Localization for the Anderson Model on Trees with Finite Dimensions

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Abstract

We introduce a family of trees that interpolate between the Bethe lattice and $\mathbb{Z}$. We prove complete localization for the Anderson model on any member of that family.

1 Introduction

The purpose of this paper is to study the spectral properties of the Anderson model on a family of graphs which interpolate in a certain sense between the Bethe lattice and $\mathbb{Z}$. The Bethe lattice can be regarded as an infinite dimensional graph because of the exponential growth (in $r$) of the volume of the ball of radius $r$ around the root (which is connected to the fact that a significant part of this volume is concentrated on the boundary of that ball). $\mathbb{Z}$ is, of-course, a one-dimensional graph. For both these domains, the Anderson model has been extensively studied. For the one-dimensional case it is known that the spectrum is pure point (with exponentially decaying eigenfunctions) for all energies and any degree of disorder (see, e.g., [6] and references therein). On the other hand, absolutely continuous spectrum is
known to occur, on the Bethe lattice, in the weak disorder regime [2, 7, 8].

We present here a family of trees whose members are all finite dimensional
in a natural sense and which have the Bethe lattice and \( \mathbb{Z} \) as extreme cases.

In this setting, we shall prove localization for the Anderson model for all
energies and any degree of disorder whenever the dimension is finite.

In order to describe the objects at the focus of our attention, we need
some terminology. By a rooted tree, \( \Gamma \), we mean a tree graph that has a
special vertex designated by the letter \( O \). We use \( \mathcal{V}(\Gamma) \) to denote the set
of vertices of \( \Gamma \). For any two vertices, \( x, y \in \mathcal{V}(\Gamma) \) it is possible to define
the distance between \( x \) and \( y \), \( d(x, y) = d(y, x) \), as the number of edges of
the unique path of minimal length connecting them. These notions allow us
to define a natural direction on the tree: For any vertex on a rooted tree,
the backward direction is the direction pointing towards the root. Any other
direction we call forward. More precisely, for \( x \in \mathcal{V}(\Gamma) \), we say that \( y \) is a
forward neighbor of \( x \) if \( d(x, y) = 1 \) and \( d(y, O) > d(x, O) \). In this case, we
shall say that \( x \) is a backward neighbor of \( y \).

The trees we construct are parametrized by a natural number \( k \geq 2 \) and a
real number \( \gamma \geq 1 \). Roughly speaking, they are obtained by taking the Bethe
lattice of coordination number \( k \), and extending its edges at an exponential
rate (determined by \( \gamma \)). This is done by replacing the edges at a distance \( n \)
from the root by a segment of \( \mathbb{Z} \) of length \( [\gamma^n] \) (where \( [\cdot] \) for a real number
denotes its integer part).

More precisely, let \( k \geq 2 \) be a natural number and \( \gamma \geq 1 \) be a real number.
We define the rooted tree \( \Gamma_{k,\gamma} \) as follows: Let \( \mathcal{S}_{k,\gamma} \subseteq \mathcal{V}(\Gamma_{k,\gamma}) \) be the set of
vertices of \( \Gamma_{k,\gamma} \) whose elements are the root \( O \), and all vertices at a distance
\( \sum_{j=1}^{N} [\gamma^j] \) from \( O \) (for any \( N \in \mathbb{N} \)). Now, \( \Gamma_{k,\gamma} \) is defined by the fact that
vertices belonging to \( \mathcal{S}_{k,\gamma} \) have \( k \) forward neighbors. All other vertices have
one forward nearest neighbor (see Figure 1). We call the elements of \( \mathcal{S}_{k,\gamma} \)
junctions. It is easy to see that by taking \( \gamma = 1 \) we get the Bethe lattice of
coordination number \( k \). On the other hand, \( \mathbb{Z} \) can be viewed as corresponding
to the case \( k = 2, \gamma = \infty \). In this sense, the family \( \{\Gamma_{k,\gamma}\}_{k \geq 2, \gamma \geq 1} \) interpolates
between the Bethe lattice and \( \mathbb{Z} \).

A straightforward computation shows:

**Proposition 1.1.** Fix \( \gamma > 1 \) and \( \mathbb{N} \ni k \geq 2 \). Let \( \Gamma = \Gamma_{k,\gamma} \) and let \( B_{\Gamma}(r) = \{x \in \mathcal{V}(\Gamma) \mid d(x, O) \leq r\} \). Then

\[
\lim_{r \to \infty} \frac{\log \#B_{\Gamma}(r)}{\log r} = \frac{\log \gamma k}{\log \gamma} = 1 + \frac{\log k}{\log \gamma} \quad (1.1)
\]
Below, we shall refer to the quantity $\log\frac{k}{\log \gamma}$ as the *dimension* of $\Gamma_{k,\gamma}$.

Since we are dealing with non-regular trees (namely, the number of nearest neighbors is not constant), there are two choices for the Laplacian:

$$(\tilde{\Delta} f)(x) = \sum_{y : d(x,y) = 1} f(y), \quad (1.2)$$

and

$$(\Delta f)(x) = \sum_{y : d(x,y) = 1} f(y) - \#\{y : d(x,y) = 1\} \cdot f(x) \quad (1.3)$$

where $\#A$, for a finite set $A$, is the number of elements in $A$. Both operators are bounded and self-adjoint on $\Gamma_{k,\gamma}$ for any $k$ and $\gamma$. Theorem 1.2 below holds, as stated, both for $\Delta$ and $\tilde{\Delta}$. Moreover, we shall give a proof that goes through for both cases. To avoid encumbrance, we shall use the notation $\Delta$ with the understanding that all statements hold for (1.2) as well as for (1.3).

Let $\Gamma$ be a tree and let $\{V_{\omega}(x)\}_{x \in \mathcal{V}(\Gamma)}$ be a family of i.i.d. random variables with common probability distribution $d\rho$. For any $\omega$, let $V_{\omega}$ stand for the corresponding multiplication operator defined over $\ell^2(\Gamma) = \ell^2(\mathcal{V}(\Gamma))$ by

$$(V_{\omega} f)(x) = V_{\omega}(x) f(x).$$

For $\lambda > 0$ we refer to the family of operators

$$H_{\omega,\lambda} = \Delta + \lambda V_{\omega}$$

as the Anderson model with coupling constant $\lambda$. For $\Gamma = \mathbb{Z}$, this model is known to exhibit almost sure pure point spectrum with exponentially decaying eigenfunctions, for any range of energies and any value of the coupling constant, for any probability distribution $d\rho$, either having an absolutely continuous component, or having some finite moment [6]. For the case of

![Figure 1: A neighborhood of the root for $\Gamma_{2,2}$.](image)
the Bethe lattice, on the other hand, it is known [2, 7, 8] that the Anderson model exhibits absolutely continuous spectrum for small values of $\lambda$ (for $d\rho$ satisfying certain regularity conditions).

We shall assume throughout that

(i) $d\rho$ has a bounded density, namely

$$d\rho(\xi) = \tilde{\rho}(\xi)d\xi$$

(1.4)

with

$$||\tilde{\rho}||_\infty < \infty.$$ (1.5)

(ii) \[ \int |\xi|^\eta \tilde{\rho}(\xi)d\xi < \infty \text{ for some } \eta > 0. \] (1.6)

Our main result is

**Theorem 1.2.** Let

$$H_{\omega,\lambda} = \Delta + \lambda V_\omega$$

be the Anderson model on $\Gamma = \Gamma_{k,\gamma}$ for some $k \geq 2$ and $\gamma > 1$. Assume that $d\rho$ satisfies requirements (i)-(ii) above. Then, for any $\lambda > 0$ and almost every realization of $V_\omega$, $H_{\omega,\lambda}$ has only pure point spectrum and the corresponding eigenfunctions decay exponentially.

**Remarks.** 1. For a function $f$ defined on $\mathscr{V}(\Gamma)$, we say that $f$ decays exponentially if there exist positive constants $A, C$ such that,

$$f(x) \leq Ae^{-C|x|}$$

where $|x| = d(x, O)$.

2. We note that the technical requirement (1.6) is also present in the proof of localization for the Anderson-Bernoulli model in one-dimension [5].

We, however, assume in addition the absolute continuity of $d\rho$ (1.4), so the question of localization for the Anderson-Bernoulli model on $\Gamma_{k,\gamma}$ is still open.

The proof of Theorem 1.2 relies on the fact that as long as $\gamma > 1$, $\Gamma_{k,\gamma}$ contains arbitrarily long one-dimensional segments. We call trees with this property **sparse**. Applying ideas of the finite-volume method developed by Aizenman, Schenker, Friedrich and Hundertmark in [1], we use a priori bounds that are known for the one-dimensional case in order to get exponential decay of
fractional moments of the Green function. Since \( \Gamma_{k,\gamma} \) has finite dimensions in the sense of (1.1), this implies localization.

We note that the behavior of the Anderson model on these sparse, finite dimensional trees is drastically different from the expected behavior on \( \mathbb{Z}^d \), where some absolutely continuous spectrum is believed to exist in the weak coupling regime. Such a difference is also manifest in the spectral properties of the Laplacian. The papers [3, 4] are devoted to the spectral analysis of \( \Delta \) on sparse trees. Examples are constructed, in these papers, of sparse trees where \( \Delta \) has singular spectral measures. In particular, it is shown in [4] that generically, in a certain probabilistic sense, the finite dimensional trees discussed here have singular spectrum and some even exhibit a component of dense point type.

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2 Proof of Theorem 1.2

Fix \( \mathbb{N} \ni k \geq 2, \gamma > 1 \) and \( \lambda > 0 \). To streamline the notation, let \( \Gamma = \Gamma_{k,\gamma} \) and \( H_\omega = H_{\omega,\lambda} \). We also use the shorthand \( |x| \equiv d(x, O) \). As mentioned earlier, for any \( x, y \in \mathcal{V}(\Gamma) \), there is a unique path of minimal length connecting them to each other. This is a finite subgraph of \( \Gamma \) which can also be embedded in \( \mathbb{Z} \). We denote this graph by \( \mathcal{L}(x, y) \).

For \( x \in \mathcal{V}(\Gamma) \), the spectral measure \( \mu_x \) is defined by the equation

\[
\int_{\mathbb{R}} \frac{d\mu_x}{x-z} = (\delta_x | (H_{\lambda,\omega} - z)^{-1}\delta_x ) \quad z \in \mathbb{C} \setminus \mathbb{R}
\]

where \( \delta_x \) is the delta function at \( x \) and \( (f | g) \) stands for the inner product in \( l^2(\Gamma) \). We shall prove Theorem 1.2 by showing that, with probability one, \( \mu_x \) is pure point for any \( x \in \mathcal{V}(\Gamma) \). Since \( \{\delta_x\}_{x \in \mathcal{V}(\Gamma)} \) is an orthogonal basis for \( l^2(\Gamma) \), the theorem is immediately implied.

We want to apply ideas of Aizenman et al. [1] to our setting. In particular, we will study \( H_\omega \) restricted to finite regions of \( \Gamma \). For any such region, \( \Omega \), we
denote by $\Theta(\Omega)$ the set of nearest-neighbor bonds reaching out of $\Omega$, that is,

$$\Theta(\Omega) = \{(x, x') \in \mathcal{V}(\Gamma) \times \mathcal{V}(\Gamma) \mid x \in \mathcal{V}(\Omega), \ x' \in \mathcal{V}(\Gamma) \setminus \mathcal{V}(\Omega), \ d(x, x') = 1\}.$$

(2.1)

We further denote by $\Omega^+$ the region containing the vertices within distance 1 from $\Omega$, and by $\mathcal{B}(\Omega)$ the boundary of $\Omega$, that is,

$$\mathcal{B}(\Omega) = \{x \in \mathcal{V}(\Omega) \mid \exists x' \in \mathcal{V}(\Gamma) \setminus \mathcal{V}(\Omega) \text{ s.t. } d(x, x') = 1\}.$$  

(2.2)

We let $\Delta_{\Omega}$ be the operator obtained by deleting the hopping terms corresponding to $\Theta(\Omega)$ (following [1] we shall call the off-diagonal matrix elements of $\Delta$ hopping terms), so that the restriction of $\Delta_{\Omega}$ to $\ell^2(\Omega)$ is just the finite volume Laplacian with Dirichlet boundary conditions on the boundary of $\Omega$. With this we may define the restriction of $H_{\omega}$ as:

$$H_{\Omega, \omega} = \Delta_{\Omega} + V_{\omega}.$$ 

For $H_{\Omega, \omega}$ as well, the restriction to $\ell^2(\Omega)$ equals the finite volume operator with Dirichlet boundary conditions on the boundary of $\Omega$.

Another kind of restriction we will consider is $H_{\Omega, \omega} (\Delta)$ which is the operator one gets from $H_{\omega}$ ($\Delta$) by deleting all hopping terms outside of $\Omega$. The restriction of this operator to $\ell^2(\Omega)$ is again just the finite volume operator with Dirichlet boundary conditions on $\Omega$.

We want to obtain decay of fractional moments of the Green function for the operators mentioned above. This is the function:

$$G_{\omega}(x, y; z) \equiv (\delta_x \left| G_{\omega}(z) \delta_y \right) \equiv (\delta_x \left| (H_{\omega} - z)^{-1} \delta_y \right) \equiv (\delta_x \left| (H_{\omega} - z)^{-1} \delta_y \right)$$  

(2.3)

defined for any $z$ in the resolvent set of $H_{\omega}$ and in particular for any $z \in \mathbb{C} \setminus \mathbb{R}$.

We also use

$$G_{\Omega, \omega}(x, y; z) \equiv (\delta_x \left| G_{\Omega, \omega}(z) \delta_y \right) \equiv (\delta_x \left| (H_{\Omega, \omega} - z)^{-1} \delta_y \right),$$  

(2.4)

and

$$G_{\omega}^{\Omega}(x, y; z) \equiv (\delta_x \left| G_{\omega}^{\Omega}(z) \delta_y \right) \equiv (\delta_x \left| (H_{\omega}^{\Omega} - z)^{-1} \delta_y \right).$$  

(2.5)

We note that $G_{\Omega, \omega}$ is a direct sum of operators, one corresponding to $\Omega$ and the other corresponding to $\Gamma \setminus \Omega$ so that if $x \in \mathcal{V}(\Omega)$ and $y \in \mathcal{V}(\Gamma) \setminus \mathcal{V}(\Omega)$, then

$$(\delta_x \left| G_{\Omega, \omega}(z) \delta_y \right) = 0.$$
The same remark goes for $G^{\omega}_{\omega}$. As mentioned in the introduction, the idea at the basis of our analysis is to somehow reduce the problem, locally, to a one-dimensional problem and to use bounds that we have on the one-dimensional Green function in order to get exponential decay of the Green function for $\Gamma$. This is possible because of the fact that there is essentially only one path between any two vertices of the tree, and because for any $\gamma > 1$ one can find one-dimensional stretches of arbitrary length in $\Gamma$.

Let $x$ and $y$ be two distinct vertices of $\Gamma$. Then, by the resolvent formula,

\begin{align}
G_{\omega}(z) &= G_{\mathcal{L}(x,y);\omega}(z) - G_{\mathcal{L}(x,y);\omega}(z) \left( H_{\omega} - H_{\mathcal{L}(x,y);\omega} \right) G_{\omega}(z) \\
&= G_{\mathcal{L}(x,y);\omega}(z) - G_{\mathcal{L}(x,y);\omega}(z) \left( \Delta - \Delta_{\mathcal{L}(x,y)} \right) G_{\omega}(z) 
\end{align}

which holds in this form since $H_{\omega}$ and $H_{\mathcal{L};\omega}$ have the same diagonal part. Writing

\begin{align}
G_{\omega}(z) &= G_{\mathcal{L}(x,y);\omega}(z) - G_{\omega}(z) \left( \Delta - \Delta_{\mathcal{L}(x,y);\omega} \right) G_{\mathcal{L}(x,y);\omega}(z) 
\end{align}

(recall that $\mathcal{L}(x,y;\omega)$ is the region in $\Gamma$ comprised of vertices of distance at most 2 from $\mathcal{L}(x,y)$) and plugging this into (2.6), we get:

\begin{align}
G_{\omega}(z) &= G_{\mathcal{L}(x,y);\omega}(z) - G_{\omega}(z) \left( \Delta - \Delta_{\mathcal{L}(x,y);\omega} \right) G_{\mathcal{L}(x,y);\omega}(z) \\
&+ G_{\mathcal{L}(x,y);\omega}(z) T_{\mathcal{L}(x,y)} G_{\omega}(z) T_{\mathcal{L}(x,y);\omega} \left( \Delta_{\mathcal{L}(x,y);\omega} \right) G_{\mathcal{L}(x,y);\omega}(z),
\end{align}

where we write

\begin{align}
T_{\Omega} = \Delta - \Delta_{\Omega}
\end{align}

for any region $\Omega$ in $\Gamma$.

Now assume $x, y, w \in \mathcal{V}(\Gamma)$ are such that $y$ is on $\mathcal{L}(x, w)$ and $w$ is outside of $\mathcal{L}(x, y;\omega)$. Then,

\begin{align}
(\delta_x | G_{\mathcal{L}(x,y);\omega}(z) \delta_w ) = 0
\end{align}

and

\begin{align}
(\delta_x | G_{\mathcal{L}(x,y);\omega}(z) T_{\mathcal{L}(x,y)} G_{\mathcal{L}(x,y);\omega}(z) \delta_w ) = 0,
\end{align}

so

\begin{align}
(\delta_x | G_{\omega}(z) \delta_w ) &= (\delta_x | G_{\mathcal{L}(x,y);\omega}(z) T_{\mathcal{L}(x,y)} G_{\omega}(z) T_{\mathcal{L}(x,y);\omega} \delta_w ) \\
&= \sum_{(u,u') \in \Theta(\mathcal{L}(x,y))} \sum_{(v,v') \in \Theta(\mathcal{L}(x,y;\omega))} (\delta_x | G_{\mathcal{L}(x,y);\omega}(z) \delta_u ) \\
&\times (\delta_{u'} | G_{\omega}(z) \delta_v ) (\delta_{v'} | G_{\mathcal{L}(x,y);\omega}(z) \delta_w ).
\end{align}
It follows that
\[
\langle |(\delta_x | G_\omega(z)\delta_w)|^s \rangle \\
\leq \sum_{(u,u') \in \Theta(\mathcal{L}(x,y))} \sum_{(v,v') \in \Theta(\mathcal{L}(x,y)))} \langle |(\delta_x | G_{\mathcal{L}(x,y);\omega}(z)\delta_u)|^s \rangle \\
\times |(\delta_{u'} | G_\omega(z)\delta_v)|^s |(\delta_{v'} | G_{\mathcal{L}(x,y);\omega}(z)\delta_w)|^s
\tag{2.10}
\]
for any \(s \in (0, 1)\) (we use \(\langle \cdot \rangle\) to denote the mean over the disorder). As in [1], variants of equations (2.9) and (2.10) are the starting point of our derivation. We shall want to focus on the Green function restricted to some finite (large) balls \(B(r)\) and not on the complete function. This is no severe limitation as long as our estimates are uniform in \(r\). We shall use the abbreviations
\[
G^r_\omega \equiv G_\omega^{B(r)},
\]
and
\[
T^r_\Omega = \Delta^{B(r)} - \Delta^{\Omega^{B(r)}}.
\]
Since the derivation of the first part of (2.9) uses only the resolvent formula, it is valid for \(G^r\), so we have
\[
(\delta_x | G^r_\omega(z)\delta_w) = (\delta_x | G^r_{\mathcal{L}(x,y);\omega}(z)T^r_{\mathcal{L}(x,y)}G^r_\omega(z)T^r_{\mathcal{L}(x,y);\omega}(z)\delta_w) \\
= \sum_{(u,u') \in \Theta(\mathcal{L}(x,y))} \sum_{(v,v') \in \Theta(\mathcal{L}(x,y)))} (\delta_x | G^r_{\mathcal{L}(x,y);\omega}(z)\delta_u) (\delta_u | T^r_{\mathcal{L}(x,y)}\delta_{u'}) \\
(\delta_{u'} | G^r_\omega(z)\delta_v) (\delta_v | T^r_{\mathcal{L}(x,y);\omega}(z)\delta_{v'}) (\delta_{v'} | G^r_{\mathcal{L}(x,y);\omega}(z)\delta_w).
\tag{2.11}
\]
Here, \(\langle |(\delta_u | T^r_{\mathcal{L}(x,y)}\delta_{u'})|^s \rangle \leq 1\) since it may vanish if \(u' \in B(r)\), and the same goes for \(\langle |(\delta_v | T^r_{\mathcal{L}(x,y);\omega}(z)\delta_{v'})|^s \rangle\). We see that
\[
\langle |(\delta_x | G^r_\omega(z)\delta_w)|^s \rangle \\
\leq \sum_{(u,u') \in \Theta(\mathcal{L}(x,y))} \sum_{(v,v') \in \Theta(\mathcal{L}(x,y)))} \langle |(\delta_x | G^r_{\mathcal{L}(x,y);\omega}(z)\delta_u)|^s \rangle \\
\times |(\delta_{u'} | G^r_\omega(z)\delta_v)|^s |(\delta_{v'} | G^r_{\mathcal{L}(x,y);\omega}(z)\delta_{v'})|^s
\tag{2.12}
\]
\[
\leq \sum_{(u,u') \in \Theta(\mathcal{L}(x,y))} \sum_{(v,v') \in \Theta(\mathcal{L}(x,y)))} \langle |(\delta_x | G^r_{\mathcal{L}(x,y);\omega}(z)\delta_u)|^s \rangle \\
\times |(\delta_{u'} | G^r_\omega(z)\delta_v)|^s |(\delta_{v'} | G^r_{\mathcal{L}(x,y);\omega}(z)\delta_{v'})|^s,
\]
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where $\Theta^r(\Omega) = \Theta(\Omega) \setminus \Theta(B(r))$.

In order to get a useful bound from (2.12) and its variants, we need an a priori bound on the finite-volume restriction of the one-dimensional Green function. This is supplied for us by the following proposition due to Minami [9] (also see [5, 11]):

**Proposition 2.1 (Minami, Proposition A.1 of [9]).** Let $H_\omega = \Delta_Z + \lambda V_\omega$ be a random Schrödinger operator on $\mathbb{Z}$ (with $\Delta_Z$ the Laplacian on $\mathbb{Z}$, $\lambda > 0$) and let $H_{\mathcal{L};\omega}$ be the restriction of $H_\omega$ to $\mathcal{L} \subset \mathbb{Z}$ defined as above. Let $G_{\mathcal{L};\omega}(x, y; z) = (\delta_x | (H_{\mathcal{L};\omega} - z)^{-1} \delta_y )$. Assume that the random potential consists of i.i.d. random variables with a common distribution $d\rho$ that satisfies (1.4)-(1.6). Then for any $E \in \mathbb{R}$, there are $s_0 \in (0, 1)$, $C > 0$, $m > 0$ and $\varepsilon > 0$ such that

$$\langle |G_{\mathcal{L};\omega}(x, y; z)|^{s_0} \rangle \leq Ce^{-m|x-y|}$$

(2.13)

for any finite segment $\mathcal{L} \subset \mathbb{Z}$, $x \in \mathcal{L}$, $y \in \mathcal{B}(\mathcal{L})$ and $z \in \{z \mid \text{Im } z > 0, |z - E| < \varepsilon\}$.

Another bound we will need is a bound on the conditional expectation of the Green function restricted to any finite volume. For this we will use Lemma B.1 of [1]. Note that the condition $R_1(\tau)$ of that paper is satisfied (with $\tau = 1$) by any probability distribution $d\rho$ satisfying (1.4)-(1.6). Thus, the hypotheses of Lemma B.1 of [1] are satisfied in our situation, and we get

**Proposition 2.2 (Aizenman et al., Lemma B.1 in [1]).** There exists $\kappa < \infty$ such that for any finite subset $\Omega$ of $\mathcal{V}(\Gamma)$, any $x, y \in \Omega$, any $z \in \mathbb{C}$ and any $s \in (0, 1)$,

$$\left\langle |(\delta_x | G_{\Omega;\omega}(z)\delta_y )|^{s} \right\rangle \{V(u)\}_{u \in \Omega \setminus \{x, y\}} \leq \frac{1}{1 - s} \frac{(4\kappa)^s}{\lambda^s},$$

(2.14)

where $\left\langle |(\delta_x | G_{\Omega;\omega}(z)\delta_y )|^{s} \right\rangle \{V(u)\}_{u \in \Omega \setminus \{x, y\}}$ is the conditional expectation of $|(\delta_x | G_{\Omega;\omega}(z)\delta_y )|^{s}$ conditioned on the values of the potential at all sites other than $x$ and $y$.

Theorem 1.2 is implied, via standard arguments, by the following

**Theorem 2.3.** Fix $x_0 \in \mathcal{V}(\Gamma)$. Then for any $E \in \mathbb{R}$, there are $s_0$, $A > 0$ and $\varepsilon > 0$ such that

$$\langle |G^r(x_0, v; z)|^{s_0} \rangle \leq Ae^{-qd(x_0, v)}$$

(2.15)
for any \( r > |x_0| \), \( v \in V(\Gamma) \) and
\[
z \in \{ z \mid \mathcal{I} z > 0, |z - E| < \varepsilon \}. \tag{2.16}
\]

**Proof of Theorem 1.2.** Given Theorem 2.3 and the fact that \( \Gamma \) has finite dimensions in the sense of (1.1), this is a straightforward application of the Simon-Wolff criterion [10, Theorem 2] (recall \( d\rho \) is absolutely continuous with respect to Lebesgue measure).

**Proof of Theorem 2.3.** Fix \( r > |x_0| \) and \( E \in \mathbb{R} \). Let \( \kappa \) be the constant from Proposition 2.2 and let
\[
C(s) = \max \left( \frac{1}{1 - s} \frac{(4\kappa)^s}{\lambda^s}, 1 \right). \tag{2.17}
\]

It follows by Proposition 2.1 that there exist \( s_0 \in (0, 1), \varepsilon > 0 \), and \( L > 0 \), such that if \( x, y \in V(\Gamma) \) are such that \( \mathcal{L}(x, y) \) has no junctions in it and \( d(x, y) \geq L \), we will have
\[
\langle |G_{\mathcal{L}(x,y)}^r(x, y; z)|^{s_0} \rangle < \frac{1}{4C(s_0)^2} \tag{2.18}
\]
for all
\[
z \in \{ z \mid \mathcal{I} z > 0, |z - E| < \varepsilon \}.
\]
This is true because \( G_{\mathcal{L}(x,y)}^r(x, y; z) \) is just the same as the one-dimensional Green function restricted to a finite segment, the boundary points of which are simply \( x \) and \( y \). The same goes for \( G^r \) if \( x \) and \( y \) are both either in or out of \( B(r) \), (otherwise \( |G_{\mathcal{L}(x,y)}^r(x, y; z)|^{s_0} = 0 \) so (2.18) is still true).

Fix
\[
z \in \{ z \mid \mathcal{I} z > 0, |z - E| < \varepsilon \}.
\]
We want to iterate (2.12) with (2.18) in order to get the exponential decay we are trying to prove.

Let \( L_0 = L + 5 \) and choose \( n_0 \) large enough so that \( [\gamma^{n_0}] > 8L_0 \). Also, choose \( R_0 > |x_0| \) such that \( R_0 = \sum_{j=1}^{n_1} [\gamma^j] \) for some \( n_1 \geq n_0 \) and \( R_1 = \sum_{j=-1}^{n_1} [\gamma^j] \) and assume that \( |v| \geq R_1 \) (for the finite number of vertices \( v \in B(R_1) \) we will bound the Green function by a constant). We may also assume that \( r \geq |v| \), since, otherwise, \( G^r(x_0, v; z) = 0 \). Let \( u_{R_0} \) be the unique vertex
on \( L(x_0, v) \) with \( |u_{R_0}| = R_0 \). Let \( L_0 = L(x_0, u_{R_0}) \). Then we have (see (2.12))

\[
\langle |(\delta_{x_0} | G^r_{\omega}(z) \delta_v)|^{s_0} \rangle \leq \sum_{(u, u') \in \Theta(L_0)} \sum_{(y, y') \in \Theta(L_0^{++})} \left\langle \left| (\delta_{x_0} | G^r_{L_0,\omega}(z) \delta_u) \right|^{s_0} \rightangle \times \left| (\delta_{y'} | G^r_{L_0^{++},\omega}(z) \delta_v) \right|^{s_0} \quad (2.19)
\]

Note that \( G^r_{L_0^{++},\omega}(z) \) is a direct sum of operators, one corresponding to a finite tree containing \( x_0 \) and the others corresponding to various (infinite) forward trees emanating from the boundary points of \( L_0^{++} \). Only one such tree contains \( v \) so there is only one element \((y_0, y'_0) \in \Theta(L_0^{++})\) for which

\[
\left| \left( \delta_{y_0} \left| G^r_{L_0^{++},\omega}(z) \delta_v \right) \right|^{s_0} \right| \neq 0.
\]

Therefore

\[
\langle |(\delta_{x_0} | G^r_{\omega}(z) \delta_v)|^{s_0} \rangle \leq \sum_{(u, u') \in \Theta(L_0)} \left\langle \left| (\delta_{x_0} | G^r_{L_0,\omega}(z) \delta_u) \right|^{s_0} \rightangle \times \left| (\delta_{y'} | G^r_{L_0^{++},\omega}(z) \delta_v) \right|^{s_0} \quad (2.20)
\]

There are three terms on the RHS of the inequality above. The terms \( \left| (\delta_{x_0} | G^r_{L_0,\omega}(z) \delta_u) \right|^{s_0} \) and \( \left| (\delta_{y_0} | G^r_{L_0^{++},\omega}(z) \delta_v) \right|^{s_0} \) are independent random variables since the first depends only on the potential in \( L_0 \) and the second only on the potential outside of \( L_0^{++} \). Moreover, neither of them depend on the potential at any of the \( u' \) and at \( y_0 \). Therefore, we may evaluate the expectation by first evaluating the conditional expectation with respect to the potential at all other points. For this we may use Proposition 2.2 to get

\[
\langle |(\delta_{x_0} | G^r_{\omega}(z) \delta_v)|^{s_0} \rangle \leq C(s_0) \sum_{u \in \Theta(L_0)} \left\langle \left| (\delta_{x_0} | G^r_{L_0,\omega}(z) \delta_u) \right|^{s_0} \rightangle \left\langle \left| \left( \delta_{y'_0} \left| G^r_{L_0^{++},\omega}(z) \delta_v \right) \right|^{s_0} \right| \right\rangle \quad (2.20)
\]

We proceed to estimate \( \left\langle \left| (\delta_{y'_0} \left| G^r_{L_0^{++},\omega}(z) \delta_v \right) \right|^{s_0} \right| \). We start by dividing the line \( L(y'_0, v) \) as follows:
• Set $x_1 = y_0$. Note that from our information about $v$ and the choice of $x_1$ (which reduces to the choice of $u_{R_0}$) $d(v, x_1) > 7L_0$.

• For any vertex $x \in \mathcal{L}(x_1, v)$ let us denote by $\mathcal{J}(x)$ the distance from $x$ to the nearest junction on $\mathcal{L}(x, v)$. Note that, by the choice of $R_0$ and since $L_0 \geq 5$, $\mathcal{J}(x_1) > L_0$. Now, if $\mathcal{J}(x_1) \geq 3L_0$, let $v_1$ be the unique vertex at a distance $L_0$ from $x_1$ in $\mathcal{L}(x_1, v)$. Otherwise, let $v_1$ be the unique vertex at a distance $5L_0$ from $x_1$ in $\mathcal{L}(x_1, v)$.

• Proceed by induction according to the following rule: Having defined $v_j$ for $j \geq 1$, let $x_{j+1}$ be the unique vertex at a distance 3 from $v_j$ in $\mathcal{L}(v_j, v)$. As long as $d(x_{j+1}, v) > 7L_0$, repeat the procedure above for choosing $v_{j+1}$, namely: If $\mathcal{J}(x_{j+1}) \geq 3L_0$, let $v_{j+1}$ be the unique vertex at a distance $L_0$ from $x_{j+1}$ in $\mathcal{L}(x_{j+1}, v)$. Otherwise, let $v_{j+1}$ be the unique vertex at a distance $5L_0$ from $x_{j+1}$ in $\mathcal{L}(x_{j+1}, v)$. If $d(x_{j+1}, v) \leq 7L_0$, let $v_{j+1} = v$.

• We terminate the construction when $v_j = v$, of course. It is easy to see that this happens after a finite number of steps, since $d(x_j, x_{j+1}) \leq 5L_0 + 3 < 7L_0$.

Thus, we get a set of vertex-pairs $\{(x_j, v_j)\}_{j=1}^l$ that satisfy:

1. For any $j$, $\mathcal{J}(x_j) \geq L_0$.

2. For any $j$, the distance between $v_j$ and the only junction (if there is one) on $\mathcal{L}(x_j, v_j)$ is at least $L_0$.

3. For any $j$, $d(x_j, v_j) \geq L_0$.

4. $l \geq \frac{d(x_1, v)}{5L_0 + 3} \geq \frac{d(u_{R_0}, v)}{6L_0}$.

Let $\mathcal{L}_j = \mathcal{L}(x_j, y_j)$. We want to repeat the analysis leading to equations (2.11) and (2.12), for $\left\langle \left( \delta_{y_0} | G_{\mathcal{L}_0^{++}, \omega}(z) \delta_v \right) \right| v_0 \rangle$. We note that $G_{\mathcal{L}_0^{++}, \omega}(z)$ is the Green function of an operator for which the hopping terms have been removed both outside of $B(r)$ and for the boundary of $\mathcal{L}_0^{++}$. Thus, setting

$$\Omega_1 = B(r) \cap (\Gamma \setminus \mathcal{L}_0^{++})$$

and recalling that both $y_0(= x_1)$ and $v$ are in $(\Gamma \setminus \mathcal{L}_0^{++})$, we get that

$$\left( \delta_{y_0} | G_{\mathcal{L}_0^{++}, \omega}(z) \delta_v \right) = \left( \delta_{x_1} | G_{\omega}^{\Omega_1}(z) \delta_v \right).$$
Note that $x_1$ is on the boundary of $\Omega_1$, so if we let

$$T^{\Omega_1}_{\mathcal{L}_1} = \Delta^{\Omega_1} - \Delta^{\Omega_1}_{\mathcal{L}_1},$$

we get that

$$T^{\Omega_1}_{\mathcal{L}_1} \delta_{x_1} = 0. \quad (2.22)$$

We have

$$\left( \delta_{x_1} \left| G^{r}_{\mathcal{L}_0^{++},o}(z) \delta_v \right. \right) = \left( \delta_{x_1} \left| G^{\Omega_1}(z) \delta_v \right. \right) = \sum_{(u,u') \in \Theta(\mathcal{L}_1)} \sum_{(y,y') \in \Theta(\mathcal{L}_1^{++})} \left( \delta_{x_1} \left| G^{\Omega_1}_{\mathcal{L}_1:1;u}(z) \delta_u \right. \right) \left( \delta_u \left| T^{\Omega_1}_{\mathcal{L}_1} \delta_{u'} \right. \right) \left( \delta_{u'} \left| G^{\Omega_1}(z) \delta_y \right. \right) \times \left( \delta_y \left| T^{\Omega_1}_{\mathcal{L}_1^{++}} \delta_{y'} \right. \right) \left( \delta_{y'} \left| G^{\Omega_1}_{\mathcal{L}_1^{++}:o}(z) \delta_v \right. \right). \quad (2.23)$$

Consider, first, $\mathcal{L}_1^{++}$. As before, there is only one element $(y, y') \in \Theta(\mathcal{L}_1^{++})$ for which $\left( \delta_{x_1} \left| G^{\Omega_1}_{\mathcal{L}_1^{++}:o}(z) \delta_v \right. \right) \neq 0$. From the construction, it follows that this element is precisely $(x_2', x_2)$ where $x_2'$ is the only point for which $\left( \delta_{x_2'} \left| T^{\Omega_1}_{\mathcal{L}_1^{++}} \delta_{x_2} \right. \right) \neq 0$. So

$$\left( \delta_{x_1} \left| G^{r}_{\mathcal{L}_0^{++}:o}(z) \delta_v \right. \right) = \sum_{(u,u') \in \Theta(\mathcal{L}_1)} \left( \delta_{x_1} \left| G^{\Omega_1}_{\mathcal{L}_1:1;u}(z) \delta_u \right. \right) \left( \delta_u \left| T^{\Omega_1}_{\mathcal{L}_1} \delta_{u'} \right. \right) \times \left( \delta_{u'} \left| G^{\Omega_1}(z) \delta_{x_2} \right. \right) \left( \delta_{x_2} \left| G^{\Omega_1}_{\mathcal{L}_1^{++}:o}(z) \delta_v \right. \right). \quad (2.24)$$

Now consider $\mathcal{L}_1$. This is a linear path which has at most three points on its boundary. One is $x_1$, another is $v_1$. If $\mathcal{L}_1$ has a junction in it (there is at most one in any case), then this is a third point on its boundary. There are no more possibilities. Because of (2.22), we have that the term corresponding to $x_1$ in the sum vanishes, so there are at most two terms in the sum above. Taking the $s_0$-moment for each of these terms and using Proposition 2.2 (by first averaging over the potential at $x_2'$ and $u'$, precisely as before) we get

$$\left\langle \left| \left( \delta_{x_1} \left| G^{r}_{\mathcal{L}_0^{++},o}(z) \delta_v \right. \right) \right|^{s_0} \right\rangle \leq C(s_0) \sum_{u \in \Theta(\mathcal{L}_1), u \neq x_1} \left\langle \left| \left( \delta_{x_1} \left| G^{\Omega_1}_{\mathcal{L}_1:1;u}(z) \delta_u \right. \right) \right|^{s_0} \right\rangle \times \left\langle \left| \left( \delta_{x_2} \left| G^{\Omega_1}_{\mathcal{L}_1^{++}:o}(z) \delta_v \right. \right) \right|^{s_0} \right\rangle. \quad (2.25)$$
We want to use (2.18) to estimate $\sum_{u \in \mathcal{B}(L_1)} u \neq x_1 \left\langle \left| \left( \delta_{x_1} | G^{\Omega_1}_{L_1^{\omega}}(z) \delta_u \right) \right|^{s_0} \right\rangle$. Indeed, if $L_1$ contains no junctions then this sum has only one element, $\left\langle \left| \left( \delta_{x_1} | G^{\Omega_1}_{L_1^{\omega}}(z) \delta_u \right) \right|^{s_0} \right\rangle$, and since it holds that
\[
(\delta_{x_1} | G^{\Omega_1}_{L_1^{\omega}}(z) \delta_u) = (\delta_{x_1} | G^{r}_{L_1^{\omega}}(z) \delta_u),
\]
for any $u \in \mathcal{V}(L_1)$, it immediately follows that
\[
\left\langle \left| (\delta_{x_1} | G^{\Omega_1}_{L_1^{\omega}}(z) \delta_u) \right|^{s_0} \right\rangle < \frac{1}{4C(s_0)^2} < \frac{1}{2C(s_0)}.
\]
(2.26)

If $L_1$ contains also a junction, $u_1$, then such a bound is not immediate from (2.18). This case has two terms appearing in the sum: $\left\langle \left| (\delta_{x_1} | G^{\Omega_1}_{L_1^{\omega}}(z) \delta_u) \right|^{s_0} \right\rangle$ and $\left\langle \left| (\delta_{x_1} | G^{\Omega_1}_{L_1^{\omega}}(z) \delta_v) \right|^{s_0} \right\rangle$. Note that, since $\mathcal{V}(L_1) \subseteq \mathcal{V}(\Omega_1)$, we have that $(\delta_{x_1} | G^{\Omega_1}_{L_1^{\omega}}(z) \delta_u) = (\delta_{x_1} | G^{r}_{L_1^{\omega}}(z) \delta_u)$, and $(\delta_{x_1} | G^{\Omega_1}_{L_1^{\omega}}(z) \delta_v) = (\delta_{x_1} | G^{r}_{L_1^{\omega}}(z) \delta_v)$.

Let $u_1'$ be the unique backward neighbor of $u_1$ and let $L'_1 = L(x_1, u_1')$. Consider, first $\left\langle \left| (\delta_{x_1} | G^{r}_{L_1^{\omega}}(z) \delta_u) \right|^{s_0} \right\rangle$. From (2.6) applied to $G^{r}_{L_1^{\omega}}$, we get that
\[
(\delta_{x_1} | G^{r}_{L_1^{\omega}}(z) \delta_u) = - (\delta_{x_1} | G^{r}_{L_1^{\omega}}(z) \delta_{u_1'}) (\delta_{u_1} | G_{L_1^{\omega}}(z) \delta_{u_1})
\]
(2.28)

so, performing first the average over $V_\omega(u_1)$ (of which $(\delta_{x_1} | G^{r}_{L_1^{\omega}}(z) \delta_{u_1'})$ is independent), we get
\[
\left\langle \left| (\delta_{x_1} | G^{\Omega_1}_{L_1^{\omega}}(z) \delta_u) \right|^{s_0} \right\rangle = \left\langle \left| (\delta_{x_1} | G^{r}_{L_1^{\omega}}(z) \delta_u) \right|^{s_0} \right\rangle \leq \frac{C(s_0)}{4C(s_0)^2} = \frac{1}{4C(s_0)}
\]
(2.29)

(recall that $L_0 = L + 5$ so $d(x_1, u_1') > L$).

As for $\left\langle \left| (\delta_{x_1} | G^{r}_{L_1^{\omega}}(z) \delta_v) \right|^{s_0} \right\rangle$, restricting $G^{r}_{L_1^{\omega}}(z)$ to $L'_1 = L(x_1, u_1')$ again, applying (2.8), taking the mean of the fractional moment and using Proposition 2.2, we get that
\[
\left\langle \left| (\delta_{x_1} | G^{r}_{L_1^{\omega}}(z) \delta_v) \right|^{s_0} \right\rangle \leq C(s_0) \left\langle \left| (\delta_{y'} | G^{r}_{L_1^{\omega}}(z) \delta_{v_1}) \right|^{s} \right\rangle \times \left\langle \left| (\delta_{y'} | G^{r}_{L_1^{\omega}}(z) \delta_{v_1}) \right|^{s} \right\rangle \times \left\langle \left| (\delta_{y'} | G^{r}_{L_1^{\omega}}(z) \delta_{v_1}) \right|^{s} \right\rangle
(2.30)
\]

where $y'$ is the only vertex on $\mathcal{B}(L_1^{++} \cap L_1$. Since $d(u_1, v_1) \geq L_0$ (see property 2 of the vertex pairs $\{x_j, v_j\}$ and $d(u_1, y') = 2$, we have that
\[ d(y', v_1) \geq L_0 - 2 > L. \] Furthermore, neither \( \mathcal{L}'_1 \) nor \( \mathcal{L}(y', v_1) \) contains a junction, so the bound \( \frac{1}{4C(s_0)^2} \) applies to both Green functions on the RHS of (2.30). Thus

\[ \langle |(\delta_{x_1} | G_{\mathcal{L}_1}(z) \delta_{v_1})|^2 \rangle \leq \frac{1}{16C(s_0)^3} < \frac{1}{4C(s_0)}. \] (2.31)

(2.29) and (2.31) give

\[ \sum_{u \in \partial(\mathcal{L}_1), u \neq x_1} \langle |(\delta_{x_1} | G_{\mathcal{L}_1}(z) \delta_{v_1})|^2 \rangle \leq \frac{1}{2C(s_0)}. \] (2.32)

Combining (2.27) and (2.32) with (2.25) we get

\[ \left\langle \left| \left( \delta_{x_1} | G_{\mathcal{L}_1}(z) \delta_{v_1} \right) \right|^2 \right\rangle \leq \frac{1}{2} \left\langle \left| \left( \delta_{x_2} | G_{\mathcal{L}_1}(z) \delta_{v_1} \right) \right|^2 \right\rangle. \] (2.33)

At this point, we note that we can repeat the procedure outlined above for \( \left\langle \left| \left( \delta_{x_2} | G_{\mathcal{L}_1}(z) \delta_{v_1} \right) \right|^2 \right\rangle \). Writing

\[ \Omega_2 = \Omega_1 \cap (\Gamma \setminus \mathcal{L}_1) \]

we note that, as before,

\[ \left( \delta_{x_2} | G_{\mathcal{L}_1}(z) \delta_{v_1} \right) = \left( \delta_{x_2} | G_{\mathcal{L}_1}(z) \delta_{v_1} \right) \]

and also

\[ T_{\mathcal{L}_2} \delta_{x_2} = 0 \]

where

\[ T_{\mathcal{L}_2} = \Delta_{\Omega_2} - \Delta_{\Omega_2}. \]

Thus, we repeat the argument above with \( \mathcal{L}_2 \) replacing \( \mathcal{L}_1 \) and \( x_2 \) replacing \( x_1 \), to get the same estimate with \( \langle \left| \left( \delta_{x_2} | G_{\mathcal{L}_1}(z) \delta_{v_1} \right) \right|^2 \rangle \) replaced by

\[ \langle \left| \left( \delta_{x_3} | G_{\mathcal{L}_2}(z) \delta_{v_1} \right) \right|^2 \rangle. \]

This can be repeated \( l - 1 \) times, so that, estimating

\[ \langle \left| \left( \delta_{x_l} | G_{\mathcal{L}_l}(z) \delta_{v_1} \right) \right|^2 \rangle \leq C(s_0), \]

\[ \text{15} \]
we get (recall (2.21))
\[
\langle |(\delta_{x_0} | G^r_{\omega}(z) \delta_v) |^{s_0} \rangle \leq C(s_0)^3 \left( \frac{1}{2} \right)^{l-1}.
\] (2.34)

d(u_{R_0}, v) = d(x_0, v) - d(u_{R_0}, x_0) \geq d(x_0, v) - 2R_0 \implies
\[
\langle |(\delta_{x_0} | G^r_{\omega}(z) \delta_v) |^{s_0} \rangle \leq A_1 e^{-q d(x_0, v)},
\] (2.35)

for any \(v\) with \(|v| \geq R_1\), where \(q = \frac{\ln 2}{6L_0}\) and \(A_1 = C(s_0)^3 \left( \#B(R_0) \right)^2 \left( \frac{2R_0}{s_0} + 1 \right)\).

Since there are only finitely many vertices in \(B(R_1)\), it is obvious that one may choose a constant, \(A > 0\) so that
\[
\langle |(\delta_{x_0} | G^r_{\omega}(z) \delta_v) |^{s_0} \rangle \leq A e^{-qd(x_0, v)},
\] (2.36)

holds for any \(v \in \mathcal{V}(\Gamma)\). Since none of our constants depended on \(r\) or \(z\), this estimate is uniform in \(r > 0\) and \(z\) (in a proper neighborhood of \(E\)) so the conclusion follows.

\[\square\]

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