Unconditionally converging polynomials on Banach spaces

BY MANUEL GONZÁLEZ†
Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, 39071 Santander, Spain

AND JOAQUÍN M. GUTIÉRREZ‡
Departamento de Matemática Aplicada, ETS de Ingenieros Industriales, Universidad
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0. Introduction

In the study of polynomials acting on Banach spaces, the weak topology is not such a good tool as in the case of linear operators, due to the bad behaviour of the polynomials with respect to the weak convergence. For example,

\[ Q: (x_n) \in l_2 \to (x_n^2) \in l_1 \]

is a continuous polynomial taking a weakly null sequence into a sequence having no weakly Cauchy subsequences. In this paper we show that the situation is not so bad for unconditional series. Recall that \( \sum_{i=1}^{\infty} x_i \) is a weakly unconditionally Cauchy series (in short a w.u.C. series) in a Banach space \( E \) if for every \( f \in E^* \) we have that \( \sum_{i=1}^{\infty} |f(x_i)| < \infty \); and \( \sum_{i=1}^{\infty} x_i \) is an unconditionally converging series (in short an u.c. series) if every subseries is norm convergent.

We prove that a continuous polynomial takes w.u.C. (u.c.) series into w.u.C. (u.c.) series. We derive this result from an estimate of the unconditional norm of the image of a sequence by a homogeneous polynomial, which is also a fundamental tool in other parts of the paper, and could be of some interest in itself.

In view of the preservation of w.u.C. (u.c.) series by polynomials, it is natural to introduce the class \( \mathcal{P}_{uc} \) of unconditionally converging polynomials as those taking w.u.C. series into u.c. series. It turns out that most of the classes of polynomials that have been considered in the literature are contained in \( \mathcal{P}_{uc} \). By means of the class of unconditionally converging polynomials we introduce the polynomial property \( (V) \) for Banach spaces, and we show that spaces with this property share some of the properties of Tsirelson’s space \( T^* \). In fact, these spaces are reflexive, and their dual spaces cannot contain copies of \( l_p, 1 < p < \infty \). This is a consequence of the following characterization: a Banach space \( E \) has the polynomial property \( (V) \) if and only if the space of scalar polynomials \( P_k(E) \) is reflexive for every positive integer \( k \). We also apply \( \mathcal{P}_{uc} \) to characterize the polynomial counterpart of other isomorphic properties of Banach spaces, like the Dieudonné property, the Schur property, and property \( (V^*) \), obtaining remarkable differences with the corresponding linear (usual) properties. This is in contrast with the results of [14], where it is proved that the polynomial Dunford–Pettis property coincides with the Dunford–Pettis property.

Throughout the paper, \( E \) and \( F \) will be real or complex Banach spaces, \( B_E \) the unit ball of \( E \), and \( E^* \) its dual space. The scalar field will always be \( \mathbb{R} \) or \( \mathbb{C} \), the real or

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the complex field, and we will write \( \mathbb{N} \) for the set of all natural numbers. Moreover, \( \mathcal{P}(E, F) \) will stand for the space of all (continuous) polynomials from \( E \) into \( F \). Any \( P \in \mathcal{P}(E, F) \) can be decomposed as a sum of homogeneous polynomials: \( P = \sum_{k=0}^{n} P_k \), with \( P_k \in \mathcal{P}(E, F) \), the space of all \( k \)-homogeneous polynomials from \( E \) into \( F \).

1. Unconditionally converging polynomials

In this section we obtain an estimate for the unconditional norm of the image of a sequence by a homogeneous polynomial, and we apply it to prove the preservation of w.u.C. series and u.c. series by homogeneous polynomials. Then we introduce the class of unconditionally converging polynomials, and compare it with other classes of polynomials that have appeared in the literature.

In the proof of the estimate, we will need the generalized Rademacher functions, denoted by \( s_n(t) \), \( n \in \mathbb{N} \), which were introduced in [4]. These functions are defined as follows.

Fix \( 2 \leq k \in \mathbb{N} \), and let \( \alpha_1 = 1, \alpha_2, \ldots, \alpha_k \) denote the \( k \)th roots of unity. Let \( s_1 : [0, 1] \to \mathbb{C} \) be the step function taking the value \( \alpha_j \) on \( ((j-1)/k, j/k) \) for \( j = 1, \ldots, k \). Then, assuming that \( s_{n-1} \) has been defined, define \( s_n \) as follows. Fix any of the \( k^{n-1} \) sub-intervals \( I_j \) of \( [0,1] \) used in the definition of \( s_{n-1} \). Divide \( I_j \) into \( k \) equal intervals \( I_{jk}, \ldots, I_{k_1} \), and set \( s_n(t) = \alpha_j \) if \( t \in I_j \).

The generalized Rademacher functions are orthogonal [4, lemma 1.2] in the sense that, for any choice of integers \( i_1, \ldots, i_k; k \geq 2 \), we have

\[
\int_0^1 s_{i_1}(t) \cdots s_{i_k}(t) dt = \begin{cases} 1 & \text{if } i_1 = \cdots = i_k; \\ 0 & \text{otherwise}. \end{cases}
\]

Lemma 1. Let \( E \) and \( F \) be Banach spaces. Given \( k \in \mathbb{N} \) there exists a constant \( C_k \) such that for every \( P \in \mathcal{P}(E, F) \), and \( x_1, \ldots, x_n \in E \) we have

\[
\sup_{|e_j| \leq 1} \left\| \sum_{j=1}^{n} e_j P x_j \right\| \leq C_k \sup_{|e_j| \leq 1} \left\| P \left( \sum_{j=1}^{n} e_j x_j \right) \right\|.
\]

In the complex case we can take \( C_k = 1 \) for every \( k \), and in the real case, \( C_k = (2k)^k/k! \).

Proof. First we assume that \( E \) and \( F \) are complex spaces. In this case, both suprema are attained for some \( |e_j| = |v_j| = 1 \).

Given \( P \in \mathcal{P}(E, F) \), we denote by \( \hat{P} \) the associated symmetric k-linear map. For any \( x_1, \ldots, x_n \in E \) and any complex numbers \( e_j \) with \( |e_j| = 1 \), we can find \( f \in F^* \), \( \|f\| = 1 \), such that

\[
\left\| \sum_{j=1}^{n} e_j P x_j \right\| = f \left( \sum_{j=1}^{n} e_j P x_j \right).
\]

Then, taking complex numbers \( \delta_j \) such that \( \delta_j^k = e_j \), we obtain

\[
\left\| \sum_{j=1}^{n} e_j P x_j \right\| = f \left( \sum_{j=1}^{n} P(\delta_j x_j) \right) = \int_0^1 \left( \sum_{j_1, \ldots, j_k=1}^{n} s_{j_1}(t) \cdots s_{j_k}(t) f \circ \hat{P}(\delta_{j_1} x_{j_1}, \ldots, \delta_{j_k} x_{j_k}) \right) dt
\]

\[
= \int_0^1 f \circ \hat{P} \left( \sum_{j_1, \ldots, j_k=1}^{n} \delta_{j_1} s_{j_1}(t) x_{j_1}, \ldots, \sum_{j_k=1}^{n} \delta_{j_k} s_{j_k}(t) x_{j_k} \right) dt
\]

\[
= \int_0^1 f \circ P \left( \sum_{j=1}^{n} \delta_j s_j(t) x_j \right) dt \leq \sup_{|v_j| = 1} \left\| P \left( \sum_{j=1}^{n} v_j x_j \right) \right\|.
\]

In this way the proof for the complex case is finished.
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Assume now that $E$ and $F$ are real Banach spaces, and denote by $\bar{E}$ and $\bar{F}$ their respective complexifications. We can extend the multilinear map $\hat{P} \in L^k(E, F)$ associated with the polynomial $P$ to a multilinear map $\hat{Q} \in L^k(\bar{E}, \bar{F})$ in a straightforward way, which in the case $k = 2$ is given by

$$\hat{Q}(x_1 + iy_1, x_2 + iy_2) = \hat{P}(x_1, x_2) + i\hat{P}(y_1, x_2) + i\hat{P}(x_1, y_2) - \hat{P}(y_1, y_2),$$

and the polynomial $Q \in \mathcal{P}(\bar{E}, \bar{F})$ associated with $\hat{Q}$ is an extension of $P$. We have

$$\left\| \sum_{j=1}^n e_j P x_j \right\| \leq \sup_{|v| = 1} \left\| Q \left( \sum_{j=1}^n v_j x_j \right) \right\|;$$

and for complex numbers $v_j = a_j + ib_j$ with $|v_j| = 1$, we obtain

$$\left\| Q \left( \sum_{j=1}^n v_j x_j \right) \right\| = \left\| \sum_{m=0}^k \binom{k}{m} i^m \hat{Q} \left( \sum_{j=1}^n b_j x_j \right)^m \left( \sum_{j=1}^n a_j x_j \right)^{k-m} \right\| = \left\| \sum_{m=0}^k \binom{k}{m} i^m \hat{P} \left( \sum_{j=1}^n b_j x_j \right)^m \left( \sum_{j=1}^n a_j x_j \right)^{k-m} \right\|.$$

Moreover, using the polarization formula [10, theorem 1.10], we obtain

$$\left\| \hat{P} \left( \sum_{j=1}^n b_j x_j \right)^m \left( \sum_{j=1}^n a_j x_j \right)^{k-m} \right\| = \frac{1}{k! 2^k} \left\| \sum_{e_1, \ldots, e_k} e_1 \cdots e_k P \left( \sum_{j=1}^n b_j x_j \right)^m \left( \sum_{j=1}^n a_j x_j \right)^{k-m} \right\| = \frac{1}{k! 2^k} \left\| \sum_{e_1, \ldots, e_k} e_1 \cdots e_k \sum_{j=1}^n c_j x_j \right\| \leq \frac{\left\| \sum_{j=1}^n c_j x_j \right\|}{k!} \sup_{|c| \leq 1} P \left( \sum_{j=1}^n c_j x_j \right),$$

where $c_j = k^{-1}((e_1 + \cdots + e_m) b_j + (e_{m+1} + \cdots + e_k) a_j)$. Hence

$$\left\| \sum_{j=1}^n e_j P x_j \right\| \leq \left\| \sum_{m=0}^k \binom{k}{m} \sum_{|c| \leq 1} P \left( \sum_{j=1}^n c_j x_j \right) \right\| = \frac{(2k)^k}{k!} \sup_{|c| \leq 1} \left\| P \left( \sum_{j=1}^n c_j x_j \right) \right\|.$$ 

Next we show that polynomials preserve w.u.C. series and u.c. series.

**Theorem 2.** Let $E$ and $F$ be Banach spaces and $P \in \mathcal{P}(E, F)$. Then $P$ takes w.u.C. (u.c.) series into w.u.C. (u.c.) series.

**Proof.** Recall that a series $\sum_{i=1}^\infty x_i$ in a Banach space is w.u.C. if and only if $\sup_{|c| \leq 1} \sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^n c_i x_i \right\|$ is finite; and it is u.c. if and only if $\sup_{|c| \leq 1} \left\| \sum_{i=1}^\infty c_i x_i \right\|$ converges to 0 when $n$ goes to infinity. Hence, the result is a direct consequence of Lemma 1, since we have

$$\sup_{|c| \leq 1} \left\| \sum_{i=n}^\infty c_i P x_i \right\| \leq C_k \| P \| \sup_{|c| \leq 1} \left\| \sum_{i=n}^\infty c_i x_i \right\|^k.$$ 

Inspired by Theorem 2, we introduce the following class of polynomials.
**Definition 3.** Let $E$ and $F$ be Banach spaces and $k \in \mathbb{N}$. A polynomial $P \in \mathcal{P}^k(E, F)$ is said to be unconditionally converging if it takes w.u.C. series into u.c. series.

We shall denote by $\mathcal{P}_{uc}^k(E, F)$ the class of all unconditionally converging $k$-homogeneous polynomials from $E$ to $F$.

Note that, in the case $E$ or $F$ contains no copies of $c_0$, we have that all the w.u.C. series in that space are u.c.; hence $\mathcal{P}^k(E, F) = \mathcal{P}_{uc}^k(E, F)$.

The prototype of w.u.C., not u.c., series is the unit vector basis $\{e_n\}$ of $c_0$. The next lemma characterizes unconditionally converging polynomials in terms of their restrictions to subspaces isomorphic to $c_0$, or the action on sequences equivalent to $\{e_n\}$.

**Lemma 4.** Given $P \in \mathcal{P}^k(E, F) \setminus \mathcal{P}_{uc}$, there exists an isomorphism $i: c_0 \to E$ such that $\{(P \circ i) e_n\}$ is equivalent to $\{e_n\}$. In particular, $P \circ i \in \mathcal{P}^k(c_0, F) \setminus \mathcal{P}_{uc}$.

**Proof.** If $P \in \mathcal{P}^k(E, F) \setminus \mathcal{P}_{uc}$, then we can find a w.u.C. series $\sum x_i$ such that $\sum P x_i$ is not u.c. Moreover, given a w.u.C., not u.c., series $\sum z_i$, we can construct suitable blocks $u_k = z_{nk+1} + \ldots + z_{nk+k}$ such that $\|u_k\|$ is bounded away from 0. Since the series $\sum u_i$ is w.u.C., the sequence $(u_k)$ is weakly null; hence by the Bessaga–Pełczynski selection theorem, it has a basic subsequence, which is equivalent to the unit vector basis of $c_0$ [6, corollary 5-7].

Now, as $\sum P x_i$ is w.u.C. but not u.c., we can construct a sequence of blocks $P x_{nk+1} + \ldots + P x_{nk+k}$ of $(P x_i)$ equivalent to the unit vector basis of $c_0$, and it follows from Lemma 1 that there exist scalars $c_i$ with $|c_i| \leq 1$ for every $i \in \mathbb{N}$, such that the vectors $P(y_k) = P(c_{nk+1} x_{nk+1} + \ldots + c_{nk+k} x_{nk+k})$ are bounded away from 0. Since $\sum y_k = \sum P x_i$ is also a w.u.C. series, we can assume, by passing to a subsequence if necessary, that both sequences $(y_k)$ and $(P x_i)$ are equivalent to $\{e_n\}$.

The map $i: c_0 \to E$ defined by $i(e_n) = y_n$ is an isomorphism, and $\sum (P \circ i) e_n$ is not u.c.; hence $P \circ i \notin \mathcal{P}_{uc}^k(c_0, F)$.

One of the remarkable properties of the class $\mathcal{P}_{uc}$ of unconditionally converging polynomials is that it includes the main classes of polynomials considered in the literature, as we shall show below.

Recall that a polynomial $P \in \mathcal{P}^k(E, F)$ is weakly compact, denoted by $P \in \mathcal{P}_{wco}^k(E, F)$, if it takes bounded subsets into relatively weakly compact subsets. Moreover, we shall say that $P$ is completely continuous, denoted by $P \in \mathcal{P}_{cc}^k(E, F)$, if it takes weakly Cauchy sequences into norm convergent sequences. These classes were considered in [13] and [14].

We shall consider also the class $\mathcal{P}_{cc}^k(E, F)$ of polynomials which are completely continuous at 0, formed by those $P \in \mathcal{P}^k(E, F)$ taking weakly null sequences into norm null sequences. Clearly $\mathcal{P}_{cc}^k(E, F) \subseteq \mathcal{P}_{cc0}^k(E, F)$, but in general (see Proposition 19) the containment is strict for $k > 1$ and $E$ failing the Schur property.

Recall that $A \subseteq E$ is said to be a Rosenthal set if any sequence $(x_n) \subseteq A$ has a weakly Cauchy subsequence. Contrary to the case of linear operators, a polynomial taking Rosenthal sets into relatively compact subsets need not take weakly null sequences into norm null sequences, as is shown by the scalar polynomial

$$P: (x_n) \in l_2 \mapsto \sum_{n=1}^{\infty} x_n^2 \in \mathbb{R}.$$
The converse implication also fails, since for the polynomial
\[ Q: (x_n) \in l_2 \rightarrow \left( \sum_{k=1}^{\infty} \frac{x_k}{k} \right) (x_n) \in l_2 \]
we have that \( Q(e_1 + e_n) = (1 + 1/n) (e_1 + e_n) \) has no convergent subsequences, although \( Q \) takes weakly null sequences into norm null sequences, because of the factor \( (\sum_{k=1}^{\infty} \frac{x_k}{k}) \).

Finally, recall that \( A \subset E \) is said to be a Dunford–Pettis set \([2]\) if for any weakly null sequence \( (f_n) \subset E^* \) we have
\[ \limsup_{n \to \infty} |f_n(x)| = 0. \]

This class of subsets, introduced in \([2]\), allowed us to define in \([7]\) the class \( \mathcal{P}_{ud} \) as follows.

A polynomial \( P \in \mathcal{P}(^kE, F) \) belongs in \( \mathcal{P}_{ud} \) if and only if its restriction to any Dunford–Pettis subset of \( E \), endowed with the inherited weak topology, is continuous.

**Proposition 5.** A polynomial \( P \in \mathcal{P}(^kE, F) \) belongs to \( \mathcal{P}_{uc} \) in the following cases: (a) \( P \in \mathcal{P}_{cc} \); (b) \( P \) takes Rosenthal subsets of \( E \) into relatively compact subsets of \( F \); (c) \( P \in \mathcal{P}_{uc} \); (d) \( P \in \mathcal{P}_{wco} \).

**Proof.** The result in cases (a) and (b) is an immediate consequence of Lemma 4, since the unit vector basis of \( c_0 \) is a weakly null sequence which forms a non-relatively compact set.

Case (c) also follows from Lemma 4, since given \( P \in \mathcal{P}(^kE, F) \) \( \setminus \mathcal{P}_{uc} \), and an isomorphism \( i: c_0 \rightarrow E \) such that \( P \circ i \notin \mathcal{P}_{uc}(^k(c_0, F)) \), we have that \( \{i e_n\} \) is a Dunford–Pettis set of \( E \) on which \( P \) is not weakly continuous; hence \( P \notin \mathcal{P}_{ud} \).

Finally, by \([7, \text{ lemma 3.11 and theorem 3.13}]\), we have \( \mathcal{P}_{wco}(^kE, F) \subset \mathcal{P}_{ud}(^kE, F) \); hence (d) follows from (c).

**2. Polynomial properties**

Banach spaces with the polynomial Dunford–Pettis property were introduced in \([13]\) as the spaces \( E \) such that weakly compact polynomials from \( E \) into any Banach space are completely continuous; i.e. \( \mathcal{P}_{wco}(^kE, F) \subset \mathcal{P}_{cc}(^kE, F) \) for any \( k \in \mathbb{N} \) and \( F \). Later Ryan \([14]\) proved that this property coincides with the usual Dunford–Pettis property, which admits the same definition in terms of linear operators \( (k = 1) \). On the other hand, Pelczynski \([12]\) introduced Banach spaces with property (V) as the spaces \( E \) such that unconditionally converging operators from \( E \) into any Banach space are weakly compact.

In this section, by means of the class \( \mathcal{P}_{uc} \) of unconditionally converging polynomials, we introduce and study the polynomial property (V) and other polynomial versions of properties of Banach spaces: Dieudonné property, Schur property, and property \((V^*)\). We show that, in contrast to the case of the Dunford–Pettis property, property (V) is very different from the polynomial property (V), since spaces with this property are analogous to Tsirelson's space \( T^* \). For the other polynomial properties, we show that sometimes the polynomial and the
linear properties coincide, and sometimes not, with a general tendency of the polynomial property to imply the absence of copies of $l_1$ in the space. Moreover, we obtain additional results relating $P_{uc}$ and other classes of polynomials.

Definition 6. A Banach space $E$ has the polynomial property (V) if for every $k$ and $F$ we have $P_{uc}(kE,F) \subseteq P_{wco}(kE,F)$.

It was shown in [12] that $C(K)$ spaces enjoy property (V). The next lemma shows that this is not the case for the polynomial property.

Lemma 7. Given a Banach space $E$, if $P_{uc}(kE,E) \subseteq P_{wco}(kE,E)$ for some $k > 1$, then $E$ contains no copies of $c_0$.

Proof. Assume $E \supset c_0$, and take a sequence $(x_n) \subset E$ equivalent to the unit vector basis of $c_0$. We select $f \in E^*$ such that $f(x_i) = 1$, $f(x_i) = 0$ for $i > 1$, and define $P : x \in E \to f(x)^{k-1}x \in E$.

Note that $P \in P_{uc}(kE,E)$, since for every w.u.C. series $\sum_{i=1}^{\infty} x_i$ in $E$, we have

$$\sum_{i=1}^{\infty} ||P(x_i)|| \leq \sup_{j \in \mathbb{N}} ||x_j|| \left( \sum_{i=1}^{\infty} ||f(x_i)||^{k-1} \right) \leq \sup_{j \in \mathbb{N}} ||x_j|| \left( \sum_{i=1}^{\infty} ||f(x_i)||^{k-1} \right) < \infty.$$

However, $P \notin P_{wco}$, since $P(x_1 + \ldots + x_n) = x_1 + \ldots + x_n$, and $(x_1 + \ldots + x_n)_{n \in \mathbb{N}}$ is a weakly Cauchy sequence having no weakly convergent subsequences.

Remark. The proof of Lemma 7 also gives the following facts:

(a) given a Banach space $E$ containing a copy of $c_0$, for every $k > 1$ there exists $P_k \in P_{uc}(kE,E)$ (even taking w.u.C. series into absolutely converging series) which does not take Rosenthal sets into relatively weakly compact sets;

(b) if $P_{wco}(kE,E) \subseteq P_{wco}(kE,E)$ for some $k > 1$, then $E$ contains no copies of $c_0$, also in contrast to the linear case ($k = 1$).

It is well known (and can be easily derived from the case $k = 1$ in Proposition 9 below) that every unconditionally converging operator $T : c_0 \to F$ is compact. In contrast, Lemma 7 shows that for $k > 1$ polynomials $P \in P_{uc}(kE,F)$ are not always weakly compact. However, these polynomials have restrictions to finite co-dimenional subspaces with arbitrarily small norm.

Proposition 9. For any $P \in P_{uc}(kE,F)$ we have

$$\lim_{n \to \infty} ||P(e_1,e_{n+1},\ldots)|| = 0.$$

Proof. If there exists $\delta > 0$ such that $||P(e_1,e_{n+1},\ldots)|| > \delta$ for all $n \in \mathbb{N}$, then we can construct blocks $u_i = a_{n_1+1}e_{n_1+1} + \ldots + a_{n_{i+1}}e_{n_{i+1}}$, with $n_1 < n_2 < \ldots$, such that $||u_i|| = 1$ and $||P(u_i)|| > \delta$.

Then $\sum_{i=1}^{\infty} u_i$ is a w.u.C. series, but $\sum_{i=1}^{\infty} Pu_i$ is not u.c.; hence $P \notin P_{uc}$.

It has been shown [1] that a Banach space $E$ such that $\mathcal{P}(kE,K) = \mathcal{P}(kE)$ is reflexive for every $k \in \mathbb{N}$ has many of the properties of Tsirelson's space $T^*$ [17]. In fact, $E$ must be reflexive, and the dual space $E^*$ cannot contain copies of $l_p$ ($1 < p < \infty$). Note also that $\mathcal{P}(kT^*)$ is reflexive for every $k \in \mathbb{N}$ [1]. Next we present in terms of the class $P_{uc}$ of polynomials a characterization of the spaces $E$ such that $\mathcal{P}(kE)$ is reflexive for some $k > 1$. 
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Given \( P \in \mathcal{P}(kE,F) \), we consider the associated conjugate operator defined by

\[
P^*: f \in F^* \rightarrow f \circ P \in \mathcal{P}(kE).
\]

Moreover, we need the fact that for every Banach space \( E \), the space \( \Delta^k_E \), defined as the closed span of \( \{ x \otimes \ldots \otimes x : x \in E \} \) in the projective tensor product \( \hat{\otimes}^k_n \), is a predual of the space of scalar polynomials \( \mathcal{P}(kE) \) [15].

**Theorem 10.** Given \( k \in \mathbb{N}, k > 1 \), and a Banach space \( E \), we have that \( \mathcal{P}_{uc}(kE,F) \subseteq \mathcal{P}_{wco}(kE,F) \) for any \( F \) if and only if \( \mathcal{P}(kE) \) is reflexive.

**Proof.** Assume \( \mathcal{P}_{uc}(kE,F) \subseteq \mathcal{P}_{wco}(kE,F) \) for any \( F \). By Lemma 7 we have that \( E \) contains no copies of \( c_0 \). Then

\[
\mathcal{P}(kE,F) = \mathcal{P}_{uc}(kE,F) = \mathcal{P}_{wco}(kE,F)
\]

for any \( k \) and \( F \). Since there exists an isomorphism between the space of polynomials \( \mathcal{P}(kE,F) \) and the space of operators \( L(\Delta^k_E,F) \) which takes the weakly compact polynomials onto the weakly compact operators [15], we obtain that all operators in \( L(\Delta^k_E,F) \) are weakly compact. Then \( \Delta^k_E \) is reflexive; hence \( \mathcal{P}(kE) \cong (\Delta^k_E)^* \) is reflexive.

Conversely, since \( P \in \mathcal{P}_{wco} \) if and only if the operator \( P^* \) is weakly compact [16, proposition 2-1], if \( \mathcal{P}(kE) \) is reflexive, then we have that any \( P \in \mathcal{P}(kE,F) \) belongs to \( \mathcal{P}_{wco} \), and the result is proved.

**Corollary 11.** A Banach space \( E \) has the polynomial property (V) if and only if \( \mathcal{P}(kE) \) is reflexive for every \( k \in \mathbb{N} \).

We will need the following well-known characterization of Banach spaces containing no copies of \( l_1 \). We include a proof for the sake of completeness.

**Lemma 12.** A Banach space \( E \) contains a copy of \( l_1 \) if and only if there exists a completely continuous surjection from \( E \) onto \( l_2 \).

**Proof.** Assume \( E \) contains a copy of \( l_1 \), and let \( q \) denote a surjective operator from \( l_1 \) onto \( l_2 \). Since \( q \) is absolutely summing [9, theorem 2-b-6], it factors through a space \( L_\infty(\mu) \), which has the extension property and the Dunford–Pettis property. Then the operator from \( l_1 \) into \( L_\infty(\mu) \) can be extended to an operator \( A \) from \( E \) into \( L_\infty(\mu) \), and the operator \( B \) from \( L_\infty(\mu) \) onto \( l_2 \) is completely continuous; hence \( BA \) is a completely continuous, surjective operator from \( E \) onto \( l_2 \).

Conversely, if \( Q \) is a completely continuous, surjective operator from \( E \) onto \( l_2 \), and we take a bounded sequence \( (x_n) \) in \( E \) such that \( \{ Qx_n \} \) is the unit vector basis of \( l_2 \), then \( (x_n) \) cannot have a weakly Cauchy subsequence; hence, by Rosenthal’s theorem, it has a subsequence equivalent to the unit vector basis of \( l_1 \).

In relation with the reflexivity if \( \mathcal{P}(kE) \), the problem of when this space contains a copy of \( l_\infty \) has received some attention. It has been considered in [3], for instance, in connection with the so called property (RP) of polynomials. We give an answer that includes the case \( l_1 \subseteq E \) (see Lemma 12).

**Proposition 13.** If \( E \) has a quotient isomorphic to \( l_2 \), then for any integer \( k > 1 \) the space \( \mathcal{P}(kE) \) contains a copy of \( l_\infty \).

**Proof.** For every \( a \equiv (a_n) \in l_\infty \), we consider the polynomial

\[
P_a: (x_n) \in l_2 \rightarrow \sum_{i=1}^{\infty} a_i x_i^k.
\]
We have $P_a \in \mathcal{P}(k l_2)$, and $\|P_a\| = \|a_n\|_\infty$. Then, the map

$$a \in l_\infty \to P_a \in \mathcal{P}(k l_2)$$

defines a linear isometry from $l_\infty$ into $\mathcal{P}(k l_2)$. Now, if $q : E \to l_2$ is a quotient map, we have that the map

$$a \in l_\infty \to P_a \circ q \in \mathcal{P}(k E)$$

is an isomorphism from $l_\infty$ into $\mathcal{P}(k E)$.

Extending the definition for operators, we shall say that a polynomial $P \in \mathcal{P}(k E, F)$ is weakly completely continuous, denoted by $P \in \mathcal{P}_{wcc}(k E, F)$, if it takes weakly Cauchy sequences into weakly convergent sequences.

A Banach space $E$ has the Dieudonné property if weakly completely continuous operators from $E$ into any Banach space are weakly compact. Grothendieck [8, 31] introduced this property and proved that $C(K)$ spaces enjoy it. The next result shows that the polynomial Dieudonné property is equivalent to the absence of copies of $l_1$ in the space.

**Proposition 14.** For a Banach space $E$ the following properties are equivalent: (a) $E$ contains no copies of $l_1$; (b) $\mathcal{P}_{wcc}(k E, F) \subseteq \mathcal{P}_{wco}(k E, F)$ for any $k$ and $F$; (c) $\mathcal{P}_{cc}(k E, F) \subseteq \mathcal{P}_{wco}(k E, F)$ for any $k$ and $F$; (d) $\mathcal{P}_{cc}(k E, F) \subseteq \mathcal{P}_{wco}(k E, F)$ for some non-reflexive $F$ and some $k > 1$.

**Proof.** (a) $\implies$ (b). Assume $E$ contains no copies of $l_1$, and let $P \in \mathcal{P}_{wcc}(k E, F)$. Since any bounded sequence $(x_n) \subseteq E$ has a weakly Cauchy subsequence, we have that $(P x_n)$ has a weakly convergent subsequence; hence $P \in \mathcal{P}_{wco}$. (b) $\implies$ (c) $\implies$ (d) is trivial. (d) $\implies$ (a). Assume $E$ contains a copy of $l_1$, and $F$ is non-reflexive. We take a sequence $(y_n) \subseteq B_F$ having no weakly convergent subsequences. By Lemma 12, we can also take a completely continuous surjection $T : E \to l_2$. Now, if $k > 1$, $Q$ is the polynomial from $l_2$ into $l_1$ defined by $Q(x) = (x^k)$, and $S : l_1 \to F$ is the operator defined by $S e_n = y_n$, where $e_n$ is the unit vector basis of $l_1$, then $S \circ Q \circ T \in \mathcal{P}_{cc}(k E, F)$, but $S \circ Q \circ T \not\in \mathcal{P}_{wco}$, because there exists a bounded sequence $(x_n) \subseteq E$ such that $S \circ Q \circ T x_n = y_n$.

**Corollary 15.** $\mathcal{P}_{wcc}(k E, F) \subseteq \mathcal{P}_{wco}(k E, F)$ for any $k \in \mathbb{N}$.

**Proof.** If $P \in \mathcal{P}(k E, F) \setminus \mathcal{P}_{wcc}$, by Lemma 4, there exists an isomorphism $i : c_0 \to E$ such that $P \circ i \in \mathcal{P}(k c_0, F) \setminus \mathcal{P}_{wcc}$. Then, by Proposition 5, $P \circ i \not\in \mathcal{P}_{wco}(k c_0, F)$, and by Proposition 14, $P \circ i \not\in \mathcal{P}_{wcc}$; hence $P \not\in \mathcal{P}_{wco}$.

**Remark 16.** It follows from Proposition 14 that, for any $k > 1$, there is a polynomial $P \in \mathcal{P}_{cc}(k l_\infty, c_0)$ which is not weakly compact. However, any operator from $l_\infty$ into $c_0$ is weakly compact and thereby completely continuous, since $l_\infty$ has the Dunford–Pettis property.

The question then arises whether every polynomial from $l_\infty$ into $c_0$ is completely continuous.

As a complement of Theorem 10 we have

**Theorem 17.** Given $k \in \mathbb{N}$, $k > 1$, and a Banach space $E$, we have that $\mathcal{P}_{cc}(k E, F) \subseteq \mathcal{P}_{wco}(k E, F)$ for any $F$ if and only if $\mathcal{P}(k-1 E)$ is reflexive.

**Proof.** First we assume that $\mathcal{P}_{cc}(k E, F) \subseteq \mathcal{P}_{wco}(k E, F)$. As in the proof of Theorem 10, it is enough to prove that $\mathcal{P}(k-1 E) = \mathcal{P}_{wco}(k-1 E, F)$. [Details of the proof]
By Proposition 14 we have that $E$ contains no copies of $l_1$. Then, if there exists $P \in \mathcal{P}(k-1,E,F)$, $P \notin \mathcal{P}_{cc}$, we can find a weakly Cauchy sequence $(x_n) \subset E$ such that $(Px_n)$ has no weakly convergent subsequence; then it is not relatively weakly compact. Since the class of relatively weakly compact sequences is closed in the space of bounded sequences, we can also assume that $(x_n)$ is not weakly null.

Taking $\phi \in E^*$ such that $\phi(x_n) \to \lambda \neq 0$, we define a polynomial $Q \in \mathcal{P}(k,E,F)$ by $Q(x) = \phi(x) P(x)$. Since $Q \in \mathcal{P}_{cc}$, we have that $Q$ is weakly compact. Then there exists a subsequence $(y_n)$ of $(x_n)$ such that $Q(y_n)$ is weakly convergent to $z \in E$; therefore $P(y_n)$ is weakly convergent to $\lambda^{-1}z$, a contradiction.

Conversely, if $\mathcal{P}(k-1,E)$ is reflexive, we have that $\mathcal{P}(l,E,F) = \mathcal{P}_{cc}(l,E,F)$ for all $l < k$. Moreover, since $E$ is reflexive, given a bounded sequence $(x_n) \subset E$, we can assume, passing to a subsequence, that it is weakly convergent to some $x \in E$. Then, given $P \in \mathcal{P}_{cc}(k,E,F)$, using the associated multilinear map $\hat{P}$ we write

$$Px_n = \hat{P}(x_n - x + x, \ldots, x_n - x + x) = \sum_{l=1}^{k-1} \hat{P}(x_n - x)^l(x)^{k-l} + P(x_n - x) + P(x).$$

Since for $l < k$ the polynomials $Q_l \in \mathcal{P}(l,E,F)$ defined by $Q_l(y) = \hat{P}(y)^l(x)^{k-l}$ are weakly compact, and $P(x_n - x)$ converges to 0, we see that $P(x_n)$ has a weakly convergent subsequence; hence $P \in \mathcal{P}_{cc}$. 

**Remark 18.** In order to compare Theorems 10 and 17, we observe that for the sequence spaces $l_p$ the space of polynomials $\mathcal{P}(l_p)$ is reflexive if and only if $k < p < \infty$.

In fact, it was proved in [11, corollary 4-3] that for $k < p$, all polynomials in $\mathcal{P}(l_p)$ are completely continuous; hence, using a result of [15] (see [1, proposition 3]), we conclude that $\mathcal{P}(l_p)$ is reflexive. For $1 < p < k$ it is not difficult to show that $\mathcal{P}(l_p)$ contains a copy of $l_\infty$.

Recall that a Banach space $E$ has the Schur property if weakly convergent sequences in $E$ are norm convergent; equivalently, weakly Cauchy sequences are norm convergent. It is an immediate consequence of the definition that $E$ has the Schur property if and only if $\mathcal{P}(k,E,F) = \mathcal{P}_{cc}(k,E,F)$ for any $k$ and $F$. Next we give some other polynomial characterizations of Schur property.

**Proposition 19.** For a Banach space $E$ the following properties are equivalent: (a) $E$ has the Schur property; (b) $\mathcal{P}_{cc}(k,E,F) \subseteq \mathcal{P}_{cc}(k,E,F)$ for any $k$ and $F$; (b') $\mathcal{P}_{cc}(k,E,F) \subseteq \mathcal{P}_{cc}(k,E,F)$ for any $k$ and $F$; (c) $\mathcal{P}_{cc}(k,E,F) \subseteq \mathcal{P}_{cc}(k,E,F)$ for some $k > 1$; (c') $\mathcal{P}_{cc}(k,E,F) \subseteq \mathcal{P}_{cc}(k,E,F)$ for some $k > 1$.

**Proof.** (a) $\Rightarrow$ (b) is immediate. (b) $\Rightarrow$ (c) $\Rightarrow$ (c') and (b) $\Rightarrow$ (b') $\Rightarrow$ (c') follow from $\mathcal{P}_{cc} \subseteq \mathcal{P}_{uc}$ (see Proposition 5). (c') $\Rightarrow$ (a). Assume $E$ fails the Schur property. We take $x_0 \in E$, with $\|x_0\| = 1$, and $f \in E^*$ such that $f(x_0) = 1$. Since the kernel of $f$ also fails the Schur property, there exists a weakly null, normalized sequence $(x_n) \subset \ker(f)$. Now, for every $k > 1$ we can define a polynomial $P \in \mathcal{P}(k,E,E)$ by

$$P: x \in E \to f(x)^{k-1}x \in E.$$

We have $P \in \mathcal{P}_{cc}$, and $x_0 + x_n \xrightarrow[w]{} x_0$, but $P(x_0 + x_n) = x_0 + x_n$ does not converge in norm to $Px_0$; hence $P \notin \mathcal{P}_{cc}$. 




Recall that a Banach space is said to have the hereditary Dunford–Pettis property if any of its subspaces has the Dunford–Pettis property.

**Proposition 20.** If a Banach space $E$ has the hereditary Dunford–Pettis property, then $\mathcal{P}_{uc}(kE,F) \subseteq \mathcal{P}_{cc0}(kE,F)$ for any $k$ and $F$.

**Proof.** Given $P \in \mathcal{P}_{uc}(kE,F)$, since $E$ has the hereditary Dunford–Pettis property, every normalized weakly null sequence in $E$ has a subsequence equivalent to the unit vector basis of $c_0$ [5, proposition 2], which is taken into a norm null sequence by $P$. Thus, every weakly null sequence $(x_n) \subset E$ has a subsequence $(x_{n_i})$ such that $(P x_{n_i})$ is norm null; hence $P \in \mathcal{P}_{cc0}$.

**Remark 21.** We do not know if the converse of Proposition 20 is true.

Another property of Banach spaces defined in terms of series is the property $(V^*)$, introduced by Pelczyński in [12]. Recall that a subset $A \subset E$ is said to be a $(V^*)$ set if for every w.u.C. series $\sum_{n=1}^{\infty} f_n$ in $E^*$ we have

$$\limsup_{n} |f_n(x)| = 0.$$ 

A Banach space $E$ has property $(V^*)$ if every $(V^*)$ set in $E$ is relatively weakly compact; equivalently, if any operator $T \in L(F,E)$, with unconditionally converging conjugate $T^*$ is weakly compact.

Next we shall show that the polynomial version of the last formulation coincides with property $(V^*)$. We shall denote by $\mathcal{P}_{uc^*}(kF,E)$ the class of all polynomials $P \in \mathcal{P}(kF,E)$ such that $P^*$ is unconditionally converging.

**Proposition 22.** Given $P \in \mathcal{P}(kF,E)$, we have that $P^*$ is unconditionally converging if and only if $P(B_F)$ is a $(V^*)$ set.

**Proof.** Assume $P^*$ is unconditionally converging and $\sum_{n=1}^{\infty} f_n$ is a w.u.C. series in $E^*$. We have that $\sum_{n=1}^{\infty} P^* f_n$ is an u.c. series; in particular, $\|P^* f_n\| \to 0$. Then

$$\limsup_{n} |f_n(x)| = \limsup_{n} |(P^* f_n)| = 0;$$

hence, $P B_F$ is a $(V^*)$ set.

Conversely, if $P B_F$ is a $(V^*)$ set, it follows in an analogous way that $\|P^* f_n\| \to 0$ for every w.u.C. series $\sum_{n=1}^{\infty} f_n$ in $E^*$; and using Lemma 4 in the case $k = 1$, we conclude that $P^*$ is unconditionally converging.

**Proposition 23.** For a Banach space $E$ the following properties are equivalent: (a) $E$ has property $(V^*)$; (b) for any $k$ and any $F$, we have $\mathcal{P}_{uc^*}(kF,E) \subseteq \mathcal{P}_{cc0}(kF,E)$; (c) for some $k$, we have $\mathcal{P}_{uc^*}(k l_1,E) \subseteq \mathcal{P}_{cc0}(k l_1,E)$.

**Proof.** (a) $\Rightarrow$ (b). Assume $E$ has property $(V^*)$ and let $P \in \mathcal{P}_{uc^*}(kF,E)$. Then $P B_F$ is a $(V^*)$ set; hence it is relatively weakly compact, and we conclude $P \in \mathcal{P}_{cc0}$. (b) $\Rightarrow$ (c) is trivial. (c) $\Rightarrow$ (a). Assume $E$ fails property $(V^*)$. Then there exists a bounded sequence $(x_n) \subset E$, having no weakly convergent subsequences, such that $(x_n)$ is a $(V^*)$ set. Now, for any $k$, we define $P \in \mathcal{P}(k l_1,E)$ by

$$P: (t_i) \in l_1 \to \sum_{i=1}^{\infty} t_i^k x_i.$$
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Since $PB_t$ is contained in the absolutely convex, closed hull of $\{x_n\}$ it is a $(V^*)$ set; hence, by Proposition 22, $P \in \mathcal{P}_{uc}$. However, as $P e_i = x_i$ for every $i \in \mathbb{N}$, where $\{e_i\}$ stands for the unit vector basis of $l_1$, we have that $P \notin \mathcal{P}_{wco}$. 

**Remark 24.** Using the Taylor expansion, it is possible to show that holomorphic mappings preserve (locally) w.u.C. series and u.c. series, and a holomorphic map $f : E \to F$ is unconditionally converging if and only if $f(0) = 0$ and the homogeneous polynomials given by the derivatives of $f$ at the origin, $d^k f(0)$ ($k \in \mathbb{N}$), are unconditionally converging. Hence, it follows that the ‘holomorphic’ property (V) coincides with the polynomial property (V), obtaining in this way a characterization of Banach spaces $E$ such that the space $\mathcal{H}_b(E)$ of holomorphic mappings of bounded type on $E$ is reflexive.

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