The Riemann zeros as energy levels of a Dirac fermion in a potential built from the prime numbers in Rindler spacetime

Germán Sierra

Instituto de Física Teórica, UAM-CSIC, Madrid, Spain

E-mail: german.sierra@uam.es

Received 30 April 2014
Accepted for publication 13 June 2014
Published 29 July 2014

Abstract
We construct a Hamiltonian $H_R$ whose discrete spectrum contains, in a certain limit, the Riemann zeros. $H_R$ is derived from the action of a massless Dirac fermion living in a domain of Rindler spacetime, in $1 + 1$ dimensions, which has a boundary given by the world line of a uniformly accelerated observer. The action contains a sum of delta function potentials that can be viewed as partially reflecting moving mirrors. An appropriate choice of the accelerations of the mirrors, provide primitive periodic orbits that are associated with the prime numbers $p$, whose periods, as measured by the observer’s clock, are $\log p$. Acting on the chiral components of the fermion $\chi$, $H_R$ becomes the Berry–Keating Hamiltonian $\pm(x\hat{p} + \hat{p}x)/2$, where $x$ is identified with the Rindler spatial coordinate and $\hat{p}$ with the conjugate momentum. The delta function potentials give the matching conditions of the fermion wave functions on both sides of the mirrors. There is also a phase shift $e^{i\theta}$ for the reflection of the fermions at the boundary where the observer sits. The eigenvalue problem is solved by transfer matrix methods in the limit where the reflection amplitudes become infinitesimally small. We find that, for generic values of $\theta$, the spectrum is a continuum where the Riemann zeros are missing, as in the adelic Connes model. However, for some values of $\theta$, related to the phase of the zeta function, the Riemann zeros appear as discrete eigenvalues that are immersed in the continuum. We generalize this result to the zeros of Dirichlet $L$-functions, which are associated to primitive characters, that are encoded in the reflection coefficients of the mirrors. Finally, we show that the Hamiltonian associated to the Riemann zeros belongs to class AIII, or chiral GUE, of the Random Matrix Theory.

Keywords: quantum mechanics, quantum chaos, Riemann zeros
1. Introduction

A century ago, Pólya and Hilbert suggested that the imaginary part of the non-trivial zeros of the Riemann zeta function would be the oscillation frequencies of a physical system. The reality of these frequencies will provide a proof of the celebrated Riemann hypothesis (RH) [1], which has deep consequences for the distribution of the prime numbers [2–7]. There is evidence that the Riemann zeros are the eigenvalues of a quantum Hamiltonian, as follows: (i) the Montgomery–Odlyzko law according to which the statistical distribution of the zeros is given, locally, by the Gaussian Unitary Ensemble (GUE) of Random Matrix Theory (RMT) [8–12]; (ii) analogies between trace formulas relating periods of classical trajectories and spectra in Quantum Chaos Theory, and explicit formulas relating prime numbers and Riemann zeros [13–18]; and, (iii) Selberg’s trace formula relating the lengths of the geodesics on a compact Riemann surface with negative curvature and the eigenvalues of the Laplace-Beltrami operator [19, 20]. The picture proposed by Berry in 1986 is that the binomium primes/zeros is similar to the binomium classical/quantum for a dynamical chaotic system [13]. Furthermore, it was conjectured that the classical Hamiltonian underlying the Riemann zeros should be quasi-one dimensional, breaking time reversal symmetry and with isolated periodic orbits whose periods are the logarithm of the prime numbers (see [16, 18] for reviews of this approach, [21, 22] for general references and [23–26] for introductions and the historical background).

In 1999 Berry and Keating (BK) [27, 28], and Connes [29], suggested that a spectral realization of the Riemann zeros could be achieved from the quantization of the simple classical Hamiltonian \( H_\text{cl} = xp \), where \( x \) and \( p \) are the position and momentum of a particle moving in the real line. The \( xp \) Hamiltonian is one dimensional, breaks the time reversal symmetry, is integrable, is not chaotic, with unbounded classical trajectories, and its quantization yields a continuum spectrum [32, 33]. The connection found by Berry, Keating and Connes between \( xp \) and the Riemann zeros was semiclassical and relied on two different regularizations schemes. Berry and Keating introduced a Planck cell regularization of the phase space imposing the constraints \( |x| \geq \ell_x \) and \( |p| \geq \ell_p \) with \( \ell_x \ell_p = 2\pi\hbar \), obtaining semiclassical energies that agree asymptotically with the average location of the Riemann zeros [27]. In Connes’s work there is a cutoff \( \Lambda \), and constraints \( |x| \leq \Lambda \) and \( |p| \leq \Lambda \), such that in the limit \( \Lambda \to \infty \) the semiclassical spectrum becomes a continuum with missing spectral lines associated to the average Riemann zeros [29]. The interpretation of the zeros as missing spectral lines would also explain a mysterious sign problem in the fluctuation term of the number of zeros [29]. The possible connection between \( xp \) and the Riemann zeros has motivated several studies in the past two decades, some of which will be discussed in more detail below [30–53].

The previous semiclassical versions of \( xp \) were formulated as consistent quantum mechanical models in references [38, 43, 44, 47]. Connes’s version was realized in terms of a charge particle moving in a box of size \( \Lambda \times \Lambda \), subject to the action of a uniform perpendicular magnetic field and an electrostatic potential \( xy \) [38]. For strong magnetic fields the dynamics are restricted to the lowest Landau level, where the \( xy \) potential acts effectively as the quantum \( xp \) Hamiltonian. In this realization, the smooth part of the counting formula of the Riemann zeros appears as a shift of the energy levels (that become a continuum in the limit \( \Lambda \to \infty \)) and not as an indication of missing spectral lines. The Landau model with \( xy \)
potential has been used in an analogue model of Hawking radiation in a quantum Hall fluid [54].

On the other hand, the Berry–Keating version of $xp$ was revisited recently using the classical Hamiltonian $H_{clp} = x(p + \ell^2 x/p)$, as defined in the half line $x \geq \ell$ [43, 44] (hereafter denoted as the $xp$ model). The role of the term $\ell^2 x/p$ is to bound the classical trajectories that become periodic, unlike the trajectories of $xp$ that are unbounded. The $xp$ Hamiltonian can be quantized in terms of the operator $\hat{H} = \hat{p}^2 + \ell^2 \hat{p}$, where $\hat{p} = i\hbar/dx$ and $(\hat{\psi})'(x) = i\hbar \int_0^\infty dy, \psi(y)$ and its spectrum agrees asymptotically with the average Riemann zeros provided by $\ell \ell_\pi = \pi \hbar$. A similar result was obtained by Berry and Keating using the Hamiltonian $H_{BK} = (x + \ell^2 x)(p + \ell^2 x/p)$, which is invariant under the exchange $x \leftrightarrow p$ [47]. These two studies have provided a spectral realization of the average Riemann zeros, but not of the actual zeros. From the Quantum Chaos perspective, the reason for this failure lies in the fact that these variants of $xp$ are non-chaotic and do not contain periodic orbits that are related to the prime numbers [47]. More generally, any one dimensional classical and conservative Hamiltonian is integrable and is therefore non-chaotic, which seems to lead to nowhere.

In this work we propose a solution of the puzzle that leads to a spectral realization of the Riemann zeros. The main ideas can be explained as follows. Let us consider a chaotic billiard in two spatial dimensions, such as the Sinai’s billiard [55–58]. A ball thrown with some energy follows chaotic trajectories that in most cases cover the entire table, except for a discrete set of periodic trajectories, whose periods, that are independent of the energy, dominate the path sum that gives rise to the Gutzwiller formula for the fluctuations of the energy levels. In contrast, a one dimensional billiard, made of two walls, will be boring since the ball will go back and forth periodically between the walls.

Let us now take semitransparent walls, so that with a certain probability the ball passes through or bounces off. In such a billiard, the particle may follow several trajectories depending on the outcome at each wall. One may say that this is a quasi-one dimensional billiard. In Sinai’s billiard, or in the motion on compact Riemann surfaces, the particle follows geodesics, which implies that the periods of the closed orbits are independent of the energy. In the quasi-1D billiard, one can achieve the same property by choosing massless particles, say photons or massless fermions, whose trajectories lie on the light cone in Minkowsky space-time. The soft walls should then be viewed as semitransparent mirrors, or beam splitters.

The last ingredient that one has to incorporate into the quasi-1D billiard, or rather the array of mirrors, is chaos. In table billiards, chaos is generated by a border that defocuses the trajectories, and in compact billiards chaos is produced by the negative curvature of the space that separates nearby trajectories exponentially fast. If the 1D mirrors stay at fix positions, then the nearby light-ray trajectories will stay close in space-time. However, if the mirrors are accelerated, then the slightly delayed light rays will generally have their reflected rays departing exponentially fast from one another after several reflections. Hence, in this model, the source of chaos is acceleration. The simplest situation is when the mirrors are uniformly accelerated, in which case they are called moving mirrors in the literature of Quantum Field Theory in curved space-times [59]. We shall then consider an infinite array of moving mirrors whose accelerations and reflection properties will be used to encode number theoretical information. In particular, we shall choose those accelerations that are inversely proportional to a power of integers, which leads to the appearance of primitive periodic orbits whose periods are the logarithms of the prime numbers. These
periods are measured by a moving observer whose acceleration sets the units of this magnitude.

This model realizes in a relativistic framework Berry’s suggestion of associating primitive periodic orbits to prime numbers [13]. Quantum mechanically, the waves propagating in the array generate an interference pattern that encodes the accelerations and reflection coefficients of the mirrors. Here, we find several situations, including destructive interference where the Riemann zeros are missing spectral lines as in the adelic Connes’ model and constructive interference where the Riemann zeros are point like spectrum embedded in a continuum. In both cases, the connection between the spectrum of the model and the Riemann zeros involves a limit where the reflection amplitudes vanish asymptotically. This limit is analogue to the semiclassical limit \( \hbar \to 0 \) that leads to the Gutzwiller formula, which was the starting point of the analogies between Quantum Chaos and Number Theory.

The rest of this paper is organized as follows. In section 2, we review the basic definitions of Rindler spacetime that describes the geometry of the model. We formulate the massive Dirac fermion in a domain of the Rindler space-time, we find the Hamiltonian, study its relation with \( xp \) and recover the spectrum of the \( xp \) model, obtaining the interpretation of the parameters \( \ell_x \) and \( \ell_p \) as inverse acceleration and fermion mass. In section 3, we construct an ideal array of moving mirrors with accelerations \( c^2/\ell_n \), and study the reflections of the light rays emitted and absorbed by an observer with acceleration \( c^2/\ell_i \). Using special relativity, we show that the proper time of the observer’s clock is proportional to \( \log \ell_n \), and that the choice \( \ell_n \propto \sqrt{n} \) singularizes the trajectories associated to the prime numbers. We also propose an array where \( \ell_n \propto e^n/2 \), which has regular trajectories, and whose discrete spectrum is proportional to the integers (this model will be denoted harmonic). In section 4, we construct the Hamiltonian of a massless Dirac fermion with delta function potentials that are associated to the moving mirrors of section 3. We derive the matching conditions for the wave functions and show that the corresponding Hamiltonian is self-adjoint for generic values of the
accelerations and the reflection coefficients. The eigenvalue problem for the Hamiltonian is formulated using transfer matrix methods, and in a semiclassical limit we find the conditions for the existence of discrete eigenvalues. So far, the construction is general, but then we analyze several examples and make contact with the Riemann zeta function and the Dirichlet $L-$ functions. In section 5, we identify the symmetry class of the Hamiltonian under time reversal (T), charge conjugation (C) and parity (P). In the conclusions, we summarize our results and discuss future developments. In appendix A we derive the spectrum of the harmonic model.

2. The massive Dirac fermion in Rindler spacetime

2.1. Rindler spacetime

We start with some basic definitions and set up our conventions. The 1 + 1 dimensional Minkowski space-time is defined by a pair of coordinates $(x^0, x^1) \in \mathbb{R}^2$, a flat metric $\eta_{\mu\nu}$, with signature $(-1, 1)$, and line element
\[ ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad -\eta_{00} = \eta_{11} = 1, \quad \eta_{01} = \eta_{10} = 0, \]
that is invariant under translations and Lorentz transformations (we set the units of the speed of light $c = 1$). The change of coordinates,
\[ x^0 = \rho \sinh \phi, \quad x^1 = \rho \cosh \phi, \]
brings (1) to the form
\[ ds^2 = d\rho^2 - \rho^2 d\phi^2. \]
$\rho$ and $\phi$ are the Rindler space and time coordinates, respectively [60, 61]. The Rindler metric (3) is invariant under shifts of $\phi$, generated by the Killing vector $\partial / \partial \phi$. Restricting $\rho$ to positive values defines the region called the right Rindler wedge [60–62],
\[ \mathcal{R}_+ = \{ (\rho, \phi) | \rho > 0, -\infty < \phi < \infty \}, \]
that in Minkowski coordinates is described by the quadrant $x^1 > |x^0|$. The value $\rho = 0$ represents a horizon of the metric (3), which is similar to the horizon of a black hole, the reason being that the Schwarzschild metric, near the black hole horizon, is approximately the Rindler metric [61].

Rindler spacetime is a natural arena to study the physical phenomena associated with accelerated observers [60], such as the Unruh effect [62, 63]. Let us consider an observer whose world line is given by (2), with $\rho = \ell > 0$ a constant value. The proper time, $\tau$, measured by the observer is defined by $ds^2 = -d\tau^2$, and its relation to $\phi$ follows from (3)
\[ \tau = \ell \phi. \]
plus an additive constant that is set to 0 in (5). The observer’s trajectory (2), written in terms of its proper time, reads
\[ x^0 = \ell \sinh (\tau / \ell), \quad x^1 = \ell \cosh (\tau / \ell), \]
and has a constant proper acceleration, $a$, defined as the Minkowski norm of the vector $a^\mu$

$$a^\mu = \frac{d^2x^\mu}{dt^2}, \quad a^\mu a_\mu = -a^2 \rightarrow a = \frac{1}{\ell'}.$$  

(7)

Restating the speed of light, $a = c^2/\ell$. This quantity plays a central role in the Unruh effect according to which an observer, with proper acceleration $a$, detects a thermal bath with temperature $T_U = a/(2\pi)$ (in units $c = h = T_B = 1$) [63]. Replacing $a$ by the surface gravity $\kappa$ of an observer near the horizon of a black hole yields the Hawking temperature $T_H = \kappa/(2\pi)$ [64, 65]. The similarity between the Hawking and Unruh formulas lies in the equivalence principle of General Relativity.

In our approach to a spectral realization of the Riemann zeros, we shall introduce an observer with acceleration, $a_1 = 1/\ell_1$. The observer’s world line, $\rho = \ell_1$, divides $\mathcal{R}_+$ into the regions $\mathcal{S}$ and $\mathcal{S}_c$, which are located to her right and left,

$$\mathcal{S} = \{(\rho, \phi) \mid \rho > \ell_1, \quad -\infty < \phi < \infty\},$$

$$\mathcal{S}_c = \{(\rho, \phi) \mid 0 < \rho \leq \ell_1, \quad -\infty < \phi < \infty\},$$

(8)

such that $\mathcal{R}_+ = \mathcal{S} \cup \mathcal{S}_c$. Let us next study the dynamics of a Dirac fermion in $\mathcal{S}$.

2.2. Dirac fermion

A representation of the Dirac’s gamma matrices in 1 + 1 dimensions is given by [66]

$$\gamma^0 = \sigma^4, \quad \gamma^1 = -i\sigma^y, \quad \gamma^5 = \gamma^0\gamma^1 = \sigma^z,$$

(9)

where $\sigma^a(a = x, y, z)$ are the Pauli matrices. $\gamma^\mu$ satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}, \quad \{\gamma^\mu, \gamma^5\} = 0, \quad (\mu = 0, 1).$$

(10)

In an abuse of notation, $\gamma^5$ denotes the 1 + 1 analogue of the 1 + 3 gamma matrix $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, and it defines the chirality of the fermions. A Dirac fermion $\psi$ is a two component spinor

$$\psi = \begin{pmatrix} \psi_- \cr \psi_+ \end{pmatrix}, \quad \psi^\dagger = \psi^\dagger\gamma^0 = \begin{pmatrix} \psi^\dagger_- \cr \psi^\dagger_+ \end{pmatrix},$$

(11)

where $\psi^\dagger_\pm$ are the conjugate of $\psi_\pm$. The fields $\psi_\pm$ are the chiral components of $\psi$, namely $\gamma^5\psi_\pm = \pm\psi_\mp$. Let us introduce the light cone coordinates $x^\pm$ and the derivatives $\partial_\pm = \partial/\partial x^\pm$

$$x^\pm = x^0 \mp x^1 = \pm\rho e^{\pm\phi},$$

$$\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1) = \pm\frac{1}{2}e^{\pm\phi}\left(\partial_\rho \pm \rho^{-1}\partial_\phi\right),$$

(12)

where the Minkowski metric (1) becomes $dx^2 = -dx^+dx^-$. Under a Lorentz transformation with boost parameter $\lambda$, that is velocity $v = \tanh\lambda$, the light cone coordinates and the Dirac spinors transform as

$$x^\pm \rightarrow e^{\mp\lambda}x^\pm, \quad \partial_\pm \rightarrow e^{\pm\lambda}\partial_\pm, \quad \psi_\pm \rightarrow e^{\pm\lambda/2}\psi_\pm,$$

(13)

and the Rindler coordinates transform as

$$\phi \rightarrow \phi - \lambda, \quad \rho \rightarrow \rho.$$
Hence, the spinors $\chi_{\pm}$

$$\chi_{\pm} = e^{\pm \phi/2} \psi_{\pm},$$

(15)

remain invariant under (14). The spaces $S, S_\pm$ and $R_\pm$ are mapped into themselves under Lorentz transformations.

2.3. Dirac action of a massive fermion

The Dirac action of a fermion with mass $m$ in the space-time domain $S$ is (units $\hbar = c = 1$)

$$S = \frac{i}{2} \int_S d^4x \bar{\psi} (\partial + im) \psi,$$

$$= \frac{i}{2} \int_S dx^+ dx^- \left[ \psi^+ \partial_+ \psi_- + \psi_- \partial_- \psi_+ + \frac{im}{2} \left( \psi^+_+ \psi_- + \psi^+_+ \psi_- \right) \right],$$

$$= \frac{i}{2} \int_0^{\infty} d\phi \int_{\ell_1}^{\infty} d\rho \left[ \chi^+(\partial_\phi + \rho \partial_\rho + \frac{1}{2}) \chi_- + \chi^+_+ (\partial_\phi - \rho \partial_\rho - \frac{1}{2}) \chi_+ + im\rho \left( \chi^+_+ x_+ + \chi^+_+ x_- \right) \right].$$

(16)

$S$ has a boundary $\partial S$ corresponding to the worldline $\rho = \ell_1 > 0$. The variational principle applied to (16), which gives the Dirac equation

$$\left( \partial + im \right) \psi(x) = 0, \quad x \in S,$$

(17)

and the boundary condition

$$\epsilon_{\mu\nu} x^\mu \dot{\psi}(x)^\nu \delta \psi(x) = 0, \quad x \in \partial S,$$

(18)

where $\delta \psi$ is an infinitesimal variation of $\psi, \epsilon_{\mu\nu}$ is the Levi-Civita tensor ($\epsilon_{01} = -\epsilon_{10} = 1, \epsilon_{\mu\nu} = 0$), and $x^\mu = dx^+ (\ell_1, \phi)/d\phi$ is the tangent to $\partial S$ in the Rindler coordinates (2). The Dirac equation (17) reads in components

$$\left( \partial_0 \mp \partial_\ell \right) \psi_\pm + im \psi_\pm = 0 \quad \implies \quad \partial_\phi \psi_\pm + \frac{im}{2} \psi_\pm = 0.$$

(19)

In the massless case, the fields $\psi_\pm$ decouple and describe a right moving fermion, $\psi_+$ ($x^+$), and a left moving fermion, $\psi_-$ ($x^-$), in terms of which one can construct a Conformal Field Theory with central charge $c = 1$ [66]. The mass term couples the two modes and therefore conformal invariance is lost. The action principle applied to the last expression of equation (16) gives

$$\left( \partial_\phi \pm \rho \partial_\rho \pm \frac{1}{2} \right) \chi_\pm + im \rho \chi_\pm = 0,$$

(20)

and the boundary condition

$$\chi^+ (\ell_1, \phi) \delta \chi_-(\ell_1, \phi) = \chi_- (\ell_1, \phi) \delta \chi_+(\ell_1, \phi), \quad \forall \phi.$$  

(21)

Equations (20) and (21) are of course equivalent to (19) and (18), respectively. The infinitesimal generator of translations of the Rindler time $\phi$, acting on the fermion wave functions is the Rindler Hamiltonian $H_R$, which can be read off from (20)

$$i \partial_\phi \chi = H_R \chi,$$

$$\chi = \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix}.$$  

(22)
\[
H_R = \begin{pmatrix}
-\left(\rho \partial_\rho + \frac{1}{2}\right) & m \rho \\
\rho m & i \left(\rho \partial_\rho + \frac{1}{2}\right)
\end{pmatrix} = \sqrt{\rho} \hat{p}_\rho \sqrt{\rho} \sigma^z + m \rho \sigma^x,
\]
(23)

where \(\hat{p}_\rho = -i \partial / \partial \rho\), is the momentum operator associated to the radial coordinate \(\rho\). The operator
\[
H_{xp} = -i \left(\rho \partial_\rho + \frac{1}{2}\right) = \frac{1}{2} \left(\rho \hat{p}_\rho + \hat{p}_\rho \rho\right) = \sqrt{\rho} \hat{p}_\rho \sqrt{\rho},
\]
(24)

coincides with the quantization of the classical \(xp\) Hamiltonian proposed by Berry and Keating [27], where \(x\) is the radial Rindler coordinate. The eigenfunctions of (24) are
\[
H_{xp} \psi_E = E \psi_E, \quad \psi_E = \frac{1}{\sqrt{2\pi}} e^{-1/2+iE},
\]
(25)

with eigenvalues \(E\) in the real numbers \(\mathbb{R}\) if \(\rho > 0\) [32, 33]. Thus, \(H_R\) consists of two copies of \(xp\), with different signs corresponding to the opposite chiralities that are coupled by the mass term. In CFT the operator \(H_R\) with \(m = 0\) corresponds to the sum of Virasoro operators \(L_0 + \bar{L}_0\) that generate the dilation transformations.

2.4. Self-adjointness of \(H_R\)

The action (16) is invariant under the \(U(1)\) transformation \(\psi \rightarrow e^{i\phi} \psi\). The corresponding Noether current is \(J^\mu = \bar{\psi} \gamma^\mu \psi\), and is conserved; that is, \(\partial_\mu J^\mu = 0\). The charge associated to \(J^\mu\) can be integrated along the space-like line (2) with constant \(\phi\). Using (15) one finds
\[
\int_{\phi=\text{cte}} dx^\mu \bar{\psi} \gamma^\mu \psi = \int d\rho \left(\chi^*_+ \chi^-_+ + \chi^*_- \chi^+_+\right),
\]
(26)

where \(\chi^*_\pm\) are the complex conjugate of \(\chi^\pm\). Hence, the scalar product of two wave functions, in the domain \(S\), can be defined as
\[
\langle \chi_1 | \chi_2 \rangle = \int_{\ell_1} d\rho \left(\chi_{1,\mp}^* \chi_{2,\mp} + \chi_{1,\pm}^* \chi_{2,\pm}\right).
\]
(27)

The Hamiltonian \(H_R\) is hermitean (or symmetric) with respect to this scalar product if
\[
\langle \chi_1 | H_R \chi_2 \rangle = \langle H_R \chi_1 | \chi_2 \rangle,
\]
(28)

when acting on a subspace of the total Hilbert space. Partial integration gives
\[
\langle \chi_1 | H_R \chi_2 \rangle - \langle H_R \chi_1 | \chi_2 \rangle = \int_{\ell_1} d\rho \partial_\rho \left[\rho \left(\chi^*_1 \chi^+_{2,\mp} - \chi^*_{1,\pm} \chi^+_{2,\mp}\right)\right].
\]
(29)

Hence, \(H_R\) is hermitean provided
\[
\lim_{\rho \rightarrow \infty} \rho^{1/2} \chi^*_+ (\rho) = 0, \quad \chi^*_{1,\mp} \chi^+_{2,\mp} - \chi^*_{1,\pm} \chi^+_{2,\pm} = 0, \quad \text{at } \rho = \ell_1.
\]
(30)

The latter condition is equivalent to equation (21). The solution of (30) is
\[
-ie^{i\beta} \chi^*_+ \quad \text{at } \rho = \ell_1,
\]
(31)
where $\theta \in [0, 2\pi)$. The quantity $-ie^{i\theta}$ has the physical meaning of the phase shift produced by the reflection of the fermion with the boundary. It will play a very important role in what follows.

$H_R$ is not only hermitean but is also self-adjoint. According to a theorem due to von Neumann, an operator $H$ is self-adjoint if the deficiency indices $n_\pm$ are equal [67, 68]. These indices are the number of linearly independent eigenfunctions of $H$ with positive and negative imaginary eigenvalues,

$$n_\pm = \dim \{ \psi_\pm | H\psi_\pm = \pm i\omega \psi_\pm, \Im \omega > 0 \}.$$  \hfill (32)

If $n_+ = n_- = 0$, then the operator $H$ is essentially self-adjoint, while if $n_+ = n_- > 0$, then $H$ admits self-adjoint extensions. In the case of $H_R$, one finds $n_+ = n_- = 1$, and the self-adjoint extensions are parameterized by the phase $e^{i\theta}$ in equation (31).

2.5. Spectrum of $H_R$  

The eigenvalues and eigenvectors of the Hamiltonian (23) are given by the solutions of the Schrödinger equation

$$H_R\chi = E\chi, \quad \chi_\pm(\rho, \phi) = e^{-iE\phi}f_{\pm}(\rho), \quad \rho \geq \ell,$$  \hfill (33)

that satisfy the boundary condition (31). $E$ is the Rindler energy. The equations for $f_{\pm}(\rho)$ that follows from (23) are

$$\left( \rho \frac{\partial}{\partial \rho} + \frac{1}{2} \pm iE \right)f_{\pm} \mp \Im \rho f_{\pm} = 0,$$  \hfill (34)

which leads to the second order differential equations

$$\left[ \rho^2 \frac{\partial^2}{\partial \rho^2} + \rho \frac{\partial}{\partial \rho} - \left( \frac{1}{2} \pm iE \right)^2 \right]f_{\pm} = 0,$$  \hfill (35)

whose general solution is a linear combination of the modified Bessel functions [69]

$$f_{\pm}(\rho) = C_1 e^{\mp i\phi/4}K_{\pm\ell}(\rho) \pm C_2 e^{\pm i\phi/4}I_{\pm\ell}(\rho).$$  \hfill (36)

The phases $e^{\pm i\phi/4}$ follows from (34). Let us compute the deficiency indices of $H_R$, for which it is enough to take $E = \pm i/2$,

$$E = \frac{i}{2} \rightarrow \begin{cases} f_+(\rho) = C_1 e^{-i\phi/4}K_0(\rho) + C_2 e^{-i\phi/4}I_0(\rho), \\ f_-(\rho) = C_1 e^{i\phi/4}K_0(\rho) - C_2 e^{i\phi/4}I_0(\rho). \end{cases}$$  \hfill (37)

$$E = -\frac{i}{2} \rightarrow \begin{cases} f_+(\rho) = C_1 e^{-i\phi/4}K_0(\rho) + C_2 e^{-i\phi/4}I_0(\rho), \\ f_-(\rho) = C_1 e^{i\phi/4}K_0(\rho) - C_2 e^{i\phi/4}I_0(\rho). \end{cases}$$  \hfill (38)

The functions $I_{0,1}(\rho)$ diverge exponentially as $\rho \to \infty$, that forces $C_2 = 0$, while $K_{0,1}(\rho)$ decreases exponentially and are normalizable provided $\ell > 0$. The deficiency indices are therefore equal, $n_+ = n_- = 1$ and the eigenfunctions are (setting $C_1 = 1$)

$$f_{\pm}(\rho) = e^{\pm i\phi/4}K_{\pm\ell}(\rho), \quad \rho \geq \ell_1,$$  \hfill (39)
which yields
\[ \chi_{\pm}(\rho, \phi) = e^{-iE\phi + i\pi/4}K_{1/2 \pm iE}(m\rho), \] (40)
up to a common normalization constant. Plugging (40) into (31) yields the equation for the
eigenenergies
\[ e^{\theta}K_{1/2 - iE}(m\ell_1) - K_{1/2 + iE}(m\ell_1) = 0. \] (41)
This equation has both positive and negative solutions, but only if \( \theta = 0 \) or \( \pi \), they come in
pairs \([E_n, -E_n]\). Moreover, if \( \theta = 0 \), then \( E_0 = 0 \) is also an eigenvalue. The imaginary part of
the Riemann zeros also form pairs \([t_n, -t_n]\) (i.e. \( \zeta(1/2 \pm it_n) = 0 \)), and \( t = 0 \) is not a zero
since \( \zeta(1/2) \neq 0 \). This situation corresponds to the choice \( \theta = \pi \).

The number of eigenvalues, \( n(E) \), in the interval \([0, E]\) (with \( E > 0 \)) is given in the
asymptotic limit \( E \gg m\ell \) by
\[ n(E) \simeq \frac{E}{\pi} \left( \log \frac{2E}{m\ell_1} - 1 \right) - \frac{\theta}{2\pi} + O\left(\frac{1}{E}\right), \quad E \gg m\ell_1. \] (42)
For \( E < 0 \) there is a similar formula with \( \theta \rightarrow -\theta \). Let us compare this expression with the
Riemann–Mangoldt formula that counts the number of zeros, \( \mathcal{N}(t) \), of the zeta function \( \zeta(s) \) that lie in the rectangle \( 0 < \text{Re} \, s < 1, 0 < \text{Im} \, s < t \) [2]
\[ \mathcal{N}(t) = \langle \mathcal{N}(t) \rangle + \mathcal{N}_0(t) \]
\[ \langle \mathcal{N}(t) \rangle = \frac{\theta(t)}{\pi} + 1 \rightarrow t \left( \log \frac{t}{2\pi} - 1 \right) + \frac{7}{8} \]
\[ \mathcal{N}_0(t) = \frac{1}{\pi} \text{Im} \log \zeta \left( \frac{1}{2} + it \right) = O(\log t) \] (43)
where \( \langle \mathcal{N}(t) \rangle \) is the average term and \( \mathcal{N}_0(t) \) is the oscillation term. This expression agrees
with (42), to order \( t \log t \) and \( t \), with the identifications
\[ E = \frac{t}{2}, \quad m\ell_1 = 2\pi. \] (44)
However, the constant term 7/8 is not reproduced by equation (42) for \( \theta = \pi \), which is the
choice for the absence of the zero mode \( E_0 = 0 \) (note that neither \( \theta = 0 \) gives the 7/8).

Comments:
- In the limit \( m\ell_1 \rightarrow 0 \), the spectrum (42) becomes a continuum. This situation arises in
three cases: 1) if \( m > 0 \) is kept constant and \( \ell_1 \rightarrow 0 \), the domain \( S \) becomes \( \mathcal{R}_+ \), and one
recovers the spectrum of a massive Dirac equation in \( \mathcal{R}_+ \); 2) if \( \ell_1 > 0 \) is kept constant,
and \( m \rightarrow 0 \), the particles become massless and the effect of the boundary \( \partial S \), is to
exchange left and right moving fermions, that constitutes a boundary CFT [66]; and, 3) \( m \) and \( \ell_1 \rightarrow 0 \) that corresponds to a massless fermion in \( \mathcal{R}_+ \).
- Equation (41) coincides with the spectrum of the quantum \( x/p \) Hamiltonian [43, 44]
\[ H_{x/p} = \sqrt{x} \left( \hat{p} + \ell_1^2 \hat{p}^{-1} \right) \sqrt{x}, \quad x \gg \ell_x, \] (45)
with the identifications
\[ E = \frac{E_{x/p}}{2}, \quad m = \ell_p, \quad \ell_1 = \ell_x. \] (46)
The origin of this coincidence lies in the fact that a classical Hamiltonian of the form
\( H = U(x)p + V(x)p \) [44] can be formulated as a massive Dirac model in a space-time metric built from the potentials \( U(x) \) and \( V(x) \) [49]. In the case of the \( xp \) model this metric is flat, which allows us to formulate this model in terms of the Dirac equation in the domain \( S \).

- Gupta et al proposed recently the Hamiltonian \( \langle xp + px \rangle / 2 \) as a Dirac variant of the \( xp \) Hamiltonian [48]. The former Hamiltonian is defined in two spatial dimensions and after compactification of one coordinate becomes 1D, with a spectrum that depends on a regularization parameter and which is similar to the one found from the Landau theory with electrostatic potential \( xy \) [38].

- Burnol has studied the causal propagation of a massive boson and a massive Dirac fermion in the Rindler right edge \( R_+ \), relating the scattering from the past light cone to the future light cone to the Hankel transform of zero order and suggesting a possible relation to the zeta function [70].

### 2.6. General Dirac action

For later purposes we shall introduce the general relativistic invariant Dirac action in 1 + 1 dimensions

\[
S = \frac{i}{2} \int_{\mathcal{S}} d^2x \bar{\psi}(\partial x + iA + im' \gamma^5)\psi,
\]  

(47)

where, in addition to the mass term, there is a chiral interaction \( \bar{\psi} \gamma^5 \psi \) and a minimal coupling \( \bar{\psi} A_\mu \psi \) to a vector potential \( A_\mu \) (\( A = \gamma^\mu A_\mu \)). Moreover, \( m, m' \) can be functions of the space-time position, in which case \( m \) becomes a scalar potential and \( m' \) becomes a pseudo scalar potential. We saw above that for a constant mass \( m \) the action (16) can be restricted to the domain \( S \), preserving the invariance under shifts of the Rindler time \( \phi \). It is clear that if \( m \) depends in \( \rho \) but not in \( \phi \), then the action remains invariant under translations of \( \phi \), and the Hamiltonian is equal to (23), with \( m \) replaced by \( m(\rho) \). The same happens with the term \( \bar{\psi} \gamma^5 \psi \) if \( m' = m'(\rho) \). Concerning the vector potential, the action (47) is invariant provided

\[
A_0 \pm A_1 = a_{\pm}(\rho)e^{\mp \phi},
\]  

(48)

in which case (47) becomes

\[
S = \frac{i}{2} \int_{-\infty}^{\infty} d\phi \int_{l_1}^{l_2} d\rho \left\{ \chi_-^\dagger \left( \partial_\phi + \rho \partial_\rho + \frac{1}{2} + ia_{+}\rho \right) \chi_- + \chi_+^\dagger \left( \partial_\phi - \rho \partial_\rho - \frac{1}{2} + ia_{-}\rho \right) \chi_+ + (im + m')\rho \chi_- \chi_+ + (im - m')\rho \chi_+ \chi_- \right\}.
\]  

(49)

By using gauge transformations, one can reduce the number of fields in (49). This issue will be consider below in a discrete realization of this model. The equations of motion derived from (49) are

\[
\left( \partial_\phi \pm \rho \partial_\rho \pm \frac{1}{2} \pm ia_{\pm}\rho \right) \chi_{\pm} + (im \pm m')\rho \chi_{\pm} = 0,
\]  

(50)
and they can be written as the Schroedinger equation (22) with Hamiltonian

\[
H_R = \begin{pmatrix}
-i \left( \rho \frac{\partial}{\partial \rho} + \frac{1}{2} \right) + a_+ \rho & (m - i m') \rho \\
(m + i m') \rho & i \left( \rho \frac{\partial}{\partial \rho} + \frac{1}{2} \right) + a_- \rho
\end{pmatrix}
\]

\[
= \sqrt{\rho} \hat{p}_\rho \sqrt{\rho} \sigma^z + \frac{a_+ + a_-}{2} \rho \sigma^z + \frac{a_+ - a_-}{2} \rho \sigma^y + m \rho \sigma^x + m' \rho \sigma^x
\]

which acts on the wave functions that satisfy the boundary condition (31).

In summary, we have shown in this section, that the spectrum of the Rindler Hamiltonian, in the domain \( S \) agrees asymptotically with the average Riemann zeros, under the identifications (44). However, there is no trace of their fluctuations that depend on the zeta function on the critical line. This observation is not surprising since the prime numbers should be included into the model. In the next section we shall take a first step in this direction.

3. Moving mirrors and prime numbers

Here, we construct an ideal optical system, in Rindler space-time, that allows us to distinguish prime numbers from composite. In the next section, we shall give a concrete realization of this optical system in the Dirac model. The system consists of an infinite array of mirrors, labeled by the integers \( n = 1, 2, ..., \infty \), that have the following properties:
The first mirror, \( n = 1 \), is perfect, while the remaining ones, \( n > 1 \), are one-way mirrors (beam splitters) that reflect and transmit the light rays partially. The light rays can be replaced by massless fermions.

The mirrors move in Minkowski space-time with uniform accelerations \( a_n \), with \( a_1 > a_2 > ... \), such that \( \lim_{n \to \infty} a_n = 0 \).

At time \( \chi^0 = 0 \), the mirrors are placed at the positions \( \ell_n = 1/a_n \) (units \( c = 1 \)).

The worldlines of the mirrors are contained in the domain \( S \), whose boundary corresponds to the first mirror, \( n = 1 \), such that \( \rho = \ell_1 \).

An observer carries the first mirror, and sends and receives light rays, whose departure and arrival times she measures with a clock.

The lengths \( \ell_n \) are given by

\[
\ell_n = \ell_1 n^{1/2}, \quad n = 1, 2, ..., \infty. \quad (52)
\]

This ansatz can be replaced by \( n^\alpha (\alpha > 0) \), but the parameter \( \alpha \) can be set to 1/2 by scaling the clock’s ticks. Figure 2 depicts the mirror’s worldlines, satisfying these conditions with \( \alpha = 1 \).

We shall next study the propagation of light rays in this optical array using the laws of special relativity. Let us consider a light ray emanating from the point \( \rho_1, \phi_1 \) and reaching the point \( \rho_2, \phi_2 \), where \( (\rho, \phi) \) are the Rindler coordinates. Along this trajectory the line element (3) vanishes,

\[
d\rho = \pm d\rho d\phi = \rho_1 e^{\pm (\phi - \phi_1)} \to \phi_2 - \phi_1 = \log \frac{\rho_2}{\rho_1}, \quad (53)
\]

corresponding to right moving \( (\rho_2 > \rho_1) \) or left moving \( (\rho_1 > \rho_2) \) rays. Suppose that the ray is emitted at the first mirror at time \( \phi_n = 0 \), i.e. \( (\rho_n, \phi_n) = (\ell_1, 0) \), moves rightwards, reflects on the \( n \)th mirror and returns to the first mirror, \( \rho_1, \phi_1 \). The value of \( \phi_n \) follows from equations (53) and (52)

\[
\phi_n = 2 \log \frac{\ell_n}{\ell_1} = \log n, \quad n > 1. \quad (54)
\]

which is twice the change in \( \phi \) from \( \rho = \ell_1 \) to \( \rho = \ell_n \). \( \phi_n \) can be measured by the observer’s clock, traveling with the perfect mirror where the ray was emitted and received. The change in the clock’s proper time is given by \( \tau_n = \ell_1 \delta \rho_n \) (see equation (5)), which in units of \( \ell_1 = 1 \), reads

\[
\tau_n = \log n, \quad n > 1. \quad (55)
\]

Hence, measuring \( \tau_n \), the observer can find the value of \( n \). If \( n = 2 \), then the ray travels forth and back between the mirrors \( n = 1 \) and \( n = 2 \). However, if \( n > 2 \), the ray must pass through the intermediate mirrors \( 2, ..., n - 1 \). Let us next analyze the case of two reflections. Now, the ray is emitted at \( (\rho_n, \phi_n) = (\ell_1, 0) \), reaches the mirror \( n_1 \), returns to the perfect mirror, reflects again, reaches the mirror \( n_2 \) and comes back finally to the perfect mirror, where the clock records the proper time, \( \tau_{n_1 n_2} \), which is given by the sum of the intermediate times (55)

\[
\tau_{n_1 n_2} = \tau_{n_1} + \tau_{n_2} = \log (n_1 n_2), \quad n_{1,2} > 1. \quad (56)
\]

Hence, the measurement of \( \tau_{n_1 n_2} \) allows the observer to compute the product \( n_1 n_2 \). There are more complicated cases, as in the one illustrated by the following sequence.
where the ray emitted by the observer is reflected by the mirror \( n_1 \), back to the mirror \( n_2 < n_1 \), that reflects the ray forwards to the mirror \( n_3 > n_2 \), that reflects the ray back to the first mirror.

The proper time recorded by the clock is

\[
\tau_{n_1,n_2,n_3} = \tau_{n_1} - \tau_{n_2} + \tau_{n_3} = \log \frac{n_1 n_3}{n_2}. 
\]

Notice that the case \( n_2 = 1 \) reproduces equation (56). It is easy to derive a general formula for the proper time elapsed for a trajectory involving \( 2k \) intervals, that starts and ends at the first mirror,

\[
1 \rightarrow n_1 \rightarrow n_2 \rightarrow \cdots \rightarrow n_{2k-1} \rightarrow n_{2k} = 1,
\]

\[
1 < n_1 > n_2 < \cdots < n_{2k-1} > n_{2k} = 1,
\]

and is given by

\[
\tau_{\{n_k\}} = \log \frac{n_1 n_3 \cdots n_{2k-1}}{n_2 n_4 \cdots n_{2k}}.
\]

The numerator of (60) corresponds to the reflections: right mover \( \rightarrow \) left mover, while the denominator corresponds to reflections: left mover \( \rightarrow \) right mover. The argument of \( \log \) in (60) will be in general a rational number. Let us suppose it is the prime \( p \). It is clear from (55) that after one reflection (i.e. \( k = 1 \)) the observer will detect one ray arriving at \( \tau_p = \log p \).

Suppose now that the argument of the log is a composite number \( n \), say 4. At time \( \tau_4 = \log 4 \), the observer will detect the ray reflected from the forth mirror, but also one ray from the two reflections on the second mirror \( \tau_{2,2} = 2\tau_2 \), which is of course the same as \( \tau_4 \). In a real experiment the two rays arriving at the observer will interfere. The study of this interference is left to the next section.

The previous example suggests that prime numbers correspond to unique paths that are characterized by observer proper times that are equal to \( \log p \). Let us prove this statement in the case of \( k = 2 \). Equation (58) becomes

\[
\frac{n_1 n_3}{n_2} = p,
\]

where \( n_{1,2,3} \) satisfy the constraints (57). According to (61), \( p \) divides the product \( n_1 n_3 \), so it is a prime factor of \( n_1 \) or \( n_3 \). In the former case, one has

\[
n_1 = pn'_{1} \left( n'_{1} \geq 1 \right) \rightarrow \frac{n'_1 n_3}{n_2} = 1,
\]

which is a contradiction because \( n_2 < n_3 \) by equation (57). The same result holds if \( p \) divides \( n_3 \). The generalization to any \( k \geq 2 \) goes as follows. Suppose that \( \tau_{\{n_k\}} = \log p \), then from equation (60)

\[
\frac{n_1 n_3 \cdots n_{2k-1}}{n_2 n_4 \cdots n_{2k-2}} = p.
\]

If \( p \) divides, say \( n_1 \), then one gets

\[
n_1 = pn'_{1} \left( n'_{1} \geq 1 \right) \rightarrow \frac{n'_1 n_3 \cdots n_{2k-1}}{n_2 n_4 \cdots n_{2k-2}} = 1,
\]

which cannot be satisfied because \( n_2 < n_3, \ldots, n_{2k-2} < n_{2k-1} \) by the conditions (59). This proves that the observer detects a single ray only when it comes from the reflection on a prime
mirror, while it detects more than one ray when the rays comes from composite mirrors. This interpretation is purely classical because it presupposes that the rays can be distinguished and disregards the interference effects. Both effects have of course to be taken into account in a realistic implementation using identical particles, such as photons or massless fermions. The interference pattern emerging for fermions in this array will be the purpose of the next section. In any case, one can easily show that this classical model can be used to implement the classical Eratosthenes’s sieve to construct prime numbers.

Comments:

- We mentioned in the introduction that the similarities between counting formulas in Number Theory and Quantum Chaos led Berry to conjecture the existence of a classical chaotic Hamiltonian whose primitive periodic orbits are labeled by the primes \( p \), with periods \( \log p \), and whose quantization will give the Riemann zeros as energy levels. Although a classical Hamiltonian with this property has not yet been constructed, the mirror system that is presented above displays some of its properties. In particular, the rays associated to prime numbers behave as primitive orbits with a period \( \log p \). Moreover, the trajectories and periods of these primitive rays are independent of their energy, which is the frequency of the light.

- One can construct an array of moving mirrors in the domain \( S \) with the same properties as the array in \( \Sigma \). The parameters \( \ell_n \) that characterize the array can be labeled with the negative integers,

\[
\ell_n = \ell_1 |n|^{1/2}, \quad n = -1, -2, \ldots, -\infty,
\]

where \( n = -1 \) is the perfect mirror located at the boundary \( \partial S = \partial \Sigma \). The proper time elapsed is now given by \( \tau_n = \log \ln n \), and is identical to (55). The relation between the arrays in \( S \) and \( \Sigma \) corresponds to the inversion transformation \( \rho \rightarrow \ell_1^2 / \rho \).

- The array of mirrors defined in \( S \) is an analogue computer for the multiplication operation, or rather the addition of \( \log \)'s. To implement the addition operation we shall define an array of mirror with positions, that is inverse accelerations,

\[
\ell_n = \ell_0 e^{n/2}, \quad n = 0, 1, 2, \ldots, \infty.
\]

Where \( n = 0 \) labels the perfect mirror. The analogue of equations (55) and (56) is (with \( \ell_0 = 1 \))

\[
\tau_n = n, \quad \tau_{n_1, n_2} = n_1 + n_2.
\]

This is an harmonic array in the sense that the proper times (67), as well as the discrete spectrum (see appendix A) are given by integer numbers. The harmonic array shows that it is not enough to have accelerated mirrors to generate chaos. The location/accelerations of the mirrors is essential. In the harmonic case, the exponential separation of the mirrors (66) compensates the exponential time dependence of the ray trajectories (53), which gives rise to a tessellation of Rindler space-time. In the case of (52), the whole set of ray trajectories do not tessellate the Rindler spacetime, which is a manifestation of the arithmetic chaos.

4. Massless Dirac fermion with delta function potentials

In this section, we present a mathematical realization of the moving mirrors in the Dirac theory. We start from the massless Dirac action and represent the mirrors by delta function
potentials that are obtained by discretizing the interacting terms in the general action (49). The mass term becomes a contact interaction that turns left moving fermions into right moving ones, and vice versa. The new action remains invariant under translations in Rindler time, so that the Rindler Hamiltonian is a conserved quantity and we look for its spectrum using transfer matrix methods. The Dirac equation with delta function potentials requires a special treatment, which we describe in detail (see [71–76] for the Dirac equation with delta interactions in Minkowski coordinates).

4.1. Discretization in Rindler variables

Let us consider the integral

$$\int_{\ell_1}^{\infty} d\rho \rho f(\rho) = \frac{1}{2} \int_{u_1}^{\infty} du \, f(u), \quad u = \rho^2, \quad u_1 = \ell_1^2, \quad (68)$$

where \( f(\rho) \) is a generic function, and \( d\rho \rho \) is the radial part of the measure \( d^2x = d\phi d\rho \). The Rindler time \( \phi \) can be easily incorporated into the equations. We shall discretize (68), using the positions \( \ell_n \) of the mirrors given in equation (52), which amounts to a partition of the half-line \( u \geq u_1 \) into segments of equal width \( \ell_1^2 \), separated by the points

$$u_k = \ell_1^2 + \ell_2^2 k, \quad k = 1, 2, \ldots \quad (69)$$

Let us discretize (68) as

$$\frac{1}{2} \int_{u_1}^{\infty} du \, f(u) \rightarrow \frac{1}{2} \sum_{k=2}^{\infty} f(u_k) \rightarrow \frac{1}{2} \int_{u_1}^{\infty} du \, f(u) \sum_{k \geq 2} \delta(u - u_k). \quad (70)$$

If \( f \) is a continuous function then the last expression of (70) coincides with the middle one, but if \( f \) is discontinuous then one gets

$$\frac{1}{2} \int_{u_1}^{\infty} du \, f(u) \rightarrow \frac{1}{2} \sum_{k=2}^{\infty} \frac{f(u_k^+)}{2} + \frac{f(u_k^-)}{2}. \quad (71)$$

where we used

$$\int_{v-\epsilon}^{v+\epsilon} du \, f(u) \delta(u - v) = \frac{1}{2} \left( f(v^+) + f(v^-) \right).$$

$$f(v^\pm) = \lim_{\epsilon > 0} f(v \pm \epsilon). \quad (72)$$

In the limit \( \ell_1 \rightarrow 0 \), the last expression in equation (70) converges towards the integral (68) for well behaved functions. Replacing (69) into (70) yields

$$\int_{\ell_1}^{\infty} d\rho \rho f(\rho) \sum_{k \geq 2} \frac{\ell_1^2 \ell_k}{2} \delta(\rho - \ell_k). \quad (73)$$

One could use this formula to discretize the mass term in the Dirac action (16). By taking \( f = m\psi \psi \) this would yield

$$\frac{i}{2} \int_{S} d\phi d\rho \psi \psi \sum_{k \geq 2} \frac{m\ell_1^2}{2} \delta(\rho - \ell_k). \quad (74)$$

However, the corresponding Dirac equation is problematic because the matching conditions are not consistent with the equations of motion, as shown in references [71–76]. This problem is solved by replacing the local delta interactions by separable delta function potentials, which
amounts to a point splitting. More concretely, the integral
\[ \int_{t_1}^{\infty} d\rho f(\rho)g(\rho) = \frac{1}{2} \int_{u_1}^{\infty} du f(u)g(u), \] (75)
should be replaced by
\[ \frac{\ell^2_k}{2} \int_{u_1}^{\infty} du \int_{u_1}^{\infty} du' f(u)g(u') \sum_{k \geq 2} \delta(u - u_k) \delta(u' - u_k), \]
\[ = \int_{t_1}^{\infty} d\rho d\rho' f(\rho)g(\rho') \sum_{k \geq 2} \frac{\ell^2_k}{2\ell^2_k} \delta(\rho - \ell_k) \delta(\rho' - \ell_k). \] (76)
If \( f \) and \( g \) are continuous functions at \( \ell_k \), then this expression is equivalent to (73).

4.2. The Dirac action with delta function potentials

By applying the discretization formula (76) to the general action (49) one obtains
\[ S_{DM} = S_0 + S_{\text{int}} \] (77)
where \( S_0 \) is the massless action
\[ S_0 = \frac{i}{2} \int_{-\infty}^{\infty} d\phi \int_{t_1}^{\infty} d\rho' \left[ \chi_\rho^+(\rho) \left( \partial_{\rho'} + \rho \partial_{\rho'} + \frac{1}{2} \chi_\rho \right) + \chi_\rho^+(\rho) \left( \partial_{\rho'} - \rho \partial_{\rho'} - \frac{1}{2} \chi_\rho \right) \right], \] (78)
and \( S_{\text{int}} \) is the discretization of the mass terms and vector potential of (49) in the gauge \( a_+ = a_- \)
\[ S_{\text{int}} = \frac{i}{2} \int d\phi d\rho d\rho' \sum_{k \geq 2} \frac{2}{\ell^2_k} \delta(\rho - \ell_k) \delta(\rho' - \ell_k) \]
\[ \times \left[ i\gamma^r \chi^r_\rho^+(\rho, \phi) \chi_\rho^- (\rho', \phi) + i\gamma^r \chi^r_\rho^+ (\rho, \phi) \chi_\rho^- (\rho', \phi) \right. \]
\[ + \left( i n_k + r'_k \right) \chi^r_\rho^+ (\rho, \phi) \chi_\rho^- (\rho', \phi) + \left( i n_k - r'_k \right) \chi^r_\rho^- (\rho, \phi) \chi_\rho^+ (\rho', \phi) \right], \] (79)
where
\[ n_k = \frac{m(\ell_k)\ell^2_k}{4\ell^2_k}, \quad r'_k = \frac{m'(\ell_k)\ell^2_k}{4\ell^2_k}, \quad r'_k = \frac{a_+(\ell_k)\ell^2_k}{4\ell^2_k}, \quad k \geq 2. \] (80)
are dimensionless parameters that are equal to the values taken by the functions \( m, m', a_+ \) in (49) multiplied by \( \ell^2_k/(4\ell^2_k) \). One can verify that the (77) is invariant under the scale transformation
\[ \rho \rightarrow \lambda \rho, \quad \chi_\pm \rightarrow \lambda^{-1/2} \chi_\pm, \quad \ell_k \rightarrow \lambda \ell_k, \] (81)
so that the physical observables only depend on the parameters \( n_k, r'_k, r'_k \). In the Dirac model considered in section 2, the mass is constant, which after discretization implies that \( r_n = m\ell^2/(4\ell^2) \). Later on we shall generalize this equation to \( r_n \propto n^{-\sigma} \), and make contact with the Riemann zeta function \( \zeta(\sigma + it) \).
4.3. The Hamiltonian

The equations of motion that follows from (77) are
\[
\left( \partial_\phi \pm \rho \partial_\rho \pm \frac{1}{2} \right) \chi_\pm + \sum_{k \geq 2} \ell_k \delta (\rho - \ell_k) \times \left[ i \rho \left( \chi_\pm (\ell_k^+) + \chi_\pm (\ell_k^-) \right) + \left( i n_k \pm r'_k \right) \left( \chi_\pm (\ell_k^+) + \chi_\pm (\ell_k^-) \right) \right] = 0,
\]
(82)
where (72) has been used to integrate around \( \rho = \ell_k \) (to simplify the notation the variable \( \phi \) has been suppressed). This equation implies that the left and right moving modes propagate freely and independently between the positions \( \ell_n \),
\[
\left( \partial_\phi \pm \rho \partial_\rho \pm \frac{1}{2} \right) \chi_\pm = 0, \quad \rho \neq \ell_k.
\]
(83)
Hence, the Hamiltonian is (recall (23)),
\[
H_R = \begin{pmatrix} \sqrt{\rho} \hat{p}_\rho & 0 \\ 0 & -\sqrt{\rho} \hat{p}_\rho \end{pmatrix}, \quad \hat{p}_\rho = -i \partial_\rho, \quad \rho \neq \ell_n, \forall n.
\]
(84)
The delta function terms yield the matching conditions between the wave functions on both sides of \( \ell_k \),
\[
\left( \pm 1 + i r_k' \right) \left( \chi_\pm (\ell_k^+) - \chi_\pm (\ell_k^-) \right) + \left( i n_k \pm r'_k \right) \left( \chi_\pm (\ell_k^+) + \chi_\pm (\ell_k^-) \right) = 0, \quad k \geq 2,
\]
(85)
which can be written in matrix form as
\[
N_{k,+} \chi_+ (\ell_k^+) = N_{k,-} \chi_- (\ell_k^-), \quad k \geq 2,
\]
(86)
where \( \chi \) is the two component vector (22) and
\[
N_{k,\pm} = \begin{pmatrix} 1 \pm i r_k' & \pm \left( i n_k + r'_k \right) \\ \pm \left( - i n_k + r'_k \right) & 1 \mp i r_k' \end{pmatrix}.
\]
(87)
These matrices are invertible provided that
\[
\det N_{k,\pm} = 1 - r_k^2 - r'_k^2 \neq 0.
\]
(88)
Otherwise, equations (85) become a decoupled set of equations; hence, hereafter we shall assume that condition (88) is satisfied and leads to
\[
\chi (\ell_k^-) = L_k \chi (\ell_k^+), \quad k \geq 2,
\]
(89)
where
\[
L_k = N_{k,-}^{-1} N_{k,+}
\]
\[
= \frac{1}{1 - r_k^2 - r'_k^2 + r_k^2} \begin{pmatrix} 1 + r_k^2 + r'_k^2 - r_k^2 + 2 i r_k' & 2 \left( i n_k + r'_k \right) \\ 2 \left( -i n_k + r'_k \right) & 1 + r_k^2 + r'_k^2 - r_k^2 - 2 i r_k' \end{pmatrix}.
\]
(90)
Finally, the variation of the action at the boundary \( \rho = \ell_1 \) gives the condition (31)
\[
- i e^{i \theta} \chi_+ (\ell_1) = \chi_+ (\ell_1) \longrightarrow \chi (\ell_1) \propto \begin{pmatrix} 1 \\ -i e^{i \theta} \end{pmatrix}, \quad \theta \in \left[ 0, 2\pi \right).
\]
(91)
4.4. Self-adjointness of the Hamiltonian

One should expect the Hamiltonian (84), acting on wave functions subject to the BC’s (89) and (91), to be self-adjoint. We shall next show that this is indeed the case. The scalar product, given by equation (27), will be written as

\[ \int \sum \chi^\dagger \rho \chi \, d\rho \] (92)

To show that \( H_R \) is a self-adjoint operator we will follow the approach of Asorey et al, which is based on the consideration of the boundary conditions that turn out to be equivalent to the von Neumann theorem [77]. The starting point is the bilinear

\[ \chi^\dagger \rho \chi \equiv i \Sigma(\chi_1, \chi_2) \] (93)

where \( \chi_{1,2} \) are two wave functions. This quantity measures the net flux or probability flowing across the boundary of the system, which for a unitary time evolution generated by \( e^{-iH_R t} \) must vanish. The self-adjoint extensions of \( H_R \) select subspaces of the total Hilbert space, where \( \Sigma(\chi_1, \chi_2) = 0 \). In the case of the Hamiltonian (84) one finds

\[ \Sigma(\chi_1, \chi_2) = \ell_1 \left( \chi^\dagger_{1,-}(\ell_1)\chi_{2,-}(\ell_1) - \chi^\dagger_{1,+}(\ell_1)\chi_{2,+}(\ell_1) \right) + \sum_{n \geq 2} \ell_n \left[ \chi^\dagger_{1,-}(\ell_n)\chi_{2,-}(\ell_n^\dagger) - \chi^\dagger_{1,+}(\ell_n^\dagger)\chi_{2,+}(\ell_n) \right] - \chi^\dagger_{1,-}(\ell_n)\chi_{2,-}(\ell_n^\dagger) + \chi^\dagger_{1,+}(\ell_n^\dagger)\chi_{2,+}(\ell_n) \], (94)

The term proportional to \( \ell_1 \) already cancels out by equation (91). Imposing the independent cancellation of the terms proportional to \( \ell_n (n \geq 2) \) yields

\[ \chi^\dagger_{1,-}(\ell_n^\dagger)\chi_{2,-}(\ell_n^\dagger) - \chi^\dagger_{1,+}(\ell_n^\dagger)\chi_{2,+}(\ell_n) = \chi^\dagger_{1,-}(\ell_n)\chi_{2,-}(\ell_n^\dagger) - \chi^\dagger_{1,+}(\ell_n^\dagger)\chi_{2,+}(\ell_n) \] (95)

which we write as

\[ (\chi_1(\ell_n^\dagger)|\sigma^\dagger|\chi_2(\ell_n)) = (\chi_1(\ell_n^\dagger)|\sigma^\dagger|\chi_2(\ell_n^\dagger)) \], (96)

where \( |\chi\rangle = (\chi_{1,-}, \chi_{1,+}) \) and \( |\phi\rangle = (\chi_{2,-}, \chi_{2,+}) \). The general solution of (96) is obtained if \( \chi_{1,2}(\ell_n^\dagger) \) and \( \chi_{1,2}(\ell_n) \) are related by a transformation

\[ \chi(\ell_n^\dagger, \phi) = U_n(\ell_n^\dagger, \phi) \], (97)

where \( U_n \) satisfies

\[ U_n^\dagger \sigma^\dagger U_n = \sigma^\dagger \], (98)

which implies that \( U_n \) belongs to the Lie group \( U(1) \otimes SU(1, 1) \). The non-compact character of \( SU(1, 1) \) arises in this problem from the relative minus sign of the \( xp \) terms in the Hamiltonian (84). The \( U(1) \) factor can be eliminated by a phase transformation of the field \( \chi(\rho) \) in the interval \( \rho \in (t_n, t_{n+1}) \), which reduces the group to \( SU(1, 1) \), that is

\[ SU(1, 1) = \left\{ \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \right\}, \quad a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1 \right\} \]. (99)

The matrices \( L_n \) (90) are of this form, which ensures that \( H_R \) is self-adjoint acting on the wave functions satisfying the conditions (89) and (91). One can further reduce the number of parameters in \( L_n \) by applying another \( U(1) \) transformation. Let us label the wave functions with an integer \( n \)
and make the transformation
\[ \chi_n(\rho) \rightarrow e^{i\alpha_n \sigma} \chi_n(\rho), \]
(101)
which induces the following changes in the boundary values
\[ \chi(\ell^+_n) \rightarrow e^{i\alpha_{n-1} \sigma} \chi(\ell^+_n), \quad \chi(\ell^-_n) \rightarrow e^{i\alpha_{n-1} \sigma} \chi(\ell^-_n), \]
(102)
and in the \( L_n \) matrices (recall (89))
\[ L_n \rightarrow e^{i\alpha_{n-1} \sigma} L_n e^{-i\alpha_n \sigma}. \]
(103)
The parameters \( \alpha_n \) can then be used to bring (90) into the form
\[ L_n = \frac{1}{1 - i r_n - r_n^2} \begin{pmatrix} 1 + r_n^2 + r_n^2 & 2(i r_n + r_n') \\ 2(-i r_n + r_n') & 1 + r_n^2 + r_n^2 \end{pmatrix}, \quad n \geq 2, \]
(104)
corresponding to an element in the coset \( SU(1, 1)/U(1) \). We use the same notation for the transformed parameters \( n, r_n \) as in (90).

In summary, we have shown that the Hamiltonian \( H_R \), defined in the domain \( S \), is self-adjoint acting on wave functions that satisfy the BC’s (89) and (91), and is characterized by the set \( \{ \ell_n, r_n \}_{n=2}^{\infty} \) and \( \theta \) (by scale invariance (81) we set \( \ell_1 = 1 \)).

### 4.5. Eigenvalue problem

We shall now consider the eigenvalue problem of the Hamiltonian (84), acting on normalizable wave functions subject to the boundary conditions (89) and (91) The eigenfunctions of \( H_R \) are given by
\[ H_R \chi = E \chi \rightarrow -i \left( \rho \frac{\partial}{\partial \rho} + \frac{1}{2} \right) \chi_\pm = \pm E \chi_\pm \rightarrow \chi_\pm \propto \rho^{-1/2 \pm i E}. \]

so for the \( n \)th interval (recall (100))
\[ \chi_{\pm, n}(\rho) = e^{\pm E \rho^{1/2} - i E \rho} \frac{A_{\pm, n}}{\rho^{1/2} \pm i E}, \quad \rho \in (\ell_n, \ell_{n+1}), \]
(106)
where \( A_{\pm, n} \) will depend, in general, on the eigenenergy \( E \) and \( \rho \) is measured in units of \( \ell_1 \) (in what follows we take \( \ell_1 = 1 \)). The phases \( e^{\pm E \rho^{1/2}} \) have been introduced by analogy with the eigenfunctions (40). The values of \( \chi \) on both sides of \( \rho = \ell_n \) are given by
\[ \chi_+(\ell_n^+) = \lim_{\epsilon \rightarrow 0^+} \chi_+(\ell_n + \epsilon) = \chi_{\pm, n}(\ell_n) = e^{\pm \frac{E}{2} \rho_n^{1/2} + i E} \frac{A_{\pm, n}}{\ell_n^{1/2} \pm i E}, \]
\[ \chi_-(\ell_n^-) = \lim_{\epsilon \rightarrow 0^-} \chi_-(\ell_n - \epsilon) = \chi_{\pm, n-1}(\ell_n) = e^{\pm \frac{E}{2} \rho_n^{1/2} + i E} \frac{A_{\pm, n-1}}{\ell_n^{1/2} \pm i E}. \]
(107)

Let us write these equations in matrix form in terms of the vector
\[ |A_n \rangle = \begin{pmatrix} A_{-, n} \\ A_{+, n} \end{pmatrix}, \]
(108)
and the matrix

$$K_{n,\pm} = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

such that (107) read

$$\chi_n(t_n) = \epsilon_n^{-1/2} K_n |A_n\rangle,$$

$$\chi_n(t_n) = \epsilon_n^{-1/2} K_n |A_{n-1}\rangle.$$ (110)

Plugging these equations into the matching condition (89) yields

$$K_n|A_{n-1}\rangle = L_n K_n |A_n\rangle,$$ (111)

so

$$|A_{n-1}\rangle = T_n |A_n\rangle, \quad T_n = K_n^{-1} L_n K_n,$$ (112)

and substituting (104)

$$T_n = \frac{1}{1 - r_n^2 - r_n'^2} \begin{pmatrix} 1 + r_n^2 + r_n'^2 & 2(r_n - ir_n')\epsilon_n^{-2}\ell^{2E} \\ 2(r_n + ir_n')\epsilon_n^{2E} & 1 + r_n^2 + r_n'^2 \end{pmatrix}.$$ (113)

To simplify the notations let us define

$$\varrho_n = r_n - ir_n',$$ (114)

so that

$$T_n = T(E, \varrho_n, \ell_n) = \frac{1}{1 - |\varrho_n|^2} \begin{pmatrix} 1 + |\varrho_n|^2 & 2\varrho_n \epsilon_n^{-2}\ell^{2E} \\ 2\varrho_n^{*}\epsilon_n^{2E} & 1 + |\varrho_n|^2 \end{pmatrix}.$$ (115)

The parameters $\varrho_n$ have the meaning of reflections amplitudes associated to the $n$th mirror. The absence of a mirror at the position $\ell_n$ is expressed by the condition $\varrho_n = 0$, so that $A_{n-1} = A_n$. Concerning the condition (91), equation (106) implies

$$e^{i\theta} A_{n-1} = A_{n+1}.$$ (116)

which can be written as

$$|A_1\rangle = |A_1(\theta)\rangle = \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix}.$$ (117)

Therefore, the eigenvalue problem has been reduced to find the energies $E$ for which the amplitudes $A_n$, satisfying (112) and (117), yield wave functions (106) that are normalized in the discrete sense (Kronecker delta function) or in the continuous sense (Dirac delta function), corresponding to the discrete or continuum spectrum of the Hamiltonian. To this aim, we shall also need the norm and scalar product of the wave functions written in terms of the amplitudes. Using equation (92), the norm of (106) is given by

$$\langle \chi | \chi \rangle = \sum_{n=1}^{\infty} \frac{\ell_{n+1}}{\ell_n} \langle A_n | A_n \rangle,$$

$$\langle A_n | A_n \rangle = |A_{-n}|^2 + |A_{n+1}|^2,$$ (118)
and the scalar product of two eigenfunctions with energies $E_1$ and $E_2$ is given by

$$\langle \chi | \chi \rangle = \sum_{n=1}^{\infty} \left\{ \left[ \epsilon_{n+1}^{E_1} - \epsilon_{n}^{E_1} \right] A_{-n}^{(1)} A_{-n}^{(2)} - \left[ \epsilon_{n+1}^{E_2} - \epsilon_{n}^{E_2} \right] A_{+n}^{(1)} A_{+n}^{(2)} \right\}$$

(119)

where $E_{12} = E_1 - E_2 \neq 0$ and $A_{\mp n}^{*}$ is the complex conjugate of $A_{\pm n}$. Since the Hamiltonian is self-adjoint, this product will vanish.

### 4.6. Semiclassical approximation

The recursion relation (112), together with the initial condition (117), gives all the vectors $A_k$ in terms of $A_1(\theta)$

$$\left| A_k \right\rangle = T_k^{-1} T_{k-1}^{-1} \cdots T_2^{-1} \left| A_1(\theta) \right\rangle, \quad k \geq 2.$$  (120)

Except for some simple cases, as the harmonic model (see appendix A), it will be impossible to find close analytic expressions of the product of matrices appearing in (120). The only hope to make progress is to evaluate (120) in the limit where the coefficients $\varrho_n$ are infinitesimally small. We shall then assume that $\varrho_n$ are proportional to a parameter $\varepsilon$ that will be taken to zero at the end of the computation. This parameter plays the role of Planck’s constant, so the limit $\varepsilon \to 0$ will be interpreted as semiclassical. This interpretation is supported by the discretization of the massive Dirac equation, which led to equation (80), according to which $r_n = m \ell / (4n^{3/2})$. The connection with the average Riemann zeros was achieved for $m \ell = 2\pi$, which corresponds in the semiclassical $xp$ Berry–Keating model to the Planck constant $\ell \hbar = 2\pi \hbar$.

Taking $\varrho_n = O(\varepsilon)$, the matrix $T_n$ given in equation (115) can be replaced in the limit $\varepsilon \to 0$, by

$$T_n \approx 1 + \tau_n \approx e^{\varepsilon \tau_n} + O(\varepsilon^2), \quad \tau_n = \begin{pmatrix} 0 & 2\varrho_n \varepsilon^{-2} \\ 2\varrho_n^* \varepsilon^2 & 0 \end{pmatrix}.$$  (121)

$T_n$ can be expressed exactly as the exponential of a matrix of the form of $\tau_0$, but in the limit $\varepsilon \to 0$ it will converge towards the expression given in (121) up to order $\varepsilon^2$. Plugging (121) into (120) yields

$$\left| A_k \right\rangle = e^{-\varepsilon \tau_1} e^{-\varepsilon \tau_2} \cdots e^{-\varepsilon \tau_k} \left| A_1(\theta) \right\rangle + O(\varepsilon^2).$$  (122)

It is convenient to define a matrix $\tau_1$, of order $\varepsilon$, corresponding to the choice $n = 1$ of $\tau_n$, which does not depend on $E$ because $\ell_1 = 1$. The vector $|A_1(\theta')\rangle$ can be replaced by $e^{-\theta^*} |A_1(\theta')\rangle$, where $\theta^*$ is equal to $\theta$ up to terms of order $\varepsilon$. Equation (122) can then be written as

$$\left| A_k \right\rangle = e^{-\varepsilon \tau_1} e^{-\varepsilon \tau_2} \cdots e^{-\varepsilon \tau_k} \left| A_1(\theta) \right\rangle + O(\varepsilon^2).$$  (123)

where $\theta'$ has been replaced by $\theta$ since they become the same quantity in the limit $\varepsilon \to 0$. The product of exponentials of matrices can be approximated by the Baker–Campbell–Haussdorf formula [68]

$$e^{iA}e^{iB} = e^{i(A+B)} + O(\varepsilon^2),$$  (124)

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which yields
\[ |A_k^\tau\rangle = |A_k^\tau\rangle + O(\epsilon^2), \quad |A_k\rangle \equiv e^{-\sum_{n=1}^k r_n} |A_k(\theta)\rangle. \] (125)

Let us next define
\[ e^{-i\Phi_k R_k} = \sum_{n=1}^k e_n e^{-2iE_n}, \quad e^{i\Phi_k R_k} = \sum_{n=1}^k e_n e^{2iE_n}, \] (126)
where \( R_k \) and \( \Phi_k \) are real for real values of \( E \). The factor 2 multiplying \( \rho_n \) in equation (121) can be absorbed into the parameter \( \epsilon \). So, without loss of generality, we can write
\[ |A_k^\tau\rangle = e^{-R_k(\cos \Phi_k \sigma^x + \sin \Phi_k \sigma^y)} |A_k(\theta)\rangle, \] (127)
whose norm is
\[ \langle A_k^\tau | A_k^\tau \rangle = e^{2R_k(1 - \cos (\Phi_k - \theta))} + e^{-2R_k(1 + \cos (\Phi_k - \theta))}. \] (128)

The discrete eigenvalues of the Hamiltonian are those for which the norm (118) is finite, which is ensured if (128) vanishes sufficiently fast when \( k \to \infty \). The approximation that we have performed above is a sort of inverse Trotter–Suzuki decomposition, \( \lim_{n \to \infty} (e^{A/n} e^{B/n})^n = e^{A+B} \), where a product of exponentials of non-commuting operators is replaced by the exponential of their sum [78, 79]. Let us consider some examples.

4.6.1. Harmonic model. This model is defined by the parameters (see equation (66) with \( \ell_0 = 1 \))
\[ \ell_n = e^{2n}, \quad q_n = \epsilon, \quad n = 0, \ldots, \infty. \] (129)
Let us remind that in this case the mirrors (delta functions) are labeled by \( n = 1, 2, \ldots \), while the boundary is located at \( \ell_0 = 1 \). \( \epsilon \) can be positive or negative and is the semiclassical parameter. This model has an exact solution (which is given in appendix A) that allows us to verify the semiclassical approximation that is done below. From (126) one finds
\[ e^{-i\Phi_k R_k} = \epsilon \sum_{n=0}^k e^{-inE} = \begin{cases} \epsilon (k + 1) & E \in 2\pi \mathbb{Z}, \\ \epsilon \frac{\sin ((k + 1)E/2)}{\sin (E/2)} e^{-iE/2} & E \notin 2\pi \mathbb{Z}. \end{cases} \] (130)

In the case \( E \in 2\pi \mathbb{Z} \), one can choose \( R_k = \epsilon (k + 1) \) and \( \Phi_k = 0 \). This identification is not unique but simplifies the calculation. The norm (128) is given by
\[ E \in 2\pi \mathbb{Z} \to \langle A_k^\tau | A_k^\tau \rangle = e^{2\epsilon (k+1)}(1 - \cos \theta) + e^{-2\epsilon (k+1)}(1 + \cos \theta), \] (131)
while the norm of the wave function \( \chi^\tau \), whose amplitudes are \( A_k^\tau \), reads (see equation (118))
\[ \langle \chi^\tau | \chi^\tau \rangle = \frac{1}{2} \sum_{n=0}^\infty \langle A_n^\tau | A_n^\tau \rangle = \frac{1}{2} \sum_{n=0}^\infty \left[ e^{2\epsilon (n+1)}(1 - \cos \theta) + e^{-2\epsilon (n+1)}(1 + \cos \theta) \right]. \] (132)
This series diverges for all values of \( \theta \) different from 0 and \( \pi \). This means that the energies \( E = 2\pi n \) do not belong to the spectrum. On the other hand, if
\[ \theta = 0 \quad \text{or} \quad \theta = \pi \quad \text{for} \quad \epsilon > 0 \] (133)
then the state has the norm
\[ \langle \chi^T | \chi^T \rangle = \frac{1}{e^{2|\varepsilon|} - 1} \rightarrow \frac{1}{2|\varepsilon|}, \quad \varepsilon \rightarrow 0, \] (134)

The parameter \( \varepsilon \) can be absorbed into the normalization constant of the state. Hence, in the two cases (133) there is an infinite number of normalized states with energies \( E_{nm} = 2\pi m (m \in \mathbb{Z}) \).

In the case \( E \notin 2\pi \mathbb{Z} \), one can choose
\[ R_k = \varepsilon \sin \left( \frac{(k+1)E}{2} \right), \quad \Phi_k = \frac{kE}{2}, \] (135)
The limit \( E \rightarrow 2\pi m \), yields \( R_k \rightarrow \varepsilon (-1)^m (k+1), \; e^{\Phi_k} \rightarrow (-1)^m \) and one recovers the previous result. If \( E \notin 2\pi \mathbb{Z} \), then equation (135) shows that \( \varepsilon \Phi \) will be bounded and the corresponding states will be normalizable in terms of Dirac delta functions; belonging therefore to the continuum spectrum.

In summary, the spectrum of Hamiltonian of the harmonic model, in the limit \( \varepsilon \rightarrow 0 \), is given by the union of a continuum part, Spec\(_c\), and a discrete part, Spec\(_d\), where
\[ \theta = 0, \; \varepsilon \rightarrow 0^+ \text{ or } \theta = \pi, \; \varepsilon \rightarrow 0^-: \text{Spec}_c \rightarrow \mathbb{R} - 2\pi \mathbb{Z}, \; \text{Spec}_d = 2\pi \mathbb{Z} \] (136)
\[ \theta \neq 0 \text{ and } \pi, \; \varepsilon \rightarrow 0^\pm \quad : \text{Spec}_c \rightarrow \mathbb{R} - 2\pi \mathbb{Z}, \; \text{Spec}_d = \emptyset \] (137)
Note that in the case \( \theta \neq 0, \pi \), the energies \( E = 2\pi m \) are missing in the continuum. For finite values of \( \varepsilon \) the spectrum is given by (see appendix A)
\[ \theta = 0, \; \varepsilon > 0 \text{ or } \theta = \pi, \; \varepsilon < 0: \]
\[ \text{Spec}_c = \bigcup_{\delta = -\infty}^{\infty} 2\pi [n + \delta, n + 1 - \delta], \; \text{Spec}_d = 2\pi \mathbb{Z}, \] (138)
where \( \sin (\pi \delta) = 2e/(1 + e^2) \) (see equation (A.16)). So, if \( \varepsilon \rightarrow 0 \), then the gap between the intervals \( 2\pi [n + \delta, n + 1 - \delta] \) closes and the continuum spectrum becomes \( \mathbb{R} - 2\pi \mathbb{Z} \) and the discrete spectrum \( 2\pi \mathbb{Z} \), as in (136). When \( \theta \neq 0, \pi \), one also recovers the spectrum in (137).

Let us consider another example with the same values of \( \ell_n \) but with exponential reflecting reflection amplitudes
\[ \ell_n = e^{\lambda/2}, \; \eta_n = \varepsilon e^{-i\lambda}, \; \lambda > 0, \quad n = 0, \ldots, \infty, \] (139)
which yields
\[ e^{-i\Phi_k} R_k = \varepsilon e^{-i(\lambda + iE)(k+1)/2} \frac{\sinh ((\lambda + iE)(k+1)/2)}{\sinh ((\lambda + iE)/2)}. \] (140)
If \( \lambda = 0 \), then one recovers equation (130). Let us see if the discrete spectrum of the previous model survives when \( \lambda > 0 \). Taking \( E \in 2\pi \mathbb{Z} \) yields
\[ E \in 2\pi \mathbb{Z} \rightarrow e^{-i\Phi_k} R_k = \frac{\varepsilon - e^{-i(k+1)}}{1 - e^{-\lambda}}. \] (141)
Hence, for any positive value of \( \lambda \), the quantities \( R_k \) are bounded, and therefore the state belongs to the continuum, that is the discrete spectrum disappears, but it can be recovered in the limit \( \lambda \rightarrow 0 \), where \( R_k \rightarrow \varepsilon(k+1) \).

4.6.2. Polylogarithm model. This model is defined by the parameters (see equation (52) with \( \ell_1 = 1 \))
The positivity conditions on $\sigma$ and $\lambda$ ensure that $q_n \to 0$ as $n \to \infty$, including the case where $\lambda = 0$. The norm of the wave function $\chi_T$ is given by (see equation (118))

$$
\left\langle \chi^T | \chi^T \right\rangle = \frac{1}{2} \sum_{n=1}^{\infty} \log \left( 1 + \frac{1}{n} \right) \left\langle A_n^T | A_n^T \right\rangle,
$$

so its convergence depends on the asymptotic behavior of $\left\langle A_n^T | A_n^T \right\rangle / n$. The ansatz (142) yields

$$
e^{-i\phi_n} R_k = \varepsilon \sum_{n=1}^{k} \frac{e^{-\frac{1}{n}}}{n^{\sigma+iE}},
$$

which in limit $k \to \infty$ becomes

$$
e^{-i\phi_n} R_\infty = \lim_{k \to \infty} e^{-i\phi_n} R_k = e Li_{\sigma+iE} \left( e^{-1} \right),
$$

where $Li(z)$ is the polylogarithm function [80]. In the limit $\lambda \to 0^+$ there is the expansion

$$
e^{-i\phi_n} R_\infty = \varepsilon \left[ \lambda^{\sigma-1+iE} \Gamma(1 - \sigma - iE) + \zeta(\sigma + iE) + O(\lambda) \right].
$$

If $\sigma > 1$, then the term proportional to $\lambda^{\sigma-1+iE}$ drops out and one is left with

$$\sigma > 1, \lambda \to 0 \to e^{-i\phi_n} R_\infty = \varepsilon \zeta(\sigma + iE),
$$

which implies that the norm of $A_k^T$ converges towards a constant, for any value of $\theta$, and therefore the series (143) diverges logarithmically, signaling a continuum of states normalizable in the Dirac delta sense. If $\sigma < 1$, then the term proportional to $\lambda^{\sigma-1+iE}$ dominates, hence

$$\sigma < 1, \lambda \to 0 \to e^{-i\phi_n} R_\infty \approx \varepsilon \lambda^{\sigma-1+iE} \Gamma(1 - \sigma - iE),
$$

leading to

$$R_\infty \approx \left| e^{\lambda^{\sigma-1}} \Gamma(1 - \sigma - iE) \right|, \quad e^{-i\phi_n} \approx \text{sign}(\varepsilon) \lambda^{-iE} \left( \frac{\Gamma(1 - \sigma - iE)}{\Gamma(1 - \sigma + iE)} \right)^{1/2},
$$

If $\lambda$ is kept fixed, then the norm of $A_k^T$ is bounded and one gets again a continuum spectrum. Let us see what happens in the limit $\lambda \to 0$. Here $R_\infty \to +\infty$, and therefore the norm of $A_k^T$ does not blow up and actually converges to zero, if and only if

$$\cos \left( \phi_n - \theta \right) = 1 \implies \text{sign}(\varepsilon) \lambda^{-iE} \left( \frac{\Gamma(1 - \sigma + iE)}{\Gamma(1 - \sigma - iE)} \right)^{1/2} = e^{i\theta},
$$

This result suggests that a regularized version of this model, involving the limit $\lambda \to 0^+$, should contain eigenstates satisfying (150). For $E \gg 1$, the Stirling formula provides the asymptotic behavior

$$n(E) \approx \frac{E}{2\pi} \log \frac{1}{\lambda} + \frac{E}{2\pi} \log \frac{E}{2\pi e} + \frac{1}{4} \left( \frac{3}{2} - \sigma - \text{sign} \varepsilon \right) - \frac{\theta}{2\pi}, E \gg 1,
$$

that in the limit $\lambda \to 0$, becomes a continuum. The parameter $\lambda$ regularizes the model and in some respects is analogue to the cutoff $\Lambda$ in Connes’s $xp$ model, with $\Lambda \propto 1/\lambda$. However, in equation (151) the term $E/(2\pi e) \log E/(2\pi e)$, appears with an opposite sign, as compared with Connes model, and it does not have the meaning of missing spectral lines but rather of a finite
energy correction. Notice also that if $\sigma = 1/2$, then the spectrum has the symmetry $E \leftrightarrow -E$, provided $\theta = 0$ if $\varepsilon > 0$ or $\theta = \pi$ if $\varepsilon < 0$, as in the harmonic model that was analyzed above.

In summary, for $\sigma > 1$ the spectrum is a continuum that is related to the Riemann zeta function $\zeta(\sigma + i\varepsilon)$. This result is consistent with the studies carried by several authors in the past where the zeta function appears in connection to the scattering states of some physical system [81–89]. However, for $0 < \sigma < 1$, the connection with the zeta function is lost and only the smoothed zeros appear as a finite size correction to the level counting formula, in analogy with Connes’s version on the $xp$ model, but with the opposite sign. We are then forced to look for another ansatz for the reflection coefficients $\varrho_n$, if the zeros are to be realized as discrete eigenvalues of the Hamiltonian $H_R$ in the semiclassical limit. A hint is provided by the harmonic model, where the eigenvalues $E_n = 2\pi n$, arises from the blow up of $R_k(E_n)$ as $k \to \infty$. This property leads us to the next example.

4.6.3. Riemann model. The Riemann model is defined by the parameters

$$\ell_n = n^{1/2}, \quad \varrho_n = i \frac{\mu(n)}{n^\sigma}(\sigma > 0), \quad n = 1, \ldots, \infty,$$

(152)

where $\mu(n)$ is the Moebius function that vanishes if $n$ contains as divisor the square of a prime, and it is equal to 1 ($-1$) if $n$ is the product of an even (odd) number of distinct primes, that is [90]

$$\mu(n) = \begin{cases} (-1)^r & \text{if } n = p_1 \cdots p_r \\ 0 & \text{if } \exists \ p, p^2 | n. \end{cases}$$

(153)

In this expression $p_1, \ldots, p_r$ are different prime numbers. Integer numbers for which $\mu(n) \neq 0$ are called square free. The Moebius function has been used to construct an ideal gas of primons with fermionic statistics [91, 92]. In our model, $\mu(n)$ appears in order to amplify the interference between waves so that the Riemann zeros are not swept out in the semiclassical limit and become visible. In the limit $k \to \infty$ one finds

$$e^{-i\varrho_n} R_\infty = e^\int_0^\infty \sum_{n=1}^\infty \frac{\mu(n)}{n^{\sigma+i\varepsilon}} = \frac{\varepsilon}{\zeta(\sigma + i\varepsilon)}, \quad \text{if } \begin{cases} \sigma \geq 1 \\ \frac{1}{2} < \sigma < 1, \quad \text{RH: True}. \end{cases}$$

(154)

The case $\sigma = 1$ is equivalent to the Prime Number Theory, which was proved by Hadamard and de la Vallée–Poussin by showing that $\zeta(1 + it) \neq 0, \forall t \in \mathbb{R}$ [2]. By repeating the analysis performed in the previous models, we find that for $\sigma \geq 1$, the spectrum is given by $\mathbb{R}$. Indeed, for all values of $E$, the norm of $A_k$ approaches a constant value in the limit $k \to \infty$, and consequently the norm (143) diverges logarithmically. The same thing occurs for $1/2 < \sigma < 1$ under the RH, according to which this region will be free of zeros.

We are then left with the model with $\sigma = 1/2$ to provide a spectral realization of the zeros. Let us first approach this case in the limit $\sigma \to 1/2$ when $E = E_n$ is a zero, that is $\zeta(1/2 + iE_n) = 0$. By expressing the zeta function on the critical line as $\zeta(1/2 + iE) = Z(E)e^{-i\theta(E)}$, where $Z(E)$ and $\theta(E)$ are the Riemann–Siegel functions [2], one finds

$$\zeta(\sigma + iE_n) = -i(\sigma - 1/2)Z'(E_n)e^{-i\theta(E_n)} + O((\sigma - 1/2)^2), \quad \sigma \to \frac{1}{2}^+, \quad (155)$$

where $Z'(E) = dZ(E)/dE$. For the sake of argument, we have assumed that $E_n$ is a simple zero of $Z(E)$, a fact that is unknown to hold for all zeros. If this is the case, then the sign of the derivative of $Z(E)$ at the zeros satisfies the rule.
where the positive zeros are labeled as \( n = 1, 2, \ldots \), and the negative zeros as \( n = -1, -2, \ldots \). Equation (156) can be derived from the continuity of \( Z(E) \) and the fact that \( Z(E_{1/2}) < 0 \). Plugging (155) and (156) into (154) gives

\[
e^{-i\vartheta_0} R_\infty \approx \frac{e^{i\vartheta(E_n)+\frac{1}{2}}}{(\sigma - 1/2) Z'(E_n)} = \frac{e^{i\vartheta(E_n)-\pi(n+\frac{1}{2}\text{sign}(n))}}{(\sigma - 1/2) Z'(E_n)}, \quad \sigma \to \frac{1}{2},
\]

which leads to the choice

\[
R_\infty \sim \frac{e}{(\sigma - 1/2) Z'(E_n)} > 0, \quad e^{-i\vartheta_0} \sim e^{i\vartheta(E_n)-\pi(n+\frac{1}{2}\text{sign}(n))}.
\]

Hence, the limit \( \sigma \to (1/2)^+ \) implies \( R_\infty \to +\infty \), whereby the amplitude \( A_{\infty}^T \) blows up unless the parameter \( \vartheta \) satisfies

\[
\cos\left(\vartheta_0 - \theta\right) = 1 \to \cos\left(\vartheta(E_n) - \pi\left(n + \frac{1}{2}\text{sign}(n)\right) + \theta\right) = 1,
\]

in which case \( A_{\infty}^T = 0 \), which corresponds to a normalizable state. This is an important result that we shall derive more rigorously a bit later on, but for now let us discuss its implications.

- The value of \( \vartheta(E_n) \) satisfying equation (159) can be written as (with \( E_n > 0 \))

\[
\vartheta(E_n) = n + \frac{1}{2} - \frac{\theta(E_n)}{\pi} = n - \left\langle N(E_n) \right\rangle + \frac{3}{2},
\]

where \( \left\langle N(E) \right\rangle \) is the average number of zeros in the range \([0, E]\) (see equation (43)). In the absence of fluctuations, the average would be exact; that is, \( \left\langle N(E_n) \right\rangle = n \), whereby \( \vartheta(E_n) = 3\pi/2, \forall n > 0 \). Hence, a single value of \( \vartheta \) would work for all of the zeros. But the existence of fluctuations makes things more interesting. To hear a given zero [87] one has to fine tune \( \vartheta \) according to equation (160), which depends on the phase of the zeta function, \( \theta(E_n) \). One thus obtains that the Riemann zeros and the phase of the zeta function both acquire a physical meaning in the model.

- Berry [13], and Badhuri et al [88] have argued that a better approximation to the average zeros is obtained if
This result can be visualized by plotting the real and imaginary parts of $\zeta(1/2 + it)$ in the complex plane for a large interval of $t$. One obtains a collection of loops that cut the real axis at the Gram points where $\sin \theta(t) = 0$ and, just before crossing the origin at $t = E_n$, the loop cuts the imaginary axis, where $\cos \theta(t) = 0$. Badhuri et al also show that $\theta(t)$ gives roughly the scattering phase shift of a non-relativistic particle in an inverted harmonic potential ($V(x) \propto -x^2$) with a hard wall at the origin. Now, replacing (161) into (160) gives that on average $\theta(E_n) \approx n \mod (2\pi)$. This result is confirmed in figure 3, which shows $\theta(E_n)$ for the first $10^3$ zeros together with a histogram.

So far, we have considered the value of $e^{-i\Phi}R_\infty$, using equation (154). To show the existence of discrete states for $\sigma = 1/2$, we need the finite sum giving $e^{-i\Phi}R_k$, which can be computed using Perron’s formula [90, 93]

$$\sum_{n \in \mathbb{Z}} \frac{\mu(n)}{n^s} = \lim_{T \to \infty} \int_{c-iT}^{c+iT} ds \frac{1}{2\pi i} \frac{x^s}{\zeta(s+z)} \delta(s), \quad c > 0, \quad c > 1 - \sigma, \quad \sigma = \text{Re} \, x \tag{162}$$

where $\sum^*$ means that the last term in the sum must be multiplied by $1/2$ when $x$ is an integer. The integral (162) can be done by residue calculus [93]

$$\lim_{T \to \infty} \int_{c-iT}^{c+iT} ds \frac{1}{2\pi i} F(s) = \sum_{\text{Re} \, s_j < c} \text{Res}_{s_j} F(s), \quad F(s) = \frac{1}{\zeta(s+z)} \frac{x^s}{s}, \tag{163}$$

where the sum runs over the poles $s_j$ of $F(s)$ located to the left of the line of integration; that is, $\text{Re} \, s_j < c$. For $z = 0$, the sum (162) is basically the Mertens functions $M(x) = \sum_{n=1}^x \mu(n)$ that play an important role in Number Theory [2–4]. Here we are interested in the values $z = iE + iE$, with $E$ a real number. Equation (162) imposes the condition $c > 1 - \text{Re} \, z = 1/2$. Hence, the poles of $F(s)$ that contribute to (163) have their real part smaller than $c$; that is, $\text{Re} \, s_j < 1/2$. The origin $s = 0$ is a simple pole $F(s)$ if $\zeta(z) \neq 0$ and a multiple pole if $\zeta(z) = 0$. In the latter case, we shall assume that $z$ is a simple zero $\zeta(z)$ so that the pole $F(s)$ is double. The corresponding residues are given by

$$\text{Res}_{s=0} F(s) = \begin{cases} 1/\zeta(z) & \text{if } \zeta(z) \neq 0, \\ \log(1/\zeta(z)) & \text{if } \zeta(z) = 0, \zeta'(z) \neq 0. \end{cases} \tag{164}$$

The remaining poles of $F(s)$ come from the zeros of $\zeta(s+z)$, except for the case $s = 0$, which is included in (164). The trivial zeros of $\zeta$ contribute with the poles $s_n = -2n - z$ ($n = 1, 2, ...$), with residue

$$\text{Res}_{s=-2n-z} F(s) = \frac{x^{-2n-z}}{-(2n+z)\zeta'(-2n)}, \quad n = 1, 2, ... \infty. \tag{165}$$

The non-trivial zeros of $\zeta$, denoted as $\rho_m$, contribute with the poles $s_m = \rho_m - z$ (note that $\text{Re} \, s_m < 1/2$), with residue

$$\text{Res}_{s=\rho_m-z} F(s) = \frac{x^\rho_m-z}{(\rho_m-z)\zeta'(\rho_m)}, \quad \rho_m \neq z \tag{166}$$
By collecting the results, we find that

\[ \sum_{\zeta \neq \xi} \mu(n) \frac{n^z}{\zeta(z)} = \sum_{r_m} \frac{\chi_{r_m}}{\zeta(r_m)} + \sum_{a=1}^{\infty} \frac{\chi_{2a-\zeta}}{\zeta(-2n)} \quad \text{if } \zeta(z) \neq 0, \]

\[ \sum_{\zeta \neq \xi} \mu(n) \frac{n^z}{\zeta'(z)} = \sum_{r_m} \frac{\chi_{r_m}}{\zeta(r_m)} \frac{\chi_{r_m}}{\zeta'(r_m)} + \sum_{a=1}^{\infty} \frac{\chi_{2a-\zeta}}{\zeta(-2n)} \quad \text{if } \zeta(z) = 0, \zeta'(z) \neq 0, \]

These equations are formally exact, so it would be interesting to prove them and find their range of validity. In what follows, we shall derive their consequences. Let us recall that the lhs of (167) gives essentially \( e^{-i\phi(E)} R_k \), where \( k \) is to be identified with \( x \). Neglecting for a while the summands in these expressions, one finds

\[ e^{-i\phi(E)} R_k \sim \frac{\epsilon e^{i\theta(E)}}{Z(E)}, \quad Z(E) \neq 0, \]

\[ e^{-i\phi(E)} R_k \sim \frac{\epsilon \log \frac{\xi}{\xi(E)}}{Z'(E)}, \quad Z(E) = 0, \quad Z'(E) \neq 0. \]

In the first case, \( R_k \) remains bounded in the limit \( k \to \infty \), which would yield a state in the continuum. In the second case, we choose

\[ R_k \sim \frac{\epsilon \log k}{Z'(E)}, \quad k \to \infty \]

which can be compared with (158). Then, imposing equation (159) yields the asymptotic behavior of the norm of \( A_k^T \) (see equation (128))

\[ \langle A_k^T \rangle \sim 2e^{-2R_k} = k^{-22z} Z(E), \quad k \to \infty. \]
and a wave function $\chi^T$ whose norm given by (143)

$$\langle \chi^T | \chi^T \rangle \sim \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^{1+2\varepsilon \theta(E)}} = \zeta \left(1 + \frac{2\varepsilon}{|Z'(E_n)|}\right),$$

(171)

which is finite for any $\varepsilon > 0$ corresponding to a discrete eigenstate with energy $E_n$.

Figure 4 shows $|A_k|^2$ in the case where $E$ corresponds to a state in the continuum, and for the first zero $E_1 = 14.13 \ldots$, corresponding to a discrete state. In the latter case we took $\theta(E) = \pi$, which guarantees that the norm converges to zero except for some jumps. For other values, like $\theta = \pi$, the norm increases with $k$. The same pattern is observed for other zeros. The values of $\theta(E_n)$ are mostly concentrated around 0 (see figure 3). Finally, we have computed the norm of the vector $\chi^T$, using equation (143) observing that, for the zeros, it converges to a finite value with the choice (160) and diverges in the remaining cases. These results are in agreement with equation (171).

Let us now return to the summands in equation (167) that we neglected in the previous computation. The last term corresponding to the trivial zeros quickly converges to 0 as $x = k \to \infty$. The term associated to the non-trivial zeros on the critical line, $\rho_m = \frac{1}{2} + i E_m$, $E_m \in \mathbb{R}$, oscillates as $\chi e^{-\varepsilon \theta(E_m) \log x}$, and we expect that it gives subleading contributions to the main term that goes with $\log x$. Finally, a zero off the critical line, say $\rho_m = 1/2 + E_m + i E_m, E_m, E_m + 0 \in \mathbb{R}$, would give a contribution $\chi e^{\varepsilon (E_m - E) \log x}$, which dominates over the remaining terms, for all values of $E$, leading to

$$R_k(E_n) \sim e^{C(\rho_m, E)} k^{E_m} > 0, \quad e^{-i\Phi_k(E_n)} \sim e^{i(E_m - E) \log k + a(E_n)}, \quad k \to \infty$$

(172)

where $C(\rho_m, E)$ and $a(\rho_m, E)$ do not depend on $k$. The expression of $R_k(E)$ diverges when $k \to \infty$ but, unlike the case of (169), the phase $\Phi_k(E)$ cannot be fixed to a value that cancels the divergent term $e^{2kE}$ in the norm (128). Hence, $|A_k^T|$ would grow typically as $\exp (\varepsilon C k^{E_m})$, so that the wave function $\chi^T$ will not be normalizable even in the continuum sense. This occurs for any value of $E$, so we arrive at the paradoxical conclusion that a zero off the critical line implies that the Hamiltonian $H_R$ does not have eigenstates! That is certainly not the case because $H_R$ is a well defined self-adjoint operator, so we must conclude that off critical zeros do not exist. Although this result seems to provide a proof of the Riemann hypothesis, one must be very cautious since it relies on several unproven assumptions.

**Comments:**

- In the spectral realization proposed above there cannot exist zeros outside the critical line in the form of resonances. As explained above, their presence leads to the non-existence of eigenvectors of the Hamiltonian $H_R$.
- von Neumann and Wigner showed that in ordinary Quantum Mechanics it is possible to have a bound state immersed in the continuum [68, 94]. They used a potential that decays as $1/r$, with oscillations that trap the particle thanks to interference effects. There is a general class of models with this property [95–97], and they all require a fine tuning of couplings. That this phenomena may happen for the Riemann zeros was suggested in [32], using an $xp$ model with non-local interactions.
- The delta function potential needed to reproduce the zeros depends on the Moebius function $\mu(n)$, which exhibits an almost random behavior. This result is reminiscent of the fractal structure of the quantum mechanical potentials that were built to reproduce the lowest values of prime numbers and the Riemann zeros. The latter potentials were built
from the smooth ones obtained by Mussardo, for the primes \([98]\), and Wu and Sprung, for the zeros \([99]\), and have a fractal dimension around 2 and 1.5, respectively \([100–102]\).

4.6.4. Dirichlet model. We shall briefly describe how the previous results can be generalized to Dirichlet \(L\)-functions. These are analytic functions constructed from a Dirichlet character \(\chi\) as \([4]\)

\[
L_{\chi}(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)p^{-s}} \quad \Re s > 1,
\]

where the product runs over the prime numbers (Euler formula). A character \(\chi\) of modulus \(q\) is an arithmetic function satisfying

\[
\chi(mn) = \chi(m)\chi(n), \quad \chi(1) = 1, \quad \chi(m + qn) = \chi(m), \quad \chi(n) = 0 \text{ if } q \text{ divides } n \quad [4].
\]

One has that \(\chi(n)\) is 1 or 0, and if \(\chi(n) = \pm 1, 0\), then the character is called real. The number of characters with modulus \(q\) is given by the Euler totient function \(\varphi(q)\), which counts the number of coprime divisors of \(q\). For \(q = 1\), the character is \(\chi(n) = 1, \forall n\), which corresponds to the Riemann zeta function \(\zeta(s)\). Primitive characters are those that cannot be written as products of characters of smaller modulus \(q\), and they can be classified into even or odd if \(\chi(-1) = 1, -1\), respectively.

The \(L\)-functions associated with primitive characters satisfy the functional relation

\[
\Lambda_{\chi}(s) \equiv \left(\frac{\pi}{q}\right)^{-(s+a)/2} \Gamma \left(s + \frac{a_\chi}{2}\right) L_{\chi}(s) = e^{i\pi \frac{a_\chi}{2}} L_{\chi}(1 - s), \quad e^{ia_\chi} = i^{-\frac{a_\chi}{q}} G_{\chi}
\]

(174)

where \(a_\chi = 0(1)\) for an even (odd) character, \(\chi^\ast\) is the character conjugate to \(\chi\), such as \(\chi^\ast(n) = \chi(n)\), and \(G_{\chi}\) is the Gauss sum

\[
G_{\chi} = \sum_{n=1}^{q} \chi(n)e^{2\pi in/q}, \quad |G_{\chi}| = \sqrt{q}.
\]

(175)

On the critical line \(s = 1/2 + it\), equation (174) reads

\[
\left(\frac{\pi}{q}\right)^{-it/2} \Gamma \left(\frac{1 + 2a_\chi}{4} + \frac{it}{2}\right) L_{\chi}(s) = e^{i\pi \frac{a_\chi}{2}} \Gamma \left(\frac{1 + 2a_\chi}{4} - \frac{it}{2}\right) L_{\chi}\left(\frac{1}{2} + it\right).
\]

(176)

For \(t\) real, it is useful to parameterize the \(L\) functions as

\[
t \in \mathbb{R} \rightarrow L_{\chi}\left(\frac{1}{2} - it\right) = L_{\chi}\left(\frac{1}{2} + it\right) = Z_{\chi}(t)e^{-i\theta_{\chi}(t)}.
\]

(177)

where \(Z_{\chi}(t)\) is real and \(e^{i\theta_{\chi}(t)}\) is a phase that can be found from equation (176)

\[
e^{2i\theta_{\chi}(t)} \equiv e^{i\pi \frac{a_\chi}{2}} \Gamma \left(\frac{1 + 2a_\chi}{4} + \frac{it}{2}\right) \Gamma \left(\frac{1 + 2a_\chi}{4} - \frac{it}{2}\right).
\]

(178)

\(\theta_{\chi}(t)\) coincides with \(\theta(t)\) when \(\chi\) is the identity character. For real characters, the zeros of \(L_{\chi}\) appear symmetrically, \(L_{\chi}(1/2 + it) = L_{\chi}(1/2 - it) = 0\), which is reflected in the properties, \(Z_{\chi}(-t) = Z(t)\) and \(\theta_{\chi}(-t) = -\theta_{\chi}(t)\). However, for complex characters, this symmetry is broken.
The Dirac model associated to $L_x$ will be defined by the parameters
\[
\ell_n = n^{1/2}, \quad q_n = \frac{\epsilon \mu(n) \chi(n)}{n^\sigma} (\sigma > 0), \quad n = 1, \ldots, \infty.
\] (179)

Note that one can deal with complex characters thanks to the existence of two types of mass terms, $\psi\psi^\dagger$ and $\psi\gamma\psi^\dagger$. The characters $\chi(n)$ acquire a physical meaning related to the reflection coefficient of the $n^{th}$ mirror. This provides a unified framework to deal with the whole family of Dirichlet $L$-functions.

We can repeat the analysis done for the zeta function to find the discrete eigenenergies in the spectrum. For example, the asymptotic limit of $\Phi(-e^R/n)\ell_k$ is given by
\[
\sum_{\epsilon = \mu} \chi(n) = \Phi(\sigma - \infty) = \infty + \infty e^{R \ln n} n \text{Li}_E(\sigma), (180)
\]
which shows that the zeros of $L_x(1/2 + iE)$ appear as poles of $R_x$, which eventually leads to discrete states provided that $\theta$ is fine tuned appropriately. Assuming that the zeros of $L_x(1/2 + iE)$ are simple, the value of $\theta(E_n)$ for which $E_n$ is a discrete state is given by
\[
\frac{\theta(E_n)}{\pi} = n + \frac{1}{2} \left(1 + b_x + \text{sign}(n)\right) \frac{\theta(E_n)}{\pi}, \quad b_x = \text{sign} Z_x(1/2). (181)
\]
For the zeta function, $b_x = -1$, equation (160) is recovered from (181). We expect the zeros of the Dirichlet $L$ functions, associated to primitive characters, to form the discrete spectrum of the corresponding Hamiltonians. This would amount to a proof of the Generalized Riemann hypothesis.

### 5. CPT symmetries and AZ classes

The statistical properties of the Riemann zeros, conjectured by Montgomery and confirmed numerically by Odlyzko, have been one of the main motivations to search for a spectral origin of these numbers. The conjecture is that the zeros satisfy, locally, the GUE law, which implies that the Riemann dynamics break the time reversal symmetry. Deviations from the GUE law were later identified by Berry and collaborators as a trace of the semiclassical origin of the zeros and a breakdown of universality. It is thus of great interest to study the discrete CPT symmetries and RMT universality classes of the Hamiltonians that were discussed in previous sections.

The action of the time reversal symmetry ($T$), charge conjugation or particle-hole symmetry ($C$), and parity or chirality ($P$) on a vector $\psi$ are defined as $[103, 104]$
\[
\psi^T = T \psi^*, \quad \psi^C = C \psi^*, \quad \psi^P = P \psi, (182)
\]
where \( T, C, P \) are unitary matrices, that is \( TT^\dagger = CC^\dagger = PP^\dagger = \mathbf{1} \), and \( \psi^* \) is the complex conjugate of \( \psi \). Here, \( \psi \) is the column vector formed by the coefficients of a pure state of a Hilbert space on an orthonormal basis. \( T \) and \( C \) are antiunitary transformations and \( P \) is unitary. A Hamiltonian \( H \) has \( T \), \( C \) or \( P \) symmetry if it satisfies the conditions

\[
H^T = TH^*T^\dagger = H, \quad H^C = C^\dagger H^\dagger C = -H, \quad H^P = PHP^\dagger = -H. \tag{183}
\]

Since \( H \) is hermitian, \( H^* = H \), the \( T \) symmetry becomes \( TH^\dagger T^\dagger = H \). Notice that \( P \) is the product of \( T \) and \( C \), and one can choose \( P = CT^\dagger \). There is a basis where \( T \) and \( C \) are real, and symmetric or antisymmetric matrices, that is \( T^\dagger = \pm T \) and \( C^\dagger = \pm C \), in which case, unitarity implies \( T^2 = \pm \mathbf{1} \) and \( C^2 = \pm \mathbf{1} \). If a symmetry is broken, say \( T \), then one writes \( T^2 = 0 \). By counting all of the possibilities, one arrives to ten symmetry classes, as found by Altland and Zirnbauer (AZ), which include the classical Wigner-Gaussian ensembles: GOE, GUE, and GSE, as well as their chiral versions chGOE, chGUE, chGSE [105]. Among the ten AZ classes there are four that break time reversal symmetry, which are candidates to describe the zeros of \( \zeta \) and other \( L \)-functions (see table 1).

Let us briefly review the main properties of the classes of table 1 and their relations with the problem at hand. Class A characterizes Hamiltonians of the form \( H = A \), where \( A \) is a hermitean matrix, with no further conditions placed on it. The statistical properties of random matrices of this form are described by GUE. As shown in table 1, all of the CPT symmetries are broken and there is no topological invariant in 1D.

Class D characterizes Hamiltonians of the form \( H = A \), where \( A \) is imaginary and antisymmetric. The eigenvalues of \( A \) appear in pairs \( \{E_n, -E_n\} \), and if the dimension of the matrix \( A \) is odd then there is a zero eigenvalue. The Berry–Keating Hamiltonian \( H = \frac{x^2 + p^2}{2} \) belongs to this class since here \( T = C = 1 \) and \( H^i = -H \) [46]. The Hamiltonian \( H = \sqrt{x} \left( \hat{p} + \frac{1}{2} \hat{p}^{-1} \right) \sqrt{x} \) also belongs to class D provided that the parameter \( \theta \), which characterizes its self-adjoint extensions, is 0 or \( \pi \). The latter choices ensure that the eigenvalues of \( H \) come in pairs \( \{E_n, -E_n\} \), and that if \( \theta = 0 \), then \( E_0 = 0 \) is eigenvalue [43, 44]. We can thus identify \( \theta/\pi = 0, 1 \) as the \( \mathbb{Z}_2 = \{0, 1\} \) topological invariant of class D in 1D.

Class C characterizes Hamiltonians of the form \( H = A + \vec{s} \cdot \vec{S} \), where \( A' = A^\dagger = -A \), \( S' = S^\dagger = S \) and \( \vec{s} \) are the Pauli matrices that act in an additional two dimensional Hilbert space, which can be seen as a spin 1/2 [46]. Here, \( C = i \sigma^3 \), so \( T^2 = -\mathbf{1} \). The eigenvalues of \( H \) come in pairs \( \{E_n, -E_n\} \), as in class D. Srednicki recently proposed that class C is associated to the zeros of the Dirichlet \( L \)-functions whose characters are real and even, which includes the zeta function [46]. He was led to this proposal by a conjecture due to Katz and Sarnak [106], according to which these \( L \)-functions form a family related by a sort of symplectic symmetry, and by the fact that the spacings of their zeros agree asymptotically with the GUE distribution.

Class AIII, is a chiral version of GUE (chGUE) and characterizes Hamiltonians with the block structure

\[
H = \begin{pmatrix}
0 & A \\
A^\dagger & 0
\end{pmatrix}, \quad P = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad PHP^\dagger = -H, \tag{184}
\]

where \( A \) is a complex matrix and \( P \) the chiral operator. The eigenvalues of \( H \) come in pairs \( \{E_n, -E_n\} \). If \( A \) is a matrix of dimension \( N_\alpha\times N_\alpha \), then the number of zeros eigenvalues is \( |N_\alpha - N_{-1}| \) which explains the \( \mathbb{Z} \) topological invariant of this class. Class chGUE, together with its relatives chGOE and chGSE, describe massless Dirac fermions and has been applied to study the QCD Dirac operator, partition functions, etc [107–110].
Let us next study the symmetry classes of the Rindler Hamiltonians that were constructed in the previous sections. First of all, the massless Hamiltonian

\[ H_R = \sqrt{\rho} \hat{p}_\rho \sqrt{\rho} \sigma^\rho, \]  

(185)

admits different representations of the CPT symmetries [104]. Indeed, \( T \) and \( P \) can be realized as \( \sigma^x \) or \( i \sigma^y \), and \( C \) as \( \sigma^z \) or \( 1 \). Adding a mass term to (185) selects a particular realization, for example

\[ H_R = \sqrt{\rho} \hat{p}_\rho \sqrt{\rho} \sigma^\rho + m \rho \sigma^\rho \rightarrow T = \sigma^\rho, \quad C = \sigma^\rho, \quad P = i \sigma^\rho. \]  

(186)

This Hamiltonian acts on the wave functions satisfying the BC (31). Although one can verify that this BC preserves \( T \) for all values of \( \theta \), the symmetry \( C \) is preserved only if \( \theta = 0, \pi \). Hence, in these two cases the Hamiltonian (186) belongs to class BDI (chGOE) since \( T^2 = C^2 = 1 \). We mentioned in section 2.6 that the Hamiltonian \( H = \sqrt{\rho} (\hat{p} + \ell^\rho \hat{p}^{-1} \sqrt{\rho} \hat{p} \sqrt{\rho} \sigma^\rho) \) (see equation (45)) has the same spectrum as (186) with the identifications (46). We show above that the former Hamiltonian belongs to class D, while we have now found that (186) belongs to class chGOE. There is no contradiction between these results. In both cases the \( C \) symmetry explains the pairing of energies \( \{ E_n, -E_n \} \), while the \( T \) symmetry appears from the doubling of degrees of freedom in the Dirac model.

Finally, let us show that the Dirac model associated with the Riemann zeros belongs to class AIII, with chiral operator \( P = i \sigma^y \). First of all notice that the matching conditions (89) are preserved by the action of \( P \) because in this model \( r' = r' \), and then

\[ \chi(L_n \ell^\rho) = L_n \chi(L^\rho \ell^\rho) \rightarrow \chi(L_n \ell^\rho) = L_n \chi(L^\rho \ell^\rho), \quad L_n = P L_n P^\dagger. \]  

(187)

On the other hand, if \( \chi \) is the eigenfunction with energy \( E_n \) it satisfies the BC (91)

\[ -ie^{i(\theta(E_n))} \chi_\rho(-t_i) = \chi_\rho(t_i) \rightarrow -ie^{i(\theta(E_n))} \chi^P_\rho(t_i) = \chi^P_\rho(t_i), \]  

(188)

where \( \chi^P_\rho \equiv \pm \chi^\rho \). The function \( \theta(E_n) \), given in equation (160) for \( n > 0 \), satisfies that \( -\theta(E_n) = \theta(\bar{E}_n) \) which, together with the equation \( PH^R P^\dagger = -H_R \), implies that if \( \chi \) is an eigenstate with energy \( E_n \), then \( \chi^P \) is an eigenstate with energy \( -E_n \). Hence, the pairing of energies is explained by the chiral symmetry and not by the charge conjugation symmetry as the Hamiltonian (186). One can perform a change of basis that brings the chiral operator \( P \) and the Hamiltonian (185) into the form (184), with \( A = i \sqrt{\rho} \hat{p}_\rho \sqrt{\rho} \) that is a real and antisymmetric matrix in some orthonormal basis. The Hamiltonian we are dealing with is, therefore, a subclass within the class chGUE.

6. Conclusions

We have proposed in this paper that a combination of Quantum Mechanics and Relativity Theory is key to the spectral realization of the Riemann zeros. The old message of Pólya and Hilbert was that Quantum Mechanics should play a central role in this realization, an idea that is behind the successful applications of RMT, and Quantum Chaos, to Number Theory. What is new is that Einstein’s theory of Relativity must also be present. The reason is that the properties of accelerated objects, in Minkowski space-time, can be used to encode and process arithmetic information. In some sense, space-time becomes an analogue computer, or simulator, that allows an accelerated observer to multiply numbers and distinguish primes from composite by measuring the proper times of events involving massless particles. In this way, the prime numbers acquire a classical relativistic meaning that is associated with
primitive trajectories of particles, along the lines suggested by the periodic orbit theory in Quantum Chaos.

The combination of Quantum Mechanics and Relativity is, of course, a Relativistic Quantum Field Theory, that in our case is the Dirac theory of fermions in Rindler space-time, the latter being the geometry associated to moving observers. The time evolution in this space is generated by the Rindler Hamiltonian that coincides with the Berry–Keating Hamiltonian \((\hat{x}p + \hat{p}x)/2\), each sign corresponding to the chirality of the fermion. This operator is also the generator of dilations of the Rindler radial coordinate \(x\), but in the presence of delta function potentials, which are associated with the prime numbers, scale invariance is broken. This has the dramatic effect that the spectrum is a continuum where the Riemann zeros are missing, in analogy with the result found by Connes in the adelic approach to the RH. However, the moving observer can fine tune the phase factor of the reflection of the fermion at the boundary in such a way that a bound state appears with an energy given by a Riemann zero. The phase factor at the boundary is given essentially by the phase of the zeta function at the corresponding zero. This result is obtained in a limit where the perturbation of the massless Dirac action by the delta potential is infinitesimally small, so that the bound state is immersed in a continuum of states. This weak coupling limit is reminiscent of the semiclassical limit that leads to the Gutzwiller formula for the fluctuations of the energy levels in chaotic systems. Although the previous scenario leads to a proof of the Riemann hypothesis, more work is required to fully support this claim.

The present work suggests the possibility of an experimental observation of the Riemann zeros as energy levels. Although the system proposed here does not look realistic at the moment, there are theoretical proposals to realize the Rindler geometry using cold atoms and optical lattices [111, 112]. Another possible route is to use the quantum Hall effect where \(xp\) arises as an effective low-energy Hamiltonian [38, 54]. On a more theoretical perspective, this work points towards a relation between Riemann zeros and black holes, whose near horizon geometry is in fact Rindler [113]. The role played by the privileged observer in our construction reminds us of the brick wall model introduced by ’t Hooft to regularize the horizon of a black hole [114]. Throughout this paper the prime numbers have been treated as classical objects. However, they turn into quantum objects in the Prime state that is formed by the superposition of primes in the computational basis of a quantum computer [115, 116]. This poses the question of whether the Prime state could be created with the same tools as the Riemann zeros become energy levels.

Acknowledgements

I am very grateful for suggestions, and conversations to José Ignacio Latorre, Belén Paredes, Paul Townsend, Michael Berry, Jon Keating, Javier Rodríguez-Laguna, Manuel Asorey, Jose Luis Fernández Barbón, Giuseppe Mussardo, André LeClair, Mark Srednicki, Javier Molina, Luis Joaquín Boya and Miguel Angel Martín-Delgado. This work has been financed by the Ministerio de Ciencia e Innovación, Spain (grant FIS2012- 33642), Comunidad de Madrid (grant QUITEMAD) and the Severo Ochoa Program.

Appendix A. The harmonic model

We first discuss some general properties of the transfer matrix \(T_k\) that will be used later on to derive the exact spectrum of the harmonic model. Let us consider the transfer matrix (115), with \(r'_k = 0\), and the condition \(|q_0| < 1\). It is easy to see that \(T_k\) can be written as
where $e^{\pm \phi_i}$ are the eigenvalues of $T_k$ (positive from the condition $|n| < 1$)

$$e^{\phi_i} = \frac{1 + n_i}{1 - n_i}, \quad e^{-\phi_i} = \frac{1 - n_i}{1 + n_i},$$

and

$$\varphi_k = \text{sign } n_i, \quad |n_i| < 1, \quad (A.2)$$

Replacing (A.1) into (112) gives the recursion relation

$$\begin{pmatrix} A_{k-1} \end{pmatrix} = e^{-i\phi_i} e^{i\phi_i^*} \begin{pmatrix} A_k \end{pmatrix}, \quad k \geq 2. \quad (A.4)$$

We now define the vector

$$\begin{pmatrix} \hat{A}_{k-1} \end{pmatrix} = e^{i\phi_i} \begin{pmatrix} A_k \end{pmatrix}, \quad k \geq 1,$$

that satisfies

$$\begin{pmatrix} \hat{A}_{k-1} \end{pmatrix} = e^{-i\Delta_k} e^{i\phi_i^*} \begin{pmatrix} \hat{A}_k \end{pmatrix}, \quad k \geq 2,$$

where

$$\Delta_k = \varphi_k - \varphi_{k-1} = E \log \frac{\ell_k}{\ell_{k-1}}, \quad k \geq 2. \quad (A.7)$$

Notice that

$$\begin{pmatrix} \hat{A}_1 \end{pmatrix} = \begin{pmatrix} A_{1}(\theta) \end{pmatrix}. \quad (A.8)$$

Equation (A.6), gives an alternative way to solve the eigenvalue problem that admits an interesting physical interpretation as the evolution of a kicked rotator with spin 1/2 [58]. In this interpretation the term $e^{-i\Delta_k \sigma}$ represents the rotation of the spin around the $z$-axis, with an energy $E$ during a time elapse $\log (\ell_k/\ell_{k-1})$, after which the spin is kicked with an imaginary magnetic field with strength $g_k$ along the $x$-direction. Kicked rotators of this sort are currently employed to analyze quantum chaos in simple situations [58].

Let us next consider the harmonic model defined by equation (129), which according to equations (A.2, A.7) correspond to constant values of $g_k = g$ and $\Delta_k = \Delta$

$$\ell_n = e^{\pi/2}, \quad q_n = r_n = e \rightarrow e^{\pi} = \frac{1 + \epsilon}{1 - \epsilon}, \quad \Delta = \frac{E}{2}. \quad (A.9)$$

In this case the recursion relation (A.6) can be iterated yielding

$$\begin{pmatrix} \hat{A}_{k+1} \end{pmatrix} = S^{k} \begin{pmatrix} A_{1}(\theta) \end{pmatrix}, \quad k \geq 0, \quad (A.10)$$

where the matrix $S$ is given by

$$S = e^{-i\phi_i^* e^{iE/2}} = \begin{pmatrix} \cosh g & e^{iE/2} \sinh g & e^{-iE/2} \\ -\sinh g & e^{-iE/2} \cosh g & e^{iE/2} \end{pmatrix}. \quad (A.11)$$

Making the replacement $E \rightarrow E + 2\pi$, the matrix $S$ changes by an overall sign, which in turn implies that the spectrum has a $2\pi$ periodicity. Let us consider the case where $E = 2\pi n$, which
Figure 5. The region in shadow represents the values of $E$ and $g$ where the matrix $S$ is elliptic, which corresponds to states in the continuum (see equation (A.15)). In the white region the matrix $S$ is hyperbolic and contains the discrete spectrum whose location depend on $g$ and $\theta$ by equation (A.14). For $\theta = 0$, $\pi$ the eigenstates are at $E_n = 2\pi n$ for all values of $g$ (see equation (A.13)). The horizontal line corresponds to a value of the coupling constant $g$. Its intersection with the shadow regions give the continuum spectrum that form the bands (A.16) separated by a gap $4\pi \delta$.

yields

$$S = (-1)^n e^{-g\tau} \begin{pmatrix} \hat{A}_{k+1} \end{pmatrix} = (-1)^n e^{-g\tau} \begin{pmatrix} 1 \\ e^{i\tau} \end{pmatrix}, \quad (A.12)$$

and which is convergent only in the following two cases

$$\theta = 0, \ g > 0 \quad \Rightarrow \quad \begin{pmatrix} \hat{A}_{k+1} \end{pmatrix} = (-1)^n e^{-g\tau} \begin{pmatrix} 1 \\ e^{i\tau} \end{pmatrix},$$

$$\theta = \pi, \ g < 0 \quad \Rightarrow \quad \begin{pmatrix} \hat{A}_{k+1} \end{pmatrix} = (-1)^n e^{-i\tau} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (A.13)$$

These are normalizable eigenstates for all values of the coupling constant $g$, even in the limit $g \approx 2\varepsilon \to 0$. Normalizable states for $\theta \neq 0, \pi$ are also possible. The relation between the energy, $E$, $g$ and $\theta$ is given by

$$\tan \left( E/2 \right) = \frac{\sin \theta}{\cos \theta + \coth g}, \quad 1 + \cos \theta \coth g > 0, \quad (A.14)$$

which in the limit $g \to 0$ gives the discrete eigenvalues $E \to 2\pi n$. By plugging $\theta = 0, \pi$ in (A.14) we recover the cases (A.13). The continuum spectrum of the model can be derived from the condition that $S$ is an elliptic matrix (note that $\det S = 1$)

$$| \text{Tr} \ S = 2 \cosh g \cos E/2 | < 2 \quad \iff \quad | \sin E/2 | > \tan g, \quad (A.15)$$

and consists of energy bands separated by a gap $4\pi \delta$, which are shown in figure 5

$$\text{Spec}_e = \bigcup_{n=-\infty}^{\infty} 2\pi [n + \delta, n + 1 - \delta], \quad \sin (\pi \delta) = \tan g = \frac{2\varepsilon}{1 + \varepsilon^2}. \quad (A.16)$$

In the limit $g \to 0$, the gap closes and the discrete energy states $E_n = 2\pi n$ become immersed in the continuum $\mathbb{R}$. 

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