KMS STATES AND THE CHEMICAL POTENTIAL FOR DISORDERED SYSTEMS

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Abstract. We extend the theory of the chemical potential associated to a compact separable gauge group to the case of disordered quantum systems. This is done in the natural framework of operator algebras. Among the other results, we show that the chemical potential does not depend on the disorder. The situation of the $n$–torus is treated in some detail. Indeed, provided that the zero–point is fixed independently on the disorder, the chemical potential is intrinsically defined in terms of the direct integral decomposition of the Connes–Radon–Nikodym cocycle associated to the KMS state $\omega$ and its trasforms $\omega \circ \rho$ by the localized automorphisms $\rho$ of the observable algebra, carrying the abelian charges of the model under consideration. This description parallels the analogous one relative to the usual (i.e. non disordered) quantum models.

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1. Introduction

In quantum physics, one often recovers the observable algebras by a principle of global gauge invariance. The reader is referred to [11, 12, 13, 14] and the reference cited therein. In order to investigate the termodynamical behavior of such physical models, the concept of chemical potential naturally arises. The algebraic description of the chemical potential is well understood, taking into account the principle of gauge invariance. Namely, suppose that we have $k$ species of particles (i.e. the chemical components). Provided that the field algebra $\mathcal{F}$ has a natural local structure, one can consider on it, infinite volume limits of states arising from the Gibbs grand canonical ensamble relative to fixed inverse temperature $\beta$ and the $k$–parametric chemical potential $\mu = (\mu_1, \ldots, \mu_k)$, one for each species of particles. It is seen that each of these states $\varphi_{\beta, \mu}$ satisfies the Kubo–Martin–Schwinger (KMS for short) boundary condition for a one parameter subgroup of the $(k + 1)$
dimensional Lie group $\mathbb{R} \times G$, the gauge group $G$ being isomorphic to the $k$–dimensional torus in this situation.\footnote{Here, in order to simplify matter, we are supposing that the asymmetry subgroup of a KMS state on the field algebra is trivial, see Section 4 for the asymmetry subgroup.} If we denote by $H$ and $\{N_i\}$ the infinitesimal generators of the time translations and the gauge transformations respectively, the one parameter subgroup mentioned above has the form

$$X = H - \sum \mu_i N_i.$$ 

As the observables are gauge invariant, the restriction $\omega_{\beta,\mu}$ of any $\varphi_{\beta,\mu}$ as above to the observable algebra $\mathcal{A} = \mathcal{F}^G$, satisfies the KMS condition at the same inverse temperature $\beta$ for the time evolution. In general, states corresponding to different values of the chemical potentials $\{\mu_i\}$, give rise to different KMS states, when restricted to the observable algebra.

In quantum physics, all the physical content of the model is encoded directly in the observable algebra. On the one hand, the chemical potential has a physical meaning, even if it naturally arises by the use of the field algebra. On the other hand, the KMS boundary condition does not refer to the local structure of the algebra of the observable.

In the seminal paper [4], all these questions are explained in detail, see also [5, 19, 20] for strictly connected questions. Namely, if $\omega$ is an extremal (or, more generally a weakly clustering) KMS state on $\mathcal{A}$, it is shown that any weakly clustering extension to all of $\mathcal{F}$ is a KMS state relative to a new evolution modified by a one parameter subgroup of the gauge group. In the simple situation described above, this one parameter subgroup uniquely determines the values of the chemical potentials.

Unfortunately, the results of [4] are not directly applicable to disordered models, the last including the very interesting examples of the spin glasses. The equilibrium statistical mechanics of models arising from spin classical glasses has been intensively studied, in order to understand the complex behaviour of the set of its temperature states. We refer the reader, for example, to [8, 15, 25, 26, 27] and the literature cited therein. Some attempts to understand the structure of the set of the KMS states of quantum disordered systems is made in [3, 6, 7, 21] by using the standard techniques of operator algebras. In the present paper, we follow the last strategy.
Namely, in order to achieve the disorder, it is natural to set for the observable algebra, $\mathfrak{A} := \mathcal{A} \otimes L^\infty(X, \nu)$, $(X, \nu)$ being the sample space for the coupling “constants” of the system.

As it is noted in [7], most of the interesting states of a disordered system are states $\omega$ whose centre $3_{\pi_\omega}$ of the GNS representation $\pi_\omega$ contains an Abelian algebra which is isomorphic to $L^\infty(X, \nu)$. In addition, the KMS states of interest for which $3_{\pi_\omega}$ is precisely $L^\infty(X, \nu)$, can be interpreted as the “pure thermodynamical phases” in the case of disordered systems. Furthermore, the phenomenon of the “weak Gibbsianess” naturally appear in this case.

Indeed, if one consider infinite volume limits of finite volume Gibbs states, one obtain states on $\mathcal{A} \otimes L^\infty(X, \nu)$ satisfying the following properties. Its marginal distribution of the couplings is the given probability measure describing the disorder, whereas the conditional distribution of the standard observable variables (the “spin” variables,), given the couplings, is some infinite-volume Gibbs state almost surely. Such a field of infinite–volume Gibbs states satisfies an equivariant property (see (2.5)). Namely, if it gives a (quantum version of a) Aizenman–Wehr metastate (see [1]), after direct integral decomposition, the last one assuming the meaning of the quantum counterpart of the classical procedure of conditioning w.r.t. the disorder variables. For recent results on metastates, we refer the reader to [26, 27] and the references cited therein. It can happen that the state so obtained is not necessarily jointly Gibbsian relatively to the standard variables and the couplings, see [16, 22] for some pivotal classical examples.

The systematic investigation of the difference between Gibbsianess and weak Gibbsianess (i.e. states which arise from infinite volume limit of finite volume Gibbs states but are not jointly Gibbsian) started in the paper [21] in the setting of operator algebras. In this paper, both equilibrium conditions are connected with some natural variational principles. As it is explained in Section 6 of [7], the KMS boundary condition for the algebra generated by spin variables and disorder variables seems to describe the weak Gibbsianess in quantum case. Moreover, it does not refer to the local structure of the observable algebra. For the rôle played by Gibbsianess and weak Gibbsianess in the description of the thermodynamical behavior of a disordered model, we refer the reader to the above mentioned papers.

In the present paper we extend the algebraic description of the chemical potential to disordered systems without referring to the difference between Gibbsianess and weak Gibbsianess. We take advantage twice by the paper [4]. First, we follow its plan. Second, we use the results of
this paper in order to describe the occurrence of the chemical potential for disordered models.

The present paper is organized as follows. After a preliminary section, in Section 3 we investigate some useful ergodic properties of states of interest of disordered systems. Section 4 is devoted to the occurrence of the chemical potential. Starting from a state $\phi$ on the field algebra $F = F \otimes L^\infty(X, \nu)$, normal when restricted to the subalgebra $L^\infty(X, \nu)$, which is weakly clustering with respect to the spatial translations, we show that the stabilizer, as well as the asymmetry subgroup coincide almost surely with the corresponding objects relative to the states $\phi_\xi$. Here, the measurable equivariant field $\{\phi_\xi\}_{\xi \in X} \subset F$ provides the direct integral decomposition of $\phi$. Then, we show that for any weakly clustering state $\varphi$ on $F$ whose restriction to $\mathfrak{A}$ is KMS, there exists a modification of the time evolution by a suitable one parameter group of the gauge group, the same for each $\varphi_\xi$, such that $\varphi_\xi$ is KMS with respect to this modified evolution almost surely. This is the content of Theorem 4.7 which is the natural generalization to our situation of Theorem II.4 of [4]. Section 5 is devoted to an intrinsic description of the chemical potential directly in terms of objects related to the algebra of observables. The case of the unit circle is treated in some detail, the case of the $n$–torus being quite similar.

Provided that its zero–point is fixed independently on the disorder, the chemical potential is intrinsically defined in terms of the Connes–Radon–Nikodym cocycle associated to the KMS state $\omega \in S(\mathfrak{A})$ under consideration, and its trasforms $\omega \circ \rho$ by the localized automorphisms $\rho$ of the observable algebra $\mathfrak{A}$, carrying the abelian charges of the model. Indeed, under suitable conditions, it is proven that $\omega \circ \rho$ is equivalent to $\omega$ also in our situation. Furthermore, the chemical potential is connected, and is independent almost surely on the disorder, with the Connes–Radon–Nikodym cocycle $(D(\omega_\xi \circ \rho_\xi) : D\omega_\xi)$ of $\omega_\xi \circ \rho_\xi$ relative to $\omega_\xi$, see Formula (5.2). Here, $\{\omega_\xi\}_{\xi \in X}$ provides the direct integral decomposition of (the normal extension of) $\omega$, and the measurable field of normal automorphisms $\{\rho_\xi\}_{\xi \in X}$ give rise the normal extension of $\rho$ to all of the $\pi_\omega(\mathfrak{A})^\pi$ which exist by Proposition 5.1.

2. PRELIMINARIES

We start by recalling the definition of the KMS boundary condition. A state $\phi$ on the $C^*$–algebra $\mathfrak{B}$ satisfies the KMS boundary condition at inverse temperature $\beta$ which we suppose to be always different from zero, w.r.t the group of automorphisms $\{\tau_t\}_{t \in \mathbb{R}}$ if

(i) $t \mapsto \phi(\tau_t(B))$ is a continuous function for every $A, B \in \mathfrak{B}$,
(ii) \[ \int \phi(\tau_t(B)) f(t) \, dt = \int \phi(\tau_t(B)A) f(t + i\beta) \, dt \] whenever \( f \in \mathcal{D} \), \( \mathcal{D} \) being the space made of all infinitely often differentiable compactly supported functions in \( \mathbb{R} \).

For the equivalent characterizations of the KMS boundary condition, the main results about KMS states, and finally the connections with Tomita theory of von Neumann algebras, see e.g. [9, 30] and the references cited therein.

It is well–known that the cyclic vector \( \Omega_\phi \) of the GNS representation \( \pi_\phi \) is also separating for \( \pi_\phi(\mathcal{B})'' \). Denote with an abuse of notation, \( \sigma_\phi \) its modular group.

According to this definition of KMS boundary condition, we have
\[
(2.1) \quad \sigma_t^\phi \circ \pi_\phi = \pi_\phi \circ \tau_{-\beta t}.
\]

Our set–up is a separable \( C^* \)–algebra \( \mathcal{A} \) with an identity \( I \), describing the physical observables.\(^2\) We suppose that \( \mathcal{A} \) is obtained as the fixed–point algebra \( \mathcal{A} = \mathcal{F}^G \) under a pointwise–norm continuous action
\[
\gamma : g \in G \mapsto \gamma_g \in \text{Aut}(\mathcal{F})
\]
of a compact second countable group \( G \) (the gauge group) on another separable \( C^* \)–algebra \( \mathcal{F} \) (the field algebra). This is a typical situation appearing in quantum field theory, when the charges present in the model are described in terms of a principle of (global) gauge invariance, see e.g. [11, 12, 13, 14]. The present description can be applied also to nontrivial models where the local algebras of observables are full matrix algebras, see e.g. [28] for a possible example along this line.

We suppose that the group \( \{\alpha_x\}_{x \in \mathbb{Z}^d} \) of spatial translations acts on \( \mathcal{F} \). We consider also a standard measure space \( (X, \nu) \) based on a compact separable space \( X \), and a Borel probability measure \( \nu \). The group \( \mathbb{Z}^d \) of the spatial translations is supposed to act on the probability space \( (X, \nu) \) by measure preserving ergodic transformations \( \{T_x\}_{x \in \mathbb{Z}^d} \).

A one parameter random group of automorphisms
\[
(t, \xi) \in \mathbb{R} \times X \mapsto \tau_t^\xi \in \text{Aut}(\mathcal{F})
\]
is acting on \( \mathcal{F} \). It is supposed to be strongly continuous in the time variable for each fixed \( \xi \in X \), and jointly strongly measurable. Consider, for \( A \in \mathcal{F} \), the strongly measurable function \( f_{A,\xi}(t) := \tau_t^\xi(A) \).

\(^2\)In order to avoid technical complications, in quantum field theory the local algebras of observables are enlarged by taking the weak operator closure in the vacuum representation. Namely, the local algebras of observables are typically von Neumann algebras with separable predual, the former being non separable \( C^* \)–algebras. This does not affect the substance of the theory, see the comments in Section 5.
We get
\[ \|f_{A,t}\|_{L^\infty(X,\nu;F)} \equiv \operatorname{esssup}_{\xi \in X} \|\tau_t^\xi(A)\|_F = \|A\|_F, \]
where the last equality follows as \(\tau_t^\xi\) is isometric. We assume further
that \(\tau\) acts locally. Namely, if \(A\) is an element of \(\mathcal{F}\), then the function
\(f_{A,t} \in L^\infty(X,\nu;F)\) belongs to the \(C^*\)-subalgebra \(\mathcal{F} \otimes L^\infty(X,\nu)\), where
the above \(C^*\)-tensor product is uniquely determined as any commu-
tative \(C^*\)-algebra is nuclear.

We assume the following commutation rules
\[
\begin{align*}
\tau_{t-x}^\xi \alpha_x &= \alpha_x \tau_t^\xi \\
\alpha_x \gamma_g &= \gamma_g \alpha_x \\
\tau_t^\xi \gamma_g &= \gamma_g \tau_t^\xi
\end{align*}
\]
for each \(x \in \mathbb{Z}^d, \xi \in X, t \in \mathbb{R}, \) and \(g \in G\).

By (2.2), it is immediate to show that \(\alpha_x\) and \(\tau_t^\xi\) leave globally stable \(\mathcal{A}\). Namely, \(\mathbb{Z}^d, \mathbb{R}\) act on \(\mathcal{A}\) as groups of automorphisms or random automorphisms, respectively.

Finally, we address also the situation when Fermion operators are
present in \(\mathcal{F}\). Namely, there exists an automorphism \(\sigma\) of \(\mathcal{F}\) commuting
with all the gauge transformations, the spatial translations and the
random time evolution, such that \(\sigma^2 = e\). We put
\[
(2.3) \quad \mathcal{F}_+ := \frac{1}{2}(e + \sigma)(\mathcal{F}), \quad \mathcal{F}_- := \frac{1}{2}(e - \sigma)(\mathcal{F}).
\]

The disordered system under consideration is described by
\[
\tilde{\mathcal{F}} := \mathcal{F} \otimes L^\infty(X,\nu).
\]

Notice that, by identifying \(\tilde{\mathcal{F}}\) with a closed subspace of \(L^\infty(X,\nu;\mathcal{F})\),
each element \(A \in \tilde{\mathcal{F}}\) is uniquely represented by a measurable essentially
bounded function \(\xi \mapsto A(\xi)\) with values in \(\mathcal{F}\).

The group \(\mathbb{Z}^d\) of all the space translations is naturally acting on the
\(C^*\)-algebra \(\tilde{\mathcal{F}}\) as
\[
\alpha_x(A)(\xi) := \alpha_x(A(T_{-x}\xi)).
\]
Further, define on \(\tilde{\mathcal{F}}\),
\[
(2.4) \quad \begin{align*}
\mathbf{t}_i(A)(\xi) &:= \tau_t^\xi(A(\xi)), \\
\mathbf{g} &:= \gamma \otimes \text{id}_{L^\infty(X,\nu)}, \\
\mathbf{s} &:= \sigma \otimes \text{id}_{L^\infty(X,\nu)}.
\end{align*}
\]

In most of the interesting physical situations, \(\sigma \in \mathcal{Z}(G), G\) being the gauge
group, see e.g. [11, 14]. The situation without Fermion operators corresponds to
\(\sigma = e\).
It is straightforward to verify that \( \{ a_x \}_{x \in \mathbb{Z}^d}, \{ t_t \}_{t \in \mathbb{R}} \) and \( \{ g_g \}_{g \in G} \) define actions of \( \mathbb{Z}^d \), \( \mathbb{R} \) and \( G \) on \( \mathcal{F} \) which are mutually commuting, and commute also with the parity automorphism \( s \). The subspaces \( \mathcal{F}_+ \) and \( \mathcal{F}_- \) are defined as in (2.3). Furthermore, taking into account (2.2) and the definition (2.4) of the action of the gauge group on the disordered field algebra \( \mathfrak{A} \). Namely, \( \{ a_x \}_{x \in \mathbb{Z}^d} \) and \( \{ t_t \}_{t \in \mathbb{R}} \) define by restriction, mutually commuting actions of \( \mathbb{Z}^d \) and \( \mathbb{R} \) on \( \mathfrak{A} \), respectively.

In order to study a class of states of interest for disordered systems, we start with *–weak measurable fields of states

\[ \xi \in X \mapsto \varphi_\xi \in \mathcal{S}(\mathcal{F}) \].

We suppose that the field \( \{ \varphi_\xi \}_{\xi \in X} \) fulfills almost surely, the equivariance condition

(2.5) \[ \varphi_\xi \circ \alpha_x = \varphi_{T_{-x} \xi} \]

w.r.t. the spatial translations, simultaneously.

A state \( \varphi \) on \( \mathfrak{F} \) is naturally defined as follows:

(2.6) \[ \varphi(A) = \int_X \varphi_\xi(A(\xi)) \nu(d\xi), \quad A \in \mathfrak{F} \].

It is immediate to verify that \( \varphi \) defined as above is invariant w.r.t. the space translations \( a_x \). Moreover, \( \varphi \mid_{I \otimes L^\infty(X, \nu)} \) is a normal state.

Equally well, one can start with a \( a \)–invariant state \( \varphi \) on \( \mathfrak{F} \), which is normal when restricted to \( I \otimes L^\infty(X, \nu) \). Then, we can recover a *–weak measurable field \( \{ \varphi_\xi \}_{\xi \in X} \subset \mathcal{F} \) fulfilling (2.5). Such a measurable fields provides the direct integral decomposition of \( \varphi \) as in (2.6), see [7], Theorem 4.1. Similar considerations can be applied to the observable algebras \( \mathfrak{A} \) as well. In the sequel, we denote by \( S_0(\mathfrak{A}) \), \( S_0(\mathfrak{F}) \) the convex closed subset of states on \( \mathfrak{A} \), \( \mathfrak{F} \) respectively, fulfilling the properties listed above.

3. Ergodic Properties of States of Disordered Systems

In this section we study some useful ergodic properties of states in \( S_0(\mathfrak{F}) \) or \( S_0(\mathfrak{A}) \). We restrict ourselves to the field algebra, the other case being similar.

Let \( C, D \in \mathfrak{F} \), and \( A, B \in \mathfrak{F}_+ \cup \mathfrak{F}_- \). Put \( \epsilon_{A,B} = -1 \) if \( A, B \in \mathfrak{F}_- \) and \( \epsilon_{A,B} = 1 \) in the three remaining possibilities. We say that the state \( \varphi \) is asymptotically Abelian w.r.t. \( a \) if

(3.1) \[ \lim_{|x| \to +\infty} \varphi \left( C(a_x(A)B - \epsilon_{A,B}B a_x(A))D \right) = 0, \]
The state $\varphi$ is weakly clustering w.r.t. $a$ if

$$(3.2) \quad \lim_N \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \varphi(Aa_x(B)) = \varphi(A)\varphi(B),$$

$\Lambda_N$ being the box with a vertex sited in the origin, containing $N^d$ points with positive coordinates.\(^4\)

Notice that, lots of interesting states are naturally asymptotically Abelian w.r.t. the spatial translations, see e.g. [7], Proposition 2.3, see also [23].

We report the following result for the sake of completeness.

**Proposition 3.1.** Suppose that $\varphi \in S(\mathfrak{F})$ is a $a$–invariant asymptotically Abelian state. Then the following assertions are equivalent.

(i) $\varphi$ is a $a$–weakly clustering,
(ii) $\varphi$ is a $a$–ergodic.

**Proof.** It is a well–known fact that (i) always implies (ii). The reverse implication follows as in Proposition 5.4.23 of [9], the last working also under the weaker condition (3.1). \(\Box\)

One sees that an asymptotically abelian invariant state is automatically $s$–invariant, that is it is an even state.

The weak clustering property for states in $S_0(\mathfrak{F})$ can be translated as a property of the corresponding equivariant fields of states on $\mathcal{F}$. Namely, Let $\varphi \in S_0(\mathfrak{F})$, and $\{\varphi_\xi\}_{\xi \in X} \subset S(\mathcal{F})$ the corresponding $\alpha$–equivariant measurable field of states. Then the results listed below hold true.

**Proposition 3.2.** The following assertions are equivalent.

(i) $\varphi$ is a $a$–weakly clustering,
(ii) we have for each $A \in \mathcal{F}$, $B \in \mathfrak{F}$,

$$(3.3) \quad \lim_N \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \varphi_\xi(Aa_x(B(T_x \xi))) = \varphi_\xi(A)\varphi(B),$$

in the $\ast$–weak topology of $L^1(X, \nu)$.

\(^4\)Taking into account natural applications to continuous disordered systems, we can consider cases when $\mathbb{R}^d$ is acting as the group of spatial translations. We use in (3.2) the natural modification $M$ of the Cesaro mean given on bounded measurable functions, by

$$M(f) := \lim_{D \to +\infty} \frac{1}{\operatorname{vol}(\Lambda_D)} \int_{\Lambda_D} f(x) d^d x,$$

$\Lambda_D$ being a box with edges of length $D$. Most of the forthcoming analysis can be applied in this situation as well.
Proof. (i) $\Rightarrow$ (ii) We compute for $f \in L^\infty(X,\nu)$, $A \in F$, $B \in \mathcal{F}$,

\[
\int_X f(\xi)(\varphi_\xi(A)\varphi(B)) = \varphi(A \otimes f)\varphi(B)
\]

which is the assertion.

(ii) $\Rightarrow$ (i) By a standard density argument, we reduce the situation to element of the form $A \otimes f, B \in \mathcal{F}$, with $f \in L^\infty(X,\nu)$, $A \in F$. We get

\[
\lim_N \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \varphi(A \otimes f\alpha_x(B))
\]

\[
= \lim_N \int_X f(\xi) \left( \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \varphi_\xi(A\alpha_x(B(T^{-x}\xi))) \right)
\]

\[
\int_X f(\xi)(\varphi_\xi(A)\varphi(B)) \equiv \varphi(A \otimes f)\varphi(B)
\]

and we are done. \qed

A sufficient condition for the weak clustering property of $\varphi$ is the pointwise–clustering property for the corresponding equivariance field $\{\varphi_\xi\}_{\xi \in X}$ of states, almost surely.

**Proposition 3.3.** Suppose that, for each $A, B \in F$

\[(3.4) \quad \lim_{|x| \to +\infty} \varphi_\xi(A\alpha_x(B)) = \varphi_\xi(A)\varphi_{T^{-x}\xi}(B) \quad \text{almost surely.} \]

Then the state $\varphi$ given by (2.6) is weakly clustering.

**Proof.** By a standard density argument, we can reduce the situation to a measurable set $F \subset X$ of full measure such that (3.4) is satisfied simultaneously for each $A, B \in F$. Define, for fixed $A, B \in F$, $g \in L^\infty(X,\nu)$ and $\xi \in F$,

\[
\delta_1(N) := \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} g(T^{-x}\xi) \left( \varphi_\xi(A\alpha_x(B)) - \varphi_\xi(A)\varphi_\xi(\alpha_x(B)) \right)
\]

\[
\delta_2(N) := \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} g(T^{-x}\xi)\varphi_{T^{-x}\xi}(B) - \varphi(B \otimes g)
\]
We have that \( \delta_1(N) \to 0 \) by hypothesis, and \( \delta_2(N) \to 0 \) by the Individual Ergodic Theorem. Then we conclude that, for fixed \( A, B \in \mathcal{F}, \ g \in L^\infty(X, \nu) \),

\[
\frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} g(T_{-x} \xi) \varphi_x(A\alpha_x(B)) - \varphi_x(A) \varphi(B \otimes g) \\
\equiv \delta_1(N) - \varphi_x(A) \delta_2(N) \to 0
\]

pointwise on \( F \), as \( N \to +\infty \). Let now \( f \in L^\infty(X, \nu) \). We have

\[
\frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \varphi(A \otimes f a_x(B \otimes g)) - \varphi(A \otimes f) \varphi(B \otimes g) \\
= \int_X f(\xi) \left( \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} g(T_{-x} \xi) \varphi_x(A\alpha_x(B)) - \varphi_x(A) \varphi(B \otimes g) \right) \nu(d\xi)
\]

which goes to 0 by the Lebesgue Dominated Convergence Theorem. \( \square \)

Finally, we point out the fact that there exist examples of weakly clustering states \( \varphi \) satisfying the properties listed above without assuming any factoriality condition about \( \varphi \), the last being not natural in the setting of disordered systems, see Section 4 and Section 5 of [7]. Indeed, it is enough to consider any quasi–local algebra \( \mathcal{F} \) admitting a state \( \omega \) which is strongly clustering (i.e. mixing) w.r.t. the space translations. Take any probability space \( (X, \nu) \) on which the space translations act ergodically. Define \( \varphi_{\xi} = \omega \), the constant field. By Proposition 3.3, the state given by \( \varphi(A) := \int_X \omega(A(\xi)) \nu(d\xi) \) satisfies all the required properties. In addition, it follows By Proposition 2.3 of [7], that the states on \( \mathfrak{F} \) constructed as above from states \( \omega \) arising from quasi local algebras describing spin models on \( \mathbb{Z}^d \) are \( \mathfrak{A} \)–asymptotically Abelian as well.

We speak, without any further mention, and if it is not otherwise specified, about asymptotic Abelianity, weak clustering, or ergodicity for states, if they satisfy these properties w.r.t. the spatial translations.

4. EXTENSION OF STATES FOR DISORDERED SYSTEMS

In this section we prove, following the line of [4], the following result. For each weakly clustering state \( \varphi \in \mathcal{S}_0(\mathfrak{F}) \) whose restriction to \( \mathfrak{A} \) satisfies the KMS boundary condition, there exists a modification of the time evolution by a suitable one parameter group of the gauge group, the same for each \( \varphi_{\xi} \), such that \( \varphi_{\xi} \) is KMS w.r.t. this modified evolution almost surely. This is done after showing that the stabilizer \( G_{\varphi} \subset G \), as well as the asymmetry subgroup \( N_{\varphi} \subset G \) coincide almost surely with the corresponding objects relative to the states \( \varphi_{\xi} \).
Let \( \varphi, \psi \in \mathcal{S}_0(\mathfrak{A}) \), and \( \{ \varphi_\xi \}_{\xi \in X}, \{ \psi_\xi \}_{\xi \in X} \) be the corresponding \( \alpha \)-equivariant measurable fields of states on \( \mathcal{F} \).

**Proposition 4.1.** If \( \varphi, \psi \in \mathcal{S}_0(\mathfrak{A}) \) are weakly clustering states whose restrictions to \( \mathfrak{A} \) are equal, then there exist \( g \in G \) and a measurable subset \( F \) of full measure, such that \( \xi \in F \) implies

\[
\psi_\xi(A) = \varphi_\xi(\gamma_g(A)),
\]

simultaneously for every \( A \in \mathcal{F} \).

**Proof.** By Theorem II.1 of [4], there exists \( g \in G \) such that \( \psi = \varphi \circ g \). We compute for each \( f \in L^\infty(X, \nu) \) and \( A \in \mathcal{F} \),

\[
\int_X \nu(d\xi)f(\xi)(\psi_\xi(A) - \varphi_\xi(\gamma_g(A))) = 0.
\]

This means that for any fixed \( A \in \mathcal{F} \) there exists a measurable set \( F_A \) of full measure such that

\[
\psi_\xi(A) = \varphi_\xi(\gamma_g(A))
\]
on \( F_A \). Choose a dense countable subset \( F_0 \subset \mathcal{F} \). Put \( F := \bigcap_{A \in F_0} F_A \).

\( F \) is a measurable subset of \( X \) of full measure. For each \( A \in \mathcal{F} \) choose a sequence \( \{ A_n \} \) in \( F_0 \) converging to \( A \). We have for \( \xi \in F \),

\[
\psi_\xi(A) = \lim_n \psi_\xi(A_n) = \lim_n \varphi_\xi(\gamma_g(A_n)) = \varphi_\xi(\gamma_g(A)).
\]

□

**Proposition 4.2.** Suppose that each state in \( \mathcal{S}_0(\mathfrak{A}) \) is asymptotically Abelian. Then any weakly clustering state \( \omega \in \mathcal{S}_0(\mathfrak{A}) \) extend to a weakly clustering state \( \varphi \in \mathcal{S}_0(\mathfrak{A}) \).

**Proof.** Let \( \psi \) be any extension of \( \omega \). Then it is normal when restricted to \( L^\infty(X, \nu) \). It could be not \( \alpha \)-invariant.\(^5\) Let \( m \) be any invariant mean on \( \mathbb{Z}^d \) which exists as \( \mathbb{Z}^d \) is amenable. Then \( m(\{ \psi(\alpha_x(A)) \}) \) defines an invariant extension of \( \omega \), that is the compact convex subset of \( \mathcal{S}_0(\mathfrak{A}) \) consisting of all the \( \alpha \)-invariant extensions of \( \omega \) is nonvoid. Take any extremal element \( \varphi \) in such compact convex set. As \( \omega \) is weakly clustering, it is extremal in the set of all \( \alpha \)-invariant states (i.e. \( \alpha \)-ergodic). As \( \varphi \) is an extremal extension of \( \omega \), we conclude that \( \varphi \) is itself \( \alpha \)-ergodic. By Proposition 3.1, \( \varphi \) is also weakly clustering under our assumptions.

\(^5\)Notice that, for the measurable field \( \{ \psi_\xi \}_{\xi \in X} \) of positive forms giving the decomposition of \( \psi \), \( \psi_\xi(I) = 1 \) almost everywhere. So we have a decomposition of \( \psi \) into a measurable, not necessarily equivariant, field of states.
Notice that, by the results contained in [7], Proposition 2.3 of [7], and the considerations in Section 5, there are disordered models satisfying the assumptions of Proposition 4.2.

We define in the usual way, the stabilizer of a state $\varphi \in S(F)$ as
$$G_{\varphi} := \{ g \in G \mid \varphi \circ g = \varphi \}.$$

The stabilizers $G_{\varphi_\xi}$ of the $\varphi_\xi \in S(F)$ are defined analogously as
$$G_{\varphi_\xi} := \{ g \in G \mid \varphi_\xi \circ \gamma_g = \varphi_\xi \}.$$

The normalizer $\mathcal{N}(H)$ and the centralizer $\mathcal{Z}(H)$ of a subgroup $H \subset G$ are defined in the usual way as
$$\mathcal{N}(H) := \{ g \in G \mid gHg^{-1} = H \},$$
$$\mathcal{Z}(H) := \{ g \in G \mid gh = hg, h \in H \}.$$

We show that the stabilizers $G_{\varphi_\xi}$ of the $\varphi_\xi$ are almost surely independent on the disorder, and coincide with the stabilizer $G_{\varphi}$ almost everywhere.

**Theorem 4.3.** Let $\varphi \in S_0(F)$. Then there exists a measurable set $F \subset X$ of full measure such that $\xi \in F$ implies $G_{\varphi_\xi} = G_{\varphi}$.

**Proof.** Without loss of generality, we can suppose that (2.5) holds true everywhere on $X$. Choose a countable dense subset $\{ h \}$ of $G$ which always exists by separability. Define
$$V_{h,n} := \{ g \in G \mid \text{dist}(g, h) \leq 1/n \},$$
$$X_{h,n} := \{ \xi \in X \mid G_{\varphi_\xi} \cap V_{h,n} \neq \emptyset \},$$
where “dist” is any metric on $G$ generating its topology. Put
$$f_A(\xi, g) := \varphi_\xi(\gamma_g(A)) - \varphi_\xi(A),$$
$$\Gamma := \bigcap_{A \in F} f_A^{-1}(\{0\}).$$

By separability, we can reduce the last intersection to a countably dense set of $F$, that is $\Gamma$ is a measurable subset of $X \times G$. Furthermore, it is immediate to check that
$$X_{h,n} = P_X((X \times V_{h,n}) \cap \Gamma),$$
$P_X$ being the projection w.r.t. the first variable. This means that the sets $X_{h,n}$ are measurable. Taking into account also (2.5), they are also

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6Let $X_0 \subset X$ be the measurable set of full measure such that (2.5) is simultaneously satisfied for $x \in \mathbb{Z}^d$. We reduce the situation to the measurable invariant set $\bigcap_{x \in \mathbb{Z}^d} T_x X_0$ of full measure.
invariant under the action of the spatial translations. So, by ergodicity, we have that $\nu(X_{h,n})$ is 0 or 1.

Let $S_n \subset G$ be the set of the $h$ such that $\nu(X_{h,n}) = 1$. Define

$$G_1 := \bigcap_{n>0} \left\{ \bigcup_{h \in S_n} V_{h,n} \right\},$$

$$F_1 := \bigcap_{n>0} \left\{ \bigcap_{h \in S_n} X_{h,n} \right\}.$$

Notice that $F_1$ is a measurable set of full measure.

We get that $\xi \in F_1$ implies that $G_{\varphi_\xi} = G_1$. Indeed, if $g \in G_1$, $g \in \bigcup_{h \in S_n} V_{h,n}$ for each $n$, which means for each $n$, $\text{dist}(G_{\varphi_\xi}, g) < 1/n$ whenever $\xi \in F_1$. Hence, $\xi \in F_1$ implies $G_1 \subset G_{\varphi_\xi}$. Conversely, if $g \in G_{\varphi_\xi}$ and $\xi \in F_1$, then $g \in \bigcup_{h \in S_n} V_{h,n}$ for every $n$, which means $G_{\varphi_\xi} \subset G_1$.

Let now $F_0 \subset F$, $G_0 \subset G_{\varphi}$ be dense countable subsets, then we obtain for each $f \in L^\infty(X, \nu)$, $A \in F_0$ and $g \in G_0$,

$$\int_X \nu(d\xi) f(\xi)(\varphi_\xi(\gamma_g(A)) - \varphi_\xi(A)) = 0$$

which means that one can choose a measurable set $F_0$ of full measure such that $\xi \in F_0$, $A \in F_0$ and $g \in G_0$ imply

$$\varphi_\xi(\gamma_g(A)) = \varphi_\xi(A),$$

see the proof of Proposition 4.1.

Let now $g \in G_{\varphi}$ be fixed. Choose a sequence $\{g_n\} \subset G_0$ converging to $g$. We have for each $\xi \in F_0$ and $A \in F_0$,

$$\varphi_\xi(\gamma_g(A)) = \lim_n \varphi_\xi(\gamma_{g_n}(A)) = \lim_n \varphi_\xi(A) \equiv \varphi_\xi(A),$$

which, by separability, implies $G_{\varphi} \subset G_{\varphi_\xi}$ whenever $\xi \in F_0$. Set $F := F_0 \cap F_1$. Taking into account the first part of the proof and the definition (2.6) of $\varphi$, we obtain that $\xi \in F$ implies

$$G_{\varphi_\xi} \subset G_{\varphi} \subset G_{\varphi_\xi} = G_1,$$

which is the assertion. \hfill \Box

**Theorem 4.4.** If $\varphi \in S_0(3)$ is a weakly clustering state whose restriction to $\mathfrak{A}$ is $t$–invariant, then there exist a continuous one–parameter subgroup $t \in \mathbb{R} \mapsto \varepsilon_t \in \mathcal{Z}(G_{\varphi})$, and a measurable subset $F$ of full measure such that $\xi \in F$ implies that $\varphi_\xi$ is invariant under the modified
time translation
\[ \varphi_\xi(A) = \varphi_\xi(\tau_1^\xi \gamma_\varepsilon(A)), \]

simultaneously for every \( A \in \mathcal{F} \) and \( t \in \mathbb{R} \).

**Proof.** We cannot directly apply Theorem II.2 of [4] as our time translations \( t \) enjoy less continuity property than strong continuity, see [7]. However, in order to apply the mentioned result, it is enough to verify that the one parameter group \( t \in \mathbb{R} \mapsto \hat{v}_t \in \mathcal{N}(G_\varphi)/G_\varphi \), in pag. 106 of [4] is continuous also in our situation (i.e. when the map \( t \mapsto \varphi(At(B)C) \) is continuous for every fixed elements \( A, B, C \in \mathfrak{H} \)). Suppose not. There exists an open neighbourhood \( U \supset G_\varphi \) of the stabilizer \( G_\varphi \) of \( \varphi \) such that \( v_{1/n} \in U^c \). Choose a subsequence \( \{v_{1/n_k}\} \) converging to some element \( v_0 \in U^c \). We get
\[ \varphi(A) = \lim_k \varphi(t_{1/n_k}(A)) = \lim_k \varphi(g_{v_{1/n_k}}(A)) = \varphi(g_{v_0}(A)) \]

which is a contradiction as the automorphism \( g_{v_0} \) is not in the stabilizer. Hence, the conclusions of Theorem II.2 of [4] hold true also in our situation. We can conclude by reasoning as in the proof of Proposition 4.1, and after choosing countable dense subsets \( \mathbb{R}_0 \subset \mathbb{R}, \mathcal{F}_0 \subset \mathcal{F} \). Namely, there exists a measurable subset \( F \subset X \) of full measure such that \( \xi \in F, t \in \mathbb{R}_0 \) and \( A \in \mathcal{F}_0 \) implies \( \varphi_\xi(A) = \varphi_\xi(\tau_1^\xi \varepsilon(A)) \). Let \( t \in \mathbb{R} \) and \( A \in \mathcal{F} \). Choose convergent sequences \( t_n \to t, A_n \to A \). We have on \( F \), taking into account that \( \tau_{t_n}^\xi \varepsilon_{t_n}(A_n) \to \tau_t^\xi \varepsilon_t(A) \),
\[ \varphi_\xi(A) = \lim_n \varphi_\xi(A_n) = \lim_n \varphi_\xi(\tau_{t_n}^\xi \varepsilon_{t_n}(A_n)) = \varphi_\xi(\tau_t^\xi \varepsilon_t(A)) \]

which is the assertion. \( \square \)

Let \( \varphi \in \mathcal{S}_0(\mathfrak{H}) \), and consider the corresponding equivariant field \( \{\varphi_\xi\}_{\xi \in X} \subset \mathcal{S}(\mathcal{F}) \). We have shown in Theorem 4.3 that \( G_{\varphi_\xi} = G_\varphi \) almost surely. This means that the GNS representations \( \pi_{\varphi_\xi} \), as well as \( \pi_\varphi \), are equipped with a strongly continuous representation \( U_{\varphi_\xi} \), or \( U_\varphi \), of the common subgroup \( G_\varphi \subset G \) implementing the gauge action of \( G_\varphi \) on \( \mathcal{F} \) or \( \mathfrak{H} \) respectively. Let \( H \subset G_\varphi \) be a closed subgroup. Denote \( \hat{H} \) the set of all irreducible representations of \( H \). It is well-known that the elements of \( \hat{H} \) act on finite dimensional Hilbert spaces (compactness of \( H \)), and \( \hat{H} \) is at most countable (second countability of \( H \)).

The restriction of \( U_{\varphi_\xi} \), or \( U_\varphi \) to \( H \) are denoted as \( U_{\varphi_\xi}^H \), or \( U_\varphi^H \) respectively. The \( H \)–spectra \( \Sigma_{\varphi_\xi}^H, \Sigma_{\varphi}^H \subset \hat{H} \) of \( \varphi_\xi, \varphi \) are the set of all irreducible representations of \( H \) contained in \( U_{\varphi_\xi}^H \), or \( U_\varphi^H \) respectively.
Following Definition II.3 of [4], we say that $\Sigma^H_\varphi$ (or equivalently $\Sigma^H_\varphi\xi$) is one-sided if it is contained in a set $\Sigma \subset \hat{H}$ which enjoys the following properties:

(i) $\sigma_1, \sigma_2 \in \Sigma$ implies that every irreducible summand of $\sigma_1 \otimes \sigma_2$ is also contained in $\Sigma$,
(ii) $\sigma, \bar{\sigma} \in \Sigma$ implies that $\sigma = id$.

**Theorem 4.5.** Let $\varphi \in S_0(\mathfrak{F})$. Then

(i) the $H$-spectrum $\Sigma^H_\varphi\xi$ is almost surely independent on $\xi \in X$,
(ii) if $\sigma \in \Sigma^H_\varphi\xi$ almost surely, its multiplicity is (almost surely) independent on $\xi \in X$.

**Proof.** We can identify the GNS triplet $(\pi_{\varphi_\xi}, \mathcal{H}_{\varphi_\xi}, \Omega_{\varphi_\xi})$ relative to $\varphi_\xi$ with $(\pi_{\varphi_\xi} \circ \alpha_x, H_{\varphi_\xi}, \Omega_{\varphi_\xi})$ almost surely. Under this identification, $U^H_{\varphi_{T-x_\xi}} = U^H_{\varphi_{T-x_\xi}} \varphi_\xi$. In other words, $U^H_{\varphi_{T-x_\xi}} \varphi_\xi = U^H_{\varphi_{T-x_\xi}} \varphi_\xi$ almost surely.

The assertion follows by ergodicity, as the measurable subsets $F_{\sigma,m} := \{\xi \in X \mid \sigma < U^H_{\varphi_\xi} \text{ with multiplicity } m\}$ give a countable partition of $X$. \qed

**Corollary 4.6.** Let $\varphi \in S_0(\mathfrak{F})$.

(i) If $\varphi$ is weakly clustering and asymptotically Abelian, then $U^H_{\varphi_\xi}$ satisfies the semigroup property of Theorem II.3 of [4] almost surely,
(ii) $\Sigma^H_\varphi$ is one-sided if and only if $\Sigma^H_\varphi\xi$ is one-sided almost surely.

**Proof.** The proof easily follows by Theorem 4.5 and the mentioned Theorem II.3 of [4]. \qed

Here, there is the main theorem of the present section concerning the appearance of the chemical potential in the setting of disordered systems.

**Theorem 4.7.** If $\varphi \in S_0(\mathfrak{F})$ is a weakly clustering asymptotically abelian state whose restriction to $\mathfrak{A}$ is $(t, \beta)$--KMS state at inverse temperature $\beta \neq 0$, then there exist a closed subgroup $N \subset G_\varphi$, a continuous one-parameter subgroup $t \in \mathbb{R} \mapsto \varepsilon_t \in Z(G_\varphi)$, a continuous one-parameter subgroup $\tau \in \mathbb{R} \mapsto \zeta_t \in G_\varphi$, and a measurable subset $F$ of full measure such that, for each $\xi \in F$,

(i) the $N$-spectrum of $\varphi_\xi$ is one-sided,
(ii) the restriction of $\varphi_\xi$ to $\mathcal{F}^N := \{A \in \mathcal{F} \mid g(A) = A, g \in N\}$ is a $(\theta^k, \beta)$--KMS state for the modified time evolution $\theta^k_{t_\xi} := \tau_{t_\xi} \gamma_{\varepsilon_t \zeta_t}$. 

(iii) the image \( [\zeta_t] := \zeta_t N \) in \( G_\varphi/N \) is in \( \mathcal{Z}(G_\varphi/N) \).

**Proof.** We start by noticing that the conclusions of Theorem II.4 of [4] hold true also in our situation. Indeed, the hypothesis of extremality w.r.t the KMS condition is not used in the proof of that theorem.\(^7\)

Furthermore, in order to apply those results to our situation, we should replace the dense subset of entire elements used in II.6, pag. 110 with the dense subset \( \mathfrak{F}_0 \) generated by elements of the form

\[ A_f(\xi) := \int f(t) \tau_t^\xi(A(\xi)) \, dt, \]

where \( A \) runs on elements of \( \mathfrak{F} \) and \( f \in \hat{D} \), together with (the image in \( \mathfrak{F} \) of) \( L^\infty(X,\nu) \). Hence, we can apply the above mentioned theorem to \( \varphi \) as above. The assertion follows with \( N := N_\rho, \varepsilon \) and \( \zeta \), the one parameter subgroups relative to \( \varphi \) as in Theorem II.4 of [4], by applying Theorem 4.3, Theorem 4.4, Corollary 4.6, and finally Proposition 3.2 of [7]. \( \square \)

The subgroup \( N_\varphi \subset G_\varphi \) of Theorem II.4 of [4] is called the asymmetry subgroup of \( \varphi \). Notice that, by an elementary application of Corollary 4.6, in the situation of Theorem 4.7 the asymmetry subgroup \( N_{\varphi_\xi} \subset G_{\varphi_\xi} \) of \( \varphi_\xi \) coincides with the asymmetry subgroup \( N_\varphi \) of \( \varphi \) almost everywhere.

5. **AN INTRINSIC CHARACTERIZATION OF THE CHEMICAL POTENTIAL FOR DISORDERED SYSTEMS**

In the present section we show how the chemical potential can arise as an object directly associated to the algebra of observables. To simplify matter, we consider the simplest non trivial case when the gauge group is the unit circle \( T \). In this situation, the charges in the model under consideration are generated by the powers \([\sigma^n] \in \text{Out}(A)\) of a single localized transportable automorphism \( \sigma \), see [11].\(^8\)

We start by proving the following

**Proposition 5.1.** Let \( \omega \in S_0(\mathfrak{H}) \) be strongly clustering asymptotically abelian \((t,\beta)-KMS \) state at inverse temperature \( \beta \neq 0 \). If \( A \) is simple, then any localized automorphism \( \rho \) of \( A \) extends to a pointwise–weak measurable field \( \{\rho_\xi\}_{\xi \in X} \) of normal automorphism of the weak closure \( \pi_\omega(\mathfrak{A})' \), almost surely.

\(^7\)See also the analogous results Theorem 4.1 of [19], and Theorem 12 of [20].

\(^8\)One can directly start from the observable algebra \( \mathfrak{A} \), and then reconstruct the field algebra \( \mathcal{F} \) by considering the localizable charges of interest of the model, see [12, 14]. This picture applies also to the case described in [28], obtaining models satisfying all the properties assumed in the present section.
Proof. Under our assumptions, we can apply Proposition IV. 1 of [4]. Then $\omega$ is quasi-equivalent to $\omega \circ (\rho \otimes \text{id})$, where $\text{id} \equiv \text{id}_{L^\infty(X, \nu)}$. Then $\pi_\omega$ is unitarily equivalent to $\pi_{\omega \circ (\rho \otimes \text{id})} \equiv \pi_\omega \circ (\rho \otimes \text{id})$. This means that there exists a spatial isomorphism of

$$
\pi_{\omega \circ (\rho \otimes \text{id})}(\mathcal{A})'' = \int_X \pi_{\omega_\xi}(\rho(\mathcal{A}))'' \nu(d\xi)
$$

onto

$$
\pi_\omega(\mathcal{A})'' = \int_X \pi_{\omega_\xi}(\mathcal{A})'' \nu(d\xi),
$$

where, under the above identification, we have a common direct integral decomposition $\{\mathcal{H}_\xi\}_{\xi \in X}$ of the same Hilbert space $\mathcal{H}_\omega$ on which $\pi_\omega$ and $\pi_{\omega \circ (\rho \otimes \text{id})}$ act simultaneously. By Theorem IV.8.23 of [31], there exists a measurable field $\{U_\xi\}_{\xi \in X}$ of unitary operators such that

$$
\pi_{\omega_\xi}(\rho(\mathcal{A}))'' = U_\xi \pi_{\omega_\xi}(\mathcal{A})'' U_\xi^*,
$$

with

$$
\pi_{\omega_\xi} \circ \rho = U_\xi \pi_{\omega_\xi}(\cdot) U_\xi^*
$$

almost everywhere. As $\mathcal{A}$ is supposed to be simple, $\{\pi_{\omega_\xi}\}_{\xi \in X}$ is a measurable field of $\ast$--isomorphism of $\mathcal{A}$ onto their ranges $\pi_{\omega_\xi}(\mathcal{A})$ almost surely.

After identifying $\mathcal{A}$ with $\pi_{\omega_\xi}(\mathcal{A})$, the measurable field of normal automorphisms are given, for $R \in \pi_{\omega_\xi}(\mathcal{A})''$, by

$$
\rho_\xi(R) := U_\xi R U_\xi^*.
$$

In order to avoid technical problems, we suppose also that the quasi-local algebra of observables $\mathcal{A}$ (or equally well the field algebra $\mathcal{F}$ is the $C^*$--inductive limit of local algebras isomorphic to a common full matrix algebra (i.e. the spin algebra).

Let $\omega \in \mathcal{S}_0(\mathfrak{A})$ be a $(t, \beta)$--KMS state such that the centre $Z_{\pi_\omega} := \pi_{\omega}(\mathfrak{A})' \wedge \pi_{\omega}(\mathfrak{A})''$ is isomorphic to $L^\infty(X, \nu)$. In [7] it is explained that this situation seems to be the right one in order to describe the “pure thermodynamical phase” in the case of disordered models. In this situation, we have that $\omega$ is weakly clustering (w.r.t. the spatial translation). Namely, as the $(\tau_\xi, \beta)$--KMS state $\omega_\xi \in \mathcal{S}(\mathcal{A})$ is a factor state almost

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9 In Proposition IV. 1 of [4], the extremality of $\omega$ cannot be dropped, as no weak clustering assumption is made there, see Remark II.2 of [4].

10 See [7], Section 2 for the last equality relative to the direct integral decomposition.
surely, it satisfies (3.4) almost surely, see [2], Lemma 10.2. Then the assertion follows by Proposition 3.3.

Take a weakly clustering extension $\varphi$ of $\omega$ to all of $\mathcal{F}$ which exists by Proposition 4.2. Furthermore, in order to avoid cases when the chemical potential is zero, we suppose that $\varphi$ is gauge–invariant (i.e. $G_{\varphi} = \mathbb{T}$). In this situation, we have that $\text{sp}(U_{\varphi}) = \mathbb{Z} \equiv \hat{G}$. Namely, the asymmetry subgroup $N_{\varphi}$ of $\varphi$ is trivial.

Let $\rho$ be a localized automorphism of $\mathcal{A}$ carrying the charge $n$ (i.e. $\rho \in [\sigma^n]$), and consider the unitary $U$ implementing $\rho$ on $\mathcal{A}$. Consider the state $\varphi_U := \varphi \circ \text{ad}_{U \otimes I}$. We have for the Connes–Radon–Nikodym cocycle ([10, 30]),

\begin{equation}
(D\varphi_U : D\varphi) = \int_X \pi_{\varphi_\xi}(U^*)\sigma^\varphi_\xi(\pi_{\varphi_\xi}(U))\nu(d\xi) = e^{in\beta\mu} \int_X \pi_{\varphi_\xi}(U^*\tau^\xi_{-\beta t}(U))\nu(d\xi),
\end{equation}

for some $\mu \in \mathbb{R}$.

Here, we have used $\gamma_\theta(U) = e^{in\theta}U$, and $\sigma^\varphi_\xi \circ \pi_{\varphi} = \pi_{\varphi} \circ \tau_{-\beta t} \circ \mathcal{G}_{\beta\mu t}$, by Theorem 4.7 taking into account (2.1).

Now, we take advantage from the fact that $\omega_\xi \circ (\rho \otimes \text{id})$ extends to a normal state on all of $\mathcal{F}$. Denote with an abuse of notation, $\{\omega_\xi \circ \rho_\xi\}_{\xi \in X}$ the (equivariant) measurable field of states providing the direct integral decomposition of such an extension. Here, the $\rho_\xi$ are the normal automorphisms of $\pi_{\varphi_\xi}$ appearing in Proposition 5.1. We have, by (5.1) and the fact that $U^*\tau^\xi_{-\beta t}(U)$ is gauge–invariant,

\begin{equation}
(D(\omega_\xi \circ \rho_\xi) : D\omega_\xi) = e^{in\beta\mu} \pi_{\omega_\xi}(U^*\tau^\xi_{-\beta t}(U))
\end{equation}

almost everywhere.

Formula (5.2) explains the occurrence of the chemical potential $\mu \in \mathbb{R}$ as an object intrinsically associated to the observable algebra. Furthermore, according with this description, it does not depend on the disorder for states $\omega$ on $\mathfrak{A}$ such that $\mathfrak{F}_{\eta} \sim L^\infty(X, \nu)$. This is in accordance with standard fact that the physically relevant quantities should not depend on the disorder.

\footnote{See [29], Proposition 3, when $\mathcal{A}$ includes Fermion operators, even if the last situation is not the standard one (e.g. [11, 14]).}

\footnote{Let $V_\sigma \in \mathcal{F}$ be the unitary implementing the automorphism $\sigma$ on $\mathcal{A}$. This means that $\gamma_\theta(V_\sigma) = e^{i\theta}V_\sigma$. The vector $\Psi_n := \pi_{\varphi}(V_\sigma^n \otimes I)\Omega_{\varphi}$ is an eigenvector of $U_{\varphi}(\theta)$ corresponding to the eigenvalue $e^{in\theta}$.}

\footnote{By the previous results, one conclude that also $N_{\varphi_\xi}$ is trivial almost surely.}
As it is explained in [4], Section IV, there is a freeness in order to define the chemical potential (see Formula (IV.6)). By this freeness, the chemical potential might be defined up to a phase factor in the centre of the GNS representation of the state. Such a phase is connected with the zero–point of the chemical potential, as that centre is trivial in the situation treated in the above mentioned paper. In our situation, such a zero–point would lie in $L^\infty(X,\nu)$. However, if one make such a choice in a measurable and invariant way, one would conclude by ergodicity, that also the freeness in the choice of the zero–point of the chemical potential can be avoided.

To conclude, the following remarks are in order. First, the situation described in the present section extends straightforwardly to the case when the gauge group is the $n$–dimensional torus. Second, one could extend the matter to more complicated situations arising from continuous disordered systems,\[^{14}\] as well as possible disordered systems arising from quantum field theory. In these cases, one could take advantage from the local normality of the objects of interest and/or the split property naturally assumed in quantum field theory.\[^{15}\] We choose not to pursue such an analysis as, at the knowledge of the author, no natural disordered model arising from quantum field theory seems to be present in literature.

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\[^{14}\text{See [17, 18] for some non trivial examples of continuous disordered models.}\]

\[^{15}\text{See [24], Section 3 for an analysis of the chemical potential for some (non disordered) models of low dimensional quantum field theory, without passing from the field algebra.}\]
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