GENERAL MOTIVIC COHOMOLOGY AND SYMPLECTIC ORIENTATION

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Abstract. In this paper, we present a general approach to establish motivic cohomology and build part of its six operations formalism. Applying this together with symplectic orientation on MW-motivic cohomology, we discuss the embedding theorem of effective Chow-Witt motives.

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1. Introduction

Suppose $E^p_C(X)$ is a kind of cohomology theory for smooth schemes, for example, $E$ is Chow rings (CH) or $E$ is Chow-Witt rings ($\tilde{CH}$, See [Fas08]). We could construct a series of bigraded cohomologies $H^{p,q}_E(X,Z)$ such that

$$H^{2p,p}_E(X,Z) = E^p(X).$$

They are called motivic cohomologies. The case $E = CH$ has been done by V. Voevodsky and the case $E = \tilde{CH}$ is developing by B. Calmès, F. Déglise and J. Fasel (See [CF14], [DF17]). The main feature in the case $E = \tilde{CH}$ is that the Thom isomorphism like

$$E^p_C(X) \longrightarrow E^{n+rk(V)}_C(V)$$

doesn’t exist any more for $V$ vector bundle over $X$ and $C$ closed in $X$. This gives an example of a non-oriented cohomology and the only solution is to make the cohomology theory dependent on the choice of vector bundles, namely, to expand the notation to $E^p_C(X,V)$ so that

$$E^p_C(X,V) \cong E^{n+rk(V)}_C(V)$$

canonically. Here $V$ is called the twist of the correspondence $E^p_C(X,V)$ by historical reasons. So this gives rise to our general definition of motivic cohomology, which still works under the non-oriented cases.

The first step is to generalize the concept of correspondence in the case of non-oriented cohomology. This requires us to understand the twist $V$. The serious approach to that is to regard it as an element in the category of virtual vector bundles. We will describe what is called a correspondence theory (See Section 3) and provide a systematic method characterizing...
behaviors of twists and doing calculation (See Section 2) on them. And as an example, we will prove that the theory of MW-correspondences is indeed a correspondence theory (See Section 4 this part is still in progress).

Then, given a correspondence theory $E$, we will establish the theory of sheaves with $E$-transfers over any smooth base and some operations, such as $f^*$, $f_!$ and $\otimes$, on those sheaves (See Section 5). And the category of effective and stabilized motives will be realized by localizing bounded above derived categories of sheaves with $E$-transfers, with the same operations above inherited, as derived functors (See Section 6). Finally we will compare our approach with the method using unbounded complexes and its model structures, for example, [CD07], [CD13] and [DF17].

Last but not the least, in the case $E = CH$, we will calculate the Thom space of symplectic bundles over any smooth base by using results in [Yan17], which will enable us to calculate the inverse Thom space of any vector bundle. Then by using the operations we defined in Section 6 together with the duality in the stable $A_1$-derived categories (See [CD13], [DF17]), we could figure out the $Hom$-group between motives of proper schemes, as shown below (See [DF17] and [MVW06] for notations):

**Theorem 1.1.** (See Theorem 4.3) Let $X,Y \in Sm/k$ with $Y$ proper, then we have

$$Hom_{DM^{eff,-}}(\widetilde{M}(X), \widetilde{M}(Y)) \cong \widetilde{CH}^{dy}(X \times Y, \omega_{X \times Y/X}).$$

This shows that the opposite category of effective Chow-Witt motives (See Definition 7.3) is a full subcategory of $DM^{eff,-}$, which is called the embedding theorem of $CH^{eff}$. A parallel result in motivic cohomology defined by V. Voevodsky says that (See [MVW06], Proposition 20.1)

$$Hom_{DM^{eff,-}}(M(X), M(Y)) \cong CH^{dy}(X \times Y),$$

for proper $Y$, hence the opposite category of Grothendieck's effective Chow motives ($CH^{eff}$) is a full subcategory of $DM^{eff,-}$.

Throughout in this article, we denote by $Sm/k$ the category of separated schemes being smooth over $k$ with some relative dimension (See [Har77], Chapter 10), where $k$ is an infinite perfect field with $char(k) \neq 2$. For any $X \in Sm/k$, we denote $dimX$ by $d_X$ and for any $f : X \rightarrow Y$ in $Sm/k$, we set $df = d_X - dy$.

## 2. Virtual Objects and Their Calculation

In this section we will introduce the category of virtual vector bundles and explain basic techniques of calculation. The definitions all come from [Del87, Section 4] and we may state them here as well for clarity.

**Definition 2.1.** (See [Del87], 4.1) A category $\mathcal{C}$ is called a commutative Picard category if

1. All morphisms are isomorphisms.
2. There is a bifunctorial pairing

$$+ : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

satisfying

(a) For every $x,y,z \in \mathcal{C}$, an associativity isomorphism

$$a(x,y,z) : (x + y) + z \rightarrow x + (y + z);$$

(b) For every $x,y \in \mathcal{C}$, a commutativity isomorphism

$$c(x,y) : x + y \rightarrow y + x.$$

And they satisfy associativity and commutativity constraints (See [Mac63]).

3. For every $P \in \mathcal{C}$, the functors $X \mapsto P + X$ and $f \mapsto X + P$ are equivalences of categories. Thus there is a unity element $0$ such that $0 + X \cong X$ for every $X \in \mathcal{C}$, there is a $-X \in \mathcal{C}$ such that $X + (-X) \cong 0$.

**Definition 2.2.** Let $X$ be a scheme, define $Vect(X)$ to be the category of vector bundles over $X$. Denote by $(Vect(X), iso)$ the subcategory of $Vect(X)$ with same objects but picking only isomorphisms as morphisms.
Definition 2.3. (See [Del87, 4.3]) Let $C$ be a commutative Picard category and $X$ be a scheme, we say it has a bracket functor on $X$ if there is a functor

$$[-] : (\text{Vect}(X), \text{iso}) \to C$$

such that

1. For any exact sequence of vector bundles
   $$0 \to E_1 \to E_2 \to E_3 \to 0,$$
   there is an isomorphism $\Sigma : [E_2] \to [E_1] + [E_3]$ being natural with respect to isomorphisms between exact sequences.

2. There is an isomorphism $z : [0] \to 0$ such that for every $E \in \text{Vect}(X)$, the composition
   $$[E] \xrightarrow{\Sigma} [0] + [E] \xrightarrow{z} 0 + [E] \xrightarrow{\Sigma} [E]$$
   is $\text{id}_{[E]}$.

3. (See Remark 2.1) For every consecutive subbundle inclusions $E_1 \subseteq E_2 \subseteq E_3$, there is a commutative diagram
   $$\begin{array}{ccc}
   [E_3] & \xrightarrow{\Sigma} & [E_1] + [E_3/E_1] \\
   \downarrow{\Sigma} & & \downarrow{\Sigma} \\
   [E_2] + [E_3/E_2] & \xrightarrow{\Sigma} & [E_1] + [E_2/E_1] + [E_3/E_2] \\
   \end{array}$$

4. For every $E_1, E_2$, there is a commutative diagram
   $$\begin{array}{ccc}
   [E_1 \oplus E_2] & \xrightarrow{\Sigma} & [E_1] + [E_2] \\
   \downarrow{\Sigma} & & \downarrow{c(E_1, E_2)} \\
   [E_2] + [E_1] & & \\
   \end{array}$$

The following comes from [Del87, 4.3]:

**Proposition 2.1.** Let $X$ be a scheme. There is a commutative Picard category $V(\text{Vect}(X))$ with a bracket functor on $X$, which is called the category of virtual vector bundles over $X$, such that for every commutative Picard category $C$ with a bracket functor on $X$, there is a unique additive functor $F : V(\text{Vect}(X)) \to C$ making the following diagram commute

$$\begin{array}{ccc}
V(\text{Vect}(X)), \text{iso} & \xrightarrow{[-]} & V(\text{Vect}(X)) \\
\downarrow{F} & & \downarrow{\cdot} \\
C & & \\
\end{array}$$

For convenience, we will denote $[E]$ still by $E$ in the sequel.

The following proposition strengthens Definition 2.3 (4) a little bit.

**Proposition 2.2.** Suppose we have a commutative diagram among vector bundles over $X$ with exact row and column

$$\begin{array}{ccc}
0 & \xrightarrow{0} & A \\
D & \xrightarrow{d} & B \\
\downarrow{a} & & \downarrow{b} \\
0 & \xrightarrow{0} & C \\
E & \xrightarrow{c} & 0 \\
\end{array}$$
Then the following diagram commutes in \( V(Vect(X)) \)

\[
\begin{array}{ccc}
B & \rightarrow & A + C \\
\downarrow & & \downarrow \\
D + E & \Rightarrow & \circ(E,D) \circ (u + v^{-1})
\end{array}
\]

Proof. Since \( v^{-1} \circ b \) splits \( d \), it’s a standard argument that there exists a unique \( \xi : E \rightarrow B \) such that

\[
\xi \circ c = 1 - \circ c \circ v^{-1} \circ b, c \circ \xi = \text{id}_E.
\]

So we have commutative diagrams with exact rows:

\[
\begin{array}{ccc}
0 & \rightarrow & D \\
\downarrow & & \downarrow \\
0 & \rightarrow & D + E \\
\downarrow & & \downarrow \\
0 & \rightarrow & E \\
\downarrow & & \downarrow \\
0 & \rightarrow & A \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

\[
\begin{array}{ccc}
D \rightarrow B & \rightarrow & E \\
\downarrow & \downarrow & \downarrow \\
D \rightarrow B & \rightarrow & E \\
\downarrow & \downarrow & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

Hence the statement follows from the commutative diagram by Definition 2.3, (4):

\[
\begin{array}{ccc}
\text{V} & \rightarrow & A + C \\
\downarrow & & \downarrow \\
\text{V} & \rightarrow & A + C \\
\downarrow & & \downarrow \\
\text{V} & \rightarrow & A + C
\end{array}
\]

The next theorem is a fundamental tool for calculations in virtual vector bundles.

Theorem 2.1. \( (1) \) Suppose we have a commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
K & \rightarrow & K \\
\downarrow & & \downarrow \\
0 & \rightarrow & V_1 \\
\downarrow & & \downarrow \\
0 & \rightarrow & V_1 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
K & \rightarrow & K \\
\downarrow & & \downarrow \\
0 & \rightarrow & V_2 \\
\downarrow & & \downarrow \\
0 & \rightarrow & V_2 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
K & \rightarrow & K \\
\downarrow & & \downarrow \\
0 & \rightarrow & V_1 + C \\
\downarrow & & \downarrow \\
V_1 + C & \rightarrow & K + W_1 + C.
\end{array}
\]
(2) Suppose we have a commutative diagram with exact rows and columns

\[
\begin{array}{cccccc}
0 & \rightarrow & V_1 & \rightarrow & V_2 & \rightarrow C & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & W_1 & \rightarrow & W_2 & \rightarrow C & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D & \rightarrow & D & \rightarrow & D & \rightarrow & 0 \\
\end{array}
\]

among vector bundles over $X$. Then we have a commutative diagram in $V(Vect(X))$:

\[
\begin{array}{cccc}
W_2 & \rightarrow & V_2 + D & \rightarrow V_1 + C + D \\
\downarrow & & \downarrow & & \downarrow \\
W_1 + C & \rightarrow & V_1 + D + C \\
\end{array}
\]

\[
\begin{array}{cccc}
W_1 + C & \rightarrow & V_1 + D + C \\
\end{array}
\]

(3) Suppose we have a commutative diagram with exact rows and columns

\[
\begin{array}{cccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & T & \rightarrow & V_1 & \rightarrow & V_2 & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & T & \rightarrow & W_1 & \rightarrow & W_2 & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

among vector bundles over $X$. Then we have a commutative diagram in $V(Vect(X))$:

\[
\begin{array}{cccc}
V_1 & \rightarrow & T + V_2 & \rightarrow T + K + W_2 \\
\downarrow & & \downarrow & & \downarrow \\
K + W_1 & \rightarrow & K + T + W_2 \\
\end{array}
\]

\[
\begin{array}{cccc}
K + W_1 & \rightarrow & K + T + W_2 \\
\end{array}
\]
Suppose we have a commutative diagram with exact rows and columns

![Diagram]

among vector bundles over \( X \). Then we have a commutative diagram in \( V(\text{Vect}(X)) \)

\[
\begin{array}{c}
W_1 \rightarrow K + W_2 \\
V_1 + C \rightarrow K + V_1 + C
\end{array}
\]

Proof. (1) We have injections \( K \rightarrow V_1 \rightarrow V_2 \), which give the diagram by Definition 2.3.

(2) We have injections \( V_1 \rightarrow V_2 \rightarrow W_2 \) and \( V_1 \rightarrow W_1 \rightarrow W_2 \). These give two commutative diagrams by Definition 2.3.

\[
\begin{array}{c}
W_2 \rightarrow V_1 + W_2/V_1 \\
V_2 + D \rightarrow V_1 + C + D
\end{array}
\]

And we have a commutative diagram with exact row and column

\[
\begin{array}{c}
\Sigma_2 : 0 \\
\downarrow \\
D \\
\Sigma_1 : 0 \rightarrow C \rightarrow W_2/V_1 \rightarrow D \rightarrow 0 \\
\downarrow \\
C \\
\downarrow \\
0
\end{array}
\]

Thus we have a commutative diagram

\[
\begin{array}{c}
W_2/V_1 \xrightarrow{\Sigma_1} C + D \\
\downarrow \Sigma_2 \\
D + C
\end{array}
\]

by Proposition 2.2. So combining the diagrams above gives the result.
(3) We denote the morphism $V_1 \to V_2 \to W_2$ by $\alpha$. There are morphisms $\ker(\alpha) \to K$ and $\ker(\alpha) \to T$ satisfying the following commutative diagrams:

\[
\begin{array}{c}
\ker(\alpha) \\ V_1 \downarrow \\
\end{array}
\begin{array}{c}
\to \\
\ker(\alpha) \\ V_2 \downarrow \\
\to \\
K \\ T \downarrow \\
\to \\
W_2 \\ W_1
\end{array}
\]

by the universal property of $K$ and $T$ as kernels. Then there is a commutative diagram with exact row and column

\[
\begin{array}{c}
\Sigma_2 : \\
0 \\
\downarrow \\
K \\
\downarrow \\
\Sigma_1 : 0 \\
T \\
\downarrow \\
ker(\alpha) \\
\downarrow \\
K \\
\downarrow \\
0.
\end{array}
\]

Hence we have a commutative diagram

\[
\begin{array}{c}
\ker(\alpha) \\
\downarrow \Sigma_1 \\
T + K \\
\downarrow \Sigma_2 \\
K + T
\end{array}
\]

by Proposition 2.2.

We have injections $T \to \ker(\alpha) \to V_1$, $K \to \ker(\alpha) \to V_1$, which induce the following commutative diagrams by Definition 2.3 (3):

\[
\begin{array}{c}
V_1 \\
\downarrow \\
\ker(\alpha) + W_2 \to T + V_2 \\
\downarrow \\
ker(\alpha) + W_2 \\
\downarrow \\
K + W_1 \\
\downarrow \\
ker(\alpha) + W_2 \to K + T + W_2.
\end{array}
\]

So combining the diagrams above gives the result.

(4) The diagram is a rotation and reflection of the diagram in (1).

\[\square\]

**Remark 2.1.** We remark that (1) in the above theorem is actually the precise meaning of Definition 2.3 (3).

**Remark 2.2.** We would like to point out that the calculation of virtual objects is not trivial, especially when judging commutativity of diagrams. We will see this point in the sections below.

3. **Correspondence Theory**

In this section, we are going to define what is a correspondence theory by axiomatic method, under the language of virtual vector bundles defined in the previous section.

**Definition 3.1.** Let $X$ be a noetherian scheme and $i \in \mathbb{N}$. We denote $Z^i(X)$ to be the set of closed subsets in $X$ whose components have codimension $i$. 

Axiom 1. (Twists) For every $X \in \text{Sm}/k$, we have a commutative Picard category (See Definition 2.1) $\mathcal{P}_X$ with an additive functor $p_X : V(\text{Vect}(X)) \to \mathcal{P}_X$ and a rank morphism $\text{rk}_X : \mathcal{P}_X \to \mathbb{Z}/2\mathbb{Z}$ such that:

1. The following diagram commutes

$$
\begin{array}{ccc}
V(\text{Vect}(X)) & \xrightarrow{\text{rk}} & \mathbb{Z} \\
\downarrow{p_X} & & \downarrow{\text{rk}_X} \\
\mathcal{P}_X & \xrightarrow{\text{rk}_X} & \mathbb{Z}/2\mathbb{Z},
\end{array}
$$

where the upper horizontal arrow is defined by $\text{rk}([E]) = \text{rk}(E)$.

2. For every $f : X \to Y$ in $\text{Sm}/k$, there is a pull-back morphism $f^* : \mathcal{P}_Y \to \mathcal{P}_X$ such that the following diagrams commute

$$
\begin{array}{c}
\mathcal{P}_Y \xrightarrow{f^*} \mathcal{P}_X \\
\mathcal{P}_X \downarrow{\text{rk}_X} \quad \downarrow{\text{rk}_Y} \\
\mathcal{P}_Y \xrightarrow{f^*} \mathcal{P}_X
\end{array}
$$

where $f^* : V(\text{Vect}(Y)) \to V(\text{Vect}(X))$ is defined by $f^*([E]) = [f^*E]$. And $f^*g^* = (g \circ f)^*$ for any morphisms $f, g$ in $\text{Sm}/k$. Moreover, $f^*(-v) = -f^*(v)$.

Remark 3.1. In application, the categories $\mathcal{P}_X$ should be chosen as ‘small’ as possible. Since this will allow more isomorphisms, such as orientations as we will see in Definition 7.1.

Axiom 2. (Correspondences) For every $X \in \text{Sm}/k$, $i \in \mathbb{N}$, $C \in \mathcal{Z}^i(X)$ and $v \in \mathcal{P}_X$, we associate an abelian group $E^i_C(X,v)$, which is called the correspondence supported on $C$ with twist $v$. They are functorial with respect to $v$. Moreover, if $C = \emptyset$, $E^i_C(X,v)$ is understood as $0$.

Remark 3.2. Note that we always assume that $C \in \mathcal{Z}^i(X)$. This has the advantage to keep the operations very concrete (in the case of Chow or Chow-Witt groups), but has the disadvantage that the group $E^i(X,v)$ with full support is in general not defined.

We are going to describe further properties these groups should satisfy.

Axiom 3. (Extension of Supports) For every $X \in \text{Sm}/k$, $C_1 \subseteq C_2 \subseteq \mathcal{Z}^i(X)$, $i \in \mathbb{N}$, $v \in \mathcal{P}_X$, we have a morphism

$$e(C_1,C_2) : E^i_{C_1}(X,v) \to E^i_{C_2}(X,v)$$

which is called the extension of support. This map is functorial with respect to $v$.

For any disjoint $C_1,C_2 \in \mathcal{Z}^i(X)$, we have

$$E^i_{C_1 \cup C_2}(X,v) \cong E^i_{C_1}(X,v) \oplus E^i_{C_2}(X,v)$$

via extension of supports. Moreover, for any $C_1 \subseteq C_2 \subseteq C_3$ we have

$$e(C_2,C_3) \circ e(C_1,C_2) = e(C_1,C_3).$$

Axiom 4. (Product) Suppose $X \in \text{Sm}/k$, $v_1,v_2 \in \mathcal{P}_X$, $C_1,C_2 \in \mathcal{Z}^i(X)$ and $i,j \in \mathbb{N}$. Suppose $C_1$ and $C_2$ intersect properly, then we have a product

$$E^i_{C_1}(X,v_1) \times E^j_{C_2}(X,v_2) \to E^{i+j}_{C_1 \cap C_2}(X,v_1 + v_2),$$

And this product is functorial with respect to twists and extension of supports.
Axiom 5. (Associativity) For any $X \in Sm/k$, $v_a \in \mathcal{P}_X$ and $C_a \in Z^i(X)$, $a = 1, 2, 3$, the following diagram commutes

$$
\begin{array}{cccc}
E_{C_1}^i(X, v_1) \times E_{C_2}^j(X, v_2) \times E_{C_3}^k(X, v_3) & \longrightarrow & E_{C_1}^i(X, v_1) \times E_{C_1 \cap C_2}^{i+j}(X, v_1 + v_2) & , \\
\times \text{id} & & & \\
 E_{C_1 \cap C_2}^i(X, v_1 + v_2) \times E_{C_3}^k(X, v_3) & \longrightarrow & E_{C_1 \cap C_2 \cap C_3}^{i+j+k}(X, v_1 + v_2 + v_3) & , \\
(a(v_1, v_2, v_3)^{-1}) & & & \\
 E_{C_1 \cap C_2 \cap C_3}^{i+j+k}(X, (v_1 + v_2) + v_3) & & & \\
\end{array}
$$

where every intersection appeared above is proper.

Axiom 6. (Conditional Commutativity) Let $X \in Sm/k$, $C_a \in Z^i(X)$, $i_a \in \mathbb{N}$, $v_a \in \mathcal{P}_X$ where $a = 1, 2$. If $(i_1 + rk_X(v_1))(i_2 + rk_X(v_2)) = 0 \in \mathbb{Z}/2\mathbb{Z}$ and $C_1$ and $C_2$ intersect properly, the following diagram commutes:

$$
\begin{array}{cccc}
E_{C_1}^i(X, v_1) \times E_{C_2}^j(X, v_2) & \longrightarrow & E_{C_1 \cap C_2}^{i+j}(X, v_1 + v_2) . \\
\times \text{id} & & & \\
 E_{C_2}^j(X, v_2) \times E_{C_1}^i(X, v_1) & \longrightarrow & E_{C_1 \cap C_2}^{i+j}(X, v_2 + v_1) . \\
\end{array}
$$

Axiom 7. (Identity) For any $X \in Sm/k$, there is an element $e$ in $E_X^0(X, 0)$ such that for any $v \in \mathcal{P}_X$, $i \in \mathbb{N}$ and $C \in Z^i(X)$, the following diagrams commute

$$
\begin{array}{cccc}
E_C^i(X, v) & \overset{e}{\longrightarrow} & E_C^i(X, 0 + v) & E_C^i(X, v) & \overset{e}{\longrightarrow} & E_C^i(X, v + 0) , \\
\text{id} & & & \text{id} & & \\
 E_C^i(X, v) & & & E_C^i(X, v) & & \\
\end{array}
$$

where $e$ are the unity isomorphisms of 0 in $\mathcal{P}_X$. We call $e$ the identity and denote it by 1.

Axiom 8. (Pull-Back) Suppose $f : X \longrightarrow Y$ is a morphism in $Sm/k$, $i \in \mathbb{N}$, $C \in Z^i(Y)$, $f^{-1}(C) \in Z^i(X)$ and $v \in \mathcal{P}_Y$. Then we have a pull-back morphism

$$
E_C^i(Y, v) \longrightarrow E_{f^{-1}(C)}^i(X, f^*v) .
$$

This morphism is functorial with respect to $v$ and extension of supports.

Axiom 9. (Functoriality of Pull-Back) Let $X \xrightarrow{g} Y \xrightarrow{f} Z$ be morphisms in $Sm/k$, $i \in \mathbb{N}$, $C \in Z^i(Z)$, $f^{-1}(C) \in Z^i(Y)$, $g^{-1}f^{-1}(C) \in Z^i(X)$ and $v \in \mathcal{P}_Z$. We have

$$(f \circ g)^* = g^* \circ f^* .$$

And the pull-back of identity morphism is just the identity morphism.

Axiom 10. (Compatibility of Pull-Back) Suppose $f : X \longrightarrow Y$ is a morphism in $Sm/k$, $i, j \in \mathbb{N}$ and $C_1 \in Z^i(Y)$, $C_2 \in Z^j(Y)$ and they intersect properly (the same for their preimages). For any $v_1, v_2 \in \mathcal{P}_Y$, we have a commutative diagram

$$
\begin{array}{cccc}
E_{C_1}^i(Y, v_1) \times E_{C_2}^j(Y, v_2) & \longrightarrow & E_{C_1 \cap C_2}^{i+j}(Y, v_1 + v_2) . \\
f^* \times f^* & & & f^* \\
E_{f^{-1}(C_1)}^i(X, f^*(v_1)) \times E_{f^{-1}(C_2)}^j(X, f^*(v_2)) & \longrightarrow & E_{f^{-1}(C_1 \cap C_2)}^{i+j}(X, f^*(v_1 + v_2)) & \\
f^* & & & \\
\end{array}
$$

And we always have $f^*(1) = 1$.

We recall some facts about tangent bundles and normal bundles.

Lemma 3.1. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in $Sm/k$.

1. If $f$, $g$ are smooth, then we have an exact sequence

$$
0 \longrightarrow T_{X/Y} \longrightarrow T_{X/Z} \longrightarrow f^*T_{Y/Z} \longrightarrow 0 .
$$
(2) If \( f \) is a closed immersion and \( g, g \circ f \) are smooth, then we have an exact sequence
\[
0 \rightarrow T_{X/Z} \rightarrow f^*T_{Y/Z} \rightarrow N_{X/Y} \rightarrow 0.
\]

(3) If \( g \) is smooth and \( f, g \circ f \) are closed immersions, then we have an exact sequence
\[
0 \rightarrow f^*T_{Y/Z} \rightarrow N_{X/Y} \rightarrow N_{X/Z} \rightarrow 0.
\]

(4) If \( f, g \) are closed immersions, then we have an exact sequence
\[
0 \rightarrow N_{X/Y} \rightarrow N_{X/Z} \rightarrow f^*N_{Y/Z} \rightarrow 0.
\]

\[\text{Proof.}\] See [Har77, Chapter II, Proposition 8.11, Proposition 8.12 and Theorem 8.17 and Chapter III, Proposition 10.4]. □

**Lemma 3.2.** Suppose we have a Cartesian square of schemes
\[
\begin{array}{ccc}
X' & \xrightarrow{u} & X \\
\downarrow{g} & & \downarrow{f} \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

Then the composition \( T_{X'/Y'} \rightarrow T_{X'/Y} \rightarrow v^*T_{X/Y} \) is an isomorphism.

\[\text{Proof.}\] See [Har77, Chapter II, Proposition 8.10]. □

**Lemma 3.3.** Suppose we have a Cartesian square in \( Sm/k \)
\[
\begin{array}{ccc}
X' & \xrightarrow{u} & X \\
\downarrow{g} & & \downarrow{f} \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

such that \( f \) is a closed immersion. If one of the following conditions holds

1. \( u \) is smooth
2. \( u \) is a closed immersion and \( \dim X' - \dim Y' = \dim X - \dim Y \),

the natural morphism \( \gamma \) defined by the following commutative diagram with exact rows
\[
\begin{array}{cccc}
0 & \rightarrow & v^*T_{X/k} & \rightarrow & v^*f^*T_{Y/k} & \rightarrow & v^*N_{X/Y} & \rightarrow & 0 \\
& & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\
0 & \rightarrow & T_{X'/k} & \rightarrow & g^*T_{Y'/k} & \rightarrow & N_{X'/Y'} & \rightarrow & 0
\end{array}
\]

is an isomorphism.

\[\text{Proof.}\] If \( u \) is smooth, then \( \alpha \) and \( \beta \) are surjective and have the same kernel by the previous two lemmas. So \( \gamma \) is an isomorphism by the snake lemma.

In the other case, the dimension condition implies \( N_{X'/Y'} \) and \( N_{X/Y} \) have the same rank. So we only have to prove \( \gamma \uparrow \) is surjective. Then we could assume all schemes are affine. Suppose \( Y = \text{Spec}A, X = \text{Spec}A/I, Y' = \text{Spec}A/I \) and \( X' = \text{Spec}A/(I + J) \). Then \( N_{X'/Y'} = I/I^2 \) and \( N_{X'/Y'} = (I + J)/(I^2 + J) \). Now the morphism \( \gamma \) is given by
\[
\begin{array}{cccc}
I/I^2 & \otimes_A & I & \rightarrow & (I + J)/(I^2 + J) \\
(\iota) & & (\alpha) & \rightarrow & (\alpha \uparrow)
\end{array}
\]

which is obviously surjective. □

**Axiom 11.** (Push-Forward for Smooth Morphisms) Suppose \( f : X \rightarrow Y \) is a smooth morphism in \( Sm/k, n \in \mathbb{N}, v \in \mathcal{P}_X \) and \( C \in Z^{n+d}f(X) \) is finite over \( Y \). Then we have a morphism
\[
f_* : E_{C}^{n+d}f(X, f^*v - T_{X/Y}) \rightarrow E_{f(C)}^{n}Y, v,
\]
which is also functorial with respect to \( v \) and extension of supports. The push-forward of the identity morphism is just the identity morphism, by using \( T_{X/Z} = 0 \).
We may also use the simplified notation
\[ f^*v - T_{X/Y} \rightarrow v \]
to denote \( f_* \). Moreover, we could talk about the push-forward of the form
\[ f_* : E^{n+d_f}_C(X, f^*v_1 - T_{X/Y} + f^*v_2) \rightarrow E^n_{f(C)}(Y, v_1 + v_2). \]
It is defined by the composition of the push-forward defined above and the commutativity isomorphism \( c(-T_{X/Y}, f^*v_2) \).

**Axiom 12.** (Functionality of Push-Forward for Smooth Morphisms) Suppose \( X \xrightarrow{g} Y \xrightarrow{f} Z \) are smooth morphisms in \( Sm/k \), \( i \in \mathbb{N} \), \( C \in Z^{i+d_x-d_z}(X) \) is finite over \( Z \) and \( v \in \mathcal{P}_Z \). Then we have a commutative diagram
\[
\begin{array}{ccc}
E^{i+d_x-d_z}_C(X, (f \circ g)^*v - T_{X/Z}) & \xrightarrow{\varphi} & E^{i+d_x-d_z}_C(X, (f \circ g)^*v - g^*T_{Y/Z} - T_{X/Y}) \\
\downarrow{(f \circ g)_*} & & \downarrow{g_*} \\
E^{i+d_y-d_z}_C(Y, f^*v - T_{Y/Z}) & \xrightarrow{f_*} & E^n_{f(g(C))}(Z, v)
\end{array}
\]
where \( \varphi \) is obtained via
\[
(f \circ g)^*v - T_{X/Z} \rightarrow (f \circ g)^*v - (T_{X/Y} + g^*T_{Y/Z}) \rightarrow (f \circ g)^*v - g^*T_{Y/Z} - T_{X/Y}.
\]

**Axiom 13.** (Push-Forward for Closed Immersions) Suppose \( f : X \rightarrow Y \) is a closed immersion in \( Sm/k \), \( v \in \mathcal{P}_Y \) and \( C \in Z^{n+d_f}(X) \). Then we have an isomorphism
\[ f_* : E^{n+d_f}_C(X, N_{X/Y} + f^*v) \rightarrow E^n_{f(C)}(Y, v), \]
And this morphism is also functorial in \( v \) and extension of supports. The push-forward of the identity is just the identity, by using \( N_{X/Y} = 0 \).

So given a vector bundle \( V \) over \( X \), the definition above gives the isomorphism \( E^n_C(X, V) \cong E^n_C(V, 0) \) (See Section 11) via the push-forward of the zero section.

We may also use the simplified notation
\[ N_{X/Y} + f^*v \rightarrow v \]
to denote \( f_* \). Moreover, we could talk about the push-forward of the form
\[ f_* : E^{n+d_f}_C(X, f^*v_1 + N_{X/Y} + f^*v_2) \rightarrow E^n_{f(C)}(Y, v_1 + v_2). \]
It is defined by the composition of the push-forward defined above and the commutativity isomorphism \( c(f^*v_1, N_{X/Y}) \).

**Axiom 14.** (Functionality of Push-Forward for Closed Immersions) Suppose \( X \xrightarrow{g} Y \xrightarrow{f} Z \) are closed immersions in \( Sm/k \), \( C \in Z^{i+d_x-d_z}(X) \) and \( v \in \mathcal{P}_Z \). Then we have a commutative diagram
\[
\begin{array}{ccc}
E^{i+d_x-d_z}_C(X, N_{X/Z} + (f \circ g)^*v) & \xrightarrow{\varphi} & E^{i+d_x-d_z}_C(X, N_{X/Y} + g^*N_{Y/Z} + (f \circ g)^*v) \\
\downarrow{(f \circ g)_*} & & \downarrow{g_*} \\
E^{i+d_y-d_z}_C(Y, N_{Y/Z} + f^*v) & \xrightarrow{f_*} & E^n_{f(g(C))}(Z, v)
\end{array}
\]
where \( \varphi \) is got by the isomorphism \( N_{X/Z} + (f \circ g)^*v \cong N_{X/Y} + g^*N_{Y/Z} + (f \circ g)^*v. \)
Axiom 15. (Base Change for Smooth Morphisms) Suppose we have a Cartesian square with all schemes being smooth

\[
\begin{array}{ccc}
X' & \xrightarrow{u} & X \\
\downarrow^g & & \downarrow^f \\
Y' & \xrightarrow{u} & Y \\
\end{array}
\]

where \( c = d_X - d_Y = d_{X'} - d_{Y'} \), \( n \in \mathbb{N} \), \( s \in \mathcal{P}_Y \), \( f \) smooth, \( C \in Z^{n+c}(X) \) is finite over \( Y \) and \( v^{-1}(C) \in Z^{n+c}(X) \). Then the following diagram commutes

\[
\begin{array}{ccc}
E^n_\mathcal{C}(X, f^*s - T_{X/Y}) & \xrightarrow{f_*} & E^n_{f(C)}(Y, s) \\
\downarrow^{v^*} & & \downarrow^{u^*} \\
E^n_{v^{-1}(C)}(X', v^*f^*s - v^*T_{X/Y}) & \xrightarrow{g_*} & E^n_{g(v^{-1}(C))}(Y', u^*s) \\
\end{array}
\]

Here we have used the canonical isomorphism \( T_{X/Y} \rightarrow v^*T_{X/Y} \in \text{Lemma 3.2} \).

Axiom 16. (Base Change for Closed Immersions) Suppose we have a Cartesian square with all schemes being smooth

\[
\begin{array}{ccc}
X' & \xrightarrow{u} & X \\
\downarrow^g & & \downarrow^f \\
Y' & \xrightarrow{u} & Y \\
\end{array}
\]

where \( c = d_X - d_Y = d_{X'} - d_{Y'} \), \( s \in \mathcal{P}_Y \), \( f \) is a closed immersion, \( C \in Z^{n+c}(X) \) and \( v^{-1}(C) \in Z^{n+c}(X) \). Then the following diagram commutes

\[
\begin{array}{ccc}
E^{n+c}_\mathcal{C}(X, N_{X/Y} + f^*s) & \xrightarrow{f_*} & E^n_{f(C)}(Y, s) \\
\downarrow^{v^*} & & \downarrow^{u^*} \\
E^{n+c}_{v^{-1}(C)}(X', v^*N_{X/Y} + v^*f^*s) & \xrightarrow{g_*} & E^n_{g(v^{-1}(C))}(Y', u^*s) \\
\end{array}
\]

Axiom 17. (Projection Formula for Smooth Morphisms) Suppose we have a smooth morphism \( f : X \rightarrow Y \) in \( \text{Sm}/k \), \( n, m \in \mathbb{N} \), \( C \in Z^{n+d_f}(X) \) being finite over \( Y \), \( D \in Z^m(Y) \), \( C \) and \( f^{-1}(D) \) intersect properly and \( v_1, v_2 \in \mathcal{P}_Y \). Then the following diagrams commute

\[
\begin{array}{ccc}
E^{n+d_f}_\mathcal{C}(X, f^*v_1 - T_{X/Y}) \times E^m_D(Y, v_2) & \xrightarrow{id \times f^*} & E^{n+d_f}_\mathcal{C}(X, f^*v_1 - T_{X/Y}) \times E^m_{f^{-1}(D)}(X, f^*v_2) \\
\downarrow^{f_* \times id} & & \downarrow^{f_* \times id} \\
E^n_{v_1} \times E^m_D(Y, v_2) & & E^{n+m+d_f}_\mathcal{C}\mathcal{D}(X, f^*v_1 - T_{X/Y} + f^*v_2) \\
\end{array}
\]

\[
\begin{array}{ccc}
E^n_D(Y, v_2) \times E^{n+d_f}_\mathcal{C}(X, f^*v_1 - T_{X/Y}) & \xrightarrow{f^* \times id} & E^n_{f^{-1}(D)}(X, f^*v_2) \times E^{n+d_f}_\mathcal{C}(X, f^*v_1 - T_{X/Y}) \\
\downarrow^{id \times f_*} & & \downarrow^{id \times f_*} \\
E^n_D(Y, v_2) \times E^n_C(Y, v_1) & & E^{n+m+d_f}_\mathcal{C}\mathcal{D}(X, f^*v_2 + f^*v_1 - T_{X/Y}) \\
\end{array}
\]

Axiom 18. (Projection Formula for Closed Immersions) Suppose we have a closed immersion \( f : X \rightarrow Y \) in \( \text{Sm}/k \), \( n, m \in \mathbb{N} \), \( C \in Z^{n+d_f}(X) \), \( D \in Z^m(Y) \), \( f^{-1}(D) \in Z^m(X) \), \( C \) and
Suppose we have a Cartesian square with all schemes being smooth. Then the following diagrams commute.

\[
E^{m+d_f}_C(X, N_{X/Y} + f^*v_1) \times E^m_D(Y, v_2) \xrightarrow{id \times f^*} E^{m+d_f}_C(X, N_{X/Y} + f^*v_1) \times E^{m-1}_m(X, f^*v_2) \\
E^n_D(Y, v_1) \times E^m_D(Y, v_2) \xrightarrow{id \times f^*} E^{n+m+d_f}_C(X, N_{X/Y} + f^*v_1 + f^*v_2) \\
E^{n+m}_D(Y, v_1 + v_2) \xrightarrow{f_*} E^{n+m+d_f}_C(X, f^*v_2 + N_{X/Y} + f^*v_1) \\
E^n_D(Y, v_2) \times E^{n+d_f}_C(X, N_{X/Y} + f^*v_1) \xrightarrow{f^* \times id} E^{n-1}_m(X, f^*v_1) \times E^{n+d_f}_C(X, f^*v_1) \\
E^n_D(Y, v_2) \times E^n_D(Y, v_1) \xrightarrow{id \times f_*} E^{n+m+d_f}_C(X, N_{X/Y} + f^*v_1) \\
E^{n+m}_D(Y, v_1 + v_2) \xrightarrow{f_*} E^{n+m+d_f}_C(X, f^*v_2 + N_{X/Y} + f^*v_1) \\
\]

We still need a compatibility between the two push-forwards introduced.

**Axiom 19.** (Compatibility between the two Push-Forwards)

1. Suppose \( X \xrightarrow{f} Z \xrightarrow{g} Y \) are morphisms in \( \text{Sm}/k, C \in \mathbb{Z}^{i+d_x-d_y} \) finite over \( Y \), \( i \in \mathbb{N} \) and \( v \in \mathcal{P}_Y \), where \( f \) is a closed immersion and \( g, g \circ f \) are smooth. Then the following diagram commutes

\[
E^{i+d_x-d_y}_C(X, N_{X/Z} + f^*g^*v - f^*T_{Z/Y}) \xrightarrow{\phi} E^{i+d_x-d_y}_C(X, f^*g^*v - T_{X/Y}) \\
E^{i}_g(X, f^*g^*v - T_{Z/Y}) \xrightarrow{\phi} E^{i}_g(X, f^*g^*v - T_{X/Y}) \\
\]

where \( \phi \) is induced by Lemma \ref{lem:compatibility} (2).

2. Suppose \( X \xrightarrow{f} Z \xrightarrow{g} Y \) are morphisms in \( \text{Sm}/k, C \in \mathbb{Z}^{i+d_x-d_y} \) finite over \( Y \), \( i \in \mathbb{N} \) and \( v \in \mathcal{P}_Y \), where \( g \) is smooth and \( f, g \circ f \) are closed immersions. Then the following diagram commutes

\[
E^{i+d_x-d_y}_C(X, N_{X/Z} + f^*g^*v - f^*T_{Z/Y}) \xrightarrow{\phi} E^{i+d_x-d_y}_C(X, f^*T_{Z/Y} + N_{X/Z} + f^*g^*v) \\
E^{i}_g(X, g^*v - T_{Z/Y}) \xrightarrow{\phi} E^{i}_g(X, N_{X/Y} + f^*g^*v) \\
\]

where \( \phi \) is induced by Lemma \ref{lem:compatibility} (3).

3. Suppose we have a Cartesian square with all schemes being smooth

\[
X' \xrightarrow{u} X \xrightarrow{g} Y' \xrightarrow{u} Y, \\
\]
Lemma 4.2. \(Q\) as a \(\mathbb{Z}\)-set.

Definition 4.3. \(E_C^{n+d_f+d_e}(X', N_{X'/Y} + g^*u^*s - g^*T_{Y'/Y}) \xrightarrow{g^*} E_{g(C)}^{n+d_f}(Y', u^*s - T_{Y'/Y})\).

\[ \begin{array}{ccc}
E_C^{n+d_f+d_e}(X', v^*N_{X'/Y} + u^*f^*s - T_{X'/X}) & \rightarrow & E_{u(g(C))}^{n}(Y, s) \\
\downarrow v & & \downarrow f_* \\
E_C^{n+d_f}(X, N_{X'/Y} + f^*s) & \rightarrow & E_{v(g(C))}^{n}(Y, s)
\end{array} \]

Axiom 20. (Étale Excision) Suppose \(f : X \rightarrow Y\) is an étale morphism in \(Sm/k, C \in \mathcal{I}^{e}(Y)\) and the morphism \(f : f^{-1}(C) \rightarrow C\) is an isomorphism (where both terms are endowed with their reduced closed subscheme structures), then for any \(i \in \mathbb{N}\) and \(v \in \mathcal{P}_Y\), the pull-back morphism

\[ f^* : E_C^{i}(Y, v) \rightarrow E_{f^{-1}(C)}^{i}(X, f^*(v)) \]

is an isomorphism between abelian groups with inverse \(f_*\).

Definition 3.3. If the categories \(\mathcal{P}_X\) and groups \(E_C^{i}(X, v)\) satisfy all the axioms above, then they are called a correspondence theory.

Remark 3.3. The first example of a correspondence theory is just the correspondence of cycles

\[ CH_C(X, v) := \text{the free abelian group generated by components of } C, \]

where we pick \(\mathcal{P}_X = \mathbb{Z}/2\mathbb{Z}\). And unfortunately, to prove that MW-correspondence is a correspondence theory needs to rewrite everything from the beginning, for example, the Rost-Schmid complexes, as we will see in the next section.

4. MW-Correspondence as a Correspondence Theory

In this section, we are going to give a plan for the proof that MW-correspondence is a correspondence theory. It’s incomplete and will be completed in the future.

We will always assume \(E = CH\) in this section.

For any scheme \(X\) and \(x \in X\), set \(\Omega_x = m_x/m_x^2\) and \(\Lambda_x = \det(m_x/m_x^2)\).

Definition 4.1. Let \(G\) be an abelian group, \(M, N\) be \(G\)-sets, define

\[ M \times_G N = M \times N/ \sim, (m, n) \sim (m', n') \iff (m, n) = (gm', g^{-1}n') \text{ for some } g \in G \]

as a \(G\)-sets.

Definition 4.2. Let \(G\) be an abelian group and \(M\) be a \(G\)-set. We denote the group algebra of \(G\) over \(\mathbb{Z}\) by \(\mathbb{Z}[G]\) and the free abelian group generated by \(M\) by \(\mathbb{Z}[M]\). Then \(\mathbb{Z}[M]\) is a \(\mathbb{Z}[G]\)-module.

The following lemma is straightforward.

Lemma 4.1. (1) Let \(M, N\) be \(G\)-sets, then

\[ \mathbb{Z}[M \times_G N] \cong \mathbb{Z}[M] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[N]. \]

(2) Let \(G \rightarrow H\) be a morphism between abelian groups, \(M\) be a \(G\)-set, then

\[ \mathbb{Z}[M \times_G H] \cong \mathbb{Z}[M] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[H] \]

as \(\mathbb{Z}[H]\)-modules.

Definition 4.3. Let \(R\) be a commutative ring, define \(Q(R) = R^*/(R^*)^2\) as an abelian group and for any one dimensional free \(R\)-module \(L\), define

\[ Q(L) = (L \setminus \{0\})/ \sim, x \sim y \iff x = r^2y \text{ for some } r \in R^* \]

as a \(Q(R)\)-sets.

The following lemma is straightforward.

Lemma 4.2. (1) Let \(L_1, L_2\) be one dimensional \(R\)-modules, then

\[ Q(L_1 \otimes_R L_2) \cong Q(L_1) \times_{Q(R)} Q(L_2). \]
(2) Let $L$ be a one dimensional $R$-module, then
$$Q(L') \cong \text{Hom}_{Q(R)}(Q(L), Q(R)).$$

(3) Let $S$ be an $R$-algebra and $L$ be a one dimensional $R$-module, then
$$Q(L \otimes_R S) \cong Q(L) \times_{Q(R)} Q(S)$$
as $Q(S)$-sets.

**Definition 4.4.** Let $X$ be a scheme, define a category $\mathcal{P}_X$ with objects of the form $(E_1, \cdots, E_n)$, namely a series of vector bundles over $X$. For any $(E_1, \cdots, E_n)$ and $(F_1, \cdots, F_m)$, if
$$rkE_1 + \cdots + rkE_n \equiv rkF_1 + \cdots + rkF_m \ (\text{mod} 2)$$
($rk = \text{rank}$) define
$$\text{Hom}_{\mathcal{P}_X}((E_1, \cdots, E_n), (F_1, \cdots, F_m))$$
to be isomorphisms in
$$\text{Hom}_{O_X}(\text{det}(E_1, \cdots, E_n), \text{det}(F_1, \cdots, F_m)),$$
where
$$\text{det}(E_1, \cdots, E_n) = \text{det}E_1 \otimes \cdots \otimes \text{det}E_n.$$
Otherwise define it to be empty.

The composition law is the same with that in the category of line bundles. (Here we define $\text{det}(0) = O_X$)

**Proposition 4.1.** The categories $\mathcal{P}_X$ for $X \in \text{Sm}/k$ satisfy the Axiom 2.

**Proof.** From the definition of $\mathcal{P}_X$, we see that for every $A = (E_1, \cdots, E_n)$, $rkE_1 + \cdots + rkE_n$ is well defined in $\mathbb{Z}/2\mathbb{Z}$, independent of isomorphisms in $\mathcal{P}_X$. Hence there is a rank morphism
$$rk_X : \mathcal{P}_X \to \mathbb{Z}/2\mathbb{Z}.$$

Define a bifunctor
$$+: \mathcal{P}_X \times \mathcal{P}_X \to \mathcal{P}_X$$
by
$$((E_1, \cdots, E_n), (F_1, \cdots, F_m)) \mapsto (E_1, \cdots, E_n, F_1, \cdots, F_m).$$
Then it makes $\mathcal{P}_X$ a Picard category with $-(E_1, \cdots, E_n) = (E_1^\vee, \cdots, E_n^\vee)$. For any $A, B \in \mathcal{P}_X$, we attach a commutativity isomorphism
$$c = c(A, B) : A \oplus B \to B \oplus A$$
by
$$(-1)^{rk_X(A)rk_X(B)}id_{\text{det}(A) \otimes \text{det}(B)}.$$
This makes $\mathcal{P}_X$ a commutative Picard category.

There is a functor $i : (\text{Vect}(X), \text{iso}) \to \mathcal{P}_X$ sending $E$ to $(E)$ and $f : E_1 \to E_2$ to $(f)$. And for every exact sequence
$$0 \to E_1 \to E_3 \to E_2 \to 0,$$we attach an isomorphism $(E_3) \to (E_1, E_2)$ by the isomorphism $\text{det}E_3 \to \text{det}E_1 \otimes \text{det}E_2$ sending $\alpha \wedge \beta$ to $\alpha \otimes \beta$ for any local base $\alpha$ (resp. $\beta$) of $E_1$ (resp. $E_3$). This functor satisfies all conditions given in Definition 233.

Finally, for any $f : X \to Y$ in $\text{Sm}/k$, we define $f^* : \mathcal{P}_Y \to \mathcal{P}_X$ by $f^*(E_1, \cdots, E_n) = (f^*E_1, \cdots, f^*E_n)$.

We set $K_n^{MW}(F, L) = K_n^{MW}(F) \otimes_{\mathbb{Z}[Q(F)]} \mathbb{Z}[Q(L)]$ (See [Mor12] Remark 2.21)) for every one dimensional $F$-vector space $L$. For every $X \in \text{Sm}/k$, $x \in X, T$ closed in $X$ and $v \in \mathcal{P}_X$, define
$$K_n^{MW}(k(x), \lambda_x \otimes v) = K_n^{MW}(k(x), \lambda_x \otimes_{k(x)} \text{det}(v)|_{k(x)})$$
and
$$C^i_{RS,T}(X; K_n^{MW}, v) = \bigoplus_{y \in X^{(n)} \cap T} K_{n-1}^{MW}(k(y), \lambda_y \otimes v),$$
where $X^{(n)}$ means the points of codimension $n$ in $X$ (See [Mor12] Chapter 4).

Now for every $X \in \text{Sm}/k$, $i \in \mathbb{N}$, $v \in \mathcal{P}_X$ and $T$ closed in $X$, we define
$$\widehat{CH}_T^i(X, v) = H^i(C^i_{RS,T}(X; K_n^{MW}, v))$$
to give Axiom 3. And the Axiom 4 just comes from the extension of supports in Chow-Witt rings.
4.1. Operations without Intersection.

**Lemma 4.3.** Let \( f : X \rightarrow X' \) be a smooth morphism in \( Sm/k \), \( x \in X \) with \( [k(x) : k(f(x))]< \infty \). Then we have an isomorphism

\[
Q(\Lambda^*_x) \cong Q(\omega_{X'/\kappa(x)}^\vee) \times Q(k(x)) Q(\Lambda^*_f(\kappa(x)) \otimes k(x))
\]

(the \( Qs \) will be ignored in the sequel for convenience).

**Proof.** If \( k(x) \) is separable over \( k(f(x)) \), then we have a commutative diagram with exact rows and columns

\[
\begin{array}{c}
0 \\
\Omega_{X/X'}|_{k(x)} \\
\downarrow \\
0 \\
0 \\
\end{array}
\quad=
\begin{array}{c}
0 \\
\Omega_{X/X'}|_{k(x)} \\
\downarrow \\
\Omega_{X/k}|_{k(x)} \\
\downarrow \\
\Omega_{f(x)} \otimes k(x) \\
\downarrow \\
0
\end{array}
\]

so we have an isomorphism

\[
\Lambda^*_x \cong \omega_{X/X'}|_{k(x)} \otimes (\Lambda^*_f(\kappa(x)) \otimes k(x))
\]

which induces an isomorphism

\[
Q(\Lambda^*_x) \cong Q(\omega_{X/X'}|_{k(x)}^\vee) \times Q(k(x)) Q(\Lambda^*_f(\kappa(x)) \otimes k(x))
\]

In general cases, we only have the horizontal exact sequences and the middle vertical arrows. But \( Q(\omega_{k(x)/k}) \cong Q(\omega_{k(f(x))/k} \otimes k(x)) \) still holds (See [Mor12] Lemma 4.1), so we have isomorphisms

\[
Q(\Lambda^*_x)
\]

\[
\cong Q(\omega_{k(x)/k}) \times Q(k(x)) Q(\omega_{X/k}^\vee) \rightarrow Q(\Lambda^*_x)
\]

\[
\cong Q(\omega_{k(x)/k}) \times Q(k(x)) Q(\omega_{X/k}^\vee) \otimes Q(\Lambda^*_x)
\]

\[
\cong Q(\omega_{k(x)/k}) \times Q(k(x)) Q(\omega_{X/k}^\vee) \otimes Q(\Lambda^*_x) \otimes k(x)
\]

This coincides with the isomorphism we got in the case of separate extension by applying Theorem 2.1 (2) to the diagram above. \( \square \)

**Lemma 4.4.** Let \( f : X \rightarrow X' \) be a closed immersion in \( Sm/k \) and \( x \in X \) (so \( k(x) = k(f(x)) \)), then we have an isomorphism

\[
\Lambda^*_f(\kappa(x)) \cong \Lambda^*_x \otimes det N_{X/X'}^\vee|_{k(x)}
\]
Proof. This follows by the following commutative diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Omega_x & \Omega_{X/k} & \Omega_{k(x)/k} & 0 & \Omega_{f(x)} & \Omega_{X'/k} & \Omega_{k(f(x))/k} & 0 & 0 \\
\Omega_{X'/k(f(x))} & \Omega_{k(f(x))/k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N^*_X/k(x) & N^*_X/k(x) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[\square\]

Lemma 4.5. Let \( f : X \longrightarrow X' \) be a smooth morphism in \( Sm/k \), \( x \in X \) with \( \text{codim}(x) = \text{codim}(f(x)) \), then we have an isomorphism

\[ \Omega_x \cong \Omega_{f(x)} \otimes_{k(f(x))} k(x). \]

Proof. Because the cotangent map

\[ \Omega_{f(x)} \otimes_{k(f(x))} k(x) \longrightarrow \Omega_x \]

of \( f \) is injective and the two cotangent spaces have the same dimension \( \text{codim}(x) \). \[\square\]

Lemma 4.6. Let \( X_1, X_2 \in Sm/k \), \( x_1 \in X_1 \), \( x_2 \in X_2 \) and \( y \) be the generic point of some component of \( \overline{x_1 \times x_2} \). Then we have an isomorphism

\[ \Omega_y \cong \Omega_{x_1} \otimes_{k(x_1)} k(y) \oplus \Omega_{x_2} \otimes_{k(x_2)} k(y). \]

Proof. This is because we have the following commutative diagram with exact rows and columns (same if we exchange \( X_1 \) and \( X_2 \))

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Omega_{x_1} \otimes k(y) & p_1^*\Omega_{X_1/k} |_{k(y)} & q_1^*\Omega_{x_1/k} |_{k(y)} & 0 \\
\Omega_{x_2} \otimes k(y) & p_2^*\Omega_{X_2/k} |_{k(y)} & q_2^*\Omega_{x_2/k} |_{k(y)} & 0 \\
\Omega_{y} & \Omega_{X_1 \times x_2/k} |_{k(y)} & \Omega_{x_1 \times x_2/k} |_{k(y)} & 0 \\
\Omega_{y} & \Omega_{x_2} \otimes k(y) & p_2^*\Omega_{X_2/k} |_{k(y)} & 0 \\
\Omega_{y} & \Omega_{x_1} \otimes k(y) & p_1^*\Omega_{X_1/k} |_{k(y)} & 0 \\
\Omega_{y} & 0 & 0 & 0 \\
\end{array}
\]

where \( p_i : X_1 \times X_2 \longrightarrow X_i \) and \( q_i : \overline{x_1 \times x_2} \longrightarrow \overline{x_i} \) are projections and \( \Omega_{\overline{x_1 \times x_2}/k} |_{k(y)} = \Omega_{\overline{x_i}/k} |_{k(y)}. \] \[\square\]

Definition 4.5. (See Axiom \([\mathbb{N}]\)) Let \( f : X \longrightarrow X' \) be a smooth morphism, \( x \in X \) with \( \text{codim}(x) = \text{codim}(f(x)) \) and \( v \in \mathcal{P}_X \), we have a (obvious) morphism

\[ K^n_{MW}(k(f(x)), \Lambda^*_f \otimes v) \longrightarrow K^n_{MW}(k(x), \Lambda^*_f \otimes f^*v) \]

by Lemma \([4.5] \). This induces a pull-back morphism (See [Fast08 Corollaire 10.4.2])

\[ f^* : \overline{CH}^n_{T}(X', v) \longrightarrow \overline{CH}^n_{T-1}(X, f^*(v)) \]

for every \( T \in Z^n(X) \). It is functorial with respect to \( v \).
Remark 4.1. The pull-back along closed immersions is much more difficult and we will discuss this in Section 4.2.

The following is obvious.

Proposition 4.2. (See Axiom [2]) The pull-back between smooth morphisms is functorial. And $f^*(1) = 1$.

Definition 4.6. (See Axiom [13]) Let $f : X \to X'$ be a smooth morphism, $C \in Z^{i+d_f}(X)$ being finite over $X'$, we define the push-forward (See Proposition [4.6])

$$f_* : CH^{i+d_f}_C(X, f^*v - T_{X/X'}v) \to CH^i_{f(C)}(X', v)$$

by the composition

$$K^MW_0(k(x), \Lambda^*_x \otimes f^*v \otimes \omega_{X/X'}) \to K^MW_0(k(x), \omega^v_{X/X'} \otimes \Lambda^*_{f(x)} \otimes f^*v \otimes \omega_{X/X'})$$

for every $x \in C \cap X^{(i+d_f)}$, where the last arrow is the cancellation morphism between the first and fourth term. Note that we have used Lemma [4.3] in case of general finite extensions.

The push-forward for smooth morphisms is functorial with respect to $v$.

It’s clear that Axiom [20] is satisfied from the definitions.

Definition 4.7. (See Axiom [13]) Let $f : X \to X'$ be a closed immersion, $C \in Z^{i+d_f}(X)$, we define the push-forward (See Proposition [4.7])

$$f_* : CH^{i+d_f}_C(X, N_{X/X'} + f^*v) \to CH^i_{f(C)}(X', v)$$

by the isomorphism

$$K^MW_0(k(x), \Lambda^*_x \otimes detN_{X/X'} \otimes f^*v) \to K^MW_0(k(f(x)), \Lambda^*_{f(x)} \otimes f^*v)$$

for every $x \in C \cap X^{(i+d_f)}$. Here we have used Lemma [4.4].

The push-forward for closed immersions is functorial with respect to $v$.

Remark 4.2. Suppose $f : X \to X'$, $C \in Z^{i+d_f}(X)$ and $C = \overline{x}$. If $f$ is a smooth morphism and $C$ is closed in $X'$, we have an exact sequence

$$0 \to T_{X/X'|k(x)} \to \Lambda^*_x \to \Lambda^*_{f(x)} \to 0;$$

if $f$ is a closed immersion, we have an exact sequence

$$0 \to \Lambda^*_x \to \Lambda^*_{f(x)} \to N_{X/X'|k(x)} \to 0.$$

So we can identify $\Lambda^*_x$ with $N_{X/X'|k(x)}$ since the latter has the same exact sequences when $C$ is smooth. Hence in the context above, the push-forward of $f$ under the support $C$ is completely determined by the composition

$$N_{C/X} + f^*v|_C - T_{X/X'}|_C \to T_{X/X'}|_C + N_{C/Y} + f^*v|_C - T_{X/X'}|_C \to N_{C/Y} + f^*v|_C$$

in case of $f$ being smooth and by the isomorphism

$$N_{C/X} + N_{X/X'}|_C + f^*v|_C \to N_{C/Y} + f^*v|_C$$

if $f$ is a closed immersion.

This inspires us to convert equations of twisted Chow-Witt groups into equations of virtual objects. And then use the method described in Section 2. This is the main idea we will use in this section.

Now let’s explain the differential maps in Rost-Schmid complexes. Suppose $X \in Sm/k$, $Y = \overline{y}, y \in X$ and $Z = \overline{z}, z \in Y^{(1)}$ and $v \in P_X$. We want to define the differential map

$$\partial^*_v : K^MW_n(k(y), \Lambda^*_y \otimes v) \to K^MW_{n-1}(k(z), \Lambda^*_z \otimes v).$$

Suppose at first $Y$ is normal. Then the exact sequence

$$I_Y/I^2_Y \to \Omega_{X/k}|_Y \to \Omega_{Y/k} \to 0$$

$$\partial^*_v : K^MW_n(k(y), \Lambda^*_y \otimes v) \to K^MW_{n-1}(k(z), \Lambda^*_z \otimes v).$$

Suppose at first $Y$ is normal. Then the exact sequence

$$I_Y/I^2_Y \to \Omega_{X/k}|_Y \to \Omega_{Y/k} \to 0$$

$$\partial^*_v : K^MW_n(k(y), \Lambda^*_y \otimes v) \to K^MW_{n-1}(k(z), \Lambda^*_z \otimes v).$$

Suppose at first $Y$ is normal. Then the exact sequence

$$I_Y/I^2_Y \to \Omega_{X/k}|_Y \to \Omega_{Y/k} \to 0$$
is also left exact at the stalk of $z$, hence we have a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & I_Y/I_Y^2 |_{k(z)} & \rightarrow & \Omega X/k |_{k(z)} & \rightarrow \Omega Y/k |_{k(z)} & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & I_Z/I_Z^2 |_{k(z)} & \rightarrow & \Omega X/k |_{k(z)} & \rightarrow \Omega Z/k |_{k(z)} & \rightarrow 0
\end{array}
\]

The map $i$ is injective with the cokernel $m_z/m_z^2$, where $m_z$ is the maximal ideal of $O_{Y,z}$, hence we have an exact sequence

\[
0 \rightarrow (m_z/m_z^2)^\vee \rightarrow (I_Z/I_Z^2)^\vee |_{k(z)} \rightarrow (I_Y/I_Y^2)^\vee |_{k(z)} \rightarrow 0.
\]

Now choose a free basis $a$ of $(I_Y/I_Y^2)^\vee$, $e$ of $(m_z/m_z^2)^\vee$ and $t$ of $v_z$. Hence $a$ is also a free basis of $\Omega_y = (I_Y/I_Y^2)^\vee |_{k(y)}$ and $(e,a)$ is a free basis of $\Omega_z = (I_Z/I_Z^2)^\vee |_{k(z)}$ by the sequence above. We define the map $\partial$ by

\[
K_{n}^{\text{MW}}(k(y), \Lambda_y^* \otimes v) \rightarrow K_{n-1}^{\text{MW}}(k(z), \Lambda_z^* \otimes v) \rightarrow \delta_{\partial}^n(s) \otimes (e \wedge a) \otimes t,
\]

where $\delta_{\partial}^n$ is the usual partial map of Milnor-Witt groups. This map is independent of the choice of $a, e, t$.

For general cases, let $\tilde{Y}$ be the normalization of $Y$ with the natural map $\pi : \tilde{Y} \rightarrow Y$ and let $\{z_i\} = \pi^{-1}(z)$. We have an isomorphism (the same for $z$)

\[
\Lambda_y^* \cong \omega_{k(y)/k} \otimes \Omega_X/k |_{k(y)}.
\]

Now fix an $i$. We find that $\Omega_{O_{Y,z_i}/k}$ satisfies

\[
\Omega_{O_{Y,z_i}/k} \otimes k(y) = \Omega_{k(y)/k}.
\]

And we also have an exact sequence

\[
0 \rightarrow m_z/m_z^2 \rightarrow \Omega_{O_{Y,z_i}/k} \otimes k(z_i) \rightarrow \Omega_{k(z_i)/k} \rightarrow 0.
\]

So choose a free basis $c_i$ of $(m_z/m_z^2)^\vee$, $c_i$ of $\Omega_{O_{Y,z_i}/k}$, $d$ of $(\Omega_X/k)_z$ and $l$ of $v_z$. We define $\partial_i$ by the following compositions

\[
K_{n}^{\text{MW}}(k(y), \Lambda_y^* \otimes v) \rightarrow K_{n-1}^{\text{MW}}(k(z), \Lambda_z^* \otimes v) \rightarrow \delta_{\partial}^n(s) \otimes c_i \otimes d \otimes l,
\]

where the second arrow is defined by

\[
s \otimes c_i \otimes d \otimes l \rightarrow \partial_{\partial}^n(s) \otimes c_i \otimes c_i \otimes 1 \otimes d \otimes l,
\]

which is independent of the choice of $c_i$. Then we define $\partial_{\partial}^n = \sum \partial_i$. This definition coincides with the definition just given when $Y$ is normal by applying Theorem 2.11 (4) to the following
Remark 4.3. Here we would like to treat a kind of linearity of the $\partial^y_\ast$. Suppose $s \in K^\ast_{nMW}(k(y), \Lambda^\ast_y \otimes v)$.

1. Suppose $f \in O^\ast_{X, z}$ and $n = 0$, we want to show that

$$\partial^y_\ast([f]s) = [\overline{f}]\partial^y_\ast(s).$$

It suffices to show the formula for each $\partial_i$. We see that $\partial_i = Tr_{k(z)}^{k(z_1)} \circ \partial_{k(z_1)}$ (the operation of twists could be ignored). And $\partial^y_{k(z)}([f]s) = e[\overline{f}]\partial^y_{k(z)}(s)$. Suppose $\partial^y_{k(z)}(s) = <a > \eta$. Then

$$Tr_{k(z)}^{k(z_1)}(e[\overline{f}] <a > \eta)$$

$$= Tr_{k(z)}^{k(z_1)}(< \overline{f} > - < 1 >) < a > \eta)$$

$$= (< \overline{f} > - < 1 >) Tr_{k(z)}^{k(z_1)}(< a > \eta)$$

Then the claim is proved.

2. If we have another line bundle $\mathcal{M}$ over $X$ and $m$ is a free basis of $\mathcal{M}_z$ (so it’s also a free basis of $\mathcal{M}_y$), we have

$$\partial^y_\ast(s \otimes m) = \partial^y_\ast(s) \otimes m.$$ 

But we have to note that this doesn’t hold for general free basis of $\mathcal{M}_y$. Since if we replace $m$ by $\lambda \cdot m$, where $\lambda \in k(y)^\ast$, then $\lambda$ has to be moved into $K^\ast_{nMW}(k(y))$ for computation and $\lambda$ may have valuation at $z_1$.

Remark 4.4. It’s obvious that any morphism $v_1 \rightarrow v_2$ in $\mathcal{P}_X$ will induce an isomorphism between corresponding Rost-Schmid complexes.

Definition 4.8. (See [CTTR]) Let $X_a \in Sm/k, a = 1, 2, x_a \in X_a, y$ be the generic point of some component of $\overline{\pi_1} \times \overline{\pi_2}$. For every $s_a \in K^\ast_{nMW}(k(x_a), \Lambda^\ast_{x_a} \otimes v_a)$, we define

$$s_1 \times s_2 = \sum_y c(p_1^!(v_1), p_2^!(\Lambda^\ast_{x_2}))(p_1^!(s_1) \otimes p_2^!(s_2)) \in \oplus_y K^\ast_{n+mMW}(k(y), \Lambda^\ast_y \otimes (p_1^!(v_1) + p_2^!(v_2))),$$

where $p_i : \overline{\pi_i} \rightarrow \overline{\pi}$ is the projection (here we’ve used Lemma 4.6). It is called the exterior product between $s_1$ and $s_2$.

The exterior product is functorial with respect to twists and extension of supports.

We will denote $p_1^!(v_1) + p_2^!(v_2)$ by $v_1 \times v_2$ for convenience.

Now we do a special case of the proof that the right exterior product with an element in supported Chow-Witt groups is a chain complex map between Rost-Schmid complexes, while the left exterior product is not.
Proposition 4.3. Let $X, X' \in \text{Sm}/k$, $v \in \mathcal{P}_X$, $v' \in \mathcal{P}_{X'}$, and $Y \in Z^1(X)$, $T \in Z^1(X')$ be smooth. Suppose $\beta \in \widetilde{CH}_T^j(X', v')$. Then the following diagram commutes

$$
\begin{array}{c}
\oplus_{y \in Y(0)} K^{MW}_n(k(y), \Lambda^*_y \otimes v) \xrightarrow{\partial} \oplus_{z \in Z \cap X^{i+1}} K^{MW}_n(k(z), \Lambda^*_z \otimes v) \\
\oplus_{y \in Y(0)} K^{MW}_n(k(s), \Lambda^*_s \otimes (v \times v')) \xrightarrow{\partial} \oplus_{u \in (X \times X')^{i+1}} K^{MW}_n(k(u), \Lambda^*_u \otimes (v \times v'))
\end{array}
$$

That is, for every $\beta \in \widetilde{CH}_T^j(X', v')$ and $\alpha \in \oplus_{y \in Y(0)} K^{MW}_n(k(y), \Lambda^*_y \otimes v)$, we have

$$\partial(\alpha \times \beta) = \partial(\alpha) \times \beta.$$

And moreover, we have

$$\partial(\beta \times \alpha) = -1 >_{(i+r_k(x,v'))} \beta \times \partial(\alpha).$$

Proof. We may assume $Y$ and $T$ are irreducible. We check the commutativity after projecting to each $u \in (X \times X')^{(i+1)}$. It suffices to let $u$ be a generic point of $\Xi \times T$, where $z \in Y \cap X^{i+1}$, since otherwise both paths vanish. Set $Z = \Xi$. We have a commutative diagram with exact columns and rows (we write $X \times Y$ by $XY'$ for short)

$$
\begin{array}{ccccccc}
0 & \to & N_{YT/XY} & \to & N_{YT/XT} & \to & N_{XT/XY'} & \to & N_{XT/XY'}|_{ZT} & \to & 0 \\
0 & \to & N_{ZX/YX} & \to & N_{ZX/XY'} & \to & N_{XT/XY'}|_{ZT} & \to & 0 \\
0 & \to & N_{YT/XY'}|_{ZT} & \to & N_{YT/XY'}|_{ZT} & \to & N_{YT/XY'}|_{ZT} & \to & 0
\end{array}
$$

We have projections maps $p_1 : ZT \to Z$ and $p_2 : ZT \to T$. By Theorem 2.1 (1), we have a commutative diagram

$$
\begin{array}{c}
p_1(N_{Z/Y} + N_{Y/X}|_Z + v|_Z) + p_2(N_{T/X'} + v'|_T) \\
p_1(N_{Z/Y} + N_{YT/X'}|_{ZT} + p_1(v|_Z) + p_2(v'|_T)
\end{array}
$$

which gives the first equation. For the second one, we could calculate directly using the first equation by using Proposition 4.3 (which still holds in this context):

$$\partial(\beta \times \alpha)$$

$$= \partial(-1 >_{(i+r_k(x,v))} \beta \times \partial(\alpha))$$

$$= -1 >_{(i+r_k(x,v))} \beta \times \partial(\alpha).$$

where $q_1, q_2$ are projections of $X \times X'$.

\[\square\]

Definition 4.9. The exterior product in Definition 4.8 induces a pairing

$$\widetilde{CH}^n_{T_1}(X_1, v_1) \times \widetilde{CH}^{n_2}_{T_2}(X_2, v_2) \to \widetilde{CH}^{n_1+n_2}_{T_1 \times T_2}(X_1 \times X_2, v_1 \times v_2)$$
for every $X_a \in \text{Sm}/k$, $T_a \in Z_{\mu a}(X_a)$ and $v_a \in \mathcal{P}_{X_a}$, $a = 1, 2$ (at least for the case when $T_a$ are smooth) by Proposition 4.3. It’s called the exterior product between Chow-Witt groups.

**Proposition 4.4.** (See Axiom 5 and 7) In the context above, the exterior product is associative and satisfies

$$s_1 \times s_2 = -1 \cdot (\text{codim}(x_1) + r_{k x_1}(v_1))(\text{codim}(x_2) + r_{k x_2}(v_2))d(p_2^*(v_2), p_1^*(v_1))(s_2 \times s_1)$$

where $s_a \in \overline{CH}^n_{T_a}(X_a, v_a)$.

**Proof.** Associativity comes from Definition 3.3 (3) and the second statement follows from the definition of commutativity isomorphism in Proposition 4.1. □

**Proposition 4.5.** (See Axiom 10) Let $f_a : Y_a \rightarrow X_a, a = 1, 2$ be a smooth morphism in $\text{Sm}/k$, then in the context above, we have

$$(f_1 \times f_2)^*(s_1 \times s_2) = f_1^*(s_1) \times f_2^*(s_2).$$

**Proof.** This follows from Lemma 4.5 and Lemma 4.6. □

**Remark 4.5.** If we have defined pull-back along any closed immersion, then the Axioms 5, 6 and 10 can be deduced from the above two propositions.

Here we would like to prove a special case that the the push-forwards defined in Definition 4.6 and Definition 4.7 form a chain complex morphism between Rost-Schmid complexes, just to explain how to treat the twists.

**Proposition 4.6.** Suppose $Z \subseteq Y \subseteq X$ are schemes with $X$ and $Y$ being smooth, $Y = T$ closed irreducible in $X$, $Z = \overline{f}$ closed irreducible in $Y$ and $Z, Y \in \mathcal{Y}^{(1)}$. Suppose $f : X \rightarrow X'$ is a smooth morphism, $v \in \mathcal{P}_{X'}$ and $Y$ is also a closed subset of $X'$. Then we have a commutative diagram

$$
\begin{array}{ccc}
K_{n}^{MW}(k(y), \Lambda^* \otimes f^* v \otimes \omega_{X/X'}) & \overset{\partial}{\rightarrow} & K_{n-1}^{MW}(k(z), \Lambda^* \otimes f^* v \otimes \omega_{X/X'}) \\
\downarrow f^* & & \downarrow f^* \\
K_{n}^{MW}(k(f(y)), \Lambda^*_{f(y)} \otimes v) & \overset{\partial}{\rightarrow} & K_{n-1}^{MW}(k(f(z)), \Lambda^*_{f(z)} \otimes v)
\end{array}
$$

**Proof.** We have the following commutative diagram with exact rows and columns

$$
\begin{array}{cccc}
0 & \rightarrow & N_{Z/Y} & \rightarrow N_{Z/Y} \\
\downarrow & & \downarrow & \\
0 & \rightarrow & T_{X/Y}|z & \rightarrow N_{Z/X} \\
\downarrow & & \downarrow & \\
0 & \rightarrow & N_{Y/X}|z & \rightarrow N_{Y/X}|z
\end{array}
$$

Now the statement is to prove the following diagram commutes

$$
\begin{array}{cccc}
N_{Z/Y} + N_{Y/X}|z + f^* v|z - T_{X/X'}|z & \rightarrow & N_{Z/Y} + T_{X/X'}|z + N_{Y/X}|z + f^* v|z - T_{X/X'}|z \\
\downarrow & & \downarrow & \\
N_{Z/X} + f^* v|z - T_{X/X'}|z & \rightarrow & N_{Z/X} + f^* v|z \\
\downarrow & & \downarrow & \\
T_{X/X'}|z + N_{Z/X'} + f^* v|z - T_{X/X'}|z & \rightarrow & N_{Z/X'} + f^* v|z
\end{array}
$$
We have the following commutative diagrams

\[ T_{X/X'}|z + N_{Z/X'} + f^*v|z - T_{X/X'}|z, \]

\[ T_{X/X'}|z + N_{Z/Y} + N_{Y/X'}|z + f^*v|z - T_{X/X'}|z, \]

\[ N_{Z/Y} + N_{Y/X}|z + f^*v|z - T_{X/X'}|z, \]

\[ N_{Z/X} + f^*v|z - T_{X/X'}|z, \]

\[ T_{X/X'}|z + N_{Z/Y} + N_{Y/X'}|z + f^*v|z - T_{X/X'}|z, \]

where the second one comes from Theorem 221, (3). Then the result follows by combining the two diagrams above. □

**Proposition 4.7.** Suppose \( Z \subseteq Y \subseteq X \) are schemes with \( X \) and \( Y \) smooth, \( Y = \overline{y} \) closed irreducible in \( X \), \( Z = \overline{z} \) closed irreducible in \( Y \) and \( Z \subseteq Y^{(1)} \). Suppose \( f : X \to X' \) is a closed immersion and \( v \in \mathcal{P}_{X'} \). Then we have a commutative diagram

\[ K_n^{MW}(k(y), \Lambda^*_g \otimes \det N_{X/X'} \otimes f^*v) \xrightarrow{\partial} K_n^{MW}(k(z), \Lambda^*_f \otimes \det N_{X/X'} \otimes f^*v) \]

**Proof.** The diagram commutes because of the following commutative diagram by Definition 230 (3)

\[ N_{Z/Y} + N_{Y/X}|z + N_{X/X'}|z + f^*v|z, \]

\[ N_{Z/X} + N_{Y/X'}|z + f^*v|z, \]

\[ N_{Z/X} + f^*v|z, \]

Since we haven’t defined pull-back along arbitrary closed immersions, the following definition is just an intention and won’t be used.

**Definition 4.10.** (See Axiom 4) Let \( X \in Sm/k, T_a \in Z^{n_a}(X), n_a \in \mathbb{N} \) and \( v_a \in \mathcal{P}_X, a = 1, 2 \). If \( T_1 \) and \( T_2 \) intersect properly, we have a product

\[ \overline{CH}^{n_1}_{T_1}(X, v_1) \times \overline{CH}^{n_2}_{T_2}(X, v_2) \to \overline{CH}^{n_1+n_2}_{T_1 \cap T_2}(X, v_1 + v_2) \]

defined by \( a \cdot b = \Delta^*(a \times b) \) where \( \Delta : X \to X \times X \) is the diagonal. It is functorial with respect to twists and extension of supports by the same property of the exterior product.

**Proposition 4.8.** Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \) be morphisms in \( Sm/k, v \in \mathcal{P}_Z \) and \( C \in Z^{n+2d_{v}}(X) \).

1. (See Axiom 22) Suppose \( f, g \) are smooth and \( C \) is a closed subset \( Z \), then the following diagram commutes

\[ \overline{CH}^{i+d_{v}+f}(X, (g \circ f)^*v - T_{X/Z}) \xrightarrow{f_*} \overline{CH}^{i+d_{v}}(X, (g \circ f)^*v - f^*T_Y - T_{X/Y}). \]
(2) (See Axiom [14]) Suppose $f$, $g$ are closed immersions, then the following diagram commutes

\[
\begin{array}{c}
\text{Proof.} \quad (1) \text{This follows by the following commutative diagram} \\
\end{array}
\]
We are going to prove that the following diagram commutes

Theorem 2.1, (2), we have a commutative diagram

Furthermore, there is a commutative diagram with exact rows and columns

And by Theorem 2.1 (2), we have a commutative diagram

Then we could easily deduce the first diagram we want.
We are going to prove that the following diagram commutes

\[ \begin{array}{ccc}
N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C & \longrightarrow & N_{C/X} - f^*T_{Y/Z}|_C + N_{X/Y}|_C + f^*g^*v|_C \\
N_{C/Y} + f^*g^*v|_C - f^*T_{Y/Z}|_C & \longrightarrow & N_{C/X} - f^*T_{Y/Z}|_C + f^*T_{Y/Z}|_C + N_{X/Y}|_C + f^*g^*v|_C \\
f^*T_{Y/Z}|_C + N_{C/Z} + f^*g^*v|_C - f^*T_{Y/Z}|_C & \longrightarrow & N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \\
N_{C/Z} + f^*g^*v|_C & \longrightarrow & N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C 
\end{array} \]

We have a commutative diagram

\[ \begin{array}{ccc}
f^*T_{Y/Z}|_C + N_{C/Z} + f^*g^*v|_C - f^*T_{Y/Z}|_C & \longrightarrow & f^*T_{Y/Z}|_C + N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \\
N_{C/Z} + f^*g^*v|_C & \longrightarrow & N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C 
\end{array} \]

Furthermore, there is a commutative diagram with exact rows and columns

\[ \begin{array}{ccc}
0 & \longrightarrow & f^*T_{Y/Z}|_C \\
n_0 & \longrightarrow & N_{C/X} \\
n_0 & \longrightarrow & N_{C/Z} \\
n_0 & \longrightarrow & N_{X/Y}|_C \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\end{array} \]

And by Theorem 2.1 (3), we have a commutative diagram

\[ \begin{array}{ccc}
N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C & \longrightarrow & N_{C/X} + f^*T_{Y/Z}|_C + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \\
N_{C/Y} + f^*g^*v|_C - f^*T_{Y/Z}|_C & \longrightarrow & N_{C/X} + f^*T_{Y/Z}|_C + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \\
f^*T_{Y/Z}|_C + N_{C/Z} + f^*g^*v|_C - f^*T_{Y/Z}|_C & \longrightarrow & f^*T_{Y/Z}|_C + N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \\
f^*T_{Y/Z}|_C + N_{C/Z} + f^*g^*v|_C - f^*T_{Y/Z}|_C & \longrightarrow & f^*T_{Y/Z}|_C + N_{C/X} + N_{X/Y}|_C + f^*g^*v|_C - f^*T_{Y/Z}|_C \\
\end{array} \]

Then we could easily deduce the first diagram we want.

\[ \square \]

**Proposition 4.9.** (See Axiom 14 (3)) Suppose we have a Cartesian square with all schemes being smooth

\[ \begin{array}{ccc}
X' & \overset{u}{\longrightarrow} & X \\
\downarrow & & \downarrow \\
Y' & \overset{v}{\longrightarrow} & Y \\
\end{array} \]
where \( u \) is smooth and \( f \) is a closed immersion, \( s \in \mathcal{P}_Y \) and \( C \in \mathbb{Z}^{n+d_f+d_u}(X') \) is closed in \( Y \).

Then the following diagram commutes

\[
\begin{array}{c}
\xymatrix{ CH_C^{n+d_f+d_u} (X', N_{X'/Y'} + g^*u^*s - g^*T_{Y'/Y}) \ar[r]^{g^*} \ar[d]_{u^*} & CH_{g(C)}^{n+d_u} (X, u^*s - T_{Y'/Y}) \ar[d]_{u^*} \\
CH_C^{n+d_f+d_u} (X', v^*N_{X/Y} + u^*f^*s - T_{X'/X}) \ar[r]_{v^*} & CH_{u(g(C))}^n (Y, s) \ar[d]_{f_*} \\
CH_{u(C)}^n (X, N_{X/Y} + f^*s) &}
\end{array}
\]

**Proof.** We are going to show the following diagram commutes

\[
\begin{array}{c}
\xymatrix{ N_{C/X'} + N_{X'/Y'}|C + g^*u^*s|C - g^*T_{Y'/Y}|C \ar[r] & N_{C/Y'} + g^*u^*s|C - g^*T_{Y'/Y}|C \\
N_{C/X'} + v^*N_{X/Y}|C + g^*u^*s|C - T_{X'/X}|C \ar[r] & g^*T_{Y'/Y}|C + N_{C/Y} + g^*u^*s|C - g^*T_{Y'/Y}|C \\
T_{X'/X}|C + N_{C/X} + v^*N_{X/Y}|C + g^*u^*s|C - T_{X'/X}|C \ar[r] & N_{C/Y} + g^*u^*s|C \\
N_{C/X} + v^*N_{X/Y}|C + g^*u^*s|C \ar[r] & }
\end{array}
\]

We have a commutative diagram with exact rows and columns

\[
\begin{array}{c}
\xymatrix{ 0 \ar[r] & T_{X'/X}|C \ar[r]^{\cong} & g^*T_{Y'/Y}|C \\
0 \ar[r] & N_{C/X'} \ar[r] & N_{C/Y'} \ar[r] & N_{X'/Y'}|C \ar[r] & 0 \\
0 \ar[r] & N_{C/X} \ar[r] & N_{C/Y} \ar[r] & v^*N_{X/Y}|C \ar[r] & 0 \\
0 & 0 & 0 & 0 & }
\end{array}
\]

So we have a commutative diagram by Theorem 2.11 (1)

\[
\begin{array}{c}
\xymatrix{ N_{C/Y'} \ar[r] & N_{C/X'} + N_{X'/Y'}|C \\
g^*T_{Y'/Y}|C + N_{C/Y} \ar[r] & T_{X'/X}|C + N_{C/X} + N_{X'/Y'}|C \\
g^*T_{Y'/Y}|C + N_{C/X} + v^*N_{X/Y}|C & }
\end{array}
\]

And the statement follows easily from the data above. \( \square \)

**Proposition 4.10.** Suppose we have a Cartesian square with all schemes being smooth

\[
\begin{array}{c}
\xymatrix{ X' \ar[r]^u \ar[d]^g & X \ar[d]^f \\
Y' \ar[r]^v & Y }
\end{array}
\]
Proposition 4.11. (See Axiom 17) Suppose \( f, u \) are smooth, \( s \in \mathcal{P}_Y \) and \( C \in Z^{n+d_f}(X) \) is a closed subset of \( Y \). Then the following diagram commutes

\[
\begin{array}{ccc}
\widehat{CH}_C^{n+d_f}(X, f^*s - T_{X/Y}) & \xrightarrow{f_*} & \widehat{CH}_f(C)(Y, s) \\
\downarrow v^* & & \downarrow u^* \\
\widehat{CH}_{v^{-1}(C)}(X', v^*f^*s - v^*T_{X/Y}) & \xrightarrow{g_*} & \widehat{CH}_{g(v^{-1}(C))}(Y', u^*s)
\end{array}
\]

(2) (See Axiom 18) Suppose \( f \) is a closed immersion, \( s \in \mathcal{P}_Y \) and \( C \in Z^{n+d_f}(X) \). Suppose \( u \) is smooth. Then the following diagram commutes

\[
\begin{array}{ccc}
\widehat{CH}_C^{n+d_f}(X, N_{X/Y} + f^*s) & \xrightarrow{f_*} & \widehat{CH}_f(C)(Y, s) \\
\downarrow v^* & & \downarrow u^* \\
\widehat{CH}_{v^{-1}(C)}(X', v^*N_{X/Y} + v^*f^*s) & \xrightarrow{g_*} & \widehat{CH}_{g(v^{-1}(C))}(Y', u^*s)
\end{array}
\]

**Proof.** (1) We have a commutative diagram by the functoriality of \( v^* \) respect to twists

\[
\begin{array}{ccc}
N_{C/X} + f^*|C - T_{X/Y}|C & \xrightarrow{T_{X/Y}|C + N_{C/Y} + f^*|C - T_{X/Y}|C} \\
\downarrow & & \downarrow \\
N_{v^{-1}(C)/X'} + v^*f^*|v^{-1}(C) - T_{X'/Y'}|v^{-1}(C) & \xrightarrow{N_{C/Y} + f^*|C}
\end{array}
\]

\[
\begin{array}{ccc}
T_{X'/Y'}|v^{-1}(C) + N_{v^{-1}(C)/Y'} + v^*f^*|v^{-1}(C) - T_{X'/Y'}|v^{-1}(C) & \xrightarrow{N_{v^{-1}(C)/Y'} + f^*|v^{-1}(C)}
\end{array}
\]

(2) We have a commutative diagram by the functoriality of \( v^* \) respect to twists

\[
\begin{array}{ccc}
N_{C/X} + N_{X/Y}|C + f^*|C & \xrightarrow{N_{C/Y} + f^*|C} \\
\downarrow & & \downarrow \\
N_{v^{-1}(C)/X'} + N_{X'/Y'}|v^{-1}(C) + v^*f^*|v^{-1}(C) & \xrightarrow{N_{v^{-1}(C)/Y'} + v^*f^*|v^{-1}(C)}
\end{array}
\]

\[
\Box
\]

**Proposition 4.11.** (1) (See Axiom 17) Suppose \( f : X \to Y \) is a smooth morphism in \( Sm/k, v \in \mathcal{P}_Y \) and \( C \in Z^{n+d_f}(X) \) is a closed subset of \( Y \). Then for any \( Z \in Sm/k \), \( v' \in \mathcal{P}_Z \) and \( D \in Z^n(Z) \), the following diagrams commute

\[
\begin{array}{ccc}
\widehat{CH}_C^{n+d_f}(X, f^*v - T_{X/Y}) \times CH_D^m(Z, v') & \xrightarrow{c} & CH_{C \times D}^{n+d_f + m}(X \times Z, (f^*v - T_{X/Y}) \times v') \\
\downarrow f_* \times id & & \downarrow (f \times id)_* \\
\widehat{CH}_f(C)(Y, v) \times \widehat{CH}_D^m(Z, v') & \xrightarrow{(f \times id)_*} & CH_{f(C) \times D}^{n+m}(Y \times Z, v \times v')
\end{array}
\]

\[
\begin{array}{ccc}
\widehat{CH}_D(Z, v')) \times \widehat{CH}_C^{n+d_f}(X, f^*v - T_{X/Y}) & \xrightarrow{id \times f_*} & CH_{D \times C}^{n+d_f + m}(X \times Z, (f^*v - T_{X/Y})) \\
\downarrow (id \times f)_* & & \downarrow (id \times f)_* \\
\widehat{CH}_D(Z, v') \times CH_f(C)(Y, v) & \xrightarrow{(id \times f)_*} & CH_{D \times f(C)}^{n+m}(Z \times Y, v' \times v)
\end{array}
\]

(2) (See Axiom 18) Suppose \( f : X \to Y \) is a closed immersion in \( Sm/k, v \in \mathcal{P}_Y \) and \( C \) is a closed subset of \( X \). Then for any \( Z \in Sm/k, v' \in \mathcal{P}_Z \) and \( D \in Z^n(Z) \), the following
Proof. We have projections $p_1: C \times D \to C$ and $p_2: C \times D \to D$.

(1) For the first diagram, we are going to prove the following diagram commutes

$$
\begin{array}{c}
\overline{CH}^{n+d_f}_C(X, N_{X/Y} + f^*v) \times \overline{CH}^{m}_D(Z, v') \longrightarrow \overline{CH}^{n+d_f+m}_{C \times D}(X \times Z, (N_{X/Y} + f^*v) \times v') \\
\downarrow_{f \times \text{id}} \\
\overline{CH}^n_{f(C)}(Y, v) \times \overline{CH}^m_D(Z, v') \longrightarrow \overline{CH}^{n+m}_{f(C) \times D}(Y \times Z, v \times v')
\end{array}
$$

$$
\begin{array}{c}
\overline{CH}^m_D(Z, v') \times \overline{CH}^{n+d_f}_C(X, N_{X/Y} + f^*v) \longrightarrow \overline{CH}^{n+d_f+m}_{D \times C}(Z \times X, v' \times (N_{X/Y} + f^*v)) \\
\downarrow_{\text{id} \times f_*} \\
\overline{CH}^m_D(Z, v') \times \overline{CH}^n_{f(C)}(Y, v) \longrightarrow \overline{CH}^{n+m}_{D \times f(C)}(Z \times Y, v' \times v)
\end{array}
$$

We have a commutative diagram

$$
\begin{array}{c}
(N_{C/X} + f^*v|_C - T_{X/Y}|_C, N_{D/Z} + v'|_D) \longrightarrow p_1^*((N_{C/X} + f^*v|_C - T_{X/Y}|_C) + p_2^*(N_{D/Z} + v'|_D)) \\
\downarrow_{f_*} \\
(N_{C/Y} + f^*v|_C, N_{D/Z} + v'|_D) \longrightarrow (N_{C \times D/X \times Z} + p_1^*(f^*v|_C) + p_2^*(v'|_D) - T_{X \times Z/Y \times Z}|_{C \times D}) \\
\downarrow_{(f \times \text{id})_*} \\
p_1^*(N_{C/Y} + f^*v|_C) + p_2^*(N_{D/Z} + v'|_D) \longrightarrow (N_{C \times D/Y \times Z} + p_1^*(f^*v|_C) + p_2^*(v'|_D))
\end{array}
$$

Hence we just have to show the following diagram commutes

$$
\begin{array}{c}
\overline{CH}^{n+d_f}_C(X, N_{X/Y} + f^*v) \longrightarrow \overline{CH}^{n+d_f+m}_{C \times D}(X \times Z, (N_{X/Y} + f^*v) \times v') \\
\downarrow_{f \times \text{id}} \\
\overline{CH}^n_{f(C)}(Y, v) \longrightarrow \overline{CH}^{n+m}_{f(C) \times D}(Y \times Z, v \times v')
\end{array}
$$

$$
\begin{array}{c}
\overline{CH}^m_D(Z, v') \longrightarrow \overline{CH}^{n+d_f}_C(X, N_{X/Y} + f^*v) \\
\downarrow_{\text{id} \times f_*} \\
\overline{CH}^m_D(Z, v') \longrightarrow \overline{CH}^n_{f(C)}(Y, v)
\end{array}
$$
This follows by Theorem 2.1 (1) from the following commutative diagram with exact rows and columns

![Diagram](image)

For the second diagram, we suppose $\alpha \in \overline{\rm{CH}}_{C}^{n+d'}(X, f^{*}v - T_{X/Y})$ and $\beta \in \overline{\rm{CH}}_{D}^{m}(Z, v')$. And we have a commutative diagram

$\begin{array}{ccc} X & \xleftarrow{p_1} & X \times Z \xrightarrow{p_2} D \\ \downarrow f & & \downarrow q_2 \\ Y & \xleftarrow{q_1} & Y \times Z \end{array}$

Then

$$(\text{id} \times f)_*(\beta \times \alpha) = \left((f \times \text{id})_*\left(<-1>^{(n+r_{k_Y}(v))(m+r_{k_Z}(v'))} c(p_1^{*}(f^{*}v - T_{X/Y}), p_2^{*}(v'))(\alpha \times \beta)\right)\right)$$

by Proposition 4.4

$$= <\alpha >^{(n+r_{k_Y}(v))(m+r_{k_Z}(v'))} (f \times \text{id})_*((p_1^{*}(f^{*}v - T_{X/Y}), p_2^{*}(v'))(\alpha \times \beta))$$

$$= <\alpha >^{(n+r_{k_Y}(v))(m+r_{k_Z}(v'))} (f \times \text{id})_*((c(p_1^{*}(f^{*}v), p_2^{*}(v'))) \circ c(-p_1^{*}T_{X/Y}, p_2^{*}(v'))(\alpha \times \beta))$$

by functoriality of push-forward with respect to twists

$$= <\alpha >^{(n+r_{k_Y}(v))(m+r_{k_Z}(v'))} c(q_1^{*}(v), q_2^{*}(v'))((f \times \text{id})_*((c(-p_1^{*}T_{X/Y}, p_2^{*}(v'))(\alpha \times \beta)))$$

by Proposition 4.4

(2) For the first diagram, we are going to prove the following diagram commutes

$$\begin{array}{ccc} (N_{C/X} + N_{X/Y}|C + f^{*}v|C, N_{D/Z} + v') & \xrightarrow{p_1^{*}(N_{C/X} + N_{X/Y}|C + f^{*}v|C) + p_2^{*}(N_{D/Z} + v')} & \end{array}$$

$$\begin{array}{ccc} (N_{C/Y} + f^{*}v|C, N_{D/Z} + v') & \downarrow & (N_{C \times D/X \times Z} + N_{X \times Z/Y \times Z}|C \times D + p_1^{*}(f^{*}v|C) + p_2^{*}(v')) \\ \downarrow & & \downarrow \\ p_1^{*}(N_{C/Y} + f^{*}v|C) + p_2^{*}(N_{D/Z} + v') & \rightarrow & N_{C \times D/Y \times Z} + p_1^{*}(f^{*}v|C) + p_2^{*}(v') \end{array}$$
We have a commutative diagram

\[
\begin{array}{ccc}
(N_{C/X} + N_{X/Y}|C + f^*v|C, N_{D/Z} + v') & \rightarrow & p_1^*(N_{C/X} + N_{X/Y}|C + f^*v|C) + p_2^*(N_{D/Z} + v') \\
(N_{C/Y} + f^*v|C, N_{D/Z} + v') & \rightarrow & p_1^*(N_{C/Y} + f^*v|C) + p_2^*(N_{D/Z} + v') \\
\end{array}
\]

Hence we just have to show the following diagram commutes

\[
\begin{array}{ccc}
p_1^*(N_{C/X} + N_{X/Y}|C + f^*v|C) + p_2^*(N_{D/Z} + v') & \rightarrow & N_{C\times D/X\times Z} + N_{X\times Z/Y\times Z}|\mathcal{C}\times D + p_1^*(f^*v|C) + p_2^*(v') \\
\end{array}
\]

This follows by Theorem 2.1 (2) from the following commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & p_1^*N_{C/X} & N_{C\times D/X\times Z} \\
0 & p_1^*N_{C/Y} & N_{C\times D/Y\times Z} \\
0 & p_1^*(N_{X/Y}|C) & N_{X\times Z/Y\times Z}|\mathcal{C}\times D \\
\end{array}
\]

The second diagram follows by the same method as in the proof of the second diagram of (1).

\[\square\]

**Remark 4.6.** If we could define pull-back for closed immersions and prove Axiom [4] then Axiom 17 and 18 can be deduced by the proposition above and the method in [4], Corollary 3.5.

### 4.2. Intersection with Divisors

Next we would like to discuss a special case of intersection, namely pull-back along a divisor of smooth support. The constructions here come basically from [4], but the treatments of push-forwards are different.

**Definition 4.11.** Let \( X \in Sm/k \), \( D = \{ (u_i, f_i) \} \) be a Cartier divisor on \( X \). Suppose \( C \in Z^n(X) \), \( s \in \overline{CH}_C(X, v) \) and \( \text{dim}(C \cap |D|) < \text{dim}C \). Suppose

\[
s = \sum_{s_i} s_a \otimes u_a \otimes v_a \in \oplus_{y_a \in X^{(n)}} K_0^{MW}(k(y_a), \Lambda^*_y \otimes v)
\]

where \( s_a \otimes u_a \otimes v_a \in K_0^{MW}(k(y_a), \Lambda^*_y \otimes v) \) and \( y_a \in X^{(n)} \). For every \( x \in X^{(n+1)} \), suppose \( x \in U_i \) for some \( i \), hence \( y_a \in U_i \) also. And since \( y_a \notin |D| \), \( f_i \in O_{X,y_a}^* \). So we have an element \( \overline{f}_i \in k(y_a) \). Define

\[
\text{ord}_x(D \cdot s) = \sum_{x \in U_i} \partial_{y_a}^x (< -\text{codim}(y_a)) [\overline{f}_i] s_a \otimes u_a \otimes f_i \otimes v_a \in K_0^{MW}(k(x), \Lambda^*_x \otimes \mathcal{L}(-D) \otimes v).
\]
And define
\[
D \cdot s = \sum_{x \in X^{(n+1)}} \text{ord}_x(D \cdot s) \in \oplus_{x \in X^{(n+1)}} K^M_0(\Lambda^*_x \otimes \mathcal{L}(-D) \otimes v).
\]

It’s functorial with respect to \(v\) by Remark 4.4.

**Lemma 4.7.** The definition of \(\text{ord}_x(D \cdot s)\) above is independent of the choice of \(i\) and \(f_i\) and \(D \cdot s \in \overline{CH}^{n+1}_{C \cap |D|}(X, \mathcal{L}(-D) + v).

**Proof.** For any other \(j\) and \(f_j\) with \(x \in U_j, f_j/f_i \in O_{X,x}^*\) holds. And we have
\[
\sum_{x \in y_a} \partial^y_a(u_a \otimes f_i \otimes v_a) = 0
\]
since \(s \in \overline{CH}^n_C(X, v)\). So we have
\[
\sum_{x \in y_a} \partial^y_a(u_a \otimes f_j \otimes v_a) = 0
\]
and
\[
\sum_{x \in y_a} \partial^y_a(u_a \otimes f_j \otimes v_a) = 0
\]
by Remark 4.3 (2). Moreover,
\[
(\overline{f}_j \cdot s) = (\overline{f}_j/f_i + (\overline{f}_j/f_i) s) = (\overline{f}_j/f_i) \cdot s \cdot u_a \otimes f_i \otimes v_a
\]
Hence
\[
\sum_{x \in y_a} \partial^y_a(\overline{f}_j \cdot s) = \sum_{x \in y_a} \partial^y_a(\overline{f}_j/f_i) + \partial^y_a(\overline{f}_j \cdot s)
\]
which shows that \(\text{ord}_x(D \cdot s)\) is well-defined.

If \(x \notin |D|\), then \(\overline{f}_i \in O_{X,x}^*\). So
\[
\text{ord}_x(D \cdot s) = \sum_{x \in y_a} \partial^y_a(-1)^{\text{codim}(y_a)}(\overline{f}_i) \cdot s \cdot u_a \otimes f_i \otimes v_a = 0.
\]

Hence the support of \(D \cdot s\) is contained in \(C \cap |D|\).

Finally let’s prove that \(\partial(D \cdot s) = 0\), where for every \(z\), we denote \(\sum_{y,z \in \overline{f}_z} \partial^y_a\) by \(\partial_z\) and the differential map \(\partial\) is then just \(\partial_z\). For this, suppose \(u \in X^{(n+2)}\), we prove that
\[
\partial(u)(D \cdot s) := \sum_{x \in X^{(n+1)}, u \in \overline{f}_x} \partial^y_a(\text{ord}_x(D \cdot s)) = 0.
\]
Suppose \(u \in U_1\). Then
\[
\text{ord}_x(D \cdot s) = \sum_{x \in \overline{f}_x} \partial^y_a(-1)^{\text{codim}(y_a)}(\overline{f}_x) \cdot s \cdot u_a \otimes f_i \otimes v_a
\]
by definition. So let \(t = \sum_{u_a < -1} \partial^y_a(\text{ord}_x(D \cdot s)) = \sum_{x \in X^{(n+1)}, u \in \overline{f}_x} \partial^y_a(\partial(t)) = \partial(u)(\partial(t)) = 0.
\]
Definition 4.12. (See Axiom 3) Let $X \in \text{Sm}/k$ and $D$ be a smooth effective Cartier divisor on $X$. Let $i : |D| \rightarrow X$ be the inclusion. So we have $N_{D/X} \cong i^*\mathcal{L}(D)$. Suppose $v \in \mathcal{P}_X$, $C \in Z^n(X), s \in \overline{CH}^n(X,v)$ and $\dim(C \cap |D|) < \dim C$. We have a push-forward isomorphism

$$i_* : \overline{CH}^n_{C\cap|D|}([D], i^*\mathcal{L}(D) + i^*\mathcal{L}(-D) + i^*v) \rightarrow \overline{CH}^{n+1}_{C\cap|D|}(X, \mathcal{L}(-D) + v).$$

Denote by $s(\mathcal{L}(D))$ the isomorphism $i^*v \rightarrow i^*\mathcal{L}(D) + i^*\mathcal{L}(-D) + i^*v$, define

$$i^*(s) \in \overline{CH}^n_{C\cap|D|}([D], i^*v)$$

to be the unique element such that

$$i_*(s(\mathcal{L}(D))(i^*(s))) = D \cdot s.$$

It’s functorial with respect to $v$.

Proposition 4.12. Suppose $a = 1, 2$. Let $X_a \in \text{Sm}/k, v_a \in \mathcal{P}_{X_a}, C_a \in Z^n(X_a)$ be smooth, $a \in \overline{CH}^n_{C_a}(X_a, v_a), p_a : X_1 \times X_2 \rightarrow X_a$ be projections and $D_a$ be smooth effective Cartier divisors on $X_a$. Then

$$(D_1 \cdot \alpha_1) \times (\alpha_2) = p_1^*(D_1) \cdot (\alpha_1 \times \alpha_2)$$

and

$$c(p_1^*v_1, p_2^*\mathcal{L}(-D_2))(\alpha_1 \times (D_2 \cdot \alpha_2)) = p_2^*(D_2) \cdot (\alpha_1 \times \alpha_2).$$

Proof. Let’s prove the first equation. Since both sides live in

$$\overline{CH}^{n+1}_{D_1 \cap |C_1| \cap p_2^{-1}(C_2)}(X_1 \times X_2, \mathcal{L}(-D_1) + (v_1 \times v_2)),$$

it suffices to check their components at any generic point $u$ in $\overline{t_1} \times \overline{t_2}$ where $t_1 \in ([D_1] \cap C_1)^{(0)}$, $t_2 \in C_2^{(0)}$. Suppose $D_1 = \{(U_i, f_i)\}, t_1 \in U_i$. Then at $u$, we have

$$(D_1 \cdot \alpha_1) \times \alpha_2$$

$$= \partial_u(< -1 >^{n_1} f_1 \times f_2 \cdot \alpha_1) \times \alpha_2$$

$$= \partial(< -1 >^{n_1} f_1 \times f_2 \cdot \alpha_1) \times \alpha_2$$

$$= \partial(< -1 >^{n_1} f_1 \times f_2 \cdot \alpha_1) \times \alpha_2$$

by Proposition 4.3

$$= \partial_a(< -1 >^{n_1} f_1 \times f_2 \cdot \alpha_1) \times \alpha_2$$

$$= \partial_a(< -1 >^{n_1} p_1^*f_1 \times p_2^*f_1 \cdot \alpha_1 \times \alpha_2)$$

$$= p_1^*(D_1) \cdot (\alpha_1 \times \alpha_2).$$

For the second equation, we exchange the role of $X_1$ and $X_2$ as before:

$$c(p_1^*v_1, p_2^*\mathcal{L}(-D_2))(\alpha_1 \times (D_2 \cdot \alpha_2))$$

$$= < -1 >^{(n_1 + r_k X_1)}(v_1)\cdot (n_2 + r_k X_2)(v_2)) c(p_2^*v_2, p_1^*v_1)((D_2 \cdot \alpha_2) \cdot \alpha_1)$$

by Proposition 4.3

$$= < -1 >^{(n_1 + r_k X_1)}(v_1)\cdot (n_2 + r_k X_2)(v_2)) c(p_2^*v_2, p_1^*v_1)(p_2^*(D_2) \cdot (\alpha_2 \times \alpha_1))$$

by the first equation

$$= < -1 >^{(n_1 + r_k X_1)}(v_1)\cdot (n_2 + r_k X_2)(v_2)) p_2^*(D_2) \cdot c(p_2^*v_2, p_1^*v_1)(\alpha_2 \times \alpha_1)$$

by the functoriality of intersection with respect to twists

$$= p_2^*(D_2) \cdot (\alpha_1 \times \alpha_2)$$

by Proposition 4.3.

Proposition 4.13. (1) (See Axiom 7) Let $f : X \rightarrow Y$ be a smooth morphism in $\text{Sm}/k$, $C \in Z^{i+d}(X)$ be smooth and closed in $Y$, $D$ be a Cartier divisor over $Y$, $\dim(|D| \cap f(C)) < \dim(f(C))$ and $a \in \overline{CH}_C^{i+d}(X, f^*v - T_{X/Y})$. Then

$$D \cdot f_*(\alpha) = f_*(f^*(D) \cdot \alpha).$$
Proof. (1) Both sides live in the same Chow-Witt group, so we check their components at any generic point \( y \) of \( f(C) \cap |D| \). Suppose \( D = \{(U_i, f_i)\}, y \in U_i \). We have a commutative diagram

\[
\begin{array}{c}
\mathcal{L}(-D)|_C, N_{C/X} + f^*v|_C - T_{X/Y}|_C \\
\downarrow \\
\mathcal{L}(-D)|_C + N_{C/X} + f^*v|_C - T_{X/Y}|_C \\
\downarrow \\
N_{C/X} + \mathcal{L}(-D)|_C + f^*v|_C - T_{X/Y}|_C \\
\end{array}
\]

Then at \( y \), we have

\[
D \cdot f_*(\alpha) = \partial_y(-1) [f_i] \otimes f_i \otimes f_*(\alpha)) = \partial_y(-1 + d) f_*(\overline{f_i} \otimes f^*(f_i) \otimes \alpha)
\]

by the diagram above

\[
= f_\alpha \partial_y(-1 + d) \overline{f_i} \otimes f^*(f_i) \otimes \alpha
\]

by Proposition 4.7

(2) Both sides live in the same Chow-Witt group, so we check their components at any generic point \( y \) of \( f(C) \cap |D| \). Suppose \( D = \{(U_i, f_i)\}, y \in U_i \). We have a commutative diagram

\[
\begin{array}{c}
\mathcal{L}(-D)|_C, N_{C/X} + N_{X/Y}|_C + f^*v|_C \\
\downarrow \\
\mathcal{L}(-D)|_C + N_{C/X} + N_{X/Y}|_C + f^*v|_C \\
\downarrow \\
N_{C/X} + N_{X/Y}|_C + \mathcal{L}(-D)|_C + f^*v|_C \\
\end{array}
\]

Then at \( y \), we have

\[
D \cdot f_*(\alpha) = \partial_y(-1) [f_i] \otimes f_i \otimes f_*(\alpha)) = \partial_y(-1 + d) f_*(\overline{f_i} \otimes f^*(f_i) \otimes \alpha)
\]

by the diagram above

\[
= f_\alpha \partial_y(-1 + d) \overline{f_i} \otimes f^*(f_i) \otimes \alpha
\]

by Proposition 4.7

\[
= f_*(e(\mathcal{L}(-f^*D), N_{X/Y})(f^*(D) \cdot \alpha)).
\]

Now we are ready for basic formulas concerning pull-back along divisors. We will use the notation in Definition 4.12
Proposition 4.14. (See Axiom [14]) Suppose $a = 1, 2$. Let $X_a \in Sm/k$, $D_a$ be effective smooth divisors over $X_a$, $v_a \in \mathcal{P}_{X_a}$, $C_a \in Z^{nu}(X_a)$ be smooth, $\dim(C_a \cap |D_a|) < \dim(C_a)$, $a \in CH^n_{C_a}(X_a, v_a)$ and $i_a : |D_a| \to X_a$ be inclusions. Then we have

$$i_1^*(\alpha_1) \times \alpha_2 = (i_1 \times id)^*(\alpha_1 \times \alpha_2)$$

$$\alpha_1 \times i_2^*(\alpha_2) = (id \times i_2)^*(\alpha_1 \times \alpha_2).$$

Proof. We denote the projection $X_1 \times X_2 \to X_a$ by $p_a$. For the first equation, it suffices to check the equation after applying the isomorphism $(i_1 \times id)_* \circ s(\mathcal{L}(p_1^*D_1))$ on both sides. We have

$$(i_1 \times id)_* s(\mathcal{L}(p_1^*D_1))(i_1^*(\alpha_1) \times \alpha_2)$$

by bifunctoriality of exterior product with respect to twists

$$= i_1_*(s(\mathcal{L}(D_1))i_1^*(\alpha_1)) \times \alpha_2$$

by Proposition 4.11

$$= (D_1 \cdot \alpha_1) \times \alpha_2$$

by Proposition 4.12

$$= (i_1 \times id)_* (s(\mathcal{L}(p_1^*D_1))((i_1 \times id)^*(\alpha_1 \times \alpha_2))).$$

The second equation follows by exchanging the roles of $X_1$ and $X_2$:

$$\alpha_1 \times i_2^*(\alpha_2)$$

$$= -(1)\circ (n_1 + rk_x(v_1))(n_2 + rk_x(v_2)) c(q_1^*i_2^*v_2, q_1^*v_1)(i_2^*(\alpha_2) \times \alpha_1)$$

$$= -(1)\circ (n_1 + rk_x(v_1))(n_2 + rk_x(v_2)) c(q_2^*i_2^*v_2, q_1^*v_1)((i_2 \times id)^*(\alpha_2 \times \alpha_1))$$

by functoriality of pull-back with respect to twists

$$= (id \times i_2)^*(\alpha_1 \times \alpha_2).$$

\[\square\]

Proposition 4.15. Suppose we have a Cartesian square with all schemes being smooth

$$\begin{array}{ccc}
X' & \xrightarrow{u} & X \\
\downarrow g & & \downarrow f \\
Y' & \xrightarrow{u} & Y
\end{array}$$

where $u$ is a closed immersion, $\dim(X') = \dim(X) - 1$ and $\dim(Y') = \dim(Y) - 1$.

1. (See Axiom [14]) If $f$ is a closed immersion, $s \in \mathcal{P}_Y$, $C \in Z^{n+d_f}(X)$ is smooth and $\dim(u^{-1}(f(C))) < \dim(f(C))$, the following diagram commutes

$$\begin{array}{ccc}
CH^{n+d_f}_C(X, N_X/Y + f^*s) & \xrightarrow{f_*} & CH^n_{f(C)}(Y, s) \\
\downarrow u^* & & \downarrow u^*
\end{array}$$

$$\begin{array}{ccc}
CH^{n+d_f}_C(X', v^*N_X/Y + v^*f^*s) & \xrightarrow{g_*} & CH^n_{g(u^{-1}(C))}(Y', u^*s)
\end{array}$$

2. (See Axiom [14]) If $f$ is smooth, $s \in \mathcal{P}_Y$ and $C \in Z^{n+d_f}(X)$ is smooth and closed in $Y$, the following diagram commutes

$$\begin{array}{ccc}
CH^{n+d_f}_C(X, f^*s - T_{X/Y}) & \xrightarrow{f_*} & CH^n_{f(C)}(Y, s) \\
\downarrow v^* & & \downarrow v^*
\end{array}$$

$$\begin{array}{ccc}
CH^{n+d_f}_C(X', v^*f^*s - v^*T_{X/Y}) & \xrightarrow{g_*} & CH^n_{g(u^{-1}(C))}(Y', u^*s)
\end{array}$$
The conditions give us a unique effective smooth divisor $D$ (resp. $D'$) over $Y$ (resp. $X$) such that $|D| = Y'$ (resp. $|D'| = X'$). And we have $D' = f^*(D)$. It suffices to check the equation after applying $u_* \circ s(D)$ on both sides.

(1) Suppose $\alpha \in \check{C}H^{n+d}_C(X, N_{X/Y} + f^*s)$, we have
\[
u_* (s(D)) (u^* f_*(\alpha)) = D \cdot f_*(\alpha) = f_*(c \mathcal{L}(-D'), N_{X/Y})(D' \cdot \alpha)
\]
by Proposition 4.13 (2)
\[
u_* (s(D) (v_*(\mathcal{L}(D'))) (v^*(\alpha))) = f_* v_* ((c(v^* \mathcal{L}(-D'), v^* N_{X/Y}) \circ s(D)) (v^*(\alpha)))
\]
\[
u_* (g_*(c(v^* \mathcal{L}(D') + v^* \mathcal{L}(-D'), v^* N_{X/Y}) \circ s(D)) (v^*(\alpha)))
\]
by Proposition 4.18
\[
u_* (s(D)) (g_*(v^*(\alpha)))
\]
by functoriality of push-forwards with respect to twists.

(2) Suppose $\alpha \in \check{C}H^{n+d}_C(X, f^*s - T_{X/Y})$, we have
\[
u_* (s(D)) (u^* f_*(\alpha)) = D \cdot f_*(\alpha) = f_*(D' \cdot \alpha)
\]
by Proposition 4.13 (1)
\[
u_* (s(D)) (v_*(\mathcal{L}(D'))) (v^*(\alpha))) = f_* v_* ((c(v^* \mathcal{L}(-D'), v^* N_{X/Y}) \circ s(D)) (v^*(\alpha)))
\]
\[
u_* (g_*(c(v^* \mathcal{L}(D') + v^* \mathcal{L}(-D'), v^* N_{X/Y}) \circ s(D)) (v^*(\alpha)))
\]
by Proposition 4.19
\[
u_* (s(D)) (g_*(v^*(\alpha)))
\]
by functoriality of push-forwards with respect to twists.

\[
\square
\]

5. Sheaves with E-Transfers and Their Operations

In this section, we develop the theory of sheaves with E-transfers over a smooth base as in [Deg] and [CF14], where $E$ is a correspondence theory.

Since there will be heavy calculation on twists, from now on, for convenience and clarity, we will use notations like $(\alpha, v)$ for $\alpha \in E^*_C(X, v)$. With this in mind, we have operations like $(\alpha, v) \cdot (\beta, u)$, $f^* ((\alpha, v))$ with obvious meaning.

Let $S \in Sm/k$ and denote the category of smooth schemes over $S$ by $Sm/S$. We need the notion of admissible subset coming from [CF14] Definition 4.1.

Definition 5.1. Let $X, Y \in Sm/S$, we denote by $\mathcal{S}(X, Y)$ the closed subsets $T$ of $X \times_S Y$ whose irreducible component are all finite over $X$ and of dimension $\dim X$. They are called admissible subsets from $X$ to $Y$ over $S$.

Lemma 5.1. In the definition above, $T$ itself is also finite over $X$.

Proof. For every affine open subset $U$ of $X$, $T \cap U$ is affine since each of its components are affine (see [Har77] Chapter III, Exercises 3.2]). Its structure ring is a submodule of a finite $O_X(U)$-module. Hence we conclude that $T \cap U$ is finite over $U$.

Definition 5.2. Let $S \in Sm/k$, $X, Y \in Sm/S$, we define
\[
\Cor_S(X, Y) = \lim_{T} \mathcal{L}^{d_T - d_S}(X \times_S Y, -T_{X \times_S Y/X})
\]
to be the group of finite $E$-correspondences between $X$ and $Y$ over $S$, where $T \in \mathcal{S}(X, Y)$.
We define a category $\widetilde{\text{Cor}}_S(X, Y)$ whose objects are smooth schemes over $S$ and morphisms between $X$ and $Y$ are just $\text{Cor}_S(X, Y)$ defined above. Let’s now study the compositions in that category. Let’s denote, for example, $X \times_S Y \times_S Z$ by $XZW$ and the projection $X \times_S Y \times_S Z \rightarrow Y \times_S Z$ by $p_{XZW}^{XY}$ if no confusion arises.

Given any $\alpha \in \text{Cor}_S(X, Y)$ and $\beta \in \widetilde{\text{Cor}}_S(Y, Z)$, we may suppose that they come from some groups with admissible supports. Then the image of

$$p_{XZW}^{XY}(\beta \cdot T_{YZ/Y}) \cdot p_X^{XZ}(\alpha, -T_{XY/X})$$

in $\widetilde{\text{Cor}}_S(X, Z)$ is just defined as $\beta \circ \alpha$. This definition is obviously compatible with extension of supports so it’s well-defined.

**Proposition 5.1.** The composition defined above is associative.

**Proof.** Suppose $X \overset{\alpha}{\longrightarrow} Y \overset{\beta}{\longrightarrow} Z \overset{\gamma}{\longrightarrow} W$ are morphisms in $\widetilde{\text{Cor}}_S$. As above, we may suppose that they come from groups with admissible supports.

We have Cartesian squares

\[
\begin{array}{ccccccc}
XZW & \longrightarrow & XZW & \longrightarrow & XYW \\
\downarrow & & \downarrow & & \downarrow \\
XZ & \longrightarrow & YZW & \longrightarrow & YW
\end{array}
\]

So

\[
\gamma \circ (\beta \circ \alpha) = p_{XZW}^{XYW}(p_{XZW}^{XZW}((\gamma, -T_{ZW/Z}))p_{XZW}^{XYZ}(\beta, -T_{YZ/Y}))p_{XY}^{XYZ}((\alpha, -T_{XY/X}))
\]

by definition

\[
= p_{XZW}^{XZW}(p_{XZW}^{XZW}((\gamma) p_{XZ}^{XYZ}p_{XY}^{XYZ}((\beta, -T_{YZ/Y}))p_{XY}^{XYZ}((\alpha, -T_{XY/X})))
\]

by definition of the product

\[
= p_{XZW}^{XZW}(p_{XZW}^{XZW}((\gamma) p_{XZ}^{XYZ}p_{XY}^{XYZ}((\beta) p_{XY}^{XYZ}((\alpha, -T_{XY/X})))
\]

by Axiom [15] for the left square above

\[
= p_{XZW}^{XZW}(p_{XZW}^{XZW}((\gamma) p_{XZW}^{XZW}((\beta) p_{XY}^{XYZ}((\alpha, -T_{XY/X})))
\]

by Axiom [17] and Axiom [10]

\[
= p_{XZW}^{XZW}(p_{XZW}^{XZW}((\gamma) p_{XZW}^{XZW}((\beta) p_{XY}^{XYZ}((\alpha)))
\]

by Axiom [17] for $p_{XZW}^{XZW}$

\[
= p_{XZW}^{XZW}(p_{XZW}^{XZW}((\delta, -p_{XZW}^{XYZ}T_{XW/X} - p_{XZW}^{XYZ}T_{XW/X} - T_{XYZW/XZW}))
\]

by definition of the product where $\delta = p_{XZW}^{XYZ}((\gamma) p_{XZW}^{XYZ}((\beta) p_{XZW}^{XYZ}((\alpha)))
\]

by Axiom [12]

\[
= p_{XZW}^{XZW}(p_{XZW}^{XZW}((\delta, -p_{XZW}^{XYZ}T_{XW/X} - p_{XZW}^{XYZ}T_{XYZW/XZW}))
\]

by Axiom [12] note that we have used $c(-T_{XYZW/XYW}, -T_{XYZW/XZW})$

\[
= p_{XZW}^{XZW}(p_{XZW}^{XZW}((\gamma) p_{XZW}^{XYZ}((\beta) p_{XZW}^{XYZ}((\alpha)))
\]

by Axiom [17] for $p_{XZW}^{XZW}$

\[
= p_{XZW}^{XZW}(p_{XZW}^{XZW}((\gamma) p_{XZW}^{XYZ}((\beta) p_{XZW}^{XYZ}((\alpha)))
\]

by Axiom [17] and Axiom [10]

\[
= p_{XZW}^{XZW}(p_{XZW}^{XZW}((\gamma) p_{XZW}^{XYZ}((\beta) p_{XZW}^{XYZ}((\alpha)))
\]

by Axiom [16] for the right square above

\[
= (\gamma \circ \beta) \circ \alpha
\]

by definition.
Lemma 3.1 is an isomorphism. So we have maps some cohomology with support. We have

\[ E^0_X(X,0) \rightarrow E^0_X(X,N_{X/X \times S} Y - \Gamma_f^*T_{X \times S} Y/X) \rightarrow E^0_X(X,Y) \rightarrow \text{Cor}_{\mathcal{S}}(X,Y) \]

and denote the image of 1 under the composition of these maps by \( \tilde{\gamma}(f) \).

The following propositions treat some easy cases of composition.

**Proposition 5.2.** Let \( f : X \rightarrow Y \) be a morphism in \( \text{Sm}/S \) and \( g : Y \rightarrow Z \) be a morphism in \( \text{Cor}_{\mathcal{S}} \). Then we have

\[ g \circ \tilde{\gamma}(f) = (f \times \text{id}_Z)^*(g) \]

where the right hand side means its image into the direct limit.

**Proof.** We have a Cartesian square

\[
\begin{array}{ccc}
X & \xrightarrow{\Gamma_f} & XY \\
\downarrow{p_X^Z} & & \downarrow{p_{XY}^Z} \\
XZ & \xrightarrow{\Gamma_f \times \text{id}_Z} & XYZ
\end{array}
\]

and denote the map \( E^0_X(X,0) \rightarrow E^0_X(X,N_{X/X \times S} Y - \Gamma_f^*T_{X \times S} Y/X) \) by \( t \). Suppose \( g \) comes from some cohomology with support. We have

\[ g \circ \tilde{\gamma}(f) = p_X^{YZ}(p_Y^Z f_{Y}(g, -T_{Y/Z} Y)) \cdot p_{XY}^{XY} \Gamma_f^*(t(1), N_{X/Y} - \Gamma_f^*T_{X/Y/X}) \]

by definition

\[ = p_X^{YZ}(p_Y^Z f_{Y}(g, -T_{Y/Z} Y)) \cdot (\Gamma_f \times \text{id}_Z)_* p_X^{XY} \cdot (t(1)) \]

by Axiom [16] for the square above

\[ = p_X^{YZ}(p_Y^Z f_{Y}(g, -T_{Y/Z} Y)) \cdot (\Gamma_f \times \text{id}_Z)_* p_X^{XY} \cdot (t(1)) \]

by Axiom [18] for \( \Gamma_f \times \text{id}_Z \)

\[ = p_X^{XY}(\Gamma_f \times \text{id}_Z)_* ((f \times \text{id}_Z)^*(g, -T_{Y/Z} Y)) \cdot p_X^{XY} \cdot (t(1))) \]

by Axiom [9]

\[ = p_X^{XY}(\Gamma_f \times \text{id}_Z)_* ((f \times \text{id}_Z)^*(g, -T_{Y/Z} Y)) \cdot p_X^{XY} \cdot (t(1))) \]

by Axiom [9]

\[ = s((f \times \text{id}_Z)^*(g, -T_{Y/Z} Y)) \cdot p_X^{XY} \cdot (t(1))) \]

by definition of the product and pull-back and Lemma [5,3]

\[ = s((f \times \text{id}_Z)^*(g, -T_{Y/Z} Y)) \cdot (1) \]

by Axiom [19] and \( s \) is the isomorphism cancelling \( N_{X/Z} \times Y \cong (\Gamma_f \times \text{id}_Z)^*T_{X/Z} \times Z \)

\[ = (f \times \text{id}_Z)^*(g, -T_{Y/Z} Y) \]

by bifunctoriality of product with respect to twists

\[ = (f \times \text{id}_Z)^*(g) \cdot p_X^{XY} \cdot (1) \]

by functoriality of pull-back with respect to twists

\[ = (f \times \text{id}_Z)^*(g) \]

by the definition of identity and Axiom [9]

\[ \square \]
Proposition 5.3. Let $f : X \to Y$ be a morphism in $\overline{\text{Cor}}_S$ and $g : Y \to Z$ be a smooth immersion in $\text{Sm}/S$. Let $t$ be the composition

$$-T_{XY/X}$$

$$\to -(id_X \times \Gamma_g)^*T_{XYZ/XY} + N_{XY/XYZ} - T_{XY/X}$$

$$\to -(id_X \times \Gamma_g)^*T_{XYZ/XY} + N_{XY/XYZ} - (id_X \times \Gamma_g)^*T_{XYZ/XZ}$$

$$\to -(id_X \times \Gamma_g)^*T_{XYZ/XY} + N_{XY/XYZ} - N_{XY/XYZ} - T_{XY/XZ}$$

$$\to -(id_X \times \Gamma_g)^*T_{XYZ/XY} - T_{XY/XZ}$$

$$\to -(id_X \times g)^*T_{XZ/X} - T_{XY/XZ}.$$  

Then we have

$$\tilde{\gamma}(g) \circ f = (id_X \times g)_*(t(f)),$$

where the right-hand side means the image into the direct limit.

Proof. We have a Cartesian square

$$\begin{array}{ccc}
Y & \xrightarrow{\Gamma_g} & YZ \\
\downarrow{p_Y^{XY}} & & \downarrow{p_Y^{YZ}} \\
XY & \xrightarrow{id_X \times \Gamma_g} & XYZ,
\end{array}$$

an isomorphism $s : 0 \to N_{Y/YZ} - \Gamma_g^*T_{YZ/Y}$ and an isomorphism $r : -T_{XY/X} \to N_{XY/XYZ} - (id_X \times \Gamma_g)^*T_{XYZ/XY} - T_{XY/X}$. Suppose $f$ comes from some group with support. So

$$\tilde{\gamma}(g) \circ f$$

$$= p_{XYZ}^{XYZ}(p_{XZ}^{XYZ} \Gamma_g^*((s(1), N_{Y/YZ} - \Gamma_g^*T_{YZ/Y})) \cdot p_{XY}^{XYZ}(f, -T_{XY/X}))$$

by definition

$$= p_{XYZ}^{XYZ}((id_X \times \Gamma_g)_*(p_{XZ}^{XYZ}((s(1), N_{Y/YZ} - \Gamma_g^*T_{YZ/Y})) \cdot (id_X \times \Gamma_g)^*p_{XY}^{XYZ}(f)))$$

by Axiom 16 for the square above

$$= p_{XYZ}^{XYZ}((id_X \times \Gamma_g)_*(r((id_X \times \Gamma_g)^*p_{XY}^{XYZ}(f))))$$

by functoriality of pull-back and product with respect to twists

$$= (id_X \times g)_*(t((id_X \times \Gamma_g)^*p_{XY}^{XYZ}(f)))$$

by Axiom 19

$$= (id_X \times g)_*(t(f))$$

by Axiom 9.

\[\square\]

Proposition 5.4. Let $f : X \to Y$ be a morphism in $\overline{\text{Cor}}_S$ and $g : Y \to Z$ be a closed immersion in $\text{Sm}/S$. Let $t'$ be the composition

$$-T_{XY/X}$$

$$\to -T_{XY/X} + N_{XY/XYZ} - (id_X \times \Gamma_g)^*T_{XYZ/XY}$$

$$\to -T_{XY/X} + (id_X \times \Gamma_g)^*T_{XYZ/XZ} + N_{XY/XZ} - (id_X \times \Gamma_g)^*T_{XYZ/XZ}$$

$$\to -T_{XY/X} + T_{XY/X} + N_{XY/XZ} - (id_X \times \Gamma_g)^*T_{XYZ/XY}$$

$$\to N_{XY/XZ} - (id_X \times \Gamma_g)^*T_{XYZ/XY}$$

$$\to N_{XY/XZ} - (id_X \times g)^*T_{XZ/X}.$$  

Then we have

$$\tilde{\gamma}(g) \circ f = (id_X \times g)_*(t'(f)),$$

where the right-hand side means the image into the direct limit.

Proof. The same as the above proposition.  

\[\square\]
Now we would like to simplify the isomorphisms $t$ and $t'$ above. This will involve more complicated calculations in the category of virtual vector bundles.

**Lemma 5.2.** Suppose we have a commutative diagram in $Sm/k$ with the square being Cartesian:

\[
\begin{array}{ccc}
Y & \xrightarrow{id_Y} & Y \\
\downarrow_{j} & & \downarrow_{f} \\
X & \rightarrow & Y \\
\downarrow_{g} & & \downarrow_{S,} \\
Z & \rightarrow & S,
\end{array}
\]

where $f$, $g$ are smooth and $i$ is a closed immersion.

1. If $j$ is a closed immersion, then the following diagram commutes

\[
\begin{array}{cccc}
T_{X/Y}\mid_Y & + & T_{Y/S} & \rightarrow & T_{X/S}\mid_Y \\
\downarrow & & \downarrow & & \downarrow \\
N_{Y/X} & + & T_{Y/S} & \rightarrow & T_{X/Z}\mid_Y + T_{Z/S}\mid_Y \\
\downarrow & & \downarrow & & \downarrow \\
T_{X/Z}\mid_Y + N_{Y/Z} & + & T_{Y/S} & \rightarrow & T_{X/Z}\mid_Y + T_{Y/S} + N_{Y/Z}.
\end{array}
\]

2. If $j$ is smooth, then the following diagram commutes

\[
\begin{array}{cccc}
T_{X/Y}\mid_Y & + & T_{Y/S} & \rightarrow & T_{X/S}\mid_Y \\
\downarrow & & \downarrow & & \downarrow \\
N_{Y/X} & + & T_{Y/S} & \rightarrow & T_{X/Z}\mid_Y + T_{Z/S}\mid_Y \\
\downarrow & & \downarrow & & \downarrow \\
N_{Y/X} & + & T_{Y/Z} & + & T_{Z/S}\mid_Y \\
\downarrow & & \downarrow & & \downarrow \\
T_{Y/Z} & + & N_{Y/X} & + & T_{Z/S}\mid_Y.
\end{array}
\]

**Proof.** In both cases, there is a commutative diagrams with exact row and column

\[
\begin{array}{cccc}
0 & \rightarrow & T_{Y/S} & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & T_{X/Y}\mid_Y & \rightarrow T_{X/S}\mid_Y & \rightarrow T_{Y/S} & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & N_{Y/X} & \rightarrow 0.
\end{array}
\]

It induces a commutative diagram

\[
\begin{array}{cccc}
T_{X/S}\mid_Y & \rightarrow & T_{X/Y}\mid_Y + T_{Y/S} \\
\downarrow & & \downarrow & & \downarrow \\
T_{Y/S} + N_{Y/X}.
\end{array}
\]

by Theorem 2.1 (3).
(1) We have a commutative diagram with exact columns and rows

\[
\begin{array}{ccccccc}
0 & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
T_{Y/S} & T_{Y/S} & & & & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & T_{X/S}|_Y & \rightarrow & T_{Z/S}|_Y & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & T_{X/S}|_Y & \rightarrow & N_{Y/X} & \rightarrow & N_{Y/Z} & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0, & & 0 & & \\
\end{array}
\]

So we get the following commutative diagram by Theorem 2.1, (3)

\[
T_{X/S}|_Y \rightarrow T_{X/S}|_Y + T_{Z/S}|_Y \rightarrow T_{X/S}|_Y + T_{Y/S} + N_{Y/Z}
\]

\[
T_{Y/S} + N_{Y/X}
\]

\[
T_{Y/S} + T_{X/S}|_Y + N_{Y/Z}.
\]

Furthermore, there is an obvious commutative diagram

\[
N_{Y/X} + T_{Y/S} \rightarrow T_{Y/S} + N_{Y/X} \rightarrow T_{Y/S} + T_{X/S}|_Y + N_{Y/Z}.
\]

\[
T_{X/Z}|_Y + N_{Y/Z} + T_{Y/S} \rightarrow T_{X/Z}|_Y + T_{Y/S} + N_{Y/Z}.
\]

So the statement follows by combining the diagrams above.

(2) We have a commutative diagram with exact columns and rows

\[
\begin{array}{ccccccc}
0 & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
T_{Y/Z} & \rightarrow & T_{Y/S} & \rightarrow & T_{Z/S}|_Y & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & T_{X/Z}|_Y & \rightarrow & T_{X/S}|_Y & \rightarrow & T_{Z/S}|_Y & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
N_{Y/X} & \rightarrow & N_{Y/X} & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

Then the statement follows by the same method as in (1) by applying Theorem 2.1, (2) to the diagram above.

\[\square\]

**Lemma 5.3.** Suppose \(X, Y, Z \in \text{Sm}/S\) and \(g: Y \rightarrow Z\) is a morphism in \(\text{Sm}/S\).

(1) If \(g\) is a closed immersion, then the isomorphism \(t\) in Proposition 5.3 is equal to

\[-T_{XY/X} \rightarrow N_{XY/XZ} - N_{XY/XZ} - T_{XY/X} \rightarrow N_{XY/XZ} - (id_X \times g)^* T_{XZ/X}.
\]
(2) If $g$ is smooth, then the isomorphism $t'$ in Proposition 5.4 is equal to

$$ -T_{XY/X} \longrightarrow -(id_X \times g)^*T_{XZ/X} - T_{XY/XZ}. $$

Proof. We have a commutative diagram in $Sm/k$ with the square being Cartesian:

\[
\begin{array}{ccc}
X & \xrightarrow{id_X \times g} & XZ \\
\downarrow & & \downarrow \\
Y & \xrightarrow{id_Y} & YX \\
\downarrow & & \downarrow \\
Z & \xrightarrow{p_X} & X.
\end{array}
\]

(1) We show that the composition

$$ -T_{XY/X} \longrightarrow -T_{XY/X} + N_{XY/XYZ} - (id_X \times \Gamma_g)^*T_{XYZ/XY} $$

$$ -T_{XY/X} + (id_X \times \Gamma_g)^*T_{XYZ/XZ} + N_{XY/XZ} - (id_X \times \Gamma_g)^*T_{XYZ/XY} $$

$$ N_{XY/XZ} - (id_X \times \Gamma_g)^*T_{XYZ/XY} $$

$$ N_{XY/XZ} - (id_X \times g)^*T_{XZ/X} $$

$$ N_{XY/XZ} - N_{XY/XZ} - T_{XY/X} $$

$$ -T_{XY/X} $$

is just $id_{-T_{XY/X}}$. It is equal to

$$ -T_{XY/X} $$

$$ -T_{XY/X} + N_{XY/XYZ} - (id_X \times \Gamma_g)^*T_{XYZ/XY} $$

$$ -T_{XY/X} - (id_X \times \Gamma_g)^*T_{XYZ/XY} + N_{XY/XYZ} $$

$$ -T_{XY/X} - N_{XY/XZ} - T_{XY/X} + N_{XY/XYZ} $$

$$ -T_{XY/X} - N_{XY/XZ} - T_{XY/X} + (id_X \times \Gamma_g)^*T_{XYZ/XZ} + N_{XY/XZ} $$

$$ -N_{XY/XZ} - T_{XY/X} + N_{XY/XZ} $$

$$ -T_{XY/X}, $$

where the sixth arrow is the cancellation map between the first and the fourth term. By Lemma 5.2 (1) and the commutative diagram above, we have a commutative diagram

\[
\begin{array}{ccc}
(id_X \times \Gamma_g)^*T_{XYZ/XY} + T_{XY/X} & \xrightarrow{(id_X \times \Gamma_g)^*T_{XYZ/X}} & (id_X \times \Gamma_g)^*T_{XYZ/X} \\
\downarrow & & \downarrow \\
N_{XY/XYZ} + T_{XY/X} & \xrightarrow{(id_X \times \Gamma_g)^*T_{XYZ/XZ} + (id_X \times g)^*T_{XZ/X}} & (id_X \times \Gamma_g)^*T_{XYZ/XZ} + T_{XY/X} + N_{XY/XZ}.
\end{array}
\]

Hence the composition above is equal to

$$ -T_{XY/X} $$

$$ -T_{XY/X} + N_{XY/XYZ} - (id_X \times \Gamma_g)^*T_{XYZ/XY} $$

$$ -T_{XY/X} - (id_X \times \Gamma_g)^*T_{XYZ/XY} + N_{XY/XYZ} $$

$$ -T_{XY/X} - N_{XY/XZ} + N_{XY/XYZ} $$

$$ -T_{XY/X} - N_{XY/XZ} - (id_X \times \Gamma_g)^*T_{XYZ/XZ} + N_{XY/XYZ} $$

$$ -T_{XY/X} - (id_X \times \Gamma_g)^*T_{XYZ/XZ} + (id_X \times \Gamma_g)^*T_{XYZ/XZ} + N_{XY/XYZ} $$

$$ -T_{XY/X}, $$

which gives the result.
(2) We show that the composition
\[ -T_{XY/X} \rightarrow -(id_X \times \Gamma_g)^* T_{XYZ/XY} + N_{XY/XYZ} - T_{XY/X} \]
\[ \rightarrow -(id_X \times \Gamma_g)^* T_{XYZ/XY} + N_{XY/XYZ} - N_{XY/XYZ} - T_{XY/XZ} \]
\[ \rightarrow -(id_X \times \Gamma_g)^* T_{XYZ/XY} - T_{XY/XZ} \]
\[ \rightarrow -(id_X \times g)^* T_{XZ/X} - T_{XY/XZ} \]
\[ \rightarrow -T_{XY/X} \]

is just \( id_{T_{XY/X}} \). By Lemma 5.2, (2) and the commutative diagram in the beginning, we get a commutative diagram
\[
\begin{array}{c}
(id_X \times \Gamma_g)^* T_{XYZ/XY} + T_{XY/X} \\
\downarrow \\
N_{XY/XYZ} + T_{XY/XZ} + (id_X \times g)^* T_{XZ/X}
\end{array}
\]
\[
\begin{array}{c}
(id_X \times \Gamma_g)^* T_{XYZ/XY} + (id_X \times g)^* T_{XZ/X} \\
\downarrow \\
N_{XY/XYZ} + T_{XY/XZ} + (id_X \times g)^* T_{XZ/X}
\end{array}
\]

Hence the composition given is equal to
\[ -T_{XY/X} \rightarrow -(id_X \times \Gamma_g)^* T_{XYZ/XY} + N_{XY/XYZ} - T_{XY/X} \]
\[ \rightarrow -(id_X \times \Gamma_g)^* T_{XYZ/XY} - T_{XY/XZ} + N_{XY/XYZ} \]
\[ \rightarrow -(id_X \times \Gamma_g)^* T_{XYZ/XY} - T_{XY/XZ} \]
\[ \rightarrow -T_{XY/X} \]

where the fifth arrow is the cancellation between the first and the fourth term. Hence the result follows.

\[ \square \]

**Proposition 5.5.** For any \( X \in \text{Sm}/S \), \( \tilde{\gamma}(id_X) \) is an identity. That is, for any \( X,Y \in \text{Sm}/S \), \( f \in \widetilde{\text{Cor}}_S(X,Y) \), \( g \in \widetilde{\text{Cor}}_S(Y,X) \), we have
\[ \tilde{\gamma}(id_Y) \circ f = f, g \circ \tilde{\gamma}(id_X) = g. \]

**Proof.** The second equation follows by Proposition 5.2 and the first one follows from Lemma 5.2 (1) and Proposition 5.3. \( \square \)

So combining Proposition 5.1 and Proposition 5.5 we have proved that \( \widetilde{\text{Cor}}_S \) is indeed a category.

Let’s prove that \( \tilde{\gamma} \) is indeed a functor.

**Proposition 5.6.** For any \( X \xrightarrow{f} Y \xrightarrow{g} Z \) in \( \text{Sm}/S \), we have
\[ \tilde{\gamma}(g \circ f) = \tilde{\gamma}(g) \circ \tilde{\gamma}(f). \]

**Proof.** Suppose at first \( f \) is a closed immersion or smooth. We have a Cartesian square
\[
\begin{array}{ccc}
XZ & \xrightarrow{f \times id_Z} & YZ \\
\downarrow \rho_{gf} & & \downarrow \rho_g \\
X & \xrightarrow{f} & Y
\end{array}
\]
and two cancelling ismorphisms \( a : N_{Y/Z} - \Gamma_1^* T_{Y/Z} \rightarrow 0 \) and \( b : N_{X/Z} - \Gamma_1^* T_{X/Z} \rightarrow 0 \). For convenience, we denote the induced morphisms between \( E \) still by \( a \) and \( b \), respectively. Then we have
\[
\tilde{\gamma}(g) \circ \tilde{\gamma}(f) = (f \times \text{id}_Z)^* (\tilde{\gamma}(g))
\]
by Proposition 5.2
\[
= (f \times \text{id}_Z)^* (\Gamma_0 (a^{-1}(1), N_{Y/Z} - \Gamma_1^* T_{Y/Z} ))
\]
by definition of \( \tilde{\gamma} \)
\[
= (\Gamma_0(f), f^*(a^{-1}(1))
\]
by Axiom 16 for the square above
\[
= (\Gamma_0(f), (b^{-1}(1))
\]
by Axiom 9 and functoriality of pull-back with respect to twists
\[
= \tilde{\gamma}(g \circ f)
\]
by definition of \( \tilde{\gamma} \).

Now suppose \( f = p \circ i \) in \( S^m/S \) where \( p \) is smooth and \( i \) is a closed immersion. Then
\[
\tilde{\gamma}(g) \circ \tilde{\gamma}(f) = \tilde{\gamma}(g) \circ \tilde{\gamma}(i) \circ \tilde{\gamma}(p) = \tilde{\gamma}(i \circ g) \circ \tilde{\gamma}(p) = \tilde{\gamma}(g \circ f)
\]
by the statements above.

**Definition 5.4.** Define \( \tilde{\mathcal{P}Sh}(S) \) to be the category of contravariant additive functors from \( \mathcal{C}or_S \) to \( \text{Ab} \) as in [DF17 Definition 1.2.1] and [MVW09 Definition 2.1], which are called presheaves with \( T \)-transfers over \( S \). Define moreover \( \mathcal{S}h(S) \) to be its full subcategory with objects the presheaves whose restriction to \( S^m/S \) via \( \tilde{\gamma} \) are Nisnevich sheaves. We call them sheaves with \( T \)-transfers over \( S \).

**Definition 5.5.** Let \( X, Y \in S^m/S \), we define \( \tilde{\mathcal{C}}_S(X) \) by \( \tilde{\mathcal{C}}_S(X)(Y) = \mathcal{C}or_S(Y, X) \). It is the presheaf with \( T \)-transfers representing \( X \).

We recall the following three propositions which are the technical heart when dealing with Nisnevich sheaves:

**Proposition 5.7.** Let \( f : X \rightarrow S \) be a morphism locally of finite type between locally noetherian schemes. Suppose \( I \) is a directed set and \( \{ T_i \} \) is an inverse system of \( S \)-schemes such that for any \( i_1 \leq i_2 \), the morphism \( T_{i_2} \rightarrow T_{i_1} \) is affine. Then \( \lim_{\leftarrow i} T_i \) exists in the category of \( S \)-schemes and we have
\[
\text{Hom}_S(\lim_{\leftarrow i} T_i, X) = \lim_{\leftarrow i} \text{Hom}_S(T_i, X).
\]

**Proof.** See [Pro Lemma 2.2] and [Pro Proposition 6.1].

Now let \( A \) be a noetherian ring and \( p \in \text{Spec}A \). Consider the set \( I \) whose elements are pairs \( (B, q) \), where \( B \) is a connected étale \( A \)-algebra, \( q \in \text{Spec}B \), \( q \cap A = p \) and \( k(p) = k(q) \). Set \( (B_1, q_1) \preceq (B_2, q_2) \) if there is an \( A \)-algebra morphism (always unique if exists) \( f : B_1 \rightarrow B_2 \) such that \( f^{-1}(q_2) = q_1 \).

**Proposition 5.8.** The set \( I \) is a directed set and we have
\[
\lim_{\leftarrow i} B \cong A_p^b,
\]
where the right hand side is the Henselization of \( A_p \).

**Proof.** See the remarks around [Mi80 Lemma 4.8] and see for example [Mi80 Theorem 4.2] for basic properties of Henselian rings.

**Proposition 5.9.** Let \( U, X, Y \) be locally noetherian schemes, \( p : U \rightarrow X \) be a Nisnevich covering and \( f : X \rightarrow Y \) be a finite morphism. Then for every \( y \in Y \), there exists a scheme \( V \) with an étale morphism \( V \rightarrow Y \) being Nisnevich at \( y \) such that the morphism \( U \times_Y V \rightarrow X \times_Y V \) has a section.
Proof. Consider the following commutative diagram with Cartesian squares:

\[
\begin{array}{ccc}
U & \xrightarrow{p} & X \\
\downarrow{\gamma} & & \downarrow{f} \\
R_2 & \xrightarrow{\alpha} & R_1 \\
& \xrightarrow{\beta} & \text{Spec}O_{Y_{\gamma}}^h.
\end{array}
\]

Since \( \beta \) is a finite morphism, \( R_1 \) is a finite direct product of Henselian rings (see [Mil80, Theorem 4.2]). Hence \( \alpha \) has a section \( s \) since it is Nisnevich at every maximal ideal of \( R_1 \). Pick an affine neighbourhood \( U_0 \) of \( y \). By [Pro, Lemma 2.3] and Proposition 5.8

\[
R_1 = ( \lim_{(B,q) \succeq (O_Y(U_0),y)} \text{Spec}B ) \times_{U_0} f^{-1}(U_0) = ( \lim_{(B,q) \succeq (O_Y(U_0),y)} \text{Spec}B \times_{U_0} f^{-1}(U_0) ),
\]

hence there exists a \((B,q) \supseteq (O_Y(U_0),y)\) such that \( \gamma \circ s \) factor through the projection

\[
\lim_{(B,q) \succeq (O_Y(U_0),y)} (\text{Spec}B \times_{U_0} f^{-1}(U_0)) \longrightarrow \text{Spec}B \times_{U_0} f^{-1}(U_0)
\]

by using Proposition 5.7 for \( p \). And we finally let \( V = \text{Spec}B \). \( \square \)

Now we are going to prove a similar result as in [DFL7 Lemma 1.2.6].

**Proposition 5.10.** Let \( X, U \in Sm/S \) and \( p : U \longrightarrow X \) be a Nisnevich covering. Denote the \( n \)-fold product \( A \times_B A \times_B \cdots A \) by \( A^n_B \) for any schemes \( A \) and \( B \). Then the following complex

\[
\cdots \longrightarrow \tilde{c}_S(U_X^n) \stackrel{d_n}{\longrightarrow} \cdots \longrightarrow \tilde{c}_S(U \times_X U) \stackrel{d_2}{\longrightarrow} \tilde{c}_S(U) \stackrel{d_1}{\longrightarrow} \tilde{c}_S(X) \stackrel{d_0}{\longrightarrow} 0,
\]

denoted by \( \tilde{C}(U/X) \), is exact after sheafifying as a complex of \( \text{PSh}(S) \). Here, if \( p_i : U_X^n \longrightarrow U_X^{n-1} \) is the projection omitting \( i \)-th factor, then \( d_n = \sum_i (-1)^{n-i} \tilde{c}_S(p_i) \).

**Proof.** For \( Y \in Sm/S \), we are to prove that the complex is exact at every point \( y \in Y \). Now assume we have an element \( a \in \text{Cor}_S(Y, U_X^n) \) such that \( d_n(a) = 0 \). We may suppose that there is a \( T \in \mathcal{O}_Y(Y, X) \) such that \( a \) comes from \( E^{p_0}_{R^n} - d_1((Y \times S U)^n_{Y \times S X}, -T_{Y \times S U^n_{Y \times S X}}) \) and \( d_n(a) = 0 \), where \( R^n \) is defined by the following Cartesian squares (\( R := R^2 \))

\[
\begin{array}{ccc}
R^n & \longrightarrow & Y \times_S U^n_X \\
\downarrow & & \downarrow \\
T & \longrightarrow & Y \times_S X \\
\downarrow & & \downarrow \\
X & \longrightarrow & X.
\end{array}
\]

By Proposition 5.9, there is a Nisnevich neighbourhood \( V \) of \( y \) such that the map \( p : R \times_Y V \longrightarrow T \times_Y V \) has a section \( s \), which is both an open immersion and a closed immersion (see [Mil80 Corollary 3.12]). And let \( D = (R \times_Y V) \setminus s(T \times_Y V) \). Then \( d_n(a|_{V \times_S U_X^n}) = 0 \). We have a commutative diagram

\[
\begin{array}{ccc}
V \times_S U^n_X & \longrightarrow & Y \times_S U^n_X \\
\downarrow & & \downarrow \\
V \times_S X & \longrightarrow & Y \times_S X,
\end{array}
\]

cartesian squares

\[
\begin{array}{ccc}
R^n \times_Y V & \longrightarrow & V \times_S U^n_X \\
\downarrow & & \downarrow \\
R^n & \longrightarrow & Y \times_S U^n_X \\
\downarrow & & \downarrow \\
R^n & \longrightarrow & Y \times_S U^n_X \\
\downarrow & & \downarrow \\
R^n & \longrightarrow & Y \times_S U^n_X.
\end{array}
\]

equations

\[
\begin{align*}
Y \times_S U^n_X &= (Y \times S U)^n_{Y \times_S X}, \\
V \times_S U^n_X &= (V \times S U)^n_{V \times_S X}, \\
R^n &= R^n_T, \\
(R \times_Y V)^n_{T \times_Y V} &= (T \times_Y \times_S X (V \times S U))^n_{T \times_Y V}, \\
(R \times_Y V)_V^{n \times V} &= (R \times_Y V)_V^{n \times V}.
\end{align*}
\]
and a diagram of Cartesian squares with right-hand vertical maps being étale:

\[
\begin{array}{ccc}
(R \times Y)^n_{T \times Y} & \longrightarrow & (V \times S U)^n_{V \times S X} \times V \times S X ((V \times S U) \setminus D) := W^{n+1} \\
\downarrow \text{id}_n \times s & & \downarrow j_{n+1} \\
(R \times Y)^{n+1}_{T \times Y} & \longrightarrow & (V \times S U)^{n+1}_{V \times S X} \\
\downarrow p_{n+1} & & \downarrow p_{n+1} \\
(R \times Y)^n_{T \times Y} & \longrightarrow & (V \times S U)^n_{V \times S X},
\end{array}
\]

where \( p_{n+1} \) denotes the projection omitting the last factor. Moreover, the maps

\[
E^{d_X - d_S}_{R \times X \times Y}((V \times S U)^n_{V \times S X}, -T_{V \times S U}^{n+1}/V) \xrightarrow{(p_{n+1} \circ j_{n+1})^*} E^{d_X - d_S}_{R \times X \times Y}((V \times S U)^{n+1}/V, -T_{V \times S U}^{n+1}/V|W^{n+1})
\]

and

\[
E^{d_X - d_S}_{R \times X \times Y}((V \times S U)^{n+1}_{V \times S X}, -T_{V \times S U}^{n+1}/V) \xrightarrow{j_{n+1}^*} E^{d_X - d_S}_{R \times X \times Y}((V \times S U)^n_{V \times S X}, -T_{V \times S U}^{n+1}/V|W^{n+1})
\]

are isomorphisms with respective inverses \((p_{n+1} \circ j_{n+1})_*\) and \((j_{n+1})_*\), by Axiom 20.

Let’s consider the element

\[
b := (j_{n+1}^* - 1 \circ (p_{n+1} \circ j_{n+1})^*)(a|_{(V \times S U)^n_{V \times S X}}) \in E^{d_X - d_S}_{R \times X \times Y}((V \times S U)^n_{V \times S X}, -T_{V \times S U}^{n+1}/V),
\]

where we have used the isomorphism

\[
p_{n+1}^* T_{V \times S U}^n/V \longrightarrow T_{V \times S U}^{n+1}/V
\]

since \( U \longrightarrow X \) is étale. Then

\[
d_{n+1}(b) = \sum_{i=1}^{n+1} (-1)^{i-1} e_i(p_{i})(b) = \sum_{i=1}^{n+1} (-1)^{i-1} p_*(t_{i,n+1}(b))
\]

by Proposition 5.3 where

\[
t_{i,n+1} : -T_{V \times S U}^{n+1}/V \longrightarrow -(id_{V \times S U}^n V \times S X)\]

is the isomorphism as in the Proposition 5.3 applying to

\[
V \longrightarrow U^{n+1} X \xrightarrow{p_i} U^n X.
\]

If \( 1 \leq i < n + 1 \), we have Cartesian squares

\[
\begin{array}{ccc}
W^{n+1} & \longrightarrow & (V \times S U)^n_{V \times S X} \\
\downarrow p_i & & \downarrow p_i \\
W^n & \longrightarrow & (V \times S U)^{n-1}_{V \times S X} \\
\end{array}
\]

So

\[
p_*(t_{i,n+1}(b)) = (p_i \circ t_{i,n+1} \circ (j_{n+1}^*)^{-1} \circ (p_{n+1} \circ j_{n+1})^*)(a|_{(V \times S U)^n_{V \times S X}}, -T_{V \times S U}^n/V)
\]

by definition

\[
= (p_i \circ (j_{n+1}^*)^{-1} \circ j_{n+1}^*(t_{i,n+1}) \circ (p_{n+1} \circ j_{n+1})^*)(a|_{(V \times S U)^n_{V \times S X}})
\]

by functoriality of pullback with respect to twists

\[
= (j_{n+1}^*)^{-1} \circ p_* \circ (j_{n+1}^*) \circ (p_{n+1} \circ j_{n+1})^*)(a|_{(V \times S U)^n_{V \times S X}})
\]

by Axiom 15 for the right square above

\[
= (j_{n+1}^*)^{-1} \circ p_* \circ (p_{n+1} \circ j_{n+1})^* \circ t_{i,n}(a|_{(V \times S U)^n_{V \times S X}})
\]

by functoriality of pull-back with respect to twists

\[
= (j_{n+1}^*)^{-1} \circ (p_n \circ j_n)^* \circ p_* \circ t_{i,n}(a|_{(V \times S U)^n_{V \times S X}})
\]

by Axiom 15 for the left square above.
Now let $i = n + 1$, we have

\[
\begin{align*}
 p_n + 1 \circ (t_{n+1,n+1}(b)) \\
 = (p_n + 1 \circ t_{n+1,n+1} \circ (j_{n+1}^*)^{-1} \circ (p_n + 1 \circ j_{n+1}^*))(a|_{(V \times S U)_{V S X}}) \ 
\end{align*}
\]

by definition

\[
\begin{align*}
 = (p_n + 1 \circ t_{n+1,n+1} \circ j_{n+1} \circ (p_n + 1 \circ j_{n+1}^*))(a|_{(V \times S U)_{V S X}}) \\
\end{align*}
\]

by Axiom \[20\]

\[
\begin{align*}
 = (p_n + 1 \circ j_{n+1} \circ j_{n+1}^*(t_{n+1,n+1}) \circ (p_n + 1 \circ j_{n+1}^*))(a|_{(V \times S U)_{V S X}}) \\
\end{align*}
\]

by functoriality of push-forwards with respect to twists

\[
\begin{align*}
 = ((p_n + 1 \circ j_{n+1}), \circ (p_n + 1 \circ j_{n+1}^*))(a|_{(V \times S U)_{V S X}}) \\
\end{align*}
\]

by Axiom \[2\] and Lemma \[5.3\] (2)

\[
\begin{align*}
 = a|_{(V \times S U)_{V S X}} \\
\end{align*}
\]

by Axiom \[20\]

Hence

\[
\begin{align*}
 d_{n+1}(b) \\
 = ((j_{n+1}^*)^{-1} \circ (p_n \circ j_n)^* \circ d_n)(a|_{(V \times S U)_{V S X}}) + (-1)^n a|_{(V \times S U)_{V S X}} \\
= (-1)^n a|_{(V \times S U)_{V S X}}. \\
\end{align*}
\]

So the complex is exact after Nisnevich sheafification.

Then by the same proofs as in the \[DF17\] 1.2.7-1.2.11, we have the following result:

**Proposition 5.11.** (1) The forgetful functor $\tilde{\alpha} : SH(S) \rightarrow \tilde{PSh}(S)$ has a left adjoint $\tilde{\alpha}$ such that the following diagram commutes:

\[
\begin{align*}
PSh(S) & \xrightarrow{\tilde{\gamma}_*} \tilde{PSh}(S) \\
\alpha \downarrow & \downarrow \tilde{\alpha} \\
SH(S) & \xrightarrow{\tilde{\gamma}_*} \tilde{SH}(S)
\end{align*}
\]

where $\alpha$ is the Nisnevich sheafication functor with respect to the smooth site over $S$.

(2) The category $\tilde{SH}(S)$ is a Grothendieck abelian category and the functor $\tilde{\alpha}$ is exact.

(3) The functor $\tilde{\gamma}_*$ appearing in the lower line of the preceding diagram, admits a left adjoint $\tilde{\gamma}^*$, and commutes with every limits and colimits.

**Proof.** The same as \[DF17\] Proposition 1.2.11.

\[\square\]

**Definition 5.6.** Given any $X \in Sm/S$, we define $\tilde{\beta}_S(X) = \tilde{\alpha}(\tilde{\alpha}_S(X))$. We denote $\tilde{\beta}_S(S)$ by $\beta_S$.

**Proposition 5.12.** Let $X \in Sm/S$ and $U_1 \cup U_2 = X$ be a Zariski covering. Then the following complex is exact as sheaves with $E$-transfers:

\[
0 \rightarrow \tilde{\beta}_S(U_1 \cap U_2) \rightarrow \tilde{\beta}_S(U_1) \oplus \tilde{\beta}_S(U_2) \rightarrow \tilde{\beta}_S(X) \rightarrow 0.
\]

**Proof.** See \[MVW06\] Proposition 6.14 with use of Proposition \[5.10\]

\[\square\]

**Definition 5.7.** Let $X_i, Y_i \in Sm/S, i = 1, 2$, for any $f_1 \in Cor_S(X_1, Y_1)$, $f_2 \in Cor_S(X_2, Y_2)$.

Define

\[
f_1 \times_S f_2 = p_1^* f_1 \cdot p_2^* f_2 \in Cor(X_1 \times_S X_2, Y_1 \times_S Y_2)
\]

to be the exterior product of $f_1$ and $f_2$, where $p_i : X_1 \times_S X_2 \rightarrow X_1 \times_S Y_1 \times_S Y_2 \rightarrow X_i \times_S Y_i, i = 1, 2$ are the projections. Here we have used the isomorphism $-T_{X_1 \times_S Y_1 \times_S Y_2} \equiv -T_{X_1 \times_S Y_2} \times X_1$.

Now we are going to show that exterior products are compatible with compositions.
**Lemma 5.4.** Let $X_i, Y_i, Z_i \in Sm/S, i = 1, 2$, we have maps $p_{i3}^1 : X_iY_iZ_i \rightarrow X_iZ_i$, $a_i : X_1X_2Y_1Y_2Z_1Z_2 \rightarrow X_1Y_1Z_i$ and $p_{i3}^1 : X_1X_2Y_1Z_2 \rightarrow X_1X_2Z_2$. Suppose $a_i \in E_{C_i}^{2Y_i+\epsilon Z_i}((p_{i3})^*v_1-T_{X_iY_iZ_i}/X_iZ_i)$ where $C_i \in S(Y_iZ_i)$ and $v_i \in S(Y_iZ_i)$. Then we have

\[ b_1^*p_{i3}^1(a_1) \cdot b_2^*p_{i3}^2(a_2) = p_{i3}^1(a_1^*(a_1) \cdot a_2^*(a_2)), \]

where we have used the isomorphism (exchanging the middle two terms and then merging the last two terms) from

\[ a_1^*(p_{i3}^1)^*v_1 - T_{X_iX_2Y_1Y_2Z_1Z_2/X_1X_2Y_1Z_2} + a_2^*(p_{i3}^2)^*v_2 - T_{X_1X_2Y_1Z_2/X_1X_2Z_2} \]

to

\[ p_{i3}^1(b_1^*(v_1) + b_2^*(v_2)) - T_{X_1X_2Y_1Y_2Z_1Z_2/X_1X_2Z_2} \]

in the right hand side.

**Proof.** We have two Cartesian squares

\[
\begin{array}{ccc}
X_1X_2Y_1Y_2Z_1Z_2 & \xrightarrow{p_{25}} & X_1X_2Y_1Z_1Z_2 \\
\downarrow & & \downarrow \\
X_1Y_1Z_1 & \xrightarrow{p_{i3}} & X_1Z_1 \\
\end{array}
\]

and equations $p_{25} = b_2 \circ p_{1245}, p \circ q = a_1$ and $p_{i3} = p_{1245} \circ q$. Then we have

\[ b_1^*p_{i3}^1((a_1^*(p_{i3}^1)^*v_1 - T_{X_iX_2Y_1Z_1Z_2})) \cdot b_2^*p_{i3}^2(a_2) \]

by Axiom 13 for the right square above

\[ = (p_{1245})_\ast (p_1^*(a_1^* \cdot b_2^*p_{i3}^2(a_2)) \]

by Axiom 17 for $p_{1245}$

\[ = (p_{1245})_\ast (p_1^*(a_1^* \cdot p_{25}^*p_{i3}^2(a_2)) \]

by Axiom 9

\[ = (p_{1245})_\ast (p_1^*(a_1^* \cdot q_\ast a_2^2(a_2))) \]

by Axiom 13 for the left square above

\[ = (p_{1245})_\ast (q_\ast (p_1^*(a_1^* \cdot a_2^2(a_2))) \]

by Axiom 17 for $q$

\[ = (p_{1245})_\ast (q_\ast (a_1^*(a_1^* \cdot a_2^2(a_2))) \]

by Axiom 9

\[ = p_{i3}^1(a_1^*(a_1^* \cdot a_2^2(a_2)) \]

by Axiom 12.

\[ \square \]

**Proposition 5.13.** Let $X_i, Y_i, Z_i \in Sm/S, f_i \in \overline{Cor}(X_i, Y_i), g_i \in \overline{Cor}(Y_i, Z_i)$ where $i = 1, 2$. Then

\[ (g_1 \times S g_2) \circ (f_1 \times S f_2) = (g_1 \circ f_1) \times S (g_2 \circ f_2). \]

**Proof.** We have a commutative diagram $(i = 1, 2)$

\[
\begin{array}{ccc}
Y_1Y_2Z_1Z_2 & \xrightarrow{p_{23}} & Y_1Z_1 \\
\downarrow & & \downarrow \\
X_1X_2Y_1Y_2Z_1Z_2 & \xrightarrow{p_{23}} & X_1Z_1 \\
\downarrow & & \downarrow \\
X_1X_2Y_1Y_2Z_1Z_2 & \xrightarrow{p_{23}} & X_1X_2Z_1Z_2 \\
\end{array}
\]

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Then
\[(g_1 \times_S g_2) \circ (f_1 \times_S f_2)\]
\[= \pi_{13} \circ p_{23}((g_1 \times r_1) \cdot (q_1 \times r_2)^* g_2) \cdot \pi_{12}((q_1 \times r_1)^* f_1 \cdot (g_2 \times r_2)^* f_2)\]
by definition
\[= \pi_{13} \cdot (a_1^*(p_{23}^* g_2) \cdot a_1^*(p_{12}^* f_1) \cdot a_2^*(p_{12}^* f_2))\]
by Axiom 10 and Axiom 9
\[= \pi_{13} \cdot (a_1^*(p_{23}^* g_2) \cdot a_1^*(p_{12}^* f_1) \cdot a_2^*(p_{12}^* f_2))\]
by Axiom 10 and Axiom 11
\[= b_1^* p_{13} \cdot ((p_{23}^* g_1) \cdot (p_{12}^* f_1) \cdot b_2^* p_{34} \cdot (p_{23}^* g_2) \cdot (p_{12}^* f_2))\]
by Lemma 5.4
\[= b_1^* (g_1 \cdot f_1) \cdot b_2^* (g_2 \cdot f_2)\]
by definition
\[= (g_1 \cdot f_1) \times_S (g_2 \cdot f_2)\]
by definition.

Here are some basic constructions of presheaves or sheaves with \(E\)-transfers.

For any \(F \in \mathcal{PSh}(S)\) and \(X \in Sm/S\), we define \(F^X \in \mathcal{PSh}(S)\) by \(F^X(Y) = F(X \times_S Y)\). If \(F \in \tilde{Sh}(S)\), then it’s clear that \(F^X \in \tilde{Sh}(S)\) also. We define \(C_* F\) for any \(F \in \tilde{Sh}(S)\) to be a complex with \((C_* F)_n = F^{\Delta^n}\) as in [MVW06, Definition 2.14].

A pointed scheme is a pair \((X, x)\) where \(X \in Sm/S\) and \(x : S \to X\) is a \(S\)-rational point.

For any pointed scheme \((X, x)\), we define
\[\tilde{Z}_S(X, x) = \text{Coker}(\pi_S \xrightarrow{x} \tilde{Z}_S(X)).\]

Then we define \(\tilde{Z}_S(q) = \tilde{Z}_S(G_{m, 1})^{\otimes q}[-q]\) for \(q \geq 0\) and we set \(\tilde{Z}_S = \tilde{Z}_S(0) = \pi_S\). The notation like \(\tilde{Z}_S(q)[2q]\) means \((\tilde{Z}_S(1)[2])^\otimes q\).

The following definitions come from [SV] Lemma 2.1.

**Definition 5.8.** Suppose \(F_i, G, G \in \mathcal{PSh}(S), i = 1, \ldots, n \geq 2\). A multilinear function \(\varphi : F_1 \times_S \cdots \times F_n \to G\) is a collection of multilinear maps of abelian groups
\[\varphi(X_1, \ldots, X_n) : F_1(X_1) \times_S \cdots \times F_n(X_n) \to G(X_1 \times_S \cdots \times X_n)\]
for every \(X_i \in Sm/S\), such that for every \(f \in CorS(X_i, X'_i)\), we have a commutative diagram
\[
\begin{array}{ccc}
\cdots \times F_i(X'_i) \times \varphi(\cdots X'_i \cdots) & G(\cdots \times_S X'_i \times \cdots) \\
\Downarrow \times_S f \\
\cdots \times F_i(X_i) \times \varphi(\cdots X_i \cdots) & G(\cdots \times_S X_i \times \cdots)
\end{array}
\]

**Definition 5.9.** Suppose \(F_i, G \in \mathcal{PSh}(S), i = 1, \ldots, n \geq 2\) (resp. \(\tilde{Sh}(S)\)). The tensor product \(F_1 \circ_S \cdots \circ_S F_n\) (resp. \(F_1 \circ_S \cdots \circ_S F_n\)) is a presheaf (resp. sheaf) with \(E\)-transfers \(G\) such that for any \(H \in \mathcal{PSh}(S)\) (resp. \(\tilde{Sh}(S)\)), we have
\[\text{Hom}(G, H) \cong \{\text{Multilinear functions } F_1 \times_S \cdots \times F_n \to H\}\]
naturally.

For any \(F, G \in \mathcal{PSh}(S)\), we can construct \(F \circ_S G \in \mathcal{PSh}(S)\) as in the discussion before [SV, Lemma 2.1]. It has the universal property above. Moreover, we define \(\text{Hom}_S(F, G)\) to be a presheaf with \(E\)-transfers which sends \(X \in Sm/S\) to \(\text{Hom}_S(F, G^X)\). And if \(F, G\) are sheaves with \(E\)-transfers, we define \(F \circ_S G = \tilde{a}(F \circ_S G)\). If \(G\) is a sheaf with \(E\)-transfers, it’s clear that \(\text{Hom}_S(F, G)\) is also a sheaf with \(E\)-transfers. Finally, it’s clear from definition that \(F \circ_S G \cong G \circ_S F\) and \(F \circ_S G \cong G \circ_S F\).
Proposition 5.14. For any $F, G, H \in \mathcal{PSh}(S)$, we have isomorphisms

\[ \text{Hom}_S(F \otimes^p_S G, H) \cong \text{Hom}_S(F, \text{Hom}_S(G, H)), \]
\[ \text{Hom}_S(F \otimes^p_S G, H) \cong \text{Hom}_S(G, \text{Hom}_S(F, H)) \]

being functorial in three variables. Similarly, for any $F, G, H \in \mathcal{S}$, we have isomorphisms

\[ \text{Hom}_S(F \otimes_S G, H) \cong \text{Hom}_S(F, \text{Hom}_S(G, H)), \]
\[ \text{Hom}_S(F \otimes_S G, H) \cong \text{Hom}_S(G, \text{Hom}_S(F, H)) \]

being functorial in three variables.

Proof. This is clear from the definition of the bilinear map. \hfill \square

Moreover, if we have $F, G, H \in \mathcal{S}$, it’s easy to see by the above proposition that $(F \otimes_S G) \otimes_S H$ and $F \otimes_S (G \otimes_S H)$ are all isomorphic to $F \otimes_S G \otimes_S H$. So the tensor product is associative. And finally one checks that $\otimes_S$ (resp. $\otimes^p_S$) gives $\mathcal{S}$ (resp. $\mathcal{PSh}(S)$) a symmetric closed monoidal structure.

Proposition 5.15. If a morphism $f : F_1 \to F_2$ of presheaves with $E$-transfers becomes an isomorphism after sheafifying, then so does the morphism $f \otimes^p_S G$ for any presheaf with $E$-transfers $G$.

Proof. The condition is equivalent to the map $\text{Hom}_S(f, H)$ is an isomorphism between abelian groups for any sheaf with $E$-transfers $H$. And

\[ \text{Hom}_S(f \otimes^p_S G, H) \cong \text{Hom}_S(f, \text{Hom}_S(G, H)) \]

by the proposition above. \hfill \square

Proposition 5.16. For any $X, Y \in \mathcal{S}_m/S$, we have

\[ \tilde{\mathcal{Z}}_S(X) \otimes_S \tilde{\mathcal{Z}}_S(Y) \cong \tilde{\mathcal{Z}}_S(X \times_S Y) \]

as sheaves with $E$-transfers.

Proof. We have $\tilde{\mathcal{Z}}_S(X) \otimes_S \tilde{\mathcal{Z}}_S(Y) \cong \tilde{\mathcal{Z}}_S(X \times_S Y)$ just by the exterior products of correspondences. Then the statement follows by Proposition 5.15. \hfill \square

Now we are going to prove some functorial properties between sheaves with $E$-transfers over different bases. Our approach is quite similar as in [Dég].

The following lemma is useful when constructing adjunctions, see [Dég 2.5.1].

Lemma 5.5. Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a functor between small categories and $\mathcal{M}$ be a category with arbitrary colimits. Then the functor

\[ \varphi^* : \text{PreShv}(\mathcal{D}, \mathcal{M}) \to \text{PreShv}(\mathcal{C}, \mathcal{M}) \]

(see [Ayo Definition 4.4.1]) defined by $\varphi^*(F) = F \circ \varphi$ has a left adjoint $\varphi^!$.

Proof. Suppose $G \in \text{PreShv}(\mathcal{C}, \mathcal{M})$. For every object $Y \in \mathcal{D}$, define $C_Y$ to be the category whose objects are $\text{Hom}_\mathcal{D}(Y, \varphi X)$ and morphisms from $a_1 : Y \to \varphi X_1$ to $a_2 : Y \to \varphi X_2$ are $b \in \text{Hom}_\mathcal{C}(X_1, X_2)$ such that $a_2 = \varphi(b) \circ a_1$. We have a contravariant functor

\[ \theta_Y : C_Y \to \mathcal{M} \]

defined by $\theta_Y(Y \to \varphi X) = GX$. Then define $(\varphi^* G)Y = \varprojlim \theta_Y$. For any morphism $c : Y_1 \to Y_2$ in $\mathcal{D}$, we define $(\varphi^* G)(c)$ by the following commutative diagram

\[ \begin{array}{ccc}
\theta_{Y_2}(a) & \xrightarrow{\lambda_{a,c}} & \varprojlim \theta_{Y_2}(\varphi^* G)(c) \\
\downarrow_{\lambda_{a,c}} & & \downarrow_{\varprojlim \theta_{Y_1}(c)} \\
\varprojlim \theta_{Y_2}(\varphi^* G)(c) & \xrightarrow{\varprojlim \theta_{Y_1}(c)} & \varprojlim \theta_{Y_1} \\
\end{array} \]

for every $a : Y_2 \to \varphi X$. One checks it is just what we want. \hfill \square
Definition 5.10. Suppose $f : S \rightarrow T$ is a morphism in $Sm/k$. For any $X \in Sm/T$, set $X^S = X \times_T S \in Sm/S$. For any $X_1, X_2 \in Sm/T$, denote by $p_f$ the projection $(X_1 \times_T X_2)^S \rightarrow X_1 \times_T X_2$. Define
\[
\varphi^f : \tilde{\text{Cor}}_T \rightarrow \tilde{\text{Cor}}_S, \\
\quad X \mapsto X^S, \\
\quad g \mapsto g^S,
\]
where $g \mapsto g^S : \tilde{\text{Cor}}_T(X_1, X_2) \rightarrow \tilde{\text{Cor}}_S(X_1^S, X_2^S)$ is the unique map such that the following diagram commutes
\[
\begin{array}{c}
\begin{array}{c}
E_{S^2}^{d_2 - d_1}(X_1 \times_T X_2, \neg T(X_1 \times_T X_2)_{X_1}) \leftarrow E_{S^2}^{d_2 - d_1}((X_1 \times_T X_2)^S, \neg T(X_1 \times_T X_2)^S_{X_1^S}) \rightarrow E_{S^2}^{d_2 - d_1}(X_1 \times_T X_2, \neg T(X_1 \times_T X_2)_{X_1^S})
\end{array}
\end{array}
\]
for any $Z \in \mathcal{A}(X_1, X_2)$.

Proposition 5.17. Suppose $X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} X_3$ are morphisms in $\tilde{\text{Cor}}_T$. Then
\[
(g_2 \circ g_1)^S = g_2^S \circ g_1^S.
\]
So $\varphi^f : \tilde{\text{Cor}}_T \rightarrow \tilde{\text{Cor}}_S$ is indeed a functor.

Proof. We have diagrams
\[
\begin{array}{c}
\begin{array}{c}
X_1 \times_T X_2 \xrightarrow{p_{12}} X_1 \times_T X_3 \\
X_1 \times_T X_3 \xrightarrow{p_{13}} X_1 \times_T X_2 \times_T X_3
\end{array}
\end{array}
\]
and three Cartesian squares
\[
\begin{array}{c}
\begin{array}{c}
X_2 \times_S X_3 \\
X_1 \times_T X_3 \xrightarrow{p_{13}} X_1 \times_T X_2 \times_T X_3 \\
X_1 \times_T X_3 \xrightarrow{p_{13}} X_1 \times_S X_2
\end{array}
\end{array}
\]
Suppose $g_1$ and $g_2$ come from cohomologies with admissible supports. We have
\[
(g_2 \circ g_1)^S = r^* p_{13*}(p_{23}^*(g_2) \cdot p_{12}^*(g_1))
\]
by definition
\[
=q_{13*} t^* (p_{23}^*(g_2) \cdot p_{12}^*(g_1))
\]
by Axiom [10] for the left square above
\[
=q_{13*} (q_{23}^* p_{12}^*(g_2) \cdot q_{12}^* g_1^S)
\]
by Axiom [10] and Axiom [9]
\[
=g_2^S \circ g_1^S
\]
by definition.

And it’s easy to verify that $\tilde{\gamma}(id_Y)^S = \tilde{\gamma}(id_Y^S)$ for any $Y \in Sm/T$. So $\varphi^f$ is a functor. 
\qed
It is straightforward to check that $\varphi_2 \circ f_2 = \varphi_2 \circ \varphi_1$.

**Proposition 5.18.** Suppose $f_i \in \text{Cor}_T(X_i, Y_i)$ where $i = 1, 2$. Then

$$(f_1 \times_T f_2)^S = f_1^S \times_S f_2^S.$$  

**Proof.** This follows from the commutative diagram

![Commutative diagram](image)

**Proposition 5.19.** In the notations above, we have an adjoint pair

$$f^* : \tilde{S}h(T) \rightleftarrows \tilde{S}h(S) : f_*,$$

where $(f_* F)X = F \circ \varphi^f$ for $F \in \tilde{S}h(S)$.

**Proof.** By Lemma [5.5](#), applying to $\varphi^f$, we obtain an adjunction $\tilde{PSh}(T) \rightleftarrows \tilde{PSh}(S)$ and we apply the sheafification functor in Proposition [5.11](#) to get desired result.

Obviously, we have $(f_1 \circ f_2)^* = f_2^* \circ f_1^*$, $(f_1 \circ f_2)_* = f_1_* \circ f_2$.

**Proposition 5.20.** Suppose $f : S \to T$ is a morphism in $Sm/k$.

1. For any $Y \in Sm/T$,

$$f^* \tilde{Z}_T(Y) \cong \tilde{Z}_S(Y \times_T S)$$

as sheaves with $E$-transfers.

2. For any $F \in \tilde{S}h(S)$ and $Y \in Sm/T$,

$$(f_* F)^Y \cong f_* (F^Y \times_S)$$

as sheaves with $E$-transfers.

3. For any $F \in \tilde{S}h(T)$ and $G \in \tilde{S}h(S)$,

$$\text{Hom}_T(F, f_* G) \cong f_* \text{Hom}_S(f^* F, G)$$

as sheaves with $E$-transfers.

4. For any $F, G \in \tilde{S}h(T)$, we have

$$f^* F \otimes_S f^* G \cong f^* (F \otimes_T G)$$

as sheaves with $E$-transfers.

**Proof.**

1. We have

$$\text{Hom}_S(f^* \tilde{Z}_T(Y), -) \cong \text{Hom}_T(\tilde{Z}_T(Y), f_* -) \cong \text{Hom}_S(\tilde{Z}_S(Y \times_T S), -).$$

2. This is because for any $Z \in Sm/T$

$$(f_* F)^Y Z = F((Y \times_T Z) \times_T S) \cong F((Z \times_T S) \times_S (Y \times_T S)) \cong (f_* (F^Y \times_S))Z$$

and Proposition [5.18](#).

3. This is because for any $Y \in Sm/T$,

$$\text{Hom}_T(F, f_* G)Y = \text{Hom}_T(F, (f_* G)^Y) \cong \text{Hom}_T(F, f_* (G^Y \times_S)) \cong \text{Hom}_S(f^* F, G^Y \times_S)$$

by (2)

$$\cong \text{Hom}_S(f^* F, G^Y \times_S) \cong (f_* \text{Hom}_S(f^* F, G))Y$$
This is because for any $H \in \mathcal{S}h(S)$,
\[
\text{Hom}_S(f^* F \otimes_S f^* G, H) \cong \text{Hom}_S(f^* G, \text{Hom}_S(f^* F, H)) \\
\cong \text{Hom}_T(f, \text{Hom}_S(f^* F, H)) \\
\cong \text{Hom}_T(F, \text{Hom}_S(f, f^* F, H)) \\
\text{by (3)} \\
\cong \text{Hom}_T(F \otimes_T f, f^* H) \\
\cong \text{Hom}_S(f^*(F \otimes_T G), H)
\]

\qed

From now on in this chapter, suppose $f : S \rightarrow T$ is a smooth morphism in $\text{Sm}/k$.

**Definition 5.11.** Suppose $X_1, X_2 \in \text{Sm}/S$, we have a closed immersion $q_f : X_1 \times_S X_2 \rightarrow X_1 \times_T X_2$. Define

$\varphi_f : \mathcal{Cor}_S \rightarrow \mathcal{Cor}_T$

$X \mapsto X$

$g \mapsto g_T$

where we see a smooth $S$-scheme as a smooth $T$-scheme via $f$ and $g \mapsto g_T : \mathcal{Cor}_S(X_1, X_2) \rightarrow \mathcal{Cor}_T(X_1, X_2)$ is the unique map such that the following diagram commutes

\[
E_{X_1 - d_S}^d(X_1 \times_S X_2 - T_{X_1 \times_S X_2} / X_1) \xrightarrow{q_f \circ t_f} E_{q_f(Z)}^{d_T}(X_1 \times_T X_2, -T_{X_1 \times_T X_2} / X_1),
\]

for any $Z \in \mathcal{S}(X_1, X_2)$ ($q_f(Z) \in \mathcal{S}(X_1, X_2)$ since $q_f$ is a closed immersion), where $t_f$ is the isomorphism

$-T_{X_1 \times_S X_2} / X_1$

$\rightarrow N_{(X_1 \times_S X_2)/(X_1 \times_T X_2)} - N_{(X_1 \times_S X_2)/(X_1 \times_T X_2)} - T_{X_1 \times_S X_2} / X_1$

$\rightarrow N_{(X_1 \times_S X_2)/(X_1 \times_T X_2)} - q_f^*T_{X_1 \times_T X_2} / X_1$

For convenience, we may denote $T_{X/Y}$ by $T_f$ for smooth $f : X \rightarrow Y$ and denote $N_{X/Y}$ by $N_f$ for a closed immersion $f$ in the following few propositions.

**Proposition 5.21.** Suppose $X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} X_3$ be morphisms in $\mathcal{Cor}_S$, we have

$(g_2 \circ g_1)_T = g_{2T} \circ g_{1T}$.

So $\varphi_f : \mathcal{Cor}_S \rightarrow \mathcal{Cor}_T$ is indeed a functor.

**Proof.** We have Cartesian squares

\[
\begin{array}{c}
X_1 \times_S X_2 \xrightarrow{i} X_1 \times_T X_2, \\
\begin{array}{c}
X_1 \times_S X_2 \\
q_{X_2}
\end{array} \xrightarrow{q_{X_2}} \\
\begin{array}{c}
X_1 \times_S X_2 \times_S X_3 \xrightarrow{\rho_{X_3}} X_1 \times_T (X_2 \times_S X_3) \\
X_1 \times_T (X_2 \times_S X_3) \xrightarrow{q} X_1 \times_T X_2 \times_T X_3, \\
\begin{array}{c}
X_2 \times_S X_3 \\
r
\end{array} \xrightarrow{r_{X_3}} \\
\begin{array}{c}
X_2 \times_S X_3 \\
\begin{array}{c}
X_1 \times_S X_2 \times_S X_3 \\
q_{X_3}
\end{array} \xrightarrow{q_{X_3}} \\
\begin{array}{c}
X_1 \times_T (X_2 \times_S X_3) \xrightarrow{i_{X_3}} X_1 \times_T X_2 \times_T X_3, \\
\begin{array}{c}
X_1 \times_T X_2 \times_T X_3 \\
j
\end{array} \xrightarrow{j_{X_3}} \\
\begin{array}{c}
X_1 \times_T (X_2 \times_S X_3) \xrightarrow{q_{X_3}} \\
\end{array}
\end{array}
\end{array}
\end{array}
\]
Suppose \( g_1 \) and \( g_2 \) come from cohomologies with admissible supports, then we have

\[
g_{2T} \circ g_{1T} = j_*t_f(g_2) \circ i_*t_f(g_1)
\]

by definition

\[
= p_{13} \cdot (p_{23}^* \circ t_f(g_2) \cdot p_{12}^* i_* t_f(g_1))
\]

by definition

\[
= p_{13} \cdot (q_r^* t_f(g_2) \cdot p_{12}^* i_* t_f(g_1))
\]

by Axiom 16 for the second square above

\[
= p_{13} \cdot q_* (r^* t_f(g_2) \cdot q_{12}^* i_* t_f(g_1))
\]

by Axiom 18 for \( q \)

\[
= p_{13} \cdot q_* (r^* t_f(g_2) \cdot q_{12}^* i_* t_f(g_1))
\]

by Axiom 16 for the first square above

\[
= p_{13} \cdot q_* i'_*(i^* r^* t_f(g_2) \cdot q_{12}^* t_f(g_1))
\]

by Axiom 18 for \( i' \)

\[
= p_{13} \cdot q_* i'_*((q_{23}^* t_f(g_2) \cdot q_{12}^* t_f(g_1), i^* N_q - i^* q^* p_{13}^* T_{X_1 \times_T X_3/X_1 + N_k} - i^* q^* T_{p_{13}^*}))
\]

by Axiom 16

\[
= (p_{13} \circ q) \cdot i'_*((q_{23}^* t_f(g_2) \cdot q_{12}^* t_f(g_1), -i^* q^* p_{13}^* T_{X_1 \times_T X_3/X_1 + N_k} - i^* q^* T_{p_{13}^*}))
\]

by Axiom 19 (1) and functoriality of push-forwards with respect to twists

\[
= k \cdot q_{13}^* ((q_{23}^* t_f(g_2) \cdot q_{12}^* t_f(g_1), -q_{13}^* k^* T_{X_1 \times_T X_3/X_1 + q_{13}^* N_k} - t_{q_{13}^*})
\]

by Axiom 19 (3) for the last square above.

Now we have to be more careful. We say a morphism \( f : A + B \to C + D \) in a Picard category contains a switch if there are morphisms \( g : A \to D \) and \( h : B \to C \) such that \( f = c(D,C) \circ (g + h) \). Conversely we say it doesn’t contains a switch if there are morphisms \( g : A \to C \) and \( h : B \to D \) such that \( f = g + h \). There is a commutative diagram with three...
squares being Cartesian

\[
\begin{array}{c}
X_1 \times_S X_2 \times_S X_3 \\
\downarrow q' \downarrow u \downarrow \downarrow i
\end{array}
\rightarrow \begin{array}{c}
(X_1 \times_S X_2) \times_T X_3 \\
\downarrow \downarrow \downarrow v
\end{array}
\rightarrow \begin{array}{c}
X_1 \times_T (X_2 \times_S X_3) \\
\downarrow q \downarrow \downarrow \downarrow i
\end{array}
\rightarrow \begin{array}{c}
X_1 \times_T X_2 \times_T X_3.
\end{array}
\]

This induces a commutative diagram (all arrows contain a switch)

\[
\begin{array}{c}
N_{q'} + q'^* N_{q'\prime} \\
\downarrow \downarrow \downarrow \downarrow N_{q''} + i'^* N_q
\end{array}
\rightarrow \begin{array}{c}
N_u + u^* N_v \\
\downarrow \downarrow \downarrow \downarrow u^* N_w
\end{array}
\]

since they all come from exact sequences related to \( N_{q\prime\prime} \circ q' = N_{q\prime \circ q} \). Then we have a commutative diagram (none arrow contains a switch except \( \varphi \))

\[
\begin{array}{c}
q'^* N_{q'\prime} + i'^* N_q \\
\downarrow \downarrow \downarrow \downarrow i'^* N_q + u^* N_w
\end{array}
\rightarrow \begin{array}{c}
N_u + N_{q'\prime} \\
\downarrow \downarrow \downarrow \downarrow N_u + u^* N_v
\end{array}
\rightarrow \begin{array}{c}
i'^* N_q + u^* N_v \\
\downarrow \downarrow \downarrow \downarrow i'^* N_q + N_{q'\prime}
\end{array}
\]

by the diagram above. Hence the composition among morphisms without switch

\[
q'^* N_{q'\prime} + i'^* N_q \rightarrow N_u + N_{q'\prime} \rightarrow i'^* N_q + u^* N_v
\]

is equal to the morphism with a switch

\[
q'^* N_{q'\prime} + i'^* N_q \rightarrow i'^* N_q + u^* N_v
\]

where the morphism \( q'^* N_{q'\prime} \rightarrow u^* N_v \) is given by the composition

\[
q'^* N_{q'\prime} \rightarrow N_{q'\prime} \rightarrow u^* N_v.
\]

So the composition among morphisms without switch

\[
q_{23}' N_i + q_{12}' N_i \rightarrow N_{q'\prime} + N_u \rightarrow q_{13}' N_k + i'^* N_q
\]

is equal to the morphism with a switch

\[
q_{23}' N_j + q_{12}' N_i \rightarrow q_{13}' N_k + i'^* N_q
\]

where the morphism \( q_{12}' N_i \rightarrow q_{13}' N_k \) is given by the composition

\[
q_{12}' N_i \rightarrow N_{q'\prime} \rightarrow q_{13}' N_k
\]

and the morphism \( q_{23}' N_j \rightarrow i'^* N_q \) is got by pulling back the morphism

\[
r'^* N_j \rightarrow N_q
\]

along \( i' \).

Moreover, there are commutative diagrams with squares being Cartesian

\[
\begin{array}{c}
X_1 \times_S X_2 \times_S X_3 \\
\downarrow q_{12} \downarrow u \downarrow \downarrow i
\end{array}
\rightarrow \begin{array}{c}
(X_1 \times_S X_3) \times_T X_2 \\
\downarrow \downarrow \downarrow \downarrow v
\end{array}
\rightarrow \begin{array}{c}
X_1 \times_T (X_2 \times_S X_3) \\
\downarrow q \downarrow \downarrow \downarrow i
\end{array}
\rightarrow \begin{array}{c}
X_1 \times_T X_3 \times_T X_2 \\
\downarrow \downarrow \downarrow \downarrow i
\end{array}
\rightarrow \begin{array}{c}
X_1 \times_T X_3 \\
\downarrow \downarrow \downarrow \downarrow i
\end{array}
\rightarrow \begin{array}{c}
X_1 \times_T X_3.
\end{array}
\]
which induce commutative diagrams where none right-hand vertical map contains a switch

\[
\begin{array}{cccc}
X_2 \times_S X_3 & \xrightarrow{j} & X_2 \times_T X_4 & \xrightarrow{k} X_1 \\
q_{13} & \downarrow & \downarrow & \downarrow \\
X_1 \times_S X_2 \times_S X_3 & \xrightarrow{q'} & (X_1 \times_S X_2) \times_T X_3 & \xrightarrow{k} X_1 \times_S X_2 \\
q_{13} & \downarrow & \downarrow & \downarrow \\
X_1 \times_S X_3 & \rightarrow & X_1 \times_T X_4 & \rightarrow X_1 \\
\end{array}
\]

From all these calculations above together with functoriality of \(q_{13}^*\) with respect to twists, we see that

\[
k_s q_{13}^* (q_{23}^* f (g_2) \cdot q_{12}^* f (g_1)) = k_s f (q_{13}^* (q_{23}^* (g_2) \cdot q_{12}^* (g_1)))
\]

by definition.

Finally, we have to show that \((id_X)_T = id_X\) for any \(X \in Sm/S\). We have the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta_s} & X \times_S X \\
\downarrow \triangle_T & \downarrow q & \downarrow \triangle_T \\
\hat{X} \times_T X & \rightarrow & X
\end{array}
\]

where \(\Delta\) means diagonal map. We have to show that the following diagram commutes

\[
\begin{array}{ccc}
N_{\Delta_s} - N_{\Delta_s} & \xrightarrow{\Delta_s^*} & \Delta_s^* T_{X \times_S X/X} \\
\downarrow & \downarrow & \downarrow \iota_T \\
0 & \xrightarrow{N_{\Delta_s} + \Delta_s^* (N_{\Delta_T} - q_{12}^* T_{X \times_T X/X})} & N_{\Delta_T} - q_{12}^* T_{X \times_T X/X} \\
\downarrow & \downarrow & \downarrow q_{12}^* \\
N_{\Delta_T} - N_{\Delta_T} & \xrightarrow{\Delta_T^*} & \Delta_T^* T_{X \times_T X/X} \\
\end{array}
\]

The right two squares come from functoriality of push-forwards with respect to twists and Axiom \([4]\). The left square comes from the following commutative diagram with exact rows

\[
\begin{array}{cccc}
0 & \rightarrow & N_{\Delta_s} & \rightarrow N_{\Delta_T} & \rightarrow \Delta_s^* N_{\Delta_T} & \rightarrow 0 \\
\downarrow \equiv & & \downarrow \equiv & & \downarrow id \\
0 & \rightarrow \Delta_s^* T_{X \times_S X/X} & \rightarrow \Delta_T^* T_{X \times_T X/X} & \rightarrow \Delta_s^* N_{\Delta_T} & \rightarrow 0
\end{array}
\]
By using Axiom [14] it is straightforward to check that $\varphi_{f_1 \circ f_2} = \varphi_{f_1} \circ \varphi_{f_2}$.

**Proposition 5.22.** Suppose $a \in \widetilde{\text{Cor}_S}(X_1, X_2)$ and $b \in \widetilde{\text{Cor}_T}(Y_1, Y_2)$. Identifying $(X_1 \times_S X_2) \times_T Y_1 \times_T Y_2$ with $X_1 \times_S X_2 \times_S (Y_1 \times_T Y_2)^S$ and $X_1 \times_T Y_1 \times_T Y_2 \times_T Y_2$ with $(X_1 \times_S Y_1^S) \times_T (X_2 \times_S Y_2^S)$, we have

$$a_{T} \times_T b = (a \times_S b^S)_{T}.$$  

**Proof.** We have a commutative diagram with the square being Cartesian

```
\begin{equation}
\begin{array}{c}
(X_1 \times_S X_2) \times_T Y_1 \times_T Y_2 \\
\downarrow p_1 & \downarrow & \downarrow q_1 \\
X_1 \times_T X_2 & \to & Y_1 \times_T Y_2 \\
\end{array}
\end{equation}
```

Suppose $a, b$ come from cohomologies with supports. Denote the isomorphism

$$-q_1^*T_{X_1 \times_T X_2/x_1} - q_2^*T_{Y_1 \times_T Y_2/y_1} \longrightarrow -T_{X_1 \times_T X_2 \times_T Y_1 \times_T Y_2}$$

by $\theta$ and the isomorphism

$$-p_1^*T_{X_1 \times_S X_2/x_1} - p_2^*T_{Y_1 \times_T Y_2/y_1} \longrightarrow -T_{X_1 \times_S X_2 \times_S Y_1^S \times_S Y_2^S}$$

by $\eta$. Then

$$a_{T} \times_T b = \theta(q_1^*t_*(t_f(a)) \cdot q_2^*\bar{b})$$

by definition

$$= \theta(r_*p_1^*(t_f(a)) \cdot q_2^*\bar{b})$$

by Axiom [10] for the square in the diagram

$$= \theta(r_*(p_1^*(t_f(a)) \cdot p_2^*b))$$

by Proposition [13] for $r$ and Axiom [10]

$$= r_*r^*(\theta((p_1^*(t_f(a)) \cdot p_2^*\bar{b}, p_1^*N_i - p_1^*t^*T_{X_1 \times_T X_2/x_1} - p_2^*T_{Y_1 \times_T Y_2/y_1})))$$

by functoriality of push-forwards with respect to twists

$$= r_*(t_f(\eta(p_1^*(a) \cdot p_2^*\bar{b})))$$

$$= (a \times_S b^S)_{T}$$

by definition.

Here the fifth equality comes from the following commutative diagram with exact rows and columns

```
\begin{equation}
\begin{array}{cccccccc}
& & & & 0 & & & \\
& & & & \downarrow & & & \\
& & & p_2^*T_{Y_1 \times_T Y_2/y_1} & & = & p_2^*T_{Y_1 \times_T Y_2/y_1} & \\
& & & \downarrow & & & \downarrow & \\
& & 0 & & \to & & 0 & \\
& \to & r^*T_{X_1 \times_T X_2/y_1} & & \to & T_{X_1 \times_T Y_1 \times_T Y_2} & & N_f & \to 0 \\
& \downarrow & & \downarrow & & \approx & & \downarrow & \\
& & 0 & & \to & p_1^*T_{X_1 \times_T X_2/x_1} & & p_1^*N_i & \to 0 \\
& \downarrow & & \downarrow & & & \downarrow & & \\
& & 0 & & 0 & & 0 & \\
\end{array}
\end{equation}
```

and Theorem [2.4] (1).

By the same proof as in Proposition [5.19] applying to $\varphi_f$, we have the following:
Proposition 5.23. There is an adjoint pair
\[ f_{\#} : \tilde{S}h(S) \cong \tilde{S}h(T) : (f_{\#})', \]
where \((f_{\#})'F = F \circ \varphi_f\) for \(F \in \tilde{S}h(T)\).

The next lemma is important when identifying \((f_{\#})'\). See also [MVW06, Exercise 1.12].

Lemma 5.6. For any \(U \in Sm/S, X \in Sm/T\), we have an adjoint pair:
\[ \tilde{Cor}_S(U, X^S) = \tilde{Cor}_T(U, X). \]

Proof. For any \(U \in Sm/S, X \in Sm/T\), we have an isomorphism
\[ \theta_{U, X} : U \times_S X^S \to U \times_T X. \]
Then define
\[ \lambda_{U, X} : \tilde{Cor}_S(U, X^S) \to \tilde{Cor}_T(U, X), \]
which is obviously an isomorphism.

Now suppose we have \(U \in Sm/S, X_1, X_2 \in Sm/T, V \in \tilde{Cor}_T(X_1, X_2), W \in \tilde{Cor}_S(U, X_1^S)\), we want to show that
\[ \lambda_{U, X_2}(V^S \circ W) = V \circ \lambda_{U, X_1}(W). \]

We have a commutative diagram
\[
\begin{array}{cccc}
X_1 \times_T X_2 & \overset{q_{13}}{\longrightarrow} & U \times_T X_1, \\
\downarrow{p} & & \\
U \times_S (X_1^S \times_S X_2^S) & \overset{p_{12} \cdot \theta_{U, X_2}}{\longrightarrow} & U \times_S X_1^S & \overset{\theta_{U, X_1}}{\cong} \\
\end{array}
\]

where we’ve identified \(U \times_S (X_1^S \times_S X_2^S)\) with \(U \times_T X_1 \times_T X_2\) for convenience.

Suppose \(V\) and \(W\) come from some elements of Chow-Witt groups with support (see Definition 5.2). We have
\[
\lambda_{U, X_2}(V^S \circ W) = \theta_{U, X_2} p_{13} (p_{23}^* p^* V \cdot p_{12}^* W)
\]
by definition
\[ = q_{23} (q_{13}^* q_{23}^* V \cdot q_{13}^* p_{12}^* W) \]
by Axiom [12]
\[ = q_{13} (q_{23}^* V \cdot q_{13}^* p_{12}^* \theta_{U, X_1} W) \]
by Axiom [9]
\[ = V \circ \lambda_{U, X_1}(W) \]
by definition.

Then suppose we have \(U_1, U_2 \in Sm/S, X \in Sm/T, V \in \tilde{Cor}_S(U_1, U_2), W \in \tilde{Cor}_S(U_2, X^S)\), we want to show that
\[ \lambda_{U_1, X}(W \circ V) = \lambda_{U_2, X}(W) \circ V_T. \]

We have a commutative diagram
where we have identified $U_1 \times_T (U_2 \times_S X^S)$ with $U_1 \times_T U_2 \times_T X$. Suppose $V$ and $W$ come from some elements of Chow-Witt groups with support like above and define $\theta$ to be the isomorphism
\[-T_{U_1 \times_T U_2 \times_T X^S}/U_1, X \rightarrow \alpha \ast T_{U_1 \times_T (U_2 \times_S X^S)}/U_1 \times_T X.
\]
We have
\[
\lambda_{U_1, X}(W \circ V) \\
= \vartheta_{U_1, X \times_P \varphi_{U_1}}((p_{23}^* W \cdot p_{12}^* V, -T_{U_1 \times_T U_2 \times_T X^S} - p_{12}^* T_{U_1 \times_S U_2 / U_1})) \\
\text{by definition} \\
= q_{13} \ast a_\ast ((a_\ast q_{23}^* \theta_{U_2, X \times_T X} W \cdot \theta(p_{12}^* V), -a_\ast q_{23}^* T_{U_2 \times_T X^S}/U_2 \times_T X^S) - \lambda_{U_1, X}(V, W)) \\
\text{by Axiom 20 and Axiom 19, (1)} \\
= q_{13} \ast ((q_{23}^* \theta_{U_2, X \times_T X} W \cdot a_\ast (p_{12}^* V), -q_{23}^* T_{U_2 \times_T X^S} / U_2 \times_T X^S) - \lambda_{U_2, X}(V, W)) \\
\text{by Axiom 16 for the leftmost square in the diagram above} \\
= \lambda_{U_2, X}(W) \circ V_T \\
\text{by definition} \\
\]
\[
\square
\]

**Proposition 5.24.**

\[(f_\#)' = f^* .\]

**Proof.** This is because for any $Y \in Sm/S$, $\gamma(id_Y) \in \overline{Cor}_T(Y, Y) = \overline{Cor}_S(Y, Y^S)$ is the initial element of $C_S$ in Lemma 5.5 by the lemma above applying to $\varphi'$ (see Definition 5.10). So for any $F \in \overline{PSh}(T)$, we have $(f^* F)Y = FY = ((f_\#)' F)Y$. And this gives an isomorphism between $f^* F$ and $(f_\#)'F$ for any presheaf with E-transfers $F$ by the lemma above. So it also gives an isomorphism after sheafication. $\square$

**Proposition 5.25.** Suppose $f : S \rightarrow T$ is smooth as before.

1. For any $X \in Sm/S$, we have
\[f_\# \mathbb{Z}_S(X) \cong \mathbb{Z}_T(X).\]
2. For any $F \in \overline{Sh}(T)$ and $Y \in Sm/T$
\[f^*(FY) \cong (f^* F)Y \times_T S\]
as sheaves with E-transfers.
3. For any $F \in \overline{Sh}(S)$ and $G \in \overline{Sh}(T)$
\[Hom_T(f_\# F, H) \cong f_\# Hom_S(F, f^* H)\]
as sheaves with E-transfers.
4. For any $F \in \overline{Sh}(S)$ and $G \in \overline{Sh}(T)$
\[f_\#(F \otimes \mathcal{S} f^* G) \cong f_\# F \otimes_T G\]
as sheaves with E-transfers.

**Proof.**

1. This is because for any $F \in \overline{Sh}(T)$
\[Hom_T(f_\# \mathbb{Z}_S(X), F) \cong Hom_S(\mathbb{Z}_S(X), f^* F) \cong (f^* F)(X) \cong F(X)\]
by the proposition before.
2. This is because for every $X \in Sm/S$
\[(f^*(HY))(X) = H(Y \times_T X)\]
and
\[(f^* H)Y \times_T S = H((Y \times_T S) \times_S X)\]
by the proposition before. Then one uses Proposition 5.22.
(3) For any $Y \in \text{Sm}/T$

\[
\text{Hom}_T(f_\# F, H) Y = \text{Hom}_T(f_\# F, H^Y) \\
\cong \text{Hom}_S(F, f^*(H^Y)) \\
\cong \text{Hom}_S(F, (f^* H)^{Y \times T S}) \\
\text{by (2)} \\
\cong \text{Hom}_S(F, f^*(H))(Y \times T S) \\
\cong (f_* \text{Hom}_S(F, f^* H)) Y.
\]

(4) This is because for any $H \in \tilde{\text{Sh}}(T)$, we have

\[
\text{Hom}_T(f_\# (F \otimes_S f^* G), H) \cong \text{Hom}_S(f \otimes_S f^* G, f^* H) \\
\cong \text{Hom}_S(f^* G, \text{Hom}_S(F, f^* H)) \\
\cong \text{Hom}_T(G, \text{Hom}_T(f_\# F, H)) \\
\text{by (3)} \\
\cong \text{Hom}_T(f_\# F \otimes_T G, H).
\]

\[\square\]

6. Operations on Localized Categories

We are going to establish the theory of effective (resp. stabilized) motives on bounded above complexes (See [MVW06]) of sheaves of $E$-transfers (resp. symmetric spectra). And we will compare our theory with respect to that of [CD07], [CD13] and [DF17], which uses unbounded complexes.

6.1. For Sheaves with $E$-Transfers.

6.1.1. On Derived Categories. Denote by $D^- (S)$ (resp. $K^- (S)$) the derived (resp. homotopy) category of bounded above complexes in $\tilde{\text{Sh}}(S)$. We are going to define $\otimes_S$ and $f_\#$ (See Section 5) over those categories. The method is inherited from [SV, Corollary 2.2] and [MVW06, Lemma 8.15].

**Definition 6.1.** We call a presheaf with $E$-transfers free if it’s a direct sum of presheaves of the form $\tilde{c}_S(X)$. We call a presheaf with $E$-transfers projective if it’s a direct summand of a free presheaf with $E$-transfers. A sheaf with $E$-transfers is called free (resp. projective) if all its terms are free (projective).

**Definition 6.2.** A projective resolution of a bounded above sheaf complex $K$ is a projective complex with a quasi-isomorphism $P \rightarrow K$. If $K$ is already projective, we take $P = K$.

Now let $Y \in \text{Sm}/k$ be an $S$-scheme and $Y \in \text{Sm}/T$. Consider in this section the functors

\[
\varphi : \tilde{\text{Cor}}_S \rightarrow \tilde{\text{Cor}}_T \\
X \mapsto (X^Y)_T \cong X \times_S Y
\]

and

\[
\psi : \text{Sm}_S \rightarrow \text{Sm}_T \\
X \mapsto X \times_S Y
\]

determined by the triple $(Y, S, T)$. We have a commutative diagram

\[
\begin{array}{ccc}
\text{Sm}/S & \xrightarrow{\psi} & \text{Sm}/T \\
\tilde{\text{Cor}}_S \downarrow \varphi \downarrow \tilde{\text{Cor}}_T
\end{array}
\]

Recall from Lemma 5.3 the definition of $\varphi^*$ and $\varphi_*$.

**Lemma 6.1.** For any $X \in \text{Sm}/S$,

\[
\varphi^*(\tilde{c}_S(X)) \cong \tilde{c}_S(\psi X)
\]

as presheaves with $E$-transfers.
Proof. For any $F \in \widetilde{PSh}(T)$, 
$$\text{Hom}_T(\varphi^*(\tilde{c}_S(X)), F) \cong \text{Hom}_S(\tilde{c}_S(X), \varphi_*F) \cong F(\psi X).$$ 

\[\square\]

Lemma 6.2. The functor $\varphi_*$ maps sheaves with $E$-transfers to sheaves with $E$-transfers.

Proof. It suffices to show that for any finite Nisnevich covering $\{U_i\}$ of $X \in Sm/S$, the following sequence is exact

$$0 \rightarrow G(X) \rightarrow \oplus_i G(U_i) \rightarrow \oplus_{i,j} G(U_i \times X U_j)$$

where $G = \varphi_*F$ for some $F \in \widetilde{Sh}(T)$. And this follows easily. \[\square\]

Lemma 6.3. Let $f : F \rightarrow G$ be morphism in $\widetilde{PSh}(S)$ such that $\tilde{a}(f)$ is an isomorphism, then $\tilde{a}(\varphi^*(f))$ is also an isomorphism.

Proof. This follows from a similar method as in Proposition 5.15. \[\square\]

Proposition 6.1. For any $F \in \widetilde{PSh}(S)$,

$$\tilde{a}(L_{i}\varphi^* \tilde{a}(F)) \cong \tilde{a}(L_{i}\varphi^* F)$$

as sheaves with $E$-transfers for any $i \geq 0$, where $L_i\varphi^*$ means the $i$th left derived functor of $\varphi^*$.

Proof. We show at first that for any presheaf with $E$-transfers $F$ with $\tilde{a}(F) = 0$

$$\tilde{a}(L_{i}\varphi^* (F)) = 0$$

for any $i \geq 0$. Then for any presheaf with $E$-transfers $F$, denote by $\theta$ the natural map $F \rightarrow \tilde{a}(F)$. We have

$$\tilde{a}(\text{coker}(\theta)) = \tilde{a}(\text{ker}(\theta)) = 0.$$

Hence for any $i \geq 0$, we have

$$\tilde{a}(L_{i}\varphi^* \tilde{a}(F)) \cong \tilde{a}(L_{i}\varphi^* \text{Im}(\theta)) \cong \tilde{a}(L_{i}\varphi^* F)$$

by using long exact sequences. Hence the statement follows.

Now we prove the claim. We do induction on $i$. The claim is true for $i = 0$ and suppose it’s true for $i < n$.

For any $F \in \widetilde{PSh}(S)$, we have a surjection

$$\oplus_{a \in F(X)} \tilde{c}_S(X) \rightarrow F$$

defined by each section of $F$ on each $X \in Sm/S$. Since $\tilde{a}(F) = 0$, for any $a \in F(X), X \in Sm/S$, there is a finite Nisnevich covering $U_a \rightarrow X$ of $X$ such that $a|_{U_a} = 0$. So the composition

$$\oplus_{a \in F(X)} \tilde{c}_S(U_a) \rightarrow \oplus_{a \in F(X)} \tilde{c}_S(X) \rightarrow F$$

is zero. Then we have got a surjection

$$\oplus_{a \in F(X)} H_0(\tilde{C}(U_a/X)) \rightarrow F$$

with kernel $K$. Proposition 5.10 implies that

$$\tilde{a}(H_p(\tilde{C}(U/X))) = 0$$

for any Nisnevich covering $U \rightarrow X$ and $p \in \mathbb{Z}$. So $\tilde{a}(K) = 0$ also. We have a hypercohomology spectral sequence

$$(L_p\varphi^*)H_q(\tilde{C}(U/X)) \Rightarrow (\bigoplus_{p+q} \varphi^*) \tilde{C}(U/X).$$

Hence

$$\tilde{a}((L_n\varphi^*) \tilde{C}(U/X)) \cong \tilde{a}((L_n\varphi^*) H_0(\tilde{C}(U/X)))$$

by induction hypothesis. But

$$\tilde{a}((L_n\varphi^*) \tilde{C}(U/X)) \cong \tilde{a}(H_n(\varphi^* \tilde{C}(U/X)))$$

by definition of hypercohomology and the latter one vanishes since we have

$$\varphi^* \tilde{C}(U/X) = \tilde{C}(\psi U/\psi X)$$

by previous lemmas. So

$$\tilde{a}((L_n\varphi^*)H_0(\tilde{C}(U/X))) = 0.$$

So

$$\tilde{a}(L_n\varphi^* F) \cong \tilde{a}(L_{n-1}\varphi^* K) = 0$$

by long exact sequence and induction hypothesis. \[\square\]
Proposition 6.2. Let functor $\varphi^*$ takes acyclic projective complexes to acyclic projective complexes.

Proof. For any projective $F \in \widehat{Sh}(S)$, $F = \widehat{a}(G)$ for some projective $G \in \widehat{PSh}(S)$ by definition. So

$$\widehat{a}((L_i \varphi^*)F) \cong \widehat{a}((L_i \varphi^*)G) = 0$$

for any $i > 0$ by the proposition above. Now given a short exact sequence of sheaves with $E$-transfers

$$0 \to K \to F \to P \to 0$$

with $\widehat{a}((L_i \varphi^*)P) = 0$ for any $i > 0$. Then the sequence is still exact as sheaves with $E$-transfers after applying $\varphi^*$ by long exact sequence. Then the statement follows easily. \hfill \Box

Proposition 6.3. We have an exact functor

$$L\varphi^* : D^- (S) \to D^- (T)$$

which maps any $K \in D^- (S)$ to $\varphi^* P$, where $P$ is a projective resolution $K$.

Proof. By the proposition above, the class of projective complexes is adapted (see [GM03 III.6.3]) to the functor $\varphi^*$. Now apply [GM03 III.6.6]. \hfill \Box

We will just write $L\varphi^*$ above by $\varphi^*$ for convenience.

Now we apply the general results above to $\otimes_S, f_#$ and $f^*$.

Proposition 6.4. (1) There is a tensor product

$$\otimes_S : \ D^- (S) \times \ D^- (S) \to \ D^- (S)$$

$$(K, L) \to P \otimes_S Q$$

where $P, Q$ are projective resolutions of $K, L$, respectively and the last tensor means taking the total complex of the bicomplex $\{P \otimes_S Q_i\}$. And for any $K \in D^- (S)$, the functor $K \otimes_s -$ is exact.

(2) Suppose $f : S \to T$ is a smooth morphism in $Sm/k$. There is an exact functor

$$f_# : \ D^- (S) \to \ D^- (T)$$

$$(K) \to f_# P$$

where $P$ is a projective resolution of $K$.

(3) Suppose $f : S \to T$ is a morphism in $Sm/k$. There is an exact functor

$$f^* : \ D^- (T) \to \ D^- (S)$$

$$(K) \to f^* P$$

where $P$ is a projective resolution of $K$.

Proof. (1) Suppose $Y \in Sm/S$. In the definition of $\varphi$, we take $(Y, S, T) := (Y, S, S)$. Then $\varphi^* F \cong F \otimes_S \mathcal{Z}_S (Y)$ for any $F \in \widehat{Sh}(S)$ by Proposition 5.13.

Now given acyclic projective complex $P$ and a projective sheaf $F$. $F \otimes_S P$ is also acyclic by applying Proposition 6.2 to $\varphi$ and definition of projectiveness. So for any projective complex $K$, the complex $P \otimes_S K$ is also acyclic by the spectral sequence of the bicomplex $\{P \otimes_S K_i\}$. Then for any projective complexes $P, Q, R$ and quasi-isomorphism $\alpha : P \to Q$, the morphism $\alpha \otimes_S R$ is still a quasi-isomorphism since we have

$$Cone (\alpha \otimes_S R) \cong Cone (\alpha) \otimes_S R$$

and the latter one is acyclic. So the statement follows easily.

(2) In the definition of $\varphi$, we take $(Y, S, T) := (S, S, T)$ and apply Proposition 6.3.

(3) In the definition of $\varphi$, we take $(Y, S, T) := (T, S, T)$ and apply Proposition 6.3. \hfill \Box

Proposition 6.5. Suppose $f : S \to T$ is a smooth morphism in $Sm/k$, we have an adjoint pair

$$f_# : D^- (S) \to D^- (T) : f^*$$.
Proof. By Proposition 5.23 it is easy to see that there is an adjunction
\[ f^\# : K^- (S) \rightleftharpoons K^- (T) : f^\ast. \]

Since \( f^\ast : \widetilde{Sh}(T) \rightarrow \widetilde{Sh}(S) \) has both a left adjoint and a right adjoint, it's an exact functor. So \( LF^\ast \cong f^\ast \) in this case. Suppose \( K \in D^- (S), L \in D^- (T) \) and \( p : P \rightarrow K \) be a projective resolution of \( K \). Hence \( f^\# K = f^\# P \) by definition.

We construct a morphism
\[ \theta : Hom_{D^- (S)} (f^\# K, L) \rightarrow Hom_{D^- (T)} (K, f^\ast L) \]
by the following: Suppose \( s \in Hom_{D^- (S)} (f^\# K, L) \) is written as a right roof (see [GM03 III.2.9])
\[ R \]
\[ \begin{array}{c} \downarrow a \cr f^\# P \end{array} \quad \begin{array}{c} \downarrow b \cr L \end{array} \]

By adjunction, \( a \) induces a morphism \( a' : P \rightarrow f^\ast R \). Then we define \( \theta(s) \) to the composition of the right roof
\[ f^\ast R \]
\[ \begin{array}{c} \downarrow a' \cr P \end{array} \quad \begin{array}{c} \downarrow f^\ast b \cr f^\ast L \end{array} \]
with \( p^{-1} \). This definition is well-defined since \( f^\ast \) is exact.

Then, we construct another morphism
\[ \xi : Hom_{D^- (T)} (K, f^\ast L) \rightarrow Hom_{D^- (S)} (f^\# K, L) \]
by the following: Suppose \( t \in Hom_{D^- (T)} (K, f^\ast L) \) and \( t \circ p \) is written as a left roof (see [GM03 III.2.8])
\[ R \]
\[ \begin{array}{c} \downarrow a \cr P \end{array} \quad \begin{array}{c} \downarrow b \cr f^\ast L \end{array} \]
where \( R \) is also projective. By adjunction, \( b \) induces a morphism \( b' : f^\# R \rightarrow L \). Then we define \( \xi(t) \) to be the left roof
\[ f^\# R \]
\[ \begin{array}{c} \downarrow f^\# a \cr f^\# P \end{array} \quad \begin{array}{c} \downarrow b' \cr L \end{array} \]
This definition is well-defined by Proposition 6.2 applied to \( f^\# \).

Finally one checks that \( \theta \) and \( \xi \) are inverse to each other and the statement follows. \( \square \)

In [CD07, Theorem 1.7], they defined a model structure \( \mathfrak{M} \) on the category of unbounded complexes of sheaves with \( E \)-transfers over \( S \). This is a cofibrantly generated model structure where the cofibrations are those \( I \)-cofibrations (See [Hov07 Definition 2.1.7]) where \( I \) consists of the morphisms \( S^{n+1}Z_S(X) \rightarrow D^nZ_S(X) \) for any \( X \in Sm/S \) (See [CD07 1.9] for notations) and weak equivalences are quasi-morphisms between complexes.

Proposition 6.6. Bounded above projective complexes are cofibrant objects in \( \mathfrak{M} \).

Proof. Suppose \( P \) is a bounded above projective complex and we have an \( I \)-injective (See [Hov07 Definition 2.1.7]) morphism \( f : A \rightarrow B \) between unbounded complexes with a morphism \( g : P \rightarrow B \). We have to show that \( g = f \circ h \) for some \( h : P \rightarrow A \).

Now we construct \( h \) by induction. Suppose for any \( m \geq n \), we have constructed a morphism \( h^m : P^m \rightarrow A^m \) such that \( g^m = f^m \circ h^m \) and \( d^A \circ h^m = h^m \circ d^P \). Since \( P \) is bounded above,
this could be done when $n$ large enough. We are going to construct an $h^{n-1} : P^{n-1} \rightarrow A^{n-1}$ satisfying the same property, that is, making the following diagram commute

\[
\begin{array}{ccc}
A^{n-1} & \xrightarrow{d^n} & A^n \\
\downarrow{f^{n-1}} & & \downarrow{h^n} \\
P^{n-1} & \xrightarrow{d^n} & P^n \\
\downarrow{g^{n-1}} & & \downarrow{u^n} \\
B^{n-1} & \xrightarrow{h^{n-1}} & B^n \\
\end{array}
\]

We have a splitting surjection $F \twoheadrightarrow P^{n-1}$ where $F$ is a free sheaf with $E$-transfers. So we may assume $P^{n-1}$ is free and it’s equal to $\oplus_i \tilde{Z}_S(X_i)$ where $X_i \in Sm/S$. For every $i$, we have two morphisms:

\[ u : \tilde{Z}_S(X_i) \rightarrow P^{n-1} \rightarrow B^{n-1} \rightarrow B^n \]

and

\[ v : \tilde{Z}_S(X_i) \rightarrow P^{n-1} \rightarrow P^n \rightarrow A^n. \]

And this two morphisms gives a commutative square with a lifting since $f$ is $I$-injective:

\[
\begin{array}{ccc}
S^n \tilde{Z}_S(X_i) & \xrightarrow{v} & A \\
\downarrow{w_i} & & \downarrow{f} \\
D^{n-1} \tilde{Z}_S(X_i) & \xrightarrow{u} & B \\
\end{array}
\]

And one checks that $\oplus_i w_i : P^{n-1} \rightarrow A^{n-1}$ is just what we want. \hfill \square

Moreover, $\mathfrak{M}$ is stable and left proper so it induces a triangulated structure $\mathfrak{T}'$ on $D(S)$ (See [Ayo, Theoreme 4.1.49]). The classical triangulated structure of $D(S)$ or $D^{-}(S)$ is denoted by $\mathfrak{T}$.

**Proposition 6.7.** The natural functor

\[ i : (D^{-}(S), \mathfrak{T}) \rightarrow (D(S), \mathfrak{T}') \]

is fully faithful exact.

**Proof.** Any distinguished triangle $T$ in $(D^{-}(S), \mathfrak{T})$ is isomorphic in $D^{-}(S)$ to a distinguished triangle in $\mathfrak{T}$ like

\[ A \xrightarrow{f} B \xrightarrow{g} \text{Cone}(f) \rightarrow A[1], \]

where all arrows come from explicit morphisms between chain complexes (See [GM03, III.3.3 and III.3.4]). By [Hir03, Proposition 8.1.23], we could find a commutative diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{g} & B' \\
\downarrow{a} & & \downarrow{b} \\
A & \xrightarrow{f} & B \\
\end{array}
\]

such that $(A', a)$ (resp. $(B', b)$) is a fibrant cofibrant approximation of $A$ (resp. $B$) and $g$ is a cofibration in $\mathfrak{M}$. So the triangle $T$ is isomorphic in $D(S)$ to the distinguished triangle

\[ A' \xrightarrow{g} B' \xrightarrow{\text{Cone}(g)} A'[1] \]

in $\mathfrak{T}$. Hence $T$ is isomorphic to the triangle above in $D(S)$. By [CD07, Lemma 1.10] and [Ayo, Theorem 4.1.38], the shift functors $-[n]$ and $-[n]'$ in $\mathfrak{T}$ and $\mathfrak{T}'$, respectively, coincide on cofibrant objects in $\mathfrak{M}$. So we have a natural isomorphism $\eta : -[n] \rightarrow -[n]'$ where $\eta_K = id_{K[n]}$ if $K$ is cofibrant in $\mathfrak{T}'$. Hence the triangle above is distinguished in $\mathfrak{T}'$ by [Ayo, Definition 4.1.45]. So the functor $i$ is exact. And it’s clearly fully faithful. \hfill \square

By in [CD07, Theorem 1.18 and Proposition 2.3], we could define $\otimes_S, f^*$ and $f_\#$ on $D(S)$. 

Proposition 6.8.  
(1) We have a commutative diagram (up to a natural isomorphism)
\[
\begin{array}{ccc}
D^-(S) \times D^-(S) & \overset{\otimes}{\longrightarrow} & D^-(S) \\
\downarrow & & \downarrow \\
D(S) \times D(S) & \overset{\otimes}{\longrightarrow} & D(S)
\end{array}
\]
(2) Suppose \(f : S \to T\) is a morphism in \(\text{Sm}/k\). We have a commutative diagram (up to a natural isomorphism)
\[
\begin{array}{ccc}
D^-(T) & \overset{f^*}{\longrightarrow} & D^-(S) \\
\downarrow & & \downarrow \\
D(T) & \overset{f^*}{\longrightarrow} & D(S)
\end{array}
\]
(3) Suppose \(f : S \to T\) is a smooth morphism in \(\text{Sm}/k\). We have a commutative diagram (up to a natural isomorphism)
\[
\begin{array}{ccc}
D^-(S) & \overset{f_\#}{\longrightarrow} & D^-(T) \\
\downarrow & & \downarrow \\
D(S) & \overset{f_\#}{\longrightarrow} & D(T)
\end{array}
\]
Proof. This follows by direct computations. \qed

6.1.2. On Categories of Effective Motives. The next definition comes from [MVW06] Definition 9.2. 

Definition 6.3. Define \(\mathcal{E}_k\) to be the smallest thick subcategory of \(D^-(S)\) such that 
(1) \(\text{Cone}(\tilde{\mathbb{Z}}_S(X \times_k \mathbb{A}^1) \to \tilde{\mathbb{Z}}_S(X)) \in \mathcal{E}_k\).
(2) \(\mathcal{E}_k\) is closed under arbitrary direct sums.
Set \(W_k\) to be the class of morphisms in \(D^-(S)\) whose cone is in \(\mathcal{E}_k\). Define
\[
\text{DM}^{\text{eff}}(S) = D^-(S)[W_k^{-1}]
\]
to be the category of motives over \(S\). And morphisms in \(D^-(S)\) becoming isomorphisms after localizing by \(W_k\) are called \(\mathbb{A}^1\)-weak equivalences.

Definition 6.4. (See [MVW06] Definition 9.17) A complex \(K \in D^-(S)\) is called \(\mathbb{A}^1\)-local if for every \(\mathbb{A}^1\)-equivalence \(f : A \to B\), the induced map
\[
\text{Hom}_{D^-(S)}(B,K) \to \text{Hom}_{D^-(S)}(A,K)
\]
is an isomorphism.

Proposition 6.9. We say a map \(p : E \to X\) in \(\text{Sm}/S\) to be an \(\mathbb{A}^n\)-bundle if there is an open covering \(\{U_i\}\) of \(X\) such that \(p^{-1}(U_i) \cong U_i \times_k \mathbb{A}^n\) over \(U_i\). In this case, \(\tilde{\mathbb{Z}}_S(p) : \tilde{\mathbb{Z}}_S(E) \to \tilde{\mathbb{Z}}_S(X)\) is an isomorphism in \(\text{DM}^{\text{eff}}(S)\).

Proof. For any \(X \in \text{Sm}/S\), the projection \(\tilde{\mathbb{Z}}_S(X \times_k \mathbb{A}^n) \to \tilde{\mathbb{Z}}_S(X)\) is an \(\mathbb{A}^1\)-weak equivalence by definition. Suppose we have two open sets \(U_1\) and \(U_2\) of \(X\) such that the statement is true over \(U_1\), \(U_2\) and \(U_1 \cap U_2\). Set \(E_i = p^{-1}(U_i)\). Then we have a commutative diagram with exact rows
\[
\begin{array}{cccc}
0 & \longrightarrow & \tilde{\mathbb{Z}}_S(E_1 \cap E_2) & \longrightarrow & \tilde{\mathbb{Z}}_S(E_1) \oplus \tilde{\mathbb{Z}}_S(E_2) & \longrightarrow & \tilde{\mathbb{Z}}_S(p^{-1}(E_1 \cup E_2)) & \longrightarrow & 0 \\
\end{array}
\]
by Proposition 6.12. So the statement is also true over \(U_1 \cup U_2\). Then we could pick an finite open covering \(\{U_i\}\) of \(X\) such that \(p^{-1}(U_i) \cong U_i \times_k \mathbb{A}^n\) for every \(i\) and do induction on the number of open sets. \qed

Proposition 6.10. Suppose \(K \in D^-(S)\).
(1) The natural map $K \to C_\ast K$ is an $A^1$-weak equivalence.
(2) The complex $C_\ast K$ is $A^1$-local.
(3) The functor $C_\ast$ induces an endofunctor of $D^-(S)$.

Proof. (1) By the same proof as in [MVW06, Lemma 9.15].
(2) By [CD13, Proposition 5.2.36].
(3) It’s easy to check that $C_\ast$ induces an endofunctor of $K^-(S)$. If $f : K \to L$ is a quasi-isomorphism, then $\text{Cone}(f)$ is acyclic. By (1), the natural morphism $\text{Cone}(f) \to C_\ast \text{Cone}(f)$ is an $A^1$-equivalence. Hence it’s a quasi-isomorphism by (2) and [MVW06, Lemma 9.21]. So $C_\ast \text{Cone}(f) = \text{Cone}(C_\ast f)$ is acyclic. So $C_\ast f$ is a quasi-isomorphism.

Definition 6.5. (See [MVW06, Definition 14.17]) Let $X \in Sm/k$ and $p, q \in \mathbb{Z}, q \geq 0$, we define the groups

$$H^p_q(E, \mathbb{Z}) = \text{Hom}_{\mathcal{DM}^\mathbb{Z}}[\mathbb{Z}(X), \mathbb{Z}(q)[p]]$$

to be $E$-motivic cohomologies of $X$.

Proposition 6.11. Let $\varphi$ be the functor as before. We have an exact functor

$$\varphi^* : \mathcal{DM}^{\mathbb{Z}}(S) \to \mathcal{DM}^{\mathbb{Z}}(T)$$

which is determined by the following commutative diagram

$$\begin{array}{ccc}
D^-(S) & \xrightarrow{\varphi^*} & D^-(T) \\
\downarrow & & \downarrow \\
\mathcal{DM}^{\mathbb{Z}}(S) & \xrightarrow{\varphi^*} & \mathcal{DM}^{\mathbb{Z}}(T)
\end{array}$$

Proof. Let $E$ be the full subcategory of $D^-(S)$ which consists of those complexes $K \in D^-(S)$ who satisfies $\varphi^* K \in E_A$. It’s a thick subcategory of $D^-(S)$. For any $X \in Sm/S$, $\varphi^*$ maps

$$\mathbb{Z}_S(X \times_k A^1) \to \mathbb{Z}_S(X)$$

to

$$\mathbb{Z}_T((\psi X) \times_k A^1) \to \mathbb{Z}_T(\psi X).$$

So $E_A \subseteq E$ by definition of $E_A$ and exactness of $\varphi^*$. So $\varphi^*$ preserves objects in $E_A$. Hence $\varphi^*$ preserves $A^1$-weak equivalences by exactness of $\varphi^*$. Then the statement follows by [Kra09, Proposition 4.6.2].

Proposition 6.12. (1) There is a tensor product

$$\otimes_S : \mathcal{DM}^{\mathbb{Z}}(S) \times \mathcal{DM}^{\mathbb{Z}}(S) \to \mathcal{DM}^{\mathbb{Z}}(S),$$

which is determined by the following commutative diagram

$$\begin{array}{ccc}
D^-(S) \times D^-(S) & \xrightarrow{\otimes_S} & D^-(S) \\
\downarrow & & \downarrow \\
\mathcal{DM}^{\mathbb{Z}}(S) \times \mathcal{DM}^{\mathbb{Z}}(S) & \xrightarrow{\otimes_S} & \mathcal{DM}^{\mathbb{Z}}(S)
\end{array}$$

And for any $K \in \mathcal{DM}^{\mathbb{Z}}(S)$, the functor $K \otimes_S -$ is exact.
(2) Suppose $f : S \to T$ is a smooth morphism in $Sm/k$. There is an exact functor

$$f^\# : \mathcal{DM}^{\mathbb{Z}}(S) \to \mathcal{DM}^{\mathbb{Z}}(T),$$

which is determined by the following commutative diagram

$$\begin{array}{ccc}
D^-(S) & \xrightarrow{f^\#} & D^-(T) \\
\downarrow & & \downarrow \\
\mathcal{DM}^{\mathbb{Z}}(S) & \xrightarrow{f^\#} & \mathcal{DM}^{\mathbb{Z}}(T)
\end{array}$$
(3) Suppose \( f : S \to T \) is a morphism in \( Sm/k \). There is an exact functor
\[
f^* : D\widetilde{\operatorname{M}}^{\operatorname{eff},-}(T) \to D\widetilde{\operatorname{M}}^{\operatorname{eff},-}(S),
\]
which is determined by the following commutative diagram
\[
\begin{array}{ccc}
D^-(T) & \xrightarrow{f^*} & D^-(S) \\
\downarrow & & \downarrow \\
\widetilde{\operatorname{D}}\operatorname{M}^{\operatorname{eff},-}(T) & \xrightarrow{f^*} & \widetilde{\operatorname{D}}\operatorname{M}^{\operatorname{eff},-}(S)
\end{array}
\]

Proof. (1) Suppose \( Y \in Sm/S \). In the definition of \( \varphi \), we take \( (Y, S, T) := (Y, S, S) \). Then \( \varphi^*F \cong F \otimes_S \tilde{\mathbb{Z}}_S(Y) \) for any \( F \in \mathcal{S}h(S) \) by Proposition 5.11.

Now given an \( \mathbb{A}^1 \)-weak equivalence \( a, \tilde{\mathbb{Z}}_S(Y) \otimes_S a \) is also an \( \mathbb{A}^1 \)-weak equivalence by applying Proposition 6.11 to \( \varphi \). Now apply a similar method as in the third paragraph of [MVW09] Lemma 9.5 to show that the functor \( K \otimes_S - : D^-(S) \to D^-(S) \) preserves \( \mathbb{A}^1 \)-weak equivalence for any \( K \in D^-(S) \). Finally we apply [Kra09] Proposition 4.6.2 to the functor \( K \otimes_S - \).

(2) In the definition of \( \varphi \), we take \( (Y, S, T) := (S, S, T) \) and apply Proposition 6.11.

(3) In the definition of \( \varphi \), we take \( (Y, S, T) := (T, S, T) \) and apply Proposition 6.11. \( \square \)

Proposition 6.13. Let \( f : S \to T \) be a smooth morphism in \( Sm/k \). We have an adjoint pair
\[
f_\# : D\widetilde{\operatorname{M}}^{\operatorname{eff},-}(S) \xrightarrow{-} D\widetilde{\operatorname{M}}^{\operatorname{eff},-}(T) \xrightarrow{\cdot f^*}.
\]

Proof. By the same method as in Proposition 6.13 since \( \varphi^* \) preserves \( E_A \) by Proposition 6.11. \( \square \)

Proposition 6.14. Suppose \( f : S \to T \) is a morphism in \( Sm/k \).

(1) For any \( K, L \in D\widetilde{\operatorname{M}}^{\operatorname{eff},-}(T) \), we have
\[
f^*(K \otimes_S L) \cong (f^*K) \otimes_S (f^*L).
\]

(2) If \( f \) is smooth, then for any \( K \in D\widetilde{\operatorname{M}}^{\operatorname{eff},-}(S) \) and \( L \in D\widetilde{\operatorname{M}}^{\operatorname{eff},-}(T) \), we have
\[
f_\#(K \otimes_S f^*L) \cong (f_\#K) \otimes_S L.
\]

Proof. Directly follows from Proposition 5.20 and Proposition 5.24 since everything works termwise. \( \square \)

In [CD07] Proposition 3.5] and [DF17] Definition 3.2.1], they defined \( D\widetilde{\operatorname{M}}^{\operatorname{eff}}(S) \) as the the Verdier localization of \( D(S) \) with respect to homotopy invariant conditions. So the localization induces a triangulated structure on \( D\widetilde{\operatorname{M}}^{\operatorname{eff}}(S) \) (See [Kra09] Lemma 4.3.1]). And this is the triangulated structure we will impose on \( D\widetilde{\operatorname{M}}^{\operatorname{eff}}(S) \).

Proposition 6.15. There is a fully faithful exact functor \( D\widetilde{\operatorname{M}}^{\operatorname{eff},-}(S) \to D\widetilde{\operatorname{M}}^{\operatorname{eff}}(S) \) which is determined by the commutative diagram (See Proposition 6.7)
\[
\begin{array}{ccc}
D^-(S) & \xrightarrow{f^*} & D^-(S) \\
\downarrow & & \downarrow \\
\widetilde{\operatorname{D}}\operatorname{M}^{\operatorname{eff},-}(S) & \xrightarrow{f^*} & \widetilde{\operatorname{D}}\operatorname{M}^{\operatorname{eff}}(S)
\end{array}
\]

Proof. The functor \( D\widetilde{\operatorname{M}}^{\operatorname{eff},-}(S) \to D\widetilde{\operatorname{M}}^{\operatorname{eff}}(S) \) is induced and exact by [Kra09] Proposition 4.6.2. And for any \( K, L \in D\widetilde{\operatorname{M}}^{\operatorname{eff},-}(S) \), we have a commutative diagram
\[
\begin{array}{ccc}
\alpha : \operatorname{Hom}_{D\widetilde{\operatorname{M}}^{\operatorname{eff},-}}(K, L) & \xrightarrow{u} & \operatorname{Hom}_{D\widetilde{\operatorname{M}}^{\operatorname{eff},-}}(C, K, C, L) \\
\downarrow & & \downarrow \\
\operatorname{Hom}_{D\widetilde{\operatorname{M}}^{\operatorname{eff},-}}(K, L) & \xrightarrow{v} & \operatorname{Hom}_{D\widetilde{\operatorname{M}}^{\operatorname{eff},-}}(C, K, C, L)
\end{array}
\]
where \( u, v, \gamma \) and \( \beta \) are isomorphisms by Proposition 6.10. So \( \alpha \) is an isomorphism. \( \square \)
6.2. For Symmetric Spectra. Now we are going to discuss operations for spectra, in order to stabilize the category $\mathcal{D}^S_{eff,-}(S)$. The main reference is [CD13, 5.3].

6.2.1. Symmetric Spectra. Let $\mathcal{A}$ be a symmetric closed monoidal abelian category with arbitrary products. We can define the category of symmetric sequences $\mathcal{A}^\otimes$ as in [CD13 Definition 5.3.5]. It is also a closed symmetric monoidal abelian category by [CD13 Definition 5.3.7] and [HSS98 Lemma 2.1.6]. Here, if we have two symmetric sequences $A$ and $B$, we define $A \otimes^\otimes B$ by

$$(A \otimes^\otimes B)_n = \oplus_{p \leq n} S_p \times S_{n-p} (A_p \otimes B_{n-p}).$$

And we define $\text{Hom}^\otimes(A, B)$ by

$$\text{Hom}^\otimes(A, B)_n = \prod_p \text{Hom}^\otimes_{S_p}(A_p, B_{n+p}),$$

where $\text{Hom}^\otimes_{S_p}(A_p, B_{n+p})$ (with an obvious $S_p$-action) is the kernel of the map

$$\text{Hom}(A_p, B_{n+p}) \xrightarrow{\langle \sigma^* - (1 \times \sigma) \rangle} \prod_{\sigma \in S_p} \text{Hom}(A_p, B_{n+p}).$$

(see [HSS98 Definition 2.1.3] and [HSS98 Theorem 2.1.11])

**Proposition 6.16.** In the context above, for any symmetric sequences $A$, $B$, $C$, we have $\text{Hom}(A \otimes^\otimes B, C) \cong \text{Hom}(A, \text{Hom}^\otimes(B, C))$ naturally.

**Proof.** Giving a morphism from $A \otimes^\otimes B$ to $C$ is equivalent to giving $S_p \times S_q$-equivariant maps $f_{p,q} : A_p \otimes B_q \rightarrow C_{p+q}$. And that is equivalent to giving $S_p$-equivariant maps $g_{p,q} : A_p \rightarrow \text{Hom}(B_q, C_{p+q})$ such that for any $\sigma \in S_q$, $\text{Hom}(\sigma, C_{p+q}) \circ g_{p,q} = \text{Hom}(B_q, id_{S_p} \times \sigma) \circ g_{p,q}$. And this just says that $g_{p,q}$ factor through $\text{Hom}^\otimes_{S_p}(B_q, C_{p+q})$. \hfill $\square$

And the abelian structure of $\mathcal{A}^\otimes$ is just defined termwise. Moreover, we have adjunctions

$$i_0 : \mathcal{A} \times \mathcal{A}^\otimes : ev_0$$

and

$$-\{i\} : \mathcal{A}^\otimes \cong \mathcal{A}^\otimes : -\{i\} (i \geq 0)$$

as in [CD13 5.3.5.1] and [CD07 6.4.1].

Now suppose $R \in \mathcal{A}$. Then $\text{Sym}(R) \in \mathcal{A}^\otimes$ is a commutative monoid object as in [CD13 5.3.8]. Define $\text{Sp}_R(\mathcal{A})$ to be the category of $\text{Sym}(R)$-modules in $\mathcal{A}^\otimes$. They are called symmetric $R$-spectra. Then it’s also a symmetric closed monoidal abelian category by [HSS98 Theorem 2.1.10] and Proposition 6.16 (The corresponding tensor product and inner-hom are just denoted by $\otimes$ and $\text{Hom}$ for convenience).

We have an adjunction $\text{Sym}(R) \otimes^\otimes - : \mathcal{A}^\otimes : \text{Sp}_R(\mathcal{A}) : U$,

where $U$ is the forgetful functor. Thus we get an adjunction

$$\Sigma^\infty : \mathcal{A} : \Omega^\infty,$$

where $\Sigma^\infty = (\text{Sym}(R) \otimes^\otimes -) \circ i_0$, $\Omega^\infty = ev_0 \circ U$ and $\Sigma^\infty$ is monoidal.

We have a canonical identification

$$A \otimes^\otimes (B\{-i\}) = (A \otimes^\otimes B)\{-i\}$$

and a morphism

$$A \otimes^\otimes (B\{i\}) \rightarrow (A \otimes^\otimes B)\{i\}$$
defined by the composition
\[ A \otimes^\mathbb{S} (B \{i\}) \to (A \otimes^\mathbb{S} (B \{i\})\{−i\}\{i\} = (A \otimes^\mathbb{S} (B \{i\}\{−i\})\{i\} \to (A \otimes^\mathbb{S} B)\{i\}. \]
Restricting the functors \(-\{−i\}\) and \(-\{i\}\) on spectra, there is also an adjunction
\[-\{i\} : \text{Sp}_R(\mathcal{A}) \cong \text{Sp}_R(\mathcal{A}^\mathbb{S}) : \mathcal{A}, \]
where the module structure \(\text{Sym}(R) \otimes^\mathbb{S} (A\{−i\}) \to A\{−i\}\) of \(A\{−i\}\) is just got by applying \(-\{−i\}\) on that of \(A\) and the module structure \(\text{Sym}(R) \otimes^\mathbb{S} (B\{i\}) \to B\{i\}\) of \(B\{i\}\) is got by the composition
\[ \text{Sym}(R) \otimes^\mathbb{S} (B\{i\}) \to (\text{Sym}(R) \otimes^\mathbb{S} B)\{i\} \to B\{i\}, \]
where the last arrow is got by applying \(-\{i\}\) on the module structure of \(B\). And we still have an isomorphism
\[ A \otimes^\mathbb{S} (B\{−i\}) \cong (A \otimes^\mathbb{S} B)\{−i\} \]
and a morphism
\[ A \otimes^\mathbb{S} (B\{i\}) \to (A \otimes^\mathbb{S} B)\{i\}\]
defined by the same way as above.

**Definition 6.6.** (See [CD13] Definition 5.3.16) For any \(S \in \text{Sm}/k\), define
\[ \mathcal{S}_S\{1\} = \text{Sym}(\text{coker}(\mathcal{Z}_S(S) \to \mathcal{Z}_S(\mathbb{G}_m))) \]
and
\[ \mathcal{S}_S\{1\} = \text{Sym}(\text{coker}(\mathcal{Z}_S(S) \to \mathcal{Z}_S(\mathbb{G}_m))). \]
Then define \(\text{Sp}(S)\) to be \(\text{Sp}_{\mathcal{S}_S\{1\}}(\mathcal{S}_h(S))\) and \(\text{Sp}'(S)\) to be \(\text{Sp}_{\mathcal{S}_S\{1\}}(\mathcal{P}_h(S))\).

We have an adjunction
\[ \tilde{\alpha} : \mathcal{P}_h(S) \to \mathcal{S}_h(S): \alpha, \]
where both functors are defined termwise (see Proposition 5.11) and \(\tilde{\alpha}\) is monoidal by definition. Restricting the above functors on modules, there is also an adjunction
\[ \tilde{\alpha} : \text{Sp}'(S) = \text{Sp}(S) : \alpha, \]
where the module structure \(\mathcal{S}_S\{1\} \otimes^\mathbb{S} \tilde{\alpha}(A) \to \tilde{\alpha}(A)\) of \(\tilde{\alpha}(A)\) is just got by applying \(\alpha\) on that of \(A\) and the module structure \(\mathcal{S}_S\{1\} \otimes^\mathbb{S} \tilde{\alpha}(B) \to \tilde{\alpha}(B)\) of \(\tilde{\alpha}(B)\) is got by precomposing that of \(B\) with the sheafication map \(\mathcal{S}_S\{1\} \otimes^\mathbb{S} \tilde{\alpha}(B) \to \mathcal{S}_S\{1\} \otimes^\mathbb{S} B\). The functor \(\tilde{\alpha}\) is again monoidal.

Now let \(f : S \to T\) be a morphism in \(\text{Sm}/k\). We have an adjunction
\[ f^* : \mathcal{S}_h(T) \to \mathcal{S}_h(S): f*, \]
where both functors are defined termwise (see Proposition 5.19) and \(f^*\) is monoidal by Proposition 5.20 (4). Restricting the above functors on spectra, there is also an adjunction
\[ f^* : \text{Sp}(T) = \text{Sp}(S) : f*, \]
where the module structure \(\mathcal{S}_S\{1\} \otimes^\mathbb{S} f^* A \to f^* A\) of \(f^* A\) is just got by applying \(f^*\) on that of \(A\) and the module structure \(\mathcal{S}_T\{1\} \otimes^\mathbb{S} f_* B \to f_* B\) of \(f_* B\) is got by the composition
\[ \mathcal{S}_T\{1\} \otimes^\mathbb{S} f_* B \to f_* (\mathcal{S}_S\{1\} \otimes^\mathbb{S} f^* B) \to f_* (\mathcal{S}_S\{1\} \otimes^\mathbb{S} B) \to f_* B, \]
where the last arrow is got by applying \(f_*\) on the module structure of \(B\). The functor \(f^*\) is also monoidal by the construction of the tensor product (see [HSS98] Lemma 2.2.2). And the same construction gives an another adjunction
\[ f^* : \text{Sp}'(T) = \text{Sp}'(S) : f*, \]
Suppose further \(f\) is smooth. We have an adjunction
\[ f^# : \mathcal{S}_h(S) \to \mathcal{S}_h(T) : f^*, \]
where both functors are defined termwise (see Proposition 5.23) and
\[ f^# (A \otimes^\mathbb{S} f^* B) \cong (f^# A) \otimes^\mathbb{S} f^* B \]
also holds by Proposition 5.26 (4). Restricting the above functors on spectra, there is also an adjunction
\[ f^# : \text{Sp}(S) = \text{Sp}(T) : f^*, \]
where the module structure \( \mathbb{I}_T \{1\} \otimes f \# A \to f \# A \) of \( f \# A \) is got by the composition
\[
\mathbb{I}_T \{1\} \otimes f \# A \cong f \# (\mathbb{I}_S \{1\} \otimes S A) \to f \# A,
\]
where the last arrow is got by applying \( f \# \) on the module structure of \( A \). And we also have
\[
f \# (A \otimes_S f^* B) \cong (f \# A) \otimes_T B
\]
for spectra by the construction of the tensor product (see [HSS98, Lemma 2.2.2]). And the same construction gives an another adjunction
\[
f_\#: Sp'(S) \Rightarrow Sp'(T) : f^*.
\]
One checks that when \( F = - \otimes_S A, f_\#, f^*, -\{i\}, -\{i\}, \Sigma^\infty \) or \( \Omega^\infty \), there is a natural isomorphism \( \tilde{a} \circ F \cong F \circ \tilde{a} \).

Suppose \( i \geq 0 \). Then for any \( F \in \widetilde{Sh}(S) \), we have
\[
(\Sigma^\infty F)\{i\} \cong \Sigma^\infty (\widetilde{\mathbb{G}} \mathbb{m}^{1})^{\otimes i} \otimes_S F).
\]
Moreover, for any \( X \in Sm/S, \)
\[
\text{Hom}_{Sp(S)}((\Sigma^\infty \widetilde{Z}_S(X))\{-i\}, A) = A_i(X)
\]
and
\[
\text{Hom}_{Sp(S)}((\Sigma^\infty \widetilde{Z}_S(X))\{-i\}, B) = B_i(X).
\]
So \( (\Sigma^\infty \widetilde{Z}_S(X))\{-i\} \) (resp. \( (\Sigma^\infty \widetilde{Z}_S(X))\{-i\} \)) are systems of generators of \( Sp(S) \) (resp. \( Sp'(S) \)) (See [CD07, 6.7] and [CD13, 5.3.11]). This enables us to imitate methods in Section 6.1.

6.2.2. On Derived Categories. We denote by \( D_{Sp}(S) \) (resp. \( D_{Sp}(S) \)) the derived category of bounded above (resp. unbounded) complex of spectra in \( Sp(S) \).

**Proposition 6.17.** Let \( X, U \in Sm/S \) and \( p: U \to X \) be a Nisnevich covering. Then the complex \( (\Sigma^\infty \tilde{C}(U/X))\{-i\} \) (defined by termise application), is exact after sheafifying as a complex of \( Sp(S) \).

**Proof.** One easily see that \( (\Sigma^\infty A)\{-i\} = \text{Sym}(\mathbb{Z}_S \{1\} \otimes_S (i_0(A)\{-i\})) \) for any \( A \in \mathbb{P}Sh(S) \). Then the statement follows by the equality
\[
\tilde{C}(U/X) \otimes_{Sp} \tilde{Z}_S(Y) = \tilde{C}(U \times_S Y/X \times_S Y)
\]
for any \( Y \in Sm/S \) and Proposition [5.10]

**Definition 6.7.** We call a spectrum \( A \in Sp'(S) \) free if it’s a direct sum of spectra of the form \( (\Sigma^\infty \widetilde{Z}_S(X))\{-i\} \). We call \( A \) projective if it’s a direct summand of a free spectrum. A spectrum in \( Sp(S) \) is called free (resp. projective) if it’s a sheafication of a free (resp. projective) spectrum in \( Sp'(S) \). A bounded above complex of spectra in \( Sp(S) \) is called free (projective) if all its terms are free (projective).

**Definition 6.8.** A projective resolution of a bounded above spectrum complex \( K \) is a projective complex with a quasi-isomorphism \( P \to K \). If \( K \) is already projective, we take \( P = K \).

Now let \( S, T \in Sm/k, j \geq 0 \) and \( Y \) be a scheme with morphisms \( S \xrightarrow{f} Y \xrightarrow{g} T \) where \( g \) is smooth. Consider in this section the adjunctions
\[
\phi^* = \{-j\} \circ g_\# \circ f^*: Sp(S) \Rightarrow Sp(T) : \phi_\# = f_\# \circ g^* \circ \{j\}
\]
\[
\varphi^* = \{-j\} \circ g_\# \circ f^*: Sp'(S) \Rightarrow Sp'(T) : \varphi_\# = f_\# \circ g^* \circ \{j\}
\]
and the functor
\[
\psi: Sm_S \to Sm_T
\]
\[
X \mapsto X \times_S Y.
\]
They are determined by the quadruple \((Y, S, T, j)\).

**Proposition 6.18.** For any \( F \in Sp'(S) \),
\[
\tilde{a}((L_i\varphi^*)\tilde{a}(F)) \cong \tilde{a}((L_i\varphi^*)F)
\]
as spectra in \( Sp(S) \) for any \( i \geq 0 \), where \( L_i\varphi^* \) means the \( i \)th left derived functor of \( \varphi^* \).
\textbf{Proof.} We show at first that for any } F \in Sp'(S) \text{ with } \tilde{a}(F) = 0
\begin{equation}
\tilde{a}(L_i \varphi^*(F)) = 0
\end{equation}
for any } i \geq 0. \text{ Then for any } F \in Sp'(S), \text{ denote by } \theta \text{ the natural map } F \to \tilde{a}(F). \text{ We have }
\begin{equation}
\tilde{a}(	ext{coker}(\theta)) = \tilde{a}(\text{ker}(\theta)) = 0.
\end{equation}
Hence for any } i \geq 0, \text{ we have }
\begin{equation}
\tilde{a}(L_i \varphi^* F) \cong \tilde{a}(L_i \varphi^* \text{Im}(\theta)) \cong \tilde{a}(L_i \varphi^* F)
\end{equation}
by using long exact sequences. Hence the statement follows.

Now we prove the claim. We do induction on } i. \text{ The claim is true for } i = 0 \text{ and suppose it’s true for } i < n.
For any } F \in Sp'(S), \text{ we have a surjection }
\begin{equation}
\oplus_{a \in F_i(X), t \geq 0}(\Sigma^\infty \check{Z}_S(X))\{ -t \} \to F
\end{equation}
defined by each section of } F_i(\text{X}), X \in S/S, \text{ there is a finite Nisnevich covering } U_n \to X \text{ of } X \text{ such that } a|_{U_n} = 0. \text{ So the composition }
\begin{equation}
\oplus_{a \in F_i(X), t \geq 0}(\Sigma^\infty \check{Z}_S(U_a))\{ -t \} \to \oplus_{a \in F_i(X), t \geq 0}(\Sigma^\infty \check{Z}_S(X))\{ -t \} \to F
\end{equation}
is zero. \text{ Then we have got a surjection }
\begin{equation}
\oplus_{a \in F_i(X), t \geq 0}H_0((\Sigma^\infty \check{C}(U_a/X))\{ -t \}) \to F
\end{equation}
with kernel } K. \text{ Proposition } \ref{prop510} \text{ implies that }
\begin{equation}
\tilde{a}(H_0((\Sigma^\infty \check{C}(U/X))\{ -t \})) = 0
\end{equation}
for any Nisnevich covering } U \to X, \text{ } t \geq 0 \text{ and } p \in \mathbb{Z}. \text{ So } \tilde{a}(K) = 0 \text{ also. We have a hypercohomology spectral sequence }
\begin{equation}
(L_p \varphi^*)H_q((\Sigma^\infty \check{C}(U/X))\{ -t \}) \Rightarrow (L_{p+q} \varphi^*)(\Sigma^\infty \check{C}(U/X))\{ -t \}).
\end{equation}
Hence
\begin{equation}
\tilde{a}((L_n \varphi^*)(\Sigma^\infty \check{C}(U/X))\{ -t \}) \cong \tilde{a}((L_n \varphi^*)H_0((\Sigma^\infty \check{C}(U/X))\{ -t \}))
\end{equation}
by induction hypothesis. \text{ But }
\begin{equation}
\tilde{a}((L_n \varphi^*)((\Sigma^\infty \check{C}(U/X))\{ -t \})) \cong \tilde{a}(H_n(\varphi^*)((\Sigma^\infty \check{C}(U/X))\{ -t \}))
\end{equation}
by definition of hypercohomology and the latter one vanishes since we have
\begin{equation}
\varphi^*((\Sigma^\infty \check{C}(U/X))\{ -t \}) = (\Sigma^\infty \check{C}(\psi U/\psi X))\{ -t - j \}.
\end{equation}
So
\begin{equation}
\tilde{a}((L_n \varphi^*)H_0((\Sigma^\infty \check{C}(U/X))\{ -t \})) = 0.
\end{equation}
So
\begin{equation}
\tilde{a}(L_n \varphi^* F) \cong \tilde{a}(L_{n-1} \varphi^* K) = 0
\end{equation}
by long exact sequence and induction hypothesis. 
\hfill \Box

\textbf{Proposition 6.19.} \textit{Let functor } \phi^* \textit{ takes acyclic projective complexes to acyclic projective complexes.}

\textbf{Proof.} \textit{The same as Proposition } \ref{prop62} \hfill \Box

\textbf{Proposition 6.20.} \textit{We have an exact functor }
\begin{equation}
L \phi^* : \mathcal{D}_{Sp}^- (S) \to \mathcal{D}_{Sp}^- (T)
\end{equation}
\textit{which maps any } K \in D_{Sp}^-(S) \textit{ to } \phi^* P, \textit{ where } P \textit{ is a projective resolution } K.

\textbf{Proof.} \textit{The same as Proposition } \ref{prop63} \hfill \Box

We will just write } L \phi^* \textit{ above by } \phi^* \textit{ for convenience.

Now we apply the general results above to } \otimes_S, f_\# \textit{ and } f^*. 
**Proposition 6.21.** (1) There is a tensor product
\[ \otimes_S : D^\circledast_S(S) \times D^\circledast_S(S) \rightarrow D^\circledast_S(S) \]
\[ (K, L) \mapsto P \otimes_S Q , \]
where \( P, Q \) are projective resolutions of \( K, L \), respectively and the last tensor means taking the total complex of the bicomplex \( \{ P_i \otimes_S Q_j \} \). And for any \( K \in D^\circledast_S(S) \), the functor \( K \otimes_S - \) is exact.

(2) Suppose \( f : S \rightarrow T \) is a smooth morphism in \( Sm/k \). There is an exact functor
\[ f^\# : D^\circledast_S(S) \rightarrow D^\circledast_T(T) \]
\[ K \mapsto f^\# P , \]
where \( P \) is a projective resolution of \( K \).

(3) Suppose \( f : S \rightarrow T \) is a morphism in \( Sm/k \). There is an exact functor
\[ f^* : D^\circledast_T(T) \rightarrow D^\circledast_S(S) \]
\[ K \mapsto f^* P , \]
where \( P \) is a projective resolution of \( K \).

(4) Suppose \( i \geq 0 \), there is an exact functor
\[ -\{ -i \} : D^\circledast_S(S) \rightarrow D^\circledast_S(S) \]
\[ K \mapsto P\{ -i \} , \]
where \( P \) is a projective resolution of \( K \).

**Proof.** In (1), (2) and (3), take \( j = 0 \) in the definition of \( \phi \) and proceed as Proposition 6.4. For (4), take the quadruple \( (S, S, S, i) \) and use Proposition 6.20. \( \square \)

**Proposition 6.22.** (1) Suppose \( f : S \rightarrow T \) is a smooth morphism in \( Sm/k \), we have an adjoint pair
\[ f^\# : D^\circledast_S(S) \rightleftarrows D^\circledast_T(T) : f^* , \]

(2) We have an adjoint pair
\[ -\{ -i \} : D^\circledast_S(S) \rightleftarrows D^\circledast_S(S) : -\{ i \} . \]

**Proof.** The same as Proposition 6.5 since \( -\{ i \} \) is an exact functor. \( \square \)

**Proposition 6.23.** The functor \( \Sigma^\infty : \hat{Sh}(S) \rightarrow Sp(S) \) takes acyclic projective complexes of sheaves to acyclic projective complexes of spectra.

**Proof.** Let \( P \) be a projective sheaf. Then
\[ (\Sigma^\infty P)_n = \mathbb{L} S\{ 1 \} \otimes^\mathbb{L} P_n \]
by definition. And a tensor product between projective sheaves is again projective. So \( \Sigma^\infty P \) is projective.

Let \( Q \) be an acyclic projective complex of sheaves. Then \( \Sigma^\infty Q \) consists of complexes like \( \mathbb{L} S\{ 1 \} \otimes^\mathbb{L} Q \). And they are all acyclic by Proposition 6.2. \( \square \)

**Proposition 6.24.** There is an exact functor
\[ L\Sigma^\infty : D^-(S) \rightarrow D^\circledast_S(S) \]
which maps \( K \) to \( \Sigma^\infty P \), where \( P \) is a projective resolution of \( K \).

**Proof.** The same as Proposition 6.3. \( \square \)

As usual, we will write \( L\Sigma^\infty \) as \( \Sigma^\infty \) for convenience.

**Proposition 6.25.** There is an adjoint pair
\[ \Sigma^\infty : D^-(S) \rightleftarrows D^\circledast_S(S) : \Omega^\infty . \]

**Proof.** The same as Proposition 6.5 since \( \Omega^\infty \) is an exact functor. \( \square \)

**Proposition 6.26.** The functor \( \Sigma^\infty : D^-(S) \rightarrow D^\circledast_S(S) \) is fully faithful.
Proof. Suppose $K,L \in D^-(S)$ with projective resolutions $P,Q$ respectively. Then there is a commutative diagram
\[
\begin{align*}
\text{Hom}_{D^-(S)}(K,L) & \xrightarrow{\Sigma^\infty} \text{Hom}_{D^-_{Sp}(S)}(\Sigma^\infty K, \Sigma^\infty L) \\
\text{Hom}_{D^-(S)}(P,Q) & \xrightarrow{\Sigma^\infty} \text{Hom}_{D^-_{Sp}(S)}(\Sigma^\infty P, \Sigma^\infty Q) \\
\end{align*}
\]
And we observe that $\Omega^\infty \Sigma^\infty Q = Q$ (The same for $P$).
\qed

Proposition 6.27. (1) We have a commutative diagram (up to a canonical isomorphism)
\[
\begin{align*}
D^-(S) \times D^-(S) & \xrightarrow{\otimes} D^-(S) \\
\Sigma^\infty \times \Sigma^\infty & \xrightarrow{\Sigma^\infty} \\
D^-_{Sp}(S) \times D^-_{Sp}(S) & \xrightarrow{\otimes} D^-_{Sp}(S)
\end{align*}
\]
(2) Suppose $f : S \rightarrow T$ is a morphism in $Sm/k$. We have a commutative diagram (up to a canonical isomorphism)
\[
\begin{align*}
D^-(T) & \xrightarrow{f^*} D^-(S) \\
\Sigma^\infty & \xrightarrow{\Sigma^\infty} \\
D^-_{Sp}(T) & \xrightarrow{f^*} D^-_{Sp}(S)
\end{align*}
\]
(3) Suppose $f : S \rightarrow T$ is a smooth morphism in $Sm/k$. We have a commutative diagram (up to a canonical isomorphism)
\[
\begin{align*}
D^-(S) & \xrightarrow{f_#} D^-(T) \\
\Sigma^\infty & \xrightarrow{\Sigma^\infty} \\
D^-_{Sp}(S) & \xrightarrow{f_#} D^-_{Sp}(T)
\end{align*}
\]
Proof. This follows by direct computations.
\qed

In [CD07, Theorem 1.7], they defined a model structure $\mathfrak{M}_{Sp}$ on the category of unbounded complexes of symmetric spectra over $S$. This is a cofibrantly generated model structure where the cofibrations are those $I$-cofibrations (See [Hov07, Definition 2.1.7]) where $I$ consists of the morphisms $S^{n+1}(\Sigma^\infty \tilde{Z}_S(X)[-i]) \rightarrow D^n(\Sigma^\infty \tilde{Z}_S(X)[-i])$ for any $X \in Sm/S$ and $i \geq 0$ and weak equivalences are quasi-morphisms between complexes.

Proposition 6.28. Bounded above projective complexes are cofibrant objects in $\mathfrak{M}_{Sp}$.
Proof. The same as Proposition 6.6.
\qed

Moreover, $\mathfrak{M}_{Sp}$ is stable and left proper so it induces a triangulated structure $\mathfrak{T}'$ on $D_{Sp}(S)$ (See [Ayo, Theoreme 4.1.49]). The classical triangulated structure of $D_{Sp}(S)$ or $D^-_{Sp}(S)$ is denoted by $\mathfrak{T}$.

Proposition 6.29. The natural functor
\[
(D^-_{Sp}(S), \mathfrak{T}) \rightarrow (D_{Sp}(S), \mathfrak{T}')
\]
is fully faithful exact.
Proof. The same as Proposition 6.7.
\qed

And there is also a compatibility result between the natural inclusion and $\otimes_S$, $f^*$, $f_#$, $\Sigma^\infty$, $-\{-i\}$, $i \geq 0$ like Proposition 6.8.
6.2.3. On Categories of Effective Motives.

**Definition 6.9.** (See [CD13] 5.2.15) Define $\mathcal{E}_A$ to be the smallest thick subcategory of $D^\sim_{Sp}(S)$ such that

1. $(\Sigma^\infty \text{Cone}(\tilde{Z}_S(X \times_A \mathbb{A}^1) \to \tilde{Z}_S(X)) \{-i\} \in \mathcal{E}_A, i \geq 0$.
2. $\mathcal{E}_A$ is closed under arbitrary direct sums.

Set $W_A$ be the class of morphisms in $D^\sim_{Sp}(S)$ whose cone is in $\mathcal{E}_A$. Define

$$\text{DM}_{Sp}^{eff,-}(S) = D^\sim_{Sp}(S)[W_A^{-1}].$$

And a morphism in $D^\sim_{Sp}(S)$ is called a levelwise $A^1$-equivalence if it becomes an isomorphism in $\text{DM}_{Sp}^{eff,-}(S)$.

**Definition 6.10.** (See [CD13] 5.3.20) A complex $K \in D^\sim_{Sp}(S)$ is called levelwise $A^1$-local if for every levelwise $k^1$-equivalence $f : A \to B$, the induced map

$$\text{Hom}_{D^\sim_{Sp}(S)}(B, K) \to \text{Hom}_{D^\sim_{Sp}(S)}(A, K)$$

is an isomorphism.

**Proposition 6.30.** A complex $K = (K_n) \in D^\sim_{Sp}(S)$ is levelwise $A^1$-local if and only if for every $n \geq 0$, the complex $K_n$ is $k^1$-local in $D^\sim(S)$.

**Proof.** By the same proof as [MVW06] Lemma 9.20, $K$ is levelwise $A^1$-local if and only if for every $X \in Sm/S$, $n \in \mathbb{Z}$ and $i \geq 0$, the map

$$\text{Hom}_{D^\sim_{Sp}(S)}((\Sigma^\infty \tilde{Z}_S(X))\{-i\}[n], K) \to \text{Hom}_{D^\sim_{Sp}(S)}((\Sigma^\infty \tilde{Z}_S(X \times A^1))\{-i\}[n], K)$$

is an isomorphism. And then one uses Proposition 6.22 and Proposition 6.25. For every $A = (A_n) \in Sp(S)$ and $X \in Sm/S$, we define $A^X$ by $(A_n)^X = (A_n^X)$. And the module structure $\mathbb{I}_S\{1\} \otimes^\mathbb{E} A^X \to A^X$ is given by the composition

$$\mathbb{I}_S\{1\} \otimes^\mathbb{E} A^X \to (\mathbb{I}_S\{1\} \otimes^\mathbb{E} A)^X \to A^X.$$

And $A^X$ is contravariant with respect to morphisms in $Sm/S$. So we could define $C_*A$ by $(C_*A)_n = C_*A_n$.

**Proposition 6.31.** Suppose $K \in D^\sim_{Sp}(S)$.

1. The natural map $K \to C_*K$ is a levelwise $A^1$-equivalence.
2. The complex $C_*K$ is levelwise $A^1$-local.
3. The functor $C_*$ induces an endofunctor of $D^\sim_{Sp}(S)$.

**Proof.** (1) We have a natural morphism $\Sigma^\infty \tilde{Z}_S(X) \otimes^\mathbb{E} A^X \to A$ defined by the composition

$$\mathbb{I}_S\{1\} \otimes^\mathbb{E} \tilde{Z}_S(X) \otimes S A_q^X \to \tilde{Z}_S(X) \otimes S (\mathbb{I}_S\{1\} \otimes^\mathbb{E} A_q^X) \to \tilde{Z}_S(X) \otimes S A_{p+q}^X \to A_{p+q}$$

for every $p, q \geq 0$. This morphism is compatible with module action so it induces an morphism

$$\Sigma^\infty \tilde{Z}_S(X) \otimes S A^X \to A.$$

Then we get a morphism

$$A^X \to \text{Hom}(\Sigma^\infty \tilde{Z}_S(X), A).$$

Then we could use the same proof as in [MVW06] Lemma 9.15] to conclude.

(2) By the proposition above and Proposition 6.10.

(3) By Proposition 6.10 since quasi-isomorphisms in $D^\sim_{Sp}(S)$ are defined levelwise. □

**Proposition 6.32.** A morphism $f : A \to B$ in $D^\sim_{Sp}(S)$ is a levelwise $k^1$-equivalence if and only if for every $n \geq 0$, the morphism

$$f_n = \Omega^\infty(f\{n\}) : A_n \to B_n$$

is an $A^1$-equivalence in $D^\sim(S)$.

**Proof.** The morphism $f$ is a levelwise $A^1$-equivalence if and only if $C_*f$ is a quasi-isomorphism by Proposition 6.31. And the latter property is levelwise. □
Proposition 6.33. Let $\phi$ be the functor as before. We have an exact functor

$$\phi^*: \tilde{DM}_{Sp}^{\text{eff},-}(S) \rightarrow \tilde{DM}_{Sp}^{\text{eff},-}(T)$$

which is determined by the following commutative diagram

$$
\begin{array}{ccc}
D_{Sp}(S) & \xrightarrow{\phi^*} & D_{Sp}(T) \\
\downarrow & & \downarrow \\
\tilde{DM}_{Sp}^{\text{eff},-}(S) & \xrightarrow{\phi^*} & \tilde{DM}_{Sp}^{\text{eff},-}(T)
\end{array}
$$

Proof. For any $X \in Sm/S$, $\phi^*$ maps

$$\Sigma^\infty(\tilde{Z}_S(X \times_k A^1) \rightarrow \tilde{Z}_S(X))\{−i\}$$

to

$$\Sigma^\infty(\tilde{Z}_T((\psi X) \times_k A^1) \rightarrow \tilde{Z}_T(\psi X))\{−i−j\}.$$ 

So the statement follows by the same method as in Proposition 6.11. □

Proposition 6.34. (1) There is a tensor product

$$\otimes_S: \tilde{DM}_{Sp}^{\text{eff},-}(S) \times \tilde{DM}_{Sp}^{\text{eff},-}(S) \rightarrow \tilde{DM}_{Sp}^{\text{eff},-}(S),$$

which is determined by the following commutative diagram

$$
\begin{array}{ccc}
D_{Sp}(S) \times D_{Sp}(S) & \xrightarrow{\otimes_S} & D_{Sp}(S) \\
\downarrow & & \downarrow \\
\tilde{DM}_{Sp}^{\text{eff},-}(S) \times \tilde{DM}_{Sp}^{\text{eff},-}(S) & \xrightarrow{\otimes_S} & \tilde{DM}_{Sp}^{\text{eff},-}(S)
\end{array}
$$

And for any $K \in \tilde{DM}_{Sp}^{\text{eff},-}(S)$, the functor $K \otimes_S −$ is exact.

(2) Suppose $f: S \rightarrow T$ is a smooth morphism in $Sm/k$. There is an exact functor

$$f_*: \tilde{DM}_{Sp}^{\text{eff},-}(S) \rightarrow \tilde{DM}_{Sp}^{\text{eff},-}(T),$$

which is determined by the following commutative diagram

$$
\begin{array}{ccc}
D_{Sp}(S) & \xrightarrow{f_*} & D_{Sp}(T) \\
\downarrow & & \downarrow \\
\tilde{DM}_{Sp}^{\text{eff},-}(S) & \xrightarrow{f_*} & \tilde{DM}_{Sp}^{\text{eff},-}(T)
\end{array}
$$

(3) Suppose $f: S \rightarrow T$ is a morphism in $Sm/k$. There is an exact functor

$$f^*: \tilde{DM}_{Sp}^{\text{eff},-}(T) \rightarrow \tilde{DM}_{Sp}^{\text{eff},-}(S),$$

which is determined by the following commutative diagram

$$
\begin{array}{ccc}
D_{Sp}(T) & \xrightarrow{f^*} & D_{Sp}(S) \\
\downarrow & & \downarrow \\
\tilde{DM}_{Sp}^{\text{eff},-}(T) & \xrightarrow{f^*} & \tilde{DM}_{Sp}^{\text{eff},-}(S)
\end{array}
$$

(4) Suppose $i \geq 0$. There is an exact functor

$$-\{−i\}: \tilde{DM}_{Sp}^{\text{eff},-}(S) \rightarrow \tilde{DM}_{Sp}^{\text{eff},-}(S),$$

which is determined by the following commutative diagram

$$
\begin{array}{ccc}
\tilde{DM}_{Sp}^{\text{eff},-}(S) & \xrightarrow{-\{−i\}} & \tilde{DM}_{Sp}^{\text{eff},-}(S) \\
\downarrow & & \downarrow \\
\tilde{DM}_{Sp}^{\text{eff},-}(S) & \xrightarrow{-\{−i\}} & \tilde{DM}_{Sp}^{\text{eff},-}(S)
\end{array}
$$
(5) Suppose $i \geq 0$. There is an exact functor

$$\{-i\} : \widetilde{DM}_{Sp}^{eff,-}(S) \to \widetilde{DM}_{Sp}^{eff,-}(S),$$

which is determined by the following commutative diagram

$$
\begin{array}{ccc}
D_{Sp}(S) & \xrightarrow{-\{i\}} & D_{Sp}(S) \\
\downarrow & & \downarrow \\
\widetilde{DM}_{Sp}^{eff,-}(S) & \xrightarrow{-\{i\}} & \widetilde{DM}_{Sp}^{eff,-}(S)
\end{array}
$$

Proof. For (1), (2), (3), take $j = 0$ in the definition of $\phi$ and proceed as Proposition 6.12 by using Proposition 6.33. For (4), take the quadruple $(S, S, S, i)$ and use Proposition 6.33. And (5) holds by Proposition 6.32.

Proposition 6.35. (1) Let $f : S \to T$ be a smooth morphism in $Sm/k$. We have an adjoint pair

$$f_\#: \widetilde{DM}_{Sp}^{eff,-}(S) \rightleftharpoons \widetilde{DM}_{Sp}^{eff,-}(T) : f^*.$$

(2) We have an adjoint pair

$$\{-i\} : \widetilde{DM}_{Sp}^{eff,-}(S) \rightleftharpoons \widetilde{DM}_{Sp}^{eff,-}(S) : \{-i\}.$$

Proof. The same as Proposition 6.35.

Proposition 6.36. (1) There is an exact functor

$$\Sigma^\infty : \widetilde{DM}_{Sp}^{eff,-}(S) \to \widetilde{DM}_{Sp}^{eff,-}(S)$$

determined by the following commutative diagram

$$
\begin{array}{ccc}
D^{-}(S) & \xrightarrow{\Sigma^\infty} & D_{Sp}(S) \\
\downarrow & & \downarrow \\
\widetilde{DM}_{Sp}^{eff,-}(S) & \xrightarrow{\Sigma^\infty} & \widetilde{DM}_{Sp}^{eff,-}(S)
\end{array}
$$

(2) There is an exact functor

$$\Omega^\infty : \widetilde{DM}_{Sp}^{eff,-}(S) \to \widetilde{DM}_{Sp}^{eff,-}(S)$$

determined by the following commutative diagram

$$
\begin{array}{ccc}
D_{Sp}(S) & \xrightarrow{\Omega^\infty} & D^{-}(S) \\
\downarrow & & \downarrow \\
\widetilde{DM}_{Sp}^{eff,-}(S) & \xrightarrow{\Omega^\infty} & \widetilde{DM}_{Sp}^{eff,-}(S)
\end{array}
$$

Proof. (1) This is because tensor products with $\mathbb{G}_m^1(S)$ preserves $A^1$-equivalences and Proposition 6.33.

(2) This follows by Proposition 6.33.

Proposition 6.37. There is an adjoint pair

$$\Sigma^\infty : \widetilde{DM}_{Sp}^{eff,-}(S) \rightleftharpoons \widetilde{DM}_{Sp}^{eff,-}(S) : \Omega^\infty.$$

Proof. The same as Proposition 6.35.

Proposition 6.38. The functor $\Sigma^\infty : \widetilde{DM}_{Sp}^{eff,-}(S) \to \widetilde{DM}_{Sp}^{eff,-}(S)$ is fully faithful.

Proof. The same as Proposition 6.26.
Proposition 6.39. (1) We have a commutative diagram (up to a canonical isomorphism)

\[
\begin{array}{c}
\mathcal{DM}_{\text{eff}}^{-}(S) \times \mathcal{DM}_{\text{eff}}^{-}(S) \xrightarrow{\otimes} \mathcal{DM}_{\text{eff}}^{-}(S) \\
\Sigma^\infty \times \Sigma^\infty \downarrow \downarrow \downarrow \downarrow \\
\mathcal{DM}_{\text{Sp}}^{-}(S) \times \mathcal{DM}_{\text{Sp}}^{-}(S) \xrightarrow{\otimes} \mathcal{DM}_{\text{Sp}}^{-}(S)
\end{array}
\]

(2) Suppose \( f : S \rightarrow T \) is a morphism in \( Sm/k \). We have a commutative diagram (up to a canonical isomorphism)

\[
\begin{array}{c}
\mathcal{DM}_{\text{eff}}^{-}(T) \xrightarrow{f^*} \mathcal{DM}_{\text{Sp}}^{-}(S) \\
\Sigma^\infty \downarrow \downarrow \downarrow \\
\mathcal{DM}_{\text{Sp}}^{-}(T) \xrightarrow{f^*} \mathcal{DM}_{\text{Sp}}^{-}(S)
\end{array}
\]

(3) Suppose \( f : S \rightarrow T \) is a smooth morphism in \( Sm/k \). We have a commutative diagram (up to a canonical isomorphism)

\[
\begin{array}{c}
\mathcal{DM}_{\text{eff}}^{-}(S) \xrightarrow{f^#} \mathcal{DM}_{\text{Sp}}^{-}(T) \\
\Sigma^\infty \downarrow \downarrow \downarrow \\
\mathcal{DM}_{\text{Sp}}^{-}(S) \xrightarrow{f^#} \mathcal{DM}_{\text{Sp}}^{-}(T)
\end{array}
\]

Proof. This follows by direct computations. \( \Box \)

In [CD07 Proposition 3.5] and [CD13 Proposition 5.2.16], they defined \( \mathcal{DM}_{\text{Sp}}^{-}(S) \) as the Verdier localization of \( D_{\text{Sp}}^{-}(S) \) with respect to homotopy invariant conditions. So the localization induces a triangulated structure on \( \mathcal{DM}_{\text{Sp}}^{-}(S) \) (See [Kra09 Lemma 4.3.1]). And this is the triangulated structure we will impose on \( \mathcal{DM}_{\text{Sp}}^{-}(S) \).

Proposition 6.40. There is a fully faithful exact functor \( \mathcal{DM}_{\text{eff}}^{-}(S) \rightarrow \mathcal{DM}_{\text{Sp}}^{-}(S) \) which is determined by the commutative diagram

\[
\begin{array}{c}
D_{\text{Sp}}^{-}(S) \longrightarrow D_{\text{Sp}}^{-}(S) \\
\downarrow \downarrow \downarrow \downarrow \\
\mathcal{DM}_{\text{eff}}^{-}(S) \longrightarrow \mathcal{DM}_{\text{Sp}}^{-}(S)
\end{array}
\]

Proof. The same as Proposition 6.15 by using Proposition 6.31. \( \Box \)

And there is also a compatibility result between the natural inclusion and \( \otimes, f^*, f^# , \Sigma^\infty, \{ -i \}, i \geq 0 \) like Proposition 6.8.

6.2.4. On Categories of Stabilized Motives.

Definition 6.11. (See [CD13 5.3.23]) Define \( \mathcal{E}_\Omega \) to be the smallest thick subcategory of \( \mathcal{DM}_{\text{Sp}}^{-}(S) \) such that

1. \( \text{Cone}(\Sigma^\infty \tilde{Z}_S(X)\{ -1 \} \rightarrow \Sigma^\infty \tilde{Z}_S(X)\{ -i \}) \in \mathcal{E}_\Omega \) for every \( X \in Sm/S, i \geq 0 \).
2. \( \mathcal{E}_\Omega \) is closed under arbitrary direct sums.

Set \( W_\Omega \) be the class of morphisms in \( \mathcal{DM}_{\text{Sp}}^{-}(S) \) whose cone is in \( \mathcal{E}_\Omega \). Define

\[
\mathcal{DM}^{-}(S) = \mathcal{DM}_{\text{Sp}}^{-}(S)[W_\Omega^{-1}]
\]

to be the category of stabilized motives over \( S \). And a morphism in \( \mathcal{DM}_{\text{Sp}}^{-}(S) \) is called a stable \( \mathbb{A}^1 \)-equivalence if it becomes an isomorphism in \( \mathcal{DM}^{-}(S) \).
Definition 6.12. A complex $K \in \overline{D}_\text{eff}^-(S)$ is called $\Omega$-local if for every stable $A^1$-equivalence $f : A \to B$, the induced map

$$\text{Hom}_{\overline{D}_\text{eff}^-(S)}(B, K) \to \text{Hom}_{\overline{D}_\text{eff}^-(S)}(A, K)$$

is an isomorphism.

By the same method as in Proposition 6.34, we have the following proposition

Proposition 6.41. (1) There is a tensor product

$$\otimes_S : \overline{D}^-(S) \times \overline{D}^-(S) \to \overline{D}^-(S),$$

which is determined by the following commutative diagram

$$\overline{D}^\text{eff}^-_S(S) \times \overline{D}^\text{eff}^-_S(S) \to \overline{D}^\text{eff}^-_S(S).$$

(2) Suppose $f : S \to T$ is a smooth morphism in $\text{Sm}/k$. There is an exact functor

$$f_* : \overline{D}^-(S) \to \overline{D}^-(T),$$

which is determined by the following commutative diagram

$$\overline{D}^\text{eff}^-_S(S) \to \overline{D}^\text{eff}^-_S(T).$$

(3) Suppose $f : S \to T$ is a morphism in $\text{Sm}/k$. There is an exact functor

$$f^* : \overline{D}^-(T) \to \overline{D}^-(S),$$

which is determined by the following commutative diagram

$$\overline{D}^\text{eff}^-_S(T) \to \overline{D}^\text{eff}^-_S(S).$$

(4) Suppose $i \geq 0$. There is an exact functor

$$\{-i\} : \overline{D}^-(S) \to \overline{D}^-(S),$$

which is determined by the following commutative diagram

$$\overline{D}^\text{eff}^-_S(S) \to \overline{D}^\text{eff}^-_S(S).$$

We denote by $\Sigma^\infty,\text{st}$ the composition

$$\overline{D}^\text{eff}^-_S(S) \to \overline{D}^\text{eff}^-_S(S) \to \overline{D}^-(S).$$

Lemma 6.4. Let $\mathcal{C}$ be a symmetric monoidal category and $T \in \mathcal{C}$. If there is a $U \in \mathcal{C}$ such that $U \otimes T \cong \mathbb{1}$, then there are isomorphisms

$$\text{ev} : U \otimes T \to \mathbb{1}, \text{coev} : \mathbb{1} \to T \otimes U$$

such that $T$ is strongly dualizable (See [CD13, 2.4.30]) with the dual $U$ under these two maps.
Proof. Let $F = - \otimes U$ and $G = - \otimes T$. Then the condition gives an endoequivalence

$$F : C \equiv C : G$$

i.e., two natural isomorphisms $a : FG \to id$ and $b : id \to GF$. So we could construct the following two morphisms

$$\theta : \text{Hom}(FX, Y) \xrightarrow{G} \text{Hom}(GFX, GY) \xrightarrow{b^*} \text{Hom}(X, GY)$$

and

$$\eta : \text{Hom}(X, GY) \xrightarrow{F} \text{Hom}(FX, FGY) \xrightarrow{a^*} \text{Hom}(FX, Y)$$

for every $X, Y \in C$. Let $\theta_1$ be the composition

$$F \xrightarrow{id \times b} FGF \xrightarrow{a \times id_F} F$$

and $\theta_2$ be

$$G \xrightarrow{b \times id_G} GFG \xrightarrow{id_G \times a} G.$$

Then $(\eta \circ \theta)(f) = \theta_1(X) \circ f$ and $(\theta \circ \eta)(g) = g \circ \theta_2(Y)$. So $\theta$ is an isomorphism, hence $F$ is a left adjoint of $G$ (vice versa).

\[\square\]

Proposition 6.42. The element $\Sigma^{\infty, st}(\mathbb{1}_S \{1\})$ has a strong dual $(\Sigma^{\infty, st} \mathbb{1}_S \{-1\})$ in $\overline{DM}^-(S)$ with the evaluation and coevaluation maps being isomorphisms.

Proof. By Definition 6.11 and the lemma above. \[\square\]

Hence we define $C(i)$ to be $C \otimes_S \Sigma^{\infty, st}(\mathbb{1}_S(i))$ and $C(-i)$ to be $C[-i][i]$ for any $C \in \overline{DM}^-$ and $i \geq 0$.

Proposition 6.43. (See [CD13, Proposition 5.3.25]) Suppose $E = \overline{CH}$. The functor

$$\Sigma^{\infty, st} : \overline{DM}^\text{eff,-} (pt) \to \overline{DM}^-(pt)$$

is fully faithful.

Proof. We first prove that for every projective $E \in \overline{DM}^\text{eff,-} (pt)$, $\Sigma^\infty E \in \overline{DM}^\text{eff,-} (pt)$ is $\Omega$-local. By the same method as in [MVW06, Lemma 9.20], this is equivalent to for any $X \in Sm/S$, $i \geq 0$ and $n \in \mathbb{Z}$, the morphism

$$\text{Hom}(\Sigma^\infty \tilde{Z}_{\text{pt}}(X)\{1\}, -1 \{-i\}, \Sigma^\infty E[n]) \to \text{Hom}(\Sigma^\infty \tilde{Z}_{\text{pt}}(X)\{-i\}, \Sigma^\infty E[n])$$

is an isomorphism. And this follows by the following commutative diagram

$$\begin{align*}
\text{Hom}(\Sigma^\infty \tilde{Z}_{\text{pt}}(X)\{1\}, -1 \{-i\}, \Sigma^\infty E[n]) & \to \text{Hom}(\Sigma^\infty \tilde{Z}_{\text{pt}}(X)\{-i\}, \Sigma^\infty E[n]) \\
\text{Hom}(\Sigma^\infty \tilde{Z}_{\text{pt}}(X)\{1\}, -1 \{-i\}, \Sigma^\infty E[n]) & \to \text{Hom}(\Sigma^\infty \tilde{Z}_{\text{pt}}(X), \Sigma^\infty \tilde{Z}_{\text{pt}}(X)\{1\}, \Sigma^\infty E[n]) \\
\text{Hom}(\Sigma^\infty \tilde{Z}_{\text{pt}}(X)\{1\}, \Sigma^\infty E[n]) & \to \text{Hom}(\Sigma^\infty \tilde{Z}_{\text{pt}}(X)\{1\}, \Sigma^\infty E[n]) \\
\text{Hom}(\Sigma^\infty \tilde{Z}_{\text{pt}}(X)\{1\}, \Sigma^\infty E[n]) & \to \text{Hom}(\Sigma^\infty \tilde{Z}_{\text{pt}}(X)\{1\}, \Sigma^\infty E[n])
\end{align*}$$

\[\square\]
and \cite[Theorem 3.3.8]{DF17}. Suppose $K, L \in \tilde{DM}^{\text{eff}, -}_{\text{Sp}}(pt)$ with projective resolutions $P, Q$ respectively. The statement follows by the following commutative diagram

$$
\begin{align*}
\text{Hom}_{\tilde{DM}^{\text{eff}, -}_{\text{Sp}}(pt)}(K, L) & \xrightarrow{\Sigma^{\infty, st}} \text{Hom}_{\tilde{DM}^{\text{eff}, -}_{\text{Sp}}(pt)}(\Sigma^{\infty, st} K, \Sigma^{\infty, st} L) \\
\text{Hom}_{\tilde{DM}^{\text{eff}, -}_{\text{Sp}}(pt)}(P, Q) & \xrightarrow{\Sigma^{\infty, st}} \text{Hom}_{\tilde{DM}^{\text{eff}, -}_{\text{Sp}}(pt)}(\Sigma^{\infty, st} P, \Sigma^{\infty, st} Q) \\
\text{Hom}_{\tilde{DM}^{\text{eff}, -}_{\text{Sp}}(pt)}(\Sigma^{\infty} P, \Sigma^{\infty} Q)
\end{align*}
$$

and Proposition \ref{prop:exact functor}. \hfill \Box

**Proposition 6.44.** Let $f : S \rightarrow T$ be a smooth morphism in $\text{Sm}/k$. We have an adjoint pair $f_{\#} : \tilde{DM}^{-}(S) \rightleftarrows \tilde{DM}^{-}(T) : f^{*}$.

**Proof.** The same as Proposition \ref{prop:adjoint pair}. \hfill \Box

**Proposition 6.45.** Suppose $f : S \rightarrow T$ is a morphism in $\text{Sm}/k$.

1. For any $K, L \in \tilde{DM}^{-}(T)$, we have $f^{*}(K \otimes_{S} L) \cong (f^{*} K) \otimes_{S} (f^{*} L)$.

2. If $f$ is smooth, then for any $K \in \tilde{DM}^{-}(S)$ and $L \in \tilde{DM}^{-}(T)$, we have $f_{\#}(K \otimes_{S} f^{*} L) \cong (f_{\#} K) \otimes_{S} L$.

**Proof.** This is because everything works termwise for spectra by discussion in Section \ref{section:triangulated structure}. \hfill \Box

In \cite[Proposition 5.3.23]{CD13}, they defined $\tilde{DM}(S)$ as the the Verdier localization of $\tilde{DM}^{\text{eff}, -}_{\text{Sp}}(S)$ with respect to $W_{\Omega}$. So the localization induces a triangulated structure on $\tilde{DM}(S)$ (See \cite[Lemma 4.3.1]{Kra09}). And this is the triangulated structure we will impose on $\tilde{DM}(S)$.

Here is a weak result which is enough for our purpose:

**Proposition 6.46.** There is an exact functor $\tilde{DM}^{-}(S) \rightarrow \tilde{DM}(S)$ which is determined by the commutative diagram

$$
\begin{align*}
\tilde{DM}^{\text{eff}, -}_{\text{Sp}}(S) & \xrightarrow{\Sigma^{\infty, st}} \tilde{DM}^{\text{eff}, -}_{\text{Sp}}(S) \\
\tilde{DM}^{-}(S) & \xrightarrow{\Sigma^{\infty, st}} \tilde{DM}(S)
\end{align*}
$$

And when $E = CH$ and $S = pt$, the morphism $\text{Hom}_{\tilde{DM}^{-}}(X, Y) \rightarrow \text{Hom}_{\tilde{DM}(S)}(L(X), L(Y))$ is an isomorphism if $X$ and $Y$ are of the form $(\Sigma^{\infty, st} A)\{i\}$, $i \geq 0$.

**Proof.** The functor is induced and exact by \cite[Proposition 4.6.2]{Kra09}. We have thus a commutative diagram (up to a natural isomorphism)

$$
\begin{align*}
\tilde{DM}^{\text{eff}, -}_{\text{Sp}}(S) & \xrightarrow{\Sigma^{\infty, st}} \tilde{DM}^{\text{eff}, -}_{\text{Sp}}(S) \\
\tilde{DM}^{-}(S) & \xrightarrow{\Sigma^{\infty, st}} \tilde{DM}(S)
\end{align*}
$$

Now let $E = CH$ and $S = pt$. We denote $\mathcal{P}(X, Y)$ be the property that the statement holds for $X$ and $Y$. Then if $\mathcal{P}(X, Y)$ is true, then for any $X' \cong X$ and $Y' \cong Y$, $\mathcal{P}(X', Y')$ is also true. And by Proposition \ref{prop:exact functor} and the fact that $L$ is monoidal, $\mathcal{P}(X\{i\}, Y\{i\})$ is also true.

By Proposition \ref{prop:exact functor} and the diagram above, $\mathcal{P}(\Sigma^{\infty, st} A, \Sigma^{\infty, st} B)$ is true for any $A, B \in \tilde{DM}^{\text{eff}, -}_{\text{Sp}}(pt)$. Hence the statement follows. \hfill \Box
And there is also a compatibility result between the natural inclusion and $\otimes_S, f^*, f^#, -\{i\}, i \geq 0$ like Proposition 6.8.

7. Orientations on Symplectic Bundles and Applications

7.1. Orientations on Symplectic Bundles. In this section, we want to extend the quaternionic projective bundle theorem and Thom isomorphism in [Yan17] to arbitrary smooth base $S$. Recall the definition of orientable bundles from [Yan17] Definition 4.1.

Definition 7.1. Let $E$ be a correspondence theory, we call it symplectic oriented if for every $X \in Sm/k$ and rank two symplectic bundle $E$ over $X$, there is an isomorphism

$$s_E : E \to 0$$

in $\mathcal{P}_X$ such that

1. For every isomorphism $\varphi : E_1 \to E_2$ between rank two symplectic bundles (preserving inner products), we have $s_{E_1} = s_{E_2} \circ \varphi$.
2. For every $f : X \to Y$ in $Sm/k$, we have $s_{f^*E} = f^*(s_E)$.

It is then clear that MW-correspondence is symplectic oriented. From now on, we assume $E = CH$ until the end of this section.

Definition 7.2. Let $E$ be an orientable bundle over $X \in Sm/S$ with an orientation $s$ and rank $n$, it has a map

$$e(E) : CH^0(X) \to CH^n(X)$$

by [Yan17] Definition 4.2. So we have an element

$$e(E)(1) \in Hom_{DM_{fl,-}(pt)}(\tilde{Z}_{pt}(X), \tilde{Z}_{pt}(n)[2n]).$$

It induces a morphism

$$\theta : \tilde{Z}_{S}(X) \to \tilde{Z}_{S}(n)[2n]$$

by Proposition 6.13 which is called the Euler class of $E$ over $S$.

If $n = 2$, then $-\theta$ is called the first Pontryagin class under the orientation $s$ of $E$ over $S$, which is denoted by $p_1(E)$.

Now let $X \in Sm/S$. Suppose we have two morphisms $f_i : \tilde{Z}_{S}(X) \to C_i, i = 1, 2$ in $\tilde{DM}_{fl,-}(S)$, we denote by $f_1 \boxtimes f_2$ the composition

$$\tilde{Z}_{S}(X) \xrightarrow{\Delta} \tilde{Z}_{S}(X) \otimes_S \tilde{Z}_{S}(X) \xrightarrow{f_1 \otimes f_2} C_1 \otimes_S C_2.$$

Suppose we have two morphisms $f_i : \tilde{Z}_{S}(X) \to \tilde{Z}_{S}(n_i)[2n_i]$ in $\tilde{DM}_{fl,-}(S)$, we denote by $f_1 f_2$ the composition

$$\tilde{Z}_{S}(X) \xrightarrow{f_1 \boxtimes f_2} \tilde{Z}_{S}(n_1)[2n_1] \otimes \tilde{Z}_{S}(n_2)[2n_2] \xrightarrow{\otimes} \tilde{Z}_{S}(n_1 + n_2)[2(n_1 + n_2)].$$

The following proposition clarifies the relationship between MW-motivic cohomologies and Chow-Witt groups.

Proposition 7.1. For any $X \in Sm/k$, and $i \geq 0$, we have an isomorphism (Set $\tilde{DM} = \tilde{DM}_{fl,-}(pt)$)

$$Hom_{\tilde{DM}}(\tilde{Z}_{pt}(X), \tilde{Z}_{pt}(i)[2i]) \cong CH^i(X)$$

which is compactible with pull-backs and sends $id_{\tilde{Z}_{pt}}$ to $1$ when $i = 0$ and $X = pt$.

Moreover, the following diagram commutes for any $i, j \geq 0$

$$Hom_{\tilde{DM}}(\tilde{Z}_{pt}(X), \tilde{Z}_{pt}(i)[2i]) \times Hom_{\tilde{DM}}(\tilde{Z}_{pt}(X), \tilde{Z}_{pt}(j)[2j]) \to CH^i(X) \times CH^j(X)$$

where the right-hand map is the intersection product on Chow-Witt groups.

Proof. See [DF17] Corollary 4.2.6. \qed
In [Yan17] Definition 3.5, we defined $H\ell^n$ (Originally defined in [PW10] Section 3]) as an open set of $Gr(2, 2n + 2)$ consisting of those two dimensional subspaces $V \subseteq k^{2n+2}$ such that the trivial symplectic form on $k^{2n+2}$ does not vanish on $V$. We denote $H\ell^n \times_k S$ by $H\ell^n_S$. And for any symplectic bundle (See [Yan17] Definition 3.3) $\mathcal{E}$ of rank two over $X \in Sm/S$, there is a canonical orientation $O_X \longrightarrow det(\mathcal{E})$ induced by the inner product $\langle \cdot, \cdot \rangle \longrightarrow O_X$ (See [Yan17] Definition 4.3). And we have a dual tautological bundle (See discussion after [Yan17] Definition 3.6]) $\mathcal{V}^\vee_S$ over $H\ell^n_S$ satisfying $\mathcal{V}^\vee_S = p^* \mathcal{V}^\vee$ as symplectic bundles where $p : H\ell^n_S \longrightarrow H\ell^n$ is the projection map.

**Theorem 7.1.** The map

$$\tilde{Z}_S(H\ell^n_S) \longrightarrow p_1(\mathcal{V}^\vee)^! \longrightarrow \oplus_{i=0}^n \tilde{Z}_S(2i)[4i]$$

is an isomorphism in $DM^{eff,-}(S)$. Here, $\mathcal{V}^\vee$ is endowed with its canonical orientation.

**Proof.** We have the projection $p : H\ell^n_S \longrightarrow H\ell^n$ as before. Now we have a commutative diagram

$$\begin{array}{ccc}
p^* \tilde{Z}_p(H\ell^n) & \xrightarrow{p^*(p_1(\mathcal{V}^\vee)^!)} & p^* \tilde{Z}_p(2)[4i] \\
\cong & & \cong \\
\tilde{Z}_S(H\ell^n_S) & \xrightarrow{p_1(\mathcal{V}^\vee)^!} & \tilde{Z}_S(2)[4i]
\end{array}$$

Hence the result follows by the commutative diagram

$$\begin{array}{ccc}
p^* \tilde{Z}_p(H\ell^n) & \xrightarrow{p^*(p_1(\mathcal{V}^\vee)^!)} & p^* \tilde{Z}_p(2i)[4i] \\
\cong & & \cong \\
\tilde{Z}_S(H\ell^n_S) & \xrightarrow{p_1(\mathcal{V}^\vee)^!} & \oplus_{i=0}^n \tilde{Z}_S(2i)[4i]
\end{array}$$

where the upper horizontal arrow is an isomorphism by [Yan17] Theorem 4.2].

Recall in [Yan17] Proposition 3.4, we defined $HGr_X(\mathcal{E})$ for any symplectic bundle $\mathcal{E}$ over $X$. It parameterizes rank two symplectic subbundles of $\mathcal{E}$. When $\mathcal{E}$ is the trivial symplectic bundle $(O_X^{2n+2}, -I)$, $HGr_X(\mathcal{E}) \cong H\ell^n \times_k X$ over $X$. It associates a dual tautological bundle $\mathcal{V}^\vee$, which is a rank two symplectic bundle.

**Theorem 7.2.** Let $X \in Sm/S$ and let $(\mathcal{E}, m)$ be a symplectic vector bundle of rank $2n+2$ on $X$. Let $\pi : HGr_X(\mathcal{E}) \longrightarrow X$ be the projection. Then, the map

$$\tilde{Z}_S(HGr_X(\mathcal{E})) \longrightarrow \tilde{Z}_S(HGr_X(\mathcal{E})) \xrightarrow{\pi^*\mathcal{V}^\vee} \oplus_{i=0}^n \tilde{Z}_S(X)(2i)[4i]$$

is an isomorphism in $DM^{eff,-}(S)$, functorial for $X$ in $Sm/k$. Here, $\mathcal{V}^\vee$ is endowed with its canonical orientation.

**Proof.** The same as [Yan17] Theorem 4.3] since we have MV-sequences (Proposition 5.10] and Theorem 7.3] over any base $S$.

**Definition 7.3.** Let $X \in Sm/S$ and $Y \subseteq X$ be a closed subset. Consider the quotient sheaf with $E$-transfers

$$\overline{M}_Y(X) := \tilde{Z}_S(X)/\tilde{Z}_S(X \setminus Y).$$

Its image in $DM^{eff,-}(S)$ will be called the relative motive of $X$ with support in $Y$ (see [Dég] Definition 2.2] and the remark before [SV] Corollary 5.3]). By abuse of notation, we still denote it by $\overline{M}_Y(X)$.

**Definition 7.4.** Suppose $X \in Sm/S$ and $E$ is a vector bundle over $X$. Define $Th_S(E) = \overline{M}_X(E)$ where $X \subseteq E$ is the zero section of $E$. 
Proposition 7.2.\quad (1) Suppose $f : S \to T$ is a morphism in $Sm/k$, $X \in Sm/T$ and $E$ is a vector bundle over $X$. Then we have
\[ f^* Th_T(E) \cong Th_S(f^*E) \]
in $\overline{DM}^{eff,-}(S)$, where $f^*E$ is a vector bundle over $X^S$.

(2) Suppose $f : S \to T$ is a smooth morphism in $Sm/k$, $X \in Sm/S$ and $E$ is a vector bundle over $X$. Then we have
\[ f_# Th_S(E) \cong Th_T(E) \]
in $\overline{DM}^{eff,-}(S)$.

(3) (See [CD13, Remark 2.4.15]) Suppose $E_1$ and $E_2$ are vector bundles over $X \in Sm/k$. Then
\[ Th_X(E_1) \otimes_X Th_X(E_2) \cong Th_X(E_1 \oplus E_2) \]
in $\overline{DM}^{eff,-}(X)$.

Proof. (1) and (2) are easy. Let’s prove (3). The total space of $E_1 \oplus E_2$ is just $E_1 \times_X E_2$. By definition, for any vector bundle $E$ over $X$, $Th_X(E)$ is just the complex
\[ \tilde{\mathcal{Z}}_S(E \setminus X) \to \tilde{\mathcal{Z}}_S(E). \]
Hence the left hand side is the total complex
\[ \tilde{\mathcal{Z}}_S((E_1 \setminus X) \times_X (E_2 \setminus X)) \to \tilde{\mathcal{Z}}_S((E_1 \setminus X) \times_X E_2) \oplus \tilde{\mathcal{Z}}_S(E_1 \times_X (E_2 \setminus X)) \to \tilde{\mathcal{Z}}_S(E_1 \times_X E_2). \]

By Proposition [5.10] the complex
\[ \tilde{\mathcal{Z}}_S((E_1 \setminus X) \times_X (E_2 \setminus X)) \to \tilde{\mathcal{Z}}_S((E_1 \setminus X) \times_X E_2) \oplus \tilde{\mathcal{Z}}_S(E_1 \times_X (E_2 \setminus X)) \]
is quasi-isomorphic to
\[ 0 \to \tilde{\mathcal{Z}}_S((E_1 \times_X E_2) \setminus X) \]
since
\[ (E_1 \times_X E_2) \setminus X = (E_1 \setminus X) \times_X E_2 \cup E_1 \times_X (E_2 \setminus X). \]
Hence we have a quasi-isomorphism
\[ \tilde{\mathcal{Z}}_S((E_1 \setminus X) \times (E_2 \setminus X)) \to \tilde{\mathcal{Z}}_S((E_1 \setminus X) \times E_2) \oplus \tilde{\mathcal{Z}}_S(E_1 \times (E_2 \setminus X)) \to \tilde{\mathcal{Z}}_S(E_1 \times E_2). \]
\[ \boxed{} \]

Proposition 7.3. Let $f : X \to Y$ be an étale morphism in $Sm/S$, $Z \subseteq Y$ be a closed subset of $Y$ such that the map $f : f^{-1}(Z) \to Z$ is an isomorphism (the schemes are endowed with their reduced structure), then the map $\tilde{M}_f^{-1}(Z) \to \tilde{M}_Z(Y)$ is an isomorphism of sheaves with $E$-transfers.

Proof. The same as [Yan17] Proposition 5.1 by using Proposition [5.10] \[\square\]

Theorem 7.3. Let $E$ be a symplectic bundle of rank $2n$ over $X \in Sm/S$. Then we have
\[ Th_S(E) \cong \tilde{\mathcal{Z}}_S(X)(2n)[4n] \]
in $\overline{DM}^{eff,-}(S)$.

Proof. The same as [Yan17] Theorem 5.1 by using Theorem [7.2] \[\square\]

The following observation is quite interesting:

Proposition 7.4. Let $E$ be a vector bundle of rank $n$ over $X \in Sm/k$. Then we have
\[ (\Sigma^{\infty,\text{st}} Th_X(E))^{-1} \cong (\Sigma^{\infty,\text{st}} Th_X(E^\vee))(-2n)[-4n] \]
in $\overline{DM}^{eff,-}(X)$.
Proof. By Proposition 5.22 and Theorem 7.3, we have

\[ \text{Th}_X(E) \otimes_X \text{Th}_X(E^\gamma) \cong \text{Th}_X(E \oplus E^\gamma) \cong \mathbb{I}_X(2n)[4n] \]

in $\tilde{DM}^{\text{eff}}(X)$. Now the statement follows from Proposition 6.42 and the fact that $\Sigma^{\infty, \text{st}}$ is monoidal.

Since we have a monoidal exact functor $L : \tilde{DM}^{\text{eff,f}}(X) \to \tilde{DM}^{\text{eff}}(X)$, by the same proof as above, we also have

**Proposition 7.5.** Let $E$ be a vector bundle of rank $n$ over $X \in Sm/k$. Then we have

\[ (\Sigma^{\infty, \text{st}}\text{Th}_X(E))^{-1} \cong (\Sigma^{\infty, \text{st}}\text{Th}_X(E^\gamma))(-2n)[-4n] \]

in $\tilde{DM}(X)$.

7.2. **Duality for Proper Schemes and Embedding Theorem.** In this section, we are going to prove $\tilde{\mathbb{Z}}_p(X)$ is strongly dualizable in $\tilde{DM}^{-}(pt)$ for proper $X \in Sm/k$. And we calculate its dual by using orientations on symplectic bundles. And finally we will prove the embedding theorem in MW-motivic cohomology. For this we need to involve the stable $A^1$-derived category $D_{A^1}(S)$ over $S$ introduced in [DF17a, Section 1] and [CD13, Example 5.3.31] and use the duality result on that category. For clarity, we describe our procedure like the following:

Duality in $D_{A^1}(S) \iff$ Duality in $\tilde{DM}(S) \iff$ Duality in $\tilde{DM}^{-}(S) \iff$ Embedding Theorem.

Let’s briefly review the construction of $D_{A^1}(S)$, the reader may also refer to [CD13, Section 5] and [DF17a, Section 1].

Define $\text{Sh}(S)$ to be category of Nisnevich sheaves of abelian groups on $Sm/S$. The Yoneda representative of $F \mapsto F(X)$ for any $X \in Sm/S$ is denoted by $\mathbb{Z}_S(X)$. The functor $\gamma : Sm/S \to \text{Cor}_S$ in Proposition 5.9 and Lemma 5.15 give us an adjunction

\[ \gamma^* : \text{Sh}(S) \leftarrow \tilde{\mathbb{Z}}_p(S) : \tilde{\gamma}_* \]

The category $\text{Sh}(S)$ is a symmetric monoidal category with $\mathbb{Z}_S(X) \otimes_S \mathbb{Z}_S(Y) \cong \mathbb{Z}_S(X \times_S Y)$ and $\gamma^*$ is a monoidal functor. For any $f : S \to T$ in $Sm/k$, there is an adjunction

\[ f^* : \text{Sh}(T) = \tilde{\mathbb{Z}}_p(S) : f_* \]

by the same method as in Proposition 5.19 and $f^* \gamma^* \cong \gamma^* f^*$ since there is an similar equality for their right adjoints. If $f$ is smooth, there is an adjunction

\[ f_# : \text{Sh}(S) = \tilde{\mathbb{Z}}_p(S) : f^* \]

as in Proposition 5.23. Then $f_# \gamma^* \cong \gamma^* f_#$ since there is an similar equality for their right adjoints.

By the same method as in Section 6.2.1 we define $SSp(S)$ to be the category of symmetric $\mathbb{I}_S[1]$-spectra by $\text{Sh}(S)$, where

\[ \mathbb{I}_S[1] = \text{Coker}(\mathbb{Z}_S(S) \to \mathbb{Z}_S(G_m)). \]

There are adjunctions

\[ \Sigma^\infty : \text{Sh}(S) = SSp(S) : \Omega^\infty \]

and

\[ \tilde{\gamma}^* : SSp(S) = Sp(S) : \tilde{\gamma}_* \]

And we could also define $\otimes_S, f^*, f_*, f_#,$ $\{-i\}$ and $\{-i\}$ ($i \geq 0$) on $SSp(S)$. Moreover, $\gamma^*$ commutes with $f^*$ and $f_#$ and it’s monoidal as above.

In [CD17, Theorem 1.7], they defined a model structure $\mathcal{M}_S$ on the category of unbounded complexes of $\text{Sh}(S)$. This is a cofibrantly generated model structure where the cofibrations are those $I$-cofibrations where $I$ consists of the morphisms $S^{n+1}(\mathbb{Z}_S(X)) \to D^n(\mathbb{Z}_S(X))$ for any $X \in Sm/S$ and weak equivalences are quasi-morphisms between complexes. The homotopy category of $\mathcal{M}_S$ is denoted by $D_S(S)$. Moreover, $\mathcal{M}_S$ is stable and left proper so it induces a triangulated structure on $D_S(S)$.

By localizing $D_S(S)$ with morphisms

\[ \mathbb{Z}_S(X \times_k k^1) \to \mathbb{Z}_S(X) \]

as in Section 6.1 we get a category $D_{A^1}^f(S)$ with the induced triangulated structure.
In \cite{CD07}, Theorem 1.7, they defined a model structure $\mathcal{M}_{SSp}$ on the category of unbounded complexes of symmetric spectra of $\text{Sh}(S)$. This is a cofibrantly generated model structure where the cofibrations are those $I$-cofibrations where $I$ consists of the morphisms $S^{n+1}\left(\Sigma^\infty Z_S(X)\{i\}\right) \rightarrow D^+(\Sigma^\infty \check{Z}_S(X)\{i\})$ for any $X \in \text{Sm}/S$ and $i \geq 0$ and weak equivalences are quasi-morphisms between complexes. The homotopy category of $\mathcal{M}_{SSp}$ is denoted by $D_{SSp}(S)$. Moreover, $\mathcal{M}_{SSp}$ is stable and left proper so it induces a triangulated structure on $D_{SSp}(S)$.

By localizing $D_{SSp}(S)$ with morphisms

$$(\Sigma^\infty Z_S(X \times_k \mathbb{A}^1) \rightarrow \Sigma^\infty Z_S(X))\{i\}, i \geq 0$$

as in Section 6.2.3, we get a category with the induced triangulated structure. And localizing that category by

$$(\Sigma^\infty Z_S(X)\{1\} \rightarrow \Sigma^\infty Z_S(X))\{-i\}$$

as in Section 6.2.3, we’ve got our category of bounded complexes of symmetric spectra of $\text{Sh}$.

Here are some further properties needed of the stable $\mathbb{A}^1$-derived categories, they come from \cite{DF17a} 1.1.7 and Theorem 1.1.10.

**Proposition 7.6.**

1. For any $f : S \rightarrow T$ in $\text{Sm}/k$, we have an adjoint pair of exact functors
   
   $$f^* : D_{\mathbb{A}^1}(T) \rightleftarrows D_{\mathbb{A}^1}(S) : f_*$$

2. For any smooth $f : S \rightarrow T$ in $\text{Sm}/k$, we have an adjoint pair of exact functors
   
   $$f_* : D_{\mathbb{A}^1}(S) = D_{\mathbb{A}^1}(T) : f^*.$$

And for any $A \in D_{\mathbb{A}^1}(S)$ and $B \in D_{\mathbb{A}^1}(T)$, we have

$$(f_* A) \otimes B \cong f_* (A \otimes f^* B).$$

3. For any $f : S \rightarrow T$ in $\text{Sm}/k$, we have a functor
   
   $$f_* : D_{\mathbb{A}^1}(S) \rightarrow D_{\mathbb{A}^1}(T).$$

If $f$ is proper, we have

$$f_* \cong f_!.$$

If $f$ is smooth, we have

$$f_* \cong f_*(- \otimes (\Sigma^\infty \text{st} \text{Th}_S(T/S/T))^{-1}).$$

**Proposition 7.7.** Let $S \in \text{Sm}/k$ and $f : X \rightarrow S$ be a smooth proper morphism. Then $\Sigma^\infty \text{st} Z_S(X) \in D_{\mathbb{A}^1}(S)$ is strongly dualizable with dual $f_*(\Sigma^\infty \text{st} \text{Th}_X(T_{X/S})^{-1})$.

**Proof.** Pick any $A, B \in D_{\mathbb{A}^1}(S)$, we have

$$\text{Hom}_{D_{\mathbb{A}^1}(S)}(\Sigma^\infty \text{st} Z_S(X) \otimes S A, B) \cong \text{Hom}_{D_{\mathbb{A}^1}(S)}(f_* f^* A, B)$$

by Proposition 7.6 (2)

$$\cong \text{Hom}_{D_{\mathbb{A}^1}(S)}(A, f_* f^* B)$$

by Proposition 7.6 (1) and (2)

$$\cong \text{Hom}_{D_{\mathbb{A}^1}(S)}(A, f_! f^* B)$$

by Proposition 7.6 (3)

$$\cong \text{Hom}_{D_{\mathbb{A}^1}(S)}(A, f_!(f^* B \otimes_X (\Sigma^\infty \text{st} \text{Th}_X(T_{X/S}))^{-1}))$$

by Proposition 7.6 (3)

$$\cong \text{Hom}_{D_{\mathbb{A}^1}(S)}(A, B \otimes_s f_!(\Sigma^\infty \text{st} \text{Th}_X(T_{X/S})^{-1}))$$

by Proposition 7.6 (2).
Proposition 7.8. Let $S \in Sm/k$ and $f : X \to S$ be a smooth proper morphism. Then $\Sigma^{\infty, st}\tilde{Z}_S(X) \in \text{DM}(S)$ is strongly dualizable with dual

$$(\Sigma^{\infty, st}Th_S(\Omega_{X/S}))(-2d)[-4d],$$

where $d = d_X - d_S$.

Proof. Since we have a monoidal exact functor $\gamma^* : \text{DM}^-(pt) \to \text{DM}(pt)$ which commutes with $f_\#$ up to a natural isomorphism, $\Sigma^{\infty, st}\tilde{Z}_S(X) \in \text{DM}(S)$ is strongly dualizable with dual $f_!(\Sigma^{\infty, st}Th_X(T_{X/Y})^{-1})$ by Proposition 6.46. And by Proposition 7.8.

Finally, we have

$$f_!(\Sigma^{\infty, st}Th_X(\Omega_{X/S}))(-2d)[-4d]) \cong (\Sigma^{\infty, st}Th_S(\Omega_{X/S}))(-2d)[-4d].$$

Now we have a monoidal exact functor $L : \text{DM}^-_{\text{eff}}(pt) \to \text{DM}^-_{\text{eff}}(pt)$ which commutes with $-\{i\}$, $i \geq 0$ up to a natural isomorphism. And by Proposition 6.46 we have Proposition 7.9. Let $X \in Sm/k$ be a proper scheme. Then $\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \in \text{DM}^-_{\text{eff}}(pt)$ is strongly dualizable with dual

$$(\Sigma^{\infty, st}Th_{pt}(\Omega_{X/k}))(-2d_X)[-4d_X].$$

Theorem 7.4. Let $X, Y \in Sm/k$ with $Y$ proper, then we have

$$\text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\tilde{Z}_{pt}(X), \tilde{Z}_{pt}(Y)) \cong CH^{-d_Y} \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(X \times Y, -T_{X \times Y/X}).$$

Proof. Let $p : Y \to pt$ be the structure map of $Y$ and $q : X \times Y \to Y$ be the second projection. We have

$$\text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\tilde{Z}_{pt}(X), \tilde{Z}_{pt}(Y))$$

by Proposition 7.2.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X), \Sigma^{\infty, st}\tilde{Z}_{pt}(Y))$$

by Proposition 7.8.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \otimes (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})), (\Sigma^{\infty, st}Th_{pt}(\Omega_{X/S}))(-2d)[-4d][-4d])$$

by Proposition 6.42.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \otimes Th_{pt}(\Omega_{Y/k}), (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})))$$

by Proposition 7.9.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \otimes (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})), (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})))$$

by Proposition 7.2.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \otimes (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})), (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})))$$

by Proposition 7.2.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \otimes Th_{pt}(\Omega_{Y/k}), (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})))$$

by Proposition 6.42.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \otimes Th_{pt}(\Omega_{Y/k}), (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})))$$

by Proposition 7.9.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \otimes Th_{pt}(\Omega_{Y/k}), (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})))$$

by Proposition 7.2.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \otimes Th_{pt}(\Omega_{Y/k}), (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})))$$

by Proposition 7.2.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \otimes Th_{pt}(\Omega_{Y/k}), (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})))$$

by Proposition 7.2.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \otimes Th_{pt}(\Omega_{Y/k}), (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})))$$

by Proposition 7.2.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \otimes Th_{pt}(\Omega_{Y/k}), (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})))$$

by Proposition 7.2.

$$\cong \text{Hom}_{\text{DM}_{\text{eff}}(pt)}(\Sigma^{\infty, st}\tilde{Z}_{pt}(X) \otimes Th_{pt}(\Omega_{Y/k}), (\Sigma^{\infty, st}Th_{pt}(\Omega_{Y/k})))$$

by Proposition 7.2.
Now we could define a category of $\overline{\text{Cor}}_k^\prime$ to be a category with objects being proper schemes in $\text{Sm}_n/k$ and

$$\text{Hom}_{\overline{\text{Cor}}_k^\prime}(X,Y) = \overline{\text{CH}}^{d_Y}(X \times Y, -T_{X \times Y/X}).$$

It’s a category because by the same proof as in Proposition [6.1] and Proposition [5.5]

**Definition 7.5.** Define the category of effective Chow-Witt motives $\overline{\text{CH}}^{eff}$ by the idempotent completion (See [BS01, Definition 1.2]) of the opposite category of $\overline{\text{Cor}}_k^\prime$.

**Proposition 7.10.** The opposite category of $\overline{\text{CH}}^{eff}$ is a full-subcategory of $\overline{\text{DM}}^{eff,-}(pt)$.

**Proof.** We have a fully faithful functor

$$\overline{\text{Cor}}_k^\prime \rightarrow \overline{\text{DM}}^{eff,-}(pt)$$

by Theorem 7.4. The category $\overline{\text{DM}}^{eff}(pt)$ has infinite direct sums so it’s idempotent complete (See [Noi] Proposition 1.6.8). Now suppose $g : K \rightarrow K$ is an idempotent in $\overline{\text{DM}}^{eff,-}(pt)$. It splits in $\overline{\text{DM}}^{eff}(pt)$ with the image morphism $f : K \rightarrow I$ and the section $s : I \rightarrow K$. Then $C_\ast(g)$ also splits with the image morphism $C_\ast(f)$ and the section $C_\ast(s)$ by [DF17, Corollary 3.2.11]. Now $f$ and $s$ comes from $D(pt)$ by loc. cit., hence $C_\ast(I)$ is isomorphic to an object in $D^-(pt)$. Hence $I$ is isomorphic to an object in $\overline{\text{DM}}^{eff,-}(pt)$. Hence $\overline{\text{DM}}^{eff,-}(pt)$ is idempotent complete thus we get a functor (See [BS01, Proposition 1.3])

$$\overline{\text{CH}}^{eff} \rightarrow \overline{\text{DM}}^{eff,-}(pt)$$

which is also fully faithful by the construction of idempotent completion. □

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