Categories of Contexts

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Abstract
Morphisms between (formal) contexts are certain pairs of maps, one between objects and one between attributes of the contexts in question. We study several classes of such morphisms and the connections between them. Among other things, we show that the category $\text{CLc}$ of complete lattices with complete homomorphisms is (up to a natural isomorphism) a full reflective subcategory of the category of contexts with so-called conceptual morphisms; the reflector associates with each context its concept lattice. On the other hand, we obtain a dual adjunction between $\text{CLc}$ and the category of contexts with so-called concept continuous morphisms. Suitable restrictions of the adjoint functors yield a categorical equivalence and a duality between purified contexts and doubly based lattices, and in particular, between reduced contexts and irreducibly bigenerated complete lattices. A central role is played by continuous maps between closure spaces and by adjoint maps between complete lattices.

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0 Introduction

Fundamental in Formal Concept Analysis is the interplay between so-called (formal) contexts, constituted by certain incidence relations, and the associated concept lattices introduced by Wille [12, 13], hence lattice-theoretical tools in the spirit of Birkhoff [2]. It is therefore of primary interest to elucidate the passage between contexts and concept lattices – and specifically, to investigate the relevant functors between the involved categories. Natural candidates on the lattice side are maps that preserve arbitrary joins, or meets, or both. Often, morphisms between contexts will be pairs of maps, because contexts always have two ground sets, one of “objects” and one of “attributes”. Since either of these sets carries a natural closure system (that of “extents” and that of “intents”), it is rather obvious that continuity will play a crucial role in that setting (see [6] for a theory of lattice representations for closure spaces). Continuity is also the defining condition for “scalings” in measurement theory (see [12]).

In the present note, we are mainly interested in complete homomorphisms between concept lattices. Since they preserve both joins and meets, we certainly have to take pairs of continuous maps between the underlying contexts – but that is not enough, as observed in [7]: one needs a certain link between the two involved mappings. This leads us to two essentially different but equally important notions, that of conceptual morphisms and that of concept continuous morphisms: given contexts \((G, M, I)\) and \((H, N, J)\), a pair of mappings \(\alpha : G \to H\) and \(\beta : M \to N\) is conceptual iff it preserves incidence (i.e. \(gI m\) implies \(\alpha(g)J\beta(m)\)) and an object \(h\) has the attribute \(n\) whenever each object \(\beta\)-forced by \(h\) has each attribute \(\alpha\)-forced by \(n\) (where \(m\) is \(\alpha\)-forced by \(n\) if \(m\) holds for each object whose \(\alpha\)-image has the attribute \(n\), and \(g\) is \(\beta\)-forced by \(h\) if \(g\) has each attribute whose \(\beta\)-image holds for \(h\)). On the other hand, \((\alpha, \beta)\) is concept continuous iff it reflects incidence (i.e. \(\alpha(j)J\beta(m)\) implies \(gI m\)), an attribute \(n\) holds for \(\alpha(g)\) whenever each \(\beta\)-generalization of \(n\) holds for \(g\), and \(\beta(m)\) holds for an object \(h\) whenever \(m\) holds for each \(\alpha\)-specialization of \(h\) (where \(g\) is an \(\alpha\)-specialization of \(h\) if \(\alpha(g)\) shares all attributes of \(h\), and \(m\) is a \(\beta\)-generalization of \(n\) if \(\beta(m)\) holds for all objects with attribute \(n\)). The category \(\text{CLc}\) of complete lattices with complete homomorphisms turns out to be a full reflective subcategory of the category of contexts with conceptual morphisms – just by passing from contexts to their concept lattices. But there is also a dual adjunction between the category \(\text{CLc}\) and the category of contexts with concept continuous morphisms. Various results on subcontexts and their concept lattices are immediate consequences. Modifying the adjoint functors, we shall arrive at a categorical equivalence and a duality between purified contexts and so-called doubly based lattices – in particular, between reduced contexts and irreducibly bigenerated complete lattices.
1 Categories of Complete Lattices

In this preliminary section, we summarize some definitions and known facts about ordered sets, complete lattices and morphisms between them.

Given an arbitrary map \( \varphi : X \rightarrow Y \), we shall denote by \( \varphi[A] \) the image of \( A \subseteq X \) under \( \varphi \), and by \( \varphi^{-1}[B] \) the preimage of \( B \subseteq Y \) under \( \varphi \).

Recall that a map \( \varphi : P \rightarrow Q \) between (partially) ordered sets is

- **order preserving** or **isotone** if \( x \leq y \Rightarrow \varphi(x) \leq \varphi(y) \),
- **order reflecting** or **antitone** if \( \varphi(x) \leq \varphi(y) \Rightarrow x \leq y \),
- **an order embedding** if \( x \leq y \Leftrightarrow \varphi(x) \leq \varphi(y) \).

Furthermore, \( \varphi \) is **join-dense** if each element in the codomain \( Q \) is a join (supremum, least upper bound) of elements in the range of \( \varphi \), or equivalently, if for \( q \not\leq r \) in \( Q \) there is a \( p \in P \) with \( \varphi(p) \leq q \) but \( \varphi(p) \not\leq r \). Caution: the composition of two join-dense (isotone) maps need not be join-dense! Meets and meet-dense maps are defined dually.

Of particular importance for our considerations are adjoint maps and functors (see [3], [4] or [9] for the order-theoretical and [1] for the categorical part). Here we only recall the basic notions and facts. A pair of maps \( \varphi : P \rightarrow Q \) and \( \psi : Q \rightarrow P \) between ordered sets is **adjoint** if

\[ \varphi(p) \leq q \iff p \leq \psi(q) \]

for all \( p \in P \) and \( q \in Q \). In that situation, \( \varphi \) is the **left** or **lower adjoint** of \( \psi \), which in turn is the **right** or **upper adjoint** of \( \varphi \). By antisymmetry of the order relations, lower and upper adjoints determine each other uniquely; we write \( \varphi^* \) for the upper adjoint of \( \varphi \), and \( \psi_* \) for the lower adjoint of \( \psi \). It is helpful to know that a lower adjoint is injective iff its upper adjoint is surjective, and **vice versa**. An injective lower adjoint \( \varphi \) (upper adjoint \( \psi \)) is always an order embedding and satisfies \( \varphi^* \circ \varphi = \text{id} \) (\( \psi_* \circ \psi = \text{id} \)). Note also that any join-dense join-preserving map and dually any meet-dense meet-preserving map is already surjective. Moreover, a map between posets is an isomorphism iff it has both an upper and a lower adjoint and these two adjoints coincide.

Given subsets \( A, B \) of a poset \( P \), we denote by \( A^\uparrow \) the collection of all upper bounds of \( A \) and by \( B^\downarrow \) that of all lower bounds of \( B \). In particular,

\[ \downarrow x = \{ x \}^\downarrow = \{ p \in P : p \leq x \} \quad \text{and} \quad \uparrow x = \{ x \}^\uparrow = \{ p \in P : p \geq x \} \]

are the **principal ideal** and the **principal filter** generated by \( x \in P \), respectively. More generally, for \( A, B \subseteq P \),

\[ A^\downarrow = \bigcap \{ \downarrow x : A \subseteq \downarrow x \} \]

is the **lower cut** generated by \( A \), and

\[ B^\uparrow = \bigcap \{ \uparrow x : B \subseteq \uparrow x \} \]
is the upper cut generated by $B$. The cuts in the sense of MacNeille [10] (generalizing Dedekind’s cuts of rational numbers) are the pairs $(A, B)$ with $A = B^\downarrow$ and $B = A^\uparrow$. Ordered by $(A, B) \leq (C, D) \iff A \subseteq B \iff D \subseteq C$, they form a complete lattice, the Dedekind-MacNeille completion, which is isomorphic to the closure system of lower cuts and dually isomorphic to the closure system of upper cuts (cf. [2], [3], [7]).

A map between posets is a lower adjoint iff it is residuated (or dually residual), i.e. preimages of principal ideals are principal ideals, and it is an upper adjoint iff it is residual (or dually residuated), i.e. preimages of principal filters are principal filters. Similarly, a map between posets is called lower (upper) cut continuous if preimages of lower (upper) cuts are again lower (upper) cuts. From [7], we cite:

**Theorem 1.1** Generally, one has the following implications:

- residuated $\Rightarrow$ lower cut continuous $\Rightarrow$ join preserving
- residual $\Rightarrow$ upper cut continuous $\Rightarrow$ meet preserving

and for maps between complete lattices, the converse implications hold, too. The completion by cuts yields a reflector from the category of posets with lower (upper) cut continuous maps to the full subcategory of complete lattices with join (meet) preserving maps.

We denote by $\text{CLc}$ the category of complete lattices and complete homomorphisms, i.e. maps preserving arbitrary joins and meets. On the other hand, we have the category $\text{CLc}_e$ of complete lattices and doubly residuated maps, i.e. maps $\varphi$ possessing an upper adjoint $\varphi^*$ which again has an upper adjoint $\varphi^{**}$, and the category $\text{CLc}^*$ of complete lattices and doubly residual maps $\psi$, having a lower adjoint $\psi_*$ which again has a lower adjoint $\psi_{**}$. Passing to upper adjoints, one obtains a dual isomorphism between the categories $\text{CLc}_e$ and $\text{CLc}$, but also one between $\text{CLc}$ and $\text{CLc}^*$. Composing both duality functors, one arrives at an isomorphism between the categories $\text{CLc}_e$ and $\text{CLc}^*$, sending any doubly residuated map $\varphi$ to $\varphi^{**}$, and in the opposite direction, any doubly residual map $\psi$ to $\psi_{**}$. Summarizing the previous remarks, we note that under the above duality functors, the following pairs of categories of complete lattices are duals of each other:

| category | morphisms | dual | morphisms |
|----------|-----------|------|-----------|
| $\text{CLc}$ | complete homomorphisms  | $\text{CLc}_e$ | doubly residuated maps |
|          |           | $\text{CLc}^c$ | doubly residual maps |
| $\text{CLc}_d$ | surjective (= dense) complete homomorphisms | $\text{CLc}_e$ | inj. doubly residuated maps |
|          |           | $\text{CLc}^e$ | inj. doubly residual maps |
| $\text{CLc}_e$ | injective (= embedding) complete homomorphisms | $\text{CLc}_d$ | surj. doubly residuated maps |
|          |           | $\text{CLc}^d$ | surj. doubly residual maps |
| $\text{CLis}$ | isomorphisms | $\text{CLis}$ | isomorphisms |
2 Closure Spaces and Continuous Maps

Since contexts and their concept lattices are always intimately related with certain closure structures, a few preliminary remarks about closure spaces and their morphisms are in order before starting the morphism theory for contexts and concept lattices. For more background concerning the interaction between closure spaces and complete lattices, we refer to [6].

A closure space \( X \) is a set together with a closure system \( \mathcal{A}(X) \), that is, a collection of subsets that is closed under arbitrary intersections. It is common use to denote the underlying set by the same letter as the space; thus, \( X = \bigcap \emptyset \in \mathcal{A}(X) \). For each subset \( A \) of \( X \), there is a last member of \( \mathcal{A}(X) \) containing \( A \), denoted by \( \overline{A} \) and called the closure of \( A \). Clearly, \( \mathcal{A}(X) \) is a complete lattice in which arbitrary meets coincide with intersections (but joins not always with unions). There is a canonical map from \( X \) to \( \mathcal{A}(X) \), \( \eta_X : X \rightarrow \mathcal{A}(X), \ x \mapsto \overline{\{x\}} \).

Let us recall several equivalent definitions of continuity for maps between closure spaces (see e.g. [6]):

**Theorem 2.1** For a map \( \alpha \) between closure spaces \( X \) and \( Y \), the following conditions are equivalent:

(a) Preimages of closed sets under \( \alpha \) are closed.
(b) \( \alpha[A] \subseteq \overline{\alpha[A]} \) for all \( A \subseteq X \).
(c) There are adjoint maps \( \alpha^- : \mathcal{A}(X) \rightarrow \mathcal{A}(Y) \) and \( \alpha^\circ \) : \( \mathcal{A}(Y) \rightarrow \mathcal{A}(X) \) with \( \alpha^- \circ \eta_X = \eta_Y \circ \alpha \).
(d) There is a join-preserving \( \alpha^- : \mathcal{A}(X) \rightarrow \mathcal{A}(Y) \) with \( \alpha^-(\{x\}) = \overline{\{\alpha(x)\}} \).

Moreover, these maps \( \alpha^- \) and \( \alpha^\circ \) are uniquely determined:
\[ \alpha^-(A) = \overline{\alpha[A]}, \ \alpha^\circ(C) = \alpha^-[C]. \]

In order to determine under what conditions the maps \( \alpha^- \) and \( \alpha^\circ \) are injective or surjective, respectively, we say \( \alpha \) is

- (strictly) dense if for each \( B \subseteq Y \) there is some \( A \subseteq X \) with \( \overline{B} = \overline{\alpha[A]} \),
- full if \( \alpha^-[\alpha[A]] \subseteq \overline{A} \) for all \( A \subseteq X \), i.e. \( \alpha(x) \in \overline{\alpha[A]} \) implies \( x \in \overline{A} \).

We shall omit the word “strictly”. The following facts are easily checked:

**Lemma 2.1** A continuous map \( \alpha \) between closure spaces \( X \) and \( Y \) is

- dense iff \( \alpha^- \) is surjective iff \( \alpha^\circ \) is injective,
- full iff \( \alpha^- \) is injective iff \( \alpha^\circ \) is surjective.

Furthermore, \( \alpha \) is full and continuous iff it is initial, i.e. \( \mathcal{A}(X) = \alpha^-[\mathcal{A}(Y)]. \)
3 Morphisms Between Contexts

A (formal) context is a triple \( \mathbb{K} = (G, M, I) \) where \( I \) is some “incidence” relation between elements of \( G \) (“objects”) and elements of \( M \) (“marks” or “attributes”), i.e. \( I \subseteq G \times M \). For \( A \subseteq G \) and \( B \subseteq M \), we put
\[
A^\uparrow = A^I = \{ m \in M : gIm \text{ for all } g \in A \}, \\
B^\downarrow = B^I = \{ g \in G : gIm \text{ for all } m \in B \}.
\]

Instead of \( \{ g \}^\uparrow \text{ and } \{ m \}^\downarrow \), we shall write \( g^\uparrow \text{ and } m^\downarrow \), respectively.

The complementary relation \( G \times M \setminus I \) will be denoted by \( I \). A (formal) concept of the context \( \mathbb{K} \) is a pair \((A, B)\) with \( A \subseteq G \), \( B \subseteq M \), \( A = B^\downarrow \) (the “extent”) and \( A^\uparrow = B \) (the “intent”). Ordered by
\[(A, B) \leq (C, D) \iff A \subseteq C \iff D \subseteq B,\]
the concepts form a complete lattice, the so-called concept lattice \( \mathbb{B}K \). By passing to the first or second components, this lattice is isomorphic to the closure system \( \mathcal{E}K \) of all extents and dually isomorphic to the closure system \( \mathcal{I}K \) of all intents. Thus, concept lattices are the natural generalization of Dedekind-MacNeille completions, replacing order relations by arbitrary relations. Notice that the corresponding closure operators are \( ^\uparrow \) and \( ^\downarrow \), respectively: indeed, \( A^\uparrow \) is the least extent containing \( A \subseteq G \), and \( B^\downarrow \) is the least intent containing \( B \subseteq M \).

As mentioned in the introduction, we are mainly interested in categorical aspects of Formal Concept Analysis, that is, in the investigation of suitable morphisms between contexts and the associated concept lattices. Naturally, context morphisms have to be certain pairs of maps, one between the objects and the other between the attributes. The choice of morphisms is not evident and may depend heavily on the intended investigations and results. From the Galois-theoretical point of view, it would be natural to consider pairs of maps with opposite directions. However, we shall not pursue that trace in the present note but focus on situations where both maps run into the same direction - an approach that leads to quite satisfactory results as well.

Given two contexts \( \mathbb{K} = (G, M, I) \) and \( \mathbb{L} = (H, N, J) \), a pair \((\alpha, \beta)\) of maps \( \alpha : G \to H \) and \( \beta : M \to N \) will be referred to as a mapping pair or (weak) concept morphism. Let us list a few natural conditions on such mappings. In accordance with the corresponding general closure-theoretical definitions, we say \( \alpha \) is
- (extent) continuous if preimages of extents under \( \alpha \) are extents,
- (extent) dense if for all \( C \subseteq H \), there is an \( A \subseteq G \) with \( C^\uparrow = \alpha[A]^\uparrow \),
- (extent) full if for all \( A \subseteq G \), \( \alpha(g) \in \alpha[A]^\uparrow \) implies \( g \in A^\uparrow \).

Dually, \( \beta \) is said to be
- (intent) continuous if preimages of intents under \( \beta \) are intents,
– (intent) dense if for all $D \subseteq N$, there is a $B \subseteq M$ with $D^{\uparrow} = \beta[B]^{\uparrow}$,
– (intent) full if for all $B \subseteq M$, $\beta(m) \in \beta[B]^{\uparrow}$ implies $m \in B^{\uparrow}$.

Extent continuous maps are often interpreted as *scalings* in the theory of measurement, in particular if the objects of the codomain are numbers or numerical functions (see e.g. [12]).

Although every closure space $X$ may be regarded as an extent space, namely of the context $(X, \mathcal{A}(X), \varepsilon)$, there is a crucial difference between arbitrary closure spaces and extent or intent spaces: in the latter situation the various types of morphisms admit descriptions in first order terms, involving quantification over objects and attributes only, but not over subsets (like extents or intents). This reduction of complexity is one of the prominent advantages of Formal Concept Analysis (where contexts are regarded as “logarithms” of their concept lattices). Note that statements like

$$h^{\uparrow} \subseteq \alpha(g)^{\uparrow} \text{ or } \alpha^{-}[n^{\downarrow}] \subseteq m^{\downarrow}$$

are expressible in first order terms (the former meaning that $hJn$ implies $\alpha(g)Jn$, and the latter that $\alpha(g)Jn$ implies $gIm$).

In the subsequent lemmas, $(\alpha, \beta)$ always denotes a mapping pair between contexts $K = (G, M, I)$ and $L = (H, N, J)$.

**Lemma 3.1** The following are equivalent:

(a) $\alpha$ is extent continuous.
(b) $\alpha^{-}[n^{\downarrow}]$ is an extent for each $n \in N$.
(c) $\alpha(g)Jn$ implies $gIm$ for some $m \in M$ with $\alpha^{-}[n^{\downarrow}] \subseteq m^{\downarrow}$.
(d) $\alpha[A^{\uparrow}]^{\uparrow} = \alpha[A]^{\uparrow}$ for all $A \subseteq G$.
(e) $\alpha[A^{\uparrow}] \subseteq \alpha[A]^{\uparrow}$ for all $A \subseteq G$.

Dual characterizations hold for intent continuous maps.

**Proof.** (a) $\Rightarrow$ (b) $\Rightarrow$ (c). $\alpha(g)Jn$ means $g \notin \alpha^{-}[n^{\downarrow}] = \alpha^{-}[n^{\downarrow}]^{\uparrow}$ (as $\alpha^{-}[n^{\downarrow}]$ is an extent). Thus, there is an $m \in M$ with $gIm$ but $\alpha^{-}[n^{\downarrow}] \subseteq m^{\downarrow}$.

(c) $\Rightarrow$ (d). If $n \notin \alpha[A^{\uparrow}]^{\uparrow}$ then $\alpha(g)Jn$ for some $g \in A^{\uparrow}$, hence $gIm$ but $\alpha^{-}[n^{\downarrow}] \subseteq m^{\downarrow}$ for some $m \in M$. It follows that $A \nsubseteq m^{\downarrow}$, a fortiori $A \nsubseteq \alpha^{-}[n^{\downarrow}]$, which means $n \notin \alpha[A]^{\uparrow}$. By contraposition, we obtain $\alpha[A]^{\uparrow} \subseteq \alpha[A^{\uparrow}]^{\uparrow}$, and the reverse inclusion is a consequence of $A \subseteq A^{\uparrow}$.

The implication (d) $\Rightarrow$ (e) is clear. For (e) $\Rightarrow$ (a), see Theorem 2.1.

Note that condition (e) may be reformulated as an “implication between implications” (cf. [S]); indeed, writing $A \rightarrow B$ for $A^{\uparrow} \subseteq B^{\uparrow}$, i.e. $B \subseteq A^{\uparrow}$ (meaning that the objects of $B$ share all common properties of objects in $A$), we see that (e) is equivalent to

$$(d') A \rightarrow B \text{ implies } \alpha[A] \rightarrow \alpha[B].$$
Lemma 3.2 The map $\alpha$ is extent dense

iff for each $h \in H$ there is a set $A \subseteq G$ with $h^\uparrow = \alpha[A]^\uparrow$

iff $h \not\in n$ implies $\alpha(g) \not\in n$ for some $g \in G$ with $h^\uparrow \subseteq \alpha(g)^\uparrow$.

Dually, the map $\beta$ is intent dense

iff for each $n \in N$ there is a set $B \subseteq M$ with $n^\downarrow = \beta[B]^\downarrow$

iff $h \not\in n$ implies $h \not\in \beta(m)$ for some $m \in M$ with $n^\downarrow \subseteq \beta(m)^\downarrow$.

Consequently, both $\alpha$ and $\beta$ are dense

iff for $h \not\in n$ there exist $g, m$ with $\alpha(g) \not\in \beta(m)$, $h^\uparrow \subseteq \alpha(g)^\uparrow$, $n^\downarrow \subseteq \beta(m)^\downarrow$.

Proof. If $\alpha$ is dense then for each $h \in H$ there is some $A \subseteq G$ such that

$h^\uparrow = \alpha[A]^\uparrow$, hence $h^\uparrow = \alpha[A]^\uparrow$. Assuming the latter equation and $h \not\in n$, we find a $g \in A$ with $\alpha(g) \not\in n$, whereas $h^\uparrow \subseteq \alpha(g)^\uparrow$.

On the other hand, assume that $h \not\in n$ implies $\alpha(g) \not\in n$ for some $g$ with $h^\uparrow \subseteq \alpha(g)^\uparrow$. In order to prove density of $\alpha$, consider an extent $C = C^\uparrow$ and put $A = \alpha^{-1}[C]$. We claim that $\alpha[A]^\uparrow = C$. If $h \not\in \alpha[A]^\uparrow$, choose $n \in N$ with $h \not\in n$ but $\alpha[A] \subseteq n^\downarrow$, and then $g \in G$ with $\alpha(g) \not\in n$ but $h^\uparrow \subseteq \alpha(g)^\uparrow$. Then we have $A \subseteq \alpha^{-1}[n^\downarrow]$ but $g \not\in A$ (otherwise $\alpha(g) \in \alpha[A] \subseteq n^\downarrow$, i.e. $\alpha(g) \not\in n$), hence $\alpha(g) \not\in C$. The assumption $h \in C$ leads to $C^\uparrow \subseteq h^\uparrow \subseteq \alpha(g)^\uparrow$ and $\alpha(g) \in C^\uparrow = C$, a contradiction. By contraposition, $C \subseteq \alpha[A]^\uparrow \subseteq C$. \qed

Note also that the map $\alpha$ is extent dense iff the set $\{\alpha(g)^\uparrow : g \in G\}$ is join-dense in the extent lattice $\mathcal{E}L$, and dually, $\beta$ is intent dense iff the set $\{\beta(m)^\downarrow : m \in M\}$ is meet-dense in the intent lattice $\mathcal{I}L$.

Lemma 3.3 The map $\alpha$ is extent full

iff $g \not\in n$ implies $\alpha(g) \not\in n$ for some $n \in N$ with $\alpha[m^\downarrow] \subseteq n^\downarrow$

iff $\alpha[A] \Rightarrow \alpha[B]$ entails $A \Rightarrow B$

iff each extent of $\mathcal{K}$ is the preimage of an extent of $\mathcal{L}$ under $\alpha$.

Dual characterizations hold for intent fullness.

Proof. If $\alpha$ is full then $g \not\in n$ implies $g \not\in m^\downarrow = m^\uparrow \not\subseteq \alpha[m^\downarrow]$, i.e. $\alpha(g) \not\in n$ for some $n \in N$ with $\alpha[m^\downarrow] \subseteq n^\downarrow$. Conversely, if the latter holds then for $g \in G$ and $A \subseteq G$ with $g \not\in A^\uparrow$, we find an $m \in A^\uparrow$ with $g \not\in m^\downarrow$ and then an $n \in \alpha[m^\downarrow]^\uparrow$ such that $\alpha(g) \not\in n$, whence $\alpha(g) \not\in n^\downarrow$ and, a fortiori, $\alpha(g) \not\in \alpha[A]^\uparrow$ (because $A \subseteq m^\downarrow$ and therefore $\alpha[A]^\uparrow \subseteq \alpha[m^\downarrow]^\uparrow \subseteq n^\downarrow$).

The last two characterizations of fullness are straightforward. \qed

Corollary 3.1 $\alpha$ is initial, i.e. extent continuous and full

iff $A \Rightarrow B$ is equivalent to $\alpha[A] \Rightarrow \alpha[B]$

iff the extents of $\mathcal{K}$ are precisely the preimages of extents of $\mathcal{L}$.

We come now to the crucial definitions, relating both partners of a mapping pair to each other. The mapping pair $(\alpha, \beta)$ is called
Lemma 3.4 The mapping pair \((\alpha, \beta)\) preserves incidence

\[ \text{iff } \alpha[A]^h \subseteq \beta[A]^i \text{ for all } A \subseteq G \]

\[ \text{iff } \beta[B]^i \subseteq \alpha[B]^h \text{ for all } B \subseteq M. \]

Proof. If \((\alpha, \beta)\) preserves incidence then \(\beta[A]^i \subseteq \alpha[A]^h\), hence \(\alpha[A]^h \subseteq \beta[A]^i\).

Conversely, that inclusion implies \(\beta[g]^i \subseteq \alpha(g)^h\) for all \(g \in G\), which means that \((\alpha, \beta)\) preserves incidence. The other equivalence is shown dually. \(\blacksquare\)

Lemma 3.5 The mapping pair \((\alpha, \beta)\) reflects incidence

\[ \text{iff } \alpha^-[C][h] \subseteq \beta^-[C][i] \text{ for all } C \subseteq H \]

\[ \text{iff } \beta^-[D][i] \subseteq \alpha^-[D][h] \text{ for all } D \subseteq N. \]

Proof. For \(g \in G \setminus \beta^-[C][i]\) we find an \(m \in M\) with \(\beta(m) \in C^i\) and \(gIm\); if \((\alpha, \beta)\) reflects incidence then \(gIm\) implies \(\alpha(g)J\beta(m)\), whence \(\alpha(g) \not\in C^h\).

By contraposition, we get \(\alpha^-[C][h] \subseteq \beta^-[C][i]\). Conversely, if that inclusion is fulfilled then \(\alpha(g)J\beta(m)\) implies \(\alpha^-\alpha(g)^h \subseteq \beta^-[\alpha(g)^i] \subseteq m^i\) (since \(m \in \beta^-[\alpha(g)^i]\)), and it follows that \(g \in \alpha^-[\alpha(g)^h] \subseteq m^i\), i.e. \(gIm\). \(\blacksquare\)

Even more important than the above properties of mapping pairs are certain strong kinds of continuity. We say a mapping pair \((\alpha, \beta)\) is

\begin{itemize}
  \item \textit{separately continuous} if both \(\alpha\) and \(\beta\) are continuous,
  \item \textit{concept preserving} if \((A, B) \in BK\) implies \((\beta[B]^i, \alpha[A]^h) \in BL,\n  \item \textit{conceptual} if it is separately continuous and concept preserving,
  \item \textit{concept continuous} if \((C, D) \in BL\) implies \((\alpha^-[C], \beta^-[D]) \in BK,\n  \item \textit{a dense context embedding} if it is an embedding and \(\alpha, \beta\) are dense,
  \item \textit{a context isomorphism} if it is an embedding and \(\alpha, \beta\) are bijective.
\end{itemize}

The next result has been shown in [7] for the case of order relations:

Lemma 3.6 The mapping pair \((\alpha, \beta)\) is conceptual

\[ \text{iff } \alpha[A]^h = \beta[A]^i \text{ for all } A \subseteq G \text{ and } \beta[B]^i = \alpha[B]^h \text{ for all } B \subseteq M \]

\[ \text{iff } (\alpha, \beta) \text{ preserves incidence, } \beta[\alpha^-[n]^i] \subseteq n^i \text{ and } \alpha[\beta^-[h]^i] \subseteq h^i \]

\[ \text{iff } (\alpha, \beta) \text{ preserves incidence and for } hIm, \text{ there are } gIm \text{ with } \alpha^-[n]^i \subseteq m^i \text{ and } \beta^-[h]^i \subseteq g^i. \]
Proof. Suppose \((\alpha, \beta)\) is conceptual. For \(A \subseteq G\), the pair \((A^\uparrow, A^\uparrow)\) is a concept; hence \((\beta[A^\uparrow], \alpha[A^\uparrow])\) is a concept, too. Thus \(\beta[A^\uparrow] = \alpha[A^\uparrow] = \alpha[A]^\uparrow\) (by continuity of \(\alpha\)). The second equation is obtained analogously.

Now, if we assume the equations \(\alpha[A]^\uparrow = \beta[A]^\uparrow\) and \(\beta[B]^\uparrow = \alpha[B]^\uparrow\) then by Lemma 3.4 \((\alpha, \beta)\) preserves incidence; furthermore, we have 
\[\beta[\alpha^-[n^\downarrow]] = \alpha[\alpha^-[n^\downarrow]] = n^\downarrow = n^\downarrow\] and \(\alpha[\beta^-[h^\uparrow]] = \beta[\beta^-[h^\uparrow]] = h^\uparrow\). 

On the other hand, if \(\beta[\alpha^-[n^\downarrow]]^\downarrow = \alpha[\alpha^-[n^\downarrow]]^\downarrow\) is contained in \(n^\downarrow\) then \(h \not\in n^\downarrow\) implies \(h \not\in \beta[\alpha^-[n^\downarrow]]^\downarrow\), so we find an \(m \in \alpha^-[n^\downarrow]^\downarrow\) with \(\beta(m) \not\in h^\uparrow\); it follows that \(\alpha^-[n^\downarrow]^\downarrow \subseteq m^\downarrow\) and \(\beta(m) \not\in \alpha[\beta^-[h^\uparrow]]^\downarrow\); therefore, we find a \(g \in \beta^-[h^\uparrow]^\downarrow\) with \(\alpha(g) \not\in \beta(m)\), hence \(g \not\in \beta[\alpha^-[n^\downarrow]]^\downarrow\) (by incidence preservation).

Finally, let us suppose that \((\alpha, \beta)\) preserves incidence and for \(h \not\in n\), there are \(g\) with \(\beta^-[n^\downarrow] = m^\downarrow\) and \(\beta^-[h^\uparrow] = g^\downarrow\). Again by Lemma 3.4 we have \(\beta[N] = \beta[\alpha^-[n^\downarrow]]^\downarrow\) and \(\beta[N] = \alpha[\beta^-[h^\uparrow]]^\downarrow\). Assume \(h \not\in \alpha[\beta^-[n^\downarrow]]^\downarrow\); then there is some \(n \in \alpha[\beta^-[n^\downarrow]]^\downarrow\) with \(h \not\in n\). Choose \(g\) as above. Then \(n \not\in g^\downarrow\), and the inclusion \(\beta^-[h^\uparrow] \subseteq g^\downarrow\) yields \(h \not\in \beta(m)\). But \(n \in \alpha[\beta^-[n^\downarrow]]^\downarrow\) means \(\alpha[\beta^-[n^\downarrow]]^\downarrow\subseteq n^\downarrow\), i.e. \(A \subseteq \alpha^-[n^\downarrow]^\downarrow \subseteq m^\downarrow\), whence \(m \in A^\uparrow\) and so \(\beta(m) \in \beta[A]^\uparrow\). This together with \(h \not\in m\), \(\alpha[\beta^-[n^\downarrow]]^\downarrow\subseteq n^\downarrow\) gives \(h \not\in \beta[A]^\uparrow\), proving the equality \(\alpha[A]^\uparrow = \beta[A]^\uparrow\). □

Lemma 3.7 The mapping pair \((\alpha, \beta)\) is concept continuous

iff \(\alpha^-[C][C^\uparrow] = \beta^-[C][C^\uparrow]\) for all \(C \subseteq H\) and \(\beta^-[D][D^\uparrow] = \alpha^-[D][D^\uparrow]\) for all \(D \subseteq N\)

iff \(\alpha^-[n^\downarrow] = \beta^-[n^\downarrow]\) for all \(n \in N\) and \(\beta^-[h^\uparrow] = \alpha^-[h^\uparrow]\) for all \(h \in H\)

iff \((\alpha, \beta)\) reflects incidence, \(\alpha[\beta^-[n^\downarrow]]^\downarrow \subseteq n^\downarrow\) and \(\beta[\alpha^-[h^\uparrow]]^\downarrow \subseteq h^\uparrow\), i.e.

for \(\alpha(g)\) \(\not\in n\), there is an \(m \in M\) with \(g \not\in m\) and \(m \subseteq \beta(m)^\downarrow\), and

for \(h \not\in \beta(m)\), there is a \(g \in G\) with \(g \not\in m\) and \(h^\uparrow \subseteq \alpha(g)^\downarrow\).

Proof. If \((\alpha, \beta)\) is concept continuous then, observing that for arbitrary \(C \subseteq H\) the pair \((C^\uparrow, C^\uparrow)\) is a concept, we infer that \((\alpha^-[C][C^\uparrow], \beta^-[C][C^\uparrow])\) is a concept, too. Hence \(\alpha^-[C][C^\uparrow] = \beta^-[C][C^\uparrow]\), and dually for \(D \subseteq N\), \(\beta^-[D][D^\uparrow] = \alpha^-[D][D^\uparrow]\). Of course, the latter two equations entail \(\alpha^-[n^\downarrow] = \beta^-[n^\downarrow]\) for all \(n \in N\) and \(\beta^-[h^\uparrow] = \alpha^-[h^\uparrow]\) for all \(h \in H\) (take \(C = n^\downarrow\) and \(D = h^\uparrow\)).

Assume in turn the validity of these equations. If \(h \not\in \beta(m)\) then \(m\) is not a member of \(\beta^-[h^\uparrow] = \alpha^-[h^\uparrow]\). Thus, there exists a \(g \in G\) with \(g \not\in m\) and \(g \not\in \alpha(g)^\downarrow\), i.e. \(h^\uparrow \subseteq \alpha(g)^\downarrow\). Dually, we find for \(\alpha(g)\) \(\not\in n\) an \(m \in M\) with \(g \not\in m\) and \(n^\downarrow \subseteq \beta(m)^\downarrow\). As in Lemma 3.5 we see that \((\alpha, \beta)\) reflects incidence.

Finally, suppose that the latter three conditions are fulfilled. In order to show that for any concept \((C, D)\) of \((H, N, J)\), the “inverse image” \((\alpha^-[C][C^\uparrow], \beta^-[D][D^\uparrow])\) is a concept of \((G, M, I)\), we have to verify the equations \(\alpha^-[D][D^\uparrow] = \beta^-[D][D^\uparrow]\) and \(\beta^-[C][C^\uparrow] = \alpha^-[C][C^\uparrow]\). If \(\alpha(g) \not\in D^\uparrow\) then \(\alpha(g)\) \(\not\in n\) for some \(n \subseteq D\). By hypothesis, there is an \(m \in M\) with \(\alpha(g)\) \(\not\in m\), whence \(m \subseteq \beta^-[D][D^\uparrow] = \beta^-[D][D^\uparrow]\) and therefore \(g \not\in \beta^-[D][D^\uparrow]\). This proves the inclusion \(\beta^-[D][D^\uparrow] \subseteq \alpha^-[D][D^\uparrow]\), and the other inclusion \(\alpha^-[D][D^\uparrow] = \alpha^-[C][C^\uparrow] \subseteq \beta^-[C][C^\uparrow] = \beta^-[D][D^\uparrow]\) follows from Lemma 3.5. □
The last two lemmas confirm our characterizations of conceptual and concept continuous pairs from the introduction: the pair \((\alpha, \beta)\) is conceptual iff it preserves incidence and an object \(h\) has the attribute \(n\) whenever each object \(g\) that is \(\beta\)-forced by \(h\) (i.e. \(\beta^{-}[h^\uparrow] \subseteq g^\uparrow\)) has each attribute \(m\) that is \(\alpha\)-forced by \(n\) (i.e. \(\alpha^{-}[n^\downarrow] \subseteq m^\downarrow\)); while \((\alpha, \beta)\) is concept continuous iff it reflects incidence, \(n\) holds for \(\alpha(g)\) whenever \(g\) has each attribute \(m\) that \(\beta\)-generalizes \(n\) (i.e. \(n^\downarrow \subseteq \beta(m)^\uparrow\)), and dually, \(\beta(m)\) holds for \(h\) whenever \(m\) holds for each \(\alpha\)-specialization \(g\) of \(h\) (i.e. \(h^\uparrow \subseteq \alpha(g)^\downarrow\)).

A rather surprising consequence of the previous results is now:

**Theorem 3.1** A mapping pair \((\alpha, \beta)\) is a dense context embedding iff it is both conceptual and concept continuous.

**Proof.** Suppose \((\alpha, \beta)\) is a dense embedding. In order to show that it is conceptual, use Lemmas 3.2 for \(h \models n\), choose \(g, m\) such that \(\alpha(g) \models \beta(m)\), \(h^\uparrow \subseteq (\alpha(g))^\downarrow\) and \(n^\downarrow \subseteq (\beta(m))^\uparrow\). Then \(g \models m\) since \((\alpha, \beta)\) preserves incidence, and \(\alpha^{-}[n^\downarrow] \subseteq \alpha^{-}[\beta(m)] \subseteq m^\downarrow\) since \((\alpha, \beta)\) reflects incidence; dually, we get \(\beta^{-}[h^\uparrow] \subseteq g^\uparrow\), and Lemma 3.6 applies. Now to concept continuity. If \(h \models \beta(m)\), then by Lemma 3.2, there is a \(g\) with \(h^\uparrow \subseteq (\alpha(g))^\uparrow\) and \(\alpha(g) \models \beta(m)\), whence \(g \models m\) (because \((\alpha, \beta)\) preserves incidence). This and a dual clue show that \((\alpha, \beta)\) is concept continuous, on account of Lemma 3.7.

Conversely, let \((\alpha, \beta)\) be conceptual and concept continuous. By Lemmas 3.6 and 3.7, \((\alpha, \beta)\) preserves and reflects incidence, so it is an embedding. For \(h \models n\), there is an \(m\) with \(h \models \beta(m)\) and \(\alpha^{-}[n^\downarrow] \subseteq m^\downarrow\) (see the proof of Lemma 3.6). By Lemma 3.7, we find a \(g\) with \(g \models m\) and \(h^\uparrow \subseteq (\alpha(g))^\downarrow\). The assumption \(\alpha(g) \models n\) leads to the contradiction \(g \models (\alpha(g))^\downarrow \subseteq m^\downarrow\); hence \(\alpha(g) \models n\). By Lemma 3.2, we conclude that \(\alpha\) (and similarly \(\beta\)) is dense. \(\square\)

Our final lemma shows that fullness or density of one partner in a concept pair implies the corresponding property of the other.

**Lemma 3.8** (1) If \((\alpha, \beta)\) is a context embedding then \(\alpha\) and \(\beta\) are full.

(2) A conceptual pair \((\alpha, \beta)\) is an embedding iff \(\alpha\) is full iff \(\beta\) is full.

(3) A conceptual pair \((\alpha, \beta)\) is dense iff \(\alpha\) is dense iff \(\beta\) is dense.

**Proof.** (1) If \((\alpha, \beta)\) is an embedding then \(g \models m\) implies \(\alpha(g) \models \beta(m)\), and \(\alpha[m^\downarrow] \subseteq \beta(m)^\uparrow\) (since incidence is preserved). Hence, Lemma 3.3 applies with \(n = \beta(m)\), showing that \(\alpha\) is full, and by a dual argument, so is \(\beta\).

(2) By (1) and Lemma 3.6, it suffices to show that a conceptual pair \((\alpha, \beta)\) for which \(\alpha\) (or \(\beta\)) is full must reflect incidence. By Lemma 3.3, \(g \models m\) implies \(\alpha(g) \models n\) for some \(n \in \alpha[m^\downarrow]\), and by Lemma 3.6, this entails that \(n\) belongs to \(\beta(m)^\uparrow\), i.e. \(\beta(m)^\uparrow \subseteq n^\downarrow\), and consequently \(\alpha(g) \models \beta(m)\).

(3) Use Lemma 2.1 and the fact that the maps \(\alpha^{-}\) and \(\beta^{-}\) agree up to the dual isomorphism between extent and intent lattices (see Section 6). \(\square\)
4 Complete Lattices as Contexts

We come now to the central part of our investigations, demonstrating that our choice of morphisms was the “right one” from a categorical point of view. Each of the previously introduced classes of mapping pairs is closed under (componentwise) composition and may, therefore, serve as the morphism class of a category of contexts. Specifically, we have the following categories of contexts and complete lattices (see the next page for comments):

Table 4.1

| contexts | morphisms                      | lattices | morphisms                      |
|----------|--------------------------------|----------|--------------------------------|
| Cmp      | mapping pairs                  | CLmp     | mapping pairs                  |
| Cip      | incidence preserving pairs     | CLip     | order preserving pairs         |
| Cir      | incidence reflecting pairs     | CLir     | order reflecting pairs         |
| Cep      | context embedding pairs        | CLep     | order embedding pairs          |
| Cjm      | separately continuous pairs    | CLjm     | join-meet preserving pairs     |
| Cc       | conceptual pairs               | CLc      | complete homomorphisms         |
| Cc^e     | concept continuous pairs       | CLc^e    | doubly residuated maps         |
| Ccd      | dense conceptual pairs         | CLcd     | doubly residuated surjections  |
| Cc^d*e   | dense concept continuous pairs | CLc^d*e  | doubly residuated surjections  |
| Cce      | conceptual embeddings          | CLce     | complete embeddings            |
| Cc^e*e   | concept continuous embeddings  | CLc^e*e  | doubly residuated embeddings   |
| Cde      | dense embeddings               | CLde     | dense complete embeddings      |
| Cis      | context isomorphisms           | CLis     | lattice isomorphisms           |
In view of the intended correspondences between contexts and complete lattices, we have included here a few less common types of morphisms: an order preserving mapping pair \((\alpha, \beta)\) between posets is characterized by the implication \(x \leq y \Rightarrow \alpha(x) \leq \beta(y)\), while order reflecting pairs are characterized by the reverse implication, and order embedding pairs by the corresponding equivalence. By a join-meet preserving pair, we mean a mapping pair \((\alpha, \beta)\) between complete lattices such that \(\alpha\) preserves arbitrary joins and \(\beta\) preserves arbitrary meets.

Each category listed under \(\text{CLjm}\), the category of complete lattices and join-meet preserving pairs, may be embedded in that category by obvious identifications: complete homomorphisms \(\varphi\) are identified with pairs \((\varphi, \varphi)\), doubly residuated maps \(\psi\) with pairs \((\psi, \psi^{**})\), and doubly residual maps \(\psi\) with pairs \((\psi^{**}, \psi)\). Thereby, the category \(\text{CLc}\) with complete homomorphisms, is identified with that subcategory of \(\text{CLjm}\) whose morphism pairs \((\alpha, \beta)\) have equal components \(\alpha = \beta\), whereas both \(\text{CLc}_*\), the category with doubly residuated morphisms, and \(\text{CLc}^*\), the category with doubly residual morphisms, are identified with that subcategory \(\text{CLc}_*^*\) of \(\text{CLjm}\) whose morphisms \((\alpha, \beta)\) satisfy the equation \(\alpha^* = \beta_*\).

For any complete lattice \(L = (L, \leq)\), the complete context \(CL = (L, L, \leq)\) is the greatest context whose concept lattice is isomorphic to the original lattice \(L\). Thus, we have a functor \(\mathcal{C} : \text{CLjm} \rightarrow \text{Cjm}\), sending \(L\) to \(CL\) and acting identically on mapping pairs. Indeed, a map \(\alpha\) between complete lattices preserves arbitrary joins iff it is residuated, i.e. preimages of principal ideals are principal ideals, and these are just the extents of the associated contexts; and dually, a map preserves arbitrary meets iff it intent continuous. Thus, up to identification of complete lattices \(L\) with their complete contexts \(CL\), the category \(\text{CLjm}\) may be regarded as a full subcategory of \(\text{Cjm}\).

Note that by Lemma 3.2, a map \(\varphi\) between complete lattices is join dense (resp. meet dense) iff it is extent (resp. intent) dense as a map between the associated contexts. Furthermore, recall that a join- or meet-preserving dense map is already surjective; in particular, dense complete embeddings are already isomorphisms. In all, we have:

**Theorem 4.1** Up to the aforementioned canonical identifications, \(\mathcal{C}\) may be regarded as a covariant full embedding functor from each of the categories of complete lattices on the right hand list of Table 4.1 into the corresponding category of contexts on the left hand list. In particular, associating with any complete homomorphism \(\varphi\) the pair \((\varphi, \varphi)\), one obtains full embeddings

- of \(\text{CLc}\) in \(\text{Cc}\), of \(\text{CLcd}\) in \(\text{Cced}\), of \(\text{CLce}\) in \(\text{Ccde}\), and of \(\text{CLis}\) in \(\text{Cde}\).

Similarly, associating with any doubly residuated map \(\varphi\) the pair \((\varphi, \varphi^{**})\) and with any doubly residual map \(\psi\) the pair \((\psi^{**}, \psi)\), one obtains full embeddings

- of \(\text{CLc}_*\) and \(\text{CLc}^*\) in \(\text{Cc}_*\), of \(\text{CLc}_d\) and \(\text{CLc}^*_d\) in \(\text{Cc}_*^d\), etc.
Proof. It remains to verify the following facts about a mapping pair \((\alpha, \beta)\) between complete contexts \(C_K\) and \(C_L\):

1. \((\alpha, \beta)\) is conceptual iff \(\alpha = \beta\) is a complete homomorphism,
2. \((\alpha, \beta)\) is dense and conceptual iff \(\alpha = \beta\) is a surjective complete homomorphism,
3. \((\alpha, \beta)\) is a conceptual embedding iff \(\alpha = \beta\) is an injective complete homomorphism,
4. \((\alpha, \beta)\) is a dense embedding iff \(\alpha = \beta\) is an isomorphism,

\((1^*)\) \((\alpha, \beta)\) is concept continuous iff \(\alpha^{**} = \beta\) (hence \(\alpha = \beta^{**}\) is doubly residuated and \(\beta\) is doubly residual),
\((2^*)\) \((\alpha, \beta)\) is concept continuous and dense iff \(\alpha^{**} = \beta\) is surjective
if \(\alpha = \beta^{**}\) is surjective,
\((3^*)\) \((\alpha, \beta)\) is a concept continuous embedding iff \(\alpha^{**} = \beta\) is injective
if \(\alpha = \beta^{**}\) is injective.

Concerning (1), note first that for a conceptual morphism \((\alpha, \beta)\) between complete contexts, both \(\alpha\) and \(\beta\) are continuous, whence \(\alpha\) preserves joins and \(\beta\) preserves meets. Furthermore, by Lemma 3.6 we have \(\alpha(x)^\uparrow = \beta(x)^\uparrow\), which means \(\alpha(x) = \wedge \beta(x)^\downarrow = \beta(x)\), because \(\beta\) is isotone. Conversely, every complete homomorphism \(\phi\) yields a conceptual morphism \((\phi, \phi)\) (see [7]).

The equivalences (2) and (3) are now immediate consequences of the remarks before the theorem. Concerning (4), recall the important fact that every dense context embedding \((\alpha, \beta)\) is conceptual (and concept continuous), whence in the present situation \(\alpha = \beta\) is a join- and meet-dense complete embedding and consequently an isomorphism.

For \((1^*)\), suppose first that \((\alpha, \beta)\) is concept continuous. Then the equation \(\alpha^-[x^\downarrow] = \beta^-[x^\downarrow]^\downarrow = \beta^-[x^\downarrow]^\uparrow\) (see Lemma 3.7) yields a map

\[\phi : L \to K, \quad \phi(x) = \bigvee \alpha^-[x^\downarrow] = \bigwedge \beta^-[x^\downarrow]\]

which is upper adjoint to \(\alpha\) and lower adjoint to \(\beta\), whence \(\alpha^{**} = \phi^* = \beta\) and \(\alpha = \phi_s = \beta_{**}\). Conversely, for a doubly residual map \(\psi : K \to L\), the lower adjoint \(\varphi = \psi_*\) is a complete homomorphism, and the pair \((\psi_{**}, \psi) = (\varphi, \varphi^*)\) is concept continuous, since each concept of \(C_L\) has the form \((x^\downarrow, x^\uparrow)\), and consequently \((\psi^-_{**}[x^\downarrow], \psi^-[x^\downarrow]) = (\varphi(x)^\downarrow, \varphi(x)^\uparrow)\) is a concept of \(C_K\).

The remaining statements are obtained as before, using the remarks on density and embedding properties of join- or meet-preserving maps.

Note that the above claims remain valid if we replace join- and meet-preserving maps with residuated and residual maps between arbitrary posets. Thus the category of posets with residuated and residual maps is a full subcategory of the category \(C_C\), etc.
5 The Concept Lattice as a Covariant Functor

The point is now that the embedding functor $C$ has a left adjoint, sending on the object level each context to its concept lattice. Thus, for any context $\mathbb{K}$, the concept lattice $B^{\mathbb{K}}$ may be viewed as the free complete lattice over $\mathbb{K}$. On the morphism level, we define for any mapping pair $(\alpha, \beta)$ between contexts $\mathbb{K} = (G, M, I)$ and $\mathbb{L} = (H, N, J)$ a “lifted” mapping pair $(\alpha^\rightarrow, \beta^\rightarrow)$ by

$\alpha^\rightarrow : B^{\mathbb{K}} \rightarrow B^{\mathbb{L}}$, $\alpha^\rightarrow (A, B) = (\alpha[A]^\uparrow, \alpha[A]^\downarrow)$,

$\beta^\rightarrow : B^{\mathbb{K}} \rightarrow B^{\mathbb{L}}$, $\beta^\rightarrow (A, B) = (\beta[B]^\downarrow, \beta[B]^\uparrow)$.

**Proposition 5.1** A mapping pair $(\alpha, \beta)$ between contexts $\mathbb{K}$ and $\mathbb{L}$ is

- separately continuous iff $\alpha^\rightarrow$ is join and $\beta^\rightarrow$ is meet preserving,
- conceptual iff $\alpha^\rightarrow = \beta^\rightarrow$ is a complete homomorphism,
- dense conceptual iff $\alpha^\rightarrow = \beta^\rightarrow$ is a surjective complete homomorphism,
- a conceptual embedding iff $\alpha^\rightarrow = \beta^\rightarrow$ is a complete embedding,
- a dense embedding iff $\alpha^\rightarrow = \beta^\rightarrow$ is an isomorphism.

On the other hand, $(\alpha, \beta)$ is concept continuous

iff $\alpha^\rightarrow$ is doubly residuated with $\alpha^{\rightarrow\rightarrow} = \beta^\rightarrow$

iff $\beta^\rightarrow$ is doubly residual with $\alpha^\rightarrow = \beta^{\rightarrow\rightarrow}$.

**Proof.** The first equivalence follows from Theorem 2.1. The second is obtained from the first one and Lemma 3.6 (cf. [7]). For the statements about density and embedding properties, apply Lemma 2.1.

In the case of a concept continuous pair $(\alpha, \beta)$, we have a well-defined complete homomorphism

$\varphi : B^{\mathbb{L}} \rightarrow B^{\mathbb{K}}$, $(C, D) \mapsto (\alpha^{-}[C], \beta^{-}[D])$

because $\alpha^-$ and $\beta^-$ preserve arbitrary intersections. The equivalences

$\alpha^\rightarrow (A, B) \leq (C, D) \iff \alpha[A] \subseteq C \iff A \subseteq \alpha^{-}[C] \iff (A, B) \leq \varphi(C, D)$

show that $\varphi$ is the upper adjoint of $\alpha^\rightarrow$. A dual argument shows that $\varphi$ is the lower adjoint of $\beta^\rightarrow$, whence $\alpha^{\rightarrow\rightarrow\rightarrow} = \varphi^* = \beta^\rightarrow$ and $\alpha^\rightarrow = \varphi_* = \beta^{-\rightarrow\rightarrow\rightarrow}$.

Conversely, if $\alpha^{\rightarrow\rightarrow\rightarrow} = \beta^\rightarrow$ then $\varphi = \alpha^{\rightarrow\rightarrow\rightarrow}$ is a complete homomorphism from $CL$ into $CK$ which is upper adjoint to $\alpha^\rightarrow$ and lower adjoint to $\beta^\rightarrow$. For any two concepts $(A, B) \in B^{\mathbb{K}}$ and $(C, D) \in B^{\mathbb{L}}$, we have the equivalences

$(A, B) \leq \varphi(C, D) \iff \alpha^\rightarrow (A, B) \leq (C, D) \iff (A, B) \leq (\alpha^{-}[C], \alpha^{-}[C]^\uparrow)$,

$\varphi(C, D) \leq (A, B) \iff (C, D) \leq \beta^\rightarrow (A, B) \iff (\beta^{-}[D]^\downarrow, \beta^{-}[D]) \leq (A, B)$,

which amount to the equations

$\varphi(C, D) = (\alpha^{-}[C], \beta^{-}[D]) = (\alpha^{-}[C], \alpha^{-}[C]^\uparrow) = (\beta^{-}[D]^\downarrow, \beta^{-}[D])$,

establishing the concept continuity of $(\alpha, \beta)$.

$\square$
Of basic importance for Formal Concept Analysis are the dense embeddings
\[ \eta_K = (\gamma_K, \mu_K) : K \to CB_K = (B_K, B_K, \leq), \]
where
\[ \gamma_K = \gamma : G \to B_K \] and \[ \mu_K = \mu : M \to B_K \]
are the natural object and attribute embeddings, respectively (cf. [12]):
\[ \gamma(g) = (g^{\uparrow \downarrow}, g^{\uparrow}) \] and \[ \mu(m) = (m^\downarrow, m^{\uparrow \downarrow}). \]

It was pointed out in the preprint version of [7] that these embeddings are the reflection morphisms for a reflector \( CB \) from the category \( Cc \) of contexts and conceptual mapping pairs to the full subcategory \( CCc \) of complete contexts and complete homomorphisms. The construction from [7] may be extended, in a straightforward manner, to the categories \( Cjm \) and \( CLjm \): for any separately continuous mapping pair \((\alpha, \beta)\) between contexts \( K = (G, M, I) \) and \( L = (L, L, \leq) \), we put \( B(\alpha, \beta) = (\alpha \rightarrow, \beta \rightarrow) \). In case \((\alpha, \beta)\) is conceptual, we have \( \alpha \rightarrow = \beta \rightarrow \), so that it is more convenient to put \( B(\alpha, \beta) = \alpha \rightarrow \) (or, alternately, \( B(\alpha, \beta) = \beta \rightarrow \)). Concerning adjoint functors, we refer to [1].

**Theorem 5.1** \( B \) is a covariant functor

- from \( Cjm \) to \( CLjm \)
- from \( Cc \) to \( CLc \)
- from \( CCc \) to \( CLc \)
- from \( Cde \) to \( Cde \)
- from \( CLc \) to \( CLc \)
- from \( CLc \) to \( CLc \)

Furthermore, \( B \) is left adjoint to the functor \( C \) in the opposite direction. The unit of the adjunctions is \( \eta \), and the counit is an isomorphism.

Proof. The functor properties of \( B \) are easily checked, using Theorem 2.1 and Proposition 5.1. Concerning adjointness, we have to show that for every complete lattice \( L \) and for every separately continuous morphism \((\alpha, \beta)\) from an arbitrary context \( K = (G, M, I) \) into \( CL = (L, L, \leq) \), there is a unique pair \((\alpha^\vee, \beta^\wedge)\) such that
\[ \alpha^\vee : B_K \to L \] preserves joins, \( \beta^\wedge : B_K \to L \) preserves meets, and
\[ (\alpha, \beta) = (\alpha^\vee, \beta^\wedge) \circ \eta_K, \] i.e. \( \alpha = \alpha^\vee \circ \gamma_K \) and \( \beta = \beta^\wedge \circ \mu_K \).

Defining for each concept \((A, B) \in B_K\)
\[ \alpha^\vee(A, B) = \bigvee \alpha[A] \] and \[ \beta^\wedge(A, B) = \bigwedge \beta[B], \]
we see that \( \alpha^\vee \) is lower adjoint to the map
\[ \alpha^\vee^* : L \to B_K, \ x \mapsto (\alpha^{-}[\downarrow x], \alpha^{-}[\downarrow x]^\downarrow), \]
and that \( \beta^\wedge \) is upper adjoint to the map
\[ \beta^\wedge^* : L \to B_K, \ x \mapsto (\beta^{-}[\uparrow x], \beta^{-}[\uparrow x]). \]
Hence, $\alpha^\vee$ preserves joins, $\beta^\wedge$ preserves meets, and
\[
\alpha^\vee \circ \gamma(g) = \bigvee \alpha'[g'^2] = \alpha(g) \quad \text{by continuity of } \alpha,
\]
\[
\beta^\wedge \circ \mu(m) = \bigwedge \beta[m'^2] = \beta(m) \quad \text{by continuity of } \beta.
\]
The uniqueness of a join-meet preserving pair $(\alpha^\vee, \beta^\wedge)$ with $\alpha = \alpha^\vee \circ \gamma_K$ and $\beta = \beta^\wedge \circ \mu_K$ follows from join-density of $\gamma_K$ and meet-density of $\mu_K$.

Now, if $(\alpha, \beta) : K \to L$ is an arbitrary $\mathbf{Cjm}$-morphism (that is, a separately continuous mapping pair) then there is a unique $\mathbf{CLjm}$-morphism $(\varphi, \psi) : B_K \to B_L$ satisfying the identity $(\varphi, \psi) \circ \eta_K = \eta_L \circ (\alpha, \beta)$, namely
\[
\varphi(A, B) = \bigvee \gamma_L[\alpha[A]] = (\alpha[A]^\uparrow, \alpha[A]^\downarrow) = \alpha^\rightarrow(A, B),
\]
\[
\psi(A, B) = \bigwedge \mu_L[\beta[B]] = (\beta[B]^\downarrow, \beta[B]^\uparrow) = \beta^\rightarrow(A, B),
\]
and consequently $(\varphi, \psi) = B(\alpha, \beta)$. This shows that $B$ is in fact left adjoint to $C$, and that $\eta$ is the unit of the adjunction. The counit $\varepsilon$ is constituted by the natural isomorphisms
\[
\varepsilon_L : BCL \to L \quad \text{with} \quad \varepsilon_L(\uparrow x, \downarrow x) = x.
\]

Invoking Theorem 2.1 once more, we conclude:

**Corollary 5.1** A mapping pair $(\alpha, \beta) : K \to L$ is separately continuous iff there exists a (unique) $\mathbf{CLjm}$-morphism $(\varphi, \psi) : B_K \to B_L$ with
\[
\varphi \circ \gamma_K = \gamma_L \circ \alpha \quad \text{and} \quad \psi \circ \mu_K = \mu_L \circ \beta,
\]
namely $(\varphi, \psi) = (\alpha^\rightarrow, \beta^\rightarrow)$. Furthermore,
\[
(\alpha, \beta) \text{ is conceptual iff } \varphi = \psi \text{ is a complete homomorphism},
\]
\[
(\alpha, \beta) \text{ is concept continuous iff } \varphi^* = \psi^* \text{ is a complete homomorphism},
\]
\[
(\alpha, \beta) \text{ is a dense embedding iff } \varphi = \psi \text{ (}= \psi^*) \text{ is an isomorphism}.
\]

**Corollary 5.2** Up to identification between complete lattices and complete contexts, we have full reflective subcategories
\[
\mathbf{CLjm} \hookrightarrow \mathbf{Cjm}, \quad \mathbf{CLc} \hookrightarrow \mathbf{Cc}, \quad \mathbf{CLcd} \hookrightarrow \mathbf{Ccd}, \quad \mathbf{CLce} \hookrightarrow \mathbf{Cce}, \quad \mathbf{CLis} \hookrightarrow \mathbf{Cde}, \quad \mathbf{CLc*} \hookrightarrow \mathbf{Cc*}, \quad \mathbf{CLc*} \hookrightarrow \mathbf{Ccd*}, \quad \mathbf{CLc}* \hookrightarrow \mathbf{Cc*e},
\]
and analogous reflections for the corresponding categories with doubly residuated morphisms.
Corollary 5.3  For all contexts $\mathbb{K}$ and complete lattices $L$, a mapping pair 
$(\alpha, \beta) : \mathbb{K} \to CL$ is conceptual iff there is a unique complete homomorphism
$\varphi : B\mathbb{K} \to L$ such that $(\alpha, \beta) = (\varphi, \varphi) \circ \eta_\mathbb{K}$, i.e. $\alpha = \varphi \circ \gamma_\mathbb{K}$ and $\beta = \varphi \circ \mu_\mathbb{K}$.
Moreover, $(\alpha, \beta)$ is a dense embedding iff $\varphi$ is an isomorphism.

These results extend known facts about the Dedekind-MacNeille completion (by cuts) of ordered sets (cf. \[4, 7\]). Another immediate application is the Fundamental Theorem on Concept Lattices (see e.g. \[12\]), saying that a concept lattice $B(G, M, I)$ is isomorphic to a given complete lattice $L$ iff there exists a join-dense map $\gamma$ from $G$ into $L$ and a meet-dense map $\mu$ from $M$ into $L$ such that $gIM \iff \gamma(g) \leq \mu(m)$.

If $(G, M, I)$ is a subcontext of a context $(H, N, J)$ (that is, $G \subseteq H$, $M \subseteq N$ and $I = (G \times M) \cap J$), we may consider the respective inclusion maps.

Corollary 5.4  For a subcontext $\mathbb{K} = (G, M, I)$ of a context $L = (H, N, J)$, the following conditions are equivalent:

(a) The inclusion maps $G \hookrightarrow H$ and $M \hookrightarrow N$ form a conceptual pair.

(b) $(A^I \cap M)J \subseteq A^J$ for $A \subseteq G$ and $(B_J \cap G)^J \subseteq B^J$ for $B \subseteq M$.

(c) If $hJn$ then there are $gIm$ with $h^I \cap M \subseteq g^I$ and $nJ \cap G \subseteq mJ$.

(d) $(A, B) \mapsto (A^I_J, A^J) = (B^I, B^J)$ is a complete homomorphism from $B\mathbb{K}$ to $BCL$.

(e) There is a complete homomorphism $\varphi : B\mathbb{K} \to B\mathbb{L}$ with $\varphi \circ \eta_\mathbb{K} = \eta_\mathbb{L}$.

Similarly, taking for $\alpha$ and $\beta$ identity maps but different incidence relations, we arrive at

Corollary 5.5  For two contexts $\mathbb{K} = (G, M, I)$ and $L = (G, M, J)$ with the same underlying sets, the following conditions are equivalent:

(a) The identity pair $(id_G, id_M)$ is conceptual.

(b) $A^I_J = A^I$ for $A \subseteq G$ and $B_J^I = B^J_I$ for $B \subseteq M$.

(c) $I \subseteq J$, and for all $hJn$ there are $gIm$ with $h^I \subseteq g^I$ and $nJ \subseteq mJ$.

(d) , (e) As in Corollary 5.4

Note that in contrast to Corollary 5.4, the complete homomorphisms in Corollary 5.5 are always surjective, because identity maps are trivially dense (but not necessarily full).
6 The Concept Lattice as a Contravariant Functor

We have already seen in the preceding section that every concept continuous context morphism \((\alpha, \beta) : K \to L\) gives rise to a complete homomorphism in the opposite direction,
\[ \varphi : BL \to BK, \quad (C, D) \mapsto (\alpha^{-}[C], \beta^{-}[D]). \]
More generally, any separately continuous context morphism \((\alpha, \beta) : K \to L\) induces a meet-preserving map
\[ \alpha^{-} = \alpha^{-\ast} : BL \to BK, \quad (C, D) \mapsto (\alpha^{-}[C], \alpha^{-}[C]^{\uparrow}) \]
and a join-preserving map
\[ \beta^{-} = \beta^{-\ast} : BL \to BK, \quad (C, D) \mapsto (\beta^{-}[D]^{\downarrow}, \beta^{-}[D]), \]
and these two maps coincide iff \((\alpha, \beta)\) is concept continuous.

The category \(\text{CLmj}\) of complete lattices and meet-join preserving pairs \((\psi, \varphi)\) (where \(\psi : K \to L\) preserves arbitrary meets and \(\varphi : K \to L\) preserves arbitrary joins) is isomorphic to the category \(\text{CLjm}\) by means of two essentially different functors: one of them exchanges the first and the second component in the mapping pairs, while the other keeps the morphisms fixed and reverses the lattice orders. But there is also a dual isomorphism between \(\text{CLjm}\) and \(\text{CLmj}\), obtained by passing to the order-theoretical adjoints:
\[ G : \text{CLjm} \to \text{CLmj}, \quad G(\psi, \varphi) = (\psi_{\ast}, \varphi_{\ast}), \]
\[ H : \text{CLmj} \to \text{CLjm}, \quad H(\psi, \varphi) = (\psi_{\ast}, \varphi_{\ast}). \]
Obviously, these mutually inverse functors induce dual isomorphisms between the categories \(\text{CLc}\) and \(\text{CLc}^{\ast}\) etc.

Now, by our previous considerations, we have a contravariant functor
\[ B^{-} : \text{Cjm} \to \text{CLmj}, \quad B^{-} K = BK, \quad B^{-}(\alpha, \beta) = (\alpha^{-}, \beta^{-}), \]
and also a contravariant functor in the other direction, namely
\[ C^{-} : \text{CLmj} \to \text{Cjm}, \quad C^{-} L = CL, \quad C^{-}(\psi, \varphi) = (\psi_{\ast}, \varphi_{\ast}). \]
Moreover, these functors are linked with the covariant functors \(B\) and \(C\) by the identities
\[ B^{-} = G \circ B, \quad B = H \circ B^{-}, \quad C^{-} = C \circ H, \quad C = C^{-} \circ G. \]
Therefore, the adjunction in Theorem 5.1 turns into a dual adjunction for the corresponding contravariant functors:

**Theorem 6.1** The contravariant functor \(B^{-} : \text{Cjm} \to \text{CLmj}\) is dually adjoint to the contravariant functor \(C^{-} : \text{CLmj} \to \text{Cjm}\). Furthermore, these functors induce dual adjunctions between
\[ \text{Cc} \quad \text{and} \quad \text{CLc}^{\ast}, \quad \text{Cc}^{\ast} \quad \text{and} \quad \text{CLc}, \quad \text{Cde} \quad \text{and} \quad \text{CLis}, \quad \text{etc.} \]
Corollary 6.1 For all contexts \( K \) and complete lattices \( L \), a mapping pair \( (\alpha, \beta) : K \to CL \) is concept continuous iff there is a unique complete homomorphism \( \varphi : L \to B^K \) such that \( \varphi_* \circ \gamma_K = \alpha \) and \( \varphi^* \circ \mu_K = \beta \).

\[
\begin{array}{ccc}
K & \xrightarrow{(\alpha, \beta)} & CL \\
\downarrow \gamma_K & & \downarrow \varphi_* \\
CBK & & CL \\
\downarrow \varphi^* & & \downarrow \varepsilon_L \\
CBK & & L
\end{array}
\]

Corollary 6.2 A mapping pair \( (\alpha, \beta) \) between contexts \( K \) and \( L \) is concept continuous iff there exists a (unique) complete homomorphism \( \varphi : B_L \to B^K \) such that \( \varphi_* \circ \gamma_K = \gamma_L \circ \alpha \) and \( \varphi^* \circ \mu_K = \mu_L \circ \beta \), namely \( \varphi = \alpha^* = \beta^* \).

Corollary 6.3 For a subcontext \( K = (G, M, I) \) of a context \( L = (H, N, J) \), the following conditions are equivalent:

(a) The inclusion morphism from \( K \) into \( L \) is concept continuous.
(b) \( K \) is compatible, that is, \( (C^J \cap M)_J \cap G \subseteq C^I_J \) for all \( C \subseteq H \) and \( (D^J \cap G)_J \cap M \subseteq D^I_J \) for all \( D \subseteq N \).
(c) For all \( g \in G \) and \( n \in N \setminus g^I \) there is an \( m \in M \setminus g^J \) with \( n_J \subseteq m_J \), and for all \( m \in M \) and \( h \in H \setminus m^J \) there is a \( g \in G \setminus m^I \) with \( h^J \subseteq g^J \).
(d) The trace map \( \varphi : B_L \to B^K, (C, D) \mapsto (C \cap G, D \cap M) \) is a complete homomorphism.
(e) There exists a unique complete homomorphism \( \varphi \) from \( B_L \) onto \( B^K \) with \( \varphi_* \circ \gamma_K = \gamma_L \) and \( \varphi^* \circ \mu_K = \mu_L \).

Another frequently used application of our results on concept continuous maps is obtained by taking for \( \alpha \) and \( \beta \) identity maps.

Corollary 6.4 For contexts \( K = (G, M, I) \) and \( L = (G, M, J) \), the following conditions are equivalent:

(a) The identity pair \( (id_G, id_M) : K \to L \) is concept continuous.
(b) \( J \subseteq I, A^J_I \subseteq A^I_J \) for \( A \subseteq G \) and \( B^J_J \subseteq B^I_J \) for \( B \subseteq M \).
(c) \( J \subseteq I \), and \( (g, m) \in I \setminus J \) implies \( h \preceq m \) for some \( h \in G \) with \( g^I \subseteq h^J \) and \( g \preceq n \) for some \( n \in M \) with \( m^I \subseteq n_J \).
(d) \( J \) is a closed relation of \( K \), that is, \( J \subseteq G \times M \) and \( B_L \subseteq B^K \).
(e) \( B_L \) is a complete sublattice of \( B^K \).

A great part of these two corollaries has been discovered earlier by Ganter, Wille and Reuter (see [13, 14] and [11]).
Diagram. Categories of Contexts and Complete Lattices

\[ B(\alpha, \beta) = \alpha^+ \]
\[ B_s(\alpha, \beta) = \beta^+ \]
\[ B^*(\alpha, \beta) = \alpha^{++} \]
\[ U\varphi = \varphi^* \]
\[ B^{**}(\alpha, \beta) = \beta^+ \]
\[ B_s^*(\alpha, \beta) = \alpha^{++} \]
\[ B^{***}(\alpha, \beta) = \alpha^+ \]

\[ C\varphi = (\varphi, \varphi) \]
\[ C_{\varphi} = (\varphi_s, \varphi_s) \]
\[ C^*\varphi = (\varphi^*, \varphi^*) \]
\[ C^{**}\varphi = (\varphi^{**}, \varphi) \]
\[ C^{***}\varphi = (\varphi^{***}, \varphi) \]
\[ C_{\varphi} = (\varphi^*, \varphi_s) \]
\[ C_{\varphi} = (\varphi, \varphi^{**}) \]
7 Purified Contexts and Doubly Based Lattices

In order to obtain not only (dual) adjunctions but even categorical (dual) equivalences between certain categories of contexts and complete lattices, we must transfer the role played by the object and attribute sets to the realm of complete lattices. To that aim, we introduce doubly based lattices as triples \((K, J, M)\) where \(K = (K, \leq)\) is a complete lattice, \(J\) is a join-dense subset (join-base) and \(M\) is a meet-dense subset (meet-base) of \(K\). Any context \(\mathcal{K} = (J, M, I)\) gives rise to a doubly based lattice \(\mathcal{B}^\circ \mathcal{K} = (\mathcal{B}^\circ \mathcal{K}, J_0, M_0)\) with \(J_0 = \gamma_{\mathcal{K}}[J]\) and \(M_0 = \mu_{\mathcal{K}}[M]\).

For any separately continuous morphism \((\alpha, \beta) : \mathcal{K} = (J, M, I) \rightarrow \mathcal{L} = (H, N, R)\) the lifted map \(\alpha^\rightarrow\) preserves not only joins but also the join-bases, i.e. \(\alpha^\rightarrow[J_0] \subseteq H_0\), and \(\beta^\rightarrow\) preserves not only meets but also the meet-bases, i.e. \(\beta^\rightarrow[M_0] \subseteq N_0\), on account of the equations \(\alpha^\rightarrow \circ \gamma_{\mathcal{K}} = \gamma_{\mathcal{L}} \circ \alpha\) and \(\beta^\rightarrow \circ \mu_{\mathcal{K}} = \mu_{\mathcal{L}} \circ \beta\).

In that way, we obtain a functor \(\mathcal{B}^\circ\) from the category \(\text{Cjm}\) of contexts with separately continuous morphisms to the category \(\text{CLjm}^\circ\) of doubly based lattices with mapping pairs \((\varphi, \psi)\) such that \(\varphi\) preserves joins and the selected join-bases, while \(\psi\) preserves meets and the selected meet-bases. In the opposite direction, we may assign to each doubly based lattice \(\mathcal{K} = (K, J, M)\) the base context \(\mathcal{C}^\circ \mathcal{K} = (J, M, \leq)\) where \(\leq\) denotes the given order relation of \(K\) but also the induced relation between \(J\) and \(M\). Then \(\mathcal{C}^\circ\) becomes a functor, acting on morphisms by restriction to the given join- and meet-bases (see the proof of Theorem 7.1). A context \(\mathcal{K}\) is said to be purified if \(\gamma_{\mathcal{K}}\) and \(\mu_{\mathcal{K}}\) are injective – in other words, if \(\eta_{\mathcal{K}}\) induces an isomorphism between \(\mathcal{K}\) and the context \(\mathcal{C}^\circ \mathcal{B}^\circ \mathcal{K} = (J_0, M_0, \leq)\). For any doubly based lattice \(\mathcal{K} = (K, J, M)\), the context \(\mathcal{C}^\circ \mathcal{K} = (J, M, \leq)\) is purified. Moreover, by join-density of \(J\) and meet-density of \(M\), we have an isomorphism

\[\varepsilon_{\mathcal{K}} : \mathcal{B}(J, M, \leq) \rightarrow \mathcal{K}, (A, B) \mapsto \bigvee A = \bigwedge B\]

with inverse

\[\iota_{\mathcal{K}} : \mathcal{K} \rightarrow \mathcal{B}(J, M, \leq), x \mapsto (J \cap \downarrow x, M \cap \uparrow x)\].

Since \(\iota_{\mathcal{K}}\) maps \(J\) onto \(J_0\) and \(M\) onto \(M_0\), we may regard that isomorphism as a morphism between \(\mathcal{K}\) and \(\mathcal{B}^\circ \mathcal{C}^\circ \mathcal{K}\) in the category \(\text{CLjm}^\circ\).

**Theorem 7.1** The augmented concept lattice functor \(\mathcal{B}^\circ\) induces an equivalence between the category \(\text{Cjm}^\circ\) of purified contexts with separately continuous morphisms and the category \(\text{CLjm}^\circ\) of doubly based lattices with join-meet preserving pairs that induce mappings between the respective bases. In the opposite direction, an equivalence is established by the functor \(\mathcal{C}^\circ\).
Similarly, the category $C^o$ of purified contexts with conceptual morphisms is equivalent to the category $CL^o$ of doubly based lattices with complete homomorphisms preserving the join- and meet-bases.

**Proof.** We have already seen that for any $CLjm^o$-morphism $(\alpha, \beta)$, the map $B^o(\alpha, \beta) = (\alpha^-, \beta^-)$ is a $CLjm^o$-morphism. The equation

$$C^oB^o(\alpha, \beta) \circ \eta_K = (\alpha^- \circ \gamma_K, \beta^- \circ \mu_K) = (\gamma_L \circ \alpha, \mu_L \circ \beta) = \eta_L \circ (\alpha, \beta)$$

shows that $\eta$ is a natural isomorphism between the identity functor on the category of purified contexts and the composite functor $C^oB^o$. On the other hand, given an arbitrary $CLjm^o$-morphism $\varphi$ between doubly based lattices $K = (K, J, M)$ and $L = (L, H, N)$, we must show that $\alpha : J \rightarrow H$, $j \mapsto \varphi(j)$ and $\beta : M \rightarrow N$, $m \mapsto \varphi(m)$ are continuous maps. The extents of $C^oL$ are of the form $N \cap \downarrow y$, and

$$\alpha^-[N \cap \downarrow y] = \{j \in J : \varphi(j) \leq y\} = \{j \in J : j \leq \varphi^*(y)\} = J \cap \downarrow \varphi^*(y)$$

is then an extent, too. Thus, $\alpha$ is extent continuous, and dually, $\beta$ is intent continuous, so that $(\alpha, \beta)$ is a $CLjm^o$-morphism. Moreover, the equation

$$(H \cap \downarrow \varphi(x), N \cap \uparrow \varphi(x)) = (H \cap \downarrow \varphi[J \cap \downarrow x], N \cap \uparrow \varphi[M \cap \uparrow x]) = (\varphi[J \cap \downarrow x]^{\uparrow}, \varphi[M \cap \uparrow x]^{\downarrow})$$

yields the naturality of the isomorphism $\iota : 1_{CLjm^o} \rightarrow B^oC^o$ and its inverse:

$$\iota_L \circ \varphi = B^oC^o \circ \iota_K, \text{ hence } \varphi \circ \varepsilon_K = \varepsilon_L \circ B^oC^o \varphi.$$  

If $\varphi$ is a $CL^o$-morphism, we have to verify that the mapping pair $C^o\varphi = (\alpha, \beta)$ from $C^oK$ to $C^oL$ is conceptual. To that aim, we compute for $A \subseteq J$ and $y = \bigvee \varphi[A]$, using join- and meet-density of the bases:

$$y = \varphi(\bigvee A) = \bigvee \varphi(M \cap \uparrow \bigvee A) = \bigvee \varphi[M \cap \uparrow \bigvee A] = \bigvee \beta[A]^\uparrow,$$

$$\alpha[A]^\downarrow = (N \cap \uparrow y)^\downarrow = H \cap \downarrow y = \beta[A]^\downarrow.$$  

A dual reasoning yields the identity $\beta[B]^\downarrow = \alpha[B]^\uparrow$ for $B \subseteq M$. \[\square\]

An obvious question is now whether the above equivalence theorem has an analogue for concept continuous maps. The answer is in the affirmative, but the choice of morphisms is a bit subtle. Let us consider the category $CL_{os}$ of doubly based lattices with join-base preserving doubly residuated maps $\psi$ whose double upper adjoint $\psi^{**}$ preserves the meet-bases. By passing to these double upper adjoints, one obtains an isomorphism between $CL_{os}$ and the category $CL_{os}^*$ of doubly based lattices with meet-base preserving doubly residual maps whose double lower adjoint preserves the join-bases. A different isomorphism between these categories results from dualization of the order relations, leaving unchanged the underlying mappings of the morphisms. But notice that $CL_{os}$ and $CL_{os}^*$ are not dual to the category $CL^o$, but to that category $CL_{ho}$ whose morphisms have a join-base preserving lower adjoint and a meet-base preserving upper adjoint.
Theorem 7.2 Assigning to each concept continuous morphism \((\alpha, \beta)\) between purified contexts \(\mathbb{K}\) and \(\mathbb{L}\) the map \(\alpha^{-} : \mathcal{B}\mathbb{K} \to \mathcal{B}\mathbb{L}\), one obtains an equivalence functor \(\mathcal{B}_{\alpha}\) between the category \(\mathcal{C}_{\alpha}\) of purified contexts with context continuous pairs and the category \(\mathcal{CL}_{\alpha}\). Similarly, sending \((\alpha, \beta)\) to \(\beta^{-}\), one obtains an equivalence functor \(\mathcal{B}_{\alpha}^*\) between the categories \(\mathcal{C}_{\alpha}\) and \(\mathcal{CL}_{\alpha}^*\). In the opposite direction, equivalence functors from \(\mathcal{CL}_{\alpha}\) and \(\mathcal{CL}_{\alpha}^*\), respectively, to \(\mathcal{C}_{\alpha}\) are given by restriction of the doubly residuated maps to the join-bases and of the doubly residual maps to the meet-bases.

Proof. We have seen earlier that for any concept continuous pair \((\alpha, \beta)\), the map \(\alpha^{-}\) is doubly residuated, with \(\alpha^{-*} = \alpha^* = \beta^*\) and \(\alpha^{-**} = \beta^{-*} = \beta^{-}\), and that \(\alpha^{-}\) preserves the join-bases, while \(\beta^{-}\) preserves the meet-bases.

Given a \(\mathcal{CL}_{\alpha}\)-morphism \(\psi = \varphi_* : \mathbb{K} \to \mathbb{L}\), we show that the mapping pair \(\mathbb{C}_{\alpha}\psi = (\alpha, \beta)\) from \(\mathcal{C}\mathbb{K}\) to \(\mathcal{C}\mathbb{L}\) with \(\alpha = \psi| J = \varphi_*| J : J \to H\) and \(\beta = \psi^*| M = \varphi^*| M : M \to N\) is concept continuous. For \(F \subseteq H\), we have:

\[
g \in \alpha^{-}[C^{\uparrow}] \Leftrightarrow \alpha(g) \in H \cap \downarrow \mathbb{V} C \Leftrightarrow \varphi_*(g) \leq \mathbb{V} C \Leftrightarrow g \leq \varphi(\mathbb{V} C)
\]
\[
\Leftrightarrow \forall m \in M (\varphi(\mathbb{V} C) \leq m \Rightarrow g \leq m)
\]
\[
\Leftrightarrow \forall m \in M (\varphi^*(m) \in N \cap \uparrow \mathbb{V} C \Rightarrow g \leq m)
\]
\[
\Leftrightarrow \forall m \in M (\beta(m) \subseteq C^\uparrow \Rightarrow g \leq m) \Leftrightarrow g \in \beta^{-}[C^\uparrow].
\]
This and the dual equation \(\beta^{-}[D^{\uparrow}] = \alpha^{-}[D^{\uparrow}]\) prove concept continuity.

That \(\eta\) and \(\varepsilon\) are natural isomorphisms between the identity functors and the functors composed by \(\mathcal{C}_{\alpha}\) and \(\mathcal{B}_{\alpha}\) is checked as in the previous proof. \(\mathcal{CL}_{\alpha}^*\)-morphisms are treated analogously. \(\square\)

Corollary 7.1 (1) Via the augmented concept lattice functor, the isomorphism classes of purified contexts bijectively correspond to the isomorphism classes of doubly based lattices. In particular, every purified context is isomorphic to one of the form \(\mathcal{C}\mathbb{K}\) for a doubly based lattice \(\mathbb{K}\).

(2) The conceptual morphisms between purified contexts \(\mathbb{K} = \mathcal{C}\mathbb{K}\) and \(\mathbb{L} = \mathcal{C}\mathbb{L}\) are exactly the restrictions of the base-preserving complete homomorphisms between the doubly based lattices \(\mathbb{K} \simeq \mathcal{B}\mathbb{K}\) and \(\mathbb{L} \simeq \mathcal{B}\mathbb{L}\).

(3) The concept continuous morphisms between purified contexts \(\mathbb{K} = \mathcal{C}\mathbb{K}\) and \(\mathbb{L} = \mathcal{C}\mathbb{L}\) are exactly the pairs formed by the base restrictions of the lower adjoint \(\varphi_*\) and the upper adjoint \(\varphi^*\) of complete homomorphisms \(\varphi\) from \(\mathbb{L} \simeq \mathcal{B}\mathbb{L}\) to \(\mathbb{K} \simeq \mathcal{B}\mathbb{K}\) such that \(\varphi_*\) preserves the join-bases and \(\varphi^*\) the meet-bases.

(4) Conversely, every base-preserving complete homomorphism between doubly based lattices is induced by a unique conceptual morphism between the underlying base contexts, and every complete homomorphism with join-base preserving lower adjoint and meet-base preserving upper adjoint is induced by a unique concept continuous morphism in the opposite direction.
8 Dualities and Galois Connections for Contexts

As expected, there are not only equivalences but also dualities between certain categories of contexts and complete lattices. In most cases, such dualities are obtained by composing the already established equivalences with the dual isomorphisms resulting from the passage between lower and upper adjoints. For example, the category $\text{CL}_\circ$ of doubly based lattices with base-preserving complete homomorphisms is dually isomorphic to the following two categories with the same objects: the morphisms of $\text{CL}_\circ^*$ are those doubly residual maps whose lower adjoint preserves join- and meet-bases (hence is a $\text{CL}_\circ^*$-morphism), and the morphisms in $\text{CL}_\circ$, are those doubly residuated maps whose upper adjoint preserves join- and meet-bases. Thus, Theorem 7.1 amounts to:

**Corollary 8.1** Sending each conceptual morphism $(\alpha, \beta)$ to the doubly residuated map $\alpha^{-} = \beta^{-}$, one obtains a dual equivalence between the category $\text{C}_\circ$ of purified contexts with conceptual morphisms and the category $\text{CL}_\circ^*$, while sending $(\alpha, \beta)$ to the doubly residuated map $\beta^{-} = \alpha^{-}$, one obtains a dual equivalence between the categories $\text{C}_\circ$ and $\text{CL}_\circ^*$.

As already observed earlier, in the same way, the category $\text{CL}_0$ of doubly based lattices with complete homomorphisms $\varphi$ having a join-base preserving lower adjoint $\varphi_*$ and a meet-base preserving upper adjoint $\varphi^*$ is dually isomorphic to the categories $\text{CL}_0^*$ and $\text{CL}_0^*$. Thus, from Theorem 7.2, we immediately derive:

**Corollary 8.2** Sending each concept continuous pair $(\alpha, \beta)$ to the complete homomorphism $\alpha^{-} = \beta^{-}$, one obtains a dual equivalence between the category $\text{C}_\circ$ of purified contexts with concept continuous morphisms and the category $\text{CL}_0$.

A basic remark about canonical order structures on contexts is now long overdue. Both the objects and the attributes of any context carry a natural "specialization order", given by

\[ j \leq k \iff k^\uparrow \subseteq j^\uparrow \quad \text{and} \quad m \leq n \iff m^\downarrow \subseteq n^\downarrow. \]

These two relations are obviously quasi-orders (reflexive and transitive), and they are partial orders (antisymmetric) iff the context is purified. The restriction to purified contexts has great structural advantages but causes no essential loss of generality, because every context $K = (J, M, I)$ has a purification $\mathcal{C}_\circ^B K = (J_0, M_0, \leq)$ whose concept lattice is isomorphic to that of the original context. Note the following implication:

\[ g \leq j I m \leq n \Rightarrow g I n. \]

It is also convenient to know that for any base context $\mathcal{C}_\circ K = (J, M, \leq)$, the specialization orders are induced by the lattice order.
The following identities connecting the partners of mapping pairs between purified contexts are easily verified with the help of Lemmas 3.6 and 3.7 (maxima and minima refer to the specialization orders):

**Proposition 8.1** Each partner of a conceptual mapping pair \((\alpha, \beta)\) between purified contexts determines the other uniquely, by the identities

\[ \alpha(g) = \max \{ h : g^\uparrow \subseteq \beta^-[h^\uparrow] \}, \quad \beta(m) = \min \{ n : m^\downarrow \subseteq \alpha^-[n^\downarrow] \}. \]

Similarly, each partner of a concept continuous mapping pair \((\alpha, \beta)\) between purified contexts determines the other uniquely, by the identities

\[ \alpha(g) = \min \{ h : \beta^-[h^\uparrow] \subseteq g^\uparrow \}, \quad \beta(m) = \max \{ n : \alpha^-[n^\downarrow] \subseteq m^\downarrow \}. \]

On account of these facts, it would suffice to consider single maps between the object or the attribute sets of purified contexts as conceptual or concept continuous morphisms. However, the approach via mapping pairs makes the interplay between object and attribute sets of contexts more transparent.

In view of the striking similarities between conceptual and concept continuous morphisms, and encouraged by the dual isomorphisms between the corresponding lattice categories \(\text{CLc}\) and \(\text{CLc}^\ast\) etc., one might wish to find similar dualities between suitable subcategories of the context categories \(\text{C}_o \subseteq \text{Cc}\) and \(\text{C}_o \subseteq \text{Cc}^\ast\). This is in fact possible, but the appropriate choice of morphisms might look a bit technical at first glance. However, from the Galois-theoretical point of view, it is rather natural. By slight abuse of language, we call a mapping pair \((\alpha, \beta)\) between purified contexts \(\mathbb{K}\) and \(\mathbb{L}\) residuated if it is concept continuous, \(\alpha\) is residuated and \(\beta\) is residual (i.e. dually residuated) with respect to the specialization orders. On the other hand, we say \((\alpha, \beta)\) is residual if it is conceptual, \(\alpha\) is residual and \(\beta\) is residuated (i.e. dually residual). The resulting categories of purified contexts are denoted by \(\text{C}_r\) and \(\text{C}_r^\ast\), respectively. As morphisms in the corresponding category \(\text{CLr}\) of doubly based lattices we take the base-preserving and *reflecting* complete homomorphisms, i.e. those maps which do not only preserve joins, meets, join-bases and meet-bases, but also have the property that their lower adjoint preserves the join-bases and their upper adjoint preserves the meet-bases, too.

**Lemma 8.1** If \((\alpha, \beta) : \mathbb{K} \rightarrow \mathbb{L}\) is a mapping pair between contexts such that \(\alpha : J \rightarrow H\) has an upper adjoint \(\alpha^*\) and \(\beta : M \rightarrow N\) has a lower adjoint \(\beta_*\), then the following statements are equivalent:

(a) \((\alpha, \beta)\) is concept continuous (residuated).
(b) \((\alpha^*, \beta_*)\) is conceptual (residual).
(c) \(\alpha(j)^\uparrow = \beta_*[^{-}[j^\uparrow]\) for all \(j \in J\) and \(\beta(m)^\downarrow = \alpha^*[^{-}[m^\downarrow]\) for all \(m \in M\).
(d) \(\alpha^*(h)^\uparrow = \beta[^{-}[h^\uparrow]\) for all \(h \in H\) and \(\beta_*(n)^\downarrow = \alpha[^{-}[n^\downarrow]\) for all \(n \in N\).
Moreover, if these conditions hold then $\alpha^\ast\beta^\ast = \beta^\ast\alpha^\ast$ and $(\alpha^\ast, \beta^\ast)$ is residuated, that is, a $\mathbf{C}_r$-morphism.

Proposition 8.2 For a mapping pair $(\alpha, \beta)$ between purified contexts $\mathbb{K}$ and $\mathbb{L}$, the following conditions are equivalent:

(a) $(\alpha, \beta)$ is residuated, that is, a $\mathbf{C}_r$-morphism.

(b) $(\alpha, \beta)$ reflects incidence, and there is an incidence preserving mapping pair $(\alpha^\ast, \beta^\ast) : \mathbb{L} \to \mathbb{K}$ with $\alpha^\ast(h)^\ast = \beta^\ast(h)^\ast$ and $\beta^\ast(n)^\ast = \alpha^\ast(n)^\ast$.

(c) There is a unique $\mathbf{CLr}$-morphism $\varphi : \mathcal{B}^\circ \mathbb{L} \to \mathcal{B}^\circ \mathbb{K}$ such that $\gamma_\mathbb{L} \circ \alpha = \varphi \circ \gamma_\mathbb{K}$ and $\mu_\mathbb{L} \circ \beta = \varphi \circ \mu_\mathbb{K}$.

(c') There is a unique $\mathbf{CLr}$-morphism $\varphi : \mathcal{B}^\circ \mathbb{L} \to \mathcal{B}^\circ \mathbb{K}$ such that $\varphi \circ \gamma_\mathbb{L} = \gamma_\mathbb{K} \circ \alpha^\ast$ and $\varphi \circ \mu_\mathbb{L} = \mu_\mathbb{K} \circ \beta^\ast$.

Moreover, if these conditions hold then $\alpha^\ast = \beta^\ast = \alpha^\ast = \beta^\ast$ and $\varphi = \alpha^\ast = \alpha^\ast = \beta^\ast = \beta^\ast = \beta^\ast$. 

\begin{tikzpicture}
  
  % Diagram
  
  % Nodes
  
  % Edges
  
  % Labels

  \end{tikzpicture}
Proof. (a) $\Rightarrow$ (b). From Lemma 6.1 we infer that $\alpha^\bullet = \alpha^*$ and $\beta^\bullet = \beta^*$ have the desired properties; $(\alpha^*, \beta^*)$ preserves incidence, because $n \in h^\uparrow$ together with $n \leq \beta(\beta^*(n))$ implies $\beta(\beta^*(n)) \in h^\uparrow$, hence $\beta^*(n) \in \beta^{-}[h^\uparrow] = \alpha^*(h)^\uparrow$. That this is the only possible choice for $\alpha^\bullet$ and $\beta^\bullet$ may be checked as follows. First, the given identities show that $\alpha$ and $\beta$ are continuous, hence isotone with respect to the specialization orders. But $\alpha^\bullet$ is isotone, too: $h \leq k$ means $k^\downarrow \subseteq h^\downarrow$, which entails $\alpha^\bullet(k)^\downarrow = \beta^-[k^\downarrow] \subseteq \beta^-[h^\downarrow] = \alpha^*(h)^\downarrow$, that is, $\alpha^*(h) \leq \alpha^\bullet(k)$. Next, we have $\alpha^\bullet(\alpha(j)) = \beta^-[\alpha(j)^\downarrow] \subseteq j^\downarrow$ since $(\alpha, \beta)$ reflects incidence; therefore, $j \leq \alpha^*(\alpha(j))$. On the other hand, the inequality $\alpha(\alpha^*(h)) \leq h$ follows from the hypothesis that $(\alpha^\bullet, \beta^\bullet)$ preserves incidence: $n \in h^\uparrow$ implies $\alpha^*(h) \in \beta^*(n)^\downarrow = \alpha^{-}[n^\downarrow]$ and then $n \in \alpha(\alpha^*(h))^\downarrow$. Thus, $\alpha^\bullet$ is the upper adjoint of $\alpha$, and similarly for $\beta^\bullet$.

(a) $\Leftrightarrow$ (a$^*$) and (b) $\Leftrightarrow$ (b$^*$) also follow from Lemma 6.1. Lemma 6.1 tells us that $(\alpha, \beta)$ is concept continuous. By Corollary 6.1 there is a unique complete homomorphism $\varphi = \alpha^\uparrow = \beta^\uparrow$ from $B_L$ to $B_K$ with $\gamma_L \circ \alpha = \varphi \circ \gamma_K$ and $\mu_L \circ \beta = \varphi \circ \mu_K$. Moreover, the equation $\varphi(\gamma_L(h)) = (\ldots, \beta^-[\varphi(h)^\downarrow]) = (\ldots, \alpha^*(h)^\downarrow) = \gamma_K(\alpha^*(h))$ and its dual yield the identities $\varphi \circ \gamma_L = \gamma_K \circ \alpha$ and $\varphi \circ \mu_L = \mu_K \circ \beta$. In particular, $\varphi$ preserves the join- and meet-bases.

(c) $\Rightarrow$ (a). By Corollary 6.1 the pair $(\alpha, \beta)$ is concept continuous. For $h \in H$, we have $(\alpha^{-}[h^\uparrow], \alpha^{-}[h^\downarrow]^\uparrow) = \alpha^{-}[h^\uparrow]^\uparrow = \varphi(\gamma_L(h)) \in \gamma_K[J]$, hence $\alpha^{-}[\downarrow h] = \alpha^{-}[h^\downarrow]^\uparrow = j^\downarrow = \downarrow j$ for some $j \in J$; thus, $\alpha$ is residuated. Dually, one shows that $\beta$ is residual.

(c$^*$) $\Rightarrow$ (a$^*$). Use Corollary 5.1 (with $K$ and $L$ exchanged).

(b$^*$) $\Rightarrow$ (c$^*$) is established in the same manner as (b) $\Rightarrow$ (c).

Let us note a few additional properties of residuated mapping pairs.

**Corollary 8.3** If $(\alpha, \beta)$ is a residuated pair then $(\alpha^*, \beta^*)$ is a residual pair, and the following identities are fulfilled:

$$\begin{align*}
\alpha^*[A^\downarrow] & = \beta[A^\downarrow]^\uparrow, \quad \beta^*[B^\downarrow] = \alpha[B^\downarrow]^\uparrow, \\
\alpha^*[C^\downarrow]^\downarrow & = \alpha^{-}[C^\downarrow]^\downarrow = \beta^*[C^\downarrow]^\downarrow = \beta^{-}[C^\downarrow]^\downarrow, \\
\alpha^*[D^\downarrow]^\uparrow & = \alpha^{-}[D^\downarrow]^\uparrow = \beta^*[D^\downarrow]^\downarrow = \beta^{-}[D^\downarrow]^\downarrow.
\end{align*}$$

**Proof.** For the first equation, observe the equivalences

$$h \in \alpha^*[A^\downarrow] \iff A^\downarrow \subseteq \alpha^*(h)^\downarrow \iff \beta[A^\downarrow] \subseteq h^\downarrow \iff h \in \beta[A^\downarrow]^\downarrow.$$  

The identity $\alpha^*[C^\downarrow]^\downarrow = \beta^*[C^\downarrow]^\downarrow$ follows from conceptuality of $(\alpha^*, \beta^*)$ (see Lemma 3.9), and the identity $\alpha^*[C^\downarrow]^\uparrow = \beta^*[C^\downarrow]^\downarrow$ from conceptuality of $(\alpha, \beta)$ (see Lemma 3.7). For $\alpha^{-}[C^\downarrow]^\downarrow = \beta^*[C^\downarrow]^\downarrow$, use the equivalences

$$j \in \alpha^{-}[C^\downarrow]^\downarrow \iff C^\downarrow \subseteq \alpha(\downarrow j)^\downarrow \iff \beta^*[C^\downarrow]^\downarrow \subseteq j^\downarrow \iff j \in \beta^*[C^\downarrow]^\downarrow.$$  

The other equations are derived analogously. \hfill $\Box$
From the equivalence of (a) and (a∗) in Proposition 8.2, we conclude:

**Corollary 8.4** By passing to adjoints, the context categories $C_r$ and $C^r$ are dually isomorphic to each other.

Now, we are in a position to establish the main result of this section:

**Theorem 8.1** Assigning to each residual pair $(\alpha, \beta)$ the complete homomorphism $\alpha^- = \beta^-$, one obtains an equivalence $B^r$ between the category $C^r$ of purified contexts and the category $\text{CLr}$ of doubly based lattices. In the opposite direction, the equivalence functor $C^r$ sends any $\text{CLr}$-morphism $\varphi$ to the mapping pair built by the restrictions of $\varphi$ to the join- and meet-bases.

Similarly, associating with any residuated pair $(\alpha, \beta)$ the complete homomorphism $\alpha^- = \beta^-$, one obtains a dual equivalence $B^∗r$ between the categories $C_r$ and $\text{CLr}$. In the opposite direction, the dual equivalence functor $C^r$ sends any $\text{CLr}$-morphism $\varphi$ to the pair constituted by the restriction of $\varphi^*$ to the join-bases and the restriction of $\varphi^*$ to the meet-bases.

**Proof.** We already know that for any context $K = (J, M, I)$, the triple $B^r K = B^∗r K = B^o K = (B^r K, J_0, M_0)$ is a doubly based lattice, and that

$$\eta_K = (\gamma_K, \mu_K) : K \rightarrow C^o B^o K, \quad x \mapsto (J \cap \downarrow x, M \cap \uparrow x)$$

is a natural isomorphism provided $K$ is purified. On the other hand, for an arbitrary doubly based lattice $K = (K, J, M)$, we have the purified context $C^o K = C^∗r K = C^o K = (J, M, \leq)$ and the natural isomorphism

$$\iota_K : K \rightarrow B^o C^o K, \quad x \mapsto (J \cap \downarrow x, M \cap \uparrow x).$$

That for any $C_r$-morphism $(\alpha, \beta)$ the pair $B^r(\alpha, \beta) = \alpha^- = \beta^-$ has the required properties of a $\text{CLr}$-morphism was shown in Proposition 8.2, and similarly, for any $C^r$-morphism $(\alpha, \beta)$, the pair $B^*(\alpha, \beta) = \alpha^- = \beta^-$ is a $\text{CLr}$-morphism, too.

Conversely, given any $\text{CLr}$-morphism $\varphi : L \rightarrow K$ between doubly based lattices $L = (L, H, N)$ and $K = (K, J, M)$, we have that the restricted maps $\alpha = \varphi_* : J \rightarrow H$ and $\beta = \varphi^* : M \rightarrow N$ form a residuated pair $C^r \varphi = (\alpha, \beta)$, hence a $C_r$-morphism. This and the remaining statements are easy consequences of earlier results. \square

Let us finally put together all pieces of the Galois duality puzzle. In the diagram on the next page, we place 13 different categories in three triangular levels; all categories of one level are mutually equivalent or dual. Each double line symbolizes a categorical equivalence, while each (non-dotted) single line stands for a duality. In the table of morphisms,

- $\vdash J$ indicates that join-bases are preserved,
- $\vdash M$ indicates that meet-bases are preserved,
- $\varphi_*$ denotes the lower adjoint and $\varphi^*$ the upper adjoint of $\varphi$. 

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Equivalent and dual categories of contexts and complete lattices

| category  | objects                  | morphisms            | additional properties |
|-----------|--------------------------|----------------------|-----------------------|
| $C^o$     | purified or reduced      | conceptual pairs     | residual              |
|           | contexts                 | concept              |                       |
| $C_r$     |                          | continuous pairs     |                       |
|           |                          | residedated          |                       |
| $CL^{o*}$ | doubly based or          | $\vdash J \vdash M$ | $\vdash J \vdash M$   |
|           | irreducibly bigenerated  |                      |                       |
| $CL^{r*}$ | lattices                 | $\vdash J \vdash M$ | $\vdash J \vdash M$   |
| $C_{o*}$  |                          | $\psi$               | $\psi^*$ $\psi$      |
|           |                          | $\psi^*$ $\psi$      | $\psi^{**}$ $\psi$   |
| $C_{r*}$  |                          | $\varphi$            | $\varphi$ $\varphi$  |
|           |                          | $\varphi^*$ $\varphi^*$ | $\varphi^{**}$ $\varphi^{**}$ |
9 Reduced Contexts and Irreducibly Bigenerated Lattices

We have seen that join- and meet-bases play a crucial role in the passage between context and complete lattices. The situation is simplified considerably if we focus on irreducibly bigenerated lattices; these have a least join-base $J(L)$, consisting of all join-irreducibles, and a least meet-base $M(L)$, consisting of all meet-irreducibles. Of course, all finite lattices have that property. Any irreducibly bigenerated lattice is isomorphic to the concept lattice of an up to isomorphism unique reduced context, viz. the standard context $SL = (J(L), M(L), \leq)$. An arbitrary context $K = (J, M, I)$ is reduced iff it is purified, each object concept $\gamma(j)$ is join-irreducible, and each attribute concept $\mu(m)$ is meet-irreducible in the concept lattice $BK$ – in other words, iff $\eta_K$ induces an isomorphism between the contexts $K$ and $SBK$. On the other hand, a complete lattice is irreducibly bigenerated iff it is isomorphic to the concept lattice of its standard context. We may regard $S$ as a covariant functor, sending each join- and meet-irreducibility preserving complete homomorphism to the pair of its restrictions to the least join- and meet-bases, respectively. But, of course, there is also a contravariant standard context functor, restricting any complete homomorphism whose lower adjoint preserves join-irreducibility and whose upper adjoint preserves meet-irreducibility, to the respective least bases. Now, the equivalences and dualities between categories of purified contexts and doubly based lattices derived in the previous sections immediately lead to the following more restricted but technically simpler results:

**Theorem 9.1** Under the concept lattice functor and the standard context functor in the reverse direction, the category of reduced contexts and conceptual morphisms is equivalent to the category of irreducibly bigenerated lattices and complete homomorphisms preserving the least join- and meet-bases. Hence, the conceptual morphisms between reduced contexts are in one-to-one correspondence with the irreducibility preserving complete homomorphisms between their concept lattices.

**Theorem 9.2** Via the contravariant concept lattice functor and the contravariant standard context functor, the category of reduced contexts with concept continuous pairs is dual to the category of irreducibly bigenerated lattices and complete homomorphisms whose lower adjoint preserves the join-bases and whose upper adjoint preserves the meet-bases.

Similarly, the category of reduced contexts with residual (respectively residuated) mapping pairs is equivalent (respectively dual) to the category of irreducibly bigenerated lattices with complete homomorphisms preserving and reflecting the least bases.
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