Nonpropagation of scalar in the deformed Hořava-Lifshitz gravity

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\textbf{Abstract}

We study the propagation of a scalar, the trace of $h_{ij}$ in the deformed Hořava-Lifshitz gravity with coupling constant $\lambda$. It turns out that this scalar is not a propagating mode in the Minkowski spacetime background. In this work, we do not choose a gauge-fixing to identify the physical degrees of freedom and instead, make it possible by substituting the constraints into the quadratic Lagrangian.

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1 Introduction

Recently Hořava has proposed a renormalizable theory of gravity at a Lifshitz point[1], which may be regarded as a UV complete candidate for general relativity. Very recently, the Hořava-Lifshitz gravity theory has been intensively investigated in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], its cosmological applications in [18, 19], and its black hole solutions in [20, 21].

We would like to mention that the IR vacuum of this theory is anti de Sitter (AdS) spacetimes. Hence, it is interesting to take a limit of the theory, which may lead to a Minkowski vacuum in the IR sector. To this end, one may modify the theory by introducing \( \mu^4 R \) and then, take the \( \Lambda_W \rightarrow 0 \) limit. This does not alter the UV properties of the theory, but it changes the IR properties. That is, there exists a Minkowski vacuum, instead of an AdS vacuum.

A relevant issue of (deformed) Hořava-Lifshitz gravity is to answer to the question of whether it can accommodate a scalar mode \( \psi \propto H \), the trace of \( h_{ij} \), in addition to two degrees of freedom for a massless graviton. Known results were sensitive to a gauge-fixing. If one chooses a gauge of \( n_i = 0 \) together with a Lagrange multiplier \( A \), then there remains a term of \( \dot{H}^2 \) in the quadratic action, which may imply that \( H \) is physical, but nonpropagating on the Minkowski background [1, 13]. On the other hand, choosing a gauge of \( A = E = 0 \) with a non-dynamical field \( B \) leads to two terms of \( c_1 \dot{\psi}^2 + c_2 (\partial_k \psi)^2 \), which implies that a gauge-invariant scalar \( \psi \) is a dynamically scalar degree of freedom [11]. If the trace \( \psi \) is really propagating on the Minkowski background, the deformed Hořava-Lifshitz gravity amounts to a scalar-tensor theory. However, it was known that a choice of gauge-fixing cannot be done, in general, by substituting the gauge condition into the action directly [22, 23, 24, 25]. Hence, we need to introduce another approach to confirm the propagation of scalar mode around Minkowski spacetimes.

In this work, we will not choose any gauge to identify physical scalar degrees of freedom.

One way to identify the physical degrees of freedom is to treat non-dynamical fields in the quadratic Lagrangian without fixing a gauge [26, 27]. In this work, we consider the Lagrangian formalism [27] only because the Hamiltonian formalism was not working for Hořava-Lifshitz gravity well, and thus, it has shown unwanted results for scalar degrees of freedom [28]. We would like to mention that there are two kinds of non-dynamical fields:

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1 In order to find propagators, first substituting the gauge-condition into the gauge-invariant bilinear action with parameter \( b^2 \), inverting, and then, taking the limit of \( b^2 \rightarrow \infty \). See Ref.[22] for the gauge-propagator in the Yang-Mills theory, Ref.[23] for graviton-propagator in general relativity, Ref.[24] for graviton-propagator in higher-derivative quantum gravity, and Ref.[25] for graviton-propagator in the Kaluza-Klein theory.
at the level of quadratic action, a non-dynamical field may enter the action either linearly or quadratically. As is shown in Eq. (39), for $\lambda \neq 1$, examples of the latter are two gauge-invariant modes $w_i$ and $\Pi$. These modes can be integrated out: their equations can be used to express these in terms of dynamical fields $\psi$ (the latter enters the action with time derivative) and then, one gets rid of these by plugging the resulting expression back into the action. Therefore, the number of dynamical fields is not reduced in this way. The other is that the action does not contain a quadratic term as a non-dynamical field. $A$ is the case for Hořava-Lifshitz gravity and $\Phi$ for general relativity. Unlike in the quadratic case, the corresponding equation is a constraint imposed on dynamical fields, and thus $A$ is a Lagrange multiplier. An important feature is that the constraint reduces the number of dynamical fields. This implies that Lagrange multipliers play the important role in finding physical degrees of freedom.

In the view of Faddeev-Jackiw constraints [26, 29], quadratic non-dynamical fields are superficial constraints and a linear non-dynamical field is a true constraint. Hence we wish to distinguish the former with notation (=) from the latter with (≈).

In order to compare the foliation-preserving diffeomorphism (FDiff) of the Hořava-Lifshitz gravity with others, we introduce transverse diffeomorphism (TDiff), full diffeomorphism (Diff), and Weyl-transverse diffeomorphism (WTDiff) for general relativity in the Appendix.

### 2 Deformed Hořava-Lifshitz gravity

First of all, we introduce the ADM formalism where the metric is parameterized

$$\text{d}s^2_{\text{ADM}} = -N^2 \text{d}t^2 + g_{ij}(dx^i - N^i \text{d}t)(dx^j - N^j \text{d}t),$$

Then, the Einstein-Hilbert action can be expressed as

$$S^{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{\mathcal{g}N} \left( K_{ij}K^{ij} - K^2 + R - 2\Lambda \right),$$

where $G$ is Newton’s constant and extrinsic curvature $K_{ij}$ takes the form

$$K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right).$$

Here, a dot denotes a derivative with respect to $t$ ("\·" = $\frac{\partial}{\partial t}$).

On the other hand, a deformed action of the non-relativistic renormalizable gravitational theory is given by [13]

$$S_{\text{dHL}} = \int \text{d}t \text{d}^3x \left( \mathcal{L}_0 + \mu^4 R + \mathcal{L}_1 \right),$$
\[ \mathcal{L}_0 = \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^4 (\Lambda_W R - 3 \Lambda_W^2)}{8(1 - 3 \lambda)} \right\}, \]  
(5)

\[ \mathcal{L}_1 = \sqrt{g} N \left\{ \frac{\kappa^2 \mu^4 (1 - 4 \lambda)}{32 (1 - 3 \lambda)} R^2 - \frac{\kappa^2}{2 w^4} \left( C_{ij} - \frac{\mu w^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu w^2}{2} R^{ij} \right) \right\}. \]  
(6)

where \( C_{ij} \) is the Cotton tensor

\[ C_{ij} = \epsilon^{ik\ell} \nabla_k \left( R_{j \ell} - \frac{1}{4} R \delta_{j \ell} \right). \]  
(7)

Comparing \( \mathcal{L}_0 \) with Eq. (2) of general relativity, the speed of light, Newton’s constant and the cosmological constant are given by

\[ c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda_W}{1 - 3 \lambda}}, \quad G = \frac{\kappa^2}{32 \pi c}, \quad \Lambda = \frac{3}{2} \Lambda_W. \]  
(8)

The equations of motion were derived in [18] and [20], but we do not write them due to the length.

In the limit of \( \Lambda_W \to 0 \), we obtain the \( \lambda \)-Einstein action from \( \mathcal{L}_0 + \mu^4 R \) as

\[ S_{EH\lambda} = \int dt d^3 x \sqrt{g} N \left[ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \mu^4 R \right]. \]  
(9)

In this case, we have Minkowski background with [13]

\[ c^2 = \frac{\kappa^2 \mu^4}{2}, \quad G = \frac{\kappa^2}{32 \pi c}, \quad \Lambda = 0. \]  
(10)

Considering the \( z = 3 \) Hořava-Lifshitz gravity, we have scaling dimensions of \( [t] = -3, \ [x] = -1, \ [\kappa] = 0, \) and \([\mu] = 1\). We wish to consider perturbations of the metric around Minkowski spacetimes, which is a solution of the full theory [4]

\[ g_{ij} = \delta_{ij} + wh_{ij}, \quad N = 1 + wn, \quad N_i = wn_i. \]  
(11)

At quadratic order the action [9] turns out to be

\[ S_{2EH\lambda} = w^2 \int dt d^3 x \left\{ \frac{1}{\kappa^2} \left[ \frac{1}{2} \dot{h}_{ij}^2 - \frac{\lambda}{2} \dot{h}^2 + (\partial_i n_j) \dot{h}^2 - (1 - 2 \lambda)(\partial \cdot n)^2 - 2 \partial_i n_j (\dot{h}_{ij} - \lambda \dot{h} \delta_{ij}) \right] \right. \]

\[ \left. + \frac{\mu^4}{2} \left\{ - \frac{1}{2} (\partial_k h_{ij})^2 + \frac{1}{2} (\partial_i h)^2 + (\partial_i h_{ij})^2 - \partial_i h_{ij} \partial_j h + 2n (\partial_i \partial_j h_{ij} - \partial^2 h) \right\} \right\}. \]  
(12)

In order to analyze the physical degrees of freedom completely, it is convenient to use the cosmological decomposition in terms of scalar, vector, and tensor modes under spatial rotations \( SO(3) \) [30]

\[ n = -\frac{1}{2} A, \]

\[ n_i = (\partial_i B + V_i), \]  
(13)

\[ h_{ij} = (\psi \delta_{ij} + \partial_i \partial_j E + 2 \partial_i F_j + t_{ij}), \]
where $\partial^i F_i = \partial^i V_i = \partial^i t_{ij} = t^i_{\ i} = 0$. The last two conditions mean that $t_{ij}$ is a transverse and traceless tensor in three dimensions. Using this decomposition, the scalar modes $(A, B, \psi, E)$, the vector modes $(V_i, F_i)$, and the tensor modes $(t_{ij})$ decouple from each other. These all amount to 10 degrees of freedom for a symmetric tensor in four dimensions.

Before proceeding, let us check dimensions. We observe $[n] = 0$, $[n_i] = 2$, and $[h_{ij}] = 0$, which imply $[A] = 0$, $[B] = 1$, $[V_i] = 2$, $[\psi] = 0$, $[E] = -2$, $[F_i] = -1$, and $[t_{ij}] = 0$.

The Lagrangian is obtained by substituting (13) into the quadratic action (12) as

$$S_2^{E\Lambda} = \int dt d^3x \left\{ \frac{w^2}{2\kappa^2} \left[ 3(1-3\lambda)\dot{\psi}^2 + 2\partial_i \omega_j \partial^i \omega^j - 4 \left( (1-3\lambda)\dot{\psi} + (1-\lambda)\partial^2 \dot{E} \right) \partial^2 B \right. \\
+ 4(1-\lambda)(\partial^2 B)^2 + 2(1-3\lambda)\dot{\psi} \partial^2 \dot{E} + (1-\lambda)(\partial^2 \dot{E})^2 + \dot{t}_{ij} t^{ij} \right\}$$

with $w_i = V_i - \dot{F}_i$.

On the other hand, the higher order action obtained from $\mathcal{L}_1$ takes the form

$$S_2 = \int dt d^3x \left\{ \frac{\kappa^2 \mu^2 w^2}{8} \left[ - \frac{1+\lambda}{2(1-3\lambda)} \psi \partial^4 \psi - \frac{1}{4} t_{ij} \partial^4 t^{ij} \\
+ \frac{1}{\mu w^2} \epsilon^{ijk} \partial_i \partial^4 \partial_j t^l_k + \frac{1}{\mu^2 w^4} \partial^4 t_{ij} \partial^4 t^{ij} \right] \right\}. \quad (15)$$

We observe that two modes of $\psi$ and $t_{ij}$ exist in the higher order action.

Now we are in a position to discuss the diffeomorphism in the $z = 3$ Hořava-Lifshitz gravity. Since the anisotropic scaling of temporal and spatial coordinates ($t \rightarrow b^z t, x^i \rightarrow b x^i$), the time coordinate $t$ plays a privileged role. Hence, the spacetime symmetry is smaller than the full diffeomorphism (Diff) in the standard general relativity (Einstein gravity). The Hořava-Lifshitz gravity of $S_2^{E\Lambda} + S_2^\perp$ should be invariant under the “foliation-preserving” diffeomorphism (FDiff) whose form is given by

$$t \rightarrow \tilde{t} = t + \epsilon^0(t), \quad x^i \rightarrow \tilde{x}^i = x^i + \epsilon^i(t, x). \quad (16)$$

Using the notation of $\epsilon^\mu = (\epsilon^0, \epsilon^i)$ and $\epsilon_\nu = \eta_{\mu\nu} \epsilon^\mu$, the perturbation of metric transforms as

$$\delta g_{\mu\nu} \rightarrow \delta \tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \quad (17)$$

Further, making a decomposition $\epsilon^i$ into a scalar $\xi$ and a pure vector $\zeta^i$ as $\epsilon^i = \partial^i \xi + \zeta^i$ with $\partial_i \zeta^i = 0$, one finds the transformation for the scalars

$$A \rightarrow \tilde{A} = A - 2\epsilon^0, \quad \psi \rightarrow \tilde{\psi} = \psi, \quad B \rightarrow \tilde{B} = B + \zeta, \quad E \rightarrow \tilde{E} = E + 2\xi. \quad (18)$$

On the other hand, the vector and the tensor take the forms

$$V_i \rightarrow \tilde{V}_i = V_i + \zeta_i, \quad F_i \rightarrow \tilde{F}_i = F_i + \zeta_i, \quad t_{ij} \rightarrow \tilde{t}_{ij} = t_{ij}. \quad (19)$$
Considering scaling dimensions of \([\epsilon^0] = -3\) and \([\epsilon^i] = -1\), we have \([\xi] = -2\) and \([\zeta^i] = -1\). For the FDiff transformations, gauge-invariant combinations are

\[
t_{ij}, \ w_i = V_i - \dot{F}_i, \tag{20}
\]

for tensor and vector, respectively and

\[
(\psi, \ \Pi = 2B - \dot{E}) \tag{21}
\]

for two scalar modes. Finally, we note scaling dimensions: \([w_i] = 2\) and \([\Pi] = 1\). We emphasize that “A” leaves a gauge-dependent quantity alone. For other gauge-invariant scalars in general relativity, see the Appendix.

3 \(n_i = 0\) gauge-fixing

Firstly, we may consider a gauge of \(n_i = 0\) \([\xi, \Pi][13]\). It amounts to the gauge-fixing:

\[
B = 0, \ V_i = 0. \tag{22}
\]

Then, the bilinear action takes the form

\[
S_2^{EH\lambda} = \int dt d^3x \left\{ \frac{\omega}{2\kappa^2} \left[ 3(1 - 3\lambda)\dot{\psi}^2 + 2\partial_i\dot{F}_j\partial^j\dot{F}^i + 2(1 - 3\lambda)\dot{\psi}\partial^2\dot{E}_i + (1 - \lambda)(\partial^2\dot{E})^2 + i_{ij}i^{ij} \right] \\
+ \frac{\mu^4\omega}{4} \left[ 2\partial_k\psi\partial^k\dot{\psi} + 4A\partial^2\dot{\psi} - \partial_k t_{ij}\partial^k t^{ij} \right] \right\}. \tag{23}
\]

It is obvious that \(A\) is a Lagrange multiplier and thus it provides a constraint

\[
\partial^2\dot{\psi} \approx 0. \tag{24}
\]

It is emphasized that the notation “\(\approx\)” is used to denote the constraint obtained by varying the Lagrange multiplier only. We may consider \(\dot{E}\) and \(\dot{F}_i\) as non-dynamical fields even though they have time derivatives. Since gauge-invariant quantities are given by \(\Pi = 2B - \dot{E}\) and \(w_i = V_i - \dot{F}_i\), it seems that canonical variables are not \(E\) and \(F_i\) but \(\dot{E}\) and \(\dot{F}_i\). Hence, in order to eliminate these fields, we use their variations

\[
\partial^2\dot{E} = -\frac{(1 - 3\lambda)}{(1 - \lambda)}\dot{\psi}, \ \dot{F}_i = 0. \tag{25}
\]

Substituting these into the quadratic action, we have the relevant one

\[
S_2^{EH\lambda} = \int dt d^3x \left\{ \frac{w^2}{2\kappa^2} \left[ \frac{(2(1 - 3\lambda))}{(1 - \lambda)}\dot{\psi}^2 + i_{ij}i^{ij} \right] - \frac{\mu^4w^2}{4} \partial_k t_{ij}\partial^k t^{ij} \right\}. \tag{26}
\]
It is clear that for \( \lambda \neq 1, 1/3 \), the scalar \( \psi \) is not a propagating mode on the Minkowski background because of the constraint (24), while \( t_{ij} \) represents for a massless graviton propagation. On the other hand, the bilinear action to \( \mathcal{L}_1 \) leads to

\[
S_2^1 = \int dt d^3x \frac{k^2 \mu^2 w^2}{8} \left\{ -\frac{1}{4} t_{ij} \partial^4 r^{ij} + \frac{1}{\mu w^2} \epsilon^{ijk} t_{il} \partial^4 \partial_j t^k + \frac{1}{\mu^2 w^4} t_{ij} \partial^6 r^{ij} \right\}.
\]

Plugging \( \psi \rightarrow \psi - \lambda \), \( t_{ij} \rightarrow \tilde{H}_{ij} \) into Eqs. (26) and (27) with \( x^0 = ct (\lfloor x_0 \rfloor = -1, \lfloor c \rfloor = 2) \), one arrives at the quadratic action exactly (13)

\[
S^H_{2L} = \int dx^0 d^3x \left\{ \frac{w^2 c^2}{2k^2} \left( \partial_0 \tilde{H}_{ij} \right)^2 - \frac{\mu^4 \kappa^2}{2c^2} \left( \partial_0 \tilde{H}_{ij} \right)^2 \right\} + \frac{w^2 c(1 - \lambda)}{4\kappa^2(1 - 3\lambda)} \left( \partial_0 H \right)^2
\]

\[
+ \frac{\kappa^2 \mu^2 w^2}{8c} \left\{ -\frac{1}{4} \tilde{H}_{ij} \partial^4 \tilde{H}^{ij} + \frac{1}{\mu w^2} \epsilon^{ijk} \tilde{H}_{il} \partial^4 \tilde{H}^{jk} + \frac{1}{\mu^2 w^4} \tilde{H}_{ij} \partial^6 \tilde{H}^{ij} \right\}.
\]

Note that for \( 1/3 < \lambda < 1 \), the kinetic term of \( H \) becomes negative, indicating a ghost instability. Thus, one may argue that either \( \lambda \) runs to \( 1^+ \) from above in the IR or \( H \) does not couple at all to matter. However, this may not be a promising way to resolve the ghost problem. A correct answer is that the scalar mode of \( H \propto \psi \) is a nonpropagating mode.

We also see from (29) that the speed of gravitational interaction is

\[
c_g^2 = \frac{\mu^4 \kappa^2}{2c^2 c_0^2},
\]

where \( c_0^2 \) is the speed of light. We know that the propagation of gravity interaction equals the velocity of light to better than \( 1 : 1000 \). Hence, we get that

\[
c^2 = \frac{\mu^4 \kappa^2}{2}
\]

with the above accuracy, independent of the value of the couplings.

Finally, we have the quadratic action

\[
S^H_{2L} = \int dx^0 d^3x \left\{ \frac{w^2 c^2}{2k^2} \left( \partial_0 \tilde{H}_{ij} \right)^2 + \frac{w^2 c(1 - \lambda)}{4\kappa^2(1 - 3\lambda)} \left( \partial_0 H \right)^2
\]

\[
+ \frac{\kappa^2 \mu^2 w^2}{8c} \left\{ -\frac{1}{4} \tilde{H}_{ij} \partial^4 \tilde{H}^{ij} + \frac{1}{\mu w^2} \epsilon^{ijk} \tilde{H}_{il} \partial^4 \tilde{H}^{jk} + \frac{1}{\mu^2 w^4} \tilde{H}_{ij} \partial^6 \tilde{H}^{ij} \right\}.
\]

4. \( A = 0 \) and \( E = 0 \) gauge-fixing

In the perturbation, the lapse function \( n \) is a function of \( t \) only, and thus, \( A \) is a function of \( t \). It may allow \( A \) to be a gauge degree of freedom by choosing a initial time \( t_0 \). Also, we
may choose $E$ as a gauge degree of freedom. In this section, we start with a gauge-fixing [11]:

$$A = 0, \ E = 0.$$  \tag{33}$$

Then, the bilinear action takes the form

$$S_{EH}^{EH} = \int dt d^3\mathbf{x} \left\{ \frac{w^2}{2\kappa^2} \left( 3(1 - 3\lambda)\dot{\psi}^2 - 4(1 - 3\lambda)\dot{\psi}\partial^2 B + 4(1 - \lambda)B\partial^4 B \right) + 2\partial_t w_i \partial^k w^i_i + \frac{\mu^4 w^2}{4} \left( 2\partial_k \partial_k \psi - \partial_k t_{ij} \partial^k t^{ij} \right) \right\}.$$  \tag{34}$$

For $\psi \rightarrow -2\Psi$, the first line (33) recovers those of Ref.[11] with $a = 1$ and $\Lambda = 0$. We observe that $B$ and $w_i$ are non-dynamical fields because they do not have time derivatives. Hence, in order to eliminate these, we use their variations

$$\partial^2 B = \frac{(1 - 3\lambda)}{2(1 - \lambda)} \dot{\psi}, \quad w_i = 0.$$  \tag{36}$$

Substituting these relations into the quadratic action, we have the relevant one

$$S_{EH}^{EH} = \int dt d^3\mathbf{x} \left\{ \frac{w^2}{2\kappa^2} \left( \frac{2(1 - 3\lambda)}{1 - \lambda} \dot{\psi}^2 + \dot{t}_{ij} t^{ij} \right) \right\}.$$  \tag{37}$$

It seems that for $\lambda \neq 1, 1/3$, the scalar “$\psi$” is propagating on the Minkowski background, in addition to $t_{ij}$ for a massless graviton propagation. This is because a kinetic term $(\partial_k \psi)^2$ survives because a gauge condition of $A = 0$ does not impose any constraint. However, the sign of $(\partial_k \psi)^2$ is opposite to that of $\partial_k t_{ij} \partial^k t^{ij}$ and thus, it may not lead to a proper scalar propagation on the Minkowski background.

On the other hand, the bilinear action to $\mathcal{L}_1$ leads to

$$S_1^{EH} = \int dt d^3\mathbf{x} \frac{\kappa^2 \mu^2 w^2}{8} \left\{ - \frac{1 + \lambda}{2(1 - 3\lambda)} \dot{\psi}^4 \dot{\psi} - \frac{1}{4} \dot{t}_{ij} \dot{t}^{ij} + \frac{1}{\mu w^2} \epsilon^{ijk} \partial_\ell \partial_\ell \partial^4 t^l_{ij} + \frac{1}{\mu^2 w^4} \dot{t}_{ij} \partial^\ell t^{ij}_\ell \right\},$$  \tag{38}$$

where the first term represents a fourth order for the scalar $\psi$. This term survives because a gauge condition of $A = 0$ was chosen.

### 5 Without gauge-fixing

One may identify physical degrees of freedom, without fixing any gauge, by treating non-dynamical fields in [14] properly. First of all, we express the quadratic action [14] in terms
of gauge-invariant quantities of the scalar, vector, and tensor modes as

\[
S_{EH\lambda}^{2} = \int dt d^{3}x \left\{ \frac{w^{2}}{2\kappa^{2}} \left[ 3(1 - 3\lambda)\dot{\psi}^{2} - 2w_{i} \Delta \omega^{i} - 2(1 - 3\lambda)\dot{\psi} \Delta \Pi + (1 - \lambda)(\Delta \Pi)^{2} + i_{ij}i^{ij} \right] + \mu A \frac{w^{2}}{4} \left[ -2\psi \Delta \psi + 4A \Delta \psi + t_{ij} \Delta t^{ij} \right] \right\} 
\]

(39)

with \( \Delta = \partial_{i}\partial^{i} = \partial^{2} \). We note that \( S_{1}^{1} \) in (15) contains only \( \psi \) and \( t_{ij} \), which are also gauge-invariant. It is emphasized again that “\( A \)” is not a gauge-invariant quantity and thus, it should be eliminated in the consistent quadratic action. Fortunately, this is possible because it belongs to a Lagrange multiplier, irrespective of any value \( \lambda \).

Before proceeding, we mention two special cases: \( \lambda = 1/3 \) and \( \lambda = 1 \). Plugging \( \lambda = 1/3 \) into the above action, we have a term like \( \dot{\psi}^{2} \). In addition, we have two non-dynamical fields \((w_{i}, \Pi)\) and one Lagrange multiplier \((A)\) which provide two relations and one constraint as, respectively

\[
\Delta w_{i} = 0, \quad \Delta \Pi = 0, \quad \Delta \psi \approx 0. 
\]

(40)

This implies that the \( t_{ij} \) are only propagating tensor modes. Similarly, for \( \lambda = 1 \), one have no scalar mode \( \psi \) definitely because of one relation and two constraints from one non-dynamical field \((w_{i})\) and two Lagrange multipliers \((\Pi, A)\):

\[
\Delta w_{i} = 0, \quad \dot{\psi} \approx 0, \quad \Delta \psi \approx 0. 
\]

(41)

Note here that for \( \lambda \neq 1/3, 1 \), \( \Pi \) and \( w_{i} \) are two non-dynamical fields to be solved to have two relations

\[
\Delta \Pi = \frac{(1 - 3\lambda)}{(1 - \lambda)} \dot{\psi}, \quad w_{i} = 0. 
\]

(42)

Substituting these into the quadratic action, we have

\[
S_{EH\lambda}^{2} = \int dt d^{3}x \left\{ \frac{\omega^{2}}{2\kappa^{2}} \left[ \frac{2(1 - 3\lambda)}{(1 - \lambda)} \ddot{\psi}^{2} + i_{ij}i^{ij} \right] + \mu A \frac{\omega^{2}}{4} \left[ -2\psi \Delta \psi + 4A \Delta \psi + t_{ij} \Delta t^{ij} \right] \right\}. 
\]

(43)

Here we observe that for \( 1/3 < \lambda < 1 \), a ghost appears because there is a negative kinetic term for \( \psi \). Also, comparing \(-2\psi \Delta \psi \) with \( t_{ij} \Delta t^{ij} \), we find a negative spatial derivative term for scalar \( \psi \). Hence it should not be a propagating mode on the Minkowski background. Since \( A \) is a Lagrange multiplier, its variation provides a constraint

\[
\Delta \psi \approx 0. 
\]

(44)

Then, we have the bilinear action without \( A \)
\[ S_{EH}^{\lambda} = \frac{\omega^2 c}{2\kappa^2} \int d^4x \left[ \frac{2(1 - 3\lambda)}{(1 - \lambda)} (\partial_0 \psi)^2 + \partial_0 t_{ij} \partial_0 t^{ij} - \frac{\mu^2 \kappa^2}{2c^2} \partial_k t_{ij} \partial^k t^{ij} \right] \]  

(45)

with \( x^0 = ct \). Using \( c^2 = \mu^4 \kappa^2 / 2 \), we may have the relativistic action for graviton

\[ S_{EH}^{\lambda} = \frac{\omega^2 c}{2\kappa^2} \int d^4x \left[ \frac{2(1 - 3\lambda)}{(1 - \lambda)} (\partial_0 \psi)^2 + t_{ij} \Box t^{ij} \right], \]

\[ (46) \]

which implies that the scalar mode is not propagating even for \( \lambda > 1 \) because it contains \((\partial_0 \psi)^2\) only, while the tensor mode (graviton) is propagating on the Minkowski background. Here \( \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu \) with \( \eta_{\mu\nu} = \text{diag}(-, +, +, +) \). Finally, the higher order action \( S_{1/2} \) is given by (26). However, this action does not determine whether a mode is propagating or not.

We would like to mention that \( \psi \) is a non-propagating mode under the \( n_i = 0 \) gauge with a Lagrange multiplier \( A \) in Section 3, but it is a propagating mode under the \( A = E = 0 \) gauge with two non-dynamical fields \( B \) and \( w_i \) in Section 4. It seems that the origin of this discrepancy is due to different gauge-fixings. However, it was known that a choice of gauge-fixing cannot be done, in general, by substituting the gauge condition into the action directly \([22, 23, 24, 25]\). Hence, our approach is a consistent mathematical formalism for checking the absence of new degrees of freedom around the Minkowski background.

### 6 Discussions

A hot issue of Hořava-Lifshitz gravity is to clarify whether it can accommodate a scalar mode as the trace of \( h_{ij} \), in addition to two degrees of freedom for a massless graviton. Actually, known results were sensitive to a gauge-fixing. If one chooses a gauge of \( n_i = 0 \) \((B = V_i = 0)\) together with a Lagrange multiplier \( A \) (equivalently, \( \partial^2 \psi \approx 0 \)) and two non-dynamical fields \((\dot{E}, \dot{F}_i)\) there remains a term of \( \dot{H}^2 \) in the action, which implies that \( H \) is nonpropagating \([1, 13]\). On the other hand, choosing a gauge of \( A = E = 0 \) together with two non-dynamical fields \((B, w_i)\) leads to two terms of \( c_1 \dot{\psi}^2 + c_2 (\partial_k \psi)^2 \), which may imply that a gauge-invariant scalar \( \psi \) is a propagating scalar degree of freedom \([11]\).

In this work, we did not choose any gauge to identify physical scalar degrees of freedom. Without fixing a gauge, one could identify physical degrees of freedom by treating two non-dynamical fields \((w_i, \Pi_i)\) and one Lagrange multiplier \( A \) appropriately. This means that Lagrange multiplier plays the important role in finding physical degrees of freedom. In the foliation-preserving diffeomorphism (FDiff), gauge-invariant scalars are \( \psi \) and \( \Pi \), while the lapse perturbation \( A \propto n \) is a gauge-dependent scalar. Thus, the latter should be eliminated from the quadratic action. It is either a function \( A(t) \) when imposing the projectability condition or a function \( A(t, x) \) without the projectability condition. Because \( A \) is a Lagrange multiplier, we could always use it to obtain a constraint \( \triangle \psi \approx 0 \) and thus,
Table 1: Summary for scalar modes. GR (HL) means general relativity (deformed Hořava-Lifshitz gravity). GIS denotes gauge-invariant scalars. SDoF and TDoF mean number of scalar and tensor degrees of freedom, respectively. Here $\Phi = A - 2\dot{B} + \ddot{E} = A - \dot{\Pi}$, $\Theta = A - \Delta E$, and $\Pi = 2B - \dot{E}$.

| diffeomorphism | TDiff | Diff | WTDiff | FDiff |
|----------------|-------|------|--------|-------|
| Theory         | GR    | GR   | GR     | HL    |
| parameters     | $a \neq 1, b \neq 1$ | $a = b = 1$ | $a = 1/2, b = 3/8$ | $\lambda \neq 1, 1/3$ |
| GIS            | $\psi, \Phi, \Theta$ | $\psi, \Phi$ | $\Xi = \psi + \Phi, \Upsilon = \psi + \Theta$ | $\psi, \Pi$ |
| SDoF           | $1(\psi)$ | 0    | 0      | 0     |
| TDoF           | $2(t_{ij})$ | $2(t_{ij})$ | $2(t_{ij})$ | $2(t_{ij})$ |

$\psi$ is not a propagating scalar mode. A gauge-invariant scalar $\Phi = A - \dot{\Pi}$ emerging in general relativity is split into a gauge-dependent scalar $A$ and a gauge-invariant scalar $\Pi$, due to the FDiff. We note that $\Phi$ is a Lagrange multiplier in TDiff and Diff as well as $A$ is Lagrange multipliers in the deformed Hořava-Lifshitz gravity.

We compare FDiff with different diffeomorphisms in general relativity in Table 1. As the general analysis was shown in the Appendix, it is not easy to have a scalar mode in four-dimensional general relativity. The TDiff case has less symmetry than Diff and WTDiff cases. One has to realize that the TDiff case has three gauge-invariant scalars, thanks to an additional condition of $\partial_\mu \epsilon^\mu = 0$. This case provides really a scalar mode which is propagating on the Minkowski background. Two cases of Diff and WTDiff correspond to enhanced diffeomorphisms. As a result, there are two gauge-invariant scalars and thus, no propagating scalar mode. The FDiff of the Hořava-Lifshitz gravity is similar to Diff and WTDiff cases, which have enhanced diffeomorphisms, compared with the TDiff. Hence, we expect to have no propagating scalar mode in the deformed Hořava-Lifshitz gravity.

We would like to mention a couple of recent works. The authors [16] have shown that $\psi$ is a scalar degree of freedom appeared when the massless limit of a massive graviton (vDVZ discontinuity [31]). Using the Hamiltonian constraints, the authors [17] have argued that a scalar mode of $\psi$ is propagating around the Minkowski space but it has a negative kinetic term, providing a ghost mode [17]. Hence, it was strongly suggested that it is desirable to eliminate this scalar mode if at all possible.

Consequently, we have shown that the deformed Hořava-Lifshitz gravity has no scalar mode which is propagating on the Minkowski background.

Note added—after the present work was released, relevant works on extra scalar mode have appeared on the arXiv. The authors [37] have shown that on the cosmological back-
ground, the extra scalar is non-dynamical. One of authors has found that $\psi$ is a scalar degree of freedom related to the massless limit of the case with Fierz-Pauli mass terms [38]. However, using the Lorentz-violating mass terms, there is no such a scalar appeared in the massless limit. Also, the authors in [39] have found that for a general background, the extra mode is propagating. The extra mode satisfies equation of motion which is first order in time derivatives. At linear level, thus, the mode is manifest only around spatially inhomogeneous and time-dependent background with two serious problems. However, the Minkowski spacetime is a singular point. Furthermore, the authors [40] have shown that the extra mode is not allowed because of its ghost-like instability around the Minkowski background.

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**Appendix: General relativity with different diffeomorphisms**

The most general relativistic Lagrangian for a massless symmetric tensor field $h_{\mu\nu}$ is given by [32, 33]

$$L_{GR} = L^I + \beta L^{II} + a L^{III} + b L^{IV},$$

(47)

where

$$L^I = \frac{1}{4} \partial_\mu h^{\nu\rho} \partial^\mu h_{\nu\rho}, \quad L^{II} = -\frac{1}{2} \partial_\mu h^{\mu\rho} \partial_\nu h^{\nu\rho},$$

$$L^{III} = \frac{1}{2} \partial^\mu h \partial_\rho h_{\mu\rho}, \quad L^{IV} = -\frac{1}{4} \partial_\mu h \partial^\mu h.$$ 

(48)

Under a general transformation of the fields $h_{\mu\nu} \rightarrow h_{\mu\nu} + \delta h_{\mu\nu}$, we have up to total derivatives

$$\delta L^I = -\frac{1}{2} \delta h_{\mu\nu} \Box h^{\mu\nu}, \quad \delta L^{II} = \delta h_{\mu\nu} \partial^\rho \partial^{(\mu} h^{\nu)\rho},$$

$$L^{III} = -\frac{1}{2} \left( \delta h \partial^\mu \partial^{\nu} h_{\mu\nu} + \delta h_{\mu\nu} \partial^\rho \partial^\mu h \partial^{\nu} h \right), \quad L^{IV} = \frac{1}{2} \delta h \Box h.$$ 

(49)

We note that the vector Lagrangian is problematic unless $\beta = 1$ because it induces a ghost problem [33]. Hence, we choose $\beta = 1$ case. It follows that the combination

$$L_{TDiff} = L^I + L^{II} + a L^{III} + b L^{IV},$$

(50)
with arbitrary $a$ and $b$ is invariant under restricted gauge transformations

$$\delta h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \tag{51}$$

with

$$\partial_\mu \epsilon^\mu = 0. \tag{52}$$

It is noted that $\epsilon^0(t, x)$ and $\epsilon^i(t, x)$. We call the transformations (51) and (52) transverse diffeomorphisms (TDiff) \[34, 35\]. We can obtain two enhanced gauge symmetries by adjusting parameters $a$ and $b$: Firstly, $a = b = 1$ leads to the Fierz-Pauli Lagrangian which is invariant under the full diffeomorphisms (Diff), where the condition (52) is dropped \[36\]. This corresponds to the standard general relativity (Einstein gravity). Secondly, $a = 1/2, b = 3/8$ provides Weyl symmetry of $h_{\mu\nu} \rightarrow h_{\mu\nu} + \frac{2}{3} \eta_{\mu\nu}$, in addition to TDiff. We call this enhanced symmetry the Weyl-transverse diffeomorphisms (WTDiff) \[32\].

Now let us investigate mode propagations when using the TDiff. Considering the decomposition (11) with (13), we have the same transformations in Eqs. (18) and (19) except replacing $B \rightarrow \tilde{B} = B + \dot{\xi}$ by

$$B \rightarrow \tilde{B} = B - \epsilon^0 + \dot{\xi} \tag{53}$$

in general relativity. In this case, using the residual gauge condition of Eq.(52) which implies $\dot{\epsilon}_0 = \partial^2 \xi$, we have three gauge-invariant scalars,

$$\left( \psi, \Phi = A - 2\dot{B} + \dot{E}, \Theta = A - \partial^2 E \right). \tag{54}$$

Substituting (11) and (13) into (50) leads to

$$\mathcal{L}_{\text{TDiff}} = \mathcal{L}_{\text{TDiff}}^t + \mathcal{L}_{\text{TDiff}}^v + \mathcal{L}_{\text{TDiff}}^s, \tag{55}$$

where

$$\mathcal{L}_{\text{TDiff}}^t = \frac{1}{4} t_{ij} \square t^{ij}, \quad \mathcal{L}_{\text{TDiff}}^v = -\frac{1}{2} w_i \triangle w^i, \tag{56}$$

for tensor and vector modes and

$$\mathcal{L}_{\text{TDiff}}^s = \frac{1}{4} \left( 3\dot{\psi}^2 + \psi \triangle \psi - \dot{\Theta}^2 - \Theta \triangle (\Theta - 2\Phi) - 2 \triangle \psi (\Phi - \Theta) \right) + \frac{a}{2} \left( (\Theta - 3\psi)(\triangle (\Theta - \psi - \Phi) - \dot{\Theta}) \right) - \frac{b}{4} \left( (\Theta - 3\psi)^2 + (\Theta - 3\psi) \triangle (\Theta - 3\psi) \right) \tag{57}$$

for all scalar modes. From this decomposition, we realize that $\Phi$ is always a Lagrange multiplier whose variation yields the constraint

$$\triangle \left[ (1 - 3a)\psi - (1 - a)\Theta \right] \approx 0. \tag{59}$$
In this case, the Lagrangian reduces to

\[ \mathcal{L}_{TDiff}^s = \frac{Z}{(a - 1)^2} \psi \Box \psi, \quad \text{with} \quad Z = \frac{3}{2} (a - \frac{1}{3})^2 - (b - \frac{1}{3}) \]  

which implies that for \( b < 1/3 \), \( \psi \) is really a propagating scalar mode on the Minkowski background. For two cases of \( a = b = 1 \) and \( a = 1/2, b = 3/8 \), we have \( Z = 0 \), which implies that these should be treated separately.

In the Diff case of \( a = b = 1 \), only two scalar combinations are gauge invariant, namely

\[ \left( \Phi, \psi \right) \]  

Then, its Lagrangian takes the form

\[ \mathcal{L}_{Diff}^s = -\frac{1}{2} \left( -2 \Phi \triangle \psi + 3 \dot{\psi}^2 + \psi \triangle \dot{\psi} \right) \]  

However, since \( \Phi \) is a Lagrange multiplier, its variation leads to \( \triangle \dot{\psi} \approx 0 \). Plugging this into the above, we have

\[ \mathcal{L}_{Diff}^s = -\frac{3}{2} \psi^2, \]  

which means that \( \psi \) is not propagating on the Minkowski background.

Finally, for Weyl transformations of \( a = 1/2 \) and \( b = 3/8 \), we have two scalar invariants which are also scalar for TDiff,

\[ \left( \Xi = \Phi + \psi, \quad \Upsilon = \Theta + \psi \right) \]  

Then, its Lagrangian is given by

\[ \mathcal{L}_{WTDiff}^s = -\frac{1}{96} \left( 2(8 \Xi - 3 \Upsilon) \triangle \Upsilon - 6 \dot{\Upsilon}^2 \right) \]  

However, since \( \Xi \) is a Lagrange multiplier, its variation leads to \( \triangle \Upsilon \approx 0 \). Plugging this into the above, we have

\[ \mathcal{L}_{WTDiff}^s = -\frac{1}{16} \dot{\Upsilon}^2, \]  

which means that \( \Upsilon \) is not propagating on the Minkowski background.

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