Cosmology with two compactification scales

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Abstract

We consider a (4 + d)-dimensional spacetime broken up into a (4 − n)-dimensional Minkowski spacetime (where n goes from 1 to 3) and a compact (n + d)-dimensional manifold. At the present time the n compactification radii are of the order of the Universe size, while the other d compactification radii are of the order of the Planck length.

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1 Introduction

In recent years, there has been growing interest and a great deal of activity (see e.g. [1], [2], [3], [4], [5]) in multidimensional cosmology. A feature common to many of those works is to assume that the Universe is a \( (4 + d) \)-dimensional manifold where, due to its evolution, only three spatial dimensions are actually observable, while the remaining \( d \) have curled up into compact spaces of unobservable small radii. This point of view is also apparent in the Kaluza-Klein spacetime of multidimensional supergravity[6], [7], [8], [9].

In this letter we consider dynamical compactification of a different sort. The \( (4 + d) \)-dimensional space is supposed to break up into a \( (4 - n) \)-dimensional Minkowski space and a compact \( (n + d) \)-dimensional manifold whose compactification radii are governed by Einstein’s field equations. Here the integer \( n \), ranging from 1 to 3, is the number of usual spatial dimensions by hypothesis compactified, like the extra \( d \) dimensions but with different radii, in a circle. Moreover we require that at the initial time \( t = 0 \) the compactification radii of the \( (n + d) \) spatial dimensions are all the same, and that each radius of the \( d \) extra dimensions at the present time equals the Planck length, while each radius of the usual \( n \) dimensions is comparable to the size of the Universe. In the simplified model we propose, only the cosmological constant \( \Lambda \) will be retained in Einstein’s equations, thus neglecting matter contributions.
as well as scalar field terms appropriate to inflationary cosmology and a scale factor for the flat dimensions. This model is admittedly not realistic, but it can prove to be useful for future developments.

The letter is organised as follows: taking as guidelines the work of Chodos and Detweiler [10], we find the solutions to the field equations and generalize those already found by Kasner when $\Lambda = 0$ [11]. Successively we consider eleven dimensional cosmological models with different values of $n$ and $\Lambda$ and give numerical estimates of some quantities of interest such as the Universe age and the evolution of the compactification radii.

# The line element

The metric suitable to our problem has the form

\[
ds^2 = dt^2 - \sum_{i=1}^{3-n} dx^i dx^i - a^2(t) \sum_{i=4-n}^{3} d\varphi^i d\varphi^i - l^2(t) \sum_{i=1}^{d} d\psi^i d\psi^i \quad (1)
\]

where $a(t)$ and $l(t)$ are respectively the compactification radii of each one of the $n$ and of the $d$ spatial dimensions and $\varphi^i$ and $\psi^i$ have a $2\pi$ period.

Einstein’s equations, with cosmological term only, can be written as

\[
R_{MN} = \frac{2\Lambda}{n + d - 1} g_{MN}, \quad M,N = 1,2, \ldots, A+d \quad (2)
\]
and the relevant ones are given explicitly by:

\[ n \ddot{a} + \frac{d}{a} \dddot{l} = \frac{2\Lambda}{n+d-1} \]  

\[ \frac{\ddot{a}}{a} + (n-1) \left( \frac{\dot{a}}{a} \right)^2 + d \frac{\dot{a}}{a} \dot{l} = \frac{2\Lambda}{n+d-1} \]  

\[ \frac{\ddot{l}}{l} + (d-1) \left( \frac{\dot{l}}{l} \right)^2 + n \frac{\dot{a}}{a} \dot{l} = \frac{2\Lambda}{n+d-1} \]

where a dot means derivative with respect to the time.

The system (3) can be solved with the conditions that at the present time \( t = t_0 \) (age of the Universe) one has

\[ a(t_0) = a_0, \quad \dot{a}(t_0) = H_0 a_0 \]

\[ l(t_0) = l_0, \quad \dot{l}(t_0) = h_0 l_0 \]

The Hubble constant \( H_0 \) and the new constant \( h_0 \) which appear in the above conditions are not independent as one can see from Eqs.(3) rewritten at \( t = t_0 \) with the introduction of the deceleration parameter \( q_0 = - (\ddot{a}/a^2)_0 \) and of its analogous \( Q_0 = - (\ddot{l}/l^2)_0 \):

\[ nq_0 H_0^2 + dQ_0 h_0^2 = - \frac{2\Lambda}{n+d-1} \]  

\[ (n - 1 - q_0) H_0^2 + dH_0 h_0 = \frac{2\Lambda}{n+d-1} \]  

\[ (d - 1 - Q_0) h_0^2 + nH_0 h_0 = \frac{2\Lambda}{n+d-1} \]
It is in fact straightforward to obtain:

\[
\frac{h_0}{H_0} = \begin{cases} 
\frac{-n}{d-1} + \sqrt{\frac{n(n + d - 1)}{d(d - 1)^2}} + \frac{2\lambda}{d(d - 1)} & \text{if } d \neq 1 \\
\frac{-n - 1}{2} + \frac{\lambda}{n} & \text{if } d = 1
\end{cases}
\]  

(6)

with

\[
\lambda = \frac{\Lambda}{H_0^2}
\]  

(7)

When \( d \neq 1 \) it must be \( \lambda > -n(n + d - 1)/2(d - 1) \) for reality.

For future convenience we define the dimensionless quantities:

\[
\tau = H_0 t, \quad \tau_0 = H_0 t_0
\]  

(8)

\[
\omega = \sqrt{\frac{n + d}{2(n + d - 1)|\lambda|}}
\]  

(9)

\[
\delta_> = \frac{1}{2} \text{arctanh} \left( \frac{2\omega}{n + d \frac{h_0}{H_0}} \right)
\]  

(10)

\[
\delta_< = \frac{1}{2} \text{arctan} \left( \frac{2\omega}{n + d \frac{h_0}{H_0}} \right)
\]  

(11)

The solutions to system (3) can then be written as

\[
a(t) = \frac{a(t)}{a_0} = \begin{cases} 
\left[ \frac{\sinh(\omega(\tau - \tau_0) + \delta_>)}{\sinh \delta_>} \right]^{\beta_+} \left[ \frac{\cosh(\omega(\tau - \tau_0) + \delta_>)}{\cosh \delta_>} \right]^{\beta_-} & \text{if } \lambda > 0 \\
\left[ 1 + (n + d \frac{h_0}{H_0})(\tau - \tau_0) \right]^{\beta_+} & \text{if } \lambda = 0
\end{cases}
\]  

(12)

\[
\left[ \frac{\sin(\omega(\tau - \tau_0) + \delta_<)}{\sin \delta_<} \right]^{\beta_+} \left[ \frac{\cos(\omega(\tau - \tau_0) + \delta_<)}{\cos \delta_<} \right]^{\beta_-} & \text{if } \lambda < 0
\]
Here

\[
\begin{align*}
\beta_+ &= 1 + \sqrt{\frac{d(n + d - 1)}{n}} \Bigg/ \frac{n + d}{n + d}, \\
\beta_- &= 1 - \sqrt{\frac{d(n + d - 1)}{n}} \Bigg/ \frac{n + d}{n + d}
\end{align*}
\]

(14)

\[
\begin{align*}
\gamma_+ &= 1 + \sqrt{n(n + d - 1)/d} \Bigg/ \frac{n + d}{n + d}, \\
\gamma_- &= 1 - \sqrt{n(n + d - 1)/d} \Bigg/ \frac{n + d}{n + d}
\end{align*}
\]

(15)

while \( l_0 \) is assumed to be of the order of the Planck length and \( a_0 \) is the radius of the circle of the actually macroscopic dimensions.

Once \( n \) and \( d \) are fixed and \( H_0 \) is taken as known, the ratios \( a(\tau)/a_0 \) and \( l(\tau)/l_0 \) result to depend only on \( \lambda \) and \( \tau_0 \). Then, due to the fact that these two quantities are not sufficiently well established, we might vary them step by step in a neighborhood, say, of \( \lambda = 0 \) and of \( \tau_0 = 1 \) to obtain numerical estimates of \( a(\tau)/a_0 \) and \( l(\tau)/l_0 \).

We find however more convenient to proceed in a different manner. Noticing that \( \beta_+ - \gamma_- = -(\beta_- - \gamma_+) \equiv 1/\alpha \) and defining

\[
\rho \equiv \left( \frac{l_0 a(0)}{a_0 l(0)} \right)^\alpha
\]

(16)

which is expected to be a quantity much less than unity, one easily obtains
from Equations (12) and (13) written at $\tau = 0$:

$$
\tau_0 = \begin{cases} 
\frac{\delta_> - \arctanh(\rho \tanh \delta_>)}{\omega} & \text{if } \lambda > 0 \\
\frac{1 - \rho}{n + dh_0/H_0} & \text{if } \lambda = 0 \\
\frac{\delta_< - \arctan(\rho \tan \delta_<)}{\omega} & \text{if } \lambda < 0
\end{cases}
$$

(17)

In this way we can calculate $\tau_0$ for a given $\lambda$ if we properly choose the parameter $\rho$ or, otherwise stated, the ratio $a(0)/l(0)$. To make an example we can recover, for $\lambda = 0$, Kasner’s solution by choosing $a(0)$ equals to zero, or equivalently $l(0)$ equals to infinity, and therefore $\rho = 0$.

In view of the smallness of the ratio $l_0/a_0$, initial values for $a(0)$ and $l(0)$ of not too much different order of magnitude does not appreciably influence the results at farther times. Our choice is therefore to have at the initial time the same compactification radii for the $(n + d)$ spatial dimensions and so we put $a(0) = l(0)$. As a consequence, for a given pair $(n, d)$, we have only $\lambda$ as the parameter left free to evaluate both the actual age of the Universe $\tau_0$ and the ratios $a(\tau)/a_0$ and $l(\tau)/l_0$.

### 3 Numerical results for eleven dimensions

We shall limit ourselves to the most popular choice of eleven dimensions and so fix the value $d = 7$.

To begin with, let us examine the values of $\tau_0 = H_0t_0$ which can be ob-
tained by our model when the adimensional cosmological constant $\lambda$ vary in the interval $(-1, 1)$. As to the Hubble constant $H_0$, it is common practice to write $H_0 = 100\eta \text{km s}^{-1} \text{Mpc}^{-1}$ where the uncertainty on it is put into the constant $\eta$, whose present value ranges from 0.50 to 0.85. Thus a characteristic time scale for the expansion of the Universe is the Hubble time $1/H_0 = (9.8/\eta) \text{Gyr}$.

If we look at the graph of Figure 1, where the above stated restriction on the negative values of $\lambda$ is apparent only for $n = 1$, we can see that $\tau_0$ can exceed unity, as one expects, only if $n = 1$ and $\lambda < 0$. Due however to the simplicity of our model, this fact does not seem a serious drawback and all other combinations of $n$ and $\lambda$ can not, in our opinion, be ruled out.

The sign of $\lambda$ and the values of $n$ appear to be of great importance for the time evolution of the radii $a(\tau)$ and $l(\tau)$ as it is shown in Figures from 2 to 7. We can summarize the various behaviours as follows:

1) When $\lambda > 0$, $a(\tau)$ is always increasing to infinity with time and so does $l(\tau)$ apart in the cases $n = 2, 3$ where $l(\tau)$ is initially decreasing in a finite time interval.

2) When $\lambda = 0$, $a(\tau)$ is always increasing to infinity with time, while $l(\tau)$ is constant if $n = 1$ and decreasing to zero if $n = 2, 3$.

3) When $\lambda < 0$, $a(\tau)$ increases to infinity and $l(\tau)$ decreases to zero until $\tau$ reaches the finite value $\tau = \tau_0 + (\pi/2 - \delta_\perp)/\omega$; whether this is a final state
or a new initial state of the Universe is a question we leave open.

Let us notice, as one can see from Figures 3, 5 and 7, that the rate of variation of the Planck length is not so dramatic in the range of times considered, and in any case still compatible with the experimental bounds due to the possible time variation of the fundamental constants involved in its definition.

4 Conclusions

The widespread belief in existing multidimensional cosmological models is that three spatial dimensions expand isotropically while the remaining $d$ are actually curled up into spaces of dimensions comparable to the Planck length. Such a behaviour is exhibited also by the $(4 + d)$-dimensional Kaluza-Klein spacetime derived from M-theory.

We instead propose that at least one of the three spatial macroscopic dimensions can undergo a compactification process with a consequent loss of isotropy. This fact would bring to important experimental consequences, for instance, with respect to the cosmic microwave background anisotropy.

When all the three usual spatial dimensions compactify, that space becomes like a flat three-dimensional one with the scale factor $a(\tau)/a_0$ describing its expansion. Of course in all the cases we have considered, the expansion in the distant future is driven by the cosmological constant.

Our model is admittedly greatly simplified, but it seems worth exploring the
possibility of a compactification process also in the large scale.
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Figure captions

Figure 1: $\tau_0$ as a function of $\lambda$ when $n = 1, 2, 3$

Figure 2: $\frac{a(\tau)}{a_0}$ as a function of $\tau$ when $n = 1$

Figure 3: $\frac{l(\tau)}{l_0}$ as a function of $\tau$ when $n = 1$

Figure 4: $\frac{a(\tau)}{a_0}$ as a function of $\tau$ when $n = 2$

Figure 5: $\frac{l(\tau)}{l_0}$ as a function of $\tau$ when $n = 2$

Figure 6: $\frac{a(\tau)}{a_0}$ as a function of $\tau$ when $n = 3$

Figure 7: $\frac{l(\tau)}{l_0}$ as a function of $\tau$ when $n = 3$
Figure 1: $\tau_0$ as a function of $\lambda$ when $n = 1, 2, 3$
Figure 2: $\frac{a(\tau)}{a_0}$ as a function of $\tau$ when $n = 1$

Figure 3: $\frac{l(\tau)}{l_0}$ as a function of $\tau$ when $n = 1$
Figure 4: \( \frac{a(\tau)}{a_0} \) as a function of \( \tau \) when \( n = 2 \)

Figure 5: \( \frac{l(\tau)}{l_0} \) as a function of \( \tau \) when \( n = 2 \)
\[ \lambda = -0.5 \]
\[ \lambda = 0.0 \]
\[ \lambda = +0.5 \]

Figure 6: \( \frac{a(\tau)}{a_0} \) as a function of \( \tau \) when \( n = 3 \)

Figure 7: \( \frac{l(\tau)}{l_0} \) as a function of \( \tau \) when \( n = 3 \)