Graph constructions for the contact process with
a prescribed critical rate

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Abstract

We construct graphs (trees of bounded degree) on which the contact process has critical rate (which will be the same for both global and local survival) equal to any prescribed value between zero and \( \lambda_c(\mathbb{Z}) \), the critical rate of the one-dimensional contact process. We exhibit both graphs in which the process at this target critical value survives (locally) and graphs where it dies out (globally).

Keywords: contact process, phase transition

1 Introduction

This paper exhibits a range of examples concerning phase transitions of the contact process. Our work can be seen as a complement to the previous works by Madras, Schinazi and Schonmann [MSS94], and by Salzano and Schonmann [SS97, SS99], where the same line of inquiry was pursued.

The contact process describes a class of interacting particle systems which serve as a model for the spread of epidemics on a graph. It was introduced by Harris [Har74]. It is defined on a locally finite graph \( G \) by the following rules for a continuous-time Markov dynamics: vertices can be healthy (state 0) or infected (state 1); infected vertices recover with rate one, and transmit the infection to each healthy neighbour with rate \( \lambda > 0 \).

We denote by \( (\xi^A_{G,\lambda,t} : t \geq 0) \) the contact process on \( G = (V,E) \) with infection rate \( \lambda \) and initially infected set \( A \subset V \) (as explained in Section 1.1, we will occasionally omit or change aspects of this notation). With a conventional abuse of notation, we treat \( \xi^A_{G,\lambda,t} \) as either an element of \( \{0,1\}^V \) or as a subset of \( V \) (the set of infected vertices). We refer the reader to [Lig85] and [Lig99] for an introduction to this process, including all the statements made without further explicit reference in this introduction.

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The contact process has as absorbing state the configuration in which all individuals are healthy; we denote this state by $\emptyset$. We define the probability of survival (of the infection)

$$\zeta_{G,\lambda}(A) := \mathbb{P}\left(\xi_{G,\lambda,t}^A \neq \emptyset \text{ for all } t\right), \quad A \subset V.$$ 

Due to an elementary monotonicity property of the process, this quantity is non-decreasing in $\lambda$, $G$, and $A$ (for the latter two, take the partial order given by graph and set inclusion, respectively). Moreover, if $G$ is connected, then for fixed $\lambda$, $\zeta_{G,\lambda}(A)$ is either equal to zero for all finite $A$ (in which case the process with parameter $\lambda$, on $G$ is said to die out) or non-zero for all finite and non-empty $A$ (the process is then said to survive, or to survive globally). We then define the critical threshold for global survival as

$$\lambda_c^{\text{glob}}(G) := \inf \{ \lambda : \zeta_{G,\lambda}(A) > 0 \text{ for all(any) finite and non-empty } A \}.$$ 

Next, define the probability of local survival

$$\beta_{G,\lambda}(A,v) := \mathbb{P}\left(\limsup_{t \to \infty} \xi_{G,\lambda,t}^A(v) = 1\right), \quad A \subset V, \ v \in V.$$ 

It is readily seen that $\beta_{G,\lambda}(A,v) \leq \zeta_{G,\lambda}(A)$. Moreover, $\beta_{G,\lambda}(A,v)$ is non-decreasing in $\lambda$, $G$, $A$, and if $G$ is connected, then for fixed $\lambda$ we either have $\beta_{G,\lambda}(A,v) = 0$ for all choices of (finite, non-empty) $A$ and $v$, or $\beta_{G,\lambda}(A,v) > 0$ for all such choices. In the latter case, we say that the process survives locally (in other sources, it is said in this case that the process survives strongly, or is recurrent). We define the critical threshold for local survival as

$$\lambda_c^{\text{loc}}(G) := \inf \{ \lambda : \beta_{G,\lambda}(A,v) > 0 \text{ for all(any) } v \text{ and finite } A \neq \emptyset \}.$$ 

Although the distinction between global and local survival will not be important for our main result and proof, we gave it here for the sake of the discussion in the rest of this introduction.

The contact process has been initially studied on $\mathbb{Z}^d$; there it holds that the two critical values coincide; we will denote their common value by $\lambda_c(\mathbb{Z}^d)$. It was proved in [BG90] that the process on $\mathbb{Z}^d$ at the critical rate dies out. Results for the contact process on the infinite regular tree with offspring number $d \geq 2$ (denoted $\mathbb{T}^d$) were obtained in the 1990’s, notably in [Lig96a], [Lig96b] and [Pem92]. There it holds that $0 < \lambda_c^{\text{glob}}(\mathbb{T}^d) < \lambda_c^{\text{loc}}(\mathbb{T}^d) < \infty$, and moreover the process at the lower critical value dies out, and the process at the upper critical value survives globally but not locally.

The main result of this paper concerns the set of values that the critical rates $\lambda_c^{\text{glob}}(G)$, $\lambda_c^{\text{loc}}(G)$ can attain, as $G$ ranges over all locally finite graphs, and also whether the critical contact process can survive for these possible values of the critical rate. Let us make some preliminary comments in this direction.

1. On a finite graph $G$, the contact process dies out regardless of $\lambda$, that is, we have $\lambda_c^{\text{glob}}(G) = \lambda_c^{\text{loc}}(G) = \infty$. 

2.
2. On an infinite graph $G$, we necessarily have $\lambda^\text{glob}(G) \leq \lambda^\text{loc}(G) \leq \lambda_c(Z)$. This follows from monotonicity: $G$ contains a copy of $\mathbb{N}$ inside it (since $G$ is locally finite), and it is known that $\lambda_c(\mathbb{N}) = \lambda_c(Z)$; see for instance Corollary 2.5 in [AMP10].

3. There are infinite graphs for which the critical rate for local (hence also global) survival is arbitrarily small, such as high-dimensional lattices and high-degree regular trees, see [Gri83, equation (1.14)] and [Pem92, Theorem 2.2].

4. There are also infinite graphs for which the critical rate for local (hence also global) survival is equal to zero, such as Galton-Watson trees with sufficiently heavy-tailed offspring distributions, see [Pem92, page 2112].

5. An example was given in [SS99] of a graph $G$ with $\lambda^\text{loc}(G) = \lambda^\text{glob}(G) = \lambda^\text{c}(Z)$ and so that the contact process with this critical rate survives locally. This is the “desert-and-oasis” example in page 863 of that paper, which is based on a construction of [MSS94] pertaining to a contact process with inhomogeneous rates.

6. In pages 859-862 of [SS99], the authors fix $d \geq 2$, then fix an arbitrary $\lambda$ with $\lambda^\text{glob}(\mathbb{Z}^d) < \lambda < \lambda^\text{loc}(\mathbb{Z}^d)$, and construct a graph $G$ for which $\lambda = \lambda^\text{glob}(G) < \lambda^\text{loc}(G)$. The class of examples obtained in this way therefore shows that
\[
\forall \lambda \in \bigcup_{d=2}^{\infty} (\lambda^\text{glob}(\mathbb{Z}^d), \lambda^\text{loc}(\mathbb{Z}^d)) \exists G : \lambda = \lambda^\text{glob}(G) < \lambda^\text{loc}(G). \quad (1)
\]

We now state our main result:

**Theorem 1.** (a) For any $\lambda \in (0, \lambda_c(Z))$ there exists a tree $G$ of bounded degree for which $\lambda^\text{glob}(G) = \lambda^\text{loc}(G) = \lambda$ and the contact process on $G$ with rate $\lambda$ survives locally.

(b) For any $\lambda \in (0, \lambda_c(Z))$ there exists a tree $G$ of bounded degree such that $\lambda^\text{glob}(G) = \lambda^\text{loc}(G) = \lambda$ and the contact process on $G$ with rate $\lambda$ dies out.

Although the construction we give here is very similar to the one in [SS99] (and [MSS94]) mentioned in item (5) above, it has novel aspects that free us from being restricted to having $\lambda_c(Z)$ as the critical rate. In essence, the graph we construct consists of an infinite half-line to which we append, in very sparse locations (say, $a_1 \ll \cdots \ll a_i \ll \cdots$), regular trees with large (but fixed) degree, truncated at height $h_i$. In the terms of the aforementioned examples of [MSS94] and [SS99], the half-line is the “desert” and the trees are the “oases”. This means that, for $\lambda$ within a certain controlled range (inside the interval $(0, \lambda_c(Z))$), the contact process stays active for a very long time in the trees, but is very unlikely to cross the line segments in between them in any single attempt. The locations and heights are chosen in a way that is increasingly sensitive to the value of $\lambda$, etc.
so that a certain target value can be guaranteed to be critical for global and local survival.

The construction uses some recent results from [CMMV14] to guarantee long-term persistence of the contact process on trees of finite height, as well as some coupling properties. We should mention that the construction would have been somewhat simplified by the use of star graphs rather than trees, but we wanted to exhibit a graph with a uniform upper bound on degrees.

Let us also mention that we believe the ideas we develop in this paper allow for graph constructions that lead to replacing the union in (1) by the full interval \((0, \lambda_c(Z))\), but we do not work out the details here. Additionally, in line with our Theorem 1, it would be interesting to know which set of pairs \((\lambda_1, \lambda_2) \in [0, \lambda_c(Z)]^2\) can occur as \((\lambda_c^{\text{glob}}(G), \lambda_c^{\text{loc}}(G))\) for some graph \(G\).

Finally, it is conceivable that glueing together graphs obtained from Theorem 1, each with a different critical value, one could prove the following. For any (finite or infinite) sequence of values \(0 < \lambda_1 < \lambda_2 < \cdots < \lambda_c(Z)\), there exists a graph \(G\) for which the function \(\lambda \mapsto \zeta_{G,\lambda}(A)\) (for any \(A\)) is discontinuous at \(\lambda_i\) for each \(i\). See the proof of Theorem 3.3.1 for an instance where glueing graphs can produce this kind of discontinuity. We leave this line of questioning for future work.

The rest of the paper is organized as follows. In the rest of this introduction, we explain the notation we use and the graphical construction of the contact process. In Section 2, we state Theorem 2, which allows us to augment graphs in a way that is favorable for the contact process with rate \(\lambda\) and unfavorable for the process with rate \(\lambda' < \lambda\), where \(\lambda\) is some prescribed infection rate. Using this theorem, we give in that section the proof of Theorem 1; the remainder of the paper is dedicated to the proof of Theorem 2. Section 3 gathers some preliminary results about the contact process on line segments and trees. Section 4 contains the key definitions of our graph augmentation construction, and states key results (Propositions 2, 3 and 4), which together readily give the proof of Theorem 4. Section 5 and the appendix are more technical and contain the proofs of the three key propositions (as well as several auxiliary results).

### 1.1 Notation, graphical construction

Let us first detail the notation we use for graphs. Let \(G = (V, E)\) be an unoriented graph with set of vertices \(V\) and set of edges \(E\). We say two vertices are neighbors if there is an edge containing both. The degree of a vertex \(v\), denoted \(\deg_G(v)\), is the number of neighbors of \(v\). All graphs we consider are locally finite, meaning that all their vertices have finite degree. Finally, graph distance in \(G\) between vertices \(u\) and \(v\) is denoted \(\text{dist}_G(u, v)\).

Next, we recall the graphical construction of the contact process. Here we will want to consider a standard monotone coupling of contact processes on the same graph with different infection rates. This is implemented by endowing transmission arrows with numerical labels, as we now explain. Fix a graph \(G\) and also \(\lambda > 0\). We take a family of independent Poisson point processes:
• for each $v \in V$, a Poisson point process $D^v$ on $[0, \infty)$ with intensity equal to Lebesgue measure; if $t \in D^u$, we say there is a recovery mark at $u$ at time $t$;

• for each ordered pair $(u, v) \in V^2$ such that $\{u, v\} \in E$, a Poisson process $D^{(u,v)}$ on $[0, \infty)^2$ with intensity equal to Lebesgue measure; if $(t, \ell) \in D^{(u,v)}$, we say there is a transmission arrow with label $\ell$ at time $t$ from $u$ to $v$.

Given $\lambda > 0$ and $u, v \in V$ and $0 \leq s < t$, a $\lambda$-infection path from $(u,s)$ to $(v,t)$ is a right-continuous function $\gamma: [s,t] \to V$ satisfying $\gamma(s) = u$, $\gamma(t) = v$, $r /\notin D^{\gamma(r)}$ for all $r$, and whenever $\gamma(r-) \neq \gamma(r)$ there is $\ell \leq \lambda$ such that $(r, \ell) \in D^{(\gamma(r-), \gamma(r))}$.

That is, a $\lambda$-infection path cannot touch recovery marks and can traverse transmission arrows with label smaller than or equal to $\lambda$.

In most places, the value of $\lambda$ will be clear from the context, so we simply speak of infection paths rather than $\lambda$-infection paths. We write $(u, s) \xrightarrow[\lambda]{\sim} (v, t)$ (sometimes omitting $\lambda$) either if $(u, s) = (v, t)$ or if there is a $\lambda$-infection path from $(u, s)$ to $(v, t)$. More generally, for $S_1, S_2 \subset V \times [0, \infty)$, we write $S_1 \rightsquigarrow S_2$ if there is an infection path from $(u, s)$ to $(v, t)$, for some $(u, s) \in S_1$, $(v, t) \in S_2$ (we write $S \rightsquigarrow (v, t)$ instead of $S \rightsquigarrow \{(v, t)\}$, and similarly for $(u, s) \rightsquigarrow S$).

Given $A \subset V$, setting

$$\xi^A_{G, \lambda,t}(v) := 1\{A \times \{0\} \xrightarrow[\lambda]{\sim} (v, t)\}, \quad t \geq 0, \quad v \in V,$$

where $1$ denotes the indicator function, we obtain that $\xi^A_{G, \lambda,t}$ is a contact process with parameter $\lambda$, started with vertices in $A$ infected and vertices in $V \setminus A$ healthy. Note that this construction readily gives the monotone relation

$$A \subset A', \ G \text{ subgraph of } G' \quad \Rightarrow \quad \xi^A_{G, \lambda,t} \leq \xi^{A'}_{G', \lambda,t}, \quad t \geq 0.$$

In case we are considering the contact process $(\xi^A_{G, \lambda,t} : t \geq 0)$ on a graph $G$ and $G'$ is a subgraph of $G$, we sometimes refer to $(\xi^A_{G', \lambda,t} : t \geq 0)$ as the process confined to $G'$.

Finally, we write

$$\bar{\xi}^A_{G, \lambda}(v) := \int_0^\infty \xi^A_{G, \lambda,t}(v) \, dt, \quad v \in V,$$

that is, $\bar{\xi}^A_{G, \lambda}(v)$ is the total amount of time that $v$ is infected in $(\xi^A_{G, \lambda,t} : t \geq 0)$.

2 Proof of main result

Our graph construction will be given by recursively applying a graph augmentation procedure, with each step taking as input a rooted graph (a tree with
bounded degree) and a prescribed value of the infection rate. The result that allows us to take each step is the following.

**Theorem 2.** For any \( \lambda \in (0, \lambda_c(Z)) \) there exist \( c_\lambda > 0 \) and \( d = d_\lambda \in \mathbb{N} \) satisfying the following. Let \((G, o) = ((V, E), o)\) be a rooted tree with degrees bounded by \( d + 1 \), and \( \deg_G(o) = 1 \). Then, there exists \( \mathcal{H} = \mathcal{H}((G, o), \lambda) \in \mathbb{N} \) such that for any \( h \geq \mathcal{H} \), there exists a rooted tree \((\tilde{G}, \tilde{o}) = ((\tilde{V}, E), \tilde{o})\) having \( G \) as a subgraph, degrees satisfying

\[
\deg_{\tilde{G}}(v) = \deg_G(v) \quad \text{for all} \quad v \in \tilde{V}\setminus\{\tilde{o}\},
\]
\[
\deg_{\tilde{G}}(o) = 2, \quad \deg_{\tilde{G}}(\tilde{o}) = 1,
\]
\[
\deg_{\tilde{G}}(v) \leq d + 1 \quad \text{for all} \quad v \in \tilde{V}\setminus\tilde{V},
\]

and such that the contact process on \( \tilde{G} \) satisfies the following properties. For all \( \lambda' \geq \lambda, A \subset V \) and \( t > 0 \),

\[
P\left(\bar{\xi}_{G,\lambda'}(\tilde{o}) > h \mid \bar{\xi}_{G,\lambda}(o) > t\right) > 1 - \exp\{-c_\lambda \cdot t\} - \frac{1}{h},
\]

and, for all \( v \in V \),

\[
P\left(\bar{\xi}_{G,\lambda'}(v) > h \mid \bar{\xi}_{G,\lambda}(o) > t\right) > 1 - \exp\{-c_\lambda \cdot t\} - \frac{1}{h}.
\]

Moreover, for all \( \lambda' < \lambda \) and \( A \subset V \),

\[
P\left(\bar{\xi}_{G,\lambda'}(\tilde{o}) > 0\right) < \exp\{-d\sqrt{\pi}\}.
\]

**Proof of Theorem 1(a).** Given a rooted tree \((G, o)\) and \( \lambda > 0 \), for each \( h \geq \mathcal{H}((G, o), \lambda) \), we denote by \( G_h((G, o), \lambda) \) the rooted graph \((\tilde{G}, \tilde{o})\) corresponding to \((G, o), \lambda, h\) as in Theorem 2.

Fix \( \lambda \in (0, \lambda_c(Z)) \). Also fix an increasing sequence \((\lambda'_n)\) with \( \lambda'_n \nearrow \lambda \).

We will define an increasing sequence of graphs \((G_n)\) by applying Theorem 2 repeatedly. We let \( G_0 \) be a graph consisting of a single vertex (its root), \( o_0 \).

Once \((G_n, o_n)\) is defined, fix

\[
h_{n+1} \geq \max\left(\mathcal{H}((G_n, o_n), \lambda), \ c_\lambda^{-1}(n + 3) \log 2, \ 2^{n+3}\right)
\]

and let \((G_{n+1}, o_{n+1}) := G_{h_{n+1}}((G_n, o_n), \lambda)\). Increasing \( h \) if necessary, by (4) we can also assume that

\[
P\left(\bar{\xi}_{G_{n+1},\lambda_{n+1}}(o_{n+1}) = 0\right) > \frac{1}{2} \quad \text{for any} \quad A \subset G_n.
\]

We then let \( G_\infty \) be the union of all these graphs, and claim that \( G_\infty \) has the desired properties.

Since each \( G_n \) is a tree, \( G_\infty \) is also a tree. The fact that \( G_\infty \) has bounded degree is an immediate consequence of the degree conditions given in the end of the statement of Theorem 2.
Let us verify that the contact process with parameter $\lambda$ on $G$ survives locally. Start noting that
\[
\P \left( \tilde{\xi}_{G_0,\lambda}^{(o_0)}(o_0) > c^{-1}_\lambda \log 4 \right) = 4^{-c^{-1}_\lambda}.
\]
Next, using (3) and (5),
\[
\P \left( \tilde{\xi}_{G_1,\lambda}^{(o_1)}(o_1) > h_1 \mid \tilde{\xi}_{G_0,\lambda}^{(o_0)}(o_0) > c^{-1}_\lambda \log 4 \right) > 1 - \frac{1}{4} - \frac{1}{h_1} \geq \frac{1}{2}
\]
and, for $n \geq 1$,
\[
\P \left( \tilde{\xi}_{G_{n+1},\lambda}^{(o_n+1)}(o_{n+1}) > h_{n+1} \mid \tilde{\xi}_{G_n,\lambda}^{(o_n)}(o_n) > h_n \right) > 1 - \exp\left\{-c_\lambda h_n \right\} - \frac{1}{h_{n+1}} > 1 - \frac{1}{2^{n+1}}
\]
and similarly,
\[
\P \left( \tilde{\xi}_{G_{n+1},\lambda}^{(o_n+1)}(o_{n+1}) > h_{n+1} \mid \tilde{\xi}_{G_n,\lambda}^{(o_n)}(o_n) > h_n \right) > 1 - \frac{1}{2^{n+1}}.
\]
From this, it follows that
\[
\P \left( \tilde{\xi}_{G,\lambda}^{(o)}(o) = \infty \right) > 0,
\]
so we have local survival.

Now fix $\lambda' < \lambda$; let us prove that the contact process on $G$ with parameter $\lambda'$ dies out. Fix $n$ such that $\lambda'_n > \lambda'$. We then have, for any $N > n$, by the Markov property and (6)
\[
\P \left( \tilde{\xi}_{G_{N+1},\lambda'}^{(o_N)}(o_N) = 0 \right) \geq \P \left( \tilde{\xi}_{G_{N+1},\lambda'}^{(o_N)}(o_N) = 0 \right)^{(4)} \geq 1 - \sum_{i=n+1}^{N} 2^{-i} > 1 - 2^{-n}.
\]
Thus, by Borel-Cantelli, we conclude that, with probability 1, there exists $n \in \mathbb{N}$ for which $\tilde{\xi}_{G_N,\lambda'}^{(o_n)}(o_n) = 0$. By our construction of the graph $G$, it follows that the infection only spreads within a finite site and that the process hence dies out.

Proof of Theorem 1(b). We fix $\lambda \in (0, \lambda_c(\mathbb{Z}))$ and again we will define an increasing sequence of graphs $(G_n)$ by applying Theorem 2 repeatedly. Only now we take a decreasing sequence $(\lambda'_n)$ with $\lambda'_n \searrow \lambda$. Like before we let $G_0$ be a graph consisting of a single vertex (its root), $o_0$ and, once $(G_n, o_n)$ is defined, fix
\[
\frac{1}{h_{n+1}} \geq \max \left( h((G_n, o_n), \lambda'_n), c_{n+2}(n + 3) \log 2, 2^{n+3} \right)
\]
and let $(G_{n+1}, o_{n+1}) := G_{h_{n+1}}((G_n, o_n), \lambda'_n+1)$. Since $\lambda < \lambda_{n+1}$, increasing $h$ if necessary, by (4) we can assume that
\[
\P \left( \tilde{\xi}_{G_{n+1},\lambda}^{(o_{n+1})} = 0 \right) > \frac{1}{2} \quad \text{for any } A \subset G_n.
\]
We then let $G_\infty$ be the union of all these graphs and note that, as in (a), $G_\infty$ is a bounded degree tree.

The fact that the contact process with parameter $\lambda$ on $G_\infty$ dies out follows similarly to the last argument in the proof of Theorem 1(a) by noting that

$$
P \left( \bar{\xi}_{G_\infty, \lambda}^{(o_0)}(a_n) \neq 0 \right) \leq 2^{-n} \quad \forall n,$$

which can be obtained by repeatedly applying the Markov property and (8).

Now, fix $\lambda' > \lambda$, and take $n$ such that $\lambda' n < \lambda n$. We then note that the event

$$\{ \bar{\xi}_{G_n, \lambda'}^{(o_0)}(a_n) > h_n \}$$

has positive probability, and that, for each $N > n$, by (3) and (7),

$$P \left( \bar{\xi}_{G_N, \lambda'}^{(o_0)}(a_N) > h_N \mid \bar{\xi}_{G_{N-1}, \lambda'}^{(o_0)}(a_{N-1}) > h_{N-1} \right) > 1 - \frac{1}{2^N},$$

and

$$P \left( \bar{\xi}_{G_n, \lambda'}^{(o_0)}(a_0) > h_N \mid \bar{\xi}_{G_{n-1}, \lambda'}^{(o_0)}(a_{n-1}) > h_{n-1} \right) > 1 - \frac{1}{2^N}.$$

From this, local survival at parameter $\lambda'$ follows as in part (a).

\[ \square \]

3 Estimates for line segments and trees

This section is devoted to listing bounds for the behavior of the contact process on finite trees and line segments which will be useful for our graph construction.

Let us first mention two results that hold on general graphs. First, if $G = (V, E)$ is a connected graph and $x, y \in V$ and we let $\text{dist}_G(x, y)$ denote the graph distance between $x$ and $y$ in $G$, we have

$$P \left( \xi_{G, \lambda}^{(x)}(y) = 1 \text{ for some } t \leq \text{dist}_G(x, y) \right) \geq (e^{-2}(1 - e^{-\lambda}))^{\text{dist}_G(x, y)}. \quad (9)$$

This is obtained by fixing a geodesic $v_0 = x, \, v_1, \ldots, v_n = y$ (with $n = \text{dist}_G(x, y)$) and prescribing that, in each time interval $[i, i+1]$ with $0 \leq i \leq n-1$, there is no recovery mark at $v_i$ or $v_{i+1}$, and there is a transmission arrow from $v_i$ to $v_{i+1}$.

Second, we have the following inequality for the extinction time of the contact process on $G$ started from full occupancy.

**Lemma 1.** For every $s > 0$, we have

$$P \left( \xi_{G, \lambda}^{G}(s) = \emptyset \right) \leq \mathbb{E} \left[ \inf \left\{ t : \xi_{G, \lambda}^{G}(t) = \emptyset \right\} \right]. \quad (10)$$

This follows from noting that for any $s$, the extinction of the process started from full occupancy is stochastically dominated by the random variable $\mathbb{P}(\xi_{G, \lambda}^{G}(s) = \emptyset)$. See Lemma 4.5 in [MMVY12] for a full proof.
3.1 Contact process on line segments

We will need some estimates involving the contact process on half-lines and line segments. From now on, we fix \( \lambda < \lambda_c (\mathbb{Z}) \). The results below are essentially all consequences of the exponential bound

\[
P \left( \xi^{0}_{\mathbb{Z}, \lambda, t} \neq \emptyset \right) \leq \exp \left( -c_\lambda \cdot t \right), \quad t \geq 0
\]

for some \( c_\lambda > 0 \); see Theorem 2.48 in Part I of [Lig99]. By simple stochastic comparison considerations and large deviation estimates for Poisson random variables, this also implies that

\[
P \left( \xi^{0}_{\mathbb{Z}, \lambda, t} = \emptyset, \bigcup_{s \leq t} \xi^{0}_{\mathbb{Z}, \lambda, s} \subset [-t, t] \right) > 1 - \exp \left( -c'_\lambda \cdot t \right), \quad t \geq 0
\]

for some \( c'_\lambda > 0 \).

For each \( \ell \in \mathbb{N} \), let \( L_\ell \) denote the subgraph of \( \mathbb{Z} \) induced by the vertex set \( \{0, \ldots, \ell\} \). The following result is an immediate consequence of (11), so we omit its proof.

**Lemma 2.** We have

\[
\lim_{\ell \to \infty} P \left( \xi^{0}_{\mathbb{Z}, \lambda, t} \neq \emptyset \right) = 0.
\]

Next, we bound the probability of existence of an infection path starting from a space-time point in the segment \( \{0\} \times [0, t] \) and crossing \( L_\ell \).

**Lemma 3.** There exists \( c_L > 0 \) such that, for \( \ell \in \mathbb{N} \) large enough, the contact process with parameter \( \lambda \) on \( L_\ell \) satisfies

\[
P \left( \{0\} \times [0, t] \ni \{\ell\} \times [0, \infty) \right) \leq (t + 1) \cdot \exp \left( -c_L \cdot \ell \right), \quad t > 0.
\]

**Proof.** Define the event

\[
A := \{0\} \times [1, t + 1] \ni \{\ell\} \times [0, \infty) \},
\]

so that the probability in the left-hand side in (14) is equal to \( P(A) \). Let \( X \) denote the Lebesgue measure of the random set of times

\[
\{s \in [0, t + 1] : (0, s) \ni \{\ell\} \times [0, \infty) \}.\]

Denote by \( \mathcal{F} \) the \( \sigma \)-algebra generated by all the Poisson processes in the graphical construction of the contact process on \( L_\ell \), and let \( \mathcal{F}' \) be similarly defined, except that it disregards all the recoveries marks at 0 that occur before time \( t + 1 \). Note that \( X \) is measurable with respect to \( \mathcal{F} \) and \( A \in \mathcal{F}' \). Moreover, we have

\[
E[X | \mathcal{F}'] \geq e^{-1} \text{ on } A,
\]
since if $A$ occurs and $s \in [1, t + 1]$ is such that $(0, s) \sim \ell \times [0, \infty)$, then with probability $e^{-1}$ there is no recovery mark on $[s - 1, s]$, so that $X \geq 1$. We thus obtain

$$\mathbb{P}(A) \leq e \cdot \mathbb{E}[X | \mathcal{F}'] \leq e \cdot \int_0^{t+1} \mathbb{P}((0, s) \sim \ell \times [0, \infty)) \, ds$$

(12)

$$\leq e(t + 1) \cdot \exp\{-c'_\lambda \cdot \ell\} < (t + 1) \cdot \exp\{-c_L \cdot \ell\}.$$  

if $c_L < c'_\lambda$ and $\ell$ is large. \hfill \Box

We now show that the subcritical contact process on $\mathbb{Z}$ started from occupation in a half-line $\{1, 2, \ldots \}$ has positive probability of never infecting the origin.

**Lemma 4.** There exists $c_L > 0$ such that

$$\mathbb{P}(\xi^{\{1,2,\ldots\}}_{\mathbb{Z},\lambda,t}(0) = 0 \text{ for all } t) > c_L.$$  

(15)

**Proof.** For $n \in \mathbb{N}$, let $A(n)$ denote the event that vertices $1, \ldots, m$ have a recovery mark and generate no transmission arrow in the time interval $[0, 1]$. We have

$$\mathbb{P}(\xi^{\{1,2,\ldots\}}_{\mathbb{Z},\lambda,t}(0) = 0 \text{ for all } t) \geq \mathbb{P}(A(n)) \cdot \mathbb{P}(\xi^{\{n,n+1,\ldots\}}_{\mathbb{Z},\lambda,t}(0) = 0 \text{ for all } t).$$

For any $n \in \mathbb{N}$, we have $\mathbb{P}(A(n)) > 0$ and

$$\mathbb{P}(\xi^{\{n,n+1,\ldots\}}_{\mathbb{Z},\lambda,t}(0) = 0 \text{ for all } t) \geq \mathbb{P}\left(\bigcap_{i=n}^{\infty} \left\{ \bigcup_{t \geq 0} \xi^{(i)}_{\mathbb{Z},\lambda,t} \subset [i/2, 3i/2] \right\} \right) \geq 1 - \sum_{i=n}^{\infty} \exp\{-c'_\lambda \cdot i\},$$

which can be made positive by taking $n$ large enough. \hfill \Box

Finally, we compare the contact process on the same graph for two different values of the infection parameter.

**Lemma 5.** For all $\lambda', \lambda > 0$ with $\lambda' < \lambda$ there exists $\eta = \eta_{\lambda',\lambda} > 1$ such that, for $l$ large enough,

$$\mathbb{P}\left(\xi^{\{0\}}_{0,1,\ldots},\alpha'(l) > 0\right) \leq \eta^{-l} \mathbb{P}\left(\xi^{\{0\}}_{0,1,\ldots},\lambda(l) > 0\right)$$  

(16)

**Proof.** Using monotonicity and the Markov property it can be proved that the limit

$$\beta(\lambda) = \lim \mathbb{P}\left(\xi^{\{0\}}_{\mathbb{Z},\lambda}(l) > 0\right)^{1/l}$$
exists (see [Lig99] for a full proof of this fact). Furthermore, it was shown in [Lal02b] that, for the contact process on a regular tree, if
\[ \lambda' < \lambda \text{ and } \beta(\lambda) < 1 / \sqrt{d} \]
then \( \beta(\lambda') < \beta(\lambda) \). Noting that the exponential bound (11) implies that \( \beta(\lambda_c(Z)) < 1 \), we have the result for the contact process on \( Z \). Finally, [Lal02a] proves that
\[ \lim P(\bar{\xi}_{Z,\lambda}^{[0]}(l) > 0) = \lim P(\bar{\xi}_{Z,\lambda}^{[0]}(l) > 0) = 1 / l. \]

\section{3.2 Contact process on finite trees}

To conclude this section, we gather a few estimates from [CMMV14] concerning the contact process on finite trees. We continue with fixed \( \lambda < \lambda_c(Z) \), and assume \( d \) is large enough that \( \lambda > \lambda_{loc,c}(T_d) \). For each \( h \in \mathbb{N} \), we let \( T_d^h \) be a rooted tree with branching number \( d \), truncated at height \( h \). This means that \( T_d^h \) is a tree with a root vertex \( \rho \) with degree \( d \), and so that vertices at graph distance between one and \( h - 1 \) from \( \rho \) have degree \( d + 1 \), and vertices at graph distance \( h \) from \( \rho \) have degree one.

**Proposition 1.** There exists \( c_T = c_T(\lambda, d) > 0 \) such that, for \( h \) large enough,
\[
P(\xi_{T_d^h,\lambda}^{[0]} \in \emptyset) = 0 \text{ and } \beta(\lambda) < 1 / \sqrt{d} \text{ implies } \lim_{h \to \infty} P(\xi_{T_d^h,\lambda}^{[0]}(l) > 0) = 0. \]

**Proof.** Theorem 1.5 in [CMMV14] states that the limit
\[
\lim_{h \to \infty} \log E \left[ \inf \left\{ t : \xi_{T_d^h,\lambda}^{[0]}(l) = \emptyset \right\} \right] = c_T.
\]
exists and is positive; denote it by \( c_1 \). Taking \( c_T < c_1 / 4 \), the inequality (17) follows from this combined with (10). Next, Corollary 4.10 in [CMMV14] implies that there exists a constant \( c_2 > 0 \) such that
\[
\inf_{A \subseteq T_d^h} P(\xi_{T_d^h,\lambda}^{[0]}(h) \neq \emptyset) > c_2,
\]
and Proposition 4.15 in [CMMV14] gives
\[
\sup_{A \subseteq T_d^h} P(\xi_{T_d^h,\lambda}^{[0]}(h) \neq \emptyset) \to 0 \text{ as } h \to \infty.
\]
Using these two facts and also (17), we obtain that, if $c_T < \min(c_1/4, c_2/2)$, then for any $A \subset \mathbb{T}_h^d$, $A \neq \emptyset$,

$$\mathbb{P}\left(\xi_A^{\mathbb{T}_h^d, \lambda, \exp\{c_T \cdot d^h\}} \geq \mathbb{P}\left(\emptyset \neq \xi_A^{\mathbb{T}_h^d, \lambda, \exp\{c_T \cdot d^h\}} \neq \emptyset \right) > c_T.\right)$$

$\square$

4 Proof of Theorem 2

Fix $\lambda \in (0, \lambda_c(\mathbb{Z}))$. The value $d = d_\lambda$ that appears in the statement of Theorem 2 will now be chosen: $d$ should be large enough that $\lambda > \lambda_{loc}^v(T^d)$, and also

$$m = m_\lambda := d \cdot (1 - e^{-\lambda}) \cdot e^{-2} > 1. \quad (20)$$

From now on, we fix $(G, o) = ((V, E), o)$ a rooted tree with degrees bounded by $d + 1$ and with $\text{deg}_G(o) = 1$, as in the statement of Theorem 2.

In this section, we will give some key definitions and state three results (Propositions 2, 3 and 4) that will immediately imply Theorem 2. The idea of our graph augmentation $(\hat{G}, o)$ of a given rooted graph $(G, o)$ is summarized by Figure 1 below: next to the root $o$ of $G$, we append a copy of $T^{d_h}$ (with $h$ large), followed by a line segment whose length is a function of $h$, denoted $L(h)$. The endpoint of this line segment that is away from the tree is the root $\hat{o}$ of $\hat{G}$.

We will be free to take $h$ large (adjusting the length $L(h)$ accordingly) so as to guarantee several desirable properties for $\hat{G}$.

Throughout this section, it will be useful to abbreviate

$$s(h) := \exp\left\{d^{\sqrt{h}}\right\}, \quad h \in \mathbb{N}. \quad (21)$$

We first define an auxiliary graph $\hat{G}$, depending on $(G, o)$ and on a positive integer $h$ (which we often omit from the notation), as follows. We let $\mathbb{T}_h^d$ be a copy of $T^d$, with root $\rho$, and let $L_\infty$ be a half-line with extremity denoted $v_\cdot$. We then let $\hat{G}$ denote the graph obtained by putting the three graphs $G, \mathbb{T}_h^d, L_\infty$ together, and connecting them by including an edge between $o$ (the root of $G$) and $\rho$ (the root of $\mathbb{T}_h^d$), and an edge between $\rho$ and $v_\cdot$ (the extremity of $L_\infty$).

For each $\ell \in \mathbb{N}_0$, let $v_\ell$ denote the vertex of $L_\infty$ at distance $\ell$ from $v_\cdot$ (in particular, $v_0 = v_\cdot$), and define

$$\mathcal{P}(\ell) = \mathcal{P}(G, o, h, \ell) := \mathbb{P}\left(\xi_{\hat{G}, \lambda}(v_\ell) > 0\right),$$

that is, $\mathcal{P}(\ell)$ is the probability that $v_\ell$ becomes infected in the contact process on $\hat{G}$ with parameter $\lambda$ and initial configuration $V \cup \mathbb{T}_h^d$. Note that $\mathcal{P}$ is non-increasing.

Lemma 6 (Properties of $\mathcal{P}$). We have

$$\lim_{\ell \to \infty} \mathcal{P}(\ell) = 0$$
and, if $h$ is large enough,

$$P(0), P(1) \geq 1 - \sigma(h)^{-1}. \tag{22}$$

**Proof.** The first statement follows from the fact that the contact process with parameter $\lambda$ on $\hat{G}$ dies out (which is in turn an easy consequence of the facts that $G, \mathcal{T}_h$ are finite graphs and $\lambda < \lambda_c(Z)$).

For the second statement, we will only treat $P(0)$, since the proof for $P(1)$ is the same. Assume $h$ is larger than the graph diameter of $G$. Then, for any non-empty $A \subset V \cup \mathcal{T}_h$ we have

$$P(\xi_0^A_{\hat{G}, \lambda_t}(v_0) = 1 \text{ for some } t \leq h) \geq (e^{-1} \cdot (1 - e^{-\lambda}))^h.$$

Iterating this, we obtain

$$P(\xi_0^A_{\hat{G}, \lambda_t}(v_0) = 0 \forall t \leq \sigma(h)) \leq \left(1 - (e^{-1} \cdot (1 - e^{-\lambda}))^h\right)^{[\sigma(h)/h]}.$$

The result now follows from noting that the right-hand side above is much smaller than $\sigma(h)^{-1}$, and moreover

$$P(\xi_{V \cup \mathcal{T}_h} = \emptyset) \leq P(\xi_{\mathcal{T}_h} = \emptyset) \leq \frac{\sigma(h)}{\exp\{c_T \cdot d^h\}} \ll \sigma(h)^{-1}$$

if $h$ is large.

With the above result at hand, for $h$ large enough we can define

$$L(h) := \inf \{ \ell \in \mathbb{N}_0 : P(\ell) < 1 - \sigma(h)^{-1} \}$$

and have $L(h) > 1$. We now define the graph $\tilde{G}$ in the same way as $\hat{G}$, with the sole exception that, instead of the half-line $L_\infty$, it includes a line segment $L_h$ with vertex set

$$v_0 = v_-, v_1, \ldots, v_{L(h)}$$

(as before, we link $L_h$ to $\mathcal{T}_h$ with an edge between $\rho$ and $v_-$). We denote by $\hat{V}$ and $\hat{E}$ the vertex and edges sets of $\hat{G}$, respectively. The vertex $v_{L(h)}$ is the root of $\tilde{G}$, denoted $\hat{o}$. The definition of $(\tilde{G}, \hat{o})$ depends on $(G, o)$ and $h$, but this dependence will be omitted from the notation. We will several times assume that $h$ is large (possibly depending on $G$).

We will now state several results about $(\tilde{G}, \hat{o})$, culminating in the proof of Theorem 2. Define the set of configurations

$$\mathcal{A}_h := \left\{ A \subset \hat{V} : \#\{v \in A \cap \mathcal{T}_h : \text{dist}_{\hat{G}}(\rho, v) = [h/2]\} \geq (m/2)^{[h/2]} \right\},$$

that is, $A \in \mathcal{A}_h$ if $A$ has at least $m^{[h/2]}$ vertices at height $[h/2]$ in $\mathcal{T}_h$. The following result is the main reason for the introduction of $\mathcal{A}_h$. 

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Lemma 7 (Persistence starting from $A_h$). If $A \in A_h$, then

$$\mathbb{P} \left( \varepsilon^A_{G,\lambda_0(h)} \neq \emptyset \right) > 1 - \delta(h)^{-2}. $$

Proof. Fix $A \in A_h$ and let $T_1, \ldots, T_{(m/2)^{(h/2)}}$ be disjoint copies of $T^{d}_{[h/2]}$ that appear as subtrees of $T_h$, rooted at a vertices $v_1, \ldots, v_{(m/2)^{(h/2)}} \in A \cap T_h$ at distance $[h/2]$ from $\rho$. We have

$$\mathbb{P} \left( \varepsilon^A_{G,\lambda_0(h)} \neq \emptyset \right) \geq \mathbb{P} \left( \bigcup_{i=1}^{(m/2)^{(h/2)}} \varepsilon^{\{v_i\}}_{T_i,\lambda_0}\exp\{c_T \cdot d^{(h/2)}\} \neq \emptyset \right) \geq 1 - (1 - c_T)^{(m/2)^{(h/2)}} \gg 1 - \delta(h)^{-1}. $$

Proposition 2 (Ignition). There exists $c_\lambda > 0$ such that for $h$ large enough, any $\lambda' \geq \lambda$ and any $A \subset V$ we have

$$\mathbb{P} \left( \varepsilon^A_{G,\lambda';s} \in A_h \text{ for some } s \geq 0 \mid \varepsilon^A_{G,\lambda'}(o) > t \right) > 1 - \exp\{-c_\lambda \cdot t\},$$

that is, given that the contact process with rate $\lambda'$, started from $A$ and confined to $G$ spends more than $t$ time units with $o$ occupied, the probability that the same process on the full graph $\tilde{G}$ reaches $A_h$ is higher than $1 - \exp\{-c_\lambda \cdot t\}$.

We interpret the conditioning in the above statement as saying that the confined process has time $t$ to attempt to “ignite” the infection on the tree $T_h$ (meaning fill it up sufficiently to enter the set $A_h$). We postpone the proof of this proposition to Section 5.1.
Proposition 3 (From $\mathcal{A}_h$ to $\partial$). If $h$ is large enough, then for any $A \in \mathcal{A}_h$ we have

$$
\mathbb{P}\left(\tilde{\xi}^A_{G,\lambda}(\partial) > h\right) > 1 - \frac{1}{h} \quad \text{and} \quad \mathbb{P}\left(\tilde{\xi}^A_{G,\lambda}(v) > h\right) > 1 - \frac{1}{h}
$$

for all $v \in V$.

The proof of this proposition will be carried out in Section 5.3.

Proposition 4. If $h$ is large enough, then for any $\lambda' < \lambda$ and for any $A \subset V \cup \mathcal{F}_h$ we have

$$
\mathbb{P}\left(\tilde{\xi}^A_{G,\lambda'}(\partial) > 0\right) < s(h)^{-1}.
$$

(23)

The proof of this proposition will be done in Section 5.4.

Proof of Theorem 2. It follows from the construction that $\tilde{G}$ satisfies the stated degree properties. The inequality (3) follows from Propositions 2 and 3, and (4) follows from Proposition 4. □

5 Proofs of results in Section 4

We now turn to the proofs of the three propositions of the previous section. In Section 5.1, we will prove Proposition 2. In Section 5.2, we will give some bounds involving the function $L(h)$, as well as a key proposition involving coupling of the contact process on $\tilde{G}$, Proposition 5. Next, Section 5.3 contains the proof of Proposition 3, and Section 5.4 contains the proof of Proposition 4.

5.1 Proof of Proposition 2

Proof of Proposition 2. We begin with some definitions. For $0 \leq i \leq h$, let $T(i)$ denote the set of vertices of $\mathcal{F}_h$ at distance $i$ from the root $\rho$. Using the graphical construction of the contact process with parameter $\lambda' \geq \lambda$, we will now define random sets $Z_{\lambda'}(0), \ldots, Z_{\lambda'}([h/2])$ with $Z_{\lambda'}(i) \subset T(i)$ for each $i$. We set $Z_{\lambda'}(0) := \{\rho\}$. Assume that $Z_{\lambda'}(i)$ has been defined, let $z$ be a vertex of $T(i+1)$ and let $z'$ be the neighbour of $z$ in $T(i)$. We include $z$ in $Z_{\lambda'}(i+1)$ if $z' \in Z_{\lambda'}(i)$ and, in the time interval $[i, i+1]$, there are no recovery marks on $z'$ or $z$, and there is a transmission arrow from $z'$ to $z$. Letting $Z_{\lambda'}(i) := |Z(i)|$ for each $i$, it is readily seen that $(Z_{\lambda'}(i) : 0 \leq i \leq [h/2])$ is a branching process. Its offspring distribution is equal to the law of $U \cdot W$, where $U \sim \text{Bernoulli}(e^{-1})$ and $W \sim \text{Binomial}(d, e^{-1}, (1 - e^{-\lambda'}))$ are independent. The expectation of this distribution is larger than $m_{\lambda'} > 1$. For this reason, there exists $\sigma_{\lambda'} > 0$ such that the event

$$
B_{\lambda'} := \left\{Z_{\lambda'}([h/2]) > (m_{\lambda'}/2)^{[h/2]}\right\}
$$

has

$$
\mathbb{P}(B_{\lambda'}) > \sigma_{\lambda'} \quad \text{for all } \lambda' \geq \lambda \text{ and } h \in \mathbb{N}.
$$

Finally note that

$$
B_{\lambda'} \subset \left\{\tilde{\xi}^{|\rho|}_{G,\lambda';[h/2]} \in \mathcal{A}_h\right\}.
$$
Now define \( B_{\lambda'}(0) := B_{\lambda'} \) and, for \( t \in [0, \infty) \), define \( B_{\lambda'}(t) \) as the time translation of \( B_{\lambda'} \), so that time \( t \) becomes the time origin (that is, \( B_{\lambda'}(t) \) is defined by using the graphical construction of the contact process on the time intervals \([t, t+1], [t+1, t+2], \ldots, \ [t+[h/2]-1, t+[h/2]]\)). We evidently have
\[
P(B_{\lambda'}(t)) = P(B_{\lambda'}) \sigma_{\lambda} \quad \text{for any } t, \tag{24}
\]
and moreover,
\[
\{ \rho \in \xi_{G,\lambda';t}^{A} \} \cap B_{\lambda'}(t) \subset \{ \xi_{G,\lambda';t+[h/2]}^{A} \in A_{h} \}
\]
for any \( A \). It will be useful to note that, if \( t_1, t_2 \geq 0 \) with \( t_2 > t_1 + 2 \), then \( B_{\lambda'}(t_1) \) and \( B_{\lambda'}(t_2) \) are independent.

Now, fix \( t > 0 \) and condition on the event \( \{ \xi_{G,\lambda';t}^{A}(o) > t \} \) occurs. Note that this event only involves the graphical construction of the contact process on \( G \); in particular, the Poisson processes involving vertices and edges of \( T_{h} \), or the edge \( \{o, \rho\} \), are still unrevealed. Then, by elementary properties of Poisson processes, there exists \( c_{\lambda} > 0 \) (depending only on \( \lambda \)) such that (uniformly on \( \lambda' \geq \lambda \)) outside probability \( \exp\{-c_{\lambda} \cdot t\}, \) we can find random times \( s_{1} < \cdots < s_{\lfloor c_{\lambda} t \rfloor} \) separated from each other by more than two units, and such that for each \( i, o \in \xi_{G,\lambda';s_{i}}^{A} \), and there is a transmission arrow from \( (o, s_{i}) \) to \( (\rho, s_{i}) \). If this is the case, and if \( B_{\lambda'}(s_{i}) \) also occurs for some \( i \), we then get \( \xi_{G,\lambda';s_{i}+h/2}^{A} \in A_{h}, \) by (25). The desired result now follows from independence between the events \( B_{\lambda'}(s_{i}) \), together with (24) and a Chernoff bound. \( \square \)

5.2 Preliminary bounds

We first give an upper bound involving the extinction time of the contact process on \( G \), in terms of the length \( L(h) \).

Lemma 8. We have
\[
\lim_{h \to \infty} P\left( \xi_{G,\lambda,\exp\{d^{2}h\} \cdot (\log L(h))^{2}}^{F} \neq \emptyset \right) = 0,
\]
that is, the extinction time of the contact process on \( G \) started from full occupancy is smaller than \( \exp\{d^{2}h\} \cdot (\log L(h))^{2} \) with high probability as \( h \to \infty \).

Proof. Let \( E_{0}' \) be the event that each vertex in \( V \cup \bar{T}_{h} \) has a recovery mark before it sends out any transmission arrow, and before time 1. Since all vertices of \( V \cup \bar{T}_{h} \) have degree at most \( d+1 \), we have
\[
P(E_{0}') \geq \left( (1 - e^{-1}) \cdot e^{-(d+1)\lambda} \right)^{|V \cup \bar{T}_{h}|} \geq \left( (1 - e^{-1}) \cdot e^{-(d+1)\lambda} \right)^{d^{2}+2},
\]
if \( h \) is large enough (since \( |\bar{T}_{h}| < d^{h+1} \) and \( |V| \) is fixed as \( h \to \infty \)). Next, let \( E_{0}'' \) denote the event that the contact process on \( G \) started from \( T_{h} \) infected dies out before time \( (\log L(h))^{2} \), and never infects the root \( \rho \) of \( T_{h} \). That is,
\[
E_{0}'' := \{ \xi_{G,\lambda,(\log L(h))^{2}}^{F} = \emptyset, \rho \notin \xi_{G,\lambda,2}^{F}, \text{ for all } t \}.
\]
The probability of $E''_0$ is the same as the probability that a contact process on the line segment $\{-1, 0, \ldots, L_G(h)\}$, with rate $\bar{\lambda}$ and initial configuration $\{0, \ldots, L(h)\}$, dies out before time $(\log L(h))^2$ and never infects vertex $-1$. Therefore, by Lemma 2 and Lemma 4, we have $P(E''_0) > \delta > 0$ for all $h$. Let $E_0 := E'_0 \cap E''_0$; since $E'_0$ and $E''_0$ are independent, we have $P(E_0) > \delta \left((1 - e^{-1}) \cdot e^{-(d+1)\lambda}\right)^{d^h+2}$.

For $i \in \{1, \ldots, \lfloor \exp\{d^3h\}\rfloor\}$, let $E_i$ be the time translation of event $E_0$ to the graphical construction on the time interval $[i(\log L(h))^2, (i+1)(\log L(h))^2]$. Finally, noting that $E_0, E_1, \ldots$ are independent and $E_i \subseteq \{\xi_{\tilde{G}, \lambda}; \exp\{d^3h\} \cdot (\log L(h))^2 = \emptyset\}$, we have

$$
P\left(\xi_{\tilde{G}, \lambda}; \exp\{d^3h\} \cdot (\log L(h))^2 \neq \emptyset\right) \leq P((E_0)^c \exp\{d^3h\}) \leq \exp\left(-[\exp\{d^3h\}] \cdot \delta \left((1 - e^{-1}) \cdot e^{-(d+1)\lambda}\right)^{d^h+2}\right) \xrightarrow{h \to \infty} 0.
$$

We now proceed to an upper bound on $L(h)$.

**Lemma 9.** If $h$ is large enough we have

$$L(h) \leq d^{2h} \tag{26}$$

**Proof.** Define

$$F_1 := \left\{\xi_{\tilde{G}, \lambda}; \exp\{d^3h\} \cdot (\log L(h))^2 = \emptyset\right\}.$$

Recall that $v_{L(h)-1}$ denotes the neighbor of $\tilde{o}$ in $L_h$, and let $F_2$ be the event that there is no infection path starting from $(\rho, s)$ for some $s \leq \exp\{d^3h\} \cdot (\log L(h))^2$, ending at $(v_{L(h)-1}, t)$ for some $t > s$, and entirely contained in $L_h \cup \{(\rho)\}$. It is easy to see that $F_1 \cap F_2 \subset \left\{\xi_{\tilde{G}, \lambda}; (v_{L(h)-1}) = 0 \text{ for all } t \geq 0\right\}$.

By Lemma 8 we have $\lim_{h \to \infty} P(F_1) = 1$ and by Lemma 3 we have

$$P(F_2) \geq 1 - \left(\exp\{d^3h\} \cdot (\log L(h))^2 + 1\right) \cdot \exp\{-c_L \cdot L(h)\}.\]
This shows that, if we had $L(h) > d_2^h$, we would get

$$\mathbb{P}\left(\xi^{V,\mathcal{F}_h}_{G,\lambda,t}(v_{L(h)}^{-1}) = 0 \text{ for all } t \geq 0\right) \geq \mathbb{P}(F_1 \cap F_2) \xrightarrow{h \to \infty} 1.$$  

On the other hand, the definition of $L(h)$ implies that

$$\mathbb{P}\left(\xi^{V,\mathcal{F}_h}_{G,\lambda,t}(v_{L(h)}^{-1}) > 0 \text{ for some } t \geq 0\right) \geq 1 - s(h)^{-1} \xrightarrow{h \to \infty} 1,$$

a contradiction. \qed

The following guarantees that if the contact process with some initial condition remains active for $s(h)$ time in $\tilde{G}$, then it is highly likely to coincide with the process started from full occupancy.

**Proposition 5.** If $h$ is large enough, for any $A \subset \tilde{V}$ we have

$$\mathbb{P}\left(\xi_{\tilde{G},\lambda,t}^A \neq \emptyset, \xi_{\tilde{G},\lambda,t}^A \neq \xi_{\tilde{G},\lambda,t}^{\tilde{V}}\right) < s(h)^{-2}.$$

The proof of this proposition is lengthy and technical, so we postpone it to the Appendix.

We are now interested in giving an upper bound for the probability that the infection crosses $L(h)$ in a single attempt. For the proof of Proposition 4, it will be important that this bound be given in terms of the extinction time of the infection on $\tilde{G}$, starting from full occupancy.

Define

$$S(h) := \mathbb{E}\left[\inf\left\{t : \xi_{\tilde{G},\lambda,t}^{\tilde{V}} = \emptyset\right\}\right],$$

that is, $S(h)$ is the expected amount of time it takes for the contact process on $\tilde{G}$ with parameter $\lambda$ started from full occupancy to die out. Also let

$$p(\ell) = p_\lambda(\ell) := \mathbb{P}\left(\xi^{(0)}_{S_N,\lambda,\ell} > 0\right), \quad (27)$$

or equivalently, $p(\ell)$ is the probability that, for the contact process with parameter $\lambda$ on a line segment of length $\ell + 1$, an infection starting at one extremity ever reaches the other extremity.

**Lemma 10.** If $h$ is large enough,

$$p(L(h)) \leq \frac{s(h)^3}{S(h)}. \quad (28)$$

**Proof.** Recall that $v_0$ is the vertex of $L_h$ neighboring $\rho$, the root of $\mathcal{F}_h$. Let $q(h)$ denote the probability that there is an infection path starting from $(v_0, 0)$, ending at $(\hat{u}, t)$ for some $t \leq s(h)$, and entirely contained in $L_h$. Note that $q(h) \leq p(L(h))$ and, by a union bound,

$$p(L(h)) \leq q(h) + \mathbb{P}\left(\xi_{L_h,\lambda,t}^{(v_0)} \neq \emptyset\right) \leq q(h) + e^{-c_3 s(h)}.$$
Next, assume that $h$ is large enough that any vertex in $V$ is at distance smaller than $h$ from $\rho$, the root of $T_h$. With this choice, we claim that for any $A \subset V, A \neq \emptyset$ we have

$$\mathbb{P}(\xi^A_{G,\lambda,t}(\partial) = 1 \text{ for some } t \leq h + s(h)) > (e^{-1}(1 - e^{-\lambda}))^h \cdot q(h). \quad (29)$$

Indeed, if $A \cap \mathcal{L}_h \neq \emptyset$ then the left-hand side is larger than $q(h)$ by the definition of $q(h)$ and simple monotonicity considerations. If $A \cap \mathcal{L}_h = \emptyset$, then by (9), with probability larger than $\delta(h) := (e^{-1}(1 - e^{-\lambda}))^h$, $\rho$ gets infected within time $h$, and conditioned on this, with probability $q(h)$, $\partial$ gets infected after at most additional $s(h)$ units of time. Applying (29) repeatedly, we have

$$\mathbb{P}\left(\xi^A_{G,\lambda,t} \neq \emptyset, \xi^A_{G,\lambda,(h) + 2} = 0 \forall r \leq t\right) \leq (1 - \delta(h) \cdot q(h))^{\left\lfloor \frac{h + s(h)}{h + s(h)} \right\rfloor}, \; t \geq h + s(h). \quad (30)$$

Now, letting $S'(h) := \frac{S(h)}{2a(h)}$, we have

$$(a(h))^{-} < \mathbb{P}\left(\xi^V_{G,\lambda,S'(h)}(\partial) = 0\right) \leq \mathbb{P}\left(\xi^V_{G,\lambda,S'(h)} = \emptyset\right) + (1 - \delta(h) \cdot q(h))^{\left\lfloor \frac{S'(h)}{h + s(h)} \right\rfloor} \leq \mathbb{P}\left(\xi^V_{G,\lambda,S'(h)} = \emptyset\right) + \mathbb{P}\left(\xi^V_{G,\lambda,S'(h)} = \emptyset\right) \cdot \left\lfloor \frac{S'(h)}{h + s(h)} \right\rfloor. \quad (31)$$

We now claim that

$$\mathbb{P}\left(\xi^V_{G,\lambda,S'(h)} = \emptyset\right) < (2a(h))^{-1} \quad (32)$$

if $h$ is large enough. Plugging this into (31), we obtain

$$q(h) < \frac{\log(2a(h)) \cdot (h + s(h))}{\delta(h) \cdot S'(h)} < \frac{4 \log(2a(h)) \cdot (h + s(h)) \cdot a(h)}{(e^{-1}(1 - e^{-\lambda}))^h \cdot S(h)} < \frac{a(h)^3}{S(h)}$$

for large enough $h$, completing the proof.

It remains to prove (32). Noting that $S'(h) \gg a(h)$ if $h$ is large, we have

$$\mathbb{P}\left(\xi^V_{G,\lambda,S'(h)} = \emptyset\right) \leq \mathbb{P}\left(\xi^V_{G,\lambda,S'(h)} = \emptyset\right) + \mathbb{P}\left(\xi^V_{G,\lambda,S'(h)} \neq \emptyset\right).$$

By Lemma 1, we have

$$\mathbb{P}\left(\xi^V_{G,\lambda,S'(h)} = \emptyset\right) \leq \frac{S'(h)}{S(h)} = (4a(h))^{-1}.$$
Now, the first term on the right-hand side is smaller than \( s(h)^{-2} \) by Lemma 7 (since \( V \cup T_h \in A_h \)), and the second term on the right-hand side is also smaller than \( s(h)^{-2} \) by Proposition 5. Putting things together gives (32) for large enough \( h \).

We end this section with a lower bound on \( L(h) \), which again will be important for the proof of Proposition 4.

**Lemma 11.** If \( h \) is large enough,

\[
L(h) \geq d^\frac{3h}{2}.
\]  

**Proof.** By (9) and (28), we have

\[
(e^{-1}(1 - e^{-\lambda}))L(h) \leq p_\lambda(L(h)) \leq \frac{s(h)^3}{S(h)}.
\]

This gives

\[
L(h) \geq \frac{1}{\log(e(1 - e^{-\lambda})^{-1})} \cdot \log \left( \frac{S(h)}{s(h)^3} \right),
\]

Recalling that \( s(h) = \exp\{d^{\sqrt{h}}\} \) and noting that

\[
S(h) \geq \inf \{ t : \xi_{T_h,\lambda,t} = \emptyset \} \geq \exp\{c_\tau \cdot d^h\},
\]

we obtain

\[
L(h) \geq \frac{c_\tau \cdot d^h - 3d^{\pi}}{\log(e(1 - e^{-\lambda})^{-1})} > d^\frac{3h}{2}
\]

if \( h \) is large enough. \( \square \)

### 5.3 Proof of Proposition 3

We begin with a simple consequence of Proposition 5.

**Lemma 12.** If \( h \) is large enough, for any \( A \in A_h \) we have

\[
P\left( \xi_{G,\lambda,\delta(h)}^A = \xi_{G,\lambda,\delta(h)}^{V \cup T_h} \right) > 1 - 4s(h)^{-2}.
\]

**Proof.** Since both \( A \) and \( V \cup T_h \) belong to \( A_h \), Lemma 7 gives

\[
P\left( \xi_{G,\lambda,\delta(h)}^A = \emptyset \right) < \frac{1}{s(h)^2}, \quad P\left( \xi_{G,\lambda,\delta(h)}^{V \cup T_h} = \emptyset \right) < \frac{1}{s(h)^2},
\]

and Proposition 5 gives

\[
P\left( \xi_{G,\lambda,\delta(h)}^A \neq \emptyset, \xi_{G,\lambda,\delta(h)}^{V \cup T_h} \neq \emptyset \right) < s(h)^{-2},
\]

\[
P\left( \xi_{G,\lambda,\delta(h)}^A \neq \emptyset, \xi_{G,\lambda,\delta(h)}^{V \cup T_h} \neq \emptyset \right) < s(h)^{-2}.
\]

The desired statement follows from these four inequalities. \( \square \)
Lemma 13. If $h$ is large enough we have, for any $v \in V$,

$$
\mathbb{P}\left( \int_{s(h)}^{\infty} \xi_{G,\lambda,t}^V \, dt > h \right) > 1 - \frac{1}{2h}.
$$

Proof. Assume $h$ is larger than the graph diameter of $G$, and fix $v \in V$. We have, for any $A \subset \tilde{V}$ with $A \cap \tilde{T}_h \neq \emptyset$,

$$
\mathbb{P}\left( \int_{0}^{4h} \xi_{G,\lambda,t}^A \, dt > h \right) \geq e^{-2h} \cdot (e^{-1} \cdot (1 - e^{-\lambda}))^{2h}.
$$

Indeed, by (9) we have that, with probability at least $(e^{-1} \cdot (1 - e^{-\lambda}))^{2h}$, $v$ becomes infected before time $2h$, and then it remains infected for time $2h$ (by having no recovery marks) with probability $e^{-2h}$. By iterating this, we obtain

$$
\mathbb{P}\left( \xi_{\tilde{T}_h}^V = \emptyset, \int_{s(h)}^{2s(h)} \xi_{G,\lambda,t}^V \, dt \leq h \right)
$$

$$
< (1 - e^{-2h} \cdot (e^{-1} \cdot (1 - e^{-\lambda}))^{2h})^{s(h)/(4h)} \ll \frac{1}{4h}.
$$

We therefore have

$$
\mathbb{P}\left( \int_{s(h)}^{\infty} \xi_{G,\lambda,t}^V \, dt \leq h \right)
$$

$$
\leq \mathbb{P}\left( \xi_{\tilde{T}_h}^V = \emptyset \right) + \mathbb{P}\left( \xi_{\tilde{T}_h}^V \neq \emptyset, \xi_{G,\lambda}^V \leq h \right)
$$

$$
\leq \frac{2s(h)}{\exp\{cT \cdot d^h\}} + (1 - e^{-2h} \cdot (e^{-1} \cdot (1 - e^{-\lambda}))^{2h})^{s(h)/(4h)} \ll \frac{1}{2h}
$$

if $h$ is large. \hfill \Box

Lemma 14. If $h$ is large enough we have

$$
\mathbb{P}\left( \int_{s(h)}^{\infty} \xi_{G,\lambda,t}^V \, dt > h \right) > 1 - \frac{1}{2h}.
$$

(34)

Proof. We will separately prove that

$$
\mathbb{P}\left( \int_{0}^{\infty} \xi_{G,\lambda,t}^V \, dt > h \right) > 1 - \frac{1}{4h}
$$

(35)

and

$$
\mathbb{P}\left( \int_{0}^{s(h)} \xi_{G,\lambda,t}^V \, dt = 0 \right) > 1 - \frac{1}{4h};
$$

(36)

the desired result will then follow.
For (35), let \( u_1 := v_{L(h)-2}, u_2 := v_{L(h)-1} \) be such that \( u_1, u_2, \tilde{\sigma} \) (in this order) are the three last vertices in \( \mathcal{L}_h \), as we move away from \( \mathcal{T}_h \). By (22) and the definition of \( L(h) \) we have
\[
\mathbb{P}\left( \xi^{V \cup \mathcal{L}_h}_{G, \lambda}(u_i) = 0 \right) < s(h)^{-1}, \quad i = 1, 2.
\] (37)

Let \( G' \) denote \( \tilde{G} \) after removing \( u_2 \) and \( \tilde{\sigma} \). Define the random set of times
\[
I := \left\{ t \geq 0 : u_1 \in \xi^{V \cup \mathcal{L}_h \backslash \{u_2\}}_{G', \lambda; t} \right\}.
\]
We have
\[
\mathbb{P}\left( \xi^{V \cup \mathcal{L}_h}_{G', \lambda}(u_2) = 0 \right) = \mathbb{E}[1 - e^{-\lambda|I|}],
\] (38)
where \( |I| \) denotes the Lebesgue measure of \( I \). Indeed, one can decide if \( u_2 \) is ever infected in the process on \( \tilde{G} \) by inspecting whether there is a point in time at which (1) \( u_1 \) is infected in process confined to \( G' \), and (2) there is a transmission arrow from \( u_1 \) to \( u_2 \). The number of such time instants is a Poisson random variable with parameter \( \lambda|I| \), justifying (38).

We bound
\[
\mathbb{P}\left( |I| < h^2 \right) \leq e^{h^2} \cdot s(h)^{-1} \ll \frac{1}{8h}.
\] (39)

for \( h \) large enough.

We next claim that
\[
\mathbb{P}\left( \xi^{V \cup \mathcal{L}_h}_{G', \lambda}(\tilde{\sigma}) \leq h \right| |I| \geq h^2) < e^{-h}.
\] (40)

To prove this, we observe that on the event \( \{|I| \geq h^2\} \), we can find an increasing sequence of times \( S_0, \ldots, S_{[h^2/2]} \in I \) with
\[
|I \cap [S_j + 1, S_{j+1}]| \geq 2 \quad \text{for each } j.
\]
Next, note that for each interval \([S_j, S_{j+1}]\), with a probability that is positive and depends only on \( \lambda \), the infection is sent to \( \tilde{\sigma} \) and remains there for one unit of time. This occurring independently in different time intervals, (40) follows from a simple Chernoff bound. Now, (35) follows from (39) and (40).

We now turn to (36). Note that the event inside the probability there is contained in the event that there is an infection path starting at some time \( s \) and ending at some time \( t \) with \( s \leq t \leq s(h) \), connecting the two endpoints of \( \mathcal{L}_h \). By Lemma 3, the probability that such a path exists is smaller than
\[
(s(h) + 1) \cdot \exp\{-c_Z \cdot L(h)\} \ll \frac{1}{4h}
\]
if \( h \) is large enough.

\[ \square \]

Proof of Proposition 3. The statements follow readily from Lemmas 12, 13 and 14. \[ \square \]
5.4 Proof of Proposition 4

Proving Proposition 4 is now just a matter of putting together bounds that were obtained earlier.

Proof of Proposition 4. Let $B$ be the event that, in the graphical construction with parameter $\lambda$, there is an infection path starting from $(v_0, s)$ for some $s \leq 2\sigma(h) \cdot S(h)$ (where $v_0$ is the vertex of $\mathcal{L}_h$ neighboring the root $\rho$ of $\mathcal{T}_h$), ending at $(\tilde{o}, t)$ for some $t > s$, and entirely contained in $\mathcal{L}_h$. Then, by a union bound, the left-hand side of (23) is smaller than

$$
P\left(\xi_{\mathcal{V}^c;2s(h) \cdot S(h)} \neq \emptyset\right) + P(B).$$

(41)

The first term is bounded using Markov’s inequality and monotonicity:

$$P\left(\xi_{\mathcal{V}^c;2s(h) \cdot S(h)} \neq \emptyset\right) \leq (2s(h))^{-1}.$$ 

Next, recalling the definition of $p$ from (27), we bound, for $h$ large enough,

$$P_{\lambda'}(B) \leq 3\sigma(h) \cdot S(h) \cdot p_{\lambda'}(L(h)) \leq 3\sigma(h) \cdot S(h) \cdot p_{\lambda}(L(h)) \cdot (\eta_{\lambda', \lambda})^{-L(h)} \leq 3\sigma(h)^4 \cdot (\eta_{\lambda', \lambda})^{-d_{4h}} \ll (2\sigma(h))^{-1}.$$ 

$\square$
Appendix: proof of Proposition 5

Proof of Proposition 5. We will first state and prove some auxiliary claims.

Claim 1. For any $A \subset \tilde{V}\setminus \mathcal{T}_h$ we have

$$
\mathbb{P}\left( \text{either } \xi_{G,\lambda,2;\sqrt{\mathcal{L}(h)}}^A = \emptyset \text{ or } \xi_{G,\lambda,t;\sqrt{\mathcal{L}(h)}}^A \cap \mathcal{T}_h \neq \emptyset \text{ for some } t \leq \sqrt{\mathcal{L}(h)} \right) \geq \frac{1}{2}.
$$

Proof. Let $G'$ be the graph obtained by removing $\mathcal{T}_h$ from $\tilde{G}$ (so that $G'$ is the disconnected union of $G$ and $\mathcal{L}_h$). The complement of the event in the probability above is

$$
\left\{ \xi_{G',\lambda,2;\sqrt{\mathcal{L}(h)}}^A \neq \emptyset \right\} \subset \left\{ \xi_{G';\lambda,2;\sqrt{\mathcal{L}(h)}}^A \neq \emptyset \right\} \subset \{ \tau_1 > \sqrt{\mathcal{L}(h)} \} \cup \{ \tau_2 > \sqrt{\mathcal{L}(h)} \},
$$

where

$$
\tau_1 = \inf \left\{ t : \xi_{G',\lambda,t}^V = \emptyset \right\}, \quad \tau_2 = \inf \left\{ t : \xi_{G',\lambda,t}^L = \emptyset \right\}.
$$

Since $G$ is fixed while $h$ can be taken arbitrarily large, we can assume

$$
\mathbb{P}(\tau_1 > \sqrt{\mathcal{L}(h)}) < \frac{1}{4}.
$$

Next, noting that

$$
\mathcal{L}(h) = \exp\{d\sqrt{h}\} \gg \left( \log \left( d^2 h \right) \right)^2 \geq \left( \log L(h) \right)^2,
$$

we have

$$
\mathbb{P}(\tau_2 > \sqrt{\mathcal{L}(h)}) \leq \mathbb{P}(\tau_2 > (\log L(h))^2) < \frac{1}{4}
$$

if $h$ is large enough. \hfill \Box

Claim 2. For any $A \subset \tilde{V}$ we have

$$
\mathbb{P}\left( \text{either } \xi_{G,\lambda,2;\sqrt{\mathcal{L}(h)}}^A = \emptyset \text{ or } \xi_{G,\lambda,2;\sqrt{\mathcal{L}(h)}}^A \cap \mathcal{T}_h \neq \emptyset \right) > c_2.
$$

Proof. Define

$$
\tau' := \inf \left\{ t : \xi_{G,\lambda,t}^A = \emptyset \right\}, \quad \tau'' := \inf \left\{ t : \xi_{G,\lambda,t}^A \cap \mathcal{T}_h \neq \emptyset \right\}
$$

and let $\tau = \min(\tau', \tau'')$. By the first claim we have

$$
\mathbb{P}(\tau \leq \sqrt{\mathcal{L}(h)}) \geq \frac{1}{2}.
$$

Next, note that

$$
\mathbb{P}\left( \xi_{G,\lambda,2;\sqrt{\mathcal{L}(h)}}^A = \emptyset \mid \tau = \tau' \leq \sqrt{\mathcal{L}(h)} \right) = 1.
$$
We will prove that
\[
\mathbb{P}(\xi^A_{G,\lambda;2\sqrt{\theta(h)}} \supset \xi^\varnothing_{h,\lambda;2\sqrt{\theta(h)}} \mid \tau = \tau'' \leq \sqrt{\theta(h)}) > c_T. \tag{44}
\]

Taken together, (42), (43) and (44) give the statement of the claim.

To prove (44), we first introduce some notation. Given \( A' \subset \mathcal{T}_h \), we write
\[
\xi^A_{g,h,\lambda_{t_1},t_2}(x) := \mathbb{1}\{A' \times \{t_1\} \rightarrow (x,t_2)\}, \quad t_1 \leq t_2, \ x \in \mathcal{T}_h.
\]

Note that \((\xi^A_{g,h,\lambda_{t_1},t_1+s} : s \geq 0)\) has same distribution as \((\xi^A_{g,h,\lambda_{t_1}} : s \geq 0)\). Next, on the event \( \{\tau'' < \infty\} \) let \( A' := \{\xi^A_{G,\lambda;\tau''} \cap \mathcal{T}_h \neq \varnothing\} \). Define the event
\[
B := \{\tau'' < \infty\} \cap \left\{\xi^A_{g,h,\lambda;\tau''},\tau'' + \sqrt{\alpha(h)} \supset \xi^\varnothing_{h,\lambda;\tau''},\tau'' + \sqrt{\alpha(h)}\right\}.
\]

By (18) and the strong Markov property we have \(\mathbb{P}(B \mid \tau'' < \sqrt{\alpha(h)}) > c_T\).

Moreover, on \( B \) we have
\[
\xi^A_{G,\lambda;2\sqrt{\alpha(h)}} \supset \xi^A_{g,h,\lambda;\tau''},\tau'' + \sqrt{\alpha(h)} \supset \xi^\varnothing_{h,\lambda;\tau''},\tau'' + \sqrt{\alpha(h)}.
\]

This completes the proof. \(\square\)

**Claim 3.** For any \( A \subset \tilde{V} \) and \( h \) large enough we have
\[
\mathbb{P}\left(\text{either } \xi^A_{G,\lambda;\sqrt{\theta(h)/2}} = \varnothing \text{ or } \xi^A_{G,\lambda;\sqrt{\theta(h)/2}} \supset \xi^\varnothing_{h,\lambda;\sqrt{\theta(h)/2}}\right) > 1 - \left(1 - \frac{c_T}{2}\right)^{\sqrt{\theta(h)/4}}. \tag{45}
\]

**Proof.** For \( 1 \leq i \leq \left\lfloor \sqrt{\theta(h)/4}\right\rfloor \), define the event
\[
F_i := \left\{\xi^A_{G,\lambda;i\cdot2\sqrt{\theta(h)}} = \varnothing\right\} \cup \left\{\xi^A_{G,\lambda;i\cdot2\sqrt{\theta(h)}} \supset \xi^\varnothing_{h,\lambda;i\cdot2\sqrt{\theta(h)}}\right\}.
\]

We then note that the event in the probability in (45) is contained in \( \bigcup F_i \), and by Claim 2,
\[
\mathbb{P}(\bigcap_i F_i^c) \leq \left(1 - \frac{c_T}{2}\right)^{\sqrt{\theta(h)/4}}.
\]

We are now ready to proceed with the proof of the proposition. Using Claim 3 together with a union bound, we have that the event
\[
E_1 := \left\{\text{for all } x \in \tilde{V}, \text{ either } \xi^A_{G,\lambda;\sqrt{\theta(h)/2}} = \varnothing \text{ or } \xi^A_{G,\lambda;\sqrt{\theta(h)/2}} \supset \xi^\varnothing_{h,\lambda;\sqrt{\theta(h)/2}}\right\}
\]
has probability larger than \(1 - |\tilde{V}| \cdot (1 - \frac{c_T}{2})^{\sqrt{\theta(h)/4}}\). We now introduce notation for the time dual of the contact process: if \( G' = (V', E') \) is a graph, we write
\[
\xi^A_{G',\lambda;x,t}(x) := \mathbb{1}\{(x, s) \rightarrow A \times \{t\}\}, \quad x \in V', \ A \subset V', \ s \leq t
\]
(as usually, we abuse notation and sometimes treat $\tilde{\xi}_{G',\lambda,\pm}$ as a subset of $V'$ rather than a configuration of 0's and 1's). By invariance of Poisson processes under time reversal, the event

$$E_2 := \left\{ \begin{array}{l}
\text{for all } x \in \tilde{V}, \text{ either } \xi^{(x)}_{G,\lambda,\pm(h)/2,\pm(h)} = \emptyset \\
\text{or } \tilde{\xi}^{(x)}_{G,\lambda,\pm(h)/2,\pm(h)} \supset \tilde{\xi}^{T_h}_{G,\lambda,\pm(h)/2,\pm(h)}
\end{array} \right\}$$

has the same probability as $E_1$. Finally, (17) implies that if $h$ is large enough, the event

$$E_3 := \{ \xi^{T_h}_{G,\lambda,\pm(h)} \neq \emptyset \}$$

has probability larger than $1 - \exp\{-c_T d^h\}$. Putting our bounds together, we have

$$\mathbb{P}(E_1^c \cup E_2^c \cup E_3^c) \leq 2|\tilde{V}| \cdot \left(1 - \frac{c_T}{2}\right) \left[\sqrt{\lambda(h)}/4\right] + \exp\{-c_T d^h\} \ll \lambda(h)^{-2}$$

if $h$ is large enough.

We now claim that for any $A \subset V$ we have

$$E_1 \cap E_2 \cap E_3 \subset \left\{ \text{either } \xi^A_{G,\lambda,\pm(h)} = \emptyset \text{ or } \xi^A_{G,\lambda,\pm(h)} = \xi^{\tilde{G}}_{G,\lambda,\pm(h)} \right\}. \tag{46}$$

Indeed, fix $A \subset \tilde{V}$ and assume $E_1 \cap E_2 \cap E_3$ occurs. In case $\xi^{\tilde{G}}_{G,\lambda,\pm(h)} = \emptyset$, then we also have $\xi^A_{G,\lambda,\pm(h)} = \emptyset$, so there is nothing to prove in this case. Now assume that $\xi^{\tilde{G}}_{G,\lambda,\pm(h)} \neq \emptyset$ and also that $\xi^A_{G,\lambda,\pm(h)} \neq \emptyset$. We can then take some $y \in \xi^{\tilde{G}}_{G,\lambda,\pm(h)}$, and some $x \in A$ such that $\xi^{(y)}_{G,\lambda,\pm(h)} \neq \emptyset$. Moreover, noting that

$$E_3 = \left\{ \xi^{T_h}_{G,\lambda,\pm(h)/2} \cap \xi^{T_h}_{G,\lambda,\pm(h)/2,\pm(h)} \neq \emptyset \right\},$$

we can take some $z \in \xi^{T_h}_{G,\lambda,\pm(h)/2} \cap \xi^{T_h}_{G,\lambda,\pm(h)/2,\pm(h)}$.

We now observe that, since $\xi^{(y)}_{G,\lambda,\pm(h)} \neq \emptyset$, we also have $\xi^{(y)}_{G,\lambda,\pm(h)/2} \neq \emptyset$ and then, since $E_1$ occurs, we have $z \in \xi^{(y)}_{G,\lambda,\pm(h)/2}$. Similarly, the occurrence of $E_2$ implies that $z \in \xi^{(y)}_{G,\lambda,\pm(h)/2}$, This proves that

$$\xi^{(x)}_{G,\lambda,\pm(h)/2} \cap \xi^{(y)}_{G,\lambda,\pm(h)/2,\pm(h)} \neq \emptyset \implies y \in \xi^{(y)}_{G,\lambda,\pm(h)} \subset \xi^A_{G,\lambda,\pm(h)}.$$ 

This concludes the proof of (46).

\[\Box\]

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