Shape design of heat convection fields based on the adjoint method

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Abstract
The utilization of shape design to improve heat transfer characteristics in heat convection fields is an important subject in engineering. This review paper explains a numerical solution for the shape design problems involving steady and unsteady heat convection fields. Reshaping for the shape design is carried out by traction method, which is proposed as an approach to solving shape optimization problems. The traction method is a shape optimization method with some advantages such as deformation analysis of pseudo-elastic bodies can be used in the shape update analysis and smoothness in shape update can be maintained. In steady-state heat convection fields, shape design problems that control the temperature distribution in the sub-domains of the heat convection field are introduced. The square integration error between the actual temperature distribution and the target temperature distribution in the specified sub-domains is employed as the objective functional for the shape design. In unsteady heat convection fields, two shape design problems are shown; these problems also involve control of the temperature distribution in sub-domains of the field. In the first problem, in a manner similar to the steady-state problem, the square integration error of the sub-domains during the specified period of time is used as the objective functional. In the second problem, a multi-objective shape design problem that utilizes a normalized objective functional is formulated for the problem to prescribe the temperature distribution and the total dissipated energy minimization problem. The shape gradient of these shape design problems for steady and unsteady fields is derived theoretically using the Lagrange multiplier method, adjoint variable method, and the formulae of the material derivative. Numerical analysis programs for shape design problems using the traction method are developed, and the validity of the demonstrated method is confirmed by results of 2D numerical analyses.

Keywords: Optimum design, Inverse problem, Computer aided design, Computational mechanics, Finite element method, Heat-convection field, Adjoint method, Traction method

1. Introduction

Shape design problems that improve the characteristics of heat transfer in thermal convection fields are an important subject in engineering. A typical example of such a problem can be seen in the design process used to create a heat exchanger. In this problem, the shape of the heat exchanger is optimized to maximize the heat discharge on the sub-boundaries of the convection field. Moreover, the problem of determining the boundary shape that can achieve the desired state distribution function of temperature or flow velocity on specified sub-boundaries, or in specified sub-domains, in a heat convection field is known as an inverse problem. If we regard the inverse problem as designing the shape needed to minimize the integrated squared error between the state distribution function of the actual temperature distribution and the target distribution function, then it can be treated as one optimization problem. This study discusses the solution of the inverse problem and the shape optimization problem with regard to the shape design of the heat convection field domain.

Research on the shape optimization of flow fields began with Pironneau (Pironneau, 1973, 1974), who proposed a method for updating shapes by treating displacements in the normal direction of the boundary as design variables in the distribution system. Further, Pironneau evaluated the shape gradient functions (sensitivities) by selecting the node
coordinates on the boundary of a finite-element model as design variables. When the finite element method (FEM) was used for numerical analysis, the mesh within the region had to be updated after the boundary was updated (Pironneau, 1984). An unstable phenomenon involving the undulation of boundary shapes has been known to occur when shape gradient functions related to the displacement of such design variables were evaluated, and the resulting sensitivities were used to move nodes (Imam, 1982). Jameson (1995) proposed a numerical method for updating the boundary shapes using an elliptical equation that was defined on the boundary. This method replaced the actual sensitivities with smooth sensitivities in order to facilitate the use of second-order differentiation (conducted during the sensitivity evaluation) when updating shapes in flow fields. Mohammadi and Pironneau (2001) also proposed a similar method.

Shape design in heat convection fields has been studied by many researchers. For example, Hilbert et al. (2006), Copiello and Fabbri (2009), and Kashani et al. (2013) have solved shape optimization problems involving heat exchangers. They have reduced the number of design variables needed to express boundary shapes to the greatest extent possible, and have proposed optimization methods based on an evolutionary method, such as the genetic algorithm. Husain and Kim (2010) analyzed the optimum shape of heat sinks using response surface approximations. Cheng and Chamg (2003) suggested shape optimization that considered the coupling problem on the common boundary of a heat convection boundary and an elastic body boundary.

Research on shape design in heat convection fields based on the gradient method and adjoint variables has been conducted by Momose et al. (2009), Park and Ku (2001), Park and Shin (2003), Morimoto et al. (2010), and Aounallah et al. (2013). The adjoint variable method has advantageous calculation efficiency in its solutions of optimization problems. The method enables sensitivity calculations of large-scale design variables for complex coupled fields, and more detailed explanations are presented in the references (Azegami, 2016; Gunzburger, 2003; Haslinger and Makinen, 2003). Momose et al. (2009) have proposed a shape sensitivity analysis method that seeks to maximize the velocity in the sub-domain in natural convection fields; however, this method was limited to steady-state problems. Park and Ku (2001) and Park and Shin (2003) used the adjoint variable method for shape identification problems in unsteady natural convection fields. They proposed the limiting of design variables that express boundary shapes to the minimum number of variables required for the steady-state problem. Aounallah et al. (2013) expressed the equations governing unsteady natural convection fields in two dimensions using flow functions and vorticity. They conducted an analysis using Bezier curves, which facilitate expressions of smooth boundary shapes with a minimal number of design variables. This method is similar to that of Park and Ku (2001) and Park and Shin (2003). As described above, many methods were shape optimization methods using parametric design variables for design boundaries. Also, Morimoto et al. (2010) presented a smoothed shape correction operation to avoid numerical unstable phenomena of boundary shape in shape updating for shape design of heat exchanger. On the other hand, a study on the topology shape optimization of heat convection fields was shown in a review paper of Dbouk (2017), and topology shape optimization analyses using the adjoint variable method have recently been conducted on heat convection fields by Yaji et al. (2015), Kametani and Hasegawa (2017a), and Alexandersen et al. (2014). Yaji et al. (2015) and Kametani and Hasegawa (2017a) optimized the topology of forced convection fields, while Alexandersen et al. (2014) used the adjoint variable method to optimize the topology of natural convection fields. In these cases, the analysis was limited to steady-state problems, but an analysis of unsteady-state problems has been performed recently (Coffin and Maute, 2016; Kametani and Hasegawa, 2017b).

One the other hand, the author has focused on a solution, which is called traction method (Azegami, 1994; Azegami et al., 1995), of shape optimization problems based on the distributed sensitivity function using adjoint method. This method has been applied to many shape optimization problems, such as elastic problems (Azegami et al., 1995; Shimoda et al., 1996), heat conduction problems (Katamine et al., 2001, 2003), and viscous flow problems (Katamine and Azegami, 1994; Katamine et al., 2005, 2007, 2009) in mechanical engineering. The traction method applies the gradient method of a distributed system and uses the shape gradient function of the domain variation that is theoretically derived from the optimization problem. The domain variations that minimize the objective functional are obtained as solutions of the pseudo-linear elastic problems of continua defined on the design domains and loaded with pseudo-distributed traction in proportion to the shape gradient function on the design domains. Therefore, this method was called traction method. In this method, it has following advantages.

1. With this method based on the distributed system (non-parametric system), the smooth optimal shapes are obtained without any shape design parameterization such as above methods (Park and Ku, 2001; Park and Shin, 2003; Aounallah et al., 2013).

2. Theoretical considerations for retaining boundary smoothness following domain variation is possible with this method, the unstable phenomenon as described above (Imam, 1982; Morimoto et al., 2010) does not occur (Azegami et
al., 1997).

(3) The numerical solutions of both the shape gradient and pseudo-linear elastic problems, which are used for evaluation of the domain variation, can be obtained using the FEM or the boundary element method.

(4) Since the traction method is implemented by the use of the FEM, it is exceptionally easy to perform. Furthermore, re-meshing is not required for ordinary shape update analyses because the domain variation of all nodes within a region can be treated as design variables in the FEM.

(5) This method enables sensitivity calculations of large-scale design variables for complex coupled fields such as steady and unsteady heat convection fields based on the adjoint variable method (Katamine et al., 2013).

This review paper explains several solutions for shape design problems involving steady and unsteady heat convection fields by using the traction method. In this paper, before the shape design problems are formulated, a method for representing domain variation and the governing equations for steady-state heat convection fields are shown. Then, a shape design problem involving the control of the temperature distribution in the sub-domain of steady-state heat convection is formulated. A shape gradient function (sensitivity function) for this problem is derived in Section 4. The shape gradient function is derived theoretically using the Lagrange multiplier method, adjoint variable method, and the formulation of the material derivative. The square integration error between the actual temperature distributions and the target temperature distributions in the sub-domains is employed as the objective functional for the shape design. In a similar manner, another shape optimization problem in a steady-state heat convection field is formulated for the problem to prescribe temperature distribution, while the total dissipation energy is constrained to less than a desired value. This problem is presented in Section 5. In Section 6, the governing equations for unsteady-state heat convection fields are shown. In addition, two shape design problems involving the control of the temperature distribution history in the sub-domain of the unsteady state heat convection is formulated in Sections 7 and 8. The traction method discussed in Section 9 is applied to examples in Section 10 based on the shape gradient functions derived for each problem.

2. Domain variation

Before formulating the shape optimization problem, a method of representing domain variation using the speed method will be discussed briefly. A more detailed explanation is presented in the references (Haug et. al., 1986; Sokolowski and Zolesio, 1992).

Assume that a bounded domain $\Omega$ of $\mathbb{R}^d$ ($d = 2, 3$) with boundary $\Gamma$ is variable. One approach to describing the domain variation is to use a one-parameter family of one-to-one mappings $\hat{T}(\Omega) : \Omega \mapsto \Omega_s$ or its inverse $\hat{T}^{-1}(\Omega_s) : \Omega_s \mapsto \Omega$, where $s$ denotes the domain variation history. When a domain functional $J_\Omega$ and a boundary functional $J_\Gamma$ of a distributed function $\psi$ are considered, their derivatives $J_\Omega$ and $J_\Gamma$ with respect to $s$ at $s = 0$ are given by the formulae of the material derivative:

$$
J_\Omega = \int_{\Omega_s} \psi \; dx, \quad J_\Gamma = \int_{\Gamma_s} \psi \; n \cdot \nabla \psi \; ds.
$$

(1)

$$
J_\Omega = \int_{\Omega_s} \psi \; dx, \quad J_\Gamma = \int_{\Gamma_s} \psi \; n \cdot \nabla \psi \; ds.
$$

(2)

where $n$ is an outward unit normal vector to the boundary, $\nabla_a(\cdot) \equiv \nabla(\cdot) \cdot n$, and $\kappa$ denotes the quantity $(d - 1)$ times the mean curvature of boundary. The shape derivative $\psi'$ of the distributed function $\psi$ indicates that the derivatives are fixed in spatial coordinates. The derivative $\nabla(\Omega_s)$ of $\hat{T}(\Omega)$ with respect to $s$ given by

$$
\nabla(\Omega_s) = \frac{\partial \hat{T}(\Omega_s)}{\partial s} = \frac{\partial \hat{T}(\hat{T}^{-1}(\Omega_s))}{\partial s}
$$

(3)

is referred to as the velocity because of the analogy between $s$ and time.

3. Governing equations for steady state heat convection fields

Consider the steady-state heat convection field in the region $\Omega$ of $\mathbb{R}^d$ ($d = 2, 3$). Consider determining the flow velocity $u(x) = u_s(x)$, pressure $p(x)$, and temperature $\theta(x)$ at $x \in \Omega$. The dimensionless forms of the Navier–Stokes equation, continuity equation, and energy equation are the governing equations for steady-state heat convection fields. They can be expressed as follows:

$$
u_s \partial_t u_s + \frac{1}{Re} \nabla^2 u_s = \frac{1}{p_s} u_s, \quad \hat{x} \in \hat{\Omega},
$$

(4)
where the boundary is \( \Gamma = \partial \Omega = \Gamma_u \cup \Gamma_{\sigma} = \Gamma_{\theta} \cup \Gamma_q \cup \Gamma_h \), as shown in Figure 1. Tensors described in this study use the Einstein summation convention and differentiation \((\cdot)_i = \partial(\cdot)/\partial x_i\).

The boundary conditions are described below:

\[
\begin{align*}
\hat{u}_i(x) &= \hat{u}_i(x), \quad x \in \Omega, \quad (5) \\
\sigma_i(x) &= \sigma_i(x) = (-p\delta_{ij} + \frac{1}{Re}\hat{u}_i\hat{u}_j)n_j = 0, \quad x \in \Gamma_{\sigma} \quad (6) \\
\theta(x) &= \hat{\theta}(x), \quad x \in \Gamma_{\theta} \quad (7) \\
\frac{1}{RePr}\theta(x)n_j &= \hat{q}(x), \quad x \in \Gamma_q \quad (8) \\
\frac{1}{RePr}\theta(x)n_j &= \hat{h}(\theta(x) - \hat{\theta}), \quad x \in \Gamma_h \quad (9)
\end{align*}
\]

Here, \( \hat{q} \) represents the heat flux, \( \hat{h} \) represents the coefficient of heat transfer, \( \hat{\theta} \) represents the external temperature, \( \delta_{ij} \) represents the Kronecker delta, \( Re \) is the Reynolds number, and \( Pr \) is Prandtl number. Finally, \((\cdot)\) represents the known function on the boundary.

The weak forms of the respective governing equations (4)-(6) can be expressed with adjoint flow velocity \( w(x) = (w_i(x))_{i=1}^2 \), adjoint pressure \( q(x) \), and adjoint temperature \( \xi(x) \) as follows:

\[
\begin{align*}
\mathcal{a}^V(u, w) + \mathcal{b}^V(u, u, w) + c(w, p) - l(w) &= 0, \quad \forall w \in W, \quad (10) \\
c(u, q) &= 0, \quad \forall q \in Q, \quad (11) \\
\mathcal{a}^H(\theta, \xi) + \mathcal{b}^H(u, \theta, \xi) + f_q^H(\theta, \xi) + f_h^H(\theta, \xi) - f_q^H(\xi) &= 0, \quad \forall \xi \in \Xi. \quad (12)
\end{align*}
\]

Furthermore, \( \mathcal{a}^V(u, w), \mathcal{b}^V(u, u, w), c(w, p), l(u), \mathcal{d}^H(u, \theta, \xi), \mathcal{d}^H(u, \theta, \xi), f_q^H(\theta, \xi), f_h^H(\theta, \xi), \) and \( f_q^H(\xi) \) are defined as follows:

\[
\begin{align*}
\mathcal{a}^V(u, w) &= \int_\Omega \frac{1}{Re} w_i u_i d\Gamma, \quad \mathcal{b}^V(u, u, w) = \int_\Omega w_i u_i u_i d\Gamma, \quad c(w, p) = -\int_\Omega w_i p d\Gamma, \quad l(u) = \int_{\Gamma_h} w_i \hat{h} \hat{\theta} d\Gamma, \quad (13) \\
\mathcal{a}^H(\theta, \xi) &= \int_\Omega \frac{1}{RePr} \xi_j \theta_i d\Gamma, \quad \mathcal{b}^H(u, \theta, \xi) = \int_\Omega \xi_i u_j \theta_i d\Gamma, \quad f_q^H(\theta, \xi) = \int_{\Gamma_h} \xi \hat{q} d\Gamma, \quad f_q^H(\xi) = \int_{\Gamma_h} \hat{h} \xi \hat{\theta} d\Gamma. \quad (14)
\end{align*}
\]

Here, the superscripts \( V \) and \( H \) represent velocity and heat, the subscripts \( q, h \) and \( hf \) denote terms about the heat flux and the heat transfers. The flow velocity \( u \), its adjoint \( w \), and the other variables are considered to be elements of the following functional spaces:

\[
\begin{align*}
U &= \{ u_i(x) \in H^1(\Omega) \mid u_i(x) = \hat{u}_i(x), \quad x \in \Gamma_u \}, \quad (15) \\
Q &= \{ q(x) \in L^2(\Omega) \mid \int_\Omega q d\Omega = 0 \ (\text{if measure}(\Gamma_{\sigma}) = 0) \}, \quad (16) \\
\Theta &= \{ \theta(x) \in H^1(\Omega) \mid \theta(x) = \hat{\theta}(x), \quad x \in \Gamma_{\theta} \}, \quad (17) \\
W &= \{ w_i(x) \in H^1(\Omega) \mid w_i(x) = 0, \quad x \in \Gamma_u \}, \quad (18) \\
\Xi &= \{ \xi(x) \in H^1(\Omega) \mid \xi(x) = 0, \quad x \in \Gamma_{\theta} \}. \quad (19)
\end{align*}
\]
Here, $H^1(\Omega)$ is a function space that is square integrable up to the first derivative, and $L^2(\Omega)$ is a square-integrable function space.

4. Prescribing temperature in a sub-domain in steady-state heat convection fields

4.1. Problem formulation

In this section, the problem of minimizing the square integration errors between the actual temperature $\theta_{\Omega_{0}}$ and the target temperature $\theta_{\Omega}$ on sub-domain $\Omega_{D} \subset \Omega$ is formulated (Katamine et al., 2007). The domain transformation of this heat convection field region $\Omega$ is denoted by $\hat{T}_s$, and the domain $\Omega$ is assumed to vary to reach $\Omega_{e} = \hat{T}_s(\Omega)$. For simplicity, we assume that the sub-domain $\Omega_{D}$ and $\Gamma_{\sigma}$ are invariable, that is $\hat{T}_s(\Omega_{D}) = \Omega_{D}$, $\hat{T}_s(\Gamma_{\sigma}) = \Gamma_{\sigma}$ for domain variation. The square integration error problem for temperature distribution is formulated as follows:

$$\int_{\Omega_{0}} dx \leq \beta_v M,$$

where $\beta_v$ is a coefficient related to the initial domain measure $M$, and

$$E_{\Omega_0}(\theta) = \int_{\Omega_{0}} (\theta - \theta_{D})^2 \, dx. \quad (21)$$

4.2. Shape gradient function

The square integration error problem can be rewritten as a retention problem without any restrictions from the Lagrange multiplier method. In this instance, the Lagrange function $L(u, p, \theta, w, q, \xi, \Lambda)$ is given as follows:

$$L = E_{\Omega_0}(\theta)$$

$$-\int_{\Omega_{0}} \left\{ a^V(u, w) + b^V(u, u, w) + c(u, p) - l(w) \right\} \, dx$$

$$+ \Lambda \left( \int_{\Omega} dx - \beta_v M \right). \quad (23)$$

Where $w \in W$, $q \in Q$, and $\xi \in \Xi$ were introduced as Lagrange multiplier functions or the adjoint functions with respect to the weak forms. The non-negative real constant number $\Lambda$ is the Lagrange multiplier with respect to the volume constraint. The derivative of $L$ with respect to domain variation is derived using the velocity field $\hat{V}(\Omega_{e}) = \partial \hat{T}_s(\Omega)/\partial s = \partial \hat{T}_s/\partial s(\hat{T}_s^{-1}(\Omega_{e}))$ in equation (3), as follows:

$$L =$$

$$-\int_{\Omega_{0}} \left\{ a^V(u, w') + b^V(u, u, w') + c(u', p) - l(w') \right\} \, dx$$

$$-\int_{\Omega_{0}} \left\{ a^H(u, \theta, \xi') + b^H(u, \theta, \xi') + f^H(\theta, \xi') - f^H_{\xi'}(\xi') \right\} \, dx$$

$$+ \Lambda \left( \int_{\Omega} dx - \beta_v M \right) + l_G(\hat{V}) \quad (24)$$

where,

$$l_G(\hat{V}) = \int_{\Gamma} G \hat{n} \cdot \hat{V} \, d\Gamma, \quad (25)$$

and

$$G = -\frac{1}{Re} w_{ij} u_{ij} + w_{i} p - \frac{1}{RePr} \hat{c}_{\theta_{ij}} - \xi u_{ij}$$

$$- \nabla_a (\hat{\xi} \hat{\theta}) - (\hat{\xi} \hat{\theta}) \kappa - \nabla_a (\hat{\kappa} \hat{\xi} \hat{\theta}) + \nabla_a (\hat{\kappa} \hat{\xi} \hat{\theta} \hat{\kappa}) + (\hat{\kappa} \hat{\xi} \hat{\theta}) \kappa + \Lambda, \quad (26)$$
where flow velocity $u$, pressure $p$, temperature $\theta$, adjoint flow velocity $w$, adjoint pressure $q$, adjoint temperature $\xi$, and Lagrange multiplier $\Lambda$ are determined by the following conditions:

\[
\begin{align*}
    &\{a'(u',w') + b'(u,u',w') + c(u',p) - l(u') + c(u,q')\} = 0 \quad \forall u' \in W, \forall q' \in Q, \\
    &\{a''(\theta,\xi') + b''(u,\theta,\xi') + f_q''(\xi') + f_p''(\theta,\xi') - f_{\theta}'(\xi')\} = 0 \quad \forall \xi' \in \Xi, \\
    &\{a''(u',w) + b''(u',u,w) + c(u',q) + b''(u',\theta,\xi) + c(w,p')\} = 0 \quad \forall u' \in U, \forall p' \in Q, \\
    &\{a''(\theta',\xi) + b''(u',\theta',\xi) + f_q''(\theta',\xi)\} - E_{\Omega_D}(\theta') = 0 \quad \forall \theta' \in \Theta, \\
    &\Lambda \geq 0, \int_{\Omega} dx \leq \beta V M, \quad \Lambda \left(\int_{\Omega} dx - \beta V M\right) = 0.
\end{align*}
\]

The derivative of the Lagrange function agrees with the derivative of the evaluation function, establishing the following relationship:

\[
\hat{L}_{u,p,\theta,w,q,\xi,\Lambda} = \hat{E}_{\Omega_D}[u,p,\theta,w,q,\xi,\Lambda] = l_g(\hat{V}).
\]

Because $\hat{g}$ in equation (25) is a coefficient function of the velocity field $\hat{V}$ that provides minute variations in the domain, $\hat{g}$ is referred to as a sensitivity function or shape gradient function. Furthermore, the scalar function $G$ is referred to as the shape gradient density function.

Equation (27) is a weak form of the Navier–Stokes equation and the continuous state equation, equation (28) is a weak form of the energy equation in the state equation. Equation (29) is a weak form of the Navier–Stokes equation and continuous state equation for the adjoint problem, equation (30) is a weak form of the energy equation in the state equation for the adjoint problem, and (31) is a constraint equation related to the Lagrange multiplier $\Lambda$. The traction method can be applied if the shape gradient function can be evaluated by analyzing $u, p, \theta, w, q, \xi,$ and $\Lambda$ based on these equations.

Furthermore, considering the continuity equation for the adjoint equation, and assuming that the flow velocity on the design boundaries satisfies $u_s = 0$, the shape gradient density function $G$ can be expressed from (26) as

\[
G = G_0 + G_1 \Lambda,
\]

\[
G_0 = -\frac{1}{Re} w_{i,j} u_{i,j} - \frac{1}{Re Pr} \xi_j \theta_j - \nabla_\theta (\xi \delta) - (\xi \delta) k - \nabla_\theta (h \xi \delta) - (h \xi \delta) k + \nabla_\theta (h \xi \delta) + (h \xi \delta) k,
\]

\[
G_1 = 1,
\]

where $G_0$ in equation (33) is the shape gradient density function for the square integration errors, and $G_1$ is the shape gradient density function for the constraint conditions.

### 5. Prescribing temperature in consideration of dissipation energy as a constraint condition

#### 5.1. Problem formulation

In this section, another shape optimization problem in a steady-state heat convection field is formulated for the problem to prescribe the temperature distribution, while the total dissipation energy is constrained to less than a desired value (Katamine et al., 2013). For simplicity, we assume that the sub-domain $\Omega_D$ and $\Gamma_D$ are invariable for domain variation. The problem is formulated as follows:

Given $\Omega$ find $\Omega_D$ that minimizes $E_{\Omega_D}(\theta)$ subject to (12) – (14) and

\[
E_{\Omega}(u) \leq E_{\Omega_D}^H,
\]

where the total dissipation energy $E_{\Omega}(u)$ is defined as follows:

\[
E_{\Omega}(u) = \int_{\Omega} \frac{2}{Re} \varepsilon_{ij}(u) \varepsilon_{ij}(u) dx, \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),
\]

and $E_{\Omega_D}^H$ is a limitation value for the dissipation energy.

#### 5.2. Shape gradient function

The Lagrange function $L(u, p, \theta, w, q, \xi, \Lambda)$ for this problem is given as follows:

\[
L = E_{\Omega_D}(\theta)
\]
for unsteady heat convection fields. They can be expressed as follows:

\[
\begin{align*}
-\{a^v(u, w) + b^v(u, u, w) + c(w, p) - l(u)\}
- c(u, q)
- \left\{ a^h(u, \theta, \xi) + b^h(u, \theta, \xi) + f_{q}^{h}(\theta, \xi) + f_{u}^{h}(\theta, \xi) - f_{q}^{h}(\xi) \right\}
+ \Lambda \left( E_{\Omega}(u) - E_{\Omega}^{M} \right).
\end{align*}
\]

(36)

In a manner similar to that shown in the previous section, the derivative of \( L \) is derived as:

\[
L = -\{a^v(u, w') + b^v(u, u, w') + c(w', p) - l(u')\}
- c(u, q')
- \left\{ a^h(u', \theta', \xi') + b^h(u', \theta, \xi') + f_{q}^{h}(\theta', \xi') + f_{u}^{h}(\theta', \xi') - f_{q}^{h}(\xi') \right\}
- \left\{ a^v(u', w) + b^v(u', u', w) + b^v(u', u', w) + c(u', q) + b^h(u', \theta, \xi) - \Lambda E_{\Omega}(u') \right\}
- c(w, p')
- \left\{ a^h(u', \theta', \xi') + b^h(u', \theta', \xi') + f_{q}^{h}(\theta', \xi') + E_{\Omega}(\theta') \right\}
+ \Lambda \left( E_{\Omega}(u) - E_{\Omega}^{M} \right) + l_{c}(\hat{V}),
\]

(37)

where,

\[
l_{c}(\hat{V}) = \int_{\Gamma} \hat{G} \hat{h} \cdot \hat{V} \, d\Gamma,
G = G_{0} + G_{1},
G_{0} = -\frac{1}{Re} \omega_{i} u_{i,j} - \frac{1}{Re Pr} \hat{\xi}, \theta \left( - \nabla_{\theta}(\xi \hat{q}) - (\xi \hat{q})_{\theta} - \nabla_{\theta}(\hat{h} \hat{\xi}) - (\hat{h} \hat{\xi})_{\theta} + \nabla_{\theta}(\hat{h} \hat{\xi} \hat{q}) + (\hat{h} \hat{\xi} \hat{q})_{\theta} \right),
G_{1} = \frac{2}{Re} \xi_{i}(u) c_{i,j}(u),
\]

(38)

(39)

\( w, q, \xi, \) and \( \Lambda \) are determined by the following adjoint equations:

\[
\left\{ a^v(u', w) + b^v(u', u, w) + b^v(u', u', w) + c(u', q) + b^h(u', \theta, \xi) + c(w, p') \right\} - \Lambda E_{\Omega}(u') = 0 \quad \forall u' \in U, \forall p' \in Q,
\]

(40)

\[
\left\{ a^h(u', \theta', \xi') + b^h(u, \theta', \xi') + f_{q}^{h}(\theta', \xi') - E_{\Omega}(\theta') = 0 \quad \forall \theta' \in \Theta,
\]

(41)

\[
\Lambda \geq 0, \quad E_{\Omega}(u) \leq E_{\Omega}^{M}, \quad \Lambda \left( E_{\Omega}(u) - E_{\Omega}^{M} \right) = 0.
\]

(42)

Because the shape gradient density function for this problem is derived in this manner, the traction method can be applied.

### 6. Governing equations for unsteady-state heat convection fields

Consider the unsteady heat convection field in the region \( \Omega \) of \( \mathbb{R}^{d} (d = 2, 3) \) in time interval \([0, T]\). Consider determining the flow velocity \( u(\hat{x}, t) = (u_{i}(\hat{x}, t))_{i=1}^{d} \), pressure \( p(\hat{x}, t) \) and temperature \( \theta(\hat{x}, t) \) at \( \hat{x} \in \Omega \) and time \( t \in [0, T] \). The dimensionless forms of the Navier–Stokes equation, continuity equation, and energy equation are the governing equations for unsteady heat convection fields. They can be expressed as follows:

\[
\frac{\partial u_{i}}{\partial t} + u_{j} u_{i,j} = -p_{j} + \frac{1}{Re} u_{i,j}, \quad (\hat{x}, t) \in \Omega \times [0, T],
\]

(43)

\[
u_{i,j} = 0, \quad (\hat{x}, t) \in \Omega \times [0, T],
\]

(44)

\[
\frac{\partial \theta}{\partial t} + u_{j} \theta_{j} = \frac{1}{Re Pr} \theta, \quad (\hat{x}, t) \in \Omega \times [0, T],
\]

(45)

where the boundary is \( \Gamma = \partial \Omega = \Gamma_{u} \cup \Gamma_{r} = \Gamma_{d} \cup \Gamma_{g} \cup \Gamma_{h} \), as shown in Figure 1. The boundary conditions and initial conditions are described below:

\[
u_{i}(\hat{x}, t) = \hat{u}_{i}(\hat{x}, t), \quad t \in [0, T], \quad \hat{x} \in \Gamma_{u}.
\]

(46)
7.1. Problem formulation

Prescribing temperature on a sub-domain in unsteady-state heat convection fields

Considered to be elements of the following functional spaces:

\[ \int_{0}^{T} \left\{ t^V(u_j, w) + a^V(u, w) + b^V(u, u, w) + c(u, p) - l(w) \right\} dt = 0, \quad \forall w \in W, \]

\[ \int_{0}^{T} \{ c(u, q) \} dt = 0, \quad \forall q \in Q, \]

\[ \int_{0}^{T} \left\{ a^H(\theta, \xi) + b^H(u, \theta, \xi) + f^H_\theta(\xi) + f^H_\xi(\theta) \right\} dt = 0, \quad \forall \xi \in \Xi. \]

Furthermore, \( t^V(u_j, w) \) and \( i^H(\theta, \xi) \) are defined as follows:

\[ t^V(u_j, w) = \int_{\Omega} w_i \frac{\partial u_j}{\partial t} dx, \quad i^H(\theta, \xi) = \int_{\Omega} \xi \frac{\partial \theta}{\partial t} dx. \]

Here, \( (\cdot)_t \) expresses the time derivative of the function. The flow velocity \( u \), its adjoint \( w \), and the other variables are considered to be elements of the following functional spaces:

\[ U = \{ u(x, t) \in H^1(\Omega \times [0, T]) | u \text{ satisfies (46) and (51)}, \} \]

\[ Q = \{ q(x, t) \in L^2(\Omega \times [0, T]) | \int_{\Omega} q dx = 0 \text{ (if measure(\Gamma_r) = 0)} \} \]

\[ \Theta = \{ \theta(x, t) \in H^1(\Omega \times [0, T]) | \theta \text{ satisfies (48) and (53)} \}, \]

\[ W = \{ w(x, t) \in H^1(\Omega \times [0, T]) | w(x, t) = 0, \quad t \in [0, T], \quad x \in \Gamma_u, \quad w(x, T) = 0, \quad x \in \Omega \}, \]

\[ \Xi = \{ \xi(x, t) \in H^1(\Omega \times [0, T]) | \xi(x, t) = 0, \quad t \in [0, T], \quad x \in \Gamma_r, \quad x \in \Omega \}. \]

7. Prescribing temperature on a sub-domain in unsteady-state heat convection fields

7.1. Problem formulation

In this section, the problem of minimizing the square integration errors between the actual temperature \( \theta_{\Omega_2 \times \{t_1, t_2\}} \) from time \( t = t_1 \in [0, T] \) to \( t = t_2 \in [0, T] \) and the target temperature \( \theta_{\Omega_2 \times [0, T]} \) on sub-domain \( \Omega_D \subset \Omega \) is formulated (Katamine et al., 2017). We assume \( t_1 < t_2 \). The domain transformation of this heat convection field region \( \Omega \) is denoted by \( T_r \), and the domain \( \Omega \) is assumed to vary to reach \( \Omega_r = T_r^{-1}(\Omega) \). For simplicity, we assume that the sub-domains \( \Omega_D \) and \( \Gamma_r \) are invariable, that is \( T_r(\Omega_D) = \Omega_D \) and \( T_r(\Gamma_r) = \Gamma_r \) or domain variation. The square integration error problem for temperature distribution from time \( t = t_1 \) to \( t = t_2 \) is formulated as follows:

Given \( \Omega \) find \( \Omega_r \) that minimizes \( \int_{t_1}^{t_2} E_{\Omega, \theta}(\theta) dt \) subject to (54) – (56) and

\[ \int_{\Omega} dx \leq \beta_r M. \] (63)

7.2. Shape gradient function

The Lagrange function \( L(u, p, \theta, w_1, q, \xi, \Lambda) \) for this problem is given as follows:

\[ L = \int_{\Omega_2} E_{\Omega, \theta}(\theta) dt \]

\[ - \int_{\Omega} \left\{ t^V(u_j, w) + a^V(u, w) + b^V(u, u, w) + c(u, p) - l(w) \right\} dt - \int_{\Omega} \{ c(u, q) \} dt \]

\[ - \int_{\Omega} \left\{ a^H(\theta, \xi) + b^H(u, \theta, \xi) + f^H_\theta(\xi) + f^H_\xi(\theta) \right\} dt \]

\[ + \Lambda \int_{\Omega} (dx - \beta_r M). \] (64)
In a manner similar to that shown in the previous section, assuming that the flow velocity satisfies \( u_t = 0 \) at the design boundary, the shape gradient density function for the problem is given by (65).

\[
G = G_0 + G_1 \Lambda,
\]
\[
G_0 = \int_0^T \left\{ -\frac{1}{Re} u_{n,i} u_{j,i} - \frac{\partial \theta}{\partial t} \xi - \frac{1}{RePr} \xi_{,j,i} - \nabla_n (\xi \tilde{q}) - (\xi \tilde{q})_\kappa - \nabla_n (\tilde{h} \xi \theta) - (\tilde{h} \xi \theta)_\kappa + \nabla_n (\tilde{h} \xi \theta_j) + (\tilde{h} \xi \theta_j)_\kappa \right\} dt
\]
\[
G_1 = 1
\]

where \( u_t, p, \theta, w, q, \xi, \) and \( \Lambda \) are determined by the following conditions:

\[
\int_0^T \left\{ r'(u, u) + a'(u, w) + b'(u, u, w) + c(w, p) - l(w) + c(u, q') \right\} dt = 0 \quad \forall u' \in W, \quad \forall q' \in Q \tag{66}
\]
\[
\int_0^T \left\{ \theta'(u, \theta, \xi') + d'(u, \theta, \xi') + f'(u, \theta, \xi') \right\} dt = 0 \quad \forall \xi' \in \Xi \tag{67}
\]
\[
\int_0^T \left\{ \theta'(u, u, w) + b'(u, u, w) + b'(u, u, w) + b(u, w, u) + c(u, q) + b(u, u, \theta, \xi) + c(w, p') \right\} dt = 0 \quad \forall u' \in U, \quad \forall p' \in Q \tag{68}
\]
\[
\Lambda \geq 0, \quad \int_{\Omega} dx \leq \beta_V M, \quad \Lambda \int_{\Omega} dx - \beta_V M = 0 \tag{70}
\]

Because the shape gradient function can be evaluated by analyzing \( u_t, p, \theta, w, q, \xi, \) and \( \Lambda \) based on these equations, the traction method can be applied in this unsteady-state heat convection problem.

8. Multi-objective shape optimization for prescribing temperature and for minimizing dissipated energy in unsteady-state heat convection fields

8.1. Problem formulation

In the second problem for the unsteady heat-convection fields, a multi-objective shape optimization problem using a normalized objective functional is formulated for the problem to prescribe the temperature distribution and the total dissipated energy minimization problem. The multi-objective problem of minimizing the square integration errors between the actual temperature \( \theta_{\text{ini}} \) and target temperature \( \theta_{\text{ini}} \) on sub-domain \( \Omega_0 \in \Omega \) and of minimizing the dissipated energy in the total domain \( \Omega \) from time \( t = t_1 \in [0, T] \) to \( t = t_2 \in [0, T] \) is formulated. For simplicity, we assume that the sub-domains \( \Omega_0 \) and \( \Gamma \) are invariable, that is \( \partial_{\text{in}} \Omega_0 = \Omega_0 \) and \( \partial_{\text{in}} \Gamma = \Gamma \) or domain variation. The multi-objective shape optimization problem based on the weighting method is formulated as follows:

Given \( \Omega \) find \( \Omega \) that minimizes \( w_o \cdot \int_{\Omega_0} E_{\text{ini}}(\theta) dt + w_u \cdot \int_{\Omega} E_{\text{ini}}(u) dt \)

subject to (54) – (56) and

\[
\int_{\Omega} dx \leq \beta_V M. \tag{71}
\]

Here, \( w_o \) and \( w_u \) are weight coefficients for prescribing temperature and dissipated energy minimization, and \( \int_{\Omega_0} E_{\text{ini}}(\theta) dt \) and \( \int_{\Omega} E_{\text{ini}}(u) dt \) are initial values for the square integration errors between the actual temperature and the target temperature and the dissipated energy, respectively. The objective functional in (71) is normalized with the initial values.

8.2. Shape gradient function

The Lagrange function \( L(u_t, p, \theta, w, q, \xi, \Lambda) \) for this problem is given as follows:

\[
L = w_o \cdot \int_{\Omega_0} E_{\text{ini}}(\theta) dt + w_u \cdot \int_{\Omega} E_{\text{ini}}(u) dt - \int_0^T \left\{ r'(u, u) + a'(u, w) + b'(u, u, w) + c(w, p) - l(w) \right\} dt - \int_0^T \left\{ c(u, q) \right\} dt
\]
\[
- \int_0^T \left\{ \theta'(u, \theta, \xi') + d'(u, \theta, \xi') + f'(u, \theta, \xi') \right\} dt
\]
\[
+ \Lambda \int_{\Omega} dx - \beta_V M \tag{72}
\]

In a manner similar to the previous section, the shape gradient density function for the problem is given by (73).

\[
G = G_0 + G_1 \Lambda.
\]
Finally, because ~Λ obtained by the traction method can be derived as a displacement field when the shape gradient function is

\[ -\nabla_n(h\xi \Theta) - (h\xi \Theta)\kappa + \nabla_n(h\xi \Theta_j) + (h\xi \Theta_j)\kappa \] dt

\[ G_1 = 1 \]  

(73)

The adjoint flow velocity \( w_i \), adjoint pressure \( q \), adjoint temperature \( \xi \), and Lagrange multiplier \( \Lambda \) related to this problem are determined by the following adjoint equations:

\[ \int_0^T \left\{ -t^i'(u', w_j) + a^i'(u', w) + b^i'(u, u', w) + b^i(u, u', w) + c(u', q) + b^H(u', \Theta, \xi) + c(w, p') \right\} dt = w_u \cdot \frac{\int_0^T G_0(u') dt}{\int_0^T G_0(u) dt_{ini}} \]  

(74)

\[ \int_0^T \left\{ -t^i(\theta', \xi_j) + a^i(\theta', \xi) + b^i(u, \theta', \xi) + f^H_{\theta}(\theta', \xi) \right\} dt = w_\theta \cdot \frac{\int_0^T G_\theta(\theta') dt}{\int_0^T G_\theta(\theta) dt_{ini}} \]  

(75)

\[ \Lambda \geq 0, \int_\Omega dx \leq \beta_V M, \quad \Lambda(\int_\Omega dx - \beta_V M) = 0 \]  

(76)

These types of shape design problems involving forced heat convection can be applied to natural convection fields (Katamine et al., 2018).

9. Solution

9.1. Traction method

The traction method has been proposed to solve the velocity field \( \vec{V} \in D \) that indicates domain variation. Domain variation is based on the following governing equation (Azegami, 1994):

\[ a^F(\vec{V}; \vec{g}) = - < G\vec{n}, \vec{g} >, \quad \forall \vec{g} \in D, \]  

(77)

where the liner form \( < G\vec{n}, \vec{g} > \) with respect to the vector \( \vec{g} \) is given by

\[ < G\vec{n}, \vec{g} > = l_G(\vec{g}) = \int_\Omega G\vec{n} \cdot \vec{g} d\Gamma, \]  

(78)

and

\[ D = \{ \vec{V} \in (H^1(\Omega))^d | \text{constraints of domain variation} \}. \]  

(79)

Here, \( a^F(\vec{V}, \vec{g}) \) is a bilinear form indicating the strain energy on a linear elastic body, which is defined by the following equation with respect to the displacement vector function \( u_i, v_j \):

\[ a^F(u, v) = \int_\Omega A_{ijkl} u_{i,j} u_{j,i} dx, \]  

(80)

where \( A_{ijkl} \) is a constant elastic tensor that is positive definite. Equation (77) indicates that the velocity field \( \vec{V} \) is analyzed as a displacement field when the negative shape gradient function \( -Gn_i \) is applied as an external force. The domain variation resulting from the traction method can be derived as a displacement field when the shape gradient function is applied to the pseudo-elasticity problem as an external force.

Furthermore, this analysis is a heat convection field analysis. For simplicity, the elastic tensor \( A_{ijkl} \) is given by

\[ A_{ijkl} = \delta_{ik} \delta_{jl}. \]  

(81)

By considering a bilinear equation (77) and \( G = G_0 + G_1 \), the domain variation is expressed as \( \vec{V} = \vec{V}_0 + \Lambda \vec{V}_1 \). We obtain

\[ a^F(\vec{V}_0, \vec{g}) = - < G_0\vec{n}, \vec{g} >, \quad \forall \vec{g} \in D, \]  

(82)

\[ a^F(\vec{V}_1, \vec{g}) = - < G_1\vec{n}, \vec{g} >, \quad \forall \vec{g} \in D. \]  

(83)

The Lagrange multiplier \( \Lambda \) is determined from

\[ \int_\Omega \vec{n} \cdot \vec{V}_0 d\Gamma + \Lambda \int_\Omega \vec{n} \cdot \vec{V}_1 d\Gamma = 0. \]  

(84)

Finally, because \( \vec{V}_0, \vec{V}_1, \Lambda \) are given, the domain \( \vec{x} \in \Omega \) is moved into \( \vec{x} + \vec{V} \in \Omega \).
Fig. 2 Numerical model: Junction channel, prescribing temperature distribution in consideration of dissipation energy as a constraint condition in steady heat convection field

(a) Initial shape
(b) Velocity distribution

(a) Pressure distribution
(b) Temperature distribution

Fig. 3 Numerical results: Initial shapes for prescribing the temperature distribution of the junction channel

9.2. Analysis procedure

If the shape optimization problem is an unsteady-state problem, the governing equations and the adjoint equations must be solved for a direction at time. In this subsection, the analytical procedure for the unsteady-state heat convection problem is shown.

The proposed shape determination analysis can be performed by repeating the following steps:

Step 1. Give initial shape.

Step 2. Analyze the flow velocity $u(\vec{x}, t)$, pressure $p(\vec{x}, t)$, and temperature $\theta(\vec{x}, t)$ based on the state equation (66) and (67) using initial conditions.

Step 3. Stop when the objective functional has converged.

Step 4. Analyze the adjoint flow velocity $w(\vec{x}, t)$, adjoint pressure $q(\vec{x}, t)$, and adjoint temperature $\xi(\vec{x}, t)$ using the given flow velocity $u(\vec{x}, t)$ and temperature $\theta(\vec{x}, t)$ based on the adjoint equations (68) and (69) or equations (74) and (75), using the conditions $w_0(\vec{x}, T) = 0$ and $\xi(\vec{x}, T) = 0$ in (61) and (62).

Step 5. Calculate the shape gradient density function $G$ based on these results using equation (65) or equation (73).

Step 6. Calculate the velocity field $\vec{V}$ based on the traction method using equation (77) to update the shape. Return to Step 2.

10. Numerical results

10.1. Prescribing temperature in consideration of constraint condition of dissipation energy in steady heat convection field

A shape design problem to prescribe the temperature distribution $\theta$ in sub-domain $\Omega_D$ of a steady-state heat convection field was analyzed for the junction model shown in Fig. 2 (Katamine et al., 2013). In this problem, the dissipation energy was considered to be a constraint condition, as shown in Section 5.

The flow boundary conditions used were a Poiseuille flow on two inlet boundaries $\Gamma_{u1}$ and $\Gamma_{u2}$, and a natural boundary on an outlet boundary $\Gamma_q$. The temperature boundary conditions are described as follows: $\hat{\theta} = 300$ on the inlet boundary $\Gamma_{\theta1}$, $\hat{\theta} = 100$ on the inlet boundary $\Gamma_{\theta2}$, an insulation boundary on the outlet boundary $\Gamma_q$, and the wall boundaries were heat transfer boundaries $\Gamma_h$, with a heat transfer coefficient $\hat{h} = 0.05$, and an external temperature $\hat{\theta}_f = 0$. The Reynolds number was $Re=40$, the Prandtl number was $Pr=2$, and the target temperature $\theta_D$ in sub-domain $\Omega_D$ in Eq.
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Fig. 3 Numerical results: Optimum shapes for prescribing the temperature distribution of the junction channel.

The iterative history of objective functional approaches zero.

The mesh, flow velocity, pressure, and temperature of the initial shape and the optimum shape are shown in Fig. 3 and Fig. 4. Figure 5 shows the iterative history for the objective functional, the dissipation energy, and the area normalized with the initial value. From these results, in the optimum shape, it was confirmed that a design boundary in the neighborhood of the sub-domain $\Omega_D$ had expanded. By the expansion of the channel, the flow velocity in the neighborhood of the sub-domain $\Omega_D$ decreased, the effect of heat transportation was weakened, and the temperature distribution decreased. As a result, the objective functional, i.e., temperature square error, was minimized with the constraint condition of dissipation energy. Based on the pressure distributions of Fig. 3 and Fig. 4, the pressure values at the inlet boundaries of the initial shape were 4.05 and 3.63. However, in the optimum shape, the values decreased to 3.74 and 3.32. The pressure loss improved as well. From these numerical results, the validity of the proposed method for a steady-state problem was confirmed.

Fig. 4 Numerical results: Optimum shapes for prescribing the temperature distribution of the junction channel.

Fig. 5 Numerical results: Iterative histories for prescribing the temperature distribution of the junction channel.

(34) was considered for near the outlet. The temperature was prescribed in three nodal points in the sub-domain $\Omega_D$ and we assumed $\theta_D = 100$, $\theta_D = 98$, $\theta_D = 96$ which only 6 lowered than each nodal point temperature in the initial shape sequentially from the left of Fig. 2. The value for a constraint condition of the dissipation energy in the initial shape was set in the limitation value $E_M$. Also, in order to confirm the fundamental validity of the proposed method, the heat transfer terms were not considered in Eq. (39) for the evaluation of the shape gradient density function.

The mesh, flow velocity, pressure, and temperature of the initial shape and the optimum shape are shown in Fig. 3 and Fig. 4. Figure 5 shows the iterative history for the objective functional, the dissipation energy, and the area normalized with the initial value. From these results, in the optimum shape, it was confirmed that a design boundary in the neighborhood of the sub-domain $\Omega_D$ had expanded. By the expansion of the channel, the flow velocity in the neighborhood of the sub-domain $\Omega_D$ decreased, the effect of heat transportation was weakened, and the temperature distribution decreased. As a result, the objective functional, i.e., temperature square error, was minimized with the constraint condition of dissipation energy. Based on the pressure distributions of Fig. 3 and Fig. 4, the pressure values at the inlet boundaries of the initial shape were 4.05 and 3.63. However, in the optimum shape, the values decreased to 3.74 and 3.32. The pressure loss improved as well. From these numerical results, the validity of the proposed method for a steady-state problem was confirmed.

Fig. 6 Numerical model: Branch channel, prescribing temperature distribution in unsteady heat convection field
10.2. Prescribing temperature in unsteady heat convection field

A shape design problem for prescribing the temperature distribution $\theta_{\Omega_D \times [t_1,t_2]}$ in an unsteady-state heat convection field was analyzed for the branch channel model shown in Fig. 6 (Katamine et al., 2017).

The hot thermal fluid flows in from a boundary $\Gamma_u$ and flows out from two boundaries $\Gamma_{\sigma}$. The purpose of shape determination in this analysis is to unify the temperature distribution history near two outlet boundaries $\Gamma_{\sigma}$ during the specified period of time. The temperature distribution history in sub-domain $\Omega_{D2}$ was set in the target distribution $\theta_D$ in Eq. (63), and the shape identification problem that the temperature distribution history in sub-domain $\Omega_{D1}$ agrees with the temperature distribution history in sub-domain $\Omega_{D2}$ was analyzed.

The flow boundary conditions included Poiseuille flow on an inlet boundary $\Gamma_u$ and a natural boundary on two outlet boundaries $\Gamma_{\sigma}$. The temperature boundary conditions were as follows: $\bar{\theta} = 1$ on the inlet boundary $\Gamma_u$, insulation boundary on the two outlet boundaries $\Gamma_{\sigma}$, and the wall boundaries were heat transfer boundaries $\Gamma_{h1}$ and $\Gamma_{h2}$, with a heat transfer coefficient $\bar{h} = 1$, and an external temperature $\bar{\theta}_f = 0$. The Reynolds number was $Re=100$ and the Prandtl number was $Pr=100$. The initial conditions of the entire domain were set to $\theta_{ini} = 0$ and $u_{ini} = 0$. The pressure was uniquely set to achieve an average of 0. The time was set to $t_1 = 0$ and $t_2 = T$, and time integration was performed from $t = 0$ to $t = T = 400$ with a $\Delta t = 0.4$ time increment. The two heat transfer boundaries $\Gamma_{h1}$ of BC and HA were considered to be design boundaries $\Gamma_{design}$. Other boundaries were constrained with respect to domain variation. The coefficient $\beta_V$, which restrains the size of the domain, was set for $\beta_V = 1$. The heat transfer terms were not considered for the evaluation of the shape gradient density function in Eq. (65).

In this numerical analysis, the flow field velocity $\bar{u}$, pressure $p$, temperature $\theta$, adjoint flow velocity $\bar{w}$, adjoint pressure $q$, adjoint temperature $\xi$, and shape updating analysis (velocity field $\bar{V}$) were all performed using FreeFem++ (Ootsuka and Takaishi, 2014; Hecht, 2012). Based on the FEM, an analysis was performed using the function "convect"
of FreeFem++ on the state equation in the time direction. The function “adaptmesh” of FreeFem++ was used to perform re-meshing for shape updates; “adaptmesh” is a function that generates an adaptive mesh for improving the solution accuracy of the state variables derived by finite-element analysis. Adaptmesh was used to improve the accuracy of the shape gradient density function $G$ and velocity field $\vec{V}$.

The mesh, flow velocity $\vec{u}$, and temperature $\theta$ at the end time $t = T = 400$ for the initial shape and the identified shape are shown in Fig. 7 and Fig. 8, respectively. Figure 9(a) shows the temperature history in the initial shape, a target temperature history, and the temperature history for the identified shape in sub-domain $\Omega_{D1}$. Figure 9(b) shows the iterative history for the objective functional. Based on a comparison between Fig. 7 and Fig. 8, it was observed that the position of branch in the channel moved to upper part in the identified shape so that the temperature history of the two outlet boundaries agreed. In fact, it was confirmed that the temperature history in the sub-domain $\Omega_{D1}$ in the identification shape agreed with the target temperature history, and the objective functional approached zero from the result of Fig. 9. According to this basic problem, the validity of the proposed method for the unsteady-state problem was confirmed.

10.3. Multi-objective shape design for prescribing temperature and dissipated energy minimization in unsteady heat convection field

A multi-objective shape design problem that prescribed the temperature distribution $\theta_{|\Omega_{D1}\Omega_{D2}\Omega_{D3}}$ in three sub-domains $\Omega_{D1}, \Omega_{D2}, \Omega_{D3}$ and minimized the total dissipated energy in an unsteady-state heat convection field was analyzed for the model shown in Fig. 10 (Katamine et al., 2017).

The hot thermal fluid flows in from a boundary $\Gamma_{in}$ and flows out from three boundaries $\Gamma_{out}$. The purpose of shape determination in this analysis is to unify the temperature distribution history near three outlet boundaries $\Gamma_{out}$ and to minimize total energy dissipation during the specified period of time. The mean of the temperature distribution histories in three sub-domains $\Omega_{D1}$, $\Omega_{D2}$, $\Omega_{D3}$ for the initial shape was set in the target temperature distribution $\theta_{|\Omega_{D1}\Omega_{D2}\Omega_{D3}}$ in Eq.(71).

The flow boundary conditions included a Poiseuille flow on three outlet boundaries $\Gamma_{out}$ and natural boundary on an inlet boundary $\Gamma_{in}$. The temperature boundary conditions are described as follows: $\hat{\theta}_i = 1$ on the inlet boundary $\Gamma_{in}$, insulation boundaries on the three outlet boundaries $\Gamma_{out}$, and the wall boundaries were heat transfer boundaries $\Gamma_h$, with a heat transfer coefficient $\hat{h} = 1$, and an external temperature $\hat{T}_j = 0$. The Reynolds number was $Re=100$ and the Prandtl number was $Pr= 100$. The initial conditions of the entire domain were set to $\theta_{ini} = 0$ and $u_{ini} = 0$. The pressure was uniquely set to achieve an average of 0. The heat transfer boundaries $\Gamma_h$ were considered to be design boundaries $\Gamma_{design}$. Other boundaries were constrained with respect to domain variation. The coefficient $\beta_V$, which restricts the size of the domain, was set for $\beta_V = 1$. The heat transfer terms were not considered in Eq.(73) for the evaluation of the shape gradient density function.

Because it was difficult to unify temperature distribution histories in three sub-domains $\Omega_{D1}$, $\Omega_{D2}$, $\Omega_{D3}$ during the time period from $t_1 = 0$ to $t_2 = T = 300$ in this analysis, the time period was set to $t_1 = 200$, $t_2 = T = 300$, and the two
shape designs were performed for Case A ($w_p=1.0$, $w_u=0$) and Case B ($w_p=0.5$, $w_u=0.5$).

The mesh, flow velocity $\bar{u}$, and temperature $\theta$ at the end time $t = T = 300$ for the initial shape and the identified shapes for Case A and B are shown in Fig. 11, Fig. 12 and Fig. 13. Figure 14 shows the iterative history for the objective functional for Case A and B. Because the dissipation energy minimization is considered in Case B, a shape that was slightly smoother than Case A was obtained. The total objective functional (Objective) in Case A became same as temperature square error integration (Objective T), the value approached zero. In Case B, the temperature square error integration (Objective T), the dissipation energy (Objective U), the total objective functional (Objective) decreased and converged. From these numerical results, the validity of the proposed method for multi-objective shape design in an unsteady-state problem was confirmed.

11. Conclusion

This paper explained a numerical method capable of solving a shape design problem. The problem examined in this study prescribes the temperature distribution in the sub-domain of steady-state and unsteady-state heat convection fields. These problems were formulated and the shape gradient density functions were derived. The verification of the proposed method was accomplished using simple numerical examples of two-dimensional problems by applying the traction method to the shape gradient distribution functions.

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Fig. 14 Iterative history of objective functional. The iterative history of the objective functional approaches zero in Case A. The iterative history of the objective functional decreases and converges to the minimum in Case B.

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