ON EXCEPTIONAL TERMINAL SINGULARITIES

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Abstract. The first examples of exceptional terminal singularities are constructed.

Introduction

The exceptional singularity study importance follows from the next observation.

1. If \((X \ni P)\) is an exceptional singularity then the linear system \(| - nK_X|\) is to have a ”good” member for small \(n\). Actually, we can take \(n \in \{1, 2\}\) [7, 5.2] in two dimensional case and \(n \in \{1, 2, 3, 4, 6\}\) [8, 7.1] in three dimensional case.

2. Exceptional singularities are ”bounded” and are to be classified.

In this paper we construct the first examples of exceptional terminal singularities.

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1. Preliminaries

All varieties are algebraic and are assumed to be defined over \(\mathbb{C}\), complex number field. We will use the terminology and notation of Log Minimal Model Program and the main properties of complements [1], [5].

Definition 1.1. Let \((X/Z \ni P, D)\) be a contraction of varieties, where \(D\) is a boundary. Then a \(\mathbb{Q}\)-complement of this contraction is an effective \(\mathbb{Q}\)-divisor \(D'\) such that \(D' \geq D\), \(K_X + D'\) is lc and \(K_X + D' \sim_\mathbb{Q} 0\).

Definition 1.2. Let \((X/Z \ni P, D)\) be a contraction of varieties, where \(D\) is a boundary.

1. Assume that \(Z\) is not a point (local case). Then \((X/Z \ni P, D)\) is said to be exceptional over \(P\) if for any \(\mathbb{Q}\)-complement of \(K_X + D\)
near the fiber over $P$ there exists at most one (not necessarily exceptional) divisor $E$ such that $a(E, D) = -1$.

2. Assume that $Z$ is a point (global case). Then $(X, D)$ is said to be exceptional if every $\mathbb{Q}$-complement of $K_X + D$ is klt.

**Example 1.3.** Let $(X \ni P)$ be a singularity. Suppose that there is an effective divisor $H$ such that $(X, H)$ is lc and $\cup H \neq 0$. Then the singularity is not exceptional. Therefore three dimensional terminal singularity is not exceptional because there is a divisor having only Du Val singularities in the anticanonical linear system $|−K_X|$ [6, 6.4].

**Definition 1.4.** Let $X$ be a normal lc variety and let $f : Y \to X$ be a blow-up such that the exceptional locus of $f$ contains only one irreducible divisor $E$ $(\text{Exc}(f) = E)$. Then $f : (Y, E) \to X$ is called a purely log terminal (plt) blow-up, if $K_Y + E$ is plt and $-E$ is $f$-ample.

**Remark 1.5.** If $(X \ni P)$ is klt singularity then there is a plt blow-up [2, 1.5].

**Theorem 1.6.** [4, 4.9] Let $(X \ni P)$ be a klt singularity and let $f : (Y, E) \to X$ be a plt blow-up of $P$. Then the following conditions are equivalent:

1. $(X \ni P)$ is an exceptional singularity;
2. $(E, \text{Diff}_E(0))$ is an exceptional log variety.

2. The examples of the exceptional terminal singularities

**Theorem 2.1.** Let $(f = 0, 0) = (x_1^{a_1} + x_2^{a_2} + x_3^{a_3} + x_4^{a_4} + x_5^{a_5} = 0, 0) \subset (\mathbb{C}^5, 0)$ be a four dimensional hypersurface singularities, where $(a_1, \ldots, a_5) = (2, 3, 11, 17, 19), (2, 3, 11, 17, 23), (2, 3, 11, 17, 25), (2, 3, 11, 17, 29), (2, 5, 7, 9, 11), (2, 5, 7, 9, 13)$. Then they are terminal and exceptional.

**Proof.** Consider the first singularity. Let us prove that it is terminal. Since it is given by non-degenerate polynomial then there exists embedded toric log resolution [9]. Therefore it is sufficient to prove that $a_p = \langle p, 1 \rangle - p(f) - 1 \geq 1$ for all $p$, where $p$ is non-zero vector with integral non-negative coordinates and $p(f) = \min_{x^m \in f} \langle p, m \rangle$. The easy way to prove this statement is following. Let $h(d) = -\frac{d}{2} + \frac{d}{3} + \frac{d}{11} + \frac{d}{17} + \frac{d}{19} - d - 1$. Let $d = p(f)$ then $h(d) \leq a_p$. It is enough to check that $h(d) \geq 1$ for all $1 \leq d \leq 2 \cdot 3 \cdot 11 \cdot 17 \cdot 19$. The later is elementary to prove with the help of the computer program.

The weighted blow-up of $\mathbb{C}^5$ with weights proportional to $(\frac{1}{2}, \frac{1}{3}, \frac{1}{11}, \frac{1}{17}, \frac{1}{19})$ induces a plt blow-up of our singularity. The obtained
log Fano variety \((E, \text{Diff}_E(0))\) is

\[
\left( \sum_{i=1}^{5} x_i \subset \mathbb{P}(1, 1, 1, 1, 1), \quad \frac{1}{2}H_1 + \frac{2}{3}H_2 + \frac{10}{11}H_3 + \frac{16}{17}H_4 + \frac{18}{19}H_5, \right)
\]

where \(H_i = \{ x_i = 0 \} \). Let \(D = \sum d_i D_i + \sum h_i H_i \) be any \(\mathbb{Q}\)-complement of this log variety. In our case it is easy to check that \(d_i < 1 \) and \(h_i < 1 \) for all \(i\). If any component \(D_i \) of \(D\) is a hyperplane section then we can easily show \(K_{\mathbb{P}^3} + D\) to be klt. Now let us prove that \((\mathbb{P}^3, D)\) is klt. This question is local. Therefore consider any chart \(\mathbb{C}^3_{y_1, y_2, y_3}\). Let \(D = \Delta + D'\), where \(\Delta\) is a sum of hyperplane sections and \(D' = \sum d_i \{ f_i = 0 \}\). Let multiplicity of \(f_i\) is equal to \(n_i\). By considering a change of coordinates of \(\mathbb{C}^3\) we can assume without loss of generality that \(f_i = y_1^{n_i} + \cdots\). Consider the deformation \(F'_i = \sum d_i t^{-n_i} f_i(ty_1, ty_2, ty_3)\). For \(t = 0\) we have \(D'_0 = \{ F'_0 = 0 \} = \sum d_i \{ y_1^{n_i} = 0 \} = \sum d_i n_i \{ y_1 = 0 \}\) and for small \(t \neq 0\) we get \(D'_t = \{ F'_t = 0 \} = D'\). By \([3, 8.6]\) it follows that \(c(\mathbb{C}^3, \Delta + D'_0) \leq c(\mathbb{C}^3, \Delta + D')\), where \(c\) is lc threshold. Since the every component \(\text{Sup}(\Delta + D'_0)\) is a hyperplane section then \((\mathbb{C}^3, \Delta + D'_0)\) is klt. Therefore \((\mathbb{C}^3, \Delta + D')\) is also klt. By criterion 1.6 the singularity is exceptional. Divisor \(\frac{1}{22}(\cup_{\text{23}} \cdot \text{Diff}_E(0))\) gives a 22-complement of minimal index. The other singularities are considered similarly. They have 24,34,34,22,28-complements of minimal index respectively. \(\square\)

Remark 2.2. A minimal index of complementarity is bounded for log Del Pezzo surfaces with standard coefficients \([7, 7.1]\). A hypothesis is that this index is not more than 66. This implies that we can take \(\mathbb{Q}\)-complement of special type for four dimensional singularities. Namely, let \(d_i = r_i/q_i\) then \(q_i \leq 66\). It allows to consider the finite number variants of \(D\).

Conjecture 2.3. Let \((X \ni P)\) be an \(n\)-dimensional \((n \geq 3)\) canonical (log canonical) hypersurface singularity. Let \(f\) be a log resolution and \(\min\{a(E, 0) | f(E) = P\} \geq n - 2\). Then there exists a hyperplane section \(H\) such that \((X, H)\) is plt (lc). Hence \((X \ni P)\) is not weakly exceptional (not exceptional) singularity \([4, 4.8]\).

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