On Fair Division with Binary Valuations Respecting Social Networks

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Abstract

We study the computational complexity of finding fair allocations of indivisible goods in the setting where a social network on the agents is given. Notions of fairness in this context are “localized”, that is, agents are only concerned about the bundles allocated to their neighbors, rather than every other agent in the system. We comprehensively address the computational complexity of finding locally envy-free and Pareto efficient allocations in the setting where the agents have binary valuations for the goods and the underlying social network is modeled by an undirected graph. We study the problem in the framework of parameterized complexity.

We show that the problem is computationally intractable even in fairly restricted scenarios, for instance, even when the underlying graph is a path. We show NP-hardness for settings where the graph has only two distinct valuations among the agents. We demonstrate W-hardness with respect to the number of goods or the size of the vertex cover of the underlying graph. We also consider notions of proportionality that respect the structure of the underlying graph and show that two natural versions of this notion have different complexities: allocating according to the notion that accounts for locality to the greatest degree turns out to be computationally intractable, while for other notions, the allocation problem can be modeled as a structured ILP which can be solved efficiently.

2012 ACM Subject Classification Theory of computation → Design and analysis of algorithms; Theory of computation → Social networks

Keywords and phrases Fair Division, Social Networks, Envy-Freeness, Parameterized Complexity

Funding Neeldhara Misra: The author would like to acknowledge the SERB ECR Grant for their support of this work

1 Introduction

The problem of fairly allocating resources among a set of agents with (possibly distinct) interests in said resources is a fundamental problem with important and varied practical applications. We focus on the problem of allocating indivisible items: in this setting, we have n agents and m resources, and every agent expresses their utilities for the resources, either as a ranking over the resources or by specifying a valuation function. The goal is to determine an allocation of the items to the agents that respects some notion of “fairness” and “efficiency”. We use the term bundle to refer to the set of items that an agent receives in an allocation.

Envy-freeness is one of the most widely used notions of fairness. Given an allocation, an agent envies another if it perceives the bundle of the other agent to be more valuable than her own. An allocation is envy-free if no agent envies another. Note that the trivial allocation that leaves every agent empty-handed is always envy-free. Therefore, one is typically interested in fair allocations that also satisfy some criteria of economic efficiency, such as completeness (every good should be allocated to some agent), non-wastefulness (no
agent receives a piece of cake that is worth nothing to her and worth something to another agent), or Pareto-efficiency (there is no other feasible agreement that would make at least one agent strictly better off while not making any of the others worse off). We remark here that just as there are trivial allocations that are fair, it is also possible to trivially achieve efficiency if we had no fairness considerations involved: for instance, the allocation that gives all goods to a single agent is Pareto-efficient assuming that the agent has a strictly monotonic utility function over the items.

The question of finding allocations that respect fairness and efficiency demands simultaneously is non-trivial: in particular, such allocations may not exist (if there are two agents and one good, and both agents have positive utility for this single resource), and can be computationally hard to find (for instance, the problem of finding a complete envy-free allocation between even two agents who hold identical valuations over m goods is equivalent to the \textsc{Partition} problem).

The focus of this work is the notion of local envy-freeness. In this setting, the agents are related by a graph, which might be thought of as modeling a social network over the agents, and we explore notions of fairness that account for the structure of this network. For instance, the notion of envy is now restricted: it only manifests between agents who are friends in the network. This is a compelling model of fairness, since agents are likely to not envy agents about whom they have little or no information. We note that the problem of fair division respecting a social network generalizes the classical notion, which can be captured by considering a complete graph on the agents. Thus, the problem of finding allocations that are “locally fair” is a generalization of the classical allocation problem.

1.1 Related Work

The model of local envy-freeness has been proposed and considered in several recent lines of work. Some of the earliest considerations for incorporating a graph structure on the agents were made in the context of the \textit{cake-cutting} problem, which is the closely related setting of allocating a divisible resource among agents \cite{Abebe11}. Abebe, Kleinberg, and Parkes \cite{Abebe11} consider both directed and undirected graphs and focus on characterizing the structure of graphs that admit algorithms with certain bounds. They also consider the issue of the \textit{price of envy-freeness} in this setting, which compares the total utility of an optimal allocation to the best utility of an allocation that is envy-free. Bei, Qiao, and Zhang \cite{Bei14}, on the other hand, propose a moving-knife algorithm that outputs an envy-free allocation on trees and an algorithm for computing a proportional allocation on descendant graphs.

We now turn to the literature in the context of indivisible items. Beynier et al \cite{Beynier14} study the fair division problem in the setting of “house allocation”: here agents have (strict) preferences over items, and each agent must receive exactly one item. An agent envies another in this setting if she prefers the item received by the other agent over her own. In the case of a complete network, for an allocation to be envy-free, each agent must get her top object, and this assignment is automatically Pareto-efficient as well. This motivates the setting of local envy-freeness with respect to a graph on the agents. The authors consider the case when the underlying graph is undirected, and they also consider a variant of the problem where agents themselves can be located on the network by the central authority. These problems turn out to be computationally intractable even on very simple graph structures.

Bredereck, Kaczmarczyk, and Niedermeier \cite{Bredereck14} consider the problem of graph-based envy-freeness in the context of \textit{directed} graphs and for various classes of valuations: including binary, identical, additive, and even valuations that are both identical and binary. They also consider the complexity of the allocation problem in the framework of parameterized
complexity. Somewhat surprisingly, it turns out that finding complete envy-free allocations in the setting of a graph is NP-hard even when the valuations are binary and identical. Note that in this setting, every agent in every strongly connected component must get the same number of items: thus, the allocation problem is trivial for directed graphs that are strongly connected, but NP-hard for general directed graphs. Also, it turns out that for general binary preferences, the problem of finding a complete envy-free allocation is NP-hard even when the graph is strongly connected. The problem is also tractable for DAGs: indeed, allocating all resources to a single source agent (corresponding to a vertex with no incoming arcs) is both complete and locally envy-free since nobody can envy a source agent, and empty-handed agents have no envy for each other.

More recently, Eiben et al. [14] consider the problem of finding locally envy-free allocations and envy-free allocations that are additionally proportional in the setting of directed graphs in the framework of parameterized complexity, and specifically considering parameters such as treewidth, cliquewidth, and vertex cover — all of these reflect the structure of the underlying network. It turns out that the problem of finding fair and efficient allocations is tractable for networks that have bounded values for these parameters with some additional assumptions that bound the number of item types or the size of the largest bundle received by an agent. The authors also show hardness results in both the parameterized and classical settings. For instance, the authors show that finding a locally envy-free allocation is NP-hard even when the underlying network is a star, but we note that this is in the setting of general utilities.

The work of Bredereck et al. [8, 9] demonstrates that the problem of finding fair and efficient allocations in various settings (including graph-based constraints) is fixed-parameter tractable in the combined parameter “number of agents” and “number of item types” for general utilities. In contrast, our work here focuses on smaller parameters for the special case of binary utilities.

In [12], Chevaleyre, Endriss, and Maudet consider distributed mechanisms for allocating indivisible goods, in which agents can locally agree on deals to exchange some of the goods in their possession. This study focuses on convergence properties for such distribution mechanisms both in the context of the classical setting and the setting involving social constraints coming from an underlying undirected graph. Here, the notions of fairness localized according to the graph, and the network also constraints the exchanges that can take place — agents can engage in an exchange only if they are friends in the network. There are also some lines of work that suggest eliminating envy by some mechanism for hiding information [15].

1.2 Our Contributions

Our focus in this paper is on the setting when agents have binary valuations over the goods and the underlying social network is modeled by an undirected graph. Our focus is on exploring the computational complexity of finding locally envy-free allocations that are also Pareto efficient (EEF) in the framework of parameterized complexity, building most closely on the works of [14, 6, 10].

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1 The terminology relevant to this framework is introduced in the next section.
Bounded Agent Types.

We begin by noting that the setting of undirected graphs can be significantly different from their directed counterparts: indeed, recall that finding a complete and locally envy-free allocation was NP-hard for even identical binary valuations for directed graphs, but the analogous question is easily seen to be tractable for undirected graphs (indeed, observe that the notions of strong connectivity and connectivity coincide). This motivates the question of whether the problem of finding locally EEF allocations is easier for undirected graphs with a bounded number of agent types. We answer this question in the negative by showing that the problem of determining locally envy-free allocations is NP-hard even when there are only two distinct binary valuations among the agents by a reduction from a graph separation problem called Cutting \( \ell \) Vertices (Theorem 3).

Sparse and Dense Graphs.

In contrast with the result for DAGs, we show that finding locally envy-free allocations that are Pareto efficient (EEF) is NP-hard even when the underlying graph is a path (Theorem 9 and Corollary 10). Although Beynier et al [5] also show hardness results for very sparse graphs, we note that our methods are significantly different since the models for the valuations are different and additionally, the allocations we seek need not give every agent exactly one item. Moving away from sparsity, we recall that finding complete envy-free allocations for binary valuations is known to be NP-hard even for complete graphs [15, 3], which justifies the need for using additional parameters in the XP algorithm for finding locally envy-free allocations shown by [14, Theorem 10].

Structural Parameters I: Treewidth and Cliquewidth

Informally speaking, the parameters treewidth and cliquewidth of graphs quantitatively capture the sparsity and density of the graph by measuring their “likeness” to trees and complete graphs. The results we have already for sparse and dense graphs demonstrate that these parameters being bounded alone is not enough to obtain tractable algorithms. On the other hand, the results of [14] imply that the problem of finding complete and locally envy-free allocations admits XP algorithms when parameterized by either the treewidth or cliquewidth of the underlying graph jointly with the number of item types and agent types. Since their model allows for bidirectional edges, these results apply to the setting of undirected graphs as well. We note that the algorithms described in [14] focus on complete allocations, but can be adapted to account for Pareto efficiency as well.

Structural Parameters II: Vertex Cover and Twin Cover

In the setting of directed graphs and general utilities, we note that the problem of finding a complete and locally envy-free allocation is NP-hard even when the underlying graph is a star. In particular, this demonstrates hardness on graphs with a constant-sized vertex cover.

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2 The algorithm referred to is XP in the cliquewidth of the underlying graph, the number of agent types and item types.
3 XP is the class of parameterized problems that can be solved in time \( n^{O(k)} \) for some computable function \( f \).
4 A vertex cover of a graph is a subset of vertices that contains at least one endpoint of every edge. A graph with a bounded vertex cover also has bounded treewidth.
It is not clear if this is the case for undirected graphs and binary utilities. We show that the problem of finding locally EEF allocations is W[1]-hard when parameterized by the vertex cover number (Theorem 8). We remark that a stronger hardness result can be observed for the closely related parameter of twin cover — indeed, the known NP-hardness of finding envy-free allocations for binary valuations on complete graphs [15, 3] implies hardness for graphs that have a twin cover of size zero.

**Few Resources or Agents.**

We also consider the cases where the number of goods or the number of agents are relatively small. When considering these parameters, the work of Bliem et al [6] shows that the computation of EEF allocations is FPT when parameterized by the number of goods or the number of agents for additive 0/1 valuations. In contrast, we show that finding EEF allocations respecting the structure of an underlying undirected graph is W[1]-hard when parameterized by the number of goods (Theorem 4). On the other hand, the FPT algorithm when parameterized by the number of agents can be extended to account for the graph constraints (noted in Observation 1).

**Other Notions of Fairness.**

Finally, we also consider notions of proportionality in the context of graphs — we refer to these as local and quasi-global proportionality concepts, representing the extent to which the definitions account for the underlying graph. We demonstrate that computing a locally proportional allocation is NP-hard (Theorem 11), while computing a proportional allocation that is quasi-global is tractable (Theorem 17). Notions of local proportionality have been proposed and studied in several of the papers that were summarized in the previous section.

### 2 Preliminaries

We use standard terminology from graph theory and fair division. Unless mentioned otherwise, the graphs we consider are simple and undirected. For a graph $G = (V, E)$, consisting of a set $V$ of vertices and a set $E$ of edges, by $N(v)$ we denote the neighborhood of vertex $v \in V$, i.e., the set $W \subset V$ of vertices such that for each vertex $w \in W$ there exists an edge $e = \{v, w\} \in E$. The closed neighborhood of a vertex $v$ is $N(v) \cup \{v\}$ and is denoted $N[v]$. The degree of a vertex $v$, denoted $d(v)$, is $|N[v]|$. A clique is a subset of vertices which are pairwise adjacent. An independent set is a subset of vertices, no two of which are adjacent. For $X \subseteq V$, the induced subgraph $G[X]$ denotes the subgraph whose vertex set is $X$ and the edge set consists of all edges whose both end points are in $X$.

**Definition 1.** Let $G$ be a graph. A tree-decomposition of a graph $G$ is a pair $\mathcal{T} = (T, (B_t)_{t \in V(T)})$, where $T$ is a rooted tree, such that $\cup_{t \in V(T)} B_t = V(G)$, $\cup_{t \in V(T)} B_t = V(G)$, for every edge $xy \in E(G)$ there is a $t \in V(T)$ such that $\{x, y\} \subseteq B_t$, and for every vertex $v \in V(G)$ the subgraph of $T$ induced by the set $\{t \mid v \in B_t\}$ is connected.

The width of a tree decomposition is $\max_{t \in V(T)} |B_t| - 1$ and the treewidth of $G$ is the minimum width over all tree decompositions of $G$ and is denoted by $\text{tw}(G)$.

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5 A twin cover of a graph is a subset of vertices $S$ such that $G \setminus S$ is a disjoint union of cliques, and further, every pair of vertices $u, v$ in any clique of $G \setminus S$ are “twins”, that is, $N[v] = N[u]$. 

For completeness, we also define here the notion of a nice tree decomposition with introduce edge nodes, as this is what we will work with in due course. We note that for a given tree decomposition can be modified in linear time to fulfill the above constraints; moreover, the number of nodes in such a tree decomposition of width \( w \) is \( O(w \cdot n) \).

\[\triangleleft\textbf{Definition 2.} \text{A tree decomposition } \mathcal{T} = (T, (B_\alpha)_\alpha \in V(T)) \text{ is a nice tree decomposition with introduce edge nodes if the following conditions hold.}\]

1. The tree \( T \) is rooted and binary.
2. For all edges in \( E(G) \) there is exactly one introduce edge node in \( T \), where an introduce edge node is a node \( \alpha \) in the tree decomposition \( \mathcal{T} \) of \( G \) labeled with an edge \((u, v) \in E(G)\) with \( u, v \in B_\alpha \) that has exactly one child node \( \alpha' \); furthermore \( B_\alpha = B_{\alpha'} \).
3. Each node \( \alpha \in V(T) \) is of one of the following types:
   \>
   - introduce edge node;
   \>
   - leaf node: \( \alpha \) is a leaf of \( T \) and \( B_\alpha = \emptyset \);
   \>
   - introduce vertex node: \( \alpha \) is an inner node of \( T \) with exactly one child node \( \beta \in V(T) \), furthermore \( B_\beta \subseteq B_\alpha \) and \( |B_\alpha \setminus B_\beta| = 1 \);
   \>
   - forget node: \( \alpha \) is an inner node of \( T \) with exactly one child node \( \beta \in V(T) \), furthermore \( B_\alpha \subseteq B_\beta \) and \( |B_\beta \setminus B_\alpha| = 1 \);
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   - join node: \( \alpha \) is an inner node of \( T \) with exactly two child nodes \( \beta, \gamma \in V(T) \), furthermore \( B_\alpha = B_\beta = B_\gamma \).

An instance of fair division for indivisible goods consists of \( n \) agents \( A = \{1, \ldots, n\} \) and \( m \) goods (also called items or resources), \( R = \{o_1, \ldots, o_m\} \). Further, we are also given valuations (also called preference functions or utilities) \( \nu_\ell : 2^R \to \mathbb{R} \) for every agent \( \ell \in A \). We will assume throughout that the valuation functions are additive, i.e., for each agent \( \ell \in A \) and any set of goods \( S \subseteq R \), \( \nu_\ell(S) := \sum_{o \in S} \nu_\ell(\{o\}) \). A 0/1 valuation is a function that takes values in \( \{0, 1\} \), while valuations are said to be identical if every agent has the same preference function. In the context of 0/1 valuations, we say that an agent values or approves a good if her utility for the good is 1. We will use \( V \) to denote the valuations of the agents \( A \) over \( R \). When considering fair division in the context of social networks, we are also given an undirected graph \( G \) over the agents \( A \).

Every subset \( S \subseteq R \) is called a bundle. An allocation is a function \( \pi : A \to 2^R \) mapping each agent to the bundle she receives, such that \( \pi(i) \cap \pi(j) = \emptyset \) when \( i \neq j \) because the items cannot be shared. When \( \bigcup_{a \in A} \pi(a) = R \), the allocation \( \pi \) is said to be complete, otherwise it is partial. An allocation is non-wasteful if every good is allocated to an agent that assigns positive utility to it.

An allocation \( \pi' \) dominates \( \pi \) if for all \( \ell \in A \) it holds that \( \nu_\ell(\pi(\ell)) \leq \nu_\ell(\pi'(\ell)) \) and for some \( a_j \in A \) it holds that \( \nu_{a_j}(\pi(a_j)) < \nu_{a_j}(\pi'(a_j)) \). An allocation \( \pi \) is Pareto-efficient if there exists no allocation \( \pi' \) that dominates \( \pi \). In the case of 0/1 preferences, we note that an allocation is Pareto-efficient if and only if it is complete and non-wasteful, assuming that each resource provides a value of 1 at least one agent.

Given an instance of fair division \((A, R, G = (A, E), V)\) as described above, we now introduce the following fairness notions:

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- **Graph Envy-Freeness (GEF).** We call allocation \( \pi \) graph-envy-free if for each pair of (distinct) agents \( i, j \in A \) such that \( j \in N(i) \), it holds that \( \nu_i(\pi(i)) \geq \nu_i(\pi(j)) \).
- **Global Proportionality (GP).** We say that an allocation \( \pi \) achieves global proportionality if for each agent \( \ell \in A \), \( \nu_\ell(\pi(\ell)) \geq \frac{1}{m} \nu_\ell(R) \).
- **Quasi-Global Proportionality (QP).** We say that an allocation \( \pi \) achieves quasi-global proportionality if for each agent \( \ell \in A \), \( \nu_\ell(\pi(\ell)) \geq \frac{1}{m^{1/2}} \nu_\ell(R) \).
Local Proportionality (LP). We say that an allocation \( \pi \) achieves local proportionality if for each agent \( \ell \in A \), 
\[ \nu_i(\pi(i)) \geq \frac{1}{d(\ell) + 1} \sum_{j \in N[i]} \nu_i(\pi(j)) . \]

Note that the graph versions of variants of envy-freeness (such as EF1 or EFX) can be defined analogously in a straightforward manner. It is easy to see that any graph envy-free allocation is also locally proportional and that if the underlying graph is complete, then local proportionality coincides with the standard notion of proportionality. For the problems we consider, we are typically given an instance of fair division on a graph, and the goal is to determine if there exists an allocation that satisfies some notion of fairness and efficiency. For instance, consider the following problems:

**GRAPH ENVY-FREE ALLOCATION (E-GEFA)**
*Input:* An instance of fair division on a graph \((A, R, G = (A, E), V)\).
*Question:* Does there exist an envy-free, Pareto-efficient allocation?

**LOCALLY PROPORTIONAL ALLOCATION (E-LPA)**
*Input:* An instance of fair division on a graph \((A, R, G = (A, E), V)\).
*Question:* Does there exist a Pareto-efficient allocation that achieves local proportionality?

For any efficiency concept (X) and fairness notion (Y), the X-YA problem is defined in a similar fashion. Although our questions are posed as decision versions, we note that most of our algorithms can be easily adapted to handle the natural “search” version of these problems. We refer the reader to the books \[7, 17\] and the article \[12\] for additional background on fair division.

A problem parameterized by \( k \) is fixed-parameter tractable if it is solvable in \( f(k)|I|^{O(1)} \) time for some computable function \( f \) and the input size \( |I| \) according to the problem’s encoding. Informally, \( W \)-hard problems are presumably not fixed-parameter tractable. The problem of finding a clique on at least \( k \) vertices is \( W[1] \)-hard when parameterized by \( k \). We call a problem para-NP-hard if it is NP-hard even for a constant value of the parameter. For a comprehensive introduction to the paradigm of parameterized complexity and algorithms, we refer the reader to the book \[13\].

## 3 Envy-Freeness

### 3.1 NP-hardness for two agent types

In this section, we show that finding \( E \)-GEFA allocations is NP-hard even in the setting of near-identical binary valuations: in particular, when all agents have one of two possible utilities over the items. Note that in the setting of identical binary valuations when the graph \( G \) is connected, it is easy to see that all agents must value all goods without loss of generality, and that desirable allocations are the ones that allocates the same number of goods to each agent, where the goods themselves may be arbitrarily chosen. Indeed, it is clear that an allocation with equal bundle sizes is \( E \)-GEFA. On the other hand, consider a \( E \)-GEFA allocation that does not allocate bundles of equal size to all agents. Let \( a_i \) and \( a_j \) be
two agents that receive bundles of different size. We can always find two adjacent agents on a path from $a_i$ to $a_j$ who have received bundles of different sizes, contradicting envy-freeness.

We now show that even a slightly more general situation is computationally intractable — in particular, if all agents have one of two valuations over the goods, the problem of identifying $\mathcal{E}$-GEFA allocations is NP-hard.

\begin{theorem}
The $\mathcal{E}$-GEFA problem is NP-complete even when there are two agent types, and further, agents have 0/1 valuations over the goods.
\end{theorem}

\begin{proof}
We reduce from the CUTTING $\ell$ VERTICES problem, where we are given a graph $G = (V, E)$, integers $\ell$ and $k$, and the question is if there exists a partition of the vertex set $V$ into $X \cup S \cup Y$ such that $|X| = \ell$, $|S| \leq k$ and there is no edge between $X$ and $Y$. It follows from \cite{11} that this problem is NP-hard. We now describe the instance of $\mathcal{E}$-GEFA. We introduce $2n + 1$ agents $w_1, \ldots, w_n, u_1, \ldots, u_n$ and $s$, where $n = |V|$. We call $s$ the trigger agent and we also refer to the $w_i$’s and $u_i$’s as greedy and happy agents, respectively. The graph structure on the agents is defined as follows. For every edge $e = (v_i, v_j) \in E$, we introduce the edges $(w_i, u_j)$ and $(w_j, u_i)$. We also make the trigger agent adjacent to every greedy agent. We also add the edges $(w_i, u_i)$ for all $1 \leq i \leq n$. We now turn to the items. We introduce $\ell$ items called the coveted items: $\{g_1, \ldots, g_\ell\}$. Every agent has a utility of one for these items. We introduce $(n - \ell + 1)$ items $\{p_1, \ldots, p_{n-\ell+1}\}$ called the w-type items and $\ell + k$ items $\{q_1, \ldots, q_{\ell+k}\}$ called the u-type items. The trigger agent and the greedy agents have utility one for all $w$-type items and the happy agents have utility one for all $u$-type items. This completes the description of the reduction.

We first argue the forward direction. Given a subset of at most $k$ vertices $S$, we allocate all the coveted goods to the $\ell$ greedy agents corresponding to vertices of $X$, and the $w$-type items to the remaining greedy agents and the trigger agent. Also, allocate the $k + \ell$ u-type items to happy agents corresponding to vertices in $X \cup S$. Observe that this allocation is both locally envy free and Pareto efficient, since the only empty-handed agents are the happy agents corresponding to vertices of $Y$, but the only agents they potentially envy are agents in $X$ (since these are the agents who got the coveted goods) — however, recall that there are no edges between $X$ and $Y$.

In the reverse direction, let a locally envy free, Pareto efficient allocation be given. To begin with, note that the trigger agent ensures that no coveted good is allocated to a happy agent: indeed, if the agent $u_i$ has a coveted good, then this generates envy in $w_i$, who must be given either a coveted good or a $w$-type good. This in turn makes the trigger agent envious, who must also be given either a coveted good or a $w$-type good. At this point, all the remaining $(n - 1)$ greedy agents are envious (of the trigger agent), but there aren’t enough goods to account for all of them: thus any allocation that does not assign all coveted goods to greedy agents is not locally envy free. Let $X$ be the subset of vertices corresponding to the greedy agents who were allocated the coveted goods, and let $S^*$ be the subset of vertices corresponding to $(k + \ell)$ happy agents who were given $u$-type goods. Note that $X \subseteq S^*$ and $|S^* \setminus X| \leq k$. It is easy to check that if we let $S := S^* \setminus X$ and $Y := V \setminus (X \cup S)$, then $(X \cup S \cup Y)$ is a partition of the desired kind. Indeed, if not, there is an edge between a vertex in $X$ and a vertex in $Y$, but since the happy agents corresponding to vertices in $Y$ are (by definition) empty-handed and the agents corresponding to vertices of $X$ have been assigned a coveted good, this edge would violate local envy-freeness, a contradiction. This concludes the argument.
\end{proof}
3.2 W-hardness parameterized by goods

In this section, we demonstrate the hardness of finding $\mathcal{E}$-GEFA allocations even when the number of goods is bounded by showing that the problem is $W[1]$-hard when parameterized by the number of goods.

▶ Theorem 4. The $\mathcal{E}$-GEFA problem is $W[1]$-hard when parameterized by the number of goods, even when agents have 0/1 valuations over the goods.

We describe a reduction from the $W[1]$-hard problem CLIQUE, given a graph $G$ and an integer $k$, does there exist a clique on $k$ vertices in $G$. Let $J := (G, k)$ be an instance of clique, where $G = (V, E)$ and further, $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. We assume, without loss of generality, that $m \geq \binom{k}{2}$, since we can always return a trivial No-instance when this is not the case. We begin by describing the construction of the reduced instance $J' := (A, R, H = (A, F), V)$. We define the set of goods $R$ as follows:

$$R = \{q_1, \ldots, q_\ell, p_1, \ldots, p_k, d_1, \ldots, d_{\ell+1}\},$$

where $\ell = \binom{k}{2}$. For ease of discussion, we call the first $\ell$ goods popular and the next $k$ goods specialized. The remaining are dummy items. We now define the set of agents as $A = V \cup E \cup S \cup W$, where:

$$S := \{s_1, \ldots, s_{\ell+1}\} \text{ and } W := \{w_{ij} \mid i \in [n], j \in [\ell + 1]\}.$$

We indulge in a mild abuse of notation and use $v_i$ to refer to both an element of $V$ from the clique instance and an agent of $A$ in the reduced instance (similarly for edges). The edges of $H$ are as follows:

$\triangleright$ $e = (u, v) \in E$ is adjacent to all vertices of $S$ and $u, v$.

$\triangleright$ $v_i \in V$ is adjacent to all vertices $w_{ij}$ for $j \in [\ell + 1]$.

$\triangleright$ For each $1 \leq i \leq n$, $H[\cup_{j=1}^{\ell+1} w_{ij}]$ induces a clique.

The preferences of the agents are as follows:

$\triangleright$ All agents have an utility of 1 for the popular goods.

$\triangleright$ All agents in $V$ have an utility of 1 for the specialized goods.

$\triangleright$ The agent $s_i \in S$ has an utility of 1 for $d_i$, for all $i \in [\ell + 1]$.

This completes the construction of the instance. Note that the number of goods is a function of $k$ alone. We now turn to the argument for equivalence, a clique $X \subseteq V$, of size $k$ exists in $G$ iff, there is an GEF allocation for the instance constructed $J' := (A, R, H = (A, F), V)$.

Proof. In the forward direction, let $X \subseteq V$ be a clique in $G$ and let $Y := G[X] \subseteq G$. Consider now the following allocation $\pi$. We let each agent corresponding to $Y$ receive one popular item, each agent corresponding to $X$ receive one specialized item, and finally allocate the item $d_i$ to $s_i$ for all $i \in [\ell + 1]$.

$\Box$ Figure 1 A sketch of the reduced instance based on an instance $G = (V, E), k$ of CLIQUE. Recall that $\ell$ denotes $\binom{k}{2}$, and only some vertices of $W$ are shown for clarity. The shaded vertices induce a complete subgraph. The edge $e_\ell = (v_i, v_j)$ is adjacent only to $v_i$ and $v_j$ among vertices in $V$. 
Claim 5. The allocation $\pi$ is Pareto-efficient and envy-free with respect to $H$.

Proof. For Pareto-efficiency, it suffices to argue that $\pi$ is complete and non-wasteful. This is evident from the definition of $\pi$ — indeed, all the goods were allocated, and further, these specialized goods were allocated only to the vertex agents who like them and any dummy good was allocated to the unique agent from $S$ who had a non-zero valuation for it. The popular goods are liked by everyone, and in particular the agents who received them.

We now address envy-freeness. Consider $v \in X$: the corresponding agent receives one of the specialized goods and its adjacent edges get at most one popular good each, thus there is no envy for $v$. If $v \not\in X$, the corresponding agent doesn’t get anything but neither do any of its adjacent edges, so again, there is no envy. Now consider $e \in E$. These agents are connected only to agents in $V \cup S$, who receive either specialized or dummy goods — since agents in $E$ do not value these goods, there is no envy.

The agents in $S$ each have received a dummy good and all the edges connected to them have received only one popular good, so they don’t envy any of their neighbors. Finally, the agents in $W$ have not been allocated anything. They are connected to vertices from $V$ who are either also empty-handed or have specialized goods that agents from $W$ don’t value. Thus no envy for agents in $W$. This concludes the argument. ◀

This concludes our description of a fair and complete allocation strategy given a clique in $G$. We now turn to the reverse direction, where we are given an allocation $\pi$ that is Pareto-efficient and envy-free with respect to $H$. It is useful to make the following observation about $\pi$ to begin with.

Claim 6. Let $H$ be defined as above, and let $\pi$ be an allocation that is Pareto-efficient and envy-free with respect to $H$. Then, any popular good is assigned by $\pi$ to an agent from $E$.

Further, no agent in $E$ can receive more than one popular good in the allocation $\pi$.

Proof. Let us assume that there exists a popular good $q$ that is not assigned to an agent from $E$. This means that, since $\pi$ is complete, $q$ is assigned to an agent from $V$, $S$, or $W$.

If $q$ was assigned to an agent $v_i$ from $V$, then all the vertices $w_{ij}$ for $j \in [\ell+1]$ will envy $v_i$. Note that there are $\ell+1$ such agents, each of which only value the popular goods, and the total number of (remaining) popular goods is at most $\ell - 1$, the allocation $\pi$ cannot account for all the envy generated in this situation.

If $q$ was assigned to an agent $w_{ij}$ from $W$, then all the other agents in $W$ belonging to the clique induced by $N(v_i)$ will envy $w_{ij}$. There are $\ell$ such agents, each of which only value the popular goods, and the total number of (remaining) popular goods is at most $\ell - 1$, and again, the allocation $\pi$ cannot account for all the envy generated in this situation.

If $q$ was assigned to an agent $s_i$ from $S$, then all the agents in $E$ will envy $q$. Since $|E| \geq \ell$, and all agents in $E$ only value the popular goods, the argument is the same as in the previous case.

From the discussion above, we conclude that $\pi$ allocates the popular goods to agents in $E$. Now suppose there exists an agent $e$ in $E$ who received more than one popular good. Since $e$ is adjacent to all the $s_i$’s in $S$, the presence of more than one good for $e$ triggers envy in all the $s_i$ agents. There are $(\ell + 1)$ many agents in $S$, each of whom must now be assigned two items that they value. However, apart from the unique dummy good that each $s_i$ values, the only goods that they value are the popular goods, of which only $(\ell - 2)$ remain — therefore,
we fall short of accounting for the envy of \((\ell + 1)\) agents. This shows that no agent in \(E\) can receive more than one popular good.

Since \(\pi\) is non-wasteful, the specialized goods must be distributed among agents corresponding to \(V\). The following is easy to see.

\(\triangleright\) Claim 7. Let \(H\) be defined as above, and let \(\pi\) be an allocation that is Pareto-efficient and envy-free with respect to \(H\). No agent in \(V\) can receive more than one specialized good in the allocation \(\pi\).

Proof. We know that \(\ell = (\binom{k}{2})\) many edge agents have each received a popular good. These edges combined span at least \(k\) many agents in \(V\) (recall that \(G\) is a simple graph). Each of those \(k\) agents suffer from envy as they are adjacent to edges who possess popular goods that the vertices also like. To satisfy this envy, each of these \(k\) vertices must be assigned a specialized good — indeed, the popular goods, the only other possibility, are all taken. The claim follows from the fact that there are only \(k\) specialized goods.

Let \(X \subseteq V\) be the subset of \(k\) agents that receive at least one specialized item and let \(Y \subseteq E\) be the subset of \(\ell\) agents that receive at least one popular item with respect to \(\pi\). We claim that \(G[X]\) is a clique. In particular, we claim that every edge of \(Y\) has both its endpoints in \(X\). Indeed, suppose not, and let \(e \in Y\) be an edge with at least one endpoint (say \(v\)) outside \(X\). Then, \(v\) envies \(e\), which contradicts our assumption about \(\pi\) being envy-free with respect to \(H\).

3.3 \(W\)-hardness parameterized by vertex cover

Recall that a vertex cover of a graph \(G = (V, E)\) is a subset \(S \subseteq V\) such that \(G \setminus S\) is an independent set (i.e., for any pair of vertices \(u, v \in G \setminus S\), \((u, v) \notin E\)). In the setting of directed graphs with arbitrary utilities, finding \(\mathcal{E}\)-GEFA allocations is \(\text{NP}\)-hard even for graphs that have a constant-sized vertex cover. Here, we show that in the setting of binary utilities, finding a \(\mathcal{E}\)-GEFA allocation is \(\text{W}[1]\)-hard when parameterized by the vertex cover of the underlying graph.

\(\triangleright\) Theorem 8. The \(\mathcal{E}\)-GEFA problem is \(\text{W}[1]\)-hard when parameterized by the vertex cover of the underlying graph, even when agents have \(0/1\) valuations over the goods.

Proof. We describe a reduction from the \(\text{W}[1]\)-hard problem \(\text{CLIQUE}\), given a graph \(G\) and an integer \(k\), does there exist a clique on \(k\) vertices in \(G\). Let \(J = (G, k)\) be an instance of clique, where \(G = (V, E)\) and further, \(V = \{v_1, \ldots, v_n\}\) and \(E = \{e_1, \ldots, e_m\}\). We begin by describing the construction of the reduced instance.

To begin with, we introduce \(k\) “key” agents \(u_1, \ldots, u_k\), and for every pair \(1 \leq i, j \leq k\), we introduce \(m\) agents \(x_{ij}^1, \ldots, x_{ij}^m\) — these agents are all adjacent to \(u_i\) and \(u_j\). We call these agents the \((i, j)\) “guard” agents. We also introduce \(n - k\) “residual” agents and impose a complete bipartite graph between the key and residual agents, that is, every residual agent is adjacent to every key agent. Note that this graph has a vertex cover of size \(k\) given by the key agents.

We now turn to the items. We introduce a “core” item \(g_i\) for each vertex \(v_i \in V\). The key agents and residual agents value all core items. We also introduce, for every pair \(1 \leq i, j \leq k\), \(m - 1\) dummy items valued by all \((i, j)\) guard agents. Finally, the guard agent \(x_{ij}^t\) values all the core items except the ones corresponding to the endpoints of the edge \(e_t\). This completes the description of the reduction.
In the forward direction, given a clique \( S \subseteq V \), we assign core items corresponding to the clique to the key agents, and the remaining core items to the residual agents. If the key agents \( a_i \) and \( a_j \) are assigned goods \( g_p \) and \( g_q \), then let \( \ell_{ij} \) denote the edge \((v_p, v_q)\). We say that the index \( \ell_{ij} \) is “special” for the \((i, j)\) guard agents. Now, for every guard agent \( x_{\ell_{ij}} \) except the special index, we assign the corresponding dummy item. Notice that when \( \ell = \ell_{ij} \), the guard \( x_{\ell_{ij}} \) does not envy either of the key agents it is adjacent to, since it has an utility of one for all goods except for \( g_p \) and \( g_q \).

In the reverse direction, it is easy to check that due to the limited number of core goods, no locally envy-free allocation gives a guard agent a core good (indeed, this can be seen to induce a chain of envy that leaves us with having to allocate \( n - 1 \) goods among \( n \) agents which is not feasible). It is also easy to check that all the key and residual agents get exactly one core good each. We claim that the vertices corresponding to the core goods assigned to the key agents, say \( S \), form a clique. Indeed, suppose not, and let \( v_p \) (the vertex corresponding to the good assigned to key agent \( a_i \)) and \( v_q \) (the vertex corresponding to the good assigned to key agent \( a_j \)) be two non-adjacent vertices in \( S \). Then it follows that all the \((i, j)\) guard agents envy both \( a_i \) and \( a_j \), but given that no core goods are allocated to the guard agents and there are only \((m - 1)\) dummy goods that can be assigned among these guards, it is inevitable that at least one of the guards will envy one of these key agents, contradicting our assumption of local envy-freeness.

\[\blacktriangleright\]

### 3.4 NP-hardness on paths

To show the hardness of \( \mathcal{E}\)-GEFA even when the underlying graph is a path, we reduce from a variant of SAT called \textsc{Linear SAT} (abbreviated LSAT). In an LSAT instance, each clause has at most three literals, and further the literals of the formula can be sorted such that every clause corresponds to at most three consecutive literals in the sorted list, and each clause shares at most one of its literals with another clause, in which case this literal is extreme in both clauses. The hardness of LSAT was shown in [2]. In fact, by studying the reduced instance, one may assume that a “hard” instance of LSAT has the following structure: the first \( 2q \) clauses have two literals each and are of the following form:

\[ A_i = \{s_i, \ell_i\}, B_i = \{\ell_i, t_i\}; 1 \leq i \leq q, \]

where \( s_i, \ell_i, \) and \( t_i \) denote literals, while the remaining \( p \) clauses have three literals each and are mutually disjoint from each other as well as the first \( 2q \) clauses. For ease of description, we will assume that the LSAT formula that we reduce from has this particular structure. We are now ready to describe our reduction — in the interest of simplicity, our proof is designed to address the case when the graph is a disjoint union of paths, although it is easy to “stitch” these components into a single, longer path, as we will explain later.

\[\blacktriangleright\textbf{Theorem 9}.\] The \( \mathcal{E}\)-GEFA problem is NP-complete even when the graph induced by the agents is a disjoint union of paths, and further, agents have 0/1 valuations over the goods.

Membership in NP is straightforward to check. We focus here on the reduction demonstrating hardness. Let \( \phi \) be an instance of LSAT over variables \( \hat{X} := \{x_1, \ldots, x_n\} \) and clauses:

\[ C := \{A_1, B_1, \ldots, A_q, B_q, C_1, \ldots, C_p\}, \]

as described above. We refer to the first \( 2q \) clauses as the \textit{coupled} clauses and the remaining as \textit{isolated} clauses. We now turn to the construction of the reduced instance.
\[ J_\Phi := (V, R, H, V). \] We define the set of goods \( R \) as \( R_X \cup R_C \), where:

\[ R_X = \{ y_1, \ldots, y_n, x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n \}, \]

and:

\[ R_C = \{ g_1, \ldots, g_q, d_1, \ldots, d_p \}. \]

The set of agents \( V \) is given by \( X \cup C \cup Y \cup G \cup D \), where \( C \) is denoted in the same way as in the LSAT instance, and further:

\[ X = \{ X_1, \ldots, X_n \}, Y = \{ Y_1, \ldots, Y_n \}, G = \{ G_1, \ldots, G_q \} \text{ and } D = \{ D_1, \ldots, D_p \}. \]

We now simultaneously describe the structure of the graph \( H \) and the preferences of the agents.

**Assignment Gadgets.** For each \( 1 \leq i \leq n \), add an edge between \( X_i \) and \( Y_i \). The agent \( X_i \) values \( \{ x_i, \bar{x}_i, y_i \} \), while \( Y_i \) values the good \( y_i \) (and nothing else).

**Isolated Clause Gadgets.** For each \( 1 \leq i \leq p \), we add an edge between agents \( C_i \) and \( D_i \). The agent \( C_i \) values the literal \( \ell \) if and only if \( \ell \in C_i \) along with \( d_i \), while \( D_i \) values the good \( d_i \) (and nothing else).

**Coupled Clause Gadgets.** For each \( 1 \leq i \leq q \), we add an edge between agents \( A_i \) and \( B_i \), and also an edge between \( G_i \) and \( A_i \). The agent \( G_i \) values the good \( g_i \) (and nothing else). Agents \( A_i \) and \( B_i \) value, respectively, the goods \( \{ g_i, s_i, t_i, \ell_i \} \) and \( \{ s_i, t_i \} \).

This completes the description of the construction.

We now discuss the equivalence. In the forward direction, given a satisfying assignment \( \tau \) for \( \Phi \), we define the allocation \( \pi \) as follows:

\[ \pi(X_i) = \begin{cases}  x_i & \text{if } \tau(x_i) = 0, \\  \bar{x}_i & \text{if } \tau(x_i) = 1. \end{cases} \]

Observe that the literals that evaluate to true under \( \tau \) are not assigned to any of the agents from \( X \).

\[ \pi(D_i) = \{ d_i \} \] and \( \pi(C_i) \) is the item corresponding to one of the literals of \( C_i \) that evaluates to 1 under \( \tau \). If there are multiple literals that evaluate to 1 under \( \tau \), allocate all items corresponding to said clause to the agent \( C_i \).

\[ \pi(G_i) = \{ g_i \}. \] Further, if \( \tau \) sets \( \ell_i \) to true, then \( \pi(A_i) = \{ \ell_i \} \) and \( \pi(B_i) = \emptyset \). However, if \( \tau \) sets \( \ell_i \) to false, then note that it must be the case that the literals \( s_i \) and \( t_i \) evaluate to true under \( \tau \), and we assign \( \pi(A_i) = \{ s_i \} \) and \( \pi(B_i) = \{ t_i \} \).

If both \( s_i \) and \( t_i \) evaluate to true under \( \tau \), then they can be distributed one each among the agents \( A_i \) and \( B_i \), and if exactly one of them is set to true then the good corresponding to the true literal can always be given to \( B_i \).
It is easy to verify that $\pi$ is well-defined: this follows from the fact that the clauses are almost disjoint. It is straightforward to check that the allocation is complete and non-wasteful (and hence Pareto-efficient). We now observe that the allocation is also envy-free: indeed, the agents involved in paths that are edges get one item each while the agents involved in paths of length three either get one item each, or in the case when $B_i$ is empty-handed, $A_i$ receives a good for which $B_i$ has zero utility (and in this case, both $A_i$ and $G_i$ receive one good each), thus there is no envy.

In the reverse direction, let $\pi$ be an envy-free, Pareto-efficient allocation. Since $\pi$ is non-wasteful, note that the item $y_i$, is allocated by $\pi$ to either $Y_i$ or $X_i$. However, if $y_i \in \pi(X_i)$, then the agent $Y_i$ envies $X_i$, but since she approves of no item other than $y_i$, it is not possible for $\pi$ to resolve this envy. Therefore, combined with the fact that $\pi$ is non-wasteful, we conclude that $\pi(Y_i) = \{y_i\}$ for all $1 \leq i \leq n$. A similar argument establishes that $\pi(D_i) = \{d_i\}$ for all $1 \leq i \leq p$ and $\pi(G_i) = \{g_i\}$ for all $1 \leq i \leq q$.

Since $\pi(Y_i) = \{y_i\}$, we have that $\pi(X_i) \neq \emptyset$. By the non-wastefulness of $\pi$, we also know that $\pi(X_i) \subseteq \{x_i, \bar{x}_i\}$. We use this to define an assignment $\tau$ as follows:

$$\tau(x_i) = \begin{cases} 1 & \text{if } \pi(X_i) = \{\bar{x}_i\}, \\ 0 & \text{if } \pi(X_i) = \{x_i\}, \\ 1 & \text{otherwise}. \end{cases}$$

It turns out that if $\pi(X_i) = \{x_i, \bar{x}_i\}$, then the setting of $\tau$ for $x_i$ is immaterial, and we set it to 1 as suggested above as a matter of convention. We now argue that $\tau$ is a satisfying assignment. Suppose not, and in particular, consider a clause that is not satisfied by $\tau$. We address the cases of isolated and coupled clauses separately.

Let $C_i$ be an isolated clause not satisfied by $\tau$. Since $C_i$ approves $d_i$, $\pi(D_i) = d_i$, and $D_i \in N(C_i)$, we know that $\pi(C_i) \neq \emptyset$. In particular, this implies that one of the literals belonging to the clause $C_i$ was allocated to the agent $C_i$. As an example, suppose this literal was $\bar{x}_i$. This implies that $\bar{x}_i \notin \pi(X_i)$, which in turn implies that $\pi(X_i) = \{x_i\}$ and that $\tau(x_i) = 0$, contradicting our assumption that $C_i$ is not satisfied by $\tau$. A symmetric argument holds for the case when the literal in question was a positive literal as opposed to a negated one.

Now, let $A_i$ be a coupled clause not satisfied by $\tau$. If $\ell_i \in \pi(A_i)$, then by an argument similar to the previous case, we can conclude that $\tau(\ell_i) = 1$, which leads to a contradiction. Therefore, assume that $\ell_i \notin \pi(A_i)$. Since $A_i$ approves $g_i$, $\pi(G_i) = g_i$, and $G_i \in N(A_i)$, we know that $\pi(A_i) \neq \emptyset$. Therefore, it must be the case that at least one of $s_i$ or $t_i$ belongs to $\pi(A_i)$. This again implies that $\tau(s_i) = 1$ or $\tau(t_i) = 1$, which leads to the desired contradiction. The same argument works if we started with the coupled clause $B_i$ instead of $A_i$. This discussion concludes the proof.

We remark that it is possible to combine the connected components in the reduced instance above by simply introducing “dummy connector agents” that each value a corresponding dummy item and nothing else. All the arguments made above will work in exactly the same fashion since nobody would have reason to envy these newly introduced agents, and vice versa. Thus, we have the following corollary.

**Corollary 10.** The $\varepsilon$-GEFA problem is NP-complete even when the graph induced by the agents is a path, and further, agents have 0/1 valuations over the goods.
4 Proportionality for Graphs

4.1 Local Proportionality: NP-hardness

Theorem 11. The $\mathcal{E}$-LPA problem is NP-complete on undirected graphs, even when all agents have 0/1 valuations over the resources.

Proof. Membership in NP follows from the fact that an allocation serves as a certificate: indeed, given an allocation, it can be checked whether the allocation is local graph proportional in polynomial time. We show NP-hardness by a reduction from $3$-COLORING, where we are given a graph $G$ and the question is to determine if there exists a coloring the vertices with 3 colors such that no two adjacent vertices are coloured the same color. Equivalently, we would like to know if the vertex set of $G$ can be partitioned into 3 independent sets.

Let $G = (V, E)$ be an instance of $3$-COLORING, where $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. We will now describe the reduced instance, which we denote by $\mathcal{G} := (A, R, H = (A, F), V)$. We have the following items:

$$R = V \cup E_1 \cup E_2 \cup E_3 \cup \{a_1, a_2, a_3\} \cup \{x_1, x_2, x_3\},$$

where $E_b = \{e_b^i \mid i \in [m]\}, b \in [3]$. We refer to the items in $V$ as the core goods, the items in $E_b$ as guard goods of type $b$. We also introduce the following agents

$$A = F_1 \cup F_2 \cup F_3 \cup \{A_1, A_2, A_3\} \cup \{B_1, B_2, B_3\} \cup C_1 \cup C_2 \cup C_3,$$

where $F_b = \{E_b^i \mid i \in [m]\}, b \in [3]$ and $C_b = \{C_b^i \mid i \in [n+1] \cup \{0\}\}, b \in [3]$. The preferences of the agents are as follows:

- For each $b \in [3]$, the agent $A_b$ approves all the core goods and the item $a_b$.
- For each $b \in [3]$, the agent $B_b$ approves $x_b$.
- For an edge $e_i = (v_p, v_q)$ and $b \in [3]$ the agent $E_b^i$ approves the goods $\{e_b^i, v_p, v_q, x_b, a_b\}$.
- For each $b \in [3]$, the agents in $C_b$ approve all the core goods and the item $a_b$.

The graph $H$ has the following structure. For each $b \in [3]$, the agent $A_b$ is adjacent to each agent in $F_b$. The agents $C_b^0$ and $B_b$ are adjacent to all agents in $F_b$. For each $i \in [m]$, the agents in $C_b$ induce a clique. We refer the reader to Figure 3 for a schematic depiction of the graph. This completes the description of the construction. We now turn to a discussion of the equivalence of these instances.

In the forward direction, let a coloring of $G$ with three colors $\{1, 2, 3\}$ be given. We propose an allocation $\pi$ as follows. For any $b \in [3]$, we assign the core goods corresponding to vertices of color $b$ to $A_b$, along with the item $a_i$. For each $b \in [3]$ and $i \in [m]$, we assign the good $e_b^i$ to the agent $E_b^i$. For each $b \in [3]$, assign the good $x_b$ to the agent $B_b$. This describes a complete and non-wasteful allocation of $R$ among the agents in $A$, and we claim that the allocation is also locally proportional. We establish this by a case analysis. In the following, let $b \in [3]$ be fixed.

- No agent in the neighborhood of $A_b$ receives a good that is valued by $A_b$, and $A_b$ receives at least one good.
- For any $i \in [m]$, the agent $E_b^i$ values five goods and receives one. Let $e_i = (v_p, v_q)$, without loss of generality, assume that $v_p$ is colored $b$. Indeed, it is possible that neither endpoint of $e_i$ is colored $b$, or that $v_q$ is colored $b$, in which case this argument works.
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in the same way, but importantly, note that it is never the case that both \( v_p \) and \( v_q \) are colored \( b \), since we started with a proper coloring.

Recall that the neighborhood of \( E^b_\pi \) has three agents, whose allocations projected on the goods valued by \( E^b_\pi \) are as follows:

\[
\pi(A_b) = \{a_b, v_p\}, \pi(B_b) = \{x_b\}, \text{ and } \pi(C_b^n) = \emptyset.
\]

Therefore, the total valuation of \( E^b_\pi \) for the items allocated in its closed neighborhood is four, and \( d(E^b_\pi) + 1 = 4 \). This implies that the allocation is locally proportional for \( E^b_\pi \).

▷ For any agent \( C_i^b \in C_b \), observe that the agents receive nothing, and no vertex in their closed neighborhood receives anything valued by them.

▷ No agent in the neighborhood of \( B_b \) receives a good that is valued by \( B_b \), and \( B_b \) receives one good.

This concludes the argument in the forward direction. In the reverse, let \( \pi \) be a Pareto-efficient allocation that is locally proportional with respect to \( H \). We make the following observations.

▷ Claim 12. For any \( b \in [3] \), \( \pi(B_b) = \{x_b\} \).

Proof. These agents don't like any other good, and are connected with the edge agents \( E^b_\pi \). If any of the edge agents receives the \( x_b \) goods, then it will be impossible to satisfy the proportionality guarantee for \( B_b \). None of the other agents value \( x_b \), so the claim follows from the non-wastefulness of \( \pi \).

▷ Claim 13. For any \( b \in [3] \), and \( \pi(C_i^n) = \emptyset \). Further, \( \pi(C_i^n) = \emptyset \) for all \( i \in [n + 1] \).

Proof. If not, then by non-wastefulness, \( C_i^b \) receives some subset of the core goods and possibly the item \( a_b \). However, since any of these goods are also valued equally by the agents \( C_i^b \), for all \( 1 \leq i \leq n + 1 \), they must also receive at least one good that they value to satisfy their local proportionality constraints. However, given that between them they only value at most \( n \) goods altogether (after excluding the goods allocated to \( C_i^n \)), it is impossible to extend this allocation to satisfy the constraints of all the agents. The second part of the claim follows by a similar argument.

▷ Claim 14. For any \( b \in [3] \), no core good is assigned to an agent from \( F_b \).

Proof. If a core good is assigned to any agent in \( F_b \), then \( \pi(C_i^n) \neq \emptyset \), which contradicts the previous claim.

▷ Claim 15. For any \( b \in [3] \), \( a_b \in \pi(A_b) \).

Proof. By non-wastefulness, \( a_b \in \pi(X) \) for some \( X \in \{A_b \} \cup F_b \cup C_b \). By Claim 13, \( X \in \{A_b \} \cup F_b \). However, if \( a_b \) is assigned to some agent in \( F_b \), then \( \pi(C_i^n) \neq \emptyset \), which again contradicts Claim 13. Therefore, it follows that \( a_b \in \pi(A_b) \).

▷ Claim 16. For any \( b \in [3] \) and \( i \in [m] \), \( \pi(E^b_\pi) = \{e_i^b\} \).

Proof. This follows from the claims above and the non-wastefulness of \( \pi \).

Consider the partition \( (V_1, V_2, V_3) \) of \( V \) given by the core goods assigned to agents \( A_1, A_2 \) and \( A_3 \), respectively — in other words, \( V_b \) is defined as the subset of vertices corresponding to the core goods assigned to \( A_b \) for each \( b \in [3] \). Note that the claims above imply that the core goods can only be assigned to agents \( A_1, A_2 \) and \( A_3 \), therefore, the proposed partition
4.2 Quasi-Global Proportionality: Efficient Algorithms

To obtain efficient algorithms for finding Pareto-efficient allocations that respect quasi-global proportionality, we model the problem of finding Pareto-efficient allocations respecting quasi-global proportionality using an integer linear program (ILP) with a structured constraint matrix. In particular, it is well-known that if the constraint matrix of an ILP is totally unimodular, then the corresponding instance can be solved in polynomial time. We turn to an explanation of our encoding.

Theorem 17. The problem of finding a Pareto-efficient allocation that is quasi-globally proportional with respect to an underlying undirected graph on the agents can be solved in polynomial time if all agents have 0/1 valuations.

Proof. Let us assume there are n agents and m goods. We will introduce a variable $x_{ij}$ which indicates whether agent $i$ gets good $j$, and $a_{ij}$ indicates whether agent $i$ likes good $j$. These constraints are as follows.

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A unimodular Matrix is a square integer matrix having determinant +1 or -1. A totally unimodular matrix is a matrix for which every square non-singular submatrix is unimodular.
We encode the fact that the allocation defined by $x$ is well-defined by introducing the following constraint for each good $j$:

For each agent $i$, let $s_i$ be the number of items that have utility 1 for agent $i$. For each agent $i$, introduce the following proportionality constraint:

$$\forall j \sum_{i=1}^{n} x_{ij} \leq 1 \text{ and } \forall i \sum_{j=1}^{m} a_{ij} x_{ij} \geq \frac{s_i}{(d_i + 1)}$$

We let the objective function be $\sum_{i=1}^{n} a_{ij} x_{ij}$. Note that any assignment for which this function achieves a value of $m$ is complete and non-wasteful, and also respects quasi-global proportionality. It is straightforward to verify that the constraint matrix for the ILP described above is totally unimodular for any underlying graph $H$.

We remark that the problem of assigning goods in a proportional fashion (for any of the notions of proportionality that we have introduced) beyond 0/1 valuations is NP-hard even when there are only two agents with identical valuations, by a standard reduction from PARTITION, with the graph being a single edge on two agents.

## 5 Concluding Remarks

We studied locally EEF allocations in the setting of binary valuations and undirected graphs, and demonstrated that the problem of finding such allocations is computationally intractable for various restricted settings. On the algorithmic front, tools based on dynamic programming and ILP can be used when the instance is structured and has a small number of agent types and item types.

It is natural to consider notions of fairness and efficiency other than the ones that we explored here. We remark that notions for which allocations can be found efficiently for complete graphs (such as EF1+PO in the setting of binary allocations), the local version is not as interesting, since the allocation that works for a complete graph will work for any graph. For this reason, fairness notions of EF1 or EQx are not relevant to this setting when combined with Pareto efficiency. It would be worth exploring locally fair and efficient allocations for more general valuations, in particular, including chores. In the context of parameterized complexity, a specific unresolved question is if the problem of finding locally EEF allocations parameterized by vertex cover in XP.

### References

1. Rediet Abebe, Jon M. Kleinberg, and David C. Parkes. Fair division via social comparison. In *Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems, AAMAS*, pages 281–289, 2017.
2. Esther M. Arkin, Aritra Banik, Paz Carmi, Gui Citovsky, Matthew J. Katz, Joseph S. B. Mitchell, and Marina Simakov. Choice is hard. In *Proceedings of the 26th International Symposium of Algorithms and Computation ISAAC 2015*, pages 318–328, 2015.
3. Haris Aziz, Serge Gaspers, Simon Mackenzie, and Toby Walsh. Fair assignment of indivisible objects under ordinal preferences. *Artif. Intell.*, 227:71–92, 2015. [doi:10.1016/j.artint.2015.06.002](https://doi.org/10.1016/j.artint.2015.06.002)
4. Xiaohui Bei, Youming Qiao, and Shengyu Zhang. Networked fairness in cake cutting. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI*, pages 3632–3638, 2017.
Aurélie Beynier, Yann Chevaleyre, Laurent Gourvès, Julien Lesca, Nicolas Maudet, and Anaëlle Wilczynski. Local envy-freeness in house allocation problems. In Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS, pages 292–300, 2018.

Bernhard Bliem, Robert Bredereck, and Rolf Niedermeier. Complexity of efficient and envy-free resource allocation: Few agents, resources, or utility levels. In Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence, IJCAI, pages 102–108, 2016.

Steven J. Brams and Alan D. Taylor. Fair division - from cake-cutting to dispute resolution. Cambridge University Press, 1996.

Robert Bredereck, Andrzej Kaczmarczyk, Dusan Knop, and Rolf Niedermeier. High-multiplicity fair allocation using parametric integer linear programming. CoRR, abs/2005.04907, 2020. URL: https://arxiv.org/abs/2005.04907.

Robert Bredereck, Andrzej Kaczmarczyk, Dušan Knop, and Rolf Niedermeier. High-multiplicity fair allocation: Lenstra empowered by n-fold integer programming. In Proceedings of the 2019 ACM Conference on Economics and Computation. Association for Computing Machinery, 2019.

Robert Bredereck, Andrzej Kaczmarczyk, and Rolf Niedermeier. Envy-free allocations respecting social networks. In Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS, 2018, pages 283–291, 2018.

Thang Nguyen Bui and Curt Jones. Finding good approximate vertex and edge partitions is np-hard. Inf. Process. Lett., 42(3):153–159, 1992. doi:10.1016/0020-0190(92)90140-Q.

Yann Chevaleyre, Ulle Endriss, and Nicolas Maudet. Distributed fair allocation of indivisible goods. Artif. Intell., 242:1–22, 2017.

Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer, 2015.

Eduard Eiben, Robert Ganian, Thekla Hamm, and Sebastian Ordyniak. Parameterized complexity of envy-free resource allocation in social networks. In Proceedings of the Thirty-Fourth AAAI Conference on Artificial Intelligence, AAAI, pages 7135–7142. AAAI Press, 2020.

Hadi Hosseini, Sujoy Sikdar, Rohit Vaish, Hejun Wang, and Lirong Xia. Fair division through information withholding. In Proceedings of the Thirty-Fourth AAAI Conference on Artificial Intelligence, AAAI, pages 2014–2021, 2020.

Ton Kloks. Treewidth, Computations and Approximations, volume 842 of Lecture Notes in Computer Science. Springer, 1994.

Jack M. Robertson and William A. Webb. Cake-cutting algorithms - be fair if you can. A K Peters, 1998.