Quantum Period Finding with a Single Output Qubit – Factoring $n$-bit RSA with $n/2$ Qubits

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Abstract. We study quantum period finding algorithms such as Simon, Shor, and Ekerå-Håstad. For a periodic function $f$ these algorithms produce – via some quantum embedding of $f$ – a quantum superposition $\sum_x |x\rangle |f(x)\rangle$, which requires a certain amount of output bits that represent $|f(x)\rangle$. We show that we can lower this amount to a single output qubit by hashing $f$ down to a single bit. Namely, we replace the embedding of $f$ in quantum period finding circuits by several embeddings of hashed versions of $f$. We show that on expectation this modification only doubles the required amount of quantum measurements, while significantly reducing the total number of qubits. For example, for Simon’s period finding algorithm in some $n$-bit function $f$:

For the Ekerå-Håstad algorithm for factoring $n$-bit RSA our hashing reduces the required qubits from $(\frac{3}{2} + o(1))n$ down to $(\frac{1}{2} + o(1))n$.

Keywords: Quantum Period Finding, Simon, Shor, Ekerå-Håstad, Minimizing Qubits

1 Introduction

Although there is steady progress in constructing larger quantum computers, within the next years the number of quantum bits seems to be too limited for tackling problems of interesting size, e.g. for period finding applications in cryptography [8,7,9,14,12,6,13] Shor’s algorithm [16] for polynomial time factorization of $n$-bit numbers computes a superposition $\sum_x |x\rangle |f(x)\rangle$ with $2n$ input qubits representing the input $|x\rangle$ to $f$ (sometimes also called control qubits) and $n$ output qubits representing the output $|f(x)\rangle$ of the function (sometimes also called arithmetic qubits).

However, it may not be necessary to implement a full-fledged $3n$-qubit Shor algorithm in order to factor numbers or compute discrete logarithms. Quantum computers with a very limited number of qubits might still serve as a powerful oracle that assists us in speeding up classical computations. For instance, Bernstein, Biasse and Mosca [2] developed an algorithm that factors $n$-bit numbers with the help of only a sublinear amount of $n^{\frac{3}{2}}$ qubits in subexponential time that is (slightly) faster than the currently best known purely classical factorization algorithm.

Several other algorithms saved on the number of qubits in Shor’s algorithm by shifting some more work into a classical post-processing, while – in contrast to [2] – still preserving polynomial run time. Interestingly, all these algorithms concentrate on reducing the input qubits, while keeping $n$ output qubits. Seifert [15] showed that – using for the classical post-process simultaneous Diophantine approximations instead of continued fractions – the number of input qubits can be reduced from $2n$ to $(1 + o(1))n$. For $n$-bit RSA numbers, which are a product of two $n/2$-bit primes, Ekerå and Hästad [5] reduced the number of input qubits down to $(\frac{1}{2} + o(1))n$, using some variant of the Hidden Number Problem [3] in the post-process. Thus, the Ekerå-Hästad version of Shor’s algorithm factors $n$-bit RSA with a total of $(\frac{3}{2} + o(1))n$ qubits.

⋆ Funded by DFG under Germany’s Excellence Strategy - EXC 2092 CASA - 390781972.
Mosca and Ekert [11] showed that in principle one can reduce the number of input qubits even down to a single one, at the cost of an increased depth of Shor’s quantum circuit. However, we currently do not see how to combine the Mosca-Ekert algorithm with our new hashing technique.

**Our contribution.** We hash \( f(x) \) in the output qubits down to a single qubit. This can be realized using quantum embeddings of \( h \circ f \) for different hash functions \( h \). Our basic observation is that hashing preserves the periodicity of \( f \). Namely, if \( f(x) = f(x + s) \) for some period \( s \) and all inputs \( x \) then also

\[
    h(f(x)) = h(f(x + s)) \quad \text{for the period } s \text{ and all inputs } x.
\]

The drawback of hashing is of course that it introduces many more undesirable collisions \( h(f(x)) = h(f(x')) \) where \( x, x' \) are not a multiple of \( s \) apart. Surprisingly, even for 1-bit range hash functions this plethora of undesirable collisions does not at all affect the correctness of our quantum period finding algorithms, and only insignificantly increases their runtimes.

More precisely, concerning correctness we show that a replacement of \( f \) by some hashed version of \( f \) has the following effects.

**Simon’s algorithm:** In the input qubits, we still measure only vectors \( y \) that are orthogonal to the period \( s \). The amplitudes of all other inputs cancel out.

**Shor’s algorithm:** Let the period be \( d = 2^r \), and let us use \( q > r \) input qubits. Then we still measure in the input qubits only numbers \( y \) that are multiples \( 2^{q-r} \). The amplitudes of all other inputs cancel out. In the case of general (not only power of two) periods and in Ekerå-Hästad’s algorithm we measure all inputs \( y \neq 0 \) with exactly half the probability as without hashing.

Our correctness property immediately implies that the original post-processing in Simon’s algorithm (Gaussian elimination) and in Shor’s algorithm (e.g. continued fractions) can still be used in the hashed version of the algorithms for period recovery.

However, this does not imply that we achieve similar runtimes. Namely, in the original algorithms of Simon and Shor we measure all \( y \) having a non-zero amplitude with a uniform probability distribution. In Simon’s algorithm for some period \( s \in \mathbb{F}_2^n \) we obtain each of the \( 2^{n-1} \) many \( y \in \mathbb{F}_2^n \) orthogonal to \( s \) with probability \( 1/2^{n-1} \). In Shor’s algorithm with period \( d = 2^r \), we measure each of the \( d \) many possible multiples \( y \) of \( 2^{q-r} \) with probability \( 1/d \).

These uniform probability distributions are destroyed by moving to the hashed version of the algorithms. Since \( h(f(x)) = h(f(x')) \) for \( x \neq x' \) happens for universal 1-bit range hash functions with probability \( 1/2 \), the undesirable collisions put a probability weight of (roughly) \( 1/2 \) on measuring \( y \neq 0 \) in the input qubits.

This seems to be bad news, since neither in Simon’s algorithm does the zero vector \( y \) provide information about \( s \), nor does in Shor’s algorithmus the zero-multiple \( y \) of \( 2^{q-r} \) provide information about \( d \). However as good news, we show that besides putting probability weight \( 1/2 \) on \( y = 0 \), hashing does not destroy the probability distribution stemming from the amplitudes of quantum period finding algorithms. Namely, we show that for the whole class of quantum period finding circuits that we consider – including Simon, Shor and Ekerå-Hästad – the following result holds. If the probability to measure \( y \) is \( p(y) \) when using \( f \), then we obtain probability \( p(y)/2 \) (taken over the random choice of \( h \) from a family of universal hash functions) to measure \( y \) when using \( h \circ f \).

Put differently, if we condition on the event that we do *not* measure \( y = 0 \) in the input bits (which happens in roughly every second measurement) in both cases – using \( f \) itself or its hashed version \( h \circ f \) – we obtain exactly the same probability distribution for the measurements of any \( y \neq 0 \). This implies that our hashing approach preserves not only the correctness but also the
runtime analysis of any processing of the measured data in a classical post-process. Thus, at
the cost of only twice as many quantum measurements we save all but one of the output qubits.

In particular, we show that the original Simon algorithm \[18\] — that recovers for a periodic
function \( f : F^n_2 \to F^n_2 \) its period in time polynomial in \( n \) with expected \( n + 1 \) measurements using
\( 2n \) qubits — admits a hashed version with expected \( 2(n + 1) \) measurements using only \( n + 1 \) qubits.

The original Ekerå-Håstad version of Shor’s algorithm that computes discrete logarithms
d\in some abelian group \( G \) in polynomial time using \( (1 + o(1)) \log d + \log(|G|) \) qubits requires in its
hashed version only \( (1 + o(1)) \log d \) qubits. Moreover, the Ekerå-Håstad algorithm computes the
factorization of an RSA modulus \( N = pq \) of bit-size \( n \) in time polynomial in \( n \) using \( (\frac{3}{2} + o(1))n \) qubits,
whereas our hashed version reduces this to only \( (\frac{1}{2} + o(1))n \) qubits.

On the downside, our hashing technique requires that we obtain quantum embeddings of \( h \circ f \)
for several \( h \) chosen uniformly at random from a family of universal hash functions. Notice that it is
of course not sufficient to compute \( f \) first, and afterwards hash the result, since this would require
qubits for representing the full range of \( f \). It remains an interesting open problem, which classes of
functions \( f \) admit the memory-efficient computation of hashed versions, which is out of the scope
of our paper. However, we believe that our independent choices of \( h \) from a universal hash function
family is mainly a theoretical proof artefact. We conjecture that in practice a single \( h \) should still
work perfectly. Even choosing \( h \) simply as the projection of \( f \) to a single bit should work for most
functions of interest.

Our paper is organized as follows. The results for Simon’s algorithm are described in Section 3.
For didactic reasons, we study in Section 4 first the simple case of Shor’s algorithm for periods
that are a power of two. In Section 5 we generalize to any quantum circuits that fall in our period
finding class. As a consequence, in Section 5 we obtain a hashed version of Shor’s algorithm with
general periods and in Section 6 a hashed version of Ekerå-Håstad.

2 Preliminaries on Period Finding Algorithms

Let us first recall some quantum notation. The reversible quantum embedding of a classical function
\( f \) is defined as
\[
U_f : |x\rangle|y\rangle \mapsto |x\rangle|y + f(x)\rangle.
\]
The 1-qubit Hadamard gate realizes the mapping \( H_1 : |x\rangle \mapsto \frac{1}{\sqrt{2}} \sum_{y \in F_2^n} (-1)^{xy} |y\rangle \). Its \( n \)-qubit
version is defined as the \( n \)-fold tensor product \( H_n = \bigotimes_{i=1}^n H_1 \). The \( n \)-qubit Quantum Fourier
Transform (QFT) realizes the mapping
\[
\text{QFT}_n : |x\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{y \in F_2^n} e^{2\pi i \frac{xy}{2^n}} |y\rangle.
\]
Notice that \( \text{QFT}_1 = H_1 \).

**Definition 2.1.** A hash function family \( \mathcal{H}_t := \{ h : D \to \{0,1\}^t \} \) is universal if for all \( x,y \in D, x \neq y \) we have
\[
P_{h \in \mathcal{H}_t} [h(x) = h(y)] = \frac{1}{2^t}.
\]

Efficient instantiation of universal hash function families exist, e.g. \( \mathcal{H}_1 = \{ h_r : F^n_2 \to \{0,1\} \mid r \in F^n_2, h_r(x) = \sum_{i=1}^{n-1} x_ir_i \mod 2 \} \) is universal. It is easy to see that strongly 2-universal hash function families as defined in \[10\] are universal in the sense of
Definition 2.1.
3 Hashed-Simon

Let us briefly recall Simon’s original algorithm. Let \( f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n \) be periodic with period \( s \in \mathbb{F}_2^n \), that is \( f(x) = f(x + s) \) for all \( x \in \mathbb{F}_2^n \). We call \( f \) a Simon function if it defines a \((2 : 1)\)-mapping, i.e.

\[
f(x) = f(y) \iff (y = x) \text{ or } (y = x + s).
\]

The use of Simon functions allows for a clean theoretical analysis, although Simon’s algorithm works also for more general periodic functions as shown in [1,4,9]. For ease of notation, we will restrict ourselves to Simon functions.

The Simon circuit \( Q^{\text{Simon}}_f \) from Figure 1 uses \( n \) input and \( n \) output qubits for realizing the embedding of \( f \). It can easily be shown that in the \( n \) input qubits we measure only \( y \in \mathbb{F}_2^n \) such that \( y \perp s \), i.e. \( ys = 0 \).

![Fig. 1: Quantum circuit Q^{Simon}_f](image)

The Simon algorithm uses \( Q^{\text{Simon}}_f \) until we have collected \( n - 1 \) linearly independent vectors, from which we compute the unique vector \( s \) that is orthogonal to all of them using Gaussian elimination in time \( O(n^3) \).

Our HASHED-SIMON (Algorithm 1) is identical to the Simon algorithm with the only difference that \( Q^{\text{Simon}}_f \) is replaced by \( Q^{\text{Simon}}_{h \circ f} \), where in each iteration we instantiate \( Q^{\text{Simon}}_{h \circ f} \) with some hash function \( h \) freshly drawn from a universal \( t \)-bit range hash function family \( \mathcal{H}_t \). Notice that Simon can be considered as special case of HASHED-SIMON, where we choose \( t = n \) and the identity function \( h = \text{id} \). This slightly abuses notation, since \( \mathcal{H}_n = \{\text{id}\} \) is not universal. However, in the following Lemma 3.1 we do not need universality. In Lemma 3.1 we show the correctness property of HASHED-SIMON that by replacing \( Q^{\text{Simon}}_f \) with \( Q^{\text{Simon}}_{h \circ f} \), we still measure only \( y \) orthogonal to \( s \).

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**Algorithm 1: HASHED-SIMON**

Input: Simon function \( f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n \), universal \( \mathcal{H}_t := \{h : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n\} \).

\[ \triangleright t = n \text{ and } \mathcal{H} = \{\text{id}\} \text{ for Simon.} \]

Output: Period \( s \) of \( f \).

begin
1. Set \( Y = \emptyset \).

2. repeat
   3. Run \( Q^{\text{Simon}}_{h \circ f} \) on \( |0^n\rangle |0^t\rangle \) for some freshly chosen \( h \in_R \mathcal{H}_t \).
   4. Let \( y \) be the measurement of the \( n \) input qubits.
   5. If \( y \notin \text{span}(Y) \), then include \( y \) in \( Y \).
   6. until \( Y \) contains \( n - 1 \) linear independent vectors

7. Compute \( \{s\} \) as \( Y^\perp \) via Gaussian elimination.

8. return \( s \).

end
Lemma 3.1 (Orthogonality). Let \( f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n \) be a Simon function with period \( s \), \( h : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n \) and \( f_h = h \circ f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n \). Let us apply \( Q_{hof} \) on \( |0^n\rangle |0^t\rangle \) and measure \( z \) in the last \( t \) qubits. Then we obtain superposition

\[
\sum_{y \in \mathbb{F}_2^n} w_{y,z} |y\rangle |z\rangle , \quad \text{where} \quad w_{y,z} = \frac{1}{(2^n \cdot |f_h^{-1}(z)|)^{1/2}} \sum_{x \in f_h^{-1}(z)} (-1)^{xy}.
\]

Proof. Since \( f \) is a Simon function we have \( f(x) = f(x + s) \) and therefore \( f_h(x) = f_h(x + s) \). This implies \( x \in f_h^{-1}(z) \) iff \( x + s \in f_h^{-1}(z) \).

An application of \( Q_{hof} \) on input \( |0^n\rangle |0^t\rangle \) yields for the operations \( H_n \otimes I_t \) and \( U_{f_h} \)

\[
|0^n\rangle |0^t\rangle \xrightarrow{H_n \otimes I_t} \frac{1}{2^{n/2}} \sum_{x \in \mathbb{F}_2^n} |x\rangle |0^t\rangle \xrightarrow{U_{f_h}} \frac{1}{2^{n/2}} \sum_{x \in \mathbb{F}_2^n} |x\rangle |f_h(x)\rangle.
\]

By assumption we measure \( z \) in the last \( t \) qubits. Therefore the first \( n \) qubits collapse to the uniform superposition over all preimages \( f_h^{-1}(z) \) of \( z \) under \( f_h \). Using \( x \in f_h^{-1}(z) \) iff \( x + s \in f_h^{-1}(z) \), we obtain in the first \( n \) qubits

\[
\frac{1}{(2^n \cdot |f_h^{-1}(z)|)^{1/2}} \sum_{x \in f_h^{-1}(z)} |x\rangle = \frac{1}{(2^n \cdot |f_h^{-1}(z)|)^{1/2}} \sum_{x \in f_h^{-1}(z)} \frac{1}{2} (|x\rangle + |x + s\rangle).
\]

An application of \( H_n \) now yields

\[
\frac{1}{(2^n \cdot |f_h^{-1}(z)|)^{1/2}} \sum_{x \in f_h^{-1}(z)} \sum_{y \in \mathbb{F}_2^n} \frac{1}{2} (-1)^{xy} (1 + (-1)^{(x+s)y}) |y\rangle
\]

\[
= \frac{1}{(2^n \cdot |f_h^{-1}(z)|)^{1/2}} \sum_{x \in f_h^{-1}(z)} \sum_{y \in \mathbb{F}_2^n} \frac{1}{2} (-1)^{xy} (1 + (-1)^{sy}) |y\rangle
\]

\[
= \frac{1}{(2^n \cdot |f_h^{-1}(z)|)^{1/2}} \sum_{x \in f_h^{-1}(z)} \sum_{y \in \mathbb{F}_2^n} (-1)^{xy} |y\rangle.
\]

The statement of the lemma follows. \( \square \)

From Lemma 3.1’s superposition

\[
\sum_{y \in \mathbb{F}_2^n} \left( \sum_{y \perp s} \frac{1}{(2^n \cdot |f_h^{-1}(z)|)^{1/2}} \sum_{x \in f_h^{-1}(z)} (-1)^{xy} |y\rangle |z\rangle \right).
\]

we see that only \( y \in \mathbb{F}_2^n \) with \( y \perp s \) have a non-vanishing amplitude \( w_{y,z} \). Recall also that Lemma 3.1 contains the analysis of Simon’s original algorithm as the special case, where \( h \) is the identity function. In this case, we know that by the definition of a Simon function \( |f_h^{-1}(z)| = 2 \) for all \( z \). Thus, all \( y \perp s \) have amplitude \( \pm \frac{1}{2(2^n - 1)^{1/2}} \). This means that a measurement yields the uniform distribution over all \( y \perp s \).

The following lemma will be useful, when we analyse superpositions over all \( z \).

Lemma 3.2. Let \( f \) be a Simon function. Then \( \sum_{z=1}^{2^n} w_{y,z} = 0 \) for all \( y \neq 0 \).
Proof. Fix \( y \neq 0^n \). If \( y \perp s \) then all \( w_{y,z} = 0 \) and thus the claim follows. Hence, in the following let \( y \perp s \). If \( z \notin f(\mathbb{F}_2^n) \) then \( w_{y,z} = 0 \). Therefore

\[
\sum_{z \in \mathbb{F}_2^n} w_{y,z} = \sum_{z \in f(\mathbb{F}_2^n)} w_{y,z}.
\]

Since \( f \) is a Simon function, \( f \) is a (2:1)-mapping. Thus

\[
\sum_{z \in f(\mathbb{F}_2^n)} w_{y,z} = \frac{1}{2} \sum_{x \in \mathbb{F}_2^n} w_{y,f(x)}.
\]

Using the definition of \( w_{y,z} \) in Eq. (1) with \( h = \text{id} \) and \( |f^{-1}(z)| = 2 \) yields

\[
\frac{1}{2} \sum_{x \in \mathbb{F}_2^n} w_{y,f(x)} = \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in \mathbb{F}_2^n} (-1)^{xy}.
\]

Since for \( y \neq 0 \) we have \( \sum_{x \in \mathbb{F}_2^n} (-1)^{xy} = 0 \), the claim follows. \( \square \)

Let us now first develop some intuition for the amplitudes \( w_{y,z} \) in Eq. (1) for hash functions \( h : \mathbb{F}_2^n \rightarrow \mathbb{F}_2 \). We expect that \( |f_h^{-1}(z)| \approx 2^{n-1} \). We first look at the amplitude of \( |y⟩ = |0^n⟩ \). Since for all \( x \in \mathbb{F}_2^n \) we have \( (-1)^{xy} = 1 \), the amplitude of \( |0^n⟩ \) adds up to \( \left( \frac{f_h^{-1}(z)}{2^n} \right)^{\frac{1}{2}} \approx \frac{1}{\sqrt{2}} \). Hence, we expect to measure the zero-vector \( 0^n \) with probability approximately \( \frac{1}{2} \). This seems to be bad news, since the zero-vector is the only one orthogonal to \( s \) that does not provide any information about \( s \).

However, we show that all \( y \perp s \) with \( y \neq 0^n \) still appear with significant amplitude. Intuitively, \( \sum_{x \in f_h^{-1}(z)} \sum_{y \in \mathbb{F}_2^n} (-1)^{xy} \) describes for \( y \neq 0^n \) a random walk with \( |f_h^{-1}(z)| \) steps. Thus, this term should contribute on expectation roughly \( |f_h^{-1}(z)|^{\frac{1}{2}} \) to the amplitude of \( |y⟩ \). So we expect for all \( y \perp s \) with \( y \neq 0^n \) an amplitude of \( \frac{1}{2^{n/2}} \). This in turn implies that conditioned on the event that we do not measure \( 0^n \) (which happens with probability roughly \( \frac{1}{2} \)), we still obtain the uniform distribution over all remaining \( y \perp s \).

We make our intuition formal in the following lemma.

**Theorem 3.1.** Let \( \mathcal{H}_1 = \{ h : \mathbb{F}_2^n \rightarrow \{0,1\} \} \) be universal, and let \( f \) be a Simon function with period \( s \). Then we measure in Algorithm Hashed-Simon in the first \( n \) qubits any \( y \perp s, y \neq 0 \) with probability \( \frac{1}{2^n} \), where the probability is taken over the random choice of \( h \in \mathcal{H}_1 \).

**Proof.** From Lemma 3.1 we conclude that Hashed-Simon gives us a superposition

\[
\sum_{z \in \mathbb{F}_2^n} \sum_{y \in \mathbb{F}_2^n} w_{y,z} |y⟩ |z⟩, \text{ where } w_{y,z} = \frac{1}{(2^n \cdot |f_h^{-1}(z)|)^{1/2}} \sum_{x \in f_h^{-1}(z)} (-1)^{xy}.
\]

In particular for \( z \notin f(\mathbb{F}_2^n) \) we have \( w_{y,z} = 0 \). Let us first look at the special case \( h = \text{id} \), that is we use the original Simon algorithm with \( |f_h^{-1}(z)| = 2 \). In this case we measure any \( y \perp s \) with probability \( \frac{1}{2^{n-1}} \). By Eq. (1), we also measure any \( y \) with probability \( \sum_{z \in \mathbb{F}_2^n} |w_{y,z}|^2 \). Since \( w_{y,z} \in \mathbb{R} \), we obtain \( |w_{y,z}|^2 = w_{y,z}^2 \) and hence the identity

\[
\sum_{z \in \mathbb{F}_2^n} w_{y,z}^2 = \frac{1}{2^{n-1}}. \tag{2}
\]
Now, let us look at Hashed-Simon with a single-bit output hash function \( h \in \mathcal{H}_1 \). Let us denote by \( p = \mathbb{P}_{h \in \mathcal{H}_1}[y] \) the probability that we measure \( y \) in the first \( n \) qubits. Our goal is to show that \( p = \frac{1}{2n} \).

For some \( h \in \mathcal{H}_1 \) we denote \( I_{h,0} = \{ z \in f(\mathbb{F}_2^n) \mid h(z) = 0 \} \) and \( I_{h,1} = \{ z \in f(\mathbb{F}_2^n) \mid h(z) = 1 \} \). Since \( I_{h,0} \cup I_{h,1} = f(\mathbb{F}_2^n) \) and \( w_{y,z} \in \mathbb{R} \), Hashed-Simon yields

\[
p = \frac{1}{|\mathcal{H}_1|} \sum_{h \in \mathcal{H}_1} \left( \left| \sum_{z \in I_{h,0}} w_{y,z} \right|^2 + \left| \sum_{z \in I_{h,1}} w_{y,z} \right|^2 \right) \]

\[
= \frac{1}{|\mathcal{H}_1|} \sum_{h \in \mathcal{H}_1} \left( \left( \sum_{z \in I_{h,0}} w_{y,z} \right)^2 + \left( \sum_{z \in I_{h,1}} w_{y,z} \right)^2 \right). \quad (3)
\]

In Eq. (3) we obtain a cross-product \( w_{y,z_1}w_{y,z_2} \) for \( z_1 \neq z_2 \) iff \( z_1, z_2 \) are in the same set \( I_{h,b} \), \( b \in \{0, 1\} \), i.e. iff \( h(z_1) = h(z_2) \). Using Definition 2.1 of a universal hash function family, we obtain \( \mathbb{P}_{h \in \mathcal{H}_1}[h(z_1) = h(z_2)] = \frac{1}{2} \) for any \( z_1 \neq z_2 \). This implies that for exactly half of all \( h \in \mathcal{H}_1 \) we obtain \( h(z_1) = h(z_2) \).

Further using \( w_{y,z} = 0 \) for \( z \notin f(\mathbb{F}_2^n) \), we conclude that

\[
p = \sum_{z \in \mathbb{F}_2^n} w_{y,z}^2 + \frac{1}{2} \sum_{z_1 \neq z_2} w_{y,z_1}w_{y,z_2}.
\]

From Lemma 3.2 we know that \( 0 = \frac{1}{2} \left( \sum_{z \in \mathbb{F}_2^n} w_{y,z} \right)^2 = \frac{1}{2} \sum_{z \in \mathbb{F}_2^n} w_{y,z}^2 + \frac{1}{2} \sum_{z_1 \neq z_2} w_{y,z_1}w_{y,z_2} \). An application of this identity together with Eq. (2) gives us

\[
p = \frac{1}{2} \sum_{z \in \mathbb{F}_2^n} w_{y,z}^2 = \frac{1}{2n},
\]

which concludes the proof.

\[\square\]

**Theorem 3.2.** Let \( \mathcal{H}_1 = \{ h : \mathbb{F}_2^n \to \{0, 1\} \} \) be universal, and let \( f : \mathbb{F}_2^n \to \mathbb{F}_2^n \) be a Simon function with period \( s \in \mathbb{F}_2^n \). Hashed-Simon recovers \( s \) with expected \( 2(n+1) \) applications of quantum circuits \( Q^{\text{Simon}}_{h \circ f} \), \( h \in \mathcal{H}_1 \), that use only \( n+1 \) qubits.

**Proof.** Let us define a random variable \( X_i, 1 \leq i < n \) for the number of applications of \( Q^{\text{Simon}}_{h \circ f} \) until Hashed-Simon finds \( i \) linearly independent \( y_1 \ldots y_i \). Let \( E_i \) be the event that we already have \( i - 1 \) linearly independent \( Y = \{ y_1, \ldots, y_{i-1} \} \) and we measure some \( y_i \notin \text{span}(Y) \). Define \( p_i = \mathbb{P}[E_i] \). Using Theorem 3.1 we obtain

\[
p_1 = \frac{2^{n-1} - 1}{2^n}.
\]

Since \( |\text{span}\{y_1 \ldots y_{i-1}\}| = 2^{i-1} \), we obtain from Theorem 3.1 more generally

\[
p_i = \frac{2^{n-1} - 2^{i-1}}{2^n}.
\]
Clearly, $X_i$ is geometrically distributed with parameter $p_i$. Let $X = X_1 + \ldots + X_{n-1}$ denote the number of required applications of $Q^{\text{Simon}}_{\text{hof}}$ in HASHED-SIMON. Then

$$
\mathbb{E}[X] = \sum_{i=1}^{n-1} \mathbb{E}[X_i] = \sum_{i=1}^{n-1} 2^n \frac{2^n - 2i - 1}{2^{n-1}} = \sum_{i=1}^{n-1} 2(2^{n-1} - 2i - 1) + 2i \\
= 2(n - 1) + \sum_{i=1}^{n-1} \frac{2^i}{2^{n-1} - 2i - 1}.
$$

Since $\lim_{n \to \infty} \sum_{i=1}^{n-1} \frac{2^i}{2^{n-1} - 2i - 1} \leq 3.2134$, the claim follows.

\[ \square \]

**Remark 3.1.** With a similar analysis as in the proof of Theorem 3.2, we obtain an upper bound of $n + 1$ for the expected number of applications of $Q^{\text{Simon}}_{f}$ in Simon’s original algorithm.

### 4 Hashed-Shor: Special Periods

Let us briefly recall Shor’s algorithm. Let $f : \mathbb{Z} \to \mathbb{Z}$ be periodic with period $d \in \mathbb{N}$, i.e. $d > 0$ is minimal with the property $f(x) = f(x + d)$ for all $x \in \mathbb{Z}$. For ease of notation, let us first focus on applying Shor’s algorithm for factorization. In Section 6 we will also see an application for discrete logarithms.

Let $N \in \mathbb{N}$ be a composite $n$-bit number of unknown factorization, and let $a$ be chosen uniformly at random from $\mathbb{Z}_N^*$, the multiplicative group modulo $N$. Let us define the function $f : \mathbb{Z} \to \mathbb{Z}_N$, $x \mapsto a^x \mod N$. Notice that $f$ is periodic with $d = \text{ord}_N(a)$, since $f(x + d) = a^{x+d} = a^x a^{\text{ord}(a)} = a^x = f(x)$. It is well-known that we can compute a non-trivial factor of $N$ in probabilistic polynomial time given $d = \text{ord}_N(a)$ [17]. We encode the inputs of $f$ with $q$ qubits.

In order to find $d$, Shor uses the quantum circuit $Q^{\text{Shor}}_f$ from Figure 2. In $Q^{\text{Shor}}_f$ we measure in the $q$ input qubits with high probability $y$ that are close to some multiple of $\frac{2^t}{q}$. The original Shor algorithm then measures sufficiently many $y$’s (a constant number is sufficient) to extract $d$ in a classical post-process.

![Fig. 2: Quantum circuit $Q^{\text{Shor}}_f$](image)

Our HASHED-SHOR (Algorithm 2) simply replaces circuit $Q^{\text{Shor}}_f$ with its hashed version $Q^{\text{Shor}}_{\text{hof}}$. Notice that SHOR is a special case of HASHED-SHOR for the choice $t = \lceil \log_2(N) \rceil$ and $\mathcal{H}_t = \{ \text{id} \}$. For this choice $\mathcal{H}_t$ is not universal, but we do not need universality in the following Lemma 4.1 about the superposition produced by $Q^{\text{Shor}}_{\text{hof}}$. From Lemma 4.1 we conclude correctness of HASHED-SHOR for any $t$-bit range hash function $h$.

**Lemma 4.1.** Let $N \in \mathbb{N}$, $a \in \mathbb{Z}_N^*$ with $d = \text{ord}_N(a)$ and $f(x) = a^x \mod N$. Let $h : \mathbb{Z}_N \to \{0,1\}^t$. Define $M_z := \{ k \in \mathbb{Z}_d \mid h(a^k \mod N) = z \}$. An application of quantum circuit $Q^{\text{Shor}}_{\text{hof}}$ on input
Algorithm 2: Hashed-Shor

**Input:** \( f : \mathbb{Z} \rightarrow \mathbb{Z}_N \), universal \( \mathcal{H}_t := \{ h : \mathbb{Z}_N \rightarrow \{0,1\}^t \} \)\>

\( \triangleright t = \lceil \log_2(N) \rceil \) and \( \mathcal{H}_t = \{ \text{id} \} \) for Shor

**Output:** Period \( d \) of \( f \)

1. Set \( Y = \emptyset \).
2. repeat
3. Run \( Q_{Shor}^{h} \) on \( |0^q\rangle |0^t\rangle \) for some freshly chosen \( h \in R \mathcal{H}_t \).
4. Let \( y \) be the measurement of the \( q \) input qubits.
5. If \( y \neq 0 \), then include \( y \) in \( Y \).
6. until \( |Y| \) is sufficiently large.
7. Compute \( d \) from \( Y \) in a classical post-process.
8. return \( d \)
9. end

\(|0^q\rangle |0^t\rangle \) yields superposition

\[
|\Phi_h \rangle = \sum_{y=0}^{2^q-1} \sum_{z \in \{0,1\}^t} \frac{1}{2^q} \sum_{k \in M_z} \sum_{c \geq 0: cd+k<2^q} e^{2\pi i \frac{cd+k}{2^q} y} |y\rangle |z\rangle .
\] \hspace{1cm} (4)

**Proof.** In \( Q_{Shor}^{h} \), we apply on input \( |0^q\rangle |0^t\rangle \) first the operation \( H_q \otimes I_t \) followed by \( U_{h,f} \). This results in superposition

\[
\frac{1}{\sqrt{2^q}} \sum_{x=0}^{2^q-1} |x\rangle |h(a^x \mod N)\rangle .
\]

Let \( x = cd + k \) with \( k \in \mathbb{Z}_d \). Since \( a^x = a^{cd+k} \equiv a^k \mod N \), the value of \( f(x) \) depends only on \( k = (x \mod d) \). Therefore, we rewrite the above superposition as

\[
\frac{1}{\sqrt{2^q}} \sum_{z \in \{0,1\}^t} \sum_{k \in M_z} \sum_{c \geq 0: cd+k<2^q} |cd+k\rangle |z\rangle ,
\]

Eventually, an application of QFT \( q \) yields

\[
|\Phi_h \rangle = \sum_{y=0}^{2^q-1} \sum_{z \in \{0,1\}^t} \frac{1}{2^q} \sum_{k \in M_z} \sum_{c \geq 0: cd+k<2^q} e^{2\pi i \frac{cd+k}{2^q} y} |y\rangle |z\rangle .
\] \hspace{1cm} (4)

\( \square \)

**Remark 4.1.** For the choice \( h = \text{id} \), Lemma 4.1 provides an analysis of Shor's original quantum circuit \( Q_{Shor}^{h} \). This choice implies \( M_z = \{ k \in \mathbb{Z}_d \mid a^k = z \mod N \} \). Therefore, we obtain the superposition

\[
|\Phi \rangle = \sum_{y=0}^{2^q-1} \sum_{k=0}^{d-1} \frac{1}{2^q} \sum_{c \geq 0: cd+k<2^q} e^{2\pi i \frac{cd+k}{2^q} y} |y\rangle |a^k \mod N\rangle .
\] \hspace{1cm} (5)
and the amplitudes of $|y⟩|z_k⟩$ with $z_k = a^k \mod N$ are

$$w_{y,z_k} = \frac{1}{2^q} \sum_{\substack{c \geq 0, \cr cd+k < 2^q}} e^{2\pi i \frac{cd+k}{2^q} y}.$$  

For didactical reasons and ease of notation, let us look in the subsequent section at the special case of periods $d$ that are powers of two. In Section 4.1 Periods that are a power of two

Let $d = 2^r$ for some $r \in \mathbb{N}$ with $r \leq q$. Then $\max\{c \in \mathbb{N} \mid cd+k < 2^q\} = \frac{2^q}{d} - 1 = 2^{q-r} - 1$, independent of $k \in \mathbb{Z}_d$. Hence, let us define $m = \frac{2^q}{d}$ and $z_k = a^k \mod N$. Using $md = 2^q$, this allows us to rewrite Eq. (5) and Eq. (4) as

$$|Φ⟩ = \frac{2^q-1}{d} \sum_{y=0}^{2^q-1} \left( \frac{1}{\sqrt{d}} \cdot e^{2\pi i \frac{k}{2^q} y} \right) \cdot \left( \frac{1}{\sqrt{m \cdot 2^q}} \sum_{c=0}^{m-1} e^{2\pi i \frac{cd}{2^q} y} \right) |y⟩|z_k⟩ \quad (6)$$

respectively for $h : \mathbb{Z}_N \to \{0,1\}$ as

$$|Φ_h⟩ = \sum_{y=0}^{2^q-1} \sum_{z \in \{0,1\}} \left( \frac{1}{\sqrt{d}} \sum_{k \in \mathbb{Z}_d} e^{2\pi i \frac{k}{2^q} y} \right) \cdot \left( \frac{1}{\sqrt{m \cdot 2^q}} \sum_{c=0}^{m-1} e^{2\pi i \frac{cd}{2^q} y} \right) |y⟩|z⟩. \quad (7)$$

Notice that the factor

$$\frac{1}{\sqrt{m \cdot 2^q}} \sum_{c=0}^{m-1} e^{2\pi i \frac{cd}{2^q} y}$$

is identical in $|Φ⟩$ and its hashed version $|Φ_h⟩$. Further notice that the factor is independent of $z_k$ and $z$. In the following lemma we show that for a measurement of any $|y⟩$, where $y$ is a multiple of $m$, this factor contributes to the probability with $\frac{1}{d}$.

**Lemma 4.2.** Let $d = 2^r \leq 2^q$ and $y = ℓm$ for some $0 \leq ℓ < d$. Then we measure $|y⟩$ in either $|Φ⟩$ or $|Φ_h⟩$ from Equation (6) or (7) with probability

$$\left| \frac{1}{\sqrt{m \cdot 2^q}} \sum_{c=0}^{m-1} e^{2\pi i \frac{cd}{2^q} y} \right|^2 = \frac{1}{d}.$$  

**Proof.** Since $y = ℓm = ℓ \frac{2^q}{d}$ we obtain

$$\left| \frac{1}{\sqrt{m \cdot 2^q}} \sum_{c=0}^{m-1} e^{2\pi i \frac{cd}{2^q} y} \right|^2 = \frac{1}{m \cdot 2^q} \left| \sum_{c=0}^{m-1} e^{2\pi icℓ} \right|^2 = \frac{m^2}{m \cdot 2^q} = \frac{m}{2^q} = \frac{1}{d}. \quad \square$$

We now show that the same common factor ensures that in both superpositions $|Φ⟩$ and its hashed version $|Φ_h⟩$ we never measure some $|y⟩$ where $y$ is not a multiple of $m$.

**Lemma 4.3.** Let $d = 2^r \leq 2^q$ and $y \in \{0,\ldots,2^q-1\}$ with $m \nmid y$. Then we measure $|y⟩$ in either $|Φ⟩$ or $|Φ_h⟩$ from Equation (6) or (7) with probability 0.
Proof. Let $y = m\ell + k$ with $0 < k < \ell$. It suffices to show that $\left| \sum_{c=0}^{m-1} e^{2\pi i \frac{cd}{m}} y \right|^2 = 0$. Using $\frac{d}{m} = \frac{1}{2}$, we obtain

$$\left| \sum_{c=0}^{m-1} e^{2\pi i \frac{cd}{m}} y \right|^2 = \left| \sum_{c=0}^{m-1} e^{2\pi i \frac{c(m\ell+k)}{m}} \right|^2 = \left| \sum_{c=0}^{m-1} \left( e^{2\pi i \frac{k}{m}} \right)^c \right|^2 = 0.$$ 

Hence, we conclude from Lemmata 4.1, 4.2 and 4.3 that in both $|\Phi\rangle$ and its hashed version $|\Phi_h\rangle$ we always measure some $|y\rangle$ for which $y = \ell m = \frac{e}{d}$. Assume that $\gcd(\ell, d) = 1$, then we directly read off $d$ from $y$. If $\ell$ is uniformly distributed in the interval $[0, d)$ this happens with sufficient probability to compute $d$ in polynomial time.

Indeed, in Shor’s original algorithm $\ell$ is uniformly distributed since the first factor in Eq. 6 satisfies for any $y$

$$\sum_{k=0}^{d-1} \frac{1}{\sqrt{d}} \cdot e^{2\pi i \frac{k}{d} y} \left| \frac{1}{\sqrt{d}} \cdot \sum_{k=0}^{d-1} e^{2\pi i \frac{k}{d} y} \right|^2 = \frac{1}{d} \sum_{k=0}^{d-1} e^{2\pi i \frac{k}{d} y} \sum_{k=0}^{d-1} 1 = 1.$$ 

Similar to the reasoning in Section 3, we show that in the case of the hashed version $|\Phi_h\rangle$ we obtain any $|y\rangle$ with $y \neq 0$ with probability $\frac{1}{2d}$, where the probability is taken over the random choice of the hash function. This implies that we measure for $|\Phi_h\rangle$ the useless $y = 0$ with probability $1 - \frac{d-1}{2d} \approx \frac{1}{2}$.

**Theorem 4.1.** Let $N \in \mathbb{N}$, $a \in \mathbb{Z}_N$ with $d = \ord_N(a)$ a power of two and $f(x) = a^x \text{ mod } N$. Let $H_1 = \{ h : \mathbb{Z}_N \rightarrow \{0, 1\} \}$ be universal. Then we measure in HASHED-SHOR in the $q$ input qubits any $y = \ell m$, $0 < \ell < d$ with probability $\frac{1}{2d}$, where the probability is taken over the random choice of $h \in H_1$.

**Proof.** Let us denote by $p_h = \mathbb{P}_{h \in H_1} |y\rangle$ the probability that we measure $y$ in HASHED-SHOR in the $q$ input qubits. By Lemma 4.1, Eq. 7 and Lemma 4.2 we know that for all $y = \ell m = \ell \frac{2\pi}{d}$ we have

$$p_h = \frac{1}{|H_1|} \sum_{h \in H_1} \sum_{z \in \{0,1\}} \left| \frac{1}{\sqrt{d}} \cdot \sum_{k=0}^{d-1} e^{2\pi i \frac{z}{d} y} \right|^2 \cdot \frac{1}{d} = \frac{1}{|H_1|} \sum_{h \in H_1} \sum_{z \in \{0,1\}} \left| \frac{1}{\sqrt{d}} \cdot \sum_{k=0}^{d-1} e^{2\pi i \frac{z}{d} \ell} \right|^2.$$ 

Recall that $M_z := \{ k \in \mathbb{Z}_d \mid h(a^k \text{ mod } N) = z \}$. Observe that for $k_1 \neq k_2$ we obtain a cross-product $e^{2\pi i \frac{k_1}{d}} \cdot e^{2\pi i \frac{k_2}{d}} = e^{2\pi i \frac{(k_1-k_2)\ell}{d}}$ iff $k_1, k_2$ are in the same set $M_b, b \in \{0,1\}$, i.e. iff $h(a^{k_1} \text{ mod } N) = h(a^{k_2} \text{ mod } N)$. Using Definition 2.1 of a universal hash function family, we obtain $\mathbb{P}_{h \in H_1} |h(a^k \text{ mod } N) = h(a^{k_2} \text{ mod } N) = \frac{1}{2}$ for any $k_1 \neq k_2$. This implies that for exactly half of all $h \in H_1$ we obtain $h(a^{k_1} \text{ mod } N) = h(a^{k_2} \text{ mod } N)$. Therefore,

$$p_h = \frac{1}{d^2} \cdot \left( \sum_{k=0}^{d-1} e^{2\pi i \frac{k}{d} \ell} \right)^2 + \frac{1}{2} \sum_{k_1 \in \mathbb{Z}_d} \sum_{k_2 \neq k_1 \in \mathbb{Z}_d} e^{2\pi i \frac{(k_1-k_2)\ell}{d}}.$$ 

Since $k_1 - k_2 \neq 0$, we can rewrite as

$$p_h = \frac{1}{d^2} \cdot \left( d + \sum_{k \in \mathbb{Z}_d \setminus \{0\}} e^{2\pi i \frac{k}{d}} \right) = \frac{1}{d^2} \cdot \left( d - \frac{d}{2} \right) = \frac{1}{2d}.$$
From Theorem 4.1 we see that in the hashed version $|\Phi_h\rangle$ we measure every $y = m\ell, y \neq 0$ with probability $\frac{1}{2d}$, whereas in comparison in $|\Phi\rangle$ we measure every $y = m\ell$ with probability $\frac{1}{d}$. It follows that in Eq. (7) the scaling factor

$$S = \sum_{z \in \{0,1\}} \left| \frac{1}{\sqrt{d}} \sum_{k \in M_z} e^{2\pi i \frac{k}{d} y} \right|^2$$

takes on expected value $\frac{1}{2}$ for $y = \ell m$, $0 < \ell < d$ taken over all $h \in \mathcal{H}_1$. Notice that $S$ is a symmetric function in $y$, i.e. $S(y) = S(2q - y)$.

Let us look at an example to illustrate how the probabilities behave. We choose $N = 51 = 3 \cdot 17$, $a = 2$ and $q = 12$. This implies $d = \text{ord}_N(a) = 8$ and $m = \frac{2q}{d} = 512$. In $|\Phi\rangle$ we measure some $y = m\ell = 512\ell, 0 \leq \ell < d = 8$ with probability $\frac{1}{8}$ each, as illustrated in Figure 3a.

Let us assume we have in Hashed-Shor $M_0 = \{2, 3, 4, 7\}$. This fully specifies the scaling function $S$ from Eq. (8). Thus, each amplitude from $|\Phi\rangle$ is multiplied by $S$, as illustrated in Figure 3b.

Fig. 3: Probability distributions for Shor and Hashed-Shor

**Theorem 4.2.** Let $N \in \mathbb{N}$, $a \in \mathbb{Z}_N^+$ with $d = \text{ord}_N(a)$ a power of two and $f(x) = a^x \mod N$. Let $\mathcal{H}_1 = \{ h : \mathbb{Z}_N \to \{0,1\} \}$ be universal, and let $x$ be represented by $q$ input qubits in $Q^{\text{Shor}}_{h \circ f}$. Then Hashed-Shor finds $f$’s period $d$ with expected 4 applications of quantum circuits $Q^{\text{Shor}}_{h \circ f}$, $h \in \mathcal{H}_1$, that use only $q + 1$ qubits.

**Proof.** In Shor we compute the fraction $\frac{y}{2^q} = \frac{\ell}{2^q}$ in reduced form. Since $d$ is a power of two, this fraction reveals $d$ in its denominator iff $\ell$ is odd. Using Theorem 4.1 we measure $y = m\ell$ with an odd $\ell$, $0 < \ell < d$ with probability $\frac{d}{2} \cdot \frac{1}{2d} = \frac{1}{4}$. Thus, we need on expectation 4 applications of $Q^{\text{Shor}}_{h \circ f}$ to find $f$’s period $d$.

Notice that we can check the validity of $d$ via testing the identity $a^d \equiv 1 \mod N$. 

□
For comparison, we need in Shor’s original algorithm with the nonhashed version of \( f \) on expectation 2 measurements until we find \( d \).

5 Hashed Shor: Finding Periods in General

Notice that we proved in Theorem 3.1 and Theorem 4.1 that when we move to the hashed version of our quantum circuits all probabilities to measure some \( y \neq 0 \) decrease exactly by a factor of \( \frac{1}{2} \) (over the random choice the hash function).

The same is true for finding arbitrary (non power of two) periods with circuit \( Q_{\text{Shor}}^{\text{Shor}} \). However, this does not immediately follow from the proof of Theorem 4.1 because the proof builds on the special form of superposition \( |\Phi_h\rangle \) from Eq. (7) that only holds if \( d \) is a power of two. Here we show a more general result for period finding algorithms that applies for Shor’s original circuit as well as for its Ekerå-Håstad variant in the subsequent section. To this end let us define a generic period finding quantum circuit \( Q_{\text{Period}}^{f} \) (see Figure 4). In Figure 4 we denote by \( Q_1, Q_2 \) any quantum circuitry that acts on the \( q \) input qubits. For example, for Simon’s circuit we have \( Q_1 = Q_2 = H_q \) (see Figure 1). For Shor’s circuit we have \( Q_1 = H_q \) and \( Q_2 = \text{QFT}_q \). In the following Theorem 5.1 we will define explicitly a cancellation criterion that this circuitry \( Q_1, Q_2 \) has to fulfill. An important feature of \( Q_{\text{Period}}^{f} \) is however that we apply \( f \) only once.

\[
\begin{array}{c}
|0^q\rangle \\
|0^n\rangle
\end{array}
\begin{array}{c}
Q_1 \\
U_f \\
Q_2
\end{array}
\]

Fig. 4: Quantum circuit \( Q_{\text{Period}}^{f} \)

Now let us use our generic period finding circuit \( Q_{\text{Period}}^{f} \) inside a generic period finding algorithm \( \text{Period} \) that uses a certain number of measurements of \( Q_{\text{Period}}^{f} \) and some classical post-processing. If we replace in \( \text{Period} \) the circuit \( Q_{\text{Period}}^{f} \) by its hashed variant \( Q_{\text{Hash}}^{f} \) then we call the resulting algorithm \( \text{Hashed-Period} \) (Algorithm 3).

Algorithm 3: (Hashed-)Period

\[
\begin{array}{l}
\text{Input : } f : \{0,1\}^q \rightarrow \{0,1\}^n, \text{ universal } \mathcal{H}_t := \{ h : \{0,1\}^n \rightarrow \{0,1\}^t \} \\
\text{Output: } \text{Period } d \text{ of } f \\
\text{begin} \\
\quad 1 \text{ Set } Y = \emptyset. \\
\quad 2 \text{ repeat} \\
\quad \quad 3 \text{ Run } Q_{\text{Hash}}^{\text{Period}} \text{ on } |0^q\rangle |0^t\rangle \text{ for some freshly chosen } h \in_R \mathcal{H}_t. \\
\quad \quad 4 \text{ Let } y \text{ be the measurement of the } q \text{ input qubits.} \\
\quad \quad 5 \text{ If } y \neq 0^q, \text{ then include } y \text{ in } Y. \\
\quad \quad 6 \text{ until } |Y| \text{ is sufficiently large.} \\
\quad 7 \text{ Compute } d \text{ from } Y \text{ in a classical post-process.} \\
\quad \text{return } d \\
\text{end}
\end{array}
\]
Theorem 5.1. Let \( f : \{0,1\}^q \to \{0,1\}^n \) and \( \mathcal{H}_1 = \{ h : \{0,1\}^n \to \{0,1\} \} \) be universal. Let \( Q_f^\text{PERIOD} \) be a quantum circuit that on input \( |0^q\rangle |0^n\rangle \) yields superposition

\[
|\Phi\rangle = \sum_{y \in \{0,1\}^q} \sum_{f(x) \in \text{Im}(f)} w_{y,f(x)} |y\rangle |f(x)\rangle \text{ satisfying } \sum_{f(x) \in \text{Im}(f)} w_{y,f(x)} = 0 \text{ for any } y \neq 0. \tag{9}
\]

Let us denote by \( p(y) \), respectively \( p_h(y) \), the probability to measure some \( |y\rangle \), \( y \neq 0 \) in the \( q \) input qubits when applying \( Q_f^\text{PERIOD} \), respectively \( Q_h^\text{PERIOD} \) with \( h \in_R \mathcal{H}_1 \), on input \( |0^q\rangle |0^n\rangle \). Then \( p_h(y) = \frac{1}{2} p(y) \).

Proof. For ease of notation let us denote \( z = f(x) \). By definition, we have \( p(y) = \sum_{z \in \text{Im}(f)} |w_{y,z}|^2 \).

Now let us find an expression for \( p_h(y) \) when using \( Q_h^\text{PERIOD} \). For \( h \in \mathcal{H}_1 \) we denote \( I_{h,0} = \{ z \in \text{Im}(f) \mid h(z) = 0 \} \) and \( I_{h,1} = \{ z \in \text{Im}(f) \mid h(z) = 1 \} \). Since \( I_{h,0} \cup I_{h,1} = \text{Im}(f) \), we obtain

\[
p_h(y) = \frac{1}{|\mathcal{H}_1|} \sum_{h \in \mathcal{H}_1} \left( \sum_{z \in I_{h,0}} w_{y,z} \right)^2 + \sum_{z \in I_{h,1}} w_{y,z} ^2. \tag{10}
\]

In Eq. (10) we obtain a cross-product \( w_{y,z_1} \overline{w_{y,z_2}} \) for \( z_1 \neq z_2 \) iff \( z_1, z_2 \) are in the same set \( I_{h,b} \), \( b \in \{0,1\} \), i.e. iff \( h(z_1) = h(z_2) \). Using Definition 2.1 of a universal hash function family, we obtain \( \mathbb{P}_{h \in \mathcal{H}} [h(z_1) = h(z_2)] = \frac{1}{2} \) for any \( z_1 \neq z_2 \). This implies that for exactly half of all \( h \in \mathcal{H} \) we obtain \( h(z_1) = h(z_2) \). We conclude that

\[
p_h(y) = \sum_{z \in \text{Im}(f)} |w_{y,z}|^2 + \frac{1}{2} \sum_{z_1 \neq z_2} w_{y,z_1} \overline{w_{y,z_2}}.
\]

Our prerequisite \( \sum_{z \in \text{Im}(f)} w_{y,z} = 0 \) for any \( y \neq 0 \) implies

\[
0 = \frac{1}{2} \sum_{z \in \text{Im}(f)} |w_{y,z}|^2 = \frac{1}{2} \sum_{z \in \text{Im}(f)} |w_{y,z}|^2 + \frac{1}{2} \sum_{z_1 \neq z_2} w_{y,z_1} \overline{w_{y,z_2}} = p_h(y) - \frac{1}{2} \sum_{z \in \text{Im}(f)} |w_{y,z}|^2.
\]

Together with the definition of \( p(y) \) we conclude that

\[
p_h(y) = \frac{1}{2} \sum_{z \in \text{Im}(f)} |w_{y,z}|^2 = \frac{1}{2} p(y).
\]

\( \square \)

We already showed in Lemma 3.2 that Simon’s circuit \( Q_f^{\text{SIMON}} \) fulfills the cancellation criterion (Equation (9)) of Theorem 5.1. Thus, the statement of Theorem 3.1 directly follows from Theorem 5.1. However, for an improved intelligibility we preferred to prove Theorem 3.1 directly.

In the following Theorem 5.2 we show that \( Q_f^{\text{SHOR}} \) also meets the cancellation criterion. Thus, going to the hashed version in Shor’s algorithm immediately halves all probabilities for \( y \neq 0 \).
Theorem 5.2. On input |0^q⟩|0^n⟩ the quantum circuit $Q_f^{\text{Shor}}$ yields superposition

$$|\Phi⟩ = \sum_{y \in \{0,1\}^q} \sum_{f(x) \in \text{Im}(f)} w_{y,f(x)} |y⟩ |f(x)⟩ \text{ satisfying } \sum_{f(x) \in \text{Im}(f)} w_{y,f(x)} = 0 \text{ for any } y \neq 0.$$ 

Proof. From Equation (5) we know that $Q_f^{\text{Shor}}$ yields superposition

$$|\Phi⟩ = \sum_{y=0}^{2^q-1} \sum_{f(x) \in \text{Im}(f)} w_{y,f(x)} |y⟩ |f(x)⟩ \text{ with } \sum_{f(x) \in \text{Im}(f)} w_{y,f(x)} = \frac{1}{2^q} \sum_{c \geq 0: \frac{cd}{d+k} < 2^q} e^{2\pi i \frac{d+k}{2^q} y}.$$ 

We conclude for $y \neq 0$ that

$$\sum_{f(x) \in \text{Im}(f)} w_{y,f(x)} = \frac{1}{2^q} \sum_{c \geq 0: \frac{cd}{d+k} < 2^q} e^{2\pi i \frac{d+k}{2^q} y} = \sum_{r=0}^{2^q-1} \frac{1}{2^q} e^{2\pi i \frac{r}{2^q} y} = \frac{1}{2^q} \sum_{r=0}^{2^q-1} \left(e^{2\pi i \frac{y}{2^q}}\right)^r = 0.$$ 

\[ \square \]

Since by Theorem 5.1 the use of hashed versions halves all probabilities for $y \neq 0$, we expect that HASHED-PERIOD requires twice as many measurements as PERIOD. This is more formally shown in the following Theorem 5.3.

Theorem 5.3. Let $f : \{0,1\}^q \rightarrow \{0,1\}^n$ have period $d$, and let $H_1 = \{h : \{0,1\}^n \rightarrow \{0,1\}\}$ be universal. Assume that PERIOD succeeds to find $d$ with probability $\rho$ with an expected number of $t$ measurements, using some $Q_f^{\text{PERIOD}}$ with $q + n$ qubits that satisfies the cancellation criterion (Equation (9)) of Theorem 5.1. Then HASHED-PERIOD succeeds to find $d$ with probability $\rho$ using $Q_h^{\text{PERIOD}}$, $h \in_R H_1$, with only $q + 1$ qubits and an expected number of $2t$ measurements.

Proof. We first show the factor of 2 difference in the expected number of measurements. In the case of $Q_f^{\text{PERIOD}}$ we measure some $y \neq 0$ with probability $\sum_{y=1}^{2^q-1} p(y)$, whereas for $Q_h^{\text{PERIOD}}$ we measure $y \neq 0$ with half the probability $\sum_{y=1}^{2^q-1} p(y)/2 = \frac{1}{2} \sum_{y=1}^{2^q-1} p(y)$ according to Theorem 5.1. This implies that on expectation we need twice as many measurements.

It remains to show that HASHED-PERIOD has the same success probability $\rho$ as PERIOD to compute the period $d$. To this end we show that conditioned on $y \neq 0$, both circuits $Q_f^{\text{PERIOD}}$ and $Q_h^{\text{PERIOD}}$ yield an identical probability distribution for the measured $|y⟩$ in the $q$ input qubits.

Let $p(y)$, respectively $p_h(y)$, be the probability that we measure $|y⟩$ in the $q$ input qubits using $Q_f^{\text{Shor}}$, respectively $Q_h^{\text{Shor}}$. Since PERIOD conditions on measuring $y \neq 0$ we obtain in the case of $Q_f^{\text{Shor}}$ the probabilities

$$\frac{p(y)}{\sum_{y=1}^{2^q-1} p(y)} \text{ for any } y \neq 0.$$ 

In the case $Q_h^{\text{Shor}}$, we obtain using Theorem 5.1 the same probabilities

$$\frac{p(y)/2}{\sum_{y=1}^{2^q-1} p(y)/2} = \frac{p(y)}{\sum_{y=1}^{2^q-1} p(y)} \text{ for any } y \neq 0.$$ 

Since both probability distributions are identical, the success probability $\rho$ is identical as well, independent of any specific post-process for computing $d$. \[ \square \]
Since by Theorem 5.2 Shor’s circuit $Q^{\text{SHOR}}$ satisfies the cancellation criterion of Theorem 5.1, Theorem 5.3 implies that we can implement Shor’s algorithm with $q+1$ instead of $q+n$ qubits at the cost of only twice as many measurements. In other words, we save all but one of the output qubits.

6 Hashed Ekerå-Håstad: Factoring $n$-bit RSA numbers with $\frac{1}{2}n$ qubits

In 2017, Ekerå and Håstad [5] proposed a variant of Shor’s algorithm for computing the discrete logarithms of $x = g^d$ in polynomial time with only $(1 + o(1)) \log d$ input qubits. The Ekerå-Håstad algorithm saves input qubits in comparison to Shor’s original discrete logarithm algorithm whenever $d$ is significantly smaller than the group order.

An interesting application of such a small discrete logarithm algorithm is the factorization of $n$-bit RSA moduli $N = pq$, where $p, q$ are primes of the same bit-size. Let $g \in R \mathbb{Z}_N^*$. Then $\text{ord}_N(g)$ divides $\phi(N)/2 = (p-1)(q-1)/2 = \frac{N+1}{2} - \frac{p+q}{2}$. Therefore

$$x := g^{-\frac{N+1}{2}} = g^\frac{p+q}{2} \mod N.$$ 

Hence, we obtain a discrete logarithm instance in $\mathbb{Z}_N^*$ where the desired logarithm $d = \frac{p+q}{2}$ is of size only roughly $\frac{n}{2}$ bits, whereas group elements have to be represented with $n$ bits. Notice that the knowledge of $d = \frac{p+q}{2}$ together with $N = pq$ immediately yields the factorization of $N$ in polynomial time.

The Ekerå-Håstad algorithm computes $d$ with $(\frac{1}{2} + \frac{1}{2})n$ input and $n$ output qubits, using a classical post-process that takes time polynomial in $n$ and $s^a$. Choosing $s = \frac{\log n}{\log \log n}$, we obtain a polynomial time factoring algorithm with a total of $(\frac{3}{2} + o(1))n$ qubits.

In the following, we show that the Ekerå-Håstad algorithm is covered by our framework of quantum period finding algorithms which fulfill the cancellation criterion of Equation (9) from Theorem 5.1. Thus, by Theorem 5.3 we can save all but 1 of the output qubits via hashing, at the cost of only doubling the number of quantum measurements. This in turn leads to a polynomial time factorization algorithm for $n$-bit RSA numbers using only $(\frac{1}{2} + o(1))n$ qubits. Concerning discrete logarithms, with our hashing approach we can quantumly compute $d$ from $g$ and $g^d$ in polynomial time using only $(1 + o(1)) \log d$ qubits.

Let $(g, x = g^d, S(G))$ be a discrete logarithm instance with $m = \log d$. Here $S(G)$ specifies how we compute in the group $G$ generated by $g$, e.g. $S(G) = N$ specifies that we compute modulo $N$ in the group $G = \mathbb{Z}_N^*$. Define

$$f_{g,x,S(G)}(a,b) = a^x \cdot x^{-b} = g^{a-bd}.$$ 

The Ekerå-Håstad quantum circuit $Q^F_{\text{EKERÅ-HÅSTAD}}$ from Figure 3 computes on input $|0^m\rangle |0^{\ell}\rangle |0^n\rangle$, where $\ell := \frac{m}{s}$, a superposition

$$|\Phi\rangle = \frac{1}{2^{m+2\ell}} \sum_{a,j=0}^{2^{\ell-1}} \sum_{b,k=0}^{2^{\ell-1}} e^{2\pi i (aj+2^b k)/2^{m+\ell}} |j,k,f_{g,x,S(G)}(a,b)\rangle.$$

(11)

The main step in the analysis of Ekerå-Håstad shows that we measure in the $m+2\ell = (1+\frac{2}{s})m = (1+\frac{2}{s}) \log d$ input qubits with high probability so-called good pairs $(j,k)$ that help us in computing $d$ via some lattice reduction technique.

In the following Theorem 6.1, we show that $Q^F_{\text{EKERÅ-HÅSTAD}}$ satisfies our cancellation criterion of Theorem 5.1. Thus, we conclude from Theorem 5.3 that by moving to the hashed version
\( Q_{\text{ho}}^{E} \) we lower the probabilities of measuring good \((j,k)\) only by a factor of \( \frac{1}{2} \) (averaged over all hash functions).

**Theorem 6.1.** Let \((g, x, S(G))\) be a discrete logarithm instance and \(f_{g, x, S(G)}(a, b) = g^a x^{-b}.\) On input \(|0^{2\ell + m}\rangle |0^n\rangle\) the quantum circuit \( Q_{f}^{E} \) yields superposition

\[
|\Phi\rangle = \sum_{y \in \{0,1\}^{m+2\ell}} \sum_{f(x) \in \text{Im}(f)} w_{y, f(x)} |y\rangle |f(x)\rangle \text{ satisfying } \sum_{f(x) \in \text{Im}(f)} w_{y, f(x)} = 0 \text{ for any } y \neq 0.
\]

**Proof.** From Eq. (11) with \(y = (j, k)\) we know that \( Q_{f}^{E} \) yields superposition

\[
|\Phi\rangle = \sum_{j=0}^{2^{m+\ell}-1} \sum_{k=0}^{2^{m+\ell}-1} \sum_{f(x) \in \text{Im}(f)} \frac{w^{(j, k, f(x))}}{2^{m+2\ell}} |j\rangle |k\rangle |f(x)\rangle
\]

with

\[
\sum_{f(x) \in \text{Im}(f)} w^{(j, k, f(x))} = \frac{1}{2^{m+2\ell}} \sum_{a=0}^{2^{m+\ell}-1} \sum_{b=0}^{2^{m+\ell}-1} e^{2\pi i (aj + 2^m bk) / 2^{m+\ell}}.
\]

Hence for \(y \neq 0\) we obtain

\[
\sum_{f(x) \in \text{Im}(f)} \frac{w^{(j, k, f(x))}}{2^{m+2\ell}} = \frac{1}{2^{m+2\ell}} \left( \sum_{a=0}^{2^{m+\ell}-1} e^{2\pi i a j / 2^{m+\ell}} \right) \cdot \left( \sum_{b=0}^{2^{m+\ell}-1} e^{2\pi i b k / 2^\ell} \right)
\]

\[
= \frac{1}{2^{m+2\ell}} \left( \sum_{a=0}^{2^{m+\ell}-1} \left( e^{2\pi i j / 2^{m+\ell}} \right)^a \right) \cdot \left( \sum_{b=0}^{2^{m+\ell}-1} \left( e^{2\pi i k / 2^\ell} \right)^b \right).
\]

Since by prerequisite \((j, k) \neq (0, 0) \in \mathbb{Z}_{2^{m+\ell}} \times \mathbb{Z}_{2^\ell},\) we have \(j \neq 0 \text{ mod } 2^{m+\ell}\) or \(k \neq 0 \text{ mod } 2^\ell.\) This implies that at least one of the factors is identical 0. \(\square\)

By Theorem 5.3, replacing in the Ekerå-Haslåd algorithm the quantum circuit \( Q_{f}^{E} \) by single output bit circuits \( Q_{\text{ho}}^{E} \) comes at the cost of only twice the number of measurements. Since the Ekerå-Haslåd algorithm finds discrete logarithms \(d\) in polynomial time using only \(m + 2\ell = \left(1 + o(1)\right) \log d\) input qubits, we obtain from Theorem 5.3 the following corollary.

**Corollary 6.1.** Ekerå-Haslåd’s Shor variant admits a hashed version that

1. computes discrete logarithms \(d\) from \(g, g^d\) in polynomial time using \((1 + o(1)) \log d\) qubits,
2. factors \(n\)-bit RSA numbers in time polynomial in \(n\) using \((\frac{1}{2} + o(1))n\) qubits.
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