Uniform perfectness for Interval Exchange Transformations with or without Flips

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UNIFORM PERFECTNESS FOR INTERVAL EXCHANGE TRANSFORMATIONS WITH OR WITHOUT FLIPS

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ABSTRACT. — Let $\mathcal{G}$ be the group of all Interval Exchange Transformations. Results of Arnoux–Fathi, Sah and Vorobets state that $\mathcal{G}_0$ the subgroup of $\mathcal{G}$ generated by its commutators is simple. Arnoux proved that the group $\mathcal{G}$ of all Interval Exchange Transformations with flips is simple.

We establish that the commutator length is at most 6 for any element of $\mathcal{G}$. Moreover, we give conditions on $\mathcal{G}$ that guarantee that the commutator lengths of the elements of $\mathcal{G}_0$ are uniformly bounded, and in this case for any $g \in \mathcal{G}_0$ this length is at most 5.

RéSUMÉ. — Soit $\mathcal{G}$ le groupe des échanges d’intervalles. Des résultats d’Arnoux–Fathi, Sah et Vorobets indiquent que $\mathcal{G}_0$ le sous-groupe de $\mathcal{G}$ engendré par ses commutateurs est simple. Arnoux prouve que le groupe $\mathcal{G}$ des échanges d’intervalles avec flips est simple.

Nous établissons que tout élément de $\mathcal{G}$ a une longueur des commutateur inférieure ou égale à 6. De plus, nous exhibons des conditions sur $\mathcal{G}$ qui garantissent que les longueurs des commutateurs des éléments de $\mathcal{G}_0$ sont uniformément bornées et dans ce cas pour tout $g \in \mathcal{G}_0$ nous montrons que cette longueur est au plus 5.

1. Introduction

Let $J = [a, b)$ be a half-open interval.

An interval exchange transformation (IET) of $J$ is a right continuous bijective map $f : J \to J$ defined by a finite partition of $J$ into half-open intervals.

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subintervals $I_i$ and a reordering of these intervals by translations. We denote by $\mathcal{G}_J$ the group consisting in all IET of $J$.

An interval exchange transformations with flips (FIET) on $J$ is a bijection $f : J \to J$ for which there exists a subdivision $a = a_1 < \cdots < a_m < a_{m+1} = b$ such that $f|_{(a_i,a_{i+1})}$ is a continuous isometry. Note that $f$ is not necessarily right continuous since the orientation of some intervals can be reversed, and there exists a flip-vector $U(f) = (u_1, \ldots, u_m) \in \{-1, 1\}^m$ such that $f|_{(a_i,a_{i+1})}$ is a direct isometry if $u_i = 1$ or an indirect one if $u_i = -1$.

It is worth noting that several authors consider FIET as piecewise isometries of $J$ with a finite number of discontinuity points reversing at least one of the intervals of continuity. This is not our case, so an IET is an FIET.

![Figure 1.1. Two equivalent FIET.](image)

We say that two FIET on $J$, $f$ and $g$ are equivalent if the set $\{x \in J : f(x) \neq g(x)\}$ is finite (see Figure 1.1). We denote by $\mathcal{G}_J$ the corresponding quotient set. Note that $\mathcal{G}_J$ is the quotient group of the FIET group by its normal subgroup consisting of elements which are trivial except possibly at finitely many points. By abuse of terminology, elements of $\mathcal{G}_J$ are also called FIET. The map $U$ is still well defined on $\mathcal{G}_J$ and the set of all elements of $\mathcal{G}_J$ such that $U(f) \subset \{1\}^m$ is identified with $\mathcal{G}_J$.

Let $f$ be an IET or an FIET on $J$. The continuity intervals of $f$ are the maximal connected subsets of $J$ on which $f$ is continuous and they are denoted by $I_1, \ldots, I_m$. The associated permutation $\pi = \pi(f) \in S_m$ is defined by $f(I_i) = J_i \pi(i)$, where the $J_i$’s are the ordered images of the $I_i$’s. By convention, $f$ is not continuous at the left endpoint of $J$ and we define $BP(f)$ to be the set of the discontinuity points of $f$. Note that $BP(f^{-1}) = f(BP(f))$ and $BP(f \circ g) \subset BP(g) \cup g^{-1}(BP(f))$. 

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Let $m$ be a positive integer, we denote by $G_m$ (resp. $G_{m,\pi}$) the set of all elements of $G$ having at most $m$ discontinuity points (resp. whose associated permutation is $\pi \in S_m$).

From now on, without mention of the defining interval $J$, an IET or an FIET is defined on $I = [0,1)$ and $G_I$ and $\bar{G}_I$ will be denoted by $G$ and $\bar{G}$ respectively.

**Remark 1.1.** — The group $G_J$ (resp. $\bar{G}_J$) is conjugated by the direct homothecy that sends $J$ to $I$ to the group $G$ (resp. $\bar{G}$). The subgroup of $G$ (resp. $\bar{G}$) consisting of elements with support in $J$ can be identified, by taking restriction, with $G_J$ (resp. $\bar{G}_J$).

Since the late seventies, the dynamics and the ergodic properties of a single interval exchange transformation were intensively studied (see e.g. Viana survey [36]). A natural extension is to consider the dynamics in terms of group actions. The most famous problem was raised by Katok: does $G$ contain copies of $\mathbb{F}_2$, the free group of rank 2? Dahmani, Fujiwara and Guirardel established that such subgroups are rare ([7, Theorem 5.2]).

More generally, one can ask for a description of possible subgroups of $G$. According to Novak there is no distortion in $G$ ([27, Theorem 1.3]) and as a standard consequence any finitely generated nilpotent subgroup of $G$ is virtually abelian (see e.g. [13]).

Among many things, Dahmani, Fujiwara and Guirardel proved that any finitely generated subgroup of $G$ is residually finite ([7, Theorem 7.1]), $G$ contains no infinite Kazhdan groups ([7, Theorem 6.2]), any finitely generated torsion free solvable subgroup of $G$ is virtually abelian ([8, Theorem 3]) and provide examples of non virtually abelian solvable subgroups of $G$ ([8, Theorem 6]). Thus, finding torsion free finitely generated non virtually abelian subgroups of $G$ seems very difficult, especially as works of Juchenko, Monod ([18]) suggest that $G$ could be amenable as it is conjectured by Cornulier.

The group $G$ shares many of the properties of the group of piecewise affine increasing homeomorphisms of the unit interval, $\text{PL}^+(I)$. For instance these two groups are not simple but have simple derived subgroups (see [10] for the PL case). As noted in Remark 1.2 of [28], they satisfy no law (i.e. there does not exist $\omega \in \mathbb{F}_2 \setminus \{e\}$ such that $\phi(\omega) = \text{Id}$ for every homomorphism $\phi : \mathbb{F}_2 \to G$), their nilpotent subgroups are virtually abelian (see [11] for the PL case) and the main result of [7] can be seen has a generic version of the Brin and Squier theorem [5] which asserts that $\text{PL}^+(I)$ does not contain non abelian free subgroups. It’s however still unknown whether $\text{PL}^+(I)$ is amenable.
Considerably less is known about $\mathcal{G}$. However, due to their connections with non oriented measured foliations on surfaces and billiards, the dynamics and ergodic properties of a single FIET were firstly explored by Gutierrez ([15]) and Arnoux ([2]).

Dealing with irreducible permutations, Keane ([19]) proved that almost all IET are minimal and Masur ([23]) and Veech ([35]) that almost all are uniquely ergodic. For FIET that reverse orientation in at least one interval, Nogueira ([25]) proved that almost all have periodic points so are nonergodic. He also exhibited an example of a minimal uniquely ergodic one (see also [16] and [22]). Recently, Skripchenko and Troubetzkoy gave bounds for the Hausdorff dimension of the set of minimal maps ([33]) and Hubert and Paris-Romaskevich described all the minimal maps having 4 continuity intervals (see [17, Theorem 6 p. 39–40]).

The decomposition into minimal and periodic components was first studied for measured surfaces flows by Mayer ([24]) and restated for IET by Arnoux ([1]) and Keane ([19]). For FIET that reverse orientation in at least one interval of the $m$ continuity intervals, Nogueira, Pires and Troubetzkoy proved that the sum of number of periodic components and twice the number of minimal components is bounded by $m$ ([26]).

As mentioned by Paris-Romaskevich in [30], one can interest ourselves in the dynamics of FIET from the point of view of geometrical group theory: describe the possible groups that can be realized as groups of FIET or establish algebraic properties of the whole group.

Here we shall be concerned solely with the structure of the whole groups $\mathcal{G}$ and $\mathcal{\bar{G}}$. This is also motivated by the algebraic study of other transformation groups, particularly groups of homeomorphisms of low dimensional manifolds, that was initiated by Schreier and Ulam in 1934 ([32]) who were interested in the simplicity of such groups.

We recall that, given $G$ a group,

- a commutator in $G$ is an element of $G$ of the form $[f, g] = f g f^{-1} g^{-1}$ with $f, g \in G$.
- $G$ is perfect if $G = [G, G]$ the subgroup of $G$ generated by its commutators.
- $G$ is simple if any normal subgroup of $G$ is either $G$ or trivial.

In the seventies, lots of homeomorphisms or diffeomorphisms groups were studied by Epstein, Herman, Thurston, Mather, Banyaga, and proved to be simple; these works are survey in the books [3] or [4]. For interval exchange transformations, it has been shown by Arnoux ([2, III §2.4]), Sah ([31]) and Vorobets ([37]) that the subgroup $\mathcal{G}_0$ of $\mathcal{G}$ generated by its commutators is
simple. In [2, III §1.4], Arnoux proved that $G$ is simple, this unpublished result has been recently recovered by Lacourte ([21]). In order to sharpen this property, it is convenient to give

**Definition 1.2.** — Let $G$ be a group and $g \in [G,G]$, the commutator length of $g$, denoted by $c(g)$, is the least number $c$ such that $g$ is a product of $c$ commutators. We set $c(G) = \sup\{c(g), g \in [G,G]\}$ and we say that $G$ is uniformly perfect if $c(G)$ is finite.

The main theorems of this paper are

**Theorem 1.3.** — $c(\mathcal{G}) \leq 6$.

For the group $\mathcal{G}$ we are not able to decide if $c(\mathcal{G})$ is finite. However, in the affirmative case, we give an explicit bound in the following

**Theorem 1.4.** — If $\mathcal{G}$ is uniformly perfect then $c(\mathcal{G}) \leq 5$.

In Section 6, we will prove stronger results, Theorems 6.1 and 6.5, which only require that commutator lengths are bounded when prescribing the number of discontinuity points or the arithmetic, that is when the elements considered belong to $\Gamma_\alpha := \{g \in \mathcal{G} : BP(g) \subset \Delta_\alpha\}$, where $\Delta_\alpha$ is the abelian subgroup of $\mathbb{R}$ generated by $p$ real numbers $\alpha_1, \ldots, \alpha_p$ and 1.

Our proofs are based on an adaptation of a result of Dennis and Vaserstein giving a criterion for uniform perfectness ([9]). This is explained in Section 4.

It is plain that a simple group $G$ is generated by any subset $S$ which is invariant under conjugation. In particular, $S$ can be the set consisting of commutators, involutions or finite order elements. Group invariants are therefore provided by considering $S$-lengths that is the least number $l_S$ such that any element can be written as product of $l_S$ elements of $S$. These lengths can be simultaneously considered by using the following

**Definition 1.5.** — A group $G$ is uniformly simple if there exists a positive integer $N$ such that for any $f, g \in G \setminus \{\text{Id}\}$, the element $g$ can be written as a product of at most $N$ conjugates of $f$ or $f^{-1}$.

Ulam and Von Neumann ([34]) showed that the group of homeomorphisms of $S^1$ is uniformly simple. Burago and Ivanov ([6]) obtained implicitly the same conclusion for $\text{PL}^+(S^1)$ and $[\text{PL}^+(I), \text{PL}^+(I)]$ (see also [12, Theorem 1.1]). The question of the uniform simplicity of $[G,G]$ and $\mathcal{G}$ is formulated in [30]. However, Cornulier communicated us that $[G,G]$ and $\mathcal{G}$ are not uniformly simple. Indeed, if the support of an IET or FIET $f$ has
length less than $\frac{1}{N}$ then any product of $N$ conjugates of $f$ or $f^{-1}$ can not have full support.

In a forthcoming paper we will prove the uniform simplicity of $\mathcal{A}$, the group of affine interval exchange transformations of $I$, i.e. bijections $I \to I$ defined by a finite partition of $I$ into half-open subintervals such that the restriction to each of these intervals is a direct affine map. More generally, we will give conditions on groups of piecewise continuous bijective maps on $I$ that ensure uniform simplicity. This study and the work of [12] suggest that most simple transformation groups are uniformly simple. The group $\mathcal{G}$ provides an example of a non uniformly simple group with bounded commutator length. As far as we know it is an open problem to determine whether $\mathcal{G}$ has bounded involution length. Recently, O. Lacourte ([20]) defined the analogues of $\Gamma_\alpha$ in $\mathcal{G}$, namely $\overline{\Gamma}_\alpha$. He proved that $[\overline{\Gamma}_\alpha, \overline{\Gamma}_\alpha]$ are simple and it would be relevant to study their uniform perfectness.

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2. Preliminaries

The aim of this section is to fix notations and terminology, to collect a few results and to prove some basic results to be used in the sequel.

2.1. Restricted rotations and periodic IET

Definition 2.1. — An IET with two continuity intervals is called a rotation and it is denoted by $R_a$, where $a$ is the image of $0$.

An IET $g$ whose support, $\text{supp}(g) = \{x \in I : g(x) \neq x\}$, is $J = [a, b) \subset [0, 1)$ is a restricted rotation if the direct homothecy that sends $J$ to $[0, 1)$ conjugates $g|_J$ to a rotation. We denote it by $R_{\alpha,J}$ where $\alpha$ is given by $R_{\alpha,J}(x) = x + \alpha \pmod{|b-a|}$ for $x \in J$.

An element $g$ of $\mathcal{G}$ (resp. $\overline{\mathcal{G}}$) is periodic if every $g$-orbit is finite.
By [2, III p. 3], [27, Lemma 6.5] or [37, Lemma 2.1], any interval exchange transformation is a product of restricted rotations (see also our Lemma 3.3 for a proof). For periodic IET, Novak showed a sharper statement.

**Lemma 2.2 ([27, Proof of Corollary 5.6]).** — Any periodic element $g$ of $G$ is conjugated in $G$ to a product of finite order restricted rotations with disjoint supports. In particular, any periodic IET has finite order.

### 2.2. Basic properties on commutators

**Definition 2.3.** — Let $G$ be a group. An element $a \in G$ is reversible in $G$ if there exists $h \in G$ such that $a = ha^{-1}h^{-1}$.

**Properties 2.4.** — Let $G$ be a group and let $a, b, a', b'$ and $h$ be elements of $G$.

1. If $a$ and $b$ commute with both $a'$ and $b'$ then $[a, b][a', b'] = [aa', bb']$.
2. If $a' = ha^{-1}h^{-1}$ then $aa' = [a, h]$.
3. If $a$ is reversible in $G$ then $a^2$ is a commutator.
4. $h[a, b]h^{-1} = [hah^{-1}, hbb^{-1}]$.

**Proof.**

1. As $a'$ commutes with $a$ and $b$, we have

$$[a, b][a', b'] = aba^{-1}b^{-1}a'b'a'^{-1}b'^{-1} = aa'ba^{-1}b^{-1}b'a'^{-1}b'^{-1}.$$ 

Repeating this process with $b'$ and then $a'^{-1}$, we get

$$[a, b][a', b'] = aa'bb'a'^{-1}b'^{-1}b^{-1} = aa'bb'a'^{-1}a^{-1}b'^{-1}b^{-1} = [aa', bb'].$$

2. $aa' = aha^{-1}h^{-1} = [a, h]$.
3. $a^2 = aha^{-1}h^{-1} = [a, h]$.
4. $[hah^{-1}, hbb^{-1}] = hah^{-1}hbb^{-1}ha^{-1}h^{-1}hb^{-1}h^{-1} = h[a, b]h^{-1}$. □

### 2.3. Periodic IET are commutators

In [14, Theorem 4], the authors proved that any periodic IET is reversible in $G$. We recall this argument briefly. Let $f$ be a periodic IET, by the Arnoux Decomposition Theorem (see [2, Proposition p. 20]), the interval $[0, 1)$ can be written as the union of finitely many $f$-periodic components $M_i$, $i = 1, \ldots, n$ of period $p_i$. In particular, $M_i = \bigsqcup_{k=1}^{p_i} J_k$, where $J_k = f^{k-1}(J_1)$ are half-open intervals and $f$ is continuous on $J_k$. 

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Eventually conjugating $f$ by an IET, we can suppose that the $J_k$’s are ordered consecutive intervals so the $M_i$’s are intervals and $\pi = \pi(f|_{M_i}) = (1, 2, \ldots, p_i)$.

We consider the IET $h$, that is defined on each $M_i$ by $h$ is continuous on $J_k$ and $h(J_k) = J_{\tau(k)}$, where $\tau \in S_p$, satisfies $\tau^{-1} \pi \tau(k) = \pi^{-1}$ (such a permutation exists by Proposition 3.4 of [29]). One has that $h^{-1} \circ f \circ h$ is continuous on $J_k$ and $h^{-1} \circ f \circ h(J_k) = J_{\tau^{-1} \pi \tau(k)} = J_{\pi^{-1}(k)}$. Therefore $h^{-1} \circ f \circ h = f^{-1}$, meaning that $f$ is reversible in $G$.

This implies

**Proposition 2.5.** — Any periodic IET is a commutator in $G$.

Indeed, we claim that any periodic IET can be written as the square of another periodic element. To see this, it is enough to consider rotations, by Lemma 2.2. This is obvious since $R_{\alpha} = R_{\frac{2\pi}{3}}$, so any periodic IET is the square of a reversible IET. Finally, the result follows from Properties 2.4 (3).

3. Generalities on commutators in $G$ and $\mathcal{G}$

3.1. Commutators in $G$

3.1.1. Fundamental examples

Let $a, b$ satisfy that $0 \leq a < b \leq 1$.

We denote by $I_{[a,b]}$ the symmetry of $[a, b]$, i.e. the FIET defined by:

$I_{[a,b]}(x) = x$ if $x \notin [a, b]$ and $I_{[a,b]}(x) = a + b - x$ if $x \in [a, b]$.

Similarly, we denote by $I_{(a,b)}$ the symmetry of $(a, b)$, i.e. the FIET defined by:

$I_{(a,b)}(x) = \begin{cases} x & \text{if } x \notin (a, b) \\ a + b - x & \text{if } x \in (a, b). \end{cases}$

\begin{align*}
\begin{array}{ccc}
I_{[a,b]} & I_{(a,b)} & S_{\theta,[a,b]} \\
\begin{picture}(50,50)
\put(0,0){a}
\put(0,50){b}
\put(0,0){$\bullet$}
\put(10,40){$\downarrow$}
\end{picture} & \begin{picture}(50,50)
\put(0,0){a}
\put(0,50){b}
\put(0,0){$\bullet$}
\put(10,40){$\downarrow$}
\end{picture} & \begin{picture}(50,50)
\put(0,0){a}
\put(0,50){\theta}
\put(0,0){\bullet}
\put(0,50){$\downarrow$}
\end{picture}
\end{array}
\end{align*}
Clearly, $I_{[a,b]}$ and $I_{[a,b)}$ are involutions and they represent the same element of $\mathcal{G}$. Therefore, given $J$ a subinterval of $I$, we define $I_J$ to be the element of $\mathcal{G}$ represented by $I_J$.

Let $\theta \in [a,b)$, we define another involution $S_{\theta,[a,b)}$ on $[a,b)$ by

$$S_{\theta,[a,b)} = I_{[a,\theta]} \circ I_{(\theta,b)}.$$ 

In particular, $S_{\theta,[0,1)} = \theta - x \pmod{1}$ and it is denoted by $S_{\theta}$.

**Property 3.1.**

(1) $S_{\theta} \circ S_{\theta'} = R_{\theta - \theta'}$.

(2) $R_{\alpha} \circ S_{\theta} \circ R_{\alpha}^{-1} = S_{\theta + 2\alpha}$.

**Lemma 3.2** ([2, III p. 3]). — The maps $I_{[a,b)}$ and $R_{\alpha,[a,b)}$ are commutators in $\mathcal{G}_{[a,b)}$ and then in $\mathcal{G}$.

**Proof.** — Taking restrictions and conjugating by a homothecy as in Remark 1.1, it is sufficient to prove that $I_{(0,1)}$ et $R_{\alpha,[0,1)}$ are commutators.

It is easy to see that $I_{(0,1)}$ is the product of the involutions $f_1$ and $f_2$ described as below:

As $f_2$ is conjugated to $f_1 = f_1^{-1}$ by $R_{\frac{1}{2}}$, Item (2) of Properties 2.4 implies that the map $I_{(0,1)}$ is a commutator.

According to Property 3.1, any rotation is the product of 2 symmetries that are conjugated by a rotation; thus $R_{\alpha,[0,1)}$ is a commutator. □

### 3.1.2. Decomposition in involutions and restricted rotations

**Lemma 3.3** ([2, III p. 3]).

(1) Any $f \in \mathcal{G}$ can be written as the product of an element of $\mathcal{G}$ and an involution that is a commutator.

(2) Any $g \in \mathcal{G}_m$ can be written as the product of $m - 1$ restricted rotations.
Proof.

(1). — Let \( f \in \mathcal{G} \), we denote by \( I_i \) its continuity intervals and by \( U = (u_i) \) its flip-vector. It is easy to check that \( f \circ \prod_{i \in U_i = -1} I_i \) belongs to \( \mathcal{G} \). Moreover the \( I_i \)'s have disjoint supports, so they commute and then \( \prod_i I_i \) is an involution and a commutator by Lemma 3.2 and Properties 2.4.

(2). — For clarity, given \( J = [a, b) \) and \( K = [b, c) \) two consecutive half-open intervals, we denote by \( R_{J,K} \) the restricted rotation of support \( J \cup K \) whose interior discontinuity point is \( b \).

Let \( g \in \mathcal{G}_{m,\pi} \) with continuity intervals \( I_1, \ldots, I_m \) and let \( g(I_i) = J_{\pi(i)} \). We consider \( R_1 = R_{J_1,K_{\pi(1)}} \), where \( J = J_1 \cup \cdots \cup J_{\pi(1)-1} \) and \( K = J_{\pi(1)} \). One directly has that \( R_1 \circ g|_{I_1} = \mathrm{Id} \) and \( \#\mathrm{BP}(g_1) \leq m - 1 \), where \( g_1 = R_1 \circ g|_{I_2 \cup \cdots \cup I_m} \).

Starting with \( g_1 \), we define similarly \( R_2 \) and we get that \( R_2 \circ g_1|_{I_2} = \mathrm{Id} \) and \( \#\mathrm{BP}(g_2) \leq m - 2 \), where \( g_2 = R_2 \circ g_1|_{I_3 \cup \cdots \cup I_m} \).

Repeating the previous argument \( m - 1 \) times leads to \( \#\mathrm{BP}(g_{m-1}) \leq 1 \) so \( g_{m-1} = \mathrm{Id} \).

Extending the restricted rotations \( R_i \) to \([0, 1]\) by the identity map, we conclude that

\[
R_{m-1} \circ \cdots \circ R_1 \circ g = \mathrm{Id}.
\]

A direct consequence of Lemmas 3.2 and 3.3 is

**Proposition 3.4 ([2, III §1.4])**. — The group \( \mathcal{G} \) is perfect and any \( g \in \mathcal{G}_m \) is the product of \( m - 1 \) commutators in \( \mathcal{G} \).

### 3.2. Commutators in \( \mathcal{G} \)

In the introduction, we have indicated a few similarities between the groups \( \mathcal{G} \) and \( \text{PL}^+(I) \). In particular, the simplicity of their derivative subgroups relies on a result of Epstein (see [10, 1.1 Theorem]). In the context of finding bounds for the commutator length, a substantial difference between these two groups is that an element \( f \) of \([\text{PL}^+(I), \text{PL}^+(J)]\) is a map whose support \( J \) satisfies \( \overline{J} \subset (0, 1) \) and \( f \) can not be written as a product of commutators of maps with support in \( J \). This contrasts with

**Remark 3.5.** — Let \( J \) be a half-open subinterval of \( I \). If \( g \in \mathcal{G} \) has support in \( J \) then \( g \in [\mathcal{G}, \mathcal{G}] \) if and only if \( g|_J \in [\mathcal{G}_J, \mathcal{G}_J] \).

Indeed, according to Theorem 1.1 of [37], there is a morphism \( \text{SAF}_J : \mathcal{G}_J \to \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R} \) such that \([\mathcal{G}_J, \mathcal{G}_J] = \text{SAF}_J^{-1}(0)\). More precisely for \( f \in \mathcal{G}_J \), \( \text{SAF}_J(f) = \sum k \lambda_k \otimes \delta_k \), where the vectors \( (\lambda_k) \), \( (\delta_k) \) encode the lengths...
of exchanged intervals and the corresponding translation constants respectively.

Let $g \in \mathcal{G}$ with support in $J$. From the previous definition, it is easy to check that $\text{SAF}_I(g) = \text{SAF}_J(g|_J)$. Therefore $g \in [\mathcal{G}, \mathcal{G}]$ if and only if $g|_J \in [\mathcal{G}_J, \mathcal{G}_J]$.

4. The adapted Dennis and Vaserstein argument

In this section, we first we recall Proposition 1(c) of Dennis and Vaserstein ([9]).

4.1. The original criterion

**Definition 4.1.** — Two subsets $S_1$ and $S_2$ of a group $G$ are commuting if any $a \in S_1$ commutes with any $a' \in S_2$.

**Dennis and Vaserstein’s criterion.** — If a group $G$ contains two commuting subgroups $H_1$ and $H_2$ such that for each finite subset $S$ of $G$ there are elements $g_i \in G$, $i = 1, 2$, such that $g_i^{-1}Sg_i \subseteq H_i$ for $i = 1, 2$, then $c(G) \leq 3$.

As an illustrating example indicated by Ghys, the group $[\text{PL}^+(I), \text{PL}^+(I)]$ consists in all $g$ of $\text{PL}^+(I)$ such that $g'(0) = g'(1) = 1$. Thus, for any finite collection $\{g_i\}$ in $[\text{PL}^+(I), \text{PL}^+(I)]$ there exist $0 < a < b < 1$ such that $(a, b)$ contains the support of all the $g_i$. The required groups $H_1$ and $H_2$ are obtained as groups of maps with disjoint supports by setting $H_1 = \langle g_i \rangle$ and $H_2 = h\langle g_i \rangle h^{-1}$ where $h$ is an element of $[\text{PL}^+(I), \text{PL}^+(I)]$ that carries $(a, b)$ into $(\frac{a}{2}, a)$.

Unfortunately, this argument doesn’t apply immediately in $[\mathcal{G}, \mathcal{G}]$ and $\mathcal{G}$. This is essentially due to the facts that both groups contain maps with full support and that if the length of the support of $g \in \mathcal{G}$ exceeds $\frac{1}{2}$ then it is impossible to find a conjugate of $g$ in $\mathcal{G}$ with a disjoint support. To avoid these difficulties, we will compose with suitable periodic maps to obtain an IET with arbitrary small support (see Propositions 5.1 and 6.1) and then we will apply the following iterated version of Dennis and Vaserstein’s criterion.
4.2. The iterated version

Let \( n \in \mathbb{N}^* \), we denote by \( H_n \) (resp. \( \overline{H}_n \)) the subgroup of \( G \) (resp. \( \overline{G} \)) consisting of elements whose support is included in \( [1 - \frac{1}{n}, 1) \). By Remark 1.1, \( H_n \) (resp. \( \overline{H}_n \)) is identified with \( G_{[1 - \frac{1}{n}, 1)} \) (resp. \( \overline{G}_{[1 - \frac{1}{n}, 1)} \)). Moreover, Remark 1.1 and Proposition 3.4 imply that \( \overline{H}_n \) is perfect and Remark 3.5 leads to \( H_n \cap [G, G] = [H_n, H_n] \).

**Lemma 4.2.**

1. If \( g \in H_2 \cap [G, G] = [H_2, H_2] \) then \( c_G(g) \leq \frac{1}{2} c_{H_2}(g) + \frac{3}{2} \).
2. If \( g \in \overline{H}_2 \) then \( c_G(g) \leq \frac{1}{2} c_{\overline{H}_2}(g) + \frac{3}{2} \).

**Proof.** — Proofs of Items (1) and (2) are similar, changing \( G \) for \( \overline{G} \) and \( H_2 \) for \( \overline{H}_2 \), so we only prove Item (1).

Let \( g \in H_2 \cap [G, G] \). We write \( c_{H_2}(g) = 2p - r \) with \( p \in \mathbb{N}^* \) and \( r = 0, 1 \).

Therefore

\[
g = (c_1 \ldots c_p)(c_{p+1} \ldots c_{2p}),
\]

where \( c_i = [a_i, b_i] \) with \( a_i, b_i \) in \( H_2 \) and the last commutator \( c_{2p} \) is eventually trivial.

Let \( R \) be the rotation of angle \( \frac{1}{2} \). We denote by \( f' = R \circ f \circ R^{-1} \).

Note that if \( f, k \in H_2 \) then \( f \) and \( k' \) have disjoint supports and they commute. We write

\[
g = (c_1 \ldots c_p)(c_{p+1} \ldots c_{2p})(c_{p+1} \ldots c_{2p})^{-1}(c_{p+1} \ldots c_{2p})
\]

\[
= (c_1 c_{p+1}) \ldots (c_p c_{2p}) C,
\]

where \( C = (c_{p+1} \ldots c_{2p})^{-1}(c_{p+1} \ldots c_{2p}) \).

On one hand, by Properties 2.4(1), we have that \( c_i c_{p+i}, i = 1, \ldots, p \), are commutators.

On the other hand, by Properties 2.4(2), it holds that

\[
C = (c_{p+1} \ldots c_{2p})^{-1}(c_{p+1} \ldots c_{2p})
\]

is a commutator since it is the product of \( (c_{p+1} \ldots c_{2p}) \) and the conjugate by \( R \) of its inverse.

Finally, we have \( c_G(g) \leq p + 1 \), thus

\[
2c_G(g) \leq 2p + 2 = c_{H_2}(g) + r + 2 \leq c_{H_2}(g) + 3.
\]

Repeatedly applying Lemma 4.2, we get

**Proposition 4.3.** — Let \( t \in \mathbb{N}^* \).

1. If \( g \in H_{2^t} \cap [G, G] = [H_{2^t}, H_{2^t}] \) then \( c_G(g) < \frac{1}{2^t} c_{H_{2^t}}(g) + 3 \).
2. If \( g \in \overline{H}_{2^t} \) then \( c_G(g) < \frac{1}{2^t} c_{\overline{H}_{2^t}}(g) + 3 \).
Proof. — As noted earlier, we only prove Item (1).

Let \( t \in \mathbb{N}^* \) and \( g \in H_{2t} \cap [G, G] \). From Remark 1.1 and Lemma 4.2, we obtain
\[
c_{H_{2t-1}}(g) \leq \frac{1}{2} c_{H_{2t}}(g) + \frac{3}{2}.
\]
It is easy to check by induction that for \( s \in \{1, \ldots, t\} \), we have
\[
(E_s)c_{H_{2t-s}}(g) \leq \frac{1}{2} s c_{H_{2t}}(g) + 3 \sum_{j=1}^{s} \frac{1}{2^j}.
\]
Indeed, for \( s = 1 \), \((E_1)\) is the first identity.

Fix \( s \in \{1, \ldots, t-1\} \) and suppose that \((E_s)\) holds. Then according to Remark 1.1 and Lemma 4.2
\[
C_{H_{2t-(s+1)}}(g) \leq \frac{1}{2} C_{H_{2t-s}}(g) + \frac{3}{2}.
\]
Thus, by induction hypothesis
\[
C_{H_{2t-(s+1)}}(g) \leq \frac{1}{2} \left( \frac{1}{2^s} C_{H_{2t}}(g) + 3 \sum_{j=1}^{s} \frac{1}{2^j} \right) + \frac{3}{2}.
\]
Therefore
\[
C_{H_{2t-(s+1)}}(g) \leq \left( \frac{1}{2^{s+1}} C_{H_{2t}}(g) + 3 \sum_{j=1}^{s+1} \frac{1}{2^{j+1}} \right) + \frac{3}{2},
\]
which leads immediately to \((E_{s+1})\).

Finally, noting that \( H_1 = G \) and \( \sum_{j=1}^{t} \frac{1}{2^j} = 1 - \frac{1}{2^t} < 1 \), the identity \((E_t)\) leads to
\[
c_G(g) < \frac{1}{2^t} c_{H_{2t}}(g) + 3. \quad \Box
\]

5. The group \( \mathcal{G} \) is uniformly perfect

The aim of this section is to prove Theorem 1.3.

5.1. Background material

Let \( g \in \mathcal{G}_m \). The combinatorial description of \( g \) is \((\lambda(g), \pi(g))\), where \( \lambda(g) \) is an \( m \)-dimensional vector whose coordinates are the lengths of \( I_1, \ldots, I_m \), the continuity intervals of \( g \) and \( \pi(g) \in S_m \) is the permutation on \( \{1, \ldots, m\} \) that tells how the intervals \( I_i \) are rearranged by \( g \).
We denote by \( a_i(g) \) the discontinuity points of \( g \). If \( g \) is continuous on a half-open interval \( J \), we define \( \delta_\lambda(x) := g(x) - x \), for \( x \in J \).

The translations of \( g \) are \( \delta_i(g) := \delta_{\lambda_i}(g) \), \( i = 1, \ldots, m \).

Note that \( a_i(g) \) and \( \delta_i(g) \) are related to \( (\lambda(g), \pi(g)) \) by

\[
(*) \quad a_i(g) = \sum_{k=1}^{i-1} \lambda_k(g) \quad \text{and} \quad \delta_i(g) = -\sum_{k=1}^{i-1} \lambda_k(g) + \sum_{k=1}^{\pi(i)-1} \lambda_{\pi^{-1}(k)}(g).
\]

The map \( g \) is said to be rational if all its discontinuity points are rational. It is easy to see that rational IET are periodic.

**Definition 5.1.** — Let \( m \) be a positive integer and \( \pi \in S_m \), we define a metric on \( G_{m, \pi} \) by

\[
d(f, g) = \sum_{i=1}^{m} |\lambda_i(f) - \lambda_i(g)|.
\]

**Properties 5.2.** — Let \( f \) and \( g \) be elements of \( G_{m, \pi} \). Then

1. \( d(f^{-1}, g^{-1}) = d(f, g) \).
2. \( |a_i(f) - a_i(g)| \leq d(f, g) \).
3. \( |\delta_i(f) - \delta_i(g)| \leq 2d(f, g) \).

**Proof.**

(1). — The first item is due to the fact that \( \lambda_{\pi(i)}(f^{-1}) = \lambda_i(f) \).

We deduce the remaining items from (\( * \)), indeed

(2). — \( |a_i(f) - a_i(g)| = |\sum_{k=1}^{i-1} \lambda_k(f) - \lambda_k(g)| \leq d(f, g) \)

(3). — \( |\delta_i(f) - \delta_i(g)| = |(-\sum_{k=1}^{i-1} \lambda_k(f) + \sum_{k=1}^{\pi(i)-1} \lambda_{\pi^{-1}(k)}(f)) - (-\sum_{k=1}^{i-1} \lambda_k(g) + \sum_{k=1}^{\pi(i)-1} \lambda_{\pi^{-1}(k)}(g))| \leq 2d(f, g) \).

**Lemma 5.3.** — Let \( g \in G_m \) and let \( l = |\text{Fix}(g)| \) be the Lebesgue measure of the fixed point set of \( g \). Then, there exists \( h \in G_m \) such that

\[
\text{Fix}(h \circ g \circ h^{-1}) = [0, l).
\]

In particular \( \#BP(h \circ g \circ h^{-1}) \leq 3m \).

**Proof.** — Denote by \( F_1, F_3, \ldots, F_{2p-1} \) the \( p \) ordered connected components of \( I \setminus \text{Fix}(g) \). We write \( F_i = [\alpha_i, \alpha_{i+1}] \), for \( i = 2k-1 \), \( k = 1, \ldots, p \).

Note that \( \alpha_i \in BP(g) \). Hence the connected components of \( \text{Fix}(g) \) are the possibly empty intervals \( F_0 = [0, \alpha_1] \), \( F_2p = [\alpha_{2p}, 1) \) and

\[
F_{2k} = [\alpha_{2k}, \alpha_{2k+1}] \quad \text{for} \quad k = 1, \ldots, p-1.
\]
The required map \( h \) is the IET whose combinatorial description is \((\lambda, \pi)\) with
\[
\begin{align*}
\lambda_i &= |F_i|, i = 0, \ldots, 2p \quad \text{and} \quad \pi \in \mathcal{S}(\{0, \ldots, 2p\}), \\
\pi(0) &= 0, \quad \pi(2k) = k \text{ and } \pi(2k - 1) = k + p, \quad k = 1, \ldots, p.
\end{align*}
\]
Finally we note that \( h \in \mathcal{G}_m \) since \( BP(h) \subset \{\alpha_i\} \subset BP(g) \).

\section*{5.2. Proof of Theorem 1.3}

For proving Theorem 1.3, we need the following

\textbf{Proposition 5.4.} — Let \( n \) be a positive integer and let \( f \in \mathcal{G}_m \). Then there exist two periodic elements \( p, p' \in \mathcal{G} \) such that
\[
|\text{supp}(p \circ f \circ p')| \leq \frac{1}{n} \quad \text{and} \quad \#BP(p \circ f \circ p') \leq 5m.
\]

\textit{Proof.} — Let \( n \) be a positive integer and \( f \in \mathcal{G}_{m, \pi} \). We set \( BP(f) = \{a_i, i = 1 \ldots m\}, I_i = [a_i, a_{i+1}] \) and \( BP(f^{-1}) = \{b_i, i = 1, \ldots, m\} \). Fix \( 0 < \epsilon < \frac{1}{2m} \) small enough \( (\epsilon \ll |I_i|) \). We consider \( p \in \mathcal{G}_{m, \pi^{-1}} \) a rational IET such that \( d(f^{-1}, p) \leq \frac{\epsilon}{2m} \) and \( BP(p) = \{b'_i, i = 1, \ldots, m\} \) satisfies \( b_i - \frac{\epsilon}{2m} < b'_i \leq b_i \). This map \( p \) is periodic.

\textbf{Claim 5.5.} — By construction, \( f_{\epsilon} = p \circ f \) satisfies \( \#BP(f_{\epsilon}) \leq 2m \), it is continuous on \([a_i, a_{i+1} - \frac{\epsilon}{2m}]\) and \( \partial_i := \delta_{[a_i, a_{i+1} - \frac{\epsilon}{2m}]}(p \circ f) \) satisfies \(|\partial_i| \leq \frac{\epsilon}{m}\).

Indeed, obviously \( \#BP(f_{\epsilon}) \leq \#BP(f) + \#BP(p) \leq 2m \).

For every \( x \in [a_i, a_{i+1} - \frac{\epsilon}{2m}] \), one has \( f(x) = x + \delta_{I_i}(f) \) and
\[
f(x) \in \left[b_{\pi(i)}, b_{\pi(i)+1} - \frac{\epsilon}{2m}\right] \subset \left[b'_{\pi(i)}, b'_{\pi(i)+1}\right],
\]
then \( p \circ f(x) = x + \delta_{I_i}(f) + \delta_{[\pi(i), \pi(i)+1]}(p) \).

Since \( d(f^{-1}, p) \leq \frac{\epsilon}{2m} \), one has:
\[
\frac{\epsilon}{m} \geq |\delta_{[\pi(i), \pi(i)+1]}(p) - \delta_{[\pi(i), \pi(i)+1]}(f^{-1})| = |\delta_{[\pi(i), \pi(i)+1]}(p) + \delta_{I_i}(f)|,
\]
therefore \(|\partial_i| = |p \circ f(x) - x| = |\delta_{I_i}(f) + \delta_{[\pi(i), \pi(i)+1]}(p)| \leq \frac{\epsilon}{m}|.
This ends the proof of the claim which is summarized by the following picture.

We turn now on to the proof of Proposition 5.4. Let \( i \in \{1, \ldots, m\} \).

If \( \partial_i = 0 \), we set \( R_i = 1d \).

In the case that \( \partial_i > 0 \), we define \( R_i \) to be the finite order restricted rotation of support \([a_i, a_i + r_i \partial_i]\) and of angle \( \partial_i \), where \( r_i \) is the greatest integer such that \( a_i + r_i \partial_i \leq \min\{a_{i+1} - (\frac{\epsilon}{2m} - \partial_i), a_{i+1}\} \).

By definition \( R_i \) and \( f_\epsilon \) coincide on the interval \([a_i, a_i + (r_i - 1) \partial_i]\) and \( ||a_i + r_i \partial_i, a_{i+1}|| \leq \frac{\epsilon}{m} \). Indeed, \( f_\epsilon \) is continuous on \([a_i, a_i + (r_i - 1) \partial_i]\), since

\[
a_i + (r_i - 1) \partial_i = a_i + r_i \partial_i - \partial_i \leq a_{i+1} - \left( \frac{\epsilon}{2m} - \partial_i \right) - \partial_i = a_{i+1} - \frac{\epsilon}{2m}.
\]

In addition, by the maximality of \( r_i \), either \( a_i + (r_i + 1) \partial_i \) is greater

- than \( a_{i+1} - (\frac{\epsilon}{2m} - \partial_i) \) and it follows that
  \[
  ||[a_i + r_i \partial_i, a_{i+1}]|| = a_{i+1} - (a_i + r_i \partial_i) < \partial_i + \left( \frac{\epsilon}{2m} - \partial_i \right) = \frac{\epsilon}{2m}
  \]

- or than \( a_{i+1} \) and then \( ||[a_i + r_i \partial_i, a_{i+1}]|| = a_{i+1} - (a_i + r_i \partial_i) < \partial_i \leq \frac{\epsilon}{m} \).

The same argument remains valid for negative \( \partial_i \) by using non positive integers \( r_i \).

Finally, the map \( g := f_\epsilon \circ \prod_{1}^{m} R_i^{-1} \) satisfies \( \#BP(g) \leq 5m \) because \( \#BP(R_i) \leq 3 \).

Since \( \text{supp}(R_i) \subset [a_i, a_{i+1}] \), the supports of the \( R_i \)'s are disjoints and \( p' = (\prod_{1}^{m} R_i)^{-1} \) is periodic and it is also a commutator in \( G \), according to Proposition 2.5.

But \( g_{|_{R_i([a_i, a_i + (r_i - 1) \partial_i])}} = 1d \) and therefore

\[
|\text{supp}(g)| \leq 1 - \sum_{i=1}^{m} ||[a_i, a_i + (r_i - 1) \partial_i]||
\]

\[
\leq 1 - \sum_{i=1}^{m} \left(||[a_i, a_{i+1}]|| - \left( \partial_i + \frac{\epsilon}{m} \right) \right) \leq 2\epsilon \leq \frac{1}{n}. \quad \square
\]

We turn now on to the proof of Theorem 1.3.

We first consider an IET \( f \in G_{m, \pi} \) viewed as an element of \( \bar{G} \). Let \( t \in N^* \).
Applying Proposition 5.4 to \( f \) and \( n = 2^t \), we get that there exist two periodic elements \( p, p' \in G \) such that the support of \( g = p \circ f \circ p' \in G_{5m} \) has measure less than or equal to \( \frac{1}{2^t} \).

By Lemma 5.3, the map \( g \) is conjugated to an element \( g' \) of \( H_{2^t} \) for which \( \# \text{BP}(g') \leq 15m \). Since \( p \) and \( p' \) are periodic and \( g \) and \( g' \) are conjugated, we have

\[
c_{G}(f) \leq c_{G}(g) + 2 = c_{G}(g') + 2.
\]

Then by Proposition 4.3(2),

\[
c_{G}(f) < \frac{1}{2^t} c_{H_{2^t}}(g') + 5.
\]

As \( \# \text{BP}(g'|_{\left[1 - \frac{1}{2^t}, 1\right)}) \leq \# \text{BP}(g') \), Remark 1.1 and Proposition 3.4 imply that

\[
c_{H_{2^t}}(g') \leq 15m - 1.
\]

Finally, for any \( t \in \mathbb{N}^* \) one has

\[
c_{G}(f) < \frac{15m - 1}{2^t} + 5
\]

and choosing \( t \) large enough, we obtain

\[
c_{G}(f) \leq 5.
\]

Thus we get \( c_{G}(f) \leq 5 \), for any \( f \in G \).

For the general case, we consider \( F \in \overline{G} \). According to Lemma 3.3, the map \( F \) can be decomposed as the product of an involution that is a commutator and an element of \( G \). Therefore, we have proved the required inequality \( c_{G}(F) \leq 1 + 5 = 6 \), for any \( F \in \overline{G} \). \( \square \)

6. Conditions for uniform perfectness of \( G \)

In this section we give two sufficient conditions for \( G \) to be uniformly perfect.

6.1. The commutator length is bounded when fixing the number of discontinuity points

We prove the following statement that directly implies Theorem 1.4.

**Theorem 6.1.** — If for any positive integer \( m \), it holds that \( C_m(G) := \text{supp}\{c_{G}(g), g \in [G, G] \cap G_m\} \) is finite, then \( G \) is uniformly perfect and \( c(G) \leq 5 \).
Proof. — Let \( f \in [G, G] \cap G_m \) and \( t \in \mathbb{N} \). Proposition 5.4 and Lemma 5.3 with \( n = 2^t \) show that there exist two periodic elements \( p', p \in G \) such that \( g = p \circ f \circ p' \in G_{5m} \) is conjugated to an element \( g' \) of \( H_{2t} \cap G_{15m} \). By Proposition 2.5, \( p \) and \( p' \) are commutators then \( g \in [G, G] \). Moreover, \( [G, G] \) is normal so \( g' \in H_{2t} \cap [G, G] \).

Therefore, according to Proposition 4.3

\[
c_G(f) \leq c_G(g) + 2 = c_G(g') + 2 < \frac{1}{2^t} c_{H_{2t}}(g') + 5.
\]

As \( c_{H_{2t}}(g') \leq C_{15m}(G) \), one has for any \( t \in \mathbb{N}^* \)

\[
c_G(f) < \frac{C_{15m}(G)}{2^t} + 5.
\]

Choosing \( t \) large enough, we get \( c_G(f) \leq 5 \). \( \square \)

6.2. The commutator length is bounded when prescribing the arithmetic

Let \( p \in \mathbb{N}^* \) and \( \alpha = (\alpha_1, \ldots, \alpha_p) \in [0, 1)^p \) such that \( \alpha_1 \notin \mathbb{Q} \).

6.2.1. Background material

We denote by \( \Delta_{\alpha} \) the abelian subgroup of \( \mathbb{R} \) generated by \( \alpha_1, \ldots, \alpha_p \) and 1. Note that the condition \( \alpha_1 \notin \mathbb{Q} \) insures that \( \Delta_{\alpha} \) is dense in \( [0, 1) \).

Let \( J \) be a half-open interval with endpoints in \( \Delta_{\alpha} \).

**DEFINITION 6.2.** — We set \( \Gamma_{\alpha} := \{ g \in G : BP(g) \subset \Delta_{\alpha} \} \) and \( \Gamma_{\alpha}(J) := \{ g \in G_J : BP(g) \subset \Delta_{\alpha} \} \).

It is plain that any \( g \in G \) is either periodic or belongs to some \( \Gamma_{\alpha} \). Indeed, if \( g \) is not periodic then its length vector \( \lambda \) has at least one irrational coordinate and \( \alpha \) is obtained from \( \lambda \) by permutation.

Note that \( \Gamma_{\alpha}(J) \) is the set of all maps \( g \in G_J \) whose extensions to \( I \) by the identity map belong to \( \Gamma_{\alpha} \). But \( \Gamma_{\alpha}(J) \) does not coincide with the set obtained by conjugating \( \Gamma_{\alpha} \) through the homothecy that carries \( J \) into \( I \). For \( J = [c, d) \), the last set is \( \{ g \in G_J : BP(g) \subset c + \frac{\Delta_{\alpha}}{d-c} \} \). For this reason, the properties of \( \Gamma_{\alpha}(J) \) are not direct consequences of the ones of \( \Gamma_{\alpha} \).

**PROPERTIES 6.3.** — Let \( g \in \Gamma_{\alpha}(J) \) and \( I_i \) be its continuity intervals.

1. The lengths of the \( I_i \) and the translations of \( g \) belong to \( \Delta_{\alpha} \).
2. \( \Gamma_{\alpha}(J) \) is a subgroup of \( G_J \).
(3) The endpoints of the connected components of $\text{Fix}(g)$ are in $\Delta_\alpha$.

Proof.

(1). — The endpoints of the $I_i$ are the discontinuity points of $g$ and the left endpoint of $J$. Therefore if $g \in \Gamma_\alpha(J)$ any length $\lambda_i = |I_i|$ belongs to $\Delta_\alpha$. The translations of $g$ also belong to $\Delta_\alpha$, as linear combinations of the $\lambda_i$’s with coefficients in $\{-1, 0, 1\}$.

(2). — According to (1), any $f \in \Gamma_\alpha(J)$ preserves $\Delta_\alpha$. Therefore the relations $BP(f^{-1}) = f(BP(f))$ and $BP(f_1 \circ f_2) \subset BP(f_2) \cup f_2^{-1}(BP(f_1))$ imply that $\Gamma_\alpha(J)$ is stable by taking inverse and composite.

(3). — From the definition of $\text{Fix}(g)$, it follows that every endpoint of a connected component of $\text{Fix}(g)$ is a discontinuity point of $g$. □

Before stating our last theorem, we give

Definition 6.4. — We say that $\mathcal{G}$ has partial uniform perfectness if for any $p \in \mathbb{N}^*$ and $\alpha \in [0, 1)^p$ it holds that $C_\alpha(\mathcal{G}) := \sup \{c_\mathcal{G}(g), g \in [\mathcal{G}, \mathcal{G}] \cap \Gamma_\alpha\}$ is finite.

Theorem 6.5. — If $\mathcal{G}$ has partial uniform perfectness then $\mathcal{G}$ is uniformly perfect.

A consequence of Theorems 6.1 and 6.5 is

Corollary 6.6. — If $\mathcal{G}$ has partial uniform perfectness then $c(\mathcal{G}) \leq 5$.

The main tool for the proof of Theorem 6.5 is

Proposition 6.7. — Let $f \in \Gamma_\alpha$. Let $n$ be a positive integer and set $s_n = \lfloor \ln(n) / \ln(1.25) \rfloor + 1$.

Then there exist $g_n \in H_n \cap \Gamma_\alpha$, a map $h \in \Gamma_\alpha$ and $s_n$ involutions $i_j \in \Gamma_\alpha$, $j = 1, 2, \ldots, s_n$ such that $f = i_1 \circ \cdots \circ i_{s_n} \circ (h \circ g_n \circ h^{-1})$.

For proving this proposition we use the following

Lemma 6.8. — Let $\epsilon \in (0, 1)$ and $J$ be a half-open interval with endpoints in $\Delta_\alpha$. If $f \in \Gamma_\alpha(J)$ then there exists an involution $i \in \Gamma_\alpha(J)$ such that

$$|\text{Fix}(i \circ f)| \geq |\text{Fix}(f)| + \frac{|J| - |\text{Fix}(f)|}{5} (1 - \epsilon).$$

Proof. — Let $f \in \Gamma_\alpha(J)$, $BP(f) = \{0 = a_1, \ldots, a_m\}$ and set $a_{m+1} = 1$.

Case 1: $\text{Fix}(f) = \emptyset$. — Fix $\delta \in \Delta_\alpha$ such that $0 < \delta < \min\{\frac{|J|}{m}; |f(x) - x|, x \in I\}$.

For every $j \in \{1, \ldots, m - 1\}$, we consider the unique integer $n_j$ such that $(n_j - 1)\delta \leq |[a_j, a_{j+1}]| < n_j\delta$. It holds that $[a_j, a_{j+1}]$ is the union of
Let $(n_j - 1)$ intervals of length $\delta$ and an eventually empty interval $F_j$ of length less than $\delta$.

Therefore $J$ can be decomposed as a finite union of pairwise disjoint half-open intervals $I_1, \ldots, I_n$ and $F_1, \ldots, F_m$ such that

- $f$ is continuous on these intervals,
- $|I_j| = \delta$ for $j = 1, \ldots, n$ and $|F_k| < \delta$ for $k = 1, \ldots, m$.

It follows that $n\delta + \sum |F_k| = |J|$. Since for any $x$ it holds that $|f(x) - x| > \delta$, one has $f(I_j) \cap I_j = \emptyset$. Therefore there exists an involution $i_1$ of support $I_1 \cup f(I_1)$ such that $i_1|_{I_1} = f|_{I_1}$ and then $i_1 \circ f|_{I_1} = \text{Id}|_{I_1}$. Now, we want to construct a similar involution $i_2$ on a second interval $I_{p_2}$ so that $i_1$ and $i_2$ have disjoint supports. This can be done if and only if $(I_{p_2} \cup f(I_{p_2})) \cap (I_1 \cup f(I_1)) = \emptyset$. This means that

$I_{p_2} \subseteq I \setminus (I_1 \cup f(I_1) \cup f^{-1}(I_1))$.

Consequently, such an interval $I_{p_2}$ and its corresponding involution $i_2$ with support $I_{p_2} \cup f(I_{p_2})$ exist provided that

$$I' \setminus (I_1 \cup D(f(I_1) \cup f^{-1}(I_1))) \neq \emptyset,$$

where $I' = I \setminus \bigcup F_k$ and $D(K) = \bigcup\{k| K \cap I_k \neq \emptyset\} I_k$.

As any half-open interval of length $\delta$ meets at most two intervals $I_k$, the condition (6.1) means that $n > 5$.

By induction, we can define $s$ involutions $i_j$ with disjoint supports $I_{p_j} \cup f(I_{p_j})$ provided that

$$I' \setminus (I_1 \cup \ldots I_{p_{s-1}}) \cup D(f(I_1 \cup \ldots I_{p_{s-1}})) \cup f^{-1}(I_1 \cup \ldots I_{p_{s-1}})) \neq \emptyset.$$

That is $n > 5(s - 1)$.

Let $s$ be the largest integer such that $n > 5(s - 1)$, we can construct the involutions $i_j$, $j = 1, \ldots, s$ but $n \leq 5s$.

By the definition of $i_j$, the map $g = i_s \ldots i_1 \circ f$ satisfies

$$g|_{I_1 \cup I_{p_2} \cup \ldots \cup I_{p_s}} = \text{Id}|_{I_1 \cup I_{p_2} \cup \ldots \cup I_{p_s}},$$

then

$$|\text{Fix}(g)| \geq \sum_{j=1}^{s} |I_{p_j}| = s \delta = \frac{s}{n}(|J| - \sum |F_k|)$$

$$\geq \frac{1}{5}(|J| - \sum |F_k|)$$

$$\geq \frac{1}{5}(|J| - m \delta) \geq \frac{1}{5}(|J| - |J| \epsilon).$$
In conclusion, since $i_j$ have disjoint supports, the map $i = i_s \ldots i_1$ is an involution and $|\text{Fix}(i \circ f)| \geq \frac{|J|}{5}(1 - \epsilon)$, this is the desired conclusion for $|\text{Fix} f| = 0$.

It remains to prove that $i \in \Gamma_\alpha$. Since the $a_i$ and $\delta$ are in $\Delta_\alpha$, the endpoints of $I_i$ and $f(I_i)$ also belong to $\Delta_\alpha$. Combining this with the fact that the discontinuity points of the involutions $i_j$ are endpoints of $I_i$ or $f(I_i)$, we get that $BP(i_j) \subset \Delta_\alpha$, for $j = 1, \ldots, s$. Therefore, by definition, the maps $i_j \in \Gamma_\alpha$ then so does $i$.

Case 2: $\text{Fix}(f) \neq \emptyset$. — We set $J = [c, d]$.

As the endpoints of the connected components of $\text{Fix}(f)$ belong to $\Delta_\alpha$, it holds that $a = |\text{Fix}(f)| \in \Delta_\alpha$. Therefore a slight adaptation of Lemma 5.3 to $f \in \Gamma_\alpha(J)$, shows that there exists $h \in \Gamma_\alpha(J)$ such that $\text{Fix}(h \circ f \circ h^{-1}) = [c, c + a]$.

Let $f_1 \in \Gamma_\alpha([c + a, d])$ be the restriction of $h \circ f \circ h^{-1}$ to $[c + a, d]$. By construction, $\text{Fix}(f_1) = \emptyset$ hence Case 1 applies to $f_1$ and provides an involution $j_1 \in \Gamma_\alpha([c + a, d])$ such that

$$|\text{Fix}(j_1 \circ f_1)| \geq \frac{|J|}{5}(1 - \epsilon).$$

Let $j \in \Gamma_\alpha(J)$ be the involution of $J$ defined by $j(x) = j_1(x)$ if $x \in [c + a, d)$ and $j(x) = x$ if $x \in [c, c + a)$. We have

$$|\text{Fix}(j \circ h \circ f \circ h^{-1})| \geq |[c, c + a]| + |\text{Fix}(j_1 \circ f_1)| \geq a + \frac{|J|}{5}(1 - \epsilon).$$

In addition, as $h$ preserves lengths, we have

$$|\text{Fix}(j \circ h \circ f \circ h^{-1})| = |\text{Fix}(h^{-1} \circ (j \circ h \circ f \circ h^{-1}) \circ h)| = |\text{Fix}((h^{-1} \circ j \circ h) \circ f)|.$$

Setting $i = h^{-1} \circ j \circ h$, we get

$$|\text{Fix}(i \circ f)| \geq a + \frac{|J|}{5}(1 - \epsilon),$$

which completes the proof.

We turn now on to the proof of Proposition 6.7.

Let $\epsilon \in (0, 1)$ small enough and such that $\frac{1}{5}(1 - \epsilon) \in \Delta_\alpha$. Consider $f \in \Gamma_\alpha$ and set $L_0 = |\text{Fix}(f)|$.

Applying Lemma 6.8 to $f$, there exists an involution $i_1 \in \Gamma_\alpha$ such that

$$|\text{Fix}(i_1 \circ f)| \geq L_0 + \frac{1 - L_0}{5}(1 - \epsilon) = \phi(L_0) := L_1,$$

where $\phi(x) := x + \frac{1-x}{5}(1 - \epsilon) = \frac{4+\epsilon}{5}(x - 1) + 1$ is a direct affine map whose fixed point is 1.
We now apply this argument again, with $f$ replaced by $i_1 \circ f$, to obtain an involution $i_2 \in \Gamma_\alpha$ such that
\[
|\text{Fix}(i_2 \circ i_1 \circ f)| \geq |\text{Fix}(i_1 \circ f)| + \frac{1 - |\text{Fix}(i_1 \circ f)|}{5}(1 - \epsilon)
\]
\[
\geq \phi(|\text{Fix}(i_1 \circ f)|)
\]
\[
\geq \phi(L_1) = \phi^2(L_0).
\]
Repeating this process $s$ times, we get $s$ involutions $i_k \in \Gamma_\alpha$ such that
\[
|\text{Fix}(i_s \circ \cdots \circ i_1 \circ f)| \geq \phi^s(L_0).
\]
We now prove that $\phi^s(L_0) \geq 1 - \frac{1}{n}$ provided that $s \geq s_n = \left\lceil \frac{\ln(n)}{\ln(1.25)} \right\rceil + 1$.
In order to get this inequality, we are looking for integers $s$ such that
\[
\phi^s(L_0) = \left(\frac{4 + \epsilon}{5}\right)^s (L_0 - 1) + 1 \geq 1 - \frac{1}{n},
\]
that is
\[
-\left(\frac{4 + \epsilon}{5}\right)^s (1 - L_0) \geq -\frac{1}{n}.
\]
Using that $L_0 \geq 0$, it suffices to determine $s$ satisfying
\[
\left(\frac{4 + \epsilon}{5}\right)^s \leq \frac{1}{n}
\]
\[
s \ln \left(\frac{4 + \epsilon}{5}\right) \leq \ln \left(\frac{1}{n}\right) = -\ln(n)
\]
\[
s \geq \frac{\ln(n)}{\ln(\frac{4 + \epsilon}{5})}
\]
Therefore, we can take $s = \left\lceil \frac{\ln(n)}{\ln(\frac{4 + \epsilon}{5})} \right\rceil + 1$. In addition, as $\frac{\ln(n)}{\ln(\frac{4}{5})} \notin \mathbb{N}$, we have
\[
\left\lceil \frac{\ln(n)}{\ln(\frac{4 + \epsilon}{5})} \right\rceil = \left\lceil \frac{\ln(n)}{\ln(\frac{4}{5})} \right\rceil
\]
for $\epsilon > 0$ small enough.
Finally $i_s \circ \cdots \circ i_1 \circ f \in \Gamma_\alpha$ has a fixed point set of length at least $1 - \frac{1}{n}$ so it is conjugated to an element of $H_n$ by some $h \in \Gamma_\alpha$, by a slight adaptation of Lemma 5.3 to $f \in \Gamma_\alpha$. \hfill $\Box$

6.2.2. Proof of Theorem 6.5

We consider $g_1 \in \Gamma_\alpha \cap [G, G]$ that realizes $C_\alpha(G)$. By Proposition 6.7 with $n = 2$ and thus $s_2 = \left\lceil \frac{\ln(2)}{\ln(1.25)} \right\rceil + 1 = 4$, there exist $g_2 \in H_2 \cap \Gamma_\alpha$, $h \in \Gamma_\alpha$ and four involutions $i_1, \ldots, i_4$ such that
\[
g_1 = i_1 \circ i_2 \circ i_3 \circ i_4 \circ (hg_2h^{-1}).
\]
We can now estimate $C_\alpha(\mathcal{G})$. By Proposition 2.5 and the normality of $[\mathcal{G}, \mathcal{G}]$,

$$C_\alpha(\mathcal{G}) = c_\mathcal{G}(g_1) \leq 4 + c_\mathcal{G}(g_2).$$

According to Lemma 4.2, we have

$$c_\mathcal{G}(g_2) \leq \frac{1}{2} c_{H_2}(g_2) + \frac{3}{2}. $$

Using Remark 1.1, the group $H_2$ inherits the partial uniform perfectness of $\mathcal{G}$ and this implies that for any $g \in [H_2, H_2] \cap \Gamma_2$, one has $c_{H_2}(g) \leq C_\alpha(\mathcal{G})$. In particular since $\Gamma_\alpha$ is a subgroup of $\Gamma_{\frac{1}{2}}$, we have $c_{H_2}(g_2) \leq C_\alpha(\mathcal{G})$. Hence,

$$C_\alpha(\mathcal{G}) \leq 4 + \frac{1}{2} C_\alpha(\mathcal{G}) + \frac{3}{2}$$

$$\frac{1}{2} C_\alpha(\mathcal{G}) \leq 4 + \frac{3}{2} = \frac{11}{2}$$

$$C_\alpha(\mathcal{G}) \leq 11.$$ 

Finally, as any IET $g$ is either periodic or it belongs to some $\Gamma_\alpha$, we get that $c_\mathcal{G}(g) \leq 11$ for all $g \in [\mathcal{G}, \mathcal{G}]$. It means that $\mathcal{G}$ is uniformly perfect. □

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