Bosonic and fermionic T-dualization of type II superstring theory in double space

Bojan Nikolić and Branislav Sazdović

Institute of Physics Belgrade, Serbia

Recent Advances in T/U-dualities and Generalized Geometries
06.-09. June 2017, Zagreb, Croatia
Outline of the talk

1. T-duality
2. Model
3. Bosonic T-duality
4. Fermionic T-duality
5. Concluding remarks
Superstrings

- There are five consistent superstring theories. They are connected by web of T and S dualities.
- There are three approaches to superstring theory: NSR (Neveu-Schwarz-Ramond), GS (Green-Schwarz) and pure spinor formalism (N. Berkovits, hep-th/0001035).
- T-duality transformation does not change the physical content of the theory.
- Well known bosonic and recently discovered fermionic T-duality.
Idea od double space

- Double space \(=\) initial coordinates plus T-dual partners - Siegel, Duff, Tseytlin about 25 years ago.

- Interest for this subject emerged again (Hull, Berman, Zwiebach) in the context of T-duality as \(O(d, d)\) transformation.

- The approach of Duff has been recently improved when the T-dualization along some subset of the initial and corresponding subset of the T-dual coordinates has been interpreted as permutation of these subsets in the double space coordinates (arXiv:1505.06044, 1503.05580). All calculations are made in full double space.

- In double space T-duality is a symmetry transformation.
General pure spinor action for type II superstring

We start from the general pure spinor action for type II superstring (arXiv: 0405072)

\[
S = \int d^2 \xi \left[ \partial_+ \theta^\alpha A_{\alpha\beta} \partial_- \bar{\theta}^\beta + \partial_+ \theta^\alpha A_{\alpha\mu} \Pi_\mu^- + \Pi_+ \bar{A}_{\mu\alpha} \partial_- \bar{\theta}^\alpha \\
+ \Pi_+ A_{\mu\nu} \Pi_-^\nu + d_\alpha E^\alpha \beta \partial_- \bar{\theta}^\beta + d_\alpha E^\alpha \mu \Pi_-^\mu + \partial_+ \theta^\alpha E^\alpha \beta \partial_- \bar{\theta}^\beta + \Pi_+ E_{\mu\beta} \partial_- \bar{\theta}^\beta \\
+ d_\alpha P^{\alpha\beta} \partial_- \bar{\theta}^\beta \right] + \frac{1}{2} N_{\mu\nu}^+ \Omega_{\mu\nu,\beta} \partial_- \bar{\theta}^\beta + \frac{1}{2} N_{\mu\nu}^+ \Omega_{\mu\nu,\rho} \Pi_-^\rho + \frac{1}{2} \partial_+ \theta^\alpha \Omega_{\alpha,\mu\nu} \bar{N}^{\mu\nu} \\
+ \frac{1}{2} \Pi_+ \Omega_{\mu,\nu\rho} \bar{N}_-^{\nu\rho} + \frac{1}{2} N_{\mu\nu}^+ \bar{C}_{\mu\nu}^\beta \partial_- \bar{\theta}^\beta + \frac{1}{2} d_\alpha C_{\mu\nu}^\alpha \bar{N}^{\mu\nu} \\
+ \frac{1}{4} N_{\mu\nu}^+ S_{\mu\nu,\rho\sigma} \bar{N}^{\rho\sigma}_- \right] + S_\lambda + S_{\bar{\lambda}}.
\]
**Bosonic T-duality - assumptions and approximations**

- **Bosonic T-dualization** - we assume that background fields are independent of $x^\mu$. In mentioned reference, expressions for background fields as well as action are obtained in an iterative procedure as an expansion in powers of $\theta^\alpha$ and $\bar{\theta}^\alpha$. Every step in iterative procedure depends on previous one, so, for mathematical simplicity, we consider only basic ($\theta$ and $\bar{\theta}$ independent) components.
Fermionic T-duality - assumptions and consistency check

- **Fermionic T-dualization** - we assume that $\theta^\alpha$ and $\bar{\theta}^\alpha$ are Killing directions. Consequently, auxiliary superfields are zero according to arXiv: 0405072. If we assume that rest of background fields are constant then their curvatures are zero. Using space-time field equations we confirmed the consistency of the choice of constant $P^{\alpha\beta}$.
Action

- In both cases, under introduced assumptions, action gets the form

\[ S = \kappa \int_{\Sigma} d^2 \xi \left[ \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu + \frac{1}{4\pi \kappa} \Phi R^{(2)} \right] \]

\[ + \int_{\Sigma} d^2 \xi \left[ -\pi_\alpha \partial_-(\theta^\alpha + \psi^\alpha_\mu x^\mu) + \partial_+(\bar{\theta}^\alpha + \bar{\psi}^\alpha_\mu x^\mu) \bar{\pi}_\alpha + \frac{\Theta^\phi_\Sigma}{2\kappa} \pi_\alpha F^{\alpha\beta} \bar{\pi}_\beta \right] \]

- Definitions: \( \Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu} \), \( \Phi \) is dilaton field, \( \psi^\alpha_\mu \) and \( \bar{\psi}^\alpha_\mu \) are NS-R fields and \( F^{\alpha\beta} \) is R-R field strength. Momента \( \pi_\alpha \) and \( \bar{\pi}_\alpha \) are canonically conjugated to \( \theta^\alpha \) and \( \bar{\theta}^\alpha \). All spinors are Majorana-Weyl ones.

- All background fields are constant.
Global shift symmetry exists $x^a \rightarrow x^a + b$, where index $a$ is subset of $\mu$.

We introduce gauge fields $v_\pm^a$ and make change in the action $\partial_\pm x^a \rightarrow \partial_\pm x^a + v_\pm^a$.

Additional term in the action

$$S_{gauge}(y, v_\pm) = \frac{1}{2} \kappa \int_\Sigma d^2 \xi \left( v_+^a \partial_- y^a - \partial_+ y^a v_-^a \right),$$

where $y^a$ is Lagrange multiplier. It makes $v_\pm^a$ to be unphysical degrees of freedom.

On the equations of motion for $y^a$ we get initial action, while, fixing $x^a$ to zero, on th equations of motion for $v_\pm^a$ we get T-dual action.
Solution of the equation of motion for $y_a$ is $v_\pm^a = \partial_\pm x^a$. Combining this solution with equations of motion for gauge fields $v_\pm^a$ we obtain T-dual transformation laws

$$\partial_\pm x^a \simeq -2\kappa \hat{\theta}_\pm^{ab} \Pi_{\mp bi} \partial_\pm x^i - \kappa \hat{\theta}_\pm^{ab} (\partial_\pm y_b - J_{\pm b}), \quad (3)$$

$$\partial_\pm y_a \simeq -2\Pi_{\mp ab} \partial_\pm x^b - 2\Pi_{\mp ai} \partial_\pm x^i + J_{\pm a}. \quad (4)$$

Here $J_{\pm \mu} = \pm \frac{2}{\kappa} \psi_\pm^\alpha \pi_{\pm \alpha}$ and $\theta_\pm^{ac} \Pi_{\mp cb} = \frac{1}{2\kappa} \delta^a_b$, where

$$\psi_+^\alpha \equiv \psi_\mu^\alpha, \quad \psi_-^\alpha \equiv \bar{\psi}_\mu^\alpha, \quad \pi_+^\alpha \equiv \pi_\alpha, \quad \pi_-^\alpha \equiv \bar{\pi}_\alpha. \quad (5)$$
Transformation laws in double space

- In double space spanned by $Z^M = (x^\mu, y_\mu)^T$ they are of the form

$$\partial_{\pm} Z^M \equiv \pm \Omega^{MN} \left( \mathcal{H}_{NP} \partial_{\pm} Z^P + J_{\pm N} \right), \quad (6)$$

where

$$\mathcal{H}_{MN} = \begin{pmatrix} G^E_{\mu\nu} & -2 B_{\mu\rho} (G^{-1})^{\rho\nu} \\ 2 (G^{-1})^\mu_\rho B_{\rho\nu} & (G^{-1})^\mu_\nu \end{pmatrix}, \quad (7)$$

is so called generalized metric, while

$$\Omega^{MN} = \begin{pmatrix} 0 & 1_D \\ 1_D & 0 \end{pmatrix}, \quad J_{\pm M} = \begin{pmatrix} 2 (\Pi_{\pm} G^{-1})^\mu_\nu J_{\pm \nu} \\ -(G^{-1})^\mu_\nu J_{\pm \nu} \end{pmatrix}. \quad (8)$$

$\Omega^{MN}$ is constant symmetric matrix and it is known as $SO(D, D)$ invariant metric. Here $G^E_{\mu\nu} = G_{\mu\nu} - 4 (B G^{-1} B)_{\mu\nu}.$
T-duality as permutation in double space

- T-dualization in double space is represented by permutation

\[
aZ^M \equiv \begin{pmatrix} y_a \\ x^i \\ x^a \\ y_i \end{pmatrix} = (\mathcal{T}^a)^M_N Z^N \equiv \begin{pmatrix} 0 & 0 & 1_a & 0 \\ 0 & 1_i & 0 & 0 \\ 1_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_i \end{pmatrix} \begin{pmatrix} x^a \\ x^i \\ y_a \\ y_i \end{pmatrix}.
\]
Demanding that $aZ^M$ has the transformation law as initial coordinates $Z^M$, we find the T-dual generalized metric

$$a\mathcal{H}_{MN} = (\mathcal{T}^a)_M^K \mathcal{H}_{KL}(\mathcal{T}^a)^L_N,$$

and T-dual current

$$aJ_{\pm M} = (\mathcal{T}^a)_M^N J_{\pm N}. $$
From (9) we obtain the T-dual NS-NS background fields which are in full agreement with those obtained by Buscher procedure

\[ a \Pi^{ab}_{\pm} = \frac{\kappa}{2} \hat{\theta}^{ab}_{\mp}, \quad a \Pi^{a}_{\pm i} = \kappa \hat{\theta}^{ab}_{\mp} \Pi^{\pm bi}_{\mp}, \]
\[ a \Pi_{\pm i}^{\pm a} = -\kappa \Pi_{\pm ib}^{\pm ba}, \quad a \Pi_{\pm ij}^{\pm} = \Pi_{\pm ij} - 2\kappa \Pi_{\pm ia}^{\pm} \hat{\theta}^{ab}_{\mp} \Pi_{\pm bj}. \]
From (10) we obtain the form of the T-dual NS-R fields

\[ a \psi^\alpha a = \kappa \hat{\theta}^{ab} \psi^\alpha_b , \quad a \bar{\psi}^\alpha a = \kappa a \Omega^\alpha_{\beta} \hat{\theta}^{ab} \bar{\psi}_b^\beta . \]  

(11)

\[ a \psi^\alpha_i = \psi^\alpha_i - 2\kappa \pi_{ib} \hat{\theta}^{ba} \psi^\alpha_a , \quad a \bar{\psi}^\alpha_i = a \Omega^\alpha_{\beta} ( \bar{\psi}_i^\beta - 2\kappa \pi_{ib} \hat{\theta}^{ba} \bar{\psi}_a^\beta ) . \]  

(12)

From transformation laws we see that two chiral sectors transform differently. Consequently, there are two sets of vielbeins in T-dual picture as well two sets of gamma matrices. This T-dual vielbeins are connected by Lorentz transformation, while spinorial representation of this Lorentz transformation, \( a \Omega^\alpha_{\beta} \), relates two sets of gamma matrices. In order to have unique set of gamma matrices, we have to multiply one fermionic index by \( a \Omega^\alpha_{\beta} \).
R-R field strength

- R-R field strength couples fermionic momenta and, consequently, its T-dual can not be read from transformation law.

- From the demand that term in the action is T-dual invariant, we obtain the form of the T-dual R-R field strength

\[ e^{\frac{a}{2}} a F^{\alpha \beta} = (e^{\frac{a}{2}} F^{\alpha \gamma} + c \psi^\alpha \hat{\theta}^{ab} \bar{\psi}^\gamma)_a \Omega^{\alpha \beta}, \]  

(13)

where \( c \) is an arbitrary constant. For the specific value of \( c \), we get the same expression as in Buscher procedure.
Basic facts

- In last years it was seen that tree level superstring theories on certain supersymmetric backgrounds admit a symmetry which is called fermionic T-duality.
- This is a redefinition of the fermionic worldsheet fields similar to the redefinition we perform on bosonic variables when we do an ordinary T-duality.
- Technically, the procedure is the same as in the bosonic case up to the fact that dualization will be done along $\theta^\alpha$ and $\bar{\theta}^\alpha$ directions.
On the equations of motion for $\pi_\alpha$ and $\bar{\pi}_\alpha$ action (2) becomes

$$S = \kappa \int_{\Sigma} d^2\xi \partial_+ x^\mu \left[ \Pi_{\mu\nu} + \frac{1}{2} \bar{\Psi}_\mu (P^{-1})_{\alpha\beta} \psi_\nu^\beta \right] \partial_- x^\nu$$

$$+ \frac{1}{4\pi} \int_{\Sigma} d^2\xi \Phi R^{(2)}$$

$$+ \frac{\kappa}{2} \int_{\Sigma} d^2\xi \left[ \partial_+ \bar{\theta}^\alpha (P^{-1})_{\alpha\beta} \partial_- \theta^\beta + \partial_+ \bar{\theta}^\alpha (P^{-1} \psi)_{\alpha\mu} \partial_- x^\nu \right]$$

$$+ \partial_+ x^\mu (\bar{\Psi} P^{-1})_{\mu\alpha} \partial_- \theta^\alpha \right] .$$
Fixing the chiral gauge invariance

- In the above action $\theta^\alpha$ appears only in the form $\partial_- \theta^\alpha$ and $\bar{\theta}^\alpha$ in the form $\partial_+ \bar{\theta}^\alpha$.
- Using the BRST formalism we fix the chiral gauge invariance adding to the action

$$S_{gf} = -\frac{\kappa}{2} \int d^2 \xi \partial_- \bar{\theta}^\alpha (\alpha^{-1})_{\alpha\beta} \partial_+ \theta^\beta, \quad (14)$$

where $\alpha^{\alpha\beta}$ is arbitrary non singular matrix.
Applying the same mathematical procedure as in the case of the bosonic T-dualization, we have

\[ \partial_+ \theta^\alpha \cong - P^{\alpha \beta} \partial_- \vartheta_\beta - \psi_\mu^\alpha \partial_- x^\mu, \partial_+ \bar{\theta}^\alpha \cong \partial_+ \bar{\vartheta}_\beta P^{\beta \alpha} - \partial_+ x^\mu \bar{\psi}_\mu^\alpha, \]  
(15)

\[ \partial_- \theta^\alpha \cong - \alpha^{\alpha \beta} \partial_+ \vartheta_\beta, \partial_- \bar{\theta}^\alpha \cong \partial_- \bar{\vartheta}_\beta \alpha^\beta \alpha, \]  
(16)

where \( \vartheta_\alpha \) and \( \bar{\vartheta}_\alpha \) are T-dual fermionic coordinates.
Transformation laws in double space

Let us introduce double fermionic coordinates

\[ \Theta^A = \begin{pmatrix} \theta^\alpha \\ \vartheta^\alpha \end{pmatrix}, \quad \bar{\Theta}^A = \begin{pmatrix} \bar{\theta}^\alpha \\ \bar{\vartheta}^\alpha \end{pmatrix}. \] (17)

Transformation laws in double space are of the form

\[ \partial_- \Theta^A \cong -\Omega^{AB} \left[ F_{BC} \partial_- \Theta^C + J_{-B} \right], \]
\[ \partial_+ \bar{\Theta}^A \cong \left[ \partial_+ \bar{\Theta}^C F_{CB} + \bar{J}_{+B} \right] \Omega^{BA}, \]
\[ \partial_+ \Theta^A \cong -\Omega^{AB} A_{BC} \partial_+ \Theta^C, \partial_- \bar{\Theta}^A \cong \partial_- \bar{\Theta}^C A_{CB} \Omega^{BA}. \]
The generalized metric and the matrix $\mathcal{A}_{AB}$ are

$$\mathcal{F}_{AB} = \begin{pmatrix} (P^{-1})_{\alpha\beta} & 0 \\ 0 & P_{\gamma\delta} \end{pmatrix}, \mathcal{A}_{AB} = \begin{pmatrix} (\alpha^{-1})_{\alpha\beta} & 0 \\ 0 & \alpha^{-1}_{\gamma\delta} \end{pmatrix}.$$  

The currents are of the form

$$\bar{J}^A_+ = \begin{pmatrix} (\bar{\Psi} P^{-1})_{\mu\alpha} \partial^+ x^\mu \\ -\bar{\Psi}^\alpha_{\mu} \partial^+ x^\mu \end{pmatrix}, \mathcal{J}^A_- = \begin{pmatrix} (P^{-1}\Psi)_{\alpha\mu} \partial^- x^\mu \\ \Psi^\alpha_{\mu} \partial^- x^\mu \end{pmatrix}.$$
Fermionic T-dualization as permutation

- T-dual coordinates are

\[ *\Theta^A = T^A_B \Theta^B, \quad *\bar{\Theta}^A = T^A_B \bar{\Theta}^B, \]

where

\[ T^A_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

is permutation matrix.

Bojan Nikolić

Bosonic and fermionic T-dualization of type II superstring theory in
Fermionic T-dualization as permutation

- Demanding that T-dual coordinates transformation laws are of the same form as those for initial coordinates we get

\[
* F_{AB} = T^C_A F_{CD} T^D_B, \quad * \bar{J}_{+A} = T^B_A \bar{J}_{+B}, \quad * J_{-A} = T^B_A J_{-B}.
\]

- The matrix $A_{AB}$ transforms as

\[
* A_{AB} = T^C_A A_{CD} T^D_B = (A^{-1})_{AB}.
\]  \hspace{1cm} (18)
From these relations we obtain the R-R and NS-R T-dual background fields in the same form as in the Buscher procedure

\[ *P_{\alpha\beta} = (P^{-1})_{\alpha\beta}, \quad (*\alpha)_{\alpha\beta} = (\alpha^{-1})_{\alpha\beta}, \]

\[ *\psi_{\alpha\mu} = (P^{-1})_{\alpha\beta} \psi_\beta^\mu, \quad \bar{\psi}^{\mu}_{\alpha\mu} = -\bar{\psi}_\beta^\mu (P^{-1})_{\beta\alpha}. \]
\[ \Pi_{\mu \nu} \] is coupled by \( x \)'s and we can not read the T-dual field from transformation laws.

As in the case of bosonic T-dualization, assuming that this term is invariant under T-dualization, we get the appropriate fermionic T-dual

\[ \star \Pi_{\mu \nu} = \Pi_{\mu \nu} + c \bar{\Psi}^\alpha (P^{-1})_{\alpha \beta} \Psi^\beta, \quad (19) \]

where \( c \) is an arbitrary constant.
Concluding remarks

- We represented both kind of T-dualizations of type II superstring as permutation symmetry in double space.
- The successive T-dualizations make a group called T-duality group. In the case of type II superstring fermionic T-duality transformations are performed by the same matrices $T^a$ as in the bosonic string case. Consequently, the corresponding T-duality group is the same.
- In the bosonic case there is an advantage of this approach. In one equation all T-dual theories (for any subset $x^a$) are contained. We do not have to repeat procedure for each specific choice of $x^a$. This kind of approach could be helpful in better understanding of M-theory.