \) either of the groups \( \text{GSpin}_{2n+1} \) or \( \text{GSpin}_{2n} \) in this section. In subsections 4.1 and 4.2 we denote by \( \text{GSpin}_{2n+1}^\sim \) the groups \( \text{GSpin}_{2n+1} \) in the odd case and \( \text{GSpin}_{2n}^\sim \) in the even case (see Remark 4.2 below) and \( G \) will denote \( \text{GSpin}_{n+1}^\sim \) in either case.

In this section we prove a key local fact, called the *stability of \( \gamma \)-factors*, which is what allows us to connect the Langlands-Shahidi \( L \)- and \( \epsilon \)-factors to those in the converse theorem. Similar results have recently been proved for the groups \( \text{SO}_{2n+1} \) in [8, 5] and other classical groups in [6] which we have followed. A more general result will appear in [10, 11].

Let \( F \) denote a non-archimedean local field of characteristic zero, i.e, one of \( k_v \)'s where \( v \) is a finite place. Composing a fixed splitting with \( \psi \) as in [38], then defines a generic character of \( U \) as well as \( U_M \) which we still denote by \( \psi \). Let \( \pi \) be an irreducible admissible \( \psi \)-generic representation of \( \text{GSpin}_{2n+1}(F) \) or \( \text{GSpin}_{2n}(F) \), and let \( \eta \) be a continuous character of \( F^\times \). The associated \( \gamma \)-factors of the Langlands-Shahidi method defined in Theorem 3.5 of [36] will be denoted by \( \gamma(s, \eta \times \pi, \psi) \). They are associated to the pair \( (\text{GSpin}_{m+2}, \text{GL}_1 \times \text{GSpin}_m) \) of the maximal Levi factor \( M = \text{GL}_1 \times \text{GSpin}_m \) in the connected reductive group \( \text{GSpin}_{m+2} \), where \( m = 2n + 1 \) or \( 2n \). Recall that the \( \gamma \)-factor is related to the \( L \)- and \( \epsilon \)-factors by

\[
\gamma(s, \eta \times \pi, \psi) = \epsilon(s, \eta \times \pi, \psi) \frac{L(1 - s, \eta^{-1} \times \tilde{\pi})}{L(s, \eta \times \pi)}. \tag{14}
\]

The main result of this section is the following:
Theorem 4.1. Let \( \pi_1 \) and \( \pi_2 \) be irreducible admissible generic representations of \( \text{GSpin}_m(F) \) with equal central characters \( \omega_{\pi_1} = \omega_{\pi_2} \). Then for every sufficiently ramified character \( \eta \) of \( F^\times \) we have
\[
\gamma(s, \eta \times \pi_1, \psi) = \gamma(s, \eta \times \pi_2, \psi).
\]

The proof of this theorem is the subject matter of this section including a review of some facts about partial Bessel functions.

4.1. \( \gamma(s, \eta \times \pi) \) as Mellin transform of a Bessel function. Recall that \( G_n \) denotes either \( \text{GSpin}_{2n+1} \) or \( \text{GSpin}_{2n} \) and \( G = \text{GSpin}_{n+1} \). We will refer to \( G = \text{GSpin}_{2n+3} \) as odd and \( G = \text{GSpin}_{2n+2} \) as even in the rest of this section.

Remark 4.2. We will need to assume that the center of our group \( G \) is connected for proof of Proposition 4.16 below. This is not true if \( G \) is taken to be \( \text{GSpin}_{2n+2} \) as we pointed out in Proposition 2.3. To remedy this we can alternatively work with the group \( \text{GSpin}_{2n+2} \) of Section 2.2. Since \( \text{GSpin}_{2n+2} \) has the same derived group as \( \text{GSpin}_{2n+2} \) its corresponding \( \gamma \)-factors are the same as those of \( \text{GSpin}_{2n+2} \) since they (and, in fact, the local coefficients via which they are defined) only depend on the derived group of our group. This has no effect on the arguments of the next few subsections as all of our crucial computations take place inside the derived group.

Let \( G \) be as above with a fixed Borel subgroup \( B = TU \) as before. We continue to denote its root data by \( (X, R, X^\vee, R^\vee) \) which we have described in detail in Section 2. Consider the maximal parabolic subgroup \( P = MN \) in \( G \), where \( N \subset U \) and the Levi component, \( M \), is isomorphic to \( \text{GL}_1 \times G_n \). The standard Levi subgroup \( M \supset T \) corresponds to the subset \( \theta = \Delta - \{ \alpha_1 \} \) of the set of simple roots \( \Delta = \{ \alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1} \} \) of \( (G, T) \) with notation as in Section 2. Let \( w \) denote the unique element of the Weyl group of \( G \) such that \( w(\theta) \subset \Delta \) and \( w(\alpha_1) < 0 \). Notice that the parabolic subgroup \( P \) is self-associate, i.e., \( w(\theta) = \theta \). We will denote by \( G, P, M, N, B \), etc., the groups \( G(F), P(F), M(F), N(F), B(F) \), etc., in what follows. Also denote the opposite parabolic subgroup to \( P \) by \( \overline{P} = MN \).

Let \( Z = Z_G \) and \( Z_M \) be the centers of \( G \) and \( M \) respectively. The following is assumption 5.1 of [38] for our cases. We will need this when dealing with Bessel functions.

Proposition 4.3. There exists an injection \( e^* : F^\times \rightarrow Z_G \backslash Z_M \) such that for all \( t \in F^\times \) we have \( \alpha_1(e^*(t)) = t \).

Proof. We define \( e^*(t) \) to be the image in \( Z_G \backslash Z_M \) of \( e_1^*(t) \) in the odd case and that of \( E_1^*(t) \) in the even case. The proposition is now clear from our explicit descriptions in Section 2. \( \square \)
Denote the image of \(e^*\) by \(Z^0_M\) as in [38]. (Note that [38] uses the notation \(\alpha^\vee\) for \(e^*\).)

We now review some standard facts about the reductive group \(G\) whose proofs could be found in either of [40] or [41], for example. For \(\alpha \in R\) let \(u_\alpha : F \to G\) be the root group homomorphism determined by the equation

\[
tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x) \quad t \in T, x \in F.
\]

(15)

Then \(u_\alpha(x)\) is additive in the variable \(x\). Moreover, define \(w_\alpha : F \times F \to G\) by

\[
w_\alpha(\lambda) = u_\alpha(\lambda)u_{-\alpha}(-\lambda^{-1})u_\alpha(\lambda).
\]

(16)

Also set \(w_\alpha := w_\alpha(1)\). Then,

\[
w_\alpha(\lambda) = \alpha^\vee(\lambda)w_\alpha = w_\alpha \alpha^\vee(\lambda^{-1}),
\]

(17)

where \(\alpha^\vee\) is the coroot corresponding to \(\alpha\). The element \(w_\alpha\) normalizes \(T\) and we will denote its image in the Weyl group by \(\tilde{w}_\alpha\).

**Remark 4.4.** Our choice of \(w_\alpha\) is indeed the same as \(n_\alpha\) on page 133 of [40]. This choice differs up to a sign from those made in (4.43), (4.19) or (4.56) of [38] requiring \(w_\alpha\) to be the image of \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) under the homomorphism from \(SL_2\) into \(G\) determined by \(\alpha\). The latter choice would introduce an occasional negative sign in some of the equations, e.g. (17) above. Of course, this choice is irrelevant to final results and we have chosen Springer’s since we will be using some detailed information on structure constants from [40] in what follows (e.g., Lemma 4.10).

Recall that

\[
w_{-\alpha}(\lambda) = w_\alpha(-1/\lambda),
\]

(18)

\[
w_\alpha^2 = \alpha^\vee(-1),
\]

(19)

\[
w_{-\alpha} = w_\alpha^{-1}.
\]

(20)

For any two linearly independent roots \(\alpha\) and \(\beta\) in \(R\) and a total order on \(R\), which we fix now, we have

\[
u_\alpha(x)v_\beta(y)v_{-\alpha}(-x) = v_\beta(y) \prod_{i,j \geq 0, i\alpha+j\beta \in R} u_{i\alpha+j\beta}(c_{ij}x^iy^j)
\]

(21)

for certain structure constants \(c_{ij} = c_{\alpha,\beta;i,j}\). In particular, if there are no roots of the form \(i\alpha + j\beta\) with \(i, j > 0\), then \(u_\alpha(x)v_\beta(y) = v_\beta(y)u_\alpha(x)\).

We recall the following result.

**Proposition 4.5.** Let \(\alpha\) and \(\beta\) be two arbitrary linearly independent roots and let \((\beta - c\alpha, \cdots, \beta + b\alpha)\) be the \(\alpha\)-string through \(\beta\). Then,

\[
w_\alpha w_\beta(x)w_\alpha^{-1} = w_{\tilde{w}_\alpha(\beta)}(d_{\alpha,\beta}x),
\]

(22)
where
\[ d_{\alpha, \beta} = \sum_{i=\max(0, c-b)}^{c} (-1)^{i} c_{-\alpha, \beta; i, 1} c_{\alpha, \beta - i \alpha; i + b - c, 1}. \tag{23} \]

Moreover,
\[ d_{-\alpha, \beta} = (-1)^{\langle \beta, \alpha \rangle} d_{\alpha, \beta} \tag{24} \]
and
\[ d_{\alpha, \beta} d_{\alpha, \tilde{\nu}(\beta)} = (-1)^{\langle \beta, \alpha \rangle}. \tag{25} \]

Proof. This is Lemma 9.2.2 of [40]. Note that Springer defines \( d_{\alpha, \beta} \) via
\[ w_{\alpha} u_{\beta}(x) w_{\alpha}^{-1} = u_{\tilde{\nu} \nu}(\beta)(d_{\alpha, \beta} x) \]
which, using (22), immediately implies (22).

Denoting the image of \( u_{\alpha} \) in \( G \) by \( U_{\alpha} \) notice that \( M \) is generated by \( U_{\alpha} \)'s with \( \alpha \) ranging over \( \Sigma(\theta) \), the set of all (positive and negative) roots spanned by \( \alpha_{2}, \alpha_{3}, \ldots, \alpha_{n}, \alpha_{n+1} \), while \( N \) is generated by \( U_{\alpha} \)'s where \( \alpha \) ranges over \( R(N) = R^{+} - \Sigma(\theta) \), the set of positive roots of \( G \) not in \( M \) (i.e., involving a positive coefficient of \( \alpha_{1} \) when written as a sum of simple roots with non-negative coefficients) and \( \tilde{N} \) is generated by \( U_{\alpha} \)'s where \( \alpha \) ranges over \( R(\tilde{N}) = R^{-} - \Sigma(\theta) \), the set of negative root of \( G \) not in \( M \) (i.e., involving a negative coefficient of \( \alpha_{1} \) when written as a sum of simple roots with non-positive coefficients). Let \( U_{M} = U \cap M \). Then \( U_{M} \) is generated by \( U_{\alpha} \)'s with \( \alpha \in \Sigma(\theta)^{+} = \Sigma(\theta) \cap R^{+} \).

The group \( M \) acts via the adjoint action on \( N \); in particular, both \( U_{M} \) and \( Z_{M}^{0} \) act on \( N \). We are interested in the orbits of the adjoint action of \( Z_{M}^{0} U_{M} \) on \( N \).

Lemma 4.6. Up to a subset of measure zero of \( N \) the following is a complete set of representatives for the orbits of \( N \) under the adjoint action of \( U_{M} \).
\[ U_{M} \backslash N \simeq \{ u_{\alpha_{1}}(a) u_{\gamma}(x) : a \in F^{\times}, x \in F \}, \]
where
\[ \gamma = \begin{cases} \alpha_{1} + 2\alpha_{2} + \cdots + 2\alpha_{n+1} & \text{if } G \text{ is odd,} \\ \alpha_{1} + 2\alpha_{2} + \cdots + 2\alpha_{n-1} + \alpha_{n} + \alpha_{n+1} & \text{if } G \text{ is even,} \end{cases} \]
is the longest positive root in \( G \).

Proof. Using the same Bourbaki notation as in Section 2 \( R(N) \) is given by (4)
\[ \left\{ \begin{array}{c} \alpha_{1}, \alpha_{1} + \alpha_{2}, \ldots, \alpha_{1} + \alpha_{2} \cdots + \alpha_{n+1}, \alpha_{1} + \alpha_{2} + \cdots + \alpha_{n} + 2\alpha_{n+1}, \\ \alpha_{1} + \alpha_{2} + \cdots + \alpha_{n-1} + 2\alpha_{n} + 2\alpha_{n+1}, \ldots, \gamma = \alpha_{1} + 2\alpha_{2} + \cdots + 2\alpha_{n} + 2\alpha_{n+1} \end{array} \right\} \tag{26} \]
for the odd case and
\[
\begin{align*}
\{ & \alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 \cdots + \alpha_{n-1} + \alpha_n, \alpha_1 + \alpha_2 \cdots + \alpha_{n-1} + \alpha_n, \\
& 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}, \alpha_1 + \alpha_2 + \cdots + 2\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}, \\
& \cdots, \gamma = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1} \}
\end{align*}
\]
(27)
for the even case.

An arbitrary element \( n \in N \) is of the form
\[
n = \prod_{\alpha \in R(N)} u_\alpha(x_\alpha)
\]
(28)
with \( x_\alpha \in F \). The ordering in the product can be arbitrarily chosen since any linear combination with positive integer coefficients of two roots in \( R(N) \) would have \( \alpha_1 \) with an integer coefficient of at least two which cannot be a root, hence by \[21\] any two terms in the above product commute. We make use of this fact in the rest of the proof.

Observe that the set \( R(N) \) has the property that if \( \alpha \) belongs to \( R(N) \), then so does \( \gamma - \alpha + \alpha_1 \). Notice that if \( \alpha' \in R(N) - \{ \alpha_1, \gamma \} \), then \( \beta = \alpha' - \alpha_1 \in \Sigma(\theta) \) and \( \beta > 0 \), hence \( g = u_\beta(x) \in U_M \) for any \( x \in F \). Fix one such \( \beta \) and consider the adjoint action of \( g \) on \( n \):
\[
gng^{-1} = \prod_{\alpha \in R(N)} gu_\alpha(x_\alpha)g^{-1} = \prod_{\alpha \in R(N)} u_\beta(x)u_\alpha(x_\alpha)u_\beta(-x).
\]
We now look at each term in this product: if \( i\beta + j\alpha \notin R \) for positive \( i \) and \( j \), then by \[21\] the term is equal to \( u_\alpha(x_\alpha) \). This is the case for all \( \alpha \in R(N) \) unless \( \alpha = \alpha_1 \) in which case \( \beta + \alpha = \beta' \) is a root or \( \alpha = \gamma - \beta = \gamma - \beta' + \alpha_1 \) (which does belong to \( R(N) \) by the above observation) in which case \( \beta + \alpha = \gamma \) is again a root. Therefore,
\[
\prod_{\alpha \in R(N)} u_\beta(x)u_\alpha(x_\alpha)u_\beta(-x) = \prod_{\alpha \in R(N)} u_\alpha(y_\alpha),
\]
where
\[
y_\alpha = \begin{cases} 
 x_\alpha + Cx_\alpha_1x_\beta & \text{if } \alpha = \alpha' \\
 x_\alpha + C'x_{\gamma' - \alpha' + \alpha_1}x_\beta & \text{if } \alpha = \gamma \\
 x_\alpha & \text{otherwise.}
\end{cases}
\]
Here, \( C, C' \in F^\times \) are the appropriate structure constants as in \[21\].

Assuming \( x_{\alpha_1} \neq 0 \), which only excludes a subset of \( n \in N \) of measure zero, we can choose \( x_\beta \in F \) appropriately in order to have \( x_\alpha + Cx_\alpha_1x_\beta = 0 \). Applying this process for all the \( \beta \) in \( \Sigma(\theta) \) described above we can make all \( x_\alpha \) in \[28\] equal to zero except for \( x_{\alpha_1} \) and \( x_\gamma \). In the process the value of \( x_{\alpha_1} \) does not change but the value of \( x_\gamma \) may change. We let \( a = x_{\alpha_1} \) and let \( x \) be the final value of \( x_\gamma \). This proves the lemma. \( \square \)
We now consider the adjoint action of $Z_0^M$.

**Lemma 4.7.** Let $n = u_{\alpha_1}(a)u_{\gamma}(x) \in N$ with $a \in F^\times$ and $x \in F$. Then, $n$ and $u_{\alpha_1}(1)u_{\gamma}(y)$ are in the same conjugacy class of the adjoint action of $Z_0^M$ on $N$ for some $y \in F$.

**Proof.** For $z = e_1^*(\lambda) \in Z_0^M$, in the odd case, we have

$$znz^{-1} = e_1^*(\lambda)u_{\alpha_1}(a)e_1^*(\lambda^{-1})e_1^*(\lambda)u_{\gamma}(x)e_1^*(\lambda^{-1})$$

$$= u_{\alpha_1}(\alpha_1(e_1^*(\lambda))a)u_{\gamma}(\alpha_1(e_1^*(\lambda))x)$$

$$= u_{\alpha_1}(\lambda^{\langle \alpha_1,e_1^* \rangle}a)u_{\gamma}(\lambda^{\langle \gamma,e_1^* \rangle}x)$$

$$= u_{\alpha_1}(\lambda a)u_{\gamma}(\lambda x).$$

In the even case $e_1^*$ above should be replaced by $E_1^*$. Take $\lambda = 1/a$ and $y = x/a$ to finish the proof. □

Lemmas 4.6 and 4.7 immediately imply the following:

**Corollary 4.8.** Up to a subset of measure zero of $N$ the following is a complete set of representatives for the orbits of $N$ under the adjoint action of $Z_0^M U_M$:

$$Z_0^M U_M \setminus N \simeq \{u_{\alpha_1}(1)u_{\gamma}(x) : x \in F\}.$$

If we set

$$w_0 = \begin{cases} w_{\alpha_1}w_{\alpha_2} \cdots w_{\alpha_n}w_{\alpha_{n+1}}w_{\alpha_n} \cdots w_{\alpha_2}w_{\alpha_1} & \text{if } G \text{ is odd,} \\ w_{\alpha_1}w_{\alpha_2} \cdots w_{\alpha_{n-1}}w_{\alpha_n}w_{\alpha_{n-1}} \cdots w_{\alpha_2}w_{\alpha_1} & \text{if } G \text{ is even.} \end{cases} \quad (29)$$

where $w_{\alpha\alpha}$ is the product of the commuting elements $w_{\alpha_1}w_{\alpha_{n+1}} = w_{\alpha_{n+1}}w_{\alpha_n}$, then $w_0$ is the representative in $G$ of the Weyl group element $w$ mentioned earlier.

**Remark 4.9.** Note that in (38) the analogue of our element $w_{\alpha\alpha}$ is denoted by $w_{\alpha_{n+1}}$ as explained in (4.45) of that article. However, our current choices, which applies equally well to $SO_{2n}$ or other cases treated in Section 4 of (38), would replace the two commuting matrices on the left hand side of (4.45) of (38) with their transposes (cf. Remark 4.4).

As in (38) we are interested in elements $n \in N$ such that

$$w_0^{-1}n = mn'\pi \in P\overline{N}. \quad (30)$$

The decomposition in (30) is clearly unique and we would like to compute the $m-$, $n'$-, and $\pi-$parts of an element $n$ as in Corollary 4.8. We will do this in Proposition 4.12. First we prove the following auxiliary lemma.
Lemma 4.10. We can normalize the $u_\alpha$’s such that the element $w_\gamma$ satisfies

$$
\gamma'(d)w_\gamma = w_\gamma(d) = \begin{cases} 
  w_{\alpha_2} \cdots w_{\alpha_n} w_{\alpha_{n+1}} w_{\alpha_n} \cdots w_{\alpha_2} w_{\alpha_1} & \text{if } G \text{ is odd}, \\
  w_{\alpha_2} \cdots w_{\alpha_n}^{-1} w_{\alpha_n}^{-1} w_{\alpha_{n-1}} \cdots w_{\alpha_2}^{-1} w_{\alpha_1}^{-1} & \text{if } G \text{ is even},
\end{cases}
$$

where

$$
d = \begin{cases} 
  (-1)^n & \text{if } G \text{ is odd}, \\
  (-1)^{n-1} & \text{if } G \text{ is even}.
\end{cases}
$$

Remark 4.11. The above $d$ in the odd case is slightly different from the corresponding value for the group $SO_{2n+3}$ carried out in 4.2.1 of [6], i.e., it differs by a factor of $-1$. The reason for this discrepancy is that the representative we have fixed above for the longest element of the Weyl group would, in the case of the group $SO_3$, lead to

$$
\begin{pmatrix} -1 & 1 \\
 1 & -1 \end{pmatrix}.
$$

This is the correct element which should have been used in 4.2.1 of [6] and [38] instead of

$$
\begin{pmatrix} -1 & 1 \\
 1 & -1 \end{pmatrix}.
$$

The latter is the corresponding Weyl group representative for the group $SL_3$. However, since the group $SO_3$ does not have a Cartan of the same dimension as that of $SL_3$ there is no natural way (i.e., not requiring a choice of basis) of embedding it in $SL_3$. Therefore, there is no reason why the representative for $SO_3$ would be the same as that of $SL_3$. Of course, both of these two matrices correspond to the same Weyl group element since they only differ by a diagonal matrix in $SO_3$. In the notation of the present paper we can fix $u_\alpha(\ )$ and $u_{-\alpha}(\ )$

in $SO_3$ such that $w_\alpha(\lambda) = \begin{pmatrix} -1 & -\lambda^2 \\
 1/\lambda^2 & 1 \end{pmatrix}$. The choice of representative above is now simply $w_\alpha(1)$.

Proof. We begin by noting that

$$
\gamma = \begin{cases} 
  \tilde{w}_{\alpha_2} \cdots \tilde{w}_{\alpha_n} \tilde{w}_{\alpha_{n+1}} \cdots \tilde{w}_{\alpha_2}(\alpha_1) & \text{if } G \text{ is odd}, \\
  \tilde{w}_{\alpha_2} \cdots \tilde{w}_{\alpha_{n-1}} \tilde{w}_{\alpha_n} \tilde{w}_{\alpha_{n+1}} \cdots \tilde{w}_{\alpha_2}(\alpha_1) & \text{if } G \text{ is even}.
\end{cases}
$$

Let $\beta_1 = \alpha_1$ and denote by $\beta_i$ the consecutive images of $\beta_1$ under the first $i-1$ Weyl group elements above for

$$
\begin{cases} 
  1 \leq i \leq 2n & \text{if } G \text{ is odd}, \\
  1 \leq i \leq 2n - 1 & \text{if } G \text{ is even}.
\end{cases}
$$
In fact, the $\beta_i$’s are precisely the roots listed in (26) and (27) and in the same order.

Now apply (22) repeatedly to conclude that the right hand side of the expression of the statement of the Lemma is equal to $w_\gamma(d)$, where

$$d = \begin{cases} 
  d_{\alpha_2, \beta_1} \cdots d_{\alpha_n, \beta_{n-1}} d_{\alpha_{n+1}, \beta_n} d_{\alpha_n, \beta_{n+1}} \cdots d_{\alpha_2, \beta_{2n-1}} & \text{if } G \text{ is odd,} \\
  d_{\alpha_2, \beta_1} \cdots d_{\alpha_n, \beta_{n-2}} d_{\alpha_{n+1}, \beta_n} d_{\alpha_n, \beta_{n+1}} \cdots d_{\alpha_2, \beta_{2n-2}} & \text{if } G \text{ is even.}
\end{cases}$$

For $1 \leq i \leq n - 1$ in the odd case and for $1 \leq i \leq n$ in the even case, the $\alpha_{i+1}$-string through $\beta_i$ is $(\beta_i, \beta_i + \alpha_{i+1})$, i.e., $c = 0$ and $b = 1$ in the notation of Proposition 4.5. In the odd case with $i = n$, the $\alpha_{i+1}$-string through $\beta_i$ is $(\beta_i, \beta_i + \alpha_{i+1}, \beta_i + 2\alpha_{i+1})$, i.e., $c = 0$ and $b = 2$. Similarly, for $1 \leq j \leq n - 1$ in the odd case the $\alpha_{n-j}$-string through $\beta_{n+j}$ is $(\beta_{n+j}, \beta_{n+j} + \alpha_{n-j-1})$, i.e., $c = 0$ and $b = 1$. Also, for $1 \leq j \leq n - 2$ in the even case the $\alpha_{n-j}$-string through $\beta_{n+j}$ is $(\beta_{n+j}, \beta_{n+j} + \alpha_{n-j})$, i.e., $c = 0$ and $b = 1$. Putting all these together and using (23) we can write

$$d_{\alpha_2, \beta_1} = c_{\alpha_2, \beta_1; 1, 1} \quad \quad d_{\alpha_2, \beta_1} = c_{\alpha_2, \beta_1; 1, 1}$$

$$d_{\alpha_n, \beta_{n-1}} = c_{\alpha_n, \beta_{n-1}; 1, 1} \quad \quad d_{\alpha_{n-1}, \beta_{n-2}} = c_{\alpha_{n-1}, \beta_{n-2}; 1, 1}$$

$$d_{\alpha_{n+1}, \beta_n} = c_{\alpha_{n+1}, \beta_n; 2, 1} \quad \quad d_{\alpha_n, \beta_{n-1}} = c_{\alpha_n, \beta_{n-1}; 1, 1}$$

$$d_{\alpha_n, \beta_{n+1}} = c_{\alpha_n, \beta_{n+1}; 1, 1} \quad \quad d_{\alpha_{n+1}, \beta_n} = c_{\alpha_{n+1}, \beta_n; 1, 1}$$

$$d_{\alpha_2, \beta_{2n-1}} = c_{\alpha_2, \beta_{2n-1}; 1, 1} \quad \quad d_{\alpha_2, \beta_{2n-2}} = c_{\alpha_2, \beta_{2n-2}; 1, 1}$$

in the odd and even cases respectively.

We can now normalize the $u_{\alpha_i}$’s such that we have $c_{\alpha_i, \alpha_{i+1}; 1, 1} = 1$ and, in the odd case, $c_{\alpha_n, \alpha_{n+1}; 1, 2} = -1$. These normalizations are motivated by the explicity matrix realizations of the related groups $SO_{2n+3}$ and $SO_{2n+2}$ such as those in [38]. The values of other structure constants are now uniquely determined by these (cf. Lemma 9.2.3 of [40]). We then get

$$c_{\alpha_2, \beta_1; 1, 1} = -1 \quad \quad c_{\alpha_2, \beta_1; 1, 1} = -1$$

$$c_{\alpha_n, \beta_{n-1}; 1, 1} = -1 \quad \quad c_{\alpha_n, \beta_{n-1}; 1, 1} = -1$$

$$c_{\alpha_{n+1}, \beta_2; 1, 1} = -1 \quad \quad c_{\alpha_{n+1}, \beta_2; 1, 1} = -1$$

$$c_{\alpha_n, \beta_{n+1}; 1, 1} = +1 \quad \quad c_{\alpha_n, \beta_{n+1}; 1, 1} = +1$$

$$c_{\alpha_2, \beta_{2n-1}; 1, 1} = +1 \quad \quad c_{\alpha_2, \beta_{2n-2}; 1, 1} = +1$$
Therefore, in the expression for $d$ the first $n$ terms in the odd case and the first $n - 1$ terms in the even case are equal to $-1$ and others are equal to $+1$. This implies that $d = (-1)^n$ in the odd case and $d = (-1)^{n-1}$ in the even case. □

Proposition 4.12. Assume that $x \in F$ and $n = u_{\alpha_1}(1) u_\gamma(x)$ satisfies (30). Moreover, assume that $x$ is non-zero (which rules out only a subset of $N$ of measure zero). Then,

$$m = w' \gamma'(d/x)$$
$$n' = u_\gamma(-x) u_{\alpha_1}(-1)$$
$$\overline{\pi} = u_{-\gamma}(1/x) u_{-\alpha_1}(1),$$

where $d$ is as in (31) and

$$w' = \begin{cases} w_{a_2}^{-1} \cdots w_{a_n}^{-1} w_{a_{n+1}}^{-1} w_{a_2}^{-1} \cdots w_{a_n}^{-1} & \text{if } G \text{ is odd}, \\ w_{a_2}^{-1} \cdots w_{a_{n-1}}^{-1} w_{a_n}^{-1} w_{a_{n-1}}^{-1} \cdots w_{a_2}^{-1} & \text{if } G \text{ is even}, \end{cases}$$

with $w_{\alpha\alpha}$ again as in Remark 4.9. Moreover, we could also write $m$ as

$$m = \alpha_1 \gamma'(d/x)w',$$  \hfill (33)

which is analogous to Propositions 4.4 and 4.8 of [38] modulo our Remark 4.4.

Proof. By uniqueness of decomposition in (30) it is enough to prove that the above values satisfy (30). This is a straightforward computation utilizing (21) multiple times.

First, observe that if $i, j > 0$ are integers, then $i \alpha_1 + j \gamma$ can not be a root. Hence, $u_\gamma(\cdot)$ and $u_{\alpha_1}(\cdot)$ commute by (21). Also, $i \alpha_1 + j(-\gamma)$ can not be a root, which again implies by (21) that $u_{-\gamma}(\cdot)$ and $u_{\alpha_1}(\cdot)$ commute. Similarly, $u_\gamma(\cdot)$ and $u_{-\alpha_1}(\cdot)$ commute.

Moreover, by (29) we have

$$w_0 = w_{\alpha_1} w'^{-1} w_{\alpha_1}$$  \hfill (34)

and by Lemma 4.10

$$w_\gamma(d) = w'^{-1} w_{\alpha_1} w'.$$  \hfill (35)
Now,

\[
\begin{align*}
    w_0^{-1}w &= w_0^{-1}w'w_0^{-1}u_{\alpha_1}(1)u_\gamma(x) \\
    &= w_0^{-1}w'u_{\alpha_1}(-1)u_{-\alpha_1}(1)u_{\alpha_1}(-1)u_{\alpha_1}(1)u_\gamma(x) \\
    &= w_0^{-1}w'u_\gamma(x)u_{\alpha_1}(-1)u_{-\alpha_1}(1) \\
    &= w_0^{-1}w'w_\gamma(x)u_{\alpha_1}(-1)u_{-\alpha_1}(1) \\
    &= w_0^{-1}w'w_\gamma(x)u_{\alpha_1}(-1)u_{\alpha_1}(1)u_{\gamma}(1/x)u_{\alpha_1}(1) \\
    &= w_0^{-1}w'w_\gamma(x)u_{\alpha_1}(-1)u_{\gamma}(1/x)u_{-\alpha_1}(1) \\
    &= w'w_{\gamma}(d)^{-1}w_\gamma(x) \cdot n' \cdot \pi \\
    &= w'(w_\gamma \gamma^\vee(1/d))^{-1}w_\gamma \gamma^\vee(1/x) \cdot n' \cdot \pi \\
    &= w' \gamma^\vee(d/x) \cdot n' \cdot \pi \\
    &= mn'n'\pi.
\end{align*}
\]

To see (33), we use (22) repeatedly to write

\[
\begin{align*}
    m &= w_0^{-1}w'w_\gamma(x) \\
    &= w_0^{-1}w'w_\gamma(x)w'^{-1} \cdot w' \\
    &= w_0^{-1}w_\alpha(Dx)w' \\
    &= w_0^{-1}w_\alpha_1(1/(Dx))w' \\
    &= \alpha_1^\vee(1/(Dx))w'.
\end{align*}
\]

where

\[
D = \begin{cases} 
    d_{-\alpha_2,\beta_2} \cdots d_{-\alpha_n,\beta_n}d_{-\alpha_{n+1},\beta_{n+1}} & \text{if } G \text{ is odd}, \\
    d_{-\alpha_2,\beta_2} \cdots d_{-\alpha_{n-1},\beta_{n-1}}d_{\alpha_n,\beta_n} & \text{if } G \text{ is even.}
\end{cases}
\]

Notice that conjugation by \(w'^{-1}\) sends \(\gamma = \beta_{2n}^\vee\) in the odd case and \(\gamma = \beta_{2n-1}\) in the even case back to \(\alpha_1\).

Finally, we claim that \(Dd = 1\) in both even and odd cases. To see this note that we can write

\[
Dd = \begin{cases} 
    \prod_{i=1}^{n} d_{\alpha_{i+1},\beta_i} d_{-\alpha_{i+1},\beta_{i+1}} \prod_{j=1}^{n-1} d_{\alpha_{j+1},\beta_{2n-j}} d_{-\alpha_{j+1},\beta_{2n+1-j}} & \text{if } G \text{ is odd}, \\
    \prod_{i=1}^{n} d_{\alpha_{i+1},\beta_i} d_{-\alpha_{i+1},\beta_{i+1}} \prod_{j=1}^{n-2} d_{\alpha_{j+1},\beta_{2n-j}} d_{-\alpha_{j+1},\beta_{2n-j-1}} & \text{if } G \text{ is even.}
\end{cases}
\]
true for case is just the compatibility of $\chi$ its conjugacy class under $G$ and notice that by (21) we know that Proposition 4.13. Let $n \in N$ satisfy (30). Then except for a subset of measure zero of $N$ we have

$$U_{M,n} = U_{M,m},$$

(37)

where notation is as in Section 4 of [38], i.e.,

$$U_{M,n} = \{u \in U_M : u n u^{-1} = n\}$$

and

$$U_{M,m} = \{u \in U_M : m u m^{-1} \in U_M \text{ and } \chi(m u m^{-1}) = \chi(m)\}.$$

Note that the condition $\chi(m u m^{-1}) = \chi(m)$ in the definition of $U_{M,m}$ in our case is just the compatibility of $\chi$ with elements of the Weyl group (cf. pages 2079–80 of [38]).

Proof. By arguments such as those on page 2085 of [38] if the proposition is true for $n \in N$, then it is also true for every member of the intersection of its conjugacy class under $G$ with $N$ provided that for the $m$-part we take the twisted conjugacy classes (cf. (4.10) of [38]). Hence, it is enough to verify the proposition for those $n$ as in Corollary 4.8. Fix one such $n = u_{\alpha_1}(1)u_\gamma(q)$ with $q \neq 0$ for the rest of this proof.

We can explicitly compute both sides of (37) as follows. Any $u \in U_M$ can be written as

$$u = \prod_{\beta \in \Sigma(\theta)^+} u_\beta(x_\beta),$$

(39)

where the order of the terms in the product is with respect to the total order of $R$ we have fixed.

We have $u n u^{-1} = n$ if and only if $u u_{\alpha_1}(1)u_\gamma(q)u^{-1} = u_{\alpha_1}(1)u_\gamma(q)$. Notice that by (21) we know that $u_\gamma(q)$ commutes with all $u_\beta(x_\beta)$ in the above product. Hence, $u \in U_{M,n}$ if and only if $u u_{\alpha_1}(1)u^{-1} = u_{\alpha_1}(1)$. Among the terms $u_\beta(x_\beta)$ the element $u_{\alpha_1}(1)$ commutes with those with $\beta \in \Sigma(\Omega)^+$ where
\( \Omega = \Delta - \{\alpha_1, \alpha_2\} \). Also, if \( \beta \in \Sigma(\theta)^+ - \Sigma(\Omega)^+ \), then so is \( \gamma = \beta - \alpha_1 \). Using \( \Sigma(\Omega) \) several times we can now write

\[
uu_{\alpha_1}(1)u^{-1} = \prod_{\beta \in \Sigma(\theta)^+} u_{\beta}(x_{\beta}) u_{\alpha_1}(1) \left( \prod_{\beta \in \Sigma(\theta)^+} u_{\beta}(x_{\beta}) \right)^{-1}
\]

\[
= u_\gamma \left( \sum c_{\alpha_1, \beta, 1, 1} c_{\alpha_1 + \beta, \alpha_1, 1} x_{\beta} x_{\delta} \right) \cdot \prod_{\beta \in \Sigma(\theta)^+ - \Sigma(\Omega)^+} u_{\alpha_1 + \beta} (-c_{\alpha_1, \beta} x_{\beta}) \cdot u_{\alpha_1}(1),
\]

where the sum in the first term is over unordered pairs \((\beta, \delta)\) of roots in \( \Sigma(\theta)^+ - \Sigma(\Omega)^+ \) such that \( \beta + \delta = \gamma - \alpha \) and \( \beta \neq \delta \). Here the order of terms is prescribed by the order we fixed in (39). This implies that \( uu_{\alpha_1}(1)u^{-1} = u_{\alpha_1}(1) \) if and only if \( x_{\beta} = 0 \) for all \( \beta \in \Sigma(\theta)^+ - \Sigma(\Omega)^+ \). Therefore,

\[
U_{M, n} = \left\{ \prod_{\beta \in \Sigma(\Omega)^+} u_{\beta}(x_{\beta}) \right\}.
\]

(40)

To compute \( U'_{M, m} \) note that with \( d \) as in (31) we have

\[
m um^{-1} = w' \gamma^\vee(d/q) \prod_{\beta \in \Sigma(\theta)^+} u_{\beta}(x_{\beta}) \gamma^\vee(d/q)^{-1} w'^{-1}
\]

\[
= w' \prod_{\beta \in \Sigma(\theta)^+} u_{\beta}(\gamma^\vee(d/q)) x_{\beta} w'^{-1}
\]

\[
= \prod_{\beta \in \Sigma(\theta)^+} w' u_{\beta}(\gamma^\vee(d/q)) x_{\beta} w'^{-1}.
\]

Conjugation by the element \( w' \) sends each positive root group with root in \( \Sigma(\theta)^+ - \Sigma(\Omega)^+ \) to a root group corresponding to a negative root and sends those with roots in \( \Sigma(\Omega)^+ \) to themselves. Therefore, again

\[
U'_{M, m} = \left\{ \prod_{\beta \in \Sigma(\Omega)^+} u_{\beta}(x_{\beta}) \right\}.
\]

(41)

Now (37) follows from (40) and (41).

We would like to have an explicit identification of \( \text{GL}_1(F) \times G_n^\sim(F) \) with \( M \) as a Levi subgroup of \( G \). Going back to our descriptions of the groups \( \text{GSpin}_{2n+1} \) and \( \text{GSpin}_{2n} \) in Section 2, note that if we consider the root datum obtained from that of \( G \) by eliminating \( e_1 \) and \( e_1^* \) in the odd case and \( E_1 \) and \( E_1^* \) in the even case as well as the root \( \alpha_1 \) and its corresponding coroot, then it corresponds to a subgroup of \( G \) isomorphic to \( G_n^\sim \). Denote the \( F \)-points of this subgroup by \( G_n^\sim \). Let \( k \in G_n^\sim \) and let \( a \in F^\times \). We claim that \( e_1^*(a) \) in the odd case and \( E_1^*(a) \) in the even case and \( k \) commute. To see this it is enough
to observe that \( e^*_1(a) \) or \( E^*_1(a) \) commutes with \( u_\beta(x) \) for all \( \beta \in \Sigma(\theta) \) since \( G_\alpha^\sim \) is generated by the corresponding \( U_\beta \)'s along with a subtorus of \( T \). By \( (15) \) we have
\[
e^*_1(a)u_\beta(x)e^*_1(a)^{-1} = u_\beta(\beta(e^*_1(a))x) = u_\beta(\alpha(\beta,e^*_1)x)
\]
and similarly for \( E^*_1(a) \). Moreover, \( \langle \beta, e^*_1 \rangle = 0 \) for all \( \beta \in \Sigma(\theta) \). Therefore, \( e^*_1(a) \) in the odd case and \( E^*_1(a) \) in the even case and \( u_\beta(x) \) commute. This implies that the map \( (a,k) \mapsto e^*_1(a)k \) in the odd case and \( (a,k) \mapsto E^*_1(a)k \) in the even case is a homomorphism which gives the identification of \( \text{GL}_1(F) \times G_\alpha^\sim(F) \) with \( M \). In particular, the element \( m = \alpha_1 \gamma(d/x)w' \) in \( (32) \) identifies with
\[
(d/x, e^*_2(x/d)w') \text{ or } (d/x, E^*_2(x/d)w') \in \text{GL}_1(F) \times G_\alpha^\sim(F)
\]
\( (42) \) since \( \alpha_1 \gamma = e^*_1 - e^*_2 \) in the odd case and \( \alpha_1 \gamma = E^*_1 - E^*_2 \) in the even case. (Notice that \( w' \in G_\alpha^\sim \).) Moreover, \( e^*_2(x/d)w' \) or \( E^*_2(x/d)w' \) is an element of a maximal Levi subgroup in \( G_\alpha^\sim(F) \) just as in the case of classical groups in \( \text{38} \).

We are now prepared to express the \( \gamma \)-factors as Mellin transforms.

**Proposition 4.14.** Let \( \sigma \) be an irreducible admissible \( \psi \)-generic representation of \( G_\alpha^\sim(F) \) (cf. Remark 4.2). Consider \( \text{GL}_1 \times G_\alpha^\sim \) as a standard Levi in \( G \) as above. Let \( \eta \) be any non-trivial character of \( F^\times \) with \( \eta^2 \) ramified. Then
\[
\gamma(s, \eta \times \sigma, \psi)^{-1} = g(s, \eta) \cdot \int_{F^\times} j_v, \underline{\mathbb{N}}_0(a(x)w')\eta(x)|x|^{s-n-\delta}dx^\times, \tag{43}
\]
where \( a(x) = e^*_2(x/d) \) or \( E^*_2(x/d) \) as in \( (42) \), \( \delta = 1/2 \) and \( g(s, \eta) = \eta(-1)^n \) in the odd case and \( \delta = 1 \) and \( g(s, \eta) = \eta(-1)^{n-1}\gamma(2s, \eta^2, \psi)^{-1} \) in the even case with \( d \) as in \( (32) \). Here \( v \in V_\sigma \) and \( W_v \in W(\sigma, \psi) \) with \( W_v(e) = 1 \), where \( W(\sigma, \psi) \) denotes the Whittaker model of \( \sigma \). Moreover, \( \underline{\mathbb{N}}_0 \subset \underline{\mathbb{N}} \) is a sufficiently large compact open subgroup of the opposite unipotent subgroup \( \mathbb{N} \) to \( N \), where \( P = MN \subset G \) is the Levi decomposition of the corresponding standard parabolic subgroup. The function \( j_{v, \underline{\mathbb{N}}_0} \) denotes the partial Bessel function defined in \( \text{38} \).

**Proof.** Given that \( \eta^2 \) is ramified this proposition is the main result of \( \text{38} \), (6.39) of Theorem 6.2, applied to our cases. Notice that the two hypotheses of that theorem, i.e., Assumptions 4.1 and 5.1 of \( \text{38} \), for our cases are our Propositions \( 14.13 \) and \( 14.13 \) respectively.

To get from (6.39) of \( \text{38} \) to \( (13) \) above note that we have
\[
\omega^{-1}_\sigma(\hat{x}_\alpha)(u_0\omega_\sigma)(\hat{x}_\alpha) = \eta(x)^2|x|^{2s}
\]
Moreover, as in Section 7 of \( \text{38} \) we have
\[
q^{(s\sigma + \rho, H_M(m))} d\tilde{n} = |x|^{-n-s}dx^\times
\]
and
\[ j_{v,\mathbb{N}_0}(\tilde{m}) = \eta(d/x)j_{v,\mathbb{N}_0}(a(x)w'), \]
where \( a(x) = e_2^*(x/d) \) in the odd case and \( E_2^*(x/d) \) in the even case. \qed\\

In the next section we first rewrite (43) in terms of Bessel functions defined as in [8] and then study their asymptotics.

4.2. Bessel functions and their asymptotics. We now briefly review some basic facts from [8]. In view of [6] and particularly [10, 11] which will study these issues in more generality, we only concentrate on the cases at hand in this article and leave out the details of the more general situation, thereby simplifying some of the notations.

We will use the same notation as [8]. Consider the group \( G \) in both even and odd cases and \( w' \) above as an element of its Weyl group. Notice that \( w' \) supports a Bessel function and is minimal (in Bruhat order) non-trivial with respect to this property. Moreover, \( A_{w'} = Z_{M_\Omega} \), where \( \Omega = \Delta - \{\alpha_1, \alpha_2\} \), \( M_\Omega \) is the standard Levi determined by \( \Omega \), and \( Z_{M_\Omega} \) denotes its center.

Let \( \sigma \) be an irreducible admissible \( \psi \)-generic representation of \( G \) and take \( v \in V_\sigma \) such that the associated Whittaker function \( W_v \in \mathcal{W}(\sigma, \psi) \) satisfies \( W_v(e) = 1 \). The associated Bessel function on \( Z_{M_\Omega} \) is defined via

\[ J_{\sigma,w',v,Y}(a) = \int_{Y} W_v(aw'y)\psi^{-1}(y)dy, \quad (45) \]

where \( a \in Z_{M_\Omega} \) and \( U_w' = \prod U_\alpha \), where the product is over all those \( \alpha \in \Sigma(\theta)^+ \) with \( w'(\alpha) < 0 \) and \( U_\alpha \) is as before.

Similar to [8, 6] we have that \( J_{\sigma,w'} \) exists and is independent of \( v \in V_\sigma \) and for convergence purposes we use a slight modification of it, namely, the partial Bessel function

\[ J_{\sigma,w',v,Y}(a) = \int_{Y} W_v(aw'y)\psi^{-1}(y)dy, \quad (45) \]

where \( Y \subset U_w' \) is a compact open subgroup.

4.2.1. Domain of Integration. We now show that the partial Bessel functions of [38] are the same as those in [8].

Recall that \( M = M_\theta = \text{GL}_1(F) \times G_n^*(F) \subset G = G(F) \) and consider \( m \in M \) as in [12], i.e., \( m = (d/x, e_2^*(x/d)w') \) in the odd case and \( m = (d/x, E_2^*(x/d)w') \) in the even case with \( d \) as in [31]. We now prove that the Bessel functions of [38] and those of [8] are actually the same.

Lemma 4.15. We can choose \( Y \) appropriately so that with \( m' = e_2^*(x/d)w' \) or \( E_2^*(x/d)w' \) as above we have

\[ j_{v,\mathbb{N}_0}(m') = \begin{cases} J_{\sigma,w',v,Y}(e_2^*(x/d)) & \text{in the odd case} \\ J_{\sigma,w',v,Y}(E_2^*(x/d)) & \text{in the even case} \end{cases} \]
Here, $m'$ is an element of a maximal Levi $M' = \text{GL}_1(F) \times G_{n-1}^\sim(F)$ in $G_n^\sim(F)$.

**Proof.** Let us first recall Theorem 6.2 of [38]. In the notation of that paper we have $j_{\psi, N_0}(m') = j_{\psi, N_0}(m', y_0)$ with $y_0 \in F^\times$ satisfying $\text{ord}_F(y_0) = -\text{cond}(\psi) - \text{cond}(\eta^2)$. Here, the function $j_{\psi, N_0}(m', y_0)$ is given by

$$
\int_{U_{M',\alpha} \setminus U_M} W_v(m'u^{-1})\phi(u\alpha^\vee(y_0)^{-1}\alpha^\vee(x_a)\overline{\alpha^\vee(x_a)}^{-1}\alpha^\vee(y_0)u^{-1})\psi(u)du,
$$

(46)

where $\phi$ is the characteristic function of $N_0$, $x_a = 1/x$, and $\overline{\alpha}$ is as in Proposition 4.2. Again as in [39, 6] it follows from Proposition 4.3 that we can take $U_{M',\alpha} \setminus U_M$ to be $U_w^-$. Notice that this only depends on $w'$. On the other hand, $u \in U_{w^-}$ is in the domain of integration if and only if

$$u\alpha^\vee(y_0)^{-1}\alpha^\vee(x_a)\overline{\alpha^\vee(x_a)}^{-1}\alpha^\vee(y_0)u^{-1} \in N_0.
$$

This condition is equivalent to $u\alpha^\vee(x_a)\overline{\alpha^\vee(x_a)}^{-1}u^{-1} \in \alpha^\vee(y_0)N_0\alpha^\vee(y_0)^{-1}$ and $\alpha^\vee(y_0)N_0\alpha^\vee(y_0)^{-1}$ is another compact open subgroup of the same type as $N_0$ which we may replace it with.

Recall that an arbitrary element of $N$ is given by

$$\overline{n}(y) = \prod_{\alpha \in R(N)} u_\alpha(y_\alpha),
$$

(47)

where $y = (y_\alpha)_{\alpha \in R(N)}$ and $y_\alpha \in F$. Also recall that $\overline{n}$ in (46) was given by $\overline{n} = u_{-\gamma}(1/x)u_{-\alpha_1}(1)$. Moreover, note that $x_\alpha = 1/x$. Hence, $\alpha^\vee(x_a)\overline{\alpha^\vee(x_a)}^{-1} = \overline{n}(y')$, where $y'_{-\gamma} = 1$ and $y'_{-\alpha_1} = x$ and all other coordinates of $y'$ are zero (cf. [38]). Of course, throughout we have a fixed ordering of the roots in the products similar to that of (28). Hence, the domain of integration is determined by $u\overline{n}(y')u^{-1} \in N_0$.

We may take $N_0 = \{\overline{n}(y) : y_\alpha \in p^M_\alpha\}$ for all $\alpha \in R(N)$ for sufficiently large integer vector $M = (M_\alpha)_{\alpha \in R(N)}$. As the $M_\alpha$ increase, $N_0$ will exhaust $N$.

On the other hand, any $u \in U_w^-$ is given by

$$u = u(b) = \prod_{\alpha \in \Sigma^+(\Omega)} u_\alpha(b_\alpha)
$$

(48)

for $b = (b_\alpha)_{\alpha \in \Sigma^+(\Omega)}$ and $b_\alpha \in F$. Now, $u\overline{n}(y')u^{-1} = \overline{n}(y'')$ where $y''$ depends linearly on $b$ and $y'$. In other words, $y''$ depends upon $x$ and $b$. Of course, we could compute $y''$ explicitly in terms of $x$ and $b$ using structure constants; however, that will have no bearing on what follows and is not needed. Now, choose $Y = \{u = u(b) : y'' \geq M\}$. This defines the domain of integration. Enlarging $N_0$ if need be will then imply that $j_{\psi, N_0}(m)$ does not depend on $m$ and we conclude the lemma. □
Therefore, we can rewrite (43) as
\[
\gamma(s, \eta \times \sigma, \psi)^{-1} = g(s, \eta) \cdot \int_{F} J_{\sigma, w', v, Y}(a(x)) \eta(x) |x|^{s-n-\delta} dx,
\]
with \(a(x) = c_2^*(x/d)\) or \(E_2^*(x/d)\).

4.2.2. Asymptotics of Bessel functions. We now study the asymptotics of our Bessel functions near zero and infinity. This will allow us to prove our result on stability given that the \(\gamma\)-factors are already written as the Mellin transform of Bessel functions.

Our starting point is the following analogue of proposition 5.1 of [8] for our groups. Note that the proposition was only proved for the group \(SO_{2n+1}\). However, as was pointed out in [6], the methods used to prove it are quite general. This was pointed out for classical groups (with finite center) in [6] but the same also holds for GSpin or GSpin\(^-\) groups which are of interest to us since the only difference is the infinite center which is already contained in the fixed Borel subgroup we are modding out with.

**Proposition 4.16.** There exists a vector \(v'_\sigma \in V_\sigma\) and a compact neighborhood \(BK_1\) of the identity in \(B \backslash G^-_n\) such that if \(\chi_1\) is the characteristic function of \(BK_1\), then for all sufficiently large compact open sets \(Y \subset U^-_w\) we have
\[
J_{\sigma, w, v, Y}(a) = \int_{Y} W_v(away) \chi_1(away) \psi^{-1}(y) dy + W_{v'_\sigma}(a).
\]

Notice that \(w\) here would be our earlier \(w'\) if we want to consider the group \(G^-_n\) as part of the Levi \(M\) in \(G\) as in (42).

We now would like to rewrite this in a way that only depends on the central character of \(\sigma\). To this end we argue similar to [6] making some necessary modifications along the way. For any positive integer \(M\) set
\[
U(M) = \langle u_\alpha(t) : \alpha \in \Delta, |t| \leq q^M \rangle.
\]
These are compact open subgroups of \(U\) and as the integer \(M\) grows, they exhaust \(U\). For any \(v \in V_\sigma\) we define
\[
v_M = \frac{1}{\text{Vol}(U(M))} \int_{U(M)} \psi^{-1}(u) \sigma(u) v du.
\]
Smoothness of \(\sigma\) implies that this is a finite sum and \(v_M \in V_\sigma\). Then just as in [8] if \(Y\) is sufficiently large relative to \(M\), then we may choose \(v'_\sigma\) and \(K_1\) such that \(K_1 \subset \text{Stab}(v_M)\) and we have
\[
\int_{Y} W_v(away) \chi_1(away) \psi^{-1}(y) dy = \int_{Y} W_{v_M}(away) \chi_1(away) \psi^{-1}(y) dy.
\]
Write \( awy = utk_1 \) with \( u \in U, t \in T \) and \( k_1 \in K_1 \). Then \( K_1 \subset \text{Stab}(v_M) \) implies that \( W_{v_M}(awy) = \psi(u)W_{v_M}(t) \). As in [3] the support of \( W_{v_M} \) is contained in

\[
T_M = \{ t \in T : \alpha(t) \in 1 + p^M \text{ for all simple } \alpha \}.
\] (52)

At this point we would like to assume that the center \( Z \) is connected which we can do (cf. Remark 4.2). This is why we chose to work with the group \( \text{GSpin}^\sim \) in the even case in this section. By connectedness of \( Z \) and since the groups are split we have the following exact sequence of \( F \)-points of tori

\[
0 \rightarrow Z \rightarrow T \rightarrow T_{ad} \rightarrow 0
\] (53)

which splits (cf. [10, 11]). Recall that \( Z = Z(F) \) and so on. Identify \( T_{ad} \) with \( (F^\times)^n \) through values of roots and let \( T^n_M \subset T \) be the image of

\[
(1 + p^M)^n \subset T_{ad},
\]

under the splitting map. Here, the rank of \( T_{ad} \) is \( n \) and \( T_M = ZT^n_M \).

Now if \( t \in T \), then we can write \( t = zt^1 \) with \( z \in Z \) and \( t \in T^n_M \). Also we have \( W_v(t) = W_{v_M}(t) \) and if we choose \( M \) large enough so that \( T^n_M \subset T \cap \text{Stab}(v) \), then

\[
W_{v_M}(t) = W_v(t) = W_v(zt^1) = \omega_\sigma(z)W_v(t^1) = \omega_\sigma(z).
\]

Therefore, in our integral, \( W_{v_M}(awy)\chi_1(awy) \neq 0 \) if and only if \( awy \in UT_MK_1 \) or \( y \in (aw)^{-1}UT_MK_1 \). Writing \( awy = utk_1 = u(awy)z(awy)t^1k_1 \) then implies that

\[
\int_{Y} W_v(awy)\chi_1(awy)\psi^{-1}(y)dy = \int_{Y \cap (aw)^{-1}UT_MK_1} \psi(u(awy))\psi^{-1}(y)\omega_\sigma(z(awy))dy.
\] (54)

Therefore, we can rewrite Proposition 4.16 as follows.

**Proposition 4.17.** Let \( v \in V_\sigma \) with \( W_v(e) = 1 \) and choose \( M \) sufficiently large so that \( T^n_M \subset T \cap \text{Stab}(v) \). There exists a vector \( v'_\sigma \in V_\sigma \) and a compact open subgroup \( K_1 \) such that for \( Y \) sufficiently large we have

\[
J_{\sigma,w,v,Y}(a) = \int_{Y \cap (aw)^{-1}UT_MK_1} \psi(u(awy))\psi^{-1}(y)\omega_\sigma(z(awy))dy + W_{v'_\sigma}(a).
\]

4.3. **Proof of Theorem 4.11** We are now prepared to prove Theorem 4.11.

**Proof.** Let \( \sigma_i = \pi_i, i = 1, 2 \) in the odd case. In the even case choose a character \( \mu \) of the center \( Z^\sim \) of \( \text{GSpin}_{2n}^\sim(F) \) (which contains the center of \( \text{GSpin}_{2n}(F) \)) such that \( \mu \) agrees with the central characters \( \omega_{\pi_1} = \omega_{\pi_2} \) on the center of \( \text{GSpin}_{2n}(F) \). Consider the representation of \( \text{GSpin}_{2n}^\sim(F) \) induced from the representation \( \mu \otimes \pi_i \) on \( Z^\sim \cdot \text{GSpin}_{2n}(F) \) (which is of finite index in \( \text{GSpin}_{2n}(F) \))
and let $\sigma_i$ be an irreducible constituent of this induced representation (cf. [12]). Note that the choice of $\sigma_i$ is irrelevant. Then,

$$\gamma(s, \eta \times \sigma_i, \psi) = \gamma(s, \eta \times \pi_i, \psi),$$

by Remark [12]. Also, the assumption $\omega_{\pi_1} = \omega_{\pi_2}$ implies $\omega_{\sigma_1} = \omega_{\sigma_2}$.

Choose $v_i \in V_{\sigma_i}, i = 1, 2$ with $W_{v_i}(e) = 1$ and let $M$ be a large enough integer so that $T_M^1 \subset T \cap \text{Stab}(v_i)$. Choose a compact open subgroup $K_0 \subset \text{Stab}(v_1) \cap \text{Stab}(v_2)$. Then in Proposition [4.17] we may take

$$K_1 = \bigcap_{u \in U(M)} u^{-1}K_0u,$$

i.e., we can take the same $K_1$ for both $\sigma_1$ and $\sigma_2$. Consequently, by Proposition [4.17] there exist $v'_{\sigma_i} \in V_{\sigma_i}$ such that

$$J_{\sigma_1,w,v,Y}(a) = \int_{M_1(w,1)} \psi(u(aw))\psi^{-1}(y)\omega_{\sigma_1}(z(aw))dy + W_{v'_{\sigma_1}}(a). \quad (55)$$

Now $\omega_{\sigma_1} = \omega_{\sigma_2}$ implies that

$$J_{\sigma_1,w,v,Y}(a) - J_{\sigma_2,w,v,Y}(a) = W_{v'_{\sigma_1}}(a) - W_{v'_{\sigma_2}}(a). \quad (56)$$

Now taking $a = a(x)$ to be $e(x/d)$ or $E_2^*(x/d)$ and $w$ to be $w'$ described before, we apply [4.19] to conclude that

$$\gamma(s, \eta \times \sigma_1, \psi)^{-1} - \gamma(s, \eta \times \sigma_2, \psi)^{-1}$$

$$= g(s, \eta) \int_{F^x} (J_{\sigma_1,w,v,Y}(a(x)) - J_{\sigma_2,w,v,Y}(a(x))) \eta(x) |x|^{s-n+\delta}d^x x$$

$$= g(s, \eta) \int_{F^x} (W_{v'_{\sigma_1}}(a(x)) - W_{v'_{\sigma_2}}(a(x))) \eta(x) |x|^{s-n+\delta}d^x x.$$

However, note that Whittaker functions are smooth and for $\Re(s) >> 0$ and $\eta$ sufficiently ramified we have

$$\int_{F^x} W_{v'_{\sigma_i}}(a(x))\eta(x)|x|^{s-n+\delta}d^x x \equiv 0.$$

Hence, for $\Re(s) >> 0$ we have $\gamma(s, \eta \times \sigma_1, \psi)^{-1} - \gamma(s, \eta \times \sigma_2, \psi)^{-1} \equiv 0$ which then implies $\gamma(s, \eta \times \sigma_1, \psi) = \gamma(s, \eta \times \sigma_2, \psi)$ for all $s$ by analytic continuation. Therefore, $\gamma(s, \eta \times \pi_1, \psi) = \gamma(s, \eta \times \pi_2, \psi)$.

4.4. Stable Form of $\gamma(s, \eta \times \pi, \psi)$. We now prove some consequences of Theorem [4.1] which are important to us later.

First, let us compute the stable form of Theorem [4.1] by taking $\pi_2$ to be an appropriate principal series representation and computing its right hand side explicitly.
Proposition 4.18. Let \( \pi \) be an irreducible generic representation of \( G_n(F) \) with central character \( \omega = \omega_\pi \). Let \( \mu_1, \ldots, \mu_n \) be \( n \) character of \( F^\times \). Then for every sufficiently ramified character \( \eta \) of \( F^\times \) we have

\[
\gamma(s, \eta \times \pi, \psi) = \prod_{i=1}^{n} \gamma(s, \eta \mu_i, \psi) \gamma(s, \eta \omega \mu_i^{-1}, \psi).
\]

Proof. Set \( \mu_0 = \omega \) and consider the character

\[
\mu = (\mu_0 \circ e_0) \otimes (\mu_1 \circ e_1) \otimes \cdots \otimes (\mu_n \circ e_n)
\]

of \( T(F) \) with \( e_i \)'s as in Section 2.1. Proposition 2.5 implies that the restriction of the character \( \mu \) to the center of \( G_n(F) \) is \( \mu_0 = \omega \). Consider the induced representation \( \text{Ind}(\mu) \) from the Borel to \( G_n(F) \). Reordering the \( \mu_i \) if necessary, we may assume that it has an irreducible admissible generic subrepresentation \( \pi_2 \) (cf. Proposition 3.2). Since \( \omega_{\pi_2} = \mu_0 = \omega = \omega_\pi \), we can apply Theorem 4.1 to get

\[
\gamma(s, \eta \times \pi, \psi) = \gamma(s, \eta \times \pi_2, \psi).
\]

Multiplicativity of \( \gamma \)-factors can now be used to compute the right hand side to get

\[
\gamma(s, \eta \times \pi_2, \psi) = \prod_{i=1}^{n} \gamma(s, \eta \mu_i, \psi) \gamma(s, \eta \omega \mu_i^{-1}, \psi)
\]

which finishes the proof. \( \square \)

Corollary 4.19. Let \( \pi \) be an irreducible generic representation of \( G_n(F) \) with central character \( \omega = \omega_\pi \). Let \( \mu_1, \ldots, \mu_n \) be \( n \) character of \( F^\times \) as in Proposition 4.18. Then for every sufficiently ramified character \( \eta \) of \( F^\times \) we have

\[
L(s, \eta \times \pi) \equiv 1
\]

and

\[
\epsilon(s, \eta \times \pi, \psi) = \prod_{i=1}^{n} \epsilon(s, \eta \mu_i, \psi) \epsilon(s, \eta \omega \mu_i^{-1}, \psi).
\]

Proof. If \( \eta \) is sufficiently ramified, then by [37] we have

\[
L(s, \eta \times \pi) \equiv 1.
\]

This implies that \( \epsilon(s, \eta \times \pi, \psi) = \gamma(s, \eta \times \pi, \psi) \). Moreover, since \( \eta \) is highly ramified so is each \( \eta \mu_i \) and \( \eta \omega \mu_i^{-1} \) which implies that \( L(s, \eta \mu_i) \equiv 1 \) and \( L(s, \eta \omega \mu_i^{-1}) \equiv 1 \). Therefore, \( \epsilon(s, \eta \mu_i, \psi) = \gamma(s, \eta \mu_i, \psi) \) and \( \epsilon(s, \eta \omega \mu_i^{-1}, \psi) = \gamma(s, \eta \omega \mu_i^{-1}, \psi) \). Now the second statement of the corollary follows from Proposition 4.18. \( \square \)
5. Analytic properties of global $L$-functions

In this section we prove the properties of global $L$-functions that we need in order to apply the converse theorem.

We again let $G_n$ denote either the group $GSpin_{2n+1}$ or $GSpin_{2n}$ as in Section 2.1. Let $k$ be a number field and let $A$ denote its ring of adeles. Let $S$ be a finite set of finite places of $k$. Let $T(S)$ denote the set of irreducible cuspidal automorphic representations $\tau$ of $GL_r(A)$ for $1 \leq r \leq N-1$ such that $\tau_v$ is unramified for all $v \in S$. If $\eta$ is a continuous complex character of $k^* \backslash A^*$, then let $T(S; \eta) = \{ \tau = \tau' \otimes \eta : \tau' \in T(S) \}$.

If $\pi$ is a globally generic cuspidal representation of $G_n(A)$ and $\tau$ is a cuspidal representation of $GL_r(A)$ in $T(S; \eta)$, then $\sigma = \tau \otimes \pi$ is a (unitary) cuspidal globally generic representation of $M(A)$, where $M = GL_r \times G_n$ is a Levi subgroup of a standard parabolic subgroup in $G_{r+n}$. The machinery of the Langlands-Shahidi method as in Section 3 now applies [34, 36]. Recall that

$$L(s, \pi \times \tau) = \prod_v L(s, \pi_v \times \tau_v), \quad (58)$$

$$\epsilon(s, \pi \times \tau) = \prod_v \epsilon(s, \pi_v \times \tau_v, \psi_v), \quad (59)$$

where the local factors are as in [53] and [6].

**Proposition 5.1.** Let $S$ to be a non-empty set of finite places of $k$ and let $\eta$ be a character of $k^* \backslash A^*$ such that $\eta_v$ is highly ramified for $v \in S$. Then for all $\tau \in T(S; \eta)$ the $L$-function $L(s, \pi \times \tau)$ is entire.

**Proof.** These $L$-functions are defined via the Langlands-Shahidi method as we outlined in Section 3. Now, the proposition is a special case of a more general result, Theorem 2.1 of [22]. Note that we have proved the necessary assumption of that theorem, Assumption 1.1 of [22], for our cases in Proposition 3.5. □

The following lemma is an immediate consequence of Proposition 3.5.

**Lemma 5.2.** The global normalized intertwining operator $N(s, \sigma, w)$ is a holomorphic and non-zero operator for $\Re(s) \geq 1/2$.

**Proposition 5.3.** For any cuspidal automorphic representation $\tau$ of $GL_r(A_F)$, $1 \leq r \leq 2n-1$, the $L$-function $L(s, \pi \times \tau)$ is bounded in vertical strips.

**Proof.** This follows as a consequence of Theorem 4.1 of [13] along the lines of Corollary 4.5 thereof, given the fact that we have proved Assumption 2.1 of [13] in our Proposition 5.2. □

**Proposition 5.4.** For any cuspidal automorphic representation $\tau$ of $GL_r(A_F)$, $1 \leq r \leq 2n$, we have the functional equation

$$L(s, \pi \times \tau) = \epsilon(s, \pi \times \tau)L(1-s, \bar{\pi} \times \bar{\tau}).$$
6. Proof of Main Theorem

As mentioned before, we will use the following variant of converse theorems of Cogdell and Piatetski-Shapiro.

**Theorem 6.1.** Let \( \Pi = \otimes \Pi_v \) be an irreducible admissible representation of \( \text{GL}_N(\mathbb{A}) \) whose central character \( \omega_{\Pi} \) is invariant under \( k^\times \) and whose \( L \)-function \( L(s, \Pi) = \prod_v L(s, \Pi_v) \) is absolutely convergent in some right half plane. With notation as in Section 5, suppose that for every \( \tau \in \mathcal{T}(S; \eta) \) we have

1. \( L(s, \Pi \times \tau) \) and \( L(s, \tilde{\Pi} \times \tilde{\tau}) \) extend to entire functions of \( s \in \mathbb{C} \),
2. \( L(s, \Pi \times \tau) \) and \( L(s, \tilde{\Pi} \times \tilde{\tau}) \) are bounded in vertical strips, and
3. \( L(s, \Pi \times \tau) = \epsilon(s, \Pi \times \tau) L(1 - s, \tilde{\Pi} \times \tilde{\tau}) \).

Then, there exists an automorphic representation \( \Pi' \) of \( \text{GL}_N(\mathbb{A}) \) such that \( \Pi_v \simeq \Pi'_v \) for all \( v \notin S \).

Here, the twisted \( L \)- and \( \epsilon \)-factors are defined via

\[
L(s, \Pi \times \tau) = \prod_v L(s, \Pi_v \times \tau_v) \quad \epsilon(s, \Pi \times \tau) = \prod_v \epsilon(s, \Pi_v \times \tau_v, \psi_v)
\]

with local factors as in [7].

This is the exact variant of converse theorems that appeared in Section 2 of [5].

We can now prove Theorem 1.1.

**Proof.** — We apply Theorem 6.1 with \( N = 2n \). We continue to denote by \( \mathbf{G}_n \) either of \( \text{GSpin}_{2n+1} \) or \( \text{GSpin}_{2n} \). First, we introduce a candidate for the representation \( \Pi \). Consider \( \pi = \otimes \pi_v \) and let \( S \) be as in the statement of the theorem, i.e., a non-empty set of non-archimedean places \( v \) such that for all finite \( v \notin S \) both \( \pi_v \) and \( \psi_v \) are unramified.

(i) \( v < \infty \) and \( \pi_v \) unramified: Choose \( \Pi_v \) as in the statement of the theorem via the Frobenius-Hecke (or Satake) parameter.

More precisely, since \( \pi_v \) is unramified, it is given by an unramified character \( \chi \) of the maximal torus \( T(k_v) \). This means that there are unramified characters \( \chi_0, \chi_1, \ldots, \chi_n \) of \( k_v^\times \) such that for \( t \in T(k_v) \)

\[
\chi(t) = (\chi_0 \circ e_0)(t)(\chi_1 \circ e_1)(t) \cdots (\chi_n \circ e_n)(t),
\]

where \( e_i \)'s form the basis of the rational characters of the maximal torus of \( \mathbf{G} \) as in Section 2.1. The character \( \chi \) corresponds to an element \( \hat{t} \) in \( \hat{T} \), the maximal torus of (the connected component) of the Langlands
dual group which is $\text{GSp}_{2n}(\mathbb{C})$ or $\text{GSO}_{2n}(\mathbb{C})$, uniquely determined by the equation

$$\chi(\phi(\varpi)) = \phi(\hat{t}),$$  \hspace{1cm} (61)

where $\varpi$ is a uniformizer of our local field $k_v$, and $\phi \in X_*(\mathcal{T}) = X^*(\hat{T})$ (cf. (I.2.3.3) on page 26 of [12]). We make this identification explicit via the correspondence $e_i^* \leftrightarrow e_i$ for $i = 0, \ldots, n$ as in Section 2.1 which gave the duality of $\text{GSpin}_{2n+1} \leftrightarrow \text{GSp}_{2n}$ and $\text{GSpin}_{2n} \leftrightarrow \text{GSO}_{2n}$.

Applying (61) with the $\phi$ on the left replaced with $e_i^*$ and the one on the right replaced with $e_i$ for $i = 0, 1, \ldots, n$ yields

$$\chi_i(\varpi) = \chi(e_i^*(\varpi)) = e_i(\hat{t}), \quad i = 0, 1, \ldots, n.$$  \hspace{1cm} (62)

We can now compute the Satake parameter explicitly as an element $\hat{t}$ in the maximal torus $\hat{T}$ of $\text{GSp}_{2n}(\mathbb{C})$ or $\text{GSO}_{2n}(\mathbb{C})$, as described in (3). If we write our unramified characters as $\chi_i(\cdot) = |s^i_v|$ for $\sigma_i \in \mathbb{C}$ and $0 \leq i \leq n$, then we get

$$\hat{t} = \begin{pmatrix} |\varpi|^{s_1} & & & \\ & \ddots & & \\ & & |\varpi|^{s_n} & \\ & & & |\varpi|^{s_0-s_n} \end{pmatrix}.$$  \hspace{1cm} (63)

Hence, $\Pi_v$ is the unique unramified constituent of the representation of $\text{GL}_{2n}(k_v)$ induced from the character

$$\chi_1 \otimes \cdots \otimes \chi_n \otimes \chi_0 \chi_n^{-1} \otimes \cdots \otimes \chi_0 \chi_1^{-1}$$  \hspace{1cm} (64)

of the $k_v$-points of the standard maximal torus in $\text{GL}_{2n}$.

A crucial point here is what the central characters of $\pi_v$ and $\Pi_v$ are. It follows from Proposition 2.3 that the central character $\omega_{\pi_v} = \chi_0$. Moreover, the central character $\omega_{\Pi_v}$ of $\Pi_v$ is $\chi_0^n$, hence we have $\omega_{\Pi_v} = \omega_{\pi_v}^n$.

Furthermore, note that $\tilde{\Pi}_v$ is the unique unramified constituent of the representation induced from

$$\chi_1^{-1} \otimes \cdots \otimes \chi_n^{-1} \otimes \chi_0^{-1} \chi_n \otimes \cdots \otimes \chi_0^{-1} \chi_1.$$  

Therefore, we have $\tilde{\Pi}_v \simeq \chi_0^{-1} \otimes \Pi_v$. In other words, $\Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v}$.

(ii) $v|\infty$: Choose $\Pi_v$ as in the statement of Theorem 1.1 [27].

To be more precise, Langlands associates to $\pi_v$ a homomorphism $\phi_v$ from the local Weil group $W_v = W_{k_v}$ to the dual group $\hat{G}$ which is $\text{GSp}_{2n}(\mathbb{C})$ or $\text{GSO}_{2n}(\mathbb{C})$ in our cases. Both of these groups have
natural embeddings \( \iota \) into \( \GL_{2n}(\mathbb{C}) \) and we take \( \Pi_v \) to be the irreducible admissible representation of \( \GL_{2n}(k_v) \) associated to \( \Phi_v = \phi_v \circ \iota \).

We want to show that again we have \( \omega_{\Pi_v} = \omega^n_{\pi_v} \) and \( \Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v} \). To do this we use some well-known facts regarding representations of \( W_v \) and local Langlands correspondence for \( \GL_n(\mathbb{R}) \) and \( \GL_n(\mathbb{C}) \). We refer to [24] for a nice survey of these results.

First assume that \( k_v = \mathbb{C} \). Then \( W_v = \mathbb{C}^\times \) and any irreducible representation of \( W_v \) is one-dimensional and of the form

\[
z \mapsto [z]^t |z|^\ell, \quad \ell \in \mathbb{Z}, t \in \mathbb{C},
\]

where \([z] = z/|z|\) and \(|z|_\mathbb{C} = |z|^2\).

The \( 2n \)-dimensional representation \( \Phi_v \) of \( W_v \) can now be written as a direct sum of \( 2n \) one-dimensional representations as above. Moreover, \( \Phi_v(z) = \phi_v(z) \), considered as a diagonal matrix in \( \GL_{2n}(\mathbb{C}) \), actually lies, up to conjugation, in \( \GSp_{2n}(\mathbb{C}) \) or \( \GSO_{2n}(\mathbb{C}) \) as in [3]. Therefore, there exist one-dimensional representations \( \phi_0, \phi_1, \ldots, \phi_n \) as above such that \( \Phi_v \) is the direct sum of \( \phi_1, \ldots, \phi_n, \phi_n^{-1} \phi_0, \ldots, \phi_1^{-1} \phi_0 \). Now the central characters of \( \Pi_v \) and \( \pi_v \) can be written as \( \omega_{\Pi_v}(z) = \det(\Phi_v(z)) \) and \( \omega_{\pi_v}(z) = e_0(\phi_v(z)) \) where \( \phi_v(z) = \Phi_v(z) \) is considered as a \( 2n \times 2n \) diagonal matrix as in [3] and \( e_0 \) is as in [4]. In other words, \( \omega_{\pi_v} = \phi_0 \) and \( \omega_{\Pi_v} = \phi_0^n \) or \( \omega_{\Pi_v} = \omega_{\pi_v} \).

Moreover, \( \tilde{\Pi}_v \) corresponds to the \( 2n \)-dimensional representation of \( W_v \) which is the direct sum of \( \phi_1^{-1}, \ldots, \phi_n^{-1}, \phi_n \phi_0^{-1}, \ldots, \phi_1 \phi_0^{-1} \) implying that the two representations \( \Pi_v \) and \( \tilde{\Pi}_v \otimes \omega_{\pi_v} \) have the same parameters, i.e., \( \Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v} \).

Next assume that \( k_v = \mathbb{R} \). Then \( W_v = \mathbb{C}^\times \cup j\mathbb{C}^\times \) with \( j^2 = -1 \) and \( jzj^{-1} = \bar{z} \) for \( z \in \mathbb{C}^\times \). Here the situation is identical and the only difference is that \( W_v \) also has two-dimensional irreducible representations. The one-dimensional representations of \( W_v \) can be described as

\[
\begin{align*}
z \mapsto |z|^t |\mathbb{R}|, & \quad j \mapsto 1, \quad t \in \mathbb{C} \\
z \mapsto |z|^t |\mathbb{R}|, & \quad j \mapsto -1, \quad t \in \mathbb{C}
\end{align*}
\]

with \(|z|_\mathbb{R} = |z|\) and the irreducible two-dimensional representations are of the form

\[
z = re^{i\theta} \mapsto \begin{pmatrix} r^{2t}e^{it\theta} & 0 \\ r^{-2t}e^{-it\theta} & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & (-1)^t \\ 1 & 0 \end{pmatrix},
\]

where \( t \in \mathbb{C} \) and \( \ell \geq 1 \) is an integer. These correspond, respectively, to representations \( 1 \otimes |\cdot|_\mathbb{R}^t \) and \( \sgn \otimes |\cdot|_\mathbb{R}^t \) of \( \GL_1(\mathbb{R}) \) and \( D_\ell \otimes |\cdot|_\mathbb{R}^t \) of \( \GL_2(\mathbb{R}) \) with notation as in [24].

Notice that again \( \Phi_v(z) = \phi_v(z) \) is a diagonal \( 2n \times 2n \) matrix in \( \GSp_{2n}(\mathbb{C}) \) or \( \GSO_{2n}(\mathbb{C}) \) as in the previous case while \( \Phi_v(j) \) may have


2 \times 2 \text{ blocks as well. Therefore, we still have } \omega_{\Pi_v} = \omega_{\pi_v}^{n_v} \text{ and } \Pi_v \simeq \widetilde{\Pi}_v \otimes \omega_{\pi_v} \text{ as above.}

(iii) \( v < \infty \) and \( \pi_v \) ramified: Choose \( \Pi_v \) to be an arbitrary irreducible admissible representation of \( \text{GL}_{2n}(k_v) \) with \( \omega_{\Pi_v} = \omega_{\pi_v}^{n_v} \).

Then \( \Pi = \bigotimes'_v \Pi_v \) is an irreducible admissible representation of \( \text{GL}_{2n}(A) \) whose central character \( \omega_\Pi \) is equal to \( \omega_{\pi_v}^{n_v} \), and hence invariant under \( k^\times \).

Moreover, for all \( v \not\in S \), we have that \( L(s, \pi_v) = L(s, \Pi_v) \) by construction.

Hence, \( L^S(s, \Pi) = L^S(s, \pi) \) where

\[
L^S(s, \Pi) = \prod_{v \not\in S} L(s, \Pi_v) \quad L^S(s, \pi) = \prod_{v \not\in S} L(s, \pi_v).
\]

Therefore, \( L(s, \Pi) = \prod_v L(s, \Pi_v) \) is absolutely convergent in some right half plane.

Choose \( \eta = \bigotimes_v \eta_v \) to be a unitary character of \( k^\times \backslash A^\times \) such that \( \eta_v \) is sufficiently ramified for \( v \in S \) in order for Theorem \( 4.4 \) to hold and such that at one place \( \eta_v^2 \) is still ramified. For \( \tau \in \mathcal{T}(S; \eta) \) we claim the following equalities (along with their analogous equalities for the contragredients):

\[
L(s, \Pi_v \times \tau_v) = L(s, \pi_v \times \tau_v), \quad (65)
\]

\[
\epsilon(s, \Pi_v \times \tau_v, \psi_v) = \epsilon(s, \pi_v \times \tau_v, \psi_v). \quad (66)
\]

Here the \( L- \) and \( \epsilon- \) factors on the left are as in \( 7 \) and those on the right are defined via the Langlands-Shahidi method \( 36, 34 \).

To see (65) and (66) we again consider different places separately.

(i) \( v < \infty \) and \( \pi_v \) unramified: Let \( \pi_v \) be again as in (60) with Satake parameter \( \epsilon \).

Then \( \Pi_v \) will be as in (64).

By \( 15 \) we have

\[
L(s, \Pi_v \times \tau_v) = \prod_{i=1}^n L(s, \tau_v \otimes \chi_i) L(s, \tau_v \otimes \chi_0 \chi_i^{-1}) \quad (67)
\]

\[
L(s, \tilde{\Pi}_v \times \tilde{\tau}_v) = \prod_{i=1}^n L(s, \tilde{\tau}_v \otimes \chi_i^{-1}) L(s, \tilde{\tau}_v \otimes \chi_0^{-1} \chi_i),
\]

and

\[
\epsilon(s, \Pi_v \times \tau_v, \psi_v) = \prod_{i=1}^n \epsilon(s, \tau_v \otimes \chi_i, \psi_v) \epsilon(s, \tau_v \otimes \chi_0 \chi_i^{-1}, \psi_v). \quad (68)
\]

On the other hand, it follows from the inductive property of \( \gamma- \) factors in the Langlands-Shahidi method (Theorem 3.5 of \( 36 \) or \( 35 \)) that

\[
\gamma(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^n \gamma(s, \tau_v \otimes \chi_i, \psi_v) \gamma(s, \tau_v \otimes \chi_0 \chi_i^{-1}, \psi_v). \quad (69)
\]
just as in (57).

Since \( \tau_v \) is generic, it is a full induced representation from generic tempered ones. Thus we can write

\[
\tau_v \simeq \text{Ind}(\nu^{b_1}\tau_{1,v} \otimes \cdots \otimes \nu^{b_p}\tau_{p,v}),
\]

where each \( \tau_{j,v} \) is a tempered representation of some \( \text{GL}_{r_j}(k_v) \), \( \nu(\cdot) = |\det(\cdot)|_{v}^{\frac{1}{2}} \) on \( \text{GL}_{r_j}(k_v) \), \( r_1 + \cdots + r_p = r \), and the \( \tau_{j,v} \) are in the Langlands order. Moreover, recall that \( \pi_v \) is the unique irreducible unramified subrepresentation of the representation of \( G_n(k_v) \) induced from the character \( \chi \) as in (60) after an appropriate reordering, if necessary.

Now by the definition of \( L \)-functions (Section 7 of [36]) and their multiplicative property (Theorem 5.2 of [35]) we have

\[
L(s, \pi_v \times \tau_v) = \prod_{j=1}^{p} L(s + b_j, \pi_v \times \tau_{j,v})
\]

\[
= \prod_{j=1}^{p} \prod_{i=1}^{n} L(s + b_j, \tau_{j,v} \otimes \chi_i) L(s + b_j, \tau_{j,v} \otimes \chi_0 \chi_i^{-1})
\]

\[
= \prod_{i=1}^{n} L(s, \tau_v \otimes \chi_i) L(s, \tau_v \otimes \chi_0 \chi_i^{-1}).
\]

(71)

and likewise

\[
L(s, \bar{\pi}_v \times \bar{\tau}_v) = \prod_{i=1}^{n} L(s, \bar{\tau}_v \otimes \chi_i^{-1}) L(s, \bar{\tau}_v \otimes \chi_0^{-1} \chi_i).\]

(72)

Note that Conjecture 5.1 of [35], which is a hypothesis of Theorem 5.2, is known in our cases by Theorem 5.7 of [2].

Equations (69), (71), and (72) in turn imply

\[
\epsilon(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{n} \epsilon(s, \tau_v \otimes \chi_i, \psi_v) \epsilon(s, \tau_v \otimes \chi_0 \chi_i^{-1}, \psi_v).
\]

(73)

Note that the product \( L \)-functions for \( \text{GL}_a \times \text{GL}_b \) of the Langlands-Shahidi method and the \( L \)-functions of [15] are known to be equal ([32]). Hence, to prove (65) and (66) all we need is to compare the right hand sides of (57) and (68) with those of (71), (72), and (73).

(ii) \( \nu|_{\infty} \): By the local Langlands correspondence [27] the representations \( \pi_v \) and \( \tau_v \) are given by admissible homomorphisms

\[
\phi : W_v \rightarrow \begin{cases} 
\text{GSp}_{2n}(\mathbb{C}) & \text{if } G_n = \text{GSpin}_{2n+1} \\
\text{GSO}_{2n}(\mathbb{C}) & \text{if } G_n = \text{GSpin}_{2n}
\end{cases}
\]
and
\[ \phi' : W_v \longrightarrow \text{GL}_r(\mathbb{C}) \]
respectively and the tensor product
\[ (\iota \circ \phi) \otimes \phi' : W_v \longrightarrow \text{GL}_{2nr}(\mathbb{C}) \]
is again admissible. Now,
\[ L(s, \Pi_v \times \tau_v) = L(s, (\iota \circ \phi) \otimes \phi') = L(s, \pi_v \times \tau_v), \]
and
\[ \epsilon(s, \Pi_v \times \tau_v, \psi_v) = \epsilon(s, (\iota \circ \phi) \otimes \phi', \psi_v) = \epsilon(s, \pi_v \times \tau_v, \psi_v), \]
where the middle factors are the local Artin-Weil factors [43] and equalities hold by [33]. (See also [3].)

(iii) \( v < \infty \) and \( \pi_v \) ramified. This is where we will need the stability of \( \gamma \)-factors. Since \( v \in S \) the representation \( \tau_v \) can be written as
\[ \tau_v \cong \text{Ind}(\nu^{b_1} \otimes \cdots \otimes \nu^{b_r}) \otimes \eta_v \cong \text{Ind}(\eta_v \nu^{b_1} \otimes \cdots \otimes \eta_v \nu^{b_r}), \quad (74) \]
where \( \nu(x) = |x|_v \). Then,
\[ L(s, \pi_v \times \tau_v) = \prod_{i=1}^r L(s + b_i, \pi_v \times \eta_v), \quad (75) \]
\[ \epsilon(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^r \epsilon(s + b_i, \pi_v \times \eta_v, \psi_v). \quad (76) \]

However, since \( \eta_v \) is sufficiently ramified (depending on \( \pi_v \)), Corollary 4.19 implies that
\[ L(s, \pi_v \times \eta_v) \equiv 1 \quad (77) \]
\[ \epsilon(s, \pi_v \times \eta_v) = \prod_{i=1}^n \epsilon(s, \eta_v \chi_i, \psi_v) \epsilon(s, \eta_v \chi_0 \mu_i^{-1}, \psi_v) \quad (78) \]
for \( n + 1 \) arbitrary characters \( \chi_0, \chi_1, \ldots, \chi_n \). We choose them to be as in (60).

On the other hand, by either [15] or [36] we have
\[ L(s, \Pi_v \times \tau_v) = \prod_{i=1}^r L(s + b_i, \Pi_v \otimes \eta_v), \quad (79) \]
\[ \epsilon(s, \Pi_v \times \tau_v, \psi_v) = \prod_{i=1}^r \epsilon(s + b_i, \Pi_v \otimes \eta_v, \psi_v). \quad (80) \]

Again since \( \eta_v \) is highly ramified (depending on \( \Pi_v \)) and \( \omega_{\Pi_v} = \omega_{\pi_v}^n = \chi_0^n \) is equal to the product of the \( 2n \) characters
\[ \chi_1, \ldots, \chi_n, \chi_0 \chi_n^{-1}, \cdots, \chi_0 \chi_1^{-1}, \]
Proposition 2.2 of [16] implies that

\[ L(s, \Pi_v \otimes \eta_v) \equiv 1 \]  \hspace{1cm} (81)

\[ \epsilon(s, \Pi_v \otimes \eta_v) = \prod_{i=1}^{n} \epsilon(s, \eta_v \chi_i, \psi_v) \epsilon(s, \eta_v \chi_0 \chi_i^{-1}, \psi_v). \]  \hspace{1cm} (82)

Comparing equations (75) through (82) now proves (65) and (66) for \( v \) non-archimedean with \( \pi_v \) ramified.

Now that we have (65) and (66) for all places \( v \) of \( k \), we conclude globally that

\[ L(s, \Pi \times \tau) = L(s, \pi \times \tau) \]  \hspace{1cm} (83)

\[ \epsilon(s, \Pi \times \tau) = \epsilon(s, \pi \times \tau) \]  \hspace{1cm} (84)

for all \( \tau \in \mathcal{T}(S; \eta) \). All that remains now is to verify the three conditions of Theorem 6.1 which we can now check for the factors coming from the Langlands-Shahidi method thanks to (83) and (84). Conditions (1) – (3) of Theorem 6.1 are Propositions 5.1, 5.3, and 5.4 respectively.

Therefore, there exists an automorphic representation \( \Pi' \) of \( GL_2n(A) \) such that for all \( v \not\in S \) we have \( \Pi_v \simeq \Pi'_v \). In particular, for all \( v \not\in S \) the local representation \( \Pi'_v \) is related to \( \pi_v \) as prescribed in Theorem 1.1. Moreover, note that for all \( v \not\in S \) we have \( \omega_{\Pi'_v} = \omega_{\Pi_v} = \omega_{\pi_v}^n \). Since \( \omega_{\Pi'_v} \) is a gr"ossencharacter which agrees with the gr"ossencharacter \( \omega_{\pi_v}^n \) at all but possibly finitely many places, we conclude that \( \omega_{\Pi'_v} = \omega_{\pi_v}^n \).

On the other hand, if \( v \) is an archimedean place or a non-archimedean place with \( v \not\in S \), then we proved earlier that

\[ \Pi'_v \simeq \Pi_v \simeq \tilde{\Pi}_v \otimes \omega_{\pi_v} \simeq \tilde{\Pi}'_v \otimes \omega_{\pi_v} \]

which means, in particular, that \( \Pi' \) is nearly equivalent to \( \tilde{\Pi}' \otimes \omega_{\pi'} \).

\[ \square \]

7. Complements

7.1. Local Consequences. Our first local result is to show that the local transfers at the unramified places remain generic. Let us first recall a general result of Jian-Shu Li which we will use. The following is a special case of Theorem 2.2 of [29].

Proposition 7.1. (J.-S. Li) Let \( G \) be a split connected reductive group over a non-archimedean local field \( F \) and let \( B = TU \) be a fixed Borel where \( T \) is a maximal torus and \( U \) is the unipotent radical of \( B \). Let \( \chi \) be an unramified character of \( T(F) \) and let \( \pi(\chi) \) be the unique irreducible unramified subquotient of the corresponding principal series representation. Then \( \pi(\chi) \) is generic if and only if for all roots \( \alpha \) of \( (G, T) \) we have \( \chi(\alpha^\vee(\varpi)) \not\equiv |\varpi|_F \). Here \( \alpha^\vee \) denotes the coroot associated to \( \alpha \) and \( \varpi \) is a uniformizer of \( F \).
Proposition 7.2. Let $\pi = \otimes'_{v} \pi_{v}$ be an irreducible globally generic cuspidal automorphic representation of $GSpin_{m}(\mathbb{A})$, $m = 2n + 1$ or $2n$, and let $\Pi = \otimes'_{v} \Pi_{v}$ be a transfer of $\pi$ to $GL_{2n}(\mathbb{A})$ (cf. Theorem 7.1). If $v < \infty$ is a place of $k$ with $\pi_{v}$ unramified, then the local representation $\Pi_{v}$ is irreducible, unramified, and we have $\Pi_{v} \simeq \tilde{\Pi}_{v} \otimes \omega_{\pi_{v}}$. Moreover, if $m = 2n + 1$ (cf. Remark 7.3), then $\Pi_{v}$ is generic (and, hence, full induced principal series).

Proof. The representation $\Pi_{v}$ is irreducible and unramified by construction (cf. (i) in the proof of Theorem 1.1). We also proved that $\Pi_{v}$ satisfies $\Pi_{v} \simeq \tilde{\Pi}_{v} \otimes \omega_{\pi_{v}}$ in the course of proof of Theorem 1.1 in Section 6. We now show that $\Pi_{v}$ is generic. Our tool will be Proposition 7.1 above.

Now assume that $m = 2n + 1$. Let $\chi$ and $\chi_{0}, \ldots, \chi_{n}$ be as in (60). Since $\pi_{v}$ is generic by Proposition 7.1 we have that $\chi(\alpha(\varpi)) \neq |\varpi|_{k_{v}}$ for all roots $\alpha$. Using the notation of Section 2, the roots in the odd case $m = 2n + 1$ are $\alpha = \pm(e_{i} - e_{j}), \pm(e_{i} + e_{j})$ with $1 \leq i \leq j \leq n$ and $\pm(2e_{i})$ with $1 \leq i \leq n$. The corresponding coroots are $\alpha^{\vee} = \pm(e_{i}^{*} - e_{j}^{*}), \pm(e_{i}^{*} + e_{j}^{*} - e_{0}^{*})$ with $1 \leq i \leq j \leq n$ and $\pm(2e_{i}^{*} - e_{0}^{*})$ with $1 \leq i \leq n$, respectively. This implies that $\chi_{i} \chi_{j}^{-1} \neq |\pm|^{1}$ for $i \neq j$ and $\chi_{i} \chi_{j} \chi_{0}^{-1} \neq |\pm|^{1}$ for all $i, j$.

The representation $\Pi_{v}$ was chosen to be the unique irreducible unramified subquotient of the representation on $GL_{2n}(F)$ induced from the $2n$ unramified characters $\chi_{1}, \ldots, \chi_{n}, \chi_{0} \chi_{n}^{-1}, \ldots, \chi_{0} \chi_{1}^{-1}$ as in (61). Therefore, the above relations imply that $\Pi_{v}$ is generic and full-induced.

Remark 7.3. The above argument does not quite work in the even case and one can easily construct local examples where the transferred local representation is the (unique) unramified subquotient of an induced representation on $GL_{2n}$ far from the generic constituent.

For example, consider $GSpin_{6}$ with $\chi_{0} = \mu^{2}$, $\chi_{1} = \mu(5/2)$, $\chi_{2} = \mu(1/5)$, and $\chi_{3} = \mu(-3/2)$, where $\mu$ is a unitary character of $F^{\times}$ and $\mu(|r|)$ means $\mu(1)|r|$. Now $\Pi_{v}$ is the unique unramified constituent of the representation on $GL_{6}(F)$ induced from $\mu(5/2)$, $\mu(3/2)$, $\mu(1/2)$, $\mu(-1/2)$, $\mu(-3/2)$, $\mu(-5/2)$ and, in fact, far from being generic. In this case, there is another constituent that is square-integrable, hence tempered and generic.

Of course, we do expect $\Pi_{v}$ in the case of $m = 2n$ to be generic as well. However, this phenomenon is not a purely local one in the case of $m = 2n$. In fact, it will be automatic that the local transfers at the unramified places are generic once we prove that the automorphic representation $\Pi$ is induced from unitary cuspidal representations (see Remark 7.3 below). As we discuss in Remark 7.3 this will follow from our future work.
7.2. Global Consequences. In this section we will make some comments about the automorphic representation \( \Pi \) which are almost immediate consequences of our main result and leave more detailed information about \( \Pi \) for a future paper.

**Proposition 7.4.** Let \( \pi \) be a globally generic cuspidal automorphic representation of \( \text{GSpin}_m(\AA) \), \( m = 2n + 1 \) or \( 2n \) and let \( \omega = \omega_\pi \). Then there exists a partition \( (n_1, n_2, \ldots, n_t) \) of \( 2n \) and (not necessarily unitary) cuspidal automorphic representations \( \sigma_1, \ldots, \sigma_t \) of \( \text{GL}_{n_i}(\AA) \), \( i = 1, \ldots, t \), and permutation \( p \) of \( \{1, \ldots, t\} \) with \( n_i = n_{p(i)} \) and \( \sigma_i \simeq \tilde{\sigma}_{p(i)} \otimes \omega \) such that any transfer \( \Pi \) of \( \pi \) as in Theorem 1.1 is a constituent of \( \Sigma = \text{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_t) \) where the induction is, as usual, from the standard parabolic of \( \text{GL}_{2n} \) having Levi subgroup \( \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_t} \).

**Proof.** Let \( \Pi \) be any transfer of globally generic cuspidal representation \( \pi \) as in Theorem 1.1. By Proposition 2 of [26] there exists a partition \( p \) and \( \sigma_i \)'s as above such that \( \Pi \) is a constituent of \( \Sigma \). Furthermore, for finite places \( v \) where \( \pi_v \) is unramified, we have that \( \Pi_v \) is the unique unramified constituent of \( \Sigma_v = \text{Ind}(\sigma_{1,v} \otimes \cdots \otimes \sigma_{t,v}) \). As part of Theorem 1.1 we showed that \( \Pi \) and \( \tilde{\Pi} \otimes \omega \) are nearly equivalent (see definition prior to Theorem 1.1). Now, \( \tilde{\Pi} \otimes \omega \) is a constituent of \( \tilde{\Sigma} \otimes \omega = \text{Ind}(\tilde{\sigma}_1 \otimes \omega) \otimes \cdots \otimes (\tilde{\sigma}_t \otimes \omega) \) and by the classification theorem of Jacquet and Shalika (Theorem 4.4 of [17]) we have that there is a permutation \( p \) of \( \{1, \ldots, t\} \) such that \( n_i = n_{p(i)} \) and \( \sigma_i \simeq \tilde{\sigma}_{p(i)} \otimes \omega \).

Now let \( \Pi' \) be another transfer of \( \pi \) as in Theorem 1.1. Then, \( \Pi' \) is again a constituent of some \( \Sigma' = \text{Ind}(\sigma'_1 \otimes \cdots \otimes \sigma'_{t'}) \), where \( \sigma'_i \) is a cuspidal automorphic representation of \( \text{GL}_{n'_i}(\AA) \) and \( (n'_1, \ldots, n'_{t'}) \) is a partition of \( 2n \). Moreover, for almost all finite places \( v \) we have that \( \Pi'_v \) is the unique unramified constituent of \( \Sigma'_v \). On the other hand, by construction, \( \Pi_v \simeq \Pi'_v \) for almost all \( v \) and, therefore, the classification theorem of Jacquet and Shalika again implies that \( t = t' \) and, up to a permutation, \( n_i = n'_i \) and \( \sigma_i \simeq \sigma'_i \) for \( i = 1, \ldots, t \). Therefore, \( \Pi' \) is also a constituent of \( \Sigma \).

**Remark 7.5.** If we write \( \sigma_i = \tau_i \otimes \det(\cdot)^{r_i} \) for \( i = 1, 2, \ldots, t \), with \( \tau_i \) unitary cuspidal and \( r_i \in \mathbb{R} \), then we expect that all \( r_i = 0 \), i.e., \( \Pi \) is an isobaric sum of unitary cuspidal representations. We will take up this issue, which will have important consequences, in our future work.

7.3. Exterior square transfer. In this section we show that exterior square generic transfer from \( \text{GL}_4 \) to \( \text{GL}_6 \) due to H. Kim ([20]) can be deduced as a special case of our main result. However, note that in this article we are only proving the weak transfer. Once we prove the strong version of transfer from \( \text{GSpin}_{2n} \) to \( \text{GL}_{2n} \) again it will have the full content of the results of [20]. A similar remark also applies to Section 7.4.
Proposition 7.6. Let $\phi : \text{GSpin}_6 \longrightarrow \text{GL}_4$ be the (double) covering map (cf. Proposition 2.2) and denote by $\hat{\phi}$ the map induced on the connected components of the $L$-groups.

\[
\begin{array}{ccc}
GSO_6(\mathbb{C}) & \overset{\iota}{\longrightarrow} & GL_6(\mathbb{C}) \\
\downarrow \hat{\phi} & & \downarrow \\
GL_4(\mathbb{C}) & & 
\end{array}
\]

Then, the $\iota \circ \hat{\phi} = \wedge^2$.

Proof. The group $\text{GSO}_6$ is of type $D_3$ and we denote its simple roots by $\alpha_1, \alpha_2, \alpha_3$ as in Section 2. Also, $\text{GL}_4$ is of type $A_3$ (or $D_3$) and we denote its corresponding simple roots by $\overline{\alpha_2}, \overline{\alpha_1}, \overline{\alpha_3}$, respectively, and similarly for other root data (cf. Section 2). Let \( A = \text{diag}(a_1, a_2, a_3, a_4) \in \text{GL}_4(\mathbb{C}) \). For a fixed appropriate choice of fourth root of unity and \( \delta = (a_1 a_2 a_3 a_4)^{1/4} \) we have

\[
\iota \circ \hat{\phi}(A) = \iota \circ \hat{\phi}(\delta \overline{a_2} \overline{a_1} (\frac{a_1 a_2}{\delta^2}) \overline{a_3} (\frac{a_1 a_2 a_3}{\delta^3}))
\]

\[
= \iota(e_0^*(\delta) e_1^*(\delta^2) e_2^*(\delta^2) e_3^*(\delta)^2) \alpha_1 \overline{\alpha_2} \overline{\alpha_3} (\frac{a_1 a_2}{\delta^2}) \alpha_3 \overline{\alpha_3} (\frac{a_1 a_2 a_3}{\delta^3}))
\]

\[
= \iota(e_0^*(\delta^4) e_1^*(a_1 a_2) e_2^*(a_1 a_3) e_3^*(a_2 a_3))
\]

\[
= \text{diag}(a_1 a_2, a_1 a_3, a_2 a_3, a_2 a_4, a_1 a_4, a_3 a_4) = \wedge^2 A.
\]

Here the third equality follows from Proposition 2.10. \( \square \)

As a corollary we see that our Theorem 1.1 in the special case of $m = 2n$ with $n = 3$ gives Kim’s exterior square transfer.

Proposition 7.7. If $\pi$ is an irreducible cuspidal automorphic representation of $GL_4(\mathbb{A})$ considered as a representation of $\text{GSpin}_6(\mathbb{A})$ via the covering map $\phi$, then the automorphic representation $\Pi$ of Theorem 1.1 is such that $\Pi_v = \wedge^2 \pi_v$ for almost all $v$.

7.4. Transfer from $\text{GSp}_4$ to $\text{GL}_4$. The special case of $m = 2n + 1$ with $n = 4$ of our Theorem 1.1 gives the following:

Proposition 7.8. Let $\pi$ be an irreducible globally generic cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A})$. Then $\pi$ can be transferred to an automorphic representation $\Pi$ of $GL_4(\mathbb{A})$ associated to the embedding $\text{GSp}_4(\mathbb{C}) \hookrightarrow GL_4(\mathbb{C})$.

Proof. Notice that $\text{GSpin}_6$ is isomorphic, as an algebraic group, to the group $GSp_4$. Now the corollary is a special case of Theorem 1.1 as mentioned above. \( \square \)

Remark 7.9. The above Proposition, in particular, proves that the spinor $L$-function of $\pi$ is entire. We understand that R. Takloo-Bighash also has a proof of this fact using a completely different method based on integral representations.
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