Non-Supersymmetric

$T^2/Z_N$ and $T^4/Z_N$ Orbifolds

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Abstract

We compute the perturbative tachyonic and massless spectra of Type II and Type 0 string theories on non-supersymmetric $T^2/Z_N$ orbifolds, and those on $T^4/Z_N$ ones. Comparing the spectra with one another, we obtain insight about the degeneracy of states and find several pairs of Type 0 orbifolds which could be identified with each other.

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1 Introduction

There have been many studies of supersymmetric both compact and non-compact orbifolds [14, 5]; however, there are many mathematically consistent non-supersymmetric orbifolds [1]. In recent years, non-compact non-supersymmetric orbifolds have been studied intensively [8], yet compact ones have not been paid much attention to; therefore we should advance studies of these. In this paper, we address only Type II and Type 0 string theories, and thus regrettably do not deal with heterotic ones.

As well known, we can regard Type 0B/0A string theory as the Type IIB/IIA orbifold with the spacetime fermion number operator \((-1)^F_S\) [2], but in this paper we treat Type 0 string theories as distinct from Type II ones, for \((-1)^F_S\) acts trivially in spacetime. The spectra of Type II and Type 0 string theories are as follows [7]:

\[
\begin{align*}
\text{IIB} & : (\text{NS}+, \text{NS}+) \quad (\text{R}+, \text{NS}+) \quad (\text{NS}+, \text{R}+) \quad (\text{R}+, \text{R}+), \\
\text{IIA} & : (\text{NS}+, \text{NS}+) \quad (\text{R}+, \text{NS}+) \quad (\text{NS}+, \text{R}+) \quad (\text{R}+, \text{R}+), \\
\text{0B} & : (\text{NS}+, \text{NS}+) \quad (\text{NS}+, \text{NS}+) \quad (\text{R}+, \text{R}+) \quad (\text{R}+, \text{R}+), \\
\text{0A} & : (\text{NS}+, \text{NS}+) \quad (\text{NS}+, \text{NS}+) \quad (\text{R}+, \text{R}+) \quad (\text{R}+, \text{R}+).
\end{align*}
\]

Of course, we can consider supersymmetric orbifolds only of Type II strings.

Although each spectrum of Type II and Type 0 string theories on a non-supersymmetric orbifold has tachyons, which cause vacuum instability, we expect that tachyon condensation scenarios rescue these from this difficulty [8]. Also for absence of fermions in Type 0 orbifolds, we anticipate that D-branes are the key to solve it, for there appear fermionic modes of open strings stretching between D-branes in a certain configuration [9]. Taking these into consideration, it is worthwhile to study non-supersymmetric orbifolds also.

For example, \(Z_2\) orbifolds of Type II and Type 0 strings have been discussed with addressing the relation between the D-brane spectra and the corresponding K-theories [4]. It is natural that one desire to analyze higher order orbifolds in the same manner. Therefore, in this paper, for the first step of the analysis, we compute and enumerate the perturbative tachyonic and massless spectra of non-supersymmetric \(T^2/Z_N\) and \(T^4/Z_N\) orbifolds of Type II and Type 0 strings. In doing that, compactifications are useful to give restrictions on the orders of orbifolds [3, 10]. We shall also enumerate the massless spectra of Type II string theories on a supersymmetric \(T^4/Z_N\) orbifold for comparison.

This paper is organized as follows. In section 2 we introduce the basic notation. We enumerate the tachyonic and massless spectra of Type II and Type 0 string theories on a \(T^2/Z_N\) orbifold in section 3, and those on a \(T^4/Z_N\) in section 4. In section 5 we extract some rules of the spectra from the results in section 4. Summary and discussion are given.
in the final section. We have included an appendix to transform the spectra in section 3 to those on a $T^2 \times T^2 / Z_N$ orbifold.

## 2 The basic notation

First, we define the complex linear combinations

$$Z^i = \frac{1}{\sqrt{2}} (X^{2i} + iX^{2i+1}),$$

$$\overline{Z}^i = \frac{1}{\sqrt{2}} (X^{2i} - iX^{2i+1}), \quad i = 2, 3, 4.$$ (2)

We can always choose the axes so that the rotation is of the form

$$\theta = \exp[2\pi i(v_2 J_{45} + v_3 J_{67} + v_4 J_{89})],$$ (3)

which acts on $Z^i$ as $\theta Z^i = e^{2\pi i v_i} Z^i$. Here, $v=(v_2, v_3, v_4)$, or $(0, v_2, v_3, v_4)$, denotes the twist vector corresponding to the generator $\theta$, and in this paper always $v_2 = 0$. The $v_i$ is of the form $k/N$, for $\theta^N=1$.

The twist vector $v$ of the generator $\theta$ has several restrictions. The restriction from geometry is that $\theta$ must act crystallographically on the torus lattice. All such twists have been obtained in [3, 10]. Modular invariance imposes the constraint

$$N \sum_i v_i = 0 \mod 2.$$ (4)

Supersymmetry gives the additional condition

$$\pm v_2 \pm v_3 \pm v_4 = 0 \mod 2.$$ (5)

We can obtain non-supersymmetric orbifolds by relaxing (5). Then, we can classify the orbifolds into two types: one does not include the $(0,0,1)=(-1)^F$ twist and the other does. Type II string theories on the former have fermionic states in the spectra. Even if the former satisfies condition (5), we can consider Type 0 string theories on that and also those are bosonic ones. For the former, the order of a Type 0 orbifold is the same as that of a Type II, as long as the generator is the same. For the latter, the orders depend on whether we consider Type II orbifold or Type 0 one; however, we can identify Type 0B/0A string theory on $Z_N$ orbifold with Type IIB/IIA one on $Z_{2N}$ orbifold, provided the generator is the same. This comes from Type 0 string theories being regarded as the Type II orbifolds with $(-1)^F$. In this paper, if a orbifold includes the $(-1)^F$ twist, we consider only Type 0 string theories, not Type II ones.
In section 3 and 4 we list the generators of $T^2/Z_N$ and $T^4/Z_N$ orbifolds, respectively. Then, following the way in the appendix of [6], we compute and enumerate the spectra of Type II and Type 0 string theories on those.

3 $T^2/Z_N$ orbifolds

In this section we concentrate our attention on $T^2/Z_N$ orbifolds of Type II and Type 0 strings. We obtain six twist vectors, which are listed in Table 1, compatible with the toroidal compactification $T^2$.

| $(v_2, v_3, v_4)$ | $Z_N$ (for Type II) | $Z_N$ (for Type 0) |
|------------------|----------------------|-------------------|
| (0,0,1)          | $Z_2$                | 1                 |
| (0,0,2/3)        | $Z_3$                | $Z_3$             |
| (0,0,1/2)        | $Z_4$                | $Z_2$             |
| (0,0,1/3)        | $Z_6$                | $Z_3$             |
| (0,0,1/4)        | $Z_8$                | $Z_4$             |
| (0,0,1/6)        | $Z_{12}$             | $Z_6$             |

Table 1: Twist vectors with one non-zero component.

As mentioned in the previous section, we identify $v=(0,0,1)$ with the spacetime fermion number operator $(-1)^F_S$, thus it is the identical operator for Type 0 string theories. We compute the tachyonic and massless spectra of Type II and Type 0 string theories on other $T^2/Z_N$ orbifolds under $SO(8) \rightarrow SO(6) \times SO(2)$, with $8_v=(6,0)+(1,1)+(1,-1)$ and $8_s=(4,-1/2)+(4,1/2)$. In each $T^2/Z_N$ orbifold, there always appear from the untwisted NS-NS sector $20+15+1$, which correspond to graviton, antisymmetric tensor and dilaton. Therefore we omit these states in the spectra. In addition, all the $T^2/Z_N$ orbifolds of Type II and Type 0 strings are non-supersymmetric, hence there always appear tachyons in the spectra. We represent those as $1_{m^2}$, where the subscript indicates the value of $m^2_R=m^2_L$.

As the first example, we consider Type II and Type 0 string theories on the $T^2/Z_3$ orbifold with $v=(0,0,2/3)$. Among $T^2/Z_N$ orbifolds, only this orbifold does not include the $(-1)^F_S=(0,0,1)$ twist. Therefore, fermionic states appear in the spectra of Type II string theories on this orbifold. In fact, we find the tachyonic and massless spectrum of Type IIB
string theory is

\[
\begin{align*}
\theta^0 & : (1 + 4 + 4 + 1 + 15) + (1 + 4 + 4 + 1 + 15), \\
\theta & : 3(1 - \frac{1}{3}) + 3(1 + 4 + 4 + 1 + 15), \\
\theta^2 & : 3(1 - \frac{1}{3}) + 3(1 + 4 + 4 + 1 + 15),
\end{align*}
\]

(6)

where \(\theta^0\) denotes the untwisted sector, each \(\theta^n \ (n \neq 0)\) does \(n\)th twisted sector. Similarly, we find that of Type IIA string theory is

\[
\begin{align*}
\theta^0 & : 2(1 + 4 + 4 + 6 + 10), \\
\theta & : 3(1 - \frac{1}{3}) + 3(1 + 4 + 4 + 6 + 10), \\
\theta^2 & : 3(1 - \frac{1}{3}) + 3(1 + 4 + 4 + 6 + 10).
\end{align*}
\]

(7)

As mentioned in section 2, we also consider Type 0 string theories on this orbifold. The tachyonic and massless spectrum of Type 0B string theory is

\[
\begin{align*}
\theta^0 & : 1 - \frac{1}{3} + 2(1) + 2(1 + 15 + 1 + 15), \\
\theta, \theta^2 & : 6(1 - \frac{1}{3} + 1 - \frac{1}{3} + 1 + 1) + 6(1 + 15 + 1 + 15).
\end{align*}
\]

(8)

And that of Type 0A string theory is

\[
\begin{align*}
\theta^0 & : 1 - \frac{1}{3} + 2(1) + 2(6 + 10 + 6 + 10), \\
\theta, \theta^2 & : 6(1 - \frac{1}{3} + 1 - \frac{1}{3} + 1 + 1) + 6(6 + 10 + 6 + 10).
\end{align*}
\]

(9)

As we expected, Type 0 string theories have doubled R-R states; \(1 + 15 + 1 + 15\) in Type 0B and \(6 + 10 + 6 + 10\) in Type 0A are twice as many as those in Type IIB and IIA respectively.

The other \(T^2/Z_N\) orbifolds include the \((-1)^F\) twist, therefore, we consider only Type 0 string theories. On the \(T^2/Z_2\) orbifold with \(v = (0, 0, \frac{1}{2})\), the tachyonic and massless spectrum of Type 0B string theory is

\[
\begin{align*}
\theta^0 & : 1 - \frac{1}{3} + 4(1) + 2(1 + 15 + 1 + 15), \\
\theta & : 8(1 - \frac{1}{3}) + 4(1 + 15 + 1 + 15),
\end{align*}
\]

(10)

and that of Type 0A string theory is

\[
\begin{align*}
\theta^0 & : 1 - \frac{1}{3} + 4(1) + 2(6 + 10 + 6 + 10), \\
\theta & : 8(1 - \frac{1}{3}) + 4(6 + 10 + 6 + 10).
\end{align*}
\]

(11)

The feature of these spectra is four scalars 1 in each untwisted sector; these are doubled compared to those of any other \(T^2/Z_N\) orbifolds of Type 0 strings. This results from \(\frac{1}{2}\) being half-integer.
Next, we consider Type 0 string theories on the $T^2/Z_3$ orbifold with $v=(0,0,\frac{1}{3})$. The tachyonic and massless spectrum of Type 0B string theory on this orbifold is

\[ \theta^0 : 1 - \frac{1}{3} + 2(1) + 2(1 + 15 + 1 + 15), \]
\[ \theta, \theta^2 : 6(1 - \frac{1}{3}^\prime + 1) + 6(1 + 15 + 1 + 15). \]  \(12\)

We also find that of Type 0A string theory is

\[ \theta^0 : 1 - \frac{1}{3} + 2(1) + 2(6 + 10 + 6 + 10), \]
\[ \theta, \theta^2 : 6(1 - \frac{1}{3} + 1 + 1) + 6(6 + 10 + 6 + 10). \]  \(13\)

We notice that these spectra are the same as those on the $T^2/Z_3$ orbifold with $v=(0,0,\frac{2}{3})$ of Type 0B and Type 0A strings, respectively. Therefore, we can identify the two $T^2/Z_3$ orbifolds of Type 0 strings, one with $v=(0,0,\frac{2}{3})$ and the other with $v=(0,0,\frac{1}{3})$, with each other at tachyonic and massless level at least.

The two following $T^2/Z_N$ orbifolds include a few distinct twists from those of generators. Let us take Type 0 string theories on the $T^2/Z_4$ orbifold with $v=(0,0,\frac{1}{4})$. We find the tachyonic and massless spectrum of Type 0B string theory on this orbifold is

\[ \theta^0 : 1 - \frac{1}{4} + 2(1) + 2(1 + 15 + 1 + 15), \]
\[ \theta, \theta^3 : 4(1 - \frac{1}{4}^\prime) + 8(1 - \frac{1}{4}) + 4(1 + 15 + 1 + 15), \]
\[ \theta^2 : 6(1 - \frac{1}{4}^\prime) + 3(1 + 15 + 1 + 15). \]  \(14\)

We notice that the number ratio of the states in the $\theta^2$ sector is the same as that in the $\theta$ sector of the $T^2/Z_2$ orbifold, for two twisted sectors have the same twist vector $(0,0,\frac{1}{2})$. However, the numbers of the states, the degeneracy of the states, are different. This difference depends on whether the twist vector $(0,0,\frac{1}{2})$ corresponds to the generator or the subgroup. We can also notice the same relation in Type 0A string theory on this orbifold: we find the tachyonic and massless spectrum of it is

\[ \theta^0 : 1 - \frac{1}{4} + 2(1) + 2(6 + 10 + 6 + 10), \]
\[ \theta, \theta^3 : 4(1 - \frac{1}{4}^\prime) + 8(1 - \frac{1}{4}) + 4(6 + 10 + 6 + 10), \]
\[ \theta^2 : 6(1 - \frac{1}{4}^\prime) + 3(6 + 10 + 6 + 10). \]  \(15\)

As the final $T^2/Z_N$ orbifold, we consider Type 0 string theories on the $T^2/Z_6$ orbifold with $v=(0,0,\frac{1}{6})$. We find the tachyonic and massless spectrum of Type 0B string theory on this orbifold is

\[ \theta^0 : 1 - \frac{1}{6} + 2(1) + 2(1 + 15 + 1 + 15), \]
\[ \theta, \theta^5 : 2(1 - \frac{1}{6}^\prime) + 2(1 - \frac{1}{6}) + 4(1 - \frac{1}{6}^\prime) + 2(1 + 15 + 1 + 15), \]
\[ \theta^2, \theta^4 : 4(1 - \frac{1}{6} + 1 - \frac{1}{6} + 1) + 4(1 + 15 + 1 + 15), \]
\[ \theta^3 : 4(1 - \frac{1}{6} + 2(1 + 15 + 1 + 15). \]  \(16\)
In this case, the number ratio of the states in the $\theta^2+\theta^4$ sector is the same as that in the $\theta$ sector of the $T^2/Z_3$ orbifold with $v=(0,0,\frac{1}{3})$. In addition, the ratio in the $\theta^3$ sector is the same as that in the $\theta$ sector of the $T^2/Z_2$ orbifold with $v=(0,0,\frac{1}{2})$. Of course, these relations are also true for Type 0A string theory on this orbifold. The tachyonic and massless spectrum of it is

$$\begin{align*}
\theta^0 & : \quad 1_{-\frac{1}{2}} + 2(1) + 2(6 + 10 + 6 + 10), \\
\theta, \theta^5 & : \quad 2(1_{-\frac{1}{2}} + 1_{-\frac{1}{2}}) + 4(1_{-\frac{1}{2}} + 1_{-\frac{1}{2}}) + 2(6 + 10 + 6 + 10), \\
\theta^2, \theta^4 & : \quad 4(1_{-\frac{1}{4}} + 1_{-\frac{1}{4}} + 1) + 4(6 + 10 + 6 + 10), \\
\theta^3 & : \quad 4(1_{-\frac{1}{4}} + 1_{-\frac{1}{4}}) + 2(6 + 10 + 6 + 10). 
\end{align*}$$

As we have seen, in the $T^2/Z_N$ orbifolds of Type 0 strings, the number ratios of the states are the same among the twisted sectors with the same twist vector. In the next section we shall observe that the ratios are not preserved among several twisted sectors of Type II and Type 0 string theories on $T^4/Z_N$ orbifolds, even if those have the same twist vector. We transform the spectra obtained in this section into $T^2\times T^2/Z_N$ orbifolds version in the appendix in order to compare these with those.

4 \quad T^4/Z_N orbifolds

This section consists of two subsections. We focus on $T^4/Z_N$ orbifolds not including the $(-1)^F$ twist and those doing in subsection 4.1 and 4.2, respectively.

We compute the spectra of Type II and Type 0 string theories on a $T^4/Z_N$ orbifold under $SO(4) \simeq SU(2)\times SU(2)$ besides $SO(8) \rightarrow SO(4)\times SO(4)$. Therefore, the vector $(\pm 1, 0)$, the spinor $(\frac{1}{2}, -\frac{1}{2})$, and the spinor $\pm(\frac{1}{2}, \frac{1}{2})$ become the $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations respectively, where underlines mean permutations. In addition, we represent tachyons as $(0, 0)_{m^2}$, where the subscript indicates the value of $m^2_R/m^2_L$ as in section 3.

4.1 \quad Orbifolds not including the $(-1)^F$ twist

In this subsection we focus on orbifolds not including the $(-1)^F$ twist, listed in Table 2. As mentioned in section 2, the order of a Type 0 orbifold is the same as that of a Type II, as long as the generator is the same.

As well known, the massless spectrum of a supersymmetric Type II orbifold forms some supersymmetric multiplets. On any supersymmetric $T^4/Z_N$ orbifolds, the states in
Table 2: twist vectors not including the $(-1)^F_S$ twist with two non-zero components. We take absolute value of two components of a twist vector. The star $*$ on a $Z_N$ means that Type II string theories on that orbifold are supersymmetric.

| $(v_2, v_3, v_4)$ | $Z_N$ | $(v_2, v_3, v_4)$ | $Z_N$ | $(v_2, v_3, v_4)$ | $Z_N$ |
|-------------------|-------|-------------------|-------|-------------------|-------|
| $(0, rac{1}{2}, rac{1}{2})$ | $Z_2^*$ | $(0, \frac{1}{5}, \frac{3}{5})$ | $Z_5^*$ | $(0, \frac{1}{8}, \frac{7}{8})$ | $Z_8$ |
| $(0, \frac{1}{3}, \frac{1}{3})$ | $Z_3^*$ | $(0, \frac{1}{6}, \frac{1}{6})$ | $Z_6^*$ | $(0, \frac{1}{12}, \frac{5}{12})$ | $Z_{10}$ |
| $(0, \frac{1}{4}, \frac{1}{4})$ | $Z_4^*$ | $(0, \frac{1}{6}, \frac{5}{6})$ | $Z_6$ | $(0, \frac{1}{12}, \frac{7}{12})$ | $Z_{12}$ |
| $(0, \frac{3}{4}, \frac{3}{4})$ | $Z_4$ | $(0, \frac{1}{6}, \frac{5}{6})$ | $Z_6$ | $(0, \frac{1}{12}, \frac{7}{12})$ | $Z_{12}$ |

The untwisted sector of Type IIB string theory form one supergravity multiplet

$$ (1, 1) + 4(1, \frac{1}{2}) + 5(0, 0) $$

and some tensor multiplets

$$ (0, 1) + 4(0, \frac{1}{2}) + 5(0, 0). $$

Several tensor multiplets (19) also appear in the twisted sectors: thus we regard the number ratio of the states in (19) as the standard one in the twisted sectors with a supersymmetric twist vector of Type IIB string theory on a $T^4/Z_N$ orbifold.

Similarly, the states in the untwisted sector of Type IIA string theory on a $T^4/Z_N$ supersymmetric orbifold form one supergravity multiplet

$$ (1, 1) + (1, 0) + (0, 1) + (0, 0) + 4(\frac{1}{2}, \frac{1}{2}) + 2(1, \frac{1}{2}) + 2(\frac{1}{2}, 1) + 2(0, \frac{1}{2}) + 2(\frac{1}{2}, 0) $$

and some vector multiplets

$$ (\frac{1}{2}, \frac{1}{2}) + 2(0, \frac{1}{2}) + 2(\frac{1}{2}, 0) + 4(0, 0), $$

and the states in the twisted sectors form several vector multiplets (21). We also regard the number ratio of the states in (21) as the standard one in the twisted sectors with a supersymmetric twist vector of Type IIA string theory on a $T^4/Z_N$ orbifold.

As the first example of $T^4/Z_N$ orbifolds, let us take the $T^4/Z_2$ orbifold with $v=(0, \frac{1}{2}, -\frac{1}{2})$. As well known, Type II string theories on this orbifold are supersymmetric, therefore we find the supersymmetric massless spectrum of Type IIB string theory on this orbifold is

$$ \theta^0 : (1 \text{ supergravity multiplet}) + (5 \text{ tensor multiplets}), $$

$$ \theta : 16 \text{ tensor multiplets}, $$

(22)
and that of Type IIA string theory is
\begin{align}
\theta^0 & : \text{(1 supergravity multiplet)+(4 vector multiplets)}, \\
\theta & : \text{16 vector multiplets.} \tag{23}
\end{align}

In addition, we can also consider Type 0 string theories on this orbifold. In General, the untwisted sectors of Type 0 string theories on a $T^4/Z_N$ orbifold always include
\begin{equation}
(0,0)_{-\frac{1}{2}} + (1,1) + (1,0) + (0,1) + (0,0): \tag{24}
\end{equation}
thus we omit these states except for the tachyon $(0,0)_{-\frac{1}{2}}$ from their spectra. That is, we omit graviton, antisymmetric tensor and dilaton. The remaining tachyonic and massless states of Type 0B string theory on this $T^4/Z_2$ orbifold are
\begin{align}
\theta^0 & : \text{(0,0)$_{-\frac{1}{2}}$} + 8(1,0) + 8(0,1) + 32(0,0), \\
\theta & : 16(1,0) + 16(0,1) + 160(0,0), \tag{25}
\end{align}
and those of Type 0A string theory are
\begin{align}
\theta^0 & : \text{(0,0)$_{-\frac{1}{2}}$} + 16(\frac{1}{2},\frac{1}{2}) + 16(0,0), \\
\theta & : 32(\frac{1}{2},\frac{1}{2}) + 128(0,0). \tag{26}
\end{align}
We regard the number ratio of the states in the $\theta$ sector of Type 0B/0A string theory as the standard one in the twisted sectors with the twist vector $(0,\frac{1}{2},-\frac{1}{2})$ of Type 0B/0A.

Next, we consider Type II and Type 0 string theories on the $T^4/Z_3$ orbifold with $v=(0,\frac{1}{3},-\frac{1}{3})$. As on the previous orbifold, Type II string theories on it are supersymmetric. Hence, we find the supersymmetric massless spectrum of Type IIB string theory is
\begin{align}
\theta^0 & : \text{(1 supergravity multiplet)+(3 tensor multiplets)}, \\
\theta, \theta^2 & : \text{18 tensor multiplets.} \tag{27}
\end{align}
We notice the total of multiplets on this $T^4/Z_3$ orbifold is the same as that on the previous $T^4/Z_2$ orbifold. The same thing also appear in Type IIA string theory as follows:
\begin{align}
\theta^0 & : \text{(1 supergravity multiplet)+(2 vector multiplets)}, \\
\theta, \theta^2 & : \text{18 vector multiplets.} \tag{28}
\end{align}

In Type 0 string theories, several tachyons appear in both the untwisted sectors and the twisted sectors. The tachyonic and massless spectrum of Type 0B string theory on this $T^4/Z_3$ orbifold is
\begin{align}
\theta^0 & : \text{(0,0)$_{-\frac{1}{2}}$} + 6(1,0) + 6(0,1) + 20(0,0), \\
\theta, \theta^2 & : 18(0,0)$_{-\frac{1}{6}}$ + 18(1,0) + 18(0,1) + 108(0,0). \tag{29}
\end{align}
Similarly, we find that of Type 0A string theory is
\[
\begin{align*}
\theta^0 & : (0, 0) - \frac{1}{4} + 12(\frac{1}{2}, \frac{1}{2}) + 8(0, 0), \\
\theta, \theta^2 & : 18(0, 0) - \frac{1}{6} + 36(\frac{1}{2}, \frac{1}{2}) + 72(0, 0).
\end{align*}
\] (30)

We regard the number ratio of the states, including the tachyons, in the \(\theta + \theta^2\) sector of Type 0B/0A string theory as the standard one in the twisted sectors with the twist vector \((0, \frac{1}{3}, -\frac{1}{3})\) of Type 0B/0A.

The \(T^4/Z_4\) orbifold with \(v=(0, \frac{1}{4}, -\frac{1}{4})\) is the first example of \(T^4/Z_N\) orbifolds which have distinct twists from that of the generator. In fact, we find the supersymmetric massless spectrum of Type IIB string theory on this orbifold is
\[
\begin{align*}
\theta^0 & : 1\text{ supergravity multiplet} + (3\text{ tensor multiplets}), \\
\theta, \theta^3 & : 8\text{ tensor multiplets}, \\
\theta^2 & : 10\text{ tensor multiplets}.
\end{align*}
\] (31)

Although this orbifold has two different twists, the total of supersymmetric multiplets on this orbifold is the same as that on a previous supersymmetric orbifolds. The same thing appear in the supersymmetric massless spectrum of Type IIA string theory on this orbifold as follows:
\[
\begin{align*}
\theta^0 & : 1\text{ supergravity multiplet} + (2\text{ vector multiplets}), \\
\theta, \theta^3 & : 8\text{ vector multiplets}, \\
\theta^2 & : 10\text{ vector multiplets}.
\end{align*}
\] (32)

We also consider Type 0 string theories on this \(T^4/Z_4\) orbifold. The tachyonic and massless spectrum of Type 0B string theory is
\[
\begin{align*}
\theta^0 & : (0, 0) - \frac{1}{4} + 6(1, 0) + 6(0, 1) + 20(0, 0), \\
\theta, \theta^3 & : 8(0, 0) - \frac{1}{4} + 8(1, 0) + 8(0, 1) + 80(0, 0), \\
\theta^2 & : 10(1, 0) + 10(0, 1) + 100(0, 0).
\end{align*}
\] (33)

We notice easily that the number ratio of the states in the \(\theta^2\) sector is the same as the standard one, i.e. that in the \(\theta\) sector of the \(T^4/Z_2\) orbifold with \(v=(0, \frac{1}{2}, -\frac{1}{2})\). We can also find the same thing in the tachyonic and massless spectrum of Type 0A string theory as follows:
\[
\begin{align*}
\theta^0 & : (0, 0) - \frac{1}{4} + 12(\frac{1}{2}, \frac{1}{2}) + 8(0, 0), \\
\theta, \theta^3 & : 8(0, 0) - \frac{1}{4} + 16(\frac{1}{2}, \frac{1}{2}) + 64(0, 0), \\
\theta^2 & : 20(\frac{1}{2}, \frac{1}{2}) + 80(0, 0).
\end{align*}
\] (34)
We regard the number ratio of the states in the $\theta + \theta^3$ sector of Type 0B/0A string theory as the standard one in the twisted sectors with the twist vector $(0, \frac{1}{4}, -\frac{1}{4})$ of Type 0B/0A.

The $T^4/Z_4$ orbifold with $v=(0, \frac{1}{4}, \frac{3}{4})$ is the first example of non-supersymmetric $T^4/Z_N$ orbifolds. Therefore, the untwisted sectors of Type II string theories on this orbifold cannot form a supergravity multiplet. However, there always appear in the untwisted sectors

$$(1, 1) + (1, 0) + (0, 1) + (0, 0),$$

which we omit also in the spectra of Type II string theories on other non-supersymmetric $T^4/Z_4$ orbifolds. And also, tachyons appear in the spectra, even if we consider Type II string theories. In fact, the tachyonic and massless spectrum of Type IIB string theory on this $T^4/Z_4$ orbifold is

$$\begin{align*}
\theta^0 & : 41(1, 0) + 2(0, 1) + 8(0, \frac{1}{2}) + 14(0, 0), \\
\theta, \theta^3 & : 8(0, 0) + 8(1, 0) + 8(\frac{1}{2}, 0) + 40(0, 0), \\
\theta^2 & : 10(0, 1) + 24(0, \frac{1}{2}) + 50(0, 0).
\end{align*}$$

As indicated in [6], there appear 8 tensor multiplets and extra 8 tachyons in the $\theta + \theta^3$ sector. We also notice that the $\theta^2$ sector wants 16$(0, \frac{1}{2})$ to fill the 10 tensor multiplets. Similarly, states appearing and disappearing are found in the tachyonic and massless spectrum of Type IIA string theory on this orbifold as follows:

$$\begin{align*}
\theta^0 & : 6(\frac{1}{2}, \frac{1}{2}) + 4(\frac{1}{2}, 0) + 4(0, \frac{1}{2}) + 8(0, 0), \\
\theta, \theta^3 & : 8(0, 0) + 8(1, 0) + 8(\frac{1}{2}, 0) + 16(0, \frac{1}{2}) + 32(0, 0), \\
\theta^2 & : 10(0, 1) + 24(0, \frac{1}{2}) + 50(0, 0).
\end{align*}$$

The $\theta^2$ sector wants $8(\frac{1}{2}, 0) + 8(0, \frac{1}{2})$ to fill the 10 vector multiplets. We regard the number ratio of the states in the $\theta + \theta^3$ sector of Type IIB/IIA string theory as the standard one in the twisted sectors with the twist vector $(0, \frac{1}{4}, \frac{3}{4})$ of Type IIB/IIA.

In addition, we consider Type 0 string theories on this $T^4/Z_4$ orbifold. The tachyonic and massless spectrum of Type 0B string theory on this orbifold is

$$\begin{align*}
\theta^0 & : (0, 0) + 6(1, 0) + 6(0, 1) + 20(0, 0), \\
\theta, \theta^3 & : 8(0, 0) + 8(1, 0) + 8(0, 1) + 80(0, 0), \\
\theta^2 & : 10(1, 0) + 10(0, 1) + 100(0, 0).
\end{align*}$$

And also, that of Type 0A string theory is

$$\begin{align*}
\theta^0 & : (0, 0) + 12(\frac{1}{2}, 0) + 8(0, 0), \\
\theta, \theta^3 & : 8(0, 0) + 16(\frac{1}{2}, 0) + 64(0, 0), \\
\theta^2 & : 20(\frac{1}{2}, 0) + 80(0, 0).
\end{align*}$$
While we notice that the tachyonic and massless spectrum of Type II B/IIA string theory on the $T^4/Z_4$ orbifold with $v=(0,\frac{1}{4},\frac{3}{4})$ is different from that with $v=(0,\frac{1}{4},-\frac{1}{4})$, we obtain the same spectrum of Type 0B/0A string theory on each $T^4/Z_4$ orbifold. Therefore the $T^4/Z_4$ orbifold with $v=(0,\frac{1}{4},\frac{3}{4})$ is equivalent to that with $v=(0,\frac{1}{4},-\frac{1}{4})$ in Type 0 string theories at tachyonic and massless level at least. On a parallel with Type II string theories, we regard the number ratio of the states in the $\theta+\theta^3$ sector of Type 0B/0A string theory as the standard one in the twisted sectors with the twist vector $(0,\frac{1}{4},\frac{3}{4})$ of Type 0B/0A.

The orbifold with $v=(0,\frac{1}{5},\frac{2}{5})$ is a non-supersymmetric $Z_5$ orbifold. As indicated in [6], the tachyonic and massless spectrum of Type IIB string theory on this orbifold is

\[
\begin{align*}
\theta^0 & : 2(1,0) + 2(0,1) + 4(0,\frac{1}{2}) + 4(\frac{1}{2},0) + 8(0,0), \\
\theta, \theta^4 & : 10(0,0) + 10(1,0) + 20(1,\frac{1}{2},0) + 20(0,0), \\
\theta^2, \theta^3 & : 10(0,0) + 10(0,1) + 20(0,\frac{1}{2}) + 20(0,0),
\end{align*}
\]

and that of Type IIA string theory is

\[
\begin{align*}
\theta^0 & : 4(\frac{1}{2},\frac{1}{2}) + 4(\frac{1}{2},0) + 4(0,\frac{1}{2}) + 4(0,0), \\
\theta, \theta^4 & : 10(0,0) + 10(1,0) + 10(\frac{1}{2},0) + 10(0,\frac{1}{2}) + 10(0,0), \\
\theta^2, \theta^3 & : 10(0,0) + 10(0,1) + 10(0,\frac{1}{2}) + 10(0,0).
\end{align*}
\]

In addition to Type II string theories, we consider Type 0 string theories. We find the tachyonic and massless spectrum of Type 0B string theory on this $T^4/Z_5$ orbifold is

\[
\begin{align*}
\theta^0 & : (0,0) + 4(1,0) + 4(0,1) + 12(0,0), \\
\theta, \theta^4 & : 10(0,0) + 10(0,0) + 10(1,0) + 10(0,1) + 30(0,0), \\
\theta^2, \theta^3 & : 10(0,0) + 10(0,0) + 10(0,1) + 10(0,1) + 30(0,0),
\end{align*}
\]

and that of Type 0A string theory is

\[
\begin{align*}
\theta^0 & : (0,0) + 8(\frac{1}{2},\frac{1}{2}) + 4(0,0), \\
\theta, \theta^4 & : 10(0,0) + 10(0,0) + 20(\frac{1}{2},\frac{1}{2}) + 10(0,0), \\
\theta^2, \theta^3 & : 10(0,0) + 10(0,0) + 20(\frac{1}{2},\frac{1}{2}) + 10(0,0).
\end{align*}
\]

The discrete rotation $Z_5$ has no subgroup, therefore, in each string theory, we can regard the number ratio of the states in each twisted sector as the standard one.

The $T^4/Z_6$ orbifold with $v=(0,\frac{1}{6},-\frac{1}{6})$ has the most highest order of all the supersymmetric $T^4/Z_N$ orbifolds. Thus, we find the supersymmetric massless spectrum of Type
IIB string theory on this orbifold is

\[ \begin{align*}
\theta^0 : & \text{(1 supergravity multiplet)+3 tensor multiplets),} \\
\theta, \theta^5 : & \text{2 tensor multiplets,} \\
\theta^2, \theta^4 : & \text{10 tensor multiplets,} \\
\theta^3 : & \text{6 tensor multiplets.} \\
\end{align*} \tag{44} \]

Although this orbifold has the three different twists, the total of the supersymmetric multiplets, one supergravity multiplet and 21 tensor multiplets, is the same as that on other supersymmetric \( T^4/Z_N \) orbifolds. Similar thing also appear in that of Type IIA string theory as follows:

\[ \begin{align*}
\theta^0 : & \text{(1 supergravity multiplet)+2 vector multiplets),} \\
\theta, \theta^5 : & \text{2 vector multiplets,} \\
\theta^2, \theta^4 : & \text{10 vector multiplets,} \\
\theta^3 : & \text{6 vector multiplets.} \\
\end{align*} \tag{45} \]

That is, we find again single supergravity multiplet and 20 vector multiplets.

We also consider Type 0 string theories. The tachyonic and massless spectrum of Type 0B string theory on this \( T^4/Z_6 \) orbifold is

\[ \begin{align*}
\theta^0 : & (0,0,\omega,\omega^0) + 6(1,0) + 6(0,1) + 20(0,0), \\
\theta, \theta^5 : & 2(0,0,\omega,\omega^0) + 8(0,0,\omega^0,\omega) + 2(1,0) + 2(0,1) + 30(0,0), \\
\theta^2, \theta^4 : & 10(0,0,\omega,\omega^0) + 10(1,0) + 10(0,1) + 60(0,0), \\
\theta^3 : & 6(1,0,\omega,\omega^0) + 6(0,1,\omega,\omega^0) + 60(0,0). \\
\end{align*} \tag{46} \]

We notice the number ratio of the states in the \( \theta^2+\theta^4 \) sector is the standard one, i.e. that in the \( \theta \) sector of the \( T^4/Z_3 \) orbifold with \( v=(0,\frac{1}{3},-\frac{1}{3}) \). And also, we notice that in the \( \theta^3 \) sector is the standard one, i.e. that in the \( \theta \) sector of the \( T^4/Z_2 \) orbifold with \( v=(0,\frac{1}{2},-\frac{1}{2}) \). Of course, we can confirm this preservation of the ratios also in Type 0A string theory as follows: the tachyonic and massless spectrum of Type 0A string theory on this orbifold is

\[ \begin{align*}
\theta^0 : & (0,0,\omega,\omega^0) + 12(1,\frac{1}{2},\omega^0,\omega) + 8(0,0), \\
\theta, \theta^5 : & 2(0,0,\omega,\omega^0,\omega) + 8(0,0,\omega^0,\omega) + 4(1,\frac{1}{2},\omega) + 26(0,0), \\
\theta^2, \theta^4 : & 10(0,0,\omega,\omega^0,\omega) + 20(1,\frac{1}{2},\omega^0,\omega) + 40(0,0), \\
\theta^3 : & 12(1,\frac{1}{2},\omega^0,\omega) + 48(0,0). \\
\end{align*} \tag{47} \]

While we find the same total massless spectrum of Type IIB/IIA string theory on any supersymmetric \( T^4/Z_N \) orbifolds, we find different that of Type 0B/0A string theory on
each supersymmetric $T^4/Z_N$ orbifold. This results from the number of the NS-NS states changing, therefore the total number of the R-R states is constant on each supersymmetric orbifold. Of course, the spectra of Type 0 string theories on a supersymmetric orbifold are not supersymmetric, but it means the twist vector $v$ corresponding to the generator of an orbifold satisfies the condition (5).

There is no supersymmetric orbifold in the following; however, this orbifold seems the most interesting of all the orbifolds treated in this paper. That is the $T^4/Z_6$ orbifold with $v=(0,\frac{1}{6},\frac{2}{6})=(0,\frac{1}{6},\frac{1}{2})$. We find the tachyonic and massless spectrum of Type IIB string theory on this orbifold is

$$\begin{align*}
\theta^0 & : \quad 2(1,0) + 2(0,1) + 4(0,\frac{1}{2}) + 10(0,0), \\
\theta, \theta^5 & : \quad 8(0,0) - \frac{1}{6} + 8(1,0) + 16(\frac{1}{2},0) + 16(0,0), \\
\theta^2, \theta^4 & : \quad 4(0,0) - \frac{1}{4} + 4(1,0) + 4(0,1) + 4(\frac{1}{2},\frac{1}{2}) + 4(\frac{1}{2},0) + 8(0,\frac{1}{2}) + 12(0,0), \\
\theta^3 & : \quad 8(0,1) + 16(0,\frac{1}{2}) + 32(0,0).
\end{align*}$$

(48)

By comparison with the standard ratio, i.e. the $\theta$ sector of the $T^2 \times T^2/Z_3$ orbifold with $v=(0,0,\frac{2}{3})$, the $\theta^2+\theta^4$ sector lacks $4(\frac{1}{2},\frac{1}{2})+4(\frac{1}{2},0)$. We should remark that disappearing $4(\frac{1}{2},\frac{1}{2})$ come from the R-R sectors. This suggests that the corresponding fractional D-branes are also absent. Although $3v$ is supersymmetric, the $\theta^3$ sector lacks $16(0,\frac{1}{2})+8(0,0)$ to fill 8 tensor multiplets. Similarly, we find the tachyonic and massless spectrum of Type IIA string theory on this orbifold is

$$\begin{align*}
\theta^0 & : \quad 4(\frac{1}{2},\frac{1}{2}) + 2(\frac{1}{2},0) + 2(0,\frac{1}{2}) + 6(0,0), \\
\theta, \theta^5 & : \quad 8(0,0) - \frac{1}{6} + 8(\frac{1}{2},\frac{1}{2}) + 8(\frac{1}{2},0) + 8(0,1) + 8(0,0), \\
\theta^2, \theta^4 & : \quad 4(0,0) - \frac{1}{4} + 2(1,0) + 2(0,1) + 8(\frac{1}{2},\frac{1}{2}) + 6(\frac{1}{2},0) + 6(0,\frac{1}{2}) + 8(0,0), \\
\theta^3 & : \quad 8(\frac{1}{2},\frac{1}{2}) + 8(\frac{1}{2},0) + 8(0,\frac{1}{2}) + 24(0,0).
\end{align*}$$

(49)

As well as Type IIB string theory, states are fewer than the standard. In the $\theta^2+\theta^4$ sector, $2(1,0)+2(0,1)+2(\frac{1}{2},0)+2(0,\frac{1}{2})+4(0,0)$ disappear from the spectrum compared with the $\theta$ sector of the $T^2 \times T^2/Z_3$ orbifold with $v=(0,0,\frac{2}{3})$. $2(1,0)+2(0,1)+4(0,0)$ out of these states come from the R-R sectors, thus we expect the corresponding fractional D-branes [14] also disappear. In the $\theta^3$ sector, $8(\frac{1}{2},0)+8(0,\frac{1}{2})+8(0,0)$ more are necessary to fill 8 vector multiplets.

The R-R states are bosonic states, therefore, we expect those also disappear in the spectra of Type 0 string theories. In fact, the tachyonic and massless spectrum of Type
0B string theory on this $T^4/Z_6$ orbifold is

\[
\begin{align*}
\theta^0 & : (0,0)_{-\frac{1}{2}} + 4(1,0) + 4(0,1) + 14(0,0), \\
\theta, \theta^5 & : 16(0,0)_{-\frac{1}{6}} + 8(1,0) + 8(0,1) + 32(0,0), \\
\theta^2, \theta^4 & : 4(0,0)_{-\frac{1}{6}} + 4(0,0)_{-\frac{1}{3}} + 8(1,0) + 8(0,1) + 8(\frac{1}{2}, \frac{1}{2}) + 20(0,0), \\
\theta^3 & : 8(1,0) + 8(0,1) + 72(0,0).
\end{align*}
\] (50)

And also, that of Type 0A string theory is

\[
\begin{align*}
\theta^0 & : (0,0)_{-\frac{1}{2}} + 8(\frac{1}{2}, \frac{1}{2}) + 6(0,0), \\
\theta, \theta^5 & : 16(0,0)_{+\frac{1}{6}} + 16(\frac{1}{2}, \frac{1}{2}) + 16(0,0), \\
\theta^2, \theta^4 & : 4(0,0)_{-\frac{1}{3}} + 4(0,0)_{-\frac{1}{6}} + 4(1,0) + 4(0,1) + 16(\frac{1}{2}, \frac{1}{2}) + 12(0,0), \\
\theta^3 & : 16(\frac{1}{2}, \frac{1}{2}) + 56(0,0).
\end{align*}
\] (51)

We can confirm that doubled R-R states as compared with Type IIB/IIA string theory disappear in the $\theta^2+\theta^4$ sector of Type 0B/0A string theory. That is, $8(\frac{1}{2}, \frac{1}{2})$ disappear in Type 0B string theory, $4(1,0)+4(0,1)+8(0,0)$ in Type 0A. The $\theta^3$ sector of both Type 0 string theories has 8 fewer $(0,0)$ than the standard i.e. the $\theta$ sector of the $T^4/Z_2$ orbifold with $v=(0,\frac{1}{6},-\frac{1}{6})$.

We consider the $T^4/Z_6$ orbifold with $v=(0,\frac{1}{6},\frac{5}{6})$. We find the tachyonic and massless spectrum of Type IIB string theory on this $T^4/Z_6$ orbifold is

\[
\begin{align*}
\theta^0 & : 4(1,0) + 2(0,1) + 14(0,0), \\
\theta, \theta^5 & : 2(0,0)_{-\frac{1}{3}} + 8(0,0)_{-\frac{1}{6}} + 2(1,0) + 12(\frac{1}{2}, 0) + 20(0,0), \\
\theta^2, \theta^4 & : 10(0,1) + 32(0, \frac{1}{2}) + 50(0,0), \\
\theta^3 & : 6(1,0) + 20(\frac{1}{2}, 0) + 28(0,0).
\end{align*}
\] (52)

In spite of $2v$ and $3v$ being supersymmetric, the $\theta^2+\theta^4$ and $\theta^3$ sectors do not form the tensor multiplets. The $\theta^2+\theta^4$ and $\theta^3$ sectors want $8(0, \frac{1}{2})$ and $4(\frac{1}{2}, 0)+2(0,0)$ to fill the tensor multiplets, respectively. And also, we find the tachyonic and massless spectrum of Type IIA string theory is

\[
\begin{align*}
\theta^0 & : 6(\frac{1}{2}, \frac{1}{2}) + 8(0,0), \\
\theta, \theta^5 & : 2(0,0)_{-\frac{1}{3}} + 8(0,0)_{-\frac{1}{6}} + 2(\frac{1}{2}, \frac{1}{2}) + 6(\frac{1}{2}, 0) + 6(0, \frac{1}{2}) + 18(0,0), \\
\theta^2, \theta^4 & : 10(\frac{1}{2}, \frac{1}{2}) + 16(\frac{1}{2}, 0) + 16(0, \frac{1}{2}) + 40(0,0), \\
\theta^3 & : 6(\frac{1}{2}, \frac{1}{2}) + 10(\frac{1}{2}, 0) + 10(0, \frac{1}{2}) + 22(0,0).
\end{align*}
\] (53)

Similar to Type IIB string theory, in order to fill the vector multiplets, the $\theta^2+\theta^4$ sector wants $4(\frac{1}{2}, 0)+4(0, \frac{1}{2})$ and the $\theta^3$ sector does $2(\frac{1}{2}, 0)+2(0, \frac{1}{2})+2(0,0)$. 

14
That the twisted sector with a supersymmetric twist vector does not form super-
symmetric multiplets is the same as the $\theta^2$ sector of the $T^4/Z_4$ orbifold with $v=(0,\frac{1}{4},\frac{3}{4})$. However, there is a different thing in the $\theta^3$ sector of each Type II string theory on this $T^4/Z_6$ orbifold. That is, 2 fewer $(0,0)$ coming from the NS-NS sector. Since the NS-NS sector is the bosonic one, we can anticipate the 2$(0,0)$ also disappear in the spectra of Type 0 string theories. Then, we shall confirm that as follows. The tachyonic and massless spectrum of Type 0B string theory on this $T^4/Z_6$ orbifold is

$$\begin{align*}
\theta^0 & : (0,0) - \frac{1}{2} + 6(1,0) + 6(0,1) + 20(0,0), \\
\theta, \theta^2 & : 2(0,0) - \frac{1}{5} + 8(0,0) - \frac{1}{6} + 2(1,0) + 2(0,1) + 30(0,0), \\
\theta^2, \theta^4 & : 10(0,0) - \frac{1}{2} + 10(1,0) + 10(0,1) + 60(0,0), \\
\theta^3 & : 6(1,0) + 6(0,1) + 58(0,0),
\end{align*}$$

and that of Type 0A string theory is

$$\begin{align*}
\theta^0 & : (0,0) - \frac{1}{2} + 12(\frac{1}{2}, \frac{1}{2}) + 8(0,0), \\
\theta, \theta^5 & : 2(0,0) - \frac{1}{3} + 8(0,0) - \frac{1}{6} + 4(\frac{1}{2}, \frac{1}{2}) + 26(0,0), \\
\theta^2, \theta^4 & : 10(0,0) - \frac{1}{3} + 20(\frac{1}{2}, \frac{1}{2}) + 40(0,0), \\
\theta^3 & : 12(\frac{1}{2}, \frac{1}{2}) + 46(0,0).
\end{align*}$$

We notice that the $\theta^3$ sector of Type 0B/0A string theory wants 2$(0,0)$ from the NS-NS sector as we expected. Therefore, we cannot identify two $T^4/Z_6$ orbifolds with each other: one with $v=(0,\frac{1}{6}, \frac{5}{6})$ and the other with $v=(0,\frac{1}{6}, -\frac{1}{6})$. This makes a contrast with the relation between two $T^4/Z_4$ orbifolds: one with $v=(0,\frac{1}{4}, \frac{3}{4})$ and the other with $v=(0,\frac{1}{4}, -\frac{1}{4})$.

We consider the $T^4/Z_8$ orbifold with $v=(0,\frac{1}{8}, \frac{3}{8})$. The tachyonic and massless spectrum of Type IIB string theory on this orbifold is

$$\begin{align*}
\theta^0 & : 2(1,0) + 2(0,1) + 4(0, \frac{1}{2}) + 8(0,0), \\
\theta, \theta^7 & : 4(0,0) - \frac{1}{8} + 4(1,0) + 8(\frac{1}{2}, 0) + 8(0,0), \\
\theta^2, \theta^6 & : 6(0,0) - \frac{1}{8} + 6(1,0) + 30(0,0), \\
\theta^3, \theta^5 & : 4(0,0) - \frac{1}{4} + 4(0,0) - \frac{1}{4} + 4(0,1) + 8(0, \frac{1}{2}) + 8(0,0), \\
\theta^4 & : 6(0,1) + 12(0, \frac{1}{2}) + 26(0,0).
\end{align*}$$

By comparison with the standard, i.e. the $\theta$ sector of the $T^4/Z_4$ orbifold with $v=(0,\frac{1}{4}, \frac{3}{4})$, the $\theta^2+\theta^6$ sector lacks 24$(\frac{1}{2},0)$. In addition, we notice that this sector has only bosonic states in spite of Type IIB string theory. Furthermore the $\theta^4$ sector lacks 12$(0,\frac{1}{2})+4(0,0)$ to fill the 6 tensor multiplets. We also find tachyonic and massless spectrum of Type IIA
string theory is

\[
\begin{align*}
\theta^0 & : 4\left(\frac{1}{2}, \frac{1}{2}\right) + 2\left(0, \frac{1}{2}\right) + 2\left(\frac{1}{2}, 0\right) + 4(0, 0), \\
\theta, \theta^7 & : 4(0, 0)_{-\frac{1}{8}} + 4\left(\frac{1}{2}, \frac{1}{2}\right) + 4\left(\frac{1}{2}, 0\right) + 4(0, \frac{1}{2}) + 4(0, 0), \\
\theta^2, \theta^6 & : 6(0, 0)_{-\frac{1}{4}} + 6\left(\frac{1}{2}, \frac{1}{2}\right) + 24(0, 0), \\
\theta^3, \theta^5 & : 4(0, 0)_{-\frac{1}{8}} + 4(0, 0)_{-\frac{1}{4}} + 4\left(\frac{1}{2}, \frac{1}{2}\right) + 4\left(\frac{1}{2}, 0\right) + 4(0, \frac{1}{2}) + 4(0, 0), \\
\theta^4 & : 6\left(\frac{1}{2}, \frac{1}{2}\right) + 6\left(\frac{1}{2}, 0\right) + 6(0, \frac{1}{2}) + 20(0, 0).
\end{align*}
\]

(57)

Similar to Type IIB string theory, the \(\theta^2 + \theta^6\) sector lacks \(12\left(\frac{1}{2}, 0\right) + 12\left(0, \frac{1}{2}\right)\), thus this sector is bosonic. And the \(\theta^4\) sector wants \(6\left(\frac{1}{2}, 0\right) + 6\left(0, \frac{1}{2}\right) + 4(0, 0)\) to fill the 6 vector multiplets.

We also consider Type 0 string theories. The tachyonic and massless spectrum of Type 0B string theory on this \(T^4/Z_8\) orbifold is

\[
\begin{align*}
\theta^0 & : (0, 0)_{-\frac{1}{2}} + 4(1, 0) + 4(0, 1) + 12(0, 0), \\
\theta, \theta^7 & : 4(0, 0)_{-\frac{1}{4}} + 8(0, 0)_{-\frac{1}{8}} + 4(1, 0) + 4(0, 1) + 16(0, 0), \\
\theta^2, \theta^6 & : 6(0, 0)_{-\frac{1}{4}} + 6(1, 0) + 6(0, 1) + 60(0, 0), \\
\theta^3, \theta^5 & : 4(0, 0)_{-\frac{1}{4}} + 8(0, 0)_{-\frac{1}{8}} + 4(1, 0) + 4(0, 1) + 16(0, 0), \\
\theta^4 & : 6(1, 0) + 6(0, 1) + 56(0, 0),
\end{align*}
\]

and that of Type 0A string theory is

\[
\begin{align*}
\theta^0 & : (0, 0)_{-\frac{1}{2}} + 8\left(\frac{1}{2}, \frac{1}{2}\right) + 4(0, 0), \\
\theta, \theta^7 & : 4(0, 0)_{-\frac{1}{4}} + 8(0, 0)_{-\frac{1}{8}} + 8\left(\frac{1}{2}, \frac{1}{2}\right) + 8(0, 0), \\
\theta^2, \theta^6 & : 6(0, 0)_{-\frac{1}{4}} + 12\left(\frac{1}{2}, \frac{1}{2}\right) + 48(0, 0), \\
\theta^3, \theta^5 & : 4(0, 0)_{-\frac{1}{4}} + 8(0, 0)_{-\frac{1}{8}} + 8\left(\frac{1}{2}, \frac{1}{2}\right) + 8(0, 0), \\
\theta^4 & : 12\left(\frac{1}{2}, \frac{1}{2}\right) + 44(0, 0).
\end{align*}
\]

(58)

The \(\theta^4\) sector of Type 0B/0A string theory has 4 fewer \((0,0)\) than the standard, i.e. the \(\theta\) sector of the \(T^4/Z_2\) orbifold with \(v=(0,0,\frac{1}{2},-\frac{1}{2})\).

Let us take the non-supersymmetric \(T^4/Z_{10}\) orbifold with \(v=(0,\frac{1}{10},\frac{9}{10})\). We find the tachyonic and massless spectrum of Type IIB string theory on this orbifold is

\[
\begin{align*}
\theta^0 & : 2(1, 0) + 2(0, 1) + 4(0, \frac{1}{2}) + 8(0, 0), \\
\theta, \theta^9 & : 2(0, 0)_{-\frac{1}{10}} + 2(1, 0) + 4\left(\frac{1}{2}, 0\right) + 4(0, 0), \\
\theta^2, \theta^8 & : 6(0, 0)_{-\frac{1}{5}} + 6(1, 0) + 12(0, 0), \\
\theta^3, \theta^7 & : 2(0, 0)_{-\frac{2}{10}} + 2(0, 0)_{-\frac{1}{5}} + 2(0, 0)_{-\frac{1}{10}} + 2(1, 0) + 8\left(\frac{1}{2}, 0\right) + 10(0, 0), \\
\theta^4, \theta^6 & : 6(0, 0)_{-\frac{1}{5}} + 6(0, 1) + 12(0, \frac{1}{2}) + 12(0, 0), \\
\theta^5 & : 4(0, 1) + 12(0, \frac{1}{2}) + 18(0, 0).
\end{align*}
\]

(60)
In comparison with the standard, i.e. the twisted sectors of the $T^4/Z_5$ orbifold with $v = (0, \frac{1}{5}, \frac{1}{5})$, the $\theta^4 + \theta^6$ sector lacks no state, in contrast, the $\theta^2 + \theta^8$ sector does $12(\frac{1}{2}, 0)$; thus this sector is bosonic. In spite of $5v$ being supersymmetric, the $\theta^5$ sector lacks $4(0, \frac{1}{2}) + 2(0, 0)$ to fill the 4 tensor multiplets. And also, we find the tachyonic and massless spectrum of Type IIA string theory is

\[
\begin{align*}
\theta^0 & : 4(\frac{1}{2}, \frac{1}{2}) + 2(0, \frac{1}{2}) + 2(\frac{1}{2}, 0) + 4(0, 0), \\
\theta, \theta^9 & : 2(0, 0) - \frac{3}{10} + 2(\frac{1}{2}, \frac{1}{2}) + 2(\frac{1}{2}, 0) + 2(0, \frac{1}{2}) + 2(0, 0), \\
\theta^2, \theta^8 & : 6(0, 0) - \frac{1}{5} + 6(\frac{1}{2}, \frac{1}{2}) + 6(0, 0), \\
\theta^3, \theta^7 & : 2(0, 0) - \frac{3}{10} + 2(0, 0) - \frac{1}{5} + 2(0, 0) - \frac{1}{10} + 2(\frac{1}{2}, \frac{1}{2}) + 4(\frac{1}{2}, 0) + 4(0, \frac{1}{2}) + 8(0, 0), \\
\theta^4, \theta^6 & : 6(0, 0) - \frac{1}{5} + 6(\frac{1}{2}, \frac{1}{2}) + 6(\frac{1}{2}, 0) + 6(0, \frac{1}{2}) + 6(0, 0), \\
\theta^5 & : 4(\frac{1}{2}, \frac{1}{2}) + 6(\frac{1}{2}, 0) + 6(0, \frac{1}{2}) + 14(0, 0).
\end{align*}
\]  

(61)

Parallel to Type IIB string theory, the $\theta^2 + \theta^8$ sector, which lacks $6(\frac{1}{2}, 0) + 6(0, \frac{1}{2})$, is bosonic and the $\theta^5$ sector lacks $2(\frac{1}{2}, 0) + 2(0, \frac{1}{2}) + 2(0, 0)$ to fill the 4 vector multiplets.

In addition, we find the tachyonic and massless spectrum of Type 0B string theory on this $T^4/Z_{10}$ orbifold is

\[
\begin{align*}
\theta^0 & : (0, 0) - \frac{1}{5} + 4(1, 0) + 4(0, 1) + 12(0, 0), \\
\theta, \theta^9 & : 2(0, 0) - \frac{3}{10} + 2(0, 0) - \frac{1}{5} + 4(0, 0) - \frac{1}{10} + 2(1, 0) + 2(0, 1) + 14(0, 0), \\
\theta^2, \theta^8 & : 6(0, 0) - \frac{1}{5} + 6(0, 0) - \frac{1}{10} + 6(1, 0) + 6(0, 1) + 18(0, 0), \\
\theta^3, \theta^7 & : 2(0, 0) - \frac{3}{10} + 2(0, 0) - \frac{1}{5} + 4(0, 0) - \frac{1}{10} + 2(1, 0) + 2(0, 1) + 14(0, 0), \\
\theta^4, \theta^6 & : 6(0, 0) - \frac{1}{5} + 6(0, 0) - \frac{1}{10} + 6(1, 0) + 6(0, 1) + 18(0, 0), \\
\theta^5 & : 4(1, 0) + 4(0, 1) + 38(0, 0),
\end{align*}
\]  

(62)

and that of Type 0A string theory is

\[
\begin{align*}
\theta^0 & : (0, 0) - \frac{1}{5} + 8(\frac{1}{2}, \frac{1}{2}) + 4(0, 0), \\
\theta, \theta^9 & : 2(0, 0) - \frac{3}{10} + 2(0, 0) - \frac{1}{5} + 4(0, 0) - \frac{1}{10} + 4(\frac{1}{2}, \frac{1}{2}) + 10(0, 0), \\
\theta^2, \theta^8 & : 6(0, 0) - \frac{1}{5} + 6(0, 0) - \frac{1}{10} + 12(\frac{1}{2}, \frac{1}{2}) + 6(0, 0), \\
\theta^3, \theta^7 & : 2(0, 0) - \frac{3}{10} + 2(0, 0) - \frac{1}{5} + 4(0, 0) - \frac{1}{10} + 4(\frac{1}{2}, \frac{1}{2}) + 10(0, 0), \\
\theta^4, \theta^6 & : 6(0, 0) - \frac{1}{5} + 6(0, 0) - \frac{1}{10} + 12(\frac{1}{2}, \frac{1}{2}) + 6(0, 0), \\
\theta^5 & : 8(\frac{1}{2}, \frac{1}{2}) + 30(0, 0).
\end{align*}
\]  

(63)

Being contrastive to Type II string theories on this orbifold, in both Type 0 string theories, there is no shortage of the states in the twisted sectors except for the $\theta^5$ sector, which has 2 fewer $(0, 0)$ than the standard, i.e. the $\theta$ sector of the $T^4/Z_2$ orbifold with $v = (0, \frac{1}{2}, -\frac{1}{2})$. 

17
The two following $T^4/Z_{12}$ orbifolds have the same orders, but a slight difference appears between each spectrum of a string theory on two orbifolds. First, we consider the non-supersymmetric $T^4/Z_{12}$ orbifold with $v=(0,\frac{1}{12},\frac{5}{12})$. We find the tachyonic and massless spectrum of Type IIB string theory on this $T^4/Z_{12}$ orbifold is

$$\begin{align}
\theta^0 & : 2(1,0) + 2(0,1) + 8(0,0), \\
\theta, \theta^5, \theta^7, \theta^{11} & : 4(0,0) + 4(0,0) + 4(1,0) + 8(\frac{1}{2},0) + 8(0,0), \\
\theta^2, \theta^{10} & : 2(0,0) + 8(0,0) + 2(1,0) + 20(0,0), \\
\theta^3, \theta^9 & : 4(0,0) + 4(0,1) + 20(0,0), \\
\theta^4, \theta^8 & : 6(0,1) + 16(0,\frac{1}{2}) + 26(0,0), \\
\theta^6 & : 4(1,0) + 8(\frac{1}{2},0) + 18(0,0).
\end{align}$$

We notice at once that the $\theta^2+\theta^{10}$ and the $\theta^3+\theta^9$ sectors are bosonic. The former wants $12(\frac{1}{2},0)$ to fill the standard number ratio of the states, i.e. the ratio in the $\theta$ sector of the $T^4/Z_6$ orbifold with $v=(0,\frac{1}{6},\frac{5}{6})$. And the latter also wants $16(0,\frac{1}{2})$ to fill the standard, i.e. that in the $\theta$ sector of the $T^4/Z_4$ orbifold with $v=(0,\frac{1}{4},-\frac{3}{4})$. Though $4v$ and $6v$ are supersymmetric twist vectors, the $\theta^4+\theta^8$ sector lacks $8(0,\frac{1}{2})+4(0,0)$ and the $\theta^6$ sector does $8(\frac{1}{2},0)+2(0,0)$ to fill the tensor multiplets. The similar things also appear in the tachyonic and massless spectrum of Type IIA string theory as follows:

$$\begin{align}
\theta^0 & : 4(\frac{1}{2},\frac{1}{2}) + 4(0,0), \\
\theta, \theta^5, \theta^7, \theta^{11} & : 4(0,0) + 4(0,0) + 4(1,0) + 4(\frac{1}{2},0) + 4(0,\frac{1}{2}) + 4(0,0), \\
\theta^2, \theta^{10} & : 2(0,0) + 8(0,0) + 2(\frac{1}{2},\frac{1}{2}) + 18(0,0), \\
\theta^3, \theta^9 & : 4(0,0) + 4(\frac{1}{2},\frac{1}{2}) + 16(0,0), \\
\theta^4, \theta^8 & : 6(\frac{1}{2},\frac{1}{2}) + 8(\frac{1}{2},0) + 8(0,\frac{1}{2}) + 20(0,0), \\
\theta^6 & : 4(\frac{1}{2},\frac{1}{2}) + 4(\frac{1}{2},0) + 4(0,\frac{1}{2}) + 14(0,0).
\end{align}$$

As well as Type IIB string theory, the $\theta^2+\theta^{10}$ and the $\theta^3+\theta^9$ sector are bosonic. By comparison with the standard, there disappear $6(\frac{1}{2},0)+6(0,\frac{1}{2})$ in the former and $8(\frac{1}{2},0)+8(0,\frac{1}{2})$ in the latter. Added to this, the $\theta^4+\theta^8$ sector wants $4(\frac{1}{2},0)+4(0,\frac{1}{2})+4(0,0)$ and the $\theta^6$ sector dose $4(\frac{1}{2},0)+4(0,\frac{1}{2})+2(0,0)$ to fill the vector multiplets.

In addition, we consider Type 0 string theories on this $T^4/Z_{12}$ orbifold. The tachyonic
and massless spectrum of Type 0B string theory on this orbifold is

\[
\begin{align*}
\theta^0 & : (0, 0) - \frac{1}{2} + 4(1, 0) + 4(0, 1) + 12(0, 0), \\
\theta, \theta^5, \theta^7, \theta^{11} & : 4(0, 0) - \frac{1}{4} + 8(0, 0) - \frac{1}{6} + 8(0, 0) - \frac{1}{12} + 4(1, 0) + 4(0, 1) + 16(0, 0), \\
\theta^2, \theta^{10} & : 2(0, 0) - \frac{1}{4} + 8(0, 0) - \frac{1}{6} + 2(1, 0) + 2(0, 1) + 30(0, 0), \\
\theta^3, \theta^9 & : 4(0, 0) - \frac{1}{4} + 4(1, 0) + 4(0, 1) + 40(0, 0), \\
\theta^4, \theta^8 & : 6(0, 0) - \frac{1}{6} + 6(1, 0) + 6(0, 1) + 32(0, 0), \\
\theta^6 & : 4(1, 0) + 4(0, 1) + 38(0, 0),
\end{align*}
\]

and that of Type 0A string theory is

\[
\begin{align*}
\theta^0 & : (0, 0) - \frac{1}{2} + 8(\frac{1}{2}, \frac{1}{2}) + 4(0, 0), \\
\theta, \theta^5, \theta^7, \theta^{11} & : 4(0, 0) - \frac{1}{4} + 8(0, 0) - \frac{1}{6} + 8(0, 0) - \frac{1}{12} + 8(\frac{1}{2}, \frac{1}{2}) + 8(0, 0), \\
\theta^2, \theta^{10} & : 2(0, 0) - \frac{1}{4} + 8(0, 0) - \frac{1}{6} + 4(\frac{1}{2}, \frac{1}{2}) + 26(0, 0), \\
\theta^3, \theta^9 & : 4(0, 0) - \frac{1}{4} + 8(\frac{1}{2}, \frac{1}{2}) + 32(0, 0), \\
\theta^4, \theta^8 & : 6(0, 0) - \frac{1}{6} + 12(\frac{1}{2}, \frac{1}{2}) + 20(0, 0), \\
\theta^6 & : 8(\frac{1}{2}, \frac{1}{2}) + 30(0, 0).
\end{align*}
\]

Several states, corresponding to disappearing bosonic states in Type II string theories, disappear in both Type 0 string theories. The \(\theta^4 + \theta^8\) sector wants \(4(0, 0)\) to fill the standard number ratio of the states, i.e. the ratio in the \(\theta\) sector of the \(T^3/Z_3\) orbifold with \(v=(0, \frac{1}{3}, -\frac{1}{3})\). And also, the \(\theta^6\) sector wants \(2(0, 0)\) to fill the standard number ratio of the states, i.e. that in the \(\theta\) sector of the \(T^4/Z_2\) orbifold with \(v=(0, \frac{1}{2}, -\frac{1}{2})\).

Second, and finally in this subsection, we consider the non-supersymmetric \(T^4/Z_{12}\) orbifold with \(v=(0, \frac{1}{12}, \frac{7}{12})\). The tachyonic and massless spectrum of Type IIB string theory on this orbifold is

\[
\begin{align*}
\theta^0 & : 2(1, 0) + 2(0, 1) + 8(0, 0), \\
\theta, \theta^5, \theta^7, \theta^{11} & : 4(0, 0) - \frac{1}{4} + 4(0, 0) - \frac{1}{6} + 4(0, 0) - \frac{1}{12} + 4(1, 0) + 8(\frac{1}{2}, 0) + 8(0, 0), \\
\theta^2, \theta^{10} & : 2(0, 0) - \frac{1}{4} + 8(0, 0) - \frac{1}{6} + 2(1, 0) + 20(0, 0), \\
\theta^3, \theta^9 & : 4(0, 1) + 8(0, \frac{1}{2}) + 16(0, 0), \\
\theta^4, \theta^8 & : 6(1, 0) + 16(\frac{1}{2}, 0) + 26(0, 0), \\
\theta^6 & : 4(0, 1) + 12(0, \frac{1}{2}) + 18(0, 0).
\end{align*}
\]

We notice only the \(\theta^2 + \theta^{10}\) sector is bosonic, which lacks \(12(0, \frac{1}{2})\) as compared with the \(\theta\) sector of the \(T^4/Z_6\) orbifold with \(v=(0, \frac{1}{6}, -\frac{1}{6})\), in contrast, Type II string theory on the previous orbifold has more bosonic sectors. \(3v, 4v\) and \(6v\) are supersymmetric twist vectors, but the \(\theta^3 + \theta^9\), \(\theta^4 + \theta^8\) and \(\theta^6\) sectors want \(8(0, \frac{1}{2}) + 4(0, 0), 8(\frac{1}{2}, 0) + 4(0, 0)\) and \(4(0, \frac{1}{2}) + 2(0, 0)\)
and massless spectrum of Type 0B string theory on this orbifold is respectively to fill the tensor multiplets. Similar things also appear in the tachyonic and massless spectrum of Type IIA string theory as follows:

\[ \begin{align*}
\theta^0 & : \quad 4(\frac{1}{2}, \frac{1}{2}) + 4(0, 0), \\
\theta, \theta^5, \theta^7, \theta^{11} & : \quad 4(0, 0) - \frac{1}{4} + 4(0, 0) - \frac{1}{6} + 8(0, 0) - \frac{1}{12} + 4(\frac{1}{2}, 0) + 4(0, \frac{1}{2}) + 4(0, 0), \\
\theta^2, \theta^{10} & : \quad 2(0, 0) - \frac{1}{4} + 8(0, 0) - \frac{1}{6} + 2(\frac{1}{2}, \frac{1}{2}) + 18(0, 0), \\
\theta^3, \theta^9 & : \quad 4(\frac{1}{2}, \frac{1}{2}) + 4(\frac{1}{2}, 0) + 4(0, \frac{1}{2}) + 12(0, 0), \\
\theta^4, \theta^8 & : \quad 6(\frac{1}{2}, \frac{1}{2}) + 8(\frac{1}{2}, 0) + 8(0, \frac{1}{2}) + 20(0, 0), \\
\theta^6 & : \quad 4(\frac{1}{2}, \frac{1}{2}) + 6(\frac{1}{2}, 0) + 6(0, \frac{1}{2}) + 14(0, 0).
\end{align*} \]  

(69)

The \( \theta^2 + \theta^{10} \) sector lacks \( 6(\frac{1}{2}, 0) + 6(0, \frac{1}{2}) \), therefore, it is bosonic sector. And also, the \( \theta^3 + \theta^9 \), \( \theta^4 + \theta^8 \) and \( \theta^6 \) sectors want \( 4(\frac{1}{2}, 0) + 4(0, \frac{1}{2}) + 4(0, 0), 4(\frac{1}{2}, 0) + 4(0, \frac{1}{2}) + 4(0, 0) \) and \( 2(\frac{1}{2}, 0) + 2(0, \frac{1}{2}) + 2(0, 0) \) respectively to fill the vector multiplets.

In addition, we consider Type 0 string theories on this \( T^4/Z_{12} \) orbifold. The tachyonic and massless spectrum of Type 0B string theory on this orbifold is

\[ \begin{align*}
\theta^0 & : \quad (0, 0) - \frac{1}{4} + 4(1, 0) + 4(0, 1) + 12(0, 0), \\
\theta, \theta^5, \theta^7, \theta^{11} & : \quad 4(0, 0) - \frac{1}{4} + 8(0, 0) - \frac{1}{6} + 8(0, 0) - \frac{1}{12} + 4(1, 0) + 4(0, 1) + 16(0, 0), \\
\theta^2, \theta^{10} & : \quad 2(0, 0) - \frac{1}{4} + 8(0, 0) - \frac{1}{6} + 2(1, 0) + 2(0, 1) + 30(0, 0), \\
\theta^3, \theta^9 & : \quad 4(0, 0) - \frac{1}{4} + 4(1, 0) + 4(0, 1) + 36(0, 0), \\
\theta^4, \theta^8 & : \quad 6(0, 0) - \frac{1}{6} + 6(1, 0) + 6(0, 1) + 32(0, 0), \\
\theta^6 & : \quad 4(1, 0) + 4(0, 1) + 38(0, 0).
\end{align*} \]  

(70)

And that of Type 0A string theory is

\[ \begin{align*}
\theta^0 & : \quad (0, 0) - \frac{1}{4} + 8(\frac{1}{2}, \frac{1}{2}) + 4(0, 0), \\
\theta, \theta^5, \theta^7, \theta^{11} & : \quad 4(0, 0) - \frac{1}{4} + 8(0, 0) - \frac{1}{6} + 8(0, 0) - \frac{1}{12} + 8(\frac{1}{2}, 0) + 8(0, \frac{1}{2}), \\
\theta^2, \theta^{10} & : \quad 2(0, 0) - \frac{1}{4} + 8(0, 0) - \frac{1}{6} + 4(\frac{1}{2}, 0) + 26(0, 0), \\
\theta^3, \theta^9 & : \quad 4(0, 0) - \frac{1}{4} + 8(\frac{1}{2}, \frac{1}{2}) + 28(0, 0), \\
\theta^4, \theta^8 & : \quad 6(0, 0) - \frac{1}{6} + 12(\frac{1}{2}, \frac{1}{2}) + 20(0, 0), \\
\theta^6 & : \quad 8(\frac{1}{2}, \frac{1}{2}) + 30(0, 0).
\end{align*} \]  

(71)

There appear the same spectrum of each Type 0 string theory on the \( T^4/Z_{12} \) orbifold with \( v=(0, \frac{1}{12}, \frac{5}{12}) \) except for the \( \theta^3 + \theta^9 \) sector, which has 4 fewer \( (0, 0) \) than the standard, i.e. the number ratio of the states in the \( \theta \) sector of the \( T^4/Z_4 \) orbifold with \( v=(0, \frac{1}{4}, -\frac{1}{4}) \). This difference depends on whether \( 3v \) is supersymmetric or not. Therefore, in Type 0 string theories, we cannot identify two \( T^4/Z_{12} \) orbifolds: one with \( v=(0, \frac{1}{12}, \frac{5}{12}) \) and the other with \( v=(0, \frac{1}{12}, \frac{7}{12}) \). This is similar to the relation between two \( T^4/Z_6 \) orbifolds: one
with \(v=(0, \frac{1}{6}, \frac{5}{6})\) and the other with \(v=(0, \frac{1}{6}, -\frac{1}{6})\). And also, this makes a contrast with the relation between two \(T^4/Z_4\) orbifolds: one with \(v=(0, \frac{1}{4}, \frac{3}{4})\) and the other with \(v=(0, \frac{1}{4}, -\frac{1}{4})\).

### 4.2 Orbifolds including the \((-1)^F_S\) twist

In this subsection, we focus on \(T^4/Z_N\) orbifolds including the \((-1)^F_S\) twist, therefore, we consider only Type 0 string theories on an orbifold whose twist vector is listed in Table 3.

#### Table 3

| \((v_2, v_3, v_4)\) | \(Z_N(\text{for Type II})\) | \(Z_N(\text{for Type 0})\) |
|---------------------|-------------------|-------------------|
| \((0, \frac{1}{3}, \frac{2}{3})\) | \(Z_6\) | \(Z_3\) |
| \((0, \frac{1}{3}, \frac{2}{3})\) | \(Z_8\) | \(Z_4\) |
| \((0, \frac{1}{3}, \frac{2}{3})\) | \(Z_{10}\) | \(Z_5\) |
| \((0, \frac{1}{6}, \frac{1}{3})\) | \(Z_{12}\) | \(Z_6\) |
| \((0, \frac{1}{6}, \frac{1}{3})\) | \(Z_{12}\) | \(Z_6\) |
| \((0, \frac{1}{12}, \frac{3}{12})\) | \(Z_{24}\) | \(Z_{12}\) |
| \((0, \frac{1}{12}, \frac{3}{12})\) | \(Z_{24}\) | \(Z_{12}\) |

Table 3: Twist vectors including the \((-1)^F_S\) twist with two non-zero components. We take absolute value of two components of the twist vectors.

Some twisted sectors of Type 0 string theories on this type \(T^4/Z_N\) orbifold want several states to fill the standard number ratio of the states. Moreover all disappearing states come from only the twisted R-R sector whose twist vector has one non-zero component. From the orbifolds considered in the previous subsection, we can give only example in which the R-R states disappear, i.e. the \(\theta^2 + \theta^4\) sector of the \(T^4/Z_6\) orbifold with \(v=(0, \frac{1}{6}, \frac{3}{6})=(0, \frac{1}{6}, \frac{1}{2})\). In this subsection, let us describe the spectrum in the same manner as the previous subsection, but the order of a orbifold indicates that for Type 0 string theories.

As the first example, let us take the \(T^4/Z_3\) orbifold with \(v=(0, \frac{1}{3}, \frac{2}{3})\). The tachyonic and massless spectrum of Type 0B string theory on this orbifold is

\[
\begin{align*}
\theta^0 &: (0, 0) - \frac{1}{2} + 6(1, 0) + 6(0, 1) + 20(0, 0), \\
\theta, \theta^2 &: 18(0, 0) - \frac{1}{6} + 18(1, 0) + 18(0, 1) + 108(0, 0).
\end{align*}
\]  

(72)

As indicated in [6], this spectrum is equivalent to that on the \(T^4/Z_3\) orbifold with
\(v=(0, \frac{1}{3}, -\frac{1}{3})\). The same thing also appear in that of Type 0A string theory as follows:
\[
\begin{align*}
\theta^0 & : (0, 0) - \frac{1}{4} + 12(\frac{1}{2}, -\frac{1}{2}) + 8(0, 0), \\
\theta, \theta^2 & : 18(0, 0) - \frac{1}{6} + 36(\frac{1}{2}, \frac{1}{2}) + 72(0, 0).
\end{align*}
\] (73)

Therefore, for Type 0 string theories we can identify two \(T^4/Z_3\) orbifolds: one with \(v=(0, \frac{1}{3}, -\frac{1}{3})\) and the other with \(v=(0, \frac{2}{3}, \frac{2}{3})\).

Let us consider the \(T^4/Z_4\) orbifold with \(v=(0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4})=(0, \frac{1}{4}, \frac{1}{2})\). This orbifold has two different twists. In fact, we find the tachyonic and massless spectrum of Type 0B string theory on this orbifold is
\[
\begin{align*}
\theta^0 & : (0, 0) - \frac{1}{2} + 4(1, 0) + 4(0, 1) + 14(0, 0), \\
\theta, \theta^3 & : 32(0, 0) - \frac{1}{8} + 16(1, 0) + 16(0, 1) + 32(0, 0), \\
\theta^2 & : 6(0, 0) - \frac{1}{4} + 6(1, 0) + 6(0, 1) + 4(\frac{1}{2}, \frac{1}{2}) + 12(0, 0).
\end{align*}
\] (74)

Since \(2v=(0, \frac{1}{2}, 0)\), the \(\theta^2\) sector lacks \(8(\frac{1}{2}, \frac{1}{2})\) in comparison with the standard number ratio of the states, i.e. the ratio in the \(\theta\) sector of the \(T^2\times T^2/Z_2\) orbifold with \(v=(0, 0, \frac{1}{2})\). As mentioned above, the disappearing states accompany the twist vector which has one non-zero component. In addition, the disappearing states are always some \((\frac{1}{2}, \frac{1}{2})\) in the twisted sector of Type 0B string theory. Similar thing also appear in the tachyonic and massless spectrum of Type 0A string theory as follows:
\[
\begin{align*}
\theta^0 & : (0, 0) - \frac{1}{2} + 8(\frac{1}{2}, \frac{1}{2}) + 6(0, 0), \\
\theta, \theta^3 & : 32(0, 0) - \frac{1}{8} + 32(\frac{1}{2}, \frac{1}{2}), \\
\theta^2 & : 6(0, 0) - \frac{1}{4} + 2(1, 0) + 2(0, 1) + 12(\frac{1}{2}, \frac{1}{2}) + 4(0, 0).
\end{align*}
\] (75)

Similar to Type 0B, the \(\theta^2\) sector wants \(4(1, 0)+4(0, 1)+8(0, 0)\) in comparison with the standard. Also in the other Type 0A orbifolds including the \((-1)^F\) twist, the disappearing states are several times \((1, 0) + (0, 1) + 2(0, 0)\), therefore we shall call this unit disappearing unit [DU]. That is, 4 DU disappear in the \(\theta^2\) sector of this Type 0A orbifold. Of course, some DU accompany the twist vector with one non-zero component.

Next, we consider the \(T^4/Z_5\) orbifold with \(v=(0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})\). This orbifold includes no \(T^2\times T^2/Z_N\) type twist. In fact, the tachyonic and massless spectrum of Type 0B string theory on this orbifold is
\[
\begin{align*}
\theta^0 & : (0, 0) - \frac{1}{4} + 4(1, 0) + 4(0, 1) + 12(0, 0), \\
\theta, \theta^2, \theta^3, \theta^4 & : 20(0, 0) - \frac{1}{10} + 20(0, 0) - \frac{4}{10} + 20(1, 0) + 20(0, 1) + 60(0, 0).
\end{align*}
\] (77)
And also, that of Type 0A string theory is

\[ \theta^0 : (0, 0) + 8(\frac{1}{3}, \frac{1}{2}) + 4(0, 0), \]

\[ \theta, \theta^2, \theta^3, \theta^4 : 20(0, 0) + 20(0, 0) + 40(\frac{1}{2}, \frac{1}{2}) + 20(0, 0). \]  
(78)

Obviously, we obtain the same spectrum of Type 0B/0A string theory on the orbifold with \(v=(0, \frac{1}{3}, \frac{3}{2})\): thus, we can identify \(v=(0, \frac{1}{3}, \frac{3}{2})\) with \(v=(0, \frac{1}{3}, \frac{3}{2})\) in Type 0 string theories.

The \(T^4/Z_6\) orbifold with \(v=(0, \frac{1}{6}, \frac{2}{6})=(0, \frac{1}{6}, \frac{1}{3})\) include three different twists. The tachyonic and massless spectrum of Type 0B string theory on this orbifold is

\[ \theta^0 : (0, 0) + 4(1, 0) + 4(0, 1) + 12(0, 0), \]

\[ \theta, \theta^5 : 6(0, 0) + 12(0, 0) + 12(0, 1) + 6(1, 0) + 6(0, 1) + 12(0, 0), \]

\[ \theta^2, \theta^4 : 12(0, 0) + 12(1, 0) + 12(0, 1) + 72(0, 0), \]

\[ \theta^3 : 4(0, 0) + 4(1, 0) + 4(0, 1) + 4(\frac{1}{2}, \frac{1}{2}) + 8(0, 0). \]  
(79)

Since \(3v=(0, \frac{1}{2}, 0)\), the \(\theta^3\) sector lacks \(4(\frac{1}{2}, \frac{1}{2})\) as compared with standard ratio, i.e. the number ratio of the state in the \(\theta\) sector of the \(T^2 \times T^2/Z_2\) orbifold with \(v=(0, 0, \frac{1}{2})\). And also, we find the tachyonic and massless spectrum of Type 0A string theory is

\[ \theta^0 : (0, 0) + 4(1, 0) + 4(0, 1) + 14(0, 0), \]

\[ \theta, \theta^5 : 6(0, 0) + 12(0, 0) + 12(0, 1) + 6(1, 0) + 6(0, 1) + 12(0, 0), \]

\[ \theta^2, \theta^4 : 12(0, 0) + 24(\frac{1}{2}, \frac{1}{2}) + 48(0, 0), \]

\[ \theta^3 : 4(0, 0) + 2(1, 0) + 2(0, 1) + 4(\frac{1}{2}, \frac{1}{2}) + 8(0, 0). \]  
(80)

Similar to Type 0B string theory, the \(\theta^3\) sector lacks 2 DU as compared with standard.

The orbifold with \(v=(0, \frac{2}{6}, \frac{3}{6})=(0, \frac{1}{3}, \frac{1}{2})\) is also a \(T^4/Z_6\) orbifold. The tachyonic and massless spectrum of Type 0B string theory on this orbifold is

\[ \theta^0 : (0, 0) + 4(1, 0) + 4(0, 1) + 14(0, 0), \]

\[ \theta, \theta^5 : 48(0, 0) + 24(1, 0) + 24(0, 1) + 48(0, 0), \]

\[ \theta^2, \theta^4 : 6(0, 0) + 6(0, 0) + 12(1, 0) + 12(0, 1) + 30(0, 0), \]

\[ \theta^3 : 8(0, 0) + 8(1, 0) + 8(0, 1) + 16(0, 0). \]  
(81)

\(2v\) and \(3v\) are one non-zero component vectors, therefore the \(\theta^2+\theta^4\) and \(\theta^3\) sectors want some \((\frac{1}{2}, \frac{1}{2})\). The former wants \(24(\frac{1}{2}, \frac{1}{2})\) as compared with the \(\theta\) sector of the \(T^2 \times T^2/Z_3\) orbifold with \(v=(0,0,\frac{1}{7})\), and the latter does \(16(\frac{1}{2}, \frac{1}{2})\) as compared with the \(\theta\) sector of the \(T^2 \times T^2/Z_2\) orbifold with \(v=(0,0,\frac{1}{7})\). We also find the tachyonic and massless spectrum of
Type 0A string theory is

\[
\begin{align*}
\theta^0 & : (0,0)_{-\frac{1}{4}} + 8(\frac{1}{2}, \frac{1}{2}) + 6(0,0), \\
\theta, \theta^5 & : 48(0,0)_{-\frac{1}{12}} + 48(\frac{1}{2}, \frac{1}{2}), \\
\theta^2, \theta^4 & : 6(0,0)_{-\frac{1}{3}} + 6(0,0)_{-\frac{1}{6}} + 24(\frac{1}{2}, \frac{1}{2}) + 6(0,0), \\
\theta^3 & : 8(0,0)_{-\frac{1}{4}} + 16(\frac{1}{2}, \frac{1}{2}).
\end{align*}
\]

As well as Type 0B string theory, the \(\theta^2 + \theta^4\) and \(\theta^3\) sectors want 12 DU and 8 DU respectively.

The last two orbifolds are \(T^4/Z_{12}\) orbifolds. Each of these has five different twists. First, we consider the \(T^4/Z_{12}\) orbifold with \(v=(0,\frac{2}{12}, \frac{3}{12})=(0,\frac{1}{6}, \frac{1}{4})\). The tachyonic and massless spectrum of Type 0B string theory on this orbifold is

\[
\begin{align*}
\theta^0 & : (0,0)_{-\frac{1}{4}} + 4(1,0) + 4(0,1) + 12(0,0), \\
\theta, \theta^5, \theta^7, \theta^{11} & : 8(0,0)_{-\frac{7}{12}} + 8(0,0)_{-\frac{1}{6}} + 8(0,0)_{-\frac{1}{12}} + 8(1,0) + 8(0,1) + 16(0,0), \\
\theta^2, \theta^{10} & : 28(0,0)_{-\frac{1}{12}} + 14(1,0) + 14(0,1) + 28(0,0), \\
\theta^3, \theta^9 & : 16(0,0)_{-\frac{1}{4}} + 8(1,0) + 8(0,1) + 16(0,0), \\
\theta^4, \theta^8 & : 4(0,0)_{-\frac{1}{4}} + 4(0,0)_{-\frac{1}{2}} + 8(1,0) + 8(0,1) + 20(0,0), \\
\theta^6 & : 6(0,0)_{-\frac{1}{4}} + 6(1,0) + 6(0,1) + 12(0,0).
\end{align*}
\]

\(4v\) and \(6v\) are one non-zero component vectors, therefore there disappear some \((\frac{1}{2}, \frac{1}{2})\). The \(\theta^4 + \theta^8\) sector has 16 fewer \((\frac{1}{2}, \frac{1}{2})\) than the standard number ratio of the states, i.e. the ratio in the \(\theta\) sector of the \(T^2\times T^2/Z_3\) orbifold with \(v=(0,0,\frac{3}{4})\). And the \(\theta^3 + \theta^9\) sector has 12 fewer \((\frac{1}{2}, \frac{1}{2})\) than the standard ratio, i.e. the ratio in the \(\theta\) sector of the \(T^2\times T^2/Z_2\) orbifold with \(v=(0,0,\frac{1}{2})\). We also find the tachyonic and massless spectrum of Type 0A string theory is

\[
\begin{align*}
\theta^0 & : (0,0)_{-\frac{1}{4}} + 8(\frac{1}{2}, \frac{1}{2}) + 4(0,0), \\
\theta, \theta^5, \theta^7, \theta^{11} & : 8(0,0)_{-\frac{7}{12}} + 8(0,0)_{-\frac{1}{6}} + 8(0,0)_{-\frac{1}{12}} + 16(\frac{1}{2}, \frac{1}{2}), \\
\theta^2, \theta^{10} & : 28(0,0)_{-\frac{1}{12}} + 28(\frac{1}{2}, \frac{1}{2}), \\
\theta^3, \theta^9 & : 16(0,0)_{-\frac{1}{4}} + 16(\frac{1}{2}, \frac{1}{2}), \\
\theta^4, \theta^8 & : 4(0,0)_{-\frac{1}{4}} + 4(0,0)_{-\frac{1}{2}} + 4(1,0) + 4(0,1) + 16(\frac{1}{2}, \frac{1}{2}) + 4(0,0), \\
\theta^6 & : 6(0,0)_{-\frac{1}{4}} + 12(\frac{1}{2}, \frac{1}{2}).
\end{align*}
\]

Similar to Type 0B string theory, the \(\theta^4 + \theta^8\) sector lacks 8 DU and the \(\theta^3 + \theta^9\) sector does 6 DU as compared with the standard number ratio of the states.

Second, we consider the \(T^4/Z_{12}\) orbifold with \(v=(0,\frac{2}{12}, \frac{4}{12})=(0,\frac{1}{6}, \frac{1}{3})\). The tachyonic and
massless spectrum of Type 0B string theory on this orbifold is

\[
\begin{align*}
\theta^0 &: (0, 0)_{-\frac{1}{2}} + 4(1, 0) + 4(0, 1) + 12(0, 0), \\
\theta, \theta^5, \theta^7, \theta^{11} &: 24(0, 0)_{-\frac{5}{24}} + 24(0, 0)_{-\frac{1}{24}} + 24(1, 0) + 24(0, 1) + 48(0, 0), \\
\theta^2, \theta^{10} &: 36(0, 0)_{-\frac{1}{12}} + 18(1, 0) + 18(0, 1) + 36(0, 0), \\
\theta^3, \theta^9 &: 4(0, 0)_{-\frac{3}{8}} + 8(0, 0)_{-\frac{1}{8}} + 8(1, 0) + 8(0, 1) + 16(0, 0), \\
\theta^4, \theta^8 &: 6(0, 0)_{-\frac{1}{4}} + 6(0, 0)_{-\frac{1}{8}} + 12(1, 0) + 12(0, 1) + 30(0, 0), \\
\theta^6 &: 6(0, 0)_{-\frac{1}{4}} + 6(1, 0) + 6(0, 1) + 12(0, 0).
\end{align*}
\]

(85)

3\nu, 4\nu and 6\nu are one non-zero component vectors, therefore the \(\theta^3+\theta^9\), \(\theta^4+\theta^8\) and \(\theta^6\) sectors want some \((\frac{1}{2}, \frac{1}{2})\); the \(\theta^3+\theta^9\) sector wants 16(\(\frac{1}{2}, \frac{1}{2}\)) as compared with the \(\theta\) sector of the \(T^2\times T^2/\mathbb{Z}_4\) orbifold with \(v=(0,0,\frac{1}{4})\), the \(\theta^4+\theta^8\) sector does 24(\(\frac{1}{2}, \frac{1}{2}\)) as compared with the \(\theta\) sector of the \(T^2\times T^2/\mathbb{Z}_3\) orbifold with \(v=(0,0,\frac{1}{3})\), and the \(\theta^6\) sector does 12(\(\frac{1}{2}, \frac{1}{2}\)) as compared with the \(\theta\) sector of the \(T^2\times T^2/\mathbb{Z}_2\) orbifold with \(v=(0,0,\frac{1}{2})\). Each \(\theta\) sector gives the standard number ratio of the states. We also find the tachyonic and massless spectrum of Type 0A string theory is

\[
\begin{align*}
\theta^0 &: (0, 0)_{-\frac{1}{2}} + 8(\frac{1}{2}, \frac{1}{2}) + 4(0, 0), \\
\theta, \theta^5, \theta^7, \theta^{11} &: 24(0, 0)_{-\frac{5}{24}} + 24(0, 0)_{-\frac{1}{24}} + 48(\frac{1}{2}, \frac{1}{2}), \\
\theta^2, \theta^{10} &: 36(0, 0)_{-\frac{1}{12}} + 36(\frac{1}{2}, \frac{1}{2}), \\
\theta^3, \theta^9 &: 4(0, 0)_{-\frac{3}{8}} + 8(0, 0)_{-\frac{1}{8}} + 16(\frac{1}{2}, \frac{1}{2}), \\
\theta^4, \theta^8 &: 6(0, 0)_{-\frac{1}{4}} + 6(0, 0)_{-\frac{1}{8}} + 24(\frac{1}{2}, \frac{1}{2}) + 6(0, 0), \\
\theta^6 &: 6(0, 0)_{-\frac{1}{4}} + 12(\frac{1}{2}, \frac{1}{2}).
\end{align*}
\]

(86)

Similar to Type 0B, the \(\theta^3+\theta^9\), \(\theta^4+\theta^8\) and \(\theta^6\) sectors want 8 DU, 12 DU and 6 DU as compared with the standard respectively.

Since on this type \(T^4/\mathbb{Z}_N\) orbifolds all of the states disappearing from the tachyonic and massless spectra come from R-R states, we expect the corresponding fractional D-branes are also absent.

5 Rules of the spectra

In this section we extract some rules of the spectra of Type II/0 string theories on a \(T^4/\mathbb{Z}_N\) orbifold from the results in the previous section. We focus on disappearing states and existence of tachyons in the twisted sectors of non-supersymmetric orbifolds.

First, we extract the rule of disappearing fermionic states from NS-R and R-NS sectors. Since Type 0 string theories are bosonic, we focus only on Type II string theories. There
disappear some fermionic states in the $\theta^n$ sector of non-supersymmetric $T^4/Z_N$ orbifolds, where $n$ is not relatively prime to $N$. However, there is an exceptional sector, that is, the $\theta^4+\theta^6$ sector of the $T^4/Z_{10}$ orbifold with $v=(0,\frac{1}{10},\frac{3}{10})$. It has no disappearing fermionic states.

Second, we consider disappearing bosonic states from NS-NS sectors in not only Type II orbifolds but also Type 0 ones, which have been treated in subsection 4.1, where orbifolds do not include the $(-1)^F S=(0,0,1)$ twist. Some NS-NS states disappear in the twisted sector with supersymmetric twist vector of a non-supersymmetric $T^4/Z_N$ orbifold. However, there is no disappearing NS-NS states in the $\theta^2+\theta^{N-2}$ sector of a non-supersymmetric $T^4/Z_N$ orbifold even if the twist vector is supersymmetric. We can take the $\theta^2$ sector of the $T^4/Z_4$ orbifold with $v=(0,\frac{1}{4},\frac{3}{4})$ and the $\theta^2+\theta^4$ sector of that with $v=(0,\frac{1}{6},\frac{5}{6})$ for instance.

Third, we extract the rule of disappearing states from R-R sectors, which we think is important for computing the D-brane spectrum. In Type II and Type 0 string theories on a $T^4/Z_N$ orbifold, they disappear in the twisted sector whose twist vector has one non-zero component. In this case disappearing does not depend on whether a $T^4/Z_N$ orbifold includes the $(-1)^F S=(0,0,1)$ twist or not.

Finally, we think about the relation between existence of tachyons and the twist vector. In a Type II orbifold, the tachyon appears in the twisted sector with non-supersymmetric twist vector. We can confirm this rule by comparing the $\theta^3+\theta^9$ sectors of the two $T^4/Z_{12}$ orbifolds with each other: one orbifold with $v=(0,\frac{1}{12},\frac{5}{12})$ and the other with $v=(0,\frac{1}{12},\frac{7}{12})$. Since the $\theta^3+\theta^9$ sector on the former has non-supersymmetric twist vector, there appear tachyons. In contrast with that, the sector on the latter has supersymmetric twist vector, therefore there appear no tachyons. In a Type 0 orbifold, there appear tachyons in any twisted sector except for in that whose twist vector is $(0,\frac{1}{2},\frac{1}{2})$, where we take absolute value of two non-zero components.

6 Summary and Discussions

In this paper, we have found the orbifolds compatible with toroidal compactification $T^2$ and $T^4$ and have enumerated the tachyonic and massless spectra of Type II and Type 0 string theories on such a orbifold for advanced studies. In Type 0 orbifolds, we have obtained four pairs of those which could be identified: $(0,0,\frac{2}{3})\sim(0,0,\frac{2}{3})$, $(0,\frac{1}{3},-\frac{1}{3})\sim(0,\frac{1}{3},\frac{2}{3})$, $(0,\frac{1}{3},-\frac{1}{3})\sim(0,\frac{1}{3},\frac{2}{3})$, and $(0,\frac{1}{3},\frac{3}{3})\sim(0,\frac{1}{3},\frac{3}{3})$. In addition, we extract some rules about disappearing states and existence of tachyons. Especially, disappearing states from R-R sectors
seem to be the key to advanced studies.

On the basis of studies in this paper, we can expand. For example, $Z_N \times Z_M$ orbifolds with and without torsion [5, 12, 13, 16], heterotic string orbifolds [11, 12], and Type II and Type 0 string theories on a $T^6/Z_N$ orbifold. Part of the last example has studied in [6], but there are many higher order non-supersymmetric $T^6/Z_N$ orbifolds not listed in it.

As another expansion along [4], we can also consider the D-brane spectra of Type II and Type 0 string theories on a $T^2/Z_N$ orbifold, and those on a $T^4/Z_N$. As mentioned above, we should remark the disappearing R-R states, because we can expect corresponding fractional D-branes also disappear. Therefore we should confirm that by computing the D-brane spectrum with the boundary state formalism [14], and also with K-theory [4, 5, 15, 16]. It also seems worthwhile to expand a sequence of this studies into $T^6/Z_N$ orbifolds and $Z_N \times Z_M$ orbifolds [17] in order to clarify the relation among the perturbative R-R states, the D-brane spectrum, and the K-theory.

Acknowledgements

The author thanks K. Inoue for reading the manuscript and discussions.

A Appendix : $T^2 \times T^2 / Z_N$ orbifolds

In this appendix we transform the spectra we have enumerated in section 3 into those of $T^2 \times T^2 / Z_N$ version. That is, those are written down as in section 4.

The tachyonic and massless spectrum of Type IIB/IA string theory on the $T^2 \times T^2 / Z_3$ orbifold with $v=(0,0,\frac{2}{3})$ is

\[
\begin{align*}
\theta^0 & : 2(1,0) + 2(0,1) + 8(1,\frac{1}{2},\frac{1}{2}) + 4(\frac{1}{2}, 0) + 4(0, \frac{1}{2}) + 10(0,0), \\
\theta, \theta^2 & : 6(0,0) - \frac{1}{3} + 6(1,0) + 6(0,1) + 12(\frac{1}{2}, \frac{1}{2}) + 12(\frac{1}{2}, 0) + 12(0, \frac{1}{2}) + 18(0,0). 
\end{align*}
\]

And that of Type 0B/0A string theory is

\[
\begin{align*}
\theta^0 & : (0,0) - \frac{1}{3} + 4(1,0) + 4(0,1) + 12(\frac{1}{2}, \frac{1}{2}) + 14(0,0), \\
\theta, \theta^2 & : 6(0,0) - \frac{1}{3} + 6(0,0) - \frac{1}{3} + 12(1,0) + 12(0,1) + 24(\frac{1}{2}, \frac{1}{2}) + 30(0,0). 
\end{align*}
\]

As done in section 3, we regard the number ratio of the states in each $\theta + \theta^2$ sector as the standard one with the twist vector $(0,0,\frac{2}{3})$. In Type 0 string theory, we regard the ratio as the standard one with the twist vector $(0,0,\frac{1}{3})$ also, for we can identify the two $T^2 \times T^2 / Z_3$ orbifolds: one with $v=(0,0,\frac{1}{3})$ and the other with $v=(0,0,\frac{2}{3})$.
The tachyonic and massless spectrum of Type 0B/0A string theory on the $T^2 \times T^2 / \mathbb{Z}_2$ orbifold with $v = (0,0,\frac{1}{2})$ is
\[
\begin{align*}
\theta^0 & : (0,0)_{-\frac{1}{2}} + 4(1,0) + 4(0,1) + 12\left(\frac{1}{2}, \frac{1}{2}\right) + 16(0,0), \\
\theta & : 8(0,0)_{-\frac{1}{4}} + 8(1,0) + 8(0,1) + 16\left(\frac{1}{2}, \frac{1}{2}\right) + 16(0,0).
\end{align*}
\] (89)

We regard the number ratio of the states in the $\theta$ sector as the standard one with the twist vector $(0,0,\frac{1}{2})$.

The tachyonic and massless spectrum of Type 0B/0A string theory on the $T^2 \times T^2 / \mathbb{Z}_4$ orbifold with $v = (0,0,\frac{1}{4})$ is
\[
\begin{align*}
\theta^0 & : (0,0)_{-\frac{1}{2}} + 4(1,0) + 4(0,1) + 12\left(\frac{1}{2}, \frac{1}{2}\right) + 14(0,0), \\
\theta, \theta^3 & : 4(0,0)_{-\frac{3}{8}} + 8(0,0)_{-\frac{1}{8}} + 8(1,0) + 8(0,1) + 16\left(\frac{1}{2}, \frac{1}{2}\right) + 16(0,0), \\
\theta^2 & : 6(0,0)_{-\frac{1}{4}} + 6(1,0) + 6(0,1) + 12\left(\frac{1}{2}, \frac{1}{2}\right) + 12(0,0).
\end{align*}
\] (90)

We also regard the number ratio of the states in the $\theta+\theta^3$ sector as standard one with the twist vector $(0,0,\frac{1}{4})$.

The tachyonic and massless spectrum of Type 0B/0A string theory on the $T^2 \times T^2 / \mathbb{Z}_6$ orbifold with $v = (0,0,\frac{1}{6})$ is
\[
\begin{align*}
\theta^0 & : (0,0)_{-\frac{1}{2}} + 4(1,0) + 4(0,1) + 12\left(\frac{1}{2}, \frac{1}{2}\right) + 14(0,0), \\
\theta, \theta^5 & : 2(0,0)_{-\frac{1}{3}} + 2(0,0)_{-\frac{1}{2}} + 4(0,0)_{-\frac{1}{3}} + 4(1,0) + 4(0,1) + 8\left(\frac{1}{2}, \frac{1}{2}\right) + 8(0,0), \\
\theta^2, \theta^4 & : 4(0,0)_{-\frac{1}{4}} + 4(0,0)_{-\frac{1}{3}} + 8(1,0) + 8(0,1) + 16\left(\frac{1}{2}, \frac{1}{2}\right) + 20(0,0), \\
\theta^3 & : 4(0,0)_{-\frac{1}{4}} + 4(1,0) + 4(0,1) + 8\left(\frac{1}{2}, \frac{1}{2}\right) + 8(0,0).
\end{align*}
\] (91)

We regard the number ratio of the states in the $\theta+\theta^5$ sector as standard one with the twist vector $(0,0,\frac{1}{6})$.

Through this appendix, we can confirm that, on $T^2 \times T^2 / \mathbb{Z}_N$ orbifolds, each number ratio of the states is the same as long as twisted sectors have the same twist vector.

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