**SINGULAR MANAKOV FLOWS AND GEODESIC FLOWS ON HOMOGENEOUS SPACES**

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**Abstract.** We prove complete integrability of the Manakov-type $SO(n)$-invariant geodesic flows on homogeneous spaces $SO(n)/SO(k_1) \times \cdots \times SO(k_r)$, for any choice of $k_1, \ldots, k_r$, $k_1 + \cdots + k_r \leq n$. In particular, a new proof of the integrability of a Manakov symmetric rigid body motion around a fixed point is presented. Also, the proof of integrability of the $SO(n)$-invariant Einstein metrics on $SO(k_1 + k_2 + k_3)/SO(k_1) \times SO(k_2) \times SO(k_3)$ and on the Stiefel manifolds $V(n,k) = SO(n)/SO(k)$ is given.

1. Introduction

It was Frahm who gave the first four-dimensional generalization of the Euler top in the second half of XIX century. [15]. Unfortunately, his paper was forgotten for more than a century. A modern history of higher-dimensional generalizations of the Euler top has more than thirty years after the paper of Manakov in 1976 [18]. Manakov used Dubrovin’s theory of matrix Lax operators (see [10] [11]) to prove that the solutions of Frahm-Manakov’s top can be given in terms of theta-functions. Although the subject has had intensive development since then, there are still few questions which in our opinion deserve additional treatment.

1.1. Liouville Integrability. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold. The equations

$$\dot{f} = \{f, H\}, \quad f \in C^\infty(M) \tag{1}$$

are called Hamiltonian equations with the Hamiltonian function $H$. A function $f$ is an integral of the Hamiltonian system (constant along trajectories of (1)) if and only if it commutes with $H$: $\{f, H\} = 0$.

One of the central problems in Hamiltonian dynamics is whether the equations (1) are completely integrable or not. The equations (1) are completely integrable or Liouville integrable if there are $l = \frac{1}{2}(\dim M + \text{corank} \{\cdot, \cdot\})$ Poisson-commuting smooth integrals $f_1, \ldots, f_l$ whose differentials are independent in an open dense subset of $M$. The set of integrals $\mathcal{F} = \{f_1, \ldots, f_l\}$ is called a complete involutive set of functions on $M$. To distinguish the situation from the case of non-commutative integrability, the last set will be called commutative as well.

If the system is completely integrable, by the Liouville-Arnold theorem there is an implicitly given set of coordinates in which the system trivializes. Moreover, from the Liouville-Arnold theorem (1) follows that regular compact connected invariant submanifolds given by integrals $\mathcal{F}$ are Lagrangian tori within appropriate
symplectic leaves of the Poisson bracket \( \{\cdot,\cdot\} \) and the dynamics over the invariant tori is quasi-periodic.

1.2. Noncommutative Integrability. Let \((M,\{\cdot,\cdot\})\) be a Poisson manifold, \(\Lambda\) be the associated bivector field on \(M\)

\[ \{f,g\}(x) = \Lambda_x(df(x),dg(x)) \]

and let \(\mathcal{F}\) be a Poisson subalgebra of \(C^\infty(M)\) (or a collection of functions closed under the Poisson bracket).

Consider the linear spaces

\[ F_x = \{df(x) \mid f \in \mathcal{F}\} \subset T^*_xM \]

and suppose that we can find \(l\) functionally independent functions \(f_1,\ldots,f_l \in \mathcal{F}\) whose differentials span \(F_x\) almost everywhere on \(M\) and that the corank of the matrix \(\{f_i,f_j\}\) is equal to some constant \(k\), i.e., \(\dim \ker \Lambda_x|_{F_x} = k\).

The numbers \(l\) and \(k\) are called differential dimension and differential index of \(\mathcal{F}\) and they are denoted by \(\text{ddim} \mathcal{F}\) and \(\text{dind} \mathcal{F}\), respectively. The set \(\mathcal{F}\) is called complete if (see [25, 24, 8]):

\[ \text{ddim} \mathcal{F} + \text{dind} \mathcal{F} = \dim M + \text{corank} \{\cdot,\cdot\}. \]

The Hamiltonian system \((M,\mathcal{H})\) is completely integrable in the noncommutative sense if it possesses a complete set of first integrals \(\mathcal{F}\). Then (under compactness condition) \(M\) is almost everywhere foliated by \((\dim \mathcal{F} - \text{corank} \{\cdot,\cdot\})\)-dimensional invariant tori. As in the Liouville-Arnold theorem, the Hamiltonian flow restricted to regular invariant tori is quasi-periodic (see Nekhoroshev [25] and Mishchenko and Fomenko [24]).

1.3. Mishchenko–Fomenko Conjecture. Let \(\mathcal{F}\) be any Poisson closed subset of \(C^\infty(M)\), then a subset \(\mathcal{F}^0 \subset \mathcal{F}\) is a complete subset if \(\text{ddim} \mathcal{F}^0 + \text{dind} \mathcal{F}^0 = \text{ddim} \mathcal{F} + \text{dind} \mathcal{F}\). In particular, a commutative subset \(\mathcal{F}^0 \subset \mathcal{F}\) is complete if \(\text{ddim} \mathcal{F}^0 = \frac{1}{2} (\text{ddim} \mathcal{F} + \text{dind} \mathcal{F})\).

Mishchenko and Fomenko stated the conjecture that non-commutative integrable systems are integrable in the usual commutative sense by means of integrals that belong to the same functional class as the original non-commutative integrals. In other words, if \(\mathcal{F}\) is a complete set, then we can always construct a complete commutative set \(\mathcal{F}^0 \subset \mathcal{F}\).

Let us mention two cases in which the Mishchenko-Fomenko conjecture has been proved. The finite-dimensional version of the conjecture is recently proved by Sadetov [29] (see also [6, 32]): for every finite-dimensional Lie algebra \(\mathfrak{g}\) one can find a complete commutative set of polynomials on the dual space \(\mathfrak{g}^*\) with respect to the usual Lie-Poisson bracket. The second case where the conjecture was proved is \(C^\infty\)-smooth case for infinite-dimensional algebras (see [8]).

Consider the homogeneous spaces \(G/H\) of a compact Lie group \(G\). Fix some \(\text{Ad}_G\)-invariant positive definite scalar product \(\langle\cdot,\cdot\rangle\) on the Lie algebra \(\mathfrak{g} = \text{Lie}(G)\).
Let \( h = \text{Lie}(H) \) and let \( g = h \oplus v \) be the orthogonal decomposition with respect to \( \langle \cdot , \cdot \rangle \). The scalar product \( \langle \cdot , \cdot \rangle \) induces a normal \( G \)-invariant metric on \( G/H \) via \( \langle \cdot , \cdot \rangle_0 = \langle \cdot , \cdot \rangle|_v \), where \( v \) is identified with the tangent space at the class of the identity element. If \( G \) is semisimple and \( \langle \cdot , \cdot \rangle \) is negative Killing form, the normal metric is called standard [4]. The geodesic flow of the normal metric is completely integrable in the non-commutative sense by means of integrals polynomial in momenta [7, 9] and the Mishchenko-Fomenko conjecture can be reduced to the following ones:

**Conjecture 1.** (9) For every homogeneous space \( G/H \) of a compact Lie group \( G \) there exist a complete commutative set of \( \text{Ad}_H \)-invariant polynomials on \( v \). Here the Poisson structure is defined by (33).

For example if \((G, H)\) is a spherical pair, the set of \( \text{Ad}_H \)-invariant polynomials is already commutative. In many examples, such as Stiefel manifolds, flag manifolds, orbits of the adjoint actions, complete commutative algebras are obtained (see [4, 9, 21]), but the general problem is still unsolved.

Note that solving the problem of commutative integrability of geodesic flows of normal metrics would allow to construct new examples of \( G \)-invariant metrics on homogeneous spaces \( G/H \) with integrable geodesic flows as well.

### 1.4. The Manakov Flows.

The Euler equations of a left-invariant geodesic flow on \( SO(n) \) have the form

\[
\dot{M} = [M, \Omega], \quad \Omega = \mathfrak{A}(M)
\]

where \( \Omega \in \mathfrak{so}(n) \) is the angular velocity, \( M \in \mathfrak{so}(n)^* \cong \mathfrak{so}(n) \) angular momentum and \( \mathfrak{A} = \mathfrak{A}^{-1} \) the positive definite operator which defines the left invariant metric (see [1]). Here we identify \( \mathfrak{so}(n) \) and \( \mathfrak{so}(n)^* \) by means of the scalar product proportional to the Killing form

\[
\langle X, Y \rangle = -\frac{1}{2} \text{tr}(XY),
\]

\( X, Y \in \mathfrak{so}(n) \). The Euler equations [3] are Hamiltonian with respect to the Lie-Poisson bracket

\[
\{f, g\}(M) = -\langle M, [\nabla f(M), \nabla g(M)] \rangle, \quad M \in \mathfrak{so}(n)
\]

with the Hamiltonian function \( H = \frac{1}{2} \langle M, \mathfrak{A}(M) \rangle \). The invariant polynomials \( \text{tr}(M^{2k}), k = 1, \ldots, \text{rank } \mathfrak{so}(n) \) are central functions, determining the regular symplectic leaves (adjoint orbits) of the Lie-Poisson brackets [3]. Thus we need half of the dimension of the generic adjoint orbit \((\frac{1}{2}(\text{dim } \mathfrak{so}(n) - \text{rank } \mathfrak{so}(n)))\) additional independent commuting integrals for the integrability of Euler equations [3]. For a generic operator \( \mathfrak{A} \) and \( n \geq 4 \) the system is not integrable.

Manakov found the Lax representation with rational parameter \( \lambda \) (see [15]):

\[
\dot{L}(\lambda) = [L(\lambda), U(\lambda)], \quad L(\lambda) = M + \lambda A, \quad U(\lambda) = \Omega + \lambda B,
\]
provided $M$ and $\Omega$ are connected by

$$[M, B] = [\Omega, A],$$

where $A$ and $B$ are diagonal matrices $A = \text{diag}(a_1, \ldots, a_n)$, $B = \text{diag}(b_1, \ldots, b_n)$.

In the case the eigenvalues of $A$ and $B$ are distinct, we have

$$\mathfrak{A} = \text{ad}_A^{-1} \circ \text{ad}_B = \text{ad}_B \circ \text{ad}_A^{-1} \iff \Omega_{ij} = \frac{b_j - b_i}{a_i - a_j} M_{ij},$$

where $M_{ij} = \langle M, E_i \wedge E_j \rangle$. Here $\text{ad}_A(M) = [A, M]$ and $\text{ad}_B(M) = [B, M]$ are considered as linear transformations from $\mathfrak{so}(n)$ to the zero-diagonal subspace of the space of symmetric matrices $\text{Sym}(n)$. They are invertible since the eigenvalues of $A$ and $B$ are distinct. Note that we take $\mathfrak{A}$ is positive definite. Formally, we can take singular $B$ (i.e., $B$ with some equal eigenvalues), but then $\mathfrak{A}$ is not invertible and represents the operator which determines the left-invariant sub-Riemannian metric on $SO(n)$.

The left invariant metric given by the operator $\mathfrak{A}$ is usually called the Manakov metric. In this case, Manakov proved that the solutions of the Euler equations (3) are expressible in terms of $\theta$-functions by using the algebro-geometric integration procedure developed by Dubrovin in [10] [11] (see [18]).

The explicit verification that integrals arising from the Lax representation

$$\mathcal{L} = \{\text{tr}(M + \lambda A)^k \mid k = 1, 2, \ldots, n, \lambda \in \mathbb{R}\},$$

form a complete Poisson-commutative set on $\mathfrak{so}(n)$ was given by Mishchenko and Fomenko in [23] in the case when the eigenvalues of $A$ are distinct (see also Bolsinov [5]). Furthermore, the system is algebraically completely integrable. Conversely, if a diagonal metrics $\Omega_{ij} = \mathfrak{A}_{ij} M_{ij}$ with distinct $\mathfrak{A}_{ij}$ has algebraically completely integrable geodesic flow then $\mathfrak{A}$ has the form $\mathfrak{A}_i$ for certain $A$ and $B$ (see [16]). An another Lax pair of the system can be found in [12].

1.5. Singular Manakov Flows. We shall describe operators $\mathfrak{A}$ satisfying the condition (7) when the eigenvalues of $A$ and $B$ are not all distinct. Suppose

$$a_1 = \cdots = a_{k_1} = \alpha_1, \ldots, a_{n+1-k_r} = \cdots = a_n = \alpha_r,$$

$$b_1 = \cdots = b_{k_1} = \beta_1, \ldots, b_{n+1-k_r} = \cdots = b_n = \beta_r,$$

$$k_1 + k_2 + \cdots + k_r = n, \quad \alpha_i \neq \alpha_j, \quad \beta_i \neq \beta_j, \quad i, j = 1, \ldots, r.$$

Let

$$\mathfrak{so}(n) = \mathfrak{so}(n)_A \oplus \mathfrak{v} = \mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \cdots \oplus \mathfrak{so}(k_r) \oplus \mathfrak{v}$$

be the orthogonal decomposition, where $\mathfrak{so}(n)_A = \{X \in \mathfrak{so}(n) \mid [X, A] = 0\}$. By $M_{\mathfrak{so}(n)_A}$ and $M_\mathfrak{v}$ we denote the projections of $M \in \mathfrak{so}(n)$ with respect to (11).

Further, let $\mathfrak{B} : \mathfrak{so}(n)_A \rightarrow \mathfrak{so}(n)_A$ be an arbitrary positive definite operator. We take $A$ and $B$ such that the sectional operator $\mathfrak{A} : \mathfrak{so}(n) \rightarrow \mathfrak{so}(n)$ defined via

$$\mathfrak{A}(M_\mathfrak{v} + M_{\mathfrak{so}(n)_A}) = \text{ad}_A^{-1} \text{ad}_B(M_\mathfrak{v}) + \mathfrak{B}(M_{\mathfrak{so}(n)_A}),$$


is positive definite. Now $\text{ad}_A$ and $\text{ad}_B$ are considered as invertible linear transformations from $v$ to $[A, v] \subset \text{Sym}(n)$.

For the given $\mathfrak A$ we have $[\Omega, A] = [M_\mathfrak A, B] = [M, B]$, and the Manakov condition (7) holds. It can be proved that $[M_\mathfrak A, \text{ad}^{-1}_A \text{ad}_B(M_\mathfrak A)] = 0$. Therefore, the system (3) takes the form

$$
\dot{M}_\mathfrak A = [M_\mathfrak A, \mathfrak B(M_\mathfrak A)],
$$

and

$$
\dot{M}_\mathfrak A = [M_\mathfrak A, \text{ad}^{-1}_A \text{ad}_B(M_\mathfrak A) + [M_\mathfrak A, \mathfrak B(M_\mathfrak A)].
$$

If $k_i \geq 4$ for some $i = 1, \ldots, r$, the equations (13) (and therefore the system (13), (14)) are not integrable for a generic $\mathfrak B$. On the other hand, since (7) holds, the system has Lax representation (9). But the integrals arising from the Lax representation do not provide complete integrability.

We refer to (13), (14) as a singular Manakov flow.

1.6. Symmetric Rigid Bodies. Consider the case $A = B^2$ and $\mathfrak A = \text{ad}^{-1}_B \text{ad}_B$. Then the angular momentum and velocity are related by

$$
\Omega = I(\Omega) = \text{ad}^{-1}_B \text{ad}_B(\Omega) = B\Omega + \Omega B,
$$

and the Euler equations (3), in coordinates $M_{ij}$, read

$$
\dot{M}_{ij} = \sum_{k=1}^n \frac{b_i - b_j}{(b_k + b_i)(b_k + b_j)} M_{ik} M_{kj}.
$$

The equations (16) describe the motion of a free $n$-dimensional rigid body with a mass tensor $B$ and inertia tensor $I$ around a fixed point $l_3$.

Now, in addition, suppose that (10) holds (the case of a $SO(k_1) \times SO(k_2) \times \cdots \times SO(k_r)$-symmetric rigid body). The operator $\mathfrak A$ given by (15) is well defined on the whole $\mathfrak{so}(n)$ and the restriction of $\mathfrak A$ to $\mathfrak{so}(k_i)$ is the multiplication by $1/2\beta_i$. Thus, the system (13) is trivial and we have the Noether conservation law $M_{\mathfrak A} = \text{const}$.

Let us denote the set of linear functions on $\mathfrak{so}(n)_A$ by $\mathcal S$. These additional integrals provide the integrability of the system. The complete integrability of the system is proved by Bolsinov by using the pencil of Lie algebras on $\mathfrak{so}(n)$ (see the last paragraph of Section 2).

1.7. Outline of the Paper. In Section 2 we prove that Manakov integrals $\mathcal L$ together with Noether integrals $\mathcal S$ form a complete noncommutative set of polynomials on $\mathfrak{so}(n)$, giving a new proof for the integrability of symmetric rigid body motion (16). We also prove that Manakov integrals induce a complete commutative set within $SO(n)$-invariant polynomials on the cotangent bundle of the homogeneous space $SO(n)/SO(k_1) \times \cdots \times SO(k_r)$ in Section 3. The complete $SO(n)$-invariant commutative sets were known before only for certain choices of numbers $k_1, \ldots, k_r$ (see [7, 9]). In particular, it is proved in Section 4 that the
construction implies the integrability of the $SO(n)$-invariant Einstein metrics on $SO(k_1 + k_2 + k_3)/SO(k_1) \times SO(k_2) \times SO(k_3)$ and on the Stiefel manifolds $V(n,k)$. These Einstein metrics have been obtained in [17] [26] [2] [3].

2. Integrability of a Symmetric Rigid Body Motion

2.1. Completeness of Manakov Integrals. Since the algebra of linear functions $S$ is not commutative if some of $k_1, \ldots, k_r$ are greater than 2, the natural framework in studying singular Manakov flows is noncommutative integration. We start with an equivalent definition of the completeness. Let $\mathcal{F}$ be a collection of functions closed under the Poisson bracket on the Poisson manifold $M$. We say that $\mathcal{F}$ is complete at $x$ if the space $F_x$ given by (2) is coisotropic:

$$F_x^\Lambda \subset F_x.$$  \hfill (17)

Here $F_x^\Lambda$ is skew-orthogonal complement of $F_x$ with respect to $\Lambda$:

$$F_x^\Lambda = \{ \xi \in T^*_x M \mid \Lambda_x(F_x, \xi) = 0 \}.$$

The set $\mathcal{F}$ is complete if it is complete at a generic point $x \in M$. In this case $\dim \mathcal{F} = \dim F_x$ and $F_x^\Lambda = \ker \Lambda_x |_{F_x}$ implying $\dim \mathcal{F} = \dim F_x^\Lambda$, for a generic $x \in M$.

Note that one can consider Hamiltonian systems restricted to symplectic leaves as well. Let $N \subset M$ be a symplectic leaf (regular or singular). The set $\mathcal{F}$ is complete on the symplectic leaf $N$ at $x \in N$ if

$$F_x^\Lambda \subset F_x + \ker \Lambda_x$$  \hfill (18)

and it is complete on the symplectic leaf $N$ if it complete at a generic point $x \in N$.

As above, let $\mathcal{S}$ be the set of linear functions on $\mathfrak{so}(n)_A$ and $L$ be the Lax pair integrals (9).

**Theorem 1.** $L + \mathcal{S}$ is a complete noncommutative set of functions on $\mathfrak{so}(n)$ with respect to the Lie-Poisson bracket (5).

**Corollary 1.** The symmetric rigid body system (10), (11) is completely integrable in the noncommutative sense. Moreover, suppose that the system (13) is completely integrable on $\mathfrak{so}(n)_A$ with a complete set of commuting integrals $S^0$. Then the singular Manakov flow (13), (14) is also completely integrable with a complete commuting set of integrals $L + S^0$.

**Proof of theorem 1.** Let $L_M = \{ \nabla_M \text{tr}(M + \lambda A)^k \mid k = 1, 2, \ldots, n, \lambda \in \mathbb{R} \}$. According to (17), $L + \mathcal{S}$ is complete at $M$ if

$$L_M + \mathfrak{so}(n)_A \Lambda \subset L_M + \mathfrak{so}(n)_A,$$

where $\Lambda$ is the canonical Lie-Poisson bivector on $\mathfrak{so}(n)$:

$$\Lambda(\xi_1, \xi_2)|_M = -\langle M, [\xi_1, \xi_2] \rangle.$$

(19)
Consider the Lie algebra $\mathfrak{gl}(n)$ of $n \times n$ real matrices equipped with the scalar product [28]. We have the symmetric pair orthogonal decomposition $\mathfrak{gl}(n) = \mathfrak{so}(n) \oplus \text{Sym}(n)$ on the skew-symmetric and symmetric matrices:

$$[\mathfrak{so}(n), \text{Sym}(n)] \subset \text{Sym}(n), \quad [\text{Sym}(n), \text{Sym}(n)] \subset \mathfrak{so}(n).$$

The scalar product $\langle \cdot, \cdot \rangle$ is positive definite on $\mathfrak{so}(n)$ while it is negative definite on $\text{Sym}(n)$.

Let us identify $\mathfrak{gl}(n)^*$ and $\mathfrak{gl}(n)$ by means of $\langle \cdot, \cdot \rangle$. On $\mathfrak{gl}(n)$ we have the pair of compatible Poisson bivectors (see Reyman [28])

$$\Lambda_1(\xi_1 + \eta_1, \xi_2 + \eta_2)|_X = -\langle X, [\xi_1, \xi_2] + [\xi_1, \eta_2] + [\eta_1, \xi_2] \rangle,$$

$$\Lambda_2(\xi_1 + \eta_1, \xi_2 + \eta_2)|_X = -\langle X + \Lambda, [\xi_1 + \eta_1, \xi_2 + \eta_2] \rangle,$$

where $X \in \mathfrak{gl}(n)$, $\xi_1, \xi_2 \in \mathfrak{so}(n)$, $\eta_1, \eta_2 \in \text{Sym}(n)$. In other words, the pencil

$$\Pi = \{ \Lambda_{\lambda_1, \lambda_2} | \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1^2 + \lambda_2^2 \neq 0 \}, \quad \Lambda_{\lambda_1, \lambda_2} = \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2$$

consist of Poisson bivectors on $\mathfrak{gl}(n)$.

The Poisson bivectors $\Lambda_{\lambda_1, \lambda_2}$, for $\lambda_1 + \lambda_2 \neq 0$ and $\lambda_2 \neq 0$, are isomorphic to the canonical Lie-Poisson bivector (in particular, their corank is equal to $n$). The union of their Casimir functions

$$\mathcal{F} = \{ f_{\lambda,k}(X) = \text{tr}(\lambda M + P + \lambda^2 A)^k | k = 1, 2, \ldots, n, \lambda \in \mathbb{R} \}$$

where $X = M + P$, $M \in \mathfrak{so}(n)$, $P \in \text{Sym}(n)$, is a commutative set with respect to the all brackets from the pencil $\Pi$ [28][5]. Also, the skew-orthogonal complement $F_{\lambda}^\perp$ does not depend on the choice $\Lambda \in \Pi$. As above, $F_X$ denotes the linear subspace of $\mathfrak{gl}(n)$ generated by the differentials of functions from $\mathcal{F}$ at $X$.

We need to take all objects complexified (see [28]). The complexification of $\mathfrak{gl}(n)$, $\mathfrak{so}(n)$, $\text{Sym}(n)$, $\mathfrak{so}(n)_A$, $\Pi$ are $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\text{Sym}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})_A \cong \mathfrak{so}(k_1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{so}(k_r, \mathbb{C})$ and $\Pi^\mathbb{C} = \{ \Lambda_{\lambda_1, \lambda_2} = \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2, \lambda_1, \lambda_2 \in \mathbb{C}, |\lambda_1|^2 + |\lambda_2|^2 \neq 0 \}$, respectively. Here, we consider $\Pi$ as complex valued skew-symmetric bilinear forms. The complexified scalar product is simply given by \langle X, Y \rangle, where now $X, Y \in \mathfrak{gl}(n, \mathbb{C})$.

At a generic point $X \in \mathfrak{gl}(n)$, the only singular bivector in $\Pi^\mathbb{C}$ with a rank smaller then $\dim \mathfrak{gl}(n) - n$ is $\Lambda_{-1,1}$. Moreover, the complex dimension of the linear space

$$K_{-1,1} = \{ \xi \in \ker \Lambda_{-1,1}(X) | \Lambda_0(\xi, \ker \Lambda_{-1,1}(X)) = 0 \}$$

is equal to $n$. Here $\Lambda_0$ is any Poisson bivector from the pencil, nonproportional to $\Lambda_{-1,1}$, say $\Lambda_0 = \Lambda_{01}$. Whence, it follows from Proposition 2.5 [5] that

$$\langle F_X^{\Lambda_1} \rangle^\mathbb{C} = F_X^\mathbb{C} + \ker \Lambda_{-1,1}(X).$$

Also, it can be proved that

$$F_X^\mathbb{C} + \ker \Lambda_{-1,1}(X) = F_X^\mathbb{C} + \mathfrak{so}(n, \mathbb{C})_A.$$
The above relations imply
\begin{equation}
(F_X + \mathfrak{so}(n)_A)^{\Lambda_1} = F_X^{\Lambda_1} \cap \mathfrak{so}(n)^{\Lambda_1}_A \subset F_X + \mathfrak{so}(n)_A
\end{equation}
and the set of functions $\mathcal{F} + \mathcal{S}$ is a complete non-commutative set on $\mathfrak{gl}(n)$ with respect to $\Lambda_1$ (Theorem 1.5 [5], for the detail proofs of the above statements, given for an arbitrary semi-simple symmetric pair, see [31], pages 234-237).

Now we want to verify the completeness of $\mathcal{F} + \mathcal{S}$ at the points $M \in \mathfrak{so}(n)$. Note that in Theorem 1.6 [5], a similar problem have been studied but for regular $A$ and singular points $M \in \mathfrak{so}(M)$, in proving that Manakov integrals provide complete commutative sets on singular adjoint orbits.

Since a regular matrix $M \in \mathfrak{so}(n)$ ($\dim \mathfrak{so}(n)_M = \text{rank } \mathfrak{so}(n) = [n/2]$), considered as an element of $\mathfrak{gl}(n, \mathbb{C})$ is also regular ($\dim \mathfrak{gl}(n, \mathbb{C})_M = n$), it can be easily proved that the only two singular brackets in $\Pi^C$ are $\Lambda_{-1,1}$ and $\Lambda_{1,0}$.

We have to estimate the complex dimensions of linear spaces
\begin{equation}
K_{-1,1} = \{ \xi \in \ker \Lambda_{-1,1}(M) \mid [M + A, [\xi, \ker \Lambda_{-1,1}(M)]] = 0 \}
\end{equation}
\begin{equation}
K_{1,0} = \{ \xi \in \ker \Lambda_{1,0}(M) \mid [M + A, [\xi, \ker \Lambda_{1,0}(M)]] = 0 \}.
\end{equation}

As for $X \in \mathfrak{gl}(n)$, repeating the arguments of [31], pages 234-237, one can prove that the dimension of (25) is $n$ for a generic $M \in \mathfrak{so}(n)$. Further
\begin{equation}
\ker \Lambda_{1,0}(M) = \ker \Lambda_1(M) = \mathfrak{so}(n, \mathbb{C})_M + \text{Sym}(n, \mathbb{C}),
\end{equation}
where $\mathfrak{so}(n, \mathbb{C})_M$ is the isotropy algebra of $M$ in $\mathfrak{so}(n, \mathbb{C})_M$.

We shall prove below that $\dim_{\mathbb{C}} K_{1,0}$ is also equal to $n$ for a generic $M \in \mathfrak{so}(n)$ (see Lemma 1). Hence, according Proposition 2.5 [5], at a generic $M \in \mathfrak{so}(n)$ we have
\begin{equation}
(F_M^{\Lambda_1})^C = F_M^C + \ker \Lambda_{-1,1}(M) + \ker \Lambda_{1,0}(M) = F_M^C + \mathfrak{so}(n, \mathbb{C})_A + \ker \Lambda_1(M).
\end{equation}

Similarly as in equation (24), we get
\begin{equation}
(F_M + \mathfrak{so}(n)_A)^{\Lambda_1} = (F_M + \mathfrak{so}(n)_A + \ker \Lambda_1(M)) \cap \mathfrak{so}(n)^{\Lambda_1}_A 
\subset F_M + \mathfrak{so}(n)_A + \ker \Lambda_1(M).
\end{equation}

Therefore the relation (18) holds for functions $\mathcal{F} + \mathcal{S}$ and the bracket $\Lambda_1$, i.e., this is a complete set on the symplectic leaf through $M$.

Notice that the symplectic leaves through $M \in \mathfrak{so}(n) \subset \mathfrak{gl}(n)$ of the bracket $\Lambda_1$ are $\mathfrak{SO}(n)$-adjoint orbit in $\mathfrak{so}(n)$ and that the restriction of $\Lambda_1$ to $\mathfrak{so}(n)$ coincides with the Lie-Poisson bracket (20). Since the restriction of the central functions (22) to $\mathfrak{so}(n)$ are Manakov integrals (9), we obtain (19). □

Remark 1. From the proof of Theorem 1 follows that the skew-orthogonal complement of $L_M$ within $\mathfrak{so}(n)$ is given by
\begin{equation}
L_M^\Lambda = L_M + \mathfrak{so}(n)_A,
\end{equation}
for a generic $M \in \mathfrak{so}(n)$.
Lemma 1. The complex dimension of the linear space (30) is equal to \(n\) for a generic \(M \in \mathfrak{so}(n)\).

Proof. For \(\xi \in \ker\Lambda_1(M)\), let \(\xi_1\) and \(\xi_2\) be the projections to \(\mathfrak{so}(n,\mathbb{C})_M\) and \(\text{Sym}(n,\mathbb{C})\), respectively. Then

\[
\langle M + A, [\xi, \ker\Lambda_1(M)] \rangle = \langle \ker\Lambda_1(M), [M + A, \xi_1 + \xi_2] \rangle = \langle \ker\Lambda_1(M), [M, \xi_2] + [A, \xi_1] + [A, \xi_2] \rangle = \langle \mathfrak{so}(n,\mathbb{C})_M, [A, \xi_2] \rangle + \langle \text{Sym}(n,\mathbb{C}), [M, \xi_2] + [A, \xi_1] \rangle
\]

Therefore \(\xi = \xi_1 + \xi_2 \in \ker\Lambda_1(M)\) belongs to \(K_{1,0}\) if and only if

\[
[M, \xi_2] + [A, \xi_1] = 0, \quad \text{pr}_{\mathfrak{so}(n,\mathbb{C})_M}[A, \xi_2] = 0.
\]

The dimension of the solutions of the system (30), for a regular \(M \in \mathfrak{so}(n)\), is \(n\). It can be directly calculated by taking the following anti-diagonal element:

\[
M = m_1E_1 \wedge E_n + m_2E_2 \wedge E_{n-1} + \cdots + m_kE_k \wedge E_{k+1}, \quad n = 2k
\]

\[
M = m_1E_1 \wedge E_n + m_2E_2 \wedge E_{n-1} + \cdots + m_kE_k \wedge E_{k+2}, \quad n = 2k + 1,
\]

when

\[
\mathfrak{so}(n,\mathbb{C})_M = \text{span}_\mathbb{C}\{E_1 \wedge E_n, E_2 \wedge E_{n-1}, \ldots, E_k \wedge E_{k+1}\}, \quad n = 2k
\]

\[
\mathfrak{so}(n,\mathbb{C})_M = \text{span}_\mathbb{C}\{E_1 \wedge E_n, E_2 \wedge E_{n-1}, \ldots, E_k \wedge E_{k+2}\}, \quad n = 2k + 1.
\]

Here \(m_1, \ldots, m_k\) are generic distinct real numbers. For example, if \(n = 2k\), then \(\xi \in \ker\Lambda_1(M)\) satisfies (30) if and only if it is of the form:

\[
\xi = \sum_{i=1}^n u_i E_i \otimes E_i + \sum_{j=1}^k v_j E_j \wedge E_{n+1-j},
\]

where parameters \(u_i, v_j\) are determined from the linear system:

\[
-m_j(u_j - u_{n+1-j}) + (a_j - a_{n+1-j})v_j = 0, \quad j = 1, \ldots, k.
\]

Thus \(\dim_\mathbb{C} K_{1,0} = n\). \(\square\)

2.2. Pencil of Lie Algebras. Bolsinov has given another proof of the integrability of Manakov flows, related to the existence of compatible Lie algebra brackets on \(\mathfrak{so}(n)\) [5]. The first Lie bracket is standard one \([M_1, M_2] = M_1M_2 - M_2M_1\), while the second is

\[
[M_1, M_2]_A = M_1AM_2 - M_2AM_1.
\]

Then \(\Lambda\) and \(\Lambda_A\) are compatible Poisson structures, where \(\Lambda\) is given by (20) and (31)

\[
\Lambda_A(\xi_1, \xi_2)|_M = -\langle M, [\xi_1, \xi_2]_A \rangle.
\]

Let \(\Lambda_{\lambda_1, \lambda_2} = \lambda_1\Lambda + \lambda_2\Lambda_A\). The central functions of the bracket \(\Lambda_{\lambda,1}\) of maximal rank \((\lambda \neq -\alpha_1, \ldots, -\alpha_r)\) are

\[
\mathcal{J} = \{\text{tr}(M(\lambda I + A)^{-1})^{2k} | k = 1, 2, \ldots, \lfloor n/2 \rfloor, \lambda \neq -\alpha_1, \ldots, \alpha_r\}.
\]
According to the general construction, these functions commute with respect to all Poisson brackets $\Lambda_{\lambda_1, \lambda_2}$. The following theorem obtained by Bolsinov can be found in [31], pages 241-244:

**Theorem 2.** (Bolsinov) The set of functions $J + S$ is a complete set on $\mathfrak{so}(n)$ with respect to the Lie-Poisson bracket (5).

The families (9) and (32) commute between each other (e.g., see [27]). Therefore, since both sets $L + S$ and $J + S$ are complete, they are equivalent, i.e., they determine the same invariant toric foliation of the phase space.

3. **Geodesic Flows on $SO(n)/SO(k_1) \times \cdots \times SO(k_r)$**

3.1. **Geodesic Flows on Homogeneous Spaces.** Consider the homogeneous spaces $G/H$ of a compact Lie group $G$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$ be the orthogonal decomposition and let $ds_0^2$ be the normal $G$-invariant metric induced by some $\text{Ad}_G$-invariant scalar product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$.

Let $\mathcal{F}^G$ be the set of $G$ invariant functions, polynomial in momenta and $\Phi : T^*(G/H) \to \mathfrak{g}^*$ be the momentum mapping of the natural Hamiltonian $G$-action. From the Noether theorem we have $\{ \mathcal{F}^G, \Phi^*(\mathbb{R}[\mathfrak{g}^*]) \} = 0$, where $\{\cdot, \cdot\}$ is the canonical Poisson bracket on $T^*(G/H)$. The Hamiltonian of the normal metric $ds_0^2$ is a central function of $\mathcal{F}^G$ so it commutes both with the Noether functions $\Phi^*(\mathbb{R}[\mathfrak{g}^*])$ and $G$-invariant functions $\mathcal{F}^G$. On the other side, the set $\mathcal{F}^G + \Phi^*(\mathbb{R}[\mathfrak{g}^*])$ is complete, implying the noncommutative integrability of the geodesic flow of the normal metric [7, 8].

The algebra $(\mathcal{F}^G, \{\cdot, \cdot\})$ can be naturally identified with $(\mathbb{R}[\mathfrak{v}]^H, \{\cdot, \cdot\}_\mathfrak{v})$, where $\mathbb{R}[\mathfrak{v}]^H$ is the algebra of $\text{Ad}_H$-invariant polynomials on $\mathfrak{v}$ and (see Thimm [30]):

$$\{f, g\}_\mathfrak{v}(x) = -\langle x, [\nabla f(x), \nabla g(x)] \rangle, \quad f, g \in \mathbb{R}[\mathfrak{v}]^H.$$  \hspace{1cm} (33)

Within the class of Noether integrals $\Phi^*(\mathbb{R}[\mathfrak{g}^*])$ one can always construct a complete commutative subset. Thus the Mishchenko–Fomenko conjecture reduces to a construction of a complete commutative subset of $\mathbb{R}[\mathfrak{v}]^H \cong \mathcal{F}^G$. This leads to Conjecture 1 stated in the Introduction.

A commutative set $\mathcal{F} \subset \mathbb{R}[\mathfrak{v}]^H$ is complete if

$$\text{ddim } \mathcal{F} = \frac{1}{2} (\text{ddim } \mathbb{R}[\mathfrak{v}]^H + \text{dind } \mathbb{R}[\mathfrak{v}]^H) = \dim \mathfrak{v} - \frac{1}{2} \dim O_G(x),$$  \hspace{1cm} (34)

for a generic $x \in \mathfrak{v}$, where $O_G(x)$ is the adjoint orbit of $G$ (see [7, 9]).

---

1It was pointed out by one of the referees that the equivalence of the integrals (9) and (32) can be derived directly, by using the matrix identity

$$\det(M + A(\alpha I + \alpha \beta I)^{-1} + \beta I) = \det(M + \beta A + \alpha / \beta) \det(\alpha I + A)^{-1}.$$
3.2. Normal Geodesic Flows on $SO(n)/SO(k_1) \times \cdots \times SO(k_r)$. Let

$$SO(n)_A = SO(k_1) \times \cdots \times SO(k_r) \subset SO(n)$$

be the isotropy group of $A$ within $SO(n)$ with respect to the adjoint action. As above, consider the normal metric $ds_A^2$ defined by the scalar product (4) and identify $SO(n)$-invariant polynomials on $T^*(SO(n)/SO(n)_A)$ with $\mathbb{R}[\mathfrak{v}]^{SO(n)_A}$ ($\mathfrak{v}$ is defined by (11)).

We shall use the following completeness criterium. Consider the space $j_M \subset \mathfrak{v}$ spanned by gradients of all polynomials in $\mathbb{R}[\mathfrak{v}]^{SO(n)_A}$. For a generic point $M \in \mathfrak{v}$ we have

$$j_M = ([M, \mathfrak{so}(n)_A]^{\perp}) \cap \mathfrak{v} = \{ \eta \in \mathfrak{v} \mid \langle \eta, [M, \mathfrak{so}(n)_A] \rangle = 0 \} = \{ \eta \in \mathfrak{v} \mid [M, \eta] \subset \mathfrak{v} \}.$$

The bracket (33) on $\mathbb{R}[\mathfrak{v}]^{SO(n)_A}$ corresponds to the restriction of the Lie-Poisson bivector (20) to $j_M$. Denote this restriction by $\bar{\Lambda}$. Then $\mathcal{F} \subset \mathbb{R}[\mathfrak{v}]^{SO(n)_A} \cong \mathcal{F}^{SO(n)}$ is a complete commutative subset in $\mathcal{F}$ if and only if

$$F_M^{\bar{\Lambda}} = F_M,$$

for a generic $M \in \mathfrak{v}$, where $F_M = \text{span} \{ \nabla_M f(M) \mid f \in \mathcal{F} \} \subset j_M$ and $F_M^{\bar{\Lambda}}$ is the skew-orthogonal complements of $F_M$ with respect to $\bar{\Lambda}$ within $j_M$. Here, for simplicity, the gradient operator with respect to the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{v}$ is also denoted by $\nabla$.

Since all polynomials in $\mathcal{L}$ commute with $\mathcal{S}$, their restrictions to $\mathfrak{v}$

$$(35) \quad \mathcal{L}_\mathfrak{v} = \{ \text{tr}(M + \lambda A)^k \mid M \in \mathfrak{v}, k = 1, 2, \ldots, n, \lambda \in \mathbb{R} \},$$

form a commutative subset of $\mathbb{R}[\mathfrak{v}]^{SO(n)_A}$ (see [7]).

Let $\Phi : T^*SO(n)/SO(n)_A \to \mathfrak{so}(n)^* \cong \mathfrak{so}(n)$ be the momentum mapping of the natural $SO(n)$-Hamiltonian action on $T^*SO(n)/SO(n)_A$ and let $\mathcal{A}$ be any commutative set of polynomial on $\mathfrak{so}(n)$ that is complete on adjoint orbits within the image $\Phi(T^*(SO(n)/SO(n)_A))$ (for example one can take Manakov integrals with regular $A$ [5]). Then $\Phi^*(\mathcal{A})$ is a complete commutative subset in $\Phi^*(\mathbb{R}[\mathfrak{so}(n)])$ and we have:

**Theorem 3.** (i) $\mathcal{L}_\mathfrak{v}$ is a complete commutative subset of $\mathbb{R}[\mathfrak{v}]^{SO(n)_A}$.

(ii) The geodesic flow of the normal metric $ds_A^2$ is Liouville integrable by means of polynomial integrals $\mathcal{L}_\mathfrak{v} + \Phi^*(\mathcal{A})$.

**Remark 2.** Note that, by using the **chains of subalgebras method**, the construction of another complete commutative algebras of $SO(n)$-invariant polynomials is performed for homogeneous spaces $SO(n)/SO(k)$ and $SO(n)/SO(k_1) \times SO(k_2)$ (see [7, 9]). Also, by using the **generalized chains of subalgebras method**, the Conjecture 1 is solved for a class of homogeneous spaces $SO(n)/SO(n)_A$ [9]. The class, say $\mathcal{C}$, is obtained by induction from $SO(n)/SO(k_1) \times SO(k_2)$, $k_1 \leq k_2 \leq \lfloor \frac{n}{2} \rfloor$ in the following way: suppose that $SO(n_1)/SO(k_1) \times \cdots \times SO(k_r)$ and $SO(n_2)/SO(l_1) \times \cdots \times SO(l_r)$ ($n_1 = n_2 \pm 0, 1$) belong in $\mathcal{C}$, then also $SO(n_1 +
\( u_2 / SO(k_1) \times \cdots \times SO(k_r) \times SO(l_1) \times \cdots \times SO(l_r) \) belongs to \( C \). Note that, for example, the homogeneous spaces \( SO(n) / SO(k_1) \times \cdots \times SO(k_r) \), where some of \( k_i \) is greater than \( \left[ \frac{n + 1}{2} \right] \) do not belong to the family \( C \).

**Proof.** Without loss of generality, suppose

\[
k_1 \leq k_2 \leq k_3 \cdots \leq k_r.
\]

If the condition

\[
k_r \leq \left\lfloor \frac{n + 1}{2} \right\rfloor
\]

is satisfied, then a generic element \( M \in v \) is regular element of \( so(n) \) and relation (29) will holds. Then it easily follows that

\[
(\bar{L}^\Lambda)_M = \{ \nabla_M f \mid f \in \mathcal{L}_v \} \subset j_M,
\]

for a generic \( M \in v \). Hence \( \mathcal{L}_v \) is complete.

Now, suppose

\[
k_r = \left\lfloor \frac{n + 1}{2} \right\rfloor + l, \quad l > 0.
\]

Let \( n' = n - 2l, k'_r = k_r - 2l, A' = \text{diag}(a_1, a_2, \ldots, a_n) \) and let

\[
so(n') = so(n')_{A'} \oplus v' = so(k_1) \oplus so(k_2) \oplus \cdots \oplus so(k'_r) \oplus v'
\]

be the orthogonal decomposition, where \( so(n')_{A'} \) is the isotropy algebra of \( A' \) within \( so(n') \).

Furthermore, we can consider Lie algebras \( so(n') \) and \( so(2l) \) embedded in \( so(n) \) as blocks:

\[
\begin{pmatrix}
so(n') & 0 \\
0 & so(2l)
\end{pmatrix}.
\]

Then the linear space \( v' \) becomes a linear subspace of \( v \):

\[
v' = so(n') \cap v.
\]

Moreover, for an arbitrary \( M \in v \) one can find a matrix \( K \in SO(n)_{A} \) such that \( M' = \text{Ad}_K(M) \) belongs to \( v' \). Indeed, consider \( M \) and \( K \) of the form

\[
M = \begin{pmatrix}
M_{11} & M_{12} \\
-M_{12}^T & 0
\end{pmatrix}, \quad K = \begin{pmatrix}
I_{n-k_r} & 0 \\
0 & U
\end{pmatrix},
\]

where \( M_{11} \in so(n - k_r) \), \( M_{12} \) is \( (n - k_r) \times (k_r) \) matrix, \( I_{n-k_r} \) is the identity \( (n - k_r) \times (n - k_r) \) matrix and \( U \in SO(k_r) \). Then

\[
M' = KMK^{-1} = \begin{pmatrix}
M_{11} & M_{12}U^T \\
-U^TM_{12}^T & 0
\end{pmatrix}.
\]

Since \( k_r - (n - k_r) \) is equal to \( 2l \) or \( 2l + 1 \), one can always find \( U \) such that the last \( 2l \) rows of \( U M_{12}^T \), i.e., the last \( 2l \) columns of \( M_{12}U^T \) are equal to zero, which implies that \( M' \) belongs to \( v' \).

Therefore, if the set \( \mathcal{L}_v \) is complete at the points of \( v' \) then it will be complete on \( v \) as well.
The Lie algebras \( \mathfrak{so}(n') \) and \( \mathfrak{so}(n')_A \) are centralizers of \( \mathfrak{so}(2l) \) in \( \mathfrak{so}(n) \) and \( \mathfrak{so}(n)_A \), respectively. Whence, from the above considerations, we can apply Theorem A1 (see Appendix) to get

\[
\{( \eta, [M', \mathfrak{so}(n')_A] ) = 0 \} = \{( \eta, [M', \mathfrak{so}(n)_A] ) = 0 \} = \{M' \},
\]

for a generic \( M' \in \mathfrak{v}' \).

In particular, \ref{eq:38} implies that Poisson tensors \( \bar{\Lambda} \) of \( \mathfrak{r}[\mathfrak{v}]^{\mathfrak{so}(n'_A)} \) and \( \bar{\Lambda}' \) of \( \mathfrak{r}[\mathfrak{v}']^{\mathfrak{so}(n')}{_A'} \) coincide on a generic \( M' \in \mathfrak{v}' \subset \mathfrak{v} \). According to the first part of the proof, the set of polynomials

\[
\mathcal{L}_{\mathfrak{v}'} = \{ \text{tr}(M' + \lambda A')^k | M' \in \mathfrak{v}', k = 1, 2, \ldots, n, \lambda \in \mathbb{R} \},
\]

is a complete commutative subset of \( \mathfrak{r}[\mathfrak{v}]^{\mathfrak{so}(n')}_A \), i.e,

\[
\bar{L}_M' = \bar{L}_M', \quad \bar{L}_M' = \{ \nabla_M | f \in \bar{L}_{\mathfrak{v}'} \} \subset \mathcal{L}_{\mathfrak{v}'},
\]

for a generic \( M' \in \mathfrak{v}' \). But from \ref{eq:38} and \ref{eq:39} we also get that \ref{eq:38} holds for a generic \( M' \in \mathfrak{v}' \).

Remark 3. An alternative proof of theorem \ref{thm:3} can be performed by using the compatibility of Poisson brackets \ref{eq:20} and \ref{eq:31}, but now considered within the algebra \( \mathfrak{r}[\mathfrak{v}]^{\mathfrak{so}(n')}_A \).

3.3. Submersion of Manakov Flows. Let \( \mathfrak{A} \) be given by \ref{eq:12} where \( \mathfrak{B} \) is the identity operator. Then the singular Manakov flow \ref{eq:13}, \ref{eq:14} represent the geodesic flow of the left \( \mathfrak{so}(n) \)-invariant metric on \( \mathfrak{so}(n) \) that is also right \( \mathfrak{so}(n)_A \)-invariant. By submersion, this metric induces \( \mathfrak{so}(n)-\)invariant metric on homogeneous space \( \mathfrak{so}(n)/\mathfrak{so}(n)_A \) that we shall denote by \( ds_{A,B}^2 \). Specially, for \( A = B \) we have the normal metric.

On \( \mathfrak{v} \), identified with the tangent space at the class of the identity element, the metric is given by the \( \mathfrak{so}(n)_A \)-invariant scalar product

\[
\langle \cdot, \cdot \rangle_{A,B} = \sum_{1 \leq i \leq j \leq r} \frac{\alpha_i - \alpha_j}{\beta_i - \beta_j} \langle \cdot, \cdot \rangle_{\mathfrak{v}_{i,j}},
\]

where

\[
\mathfrak{v} = \bigoplus_{1 \leq i < j \leq r} \mathfrak{v}_{i,j}
\]

is the decomposition into a sum of \( \mathfrak{so}(n)_A \)-invariant subspaces defined by \( \mathfrak{so}(k_i + k_j) = \mathfrak{so}(k_i) \oplus \mathfrak{so}(k_j) \oplus \mathfrak{v}_{i,j} \).

As before the formulation of theorem \ref{thm:3} let \( \mathcal{A} \) be any commutative set of polynomial on \( \mathfrak{so}(n) \) that is complete on adjoint orbits within the image of the momentum mapping \( \Phi \). Since the Hamiltonian \( H_{A,B}(M) = \frac{1}{2}(\text{ad}^{-1} \circ \text{ad}_B(M), M) \in \mathfrak{r}[\mathfrak{v}]^{\mathfrak{so}(n'_A)} \cong \mathcal{F}^{\mathfrak{so}(n)} \) of the geodesic flow of the metric \( ds_{A,B}^2 \) Poisson commute with \( \mathcal{L}_{\mathfrak{v}} \), from theorem \ref{thm:3} we get
Corollary 2. The geodesic flows of the metrics $ds^2_{A,B}$ on the homogeneous spaces $SO(n)/SO(n)_A$ are completely integrable in the noncommutative sense. The complete set of integrals is given by (35) and Noether integrals $\Phi^*(\mathbb{R}[\mathfrak{so}(n)])$. The geodesic flows is also Liouville integrable by means of polynomial integrals $\mathcal{L}_\nu + \Phi^*(\mathcal{A})$.

4. Examples: Einstein Metrics

Among $SO(n)$-invariant metrics on $SO(n)/SO(k_1) \times \cdots \times SO(k_r)$ the specific geometric significance have Einstein metrics (see [3]). It is well known that the unique (up to homotheties) $SO(n)$-invariant metrics on symmetric spaces $SO(n)/SO(n-k) \times SO(k)$ are Einstein. Further examples are given by Jensen [17], Arvanitoyeorgos, Dzhepko and Nikonorov [2, 3] and Nikonorov [26] on Stiefel manifolds $V(n,k) = SO(n)/SO(k)$ and spaces $SO(k_1+k_2+k_3)/SO(k_1) \times SO(k_2) \times SO(k_3)$, respectively.

The geodesic flows on symmetric spaces are completely integrable (see Mishcheenko [22]). It is very interesting that the geodesic flows of Einstein metrics given in [17, 26, 2, 3] are also integrable.

Firstly, note that the metrics on $SO(k_1+k_2+k_3)/SO(k_1) \times SO(k_2) \times SO(k_3)$ constructed by Nikonorov in [26] are already of the form (40). On the other side, to prove the integrability of geodesic flows of Einstein metrics on Stiefel manifolds $V(n,k)$ obtained in [17, 2, 3] we need the following simple modification of theorem 3.

Let us fix $l$, $1 \leq l \leq r$ and consider products $SO(n)_A = H \times K$ and $\mathfrak{so}(n)_A = \mathfrak{h} \oplus \mathfrak{k}$, where

$$H = SO(k_1) \times \cdots \times SO(k_l), \quad K = SO(k_{l+1}) \times \cdots \times SO(k_r),$$

$$\mathfrak{h} = \mathfrak{so}(k_1) \oplus \cdots \oplus \mathfrak{so}(k_l), \quad \mathfrak{k} = \mathfrak{so}(k_{l+1}) \oplus \cdots \oplus \mathfrak{so}(k_r).$$

Let $p = \mathfrak{t} \oplus \mathfrak{v}, \mathbb{R}[p]^H$ be the algebra of Ad$_H$-invariant functions on $p$ identified with the algebra of $SO(n)$ functions on $T^*(SO(n)/H)$. $K$ be the algebra of linear functions on $\mathfrak{t}$ lifted to the functions in $\mathbb{R}[p]^H$ and

$$\mathcal{L}_\mathfrak{p} = \{ \text{tr}(M + \lambda A)^k \mid M \in \mathfrak{p}, k = 1, 2, \ldots, n, \lambda \in \mathbb{R} \}.$$  

Theorem 4. (i) $\mathcal{L}_\mathfrak{p} + K$ is a complete subset of $\mathbb{R}[p]^H$.

(ii) If $K^0$ is any complete commutative set of functions on $\mathfrak{t}$ lifted to the functions on $\mathfrak{p}$, then $\mathcal{L}_\mathfrak{p} + K^0$ will be a complete commutative subset of $\mathbb{R}[p]^H$.

Now, consider the case $r = 2$, $l = 1$, $H = SO(k)$, $K = SO(n-k)$. Then $\mathfrak{v}_{1,2} = \mathfrak{v}$ and $p = \mathfrak{so}(n-k) \oplus \mathfrak{v}$. Define the $SO(n)$-invariant metric $ds^2_{2,\kappa}$ on $V(n,k) = SO(n)/SO(k)$ by its restrictions to $p$:

$$(\cdot, \cdot)_{2,\kappa} = (\cdot, \cdot)_{\mathfrak{so}(n-k)} + \kappa (\cdot, \cdot)|_0,$$

where $\mathcal{J}: \mathfrak{so}(n-k) \rightarrow \mathfrak{so}(n-k)$ is positive definite and $\kappa > 0$.  

Note that Manakov integrals are integrals of the geodesic flow of the metric. Thus, if Euler equations on $so(n-k)$

\begin{equation}
\dot{M} = [M, J^{-1}(M)]
\end{equation}

are integrable, the geodesic flow of the metric $ds^2_{g,\kappa}$ will be completely integrable.

Let $J = \chi \cdot \mathrm{Id}_{so(n-k)}$. In [17], Jensen proved that for $n-k = 2$ there is a unique value, while for $n-k > 2$ there are exactly two values of $(\chi, \kappa) \in \mathbb{R}^2$ (up to homotheties), such that $ds^2_{g,\kappa}$ is Einstein metric. Since then equations are integrable, the geodesic flow of the metric $\mathcal{L}_p + \mathcal{K}$ are integrals of the geodesic flow.

Arvanitoyeorgos, Dzhepko and Nikonorov found two new Einstein metrics within the class of metrics with $n-k = sl$, $s > 1$, $k > l \geq 3$. It appears that the integrability of corresponding Euler equations can be easily proved by using the chain method developed by Mykytyuk [19].

**Corollary 3.** The geodesic flows of Einstein metrics on Stiefel manifolds $SO(n)/SO(k)$ and homogeneous spaces $SO(k_1 + k_2 + k_3)/SO(k_1) \times SO(k_2) \times SO(k_3)$ constructed in [17, 2], [3, 26] are completely integrable.

Note that the integrability of the geodesic flows of Einstein metrics on Stiefel manifolds $V(k, n)$ can be proved in a different way, starting from the analogue of the Neumann system on $V(n, r)$ (see [14]).

**Appendix: Pairs of Reductive Lie Algebras**

Let $g$ be a reductive real (or complex) Lie algebra. Take a faithful representation of $g$ such that its associated bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerate on $g$. Let $\mathfrak{t} \subset g$ be a reductive in $g$ subalgebra and $v = \mathfrak{t}^\perp = \{\eta \in g \mid \langle \eta, \mathfrak{t} \rangle = 0\}$.

For any $\xi \in v$ define the subspace $j_\xi \subset v$ by

\begin{equation}
\mathfrak{j}_\xi = \{\eta \in v \mid [\xi, \eta] \in v\} = \{\eta \in v \mid [\eta, [\xi, v]] = 0\}.
\end{equation}

Consider a Zariski open subset of $R$-elements in $v$ defined by

\begin{equation}
R(v) = \{\xi \in v \mid \dim g_\xi \leq \dim g_\eta, \dim \mathfrak{t}_\xi \leq \dim \mathfrak{t}_\eta, \dim g^0_\xi \leq \dim g^0_\eta, \eta \in v\},
\end{equation}

where $g_\eta$ and $\mathfrak{t}_\eta$ are centralizers of $\eta$ in $g$ and $\mathfrak{t}$, respectively, and $g^0_\eta$ denote the set of all $\zeta \in g$ which satisfy $(\text{ad} \eta)^n(\zeta) = 0$ for sufficiently large $n$.

Assume that $\xi_0 \in R(v)$ and $a$ is a reductive (in $g$) subalgebra of $\xi_0$. Let $g'$ and $\mathfrak{t}'$ be the centralizers of $a$ in $g$ and $\mathfrak{t}$, respectively. Then algebras $g'$ and $\mathfrak{t}'$ are subalgebras reductive in $g$ and the restriction of $\langle \cdot, \cdot \rangle$ to $g'$ and $\mathfrak{t}'$ are nondegenerate (for more details, see Mykytyuk [20]).

Let $v' = \mathfrak{t}' \cap v$. Then $v' = g' \cap v$. As above, define

\begin{equation}
\mathfrak{j}'_\xi = \{\zeta \in v' \mid [\xi, \zeta] \in v'\}, \quad \xi \in v'
\end{equation}

and the set of $R$-elements $R(v')$ in $v'$. 

The following result is contained in the proof of theorem 11 [20] (see also proposition 2.3 given in [21]).

**Theorem A 1.** (Mykytyuk [20]) The relation

\[ i_{\xi} = i_{\xi}' \]

is satisfied for any element \( \xi \) in a Zariski open subset \( R(\nu') \cap R(\nu) \) of \( \nu' \).

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