Abstract

We study small Seifert possibly chiral cosmetic surgeries on not necessarily null-homologous knot in rational homology spheres. Using $\text{PSL}_2(\mathbb{C})$-character variety theory we give a sharp bound on the number of slopes producing the same small Seifert manifold if the ambient manifold satisfies some representation theoretic conditions.

Keywords: Dehn surgeries, cosmetic surgeries, character variety.

1 Introduction

Let $Y$ be a rational homology sphere and $K$ be a knot in $Y$ such that $Y_K := Y \setminus \text{int}(\mathcal{N}(K))$ is boundary irreducible and irreducible. Two Dehn surgeries $Y_K(r)$ and $Y_K(s)$ with distinct slopes are called “cosmetic” if they are homeomorphic. They are called “truly cosmetic” if the homeomorphism preserve orientation and “chirally cosmetic” if the homeomorphism reverse orientation. It is conjectured that truly cosmetic surgery on such knot $K$ does not exist [9]. In this paper we will focus on “general” cosmetic surgery and will not distinguish between chiral and truly cosmetic surgeries.

Let’s fix a slope $s$ and define

$$C(s) = \{\text{slope } r \neq s | \text{ } Y_K(r) \cong Y_K(s)\}.$$ 

Here we allow the homeomorphism to reverse the orientation. If $C(s) \neq \emptyset$ then we have a cosmetic surgery (possibly chiral). The following theorem then gives a bound on the number of element in $C(s)$.

**Theorem 1.1.** Let $K$ be a small knot in $Y$ and $Y_K(s)$ be small-Seifert. If $\text{Hom}(\pi_1(Y), \text{PSL}_2(\mathbb{C}))$ contains only diagonalisable representations and $||s||$ is not a multiple of $s \cdot \lambda$. Then $\sharp C(s) \leq 1$.

Here $|| \cdot ||$ is a semi-norm on $H_1(\partial Y_K; \mathbb{R})$ similar to the Culler-Shalen semi-norm, $\lambda$ is the rational longitude of $K$ and $s \cdot \lambda$ is the algebraic intersection. The knot $K$ being small means that its exterior does not contain closed “incompressible surfaces”.

The bound in the theorem is sharp since amphicheiral knot in $S^3$ admits “chiral” cosmetic surgeries. Moreover in [2], we can find a construction of a one-cusped hyperbolic 3-manifold with a pair of distinct slopes which gives oppositely oriented copies of the lens space $L(49/18)$. An infinite family of hyperbolic manifolds which admit pairs $\{\alpha, \beta\}$ of reducible filling slopes, of which some pairs yield homeomorphic manifolds are presented in [10]. A straightforward consequence of the theorem which relates to the cosmetic surgery conjecture is the following.
Corollary 1.2. Under the hypothesis of the theorem, there are at most two distinct slopes which can produce the same oriented manifold after surgery.

Result similar to this corollary has been proven for null homologous knot in 3-manifold with positive first Betti number [11], for knot in $S^3$ [13] [12] and for knot in L-space integer homology sphere, [10].

The main ingredient for the proof of Theorem 1.1 is the theory of 3-manifold character variety started by Culler and Shalen [8] and which was essential in the proof of the “cyclic surgery theorem” [7], the “finite surgery theorem” [5] and is also an useful tool for studying topological properties of knot exterior. We will use results from [1] together with a norm derived from $PSL_2(C)$-character variety similar to the “Culler Shalen norm”.

Organization. In section 2 we give some background on character varieties and the Culler-Shalen norm. The proof of Theorem 1.1 will be given in section 3.

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2 Character varieties

Let $M$ denote a compact orientable 3-manifold which is irreducible and boundary irreducible with boundary consisting of a single torus. Typically $M$ would be the knot exterior $Y_K$ in the introduction. We recall that irreducible means every embedded 2-sphere bounds a 3-ball and boundary irreducible means every simple closed curve on $\partial M$ which bounds a disk in $M$ bounds a disk in $\partial M$. A properly embedded surface in $M$ will be called essential if it is not boundary parallel, it is not a 2-sphere and is $\pi_1$-injective. We say that $M$ is small if it does not contain closed essential surfaces. A slope is the isotopy class of a simple closed curve on $\partial M$. Since $M$ is boundary irreducible, $\pi_1(\partial M) \to \pi_1(M)$ therefore using the fact that $H_1(M) \cong \pi_1(\partial M)$ we will think of a slope as both being an element of $H_1(\partial M)$, $\pi_1(\partial M)$ or $\pi_1(M)$. A slope will be called “boundary slope” if it corresponds to the boundary of an essential surface. It will be called strict boundary slope” if it corresponds to the boundary of an essential surface which is not a (semi) fibre in any (semi) fibration of $M$ over $S^1$.

The $PSL_2(C)$-representation variety of $M$ is the set $\overline{R}(M) := \text{hom}(\pi_1(M), PSL_2(C))$ equipped with the compact-open topology. It consists of representations of $\pi_1(M)$ to $PSL_2(C)$. The space $\overline{R}(M)$ has the structure of an affine complex algebraic set [8]. The group $PSL_2(C)$ acts algebraically on $\overline{R}(M)$ by conjugation. Two representations are called equivalent if they are conjugate to each other. If two representations are equivalent then they belong to the same irreducible component of $\overline{R}(M)$ [8]. The set of equivalence classes of representations corresponds to the quotient $\overline{R}(M) // PSL_2(C)$, where the quotient is taken in the algebraic geometric category. In order to understand this set, Culler and Shalen introduced the $PSL_2(C)$-character variety of $M$ using the trace function. For each representation $\rho \in \overline{R}(M)$, the $PSL_2(C)$-character of $\rho$ is the map $\chi_\rho$ defined by

$$\chi_\rho : \pi_1(M) \to \mathbb{C}, \quad \chi_\rho(g) = \text{trace}(\rho(g))^2.$$
The set of all characters $\overline{X}(M) = \{ \chi_\rho \mid \rho \in \overline{R}(M) \}$ is also a complex algebraic set in a natural way such that the following map is regular, in the sense of algebraic geometry,

$$\overline{\tau} : \overline{R}(M) \longrightarrow \overline{X}(M), \quad \overline{\tau}(\rho) = \chi_\rho.$$ 

Moreover its corresponds to the quotient $\overline{R}(M)//PSL_2(\mathbb{C})$ [8]. Let $\overline{R}^{irr}(M)$ be the subset of irreducible representations and let $\overline{X}^{irr}(M) = \overline{\tau}(\overline{R}^{irr}(M))$. The spaces $\overline{R}^{irr}(M)$ and $\overline{X}^{irr}(M)$ are Zariski open subsets of $\overline{R}(M)$ and $\overline{X}(M)$ respectively [8].

For each $\gamma \in \pi_1(M)$ we consider the function defined by

$$f_\gamma : \overline{X}(M) \longrightarrow \mathbb{C}, \quad f_\gamma(\chi) = \text{trace}(\rho(\gamma))^2 - 4 = \chi(\gamma) - 4.$$ 

The function $f_\gamma$ is a regular function and the zeros of $f_\gamma$ are the characters of representations $\pi$ for which $\pi(\gamma)$ is parabolic or $\pi(\gamma) = [\pm \text{Id}]$. We will use the same notation $f_\gamma$ for the restriction of $f_\gamma$ to a curve $X_0 \subset \overline{X}(M)$.

Let $X_0 \subset \overline{X}(M)$ be a non-trivial irreducible curve component. Here non-trivial means that it contains the character of an irreducible representation. Let $\widehat{X}_0$ be the normalized projective completion of $X_0$. There is an isomorphism between function fields

$$\mathbb{C}(X_0) \cong \mathbb{C}(\widehat{X}_0), \quad f \mapsto \widehat{f}.$$ 

We can then define the degree of $f$ to be the degree of $\widehat{f}$. For $x \in \widehat{X}$ we denote by $Z_x(\widehat{f}_\gamma)$ the multiplicity of $x$ as a zero of $\widehat{f}_\gamma$. By convention $Z_x(\widehat{f}_\gamma) = \infty$ if $\widehat{f}_\gamma \equiv 0$. Now we denote $\Lambda = \pi_1(\partial M)$ seen as a subgroup of $\pi_1(M)$. We can also think of $\Lambda$ as a lattice in $H_1(\partial M, \mathbb{R})$. An element $\gamma \in \Lambda$ satisfies [14],

$$\deg(\widehat{f}_\gamma) = \sum_{x \in \widehat{X}_0} Z_x(\widehat{f}_\gamma).$$

The degree is finite if $\widehat{f}_\gamma$ is non-constant on $\widehat{X}_0$. The key property of $\deg(\widehat{f}_\gamma)$ is that for each curve $X_0 \subset X(M)$ it defines a semi-norm $||.||_{X_0}$ on $H_1(\partial M, \mathbb{R})$ which for each $\gamma \in \Lambda$ satisfies [3, 14]

$$||\gamma||_{X_0} = \left\{ \begin{array}{ll} 0 & \text{if } f_\gamma|X_0 \text{ is constant} \\ \deg(\widehat{f}_\gamma) & \text{if } f_\gamma|X_0 \neq 0. \end{array} \right.$$ 

This semi-norm is called the Callier-Shalen semi-norm associated to the curve $X_0$.

Note that if $B_k$ is the ball of radius $k$ centred at the origin, then $B_k$ can be viewed as the unit ball for the norm $\frac{1}{k}||.||_{X_0}$, therefore $B_k$ has the same properties as the unit ball. Some of these properties are [3, Proposition 5.2, Proposition 5.3]: if it is a norm then the unit ball is a balanced convex polygon and if it is not a norm but is a non-zero semi-norm then the unit ball is an infinite band.

Let $X_1, \cdots, X_k$ be all the non-trivial irreducible curve components in $\overline{X}(M)$. We can define an “absolute” semi-norm $||.||$ on $H_1(\partial M, \mathbb{R})$ by

$$||.|| = ||.||_{X_1} + \cdots + ||.||_{X_k}.$$ 

We will call this semi-norm the absolute semi-norm.

There is a unique 4-dimensional subvariety $R_0 \subset \overline{R}(M)$ for which $t(R_0) = X_0$, see [8, Lemma 4.1]. If $\overline{\pi}(\alpha) = [\pm \text{Id}]$ for some slope $\alpha$, then we have an induced representation $\overline{\pi}' : \pi_1(M(\alpha)) \longrightarrow PSL_2(\mathbb{C})$ and a cohomology group $H^1(M(\alpha); Ad_{\overline{\pi}'})$. 

3
Let \( \nu : X_0' := \widehat{X}_0 \setminus \{ \text{ideal points} \} \to X_0 \) be the map which corresponds to the affine normalization of \( X_0 \). There is an affine normalization \( R_0' \to R_0 \), which we still denote by \( \nu \), such that the following diagram commutes.

\[
\begin{array}{ccc}
R_0' & \xrightarrow{\nu} & R_0 \\
\downarrow \iota' & & \downarrow \iota \\
X_0' & \xrightarrow{\nu} & X_0
\end{array}
\]

The map \( \iota' \) and \( \nu \) are all surjective, see [7].

Let \( N \subset PSL_2(\mathbb{C}) \) denote the subgroup
\[
\left\{ \pm \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \pm \begin{pmatrix} 0 & w \\ -w^{-1} & 0 \end{pmatrix} \mid z, w \in \mathbb{C}^* \right\}
\]

For each \( \gamma \in \pi_1(M) \) we are going to consider the following subset of \( \overline{X}(M) \):
\[
A(\gamma) = \left\{ \chi_\varpi \in \overline{X}(M) \mid \varpi(\gamma) = \pm I; \quad \varpi \text{ is non-abelian and conjugates into } N \right\}.
\]
\[
B(\gamma) = \left\{ \chi_\varpi \in \overline{X}(M) \mid \varpi(\gamma) = \pm I; \quad \varpi \text{ is non-abelian and does not conjugates into } N \right\}.
\]

Note that elements of \( A(\gamma) \) must be irreducible but not necessarily those of \( B(\gamma) \).

The following Theorem and proposition relate \( Z_x(\widehat{f_\alpha}) \) and \( Z_x(\widehat{f_r}) \) for two slopes \( \alpha \) and \( r \) when \( x \in \widehat{X}_0 \) is a regular point.

**Theorem 2.1.** [1] Fix a slope \( \alpha \) on \( \partial M \) and consider a non-trivial, irreducible curve \( X_0 \subset \overline{X}(M) \). Suppose that \( x \in \widehat{X}_0 \) is not an ideal point and corresponds to a character \( \chi_\rho \) for some representation \( \varpi \in R_0 \) with non-abelian image and which satisfies \( \rho(\alpha) \in \{ \pm I \} \). Assume that \( H^1(M(\alpha); Ad_\rho) = 0 \) and \( \rho(\pi_1(\partial M)) \not\subseteq \{ \pm I \} \).

1. If \( \beta \in \pi_1(\partial M) \) and \( \rho(\beta) \neq \pm I \), then
   \[
   Z_x(\widehat{f_\alpha}) \geq \begin{cases} 
   Z_x(\widehat{f_\beta}) + 1 & \text{if } \varpi \text{ conjugates into } N, \\
   Z_x(\widehat{f_\beta}) + 2 & \text{otherwise}.
   \end{cases}
   \]

2. If \( r \in \pi_1(\partial M) \) and \( Z_x(\widehat{f_\alpha}) > Z_x(\widehat{f_\beta}) \), then \( \widehat{f_\alpha}|X_0 \neq 0 \), \( \rho(\alpha) \neq \pm I \) and
   \[
   Z_x(\widehat{f_\alpha}) = \begin{cases} 
   Z_x(\widehat{f_\beta}) + 1 & \text{if } \varpi \text{ conjugates into } N, \\
   Z_x(\widehat{f_\beta}) + 2 & \text{otherwise}.
   \end{cases}
   \]

The condition \( \rho(\pi_1(\partial M)) \not\subseteq \{ \pm I \} \) may not be satisfied in general. For it to be true we will assume the auxiliary assumption that the manifold \( Y \) has only diagonalisable \( PSL_2(\mathbb{C}) \) representations.
Proposition 2.2. \[7\] Let $\alpha$ and $\beta$ be non-zero elements of $\Lambda$. Suppose that $x$ is a point of $X'_0$ such that $Z_x(f_{\alpha}) > Z_x(f_{\beta})$. Then for every $\hat{\rho} \in R'_0$ with $t'^*(\hat{\rho}) = x$, the representation $\rho = \nu(\hat{\rho})$ satisfies $\rho(\alpha) = \pm I$.

When we have zeros at ideal points we have the following property.

Proposition 2.3. \[7\] Let $x$ be an ideal point of $\hat{X}_0$. Let $\alpha$ and $\beta$ be non-zero elements of $\Lambda$. Suppose that $\alpha$ is primitive and is not a boundary class, and that

$$Z_x(f_{\alpha}) > Z_x(f_{\beta}).$$

Then there is a closed essential surface in $M$ which is incompressible in $M(\alpha)$.

3 Proof of theorem 1.1

From now on we consider the case $M = Y_K$ the knot exterior described in the introduction.

Lemma 3.1. Assume that $\operatorname{rank}_{\mathbb{Z}}(H_1(Y_K)) = 1$. Then for each ordinary point $x \in \hat{X}_0$ there is a representation $\overline{\rho} \in R_0$, with non-abelian image, such that $\chi_{\overline{\rho}} = \nu(x)$.

Proof. Let $Z_0 \subset X(Y_K)$ ($SL_2(\mathbb{C})$-character variety) be an irreducible curve component of $\pi^{-1}(X_0)$, and $S_0$ a component of $t^{-1}(Z_0)$.

Let $X(\Gamma)$ be the $SL_2(\mathbb{C})$-character variety of a finitely generated group $\Gamma$. In [4, Proposition 2.8] it is shown that if $x$ is a reducible trivial character in a non-trivial curve inside $X(\Gamma)$ then the first Betti number satisfies $b_1(\Gamma) \geq 2$. Since $\operatorname{rank}_{\mathbb{Z}}(H_1(Y_K)) = 1$ by assumption and $b_1(\pi_1(Y_K)) = \operatorname{rank}_{\mathbb{Z}}(H_1(Y_K))$, any character in a non-trivial curve inside $X(Y_K)$ is non-trivial, in particular any element of $Z_0$ is non-trivial. The same Proposition 2.8 of [4] applied to $\pi_1(Y_K)$ implies that if a character $z \in Z_0$ is non-trivial then there is a representation $\rho \in S_0 \cap t^{-1}(z)$ with non-abelian image. Since for each $x \in \hat{X}_0$, $\nu(x) \in X_0$ we can take $z \in \pi^{-1}(\nu(x))$ to get such a representation $\rho$ and then take the corresponding $PSL_2(\mathbb{C})$ representation $\overline{\rho}$.

Lemma 3.2. If $Y_K$ is small then for each curve $X_0 \subset X(Y_K)$, $\|\|_{X_0}$ is not identically zero.

Proof. By [3] Proposition 5.5 if $\|\|_{X_0} \equiv 0$ then $Y_K$ contains a closed essential surface, this is not possible if $Y_K$ is small.

Lemma 3.3. If $s$ and $r$ be two slopes on $\partial Y_K$ such that $\pi_1(Y_K(s)) \cong \pi_1(Y_K(r))$. Then there is a one to one correspondence between $A(s)$ and $A(r)$, and between $B(s)$ and $B(r)$.

Proof. Let $\Psi : \pi_1(Y_K(r)) \to \pi_1(Y_K(s))$ be an isomorphism, $\rho_s : \pi_1(Y_K) \to \pi_1(Y_K(s))$, and $\rho_r : \pi_1(Y_K) \to \pi_1(Y_K(r))$ be the obvious projections. Let $\chi_{\overline{\rho}} \in A(s)$, we have a representation $\Phi_s(\overline{\rho}) : \pi_1(Y_K(s)) \to PSL_2(\mathbb{C})$ obtained via the following factorisation of $\overline{\rho}$.
We also have an equivalent representation $\Phi_r(\mathcal{P}) : Y_K(r) \to PSL_2(\mathbb{C})$ for the $r$-case. Let $\mathcal{P}'$ be the composition $\mathcal{P}' := \Phi(\mathcal{P}) \circ \Psi \circ p_r$.

The maps $p_r, p_s$ and $\Psi$ are all surjective so $\text{im} \mathcal{P} = \text{im} \mathcal{P}'$. In particular if $\mathcal{P}$ does not conjugate into $N$ then neither does $\mathcal{P}'$, and if $\mathcal{P}$ is irreducible then so is $\mathcal{P}'$. The representation $\mathcal{P}'$ satisfies $\mathcal{P}'(r) = \pm I$ by construction. Next we need to check that if $\chi_{\mathcal{P}_1} = \chi_{\mathcal{P}_2}$ then $\chi_{\mathcal{P}_1} = \chi_{\mathcal{P}_2}$.

We first assume that $\chi_{\mathcal{P}_1}$ is an irreducible character. Therefore $\mathcal{P}_2 = g \mathcal{P}_1 g^{-1}$ for some $g \in SL_2(\mathbb{C})$. Then we deduce that

$$\mathcal{P}_2 = \Phi_s(\mathcal{P}_2) \circ \Psi \circ p_r$$

$$= \Phi_s(g \mathcal{P}_1 g^{-1}) \circ \Psi \circ p_r$$

$$= (g \Phi_s(\mathcal{P}_1) g^{-1}) \circ \Psi \circ p_r$$

$$= g(\Phi_s(\mathcal{P}_1) \circ \Psi \circ p_r) g^{-1}$$

$$= g \mathcal{P}_1 g^{-1}$$

which implies $\mathcal{P}_2 = g \mathcal{P}_1 g^{-1}$. Therefore $\chi_{\mathcal{P}_1} = \chi_{\mathcal{P}_2}$. If $\gamma \in \pi_1(\partial Y_K)$ we denote $X^{irr}(\gamma)$ and $X^{red}(\gamma)$ the sets

$$X^{irr}(\gamma) = \{ \chi_{\mathcal{P}} \in X(Y_K) \mid \mathcal{P}(\gamma) = \pm I, \text{ and } \mathcal{P} \text{ is irreducible} \}$$

$$X^{red}(\gamma) = \{ \chi_{\mathcal{P}} \in X(Y_K) \mid \mathcal{P}(\gamma) = \pm I, \text{ and } \mathcal{P} \text{ is reducible} \}.$$
Finally, we show that $Y$ is small. Hence  

$$R_x^a = \left\{ A \varphi A^{-1} | A \in PSL_2(\mathbb{C}), \varphi \in U_x^{-1} \right\}$$

$$U_x^a = \left\{ \text{representation } \varphi = \pm \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\},$$

$$U_x^{-a} = \left\{ \text{representation } \varphi = \pm \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix} \right\}$$

If $\varphi_1$ and $\varphi_2$ are conjugate, by the same argument as for irreducible characters $\chi_{\varphi_1} = \chi_{\varphi_2}$. Assume that $\varphi_1$ and $\varphi_2$ are not conjugate. Without loss of generality we can suppose that

$$\varphi_1 = \pm \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad \varphi_2 = \pm A \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix} A^{-1}, \quad \text{for some } A \in PSL_2(\mathbb{C}).$$

Since $\Phi(\varphi_i)$ and $\varphi_i$, $i = 1, 2$ have the same image we can use the same matrices to represent $\Phi(\varphi_i)$, $i = 1, 2$. Hence

$$\varphi_1 = \Phi(\varphi_i) \circ \Psi \circ p_r = \pm \begin{pmatrix} a \circ \Psi \circ p_r & b \circ \Psi \circ p_r \\ 0 & a^{-1} \circ \Psi \circ p_r \end{pmatrix}$$

and

$$\varphi_2 = \Phi(\varphi_2) \circ \Psi \circ p_r = \pm A \begin{pmatrix} a^{-1} \circ \Psi \circ p_r & b \circ \Psi \circ p_r \\ 0 & a \circ \Psi \circ p_r \end{pmatrix} A^{-1}.$$

Therefore

$$\text{trace}(\rho'_1) = \pm [a \circ \Psi \circ p_r + a^{-1} \circ \Psi \circ p_r]$$

$$\text{trace}(\rho'_2) = \pm [a^{-1} \circ \Psi \circ p_r + a \circ \Psi \circ p_r]$$

It follows that $\text{trace}(\rho'_1)^2 = \text{trace}(\rho'_2)^2$, that is $\chi_{\rho'_1} = \chi_{\rho'_2}$. Thus $F$ is well defined on the set of reducible characters.

Finally, we show that $F$ is bijective. If $\varphi \in X^{irr}(r)$ (resp. $X^{red}(r)$), we get $\varphi \in X^{irr}(s)$ (resp. $X^{red}(s)$) as follow: we first define $\Phi_s(\varphi)$ to be $\Phi_s(\varphi) = \Phi_s(\varphi) \circ \Psi^{-1}$ then $F = \Phi_s(\varphi) \circ p_s$. This uniquely determine $\varphi$, therefore the map $F$ is bijective. 

\[ \Box \]

**Lemma 3.4.** Let $s$ be a slope on $\partial Y_k$ which is not a boundary slope. Assume that $Y_k(s)$ is small-Seifert, $Y_k$ is small, $Y$ has only diagonalisable $PSL_2(\mathbb{C})$-representations and $b_1(Y_k(s)) = 0$. If $r$ is a slope such that $Y_k(r) \cong Y_k(s)$, then $||r|| = ||s||$.

**Proof.** It is known from \[ \Box \] that $X(Y_k)$ has no 0-dimensional component. Let $X_1, \cdots, X_k$ be the curve components of $X(Y_k)$. Since $Y_k$ is small $X(Y_k)$ is the union of these components. If $s \in \Lambda$ is not a boundary slope, then Lemma 3.2 allows us to write the $X_i$-norm of $s$ in terms of the zeros of $f_s|X_i$ for each $i \in \{1, \cdots, k\}$

$$||s||_X = \sum_{x \in \tilde{X}_i} Z_x(f_s|_{\tilde{X}_i}).$$
Let $\hat{X}$ be the abstract disjoint union of all the $\hat{X}_i$, $i \in \{1, \cdots, k\}$, then we have the following formula for the absolute semi-norm
\[
\|s\| = \sum_{x \in \hat{X}} Z_x(\hat{f}_s)
\]
where $\hat{f}_s$ is understood to be the restriction to the appropriate component. Let $x \in \hat{X}$, we define the number $m_x$ and $m_0$ to be
\[
m_x = \min \left\{ Z_x(\hat{f}_r) \mid \gamma \in \Λ \setminus \{0\} \right\}, \quad \text{and} \quad m_0 = \sum_{x \in \hat{X}} m_x.
\]
We can then deduce
\[
\|s\| = m_0 - m_0 + \sum_{x \in \hat{X}} Z_x(\hat{f}_s) = m_0 + \sum_{x \in \hat{X}} \left( Z_x(\hat{f}_s) - m_x \right).
\]

Let us suppose that $x$ is an ideal point of $\hat{X}_0 \subset \hat{X}$. If $Z_x(\hat{f}_s) - m_x > 0$ then $Z_x(\hat{f}_s) > Z_x(\hat{f}_r)$ for some $\gamma \in \Lambda \setminus \{0\}$. Since $s$ is primitive and is not a boundary class, Lemma 2.3 implies that there is a closed surface in $Y_K$ which is incompressible in $Y_K(s)$. This situation does not occur if we assume $Y_K(s)$ is small-Seifert with $b_1(M(s)) = 0$. Therefore we always have $Z_x(\hat{f}_s) - m_x = 0$ at an ideal point.

Let $x \in \hat{X}$ be an ordinary point, $x$ is contained in some $\hat{X}_0$ and by Lemma 3.1 there is a representation $\hat{\rho} \in R_0$ with non-abelian image, such that $\chi(\hat{\rho}) = \nu(x)$. Let $\hat{\rho} = \nu^{-1}(\rho)$, we have the following equality
\[
\nu(t^\nu(\hat{\rho})) = t(\rho) = \nu(x).
\]
The normalization map $\nu : X^s_0 \to X_0$ is an “isomorphism” outside singular points, so if $x$ is a smooth point then $t^\nu(\hat{\rho}) = x$. This smoothness is provided by Theorem A of [4]. A direct consequence of this is that for an ordinary point $x$, $\nu(x)$ is contained in only one irreducible component. Therefore if we consider instead of $\hat{X}$, the “natural” union $\hat{X}_1 \cup \cdots \cup \hat{X}_k$, we can write the absolute semi-norm of $s$ as
\[
\|s\| = \sum_{x \in \hat{X}_1 \cup \cdots \cup \hat{X}_k} Z_x(\hat{f}_s).
\]

Now if we assume that $Z_x(\hat{f}_s) > m_x$ then by Lemma 2.2 the representation $\rho = \nu(\hat{\rho})$ satisfies $\rho(s) = \pm I$. Since $Y_K(s)$ is Small seifert, $H^1(Y_K(s); Ad_{\rho}) = 0$. If we add the extra condition that $Y$ have only Abelian $PSL_2(\mathbb{C})$-representations then $\rho(\Lambda) \not\subset \{ \pm I \}$ and all the hypothesis of Theorem 2.1 are satisfied. In particular if $m_x = Z_x(\hat{f}_r)$ for some $r \in \Lambda \setminus \{0\}$ then $\hat{f}_s|_{X_0} \neq 0$, $\rho(r) \neq \pm I$ and
\[
\begin{cases}
Z_x(\hat{f}_s) - m_x = Z_x(\hat{f}_s) - Z_x(\hat{f}_r) = 1 \quad \text{if} \quad \hat{\rho} \text{ conjugates into} \quad D, \\
Z_x(\hat{f}_s) - m_x = Z_x(\hat{f}_s) - Z_x(\hat{f}_r) = 2 \quad \text{otherwise}.
\end{cases}
\]
Since $X(Y_K)$ is the union of its 1-dimensional components we have
\[
\|s\| = m_0 + \sum_{x \in \hat{X}} \left( Z_x(\hat{f}_s) - m_x \right) = m_0 + A(s) + 2B(s).
\]
Finally $\sharp A(s) = \sharp A(r)$ and $\sharp B(s) = \sharp B(r)$ from Lemma 3.3 since $Y_K(s) \cong Y_K(r)$. Therefore $\|s\| = \|r\|$. 
\qed
Proof of Theorem 1.1 Let $s$ be an exceptional slope on $\partial Y_K$ and $r \in C(s)$.

We can assume that $b_1(Y_K(s)) = 0$ since there is only one slope which produces a manifold with positive first Betti number. Let us suppose that $s$ is a boundary slope. From [7, Theorem 2.0.3] we have the following possibilities:

(i) $Y_K(s)$ contains a closed essential surface of strictly positive genus.

(ii) $Y_K(s)$ is the connected sum of two lens spaces.

(iii) There is a closed essential surface $S \subset Y_K$ which compresses in $Y_K(s)$ but which remains incompressible in $Y_K(\delta)$ as long as $\Delta(s, \delta) > 1$.

(iv) $Y_K(s) \cong S^1 \times S^2$.

Since $Y_K(s)$ is small-Seifert with $b_1(Y_K(s)) = 0$, only (iii) can occur. Then the fact that $Y_K(s) \cong Y_K(r)$ implies that $S$ also compresses in $Y_K(r)$ so $\Delta(s, r) \leq 1$. The condition $\Delta(s, r) \leq 1$ implies that there are at most three of such slopes. Assume $r \in C(s)$ is distinct from $s$, then $\Delta(s, r) = 1$ and we have either

$$C(s) = \{r, s + r\}, \quad \text{or} \quad C(s) = \{r\}.$$

Now by homological reason there is a constant $c_K$ independent of $s$ such that

$$|H_1(Y_K(s); \mathbb{Z})| = c_K \Delta(s, \lambda)$$

and similarly for $r$, see [15] for details. Therefore since $H_1(Y_K(s); \mathbb{Z}) = H_1(Y_K(r); \mathbb{Z}) = H_1(Y_K(s + r); \mathbb{Z})$, we must have

$$\Delta(s, \lambda) = \Delta(r, \lambda) = \Delta(s + r, \lambda).$$

Thus all three elements $s, r, s + r$ lie on the same line $l$ in $\mathbb{R}^2$ which is at fixed distance from the rational longitude $\lambda$. This is impossible since $s$ and $r$ are linearly independent. It follows that the first case cannot occur and we have:

$$C(s) = \{r\}.$$

Now if $s$ is not a boundary slope we can apply Lemma 3.4 to obtain

$$\|s\| = ||r||.$$

Let $k = ||s||$, then every element of $C(s)$ lies on the boundary of the ball $B(0, k)$ of the absolute semi-norm. On the other hand if $r \in C(s)$ then $s$ and $r$ must have the same $\mu$ component since they give homeomorphic manifolds. Hence they also lie on a line $l$ parallel to $\lambda$. From [3, Proposition 5.2, Proposition 5.3] $B(0, k)$ is either a convex polygon or an infinite band. In each cases the line $l$ intersect $\partial B(0, r)$ twice (at $s$ and $r$) unless $k$ is a multiple of $s \cdot \lambda$. This prove $\sharp C(s) \leq 1$.

References

[1] L. Ben Abdelghani and Steven Boyer. A calculation of the Culler-Shalen seminorms associated to small Seifert Dehn fillings. Proc. London Math. Soc. (3), 83(1):235–256, 2001.

[2] Steven A. Bleiler, Craig D. Hodgson, and Jeffrey R. Weeks. Cosmetic surgery on knots. In Proceedings of the Kirbyfest (Berkeley, CA, 1998), volume 2 of Geom. Topol. Monogr., pages 23–34 (electronic). Geom. Topol. Publ., Coventry, 1999.
[3] S. Boyer and X. Zhang. On Culler-Shalen seminorms and Dehn filling. *Annals of Mathematics*, 148(3):737–801, 1998.

[4] Steven Boyer. On the local structure of SL(2, C)-character varieties at reducible characters. *Topology Appl.*, 121(3):383–413, 2002.

[5] Steven Boyer and Xingru Zhang. A proof of the finite filling conjecture. *J. Differential Geom.*, 59(1):87–176, 2001.

[6] D. Cooper, M. Culler, H. Gillet, D. D. Long, and P. B. Shalen. Plane curves associated to character varieties of 3-manifolds. *Invent. Math.*, 118(1):47–84, 1994.

[7] Marc Culler, C. McA. Gordon, J. Luecke, and Peter B. Shalen. Dehn surgery on knots. *Annals of Mathematics*, 125(2):237–300, 1987.

[8] Marc Culler and Peter B. Shalen. Varieties of group representations and splittings of 3-manifolds. *Annals of Mathematics*, 117(1):109–146, 1983.

[9] R. Kirby. Problems in low-dimensional topology. In R. Kirby, editor, *Geometric topology (Athens, GA, 1993)*, volume 2 of *AMS/IP Stud. Adv. Math.*, pages 35–473. Amer. Math. Soc., Providence, RI, 1997.

[10] D. Matignon and James A. Hoffman. Examples of bireducible Dehn fillings. *Pacific J. Math.*, 209(1):67–83, 2003.

[11] Yi Ni. Thurston norm and cosmetic surgeries. In *Low-dimensional and symplectic topology*, volume 82 of *Proc. Sympos. Pure Math.*, pages 53–63. Amer. Math. Soc., Providence, RI, 2011.

[12] Yi Ni and Zhongtao Wu. Cosmetic surgeries on knots in $S^3$. *J. reine angew. Math.*, 2013.

[13] Huygens C. Ravelomanana. Exceptional cosmetic surgeries on $S^3$. *Topology and its Applications*, 204:217–229, 2016.

[14] Peter B. Shalen. Representations of 3-manifold groups. In *Handbook of geometric topology*, pages 955–1044. North-Holland, Amsterdam, 2002.

[15] L. Watson. *Involutions on 3-manifolds and Khovanov homology*. PhD thesis, Université du Québec à Montréal, 2009. Ph. D. thesis.

[16] Zhongtao Wu. Cosmetic surgery in L-space homology spheres. *Geom. Topol.*, 15(2):1157–1168, 2011.

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