Nonlinear dynamics and Kerr frequency comb formation in lattices of coupled microresonators

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Recently, substantial progress has been made in the understanding of microresonators frequency combs based on dissipative Kerr solitons (DKSs). However, most of the studies have focused on the single-resonator level. Coupled resonator systems can open new avenues in dispersion engineering and exhibit unconventional four-wave mixing (FWM) pathways. However, these systems still lack theoretical treatment. Here, starting from general considerations for the N-(spatial) dimensional case, we derive a model for a one-dimensional lattice of microresonators having the form of the two-dimensional Lugiato-Lefever equation (LLE) with a complex dispersion surface. Two fundamentally different dynamical regimes can be identified in this system: elliptic and hyperbolic. Considering both regimes, we investigate Turing patterns, regularized wave collapse, and 2D (i.e., spatio-temporal) DKSs. Extending the system to the Su-Schrieffer-Heeger model, we show that the edge-state dynamics can be approximated by the conventional LLE and demonstrate the edge-bulk interactions initiated by the edge-state DKS.
Over the past decade, it has been shown that continuous wave-driven Kerr nonlinear resonators host a variety of coherent dissipative structures. In the anomalous dispersion regime, they give rise to dissipative Kerr solitons (DKS), while in the normal dispersion regime, platicons or interlocked dispersive waves that can strongly disturb soliton coherence and shorten its existence range. 1D and 2D lattices are particularly attractive as they allow arbitrary N-dimensional lattice of resonators (Fig. 1a, b), following by a detailed study of a 1D chain of equally coupled resonators. We demonstrate that this systems has a 2D dispersion surface that defines two fundamentally distinct nonlinear regimes: elliptic and hyperbolic. In these regimes, we numerically investigate stable and chaotic dynamics. For the former, we study coherent dissipative structures, such as Turing patterns and 2D dissip灶tes. Furthermore, we study a dimerized chain of coupled resonators described by the simplest topological model, the SSH model (Fig. 1c). We focus on the aspect of DKS generation in the edge state that is localized in the middle of the photonic bandgap. Specifically, we show that edge-state solitons, in the absence of interactions with the bulk, can be approximated by the conventional single-resonator DKS (Fig. 1d). However, for the experimentally accessible set of parameters, the generated DKS in the edge states can induce edge-to-bulk scattering and generation of the dispersive waves that can strongly disturb soliton coherence and shorten its existence range.

In this manuscript, we derive an effective (N+1)D coupled-mode equation for the general case of nonlinear dynamics in an arbitrary N-dimensional lattice of resonators (Fig. 1a, b), following by a detailed study of a 1D chain of equally coupled resonators. We demonstrate that this systems has a 2D dispersion surface that defines two fundamentally distinct nonlinear regimes: elliptic and hyperbolic. In the latter, we compare the chaos in both dynamical regimes and attribute the observed difference to the presence of the regularized wave collapse in the elliptic regime. We conclude by considering DKS generation in the edge state of the SSH model (Fig. 1c). We demonstrate nonlinearly induced edge-to-bulk scattering and generation of the dispersive waves that can strongly disturb soliton coherence and shorten its existence range, which can be deduced from single resonator approximation (Fig. 1d).

Results

Coupled Lugiato-Lefever equations in lattices of resonators.

We start with a general description of a system of weakly coupled (coupling rate ≪ free spectral range (FSR)) identical optical resonators that is shown to be governed by a set of linearly coupled LLEs, which can be presented in matrix form as

$$\frac{d}{dt}A = \bar{D}A + i\bar{M}A + i\bar{g}_0 |A|^2 A + F,$$

(1)

where vector $A = [A_0, A_1, \ldots, A_{N-1}]^T$ contains optical field envelopes of each resonator as a function of the azimuthal angle in the moving frame $\varphi$ in the lattice, matrix

$$\bar{D} = \text{diag}\left[-\frac{\kappa_T}{2} + i\delta\omega_0, \frac{D_0}{2}, \varphi, \ldots, \frac{-\kappa_T}{2} + i\delta\omega_0 + i\frac{D_0}{2}, \varphi, \ldots\right]$$

contains detuning ($\delta\omega_0$), losses ($\kappa_0$), dispersion of each resonator ($D_0$), and coupling to the bus waveguides ($\kappa_{ex}$). The coupling between different rings is introduced in matrix $M$, the nonlinear term $|A|^2 A = [A_0^2 A_0, \ldots, A_{N-1}^2 A_{N-1}]^T$ describes the conventional Kerr nonlinearity with single photon Kerr frequency shift $\delta\omega_0$, and $F = [\sqrt{\kappa_{ex}} \theta_{a,0} \theta_{b,0} \ldots, \sqrt{\kappa_{ex,N-1}} \theta_{a,N-1} \theta_{b,N-1}]^T$ represents the pump. Usually, the coupling matrix $M$ is diagonalizable and possesses a set of eigenvectors $\{V_j\}$ and associated eigenvalues $\lambda_j$. 

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where coefficients $c_j = \langle \Lambda | \mathcal{V}_j \rangle$ correspond to the amplitude of the supermode $\mathcal{V}_j$ and $\langle \cdot | \cdot \rangle$ indicates the scalar product. Therefore, Eq. (1) can be rewritten for the amplitudes $c_j$ in the basis of eigenvectors $\{ \mathcal{V}_j \}$, where the linear part of the equation will take a form of a matrix with eigenvalues $\lambda_j$ on the diagonals corresponding to the resonance frequencies of the collective excitations. However, the nonlinear term will no longer be diagonal on this basis. In the direct space, the nonlinear term takes form

$$|\Lambda|^2 \Lambda = \sum_{j_1,j_2,j_3} c_{j_1}^* c_{j_2} c_{j_3}^* \mathcal{V}_{j_1} \mathcal{V}_{j_2} \mathcal{V}_{j_3}.$$

Projecting this expression onto the state $\mathcal{V}_{j'}$, one obtains the coupled-mode equations for the amplitudes $c_j$:

$$\frac{\partial c_j}{\partial t} = -\left( \frac{k_0 + k_{\text{FSR}}}{2} + i(\delta \omega_j - \lambda_j) \right) c_j + iD \frac{\partial^2 c_j}{\partial \phi^2} + ig_0 \sum_{j_1,j_2,j_3} c_{j_1}^* c_{j_2} c_{j_3}^* \langle \mathcal{V}_{j_1} \mathcal{V}_{j_2} \mathcal{V}_{j_3} | \mathcal{V}_{j'} \rangle + \tilde{f}_j,$$

where $\tilde{f}_j = \langle F | \mathcal{V}_j \rangle$ is the projection of the pump on the eigenstate $\mathcal{V}_j$, the nonlinear term represents the conventional FWM process with the conservation law dictated by the product $\langle \mathcal{V}_{j_1} \mathcal{V}_{j_2} \mathcal{V}_{j_3} | \mathcal{V}_j \rangle$. The eigenvalues $\lambda_j$, showing the dependence of supermode frequency on supermode number, naturally start to play a role of dispersion, similar to the conventional LLE in a single resonator. In general, the eigenvalues $\lambda_j$ are not equidistantly separated, and the supermode dispersion can be introduced similar to the integrated dispersion of a single resonator $D_{\text{int}}(k) = \lambda_k - \lambda_{k_0} - f_1(k - k_0)$, where $f_1$ is the local FSR of the spatial supermodes in the vicinity of $k_0$. Depending on the system, the supermode dispersion has the same dimensionality $D$ as the system’s band structure. Thus, the total hybridized dispersion (including the chromatic dispersion) profile for photons in the system has $D + 1$ dimensionality.

We note, however, that the key requirement for the validity of the reasoning presented above is the diagonalizability of the coupling matrix $M$ which does not impose any restriction on the dimensionality of the system.

**Chains of coupled microresonators**

Two-dimensional hybridized dispersion. We continue our analysis by considering a system of the equally coupled chain of...
resonators. First, we explicitly write Eq. (1) in the case of constant coupling in a chain
\[
\frac{\partial A_\ell}{\partial t} = - \left( \frac{\kappa_{ex.\ell} + \kappa_0}{2} + i \delta \omega_0 \right) A_\ell + i f \left( A_{\ell - 1} + A_{\ell + 1} \right) + i \frac{D_2}{2} \frac{\partial^2 A_\ell}{\partial \varphi^2} + ig_0 |A_\ell|^2 A_\ell + \sqrt{\kappa_{ex.\ell} \kappa_0} e^{i \delta \omega_0} \frac{\partial A_\ell}{\partial \varphi}.
\] (4)

For simplicity, in the case of constant couplings to the bus waveguides \( \kappa_{ex.\ell} \), we introduce normalized variables \( d_2 = D_2/\kappa \), \( \kappa = \kappa_0 + \kappa_{ex.\ell} \), \( \zeta_0 = 2 \delta \omega_0 / \kappa \), \( f_\ell = \sqrt{8 \kappa_{ex.0} / \kappa^3} e^{i \delta \omega_0} \), \( \Psi_\ell = \sqrt{2 \kappa_0 / \kappa} A_\ell \). In the normalized units, Eq. (4) reads
\[
\frac{\partial \Psi_\ell}{\partial \tau} = - (1 + i \zeta_0) \Psi_\ell + i d_2 \frac{\partial^2 \Psi_\ell}{\partial \varphi^2} + i f \left( \Psi_{\ell - 1} + \Psi_{\ell + 1} \right) + i |\Psi_\ell|^2 \Psi_\ell + f_\ell.
\] (5)

Further, we can readily diagonalize the linear part by taking the Fourier transform
\[
\psi_{\mu k} = \frac{1}{2 \pi \sqrt{N}} \int \sum_{\ell = 1}^{N} \psi_{\mu \ell} e^{2 \pi i (k \mu + \varphi_\ell)} d \varphi,
\] (6)

where \( k \) is the supermode index and \( \mu \) is the comb line index. With the Kerr term, Eq. (5) transforms to
\[
\frac{\partial \psi_{\mu k}}{\partial \tau} = - (1 + i \zeta_0) \psi_{\mu k} - i \left[ d_2 \mu^2 - 2 j \cos \frac{2 \pi k}{N} \right] \psi_{\mu k} + \delta_{\mu 0} \hat{f}_k \\
+ i \frac{1}{N} \sum_{k_1, k_2, k_3} \psi_{\mu k_1 k_2 k_3} \psi_{\mu k_1 k_2} \delta_{\mu + k_3 - \mu - k_2 - k_1} \psi_{\mu k_2 k_3} \psi_{\mu k_1} \delta_{\mu + k_1 - \mu - k_2 - k_3}.
\] (7)

In this form, we obtain the analytical expression for the hybridized 2D dispersion surface
\[
d_{int}(\mu, k) = 2 \omega_{pk} - \omega_0 + D_1 \mu / \kappa = d_2 \mu^2 - 2 j \cos(2 \pi k / N).
\] (8)

In the case of anomalous group velocity dispersion (GVD) \((d_2 > 0)\), Fig. 1d) of the individual resonator, this surface with parabolic and cosine cross-sections is shown in Fig. 1a. Local dispersion topography changes along the \( k \) axis, revealing different regions with parabolic and saddle shapes. The pump term \( f_\ell \) stands for the projection of the pump on the \( k \)-th supermode
\[
f_k = \frac{1}{N \sqrt{N}} \sum_{\ell = 1}^{N} f_\ell e^{2 \pi i k / N}.
\] (9)

Spatial eigenstates and pump projection on the chain. The supermode dispersion (i.e., band structure) has regions of anomalous and normal supermode group velocity dispersion (sGVD). For a given supermode index \( k_0 \), the linear term in the Taylor series of the cosine gives the supermode FSR equal to
\[
J_1 / 2 = 2 \pi / N \sin(2 \pi k_0 / N) \text{ and the corresponding quadratic term yields sGVD } J_2 = 2 \pi / N \cos(2 \pi k_0 / N) \text{ for Eq. (4).}
\]

The excitation of the individual supermode requires an accurate pump projection on its spatial profile. In case of imperfect projection of the pump, the number of the excited modes will depend on the local density of states within the width of the band. Moreover, the single-resonator pump scheme always leads to the excitation of supermodes in pairs due to their twofold degeneracy, except for the modes from the very top and bottom of the band. According to Eq. (9), if the resonator \( \ell = 0 \) is pumped, all the supermodes have a pump term with the projection amplitude \( 1 / \sqrt{N} \). With the increasing number of resonators, a pumping scheme with a single resonator excitation becomes less efficient, and more sophisticated schemes are required. For simplicity of the further analysis, in the following we focus on the ideal case of a single supermode excitation. Accurate projection to the supermode with index \( k_0 \) requires accurate adjustment the relative phases of the pump lasers according to
\[
f = f^{(0)} \left[ 1, e^{2 \pi \imath k_0 / N}, e^{4 \pi \imath k_0 / N}, \ldots, e^{2 (N-1) \pi \imath k_0 / N} \right],
\] (10)

where \( f^{(0)} = \sqrt{8 \kappa_0 \kappa_{ex} P / \kappa^3 \nu_0 N} \) is normalized pump for a single resonator.

Modulation instability gain lobes. Further, we investigate the stability of plane wave solutions \( \psi_{\mu 0} \). Considering the pump at \( \mu_0 = 0 \) and at the parabolic region \( k_0 = 0 \) (saddle point \( k_0 = N/2 \)), we investigate FWM processes between the pump mode and the modes with indexes \( \mu, k \). Linearizing the system, we identify the modes with positive parametric gain. Our analysis, similar to \( \psi^{(0)} \), shows that the modulationally unstable solutions form an ellipse (hyperbola) in the \( \mu - k \) space \((k = 0, \mu = 0)\).

\[
d_2 \mu^2 \pm j_2 k^2 = 4 |\psi_{\mu 0}|^4 + \sqrt{\pi} |\psi_{\mu 0}|^4 - 1 - (\zeta_0 + 2 j),
\] (11)

here \((+ \) stands for the excitation of \( k - k_0 = 0 \) \((k = k_0 = N/2)\).

An example of the modulation instability (MI) gain lobes [Eq. (11)] is presented in Fig. 2a, b for both regions in case of \( d_2 = 0.04 \) and \( j_2 = |J_2| / k = 1 \). Figure 2a reveals that the supermode corresponding to the excitation of all the resonators in-phase (anomalous sGVD) is unstable against small perturbations with \( \mu \) and \( k \) indexes that form an ellipse. The width and height of the ellipse are defined by pump power, \( d_2 \), and \( j_2 \) coefficients that correspond to GVD and sGVD. In contrast, the state corresponding to the excitation of the neighboring resonators in the opposite phase (normal sGVD) is unstable against the perturbations with \( \mu \) and \( k \).

Fig. 2 Modulation instability gain lobes in chains of coupled resonators. Hybridized dispersion profile shown as a surface in panel (a) for the elliptic and (b) for the hyperbolic regions. Contour plots in the \( k - \mu \) plane at \( D_{int} = 0 \) highlight the different local topographies that result in the elliptic (a) or hyperbolic (b) modulation instability gain lobes depicted in red.

[Image 433x81 to 452x199]
forming a hyperbola (see Fig. 2b), showing that all the supermodes can experience positive parametric gain.

Wave collapse. We continue with the simulation of the coupled LLEs in Eq. (4) for 20 resonator chain and constant normalized coupling \( j = 10 \) \((j_2 = 1)\). To simulate the temporal dynamics, we employ the step-adaptive Dormand-Prince Runge-Kutta method of Order 8(5,3)\(^{39}\) and approximate the dispersion operator by the second-order finite difference scheme. We deliberately choose the pumping scheme allowing for exciting only a given mode. To trigger the FWM processes, we numerically scan the resonance with a fixed pump power and track field dynamics in all the resonators.

First, we focus on the investigation of the unstable behavior of the system pumping the elliptic \((k_0 = 0)\) and hyperbolic regions \((k_0 = N/2)\) with the pump \( f_p = 2.35 \) and corresponding detunings \( \zeta_0 = 22.1 \) and \( \zeta_0 = -17.0 \). In the former case, at a single resonator level we observe the random appearance of the pulses in different parts of the cavity and their further rapid compression, during which the peak amplitude significantly exceeds (60 times) the background level (cf. Fig. 3a). Computing the nonlinear dispersion relation (NDR)\(^{40}\), we observe the high photon occupancy of the pump region beneath the parabolas (cf. Fig. 3b), which indicates the presence of 2D dissipative nonlinear structures. Furthermore, all the hybridized parabolas are populated by the photons, meaning that supermodes from both dispersion regions are excited (please also refer to the Supplementary Movie 1 demonstrating resolved dynamics in time of the field in all resonators and the corresponding 2D \( k - \mu \) spectrum). To further confirm it, we reconstruct the supermode NDR (sNDR) for the 0th comb line \((\mu = 0)\) for all resonators in the following way

\[
sNDR(\Omega, \mu_0, k) = \frac{1}{\sqrt{N \sum_{n=1}^{N} |\psi_n(0)^{\langle}\rangle|}} e^{(2\pi i / N - \Omega t_n)}, \tag{12}
\]

where \( \Omega \) is slow frequency, \( t_n = \Delta t n \) with \( \Delta t = T/N \), \( T \) is time simulation with \( N \) number of discretization points. The result is shown in Fig. 3c. The whole cosine band structure is populated, including the region of the normal dispersion. In the opposite case, the spatiotemporal diagram (Fig. 3d) in hyperbolic region does not demonstrate any extreme events, showing slow (with respect to the elliptic case) coherent dynamics (cf. Supplementary Movie 2). Comparing the NDR (Fig. 3e) with the elliptic case, we show less supermode occupancy. In the vicinity of \( \mu = 0 \), the normal sGVD suppresses the photon transfer along the \( k \) axis. Nevertheless, the photon transfer to other supermodes is stimulated with respect to eq. (11) that depicts MI gain lobes position, resulting in the generation of dispersive waves\(^{41,42}\). Reconstructing the supermode NDR (Fig. 3f) for \( \mu = 25 \) comb line [the average crossing position in Fig. 3e], we observe the predominant population of the center of the band.

We attribute this drastic difference in the chaotic dynamics to the effect called wave collapse\(^{41,42}\) that plays an important role in physics and leads to an effective mechanism of local energy dissipation. Our system, in the long-wavelength limit, can be modeled by 2D LLE with elliptic \((\partial_{\varphi}^2 + \partial_{\theta}^2)\) or hyperbolic \((\partial_{\varphi}^2 - \partial_{\theta}^2)\) dispersion (here \( \theta \) stands for the continuous coordinate along the circumference of the chain). Neglecting the pump and damping terms, we obtain the conservative 2D NLSE, which in the elliptic case can result in full compression of a pulse to an infinitely small area concentrating there a finite amount of energy\(^{43,44}\). Such pulse becomes ultra-broad in the spectral domain, and even the presence of dissipation in 2D LLE does not restrict this effect\(^ {45}\). On the contrary, wave collapses do not occur in the 2D focusing NLSE with hyperbolic dispersion\(^ {44}\), signifying that it is the dispersion curvature that is responsible for the effect. Moreover, higher dispersion orders of the cosine limit the pulse compression in the elliptic region, regularizing the singularity\(^ {46}\).

Coherent dissipative structures (Turing patterns and 2D dissipative solitons). Turing patterns. As the different dispersion topographies result in completely different chaotic dynamics, the Turing patterns in the elliptic and hyperbolic regions differ in the same way. To observe the coherent structures, we first bring the system to into an unstable state. Stimulating the incoherent patterns, we further tune towards the monostable region \((\zeta_0 = \zeta_0 = 2 < \sqrt{3}, + (-) \) stands for \( k_0 = 0 \) (N/2)), pass through breathers (e.g., Supplementary Movie 3 indicating distorted breathing Turing pattern in the elliptic regime), and obtain stable coherent structures in both regimes (Fig. 4). One can see that in the elliptic regime at \(|f| = 1.05\) and \( \zeta_0 = 20.5 \), we observe the

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**Fig. 3** Numerical reconstruction of the nonlinear dispersion relation in the elliptic and hyperbolic regions in the unstable regime. Panels (a–c) correspond to the elliptic region \((k_0 = 0, d_2 > 0, j_2 > 0)\), panels (d–f) to the hyperbolic \((k_0 = N/2, d_2 > 0, j_2 < 0)\). Spatiotemporal diagrams of unstable states in 0th resonator are shown in (a) and (d); The corresponding nonlinear dispersion relation (NDR) in the elliptic region (b) demonstrates excitation of all the optical and spatial modes, whereas the NDR in the hyperbolic region (c) reveals that photon transfer between the spatial supermodes is suppressed in the vicinity of the pump mode \( \mu = 0 \). The panels (c) and (f) represent the nonlinear supermode dispersion relation [Eq. (12)] of 0th comb line for the state in (a) and 25th comb line for the state in (d).
formation of a Turing pattern (Fig. 4a)\textsuperscript{47–49}. On a single resonator level, this corresponds to locked pulses (Fig. 4b) with a typical comb spectrum shown in Fig. 4d. The corresponding 2D \(k\)-\(\mu\) spectral profile in Fig. 4c shows that the sidebands form a disk, occupying the supermodes from both anomalous (\(|k−k_0|<5\)) and normal dispersion regimes. In the hyperbolic regime, at \(|f_d| = 2.35\) and \(\zeta_0 = −20.3\), we observe a train of pulses in each resonator locked to each other (Fig. 4e, f). The corresponding 2D spectral profile (Fig. 4g) forms a line in \(k\)-\(\mu\) space, that qualitatively follows one of the asymptotes of the hyperbola that depicts modulation instability gain lobes in Fig. 2b (also similar to\textsuperscript{50}).

The 2D spectral profiles of the states (a) and (e) obtained via Eq. (6) are presented in (c) and (g), respectively. The spectral profile in elliptic regime (c) forms a disk, whereas the spectrum of the pattern in hyperbolic regime (g) tends to align one of the asymptotes of the hyperbola depicting modulation instability gain. The Fourier spectra of the states (b) and (f) are presented in (d) and (h).

**Fig. 4** Coherent dissipative structures in a driven nonlinear photonic ring lattice. Panels (a–d) correspond to the elliptic region (\(k_0 = 0, d_2 > 0, j_2 > 0\)), and panels (e–h) correspond to the hyperbolic (\(k_0 = N/2, d_2 > 0, j_2 < 0\)). Spatiotemporal profiles of the mode-locked structures are shown in panels (a, e) with the corresponding field profile on a single resonator level in panels (b, f). The 2D spectral profiles of the states (a) and (e) obtained via Eq. (6) are presented in (c) and (g), respectively. The spectral profile in elliptic regime (c) forms a disk, whereas the spectrum of the pattern in hyperbolic regime (g) tends to align one of the asymptotes of the hyperbola depicting modulation instability gain. The Fourier spectra of the states (b) and (f) are presented in (d) and (h).
1D dispersion curve. To generate an edge soliton, one needs to pump the edge state ($\xi_0$ is the corresponding amplitude). Neglecting the presence of the bulk states ($\xi_j \ll \xi_0$), according to Eq. (3), the governing equation takes the form of a simple LLE. This analogy helps further to understand the soliton interaction with the bulk states. Generation of the edge soliton corresponds to the formation of the dispersionless line below the edge state parabola (schematically shown in Fig. 1c). If the width of the bandgap is large enough (effectively corresponds to the limit $J_{\text{inter}}/J_{\text{intra}} \rightarrow 0$, $J_{\text{inter}} \gg k_0$), the dynamics of the soliton will be similar to the single-resonator dynamics, because the field will be still localized in the first ring. However, if the soliton line crosses the lower bulk band, additional photon transfer to the bulk modes will occur (a similar effect has already been observed in the system of just two coupled resonators considered in24). The photons scattered to the bulk will experience now 2D dynamics and can drastically affect the soliton stability and existence range.

To demonstrate this effect, we simulated an SSH chain of 10 resonators with typical parameters of Si$_3$N$_4$ photonic microcombs: $\kappa_0/2\pi = 50$ MHz, $D_1/2\pi = 4.1$ MHz, $J_{\text{intra}}/2\pi = 5$ GHz, $D_2/2\pi = 182$ GHz, $J_{\text{inter}}/2\pi = 1$ GHz under 100 mW pump power (that corresponds to the normalized pump $f^2 = 22.5$ for the normalization) with critically coupled ($\xi_0 = \kappa_0$) first resonator. We excited the edge state resonance in the conventional way, scanning the pump laser from the blue- to red-detuned zone. We show the intracavity power in the first (blue line) and second (red line) resonators as a function of detuning in Fig. 6a along with the limit case of a decoupled single resonator (black line). As one can clearly see, the power dynamics in the first resonator generally has the same features as the decoupled resonator, but the soliton existence range ($\delta \omega = 27.2\pi/2$) is shortened in the case of the SSH chain ($\delta \omega = 21.9\pi/2$). In fact, the power in the second resonator reveals several resonance features with increased detuning. Investigating the field and spectral profiles of the DKS (detuning $\delta \omega = 2\pi \times 1.48$ GHz = $14.8\pi/2$) itself in these resonators (cf. Fig. 6b, c), one can see that the soliton has a smooth sech profile in the first cavity, while there is a strong background modulation in the second with the soliton amplitude 200 times smaller. The reconstructed NDRs (Fig. 6d, e) reveal that the soliton, formed below the edge parabola crosses the lower bulk modes that lead to the generation of the dispersive waves in the second resonator, mixing the edge and bulk states. While here we presented a case of moderate edge-bulk interaction employing a single set of parameters used in recent experiments24,25,53, stronger interactions can occur for spectrally broader solitons (smaller ratio $D_2/k_0$) or narrow bandgaps. Nevertheless, our conclusions remain valid for a smaller gap size and can be generalized for higher dimension topological lattices: due to the presence of the other bands, the generated edge-state soliton (be it DKS or 2D-DKS) induces edge-to-bulk scattering that influences soliton stability and can result in its temporal decoherence. However, it should be noted that according to the nonlinear term in Eq. (3), rigorous estimation of the efficiency of the nonlinear interactions between the edge and bulk states depends on the spatial overlap between the eigenstates $\psi_j$. Thus, in order to quantitatively characterize the scattering mechanism, one needs to compute the eigenstates and the corresponding overlap, but this research is beyond the scope of the current manuscript.

**Conclusions**

We theoretically described nonlinear interactions via four-wave mixing (i.e. parametric oscillations) in lattices of driven photonic microresonators. We showed the hybridization of the chromatic dispersion with the $N$-spatial dimensional band structure gives rise to an effective $(N+1)$-dimensional dispersion surface that governs the FWM processes with the conservation law defined by the structure of the lattice’s eigenfunctions. Further, we analytically and numerically explored the nonlinear dynamics of the 1D band in a chain of equally coupled resonators. We demonstrated that this system possesses a 2D dispersion surface and can be described in the long wavelength limit by the 2D LLE at its local extremum. Different parts of the dispersion surface correspond to two fundamentally different regimes of operation: elliptic and hyperbolic. This results in different local dispersion topography. Simulating the full set of coupled LLEs, we demonstrated nonlinear effects inherent to 2D systems which include Turing pattern formation, 2D spatial-temporal dissipative Kerr solitons, and wave collapses in the chaotic state. Considering the SSH model, we demonstrated that 0D bands that correspond to the edge state can be approximated by the conventional 1D-LLE. However, within the range of experimentally accessible parameters for state of the art nonlinear integrated platforms that support generation sufficiently broad DKS, the overlap of the generated edge-state DKS with the bulk states can lead to the DKS-induced edge-bulk mixing and consequently perturb the soliton stability. In summary, our theory sheds light on nonlinear interactions in integrated photonic lattices and will be helpful to guide future experimental investigations of multi-mode systems with complex band structures, and highlight the
 limitation of topological protection when it comes to the formation of Kerr frequency combs in lattices.

**Methods**

**Numerical simulations.** The nonlinear dynamics in the chain of 20 coupled resonators is modeled using step-adaptive Dormand-Prince Runge-Kutta method that is implemented in Python-based library PyCORE [https://github.com/ElKosto/PyCORE/tree/PyCORE](https://github.com/ElKosto/PyCORE/tree/PyCORE)++ with included integrator from Numerical Recipes 3. The normalized parameters of the simulated system in Eq. (5) are: $d_0 = 0.04$, $f_0 = 10.00$. The corresponding realistic Si$_3$N$_4$ parameters are: $\kappa/2\pi = \kappa_{\text{ex}}/2\pi = 50$ MHz, $J/2\pi = 0.5$ GHz, $D_2/2\pi = 4.1$ MHz, FSR $= 182$ GHz.

**Wave collapse.** To observe the wave collapse in the elliptic regime, we fix the detuning at $\zeta_0 = 22.1$. Each resonator is pumped with a power of $f_0 = 2.35$ with equal relative phases. To reconstruct the incoherent dynamics in the hyperbolic regime, we fix the detuning at $\zeta_0 = -17.0$ and use the same pump power. However, we alternate the phases of the pump by $\pi$ to excite only the supermode $k = N/2$. In both cases, we employ 1024 sampling points to resolve the angular dynamics within each resonator and 20,000 sampling points to resolve the temporal dynamics.

**Turing patterns.** To excite the Turing patterns, we first induce incoherent dynamics in the cavity and then tune the pump laser from the blue to the red-detuned side of the resonance (i.e., decreasing $\zeta_0$). For the elliptic regime, we observe stable pulses at $f_0 = 1.05$ and $\zeta_0 = 20.5$, and for the hyperbolic regime at $f_0 = 2.35$ and $\zeta_0 = -20.3$. We employ the Newton-Raphson method for finding stationary solutions to verify the stability of such structures. Using the simulation data as an initial seed solution, we then apply the gradient descent method to converge to the stationary solutions shown in Fig. 4.

**2D spatio-temporal dissipative soliton.** To observe the generation of the 2D soliton, we fix the pump power at $f_0 = 2.35$ and set the relative phase to $2\pi/5$ to excite the 4th spatial supermode in Fig. 1a. We set the laser detuning $\zeta_0$ such that the system is in a chaotic state, then we gradually reduce the detuning value to $\zeta_0 = 10.92$, where we observe the formation of the soliton.

**Edge state soliton in the Su-Schrieffer-Heeger model.** We use modified Eq. (5) to simulate the dynamics of the 10 resonator Su-Schrieffer-Heeger (SSH) chain. To take into account the chain with open boundary conditions, we consider 1st and 10th resonators coupled only to one of its neighbors, but decoupled from each other. The resonator parameters, such as $\kappa_0$, $\kappa_{\text{ex}}$, $D_2$, and FSR, are to be the same as for the closed chain. The inter-cell coupling equals to $J_{\text{inter}}/2\pi = 5$ GHz, the intra-cell $J_{\text{intra}}/2\pi = 1$ GHz. The chain consists of 10 resonators, and the bus waveguide is only coupled to the first and last resonators. To excite the edge state, we fix the pump power at 100 mW which is equivalent to $f_0 = 22$ in the normalized units. The edge state resonance is located at $\zeta_0 = 0$, and we sweep the laser frequency from the blue to the red-detuned side (increasing $\zeta_0$). The single-resonator transmission trace presented in Fig. 6a is obtained using the standard Lugiato-Lefever equation (i.e., Eq. (5)) with zero coupling terms. The resonator and pump laser parameters are chosen to be the same as for the excitation of the edge state in the SSH model.

**Data availability**

All data that support the plots within this paper and other findings of this study are available from the corresponding author upon reasonable request.

**Code availability**

Numerical codes used in this study are available from the corresponding author upon reasonable request.
