A TIGHT CLOSURE APPROACH
TO A RESULT OF G. FALTINGS

TIRDAD SHARIF

ABSTRACT. Using a result of M. Hochster and C. Huneke on $F$-rational rings a criterion for complete intersection rings of characteristic $p > 0$ is presented. As an application, we give a completely different proof for an algebraic result of G. Faltings that was used by Taylor and Wiles in [9] for a simplification of the proof of the minimal deformation problem.

1. Introduction

The theory of tight closure was created by Melvin Hochster and Craig Huneke. An important notion in the theory of tight closure is the notion of $F$-rationality. There is a close connection between this notion and rational singularity, see [8].

The main purpose of this note is to use a result of M. Hochster and C. Huneke in [3] on $F$-rational rings, to give a criterion for complete intersection rings of characteristic $p > 0$, see Theorem 2.5 for the precise statement.

In [9] G. Faltings proved an algebraic result that was used by Taylor and Wiles to reprove the minimal deformation problem by a simpler method. As an application of Theorem 2.5 we prove Proposition 2.8 which is an extension of Faltings result. Our approach to prove this proposition is fundamentally different from Faltings method where he simplifies Taylor and Wiles argument, see Remark 2.6.

In order to give a view for the reader we recall the minimal deformation problem very briefly.

The modularity conjecture for semistable elliptic curves depends on a critical conjecture of Wiles that was proved in [9, 10], see Conjecture (2.16) of [10].

Let $\mathcal{D}$ be a deformation theory and let $R_\mathcal{D}$ and $T_\mathcal{D}$ be the universal deformation and Hecke algebras associated to $\mathcal{D}$, respectively. The universal property of $R_\mathcal{D}$ implies that there is a homomorphism $\varphi_\mathcal{D} : R_\mathcal{D} \rightarrow T_\mathcal{D}$. Wiles conjecture asserts that $\varphi_\mathcal{D}$ is an isomorphism. In the special case, if $\mathcal{D}$ is a minimal deformation theory, then the above conjecture is called the minimal deformation problem. For different types of deformation theories and their relations see Chapter 1 and 2 of [10].

Wiles method in [10] to prove the above conjecture, was closely related to show that certain Hecke algebras are complete intersection that was shown in [9].

There is no need for the reader to know the number theoretic materials of Wiles proof in detail. We refer the interested readers to [9] for an elegant exposition of the above matters.

2000 Mathematics Subject Classification. 13A35, 13H10.

Key words and phrases. Complete Intersection Algebras, Deformation Algebras, Hecke Algebras, Tight Closure.

The author was supported by a grant from IPM, (No. 83130311).
2. Definitions, Notations and The Main Theorem

Throughout this note all rings are commutative and Noetherian of characteristic $p > 0$ and all of modules are finite (that is, finitely generated).

In this note we use the notations $\nu_R(M)$ and $\ell_R(M)$ respectively for the minimal number of generators of $M$ and the length of $M$ over $R$.

**Definition 2.1.** Let $(R, m, k)$ be a local ring and let $R[[X]] = R[[X_1, \ldots, X_n]]$ be the formal power series of $n$ variables over $R$. Write $A = R[[X]]/J$ for an ideal $J$ of $R[[X]]$. If the local ring $A \otimes_R k$ is complete intersection, then $A$ is called a complete intersection $R$-algebra.

**Definition 2.2.** Let $R$ be a ring and let $J = (a_1, \ldots, a_\ell)$ be an ideal of $R$. An element $x \in R$ is said to be in the tight closure of $J$ and write $x \in J^*$ if there is an element $c \in R^0$ such that for all large $q = p^n$ we have $cx^q \in J[q]$, wherein $R^0$ is the complement of the union of all minimal primes of $R$ and $J[q] = (a_1^q, \ldots, a_\ell^q)$.

An ideal $J$ is called tightly closed if $J = J^*$. The ring $R$ is called $F$-rational if every parameter ideal of $R$ is tightly closed and in particular, if every ideal of $R$ is tightly closed, then it is called $F$-regular.

**Fact 2.3.** Let $Q$ be a regular local ring, then it is $F$-regular, see [2] (4.4)].

To state the main theorem of this note we will make use of the following simple lemma the proof of which is omitted.

**Lemma 2.4.** Let $I$ be an ideal of $R$, then $(I^*)^n \subseteq (I^n)^*$ for $n \geq 1$.

**Theorem 2.5.** Let $(A, n, k)$ be a local ring of dimension $t$ and let $\mathfrak{q}$ be an ideal of $A$, generated by a system of parameters. Let $\eta: A \rightarrow Q$ and $\theta: Q \rightarrow B$ be local ring homomorphisms such that $Q$ is regular of dimension $t$, $\theta$ is surjective, and $\eta$ induces an isomorphism between the residue fields of $A$ and $Q$. Set $\delta = \theta \eta: A \rightarrow B$ and $\ell(B/q^*B) = d$. Assume for an integer $q = p^n > d^t t^{-1}$ we have $\ell(B/q^*B) \geq \ell(A/q^*A)$. If the ideal $q^*A$ is tightly closed, then the canonical homomorphism $\pi: Q/q^*Q \rightarrow B/q^*B$ is an isomorphism.

In particular, if $A$ is equidimensional homomorphic image of a Cohen-Macaulay ring, then $\pi$ is an isomorphism between complete intersection rings.

**Proof.** Write $m$ for the maximal ideal of $Q$ and $J = \text{Ker} \theta$. We may assume that $B = Q/J$. From the assumptions it follows that $m^d \subseteq J + q^*Q$. Therefore we have the next inclusions $m^{qd} \subseteq (J + q^*Q)^{tq} \subseteq (J + q^*Q)^{tq}Q$. Lemma 2.4 implies that $m^{qd} \subseteq J + (q^*Q)^t Q$. Since $\mathfrak{q}$ is generated by $t$ elements it is easy to see that $q^\mathfrak{q} \subseteq q^*Q$. This yields that $m^{qd} \subseteq J + (q^*Q)^t Y$ and from our assumptions we have $m^{qd} \subseteq J + q^\mathfrak{q} Q$. We claim that $J \subseteq m^{d+1}$. Otherwise choose $u$ in $J$ not in $m^{d+1}$ and consider the following exact sequence

$$0 \rightarrow \text{Ker}(\alpha) \rightarrow Q/m^c \rightarrow Q/m^c \rightarrow \text{Coker}(\alpha) \rightarrow 0 \quad (2.5.1)$$

wherein the map $\alpha$ is the homothety by element $u$ and $c = tdq$. Obviously, there is a surjective map as the following

$$\text{Coker}(\alpha) = Q/(uQ + m^c) \rightarrow Q/(J + q^\mathfrak{q} Q) = B/q^\mathfrak{q} B.$$

Therefore $\ell_Q(\text{Coker}(\alpha)) \geq \ell_Q(B/q^\mathfrak{q} B)$. It is clear that $B/q^\mathfrak{q} B$ has finite length over $Q$. The ideal $q$ is an $n$-primary ideal of $A$, therefore $\ell_A(B/q^\mathfrak{q} B)$ is finite and
since \( k = A/n \simeq Q/m \), clearly the length of \( B/q^tB \) over both of \( Q \) and \( A \) are equal. By assumptions \( \ell(B/q^tB) \geq \ell(A/q^t) \), therefore we have \( \ell_Q(\text{Coker}(\alpha)) \geq \ell(B/q^tB) \geq \ell(A/q^t) \) and from \([5, (14.10)]\) we find that \( \ell(A/q^t) \geq q^t \ell(A) \geq q^t \) wherein the symbol \( \ell(A) \) is the multiplicity of \( A \). Thus \( \ell_Q(\text{Coker}(\alpha)) \geq q^t \).

Let \( x_1, x_2, \ldots, x_l \) be a minimal generator for \( m \). The local ring \( Q \) is regular, so the associated graded ring \( gr_m(Q) \) is a polynomial ring in \( t \) variables over \( k \), see \([7, \text{Page 76}, \text{Theorem 9(d)}]\). This fact implies that \( x_j \) for \( 1 \leq j \leq t \) are analytically independent in \( m \) and so that \( \nu_Q(m^{e-i}) = (c_{i+1}^{i-1}) \) for \( i \geq 1 \). Since \( u \) is not an element of \( m^{d+1} \) therefore \( \text{Ker}(\alpha) \subseteq m^{c-d}/m^c \) and hence \( \ell_Q(\text{Ker}(\alpha)) \leq \ell_Q(m^{c-d}/m^c) \).

The following equality is clear

\[
\ell_Q(m^{c-d}/m^c) = \sum_{i=1}^{d} \nu_Q(m^{c-i}).
\]  (2.5.2)

Now from (2.5.1) we have \( \ell_Q(\text{Ker}(\alpha)) = \ell_Q(\text{Coker}(\alpha)) \) and from (2.5.2) we find that \( \ell_Q(m^{c-d}/m^c) \leq d^{e(t-2)} \leq d^{e-1} = d(tdq)^{t-1} \). Therefore \( q^t \leq d^{e-1}q^{-1} \) and this contradicts the choice of \( q \). Hence we must have \( J \subseteq m^{d+1} \) and consequently \( m^d \subseteq J + q^tQ \subseteq m^{d+1} + q^tQ \). Thus \( m^d + q^tQ = m^{d+1} + q^tQ \) and by using Nakayama’s Lemma \( m^{d+1} \subseteq q^tQ \). This yields that \( J \subseteq q^tQ \) and this shows that the canonical homomorphism \( \pi : Q/q^tQ \to B/q^tB \) is an isomorphism.

Now let \( A \) be an equidimensional ring which is homomorphic image of a Cohen-Macaulay ring. In this case, since \( q^{|t|} \) is generated by a system of parameters and it is tightly closed it follows from \([4, (4.2.e)]\) that \( A \) is \( F \)-rational and hence \( q = q^* \). Now it is easy to see that \( \pi \) is an isomorphism between complete intersection rings.

\( \square \)

**Remark 2.6.** Let \( (O, m, k) \) be a complete discrete valuation ring with finite residue field \( k \) and let \( D \) be a minimal deformation theory. Let \( R \) and \( T \) be the universal deformation and Hecke \( O \)-algebras associated to \( D \), respectively and let \( \varphi = \varphi_D \) be the homomorphism described in Section 1. Taylor and Wiles in the appendix of \([9]\) for a simplification of some arguments in Section 3 of \([9]\) and Chapter 3 of \([10]\) use a commutative algebraic result of G. Faltings to reprove that \( \varphi \) is an isomorphism between complete intersection algebras. In Proposition 2.8 and Corollary 2.11 we give an extension of corresponding results due to Taylor–Wiles and Faltings without the assumption of finiteness of \( k \). Our method to prove the above results is totally different from their methods.

In the following we refine the definition of a (level) \( n \)-structure due to Wiles and Faltings in \([9]\).

**Definition 2.7.** Let \( (O, m, k) \) be a local ring and let \( O[[Y]] = O[[Y_1, \ldots, Y_l]] \). Set \( q = (Y_1, \ldots, Y_l) \) and assume that for an integer \( n \geq 1 \) we have a commutative diagram of local \( O \)-algebras as the following in which \( \varphi \) is surjective and \( T \) is a finite free \( O \)-module.

\[
\begin{align*}
O[[Y]] & \xrightarrow{\phi_n} R_n \xrightarrow{\phi_n} T_n \xrightarrow{T} T \\
\downarrow & \downarrow & \downarrow \\
R & \xrightarrow{\varphi} & T
\end{align*}
\]

By a level \( n \)-structure we mean the above diagram with the following properties:

1. There is a surjective homomorphism \( \lambda_n : O[[X_1, X_2, \ldots, X_l]] \to R_n \).
2. The ring homomorphism \( R_n \to T_n \) is surjective.
we can lift the local homomorphism \( O \) to use the fifth property directly. However, in the following corollary as a simple completeness of application of the level \( n \) structure, we make clear the relation between the completeness of \( O \) and the fifth property. In a sense, we can say that the assumption

(3) From the vertical homomorphisms we get the isomorphisms \( R_n/qR_n \to R \) and \( T_n/qT_n \to T \).

(4) The ring \( T_n/q[p^n]T_n \) is finite and free as a module over \( O[[Y]]/q[p^n] \).

For convenience we represent a level \( n \)-structure by the notation \( L_n(O, \varphi, \varphi_n, \psi_n, \lambda_n) \).

**Proposition 2.8.** Let \( m \geq 1 \) be an integer such that for every \( n \geq m \), the level \( n \)-structure \( L_n(O, \varphi, \varphi_n, \psi_n, \lambda_n) \) has the following additional property:

(5) For each \( x \in m \) we have \( \psi_n(x) \in mR_n \).

Then \( \varphi \) is an isomorphism between complete intersection \( O \)-algebras.

**Proof.** Set \( \hat{\square} = \square \otimes_O k, \ q_n = q[p^n] \) and \( O[[X]] = O[[X_1, X_2, \ldots, X_t]] \). From (1) there is a surjective homomorphism \( k[[X]] \to R_n \). The fifth property implies that \( \psi_n(m[[Y]]) \subseteq mR_n \). Thus \( \psi_n \) induces a local homomorphism \( \tilde{\psi}_n : k[[Y]] \to \tilde{R}_n \).

Since \( k[[X]] \) is a complete local ring, using [1, (7.16)] we can lift the homomorphism \( k[[Y]] \to \tilde{R}_n \) to a unique homomorphism \( k[[Y]] \to k[[X]] \). Thus from (1), (2) and (3) we find that there is a surjective homomorphism as the following

\[
\alpha : k[[X]]/qk[[X]] \to \tilde{T}_n/q\tilde{T}_n \simeq \tilde{T}.
\]

From (3) it follows that \( \tilde{T}_n/q\tilde{T}_n \) is a finite vector space over \( k \). We may assume that the length of \( \tilde{T}_n/q\tilde{T}_n \) over \( k[[X]] \) is equal \( d \). Now choose \( n \geq m \) such that \( p^n > \ell^{-1}d \). From the fourth property it is clear that the length of \( \tilde{T}_n/q\tilde{T}_n \) over \( k[[Y]] \) is equal or greater than the length of \( k[[Y]]/qk[[Y]] \).

Now by writing \( A = k[[Y]], Q = k[[X]] \) and \( B = \tilde{T}_n \) from Theorem 2.5 and Fact 2.6 we get that \( \alpha \) is an isomorphism between complete intersection rings.

On the other hand, \( \hat{\varphi}_n \) is surjective, from this and the third property we have the following surjective homomorphisms

\[
\beta : k[[X]]/qk[[X]] \to \tilde{R}_n/q\tilde{R}_n \quad \gamma : \tilde{R}_n/q\tilde{R}_n \to \tilde{T}_n/q\tilde{T}_n
\]

such that \( \alpha = \gamma \beta \). It was shown that \( \alpha \) is an isomorphism thus from (3) it is clear that \( \hat{\varphi} \) is an isomorphism between \( \hat{R} \) and \( \hat{T} \) as complete intersection rings. Since \( T \) is a finite free module and \( \varphi \) is surjective, Nakayama’s Lemma implies that \( \varphi \) is an isomorphism. It is obvious that the local ring \( T \) is a homomorphic image of \( O[[X]] \).

Since \( \hat{T} \) is a complete intersection ring from Definition 2.4 it follows that \( T \) and so \( R \) are complete intersection \( O \)-algebras. \( \square \)

**Remark 2.9.** In the proof of the above proposition the fifth property enables us to reduce the level \( n \)-structure \( L_n(O, \varphi, \varphi_n, \psi_n, \lambda_n) \) to the level \( n \)-structure \( L_n(k, \hat{\varphi}, \hat{\varphi}_n, \hat{\psi}_n, \hat{\lambda}_n) \) and then by lifting property which is based on [1, (7.16)] we can prove the proposition as a simple corollary of our main theorem. If we assume that \( O \) is \( m \)-adic complete, then \( O[[X]] = (m[[X]] + (X)) \)-adic complete. From (1) we have a surjective homomorphism \( O[[X]] \to R_n \). Now again by using [1, (7.16)] we can lift the local homomorphism \( \psi_n \) to a unique local homomorphism \( \tilde{O}[[Y]] \to \tilde{O}[[X]] \). Now by a similar method to our proof we can show that \( \varphi \) is an isomorphism between complete intersection \( O \)-algebras. In this case, we don’t need to use the fifth property directly. However, in the following corollary as a simple application of the level \( n \)-structures, we make clear the relation between the completeness of \( O \) and the fifth property. In a sense, we can say that the assumption
of completeness of our base ring implies the fifth property. We use the next simple lemma the proof of which is omitted.

**Lemma 2.10.** Let \( L_n(O, \varphi, \varphi'_n, \psi_n, \lambda_n) \) be a level \( n \)-structure, then there is a level \( n \)-structure \( L'_n(O, \varphi, \varphi'_n, \psi'_n, \lambda'_n) \) such that \( \lambda'_n \) is the canonical epimorphism.

**Corollary 2.11.** Let \( (O, m, k) \) be an \( m \)-adic complete local ring, then \( \varphi \) is an isomorphism between complete intersection \( O \)-algebras.

**Proof.** From Lemma 2.10 it follows that there is a level \( n \)-structure such that the homomorphism \( \lambda_n : O[[X]] \to O[[X]]/J_n = R_n \) is a canonical epimorphism. Therefore for each \( g \in O[[X]] \) we have \( \lambda_n(g) = g + J_n \), where \( J_n = \text{Ker} \lambda_n \). Since \( O \) is \( m \)-adic complete hence \( O[[X]] \) is \((m[[X]] + (X))\)-adic complete. Thus from [1, (7.16)] we can lift \( \psi_n \) to a unique homomorphism \( \xi : O[\{Y\}] \to O[[X]] \) such that \( \xi(x) = x \) for each \( x \in m \). On the other hand, we have \( \lambda_n \xi = \psi_n \). Hence \( \psi_n(x) = x + J_n \) and this shows that \( \psi_n(x) \in mR_n \). Now from Proposition 2.8 assertion holds.

**Acknowledgments**

The author is grateful to Sean Sather-Wagstaff and Irena Swanson for their useful comments on this note.

**References**

1. D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer, (1995).
2. M. Hochster, C. Huneke, *Tight closure, invariant theory and the Briançon-Skoda theorem*, J. Amer. Math. Soc. 3 (1990), 31–116.
3. M. Hochster, C. Huneke, *Tight closure of parameter ideals and splitting in module-finite extensions*, J. Algebraic Geom. 3 (1994), no. 4, 599–670.
4. H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, (1989).
5. K. A. Ribet, *Galois representations and modular forms*, Bull. AMS. 32 (1995), no. 4, 375–402.
6. J. P. Serre, *Local Algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, (2000).
7. K. Smith, *F-rational rings have rational singularities*, Amer. J. Math. 119 (1997), 159–180.
8. R. Taylor, A. Wiles, *Ring theoretic properties of certain Hecke Algebras*, Annals of Math. 141 (1995), 553–572.
9. A. Wiles, *Modular elliptic curves and Fermat's last theorem*. Ann. of Math. (2) 141 (1995), no. 3, 443–551.