Study on Constitutive Relations of Unsaturated Sand

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Abstract. The development of a coupled hydraulic-mechanical bounding-surface constitutive model for unsaturated sand based on phase transformation state. The key point of the model is the state parameter which is introduced based on the phase transformation state considering the fabric change effects, and with the Soil-Water Characteristic Curve and taking into account of the hydraulic-mechanical coupling effects, a bounding-surface model is established. The model has smooth transition between unsaturated state, saturated state and dry state. The model parameters are easily calibrated via common triaxial tests and some other simple tests.

1. Hydro-mechanical process of unsaturated sands
For saturated or dry medium-dense and dense sand, especially coarse-grained sand, it is easy to undergo dilatation under the effect of shear stress, but the dilatancy characteristics of loose sand or silty sand are not obvious which is related to the initial porosity ratio and the soil quality. But under the action of suction, additional "effective pressure" is generated between the particles which increases the cohesive force between the particles. Due to the surface tension generated at the meniscus of the particle contact point, the particles are more tightly combined, and the fine sand particles are combined into large particles. Under the effect of shear stress, the particles at the shear surface are more likely to be lifted, and the volume changes due to rotation. The dilatancy characteristics are obvious, and the shear strength is also greatly improved.

As shown in Figure1, the confining pressure is 20kPa and 75kPa, respectively, For ASU compact silty sand under different suction[1], With the increase of the axial strain, the shear stress continues to increase until a significant peak point appears, and then enters the softening stage and eventually fails, taking on obvious strain strengthening and softening characteristics; The volumetric strain is in the shrinking state in the initial stage. Then the phase transition point is quickly reached and enters the dilatancy stage; When the suction is zero which means the saturation state, the silty sand does not show the above characteristics but the strain strengthening and volume shrinkage characteristics; Further, from the change curve, it reflects that under the same confining pressure, the greater the suction, the greater the shear stiffness, and the faster the speed at which the peak point is reached. While under the same confining pressure, the smaller the confining pressure, the easier it is to enter the dilatancy state.
Fig. 1 The stress-strain curves of unsaturated dense sands with $\sigma_u - u_a = 20.75\text{kPa}$ under traxial tests$[^{[1]}]$

Experimental data$[^{[1]}]$ shows that due to the effect of suction, the loose sand is transformed into dense sand, the dilatancy characteristics are more obvious, and the phase transition point is more likely to occur during the deformation process. After the various soil samples reach the peak intensity, the intensity enters the descending section has been destroyed when the critical state has not been reached. Therefore, it is more appropriate for unsaturated sand to establish a constitutive relationship with the phase transition state as its starting point.

2. Elastoplastic model

Based on the common point, for the dilatancy of saturated sand and unsaturated fine sand, this paper proposes a constitutive model for unsaturated sand with different densities. This model must be able to reflect the hydraulic-mechanical behavior of unsaturated sand. In order to obtain a smooth transition between saturated and unsaturated states, this unsaturated soil model should be converted to a saturated soil model when the suction is less than the air intake value AEV. The air intake value of sand is generally small because the sand particles are relatively very large compared to clay, and the typical air intake value is only a few kPa.

First, the stress variables of unsaturated sand are selected as:

$$\sigma_y = (\sigma_{ij} - u_a \delta_{ij}) + s S_{ij} \delta_{ij}; \bar{s} = ns = n(u_q - u_a)$$  \hspace{1cm} (1)

Here $\sigma_y$ is the net stress of unsaturated soil, $u_a$ is atmospheric pressure, $\sigma'$ is the average stress of the soil skeleton, that is, the effective stress of the soil skeleton, $\bar{s} = ns$ is the corrected suction, $n$ is sand porosity.

2.1 Elastic incremental response

The elastic relationship with saturated sand, irrespective of elastoplastic coupling, is considered that there is no relationship between elastic modulus and plasticity. Therefore, using the isotropic elasticity assumption, the elastic relationship between soil skeleton and water can be expressed as follows:

$$d\epsilon_i^e = \frac{dp}{K}; \quad d\epsilon_t^e = \frac{dS}{2G}; \quad dS = -\frac{1}{\Gamma \bar{s}} \frac{d\bar{s}}{\bar{s}}$$  \hspace{1cm} (2)
Among them, $p'$ and $S$ are the effective mean stress and the deviatoric stress tensor of unsaturated sand, respectively, and $p' = (\mathbf{tr} \mathbf{\sigma}) / 3$. Deviatoric stress tensor $\mathbf{S} = \mathbf{\sigma} - p' \mathbf{I}$. $\mathbf{I}$ is the unit tensor, $e_v$ with $e$ are the volumetric strain and deviatoric strain tensor, here $G$, $K$, $G' = (\mathbf{I} + \mathbf{e})$ are the elastic shear modulus, bulk modulus for soil skeleton and elastic modulus of water. The equation Richart$^2$ proposed is used for the elastic shear modulus of soil skeleton which is expressed as a function of effective stress and current void ratio:

$$G = G_0 \left(\frac{2.97 - e}{1 + e}\right)^2 \left(p'_p\right)^{3/2}$$

$$K = \frac{G}{3(1 - 2\nu)}$$

(3)

Here $G_0$ is the material constant, $e$ is the current void ratio, $p'_a$ is for reference pressure, for convenience, Atmospheric pressure is often set as $p'_a = 101kPa$. $\nu$ is Poisson's ratio and is a constant.

2.2 Yield, Dilatancy, Boundary Surface Functions

According to the research work$^{[3][4]}$, the defined yield surface should consider both isotropic hardening and kinematic hardening. This article only considers monotonic loading. For ease of understanding, it is first described in triaxial space and then extended to general stress space, as shown in Figure 2.

![Fig. 2 Schematic illustration of the yield, dilatancy, critical state, bounding lines in the general stress space and $p', q$ space](image)

Therefore, solid-phase yielding uses a similar expression on saturated sand:

$$f = q - (\alpha \pm m) p = 0$$

(4)

Here $p'$ is the effective mean stress. Because the expression is the same under triaxial compression and triaxial tension, for the sake of description, it is only discussed in the triaxial compression space. In the triaxial compression space, the principal stress $\sigma_1 > \sigma_2 = \sigma_3$ (axisymmetric condition), $p' = (\sigma_1 + 2\sigma_3)/3$, $q = \sigma_1 - \sigma_3$, $\alpha$ is the back stress ratio, in general space, $\alpha$ is the reaction force ratio tensor. Correspondingly, the phase transition state parameters of unsaturated sand are introduced in this paper. $\beta$, $\beta = (e(p')/e_p, (p' - 1)$, dilatancy stress ratio $M_{\text{pl}}$, peak boundary stress ratio $M_{\text{plp}}$ and back stress ratio $\alpha_{\text{pl}}$, $\alpha_{\text{plp}}$ which are denoted as

$$M_{\text{pl}} = M_p \exp(k_p \beta); M_{\text{plp}} = M_p / \gamma \exp(-k_p \beta)$$

$$\alpha_{\text{pl}} = M_{\text{pl}} - m = M_p \exp(k_p \beta) - m;$$

$$\alpha_{\text{plp}} = M_{\text{plp}} - m = M_p / \gamma \exp(-k_p \beta) - m$$

(5)

The above variables are described in the triaxial stress space. In order to describe the behavior of unsaturated sand in the general stress space, the yield surface should be converted from the triaxial stress space to the general stress space. Therefore, in the general stress space, the yield surface equation is
Here $\alpha$ is the back stress tensor, we will use the same method, that is, with the help of the modified Lord’s angle $\theta$ to complete the transformation from triaxial to general stress space, and set the deviatoric stress ratio tensor $r = S/p'$. The loading direction of the yield surfacen at $r$ is:

$$L = \frac{\delta f}{\delta \sigma} = n - \frac{1}{3} (n : r) \mathbf{n} = \frac{r - \alpha}{[(r - \alpha) : (r - \alpha)]^{1/2}}$$

We use the method similar to saturated sand, in the general stress space (here considered as compression space), the dilatancy and boundary surfaces, and the corresponding back stress tensor corresponding to them $\alpha^b$ is set as

$$M^a = g(\theta, c_x) M^a; \alpha^a = \sqrt{2/3} \alpha^b n; \quad \alpha^p = g(\theta, c_x) M_a - m$$

The liquid phase yield function can be described using the soil-water characteristic curve [5], where the slopes of the main drying line and the main infiltration line are $\lambda_u$, and the slope of the scan line $\lambda_w$. Assuming $\tilde{S}$, as the liquid phase yield stress, when $y = I$, it means yield due to increased suction; and $y = D$ means the yield generated from a decrease in suction. The elastic change for liquid degree of saturation is denoted as

$$dS_y = -\lambda_u - \lambda_w$$

That is, the liquid phase plastic modulus is:

$$\Gamma_p = \frac{1}{\lambda_u - \lambda_w}$$

### 2.3 Plastic relationship

#### Dilatancy ratio $D$

Same as dilatancy ratio of saturated sand, the mapping point on the dilatant surface for the back stress tensor $\alpha$ is $\alpha^b_{\mu}$, and the corresponding dilatancy equation is:

$$D = \sqrt{\frac{3}{2} D_t (\alpha^b_{\mu} - \alpha) : n = D_t (\alpha^b_{\mu} - \sqrt{\frac{3}{2}} \alpha : n)}$$

#### Back stress ratio tensor $\alpha$

The back stress ratio tensor $\alpha$ is usually set to the function $\alpha = \alpha(\sigma, e^p)$ for the current stress $\sigma$ of soil skeleton and plastic deviatoric strain $e^p$. For unsaturated silt, both liquid phase modified suction $\tilde{S}$ and saturation $S$, affect the follow-up hardening by contributing to the effective stress $\sigma$. It is usually considered $\alpha$ pointing to boundary map $\alpha_0$, and $\alpha$ depends on "distance" $\alpha^p - \alpha$ [6,7] which conform to the concept of plasticity of the boundary surface, so the following equation shows the development of the back stress ratio tensor $\alpha$

$$d\alpha = (L) h(\alpha^p_{\mu} - \alpha) = \sqrt{\frac{3}{2}} h(\frac{2}{3} \alpha^p_{\mu} de^p - d\alpha)$$

Here $h$ is a state variable function, the same method used for saturated sand[8], but the effective stress here $p'$ contains the effect of suction, so, $h$ is expressed as
\[
\begin{align*}
h = G_0 h_0 (1 - e) \left( 2.97 - e \right)^2 \left( 1 + e \right) (p_\nu / p)^{3/2} \\
\sqrt{3/2} \sqrt{(r - \alpha_m) : (r - \alpha)}
\end{align*}
\]

Here \( G_0 \) is the material constant, \( \alpha_m \) is the initial value of the back stress tensor.

**Isotropic hardening**

Isotropic parameter \( m \) also defines the range of yield surface elasticity, but different from the parameters in the saturated sand constitutive model \( m \). It is not only a function of stress state, but also the plastic function of volumetric strain \( \varepsilon_v \) and plastic saturation \( S_p \). The determination of isotropic parameters is based on experiments, and the porosity ratio \( \varepsilon \) is generally considered to decrease and then \( m \) increases. At the same time, the change of suction \( s \) also affects the size of the yield surface of the soil skeleton, but the literature \([5]\) pointed out that plastic saturation can directly change the fluid in the pores, and its effect on the meniscus is more important than suction, so plastic saturation \( \Delta S_p \) should be selected as the role of the hardening parameter, the yield stress of the solid phase changes due to the deformation of the liquid phase, so only the plastic volume strain of the soil skeleton \( \Delta \varepsilon_v \) and plastic saturation \( \Delta S_p \) will affect the isotropic hardening parameters. Refer to the definition from the reference \([3]\). The change for \( m \) following the changes of the plastic volume strain \( \Delta \varepsilon_v \) can be expressed as \( \partial m / \partial \varepsilon_v = c_v (1 + e_0) \), and following the change of the irreversible water content \( \Delta S_p \) is defined as \( \partial m / \partial S_p = c_s (\nu S / \nu_0) \). In general, following the changes of \( \Delta \varepsilon_v \) with \( \Delta S_p \), \( m \) can be defined as the sum of these two terms, namely

\[
dm = c_v (1 + e_0) \Delta \varepsilon_v + c_s \left( \frac{\nu S}{\nu_0} \right) \Delta S_p
\]

Here \( e_0 \) is the initial void ratio, \( c_s, c_v, e \) are all model constants. The suction-induced hardening effect is considered by the above formula.

**Liquid hardening**

The hardening law of the liquid phase usually needs to consider the effect of solid deformation on the soil-water characteristic curve. The soil-water characteristic curve SWCC characterizes the change of the liquid phase, so the hardening equation of the liquid phase adopted the same form as in reference \([9]\)

\[
\frac{d \tilde{s}_y}{s_y} = -\Gamma_p d S_p + \Gamma_s d \varepsilon_v, \Gamma_p = \frac{\kappa_v (1 + e)}{\lambda - k}
\]

Here \( \tilde{s}_y \) means liquid phase drying or moisturizing correction suction, \( \lambda, k \) for the slope of normal consolidation line and rebound line of \( v \cdot \ln p' \), \( v=1+e \) for liquid phase plastic modulus \( \Gamma_p = \frac{1}{(\lambda_s - \kappa_s)} \), \( \Gamma_p \) indicates the effect of the solid phase on the liquid phase, \( \kappa_m \) is the coupling coefficient. According to the article \([5]\), the coupling coefficient \( \kappa_m \) reflects the magnitude of the liquid yield surface movement during solid-phase yielding which control the position of the intersection of the two yield surfaces in the stress space, and it is usually assumed to be constant and can be determined experimentally or empirically. According to the above description, the change of air intake value AEV with volumetric strain is

\[
\frac{d \tilde{s}_a}{s_a} = \Gamma_p d \varepsilon_v = \kappa_m (1 + e_0) \ln p' / p_0
\]

Among them, \( \tilde{s}_a \) denotes air intake value AEV of the material, \( p' \) and \( p_0 \) are respectively corresponding to the effective stresses for different air intake values AEV.

5
2.4 Consistency conditions and plastic modulus

Solid phase consistency conditions
According to the classic plasticity theory, in order to ensure that the stress remains on the yield surface during plastic loading, the yield surface should meet the uniform conditions, \( df = 0 \), namely
\[
df = \frac{\partial f}{\partial \mathbf{\sigma}} : \mathbf{\sigma} + \frac{\partial f}{\partial \mathbf{a}} : \mathbf{a} + \frac{\partial f}{\partial \mathbf{m}} : \mathbf{m} = \mathbf{d} \cdot \mathbf{h} (\mathbf{a} : \mathbf{a} - \mathbf{a}) (L) + \frac{\partial f}{\partial \mathbf{m}} : \mathbf{D} (L) + \frac{\partial f}{\partial \mathbf{S}^p} : \mathbf{d} \mathbf{S}^p = 0
\]
(17)

Here \( K_r, K_p \) represent two kinds of plastic modulus of two mechanical mechanisms (hydraulic, mechanical) related to the yield surface. \( K_p \) is the plastic modulus of the soil skeleton, \( K_p \) is a new plastic modulus which shows the effect of hydraulic mechanism on the yield surface of the solid phase. The above formula can clearly express that the loading index \( L \) is affected by both net stress and suction. According to plastic theory, the strain tensor of the soil skeleton is
\[
d_{\varepsilon} = < L > \mathbf{R}; \mathbf{R} = \mathbf{n} + \frac{1}{3} \mathbf{D} \mathbf{I} ; \mathbf{d}_{\varepsilon} = < L > \mathbf{n} \mathbf{d}_{\varepsilon} = < L > \mathbf{D}
\]
(18)

Combined with equation (17) can get
\[
\frac{\partial f}{\partial \mathbf{\sigma}} = \frac{\partial f}{\partial \mathbf{S}} + \frac{1}{3} \frac{\partial f}{\partial \mathbf{p}} ; \frac{\partial f}{\partial \mathbf{S}} = \frac{\mathbf{r} - \mathbf{\alpha}}{\sqrt{2/3 m}} ; \frac{\partial f}{\partial \mathbf{p}} = \left[ (\mathbf{r} - \mathbf{\alpha}) \cdot \mathbf{\alpha} \right] \frac{2}{\sqrt{2/3 m}} + \frac{2}{\sqrt{2/3 m}} m \mathbf{f}_{\alpha} = - \frac{(S - p^{\prime} \mathbf{\alpha})}{\sqrt{2/3 m}}
\]
(19)

According to the above equation, we get
\[
L = \frac{1}{K_r} \left( \frac{\partial f}{\partial \mathbf{S}} : \mathbf{d} \mathbf{S} + \frac{\partial f}{\partial \mathbf{p}} : \mathbf{d} \mathbf{p} \right) \frac{K_r}{K_p}
\]
(20)
\[
K_r = p^{\prime} \mathbf{h} (\sqrt{2/3 a^0 - \mathbf{\alpha}} : \mathbf{n}) + \frac{2}{3} m \mathbf{c} (1 + e_s) \mathbf{D} (\alpha^0 - \sqrt{3/2 \mathbf{\alpha}} : \mathbf{n})
\]
\[
K_p = - \frac{\partial f}{\partial \mathbf{m}} : \mathbf{d} \mathbf{m} \mathbf{S}^p = \left[ \frac{2}{3} p^{\prime} \mathbf{c} (m \mathbf{S}^p) \right] \frac{c_s}{p_s}
\]
(21)

From formula (21), It can be seen that the plastic modulus \( K_p \) depends on the distance of stress ratio \((\sqrt{2/3 a^0 - \mathbf{\alpha}} : n)\) and \((\alpha^0 - \sqrt{3/2 \mathbf{\alpha}} : \mathbf{n})\), the sign of \( K_p \) is completely determined by \((\sqrt{2/3 a^0 - \mathbf{\alpha}} : \mathbf{n})\) with \((\alpha^0 - \sqrt{3/2 \mathbf{\alpha}} : \mathbf{n})\). If the isotropic hardening of the yield surface is neglected (for dense sand, \( c_s \) is very small and can be set to a constant value, i.e. \( c_s = 0 \)), Then the plastic modulus will be a very simple form \( K_p = p^{\prime} \mathbf{h} (\sqrt{2/3 a^0 - \mathbf{\alpha}} : \mathbf{n}) \).

Liquid consistency conditions
For the liquid phase, it also conforms to the classical plasticity theory, For liquid yield surface function \( f_w = 0 \), it should also meet the consistency conditions, i.e. \( df_w = 0 \)
\[
df_w = \frac{\partial f_w}{\partial \mathbf{S}} : \mathbf{d} \mathbf{S} + \frac{\partial f_w}{\partial \mathbf{E}_v} : \mathbf{d} \mathbf{E}_v + \frac{\partial f_w}{\partial \mathbf{S}^p} : \mathbf{d} \mathbf{S}^p = 0
\]
(22)

According to (15), it can get
\[
\frac{\partial \mathbf{S}^p}{\partial \mathbf{E}_v} = \Gamma_p \mathbf{S}^v, \frac{\partial \mathbf{S}^p}{\partial \mathbf{S}^p} = - \Gamma_p \mathbf{S}^v
\]
(23)
Combined with formula (17), (22), (23), it can get

\[ K_p (L) + K_p dS^p = \frac{\partial f}{\partial \sigma^*} : d\sigma^* + \frac{\partial f}{\partial s} \sigma_r^* D(L) + \frac{\partial f}{\partial s} \sigma_r^* dS^p = \frac{\partial f}{\partial s} dS^p \]  

(24)

And

\[ \frac{\partial f}{\partial s} = 1 \frac{\partial f}{\partial s} + \frac{\partial f}{\partial s} \sigma_r^* = \Gamma_p s, \frac{\partial f}{\partial s} \sigma_r^* = -\Gamma_p s, \]  

(25)

Substituting equation (25) into equation (24), it can be simplified as

\[ K_p (L) + K_p dS^p = \frac{\partial f}{\partial s} d\sigma^* + \Gamma_p D(L) - \Gamma_p s, dS^p = ds \]  

(26)

Synthesizing the above equation, the increment of liquid phase plasticity is

\[ dS^p = \frac{\Gamma_p D}{\Pi_p} \frac{\partial f}{\partial \sigma^*} : d\sigma^* + \frac{\Gamma_p}{s, \Pi_p} ds ; \Pi_p = K_p \Gamma_p D + K_p \Gamma_p \]  

(27)

Substituting equation (27) into equation (24), then Solid-phase loading index \( L \) can be expressed as

\[ L = \frac{B}{K_p} \frac{\partial f}{\partial \sigma^*} : d\sigma^* + Cds \]  

(28)

Here \( B = 1 - K_p \Gamma_p D / \Pi_p \), \( C = K_p / s, \Pi_p \).

So integrated with formula (18), The plastic strain increment of the soil skeleton can be obtained as

\[ \text{de}^e = \langle L \rangle \text{R} = \frac{BR}{K_p} \frac{\partial f}{\partial \sigma^*} : d\sigma^* + CRds + \frac{B}{3} (n + 1-DI) \frac{\partial f}{\partial \sigma^*} : d\sigma^* + C(n + 1-DI)ds \]

\[ = \frac{B}{K_p} n \frac{\partial f}{\partial \sigma^*} : d\sigma^* + Cn + \frac{1}{3} DB \frac{\partial f}{\partial \sigma^*} : d\sigma^* + \frac{1}{3} CDI ds \]

\[ = \left( \frac{B}{K_p} \frac{\partial f}{\partial \sigma^*} : d\sigma^* + Cds \right) n + \frac{1}{3} \left( \frac{B}{K_p} \frac{\partial f}{\partial \sigma^*} : d\sigma^* + Cds \right) DI \]  

(29)

2.5 Incremental Constitutive Equation

In summary, the strain increments of unsaturated soils are

\[ \text{de} = \text{de}^e + \text{de}^s \]

(30)

The soil skeleton strain increment relationship is

\[ \text{de} = \text{de}^e + \text{de}^s \]

\[ = \frac{\partial S}{\partial \sigma^*} + \frac{1}{3} \left( \frac{B}{K_p} \frac{\partial f}{\partial \sigma^*} : dS + \frac{\partial f}{\partial \sigma^*} : d\sigma^* + Cds \right) n \]

\[ + \frac{1}{3} \left( \frac{B}{K_p} \frac{\partial f}{\partial \sigma^*} : dS + \frac{\partial f}{\partial \sigma^*} : d\sigma^* + Cds \right) DI \]  

(31)

The liquid phase increment relationship is

\[ dS^l = dS^p + dS^l \]

\[ = \frac{\Gamma_p D}{\Pi_p} \frac{\partial f}{\partial \sigma^*} : d\sigma^* + (1 + K_p/s) ds \]  

(32)

To make the description simple, we will transform the general stress space to give the constitutive relationship of the phases of unsaturated sand in the triaxial stress space, and express it as a matrix
\[ \begin{align*}
\frac{d\sigma}{d\varepsilon} &= \begin{bmatrix} \frac{1}{1 + B \frac{\partial f}{\partial \sigma}} & B \frac{\partial f}{\partial \sigma} & CD \frac{\partial f}{\partial \sigma} \\
\frac{B \frac{\partial f}{\partial \sigma}}{K_r} & 1 + B \frac{\partial f}{\partial \sigma} & C \\
\frac{\Gamma D \frac{\partial f}{\partial \sigma}}{\Pi_r} & \frac{\Gamma D \frac{\partial f}{\partial \sigma}}{\Pi_q} & 1 - \frac{1}{\Pi_s} \end{bmatrix} \begin{bmatrix} dp' \\
dq \\
ds 
\end{bmatrix}
\end{align*} \]

Seen from formula (33), when saturation \( S_s = 100\% \), it indicates the constitutive relation of saturated sand or dry soil respectively which realizes the smooth transition of three states of unsaturated, saturated and dry soil.

In actual engineering applications, the total external stress, water pressure and air pressure are generally known, so the stress variables in it also need to be converted. According to the definition of effective stress and modified suction, we can get

\[ dp' = dp - (1 - S_s) du_a - S_s du_w + s d\sigma_s, \]
\[ ds = ndu_a - ndu_w + s (n - 1) d\varepsilon, \]

Substitute formula (34) into (33), the stress increment becomes \( dp, dq, du_a, du_w \). Strain variable in formula (33) becomes the function of stress increment \( dp, dq, du_a, du_w \). Normally the atmospheric pressure is maintained during the experiment as constant \( u_a \). Only three other quantities need to be directly controlled in the triaxial test. So as long as the stress increment is known, the formula (33) can be used to find out the corresponding strain increment.

3. Model verification

3.1 Methods for determining model parameters

The object of this article is mainly for medium or dense sand. For sand, dilatancy is its most significant feature, but its critical state is difficult to measure. Therefore, this article considers another aspect of dilatancy, "phase transition "To build the soil skeleton part of the model. The above model parameters totaled 15 and all relevant parameters are as follows:

Elastic constant: \( G, \nu \), Phase change state parameters: \( M, k_0, k_a, \gamma \); Hardening parameters: \( h_0, m, c, c_w, c \); Dilatation parameters: \( D_b \); swcc model parameters: \( \lambda_w, k_w, \kappa_{sw} \).

Initial value \( m \) and coefficient \( c \) are related to isotropic hardening. If \( c_w = 0 \), it means that the yield surface size has nothing to do with the plastic volume strain of the soil skeleton. Generally speaking, for saturated sand, the condition is satisfied for the constant \( m \), but for unsaturated sand, the coupling between the solid phase and the liquid phase also needs to be considered. Assuming \( c_w = 0 \), Parameters \( c_w \) and \( c \) must be determined related to suction. The change in the corresponding isotropic yield surface also changes as \( c_w \) change, but from the triaxial drainage test of unsaturated sand, the effect of suction on the yield surface is not obvious.

Confirmation for Swcc model parameters \( \lambda_w, k_w, \kappa_{sw} \), SWCC model parameters are easily obtained from any drying or wetting process. This article uses bilinear soil-water characteristic curve model from the literature \[10\]. For silty dense sand, the soil-water characteristic curve is simple, and the air intake value AEV is small which can be easily determined. \( \lambda_w, k_w, \kappa_{sw} \) are the coupling parameters, indicating the effect of soil skeleton deformation on the soil-water characteristic curve. At least two sets of soil-water characteristic curves under different confining pressures are required.
3.2 Test data

In order to verify the effectiveness of the proposed model, the soil sample used in this paper is ASU silt and dense sand (data from Natalia Perez (2006) [1] - ASU sand). According to ASTM standard, ASU sand is a dense and non-cohesive silt sand with a specific gravity of 2.71, \(c\) with \(\phi\) are 2.3kPa and 36.1 \(^{\circ}\), respectively, with sandy soil 67%, 19.6% silt, 7.4% clay, maximum dry density 19.5kN/m\(^3\), optimal moisture content 10.5%, initial porosity ratio \(e_0 = 0.44~0.46\).

Test data [1] shows that the soil-water characteristic curve swcc (including the air intake value (AEV)) is not sensitive to changing confining pressures, so the bilinear soil-water characteristic curve model shown in Figure 3 is used.

![Fig. 3 SWCC Skematic illustration of ASU sands](AEV=2kpa)

| After compaction | After evaporation/shear |
|-----------------|-------------------------|
| sample          | davg cm | havg cm | Sample weight | W% | \(\gamma_d\) gr/cm\(^3\) | ms gr/cm\(^3\) | mw gr/cm\(^3\) | \(\nu\) | Sample weight | \(\Delta w\)% | \(\rho\) gr/cm\(^3\) | e0 | \(w\)% |
| 75-125          | 10.16    | 22.3    | 3731.98       | 11.56 | 1.8 50  | 3342.97  | 389.00  | 1807.93 | 3648.05 | 0.091257 | 2.01 2  | 0.465 | 0.0893 |
| 250-95          | 10.15    | 22.4    | 3733.16       | 11.64 | 1.8 44  | 3340.49  | 392.66  | 1800.82 | 3648.13 | 0.092093 | 2.02 5  | 0.460 | 0.0871 |
| 75-360          | 10.11    | 22      | 3736.32       | 11.53 | 1.8 89  | 3334.46  | 401.85  | 1772.01 | 3602.49 | 0.080379 | 2.05 2  | 0.440 | 0.0769 |
| 250-35 5        | 10.15    | 22.2    | 3731.6        | 11.6  | 1.8 61  | 3341.18  | 390.41  | 1799.82 | 3601.37 | 0.077871 | 2.00 0  | 0.459 | 0.0789 |
| 250-35 5        | 10.12    | 22.6    | 3733.57       | 11.58 | 1.8 41  | 3344.97  | 388.59  | 1812.47 | 3601.86 | 0.076797 | 1.98 2  | 0.468 | 0.0757 |
| 75-23           | 10.12    | 22.2    | 3732.97       | 11.3  | 1.8 79  | 3353.59  | 379.37  | 1816.04 | 3693.23 | 0.101275 | 2.03 4  | 0.467 | 0.1054 |
| 250-25          | 10.13    | 22.4    | 3732.94       | 11.55 | 1.8 54  | 3345.38  | 387.55  | 1804.37 | 3696.33 | 0.104904 | 2.08 8  | 0.461 | 0.1038 |

It can be seen from Table 1 that under the same initial conditions, the triaxial test, the water content only changes very slightly. For the suction 95-135 group, the maximum change in water content is 0.1%; for the suction in the 320-355 group, The maximum change of water content is 0.1%; According to the water content formula \(e_S = wG\), the effect of the entire shear body strain on the movement of the liquid yield surface is not obvious.

3.3 Model parameter determination

All model parameters are determined as follows. From the data analysis in the table, when \((\sigma_3-u_n)=75kPa, 250kPa, k_d, k_h, \gamma\) is very close and almost the same (except for very small suction) which can more accurately reflect the characteristics of the phase change point and peak point of dense
sand; And under the same confining pressure, the phase change stress ratio \( M_p \) increases with the increase of suction. When the suction is greater than 300kPa, the magnitude of the phase change stress ratio changes is small, indicating that the suction has a certain contribution to the confining pressure. When the suction exceeds a certain value, the contribution of the suction to the normal stress weakens.

| Table 2 Model parameters |
|--------------------------|
| \( G_0 \) | \( V \) | \( m \) | \( c_s \) | \( c_w \) | \( c \) |
| 125 | 0.15 | 0.01 | 0 | 0.5 | 10 |

| Table 3 Model parameters under different confining pressure |
|--------------------------|
| \( (\sigma_{x}-u_{x})-v(kPa) \) | \( k_d \) | \( D_b \) | \( k_b \) | \( \gamma \) | \( h_s \) | \( M_p \) |
| 75-23 | 0.9 | 1.55 | 3.5 | 0.89 | 2.5 | 1.35 |
| 75-125 | -3.5 | 1.55 | 3.5 | 0.89 | 3.0 | 1.40 |
| 75-360 | -3.5 | 1.55 | 3.5 | 0.89 | 5.5 | 1.44 |
| 75-600 | -3.5 | 1.55 | 3.5 | 0.93 | 5.9 | 1.45 |
| 250-25 | -3.5 | 1.55 | 8.0 | 0.96 | 2.0 | 1.48 |
| 250-95 | -1.5 | 1.55 | 8.0 | 0.96 | 1.75 | 1.49 |
| 250-353 | -1.5 | 1.55 | 8.0 | 0.92 | 1.25 | 1.49 |

3.4 Comparison of calculation results and test results

Figure 4-5 (see below) is the experimental results and simulation of the changes in shear stress and body strain of ASU sand under different confining pressures (For space limitation, only two confining pressures of 75 and 250 kPa are listed) and different suction forces. The comparison of the results shows that the two have a good fit. It shows that the model in this paper can well describe the stress-strain behavior of silty dense sand, and capture the significant characteristics of its phase transition and peak. The test results show that for silty sand At the beginning of shear, like saturated dense sand, it is always compressed first to reach the phase transition point. If the confining pressure is small, such as 75kPa in this article, the speed of reaching the phase transition point is very fast (the corresponding axial strain is 3 %, 2%), but when the confining pressure is 250kPa, the phase strain will only occur when the axial strain reaches 8% and 6% respectively; then, as the shear strain develops, the soil sample further dilates, and when it reaches dilatancy At the maximum value, that is, the peak strength, the soil sample enters the softening stage until it breaks. The above test results also clearly show that at a given confining pressure, the dilatancy increases with increasing suction, rotates, and accompanies volume increase. The shear strength increases, and the larger the suction, the easier it is to reach the phase transition. This type of dilatancy is very significant under the action of low stress and high suction, Indicating that confining pressure plays a very important role in dilatancy. As far as silty sand is non-cohesive soil, the coupling between solid and liquid is not obvious. This can be explained by their soil-water characteristic curve SWCC. Under different confining pressures (75, 250Pa), its air intake value has remained almost unchanged, all staying at 2kPa, and only a slight change in water content occurs when the suction is kept constant throughout the shearing process.

4. Summary

For unsaturated sand, under low confining pressure, due to the presence of suction, the silt dense sand undergoes a large volume change during shearing-dilatancy, and obvious dilatancy and peak characteristics appear throughout the stress-strain behavior. Point, the critical state is often destroyed before reaching it. Based on the above situation, this article is based on the phase transition state and considers the hydraulic-mechanical coupling. Model parameters are introduced to establish the boundary surface of unsaturated sand in general stress space and triaxial stress space. Innovative points include: 1. Considering stress-induced anisotropy and its development and change, the establishment of a kinematic hardening yield surface and corresponding dilatancy and boundary surfaces, highlighting
the phase transition and peak characteristics of unsaturated dense sand. 2. Propose a special hardening rule, considering the coupling of hydraulic and mechanical behavior. Based on the experimental data and the model in this paper, it is shown that for unsaturated dense sand or non-cohesive soil, this coupling has little effect on its stress-strain properties, but this is a supplement and improvement to the constitutive model.

Fig. 4 Shear stress, volumetric strain versus axial strain for ASU sands with \( \sigma_3-u_a = 75\text{kPa}, \ s=23,125\text{kPa} \)

Fig. 5 Shear stress, volumetric strain versus axial strain for ASU sands with \( \sigma_3-u_a = 250\text{kPa}, \ s=25,95\text{kPa} \)
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