QCD AND MULTIPLICITY SCALING

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In QCD, the similarity of multiplicity distributions is violated i) by the running of the strong coupling constant $\alpha_s$ and ii) by the self-similar nature of parton cascades. It will be shown that the data collapsing behavior of $P_n$ onto a unique scaling curve can be restored by performing the original scaling prescription (translation and dilatation) in the multiplicity moments’ rank.

1 Similitude

The notion of scaling is hardly new. One of the earliest scaling arguments dates back to 1638 when Galileo Galilei published his infamous masterpiece entitled “Dialogues Concerning Two New Sciences”. Among other fundamental observations he examined the principle of similitude, the elementary properties of similar physical/biological structures. Galileo realized that the strength $S$ of a bone increases in direct proportion to its cross-sectional area ($S \sim l^2$, if $l$ is the linear size), whereas the weight of a bone increases in direct proportion to its volume ($W \sim l^3$). Thus, there will be a characteristic point where a bone has insufficient strength to support its own weight: the intersection point of the quadratic and cubic curves denoting the strength and weight of a bone, respectively. This general engineering consideration implies that terrestrial bodies can not exceed a certain maximum size. The classical scaling argument of Galileo teaches us an important lesson: the physical laws are not invariant under a uniform change of the size of macroscopic objects. The gravitational force, governed by Newton’s constant $G_N$ with dimension of $(\text{mass})^{-2}$, inevitably leads to the breakdown of dilatation symmetry.

Classical scaling principles of the above sort are based on the key assumption that the physical bodies or processes are uniform, filling an interval in a smooth, continuous fashion. In the example given by Galileo, the strength of a bone was assumed to be uniformly distributed over the cross-sectional area with its weight having a similar uniformity. This is a major limitation of the principle of similitude because such assumptions are not necessarily accurate. In reality a vast number of biological and physical systems, the so-called fractals, exhibit highly irregular appearance as the result of their self-similar...
structure. Let us consider a well-known example, the architecture of the human lung. If one unfolds the cca. 300 million air sacs of the self-similar bronchial tree and merges them into one continuous flat surface, its area will be as large as a tennis court. This anomalous surface-to-volume ratio cannot be explained by classical scaling arguments based on Galileo’s principle of similitude. Only the modern, more powerful scaling ideas of fractal geometry can properly characterize self-similar geometric forms.

2 Similar Distributions

Is it meaningful to speak about the concept of similarity with regard to multiplicity fluctuations? Of course, yes. Counting the number of particles created in a certain collision process, one of the most basic observables is the distribution of counts: the multiplicity distribution $P_n$. It is a discrete distribution but at high energies we can approximate $P_n$ by a continuous probability density $f(x)$ either via $P_n \approx f(x = n)$ or via $P_n \approx \int_{x=n}^{x=n+1} f(x) \, dx$ where $f(x)$ is called similar if it satisfies

$$f(x) = \frac{1}{\lambda} \psi \left( \frac{x - c}{\lambda} \right)$$

with $\lambda > 0$ being a scale parameter. In multiparticle physics one often sets $c = 0$, $\lambda = \langle n(s) \rangle$ and uses $P_n(s) \approx f(x = n, s)$ to approximate the shape of $P_n(s)$ measured at different collision energies $s$. Then Eq. (1) means that expressing the multiplicities $n$ in units of $\langle n(s) \rangle$, the properly rescaled data points, preserving normalization, fall onto the universal curve $\psi(z)$ which depends only on the dimensionless ratio $z = n/\langle n(s) \rangle$. This behavior is called KNO scaling after the work of Koba, Nielsen and Olesen. Two years earlier it was obtained by Polyakov too. Sometimes people try to improve on the scaling via shifting the multiplicity distributions by a factor $c(s) \sim 1$. Usually this number is interpreted as the average of produced leading particles.

Can we extend the similarity property (1) for multiplicity distributions $P_n(\delta)$ measured at different bin-sizes $\delta$ in phase space? Not quite. The experimental data collected in the past 15 years or so revealed a dominant feature of multiplicity fluctuations: in a wide range of collision energies, bin-sizes, and for a large variety of reaction types, the observed fluctuation pattern proved to be self-similar. This so-called intermittent behavior manifests through the power-law dependence of the normalized factorial moments of $P_n(\delta)$ as the resolution scale $\delta$ is varied, whereas Eq. (1) expresses the constancy of normalized moments. The breakdown of the similarity feature Eq. (1) due to self-similar multiplicity fluctuations is analogous to the incompatibility of Galileo’s principle of similitude and the properties of fractal geometric forms.
3 Scaling and Quantum Mechanics

As we have seen previously, an obvious reason of the breakdown of dilatation symmetry of physical laws is the appearance of explicit scales, such as the masses of macroscopic bodies or of elementary particles. But there is another source of non-scale-invariance, related to the properties of the quantum mechanical vacuum. In quantum mechanics the physical vacuum is a polarizable medium. Virtual pairs of charges are always present as quantum mechanical fluctuations whose effect can not be switched off. They partially screen or antiscreen a test charge. Therefore its effective value depends on the distance or energy scale at which it is measured. In other words, the effective coupling strength is running in quantum theory. This fundamental effect has important consequences for multiplicity fluctuations, too: the various scaling behaviors inevitably break down at certain energy and resolution scales. For example, in $e^+e^-$ annihilation the $s$-dependence of the QCD coupling constant $\alpha_s$ cannot be compensated by a suitable change of $\lambda$ and $c$ in Eq. (1). The running of $\alpha_s$ is expected to cause violation of KNO scaling at high energies.

4 New Multiplicity Scaling Law

The multiplicity moments $\langle n^q \rangle$ provide another very useful representation of the information encoded in $P_n$. Our variable in this case is the rank $q$. Is it meaningful to perform a scaling transformation of type (1) in the moments’ rank? If so, what kind of dynamics yield a shifting or rescaling in $q$-space? The Mellin transform of a probability density $f(x)$ is defined by

$$M\{f(x); q\} = \int_0^\infty x^{q-1} f(x) \, dx$$

and it provides the moment $\langle x^{q-1} \rangle$ (for simplicity we make use of $P_n \approx f(x=n)$). Via the functional relation

$$\frac{1}{\mu} M\left\{ f(x); q + \frac{r}{\mu} \right\} = M\left\{ x^r f(x^\mu); q \right\} \tag{2}$$

one can introduce translation and dilatation in the moments’ rank $q$ by performing the transformation $f(x) \rightarrow x^r f(x^\mu)$ of the probability density $f(x)$ approximating the shape of $P_n$. The above scaling relation in $M$-space is our main concern in the remaining sections.

5 Dilatation in Mellin Space

The most important source of dilatation in Mellin space is related to QCD. In higher-order pQCD calculations, allowing more precise account of energy conservation in the course of multiple parton splittings, the natural variable of
the multiplicity moments is the rescaled rank \( q\gamma \) instead of rank \( q \) itself. Here \( \gamma(\alpha_s) \) is the so-called QCD multiplicity anomalous dimension. Because of the running of the strong coupling constant \( \alpha_s \), it is inevitable to adjust an energy dependent scale factor in Mellin space if we want to arrive at data collapsing of \( P_n(s) \) onto a universal scaling curve.

Let us consider in detail the shape change of \( P_n(s) \) in \( e^+e^- \) annihilation. Taking into account MLLA corrections responsible for energy-momentum conservation in parton jets, the analytic form of the KNO scaling function becomes a gamma distribution in the power-transformed variable \( z^\mu \):

\[
\psi(z) = N z^{\mu k - 1} \exp \left( - [Dz]^\mu \right)
\]

where \( k = 3/2 \), \( N = \mu D^{\mu k}/\Gamma(k) \), \( \mu = (1 - \gamma)^{-1} \approx 5/3 \) and \( D \) is a scale parameter depending on \( \gamma(\alpha_s) \); \( \gamma \approx 0.4 \) at LEP-1 energy. Thus, the MLLA calculation predicts violation of KNO scaling, see Fig. 1a, since \( \mu \) varies with collision energy \( s \) due to the running of \( \alpha_s \). Note, however, that data collapsing can be restored in a simple manner using logarithmic scaling variable; for the KNO function Eq. (3) we get

\[
\psi(x) = \mu \exp \left( k\mu x - e^{\mu x} \right)/\Gamma(k), \quad x = \ln(Dz).
\]

Because only the exponent \( \mu \) and scale parameter \( D \) of (3) are expected to depend on collision energy \( s \) through the variation of \( \gamma(\alpha_s) \), data collapsing is recovered by plotting \( \mu^{-1}\psi(\mu x) \) as displayed in Fig. 1b. The scale change in logarithmic multiplicity is governed by the multiplicity anomalous dimension of QCD, which sets the scale in Mellin space, too – see our basic relation (2). This type of scaling of \( P_n(s) \) is called log-KNO scaling since one observes the behavior of type (1) but now the distribution of logarithmic multiplicity turns out to be similar.
In $e^+e^-$ annihilation the breakdown of ordinary KNO scaling at high energies is only expected to arise. In hh collision, however, this proved to be a dominant feature of observations already in the mid-80s when the exploration of SPS energies started. With the log-KNO law in our hands it is challenging to test its validity using real data. The violation of Eq. (1) is most visible for multiplicities measured by the E735 Collaboration. The full phase space multiplicity distributions were obtained in pp and $p\bar{p}$ collisions at c.m. energies $\sqrt{s} = 300, 546, 1000$ and 1800 GeV at the Tevatron collider.

At Tevatron energies, bimodal shapes of the distributions show up having shoulder structure – like at SPS. It was argued that the low multiplicity regimes are influenced mainly by single parton collisions and exhibit KNO scaling, whereas the large-$n$ tails of the distributions are influenced more heavily by double parton interactions and violate (1) considerably. This part of the 4 data sets was analyzed in log-KNO fashion and, as shown in Fig. 2, scaling holds with good accuracy. Our (still preliminary) investigation suggests that double parton collisions yield a scale change not only in multiplicity but in the multiplicity moments’ rank as well, whereas single parton collisions do not produce the latter effect.

6 Translation in Mellin Space

The other major source of the breakdown of Eq. (1) is the self-similarity of multiplicity fluctuations. This can be observed through the power-law scaling $C_q \propto \delta^{-\varphi_q}$ of the normalized moments $C_q = \langle n^q \rangle / \langle n \rangle^q$ of $P_n(\delta)$ as the bin-size $\delta$ in phase space is varied (we neglect the influence of low count rates). The simplest possibility is the monofractal fluctuation pattern. Then, the so-called intermittency exponents $\varphi_q$ are given by $\varphi_q = \varphi_2(q - 1)$ and the anomalous fractal dimensions $D_q = \varphi_q / (q - 1)$ are $q$-independent, $D_q = D_2$. The normalized moments $C_q$ of $P_n(\delta)$ take the form

$$C_q = A_q [C_2]^{q-1} \quad \text{for} \quad q > 2,$$

with coefficients $A_q$ independent of bin-size $\delta$. Eq. (1) is obviously violated since one measures $\delta$-dependent second moment, $C_2 \propto \delta^{-D_2}$. 

Figure 2. Log-KNO scaling of the E735 data.
In the restoration of the similarity feature (1) for self-similar fluctuations, the basic idea is the investigation of the higher-order moment distributions \( P_{n,r} \equiv n^r P_n / \langle n^r \rangle \). Their moments are \( \langle n^q \rangle_r = \langle n^{q+r} \rangle / \langle n^r \rangle \), i.e. the moments of the original \( P_n \) are transformed out up to \( r \)-th order by performing a shift in Mellin space, see Eq. (2). For \( r = 1 \), the normalized moments of the first moment distribution \( P_{n,1} \) are found to be \( C_{q,1} = C_{q+1} / [C_2]^q \) in terms of the original \( C_q \) and comparison to Eq. (5) yields \( C_{q,1} = A_{q+1} \) for monofractal multiplicity fluctuations. Since the coefficients \( A_q \) are independent of bin-size \( \delta \), we see that monofractality yields not only power-law scaling of the normalized moments of \( P_n \) but also data collapsing behavior of the first moment distributions \( P_{n,1} \) measured at different resolution scales \( \delta \). The effect of low multiplicities (Poisson noise) can be taken into account via the study of factorial moment distributions \( P_{n,r} \equiv n^r P_n / \langle n^r \rangle \) and their factorial moments.

Increasing the rank of the moment distributions allows the restoration of data collapsing behavior in the presence of an increasing degree of multifractality of self-similar fluctuations. This feature is best seen for random multiplicative cascades which interpolate between monofractals and fully developed multifractals. The fluctuations give rise to the log-Lévy law having a characteristic parameter, the Lévy index \( 0 \leq \nu \leq 2 \). The moments obey the same structure as in Eq. (5) with exponent \( (q^\nu - q) / (2^\nu - 2) \). For \( \nu = 0 \) this gives back the monofractal case, whereas the upper limit of the Lévy index, \( \nu = 2 \), corresponds to the log-normal law resulting from fully developed multifractal fluctuations. Log-normal distributions exhibit two remarkable properties: the higher-order moment distributions are also log-normals (form invariance), further, they differ from each other only up to a change of scale (scale invariance). Hence, for fully developed multifractals it is impossible to arrive at data collapsing behavior via translation in Mellin space, no matter how large is \( r \), because the normalized moments remain unaltered. In the other limit, monofractals produce data collapsing already for \( r = 1 \). Fig. 3 illustrates the changing degree of fractality with increasing \( r \) through the variation of the ratio \( D_{2,r}/D_2 \): the larger is the value of the Lévy index \( \nu \), the harder is to arrive at \( D_{2,r} = 0 \). The fixed-point at \( \nu = 2 \) is apparent (the mathematically disallowed values \( \nu > 2 \) bring farther away from scaling).
7 Summary

In QCD, the similarity feature Eq. (1) of multiplicity distributions $P_n$ breaks down. For $P_n(s)$, the running of the strong coupling constant $\alpha_s$ gives rise to the scale breaking. For $P_n(\delta)$, the self-similar nature of multiplicity fluctuations in parton jets results in the violation of KNO scaling. (Due to running coupling effects, self-similarity itself also breaks down at very small $\delta$). But if we switch from $P_n$ to $\langle n^q \rangle$, it turns out that both QCD effects can be compensated by a suitably chosen shifting and rescaling in the moments’ rank $q$. That is, in order to arrive at data collapsing of the multiplicity distributions onto a unique scaling curve, the original similarity prescription (translation and dilatation) is still satisfactory, only the mathematical representation of fluctuations should be changed from distributions to their moments – in the intermittency era this is the dominant practice, anyway. The functional relation Eq. (2) tells everything about how the scaling behavior manifests for the distributions themselves: $\langle x^{q/\mu} \rangle$ corresponds to $f(x^\mu)$ and therefore log-KNO scaling of the form $\mu^{-1} f(\mu \ln x)$ shows up, whereas $\langle x^{q+r} \rangle$ implies that the moment distributions $x^r f(x)/\langle x^r \rangle$ exhibit similarity.

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