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Second Hankel determinant for certain subclasses of bi-univalent functions

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Abstract: In the present paper, we obtain the upper bounds for the second Hankel determinant for certain subclasses of analytic and bi-univalent functions. Moreover, several interesting applications of the results presented here are also discussed.

Key words: Analytic functions, univalent functions, bi-univalent functions, subordination between analytic functions, Hankel determinant

1. Introduction and definitions

Let \( \mathcal{A} \) denote the family of functions \( f \) analytic in the open unit disk

\[
\mathcal{U} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}
\]

of the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}
\]

Let \( \mathcal{S} \) denote the class of all functions in \( \mathcal{A} \) that are univalent in \( \mathcal{U} \). The Koebe one-quarter theorem (see, for example, [9]) ensures that the image of \( \mathcal{U} \) under every \( f \in \mathcal{S} \) contains a disk of radius \( 1/4 \). Clearly, every \( f \in \mathcal{S} \) has an inverse function \( f^{-1} \) satisfying \( f^{-1}(f(z)) = z \) (\( z \in \mathcal{U} \)) and \( f(f^{-1}(w)) = w \) (\( |w| < r_0(f); \ r_0(f) \geq 1/4 \)),

where

\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots.
\]

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathcal{U} \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \mathcal{U} \). Let \( \sigma \) denote the class of bi-univalent functions in \( \mathcal{U} \) given by (1.1).

In 1967, Lewin [21] showed that, for every function \( f \in \sigma \) of the form (1.1), the second coefficient of \( f \) satisfies the estimate \( |a_2| < 1.51 \). In 1967, Brannan and Clunie [2] conjectured that \( |a_2| \leq \sqrt{2} \) for \( f \in \sigma \). Later, Netanyahu [22] proved that \( \max_{f \in \sigma} |a_2| = \frac{4}{3} \). In 1985, Kedzierawski [17] proved the Brannan–Clunie conjecture for bi-starlike functions. In 1985, Tan [31] obtained the bound for \( a_2 \), namely that \( |a_2| < 1.485 \), which is the best

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known estimate for functions in the class $\sigma$. Brannan and Taha [3] obtained estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions in the classes of bi-starlike functions of order $\beta$ and bi-convex functions of order $\beta$.

The study of bi-univalent functions was revived in recent years by Srivastava et al. [30] and a considerably large number of sequels to the work of Srivastava et al. [30] have appeared in the literature since then. In particular, several results on coefficient estimates for the initial coefficients $|a_2|$, $|a_3|$, and $|a_4|$ were proved for various subclasses of $\sigma$ (see, for example, [1, 4, 5, 10, 12, 14, 16, 25, 28, 29, 32, 33]).

Recently, Deniz [7] and Kumar et al. [19] both extended and improved the results of Brannan and Taha [3] by generalizing their classes by means of the principle of subordination between analytic functions. The problem of estimating the coefficients $|a_n|$ ($n \geq 2$) is still open (see also [29] in this connection).

Among the important tools in the theory of univalent functions are Hankel determinants, which are used, for example, in showing that a function of bounded characteristic in $U$, that is, a function that is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral coefficients, is rational [6]. The Hankel determinants $H_q(n)$ ($n = 1, 2, 3, \cdots, q = 1, 2, 3, \cdots$) of the function $f$ are defined by (see [23])

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} (a_1 = 1).$$

This determinant was discussed by several authors with $q = 2$. For example, we know that the functional $H_2(1) = a_3 - a_2^2$ is known as the Fekete–Szegő functional and one usually considers the further generalized functional $a_3 - \mu a_2^2$ where $\mu$ is some real number (see [11]). Estimating for the upper bound of $|a_3 - \mu a_2^2|$ is known as the Fekete–Szegő problem. In 1969, Keogh and Merkes [18] solved the Fekete–Szegő problem for the classes of starlike and convex functions. One can see the Fekete–Szegő problem for the classes of starlike functions of order $\beta$ and convex functions of order $\beta$ in special cases in the paper of Orhan et al. [24]. On the other hand, quite recently, Zaprawa (see [34, 35]) studied the Fekete–Szegő problem for some classes of bi-univalent functions. In special cases, he gave the Fekete–Szegő problem for the classes of bi-starlike functions of order $\beta$ and bi-convex functions of order $\beta$.

The second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2a_4 - a_3^2$. The bounds for the second Hankel determinant $H_2(2)$ were obtained for the classes of starlike and convex functions in [15]. Lee et al. [20] established the sharp bound for $|H_2(2)|$ by generalizing their classes by means of the principle of subordination between analytic functions. In their paper [20], one can find the sharp bound for $|H_2(2)|$ for the functions in the classes of starlike functions of order $\beta$ and convex functions of order $\beta$. Recently, Deniz et al. [8] and Orhan et al. [26] found the upper bound for the functional $H_2(2) = a_2a_4 - a_3^2$ for the subclasses of bi-univalent functions.

The object of the present paper is to seek the upper bound for the functional $|a_2a_4 - a_3^2|$ for $f \in N_\sigma(\beta)$ and $f \in N_\sigma(\beta)$, which are defined as follows.

**Definition 1** (see [30]) A function $f(z)$ given by (1.1) is said to be in the class $f \in N_\sigma(\beta)$ ($0 \leq \beta < 1$) if the following conditions are satisfied:

$$f \in \sigma \quad \text{and} \quad \Re (f'(z)) > \beta \quad (z \in \mathbb{U}; \quad 0 \leq \beta < 1) \quad (1.2)$$
and
\[ \Re (g'(w)) > \beta \quad (w \in \mathbb{U}; 0 \leq \beta < 1), \]  
(1.3)

where the function \( g \) is given by
\[ g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \]  
(1.4)

**Definition 2** *(see [30])* A function \( f(z) \) given by *(1.1)* is said to be in the class \( \mathcal{N}_\alpha^\sigma \) \((0 < \alpha \leq 1)\) if the following conditions are satisfied:
\[ f \in \sigma \quad \text{and} \quad |\arg (f'(z))| \leq \frac{\alpha \pi}{2} \quad (z \in \mathbb{U}; 0 < \alpha \leq 1) \]  
(1.5)

and
\[ |\arg (g'(w))| < \frac{\alpha \pi}{2} \quad (w \in \mathbb{U}; 0 < \alpha \leq 1), \]  
(1.6)

where the function \( g \) is defined by *(1.4).*

For special values of the parameters \( \alpha \) and \( \beta \), we have
\[ \mathcal{N}_\alpha(0) = \mathcal{N}_\alpha = \mathcal{N}_\sigma. \]

Let \( \mathcal{P} \) be the class of functions with positive real part consisting of all analytic functions \( \mathcal{P} : \mathbb{U} \rightarrow \mathbb{C} \) satisfying \( p(0) = 1 \) and \( \Re (p(z)) > 0 \).

To establish our main results, we shall require the following lemmas.

**Lemma 1** *(see, for example, [27])* If the function \( p \in \mathcal{P} \) is given by the following series:
\[ p(z) = 1 + c_1 z + c_2 z^2 + \cdots, \]  
(1.7)

then the sharp estimate given by
\[ |c_k| \leq 2 \quad (k = 1, 2, 3, \ldots) \]  
holds true.

**Lemma 2** *(see [13])* If the function \( p \in \mathcal{P} \) is given by the series *(1.7)*, then
\[ 2c_2 = c_1^2 + x(4 - c_1^2), \]  
(1.8)

\[ 4c_3 = c_1^3 + 2(4 - c_1^3)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2) \left( 1 - |x|^2 \right) z \]  
(1.9)

for some \( x \) and \( z \) with \(|x| \leq 1 \) and \(|z| \leq 1 \).

2. **Main results**

Our first main result for the class \( f \in \mathcal{N}_\alpha(\beta) \) is stated as follows:

**Theorem 1** Let \( f(z) \) given by *(1.1)* be in the class \( \mathcal{N}_\alpha(\beta) \) \((0 \leq \beta < 1)\). Then
\[ |a_2a_4 - a_3^2| \leq \begin{cases} \frac{(1-\beta)^2}{2} \left( 2(1 - \beta)^2 + 1 \right) & (\beta \in \left[ 0, \frac{11 - \sqrt{37}}{12} \right]), \\ \frac{(1-\beta)^2}{16} \left( \frac{60\beta^2 - 84\beta - 25}{93\beta^2 - 15\beta + 1} \right) & (\beta \in \left( \frac{11 - \sqrt{37}}{12}, 1 \right)). \end{cases} \]  
(2.1)
Proof Let $f \in \mathcal{N}_\sigma(\beta)$ and $g = f^{-1}$. Then

\[ f'(z) = \beta + (1 - \beta)p(z) \quad \text{and} \quad g'(w) = \beta + (1 - \beta)q(w) \quad (2.2) \]

where the functions $p(z)$ and $q(z)$ given by

\[ p(z) = 1 + c_1z + c_2z^2 + \cdots \]

and

\[ q(w) = 1 + d_1w + d_2w^2 + \cdots \]

are in class $\mathcal{P}$.

Comparing the coefficients in (2.2), we have

\[ 2a_2 = (1 - \beta)c_1, \quad (2.3) \]
\[ 3a_3 = (1 - \beta)c_2, \quad (2.4) \]
\[ 4a_4 = (1 - \beta)c_3, \quad (2.5) \]

and

\[ -2a_2 = (1 - \beta)d_1, \quad (2.6) \]
\[ 3 \left( 2a_2^2 - a_3 \right) = (1 - \beta)d_2, \quad (2.7) \]
\[ -4 \left( 5a_3^2 - 5a_3a_2 + a_4 \right) = (1 - \beta)d_3. \quad (2.8) \]

From (2.3) and (2.6), we find that

\[ c_1 = -d_1 \quad (2.9) \]

and

\[ a_2 = \frac{(1 - \beta)}{2} c_1. \quad (2.10) \]

Now, from (2.4), (2.7) and (2.10), we get

\[ a_3 = \frac{(1 - \beta)^2}{4} c_1^2 + \frac{(1 - \beta)}{6} (c_2 - d_2). \quad (2.11) \]

Also, from (2.5) and (2.8), we find that

\[ a_4 = \frac{5 (1 - \beta)^2}{24} c_1 (c_2 - d_2) + \frac{(1 - \beta)}{8} (c_3 - d_3). \quad (2.12) \]

Thus, we can easily establish that

\[
|a_2a_4 - a_3^2| = \left| \frac{(1 - \beta)^4}{16} c_1^4 + \frac{(1 - \beta)^3}{48} c_1^2 (c_2 - d_2) + \frac{(1 - \beta)^2}{16} c_1 (c_3 - d_3) - \frac{(1 - \beta)^2}{36} (c_2 - d_2)^2 \right|. \quad (2.13)
\]
According to Lemma 2 and (2.9), we write

\[
2c_2 = c_1^2 + x(4 - c_1^2) \quad 2d_2 = d_1^2 + y(4 - d_1^2) \quad \Rightarrow \quad c_2 - d_2 = \frac{4 - c_1^2}{2}(x - y)
\]

(2.14)

and

\[
4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)\left(1 - |x|^2\right)z,
\]

\[
4d_3 = d_1^3 + 2(4 - d_1^2)d_1y - d_1(4 - d_1^2)y^2 + 2(4 - d_1^2)\left(1 - |y|^2\right)w.
\]

Moreover, we have

\[
c_3 - d_3 = \frac{c_1^3}{2} + \frac{c_1(4 - c_1^2)}{2}(x + y) - \frac{c_1(4 - c_1^2)}{4}(x^2 + y^2)
\]

\[
+ \frac{4 - c_1^2}{2}\left(\left(1 - |x|^2\right)z - \left(1 - |y|^2\right)w\right),
\]

(2.15)

\[
c_2 + d_2 = c_1^2 + \frac{4 - c_1^2}{2}(x + y)
\]

(2.16)

for some \(x, y\) and \(z, w\) with \(|x| \leq 1, |y| \leq 1, |z| \leq 1\) and \(|w| \leq 1\). Using (2.14) and (2.15) in (2.13), and applying the triangle inequality, we have

\[
|a_2a_4 - a_3^2| = \left| -\frac{(1 - \beta)^4}{16}c_1^4 + \frac{(1 - \beta)^3}{96}c_1^2(4 - c_1^2)(x - y)
\]

\[
+ \frac{(1 - \beta)^2}{16}c_1 \left[ \frac{c_1^3}{2} + \frac{4 - c_1^2}{2}(x + y) - \frac{(4 - c_1^2)c_1}{4}(x^2 + y^2) + \frac{4 - c_1^2}{2}\left(1 - |x|^2\right)z - \left(1 - |y|^2\right)w\right]
\]

\[
- \frac{(1 - \beta)^2}{144}(4 - c_1^2)^2(x - y)^2 \right|
\]

\[
\leq \frac{(1 - \beta)^4}{16}c_1^4 + \frac{(1 - \beta)^2}{32}c_1^4 + \frac{(1 - \beta)^2}{16}c_1(4 - c_1^2)
\]

\[
+ \left[ \frac{(1 - \beta)^3}{96}c_1^2(4 - c_1^2) + \frac{(1 - \beta)^2}{32}c_1^2(4 - c_1^2) \right] (|x| + |y|)
\]

\[
+ \left[ \frac{(1 - \beta)^2}{64}c_1^2(4 - c_1^2) - \frac{(1 - \beta)^2}{32}c_1(4 - c_1^2) \right] (|x|^2 + |y|^2) + \frac{(1 - \beta)^2}{144}(4 - c_1^2)^2(|x| + |y|)^2.
\]

Since \(p \in \mathcal{P}\), we have \(|c_1| \leq 2\). Letting \(c_1 = c\), we may assume without loss of generality that \(c \in [0, 2]\). Thus, for \(\lambda = |x| \leq 1\) and \(\mu = |y| \leq 1\), we obtain

\[
|a_2a_4 - a_3^2| \leq T_1 + T_2(\lambda + \mu) + T_3(\lambda^2 + \mu^2) + T_4(\lambda + \mu)^2 = F(\lambda, \mu),
\]

where
where

\[
T_1 = T_1(c) = \frac{(1-\beta)^2}{32} c \left( 1 + 2 (1-\beta)^2 \right) c^3 + 2(4-c^2) \gtrless 0,
\]

\[
T_2 = T_2(c) = \frac{(1-\beta)^2}{96} c^2(4-c^2)(4-\beta) \gtrless 0,
\]

\[
T_3 = T_3(c) = \frac{(1-\beta)^2}{64} c(4-c^2)(c-2) \lesssim 0,
\]

\[
T_4 = T_4(c) = \frac{(1-\beta)^2}{144} (4-c^2)^2 \gtrless 0.
\]

Now we need to maximize \( F(\lambda, \mu) \) in the closed square \( S = \{(\lambda, \mu) : 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\} \) for \( c \in [0, 2] \). We must investigate the maximum of \( F(\lambda, \mu) \) according to \( c = (0, 2), c = 0 \) and \( c = 2 \), keeping in view the sign of \( F_{\lambda\lambda} F_{\mu\mu} - (F_{\lambda\mu})^2 \).

First, let \( c \in (0, 2) \). Since \( T_3 < 0 \) and \( T_3 + 2T_4 > 0 \) for \( c \in (0, 2) \), we conclude that

\[
F_{\lambda\lambda} F_{\mu\mu} - (F_{\lambda\mu})^2 < 0.
\]

Thus, the function \( F \) cannot have a local maximum in the interior of the square \( S \). Now we investigate the maximum of \( F \) on the boundary of the square \( S \).

For \( \lambda = 0 \) and \( 0 \leq \mu \leq 1 \), we obtain

\[
F(0, \mu) = G(\mu) = (T_3 + T_4) \mu^2 + 2\mu + T_1.
\]

We consider the following two cases separately.

**Case 1.** Let \( T_3 + T_4 \gtrless 0 \). In this case, for \( 0 < \mu < 1 \) and for any fixed \( c \) with \( 0 < c < 2 \), it is clear that

\[
G'(\mu) = 2 (T_3 + T_4) \mu + T_2 > 0 \quad (0 < \mu < 1),
\]

that is, that \( G(\mu) \) is an increasing function. Hence, for fixed \( c \in (0, 2) \), the maximum of \( G(\mu) \) occurs at \( \mu = 1 \), and

\[
\max G(\mu) = G(1) = T_1 + T_2 + T_3 + T_4.
\]

**Case 2.** Let \( T_3 + T_4 < 0 \). Since

\[
T_2 + 2 (T_3 + T_4) \gtrsim 0
\]

for any fixed \( c \) with \( 0 < c < 2 \), it is clear (in this case) that

\[
T_2 + 2 (T_3 + T_4) < 2 (T_3 + T_4) \mu + T_2 < T_2 \quad (0 < \mu < 1),
\]

which shows that \( G'(\mu) > 0 \). Hence, for fixed \( c \in (0, 2) \), the maximum of \( G(\mu) \) occurs at \( \mu = 1 \). Similarly, for \( \mu = 0 \) and \( 0 \leq \lambda \leq 1 \), we get

\[
\max F(\lambda, 0) = \max G(\lambda) = G(1) = T_1 + T_2 + T_3 + T_4.
\]

For \( \lambda = 1 \) and \( 0 \leq \mu \leq 1 \), we obtain

\[
F(1, \mu) = H(\mu) = (T_3 + T_4) \mu^2 + (T_2 + 2T_4) \mu + T_1 + T_2 + T_3 + T_4.
\]
Thus, from the above Case 1 and Case 2 for \( T_3 + T_4 \), we get
\[
\max\{H(\mu)\} = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4.
\]
Similarly, for \( \mu = 1 \) and \( 0 \leq \lambda \leq 1 \), we have
\[
\max\{F(\lambda, 1)\} = \max\{H(\lambda)\} = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4.
\]
Since \( G(1) \leq H(1) \) for \( c \in (0, 2) \), we have
\[
\max\{F(\lambda, \mu)\} = F(1, 1)
\]
on the boundary of the square \( S \). Thus, clearly, the maximum of the function \( F(\lambda, \mu) \) occurs when \( \lambda = 1 \) and \( \mu = 1 \) in the closed square \( S \) and for \( c \in (0, 2) \).

Let \( K : (0, 2) \to \mathbb{R} \) be given by
\[
K(c) = \max\{F(\lambda, \mu)\} = F(1, 1) = T_1 + 2T_2 + 2T_3 + 4T_4. \tag{2.17}
\]
Substituting the values of \( T_1, T_2, T_3, \) and \( T_4 \) into the function \( K(c) \) defined by (2.17) yields
\[
K(c) = \frac{(1 - \beta)^2}{144} \left[ (9\beta^2 - 15\beta + 1) c^4 + (34 - 12\beta)c^2 + 64 \right].
\]
We now investigate the maximum value of \( K(c) \) in the interval \( (0, 2) \). By elementary calculation, we find that
\[
K'(c) = \frac{(1 - \beta)^2}{36} \left[ (9\beta^2 - 15\beta + 1) c^3 + (17 - 6\beta)c \right]. \tag{2.18}
\]
As a result of some calculations, we can accomplish the following results.

**Result 1.** Let
\[
9\beta^2 - 15\beta + 1 \geq 0,
\]
that is,
\[
\beta \in \left[ 0, \frac{5 - \sqrt{21}}{6} \right].
\]
Then \( K'(c) > 0 \) for every \( c \in (0, 2) \). Furthermore, since \( K(c) \) is an increasing function in the interval \( (0, 2) \), it has no maximum value in this interval.

**Result 2.** Let
\[
9\beta^2 - 15\beta + 1 < 0,
\]
that is,
\[
\beta \in \left( \frac{5 - \sqrt{21}}{6}, 1 \right).
\]
Then $K'(c) = 0$ implies the real critical point given by
\[ c_0 = \sqrt[9]{\frac{6\beta - 17}{9\beta^2 - 15\beta + 1}}. \]

In the case when
\[ \beta \in \left(\frac{5 - \sqrt{21}}{6}, \frac{11 - \sqrt{37}}{12}\right), \]
then $c_0 \geq 2$, that is, $c_0$ lies outside of the interval $(0, 2)$. In the case when
\[ \beta \in \left(\frac{11 - \sqrt{37}}{12}, 1\right), \]
then $c_0 < 2$, that is, $c_0$ is in the interior of the interval $[0, 2]$. Furthermore, since $K''(c_0) < 0$, the maximum value of $K(c)$ occurs at $c = c_0$ for
\[ \beta \in \left(\frac{11 - \sqrt{37}}{12}, 1\right). \]

Thus, clearly, it is observed that
\[ \max_{0 < c < 2} \{K(c)\} = K(c_0) = K\left(\sqrt[9]{\frac{6\beta - 17}{9\beta^2 - 15\beta + 1}}\right) = \frac{(1 - \beta)^2}{2} \left(\frac{15\beta^2 - 21\beta - \frac{25}{4}}{18\beta^2 - 30\beta + 2}\right) \] (2.19)
for
\[ \beta \in \left(\frac{11 - \sqrt{37}}{12}, 1\right). \]

Secondly, let $c = 2$ and $(\lambda, \mu) \in \mathbb{S}$. We then obtain a constant function of the dependent variables $\lambda$ and $\mu$ as follows:
\[ F(\lambda, \mu) = \frac{(1 - \beta)^2}{2} (2\beta^2 - 4\beta + 3) \] (2.20)
for every $0 \leq \beta < 1$.

Finally, let $c = 0$ and $(\lambda, \mu) \in \mathbb{S}$. We then find that
\[ F(\lambda, \mu) = \frac{(1 - \beta)^2}{9} (\lambda + \mu)^2. \]

We can easily see that the maximum of $F(\lambda, \mu)$ occurs at $\lambda = \mu = 1$ and we have
\[ \max\{F(\lambda, \mu)\} = F(1, 1) = \frac{4(1 - \beta)^2}{9} \] (2.21)
for every $\beta \ (0 \leq \beta < 1)$. 

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From (2.19), (2.20), and (2.21), it is easily seen that
\[
\frac{4(1 - \beta)^2}{9} < \frac{(1 - \beta)^2}{2}(2\beta^2 - 4\beta + 3) < \frac{(1 - \beta)^2}{2}\left(\frac{15\beta^2 - 21\beta - \frac{25}{4}}{18\beta^2 - 30\beta + 2}\right)
\]
for
\[
\beta \in \left(\frac{11 - \sqrt{37}}{12}, 1\right).
\]
We thus obtain the second inequality of (2.1) for
\[
\beta \in \left(\frac{11 - \sqrt{37}}{12}, 1\right).
\]
On the other hand, since the following inequality:
\[
\frac{4(1 - \beta)^2}{9} < \frac{(1 - \beta)^2}{2}(2\beta^2 - 4\beta + 3)
\]
is satisfied for every \(\beta\ (0 \leq \beta < 1)\), we obtain the first inequality of (2.1) for
\[
\beta \in \left[0, \frac{11 - \sqrt{37}}{12}\right].
\]

This completes the proof of Theorem 1.

Our second main result for the class \(N_\sigma\) is given by Theorem 2 below.

**Theorem 2** Let the function \(f(z)\) given by (1.1) be in the class \(N_\sigma^\alpha\) \((0 < \alpha \leq 1)\). Then
\[
|a_2a_4 - a_3^2| \leq \begin{cases} 
\frac{4\alpha^2}{9} & \text{if } (0 < \alpha \leq \frac{7}{24}), \\
\frac{\alpha^2}{38}\left(\frac{64\alpha^2 - 144\alpha + 5}{12\alpha^2 - 12\alpha + 1}\right) & \text{if } \left(\frac{7}{24} \leq \alpha \leq \frac{1 + \sqrt{2}}{4}\right), \\
\frac{\alpha^2(8\alpha^2 + 1)}{6} & \text{if } \left(\frac{1 + \sqrt{2}}{4} \leq \alpha \leq 1\right).
\end{cases}
\]
(2.22)

**Proof** Let \(f \in N_\sigma^\alpha\), \(0 < \alpha \leq 1\), and \(g = f^{-1}\). Then
\[
f'(z) = [p(z)]^\alpha \quad \text{and} \quad g'(w) = [q(w)]^\alpha,
\]
(2.23)
where the functions \(p(z)\) and \(q(z)\) given by
\[
p(z) = 1 + c_1z + c_2z^2 + \cdots \quad \text{and} \quad q(w) = 1 + d_1w + d_2w^2 + \cdots
\]
are in class \(\mathcal{P}\).
Now, upon equating the coefficients in (2.23), we have
\[2a_2 = \alpha c_1, \quad (2.24)\]
\[3a_3 = \alpha c_2 + \frac{\alpha (\alpha - 1)}{2} c_1^2, \quad (2.25)\]
\[4a_4 = \alpha c_3 + \alpha (\alpha - 1) c_1 c_2 + \frac{\alpha (\alpha - 1) (\alpha - 2) c_1^3}{6}, \quad (2.26)\]
and
\[-2a_2 = \alpha d_1, \quad (2.27)\]
\[3 (2a_2^2 - a_3) = \alpha d_2 + \frac{\alpha (\alpha - 1)}{2} d_1^2, \quad (2.28)\]
\[-4 (5a_2^3 - 5a_2 a_3 + a_4) = \alpha d_3 + \alpha (\alpha - 1) d_1 d_2 + \frac{\alpha (\alpha - 1) (\alpha - 2) d_1^3}{6}. \quad (2.29)\]

From (2.24) and (2.27), we obtain
\[c_1 = -d_1 \quad (2.30)\]
and
\[a_2 = \frac{\alpha c_1}{2}. \quad (2.31)\]

Now, from (2.25), (2.28), and (2.31), we find that
\[a_3 = \frac{\alpha^2 c_1^2}{4} + \frac{\alpha (c_2 - d_2)}{6}. \quad (2.32)\]

Also, from (2.26) and (2.29), we get
\[a_4 = \frac{\alpha (\alpha - 1) (\alpha - 2) c_1^3}{24} + \frac{5\alpha^2 c_1 (c_2 - d_2)}{24} + \frac{\alpha (c_3 - d_3)}{8} + \frac{\alpha (\alpha - 1) c_1 (c_2 + d_2)}{8}. \quad (2.33)\]

We can thus easily establish that
\[|a_2 a_4 - a_3^2| = \left| \frac{\alpha^2 (\alpha - 1) (\alpha - 2) c_1^4}{48} - \frac{\alpha^4 c_1^4}{16} + \frac{\alpha^3 c_1^2 (c_2 - d_2)}{48} \right| + \frac{\alpha^2 c_1 (c_3 - d_3)}{16} - \frac{\alpha^2 (c_2 - d_2)^2}{36} + \frac{\alpha^2 (\alpha - 1) c_1^2 (c_2 + d_2)}{16}. \quad (2.34)\]

Using (2.14), (2.15), and (2.16) in (2.34), we have
\[|a_2 a_4 - a_3^2| \leq \frac{\alpha^2 (\alpha - 1) (\alpha - 2) c_1^4}{48} + \frac{\alpha^4 c_1^4}{16} + \frac{\alpha^2 (\alpha - 1) c_1^4}{16} + \frac{\alpha^2 c_1 (4 - c_1^2)}{16} \]
\[+ \frac{\alpha^2 c_1^2 (4 - c_1^2)}{24} (|x| + |y|) + \frac{\alpha^2 c_1 (4 - c_1^2) (c_1 - 1)}{64} (|x|^2 + |y|^2) + \frac{\alpha^2 (4 - c_1^2)^2}{144} (|x| + |y|)^2. \]

Since \( p(z) \in \mathcal{P} \), we obtain \(|c_1| \leq 2\). Taking \( c_1 = c \), we may assume without any loss of generality that \( c \in [0, 2] \). Thus, for
\[\lambda = |x| \leq 1 \quad \text{and} \quad \mu = |y| \leq 1, \quad 703\]
we obtain

\[ |a_2a_4 - a_3^2| \leq M_1 + M_2(\lambda + \mu) + M_3(\lambda^2 + \mu^2) + M_4(\lambda + \mu)^2 = \Psi(\lambda, \mu), \]

where

\[
M_1 = M_1(c) = \frac{\alpha^2}{96} [(8\alpha^2 + 1) c^4 - 6c^3 + 24c] \geq 0, \\
M_2 = M_2(c) = \frac{\alpha^3}{24}c^2(4 - c^2) \geq 0, \\
M_3 = M_3(c) = \frac{\alpha^2}{64}c(4 - c^2)(c - 2) \leq 0, \\
M_4 = M_4(c) = \frac{\alpha^2}{144}(4 - c^2)^2 \geq 0.
\]

Therefore, we need to maximize \( \Psi(\lambda, \mu) \) in the closed square \( S \) given by

\[ S = \{ (\lambda, \mu) : 0 \leq \lambda \leq 1 \text{ and } 0 \leq \mu \leq 1 \}. \]

In order to determine the maximum of \( \Psi(\lambda, \mu) \), we can analogously follow the derivation of the maximum of \( F(\lambda, \mu) \) in Theorem 1. Thus, clearly, the maximum of \( \Psi(\lambda, \mu) \) occurs at \( \lambda = 1 \) and \( \mu = 1 \) in the closed square \( S \). Let \( \Phi : (0,2) \rightarrow \mathbb{R} \) defined by

\[ \Phi(c) = \max \{ \Psi(\lambda, \mu) \} = \Psi(1,1) = M_1 + 2(M_2 + M_3) + 4M_4. \quad (2.35) \]

Substituting the values of \( M_1, M_2, M_3, \) and \( M_4 \) into the function \( \Phi(c) \) given by (2.35), we get

\[ \Phi(c) = \frac{\alpha^2}{144} [(12\alpha^2 - 12\alpha + 1) c^4 + (48\alpha - 14)c^2 + 64]. \]

Let

\[ P = 12\alpha^2 - 12\alpha + 1, \quad Q = 48\alpha - 14, \quad \text{and} \quad R = 64. \quad (2.36) \]

Then, since

\[
\max_{0 \leq t \leq 4} \{ (Pt^2 + Qt + R) \} = \begin{cases} 
R & (Q \leq 0; P \leq -\frac{Q}{4}), \\
16P + 4Q + R & \left( Q \geq 0 \text{ and } P \geq -\frac{Q}{8} \text{ or } Q \leq 0 \text{ and } P \geq -\frac{Q}{4} \right), \\
\frac{4PR - Q^2}{4P} & (Q > 0; P \leq -\frac{Q}{8}),
\end{cases} \quad (2.37)
\]

we have

\[
|a_2a_4 - a_3^2| \leq \frac{\alpha^2}{144} \begin{cases} 
R & (Q \leq 0; P \leq -\frac{Q}{4}), \\
16P + 4Q + R & \left( Q \geq 0 \text{ and } P \geq -\frac{Q}{8} \text{ or } Q \leq 0 \text{ and } P \geq -\frac{Q}{4} \right), \\
\frac{4PR - Q^2}{4P} & (Q > 0; P \leq -\frac{Q}{8}),
\end{cases}
\]
where $P, Q,$ and $R$ are given by (2.36).

This completes the proof of Theorem 2.

For $\beta = 0$ in Theorem 1 or for $\alpha = 1$ in Theorem 2, we obtain the coefficient estimate given by the corollary below.

**Corollary.** Let $f(z)$ given by (1.1) be in the class $N_{\alpha}$. Then

$$|a_2a_4 - a_3^2| \leq \frac{3}{2}.$$

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