Equivalence between tails, Grand Lebesgue Spaces and Orlicz norms for random variables without Cramer’s condition

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Abstract

We offer in this paper the non-asymptotical pairwise bilateral exact up to multiplicative constants interrelations between the tail behavior, moments (Grand Lebesgue Spaces) norm and Orlicz’s norm for random variables (r.v.), which does not satisfy in general case the Cramer’s condition.

Key words and phrases: Random variable and random vector (r.v.), centered (mean zero) r.v., saddle-point method, tail and bilateral tail estimates, rearrangement invariant Banach space of random variables, tail of distribution, moments, Lebesgue-Riesz, Orlicz and Grand Lebesgue Spaces (GLS); slowly varying functions, Tchebychev-Markov inequality, Young-Fenchel transform, theorem and inequality of Fenchel-Moreau, Young-Orlicz function, norm, Markov-Tchnerov’s estimate, non-asymptotical estimates, Cramer’s condition.

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1 Definitions. Notations. Previous results. Statement of problem.

Let $(\Omega, B, P)$ be certain probability space with non-trivial probability measure $P$ and correspondent expectation $E$, and let $\xi = \xi(\omega), \omega \in \Omega$ be numerical valued random variable. We denote as usually by $|\xi|_p, p \in [1, \infty]$ its classical Lebesgue - Riesz $L_p = L(p) = L_p(\Omega)$ norm

$$|\xi|_p := [E|\xi|^p]^{1/p}, \ 1 \leq p < \infty; \ |\xi|_\infty := \operatorname{vraisup}_{\omega \in \Omega} |\xi(\omega)|.$$
The so-called tail-function \( T_\xi(u), \ u \geq 0 \) for this random variable \( \xi \) is defined by a formula

\[
T_\xi(u) \overset{\text{def}}{=} \max \{ \mathbb{P}(\xi > u), \ \mathbb{P}(\xi < -u) \}, \ u \geq 0.
\]  

(1.1)

An equivalent version:

\[
T_\xi(u) := \mathbb{P}(|\xi| > u), \ u \geq 0.
\]  

(1.1a)

Obviously,

\[
\mathcal{T}_\xi(u) \leq T_\xi(u) \leq 2 \mathcal{T}_\xi(u), \ u \geq 0,
\]

the equivalence.

The aim of this report is to establish the reciprocal non-asymptotic interrelations separately mutually possibly exact up to multiplicative constant between tail functions, suitable Orlicz and Grand Lebesgue Spaces norms for random variables.

We do not suppose that the considered in this article r.v. satisfy the famous Cramer’s condition:

\[
\exists \epsilon_0 > 0 \ \forall \lambda : \ |\lambda| < \epsilon_0 \ \Rightarrow \mathbb{E}\exp(\lambda \xi) < \infty,
\]

in contradiction with previous works, see, for example, works [2], [6], [7], [8], [19].

Throughout this paper, the letters \( C, C_j(\cdot) \) etc. will denote a various positive finite constants which may differ from one formula to the next even within a single string of estimates and which does not depend on the essentially variables \( p, x, \lambda, y, u \) etc.

We make no attempt to obtain the best values for these constants.

The immediate predecessor of offered report is the article [7], in which was considered the case when the considered r.v. satisfy the famous Cramer’s condition. See also [6], [8], chapters 1, 2 etc.

We will use in this report in general at the same techniques as in [7].

Recall that the so-called Young-Fenchel transform \( g \rightarrow g^* \) of arbitrary real valued function \( g = g(x) \) is defined as follows

\[
g^*(y) \overset{\text{def}}{=} \sup_{x \in \text{Dom}(g)} (xy - g(x)).
\]  

(1.2)

The symbol \( \text{Dom}(g) \) denotes as ordinary the domain of definition (in particular, finiteness) of the function \( g(\cdot) \).

Let us bring some used further examples. Define the function

\[
\phi_{m,L} = \phi_{m,L}(\lambda) := m^{-1} \lambda^m L(\lambda), \ \lambda > 0, \ m = \text{const} > 1,
\]  

(1.3)

where \( L = L(\lambda) \) is positive slowly varying at infinity, i.e. as \( \lambda \rightarrow \infty \) function. Then as \( x \rightarrow \infty \)
\[ \phi_{m,L}^*(x) \sim (m')^{-1} x^{m'} L^{-1/(m-1)} \left( x^{1/(m-1)} \right), \quad (1.4) \]
and as ordinary for arbitrary value \( m > 1 \)
\[ m' \overset{def}{=} \frac{m}{m-1}. \]
If for instance \( L(\lambda) = [\ln(\lambda + e)]^r, \ r = \text{const} \in \mathbb{R}, \ i.e. \)
\[ \phi(\lambda) := \phi_{m,r}(\lambda) = m^{-1} \lambda^m \ [\ln(\lambda + e)]^r, \ \lambda \geq 0, m = \text{const} > 1, \ r \in \mathbb{R}, \]
then as \( x \to \infty \)
\[ \phi_{m,r}^*(x) \sim (m')^{-1} x^{m'} \ [\ln(x + e)]^{-r/(m-1)}, \quad (1.5) \]
see, e.g. [18], p. 40 - 42.
Analogously if \( \phi(\lambda) := \phi_{m,r,q}(\lambda) = \)
\[ m^{-1} \lambda^m \ [\ln(\lambda + e)]^r \ [\ln \ln(\lambda + e^e)]^q, \ \lambda > 0, m = \text{const} > 1, \ r, q \in \mathbb{R}, \]
then similarly as \( x \to \infty \)
\[ \phi_{m,r,q}^*(x) \sim (m')^{-1} x^{m'} \ [\ln(x + e)]^{-r/(m-1)} \ [\ln \ln(x + e^e)]^{-q/(m-1)}. \quad (1.6) \]
More generally, if
\[ L(\lambda) = [\ln \lambda]^r \ M(\ln \lambda), \]
where \( M = M(\lambda) \) is positive slowly varying function as \( \lambda \to \infty, \) then as \( x \to \infty \)
\[ \phi_{m,L}^*(x) \sim (m')^{-1} x^{m'} \ [\ln \lambda]^{-r/(m-1)} M^{-1/(m-1)}(\ln x). \quad (1.7) \]
The case \( m = 1 \) is more complicated. Define the \( \Psi \) function \( \psi(L) = \psi(L)(p) \) as follows
\[ \psi(L)(p) \overset{def}{=} \frac{p}{L(p)}, \quad (1.8) \]
where as before \( L = L(\lambda) \) is positive continuous slowly varying function as \( \lambda \to \infty \) tending to infinity as \( \lambda \to \infty. \) Let also the r.v. \( \xi \) be from the Grand Lebesgue Space \( G_{\psi(L)} \) with unit norm:
\[ ||\xi||_{G_{\psi(L)}} \overset{def}{=} \sup_{p \geq 1} \left\{ \frac{||\xi||_p}{\psi(L)(p)} \right\} = 1, \]
then by the direct definition of these norms
\[ ||\xi||_p \leq \frac{p}{L(p)}, \ p \geq 1. \quad (1.9) \]
We deduce by means of Tchebychev-Markov inequality
\[ T_\xi(x) \leq \exp\left(-C(L)\ x \ \ln L(x)\right). \] (1.10)

Conversely, let the estimate (1.10) be a given for some r.v. $\xi$ and some such a positive function $L = L(\cdot)$ for which $L(x) \uparrow \infty$, then
\[ |\xi|_p \leq C(L) \frac{p}{L(p)}; \quad p \geq 1. \] (1.11)

2 Grand Lebesgue Spaces (GLS).

Let $(\Omega, B, \mathbb{P})$ be again at the same probability space. Let also $\psi = \psi(p)$, $p \in [1, b]$, $b = \text{const} \in (1, \infty)$ (or $p \in [1, b]$) be certain bounded from below: $\inf \psi(p) > 0$ continuous inside the semi-open interval $p \in [1, b]$ numerical function such that the function
\[ h(p) = h[\psi](p) \overset{\text{def}}{=} p \ \ln \psi(p) \] (2.0)
is convex.

An important example. Let $\eta$ be a random variable such that there exists $b = \text{const} > 1$ so that $|\xi|_b < \infty$. The natural $G\Psi$ function $\psi_\eta = \psi_\eta(p)$ for the r.v. $\eta$ is defined by a formula
\[ \psi_\eta(p) \overset{\text{def}}{=} |\eta|_p. \]

We can and will suppose $b = \sup\{p, \psi(p) < \infty\}$, so that $\text{supp } \psi = [1, b)$ or $\text{supp } \psi = [1, b]$. The set of all such a functions will be denoted by $\Psi(b) = \{\psi(\cdot)\}; \quad \Psi := \Psi(\infty)$.

We will consider in this article only the case when $b = \infty$; i.e. $\Psi := \Psi(\infty)$.

By definition, the (Banach) Grand Lebesgue Space (GLS) space $G\psi = G\psi(b)$ consists on all the numerical valued random variables (measurable functions) $\zeta$ defined on our measurable space and having a finite norm
\[ |||\zeta||| = |||\zeta|||_{G\psi} \overset{\text{def}}{=} \sup_{p \in [1, b]} \left\{ \frac{|\zeta|_p}{\psi(p)} \right\}. \] (2.1)

The function $\psi = \psi(p)$ is named generating function for the Grand Lebesgue Spaces.

These spaces are Banach functional space, are complete, and rearrangement invariant in the classical sense, see [1], chapters 1, 2; and were investigated in particular in many works, see e.g. [3], [4], [5], [6], [7], [8], [15], [16], [17]. We refer here some used in the sequel facts about these spaces and supplement more.
It is known that if $\zeta \neq 0$, and $\zeta \in G\psi(b)$, then
\[ T_\zeta(y) \leq \exp \left( -h^*_{\psi}(\ln(y/||\zeta||)) \right), \ y \geq ||\zeta||, \]  
where
\[ h(p) = h[\psi](p) \overset{def}{=} p \ln \psi(p), \ 1 \leq p < b. \]

Namely, let $||\zeta||_{G\psi(b)} = 1$; therefore by means of Tchebychev-Markov inequality
\[ T_\zeta(y) \leq \psi^p(p) = \exp (-p \ln y + p \psi(p)), \]
following
\[ T_\zeta(y) \leq \inf_{p \in [1,b]} \exp (-p \ln y + p \psi(p)) = \exp (-h[\psi]^*(\ln(y/||\zeta||)))), \ y \geq e \cdot ||\zeta||. \]

Conversely, the last inequality may be reversed in the following version: if the r.v. $\zeta$ satisfies the Cramer’s condition and
\[ P(|\zeta| > y) \leq \exp \left( -h^*_{\psi}(\ln(y/K)) \right), \ y \geq e \cdot K, \ K = \text{const} \in (0, \infty), \]
and if the function $h_{\psi}(p), \ 1 \leq p < \infty$ is positive, continuous, convex and such that
\[ \lim_{p \to \infty} \psi(p)/p = 0, \]
then $\zeta \in G\psi$, herewith $||\zeta|| \leq C(\psi) \cdot K$ and conversely
\[ ||\zeta||_{G\psi} \leq C(\psi)K \leq C_2(\psi)||\zeta||_{G\psi}, \ 0 < C_1(\psi) < C_2(\psi) < \infty. \]  

Introduce the following exponential Young-Orlicz function
\[ N_{\psi}(u) = \exp \left( h^*_{\psi}(\ln |u|) \right), \ |u| \geq 1; \ N_{\psi}(u) = Cu^2, \ |u| < 1, \]
and the correspondent Orlicz norm will be denoted by $|| \cdot ||_{L(N_{\psi})} = || \cdot ||_{L(N)}$. It was done
\[ ||\zeta||_{G\psi} \leq C_1||\zeta||_{L(N)} \leq C_2||\zeta||_{G\psi}, \ 0 < C_1 < C_2 < \infty. \]  

If for instance $\psi(p) = \psi_m(p) \overset{def}{=} p^{1/m}, \ p \in [1,\infty)$, where $m = \text{const} > 1$, then
\[ 0 \neq \xi \in G\psi_m \iff T_\xi(u) \leq \exp (-C(m)u^m). \]

Define also the correspondent Young-Orlicz function
\[ N_m(u) := \exp (|u|^m), \ |u| \geq 1; \ N_m(u) = Cu^2, \ |u| \leq 1. \]

The relation (2.3) means in addition in this case
\[ ||\zeta||_{G\psi_m} \leq C_1(m)||\zeta||_{L(N_m)} \leq C_2||\zeta||_{G\psi_m}, \quad 0 < C_1(m) < C_2(m) < \infty. \quad (2.5) \]

Notice that in the case when \( m \in (0, 1) \) the correspondent random variable \( \xi \) does not satisfy the Cramer’s condition. We intend to generalize the last propositions further on the case just in particular \( m \in (0, 1) \).

Define as an example the following degenerate \( G\psi \) function

\[ \psi_{(r)}(p) = 1, \quad 1 \leq p \leq r; \quad \psi_{(r)}(p) = \infty, \quad p > r; \quad r = \text{const} > 1. \]

The \( G\psi_{(r)} \) norm of an arbitrary r.v. \( \eta \) is quite equivalent to the classical Lebesgue-Riesz \( L_r \) norm

\[ ||\eta||_{G\psi_{(r)}} = |\eta|_r. \quad (2.6) \]

Thus, the Grand Lebesgue Spaces are direct generalizations of the Lebesgue-Riesz spaces.

### 3 Auxiliary estimates from the saddle-point method.

We must investigate in advance one interest and needed further integrals. Namely, let \((X, M, \mu), \ X \subset R\) be non-trivial measurable space with non-trivial sigma finite measure \( \mu \).

We assume at once \( \mu(X) = \infty \), as long as the opposite case is trivial for us. We intend to estimate for sufficiently greatest values of real parameter \( \lambda \), say \( \lambda > e \), the following integral

\[ I(\lambda) := \int_X e^{\lambda x - \zeta(x)} \mu(dx). \quad (3.1) \]

assuming of course its convergence for all the sufficiently great values of the parameter \( \lambda \). The offered below estimates may be considered as a some generalizations of the saddle-point method.

Here \( \zeta = \zeta(x) \) is non-negative measurable function, not necessary to be convex.

We represent now two methods for upper estimate \( I(\lambda) \) for sufficiently greatest values of the real parameter \( \lambda \).

Note first of all that if in contradiction the measure \( \mu \) is finite: \( \mu(X) = M \in (0, \infty) \); then the integral \( I(\lambda) \) allows a very simple estimate

\[ I(\lambda) \leq M \cdot \sup_{x \in X} \exp (\lambda x - \zeta(x)) = M \cdot \exp (\zeta^*(\lambda)). \quad (3.2) \]

Let now \( \mu(X) = \infty \) and let \( \epsilon = \text{const} \in (0, 1) \); let us introduce the following auxiliary integral
\[ K(\epsilon) := \int_X e^{-\epsilon \zeta(x)} \mu(dx). \]  

(3.3)

It will be presumed its finiteness at last for some positive value \( \epsilon_0 \in (0, 1) \); then \( \forall \epsilon \geq \epsilon_0 \Rightarrow K(\epsilon) < \infty \).

Then the following measures are probabilistic:

\[ \nu_\epsilon(A) := \frac{\int_A \exp(-\epsilon \zeta(x)) \mu(dx)}{K(\epsilon)}, \; \epsilon \geq \epsilon_0. \]  

(3.4)

We have

\[ \frac{I(\lambda)}{K(\epsilon)} = \int_X \exp(\lambda x - (1 - \epsilon)\zeta(x)) \nu_\epsilon(dx) \leq \exp\left\{ \sup_{x \in X} [\lambda x - (1 - \epsilon)\zeta(x)] \right\} = \exp\left\{ (1 - \epsilon)\zeta^* \left( \frac{\lambda}{1 - \epsilon} \right) \right\}. \]

Following,

\[ I(\lambda) \leq K(\epsilon) \cdot \exp\left\{ (1 - \epsilon)\zeta^* \left( \frac{\lambda}{1 - \epsilon} \right) \right\} \]  

(3.5)

and hence:

**Theorem 3.1** We assert actually under formulated here conditions, in particular, the condition of the finiteness of \( K(\epsilon) \) for some value \( \epsilon_0 \in (0, 1) \):

\[ I(\lambda) \leq \inf_{\epsilon \in (0, 1)} \left[ K(\epsilon) \cdot \exp\left\{ (1 - \epsilon)\zeta^* \left( \frac{\lambda}{1 - \epsilon} \right) \right\} \right]. \]  

(3.6)

We can detail the choice of the value \( \epsilon \) in the estimates (3.5) - (3.6). Namely, denote

\[ \theta = \theta(\lambda) := \frac{c_1}{\lambda \zeta^*(2\lambda)}, \; \lambda \geq \lambda_0 = \text{const} > 0. \]  

(3.7)

The value \( \lambda_0 \) is selected such that \( \theta(\lambda) \leq 1/2, \; \lambda \geq \lambda_0 \). Then

\[ \frac{\lambda}{1 - \epsilon} \leq \lambda(1 + 2\epsilon), \]

and we have taking into account the convexity of the function \( \zeta^*(\cdot) \) and denoting \( \phi(\lambda) = \zeta^*(\lambda) \):

\[ \phi \left( \frac{\lambda}{1 - \theta} \right) \leq \phi(\lambda + 2\lambda\theta) \leq \phi(\lambda) + 2\theta\lambda \phi'(2\lambda) \leq c_2 + \phi(\lambda). \]

To summarize:
\[ I(\lambda) \leq c_2 \ K(\theta(\lambda)) \ \exp(\zeta^*(\lambda)). \] (3.8)

As regards the function \( K = K(\theta(\lambda)) \), note that if \( X = R^+ \), \( \mu(dx) = dx \), and if

\[ \zeta(x) \geq c_4 \ x, \ x \geq 0, \] (3.9)

then

\[ K(\theta(\lambda)) \leq c_5 \ \lambda \ \zeta^*(2\lambda), \]

giving

\[ I(\lambda) \leq c_6 \ \lambda \ \zeta^*(2\lambda) \cdot \exp(\zeta^*(\lambda)), \ \lambda > \lambda_0. \] (3.10)

If in turn instead (3.9) there holds

\[ \zeta(x) \geq c_7 \ x^\alpha, \ \alpha = \text{const} > 0, \ X = R^+, \ \mu(dx) = dx, \]

then

\[ I(\lambda) \leq c_8 \ \left[ \lambda \ \zeta^*(2\lambda) \right]^{1/\alpha} \cdot \exp(\zeta^*(\lambda)), \ \lambda > \lambda_0. \] (3.11)

**Theorem 3.2.** Suppose in addition \( X = (a, \infty) \), \( a = \text{const} \in \mathbb{R} \), or \( X = R \), and that

\[ \exists C = \text{const} \in (0, \infty), \exists \alpha = \text{const} > 1 \Rightarrow \zeta(x) \geq Cx^\alpha, \ x \geq 1. \] (3.12)

Then there exists a finite positive constant \( C = C(\zeta, a) \) such that for sufficiently values \( \lambda \), say for \( \lambda \geq 1 \)

\[ I(\lambda) \leq \exp(\zeta^*(C\lambda)). \] (3.13)

**Proof,** in particular, the finiteness of \( K(\epsilon) \), \( \epsilon \in (0, 1) \) contains in fact in [9], chapter 2.1.

We represent now an opposite method, which was introduced in particular case in [7], [8], sections 1.2. Indeed, let \( \gamma = \text{const} \in (0, 1) \). We apply the Young’s inequality

\[ \lambda x \leq \zeta(\gamma x) + \zeta^*(\lambda/\gamma), \]

therefore

\[ I(\lambda) \leq e^{\zeta^*(\lambda/\gamma)} \cdot \int_X e^{\zeta(\gamma x) - \zeta(x)} \ \mu(dx) = R(\gamma) \ e^{\zeta^*(\lambda/\gamma)}, \] (3.14)
\[ R(\gamma) := \int_X e^{\zeta(\gamma x) - \zeta(x)} \mu(dx). \]  

(3.15)

We obtained really the following second estimate.

Lemma 3.2.

\[ I(\lambda) \leq \inf_{\gamma \in (0,1)} \left[ R(\gamma) e^{\xi^*(\lambda/\gamma)} \right]. \]  

(3.16)

4 Main results: connection between tail behavior and Grand Lebesgue Space norm.

Statement of problem: given a tail function \( T_\xi(y) \) for the certain (non-zero) random variable \( \xi \) of the form

\[ T_\xi(y) \leq \exp(-h^*[\psi](\ln y)), \quad y \geq 1, \]  

(4.1)

where \( \psi(\cdot) \in G\Psi \). It is required to prove \( \xi \in G\psi \), or on the other words to obtain an estimate of the form \( ||\xi||_{G\psi} < \infty \).

Recall that the inverse conclusion: \( ||\xi||_{G\psi} = 1 \Rightarrow (4.1) \) is known, see (2.2).

So, let the estimate (4.1) be a given. We have for the values \( p \geq e \)

\[ p^{-1}||\xi||_p^p \leq \int_0^\infty x^{p-1} \exp(-h^*[\psi](\ln x)) \, dx = \int_{-\infty}^\infty \exp(p \, y - h^*(y)) \, dy. \]  

(4.2)

It remains to use the proposition of theorem 3.1.

**Theorem 4.1.** Suppose

\[ C(h) := \sup_{p \in [1,\infty)} \left[ h^*[\psi](p) \right]^{1/p} < \infty. \]  

(4.3)

If the r.v. \( \xi \) satisfies the inequalities (4.1) and (4.3), then \( \xi \in G\psi : \)

\[ ||\xi||_{G\psi} \leq 2 \, C[h] \, e^{1/e} < \infty. \]  

(4.4)

**Proof.** It is sufficient to note that the function \( p \to h[\psi(p)] \) is continuous and convex and that

\[ (h^*)^* = h^* = h \]

by virtue of theorem of Fenchel-Moreau.

Let us bring some examples.
Example 4.1. Put as before

$$\psi_m(p) = p^{1/m},$$

but here $m = \text{const} \in (0, \infty)$. Let $\xi \in G\psi_m$ and $||\xi||G\psi_m = 1$.

Note that in the case $m \in (0, 1)$ the r.v. $\xi$ does not satisfy in general case the Cramer’s condition. But we conclude on the basis of theorem 3.1 $||\xi||G\psi_m \in (0, \infty) \iff$

$$\exists C(m) \in (0, \infty), \ T_\xi(u) \leq \exp (-C(m) u^m), \ u \geq 0. \quad (4.5)$$

More precisely, if $||\xi||G\psi_m = 1$, then

$$T_\xi(u) \leq \exp \left( -(me)^{-1} y^m \right), \ y > 0.$$ Inversely, assume

$$T_\xi(u) \leq \exp (-y^m), \ y > 0.$$ Then it follows from theorem 3.1

$$||\xi||G\psi_m \leq e^{m+1/e}$$
or equally

$$||\xi||G\psi_m \leq e^{m+1/e} p^{1/m}, \ p \geq 1.$$ Let us consider a more general case, indeed, introduce as above the following $\Psi$ function

$$\psi_{m,L}(p) \overset{def}{=} p^{1/m} L(p), \ m = \text{const} > 0, \quad (4.6)$$

where $L = L(p), \ p \geq 1$ is some positive continuous slowly varying as $p \to \infty$ function. We impose for simplicity the following condition on this function:

$$\forall \theta > 0 \Rightarrow \sup_{p \geq 1} \left[ \frac{L(p^\theta)}{L(p)} \right] =: C(\theta) < \infty. \quad (4.7)$$

This condition is satisfied, if for example $L(p) = [\ln(p + 1)]^r, \ r = \text{const}.$ It follows again from theorem 3.1 that the r.v. $\xi$ belongs to the space $G\psi_{m,L}$:

$$||\xi||G\psi_{m,L} = \sup_{p \geq 1} \left[ \frac{||\xi||p}{\psi_{m,L}(p)} \right] = 1 \quad (4.8)$$

if and only if

$$T_\xi(y) \leq \exp \left( -C(m, \ L) \ y^m / L(y) \right), \ y \geq e. \quad (4.9)$$

As a particular case: define the $\Psi$ — function
\[
\psi_{(m,r)}(p) := p^{1/m} \ln^r (p+1), \; p \geq 1; \; m = \text{const} > 0, \; r = \text{const} \in \mathbb{R}. \quad (4.10)
\]

The random variable \( \xi \) belongs to the space \( G\psi_{(m,r)} \):

\[
||| \xi |||_{G\psi_{(m,r)}} = \sup_{p \geq 1} \left[ \frac{||\xi||_p}{\psi_{(m,r)}(p)} \right] = l \in (0, \infty) \quad (4.11a)
\]

if and only if

\[
T_\xi(u) \leq \exp \left( -C(m,r) \left( \frac{u}{l} \right)^m \ln^r \left( \frac{u}{l} \right) \right), \; u \geq e \; l. \quad (4.11b)
\]

**Example 4.2.** A boundary case.

We introduce the following \( G\Psi \) function

\[
\psi^{(s)}(p) = p \left( \ln (p+1) \right)^s, \; s = \text{const} \in \mathbb{R}, \; p \in [1, \infty). \quad (4.12a)
\]

Then the non-zero r.v. \( \nu \) belongs to the \( G\psi^{(s)} \) space if and only if

\[
T_\nu(y) \leq \exp \left( -C(s) \left( y \ln^{s} (y+1) \right) \right), \; y \geq 0. \quad (4.12b)
\]

Note that the r.v. \( \nu \) satisfies the Cramer’s condition if and only if \( s \leq 0 \). The case \( s = 0 \) correspondent to the exponential distribution for the r.v. \( \nu \); the case \( s = -1 \) take place in particular when the r.v. \( \nu \) has a Poisson distribution, which obey’s but the exponential moments.

**Example 4.3.**

Let us consider the following \( \psi_\beta(p) \) function

\[
\psi_{\beta,C}(p) := \exp \left( Cp^\beta \right), \; C, \; \beta = \text{const} > 0. \quad (4.13)
\]

Obviously, the r.v. \( \tau \) for which

\[
\forall p \geq 1 \quad |\tau|_p \geq \psi_{\beta,C}(p)
\]
does not satisfy the Cramer’s condition.

Let \( \xi \) be a r.v. belongs to the \( G\psi_{\beta,C}(\cdot) \) space:

\[
||| \xi |||_{G\psi_{\beta,C}} = 1, \quad (4.14a)
\]
or equally

\[
||\xi||_p \leq \exp \left\{ Cp^\beta \right\}, \; p \in [1, \infty). \quad (4.14b)
\]

The last restriction is quite equivalent to the following tail estimate

\[
T_\xi(y) \leq \exp \left( -C_1(C, \beta) \left[ \ln (1 + y) \right]^{1+1/\beta} \right), \; y > 0. \quad (4.15)
\]
5 Main results: connection between tail behavior and Orlicz’s space norm.

We retain the notations and definitions of the previous sections, in particular,

\[ G(u) = G[\psi](u) = h^\ast[\psi](\ln u), \ \psi \in G\Psi \]  \hspace{1cm} (5.0)

etc. Define also the following Young-Orlicz function

\[ N[\psi](u) := \exp[\psi(u)] = \exp[ h^\ast[\psi](\ln |u|) ], \ u \geq e; \ N[\psi](u) = C \ u^2, |u| < e. \]  \hspace{1cm} (5.1)

We will prove in this section that the tail estimate (2.2) of the r.v. \( \xi \) is completely equivalent under some simple conditions to the finiteness of its Orlicz’s norm \( ||\xi||_{LN[\psi]} \).

Recall that we do not suppose that the r.v. \( \xi \) satisfies the Cramer’s condition.

**Proposition 5.1.** If for some r.v. \( \xi \) there holds \( ||\xi||_{LN[\psi]} = K \in (0, \infty) \), then

\[ T_\xi(y) \leq \exp \left( -h^\ast[\psi](\ln(y/(C \ K))) \right), \ y \geq e \cdot ||\xi||, \]  \hspace{1cm} (5.2)

**Proof** basing only on the Tchebychev-Markov inequality is at the same as before in the inequality (2.2), see [2], chapters 2,3; [6], [7], [19]. Namely, we deduce that for some positive finite constant \( C_1 \)

\[ E \exp(G(||\xi||/C_1)) < \infty. \]

It remains to use the Tchebychev-Markov inequality.

**Proposition 5.2.** Assume in addition to the foregoing conditions on the function \( \psi(\cdot) \) that the function \( G[\psi](u) \) satisfies the following restriction:

\[ \exists \ \alpha = \text{const} \in (0,1), \ \exists K = \text{const} > 1, \forall x \in (0, \infty) \Rightarrow G(x/K) \leq \alpha \ G(x). \]  \hspace{1cm} (5.3)

If for some r.v. \( \xi \)

\[ T_\xi(y) \leq \exp \left( -h^\ast[\psi](\ln(y)) \right), \ y \geq e, \]  \hspace{1cm} (5.4)

then the r.v. \( \xi \) belongs to the Orlicz space \( LN[\psi] : \)

\[ ||\xi||_{LN[\psi]} \leq C(\psi, \alpha, K) < \infty. \]  \hspace{1cm} (5.5)

**Proof** is more complicated than one for proposition 5.1. It used the following auxiliary fact.
Lemma 5.1. Let a function \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be monotonically increasing, \( T = T_\xi(x), \ S = S_\eta(x), \ x \geq 0 \) be two tail functions correspondingly for non-negative r.v. \( \xi, \eta \) and such that

\[ T_\xi(x) \leq S_\eta(x), \ x \geq 0. \]

We assert:

\[ \int_0^\infty g(x) |dT_\xi(x)| \leq \int_0^\infty g(x) |dS_\eta(x)|. \]

Proof of lemma 5.1. One can suppose without loss of generality that both the tail functions \( T \) and \( S \) are continuous and strictly decreasing. Further, one can realize both the r.v. \( \xi, \eta \) on the classical probability space \( \Omega = \{\omega\} = [0, 1] \) equipped with ordinary Lebesgue measure:

\[ \xi = \xi(\omega) = (1 - T)^{-1}(\omega), \ \eta = \eta(\omega) = (1 - S)^{-1}(\omega), \]

where \( f^{-1} \) denotes the inverse function.

We have \( \xi(\omega) \leq \eta(\omega) \) a.e., therefore \( g(\xi) \leq g(\eta) \) a.e., and all the more so

\[ E g(\xi) = -\int_0^\infty g(x)dT_\xi(x) \leq -\int_0^\infty g(x)dS_\xi(x) = E g(\eta), \]

Q.E.D.

Proof of proposition 5.2. Let the pair of numbers \((\alpha, K)\) be from the condition (5.3). We have relaying the proposition of Lemma 5.1

\[ E \exp(G(\xi/K)) = \int_0^\infty \exp G(x/K) \ |dT_\xi(x)| \leq \int_0^\infty \exp G(x/K) \ |d\exp(-G(x))| \leq \int_0^\infty \exp[\alpha G(x)] \ |d\exp(-G(x))| = \int_0^1 z^{-\alpha}dz = \frac{1}{1 - \alpha} < \infty. \]

We used by passing (5.6) \( \rightarrow \) (5.7) the fact quite thin from an article [14]. It follows immediately from this estimates that \( \xi \in L(N[\psi]), \) see for example [19], p. 31 - 33.

Examples. The condition (5.3) is satisfied for example for the functions of the form

\[ \psi(p) = p^{1/m} L(p), \ \psi(p) = cp \ [\ln(p + 1)]^r \ L(\ln p), \]

where \( m,c = \text{const} \in (0, \infty), \ r = \text{const} \in \mathbb{R} \) and \( L(\cdot) \) is positive continuous slowly varying at infinity function.

Counterexample. The function

\[ \psi(r)(p) = 1, \ 1 \leq p \leq r; \ \psi(r)(p) = \infty, \ p > r; \ r = \text{const} > 1, \]

for which the correspondent function has a form
\[ h^*(\ln u) = r \ln u \]
does not satisfy the condition (5.3). Actually, for the r.v. \( \eta \) from the space \( L_r = L_r(\Omega) \) the correspondent tail estimate has a form

\[ T_\eta(u) \leq cu^{-r}, \]

but the inverse conclusion is not true.

### 6 Concluding remarks.

A. It is interest by our opinion to obtain the generalization of results of this report into multidimensional case, i.e. into random vectors, alike in the article [7].

B. We mention even briefly an important possible application of obtained results: a Central Limit Theorem in Banach spaces, in the spirit of [8], section 4.1.

C. The case of finite support \( \psi : b < \infty \).

In this case approvals 4.1, 5.1, and 5.2 are in general case incorrect. The correspondent counterexamples may be found in the article [7]. Thus, the problem of description of correspondence between tail behavior and Grand Lebesgue Space norm is in this case an open problem.

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