POLAR FOLIATIONS ON SYMMETRIC SPACES AND MEAN CURVATURE FLOW

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Abstract. In this paper, we study polar foliations on simply connected symmetric spaces with non-negative curvature. We will prove that all such foliations are isoparametric as defined in [11]. We will also prove a splitting theorem which reduces the study of such foliations to polar foliations in compact simply connected symmetric spaces. Moreover, we will show that solutions to mean curvature flow of regular leaves in such foliations are always ancient solutions. This generalizes part of the results in [14] for mean curvature flows of isoparametric submanifolds in spheres.

1. Introduction

In this paper we consider polar foliations \((M, F)\) in a simply connected, non-negatively curved symmetric space \(M\). Recall that polar foliation \(F\) on a complete Riemannian manifold \(M\) is a singular Riemannian foliation such that each point \(x \in M\) is contained in a totally geodesic submanifold, called a section, which meets all leaves of \(F\) and intersects them orthogonally. Polar foliations with flat sections are called hyperpolar foliations. Foliations given by orbits of polar actions by Lie groups are homogeneous examples of polar foliations. Other typical examples include the foliations by parallel and focal submanifolds of any isoparametric submanifold in a space form (cf. [25]). Each equifocal submanifold in a compact symmetric space gives a hyperpolar foliation with leaves the images of parallel normal vector fields under the exponential map (cf. [27]).

The study of isoparametric submanifolds can be traced back to Cartan’s work on isoparametric hypersurfaces in 1930’s. Such manifolds have become an important subject in submanifold geometry and have been extensively studied since then. A nice survey article on this subject can be found in [28]. For a general Riemannian manifold \(M\), a submanifold \(L\) in \(M\) is called isoparametric if the normal bundle \(\nu L\) is flat, \(\exp(\nu_p L)\) is totally geodesic in a neighbourhood of \(p\) for every \(p \in L\), and locally parallel submanifolds of \(L\) have parallel mean curvature vector fields (cf. [11]). Here parallel submanifolds of \(L\) mean images of parallel normal vector fields along \(L\) under the exponential map. When \(M\) is a space form, this notion coincides with Terng’s definition of isoparametric submanifolds in [25]. Equifocal submanifolds \(L\) in a compact symmetric space defined by Terng and Thorbergsson in [27] are precisely isoparametric submanifolds with \(\exp(\nu_p L)\) flat in a neighbourhood of \(p\) for every \(p \in L\). The definition of an isoparametric submanifold \(L\) given in [11] is in purely local terms. In particular, one can not expect parallel submanifolds of \(L\) to give a global foliation of the ambient space in general. In case that parallel submanifolds

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of $L$ do give a global foliation of the ambient space, such a foliation is called an isoparametric foliation. It turns out that each regular leaf of an isoparametric foliation is always an isoparametric submanifold (cf. Corollary 2.5 in [11]).

Polar foliations share many similar properties as isoparametric foliations. For example, Alexandrino and Toeben have proved in [5] that for polar foliations with compact leaves in a complete simply connected Riemannian manifold, each regular leaf has trivial normal holonomy. This implies that the normal bundle of each regular leaf is flat. The existence of sections for polar foliations also implies that $\exp(\nu_p L)$ is totally geodesic for all $p$ in any regular leaf $L$. However, unlike in the isoparametric case, there is no restriction for the mean curvature of the leaves of polar foliations.

It is an interesting question when a polar foliation is indeed isoparametric. When the ambient manifold has negative sectional curvature, then in the compact case there are no nontrivial polar or isoparametric foliations (cf. [29, 17]), while in the simply connected case one can easily produce examples of polar foliations that are not isoparametric (cf. the discussion in the first page of [29]).

The first main result of this paper shows that the situation is entirely different when the symmetric space has non-negative curvature:

**Theorem 1.1.** Every polar foliation $(M, F)$ on a simply connected symmetric space with non-negative curvature is isoparametric.

Although this will not be used in the sequel, we remark that Theorem 1.1 implies that for such a foliation, the mean curvature vector field along all regular leaves is basic in the sense that it projects to a vector field on the manifold part of the leaf space $M/F$. It was proved in [19] that, given a foliation with basic mean curvature vector field, there is an “averaging operator” projection $\text{Av}: C^\infty(M) \rightarrow C^\infty(M)^F$ (where $C^\infty(M)^F$ denotes the algebra of smooth functions constant along the leaves of $F$) which commutes with Laplacian. This opens the possibility of studying polar foliations on symmetric spaces in terms of the algebra $C^\infty(M)^F$, together with the action of the Laplacian, as was done in [21, 22] for singular Riemannian foliations on spheres.

Splitting theorems play an important role in the classification of isoparametric and equifocal submanifolds (cf. [25, 10, and 8]). These theorems assert that such submanifolds decompose into products of lower dimensional submanifolds if their associated Coxeter groups decompose. In [18], Lytchak proved that every polar foliation $(M, F)$ on a simply connected symmetric space with non-negative curvature splits as product of hyperpolar foliations, polar foliations with spherical sections, and trivial foliations. Here a trivial foliation means the foliation given by fibers of the projection from a product of two manifolds to one of its components. In this paper, we will prove a splitting theorem of another type.

**Theorem 1.2.** Let $(M, F)$ be a polar foliation without trivial factors on a simply connected symmetric space with non-negative curvature. Then the foliation splits as the product of a polar foliation on the compact factor of $M$, and an isoparametric foliation on the Euclidean factor.

Isoparametric foliations on Euclidean spaces have been completely classified (see, for example, survey articles [28] and [7]). Hence Theorem 1.2 reduces the study of corresponding polar foliation to those in compact simply connected symmetric
spaces. Note that canonical metrics on compact simply connected symmetric spaces have non-negative sectional curvature.

The mean curvature flow (abbreviated as MCF) of a submanifold \( L \) in a Riemannian manifold \( M \) is a map \( f : I \times L \rightarrow M \) satisfying

\[
\frac{\partial f}{\partial t} = H(t, \cdot),
\]

where \( I \) is an interval and \( H(t, \cdot) \) is the mean curvature vector field of \( L_t := f(t, \cdot) \).

It was proved in [13] that the solution to MCF for any compact isoparametric submanifold in a Euclidean space or in a sphere always exists over a finite interval \([0, T)\) with each \( L_t \) an isoparametric submanifold for \( t \in [0, T) \) and it converges to a focal submanifold as \( t \) goes to \( T \). This result was generalized to MCF flow for equifocal submanifolds in [12] and MCF for regular leaves of an isoparametric foliation on a compact non-negatively curved space in [4]. It was also proved in [4] that such mean curvature flows always have type I singularity. An immediate consequence of Theorem 1.1, Theorem 1.2, and results in [4] is that the same result holds for MCF of regular leaves of any polar foliation on a simply connected symmetric space with non-negative curvature.

If a solution to MCF exists for all \( t \in (-\infty, T) \) for some \( T \geq 0 \), then it is called an ancient solution. Ancient solutions to MCF have been extensively studied in recent years since they are important in studying singularities of general MCF. So far most results about ancient solutions are for MCF in Euclidean spaces and spheres. We refer to the reference in [14] for some of these results. In [14], it was proved that MCF for isoparametric submanifolds in Euclidean spaces and spheres always have ancient solutions. Moreover, in each isoparametric foliation on a sphere, there is a unique minimal regular leaf and MCF of any other regular leaves always converge to the unique minimal regular leaf as \( t \) goes to \(-\infty\). Another main result of this paper is that MCF of regular leaves of any polar foliation on a simply connected symmetric space with non-negative curvature always have ancient solutions. By Theorem 1.2 we only need to consider MCF for polar foliations on compact simply connected symmetric spaces. More precisely, we have

**Theorem 1.3.** Let \((M,F)\) be a polar foliation on a compact simply connected symmetric space. Then there is a unique minimal regular leaf \( L_{\min} \) in \( F \). For any regular leaf \( L \) in \( F \), the solution of MCF \( L_t \) with initial data \( L_0 = L \) is always an ancient solution and \( L_t \) converges to \( L_{\min} \) as \( t \) goes to \(-\infty\).

This theorem will give many examples of ancient solutions of MCF in compact symmetric spaces. The proof of Theorem 1.3 is based on estimates of Jacobi fields using comparison theorem for solutions to the Riccati equation. This is completely different from the approach in [14] which relies on structure of Coxeter groups associated to isoparametric submanifolds and representations of mean curvature vectors in terms of curvature normals.

This paper is organized in the following way: In Section 2 we collect some known results about polar foliations and holonomy Jacobi fields which will be needed in the proof of above theorems. In Section 3 we prove a splitting result for hyperpolar foliations, i.e. Proposition 3.8 which is the essential part of Theorem 1.2. The proof of Proposition 3.8 needs a result, i.e. Proposition 3.4 which is similar to Ewert’s splitting theorem in [8] but under a weaker assumption, i.e. the ambient space may not be compact. We will give a proof of Proposition 3.4 in the appendix.
In Section 4, we study polar foliations with spherical sections and complete the proof of Theorems 1.1 and 1.2. Finally, we prove Theorem 1.3 in Section 5.

2. Preliminaries

2.1. Decomposition theorem. We will use in a fundamental way the following classification of polar foliations by Lytchak ([18], Theorem 1.2):

**Theorem 2.1 (Decomposition theorem).** Let \((M, F)\) be a polar foliation on a simply connected non-negatively curved symmetric space \(M\). Then we have a splitting
\[
(M, F) = (M_{-1}, F_{-1}) \times (M_0, F_0) \times \prod_i (M_i, F_i)
\]
where:
1. \((M_{-1}, F_{-1})\) is given by the fibers of the projection of \(M_{-1}\) onto a direct factor.
2. \((M_0, F_0)\) is hyperpolar.
3. \((M_i, F_i)\) are polar foliations, whose section has constant positive sectional curvature (these were called spherical polar in [9]).

We will refer to the factors in the decomposition of \((M, F)\) as factors of type 1, 2, 3.

2.2. Structure of polar foliations on simply connected manifolds. We collect here a number of results, about the structure of polar foliations on simply connected spaces.

Let \((M, F)\) be a polar foliation on a simply-connected space. Then:
1. The leaves of \(F\) are closed, and the leaf space \(M/F\) is a Hausdorff space (Theorem 1.2 of [15]).
2. The leaf space \(M/F\) has boundary. Furthermore, the boundary entirely consists of singular points, while the interior \((M/F)_0\) consists of principal leaves (Theorem 1.6 of [15]).
3. Given a section \(\Sigma\), there is a discrete group \(W\) of isometries of \(\Sigma\) (called the Weyl group) such that \(\Sigma/W\) is isometric to \(M/F\) (Proposition 4.16 of [30]). Furthermore, for \(M\) simply connected, this group is generated by reflections, i.e. isometries that fix a codimension 1 submanifold of \(\Sigma\) called wall (Theorem 1.1 of [1]).

It follows that the leaf space is a smooth compact manifold with corners, and the interior is convex.

2.3. Lagrangian families of Jacobi fields. We collect here the main definitions and results about Lagrangian families of Jacobi fields. The interest reader can find more information and proofs about the statements below, in [14] and [21].

Let \(V\) be a vector bundle over an interval \(I\), endowed with a Euclidean product \(\langle \cdot, \cdot \rangle\) and a metric connection \(\nabla\). A vector fields is then simply a function \(X : I \to V\) such that \(X(t) \in V_t\), and we will write \(\nabla X(t)\) simply as \(X'(t)\). Given a section \(R \in \text{Sym}^2(V^*)\), a \(R\)-Jacobi field is a vector field \(J : I \to V\) such that \(J''(t) + R_t J(t) = 0\) for \(t \in I\).

A space \(\Lambda\) of \(R\)-Jacobi fields is called isotropic if
\[
\langle J_1'(t), J_2(t) \rangle = \langle J_1(t), J_2'(t) \rangle = 0 \quad \forall J_1, J_2 \in \Lambda, t \in I.
\]
Notice that the quantity is constant in \( t \), so it is enough to check that it holds for some \( t_0 \in I \). An isotropic space of Jacobi fields is called Lagrangian if furthermore \( \dim \Lambda = \dim V \).

Given an isotropic space of Jacobi fields \( \Lambda \), the dimension of \( \Lambda(t) = \{ J(t) \mid J \in \Lambda \} \) is constant and equal to \( \dim \Lambda \) for all but discretely many values \( t_i \), where the dimension can drop. In this case, \( t_i \) is called a focal distance and the quantity \( \dim \Lambda - \dim \Lambda(t_i) \) is the corresponding multiplicity. If \( \Lambda(t) \) has maximal dimension, we say that \( t \) is regular otherwise it is singular.

If \( \Lambda \) is Lagrangian, then it is possible to define the Riccati operator \( S_t \in \text{Sym}^2(V_t) \) defined on regular times by \( S_tJ(t) = J'(t) \) for \( J \in \Lambda \). Such operator satisfies the Riccati equation

\[
S'_t + S^2_t + R_t = 0.
\]

Given an isotropic space \( \Lambda \) of Jacobi fields along a geodesic \( \gamma : \mathbb{R} \to M \) and some interval \([a, b] \), let the index of \( \Lambda \) over \([a, b] \) be

\[
\text{ind}_{[a, b]} \Lambda = \sum_{t \in [a, b]} (\dim \Lambda - \dim \Lambda(t)).
\]

By the discussion above, the sum is actually finite for \([a, b] \) compact interval.

### 2.4. Holonomy Jacobi fields in a polar foliation

Let \((M, F)\) be a polar foliation, and let \(L_0\) be a regular leaf. Given a horizontal vector \( x \) at a point \( p \in L \), it is possible to extend \( x \) to a parallel vector field \( X \) along \( L_0 \), and this induces an end-point map

\[
\phi_X : L_0 \to M, \quad \phi_X(q) = \exp_q X_q.
\]

The image of \( \phi_X \) is the leaf through \( \phi_X(p) \).

Fix a point \( p \in L_0 \). Then rescaling \( X \) induces a family of maps \( \phi_{tX} : L_0 \times \mathbb{R} \to M \) such that \( \phi_{tX}(p) \) is the horizontal geodesic from \( p \) with \( \gamma'(0) = X(p) \), and for every \( v \in T_pL_0 \) the vector field \( J_v(t) := d_p\phi_{tX}(v) \) is the Jacobi field along \( \gamma(t) \) (called holonomy Jacobi field with \( J_v(0) = v, J'_v(0) = S_{\gamma'(0)}v \)).

Along \( \gamma \), define \( \mathcal{V}_t = \nu_{\gamma(t)} \Sigma \) with the Euclidean structure induced by the metric on \( M \). Since \( \Sigma \) is totally geodesic, \( \mathcal{V} \) is parallel and in particular the Levi Civita connection restricts to a connection on \( \mathcal{V} \). Letting \( R_t \in \text{Sym}^2(\mathcal{V}^*) \) be \( R_t(v) = R(v, \gamma'(t))\gamma'(t) \), the \( R \)-Jacobi fields are simply the Jacobi fields in \( M \) along \( \gamma \), which stay in \( \mathcal{V} \) the whole time.

Let \( \Lambda_h \) denote the vector space spanned by holonomy Jacobi fields along \( \gamma \). This can be seen as a Lagrangian space of \( R \)-vector fields in \( \mathcal{V} \) along \( \gamma \). For all regular times \( t \), one has \( \Lambda_h(t) = \mathcal{V}_t = T_{\gamma(t)} L_t \). Furthermore, the Riccati operator \( S_t \) for \( \Lambda_h \) coincides with the shape operator \( S_{\gamma'(t)} \) of the leaves along \( \gamma(t) \).

### 2.5. Lifting hyperpolar foliations to Hilbert spaces

Let \((M, F)\) be a hyperpolar foliation on a simply connected symmetric space of compact type. From work of Terng and Thorbergsson [27], one can lift \( F \) to a hyperpolar foliation on a Hilbert space, as follows: first, writing \( M = G/H \) for some Lie groups \( G, H \), one can lift \( F \) to \((G, F_G)\) by taking preimages of the projection map \( G \to M \). Secondly, one can define the Hilbert space

\[
V = H^0([0, 1], g) = \left\{ x : [0, 1] \to g \mid \int_0^1 \| x(t) \|^2 dt < \infty \right\}
\]
which comes with a projection \( \psi : H^0([0, 1], \mathfrak{g}) \to G \) sending \( x(t) \) to the endpoint \( E(1) \) of the curve \( E : [0, 1] \to G \) with \( E(0) = e, E'(t) = E(t) \cdot x(t) \).

The map \( x(t) \mapsto E(t) \) gives an isomorphism \( V \to P(G, e \times G) \), where \( P(G, e \times G) \) is the space of absolutely continuous paths in \( G \) starting at \( e \), with square-integrable first derivative.

Via this identification, the map \( \psi : P(G, e \times G) \to G \) is evaluation at time 1.

The map \( \psi \) is a Riemannian submersion, and it is possible to lift the foliation \( \mathcal{F} \) to a hyperpolar foliation \((V, \mathcal{F}_V)\).

Despite \( V \) being infinite dimensional, the foliation behaves very much like a foliation in finite-dimensional Euclidean spaces. In particular, given a regular leaf \( L \) of \( \mathcal{F}_V \) it makes sense to define the shape operator \( S_x : T_p L \to T_p L \) for any horizontal vector \( x \in \nu_p L \). Furthermore, all shape operators commute, hence they can be simultaneously diagonalized, i.e. there are orthogonal subspaces \( E_i(p) = \{ v \in T_p L \mid S_x v = \lambda_i(x) v \ \forall x \in \nu_p L \} \), where \( \lambda_i \in (\nu_p L)^\ast \), such that \( T_p L = \bigoplus_i E_i(p) \).

In addition, the functional \( \lambda_i \) can be written as \( \lambda_i(x) = \langle x, \xi_i(p) \rangle \) for some vectors \( \xi_i(p) \in \nu_p L \) called the curvature normals. These vectors are related to the Weyl group \( W \) acting on the section \( \Sigma = p + \nu_p L \) as follows: the group \( W \) is generated by isometries of \( \Sigma \) called reflections, which fix the affine subspaces given by \( w_i = \{ p + y \in \Sigma \mid y \in \nu_p L, \langle y, \xi_i(p) \rangle = 1 \} \), called walls of \( W \), cf. [26].

Letting \( \Gamma \) denote the set of walls for \( W \), we can write \( E_v(p) \) the corresponding eigenspace of the shape operators in \( T_p L \), letting \( E_0(p) \) the intersection of kernels of all shape operators, we can rewrite

\[
T_p L = E_0(p) \oplus \bigoplus_{w \in \Gamma} E_w(p).
\]

3. Factors of type 2: Hyperpolar foliations

In this section we focus our attention to factors of type 2, i.e. hyperpolar foliations \((M, \mathcal{F})\) on a simply connected symmetric space \( M \) with non-negative curvature, without trivial factors.

The main goal is to prove Theorem [122] for the factors of type 2. That is, any factor of type 2 splits as a product of a hyperpolar foliation on a compact symmetric space, and an isoparametric foliation in Euclidean space.

We divide the section in three parts: First, given a polar foliation of type 2 \((M, \mathcal{F})\), we show that it splits as a product of foliations \((M_1, \mathcal{F}_1) \times (M_2, \mathcal{F}_2)\) such that the curvature operator on \( M_1 \) along \( \mathcal{F}_1 \)-horizontal directions is zero, and the curvature operator of \( M_2 \) along \( \mathcal{F}_2 \)-horizontal directions is only zero along the sections. Second, we show that \( M_1 \) is the Euclidean space. And finally, we show that \( M_2 \) is compact.

3.1. Splitting of the foliation.

Lemma 3.1. Let \((M, \mathcal{F})\) be a factor of type 2, \( p \in M \) a regular point, \( \Sigma \) the section through \( p \), \( x \in T_p \Sigma \), and \( \gamma(t) = \exp_p(tx) \) the corresponding horizontal geodesic. Finally, let \( \mathcal{V}_t = \nu_{\gamma(t)} \Sigma \). The following are equivalent:

1. \( \text{tr} |_{\mathcal{V}_0} R(\cdot, x)x = 0 \).
2. \( R(\cdot, x)x = 0 \).
3. \( R_t = R(\cdot, \gamma'(t))\gamma'(t) = 0 \) for all \( t \).
4. \( \text{tr} |_{\mathcal{V}_t} R_t = 0 \) for all \( t \).
The space $\Lambda_h$ of holonomy Jacobi fields along $\gamma$ satisfies $\text{ind}_{(-\infty,\infty)} \Lambda_h < \infty$.

Proof. (1 $\Rightarrow$ 2) Follows from the fact that the eigenvalues of $R(\cdot, x)x$ are non-negative, hence $R(v, x)x = 0$ for $v \in V_0$. But since $\Sigma$ is flat, one has that $R(y, x)x = 0$ for $y \in T_p\Sigma$ as well.

(2 $\Rightarrow$ 3) Follows from the fact that $R_t$ is parallel along $\gamma$ hence the eigenvalues of $R_t$ are constant along $\gamma$.

(3 $\Rightarrow$ 4) and (4 $\Rightarrow$ 1) are obvious.

(3 $\Rightarrow$ 5) Let $e_1, \ldots, e_n \in V_0$ be an orthonormal basis of eigenvectors for the Riccati operator $S_0$ of $\Lambda_h$ (cf. section 2.4), with eigenvalues $\mu_1 \ldots \mu_n$. Then since $R_t = 0$, the Jacobi fields in $\Lambda_h$ with $J_i(0) = e_i$ and $J_i'(0) = S_0 e_i = \mu_i e_i$ are given by $J_i(t) = (1 + \mu_i t)E_i$. In particular, the $J_i(t)$ are everywhere orthogonal to one another, and the singular times for $\Lambda_h$ are $t_i = -\frac{1}{\mu_i}$ whenever $\mu_i \neq 0$. In particular, $\text{ind}_{(-\infty,\infty)} \Lambda_h \leq n < \infty$.

(5 $\Rightarrow$ 4) Suppose by contradiction that (4) does not hold, $\text{tr} |\gamma| R_t > 0$. Since the trace of $R_t$ is constant along $\gamma$, it follows that $\text{tr} |\gamma| R_t > n\delta > 0$ for some $\delta$. Fix a regular point $q = \gamma(t*)$ along $\gamma$, and consider the function $a(t) = \frac{1}{n} \text{tr} |\gamma| S_{t+t*}$, where $S_t$ is as usual the Riccati operator $S_t$ of $\Lambda_h$. Since $\text{tr} |\gamma| R_t > n\delta > 0$ we can apply the Average Comparison Theory for the Riccati operator, to obtain that $a(t) \leq \bar{a}(t)$ where $\bar{a}(t)$ is the solution of the model equation $a' + \bar{a}^2 + \delta = 0$, with initial condition $\bar{a}(0) = a(0)$. Such a solution is given by $\bar{a}(t) = \sqrt{\delta} \tan(\sqrt{\delta}(t_0 - t))$ for some $t_0$. As a consequence of the Comparison Theorem, it follows that the first positive singular time of $\Lambda_h$, which coincides with the first time $t_1$ such that $\lim_{t \to t_1} a(t) = -\infty$, is bounded above by $\frac{1}{\sqrt{n}}$. That is, any two singular times of $\Lambda_h$ are at most $\frac{1}{\sqrt{n}}$ apart. Since every singular time contributes at least 1 to the index, it follows that $\text{ind}_{(-\infty,\infty)} \Lambda_h = \infty$. \hfill \qed

Given a type 2 factor $(M, F)$, let $\Sigma$ be a section. Again, we will think of $\Sigma \simeq \mathbb{R}^n$ as a flat space, (possibly not injectively) immersed in $M$.

For each $p \in \Sigma$, define $R : T_p\Sigma \to \text{Sym}^2(T_pM)$ given by $x \mapsto R(\cdot, x)x$, and let $\mathcal{D}_p$ denote the kernel of $R$. Since $R$ maps $T_p\Sigma$ into the set of positive semidefinite self-adjoint endomorphisms of $T_pM$, it follows that $\mathcal{D}_p$ is a vector space: in fact given $x_0, x_1 \in \mathcal{D}_p$, let $x_t = tx_0 + (1 - t)x_0$ and $f(t) = \text{tr} R(\cdot, x_t)x_t$. Then $f(t)$ is a quadratic polynomial, everywhere non-negative and equal to 0 at $t = 0, 1$. Then $f(t) \equiv 0$ that is $x_t \in \mathcal{D}_p$ for every $t$.

Lemma 3.2. Let $(M, F)$ be a hyperpolar foliation on a simply connected symmetric space with nonnegative curvature. Given a section $\Sigma$, the distribution $\mathcal{D} \subseteq T\Sigma$ defined above is parallel (in particular integrable with totally geodesic integral manifolds), and contained in the Euclidean factor of $M$.

Proof. Let $\gamma$ be a path in $\Sigma$ and let $X(t)$ be a parallel vector field along $\gamma$ with $X(0) \in \mathcal{D}_{\gamma(0)}$ (hence $R(\cdot, X(0))X(0) = 0$). Since $R$ is parallel, we then have that the 1-form $R(\cdot, X(t))X(t)$ is parallel as well, and in particular zero everywhere since it is zero for $t = 0$. Therefore $X(t) \in \mathcal{D}_{\gamma(t)}$ hence $\mathcal{D}$ is parallel.

Write now $M = G/H$ for some symmetric pair $(G, H)$ with $H$ compact. Furthermore, assume $eH = p$, and let $\pi : G \to M$ denote the canonical projection. Letting $g, h$ be the Lie algebras of $G$ and $H$ respectively, there is a splitting $g = h \oplus m$ where $m$ can be identified via $d_e\pi$ with $T_pM$. Recall that, with respect to this identification, the curvature operator of $M$ can be expressed as $R(x, y)z = [\{x, y\}, z]$.
for all $x, y, z \in \mathfrak{m}$. Let $x \in D_p$, so that

$$R(v, x)x = [[v, x], x] = 0 \Rightarrow [x, v] = 0 \ \forall v \in \nu_p \Sigma.$$ 

Since $\Sigma$ is flat and totally geodesic, for every $y \in T_p \Sigma$ one has

$$R(y, x)x = [[y, x], x] = 0 \Rightarrow [x, y] = 0 \ \forall y \in T_p \Sigma.$$ 

Altogether, we have that $[u, x] = 0$ for every $u \in \mathfrak{h}$. Given $w \in \mathfrak{h}$, we have that $y = [x, w] \in \mathfrak{m}$ hence, using the bi-invariant metric in $\mathfrak{g}$, we get

$$||[x, w]||^2 = \langle [x, w], [x, w] \rangle = -\langle [x, y], w \rangle = 0$$

and therefore $x$ belongs to the center of $\mathfrak{g}$. In particular, there is a splitting $G = \mathbb{R}^n \times G_c$ for some $n > 0$ and $G_c$ some compact simply connected group (possibly $G_c = \{e\}$). Since $H$ is compact it is contained in $G_c$, hence $M = \mathbb{R}^n \times G_c/H$, with $x$ contained in the Euclidean factor.

**Lemma 3.3.** Suppose that $(M, \mathcal{F})$ is a factor of type 2 whose distribution $\mathcal{D}$ is neither trivial nor it contains $T \Sigma$. Then the sections split as a product $\Sigma_1 \times \Sigma_2$ with $\Sigma_1$ an integral manifold for $\mathcal{D}$, and the Weyl group $W$ splits as a product $W_1 \times W_2$ where $W_i$ acts on $\Sigma_i$ and fixes $\Sigma_{2-i}$, $i = 1, 2$.

**Proof.** Let $\Sigma \simeq \mathbb{R}^k$ be a (simply connected, immersed but possibly non-injectively) section of $\mathcal{F}$, and denote by $\Gamma$ the set of codimension 1 affine subspaces of $\Sigma$ fixed by some reflection in the Weyl group $W$ (the walls of $W$).

Fix a point $0 \in \Sigma$ as the origin of $\Sigma$, denote with $\Sigma_1$ the integral submanifold of $\mathcal{D}$ through 0, and denote $\Sigma_2$ the affine subspace of $\Sigma$ through 0 perpendicular to $\Sigma_1$. By Proposition 3.6 in [29], the union of all walls for the Weyl group is precisely the set of singular points on $\Sigma$. Hence a geodesic starting at regular point in $\Sigma$ passes a wall if and only if there is an increase for the index of $\Lambda_h$. By Lemma 3.1, a geodesic starting at a regular point in $\Sigma$ which is not tangent to $\Sigma_1$ must intersect infinitely many walls. In particular, the number of walls must be infinite.

We claim that $\Sigma_1$ intersect finitely many walls. In fact, assume that there is a sequence of walls $w_i$ intersecting $\Sigma_1$, and let $v_i$ be a unit normal vector for $w_i$. Then $v_i$ can be written as $a_ix_i + b_iy_i$ where $x_i$ and $y_i$ are unit vectors tangent to $\Sigma_1$ and $\Sigma_2$ respectively and $a_i \neq 0$. Without loss of generality, we can assume that the $x_i$'s converge to a unit vector $x$ tangent to $\Sigma_1$. For $i$ sufficiently large, $\langle v_i, x \rangle = \langle a_ix_i, x \rangle \neq 0$. Hence the geodesic $\exp(tx)$ intersects infinitely many walls $w_i$, which contradicts Lemma 3.1.

Since a wall intersects $\Sigma_1$ if and only if its normal vector is not perpendicular to $\Sigma_1$, there are only finitely many walls whose normal vector can be written as $v = v_1 + v_2$ with $v_1$ tangent to $\Sigma_1$ and $v_1 \neq 0$. For infinitely many other walls, their normal vectors must be tangent to $\Sigma_2$.

Since the action of the Weyl group preserves the set of walls, we claim that the normal vector to every wall is either tangent to $\Sigma_1$ or to $\Sigma_2$. In fact, assume by contradiction that there is a wall with normal vector $u = u_1 + u_2$ and both $u_1, u_2$ are non-zero. Notice that, since a geodesic $\gamma(t) = \exp_p t u_2$ from a regular point $p$ in $\Sigma$ must intersect infinitely many walls $w_i$, which means that their normal vectors $v_i$ satisfy $\langle v_i, u_2 \rangle \neq 0$. Furthermore, for all but finitely many of these the normal vector $v_i$ is tangent to $\Sigma_2$.

The reflection $r$ through this wall will map a wall to another wall. It is easy to check that a reflection $r$ fixing a wall $w$ with unit normal $v$, takes a wall $w'$ with
normal vector \( v' \) to a wall with normal vector

\[ r_*(v') = v' - 2\langle v', v \rangle v. \]

Assuming that there is a wall with normal vector \( u = u_1 + u_2 \), apply the corresponding reflection \( r \) to the infinite walls \( w_i \) above, whose normal vector \( v_i \) is tangent to \( \Sigma_2 \) and such that \( \langle v_i, u_2 \rangle \neq 0 \). Each wall \( r(w_i) \) has now normal vector \( r_*(v_i) = v_i - 2\langle v_i, u_2 \rangle u \) and in particular its component tangent to \( \Sigma_1 \) is \( -2\langle v_i, u_2 \rangle u_1 \neq 0 \). Therefore, the infinitely many walls \( r(w_i) \) intersect \( \Sigma_1 \), contradicting the fact that there are only finitely many such walls.

In particular \( \Gamma = \Gamma_1 \cup \Gamma_2 \) where \( \Gamma_i \) denotes the set of walls whose normal vector is tangent to \( \Sigma_i \). By Lemma 2.4 in [10] the Weyl group splits as a product \( W = W_1 \times W_2 \), where \( W_i \) is generated by the reflections in \( \Gamma_i \), and it acts on \( \Sigma_i \) while fixing \( \Sigma_{2-i} \) \((i = 1, 2)\).

The splitting of the Weyl group action induces a splitting of the symmetric space itself.

**Proposition 3.4.** Assume that \((M, F)\) is a factor of type 2, whose section splits \( \Sigma = \Sigma_1 \times \Sigma_2 \) so that the Weyl group \( W \) splits as \( W = W_1 \times W_2 \), with \( W_i \) acting on \( \Sigma_i \) and fixing \( \Sigma_{2-i} \). Then there is a splitting of the foliation \((M, F) = (M_1, F_1) \times (M_2, F_2)\) such that \((M_i, F_i)\) is a factor of type 2 with section \( \Sigma_i \).

This result was proved by Ewert in [3] under the slightly stronger assumption that \( M \) does not have Euclidean factors. Proposition 3.4 can be proved along the same lines with appropriate modifications. For the sake of completeness, we will include a proof of this fact in the appendix.

3.2. **The case \( D \supseteq T\Sigma \).**

**Proposition 3.5 (If \( D \supseteq T\Sigma \)).** Suppose \((M, F)\) is a hyperpolar foliation on a simply connected symmetric space with non-negative curvature, let \( \Sigma \) be a section and assume that the distribution \( D \) contains \( T\Sigma \). Then there is a splitting \( M = \mathbb{R}^n \times M' \) such that \((M, F)\) splits as \((\mathbb{R}^n, F_0) \times M'\). In particular, if \((M, F)\) is of type 2 (no non-trivial factors) then \( D \supseteq T\Sigma \) implies \( M = \mathbb{R}^n \).

**Proof.** Write \( M = \mathbb{R}^n \times (G_c/K) \) where \( G_c/K \) is a symmetric space of compact type. Let \( \Sigma \) be a section. By Lemma 3.2 the distribution \( D \) of \( \Sigma \) is everywhere tangent to the Euclidean factor of \( M \). The assumption that \( D = T\Sigma \) implies that \( \Sigma \) is contained in the Euclidean factor. We now claim that in fact every section is contained in the Euclidean factor. Let \( \Sigma' \) denote any other section. Given a regular point \( p' \in \Sigma' \), let \( L \) denote the leaf through \( p' \) and let \( p \in L \cap \Sigma \). Given \( x' \in T_{p'} \Sigma' \), let \( X \) the corresponding parallel vector field along \( L \) and let \( x = X(p) \). By equifocality, the geodesic \( \exp_p tx' \) meets the same singular leaves as \( \exp_p tx \) at the same times, and in particular the families \( \Lambda_h, \Lambda_h' \) of holonomy Jacobi fields along \( \exp_p tx, \exp_p' tx' \) respectively, satisfy \( \text{ind}_{(-\infty, \infty)} \Lambda_h = \text{ind}_{(-\infty, \infty)} \Lambda_h' \). Since \( x \in D \) by assumption, it follows from Lemma 3.1 that the latter index is finite, hence \( \text{ind}_{(-\infty, \infty)} \Lambda_h' < \infty \) and thus \( x' \in D \) as well. Since \( x' \) was arbitrary, \( T\Sigma' \subseteq D \) and therefore \( \Sigma' \) is contained in \( \mathbb{R}^n \) as well.

This ends the proof, since it follows by the discussion above that \( G_c/H \) is vertical.
3.3. The case $\mathcal{D} = 0$. In this section we will show next that when $\mathcal{D} = 0$ then $M$ is compact. For this, we first need a preliminary result.

**Lemma 3.6.** Let $(\mathcal{M}, \mathcal{F})$ be a polar foliation on a non negatively curved symmetric space, and let $\gamma: \mathbb{R} \to \mathcal{M}$ a horizontal geodesic through the regular part of $\mathcal{F}$ whose projection $\pi(\gamma): \mathbb{R} \to \mathcal{M}/\mathcal{F}$ is periodic. Then the kernel of $R_0 := R(\cdot, \gamma'(0))\gamma'(0)$ is contained in the kernel of the shape operator $S_0 := S_{\gamma'(0)}$.

**Proof.** Let $L$ be the leaf through $\gamma(0)$ (which we assume to be regular), and let $X$ be the parallel vector field along $L$ with $X_{\gamma(0)} = \gamma'(0)$.

Say that the period of $\pi(\gamma)$ is 1. For any integer $k$, let $p_k = \gamma(k) \in L$. For $e_1, \ldots, e_n$ orthonormal frame of eigenvectors for $R_0$ in $T_{p_k}L$, chosen so that $e_1, \ldots, e_r$ span the kernel of $R_0$, let $E_i$ be the parallel extensions of $e_i$ along $\gamma$, which induce orthonormal bases $E_1(k), \ldots, E_n(k)$ of $T_{p_k}L$.

Since $\gamma$ closes up at time 1 in the quotient, it follows that for any $q \in L$, $\frac{d}{dt} \exp_q tX_q = X_{\phi_X(q)}$ and therefore $\phi_X = (\phi_X)^2$. More in general, $\phi_X = (\phi_X)^N$ for every $N$. In particular,

$$d_{p_0} \phi_{NX} = d_{p_{N-1}} \phi_X \circ \ldots \circ d_{p_1} \phi_X \circ d_{p_0} \phi_X.$$ Identifying each $T_{p_k}L$ with $T_{p_0}L$ by identifying the bases $E_1(k), \ldots, E_n(k)$ with $e_1, \ldots, e_n$, all the maps $d_{p_k} \phi_X$ can be all identified with $A := d_{p_0} \phi_X$.

It follows that for all $N$, one has $d_{p_0} \phi_{NX} = A^N$. Therefore, for any matrix norm $\| \cdot \|$ chosen, either $\|d_{p_0} \phi_{NX}\|$ grows exponentially with $N$, or $\|d_{p_0} \phi_{NX}\|$ remains bounded for all $N$.

On the other hand, letting $J_i(t)$ be the Jacobi fields with $J_i(0) = e_i$ and $J_i'(0) = S_0 e_i = \sum_j b_{ij} e_j$, we have:

$$J_i(t) = \sum_{j=1}^r (\delta_{ij} + b_{ij} t) E_j(t) + \sum_{j=r+1}^n \left( \delta_{ij} \cos \lambda_j t + \frac{b_{ij}}{\lambda_j} \sin \lambda_j t \right) E_j(t).$$

Since $d_p \phi_{NX}(e_i) = J_i(N)$, we have for $v = \sum_i a_i e_i$:

$$\|d_p \phi_{NX} v\|^2 = \sum_{j=1}^r \left( \sum_i a_i (\delta_{ij} + b_{ij} N) \right)^2 + \sum_{j=r+1}^n \left( a_j \cos \lambda_j N + \sum_i a_i \frac{b_{ij}}{\lambda_j} \sin \lambda_j N \right)^2 \leq (c_1 N^2 + c_2) \|v\|^2.$$ In particular, the norm $\|d_p \phi_{NX}\| = \sup_v \frac{\|d_p \phi_{NX} v\|}{\|v\|}$ grows sub-exponentially in $N$, and therefore it must be bounded.

Assume by contradiction that $\ker R_0 \not\subseteq \ker S_0$ for some $t$. Then there would be some vector $e_i$ such that the projection of $S_0 e_i$ into $\ker R_0 = \text{span}(e_1, \ldots, e_r)$ is nonzero. Up to rearranging $e_1, \ldots, e_r$ we can assume $S_0 e_1 = \mu e_1 + \sum_{j=r+1}^n b_{ij} e_j$.

$$d_{p_0} \phi_{NX}(e_i) = J_i(N) = t \mu E_1(t) + \sum_{j=1}^r \delta_{ij} E_j(t) + \sum_{j=r+1}^n \left( \delta_{ij} \cos \lambda_j t + \frac{b_{ij}}{\lambda_j} \sin \lambda_j t \right) E_j(t)$$

and its norm grows unbounded as $t \to \infty$, contradicting the boundedness of $\|d_p \phi_{NX}\|$. \[\square\]

**Proposition 3.7 (If $\mathcal{D} = 0$).** Suppose $(\mathcal{M}, \mathcal{F})$ is a factor of type 2, such that for every horizontal vector $x$, $\text{tr} \mid_{T_x L} R(\cdot, x)x > 0$. Then $\mathcal{M}$ is a compact symmetric space.
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Proof. Since $\text{tr} | R(\cdot, x)x > 0$ for every $x$, it follows by the proof of Lemma [3.1] that every horizontal geodesic meets some singular leaf at some positive time, and therefore the quotient $M/F$ is a compact flat orbifold.

Suppose by contradiction that $M = M_c \times \mathbb{R}^k$ with $M_c$ symmetric space of compact type. Let $L$ be a leaf of $F$, and let $\pi_c : L \to M_c$, $\pi : L \to \mathbb{R}^k$ the projections of $L \subseteq M_c \times \mathbb{R}^k$ onto the compact and Euclidean factor, respectively. We claim that $\pi_c$ is a submersion. In fact, if not then there exists a point $p = (p_c, p_\varepsilon) \in L$ and $x_2 \in T_p \mathbb{R}^k$ perpendicular to $d_p \pi_c(T_p L)$. Then in particular $x = (0, x_2) \in T_p M_c \times T_p \mathbb{R}^k$ is perpendicular to $L$, hence horizontal, but $\text{tr} R(\cdot, x)x = 0$ contradicting the hypothesis.

In particular, we can split $V_p = T_p L$ as a sum $V_p = V_p^1 \oplus V_p^2$ by letting $V_p^1$ be the kernel of $d_p \pi_c$, and $V_p^2$ the orthogonal complement of $V_p^1$ in $V_p$. By construction, $d_p \pi_c | V_p^2 : V_p^2 \to \mathbb{R}^k$ is an isomorphism, and we can define $\phi_p = d_p \pi_c \circ (d_p \pi_c | V_p^2)^{-1} : \mathbb{R}^k \to T_p M_c$. The image of $\phi_p$ is contained in the orthogonal complement of $d_p \pi_c(V_p^2) \cong \nu_p^1$ inside $T_p M_0$. We call such space $\nu(V_p^1 \subseteq T_p M_c)$. The dimensions of $V_p^1$ and $V_p^2$ are constant on the regular part of $F$: $\dim V_p^1 = k$, $\dim V_p^2 = \dim F - k$.

The tangent space to the section $\Sigma$ through $p$ is then given by $T_p \Sigma = \{x = (x_1, -\phi_p^*(x_1)) | x_1 \in \nu(V_p^2 \subseteq T_p M_c)\}$. The projection of the regular part of $\Sigma$ to $M_c$ (which we call $\pi_c$ as well) is then an immersion, hence its image $\Sigma_1$ is a smooth manifold, with $T_p \Sigma_1 = (V_p^1 \subseteq T_p M_c)$. It is easy to check that:

- $\Sigma_1$ is totally geodesic: in fact, since $\Sigma$ is totally geodesic, for every vector $x$ tangent to $\Sigma$ there is a geodesic $\gamma(t) = (\gamma_c(t), \gamma_\varepsilon(t))$ of $M$ in $\Sigma$ with the initial vector $x$. But then $\pi_c(\gamma(t)) = \gamma_\varepsilon(t)$ is a geodesic of $M_c$ contained in $\Sigma_1$, with initial velocity $(\pi_c)_*(x)$, which is arbitrary.
- $\Sigma_1$ is flat: in fact, for any closed contractible curve $\gamma$ in $\Sigma_1$, take the corresponding curve $\tilde{\gamma}$ in $\Sigma$. Since $\Sigma$ is flat, the parallel transport $P_{\tilde{\gamma}}$ around $\tilde{\gamma}$ is the identity. But since $(\pi_c)_* P_{\tilde{\gamma}}$ is the parallel transport in $\Sigma_1$ along $\gamma$, it follows that this is the identity as well, and hence $\Sigma_1$ is flat.

It follows that given a horizontal vector $x = (x_1, \phi_p^*(x_1)) \in T_p M$ and a vertical vector $v = (\phi_p(v_1), v_2) \in V_p^2$ one has $R^M(v, x)x = R^{M_c}(\phi_p(v_2), x_1)x_1 = 0$ since $x_1, \phi_p(v_2) \in \nu(V_p^1 \subseteq T_p M_c) = T_p \Sigma_1$.

Since $M/F$ is a compact flat orbifold, there projection $\Sigma \to M/F$ factors through a flat torus $\Sigma \to T \to M/F$. In particular, there is a dense set of directions $x_c$ tangent to the section $\Sigma_c$, such that the geodesic from $x_c$ is closed in $M/F$. For any such direction, we have that $\ker(R(\cdot, x_c)x_c) \subseteq \ker(S_{x_c})$ by Lemma [3.6] and in particular $S_{x_c}v = 0$ for any $v \in V_p^2$. Since the set of such directions is dense, one has $S_{x_c}v = 0$ for all horizontal vectors $x$ and all vectors $v \in V_p^2$. Therefore $S_x$ takes $V_p^1$ to itself, and therefore $(\pi_c)_* S_x = 0$.

Let now $X = (X_1, X_2)$ be a parallel normal vector field along $L$ (with respect to the normal connection). In particular, for any vertical $v = (u_1, u_2)$ one has $\nabla_v X = -S_{x}v$ and in particular $0 = (\pi_c)_*(\nabla_v X) = \nabla_v X_2$. This implies that $X_2$ is parallel, hence constant, along $L$. In particular, for $\|X_1\|^2$ and $\|X_2\|^2$ are both constant along $L$.

Finally, fix a regular leaf $L$ and a point $p \in L$. Given a wall $w$, let $\xi \in T_p \Sigma$ be a vector perpendicular to $w$. The geodesic $\gamma(t) = \exp_p t\xi$ then intersects the wall perpendicularly at some time $T$. Furthermore, the reflection $r$ fixing $w$ satisfies
\[ r(\gamma(T - t)) = \gamma(T + t). \] In particular, \( \gamma \) projects to a curve \( \pi(\gamma) \) in \( M/F \) which meets a wall at \( T \), and “bounces back on itself”, in such a way that \( \pi(\gamma(T + t)) = \pi(\gamma(T - t)) \). In particular \( \pi(\gamma(2T)) = \pi(\gamma(0)) \) and \( \pi_\gamma(2T) = -\pi_\gamma(0) \). Since \( M/F \) is compact, one can find a basis of \( T_p\Sigma \) consisting of vectors \( \xi \) perpendicular to some wall.

Assume \( p = (p_c, p_e) \) and \( \xi = (\xi_c, \xi_e) \). Then \( \gamma(t) = (\exp_{p_e}(t\xi_c), p_e + t\xi_c) \), and \( \gamma'(2T) = (u, \xi_2) \) for some \( u \in T_{\gamma(2T)} M_c \). The fact that \( \pi_\gamma(2T) = -\pi_\gamma(0) \) implies that the parallel vector normal fields extending \( \xi \) and \( \gamma'(2T) \) are opposite of one another. However, since the second component of a parallel normal vector is constant along the leaf, it follows that \( 0 = (\pi_2(\xi) + \gamma'(2T)) = 2\xi_e \). Therefore, \( \xi = (\xi_c, 0) \), but since such vectors form a basis of \( \nu_p L \), it follows that the normal space of a regular leaf is contained in \( T_p M_0 \). This implies that the \( \mathbb{R}^k \) factor is tangent to regular leaves. An integration argument shows that each regular leaf contains the \( \mathbb{R}^k \) factor. Since singular leaves are Hausdorff limits of regular ones, it follows that the \( \mathbb{R}^k \) factor is contained in all leaves, that is, the foliation splits as \( (M_c, F_{M_c}) \times \mathbb{R}^k \). Since by assumption \( F \) does not contain trivial factors, it must be \( M = M_c \), hence \( M \) is compact.

We can now sum up the results in this section, to prove Theorem 1.2 for factors of type 2.

**Proposition 3.8.** Let \( (M, F) \) be a hyperpolar foliation on a simply connected symmetric space with nonnegative curvature. Then there is a splitting \( (M_c, F_c) = (M_c, F_{M_c}) \times \mathbb{R}^k \) with \( M_c \) compact.

**Proof.** Let \( (M, F) \) be a polar foliation on a simply connected symmetric space with non-negative curvature. It is enough to prove the statement assuming that there are no trivial factors. By Lemma 3.5, the section \( \Sigma \) splits as a product \( \Sigma_1 \times \Sigma_2 \) where \( \operatorname{tr}_{|V_p} R(\cdot, x_1) x_1 > 0 \) for every \( x_1 \in T_p \Sigma_1 \) and \( R(\cdot, x_2) x_2 = 0 \) for every \( x_2 \in T_p \Sigma_2 \). By Proposition A.6 there is a splitting \( (M, F) = (M_1, F_1) \times (M_2, F_2) \) where \( (M_1, F_1) \) is a polar foliation with section \( \Sigma_1 \).

By Proposition 3.5 \( M_2 = \mathbb{R}^k \) for some \( k \). By Proposition 3.7 \( M_1 \) is compact.

4. FACTORS OF TYPE 3: SPHERICAL POLAR FOLIATIONS

In this section, we focus our attention to factors of type 3. That is, a polar foliation \( (M, F) \) on a simply connected symmetric space \( M \), with sections of constant positive curvature. The main goal of this section is to prove that factors of type 3 are compact and isoparametric, i.e. they have parallel mean curvature vector field.

We start by proving compactness.

**Proposition 4.1.** Let \( (M, F) \) be a factor of type 3. Then \( M \) is compact.

**Proof.** Since \( M \) is simply connected and non-negatively curved, it splits as \( M = M_c \times \mathbb{R}^k \) where \( M_c \) is compact. Fix a regular leaf \( L \) of \( F \), a point \( p \in L \), and a vector \( v \) tangent to the \( \mathbb{R}^k \) factor. Decompose \( v = v^h + v^i \) into \( v^h \in \nu_p L \) and \( v^i \in T_p L \). Since the dimension of \( M/F \) is at least 2, if \( v^h \neq 0 \) it is possible to find a horizontal vector \( x \) not parallel to \( v^h \) and

\[ 0 = \langle R(x, v) v, x \rangle = \langle R(x, v^h) v^h, x \rangle + 2 \langle R(x, v^h) v^i, x \rangle + \langle R(x, v^i) v^i, x \rangle \geq \langle R(x, v^h) v^h, x \rangle \]

where the last inequality follows from the fact that, on the one hand \( \langle R(x, v^h) v^i, x \rangle = -\langle R(x, v^h) x, v^i \rangle = 0 \) since \( x, v^h \in T_p \Sigma \) with \( \Sigma \) totally geodesic, and on the other
hand \( \langle R(x, v^i) v^i, x \rangle \geq 0 \). Since \( \Sigma \) has constant sectional curvature, \( \langle R(x, v^h) v^h, x \rangle = 0 \) implies \( v^h = 0 \). In other words, every \( v \) tangent to the \( \mathbb{R}^k \)-factor must be tangent to \( L \), and thus \( L \) must split as \( L_c \times \mathbb{R}^k \) where \( L_c = L \cap M_c \). Since any other leaf of the foliation is determined by \( L \) as the exponential image of parallel normal vector fields along \( L \), it follows that the whole foliation splits as \((M_c, F \cap M_c) \times \mathbb{R}^k \). Since \((M, F)\) does not have trivial factors by assumptions, it follows that \( k = 0 \) and \( M = M_c \) is compact. \( \square \)

Rescale the metric so that sections have positive sectional curvature 1, and we will consider the section as the (possibly non injectively) immersed round sphere. In particular, all horizontal geodesics are closed with common (not necessarily smallest) period \( 2\pi \), and the end-point map \( \phi_{tX} \) has period \( 2\pi \) whenever \( X \) is a parallel normal vector of unit length along a regular leaf.

Finally, since \( M \) is a symmetric space of non-negative curvature, along any horizontal geodesic \( \gamma(t) \) the eigenvalues of \( R_t \) are constant and non-negative, and we call them \( 0 = \lambda_0 < \lambda_1^2 < \ldots < \lambda_N^2 \).

**Lemma 4.2.** Given a factor \((M, F)\) of type 3, with metric rescaled so that the section \( \Sigma \) has sectional curvature 1. Then fixing a regular point \( p \) in \( \Sigma \) and a unit-speed horizontal geodesic \( \gamma(t) = \exp_p tX \), the eigenvalues of the curvature operator \( R_t \) along \( \gamma \) are squares of integers. Furthermore, the kernel of \( R_t \) is contained in the kernel of the shape operator \( S_{\gamma(t)} \).

**Proof.** Recall that a holonomy Jacobi field \( J(t) \) along \( \gamma \) is given by \( d_{\gamma} \phi_{tX}(v) \) for some \( v \in \mathcal{V}_p \). In particular, since \( \phi_{tX} \) is periodic with period \( 2\pi \), so is any holonomy Jacobi field.

Decompose \( \mathcal{V} \) into a sum \( \bigoplus_{i=0}^{N} \mathcal{V}^i \), where \( \mathcal{V}^0_t = \ker R_t \) and each \( \mathcal{V}^i_t \) is the eigenspace of \( R_t \) with eigenvalue \( \lambda_i^2 > 0 \). Since \( R_t \) is parallel, so are the subspaces \( \mathcal{V}^i \). Furthermore, since \( R_{2\pi} = R_0 \), then \( \mathcal{V}_{2\pi} = \mathcal{V}_0 \) and in particular the projection of a Jacobi field onto a subspace \( \mathcal{V}^i \) is again a periodic Jacobi field.

The fact that the kernel of \( R_t \) is contained in the kernel of \( S_{\gamma(t)} \) follows from Lemma 4.3. Consider now a positive eigenspace \( \mathcal{V}^i \) of \( R_t \). We want to show that the corresponding eigenvalue \( \lambda_i^2 \) is the square of an integer. Assume first, that \( \mathcal{S}_t(\mathcal{V}^i) \) is not contained in \( \mathcal{V}^i \). Arguing as above, there is some \( j \neq i \) and a holonomy Jacobi field with \( J(0) \in \mathcal{V}^j \) and \( (\pi_{\mathcal{V}^i} J'_t)(0) \neq 0 \). Therefore, the Jacobi field \( J_t = \pi_{\mathcal{V}^i} J \) satisfies \( J_t(0) = 0, J'_t(0) = v \neq 0 \in \mathcal{V}^i \), hence \( J(t) = \sin(\lambda_i t) V(t) \) for \( V(t) \) the parallel transport of \( v \). Since \( J(t) \) is periodic with period \( 2\pi \), so is \( \|J(t)\|^2 = \sin^2(\lambda_i t) \), which implies that \( \lambda_i \) is an integer.

Finally, assume that \( \mathcal{V}^i \) of \( R_0 \) satisfies \( S_0(\mathcal{V}^i) \subseteq \mathcal{V}^i \). Fixing an eigenvector \( v \in \mathcal{V}^i \) of \( S_0 \) with eigenvalue \( \mu \), the holonomy Jacobi field \( J(t) \) with \( J(0) = v \) satisfies

\[
J(t) = (\cos(\lambda_i t) + \frac{\mu}{\lambda_i} \sin(\lambda_i t)) V(t)
\]

and again since \( \|J(t)\|^2 \) is periodic with period \( 2\pi \), it follows that \( \lambda_i \) is an integer. \( \square \)

**Lemma 4.3.** Let \( \Lambda_h \) be the space of holonomy Jacobi fields along a horizontal geodesic \( \gamma \), and let \( m = \text{ind}_{[0,2\pi]} \Lambda_h \). This index does not depend on the choice of initial regular point \( p \) or horizontal geodesic \( \gamma \), and it equals \( 2 \sum \lambda_i \).

**Proof.** The independence on the choice of geodesic, or on the regular point \( p \), follows from the continuity of the index for Lagrangian spaces of Jacobi fields proved in
Proposition 1.4 of [16], and the fact that in this case, any two horizontal closed geodesics of the same length can be connected via a path of horizontal closed geodesics of constant length.

For the second statement, fix a horizontal geodesic \( \gamma \) and notice that for any integer \( k \), \( \text{ind}_{[0,2\pi k]} \Lambda_h = mk \). We now consider a different Lagrangian space of Jacobi fields \( \mathcal{V} \) along \( \gamma \), namely

\[ \Lambda_0 = \{ J \mid J(0) = 0, J'(0) \in \mathcal{V}_{\gamma(0)} \} \]

If \( e_1, \ldots, e_n \in \mathcal{V}_{\gamma(0)} \) is a basis of eigenvectors of \( R_0 \), with eigenvalues \( \lambda_i^2 \), then the Jacobi field \( J_i \in \Lambda_0 \) with \( J_i(0) = 0 \), \( J'_i(0) = e_i \) is given by \( J_i(t) = \sin(\lambda_it)E_i(t) \), which vanishes \( 2\lambda_i \) times for every period \( 2\pi \) of the geodesic. Since the \( J_i \) are everywhere linearly independent, it follows that

\[ \text{ind}_{[0,2\pi]} \Lambda_0 = 2 \sum \lambda_i \quad \Rightarrow \quad \text{ind}_{[0,2\pi]} \Lambda_0 = 2k \sum \lambda_i. \]

On the other hand, it follows by Proposition 1.4 of [16] that given two Lagrangian spaces of Jacobi fields \( \Lambda_1, \Lambda_2 \) along \( \gamma \), then for any interval \( I \) one has

\[ |\text{ind}_I \Lambda_1 - \text{ind}_I \Lambda_2| \leq \dim \mathcal{V}. \]

Applying this to the case of \( \Lambda_h \) and \( \Lambda_0 \), one has that for any positive integer \( k \), \( k|m - 2 \sum \lambda_i| < n \). The only way this can be true for all \( k \) is that \( m - 2 \sum \lambda_i = 0 \).

**Remark 4.4.** It follows from the previous lemma that \( m \) is even, but this should not be a surprise. Consider, in fact, the section \( \Sigma = S^{\omega} \), cut by the walls \( w_i \) fixed by reflections in the Weyl group. Given a horizontal geodesic \( \gamma \) along \( \Sigma \), the singular times for the family \( \Lambda_h \) of holonomy Jacobi fields along \( \gamma \), i.e. the times in which \( \dim \Lambda_h(t_i) < \dim \Lambda_h \) (cf. Section 2.3), coincide with the times in which \( \gamma(t_i) \) meets a wall \( w_i \) of \( \Sigma \). Furthermore, the multiplicity \( m_i = \dim \Lambda_h - \dim \Lambda_h(t_i) \) corresponds to the multiplicity of the wall \( w_i \). Since \( \gamma \) meets each wall \( w_i \) twice, at times \( t_i \) and \( t_i + \pi \), each wall contributes \( 2m_i \) to the index \( \text{ind}_{[0,2\pi]} \Lambda_h \).

**Corollary 4.5.** Given a horizontal geodesic \( \gamma \), the curvature operator \( R_t \in \text{Sym}^2(\mathcal{V}^*) \) satisfies \( \text{tr}(R_t) > 0 \).

**Proof.** It follows by Lemma 2.3 of [18] that a polar foliation on a simply connected, non-negatively curved symmetric space must contain singular leaves, unless the foliation is of type 2. Therefore we have at least one singular leaf \( L \), and there is at least one horizontal closed geodesic \( \gamma \) through \( L \), which must satisfy \( \text{ind}_{[0,\pi]} \Lambda_h =: m > 0 \).

By Lemma 4.3 this index is the same for any horizontal closed geodesic in the section. Since \( \sum \lambda_i = 2m \) by the previous lemma, then \( \text{tr}(R_t) = \sum \lambda_i^2 > 0 \). □

**Lemma 4.6.** Fix a factor \((M, F)\) of type 3, with the metric normalized as above. Then for any horizontal geodesic \( \gamma(t) \), the function \( \det(\tilde{d}F_{\gamma(t)}) \) can be written as a linear combination

\[ f(t) = \sum a_i \sin(s_it) + b_i \cos(s_it), \]

where \( s_i \) are integers of the same parity between \( \frac{\pi}{2} = \text{ind}_{[0,\pi]} \Lambda_h \) and \( -\frac{\pi}{2} \).
Proof. As usual let $V_i$ be the bundle along $\gamma$ perpendicular to $\Sigma$. We call the eigenvalues of $R_\gamma \nu = 0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n$. Here the numbering of the eigenvalues is different from what used previously, since here we allow repetitions of eigenvalues. Let $e_1, \ldots, e_n$ be an orthonormal basis of eigenvectors of $R_\gamma$, with eigenvalues $\lambda^2_i$, and let $E_i(t)$ be the parallel transport of $e_i$ along $\gamma$. Assume that $\lambda_i = 0$ for $i = 1, \ldots, r$ and $\lambda_i > 0$ for $i = r+1, \ldots, n$. Let $A_i$ be the family of holonomy Jacobi fields along $\gamma$, with a basis $J_1, \ldots, J_n$ with

$$J_i(0) = e_i, \quad J_i'(0) = \sum_j b_{ij} e_j.$$ 

Since $M$ is a symmetric space, $R_\gamma$ is parallel and the Jacobi fields can be explicitly computed as

$$J_i(t) = \sum_{j=1}^r (\delta_{ij} + b_{ij} t) E_j(t) + \sum_{j=r+1}^n \left( \delta_{ij} \cos \lambda_j t + \frac{b_{ij}}{\lambda_j} \sin \lambda_j t \right) E_j(t).$$

From Lemma 4.2 that for $i = 1, \ldots, r$ one has $J_i(t) = E_i(t)$, and for $i = r+1, \ldots, n$

$$J_i(t) = \sum_{j=r+1}^n \left( \delta_{ij} \cos \lambda_j t + \frac{b_{ij}}{\lambda_j} \sin \lambda_j t \right) E_j(t).$$

Since $\Sigma = S^n$ is simply connected, the normal holonomy of $\Sigma$ is contained in $SO(n)$ and in particular $E_1(t), \ldots, E_n(t)$ represent at each point an oriented orthonormal basis of $V_i$. Hence, it makes sense to define $f(t) := \det(dJ_{\phi_{1X}}(t) = \det((J_i, E_j))$, which is given by a linear combination of terms of the form

(1) $$\prod_{i=r+1}^n \sin(\lambda_i t), \quad \sin \lambda_i \in \{\sin, \cos\}.$$

Using the product formulas for trigonometric functions, it follows that $f(t)$ is a linear combination

$$f(t) = \sum_i a_i \sin(s_i t) + b_i \cos(s_i t),$$

where each $s_i$ is a linear combination $s_i = \epsilon_{r+1} \lambda_{r+1} + \ldots + \epsilon_n \lambda_n$ with coefficients $\epsilon_r, \ldots, \epsilon_n \in \{\pm 1\}$. In particular, $s_i$ are integers, bounded between $\sum \lambda_i = \frac{m}{2}$ and $-m/2$. All $s_i$ have the same parity, since their difference is a linear combination of the $\lambda_i$’s with coefficients in $\{-2, 0, 2\}$. \hfill \Box

Proposition 4.7. Let $(M, F)$ be a factor of type 3. Then the mean curvature is basic.

Proof. Fix a regular leaf $L_0$ and a basic horizontal vector field $X$ along $L_0$. For $p \in L_0$, let $\Sigma_p$ be the section through $p$, $\gamma_p(t) = \exp_p tX_p$, $V_p = \nu \Sigma_p |_{\gamma_p}$, $A_p$ the space of holonomy Jacobi fields along $\gamma_p$, and $E_1(t), \ldots, E_n(t)$ a frame of parallel vector fields along $\gamma_p$, tangent to $V_p$. Finally, let $f_p(t) = \det((J_i(t), E_j(t)))$.

Once again, we normalize the metric so that the section is a round sphere of curvature 1. By Lemma 4.2 the eigenvalues $\lambda^2_i$ of $R_\gamma$ are squares of integers, and by Lemma 4.3 $\sum_i \lambda_i = \frac{m}{2} = \text{ind}_{\{0, \pi\}} A_{L_0}$. Furthermore, by Lemma 4.4

$$f_p(t) = \sum_i a_i \sin(s_i t) + b_i \cos(s_i t), \quad s_i = \lambda_1 \pm \lambda_2 \pm \ldots \pm \lambda_n.$$
where \( s_i \) can range within the integers from \( \sum \lambda_i = \frac{m}{2} \) and \( -\frac{m}{2} \). Notice furthermore, that all \( s_i \) must have the same parity. Taking into account that \( \cos(s,t) = \cos(-s,t) \), \( \sin(-s,t) = -\sin(s,t) \) and \( \sin(0) = 0 \), it follows that depending on the parity of \( \frac{m}{2} \), the functions \( \sin(s,t), \cos(s,t) \) are contained in the space \( T \) of functions:

\[
T = \begin{cases} \text{span}\{1, \cos(2t), \sin(2t), \cos(4t), \sin(4t), \ldots, \cos(\frac{m}{2}t), \sin(\frac{m}{2}t)\} & \text{if } \frac{m}{2} \text{ is even} \\
\text{span}\{\cos(t), \sin(t), \cos(3t), \sin(3t), \ldots, \cos(\frac{m}{2}t), \sin(\frac{m}{2}t)\} & \text{if } \frac{m}{2} \text{ is odd}
\end{cases}
\]

in either case of dimension \( \frac{m}{2} + 1 \), which does not depend on \( p \). The projection of \( \gamma_p \) to \( M/F \) will intersect singular strata at singular times \( t_1, \ldots, t_k \), and for each \( j = 1, \ldots, k \) we can let \( m_j := \dim L_0 - \dim L_{t_j} = \dim \Lambda_p - \dim \Lambda_p(t_j) \). Notice that, by the equifocality of singular Riemannian foliations (cf. Proposition 4.3 of [20], Theorem 2.9 in [2] or Proposition 2.26 of [24]) the data \( t_j, m_j \) do not depend on the choice of point \( p \in L_0 \) but only on the choice of basic vector field \( X \): in fact, if we chose a different point \( q \) and let \( \gamma_q(t) = \exp_q(tX_q) \) then \( \gamma_p \) and \( \gamma_q \) would meet the same leaf at each time \( t \).

The fact that \( \gamma_p(t) \) meets the singular leaves \( L_{t_j} \) with \( \dim \Lambda_p - \dim \Lambda_p(t_j) = m_j \) can be restated by saying that \( f_p(t) \) vanishes with order \( m_j \). This imposes, for every singular time \( t_j \), exactly \( m_j \) conditions:

\[
f_p(t_j) = f_p'(t_j) = \ldots = f_p^{(m_j - 1)}(t_j) = 0.
\]

These conditions form a system of \( \sum_{j=1}^k m_j = \frac{m}{2} \) linear equations on \( T \), which are easily seen to be linearly independent: in fact, consider the subspace \( T' \subseteq T \) spanned by the \( m \) linearly independent functions:

\[
\cos(t - t_j) \prod_{i=1}^k \sin^{m_i}(t - t_i), \quad j = 1, \ldots, k, l = 1, \ldots m_j.
\]

Then the linear map \( T \to \mathbb{R}^{\frac{m}{2}} \) which sends a function \( h(t) \in T \) to

\[ (h(t_j))_{j=0,\ldots,m_j-1, j=1,\ldots,k} \]

is invertible when restricted to \( T' \) (as the matrix for this map is triangular with non-zero diagonals with respect to that basis). Hence the kernel of this map has dimension 1, and \( f_p \) is the unique function in the kernel satisfying \( f_p(0) = 1 \).

In particular, \( f_p(t) \) is uniquely determined by \( X \) from information on the leaf space \( M/F \), and it is independent of \( p \). Since

\[
f_p'(0) = \text{tr} S_{\gamma_p(0)} f_p(0) = \langle H_p, X_p \rangle,
\]

it follows in particular that the inner product of \( H \) with any basic horizontal vector field \( X \) along \( L_0 \) is constant. Thus, \( H \) is basic as well.

We end this section with a proof of Theorems [1.1] (that is, polar foliations on symmetric spaces with non negative curvature are isoparametric) and [1.2] (polar foliations on symmetric spaces with nonnegative curvature split into a compact factor and an Euclidean one).

**Proof of Theorem [1.1]** We check that this is true on every factor of the foliation.

This is trivially true for factors of type 1. For factors \( (M_0, \mathcal{F}_0) \) of type 2 (hyperpolar foliations) this fact follows from Theorems 2.4 and 6.5 in [11]. Finally, we proved that factors of type 3 are isoparametric in Proposition [1.7].
Proof of Theorem 5.1. Let \((M, \mathcal{F})\) be a polar foliation on a symmetric space with nonnegative curvature. Let \((M, \mathcal{F}) = (M_{-1}, \mathcal{F}_{-1}) \times (M_0, \mathcal{F}_0) \times \prod_i (M_i, \mathcal{F}_i)\) be Lytchak's decomposition (cf. Theorem 2.4). Let

\[(1) \quad M_{-1} = M_{-1}' \times \mathbb{R}^{k_1} \text{ be the splitting of } M_{-1} \text{ with } M_{-1}' \text{ compact, and } M_{-1}, \mathcal{F}_{-1} = (M_{-1}', \mathcal{F}_{-1}') \times (\mathbb{R}^{k_1}, \mathcal{F}_{-1}') \text{ where } \mathcal{F}_{-1}' \text{ and } \mathcal{F}_{-1}' \text{ are the fibers of the projections of } M_{-1}', \mathbb{R}^{k_1}\text{ onto some of their direct factors.}
\]

\[(2) \quad (M_0, \mathcal{F}_0) = (M_0^c, \mathcal{F}_c^c) \times (\mathbb{R}^{k_0}, \mathcal{F}_0^c) \text{ be the splitting of } \mathcal{F}_0 \text{ from Proposition 3.8.}
\]

\[(3) \quad (M^c, \mathcal{F}^c) \text{ be the product foliation } (M^c, \mathcal{F}^c) = (M_{-1}', \mathcal{F}_{-1}') \times (M^c_0, \mathcal{F}^c_0) \times \prod_i (M_i, \mathcal{F}_i). \text{ Notice that by Proposition 4.1, } M_i \text{ is compact for } i > 0 \text{ and therefore } M_i \text{ is compact.}
\]

\[(4) \quad (\mathbb{R}^{k_i}, \mathcal{F}^c_i) \text{ be the splitting of } \mathcal{F}^c_i \text{ along the horizontal parallel vector field along } \gamma_i \text{ at } x = 0 \text{ for any principal leaf } \gamma_i \text{ such that for any principal leaf } \gamma_i \text{ and any } x \in \nu_p \mathcal{L}, \text{ the curvature operator } R \text{ on } \mathcal{L} \text{ satisfies } \text{tr}_{T_p \mathcal{L}} R(\cdot, x) > 0. \text{ Then the function}
\]

\[V : M/\mathcal{F} \to \mathbb{R} \quad V(p_* \gamma) = \text{vol}(\pi^{-1}(p_* \gamma))^{\frac{1}{n}}
\]

is strictly concave on the regular part of \(M/\mathcal{F}\), and equal to 0 on the singular part. In particular, if \(M/\mathcal{F}\) is compact, there is a unique leaf achieving the maximum volume.

Proof. Let \(\gamma_* : [-a, b] \to M/\mathcal{F}\) a geodesic segment on the regular part of \(M/\mathcal{F}\). It is enough to prove that \(V(\gamma_*(t))\) is concave.

Let \(L_t = \pi^{-1}(\gamma_*(t))\), and let \(X\) the horizontal parallel vector field along \(L_0\) projecting to \(\gamma'_*(0)\), and let \(\phi_t : L_0 \times [-a, b] \to M\) the end-point map defined in Section 2.4. Then for every \(p \in L_0\), \(\gamma_p(t) = \phi_t(p)\) is a horizontal geodesic in \(M\) projecting to \(\gamma_*\), and letting \(\omega_t\) the volume form of \(L_t\), one has that \(\omega_t \omega_t(\gamma_*(p), p) = f(p, t)\omega_0\) where

\[f(p, t) = \det(d_p \phi_t X) = \det((J_1(t), E_i(t)))
\]

where \(J_i(t)\) are holonomy Jacobi fields and \(E_i(t)\) are parallel vector fields, with \(J_i(0) = E_i(0) = e_i\) a basis of orthonormal vectors in \(T_p L_0\). Then

\[V(\gamma_*(t)) = \text{vol}(L_t)^{\frac{1}{n}} = \left(\int_{L_t} \omega_t\right)^{\frac{1}{n}} = \left(\int_{L_0} f(p, t)\omega_0\right)^{\frac{1}{n}}.
\]

Fixing a \(p \in L_0\), recall from Section 2.4 that the holonomy Jacobi fields form a Lagrangian space \(\Lambda_p\) of Jacobi fields of the bundle \(V\) given by \(V_1 = T_{\gamma_p(t)} L_t\). In particular there is a Riccati operator \(S \in \text{Sym}^2(V^*)\) along \(\gamma_p(t) L_t\) such that \(S(J_i(t)) = J'_i(t)\), which solves the ODE \(S h + h^2 + R_t = 0\). Since by assumption \(\text{tr}_{V} R_t > 0\), let \(\delta > 0\) be such that along \(\gamma_p\), \(\text{tr}_{V} R_t > n\delta\).
Then by comparison theory of the Riccati operator, letting \( s_0 = \frac{1}{n} \operatorname{tr}(S_0) \), one has that

\[
\left\{ \begin{array}{l}
\hat{s}'(t) + \hat{s}^2(t) + \delta = 0 \\
\hat{s}(0) = s_0
\end{array} \right.
\]

that is, \( \hat{s}(t) = -\sqrt{\delta} \tan(\sqrt{\delta}(t - t_0)) \), where \( t_0 = \frac{1}{\sqrt{\delta}} \arctan(s_0/\sqrt{\delta}) \). Finally, \( f(p, t) \) solves the ODE \( \frac{d}{dt}(\ln f(p, t)) = \operatorname{tr}(S_t) \leq -n\sqrt{\delta} \tan(\sqrt{\delta}(t - t_0)) \). Hence for any \( t > 0 \)

\[
\ln \left( \frac{f(p, t)}{f(p, 0)} \right) \leq \int_0^t -n\sqrt{\delta} \tan(\sqrt{\delta}(t - t_0)) dt = \ln \left( \frac{\cos^n(\sqrt{\delta}(t - t_0))}{\cos^n(\sqrt{\delta}t_0)} \right).
\]

Since \( f(p, 0) = 1 \),

\[
f(p, t) \leq \frac{\cos^n(\sqrt{\delta}(t - t_0))}{\cos^n(\sqrt{\delta}t_0)}, \quad t > 0.
\]

For negative values of \( t \), we can repeat the same argument for \( \hat{\gamma}(t) := \gamma(-t) \). In this case, \( \operatorname{tr}(S_{\hat{\gamma}(0)}) = -\operatorname{tr}(S_{\gamma(0)}) = -s_0 \), and one can apply the comparison theory to obtain

\[
\frac{1}{n} \operatorname{tr}(S_{\hat{\gamma}(t)}) \leq \hat{s}(t) \quad \text{where } \hat{s}(t) \text{ now solves}
\]

\[
\left\{ \begin{array}{l}
\hat{s}'(t) + \hat{s}^2(t) + \delta = 0 \\
\hat{s}(0) = -s_0
\end{array} \right.
\]

that is, \( \hat{s}(t) = -\sqrt{\delta} \tan(\sqrt{\delta}(t + t_0)) \). Now, \( \hat{f}(p, t) := f(p, -t) \) solves the ODE

\[
\frac{d}{dt}(\ln \hat{f}(p, t)) = \operatorname{tr}(S_{\hat{\gamma}(t)}) \leq -n\sqrt{\delta} \tan(\sqrt{\delta}(t + t_0))
\]

and again since \( \hat{f}(p, 0) = 1 \), one obtains for any \( t > 0 \)

\[
\hat{f}(p, t) \leq \frac{\cos^n(\sqrt{\delta}(t + t_0))}{\cos^n(-\sqrt{\delta}t_0)}.
\]

Substituting \( \hat{f}(p, t) = f(p, -t) \) one gets, now for negative values of \( t \), that

\[
f(p, t) \leq \left( \frac{\cos(\sqrt{\delta}(-t + t_0))}{\cos(-\sqrt{\delta}t_0)} \right)^n = \left( \frac{\cos(\sqrt{\delta}(t - t_0))}{\cos(\sqrt{\delta}t_0)} \right)^n.
\]

Therefore, the same inequality for \( f(p, t) \) applies to both sides of \( t = 0 \). In particular, we have

\[
V(\gamma_*(t)) = \left( \int_{L_0} f(p, t) \omega_0 \right)^{1/n} \leq \frac{\cos(\sqrt{\delta}(t - t_0))}{\cos(\sqrt{\delta}t_0)} \left( \int_{L_0} \omega_0 \right)^{1/n}
\]

\[
= \frac{\cos(\sqrt{\delta}(t - t_0))}{\cos(\sqrt{\delta}t_0)} V(\gamma_*(0))
\]

with equality at \( t = 0 \). In particular,

\[
\frac{d^2}{dt^2} \bigg|_{t=0} V(\gamma_*(t)) \leq \frac{d^2}{dt^2} \bigg|_{t=0} \left( \frac{\cos(\sqrt{\delta}(t - t_0))}{\cos(\sqrt{\delta}t_0)} V(\gamma_*(0)) \right) = -\delta V(\gamma_*(0)) < 0.
\]

Hence \( V \) is strictly concave in the interior of \( \mathcal{M}/\mathcal{F} \). Since points on the boundary of \( \mathcal{M}/\mathcal{F} \) corresponds to lower dimensional leaves, \( V \) is 0 on the boundary. Moreover if \( \mathcal{M}/\mathcal{F} \) is compact, \( V \) must have a maximum in the interior and this is the only critical point in the interior since the interior of \( \mathcal{M}/\mathcal{F} \) is convex (cf. Section 2.2). \( \square \)
Remark 5.2. The conditions in Proposition 5.1 are easily seen to be satisfied in the following situations:

- $M$ is compact with $\text{Ric}_M > 0$ and $(M, \mathcal{F})$ is hyperpolar.
- $M$ is compact with $\text{sec}_M > 0$ and $(M, \mathcal{F})$ is polar.

Furthermore, Proposition 4.4 and Corollary 4.5 show that the condition above is satisfied for factors of type 3.

Recall that, by Theorems 1.18 and 1.20 of [25], the leaves of an isoparametric foliation $(\mathbb{R}^k, \mathcal{F})$ without trivial factors must be compact, and contained in concentric spheres. Furthermore, restriction of $\mathcal{F}$ to each sphere $S$ is still isoparametric, and by Theorem 1.1(2) of [14] there is a unique regular leaf that is minimal in $S$. The following proposition is a generalization of this result.

**Proposition 5.3** (Minimal leaves of polar foliations). Let $(M, \mathcal{F})$ be a polar foliation on a simply connected symmetric space with non-negative curvature, and let $(M, \mathcal{F}) = (M_{-1}, \mathcal{F}_{-1}) \times (M_0, \mathcal{F}_0) \times \prod_i (M_i, \mathcal{F}_i)$ its decomposition into factors. Then:

1. All leaves of $\mathcal{F}_{-1}$ are minimal.
2. $(M_0, \mathcal{F}_0)$ has either one or no minimal regular leaves, depending on whether $M_0$ is compact or not.
3. Each of $(M_i, \mathcal{F}_i)$ has exactly one minimal regular leaf.

**Proof.** The first point is obvious. By Proposition 3.8, $(M_0, \mathcal{F}_0)$ splits as a product of hyperpolar foliations $(M_0^c, \mathcal{F}_0^c) \times (\mathbb{R}^k, \mathcal{F}_0^w)$, where $M_0^c$ is compact and for every $\mathcal{F}_0^c$-horizontal direction $x$ in $(M_0^c, \mathcal{F}_0^c)$, $\tr_{\mathcal{F}} R > 0$. If $k = 0$ then $M_0 = M_0^c$ has a minimal leaf by Proposition 5.1. If $k > 0$, then it is well known that the leaves of $(\mathbb{R}^k, \mathcal{F}_0^w)$ are not minimal, and so neither are the leaves of $M_0$. Finally, point 3) follows from Proposition 5.1 and Remark 5.2 since factors of type 3 have positive Ricci curvature.

**Proposition 5.4.** Let $(M, \mathcal{F})$ be a polar foliation without trivial factors on a compact simply connected symmetric space. Then the mean curvature flow $f(t, \cdot) : L \rightarrow M$ starting at a regular leaf $L$ is an ancient solution. Furthermore:

1. For all $t < 0$, $L_t := f(t, L)$ is a regular leaf of $\mathcal{F}$.
2. The limit $\lim_{t \rightarrow -\infty} L_t$ exists, and it is the unique minimal regular leaf of $(M, \mathcal{F})$.

**Proof.** Since by Theorem 1.4 the mean curvature vector of the regular leaves of $(M, \mathcal{F})$ is parallel, it projects to a vector field $H_\ast$ on the regular part of $M/\mathcal{F}$. Furthermore the mean curvature flow $f(t, \cdot)$ starting from a regular leaf $L = \pi^{-1}(p_\ast)$ of $\mathcal{F}$ flows through regular leaves of $\mathcal{F}$, and in fact $L_t := f(t, L) = \pi^{-1}(\gamma_\ast(t))$ where $\gamma_\ast$ is the integral curve of $H_\ast$ with $\gamma_\ast(0) = p_\ast$.

Since the ambient manifold $M$ is compact, so are the leaves and the leaf space (cf. Section 2.2). In this cases, we analyze the integral curves $c_\ast(t, \cdot) = \pi(f(t, \cdot))$ of the vector field $H_\ast$ on the manifold part of $M/\mathcal{F}$ for $t < 0$. In particular, studying the behaviour of $f(t, \cdot)$ as $t \rightarrow -\infty$ reduces to studying the integral curves of $-H_\ast$ for positive times. By Proposition 3.3 of [3], as $t \rightarrow -\infty$ the flow $f(t, \cdot)$ escapes small tubular neighbourhoods of any singular leaf. By compactness, there is a tubular neighborhood $U$ of the singular set of $M/\mathcal{F}$ such that the integral curves of $-H_\ast$ starting from $U^c$ stay in $U^c$ for all time $t > 0$. Furthermore, the function
\[ V : M/F \to \mathbb{R}, \quad V(p_\ast) = \text{vol}(\pi^{-1}(p_\ast))^{1/n} \]
from Proposition 5.1 is a Lyapunov function for the flow of \(-H_\ast\). In particular, the flow has a unique global attractor, that is the projection of the unique minimal regular leaf of \(F\). \(\Box\)

Theorem 1.3 follows from Propositions 5.3 and 5.4.

**Remark 5.5.** By Theorem 1.2 any polar foliation on a symmetric space with non-negative curvature splits isometrically as a product of a polar foliation on a compact symmetric space, and an isoparametric foliation in Euclidean space. The behaviour of solutions of the mean curvature flow starting from isoparametric submanifolds in Euclidean space was studied in [13] and [14], where in particular it was shown that solutions to the mean curvature flow with isoparametric leaves as initial data are ancient. Moreover, since all leaves in a trivial factor are minimal, MCF of such leaves are stationary. Together with Proposition 5.4 one has a complete picture of the mean curvature flow with regular leaves as initial data in a polar foliation on complete simply connected symmetric spaces with non-negative curvature. In particular, we have:

**Corollary 5.6.** Let \((M, F)\) be a polar foliation on a complete simply connected symmetric space with non-negative curvature. Then the solutions of the mean curvature flow starting at regular leaves of \(F\) are ancient.

**Appendix A. Splitting theorem for factors of type 2**

Let \((M, F)\) be a factor of type 2, that is, a hyperpolar foliation on a simply connected symmetric space with nonnegative curvature, without trivial factors. The goal of this appendix is to prove Proposition 3.4, that is, whenever the action of the Weyl group \(W\) on the section \(\Sigma\) of \((M, F)\) splits as a product action of \(W = W_1 \times W_2\) on \(\Sigma = \Sigma_1 \times \Sigma_2\), then the foliation splits accordingly. Most arguments are similar to the ones in Ewert’s paper [8], who treated the case of \(M\) compact. In our case, we need to show that having a Euclidean factor does not change the result.

Write \(M = M_c \times \mathbb{R}^k = (G_c/K) \times \mathbb{R}^k\) where \(M_c\) is compact simply connected, \(G_c\) is a compact Lie group and \(K\) is a closed Lie subgroup of \(G\). The choice of \(G_c\) is not unique, and the only property important for us is that there is a bi-invariant metric on \(G_c\) such that \(G_c \to M_c\) is a Riemannian submersion with minimal fibers, cf. Corollary 3.5 in [11]. We choose \(G_c\) as follows: we split \(M_c\) into a product \(M_c = \prod_{i=1}^r M_i\) of irreducible factors. Each factor \(M_i\) is either a simply connected simple Lie group, or the quotient \(G_i/K_i\) of a simply connected simple Lie group \(G_i\). We then choose \(G_c = \prod_{i=1}^r H_i\) where \(H_i = M_i\) if \(M_i\) is a Lie group, otherwise \(H_i = G_i\).

Finally, let \(V = V_c \times \mathbb{R}^k\) where \(V_c = H^0([0, 1], g_c)\) (cf. Section 2.5). The projection maps \(V_c \overset{\psi_c}{\to} G_c \overset{\alpha_c}{\to} M_c\) induce corresponding maps
\[ V \overset{\psi}{\to} G \overset{\rho}{\to} M, \quad \varphi = \rho \circ \psi : V \to M \]
and by Theorem 6.5 in [11] one can lift the hyperpolar foliation \((M, F)\) to a hyperpolar foliation \((G, \tilde{F})\) and an isoparametric foliation \((V, \tilde{F})\), whose sections \(\Sigma \subset G, \tilde{\Sigma} \subset V\) are the horizontal lifts of the sections \(\Sigma\) in \(M\), and with respect to the identifications \(\rho : \tilde{\Sigma} \to \Sigma, \varphi : \tilde{\Sigma} \to \Sigma\) the Weyl group actions coincide.
A.1. Notation. In what follows, we will use the following notation: given a leaf $L \in \mathcal{F}$, we will denote $\tilde{L} = \rho^{-1}(L) \subseteq G$ and $\tilde{L} = \varphi^{-1}(L) \in \tilde{\mathcal{F}}$ the corresponding leaves in $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}$. It will be convenient to assume that the regular leaf $\tilde{L}$ we consider, passes through the origin of $V$ – we can always arrange so by possibly translating the foliation. Given a point $p \in M$, we will let $\tilde{p} \in G, \tilde{p} \in V$ denote points in the corresponding preimages of $p$. We will also use letters as $g,h$ to indicate elements of $G$. By abuse of notation, any of their preimage in $V$ will be denoted $\tilde{g}, \tilde{h}$ respectively.

Finally, given a vector $x \in T_p M_c$, we will let $\tilde{x} \in T_{\tilde{p}} G_c, \tilde{x} \in T_{\tilde{p}} V_c$ denote the horizontal lifts of $x$.

**Proposition A.1.** Let $(M, \mathcal{F})$ be a factor of type 2. Then the preimage $\tilde{L} = \varphi^{-1}(L) \subseteq V$ of a principal leaf $L \in \mathcal{F}$ is full in $V$, i.e. at each point $\tilde{p} \in \tilde{L}$ the normal space $\nu_{\tilde{p}} \tilde{L}$ is spanned by the curvature normals.

**Proof.** Suppose by contradiction that there is a normal vector $\tilde{x}$ to $\tilde{L}$ perpendicular to all curvature normals, that is, such that the shape operator $S_{\tilde{x}} \equiv 0$. Then every holonomy Jacobi field along the geodesic $\tilde{\gamma}(t)$ with $\tilde{\gamma}'(0) = \tilde{x}$ is parallel, and in particular the projection $\gamma_\ast(t) = \pi(\varphi(\tilde{\gamma}(t)))$ of $\tilde{\gamma}(t)$ onto the leaf space $M/\mathcal{F}$ never meets the singular part of the quotient.

We claim that $\gamma_\ast$ is a line. In fact, given any two points $p^\ast, q^\ast \in M/\mathcal{F}$, recall that a length minimizing geodesic between $p^\ast$ and $q^\ast$ exists (because the universal cover $\tilde{M}/\tilde{\mathcal{F}}$ of $M/\mathcal{F}$, being isometric to the section of $\mathcal{F}$, is a complete manifold) and it is an orbifold geodesic. Because $M/\mathcal{F}$ is flat hence with empty cut locus, there are as many orbifold geodesics between $p^\ast$ and $q^\ast$ as there are elements in the Weyl group $W$: fixing preimages $\tilde{p}^\ast$ and $\tilde{q}^\ast$ of $p^\ast$ and $q^\ast$ in the same Weyl chamber, for any $c \in W$ define the orbifold geodesic $\gamma_c$ as the projection to $M/\mathcal{F}$ of the segment from $\tilde{p}^\ast$ to $c \cdot \tilde{q}^\ast$. Of these orbifold geodesics, only $\gamma_c$ does not intersect any wall. Since the conjugacy class of the isotropy group does not change along a length minimizing geodesic in the orbifold (cf. [6], Prop. 7) it follows that there is only one length minimizing geodesic between $p^\ast$ and $q^\ast$, and it coincides with the unique orbifold geodesic which stays in the manifold part. Since $\gamma_\ast$ is entirely contained in the manifold part of $M/\mathcal{F}$, then it must minimize the length of any two points in it, hence it is a line.

By the splitting theorem for Riemannian orbifolds ([6], Theorem 1) the quotient splits isometrically as $Q \times \mathbb{R}$ for some flat Riemannian orbifold $Q$. Letting $L = \varphi(\tilde{L})$ and, for any $p \in L \subseteq M$, let $\gamma_p(t)$ be the horizontal geodesic in $M$ from $p$ projecting to $\gamma_c$. Each of them is a line: In fact, fixing one such $\gamma_p$, then for any two points $\gamma_p(t_1), \gamma_p(t_2)$ one has

$$\ell(\gamma_p|_{[t_1,t_2]}) = d(\gamma(t_1), \gamma(t_2)) \geq d(p, \gamma(t_1), \pi(\gamma(t_2))) = \ell(\gamma^\ast|_{[t_1,t_2]}) = \ell(\gamma_p|_{[t_1,t_2]})$$

hence all inequalities are in fact equalities, and from the first one it follows that $\gamma_p$ minimizes the distance between any two points in it.

One then considers the sets

$$C^\pm_p = \left( \bigcap_{t > 0} M \setminus B_t(\gamma_p(\pm t)) \right), \quad C^\pm = \bigcap_{p \in L} C^\pm_p, \quad C = C^+ \cap C^-$$

where $B_t(q)$ denotes the open ball of radius $t$ around $q$. Since $\pi : M \to M/\mathcal{F}$ is a submetry (that is, metric balls project to metric balls of the same radius) it follows
that for any \( t \in \mathbb{R} \),
\[
\bigcap_{p \in L} \left( M \setminus B_t(\gamma_p(t)) \right) = \pi^{-1} \left( M / F \setminus B_t(\gamma_+(t)) \right).
\]

On the one hand, the inclusion \( \supseteq \) is clear. On the other hand, notice that \( \{ \gamma_p(t) \mid p \in L \} = \pi^{-1}(\gamma^+(t)) \), hence if \( q \in \bigcap_{p \in L} \left( M \setminus B_t(\gamma_p(t)) \right) \) this means that \( d(q, \gamma_p(t)) \geq t \) for all \( p \in L \) and thus \( d(q, \pi^{-1}(\gamma^+(t))) \geq t \). In particular, \( d(\pi(q), \gamma^+(t)) = \inf_{q' \in \pi^{-1}(\gamma^+(t))} d(q, q') \geq t \) and \( \pi(q) \in M / F \setminus B_t(\gamma_+(t)) \), thus proving the other inclusion.

It follows in particular that \( C^\pm = \pi^{-1} \left( \bigcap_{t>0} M / F \setminus B_t(\gamma_+(\pm t)) \right) \) and thus
\[
C = \pi^{-1} \left( \bigcap_{t \neq 0} M / F \setminus B_t(\gamma_+(t)) \right) = \pi^{-1}(Q).
\]

In particular, \( C \) is a union of leaves.

On the other hand, since \( M \) has non-negative sectional curvature, by Toponogov Comparison Theorem the sets \( C_p^\pm \) are totally convex (i.e. any geodesic between two points in \( C_p^\pm \) is contained in \( C_p^\pm \)) and thus so are \( C^+, C^-, C \), since they are intersections of totally convex sets.

Since \( C \) is a totally convex set, it is a totally geodesic submanifold, possibly with boundary (cf. Lemma 3.34 of [23]). Furthermore, at each point \( p \in C \), the tangent cone \( T_pC \) is a convex cone in \( T_pM \). Given a point \( p \in C \), on the one hand the leaf \( L \) through \( p \) is contained in \( C \), and thus \( T_pL \subseteq T_pC \). On the other hand, given any horizontal vector \( x \in T_pC \), the whole geodesic \( \exp_p(tx), t \in \mathbb{R} \), projects to an orbifold geodesic in \( M / F \) which is initially tangent to \( Q \), and thus it is contained in \( Q \) for all \( t \). In particular, \( -x \in T_pC \) as well. It then follows that \( T_pC \) is not contained in any half space, and therefore it is a linear subspace of \( T_pM \) which implies that \( p \) is not a boundary point of \( C \). Since \( p \) was arbitrary, it follows that \( C \) has no boundary, and hence it is a totally convex, foliated submanifold of \( M \).

We claim that \( (M, F) = (C, F|_c) \times (\mathbb{R}, \{ \text{pts} \}) \) which gives a contradiction with the fact that, since \( (M, F) \) is of type 2, it does not contain trivial factors.

Notice first that since \( C \) projects to \( Q \), it is a totally geodesic hypersurface of \( M \). Furthermore, since any geodesic \( \gamma_p \) as above is a line in \( M \), by the Splitting Theorem it follows that \( M = C' \times \mathbb{R} \) for some totally geodesic hypersurface \( C' \) through \( p \), perpendicular to \( \gamma_p \). Since \( C, C' \) are both totally geodesic hypersurfaces through \( p \) with the same tangent space, it follows that \( C = C' \) and \( M = C \times \mathbb{R} \).

Since leaves of polar foliations are given by images of parallel normal vector fields under the normal exponential map of one regular leaf it follows that the foliation on any other \( C \times \{ t \} \) is simply obtained by shifting the leaves on \( C \) by \( t \). \( \square \)

Let \( M = M_c \times \mathbb{R}^k \) with \( M_c \) compact. Given a point \( p \in M \) we will denote by \( p_c \) and \( p_\mathbb{R} \) the projection of \( p \) onto the compact factor \( M_c \) and the Euclidean factor \( \mathbb{R}^k \), respectively. Similarly, we will denote a tangent vector \( x \) in \( M \) as \( x_c + x_\mathbb{R} \), where \( x_c \in TM_c \) and \( x_\mathbb{R} \in T\mathbb{R}^k \). Clearly the horizontal lifts of \( x_c + x_\mathbb{R} \) in \( G \) and \( V \) are \( \tilde{x}_c + x_\mathbb{R} \) and \( \hat{x}_c + x_\mathbb{R} \) respectively.

Fix a regular leaf \( L \in \mathcal{F} \). Up to translations in \( G \), we can assume that \( \tilde{L} = \rho^{-1}(L) \) passes through the identity \( e \in G \), and thus \( \tilde{L} \) contains the origin of \( V \).
Recall from [27] that, for a point $\tilde{p} = (\tilde{p}_c, \tilde{p}_e) \in V$ with $p = \varphi(\tilde{p})$ and a vector $x = x_c + x_e \in T_p M$, its horizontal lift to $\tilde{p}$ is $\tilde{x}_c + x_e$ with

\[(2) \quad \tilde{x}_c(t) = \text{Ad}_{\tilde{u}(t)}(\tilde{x}_c\tilde{p}_c^{-1}), \quad \text{where } \tilde{u}(0) = e, \quad \tilde{u}'(t) = -\tilde{p}_c(t)\tilde{u}(t) \quad \forall t \in [0, 1].\]

Here $\tilde{u} : [0, 1] \to G_c$ and $\tilde{x}_c + x_e$ is the horizontal lift of $x$ to $T_{\tilde{p}} G$ with $\hat{p} = (\hat{p}_c, \hat{p}_e) = \psi(\tilde{p})$. Notice $\tilde{u}(t) = E_c(t)^{-1}$ where $E_c \in P(G_c, e \times G_c)$ is the path corresponding to $\hat{p}_c$ via the identification in Section 2.5. In particular $\tilde{u}(1) = \tilde{p}_c^{-1}$.

Define $V' = \text{span}\{x \mid x \in \nu L\}$ the span of the horizontal lifts of all vectors $x \in \nu L$ at all points in $\hat{L}$, and let $V_0 := (V')^\perp$. Furthermore, define the following subspace $\mathfrak{h} \subseteq \mathfrak{g}$:

$$\mathfrak{h} := \text{span}\{\text{Ad}_g(\hat{x}\hat{p}^{-1}) \mid \hat{p} \in \hat{L}, \hat{x} \in \nu_p(\hat{L}), g \in G\} = \text{span}\{\text{Ad}_{g_c}(\tilde{x}_c\tilde{p}_c^{-1}) + x_e \mid (\tilde{x}_c, x_e) \in \nu_{\tilde{p}}(\tilde{L}), g_c \in G_c\}.$$

This is an ideal ([8], Lemma 3.2) and in particular, since $\mathfrak{g}_c$ is centerless, it splits as $\mathfrak{h}_c \oplus \mathfrak{h}_e$ where $\mathfrak{h}_e \subseteq \mathbb{R}^k$ and $\mathfrak{h}_c = \text{span}\{\text{Ad}_{g_c}(\tilde{x}_c\tilde{p}_c^{-1})\} \subseteq \mathfrak{g}_c$. Let $\mathfrak{h}_c^\perp$ and $\mathfrak{h}_e^\perp$ the orthogonal complements of $\mathfrak{h}_c$ in $\mathfrak{g}_c$ and of $\mathfrak{h}_e$ in $\mathbb{R}^k$, respectively. Finally, let $\mathfrak{h}^\perp = \mathfrak{h}_c^\perp \oplus \mathfrak{h}_e^\perp$.

**Lemma A.2.** $V' = H^0([0, 1], \mathfrak{h}_c) \oplus \mathfrak{h}_e$ and $V_0 = H^0([0, 1], \mathfrak{h}_c^\perp) \oplus \mathfrak{h}_e^\perp$.

**Proof.** We prove that $V' = H^0([0, 1], \mathfrak{h}_c) \oplus \mathfrak{h}_e$, from which it trivially follows that $V_0 = H^0([0, 1], \mathfrak{h}_c^\perp) \oplus \mathfrak{h}_e^\perp$. By definition $V'$ is the closure of the the space spanned by vectors of the form $\tilde{x}_c + x_e$ with $\tilde{x}_c(t) + x_e \in \mathfrak{h}$ for all $t$, hence $V' \subseteq H^0([0, 1], \mathfrak{h}_c) \oplus \mathfrak{h}_e$.

Next we prove that $\mathfrak{h}_c \subseteq V'$: Fix an element $\hat{x} = \tilde{x}_c + x_e \in \nu_{\tilde{p}} \hat{L}$. The space $\mathfrak{h}_{\hat{x}} = \text{span}\{\text{Ad}_g(\hat{x}\hat{p}^{-1}) \mid g \in G\} = \text{span}\{\text{Ad}_{g_c}(\tilde{x}_c\tilde{p}_c^{-1}) + x_e \mid g_c \in G_c\}$ is an ideal of $\mathfrak{h}$.

Since $G_c$ is compact, the element $\int_{G_c} \text{Ad}_h(\tilde{x}_c\tilde{p}_c^{-1}) dh \in \mathfrak{g}_c$ is well-defined and $\text{Ad}_{G_c}$-invariant, hence it belongs to the center of $\mathfrak{g}_c$. Since $\mathfrak{g}_c$ is centerless, the integral is zero. Thus for any $\epsilon > 0$ it is possible to find an integer $C = C(\epsilon)$, elements $g_1, \ldots, g_C \in G_c$, and positive coefficients $a_1, \ldots, a_C \in \mathbb{R}$ such that $\sum_{i=1}^C a_i \text{Ad}_{g_i}(\tilde{x}_c\tilde{p}_c^{-1})$ is $\epsilon$-close to $\int_{G_c} \text{Ad}_h(\tilde{x}_c\tilde{p}_c^{-1}) dh = 0$ in $\mathfrak{g}_c$, i.e.

$$\left\| \sum_{i=1}^C a_i \text{Ad}_{g_i}(\tilde{x}_c\tilde{p}_c^{-1}) \right\| < \epsilon, \quad A := \sum a_i = \text{volume}(G_c).$$

Thus,

$$\left\| x_e - \sum_{i=1}^C \frac{a_i}{A} \text{Ad}_{g_i}(\tilde{x}_c\tilde{p}_c^{-1}) \right\| < \epsilon.$$ 

Let $\tilde{u}_i \in P(G_c, e \times G_c)$ be paths from $e$ to $\tilde{p}_c^{-1}$ such that $\tilde{u}_i|_{[\epsilon, 1-\epsilon]} \equiv g_i$, and let

$$\tilde{x}_i = \text{Ad}_{\tilde{u}_i(t)}(\tilde{x}_c\tilde{p}_c^{-1}) + x_e \in V'$$
(notice that each $\hat{x}_i$ is a horizontal lift of $\hat{x} \in \nu_\rho \hat{L}$). Then

$$\left\| \sum_i \frac{a_i}{A} \hat{x}_i - x_e \right\|_V^2 = \int_0^1 \left\| \sum_i \frac{a_i}{A} \text{Ad}_{\hat{u}_i(t)}(\hat{x}_e \hat{p}_e^{-1}) \right\|^2 dt$$

$$\leq \left( \int_0^\epsilon + \int_1^{1-\epsilon} \left\| \sum_i \frac{a_i}{A} \text{Ad}_{\hat{u}_i(t)}(\hat{x}_e \hat{p}_e^{-1}) \right\|^2 dt \right) + \int_\epsilon^{1-\epsilon} \left\| \sum_i \frac{a_i}{A} \text{Ad}_{\hat{g}_i}(\hat{x}_e \hat{p}_e^{-1}) \right\|^2 dt$$

$$\leq 2\epsilon \left\| \hat{x}_e \hat{p}_e^{-1} \right\|^2 + \frac{1-2\epsilon}{A^2}.\epsilon^2.$$

Since $\epsilon$ can be taken arbitrarily small, it follows that $x_e \in V'$ whenever $\hat{x} = \hat{x}_e + x_e \in \nu L$. In particular, $\mathfrak{h}_e \subseteq V'$ and, for any $\hat{x}_e + x_e \in \nu_\rho L$ and any $\hat{u} \in P(G_c,e \times G_c)$ with $\hat{u}(1) = \hat{p}^{-1}$, the element $\hat{x}_e = \text{Ad}_{\hat{u}}(\hat{x}_e \hat{p}_e^{-1})$ belongs to $V'$.

We now prove that $H^0([0,1], \mathfrak{h}_c) \subseteq V'$. To do so, recall that $H^0([0,1], \mathfrak{h}_c)$ contains a dense subspace spanned by functions

$$y(t) = y_{t_0,v}(t) = \begin{cases} 0 & t \leq t_0 \\ v & t > t_0 \end{cases}$$

for any $v \in \mathfrak{h}_c$ and $t_0 \in [0,1]$. In particular, it is enough to show that any element $y_{t_0,v}(t)$ can be approximated arbitrarily well, in the $L^2$ norm, by some element in $V'$. Of course, it is enough to consider elements $y_{t_0,v}(t)$ with $v$ in some generating set of $\mathfrak{h}_c$.

Fixing $g_c \in G_c$, $\hat{x}_e \in \nu_\rho \hat{L}$ and $t_0 \in [0,1]$ we want to approximate $y = y_{t_0,v}$ where $v = \text{Ad}_{g_c}(\hat{x}_c \hat{p}_c^{-1})$. Again consider the equation $\| \sum_{i=1}^C a_i \text{Ad}_{g_i}(\hat{x}_c \hat{p}_c^{-1}) \| < \epsilon$. For $i = 1, \ldots, C$, let $\hat{x}_i(t) = \text{Ad}_{\hat{u}_i(t)}(\hat{x}_c \hat{p}_c^{-1}) \in V'$ with $\hat{u}_i \in P(G_c,e \times G_c)$ such that $\hat{u}_1(1) = \hat{p}^{-1}$, $\hat{u}_i|_{(t_0,\epsilon)} = g_i$ and $\hat{u}_i|_{(t_0,1-\epsilon)} = g_c$. Finally, define $\tilde{y} = \sum_{i=1}^C \frac{a_i}{A} \hat{x}_i \in V'$. Then

$$\| y - \tilde{y} \|_V^2 = \int_0^{t_0} \left\| y(t) - \sum_{i=1}^C a_i \hat{x}_i(t) \right\|^2 dt + \int_{t_0}^{1-\epsilon} \left\| y(t) - \sum_{i=1}^C a_i \hat{x}_i(t) \right\|^2 dt$$

$$= \left( \int_0^\epsilon \left\| \sum_{i=1}^C \frac{a_i}{A} \hat{x}_i(t) \right\|^2 dt \right) + \int_{t_0}^{1-\epsilon} \left\| \sum_{i=1}^C \frac{a_i}{A} \text{Ad}_{g_i}(\hat{x}_c \hat{p}_c^{-1}) \right\|^2 dt$$

$$\leq 2\epsilon \left\| \hat{x}_c \hat{p}_c^{-1} \right\|^2 + \frac{1-2\epsilon}{A^2} \epsilon^2 + 8\epsilon \left\| \hat{x}_c \hat{p}_c^{-1} \right\|^2.$$

This can be made arbitrarily small, hence every $y \in H^0([0,1], \mathfrak{h}_c)$ locally constant belongs to $V'$, and thus $H^0([0,1], \mathfrak{h}_c) \subseteq V'$.

\begin{remark}
It seems that the proof of Lemma 3.3 in [8], to which Lemma [A.2] corresponds, contained some gaps. First it assumed that given an element $\alpha(t) \in V$ and a function $\hat{u}(t) \in P(G_c,e \times G_c)$ with $\hat{u}(1) = \hat{p}^{-1}$ the function $\varphi(t) = \langle \alpha(t), \text{Ad}_{\hat{u}(t)}(\hat{x}_c \hat{p}_c^{-1}) \rangle$ was continuous, even though $\alpha(t)$ is only assumed to
\end{remark}
be $L^2$. Secondly, it used the wrong identity $\langle \alpha(t), \text{Ad}_{\tilde{x}(\lambda(t))}(\tilde{x}^{-1}) \rangle = \varphi(\lambda(t))$ where $\lambda : [0, 1] \to [0, 1]$ is some continuous, piecewise linear function.

For these reasons, the proof here takes a rather different, more direct approach.

Since $\mathfrak{h}, \mathfrak{h}^\perp$ are ideals with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ and $G$ is simply connected, it follows $G = H \times H^\perp$ where $H, H^\perp$ are the subgroups of $G$ with Lie algebras $\mathfrak{h}, \mathfrak{h}^\perp$. Furthermore, the map $\psi : V \to G$ splits as a product $\psi_H \times \psi_{H^\perp}$, where $\psi_H = \psi|_{H^\psi([0, 1], h_\mathfrak{e}) \oplus \mathfrak{h}_s} = \psi|_V$ and, similarly, $\psi_{H^\perp} = \psi|_{V_0}$.

**Lemma A.4.** If $(M, F)$ is a factor of type 2, then $V_0 = 0$.

**Proof.** By Lemma 3.1 of [10], if $V_0 \neq 0$ then $\tilde{L}$ splits isometrically as $\tilde{L}' \times V_0$, with $\tilde{L}' = \tilde{L} \cap V'$. In particular,

$$\tilde{L} = \psi(\tilde{L}) = \psi(\tilde{L}' \times V_0) = \psi_H(\tilde{L}') \times \psi_{H^\perp}(V_0) = \psi_H(\tilde{L}') \times H^\perp.$$ 

Therefore, the regular leaf $\tilde{L}$ splits off the factor $H^\perp$ and, since the whole foliation is determined by a unique regular leaf, the whole foliation $(G, \tilde{F})$ splits as $(H, F|_H) \times H^\perp$. Recall that we chose $G$ (cf. the beginning of Section A) to be a product $G = (\prod_i H_i) \times \mathbb{R}^k$ where $H_i$ are compact, simple, simply connected Lie groups, and each irreducible factor of $G$ is the preimage of an irreducible factor of $M$. In particular, $\rho(H) = \rho(H_c \times H_\mathfrak{e}) = \rho_c(H_c) \times H_\mathfrak{e}$ is a direct factor of $M$, call it $N$, and the same holds for $N^\perp := \rho(H^\perp)$. In particular, $M$ splits as $M = N \times N^\perp$ and $L$ splits as the product of $\rho(\psi_H(\tilde{L}'))$ and $N^\perp$. Again because the foliation is uniquely determined by a single leaf, it follows that the whole foliation splits as $(N, F|_N) \times N^\perp$. However since $F$ is of type 2, it does not contain trivial factors, contradicting the fact that $N^\perp = \rho(H^\perp) = \varphi(V_0)$ is not a point. \hfill $\square$

Define now subspaces $V_1, V_2 \subseteq V$ as follows: Along $L \subset M$ define $\Sigma = \Sigma_1 \times \Sigma_2$ the splitting of $\Sigma$ as in Lemma 3.3 and let $P_1, P_2$ the subbundles of $\nu L$ given by taking the parallel transport of $T_p \Sigma_1$ and $T_p \Sigma_2$ respectively. Finally for $i = 1, 2$ define

$$V_i = \text{span}\{ \tilde{x} | x \in P_i\}.$$ 

Clearly $V_1 \oplus V_2 = V'$ which, by the previous lemma, equals $V$. Furthermore, by Lemma 3.5 of [10], one has $V_1 \perp V_2$. Therefore $V = V_1 \times V_2$ and by Corollary 3.12 of [10] the leaf $\tilde{L}$ splits accordingly as $\tilde{L}_1 \times \tilde{L}_2$ with $\tilde{L}_i = \tilde{L} \cap V_i$. Again, this implies that $(V, \tilde{F})$ splits as a product $(V_1, \tilde{F}|_{V_1}) \times (V_2, \tilde{F}|_{V_2})$.

**Proposition A.5.** Let $(M, F)$ be a factor of type 2, and $(G, \tilde{F})$ the lift of $F$ to the Lie group $G$. Then the foliation $(G, \tilde{F})$ splits as a product of isoparametric foliations $(G_1, \tilde{F}_1) \times (G_2, \tilde{F}_2)$ where the section of $\tilde{F}_i$ is the lift of the section $\Sigma_i$ from Lemma 3.3.

**Proof.** We proceed along the lines of Section 3 in [8], and we keep using the notation in A.1 For $i = 1, 2$ and $\tilde{p} \in \tilde{L}$, define $\tilde{P}_i(\tilde{p})$, $\tilde{P}_i(\tilde{p})$ the horizontal lifts of $P_i(p)$ to $\tilde{p}$ and $\tilde{p}$ respectively. Clearly $V_i = \text{span}\{ \tilde{x} \in \tilde{P}_i(\tilde{p}) | \tilde{p} \in L\}$.

Take two points $\tilde{x}, \tilde{y} \in \tilde{L}$, and two vectors $\tilde{x}_c + x_c \in \tilde{P}_1(\tilde{p})$, $\tilde{y}_c + y_c \in \tilde{P}_2(\tilde{q})$. Recall (cf. Equation 23) that $\tilde{u}(t)$ equals $\tilde{z}_c(t) = \text{Ad}_{\tilde{u}(t)}(\tilde{x}_c \tilde{p}_c^{-1})$, where $\tilde{u}(t)$ satisfies $\tilde{u}(0) = e$ and $\tilde{u}'(t) = -\tilde{p}_c(t)\tilde{u}(t)$, and an analogous formula holds for $\tilde{y}_c$. 
The fact that $V_1$, $V_2$ are perpendicular, implies that $\hat{P}_1(\hat{p}) \perp \hat{P}_2(\hat{q})$ for any $p, q \in \hat{L}$, that is
\begin{equation}
\langle x_e, y_e \rangle + \int_0^1 \langle \text{Ad}_{\eta(t)}(\hat{x}_c\hat{p}_c^{-1}), \text{Ad}_{\eta(t)}(\hat{y}_c\hat{q}_c^{-1}) \rangle \, dt = 0.
\end{equation}

Using a reparametrization technique completely analogous to the one used by Ewert in the proofs of Lemma 3.7 in [3], Equation (3) gives the following pointwise condition:
\begin{equation}
\langle x_e, y_e \rangle + \langle \text{Ad}_{h_1}(\hat{x}_c\hat{p}_c^{-1}), \text{Ad}_{h_2}(\hat{y}_c\hat{q}_c^{-1}) \rangle = 0 \quad \forall h_1, h_2 \in G_c.
\end{equation}

Equation (4) can be interpreted by saying that for every $\hat{x} \in \hat{P}_1(\hat{p})$, $\hat{y} \in \hat{P}_2(\hat{q})$, it holds
\begin{equation}
\langle \text{Ad}_{h_1}(\hat{x}\hat{p}^{-1}), \text{Ad}_{h_2}(\hat{y}\hat{q}^{-1}) \rangle = 0 \quad \forall h_1, h_2 \in G
\end{equation}
which implies that the two subspaces
\[ g_i := \text{span}\{\text{Ad}_{\eta}(\hat{x}\hat{p}^{-1}) \mid \hat{p} \in \hat{L}, \hat{x} \in \hat{P}_1(\hat{p}), h \in G\}
\]
are mutually perpendicular to one another. Furthermore, since by definition $\text{Ad}_{h}(g_i) \subseteq g_i$ for all $h \in G$, it follows that $g_i$ are ideals of $g$. In particular, $g_i$ is a sum of simple ideals of $g_c$, plus some abelian subspace of $\mathbb{R}^k$, hence $g_i = (g_i)_c \oplus (g_i)_e$ where $(g_i)_c = g_i \cap g_c$ and $(g_i)_e = g_i \cap \mathbb{R}^k$.

Given $\hat{x} = \hat{x}_c(t) + x_e \in \hat{P}_1(\hat{p})$, then its projection to $G$ is $\hat{x} = \hat{x}_c + x_e \in \hat{P}_1(\hat{p})$, and in particular $\hat{x}\hat{p}^{-1} = \hat{x}_c\hat{p}_c^{-1} + x_e \in (g_i)_c \oplus (g_i)_e$. It follows that $x_e \in (g_i)_e$ and for all $t$, $\hat{x}(t) = \text{Ad}_{\eta(t)}(\hat{x}_c\hat{p}_c^{-1}) \in (g_i)_e$, hence $\hat{x} = \hat{x}_c(t) + x_e \in H^0([0,1], (g_i)_c) \oplus (g_i)_e$.

In other words, $V_i \subseteq H^0([0,1], (g_i)_c) \oplus (g_i)_e$ for $i = 1, 2$ but, since $V = V_1 \oplus V_2$ and $V = H^0([0,1], g_c) \oplus g_e = (H^0([0,1], (g_i)_c) \oplus (g_i)_e) \oplus (H^0([0,1], (g_2)_c) \oplus (g_2)_e)$.

It follows that $V_i = H^0([0,1], (g_i)_c) \oplus (g_i)_e$. In particular, letting $G = G_1 \times G_2$ the splitting of $G$ corresponding to $g = g_1 \oplus g_2$, it follows that $\psi(V_i) = G_i$ and $\psi|\psi|_{V_1} \times \psi|_{V_2}$. In particular
\[ \hat{L} = \psi(\hat{L}) = \psi|V_1(\hat{L}_1) \times \psi|V_2(\hat{L}_2) \]
gives a splitting of $\hat{L}$ into two factors, contained in $G_1$ and $G_2$ respectively. Once again because the foliation is determined by one regular leaf, it follows that $(G, \hat{F})$ splits as $(G_1, \hat{F}_1) \times (G_2, \hat{F}_2)$. By construction, the sections of $\hat{F}_i$ are $\Sigma_i$, for $i = 1, 2$. \hfill \Box

**Corollary A.6.** Let $(M, \mathcal{F})$ be a factor of type 2, and $(G, \hat{F})$ the lift of $\mathcal{F}$ to the Lie group $G$. Then the splitting $(G, \hat{F}) = (G_1, \hat{F}_1) \times (G_2, \hat{F}_2)$ from Proposition A.2 induces a splitting $(M, \mathcal{F}) = (M_1, \mathcal{F}_1) \times (M_2, \mathcal{F}_2)$ where the section of $\mathcal{F}_i$ is the the section $\Sigma_i$ from Lemma A.3.

**Proof.** Recall that we chose $G$ (cf. the beginning of Section A) to be a product $G = (\prod_i H_i) \times \mathbb{R}^k$ where $H_i$ are compact, simple, simply connected Lie groups, and each irreducible factor of $G$ is the preimage of an irreducible factor of $M$.

In particular, $\rho = \rho_1 \times \rho_2$ where $\rho_i = \rho|_{G_i} : G_i \to M_i := \rho(G_i)$. In particular, $M = M_1 \times M_2$ and
\[ L = \rho(\hat{L}) = \rho(\hat{L}_1 \times \hat{L}_2) = \rho_1(\hat{L}_1) \times \rho_2(\hat{L}_2) \]
and therefore $(M, \mathcal{F})$ splits as $(M_1, \mathcal{F}_1) \times (M_2, \mathcal{F}_2)$. \hfill \Box

This completes the proof of Proposition B.3.
References

[1] M.M. Alexandrino, On polar foliations and fundamental group, Results in Mathematics 60 (2011), no. 1, 213–223.
[2] M.M. Alexandrino, Desingularization of singular Riemannian foliation, Geom. Dedicata 149 (2010), 397–416.
[3] M. Alexandrino and M. Radeschi, Isometries between leaf spaces, Geom. Dedicata 174 (2015), no. 1, 193–201.
[4] M. Alexandrino and M. Radeschi, Mean Curvature Flow of Singular Riemannian Foliations, Journal of geom. anal. 26 (2016), no. 3, 2204–2220.
[5] M. Alexandrino and D. Toeben, Singular Riemannian foliations on simply connected spaces, Diff. Geom. Appl. 24 (2006), 383C397
[6] J.E. Borzellino and S.-H. Zhu, A splitting theorem for orbifolds, Ill. J. Math. 38 (1994), no. 4, 679–691.
[7] Q.-S. Chi, Classification of isoparametric hypersurfaces, preprint.
[8] H. Ewert, A splitting theorem for equifocal submanifolds in simply connected compact symmetric spaces, Proc. Amer. Math. Soc. 126 (1998), no. 8, 2443–2452.
[9] Karsten Grove and Wolfgang Ziller, Polar manifolds and actions, Journal of Fixed Point Theory and Applications 11 (2012), 279–313.
[10] E. Heintze and X. Liu, A splitting theorem for isoparametric submanifolds in Hilbert Space, Journal of Diff. Geom. 45 (1997), 319–335.
[11] E. Heintze, X. Liu and C. Olmos, Isoparametric submanifolds and a Chevalley-type restriction theorem, Integrable systems, geometry, and topology, editor C.L. Terng (2006), ISBN:0-8218-4048-7.
[12] N. Koike, Collapse of the mean curvature flow for equifocal submanifolds, Asian J. Math 15 (2011), no. 1, 101-128.
[13] X. Liu and C.-L. Terng, The mean curvature flow for isoparametric submanifolds, Duke Math. J. 147 (2009), no. 1, 157–179.
[14] X. Liu and C.-L. Terng, Ancient solutions to mean curvature flow for isoparametric submanifolds, preprint arXiv:1911.12535.
[15] A. Lytchak, Geometric resolution of singular Riemannian foliations, Geom. Dedicata 149 (2010), 379–395.
[16] A. Lytchak, Notes on the Jacobi equation, Differential Geom. Appl. 27 (2009), 329–334.
[17] A. Lytchak, Singular Riemannian foliations on spaces without conjugate points, Proceedings of VIII International Colloquium on Differential Geometry (2009), 75–82.
[18] A. Lytchak, Polar foliations of symmetric spaces, Geom. Funct. Anal. 24 (2014), 1298–1315.
[19] A. Lytchak and M. Radeschi, Algebraic nature of singular Riemannian foliations in spheres, J. Reine Ang. Math. (Crelle) 744 (2018), 265–273.
[20] A. Lytchak and G. Thorbergsson, Curvature explosion in quotients and applications, J. Differential Geom., 85 (2010), no. 1, 117–139.
[21] R. Mendes and M. Radeschi, Singular Riemannian foliations and their quadratic basic polynomials, Transf. Groups 25 (2019), no. 1, 251–277.
[22] R. Mendes and M. Radeschi, Laplacian algebras, manifold submetries and the Inverse Invariant Theory Problem, to appear in Geom. Funct. Anal. (GAFA) (2020), preprint arXiv:1908.05796.
[23] W. Meyer, Toponogov’s theorem and applications, lecture notes available online at https://www.math.upenn.edu/~wziller/math660/ToponogovTheorem-Meyer.pdf
[24] M. Radeschi, Singular Riemannian foliations, lecture notes available online at https://www.marcoradeschi.com
[25] C.-L. Terng, Isoparametric submanifolds and their Cozeter groups, J. Differential Geom. 1 (1985), 79–107.
[26] C.-L. Terng, Proper Fredholm Submanifolds of Hilbert Spaces, J. Diff. Geom. 29 (1989), 9–47.
[27] C.-L. Terng and G. Thorbergsson, Submanifold geometry in symmetric spaces, J. Differential Geom. 42 (1995), 665–718.
[28] G. Thorbergsson, A survey on isoparametric hypersurfaces and their generalizations. In: Handbook of Differential Geometry, Vol. I, Chap. 10. Elsevier Science, London (2000).
[29] D. Toeben, Singular Riemannian foliations on nonpositively curved manifolds, Math. Z 255 (2007), 427–436.
[30] D. Toeben, *Parallel Focal Structure and Singular Riemannian Foliations*, Transactions of the Am. Math. Soc. **358** (2006), no. 4, 1677–1704.

[31] L. Verdiani and W. Ziller, *Concavity and rigidity in non-negative curvature*, J. Differential Geom. **97** (2014), 349–375.

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