Multidimensional Einstein-Yang-Mills cosmological models

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Abstract

We study the process of the evolution of the space of extra dimensions in the framework of Einstein-Yang-Mills cosmological models. It is shown that, for certain classes of models, the static compact space of extra dimensions is the attractor for a wide range of initial conditions. Also the effect of isotropization of extra dimensions in the course of evolution is demonstrated.

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1. The Kaluza-Klein theories, i.e. theories formulated on the space-time with \((1 + 3 + d)\) dimensions, is an important framework for the unification of particle interactions \([1]\) and, probably, could be relevant in cosmology for generation of anisotropic and inhomogeneous generalizations of the radiative Friedmann - Lemaître - Robertson - Walker (FLRW) models in four dimensions \([2]\). When considered in the cosmological setting a Kaluza-Klein model should possess certain classical properties in order not to contradict the observations. The most important of them are the following:

a) The model should possess a solution describing an expanding three-dimensional space to the present size of the Universe.

b) The scale factor(s) \(L\) of the space of extra dimensions should decrease up to a size \(L \leq (0.1 \div 1) \text{TeV}^{-1}\) in the course of the evolution, described by the same solution, and the model should have a mechanism for stopping the contraction or at least slowing it down considerably at this value of \(L\). The bounds on the variation of \(L(t)\) following from the yield of primordial \(^4\text{He}\) are: \(0.99 \leq L_0/L_N \leq 1.01\), where \(L_0\) is the present size of the space of extra dimensions and \(L_N\) its size at the time of the nucleosynthesis \([3]\).

c) The solution should be an attractor for a wide range of initial conditions (naturalness of the solution).

d) The solution should satisfy restrictions imposed by the limits on the energy density of particles produced during the contraction of the space of extra dimensions (see Refs. \([4]\)).

We would like to remark that many of the multidimensional cosmological models considered in the literature do not exhibit some and in some cases none of these basic properties. Examples of these are Kasner-type solutions in \((4 + d)\) dimensions and models dominated by the energy density of radiation (see eg. \([3]\)).

2. In this contribution we consider a class of Einstein-Yang-Mills models which will be shown to satisfy the conditions (b)-(c). The isotropization of the space of extra dimensions metric will be also shown.

The models are formulated on the manifold \(M_{(4+d)} = M_{(4)} \times K\), where \(M_{(4)}\) is the four-dimensional part of the space-time and \(K = S/R\) is a compact homogeneous \(d\)-dimensional space on which the group \(S\) acts transitively. The action is given by

\[
S = \int_{M_{(4+d)}} d\hat{\tau} \sqrt{-\hat{g}} \left( \frac{\hat{R}}{16\pi \hat{\kappa}} - \frac{1}{8\hat{e}^2} Tr \hat{F}_{MN} \hat{F}^{MN} - \frac{\hat{\Lambda}}{16\pi \hat{\kappa}} \right),
\]

where \(M, N = 0, 1, \ldots 3 + d\), \(\hat{g} = \det(\hat{g}_{MN})\), \(\hat{R}\) is the multidimensional scalar curvature, \(\hat{\kappa}, \hat{\Lambda}\) and \(\hat{e}\) are the gravitational, cosmological and gauge constants.
in \((4 + d)\) dimensions respectively and \(\hat{F}_{MN}\) is the stress tensor of the gauge field with a gauge group \(G\). We consider the class of models with metric \(\hat{g}_{MN}\) given by:

\[
ds^2 = -dt^2 + a(t)^2d\Omega_3^2 + L_1^2(t)d\omega_{(d_1)}^2 + \ldots + L_k^2(t)d\omega_{(d_k)}^2.
\]

The four-dimensional part of the metric is the one corresponding to a FLRW model and the metric on \(S/R\) is the most general \(S\)-invariant metric. The terms \(L_p^2d\omega_{(d_p)}^2\) determine the invariant metric on the irreducible invariant \(d_p\)-dimensional subspaces in the tangent space decomposition of \(S/R\) under the action of the isotropy group \(R\).

We assume that there is a non-trivial background gauge field, which is described by the so-called \(S\)-symmetric gauge field configuration, satisfying the equations of motion. An important solution for the Kaluza-Klein cosmology of this type, is the one corresponding to magnetic monopole like configurations on \(S/R\), and this was first proposed to drive the compactification of the extra dimensions in [6]. The dimensional reduction of the theory (1) can be carried out consistently and gives (see [7]) an effective theory in four dimensions describing Einstein gravity with scalar fields associated to the scales \(L_p(t)\) with a certain potential \(W\). For the models considered here the potential posseses at least one minimum due to the non-trivial monopole contribution. In general, depending on the initial conditions, the evolution leads to one of the two situations:

i) All or some of the scales \(L_p\) tend to \(+\infty\). This corresponds to decompactification, i.e. unlimited growth of the size of extra dimensions, and is physically unacceptable.

ii) The scales approach the minimum, \(L_p \to L_p^0\), through damped oscillations about it (see below). This corresponds to a successful compactification of the extra dimensions.

The question we address here is how large is the region of initial values \((L_p(0), \dot{L}_p(0))\) for which compactification is successful, or in other words, for which set of initial conditions the minimum is an attractor.

3. We consider first the case where \(S/R\) is a symmetric space [8, 9]. Then the metric is parametrized by one single scale, the scalar curvature on \(S/R\) is \(R^{(d)} = \mathcal{R}_0/L^2\) and the monopole contribution is given by \(TrF^2/(8\epsilon^2) = v_0/L^4\), where \(\mathcal{R}_0\) and \(v_0\) are some constants.

For the range of values of \(\Lambda\) that we are of interest here the potential has a minimum at \(L = L_{\text{min}}\) and a maximum at \(L = L_{\text{max}}\). We introduce the scalar field \(b(t) = \ln(L(t)/L_{\text{min}})\) and choose the cosmological constant such that \(W_{\text{min}} \equiv W(b = 0) = 0\). Then \(L_{\text{min}} = 2v_0/\mathcal{R}_0\), \(L_{\text{max}}^2 = (d + 4)L_{\text{min}}^2/d\).
and the potential is equal to

\[ W(b) = \frac{\mathcal{A}}{16\pi\kappa} e^{-db} \left(1 - e^{-2b}\right)^2, \]

where \( \mathcal{A} = \mathcal{R}_{00}^2/(4v_0) \). The equations of motion are

\[ h^2 \equiv \left(\frac{a'}{a}\right)^2 = \frac{d(d+2)}{12} b'^2 + \frac{(16\pi\kappa)}{6} W(b), \]

\[ b'' + 3h(t)b' + \frac{16\pi\kappa}{d(d+2)} \frac{dW(b)}{db} = 0, \]

where the prime means derivative with respect to \( t \). This system was studied qualitatively in refs. [8, 9]. It describes the evolution of a homogeneous scalar field in the potential (3) with a growing scale factor \( a(t) \). The quantity

\[ E = h^2/M_{Pl}^4 = d(d+2)b'^2/2M_{Pl}^2 + W(b)/M_{Pl}^4 \]

can be interpreted as the energy of a 1-dimensional particle (in the units of \( M_{Pl} \)). The motion of the particle is dissipative due to the viscosity term in eq. (5). Indeed, that equation can be rewritten as \( \dot{E} = -3hb'^2 \leq 0 \).

In Fig. 1 we show part of the phase plane \((b, b')\) around the minimum for \( d = 6 \). The dashed lines represent the levels of constant \( E \): \( E = 1 \) and \( E = 10 \). The former one corresponds to the Planckian energy densities and thus limits the sector of the classical dynamics, i.e. the region of values of \( b \) and \( b' \) where the energy density of the field is smaller than \( M_{Pl}^4 \) and hence the classical equations of motion make sense. We suppose that the Universe started its classical evolution from small values of \( L \) and we consider the case where the initial value \( b(0) = b_0 < 0 \). Fig. 1 shows two typical trajectories (thin lines) corresponding to the cases (i) and (ii) of Sect. 2.

The thick line in Fig. 1 is the boundary of the trapped region \( T \), i.e. the part of the phase space for which the minimum \( b = 0 \) is the attractor. Lacking a satisfactory quantum theory of gravity it is not possible to know the initial conditions for the classical evolution of the Universe. If we follow the conventional Hot Big Bang scenario, the trapped region in Fig. 1 corresponds to the region where the initial conditions at the moment, when the Universe enters the low temperature regime, should lie for the compactification to be successful. We see that the large part of the classical region lies inside \( T \).

Consider now scenarios based on the quantum creation of the Universe. In this case, according to the Vilenkin distribution for \( b \) [10] or the Hartle-Hawking distribution function [11], the Universe was probably created with
\[ b' = 0 \] away from the origin of the potential, i.e. where \( E = W(b)/M^4_{Pl} \sim 1 \), or on the top of the local maximum of the potential. (These statements, however, should be taken with caution since the potential \( (3) \) strictly speaking does not satisfy the condition of slow variation.) Fig. 1 shows that in both cases the initial state is within the trapped region and the Universe always approaches the \( M^4 \times S/R \) space-time with \( L = L_{\text{min}} \). For other numbers of the extra dimensions the situation is qualitatively the same, but for increasing \( d \) the region \( T \) extends itself towards the direction of negative values for \( b \). Similar analysis for the Candelas - Weinberg model and the Calabi-Yau compactification in a superstring model can be found in [12].

4. \( S/R = SU(5)/SU(2) \times U(1) \). The space of extra dimensions is a six-dimensional non-symmetric homogeneous space, known as the \( CP^3 \) manifold, with the anisotropic metric \( (2) \) parametrized by two scales \( (k = 2) \). After fine tuning \( \Lambda \), as in Sect. 3, the potential, \( W \), has a minimum at \( (L_{1\text{min}}, L_{2\text{min}}) \) with \( L^2_{2\text{min}} = 2L^2_{1\text{min}} \) and a saddle point with \( L^2_{2s} = 2L^2_{1s} = 10L^2_{1\text{min}}/3 \). Its explicit form as a function of the two scalar fields, \( b_p(t) = \ln L_p(t)/L_{p\text{min}} \) \( (p = 1, 2) \), was calculated in [7]. Fig. 2 represents the contour plot of the potential in the \( (b_1, b_2) \)-plane and the levels \( E = 1 \) and \( E = 10 \), where, \( E \equiv W(b_1, b_2)/M^4_{Pl} \). \( W \) decreases exponentially for \( b_1, b_2 \to +\infty \) and grows exponentially when \( b_1, b_2 \to -\infty \). The line \( b_1 = b_2 \) corresponds to the isotropic \( SU(4) \)-invariant metric on \( SU(4)/SU(3) \times U(1) \) (unsquashed \( CP^3 \)), which is of course topologically equivalent to \( S/R \). The thick closed curve is the boundary of the trapped region, which in this case, is the region of initial conditions \( (b_{10}, b'_{10} = 0; b_{20}, b'_{20} = 0) \) for which compactification is successful. For all initial conditions of this type, with \( b_{10}, b_{20} < 0 \) situated in the classical region \( E \leq 1 \), the minimum of the potential \( b_1 = b_2 = 0 \) is the attractor. The same holds for initial conditions favoured by scenarios of the quantum creation of the Universe. It appears that, in this model, isotropy in the space of extra dimensions is restored dynamically. Two typical trajectories with highly anisotropic initial metric (i.e. with \( b_1 \neq b_2 \)) are depicted in Fig. 2. We see that after a few oscillations the trajectories approach the line of isotropic metrics, for which \( b_1 = b_2 \), and the main part of evolution occurs in the narrow vicinity of this line.

5. Thus we have shown that the Einstein-Yang-Mills cosmological models considered here satisfy conditions (b) and (c) of Sect 1. Indeed, a \( M^4 \times S/R \) Universe with constant radii for the space of extra dimensions is achieved for a wide range of initial conditions in the Hot Big Bang scenario and for the initial states suggested by scenarios of quantum creation of the Universe. Furthermore, in the model with two scales in the space \( K \), it was shown that
isotropization occurs, i.e. the metric tends to the one with higher symmetry in the course of its evolution.

What about conditions (a) and (d) above? During the evolution of $L_p(t)$, the scale factor of the 3-dimensional space expands according to the law $a(t) \sim t^\lambda$ with $\lambda < 1$ [8]. Hence, our models do not describe inflationary expansion without, for instance, additional scalar fields [8, 13]. The inclusion of effects due to particle production (see [14]) and vacuum polarization might also help in achieving power law inflation in these models. Once the scales of the extra dimensions have reached their minimum, then the main contribution to expansion comes from radiation and further evolution of $a(t)$ follows the standard radiation-dominated scenarios. To check whether the models satisfy condition (d) further calculations are needed. This work is in progress now.

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References
[1] For reviews see M.J. Duff, B. Nilsson and C. Pope, Phys. Rep. C130 (1966) 1; Yu.A. Kubyshin, J.M. Mourão, G. Rudolph and I.P. Volobujev, Dimensional Reduction of Gauge Theories, Spontaneous Compactification and Model Building, Lecture Notes in Physics, Vol. 349 (Springer-Verlag, Berlin, 1989); D. Kapetanakis and G. Zoupanos, Phys. Rep. C219 (1992) 1.
[2] J. Ibáñez and E. Verdaguer, Astrophys. J. 306 (1986) 401.
[3] E. Kolb and M. Turner. The Early Universe. (Addison-Wesley Publishing Co, Redwood City, Ca, 1990).
[4] K. Maeda, Phys. Lett. 138B (1984) 269; J. Garriga and E. Verdaguer, Phys. Rev. D39 (1989) 1072.
[6] E. Cremmer and J. Scherk, *Nucl. Phys.* B118 (1977) 61; J.F. Luciani, *Nucl. Phys.* B135 (1978) 11.

[7] Yu.A. Kubyshin and J.I. Pérez Cadenas, *Compactification to non-symmetric homogeneous space in multidimensional Einstein-Yang-Mills theory.* Preprint UB-ECM-PF-93/11, University of Barcelona, 1993. To appear in *Comm. Theor. Phys.* 3 (1994).

[8] Yu.A. Kubyshin, V.A. Rubakov and I.I. Tkachev, *Int. J. Mod. Phys.* A 4 (1989) 1409.

[9] O. Bertolami, Yu.A. Kubyshin and J.M. Mourão, *Phys. Rev.* D45 (1992) 3405.

[10] A. Vilenkin, *Phys. Lett.* 117B (1982) 25; *Phys. Rev.* D27 (1983) 848; *Phys. Rev.* D37 (1988) 888. A. Linde, *Lett. Nuovo Cimento* 39 (1984) 401.

[11] S.W. Hawking, *Pontif. Accad. Sci. Scr. Varia* 48 (1982) 563; J.B. Hartle and S.W. Hawking, *Phys. Rev.* D28 (1983) 2960.

[12] K. Maeda and P.Y.T. Pang, *Phys. Lett.* B180 (1986) 29.

[13] M.C. Bento, O. Bertolami and P.M. Sá, *Phys. Lett.* B262 (1991) 11.

[14] J. Yokoyama and K. Maeda, *Phys. Lett.* B207 (1988) 31.
Figure captions

Fig. 1 Phase plane \((b', b)\) of the model with symmetric space of extra dimensions. The dashed curves represent lines of constant "energy" defined by eq. \(\text{(3)}\). Thick solid line is the boundary of the trapped region \(T\), where lie the points for which the Universe \(M^4 \times S/R\), with \(L = L_{\text{min}}\), is an attractor. The two characteristic trajectories are shown as thin solid lines.

Fig. 2 Contour plot of the potential, \(W(b_1, b_2)\), of the model of Sect. 4. Curves of constant \(E = W/M^4_{pl}\) are shown as dashed lines. \(E = 1\) marks the limit of application of the classical description. The thick closed curve is the boundary of the trapped region, the region for which the minimum of the potential is an attractor. Two characteristic trajectories of the evolution of the scale factors are shown as thin solid lines. The initial conditions are anisotropic \((b_1 \neq b_2)\), however the metric isotropizes after a few oscillations.
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