Cartan Maps and Projective Modules

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Abstract. Let $R$ be a commutative ring, $\pi$ be a finite group, $R\pi$ be the group ring of $\pi$ over $R$. Then the Cartan map $c : K_0(R\pi) \to G_0(R\pi)$ is injective. Suppose that $R$ is a Dedekind domain with char $R = p > 0$ and $\pi$ is a $p$-group. Then every finitely generated projective $R\pi$-module is isomorphic to $F \oplus A$ where $F$ is a free module and $A$ is a projective ideal of $R\pi$. Moreover, $R$ is a principal ideal domain if and only if every finitely generated projective $R\pi$-module is isomorphic to a free module. Let $R$ be a commutative noetherian ring with total quotient ring $K$, $A$ be an $R$-algebra which is a finitely generated $R$-projective module. Suppose that $I$ is an ideal of $R$ such that $R/I$ is artinian. Let $\{M_1, \ldots, M_n\}$ be the set of all maximal ideals of $R$ containing $I$. Assume that the Cartan map $c_i : K_0(A/M_iA) \to G_0(A/M_iA)$ is injective for all $1 \leq i \leq n$. If $P$ and $Q$ are finitely generated $A$-projective modules with $KP \simeq KQ$, then $P/IP \simeq Q/IQ$.

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§1. Introduction

Throughout this note, \( R\pi \) denotes the group ring where \( \pi \) is a finite group and \( R \) is a commutative ring; all the modules we consider are left modules. The present article arose from an attempt to understand the following theorem of Swan.

**Theorem 1.1** (Swan [Sw1]) Let \( R \) be a Dedekind domain with quotient field \( K \) and \( \pi \) be a finite group. Assume that \( \text{char} R = 0 \) and no prime divisor of \(|\pi|\) is a unit in \( R \). If \( P \) is a finitely generated projective \( \mathbb{R} \pi \)-module, then \( K \otimes R P \) is a free \( K \pi \)-module and \( P \) is isomorphic to \( F \oplus A \) where \( F \) is a free \( \mathbb{R} \pi \)-module and \( A \) is a left ideal of \( \mathbb{R} \pi \). Moreover, for any non-zero ideal \( I \) of \( R \), we may choose \( A \) such that \( I + (R \cap A) = R \).

Several alternative approaches to the proof of some parts of Theorem 1.1 were proposed; see, for examples, [Ba1], [Gi], [Ri2], [Ha], [Gr, page 20], [Sw3, page 57, Theorem 4.2]; also see [Sw2, page 171, Theorem 11.2]. Using the injectivity of the Cartan map (see Definition 2.4), Bass recast a crucial step of the proof of Theorem 1.1 as follows.

**Theorem 1.2** (Bass [Ba1, Theorem 1]) Let \( R \) be a commutative noetherian ring with total quotient ring \( K \) and denote by \( m - \text{spec}(R) \) the space of all the maximal ideals of \( R \) (under Zariski topology) with \( d = \dim(m - \text{spec}(R)) \). Let \( A \) be an \( R \)-algebra which, as an \( R \)-module, is a finitely generated projective \( R \)-module. Suppose that \( P \) is a finitely generated projective \( A \)-module satisfying that (i) \( K \otimes R P \) is a free \( K A \)-module of rank \( r \), and (ii) the Cartan map \( c_\mathcal{M} : K_0(A/\mathcal{M}A) \to G_0(A/\mathcal{M}A) \) is injective for any \( \mathcal{M} \in m - \text{spec}(R) \). Then \( P \) is isomorphic to \( F \oplus Q \) where \( F \) is a free \( A \)-module of rank \( r' \) and \( Q/\mathcal{M}Q \) is a rank \( d' \) free module over \( A/\mathcal{M}A \) for any \( \mathcal{M} \in m - \text{spec}(R) \) with \( d' = \min\{d, r\} \) and \( r' = r - d' \).

Note that the assumption about the Cartan map in Theorem 1.2 is valid when \( A = R\pi \) where \( \pi \) is a finite group, thanks to the following theorem of Brauer and Nesbitt.

**Theorem 1.3** (Brauer and Nesbitt [BN1, BN2, Br, CR, page 442]) Let \( k \) be a field, \( \pi \) be a finite group. Then the Cartan map \( c : K_0(k\pi) \to G_0(k\pi) \) is injective.

It is known that the Cartan map \( c : K_0(A) \to G_0(A) \) is an isomorphism if the (left) global dimension of \( A \) is finite [Ei, Proposition 21; Sw2, page 104, Corollary 4.7]. However, it is possible that the global dimension of \( A \) is infinite while the Cartan map is injective. By Lemma 2.11 the global dimension of the group ring \( k\pi \) (\( k \) is a field) is infinite if \( \text{char} k = p > 0 \) and \( p \mid |\pi| \). Thus Theorem 1.3 provides plenty of
such examples. For examples other than the group rings, see [EIN, Section 5], [La3, Example 5.76], [BFVZ] and also [La1, Theorem 2.4; St].

In this article we will prove the following result which generalizes Theorem 1.3.

**Theorem 1.4** Let $R$ be a commutative artinian ring and $\pi$ be a finite group. Then the Cartan map $c : K_0(R\pi) \to G_0(R\pi)$ is injective.

The main idea of the proof of Theorem 1.4 is to use the Frobenius functors as in Lam’s paper [La1]. For a generalization of this theorem, see Theorem 1.3.

We will also study a variant of Theorem 1.1, i.e. finitely generated $R\pi$-projective modules where $R$ is a Dedekind domain with char $R = p > 0$. One of our results is the following (see Theorem 3.1 and Theorem 3.3).

**Theorem 1.5** Let $R$ be a Dedekind domain with quotient field $K$ such that char $R = p > 0$. Let $\pi$ be a finite group with $p \mid |\pi|$, and $\pi_p$ be a $p$-Sylow subgroup of $\pi$.

1. Let $M$ be a finitely generated $R\pi$-module. Then
   
   $M$ is a projective $R\pi$-module,
   
   $\Leftrightarrow$ The restriction of $M$ to $R\pi_p$ is a projective $R\pi_p$-module,
   
   $\Leftrightarrow$ The restriction of $M$ to $R\pi'$ is a projective $R\pi'$-module where $\pi'$ is any elementary abelian subgroup of $\pi_p$.

2. If $\pi$ is a $p$-group and $P$ is a finitely generated projective $R\pi$-module, then $K \otimes_R P$ is a free $K\pi$-module and $P$ is isomorphic to $F \oplus A$ where $F$ is a free module and $A$ is a projective ideal of $R\pi$. Moreover, for any non-zero ideal $I$ of $R$, we may choose $A$ such that $I + (R \cap A) = R$.

In the situation of Part (2) of the above theorem, we will show in Theorem 3.5 that $R$ is a principal ideal domain if and only if every finitely generated $R\pi$-projective module is free. For more cases, see Lemma 3.6, Lemma 4.6 and Lemma 4.9.

Terminology and notations. For the sake of brevity, a projective module over a ring $A$ will be called an $A$-projective module (or simply $A$-projective). A projective ideal $A$ of $A$ is a left ideal of the ring $A$ such that $A$ is $A$-projective. An $A$-module $M$ is called indecomposable if $M \cong M_1 \oplus M_2$ implies either $M_1 = 0$ or $M_2 = 0$; similarly for indecomposable projective modules. If $A$ is a ring we will denote by rad($A$) the Jacobson radical of $A$. If $A$ is an $R$-algebra where $R$ is a commutative ring with total quotient $K$, we denote $KA := K \otimes_R A$, $KM := K \otimes_R M$ if $M$ is an $A$-module; similarly, $R_M$ denotes the localization of $R$ at the maximal ideal $M$ and $M_M := R_M \otimes_R M$ if $M$ is an $A$-module.
An $R\pi$-lattice $M$ is a finitely generated $R\pi$-module which is an $R$-projective module as an $R$-module (see Definition 2.6). Two $R\pi$-lattices $M$ and $N$ belong to the same genus if $R_M \otimes_R M$ is isomorphic to $R_M \otimes_R N$ for any maximal ideal $M$ of $R$ [CR page 643].

If $M$ is an $R\pi$-module and $\pi'$ is a subgroup of $\pi$, then we may regard $M$ as an $R\pi'$-module through the ring homomorphism $R\pi' \rightarrow R\pi$; such an $R\pi'$-module is called the restriction of $M$ to $R\pi'$ and is denoted by $M_{\pi'}$. On the other hand, if $N$ is an $R\pi'$-module and $\pi'$ is a subgroup of $\pi$, then the $R\pi$-module $R\pi \otimes_{R\pi'} N$ is called the induced module of $N$ and is denoted by $N^{\pi}$. For details, see [CR page 228].

§2. The Cartan map

Recall the definitions of the Grothendieck groups $K_0(A)$ and $G_0(A)$. Let $A$ be a ring. Then $K_0(A)$ is the abelian group defined by generators $[P]$ where $P$ is a finitely generated $A$-projective module, with relations $[P] = [P'] + [P'']$ whenever there is a short exact sequence of projective $A$-modules $0 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 0$. In a similar way, if $A$ is a left noetherian ring, then $G_0(A)$ is the abelian group defined by generators $[M]$ where $M$ is a finitely generated $A$-module, with relations $[M] = [M'] + [M'']$ whenever an exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ exists. For details, see [Sw3, Chapter 1].

**Definition 2.1** ([Sw2, page 86]) Let $A$ be a ring, $I$ be a two-sided ideal of $A$. We say that $A$ is I-complete if the natural map $A \rightarrow \lim_{\leftarrow n \in \mathbb{N}} A/I^n$ is an isomorphism.

**Lemma 2.2** ([Sw2, page 89, Theorem 2.26]) If $I$ is a two-sided ideal of a ring $A$ such that $A$ is I-complete, then there is a one-to-one correspondence between the isomor-
H"older composition series of finitely generated \(A\)-projective modules and the isomorphism classes of finitely generated \(A/I\)-projective modules given by \(P \mapsto P/IP\).

**Lemma 2.3** Let \(A\) be a left artinian ring with Jacobson radical \(J\). Then \(K_0(A)\) and \(G_0(A)\) are free abelian groups of the same rank. In fact, it is possible to find finitely generated indecomposable projective \(A\)-modules \(P_1, P_2, \ldots, P_n\) and simple \(A\)-modules \(M_1, M_2, \ldots, M_n\) such that, for \(1 \leq i \leq n\), \(M_i \simeq P_i/JP_i\) and \(K_0(A) = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot [P_i]\), \(G_0(A) = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot [M_i]\). Moreover, each \(P_i\) is a left ideal generated by some idempotent element of \(A\).

**Proof.** Since \(J\) is a nilpotent ideal [La2 page 56], \(A\) is \(J\)-complete. On the other hand, if \(M\) is a simple \(A\)-module, then \(J \cdot M = 0\) [La2 page 54]; thus the family of simple \(A\)-modules is identical to that of \(A/J\)-modules.

Note that \(A/J\) is left artinian and \(\text{rad}(A/J) = 0\). It is semisimple by the Artin-Wedderburn Theorem [La2 page 57]. Since any \(A/J\)-module is projective [La2 page 29], a simple \(A/J\)-module is an indecomposable \(A/J\)-projective module. If \(Q\) is a finitely generated indecomposable \(A/J\)-projective module, then \(Q \oplus Q' \simeq (A/J)^{(t)}\) for some module \(Q'\) and some integer \(t\). By the Krull-Schmidt-Azumaya Theorem [CR, page 128], \(Q\) is isomorphic to some minimal left ideal of \(A/J\). It follows that every finitely generated indecomposable \(A/J\)-projective module is isomorphic to a minimal left ideal of \(A/J\) (which is generated by some idempotent of \(A/J\)). Thus the family of simple \(A/J\)-modules is identical to that of finitely generated indecomposable \(A/J\)-projective modules.

Apply the correspondence of Lemma 2.2. Since \(A\) is \(J\)-complete, any idempotent in \(A/J\) can be lifted to one in \(A\) [Sw2 page 86, Proposition 2.19], which gives rise to an indecomposable \(A\)-projective module. \(\blacksquare\)

**Definition 2.4** Let \(A\) be a left artinian ring. The Cartan map \(c : K_0(A) \to G_0(A)\) is defined as follows. For any finitely generated \(A\)-projective module \(P\), find a Jordan-H"older composition series of \(P\): \(M_0 = P \supset M_1 \supset M_2 \supset \cdots \supset M_t = \{0\}\), where each \(M_i/M_{i+1}\) is a simple \(A\)-module. Define \(c([P]) = \sum_{0 \leq i \leq t-1} [M_i/M_{i+1}] \in G_0(A)\). It is easy to see that \(c\) is a well-defined group homomorphism.

By Lemma 2.3, write \(K_0(A) = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot [P_i]\), \(G_0(A) = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot [M_i]\). If \(c([P_i]) = \sum_{1 \leq j \leq n} a_{ij}[M_j]\) where \(a_{ij} \in \mathbb{Z}\), the matrix \((a_{ij})_{1 \leq i, j \leq n}\) is called the Cartan matrix. Clearly the Cartan map is injective if and only if \(\det(a_{ij}) \neq 0\).

In general, the Cartan map \(c : K_0(A) \to G_0(A)\) may be defined for a left noetherian ring \(A\) by sending \([P] \in K_0(A)\) (where \(P\) is a finitely generated \(A\)-projective module) to \([P] \in G_0(A)\) by regarding \(P\) as a finitely generated \(A\)-module. As noted before, if \(A\) is a left noetherian ring with finite global dimension, then the Cartan map \(c : K_0(A) \to \)
$G_0(A)$ is an isomorphism [Sw2, page 104]. In this article we will restrict our attention only to Cartan maps of left artinian rings.

**Lemma 2.5** Let $A$ be a left artinian ring with Jacobson radical $J$. Then $A$ contains finitely many indecomposable projective ideals, $P_1, P_2, \ldots, P_n$, satisfying the following properties,

(i) $P_i \neq P_j$ if $i \neq j$;
(ii) each $P_i$ is generated by an idempotent element of $A$;
(iii) every finitely generated $A$-projective module is isomorphic to $\bigoplus_{1 \leq i \leq n} P_i^{(m_i)}$ for some non-negative integers $m_i$;
(iv) $\{P_i/JP_i : 1 \leq i \leq n\}$ forms the family of all the isomorphism classes of simple $A$-modules. In fact, $P_i$ is the projective cover of $P_i/JP_i$.

**Proof.** The proofs of (i), (ii) and (iii) are implicit in the proof of Lemma 2.3. As to the definition of projective covers, see [Sw2, page 88]. The proof of (iv) follows from [Sw2, page 89, Corollary 2.25].

For the proof of Theorem 1.4 recall the definitions of $G_0^R(R\pi)$ and Frobenius functors. Note that the definition of Frobenius functors in Definition 2.7 is that given in [Sw3] and is slightly different from that in [La1].

**Definition 2.6** ([Sw3, page 2]) Let $R$ be a commutative ring, $A$ be an $R$-algebra which is a finitely generated $R$-module. Define $G_0^R(A)$ to be the abelian group with generators $[M]$ where $M$ is a finitely generated $A$-module which is $R$-projective as an $R$-module, with relations $[M] = [M'] + [M'']$ whenever there is a short exact sequence of $A$-modules $0 \to M' \to M \to M'' \to 0$ such that $M', M, M''$ are $R$-projective as $R$-modules. Note that $G_0^R(R\pi)$ is a commutative ring if $\pi$ is a finite group [Sw3, page 7].

**Definition 2.7** ([La1, Sw3, page 15]) Let $\pi$ be a finite group, $\text{Grp}_\pi$ be the category whose objects are all the subgroups of $\pi$ with morphisms $\text{hom}(\pi_1, \pi_2)$ consisting of the unique injection if $\pi_1 \subset \pi_2 \subset \pi$ with the understanding that $\text{hom}(\pi_1, \pi_2) = \emptyset$ if $\pi_1 \not\subset \pi_2$. Let $\text{Ring}$ be the category of commutative rings. A Frobenius functor consists of the following data,

(i) for each subgroup $\pi'$ of $\pi$, there corresponds a commutative ring $F(\pi')$,
(ii) for subgroups $\pi_1 \subset \pi_2 \subset \pi$ and the injection $i : \pi_1 \to \pi_2$, there exist the ring homomorphism $i^* : F(\pi_2) \to F(\pi_1)$ and the additive group homomorphism $i_* : F(\pi_1) \to F(\pi_2)$ satisfying the properties that $i^* : \text{Grp}_\pi \to \text{Ring}$ is a contravariant functor and $i_*$ from finite groups to abelian groups is a covariant functor,
(iii) (Frobenius identity) for each injection \( i : \pi_1 \to \pi_2 \), if \( x \in F(\pi_1), \ y \in F(\pi_2) \), then 
\[ i_*(x) \cdot y = i_*(x \cdot (i^*y)). \]

It is not difficult to see that \( \pi' \mapsto G^R_0(R\pi') \) is a Frobenius functor where \( R \) is a commutative ring and \( G^R_0(R\pi') \) is defined in Definition [2.6].

**Definition 2.8** Given a finite group \( \pi \) and a Frobenius functor \( F : \text{Grp}_\pi \to \text{Ring} \), a Frobenius module \( M \) over \( F \) consists of the data

(i) for each subgroup \( \pi' \) of \( \pi \), there corresponds an \( F(\pi') \)-module \( M(\pi') \);

(ii) for each injection \( i : \pi_1 \to \pi_2 \), there exist the contravariant additive functor 
\( i^* : M(\pi_2) \to M(\pi_1) \) and the covariant additive functor \( i_* : M(\pi_1) \to M(\pi_2) \) such that if \( x \in F(\pi_2), \ u \in M(\pi_2) \), then \( i^*(x \cdot u) = i^*(x) \cdot i^*(u) \);

(iii) for any injection \( i : \pi_1 \to \pi_2 \) and \( x \in F(\pi_1), \ v \in M(\pi_2), \) then 
\[ i_*(x) \cdot v = i_*(x \cdot i^*(v)) ; \]
if \( y \in F(\pi_2), \ u \in M(\pi_1) \), then 
\[ y \cdot i_*(u) = i_*(i^*(y) \cdot u). \]

Let \( R \) be a commutative ring, \( \pi \) be a finite group. Let \( F \) be the Frobenius functor defined by \( \pi' \mapsto G^R_0(R\pi') \). It is easy to show that \( \pi' \mapsto G_0(R\pi') \) and \( \pi' \mapsto K_0(R\pi') \) are Frobenius modules over \( F \).

The morphism of Frobenius modules over a given Frobenius functor can be defined in an obvious way. For details, see [Sw3, pages 16–18]. If \( M_1 \) and \( M_2 \) are Frobenius modules over a Frobenius functor \( F \) and \( \varphi : M_1 \to M_2 \) is a morphism over \( F \), then \( \text{Ker}(\varphi) \) and \( \text{Coker}(\varphi) \), defined in the obvious way, are also Frobenius modules over \( F \).

If \( R \) is a commutative artinian ring, the Cartan map of Definition [2.3] defined by 
\( K_0(R\pi') \to G_0(R\pi') \) is a morphism of Frobenius modules over the Frobenius functor 
\( \pi' \mapsto G^R_0(R\pi') \).

**Definition 2.9** ([Sw3, pages 22–23]) Let \( \pi \) be a finite group, \( \mathcal{C} \) be a class of certain subgroups of \( \pi \). If \( F : \text{Grp}_\pi \to \text{Ring} \) is a Frobenius functor and \( M \) is a Frobenius module over \( F \). We define 
\[
F(\pi)_\mathcal{C} = \sum_{\pi' \in \mathcal{C}} i_*(F(\pi')),
\]
\[
M(\pi)_\mathcal{C} = \sum_{\pi' \in \mathcal{C}} i_*(M(\pi')),
\]
\[
M(\pi)^\mathcal{C} = \bigcap_{\pi' \in \mathcal{C}} \text{Ker}\{i^* : M(\pi) \to M(\pi')\}.
\]

It can be shown that \( F(\pi)_\mathcal{C} \) is an ideal of \( F(\pi) \), \( M(\pi)_\mathcal{C} \) and \( M(\pi)^\mathcal{C} \) are submodules of \( M(\pi) \) over \( F(\pi) \), both of \( M(\pi)/M(\pi)_\mathcal{C} \) and \( M(\pi)^\mathcal{C} \) are modules over \( F(\pi)/F(\pi)_\mathcal{C} \), (see [Sw3, pages 22-23, Lemma 2.6 and Lemma 2.7]).
Now we turn to the proof of Theorem 1.4. Our proof is an adaptation of the proof in [Sw3, page 36, Theorem 2.20].

Suppose that $R$ is a commutative artinian ring and $\pi$ is a finite group. We will show that the Cartan map $c_\pi : K_0(R\pi) \to G_0(R\pi)$ is injective.

Step 1. We claim that if $c_{\pi'}$ is injective for any cyclic subgroup $\pi'$ of $\pi$, then $c_\pi$ is injective for the group $\pi$.

Consider the Frobenius functor $F : \text{Grp}_\pi \to \text{Ring}$ defined by $F(\pi') = G_0^R(R\pi')$ where $\pi'$ is any subgroup of $\pi$. Note that the Cartan map $c_{\pi'} : K_0(R\pi') \to G_0(R\pi')$ is a morphism of Frobenius modules $K_0(R\pi') \to G_0(R\pi')$ over the Frobenius functor $F$. Define a Frobenius module by $M(\pi') = \text{Ker}\{c_{\pi'} : K_0(R\pi') \to G_0(R\pi')\}$. Note that $M(\pi') = 0$ if $\pi'$ is a cyclic subgroup by the assumption at the beginning of this step.

Let $\mathcal{C}$ be the class of all the cyclic subgroups of $\pi$. Thus $M(\pi)^\mathcal{C} = M(\pi)$ since $M(\pi') = 0$ if $\pi'$ is cyclic.

Let $|\pi| = n$. Then $n^2 \cdot (G_0^R(R\pi)/G_0^R(R\pi)^{\mathcal{C}}) = 0$ by Artin’s induction theorem [Sw3, page 24, Corollary 2.12]. Since $M(\pi)^\mathcal{C}$ is a module over $G_0^R(R\pi)/G_0^R(R\pi)^{\mathcal{C}}$, it follows that $n^2 \cdot M(\pi)^\mathcal{C} = 0$ by [Sw3, page 23, Lemma 2.10].

As $M(\pi)^\mathcal{C} = M(\pi)$ and $M(\pi)$ is a subgroup of $K_0(R\pi)$ which is a free abelian group of finite rank by Lemma 2.3, we find that $M(\pi)^\mathcal{C}$ is a torsion subgroup of $K_0(R\pi)$. It follows that $M(\pi)^\mathcal{C} = 0$.

Note that the above arguments was formalized in [La1, Corollary 3.5].

Step 2. It remains to show that $c_\pi : K_0(R\pi) \to G_0(R\pi)$ is injective if $\pi$ is a cyclic group.

Without loss of generality, we may assume that $R$ is a commutative artinian local ring. Write $R = (R, \mathcal{M})$ where $\mathcal{M}$ is the maximal ideal of $R$ and $k = R/\mathcal{M}$ is the residue field.

Let $\pi = \langle \sigma \rangle$ be a cyclic group of order $m$. We may write $k\pi = k[\sigma] \simeq k[X]/(X^m - 1)$ where $k[X]$ is the polynomial ring. Note that $\text{rad}(R) = \mathcal{M}$ and $\text{rad}(R) \cdot R\pi \subset \text{rad}(R\pi)$ (see, for examples, [La2, page 74, Corollary 5.9; Sw2, page 170, Lemma 11.1]). Thus

$$\frac{R\pi/ \text{rad}(R\pi)}{\text{rad}(R\pi)/R\pi} \simeq \frac{R\pi/ \mathcal{M} \cdot R\pi}{\text{rad}(R\pi)/R\pi} \simeq \frac{k\pi}{\text{rad}(k\pi)} \simeq k[X]/\langle f(X) \rangle$$

with $f(X) = \prod_{1 \leq i \leq t} f_i(X)$ where $f_1(X), \ldots, f_t(X)$ are all the distinct monic irreducible factors of $X^m - 1$ in $k[X]$.

It follows that $S_i = k[X]/\langle f_i(X) \rangle$, $1 \leq i \leq t$, are all the simple modules over $k[X]/\langle f(X) \rangle \simeq R\pi/ \text{rad}(R\pi)$. By Lemma 2.3, $S_1, \ldots, S_t$ are all the non-isomorphic simple $R\pi$-modules and their projective covers $P_1, \ldots, P_t$ are all the non-isomorphic indecomposable $R\pi$-projective modules. Consequently, $K_0(R\pi) = \bigoplus_{1 \leq i \leq t} \mathbb{Z} \cdot [P_i]$ and $G_0(R\pi) = \bigoplus_{1 \leq i \leq t} \mathbb{Z} \cdot [S_i]$. We will consider the Cartan map $c_\pi : K_0(R\pi) \to G_0(R\pi)$. 

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Write $J = \text{rad}(R\pi)$. For $1 \leq i \leq t$, consider the filtration $P_i \supset JP_i \supset J^2P_i \supset \cdots \supset J^sP_i = \{0\}$ (note that $J$ is a nilpotent ideal). Each quotient module $J^jP_i/J^{j+1}P_i$ can be regarded as a module over $R\pi/J \simeq k[X]/\langle f_i(X) \rangle$. Note that $\bar{f}_i \cdot J^jP_i/J^{j+1}P_i = 0$ because $0 = \bar{f}_i \cdot S_i \simeq \bar{f}_i \cdot P_i/JP_i$ (remember that $R\pi$ is a commutative ring as $\pi$ is cyclic). Thus $J^jP_i/J^{j+1}P_i$ becomes a module over $k[X]/\langle f_i(X) \rangle$. It follows that the only simple $R\pi$-module which may arise as a Jordan-Hölder composition factor of $P_i$ is $S_i = k[X]/\langle f_i(X) \rangle$. We conclude that $c_\pi([P_i]) = a_i[S_i]$ for some positive integer $a_i$. Hence the determinant of the Cartan matrix is non-zero. 

**Example 2.10** Let $R$ be a commutative artinian ring, $\pi$ be a finite group. By Lemma 2.5 every finitely generated projective $R\pi$-module is a direct sum of projective ideals generated by some idempotent of $R\pi$. If $P$ and $Q$ are finitely generated $R\pi$-projective modules, we will show that $P \simeq Q$ if and only if $P$ and $Q$ have the same composition factors. For, if $[P] = [Q]$ in $G_0(R\pi)$, then $c([P] - [Q]) = 0$ where $c : K_0(R\pi) \to G_0(R\pi)$ is the Cartan map. By Theorem 1.4 $[P] = [Q]$ in $K_0(R\pi)$. Thus $P \oplus F \simeq Q \oplus F$ for some free $R\pi$-module $F$ of finite rank. By the Krull-Schmidt-Azumaya Theorem [CR, page 128], we find that $P \simeq Q$.

On the other hand, let $A$ be a left artinian ring such that the Cartan map $c : K_0(A) \to G_0(A)$ is not injective (such an artinian ring does exist by [BFVZ, Lemma 2]). By Lemma 2.3 choose indecomposable $A$-projective modules $P_1, P_2, \ldots, P_n$ and simple $A$-modules $M_1, M_2, \ldots, M_n$ such that, for $1 \leq i \leq n$, $M_i \simeq P_i/JP_i$. Then there is some $1 \leq i \leq n$ such that $M_j$ arises in the composition factor of $P_i$ for some $j \neq i$; otherwise, the determinant of the Cartan matrix would be positive. In general the Cartan matrix is a diagonal matrix (as in the proof of the Theorem 1.4) if and only if $\text{Hom}_A(P_i, P_j) = 0$ for any $1 \leq i, j \leq n$ with $i \neq j$ by [La2, page 325, Proposition (21.19)].

The following lemma is a folklore among experts (see, for example, [La3, page 190]). We include it here for completeness.

**Lemma 2.11** Let $k$ be a field, $\pi$ be a finite group.

(i) If $\text{char } k = 0$ or $\text{char } k = p > 0$ with $p \nmid |\pi|$, then the global dimension of $k\pi$ is zero.

(ii) If $\text{char } k = p > 0$ with $p \mid |\pi|$, then the global dimension of $k\pi$ is infinite.

**Proof.** (i) $k\pi$ is semisimple by Maschke’s Theorem. Thus every $k\pi$-module is projective [La2, page 29].

(ii) $k\pi$ is right self-injective by [La3, page 420, Exercise 14]. Hence it is right Kasch [La3, page 411]. By [La3, page 189, Corollary 5.74] the global dimension of $k\pi$ is either zero or infinite.
Now suppose that \( \text{char } k = p > 0 \) and \( p \mid |\pi| \). Once we find a \( k\pi \)-module which is not projective, then we are done (because of the assertion of the above paragraph).

Define \( u = \sum_{\sigma \in \pi} \sigma \in k\pi \). Then \( u^2 = 0 \) and \( u \) belongs to the center of \( k\pi \).

Write \( I = k\pi \cdot u \), the ideal generated by \( u \). We claim that \( k\pi/I \) is not a projective \( k\pi \)-module.

Otherwise, \( I \) is a direct summand of \( k\pi \). It follows that \( I = k\pi \cdot e \) for some idempotent \( e \) of \( k\pi \). Write \( e = \alpha u \) where \( \alpha \in k\pi \). Then \( e = e^2 = (\alpha u)(\alpha u) = \alpha^2 u^2 = 0 \).

This is impossible. \( \blacksquare \)

§3. Projective modules

**Theorem 3.1** Let \( R \) be a Dedekind domain with \( \text{char } R = p > 0 \). Let \( \pi \) be a finite group, \( M \) be an \( R\pi \)-module. Assume that \( p \mid |\pi| \) and choose a \( p \)-Sylow subgroup \( \pi_p \) of \( \pi \). Then \( M \) is an \( R\pi \)-projective module. \( \iff \) The restriction of \( M \) to \( R\pi_p \) is an \( R\pi_p \)-projective module. \( \iff \) The restriction of \( M \) to \( R\pi' \) is an \( R\pi' \)-projective module where \( \pi' \) is any elementary abelian subgroup of \( \pi_p \).

**Proof.** Suppose \( M \) is an \( R\pi_p \)-projective module. We will show that \( M \) is a projective module over \( R\pi \). Since \( [\pi : \pi_p] \) is a unit in \( R \), it follows that \( M \) is \( (\pi, \pi_p) \)-projective and \( M \) is a direct summand of \( (M_{\pi_p})^\pi \) by [CR, page 452, Proposition 19.5] (where \( M_{\pi_p} \) is the restriction of \( M \) to \( R\pi_p \), and \( (M_{\pi_p})^\pi := R\pi \otimes_{R\pi_p} (M_{\pi_p}) \)).

Since \( M_{\pi_p} \) is an \( R\pi_p \)-projective module, it follows that \( (M_{\pi_p})^\pi \) is an \( R\pi \)-projective module. So is its direct summand \( M \).

Now assume that \( M \) is an \( R\pi' \)-projective module for all elementary abelian \( p \)-group \( \pi' \) of \( \pi_p \). By [Ch, Corollary 1.1], \( M \) is an \( R\pi_p \)-projective module.

Note that, by a theorem of Rim [R1, Proposition 4.9], a module \( M \) is \( R\pi \)-projective if and only if so is it when restricted to all the Sylow subgroups of \( \pi \). But the situation of our theorem requires that \( \text{char } R = p > 0 \); thus only a \( p \)-Sylow subgroup is sufficient to guarantee the projectivity over \( R\pi \). \( \blacksquare \)

**Remark.** If \( \pi \) is a \( p \)-group, recall the definition of the Thompson subgroup of \( \pi \), which is denoted by \( J(\pi) \) [Is, page 202]: \( J(\pi) \) is the subgroup of \( \pi \) generated by all the elementary abelian subgroups of \( \pi \).

With the definition of \( J(\pi) \), we may rephrase Chouinard’s theorem [Ch, Corollary 1.1] as follows: Let \( \pi \) be a \( p \)-group and \( M \) be an \( R\pi \)-module where \( R \) is any commutative ring. Then \( M \) is an \( R\pi \)-projective module if and only if so is its restriction to the group
ring of \( J(\pi) \) over \( R \). Similarly, Theorem 3.1 may be formulated via the Thompson subgroup of \( \pi_p \).

Recall the following well-known lemma, which will be used in the sequel.

**Lemma 3.2** ([Ba1, Lemma 2.4; Sw3, page 13]) Let \( A \) be a ring, \( I \) be a two-sided ideal of \( A \) with \( I \subset \text{rad}(A) \). If \( P \) and \( Q \) are finitely generated \( A \)-projective modules satisfying that \( P/IP \cong Q/IQ \), then \( P \cong Q \).

**Theorem 3.3** Let \( p \) be a prime number, \( \pi \) be a \( p \)-group, and \( R \) be a Dedekind domain with quotient field \( K \) such that \( \text{char}(R) = p \). If \( P \) is a finitely generated \( R\pi \)-projective module, then \( KP \) is a free \( K\pi \)-module, and \( P \cong F \oplus A \) where \( F \) is a free \( R\pi \)-module and \( A \) is a projective ideal of \( R\pi \). Moreover, for any non-zero ideal \( I \) of \( R \), we may choose \( A \) such that \( I + (R \cap A) = R \).

On the other hand, if it is assumed furthermore that \( R \) is semilocal, then every finitely generated \( R\pi \)-projective module \( P \) is a free module.

**Proof.** By [CR, page 114, Theorem 5.24] \( \text{rad}(K\pi) = \sum_{\lambda \in \pi} K \cdot (\lambda - 1) \). Thus \( K\pi/\text{rad}(K\pi) \cong K \). By Lemma 2.2 (with \( I = \text{rad}(K\pi) \)), all the finitely generated \( K\pi \)-projective modules are free modules, because all the finitely generated projective modules over \( K\pi/\text{rad}(K\pi) \) are the free modules \( K^{(n)} \).

Consequently, if \( P \) is a finitely generated \( R\pi \)-projective module, then \( KP \) is a free \( K\pi \)-module. Thus we may apply Theorem 1.2 to \( P \) because the second assumption of Theorem 1.2 is valid by Theorem 1.3 (note that \( \dim(m \text{- spec}(R)) \leq 1 \)). Thus \( P \cong F \oplus A \) where \( F \) is a free \( R\pi \)-module and \( A \) satisfies that, for any maximal ideal \( \mathcal{M} \) of \( R \), \( A/\mathcal{M}A \) is isomorphic to \( R'\pi \) where \( R' = R/\mathcal{M} \). In case \( R \) is semilocal, then \( \dim(m \text{- spec}(R)) = 0 \) and therefore finitely generated \( R\pi \)-projective modules are free by Theorem 1.2. We remark that the result when \( R \) is semilocal may be deduced also from Theorem 3.5.

From \( P \cong F \oplus A \), we find that \( KF \oplus KA \cong KP \) is \( K\pi \)-free. By the Krull-Schmidt-Azumaya’s Theorem [CR, page 128] it follows that \( KA \cong K\pi \). Thus \( A \) is a projective ideal of \( R\pi \). It remains to show that \( A \) may be chosen such that \( I + (R \cap A) = R \) for any non-zero ideal \( I \) of \( R \).

First we will show that \( A \) and the free module \( R\pi \) belong to the same genus. For any maximal ideal \( \mathcal{M} \) of \( R \), consider the projective \( R\mathcal{M}\pi \)-modules \( A_{\mathcal{M}} \) and \( R_{\mathcal{M}}\pi \). As \( \mathcal{M}R_{\mathcal{M}}\pi \subset \text{rad}(R_{\mathcal{M}}\pi) \) by [La2, page 74, Corollary 5.9] and \( A_{\mathcal{M}}/\mathcal{M}A_{\mathcal{M}} \cong A/\mathcal{M}A \cong R_{\mathcal{M}}\pi/\mathcal{M}R_{\mathcal{M}}\pi \), we may apply Lemma 3.2. It follows that \( A_{\mathcal{M}} \) and \( R_{\mathcal{M}}\pi \) are isomorphic.
Once we know that $A$ and $R_{\pi}$ belong to the same genus, we may apply Roiter’s Theorem [Sw3, page 37]. Thus we have an exact sequence of $R_{\pi}$-modules $0 \to A \to R_{\pi} \to X \to 0$ such that $I + \text{Ann}_{R}X = R$ where $\text{Ann}_{R}X = \{r \in R : r \cdot X = 0\}$. Note that $R \cap A = \text{Ann}_{R}R_{\pi}/A$ and $R_{\pi}/A \cong X$. Hence the result.

Remark. The assumption that no prime divisor of $|\pi|$ is a unit in $R$ is crucial in the above Theorem 3.3 and in Theorem 1.1. In fact, if some prime divisor of $|\pi|$ is invertible in $R$, then $R_{\pi}$ contains a non-trivial idempotent element (and thus $KP$ will not be a free $K_{\pi}$-module for some projective module $P$); Coleman shows that the converse is true also [CR, page 678].

The following theorem, due to S. Endo, provides an alternative proof of Theorem 3.3.

**Theorem 3.4** Let $R$ be a Dedekind domain with $\text{char } R = p > 0$, and $\pi$ be a $p$-group. If $P$ is a finitely generated $R_{\pi}$-projective module, then $P$ is isomorphic to $R_{\pi} \otimes_{R} P_{0}$ for some $R$-projective module $P_{0}$, and is also isomorphic to a direct sum of a free module and a projective ideal of the form $R_{\pi} \otimes_{R} I$ where $I$ is some non-zero ideal of $R$. Moreover, for any non-zero ideal $I'$ of $R$, the ideal $I$ may be chosen so that $I + I' = R$.

**Proof.** Let $\phi : R_{\pi} \to R$ be the augmentation map defined by $\phi(\lambda) = 1$ for any $\lambda \in \pi$. Let $J$ be the kernel of $\phi$. Define $J_{0} = \sum_{\lambda \in \pi} R \cdot (\lambda - 1)$. Then $J = J_{0} \cdot R_{\pi}$.

Let $K$ be the quotient field of $R$. Then $\text{rad}(K_{\pi}) = J_{0} \cdot K_{\pi}$ by [CR, page 114]. Since $\text{rad}(K_{\pi})$ is nilpotent, so is the ideal $J$ of $R_{\pi}$. It follows that $R_{\pi}$ is $J$-complete and $J \subset \text{rad}(R_{\pi})$.

Apply Lemma 2.2 to get a one-to-one correspondence of finitely generated projective modules over $R_{\pi}$ and over $R$. For any finitely generated projective module $P$ over $R_{\pi}$, define $P_{0} = P/JP$. Since both $P$ and $R_{\pi} \otimes_{R} P_{0}$ descend to $P_{0}$, it follows that $P$ is isomorphic to $R_{\pi} \otimes_{R} P_{0}$.

Every finitely generated projective $R$-module is isomorphic to $R^{(n)} \oplus I$ where $n$ is a non-negative integer and $I$ is a non-zero ideal of $R$ (see [Sw3, page 219, Theorem A15]). Thus a finitely generated projective $R_{\pi}$-module is isomorphic to a direct sum of a free module and a projective ideal of the form $R_{\pi} \otimes_{R} I$. If $I'$ is any non-zero ideal of $R$, we can find a non-zero ideal $I_{0}$ of $R$ such that $I \cong I_{0}$ and $I_{0} + I' = R$ by [Sw3, page 218, Theorem A12].

A corollary of Theorem 3.4 is the following.

**Theorem 3.5** Let $R$ be a Dedekind domain with $\text{char } R = p > 0$, and $\pi$ be a $p$-group. Then $R$ is a principal ideal domain if and only if every finitely generated $R_{\pi}$-projective module is isomorphic to a free module.
The following lemma is a partial generalization of Theorem 3.3 from $p$-groups to finite groups $\pi$ with $p \mid |\pi|$.

**Lemma 3.6** Let $R$ be a Dedekind domain with char $R = p > 0$ and with quotient field $K$. Let $\pi$ be a finite group such that $p \mid |\pi|$, and $\pi_p$ be a $p$-Sylow subgroup of $\pi$. Let $P$ be a finitely generated $R\pi$-projective module, $P_{\pi_p}$ be the restriction of $P$ to $R\pi_p$, and $(P_{\pi_p})^\pi := R\pi \otimes_{R\pi_p} (P_{\pi_p})$ be the induced module of $P_{\pi_p}$. Then $K(P_{\pi_p})^\pi$ is $K\pi$-free and $(P_{\pi_p})^\pi$ is isomorphic to $F \oplus A$ where $F$ is a free $R\pi$-module and $A$ is a projective ideal of $R\pi$.

**Proof.** If $K(P_{\pi_p})^\pi$ is $K\pi$-free, then we may apply Theorem 1.2 to finish the proof. It remains to show that $K(P_{\pi_p})^\pi$ is $K\pi$-free.

By Theorem 3.3, $K P_{\pi_p}$ is $K\pi_p$-free. It follows that $K(P_{\pi_p})^\pi$ is $K\pi$-free. Done.

Note that $P$ is a direct summand of $(P_{\pi_p})^\pi$ by [CR, pages 449-450].

**Example 3.7** A different proof of Theorem 1.1 other than that in [Sw1] is given in [Gr, Lecture 4]. It is proved first that, if $R$ is a semilocal Dedekind domain with char $R = 0$ and no prime divisor of $|\pi|$ is a unit in $R$, then every finitely generated $R\pi$-projective module is a free module [Gr] page 21,Theorem 4.7).

We remark that we may derive the above result directly from Theorem 1.1. For, if all the maximal ideals of $R$ are $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_t$, define $I = \mathcal{M}_1 \cap \mathcal{M}_2 \cap \cdots \cap \mathcal{M}_t$ and apply Theorem 1.1. Then every finitely generated $R\pi$-projective module $P$ is isomorphic to $F \oplus A$ where $F$ is free and $A$ is a projective ideal of $R\pi$ with $I + (R \cap A) = R$. It follows that $R \cap A = R$, i.e. $1 \in A \subset R\pi$. Thus $A = R\pi$ is also a free module.

Note that, when $\pi = \{1\}$ is the trivial group, the similar statement as the above result (for semilocal rings) is not true in general. It is well-known that projective modules over a quasi-local ring are free modules (Kaplansky’s Theorem; see [Sw2, page 82, Corollary 2.14] for the case of finitely generated projective modules).

When $R$ is a commutative ring with only finitely many maximal ideals (e.g. a semilocal ring) having no non-trivial idempotent elements, then every projective $R$-module (which may not be finitely generated) is a free module, an analogy of Kaplansky’s Theorem proved by Hinohara [Hi]; a similar result for finitely generated $R$-projective modules was proved independently by S. Endo.

Thus if $R$ is a commutative ring with only finitely many maximal ideals, say, $t$ is the number of distinct maximal ideals, we will show that there are at most $t$ primitive idempotents in $R$. Write $R/\text{rad}(R) = \prod_{1 \leq i \leq t} K_i$ where each $K_i$ is a field (and is indecomposable). If $R = \prod_{1 \leq j \leq s} R_j$, from $\text{rad}(R) = \prod_{1 \leq j \leq s} \text{rad}(R_j)$, we find that $R/\text{rad}(R)$ has at least $s$ maximal ideals and therefore $s \leq t$. Thus we may write $R = \prod_{1 \leq j \leq s} R_j$ where each $R_j$ has no non-trivial idempotent elements; obviously $s \leq t$. 13
Although a projective $R$-module is not necessarily free, it is isomorphic to a direct sum of free modules over these $R_j$'s by applying Hinohara's Theorem.

**Example 3.8** We remind the reader that Theorem 3 in [Ba1, page 533] is generalized as Theorem 8.2 in [Ba2, page 24] (see also [Sw2, page 171, Theorem 11.2]). We reproduce these two theorems as follows.

**Theorem A** ([Ba1, Theorem 3]) Let $R$ be a commutative noetherian ring, $A$ be an $R$-algebra which is a finitely generated $R$-module and $d = \dim(m\text{-}\text{spec}(R))$. Let $P$ be a finitely generated $A$-projective module such that there is an integer $r$ such that $P/M \simeq (A/MA)^{(r)}$ for all maximal ideals $M$ in $R$, then $P \simeq F \oplus Q$ where $F$ is a free module of rank $r'$, $Q/M \simeq (A/MA)^{(d')}$ for all maximal ideals $M$ in $R$ with $d' = \min\{d, r\}$ and $r' = r - d'$.

**Theorem B** ([Ba2, page 24, Theorem 8.2]) Let $R$, $A$, $d$ be the same as above. Let $P$ be a finitely generated $A$-projective module such that $P_M$ contains a direct summand isomorphic to $A_M^{(d+1)}$ for all maximal ideals $M$ in $R$. Then $P \simeq A \oplus Q$ for some projective module $Q$.

Let $P$ be a finitely generated $A$-projective module in Theorem A. Note that the assumption for $P$ (in the above Theorem A and also in Theorem 1.2) that $P/M \simeq (A/MA)^{(r)}$ for all maximal ideals $M$ in $R$ is equivalent to the assumption that $P$ and $A^{(r)}$ are locally isomorphic, i.e. $P_M \simeq (A_M)^{(r)}$ for any maximal ideal $M$ in $R$. The proof is the same as that in Theorem 3.3 for the projective ideal $A$. Thus $P$ satisfies the assumption of Theorem B.

When $r \geq d+1$, we find $P \simeq A \oplus Q$ by Theorem B. Since $A_M$ is a (non-commutative) semilocal ring, the cancelation law is valid for finitely generated projective $A_M$-modules [Sw2, page 176]. Thus $Q_M \simeq A_M^{(r-1)}$ for any maximal ideal $M$ in $R$. Proceed by induction on $r$ to obtain the conclusion of Theorem A.



§4. A local criterion

Finally we will discuss the following question. Let $R$ be a commutative noetherian ring with total quotient ring $K$, $A$ be an $R$-algebra which is a finitely generated projective $R$-module. Let $P$ and $Q$ be finitely generated $A$-projective modules. If $KP \simeq KQ$, under what situation, can we conclude that $P \simeq Q$?
The prototype of this question is a theorem of Brauer and Nesbitt [BN1 page 12, Theorem 2; CR, page 424, Corollary 17.10]: Let \((R, \mathcal{M})\) be a discrete valuation ring with quotient field \(K\) such that \(\text{char}(\overline{R}/\mathcal{M}) = p > 0\). If \(\pi\) is a finite group, \(M\) and \(N\) are \(R\pi\)-lattices with \(KM \simeq KN\), then \([M/\mathcal{M}M] = [N/\mathcal{M}N]\) in \(G_0(\overline{R}/\mathcal{M}\pi)\). A generalization of this theorem by Swan is given in [Sw1 Corollary 6.5]; see [CR page 436, Corollary 18.16] also.

The above results of Brauer-Nesbitt and Swan are generalized furthermore by Bass as follows.

**Theorem 4.1** (Bass [Ba1 Theorem 2; Sw3, page 12, Theorem 1.10; CR, page 671])

Let \((R, \mathcal{M})\) be a local ring with total quotient ring \(K\), \(A\) be an \(R\)-algebra which is a finitely generated \(R\)-projective module. Assume that the Cartan map \(c : K_0(A/\mathcal{M}A) \to G_0(A/\mathcal{M}A)\) is injective. If \(P\) and \(Q\) are finitely generated \(A\)-projective modules such that \(KP \simeq KQ\), then \(P \simeq Q\).

**Proof.** Step 1. Let \(S = R \setminus \bigcup_{1 \leq i \leq n} \mathcal{M}_i\), \(J = \bigcap_{1 \leq i \leq n} \mathcal{M}_i\). Then \(S^{-1}R\) is a semilocal ring with maximal ideals \(S^{-1}\mathcal{M}_1, S^{-1}\mathcal{M}_2, \ldots, S^{-1}\mathcal{M}_n\). Consider the projective modules \(S^{-1}P\) and \(S^{-1}Q\) over the algebra \(S^{-1}A\). We will show that \(S^{-1}P/S^{-1}J\) is injective in \(S^{-1}Q/S^{-1}JQ\). In Step 2. Assume this result (which will be proved in Step 2). Then we apply Lemma 3.2 (note that \(S^{-1}JA \subset \text{rad}(S^{-1}A)\) by [La2 page 74, Corollary 5.9]). We get \(S^{-1}P \simeq S^{-1}Q\), and therefore \(S^{-1}P/S^{-1}IP \simeq S^{-1}Q/S^{-1}IQ\).

Write the primary decomposition of \(I\) as \(I = \bigcap_{1 \leq i \leq n} I_i\) where \(I_i\) is an \(\mathcal{M}_i\)-primary ideal. Then \(S^{-1}I = \bigcap_{1 \leq i \leq n} S^{-1}I_i\). For \(1 \leq i \leq n\), since \(\langle S, I_i \rangle = R\), it follows that \(S^{-1}(R/I_i) \simeq R/I_i\). Thus \(S^{-1}(A/I_iA) \simeq A/I_iA\) and \(S^{-1}(P/I_iP) \simeq P/I_iP\), \(S^{-1}(Q/I_iQ) \simeq Q/I_iQ\). Since \(R/I = \bigcap_{1 \leq i \leq n} R/I_i\), we get \(P/IP \simeq \bigoplus_{1 \leq i \leq n} P/I_iP\), \(S^{-1}P/S^{-1}IP \simeq \bigoplus_{1 \leq i \leq n} S^{-1}P/S^{-1}IP\) and similarly for \(Q\) and \(S^{-1}Q\).

Now we have \(P/IP \simeq \bigoplus_{1 \leq i \leq n} P/I_iP \simeq \bigoplus_{1 \leq i \leq n} S^{-1}(P/I_iP) \simeq S^{-1}P/S^{-1}IP\) and \(Q/IQ \simeq S^{-1}Q/S^{-1}IQ\). Because we have shown that \(S^{-1}P/S^{-1}IP \simeq S^{-1}Q/S^{-1}IQ\), we find that \(P/IP \simeq Q/IQ\). If \(R\) is semilocal with \(I \subset \text{rad}(R)\), then \(P \simeq Q\) by Lemma 3.2.
In summary, define $S = R \setminus \bigcup_{1 \leq i \leq n} M_i$, $J = \bigcap_{1 \leq i \leq n} M_i$ and consider the $S^{-1}A$-projective modules $S^{-1}P$ and $S^{-1}Q$. In the next paragraph, we will show that the assumption $KP \simeq KQ$ carries over to the ring $S^{-1}A$.

Let $K_S$ be the total quotient ring of $S^{-1}R$ and let $\phi : R \to S^{-1}R$ be the canonical ring homomorphism. For any element $a \in R$, if $a$ is not a zero-divisor, then $\phi(a)$ is not a zero-divisor in $S^{-1}R$. Thus the map $\phi$ may be extended to $K \to K_S$. It follows that $K_S \otimes_{S^{-1}R} S^{-1}P \simeq K_S \otimes_R P \simeq K_S \otimes_K KP$. Similarly, $K_S \otimes_{S^{-1}R} S^{-1}Q \simeq K_S \otimes_K KQ$. Since $KP \simeq KQ$ by assumption, it follows that $K_S \otimes_{S^{-1}R} S^{-1}P$ is also isomorphic to $K_S \otimes_{S^{-1}R} S^{-1}Q$.

It remains to prove that $S^{-1}P/S^{-1}JP \simeq S^{-1}Q/S^{-1}JQ$.

Step 2. To simplify the notation, we may assume, without loss of generality, that $R$ is a semilocal ring with maximal ideals $M_1, \ldots, M_n$ and $J = \bigcap_{1 \leq i \leq n} M_i$. Let $K$ be the total quotient ring of $R$. If $KP \simeq KQ$, we will prove that $P/JP \simeq Q/JQ$.

Define $S_i = R \setminus M_i$ for $1 \leq i \leq n$. Let the total quotient ring of $S_i^{-1}R$ be $K_i$ and $\phi_i : R \to S_i^{-1}R$ be the canonical ring homomorphism. As in the last two paragraphs of Step 1, the map $\phi_i$ may be extended to a map $K \to K_i$ and we obtain an isomorphism of $K_i \otimes_{S_i^{-1}R} S_i^{-1}P$ with $K_i \otimes_{S_i^{-1}R} S_i^{-1}Q$.

Now we may apply Theorem 4.1 to the projective modules $S_i^{-1}P$ and $S_i^{-1}Q$ over the algebra $S_i^{-1}A$ for $1 \leq i \leq n$.

We find that $S_i^{-1}P \simeq S_i^{-1}Q$. Thus $S_i^{-1}P/S_i^{-1}JP \simeq S_i^{-1}Q/S_i^{-1}JQ$ for $1 \leq i \leq n$.

The remaining proof is analogous to that in Step 1. Note that $R/M_i \simeq S_i^{-1}(R/M_i)$. Thus $P/JP \simeq \bigoplus_{1 \leq i \leq n} P/M_i P \simeq \bigoplus_{1 \leq i \leq n} S_i^{-1}P/S_i^{-1}M_i P \simeq \bigoplus_{1 \leq i \leq n} S_i^{-1}P/S_i^{-1}JP \simeq \bigoplus_{1 \leq i \leq n} S_i^{-1}Q/S_i^{-1}JQ \simeq \cdots \simeq Q/JQ$. 

**Remark.** When $R$ is a semilocal ring and $A$ is a maximal $R$-order, an analogous result of Theorem 4.1 can be found in [Sw3, page 102, Corollary].

The following theorem is communicated to us by S. Endo. It provides a generalization of Theorem 1.4 (with the aid of Theorem 1.3).

**Theorem 4.3** Let $(R, \mathcal{M})$ be a commutative artinian local ring, $A$ be an $R$-algebra which is a finitely generated free $R$-module. Then the Cartan map $K_0(A) \to G_0(A)$ is injective if and only if so is the Cartan map $K_0(A/\mathcal{M}A) \to G_0(A/\mathcal{M}A)$.

**Proof.** Since $R$ satisfies the ACC condition and the DCC condition on ideals, we can find a filtration of ideals of $R$ as follows: $R = J_0 \supset J_1 \supset \ldots \supset J_t = 0$ where $t$ is some positive integer and $J_{i-1}/J_i \simeq R/\mathcal{M}$ for $1 \leq i \leq t$.

As $A$ is a free $R$-module, every finitely generated $A$-projective module $P$ is also $R$-free. Tensor the exact sequence $0 \to J_i \to J_{i-1} \to R/\mathcal{M} \to 0$ with $P$ over $R$. Note
that \( J_i \otimes_R P \simeq J_i P \) as \( A \)-modules (because we may tensor the injection \( 0 \to J_i \to R \) with \( P \)). It follows that we obtain a filtration of \( A \)-modules \( P = P_0 \supset P_1 = J_1 \supset \ldots \supset P_t = J_t P = 0 \) where \( P_{t-1}/P_t \simeq P/MP \). We conclude that \( [P] = t[P/MP] \) in \( G_0(A) \).

By Lemma \( 2.3 \) find projective \( A \)-modules \( P_1, P_2, \ldots, P_n \) and simple \( A \)-modules \( M_1, M_2, \ldots, M_n \) such that \( K_0(A) = \bigoplus_{1 \leq i \leq n} Z \cdot [P_i] \) and \( G_0(A) = \bigoplus_{1 \leq i \leq n} Z \cdot [M_i] \). The same simple \( A \)-modules \( M_i \) satisfies that \( G_0(A/MA) = \bigoplus_{1 \leq i \leq n} Z \cdot [M_i] \). Moreover, \( K_0(A/MA) = \bigoplus_{1 \leq i \leq n} Z \cdot [P_i/MP] \) by Lemma \( 2.2 \).

Now if \( [P_i/MP_i] = \sum_{1 \leq j \leq n} a_{ij} [M_j] \) in \( G_0(A/MA) \) where \( a_{ij} \) are some integers, then \( [P_i] = \sum_{1 \leq j \leq n} t a_{ij} [M_j] \) in \( G_0(A) \) (note that \( G_0(A/MA) \) is naturally isomorphic to \( G_0(A) \) by [Sw2] page 94, Theorem 3.4). Thus the determinant of the Cartan matrix \( (a_{ij})_{1 \leq i, j \leq n} \) is non-zero if and only if so is that of \( (t a_{ij})_{1 \leq i, j \leq n} \).

The following theorem of Rim is a generalization of Theorem 4.1. However, its proof was omitted in [Ri2]. For the convenience of the readers, we supply a proof of it as an application of Theorem 4.2 and Theorem 4.3.

**Theorem 4.4** (Rim [Ri2] Theorem 7) Let \( R \) be a commutative noetherian ring with total quotient ring \( K \), \( A \) be an \( R \)-algebra which is a finitely generated \( R \)-projective module. Suppose that \( I \) is an ideal of \( R \) such that \( R/I \) is artinian. Assume that the Cartan map \( c : K_0(A/IA) \to G_0(A/IA) \) is injective. If \( P \) and \( Q \) are finitely generated \( A \)-projective modules with \( KP \simeq KQ \), then \( P/IP \simeq Q/IQ \).

**Proof.** Write the primary decomposition of \( I \) as \( I = \bigcap_{1 \leq i \leq n} I_i \) where \( I_i \) is an \( M_i \)-primary ideal and each \( M_i \) is a maximal ideal of \( R \). Then \( A/IA \simeq \prod_{1 \leq i \leq n} A/I_i A \). It follows that this isomorphism induces isomorphisms \( K_0(A/IA) \simeq \bigoplus_{1 \leq i \leq n} K_0(A/I_i A) \) and \( G_0(A/IA) \simeq \bigoplus_{1 \leq i \leq n} G_0(A/I_i A) \). Note that, for \( 1 \leq i \leq n \), \( A/I_i A \) is a \( R/I_i \)-free module and the Cartan map \( K_0(A/I_i A) \to G_0(A/I_i A) \) is injective. Apply Theorem 4.3. We find that the Cartan map \( K_0(A/MA) \to G_0(A/MA) \) is injective. Now we may apply Theorem 4.2 to finish the proof.

**Example 4.5** With the aid of Theorem 4.1, we will show that Theorem 3.8 of Example 3.8 implies Theorem 1.2. Let \( A, R \) and \( d \) be given as in Theorem 1.2 and \( P \) be a finitely generated \( A \)-projective module. Suppose \( KP \) is free of rank \( r \). For any maximal ideal \( M \) in \( R \), consider \( P_M \). Now the (new!) base ring is the local ring \( R_M \). We will compare \( P_M \) with \( P^r = A^r_M \).

Let \( \phi : R \to R_M \) be the canonical ring homomorphism, and let \( K_M \) be the total quotient ring of \( R_M \). For any element \( a \in R \), if \( a \) is not a zero-divisor, then \( \phi(a) \) is not a zero-divisor in \( R_M \). Thus the map \( \phi \) may be extended to \( K \to K_M \). It follows
that $K_M \otimes_{R_M} P_M \simeq K_M \otimes_K K P$ is a free module and $K_M \otimes_{R_M} P_M$ is isomorphic to $K_M \otimes_{R_M} P'$.

Apply Theorem 4.1. We find that $P_M \simeq P' = A(r)^M$. Since $P$ is locally free, we may apply Theorem 3 of Example 3.8 so that $P \simeq A \oplus Q$ where $Q$ is locally free of rank $r - 1$ if $r \geq d + 1$ as in Example 3.8. The proof of Theorem 1.2 is finished by induction on $r$.

In general, a finitely generated $R\pi$-projective module may be written as a direct sum of indecomposable $R\pi$-projective modules. The following lemma tells what an indecomposable $R\pi$-projective module looks like in case $|\pi|$ is invertible in $R$.

**Lemma 4.6** Let $R$ be a Dedekind domain with quotient field $K$, $\pi$ be a finite group such that $|\pi|$ is invertible in $R$. If $P$ is a finitely generated indecomposable $R\pi$-projective module, then $P$ is isomorphic to a projective ideal of $R\pi$; moreover, there is some projective ideal $A$ generated by a primitive idempotent of $R\pi$ such that $P$ and $A$ belong to the same genus.

**Proof.** Note that $R\pi$ becomes a maximal $R$-order because $|\pi|$ is invertible in $R$ [CR, page 582]. As such, it is known that (i) $R\pi$ is left hereditary; (ii) a finitely generated $R\pi$-module $P$ is $R\pi$-projective if and only if it is an $R\pi$-lattice; (iii) the module $P$ is an indecomposable $R\pi$-projective module if and only if $K P$ is a simple $K\pi$-module [CR, page 565].

Now we come to the proof. By a theorem of Kaplansky every projective module over a left hereditary ring is a direct sum of projective ideals (see [CE, page 13, Theorem 5.3]). Since the projective module $P$ we consider is indecomposable, it is isomorphic to a projective ideal of $R\pi$. It remains to find some projective ideal $A$ such that $A$ is a direct summand of $R\pi$ satisfying that $P$ and $A$ belong to the same genus. Since $K P$ is a simple $K\pi$-module, it is isomorphic to a minimal left ideal $V$ of $K\pi$ by the Artin-Wedderburn Theorem. Since $K\pi$ is semi-simple, write $K\pi = V \oplus V'$ where $V'$ another left ideal of $K\pi$.

From the embedding $R\pi \to K\pi$, define $A = R\pi \cap V$ and define $A'$ by the exact sequence $0 \to A \to R\pi \to A' \to 0$. Hence $K A = V$ and $A'$ is $R$-torsion free. It follows that $A'$ is $R\pi$-projective and the exact sequence $0 \to A \to R\pi \to A' \to 0$ splits. Thus $A$ is generated by an idempotent element $u$ of $R\pi$. This idempotent element $u$ is primitive because $A$ is indecomposable (remember that $K A = V$ which is a simple $K\pi$-module).

Note that, if $i : P \to R\pi \cap V$ ($= A$) is the embedding of $P$ via Kaplansky’s Theorem and $K P \simeq V$ (see the proof of [CE, page 13, Theorem 5.3]), it is not true in general that $i(P)$ should be equal to $A$. 18
Finally we will show that $P$ and $A$ belong to the same genus. Both $KP$ and $KA$ are isomorphic to $V$. Because $R\pi$ is a maximal order, we may apply [CR, page 643, Proposition 31.2] to finish the proof. Note that this result may be proved alternatively by applying Theorem 4.1.

**Remark.** In the above lemma, $KA$ is not free if $|\pi| > 1$. In case $KP$ is free, the following result is known: Let $R$ be a Dedekind domain with quotient field $K$ and $\pi$ be a finite group. If $\gcd(|\pi|, \text{char } R) = 1$ and $P$ is a finitely generated $R\pi$-projective module such that $KP$ is free, then $P \simeq F \oplus A$ where $F$ is a free module and $A$ is some projective ideal (see [Sw1, Theorem 7.2]).

**Lemma 4.7** (Villamayor [Vi]) Let $\pi$ be a finite group and $R$ be a commutative ring such that $\text{rad}(R) = 0$ and $|\pi|$ is a unit in $R$. Then $\text{rad}(R\pi) = 0$.

**Proof.** This theorem is proved essentially in [Vi, page 626, Theorem 3]; as noted in [Vi, page 627, Remark 1], if $R$ is a commutative ring such that $|\pi|$ is a unit in $R$, the proof of Theorem 3 in [Vi, page 626] remains valid (as $\pi$ is a finite group). Be aware that, according to the convention of [Vi, page 621], a ring $A$ is called semisimple if $\text{rad}(A) = 0$. Villamayor’s Theorem can be found also in [Pa, page 278]; it is easy to check that the proof of this theorem in [Pa, page 278] works as well so long as $R$ is any commutative ring such that $|\pi|$ is a unit in $R$ (in other words, the assumption that $R$ is a field may be relaxed).

**Example 4.8** Let $\pi$ be a finite group. Choose a Dedekind domain $R$ such that $R$ is not semilocal and $|\pi|$ is a unit in $R$. Then $\text{rad}(R) = 0$. Thus $\text{rad}(R\pi) = 0$ by Villamayor’s Theorem. It follows that $R\pi/\text{rad}(R\pi) \simeq R\pi$ is not left artinian. Hence $R\pi$ is not semiperfect [La2, page 346]. By Theorem (25.3) of [La2, page 371], any finitely generated indecomposable projective module over a semiperfect ring is isomorphic to a projective ideal generated by a primitive idempotent (compare this result with Lemma 4.6).

**Lemma 4.9** Let $R$ be a commutative noetherian integral domain with $\text{char } R = p > 0$, $\pi$ be a finite group such that $p \mid |\pi|$. Assume that the $p$-Sylow subgroup $\pi_p$ is normal in $\pi$. Write $\pi' = \pi/\pi_p$.

1. Define a right ideal $I := \sum_{\lambda \in \pi_p} (\lambda - 1) \cdot R\pi$. Then $I$ is a nilpotent two-sided ideal of $R\pi$, $R\pi/I \simeq R\pi'$, and $\text{rad}(R\pi) = (I, \text{rad}(R))$.

2. There is a one-to-one correspondence between the isomorphism classes of finitely generated $R\pi'$-projective modules and the isomorphism classes of finitely generated $R\pi'$-projective modules given by $P \leadsto P/IP$ where $I$ is defined in (1). Note that $|\pi'|$ is a unit in $R$. 

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Proof. Step 1. For any $\sigma \in \pi$ and any $\lambda \in \pi_p$, $\sigma(\lambda - 1)\sigma^{-1} \in I$, because $\pi_p$ is a normal subgroup of $\pi$. Thus $I$ is a two-sided ideal of $R\pi$. Clearly $R\pi/I \simeq R\pi'$.

Let $K$ be the quotient field of $R$. As in the proof of Theorem 3.3, we find that $\text{rad}(K\pi_p) = \sum_{\lambda \in \pi_p} K \cdot (\lambda - 1)$. Since $\text{rad}(K\pi_p)$ is nilpotent, so is the ideal $I_0 := \sum_{\lambda \in \pi_p} R \cdot (\lambda - 1)$ in $R\pi_p$. It follows that $I = I_0 \cdot R\pi$ and $I^n = I_0^n \cdot R\pi$. Thus $I$ is nilpotent and is contained in $\text{rad}(R\pi)$. Note that $|\pi'|$ is a unit in $R$.

Using the fact that $|\pi'|$ is a unit in $R$, we will show that $\text{rad}(R\pi') = \text{rad}(R) \cdot R\pi'$. Because $\text{rad}(R) \cdot R\pi' \subset \text{rad}(R\pi')$, the fact that $\text{rad}(R\pi') = \text{rad}(R) \cdot R\pi'$ is equivalent to $\text{rad}(R'\pi') = 0$ where $R' = R/\text{rad}(R)$. The latter assertion is true by Lemma 4.7. Hence $\text{rad}(R\pi') = \text{rad}(R) \cdot R\pi'$.

From $\text{rad}(R\pi/I) \simeq \text{rad}(R\pi')$ and $\text{rad}(R\pi/I) = \text{rad}(R\pi)/I$ [La2 page 55], we find that $\text{rad}(R\pi) = \langle I, \text{rad}(R) \rangle$.

Step 2. Since $I$ is nilpotent, $R\pi$ is $I$-complete. Apply Lemma 2.2 to get the one-to-one correspondence of finitely generated projective modules over $R\pi$ and $R\pi'$.

Remark. Let the notations be the same as the above lemma. Assume furthermore that the group extension $1 \to \pi_p \to \pi \to \pi' \to 1$ splits. Then the composite of the imbedding $R\pi' \to R\pi$ and the canonical projection $R\pi \to R\pi'$ is the identity map on $R\pi'$. By the same idea of Theorem 3.4, it can be shown that every finitely generated $R\pi$-projective module is of the form $R\pi \otimes_{R\pi'} P_0$ for some $R\pi'$-projective module $P_0$. 20
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