KOSZUL COMPLEXES AND SPECTRA OF PROJECTIVE HYPERSURFACES WITH ISOLATED SINGULARITIES

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Abstract. For a projective hypersurface with isolated singularities, we generalize some well-known results in the nonsingular case due to Griffiths, Scherk, Steenbrink, Varchenko, and others. They showed, for instance, a relation between the mixed Hodge structure on the vanishing cohomology and the Gauss-Manin system filtered by shifted Brieskorn lattices of a defining homogeneous polynomial by using the V-filtration of Kashiwara and Malgrange. Numerically this implied an identity between the Steenbrink spectrum and the Poincaré polynomial of the Milnor algebra. In our case, however, we have to replace these with the pole order spectrum and the alternating sum of the Poincaré series of certain subquotients of the Koszul cohomologies respectively, and then study the pole order spectral sequence which does not necessarily degenerate at $E_2$. This non-degeneration is closely related with the torsion of the Brieskorn module which vanished in the classical case.

Introduction

Let $f$ be a homogeneous polynomial in the graded $\mathbb{C}$-algebra $R := \mathbb{C}[x_1, \ldots, x_n]$ where $\deg x_i = 1$ and $n \geq 2$. Set $d = \deg f$. Consider the shifted Koszul complex

$$sK^+_j := K^+_j[n] \quad \text{with} \quad K^+_j = (\Omega^j, df \wedge).$$

Here $\Omega^j := \Gamma(\mathbb{C}^n, \Omega^j_{\mathbb{C}^n})$ with $\Omega^j_{\mathbb{C}^n}$ algebraic so that the $\Omega^j$ are finite free graded $R$-modules, and the degree of $\Omega^j$ in $sK_j$ is shifted so that

$$sK^+_j = \Omega^{j+n}(jd) \quad (\text{i.e. } sK^+_j, k = \Omega^{j+n}_{jd+k}) \quad \text{for } j \in \mathbb{Z}.$$ 

In general the shift of degree by $p$ of a graded module $M$ will denoted by $M(p)$, where the latter is defined by $M(p)_k = M_{k+p}$. Since the dualizing complex for complexes of $R$-modules is given by $\Omega^n[n]$, we have the self-duality

$$D(sK^+_j) := R\text{Hom}_R(sK^+_j, \Omega^n[n]) = sK^+_j(nd).$$

In this paper we assume

$$(A) \quad \dim \text{Sing } f^{-1}(0) \leq 1.$$ 

It is well-known, and is easy to show (see e.g. Remark (1.9)(iv) below) that this implies

$$H^j(sK^+_j) = 0 \quad \text{if } j \neq -1, 0.$$ 

Define

$$M := H^0(sK^+_j), \quad N := H^{-1}(sK^+_j).$$ 

Let $m = (x_1, \ldots, x_n) \subset R$, the maximal graded ideal. Set

$$M' := H^0_m M, \quad M'' := M/M'.$$

These are finitely generated graded $R$-modules having the decompositions $M = \bigoplus_k M_k$, etc. In the isolated singularity case we have $M'' = N = 0$, and $M = M'$. Generalizing a well-known assertion in the isolated singularity case, one may conjecture that the canonical morphism from $M'$ to the graded quotient of the pole order filtration on the Gauss-Manin
system is injective, see Proposition (3.5) below for a partial evidence. This is closely related with Question 2 and Remark (5.9) below, see also [Br1], [Br2], etc.

Let \( y := \sum_{i=1}^n c_i x_i \) with \( c_i \in \mathbb{C} \) sufficiently general so that \( \{ y = 0 \} \subset \mathbb{C}^n \) is transversal to any irreducible component of \( \operatorname{Sing} f^{-1}(0) \). Then \( M' \) is the \( y \)-torsion subgroup of \( M \), and \( M'' \), \( N \) are finitely generated free graded \( \mathbb{C}[y] \)-modules of rank \( \tau_Z \), where \( \tau_Z \) is the total Tjurina number as in (0.4) below. Note that there is a shift of the grading on \( N \) by \( d \) between this paper and [DiSt1], [DiSt2].

Define the (higher) dual graded \( R \)-modules by

\[
D_i(M) := \operatorname{Ext}_R^{n-i}(M, \Omega^n) \quad (i \in \mathbb{Z}),
\]

and similarly for \( D_i(N) \), etc. From the above self-duality of the Koszul complex \( \wedge^* K_f \), we can deduce the following duality (which is known to the specialists at least by forgetting the grading, see [Pa], [StWa] and also [EyMe], [Sc], etc.):

**Theorem 1.** There are canonical isomorphisms of graded \( R \)-modules

\[
\begin{align*}
D_0(M') &= D_0(M) = M'(nd), \\
D_1(M'') &= D_1(M) = N(nd), \\
D_1(N) &= M''(nd),
\end{align*}
\]

and \( D_i(M), D_i(M'), D_i(M''), D_i(N) \) vanish for other \( i \).

This generalizes a well-known assertion in the isolated singularity case where \( M'' = N = 0 \). Theorem 1 implies that \( M', M'' \) and \( N \) are Cohen-Macaulay graded \( R \)-modules of dimension 0, 1 and 1 respectively (but \( M \) itself is not Cohen-Macaulay). Moreover \( M' \) is graded self-dual, and \( M'' \) and \( N \) are graded dual of each other, up to a shift of grading.

For \( k \in \mathbb{Z} \), set

\[
\mu_k = \dim M_k, \quad \mu'_k = \dim M'_k, \quad \mu''_k = \dim M''_k, \quad \nu_k = \dim N_k.
\]

Let \( g := \sum_{i=1}^n x_i^d \), and \( \gamma_k := \dim(H^k K_f)_k = \dim(\Omega^n / \sum_{i=1}^n x_i^{d-1} \Omega^n)_k \), so that

\[
\sum_k \gamma_k t^k = (t^d - t)^n / (t - 1)^n,
\]

(Here \( g \) can be any homogeneous polynomial of degree \( d \) with an isolated singular point.) It is known (see [Di2], [DiSt1], [DiSt2]) that

\[
\mu_k = \mu'_k + \mu''_k = \nu_k + \gamma_k \quad (k \in \mathbb{Z}),
\]

since the Euler characteristic of a bounded complex is independent of its differential if the components of the complex are finite dimensional.

By the first assertion of (0.1) together with (1.1.4) for \( i = 1 \) and by (0.2), we get the following symmetries:

**Corollary 1.** \( \mu'_k = \mu'_{nd-k}, \quad \gamma_k = \gamma_{nd-k} \quad (k \in \mathbb{Z}). \)

Let \( Z := \{ f = 0 \} \subset Y := \mathbb{P}^{n-1} \), and \( \Sigma := \operatorname{Sing} Z \). The total Tjurina number \( \tau_Z \) is defined by

\[
\tau_Z := \sum_{z \in \Sigma} \tau_z \quad \text{with} \quad \tau_z := \dim_O \mathcal{O}_{Y,z} / (h_z, \partial h_z),
\]

where \( h_z \) is a local defining equation of \( Z \) at \( z \), and \( \partial h_z \) is the Jacobian ideal of \( h_z \) generated by its partial derivatives. By Theorem 1, \( M'' \) and \( N \) are Cohen-Macaulay, and are dual of each other up to a shift of grading. Combining this with the graded local duality (1.1.4) for \( i = 1 \) (see [BrHe], [Eil], etc.) together with (1.9.3) below, we get the following:

**Corollary 2.** \( \mu''_k + \nu_{nd-k} = \tau_Z \quad (k \in \mathbb{Z}). \)
Here the calculation of the local cohomology in the local duality is not so trivial (see Remark (1.7) below), and we can also use an exact sequence as in [SI] Prop. 2.1] (see also [Gro, Prop. 2.1.5] and [SaSt], etc.) Note that Corollary 2 can also be deduced from Thm. 3.1 in [Di2], see Remark (1.9)(i) below. By Corollaries 1 and 2 together with (0.3), we get the following.

**Corollary 3.** $\mu_k' = \mu_k + \mu_{nd-k} - \gamma_k - \tau_z, \quad \mu_k'' = \tau_z - \mu_{nd-k} + \gamma_k \quad (k \in \mathbb{Z}).$

This means that $\mu_k'$ and $\mu_k''$ are essentially determined by $\mu_k$ and $\mu_{nd-k}$. Note that $\{\mu_k''\}$ and $\{\nu_k\}$ are weakly increasing sequences of non-negative integers. It is shown that $\{\mu_k'\}$ is log-concave in a certain case, see [Sti]. Assuming $\text{Sing} Z \neq \emptyset$, we have $\mu_k' = \nu_k = \tau_z > 0$ for $k \gg 0$, hence $M''$ is nonzero, although $M'$ may vanish, see Remark (1.9)(iii) below. By Corollary 2 and (0.3) we get the following.

**Corollary 4.** $\gamma_k - \mu_k' = \mu_k'' + \mu_{nd-k} - \tau_z \quad (k \in \mathbb{Z}).$

Here a fundamental question seems to be the following.

**Question 1.** Are both sides of the above equality non-negative?

This seems to be closely related to the subject treated in [ChDi], [Di2], [DiSt1], [DiSt2], etc. We have a positive answer to Question 1 if $n = 3$ and $\Sigma$ is a complete intersection in $\mathbb{P}^2$ (see [Sti]) or if $f$ has type (I), where $f$ is called type (I) if the following condition is satisfied (and type (II) otherwise):

\[(0.5) \quad \mu_k'' = \tau_z \quad \text{for} \quad k \geq nd/2, \quad \text{i.e.} \quad \nu_k = 0 \quad \text{for} \quad k \leq nd/2.\]

By the definition of $N$, the last condition in (0.5) cannot hold if there is a nontrivial relation of very low degree between the partial derivatives of $f$, e.g. in case $f$ is a polynomial of $n-1$ variables (or close to it), see Remark (2.9) below. However, it holds in relatively simple cases, including the nodal hypersurface case by [DiSt2] Thm. 2.1, see Remark (2.10) below.

In the type (I) case, we get the $\mu_k'$ by restricting to $k \leq nd/2$ (where $\mu_k' + \mu_k'' = \mu_k = \gamma_k$ holds) if we know the $\mu_k''$. This can be done for instance in the following case.

**Proposition 1.** Assume $Z$ has only ordinary double points $z_1, \ldots, z_{\tau_Z}$, and moreover the $z_i$ correspond to linearly independent vectors in $\mathbb{C}^n$ so that $\tau_Z = r \leq n$. Then

$\mu''_k = \begin{cases} 0 & (k < n), \\ 1 & (k = n), \\ \tau_Z & (k > n), \end{cases}$

$\nu_k = \begin{cases} 0 & (k < n(d-1)), \\ \tau_Z - 1 & (k = n(d-1)), \\ \tau_Z & (k > n(d-1)). \end{cases}$

$\mu'_k = \begin{cases} 0 & (k \notin (n, n(d-1))), \\ \gamma_k - \tau_Z & (k \in (n, n(d-1))), \end{cases}$

where $(n, n(d-1)) \subset \mathbb{R}$ denotes an open interval.

This follows from Lemma (2.1) below together with Corollary 2 and (0.3). It can also be deduced from the results in [Di2], and seems to be closely related with [DiSaWo] Thm. 2.2.

The situation becomes, however, rather complicated if the number of singular points is large, see [ChDi], [Di2], [DiSt1], [DiSt2].

Let $\text{Sp}(f) = \sum_a n_{f, a} t_a^\alpha \in \mathbb{Q}[t^{1/d}]$ be the Steenbrink spectrum of $f$ (see [St2], [St3]) which is normalized as in [St2]. To study the relation with the Koszul cohomologies $M, N$ by generalizing the well-known assertion in the isolated singularity case where $M'' = N = 0$ and $M = M'$ (see [St1] and also [Grî, SkSt, Va], etc.), we have to introduce the pole order spectrum $\text{Sp}_P(f)$ by replacing the Hodge filtration $F$ with the pole order filtration $P$ in [Di1], [Di3], [DiSa2], [DiSt1]. There are certain shifts of the exponents coming from the
difference between $F$ and $P$. Here we have the inclusion $F \subset P$ in general, and the equality holds in certain cases (see [DiST]). We can calculate these spectra explicitly in the case $n = 2$, see Propositions (3.3) and (3.4). The relation between the two spectra is, however, quite nontrivial in general (see for instance Example (3.7) below).

The reason for which we introduce $Sp_\Psi(f)$ is that it is related with the Poincaré series of $M$, $N$ as follows: The differential of the de Rham complex $(\Omega^\bullet, d)$ induces a morphism of graded $\mathbf{C}$-vector spaces of degree $-d$:

$$d^{(1)} : N \to M,$$

i.e. preserving the degree up to the shift by $-d$. Let $H^n A^*_f$ denote the Brieskorn module (in a generalized sense) which is a graded $\mathbf{C}$-module endowed with actions of $t, \partial_t^{-1}$, and $t\partial_t$, see (4.2) below. Let $(H^n A^*_f)_{\text{tor}}$ be its $t$-torsion (or equivalently, $\partial_t^{-1}$-torsion) subspace. It has the kernel filtration $K_\bullet$ defined by

$$(0.6) \quad K_i(H^n A^*_f)_{\text{tor}} := \text{Ker} t^i \subset (H^n A^*_f)_{\text{tor}} \quad (i \geq 0).$$

The main theorem of this paper is as follows:

**Theorem 2.** There are inductively defined morphisms of graded $\mathbf{C}$-vector spaces of degree $-rd$:

$$d^{(r)} : N^{(r)} \to M^{(r)} \quad (r \geq 2),$$

such that $N^{(r)}$, $M^{(r)}$ are the kernel and the cokernel of $d^{(r-1)}$ respectively, and are independent of $r \gg 0$ (that is, $d^{(r)} = 0$ for $r \gg 0$), and we have

$$(0.7) \quad Sp_\Psi(f) = S(M^{(r)})(t^{1/d}) - S(N^{(r)})(t^{1/d}) \quad (r \gg 0),$$

where $S(M^{(r)})(t)$, $S(N^{(r)})(t)$ denote the Poincaré series of $M^{(r)}$, $N^{(r)}$ for $r \geq 2$.

Moreover, there are canonical isomorphisms

$$(0.8) \quad \text{Im} d^{(r)} = \text{Gr}^K_{r-1}(\text{Coker} \, t) \quad (r \geq 2),$$

where $K_\bullet$ is the kernel filtration on $(H^n A^*_f)_{\text{tor}}$, and the right-hand side of (0.8) is a subquotient of $(H^n A^*_f)_{\text{tor}}$. In particular, $d^{(r)}$ vanishes for any $r \geq 2$ (that is, $M^{(r)} = M^{(2)}$, $N^{(r)} = N^{(2)}$ for any $r \geq 2$) if and only if $H^n A^*_f$ is torsion-free.

Note that $\text{Ker} \, t^i$ in (0.6) and $\text{Coker} \, t$ in (0.8) can be replaced respectively with $\text{Ker} \, \partial_t^{-i}$ and $\text{Coker} \, \partial_t^{-1}$ by using (4.2.2) below. For the proof of Theorem 2 we use the spectral sequence associated with the pole order filtration on the algebraic microlocal Gauss-Manin complex (see (4.4.4) below), and the morphisms $d^{(r)}$ are induced by the differentials $d_i$ of the spectral sequence. (We can also use the usual Gauss-Manin complex instead of the microlocal one.) The last equivalent two conditions in Theorem 2 are further equivalent to the $E_2$-degeneration of the (microlocal) pole order spectral sequence, see Corollary (4.7) below (and also [vST]). Moreover $(H^n A^*_f)_{\text{tor}}$ is finite dimensional if and only if $Z$ is analytic-locally defined by a weighted homogeneous polynomial at any singular point, see Theorems (5.2) and (5.3) below. (In fact, the if part in the analytic local setting was shown in the second author’s master thesis, see e.g. [BaSa] Thm. 3.2 and also [vST].) Here Theorem (5.3) gives rather precise information about the kernel of $d^{(1)}$. This is a refinement of [DiST1 Thm. 2.4(ii)], and is used in an essential way in [DiSa3]. Theorem (5.3) implies a sharp estimate for $\max\{k \mid v_k = 0\}$ when $n = 3$, see Corollary (5.5) below. This assertion is used in an essential way in [DiSa3], and is generalized to the case $n > 3$ in [DiSa3, Theorem 9] (see [Di1] for another approach to the case $n > 3$).

In case $(H^n A^*_f)_{\text{tor}} = 0$, we can determine the pole order spectrum if we can calculate the morphism $d^{(1)} : N \to M$, although the latter is not so easy in general unless the last
conditions in Theorem (5.3) are satisfied (see also Remark (5.9) below). Note that the pole order spectral sequence was studied in [vSt] from a slightly different view point in the (non-graded) analytic local case.

For the moment there are no examples such that the singularities of $Z$ are weighted homogeneous and $(H^nA^*_f)_{\text{tor}} \neq 0$. We have the following.

**Question 2.** Assume all the singularities of $Z$ are weighted homogeneous. Then, is $H^nA^*_f$ torsion-free so that the pole order spectral sequence degenerates at $E_2$ and the equality (0.7) holds with $r = 2$?

We have a positive answer in certain cases; for instance, if $n = 2$ or $1$ is not an eigenvalue of $T^d_z$ for any $z \in \text{Sing} Z$ where $T^d_z$ is the monodromy of a local defining polynomial $h_z$ of $(Z, z)$, see Corollary (5.4) below for a more general condition. However, the problem is quite nontrivial in general. In the above second case, Theorem (5.3) actually implies the injectivity of $d(1): N \to M$ (which is a morphism of degree $-d$), and we get the following.

**Theorem 3.** If $(Z, z)$ is weighted homogeneous and $1$ is not an eigenvalue of $T^d_z$ for any $z \in \text{Sing} Z$, then $H^nA^*_f$ is torsion-free and we have

$$\text{Sp}_P(f) = S(M)(t^{1/d}) - S(N)(t^{1/d}) t^{-1}.$$ 

Here the second condition is satisfied if $1$ is not an eigenvalue of $T^d_z$ and moreover the order of $T^d_z$ is prime to $d$ for any $z \in \text{Sing} Z$. Note that the second assumption can be replaced with $H^{n-2}(f^{-1}(1), C) = 0$ if Question 2 is positively solved in this case (see Remark (5.9) below for a picture in the optimal case).

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In Section 1 we prove Theorem 1 after reviewing graded local duality for the convenience of the reader. In Section 2 we explain some methods to calculate the Koszul cohomologies in certain cases. In Section 3 we recall some basics from the theory of spectra, and prove Proposition (3.3), (3.4), and (3.5). In Section 4 we prove Theorem 2 after reviewing some facts from Gauss-Manin systems and Brieskorn modules. In Section 5 we calculate $d(1)$ in certain cases, and prove Theorems (5.2) and (5.3).

1. Graded local cohomology and graded duality

In this section we prove Theorem 1 after reviewing graded local duality for the convenience of the reader.

1.1. Graded local duality. Let $R = \mathbb{C}[x_1, \ldots, x_n]$, and $m = (x_1, \ldots, x_n) \subset R$. Set

$$\Omega^k = \Gamma(\mathbb{C}^n, \Omega^n_{\mathbb{C}^n}) \quad (k \in \mathbb{Z}).$$

Here $\Omega^n_{\mathbb{C}^n}$ is algebraic, and $\Omega^k$ is a finite free graded $R$-module with $\deg x_i = \deg dx_i = 1$.

For a bounded complex of finitely generated graded $R$-modules $M^*$, define

$$D_i(M^*) := \text{Ext}_R^{n-i}(M^*, \Omega^n) = H^{-i}(D(M^*))$$

with $D(M^*) := R\text{Hom}_R(M^*, \Omega^n[n])$,

where $D(M^*)$ can be defined by taking a graded free resolution $P^* \to M^*$.

For a finitely generated graded $R$-module $M$, set

$$H^0_{m^k}M := \{a \in M \mid m^k a = 0 \text{ for } k \gg 0\}.$$
Let $H^i_mM$ be the cohomological right derived functors ($i \in \mathbb{N}$). These are defined by taking a graded injective resolution of $M$. We can calculate them by taking a graded free resolution of $M$ as is explained in textbooks of commutative algebra, see e.g. [BrHe], [Ei]. Indeed, $H^i_mR = 0$ for $i \neq n$, and
\[ H^i_mR = C[x_1^{-1}, \ldots, x_n^{-1}]_{x_1 \cdots x_n}, \]
where the right-hand side is identified with a quotient of the graded localization of $R$ by $x_1 \cdots x_n$. We then get the graded local duality for finitely generated graded $R$-modules $M$:
\[ D_i(M)_k = \text{Hom}_C((H^i_mM)_{-k}, C) \quad (k \in \mathbb{Z}, i \geq 0), \]
see loc. cit. (Indeed, this can be reduced to the case $M = R$ by the above argument.)

Remarks 1.2. (i) The functors $H^i_m$ and $D_i$ are compatible with the corresponding functors for non-graded $R$-modules under the forgetful functor, and moreover, the latter functors are compatible with the corresponding sheaf-theoretic functors as is well-known in textbooks of algebraic geometry, see e.g. [Ha]. However, the information of the grading is lost by passing to the corresponding sheaf unless we use a sheaf with $C^*$-action.

(ii) If $M$ is a finitely generated graded $R$-module, then it is well-known that
\[ D_i(M) = 0 \quad \text{for} \quad i < 0. \]

1.3. Spectral sequences. For a bounded complex of finitely generated graded $R$-modules $M^\bullet$, we have a spectral sequence
\[ E_2^{p,q}(M^\bullet) = D_{-p}(H^{-q}M^\bullet) \Rightarrow D_{-p-q}(M^\bullet). \]
This can be defined for instance by taking graded free resolutions of $H^iM^\bullet$ and $\text{Im} d^i$ for $i \in \mathbb{Z}$, and then extending these to a graded free resolution of $M^\bullet$ by using the short exact sequences
\[ 0 \rightarrow \text{Im} d^{i-1} \rightarrow \text{Ker} d^i \rightarrow H^iM^\bullet \rightarrow 0, \quad 0 \rightarrow \text{Ker} d^i \rightarrow M^i \rightarrow \text{Im} d^i \rightarrow 0, \]
as is explained in classical books about spectral sequences. We can also construct (1.3.1) by using the filtration $\tau_{\leq -q}$ on $M^\bullet$ as in [De].

Applying (1.3.1) to $D(M^\bullet)$ and using $D(D(M^\bullet)) = M^\bullet$, we get
\[ E_2^{p,q}(M^\bullet) = D_{-p}(D_q(M^\bullet)) \Rightarrow H^{p+q}M^\bullet. \]

Lemma 1.4. Let $\text{Sh}(M)$ denote the coherent sheaf on $X := C^n$ corresponding to a finitely generated graded $R$-module $M$. Then we have the following equivalence.
\[ H^0_mM = M \iff \text{supp} \text{Sh}(M) \subset \{0\} \]
\[ \iff M \text{ is finite dimensional over } C, \]
\[ \iff D_i(M) = 0 \text{ for any } i \neq 0. \]

Proof. This is almost trivial except possibly for the last equivalence. It can be shown by restricting to a sufficiently general point of the support of $\text{Sh}(M)$ in case the support has positive dimension. Here we use the fact that the dual $D(\text{Sh}(M))$ is compatible with the direct image under a closed embedding, and this follows from Grothendieck duality for closed embeddings as is well-known, see e.g. [Ha]. This finishes the proof of Lemma (1.4).

The following is well-known, see [BrHe], [Ei], etc. We note here a short proof for the convenience of the reader.
Proposition 1.5. Let $M$ be a finitely generated $R$-module. Set $m := \dim \operatorname{supp} \mathcal{S}h(M)$. Then

\[ D_i(M) = 0 \text{ for } i > m. \tag{1.5.1} \]

Proof. There is a complete intersection $Z$ of dimension $m$ in $X = \operatorname{Spec} R$ such that $M$ is annihilated by the ideal $I_Z$ of $Z$, i.e. $M$ is an $R_Z$-module with $R_Z := R/I_Z$, and $I_Z$ is generated by a regular sequence $(g_i)_{i \in [1, n-m]}$ of $R$ with $g_iM = 0$. (Here $M$ is not assumed graded.) Set

\[ \omega_Z = \operatorname{Ext}^{n-m}_R(R_Z, \Omega^n). \]

This is called the canonical (or dualizing) module of $Z$. We then get

\[ D_i(M) = \operatorname{Ext}^{-i}_R(M, \omega_Z[m]), \tag{1.5.2} \]

by Grothendieck duality for the closed embedding $i_Z : Z \hookrightarrow X$, see e.g. \textit{Ha}, etc. In fact, taking an injective resolution $G$ of $\Omega^n[n]$, one can show (1.5.2) by using the canonical isomorphism

\[ \operatorname{Hom}_{R_Z}(M, \operatorname{Hom}_R(R_Z, G)) = \operatorname{Hom}_R(M, G). \]

Since the right-hand side of (1.5.2) vanishes for $i > m$, the assertion follows.

Corollary 1.6. Let $M$ be a finitely generated graded $R$-module with $\dim \operatorname{supp} \mathcal{S}h(M) = 1$. Then we have a short exact sequence

\[ 0 \to D_0(D_0(M)) \to M \to D_1(D_1(M)) \to 0, \tag{1.6.1} \]

together with

\[ D_0(D_1(M)) = 0, \quad D_1(D_0(M)) = 0. \tag{1.6.2} \]

Proof. By Lemma (1.5) we get

\[ ^nE_2^{p,q}(M) = 0 \text{ if } (p, q) \notin [-1, 0] \times [0, 1]. \]

So the spectral sequence (1.3.2) degenerates at $E_2$ in this case, and the assertion follows.

Remark 1.7. Let $M$ be a graded $R$-module of dimension 1, i.e. $C := \operatorname{supp} \mathcal{S}h(M)$ is one-dimensional. Let $I_M \subset R$ be the annihilator of $M$. Set $\overline{R} := R/I_M$. Let $y \in R$ be a general element of degree 1 whose restriction to any irreducible component of $C$ is nonzero. Set $R' := C[y] \subset R$. Let $\overline{m}, \overline{m}'$ be the maximal graded ideals of $\overline{R}, R'$. Let $H^i_{(R, m)}M$ denote $H^i_{\overline{m}}M$, and similarly for $H^i_{(\overline{R}, \overline{m})}M$, etc. (to avoid any confusion). There are canonical morphisms

\[ (R, m) \to (\overline{R}, \overline{m}) \leftarrow (R', m'), \tag{1.7.1} \]

and they imply canonical morphisms

\[ H^i_{(R, m)}M \leftarrow H^i_{(\overline{R}, \overline{m})}M \to H^i_{(R', m')}M. \]

Indeed, any graded injective resolution of $M$ over $\overline{R}$ can be viewed as a quasi-isomorphism over $R$ or $R'$, and we can further take its graded injective resolution over $R$ or $R'$, which induces the above morphisms.

These morphisms are isomorphisms since they are isomorphisms by forgetting the grading as is well-known. (Note that the morphisms $\operatorname{Spec} R \leftarrow \operatorname{Spec} \overline{R} \to \operatorname{Spec} R'$ are proper. Here it is also possible to use the graded local duality together with Grothendieck duality.) Using the long exact sequence associated with the local cohomology and the localization, we can show

\[ H^1_{(R', m')}M = M[y^{-1}]/M. \tag{1.7.2} \]
1.8. Proof of Theorem 1. As is explained in the introduction, we have the self-duality

$$D(K_f^*) = K_f^*(nd),$$

which implies the isomorphisms of graded $R$-modules

$$D_i(K_f^*) = H^{-i}(K_f^*)(nd).$$

Consider the spectral sequence (1.3.1) for $M$. This degenerates at $E_2$. Combining this with (1.8.1), we thus get

$$D_1(M) = N(nd), \quad D_0(N) = 0,$$

together with a short exact sequence

$$0 \to D_0(M) \to M(nd) \to D_1(N) \to 0.$$  

By (1.6.2) in Corollary (1.6) and Lemma (1.5) applied to $M$, $N$, the proof of Theorem 1 is then reduced to showing that (1.8.3) is naturally identified, up to the shift of grading by $nd$, with

$$0 \to M' \to M \to M'' \to 0.$$  

For this, it is enough to show

$$H^0_mD_0(M) = D_0(M), \quad H^0_mD_1(N) = 0.$$  

However, the first equality is equivalent to the vanishing of $D_i(D_0(M))$ for $i \neq 0$ by Lemma (1.4), and follows from (1.6.2) in Corollary (1.6) together with Lemma (1.5) applied to $D_0(M)$. The second equality follows for instance from the local duality (1.1.4) for $i = 0$ together with (1.6.2) in Corollary (1.6) applied to $N$. Thus (1.8.5) is proved. This finishes the proof of Theorem 1.

Remarks 1.9. (i) Corollary 2 can also be deduced from [Di2, Thm. 3.1]. Indeed, by the argument in Section 2 in loc. cit., we can deduce

$$\text{def}_{k-n}\Sigma_f = \tau_Z - \mu_k^n,$$

where $\text{def}_{k-n}\Sigma_f$ is as in loc. cit. Moreover, Thm. 3.1 in loc. cit. gives

$$\text{def}_{k-n}\Sigma_f = \mu_{nd-k} - \gamma_{nd-k} = \nu_{nd-k}.$$  

So Corollary 2 follows.

(ii) It is well-known that

$$\dim_C M_k'' = \dim_C M_k = \tau_Z \quad \text{if} \quad k \gg 0.$$  

Indeed, the first equality of (1.9.3) is trivial, and it is enough to show the last equality. Changing the coordinates, we may assume $x_n = y$, where $y$ is as in the introduction. On $\{x_n \neq 0\} \subset \mathbb{C}^n$, we have the the coordinates $x'_1, \ldots, x'_n$ defined by $x'_j = x_j/x_n$ for $j \neq n$, and $x'_n = x_n$. Using these, we have $f(x) = x''_n h(x')$, where $x' = (x'_1, \ldots, x'_{n-1})$. This implies that the restriction of $Sh(M)$ to the generic point of an irreducible component of the support of $M$ corresponding to $z \in Z$ has rank $\tau_z$ in the notation of the introduction. So (1.9.3) follows.

(iii) Assume $\dim \text{Sing } f^{-1}(0) = 1$, i.e. $\Sigma = \text{Sing } Z \neq \emptyset$. Let $(\partial f) \subset R$ denote the Jacobian ideal of $f$ (generated by the partial derivatives $\partial f/\partial x_i$ of $f$). Then the Jacobian
ring $R/(\partial f)$ (which is isomorphic to $M$ as a graded $R$-module up to a shift of grading) is a Cohen-Macaulay ring if and only if $M' = 0$. Indeed, these are both equivalent to the condition that $M$ is a Cohen-Macaulay $R$-module (since $\tau_Z \neq 0$ and hence $M'' \neq 0$). Here Grothendieck duality for closed embeddings is used to show the equivalence with the condition that $R/(\partial f)$ is a Cohen-Macaulay ring. Note that $M'$ might vanish in general, for instance if $f$ is as in Example (2.7) below or even in case $f = xyz$.

(iv) Assume $\bigcap_{i=1}^m g_i^{-1}(0) \subset \mathbb{C}^n$ has codimension $\geq r$, where $g_i \in R$ ($i \in [1, m]$). Then there is a regular sequence $(h_j)_{j \in [1, r]}$ of $R$ with $h_j \in V := \sum_{i=1}^m C g_i$ by increasing induction on $r$ or $m$. This implies the vanishing of the cohomology of the Koszul complex:

$$H^k K^*(R; g_1, \ldots, g_m) = 0 \quad (k < r),$$

by using the $n$-ple complex structure of the Koszul complex as is well-known (see Remark (v) below). In fact, we can replace the basis $(g_i)$ of the vector space $V$ so that a different expression of the Koszul complex can be obtained. (However, it is not always possible to choose $h_j$ so that $\sum_{i=1}^m R g_i = \bigcap_{j=1}^r R h_j$ even if $\bigcap_{i=1}^m g_i^{-1}(0)$ has pure codimension $r$ unless $(g_i)$ is already a regular sequence, i.e. $r = m$.)

(v) For $g_i \in R$ ($i \in [1, m]$), the Koszul complex $K^*(R; g_1, \ldots, g_m)$ can be identified with the associated single complex of the $m$-ple complex whose $(j_1, \ldots, j_m)$-component is $R$ for $(j_1, \ldots, j_m) \in [0, 1]^m$, and 0 otherwise, where its $i$-th differential $d_i$ is defined by the multiplication by $g_i$ on $R$.

(vi) Theorem 1 holds with $df$ in the definition of the Koszul complex replaced by a 1-form $\omega = \sum_{i=1}^n g_i dx_i$ if the $g_i$ are homogeneous polynomials of degree $d - 1$ such that $\bigcap_i g_i^{-1}(0) \subset \mathbb{C}^n$ is at most 1-dimensional. See [Pe], [VStWa] for the (non-graded) analytic local case.

2. Calculation of the Koszul cohomologies

In this section we explain some methods to calculate the Koszul cohomologies in certain cases.

Lemma 2.1. Let $r$ be the dimension of the vector subspace of $\mathbb{C}^n$ generated by the one-dimensional vector subspaces corresponding to the singular points of $Z$. Then

$$\mu''_n = 1, \quad \mu''_{n+1} \geq r.$$  

Proof. Let $\Sigma'$ be a subset of $\Sigma (= \text{Sing} Z)$ corresponding to linearly independent $r$ vectors of $\mathbb{C}^n$. Let $I_{\Sigma'}$ be the (reduced) graded ideal of $R$ corresponding to $\Sigma'$. There is a canonical surjection

$$M \rightarrow \overline{M} := \Omega^n / I_{\Sigma'} \Omega^n. \tag{2.1.1}$$

The target is a free graded $\mathbb{C}[y]$-module of rank $r$, where $y$ is as in the introduction, and it has free homogeneous generators $w_i$ ($i \in [1, r]$) with $\deg w_1 = n$ and $\deg w_i = n + 1$ for $i > 1$. So the surjection (2.1.1) factors through $M''$, and the assertion follows.

Proposition 2.2. Let $f = f_1 + f_2$ with $f_1 \in \mathbb{C}[x_1, \ldots, x_{n_1}]$, $f_2 \in \mathbb{C}[x_{n_1+1}, \ldots, x_n]$ where $1 < n_1 < n - 1$. Assume the dimensions of the singular loci of $f_1^{-1}(0) \subset \mathbb{C}^n$ and $f_2^{-1}(0) \subset \mathbb{C}^{n-n_1}$ are respectively 1 and 0. Then there are isomorphisms of graded $R$-modules

$$M' = M'_{(1)} \otimes \mathbb{C} M_{(2)}, \quad M'' = M''_{(1)} \otimes \mathbb{C} M_{(2)}, \quad N = N_{(1)} \otimes \mathbb{C} M_{(2)},$$

and, setting $S(\mu) := \sum_k \mu_k \tau^k \in \mathbb{Z}[t]$, etc., we have the equalities

$$S(\mu') = S(\mu'_{(1)}) S(\mu'_{(2)}), \quad S(\mu'') = S(\mu''_{(1)}) S(\mu''_{(2)}), \quad S(\nu) = S(\nu_{(1)}) S(\nu_{(2)}).$$
where $M'_i$, $M''_i$, $N_i$, and $\mu'_i(k), \mu''_i(k), \nu_i(k)$ ($k \in \mathbb{Z}$) are defined for $f_i$ ($i = 1, 2$).

**Proof.** Using the $n$-ple complex structure of the Koszul complex as in Remark (1.9)(v), we get the canonical isomorphism

$\ast K^*_f = \ast K^*_{f_1} \otimes_{\mathbb{C}} \ast K^*_{f_2},$

where $\ast K^*_{f_1}$ is defined by using the subring $\mathbb{C}[x_1, \ldots, x_m]$, and similarly for $\ast K^*_{f_2}$. Since $f_2^{-1}(0)$ has an isolated singularity, $K^*_{f_2}$ is naturally quasi-isomorphic to $M'_2$. We get hence

$$M = M(1) \otimes_{\mathbb{C}} M'_2, \quad N = N(1) \otimes_{\mathbb{C}} M'_2.$$  

Moreover, the freeness of $M''_1 \otimes_{\mathbb{C}} M'_2$ over $\mathbb{C}[y]$ can be shown by using an appropriate filtration of $M'_2$; where $y$ is as in the introduction. These imply that the following two short exact sequences are identified with each other:

$$0 \to M' \to M \to M'' \to 0,$$

$$0 \to M'(1) \otimes_{\mathbb{C}} M'_2 \to M(1) \otimes_{\mathbb{C}} M'_2 \to M''_1 \otimes_{\mathbb{C}} M'_2 \to 0.$$

So the assertion follows.

For the proof of Proposition (2.6) below, we need the following lemma. Essentially this may be viewed as a special case of Prop. 13 in [ChDi], see Remark (2.5) below. We note here a short proof of the lemma using Corollaries 1 and 2 and (0.3) for the convenience of the reader.

**Lemma 2.3.** Assume $n = 2$. Let $r$ be the number of the irreducible components of $f^{-1}(0) \subset \mathbb{C}^2$. Then $\tau_Z = d - r$, and we have for $k \in \mathbb{Z}$

$$\mu'_k = \max(r - 1 - |d - k|, 0),$$

$$\mu''_k = (k - 1)[0, \tau_Z],$$

$$\nu_k = (k - d - r + 1)[0, \tau_Z],$$

where $x_{[\alpha, \beta]}$ for $x, \alpha, \beta \in \mathbb{Z}$ with $\alpha < \beta$ is defined by

$$x_{[\alpha, \beta]} = \begin{cases} 
\alpha & \text{if } x \leq \alpha, \\
\beta & \text{if } \beta \leq x.
\end{cases}$$

**Proof.** We have the decomposition

$$f = \prod_{i=1}^r g_i^{m_i},$$

with $\deg g_i = 1$ and $m_i \geq 1$. For $z \in \mathbb{P}^1$ corresponding to $g_i^{-1}(0) \subset \mathbb{C}^2$, we have

$$\tau_z = m_i - 1, \quad \text{and hence} \quad \tau_Z = d - r.$$

Setting

$$f' := \prod_{i=1}^r g_i^{m_i - 1},$$

we get

$$M'' = \Omega^2 / f' \Omega^2.$$  

Indeed, the right-hand side is a quotient graded $R$-module of $M$, and is a free graded $\mathbb{C}[y]$-module of rank $\tau_Z$. Since $\deg f' = \tau_Z$, this implies

$$\mu''_k = (k - 1)[0, \tau_Z].$$

Using Corollary 2, we then get

$$\nu_k = d - r - (2d - k - 1)[0, \tau_Z] = (k - d - r + 1)[0, \tau_Z].$$
Here note that
\[ \nu_k = 0 \text{ if } k \leq d. \]
For \( n = 2 \) and \( k \leq d \), we have
\[ \gamma_k = \max(k - 1, 0). \]
By (0.3) we then get for \( k \leq d \)
\[ \mu_k' = \gamma_k - \mu_k'' = \max(k - 1 - \tau_Z, 0). \]
The formula for \( k \geq d \) follows by using the symmetry in Corollary 1. This finishes the proof of Lemma (2.3).

By an easy calculation we see that Lemma (2.3) is equivalent to the following.

**Corollary 2.4.** With the notation and the assumption of Lemma (2.3), we have
\begin{align*}
S(\mu') &= S(1, r - 1) S(d - r + 1, d - 1), \\
S(\mu'') &= S(1, \infty) S(1, d - r), \\
S(\nu) &= S(d + r - 2, \infty) S(1, d - r),
\end{align*}
where \( S(\mu') \) is as in Proposition (2.2), and \( S(a, b) \) for \( a \in \mathbb{N}, b \in \mathbb{N} \cup \{\infty\} \) is defined by
\begin{equation}
S(a, b) := \sum_{k=a}^{b} t^k \in \mathbb{Z}[[t]] \text{ if } a \leq b, \text{ and } 0 \text{ otherwise.}
\end{equation}

**Remark 2.5.** With the notation and the assumption of Corollary (2.4), the following is shown in [Di2, Example 14 (i)] as a corollary of Prop. 13 (loc. cit.)
\begin{equation}
S(\mu) = t^2 (1 - 2t^{d-1} + t^{d+r-2})/(1 - t)^2.
\end{equation}

By Corollaries 2 and 3 together with (0.3), this is essentially equivalent to the equalities in (2.4.1). In fact, it seems rather easy to deduce (2.5.1) from (2.4.1). For the converse some calculation seems to be needed. (The details are left to the reader.)

In case \( n_1 = 2 \), we can calculate \( \mu_1'(1), \mu_2'(1), \nu(1), \) for \( f_1 \) by Lemma (2.3), and get the following.

**Proposition 2.6.** Assume \( f = f_1 + f_2 \) as in Proposition (2.2) with \( n_1 = 2 \). Let \( r \) be the number of the irreducible components of \( f_1^{-1}(0) \subset \mathbb{C}^2 \). Then, under the assumption of Proposition (2.2), we have
\begin{align*}
S(\mu') &= S(1, r - 1) S(d - r + 1, d - 1) S(1, d - 1)^{n-2}, \\
S(\mu'') &= S(1, \infty) S(1, d - r) S(1, d - 1)^{n-2}, \\
S(\nu) &= S(d + r - 2, \infty) S(1, d - r) S(1, d - 1)^{n-2},
\end{align*}
where \( S(a, b) \) is as in (2.4.2).

**Proof.** The assertion follows from Corollary (2.4) and Proposition (2.2), since \( S(\mu') \) in the isolated singularity case is invariant by \( \mu \)-constant deformation, and is given by (0.2).

**Example 2.7.** Let \( f \) be as in Theorem 1, and assume further
\[ f \in \mathbb{C}[x_1, \ldots, x_{n-1}] \subset \mathbb{C}[x_1, \ldots, x_n]. \]
Then \( f \) has an isolated singularity at the origin of \( \mathbb{C}^{n-1} \). Set
\[ \gamma_j' := \dim_{\mathbb{C}}(\Omega^{n-1}/df \wedge \Omega^{n-2}) \quad \text{with} \quad \Omega^k := \Gamma(\mathbb{C}^{n-1}, \Omega_{\mathbb{C}^{n-1}}^k). \]
We have the symmetry
\begin{equation}
\gamma_j' = \gamma_{(n-1)d-j}'.
\end{equation}
Remark 2.10. Conditions (0.5) hold in the nodal hypersurface case by [DiSt2, Thm. 2.1]. Indeed, it is shown there that

(2.9.1) \( \nu_{d+n+k-1} \neq 0 \),

and hence

(2.9.2) \( \text{Condition (0.5) does not hold if } (n-2)(d-2) \geq 2(k+1). \)

Indeed, (2.9.1) follows from the definition \( N := H^{-1}(^*K) \) since \( \deg f_i = d - 1 \).

This applies to \( f \) in Example (2.7) with \( k = 0 \) since \( f_n = 0 \), and to \( f \) in Example (2.8) with \( k = 1 \) since

\[
(d-a)x_1 f_1 = ax_2 f_2.
\]

Remark 2.10. Conditions (0.5) hold in the nodal hypersurface case by [DiSt2] Thm. 2.1. Indeed, it is shown there that

(2.10.1) \( \nu_k = 0 \) if \( k \leq (n_1+1)d \) with \( n \) even or \( k \leq (n_1+1)d-1 \) with \( n \) odd,

In this case, we have \( M' = 0 \), and

\[
(2.7.2) \quad \mu_k = \mu_k'' = \sum_{j \leq k-1} \gamma_j', \quad \nu_k = \sum_{j \leq k-d} \gamma_j' = \sum_{j \geq nd-k} \gamma_j',
\]

where the last equality follows from the symmetry (2.7.1), and Corollary 2 is verified directly in this case.

Equivalently, \( \mu_k'' = \mu_k \) and \( \nu_k \) are given as follows:

\[
(2.7.3) \quad S(\mu) = S(1, \infty) S(1, d-1)^{n-1}, \quad S(\nu) = S(d, \infty) S(1, d-1)^{n-1},
\]

where \( S(\mu) \), etc. are as in Proposition (2.2), and the order of the coordinates are changed.

Example 2.8. Assume \( n, d \geq 3 \). Let

\[
(2.8.1) \quad f = x_1^a x_2^{d-a} + \sum_{i=3}^n x_i^d \quad \text{with } 0 < a < d.
\]

We can apply Proposition (2.6) to this example. More precisely, the calculation of \( \mu_k', \mu_k'' \) and \( \nu_k \) are reduced to the case \( n = 2 \) by Proposition (2.2), where \( n_1 = 2 \) and

\[
\mu_1 = x_1^a x_2^{d-a}, \quad \mu_2 = x_2^d, \quad f_1 = x_1^a x_2^{d-a}, \quad \mu_2 = \sum_{i=3}^n x_i^d.
\]

The calculation for \( \mu_1 \) follows from Lemma (2.3) or Corollary (2.4) where \( r = 2 \). For instance, we get in the notation of Proposition (2.2)

\[
\mu_1'(1, k) = \begin{cases} 
1 & \text{if } k = d, \\
0 & \text{if } k \neq d,
\end{cases}
\]

and hence

\[
\mu_k' = \mu_2'(k+d) = \gamma_k'' (k \in \mathbb{Z}),
\]

where \( \gamma_k'' \) is as in (0.2) with \( n \) replaced by \( n-2 \). By Proposition (2.6), we have

\[
(2.8.2) \quad \begin{align*}
S(\mu') &= t^d S(1, d-1)^{n-2}, \\
S(\mu'') &= S(1, \infty) S(1, d-2) S(1, d-1)^{n-2}, \\
S(\nu) &= S(d, \infty) S(1, d-2) S(1, d-1)^{n-2},
\end{align*}
\]

where \( S(\mu') \), etc. are as in Proposition (2.2).

Remark 2.9. If there is a nontrivial relation of degree \( k \leq d-2 \) among the partial derivatives \( f_i := \partial f/\partial x_i \), i.e. if there are homogeneous polynomials \( g_i \) of degree \( k \leq d-2 \) with \( \sum_i g_i f_i = 0 \) and \( g_i \neq 0 \) for some \( i \), then we have

(2.9.1) \( \nu_{d+n+k-1} \neq 0 \),

and hence

(2.9.2) \( \text{Condition (0.5) does not hold if } (n-2)(d-2) \geq 2(k+1). \)

Indeed, (2.9.1) follows from the definition \( N := H^{-1}(^*K) \) since \( \deg f_i = d - 1 \).

This applies to \( f \) in Example (2.7) with \( k = 0 \) since \( f_n = 0 \), and to \( f \) in Example (2.8) with \( k = 1 \) since

\[
(d-a)x_1 f_1 = ax_2 f_2.
\]
where \( n_1 := [(n - 1)/2] \). (There is a difference in the grading on \( N \) by \( d \) between this paper and loc. cit., and \( n \) in this paper is \( n + 1 \) in loc. cit.)

3. Spectrum

In this section we recall some basics from the theory of spectra, and prove Proposition (3.3), (3.4), and (3.5).

3.1. Hodge and pole order filtrations. Let \( f \) be a homogeneous polynomial of \( n \) variables with degree \( d \). It is well-known that there is a \( \mathbb{C} \)-local system \( L_k \) (\( k \in [1, d] \)) of rank 1 on \( U := Y \setminus Z \) such that

\[
H^j(U, L_k) = H^j(f^{-1}(1), \mathbb{C})_\lambda \quad (\lambda = \exp(-2\pi ik/d), \; k \in [1, d]),
\]

where \( H^j(f^{-1}(1), \mathbb{C})_\lambda \) is the \( \lambda \)-eigenspace of the cohomology for the semisimple part of the monodromy, see e.g. [Di1], etc. (Note that monodromy in our paper means the one as a local system, see also [BuSa, Section 1.3], etc.) Let \( \mathcal{L}_k \) be the meromorphic extension of \( L_k \otimes \mathcal{O}_U \). This is a regular holonomic \( \mathcal{D}_Y \)-module, and

\[
H^j(Y, \Omega^*_Y(\mathcal{L}_k)) = H^j(f^{-1}(1), \mathbb{C})_\lambda \quad (\lambda = \exp(-2\pi ik/d), \; k \in [1, d]),
\]

where \( \Omega^*_Y(\mathcal{L}_k) \) denotes the de Rham complex of \( \mathcal{L}_k \). We have the Hodge and pole order filtrations \( F_i \) and \( P_i \) on \( \mathcal{L}_k \) such that

\[
(3.1.3) \quad F_i \subset P_i,
\]

where the equality holds outside the singular points of \( Z \), and

\[
P_i \mathcal{L}_k = \begin{cases} \mathcal{O}_Y(id + k) & \text{if } i \geq 0, \\ 0 & \text{if } i < 0, \end{cases}
\]

see e.g. [Sa4] Section 4.8]. (Note that \( F \) comes from the Hodge filtration of a mixed Hodge module.) Set \( F^i = F_{-i} \), \( P^i = P_{-i} \). They induces the Hodge and pole order filtrations on \( \Omega^*_Y(\mathcal{L}_k) \) such that the \( j \)-th components of \( F^i \Omega^*_Y(\mathcal{L}_k), P^i \Omega^*_Y(\mathcal{L}_k) \) are respectively given by

\[
\Omega^j \otimes \Omega_Y P^{i-j} \mathcal{L}_k, \quad \Omega^j \otimes \Omega_Y F^{i-j} \mathcal{L}_k.
\]

By the isomorphism (3.1.2) they further induce the Hodge and pole order filtrations on the Milnor cohomology \( H^j(f^{-1}(1), \mathbb{C}) \). Here \( F \) coincides with the Hodge filtration of the canonical mixed Hodge structure. By using the Bott vanishing theorem, \( H^\bullet(Y, P^i \Omega^*_Y(\mathcal{L}_k)) \) can be calculated by the complex whose \( j \)-th component is

\[
(3.1.3) \quad \Gamma(Y, \Omega^j \otimes \Omega_Y P^{i-j} \mathcal{L}_k) = \begin{cases} \Gamma(Y, \Omega^j \otimes ((j - i)d + k)) & \text{if } j \geq i, \\ 0 & \text{if } j < i. \end{cases}
\]

But it does not give a strict filtration, and it is not necessarily easy to calculate it.

Note that the pole order filtration coincides with the one defined by using the Gauss-Manin system, see (4.4.7) and (4.5.7) below.

3.2. Spectrum. For \( f \) as in (3.1), the spectrum \( \text{Sp}(f) = \sum_{\alpha \in \mathbb{Q}} n_{f, \alpha} t^\alpha \) is defined by

\[
(3.2.1) \quad n_{f, \alpha} := \sum_j (-1)^j n_{j, \alpha} \dim \text{Gr}_F^{\rho} \tilde{H}^j(f^{-1}(1), \mathbb{C})_\lambda \quad \text{with} \quad p = |n - \alpha|, \; \lambda = \exp(-2\pi i \alpha),
\]

(see [St2], [St3]). Here \( \tilde{H}^j(f^{-1}(1), \mathbb{C}) \) is the reduced cohomology, and we have by definition

\[
(3.2.2) \quad \lfloor \alpha \rfloor := \max \{ i \in \mathbb{Z} \mid i \leq \alpha \} , \quad \lceil \alpha \rceil := \min \{ i \in \mathbb{Z} \mid i \geq \alpha \} \quad (\alpha \in \mathbb{R}).
\]
The pole order spectrum $\text{Sp}_p(f)$ is defined by replacing $F$ with $P$.

For $j \in \mathbb{N}$, we define $\text{Sp}^j(f) = \sum_{\alpha \in \mathbb{Q}} n_{f,\alpha}^j t^\alpha$ by

\begin{equation}
(3.2.3) \quad n_{f,\alpha}^j := \dim \text{Gr}_F^p \widetilde{H}^{n-1-j}(f^{-1}(1), C)_\lambda
\end{equation}

with $p = [n - \alpha]$, $\lambda = \exp(-2\pi i \alpha)$, so that

$$\text{Sp}(f) = \sum_j (-1)^j \text{Sp}^j(f).$$

Similarly $\text{Sp}^j_p(f) = \sum_{\alpha \in \mathbb{Q}} n_{f,\alpha}^{P,j} t^\alpha$ is defined by replacing $F$ with $P$.

Set $Z := \{ f = 0 \} \subset Y := \mathbb{P}^{n-1}$. Let $\pi : (\widetilde{Y}, \widetilde{Z}) \to (Y, Z)$ be an embedding resolution, and $E_i$ be the irreducible components of $\widetilde{Z}$ with $m_i$ the multiplicity of $E_i$. Let $\alpha = k/d + q \in (0, n]$ with $k \in [1, d], q \in [0, n - 1]$. We have by [BuSa 1.4.3]

\begin{equation}
(3.2.4) \quad n_{f,\alpha}^j = \dim H^{p-j}(\widetilde{Y}, \text{O}_{\widetilde{Y}}^{n-1-q}(\log \widetilde{Z}) \otimes \text{O}_{\widetilde{Y}}(-\ell \widetilde{H} + \sum_i [\ell m_i/d]) E_i),
\end{equation}

where $\ell := d - k$, and $\widetilde{H}$ is the pull-back of a sufficiently general hyperplane $H$ of $Y$.

In a special case we get the following.

**Proposition 3.3.** Assume $n = 2$. Set $e := \gcd(m_i)$ with $m_i$ the multiplicities of the irreducible factors of $f$. Then, for $\alpha = k/d + q \in (0, 2)$ with $k \in [1, d], q \in [0, 1)$, we have

\begin{equation}
(3.3.1) \quad n_{f,\alpha}^j = \begin{cases} 
  r - 1 + k - \sum_i \lfloor km_i/d \rfloor & \text{if } j = 0, q = 0, \\
  \max(-k - 1 + \sum_i \lfloor km_i/d \rfloor, 0) & \text{if } j = 0, q = 1, \\
  1 & \text{if } j = 1, q = 1, ke/d \in \mathbb{Z}, \\
  0 & \text{otherwise},
\end{cases}
\end{equation}

where $\lfloor \alpha \rfloor$ is as in (3.2.2).

**Proof.** We have $\text{O}_{\mathbb{P}^1}(\log Z) = \text{O}_{\mathbb{P}^1}(r - 2)$ with $\widetilde{Y} = Y = \mathbb{P}^1$, $\widetilde{Z} = Z$, $\widetilde{H} = H$, Hence (3.2.4) in this case becomes

$$n_{f,\alpha}^j = \begin{cases} 
  \dim H^0(\mathbb{P}^1, \text{O}_{\mathbb{P}^1}(\log Z)(-\ell + \sum_i \lfloor \ell m_i/d \rfloor)) & \text{if } j = 0, q = 0, \\
  \dim H^1(\mathbb{P}^1, \text{O}_{\mathbb{P}^1}(-\ell + \sum_i \lfloor \ell m_i/d \rfloor)) & \text{if } j = 0, q = 1, \\
  \dim H^0(\mathbb{P}^1, \text{O}_{\mathbb{P}^1}(-\ell + \sum_i \lfloor \ell m_i/d \rfloor)) & \text{if } j = 1, q = 1, \\
  0 & \text{otherwise},
\end{cases}$$

and then

\begin{equation}
(3.3.2) \quad n_{f,\alpha}^j = \begin{cases} 
  r - 1 - \ell + \sum_i \lfloor \ell m_i/d \rfloor & \text{if } j = 0, q = 0, \\
  \max(\ell - 1 + \sum_i \lfloor \ell m_i/d \rfloor, 0) & \text{if } j = 0, q = 1, \\
  \max(-\ell + 1 + \sum_i \lfloor \ell m_i/d \rfloor, 0) & \text{if } j = 1, q = 1, \\
  0 & \text{otherwise}.
\end{cases}
\end{equation}

Since $\sum_i m_i = d$ and $e = \gcd(m_i)$, we have

$$\ell > \sum_i \lfloor \ell m_i/d \rfloor \iff \ell m_i/d \notin \mathbb{Z} \ (\exists i) \iff \ell e/d \notin \mathbb{Z}.$$

So (3.3.1) follows (since $\ell = d - k$). This finishes the proof of Proposition (3.3).

We note here an application of Theorem 2, Theorem (5.3) and Corollary (5.4) below. (This will not be used in their proofs.)
Proposition 3.4. Assume $n = 2$. Then $\text{Sp}_P(f) = \text{Sp}_P^0(f) - \text{Sp}_P^1(f)$ is given by

$$\text{Sp}_P^j(f) = \begin{cases} \sum_k (\mu_k - \nu_{k+d}) t^{k/d} + (t^{1/e} + \cdots + t^{(e-1)/e}) & \text{if } j = 0, \\ t (t^{1/e} + \cdots + t^{(e-1)/e}) & \text{if } j = 1, \end{cases}$$

with $\mu_k, \nu_k$ explicitly expressed in Lemma (2.3), and $e = \gcd(m_i)$ as in Proposition (3.3).

Proof. The pole order spectral sequence degenerates at $E_2$ by Corollary (5.4) below. So the assertion is shown in the case $e = 1$, since the last condition implies that $\nu_k^{(2)} = 0$. In the general case it is well-known that

$$\tilde{H}^0(f^{-1}(1), C)_\lambda = \begin{cases} C & \text{if } \lambda^e = 1 \text{ with } \lambda \neq 1, \\ 0 & \text{otherwise} \end{cases}.$$  

By using Theorem (5.3) and Lemma (2.3), this implies

$$N^{(2)}_{k+d} = \begin{cases} C & \text{if } k = i (d/e) \text{ with } i \in \{1, \ldots, e - 1\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $N^{(2)} \subset N$ is the kernel of $d^{(1)}$. This gives also the information of the coimage of $d^{(1)}$ which is a morphism of degree $-d$. So the correction terms for $\text{Sp}_P^0(f)$ and $\text{Sp}_P^1(f)$ coming form the non-vanishing of $d^{(1)}$ are given respectively by

$$t^{1/e} + \cdots + t^{(e-1)/e} \quad \text{and} \quad t (t^{1/e} + \cdots + t^{(e-1)/e}).$$

So (3.4.1) follows. This finishes the proof of Proposition (3.4).

Proposition 3.5. Assume $f = f_1 + f_2$ as in Proposition (2.2) with $n_1 = 2$. Then, under the assumption of Proposition (2.2), we have

$$\mu'_k \leq n^0_{f,k/d}, \quad \mu'_k \leq n^P_{f,k/d} \quad (k \in \mathbb{Z}),$$

where $n^0_{f,k/d}$, $n^P_{f,k/d}$ are as in (3.2).

Proof. The Thom-Sebastiani type theorem holds for $\text{Sp}_P^0(f)$, $\text{Sp}_P^0(f)$ under the assumption of Proposition (2.2), see (4.9) below. So the assertion is reduced to the case $f = f_1$ with $n = 2$. The assertion for $\text{Sp}_P^0(f)$ then follows from Proposition (3.4) and Lemma (2.3), where we may assume $r \geq 2$ since $\mu_k = 0$ otherwise. By using Lemma (2.3) and Proposition (3.3) (more precisely, (3.3.2) for $q = 0$ and (3.3.1) for $q = 1$), the assertion for $\text{Sp}_P^0(f)$ is reduced to the following trivial inequalities

$$r - 1 - (d - k) \leq r - 1 - \ell + \sum_{i=1}^{\ell} \lfloor \ell m_i / d \rfloor \quad (\ell \in [0, d - 1], q = 0),$$

$$r - 1 + d - (k + d) \leq -k - 1 + \sum_{i=1}^{k} \lfloor km_i / d \rfloor \quad (k \in [1, d - 1], q = 1),$$

where $\ell = d - k$. (Note that $k$ in Lemma (2.3) is $k + d$ in the case $q = 1$.) This finishes the proof of Proposition (3.5).

Remarks 3.6. (i) If $f$ has an isolated singularity, the equality holds in (3.5.1), and $S(\mu')$ (with $t$ replaced by $t^{1/d}$) coincides with the spectrum $\text{Sp}(f)$, see [St1] and also [Gr1], [SkSt], [Va], etc. It would be interesting if (3.5.1) holds in a more general case.

(ii) Let $f$ be as in (3.1). Assume $Z \subset \mathbb{P}^{n-1}$ has only isolated singularities. Let $\alpha'_f$ be the minimal of the exponents of the spectrum for all the singularities of $Z$ (see also Corollary (5.5) below). Then the multiplicity $n_{f,\alpha}$ of the spectrum $\text{Sp}(f)$ for $\alpha = p/d < \min(\alpha'_f, 1)$ can be given by

$$n_{f,p/d} = \binom{p-1}{n-1} \quad (p/d < \min(\alpha'_f, 1)).$$
This follows from a formula for multiplier ideals \([\text{Sa4}, \text{Prop. 1}]\) together with \([\text{Bu}]\) (see also a remark before \([\text{Sa4}, \text{Cor. 1}]\)). This equality holds also for the pole order spectrum since \(\mu_p\) is at most the right-hand side of the equality and \(F \subset P\) (and \(\nu_p = 0\) for \(p < d\)).

**Example 3.7.** Let \(f = (x^m + y^m) x^m y^n (m \geq 2)\), where \(d = 3m, r = m + 2, \tau_z = 2m - 2\). For \(\alpha = k/3m + q\) with \(k \in [1, 3m], q = 0, 1\), we have by Proposition (3.3)

\[
(3.7.1) \quad n_{f, \alpha} = \begin{cases} 
  k + 1 - 2[k/3] & \text{if } \alpha \in (0, 1), \ q = 0, \\
  m - k - 1 + 2[k/3] & \text{if } \alpha \in (1, 2), \ q = 1.
\end{cases}
\]

In fact, \(m_i = 1\) for \(i \in [1, m]\), and \(m_i = m\) for \(i = m + 1, m + 2\). Here \(e = 1\) in the notation of Proposition (3.3), and hence \(\text{Sp}^1(f) = 0, \text{Sp}(f) = \text{Sp}^0(f)\) (similarly for \(\text{Sp}_p(f)\)).

On the other hand, Lemma (2.3) and Proposition (3.4) imply that

\[
(3.7.2) \quad n_{f, k/3m}^p = \mu_k^{(2)} = \mu_k' + \mu_k'' - \nu_{k + 3m} = \begin{cases} 
  0 & (k \leq m), \\
  k - 1 & (1 \leq k \leq m + 1), \\
  m & (m + 1 \leq k \leq 3m - 1), \\
  4m + 1 - k & (3m \leq k \leq 4m + 1), \\
  0 & (4m + 1 \leq k).
\end{cases}
\]

(Note that \(\mu_k^{(2)} + \nu_{k + 3m}^{(2)} = m\ (k \in [1, 3m - 1])\) and \(\mu_{3m}^{(2)} = m + 1\).) In fact, we have by Lemma (2.3)

\[
\mu_k' = \begin{cases} 
  0 & (k \leq 2m - 1), \\
  k - 2m + 1 & (2m - 1 \leq k \leq 3m), \\
  4m + 1 - k & (3m \leq k \leq 4m + 1), \\
  0 & (4m + 1 \leq k),
\end{cases}
\]

\[
\mu_k'' = \begin{cases} 
  0 & (k \leq 1), \\
  k - 1 & (1 \leq k \leq 2m - 1), \\
  2m - 2 & (2m - 1 \leq k),
\end{cases}
\]

\[
\nu_{k + 3m} = \begin{cases} 
  0 & (k \leq m + 1), \\
  k - m - 1 & (m + 1 \leq k \leq 3m - 1), \\
  2m - 2 & (3m - 1 \leq k).
\end{cases}
\]

These formulas show that the relation between the Steenbrink spectrum \(\text{Sp}(f)\) and the pole order spectrum \(\text{Sp}_p(f)\) is rather complicated even for \(n = 2\) in general.

### 4. Gauss-Manin systems and Brieskorn modules

In this section we prove Theorem 2 after recalling some facts from Gauss-Manin systems and Brieskorn modules.

#### 4.1. Graded Gauss-Manin complexes.

Let \(f\) be a homogeneous polynomial in \(R\) with degree \(d\). In the notation of (1.1), the graded Gauss-Manin complex \(C_f^*\) associated with \(f\) is defined by

\[
C_f^j := \Omega^j[\partial_t] \quad (j \in \mathbb{Z}),
\]

where \(\partial_t\) has degree \(-d\). This means that

\[
\Omega^j \partial_t^p = \Omega^j(pd),
\]
where \((pd)\) denotes the shift of the grading as in the introduction. Its differential \(d\) is defined by
\[
(d\omega \partial_t^p) = (d\omega) \partial_t^p - (df \land \omega) \partial_t^{p+1} \quad \text{for } \omega \in \Omega^k.
\]
where \(d\omega\) denotes the differential of the de Rham complex. It has a structure of a complex of \(C[t]\langle \partial_t \rangle\)-modules defined by
\[
t(\omega \partial_t^p) = (f\omega) \partial_t^p - p\omega \partial_t^{p-1}, \quad \partial_t(\omega \partial_t^p) = \omega \partial_t^{p+1} \quad \text{for } \omega \in \Omega^j.
\]
The Gauss-Manin systems are defined by the cohomology groups \(H^j C_f^*\) \((j \in \mathbb{Z})\). These are regular holonomic graded \(C[t]\langle \partial_t \rangle\)-modules. By the same argument as in \([\text{BaSa}]\), we have
\[
\text{(4.3.3)} \quad \text{The action of } \partial_t \text{ on } H^j C_f^* \text{ is bijective for } j \neq 1.
\]

4.2. Brieskorn modules. Let \((A_f^*, d)\) be a graded subcomplex of the de Rham complex \((\Omega^*, d)\) defined by
\[
A_f^j := \ker(df \land : \Omega^j \rightarrow \Omega^{j+1}(d)).
\]
The Brieskorn modules are graded \(C[t]\langle \partial_t \rangle\)-modules defined by its cohomology groups
\[
H^j A^*_f \quad (j \in \mathbb{Z}).
\]
The actions of \(t, \partial_t^{-1}, \partial_t\) are respectively defined by the multiplication by \(f\);
\[
\partial_t^{-1}[\omega] = [df \land \xi] \quad \text{with} \quad df = \omega,
\]
\[
\partial_t t[\omega] = [d\eta] \quad \text{with} \quad df \land \eta = f\omega,
\]
where \([\omega]\) denotes the cohomology class, see \([\text{Bri}]\), \([\text{BaSa}]\), etc. (In case \(j = 1\), we have to choose a good \(\xi\) for the action of \(\partial_t^{-1}\), see \([\text{BaSa}]\).) Moreover, we have
\[
\text{(4.2.1)} \quad \partial_t t[\omega] = (k/d)[\omega] \quad \text{for } \omega \in (H^j A_f^*)_k,
\]
where \((H^j A_f^*)_k\) denotes the degree \(k\) part. (This follows from the definition by using the contraction with the Euler vector field \(\xi := \sum x_i \partial/\partial x_i\).) This implies
\[
\text{(4.2.2)} \quad t[\omega] = (k/d) \partial_t^{-1}[\omega] \quad \text{for } \omega \in (H^j A_f^*)_k.
\]
Since \((H^j A_f^*)_k = 0\) for \(k \leq 0\), this implies that \(\text{Coker } t\) in Theorem 2 can be replaced with \(\text{Coker } \partial_t^{-1}\).

There is a natural inclusion
\[
A^*_f \hookrightarrow C^*_f.
\]
This is compatible with the actions of \(t, \partial_t^{-1}, \partial_t\) on the cohomology by definition. So (4.2.1) holds also for \(\omega \in (H^j C_f^*)_j\), since the image of \(H^j A^*_f\) generates \(H^j C^*_f\) over \(C[\partial_t]\). The last assertion is well-known in the analytic case, see e.g. \([\text{BaSa}]\), and is reduced to this case by using the scalar extensions
\[
R \hookrightarrow C\{x_1, \ldots, x_n\}, \quad C[t] \hookrightarrow C\{t\}.
\]

For \(j \in \mathbb{Z}\), we then get
\[
\text{(4.2.3)} \quad H^{j+1}(C_f^*)_k = \begin{cases} H^j(f^{-1}(1), C)_\lambda & \text{if } k/d \notin \mathbb{Z}_{\leq 0}, \\ \overline{H}^j(f^{-1}(1), C)_\lambda & \text{if } k/d \notin \mathbb{Z}_{> 1}, \end{cases}
\]
in the notation of (3.2), where \(\lambda = \exp(-2\pi ik/d)\), see also \([\text{DT}]\).

We have moreover
\[
\text{(4.2.4)} \quad \ker(H^j A_f^* \rightarrow H^j C_f^*) = (H^j A_f^*)_{\text{tor}},
\]
where the last term denotes the $t$-torsion subspace of $H^jA^*_f$, which coincides with the $\partial_t^{-1}$-torsion, and is annihilated by $\partial_t^{-p}$ for $p \gg 0$, see [BaSa].

### 4.3. Relation with the Koszul cohomologies

Set

\[ (4.3.1) \quad A^j_d := df \wedge \Omega^{j-1} \xrightarrow{\partial} A^j_f \quad (j \in \mathbb{Z}). \]

Using the short exact sequence of complexes

\[ 0 \to (A^*_f, d) \to (\Omega^*, d) \to (A^*_f, d)[1] \to 0, \]

we get isomorphisms

\[ (4.3.2) \quad \partial : H^jA^*_f \xrightarrow{\sim} H^jA^*_f \quad (j \neq 1), \]

together with a short exact sequence

\[ 0 \to C \to H^1A^*_f \to H^1A^*_f \to 0. \]

By (4.3.1) and (4.3.2), we get an action of $\partial_t^{-1}$ on $H^jA^*_f$, $H^jA^*_f$ defined respectively by

\[ \partial_t^{-1} := \partial^{-1} \circ H^j \iota, \quad \partial_t^{-1} := H^j \partial^{-1} \circ \iota \quad (j \neq 1). \]

Let

\[ (4.3.3) \quad (\mathcal{K}^*_f, d) := (A^*_f/A^*_f, d). \]

The relation with the shifted Koszul complex $(^*K^*_f, df \wedge)$ in the introduction is given by

\[ \mathcal{K}^{j+n}_f = H^j(^*K^*_f)(-jd) \quad (j \in [-n, 0]). \]

By the short exact sequence of complexes

\[ 0 \to (A^*_f, d) \xrightarrow{\iota} (A^*_f, d) \to (\mathcal{K}^*_f, d) \to 0, \]

we get a long exact sequence

\[ (4.3.4) \quad \to H^{j-1}\mathcal{K}^*_f \to H^jA^*_f \xrightarrow{\iota_j} H^jA^*_f \to H^j\mathcal{K}^*_f \to, \]

where the middle morphism $\iota_j := H^j \iota$ can be identified by (4.3.2) with

\[ \partial_t^{-1} : H^jA^*_f \to H^jA^*_f \quad \text{if} \quad j > 1. \]

In particular we get for $j = n$

\[ (4.3.5) \quad H^n\mathcal{K}^*_f = \text{Coker}(\partial_t^{-1} : H^nA^*_f \to H^nA^*_f). \]

By the above argument, the $\partial_t^{-1}$-torsion of $H^jA^*_f$ contributes to $H^{j-1}\mathcal{K}^*_f$, and we get in particular

\[ (4.3.6) \quad H^cA^*_f \text{ is torsion-free if } c \text{ is the codimension of } \text{Sing } f^{-1}(0) \subset \mathbb{C}^n. \]

Note that $c = n - 1$ under the assumption of the introduction. By Theorems (5.2) and (5.3) below, the $\partial_t^{-1}$-torsion of $H^nA^*_f$ is finite dimensional if and only if all the singularities of $Z$ are weighted homogeneous.

### 4.4. Filtrations $P'$ and $G$

There are two filtrations $P'$, $G$ on $C^*_f$ defined by

\[ (4.4.1) \quad P'^k_c:= \bigoplus_{i \leq k+p} \Omega^k \partial_t^i, \]

\[ G^k_p := \bigoplus_{i \leq p} \Omega^k \partial_t^i \bigoplus A^*_f \partial_t^p. \]

These are exhaustive increasing filtrations. Set $P'^p = P'_{-p}$, $G^p = G_{-p}$. By definition, we have

\[ (4.4.2) \quad \text{Gr}_{p'} \sigma^*_f = \sigma_{\geq p}\big(K^*_f((n-p)d)\big), \]
see [De] for the truncation $\sigma_{>p}$. Let $\text{Dec} P'$ be as in loc. cit. Then we have

\[(4.4.3) \quad G = \text{Dec} P'.\]

Since the differential of $C_f^*$ respect the grading, we have the pole order spectral sequence in the category of graded $C$-vector spaces

\[(4.4.4) \quad p' E_1^{p,j-p} = H^j \text{Gr}_{p'}^p C_f^* \implies H^j C_f^*,\]

with

\[(4.4.5) \quad p' E_2^{p,j-p} = \begin{cases} 0 & \text{if } j < p, \\ A_f^p & \text{if } j = p, \\ H^j K^*_f((n-p)d) & \text{if } j > p, \end{cases}\]

\[(4.4.6) \quad p' E_2^{p,j-p} = \begin{cases} 0 & \text{if } j < p, \\ H^p A_f^* & \text{if } j = p, \\ H^j K^*_f((j-p)d) & \text{if } j > p, \end{cases}\]

where $K^*_f$ is as in (4.3.3).

Note that the degeneration at $E_2$ of the pole order spectral sequence is equivalent to the strictness of $\text{Dec} P'$ by [De], and the latter condition is equivalent to the torsion-freeness of the $H^j A_f^*$ by using (4.2.4) and (4.4.3). The obtained equivalence seems to be known to the specialists (see e.g. [VSh]), and the above argument may simplify some argument in loc. cit.

By the isomorphism (4.2.3) for $k \in [1,d]$, the filtration $P'$ on the left-hand side of (4.2.3) induces a filtration $P'$ on the right-hand side. This corresponds to the filtration $P$ by the isomorphism (3.1.2) up to the shift of the filtration by 1, and we get the isomorphisms

\[(4.4.7) \quad P'^{p+1} H^{j+1}(C_f^*)_k \simeq P^p H^j(f^{-1}(1), C)_\lambda \quad (\lambda = \exp(-2\pi i k/d), k \in [1,d]),\]

see [Di] Ch. 6, Thm. 2.9] (and also [DiSa2] Section 1.8 in case $j = n - 1$). By (3.1.3), we have the inclusions

\[(4.4.8) \quad F^p \subset P^p \quad \text{on} \quad H^j(f^{-1}(1), C)_\lambda,\]

Here it is possible to show (4.4.8) by calculating the direct image of $(\mathcal{O}_X, F)$ by $f$ as a filtered $\mathcal{D}$-module underlying a mixed Hodge module, see [Sa1], [Sa2], where a compactification of $f$ must be used. (The shift of the filtration by 1 comes from the direct image of $\mathcal{O}_X$ as a left $\mathcal{D}$-module by the graph embedding of $f$.)

The inclusion (4.4.8) implies some relation between the spectrum and the Poincaré series of the Koszul cohomologies via the spectral sequence (4.4.4), and the difference between $F^p$ and $P^p$ implies also their difference in certain cases, see also [Di], [Di3], [DiSt1], etc.

4.5. **Algebraic microlocal Gauss-Manin complexes.** For a homogeneous polynomial $f$, let $\tilde{C}_f^*$ be the algebraic microlocal Gauss-Manin complex (i.e. $\tilde{C}_f^* = \Omega^j[\partial_t, \partial_t^{-1}]$). The algebraic microlocal Gauss-Manin systems $H^j \tilde{C}_f^*$ are free graded $C[\partial_t, \partial_t^{-1}]$-modules of finite type. Replacing $C_f^*$ with $\tilde{C}_f^*$ in (4.4.1) and (4.4.4), we have the filtrations $P'$, $G$ on $\tilde{C}_f^*$ together with the microlocal pole order spectral sequence

\[(4.5.1) \quad p' E_1^{p,j-p} = H^j \text{Gr}_{p'}^p \tilde{C}_f^* \implies H^j \tilde{C}_f^*,\]
where (4.4.3) holds again (i.e. \( G = \text{Dec} P' \)), and the last equalities of (4.4.5) and (4.4.6) hold for any \( j, p \in \mathbb{Z} \), i.e.

\[
(4.5.2) \quad p^r E^{p,j-p}_r = \begin{cases} 
H^j K^1_p((n-p)d) & \text{if } r = 1, \\
H^j K^1_p((j-p)d) & \text{if } r = 2.
\end{cases}
\]

Moreover the last equality of (4.2.3) holds for any \( k \), i.e.

\[
(4.5.3) \quad H^{j+1}(\widetilde{C}^*_j)_{k} = \mathcal{H}^j(f^{-1}(1), C)_{\lambda} \quad \text{with} \quad \lambda = \exp(-2\pi ik/d),
\]
(Note that the Gauss-Manin complex \( C^*_j \) can be defined also as the single complex associated with the double complex having two differentials \( d \) and \( df, \text{see [Di1], [Di3], etc.} \)

Let \( P', G \) denote also the induced filtrations on \( H^j(C^*_j) \), \( H^j(\widetilde{C}^*_j) \). Then there is a canonical inclusion

\[ C^*_j \hookrightarrow \widetilde{C}^*_j. \]

Set

\[ \omega_0 := df \in H^1(G_0 C^*_j) (= H^1 A^*_j). \]

By the same argument as in [BaSa], it generates a free \( C[t] \)-module for \( p \in \mathbb{N} \cup \{\infty\} \)

\[ C[t] \omega_0 \subset H^1(G_p C^*_j), \]

where \( G_\infty C^*_j := C^*_j \). Set

\[ \mathcal{H}^j(G_p C^*_j) = \begin{cases} 
H^j(G_p C^*_j) & \text{if } j \neq 1, \\
H^j(G_p C^*_j)/C[t] \omega_0 & \text{if } j = 1.
\end{cases} \]

Then the above inclusion induces the canonical isomorphisms

\[
(4.5.4) \quad \mathcal{H}^j(G_p C^*_j) \xrightarrow{\sim} H^j(G_p \widetilde{C}^*_j) \quad (p \in \mathbb{N} \cup \{\infty\}, \ j \in \mathbb{Z}).
\]

In fact, the assertion for \( p = \infty \) follows from the same argument as in loc. cit. This implies the assertion for \( p \in \mathbb{N} \) by using the canonical morphism of long exact sequences

\[
\begin{array}{cccccccc}
\longrightarrow & \mathcal{H}^j(G_p C^*_j) & \longrightarrow & \mathcal{H}^j(C^*_j) & \longrightarrow & H^j(C^*_j/G_p C^*_j) & \longrightarrow \\
\downarrow & & \downarrow & & & & \\
\longrightarrow & H^j(G_p \widetilde{C}^*_j) & \longrightarrow & \mathcal{H}^j(\widetilde{C}^*_j) & \longrightarrow & H^j(\widetilde{C}^*_j/G_p \widetilde{C}^*_j) & \longrightarrow
\end{array}
\]

From the canonical isomorphisms (4.5.4), we can deduce

\[
(4.5.5) \quad G_p \mathcal{H}^j(C^*_j) \xrightarrow{\sim} G_p H^j(\widetilde{C}^*_j) = \partial^p G_0 H^j(\widetilde{C}^*_j) \quad (p \in \mathbb{N}, \ j \in \mathbb{Z}).
\]

This implies

\[
(4.5.6) \quad \partial_t : \text{Gr}^G_p \mathcal{H}^j(C^*_j)_{k} \xrightarrow{\sim} \text{Gr}^G_{p+1} \mathcal{H}^j(C^*_j)_{k-d} \quad (p \in \mathbb{N}, \ j, k \in \mathbb{Z}).
\]

Note that these hold with \( G \) replaced by \( P' \) by (4.4.3). We then get by (4.4.7)

\[
(4.5.7) \quad P^{\nu+1} H^{j+1}(\widetilde{C}^*_j)_{k} \cong P^p \mathcal{H}^j(f^{-1}(1), C)_{\lambda} \quad (\lambda = \exp(-2\pi ik/d), \ k \in [1, d]),
\]
Proposition 4.6. With the notation of (4.4) and (4.5), there are canonical isomorphisms for \( r \geq 2 \)

\[
\text{Im}(d_r : p^p E_{r}^{p-r, n-p+r-1} \to p^p E_{r}^{p, n-p}) = \begin{cases} 
0 & \text{if } p > n, \\
Gr_{r-1}^K(H^n A_f^*)_{\text{tor}} & \text{if } p = n, \\
Gr_{r-1}^K(\text{Coker } \partial_t^{-1})((n-p)d) & \text{if } p < n,
\end{cases}
\]

\[
\text{Im}(d_r : p^p \tilde{E}_{r}^{p-r, n-p+r-1} \to p^p \tilde{E}_{r}^{p, n-p}) = Gr_{r-1}^K(\text{Coker } \partial_t^{-1})((n-p)d),
\]

where \( K_* \) is the kernel filtration, and \( \text{Coker } \partial_t^{-1} \) is a quotient of \( (H^n A_f^*)_{\text{tor}} \) as in Theorem 2.

Proof. We first show the assertion for the microlocal pole order spectral sequence, i.e. for the second isomorphism. Since \( p^p \hat{E}_r^{p-j-p} = 0 \) for \( j > n \), the images of the differentials

\[
d_r : p^p \hat{E}_{r}^{p-r, n-p+r-1} \to p^p \hat{E}_{r}^{p, n-p} \quad (r \geq 2)
\]
correspond to an increasing sequence of subspaces (with \( p \) fixed):

\[
(\text{4.6.1}) \quad \tilde{I}_r^{p,n-p} \subset p^p \hat{E}_2^{p,n-p} = (\text{Coker } \partial_t^{-1})((n-p)d) \quad (r \geq 2),
\]

such that

\[
\text{Im}(d_r : p^p \tilde{E}_{r}^{p-r, n-p+r-1} \to p^p \tilde{E}_{r}^{p, n-p}) = \tilde{I}_r^{p,n-p}/\tilde{I}_{r-1}^{p,n-p} \quad (r \geq 2),
\]

with \( \tilde{I}_{r}^{p,n-p} := 0 \). Here \( \text{Coker } \partial_t^{-1} \) is a quotient of \( H^n A_f^* \) (and not \( (H^n A_f^*)_{\text{tor}} \)), and (4.3.5) is used for the last isomorphism of (4.6.1).

By the construction of the spectral sequence (see e.g. [De]), we have

\[
(\text{4.6.2}) \quad \tilde{I}_r^{p,n-p} = Gr_{r-1}^K(\text{Coker } \partial_t^{-1})((n-p)d),
\]

where \( K_* \) is the kernel filtration defined just before Theorem 2. (More precisely, \( K_* \) defines a non-exhaustive filtration of \( H^n A_f^* \), and its union is \( (H^n A_f^*)_{\text{tor}} \).) In fact, the left-hand side is given by the classes of \( \omega \in \Omega^n \) such that there are

\[
\eta_i \in \Omega^n \quad (i \in [0, r - 1])
\]

satisfying

\[
d\eta_0 = \omega, \quad d\eta_{i+1} = df \wedge \eta_i \quad (i \in [0, r - 2]), \quad df \wedge \eta_{r-1} = 0.
\]

However, this condition is equivalent to that the class of \( \omega \) in the Brieskorn module is contained in \( K_{r-1}(H^n A_f^*)_{\text{tor}} \). (Note that \( [df \wedge \eta_{r-2}] \) gives \( \partial_t^{l-r}[\omega] \) and vanishes in \( H^n A_f^* \).) So the second isomorphism follows.

The argument is essentially the same for the first isomorphism by replacing (4.6.2) with

\[
I_r^{p,n-p} = \begin{cases} 
0 & \text{if } p > n, \\
Gr_{r-1}(H^n A_f^*)_{\text{tor}} & \text{if } p = n, \\
Gr_{r-1}(\text{Coker } \partial_t^{-1})((n-p)d) & \text{if } p < n.
\end{cases}
\]

This finishes the proof of Proposition (4.6).

As a corollary of Proposition (4.6), we get the following.

Corollary 4.7. The following three conditions are equivalent to each other:

(a) The pole order spectral sequence (4.4.4) degenerates at \( E_2 \).

(b) The algebraic microlocal pole order spectral sequence (4.5.1) degenerates at \( E_2 \).

(c) The torsion subgroup \( (H^n A_f^*)_{\text{tor}} \) vanishes.

4.8. Proof of Theorem 2. By (4.5.7) the assertion follows from the second isomorphism in Proposition (4.6) by choosing any \( p \in \mathbb{Z} \), where the obtained isomorphism is independent
of the choice of \( p \) by using the bijectivity of the action of \( \partial_t \). (It is also possible to use the first isomorphism in Proposition (4.6) by choosing some \( p < n \) although the independence of the choice of \( p \) is less obvious unless the relation with the algebraic microlocal pole order spectral sequence is used.) This finishes the proof of Theorem 2.

4.9. Thom-Sebastiani type theorem for \( P' \). Let \( f, f_1, f_2 \) be as in Proposition (2.2). In the notation of (4.5), we have a canonical isomorphism

\[(\tilde{C}_j^*, P') = (\tilde{C}_j^*, P') \otimes_{\mathbb{C}[\partial_t, \partial_t^{-1}]} (\tilde{C}_j^*, P').\]

Assume \( f_2 \) has an isolated singularity at the origin as in Proposition (2.2). Then

\[H^jGr_k^P\tilde{C}_j^* = 0 \quad (j \neq n_2, \, k \in \mathbb{Z}).\]

Hence \((\tilde{C}_j^*, P')\) is strict, and we get a filtered quasi-isomorphism

\[(\tilde{C}_j^*, P') \xrightarrow{\sim} H^{n_2}(\tilde{C}_j^*, P')[-n_2].\]

This implies a filtered quasi-isomorphism

\[(4.9.1) \quad (\tilde{C}_j^*, P') \xrightarrow{\sim} (\tilde{C}_j^*, P') \otimes_{\mathbb{C}[\partial_t, \partial_t^{-1}]} H^{n_2}(\tilde{C}_j^*, P')[-n_2],\]

which is compatible with the action of \( t \). More precisely, the action of \( t \) on the left-hand side corresponds to \( t \otimes \text{id} + \text{id} \otimes t \) on the right-hand side (since \( f = f_1 + f_2 \)).

Combining (4.9.1) with (4.5.7), we get the Thom-Sebastiani type theorem for the pole order spectrum:

\[(4.9.2) \quad \text{Sp}_P(f) = \text{Sp}_P(f_1) \text{Sp}_P(f_2), \quad \text{Sp}_P^j(f) = \text{Sp}_P^j(f_1) \text{Sp}_P^0(f_2) \quad (j \in \mathbb{N}),\]

assuming that \( f_2 \) has an isolated singularity as above so that \( \text{Sp}_P(f_2) = \text{Sp}_P^0(f_2) \), see [SkSt] for the case where \( f_1 \) has also an isolated singularity. Note that the Thom-Sebastiani type theorem holds for the Steenbrink spectrum by [Sa5].

**Remarks 4.10.** (i) With the notation and the assumption of (4.9), the pole order spectral sequences degenerate at \( E_2 \) for \( f \) if and only if they do for \( f_1 \). This follows from (4.9.1) together with Corollary (4.7).

(ii) The equivalence between the \( E_2 \)-degeneration of the pole order spectral sequence (4.4.4) and the vanishing of \((H^nA^*)_{\text{tor}}\) was shown in [vSt] in the (non-graded) analytic local case.

(iii) Assuming \( \dim \operatorname{Sing} f^{-1}(0) = 1 \), we have by (4.3.4) the following exact sequence:

\[0 \to \tilde{H}^{n-1}A^*_f(-d) \xrightarrow{\partial_{\omega}^{-1}} \tilde{H}^{n-1}A^*_f \to H^{n-1}K^*_f \]

\[\to H^nA^*_f(-d) \xrightarrow{\partial_{\omega}^{-1}} H^nA^*_f \to H^nK^*_f \to 0,
\]

where \( \tilde{H}^{n-1}A^*_f \) is defined by \( H^{n-1}A^*_f \) if \( n \neq 2 \), and by its quotient by \( \mathbb{C}[t] \omega_0 \) if \( n = 2 \). (For \( \omega_0 \), see the definition of \( \tilde{H}^j(G_pC^*_f) \) in (4.5).) This exact sequence has sufficient information about the torsion subgroup \((H^nK^*_f)_{\text{tor}}\) to give another proof of Theorem 2.

(iv) By forgetting the grading, Proposition (4.6) and Corollary (4.7) can be extended to the analytic local case where \( f \) is a germ of a holomorphic function on a complex manifold with \( \dim \operatorname{Sing} f^{-1}(0) = 1 \).

The following will be used in the proof of Theorem (5.2) below.

4.11. Multiplicity of the minimal exponent. Let \( g \) be a germ of holomorphic function on a complex manifold \((Y, 0)\) having an isolated singularity. We have the direct image
Consider $\text{Gr}_V^\alpha(B_g, F)$ for $\alpha < 1$. These underlie mixed Hodge modules supported at 0, and are the direct images of filtered vector spaces by the inclusion $\{0\} \hookrightarrow Y$ as filtered $\mathcal{D}$-modules. (This is shown by using [Sa1, Lemma 3.2.6] applied to any function vanishing at 0.) So we get

$$
(4.11.1) \quad \text{The } \text{Gr}_p^F \text{Gr}_V^\alpha B_g \text{ are annihilated by } m_{Y,0} \subset \mathcal{O}_{Y,0} \text{ for } \alpha < 1,
$$

where $m_{Y,0} \subset \mathcal{O}_{Y,0}$ is the maximal ideal.

Let $\tilde{B}_g := \mathcal{O}_{Y,0}[\partial_t, \partial_t^{-1}]$ be the algebraic microlocalization of $B_g$. By [Sa2, Sections 2.1-2], it has the Hodge filtration $F$ by the order of $\partial_t$ and also the filtration $V$ such that

$$
\partial_t : F_p V^\alpha \tilde{B}_g \longrightarrow F_{p+1} V^{\alpha-1} \tilde{B}_g \quad (\forall p, \alpha).
$$

Then $(4.11.1)$ implies

$$
(4.11.2) \quad \text{The } \text{Gr}_p^F \text{Gr}_V^\alpha \tilde{B}_g \text{ are annihilated by } m_{Y,0} \subset \mathcal{O}_{Y,0} \text{ for any } \alpha.
$$

Consider the (relative) de Rham complexes

$$
\mathcal{C}_g := \text{DR}_Y(B_g), \quad \tilde{\mathcal{C}}_g := \text{DR}_Y(\tilde{B}_g).
$$

Up to a shift of complexes, these are the Koszul complexes associated with the action of $\partial_{y_i}$ on $B_g$ and $\tilde{B}_g$ where the $y_i$ are local coordinates of $Y$. It has the filtrations $F$ and $V$ induced by those on $B_g$ and $\tilde{B}_g$. Here $V$ is stable by the action of $\partial_{y_i}$, but we need a shift for $F$ depending on the degree of the complexes $\mathcal{C}_g$, $\tilde{\mathcal{C}}_g$. By the above argument we have

$$
(4.11.3) \quad H^j \text{Gr}_p^F \text{Gr}_V^\alpha \tilde{\mathcal{C}}_g = H^j \text{Gr}_p^F \mathcal{C}_g = H^j \text{Gr}_p^F \mathcal{C}_g = 0 \quad (j \neq 0),
$$

where we also use the fact that $\text{Gr}_p^F \tilde{\mathcal{C}}_g$ is the Koszul complex for the regular sequence $\{\partial g/\partial y_j\}$. These imply the vanishing of $H^j F_p \text{Gr}_V^\alpha \tilde{\mathcal{C}}_g$, etc. for $j \neq 0$, and we get

$$
(4.11.4) \quad (\tilde{\mathcal{C}}_g, F, V) \text{ is strict,}
$$

by showing the exactness of some commutative diagram appearing in the definition of strict complex [Sa1].

It is known that the filtration $V$ on $\mathcal{C}_g$ is strict, and induces the filtration $V$ of Kashiwara and Malgrange on the Gauss-Manin system $H^0 \mathcal{C}_g$ (by using the arguments in the proof of [Sa1, Prop. 3.4.8]). This assertion holds by replacing $\mathcal{C}_g$ with $\tilde{\mathcal{C}}_g$, since $\mathcal{C}_g/V^\alpha \mathcal{C}_g = \tilde{\mathcal{C}}_g/V^\alpha \tilde{\mathcal{C}}_g$ for $\alpha \leq 1$ and $H^0 \mathcal{C}_g = H^0 \tilde{\mathcal{C}}_g$ (see e.g. [BaSa]). Here we also get the canonical isomorphism

$$
(4.11.5) \quad (H^0 \mathcal{C}_g, V) = (H^0 \tilde{\mathcal{C}}_g, V).
$$

Consider now $(\text{Gr}_0^F \tilde{\mathcal{C}}_g, V)$. This is a complex of filtered $\mathcal{O}_{Y,0}$-modules, and is strict. By the above argument we get the canonical isomorphism of filtered $\mathcal{O}_{Y,0}$-modules

$$
(4.11.6) \quad H^0(\text{Gr}_0^F \tilde{\mathcal{C}}_g, V) = (\mathcal{O}_{Y,0}/(\partial g), V).
$$

Combining this with $(4.11.2)$, $(4.11.4)$ and using $\text{Gr}_V^\alpha \text{Gr}_p^F \tilde{\mathcal{C}}_g = \text{Gr}_p^F \text{Gr}_V^\alpha \mathcal{C}_g$, we get

$$
(4.11.7) \quad \text{The } \text{Gr}_V^\alpha(\mathcal{O}_{Y,0}/(\partial g)) \text{ are annihilated by } m_{Y,0} \subset \mathcal{O}_{Y,0} \text{ for any } \alpha.
$$

In particular, the multiplicity of the minimal exponent is 1.
5. Calculation of $d^{(1)}$.

In this section we calculate $d^{(1)}$ in certain cases, and prove Theorems (5.2) and (5.3).

5.1. Relation with the isolated singularities in $\mathbb{P}^{n-1}$. Let $\rho : \tilde{X} \to X$ be the blow-up of the origin of $X := \mathbb{C}^n$. Let $y = \sum c_i x_i$ be as in the introduction (i.e. $(c_i) \in \mathbb{C}^n$ are sufficiently general). We may assume that

$$y = x_n,$$

replacing the coordinates $x_1, \ldots, x_n$ of $X = \mathbb{C}^n$. Let $\tilde{X}'$ be the complement of the proper transform of $\{x_n = 0\}$. It has the coordinates $\tilde{x}_1, \ldots, \tilde{x}_n$ such that

$$\rho^* x_i = \begin{cases} \tilde{x}_i \tilde{x}_n & \text{if } i \neq n, \\ \tilde{x}_n & \text{if } i = n. \end{cases}$$

Set $n' := n - 1$. Define the complex $s' K^*_f$, with $R$ and $f$ respectively replaced with

$$\mathcal{C} [\tilde{x}_1, \ldots, \tilde{x}_{n'}][\tilde{x}_n, \tilde{x}_n^{-1}]$$

and $f' := \rho^* f |_{\tilde{X}'} = \tilde{x}_n^d h(\tilde{x}_1, \ldots, \tilde{x}_{n'})$, where $h(\tilde{x}_1, \ldots, \tilde{x}_{n'}) := f(\tilde{x}_1, \ldots, \tilde{x}_{n'}, 1)$, and the grading is given only by the degree of $\tilde{x}_n$. This is compatible with $s' K^*_f$ via $\rho^*$. We have the canonical graded morphism

$$H^j (s' K^*_f) \to H^j (s' K^*_f'),$$

in a compatible way with the differential $d$. This morphism induces injective morphisms

$$(5.1.1) \quad N \hookrightarrow H^{-1}(s' K^*_f), \quad M'' \hookrightarrow H^0(s' K^*_f'),$$

where the image of $M'$ in $H^0(s' K^*_f)$ vanishes. We have the inclusion

$$(5.1.2) \quad N_{p+d}^{(2)} \subset \text{Ker} (d : H^{-1}(s' K^*_f) \to H^0(s' K^*_f')) \cap N_{p+d},$$

under the first injection of $(5.1.1)$, and the equality holds if $M'_p = 0$.

Let $Y'$ be the complement of $\{\tilde{x}_n = 0\}$ in $Y := \mathbb{P}^{n'}$. Then

$$\tilde{X}' = Y' \times \mathbb{C},$$

where $\tilde{x}_1, \ldots, \tilde{x}_{n'}$ and $\tilde{x}_n$ are respectively coordinates of $Y'$ and $\mathbb{C}$. Moreover $s' K^*_f'$ is quasi-isomorphic to the mapping cone of

$$\partial f'/\partial \tilde{x}_n = d \tilde{x}_n^{-1} h : (\Omega^*_Y, d/\Omega^*_Y) \to (\Omega^*_Y, d/\Omega^*_{Y'}) [\tilde{x}_n, \tilde{x}_n^{-1}],$$

where $\Omega^*_Y$ is identified with the group of global sections.

Let $\{z_k\}$ be the singular points of the morphism $h : Y' \to \mathbb{C}$. These are all isolated singular points. (In fact, they are the union of the singular points of $\{h = c\}$ for $c \in \mathbb{C}$. But $\{h = c\}$ is the intersection of $\{f = c\}$ and $\{x_n = 1\}$ in $\mathbb{C}^n$, and the intersection of its closure in $\mathbb{P}^n$ with the boundary $\mathbb{P}^{n'} = \mathbb{P}^n \setminus \mathbb{C}^n$ is the intersection of $\{f = 0\}$ and $\{x_n = 0\}$ in $\mathbb{P}^{n'}$, which is smooth by hypothesis. So the assertion follows.)

Since the support of the $\mathcal{C}[\tilde{x}_1, \ldots, \tilde{x}_{n'}]$-module $\Omega^*_Y/dh \wedge \Omega^*_{Y'}$ is $\{z_k\}$, we have the canonical isomorphism

$$\Omega^*_Y/dh \wedge \Omega^*_{Y'} = \bigoplus_k \Omega^*_{h_k}$$

with $\Omega^*_{h_k} := \Omega^*_Y [z_k]/dh_k \wedge \Omega^*_{Y', z_k}$, where $Y_{an}$ is the associated analytic space, and $h_k$ is the germ of an analytic function defined by $h$ at $z_k$. 


Let $z_k (k \leq r_0)$ be the singular points contained in $\{ h = 0 \}$. These are the singular points of $Z := f^{-1}(0) \subset P^n$ since $x_n$ is sufficiently general. Since $h_k$ is invertible for $k > r_0$, we have
\begin{equation}
H^n(\ast' K_f^\bullet) = \bigoplus_{k \leq r_0} (\Omega^n / h_k \Omega^n' \wedge C[\tilde{x}_n, \tilde{x}_n^{-1}] d\tilde{x}_n,
\end{equation}
and there is a similar formula for $H^{n-1}(\ast' K_f^\bullet)$ (with $d\tilde{x}_n$ on the right-hand side deleted and $\wedge$ replaced by $\otimes$). So the $z_k$ for $k > r_0$ may be forgotten from now on.

Note that, via (5.1.1) and (5.1.3), we have for $p \gg 0$
\begin{equation}
M'' \supset (\Omega^n / h_k \Omega^n') \wedge C[\tilde{x}_n] \tilde{x}_n^p d\tilde{x}_n.
\end{equation}

Take an element of pure degree $p$ of
\begin{equation}
\text{Ker}(h_k : \Omega^n / h_k \Omega^n' \rightarrow \Omega^n / h_k \Omega^n' \wedge \tilde{x}_n^{-1}) \quad (k \leq r_0).
\end{equation}
It is represented by $\psi := \frac{1}{d} \tilde{x}_n^p \eta$ where $\xi \in \Omega^n_{Y_{an}, z_k}$ satisfies
\begin{equation}
h_k \xi = dh_k \wedge \eta \quad \text{with} \quad \eta \in \Omega^n_{Y_{an}, z_k}.
\end{equation}
The corresponding element of $H^n(\ast' K_f^\bullet)$ is represented by
\begin{equation}
\psi' := \frac{1}{d} \tilde{x}_n^p \xi + \tilde{x}_n^{-1} d\tilde{x}_n \wedge \eta.
\end{equation}
Its image in $H^n(\ast' K_f^\bullet)$ by the differential $d$ is given by
\begin{equation}
d[\psi'] = \pm \left[ \left( \frac{1}{d} \xi - d\eta \right) \wedge \tilde{x}_n^{-1} d\tilde{x}_n \right],
\end{equation}
and we have by (5.1.5)
\begin{equation}
[d\eta] = \partial_t [\xi] \quad \text{in} \quad H''_{h_k}.
\end{equation}

Let $V$ be the $V$-filtration of Kashiwara [Ka] and Malgrange [Ma] on the Gauss-Manin system $\mathcal{G}_{h_k}$ indexed by $Q$, see e.g. [SkSt2] (It is closely related with the theory of asymptotic Hodge structure [Ya]). We denote also by $V$ the induced filtration on the Brieskorn module $H^n_{h_k}$ and also on $\Omega^n_{h_k}$ via the canonical inclusion and the surjection
\begin{equation}
\mathcal{G}_{h_k} \supset H^n_{h_k} \rightarrow \Omega^n_{h_k},
\end{equation}
see [Bri] for the latter. In this paper we index $V$ so that $\partial_t - \alpha$ is nilpotent on $Gr^V_0 \mathcal{G}_{h_k}$.

Let $\{ \alpha_{h_k,j} \}$ be the exponents of $h_k$ counted with multiplicity; more precisely
\begin{equation}
\# \{ j : \alpha_{h_k,j} = \alpha \} = \dim \text{Gr}^V_0 \Omega^n_{h_k} \quad \text{and} \quad \text{Sp}_{h_k}(t) = \sum_j t^{\alpha_{h_k,j}}.
\end{equation}
Here we may assume the $\alpha_{h_k,j}$ are weakly increasing (i.e. $\alpha_{h_k,j} \leq \alpha_{h_k,j+1}$) for each $k$. We have the symmetry $\{ \alpha_{h_k,j} \}_j = \{ n - \alpha_{h_k,j} \}_j$ (counted with multiplicity) by [St2].

**Theorem 5.2.** With the notation of (5.1), assume $h_k$ is non-quasihomogeneous (i.e. $h_k \notin (\partial h_k)$) for some $k \leq r_0$. Then the kernel and cokernel of $d^{(1)} : N \rightarrow M$ (i.e. $N^{(2)}$ and $M^{(2)}$ in Theorem 2) and $(H^n A_f^\ast)_{\text{tor}}$ are all infinite dimensional over $\mathbb{C}$.

**Proof.** Since the minimal exponent $\alpha_{h_k,1}$ has multiplicity 1 (see (4.11)), we have
\begin{equation}
V^{>\alpha_{h_k,1}} \Omega^n_{h_k} = \text{my}_{z_k} \Omega^n_{h_k} \supset \text{Ker}(h_k : \Omega^n_{h_k} \rightarrow \Omega^n_{h_k}).
\end{equation}
Combined with (5.1.7), this implies for $\xi$ as in (5.1.5)
\begin{equation}
\left[ \frac{1}{d} \xi - d\eta \right] \in V^{>\alpha_{h_k,1}} \Omega^n_{h_k}.
\end{equation}
So the infinite dimensionality of $M^{(2)}$ follows from (5.1.4), (5.1.6) and (5.2.1). It implies the assertion for $N^{(2)}$ since the morphisms
\[
d^{(1)} : N_{p+d} \to M_{p}
\]
are morphisms of finite dimensional vector spaces of the same dimension for $p \gg 0$. The assertion for the torsion part $(H^n A^*_f)_\text{tor}$ then follows from Theorem 2. This finishes the proof of Theorem (5.2).

**Theorem 5.3.** With the notation of (5.1), assume the $h_k$ are quasihomogeneous (i.e. $h_k \in (\partial h_k)$) for any $k \leq r_0$. Then the kernel and cokernel of $d^{(1)} : N \to M$ (i.e. $N^{(2)}$ and $M^{(2)}$ in Theorem 2) and $(H^n A^*_f)_\text{tor}$ are finite dimensional over $\mathbb{C}$. More precisely we have
\[
\nu^{(2)}_{p+d} := \dim N^{(2)}_{p+d} \leq \# \{(k, j) \mid \alpha_{h_k, j} = \frac{p}{d} (k \leq r_0) \},
\]
and the equality holds in the case where $\mu'_p = 0$ and either $\nu_{p+d} = 0$ or all the singularities of $Z$ are ordinary double points.

**Proof.** By Theorem 2, it is enough to show the inequality (5.3.1) together with the equality in the special case as above. Take any $k \leq r_0$. In the notation of (5.1) there is a local analytic coordinate system $(y_1, \ldots, y_n)$ of $Y'$ around $z_k$ together with positive rational numbers $w_1, \ldots, w_n$ such that $h_k$ is a linear combination of monomials $\prod_i y_i^{m_i}$ with $\sum_i w_i m_i = 1$ (see [SaK]). Then
\[
v(h_k) = h_k \quad \text{with} \quad v := \sum_i w_i y_i \partial y_i.
\]
We will denote the contraction of $d y_1 \wedge \cdots \wedge d y_{n'}$ and $v$ by $\zeta$.

Take a monomial basis $\{\xi_j\}$ of $\Omega^m_{h_k}$, where monomial means that
\[
\xi_j = \prod_i y_i^{m_{j,i}} dy_1 \wedge \cdots \wedge dy_{n'} \quad \text{with} \quad m_{j,i} \in \mathbb{N}.
\]
Set
\[
\eta_j := \prod_i y_i^{m_{j,i}} \xi_j, \quad w(\xi_j) := \sum_i w_i (m_{j,i} + 1).
\]
Then
\[
d f \wedge \eta_j = f \xi_j, \quad d \eta_j = w(\xi_j) \xi_j.
\]
So we get
\[
\partial t [\xi_j] = w(\xi_j) [\xi_j] \quad \text{in} \quad H^m_{h_k}.
\]
In particular
\[
w(\xi_j) = \alpha_{h_k, j},
\]
by changing the ordering of the $\xi_j$ if necessary. The inequality (5.3.1) then follows from (5.1.6) and (5.1.7) together with the inclusion (5.1.2). In case the assumption after (5.3.1) is satisfied, we have the equality by using the remark after (5.1.2) together with the fact that $\alpha_{h_k, j} = n'/2$ if $z_k$ is an ordinary double point of $Z$. This finishes the proof of Theorem (5.3).

**Corollary 5.4.** With the hypothesis of Theorem (5.3), assume $n = 2$ or more generally
\[
(5.4.1) \quad \max \{ \alpha_{h_k, j} \mid d \alpha_{h_k, j} \in \mathbb{N} \quad (k \leq r_0) \} < 1 + \frac{n}{2},
\]
(for instance, $d \alpha_{h_k, j} \notin \mathbb{N}$ for any $j$ and $k \leq r_0$). Then the pole order spectral sequences (4.4.4) and (4.5.1) degenerate at $E_2$, and $(H^n A^*_f)_\text{tor} = 0$.

**Proof.** This follows from Theorem (5.3) together with Corollary (4.7) and Theorem 2 since $d^{(1)}$ is a graded morphism of degree $-d$.

**Corollary 5.5.** With the first hypothesis of Theorem (5.3), assume $n = 3$. Let $\alpha'_f := \min \{ \alpha_{h_k, j} \}$ in the notation of (5.1.8). Then
\[
(5.5.1) \quad \nu_{p+d} = 0 \quad \text{for} \quad p < d \alpha'_f.
\]
Proof. Note first that $\alpha'_f \leq 1$ since $\dim Z = 1$. Assume $\nu_{p+d} \neq 0$ with $p < d\alpha'_f$. Then the image of $d^{(1)} : N_{p+d} \to M_p$ is nonzero by Theorem (5.3). We get hence by Theorem 2

$$n'_{f,p/d} < \mu_p \leq \left(\frac{p-1}{n-1}\right),$$

where $n'_{f,p/d}$ is the coefficient of the pole order spectrum $\text{Sp}_p(f)$ (i.e. $\text{Sp}_p(f) = \sum_{\alpha} n'_{f,\alpha} t^\alpha$). However, this contradicts Remark (3.6)(ii). So Corollary (5.5) follows.

Remarks 5.6. (i) In Theorem (5.3), the inequality (5.3.1) holds with the left-hand side replaced by the dimension of the kernel of the composition

$$N_{p+d} \xrightarrow{d^{(1)}} M_p \to M''_p.$$ 

In fact, (5.1.1) implies that (5.1.2) holds with $N''_{p+d}$ replaced by this kernel.

(ii) Corollary (5.4) seems to be closely related with the short exact sequence in [DiSa] Thm. 1.

(iii) If all the singularities of $Z$ are nodes, then $\alpha'_f = 1$, and the estimation obtained by Corollary (5.5) coincides with the one in [DiSa] Thm. 4.1, which is known to be sharp. It is also sharp for instance if the singularities are $A_1$ or $D_4$ (e.g. $f = (x^2 - y^2)(x^2 - z^2)(y^2 - z^2)$).

(iv) The proof of the finiteness of $(H^n \Lambda^n)_{\text{tor}}$ can be reduced to the analytic local case by considering the formal completion where the direct sum is replaced with the infinite direct product and the convergent power series factors through the formal completion.

(v) The argument in (5.1) can be extended to the analytic local case if there is a projective morphism of complex manifolds $\rho : \tilde{X} \to X$ such that the restriction of $\rho$ over $X \setminus \{0\}$ is an isomorphism and the following two conditions are satisfied:

(a) The proper transform of each irreducible component of $\text{Sing} f$ transversally intersects $\rho^{-1}(0)_{\text{red}}$ at a smooth point $z_k$, and $z_k \neq z_{k'}$ for $k \neq k'$.

(b) We have $\rho^* f = \tilde{x}^{a_k}_n h_k(\tilde{x}_1, \ldots, \tilde{x}_{n-1})$ around each $z'_k$ where $\tilde{x}_1, \ldots, \tilde{x}_n$ are local coordinates of $\tilde{X}$ around $z_k$, $h_k$ is a germ of a holomorphic function of $n - 1$ variables, and $a_k \in \mathbb{N}$.

Note that condition (b) implies that $\rho^{-1}(0)_{\text{red}}$ is locally defined by $\{\tilde{x}_n = 0\}$. Let $y$ be a sufficiently general linear combination of local coordinates of $(X, 0)$. Here we assume that $\rho$ factors through the blow-up along $0 \in X$, and moreover the intersection of the exceptional divisor $E_0$ of the blow-up along 0 with the proper transform of each irreducible component of $\text{Sing} f$ is not contained in the hyperplane of $E_0$ defined by $h$ (by replacing $h$ if necessary). Then $\rho^* h = u_k \tilde{x}^{b_k}_n$ with $b_k \in \mathbb{N}$ and $u_k$ an invertible function. Here we may assume $\rho^* h = \tilde{x}^{b_k}_n$ by replacing $\tilde{x}_n$, but the equality in condition (b) is replaced by $\rho^* f = u'_k \tilde{x}^{a_k}_n h_k(\tilde{x}_1, \ldots, \tilde{x}_{n-1})$ where $u'_k$ is an invertible function. Then condition (b) can be replaced with

(b' The restriction of $\rho^* y$ to the proper transform $\tilde{D}$ of $f^{-1}(0)$ gives an analytically trivial deformation on a neighborhood of each $z'_k$ by replacing $\rho^* y : \tilde{D} \to C$ with the normalization of the base change by an appropriate ramified covering of $C$ if necessary.

Under these assumptions, Theorem (5.2) can be extended to the analytic local case where $h_k$ is as in condition (b) above. However, it does not seem easy to generalize Theorem (5.3) unless $f$ admits a $\mathbb{C}^*$-action (or the arguments related to the grading are completely ignored).

(vi) If $n = 3$ and $Z$ has only ordinary double points as singularities, then the coefficients $n'_{f,\alpha}$ of the Steenbrink spectrum for $\alpha \notin \mathbb{Z}$ are the same as that of a central hyperplane arrangement in $\mathbb{C}^3$ having only ordinary double points in $\mathbb{P}^2$. (Note that its formula can be found in [BuSa].) In fact, the vanishing cycle sheaf $\varphi_{f,\not\in \mathbb{Q} X}$ is supported at the origin so
that we have the symmetry of the coefficients $n_{f,\alpha}$ for $\alpha \notin \mathbb{Z}$. Moreover $n_{f,\alpha}$ for $\alpha < 1$ can be obtained by Remark (3.6)(ii), and $n_{f,\alpha}$ for $\alpha \in (1, 2)$ can be calculated from the $n_{f,\alpha}$ for $\alpha \notin (1, 2)$ by using the relation with the Euler characteristic of the complement of $Z \subset \mathbb{P}^2$. (The latter follows from (3.1.2).) Note also that the $n_{f,\alpha}$ for $\alpha \in \mathbb{Z}$ can be obtained from the Hodge numbers of the complement of $Z \subset \mathbb{P}^{n-1}$.

**Examples 5.7.** We first give some examples where the assumptions of Corollary (5.4) and the last conditions of Theorem (5.3) are all satisfied, and moreover Remark (5.6)(vi) can also be applied. These are also examples of type (I) singularities (i.e. (0.5) is satisfied).

(i) $f = xyz$ (three $A_1$ singularities in $\mathbb{P}^2$) $n = d = 3$.

\[
\begin{array}{cccccccccccc}
 k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
\gamma_k & 1 & 3 & 3 & 1 \\
\mu_k & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \ldots \\
\nu_k & 2 & 3 & 3 & 3 & \ldots \\
\end{array}
\]

(ii) $f = x^2y^2 + x^2z^2 + y^2z^2$ (three $A_1$ singularities in $\mathbb{P}^2$) $n = 3$, $d = 4$.

\[
\begin{array}{cccccccccccc}
 k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \ldots \\
\gamma_k & 1 & 3 & 6 & 7 & 6 & 3 & 1 \\
\mu'_k & 3 & 4 & 3 \\
\mu''_k & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \ldots \\
\mu_k & 1 & 3 & 6 & 7 & 6 & 3 & 3 & 3 & 3 & \ldots \\
\nu_k & 2 & 3 & 3 & 3 & \ldots \\
\mu^{(2)}_k & 1 & 3 & 4 & 4 & 3 \\
\nu^{(2)}_k & \ \\
\mathrm{Sp}_P & 1 & 3 & 4 & 4 & 3 & 0 & 0 \\
\mathrm{Sp} & 1 & 3 & 3 & 4 & 3 & 0 & 1 \\
\end{array}
\]

(iii) $f = xyz(x + y + z)$ (six $A_1$ singularities in $\mathbb{P}^2$) $n = 3$, $d = 4$.

\[
\begin{array}{cccccccccccc}
 k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \ldots \\
\gamma_k & 1 & 3 & 6 & 7 & 6 & 3 & 1 \\
\mu'_k & 1 \\
\mu''_k & 1 & 3 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & \ldots \\
\mu_k & 1 & 3 & 6 & 7 & 6 & 6 & 6 & 6 & 6 & \ldots \\
\nu_k & 3 & 5 & 6 & 6 & 6 & \ldots \\
\mu^{(2)}_k & 1 & 3 & 1 & 1 \\
\nu^{(2)}_k & 3 \\
\mathrm{Sp}_P & 1 & 3 & 1 & 1 & 0 & -3 & 0 \\
\mathrm{Sp} & 1 & 3 & 0 & 1 & 0 & -3 & 1 \\
\end{array}
\]

Here we have $\mu''_4 = 3$ by Lemma (2.1), but the proof of $\mu''_5 = 6$ is not so trivial. In fact, if $\mu''_5 < 6$, then we have $\nu_7 \neq 0$ by Corollary 2. However, this contradicts Corollary (5.5).

**Examples 5.8.** (i) $f = x^2y^2 + z^2$ (two $A_3$ singularities in $\mathbb{P}^2$) $n = 3$, $d = 4$. 
The calculation of this example does not immediately follow from Corollary (5.4) since the last conditions of Theorem (5.3) are not satisfied and Remark (5.6)(vi) does not apply to this example. This example can be calculated by using the Thom-Sebastiani type theorems in (2.2) and (4.9).

\[
\begin{array}{cccccccccccc}
  k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\
\gamma_k & 1 & 3 & 6 & 7 & 6 & 3 & 1 & & & & & & \\
\mu'_k & 1 & 1 & 1 & & & & & & & & & & \\
\mu''_k & 1 & 3 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & \cdots & & & \\
\mu_k & 1 & 3 & 6 & 7 & 7 & 6 & 6 & 6 & 6 & 6 & \cdots & & \\
\nu_k & 1 & 3 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & \cdots & & & \\
\mu^{(2)}_k & 1 & 1 & 2 & 1 & 1 & & & & & & & & \\
\nu^{(2)}_k & 1 & 1 & 1 & & & & & & & & & & \\
Sp_P & 1 & 1 & 2 & 1 & 0 & -1 & -1 & & & & & & \\
Sp & 1 & 1 & 2 & 1 & 0 & -1 & -1 & & & & & & \\
\end{array}
\]

We note the calculation in the case \( f = x^2 y^2 \) for the convenience of the reader.

(ii) \( f = x^2 y^2 \) (two A\(_1\) singularities in \( P^1 \)) \( n = 2, d = 4 \).

\[
\begin{array}{cccccccccccc}
  k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & & & \\
\gamma_k & 1 & 2 & 3 & 2 & 1 & & & & & & & \\
\mu'_k & 1 & 1 & & & & & & & & & & \\
\mu''_k & 1 & 2 & 2 & 2 & 2 & 2 & 2 & \cdots & & & & \\
\mu_k & 1 & 2 & 3 & 2 & 2 & 2 & 2 & \cdots & & & & \\
\nu_k & 1 & 2 & 2 & \cdots & & & & & & & & \\
\mu^{(2)}_k & 1 & 1 & 1 & & & & & & & & & & \\
\nu^{(2)}_k & 1 & & & & & & & & & & & & \\
Sp_P & 1 & 0 & 1 & 0 & -1 & & & & & & & \\
Sp & 1 & 0 & 1 & 0 & -1 & & & & & & & \\
\end{array}
\]

**Remark 5.9.** We have the \( V \)-filtration of Kashiwara and Malgrange on \( N_p, M''_p \) by using the injections in (5.1.1). Assume all the singularities of \( Z \) are weighted homogeneous. It seems that the following holds in many examples:

\[
(5.9.1) \quad \dim \text{Gr}^\alpha V N_{p+d} = \begin{cases} 
\nu^{(2)}_{p+d} = \nu^{(\infty)}_{p+d} = n^1_{f,\alpha+1} & \text{if } p/d = \alpha, \\
0 & \text{if } p/d < \alpha,
\end{cases}
\]

where \( n^j_{f,\alpha} \) is as in (3.2.3), and \( \nu^{(\infty)}_{p+d} := \nu^{(r)}_{p+d} (r \gg 0) \). Note that (5.9.1) would imply the \( E_2 \)-degeneration of the pole order spectral sequence in Question 2.

As for \( M''_p \), (5.9.1) seems to correspond by duality to the following:

\[
(5.9.2) \quad \dim \text{Gr}^\alpha V M''_p = \begin{cases} 
n_{Z,\alpha} - n^1_{f,\alpha} & \text{if } p/d = \alpha, \\
n_{Z,\alpha} & \text{if } p/d > \alpha,
\end{cases}
\]

where \( n_{Z,\alpha} := \sum_{k\leq r_0} n_{h_k,\alpha} \) with \( n_{h_k,\alpha} \) defined for the isolated singularities \( \{h_k = 0\} \) \( (k \leq r_0) \) as in (3.2.1). In fact, we have the symmetries

\[
n_{Z,\alpha} = n_{Z,n-\alpha}, \quad n^1_{f,\alpha} = n^1_{f,n-\alpha} \ (\alpha \in \mathbb{Q}),
\]
and it is expected that the duality isomorphism in Theorem 1 is compatible with the filtration $V$ on $N_p, M''_p$ in an appropriate sense so that we have the equality
\[(5.9.3) \dim \text{Gr}_V^\alpha N_p + \dim \text{Gr}_V^{1-\alpha} M''_{nd-p} = n_{Z, \alpha} \quad (\alpha \in \mathbb{Q}, \ p \in \mathbb{Z}),\]
giving a refinement of Corollary 2. Note that (5.9.2) for $\alpha = p/d$ is closely related with [KI].

If the above formulas hold, these would imply a refinement of Corollary 5.5 (and also its generalization to the case $n > 3$ in [DiSa3, Theorem 9]). However, it is quite nontrivial whether (5.9.2) holds, for instance, even for $p/d > \alpha$, since this is closely related to the independence of the $V$-filtrations associated to various singular points of $Z$. In the case where the Newton boundaries of $f$ are non-degenerate, the formula for $M''_p$ with $\alpha \leq 1$ seems to follow from the theories of multiplier ideals and microlocal $V$-filtrations.

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