Characterization of Exponential Distribution through Equidistribution Conditions for Consecutive Maxima

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Abstract: A characterization of the exponential distribution based on equidistribution conditions for maxima of random samples with consecutive sizes \( n - 1 \) and \( n \) for an arbitrary and fixed \( n \geq 3 \) is proved. This solves an open problem stated recently in Arnold and Villasenor [3].

Keywords: characterizations, exponential distribution, order statistics, maxima

1 Introduction

Characterizations of the exponential distribution are abundant. Comprehensive surveys can be found in Ahsanullah and Hamedani [1], Arnold and Huang [2], and Johnson, Kotz, and Balakrishnan [5]. Recently, Arnold and Villasenor [3] obtained a series of characterizations based on random sample of size two. They also identified a list of conjectures for possible extensions of their results to larger samples. In this work we confirm that one of these conjectures is true for a sample of any fixed size \( n \geq 2 \). Note that in Yanev and Chakraborty [8] the case of random sample of size three was considered.

Let \( X_1, X_2, \ldots, X_n, n \geq 2 \) be a random sample from an exponentially distributed parent \( X \). It is known that

\[
\max \{X_1, X_2, \ldots, X_{n-1}\} + \frac{1}{n} X_n \overset{d}{=} \max \{X_1, X_2, \ldots, X_n\},
\]

where \( \overset{d}{=} \) denotes equality in distribution. We write \( X \sim \exp(\lambda) \) if the probability density function (pdf) of \( X \) equals \( f_X(x) = \lambda e^{-\lambda x} I(x > 0) \). Our goal is to prove that (1), under analyticity assumptions on the cumulative distribution function (cdf) \( F \) of \( X \), is a sufficient condition for \( X \) to be exponential.

Theorem Let \( X \) be a non-negative continuous random variable with pdf \( f \). If \( f \) is analytic in a neighborhood of zero and (1) holds true, then \( X \sim \exp(\lambda) \) with some \( \lambda > 0 \).

Wesołowski and Ahsanullah [7] and more recently Castaño-Martínez et al. [4] proved characterizations of probability distributions in the context of random translations. The characterization (1) above can be deduced from their results (see Corollary 1 in Wesołowski and Ahsanullah [7] and Corollary 3 in Castaño-Martínez et al. [4]). However, our proof is different from theirs in not referring to uniqueness results for integral equations. The direct approach we follow may also be used in obtaining some more general results, a possibility which we will explore in the future.

2 Preliminaries

Define for all non-negative integers \( n, i \), and any real number \( x \)

\[
H_{n,i}(x) := \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x-j)^i.
\]
It is known, (e.g., Ruiz [6]) that for all integers \( n \geq 0 \) and all real \( x \)

\[
H_{n,i}(x) = \begin{cases} 
  n! & \text{if } i = n; \\
  0 & \text{if } 0 \leq i \leq n - 1.
\end{cases}
\] (2)

Define \( G_m(x) := F^m(x) f(x) \) for \( m \geq 1 \) and denote by \( g^{(i)}(x) \) for \( i \geq 1 \) the \( i \)th derivative of a function \( g(x) \); \( g^{(0)}(x) := g(x) \).

**Lemma 1** Let \( X \) be a continuous random variable with cdf \( F \) satisfying \( F(0) = 0 \). If for \( 0 \leq r \leq m - 1 \)

\[
f^{(r)}(0) = \left[ \frac{f'(0)}{f(0)} \right]^{r-1} f'(0),
\] (3)

then for \( 0 \leq i \leq 2m \)

\[
G^{(i)}_m(0) = \left[ \frac{f'(0)}{f(0)} \right]^{i-m} f^{m+1}(0) H_{m,i}(m + 1).
\] (4)

**Proof. Case 0 \leq i \leq m - 1.** In this case (2) implies \( H_{m,i}(m + 1) = 0 \). On the other hand, in the left-hand side of (4), we have \( G^{(i)}_m(0) = 0 \) because each term in the expansion of \( G^{(i)}_m(0) \) has a factor \( F(0) = 0 \).

**Case i = m.** From (2) it follows that (4) is equivalent to

\[
G^{(m)}_m(0) = m! f^{m+1}(0).
\] (5)

We shall prove (5) by induction. If \( m = 1 \), then (5) follows from the definition of \( G(x) \) and the assumption \( F(0) = 0 \). Assuming that (5) is true for \( m = k \), we will prove it for \( m = k + 1 \). Since \( G_{k+1}(x) = F(x) G_k(x) \) and \( F(0) = 0 \), we have

\[
G^{(k+1)}_{k+1}(0) = \sum_{j=0}^{k+1} \binom{k+1}{j} F^{(j)}(0) G^{(k+1-j)}_k(0)
\]

\[
= F(0) G^{(k+1)}_k(0) + (k+1) F^{(1)}(0) G^{(k)}_k(0)
\]

\[
= (k+1) f(0) k! f^{k+1}(0)
\]

\[
= (k+1)! f^{k+2}(0),
\]

where we have used that \( G^{(r)}(0) = 0 \) for \( 0 \leq r \leq k - 1 \) and the induction assumption \( G^{(k)}_k(0) = k! f^{k+1}(0) \).

**Case m < i \leq 2m.** Suppose we have proved (4) for \( m = 1, 2, \ldots, k \). We want to prove it for \( m = k + 1 \). Observe that

\[
G^{(i)}_{k+1}(0) = \sum_{j=0}^{k} \binom{k}{j} F^{(j)}(0) G^{(i-j)}_k(0).
\]

Since \( G^{(j)}_k(0) = 0 \) for \( 0 \leq r \leq k - 1 \), making use of (3) and the induction assumption, we obtain

\[
G^{(i)}_{k+1}(0) = \sum_{j=1}^{k} \binom{k}{j} F^{(j-1)}(0) G^{(i-j)}_k(0) + \sum_{j=k+1}^{i} \binom{k}{j} F^{(j-1)}(0) G^{(i-j)}_k(0)
\]

\[
= \sum_{j=1}^{k} \binom{k}{j} \left[ \frac{f'(0)}{f(0)} \right]^{j-2} f'(0) \left[ \frac{f'(0)}{f(0)} \right]^{i-j-k} f^{k+1}(0) H_{k,i-j}(k+1)
\]

\[
= \left[ \frac{f'(0)}{f(0)} \right]^{i-k-1} f^{k+2}(0) \sum_{j=1}^{i} \binom{i}{j} H_{k,i-j}(k+1),
\]

\[
\]
where in the last equality we used that \((2)\) implies \(H_{k,j-r}(k+1) = 0\) for \(j = i + 1, \ldots, k\). Further, we have
\[
\sum_{j=1}^{i} \binom{i}{j} H_{k,j-r}(k+1) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \sum_{j=1}^{i} \binom{i}{j} (k+1-r)^{i-j}
\]
\[
= \sum_{r=0}^{k} (-1)^r \binom{k}{r} \left[(k+2-r)^i - (k+1-r)^i\right]
\]
\[
= (k+2)^i \left[ (k+1)^i + \frac{k}{1} (k+1)^i \right] + \left[ \frac{k}{1} k^i + \frac{k}{2} k^i \right] + \ldots + (-1)^k \left[ \frac{k}{k-1} 2^i + 2^i \right] + (-1)^{k+1}
\]
\[
= (k+1)^i \left[ (k+1)^i + \ldots + (-1)^k \left( \frac{k+1}{k} \right) 2^i + (-1)^{k+1} \right]
\]
\[
= \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (k+2-j)^i = H_{k+1,j}(k+2).
\]

The lemma’s claim follows by induction, taking into account (6).

The identity below may be of independent interest.

**Lemma 2** For any integers \(m \geq 0\) and \(k \geq 0\)
\[
\sum_{j=0}^{m} (k+2)^{m-j} H_{k,j}(k+1) = \sum_{j=0}^{m} \binom{m+1}{j+1} H_{k,j}(k+1).
\]

**Proof.** The left-hand side of (7) equals
\[
\sum_{j=0}^{m} (k+2)^{m-j} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k+1-i)^j = \sum_{j=0}^{m} (-1)^j \binom{k}{j} (k+2)^m \sum_{j=0}^{m} \binom{k+1-j}{k+2-j}^j
\]
\[
= \sum_{i=0}^{k} (-1)^i \binom{k+1}{i} \frac{1}{i+1} \left[ (k+2)^{m+1} - (k+1-i)^{m+1} \right]
\]
\[
= \sum_{i=0}^{k} (-1)^i \binom{k+1}{i} \frac{1}{i+1} \left[ (k+2)^{m+1} - (k+1-i)^{m+1} \right]
\]
\[
= - \frac{(k+2)^{m+1}}{k+1} \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (k+2-r)^{m+1} - \frac{1}{k+1} \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} (k+2-r)^{m+1} - (k+2)^{m+1}
\]
\[
= \frac{1}{k+1} \sum_{i=0}^{k} (-1)^i \binom{k+1}{i} (k+2-i)^{m+1}.
\]

For the right-hand side of (7) we obtain
\[
\sum_{j=0}^{m} \binom{m+1}{j+1} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k+1-i)^j = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \sum_{j=0}^{m} \binom{m+1}{j+1} (k+1-i)^j
\]
\[
= \sum_{i=0}^{k} (-1)^i \binom{k+1}{i} \frac{1}{k+1-i} \sum_{j=0}^{m} \binom{m+1}{j+1} (k+1-i)^j + 1
\]
\[
= \frac{1}{k+1} \sum_{i=0}^{k} \left[ (-1)^i \binom{k+1}{i} \binom{m+1}{r} (k+1-i)^r - 1 \right]
\]
\[
= \frac{1}{k+1} \sum_{i=0}^{k} (-1)^i \binom{k+1}{i} (k+2-i)^{m+1} - \frac{1}{k+1} \sum_{i=0}^{k} (-1)^i \binom{k+1}{i} - (1)^{k+1}
\]
\[
= \frac{1}{k+1} \sum_{i=0}^{k} (-1)^i \binom{k+1}{i} (k+2-i)^{m+1},
\]
which equals (8). The proof of the lemma is complete.

Next lemma (see also Arnold and Villaseñor [3]) will play a crucial role in the proof of the theorem. In private communications, P. Fitzsimmons pointed out to us that the assumption of analyticity of the density function \( f \) is missing in [3].

**Lemma 3** If \( F(0) = 0 \), the pdf \( f \) is analytic in a neighborhood of 0, and

\[
f^{(k)}(0) = \left[ \frac{f'(0)}{f(0)} \right]^{k-1} f'(0), \quad k = 1, 2, \ldots,
\]

(9)

then \( X \sim \exp\{\lambda\} \) for some \( \lambda > 0 \).

**Proof.** For the Maclaurin series of \( f(x) \), we have for \( x > 0 \)

\[
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + \sum_{k=1}^{\infty} \frac{f'(0)}{f(0)} \left[ \frac{1}{k!} f'(0) \right]^{k-1} f(0) \exp \left\{ \frac{f'(0)}{f(0)} x \right\}.
\]

(10)

Since \( f(x) \) is a pdf, we have \( f'(0)/f(0) < 0 \). Denoting \( \lambda = -f'(0)/f(0) > 0 \) and setting the integral of (10) from 0 to \( \infty \) to be 1, we obtain \( \lambda = f(0) \). Therefore, \( f(x) = \lambda e^{-\lambda x} \) for \( x > 0 \), i.e., \( X \sim \exp\{\lambda\} \).

### 3 Proof of the theorem

Equation (1) can be written as

\[
\int_{0}^{x} f_{n/n}(y) f_{max(x_1, \ldots, x_{n-1})}(x-y) dy = n(n-1)f(x) \int_{0}^{x} G_{n-2}(y) dy.
\]

This is equivalent to

\[
\int_{0}^{x} n f(ny)(n-1)F^{n-2}(x-y)f(x-y) dy = n(n-1)f(x) \int_{0}^{x} G_{n-2}(y) dy,
\]

which simplifies to

\[
\int_{0}^{x} f(ny) G_{n-2}(x-y) dy = f(x) \int_{0}^{x} G_{n-2}(y) dy.
\]

(11)

Differentiating the left-hand side of (11) with respect to \( x \), we obtain

\[
\frac{d}{dx} \int_{0}^{x} f(ny) G_{n-2}(x-y) dy = f(nx) G_{n-2}(0) + \int_{0}^{x} f(ny) G_{n-2}'(x-y) dy.
\]

Differentiating the last equation \( 2n - 3 \) times, we obtain

\[
\frac{d^{2n-2}}{dx^{2n-2}} \int_{0}^{x} f(ny) G_{n-2}(x-y) dy = \sum_{i=0}^{2n-3} n^{2n-3-i} f^{(2n-3-i)}(x) G_{n-2}^{(i)}(0) + \int_{0}^{x} f(ny) G_{n-2}^{(2n-2)}(x-y) dy.
\]

(12)

On the other hand, applying to the right-hand side of (11) the Leibnitz product rule of differentiation, we have

\[
\frac{d^{2n-2}}{dx^{2n-2}} \left[ f(x) \int_{0}^{x} G_{n-2}(y) dy \right] = \sum_{i=0}^{2n-3} \binom{2n-2}{i+1} f^{(2n-3-i)}(x) G_{n-2}^{(i)}(x) + f^{(2n-2)}(x) \int_{0}^{x} G_{n-2}(y) dy.
\]

(13)

Therefore, the equation (11), taking into account (12) and (13), becomes

\[
\sum_{i=0}^{2n-3} n^{2n-3-i} f^{(2n-3-i)}(x) G_{n-2}^{(i)}(0) + \int_{0}^{x} f(ny) G_{n-2}^{(2n-2)}(x-y) dy = \sum_{i=0}^{2n-3} \binom{2n-2}{i+1} f^{(2n-3-i)}(x) G_{n-2}^{(i)}(x) + f^{(2n-2)}(x) \int_{0}^{x} G_{n-2}(y) dy.
\]

(14)
Setting \( x = 0 \) and taking into account that \( G^{(i)}_{n-2}(0) = 0 \) for \( 0 \leq i \leq n - 3 \), we obtain that (14) is equivalent to

\[
\sum_{i=n-2}^{2n-4} n^{2n-3-i} f^{(2n-3-i)}(0) G^{(i)}_{n-2}(0) = \sum_{i=n-2}^{2n-4} \binom{2n-2}{i+1} f^{(2n-3-i)}(0) G^{(i)}_{n-2}(0).
\]

For \( i = n - 2 \), we have \( f^{(n-1)}(0) G^{(n-2)}_{n-2}(0) = f^{(n-1)}(0) f^{n-1}(0) (n-2)! \). Thus, the equation above can be written as

\[
\left[n^{n-1} - \binom{2n-2}{n-1}\right] f^{(n-1)}(0) f^{n-1}(0) (n-2)! = \sum_{i=n-1}^{2n-4} \binom{2n-2}{i+1} - n^{2n-3-i} f^{(2n-3-i)}(0) G^{(i)}_{n-2}(0).
\]

(15)

In view of Lemma 3, to complete the proof it suffices to show

\[
f^{(r)}(0) = \left[\frac{f'(0)}{f(0)}\right]^{r-1} f'(0), \quad r = 1, 2, \ldots
\]

(16)

Assume (16) for all \( 1 \leq r \leq n - 2 \). We shall prove it for \( r = n - 1 \), i.e.,

\[
f^{(n-1)}(0) = \left[\frac{f'(0)}{f(0)}\right]^{n-2} f'(0), \quad r = 1, 2, \ldots
\]

(17)

It follows from Lemma 1 with \( m = n - 2 \) that for \( n - 1 \leq i \leq 2n - 4 \)

\[
f^{(2n-3-i)}(0) G^{(i)}_{n-2}(0) = \left[\frac{f'(0)}{f(0)}\right]^{i+2} f^{n-1}(0) H_{n-2,i}(n-1).
\]

(18)

Substituting (18) in the right-hand side of (15) we obtain

\[
\left[n^{n-1} - \binom{2n-2}{n-1}\right] f^{(n-1)}(0) (n-2)! = \left[\frac{f'(0)}{f(0)}\right]^{n-2} f'(0) \sum_{i=n-1}^{2n-4} \binom{2n-2}{i+1} - n^{2n-3-i} H_{n-2,i}(n-1).
\]

To establish (18) we need to prove

\[
\left[n^{n-1} - \binom{2n-2}{n-1}\right] = \sum_{i=n-1}^{2n-4} \binom{2n-2}{i+1} - n^{2n-3-i} H_{n-2,i}(n-1)
\]

or equivalently

\[
\sum_{i=n-2}^{2n-4} n^{2n-3-i} H_{n-2,i}(n-1) = \sum_{i=n-2}^{2n-4} \binom{2n-2}{i+1} H_{n-2,i}(n-1).
\]

(19)

Since (2) implies \( H_{n-2,i}(n-1) = 0 \) for \( 0 \leq i \leq n - 3 \) and for \( i = 2n - 3 \) we have \( n^{2n-3-i} = (2n-2)_{i+1} = 1 \), we obtain that (19) is equivalent to

\[
\sum_{i=0}^{2n-3} n^{2n-3-i} H_{n-2,i}(n-1) = \sum_{i=0}^{2n-3} \binom{2n-2}{i+1} H_{n-2,i}(n-1),
\]

which follows from Lemma 3 with \( m = 2n - 3 \). This completes the induction argument and thus proves (16). Referring to (16) and Lemma 2 we complete the proof of the theorem.

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