Divisorial contractions in dimension 3 which contract divisors to smooth points

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Abstract

We deal with a divisorial contraction in dimension 3 which contracts its exceptional divisor to a smooth point. We prove that any such contraction can be obtained by a suitable weighted blow-up.

0 Introduction

Divisorial contractions play a major role in the minimal model program ([KMM87]). Now that we know this program works in dimension 3 ([MS88]), it is desirable to describe them explicitly in dimension 3. Moreover also in view of the Sarkisov program ([Co95]) and its applications (for example [CPR99]), we can recognize the importance of such description since Sarkisov links of types I and II in this program start from the converse of divisorial contractions.

Now we concentrate on divisorial contractions in dimension 3. Let $f: (Y \supset E) \rightarrow (X \ni P)$ be such a contraction. There are two ways to deal with $f$, that is to say, one starting from $Y$, and the other from $X$. From the former standpoint, S. Mori classified them in the case when $Y$ is smooth ([M82]), and S. Cutkosky extended this result to the case when $Y$ has only terminal Gorenstein singularities ([Cu88]). On the other hand, from the latter standpoint, Y. Kawamata showed that $f$ must be a certain weighted blow-up when $P$ is a terminal quotient singularity ([K96]), and A. Corti showed that $f$ must be the blow-up when $P$ is an ordinary double point ([Co99, Theorem 3.10]).

While it seems that singularities on $Y$ make it hard to tackle the problem in the former case, the singularity of $P$ may be useful in the latter case because it gives a special filtration in the tangent space at $P$. In this paper we treat the case when $P$ is a smooth point and prove the following theorem:

**Theorem 1.2.** Let $Y$ be a 3-dimensional $\mathbb{Q}$-factorial normal variety with only terminal singularities, and let $f: (Y \supset E) \rightarrow (X \ni P)$ be an algebraic germ of a divisorial contraction which contracts its exceptional divisor $E$ to a smooth point $P$. Then we can take local parameters $x, y, z$ at $P$ and coprime positive integers $a$ and $b$, such that $f$ is the weighted blow-up of $X$ with its weights $(x, y, z) = (1, a, b)$.
Now we explain our approach to the problem. Y. Kawamata adopted the method of comparing discrepancies of exceptional divisors, and A. Corti applied Shokurov’s connectedness lemma ([K+92, Theorem 17.4]). But in the case when \( P \) is a smooth point, these methods do not work well if the center of \( E \) on \( \text{Bl}_P(X) \) is a point. Our main tools are the singular Riemann-Roch formula ([R87, Theorem 10.2]) on \( Y \) and a relative vanishing theorem ([KMM87, Theorem 1-2-5]) with respect to \( f \). First with them we derive a rather simple formula for \( \dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-iE) \)'s and an upper-bound of the number of fictitious non-Gorenstein points of \( Y \) (Proposition 2.7). Next using this upper-bound, we show that the coefficient of \( E \) in the pull-back of a general prime divisor through \( P \) is 1 (Subsection 2.3). And finally investigating the values of \( \dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-iE) \)'s more carefully, we prove the theorem (Subsection 2.4).

I wish to express my gratitude to Professor Yujiro Kawamata for his valuable comments and warm encouragement. He also recommended me to read the papers [CPR99] and [Co99]. In fact I found the problem treated here as [Co99, Conjecture 3.11].

1 Statement of the theorem

We work over an algebraically closed field \( k \) of characteristic zero. A variety means an integral separated scheme of finite type over \( \text{Spec } k \). We use basic terminologies in [K+92, Chapters 1, 2].

Before we state the theorem, we have to define a divisorial contraction. In this paper it means a morphism which may emerge in the minimal model program (see [KMM87]).

Definition 1.1. Let \( f: Y \to X \) be a morphism with connected fibers between normal varieties. We call \( f \) a divisorial contraction if it satisfies the following conditions:

1. \( Y \) is \( \mathbb{Q} \)-factorial with only terminal singularities.
2. The exceptional locus of \( f \) is a prime divisor.
3. \(-K_Y\) is \( f \)-ample.
4. The relative Picard number of \( f \) is 1.

Now it is the time when we state the theorem precisely.

Theorem 1.2. Let \( Y \) be a 3-dimensional \( \mathbb{Q} \)-factorial normal variety with only terminal singularities, and let \( f: (Y \supset E) \to (X \supset P) \) be an algebraic germ of a divisorial contraction which contracts its exceptional divisor \( E \) to a smooth point \( P \). Then we can take local parameters \( x, y, z \) at \( P \) and coprime positive integers \( a \) and \( b \), such that \( f \) is the weighted blow-up of \( X \) with its weights \( (x, y, z) = (1, a, b) \).
2 Proof of the theorem

2.1 Strategy for its proof

We may assume that $X$ is projective and smooth, and consider its algebraic germ if necessary. First we construct a series of birational morphisms.

**Construction 2.1.** We construct birational morphisms $g_i: X_i \to X_{i-1}$ between smooth varieties, integral closed subschemes $Z_i \subset X_i$, and prime divisors $F_i$ on $X_i$ inductively, and define positive integers $n, m$, with the following procedure:

1. Define $X_0$ as $X$ and $Z_0$ as $P$.
2. Let $b_i: \text{Bl}_{Z_i-1}(X_{i-1}) \to X_{i-1}$ be the blow-up of $X_{i-1}$ along $Z_{i-1}$, and let $b'_i: X_i \to \text{Bl}_{Z_i-1}(X_{i-1})$ be a resolution of $\text{Bl}_{Z_i-1}(X_{i-1})$, that is, a proper birational morphism from a smooth variety $X_i$ which is isomorphic over the smooth locus of $\text{Bl}_{Z_i-1}(X_{i-1})$. We note that $b'_i$ is isomorphic at the generic point of the center of $E$ on $\text{Bl}_{Z_i-1}(X_{i-1})$. We define $g_i = b_i \circ b'_i: X_i \to X_{i-1}$.
3. Define $Z_i$ as the center of $E$ on $X_i$ with the reduced induced closed subscheme structure, and $F_i$ as the only $g_i$-exceptional prime divisor on $X_i$ which contains $Z_i$.
4. We stop this process when $Z_n = F_n$. This process must terminate after finite steps (see Remark 2.1.2) and thus we get the sequence $X_n \to \cdots \to X_0$.
5. We define $m \leq n$ as the largest integer such that $Z_{m-1}$ is a point.
6. We define $g_{ji}$ ($j \leq i$) as the morphism from $X_i$ to $X_j$.

**Remark 2.1.1.** We remark that $f_* \mathcal{O}_Y(-iE) = g_{0*} \mathcal{O}_{X_0}(-iF_n)$ for any $i$ because $E$ and $F_n$ are the same as valuations.

**Remark 2.1.2.** We prove the termination of the process. Assume that we have the sequence $X_l \to \cdots \to X_0$ and $Z_l \neq F_l$. We take common resolutions of $X_l$ and $Y$ over $X$, that is, birational morphisms $h: W \to X_l$ and $h': W \to Y$ from a smooth variety $W$ such that $g_{0l} \circ h = f \circ h'$. We put

\[ K_Y = f^*K_X + aE, \]
\[ K_{X_l} = g_{0l}^*K_X + sF_l + \text{(others)}, \]
\[ K_W = h^*K_{X_l} + c(h'^{-1})_*E + \text{(others)}, \]
\[ h^*F_l = (h^{-1})_*F_l + t(h'^{-1})_*E + \text{(others)}. \]
We note that \( a, s, c \) and \( t \) are positive integers. Then
\[
K_W = h'^*(f^*K_X + aE) + (\text{others}) \\
= h'^*(g_{0l}^*K_X + sF_l + (\text{others})) + c(h'^{-1})_*E + (\text{others}) \\
= h'^*g_{0l}^*K_X + s(h^{-1})_*F_l + (st + c)(h'^{-1})_*E + (\text{others}).
\]
Comparing the coefficients of \((h'^{-1})_*E\), we have \( a = st + c \) and especially \( a > s \). On the other hand because we know \( s \geq l + 1 \) by the construction of \( F_l \), we get \( a > l + 1 \). It shows that the above process terminates with \( n \leq a - 1 \).

We state an easy lemma.

**Lemma 2.2.** Let \( f_i : (Y_i \supset E_i) \rightarrow (X \supset f_i(E_i)) \) with \( i = 1, 2 \) be algebraic germs of divisorial contractions. Assume that \( E_1 \) and \( E_2 \) are the same as valuations. Then \( f_1 \) and \( f_2 \) are isomorphic as morphisms over \( X \).

**Proof.** Let \( g_i : Z \rightarrow Y_i \) with \( i = 1, 2 \) be common resolutions and \( h = f_i \circ g_i \). We choose \( g_i \)-exceptional effective \( \mathbb{Q} \)-divisors \( F_i \) \((i = 1, 2) \) and a \( \mathbb{Q} \)-divisor \( G \) on \( Z \) such that \( G = -g_1^*E_1 + F_1 = -g_2^*E_2 + F_2 \). Then,
\[
Y_i = \text{Proj}_X \oplus_{j \geq 0} f_i_*\mathcal{O}_{Y_i}(-jE_i) = \text{Proj}_X \oplus_{j \geq 0} h_*\mathcal{O}_Z(jG).
\]

For weighted blow-ups in dimension 3, we have a criterion on terminal singularities.

**Theorem 2.3.** Let \( X \ni P \) be an algebraic germ of a smooth 3-dimensional variety with local parameters \( x, y, z \) at \( P \), let \( r, a, b \) be positive integers with \( r \leq a \leq b \), and let \( Y \rightarrow X \) be the weighted blow-up of \( X \) with its weights \( (x, y, z) = (r, a, b) \). Then \( Y \) has only terminal singularities if and only if \( r = 1 \) and \( a, b \) are coprime.

By the above lemma and theorem, the problem is reduced to proving that \( F_n \) equals, as valuations, an exceptional divisor obtained by a weighted blow-up of \( X \). We restate this in terms of ideal sheaves of \( \mathcal{O}_X \).

**Proposition 2.4.** (Notation as above). \( F_n \) equals, as valuations, an exceptional divisor obtained by a weighted blow-up of \( X \) with its weights \( (x, y, z) = (1, m, n) \) for suitable local parameters \( x, y, z \) at \( P \), if and only if the following conditions hold:

1. \( f_*\mathcal{O}_Y(-2E) \neq m_P \), that is, \( g_{0m}^*\mathcal{O}_X,(-2F_n) \neq m_P \).
2. \( f_*\mathcal{O}_Y(-nE) \not\subseteq m_P^2 \), that is, \( g_{0n}^*\mathcal{O}_X,(-nF_n) \not\subseteq m_P^2 \).

Here \( m_P \subset \mathcal{O}_X \) is the ideal sheaf of \( P \).
Proof. The “only if” part is obvious taking it into account that for any \( i \) \( g_{0n^2}\mathcal{O}_{X_n}(-iF_n) = (x^sy^t/z^n|s + mt + nu \geq i) \). Actually \( x \notin g_{0n^2}\mathcal{O}_{X_n}(-2F_n) \) and \( z \in g_{0n^2}\mathcal{O}_{X_n}(-nF_n) \).

Now we prove the “if” part. The condition 1 means that the coefficient of \( F_n \) in \( g_{1n}^iF_1 \) is 1. This says that for any \( i \geq 1 \), \( F_i \) is the only \( g_0 \)-exceptional prime divisor on \( X_i \) containing \( Z_i \) and the coefficient of \( F_n \) in \( g_{1n}^iF_i \) is 1.

We consider a prime divisor \( D \ni P \) on \( X \) which is smooth at \( P \) and define \( 1 \leq l \leq n \) as the largest integer such that \( Z_{l-1} \subseteq (g_{1n}^{i-1})_sD \). Then \( (g_{1n}^{-1})_sD \) is smooth at the generic point of \( Z_i \) for any \( i < l \), and so we get \( g_{1n}^{-1}_iD = (g_{1n}^{-1})_sD + \sum_{i=1}^l i(g_{1n}^{-1})_sF_i + \text{others} \). Therefore the coefficient of \( F_n \) in \( g_{1n}^{-1}D \) is \( l \). By the condition 2, we can choose \( z \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2 \) such that \( g_{1n}^{-1}_i\text{div}(z) \geq nF_n \), that is, \( Z_{n-1} \subseteq (g_{1n}^{-1})_s\text{div}(z) \) because of the above argument. Adding \( x, y \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2 \) such that \( Z_{m-1} \subseteq (g_{1n}^{-1})_s\text{div}(y) \), we can take local parameters \( x, y, z \) at \( P \). Then \( F_i (1 \leq i \leq n) \) equals, as valuations, the exceptional divisor obtained by the weighted blow-up of \( X \) with its weights \( (x, y, z) = (1, \min\{i, m\}, i) \), and especially \( F_n \) is obtained by the weighted blow-up of \( X \) with its weights \( (x, y, z) = (1, m, n) \). \( \square \)

So we prove the above two conditions.

2.2 Preliminaries

Let \( K_Y = f^*K_X + aE \), and let \( r \) be the global Gorenstein index of \( Y \), that is, the smallest positive integer such that \( rK_Y \) is Cartier. Since \( a \) equals the discrepancy of \( F_n \) with respect to \( K_X \), \( a \in \mathbb{Z}_{\geq 2} \).

Lemma 2.5. (Notation as above). \( a \) and \( r \) are coprime.

Proof. Let \( s \) be the greatest common divisor of \( a \) and \( r \), and let \( a = so', r = sr' \). Since \( sr'E = ar'E \) is Cartier by [K88 Corollary 5.2], so is \( r'K_Y \). Hence \( r' = r \) and \( s = 1 \). \( \square \)

We recall the singular Riemann-Roch formula ([R87, Theorem 10.2]).

Theorem 2.6. Let \( X \) be a projective 3-dimensional variety with only canonical singularities, and let \( D \) be a Weil divisor on \( X \) such that for any \( P \in X \) there exists an integer \( i_P \) satisfying \( (\mathcal{O}_X(D))_P \cong (\mathcal{O}_X(i_PK_X))_P \). Then there is a formula of the form

\[
\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12}D(D - K_X)(2D - K_X)
+ \frac{1}{12}D \cdot c_2(X) + \sum_P c_P(D),
\]

where the summation takes place over singular points of \( X \), and \( c_P(D) \in \mathbb{Q} \) is a contribution depending only on the local analytic type of \( P \in X \) and \( \mathcal{O}_X(D) \).
If $P$ is a terminal quotient singularity of type $\frac{1}{r_P}(1,-1,b_P)$, then
\[
c_P(D) = -i_P r_P^2 - 1 + \frac{i_P-1}{12r_P} \sum_{j=1} jb_P(r_P - jb_P),
\]
where $\overline{j}$ denotes the smallest residue modulo $r_P$, that is, $\overline{j} = j - \lfloor \frac{j}{r_P} \rfloor r_P$ in terms of the round down \([\ ]\). The definition of the round down \([\ ]\) is $[j] = \max\{k \in \mathbb{Z} | k \leq j\}$.

And for any terminal singularity $P$,
\[
c_P(D) = \sum_{\alpha} c_{P_{\alpha}}(D_{\alpha}),
\]
where $\{(P_{\alpha}, D_{\alpha})\}_\alpha$ is a flat deformation of $(P, D)$ to terminal quotient singularities.

Remark 2.6.1. If $X$ has only terminal singularities, then we can write the contribution term $\sum_P c_P(D)$ as $\sum_Q c_Q(D)$, where
\[
c_Q(D) = -\overline{i_Q} r_Q^2 - 1 + \frac{i_Q-1}{12r_Q} \sum_{j=1} jb_Q(r_Q - jb_Q).
\]
For its summation takes place over points which need not lie on $X$ but may lie on deformed varieties of $X$, $Q$’s are called “fictitious” points in the sense of M. Reid. This description holds even though $X$ has canonical singularities, but in this case $Q$’s may lie on deformed varieties of crepant blown-up varieties of $X$ (see [R87] for details).

By Lemma 2.3, we can take an integer $e$ such that $ae \equiv 1$ modulo $r$. Then $(O_Y(iE))_Q \cong (O_Y(eK_Y))_Q$ for any $Q \in E$. Using the singular Riemann-Roch formula, we get
\[
\chi(O_Y(iE)) = \chi(O_Y) + \frac{1}{12} i(i - a)(2i - a)E^3 + \frac{1}{12} iE \cdot c_2(Y) + A_i,
\]
where $A_i$ is the contribution term and has the below description:
\[
A_i = \sum_{Q \in I} c_Q(iE),
\]
\[
c_Q(iE) = -\overline{i_Q} r_Q^2 - 1 + \frac{i_Q-1}{12r_Q} \sum_{j=1} jb_Q(r_Q - jb_Q).
\]
Here $Q \in I$ are fictitious singularities. The type of $Q$ is $\frac{1}{r_Q}(1,-1,b_Q)$, $(O_{Y_Q}(E_Q))_Q \cong (O_{Y_Q}(eK_{Y_Q}))_Q$ where $(Y_Q, E_Q)$ is the fictitious pair for $Q$. 

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and $\bar{\cdot}$ denotes the smallest residue modulo $r_Q$. We note that $b_Q$ is coprime to $r_Q$ and also $e$ is coprime to $r_Q$ because $r|(ae - 1)$. So $v_Q = \overline{eb_Q}$ is coprime to $r_Q$. With this description, $r = 1$ if $I$ is empty, and otherwise $r$ is the lowest common multiple of $\{r_Q\}_{i \in I}$. We note that $c_Q(iE)$ depends only on $i \mod r_Q$ and equals $0$ if $r_Q | i$. Especially $A_i$ depends only on $i \mod r_Q$ and equals $0$ if $r_Q | i$.

We put $B_i = -(A_i + A_{-i})$. Because

$$c_Q(iE) + c_Q(-iE) = \left(\frac{-i^2 e^2 - 1}{12r_Q} + \sum_{j=1}^{e-1} \frac{jv_Q(r_Q - jv_Q)}{2r_Q}\right)$$

$$+ \left(-\frac{i^2 e - 1}{12r_Q} + \sum_{j=1}^{e-1} \frac{jv_Q(r_Q - jv_Q)}{2r_Q}\right)$$

$$= \frac{-r^2 Q - 1}{12} + \sum_{j=1}^{r_Q} \frac{jv_Q(r_Q - jv_Q)}{2r_Q} = \frac{-iv_Q(r_Q - iv_Q)}{2r_Q}$$

where the third equality comes from the property that $b_Q$ and $r_Q$ are coprime, we have

$$(2.2) \quad B_i = -\sum_{Q \in I} (c_Q(iE) + c_Q(-iE)) = \sum_{Q \in I} \frac{iv_Q(r_Q - iv_Q)}{2r_Q}.$$  

**Proposition 2.7.** (Notation as above).

(A) $rE^3 \in \mathbb{Z}_{>0}$.

(B) $1 = \frac{1}{2}aE^3 + \sum_{Q \in I} \frac{v_Q(r_Q - v_Q)}{2r_Q}$.

(C) $\dim_k \mathcal{O}_X / f_* \mathcal{O}_Y (-iE) = i^2 - \frac{1}{2} \sum_{Q \in I} \min \{(1 + j)Q + i(i - 1 - 2j)v_Q \} \quad (1 \leq i \leq a)$.

(D) $\sum_{Q \in I} \min \{v_Q, r_Q - v_Q \} = \dim_k f_* \mathcal{O}_Y (-2E) / m_P^2$.

**Remark 2.7.1.** In particular (A), (C) and (D) are essential. We use (A) to bound the value of $a$ from above and use (C) to control the values of $r_Q$'s.
(1) shows that the number of fictitious non-Gorenstein points of $Y$ is at most 3. We prove the conditions 1 and 2 in Proposition 2.4 according to the value of $\dim k f_*\mathcal{O}_Y(-2E)/m_f^2$.

Remark 2.7.2. In fact, because of (2.2) and (2.9) the right hand side of (3) is the same if we replace $v_Q$ by $r_Q - v_Q$.

Proof. We consider the exact sequence:

$$0 \to \mathcal{O}_Y((i - 1)E) \to \mathcal{O}_Y(iE) \to \mathcal{Q}_i \to 0.$$  

By (2.1), we get

$$\chi(\mathcal{Q}_i) = \chi(\mathcal{O}_Y(iE)) - \chi(\mathcal{O}_Y((i - 1)E))$$

$$= \frac{1}{12} \{2(3i^2 - 3i + 1) - 3(2i - 1)a + a^2\}E^3$$

$$+ \frac{1}{12} E \cdot c_2(Y) + A_i - A_{i-1}.$$

Since $\chi(\mathcal{Q}_i) - \chi(\mathcal{Q}_{r+i}) = \frac{r}{2}(a + 1 - r - 2i)E^3$ is an integer for any $i$ and $E^3$ is positive, we have (A).

By (2.4),

$$\chi(\mathcal{Q}_{i-1}) - \chi(\mathcal{Q}_{i+1}) = (i + \frac{1}{2})aE^3 + B_{i+1} - B_i.$$

Let $d(i) = \dim k f_*\mathcal{O}_Y(iE)/f_*\mathcal{O}_Y((i - 1)E)$. We note that $d(i) = 0$ if $i \geq 1$, and $d(0) = 1$. Because $(Y, \varepsilon E)$ is weak KLT and $iE - (K_Y + \varepsilon E)$ is $f$-ample for a sufficiently small positive rational number $\varepsilon$ and an integer $i \leq a$, using [KMM87, Theorem 1-2-5], we have $R^j f_*\mathcal{O}_Y(iE) = 0$ for $i \leq a$, $j \geq 1$. So by (2.3), for any $i \leq a$,

$$H^0(Y, \mathcal{Q}_i) = f_*\mathcal{Q}_i = f_*\mathcal{O}_Y(iE)/f_*\mathcal{O}_Y((i - 1)E),$$

$$H^j(Y, \mathcal{Q}_i) = R^j f_*\mathcal{Q}_i = 0 \quad \text{for} \quad j \geq 1,$$

and therefore $d(i) = \chi(\mathcal{Q}_i)$.

Putting $i = 0$ in (2.5), we get

$$1 = \frac{1}{2}aE^3 + B_1.$$  

Combining this and (2.2) with $i = 1$, we get (B).

With (2.5), we obtain for $1 \leq i \leq a$,

$$\sum_{1 \leq j < i} d(-j) = \sum_{1 \leq j < i} \{\chi(\mathcal{Q}_{i-j}) - \chi(\mathcal{Q}_{j+1})\}$$

$$= \sum_{1 \leq j < i} \{(j + \frac{1}{2})aE^3 + B_{j+1} - B_j\}$$

$$= \frac{1}{2}(i^2 - 1)aE^3 + B_i - B_1.$$
Since \( \dim k \) with (2.8) and (2.9), we obtain (C).

Of course because \( \dim k \) (2.9) for

2.3 Proof of \( f_\ast \mathcal{O}_Y(-2E) \neq \mathfrak{m}_p \)

Assuming that \( f_\ast \mathcal{O}_Y(-2E) = \mathfrak{m}_p \), we will derive a contradiction. The assumption means that the coefficient of \( F_n \) in \( g_{1n}F_1 \) is bigger than 1, so there exists a \( Z_1 \) which is contained in at least two \( g_{0n} \)-exceptional prime divisors on \( X_i \). The minimum value of \( a \) in this case occurs when \( Z_1 \) is a curve, \( Z_2 = (g_{2}^{-1})_\ast F_1 \cap F_2 \), and \( n = 3 \), and the minimum value is 6. So we get \( a \geq 6 \). By the assumption and (2.8), we obtain (2.6).

Thus we have only to consider the three cases:

Case 1. \( \{(r_Q, \overline{r_Q}) \}_{Q \in I} = \{(r, \overline{r}), (r, \overline{r})\} \), \( r \geq 7 \).

Case 2. \( \{(r_Q, \overline{r_Q}) \}_{Q \in I} = \{(r_1, \overline{r_1}), (r_2, \overline{r_2})\} \), \( r_1 \geq 2 \), \( r_2 \geq 5 \).

Case 3. \( \{(r_Q, \overline{r_Q}) \}_{Q \in I} = \{(r_1, \overline{r_1}), (r_2, \overline{r_2}), (r_3, \overline{r_3})\} \), \( 2 \leq r_1 \leq r_2 \leq r_3 \).
Here $\pm$ means that one of these occurs for each $\nu_Q$. We remark that $\nu_Q$ is coprime to $r_Q$.

Since $\sum_{Q \in I} ^{v_Q(r_Q - v_Q)}/2r_Q < 1$ from (3), we have the below inequalities:

Case 1. $3/2 - 9/2r < 1$.
Case 2. $3/2 - (1/2r_1 + 2/r_2) < 1$.
Case 3. $3/2 - (1/2r_1 + 1/2r_2 + 1/2r_3) < 1$.

Using this evaluation, we can restrict possible values of $r_Q$’s. Below we show all the possible values and the corresponding values of $aE^3$:

Case 1. $r$ : 7 8
$aE^3$ : 2/7 1/8
Case 2. $(r_1, r_2)$ : (2, 5) (3, 5) (4, 5) (2, 7)
$aE^3$ : 3/10 2/15 1/20 1/14
Case 3. $(r_1, r_2, r_3)$ : (2, 2, r_3) (2, 3, 3) (2, 3, 4) (2, 3, 5)
$aE^3$ : 2/2r_3 1/6 1/12 1/30

Recalling that $r$ is the lowest common multiple of $\{r_Q\}_{Q \in I}$, with (A) we have $a \leq 3$ for all the above cases. This contradicts $a \geq 6$.

2.4 Proof of $f_* \mathcal{O}_Y(-nE) \not\subseteq m_P^2$

Because $f_* \mathcal{O}_Y(-2E) \neq m_P$, we have $g_{1n}^* F_1 = \sum_{i=1}^{n} (g_{1n}^{-1})_* F_i + \text{(others)}$ and,

\((*)\) $F_1 (1 \leq i \leq m)$ is obtained as a valuation by the weighted blow-up of $X$ with its weights $(x, y, z) = (1, i, i)$ for local parameters $x, y, z$ at $P$ such that $Z_{m-1} \subseteq (g_{0,m-1}^{-1})_* \text{div}(y) \cap (g_{0,m-1}^{-1})_* \text{div}(z)$.

We divide the proof according to the value of $\dim_k f_* \mathcal{O}_Y(-2E)/m_P^2 \leq 2$.

Case 1. $\dim_k f_* \mathcal{O}_Y(-2E)/m_P^2 = 0$.
This is the case when $Z_1 \subseteq F_1$ is neither a line nor a point.

Case 2. $\dim_k f_* \mathcal{O}_Y(-2E)/m_P^2 = 1$.
This is the case when $Z_1 \subseteq F_1$ is a line.

Case 3. $\dim_k f_* \mathcal{O}_Y(-2E)/m_P^2 = 2$.
This is the case when $Z_1 \subseteq F_1$ is a point.

Since

\[
\dim_k f_* \mathcal{O}_Y(-2E)/m_P^2 = \dim_k \text{Im}[\{v \in m_P|Z_1 \subseteq (g_1^{-1})_* \text{div}(v)\} \rightarrow m_P/m_P^2]
= \dim_k \{v \in \Gamma(F_1, \mathcal{O}_{F_1}(1))|v = 0 \text{ or } Z_1 \subseteq \text{div}(v)\},
\]

the value of $\dim_k f_* \mathcal{O}_Y(-2E)/m_P^2$ decides the type of $Z_1 \subseteq F_1 \cong \mathbb{P}_k^2$ as above.

In Case 1, $\sum_{Q \subseteq I} \min\{\nu_Q, r_Q - \nu_Q\} = 0$ by (3). Therefore $I$ is empty and thus $Y$ is Gorenstein. By [Cu88, Theorem 5], $f$ must be the blow-up of $X$ along $P$, that is, $f = g_1$, and so we have nothing to do. Thus we have only to consider Cases 2 and 3. In these cases we investigate the values of $\dim_k \mathcal{O}_X/f_* \mathcal{O}_Y(-iE)$’s carefully.
Proposition 2.8. (Notation as above). Let $2 \leq l \leq n$ be an integer such that $g_{0i}((-iF_i)) \not\subset m_P^2$, for any $i < l$.

(1) If $g_{0i}((-iF_i)) \not\subset m_P^2$, then
\[
\dim_k \mathcal{O}_X/f_s\mathcal{O}_Y(-lE) \leq l - \frac{1}{2} \min_{0 \leq j < l} \{(1+j)m - 2l|j\}.
\]

(2) If $g_{0i}((-iF_i)) \subset m_P^2$, (in this case we have $l > m$ by (*)), then
\[
\dim_k \mathcal{O}_X/f_s\mathcal{O}_Y(-lE) > l - \frac{1}{2} \min_{0 \leq j < l} \{(1+j)m - 2l|j\}.
\]

Remark 2.8.1. In the case when $m = 1$ because
\[
\min_{0 \leq j < l} \{(1+j)m - 2l|j\} = \min_{0 \leq j < l} \{(j-(2l-1))j\} = -l(l-1),
\]
we can simplify the above inequalities:

(1) $\dim_k \mathcal{O}_X/f_s\mathcal{O}_Y(-lE) \leq \frac{1}{2}l(l+1)$.

(2) $\dim_k \mathcal{O}_X/f_s\mathcal{O}_Y(-lE) > \frac{1}{2}l(l+1)$.

Proof. (1) By the assumption and $f_s\mathcal{O}_Y(-2E) \neq m_P$, the proof of Proposition 2.4 says that we can take local parameters $x, y, z$ at $P$ such that $Z_{\min\{l,m\}-1} \subseteq (g_{0\min\{l,m\}-1})_s\text{div}(y)$ and $Z_{l-1} \subseteq (g_{0,l-1})_s\text{div}(z)$. Then for $1 \leq i \leq l$, $F_i$ equals, as valuations, the exceptional divisor obtained by the weighted blow-up of $X$ with its weights $(x, y, z) = (1, \min\{i, m\}, i)$.

Hence
\[
f_s\mathcal{O}_Y(-lE) = g_{0m}\mathcal{O}_{X_m}(-lF_n)
\]
\[
\supseteq g_{0i}\mathcal{O}_{X_i}(-lF_i) = (x^iy^iz^u|s + \min\{l, m\}t + lu \geq l),
\]
and so
\[
\dim_k \mathcal{O}_X/f_s\mathcal{O}_Y(-lE) \leq \dim_k \mathcal{O}_X/(x^iy^iz^u|s + \min\{l, m\}t + lu \geq l)
\]
\[
= l - \frac{1}{2} \min_{0 \leq j < l} \{(1+j)m - 2l|j\}.
\]

Here we used Lemma 2.3 proved later.

(2) As in the proof of (1), we can take local parameters $x, y, z$ at $P$ such that $Z_{m-1} \subseteq (g_{0,m-1})_s\text{div}(y)$ and $Z_{l-2} \subseteq (g_{0,l-2})_s\text{div}(z)$. Then for $1 \leq i < l$, $F_i$ equals, as valuations, the exceptional divisor obtained by the weighted blow-up of $X$ with its weights $(x, y, z) = (1, \min\{i, m\}, i)$.

We have
\[
f_s\mathcal{O}_Y(-lE) = g_{0m}\mathcal{O}_{X_m}(-lF_n)
\]
\[
\supseteq g_{0,l-1}\mathcal{O}_{X_{l-1}}(-lF_{l-1}) + (v \in m_P|Z_{l-1} \subseteq (g_{0,l-1})_s\text{div}(v)).
\]
But since 
\[(v \in m_P | Z_{l-1} \subseteq (g_{0,l-1}^{-1})^* \text{div}(v)) \subseteq g_{0,l}^* O_{X_l}(-lF_l) \subseteq m_P^2,\]
for any \(v \in m_P\) such that \(Z_{l-1} \subseteq (g_{0,l-1}^{-1})^* \text{div}(v)\) we have 
\[g_{0,l-1}^* \text{div}(v) \geq g_{1,1}^* (2F_1 + (g_1^{-1})^* \text{div}(v)) \geq (2 + (l - 2))F_{l-1} = lF_{l-1}.\]
Thus 
\[(v \in m_P | Z_{l-1} \subseteq (g_{0,l-1}^{-1})^* \text{div}(v)) \subseteq g_{0,l-1}^* O_{X_{l-1}}(-lF_{l-1}),\]
and hence 
\[f_* O_Y(-lE) \subseteq g_{0,l-1}^* O_{X_{l-1}}(-lF_{l-1}) = (x^s y^t z^u | s + mt + (l - 1)u \geq l).\]
Therefore with Lemma 2.9,
\[
\dim_k O_X / f_* O_Y(-lE) \geq \dim_k O_X / (x^s y^t z^u | s + \min\{l, m\}t + lu \geq l) > \dim_k O_X / (x^s y^t z^u | s + mt + lu \geq l) = l - \frac{1}{2} \min_{0 \leq j < l} \{(1 + j)m - 2l)j\}.\]

We used the following lemma in the above proof.

**Lemma 2.9.** Let \(X \ni P\) be an algebraic germ of a smooth 3-dimensional variety with local parameters \(x, y, z\) at \(P\), and let \(l, m\) be positive integers. Then

\[
\dim_k O_X / (x^s y^t z^u | s + \min\{l, m\}t + lu \geq l) = l - \frac{1}{2} \min_{0 \leq j < l} \{(1 + j)m - 2l)j\}.\]

**Proof.**

\[
\dim_k O_X / (x^s y^t z^u | s + \min\{l, m\}t + lu \geq l) = \dim_k \text{Span}_k (x^s y^t | s + \min\{l, m\}t < l) \\
= \sum_{0 \leq t < \frac{1}{\min\{l, m\}}} (l - \min\{l, m\}t) \\
= \sum_{0 \leq t < \frac{1}{m}} (l - mt) \\
= l - \frac{m}{2} \left\{ \left( \left\lfloor \frac{l}{m} \right\rfloor + \frac{1}{2} - \frac{l}{m} \right)^2 - \left( \frac{1}{2} - \frac{l}{m} \right)^2 \right\} \\
= l - \frac{m}{2} \min_{0 \leq j < l} \left\{ \left( j + \frac{1}{2} - \frac{l}{m} \right)^2 - \left( \frac{1}{2} - \frac{l}{m} \right)^2 \right\} \\
= l - \frac{1}{2} \min_{0 \leq j < l} \{(1 + j)m - 2l)j\}.\]
Now we prove $f_*O_Y(-nE) \not\subset m_P^2$ in Cases 2 and 3.

**Proof in Case 2.** For $Z_1 \subseteq F_1$ is a line in this case and $a$ is the discrepancy of $F_n$ with respect to $K_X$, we get $m = 1$ and

$$a = n + 1 \quad (n \geq 2).$$

By (2.10), $\sum_{Q \in I} \min\{v_Q, r_Q - v_Q\} = 1$ and thus $\{(r_Q, v_Q)\}_{Q \in I} = \{(r, \pm 1)\}$. From (A), we obtain $aE^3 = (r + 1)/r$. By (A),

$$a \leq r + 1.$$

From (2.10), Remark 2.7.2, and (2.11), for $1 \leq i \leq n + 1$ we have

$$\dim_k O_X / f_* O_Y(-iE) = i^2 - \frac{1}{2} \min\{(1 + j)jr + i(i - 1 - 2j)\}$$

$$= \frac{1}{2}i(i + 1) - \frac{1}{2} \min\{(1 + j)r - 2i j\}.$$

Hence for $1 \leq i \leq n + 1$,

$$\dim_k O_X / f_* O_Y(-iE) \geq \frac{1}{2}i(i + 1),$$

where the equality holds if and only if $i \leq r$.

If there exists a positive integer $2 \leq l \leq n$ such that $g_{0l}O_X(-lF_l) \subseteq m_P^2$ and $g_{0l}O_X,(-iF_l) \subseteq m_P^2$ for any $i < l$, then by Proposition 2.8, Remark 2.8.1, and the condition of the equality in (2.12), we obtain $l = r + 1$. Thus with (2.10), we have $r + 1 = l \leq n = a - 1$, that is, $a \geq r + 2$. This contradicts (2.11) and hence we get $g_{0n}O_X,(-nF_n) \not\subset m_P^2$. □

**Proof in Case 3.** In this case we use essentially the same idea as in Case 2, but it is a little more complicated. By (2.10), $\sum_{Q \in I} \min\{v_Q, r_Q - v_Q\} = 2$. Thus we have only to consider the two subcases:

Subcase 1. $\{(r_Q, v_Q)\}_{Q \in I} = \{(r, \pm 2)\}$, $r \geq 5$.

Subcase 2. $\{(r_Q, v_Q)\}_{Q \in I} = \{(r_1, \pm 1), (r_2, \pm 1)\}$, $2 \leq r_1 \leq r_2$.

In Subcase 1, we have $aE^3 = 4/r$ by (B) and thus $a \leq 4$ from (A). But since $Z_1 \subseteq F_1$ is a point, we get $n = 2$ and $a = 4$. Then choosing local parameters $x, y, z$ at $P$ such that $Z_1 \subseteq (g_1^{-1})_*, \text{div}(y) \cap (g_1^{-1})_* \text{div}(z)$, $F_2$ equals, as valuations, the exceptional divisor obtained by the weighted blow-up of $X$ with its weights $(x, y, z) = (1, 2, 2)$. So we have only to investigate Subcase 2.

Recalling that $a$ is the discrepancy of $F_n$ with respect to $K_X$, we have

$$\dim_k O_X / f_* O_Y(-iE) = \frac{1}{2}i(i + 1),$$

where the equality holds if and only if $i \leq r$.
Calculating with (B) we obtain \( aE^3 = (r_1 + r_2)/r_1r_2 \), and thus by (A),
\[
(2.14) \quad a \leq r_1 + r_2.
\]
From (C), Remark 2.7.2, and (2.13), for \( 1 \leq i \leq m+n \) we have
\[
\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-iE) = i^2 - \frac{1}{2} \min \{(1 + j)r_1 + i(i - 1 - 2j)\} - \frac{1}{2} \min \{(1 + j)r_2 + i(i - 1 - 2j)\} = i - \frac{1}{2} \left( \min\{((1 + j)r_1 - 2i)j\} + \min\{((1 + j)r_2 - 2i)j\} \right).
\]
Hence for \( 1 \leq i \leq m+n \),
\[
(2.15) \quad \dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-iE) \geq i - \frac{1}{2} \min\{((1 + j)r_1 - 2i)j\} \geq i,
\]
where the equality of the first inequality holds if and only if \( i \leq r_2 \), and the second holds if and only if \( i \leq r_1 \).

Claim 2.10. \( r_1 = m \).

Proof of the claim. Utilizing Proposition 2.8 (1) with \( l = m \), we have
\[
(2.16) \quad \dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-mE) \leq m - \frac{1}{2} \min\{(j - 1)jm\} = m.
\]
We take local parameters \( x, y, z \) at \( P \) as in (*), satisfying \( Z_m \subseteq (g_{0m}^{-1})_*\text{div}(z) \) if \( Z_m \subseteq F_m \cong \mathbb{P}^2 \) is a line. We have
\[
f_*\mathcal{O}_Y(-(m+1)E) = g_{0m}^*\mathcal{O}_{X_m}(-(m+1)F_m) \\
\subseteq g_{0m}^*\mathcal{O}_{X_m}(-(m+1)F_m) + (v \in \mathfrak{m}_P|Z_m \subseteq (g_{0m}^{-1})_*\text{div}(v)).
\]
But since
\[
(v \in \mathfrak{m}_P|Z_m \subseteq (g_{0m}^{-1})_*\text{div}(v)) \subseteq g_{0m}^*\mathcal{O}_{X_m}(-mF_m) = (x^m, y, z),
\]
we get
\[
(v \in \mathfrak{m}_P|Z_m \subseteq (g_{0m}^{-1})_*\text{div}(v)) \subseteq (z) + g_{0m}^*\mathcal{O}_{X_m}(-(m+1)F_m),
\]
and thus
\[
f_*\mathcal{O}_Y(-(m+1)E) \subseteq (z) + g_{0m}^*\mathcal{O}_{X_m}(-(m+1)F_m) \\
= (z) + (x^s y^t z^u |s + mt + mu \geq m + 1).
\]

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Hence,

\[(2.17) \quad \dim_k \mathcal{O}_X / f_* \mathcal{O}_Y (- (m + 1)E) \geq \dim_k \mathcal{O}_X / ((z) + \{x^y y^z | s + mt + mu \geq m + 1\}) = \dim_k \text{Span}_k \langle x^s, y | s \leq m \rangle = m + 2.\]

From (2.16), (2.17), and the condition of the second equality in (2.15), we have \(r_1 = m\). \qed

If there exists a positive integer \(l \leq n\) such that \(g_{0l} \mathcal{O}_{X_l} (-lF) \subseteq m^2\), and \(g_{0i} \mathcal{O}_{X_i} (-iF) \nsubseteq m^2\) for any \(i < l\), then by Proposition 2.8, Claim 2.10, and the condition of the first equality in (2.15), we obtain \(l = r_2 + 1\). Thus with (2.13) and Claim 2.11, we have \(r_1 + r_2 + 1 = m + l \leq m + n = a\). This contradicts (2.14) and hence we get \(g_{0n} \mathcal{O}_{X_n} (-nF_n) \nsubseteq m^2\). \qed

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