Defeating entanglement sudden death by a single local filtering

Michael Siomau and Ali A. Kamli

Physics Department, Jazan University, P.O. Box 114, 45142 Jazan, Kingdom of Saudi Arabia

(Dated: May 1, 2014)

Genuine multipartite entanglement of a quantum system can be partially destroyed by local decoherence. Is it possible to retrieve the entanglement to some extent by a single local operation? The answer to this question depends very much on the type of initial genuine entanglement. For initially pure W and cluster states and if the decoherence is given by generalized amplitude damping, the answer is shown to be positive. In this case, the entanglement retrieving is achieved just by redistributing the remaining entanglement of the system.

PACS numbers: 03.67.Mn, 03.67.Bg, 03.67.Hk, 03.67.Lx

Being a cornerstone for quantum technologies, entanglement remains maybe the most mysterious feature of quantum theory. Despite many remarkable results devoted to quantifying[1] and measuring[2], the deep understanding of fundamental laws of entanglement evolution, and especially for complex multipartite systems, is still challenging. Nevertheless, it is known for certain that entanglement of a quantum system is very fragile and can be seriously harmed by interaction of the system with its environment[10,11]. This effect, which is known today as entanglement sudden death (ESD), establishes serious limitations on efficient practical utilization of entanglement in communication and computing[12]. Therefore, it is highly desired to find a way to reverse ESD by local manipulations when it is possible.

For bipartite entangled systems, reversing ESD by any local manipulations is impossible[13], unless one is able to control local environment of the system[3] or acquire information about the whole system prior to its interaction with the environment[8]. In this paper we show that, for systems exhibiting certain types of multipartite entanglement, reversing ESD may be successful probabilistically and without mentioned additional assumptions. Focusing on qubits, the two-dimensional quantum systems, we shall analyze entanglement dynamics of different locally unitary inequivalent entangled states, such as W, Greenberger-Horne-Zeilinger (GHZ)[14] and cluster[15] states. We shall only assume that a multiqubit system is partially affected by decoherence, i.e. some qubits are preserved from detrimental influence of environment. The consequence of this assumption is that entanglement among decohering qubits is lost much faster comparing to the entanglement between decohering and non-decohering qubits. This makes possible to retrieve genuine multiqubit entanglement after partial ESD by redistributing the entanglement which remained in the system. To achieve such redistribution we request that just a single non-decohering qubit is available for a local operation. This is indeed the minimal demand on accessibility of the multiqubit system in practice.

As noise model for decoherence we chose generalized amplitude damping (GAD) – a reservoir in thermal equilibrium with a qubit at a finite temperature[10]. The GAD can be represented in terms of four (Kraus) operators

\[ K_1 = \sqrt{1 - p} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1 - \gamma} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \sqrt{1 - p} \begin{pmatrix} 0 & 0 & \sqrt{\gamma} \\ 0 & 1 & 0 \\ \sqrt{\gamma} & 0 & 0 \end{pmatrix}, \]

where the dynamical parameter \( \gamma \) can be expressed as \( 1 - e^{-t} \) through the coupling constant \( \Gamma \) (which defines the temperature of the reservoir, for instance) and the time of interaction \( t \). In order to reduce number of involved parameters we shall later assume \( p = 1/2 \), although our results remain true without this latter assumption.

Let us start our discussion with the maximally entangled three-qubit W state which can be written in a computational basis as

\[ |W_3\rangle = \frac{1}{\sqrt{3}} (|0_00_1c_1\rangle + |a_1b_0c_0\rangle + |1_0b_00_c\rangle). \]

Suppose, two qubits (let say b and c) of the three qubit state undergo GAD simultaneously as drawn in Fig. 1

FIG. 1: (Color online) If two qubits b and c of a three-qubit system, which is initially prepared in W state[3], undergo GAD simultaneously, ESD occurs between these qubits faster than between qubits a and b (a and c). A local filtering on the qubit a can retrieve entanglement between qubits b and c with some probability.
Following quantum operation formalism [16], the final state of the system is given by
\[ \rho_{\text{fin}} = \sum_{ij} S_{ij} \rho_{\text{ini}} S_{ij}^\dagger, \]
where \( S_{ij} = I \otimes K_i \otimes K_j \) for \( i, j = 1, 4 \), \( I \) is the identity matrix, \( \rho_{\text{ini}} = \ket{W_3} \bra{W_3} \) and the condition \( \sum_{ij} S_{ij}^\dagger S_{ij} = I \) is fulfilled.

Entanglement between any two qubits of the three-qubit system can be quantified with Wootters concurrence [17] ignoring the third qubit. The concurrence is given by \( C = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \} \), where \( \lambda^i \) are the square roots of the four nonvanishing eigenvalues of the non-Hermitian matrix \( \rho (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y) \), if taken in decreasing order. Because of the symmetry of the process in Fig. 1 concurrences \( C_{ab} \) and \( C_{ac} \), which are defined between corresponding qubits, are equal for all parameters \( \gamma \) of the GAD. At the same time, concurrence \( C_{bc} \) differs significantly from concurrences \( C_{ab} \) and \( C_{ac} \) as shown in Fig. 2. When \( C_{bc} \) vanish for \( \gamma \approx 0.41 \), each pair of qubits \( ab \) and \( ac \) preserve significant amount of quantum information.

Let us now perform a filtering operation on qubit a. This operation can be written in a computational basis as
\[ F = \sqrt{1 - \kappa} \ket{0} \bra{0} + \sqrt{\kappa} \ket{1} \bra{1}, \quad 0 < \kappa < 1. \]
Filtering is a non-trace-preserving map which is known to be capable of increasing entanglement with some probability [18]. Practically, this map can be realized as a null-result weak measurement [19].

When filtering (4) is applied to the qubit a, the final three-qubit state \( [F \otimes \text{GAD} \otimes \text{GAD}] \rho_{\text{ini}} \) as well as the bipartite concurrences \( C_{ab}, C_{ac} \) and \( C_{bc} \) are dependent of two parameters \( \gamma \) and \( \kappa \). For fixed \( \gamma \), concurrences \( C_{ab} \) and \( C_{ac} \) decrease with increasing \( \kappa \), while concurrence \( C_{bc} \) simultaneously increase. Let us assume that \( \gamma = 0.41 \), then concurrence \( C_{bc} \) vanish before filtering as we mentioned above. Demanding that after the filtering all bipartite concurrences are equal, we can find parameter \( \kappa \approx 0.24 \). For parameters \( \gamma \approx 0.41 \) and \( \kappa \approx 0.24 \), the entanglement between qubits \( b \) and \( c \) is retrieved after ESD and \( C_{ab} = C_{ac} = C_{bc} \approx 0.14 \) ebit. This retrieving is, however, probabilistic with probability \( p = 0.37 \). The probability is calculated from the norm of the final three-qubit state \( [F \otimes \text{GAD} \otimes \text{GAD}] \rho_{\text{ini}} \).

Our consideration can be generalized to the case of a \( N \)-qubit W state. If \( N - 1 \) qubits undergo GAD simultaneously as drawn in Fig. 3, entanglement between the decohering qubits can be retrieved after ESD by a single filtering. However, the amount of retrieved entanglement as well as the probability of the retrieving decrease rapidly with number of decohering qubits. This situation can be changed dramatically, if \( k > 1 \) qubits of the \( N \)-qubit W state are preserved from decoherence as shown in Fig. 3b. In this case, one can increase significantly both the amount of retrieved entanglement and the probability of retrieving by performing filtering on the \( k \) qubits simultaneously. We observed significant increase of pairwise entanglement among non-decohering qubits after the filtering, in the case if \( k > 1 \) qubits are preserved from decoherence and become the subject of the filtering.

Having completed our analysis of W states and before moving to cluster states we would like to note that genuine GHZ entanglement can not be retrieved if ESD happened between two qubits of the multiqubit GHZ state.
In fact, if ESD appears between any two qubits of the multiqubit GHZ state, all the qubits suddenly become disentangled and no local operations can retrieve GHZ entanglement.

We are now at the position to consider entanglement dynamics of cluster states under GAD. Because three-qubit cluster state is locally unitary equivalent to the GHZ state, the simplest nontrivial member of the family of cluster states is the four-qubit state \( \ket{5} \)

\[
|C_4⟩ = \frac{1}{2}(|0_0 0_c 0_d⟩ + |0_0 b_1 c_1 d⟩ + |1_a b_1 0_c 0_d⟩ - |1_a 1 b_1 c_1 d⟩). \tag{5}
\]

This state is less symmetric than four qubit GHZ and W states. For the cluster state, only permutations of qubits ab and cd leave the state unchanged, while for the GHZ and W states any two qubits can be permuted without changing the state. The consequence of the low symmetry of the cluster state is that it does not have pairwise entanglement as W states: any two qubits are disentangled if the other two qubits are discarded. At the same time, any three qubits are entangled, if the remaining one is ignored. This differs cluster states from GHZ states.

Since even pure cluster state \( \ket{5} \) does not contain pairwise entanglement, Wootters concurrence is not suitable to describe its entanglement dynamics. Another entanglement measure has to be chosen to quantify entanglement of the cluster state properly. We shall use a generalization of Wootters concurrence to bipartite finite-dimensional systems \( [19, 20] \). The bipartite concurrence of \( d_1 \otimes d_2 \)-dimensional system is given by \( BC = \sqrt{\sum_{{m,n}} C_{mn}^2} \) where \( C_{mn} = \max\{0, \lambda_{mn}^1 - \lambda_{mn}^2 - \lambda_{mn}^3 - \lambda_{mn}^4\} \) and the \( \lambda_{mn}^k \), \( k = 1..4 \) are the square roots of the four nonvanishing eigenvalues of the matrix \( P \rho_{mn} \), if taken in decreasing order. These matrices \( \rho_{mn} \) are formed by means of the density matrix \( \rho \) and its complex conjugate \( \rho^* \), and are further transformed by the operators \( \{S_{mn} = L_m \otimes L_n, m = 1,...,M, n = 1,...,N\} \) as: \( \tilde{\rho}_{mn} = S_{mn} \rho^* S_{mn} \). In this notation, moreover, \( L_m \) and \( L_n \) are generators of groups \( SO(M) \) and \( SO(N) \) respectively, where \( N = d_1(d_1 - 1)/2 \) and \( M = d_2(d_2 - 2)/2 \).

In the following analysis we shall focus on bipartite concurrences of the form \( BC_{ijkl} \), where \( i,j,k = a,b,c,d \) and \( i \neq j \neq k \neq i \). These bipartite concurrences quantify entanglement between a single qubit \( k \) and a pair of qubits \( ij \) discarding the fourth qubit of the cluster state \( \ket{5} \). If concurrence \( BC_{ijkl} \) vanishes, we say that ESD has occurred for the partition \( k|ij \), i.e. between the qubit \( k \) and the pair of qubits \( ij \).

Let us assume as before that just two qubits of the four-qubit cluster state \( \ket{5} \) undergo GAD simultaneously. It is important to note that if qubits ab or cd are the subject of GAD, the entanglement between different parts of the four qubit system decreases asymptotically with parameter \( \gamma \) and ESD never occurs between partitions of the cluster state. This is the consequence of the permutational symmetry of the cluster state \( \ket{5} \). Nevertheless, if qubits bc or ad are influenced by GAD, ESD is present in the system for some coupling parameters \( \gamma \). Let qubits bc and \( d \) undergo GAD simultaneously as shown in Fig. 4. The dynamics of the cluster state \( \ket{5} \) is given by Eq. (3) with \( \gamma = 0.59 \) for \( i,j,k = 1,2,3,4 \) and \( \rho_{mn} = \langle C_i | C_i \rangle \). By computing bipartite concurrences for different partitions of the four-qubit system, we have found that ESD appears first for the partition \( ab|c \) for \( \gamma \approx 0.57 \) and later for partition \( cd|b \) for \( \gamma \approx 0.69 \) as displayed in Fig. 5.

For \( 0.69 > \gamma \geq 0.57 \), entanglement between qubits ab and c can be retrieved by the local filtering on qubit a as shown in Fig. 4. Parameter of the filtering \( \kappa \) can be chosen freely, unless we make additional demands for entanglement retrieving as discussed below. The probability of the entanglement retrieving is given as before by the norm of the final state \( [F \otimes \text{GAD} \otimes \text{GAD} \otimes I] \rho_{ab} \).

It is important to note that the entanglement between
qubits $cd$ and $b$ is not influenced by the filtering, since the qubit $a$ is discarded in the bipartite concurrence $BC_{cd|b}$. From one hand, it means that if ESD occurred for partition $cd|b$ (for $\gamma \geq 0.69$), entanglement between these qubits cannot be retrieved by the local filtering. Moreover, after $cd|b$ entanglement is destroyed, the filtering cannot retrieve $ab|c$ entanglement any more. From the other hand, invariance of $cd|b$ entanglement with regard to the filtering on qubit $a$ means that some bipartite entanglement in the system decreased after the filtering. Indeed, there must be some price for retrieving entanglement between qubits $ab$ and $c$ while $cd|b$ entanglement remains the same. The price for the entanglement retrieving is the decrease of $ad|b$ and $ad|c$ entanglement.

As we have seen, the analysis of the entanglement retrieving for the four-qubit cluster state [5] is much more complicated than for $N$-qubit W states. This analysis involves computation of twelve bipartite concurrences $BC_{ij|k}$ (for $i, j, k = a, b, c, d$ and $i \neq j \neq k \neq i$) which, moreover, behave differently with respect to each other because of the low symmetry of cluster state. This makes difficult a straightforward generalization of our results to the case on $N$-qubit cluster states for $N > 4$ using approach presented in this paper, i.e. by analyzing bipartite concurrences between different partitions of the state. Nevertheless, it seems possible to use a single filtering for entanglement retrieving in high order cluster states.

Our proposal for entanglement retrieving after ESD by a single local filtering may find its applications in quantum communication and quantum computing. It is known, in particular, that W states can be used for communication between several remote recipients [21]. Cluster states, in turn, are the resource for measurement-based quantum computation [22]. In both these applications, protection of genuine multipartite entanglement against partial disentanglement is highly desired to ensure correct and rigorous quantum information processing.

In conclusion, we have shown that genuine multipartite entanglement of W and cluster states, which has been partially lost due to detrimental influence of local environments, can be probabilistically retrieved to some extent by a single local filtering. This retrieval is achieved only by redistributing entanglement which remained in the system after the decoherence and does not rely on ability to control environmental degrees of freedom [3] or acquiring information about the whole system before the decoherence [4]. Therefore, we believe that our scheme for entanglement retrieving can be successfully combined with these alternative schemes.

Acknowledgments

We thank Mohammad Al-Amri and Suhail Zubairy for fruitful discussions during our meetings in Jazan University and KACST. We are also grateful to Luiz Davidovich for his comments.