On channels with positive quantum zero-error capacity having vanishing $n$-shot capacity

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Abstract

We show that unbounded number of channel uses may be necessary for perfect transmission of a quantum state. For any $n$ we explicitly construct low-dimensional quantum channels ($d_A = 4, d_E = 2$ or $4$) whose quantum zero-error capacity is positive but the same $n$-shot capacity is zero. We give estimates for quantum zero-error capacity of such channels (as a function of $n$) and show that they can be chosen in any small vicinity (in the cb-norm) of a classical-quantum channel.

Mathematically, this property means appearance of an ideal (noiseless) subchannel only in sufficiently large tensor power of a channel.

Our approach (using special continuous deformation of a maximal commutative $*$-subalgebra of $M_4$) also gives low-dimensional examples of superactivation of 1-shot quantum zero-error capacity. We note that such superactivation is possible if one of two channels is arbitrarily close to a classical-quantum channel.

1 Introduction

It is well known that the ultimate rate of information transmission over classical and quantum communication channels can be increased by simultaneous use of many copies of a channel. It is this fact that implies necessity of regularization in definitions of different capacities of a channel [7 11].

In this paper it is shown that perfect transmission of a quantum state over simple low-dimensional channels may require unbounded number of channel uses.

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uses. This means that the $n$-shot quantum zero-error capacity of these channels becomes positive only for arbitrary large $n$ (depending on a channel).

This effect is closely related to the recently discovered superactivation of zero-error capacities \cite{1, 4, 5}, since it means that for any given $n$ there is a channel $\Phi_n$ such that

\[
\tilde{Q}_0(\Phi_n) = \tilde{Q}_0(\Phi_n \otimes 2) = \ldots = \tilde{Q}_0(\Phi_n \otimes (m-1)) = 0, \quad \text{but} \quad \tilde{Q}_0(\Phi_n \otimes m) > 0 \quad (1)
\]

for some $m > n$, where $\tilde{Q}_0$ denotes the 1-shot quantum zero-error capacity.

Mathematically, (1) means that all the channels $\Phi_n, \Phi_n \otimes 2, \ldots, \Phi_n \otimes (m-1)$ have no ideal (noiseless) subchannels but the channel $\Phi_n \otimes m$ has.

We show how to explicitly construct for any given $n$ a low-dimensional pseudo-diagonal quantum channel $\Phi_n$ satisfying (1) by determining its non-commutative graph. We also obtain the upper bound for $m$ as a function of $n$, which gives the lower estimate for the maximal quantum zero-error capacity of $\Phi_n$. It is observed that the channel $\Phi_n$ can be obtained by arbitrary small deformation (in the $cb$-norm) of some classical-quantum channel with $d_A = d_E = 4$.

The main problem in finding the channel $\Phi_n$ is to show nonexistence of error correcting codes for the channel $\Phi_n \otimes n$ (provided the existence of such codes is proved for $\Phi_n \otimes m$). We solve this problem by using the special continuous deformation of a maximal commutative $*$-subalgebra of $4 \times 4$ matrices as the noncommutative graph of $\Phi_n$ and by noting that the Knill-Laflamme error-correcting conditions are violated for any maximal commutative $*$-subalgebra with the positive dimension-independent gap (Lemma 3).

Our construction also gives low-dimensional examples of superactivation of 1-shot quantum zero-error capacity. In particular, it makes it possible to obtain an example of symmetric superactivation with $d_A = 4, d_E = 2$ (simplifying the example in \cite{14}) and to show that such superactivation is possible if one of two channels is arbitrarily close (in the $cb$-norm) to a classical-quantum channel.

2 Preliminaries

Let $\Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ be a quantum channel, i.e. a completely positive trace-preserving linear map \cite{7, 11}. Stinespring’s theorem implies the existence of a Hilbert space $\mathcal{H}_E$ and of an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such
that
\[ \Phi(\rho) = \text{Tr}_{H_E} V \rho V^*, \quad \rho \in \mathcal{S}(H_A). \] (2)

The minimal dimension of $H_E$ is called Choi rank of $\Phi$ and denoted $d_E$.

The quantum channel
\[ \mathcal{S}(H_A) \ni \rho \mapsto \hat{\Phi}(\rho) = \text{Tr}_{H_B} V \rho V^* \in \mathcal{S}(H_E) \] (3)
is called complementary to the channel $\Phi$ [7, 8]. The complementary channel is defined uniquely up to isometrical equivalence [8, the Appendix].

The 1-shot quantum zero-error capacity $\bar{Q}_0(\Phi)$ of a channel $\Phi$ can be defined as $\sup_{H \in q_0(\Phi)} \log \dim H$, where $q_0(\Phi)$ is the set of all subspaces $H_0$ of $H_A$ on which the channel $\Phi$ is perfectly reversible (in the sense that there is a channel $\Theta$ such that $\Theta(\Phi(\rho)) = \rho$ for all states $\rho$ supported by $H_0$). Any subspace $H_0 \in q_0(\Phi)$ is called error correcting code for the channel $\Phi$ [6, 7].

The (asymptotic) quantum zero-error capacity is defined by regularization:
\[ Q_0(\Phi) = \sup_n n^{-1} \bar{Q}_0(\Phi \otimes^n) \] [5, 6].

It is well known that a channel $\Phi$ is perfectly reversible on a subspace $H_0$ if and only if the restriction of the complementary channel $\hat{\Phi}$ to the subset $\mathcal{S}(H_0)$ is completely depolarizing, i.e. $\hat{\Phi}(\rho_1) = \hat{\Phi}(\rho_2)$ for all states $\rho_1$ and $\rho_2$ supported by $H_0$ [7, Ch.10]. It follows that the 1-shot quantum zero-error capacity $\bar{Q}_0(\Phi)$ of a channel $\Phi$ is completely determined by the set $\mathcal{G}(\Phi) = \hat{\Phi}^*(\mathcal{B}(H_E))$ called the noncommutative graph of $\Phi$ [6].

In particular, the following simple lemma is valid (Lemma 3 in [13]).

**Lemma 1.** A channel $\Phi : \mathcal{S}(H_A) \to \mathcal{S}(H_B)$ has positive 1-shot quantum zero-error capacity if and only if there are unit vectors $\varphi$ and $\psi$ in $H_A$ such that
\[ \langle \psi \rvert A \lvert \varphi \rangle = 0 \quad \text{and} \quad \langle \varphi \rvert A \lvert \varphi \rangle = \langle \psi \rvert A \lvert \psi \rangle \quad \forall A \in \mathcal{L}, \] (4)

where $\mathcal{L}$ is any subset of $\mathcal{B}(H_A)$ such that $\text{lin}\mathcal{L} = \mathcal{G}(\Phi)$.

Condition (4) is equivalent to perfect reversibility of the channel $\Phi$ on the subspace of $H_A$ spanned by the vectors $\varphi$ and $\psi$.

**Remark 1.** Since a subspace $\mathcal{L}$ of the algebra $\mathcal{M}_n$ of $n \times n$-matrices is a noncommutative graph of a particular channel if and only if
\[ \mathcal{L} \text{ is symmetric } (\mathcal{L} = \mathcal{L}^*) \text{ and contains the unit matrix} \] (5)
\[ ^1\text{This lemma can be derived from the Knill-Laflamme error-correcting condition} \] [9].
(see Lemma 2 in [4] or Proposition 2 in [13]), Lemma shows that one can "construct" a channel Φ with dim HA = n having positive (correspondingly, zero) 1-shot quantum zero-error capacity by taking a subspace L ⊂ Mn satisfying (5) for which the following condition is valid (correspondingly, not valid)

∃ϕ,ψ ∈ [C^n]_1 s.t. ⟨ψ|A|ϕ⟩ = 0 and ⟨ϕ|A|ψ⟩ = ⟨ψ|A|ψ⟩ ∀A ∈ L, (6)

where [C^n]_1 is the unit sphere of C^n. □

We will use the following two notions.

Definition 1. A finite-dimensional channel Φ : S(H_A) → S(H_B) is called classical-quantum if it has the representation

Φ(ρ) = ∑_k ⟨k|ρ|k⟩σ_k, (7)

where \{|k⟩\} is an orthonormal basis in H_A and \{|σ_k⟩\} is a collection of states in S(H_B).

Definition 2. [3] A finite-dimensional channel Φ : S(H_A) → S(H_B) is called pseudo-diagonal if it has the representation

Φ(ρ) = ∑_{i,j} c_{ij}⟨ψ_i|ρ|ψ_j⟩|i⟩⟨j|,

where \{c_{ij}\} is a Gram matrix of a collection of unit vectors, \{|ψ_i⟩\} is a collection of vectors in H_A such that \(\sum_i |ψ_i⟩⟨ψ_i| = I_{H_A}\) and \{|i⟩\} is an orthonormal basis in H_B.

Pseudo-diagonal channels are complementary to entanglement-breaking channels and vice versa [3, 8].

3 Basic results

For any given \(θ ∈ T = (-π, π]\) consider the subspace

\(L_θ = \left\{ M = \begin{bmatrix} a & b & γc & d \\ b & a & d & γc \\ γc & d & a & b \\ d & γc & b & a \end{bmatrix}, a, b, c, d ∈ C, γ = \exp \left[ \frac{i}{2}θ \right] \right\} \) (8)

In infinite dimensions there exist channels naturally called classical-quantum, which have no representation [7].
of $\mathcal{M}_4$ satisfying condition (5). Denote by $\hat{\mathcal{L}}_\theta$ the set of all channels whose noncommutative graph coincides with $\mathcal{L}_\theta$. For each $\theta$ the set $\hat{\mathcal{L}}_\theta$ contains infinitely many different channels with $d_A = \dim \mathcal{H}_A = 4$.

**Lemma 2.** 1) There is a family $\{\Phi_1^\theta\}$ of pseudo-diagonal channels (see Def.3) with $d_E = 2$ such that $\Phi_1^\theta \in \hat{\mathcal{L}}_\theta$ for each $\theta$.

2) There is a family $\{\Phi_0^\theta\}$ of pseudo-diagonal channels with $d_E = 4$ such that $\Phi_0^\theta \in \hat{\mathcal{L}}_\theta$ for each $\theta$ and $\Phi_0^\theta$ is a classical-quantum channel (see Def.7).

The families $\{\Phi_1^\theta\}$ and $\{\Phi_0^\theta\}$ can be chosen continuous in the following sense:

$$\Phi_0^k(\rho) = \text{Tr}_{H_E}V_\theta^k\rho[V_\theta^k]^*, \quad \rho \in \mathcal{S}(\mathcal{H}_A), \quad k = 1, 2,$$

(9)

where $V_\theta^1$, $V_\theta^2$ are continuous families of isometries, $\mathcal{H}_E = \mathbb{C}^2$, $\mathcal{H}_E = \mathbb{C}^4$.

Lemma 2 is proved in the Appendix by explicit construction of representations (9).

**Theorem 1.** Let $\Phi_\theta$ be an arbitrary channel in $\hat{\mathcal{L}}_\theta$ then

A) $\hat{Q}_0(\Phi_\theta) > 0$ if and only if $\theta = \pi$ and $\hat{Q}_0(\Phi_\pi) = \log 2$.

B) If $\theta_1, \ldots, \theta_n$ is a subset of $T$ such that $\theta_1 + \ldots + \theta_n = \pi (\text{mod } 2\pi)$ then $\hat{Q}_0(\Phi_{\theta_1} \otimes \ldots \otimes \Phi_{\theta_n}) > 0$ and 2D error correcting code for the channel $\Phi_{\theta_1} \otimes \ldots \otimes \Phi_{\theta_n}$ is spanned by the vectors

$$|\varphi\rangle = \frac{1}{\sqrt{2}} \left[ |1\ldots1\rangle + i|2\ldots2\rangle \right], \quad |\psi\rangle = \frac{1}{\sqrt{2}} \left[ |3\ldots3\rangle + i|4\ldots4\rangle \right],$$

where $\{|1\rangle, \ldots, |4\rangle\}$ is the canonical basis in $\mathbb{C}^4$.

C) $\hat{Q}_0(\Phi_\theta^0) = 0$ if $|\theta| \leq 2(\sqrt{3}/2 - 1)$ and $\hat{Q}_0(\Phi_\theta^0) > 0$ if $|\theta| = \pi/n$.

**Remark 2.** It is easy to show that $\hat{Q}_0(\Phi_{\theta_1}^0) = \hat{Q}_0(\Phi_{\theta_2}^0)$ and that the set of all $\theta$ such that $\hat{Q}_0(\Phi_{\theta_1}^0) = 0$ is open. Hence for each $n$ there is $\varepsilon_n > 0$ such that $\hat{Q}_0(\Phi_{\theta_1}^0) = 0$ if $|\theta| < \varepsilon_n$ and $\hat{Q}_0(\Phi_{\theta_1}^0) > 0$. Assertions A and C of Theorem 1 show respectively that $\varepsilon_1 = \pi$ and $2(\sqrt{3}/2 - 1) < \varepsilon_n \leq \pi/n$ for $n > 1$. Since $2(\sqrt{3}/2 - 1) \approx 1/n$ for $n \gg 1$ and the first part of assertion C is proved by using quite coarse estimates, one can conjecture that $\varepsilon_n = \pi/n$ for $n > 1$. There exist some arguments confirming validity of this conjecture for $n = 2$.  

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3This implies continuity of these families in the cb-norm [10].

4By using the arguments from the proof of this assertion one can show existence of a continuous family of different 2D error correcting codes for the channel $\Phi_\pi$. 

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5
Theorem 1 implies the main result of this paper.

**Corollary 1.** Let $n$ be arbitrary and $m$ be a natural number such that $\theta_* = \pi/m \leq 2(\sqrt{3}/2 - 1)$. Then

$$\bar{Q}_0(\Phi_{\theta_*}^\otimes n) = 0$$

but

$$\bar{Q}_0(\Phi_{\theta_*}^{\otimes m}) \geq \log 2 \quad \text{and hence} \quad Q_0(\Phi_{\theta_*}) \geq m^{-1} \log 2.$$ 

(10)

Relation (10) means that it is not possible to transmit any quantum state with no errors by using $\leq n$ copies of the channel $\Phi_{\theta_*}$, but such transmission is possible if the number of copies is $\geq m$.

**Remark 3.** In (10) one can take $\Phi_{\theta_*} = \Phi_{\theta_*}^1$ – a channel from the family described in the first part of Lemma 2. So, Corollary 1 shows that for any $n$ there exists a channel $\Phi_n$ with $d_A = 4$ and $d_E = 2$ such that

$$\bar{Q}_0(\Phi_{n}^\otimes n) = 0 \quad \text{and} \quad Q_0(\Phi_{n}) \geq \log 2 \left( \frac{\pi}{2(\sqrt{3}/2 - 1)} + 1 \right)^{-1} \approx \frac{\log 2}{[\pi n] + 1},$$

where $[x]$ is the integer part of $x$ and " $\approx$ " holds for $n \gg 1$.

Since the parameter $\theta_*$ in Corollary 1 can be taken arbitrarily close to zero, the second part of Lemma 2 shows that the channel $\Phi_{\theta_*}$ in (10) can be chosen in any small vicinity (in the cb-norm) of the classical-quantum channel $\Phi_0^2$.

Theorem 1 also gives examples of superactivation of 1-shot quantum zero-error capacity.

**Corollary 2.** If $\theta \neq 0, \pi$ then the following superactivation property

$$Q_0(\Phi_{\theta}) = Q_0(\Phi_{\pi-\theta}) = 0 \quad \text{and} \quad Q_0(\Phi_{\theta} \otimes \Phi_{\pi-\theta}) > 0$$

holds for any channels $\Phi_{\theta} \in \mathcal{L}_\theta$ and $\Phi_{\pi-\theta} \in \mathcal{L}_{\pi-\theta}$. For any $\theta$ error correcting code for the channel $\Phi_{\theta} \otimes \Phi_{\pi-\theta}$ is spanned by the vectors

$$|\varphi\rangle = \frac{1}{\sqrt{2}} \left[ |11\rangle + i |22\rangle \right], \quad |\psi\rangle = \frac{1}{\sqrt{2}} \left[ |33\rangle + i |44\rangle \right].$$

(11)

**Remark 4.** Corollary 2 shows that the channel $\Phi_{\pi/2}^1$ (taken from the fist part of Lemma 2) is an example of symmetric superactivation of 1-shot quantum zero-error capacity with the Choi rank $2^5$.

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5This strengthens the result in [14], where a similar example with Choi rank 3 and the same input dimension was constructed.
By taking the family \( \{ \Phi^2 \} \) from the second part of Lemma 2 and tending \( \theta \) to zero we see from Corollary 2 that the superactivation of 1-shot quantum zero-error capacity may hold for two channels with \( d_A = d_E = 4 \) if one of them is arbitrarily close (in the cb-norm) to a classical-quantum channel.

It is interesting also to note that the entangled subspace spanned by the vectors (11) is an error correcting code for the channel \( \Phi^2_0 \otimes \Phi^2_\pi \) (and hence for the channel \( \Phi^2_0 \otimes \text{Id}_{C^4} \)) despite the fact that \( \Phi^2_0 \) is a classical-quantum channel.

**Proof of Theorem 1.** A) It is easy to verify that the subspace \( \mathfrak{L}_\pi \) satisfies condition (6) with the vectors

\[ |\varphi\rangle = [1, i, 0, 0]^T, \quad |\psi\rangle = [0, 0, 1, i]^T. \]

To show that \( \bar{Q}_0(\Phi^2_\theta) = 0 \) for all \( \theta \neq \pi \) represent the matrix \( M \) in (8) as

\[ M = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, \quad \tau = \gamma - 1. \]

Let

\[ S = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \]

then

\[ S^{-1} \begin{bmatrix} 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \bar{\tau} \\ \bar{\tau} & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \]

and

\[ S^{-1} AS = \begin{bmatrix} \tilde{a} & 0 & 0 & 0 \\ 0 & \tilde{b} & 0 & 0 \\ 0 & 0 & \tilde{c} & 0 \\ 0 & 0 & 0 & \tilde{d} \end{bmatrix}, \quad S^{-1} BS = \begin{bmatrix} u & 0 & 0 & v \\ 0 & u & v & 0 \\ 0 & -v & -u & 0 \\ -v & 0 & 0 & -u \end{bmatrix}, \]

where

\[ \tilde{a} = a - b - c + d, \quad \tilde{b} = a + b - c - d, \quad u = -\Re \tau = 1 - \Re \gamma \]
\[ \tilde{c} = a - b + c - d, \quad \tilde{d} = a + b + c + d, \quad v = i \Im \tau = i \Im \gamma. \]

Thus the subspace \( \mathfrak{L}_\theta \) is unitary equivalent to the subspace

\[ \mathfrak{L}_\theta = \left\{ M = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} + \frac{1}{4}(d + c - b - a) T_\theta, \quad a, b, c, d \in \mathbb{C} \right\} \quad (12) \]
where \( T_\theta = S^{-1}BS \) is the above-defined matrix. Hence it suffices to show that condition (13) is invariant under the rotation \( \gamma \). Assume existence of unit vectors \(|\varphi\rangle = [x_1, x_2, x_3, x_4]^T \) and \(|\psi\rangle = [y_1, y_2, y_3, y_4]^T \) in \( \mathbb{C}^4 \) such that

\[
\langle \psi|M|\varphi \rangle = 0 \quad \text{and} \quad \langle \psi|M|\psi \rangle = \langle \varphi|M|\varphi \rangle \quad \text{for all} \quad M \in \mathfrak{L}_d^x
\]

(13)

Since condition (13) is invariant under the rotation

\[
|\varphi\rangle \mapsto |p\varphi - q\psi\rangle, \quad |\psi\rangle \mapsto |\bar{q}\varphi + \bar{p}\psi\rangle, \quad |p|^2 + |q|^2 = 1,
\]

we may consider that \( y_1 = 0 \).

By taking successively \((a = -1, b = c = d = 0), (b = -1, a = c = d = 0), (c = 1, a = b = d = 0)\) and \((d = 1, a = b = c = 0)\) we obtain from (13) the following equations

\[
\bar{y}_1x_1 = -\bar{y}_2x_2 = -\bar{y}_3x_3 = -\bar{y}_4x_4 = \frac{1}{4}\langle \psi|T_\theta|\varphi \rangle,
\]

\[
|x_1|^2 - |y_1|^2 = |x_2|^2 - |y_2|^2 = |y_3|^2 - |x_3|^2 = |y_4|^2 - |x_4|^2 = \frac{1}{4}[\langle \varphi|T_\theta|\varphi \rangle - \langle \psi|T_\theta|\psi \rangle],
\]

Since \( y_1 = 0 \) and \( ||\varphi|| = ||\psi|| = 1 \), the above equations imply

\[
y_1 = y_2 = x_3 = x_4 = 0
\]

and

\[
|x_1|^2 = |x_2|^2 = |y_3|^2 = |y_4|^2 = \frac{1}{4}[\langle \varphi|T_\theta|\varphi \rangle - \langle \psi|T_\theta|\psi \rangle] = 1/2. \tag{14}
\]

So, \(|\varphi\rangle = [x_1, x_2, 0, 0]^T\) and \(|\psi\rangle = [0, 0, y_3, y_4]^T\), where \([x_1, x_2]^T\) and \([y_3, y_4]^T\) are unit vectors in \( \mathbb{C}^2 \). It follows from (14) that

\[
2 = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_4 \end{pmatrix} - \begin{pmatrix} y_3 & 0 \\ -u & y_4 \end{pmatrix} = 2u,
\]

which can be valid only if \( \gamma = i \), i.e. \( \theta = \pi \).

The above arguments also show that \( \tilde{Q}_0(\Phi_\pi) = \log 2 \), since the assumption \( \tilde{Q}_0(\Phi_\pi) > \log 2 \) implies existence of orthogonal unit vectors \( \phi_1, \phi_2, \phi_3 \) such that condition (13) with \( \varphi = \phi_i, \psi = \phi_j \) is valid for all \( i \neq j \).

B) By Lemma [Lemma] it suffices to show that

\[
\langle \psi|M_1 \otimes \ldots \otimes M_n|\varphi \rangle = 0 \quad \forall M_1 \in \mathfrak{L}_{d_1}, \ldots, M_n \in \mathfrak{L}_{d_n}, \tag{15}
\]
Thus (15) and (16) are valid.

\[
\langle \psi | M_1 \otimes \cdots \otimes M_n | \psi \rangle = \langle \varphi | M_1 \otimes \cdots \otimes M_n | \varphi \rangle \quad \forall M_1 \in \mathcal{L}_{\theta_1}, \ldots, M_n \in \mathcal{L}_{\theta_n}. \tag{16}
\]

Let \( M_1 \in \mathcal{L}_{\theta_1}, \ldots, M_n \in \mathcal{L}_{\theta_n} \) be arbitrary and \( X = M_1 \otimes \cdots \otimes M_n \). Then

\[2 \langle \psi | X | \varphi \rangle = \langle 3 \cdots 3 | X | 1 \cdots 1 \rangle + i \langle 3 \cdots 3 | X | 2 \cdots 2 \rangle - i \langle 4 \cdots 4 | X | 1 \cdots 1 \rangle \]

\[+ \langle 4 \cdots 4 | X | 2 \cdots 2 \rangle = c_1 \cdots c_n (\bar{\gamma}_1 \cdots \bar{\gamma}_n + \gamma_1 \cdots \gamma_n) + d_1 \cdots d_n (i - i) = 0,
\]

since \( \gamma_1 \cdots \gamma_n = \pm i \),

\[2 \langle \varphi | X | \varphi \rangle = \langle 1 \cdots 1 | X | 1 \cdots 1 \rangle + i \langle 1 \cdots 1 | X | 2 \cdots 2 \rangle - i \langle 2 \cdots 2 | X | 1 \cdots 1 \rangle \]

\[+ \langle 2 \cdots 2 | X | 2 \cdots 2 \rangle = a_1 \cdots a_n (1 + 1) + b_1 \cdots b_n (i - i) = 2a_1 \cdots a_n
\]

and

\[2 \langle \psi | X | \psi \rangle = \langle 3 \cdots 3 | X | 3 \cdots 3 \rangle + i \langle 3 \cdots 3 | X | 4 \cdots 4 \rangle - i \langle 4 \cdots 4 | X | 3 \cdots 3 \rangle \]

\[+ \langle 4 \cdots 4 | X | 4 \cdots 4 \rangle = a_1 \cdots a_n (1 + 1) + b_1 \cdots b_n (i - i) = 2a_1 \cdots a_n.
\]

Thus (15) and (16) are valid.

C) The positivity of \( \bar{Q}_0(\Phi_{\pm \pi/n}^{\otimes n}) \) follows from assertion B.

To show that \( \bar{Q}_0(\Phi_{\pm \pi/n}^{\otimes n}) = 0 \) if \( |\theta| \leq 2(\sqrt{3}/2 - 1) \) note that \( \mathcal{L}_\theta = \Psi_\theta(\mathcal{L}_0) \) and \( \mathcal{L}_\theta^{\otimes n} = \Psi_\theta^{\otimes n}(\mathcal{L}_0^{\otimes n}) \), where \( \Psi_\theta(\cdot) \) is the Schur multiplication by the matrix

\[
\begin{bmatrix}
1 & 1 & \gamma & 1 \\
1 & 1 & \bar{\gamma} & 1 \\
\bar{\gamma} & 1 & 1 & 1 \\
1 & \gamma & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & \tau & 0 \\
0 & 0 & 0 & \bar{\tau} \\
\bar{\tau} & 0 & 0 & 0 \\
0 & \tau & 0 & 0
\end{bmatrix}, \quad \tau = \gamma - 1. \tag{17}
\]

Let \( x_n = \| \Psi_\theta^{\otimes n} - \text{Id}_{\mathcal{V}}^{\otimes n} \|_{cb} \). Multiplicativity of the \( cb \)-norm implies

\[x_n \leq \| \Psi_\theta \|_{cb}^{n-1} x_1 + x_{n-1} \leq (1 + x_1)^{n-1} x_1 + x_{n-1} \leq \cdots \leq (x_1 + 1)^n - 1 \tag{18}
\]

By using (17) and Theorem 8.7 in [12] it is easy to show that

\[x_1 = \| \Psi_\theta - \text{Id}_{\mathcal{V}} \|_{cb} \leq |\tau| = |1 - \gamma| = |1 - e^{i\theta/2}| \leq |\theta/2|.
\]
Assume \( \bar{Q}_0(\Phi^{\otimes n}_\theta) > 0 \) for some \( \theta \in [-a_n, a_n] \), where \( a_n = 2(\sqrt{3}/2 - 1) \).
Then Lemma 4 implies existence of unit vectors \( \varphi \) and \( \psi \) such that
\[
\langle \psi \| \Psi^{\otimes n}_\theta (A) \| \varphi \rangle = 0 \quad \text{and} \quad \langle \varphi \| \Psi^{\otimes n}_\theta (A) \| \varphi \rangle = \langle \psi \| \Psi^{\otimes n}_\theta (A) \| \psi \rangle \quad \forall A \in \mathfrak{L}^{\otimes n}.
\]
Hence for any \( A \) in the unit ball of \( \mathfrak{L}^{\otimes n} \) we have
\[
|\langle \psi A \varphi \rangle| \leq x_n \quad \text{and} \quad |\langle \varphi A \varphi \rangle - \langle \psi A \psi \rangle| \leq 2x_n.
\]
It follows from (18), (19) and the assumption \( |\theta| \leq a_n \), that \( x_n \leq 1/2 \). So, the above relations can not be valid by the below Lemma 3, since \( \mathfrak{L}^{\otimes n} \) is a maximal commutative *-subalgebra of \( \mathfrak{M}_n \). □

**Lemma 3.** Let \( \mathfrak{A} \) be a maximal commutative *-subalgebra of \( \mathfrak{M}_n \). Then

either \( 2 \sup_{A \in \mathfrak{A}} |\langle \psi A \varphi \rangle| > 1 \) or \( \sup_{A \in \mathfrak{A}} |\langle \varphi A \varphi \rangle - \langle \psi A \psi \rangle| > 1 \)

for any two unit vectors \( \varphi \) and \( \psi \) in \( \mathbb{C}_n \), where \( \mathfrak{A}_1 \) is the unit ball of \( \mathfrak{A} \).

**Proof.** Let \( \{x_i\}_{i=1}^n \) and \( \{y_i\}_{i=1}^n \) be the coordinates of \( \varphi \) and \( \psi \) in the basis in which the algebra \( \mathfrak{A} \) consists of diagonal matrices. Then
\[
\sup_{A \in \mathfrak{A}_1} |\langle \psi A \varphi \rangle| = \sum_{i=1}^n |x_i||y_i|, \quad \sup_{A \in \mathfrak{A}_1} |\langle \varphi A \varphi \rangle - \langle \psi A \psi \rangle| = \sum_{i=1}^n |x_i|^2 - |y_i|^2.
\]
Let \( d_i = |y_i| - |x_i| \). Assume that
\[
2 \sum_{i=1}^n |x_i||y_i| \leq 1 \quad \text{and} \quad \sum_{i=1}^n |x_i|^2 - |y_i|^2 \leq 1.
\]
Since \( \sum_{i=1}^n |x_i|^2 = \sum_{i=1}^n |y_i|^2 = 1 \), the first of these inequalities implies
\[
\left| \sum_{i=1}^n d_i|x_i| \right| \geq 1/2 \quad \text{and} \quad \left| \sum_{i=1}^n d_i|y_i| \right| \geq 1/2.
\]
Hence
\[
\sum_{i=1}^n |x_i|^2 - |y_i|^2 = \sum_{i=1}^n |d_i||x_i| + |y_i| > \sum_{i=1}^n |d_i|x_i| + \sum_{i=1}^n |d_i|y_i| \geq 1,
\]
where the strict inequality follows from existence of negative and positive numbers in the set \( \{d_i\}_{i=1}^n \). This contradicts to the above assumption. □
Appendix: Stinespring representations for the channels $\Phi^1_\theta$ and $\Phi^2_\theta$

Proof of Lemma 2. Show first that for each $\theta$ one can construct basis $\{A^\theta_i\}_{i=1}^4$ of $\mathfrak{L}_\theta$ consisting of positive operators with $\sum_{i=1}^4 A^\theta_i = I_4$ such that:

1) the function $\theta \mapsto A^\theta_i$ is continuous for $i = 1, 4$;
2) $\{A^\theta_i\}_{i=1}^4$ consists of mutually orthogonal 1-rank projectors.

Note that $\mathfrak{L}_\theta$ is unitary equivalent to the subspace $\mathfrak{L}^\theta_\theta$ defined by (12).

Denote by $\|T_\theta\|$ the operator norm of the matrix $T_\theta$ involved in (12). Note that the function $\theta \mapsto T_\theta$ is continuous, $T_0 = 0$ and $\|T_\theta\| \leq \|T_\pi\| = 2$. Let

$$\tilde{A}^\theta_1 = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} - \frac{1}{4}(\alpha - \beta)T_\theta, \quad \tilde{A}^\theta_2 = \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} - \frac{1}{4}(\alpha - \beta)T_\theta,$$

$$\tilde{A}^\theta_3 = \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} + \frac{1}{4}(\alpha - \beta)T_\theta, \quad \tilde{A}^\theta_4 = \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix} + \frac{1}{4}(\alpha - \beta)T_\theta$$

be operators in $\mathfrak{L}^\theta_\theta$, where $\beta = \min \left\{ \frac{3}{16}, \frac{1}{4}\|T_\theta\| \right\}$ and $\alpha = 1 - 3\beta$. It is easy to verify that $\tilde{A}^\theta_i \geq 0$ for all $i$ and $\sum_{i=1}^4 \tilde{A}^\theta_i = I_4$. Then $\{A^\theta_i = S\tilde{A}^\theta_i S^{-1}\}_{i=1}^4$, where $S$ is the unitary matrix defined before (12), is a required basis of $\mathfrak{L}_\theta$.

Let $m \geq 2$ and $\{|\psi_i\rangle\}_{i=1}^4$ be a collection of unit vectors in $\mathbb{C}^m$ such that $\{|\psi_i\rangle\langle\psi_i|\}_{i=1}^4$ is a linearly independent subset of $\mathfrak{M}_m$. It is easy to show (see the proof of Corollary 1 in [13]) that $\mathfrak{L}_\theta$ is a noncommutative graph of the pseudo-diagonal channel

$$\Phi_\theta(\rho) = \text{Tr}_{\mathbb{C}^m} V_\theta \rho V_\theta^*,$$

where

$$V_\theta : |\varphi\rangle \mapsto \sum_{i=1}^4 \sqrt{A^\theta_i^{1/2}} |\varphi\rangle \otimes |i\rangle \otimes |\psi_i\rangle$$

is an isometry from $\mathcal{H}_A = \mathbb{C}^4$ into $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^m$ ($\{|i\rangle\}$ is the canonical basis in $\mathbb{C}^4$). By property 1 of the basis $\{A^\theta_i\}_{i=1}^4$, the function $\theta \mapsto V_\theta$ is continuous.
The first part of Lemma 2 follows from this construction with \( m = 2 \).

To prove the second part assume that \( m = 4 \) and \( |\psi_i\rangle = |i\rangle \), \( i = 1, 2, 3, 4 \). Property 2 of the basis \( \{A_i^\theta\}_{i=1}^4 \) implies

\[
V_0|\varphi\rangle = \sum_{i=1}^{4} \langle e_i | \varphi \rangle |e_i\rangle \otimes |i\rangle \otimes |i\rangle,
\]

where \( \{ |e_i\rangle \}_{i=1}^4 \) is an orthonormal basis in \( \mathbb{C}^4 \). Hence \( \Phi_0(\rho) = \sum_{i=1}^{4} \langle e_i | \rho | e_i \rangle \sigma_i \), \( \sigma_i = |e_i \otimes i\rangle \langle e_i \otimes i| \), is a classical-quantum channel.

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