FAST SPHERICAL HARMONIC ANALYSIS: A QUICK ALGORITHM FOR GENERATING AND/OR INVERTING FULL-SKY, HIGH-RESOLUTION COSMIC MICROWAVE BACKGROUND ANISOTROPY MAPS

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ABSTRACT

We present a fast algorithm for generating full-sky, high-resolution (∼5') simulations of the cosmic microwave background anisotropy pattern. We also discuss the inverse problem, that of evaluating from such a map the full set of $a_{lm}$ values and the spectral coefficients $C_l$. We show that using an equidistant cylindrical projection of the sky substantially speeds up the calculations. Thus, generating and/or inverting a full-sky, high-resolution map can be easily achieved with present-day computer technology.

Subject heading: cosmic microwave background

1. INTRODUCTION

The angular power spectrum of cosmic microwave background (CMB) anisotropies is a gold mine of cosmological information. It sensitively depends upon a number of parameters: the total and baryonic density parameters, $\Omega_x$ and $\Omega_\rm{b}$; the cosmological constant, $\Lambda$; the Hubble constant, $H_0$; the spectral indices, $n_x$ and $n_r$, and the amplitudes of scalar and tensor metric fluctuations; and the redshift, $z_{\rm{eh}}$, at which the universe could have been reionized. Because of their planned high sensitivity and high angular resolution, future space missions will measure the anisotropy power spectrum with great accuracy. Thus, all these cosmological parameters will be determined with an unprecedented precision (Bersanelli et al. 1996; Jungman et al. 1996). To achieve these goals, Monte Carlo simulations of the CMB anisotropy pattern have been and will be more and more crucial in this game. On the one hand, they allow us to prepare a mission, to optimize the observational strategy, and to test for different payload configurations. On the other hand, they are important for the data analysis and, for example, to look for systematics.

Up to now, these simulations were realized without problems. Experiments with high angular resolution observed only small (and hence flat) patches. Experiments with large sky coverage, such as the COBE/Differential Microwave Radiometer, had low resolution: several full-sky, CMB anisotropy maps were easily generated through a spherical harmonic expansion with a low ($\sim 100$) number of harmonics.

A potential problem for present computer technology is generating high-resolution, full-sky maps (Saez, Holtmann, & Smoot 1996; see also Hinshaw, Bennett, & Kogut 1995). Also generally perceived as a heavy, almost impossible, computational task is the inverse problem, that is, extracting out of an observed full-sky, high-resolution map the anisotropy spectrum up to $l \sim 1000$.

The purpose of this Letter is quite technical, but of interest, we believe, to the large community involved in future CMB anisotropy experiments. We want to present a fast algorithm for (1) generating high-resolution ($\sim 10'$), full-sky maps and (2) reconstructing all the coefficients of a spherical harmonic expansion, and hence the spectral coefficient $C_l$, up to $l \sim 1000$. Both of these tasks can be easily achieved on currently available workstations. Thus, the plan of this Letter is as follows. In § 2 we will describe the method. In § 3 we will discuss numerical results and the efficiency of the method. Finally, in § 4 we will present a brief summary of our main findings.

2. METHOD

Generating a CMB anisotropy map is in principle very simple. The temperature fluctuation observed along a line of sight, $\delta_T$, can be conveniently described by a spherical harmonic expansion:

$$\Delta T \over T \equiv \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} a_{lm}(\theta, \phi) Y_{lm}(\theta, \phi),$$

where $a_{lm} = (-1)^{m} a_{lm}^\ast$. The $a_{lm}$ values are random variables of the observer position, $x$, Gaussian distributed (at least in most of the inflation-based scenarios), with zero mean and variances $\langle |a_{lm}|^2 \rangle = C_l$. In simulating the CMB primary anisotropy pattern, the sum over $l$ usually starts from 2. In fact, on the one hand, the monopole vanishes by construction, being the mean (over the sky) CMB anisotropy. On the other hand, the dipole components are dominated by the Doppler anisotropy, induced by our peculiar motion relative to the comoving frame (see, e.g., Kogut et al. 1994). On the same line, we will keep the sum over $l$ from 0 to $l_{\max}$, but we set $a_{l0} = a_{l-1} = a_{l0} = a_{l,1} = 0$.

The $C_l$ values are the main prediction of a theory of structure formation (see, e.g., Hu & Sugiyama 1995). Thus, for a given scenario (i.e., for given $C_l$ values) and for a given statistics, we can generate a random set of $a_{lm}$ values and, from equation (1), a CMB anisotropy map. In practice, using the spherical harmonic expansion as in equation (1) is not very efficient: for each line of sight $\gamma$ we should evaluate $Y_{lm}(\gamma)$ for each value of $l$ and $m$. Fortunately, it is possible to rewrite equation (1) in a form more suitable for numerical implementation.

First, it is easy to verify that the double sum in equation (1), $\sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\gamma)$, is completely equivalent to $\sum_{m=-l_{\max}}^{l_{\max}} \sum_{l=0}^{l_{\max}} a_{lm} Y_{lm}(\gamma)$. We sample exactly the same region of the $l$-$m$ space, by columns (in the former case) or by rows (in the latter case; see Fig. 1). Second,
It is then possible to rewrite equation (1) as follows:

\[ \frac{\Delta T}{T} (\phi, \theta) = \sum_{m=-l_{\text{max}}}^{l_{\text{max}}} b_m(\theta)e^{\imath m\phi}, \quad (4) \]

where

\[ b_m = \sum_{l=|m|} [a_{lm}L_l^m], \quad (5) \]

and \( b_m = b_m^* \). Written in this way, equation (4) highlights a couple of attractive features. If we use an equidistant cylindrical projection (hereafter ECP) of the sky, which conserves distances along meridians and along the equator, the anisotropy map can be thought as a rectangular matrix of \( N_{\text{pix}} = N_y \times N_x \) squared pixels (\( N_x = N_y/2 \)), each of dimension \( \approx 20' \times 20'(524,288/N_{\text{pix}}) \). In this projection, the temperature anisotropy along parallels (i.e., at fixed \( \theta \)) is nothing more than the one-dimensional Fourier transform of the coefficient \( b_m(\theta) \) values (see eq. [4]), very efficiently computed with FFT techniques. If we regard the sum of equation (4) as a Fourier expansion, then \( l_{\text{max}} \) must be fixed to be \( N_y/2 \), as it plays the role of the Nyquist critical frequency of the problem. Second, the \( \lambda_l^n \) are evaluated by standard recurrence relations:

\[
\lambda_l^n = (-1)^m \sqrt{\frac{2m + 1}{4\pi}} \frac{(2m - 1)!!}{(2m)!!} (1 - x^2)^{m/2},
\]

\[
\lambda_{l+1}^n = x \sqrt{2m + 3} \lambda_l^n,
\]

\[
\lambda_l^n = \left[ x \lambda_{l-1}^n - \sqrt{\frac{(l + m - 1)(l - m - 1)}{(2l - 3)(2l - 1)}} \lambda_{l-2}^n \right] \frac{4l^2 - 1}{4l^2 - m^2},
\]

where \( x = \cos \theta \). Because of these relations, the \( b_m(\theta) \) values can be computed very efficiently, as we can perform the sum in equation (5) while computing the \( \lambda_l^n \) values. Such a computation is further simplified because \( \lambda_l^n(\cos \theta) = \pm \lambda_l^n(\cos \pi - \theta) \), the plus (minus) sign holding if \( l \) and \( m \) are (are not) both even or both odd.

Generating the \( b_m(\theta) \) values at fixed \( \theta \) requires evaluating \( \approx l_{\text{max}} \propto N_y^2 \propto N_{\text{pix}} \) recurrence relations \( (l_{\text{max}} - m) \) recurrence relations for each value of \( m \). “FFTing” the \( b_m(\theta) \) values requires \( N_r \ln \sqrt{N_r} \) operations. The \( b_m(\theta) \) values are then evaluated \( N_r \) times. Thus, at the end, the number of operations needed for generating an ECP of the anisotropy pattern is expected to scale as \( N_r^2 \ln \sqrt{N_r} \) or, equivalently, \( N_{\text{pix}}^2 \ln \sqrt{N_{\text{pix}}} \).

At this point it is quite easy to address also the inverse problem, that of evaluating from an observed high-resolution, full-sky map the set of coefficients \( a_{lm} \). It is very well known that the orthonormality of the spherical harmonics allows us to invert equation (1) and write

\[ a_{lm} = \int d\Omega \frac{\Delta T}{T}(\hat{\gamma}) Y_{lm}(\hat{\gamma}). \quad (7) \]

This sounds awful: in principle, for each \( l \) and \( m \) we should
evaluate $Y_{lm} (\hat{\mathbf{r}})$ for a given $\hat{\mathbf{r}}$ and integrate over the whole sky. Fortunately, after substituting equation (2) in equation (7) we can write

$$a_m = \int \sin \theta d\theta \lambda_l^2 (\theta) b_m (\theta),$$

(8)

where

$$b_m (\theta) = \int_0^{2\pi} d\phi \Delta (\phi, \theta) \exp (-im \phi).$$

(9)

Thus, equation (8) is the conjugate of equation (4); the $b_m$ values are the Fourier antitransform of the anisotropy pattern along a parallel in the ECP of the sky and are easily computed at fixed $\theta$ with an FFT. In conclusion, inverting a map to obtain the $a_m$ values requires $\approx l_{max}^2 \propto N_p^2$ recurrence relations for evaluating the $\lambda_l^2$ values plus an FFT (which scales as $N_p \ln N_p$) to evaluate the $b_m$ values. All of this must be done $N_p (\propto N_p)$ times to be able to perform the integral in equation (8). Using these tricks, we can invert a full-sky, high-resolution map with a number of operations that are, in principle, comparable with those needed for generating a map, i.e., $N_p^2 \ln N_p$. As in that case, we can exploit the symmetries of the $\lambda_{lm}$ values evaluated at $\theta$ and $\pi - \theta$, respectively.

3. NUMERICAL RESULTS

In the previous section we described an algorithm for generating and/or inverting a high-resolution, full-sky map of the CMB anisotropy. In this section we will briefly discuss the actual performances of this algorithm on a DEC 1000/200. Hereafter, we will consider the standard cold dark matter model.

In Figure 2 we plot the CPU time needed for generating a full-sky, ECP of the CMB anisotropy pattern as a function of the angular resolution, the only free parameter with which we can play. In fact, we sample the ECP of the sky with $N_{pix}$ squared pixels. This fixes $N_p = l_{max} = N_{pix} / 2$. The CPU time scales as $N_{pix}^2 = N_p^2$, consistently with the fact that FFT packages are highly optimized. We want to stress that only 1 hr of CPU time is needed to generate a full-sky, ECP of the CMB anisotropy with a resolution of 5' (i.e., $N_p = 4096$).

Using an ECP is not a limitation. In fact, once we obtain an ECP of the anisotropy pattern, we can reproduce it in any given projection. As an example, we show in Figure 3 (Plate L5) an ECP, 10' resolution map (obtained in only 8 minutes of CPU time) and the corresponding equal area projection (hereafter EAP). The latter is obtained by the former using standard spherical trigonometry. We verified that this procedure is numerically stable. In fact, if we transform an ECP to an EAP and the obtained EAP back to an ECP, we reproduce the initial anisotropy pattern exactly. In Figure 2 we also show the CPU time needed for inverting an ECP map as a function of the map resolution. The CPU time scales roughly as $N_p^3$. Again, ~1 hr of CPU time is needed to recover from a 5' resolution map the entire set of $a_m$ values, roughly $4 \times 10^6$ coefficients. The percentage error between the recovered $a_m$ values and the input ones is large only for $l \sim l_{max}$ and $m \sim$ a few, a very small
portion of the allowed region of the \( l-m \) space. This is due to the fact that for \( m \to 0 \), \( \lambda_l^m \propto P_l(\cos \theta) \): for large values of \( l \), this is a highly oscillating function of \( \theta \). Contrarily, for \( m \to l \), \( \lambda_l^m \propto (1 - \cos \theta)^{l/2} \), a quite smooth function of the azimuthal angle (see eq. [6]). So, a simple trapezoidal rule for performing the integral along meridians in equation (8) gives a poor result only for very large values of \( l \) and quite small values of \( m \). This is not a problem if we are interested in evaluating the spectral coefficients \( C_l \). In fact, these are obtained from the recovered \( a_{lm} \) values as follows:

\[
C_l^{\text{estimated}} = \frac{1}{2l + 1} \sum_{m=-l}^{l} |a_{lm}|^2. \tag{10}
\]

It is clear that the error we make in recovering the \( a_{lm} \) values for large \( l \) and small \( m \) is highly diluted in the sum of equation (10). In Figure 4 we show the (percentage) error between the recovered and the input \( C_l \) values as a function, again, of the map resolution. The recovered spectrum has a maximum error of \( \sim 0.1\% \) up to \( l \leq 1500 \) for \( N_o = 4096 \), i.e., for a pixel size of \( \sim 5' \times 5' \).

4. CONCLUSIONS

We present a fast algorithm for (1) generating high-resolution, full-sky maps of the CMB anisotropy and (2) evaluating out of an observed map the \( a_{lm} \) values and then the spectral coefficient \( C_l \) values. The basic trick for speeding up the calculation consists in generating and/or inverting a full-sky map using an ECP. It is this projection that allows the use of an FFT either in equation (4) and/or in equation (9). If we are interested in probing the anisotropy power spectrum up to \( l_{\text{max}} \approx 1000 \), then \( N_o = 2l_{\text{max}} = 2048 \), and we need only 8 minutes of CPU time for either generating or inverting a 10' resolution, full-sky map. Pushing the sampling down to 5' boosts the needed CPU time up to 1 hr. Our algorithm also allows us to fully exploit a parallel architecture, such as the one of APEmille (Bartoloni et al. 1995). Details about this application will be discussed elsewhere.

In addressing the problem of inverting a full-sky map, we assumed full-sky coverage, and we fully exploited the orthonormality of the spherical harmonics. We test our algorithm against a pure CMB anisotropy pattern. In the realistic case, a \( \mu \)-wave map will be the superposition of different processes (CMB anisotropy, Galactic foregrounds, secondary anisotropy due to clusters of galaxies, point sources, etc.), and the sky coverage can be incomplete. The problem of separating the CMB anisotropy pattern from Galactic and extragalactic foregrounds has been addressed by several authors (see, e.g., Gutierrez et al. 1995; Kogut et al. 1996; Bersanelli et al. 1996). We will discuss an application of our algorithm to the problem of foreground subtraction and incomplete sky coverage in a forthcoming paper.

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Fig. 3.—A realization of the CMB anisotropy pattern in a standard cold dark matter model (top) in the ECP and (bottom) in the EAP.

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