Anderson Localization for Jacobi Matrices Associated with High-Dimensional Skew Shifts*

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Abstract In this paper, the authors establish Anderson localization for a class of Jacobi matrices associated with skew shifts on $T^d$, $d \geq 3$.

Keywords Anderson localization, Jacobi matrices, Skew shifts

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1 Introduction and Main Result

Over the past thirty years, there are many papers on the topic of Anderson localization for lattice Schr"{o}dinger operators

$$H = v_n \delta_{nn'} + \Delta,$$

where $v_n$ is a quasi-periodic potential, $\Delta$ is the lattice Laplacian on $\mathbb{Z}$,

$$\Delta(n, n') = \begin{cases} 1, & |n - n'| = 1, \\ 0, & |n - n'| \neq 1. \end{cases}$$

Anderson localization means that $H$ has pure point spectrum with exponentially localized states $\varphi = (\varphi_n)_{n \in \mathbb{Z}}$,

$$|\varphi_n| < e^{-c|n|}, \quad |n| \to \infty. \quad (1.2)$$

We may associate the potential $v_n$ to a dynamical system $T$ as follows:

$$v_n = \lambda v(T^n x), \quad (1.3)$$

where $v$ is real analytic on $\mathbb{T}^d$ and $T$ is a shift on $\mathbb{T}^d$:

$$Tx = x + \omega. \quad (1.4)$$

Fix $x = x_0$. If $\lambda$ is large and $\omega$ outside set of small measure, $H$ will satisfy Anderson localization.

The proof of Anderson localization is based on multi-scale analysis and semi-algebraic set theory. In this line, Bourgain and Goldstein [6] proved Anderson localization for Schrödinger

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operators (1.1) with the help of fundamental matrix and Lyapounov exponent. By multi-scale method, Bourgain, Goldstein and Schlag [8] proved Anderson localization for Schrödinger operators on $\mathbb{Z}^2$,

$$H(\omega_1, \omega_2; \theta_1, \theta_2) = \lambda v(\theta_1 + n_1 \omega_1, \theta_2 + n_2 \omega_2) + \Delta.$$  \hspace{1cm} (1.5)

Later, Bourgain [5] proved Anderson localization for quasi-periodic lattice Schrödinger operators on $\mathbb{Z}^d$, $d$ arbitrary. Recently, using more elaborate semi-algebraic arguments, Bourgain and Kachkovskiy [10] proved Anderson localization for two interacting quasi-periodic particles.

More generally, we can study the long range model

$$H = v(x + n \omega) \delta_{nn'} + S_\phi$$ \hspace{1cm} (1.6)

with $\Delta$ replaced by a Toeplitz operator

$$S_\phi(n, n') = \hat{\phi}(n - n'),$$ \hspace{1cm} (1.7)

where $\phi$ is real analytic, and $\hat{\phi}(n)$ is the Fourier coefficient of $\phi$. Bourgain [4] proved Anderson localization for the long-range quasi-periodic operators (1.6). Note that in this case, we cannot use the fundamental matrix formalism as (1.1). Bourgain’s method in [4] also permits us to establish Anderson localization for band Schrödinger operators (cf. [9]),

$$H(n,s), (n', s') (\omega, \theta) = \lambda v_s(\theta + n \omega) \delta_{nn'} \delta_{ss'} + \Delta,$$ \hspace{1cm} (1.8)

where $\{v_s | 1 \leq s \leq b\}$ are real analytic. Recently, this method was used in [13] to prove Anderson localization for quasi-periodic block operators with long-range interactions.

If the transformation $T$ is a skew shift on $\mathbb{T}^2$:

$$T(x_1, x_2) = (x_1 + x_2, x_2 + \omega),$$ \hspace{1cm} (1.9)

using transfer matrix and Lyapounov exponent, Bourgain, Goldstein and Schlag [7] proved Anderson localization for

$$H = \lambda v(T^n x) + \Delta.$$ \hspace{1cm} (1.10)

In order to study quantum kicked rotor equation

$$i \frac{\partial \Psi(t, x)}{\partial t} = a \frac{\partial^2 \Psi(t, x)}{\partial x^2} + ib \frac{\partial \Psi(t, x)}{\partial x} + V(t, x) \Psi(t, x), \hspace{1cm} x \in \mathbb{T},$$ \hspace{1cm} (1.11)

where

$$V(t, x) = \kappa \left[ \sum_{n \in \mathbb{Z}} \delta(t - n) \right] \cos(2\pi x),$$ \hspace{1cm} (1.12)

using multi-scale method, Bourgain [3] proved Anderson localization for the operator

$$W = \phi_{m-n}(T^m x),$$ \hspace{1cm} (1.13)

where $\phi_k$ are trigonometric polynomials and $T$ is a skew shift on $\mathbb{T}^2$.

However, there are few results on high-dimensional skew shifts. When $d \geq 3$, the skew shift $T : \mathbb{T}^d \rightarrow \mathbb{T}^d$ is given by

$$(Tx)_i = x_i + x_{i+1}, \hspace{1cm} 1 \leq i \leq d - 1,$$ \hspace{1cm} (1.14)
High-Dimensional Skew Shifts

\[(Tx)_d = x_d + \omega, \quad x = (x_1, \cdots, x_d). \quad (1.15)\]

In [14], Krüger proved positivity of Lyapounov exponents for the Schrödinger operator

\[H = \lambda f((T^n x)_1)\delta_{nn'} + \Delta, \quad (1.16)\]

where \(T\) is a skew shift on \(\mathbb{T}^d\), \(d\) is sufficiently large, and \(f\) is a real, nonconstant function on \(\mathbb{T}\).

In this paper, we generalize Bourgain’s result on skew shifts on \(\mathbb{T}^2\) (cf. [3]) to higher dimensional ones on \(\mathbb{T}^d\), \(d \geq 3\). More precisely, we consider matrices \((A_{mn}(x))_{m,n \in \mathbb{Z}}, x \in \mathbb{T}^d\) associated with a skew shift \(T: \mathbb{T}^d \to \mathbb{T}^d\) of the form

\[A_{mm}(x) = v(T^m x), \quad (1.17)\]

\[A_{mn}(x) = \phi_{m-n}(T^m x) + \phi_{n-m}(T^n x), \quad m \neq n, \quad (1.18)\]

where

\[v\] is a real, nonconstant, trigonometric polynomial, \(\phi_k\) is a trigonometric polynomial of degree \(< |k|^{C_1}\), \(\|\phi_k\|_\infty < \gamma e^{-|k|}\), \(\|k\omega\| > c|k|^{-2}, \quad \forall k \in \mathbb{Z} \setminus \{0\}\), \(\|G_{[0,N]}(E, x)\| < e^{N^{1-}}, (1.23)\]

\[|G_{[0,N]}(E, x)(m, n)| < e^{-\frac{\text{mes} \Omega_N(E)}{10}|m-n|}, \quad 0 \leq m, n \leq N, \ |m - n| > \frac{N}{10} (1.24)\]

for \(x \notin \Omega_N(E)\), where \(\text{mes} \Omega_N(E) < e^{-N^{\sigma}}, \quad \sigma > 0\).

The main difficulty here is to study the intersection of \(\Omega_N(E)\) and skew shift orbits. We need to prove

\[\#\{n = 1, \cdots, M : T^n x \in \Omega_N(E)\} < M^{1-\delta}, \quad \delta > 0, \quad (1.26)\]
where

$$\log \log M \ll \log N \ll \log M.$$ \hspace{1cm} (1.27)

To obtain (1.26), we study the ergodic property of skew shifts on $T^d$ in Section 2.

Next, in Section 4, we use decomposition of semi-algebraic set to estimate

$$\text{mes}\{\omega \in T : (\omega, T^j \omega) \in A, \exists j \sim M\} < M^{-c}, \ c > 0,$$

where $x \in T^d, A \subset T^{d+1}$ is a semi-algebraic set of degree $B$ and measure $\eta$, satisfying

$$\log B \ll \log M \ll \log \frac{1}{\eta}.$$ 

This is a key point to eliminate the energy $E$ in the proof of Anderson localization.

Finally, using Green’s function estimates and semi-algebraic set theory, we prove Anderson localization of the operator $H_\omega(x)$ in Section 5 as in [6–7].

We will use the following notations. For positive numbers $a, b$, $a \lesssim b$ means $C a \leq b$ for some constant $C > 0$. $a \ll b$ means $a \lesssim b$ and $b \lesssim a$. $N^{1-\epsilon}$ means $N^{1-\epsilon}$ with some small $\epsilon > 0$. For $x \in T$, $\|x\| = \inf_{m \in \mathbb{Z}} |x - m|$ for $x = (x_1, \cdots, x_d) \in T^d$, $\|x\| = \sum_{i=1}^d \|x_i\|$.

2 An Ergodic Property of Skew Shifts on $T^d$

In this section, we prove the following ergodic property of skew shifts on $T^d$.

**Lemma 2.1** Assume that $\omega \in DC, T = T_\omega$ is the skew shift on $T^d$, $\epsilon > L^{-\frac{1}{(d+1)^2+d+1}}$. Then

$$\# \{n = 1, \cdots, L : \|T^n x - a\| < \epsilon\} < Ce^d L, \ C = C(d).$$

**Proof** We assume $a = 0$. Let $\chi$ be the indicator function of the ball $B(0, \epsilon), R = \frac{1}{\epsilon}$, and $F_R$ be the Fejer kernel. Then $\chi \leq Ce^d \prod_{j=1}^d F_R(x_j)$.

Let $f(x) = \prod_{j=1}^d F_R(x_j)$. Then

$$\sum_{n=1}^L \chi(T^n x) \leq Ce^d \sum_{n=1}^L f(T^n x) \leq Ce^d \sum_{n=1}^L \sum_{0 \leq |l| < R} \tilde{f}(l_1, \cdots, l_d) e^{2\pi i (T^n x, l)}$$

$$\leq Ce^d \left(L + \sum_{0 < |k| < \frac{1}{\epsilon}} \left| \sum_{n=1}^L e^{2\pi i (T^n x, k)} \right| \right).$$

Let

$$S_k = \left| \sum_{n=1}^L e^{2\pi i (T^n x, k)} \right|, \ 0 < |k| < \frac{1}{\epsilon}. \hspace{1cm} (2.1)$$

We only need to prove

$$\sum_{0 < |k| < \frac{1}{\epsilon}} S_k \leq CL. \hspace{1cm} (2.2)$$
From the skew shift, we have

\[
(T^n x)_i = x_i + nx_{i+1} + \cdots + \left( \frac{n}{d-i} \right) x_d + \left( \frac{n}{d-i+1} \right) \omega, \quad i = 1, \ldots, d, \ x = (x_1, \ldots, x_d).
\]

(2.3)

If \( k_1 = \cdots = k_{d-1} = 0 \), then

\[
S_k = \left| \sum_{n=1}^L e^{2\pi i n k d \omega} \right| \leq \frac{1}{\| k_d \omega \|} \leq C|k_d|^2. \quad (2.4)
\]

If \( k_1 = \cdots = k_{d-2} = 0, k_{d-1} \neq 0 \), then \( S_k = \left| \sum_{n=1}^L e^{2\pi i f(n)} \right| \), where \( f(n) = \frac{1}{2} n^2 k_{d-1} \omega + cn, \ c \) is independent of \( n \).

So

\[
S_k^2 = \left( \sum_{n=1}^L e^{2\pi i f(n)} \right) \left( \sum_{n=1}^L e^{-2\pi i f(n)} \right) \lesssim L + \sum_{h=1}^{L-1} \left| \sum_{n=1}^{L-h} e^{2\pi i (f(n+h)-f(n))} \right|
\]

\[
\lesssim L + \sum_{h=1}^{L-1} \min \left( L, \frac{1}{\|hk_{d-1} \omega\|} \right) \lesssim L + \sum_{m=1}^{\min \left( L, \frac{1}{\|k_{d-1} \omega\|} \right)} \min \left( L, \frac{1}{\|m \omega\|} \right).
\]

Since \( \omega \in \text{DC} \), we may find an approximant \( q \) of \( \omega \) satisfying

\[
L^{\frac{1}{2}} < q < L.
\]

(2.5)

Using

\[
\# \left\{ M + 1 \leq n \leq M + q : \| n \omega - u \| \leq \frac{1}{2q} \right\} \leq 3, \ \forall M \in \mathbb{Z}, \ u \in \mathbb{R},
\]

we get

\[
\sum_{n=M+1}^{M+q} \min \left( L, \frac{1}{\|n \omega\|} \right) \lesssim L + q \log q. \quad (2.6)
\]

By (2.5)–(2.6), we have

\[
S_k^2 \lesssim \frac{|k_{d-1}| L}{q} (L + q \log q) \lesssim |k_{d-1}| L^{\frac{3}{2}}.
\]

Hence

\[
S_k \leq C|k_{d-1}|^{\frac{1}{2}} L^{\frac{3}{2}}. \quad (2.7)
\]

If \( k_1 = \cdots = k_{d-3} = 0, k_{d-2} \neq 0 \), then \( S_k = \left| \sum_{n=1}^L e^{2\pi i g(n)} \right| \), where \( g(n) = \frac{1}{6} n^3 k_{d-2} \omega + bn^2 + cn, \ b, c \) is independent of \( n \).

So

\[
S_k^2 \lesssim L + \sum_{h=1}^{L-1} \left| \sum_{n=1}^{L-h} e^{2\pi i g(n)} \right|, \quad g_{h_1}(n) = g(n + h_1) - g(n).
\]
We have
\[ S_k^4 \lesssim L^2 + L \sum_{h_1=1}^{L-h_1} \left| \sum_{n=1}^{L-h_1} e^{2\pi i g_{h_1}(n)} \right|^2 \]
\[ \lesssim L^3 + L \sum_{h_1=1}^{L-h_1} \sum_{h_2=1}^{L-h_1-h_2} \left| \sum_{n=1}^{L-h_1-h_2} e^{2\pi i (g_{h_1}(n+h_2)-g_{h_1}(n))} \right| \]
\[ \lesssim L^3 + L \sum_{h_1=1}^{L} \sum_{h_2=1}^{L} \min \left( L, \frac{1}{|h_1 h_2 k_{d-2} \omega|} \right) \].

Using
\[ \# \{(h_1, h_2) \in \mathbb{Z}^2 : h_1 h_2 = N \} \lesssim N^{0+}, \]
we get
\[ S_k^4 \lesssim L^3 + L^{1+} \sum_{n=1}^{k_{d-2}|L|^2} \min \left( L, \frac{1}{|m\omega|} \right) \lesssim L^3 + L^{1+} \frac{|k_{d-2}|L^2}{q} (L + q \log q) \lesssim |k_{d-2}| L^{\frac{7}{2}}. \]

Hence
\[ S_k \lesssim C |k_{d-2}| L^{\frac{7}{2}}. \quad (2.8) \]

Repeating the argument above, we get
\[ S_k \leq C |k_{d-j}| \frac{1}{\epsilon} L^{1-\frac{1}{2^{j+1}}} + \frac{1}{\epsilon} L^{\frac{7}{2}} + \sum_{j=2}^{d-1} \frac{1}{\epsilon} \left( \sum_{|k_{d-j}| < \frac{1}{\epsilon}} \frac{|k_{d-j}|}{L^{1-\frac{1}{2^{j+1}}}} \right) \]
\[ \lesssim \left( \frac{1}{\epsilon} \right)^3 + \frac{1}{\epsilon} \left( \frac{1}{\epsilon} \right)^{\frac{7}{2}} L^{\frac{7}{2}} + \sum_{j=2}^{d-1} \left( \frac{1}{\epsilon} \right)^{\frac{7}{2}+j+1} L^{1-\frac{1}{2^{j+1}}} \lesssim L. \]

This proves (2.2) and Lemma 2.1.

**Remark 2.1** In the proof of Lemma 2.1, we only need to assume
\[ \|k\omega\| > c |k|^{-2}, \quad \forall 0 < |k| \leq L. \quad (2.10) \]

**3 Green’s Function Estimates**

In this section, we will prove the Green’s function estimates by using multi-scale analysis in [3].

We need the following lemma.

**Lemma 3.1** (cf. [3, Lemma 3.16]) Let \( A(x) = \{A_{mn}(x)\}_{1 \leq m, n \leq N} \) be a matrix valued function on \( \mathbb{T}^d \) such that
\[ A(x) \] is self-adjoint for \( x \in \mathbb{T}^d \),
\[ A(x) \] is self-adjoint for \( x \in \mathbb{T}^d \),
\[ (3.1) \]
$A_{mn}(x)$ is a trigonometric polynomial of degree $< N^{C_1}$, \[(3.2)\]

$|A_{mn}(x)| < C_2 e^{-c_2 |m-n|}$, \[(3.3)\]

where $c_2, C_1, C_2 > 0$ are constants.

Let $0 < \delta < 1$ be sufficiently small, $M = N^{\delta^6}$, $L_0 = N^{\frac{100}{c_3}}$, $0 < c_3 < \frac{1}{10} c_2$.

Assume that for any interval $I \subset [1, N]$ of size $L_0$, except for $x$ in a set of measure at most $e^{-L_0^{\delta^5}}$,

$$
\|(R_I A(x) R_I)^{-1}\| < e^{L_0^{\delta^5}},
$$

$$
\|(R_I A(x) R_I)^{-1} (m,n)| < e^{-c_3 |m-n|}, \quad m,n \in I, |m-n| > \frac{L_0}{10}.
$$

Fix $x \in \mathbb{T}^d, n_0 \in [1, N]$ is called a good site if $I_0 = [n_0 - \frac{M}{2}, n_0 + \frac{M}{2}] \subset [1, N]$, \[(3.4)\]

$$
\|(R_{I_0} A(x) R_{I_0})^{-1}\| < e^{M^{\delta^5}},
$$

$$
\|(R_{I_0} A(x) R_{I_0})^{-1} (m,n)| < e^{-c_3 |m-n|}, \quad m,n \in I_0, |m-n| > \frac{M}{10}.
$$

Denote $\Omega(x) \subset [1, N]$ the set of bad sites. Assume that for any interval $J \subset [1, N], |J| > N^{\frac{d}{C(d)}}$, we have $|J \cap \Omega(x)| < |J|^{1-\delta}$.

Then

$$
\|A(x)^{-1}\| < e^{N^{1-\frac{\delta}{C(d)}}},
$$

$$
|A(x)^{-1} (m,n)| < e^{-c_3' |m-n|}, \quad m,n \in [1, N], |m-n| > \frac{N}{10}
$$

except for $x$ in a set of measure at most $e^{-N^{\frac{\delta^2}{C(d)}}}$, where $C(d)$ is a constant depending on $d$, and $c_3' > c_3 - (\log N)^{-8}$.

By Lemmas 2.1 and 3.1, we can prove the Green’s function estimates.

**Proposition 3.1** Let $T = T_\omega : \mathbb{T}^d \rightarrow \mathbb{T}^d$ be the skew shift with frequency $\omega$ satisfying

$$
\|k\omega\| > c |k|^{-2}, \quad \forall 0 < |k| \leq N.
$$

$A_{mn}(x)$ is the form (1.17)–(1.21), and $\gamma$ in (1.21) is small.

Then for all $N$ and energy $E$,

$$
\|G_{[0,N]}(E,x)\| < e^{N^{\gamma}},
$$

$$
|G_{[0,N]}(E,x)(m,n)| < e^{-\frac{\log \gamma}{C_1} |m-n|}, \quad 0 \leq m,n \leq N, |m-n| > \frac{N}{10}
$$

for $x \notin \Omega_N(E)$, where

$$
\text{mes} \Omega_N(E) < e^{-N^\sigma}, \quad \sigma > 0.
$$

**Proof** Since $T^n(x_1, \cdots, x_d) = (x_1 + nx_2 + \cdots + \binom{n}{d-1} x_d + \binom{n}{d} \omega, \cdots, x_d + n\omega)$, $A_{mn}(x)$ is a trigonometric polynomial in $x$ of degree $< (|m| + |n|)^{C_1 + d}$, $\{A_{mn}(x) - E\}_{0 \leq m,n \leq N}$ satisfy (3.1)–(3.3) with $c_2 = 1, C_2 = \gamma$. 
First fix any large initial scale $N_0$ and choose $\gamma = \gamma(N_0)$ small. Using Lojasiewicz’s inequality (cf. [3, Section 4]), we get
\[
|G_{[0,N_0]}(E,x)(m,n)| < e^{N_0^{\frac{3}{2}} - \frac{1}{2}|m-n|}, \quad 0 \leq m,n \leq N_0 \tag{3.14}
\]
except for $x$ in a set of measure $< e^{-N_0^{\frac{3}{2}}}$. Then we establish inductively on the scale $N$ that
\[
\text{mes}\{x \in \mathbb{T}^d : |G_{[0,N]}(E,x)(m,n)| > e^{N^{1-3_{2}}-c_3|m-n|\chi_{|m-n|>\frac{M}{10}}}, \quad \exists 0 \leq m,n \leq N\} < e^{-N^{\delta_3}}, \tag{3.15}
\]
where $c_3 > \frac{1}{100}$, $0 < \delta < 1$ is a fixed small number. (3.14) implies (3.15) for an initial large scale $N_0$.

Assume that (3.15) holds up to scale $L_0 = N_0^{\frac{3}{4}}\delta^2$. Since $A_{m+1,n+1}(x) = A_{mn}(Tx)$, we have
\[
R_I(A(x) - E)R_I = R_{[0,N]}(A(T^n x) - E)R_{[0,N]}, \quad G_I(E,x) = G_{[0,N]}(E,T^n x), \quad I = [n,n+N].
\]
Since $T$ is measure preserving, (3.4)–(3.5) will hold for $x$ outside a set of measure at most $e^{-L_0^{\delta_3}}$. Denote $\Omega(x) \subset [0,N]$ the set of bad sites with respect to scale $M = N_0^{\delta_0}$. $n_0 \notin \Omega(x)$ means
\[
|G_{[0,M]}(E,T^{n_0-\frac{M}{2}}x)(m,n)| = |G_{[n_0-\frac{M}{2},n_0+\frac{M}{2}]}(E,x)\left(m + n_0 - \frac{M}{2}, n + n_0 - \frac{M}{2}\right)| < e^{M^{1-3_{2}}-c_3|m-n|\chi_{|m-n|>\frac{M}{10}}}. \tag{3.16}
\]
From the inductive hypothesis, we have
\[
|G_{[0,M]}(E,x)(m,n)| < e^{M^{1-3_{2}}-c_3|m-n|\chi_{|m-n|>\frac{M}{10}}}, \quad 0 \leq m,n \leq M, \quad \forall x \notin \Omega, \quad \text{mes} \Omega < e^{-M^{\delta_3}}. \tag{3.17}
\]

By (3.16)–(3.17) and Lemma 3.1, we only need to show that for any $x \in \mathbb{T}^d$, $N_0^{\frac{3}{4}} < L < N$,
\[
\#\{1 \leq n \leq L : T^n x \in \Omega\} < L^{1-\delta}. \tag{3.18}
\]
Since $A_{mn}(x)$ is a trigonometric polynomial of degree $< (|m| + |n|)^C$, we can express $G_{[0,M]}(E,x)(m,n)$ as a ratio of determinants to write (3.17) in the form
\[
P_{mn}(\cos x_1, \sin x_1, \cdots, \cos x_d, \sin x_d) \leq 0, \tag{3.19}
\]
where $P_{mn}$ is a polynomial of degree at most $M^C$. Replacing cos, sin by truncated power series, permits us to replace (3.19) by
\[
P_{mn}(x_1, \cdots, x_d) \leq 0, \quad \deg P_{mn} < M^C. \tag{3.20}
\]
So, $\Omega$ may be viewed as a semi-algebraic set of degree at most $M^C$. (For properties of semi-algebraic sets, see Section 4.) When $\epsilon > e^{-\frac{1}{4}M^{\delta_3}}$, by Corollary 4.1, $\Omega$ may be covered by at most $M^C(\frac{1}{\epsilon})^{d-1}e^d$-balls. Choosing $\epsilon = L^{-\frac{1}{(d+1)^2\alpha + 1}} > N^{-1} > e^{-\frac{1}{4}M^{\delta_3}}$, by (3.10), using Lemma 2.1 and Remark 2.1, we have
\[
\#\{1 \leq n \leq L : T^n x \in \Omega\} < M^C\left(\frac{1}{\epsilon}\right)^{d-1}e^dL < L^{C\delta^5 + \frac{1}{(d+1)^2\alpha + 1}} < L^{1-\delta},
\]
when $\delta$ is small enough.

This proves (3.18) and Proposition 3.1.
4 Semi-algebraic Sets

We recall some basic facts of semi-algebraic sets. Let \( P = \{ P_1, \cdots, P_s \} \subset \mathbb{R}[X_1, \cdots, X_n] \) be a family of real polynomials whose degrees are bounded by \( d \). A semi-algebraic set is given by

\[
S = \bigcup_j \bigcap_{i \in L_j} \{ \mathbb{R}^n : P_i s_j i 0 \}, \tag{4.1}
\]

where \( L_j \subset \{ 1, \cdots, s \}, s_j i \in \{ \leq, \geq, = \} \) are arbitrary. We say that \( S \) has degree at most \( s d \) and its degree is the inf of \( s d \) over all representations as in (4.1).

The projection of a semi-algebraic set of \( \mathbb{R}^n \) onto \( \mathbb{R}^m \) is semi-algebraic.

**Proposition 4.1** (cf. [2]) Let \( S \subset \mathbb{R}^n \) be a semi-algebraic set of degree \( B \). Then any projection of \( S \) has degree at most \( B^C, C = C(n) \).

We need the following bound on the number of connected components.

**Proposition 4.2** (cf. [1]) Let \( S \subset \mathbb{R}^n \) be a semi-algebraic set of degree \( B \). Then the number of connected components of \( S \) is bounded by \( B^C, C = C(n) \).

A more advanced part of the theory of semi-algebraic sets is the following triangulation theorem.

**Theorem 4.1** (cf. [11]) For any positive integers \( r, n \), there exists a constant \( C = C(n, r) \) with the following property: Any semi-algebraic set \( S \subset [0, 1]^n \) can be triangulated into \( N \lesssim (\deg S + 1)^C \) simplices, where for every closed \( k \)-simplex \( \Delta \subset S \), there exists a homeomorphism \( h_\Delta \) of the regular simplex \( \Delta^k \subset \mathbb{R}^k \) with unit edge length onto \( \Delta \) such that \( \|D_r h_\Delta\| \leq 1 \).

**Corollary 4.1** (cf. [4, Corollary 9.6]) Let \( S \subset [0, 1]^n \) be semi-algebraic of degree \( B \). Let \( \epsilon > 0 \), \( \mes_n S < \epsilon^n \). Then \( S \) may be covered by at most \( B^C \left( \frac{1}{\epsilon} \right)^n - 1 \epsilon \)-balls.

Finally, we will make essential use of the following transversality property.

**Lemma 4.1** (cf. [5, (1.5)]) Let \( S \subset [0, 1]^{n_1 + n_2} \) be a semi-algebraic set of degree \( B \) and

\[
\mes_{n_1}(\Proj_x S_1) < B^C \epsilon \tag{4.3}
\]

and \( S_2 \) satisfying the transversality property

\[
\mes_{n_2}(S_2 \cap L) < B^C \epsilon^{-1} \eta^{\frac{1}{2}} \tag{4.4}
\]

for any \( n_2 \)-dimensional hyperplane \( L \) such that \( \max_{1 \leq j \leq n_1} |\Proj_L(e_j)| < \frac{\epsilon}{100} \), where \( \{ e_j \mid 1 \leq j \leq n_1 \} \) are \( x \)-coordinate vectors.

Now we can prove the following lemma.
Lemma 4.2 Let $S \subset [0, 1]^{d+1}$ be a semi-algebraic set of degree $B$ such that
\[ \text{mes } S < e^{-B\sigma}, \quad \sigma > 0. \] (4.5)

Let $M$ satisfy
\[ \log \log M \ll \log B \ll \log M. \] (4.6)

Then for all $x \in \mathbb{T}^d$,
\[ \text{mes} \{ \omega \in \mathbb{T} : (\omega, T_j^x) \in S, \exists j \sim M \} < M^{-c}, \quad c > 0. \] (4.7)

Proof For $x^0 = (x^0_1, \ldots, x^0_d) \in \mathbb{T}^d$, we study the intersection of $S \subset [0, 1]^{d+1}$ and sets
\[ \{(\omega, x_1, \ldots, x_d) : \omega \in [0, 1]\}, \] (4.8)
where $x_i = (T_j^x)^i = x^0_i + jx^0_{i+1} + \cdots + \left( \frac{j}{d-i} \right)x^0_d + \left( \frac{j}{d-i+1} \right)\omega$, $1 \leq i \leq d$ are considered (mod 1).

By (4.5)–(4.6), we have
\[ \text{mes}_{d+1} S < \eta = e^{-B\sigma}, \quad \log B \ll \log M \ll \log \frac{1}{\eta}. \] (4.9)

Take $\epsilon = M^{-1+}$ and apply Lemma 4.1, then $S = S_1 \cup S_2$. Since $\text{mes}_1(\text{Proj}_x S_1) < B^C M^{-1+} = M^{-1+}$, restriction of $\omega$ permits us to replace $S$ by $S_2$ satisfying
\[ \text{mes}_d(S_2 \cap L) < B^C \epsilon^{-1} \eta^{\frac{m_1}{d+1}} < \eta^{\frac{1}{d+2}}, \] (4.10)
whenever $L$ is a $d$-dimensional hyperplane satisfying $|\text{Proj}_L(e_0)| < \frac{\epsilon}{100}$, where $e_0$ is the $\omega$-coordinate vector.

Fixing $j$, (4.8) can be considered as a subset of $[0, 1]^{d+1}$ lying in the union of the parallel $d$-dimensional hyperplanes
\[ Q_{m_1}^{(j)} = \left[ \omega = \frac{x_d}{j} - \frac{m_1 + x^0_d}{j} \epsilon_0, \quad |m_1| < M. \right. \] (4.11)

By (4.10), we have
\[ \text{mes}_d(S \cap Q_{m_1}) < \eta^{\frac{1}{d+2}}. \] (4.12)

Fixing $m_1$, consider the semi-algebraic set $S \cap Q_{m_1}$ and its intersection with the parallel $(d-1)$-dimensional hyperplanes
\[ Q_{m_1, m_2}^{(j)} = Q_{m_1} \bigcap \left[ x_d = \frac{2}{j-1} x_{d-1} - \frac{2}{j-1} \left( x^0_{d-1} + \frac{j+1}{2} x^0_d + m_2 \right), \quad |m_2| < M. \right. \] (4.13)

Take $\epsilon = M^{-1+}$ and apply Lemma 4.1 in $Q_{m_1}$, then $S \cap Q_{m_1} = S_{m_1}^{S_1} \cup S_{m_1}^{S_2}$, where
\[ \text{Proj}_{x_d} S_{m_1}^{S_1} \] is a union of at most $B^C$ intervals of measure at most $B^C M^{-1+}$, (4.14)
and by (4.12), we have
\[ \text{mes}_{d-1}(S_{m_1}^{S_1} \cap Q_{m_1, m_2}) < B^C M \eta^{\frac{1}{d+2}} < \eta^{\frac{1}{(d+2)^2}}. \] (4.15)
Fixing $m_2$, consider the semi-algebraic set $S^2_{m_1} \cap Q_{m_1, m_2}$ and its intersection with the parallel $(d - 2)$-dimensional hyperplanes

$$Q^{(j)}_{m_1, m_2, m_3} = Q_{m_1, m_2} \cap \bigg[ x_{d-1} = \frac{3}{j-2} x_{d-2} - \frac{3}{j-2} \left( x^0_{d-2} + \cdots + \frac{j(j+1)}{6} x^0_d + m_3 \right) \bigg],$$

where $|m_3| < M$.

Take $\epsilon = M^{-1+}$ and apply Lemma 4.1 in $Q_{m_1, m_2}$, then $S^2_{m_1} \cap Q_{m_1, m_2} = S^1_{m_1, m_2} \cup S^2_{m_1, m_2}$, where

$$\text{Proj}_{x_{d-1}} S^1_{m_1, m_2}$$

is a union of at most $B^C$ intervals of measure at most $B^C M^{-1+}$, and by (4.15), we have

$$\text{mes}_{d-2}(S^2_{m_1, m_2} \cap Q_{m_1, m_2, m_3}) < \eta \frac{1}{(d+2)^\omega}.$$  

Repeat the argument above. Fixing $m_i, 2 \leq i \leq d - 1$, consider the semi-algebraic set $S^2_{m_1, \ldots, m_{i-1}} \cap Q_{m_1, \ldots, m_i}$ and its intersection with the parallel $(d - i)$-dimensional hyperplanes

$$Q^{(j)}_{m_1, \ldots, m_{i+1}} = Q_{m_1, \ldots, m_i} \cap \bigg[ x_{d-i+1} = \frac{i+1}{j-i} x_{d-i} - \frac{i+1}{j-i} \left( x^0_{d-i} + \cdots + \frac{j+1}{i+1} \frac{d}{j} x^0_d + m_{i+1} \right) \bigg],$$

where $|m_{i+1}| < M$.

Take $\epsilon = M^{-1+}$ and apply Lemma 4.1 in $Q_{m_1, \ldots, m_i}$, then $S^2_{m_1, \ldots, m_{i-1}} \cap Q_{m_1, \ldots, m_i} = S^1_{m_1, \ldots, m_i} \cup S^2_{m_1, \ldots, m_i}$, where

$$\text{Proj}_{x_{d-i+1}} S^1_{m_1, \ldots, m_i}$$

is a union of at most $B^C$ intervals of measure at most $B^C M^{-1+}$, (4.17) and

$$\text{mes}_{d-i}(S^2_{m_1, \ldots, m_i} \cap Q_{m_1, \ldots, m_{i+1}}) < \eta \frac{1}{(d+2)^\omega}. \quad (4.18)$$

Finally, fixing $m_{d-1}$, consider the semi-algebraic set $S^2_{m_1, \ldots, m_{d-2}} \cap Q_{m_1, \ldots, m_{d-1}}$ and its intersection with the parallel lines

$$Q^{(j)}_{m_1, \ldots, m_d} = Q_{m_1, \ldots, m_{d-1}} \cap \bigg[ x_2 = \frac{d}{j-d+1} x_1 - \frac{d}{j-d+1} \left( x^0_1 + \cdots + \frac{(j+1)}{d(j-d)} x^0_d + m_d \right) \bigg],$$

where $|m_d| < M$.

Take $\epsilon = M^{-1+}$ and apply Lemma 4.1 in $Q_{m_1, \ldots, m_{d-1}}$, then $S^2_{m_1, \ldots, m_{d-2}} \cap Q_{m_1, \ldots, m_{d-1}} = S^1_{m_1, \ldots, m_{d-1}} \cup S^2_{m_1, \ldots, m_{d-1}}$, where

$$\text{Proj}_{x_2} S^1_{m_1, \ldots, m_{d-1}}$$

is a union of at most $B^C$ intervals of measure at most $B^C M^{-1+}$, (4.20) and

$$\text{mes}(S^2_{m_1, \ldots, m_{d-1}} \cap Q_{m_1, \ldots, m_d}) < \eta \frac{1}{(d+2)^\omega}. \quad (4.21)$$

Summing (4.21) over $j, m_1, \ldots, m_d$, the collected contribution in the $\omega$-parameter is less than $M^{-d} M^d B^C M^\eta (d+2)^\omega < \eta (d+2)^\omega$. So, we only need to consider the contribution of
$S_{m_1, \ldots, m_d}$ in (4.17). We just deal with $S_{m_1, \ldots, m_d-1}$ below, since for other sets, the method is similar.

If (4.7) fails, we have

$$\sum_{j \sim M, |m_1| \ldots, |m_d| < M} \mes[\text{Proj}_x \text{Proj}_y (S_{m_1, \ldots, m_d-1} \cap Q_{m_1, \ldots, m_d})] > M^{0-},$$

(4.22)

$$\sum_{j \sim M, |m_1| \ldots, |m_d| < M} \mes[\text{Proj}_x (S_{m_1, \ldots, m_d-1} \cap Q_{m_1, \ldots, m_d})] > M^{d-1-}.$$

(4.23)

So, there is a set $J \subset \mathbb{Z} \cap [j \sim M, |J| > M^{1-}$ such that for each $j \in J$, there are at least $M^{d-1-}$ values of $(m_1, \ldots, m_d)$ satisfying

$$\sum_{|m_d| < M} \mes[\text{Proj}_x (S_{m_1, \ldots, m_d-1} \cap Q_{m_1, \ldots, m_d})] > M^{-1}.$$  

(4.24)

By (4.20), $S_{m_1, \ldots, m_d-1} \cap Q_{m_1, \ldots, m_d} \neq \emptyset$ for at most $M^{0+}$ values of $m_d$. Hence

$$\max_{m_d} \mes(S \cap Q_{m_1, \ldots, m_d}) > M^{0-}.$$  

(4.25)

For fixed $j$,

$$Q_{m_1, \ldots, m_d} / \xi_j / \left(1, \left(\frac{j}{d}\right), \ldots, j\right)^T, \quad \|\xi_j\| = 1.$$  

(4.26)

Denote $S_x$ the intersection of $S$ and the $d$-dimensional hyperplane $[x'] = x$. From (4.24), to each $(m_1, \ldots, m_d-1)$ we can associate some $m_d$, such that

$$\int_0^1 \#\{|m_1|, \ldots, |m_d-1| < M | S_x \cap Q_{m_1, \ldots, m_d} \neq \emptyset\} dx > M^{d-1-}.$$  

(4.27)

If $\mes_S S_x < \eta^{d+}$, then $S_x \cap Q_{m_1, \ldots, m_d} \neq \emptyset$ implies $\text{dist}(Q_{m_1, \ldots, m_d}, \partial S_x) < \eta^{\frac{d-1}{2}}$, where $\partial S_x$ is a union of at most $B^C$ connected $(d-1)$-dimensional algebraic set of degree at most $B^C$. From (4.26), it follows that there is a fixed $(d-1)$-dimensional algebraic set $\Gamma = \Gamma^{(j)}$ of degree at most $B^C$ such that for $x \in [0, 1]$ in a set of measure $> M^{0-}$, there are at least $M^{d-1-}\frac{1}{M}$-separated points that are $\eta^{\frac{d-1}{2}}$-close to both $\partial S_x$ and $\Gamma + x\xi_j$. Hence $(\Gamma + x\xi_j) \cap S_{\eta_1}$ $(\eta_1$-neighborhood of $S$, $\eta_1 = 2\eta^{\frac{d-1}{2}}$) contains at least $M^{d-1-}\frac{1}{M}$-separated points. So, $\mes_{d-1}((\Gamma + x\xi_j) \cap S_{\eta_1}) > M^{0-}$.

The hypercylinder $C^{(j)} = t\xi_j + \Gamma^{(j)}$ satisfies

$$\mes_d(C^{(j)} \cap S_{\eta_1}) > M^{0-}.$$  

(4.28)

By Corollary 4.1, we have

$$\mes_{d+1} S_{\eta_1} < B^C \eta_1.$$  

(4.29)

Since (4.27) holds for all $j \in J$, by (4.27)–(4.28), we have

$$\sum_{j_1, \ldots, j_{d+1} \in J} \mes_{d+1} \left[ \bigcap_{1 \leq i \leq d+1} C_{\eta_1}^{(j_i)} \right] > \eta_1 M^{d+1-}.$$  

(4.29)

So, there are distinct $j_1, \ldots, j_{d+1} \sim M$ such that

$$\mes_{d+1} \left[ \bigcap_{1 \leq i \leq d+1} C_{\eta_1}^{(j_i)} \right] > \eta_1 M^{0-}.$$  

(4.29)
By (4.25), using Vandermonde determinant, we have
\[ \det[\xi_{j_1}, \cdots, \xi_{j_{d+1}}] \neq 0 \quad (4.30) \]
for distinct \( j_1, \cdots, j_{d+1} \). So, the vectors \( \xi_{j_1}, \cdots, \xi_{j_{d+1}} \) are not in any \( d \)-dimensional hyperplane. Since \( \log M \ll \log \frac{1}{m} \), this leads to a contradiction to (4.29).
This proves Lemma 4.2.

5 Proof of Anderson Localization

In this section, we give the proof of Anderson localization as in [6].

By application of the resolvent identity, we have the following lemma.

**Lemma 5.1** (cf. [4, Lemma 10.33]) Let \( I \subset \mathbb{Z} \) be an interval of size \( N \) and \( \{I_\alpha\} \) be subintervals of size \( M \ll N \). Assume that
(i) if \( k \in I \), then there is some \( \alpha \) such that \( [k - M/4, k + M/4] \cap I \subset I_\alpha \).
(ii) For all \( \alpha \),
\[ \|G_{I_\alpha}\| < e^{M^{1-}}, \quad |G_{I_\alpha}(n_1, n_2)| < e^{-c_0|n_1-n_2|}, \quad n_1, n_2 \in I_\alpha, \quad |n_1 - n_2| > M/10. \]
Then
\[ |G_I(n_1, n_2)| < e^{-(c_0-)|n_1-n_2|}, \quad n_1, n_2 \in I, \quad |n_1 - n_2| > N/10. \]

Let \( T = T_\omega \) be the skew shift on \( \mathbb{T}^d \) with frequency \( \omega \) satisfying
\[ \|k\omega\| > c\|k\|^{-2}, \quad \forall k \in \mathbb{Z} \setminus \{0\}. \quad (5.1) \]
Fix \( x_0 \in \mathbb{T}^d \),
\[ H(x_0)(m, m) = v(T^m x_0), \quad (5.2) \]
\[ H(x_0)(m, n) = \phi_{m-n}(T^m x_0) + \overline{\phi_{n-m}(T^n x_0)}, \quad m \neq n \quad (5.3) \]
with \( v \) and \( \phi_k \) satisfying (1.19)–(1.21) and \( \gamma \) taken small enough. Then we have the following theorem.

**Theorem 5.1** For almost all \( \omega \) satisfying (5.1), the lattice operator \( H_\omega(x_0) \) satisfies Anderson localization.

**Proof** By Shnol’s theorem (cf. [12]), to establish Anderson localization, it suffices to show that if \( \xi = (\xi_n)_{n \in \mathbb{Z}}, E \in \mathbb{R} \) satisfy
\[ \xi_0 = 1, \quad |\xi_n| < C|n|, \quad |n| \to \infty, \quad (5.4) \]
\[ H(x_0)\xi = E\xi, \quad (5.5) \]
then
\[ |\xi_n| < e^{-c|n|}, \quad |n| \to \infty. \quad (5.6) \]
Let \( M = N^{C_0}, \quad L = M^C \). Denote \( \Omega \subset \mathbb{T}^d \) the set of \( x \) such that
\[ |G_{[-M,M]}(E, x)(m, n)| < e^{M^{1-} - \frac{1}{100}|m-n|\|x\|_{\mathbb{T}^d} > \frac{M}{10}} \]
fails for some $|m|, |n| \leq M$. It was shown in Section 3 that
\[ \# \{1 \leq |n| \leq L : T^m x_0 \in \Omega \} < L^{1 - \delta}. \]
So, we may find an interval $I \subset [0, L]$ of size $M$, such that
\[ T^{m_0} x_0 \notin \Omega, \quad \forall n_0 \in I \cup (-I). \]
Hence
\[ |G_{[n_0 - M, n_0 + M]}(E, x_0)(m, n)| < e^{M^1 - \frac{M}{100} |m-n| |x_{m-n}> M}, \quad m, n \in [n_0 - M, n_0 + M]. \quad (5.7) \]
By (5.4)–(5.5) and (5.7), we have
\[ |\xi_{n_0}| \leq \sum_{n' \in [n_0 - M, n_0 + M], n'' \notin [n_0 - M, n_0 + M]} e^{M^1 - \frac{M}{100} |n_0 - n'| |x_{n_0 - n'}| + M |n - n''| |\xi_{n''}| | < e^{\frac{M}{100}}. \quad (5.8) \]
Denoting $j_0$ the center of $I$, we have
\[ 1 = |\xi_0| \leq \| G_{[-j_0, j_0]}(x_0, E) \| \| R_{[-j_0, j_0]} H(x_0) R_{[-j_0, j_0]} \| \xi. \quad (5.9) \]
By (5.4) and (5.8), we have for $|n| \leq j_0$,
\[ |(H(x_0) R_{[-j_0, j_0]} \xi)| \leq \sum_{|n'| > j_0} e^{-|n-n'|} |\xi_{n'}| < e^{-\frac{M}{100}} + \sum_{|n'| > j_0 + \frac{M}{2}} e^{-|n-n'|} |\xi_{n'}| < e^{-\frac{M}{100}}. \quad (5.10) \]
By (5.9)–(5.10), we have
\[ \| G_{[-j_0, j_0]}(x_0, E) \| > e^{N}. \quad (5.11) \]
So there is some $j_0$, $|j_0| < N_1 = N C_1$ ($C_1$ is a sufficiently large constant), such that by (5.11)
\[ \text{dist}(E, \text{spec } H_{[-j_0, j_0]}(x_0)) < e^{-N}. \quad (5.12) \]
Denote $\Omega(E) \subset \mathbb{T}^d$ the set of $x$ such that
\[ |G_{[-N, N]}(E, x)(m, n)| < e^{N^1 - \frac{M}{100} |m-n| |x_{m-n}> M}, \quad |m|, |n| \leq N. \quad (5.13) \]
Consider the set $S = S_N \subset \mathbb{T}^{d+1} \times \mathbb{R}$ of $(\omega, x, E')$, where
\[ \|k\omega\| > c|k|^{-2}, \quad \forall 0 < |k| \leq N, \quad (5.14) \]
\[ x \in \Omega(E'), \quad (5.15) \]
\[ E' \in \mathcal{E}_\omega. \quad (5.16) \]
By (5.14)-(5.16),
\[ \text{Proj}_{T^{d+1}} S \text{ is a semi-algebraic set of degree } < N^C, \]  
(5.17)
and by Proposition 3.1,
\[ \text{mes}(\text{Proj}_{T^{d+1}} S) < e^{-\frac{1}{4}N^c}. \]  
(5.18)

Let \( N_2 = e^{(\log N)^2} \),
\[ B_N = \{ \omega \in T : (\omega, T^j x_0) \in \text{Proj}_{T^{d+1}} S_N, \exists |j| \sim N_2 \}. \]  
(5.19)

By (5.17)-(5.19), using Lemma 4.2, \( \text{mes} B_N < N^{-c}, c > 0 \). Let
\[ B = \bigcap_{N_0} \bigcup_{N > N_0} B_N. \]  
(5.20)

Then by Borel-Cantelli theorem, \( \text{mes} B = 0 \). We restrict \( \omega \notin B \).

If \( \omega \notin B_N \), we have for all \( |j| \sim N_2 \), \( (\omega, T^j x_0) \notin \text{Proj}_{T^{d+1}} S_N \), by (5.13),
\[ |G_{[j-N,j+N]}(E, x_0)(m,n)| < e^{-\frac{1}{100}|m-n|} \chi_{|m-n| > \frac{N}{100}}, \]  
(5.21)

Let \( \Lambda = \bigcup_{\frac{1}{4}N_2 < j < 2N_2} \frac{1}{4}N_2 < j < 2N_2 \]. By Lemma 5.1, we deduce from (5.21) that
\[ |G_{\Lambda}(E, x_0)(m,n)| < e^{-\frac{1}{100}|m-n|}, |m-n| > \frac{N_2}{10}, \]  
(5.22)
and therefore
\[ |\xi_j| < e^{-\frac{1}{100}|j|}, \frac{1}{2}N_2 \leq j \leq N_2. \]  
(5.23)

Since \( \omega \notin B \), by (5.20), there is some \( N_0 > 0 \), such that for all \( N \geq N_0 \), \( \omega \notin B_N \). So (5.23) holds for \( j \in \bigcup_{N \geq N_0} \left[ \frac{1}{2}e^{(\log N)^2}, e^{(\log N)^2} \right] = \left[ \frac{1}{2}e^{(\log N_0)^2}, \infty \right) \). This proves (5.6) for \( j > 0 \), similarly for \( j < 0 \). Hence Theorem 5.1 follows.

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