CONSTRUCTION OF $\mu$-NORMAL NUMBERS

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Abstract. In the present paper we want to extend Champernowne’s construction of normal numbers to provide numbers which are generic for a given invariant probability measure, which need not be the maximal one. We present a construction together with estimates and examples for normal numbers with respect to Lüroth series, continued fractions expansion or $\beta$-expansion.

1. Introduction

Let $q \geq 2$ be a positive integer, then every real $x \in [0,1]$ has a $q$-adic representation of the form

$$x = \sum_{h=1}^{\infty} d_h(x) q^{-h}$$

with $d_h(x) \in D := \{0,1,\ldots,q-1\}$. We call a number $x \in [0,1]$ normal with respect to the base $q$ if for any $k \geq 1$ and any block $b = b_1 \ldots b_k$ of $k$ digits the frequency of occurrences of this block tends to the expected one, namely $q^{-k}$. In particular, let $N_n(b,x)$ be the number of occurrences of $b$ among the first $n$ digits, i.e.

$$N_n(b,x) = \# \{0 \leq h < n : d_{h+1}(x) = b_1, \ldots, d_{h+k}(x) = b_k \}.$$

Then we call $x \in [0,1]$ normal of order $k$ in base $q$ if for every block $b$ of length $k$ we have

$$\lim_{n \to \infty} \frac{N_n(b,x)}{n} = q^{-k}.$$

Furthermore, we call a number absolutely normal if it is normal in every base $q \geq 2$.

In 1909 Borel [7] showed that Lebesgue almost every real is absolutely normal. This motivated people to look for a concrete example of a normal number. It took more than 20 years until 1933 when Champernowne [8] provided the first explicit construction by showing that the number

$$0.12345678910111213141516$$

is normal to base 10.

This construction was generalized to different numeration systems. Normal sequences for Bernoulli shifts and continued fractions were already investigated by Postnikov and Pyateckii [17,18], see also Postnikov [16]. A different construction for continued fractions is due to Adler et al. [1]. Generalizations to $\beta$-expansion are due to Bertrand-Mathis and Volkmann [4] and Ito and Shiokawa [12]. The normality of the Champernowne number with respect to numeration systems in the Gaussian integers was investigated by Dumont et al. [11]. The generalization

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of the Champernowne construction in dynamical systems fulfilling the specification property provides generic points for the maximal measure as was shown by Bertrand-Mathis [3].

2. Definitions and statement of results

All the above constructions have in common, that they are aiming for normal numbers or equivalently numbers that are generic for the maximum measure. In the present paper, however, we are interested in a different aspect. In particular, we modify the Champernowne construction such that we get arbitrary close to any given shift invariant measure. This is motivated by recent constructions by Altomare and Mance [2] and Mance [13, 14]. In their constructions they need that a certain block of digits occurs much more often than the others. By this imbalance they could construct numbers whose Cantor series expansion is block-normal but not distribution-normal.

Our common basis is a symbolic dynamical system that fulfills the specification property. In our definitions we mainly follow the articles of Bertrand-Mathis and Volkmann [3,4]. Let $A$ be a fixed (possibly infinite) alphabet. We denote by $A^+$ the semigroup generated by $A$ under catenation. Let $\varepsilon$ denote the empty word and $A^* = A^+ \cup \{\varepsilon\}$. The length of a word $\omega = a_1a_2 \ldots a_k$ with $a_i \in A$ for $1 \leq i \leq k$ is denoted by $|\omega| = k$ and we write $A^k$ for the set of words of length $k$ (over $A$).

A set $L \subset A^*$ is called a language. We say that a language $L$ fulfills the specification property if there exists a positive integer $j$ such that for any two words $a, b \in L$ there exists a word $u \in L$ with $|u| \leq j$ such that $aub \in L$. For any pair of finite words $a$ and $b$ we fix a $u_{a,b}$ with $|u_{a,b}| \leq j$ such that $au_{a,b}b \in L$. Then for $a, a_1, \ldots, a_n \in L$ and $n \in \mathbb{N}$ we write

$$a_1 \odot a_2 \odot \cdots \odot a_n := a_1u_{a_1,a_2}a_2u_{a_2,a_3}a_3 \cdots a_{n-1}u_{a_{n-1},a_n}a_n$$

and

$$a^{\odot n} := \underbrace{a \odot a \odot \cdots \odot a}_{n \text{ times}}.$$

For a language $L \subset A^*$ let $W^\infty = W^\infty(L)$ be the set of infinite words generated by $L$, i.e. the set of sequences $\omega = (a_i)_{i \geq 1}$ with $a_i a_{i+1} \cdots a_k \in L$ for any $1 \leq i < k < \infty$.

We introduce the discrete topology on $A$ and the corresponding product topology on $A^\mathbb{N}$. Let $\omega = (a_i)_{i \geq 1} \in A^\mathbb{N}$, then we define the shift operator $T$ as the mapping $(T(\omega))_i = a_{i+1}$ for $i \geq 1$. We associate with each language $L$ the symbolic dynamical system

$$S_L = (W^\infty, \mathcal{B}, T, I),$$

where $W^\infty = W^\infty(L)$; $\mathcal{B}$ is the $\sigma$-algebra generated by all cylinder sets of $A^\mathbb{N}$, i.e. sets of the form

$$c(\omega) = c_n(\omega) = \{a_1a_2a_3 \ldots a_n \in A^\mathbb{N} : a_1a_2 \ldots a_n = \omega\}$$

for some word $\omega \in A^n$ of length $n$; $T$ is the shift operator; and $I$ is the set of all $T$-invariant probability measures $\mu$ on $\mathcal{B}$. We will also write $\mu(\omega)$ for $\mu(c(\omega))$. Note that $W^\infty$ is invariant under $T$ and closed with respect to this topology.

With each symbolic dynamical system $S$ we associate the entropy

$$h(W^\infty) = \sup_{\mu \in I} h(\mu),$$
where $h(\mu)$ denotes the entropy\footnote{For a definition see chapter 2 of Billingsley \cite{10}} of the measure $\mu$. For finite alphabets it is known (cf. Proposition 19.13 of Denker et al. \cite{10}) that there always exists a unique measure $\chi_\omega \in I$, called measure of maximal entropy or equilibrium state, such that $h(W^{\infty}) = h(\chi)$. Bertrand-Mathis \footnote{\cite{3}} has shown, that this measure can be generated by a Champernowne type construction.

Now we fix a $T$-invariant measure $\mu \in I$. A word $b \in W(L^*)$ is $\mu$-admissible if $\mu(b) \neq 0$. Let $D_\mu$ denote the set of $\mu$-admissible words and let $D_{\mu,k}$ denote the set of $\mu$-admissible words of length $k$. Given words $b$ and $\omega$ we will let $N_n(b,\omega)$ denote the number of times the word $b$ occurs starting in position no greater than $n$ in the word $\omega$, i.e.

$$N_n(b,\omega) = \# \{0 \leq i < n: a_i+1a_{i+2}\cdots a_{i+k} = b \}.$$  

If $\omega$ is finite we will often write $N(b,\omega)$ in place of $N|\omega|(b,\omega)$. When we say that a sequence of $T$-invariant measures $(\nu_i)_{i=1}^\infty$ in $I$ converges weakly to $\mu \in I$ (written $\nu_i \to \mu$), we silently make the additional assumptions that $D_\nu \subset D_\mu$ \footnote{\cite{3}} and that $\mu_i(b)$ is eventually non-increasing in $i$.

**Definition 2.1.** Suppose that $0 < \epsilon < 1$, $k$ is a positive integer and $\mu \in I$. A word $\omega$ is called $(\epsilon, k, \mu)$-normal\footnote{A version of our main theorem is still true if we drop the condition $D_{\nu_i} \subset D_\mu$, but every example we will consider has this property.} if for all $t \leq k$ and words $b$ in $D_{\mu,t}$, we have

$$\mu(b)|\omega|(1 - \epsilon) \leq N(b, \omega) \leq \mu(b)|\omega|(1 + \epsilon).$$

An infinite word $\omega \in A^\mathbb{N}$ is called $\mu$-normal of order $k$ if for every admissible word $b$ of length $k$ we have

$$\lim_{n \to \infty} \frac{N_n(b, \omega)}{n} = \mu(b).$$

We write $\omega \in N_{\mu,k}$. Furthermore we call $\omega$ $\mu$-normal (or equivalently generic for $\mu$) if $\omega \in N_{\mu} := \bigcap_{k=1}^\infty N_{\mu,k}$.

Now we state a condition, which we call $(W, \mu)$-good, on the sequence of blocks $(\omega_i)_{i=1}^\infty$, which are $(\epsilon_i, k_i)$-normal, such that we can successfully concatenate them and get a $\mu$-normal word. In particular, let $F$ be the set of all sequences of 4-tuples $W = (\epsilon_i, k_i, \nu_i)_{i=1}^\infty$ with non-decreasing sequences of non-negative integers $(\epsilon_i)_{i=1}^\infty$ and $(k_i)_{i=1}^\infty$, such that $(\nu_i)_{i=1}^\infty$ is a sequence of $T$-invariant measures and $(\epsilon_i)_{i=1}^\infty$ strictly decreases to 0. Let $\mu \in I$ be a shift invariant measure and $R(W) = [1, \lim_{i \to \infty} k_i] \cap \mathbb{N}$ be the set of supported block-lengths. If $(\omega_i)_{i=1}^\infty$ is a sequence of words such that $|\omega_i|$ is non-decreasing and $\omega_i$ is $(\epsilon_i, k_i, \nu_i)$-normal, then $(\omega_i)_{i=1}^\infty$ is said to be $(W, \mu)$-good if $\nu_i \to \mu$ and for all $i \geq 2$ the following hold:

\begin{align}
\frac{1}{\epsilon_{i-1} - \epsilon_i} &= o(|\omega_i|); \\
\frac{l_{i-1}}{l_i} \cdot \frac{|\omega_{i-1}|}{|\omega_i|} &= o(i^{-1});
\end{align}

\footnote{$\epsilon, k, \mu$-normality is a generalization of the concept of $(\epsilon, k)$-normality, originally due to Besicovitch \cite{3}.}
Now we are able to state our main theorem.

**Main Theorem 2.1.** Let $W \in \mathcal{F}$ and suppose that $(\omega_i)_{i=1}^{\infty}$ is a $(W, \mu)$-good sequence. Then, for each $k_i \in R(W)$, the infinite word $\omega = \omega_1 \circ \omega_2 \circ \cdots \in N_{\mu, k_i}$. Moreover, if $k_i \to \infty$, then $\omega \in N_{\mu}$.

In the following section we build our toolbox and prove Main Theorem 2.1. For the applications we present our construction of a $(W, \mu)$-good sequence in Section 4. Finally we apply this constructed sequence of blocks in Section 5 to different numbers systems. Since in these different number systems we have different requirements (finite or infinite digit set, restrictions on the digit set, etc.), we need different variants of members of $\mathcal{F}$.

### 3. Proof of Main Theorem 2.1

Since the proof follows along similar lines to the proof of Main Theorem 1.15 in [13], we will only include those parts that differ significantly and omit the proofs, which are similar to proofs of lemmas in [13]. Throughout this section, we will fix $W = ((l_i, \epsilon_i, k_i, \nu_i))_{i=1}^{\infty} \in \mathcal{F}$ and a $(W, \mu)$-good sequence $(\omega_i)$. Set $\omega = \omega_1 \circ \omega_2 \circ \cdots$ the constructed infinite word. Let

$$\sigma_k = \omega_k \circ \nu_k \circ \omega_{k+1}$$

be the $k$th block and

$$M_k = |\sigma_k| = l_k |\omega_k| + k_{k-1} |u_{\omega_k, \omega_k}| + |u_{\omega_k, \omega_{k+1}}|$$

its length. Furthermore let $L_i = \sum_{k=1}^{i} M_k$ be the concatenated length up to the $i$th block. For a given $n$, the letter $i = i(n)$ will always be understood to be the positive integer that satisfies $L_i < n \leq L_{i+1}$, i.e. position $n$ lies in the block $\sigma_{i+1}$. This usage of $i$ will be made frequently and without comment. Let $m = n - L_i$, which allows $m$ to be written in the form

$$m = x(\omega_{i+1}) + |u_{\omega_{i+1}, \omega_{i+1}}| + y$$

where $x = x(n)$ and $y = y(n)$ are integers satisfying

$$0 \leq x < l_{i+1} \text{ and } 0 \leq y < |\omega_{i+1}| + 1.$$ 

We note that $x$ and $y$ are not unique. However, we suppose them to be chosen in the natural way making them unique. By carefully distinguishing if $x = l_i - 1$ or not, we could have get better bounds for $y$. The given one, however, is sufficient for our purpose.

Thus, we can write the first $n$ digits of $\omega$ in the form

$$\omega|_n = \omega_1 \circ \omega_2 \circ \cdots \circ \omega_{i-1} \circ \omega_i \circ \cdots \circ \omega_{i+1} \circ \gamma$$

with $y = |\gamma|$.

For a word $b$, let

$$\phi_n(b) = \sum_{k=1}^{i} M_k \nu_k(b) + m \nu_{i+1}(b).$$
Since \((\omega_i)_{i=1}^\infty\) is \((W, \mu)\)-good, we have that \(\lim_{n \to \infty} \frac{\phi_n(b)}{n} = \mu(b)\). Therefore \(\omega\) is 
\(\mu\)-normal if and only if
\[
(3.1) \quad \lim_{n \to \infty} \frac{N_n(b, \omega)}{\phi_n(b)} = 1
\]
for all words \(b \in \mathcal{D}_\mu\).

For a given word \(b\) of supported length \(k \in R(W)\), the following lemma, which
is proved identically to Lemma 2.1 and Lemma 2.2 in [13], provides us with upper
and lower bounds for \(N_n(b, \omega)\).

**Lemma 3.1.** If \(k \leq k_i\) and \(b \in \mathcal{D}_{v_i,k}\), then
\[N_n(b, \omega) \leq L_{i-1} + (1 + \epsilon_i) \nu_i(b) l_i |\omega_i| + (k + j)(l_i + 1) + (1 + \epsilon_{i+1}) \nu_{i+1}(b) |\omega_{i+1}| + k + j) x + y\]
and
\[N_n(b, \omega) \geq (1 - \epsilon_i) \nu_i(b) l_i |\omega_i| + (1 - \epsilon_{i+1}) \nu_{i+1}(b) a |\omega_{i+1}|.\]

We will use the following rational functions, defined on \(\mathbb{R}_0^+ \times \mathbb{R}_0^+\), to estimate
the distance from the limit in (3.1) with respect to \(i, x\) and \(y\) (all depending on \(n\))
from below and above, respectively:
\[f_{i,b}(x, y) = \frac{\phi_{l_{i+1}}(b) + \epsilon_i \nu_i(b) l_i |\omega_i| + \nu_{i+1}(b) (\epsilon_{i+1} |\omega_{i+1}| + \{u_{\omega_{i+1}, \omega_{i+1}} \} x + \nu_{i+1}(b) y)}{\phi_{L_i}(b) + \nu_{i+1}(b) (|\omega_{i+1} u_{\omega_{i+1}, \omega_{i+1}}| x + y)}\]
\[g_{i,b}(x, y) = \frac{L_{i-1} + (\epsilon_i \nu_i(b) |\omega_i| + (k + j) l_i + (\epsilon_{i+1} \nu_{i+1}(b) |\omega_{i+1}| + (k + j)) x + y}{\phi_{L_i}(b) + \nu_{i+1}(b) (|\omega_{i+1} u_{\omega_{i+1}, \omega_{i+1}}| x + y)}\]

**Lemma 3.2.** Let \(k \in R(W)\), \(k \leq k_i\), and \(b \in \mathcal{D}_{v_i,k}\). Then
\[
(3.2) \quad \left| \frac{N_n(b, \omega)}{\phi_n(b)} - 1 \right| < g_{i,b}(a, b).
\]

**Proof.** Using the upper and lower bounds from Lemma 3.1 on \(N_n(b, \omega)\), we arrive
at the bound
\[-f_{i,b}(x, y) \leq \frac{N_n(b, \omega)}{\phi_n(b)} - 1 \leq g_{i,b}(x, y).
\]
So,
\[
\left| \frac{N_n(b, \omega)}{\phi_n(b)} - 1 \right| < \max(f_{i,b}(x, y), g_{i,b}(x, y)).
\]
However, since the numerator of \(g_{i,b}(x, y)\) is clearly greater than the numerator
of \(f_{i,b}(x, y)\) and their denominators are the same we conclude that \(f_{i,b}(x, y) < g_{i,b}(x, y)\). Therefore,
\[
\left| \frac{N_n(b, \omega)}{\phi_n(b)} - 1 \right| < g_{i,b}(x, y).
\]
\[\square\]

We will want to find a good bound for \(g_{i,b}(x, y)\) where \((x, y)\) ranges over values
in \(\{0, 1, \ldots, l_{i+1}\} \times \{0, 1, \ldots, |\omega_{i+1}| - 1\}\). Put
\[
\tau_{W, b, i} = \sup \{0, \sup \{t : \nu_{i+1}(b) \geq \nu_t(b)\}\}.
\]
Note that \(\tau_{W, b, i} \leq \infty\) as \((\nu_t(b))\) is eventually non-increasing. Set
\[
\eta_{W, b, i} = \max(0, L_{\tau_{W, b, i}}, \nu_{i+1}(b) - \phi_{L_i}).
\]
Thus,
\begin{equation}
\nu_{i+1}(b)l_{i-1} \leq \phi_{L_{i-1}}(b) + \eta_{W,b,i}.
\end{equation}

Lemma 3.3. If \( k \in R(W), |\omega_i| > 2(k + j) + \frac{2\nu_i(b)}{\nu_i(b)}, |\omega_{i+1}| > \frac{k+j}{\nu_{i+1}(b)(\epsilon_i - \epsilon_{i+1})}, \epsilon_i < 1/2, l_i > 0, b \in \mathcal{D}_{\nu_{i+1}(b)} \leq \mu_i(b), \) and
\begin{equation}
(x, y) \in \{0, 1, \ldots, l_i\} \times \{0, 1, \ldots, |\omega_i| + j - 1\},
\end{equation}
then
\begin{equation}
g_{i,b}(x, y) < g_{i,b}(0, |\omega_{i+1}| + j) = \frac{(L_{i-1} + \nu_i(b)_l_i |\omega_i| + (k+j)l_i) + |\omega_{i+1}| + j}{\phi_{L_{i}}(b) + \nu_{i+1}(b) (|\omega_{i+1}| + j)}. \tag{3.5}
\end{equation}

Proof. We note that \( g_{i,b}(x, y) \) is a rational function of \( x \) and \( y \) of the form
\begin{equation}
g_{i,b}(x, y) = \frac{C + Dx + Ey}{F + Gx + Hy}
\end{equation}
where
\begin{align*}
C &= L_{i-1} + \nu_i(b)_l_i |\omega_i| + (k+j)l_i, \\
D &= \nu_{i+1}(b)_l_i |\omega_{i+1}| + (k+j), \\
E &= 1, \\
F &= \phi_{L_{i}}(b), \\
G &= \nu_{i+1}(b)_l_i |\omega_{i+1}|, \\
H &= \nu_{i+1}(b).
\end{align*}

We will show that if we fix \( y \), then \( g_{i,b}(x, y) \) is a decreasing function of \( x \) and if we fix \( x \), then \( g_{i,b}(x, y) \) is an increasing function of \( y \). To see this, we compute the partial derivatives:
\begin{align*}
\frac{\partial g_{i,b}}{\partial x}(x, y) &= \frac{D(F + Gx + Hy) - G(C + Dx + Ey)}{(F + Gx + Hy)^2} = \frac{D(F + Hy) - G(C + Ey)}{(F + Gx + Hy)^2}, \\
\frac{\partial g_{i,b}}{\partial y}(x, y) &= \frac{E(F + Gx + Hy) - H(C + Dx + Ey)}{(F + Gx + Hy)^2} = \frac{E(F + Gx) - H(C + Dx)}{(F + Gx + Hy)^2}.\end{align*}

Thus, the sign of \( \frac{\partial g_{i,b}}{\partial x}(x, y) \) does not depend on \( x \) and the sign of \( \frac{\partial g_{i,b}}{\partial y}(x, y) \) does not depend on \( y \). We will first show that \( g_{i,b}(x, y) \) is an increasing function of \( y \) by verifying that
\begin{equation}
E(F + Gx) > H(C + Dx). \tag{3.6}
\end{equation}

Let \( \phi^*_L(b) = HC = \nu_{i+1}(b)(L_{i-1} + \nu_i(b)_l_i |\omega_i| + (k+j)l_i) \). Then Equation (3.6) can be written as
\begin{equation}
\phi_{L_{i}}(b) + \left[ \nu_{i+1}(b)|\omega_{i+1}|x \right] > \phi^*_L(b) + \left[ \nu_{i+1}(b)(\epsilon_i + \nu_{i+1}(b)|\omega_{i+1}| + (k+j)x \right]. \tag{3.7}
\end{equation}

We will verify this inequality in two steps by showing
\begin{align*}
\phi_{L_{i}}(b) > \phi^*_L(b) \quad \text{and} \quad \nu_{i+1}(b)|\omega_{i+1}|x > \nu_{i+1}(b)(\epsilon_i + \nu_{i+1}(b)|\omega_{i+1}| + (k+j)x).
\end{align*}

In order to show that \( \phi_{L_{i}}(b) > \phi^*_L(b) \), we first note that
\begin{equation}
\phi_{L_{i}}(b) = \phi_{L_{i-1}}(b) + \nu_{i}(b) ((l_i - 1)|\omega_{i}|u_{i-1,\omega_i,\omega_{i+1}} + |\omega_{i}u_{i,\omega_i,\omega_{i+1}}|) \geq \phi_{L_{i-1}}(b) + \nu_{i}(b)|\omega_i|. \tag{3.8}
\end{equation}

But by (3.3), we only need to show that
\begin{equation}
\nu_{i}(b)|\omega_i| > \nu_{i+1}(b)(\epsilon_i + \nu_{i}(b)|\omega_i| + (k+j)l_i) + \eta_{W,b,i}. \tag{3.8}
\end{equation}

However, by rearranging terms, (3.8) is equivalent to
\begin{equation}
|\omega_i| > \frac{\nu_{i+1}(b)}{\nu_{i}(b)} \cdot \frac{1}{1 - \nu_{i+1}(b)\epsilon_i} \cdot (k + j) + \frac{\eta_{W,b,i}}{l_i\nu_{i}(b)(1 - \nu_{i+1}(b)\epsilon_i)}. \tag{3.9}
\end{equation}
Since \( \epsilon_i < 1/2 \), we know that 
\((1 - \nu_{i+1}(b)\epsilon_i)^{-1} < 2 \). Additionally, \( \nu_{i+1}(b) \leq \nu_i(b) \) implies
\[
\frac{\nu_{i+1}(b)}{\nu_i(b)} \leq 1.
\]
Therefore,
\[
\begin{align*}
\frac{l_i + 1}{l_i} \cdot \frac{\nu_{i+1}(b)}{\nu_i(b)} \cdot \frac{1}{1 - \nu_{i+1}(b)\epsilon_i} \cdot (k + j) + \frac{\eta_{i+1,b}}{\nu_i(b)(1 - \nu_{i+1}(b)\epsilon_i)} & \leq 2 \cdot (k + j) + \frac{2\eta_{i+1,b}}{\nu_i(b)}.
\end{align*}
\]
But, \(|\omega_i| > 2(k + j) + \frac{2\eta_{i+1,b}}{\nu_i(b)} \). So (3.9) is satisfied and thus \( \phi_{L_i}(b) > \phi_i^*(b) \).

The last step one the way to verifying (3.7) is to show that\[
\nu_{i+1}(b)|\omega_{i+1}| x \geq \nu_{i+1}(b)(\epsilon_{i+1}\nu_{i+1}(b)|\omega_{i+1}| + (k + j)) x.
\]
However, this is equivalent to
\[
|\omega_{i+1}| \geq \frac{1}{1 - \nu_{i+1}(b)\epsilon_{i+1}} \cdot (k + j).
\]
Similar to above, we will verify this in two steps:

- \( (L_{i-1} + \epsilon_i \nu_i(b) + (k + j)\nu_{i+1}(b)|\omega_{i+1}| + (k + j)) \phi_{L_{i-1}}(b) > (\epsilon_{i+1}\nu_{i+1}(b)|\omega_{i+1}| + (k + j)) \phi_{L_i}(b) \)

Since \( 0 \leq |u_{\mathbf{a},b}| \leq j \) it suffices to show that
\[
L_{i-1}\nu_{i+1}(b)|\omega_{i+1}| + (k + j)\nu_{i+1}(b)|\omega_{i+1}| + (k + j)\nu_{i+1}(b)|\omega_{i+1}| + (k + j) \nu_{i+1}(b)|\omega_{i+1}| + (k + j).
\]

Similar to above, we will verify this in two steps:

\[
(\epsilon_{i+1}\nu_{i+1}(b)|\omega_{i+1}| + (k + j)) \phi_{L_{i-1}}(b) \]

Since \( L_{i-1} > \phi_{L_{i-1}}(b) \), in order to prove the first inequality of (3.11), it is enough to show that
\[
\nu_{i+1}(b)|\omega_{i+1}| > \epsilon_{i+1}\nu_{i+1}(b)|\omega_{i+1}| + (k + j),
\]
which is equivalent to
\[
|\omega_{i+1}| > \frac{k + j}{\nu_{i+1}(b)(1 - \epsilon_{i+1})}.
\]
But $\epsilon_i < 1/2$, so

$$\frac{k + j}{\nu_{i+1}(b)(1 - \epsilon_i)} < \frac{k + j}{\nu_{i+1}(b)(\epsilon_i - \epsilon_{i+1})} < |\omega_{i+1}|.$$  

To verify the second inequality of (3.11) we note that this is equivalent to

$$|\omega_{i+1}| > \frac{k + j}{\nu_{i+1}(b)(\epsilon_i - \epsilon_{i+1})},$$

which is given in the hypotheses.

So, we may conclude that $g_{i, b}(x, y)$ is a decreasing function of $x$ and an increasing function of $y$. Since $x \geq 0$ and $y < |\omega_{i+1}| + j$, we achieve the given upper bound by setting $x = 0$ and $y = |\omega_{i+1}| + j$. □

Set

$$\epsilon'_i = g_{i, b}(x, y) < g_{i, b}(0, |\omega_{i+1}| + j) = \left(\frac{L_{i} - \epsilon_i \nu_i(b)}{\nu_{i+1}(b)} \right) |\omega_{i}| + (k + j) \frac{|\omega_{i}| + j)}{\phi_{i} L_{i}(b) + \nu_{i+1}(b)}.$$  

Thus, under the conditions of Lemma 3.2 and Lemma 3.3

(3.12) $$\left| \frac{N_n(b, \omega)}{\phi_n(b)} - 1 \right| < \epsilon'_i$$

The proof of the following lemma is essentially identical to the combined proofs of Lemma 2.6, Lemma 2.7, and Lemma 2.8 in [13] so the proof has been omitted.

**Lemma 3.4.** If $k \in R(W)$, then $\lim_{i \to \infty} \epsilon'_i = 0$.

**Proof of Main Theorem** Let $b \in D_{\mu, k}$ for $k \in R(W)$. Since $\frac{1}{\epsilon_i - \epsilon_{i+1}} = o(|\omega_i|)$, there exists $n$ large enough so that $|\omega_i|$ and $|\omega_{i+1}|$ satisfy the hypotheses of Lemma 3.3.

Since $\lim_{n \to \infty} i(n) = \infty$, we conclude by applying Lemma 3.4 in (3.12) that

$$\lim_{n \to \infty} \left| \frac{N_n(b, \omega)}{\phi_n(b)} - 1 \right| = 0$$

which implies that

$$\lim_{n \to \infty} \frac{N_n(b, \omega)}{n} = \mu(b).$$

On the contrary let $b \in A^k \setminus D_{\mu, k}$. Since

$$1 = \lim_{n \to \infty} \sum_{b' \in A^k} \frac{N_n(b', \omega)}{n}$$

$$= \sum_{b' \in D_{\mu, k}} \lim_{n \to \infty} \frac{N_n(b', \omega)}{n} + \sum_{b' \in A^k \setminus D_{\mu, k}} \lim_{n \to \infty} \frac{N_n(b', \omega)}{n}$$

$$= \sum_{b' \in D_{\mu, k}} \mu(b') + \sum_{b' \in A^k \setminus D_{\mu, k}} \lim_{n \to \infty} \frac{N_n(b', \omega)}{n}$$

$$= 1 + \sum_{b' \in A^k \setminus D_{\mu, k}} \lim_{n \to \infty} \frac{N_n(b', \omega)}{n}$$

and $N_n(b', \omega) \geq 0$ we get that

$$\lim_{n \to \infty} \frac{N_n(b, \omega)}{n} = 0 = \mu(b).$$
Therefore combining the two limits from above we get for \( b \in A^k \) that
\[
\lim_{n \to \infty} \frac{N_n(b, \omega)}{n} = \mu(b),
\]
which implies that \( \omega \in N_{\mu,k} \).

\[\square\]

4. The construction

At first sight our construction is very similar to the Champernowne type construction of Bertrand-Mathis and Volkmann [3, 4]. However, in our case we construct an infinite word which is generic for an arbitrary measure, instead of the maximal one.

Therefore we have to face two main issues in our construction. The first one is that our digit set might be infinitely large. This we can easily circumvent by increasing the digit set in every step (i.e. in every \( w_i \)). The other issue we have to face is that there might be restrictions on the concatenation of words. For example, if we take the golden mean as basis of a \( \beta \)-expansion, two successive ones are forbidden in the expansion. However, concatenating 1001 and 1010, which are admissible as such, yields the word 10011010, which is not admissible. Therefore similar to above we want to use the specification property in order to glue the words together.

Therefore, as above, let \( j \) be the maximum size of the padding given by the specification property and let \( P_{b,w} = \{p_1, \ldots, p_{b^w}\} \) be the set of all possible words of length \( w \) of the alphabet \( A = \{0, 1, \ldots, b-1\} \) of digits in base \( b \). Furthermore let \( m_k = \min\{\mu(b) : b \in D_{\mu,k}\} \) for \( k \geq 1 \) and \( M \) be an arbitrary large constant such that \( M \geq \frac{1}{m_w} \).

The central tool for our construction will be a weighted concatenation of the words \( \tilde{p}_i \), i.e.,
\[ p_{b,w,M} := p_1 \circ [M\mu(p_1)] \circ p_2 \circ [M\mu(p_2)] \circ \cdots \circ p_{b^w} \circ [M\mu(p_{b^w})]. \]

In the following we want to show the \((\varepsilon, k)\)-normality of \( p_{b,w,M} \). Thus it suffices to show that for all words \( b \) of length \( m \leq k \) we have
\[
(1 - \varepsilon)\mu(b) \leq \frac{N(b, p_{b,w,M})}{|p_{b,w,M}|} \leq (1 + \varepsilon)\mu(b) \tag{4.1}
\]

To this end we need lower and upper bounds for the length of \( p_{b,w,M} \) as well as lower and upper bounds for the number of occurrences of a fixed block within \( p_{b,w,M} \).

Starting with the estimation of the length of \( p_{b,w,M,j} \) we get as upper bound
\[
|p_{b,w,M}| \leq \sum_{i=1}^{b^w} \lfloor M\mu(p_i) \rfloor (j + w) \leq M(j + w) \sum_{i=1}^{b^w} \mu(p_i) + (j + w)b^w = (j + w)(M + b^w).
\]

On the other hand we obtain as lower bound
\[
|p_{b,w,M}| \geq \sum_{i=1}^{b^w} \lfloor M\mu(p_i) \rfloor w \geq Mw \sum_{i=1}^{b^w} \mu(p_i) = Mw.
\]

Now we want to give upper and lower bounds for the number of occurrences of a word \( b \) of length \( k \) in \( p_{b,w,M} \).
• **Lower bound.** For the lower bound we only count the possible occurrences within a $p_i$. If there is an occurrence then we can write $p_i$ as $c_1b_{c_2}$ with possible empty $c_1$ or $c_2$. Since the word $b$ is fixed, we let $c_1$ and $c_2$ vary over all possible words. Thus

$$N(b, p_{b,w,M}) \geq \sum_{m=0}^{w-k} \sum_{|c_1|=m} \sum_{|c_2|=w-k-m} \left[ M\mu(c_1b_{c_2}) \right]$$

$$\geq M \sum_{m=0}^{w-k} \sum_{|c_1|=m} \sum_{|c_2|=w-k-m} \mu(c_1b_{c_2})$$

$$= M \sum_{m=0}^{w-k} \sum_{|c_1|=m} \sum_{|c_2|=w-k-m-1} \sum_{d=0}^{b-1} \mu(c_1b_{c_2}d)$$

$$= \cdots = M \sum_{m=0}^{w-k} \mu(b) = (w-k+1)M\mu(b),$$

where we have used the shift invariance of $\mu$, i.e. $\sum_{d=0}^{b-1} \mu(da) = \sum_{d=0}^{b-1} \mu(ad) = \mu(a)$.

• **Upper bound.** For the upper bound we have to consider several different possibilities: The word $b$ can occur

1. within $p_i$,
2. between two similar words $p_i \odot p_i$ or
3. between two different words $p_i \odot p_{i+1}$.

If the word $b$ is completely within $p_i$, then we have that $p_i = c_1b_{c_2}$ with possible empty $c_1$ or $c_2$. By using similar means as above we get that

$$\sum_{c_1, c_2} \left[ M\mu(c_1b_{c_2}) \right] \leq \sum_{c_1, c_2} (M\mu(c_1b_{c_2}) + 1) = \cdots = (w-k+1) (M\mu(b) + b^{\ell-k}),$$

Now we turn our attention to the number of occurrences between two consecutive words. First we assume that these words are equal. Let $\ell = |p_i \odot p_i|$ be the length of the resulting word. Then $p_i \odot p_i = c_1b_{c_2}$ with $w-k+1 \leq |c_1| \leq \ell - w + k - 1$. Thus similar to above we get that there are

$$\sum_{m=w-k+1}^{\ell-w+k-1} \sum_{|c_1|=m} \sum_{|c_2|=\ell-k-m} \left[ M\mu(c_1b_{c_2}) \right]$$

$$\leq M \sum_{m=w-k+1}^{\ell-w+k-1} \sum_{|c_1|=m} \sum_{|c_2|=\ell-k-m-1} \sum_{d=0}^{b-1} \mu(c_1b_{c_2}d) + \sum_{m=w-k+1}^{\ell-w+k-1} b^{\ell-k}$$

$$= \cdots = M \sum_{m=w-k+1}^{\ell-w+k-1} \mu(b) + (\ell - 2w + k - 1)b^{\ell-k}$$

$$= (\ell - 2w + k - 1) (M\mu(b) + b^{\ell-k})$$

$$\leq (j + k - 1) (M\mu(b) + b^{2w+j-k})$$

occurrences between two identical words.
Finally, we trivially estimate the number of occurrences between two different words by their total amount, which is \( \leq (j + k - 1) b^w \).

Combining these three bounds and using \( k \leq w \) we get as upper bound for the number of occurrences

\[
N(b, p_{b,w,M}) \leq (w - k + 1) \left( M \mu(b) + b^{w-k} \right) + (j + k - 1) \left( M \mu(b) + b^{2w+j-k} \right) + (j + k - 1) b^w.
\]

Now we calculate \( \varepsilon \) such that (5.1) holds. Using our lower bound for the number of occurrences together with our upper bound for length we get that

\[
\frac{N(b, p_{b,w,M})}{|p_{b,w,M}|} \geq \frac{(w - k + 1) M \mu(b)}{(w + j) (M + b^w)} \geq \mu(b) \left( 1 - \frac{\varepsilon}{w} \right) \left( 1 - \frac{b^w}{M + b^w} \right)
\]

which implies for \( \varepsilon \) the upper bound

\[
\varepsilon \leq \frac{j + k - 1}{w + j} + \frac{b^w}{M + b^w}.
\]

On the other side an application of the upper bound for the number of occurrences together with the lower bound for the length yields

\[
\frac{N(b, p_{b,w,M})}{|p_{b,w,M}|} \leq \mu(b) \left( 1 + \frac{j}{w} \right) \left( 1 + \frac{j + k - 1}{w + j} \right) \left( 1 + \frac{b^w}{M + b^w} \right) \left( 1 + \frac{j}{w} \right)
\]

Putting these together we get that \( p_{b,w,M} \) is \((\varepsilon, k)\)-normal for

\[
k \leq w \quad \text{and} \quad \varepsilon \leq \max \left( \frac{j + k - 1}{w + j} + \frac{b^w}{M + b^w}, \frac{j}{w} \right)
\]

5. Applications

In the following subsections we show different number systems in which our construction provides normal numbers. In particular, we consider the \( q \)-ary expansions, Lüroth series expansion, beta-expansions and continued fraction expansion. We only have restrictions on the concatenation in the case of \( \beta \)-expansions; all other examples are in the full-shift. Moreover the authors are not aware of any example with an infinite digit set and restrictions on the concatenation.

5.1. \( q \)-expansion. Let \( A = \{0, 1, \ldots, q-1\} \). In this example we take as language the full-shift \( A^* \) and therefore we do not have any restrictions on the concatenation, \( i.e. j = 0 \). We will use the following lemma which follows immediately from Main Theorem 2.1 and the previous section:

Lemma 5.1. Let \( W = (\{l_i, \epsilon_i, k_i, \mu_i\})_{i=1}^\infty \in \mathcal{F} \) and \( (\omega_i)_{i=1}^\infty \) be a \((W, \mu)\)-good sequence. Suppose that \( q_i \geq 2 \) and \( M_i \) are sequences of positive integers such that \( M_i \geq \min \{ \mu(b) : b \in D_{\mu, i} \} \) and

\[
q_i^{l_i} = o(M_i)
\]

If \( \omega_i = p_{q_i, l_i, M_i} \), then \( \omega = \omega_1^{\otimes l_1} \otimes \omega_2^{\otimes l_2} \otimes \cdots \in N_\mu \).

Let

\[
\nu_i(t) = \begin{cases} \frac{4}{b} & \text{if } 0 \leq t \leq b - 1 \\ 0 & \text{if } t \geq b \end{cases}
\]

.
For $b = b_1 \ldots b_k$, define $\nu_i(b) = \prod_{l=1}^k \nu_i(b_l)$ and let $\mu = \nu_1$. Let $q_i = b$, $M_i = b^{2i} \log i$, $l_i = i^{2i}$, and put $\omega_i = \nu_{b_i,i,M_i}$, so $ib^{2i} \log i \leq |\omega_i| \leq ib^{2i} \log i + ib^i$. A short computation shows that (2.1), (2.2), (2.3), and (5.1) hold with $\epsilon_i = 1/\sqrt{i}$. Thus, by Lemma 5.1 the numbers whose digits of its $b$-ary expansion are formed by $\omega_1^{(i_1)} \odot \omega_2^{(i_2)} \odot \cdots$ is normal in base $b$.

We note that the constructions of Bertrand-Mathis and Volkmann [3,4] are more effective (no repetitions) than ours. However, the main aim of our construction is to generate arbitrary shift invariant measures. We think that a more careful control of the available words and their distribution, would lead to a reduction in the number of copies $l_i$.

5.2. Lüroth series expansion. \footnote{This example may be modified to construct normal numbers with respect to Generalized Lüroth series expansions (see \cite{9} for a definition of these expansions.)} Put

$$\nu_i(t) = \begin{cases} 0 & t = 0, 1 \\ \frac{1}{t(i-1)} & 2 \leq t \leq i + 1 \\ \frac{1}{i(i-1)} & t = i + 2 \\ 0 & t > i + 2 \end{cases}$$

and

$$\mu(t) = \begin{cases} 0 & i = 0, 1 \\ \frac{1}{t(i-1)} & t \geq 2 \end{cases}$$

For $b = b_1 \ldots b_k$, define $\nu_i(b) = \prod_{l=1}^k \nu_i(b_l)$ and $\mu(b) = \prod_{l=1}^k \mu(b_l)$. Clearly, $\nu_i \to \mu$. Next, we let $j = 0$, $q_i = i + 2$, $M_i = \max(3t^2, i^{2i} \log i)$, $l_i = \lfloor i^2 \log i \rfloor$, and $\omega_i = \nu_{i+2,i,M_i}$. Note that for all $i \geq 1$

$$M_i \geq (i + 1)!^2 \left(\min\{\mu(b) : b \in D_{\nu_i,i}\}\right)^{-1}.$$  

Conditions (2.1), (2.2), (2.3), and (5.1) hold. Thus, by Lemma 5.1 the numbers whose digits of its Lüroth series expansion are formed by $\omega_1^{(i_1)} \odot \omega_2^{(i_2)} \odot \cdots$ is normal with respect to the Lüroth series expansion.

5.3. $\beta$-expansions. Let $\beta > 1$. Then every number $x \in [0, 1)$ has a greedy $\beta$-expansions given by the greedy algorithm (cf. Renyi [19]): set $r_0 = x$, and for $n \geq 1$, let $d_n = \lfloor \beta r_{n-1} \rfloor$ and $r_n = \{\beta r_{n-1}\}$. Then

$$x = \sum_{n \geq 1} d_n \beta^{-n},$$

where the $d_n$ are integer digits in the alphabet $A_\beta = \{0, 1, \ldots, \lfloor \beta \rfloor - 1\}$. We denote by $d(x) = d_1 d_2 d_3 \ldots$ the greedy $\beta$-expansion of $x$.

Let $D_\beta$ denote the set of greedy $\beta$-expansions of numbers in $[0, 1)$. A finite (resp. infinite) word is called $\beta$-admissible if it is a factor of an element (resp. an element) of $D_\beta$. Not every number is $\beta$-admissible and the $\beta$-expansion of 1 plays a central role in the characterization of all admissible sequences. Let $d_\beta(1) = b_1 b_2 \ldots$ be the greedy $\beta$-expansion of 1. Since the expansion might be finite we define the quasi-greedy expansion $d_\beta^*(1)$ by

$$d_\beta^*(1) = \begin{cases} (b_1 b_2 \ldots b_{t-1} (b_t - 1))^w & \text{if } d_\beta(1) = b_1 b_2 \ldots b_t \text{ is finite,} \\ d_\beta(1) & \text{otherwise.} \end{cases}$$

Then Parry [15] could show the following
Lemma 5.1. Let \( \beta > 1 \) be a real number, and let \( s \) be an infinite sequence of non-negative integers. The sequence \( s \) belongs to \( D_\beta \) if and only if for all \( k \geq 0 \)

\[
\sigma^k(s) < d_\beta^*(1),
\]

where \( \sigma \) is the shift transformation.

According to this result we call a number \( \beta \) such that \( d_\beta(1) \) is eventually periodic a Parry number. In the present example we assume that \( \beta \) is such a number.

For the padding size we denote by \( d_\beta(1) = b_1 \ldots b_k(b_{k+1} \ldots b_{t+p})^w \) the \( \beta \)-expansion of \( 1 \). If \( 1 \) has a finite expansion then we set \( p = 0 \). We are looking for the longest possible sequence of zeroes occurring in the expansion of \( 1 \). As one easily checks, the longest occurs if \( b_1 = \cdots = b_{t+p-1} = 0 \) and \( b_{t+p} \neq 0 \). Thus we can set the padding size \( j \) to be

\[
j = t + p.
\]

We wish to minimize the length of a cylinder set defined by a word of length \( w \). Define

\[
\phi_\beta(w) = \begin{cases} 1 & \text{if } 1 \leq w \leq t \\ r & \text{if } t + (r - 2)p \leq w \leq t + (r - 1)p \end{cases}
\]

Then the length of this interval is at least \( \beta^{-(t + \phi_\beta(w))p} \). We use the fact that \( \mu_\beta(I) \geq (1 - 1/\beta)\lambda(I) \) and put

\[
M_i = \max \left( \frac{\beta^{t + \phi_\beta(i)p}}{1 - \frac{1}{\beta}}, [\beta]^{2i} \log i \right).
\]

Put \( \omega_i = p_{[\beta],i,M_i} \) and \( q_i = [\beta] \). Note that \( \lim_{i \to \infty} \frac{\phi(i)}{it} = 1 \), so for large \( i \)

\[
(i + j)[\beta]^{2i} \log i \leq |\omega_i| \leq (i + j) ([\beta]^{2i} \log i + [\beta]^i)
\]

Thus, for large \( i \)

\[
|\omega_i| \approx i[\beta]^{2i} \log i.
\]

Put \( l_i = i^{2i} \) and the computation follows the same lines as above.

5.4. **Continued fraction expansion.** For a word \( b = b_1 \ldots b_k \), let \( \Delta_b \) be the set of all real numbers in \((0,1)\) whose first \( k \) digits of it’s continued fraction expansion are equal to \( b \). Put

\[
\mu(b) = \frac{1}{\log 2} \int_{\Delta_b} \frac{dx}{1 + x}.
\]

If there is an index \( n \) such that \( b_n > i \), then let \( \nu_i(b) = 0 \). Let \( S = \{ n : b_n = i \} \).

For \( i < 8 \), set \( \nu_i(b) = \mu(b) \). For \( i \geq 8 \), if \( S = \emptyset \), then let \( \nu_i(b) = \mu(b) \). If \( S \neq \emptyset \), then let

\[
\nu_i(b) = \sum_{b'} \mu(b'),
\]

where the sum is over all words \( b' = b'_{1} \ldots b'_{k} \) such that for each index \( n \) in \( S \), \( b'_{n} \geq i \).

Put \( m_i = \min_{b \in D_i} \nu_i(b) \). We wish to find a lower bound for \( m_i \). If \( b = b_1 \ldots b_k \), then let

\[
\frac{p_k}{q_k} = \frac{1}{b_1 + \frac{1}{b_2 + \ldots + \frac{1}{b_k}}}.
\]
It is well known that $\lambda(\Delta_b) = \frac{1}{q_k(q_k + q_{k-1})}$ and $\mu(b) > \frac{1}{2\log 2}\lambda(\Delta_b)$.

Thus, we may find a lower bound for $m_i$ by minimizing $\frac{1}{q_i(q_i + q_{i-1})}$ for words $b$ in $D_\nu$. The minimum will occur for $b = ii\ldots i$. It is known that $q_n = iq_{n-1} + q_{n-2}$ if we set $q_0 = 1$ and $q_1 = i$. Set

$$r_1 = \frac{i + \sqrt{i^2 + 4}}{2}, r_2 = \frac{i - \sqrt{i^2 + 4}}{2}.$$ 

Then

$$q_n = \frac{r_1^{n+1} - r_2^{n+1}}{i^2 + 4}.$$ 

Thus,

$$\frac{1}{q_i(q_i + q_{i-1})} = \frac{i^2 + 4}{(r_1^{i+1} - r_2^{i+1})(r_1^{i+1} + r_1^i) - (r_2^{i+1} - r_2^i)} > \frac{\log 2}{2i^2}$$

for $i \geq 8$.

Thus, $m_i > \frac{1}{2\log 2} \frac{\log 2}{2i^2} = \frac{1}{2i^2}$. Let $M_i = 2i^{2l} \log i$, $j = 0$, $\omega_j = p_{i+1,i,M_i,0}$. Set $l_i = 0$ for $i < 8$ and $l_i = \lfloor i^2 \log i \rfloor$ for $i \geq 8$. Then for $i \geq 9$

$$\frac{l_i - 1}{i} \leq \frac{2(i - 1)^{2i-1} + i^{i-1}}{2i^{2l_i}} = \left(1 - \frac{1}{i}\right)^{i^2} \frac{1}{i - 1} + \frac{1}{2i^{l+1}} \to 0$$

and

$$\frac{l_i + 1}{i} \leq \frac{2(i + 1)^{2i+3} + (i + 2)^{i+1}}{i^2 \log i \cdot 2i^{2l_i}} = \left(1 + \frac{1}{i}\right)^{i^2} \frac{(i + 1)^3}{i^2 \log i} + o(1^{-i}) \to 0.$$ 

By Lemma 5.1, the number whose digits of its continued fraction expansion are formed by $\omega_1^{\omega_1^{\omega_1}} \circ \omega_2^{\omega_2^{\omega_2}} \circ \ldots$ is normal with respect to the continued fraction expansion.

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**References**

[1] R. Adler, M. Keane, and M. Smorodinsky, *A construction of a normal number for the continued fraction transformation*, J. Number Theory 13 (1981), no. 1, 95–105. MR 602450 (82k:10070)

[2] C. Altomare and B. Mance, *Cantor series constructions contrasting two notions of normality*, Monatsh. Math. 164 (2011), no. 1, 1–22. MR 2827169 (2012i:11077)

[3] Anne Bertrand-Mathis, *Points génériques de Champernowne sur certains systèmes codes; application aux $\theta$-shifts*, Ergodic Theory Dynam. Systems 8 (1988), no. 1, 35–51. MR 93059 (89d:94032)

[4] Anne Bertrand-Mathis and Bodo Volkmann, *On $(\epsilon, k)$-normal words in connecting dynamical systems*, Monatsh. Math. 107 (1989), no. 4, 267–279. MR 1012458 (90m:11115)

[5] A. S. Besicovitch, *The asymptotic distribution of the numerals in the decimal representation of the squares of the natural numbers*, Math. Z. 39 (1935), no. 1, 146–156. MR MR1545494

[6] Patrick Billingsley, *Ergodic theory and information*, John Wiley & Sons Inc., New York, 1965. MR 0192027 (33 #254)

[7] E. Borel, *Les probabilités dénombrables et leurs applications arithmétiques*, Palermo Rend. 27 (1909), 247–271 (French).
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[8] D.G. Champernowne, *The construction of decimals normal in the scale of ten*, J. Lond. Math. Soc. 8 (1933), 254–260 (English).

[9] K. Dajani and C. Kraaikamp, *Ergodic theory of numbers*, Carus Mathematical Monographs, vol. 29, Mathematical Association of America, Washington, DC, 2002. MR 1917322 (2003f:37014)

[10] Manfred Denker, Christian Grillenberger, and Karl Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Mathematics, Vol. 527, Springer-Verlag, Berlin, 1976. MR 0457675 (56 #15879)

[11] J. M. Dumont, P. J. Grabner, and A. Thomas, *Distribution of the digits in the expansions of rational integers in algebraic bases*, Acta Sci. Math. (Szeged) 65 (1999), no. 3-4, 469–492. MR 1737265 (2001f:11132)

[12] S. Ito and I. Shiokawa, *A construction of $\beta$-normal sequences*, J. Math. Soc. Japan 27 (1975), 20–23. MR 0357361 (50 #9829)

[13] Bill Mance, *Construction of normal numbers with respect to the $Q$-Cantor series expansion for certain $Q$*, Acta Arith. 148 (2011), no. 2, 135–152. MR 2786161 (2012c:11153)

[14] , *Cantor series constructions of sets of normal numbers*, Acta Arith. 156 (2012), no. 3, 223–245. MR 2999070

[15] W. Parry, *On the $\beta$-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. 11 (1960), 401–416. MR 0142719 (26 #288)

[16] A. G. Postnikov, *Arithmetic modeling of random processes*, Trudy Mat. Inst. Steklov. 57 (1960), 84. MR 0148639 (26 #6146)

[17] A. G. Postnikov and I. I. Pyateckii, *A Markov-sequence of symbols and a normal continued fraction*, Izv. Akad. Nauk SSSR. Ser. Mat. 21 (1957), 729–746. MR 0101857 (21 #664)

[18] , *On Bernoulli-normal sequences of symbols*, Izv. Akad. Nauk SSSR. Ser. Mat. 21 (1957), 501–514. MR 0101856 (21 #663)

[19] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar 8 (1957), 477–493. MR 0097374 (20 #3843)

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