K-stability of constant scalar curvature Kähler manifolds

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Abstract

We show that a polarised manifold with a constant scalar curvature Kähler metric and discrete automorphisms is K-stable. This refines the K-semistability proved by S. K. Donaldson.

1 Introduction

Let \((X, L)\) be a polarised manifold. One of the more striking realisations in Kähler geometry over the past few years is that if one can find a constant scalar curvature Kähler (cscK) metric \(g\) on \(X\) whose \((1,1)\)-form \(\omega_g\) belongs to the cohomology class \(c_1(L)\) then \((X, L)\) is semistable, in a number of senses. The seminal references are Yau [16], Tian [13], Donaldson [5, 7].

In this note we are concerned with Donaldson’s algebraic K-stability [7], see also Definition 2.5 below. This notion generalises Tian’s K-stability for Fano manifolds [13]. It should play a role similar to Mumford-Takemoto slope stability for bundles. The necessary general theory is recalled in Section 2.

Asymptotic Chow stability (which implies K-semistability, see e.g. [12] Theorem 3.9) for a cscK polarised manifold was first proved by Donaldson [6] in the absence of continuous automorphisms. Important work in this connection was also done by Mabuchi, see e.g. [10]. From the analytic point of view the fundamental result is the lower bound on the K-energy proved by Chen-Tian [4].

The neatest result in the algebraic context seems to be Donaldson’s lower bound on the Calabi functional, which we now recall.

For a Kähler form \(\omega\) let \(S(\omega)\) denote the scalar curvature, \(\hat{S}\) its average (a topological quantity). Denote by \(F\) the Donaldson-Futaki invariant of a test configuration (Definitions 2.1, 2.2). The precise definition of the norm \(\|\mathcal{X}\|\) appearing below will not be important for us.
Theorem 1.1 (Donaldson [8]) For a polarised manifold \((X, L)\):

\[
\inf_{\omega \in c_1(L)} \int_X (S(\omega) - \hat{S})^2 \omega^n \geq -\frac{\sup_X F(\mathcal{X})}{\|\mathcal{X}\|},
\]

(1.1)

where the supremum is taken with respect to all test configurations \((\mathcal{X}, \mathcal{L})\) for \((X, L)\).

Thus if \(c_1(L)\) admits a cscK representative \((X, L)\) is K-semistable.

There is a strong analogy here with Hermitian Yang-Mills metrics on vector bundles. By the celebrated results of Donaldson and Uhlenbeck-Yau these are known to exist if and only if the bundle is slope polystable, namely a semistable direct sum of slope stable vector bundles.

In particular a simple vector bundle endowed with a HYM metric is slope stable. In this note we will prove the corresponding result for polarised manifolds.

Theorem 1.2 If \(c_1(L)\) contains a cscK metric and \(\text{Aut}(X, L)\) is discrete then \((X, L)\) is K-stable.

Theorem 1.2 fits in a more general well known conjecture.

Conjecture 1.3 (Donaldson [7]) If \(c_1(L)\) contains a cscK metric then \((X, L)\) is K-polystable (Definition 2.6).

Thus our result confirms this expectation when the group \(\text{Aut}(X, L)\) is discrete. From a differential-geometric point of view this means that \(X\) has no nontrivial Hamiltonian holomorphic vector fields - holomorphic fields that vanish somewhere.

Remark 1.4 Conjecture 1.3 and its converse are known as Yau - Tian - Donaldson Conjecture, and sometimes called the Hitchin-Kobayashi correspondence for manifolds.

For the rest of the note we will assume \(\dim(X) > 1\) in all our statements.

K-stability for Riemann surfaces is completely understood thanks to the work of Ross and Thomas [12] Section 6. In particular Conjecture 1.3 is known to hold for Riemann surfaces.

Our proof of Theorem 1.2 rests on the general principle that one should be able to perturb a semistable object (in the sense of geometric invariant theory) to make it unstable - although this necessarily involves perturbing the GIT problem too, since the locus of semistable points for an action on a fixed variety is open. Conversely in the absence of continuous automorphisms, the
The cscK property is open - at least in the sense of small deformations - so cscK should imply stability. Of course we need to make this rigorous; in particular testing small deformations is not enough to prove K-stability.

Thus suppose that \((X, L)\) is properly K-semistable (Definition 2.7). We will find a natural way to construct from this a family of K-unstable small perturbations \((X_\varepsilon, L_\varepsilon)\) for small \(\varepsilon > 0\). Our choice for \(X_\varepsilon\) is actually constant, the blowup \(\hat{X} = \text{Bl}_qX\) at a very special point \(q\) with exceptional divisor \(E\). Only the polarisation changes, and quite naturally \(L_\varepsilon = \pi^*L - \varepsilon \mathcal{O}(E)\). This would involve taking \(\varepsilon \in \mathbb{Q}^+\) and working with \(\mathbb{Q}\)-divisors, but in fact we rather take tensor powers and work with \(\hat{X}\) polarised by \(L_\gamma = \pi^*L^\gamma - \mathcal{O}(E)\) for integer \(\gamma \gg 0\). K-(semi, poly, in)stability is unaffected by Definition 2.7.

**Proposition 1.5** Let \((X, L)\) be a properly K-semistable polarised manifold. Then there exists a point \(q \in X\) such that the polarised blowup \((\text{Bl}_qX, \pi^*L^\gamma \otimes \mathcal{O}(-E))\) is K-unstable for \(\gamma \gg 0\).

**Remark 1.6** It is interesting to note that the corresponding result for vector bundles follows from Buchdahl [3]. Let \((X, L)\) be a polarised manifold and \(E \to X\) a properly slope semistable vector bundle. Then the pullback \(\pi^*E\) to the blowup \(\text{Bl}_{q_1,...,q_m}X\) in a finite number of suitably chosen points is slope unstable with respect to the polarisation \(\pi^*L^\gamma \otimes \mathcal{O}_{\text{Bl}_{q_i}X}(1)\) for \(\gamma \gg 0\).

Assume now that a properly semistable \((X, L)\) also admits a cscK metric \(\omega \in c_1(L)\). If \(\text{Aut}(X, L)\) is discrete the blowup perturbation problem for \(\omega\) is unobstructed by a theorem of Arezzo and Pacard [4], so we would get cscK metrics in \(c_1(\pi^*L^\gamma \otimes \mathcal{O}(E))\) for \(\gamma \gg 0\), a contradiction.

**Remark 1.7** This perturbation strategy for proving 1.3 is very general, and was first pointed out to the author by S. Donaldson and G. Székelyhidi. Different choices for \((X_\varepsilon, L_\varepsilon)\) lead to different perturbation problems for \(\omega\), which may settle Conjecture 1.3 in the presence of continous automorphisms. A possible variant is to perturb the cscK equation with \(\varepsilon\) at the same time, but one would then need to develop the relevant K-stability theory for a more general equation.

To sum up the main ingredients for our proof (besides Theorem 1.1) are:

1. A well known embedding result for test configurations (Proposition 2.9), together with the algebro-geometric estimate Proposition 3.3.
2. A blowup formula for the Donaldson-Futaki invariant proved by the author [14] Theorem 1.3;

3. A special case of the results of Arezzo and Pacard on blowing up cscK metrics [1].

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2 Some general theory

Let $n$ denote the complex dimension of $X$.

Definition 2.1 (Test configuration.) A test configuration for a polarised manifold $(X, L)$ is a polarised flat family $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ with $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, L)$ and which is $\mathbb{C}^*$-equivariant with respect to the natural action of $\mathbb{C}^*$ on $\mathbb{C}$.

Given a test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$ denote by $A_k$ the matrix representation of the induced $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$. By (equivariant) Riemann-Roch we can find expansions

\begin{align}
    h^0(\mathcal{X}_0, \mathcal{L}_0^k) &= a_0k^n + a_1k^{n-1} + O(k^{n-2}), \quad (2.1) \\
    \text{tr}(A_k) &= b_0k^{n+1} + b_1k^n + O(k^{n-1}). \quad (2.2)
\end{align}

Definition 2.2 (Donaldson-Futaki invariant.) This is the rational number

$$F(\mathcal{X}) = a_0^{-2}(b_0a_1 - a_0b_1)$$

which is independent of the choice of a lifting of the action to $\mathcal{L}_0$.

Equivalently $F(\mathcal{X})$ is the coefficient of $k^{-1}$ in the Laurent series expansion of the quotient

$$\frac{\text{tr}(A_k)}{kh^0(\mathcal{X}_0, \mathcal{L}_0^k)}.$$

Note moreover that $F$ is invariant under taking tensor powers, i.e.

$$F(\mathcal{X}, \mathcal{L}) = F(\mathcal{X}, \mathcal{L}^r).$$

Therefore for the rest of this note we will assume without loss of generality that $\mathcal{L}$ is relatively very ample.
Remark 2.3 (Coverings) Given a test configuration $(X, \mathcal{L})$ we can construct a new test configuration for $(X, L)$ by pulling $X$ and $L$ back under the $d$-fold ramified covering of $\mathbb{C}$ given by $z \mapsto z^d$. This changes $A_k$ to $d \cdot A_k$ and consequently $F$ to $d \cdot F$.

Definition 2.4 A test configuration $(X, \mathcal{L})$ is called a product if it is $\mathbb{C}^*$-equivariantly isomorphic to the product $(X \times \mathbb{C}, p_X^* L)$ endowed with the composition of a $\mathbb{C}^*$-action on $(X, L)$ with the natural action of $\mathbb{C}^*$ on $\mathbb{C}$.

A product test configuration is called trivial if the associated action on $(X, L)$ is trivial.

The Donaldson-Futaki invariant $F(X)$ in this case coincides with the classical Futaki invariant for holomorphic vector fields.

Definition 2.5 (K-stability) A polarised manifold $(X, L)$ is K-semistable if for all test configurations $(X, L)$

$$F(X) \geq 0.$$ 

It is K-stable if the strict inequality holds for nontrivial test configurations.

In particular if $(X, L)$ is K-stable Aut$(X, L)$ must be discrete. The correct notion to take care of continuous automorphisms is K-polystability.

Definition 2.6 A polarised manifold $(X, L)$ is K-polystable if it is K-semistable and moreover any test configuration $(X, L)$ with $F(X) = 0$ is a product.

Definition 2.7 A polarised manifold $(X, L)$ is properly K-semistable if it is K-semistable and it admits a nonproduct test configuration with vanishing Donaldson-Futaki invariant.

Remark 2.8 The terminology strictly K-semistable is also found in the literature with the same meaning.

Test configurations are well known to be equivalent to 1-parameter flat families induced by projective embeddings.

Proposition 2.9 (see e.g. Ross-Thomas [12] 3.7) A test configuration for $(X, L)$ is equivalent to a 1-parameter subgroup of $GL(H^0(X, L)^*)$.

In [13] the author proved a blowup formula for the Donaldson-Futaki invariant. The statement involves some more terminology.
Definition 2.10 (Hilbert-Mumford weight.) Let $\alpha$ be a 1-parameter subgroup of $\text{SL}(N+1)$, inducing a $\mathbb{C}^*$-action on $\mathbb{P}^N$. Choose projective coordinates $[x_0 : \ldots : x_N]$ such that $\alpha$ is given by $\text{Diag}(\lambda^{m_0}, \ldots, \lambda^{m_N})$. The Hilbert-Mumford weight of a closed point $q \in \mathbb{P}^N$ is defined by
\[
\mu(q, \alpha) = -\min\{m_i : q_i \neq 0\}.
\]
Note that this coincides with the weight of the induced action on the fibre of the hyperplane line bundle $\mathcal{O}(1)$ over the specialisation $\lim_{\lambda \to 0} \lambda \cdot q$.

Definition 2.11 (Chow weight.) Let $(Y, L)$ be a polarised scheme, $y \in Y$ a closed point, and $\alpha$ a $\mathbb{C}^*$-action on $(Y, L)$. Suppose that $L$ is very ample and $\alpha \hookrightarrow \text{SL}(H^0(Y, L)^*)$. The Chow weight $\mathcal{CH}_{(Y, L)}(q, \alpha)$ is defined to be the Hilbert-Mumford weight of $y \in \mathbb{P}(H^0(Y, L)^*)$ with respect to the induced action. The definition extends to 0-dimensional cycles on $Y$, that is effective linear combinations of closed points.

Theorem 2.12 (S. [14] 1.3) For points $q_i \in X$ and integers $a_i > 0$ let $Z \subset X$ be the 0-dimensional closed subscheme $Z = \cup_i a_i q_i$. Let $\Lambda$ be the 0-cycle on $X$ given by $\sum_i a_i^{n-1} q_i$.

A 1-parameter subgroup $\alpha \hookrightarrow \text{Aut}(X, L)$ induces a test configuration $(\hat{X}, \hat{L})$ for $(\text{Bl}_Z X, \pi^* L^\gamma \otimes \mathcal{O}_{\text{Bl}_Z X}(1))$, where $\mathcal{O}_{\text{Bl}_Z X}(1)$ denotes the exceptional invertible sheaf. More precisely let $O(Z)^-$ be the closure of the orbit of $Z$. Then $\hat{X} = \text{Bl}_{O(Z)^-} X$ and $\hat{L} = \pi^* L^\gamma \otimes \mathcal{O}_{\hat{X}}(1)$.

Suppose that $\alpha$ acts through $\text{SL}(H^0(X, L)^*)$ with Futaki invariant $F(X)$. Then the following expansion holds as $\gamma \to \infty$
\[
F(\hat{X}) = F(X) - \mathcal{CH}_{(X, L)}(\Lambda, \alpha) \frac{\gamma^{1-n}}{2(n-1)!} + O(\gamma^{-n}).
\]

We will need a slight generalisation of this result, covering blowups of non-product test configurations.

Proposition 2.13 Let $(\mathcal{X}, \mathcal{L})$ be a test configuration for $(X, L)$, $Z = \cup_i a_i q_i$ as above. There is a test configuration $(\hat{X}, \hat{L})$ for $(\text{Bl}_Z X, \pi^* L^\gamma \otimes \mathcal{O}_{\text{Bl}_Z X}(1))$ with total space $\hat{X}$ given by the blowup of $X$ along $O(Z)^-$. The linearisation is the natural one induced on $\hat{L} = \pi^* L^\gamma \otimes \mathcal{O}_{\hat{X}}(1)$.

Let $q_{i,0} = \lim_{\lambda \to 0} \lambda \cdot q_i$ be the specialisation, $\Lambda_0$ the 0-cycle on $X_0$ given by $\sum_i a_i^{n-1} q_{i,0}$.

Let $\alpha$ denote the induced action on $(\mathcal{X}_0, \mathcal{L}_0)$ and suppose that $\alpha$ acts through $\text{SL}(H^0(X_0, L_0)^*)$. Then the expansion
\[
F(\hat{X}) = F(X) - \mathcal{CH}_{(X_0, L_0)}(\Lambda_0, \alpha) \frac{\gamma^{1-n}}{2(n-1)!} + O(\gamma^{-n})
\]
holds as $\gamma \to \infty$.

We emphasise that the relevant Chow weight is computed on the central fibre $(X_0, L_0)$ with its induced $\mathbb{C}^*$-action.

**Proof.** The argument of [14] Section 4 goes over verbatim to non-product test configurations, with only two exceptions:

1. The proof of flatness of the composition $\hat{X} \to X \to \mathbb{C}$;

2. The identification of the weight $\mathcal{CH}(X_0, L_0)(\Lambda_0)$ (with respect to the induced action on $X_0$) with $\mathcal{CH}(X, L)(\Lambda, \alpha)$.

We do not need the latter identification, and indeed it does not make sense in this case since the general fibre is not preserved by the $\mathbb{C}^*$-action.

To prove flatness we use the criterion [9] III Proposition 9.7. Thus we need to prove that all associated points of $\hat{X}$ (i.e. irreducible components and their thickenings) map to the generic point of $\text{Spec}(\mathbb{C})$.

By flatness this is true for the morphism $X \to \mathbb{C}$, and blowing up $O(\Lambda)^-$ does not contribute new associated points, only the Cartier exceptional divisor $\pi^{-1}O(\Lambda)^-$. More precisely let $I$ denote the ideal sheaf of $O(q)^- \subset X$, and recall $\hat{X}$ is defined as $\text{Proj} \bigoplus_{d \geq 0} I^d$. Any homogeneous zero divisor in the graded sheaf $\bigoplus_{d \geq 0} I^d$ is already a zero divisor when regarded as an element of $O_X$. On the other hand an associated point $\hat{x} \in \hat{X}$ is by definition (following [9] III Corollary 9.6) a point for which every element of $m_{\hat{x}}$ is a zero divisor. The natural map $\hat{X} \to X$ maps $m_{\hat{x}}$ to its degree 0 piece. Thus by the above remark $\hat{x}$ necessarily maps to an associated point $x \in X$. But $x$ maps to the generic point of $\text{Spec}(\mathbb{C})$ by flatness, so the same is true for $\hat{x}$.

Q.E.D.

**Remark 2.14** In both cases the assumption that $\alpha$ acts through SL is not really restrictive. This can always be achieved by replacing $L$ by some power and pulling back $X$ by $z \mapsto z^d$ for some $d$. This gives a new test configuration for which $\alpha$ can be rescaled to act through SL and for which the Futaki invariant is only multiplied by $d$, by Remark [2.3].

This property of the Futaki invariant turns out to be important in our proof of Theorem 1.2.

### 3 Proof of Theorem 1.2

It will be enough to prove Proposition 1.5 and to apply the result of Arezzo and Pacard recalled as Theorem 3.1 below.
Thus let 
\[(\hat{X}, L_\gamma) = (\text{Bl}_q X, \pi^* L_\gamma \otimes \mathcal{O}(-E)).\]

We need to show that \((\hat{X}, L_\gamma)\) is K-unstable for \(\gamma \gg 0\). We will construct test configurations \((X_\gamma, L_\gamma)\) for \((\hat{X}, L_\gamma)\) which have strictly negative Donaldson-Futaki invariant for \(\gamma \gg 0\).

By assumption \((X, L)\) is properly semistable, so it admits a nontrivial test configuration \((X, L)\) with \(F(X) = 0\). Moreover we can assume that the induced \(\mathbb{C}^*\)-action on \(H^0(X_0, L_0)^*\) is special linear. Indeed this can be achieved by taking some power \(L^{r}\) and a ramified cover \(z \mapsto z^d\). The new Futaki invariant \(F'\) still vanishes since \(F' = d \cdot F = 0\).

We blow \(X\) up along the closure \(O(q)^-\) of the orbit \(O(q)\) of \(q \in X_1\) under the \(\mathbb{C}^*\)-action on \(X\), i.e. define
\[X_\gamma = \hat{X} = \text{Bl}_{O(q)^-} X.\] (3.1)

Let \(\mathcal{O}_{\hat{X}}(1)\) denote the exceptional invertible sheaf on \(\hat{X}\). We endow \(\hat{X}\) with the polarisation
\[L_\gamma = \pi^* L_\gamma \otimes \mathcal{O}_{\hat{X}}(1).\] (3.2)

Define the closed point \(q_0 \in X_0\) to be the specialisation
\[q_0 = \lim_{\lambda \to 0} \lambda \cdot q.\]

Applying the blowup formula 2.13 in this case gives
\[F(\hat{X}_0, \pi^* L_0 \otimes \mathcal{O}_{\hat{X}_0}(1)) = F(X_0, L_0) - \mathcal{CH}(X_0, L_0)(q_0) \frac{\gamma^{1-n}}{2(n-2)!} + O(\gamma^{-n})\]
\[= -\mathcal{CH}(X_0, L_0)(q_0) \frac{\gamma^{1-n}}{2(n-2)!} + O(\gamma^{-n}).\]

In Proposition 3.3 below we will prove that for a very special \(q \in X_1 \cong X\),
\[\mathcal{CH}(X_0, L_0)(q_0) > 0.\]

This holds thanks to the assumption \(F(X) = 0\), or more generally \(F(X) \leq 0\). This is enough to settle Proposition 1.5.

The final step for Theorem 1.2 is to show that the perturbation problem is unobstructed provided \(\text{Aut}(\hat{X}, L)\) is discrete. This is precisely the content of a beautiful result of C. Arezzo and F. Pacard.
Theorem 3.1 (Arezzo-Pacard \[1\]) Let $(X, L)$ be a polarised manifold with a cscK metric in the class $c_1(L)$. Suppose $\text{Aut}(X, L)$ is discrete and let $q \in X$ be any point. Then the blowup $\text{Bl}_q X$ with exceptional divisor $E$ admits a cscK metric in the class $\gamma \pi^* c_1(L) - c_1(\mathcal{O}(E))$ for $\gamma \gg 0$.

Remark 3.2 The Arezzo-Pacard theorem also holds in the Kähler case and, more importantly, even when $\text{Aut}(X, L) \neq 0$, provided a suitable stability condition is satisfied. We refer to \[2\], \[14\] for further discussion.

Thus the following Proposition will complete our proof(s). We believe it may also be of some independent interest.

Proposition 3.3 Let $(X, L)$ be a nonproduct test configuration for a polarised manifold $(X, L)$ with nonpositive Donaldson-Futaki invariant and suppose the induced $\mathbb{C}^*$-action on $H^0(X_0, L_0)^*$ is special linear. Then there exists $q \in X_1 \cong X$ such that $\text{CH}(X_0, L_0)(q_0) > 0$.

Proof. By the embedding Theorem 2.9 we reduce to the case of a nontrivial $\mathbb{C}^*$ acting on $\mathbb{P}^N$ for some $N$, of the form $\text{Diag}(\lambda^{m_0} x_0, ... \lambda^{m_N} x_N)$, ordered by $m_0 \leq m_1 \leq ... \leq m_N$.

Let $\{Z_i\}_{i=1}^k$ be the distinct projective weight spaces, where $Z_i$ has weight $m_i$ (i.e. the induced action on $Z_i$ is trivial with weight $m_i$). Each $Z_i$ is a projective subspace of $\mathbb{P}^N$, and the central fibre with its reduced induced structure $X_0^\text{red}$ is a contained in $\text{Span}(Z_{i_1}, ... Z_{i_l})$ for some minimal flag $0 = i_1 < i_2 < ... < i_l$.

The case $1 = l$. In this case the induced action on closed points of $X_0$ is nontrivial. Let $q \in X_1$ be any point with

$$\lim_{\lambda \to 0} \lambda \cdot q = q_0 \in Z_{i_l}.$$  

Such a point exists by minimality and because the specialisation of every point must lie in some $Z_j$. Since the action on $X_0$ is induced from that on $\mathbb{P}^N$, $q_0$ belongs to the totally repulsive fixed locus $R = X_0 \cap Z_{i_l} \subset X_0$. By this we mean that every closed point in $X_0 \setminus R$ specialises to a closed point in $X_0 \setminus R$. In particular the natural birational morphism $X_0 \to \text{Proj}(\bigoplus_d H^0(X_0, L_0^\otimes d)^{\mathbb{C}^*})$ blows up along $R$. So $q_0 \in R$ is an unstable point for the $\mathbb{C}^*$-action in the sense of geometric invariant theory. By the Hilbert-Mumford criterion the weight of the induced action on the line $L_0|_{q_0}$ must be strictly positive. Since we are assuming that the induced action on $H^0(X_0, L_0)^*$ is special linear this
weight coincides with the Chow weight, so $\mathcal{C}(\mathcal{X}_0, \mathcal{L}_0)(q_0) > 0$.

Degenerate case. In the rest of the proof we will show that in the degenerate case $X_0^{\text{red}} \subseteq Z_0$ the Donaldson-Futaki invariant is strictly positive. Note that since by assumption the original $\mathbb{C}^*$-action on $\mathbb{P}^N$ is nontrivial, $Z_0 \subseteq \mathbb{P}^N$ is a proper projective subspace.

We digress for a moment to make the following observation: for any $\mathbb{C}^*$-action on $\mathbb{P}^N$ with ordered weights $\{m_i\}$, and a smooth nondegenerate manifold $Y \subseteq \mathbb{P}^N$, the map $\rho : Y \ni y \mapsto y_0 = \lim_{\lambda \to 0} \lambda \cdot y$ is rational, defined on the open dense set $\{y \in Y : \mu(y) = m_0\}$ of points with minimal Hilbert-Mumford weight. Indeed, in the above notation, generic points specialise to some point in the lowest fixed locus $Z_0$. In any case the map $\rho$ blows up exactly along loci where the Hilbert-Mumford weight jumps.

Going back to our discussion of the case $X_0^{\text{red}} \subseteq Z_0$, we see that this means precisely that all points of $X_1$ have minimal Hilbert-Mumford weight $m_0$, so there is a well defined morphism

$$\rho : X_1 \to Z_0.$$  

Moreover $\rho$ is a finite map: the pullback of $L_0$ under $\rho$ is $L$ which is ample, therefore $\rho$ cannot contract a positive dimensional subscheme. If $\rho$ were an isomorphism on its image, it would fit in a $\mathbb{C}^*$-equivariant isomorphism $X \cong X_1 \times \mathbb{C}$. Therefore $\rho$ cannot be injective, either on closed points or tangent vectors. If, say, $\rho$ identifies distinct points $x_1, x_2$, this means that the $x_i$ specialise to the same $x$ under the $\mathbb{C}^*$-action; by flatness then the local ring $O_{X_0,x}$ contains a nontrivial nilpotent pointing outwards of $Z_0$, i.e. the sheaf $\mathcal{I}_{X_0 \cap Z_0} / \mathcal{I}_{X_0}$ is nonzero. In other words $X_0$ is not a closed subscheme of $Z_0$. The case when $\rho$ annihilates a tangent vector produces the same kind of nilpotent in the local ring of the limit, by specialisation.

To sum up, the central fibre $X_0$ is nonreduced, containing nontrivial $Z_0$-orthogonal nilpotents. Equally important, the induced action on the closed subscheme $X_0 \cap Z_0 \subset X_0$ is trivial. The proof will be completed by a weight computation.

Donaldson-Futaki invariant. Suppose $Z_0 \subset \mathbb{P}^N$ has projective coordinates $[x_1 : ... : x_r]$, i.e. it is cut out by $\{x_{r+1} = ... = x_N = 0\}$. We change the linearisation by changing the representation of the $\mathbb{C}^*$-action, to make it of
the form

\[ [x_0 : ...x_r : x_{r+1} : ... : x_N] \mapsto [x_0 : ...x_r : \lambda^{m_{r+1} - m_0}x_{r+1} : ... : \lambda^{m_N - m_0}x_N], \]

and recall \( m_{r+i} > m_0 \) for all \( i > 0 \). It is possible that the induced action on \( H^0(\mathcal X_0, \mathcal L_0)^* \) will not be special linear anymore, however this does not affect the Donaldson-Futaki invariant.

Note that for all large \( k \),

\[ H^0(\mathbb P^N, \mathcal O(k)) \to H^0(\mathcal X_0, \mathcal L_0^k) \to H^1(\mathcal I_{\mathcal X_0}(k)) \]

(3.4)

By (3.4) our geometric description of \( \mathcal X_0 \) and the choice of linearisation 3.3 we see that any section \( \xi \in H^0(\mathcal X_0, \mathcal L_0^k) \) has nonnegative weight under the induced \( \mathbb C^* \)-action. The section \( \xi \) can only have strictly positive weight if it is of the form \( x^{r+i} \cdot f \) for some \( i > 0 \). Moreover we know there exists an integer \( a > 0 \) such that \( x^{a+i}|_{\mathcal X_0} = 0 \) for all \( i > 0 \). Let \( w(k) \) denote the total weight of the action on \( H^0(\mathcal X_0, \mathcal L_0^k) \), i.e. the induced weight on the line \( \Lambda^{P(k)}H^0(\mathcal X_0, \mathcal L_0^k) \), where \( P(k) = h^0(\mathcal X_0, \mathcal L_0^k) \) is the Hilbert polynomial. Our discussion implies the upper bound

\[ w(k) \leq C(P(k - 1) + ... + P(k - a)) \]

(3.5)

for some \( C > 0 \), independent of \( k \). In particular,

\[ w(k) = O(k^n). \]

(3.6)

On the other hand we can look at just one section \( x^{r+i}, i > 0 \) with \( x^{r+i}|_{\mathcal X_0} \neq 0 \). This gives a lower bound

\[ w(k) \geq C \cdot P(k - 1) \]

(3.7)

for some \( C > 0 \), independent of \( k \). So we see that

\[ \frac{w(k)}{kP(k)} \geq \frac{C'}{k}. \]

(3.8)

holds for \( k \gg 0 \) and some \( C' > 0 \) independent of \( k \). Together with

\[ \frac{w(k)}{kP(k)} = O(k^{-1}) \]

(3.9)

which follows from (3.6) this implies

\[ \frac{w(k)}{kP(k)} = \frac{C''}{k} + O(k^{-2}) \]

(3.10)
for some $C'' > 0$ independent of $k$.

By definition of Donaldson-Futaki invariant, this immediately implies

$$F(\mathcal{X}) \geq C'' > 0,$$

a contradiction.

Q.E.D.

**Remark 3.4** One can characterise the degenerate case in the above proof more precisely.

As observed by Ross-Thomas [12] Section 3 a result of Mumford implies that any test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$ is a contraction of some blowup of $X \times \mathbb{C}$ in a flag of $\mathbb{C}^*$-invariant closed subschemes supported in some thickening of $X \times \{0\}$.

The existence of the map $\rho : \mathcal{X}_1 \to Z_0$ means precisely that in this Mumford representation of $\mathcal{X}$ no blowup occurs, i.e. $\mathcal{X}$ is a contraction of the product $X \times \mathbb{C}$.

Define a map $\nu : X \times \mathbb{C} \to X$ by $\nu(x, \lambda) = \lambda \cdot x$ away from $X \times \{0\}$, $\nu = \rho$ on $X \times \{0\}$. This is a well defined morphism, and since $\rho$ is finite, $\nu$ is precisely the *normalisation* of $\mathcal{X}$.

So in the degenerate case $X^\text{red}_0 \subset Z_0$ the normalisation of $\mathcal{X}$ is $X \times \mathbb{C}$.

Ross-Thomas [12] Proposition 5.1 proved the general result that normalising a test configuration reduces the Donaldson-Futaki invariant. This already implies $F \geq 0$ in the degenerate case, since the induced action on $X \times \mathbb{C}$ must have vanishing Futaki invariant. In our special case our direct proof yields the strict inequality we need.

**Remark 3.5** The result of Mumford mentioned above states more precisely that any test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$ is a contraction of the blowup of $X \times \mathbb{C}$ in an ideal sheaf

$$I_r = I_0 + tI_1 + \ldots + t^{r-1}I_{r-1} + (t^r)$$

where $I_0 \subseteq \ldots \subseteq I_{r-1} \subseteq \mathcal{O}_X$ correspond to a descending flag of closed subschemes $Z_0 \supseteq \ldots \supseteq Z_{r-1}$. The action on $(\mathcal{X}, \mathcal{L})$ is the natural one induced from the trivial action on $X \times \mathbb{C}$.

Suppose now that $F(\mathcal{X}) = 0$ and that no contraction occurs in Mumford’s representation.

Then in Proposition 3.3 we can simply choose any closed point $q \in Z_{r-1}$. This is because the proper transform of $Z_{r-1} \times \mathbb{C}$ cuts $\mathcal{X}_0$ in the totally repulsive locus for the induced action, i.e. the action flows every closed point in $\mathcal{X}_0$ outside this locus to the proper transform of $X \times \{0\}$. 

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Conversely blowing up $q \in X \setminus Z_0$ only increases the Donaldson-Futaki invariant (at least asymptotically).

For example K-stability with respect to test configurations with $r = 1$ and no contraction is known as Ross-Thomas slope stability [12] and has found interesting applications to cscK metrics. In particular this discussion gives a simpler proof that a cscK polarised manifold with discrete automorphisms is slope stable.

**Remark 3.6** A refinement of Conjecture [13] was proposed by G. Székelyhidi. If $\omega \in c_1(L)$ is cscK there should be a strictly positive lower bound for a suitable normalisation of $F$ over all nonproduct test configurations. This condition is called uniform K-polystability. In [15] Section 3.1.1 it is shown that the correct normalisation in the case of algebraic surfaces coincides with that of Theorem [14] namely $\frac{F(X)}{\|X\|}$. For toric surfaces K-polystability implies uniform K-polystability with respect to toric test configurations; this is shown in [15] Section 4.2. It seems clear however that the proof presented here cannot be refined to yield uniform K-stability for surfaces.

**Example 3.7 (Del Pezzo surfaces)** Del Pezzo surfaces played an important role in the development of the subject. By the work of Tian and others all smooth Del Pezzo surfaces $V_d$ of degree $d \leq 6$ admit a Kähler-Einstein metric. For $d \leq 5$, $V_d$ has discrete automorphism group. K-stability in the sense of Tian for $V_d$, $d \leq 5$ follows from [13] Theorem 1.2. K-stability with respect to “good” test configurations follows from [11] Theorem 2.

Our Theorem 1.2 refines this to K-stability in the sense of Donaldson.

Moreover Theorem 1.2 also applies to polarisations on $V_d$, $d \leq 5$ for which the exceptional divisors have sufficiently small volume, thanks to the results of Arezzo and Pacard [2].

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