STABILITY ESTIMATES FOR A MAGNETIC SCHRÖDINGER OPERATOR WITH PARTIAL DATA

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(Communicated by Mikko Salo)

Abstract. In this paper we study local stability estimates for a magnetic Schrödinger operator with partial data on an open bounded set in dimension \( n \geq 3 \). This is the corresponding stability estimates for the identifiability result obtained by Bukhgeim and Uhlmann [2] in the presence of a magnetic field and when the measurements for the Dirichlet-Neumann map are taken on a neighborhood of the illuminated region of the boundary for functions supported on a neighborhood of the shadow region. We obtain log log-estimates for the magnetic fields and log log log-estimates for the electric potentials.

1. Introduction. Let \( \Omega \subset \mathbb{R}^n \) \( (n \geq 3) \) be a simply connected open bounded set with \( C^\infty \) boundary, denoted by \( \partial \Omega \). We consider the following magnetic Schrödinger operator

\[
\mathcal{L}_{A,q}(x,D) := \sum_{j=1}^{n} (D_j + A_j(x))^2 + q(x) = D^2 + A \cdot D + D \cdot A + A^2 + q,
\]

where \( D = -i \nabla \), \( A = (A_j)_{j=1}^{n} \in C^2(\overline{\Omega}; \mathbb{R}^n) \) is a magnetic potential and \( q \in L^\infty(\Omega; \mathbb{R}) \) is an electric potential. Here \( Z^2 \) denotes \( Z \cdot Z \) for operators as well as for vector-valued functions. The inverse boundary value problem, IBVP for short, under consideration in this article is to recover information (inside \( \Omega \)) about the magnetic and electric potentials from measurements on subsets of the boundary. Roughly speaking, we divide the boundary \( \partial \Omega \) in two open subsets, \( F \) and \( B \). In this setting and if 0 is not an eigenvalue in \( L^2(\Omega) \) of \( \mathcal{L}_{A,q} \), we define the so called Dirichlet-Neumann map (DN map) \( \Lambda_{A,q}^{B \rightarrow F} \) as follows:

\[
\Lambda_{A,q}^{B \rightarrow F} : H^\frac{1}{2}_B(\partial \Omega) \to H^{-\frac{1}{2}}(\partial \Omega)
\]

\[
f \to (\partial_\nu + i A \cdot \nu)u|_F,
\]

2010 Mathematics Subject Classification. Primary: 35R30; Secondary: 42B37.

Key words and phrases. Inverse problems, magnetic Schrödinger operator, Dirichlet-Neumann map, complex geometric optic solutions, Carleman estimates, Radon transform.

The first author is supported by the Project MTM2011-28198 of Ministerio de Economía y Competitividad de España.

The second author is supported by the Project 00MTM2014-57769-C3-1-P of Ministerio de Economía y Competitividad de España.

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where $\nu$ is the exterior unit normal of $\partial \Omega$, the set $H^\frac{1}{2}(\partial \Omega)$ consists of all $f \in H^\frac{1}{2}(\partial \Omega)$ such that $\text{supp } f \subset \overline{B}$ (we will call this condition “support constraint”) and $u \in H^1(\Omega)$ is the unique solution of the Dirichlet problem:

$$
\begin{cases}
\mathcal{L}_{A,q} u = 0 & \text{in } \Omega \\
 u|_{\partial \Omega} = f.
\end{cases}
$$

Moreover, due to elliptic estimates, it follows $u$ belongs to $H^2(\Omega)$. Consequently, $(\partial_{\nu} + i A \cdot \nu)|_{\partial \Omega}$ belongs to $L^2(\partial \Omega)$. This fact will be used several times later.

In the case that $F$ or $B$ is not equal to $\partial \Omega$, we say that the IBVP has partial data. According to the choice of the sets $F$ and $B$, we can distinguish several types of partial data results that we briefly describe.

In the absence of a magnetic potential ($A \equiv 0$), the pioneering work, which we describe as illuminating $\Omega$ from infinity was studied by Bukhgeim and Uhlmann [2]. They consider a direction $\xi \in S^{n-1}$ and $F \subset \partial \Omega$ to be a neighborhood of the $\xi$-illuminated face or front region, defined as

$$
\partial \Omega_{-0}(\xi) = \{ x \in \partial \Omega : \xi \cdot \nu(x) < 0 \}
$$

and we also define the $\xi$-shadowed face or back region as

$$
\partial \Omega_{+0}(\xi) = \{ x \in \partial \Omega : \xi \cdot \nu(x) > 0 \},
$$

where $\nu(x)$ denotes the exterior unit normal vector at $x$. They considered $B = \partial \Omega$ obtaining the following identifiability result: if $\Lambda_{0,q_1} = \Lambda_{0,q_2}$ then $q_1 = q_2$. The corresponding stability estimates were derived by Heck and Wang [10]. Later, Kenig, Sjöstrand and Uhlmann [11] obtained a similar result when $F$ and $B$ are neighborhoods respectively of the illuminated and shadowed boundary regions of $\Omega$ from a point $x_0$ (out of the convex hull of $\Omega$), $\partial \Omega_{-0}(x_0)$ and $\partial \Omega_{+0}(x_0)$, respectively; which now are defined by

$$
\partial \Omega_{-0}(x_0) = \{ x \in \partial \Omega : (x - x_0) \cdot \nu(x) < 0 \},
$$

$$
\partial \Omega_{+0}(x_0) = \{ x \in \partial \Omega : (x - x_0) \cdot \nu(x) > 0 \}.
$$

Notice that in this case, if the domain is strictly convex then $F$ could be arbitrarily small.

In the case of illuminating from infinity the supporting set $B$ could also be restricted to a neighborhood of the shadow region from infinity. In the case of $A \equiv 0$, stability estimates with the support constraint were derived by Caro, Dos Santos Ferreira and Ruiz [3], using Radon transform and for illumination from a point without the support constraint in [4] by using the geodesic ray transform on the sphere. In both cases, they obtained log log-estimates.

In the presence of a magnetic potential ($A \not\equiv 0$), as it was noted in [19], there exists a gauge invariance of the DN map. To be specific, if $\varphi \in C^1(\overline{\Omega})$ is a real valued function with $\varphi|_{\partial \Omega} = 0$ then $\Lambda_{A,q} = \Lambda_{A + \nabla \varphi,q}$. Hence, for the identifiability problem we only expect to recover the magnetic fields, $dA_1 = dA_2$, and the electric potentials, $q_1 = q_2$. Here we consider the magnetical potential $A$ as a 1-form and so $dA$ as a 2-form, as follow:

$$
A = \sum_{j=1}^{n} A_j dx_j, \quad A = (A_1, A_2, \ldots, A_n),
$$

$$
dA = \sum_{1 \leq j < k \leq n} (\partial_{x_j} A_k - \partial_{x_k} A_j) dx_j \wedge dx_k.
$$
For the magnetic case, we briefly mention the results concerning to full data, that is when \( F = B = \partial \Omega \). Sun [19] proved identifiability under the assumption of the smallness of the magnetic potential in a suitable space. Together with Nakamura and Uhlmann [14], they removed the smallness condition by assuming \( C^2 \)-compactly supported magnetic potential and \( L^\infty \) electric potential. For these cases, stability estimates were derived by Tzou [20]. The previous full data identifiability results were extended by Krupchyk and Uhlmann [12] for both, magnetic and electric potentials in \( L^\infty \). The corresponding stability estimates were derived by Caro and Pohjola [5].

The aforementioned full data results were extended to the partial data case. In the case of illumination from a point, the identifiability result was obtained by Dos Santos Ferreira, Kenig, Sjöstrand and Uhlmann [8]. They considered the knowledge of the DN map on a neighborhood of the illuminated face \( F \) for functions without any support constraint on the boundary. This result was extended by Chung [6], assuming an extra condition, that is the support constraint is on a neighborhood \( B \) of the shadowed face. For both previous cases, the issue of stability still remains open. On the other hand, in the case of illumination from infinity, the identifiability result and the corresponding stability estimates were obtained by Tzou [20]. Analogously to [8], Tzou did not consider any support constraint on the boundary.

In this article we extend Tzou’s result [20] in the following sense: we consider the case of illumination from infinity with an additional condition similar in spirit to [6], that is, we consider the support constraint on a neighborhood \( B \) of the shadowed face of the boundary. In this case, our main goal is to derive stability estimates when recovering \( dA \) and \( q \).

This paper is organized as follows. In Section 2 we state the stability estimates for the magnetic and electric potentials. In Section 3 we prove the stability estimate for the magnetic potential, see Theorem 2.3. In Section 4, we prove the stability estimate for the electric potential, see Theorem 2.4. In Section 5 we deduce a new Carleman estimate with linear weight, see Lemma 5.5, and also it contains the proof of Proposition 1, which ensures the existence of special solutions for the magnetic Schrödinger equation, supported on the shadowed part of the boundary.

2. Stating our stability estimates. Before stating our results we introduce some notation following [3]. Let \( N \) be a non empty open subset of \( S^{n-1} \) representing the set of directions from where we are illuminating \( \Omega \), and define the sets

\[
F_N = \bigcup_{\xi \in N} \partial \Omega_{-0}(\xi), \quad B_N = \bigcup_{\xi \in N} \partial \Omega_{+0}(\xi).
\]

Now let \( F \) and \( B \) be open neighborhoods on \( \partial \Omega \) of \( F_N \) and \( B_N \), respectively. In this geometric setting we define the partial DN map in the following way. Let \( \chi \) be a cutoff function supported on \( F \) such that it equals to 1 on \( F_N \). Here we denote by \( H^{1/2}_B(\partial \Omega) \) the class consisting of all \( f \in H^{1/2}(\partial \Omega) \) such that \( \text{supp} \ f \subset B \). Finally, we define the partial DN map \( \Lambda_{A,q}^f : H^{1/2}_B(\partial \Omega) \rightarrow H^{-1/2}(\partial \Omega) \) as follows:

\[
\Lambda_{A,q}^f f = \chi A_{A,q} f.
\]

We consider its associated operator norm defined by
Definition 2.1. Given $M > 0$ and $\gamma \in [0, 1)$, we define the class of admissible magnetic potentials $\mathcal{A}(\Omega, M, \gamma)$ by

$$\mathcal{A}(\Omega, M, \gamma) = \left\{ A \in C^{2, \gamma}(\overline{\Omega}; \mathbb{R}^n) : \| A \|_{C^{2, \gamma}(\overline{\Omega})} \leq M \right\}.$$  

Here and throughout this paper, we denote the characteristic function of $\Omega$ by $\chi_{\Omega}$. For a function $h : \Omega \to \mathbb{R}$ (or $\mathbb{R}^n$) we denote by $\chi_{\Omega}h$ its extension by zero out of $\Omega$. According to Proposition 3.6 in [18] (see also Lemma 1.1 in [9]), we have $\chi_{\Omega}$ belongs to $H^s(\mathbb{R}^n)$ with $s \in (0, 1/2)$. Motivated by this fact, we also introduce the class of admissible electric potentials.

Definition 2.2. Given $M > 0$ and $\sigma \in (0, 1/2)$, we define the class of admissible electric potentials $\mathcal{D}(\Omega, M, \sigma)$ by

$$\mathcal{D}(\Omega, M, \sigma) = \left\{ q \in L^\infty(\Omega) : \| q \|_{L^\infty(\Omega)} + \| \chi_{\Omega}q \|_{H^\sigma(\mathbb{R}^n)} \leq M \right\}.$$  

With these definitions at hand, we can now formulate our stability results.

Theorem 2.3. Let $\Omega \subset \mathbb{R}^n$ be a simply connected open set with smooth boundary. Consider two positive constants $M$ and $\gamma \in (0, 1)$. Let $N$ be a non empty open subset of $S^{n-1}$ and consider $F$ an open neighborhood of $F_N$, where $F_N$ is defined as 3. Then there exist $C > 0$ (depending on $n, \Omega, M, \gamma$) and $\lambda \in (0, 1/2)$ (depending on $n$) such that the following estimate

$$\| dA_1 - dA_2 \|_{L^2(\Omega)} \leq C \log \| \log \| A_1^T - A_2^T \| \|^{-\lambda/2},$$

holds true for all $A_1 \in \mathcal{A}(\Omega, M, \gamma)$ and for all $A_2 \in \mathcal{A}(\Omega, M, 0)$ satisfying $A_1 = A_2$ and $\partial_n A_1 = \partial_n A_2$ both on $\partial\Omega$; and for all $q_1, q_2 \in L^\infty(\Omega)$.

Theorem 2.4. Let $\Omega \subset \mathbb{R}^n$ be a simply connected open set with smooth boundary. Consider three positive constants $M$, $\sigma \in (0, 1/2)$ and $\gamma \in (0, 1)$. Let $N$ be a non empty open subset of $S^{n-1}$ and consider $F$ an open neighborhood of $F_N$, where $F_N$ is defined as 3. Then there exist $C > 0$ (depending on $n, \Omega, M, \sigma, \gamma$) and $\lambda \in (0, 1/2)$ (depending on $n$) such that the following estimate

$$\| q_1 - q_2 \|_{L^2(\Omega)} \leq C \log \| \log \| A_1^T - A_2^T \| \|^{-\lambda/2},$$

holds true for all $A_1 \in \mathcal{A}(\Omega, M, \gamma)$ and for all $A_2 \in \mathcal{A}(\Omega, M, 0)$ satisfying $A_1 = A_2$ and $\partial_n A_1 = \partial_n A_2$ both on $\partial\Omega$; and for all $q_1, q_2 \in \mathcal{D}(\Omega, M, \sigma)$.

Remark 1. In previous works on stability from infinity, different norms of the DN map have been used. For instance, in [20] Tzou considered $\Lambda : H^{3/2}(\partial\Omega) \to H^{1/2}(F)$ and the corresponding distance $\| \Lambda_1 - \Lambda_2 \|_{H^{3/2}(\partial\Omega) \to H^{1/2}(F)}$. In contrast
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and following [13], Caro, Dos Santos Ferreira and Ruiz [3] considered the difference of DN maps $\Lambda_1 - \Lambda_2 : \mathcal{H}(\partial \Omega) \to \mathcal{H}(\partial \Omega)^*$, where $\mathcal{H}(\partial \Omega)$ denotes the range of the trace map $Tr : \{ u \in L^2(\Omega) : \Delta u \in L^2(\Omega) \} \to H^{-1/2}(\partial \Omega)$, and the corresponding distance as:

$$\| \Lambda_1 - \Lambda_2 \|_{B \to F} = \sup_Y |\langle (\Lambda_1 - \Lambda_2)u_B, u_F \rangle|,$$

where

$$Y = \{ (u_B, u_F) \in \mathcal{H}(\partial \Omega)^2 : \|u_B\| = \|u_F\| = 1, u_B \in \mathcal{E}'(B), u_F \in \mathcal{E}'(F) \}.$$

In the present work we use the natural distance given by 5.

The proofs of Theorem 2.3 and Theorem 2.4, will be carried out by proving an integral identity relating the partial boundary data, i.e. the partial DN maps, with the unknown magnetic and electric potentials, by means of solutions $u \in H^1(\Omega)$ of the magnetic Schrödinger equation $L_{A,q}u = 0$ in $\Omega$. In order to decode the information hidden into the integral identity, we use two classes of special solutions, the so-called complex geometric optic solutions (CGO solutions). The first class will be obtained by using a suitable Carleman estimate with a linear weight and following the arguments used in [6] in order to have the required support constraint on the boundary. In the literature, all previous results do not consider this support constraint on the boundary. We emphasize that one of the main difficulty to construct such required solutions is the derivation of a suitable Carleman estimate. The second class of solutions have already been constructed in [8] and need not have the support constraint. By plugging these solutions into the integral identity leads us to obtain two Radon transforms, one for the difference of the magnetic fields $dA_1 - dA_2$ and other for the difference of the electric potentials $q_1 - q_2$. At this point, we apply a quantitative estimate derived in [3] to the Radon transform of $dA_1 - dA_2$ to end up the proof of Theorem 2.3. The quantitative estimate involves a logarithm of the difference of the DN maps. To prove Theorem 2.4, we applied the same quantitative estimate for the Radon transform now for $q_1 - q_2$, the Hodge decomposition derived by Tzou [20] (here we require the connectedness hypothesis) and the gauge invariance of the DN map in order to use the already established stability estimate for the magnetic fields. This step involves two logarithms of the difference of the partial DN maps, and by the quantitative estimate for the Radon transform, an extra logarithm has to be added.

3. Stability estimate for the magnetic potential. The goal of this section is to prove Theorem 2.3. According to this, throughout this section we use the notations and hypotheses from Theorem 2.3 for the sets $\Omega, N$ and $F_N$ and the corresponding regularities for the magnetic and electric potentials.

3.1. Construction of special solutions - CGO solutions. In this section we shall establish the existence of CGO solutions $u \in H^1(\Omega)$ (with and without the required boundary support constraint on $B$) for the magnetic Schrödinger equation $L_{A,q}u = 0$. The following proposition is the analogous of Proposition 9.2 in Chung article [6]. However, we mention that there is a slight difference with respect to Chung’s proof. By our method, we required a little bit more regularity than $C^2$ for the magnetic potential while Chung only consider $C^2$.

Proposition 1. Let $\xi, \zeta \in S^{n-1}$ be a pair of orthogonal vectors and let $\gamma \in (0,1)$. If $A \in C^{2,\gamma}(\overline{\Omega}; \mathbb{R}^n)$ and $q \in L^{\infty}(\Omega; \mathbb{R})$ then there exist three positive constants, $\tau_0$
and $C$ (both depending on $n, \Omega, \|A\|_{C^{2, \gamma}}, \|q\|_{L^\infty}$) and $\gamma$ with $0 < \gamma < \gamma$ such that the equation
\[
\begin{cases}
L_{A,q} u = 0 & \text{in } \Omega \\
|u|_{\partial \Omega \setminus B} = 0
\end{cases}
\]
has a solution $u \in H^1(\Omega)$ of the form
\[
 u = e^{\tau(\xi \cdot x + i \zeta \cdot x)} (e^\Phi + r) - e^{\tau^* b},
\]
with the following properties:
(i). Let $\tilde A \in C^{2, \gamma}(\mathbb{R}^n)$ be any compactly supported extension of $A$ in $\mathbb{R}^n$. If $\Phi := C_{\xi + i \zeta} \tilde A$ then $\Phi \in C^{3, 2}(\Omega)$ and satisfies in $\mathbb{R}^n$
\[
(\xi + i \zeta) \cdot \nabla \Phi + i(\xi + i \zeta) \cdot \tilde A = 0,
\]
(6) \[\|\Phi\|_{W^{0, \infty}(\Omega)} \leq C \|A\|_{C^\gamma(\Omega)}, \quad |\alpha| \leq 2\]
and
(7) \[\|\Phi\|_{C^{3, 2}(\Omega)} \leq C \|A\|_{C^{2, \gamma}(\Omega)}.
\]
Here $C_{\xi + i \zeta} \tilde A$ denotes the Cauchy transform in $\mathbb{R}^n$ of $\tilde A$, with respect to $\xi$ and $\zeta$. The definition of $C_{\xi + i \zeta}$ is given in Appendix, see Definition 5.1.
(ii). The function $l$ depends on the a priori bounds of $A$ and $q$, and satisfies
(8) \[\Re l(x) = \xi \cdot x - \tilde k(x),\]
where the positive function $\tilde k(x) \simeq \text{dist}(x, \partial \Omega \setminus B)$ in $G$, a neighborhood of $\partial \Omega \setminus B$ in $\mathbb{R}^n$. Here $\Re l$ denotes the real part of $l$.
(iii). The function $b$ belongs to $C^{1, 2}(\Omega)$ with supp $b \subset G$; and it depends on the a priori bounds of $A$ and $q$. Moreover, we have
(9) \[\|l\|_{H^1(\Omega)} \leq C, \quad \|b\|_{H^1(\Omega)} \leq C\]
and
(10) \[\left\| e^{-\tau \tilde k} \right\|_{L^2(\Omega)} \leq C \tau^{-1/2}, \quad \left\| e^{-\tau \tilde k} \right\|_{L^\infty(\Omega)} \leq C.
\]
(iv). Finally, $r \in H^1(\Omega)$ satisfies $r|_{\partial \Omega \setminus B} = 0$ and for all $\tau \geq \tau_0$ the following estimates hold true
(11) \[\|\partial^\alpha r\|_{L^2(\Omega)} \leq C \tau^{1-|\alpha|}, \quad |\alpha| \leq 1,
\]
\[\|r\|_{L^2(\partial \Omega)} \leq C \tau^{-1/2}.
\]

For expository convenience, we leave the proof to the Appendix. We will also use the solutions constructed by Dos Santos Ferreira, Kenig, Sjöstrand and Uhlmann, see Lemma 3.4 in [8]. These solutions do not require the support constraint on $B$.

**Proposition 2.** Let $\xi, \zeta \in S^{n-1}$ be a pair of orthogonal vectors. If $A \in C^2(\overline{\Omega}; \mathbb{R}^n)$ and $q \in L^\infty(\Omega; \mathbb{R})$ then there exists two positive constants, $\tau_0$ and $C$ (both depending on $n, \Omega, \|A\|_{C^2}, \|q\|_{L^\infty}$) such that the equation $L_{A,q} u = 0$ has a solution $u \in H^1(\Omega)$ of the form
\[
 u = e^{-\tau(\xi \cdot x - i \zeta \cdot x)} (e^\Phi g + r),
\]
with the following properties:

(i). Let $\tilde{A} \in C^2_{c}(\mathbb{R}^n)$ be any compactly supported extension of $A$ in $\mathbb{R}^n$. If $\Phi := C\xi - i\zeta \tilde{A}$ then $\Phi \in C^2(\Omega)$ and satisfies in $\mathbb{R}^n$

$$(\xi - i\zeta) \cdot \nabla \Phi + i(\xi - i\zeta) \cdot \tilde{A} = 0.$$ 

and

$$\|\Phi\|_{W^{\alpha, \infty}(\Omega)} \leq C \|A\|_{C^n(\Omega)}, \quad |\alpha| \leq 2.$$ 

Here $C\xi - i\zeta \tilde{A}$ denotes the Cauchy transform in $\mathbb{R}^n$ of $\tilde{A}$, with respect to $\xi$ and $-\zeta$.

The definition of $C\xi - i\zeta$ is given in Appendix, see Definition 5.1.

(ii). The function $g$ is smooth and satisfies in $\mathbb{R}^n$

$$(\xi - i\zeta) \cdot \nabla g = 0.$$ 

(iii). The function $r$ belongs to $H^1(\Omega)$ and for all $\tau \geq \tau_0$ satisfies the following estimate

$$\|\partial^\alpha r\|_{L^2(\Omega)} \leq C\tau^{|\alpha|-1} \|q\|_{H^2(\Omega)}, \quad |\alpha| \leq 1.$$ 

3.2. Relating the partial DN maps with the magnetic and electric potentials. We state an integral estimate which involves a relation between the partial DN maps with the magnetic and electric potentials, see Proposition 4. From now on, for $j = 1, 2$, we denote the global and partial DN maps $\Lambda_{A,q}$ by $\Lambda_j$ and $\Lambda_{A,q}^j$, by $A^j_\Omega$, respectively. The following Carleman estimate with boundary terms was derived by Dos Santos Ferreira et al., see Proposition 2 in [8].

**Proposition 3** (A Carleman estimate with boundary terms). Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let $\xi \in S^{n-1}$, and define $\varphi(x) = \xi \cdot x$. If $A \in C^1(\overline{\Omega}; \mathbb{R}^n)$ and $q \in L^\infty(\Omega; \mathbb{R})$ then there exist two positive constants $\tau_0$ and $C$ (both depending on $n, \Omega, A, \|A\|_{C^1}, \|q\|_{L^\infty}$) such that for all $u \in C^\infty(\overline{\Omega}) \cap H^1_0(\Omega)$ the following estimate

$$\left\|\sqrt{\xi \cdot \varphi} e^{\tau \varphi} \partial_{\nu} u\right\|_{L^2(\partial \Omega, +, 0(\xi))} + \tau^{1/2} \left\|e^{\tau \varphi} u\right\|_{L^2(\Omega)} + \tau^{-1/2} \left\|\sqrt{-\xi \cdot \varphi} e^{\tau \varphi} \partial_{\nu} u\right\|_{L^2(\partial \Omega, -0(\xi))} \leq C \left(\tau^{-1/2} \left\|e^{\tau \varphi} \mathcal{L}_{A,q} u\right\|_{L^2(\Omega)} + \left\|\sqrt{-\xi \cdot \varphi} e^{\tau \varphi} \partial_{\nu} u\right\|_{L^2(\partial \Omega, -0(\xi))}\right)$$

holds true for all $\tau \geq \tau_0$. Here $\nu$ denotes the exterior unit normal of $\partial \Omega$ and $\partial_{\nu} = \nu \cdot \nabla$. The sets $\partial \Omega, 0(\xi)$ and $\partial \Omega, +0(\xi)$ are defined by 1 and 2, respectively.

**Remark 2.** The Carleman estimate 12 is still true for all $u \in H^1_0(\Omega)$ such that $\mathcal{L}_{A,q} u \in L^2(\Omega)$, which could be seen by a standard regularization method. The vanishing condition on the boundary of $u$ is essential for deriving this Carleman estimate with boundary terms. In short, this estimate said us that it is possible to bound the $L^2(\partial \Omega, +0(\xi))-\text{norm}$ by the $L^2(\partial \Omega, -0(\xi))-\text{norm}$ plus remainder terms in $L^2(\Omega)$-norm. In other words, we can bound the measurements of the shadowed face of $\partial \Omega$ by measurements of the illuminated face but paying with quantities in $L^2(\Omega)$-norm.

The following lemma it is a well known result which relates the global DN maps with the magnetic and electric potentials, see Proposition 3.1 in [19].
Lemma 3.1. Assume that the functions $u_1, u_2 \in H^1(\Omega)$ satisfy $\mathcal{L}_{A_1,q_1}u_1 = 0$ and $\mathcal{L}_{A_2,q_2}u_2 = 0$. Then

$$\langle (A_1 - A_2)u_1, u_2 \rangle_{L^2(\partial\Omega)}$$

(13)

$$= \int_{\Omega} \left[ (A_1 - A_2) \cdot (Du_1 \nu_2 + u_1 Du_2) + (A_1^2 - A_2^2 + q_1 - q_2)u_1 \nu_2 \right] dx.$$

In particular, this identity holds true for every $u_1 \in H^1(\Omega)$ satisfying both $\mathcal{L}_{A_1,q_1}u_1 = 0$ and $\text{supp}(u_1)_{\partial\Omega} \subset B$. For technical reasons to prove Proposition 4, we need to introduce the following subsets of the boundary. Given a direction $\xi \in S^{n-1}$ and $\epsilon > 0$, we define the $(\xi, \epsilon)$-illuminated face of $\partial\Omega$ as

$$\partial\Omega_{-,\epsilon}(\xi) = \{ x \in \partial\Omega : \xi \cdot \nu(x) < \epsilon \},$$

and the $(\xi, \epsilon)$-shadowed face as

$$\partial\Omega_{+,\epsilon}(\xi) = \{ x \in \partial\Omega : \xi \cdot \nu(x) > -\epsilon \}.$$

Note that these sets are open neighborhoods on $\partial\Omega$ of $\partial\Omega_{-,0}(\xi)$ and $\partial\Omega_{+,0}(\xi)$, respectively; see 1 and 2. Hence, according to 3, the sets defined by

$$F_{N,\epsilon} = \bigcup_{\xi \in N} \partial\Omega_{-,\epsilon}(\xi), \quad B_{N,\epsilon} = \bigcup_{\xi \in N} \partial\Omega_{+,\epsilon}(\xi)$$

are open neighborhoods on $\partial\Omega$ of $F_N$ and $B_N$, respectively. Finally, since $F$ is an open neighborhood of $F_N$, it follows that the following inclusions

$$F_N \subset F_{N,\epsilon} \subset F, \quad B_N \subset B_{N,\epsilon} \subset B,$$

hold true whenever $\epsilon > 0$ is small enough.

Now in order to exploit the information about $A_1 - A_2$ encoded into the previous integral identity, we start by obtaining an estimative of the left-hand side of 13.

Proposition 4. Consider a direction $\xi \in N$. Assume that the functions $u_1, u_2 \in H^1(\Omega)$ satisfy $\mathcal{L}_{A_1,q_1}u_1 = 0$ with $\text{supp}(u_1)_{\partial\Omega} \subset B$ and $\mathcal{L}_{A_2,q_2}u_2 = 0$. Then there exist two positive constants $\tau_0 > 0$ and $C > 0$ (both depending on $n, \Omega, M$) such that the estimate

$$\left| \langle (A_1 - A_2)u_1, u_2 \rangle_{L^2(\partial\Omega)} \right|$$

(15)

$$\leq C \left\| A_1^2 - A_2^2 \right\| \left( \| u_1 \|_{H^1(\Omega)} \| u_2 \|_{H^1(\Omega)} + e^\tau c \| u_1 \|_{H^1(\Omega)} \| e^{\tau \xi \cdot x}u_2 \|_{L^2(\partial\Omega)} \right)$$

$$+ C \tau^{-\frac{1}{2}} \left\| e^{-\tau \xi \cdot x} (\mathcal{L}_{A_1,q_1} - \mathcal{L}_{A_2,q_2})u_1 \right\|_{L^2(\Omega)} \left\| e^{\tau \xi \cdot x}u_2 \right\|_{L^2(\partial\Omega)}$$

holds true for all $\tau \geq \tau_0$. Here $c = \sup \{ |x| : x \in \overline{\Omega} \}$.

Proof. We start by choosing $\epsilon > 0$ being small enough such that 14 holds true and consider $\chi \in C^\infty(\partial\Omega)$ be a cutoff function supported in $F$ such that it equals to 1 on $F_{N,\epsilon}$. We consider the decomposition

$$\langle (A_1 - A_2)u_1, u_2 \rangle_{L^2(\partial\Omega)} = \langle \chi (A_1 - A_2)u_1, u_2 \rangle_{L^2(\partial\Omega)}$$

(16)

$$+ \langle (1 - \chi) (A_1 - A_2)u_1, u_2 \rangle_{L^2(\partial\Omega)}.$$
We now estimate each of the right-hand side terms. By the definition of the partial DN map and Cauchy-Schwarz inequality, the first term can be estimated as follows
\[
\int_{\partial \Omega} \chi(\Lambda_1 - \Lambda_2) u_1 \bar{u}_2 \, dS = \int_{\partial \Omega} (\Lambda_1^\sharp - \Lambda_2^\sharp) u_1 \bar{u}_2 \, dS \\
\leq \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\|_{H^\frac{1}{2}(\partial \Omega)} \| u_1 \|_{H^\frac{1}{2}(\partial \Omega)} \| u_2 \|_{H^\frac{1}{2}(\partial \Omega)} \\
\leq C_1 \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\|_{H^1(\Omega)} \| u_1 \|_{H^1(\Omega)} \| u_2 \|_{H^1(\Omega)} .
\]
(17)

The estimation of the second term requires a more refined analysis. It can be done by using Proposition 4 as follows.
\[
\int_{\partial \Omega} (1 - \chi)(\Lambda_1 - \Lambda_2) u_1 \bar{u}_2 \, dS \\
= \int_{\Omega \setminus (\xi) \setminus (\partial \Omega \setminus (\xi))} (1 - \chi)(\Lambda_1 - \Lambda_2) u_1 \bar{u}_2 \, dS \\
= \int_{\partial \Omega \setminus (\Omega \setminus (\xi))} (1 - \chi)(\Lambda_1 - \Lambda_2) u_1 \bar{u}_2 \, dS \\
\leq C_2 \left\| e^{-\tau \xi} (\Lambda_1 - \Lambda_2) u_1 \right\|_{L^2(\partial \Omega) \setminus (\Omega \setminus (\xi))} \left\| e^{\tau \xi} u_2 \right\|_{L^2(\partial \Omega) \setminus (\Omega \setminus (\xi))} .
\]
(18)

It is easy to see that
\[
\left\| e^{\tau \xi} u_2 \right\|_{L^2(\partial \Omega) \setminus (\Omega \setminus (\xi))} \leq \left\| e^{\tau \xi} u_2 \right\|_{L^2(\Omega)} .
\]
(19)

It remains to estimate the other \( L^2(\partial \Omega \setminus (\Omega \setminus (\xi)))\)-norm. To do this, we introduce an auxiliary function \( w \in H^1(\Omega) \) satisfying
\[
\begin{cases}
\mathcal{L}_{A_2,q_2} w = 0, & \text{in } \Omega \\
|w|_{\partial \Omega} = u_1 |_{\partial \Omega} .
\end{cases}
\]
Thus, since \( u_1 \in H^1(\Omega) \) and \( \mathcal{L}_{A_2,q_2}(w - u_1) = (\mathcal{L}_{A_1,q_1} - \mathcal{L}_{A_2,q_2}) u_1 \), it follows that \( \mathcal{L}_{A_2,q_2}(w - u_1) \in L^2(\Omega) \). Moreover, we have that \( w - u_1 \in H^0(\Omega) \). Hence, taking into account Remark 2, \( A_1 = A_2 \) on \( \partial \Omega \) and applying Proposition 3 with \( u = u_1 - w \), we obtain
\[
\left\| e^{-\tau \xi} (\Lambda_1 - \Lambda_2) u_1 \right\|_{L^2(\partial \Omega) \setminus (\Omega \setminus (\xi))} = \left\| e^{-\tau \xi} \partial_\nu (u_1 - w) \right\|_{L^2(\partial \Omega) \setminus (\Omega \setminus (\xi))} \\
\leq \frac{1}{\sqrt{\epsilon}} \left\| \sqrt{\xi \cdot \nu(\cdot)} e^{-\tau \xi} \partial_\nu (u_1 - w) \right\|_{L^2(\Omega, \partial \Omega)} \\
\leq \frac{C_3}{\sqrt{\epsilon}} \left( \left\| \sqrt{\xi \cdot \nu(\cdot)} e^{-\tau \xi} \partial_\nu (u_1 - w) \right\|_{L^2(\Omega, \partial \Omega)} + \tau^{-\frac{1}{2}} \left\| e^{-\tau \xi} (\mathcal{L}_{A_2,q_2}(w - u_1)) \right\|_{L^2(\Omega)} \right) \\
\leq \frac{C_3}{\sqrt{\epsilon}} \left( \left\| e^{-\tau \xi} \partial_\nu (u_1 - w) \right\|_{L^2(\partial \Omega)} + \tau^{-\frac{1}{2}} \left\| e^{-\tau \xi} (\mathcal{L}_{A_1,q_1} - \mathcal{L}_{A_2,q_2}) u_1 \right\|_{L^2(\Omega)} \right) .
\]
(20)

The \( L^2(\partial \Omega \setminus (\Omega \setminus (\xi)))\)-norm in the last inequality can be estimated as follows
\[
\left\| e^{-\tau \xi} \partial_\nu (u_1 - w) \right\|_{L^2(\partial \Omega \setminus (\Omega \setminus (\xi)))} = \left\| e^{-\tau \xi} (\Lambda_1 - \Lambda_2) u_1 \right\|_{L^2(\partial \Omega \setminus (\Omega \setminus (\xi)))} \\
\leq \left\| e^{-\tau \xi} \chi(\Lambda_1 - \Lambda_2) u_1 \right\|_{L^2(\partial \Omega)} \leq e^{\tau \epsilon} \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\|_{H^\frac{1}{2}(\partial \Omega)} \| u_1 \|_{H^\frac{1}{2}(\partial \Omega)} .
\]
(21)
Thus, replacing 20 and 21 into 18, we get
\[
\int_{\partial\Omega} (1 - \chi)(A_1 - A_2) u_1 \nu_2 \, dS \\
\leq C_4 \left( e^{\tau} \left\| A_1^\sharp - A_2^\sharp \right\|_{H^1(\Omega)} + \tau^{-\frac{1}{2}} \left\| e^{-\tau \xi \cdot x} (\mathcal{L}_{A_1,q_1} - \mathcal{L}_{A_2,q_2}) u_1 \right\|_{L^2(\Omega)} \right) \left\| e^{\tau \xi \cdot x} u_2 \right\|_{L^2(\partial\Omega \setminus \partial N_\gamma(\xi))}.
\]
We conclude the proof by replacing 17 and 22 into 16 and considering 19. □

Since 15 holds for every pair of solutions \( u \in H^1(\Omega) \) corresponding to the magnetic Schrödinger equation \( \mathcal{L}_{A,q} u = 0 \), we use the CGO solutions constructed in Section 3.1. More precisely, we consider \( \xi \in N \) and \( \zeta \in S^{n-1} \) such that it still belongs to \( C^2 \gamma(\mathbb{R}^n) \). By Proposition 1 there exist \( u_1 \in H^1(\Omega) \) satisfying
\[
\begin{cases}
\mathcal{L}_{A_1,q_1} u_1 = 0 & \text{in } \Omega \\
u_1|_{\partial N_\gamma} = 0
\end{cases}
\]
of the form
\[
u_1 = e^{\tau (\xi \cdot x + i \zeta \cdot x)} (e^{\Phi_1 + r_1}) - e^{\tau b}
\]
with \( \Phi_1 := C_\xi + i \zeta \cdot (\xi + i \zeta) \cdot A_1^{\text{ext}} \) satisfying in \( \mathbb{R}^n \):
\[
(\xi + i \zeta) \cdot \nabla \Phi_1 + i (\xi + i \zeta) \cdot A_1^{\text{ext}} = 0,
\]
and two positive constants \( C_1 \) and \( \tau_1 \) such that
\[
\| \partial^\alpha r_1 \|_{L^2(\Omega)} \leq C_1 \tau^{1-|\alpha|}, \quad |\alpha| \leq 1,
\]
holds true for all \( \tau \geq \tau_1 \). Moreover, we have the estimate
\[
\| e^{\tau} \|_{L^\infty(\Omega)} = \left\| e^{\tau (\xi \cdot x - k(x))} \right\|_{L^\infty(\Omega)} \leq \left\| e^{\tau \xi \cdot x} \right\|_{L^\infty(\Omega)} \leq e^{\tau |\xi|}.
\]
On the other hand, we consider the following compactly supported extensions of \( A_2 \)
\[
A_2^{\text{ext}} = \begin{cases}
A_2 & \text{in } \Omega, \\
A_1^{\text{ext}} & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}
\]
Since \( A_1 = A_2 \) and \( \partial_n A_1 = \partial_n A_2 \) on \( \partial \Omega \), it follows that \( A_2^{\text{ext}} \in C^2(\mathbb{R}^n) \). Thus, by Proposition 2 there exist \( u_2 \in H^1(\Omega) \) satisfying \( \mathcal{L}_{A_2,q_2} u_2 = 0 \) of the form
\[
u_2 = e^{-\tau (\xi \cdot x - i \zeta \cdot x)} (e^{\Phi_2 + r_2})
\]
with \( \Phi_2 := C_\xi - i \zeta \cdot (\xi - i \zeta) \cdot A_2^{\text{ext}} \) satisfying in \( \mathbb{R}^n \):
\[
(\xi - i \zeta) \cdot \nabla \Phi_2 + i (\xi - i \zeta) \cdot A_2^{\text{ext}} = 0,
\]
and two positive constants \( C_2 \) and \( \tau_2 \) such that the following estimate
\[
\| \partial^\alpha r_2 \|_{L^2(\Omega)} \leq C_2 \tau^{1-|\alpha|} \left\| g \right\|_{H^2(\Omega)}, \quad |\alpha| \leq 1,
\]
holds true for all \( \tau \geq \tau_2 \).

**Remark 3.** With the previous compactly supported extensions at hand, it is easy to check \( A_1^{\text{ext}} - A_2^{\text{ext}} = \chi_\Omega (A_1 - A_2) \in C^2(\mathbb{R}^n) \). Consequently, we obtain
\[
(\xi + i \zeta) \cdot \nabla (\Phi_1 + \Phi_2) + i (\xi + i \zeta) \cdot [\chi_\Omega (A_1 - A_2)] = 0, \text{ in } \mathbb{R}^n.
\]
We leave aside for a moment this equation. It will play a crucial role in the proof of Proposition 5.
According to Proposition 4, our next task will be to estimate the terms coming from the right-hand side of 15. For simplicity, we denote \( a_1 = e^{b_1 \xi} \), \( \varphi(x) = \xi \cdot x \) and \( \psi(x) = \zeta \cdot x \). Then, since \( Rl(x) = \xi \cdot x - \ell(x) \) and taking into account 9 and 23-25, we obtain
\[
\|u_1\|_{H^1(\Omega)} = \|u_1\|_{L^2(\Omega)} + \|\nabla u_1\|_{L^2(\Omega)} = \|e^{\tau(\varphi+i\psi)}(a_1 + r_1) - e^{\tau L_2(\Omega)}l \|_{L^2(\Omega)} + \|\nabla \left[ e^{\tau(\varphi+i\psi)}(a_1 + r_1) - e^{\tau L_2(\Omega)}l \right] \|_{L^2(\Omega)} \leq C_1 \|e^{\tau L_2(\Omega)}l\|_{L^\infty(\Omega)} \|a_1 + r_1\|_{L^2(\Omega)} + \|e^{\tau L_2(\Omega)}l\|_{L^\infty(\Omega)} \|b\|_{L^2(\Omega)} + \tau \|e^{\tau L_2(\Omega)}l\|_{L^\infty(\Omega)} \|b\|_{H^1(\Omega)} + C_1 \tau \|e^{\tau L_2(\Omega)}l\|_{L^\infty(\Omega)} \|a_1 + r_1\|_{L^2(\Omega)} + \|e^{\tau L_2(\Omega)}l\|_{L^\infty(\Omega)} \|\nabla(a_1 + r_1)\|_{L^2(\Omega)} \leq C_2 \tau e^{\tau L_2(\Omega)} \|a_1 + r_1\|_{H^1(\Omega)} + C_2 \tau e^{\tau L_2(\Omega)} \|b\|_{H^1(\Omega)} \leq C_3 \tau e^{\tau L_2(\Omega)}.
\]

We continue in this fashion to compute the \( L^2 \)-norm of \( e^{-\tau(\varphi+i\psi)}(L_{A_1,q_1} - L_{A_2,q_2})u_1 \). Denote \( V = a_1 + r_1 + e^{-\tau(\varphi+i\psi)}e^{\tau L_2(\Omega)}l \). Then, we have
\[
\left\| e^{-\tau(\varphi+i\psi)}(L_{A_1,q_1} - L_{A_2,q_2})u_1 \right\|_{L^2(\Omega)} = \left\| e^{-\tau(\varphi+i\psi)}(L_{A_1,q_1} - L_{A_2,q_2}) \left[ e^{\tau(\varphi+i\psi)}V \right] \right\|_{L^2(\Omega)} \leq \left\| (A_1 - A_2) \cdot (\nabla \varphi + \tau \nabla \psi) V \right\|_{L^2(\Omega)} + \left\| (A_1 - A_2) \cdot \nabla V \right\|_{L^2(\Omega)} + \left\| (\nabla \cdot (A_1 - A_2) V \right\|_{L^2(\Omega)} + \left\| (A_1^2 - A_2^2 + q_1 - q_2) V \right\|_{L^2(\Omega)} \leq C_6 \tau \|a_1 + r_1 + b\|_{L^2(\Omega)} + C_6 \|\nabla(a_1 + r_1 + b)\|_{L^2(\Omega)} \leq C_6 \tau \|a_1 + r_1 + b\|_{H^1(\Omega)} \leq C_7 \tau.
\]

Repeating similar previous arguments and using 26-27, leads us to get the following estimates for \( u_2 \)
\[
\|u_2\|_{H^1(\Omega)} \leq C_4 \tau e^{\tau L_2(\Omega)} \|\Phi\|_{H^2(\Omega)}
\]
and
\[
\|e^{\tau L_2(\Omega)}u_2\|_{L^2(\Omega)} \leq C_5 \|\Phi\|_{H^2(\Omega)}.
\]

By combining the estimates 28-31 into 15, and taking into account that there exists \( \tau_3 > 0 \) such that \( \tau \leq e^{2\tau L_2(\Omega)} \) for all \( \tau \geq \tau_3 \), we get
\[
\left\| (A_1 - A_2)u_1, u_2 \right\|_{L^2(\Omega)} \leq C \left\| A_1^2 - A_2^2 \right\| \left( e^{2\tau L_2(\Omega)} + \tau e^{2\tau L_2(\Omega)} \right) \|\Phi\|_{H^2(\Omega)} + C \left( \tau^{1/2} + 1 \right) \|\Phi\|_{H^2(\Omega)} \leq C \left( \tau e^{4\tau L_2(\Omega)} \right) \left\| A_1^2 - A_2^2 \right\| + \tau \left\| \Phi\|_{H^2(\Omega)} \right. \]

Taking \( \tau_0 = \max(\tau_1, \tau_2, \tau_3) \) and multiplying by \( \tau^{-1} \) both sides of the previous inequality, we deduce the following estimate for all \( \tau \geq \tau_0 \)
\[
\tau^{-1} \left\| (A_1 - A_2)u_1, u_2 \right\|_{L^2(\Omega)} \leq C \left( \tau e^{4\tau L_2(\Omega)} \right) \left\| A_1^2 - A_2^2 \right\| + \tau \left( \right) \|\Phi\|_{H^2(\Omega)}
\]
Note that by 13, the expression $\tau^{-1} \langle (\Lambda_1 - \Lambda_2)u_1, u_2 \rangle_{L^2(\partial \Omega)}$ is equal to

$$
\tau^{-1} \int_\Omega \left[ (A_1 - A_2) \cdot (Du_1 \vec{\nu}_2 + u_1 \partial u_2) + (A_1^2 - A_2^2 + q_1 - q_2)u_1 \vec{\nu}_2 \right] dx,
$$

which in turn can be decomposed in the following way. For simplicity, we denote $\rho(x) = (\xi + i\zeta) \cdot x$, $u_r = e^{\tau (-\rho + i)l}b$, $a_2 = e^\Omega g$ and recall that we have already denoted $a_1 = e^{b_1}$. Hence the solutions $u_1$ and $u_2$, given by 23 and 26, respectively; have the form

$$(33) \quad u_1 = e^\tau (a_1 + r_1 - u_r), \quad u_2 = e^{-\tau} (a_2 + r_2).$$

An easy computation shows that

$$
\tau^{-1} Du_1 \vec{\nu}_2 = (e^{\tau \rho}(D\rho(a_1 + r_1 - u_r) + \tau^{-1} D(a_1 + r_1 - u_r))) (e^{-\tau \rho}(\vec{\nu}_2 + r_2))
$$

$$
= D\rho a_1 \vec{\nu}_2 + M_1
$$

and

$$
\tau^{-1} u_1 \partial u_2 = (e^{\tau \rho}(D\rho(a_1 + r_1 - u_r)) (e^{-\tau \rho}(D\rho(\vec{\nu}_2 + r_2)))
$$

$$
= D\rho a_1 \vec{\nu}_2 + M_2,
$$

where

$$
M_1 = D\rho a_1 \vec{\nu}_2 + \tau^{-1} D\rho(\vec{\nu}_2 + r_2) + \tau^{-1} D\rho a_1 (\vec{\nu}_2 + r_2) + \tau^{-1} e^{-\tau \rho} Du_r (\vec{\nu}_2 + r_2),
$$

$$
M_2 = D\rho(a_1 + r_1) \vec{\nu}_2 + \tau^{-1} D\rho a_1(\vec{\nu}_2 + r_2) + \tau^{-1} D\rho a_1(\vec{\nu}_2 + r_2) + \tau^{-1} e^{-\tau \rho} u_r (\vec{\nu}_2 + r_2),
$$

Hence, by 13 and 34-35, we obtain

$$
\tau^{-1} \langle (\Lambda_1 - \Lambda_2)u_1, u_2 \rangle_{L^2(\partial \Omega)}
$$

$$
= \tau^{-1} \int_\Omega \left[ (A_1 - A_2) \cdot (Du_1 \vec{\nu}_2 + u_1 \partial u_2) + (A_1^2 - A_2^2 + q_1 - q_2)u_1 \vec{\nu}_2 \right] dx
$$

$$
= 2 \int_\Omega (A_1 - A_2) \cdot D\rho a_1 \vec{\nu}_2 dx + \int_\Omega (A_1 - A_2) \cdot (M_1 + M_2) dx
$$

$$
+ \tau^{-1} \int_\Omega (A_1^2 - A_2^2 + q_1 - q_2)u_1 \vec{\nu}_2 dx
$$

Now by 10 and similar analysis to that in the proof of the estimates 28-31 shows

$$
\| e^{-\tau \rho} u_r \|_{L^2(\Omega)} \leq C_1 \tau^{-1}, \quad \| e^{-\tau \rho} Du_r \|_{L^2(\Omega)} \leq C_1
$$

and

$$
\| M_j \|_{L^2(\Omega)} \leq C_2 \tau^{-1} \| \vec{\gamma} \|_{H^2(\Omega)}, \quad j = 1, 2, \quad \tau \geq \tau_4
$$

for some positive constants $C_1, C_2$ and $\tau_4$. Thus, by combining 32, and 36-38, we get

$$
2 \int_\Omega (A_1 - A_2) \cdot D\rho a_1 \vec{\nu}_2 dx
$$

$$
\leq \tau^{-1} \left| (\Lambda_1 - \Lambda_2)u_1, u_2 \right|_{L^2(\partial \Omega)} + C_3 \| M_1 + M_2 \|_{L^2(\Omega)}
$$

$$
+ C_4 \tau^{-1} \| e^{-\tau \rho} u_1 \|_{L^2(\Omega)} \| e^{\tau \rho} u_2 \|_{L^2(\Omega)}
$$

$$
\leq C_5 \left( e^{4\tau c} \| A_2^e - A_2 \| + \tau^{-\frac{1}{2}} \right) \| \vec{\gamma} \|_{H^2(\Omega)}.
$$
Finally, by recalling that $a_1 = e^{\Phi_1}$ and $a_2 = e^{\Phi_2} g$, we obtain

$$(39) \quad (\xi + i\zeta) \cdot \int_{\Omega} (A_1 - A_2) e^{\Phi_1 + \Phi_2} g dx \leq C_0 \left( e^{4\tau e} \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| + \tau^{-\frac{1}{2}} \right) \| g \|_{H^2(\Omega)}.$$ 

Next we show that it is possible to remove the exponential term $e^{\Phi_1 + \Phi_2}$.

**Remark 4.** Let $\xi, \zeta \in S^{n-1}$ ($n \geq 3$) be unit orthogonal vectors. Then, every $x \in \mathbb{R}^n$ can be written as follows

$$x = t\xi + r\zeta + x',$$

it leads to parameterize $\mathbb{R}^n$ in the following way $x \mapsto (t, r, x')$.

**Proposition 5.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let $\xi \in N \subset S^{n-1}$ and $\zeta \in S^{n-1}$ such that $\xi \cdot \zeta = 0$. Consider two positive constants $M$ and $\gamma \in (0, 1)$. Consider $A_1 \in \mathcal{A}^\gamma(\Omega, M, \gamma)$, $A_2 \in \mathcal{A}^\gamma(\Omega, M, 0)$ and $q_1, q_2 \in L^\infty(\Omega)$. If $A_1 = A_2$ and $\partial_r A_1 = \partial_r A_2$ both on $\partial \Omega$, then there exist two positive constants $\tau_0$ and $C > 0$ (both depending on $n, \Omega, M, \gamma$) such that

$$(40) \quad \left\| \mu \cdot \int_{\Omega} (A_1 - A_2) g dx \right\| \leq C |\mu| \left\| \log \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| \right|^{-1/2} \| g \|_{H^2(\Omega)}$$

holds true for all $\mu \in \text{span}\{\xi, \zeta\}$, provided that $\left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| \leq e^{-4\tau_0}$. Here $g$ is any smooth function depending only on $x'$ in the sense of Remark 4.

**Proof.** The proof is a direct consequence of Remark 3 and Lemma 5.3 with $W = -i\chi_\Omega (A_1 - A_2)$ and $\Phi = \Phi_1 + \Phi_2$. Thus, we get

$$-i(\xi + i\zeta) \cdot \int_{\mathbb{R}^n} \chi_\Omega (A_1 - A_2) g e^{\Phi_1 + \Phi_2} dx = -i(\xi + i\zeta) \cdot \int_{\mathbb{R}^n} \chi_\Omega (A_1 - A_2) g dx$$

which implies

$$(41) \quad (\xi + i\zeta) \cdot \int_{\Omega} (A_1 - A_2) g e^{\Phi_1 + \Phi_2} dx = (\xi + i\zeta) \cdot \int_{\Omega} (A_1 - A_2) g dx$$

for any smooth function $g$ depending only on $x'$. In particular, this kind of smooth functions $g$ satisfies $(\xi + i\zeta) \cdot \nabla g = 0$ in $\mathbb{R}^n$. Hence, by 39 and 41, we obtain

$$\left\| (\xi + i\zeta) \cdot \int_{\Omega} (A_1 - A_2) g dx \right\| \leq C_0 \left( e^{4\tau e} \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| + \tau^{-\frac{1}{2}} \right) \| g \|_{H^2(\Omega)}.$$ 

Now we consider $\tau_0 > 0$ such that for all $\tau \geq \tau_0$ the previous inequality and $e^{-2\tau e} \leq \tau^{-1/2}$ are satisfied. It is a simple matter to check that

$$\tau := \frac{1}{8} e^{-1} \left\| \log \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| \right\| \geq \tau_0,$$

whenever

$$\left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| \leq e^{-4\tau_0}.$$ 

Hence, we get

$$\left\| (\xi + i\zeta) \cdot \int_{\Omega} (A_1 - A_2) g dx \right\| \leq C_1 \left\| \log \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| \right\|^{-1/2} \| g \|_{H^2(\Omega)}.$$ 

By a similar analysis, with $(\xi + i\zeta)$ instead of $(\xi - i\zeta)$, we obtain

$$\left\| (\xi - i\zeta) \cdot \int_{\Omega} (A_1 - A_2) g dx \right\| \leq C_2 \left\| \log \left\| \Lambda_1^\sharp - \Lambda_2^\sharp \right\| \right\|^{-1/2} \| g \|_{H^2(\Omega)}.$$ 

By combining the two previous inequalities, we conclude the proof.  

$\Box$
3.3. Radon transform and its applications. Let $f$ be a function on $\mathbb{R}^n$, integrable on each hyperplane in $\mathbb{R}^n$. Each of these hyperplanes can be parametrized by its unit normal vector and distance to the origin: $\theta \in S^{n-1}$ and $s \in \mathbb{R}$, respectively. Thus, each hyperplane $H$ can be defined in the following way:

$$ H(s, \theta) = \{ x \in \mathbb{R}^n : \langle x, \theta \rangle = s \}. $$

In this setting, the Radon transform of $f$ is defined by

$$ (RRRf)(s, \theta) = \int_{H(s, \theta)} f(x) d\mu_H = \int f(s\theta + y) dy, $$

whenever the integral exists. Here $\theta^\perp$ denotes the set of orthogonal vectors to $\theta$.

Analogously, we define the parametrized hyperplane with respect to an arbitrary $y_0 \in \mathbb{R}^n$ and its corresponding Radon transform as follows

$$ H_{y_0}(s, \theta) = \{ x \in \mathbb{R}^n : \langle x - y_0, \theta \rangle = s \} $$

and

$$ R_{y_0}f(s, \theta) = \int_{H_{y_0}(s, \theta)} f(x) d\mu_{H_{y_0}}, $$

respectively. It is easy to check the relation

$$ R_{y_0}f(s, \theta) = (RRf)(s + \langle y_0, \theta \rangle, \theta). $$

On the other hand, we define the Fourier transform with respect to the first variable of a function $F : \mathbb{R} \times S^{n-1} \rightarrow \mathbb{R}$ as

$$ \hat{F}(\sigma, \theta) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-i s \sigma} F(s, \theta) ds. $$

Thus, for $\alpha \geq 0$, we denote by $H^\alpha(\mathbb{R} \times S^{n-1})$ the space consisting of all functions $F \in L^2(\mathbb{R} \times S^{n-1})$ such that

$$ \| F \|_{H^\alpha(\mathbb{R} \times S^{n-1})} = \left( \int_{S^{n-1}} \int_{\mathbb{R}} (1 + \sigma^2)^\alpha |\hat{F}(\sigma, \theta)|^2 d\sigma d\theta \right)^{1/2}. $$

is finite. It is well known that the Radon transform is continuous in suitable spaces and there exits a relation between its $s$-derivative and the partial derivatives with respect to $x$. More precisely, for every compactly supported function $f$ in $\mathbb{R}^n$, there exists a positive constant $C_0$, depending on $\alpha$ and $n$, such that

$$ \| Rf \|_{H^{n+(n-1)/2}(\mathbb{R} \times S^{n-1})} \leq C_0 \| f \|_{H^{\alpha}(\mathbb{R}^n)}, $$

and the following identity holds in the sense of distributions in $C_0^\infty(\mathbb{R})$

$$ \theta_i \frac{\partial}{\partial s}(RF)(\cdot, \theta) = R(\partial_{x_i} f)(\cdot, \theta), \quad i = 1, 2, \ldots, n. $$

Here $\theta_i$ denotes the $i$-th coordinate of $\theta$. Both previous results can be found in [15]. The next result plays an important role in deriving stability estimates for the magnetic and electric potentials. It was proved by Caro, Dos Santos Ferreira and Ruiz, see Theorem 2.5 in [3]. Roughly speaking, it gives us a quantitative relation between a function and its Radon transform: it is enough to have knowledge of the Radon transform $RF$ in a proper bounded subset of $\mathbb{R} \times S^{n-1}$ to obtain information...
about $F$ but we have to pay with a logarithmic term. Before stating Theorem 3.2, we introduce the set $X$ consisting of all $F \in L^1(\mathbb{R}^n)$ such that

$$
\|F\|_X := \int_{\mathbb{R}} (1 + |s|)^n \|RF(s, \cdot)\|_{L^1(S^{n-1})} ds
$$

is finite. By means of Fubini-Tonelli theorem, a sufficient condition for a function $F$ to be in $X$, is given by the following estimate

$$
\|F\|_X \leq |S^{n-1}| \int_{\mathbb{R}^n} (1 + |x|)^n |F(x)| dx,
$$

where $|S^{n-1}|$ denotes the measure of $S^{n-1}$. We also recall the distance on the unit sphere

$$
d_{S^{n-1}}(x, y) = \arccos((x, y)).
$$

**Theorem 3.2.** Let $M \geq 1$, $\alpha > 0$ and $\beta \in (0, 1)$. Given $y_0 \in \mathbb{R}^n$ and $\theta_0 \in S^{n-1}$, consider the set

$$
\Gamma = \{ \theta \in S^{n-1} : d_{S^{n-1}}(\theta_0, \theta) < \arcsin \beta \}
$$

and the domain of dependence of the Radon transform given by

$$
E = \{ x \in \mathbb{R}^n : \langle \theta, x - y_0 \rangle = s, s \in (-\alpha, \alpha), \theta \in \Gamma \}.
$$

Assume that there exist two positive constants $p$ and $\lambda$ with $1 \leq p < \infty$ and $0 < \lambda < p^{-1}$, respectively; and a function $F$ satisfying the following conditions:

(a). $\chi_E F \in X \cap L^\infty(\mathbb{R}^n)$, where $\chi_E$ denotes the characteristic function of the set $E$. Moreover

$$
\|\chi_E F\|_X + \|\chi_E F\|_{L^\infty(\mathbb{R}^n)} \leq M.
$$

(b). $y_0 \in \text{supp } F$ and supp $F \subset \{ x \in \mathbb{R}^n : \langle x - y_0, \theta_0 \rangle \leq 0 \}$.

(c). The $(\lambda, p, p)$-Besov regularity

$$
\int_{\mathbb{R}^n} \frac{\|\chi_E F(\cdot) - (\chi_E F)(\cdot - y)\|_{L^p(\mathbb{R}^n)}^p}{|y|^{n+\lambda p}} dy \leq M^p.
$$

Then there exists a positive constant $C$ (depending on $G, M, \alpha, \beta, \lambda$), such that

$$
\|F\|_{L^p(G)} \leq C \left( \log \int_{-\alpha}^{\alpha} (1 + |s|)^n \|R_{y_0} F(s, \cdot)\|_{L^1(\Gamma)} ds \right)^{-\lambda/2},
$$

where

$$
G = \left\{ x \in \mathbb{R}^n : |x - y_0| < \frac{\alpha}{8 \cosh(8\pi/\beta)} \right\}.
$$

**Remark 5.** In our context, the constant $\beta$ stands for the size of the set $N \subset S^{n-1}$ and $\langle -\alpha, \alpha \rangle$ is the interval where we have control of the Radon transform $RF(\cdot, \theta)$. Note that for fixed $y_0 \in \mathbb{R}^n$ and $\beta > 0$ we can take $\alpha$ large enough so that $\Omega \subset G$. We take advantage of this last fact to prove Theorem 2.3.

3.4. **Proof of Theorem 2.3.** We continue the proof of Theorem 2.3 by writing Proposition 5 in the context of the Radon transform of $\chi_{\Omega}(A_1 - A_2)$.

**Proposition 6.** Consider the open set in $S^{n-1}$

$$
P = \bigcup_{\xi \in N} [\xi]^{-},
$$

where $[\xi]^{-}$ denotes the subspace of $\mathbb{R}^n$ consisting of all the unit vectors orthogonal to $\xi$. Then for any $\tilde{g} \in C^\infty(\mathbb{R})$ there exist two positive constants $C$ and $\tau_0$ (both
depending on \( n, \Omega \) and the a priori bounds of \( \|A_j\|_{C^2(\Omega)} \) and \( \|q_j\|_{L^\infty(\Omega)} \) such that the following estimate

\[
\left| \mu \cdot \int_R \tilde{g}(s) \left( R \left[ \chi_\Omega(A_1 - A_2) \right] \right)(s, \theta)ds \right| \leq C' |\mu| \left| \log \|A_1 - A_2\| \right|^{-1/2} \|\tilde{g}\|_{L^2(R)},
\]

holds true for all \( \theta \in \mathcal{P} \) and \( \mu \in \theta^\perp \).

**Proof.** We consider \( \xi \in N \subset S^{n-1} \) and \( \zeta \in S^{n-1} \) such that \( \xi \cdot \zeta = 0. \) Since \( n \geq 3 \), it follows that there exists an unit vector \( \theta \in [\xi, \zeta]^\perp := (\text{span } \{\xi, \zeta\})^\perp. \) Thus, every \( x \in \mathbb{R}^n \) can be written in the following way

\[
x = t\xi + r\zeta + s\theta + y', \quad y' \in [\xi, \zeta]^\perp.
\]

Hence, \( \mathbb{R}^n \) can be parametrized as \( \Psi : x \mapsto (t, r, s, y'). \) On the one hand, according to Remark 4 and Proposition 5, in terms of \( \Psi \)-coordinates, 40 holds true for every smooth function \( \tilde{g} := R \circ \Psi^{-1} \) depending only on \( s \) and \( y'. \) For our purpose, we consider any smooth function \( \bar{g} := \tilde{g}(s) \) depending only on \( s. \) On the other hand, since \( \xi, \zeta, \) and \( \theta \) are unit orthogonal vectors, we have \( dx = dy'dt'dr'ds. \) Thus, for every \( \mu \in [\xi, \zeta] := \text{span } \{\xi, \zeta\}, \) we have

\[
\mu \cdot \int_\Omega (A_1 - A_2)\tilde{g}dx = \mu \cdot \int_{\mathbb{R}^n} \chi_\Omega(A_1 - A_2)\tilde{g}dx
\]

\[
= \mu \cdot \int_{\mathbb{R}^2} \chi_\Omega(A_1 - A_2) \circ \Psi^{-1} \left[ \tilde{g} \circ \Psi^{-1} \right] dy'dt'dr'ds
\]

\[
= \mu \cdot \int_{\mathbb{R}} \bar{g}(s) \left( \int_{\mathbb{R}^2} \chi_\Omega(A_1 - A_2) \left( t\xi + r\zeta + s\theta + x' \right) dy'dt \right) ds
\]

\[
= \mu \cdot \int_{\mathbb{R}} \bar{g}(s) \left( \int_{\theta^\perp} \chi_\Omega(A_1 - A_2) \left( s\theta + y \right) dy \right) ds
\]

\[
= \mu \cdot \int_{\mathbb{R}} \bar{g}(s) \left( R \left[ \chi_\Omega(A_1 - A_2) \right] \right)(s, \theta)ds.
\]

By combining this equality and 40, we conclude the proof. \( \square \)

It is easy to see that for any \( \theta \in \mathcal{P} \subset S^{n-1}, \) the vectors of the form \( \mu_{j,k} := \theta_j e_k - \theta_k e_j \) \( (j, k = 1, 2, \ldots, n) \) belong to \( \theta^\perp. \) Here \( \left( e_j \right)_{j=1}^n \) denotes the canonical basis of \( \mathbb{R}^n \) and \( \theta_j \) the \( j \)-th component of \( \theta. \) For simplicity in computations, we denote \( \tilde{A} = \chi_\Omega(A_1 - A_2) \) and \( \tilde{A}_j = \tilde{A} \cdot e_j. \) Hence, by 43, the following equality

\[
\mu_{j,k} \cdot \int_{R} \frac{\partial}{\partial s} \bar{h}(s) \left( R \left[ \chi_\Omega(A_1 - A_2) \right] \right)(s, \theta)ds
\]

\[
= \int_{R} \frac{\partial}{\partial s} \bar{h}(s) \left[ \theta_j e_k - \theta_k e_j \right] \cdot (R\tilde{A})(s, \theta)ds
\]

\[
= \int_{R} \frac{\partial}{\partial s} \bar{h}(s) \left[ \theta_j \left( R\tilde{A}_k \right)(s, \theta) - \theta_k \left( R\tilde{A}_j \right)(s, \theta) \right] ds
\]

\[
= - \int_{R} \bar{h}(s) \left[ \theta_j \frac{\partial}{\partial s} \left( R\tilde{A}_k \right)(s, \theta) - \theta_k \frac{\partial}{\partial s} \left( R\tilde{A}_j \right)(s, \theta) \right] ds
\]

\[
= - \int_{R} \bar{h}(s) \left[ R \left( \partial_x \tilde{A}_k - \partial_x \tilde{A}_j \right) \right](s, \theta)ds
\]
holds true for every compactly supported smooth function $\tilde{h}$ in $\mathbb{R}$. This equality and applying Proposition 6 with $\mu := \mu_{j,k}$ and $\tilde{g} = \partial_x \tilde{h}$, give us

$$\int_{\mathbb{R}} \tilde{h}(s) \left[ R \left( \frac{\partial_x \tilde{A}_k - \partial_x \tilde{A}_j}{\partial_x \tilde{A}_k} \right) \right] (s, \theta) ds \leq C |\mu_{j,k}| \left| \log \left( \frac{\Lambda_1^x - \Lambda_2^x}{\Lambda_2^x - \Lambda_1^x} \right) \right|^{-1/2} \left\| \partial_x \tilde{h} \right\|_{H^2(\mathbb{R})}$$

which implies for all $\theta \in \mathcal{P}$

$$\int_{\mathbb{R}} \tilde{h}(s) \left[ R \left( \frac{\partial_x \tilde{A}_k - \partial_x \tilde{A}_j}{\partial_x \tilde{A}_k} \right) \right] (s, \theta) ds \leq C \left| \log \left( \frac{\Lambda_1^x - \Lambda_2^x}{\Lambda_2^x - \Lambda_1^x} \right) \right|^{-1/2} \left\| \tilde{h} \right\|_{H^3(\mathbb{R})},$$

and then

$$\left\| R \left( \frac{\partial_x \tilde{A}_k - \partial_x \tilde{A}_j}{\partial_x \tilde{A}_k} \right) \right\|_{L^p(\mathcal{P}; \mathcal{H}^{-3}(\mathbb{R}))} \leq C \left| \log \left( \frac{\Lambda_1^x - \Lambda_2^x}{\Lambda_2^x - \Lambda_1^x} \right) \right|^{-1/2},$$

where $L^\infty(\mathcal{P}; \mathcal{H}^{-3}(\mathbb{R}))$ denotes the functions on $\mathcal{P}$ which are vector valued in $\mathcal{H}^{-3}(\mathbb{R})$, whose norm is given by

$$\|F\|_{L^\infty(\mathcal{P}; \mathcal{H}^{-3}(\mathbb{R}))} = \sup_{\theta \in \mathcal{P}} \int_{\mathbb{R}} |\tilde{F}(\sigma, \theta)|^2 (1 + |\sigma|^2)^{-3} d\sigma.$$
is closely related with \(47\) because we have the control of the Radon transform of \(F_{j,k}\) in the whole real line in the \(s\)-variable.

**Condition (b).** One sees immediately that for every fixed \(\theta_0 \in \mathcal{P}\), by translation there exists \(y_0 \in \text{supp } F_{j,k}\) such that the hyperplane \(H_{y_0}(0, \theta_0)\) stands on one side of supp \(F_{j,k}\).

**Condition (c).** It is well known that a function \(Y\) has \((\lambda, 2, 2)\)-Besov regularity if and only if \(Y \in H^\lambda(\mathbb{R}^n)\) with \(0 < \lambda < 1\). According to Proposition 3.6 in [18] (see also Lemma 1.1 in [9]), we have \(\chi_E\) belongs to \(H^\lambda(\mathbb{R}^n)\) with \(\lambda \in (0, 1/2)\). Hence \(\chi_E F_{j,k} \in H^\lambda(\mathbb{R}^n)\) and then it has \((\lambda, 2, 2)\)-Besov regularity. So the condition (c) is satisfied with \(p = 2\) and \(\lambda \in (0, 1/2)\).

As we have discussed already and recalling Remark 5, although \(\beta \in (0, 1)\), which is related closely with the size of the set \(N\), could be very small; we can take \(\alpha > 0\) large enough such that \(\text{supp } F_{i,j} \subset G\), where \(G\) is defined by \(45\). Thus, Theorem 3.2 with \(\Gamma = \mathcal{P}\), ensures that there exists \(C > 0\) such that

\[
\|F_{j,k}\|_{L^2(\mathbb{R}^n)} \leq C \left| \log \int_{-\infty}^{\infty} (1 + |s|) \| R_{y_0} F_{i,j}(s, \cdot) \|_{L^1(\mathcal{P})} \, ds \right|^{-\lambda/2}.
\]

Now we set

\[
L := \sup_{\theta \in \mathcal{P}} \|(1 + |\cdot - \langle \theta, y_0 \rangle|)^n \|_{L^2(|s| \leq \alpha + |y_0|)}
\]

and denote by \(|\mathcal{P}|\) the Lebesgue measure of \(\mathcal{P}\). Then \(47\), Fubini-Tonelli theorem and Hölder’s inequality, give us

\[
\int_{-\alpha}^{\alpha} (1 + |s|)^n \| R_{y_0} F_{j,k}(s, \cdot) \|_{L^1(\mathcal{P})} \, ds
= \int_{-\alpha}^{\alpha} (1 + |s|)^n \int_{\mathcal{P}} |(RF_{j,k})(s + \langle \theta, y_0 \rangle, \theta)| \, d\theta \, ds
= \int_{\mathcal{P}} \int_{-\alpha}^{\alpha} (1 + |s - \langle \theta, y_0 \rangle|)^n \|RF_{j,k}(s, \theta)\| \, ds \, d\theta
\leq \int_{\mathcal{P}} \|(1 + |\cdot - \langle \theta, y_0 \rangle|)^n \|_{L^2(|s| \leq \alpha + |y_0|)} \|RF_{j,k}(\cdot, \theta)\|_{L^2(|s| \leq \alpha + |y_0|)} \, d\theta
\leq L \int_{\mathcal{P}} \left( \int_{\mathbb{R}} \|RF_{j,k}(s, \theta)\|^2 \, ds \right)^{1/2} \, d\theta
\leq L |\mathcal{P}|^{\frac{1}{2p}} \left( \int_{\mathcal{P}} \left( \int_{\mathbb{R}} \|RF_{j,k}(s, \theta)\|^2 \, ds \right)^{p_1/2} \, d\theta \right)^{1/p_1}
= L |\mathcal{P}|^{\frac{1}{2p}} \left\| R \left( \partial_{x_j} A_k - \partial_{x_k} A_j \right) \right\|_{L^{p_1}(\mathcal{P}, L^2(\mathbb{R}))} \leq C_4 \log \left\| \Lambda_1^k - \Lambda_2^j \right\|^{-s_1}.
\]

We conclude the proof by taking logarithm to both sides of this inequality and taking into account \(48\).

4. **Stability estimate for the electric potential.** The goal of this section is to prove Theorem 2.4, we use its notations and hypotheses for the sets \(\Omega, N\) and \(F_N\) and the corresponding regularities for the magnetic and electric potentials. The idea will be to combine the gauge invariance of the DN map, a Hodge type decomposition and the stability result already proved in Theorem 2.3. Although the following result has not been explicitly declared by Tzou [20], it can be easily
deduced by combining his Lemma 6.2 (applied to $A_1 - A_2$) with the discussion just after of its proof. See also estimate 23 in [20].

**Lemma 4.1.** Consider $p > n$. Then there exist $\omega \in W^{3,p}(\Omega) \cap H^1_0(\Omega)$ and a positive constant $C$ such that

$$\|A_1 - A_2 - \nabla \omega\|_{W^{1,p}(\Omega)} \leq C \|d(A_1 - A_2)\|_{L^p(\Omega)}$$

and

$$\|\omega\|_{W^{3,p}(\Omega)} \leq C \|A_1 - A_2\|_{W^{2,p}(\Omega)}. $$

**Remark 6.** As we mentioned previously, we are considering a magnetic potential $A$ as a 1-form and so $dA$ as a 2-form. In this sense, we have

$$\|dA\|_{L^p(\Omega)} := \sum_{1 \leq j < k \leq n} \|\partial_j A_k - \partial_k A_j\|_{L^p(\Omega)}. $$

Applying now Morrey’s inequality, we obtain

$$\|A_1 - A_2 - \nabla \omega\|_{W^{1,p}(\Omega)} \leq C \|d(A_1 - A_2)\|_{L^p(\Omega)}$$

and

$$\|\omega\|_{W^{3,p}(\Omega)} \leq C \|A_1 - A_2\|_{W^{2,p}(\Omega)}. $$

**Lemma 4.2.** Define $\tilde{A}_1 = A_1 - \nabla \omega/2$ and $\tilde{A}_2 = A_2 + \nabla \omega/2$. Consider $u_1, u_2 \in H^1(\Omega)$ being the solutions of $L_{A_1, q_1} u_1 = 0$ with supp($u_1|_{\partial \Omega}$) $\subset B$ and $L_{A_2, q_2} u_2 = 0$ defined by 23 and 26, respectively. Then $U_1 := e^{-i\omega/2} u_1 \in H^1(\Omega)$ is a solution of $L_{\tilde{A}_1, q_1} U_1 = 0$ with supp($U_1|_{\partial \Omega}$) $\subset B$ and $U_2 := e^{i\omega/2} u_2 \in H^1(\Omega)$ is a solution of $L_{\tilde{A}_2, q_2} U_2 = 0$. Moreover, we have $\Lambda_{A_j, q_j} = \Lambda_{\tilde{A}_j, q_j}$ with $j = 1, 2$.

**Proof.** By Lemma 3.1 in [12], we have the identities

$$e^{i\omega/2} L_{A_1, q_1} e^{-i\omega/2} = L_{\tilde{A}_1, q_1}, \quad e^{-i\omega/2} L_{A_2, q_2} e^{i\omega/2} = L_{\tilde{A}_2, q_2}, $$

which immediately imply that $L_{\tilde{A}_j, q_j} U_j = 0$ for $j = 1, 2$. From 50, it follows that $U_1, U_2 \in H^1(\Omega)$. Moreover, supp($U_1|_{\partial \Omega}$) $\subset B$, since $u_1$ has the same property. Finally, due to the gauge invariance of the DN map and since $\omega|_{\partial \Omega} = 0$, we have $\Lambda_{A_j, q_j} = \Lambda_{\tilde{A}_j, q_j}$ for $j = 1, 2$.  

From 23 and 26, we have

$$U_1 = e^{-i\omega/2} \left[ e^{\tau(x+C_x \cdot x)} (e^{\Phi_1} + r_1) - e^{\tau b} \right],$$

$$U_2 = e^{i\omega/2} \left[ e^{-\tau (x-C_x \cdot x)} (e^{\Phi_2} g + r_2) \right],$$

where $l, b$ and $\Phi_j, r_j$ ($j = 1, 2$) satisfy the equations, regularities and estimates discussed in Section 3.2. With these solutions at hand, we use the gauge invariance and 13 with $\tilde{A}_j$ and $U_j$ instead of $A_j$ and $u_j$, respectively; to get

$$\langle (A_1 - A_2) U_1, U_2 \rangle_{L^2(\partial \Omega)}$$

$$= \int_\Omega \left[ (\tilde{A}_1 - \tilde{A}_2) \cdot (DU_1 U_2 + U_1 \overline{DU_2}) + (\tilde{A}_1^2 - \tilde{A}_2^2 + q_1 - q_2) U_1 U_2 \right] dx.$$

From this identity, we will now try to isolate $q_1 - q_2$ and hence, we have to deal with terms of the form $(\tilde{A}_1 - \tilde{A}_2) \cdot (DU_1 U_2 + U_1 \overline{DU_2})$ and $(\tilde{A}_1^2 - \tilde{A}_2^2) U_1 U_2$. Both
terms have the common factor $\tilde{A}_1 - \tilde{A}_2 = A_1 - A_2 - \nabla \omega$, which can be managed by using Lemma 4.1 and an elementary interpolation as follows

$$
\|d(A_1 - A_2)\|_{L^p(\Omega)} \leq \|d(A_1 - A_2)\|_{L^t(\Omega)}^{1-t} \|d(A_1 - A_2)\|_{L^t(\Omega)},
$$

where $n < p < z$ and $t \in (0, 1)$ such that $1/p = t/2 + (1 - t)/z$. Thus, by Theorem 2.3 and 49, we get

$$
\|A_1 - A_2 - \nabla \omega\|_{C^{0,1-\frac{1}{p}}(\Omega)} \leq C_1 \left|\log \left|\log \left\|\Lambda \tilde{e} - \Lambda \tilde{e}_t\right\|\right|\right|^\frac{-1}{2}
$$

By an analysis similar to that in Section 3.2, we deduce that

$$
\left|\langle (A_1 - A_2)U_1, U_2 \rangle_{L^2(\Omega)}\right| 
\leq C \left( e^{4\tau c} \left\|\Lambda \tilde{e} - \Lambda \tilde{e}_t\right\| + \tau^{1/2} \|A_1 - A_2 - \nabla \omega\|_{L^\infty(\Omega)} + \tau^{-1/2}\right) \|\bar{g}\|_{H^2(\Omega)}.
$$

From identity 51, we get

$$
\int_{\Omega} (q_1 - q_2) U_1 \bar{U}_2 dx \leq \left|\langle (A_1 - A_2)U_1, U_2 \rangle_{L^2(\Omega)}\right| 
+ C \|A_1 - A_2 - \nabla \omega\|_{L^\infty} \left( \|DU_1 U_2\|_{L^1(\Omega)} + \|U_1 \bar{U}_2\|_{L^1(\Omega)} \right),
$$

and 52-54, give us

$$
\left|\int_{\Omega} (q_1 - q_2) e^{\Phi_1 + \Phi_2 + i\omega} \bar{g} dx\right| 
\leq C \left( e^{4\tau c} \left\|\Lambda \tilde{e} - \Lambda \tilde{e}_t\right\| + \tau \left|\log \left|\log \left\|\Lambda \tilde{e} - \Lambda \tilde{e}_t\right\|\right|\right|\right)^\frac{-1}{2} + \tau^{-1/2}\right) \|\bar{g}\|_{H^2(\Omega)}.
$$

The next step will be to remove the exponential term $e^{\Phi_1 + \Phi_2 + i\omega}$. To do this, we start with the following identity

$$
\int_{\Omega} (q_1 - q_2) \bar{g} dx = \int_{\Omega} (1 - e^{\Phi_1 + \Phi_2 + i\omega}) (q_1 - q_2) \bar{g} dx + \int_{\Omega} (q_1 - q_2) e^{\Phi_1 + \Phi_2 + i\omega} \bar{g} dx.
$$

By Lemma 4.1 and Morrey’s inequality we have $\omega \in C^{1,1-\frac{1}{p}}(\Omega)$ and $\omega|_{\partial \Omega} = 0$. Combining Remark 3 and Lemma 5.2, we obtain in $\mathbb{R}^n$

$$
(\xi + i\zeta) \cdot \nabla (\Phi_1 + \Phi_2 + i\chi \Omega \omega) + i(\xi + i\zeta) \cdot [\chi \Omega (A_1 - A_2) - \nabla (\chi \Omega \omega)] = 0
$$

and

$$
\|\Phi_1 + \Phi_2 + i\omega\|_{L^\infty(\Omega)} \leq \|\Phi_1 + \Phi_2 + i\chi \Omega \omega\|_{L^\infty(\mathbb{R}^n)} 
\leq C_1 \|\chi \Omega (A_1 - A_2) - \nabla (\chi \Omega \omega)\|_{L^\infty(\mathbb{R}^n)}
= C_1 \|\chi \Omega (A_1 - A_2) - \chi \Omega \nabla \omega\|_{L^\infty(\mathbb{R}^n)}
= C_1 \|A_1 - A_2 - \nabla \omega\|_{L^\infty(\Omega)}.
$$

Taking into account the following well know inequality for complex numbers

$$
|e^a - e^b| \leq |a - b| e^{\max\{R_a, R_b\}}, \quad a, b \in \mathbb{C},
$$

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and Hölder’s inequality, we deduce that
\[
\left| \int_{\Omega} (1 - e^{\Phi + i\omega}) (q_1 - q_2) \bar{\varphi} dx \right| \leq \| (\Phi + i\omega) \|_{L^\infty} \| (q_1 - q_2) \bar{\varphi} \|_{L^1} \\
\leq C_2 \| A_1 - A_2 - \nabla \omega \|_{L^\infty} \| \bar{\varphi} \|_{L^2}.
\]
Hence, 52 and 55-56, imply that
\[
\left| \int_{\Omega} (q_1 - q_2) \bar{\varphi} dx \right| \\
\leq C_3 \left( e^{4\gamma} \| A_1^q - A_2^q \| + \tau \log \| \Lambda_1^q - \Lambda_2^q \| \| \bar{\varphi} \|_{H^2} \right),
\]
for all \( \tau \geq \tau_0 \). Choosing \( \tau \) as follows
\[
\tau = \frac{1}{8c} \log \| \Lambda_1^q - \Lambda_2^q \| \| \bar{\varphi} \|_{H^2} \geq \tau_0,
\]
which is true whenever
\[
\| \Lambda_1^q - \Lambda_2^q \| \leq e^{-e^{(8c\tau_0)^3(\lambda/\gamma)}}
\]
we obtain
\[
\left| \int_{\Omega} (q_1 - q_2) \bar{\varphi} dx \right| \leq C \log \| \Lambda_1^q - \Lambda_2^q \| \| \bar{\varphi} \|_{H^2}.
\]

4.1. Proof of Theorem 2.4. Analysis similar to that in the proof of Theorem 2.3 shows that estimate 57 can be written in term of the Radon transform as
\[
\left| \int_{\mathbb{R}} \bar{g}(s) \langle R \chi_{\Omega}(q_1 - q_2) \rangle (s, \theta) ds \right| \leq C_1 \log \| \Lambda_1^q - \Lambda_2^q \| \| \bar{\varphi} \|_{H^2}.
\]
for all \( \theta \in \mathcal{P} \) and \( \bar{\varphi} \in C_c^\infty(\mathbb{R}^n) \), where \( \mathcal{P} \) is defined by 46. Hence, we have
\[
\| R \langle \chi_{\Omega}(q_1 - q_2) \rangle \|_{L^\infty(\mathcal{P}; H^{-2}(\mathbb{R}))} \leq C_2 \log \| \Lambda_1^q - \Lambda_2^q \| \| \bar{\varphi} \|_{H^2}.
\]
Applying 42 with \( \alpha = 0 \) and \( f = \chi_{\Omega}(q_1 - q_2) \), we get
\[
\| R \langle \chi_{\Omega}(q_1 - q_2) \rangle \|_{L^2(\mathcal{P}; H^{s_2}(\mathbb{R}))} \leq C_3 \| \chi_{\Omega}(q_1 - q_2) \|_{L^2(\mathbb{R}^n)} \leq C_4.
\]
We can now proceed analogously to the proof of the magnetic case. Consider \( p_0 > 0 \) large enough and a complex interpolation between \( L^{p_0}(\mathcal{P}; H^{-2}(\mathbb{R})) \) and \( L^2(\mathcal{P}; H^{2s_2}(\mathbb{R})) \), give us
\[
\| R \langle \chi_{\Omega}(q_1 - q_2) \rangle \|_{L^p(\mathcal{P}; L^2(\mathbb{R}))} \leq C_5 \| \chi_{\Omega}(q_1 - q_2) \|_{L^2(\mathbb{R}^n)} \leq C_4.
\]
for any \( p_2 < (n + 3)/2 \) and \( s_2 < \frac{1}{2} (n - 1)/(n + 3) \). For simplicity, we now denote \( q = \chi_{\Omega}(q_1 - q_2) \). Our next objective is to apply Theorem 3.2 in order to extract information about \( q \) by means of its Radon transform estimate 58. The conditions (a) and (b) are verified similar to the magnetic case. The condition (c) is also satisfied since by hypothesis we have \( q \in H^s(\mathbb{R}^n) \). So \( q \) has \((\sigma, 2, 2)\)-Bessel regularity. Hence, we can take \( \alpha > 0 \) large enough such that \( \supp q \subset G \), where \( G \) is defined by 45. Thus, Theorem 3.2 with \( \Gamma = \mathcal{P} \), ensures that there exists \( C_6 > 0 \) such that
\[
\| q \|_{L^2(\mathbb{R}^n)} \leq C_6 \log \int_{-\alpha}^\alpha (1 + |s|)^n \| R_{g_0} q(s, \cdot) \|_{L^1(\Gamma)} ds \leq \lambda/2.
\]
Then \(58\), Fubini-Tonelli theorem and Hölder’s inequality, give us

\[
\int_{-\alpha}^{\alpha} (1 + |s|)^n \| R_{y_0}q(s, \cdot) \|_{L^1(\mathcal{P})} \, ds \\
= \int_{-\alpha}^{\alpha} (1 + |s|)^n \int_{\mathcal{P}} |(Rq)(s + \langle \theta, y_0 \rangle, \theta)| \, d\theta \, ds \\
\leq \int_{\mathcal{P}} \int_{-\alpha+|y_0|}^{\alpha+|y_0|} (1 + |s - \langle \theta, y_0 \rangle|)^n \| (Rq)(s, \theta) \| \, ds \, d\theta \\
\leq \left( \int_{\mathcal{P}} \left( \int_{\mathcal{R}} |Rq(s, \theta)|^2 \, ds \right)^{p_2/2} \, d\theta \right)^{1/p_2} \\
\leq L |\mathcal{P}|^{\frac{p_2-1}{p_2}} \left( \int_{\mathcal{P}} \left( \int_{\mathcal{R}} |Rq(s, \theta)|^2 \, ds \right)^{p_2/2} \, d\theta \right)^{1/p_2} \\
= L |\mathcal{P}|^{\frac{p_2-1}{p_2}} \| Rq \|_{L^{p_2}(\mathcal{P}; L^2(\mathcal{R}))} \leq C_0 \log \log \| \Lambda_1^\sharp - \Lambda_2^\sharp \|^{-s_2}.
\]

We conclude the proof by taking logarithms in both sides of the above inequality and taking into account the estimate \(59\).

5. Appendix. This section is mainly focused on proving Proposition 1. For more details, see also Appendix in [16].

5.1. The Cauchy transform. We first discuss the existence of solutions \(\Phi\) in \(\mathbb{R}^n\) of the following equation:

\[
(\xi + i\zeta) \cdot \nabla \Phi = W,
\]

where \(\xi, \zeta \in S^{n-1}\) are orthogonal vectors and \(W\) is a prescribed function.

**Definition 5.1.** Let \(\xi, \zeta \in S^{n-1}\) be orthogonal vectors. The Cauchy transform in \(\mathbb{R}^n\) of \(W\), with respect to \(\xi\) and \(\zeta\), is defined by

\[
(C_{\xi+i\zeta} W)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{y_1 + iy_2} W(x - y_1 \xi - y_2 \zeta) dy_1 dy_2,
\]

whenever the integral exists.

This transform allows us to obtain a solution for the equation \(60\). More precisely, we have the following result. See Lemma 2.1 in [19] and also Lemma 4.6 in [17].

**Lemma 5.2.** Let \(\xi, \zeta \in S^{n-1}\) be orthogonal vectors. Let \(W \in W^{m, \infty}(\mathbb{R}^n)\) with \(m \geq 0\) and \(\text{supp}(W) \subset B_R(0), R > 0\). Then \(\Phi := C_{\xi+i\zeta} W \in W^{m, \infty}(\mathbb{R}^n)\) solves \(60\) and satisfies

\[
\| C_{\xi+i\zeta} W \|_{W^{m, \infty}(\mathbb{R}^n)} \leq C \| W \|_{W^{m, \infty}(\mathbb{R}^n)}.
\]

Here \(C > 0\) only depends on \(R\). Moreover, if \(W \in C_c(\mathbb{R}^n)\) then \(\Phi \in C(\mathbb{R}^n)\).

The following is an already well known result in the case \(g \equiv 1\), see Proposition 3.3 in [12] and also Lemma 2.6 in [20].
Lemma 5.3. Let $\xi, \zeta \in S^{n-1}$ $(n \geq 3)$ be orthogonal vectors. Let $W \in C^2(\mathbb{R}^n; \mathbb{C}^n)$ with $\text{supp}(W) \subset B_R(0)$, $R > 0$. Then $\Phi := C_{\xi + i\zeta} \left( ((\xi + i\zeta) \cdot W) \right)$ satisfies in $\mathbb{R}^n$

(62) 

$$(\xi + i\zeta) \cdot \nabla \Phi = (\xi + i\zeta) \cdot W.$$ 

Moreover, the following identity:

$$(\xi + i\zeta) \cdot \int_{\mathbb{R}^n} W(x)e^{\Phi(x)}g(x')dx = (\xi + i\zeta) \cdot \int_{\mathbb{R}^n} W(x)g(x')dx,$$

holds true for all smooth function $g$ depending only on $x'$ in the sense of Remark 4.

Proof. Without loss of generality we can assume that $\xi = e_1$ and $\zeta = e_2$. Let $g$ be a smooth function depending only on $x'$. Now Lemma 5.2 ensures that $\Phi := C_{\xi + i\zeta} \left( ((\xi + i\zeta) \cdot W) \right)$ satisfies 62 and so we have

(63) 

$$(\xi + i\zeta) \cdot \int_{\mathbb{R}^n} W(x)e^{\Phi(x)}g(x')dx = \int_{\mathbb{R}^n} (e_1 + ie_2) \cdot W(x)e^{\Phi(x)}g(x')dx$$



$$= \int_{\mathbb{R}^n} (e_1 + ie_2) \cdot \nabla \Phi e^{\Phi(x)}g(x')dx = \int_{\mathbb{R}^{n-2}} g(x')h(x')dx',$$

where

$$h(x') = \int_{\mathbb{R}^2} (\partial_1 + i\partial_2)e^{\Phi(x_1,x_2,x')}dx_1dx_2.$$

Green’s theorem give us

$$h(x') = \lim_{T \to \infty} \int_{|x_1e_1\pm x_2e_2| \leq T} (\partial_1 + i\partial_2)e^{\Phi(x_1,x_2,x')}dx_1dx_2$$

$$= \lim_{T \to \infty} \int_{|x_1e_1\pm x_2e_2| = T} (\nu_1 + i\nu_2)e^{\Phi(x_1,x_2,x')}dS(x_1,x_2)$$

$$= \lim_{T \to \infty} \int_{|x_1e_1\pm x_2e_2| = T} (\nu_1 + i\nu_2) \left[ 1 + O(|x_1e_1\pm x_2e_2|^{-2}) \right] dS(x_1,x_2)$$

$$= \lim_{T \to \infty} \int_{|x_1e_1\pm x_2e_2| \leq T} (\partial_1 + i\partial_2)\Phi(x_1,x_2,x')dx_1dx_2$$

$$= \int_{\mathbb{R}^2} (\partial_1 + i\partial_2)\Phi(x_1,x_2,x')dx_1dx_2$$

$$= \int_{\mathbb{R}^2} (\xi + i\zeta) \cdot \nabla \Phi(x_1,x_2,x')dx_1dx_2.$$ 

By combining this identity and 63, the lemma follows. 

To construct solutions for the magnetic Schrödinger operator with the support constraint on the shadowed face of the boundary, we need a more refined estimate than 61. The proof of the following result can be found in Proposition B.3.1 in [16].

Proposition 7. Let $\xi, \zeta \in S^{n-1}$ be orthogonal vectors and $\gamma \in (0,1)$. Consider $F \in C^{m,\gamma}(\mathbb{R}^n)$ $(m \geq 0)$ such that $\text{supp}(F) \subset B_R(0)$ with $R > 0$. Then there exists a constant $\gamma$ with $0 < \gamma < \gamma$ such that the function $\Phi := C_{\xi + i\zeta} F \in C^{m+1,\gamma}(\mathbb{R}^n)$ solves 60 in $\mathbb{R}^n$, and satisfies

$$\|C_{\xi + i\zeta} F\|_{C^{m+1,\gamma}(\mathbb{R}^n)} = \|\Phi\|_{C^{m+1,\gamma}(\mathbb{R}^n)} \leq C \|F\|_{C^{m,\gamma}(\mathbb{R}^n)}.$$ 

Here $C > 0$ only depends on $R$. 


5.2. **Proof of Proposition 1.** We only give the main ideas to prove Proposition 1 in order to convince the reader that it is essentially similar to Proposition 9 in [6] with a linear limiting Carleman weight \( \varphi(x) = \xi \cdot x \) with \( \xi \in S^{n-1} \), instead of the logarithmic weight \( \varphi(x) = \log |x - x_0| \). To construct solutions with the desired support constraint on the shadowed face of the boundary, Chung derived a novel Carleman estimate with logarithmic weight in an extended domain of \( \Omega \), see Theorem 1.4 in [6]; which for us is the heart of his article. Actually, he dedicated several sections to prove this estimate, employing original, elegant and sophisticated arguments which can be easily adapted to our case. The key point in his paper is to prove that one can pass from a Carleman estimate \( L_2 \rightarrow H^1 \) to a Carleman estimate \( H^{-1} \rightarrow L_2 \) with some condition on the support of the functions adapted by duality to the case with the support constrains. Fortunately there is a result in Chung’s works, not explicitly stated, that avoid a repetition of his argument and computations for our linear limiting Carleman weight. We state this result in Lemma 5.9. This lemma could be interesting by itself for future applications. After deriving such estimate, a standard Hahn-Banach argument completes our proof. For a complete version of the proof of Proposition 1, we refer the reader to Appendix C in [16].

**Definition 5.4.** A real-valued function \( \varphi \) is called a limiting Carleman weight in \( \Omega \) if has nonvanishing gradient in \( \Omega \) and also satisfies
\[
\langle \varphi'' \nabla \varphi, \nabla \varphi \rangle + \langle \varphi'' \varrho, \varrho \rangle = 0,
\]
whenever \( |\varrho| = |\nabla \varphi| \) and \( \nabla \varphi \cdot \varrho = 0 \). Here \( \varphi'' \) and \( \langle \cdot, \cdot \rangle \) denote the Hessian matrix of \( \varphi \) and the usual inner product in \( \mathbb{R}^n \), respectively.

We mention that the limiting Carleman weights defined in any open subset of the \( \mathbb{R}^n \) are completely characterized. There are infinitely many of them, which can be classified in 6 categories as it was shown in [7]. Aside from this, for a large parameter \( \tau > 0 \) and a limiting Carleman weight \( \varphi \) in \( \Omega \), we set
\[
\mathcal{L}_{A,q,\varphi} = \tau^{-2} e^{\tau \varphi} \mathcal{L}_{A,q} e^{-\tau \varphi}.
\]
For \( \epsilon > 0 \), we set
\[
\mathcal{L}_{A,q,\varphi,\epsilon} = e^{\varepsilon^2/2 \tau} \mathcal{L}_{A,q,\varphi} e^{-\varepsilon^2/2 \tau}.
\]
For a bounded open subset \( V \subset \mathbb{R}^n \), we denote by \( H^1_{scl}(V) \) the \( H^1 \)-Sobolev space with semiclassical parameter \( \tau^{-1} \) and its dual space by \( H^{-1}_{scl}(V) \). Their norms are respectively defined by
\[
\|u\|_{H^1_{scl}(V)} = \|u\|_{L^2(V)} + \|\tau^{-1} \nabla u\|_{L^2(V)}
\]
and
\[
\|u\|_{H^{-1}_{scl}(V)} = \sup_{\psi \in C_0^\infty(V) \setminus \{0\}} \frac{|\langle u, \psi \rangle_V|}{\|\psi\|_{H^1_{scl}(V)}}.
\]
Here \( \langle \cdot, \cdot \rangle_V \) denotes the distribution duality in \( V \).

From here we divide the proof into two steps. For expository convenience, in the first step, we only state our Carleman estimate with linear weight. As a consequence, we prove the existence of CGO solutions with the desired support constraint on the boundary. In the second step, we formulate the hidden implicit result in [6] and how to use it in order to prove the Carleman estimate aforementioned in the first step.
First step. With the above notation and definition at hand, we have:

**Theorem 5.5.** Let \( A \in C^2(\Omega; \mathbb{R}^n) \) and \( q \in L^\infty(\Omega; \mathbb{R}) \). Let \( \xi \in S^{n-1} \) and set \( \varphi(x) = \xi \cdot x \). Suppose that \( \Omega' \) is a smooth domain with \( \Omega \subset \Omega' \) such that \( \partial \Omega' \cap \partial \Omega = E \), where \( E \) is a compact subset of \( \partial \Omega_{-0}(\xi) \) on \( \partial \Omega \). Then there exist two positive constants \( C \) and \( \tau_0 \) (depending on \( n, \Omega \) and priori bounds on \( A \) and \( q \)) such that the following estimate

\[
\tau^{-1} \| w \|_{L^2(\Omega)} \leq C \| \mathcal{L}_{A,q,\varphi} w \|_{H^{-1}_{sc}(\Omega')} , \quad w \in C_0^\infty(\Omega),
\]

holds true for all \( \tau \geq \tau_0 \).

By a standard Hahn-Banach argument (see for instance Proposition 9.1 in [6]), it follows immediately the next existence result.

**Proposition 8.** Assume the hypotheses of Theorem 5.5. Then for every \( v \in L^2(\Omega) \), there exists \( u \in H^1(\Omega) \) satisfying

\[
\begin{cases}
\mathcal{L}_{A,q,\varphi} u = v, & \text{in } \Omega, \\
|u|_{E} = 0.
\end{cases}
\]

Moreover, there exist two positive constants \( C \) and \( \tau_0 \) such that

\[
\| u \|_{H^1_{sc}(\Omega)} \leq C \tau \| v \|_{L^2(\Omega)}, \quad \tau \geq \tau_0.
\]

This result will be used to ensure the existence of a remainder term \( r \), which it is part of our solution. More precisely, our next task is to provide the necessary conditions for the functions \( a, r, l \) and \( b \) such that \( u \) of the form

\[
u = e^{\tau(\xi \cdot x + i\zeta \cdot x)}(a + r) - e^{\tau l} b
\]

belongs to \( H^1(\Omega) \) and satisfies

\[
\begin{cases}
\mathcal{L}_{A,q} u = 0 & \text{in } \Omega \\
|u|_{\partial \Omega_B} = 0,
\end{cases}
\]

where \( B \) is an open neighborhood of \( B_N \), see Section 2. In this way, a straightforward computation give us

\[
e^{-\rho} \mathcal{L}_{A,q}(e^\rho v) = \mathcal{L}_{A,q} v + (D\rho \cdot D\rho + D^2 \rho + 2A \cdot D\rho) v + 2D\rho \cdot Dv.
\]

Applying this formula twice, first with \( \rho(x) = \tau(\xi \cdot x + i\zeta \cdot x) \) and \( v = a + r \) and later with \( \rho(x) = \tau l(x) \) and \( v = b \), we deduce that \( u \) defined by (64) satisfies \( \mathcal{L}_{A,q} u = 0 \) in \( \Omega \) whenever the following identity holds true:

\[
0 = \tau^{-2} e^{-\tau(\xi \cdot x + i\zeta \cdot x)} \mathcal{L}_{A,q} u
\]

\[
= \tau^{-2} e^{-\tau(\xi \cdot x + i\zeta \cdot x)} \left[ \mathcal{L}_{A,q} (e^{\tau(\xi \cdot x + i\zeta \cdot x)}(a + r) - e^{\tau l} b) \right]
\]

\[
= \tau^{-2} \mathcal{L}_{A,q} a - 2\tau^{-1} [(\xi + i\zeta) \cdot \nabla a + i(\xi + i\zeta) \cdot A a]
\]

\[
+ \tau^{-2} e^{-\tau(\xi \cdot x + i\zeta \cdot x)} \mathcal{L}_{A,q} \left( e^{\tau(\xi \cdot x + i\zeta \cdot x)} r \right)
\]

\[
- e^{-\tau(\xi \cdot x + i\zeta \cdot x)} e^{\tau l} (\tau^{-2} \mathcal{L}_{A,q} b + Dl \cdot Db + \tau^{-1}(2Dl \cdot Db + \tau^{-2}(2Dl \cdot Dl + 2Dl \cdot A + D^2 l) b))
\]

This naturally suggests the following construction.

**Equation for \( a \).** In order to reduce the right-hand side of the above expression, we start by despising the first term of \( \tau^{-1} \) order. We consider \( a \) satisfying in \( \Omega \)

\[
(\xi + i\zeta) \cdot \nabla a + i(\xi + i\zeta) \cdot A a = 0.
\]
We try solutions of the form \( a = e^\phi \), so the function \( \Phi \) must satisfy

\[
(\xi + i\zeta) \cdot \nabla \Phi + i(\xi + i\zeta) \cdot A = 0 \quad \text{in } \Omega.
\]

This can be solved by means of the Cauchy transform as follows. Let \( \tilde{A} \) be a compactly supported extension in \( \mathbb{R}^n \) of \( A \) such that belongs to \( C^{2,\gamma}_{\xi}(\mathbb{R}^n) \) and satisfying

\[
\|\tilde{A}\|_{C^{2,\gamma}(\mathbb{R}^n)} \leq C_{10} \|A\|_{C^{2,\gamma}(\mathbb{R}^n)}.
\]

Then, applying Proposition 7 with \( (68) \), we deduce that \( \Phi := C_{\xi+i\zeta}(-i(\xi + i\zeta) \cdot A) \) belongs to \( C^{3,\gamma}(\mathbb{R}^n) \) with \( 0 < \gamma < \gamma \), satisfy in \( \mathbb{R}^n \)

\[
(\xi + i\zeta) \cdot \nabla \Phi + i(\xi + i\zeta) \cdot \tilde{A} = 0,
\]

and the following estimate

\[
\|\Phi\|_{C^{3,\gamma}(\mathbb{R}^n)} \leq C_{11} \|\tilde{A}\|_{C^{2,\gamma}(\mathbb{R}^n)}.
\]

In particular, the restriction \( \Phi|_{\Omega} \in C^{3,\gamma}(\Omega) \), it still denotes by \( \Phi \), is a solution of the equation \( 67 \). Hence, we deduce that \( a = e^\phi \) also belongs to \( C^{3,\gamma}(\Omega) \) and satisfies \( 66 \). The estimates \( 6-7 \) follow by combining \( 68-69 \) and Proposition 7. Note that \( a \) is independent of \( \tau \). The next paragraph is intended to motivate the equations that have to satisfy both \( l \) and \( b \).

By 65 and once we had proved the existence of \( a \), we deduce that \( R := e^{i\tau \zeta \cdot x} \) has to satisfy

\[
L_{A,q,-\varphi} R = -\tau^{-2} e^{i\tau \zeta \cdot x} \mathcal{L}_{A,q} a + e^{-\tau \zeta \cdot x} e^{\tau l} \left[ \tau^{-2} \mathcal{L}_{A,q} b + Dl \cdot Db \right.
\]
\[
\left. + \tau^{-1} \left( 2Dl \cdot Db + (2Dl \cdot A + D2l) b \right) \right].
\]

This equation for \( R \) can be solved by using Proposition 8 with \( E = \partial \Omega \setminus B \). In particular, \( \|R|_{\partial \Omega \setminus B} = 0 \) and so \( r|_{\partial \Omega \setminus B} = 0 \) (these facts will be verified later on). Since we would like to have \( u|_{\partial \Omega \setminus B} = 0 \), from 64 we deduce that

\[
(e^{\tau \zeta \cdot (x+i\zeta \cdot x)})|_{\partial \Omega \setminus B} = (e^{\tau l})|_{\partial \Omega \setminus B}.
\]

One way to achieve this condition is considering the functions \( l \) and \( b \) with the boundary conditions: \( l(x)|_{\partial \Omega \setminus B} = (\xi \cdot x + i\zeta \cdot x)|_{\partial \Omega \setminus B} \) and \( b|_{\partial \Omega \setminus B} = u|_{\partial \Omega \setminus B} \). Moreover, in order to have a decay of \( R \) in \( \tau \), we have to ensure the decay in \( \tau \) of the right-hand side of 70. It can be done by assuming that \( Dl \cdot Db \) and \( 2Dl \cdot Db + (2Dl \cdot A + D2l) b \) are small enough in a suitable sense.

**Equation for \( l \).** We claim that for every \( p \in \mathbb{N} \) there exist a function \( l \) satisfying

\[
\begin{aligned}
&\{ Dl \cdot Db = 0 \text{ (dist}(x, \partial \Omega \setminus B)^p) , \\
&l|_{\partial \Omega \setminus B} = (\xi \cdot x + i\zeta \cdot x)|_{\partial \Omega \setminus B} , \\
&((\partial_{\nu} l)|_{\partial \Omega \setminus B} = -\nu \cdot (\xi + i\zeta)|_{\partial \Omega \setminus B}.
\end{aligned}
\]

Indeed, first note that the function \( (\xi + i\zeta) \cdot x \) satisfies the first two conditions of 71. The reason to consider the third boundary condition is to avoid the repetition of this solution. We start by picking coordinates \((t, s)\) in a neighborhood of \( \partial \Omega \setminus B \), where \( t \) is the coordinate over \( \partial \Omega \setminus B \) and \( s \) is perpendicular to \( \partial \Omega \setminus B \) and then stands for \( \text{dist}(x, \partial \Omega \setminus B) \). Formally, we consider \( \tilde{l} \) as follows

\[
\tilde{l}(t, s) = \sum_{j=0}^{\infty} a_j(t)s^j,
\]
where the smooth functions $a_j$ can be determined by imposing

\[
\begin{aligned}
\nabla \tilde{l} \cdot \nabla \tilde{l} &= 0, \\
\tilde{l} |_{\partial \Omega \setminus B} &= (\xi \cdot x + i \zeta \cdot x)|_{\partial \Omega \setminus B}, \\
(\partial_{\nu} \tilde{l}) |_{\partial \Omega \setminus B} &= -\nu \cdot (\xi + i \zeta)|_{\partial \Omega \setminus B}.
\end{aligned}
\]

From the boundary conditions, it follows immediately that

\[
(72) \quad a_0(t) = (\xi \cdot x + i \zeta \cdot x)|_{\partial \Omega \setminus B}, \quad a_1(t) = -\nu \cdot (\xi + i \zeta)|_{\partial \Omega \setminus B}.
\]

By a recursive relation, we can determine $a_j$ for $j \geq 2$. In $(t, s)$-coordinates, the gradient of $\tilde{l}$ has the form

\[
\nabla \tilde{l} = \left( \sum_{j=0}^{\infty} \nabla_t a_j(t)s^j, \sum_{j=0}^{\infty} a_j(t)j s^{j-1} \right).
\]

Thus

\[
0 = \nabla \tilde{l} \cdot \nabla \tilde{l} = \sum_{m=0}^{\infty} \left( \sum_{j+k=m} \nabla_t a_j \cdot \nabla_t a_k + (j+1)(k+1)a_{j+1}a_{k+1} \right) s^m,
\]

which implies that

\[
(73) \quad \sum_{j+k=m} \nabla_t a_j \cdot \nabla_t a_k + (j+1)(k+1)a_{j+1}a_{k+1} = 0, \quad m \in \mathbb{N}.
\]

From this recursive formula, we have for $m \geq 1$

\[
(m+1)a_1 a_{m+1} = - \sum_{j+k=m} \nabla_t a_j \cdot \nabla_t a_k - \sum_{j+k=m} (j+1)(k+1)a_{j+1}a_{k+1}.
\]

Thus, to determine $a_{m+1}$ we need to verify that $a_1 \neq 0$ on $\partial \Omega \setminus B$. On the one hand, $\nu(x) \cdot \xi > 0$ on $\partial \Omega \setminus B$, since $\partial \Omega \setminus B \subset \partial \Omega_{-0}(\xi)$. On the other hand, $\partial \Omega \setminus B$ is a compact subset of the boundary, thus taking into account 72, we deduce that $|a_1| > \epsilon_0 > 0$. Hence, we can divide by $a_1$ the last above recursive identity to know $a_m$ for all $m \in \mathbb{N}$. From here, it is immediate to see that the $p$-truncation of the serie corresponding to $\tilde{l}$ give us a solution of 71. Indeed, consider the $p$-truncation of $\tilde{l}$ defined by

\[
(74) \quad l_p(t, s) = \sum_{j=0}^{p} a_j(t)s^j.
\]

By 73 and a straightforward computation, we obtain

\[
\nabla l_p \cdot \nabla l_p = \sum_{m=0}^{p-1} \left( \sum_{j+k=m} \nabla_t a_j \cdot \nabla_t a_k + (j+1)(k+1)a_{j+1}a_{k+1} \right) s^m + O(s^p)
\]

\[
= O(s^p).
\]

Moreover, by 72 we deduce that $l_p(t, 0) = (\xi \cdot x + i \zeta \cdot x)|_{\partial \Omega \setminus B}$ and $\partial_{\nu} l_p(t, 0) = -\nu \cdot (\xi + i \zeta)|_{\partial \Omega \setminus B}$; so $l_p$ satisfies 71. Thus, one gets...
\[ l_p(t, s) = a_0(t) + s \left[ a_1(t) + \sum_{j=1}^{p-1} a_{j+1}(t)s^j \right] \]

\[ = (\xi \cdot x + i\zeta \cdot x)|_{\partial \Omega \setminus B} + s \left[ -\nu \cdot (\xi + i\zeta)|_{\partial \Omega \setminus B} + \sum_{j=1}^{p-1} a_{j+1}(t)s^j \right]. \]

Hence, we have

\[ \Re l_p = \xi \cdot x|_{\partial \Omega \setminus B} - s \left[ \nu \cdot \xi|_{\partial \Omega \setminus B} - \sum_{j=1}^{p-1} \Re a_{j+1}(t)s^j \right]. \]

Taking \( s \) small enough and since \( \nu \cdot \xi|_{\partial \Omega \setminus B} > \epsilon_0 > 0 \), we conclude that \( \Re l_p = \xi \cdot x - s(x)k(x) \) with \( 0 < \rho_0 \leq k(x) \leq \rho_1 \) in a neighborhood of \( \partial \Omega \setminus B \). Then 8-9 are satisfied with \( k = sk \). Here \( \Re Z \) denotes the real part of the function \( Z \). Thus, we conclude that \( l = l_p \) satisfies 71. We emphasize that this solution is independent of \( \tau \).

**Equation for \( b \).** Once proved the existence of the function \( l \), we consider \( b \) being a solution of the equation

\[
\begin{align*}
2Dl \cdot Db + (2Dl \cdot A + D^2l)b &= O \left( dist(x, \partial \Omega \setminus B)^2 \right) \quad \text{in } \Omega, \\
b|_{\partial \Omega \setminus B} &= a|_{\partial \Omega \setminus B}.
\end{align*}
\]

We try a solution of the form

\[ b(t, s) = b_0(t) + b_1(t)s + b_2(t)s^2, \]

where the functions \( b_j \) will be determined later on. By boundary condition, we immediately deduce that \( b_0(t) = a|_{\partial \Omega \setminus B} \). It remains to determine \( b_1 \) and \( b_2 \). At this point, there is a slight difference with respect to the construction of \( l \). This is because the magnetic potential \( A \) has only integer derivative until the second order, so its Taylor series is not well-defined. For this reason, we consider its residual approximation until the second order in the \((t, s)\)-coordinates as follows

\[ A(t, s) = (A_0'(t) + A_1'(t)s + R_A'(t, s), A_0''(t) + A_1''(t)s + R_A''(t, s)), \]

where \( A_j' \) and \( R_A' \) are vector-valued functions in \( \mathbb{R}^{n-1} \), \( A_j'' \) and \( R_A'' \) are real-valued functions with \( j = 0, 1 \). Since \( A \in C^{2, \gamma}(\bar{\Omega}) \), in particular belongs to \( C^2(\bar{\Omega}) \), we deduce that

\[ (A_0', A_0'') \in C^2(\bar{\Omega}), \quad (A_1', A_1'') \in C^1(\bar{\Omega}), \quad (R_A', R_A'') \in C(\bar{\Omega}). \]

Moreover, by residual approximation, we have

\[ \|(R_A'(t, s); R_A''(t, s))\| \leq Cs^2, \]

where the constant \( C > 0 \) only depends on \( \Omega \) and \( \|A\|_{C^{2, \gamma}(\bar{\Omega})} \). Now, for a fixed \( p \in \mathbb{N} \), we consider \( l = l_p \) defined by 74. Thus, by an easy computation, we obtain

\[ 2Dl \cdot Db + (2Dl \cdot A + D^2l)b = d_0(t, s) + d_1(t, s)s + O(s^2), \]

where we have used 77 to get the term \( O(s^2) \). Here, the function \( d_0 \) and \( d_1 \) are defined by

\[
\begin{align*}
d_0(t, z) &= -2(\nabla_t a_0 \cdot \nabla_t b_0 + a_1 b_1) \\
&\quad - [2i(\nabla_t a_0 \cdot A_0' + a_1 A_0'') + \Delta_t a_0 + 2a_2] b_0
\end{align*}
\]
and
\[
d_1(t, z) = -2(\nabla_t a_0 \cdot \nabla_t b_1 + \nabla_t a_1 \cdot \nabla_t b_0 + 2a_1 b_2 + 2a_2 b_1)
\]
(80)
\[-2i(\nabla_t a_0 \cdot A_0' + a_1 A_0' + \Delta_t a_0 + 2a_2 b_1) b_1
\]
\[-2i(\nabla_t a_0 \cdot A_1' + \nabla_t a_1 \cdot A_0' + a_1 A_1' + 2a_2 A_0' + \Delta_t a_1) b_0.
\]
According to 78, the function \(b\) defined by 76 satisfies 75 whenever \(d_0 = d_1 = 0\). In this way, since \(|a_1| > \epsilon_0 > 0\) on \(\partial \Omega \setminus \Omega\), we can divide by \(a_1\) both sides of 79 to obtain \(b_1 \in C^{2,2}\). Once known \(b_1\), we divide by \(a_1\) now both sides of 80 to obtain \(b_2 \in C^{1,2}(\Omega)\).

**Equation for \(r\).** Finally, we prove the existence of \(r \in H^1(\Omega)\). We claim that \(w\) defined by
\[
w = -\tau^{-2}e^{i\tau \zeta x}L_{A,q} a - e^{-\tau \zeta x}e^{\tau l} \left[\tau^{-2}L_{A,q} b + Dl \cdot Dl b \right.
\]
\[+ \tau^{-1} (2Dl \cdot Db + (2Dl \cdot A + D^2 l)b) \]
belongs to \(L^2(\Omega)\). More precisely, \(\|w\|_{L^2(\Omega)} = O(\tau^{-2})\). Indeed, we divide the analysis in two cases.

**First case.** When \(dist(x, \partial \Omega \setminus B) \leq \tau^{-1/2}\). In this case, we consider 71 with \(p = 4\) to obtain
\[|Dl \cdot Dl| \leq Cs^4 \leq C\tau^{-2}.
\]
Therefore, by 75 we also obtain
\[|2Dl \cdot Db + (2Dl \cdot A + D^2 l)b| \leq Cs^2 \leq C\tau^{-1}.
\]
Since \(e^{-\tau \zeta x}e^{\tau l} = e^{-\tilde{k}} \leq 1\), we deduce that \(|w(x)| \leq C_{12}\tau^{-2} - 2\).

**Second case.** When \(dist(x, \partial \Omega \setminus B) > \tau^{-1/2}\). In this case, we consider 71 with \(p = 2\) to obtain
\[|Dl \cdot Dl| \leq C_{13}s^2.
\]
Analogously to the previous case, we also have
\[|2Dl \cdot Db + (2Dl \cdot A + D^2 l)b| \leq C_{14}s^2.
\]
On the one hand, since \(\tilde{k}(x) \simeq s(x) = dist(x, \partial \Omega \setminus B)\) and \(s > \tau^{-1/2}\), we have
\[|e^{-\tau \zeta x}e^{\tau l} \left[\tau^{-2}L_{A,q} b + Dl \cdot Dl b + \tau^{-1} (2Dl \cdot Db + (2Dl \cdot A + D^2 l)b) \right] - C_{15}e^{-\tau \kappa}(\tau^{-2} + s^2 + \tau^{-1}s^2) \leq C_{16}\tau^{-2}(\tau^{-2} + s^2 + \tau^{-1}s^2) \leq C_{17}\tau^{-2}.
\]
On the other hand, we have
\[|\tau^{-2}e^{i\tau \zeta x}L_{A,q} a| \leq C_{18}\tau^{-2}.
\]
Combining these inequalities, we deduce that \(|w(x)| \leq C_{18}\tau^{-2}\). This proves the claim. Hence, according to Proposition 8, there exists \(R \in H^1(\Omega)\) satisfying \(L_{A,q} - \phi R = w\), \(R|_{\partial \Omega \setminus B} = 0\) and the following estimate
\[\|R\|_{H^2(\Omega)} \leq C_{11} \tau \|w\|_{L^2(\Omega)} \leq C_{12} \tau^{-1}.
\]
It is a simple matter to verify that \(r = e^{-i\tau \zeta x}R\) belongs to \(H^1(\Omega)\) and satisfies 11. This completes the proof of Proposition 1. As it was already mentioned, we now turn to give the main ideas to prove Theorem 5.5.

**Second step.** We briefly outline the main steps followed by Chung [6] to prove a Carleman estimate with a logarithmic weight, see his Theorem 1.4.
A1. By means of a change to spherical coordinates, the starting point is the derivation of a Carleman estimate for a particular case, when \( \Omega \) lies entirely in the upper part of a region determined by the graph of a positive real-valued smooth function \( f \) defined in \( S^{n-1} \) and such that \( E \) is a subset of the graph of \( f \). Moreover, \( f \) is small enough in a suitable sense, see his Proposition 3.1.

A2. The smallness condition over \( f \) is removed while still maintaining that \( E \) is a subset of its graph. This is done by means of a partition of unity argument applied to a finite open cover of \( \Omega \), see his Proposition 8.1.

A3. Finally, by combining the previous results with arguments of compactness and cutoff functions, the graph condition over \( E \) is removed and so the proof of Theorem 1.4 is completed, see Section 8 in [6].

Let us explain with more details the step A1. Under a standard change of coordinates, Chung first flatten out the boundary of \( \Omega \), in particular the set \( E \), to work in \( \mathbb{R}^{n+1}_+ = \{ (\theta, r) \in \mathbb{R}^{n-1} \times \mathbb{R} : r \geq 1 \} \). Thus, he obtained a new second order operator defined by 3.5 in [6], instead of \( \mathcal{L}_{0,0,\varphi,\epsilon} \). Due to the following result obtained by Dos Santos Ferreira et al. [8], see their Equation 2.12; the estimate 3.4 is satisfied for every limiting Carleman weight, in particular it holds for our linear case.

**Proposition 9.** If \( \varphi \) is a limiting Carleman weight in \( V \), an open subset of \( \mathbb{R}^n \), then there exist two positive constant \( \tau_0 \) and \( C \) such that the following estimate

\[
\frac{\tau^{-1}}{\sqrt{\epsilon}} \| w \|_{H^1_{\tau,\epsilon}(V)} \leq C \| \mathcal{L}_{0,0,\varphi,\epsilon} w \|_{L^2(V)}, \quad \tau \geq \tau_0
\]

holds true for every \( w \in C_0^\infty(V) \).

Now we return to [6]. We have noted that Lemma 4.1 and Lemma 4.2 hold true whenever the conditions 3.4-3.8 are satisfied. Combining these lemmas and changing coordinates back to the original ones, Chung ends the proof of Proposition 3.1, see pages 128-130. Just before changing back coordinates, we have detected the implicit result. This is a Carleman estimate in \( \mathbb{R}^{n+1}_+ \) contained on page 130. It can be stated as follows.

**Lemma 5.6.** Let \( \mathcal{L}_{\tau,\epsilon} \) be a second-order semiclassical operator on

\[
\mathbb{R}^n_{+1} = \{ (\theta, r) \in \mathbb{R}^{n-1} \times \mathbb{R} : r \geq 1 \}
\]

of the form

\[
\mathcal{L}_{\tau,\epsilon} = \left( 1 + |F|^2 \right) \tau^{-2} \partial_r^2 - \frac{2}{r} \left( a_{\tau,\epsilon} + G \cdot \tau^{-1} \nabla_\theta \right) \tau^{-1} \partial_r
\]

\[
+ \frac{1}{r^2} \left( a_{\tau,\epsilon}^2 + \tau^{-2} L_\theta \right),
\]

where \( F \) and \( G \) are smooth vector fields, \( \partial_r \) denotes the partial derivative with respect to \( r \), \( \nabla_\theta \) denotes the gradient operator with respect to \( \theta \), \( a_{\tau,\epsilon} \) is a smooth real-valued function and \( L_\theta \) is a second-order differential operator of the form

\[
L_\theta = a_1 \partial_{\theta_1}^2 + a_2 \partial_{\theta_2}^2 + \ldots + a_{n-1} \partial_{\theta_{n-1}}^2 + \text{first and zero order terms.}
\]

Here \( \theta = (\theta_1, \ldots, \theta_{n-1}) \) and \( (a_j)_{j=1}^{n-1} \) are smooth real-valued functions. Let \( U \) and \( U_2 \) be two bounded open sets on \( \mathbb{R}^n_{+1} \setminus \{ e_n \} \) (\( e_n \) denotes the n-th canonical unit vector of \( \mathbb{R}^n \)) with smooth boundaries such that \( U \subseteq U_2 \) and \( \emptyset \neq \partial U \cap \partial U_2 \subseteq \partial \mathbb{R}^n_{+1} \). Finally, assume that there exist positive constants \( C, \tau_0, \delta \) small enough; and \( K \in \mathbb{R}^n \) such
that:

(i) The operator $L_{\tau,\epsilon}$ satisfies the following estimate

$$\tau^{-1/\sqrt{\epsilon}} \|w\|_{H^{1}_{\text{ext}}(U_2)} \leq C \|L_{\tau,\epsilon}w\|_{L^2(U_2)}, \quad \tau \geq \tau_0,$$

for all $w \in C_0^\infty(U_2)$.

(ii) The coefficients of the operator $L_{\tau,\epsilon}$ satisfy

$$|G - K| \leq \delta, \quad |F - K| \leq \delta, \quad |a_j - 1| \leq \delta, \quad j = 1, \ldots, n - 1,$$

in $U_2$, and

$$|a_{\tau,\epsilon} - 1| \leq C \frac{\tau^{-1}}{\epsilon}, \quad \text{in } U_2,$$

for all $\tau \geq \tau_0$. Then there exist two positive constants $C_1$ and $\tau_1$ such that

$$\tau^{-1/\sqrt{\epsilon}} \|w\|_{L^2(U)} \leq C_1 \|L_{\tau,\epsilon}w\|_{H^{1}_{\text{ext}}(\mathbb{R}^{n+1})}, \quad \tau \geq \tau_1,$$

holds true for all $w \in C_0^\infty(U)$.

Using this result and following Chung’s ideas according to step A1, we first obtain an initial Carleman estimate for our linear case, which is analogous to Proposition 3.1 in [6].

**Proposition 10.** Consider the linear limiting Carleman weight in $\Omega$, $\varphi(x) = \xi \cdot x$ with $\xi \in S^{n-1}$. Let $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ be a smooth function such that

$$E \subset \{ x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n = f(x') \}$$

and $\Omega \subset \Xi_f$, where $\Xi_f$ is defined by

$$\Xi_f := \{ x = (x', x_n) : x_n \geq f(x') \}.$$

Additionally, assume that there exist $K \in \mathbb{R}^n$ and $\delta > 0$ such that

$$|\nabla x' \cdot f - K| < \delta.$$

Then there exist two positive constants $\tau_0$ and $C > 0$ (both depending on $K$, $\delta$, $f$ and $\Omega$), such that the following estimate

$$\tau^{-1/\sqrt{\epsilon}} \|w\|_{L^2(\Omega)} \leq C \|L_{0,0,\varphi,\epsilon}w\|_{H^{1}_{\text{ext}}(\Xi_f)}, \quad \tau \geq \tau_0,$$

holds true for all $w \in C_0^\infty(\Omega)$.

**Proof.** Without loss of generality, we assume that $\xi = e_n$, and hence $\varphi(x) = x_n$. For the general case, it is enough to consider a rotational transformation linear leading $\xi$ to $e_n$. In order to apply Lemma 5.6, we first flatten out $E$ into the hyperplane $\mathbb{R}^{n+1}_{\tau,1}$. To do that, we make two changes of variables $\Upsilon_1$ and $\Upsilon_2$ defined by:

$$\Upsilon_1 : \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}_{+0}, \quad \Upsilon_1(x', x_n) = (x', x_n - f(x'))$$

and

$$\Upsilon_2 : \mathbb{R}^{n+1}_{+0} \rightarrow \mathbb{R}^{n+1}_{+1}, \quad \Upsilon_2(x', x_n) = (x', e_{x_n}) := (x', r),$$
bounded sets in $\mathbb{R}^n_{+0} = \{(x',x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n \geq 0\}$. By a straightforward computation, in $(x',r)$-coordinates the operator $L_{0,0,\varphi,\epsilon}$ has the form:

$$L_{0,0,\varphi,\epsilon,Y_1,Y_2} := r^2 L_{\tau,\varphi,Y_2 \circ Y_1} + \tau^{-1} J_{Y_2 \circ Y_1},$$

where

$$L_{\tau,\varphi,Y_2 \circ Y_1} = \left(1 + |\nabla x'|^2\right)^{-\frac{1}{2}} \frac{r}{r} (\alpha_{Y_2 \circ Y_1} + \nabla x' \cdot \tau^{-1} \nabla x') \tau^{-1} \partial_r + \frac{1}{r^2} (\alpha_{Y_2 \circ Y_1} + \tau^{-2} \Delta x')$$

with

$$\alpha_{Y_2 \circ Y_1} = 1 + \frac{\tau^{-1}}{\epsilon} (\log r + f(x'))$$

and $J_{Y_2 \circ Y_1}$ is a first-order operator defined by

$$J_{Y_2 \circ Y_1} = \tau^{-1} r \Delta x' f \partial_r - \frac{\tau^{-1}}{\epsilon}.$$

**Claim.** There exist two positive constants $C_2$ and $\tau_2$ such that

$$\frac{\tau^{-1}}{\epsilon} \|w\|_{L^2(Y_2 \circ Y_1(\Omega))} \leq C_2 \|L_{\tau,\varphi,Y_2 \circ Y_1} w\|_{H^{-1}_{scl}(\mathbb{R}^n_{+0})}, \quad \tau \geq \tau_2,$$

holds true for all $w \in C_0^\infty(Y_2 \circ Y_1(\Omega))$. Indeed, on the one hand, $L_{\tau,\varphi,Y_2 \circ Y_1}$ is a second-order semiclassical operator of the form $82$ with $x'$ instead of $\theta$ and

$$F = G = \nabla x' f, \quad a_{\tau,\epsilon} = \alpha_{Y_2 \circ Y_1}, \quad L\theta = \Delta x'.$$

Thus $a_j = 1$ for all $j = 1,2,\ldots,n-1$. On the other hand, we consider the open bounded sets in $\mathbb{R}^n_{+1} \setminus \{(0,1)\}: U = (Y_2 \circ Y_1)(\Omega)$ and $U_2 = (Y_2 \circ Y_1)(\Omega')$. It is evident that both satisfy

$$U \subset U_2, \quad \emptyset \neq \partial U \cap \partial U_2 \subset \partial \mathbb{R}^n_{+1}.$$

By hypothesis and with the previous identification at hand, the conditions $84-87$ are immediately satisfied. It only remains to verify $83$. To do that, we apply Proposition $9$ with $V = \Omega'$ to obtain

$$\frac{\tau^{-1}}{\epsilon} \|v\|_{H^{1}_{scl}(\Omega')} \leq C_3 \|L_{0,0,\varphi,\epsilon} v\|_{L^2(\Omega')}, \quad v \in C_0^\infty(\Omega').$$

From $90$, it is immediate to see that

$$\frac{\tau^{-1}}{\epsilon} \|J_{Y_2 \circ Y_1} w\|_{L^2(U_2)} \leq C_4 \|w\|_{H^1_{scl}(U_2)}, \quad w \in C_0^\infty(U_2).$$

For a fixed $w \in C_0^\infty(U_2)$, we set $v = (Y^{-1} \circ Y^{-2})(w) \in C_0^\infty(\Omega')$. Then

$$\|L_{0,0,\varphi,\epsilon,Y_1,Y_2} w\|_{L^2(U_2)} \simeq \|L_{0,0,\varphi,\epsilon} v\|_{L^2(\Omega')} \simeq \|w\|_{H^1_{scl}(U_2)} \simeq \|v\|_{H^1_{scl}(\Omega')}.$$

Hence, combining this fact with $89$ and $92-93$, we obtain

$$\|L_{\varphi,\epsilon,Y_1 \circ Y_2} w\|_{L^2(U_2)} \geq \frac{\tau^{-1}}{\epsilon} \|v\|_{H^1_{scl}(\Omega')} \geq C_8 \frac{\tau^{-1}}{\epsilon} \|w\|_{H^1_{scl}(U_2)}.$$
where in the last line we have taken $\epsilon$ small enough. Thus, (83) is verified and Claim follows by applying Lemma 5.6 to $L_{\tau, \epsilon, \Upsilon_2 \circ \Upsilon_1}$. Now combining (89) and (91), it follows that

$$\frac{1}{\sqrt{\epsilon}} \|w\|_{L^2(\Upsilon_2 \circ \Upsilon_1(\Omega))} \leq C_{10} \|L_{0,0,\varphi,\epsilon,\Upsilon_1,\Upsilon_2} w\|_{H^{-1}_{scl}(\mathbb{R}^N_{+1})}, \ w \in C_0^\infty(\Upsilon_2 \circ \Upsilon_1(\Omega)).$$

Changing back coordinates to the original ones, the proof is complete.

Finally, we mention that similar arguments employed in steps A2 and A3 also work in our linear case. That is, we can first remove the smallness condition for the function $f$ and later, the graph condition for the set $E$. This completes the proof of Theorem 5.5.

Acknowledgments. We would like to thank Pedro Caro, Yavar Kian and Mikko Salo for fruitful conversations, comments and suggestions. Leyter PM would also want to thank Mikko Salo for his hospitality during his time in Jyväskylä-Finland, as well as Iason Efraimidis for his valuable help with the presentation of the first version of this manuscript. We also thank anonymous referees for very helpful suggestions and comments on a previous version of this work. This article is part of the PhD dissertation of Leyter PM and it is supported by the Project MTM2011-28198 of Ministerio de Economía y Competitividad de España.

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Received July 2016; 1st revision March 2017; 2nd revision February 2018.

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