DIVERGENT SOLUTIONS TO THE 5D HARTREE EQUATIONS

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Abstract. We consider the Cauchy problem for the focusing Hartree equation \( iu_t + \Delta u + (| \cdot |^{-3} * |u|^2)u = 0 \) in \( \mathbb{R}^5 \) with the initial data in \( H^1 \), and study the divergent property of infinite-variance and nonradial solutions. Letting \( Q \) be the ground state solution of \( -Q + \Delta Q + (| \cdot |^{-3} * |Q|^2)Q = 0 \) in \( \mathbb{R}^5 \), we prove that if \( u_0 \in H^1 \) satisfying \( M(u_0)E(u_0) < M(Q)E(Q) \) and \( \| \nabla u_0 \|_2 \cdot \| u_0 \|_2 > \| \nabla Q \|_2 \cdot \| Q \|_2 \), then the corresponding solution \( u(t) \) either blows up in finite forward time, or exists globally for positive time and there exists a time sequence \( t_n \to +\infty \) such that \( \| \nabla u(t_n) \|_2 \to +\infty \). A similar result holds for negative time.

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1. Introduction

In this paper, we consider the following Cauchy problem for the 5D Hartree equation

\[
\begin{cases}
  iu_t + \Delta u + (V * |u|^2)u = 0, & (x, t) \in \mathbb{R}^5 \times \mathbb{R}, \\
  u(x, 0) = u_0(x) \in H^1(\mathbb{R}^5),
\end{cases}
\]

where \( V(x) = |x|^{-3} \) and \( * \) denotes the convolution in \( \mathbb{R}^5 \).

Hartree type nonlinearity \((| \cdot |^{2-N} * |u|^2)u \) in \( \mathbb{R}^N \) describes the dynamics of the mean-field limits of many-body quantum systems such as coherent states and condensates. The case \( N = 4 \) gives the \( L^2 \)-critical Hartree equation, the solution of which, by the authors in [21], scatters when the mass of the initial data is strictly less than that of the ground state. A large amount of work has been devoted to the theory of scattering for the Hartree equation, see for example [20], [5], [6], [22], [3].

It is well known from Ginibre and Velo [4] that, (1.1) is locally well-posed in \( H^1 \). Namely, for \( u_0 \in H^1 \), there exist \( 0 < T \leq \infty \) and a unique solution \( u(t) \in C([0, T); H^1) \) to (1.1). When \( T < \infty \), we have \( \lim_{t \uparrow T} \| \nabla u(t) \|_2 \to \infty \) and say that solution \( u \) blows up in finite positive time. On the other hand, when \( T = \infty \), the solution is called positively global. Note that the local theory gives nothing about the behavior of \( \| \nabla u(t) \|_2 \) as \( t \uparrow +\infty \). Solutions of
admits the following conservation laws in energy space $H^1$:

- **$L^2$-norm**: $M(u)(t) \equiv \int |u(x,t)|^2 dx = M(u_0)$;

- **Energy**: $E(u)(t) \equiv \frac{1}{2} \int |\nabla u(x,t)|^2 dx - \frac{1}{4} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x,t)|^2|u(y,t)|^2}{|x-y|^3} dx dy = E(u_0)$;

- **Momentum**: $P(u)(t) \equiv \text{Im} \int \pi(x,t) \nabla u(x,t) dx = P(u_0)$.

In [3], it is proved that if $u_0 \in H^1$, $M(u_0)E(u_0) < M(Q)E(Q)$ and $\|\nabla u_0\|_2 \|u_0\|_2 > \|\nabla Q\|_2 \|Q\|_2$, then the solution $u(t)$ to (1.1) blows up in finite time provided $\|xu_0\|_{L^2} < \infty$ or $u_0$ is radial. Note that it is sharp in the sense that $u(t) = e^{it}Q(x)$ solves (1.1) and does not blow-up in finite time.

In this paper, in the spirit of Holmer and Roudenko [9] dealing with the cubic 3D Schrödinger equation, without assuming finite variance and radiality we obtain the following result:

**Theorem 1.1.** Suppose that $u_0 \in H^1$, $M(u_0)E(u_0) < M(Q)E(Q)$ and $\|\nabla u_0\|_2 \|u_0\|_2 > \|\nabla Q\|_2 \|Q\|_2$. Then either $u(t)$ blows up in finite forward time, or $u(t)$ is forward global and there exists a time sequence $t_n \rightarrow +\infty$ such that $\|\nabla u(t_n)\|_2 \rightarrow +\infty$. A similar statement holds for negative time.

**Remark 1.2.** Using the same argument as in the introduction of [9] (see more details in Appendix B there), via the Galilean transformation, we will always assume in this paper that $P(u) = 0$. That is, we need only show Theorem 1.1 under the condition $P(u) = 0$. In fact, on the one hand, by [9], the dichotomy result of Proposition 2.1 and Proposition 2.2 in section 2 below is preserved by the Galilean transformation; On the other hand, we get from the relationship between $u(t)$ with nonzero momentum and its Galilean transformation $\tilde{u}(t)$ satisfying

$$\tilde{u}(x,t) = e^{ix\xi}e^{-it|\xi|^2}u(x-2\xi t, t) \quad \text{with} \quad \xi = \frac{P(u)}{M(u)}$$

that

$$P(\tilde{u}) = 0, \quad M(\tilde{u}) = M(u) = M(Q), \quad \|\nabla \tilde{u}\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 - \frac{P(u)^2}{2M(u)}.$$ 

Thus, Theorem 1.1 is also true by Galilean transformation.

In this paper, $H^1$ denotes the usual Sobolev space $W^{1,2}(\mathbb{R}^3)$ and

$$\|u\|_{L^V} = \left( \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} |u(x)|^2V(x-y)|u(y)|^2 dxdy \right)^{\frac{1}{2}}.$$

As usual, we denote the $L^p$ norm as $\| \cdot \|_p$ and use the convention that $c$ always stands for the variant absolute constants.

The rest of this paper is organized as follows. In section 2 we recall the dichotomy and scattering results. In section 3, we discuss blow-up of solutions based on the virial identity and its localized versions. Section 4 is devoted to the variational characterization.
of the ground state and can be taken as a preparation for section 5, in which we set up
the inductive argument that will be continued in section 7 and section 8. In section 6
we introduce the linear and nonlinear profile decomposition lemmas that needed in the
argument in section 7 and section 8, where we give proof of Theorem 1.1.

2. Ground state and dichotomy

As in [25], let $C_{HLS}$ be the best constant in the following Hardy-Littlewood-Sobolev
inequality

$$
\int \int \frac{|u(x)|^2|u(y)|^2}{|x-y|^3}dxdy \leq C_{HLS}\|u\|_2\|\nabla u\|_2^3.
$$

(2.1)

Then it is attained at $Q$, which is the unique radial positive solution to

$$
Q - \Delta Q = (V * Q^2)Q.
$$

(2.2)

The uniqueness of the ground state of (2.2) can be obtain by the same method as in the
cases of dimension three and four ([17] and [16]) by means of Newton’s theorem [18]. In
fact, it suffices to note that the convolution term in (1.1) is none other than the Newton
potential in $\mathbb{R}^5$.

From (2.2) we have

$$
\int |Q|^2dx + \int |\nabla Q|^2dx - \|Q\|_{L^4}^4 = 0,
$$

and the Pohozaev identity

$$
\frac{5}{2} \int |Q|^2dx + \frac{3}{2} \int |\nabla Q|^2dx - \frac{7}{4}\|Q\|_{L^4}^4 = 0.
$$

These two equalities imply that

$$
\|Q\|_{L^4}^4 = \frac{4}{3}\|\nabla Q\|_2^2 = 4\|Q\|_2^2.
$$

As a consequence,

$$
C_{HLS} = \frac{\|Q\|_{L^4}^4}{\|Q\|_2\|\nabla Q\|_2^3} = \frac{4}{3\|Q\|_2\|\nabla Q\|_2},
$$

(2.3)

and therefore

$$
E(Q) = \frac{1}{6}\|\nabla Q\|_2^2.
$$

(2.4)

Let

$$
\eta(t) = \frac{\|\nabla u\|_2\|u\|_2}{\|\nabla Q\|_2\|Q\|_2}.
$$

(2.5)

By (2.1), (2.3) and (2.4) we have

$$
3\eta(t)^2 \geq \frac{E(u)M(u)}{E(Q)M(Q)} \geq 3\eta(t)^2 - 2\eta(t)^3.
$$

(2.6)
Thus it is not difficult to observe that if $0 \leq M(u)E(u)/M(Q)E(Q) < 1$, then there exist two solutions $0 \leq \lambda_- < 1 < \lambda$ of the following equation of $\lambda$

\[
\frac{E(u)M(u)}{E(Q)M(Q)} = 3\lambda^2 - 2\lambda^3. \tag{2.7}
\]

On the other hand, if $E(u) < 0$, there exists exactly one solution $\lambda > 1$ to (2.7).

By the $H^1$ local theory [4], there exist $-\infty \leq T_- < 0 < T_+ \leq +\infty$ such that $(T_- , T_+)$ is the maximal time interval of existence for $u(t)$ solving (1.1), and if $T_+ < +\infty$ then

\[
\|\nabla u(t)\|_2 \to +\infty \text{ as } t \uparrow T_+.
\]

A similar conclusion holds if $T_- > -\infty$. Moreover, as a consequence of the continuity of the flow $u(t)$, we have the following dichotomy proposition:

**Proposition 2.1.** (Global versus blow-up dichotomy) Let $u_0 \in H^1$, and let $I = (T_- , T_+)$ be the maximal time interval of existence of $u(t)$ solving (1.1). Suppose that

\[
M(u)E(u) < M(Q)E(Q). \tag{2.8}
\]

If (2.8) holds and

\[
\|u_0\|_2\|\nabla u_0\|_2 < \|Q\|_2\|\nabla Q\|_2, \tag{2.9}
\]

then $I = (-\infty , +\infty)$, i.e., the solution exists globally in time, and for all time $t \in \mathbb{R}$,

\[
\|u(t)\|_2\|\nabla u(t)\|_2 < \|Q\|_2\|\nabla Q\|_2. \tag{2.10}
\]

If (2.8) holds and

\[
\|u_0\|_2\|\nabla u_0\|_2 > \|Q\|_2\|\nabla Q\|_2, \tag{2.11}
\]

then for $t \in I$,

\[
\|u(t)\|_2\|\nabla u(t)\|_2 > \|Q\|_2\|\nabla Q\|_2. \tag{2.12}
\]

**Proof.** Multiplying the formula of energy by $M(u)$ and using the Hardy-Littlewood-Sobolev inequality (2.1), we obtain

\[
E(u)M(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2\|u\|_{L^2}^2 - \frac{1}{4}\|u\|_{L^4}^4\|u\|_{L^2}^2 \\
\geq \frac{1}{2}\|\nabla u\|_{L^2}^2\|u\|_{L^2}^2 - \frac{1}{4}C_{HLS}\|\nabla u\|_{L^2}^3\|u\|_{L^2}^3.
\]

Define $f(x) = \frac{1}{2}x^2 - \frac{1}{4}C_{HLS}x^3$. Then $f'(x) = x(1 - \frac{3C_{HLS}}{4}x)$, and $f'(x) = 0$ when $x_0 = 0$ and $x_1 = \|\nabla Q\|_2\|Q\|_2 = \frac{4}{3}\frac{1}{C_{HLS}}$ by (2.3). Note that $f(0) = 0$ and $f(x_1) = \frac{1}{3}\|\nabla Q\|_2^3\|Q\|_2^2$.

Thus $f$ has two extrema: a local minimum at $x_0$ and a local maximum at $x_1$. (2.8) implies that $E(u_0)M(u_0) < f(x_1)$, which combined with energy conservation deduces that

\[
f(\|\nabla u\|_2\|u\|_2) \leq E(u)M(u_0) = E(u)M(u) < f(x_1). \tag{2.13}
\]

If initially $\|\nabla u_0\|_2\|u_0\|_2 < x_1$, i.e., (2.9) holds, then by (2.13) and the continuity of $\|\nabla u(t)\|_2$ in $t$, we have $\|\nabla u(t)\|_2\|u(t)\|_2 < x_1$ for all $t \in I$. In particular, the $H^1$ norm of the solution is bounded, which implies the global existence and (2.10) in this case.

If initially $\|\nabla u_0\|_2\|u_0\|_2 > x_1$, i.e., (2.11) holds, then by (2.13) and the continuity of $\|\nabla u(t)\|_2$ in $t$, we have $\|\nabla u(t)\|_2\|u(t)\|_2 > x_1$ for all $t \in I$, which proves (2.12). \qed
The following is another statement of the Dichotomy Proposition in terms of \( \lambda \) and \( \eta(t) \) defined by (2.7) and (2.5) respectively, which will be useful in the sequel.

**Proposition 2.2.** Let \( M(u)E(u) < M(Q)E(Q) \) and \( 0 \leq \lambda_- < 1 < \lambda \) be defined as (2.7). Then exactly one of the following holds:

1. The solution \( u(t) \) to (1.1) is global and
   
   \[
   \frac{1}{3} \frac{E(u)M(u)}{E(Q)M(Q)} \leq \eta(t)^2 \leq \lambda_-^2, \quad \forall \ t \in (-\infty, +\infty)
   \]

2. \( 1 < \lambda \leq \eta(t), \forall \ t \in (T_-, T_+). \)

To easily understand, one can refer to the figure in [9] describing the relationship between \( M(u)E(u)/M(Q)E(Q) \) and \( \eta(t) \). Whether the solution is of the first or second type in Proposition 2.2 is determined by the initial data. Note that the second case does not assert finite-time blow-up.

In the remainder of this section, we will review the Strichartz estimates and some facts about the scattering. It is well-known that a pair of exponents \((q, r)\) is Strichartz admissible if

\[
\frac{2}{q} + \frac{5}{r} = \frac{5}{2}, \quad 2 \leq q \leq \infty, \quad 2 \leq r \leq \frac{10}{3}.
\]

Similarly for \( s > 0 \), we say that \((q, r)\) is \( \dot{H}^s(\mathbb{R}^5) \) admissible and denote it by \((q, r) \in \Lambda_s\) if

\[
\frac{2}{q} + \frac{5}{r} = \frac{5}{2} - s, \quad 4 < q \leq \infty, \quad \frac{10}{5 - 2s} \leq r < \frac{10}{3}.
\]

Correspondingly, we denote \((q', r')\) the dual \( \dot{H}^s(\mathbb{R}^5) \) admissible by \((q', r') \in \Lambda'_s\) if \((q, r) \in \Lambda_{-s}\) with \((q', r')\) is the Hölder dual to \((q, r)\). We define the following Strichartz norm

\[
\|u\|_{S(\dot{H}^{\frac{s}{2}})} = \sup_{(q, r) \in \Lambda_{\frac{s}{2}}} \|u\|_{L^q_tL^r_x}
\]

and the dual Strichartz norm

\[
\|u\|_{S'(\dot{H}^{-\frac{s}{2}})} = \inf_{(q', r') \in \Lambda'_{-\frac{s}{2}}} \|u\|_{L^{q'}_tL^{r'}_x} = \inf_{(q, r) \in \Lambda_{-\frac{s}{2}}} \|u\|_{L^{q'}_tL^{r'}_x},
\]

where \((q', r')\) is the Hölder dual to \((q, r)\).

So we have the following Strichartz estimates

\[
\|e^{it\Delta}\phi\|_{S(L^2)} \leq c\|\phi\|_2 \quad \text{and} \quad \left\| \int_0^t e^{i(t-t')\Delta}f(\cdot, t')dt \right\|_{S(L^2)} \leq c\|f\|_{S'(L^2)}.
\]

Together with Sobolev embedding, we obtain

\[
\|e^{it\Delta}\phi\|_{S(\dot{H}^{\frac{s}{2}})} \leq c\|\phi\|_{\dot{H}^{\frac{s}{2}}} \quad \text{and} \quad \left\| \int_0^t e^{i(t-t')\Delta}f(\cdot, t')dt \right\|_{S(\dot{H}^{\frac{s}{2}})} \leq c\|D^{\frac{s}{2}}f\|_{S'(L^2)}.
\]
In fact, we also have the following Kato inhomogeneous Strichartz estimate \[^{10}\]

\[
\left\| \int_0^t e^{i(t-t')\Delta} f(\cdot, t') dt' \right\|_{S(\dot{H}^{1/2})} \leq c \| f \|_{S'((\dot{H}^{1/2})').}
\]  

(2.14)

In the sequel we will write \( S(\dot{H}^{1/2}; I) \) to indicate a restriction to a time subinterval \( I \subset (-\infty, +\infty) \).

For the first case of the dichotomy proposition (Proposition \[^{2.2}\]), we have furthermore scattering results that will be used in the future discussion. We omit the proofs since they are similar to those in \[^{3}\].

**Lemma 2.3.** (Small data) Let \( \| u_0 \|_{\dot{H}^{1/2}} \leq A \), then there exists \( \delta_{sd} = \delta_{sd}(A) > 0 \) such that \( \| e^{it\Delta} u_0 \|_{S(\dot{H}^{1/2})} \leq \delta_{sd} \), then \( u \) solving \((1.1)\) is global and

\[
\| u \|_{S(\dot{H}^{1/2})} \leq 2 \| e^{it\Delta} u_0 \|_{S(\dot{H}^{1/2})},
\]

(2.15)

\[
\| D^{1/2} u \|_{S(L^2)} \leq 2c \| u_0 \|_{\dot{H}^{1/2}}.
\]

(2.16)

(Note that by Strichartz estimates, the hypotheses are satisfied if \( \| u_0 \|_{\dot{H}^{1/2}} \leq c\delta_{sd} \).)

**Theorem 2.4.** (Scattering). Suppose that \( 0 < M(u)E(u)/M(Q)E(Q) < 1 \) and the first case of Proposition \[^{2.2}\] holds, then \( u(t) \) scatters as \( t \to +\infty \) or \( t \to -\infty \). That is, there exist \( \phi_{\pm} \in H^1 \) such that

\[
\lim_{t \to \pm \infty} \| u(t) - e^{it\Delta} \phi_{\pm} \|_{H^1} = 0.
\]

(2.17)

Consequently,

\[
\lim_{t \to \pm \infty} \| u(t) \|_{L^V} = 0
\]

(2.18)

and

\[
\lim_{t \to \pm \infty} \eta(t)^2 = \frac{1}{3} \frac{E(u)M(u)}{E(Q)M(Q)}.
\]

(2.19)

**Lemma 2.5.** (Existence of wave operators) Suppose that \( \phi^+ \in H^1 \) and

\[
\frac{1}{2} \| \phi^+ \|_2^2 \| \nabla \phi^+ \|_2^2 < E(Q)M(Q).
\]

(2.20)

Then there exists \( v_0 \in H^1 \) such that the corresponding solution \( v \) to \((1.1)\) exists globally and satisfies

\[
\| \nabla v(t) \|_2 \| v_0 \|_2 \leq \| \nabla Q \|_2 \| Q \|_2, \quad M(v) = \| \phi^+ \|_2^2, \quad E(v) = \frac{1}{2} \| \nabla \phi^+ \|_2^2,
\]

and

\[
\lim_{t \to +\infty} \| v(t) - e^{it\Delta} \phi^+ \|_{H^1} = 0.
\]

Moreover, if \( \| e^{it\Delta} \phi^+ \|_{S(\dot{H}^{1/2})} \leq \delta_{sd} \), then

\[
\| v \|_{S(\dot{H}^{1/2})} \leq 2 \| e^{it\Delta} \phi^+ \|_{S(\dot{H}^{1/2})}, \quad \| D^{1/2} v \|_{S(L^2)} \leq 2c \| \phi^+ \|_{\dot{H}^{1/2}}.
\]
3. Virial Identity and Blow-Up Conditions

From now on we will focus on the second case of Proposition 2.2. Using the classical virial identity we first derive the upper bound of the finite blow-up time under the finite variance hypothesis.

Proposition 3.1. Suppose that $\|xu\|_2 < +\infty$. Let $M(u) = M(Q)$, $E(u) < E(Q)$ and suppose that the second case of Proposition 2.2 holds ( $\lambda > 1$ is defined in (2.7)). Let $r(t)$ be the scaled variance given by

$$r(t) = \frac{\|xu\|_2^2}{48\lambda^2(\lambda - 1)E(Q)}.$$

Then blow-up occurs in forward time before $t_b$, where $t_b = r'(0) + \sqrt{r'(0)^2 + 2r(0)}$.

Proof. The virial identity gives

$$r''(t) = \frac{24E(u) - 4\|\nabla u\|_2^2}{48\lambda^2(\lambda - 1)E(Q)}.$$

Using (2.3) we obtain

$$r''(t) = \frac{1}{2\lambda^2(\lambda - 1)} \left( \frac{E(u)}{E(Q)} - \frac{\|\nabla u\|_2^2}{\|\nabla Q\|_2^2} \right).$$

By the definition of $\lambda$ and $\eta$,

$$r''(t) = \frac{3\lambda^2 - 2\lambda^3 - \eta(t)^2}{2\lambda^2(\lambda - 1)}.$$

Since $\eta(t) \geq \lambda > 1$, we have

$$r''(t) \leq -1,$$

which by integrating in time twice gives

$$r(t) \leq -\frac{1}{2}t^2 + r'(0)t + r(0).$$

Note that $t_b$ is the positive root of the polynomial on the right hand side, which deduces that $r(t) \leq t_b$. □

The next result is related to the local virial identity. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be radial such that $\varphi'' \leq 2$ and

$$\varphi(x) = \begin{cases} |x|^2, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

For $R > 0$ define

$$z_R(t) = \int R^2 \varphi \left( \frac{x}{R} \right) |u(x,t)|^2 dx. \tag{3.1}$$
Then by direct calculations we obtain the following local virial identity:

$$z''_R(t) = 4 \sum_{j,k} \int \partial_j \partial_k \varphi \left( \frac{x}{R} \right) \partial_j \bar{u} \partial_k u dx - \frac{1}{R^2} \int \Delta^2 \varphi \left( \frac{x}{R} \right) |u|^2 dx$$

$$+ R \int \int \left( \nabla \varphi \left( \frac{x}{R} \right) - \nabla \varphi \left( \frac{y}{R} \right) \right) \nabla V(x-y)|u(x)|^2 |u(y)|^2 dxdy.$$ 

Set

$$I = 3 \sum \int \int \left[ (2x_j - R \partial_j \varphi \left( \frac{x}{R} \right)) - (2y_j - R \partial_j \varphi \left( \frac{y}{R} \right)) \right] \frac{x_j - y_j}{|x-y|^3} |u(x)|^2 |u(y)|^2 dxdy,$$

and by the definition of $\varphi$, we have

$$z''_R(t) = 24E(u) - 4\|\nabla u\|_2^2 + A_R(u(t)),$$

where

$$A_R(u(t)) = 4 \sum_{j \neq k} \int_{|x| > R} |x| \partial_j \partial_k \varphi \left( \frac{x}{R} \right) \partial_j \bar{u} \partial_k u dx + 4 \sum \int_{|x| \leq R} |\partial_j^2 \varphi \left( \frac{x}{R} \right) - 2 |\nabla u|^2 dx$$

$$- \frac{1}{R^2} \int \Delta^2 \varphi \left( \frac{x}{R} \right) |u|^2 dx + I.$$ 

Observe that $I$ vanishes in the region $|x|, |y| \leq R$, while in the region $|x|, |y| \geq R$, $I$ becomes

$$6 \int_{|x| \geq 2R} \int_{|y| \geq 2R} V(x-y)|u(x)|^2 |u(y)|^2 dxdy.$$

In other cases, since the integral is symmetric with respect to $x$ and $y$, $I$ is bounded by

$$6 \sum \int \int_{|x| \geq R} \left[ (2x_j - R \partial_j \varphi \left( \frac{x}{R} \right)) - (2y_j - R \partial_j \varphi \left( \frac{y}{R} \right)) \right] \frac{x_j - y_j}{|x-y|^3} |u(x)|^2 |u(y)|^2 dxdy,$$

which is bounded by $c \int \int_{|x| \geq R} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^3} dxdy$. Thus, for a suitable radial function $\varphi$ such that $\varphi'' \leq 2$, we have the following control

$$A_R(u(t)) \leq c \left( \frac{1}{R^2} \|u\|_{L^2(|x| > R)}^4 + \|u\|^4_{L^4(|x| > R)} \right).$$

The local virial identity will give another version of Proposition 3.1, for which, without the assumption of finite variance, we will assume that the solution is suitably localized in $H^1$ for all times.

**Proposition 3.2.** Let $M(u) = M(Q)$, $E(u) < E(Q)$ and suppose that the second case of Proposition 2.2 holds ( $\lambda > 1$ is defined in (2.7)). Select $\gamma$ such that $0 < \gamma < \min \{ \lambda - 1, 1 \}$. Suppose that there is a radius $R \geq \frac{\sqrt{\gamma}}{\sqrt{\theta_0}}$ such that for all $t$, there holds that

$$\|u\|^4_{L^4(|x| \geq R)} \leq \frac{6 \gamma E(Q)}{c},$$

(3.4)
where the absolute constant $c$ is determined in (3.3). Let $\tilde{r}(t)$ be the scaled local variance given by

$$\tilde{r}(t) = \frac{z_R(t)}{48\lambda^2(\lambda - 1 - \gamma) E(Q)}.$$ 

Then blow-up occurs in forward time before $t_b$, where $t_b = \tilde{r}'(0) + \sqrt{\tilde{r}'(0)^2 + 2\tilde{r}(0)}$.

Proof. In view of the assumptions, by the local virial identity and the same steps as in the proof of Proposition 3.1,

$$\tilde{r}''(t) = \frac{1}{48\lambda^2(\lambda - 1 - \gamma) E(Q)} \left(24E(u) - 4\|\nabla u\|^2 + A_R(u(t))\right)$$

$$= \frac{1}{2\lambda^2(\lambda - 1 - \gamma)} \left(3\lambda^2 - 2\lambda^3 - \eta(t)^2 + \frac{A_R(u(t))}{24E(Q)}\right)$$

$$\leq \frac{3\lambda^2 - 2\lambda^3 - \eta(t)^2}{2\lambda^2(\lambda - 1 - \gamma)} + \frac{c\|u\|_{L^2(|x|>R)}}{48E(Q)\lambda^2(\lambda - 1 - \gamma)} + \frac{c\|u\|_{L^V(|x|>R)}^4}{48E(Q)\lambda^2(\lambda - 1 - \gamma)}$$

$$\leq \frac{1}{2\lambda^2(\lambda - 1 - \gamma)} \left(3\lambda^2 - 2\lambda^3 - \eta(t)^2 + \gamma\eta(t)^2\right)$$

$$\leq 1.$$ 

Finally, we complete our proof just the same as in the proof of Proposition 3.1.

Remark 3.3. Note that by Hardy-Littlewood-Sobolev inequalities, Hölder estimates and Sobolev embedding, the assumption (3.4) is satisfied by $u$ which is $H^1$ bounded and $H^1$ localized, i.e. for any $\epsilon > 0$ there exists $R > 0$ large enough such that $\|u\|_{H^1(|x|\geq R)} \leq \epsilon$.

We will finally give a quantified proof of finite-time blow-up for radial solutions, for which we need the following radial Sobolev embedding: Let $u \in H^1(\mathbb{R}^d)$ be radially symmetric, then

$$\|x|^{d-1} u\|_{L^\infty} \leq c\|u\|_{L^2} \|\nabla u\|_{L^2}.$$ 

(3.5)

Proposition 3.4. Let $M(u) = M(Q)$, $E(u) < E(Q)$. Suppose $u$ is radial and the second case of Proposition 2.2 holds ($\lambda > 1$ is defined in (2.7)). Select $\gamma$ such that $0 < \gamma < \min\{\lambda - 1, 1\}$. Suppose that $R \geq \max\left\{\sqrt{\frac{c}{6\gamma}}\left(\frac{cE(Q)}{12\gamma}\right)^{\frac{4}{d}}\right\}$, where the absolute constant $c$ is determined by the two in (3.3) and (3.5). Let $\tilde{r}(t)$ be the scaled local variance given by

$$\tilde{r}(t) = \frac{z_R(t)}{48\lambda^2(\lambda - 1 - \gamma) E(Q)}.$$ 

Then blow-up occurs in forward time before $t_b$, where $t_b = \tilde{r}'(0) + \sqrt{\tilde{r}'(0)^2 + 2\tilde{r}(0)}$. 


Following holds: suppose there is \( \lambda > 0 \), there exists a function \( \eta(t) \) such that \( c \left( \frac{\|u\|_{L^4(|x|>R)}^4}{\lambda^2(\gamma-\frac{\gamma^2}{2})} \right) \leq cE(Q) \), and \( \left( 3\lambda^2 - 2\lambda^3 - \eta(t)^2 \right) \leq 1 \). Arguing the same as in the proof of the preceding propositions we can complete our proof.

**4. Variational Characterization of the Ground State**

In this section we deal with the variation characterization of \( Q \) defined in section 2. It is an important preparation for the “near boundary case” in section 5. Since the time dependence plays no role in this section, we will write \( u = u(x) \) for now.

**Proposition 4.1.** There exists a function \( \epsilon(\rho) \) with \( \epsilon(\rho) \to 0 \) as \( \rho \to 0 \) such that the following holds: suppose there is \( \lambda > 0 \) satisfying

\[
\left| \frac{M(u)E(u)}{M(Q)E(Q)} - \left( 3\lambda^2 - 2\lambda^3 \right) \right| \leq \rho \lambda^3, \quad (4.1)
\]

and

\[
\left| \frac{\|u\|_2\|\nabla u\|_2}{\|Q\|_2\|\nabla Q\|_2} - \lambda \right| \leq \rho \begin{cases} \lambda, & \lambda \geq 1, \\ \lambda^2, & \lambda \leq 1. \end{cases} \quad (4.2)
\]

Then there exists \( \theta \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^5 \) such that

\[
\left\| u - e^{i\theta} \lambda^{\frac{2}{\beta}} \lambda^{\frac{2}{\beta}} Q \left( \lambda(\beta^{-1} \cdot -x_0) \right) \right\|_2 \leq \beta^2 \epsilon(\rho) \quad (4.3)
\]

and

\[
\left\| \nabla \left[ u - e^{i\theta} \lambda^{\frac{2}{\beta}} \lambda^{\frac{2}{\beta}} Q \left( \lambda(\beta^{-1} \cdot -x_0) \right) \right] \right\|_2 \leq \lambda \beta^{-\frac{1}{2}} \epsilon(\rho), \quad (4.4)
\]

where \( \beta = \frac{M(u)}{M(Q)} \).
Remark 4.2. If we let \( v(x) = \beta^2 u(\beta x) \), then \( M(v) = \beta^{-1} M(u) = M(Q) \), and we can then restate Proposition 4.1 as follows:

Suppose \( \|v\|_2 = \|Q\|_2 \) and there is \( \lambda > 0 \) such that

\[
\left| \frac{E(v)}{E(Q)} - (3\lambda^2 - 2\lambda^3) \right| \leq \rho\lambda^3, \tag{4.5}
\]

and

\[
\left| \frac{\|\nabla v\|_2}{\|\nabla Q\|_2} - \lambda \right| \leq \rho \left\{ \begin{array}{ll}
\lambda, & \lambda \geq 1; \\
\lambda^2, & \lambda \leq 1.
\end{array} \right. \tag{4.6}
\]

Then there exists \( \theta \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^5 \) such that

\[
\left\| v - e^{i\theta} \lambda \hat{Q}(\lambda(\cdot - x_0)) \right\|_2 \leq \epsilon(\rho) \tag{4.7}
\]

and

\[
\left\| \nabla \left[ v - e^{i\theta} \lambda \hat{Q}(\lambda(\cdot - x_0)) \right] \right\|_2 \leq \lambda \epsilon(\rho). \tag{4.8}
\]

Thus it suffices to prove the scaled statement equivalent to Proposition 4.1. We will carry it out by means of the following result from Lions [19].

**Lemma 4.3.** There exists a function \( \epsilon(\rho) \), defined for small \( \rho > 0 \) such that

\[
\lim_{\rho \to 0} \epsilon(\rho) = 0,
\]

such that for all \( u \in H^1 \) with

\[
\|u\|_{L^4} - \|Q\|_{L^4} + \|u\|_2 - \|Q\|_2 + \|\nabla u\|_2 - \|\nabla Q\|_2 \leq \rho, \tag{4.9}
\]

there exist \( \theta_0 \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^N \) such that

\[
\left\| u - e^{i\theta_0} Q(\cdot - x_0) \right\|_{H^1} \leq \epsilon(\rho). \tag{4.10}
\]

**Proof of Proposition 4.1.** As a result of Remark 4.2, we will just prove the equivalent version rescaling off the mass. Set \( \tilde{u}(x) = \lambda^{-\frac{2}{5}} v(\lambda^{-1} x) \), and then (4.6) gives

\[
\left| \frac{\|\nabla \tilde{u}\|_2}{\|\nabla \tilde{Q}\|_2} - 1 \right| \leq \rho. \tag{4.11}
\]

On the other hand, by (2.3), (4.3) and (4.6) imply

\[
2 \left| \frac{\|v\|_4^4}{\|Q\|_4^4} - \lambda^3 \right| \leq \left| \frac{E(v)}{E(Q)} - (2\lambda^3 - 3\lambda^2) \right| + 3 \left| \frac{\|\nabla v\|_2^2}{\|\nabla Q\|_2^2} - \lambda^2 \right| \leq \rho \lambda^3 + 3\rho \left\{ \begin{array}{ll}
\lambda^2, & \lambda \geq 1; \\
\lambda^4, & \lambda \leq 1.
\end{array} \right. \leq 4\rho \lambda^3.
\]

Thus in terms of \( \tilde{u} \), we obtain

\[
\left| \frac{\|	ilde{u}\|_4^4}{\|Q\|_4^4} - 1 \right| \leq 2\rho. \tag{4.12}
\]

Thus (4.11) and (4.12) imply that \( \tilde{u} \) satisfies (4.3) (\( \rho \) may be different). By Lemma 4.3 and rescaling back to \( v \), we obtain (4.7) and (4.8).

\( \square \)
5. Near-Boundary Case

We know from Proposition 2.2 that if $M(u) = M(Q)$ and $E(u)/E(Q) = 3\lambda^2 - 2\lambda^3$ for some $\lambda > 1$ and $\|\nabla u_0\|_2/\|\nabla Q\|_2 \geq \lambda$, then $\|\nabla u(t)\|_2/\|\nabla Q\|_2 \geq \lambda$ for all $t$. Now in this section, we will claim that $\|\nabla u(t)\|_2/\|\nabla Q\|_2$ cannot remain near $\lambda$ globally in time.

**Proposition 5.1.** Let $\lambda_0 > 1$. There exists $\rho_0 = \rho_0(\lambda_0) > 0$ with the property that $\rho_0(\lambda_0) \rightarrow 0$ as $\lambda_0 \rightarrow 1$, such that for any $\lambda \geq \lambda_0$, the following holds: There does not exist a solution $u(t)$ of problem (1.1) with $P(u) = 0$ satisfying $M(u) = M(Q)$,

$$\frac{E(u)}{E(Q)} = 3\lambda^2 - 2\lambda^3,$$

and for all $t \geq 0$

$$\lambda \leq \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \leq \lambda(1 + \rho_0). \tag{5.2}$$

We would like to give another equivalent statement implied by this assertion: For any solution $u(t)$ to (1.1) with $P(u) = 0$ satisfying $M(u) = M(Q)$, (5.1), and $\|\nabla u(t)\|_2/\|\nabla Q\|_2 \geq \lambda$ for all $t \geq 0$, there exist a time $t_0 \geq 0$ such that $\frac{\|\nabla u(t_0)\|_2}{\|\nabla Q\|_2} \geq \lambda(1 + \rho_0)$.

Before proving Proposition 5.1, following the idea of [2], we introduce a useful lemma.

**Lemma 5.2.** Suppose that $u(t)$ with $P(u) = 0$ solving (1.1) satisfies, for all $t$

$$\|u(t) - e^{i\theta(t)}Q(\cdot - x(t))\|^2_{H^1} \leq \epsilon \tag{5.3}$$

for some continuous functions $\theta(t)$ and $x(t)$. Then if $\epsilon > 0$ is sufficiently small, we have

$$\frac{|x(t)|}{t} \leq c\epsilon \quad \text{as} \quad t \to +\infty.$$

**Proof.** We argue by contradiction. If not, (5.3) holds for any small $\epsilon > 0$ while there exists a time sequence $t_n \to +\infty$ such that $|x(t_n)|/t_n \geq \epsilon_0$ with some $\epsilon_0 > 0$. Without loss of generality we assume $x(0) = 0$. For $R > 0$ we define $t_0(R) = \inf\{t \geq 0 : x(t) \geq R\}$ and then by the continuity of $x(t)$ there holds that 1), $t_0(R) > 0$; 2), $|x(t)| < R$ for $0 \leq t \leq t_0(R)$; and 3), $|x(t_0(R))| = R$. If we set $R_n = |x(t_n)|$ and $\tilde{t}_n = t_0(R_n)$, then $t_n \geq \tilde{t}_n$ which implies that $R_n/\tilde{t}_n \geq \epsilon_0$. We get from $|x(t_0)|/t_0 \geq \epsilon_0$ and $t_0 \to +\infty$ that $R_n = |x(t_n)| \to +\infty$. Thus, $\tilde{t}_n \to t_0(R_n) \to +\infty$. In the sequel, we will work on the time interval $[0, \tilde{t}_n]$ to get a contradiction.

For that purpose we need a uniform localization. That is for any $\epsilon > 0$ there exists $R_0(\epsilon) \geq 0$ such that for all $t \geq 0$, there holds that

$$\int_{|x-x(t)| \geq R_0(\epsilon)} |u|^2 + \|\nabla u\|^2 dx \leq 2\epsilon. \tag{5.4}$$

In fact, since the ground state $Q \in H^1$, there must exist $R(\epsilon) > 0$ such that

$$\int_{|x| \geq R(\epsilon)} |Q|^2 + \|\nabla Q\|^2 + (V \ast |Q|^2)|Q|^2 dx \leq \epsilon. \tag{5.5}$$
Thus, take $R_0(\epsilon) = R(\epsilon)$, we have
\[
\int_{|x-x(t)| \geq R_0(\epsilon)} |u|^2 + |\nabla u|^2dx \leq \int |u - e^{i\theta(t)}Q(\cdot - x(t))|^2 + |\nabla(u - e^{i\theta(t)}Q(\cdot - x(t)))|^2dx \\
+ \int_{|x-x(t)| \geq R(\epsilon)} |Q(\cdot - x(t))|^2 + |\nabla Q(\cdot - x(t))|^2dx \leq 2\epsilon.
\]

For $x \in \mathbb{R}$, let $\theta(x) \in C^\infty_c$ such that $\theta(x) = x$ for $-1 \leq x \leq 1$, $\theta(x) = 0$ for $|x| \geq 2\hat{t}$, $|\theta(x)| \leq |x|$, $\|\theta(x)\|_\infty \leq 2$ and $\|\theta'(x)\|_\infty \leq 4$. For $x \in \mathbb{R}^5$, let $\phi(x) = (\theta(x_1), \ldots, \theta(x_5))$ and then $\phi(x) = x$ for $|x| \leq 1$ and $\|\phi(x)\|_\infty \leq 2$. For $R > 0$, set $\phi_R = R\phi(x/R)$. We consider the truncated center of mass: $z_R(t) = \int \phi_R(x)|u(x,t)|^2dx$ and $[z'_R(t)]_j = 2Im \int \theta'(x_j/R)\partial_j u\hat{u}dx$.

By the zero momentum property we obtain $|z'_R(t)| \leq 5\int_{|x| \geq R} |u|^2 + |\nabla u|^2dx$. Setting $\tilde{R}_n = R_n + R_0(\epsilon)$, we then have for $0 \leq t \leq \tilde{t}_n$ and $|x| > \tilde{R}_n$, $|x - x(t)| \geq R_0(\epsilon)$. Then by the uniform localization (5.4), we obtain
\[
|z'_{\tilde{R}_n}(t)| \leq 5\epsilon. \tag{5.6}
\]

Now we claim that
\[
|z_{\tilde{R}_n}(0)| \leq R_0(\epsilon)M(u) + 2\tilde{R}_n\epsilon \tag{5.7}
\]
and
\[
|z_{\tilde{R}_n}(\tilde{t}_n)| \geq \tilde{R}_n(M(u) - 3\epsilon) - 2R_0(\epsilon)M(u). \tag{5.8}
\]

In fact, firstly, the upper bound for $z_{\tilde{R}_n}(0)$ can be obtained by
\[
z_{\tilde{R}_n}(0) = \int_{|x| < R_0(\epsilon)} \phi_{\tilde{R}_n}(x)|u_0(x)|^2dx + \int_{|x| \geq R_0(\epsilon)} \phi_{\tilde{R}_n}(x)|u_0(x)|^2dx
\]
and (5.4) immediately. We next show the lower bound for $z_{\tilde{R}_n}(t)$ as follows. For $0 \leq t \leq \tilde{t}_n$, we split $z_{\tilde{R}_n}(t)$ as
\[
z_{\tilde{R}_n}(t) = \int_{|x-x(t)| < \tilde{R}_n(\epsilon)} \phi_{\tilde{R}_n}(x)|u(x,t)|^2dx + \int_{|x-x(t)| \geq \tilde{R}_n(\epsilon)} \phi_{\tilde{R}_n}(x)|u(x,t)|^2dx \equiv I + II.
\]

Again from (5.4), we obtain that $|II| \leq 2\tilde{R}_n\epsilon$. For $I$, since $|x| \leq |x - x(t)| + |x(t)| \leq R_0(\epsilon) + \tilde{R}_n = \tilde{R}_n\epsilon$, we can rewrite $I$ as
\[
I = \int_{|x-x(t)| < \tilde{R}_n(\epsilon)} (x - x(t))|u(x,t)|^2dx + (x(t)\int_{|x-x(t)| < \tilde{R}_n(\epsilon)} |u(x,t)|^2dx \\
= \int_{|x-x(t)| < \tilde{R}_n(\epsilon)} (x - x(t))|u(x,t)|^2dx + x(t)\int_{|x-x(t)| \geq \tilde{R}_n(\epsilon)} |u(x,t)|^2dx \\
\equiv I_1 + I_2 + I_3.
\]

Since $|I_1| \leq R_0(\epsilon)M(u)$, and by (5.4), $|I_3| \leq |x(t)|\epsilon$, thus we have
\[
|z_{\tilde{R}_n}(t)| \geq |I_2| - |I_1| - |I_3| - |II| \geq |x(t)|M(u) - R_0(\epsilon)M(u) - 3\tilde{R}_n\epsilon,
\]
which gives (5.8) since $|x(t_n)| = R_n$.

Combining (5.6), (5.7) and (5.8), we obtain

$$5\epsilon \bar{t}_n \geq \int_0^{\bar{t}_n} z'_{R_n}(t) dt \geq |z_{R_n}(\bar{t}_n) - z_{R_n}(0)| \geq \bar{R}_n(M(u) - 5\epsilon) - 3R_0(\epsilon)M(u).$$

Thus assuming $\epsilon \leq \frac{M(u)}{5}$, since $\bar{R}_n \geq R_n$ and $R_n/\bar{t}_n \geq \epsilon_0$, we finally obtain

$$5\epsilon \geq \epsilon_0(M(u) - 5\epsilon) - \frac{3R_0(\epsilon)M(u)}{\bar{t}_n}.$$

If taking $\epsilon < M(u)\epsilon_0/20$ and letting $n \to \infty$ ($\bar{t}_n \to \infty$ therefore), we get a contradiction. \hfill \Box

We shall prove Proposition 5.1 using the above lemma, and our arguments will not use any exponential decay property of the Ground State $Q$, which is different from those when dealing with the Schrödinger equation.

**Proof of Proposition 5.1.** To the contrary, we suppose that there exists a solution $u(t)$ satisfying $M(u) = M(Q)$, $E(u)/E(Q) = 3\lambda^2 - 2\lambda^3$ and

$$\lambda \leq \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \leq \lambda(1 + \rho_0). \tag{5.9}$$

Since $\|\nabla u(t)\|_2^2 \geq \lambda^2 \|\nabla Q\|_2^2 = 6\lambda^2 E(Q)$, we have

$$24E(u) - 4\|\nabla u(t)\|_2^2 \leq -48E(Q)\lambda^2(\lambda - 1).$$

By Proposition 4.1, there exist functions $\theta(t)$ and $x(t)$ such that for $\rho = \rho_0$

$$\left\|u(t) - e^{i\theta(t)}\frac{\lambda}{n}Q(\lambda(\cdot - x(t)))\right\|_2 \leq \epsilon(\rho) \tag{5.10}$$

and

$$\left\|\nabla \left[u(t) - e^{i\theta(t)}\frac{\lambda}{n}Q(\lambda(\cdot - x(t)))\right]\right\|_2 \leq \lambda\epsilon(\rho). \tag{5.11}$$

By the continuity of the $u(t)$ flow, we may assume $\theta(t)$ and $x(t)$ are continuous. Let

$$R(T) = \max \left(\max_{0 \leq t \leq T} |x(t)|, R(\epsilon(\rho))\right),$$

where $R(\epsilon(\rho))$ is given by (5.5) with $R(\epsilon(\rho)) \to +\infty$ as $\rho \to 0$. For fixed $T$, take $R = 2R(T)$ in the local virial identity (5.2). Then we claim

$$|A_R(u(t))| \leq c\lambda^2\epsilon(\rho)^2.$$

In fact,

$$\|u\|_{L^V(|x| \geq R)} \leq \|u - e^{i\theta(t)}\frac{\lambda}{n}Q(\lambda(\cdot - x(t)))\|_{L^V} + \|e^{i\theta(t)}\frac{\lambda}{n}Q(\lambda(\cdot - x(t)))\|_{L^V(|x| \geq R)}.$$

By Hardy-Littlewood-Sobolev inequality (2.1), (5.10) and (5.11) imply that

$$\|u - e^{i\theta(t)}\frac{\lambda}{n}Q(\lambda(\cdot - x(t)))\|_{L^V}^4 \leq \lambda^3\epsilon(\rho)^4.$$

Therefore, we have

$$\|u\|_{L^V(|x| \geq R)} \leq \|u - e^{i\theta(t)}\frac{\lambda}{n}Q(\lambda(\cdot - x(t)))\|_{L^V} + \|e^{i\theta(t)}\frac{\lambda}{n}Q(\lambda(\cdot - x(t)))\|_{L^V(|x| \geq R)}.$$
On the other hand, by (5.5), we have

\[ \| e^{i\theta(t)} \lambda^\frac{5}{2} Q (\lambda \cdot -x(t)) \|_{L^4 [|x| \geq R]} \leq \| \lambda^\frac{5}{2} Q (\lambda (\cdot)) \|_{L^4 [|x| \geq R_{\max \leq t \leq T} |x(t)|]} \]

\[ \leq \| \lambda^\frac{5}{2} Q (\lambda (\cdot)) \|_{L^4 [|x| \geq R(T)]} \leq \| \lambda^\frac{5}{2} Q (\lambda (\cdot)) \|_{L^4 [|x| \geq R(\epsilon(\rho))]} \leq \lambda^3 \epsilon(\rho)^4. \]

Similarly but more easily, we also have \( \| u \|_{L^2 [|x| > R]} \leq c \epsilon(\rho)^2 \). Thus (3.3) implies the claim.

Taking \( \rho_0 \) small enough to make \( \epsilon(\rho)^2 \) small such that for all \( 0 \leq t \leq T \),

\[ z_0^R(t) \leq -24E(Q)\lambda^2 (\lambda - 1), \]

and so

\[ \frac{z_R(T)}{T^2} \leq \frac{z_R(0)}{T^2} + \frac{z_R'(0)}{T} - 12E(Q)\lambda^2 (\lambda - 1). \]

By definition of \( z_R(t) \) we have

\[ |z_R(0)| \leq cR^2 \| u_0 \|_2^2 = c \| Q \|_2^2 R^2 \]

and

\[ |z_R'(0)| \leq cR \| u_0 \|_2 \| \nabla u_0 \|_2 \leq c \| Q \|_2 \| \nabla Q \|_2 R(1 + \rho_0)\lambda. \]

Consequently,

\[ \frac{z_{2R(T)}(T)}{T^2} \leq c \left( \frac{R(T)^2}{T^2} + \frac{\lambda R(T)}{T} \right) - 12E(Q)\lambda^2 (\lambda - 1). \]

Taking \( T \) sufficiently large, from Lemma 5.2 we have

\[ 0 \leq \frac{z_{2R(T)}(T)}{T^2} \leq c (\lambda \epsilon(\rho)^2 - \lambda^2 (\lambda - 1)) < 0 \]

provided \( \rho_0 \) is small enough.

Note that \( \rho_0 \) is independent of \( T \). We then get a contradiction and complete our proof. \( \Box \)

6. Profile Decomposition

The following Keraani-type profile decomposition will play an important role in our future discussion.

**Lemma 6.1.** (Profile expansion). Let \( \phi_n(x) \) be an uniformly bounded sequence in \( H^1 \), then for each \( M \) there exists a subsequence of \( \phi_n \), also denoted by \( \phi_n \), and (1) for each \( 1 \leq j \leq M \), there exists a (fixed in \( n \)) profile \( \tilde{\psi}^j(x) \) in \( H^1 \), (2) for each \( 1 \leq j \leq M \), there exists a sequence (in \( n \)) of time shifts \( t_n^j \), (3) for each \( 1 \leq j \leq M \), there exists a sequence (in \( n \)) of space shifts \( x_n^j \), (4) there exists a sequence (in \( n \)) of remainders \( \tilde{W}_n^M(x) \) in \( H^1 \), such that

\[ \phi_n(x) = \sum_{j=1}^M e^{-i t_n^j} \Delta \tilde{\psi}^j (x - x_n^j) + \tilde{W}_n^M(x), \]

The time and space sequences have a pairwise divergence property, i.e., for \( 1 \leq j \neq k \leq M \), we have

\[ \lim_{n \to +\infty} (|t_n^j - t_n^k| + |x_n^j - x_n^k|) = +\infty. \]
The remainder sequence has the following asymptotic smallness property:

$$\lim_{M \to +\infty} \left[ \lim_{n \to +\infty} \| e^{it\Delta} \tilde{W}_n^M \|_{S(\dot{H}^1)} \right] = 0.$$ (6.2)

For fixed $M$ and any $0 \leq s \leq 1$, we have the asymptotic Pythagorean expansion:

$$\| \phi_n \|^2_{\dot{H}^s} = \sum_{j=1}^{M} \| \tilde{\psi}^j \|^2_{\dot{H}^s} + \| \tilde{W}_n^M \|^2_{\dot{H}^s} + o_n(1).$$ (6.3)

Remark 6.2. By refining the subsequence for each $j$ and using a standard diagonalization argument, we may assume that for each $j$ that the sequence $t^j_n$ is convergent to some time in the compactified time interval $[-\infty, +\infty)$. If $t^j_n$ converges to some finite time $t^j \in (-\infty, +\infty)$, we may shift $\tilde{\psi}^j$ by the linear propagator $e^{-it^j \Delta}$ to assume without loss of generality that $t^j_n$ converges either to $-\infty$, $0$, or $+\infty$. If $t^j_n$ converges to 0, we may absorb the error $e^{-it^j \Delta} \tilde{\psi}^j - \tilde{\psi}^j$ to the remainder $\tilde{W}_n^M$ without significantly affecting the scattering size of the linear evolution of $\tilde{W}_n^M$ and so assume, without loss of generality, in this case that $t^j_n \equiv 0$.

Since the profile decomposition corresponds to the linear equation and there is no difference in the linear terms between the Hartree equation and the Schrödinger equation, there is no essential difference in the proof as in [2] for the 3D cubic Schrödinger equation, and one can find similar proof there. Furthermore, we have also the following energy expansion.

Lemma 6.3. (Energy pythagorean expansion) Under the same assumptions of Lemma 6.1, we have

$$E(\phi_n) = \sum_{j=1}^{M} E(e^{-it^j_n \Delta} \tilde{\psi}^j) + E(\tilde{W}_n^M) + o_n(1).$$ (6.4)

Similar to Keraani [14] and [12], we give the following definition of the nonlinear profile:

Definition 6.4. Let $V$ be a solution to the linear Schrödinger equation. We say that $U$ is the nonlinear profile associated to $(V, \{t_n\})$, if $U$ is a solution to (1.1) satisfying

$$\|(U - V)(-t_n)\|_{H^1} \to 0 \quad as \quad n \to \infty.$$
Proposition 6.5. Let \( \phi_n(x) \) be an uniformly bounded sequence in \( H^1 \). There exists a subsequence of \( \phi_n \), also denoted by \( \phi_n \), profiles \( \psi^j(x) \) in \( H^1 \), and parameters \( x^j_n, t^j_n \) so that for each \( M \),
\[
\phi_n(x) = \sum_{j=1}^{M} \text{NLH}(-t^j_n)\psi^j(x - x^j_n) + W^M_n(x),
\]
where as \( n \to \infty \)

- For each \( j \), either \( t^j_n = 0 \), \( t^j_n \to +\infty \) or \( t^j_n \to -\infty \).
- If \( t^j_n \to +\infty \), then \( \|\text{NLH}(-t)\psi^j\|_{S(H^1_x;[0,\infty))} < \infty \); If \( t^j_n \to -\infty \), then
  \[\|\text{NLH}(-t)\psi^j\|_{S(H^1_x;(-\infty,0])} < \infty.\]
- For \( j \neq k \),
  \[
  \lim_{n \to +\infty} (|t^j_n - t^k_n| + |x^j_n - x^k_n|) = +\infty.
  \]
- \( \text{NLH}(t)W^M_n \) is global for \( M \) large enough with
  \[
  \lim_{M \to +\infty} \left( \lim_{n \to +\infty} \|\text{NLH}(t)W^M_n\|_{S(H^1_x)} \right) = 0.
  \]

We also have the \( H^s \) Pythagorean decomposition: for fixed \( M \) and \( 0 \leq s \leq 1 \),
\[
\|\phi_n\|^2_{H^s} = \sum_{j=1}^{M} \|\text{NLH}(-t^j_n)\psi^j\|^2_{H^s} + \|W^M_n\|^2_{H^s} + o_n(1),
\]
and, by energy conservation \( E(\text{NLH}(-t^j_n)\psi^j) = E(\psi^j) \), the energy Pythagorean decomposition
\[
E(\phi_n) = \sum_{j=1}^{M} E(\psi^j) + E(W^M_n) + o_n(1).
\]

Remark 6.6. As is stated in [9], (6.7) was proven by establishing the following orthogonal decomposition first
\[
\|\phi_n\|^4_{L^4} = \sum_{j=1}^{M} \|\text{NLH}(-t^j_n)\psi^j\|^4_{L^4} + \|W^M_n\|^4_{L^4} + o_n(1),
\]
and we will find a similar one in the proof of Lemma 6.8.

The next perturbation lemma is essential to get our main theorem.

Lemma 6.7. (Long time perturbation theory) For any given \( A \gg 1 \), there exist \( \epsilon_0 = \epsilon_0(A) \ll 1 \) and \( c = c(A) \) such that the following holds: Fix \( T > 0 \). Let \( u = u(x,t) \in L^\infty([0,T];H^1) \) solves
\[
iu_t + \Delta u + (V * |u|^2)u = 0
\]

\(^1\)This property is obtained by solving an asymptotic problem similar to that in the proof of the existence of the wave operator. In fact, we obtain further that \( \|D^\frac{1}{2} \text{NLH}(-t)\psi^j\|_{S(L^2;[0,\infty))} < \infty \) in the case of \( t^j_n \to +\infty \), and a similar result for the case \( t^j_n \to -\infty \).
on $[0, T]$. Let $\tilde{u} = \tilde{u}(x, t) \in L^\infty([0, T]; H^1)$ and set
\[ e \equiv i\tilde{u}_t + \Delta \tilde{u} + (V * |\tilde{u}|^2)\tilde{u}. \]

For each $\epsilon \leq \epsilon_0$, if
\[ \|\tilde{u}\|_{S(H^{1/2}; [0, T])} \leq A, \quad \|e\|_{S(H^{-1/2}; [0, T])} \leq \epsilon \quad \text{and} \quad \|e^{it\Delta}(u(0) - \tilde{u}(0))\|_{S(H^{1/2}; [0, T])} \leq \epsilon, \]
then
\[ \|u - \tilde{u}\|_{S(H^{1/2}; [0, T])} \leq c(A)\epsilon. \]

**Proof.** Define $w = u - \tilde{u}$. Then $w$ solves the equation
\[ iw_t + \Delta w + (V * |w + \tilde{u}|^2)w + (V * |w + \tilde{u}|^2)\tilde{u} - (V * |\tilde{u}|^2)\tilde{u} + e = 0. \]

That is
\[
\begin{align*}
iw_t &+ \Delta w + (V * |w|^2)w + (V * (\tilde{w}\tilde{u}))w + (V * (\tilde{w}\tilde{u}))w \\
&\quad + (V * |w|^2)\tilde{u} + (V * |\tilde{u}|^2)w + (V * (\tilde{w}\tilde{u}))\tilde{u} + (V * (\tilde{w}\tilde{u}))\tilde{u} + e = 0.
\end{align*}
\]
Since $\|\tilde{u}\|_{S(H^{1/2}; [0, T])} \leq A$, we can divide $[0, T]$ into $N = N(A)$ intervals $I_j = [t_j, t_{j+1}]$, such that, for each $0 \leq j \leq N-1$, $\|\tilde{u}\|_{S(H^{1/2}; [I_j])} < \delta$ with the sufficiently small $\delta$ to be specified later. The integral equation of (6.9) with initial time $t_j$ is
\[ w(t) = e^{i(t-t_j)\Delta}w(t_j) + i \int_{t_j}^t e^{i(t-s)\Delta}W(\cdot, s)ds, \quad (6.10) \]
where
\[
W = (V * |w|^2)w + (V * (\tilde{w}\tilde{u}))w + (V * (\tilde{w}\tilde{u}))w \\
+ (V * |w|^2)\tilde{u} + (V * |\tilde{u}|^2)w + (V * (\tilde{w}\tilde{u}))\tilde{u} + (V * (\tilde{w}\tilde{u}))\tilde{u} + e.
\]

Applying the Kato Strichartz estimate (2.14) on $I_j$ we have
\[
\|w\|_{S(H^{1/2}; [I_j])} \leq \|e^{i(t-t_j)\Delta}w(t_j)\|_{S(H^{1/2}; [I_j])} + c\|V * |w|^2w\|_{L^{12}_{t_j}L^{12}_x} + c\|(V * (\tilde{w}\tilde{u}))w\|_{L^{12}_{t_j}L^{12}_x} + c\|(V * (\tilde{w}\tilde{u}))\tilde{u}\|_{L^{12}_{t_j}L^{12}_x} + c\|(V * |\tilde{u}|^2w\|_{L^{12}_{t_j}L^{12}_x} + c\|(V * |\tilde{u}|^2)\tilde{u}\|_{L^{12}_{t_j}L^{12}_x} \quad (6.11)
\]
In fact, we can easily check that $(\frac{24}{13}, \frac{12}{7}) \in \Lambda'$ and $(\frac{24}{5}, \frac{60}{19}), (8, \frac{20}{7}) \in \Lambda_2$. And by Hardy-Littlewood-Sobolev inequalities and Hölder estimates we have
\[
\|(V * |\tilde{u}|^2)w\|_{L^{12}_{t_j}L^{12}_x} \leq \|\tilde{u}\|_{S(H^{1/2}; [I_j])}^2 \quad \text{and} \quad \|w\|_{S(H^{1/2}; [I_j])} \leq \delta^2 \|w\|_{S(H^{1/2}; [I_j])},
\]
\[
\|(V * |w|^2)\tilde{u}\|_{L^{12}_{t_j}L^{12}_x} \leq \|w\|_{L^{24}_{t_j}L^{12}_x} \quad \text{and} \quad \|\tilde{u}\|_{S(H^{1/2}; [I_j])} \leq \delta \|w\|_{S(H^{1/2}; [I_j])}. 
\]
Similarly, we can estimate other terms in (6.11) and get
\[\|w\|_{S(\dot{H}^1_x, I_j)} \leq \|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\dot{H}^1_x, I_j)} + c\delta^2 \|w\|_{S(\dot{H}^1_x, I_j)} + c\|w\|_3^{3}\|w\|_{S(\dot{H}^1_x, I_j)} + c\|w\|_{S(\dot{H}^1_x, I_j)} + c\epsilon.\] (6.12)

Now if \(\delta \leq \min(1, \frac{1}{6c})\) and
\[\|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\dot{H}^1_x, I_j)} + c\epsilon \leq \min(1, \frac{1}{2\sqrt{6c}}),\] (6.13)
we obtain
\[\|w\|_{S(\dot{H}^1_x, I_j)} \leq 2\|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\dot{H}^1_x, I_j)} + 2c\epsilon.\] (6.14)

Next, taking \(t = t_j\) in (6.10) and applying \(e^{i(t-t_{j+1})\Delta}\) to both sides, we obtain
\[e^{i(t-t_{j+1})\Delta} w(t_{j+1}) = e^{i(t-t_j)\Delta} w(t_j) + i \int_{t_j}^{t_{j+1}} e^{i(t-s)\Delta} W(\cdot, s)ds.\] (6.15)

Note that the Duhamel integral is confined to \(I_j\), similar to (6.12) we have the estimate
\[\|e^{i(t-t_{j+1})\Delta} w(t_{j+1})\|_{S(\dot{H}^1_x; [0, T])} \leq 2\|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\dot{H}^1_x; [0, T])} + c\delta^2 \|w\|_{S(\dot{H}^1_x, I_j)} + c\|w\|_3^{3}\|w\|_{S(\dot{H}^1_x, I_j)} + c\epsilon.\]

Then (6.13) and (6.14) imply
\[\|e^{i(t-t_{j+1})\Delta} w(t_{j+1})\|_{S(\dot{H}^1_x; [0, T])} \leq 2\|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\dot{H}^1_x; [0, T])} + 2c\epsilon.\]

Now beginning with \(j = 0\) we get by iteration
\[\|e^{i(t-t_j)\Delta} w(t_j)\|_{S(\dot{H}^1_x; [0, T])} \leq 2^j\|e^{i(t-t_0)\Delta} w(t_0)\|_{S(\dot{H}^1_x; [0, T])} + (2^j - 1)2c\epsilon \leq 2^{j+2}c\epsilon.\]

Since the second part of (6.13) is needed for each \(I_j, \ 0 \leq j \leq N - 1\), we require
\[2^{N+2}c\epsilon_0 \leq \min(1, \frac{1}{2\sqrt{6c}}).\] (6.16)

Recall that, \(\delta\) is an absolute constant satisfying (6.13); the number of intervals \(N\) is determined by the given \(A\); and then by (6.16) \(\epsilon_0\) was determined by \(N = N(A)\). Thus the iteration complete our proof.

\[\square\]

Note that from the proof above the parameters in Lemma 6.7 is independent of \(T\). As is stated in [2], besides the \(H^1\) asymptotic orthogonality (6.6) at \(t = 0\), this property can be extended to the nonlinear flow for \(0 \leq t \leq T\) as an application of Lemma 6.7 with a constant \(A = A(T)\) depending on \(T\) (but only through \(A\)). As for the Hartree equation (1.1), we will show a similar result:
Lemma 6.8. Suppose \( \phi_n(x) \) be a uniformly bounded sequence in \( H^1 \). Fix any time \( 0 < T < \infty \). Suppose that \( u_n(t) \equiv NLH(t)\phi_n \) exists up to time \( T \) for all \( n \) and

\[
\lim_{n \to \infty} \| \nabla u_n(t) \|_{L^\infty([0, T]; L^2)} < \infty.
\]

Let \( W_n^M(t) \equiv NLH(t)W_n^M \). Then, for all \( j \), \( v^j(t) \equiv NLH(t)\psi^j \) exist up to time \( T \) and for all \( t \in [0, T] \),

\[
\| \nabla u_n \|^2 = \sum_{j=1}^M \| \nabla v^j(t - t^j_n) \|^2 + \| \nabla W_n^M(t) \|^2 + o_n(1), \tag{6.17}
\]

where, \( o_n(1) \to 0 \) uniformly for \( 0 \leq t \leq T \).

Proof. Let \( M_0 \) be such that for \( M \geq M_0 \) and for \( \delta_{sd} \) in Lemma 2.3, we have

\[
\| NLH(t)W_n^M \|_{S(H^{\frac{1}{2}})} \leq \delta_{sd}/2
\]

and \( \| v^j \|_{S(H^{\frac{1}{2}})} \leq \delta_{sd} \) for \( j > M_0 \). Reorder the first \( M_0 \) profiles and introduce an index \( M_2 \), \( 0 \leq M_2 \leq M_0 \), such that

- For each \( 0 \leq j \leq M_2 \) we have \( t^j_n = 0 \) (There is no \( j \) in this case if \( M_2 = 0 \)).
- For each \( M_2 + 1 \leq j \leq M_0 \) we have \( |t^j_n| \to \infty \). (There is no \( j \) in this case if \( M_2 = M_0 \)).

By definition of \( M_0 \), \( v^j(t) \) for \( j > M_0 \) scatters in both time directions. We claim that for fixed \( T \) and \( M_2 + 1 \leq j \leq M_0 \), \( \| v^j(t - t^j_n) \|_{S(H^{\frac{1}{2}}; [0, T])} \to 0 \) as \( n \to \infty \). Indeed, take the case \( t^j_n \to +\infty \) for example. By Proposition 6.3, \( \| v^j(-t) \|_{S(H^{\frac{1}{2}}; [0, \infty))} < \infty \). Then for \( q < \infty \), \( \| v^j(-t) \|_{L^q([0, \infty); L^r)} < \infty \) implies \( \| v^j(t - t^j_n) \|_{L^q([0, T]; L^r)} \to 0 \). On the other hand, since \( v^j(t) \) in Proposition 6.3 is constructed by the existence of wave operators which converge in \( H^1 \) to a linear flow at \( -\infty \), then the \( L^{\frac{2}{r'}} \) decay of the linear flow implies immediately that \( \| v^j(t - t^j_n) \|_{L^\infty([0, T]; L^{\frac{2}{r'}})} \to 0 \). Similarly, we can obtain further that for \( M_2 + 1 \leq j \leq M_0 \), \( \| D_t^\frac{1}{2} v^j(t - t^j_n) \|_{L^2([0, T]; L^2)} \to 0 \) as \( n \to +\infty \).

Let \( B = \max\{1, \lim_n \| \nabla u_n \|_{L^\infty([0, T]; L^2)} \} \). For each \( 1 \leq j \leq M_2 \), define \( T^j \leq T \) to be the maximal forward time on which \( \| \nabla v^j \|_{L^\infty([0, T^j]; L^2)} \leq 2B \). Let \( \bar{T} = \min_{1 \leq j \leq M_2} T^j \), and if \( M_2 = 0 \), just take \( \bar{T} = T \). Note that if we have proved (6.17) holds for \( T = \bar{T} \), then by definition of \( T^j \), using the continuity arguments, it follows from (6.17) that for each \( 1 \leq j \leq M_2 \), we have \( T^j = T \). Hence \( \bar{T} = T \). Thus, for the remainder of the proof, we just work on \([0, \bar{T}]\).

For each \( 1 \leq j \leq M_2 \), \( \| v^j \|_{L^\infty([0, \bar{T}]; L^2)} = \| \psi^j \|_2 \leq \lim_n \| \phi_n \|_2 \) by (6.3), thus we have

\[
\| v^j(t) \|_{S(H^{\frac{1}{2}}; [0, \bar{T}])} \leq c \left( \| v^j \|_{L^\infty([0, \bar{T}]; L^2)} + \| v^j \|_{L^4([0, \bar{T}]; L^\frac{16}{5})} \right), \tag{6.18}
\]

\[
\leq c \left( \| v^j \|_{L^\infty([0, \bar{T}]; L^2)} + \| \nabla v^j \|_{L^\infty([0, \bar{T}]; L^2)} + \bar{T} \| \nabla v^j \|_{L^\infty([0, \bar{T}]; L^2)} \right) \leq c(1 + \bar{T}^\frac{7}{3}) B.
\]
In fact, from the local theory (see chapter 4 in [1]), we obtain further that for each \(1 \leq j \leq M_2\) \[
\|D^\frac{j}{2} v^j(t)\|_{S(L^2;[0,T])} \leq C(T, B). \tag{6.19}
\]

For fixed \(M\), let \(\tilde{u}_n(x, t) = \sum_{j=1}^{M} v^j(x - x^j_n, t - t^j_n)\), and let \(\epsilon_n = i\partial_t \tilde{u}_n + \Delta \tilde{u}_n + (V * |\tilde{u}_n|^2)\tilde{u}_n\).

Claim 1. There exists \(A = A(\tilde{T})\) (independent of \(M\)) such that for all \(M > M_0\), there exists \(n_0 = n_0(M)\) such that for all \(n > n_0\), \[
\|\tilde{u}_n\|_{S(\dot{H}^\frac{1}{2};[0,\tilde{T}])} \leq A.
\]

Claim 2. For each \(M > M_0\) and \(\epsilon > 0\), there exists \(n_1 = n_1(M, \epsilon)\) such that for \(n > n_1\) and for some \((q, r)\dot{H}^{-1/2}\) admissible, \[
\|\epsilon_n\|_{L^q([0,\tilde{T}];L^r')} \leq \epsilon.
\]

We postpone the proof of the claims to the end of our proof and suppose the two claims hold. Since \(u_n(0) - \tilde{u}_n(0) = W_n^M\), there exists \(M' = M'(\epsilon)\) large enough such that for each \(M > M'\) there exists \(n_2 = n_2(M')\) such that for \(n > n_2\), \[
\|\epsilon_{u\Delta}(u(0) - \tilde{u}(0))\|_{S(\dot{H}^\frac{1}{2};[0,\tilde{T}])} \leq \epsilon.
\]

For \(A = A(\tilde{T})\) in the first claim, Lemma [6.7] gives us \(\epsilon_0 = \epsilon_0(A) \ll 1\). We select an arbitrary \(\epsilon \leq \epsilon_0\) and obtain from above arguments an index \(M' = M'(\epsilon)\). Now select an arbitrary \(M > M'\), and set \(n' = \max(n_0, n_1, n_2)\). Then by Lemma [6.7] and the above arguments, for \(n > n'\), we have \[
\|u_n - \tilde{u}_n\|_{S(\dot{H}^\frac{1}{2};[0,\tilde{T}])} \leq c(\tilde{T})\epsilon. \tag{6.20}
\]

In order to obtain the \(\|\nabla \tilde{u}_n\|_{L^\infty([0,\tilde{T}];L^2)}\) bound, we also have to discuss \(j \geq M_2 + 1\). As is noted in the first paragraph of the proof, \(\|v^j(t - t^j_n)\|_{S(\dot{H}^\frac{1}{2};[0,\tilde{T}])} \to 0\) as \(n \to \infty\). By Strichartz estimate we can easily get \(\|\nabla v^j(t - t^j_n)\|_{L^\infty([0,\tilde{T}];L^2)} \leq c\|\nabla v^j(-t^j_n)\|_2\). By the pairwise divergence of parameters, \[
\|\nabla \tilde{u}_n\|_{L^\infty([0,\tilde{T}];L^2)}^2 = \sum_{j=1}^{M_2} \|\nabla v^j(t)\|_{L^\infty([0,\tilde{T}];L^2)}^2 + \sum_{j=M_2+1}^{M} \|\nabla v^j(t - t^j_n)\|_{L^\infty([0,\tilde{T}];L^2)}^2 + o_n(1)
\leq c \left( M_2 B^2 + \sum_{j=M_2+1}^{M} \|\nabla N L H (-t^j_n)\phi^j\|_2^2 + o_n(1) \right)
\leq c \left( M_2 B^2 + \|\nabla \phi_n\|_2^2 + o_n(1) \right) \leq c \left( M_2 B^2 + B^2 + o_n(1) \right).
\]
Note that for $\frac{5}{2} < p < \frac{10}{3}$, from (6.20) we have for some $0 < \theta < 1$

\[
\|u_n - \tilde{u}_n\|_{L^\infty([0,\bar{T}];L^p)} \leq c \left( \|u_n - \tilde{u}_n\|_{L^\infty([0,\bar{T}];L^5)} \right)^\theta \|\nabla (u_n - \tilde{u}_n)\|_{L^\infty([0,\bar{T}];L^2)}^{1-\theta} \\
\leq c(\bar{T})^\theta (M_2 B^2 + B^2 + o_n(1))^{\frac{1-\theta}{2}} \varepsilon^\theta.
\]

Thus, by Hardy-Littlewood-Sobolev inequalities and Hölder estimates, we in fact obtain

\[
\sup_{t \in [0,\bar{T}]} \|u_n - \tilde{u}_n\|_{L^4}^4 \leq c(\bar{T})^2 (M_2 B^2 + B^2 + o_n(1)) \varepsilon^2.
\]  

(6.21)

Now in the sequel we first replace the large parameter $M$ in the notation $\tilde{u}_n$ and all other arguments above for $M_1$. Then for any fixed $M$, we will prove (6.17) on $[0,\bar{T}]$. In fact, we need only to establish that, for each $t \in [0,\bar{T}]$,

\[
\|u_n\|_{L^4}^4 = \sum_{j=1}^{M} \|v^j(t - t_n^j)\|_{L^4}^4 + \|W^M_n(t)\|_{L^4}^4 + o_n(1).
\]  

(6.22)

Since then by (6.7) and the energy conservation we have

\[
E(u_n(t)) = \sum_{j=1}^{M} E(v^j(t - t_n^j)) + E(W^M_n(t)) + o_n(1).
\]  

(6.23)

Thus (6.22) combined with (6.23) gives (6.17), which completes our proof. So now we are to establish (6.22).

Firstly, for now, we again apply the perturbation theory Lemma 6.7 to $u_n(t) = W^M_n(t)$ and $\tilde{u}_n = \sum_{j=M+1}^{M_1} v^j(t - t_n^j)$. For any fixed $M < M_1$, since $u_n(0) - \tilde{u}_n(0) = W^M_n$, similar to the above two claims and the arguments followed, we obtain

\[
\|W^M_n(t) - \sum_{j=M+1}^{M_1} v^j(t - t_n^j)\|_{L^4}^4 \to 0 \text{ as } n \to \infty.
\]

From all arguments above and by the pairwise divergence of parameters,

\[
\|u_n\|_{L^4}^4 = \|\tilde{u}_n\|_{L^4}^4 + o_n(1) \\
= \sum_{j=1}^{M_1} \|v^j(t - t_n^j)\|_{L^4}^4 + o_n(1) \\
= \sum_{j=1}^{M} \|v^j(t - t_n^j)\|_{L^4}^4 + \sum_{j=M+1}^{M_1} \|v^j(t - t_n^j)\|_{L^4}^4 + o_n(1) \\
= \sum_{j=1}^{M} \|v^j(t - t_n^j)\|_{L^4}^4 + \|W^M_n(t)\|_{L^4}^4 + o_n(1).
\]

If on the other hand $M \geq M_1$, we then easily get from the selection of $M_1$ (see above analysis) that $\|W^M_n(t)\|_{L^4} = o_n(1)$ and (6.21) implies (6.22).
What the remainder is to establish the two claims. Recall that $M_0$ is sufficiently large such that $\| e^{it\Delta} W_n^M \|_{S(\dot{H}^1)} \leq \tilde{\delta}_d/2$ and for each $j > M_0$, it holds that $\| e^{it\Delta} v^j(-t_n^i) \|_{S(\dot{H}^1)} \leq \delta_d$. Similar to the small data scattering and Proposition 2.5, we obtain
\[
\| v^j(t - t_n^i) \|_{S(\dot{H}^1)} \leq 2 \| e^{it\Delta} v^j(-t_n^i) \|_{S(\dot{H}^1)} \leq 2\delta_d,
\]
and
\[
\| D^{1/2} v^j(t - t_n^i) \|_{S(L^2)} \leq c \| v^j(-t_n^i) \|_{H^1} \quad \text{for } j > M_0. 
\]
Thus by elementary inequality: for $a_j > 0$
\[
\left| \sum_{j=1}^{M} a_j \right|^{\frac{7}{2}} - \sum_{j=1}^{M} |a_j|^{\frac{7}{2}} \leq C_M \sum_{j \neq k} |a_j| |a_k|^{\frac{7}{2}}
\]
we have
\[
\| \tilde{u}_n \|_{L^\infty([0,T];L^\frac{7}{2})} \leq \sum_{j=1}^{M_0} \| v^j \|_{L^\infty([0,T];L^\frac{7}{2})} + \sum_{j=M_0+1}^{M} \| v^j(t - t_n^i) \|_{L^\frac{7}{2}([0,T];L^\frac{7}{2})} + crossterms \leq M_0 C(\bar{T}, B) + M_0 \epsilon^{\frac{7}{2}} + c \sum_{j=M_0+1}^{M} \| v^j(-t_n^i) \|_{H^1} + crossterms
\]
where we have used (6.19) and the analysis in the second paragraph. Now by (6.6)
\[
\| u_{n,0} \|_{H^\frac{1}{2}} = \sum_{j=1}^{M_0} \| v^j(-t_n^i) \|_{H^\frac{1}{2}} + \sum_{j=M_0+1}^{M} \| v^j(-t_n^i) \|_{H^\frac{1}{2}} + \| W_n^M \|_{H^\frac{1}{2}} + o_n(1),
\]
we know that the quantity $\sum_{j=M_0+1}^{M} \| v^j(-t_n^i) \|_{H^\frac{1}{2}}$ and so $\sum_{j=M_0+1}^{M} \| v^j(-t_n^i) \|_{H^\frac{1}{2}}$ is bounded independently of $M$ provided $n > n_0$ is sufficiently large. On the other hand, the crossterms can also be made bounded by taking $n_0$ large owing to the pairwise divergence of parameters. Above all, we have shown that $\| \tilde{u}_n \|_{L^\infty([0,T];L^\frac{7}{2})}$ is bounded independent of $M$ for $n > n_0$. A similar argument give the conclusion that $\| \tilde{u}_n \|_{L^\infty([0,T];L^\frac{7}{2})}$ is also bounded independent of $M$ for $n > n_0$ and the first claim holds true since the Strichartz norm $\| \tilde{u}_n \|_{S(H^\frac{1}{2};[0,T])}$ can be bounded by interpolation between the time-space norms with the above two exponents.
Now we turn to prove the second claim. We easily have the following expansion of $e_n$ which consists of $O(M^3)$ terms involving $V * |v^j(t-t_n^j)|^2 v^k(t-t_n^k)(k \neq j)$ (we will call such term cross term in the sequel).

$$e_n = \left( V * \left| \sum_{j=1}^{M} v^j(t-t_n^j) \right|^2 \right) \sum_{j=1}^{M} v^j(t-t_n^j) - \sum_{j=1}^{M} \left( V * |v^j(t-t_n^j)|^2 \right) v^j(t-t_n^j)$$

$$= \left( V * \left( \left| \sum_{j=1}^{M} v^j(t-t_n^j) \right|^2 - \sum_{j=1}^{M} |v^j(t-t_n^j)|^2 \right) \right) \sum_{j=1}^{M} v^j(t-t_n^j)$$

$$+ \sum_{j=1}^{M} \left( V * |v^j(t-t_n^j)|^2 \right) \sum_{k \neq j} v^k(t-t_n^k).$$

The point is how to estimate those cross terms. Assume first that $j \neq k$ and $|t_n^j - t_n^k| \to +\infty$, then at least one index $\geq M_2 + 1$. Take the Strichartz estimate of one of the cross terms for example, we have

$$\left\| (V * |v^j|^2)(t-t_n^j)v^k(t-t_n^k) \right\|_{L_{t}^{24}(0,T;L_{x}^{12})} \leq \left\| (V * |v^j|^2)(t)v^k(t-\tau_n^j) \right\|_{L_{t}^{24}(0,T;L_{x}^{12})}.$$  

Similar to the analysis in the second paragraph, this term goes to zero since $v^j, v^k \in L_{t}^{24} L_{x}^{60} \cap L_{t}^{8} L_{x}^{20}$ and

$$\left\| (V * |v^j|^2)(t)v^k(t+\tau_n^j - t_n^k) \right\|_{L_{t}^{24}(0,T;L_{x}^{12})} \leq \left\| |v^j|^2 \right\|_{L_{t}^{24}(0,T;L_{x}^{60})} \left\| v^k(t+\tau_n^j - t_n^k) \right\|_{L_{t}^{8}(0,T;L_{x}^{20})}.$$  

Then if $j \neq k$ and $\tau_n^j = t_n^k$, then by \((6.1)\), $|x_n^j - x_n^k| \to +\infty$. Take the same one of the cross terms for example, we have

$$\left\| \int \frac{|v^j(y-x_n^j)|^2 v^k(x-x_n^k)}{|x-y|^3} dy \right\|_{L_{t}^{24}(0,T;L_{x}^{12})} = \left\| \int \frac{|v^j(y')|^2 v^k(x-x_n^k)}{|x-x_n^j - y'|^3} dy' \right\|_{L_{t}^{24}(0,T;L_{x}^{12})}$$

$$= \left\| \int \frac{|v^j(y')|^2 v^k(x'+x_n^k-x_n^k)}{|x'-y'|^3} dy' \right\|_{L_{t}^{24}(0,T;L_{x}^{12})} = \left\| (V * |v^j|^2)v^k(\cdot + x_n^j - x_n^k) \right\|_{L_{t}^{24}(0,T;L_{x}^{12})}.$$  

In the same way, we obtain that it must go to zero again. Observe that all other cross terms will have the same property through similar estimates, and we in fact have proved the second claim.

\[ \square \]

**Lemma 6.9. (Profile Reordering).** Let $\phi_n(x)$ be a bounded sequence in $H^1$ and let $\lambda_0 > 1$. Suppose that $M(\phi_n) = M(Q), E(\phi_n)/E(Q) = 3\lambda_n^2 - 2\lambda_n^3$ with $\lambda_n \geq \lambda_0 > 1$ and $\|\nabla \phi_n\|_2/\|\nabla Q\|_2 \geq \lambda_n$ for each $n$. Then, for a given $M$, the profiles can be reordered so that there exist $1 \leq M_1 \leq M_2 \leq M$ and

1. For each $1 \leq j \leq M_1$, we have $t_n^j = 0$ and $\psi(t) \equiv NLH(t)\psi^j$ does not scatter as $t \to +\infty$. (We in fact assert that at least one $j$ belongs to this category.)
2. For each $M_1 + 1 \leq j \leq M_2$, we have $t_n^j = 0$ and $\psi(t)$ scatters as $t \to +\infty$. (There is no $j$ in this category if $M_2 = M_1$.)


For each $M_2 + 1 \leq j \leq M$ we have $|t_n^j| \to \infty$. (There is no $j$ in this category if $M_2 = M$.)

**Proof.** Firstly, we claim that there exists at least one $j$ such that $t_n^j$ converges as $n \to \infty$. In fact,

$$\frac{\|\phi_n\|^4_{L^4}}{\|Q\|^4_{L^4}} = -\frac{1}{2} \frac{E(\phi_n)}{E(Q)} + \frac{3}{2} \frac{\|\nabla \phi_n\|^2}{\|\nabla Q\|^2} \geq -\frac{1}{2} \left(3\lambda_n^2 - 2\lambda_0^3\right) + \frac{3}{2} \lambda_n^2 = \lambda_n^3 \geq \lambda_0^3 > 1. \quad (6.28)$$

If $|t_n^j| \to \infty$, then $\|NLH(-t_n^j)\psi|_{L^4} \to 0$ and $(6.8)$ implies our conclusion. Now if $j$ is such that $t_n^j$ converges as $n \to \infty$, then we might as well assume $t_n^j = 0$.

Reordering the profiles $\psi^j$ so that for $1 \leq j \leq M_2$, we have $t_n^j = 0$, and for $M_2 + 1 \leq j \leq M$ we have $|t_n^j| \to \infty$. It remains to show that there exists one $j$, $1 \leq j \leq M_2$, such that $\psi^j(t)$ does not scatter as $t \to +\infty$. To the contrary, if for all $1 \leq j \leq M_2$, $\psi^j(t)$ scatters, then we have that $\lim_{t \to +\infty} \|\psi^j(t)\|_{L^4} = 0$. Let $t_0$ be sufficiently large so that for all $1 \leq j \leq M_2$, we have $\|\psi^j(t_0)\|^4_{L^4} \leq \epsilon/M_2$. The $L^4$ orthogonality $(6.22)$ along the NLH flow and an argument as $(6.28)$ imply

$$\lambda_0^3 \|Q\|^4_{L^4} \leq \|u_n(t_0)\|^4_{L^4} = \sum_{j=1}^{M_2} \|\psi^j(t_0)\|^4_{L^4} + \sum_{j=M_2+1}^{M} \|\psi^j(t_0 - t_n^j)\|^4_{L^4} + \|W_n^M(t_0)\|^4_{L^4} + o_n(1).$$

We know from Proposition $(6.5)$ that, as $n \to +\infty$, $\sum_{j=M_2+1}^{M} \|\psi^j(t_0 - t_n^j)\|^4_{L^4} \to 0$, and thus we have

$$\lambda_0^3 \|Q\|^4_{L^4} \leq \epsilon + \|W_n^M(t_0)\|^4_{L^4} + o_n(1).$$

This gives a contradiction since $W_n^M(t)$ is a scattering solution. 

## 7. Inductive Argument and Existence of a Critical Solution

We now begin to prove Theorem $(1.1)$. By Remark $(1.2)$ we only need to deal with the case that $P(u) = 0$. We will use the notations from $(9)$ and give some definitions first.

**Definition 7.1.** Let $\lambda > 1$. We say that $\exists GB(\lambda, \sigma)$ holds if there exists a solution $u(t)$ to $(1.1)$ such that

$$P(u) = 0, \quad M(u) = M(Q), \quad \frac{E(u)}{E(Q)} = 3\lambda^2 - 2\lambda^3$$

and

$$\lambda \leq \frac{\|\nabla u(t)\|^2}{\|\nabla Q\|^2} \leq \sigma \quad \text{for all } t \geq 0.$$

$\exists GB(\lambda, \sigma)$ means that there exist solutions with energy $3\lambda^2 - 2\lambda^3$ globally bounded by $\sigma$. Thus by Proposition $(5.1)$, $\exists GB(\lambda, \lambda(1 + \rho_0(\lambda_0)))$ is false for all $\lambda \geq \lambda_0 > 1$.

The statement $\exists GB(\lambda, \sigma)$ is false is equivalent to say that for every solution $u(t)$ to $(1.1)$ with $M(u) = M(Q)$ and $E(u)/E(Q) = 3\lambda^2 - 2\lambda^3$ such that $\|\nabla u(t)\|^2/\|\nabla Q\|^2 \geq \lambda$ for
all \( t \), there must exists a time \( t_0 \geq 0 \) such that \( \| \nabla u(t_0) \|_2/\| \nabla Q \|_2 \geq \sigma \). By resetting the initial time, we can find a sequence \( t_n \to \infty \) such that \( \| \nabla u(t_n) \|_2/\| \nabla Q \|_2 \geq \sigma \) for all \( n \).

Note that if \( \lambda \leq \sigma_1 \leq \sigma_2 \), then \( \exists GB(\lambda, \sigma_2) \) is false implies \( \exists GB(\lambda, \sigma_1) \) is false. We will induct on the statement and define a threshold.

**Definition 7.2.** (The Critical Threshold.) Fix \( \lambda_0 > 1 \). Let \( \sigma_c = \sigma_c(\lambda_0) \) be the supremum of all \( \sigma > \lambda_0 \) such that \( \exists GB(\lambda, \sigma) \) is false for all \( \lambda \) such that \( \lambda_0 \leq \lambda \leq \sigma \).

Proposition 5.1 implies that \( \sigma_c(\lambda_0) > \lambda_0 \). Let \( u(t) \) be any solution to (1.1) with \( P(\lambda) = 0 \), \( M(\lambda) = M(Q) \), \( E(\lambda)/E(Q) \leq 3\lambda_0^2 - 2\lambda_0^3 \) and \( \| \nabla u(0) \|_2/\| \nabla Q \|_2 \geq 1 \). If \( \lambda_0 > 1 \) and \( \sigma_c = \infty \), we claim that there exists a sequence of times \( t_n \) such that \( \| \nabla u(t_n) \|_2 \to \infty \). In fact, if not, and let \( \lambda \geq \lambda_0 \) be such that \( E(u)/E(Q) \geq 3\lambda^2 - 2\lambda^3 \). Since there is no sequence \( t_n \) such that \( \| \nabla u(t_n) \|_2 \to \infty \), there must exists \( \sigma < \infty \) such that \( \lambda \leq \| \nabla u(t) \|_2/\| \nabla Q \|_2 \leq \sigma \) for all \( t \geq 0 \), which means that \( \exists GB(\lambda, \sigma) \) holds true. Thus \( \sigma_c \leq \sigma < \infty \) and we get a contradiction.

In view of the above results, if we can prove that for every \( \lambda_0 > 1 \) then \( \sigma_c(\lambda_0) = \infty \), we then have in fact proved Theorem 1.1. Thus, in the sequel, we shall carry it out by contradiction. More precisely, fix \( \lambda_0 > 1 \) and assume \( \sigma_c < \infty \), we shall work toward an absurdity. (It, of course, suffices to do this for \( \lambda_0 \) close to 1, so we might as well assume that \( \lambda_0 < \frac{3}{2} \), which will be convenient in the sequel.) For that purpose, we need first to obtain the existence of a critical solution:

**Lemma 7.3.** \( \sigma_c(\lambda_0) < \infty \). Then there exist initial data \( u_{c,0} \) and \( \lambda_c \in [\lambda_0, \sigma_c(\lambda_0)] \) such that \( u_c(t) \equiv NLH(t)u_{c,0} \) is global, \( P(u_c) = 0 \), \( M(u_c) = M(Q) \), \( E(u_c)/E(Q) = 3\lambda_c^2 - 2\lambda_c^3 \), and

\[
\lambda_c \leq \frac{\| \nabla u_c(t) \|_2}{\| \nabla Q \|_2} \leq \sigma_c \quad \text{for all } t \geq 0.
\]

**Proof.** By definition of \( \sigma_c \), there exist sequence \( \lambda_n \) and \( \sigma_n \) such that \( \lambda_0 \leq \lambda_n \leq \sigma_n \) and \( \sigma_n \downarrow \sigma_c \) for which \( \exists GB(\lambda_n, \sigma_n) \) holds. This means that there exists \( u_{n,0} \) such that \( u_n(t) \equiv NLH(t)u_{n,0} \) is global with \( P(u_n) = 0 \), \( M(u_n) = M(Q) \), \( E(u_n)/E(Q) = 3\lambda_n^2 - 2\lambda_n^3 \), and

\[
\lambda_n \leq \frac{\| \nabla u_n(t) \|_2}{\| \nabla Q \|_2} \leq \sigma_n \quad \text{for all } t \geq 0.
\]

The boundedness of \( \lambda_n \) make us pass to a subsequence such that \( \lambda_n \) converges with a limit \( \lambda' \in [\lambda_0, \sigma_c] \).

According to Lemma 6.9, where we take \( \phi_n = u_{n,0} \), for \( M_1 + 1 \leq j \leq M_2 \), \( \psi_j(t) \equiv NLH(t)\psi_j \) scatter as \( t \to +\infty \) and combined with Proposition 6.5, for \( M_2 + 1 \leq j \leq M \), \( \psi_j \) also scatter in one or the other time direction. Thus by the scattering theory, for \( M_1 + 1 \leq j \leq M \), we have \( E(\psi_j) = E(\psi_j) \geq 0 \) and then by (6.7)

\[
\sum_{j=1}^{M_1} E(\psi_j) \leq E(\phi_n) + o_n(1).
\]
Thus there exists at least one $1 \leq j \leq M_1$ with
\[ E(\psi^j) \leq \max \{ \lim_n E(\phi_n), 0 \}, \]
Without loss of generality, we might take $j = 1$. Since, by the profile composition, also $M(\psi^1) \leq \lim_n M(\phi_n) = M(Q)$, we then have
\[ \frac{M(\psi^1)E(\psi^1)}{M(Q)E(Q)} \leq \max \left\{ \lim_n \frac{E(\phi_n)}{E(Q)}, 0 \right\}. \]
Thus, there exist $\tilde{\lambda} \geq \lambda_0$\footnote{If $\lim_n E(\phi_n) \geq 0$, we have $\tilde{\lambda} \geq \lambda' \geq \lambda_0$; while in the case $\lim_n E(\phi_n) < 0$, we will have $\tilde{\lambda} \geq \frac{\lambda}{2} > \lambda_0$ though we might not have $\tilde{\lambda} \geq \lambda'$.} such that
\[ \frac{M(\psi^1)E(\psi^1)}{M(Q)E(Q)} = 3\tilde{\lambda}^2 - 2\tilde{\lambda}^3. \]
Note that by Lemma 6.9, $v^1$ does not scatter, so it follows from Theorem 2.4 that $\|\psi^1\|_2 \|\nabla \psi^1\|_2 < \|Q\|_2 \|\nabla Q\|_2$ cannot hold. Then by the dichotomy Proposition 2.2, we must have $\|\psi^1\|_2 \|\nabla \psi^1\|_2 \geq \tilde{\lambda}\|Q\|_2 \|\nabla Q\|_2$.

Now if $\tilde{\lambda} > \sigma_c$ and recall that $t_n^1 = 0$, then for all $t$ we know that
\[ \tilde{\lambda}^2 \leq \frac{\|v^1(t)\|_2^2 \|\nabla v^1(t)\|_2^2}{\|Q\|_2^2 \|\nabla Q\|_2^2} \leq \frac{1}{\|\nabla Q\|_2^2} \sum_{j=1}^M \|\nabla v^j(t - t_k^j)\|_2^2 + \|\nabla W_n^M(t)^2\|_2. \quad (7.1) \]
Taking $t = 0$, for example, by Lemma 6.8 we have
\[ \tilde{\lambda}^2 \leq \frac{\sum_{j=1}^M \|\nabla v^j(-t_k^j)\|_2^2 + \|W_n^M\|_2^2}{\|\nabla Q\|_2^2} \leq \frac{\|\nabla u_n(0)\|_2^2}{\|\nabla Q\|_2^2} + o_n(1) \leq \sigma_c^2 + o_n(1), \]
which contradicts the assumption $\tilde{\lambda} > \sigma_c$. Hence we must have $\tilde{\lambda} \leq \sigma_c$.

Now if $\tilde{\lambda} < \sigma_c$, we know from the definition of $\sigma_c$ that $\exists GB(\tilde{\lambda}, \sigma_c - \delta)$ is false for any $\delta > 0$ sufficiently small, and then there exists a nondecreasing sequence $t_k$ of times such that
\[ \lim_k \frac{\|v^1(t_k)\|_2 \|\nabla v^1(t_k)\|_2}{\|Q\|_2 \|\nabla Q\|_2} \geq \sigma_c. \]
Note that $t_n^1 = 0$, then
\[ \sigma_c^2 - o_k(1) \leq \frac{\|v^1(t_k)\|_2^2 \|\nabla v^1(t_k)\|_2^2}{\|Q\|_2^2 \|\nabla Q\|_2^2} \leq \frac{\|\nabla v^1(t_k)\|_2^2}{\|\nabla Q\|_2^2} \leq \frac{\sum_{j=1}^M \|\nabla v^j(t_k - t_k^j)\|_2^2 + \|\nabla W_n^M(t_k)\|_2^2}{\|\nabla Q\|_2^2} \leq \frac{\|\nabla u_n(t_k)\|_2^2}{\|\nabla Q\|_2^2} + o_n(1) \leq \sigma_c^2 + o_n(1). \quad (7.2) \]
Then by the same way as in the proof of Lemma 7.3, we get that
\[ W \]
exists a sequence
\[ x \]
Hence, we take
\[ u \]
Lemma 8.1.
Note that by Lemma 6.9, compactness of the flow of the critical solution. For convenience, we take 0 for all \( j \) such that, by passing to a subsequence, \( n \to \infty \) .

Moreover, by Lemma 6.8, for all \( t \)
\[ \frac{\|\nabla u(t)\|^2}{\|\nabla Q\|^2} \leq \lim_n \frac{\|\nabla u_n(0)\|^2}{\|\nabla Q\|^2} \leq \sigma_c^2. \]

Hence, we take \( u_{c,0} = v^1(0) = \psi^1 \) and \( \lambda_c = \tilde{\lambda} \) to complete our proof.

\[ \square \]

8. Concentration of Critical Solutions and Proof of Theorem 1.1

In this section, we will finally complete our proof of Theorem 1.1 by virtue of the pre-compactness of the flow of the critical solution. For convenience, we take \( u(t) = u_c(t) \) in the sequel.

**Lemma 8.1.** There exists a path \( x(t) \) in \( \mathbb{R}^N \) such that
\[ K \equiv \{ u(t, \cdot - x(t)) | t \geq 0 \} \subset H^1 \]
is precompact in \( H^1 \).

**Proof.** As is showed in [2], it suffices to prove that for each sequence of times \( t_n \to \infty \), there exists a sequence \( x_n \) such that, by passing to a subsequence, \( u(t_n, \cdot - x_n) \) converges in \( H^1 \).

Taking \( \phi_n = u(t_n) \) in Lemma 6.9 and by definition of \( u(t) = u_c(t) \), similar to the proof of Lemma 7.3 we obtain that there exists at least one \( 1 \leq j \leq M_1 \) with
\[ E(\psi^j) \leq \max(\lim_n E(\phi_n), 0). \]

Without loss of generality, we can take \( j = 1 \). Since, also \( M(\psi^1) \leq \lim_n M(\phi_n) = M(Q) \), there exist \( \tilde{\lambda} \geq \lambda_0 \) such that
\[ \frac{M(\psi^1)E(\psi^1)}{M(Q)E(Q)} = 3\tilde{\lambda}^2 - 2\tilde{\lambda}^3. \]
Note that by Lemma 6.9, \( v^1 \) does not scatter, so we must have \( \|\psi^1\|^2_2 \|\nabla \psi^1\|^2_2 \geq \tilde{\lambda} \| Q \|^2_2 \|\nabla Q\|^2_2 \).
Then by the same way as in the proof of Lemma 7.3, we get that \( W_n^{M}(t_k) \to 0 \) in \( H^1 \) and \( v^j \equiv 0 \) for all \( j \geq 2 \). Since we know that \( W_n^{M}(t) \) is a scattering solution , this implies that
\[ W_n^{M}(0) = W_n^{M} \to 0 \text{ in } H^1. \] (8.1)

Consequently, we have
\[ u(t_n) = NLH(-t^1_n)^1(x - x^1_n) + W_n^{M}(x). \]
Note that by Lemma 6.9, \( t_n^1 = 0 \), and thus
\[
u(t_n, x + x_n^1) = \psi^1(x) + W_n^M(x + x_n^1).
\]
This equality and (8.1) imply our conclusion.

Using the uniform-in-time \( H^1 \) concentration of \( u(t) = u_c(t) \) and by changing variables, we can easily get

**Corollary 8.2.** For each \( \epsilon > 0 \), there exists \( R > 0 \) such that for all \( t \),
\[
\|u(t, \cdot - x(t))\|_{H^1(|x| \geq R)} \leq \epsilon.
\]

With the localization property of \( u_c \), we show, similar to [9], that \( u_c \) must blow up in finite time using the same method as that in the proof of Proposition 3.2 and Remark 3.3. However, this contradicts the boundedness of \( u_c \) in \( H^1 \). Hence, \( u_c \) cannot exist and \( \sigma_c = \infty \). As is argued in section 7, this indeed completes the proof of Theorem 1.1.

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