Lifshitz Scaling, Microstate Counting from
Number Theory and Black Hole Entropy

Dmitry Melnikov, Fábio Novaes, Alfredo Pérez and Ricardo Troncoso

Abstract: Non-relativistic field theories with anisotropic scale invariance in (1+1)–d are typically characterized by a dispersion relation $E \sim k^z$ and dynamical exponent $z > 1$. The asymptotic growth of the number of states of these theories can be described by an extension of Cardy formula that depends on $z$. We show that this result can be recovered by counting the partitions of an integer into $z$th powers, as proposed by Hardy and Ramanujan a century ago. This gives a novel relationship between the characteristic energy of the dispersion relation with the cylinder radius and the ground state energy. We show how these results are connected to free bosons with Lifshitz scaling and the quantum Benjamin-Ono ($BO_2$) integrable system, relevant for the AGT correspondence. We provide boundary conditions for which the reduced phase space of Einstein gravity with a couple of $U(1)$ fields on AdS$_3$ is described by the $BO_2$ equations. This suggests that the phase space can be quantized in terms of quantum $BO_2$ states. In the semiclassical limit, the ground state energy of $BO_2$ coincides with the energy of global AdS$_3$, and the Bekenstein-Hawking entropy for BTZ black holes is recovered from the anisotropic extension of Cardy formula.
1 Introduction

Non-relativistic field theories in (1 + 1)-dimensions, possessing anisotropic Lifshitz scaling of the form

\[ t \to \lambda^z t, \quad x \to \lambda x, \]  

are typically characterized by modes with a dispersion relation \( E \sim k^z \) and entropy \( S \sim E^{\frac{1}{z-1}} \), with dynamical exponent \( z > 1 \). They have been extensively studied in the context of the holographic AdS/CMT correspondence, see e.g. [1–4]. A number of condensed matter systems are known to enjoy this type of scaling. For instance, the quantum Hall fluid has been proposed to possess a non-linear dispersion with \( z = 2 \) [5, 6], while the (1+1)-dimensional Bose gas in cold atom systems can be related to \( z = 3 \) dispersion[7]. The non-relativistic effective field theory description of such systems is invariant under the Lifshitz group, which in (1+1)-D is generated by translations in space and time and the anisotropic scale transformation (1.1). At finite temperature, chiral movers with Lifshitz scaling can be described in terms of the torus partition function

\[ Z[\tau; z] = \sum_E \rho_z(E) e^{2\pi i \tau E}, \]  

where \( \tau = i\beta/(2\pi \ell) \) is the modular parameter, defined in terms of the inverse temperature \( \beta \) and the radius of the cylinder \( \ell \), and \( \rho_z(E) \) is the density of states at fixed energy \( E \). As
explained in [8, 9], assuming modular invariance of the partition function in the anisotropic case implies that

\[ Z[\tau; z] = Z[i^{1+\frac{k}{2}}\tau^{-\frac{k}{2}}; z^{-1}], \]

which connects the spectrum at high and low temperatures and reduces to the well-known modular invariance in CFT\(_2\) for \(z = 1\) [10]. If the spectrum has a gap with a non-vanishing ground state energy given by \(-E_0[z]\), then the asymptotic growth of the number of states at fixed energy \(E \gg |E_0[z]|\) can be obtained from the Cauchy transform of (1.3) in the steepest descent approximation

\[ \rho_z(E) \approx \exp \left[ 2\pi \ell (1 + z) \left( \frac{|E_0[z^{-1}]|}{z} \right)^{\frac{1}{1+z}} E^{\frac{1}{1+z}} \right], \]

and hence

\[ S = \log \rho_z(E) = 2\pi \ell (1 + z) \left( \frac{|E_0[z^{-1}]|}{z} \right)^{\frac{1}{1+z}} E^{\frac{1}{1+z}}, \]

stands for the leading term of the microcanonical entropy. Note that for \(z = 1\), the entropy reduces to the well-known Cardy formula in CFT\(_2\) [11]. The logarithmic correction to (1.5) was discussed in [12] (see also [13]).

2 Non-Relativistic Microstate Counting from Number Theory

In this section, we show that for (1+1)-dimensional weakly-coupled systems at high temperatures on a cylinder of radius \(\ell\), the leading term of the asymptotic growth of the number of states \(\rho_z (E)\) in (1.5) agrees with the asymptotic growth of the number of partitions of an integer \(N\) into \(z\)-th powers \(p_z(N)\). This is true provided that the ground state energy and the radius of the cylinder are precisely linked with the characteristic energy of the quasiparticles. Indeed, if the interactions are weak enough so that at high temperature the system behaves as a gas of free quasiparticles, as pointed out in the introduction, the dispersion relation has to be of the form

\[ E_n = \varepsilon_z (k_n \ell)^{z} = \varepsilon_z n^z, \]

where \(k_n = n/\ell\) is the quasiparticle momentum, \(n\) is a non-negative integer and \(\varepsilon_z\) stands for the characteristic energy of the modes. The total energy is then given by

\[ E = \sum_i E_{n_i} = \varepsilon_z \sum_i n_i^z = \varepsilon_z N. \]

Therefore, assuming the ordering \(n_1 \geq n_2 \geq \ldots \geq 0\) to count only indistinguishable configurations, the number of states with fixed energy \(E\) corresponds to the combinatorial problem
of finding the number of power partitions \( p_z(N) \) for fixed \( N = \sum_i n_i^z = E/\varepsilon_z \). Quite remarkably, this problem was solved in 1918 by Hardy and Ramanujan [14]. Indeed, in one of the last formulas of their paper, one finds that for large \( N \), the leading term of the asymptotic growth of power partitions is given by

\[
p_z(N) \approx \exp \left[ (1 + z) \left( \frac{\Gamma(1 + 1/z) \zeta(1 + 1/z)}{z} \right)^{1/z} N^{1/z} \right]. \tag{2.2}
\]

Surprisingly, for \( z > 1 \) the result was actually a conjecture, proven later by Wright in 1934 using generalized Bessel functions [15]. A simplified proof has been recently given in [16] for \( z = 2 \), and extended to \( z \geq 2 \) in [17], both using the Hardy-Littlewood circle method. Hence, at high temperature, the leading term of the entropy can be read from (2.2), and it is given by

\[
S = \log p_z(N) = (1 + z) \left( \frac{\Gamma(1 + \frac{1}{z}) \zeta(1 + \frac{1}{z})}{z} \right)^{1/z} \left( \frac{E}{\varepsilon_z} \right)^{1/z}. \tag{2.3}
\]

One then concludes that at high temperature, the asymptotic growth of the number of states obtained from anisotropic modular invariance, given by \( p_z(E) \) in (1.4), agrees with the one from number theory given by \( p_z(N) \) in (2.2), provided that the characteristic energy of the dispersion relation is related to the radius of the cylinder and the non-vanishing ground state energy according to

\[
\left( \frac{\varepsilon_1}{z} \right)^z = \frac{\Gamma(1 + z)\zeta(1 + z)}{(2\pi \ell)^{1+z}} \frac{1}{|E_0[z]|}. \tag{2.4}
\]

Expressing the leading term of the entropy in terms of the characteristic energy as in (2.3) possesses an advantage, since the formula holds even if the ground state energy vanishes. Note that, for \( z = 1 \), the characteristic energy is related to the effective central charge as

\[
c_{\text{eff}} = (\varepsilon_1 \ell)^{-1},
\]

so that according to (2.4), the energy of the ground state acquires the expected form for chiral movers

\[
|E_0[1]| = \frac{c_{\text{eff}}}{24\ell} = \frac{1}{24\ell^2 \varepsilon_1}. \tag{2.5}
\]

Besides, one of the advantages of expressing the entropy as in (1.5) is that its value can be directly obtained from the ground state energy, which certainly helps in cases where the microscopic counting cannot be explicitly performed.

It is also worth pointing out that the asymptotic growth of the number of states in (1.4) was recovered from an anisotropic extension of modular invariance which actually holds for real values of \( z > 0 \). Therefore, by virtue of the equivalence between (1.5) and (2.3), one

\[\text{As pointed out in [9, 12, 18], in the limit } z \to 0 \text{ there is a very intriguing link with the results in [19] about “soft hair” in the sense of Hawking, Perry and Strominger [20, 21].}\]
is naturally led to conjecture that the expression for the asymptotic growth of the power partitions of Hardy and Ramanujan can actually be extended to hold for positive real values of \( z \). Indeed, very recent results in number theory give support to this conjecture, since it has already been proved for \( z = \frac{1}{2} \) in [22] and for \( 0 < z < 1 \) in [23]. Remarkably, Li and Chen in [23] have also arrived to the same conjecture, but following a completely different line of reasoning.

3 Free Boson with Lifshitz Scaling

In order to test the results of the previous section, it is instructive to consider the simple case of a free boson with Lifshitz scaling, described by

\[
I = \frac{1}{2} \int dtdx \left[ (\partial_t \varphi)^2 - \sigma^{2(z-1)} (\partial_x^z \varphi)^2 \right],
\]

(3.1)

with \( 0 \leq x < 2\pi \ell \). Here \( \sigma \) is an arbitrary parameter with unit of length, and the units have been chosen such that, for \( z = 1 \), the speed of light is unity. The dispersion relation of the modes then reads

\[
E_n = \pm \varepsilon_z |n|^z, \quad \varepsilon_z = \frac{\sigma^{z-1}}{\ell^z},
\]

(3.2)

so that the characteristic energy of left and right movers matches eq. (2.1). The Hamiltonian of chiral movers can then be written as

\[
H_z = 2 \frac{\sigma^{z-1}}{\ell^z} \sum_{n>0} n^{z-1} a_{-n} a_n - E_0[z],
\]

(3.3)

where \([a_n, a_m] = \frac{\pi}{2} \delta_{n,-m}\), and by virtue of \( \zeta \)-function regularization, the ground state energy is

\[
E_0[z] = -\frac{1}{2} \frac{\sigma^{z-1}}{\ell^z} \zeta(-z).
\]

(3.4)

In the case of even values of \( z \) the ground state energy vanishes, and hence the entropy of chiral movers can be obtained from (2.3) with \( \varepsilon_z \) given by (3.2).

When \( z \) takes odd values, the ground state energy (3.4) becomes non-trivial, and remarkably, the duality relation between the characteristic energy \( \varepsilon_z \), the energy of the ground state \( E_0[z] \) and the radius of the cylinder \( \ell \) in (2.4) becomes identically fulfilled by the reflection property of the Riemann \( \zeta \)-function

\[
\zeta(-z) = -\frac{1}{2 \pi^{1+z}} \sin \left( \frac{\pi z}{2} \right) \Gamma(1+z) \zeta(1+z).
\]

Therefore, for odd values of \( z \), the leading term of the entropy can be either obtained from (2.3) with \( \varepsilon_z = \frac{\sigma^{z-1}}{\ell^z} \), or equivalently by virtue of (1.5) with \( E_0[z] \) given by (3.4).
Interestingly, according to number theory, the sequence of power partitions possesses the following generating function (see e.g. [14, 17])

$$\sum_{N=0}^{\infty} p_z(N)q^N = \prod_{n=1}^{\infty} \frac{1}{1 - q^{n^z}},$$

and hence, the partition function for free bosons with Lifshitz scaling acquires the form

$$Z[\tau; z] = \mathcal{N}(\tau; z) \left| q^{-E_0[z]} \prod_{n=1}^{\infty} \frac{1}{1 - q^{n^z}} \right|^2,$$

(3.5)

with $q = e^{2\pi i \tau}$. Here $\mathcal{N}(\tau; z)$ stands for a non-exponential one-loop correction, coming from the contribution of zero modes, so that it does not modify the leading high temperature asymptotics of $Z[\tau; z]$. Subleading corrections and further details about its precise form in connection to the anisotropic modular invariance will be addressed in [24].

Note that for $z = 1$, the action (3.1) reduces to the one of a free boson in CFT$_2$, while the energy of the ground state energy is recovered from (3.4) to be given by $-E_0[1] = -\frac{1}{24\ell}$, in agreement with the known result for chiral movers. Thus, the duality relation in (2.4) reduces to (2.5), which is consistent with the fact that $c_{eff} = 1$. The suitable factor of the partition function in this case is given by $\mathcal{N}(\tau; 1) = \text{Im}(\tau)^{-1/2}$ (see e.g. [10]).

4 Microstate Counting and the Quantum Benjamin-Ono$_2$ Hierarchy

In the previous section, we discussed a simple free bosonic model with Lifshitz scaling $z$ and how its spectrum is connected to the partitions of integers into $z$-th powers. Here we describe a quantum integrable system, with an infinite set of conserved quantities, presenting Lifshitz scaling in the semiclassical limit, the quantum Benjamin-Ono$_2$ (BO$_2$) model. This will give an interesting link between the semiclassical limit of quantum systems, microstate counting of models with Lifshitz scaling and gravitation on AdS$_3$.

4.1 Classical Formulation of the BO$_2$ Hierarchy

The Benjamin-Ono equation describes deep inner waves in a stratified fluid, being then a counterpart of the KdV equation for propagation in a shallow depth channel [25]. Both equations possess solitonic solutions and an infinite set of commuting conserved quantities, so that they belong to a hierarchy of integrable systems.

The Benjamin-Ono$_2$ (BO$_2$) hierarchy is a generalization of these integrable systems [26–28], describing non-linear perturbations and solitonic waves on the edge of the quantum Hall
fluid [5, 6, 29, 30], as well as in further applications of one-dimensional condensed matter systems [5, 30, 31]. It is defined in terms of two dynamical fields, $\mathcal{L}(t,x)$ and $\mathcal{J}(t,x)$, which we assume to be $2\pi$-periodic in the $x$ coordinate.

The BO$_2$ hierarchy also possesses solitonic solutions and an infinite set of commuting integrals of motion $H_z = \frac{c}{12\pi^2} \int H_z(x) dx$, with $z \in \mathbb{Z}_{>0}$, so that the field equations of the $z$-th representative can be written in Hamiltonian form as

$$\dot{\mathcal{L}} = \{\mathcal{L}, H_z\}, \quad \dot{\mathcal{J}} = \{\mathcal{J}, H_z\},$$

where the Poisson brackets are given by

$$\{\mathcal{L}(x), \mathcal{L}(y)\} = -\frac{48\pi}{c} \mathcal{D}_z \delta(x-y), \quad \{\mathcal{L}(x), \mathcal{J}(y)\} = 0,$$

$$\{\mathcal{J}(x), \mathcal{J}(y)\} = \frac{6\pi}{c} \partial_x \delta(x-y),$$

and $\mathcal{D}_z = \partial_x \mathcal{L} + 2\mathcal{L} \partial_x - 2\partial_x^3$. The first three densities $H_z$ read

$$H_1 = \frac{1}{4} \mathcal{L} - \mathcal{J}^2, \quad H_2 = \frac{2}{3} \left[\frac{1}{4} \mathcal{L} \mathcal{J} - \frac{1}{3} \mathcal{J}^3 - \mathcal{J} \partial_x \mathcal{J}\right],$$

$$H_3 = \frac{1}{8} \mathcal{L}^2 - 3\mathcal{L} \mathcal{J}^2 - 3\mathcal{L} \mathcal{H} \partial_x \mathcal{J} + 10 (\partial_x \mathcal{J})^2 + 12 \mathcal{J}^2 \mathcal{H} \partial_x \mathcal{J} + 2 \mathcal{J}^4,$$

where $\mathcal{H}$ is the Hilbert transform defined by the principal value integral

$$\mathcal{H} F(x) = \frac{1}{2\pi} \mathcal{P} \int_0^{2\pi} F(y) \cot \frac{1}{2} (y-x) dy.$$

The remaining conserved quantities of the hierarchy can be obtained recursively by imposing commutativity, $\{H_k, H_l\} = 0$. More powerful methods to obtain the integrals of motion can be found in [26–28, 32, 33].

For our purposes, it is worth stressing that the BO$_2$ equations are invariant under anisotropic scaling of Lifshitz type with dynamical exponent $j$ in (1.1), provided that the fields scale as $\mathcal{L} \to \lambda^{-2} \mathcal{L}$, $\mathcal{J} \to \lambda^{-1} \mathcal{J}$. Indeed, the conserved charges scale according to $H_z \to \lambda^{-z} H_z$, and thus, the Hamiltonian of the $z$-th representative of the hierarchy, $H_z$, becomes labeled in terms of its scaling dimension $z$, which matches the dynamical exponent used before.

In spite of the non–locality introduced through the Hilbert transform, it is remarkable that BO$_2$ can be quantized.

$^2$In our conventions, the fields are expanded in modes according to (4.5), so that $c$ stands for the Virasoro central charge.
4.2 Quantum BO$_2$ Hierarchy

The quantum BO$_2$ hierarchy surprisingly emerges in the context of the AGT correspondence, which describes a relationship between 4d $\mathcal{N} = 2$ supersymmetric gauge theories and 2d conformal field theories [34]. The partition function of certain type of supersymmetric models, called class-$\mathcal{S}$ models, is given by the Nekrasov partition function $Z_{\text{inst}}$ [35]. The AGT correspondence states that $Z_{\text{inst}} \propto \mathcal{F}_c$, where $\mathcal{F}_c$ is a Liouville CFT conformal block. For the detailed map between the two sides, see [34]. Here, we just sketch the minimal information about the correspondence that is useful for our requirements.

The proof of the AGT expansion relies on the introduction of a new basis of descendant CFT$_2$ states, which we call the AFLT basis [36]. It starts by considering the tensor product algebra $\mathcal{A} = \text{Vir} \otimes \mathcal{H}$, spanned by the modes $L_n$ of the Virasoro algebra (Vir) and the modes $a_n$ of the Heisenberg algebra ($\mathcal{H}$), so that $[a_n, L_m] = 0$. In CFT, the generators of $\mathcal{A}$ are given in terms of the energy-momentum tensor $T$ and the $U(1)$ current $J$. Here we use an alternative normalization, with respect to CFT, for the mode expansion of these currents

\[
\hat{T}(x) = \frac{24}{c} \sum_{n=-\infty}^{\infty} L_n e^{-inx} - 1,
\]

\[
\hat{J}(x) = i \sqrt{\frac{24}{c}} \sum_{n=-\infty}^{\infty} a_n e^{-inx},
\]

to match the conventions set in the classical formulation in section 4.1. As in [33, 36], we also discard the zero mode of the $u(1)$ current. Notice that, for $\hat{\mathcal{L}}$ and $\hat{\mathcal{J}}$ to be Hermitian, we set $L_n^\dagger = L_{-n}$ and $a_n^\dagger = -a_{-n}$.

To proceed, we introduce the standard Liouville notation for the central charge and conformal dimensions [37]

\[
c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}, \quad \Delta(P) = \frac{Q^2}{4} - P^2,
\]

where $b$ is the Liouville parameter and the momentum $P$ labels the primary states. The orthogonal AFLT basis reads

\[
|P\rangle_{\tilde{\lambda}} = \sum_{|\mu| = |\tilde{\lambda}|} C^{\mu_1, \mu_2}_{\tilde{\lambda}}(P) a_{-\mu_1} L_{-\mu_2} |\Delta(P)\rangle,
\]

where the first few coefficients $C^{\mu_1, \mu_2}_{\tilde{\lambda}}(P)$ are given in [36]. Here $\tilde{\lambda} = (\lambda_1, \lambda_2)$ corresponds to two integer partitions $\lambda_k = \{(\lambda_k)_1, (\lambda_k)_2, \ldots, (\lambda_k)_n\}$, with $k = 1, 2$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. One of the main conclusions of [36] is that the insertion of the completeness relation of the basis (4.8) in a CFT correlator gives the AGT conformal block expansion.
The AFLT basis $|P\rangle_{\tilde{\lambda}}$ also diagonalizes an infinite set of mutually commuting operators $H_z$, $z \in \mathbb{Z}_{>0}$, given by the quantum $BO_2$ integrals of motion and their eigenvalues can be explicitly obtained [33, 36]. The quantum integrals of motion $H_z$ lie in the universal enveloping algebra of $A$. The first two of them read

$$H_1 = \frac{1}{\ell} \left( L_0 + 2 \sum_{k=1}^{\infty} a_{-k} a_{k} - \frac{c+1}{24} \right),$$

$$H_2 = \frac{4i}{3\ell} \sqrt{\frac{6}{c}} \left( \sum_{k=-\infty, k\neq 0}^{\infty} a_{-k} L_k + 2iQ \sum_{k=1}^{\infty} ka_{-k} a_k + \frac{1}{3} \sum_{i+j+k=0} a_i a_j a_k \right),$$

while for odd values of $z = 2n - 1$ one obtains

$$H_{2n-1} = \frac{1}{n\ell} \left( \frac{24}{c} \right)^{n-1} (L_0^n + \cdots),$$

where the ellipsis stands for non-zero modes. Note that we have defined the operators $H_j$ to be Hermitian, so that the spectrum is real, and the classical integrals of motion $H_j$ obtained from the densities (4.3), can be recovered from (4.9) in the semiclassical limit $b \to 0$.

The BO$_2$ eigenstates $|P\rangle_{\tilde{\lambda}}$ obey $H_z |P\rangle_{\tilde{\lambda}} = E^{(z)}(P) |P\rangle_{\tilde{\lambda}}$, with eigenvalues given by $E_{\lambda_1,\lambda_2}(P) = E^{(z)}_{\lambda_1}(P) + E^{(z)}_{\lambda_2}(-P)$. For a generic Hamiltonian $H_z$, it was conjectured in [36] and [33] that the spectrum can be written as a sum of two eigenvalues of the Calogero-Sutherland model plus some extra terms depending only on $\Delta$ and $c$. The Calogero-Sutherland eigenvalues are given by $h^{(z)}_{\lambda} = h^{(z)}_{\lambda}(P) + h^{(z)}_{\mu}(-P)$, where

$$h^{(z)}_{\lambda}(P) = \frac{2z}{(1+z)\ell} \left( \frac{6}{b^2 c} \right)^{1/z} \sum_{j=1}^{N_z} \left[ \left( bP - \frac{b^2}{2} + \lambda_j + jb^2 \right)^z - \left( bP - \frac{b^2}{2} + jb^2 \right)^{z} \right].$$

(4.10)

We call $h^{(z)}_{\lambda,\mu}$ the descendant part of the spectrum. The descendant part can be obtained from the Bethe ansatz equations conjectured in [33] and proven by [38]. We denote the contribution to the energy due to primary states as $E^{(z)}(P)$, so that the ground state energy is given by $E^{(z)}_0 \equiv E^{(z)}(\pm \frac{Q}{2})$. The eigenvalues of the first four operators $H_z$, in our normalization, are given by

$$E^{(1)}_{\lambda,\mu}(P) = h^{(1)}_{\lambda,\mu}(P) + E^{(1)}(P), \quad E^{(2)}_{\lambda,\mu}(P) = \frac{1}{2} h^{(2)}_{\lambda,\mu}(P),$$

$$E^{(3)}_{\lambda,\mu}(P) = -2h^{(3)}_{\lambda,\mu}(P) - \frac{6}{c\ell} \left( 1 + b^2 \right) N_{\lambda,\mu} + E^{(3)}(P),$$

$$E^{(4)}_{\lambda,\mu}(P) = \frac{1}{2} h^{(4)}_{\lambda,\mu}(P) + \frac{18}{5} \frac{(1+b^2)}{c} h^{(2)}_{\lambda,\mu}(P),$$

(4.11)
where \( N_{\lambda, \mu} = |\lambda| + |\mu| \), and
\[
E^{(1)}(P) = \frac{1}{\ell} \left[ \Delta(P) - \frac{c + 1}{24} \right],
\]
\[
E^{(3)}(P) = \frac{12}{c\ell} \left[ \Delta(P)^2 - \frac{c + 5}{12} \Delta(P) + \frac{5c^2 + 52c + 15}{2880} \right].
\]

Note that for a generic heavy state, where \( \Delta(P) \approx \frac{c}{24} \mathcal{L}(P) \) in the semiclassical limit, for \( z \) odd we have
\[
E^{(z)}(P) = \frac{c}{12\ell} \left[ \mathcal{L}(P) \right]^{\frac{1+2z}{2}}.
\]

This corresponds to a classical state with energy \( E^{(z)}(P) = \langle \Delta(P) | \hat{H}_z | \Delta(P) \rangle \).

Knowing the spectrum then allows us to obtain the leading term of the entropy in the semiclassical limit. In the case of even values of \( z \), the leading entropy can be obtained along the lines of number theory, while for odd values of \( z \) it can be done through anisotropic modular invariance.

**Entropy for \( z = 2n \):** In the semiclassical limit \( b \to 0 \), as it occurs for \( z = 2, 4 \), we assume that the energy levels have only contributions from the descendant part, so that \( E^{(z)}_{\lambda, \mu}(P) \sim h^{(z)}_{\lambda, \mu}(P) \) to leading order in \( c \). From (4.10), if \( P \ll b^{-1} \), we have that
\[
E^{(z)}_{\lambda, \mu} = \varepsilon_z \sum_k \left( \lambda_k^z + \mu_k^z \right).
\]

This corresponds to energies close to the CFT gap. For states in which \( P \sim \sqrt{c} \), the situation is more complicated, but, for large enough partitions, the energies are still dominated by (4.14). The explicit values of the characteristic energies for \( z = 2, 4 \) can be read from (4.11) to be given by \( \varepsilon_2 = \frac{2}{3\ell} \) and \( \varepsilon_4 = \frac{8}{5\ell} \).

According to (4.14), the asymptotic growth of the number of states then goes as in section 2, but extended to a two-colored system. Indeed, the \( N \)-colored entropy for systems with Lifshitz scaling in \((1+1)d\) can be readily obtain from (2.3) by the replacement \( \varepsilon_z \to \varepsilon_z/N^z \). [24]. Therefore, in this case the entropy is given by (2.3) with \( \varepsilon_z \to 2^{-z} \varepsilon_z \).

**Entropy for \( z = 2n - 1 \):** In this case, the energies are no longer dominated by the descendant part, but instead by the primary part \( E^{(z)}(P) \). In the semiclassical limit, assuming that \( \Delta \ll c \), the leading contribution of \( E^{(z)}(P) \) comes from normal ordering of the leading term of the Hamiltonian, \( \mathcal{H}_z \sim \hat{\mathcal{L}}^{\frac{1+z}{2}} \), so that the ground state energy reads
\[
E^{(z)}_0 = \frac{(-1)^{\frac{1+z}{2}} c}{1 + z} \frac{\varepsilon_z}{12\ell}.
\]

Hence, in this case the entropy is determined by (1.5) with \( E_0 [z] = E^{(z)}_0 \).

In the next section, we connect the present discussion of the semiclassical BO\(_2\) spectrum with gravitation on AdS\(_3\) and black holes.
5  Geometrization of Benjamin-Ono $^2$ and Black Hole Entropy in 3D

Following the lines of [9], here we show that the BO$^2$ hierarchy of integrable systems can be fully geometrized, in the sense that its dynamics can be equivalently understood in terms of the evolution of spacelike surfaces and $U(1)$ fields with vanishing field strength embedded in locally AdS$_3$ spacetimes. Let us then consider the Einstein-Hilbert action with negative cosmological constant in 3D, endowed with a couple of noninteracting $U(1)$ fields

$$I = \frac{1}{16\pi G} \int d^3x \left[ \sqrt{-g} (R + 2\ell^{-2}) - 2\ell \epsilon^{\mu\nu\lambda} \left( A^+_\mu \partial_\nu A'^+_\lambda - A^-_\mu \partial_\nu A'^-_\lambda \right) \right], \quad (5.1)$$

which agrees with the bosonic sector of $\mathcal{N} = (2, 2)$ supergravity [39]. Note that since the $U(1)$ fields are described by Chern-Simons actions, the spacetime metric does not acquire a back reaction due to their presence. Therefore, the field equations imply that spacetime is of negative constant curvature, carrying two independent $U(1)$ fields of vanishing field strength.

As done in [9] (see also [40]), it can be shown that there exists a precise set of boundary conditions, being such that, in the reduced phase space, the field equations obtained from (5.1) exactly reduce to (left and right copies of) BO$^2$. This can be seen as follows. According to [39, 41], up to boundary terms, the action (5.1) can be written as the difference of two Chern-Simons actions, both with level $k = \ell/4G$, for independent $SL(2,R) \times U(1)$ gauge fields, so that the dreibein and the (dualized) spin connection are related to the $SL(2,R)$ gauge fields as $A^\pm_{SL(2,R)} = \omega \pm \ell^{-1}$. We then have to specify the asymptotic structure of the fields. For simplicity we restrict the analysis to the left copy, since the extension to the remaining one is straightforward. It is useful to make a gauge choice as in [42–44], so that the $SL(2,R) \times U(1)$ connection reads

$$\mathcal{A} = g^{-1} (d + a) g, \quad (5.2)$$

with $g = e^{\log(\tau/\ell) L_0}$. This gauge choice certainly simplifies our task, since the remaining analysis can be performed in terms of the auxiliary gauge field

$$a = a_t dt + a_\phi d\phi, \quad (5.3)$$

which exclusively depends on $t, \phi$.

Thus, the asymptotic form of (5.3) is proposed to be given by

$$a_\phi = L_1 - \frac{1}{4} \mathcal{L} L_{-1} + \mathcal{J} J_0, \quad (5.4)$$

$$a_t = \mu L_1 - \frac{1}{4} \mu \mathcal{L} L_{-1} - \mu' L_0 + \frac{1}{2} \mu'' L_{-1} - \frac{1}{8} \xi J_0,$$
where $\mu, \xi$ stand for Lagrange multipliers associated to the dynamical fields $\mathcal{L}, \mathcal{J}$ respectively. The boundary conditions then become fully specified only once the Lagrange multipliers are kept fixed at the boundary, located at a fixed value of the radial coordinate. Our choice of boundary conditions then consists in precisely fixing $\mu$ and $\xi$ in terms of the dynamical fields and their derivatives along $\phi$ according to

$$\mu = \frac{48 \pi \delta H_z}{c} \delta \mathcal{L}, \quad \xi = \frac{48 \pi \delta H_z}{c} \delta \mathcal{J},$$

where $H_z$ stands for the $z$-th conserved charge of $\text{BO}_2$, with $c$ given by the Brown-Henneaux central charge $c = 3\ell/2G$ [45].

Since we are dealing with a Chern-Simons theory, the field equations imply that the $\text{SL}(2,\mathbb{R}) \times U(1)$ connection $\mathcal{A}$ is locally flat, and by virtue of the gauge choice in (5.2), the field strength of the auxiliary gauge field (5.3) also vanishes. Therefore, the components of $a$ in (5.4) reduce to an $\text{SL}(2,\mathbb{R}) \times U(1)$-valued Lax pair formulation of the $\text{BO}_2$ hierarchy, so that the field equations in (4.1) can be compactly written as

$$f = da + a^2 = 0. \quad (5.5)$$

Therefore, two independent copies of the $\text{BO}_2$ equations are precisely recovered from the reduced phase space of the three-dimensional field equations of (5.1) endowed with our choice of boundary conditions.

Furthermore, according to [9], the symmetries of the $\text{BO}_2$ equations, spanned by the conserved quantities $H_j$, now emerge from the set of diffeomorphisms that preserve the asymptotic form of the gauge field. Noteworthy, in the geometric framework, the symmetries of $\text{BO}_2$ become Noetherian, and hence, the infinite set of commuting conserved charges $H_j$ is precisely obtained from the corresponding surface integrals in the canonical approach [46]. In particular, the total energy of a three-dimensional configuration that fulfills our boundary conditions, including gravitation and the $U(1)$ fields, is then given by the sum of left and right Hamiltonians of $\text{BO}_2$, i.e., $E = Q[\partial_t] = H^+_z + H^-_z$.

In sum, the whole structure of classical $\text{BO}_2$, including its phase space, the infinite number of commuting charges and its field equations, emerges from the reduced phase space of gravitation on $\text{AdS}_3$ coupled to two $U(1)$ fields with our boundary conditions. Hence, this construction provides a gravitational dual of two noninteracting left and right $\text{BO}_2$ movers, describing locally $\text{AdS}_3$ spacetimes with anisotropic scaling induced by the choice of boundary conditions. Consequently, any solution of the $\text{BO}_2$ equations can be mapped into a locally $\text{AdS}_3$ spacetime with suitable $U(1)$ fields of vanishing field strength. In particular,

\footnote{In the special case of $z = 1$ our boundary conditions reduce to the bosonic part of the ones in [47], and the asymptotic symmetry algebra corresponds to two copies of the direct sum of Virasoro with the Brown-Henneaux central extension and a $u(1)$ current.}
one of the most trivial $BO_2$ configurations, given by $J^\pm = 0$ and $L^\pm = \ell^{-2}(r_+ \pm r_-)^2$ constants, corresponds to the geometry of a BTZ black hole in vacuum [48, 49]. Note that in the geometric picture this configuration is clearly non-trivial because the event horizon has Hawking temperature and entropy, and its mass and angular momentum become well defined in terms of left and right $BO_2$ energies

$$H^\pm_z[L^\pm] = \frac{c}{12\ell} \frac{1}{1 + z} L^\frac{1+z}{2}, \quad (5.6)$$

provided that $z = 2n - 1$. Note that (5.6) agrees with (4.13).

This geometric realization suggests that the reduced gravitational phase space could be quantized in terms of two copies of $BO_2$, so that the states would be given by the AFLT ones in (4.8). Indeed, two points are worth to be emphasized:

(i) The ground state energy of quantum $BO_2$ in the semiclassical limit, given by $E_0^{(z)}$ in (4.15), exactly coincides with the one of the geometric configuration of lowest energy, determined by global $AdS_3$ spacetime. Indeed, left and right energies of $AdS_3$ correspond to (5.6) with $L^\pm = -1$, and hence

$$H^\pm_z[-1] = E_0^{(z)} = \left(\frac{-1}{1 + z}\right) \frac{c}{12\ell}. \quad (5.7)$$

(ii) The leading term of the asymptotic growth of the number of states is then obtained from (1.5) for both copies, i.e.,

$$S = 2\pi\ell(1 + z) \left[ \left( \frac{|E_0^+|}{z} \right)^\frac{1+z}{2} E_0^+ \left( \frac{|E_0^-|}{z} \right)^\frac{1+z}{2} E_0^- \right], \quad (5.8)$$

where $E_0^+ = E_0^- = E_0^{(z)}$ stand for left and right energies of the ground state, determined by (5.7). Hence, for left and right energies given by the ones of the black hole, i.e., $E^\pm = H^\pm_z[L^\pm]$ in (5.6), noteworthy, the entropy obtained from the anisotropic extension of Cardy formula (5.8) exactly reduces to the one of Bekenstein and Hawking, given by

$$S = \frac{A}{4G}. \quad S$$

Acknowledgments

The authors thank valuable discussions with Sebas Eliens, Hernán González, Rodrigo Pereira, Pablo Rodríguez, David Tempo, Jacopo Viti, Paul Wiegmann and Alexander B. Zamolodchikov. The work of DM was supported by the grant No. 16-12-10344 of the Russian Science
Foundation. FN thanks Máté Lencsés for pointing out the Hardy-Ramanujan paper. FN also thanks Jun’ichi Shiraishi and, specially, Yohei Tutiya for the initial discussions on this project and kind hospitality at the Komaba Mathematics section of the University of Tokyo, where part of this work was developed. FN thanks the organizers of the *Latin-American Workshop on Gravity and Holography* held in São Paulo in June, 2018 for the opportunity to present the main results of this work and for financial support. FN acknowledges the Brazilian Ministry of Education for the financial support. AP thanks Stefan Theisen for his kind hospitality at the MPI für Gravitationsphysik in Golm, and the German Academic Exchange Service (DAAD) for financial support through the “Re-invitation Programme for Former DAAD Scholarship Holders”. This research has been partially supported by Fondecyt grants N° 1161311, 1171162 and 1181496. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt.

**References**

[1] M. Taylor, *Non-relativistic holography*, [arXiv:0812.0530](https://arxiv.org/abs/0812.0530).

[2] S. A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, *Class. Quant. Grav.* **26** (2009) 224002, [arXiv:0903.3246](https://arxiv.org/abs/0903.3246).

[3] S. A. Hartnoll, *Horizons, holography and condensed matter*, in *Black holes in higher dimensions* (G. T. Horowitz, ed.), pp. 387–419. 2012. [arXiv:1106.4324](https://arxiv.org/abs/1106.4324).

[4] M. Taylor, *Lifshitz holography*, *Class. Quant. Grav.* **33** (2016), no. 3 033001, [arXiv:1512.03554](https://arxiv.org/abs/1512.03554).

[5] E. Bettelheim, A. G. Abanov, and P. Wiegmann, *Quantum Shock Waves: The case for non-linear effects in dynamics of electronic liquids*, *Phys. Rev. Lett.* **97** (2006) 246401, [cond-mat/0606778](https://arxiv.org/abs/cond-mat/0606778).

[6] P. Wiegmann, *Non-Linear hydrodynamics and Fractionally Quantized Solitons at Fractional Quantum Hall Edge*, *Phys. Rev. Lett.* **108** (2012) 206810, [arXiv:1112.0810](https://arxiv.org/abs/1112.0810).

[7] S. Sotiriadis, *Equilibration in one-dimensional quantum hydrodynamic systems*, *J. Phys. A* **50** (2017), no. 42 424004, [arXiv:1612.00373](https://arxiv.org/abs/1612.00373).

[8] H. A. Gonzalez, D. Tempo, and R. Troncoso, *Field theories with anisotropic scaling in 2D, solitons and the microscopic entropy of asymptotically Lifshitz black holes*, *JHEP* **11** (2011) 066, [arXiv:1107.3647](https://arxiv.org/abs/1107.3647).

[9] A. Pérez, D. Tempo, and R. Troncoso, *Boundary conditions for General Relativity on AdS$_3$ and the KdV hierarchy*, *JHEP* **06** (2016) 103, [arXiv:1605.04490](https://arxiv.org/abs/1605.04490).
[10] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[11] J. L. Cardy, *Operator Content of Two-Dimensional Conformally Invariant Theories*, Nucl. Phys. **B270** (1986) 186–204.

[12] D. Grumiller, A. Perez, D. Tempo, and R. Troncoso, *Log corrections to entropy of three dimensional black holes with soft hair*, JHEP **08** (2017) 107, [arXiv:1705.10605].

[13] E. Shaghoulian, *A Cardy formula for holographic hyperscaling-violating theories*, JHEP **11** (2015) 081, [arXiv:1504.02094].

[14] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proceedings of the London Mathematical Society **s2-17** (1918), no. 1 75–115.

[15] E. M. Wright, *Asymptotic partition formulae. iii. partitions into k-th powers*, Acta Mathematica **63** (1934), no. 1 143–191.

[16] R. C. Vaughan, *Squares: additive questions and partitions*, International Journal of Number Theory **11** (2015), no. 05 1367–1409.

[17] A. Gafni, *Power partitions*, Journal of Number Theory **163** (2016) 19–42.

[18] H. Afshar, D. Grumiller, W. Merbis, A. Perez, D. Tempo, and R. Troncoso, *Soft hairy horizons in three spacetime dimensions*, Phys. Rev. **D95** (2017), no. 10 106005, [arXiv:1611.09783].

[19] H. Afshar, S. Detournay, D. Grumiller, W. Merbis, A. Perez, D. Tempo, and R. Troncoso, *Soft Heisenberg hair on black holes in three dimensions*, Phys. Rev. **D93** (2016), no. 10 101503, [arXiv:1603.04824].

[20] S. W. Hawking, M. J. Perry, and A. Strominger, *Soft Hair on Black Holes*, Phys. Rev. Lett. **116** (2016), no. 23 231301, [arXiv:1601.00921].

[21] S. W. Hawking, M. J. Perry, and A. Strominger, *Superrotation Charge and Supertranslation Hair on Black Holes*, JHEP **05** (2017) 161, [arXiv:1611.09175].

[22] F. Luca and D. Ralaivaosaona, *An explicit bound for the number of partitions into roots*, Journal of Number theory **169** (2016) 250–264.

[23] Y.-L. Li and Y.-G. Chen, *On the r-th root partition function, ii*, Journal of Number Theory **188** (2018) 392–409.

[24] D. Melnikov, F. Novaes, A. Pérez, and R. Troncoso, Working in progress.

[25] Y. Matsuno, *Bilinear transformation method*, vol. 174. Academic Press New York, 1984.

[26] D. R. Lebedev and A. O. Radul, *Generalized Internal Long Waves Equations: Construction, Hamiltonian Structure and Conservation Laws*, Commun. Math. Phys. **91** (1983) 543.
[27] A. Degasperis, D. Lebedev, M. Olshanetsky, S. Pakuliak, A. Perelomov, and P. Santini, Nonlocal integrable partners to generalized MKdV and two-dimensional Toda lattice equations in the formalism of a dressing method with quantized spectral parameter, Commun. Math. Phys. 141 (1991) 133–152.

[28] A. Degasperis, D. Lebedev, M. Olshanetsky, S. Pakuliak, A. Perelomov, and P. Santini, Generalized intermediate long-wave hierarchy in zero-curvature representation with noncommutative spectral parameter, Journal of mathematical physics 33 (1992), no. 11 3783–3793.

[29] A. G. Abanov and P. B. Wiegmann, Quantum hydrodynamics, the quantum benjamin-ono equation, and the calogero model, Physical review letters 95 (2005), no. 7 076402.

[30] A. G. Abanov, E. Bettelheim, and P. Wiegmann, Integrable hydrodynamics of Calogero-Sutherland model: Bidirectional Benjamin-Ono equation, J. Phys. A42 (2009) 135201, [arXiv:0810.5327].

[31] A. Imambekov, T. L. Schmidt, and L. I. Glazman, One-dimensional quantum liquids: Beyond the Luttinger liquid paradigm, Reviews of Modern Physics 84 (July, 2012) 1253–1306, [arXiv:1110.1374].

[32] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz, Commun. Math. Phys. 177 (1996) 381–398, [hep-th/9412229].

[33] A. V. Litvinov, On spectrum of ILW hierarchy in conformal field theory, JHEP 11 (2013) 155, [arXiv:1307.8094].

[34] L. F. Alday, D. Gaiotto, and Y. Tachikawa, Liouville Correlation Functions from Four-dimensional Gauge Theories, Lett. Math. Phys. 91 (2010) 167–197, [arXiv:0906.3219].

[35] N. A. Nekrasov, Seiberg-Witten Prepotential From Instanton Counting, Adv. Theor. Math. Phys. 7 (2003), no. 5 831–864, [hep-th/0206161].

[36] V. A. Alba, V. A. Fateev, A. V. Litvinov, and G. M. Tarnopolskiy, On combinatorial expansion of the conformal blocks arising from AGT conjecture, Lett.Math.Phys. 98 (2011) 33–64. 00062.

[37] S. Ribault, Conformal field theory on the plane, arXiv:1406.4290.

[38] B. Feigin, M. Jimbo, and E. Mukhin, Integrals of motion from quantum toroidal algebras, J. Phys. A50 (2017), no. 46 464001, [arXiv:1705.07984].

[39] A. Achucarro and P. K. Townsend, Extended Supergravities in $d = (2+1)$ as Chern-Simons Theories, Phys. Lett. B229 (1989) 383–387.

[40] O. Fuentealba, J. Matulich, A. P´erez, M. Pino, P. Rodr´ıguez, D. Tempo, and R. Troncoso, Integrable systems with BMS3 Poisson structure and the dynamics of locally flat spacetimes, JHEP 01 (2018) 148, [arXiv:1711.02646].
[41] E. Witten, (2+1)-Dimensional Gravity as an Exactly Soluble System, Nucl. Phys. B311 (1988) 46.

[42] O. Coussaert, M. Henneaux, and P. van Driel, The Asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant, Class. Quant. Grav. 12 (1995) 2961–2966, [gr-qc/9506019].

[43] M. Henneaux, A. Perez, D. Tempo, and R. Troncoso, Chemical potentials in three-dimensional higher spin anti-de Sitter gravity, JHEP 12 (2013) 048, [arXiv:1309.4362].

[44] C. Bunster, M. Henneaux, A. Perez, D. Tempo, and R. Troncoso, Generalized Black Holes in Three-dimensional Spacetime, JHEP 05 (2014) 031, [arXiv:1404.3305].

[45] J. D. Brown and M. Henneaux, Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity, Commun. Math. Phys. 104 (1986) 207–226.

[46] T. Regge and C. Teitelboim, Role of Surface Integrals in the Hamiltonian Formulation of General Relativity, Annals Phys. 88 (1974) 286.

[47] M. Henneaux, L. Maoz, and A. Schwimmer, Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity, Annals Phys. 282 (2000) 31–66, [hep-th/9910013].

[48] M. Banados, C. Teitelboim, and J. Zanelli, The Black hole in three-dimensional space-time, Phys. Rev. Lett. 69 (1992) 1849–1851, [hep-th/9204099].

[49] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, Geometry of the (2+1) black hole, Phys. Rev. D48 (1993) 1506–1525, [gr-qc/9302012]. [Erratum: Phys. Rev.D88,069902(2013)].