On bipartite consensus of linear MASs with input saturation over directed signed graphs: Fully distributed adaptive approach

Xiaoya Nan1 | Yuezu Lv2 | Zhisheng Duan1

Abstract
This paper studies the bipartite consensus problem with input saturation for general linear represents multi-agent systems (MASs) with signed digraphs. Based on relative state information among neighbour agents, distributed adaptive protocols and a compensation observer are proposed, wherein both leaderless bipartite consensus and tracking bipartite consensus problems are addressed. For the case only relative output information is available, observer-based distributed adaptive protocols are also designed, where for each agent, a local observer is given to estimate signed consensus error and a distributed observer is presented to achieve bipartite consensus and generate control input and a compensation observer is designed to handle input saturation. Simulation illustrations are given to explain the feasibility. The control protocols presented in this article depend on only local relative state or output information, without any eigenvalue information of Laplacian matrices associated with signed digraphs, which can be practically used for each agent in a fully distributed way.

1 | INTRODUCTION

Recently, the consensus problem of MASs has attracted wide study for its wide applications [1, 2]. The consensus problem can be partitioned into leaderless and tracking consensus problem from the graph with leader or not.

Plenty of researchers have studied the consensus problem in various aspects. The consensus problem is studied in [3] for heterogeneous MASs in directed communication graphs, in which both first- and second-order cases are discussed. Ren extends some consensus works for second-order integrator dynamics in several different cases in [4]. Finite-time leaderless and tracking consensus for MASs with second-order integrator are considered in [5] via the proposed linear matrix inequality (LMI) distributed coordination protocols. The continuous and discrete time consensus conditions of high-order general linear systems over random switched topologies are introduced in [6]. Distributed leaderless and tracking consensus for linear MASs are discussed in [7] by adaptive protocols, and the results are developed to directed graph in [8]. Yu et al. [9] discuss the tracking problem for linear undirected MASs with exogenous interferences. Two types of optimization approaches are introduced in [10, 11]. Duan et al. introduce the state estimation problem for discrete nonlinear system via sensor network in [12]. Distributed consensus of nonlinear MASs and external interferences is achieved in [13] via the proposed fixed-time nonlinear control protocols. The high-order leaderless consensus problem for MASs with Lipschitz nonlinear entries is considered in [14]. The leaderless consensus control problem for MASs with quadratic inner boundedness nonlinear dynamics and one-side Lipschitz are addressed in [15] based on relative state information.

What should be pointed out is that the previous consensus studies primarily focused on MASs with a unity interaction between nodes such as cooperative. However, in social relationships, two or more different relationships and interactions are allowed among agents, such as competition and cooperation network with two different interactions or averager–copier–voter network with three types agents, see [16–18]. Taken the consensus of averager–copier–voter network as an example, Shang studies the case with three types agents walking random on the fixed communication graph in [18], where averagers average the state of its neighbours, while copiers and voters copy and vote stochastically from their neighbours’ information.
This paper is devoted to bipartite consensus problems over competition and cooperation network as described in [16, 17]. Some necessary conditions are discussed in [19] to achieve uniform bipartite tracking consensus for first-order integrators under switching directed signed topology. Meng et al. [20] provide finite-time consensus with undirected signed graph under first-order case, and Wang et al. [21] discuss four cases on finite-time bipartite consensus with signed digraph under first-order and second-order systems. Zhang studies output feedback protocols in [22] to solve bipartite consensus problem, and the results are extended in [23], in which the bipartite consensus by the proposed control laws based on state and output information with signed graph is solved. What should be pointed out is that, Zhang et al. present an equivalence between bipartite consensus problems and the conventional consensus problems by gauge transformation in [23]. In [24], Hu et al. consider average bipartite consensus of linear MASs with signed interactions and communication noises. The bipartite consensus of signed directed topology with time delay is discussed in [25]. The input saturation is further considered in [26] to achieve saturated bipartite consensus for linear MASs. Moreover, the asymmetric bipartite consensus problem is discussed for MASs associated with signed strongly connected digraph in [27].

It should be noticed that the aforementioned protocols all rely on the connectivity of Laplacian matrix, requiring certain global information about the entire signed graph when implementing in practice. Such limitation can be conquered by introducing adaptive control method to propose fully distributed adaptive protocols. The fully distributed adaptive protocols for cooperative consensus problem can be found in [7] for undirected graphs and [8, 28, 29] for directed graphs. The bipartite adaptive consensus tracking problem of second-order integrator systems with bounded interferences is discussed in [30]. In [31], Wen et al. investigate a distributed bipartite tracking consensus problem with a leader based on relative state information, in which the signed graph is undirected. The researchers consider the adaptive bipartite consensus problem with external disturbances under undirected graph in [32, 33]. And fully distributed adaptive bipartite consensus protocols for directed signed graphs can be easily derived from the adaptive protocols in [8, 28, 29] by using the gauge transformation to transfer the adaptive bipartite consensus problem under directed signed graphs into the equivalent adaptive consensus problem under directed graphs. In addition, it is rarely to see literature on bipartite consensus with input saturation. In [26], the input saturation is considered for bipartite consensus of linear MASs with state feedback, where global information is still needed.

This paper intends to solve the bipartite consensus with input saturation over fully distributed adaptive protocols under signed digraphs. Both the leaderless consensus and the leader–follower tracking problems with input saturation are considered in this paper. Based on relative state information, we first provide a distributed adaptive state feedback protocol to realize bipartite consensus for strongly connected signed digraphs, where a compensation observer is introduced for each agent to tackle the effect of input saturation. The leader–follower bipartite consensus tracking problem for signed digraph containing a directed spanning tree is also well resolved by the modified distributed adaptive saturated protocol. For the case only relative output information is available among neighbour agents, local observer, compensator and distributed observer are designed separately to estimate the consensus error and achieve bipartite consensus. To illustrate the effectiveness of the proposed protocols, novel properties of the signed Laplacian matrix associated with the structural balanced graph are presented, which make the demonstration of the achievement of bipartite consensus much more simplified.

The main contributions of this paper, compared with existing literature on bipartite consensus, lies in at least two aspects. First, different from the most existing bipartite consensus literature, this paper considers the bipartite consensus problem from a novel perspective. Specifically, the gauge transformation is no longer used to the bipartite consensus achievement in this paper. Instead, two properties on signed Laplacian matrix are presented from gauge transformation, which play key roles in subsequent proofs.

Second, compared with the existing results on bipartite consensus [23, 26, 31], the proposed bipartite control protocols are applicable to general directed graphs, without using any global eigenvalue information of Laplacian matrix. Besides, it appears the first to present fully distributed adaptive control protocols of bipartite consensus with input saturation under signed digraphs.

The remainder of this paper is as follows. The notations, preliminaries and some useful propositions are stated in Section 2. The fully distributed state feedback adaptive bipartite control protocols for MASs with leaderless and leader–follower topologies are proposed in Section 3. The relative output feedback adaptive protocols for signed digraph are presented in Section 4. Section 5 gives simulation illustrations in two different cases. Finally, the conclusion of full paper is summarized in Section 6.

2 PROBLEM FORMULATIONS

Notation: $\mathbb{R}^n$ means a set of $n$-dimensional real matrices. $\mathbb{R}^{\mathbb{N}}$ denotes a set of $n \times m$ real matrices. And $I_n$ is the $n$th-order identity matrix. $\mathbb{1}_n = [1, 1, ..., 1]^T \in \mathbb{R}^n$ means the vector with each of this entries equaling to 1, where the superscript $T$ is the transpose of the matrix. $\otimes$ is the Kronecker product. $\cdot | \cdot$ is the absolute value of a constant. $\|Q\|$ denotes the 2-norm of matrix $Q$. The matrix inequality $A > 0$ represents the matrix $A$ is positive definite and the inequality $A \succeq B$ means that $A$ and $B$ are both symmetric matrices and $A - B > 0$. $\mathcal{M} = [m_{ij}]_{n \times n} \in \mathbb{R}^{n \times n}$ is called a nonsingular $M$-matrix, if each eigenvalue of $M$ has positive real part, and $m_{ij} < 0$, for any $i \neq j$.

Consider a class of linear MASs with $N$ agents. The dynamics of $i$th agent are given by

$$
\dot{x}_i = A x_i + B \delta (\eta_i) \\
y_i = C x_i, \ i = 1, ..., N
$$

(1)

where $x_i \in \mathbb{R}^n$ represents the system state, $y_i \in \mathbb{R}^m$ represents the system output and $\eta_i \in \mathbb{R}^r$ represents the system control input, $A$, $B$ and $C$ are the coefficient matrices with proper
dimension $\delta_\Delta(\cdot)$ is a saturation function as follows:

$$\delta_\Delta(u_{ij}) = \begin{cases} \text{sgn}(u_{ij})\Delta, & \text{if } |u_{ij}| > \Delta \\
0, & \text{if } |u_{ij}| \leq \Delta \end{cases}$$

and $\delta_\Delta(u_i) \triangleq [\delta_\Delta(u_{i1}), \ldots, \delta_\Delta(u_{iN})]^T$.

For linear MASs, the following assumption is needed.

Assumption 1 ([36]). The pair $(A,B)$ satisfies ABCBC condition, i.e., $(A,B)$ is stabilizable and $A$ has no eigenvalue in the open right-half plane.

The signed graph among the agents is represented by a signed directed graph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ is a node set of $G$, and $\mathcal{E} = \mathcal{V} \times \mathcal{V} = \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\}$ is an edge set of $G$, $(v_i, v_j)$ represents a directed edge from $v_j$ to $v_i$, $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ represents the adjacency weight matrix of signed graph, and $a_{ij} \neq 0$ is equivalent to $(v_i, v_j) \in \mathcal{E}$, where $a_{ij}$ may be positive or negative. Signed graph is used to describe competitive and cooperative network, where positive adjacency weights represent cooperation interaction between the two linked agents, and negative weights represent the competition interaction. Specially, if all entries of $\mathcal{A}$ are nonnegative, the topology degenerates into ordinary graph. The Laplacian matrix of signed graph $G$ is denoted as $L_s = [l_{ij}] \in \mathbb{R}^{N \times N}$, and defined as $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}, i \neq j$.

Definition 1 ([16]). A signed topology graph $G(A)$ is called structurally balanced if it admits a bipartition of the nodes $\mathcal{V}_1, \mathcal{V}_2$ satisfying $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ such that $a_{ij} \geq 0, \forall v_i, v_j \in \mathcal{V}_1, (q \in \{1,2\})$ and $a_{ij} \leq 0, \forall v_i, v_j \in \mathcal{V}_2, (\ell \in \{1,2\}, \ell_1 \neq \ell_2)$. Otherwise, we believe it structurally unbalanced.

Remark 1. A structural balanced graph means that the nodes in the graph can be partitioned into two groups, where the nodes on the same group cooperate with each other in the sense that the edges are of positive weights, and the nodes between these two groups compete with each other in the sense that the corresponding edges are of negative weights.

Definition 2. A directed topology graph is said to be strongly connected if there always exists a directed path from $v_i$ to $v_j$ for all $v_i \neq v_j \in \mathcal{V}$.

The signed graph $G$ in this paper is required to fulfill the following assumption.

Assumption 2. The signed graph $G$ is structurally balanced.

Denote $D = \{D = \text{diag}(d_1, \ldots, d_N), d_i \in \{\pm 1\}\}$ the set of gauge transformation matrices.

Lemma 1 ([16]). A signed graph $G(A)$ is said to be structurally balanced if and only if $\exists D \in D$ such that $DAD$ has all nonnegative entries.

$$L_D = DL_AD$$ is a new Laplacian matrix of the gauge transformed system. Denote $L_{Dij} = [l_{Dij}]$, where $l_{Dij} = \sum_{j=1}^{N} |a_{ij}|$, $l_{ij} = -d_id_j, i \neq j$.

Lemma 2 ([34]). For a strongly connected topology $G$, there exists a diagonal matrix $R = \text{diag}(r_1, r_2, \ldots, r_N)$ such that the new Laplacian matrix $L = RL + L^T R$ is symmetric. Let $r = [r_1, r_2, \ldots, r_N]^T$ be the left zero feature vector of $L$, i.e., $r^T L = 0$. Moreover, $\min_{\xi, x} \xi^T L_s x \geq \lambda_2(L_s) \|x\|^2$, where $\lambda_2$ denotes the second smallest eigenvalue of $L_s$ and $\xi$ is any vector with positive entries.

Proposition 1. For any strongly connected signed topology $G$ with $L_s \in \mathbb{R}^{N \times N}$ as the associated signed Laplacian matrix, there exists a positive diagonal matrix $R = \text{diag}(\xi_1, \xi_2, \ldots, \xi_N)$ such that $\min_{\xi, x} \xi^T L_s x \geq \lambda_2(L_s) \|x\|^2 > \hat{\lambda}_2(L_s) \|x\|^2$, where $L_s = RL_s + L_s^T R$, and $\hat{\lambda}_2$ is the left zero feature vector of $L_s$. The proposition can be proved by the following procedure.

Proof. Based on Remark 1, we know that $L_{D} = DL_{D}sD$ is a Laplacian matrix of a strongly connected cooperative topology. Using Lemma 1, we can find a positive definite diagonal matrix $R = \text{diag}(r_1, \ldots, r_N)$ such that for any $\xi$ with positive entries, $\min_{\xi, x} \xi^T L_s x \geq \lambda_2(L_s) \|x\|^2$, where $\lambda_2 = \lambda_2(L_s)$. Let $R = \text{diag}(r_1, \ldots, r_N)$ denote the left zero feature vector of $L_s$, and $\|x\|^2 = \xi$ denotes any vector with positive entries.

Lemma 3 ([35]). For a nonsingular $M$-matrix $M$, there exists a positive definite diagonal matrix $G = \text{diag}(g_1, \ldots, g_N)$ such that $GM + M^T G > 0$.

For the leader–follower signed topology containing a directed spanning tree where the leader is the root node, we have $L_{D} = DL_{D}sD$, and $L_{D1}$ is a nonsingular $M$-matrix. We can write the signed Laplacian matrix as $L_s = \begin{bmatrix} 0 & 0 \\ L_{D2} & L_{D1} \end{bmatrix}$, and we have Proposition 2.

Proposition 2. There exists a diagonal positive matrix $G = \text{diag}(g_1, \ldots, g_N)$ such that $G_{DL} + L_{D1}^T G > 0$.

Proof. Since $L_{D1} = D_1L_{D1}D_1$ is a nonsingular $M$-matrix, there is a diagonal positive definite matrix $G = \text{diag}(g_1, \ldots, g_N)$ satisfying $GL_{D1} + L_{D1}^T G > 0$, which can be rewritten as $G_{DL}L_{D1} + L_{D1}^T L_{D1} > 0$. Since $D_1 = D_1^{-1}$ and $L_{D1}^T L_{D1} = D_1$, we can take the congruent transformation to the matrix inequality: $D_1G_{DL}D_1 + L_{D1}^T D_1G_{DL}D_1 > 0$. Let $G \equiv D_1G_{DL}D_1 = G_{DL}$, we have $G_{DL} + L_{D1}^T G > 0$.
This paper intends to propose fully distributed adaptive protocols to solve the bipartite consensus of the agents with input saturation in the sense that \( \lim_{t \to \infty} \| x_i - \text{sgn}(a_{ij}) x_j \| = 0 \), \( \forall i, j = 1, \ldots, N \), where only relative state information \( x_i - \text{sgn}(a_{ij}) x_j \) or relative output information \( y_i - \text{sgn}(a_{ij}) y_j \) among neighbouring agents can be used.

**Remark 3.** As is mentioned in [28], in many circumstance such as the agents equipped with radar or ultrasonic sensors to receive neighbourhood information, only relative information can be obtained. On the other hand, the agents can calculate the relative information if they can measure their absolute information directly and send to their neighbours. Therefore, it is more meaningful to study the bipartite consensus with relative information.

The following lemma presents a multi-level saturation algorithm to tackle the input saturation.

**Lemma 4 ([36]).** For a system \( \dot{x} = A x + B \delta(x + g(t)) - g(t) \) satisfying ANCBC condition, where \( f(x) \) is a multilevel saturation feedback controller satisfying designed as Algorithm 1. Then, \( z(t) \) can be ensured to be bounded and \( z(t) \in L_2 \) if \( g(t) \in L_2 \).

## 3 Bipartite Consensus with Input Saturation over State Feedback Adaptive Protocols

In this section, we will consider bipartite consensus with relative state feedback adaptive protocols of linear MASs over leaderless or leader–follower signed graphs.

### 3.1 Leaderless Bipartite Consensus

For the leaderless signed graph, we have the following assumption.

**Assumption 3.** The signed graph \( G \) is strongly connected.

To achieve the leaderless bipartite consensus in (1), the following distributed adaptive bipartite consensus protocol is designed based on neighbouring agents’ relative information:

\[
\dot{u}_i = f(w_i) + (b_i + \varphi_i) E \Gamma (z_i - \eta_i)
\]

\[
\dot{w}_i = A w_i + B \delta \left( z_i - \eta_i \right) - (b_i + \varphi_i) E \Gamma (z_i - \eta_i)
\]

\[
b_i = (z_i - \eta_i)^T \Gamma (z_i - \eta_i)
\]

(2)

where \( \xi_i = \sum_{j=1}^{N} |a_{ij}| (x_i - \text{sgn}(a_{ij}) x_j) \) is the bipartite consensus error, \( b_i \) is the coupling time varying weight of \( i \)th agent satisfying \( b_i(0) \geq 0 \), \( w_i \) is a designed compensation observer and denote \( \eta_i = \sum_{j=1}^{N} |a_{ij}| (w_i - \text{sgn}(a_{ij}) w_j) \). \( E \) and \( \Gamma \) are feedback gain matrices, and \( \varphi_i \) is a smooth function defined later.

**Algorithm 1 Multi-level Saturation Feedback [36]**

1. **Linear transformation.** For a system \( \dot{z} = A z + B \delta(x + g(t)) - g(t) \) satisfying ANCBC condition, make a linear transformation \( T \) on \( z \) to obtain the variable \( \tilde{z} = T z \) with \( \ddot{\tilde{z}} = \ddot{z} + B \delta (f(x + g(t)) - g(t)) \), where \( A = \dot{T} T^{-1} = \text{diag}(f_1, \ldots, f_N) \) is the Jordan canonical form of \( A \) with \( f_i \in \mathbb{R}^p, i \leq q \) being blocks associated with zero real part eigenvalues of \( A \) satisfying \( m_1 \geq \cdots \geq m_q \), and \( x \in \mathbb{R}^{Np} \), including all the Jordan blocks of negative real part eigenvalues of \( A \).

2. **System reformulation.** Reformulate the system \( \ddot{\tilde{z}} = \ddot{z} + B \delta (f(x + g(t)) - g(t)) \) into \( \ddot{\tilde{z}} = \ddot{z} + B \delta (f(x + g(t)) - g(t)) \) with \( \tilde{z}_i = \begin{bmatrix} z_1 \cdots z_{m_i} \\ \vdots \\ z_{m_i} \end{bmatrix} \) with \( m_i \) is null if \( m_j - i \leq 0 \). We have \( \ddot{\tilde{z}} = \ddot{z} + B \delta (f(x + g(t)) - g(t)) \) with

\[
\begin{bmatrix}
\ddot{z}_i \\
\vdots \\
\ddot{z}_m
\end{bmatrix} = \begin{bmatrix}
\ddot{A}_i \\
\vdots \\
\ddot{A}_m
\end{bmatrix} \in \mathbb{R}^{mp} \times \mathbb{R}^{mp}
\]

is a block upper triangle matrix with each diagonal submatrix \( \ddot{A}_i \) being critically stable.

3. **Subsystem construction.** Let \( \tilde{Z}_i = \begin{bmatrix} z_{1i} \cdots z_{mi} \end{bmatrix} \), \( i = 1, \ldots, m_i \in \mathbb{R}^{mp} \), and we have \( \ddot{\tilde{z}} = \ddot{A}_i \ddot{\tilde{z}} + B \delta (f(x + g(t)) - g(t)) \) with

\[
\begin{bmatrix}
\ddot{A}_i \\
\vdots \\
\ddot{A}_m
\end{bmatrix} \in \mathbb{R}^{mp} \times \mathbb{R}^{mp}
\]

as a submatrix of \( \ddot{A} \) and \( \ddot{B} \in \mathbb{R}^{mp} \times \mathbb{R}^{mp} \) as the corresponding submatrix of \( \ddot{B} \).

4. **Feedback function design.** Let \( \tau_m = -\ddot{B}_m P_m \ddot{z}_m \in \mathbb{R}^p \), where \( P_m \) is a positive definite solution of the LMI \( \dddot{A}_m \dddot{B}_m + P_m \dddot{B}_m \dddot{A}_m \leq 0 \). Let \( \tau_i = -\ddot{B}_i P_i \ddot{z}_i \in \mathbb{R}^p \), where \( P_i \) is a positive definite solution of the LMI \( \dddot{A}_i \dddot{B}_i + P_i \dddot{B}_i \dddot{A}_i \leq 0 \), for \( i = m_i - 1, \ldots, 1 \).

5. **Multi-level saturation controller design.** Choose

\[
f_i = \mu_i (x_i, \eta_i) \in \mathbb{R}^p \), \( \mu_i \) is a sufficiently small constant.

Choose \( f_i = \mu (x_i, \eta_i) \in \mathbb{R}^{mp} \), \( i = 2, \ldots, m_1 \), where \( \mu_i \) is sufficiently small constant. Then, \( f_i = f_i \).

Denote \( \xi = [\xi_1 \cdots \xi_N]^T \), \( \eta = [\eta_1 \cdots \eta_N]^T \) and \( x = [x_1 \cdots x_N]^T \), \( w = [w_1 \cdots w_N]^T \). Then, we have

\[
\begin{align*}
\xi &= (L_x \otimes I_p) x \\
\eta &= (L_x \otimes I_p) w
\end{align*}
\]

(3)

Under Assumptions 1–3, it is obvious that the leaderless bipartite consensus can be achieved if and only if \( \xi \) asymptotically converges to 0. Let \( \xi = [\xi_1 \cdots \xi_N]^T \). Then, the dynamics in the error phase are as follows:

\[
\begin{align*}
\dot{\xi} &= (L_x \otimes A) \xi + (L_x \otimes B) \delta(n) \\
\eta &= (L_x \otimes A) \eta + (L_x \otimes B) \delta(n) - [L_x (H + g) \otimes BE] \xi \\
\dot{\xi} &= (L_x \otimes A) \xi + [L_x (H + g) \otimes BE] \xi \\
b_i &= \xi_i^T \Gamma \xi_i
\end{align*}
\]

(4)
where \( H \triangleq \text{diag}(b_1, \ldots, b_N) \), \( \varphi \triangleq \text{diag}(\varphi_1, \ldots, \varphi_N) \) and \( \delta(u) \triangleq [\delta^T(u_1), \ldots, \delta^T(u_N)]^T \).

On the basis of above analysis, we have the following theorem.

**Theorem 1.** Suppose that Assumptions 1–3 hold. Then, the leaderless bipartite consensus with input saturation of MAS: (1) can be achieved under the state feedback adaptive protocol (2) with the control parameters designed as \( E = -B^T P, \Gamma = PB^T P \) and \( \varphi_i = \xi_i^T P \xi_i \), where \( P \) is a positive definite solution to the following algebraic Riccati inequality (ARI):

\[
A^T P + PA - 2PB^TP < 0
\]

By the way, each \( h_i \) converges to a certain finite state value.

**Proof.** Consider a feasible Lyapunov function candidate

\[
V_1 = \frac{1}{2} \sum_{i=1}^{N} r_i(2b_i + \varphi_i) \varphi_i + \frac{1}{2} \sum_{i=1}^{N} r_i(b_i - \alpha_1)^2
\]

where \( r = (r_1, \ldots, r_N) \) is the left zero feature vector of \( DL_s D \). \( \alpha_1 > 0 \) is an appropriate constant to be determined later. As mentioned in Lemma 2, \( R \triangleq \text{diag}(r_1, \ldots, r_N) \) > 0 holds under Assumption 3. Clearly, \( b_i \geq 0 \) and \( b_i(0) > 0 \), so \( b_i > 0 \) at any time. And it is not hard to verify that the Lyapunov function \( V_1 \) is positive definite.

The time derivative of \( V_1 \) along the trajectory (4) is described by

\[
\dot{V}_1 = \sum_{i=1}^{N} r_i(b_i + \varphi_i) \dot{\varphi}_i + \sum_{i=1}^{N} r_i(\varphi_i + b_i - \alpha_1) \dot{b}_i
\]

Notice that

\[
\sum_{i=1}^{N} r_i(\varphi_i + b_i - \alpha_1) \dot{b}_i = \sum_{i=1}^{N} r_i(\varphi_i + b_i - \alpha_1) \dot{\varphi}_i \Gamma \xi_i
\]

\[
= \xi^T [(\varphi + \alpha_1 L)R \otimes \Gamma] \xi
\]

\[
= \xi^T [(\varphi + \alpha_1 L)R \otimes \Gamma] \xi
\]

\[
- \xi^T [(\varphi + \alpha_1 L)R^T P \otimes \Gamma] \xi
\]

We then have

\[
\dot{V}_1 = \xi^T [(\varphi + \alpha_1 L)R \otimes (A^T P + B4 + \Gamma) - ((\varphi + \alpha_1 L)R + \alpha_1 R) \otimes \Gamma] \xi
\]

where \( L_s \triangleq RL_s + R_s^T R \).

Let \( \xi = ((H + \varphi) \otimes I_o) \xi. \) Then, we obtain that

\[
\xi^T (D(H + \varphi)^{-1}r \otimes 1) = \xi^T Dr \otimes 1
\]

\[
= (\chi - w)^T [D(DL_s^T D)Dr \otimes 1] = 0
\]

where the last equality is ensured by the fact \( r^T DL_s D = 0 \). Obviously, all entries of \( (H + \varphi)^{-1}r \otimes 1 \) are positive. By Proposition 1, we can get that

\[
\xi^T (L_s \otimes I_o) \xi \geq \frac{\lambda_2(L_s)}{N} \xi^T \xi
\]

where \( \lambda_2(L_s) \) means the second smallest eigenvalue of \( L_s \). By substituting (11) into (10), we obtain

\[
\dot{V}_1 \leq \xi^T [(\varphi + \alpha_1 L)R \otimes (A^T P + B4 + \Gamma) - ((\varphi + \alpha_1 L)R + \alpha_1 R) \otimes \Gamma] \xi
\]

\[
\leq - \xi^T \sqrt{\alpha_1 R^{1/2} \otimes B^T P} \xi
\]

\[
\leq - \xi^T \sqrt{\alpha_1 R^{1/2} \otimes B^T P} \xi
\]

\[
\leq - 2 \xi^T \sqrt{\frac{\lambda_2(L_s)\alpha_1}{N}(H + \varphi)R^T L_s^T R \otimes \Gamma} \xi
\]

Substituting (13) into (12) and choosing \( \alpha_1 \geq \frac{9 \max(r)}{4 \lambda_2(L_s)} \), we have

\[
\dot{V}_1 \leq \xi^T [(\varphi + \alpha_1 L)R \otimes (A^T P + B4 + 2\Gamma) - ((\varphi + \alpha_1 L)R + \alpha_1 R) \otimes \Gamma] \xi
\]

\[
< 0.
\]

Hence, we can see that \( \dot{V}_1(t) \) is bounded, so are \( \varphi_i \) and \( \xi \). Since \( \varphi_i \) is a quadratic form of \( \xi_i \), which is nonnegative, we have that \( (b_i - \alpha_1)^2 \) is also bounded. Note that \( \alpha_1 \) is a constant, which implies the boundedness of \( b_i \). Notice that \( b_i \geq 0 \), we can get the conclusion that each \( b_i \) converges to a certain finite value. Integrating formula (14),

\[
\int_0^\infty - \xi^T [(\varphi + \alpha_1 L)^T R \otimes (A^T P + B4 - 2\Gamma)] \xi \leq \dot{V}_1 - V(0)
\]


That is, $\xi_i \in L_2$. It is not difficult to show the boundedness of $(b_i + \bar{\xi}_i)BE_\xi \xi_i$ and $(b_i + \bar{\xi}_i)BE_\xi \xi_i \in L_2$. By Lemma 4, we get that $w_i \in L_2$ and $\eta_i \in L_2$. Considering the completeness of $L_2$ space, $\xi_i$ is also bounded and $\zeta_i \in L_2$. The dynamic of $\zeta_i$ can be given as

$$\dot{\zeta}_i = A\zeta_i + B \sum_{j=1}^{N} |a_{ij}|[\Delta (u_i) - \text{sgn}(a_{ij})\Delta (u_j)]$$

Since $\zeta_i$ and $\Delta (u_i)$ are bounded, $\bar{\zeta}_i$ is also bounded. In light of Barbalat's lemma, we can draw the conclusion that $\bar{\zeta}_i$ converges to 0. Hence, the bipartite consensus with input saturation is achieved. (4)

**Remark 4.** Theorem 1 indicates that the proposed control protocol (2) can solve bipartite consensus with input saturation for linear MASs under signed digraph, where only relative state information is involved, overcoming the limits of asking for global connectivity information of signed graph. The adequate assumption for the existence of such protocol (2) is that $(A, B)$ satisfies ANCBC condition and the signed digraph is strongly connected and structurally balanced, which is similar to previous works, meaning that such fully distributed design approach does not present further requirement for communication graph or the dynamics of MASs.

**Remark 5.** Fully distributed adaptive control protocols have been provided in [7] and [8] to achieve the consensus problems for linear MASs under communication topology containing only cooperative interaction, in which the adjacency weights are positive, and non-diagonal entries of Laplacian matrix are non-positive, which possesses some useful properties; see Lemmas 2 and 3. This paper mainly considers bipartite consensus of linear MASs with signed digraph. The main distinction from the above results is that the weights in signed graph are allowed to be negative. In this case, some useful properties of Laplacian matrix is no longer valid, which results in the main difficulty compared with the cooperative interaction case. As is revealed in [23], the bipartite consensus problem can be turned into the consensus problem by using gauge transformation. However, Theorem 1 presents a direct perspective on bipartite consensus realization by introducing Property 1 of the signed Laplacian matrix.

**Remark 6.** As defined in [37], the agents are said to achieve scaled consensus if the rates between the states come to the mentioned constants in the asymptote, i.e. $\lim_{t\to \infty}(\alpha_i x_i(t) - \alpha_j x_j(t)) = 0$, while the bipartite consensus is to make $\alpha_i$ be chosen in the set $\{1, -1\}$. In this sense, it is a special case of scaled consensus. However, the protocol designed in this paper can be simply developed to scaled consensus with certain corresponding assumption.

### 3.2 Leader–follower bipartite consensus tracking

In this subsection, we discuss the bipartite tracking consensus with input saturation in a signed topology with a leader, where there are $N + 1$ agents containing one leader indexed as 0 and $N$ followers indexed as 1, ..., $N$. The dynamic of $i$th agent is given as (1), where we have $x_i = 0$. The signed topology $G$ among $N + 1$ agents is required to satisfy the following assumption.

**Assumption 4.** Assume that the signed digraph $G$ has a directed spanning tree with the leader being the root.

From Proposition 2, we can find a positive definite matrix $G = \text{diag}(\bar{\xi}_1, ..., \bar{\xi}_N)$, such that $\bar{G} L_{s1} + L_{s1}^T \bar{G} > 0$.

The tracking bipartite consensus problem with input saturation is said to be solved if we can design a distributed adaptive consensus protocol such that the following condition holds: $\lim_{t\to \infty} \|x_i - d x_0\| = 0, \forall i = 1, ..., N$.

To achieve the leader–follower bipartite tracking consensus in (1), we design the distributed consensus control protocol as follows:

$$u_i = f(\bar{w}_i) + (\bar{b}_i + \bar{\xi}_i)E_\xi (\bar{\zeta}_i - \bar{\eta}_i)$$
$$\dot{\bar{w}}_i = A \bar{w}_i + B [\Delta (u_i) - (\bar{b}_i + \bar{\xi}_i)E_\xi (\bar{\zeta}_i - \bar{\eta}_i)]$$
$$\dot{\bar{\zeta}}_i = (\bar{\zeta}_i - \bar{\eta}_i)^T \Gamma (\bar{\zeta}_i - \bar{\eta}_i)$$

where $\bar{\zeta}_i \triangleq \sum_{j=1}^{N} |a_{ij}|(x_j - \text{sgn}(a_{ij})x_j)$ is the relative states of neighbouring agents of the $i$th follower, $\bar{w}_i$ is a designed compensator observer and $\bar{\eta}_i \triangleq \sum_{j=0}^{N} |a_{ij}|(\bar{w}_j - \text{sgn}(a_{ij})\bar{w}_j)$. $\bar{b}_i(t)$ represents the coupling time-varying weight of the $i$th follower with $\bar{b}_i(0) > 0$, $\bar{\zeta}_i$ represents smooth function to be designed later, $E_\xi$ is a feedback gain matrix with proper dimensions.

**Definition**

$$\bar{\xi}_i \triangleq (\bar{\xi}_i^T, ..., \bar{\xi}_N^T)^T, \quad \bar{\eta}_i \triangleq (\bar{\eta}_i^T, ..., \bar{\eta}_N^T)^T, \quad \bar{x}_i \triangleq (x_1^T, ..., x_N^T)^T, \quad \bar{w}_i \triangleq (\bar{w}_1^T, ..., \bar{w}_N^T)^T.$$ Then, we have

$$\bar{\xi}_i = (L_{s1} \otimes L_i)(x_i - d \otimes x_0)$$
$$\bar{\eta}_i = (L_{s1} \otimes L_i)\bar{w}_i$$

where $d \triangleq [d_1, ..., d_N]^T$. Notice that the bipartite consensus tracking is achieved if and only if $\bar{\xi}_i$ asymptotically converges. Obviously, $\bar{\xi}_i$ is considered the consensus error of the leader–follower bipartite consensus.

Denote $\bar{\xi}_0 \triangleq \bar{\xi}_0 - \bar{\eta}_i$. Taking the time derivative with respect to (16), and substituting (1) and (15) into it, we can get the error planar dynamics as follows:

$$\dot{\bar{\xi}}_i = (L_N \otimes A)\bar{\xi}_i + (L_{s1} \otimes B)\Delta (u)$$
$$\dot{\bar{\eta}}_i = (L_N \otimes A)\bar{\eta}_i + (L_{s1} \otimes B)\Delta (u) - [L_{s1}(FL + \bar{\phi}) \otimes BE]\bar{\xi}_i$$
$$\dot{\bar{\xi}}_i = (L_N \otimes A)\bar{\xi}_i + [L_{s1}(FL + \bar{\phi}) \otimes BE]\bar{\xi}_i$$
$$\dot{\bar{\eta}}_i = \bar{\xi}_i^T \Gamma \bar{\xi}_i$$

where $FL \triangleq \text{diag}(\bar{b}_1, ..., \bar{b}_N)$ and $\bar{\phi} \triangleq \text{diag}(\bar{\phi}_1, ..., \bar{\phi}_N)$. 


Based on the designed bipartite adaptive consensus protocol (15), the above discussion is summarized as a theorem.

**Theorem 2.** Suppose that the signed digraph $G$ satisfies Assumptions 1, 2 and 4. Then, the bipartite consensus tracking with input saturation of MASs (1) can be achieved under the state feedback adaptive protocol (15), where $\tilde{\xi}_i = s_i^T P_{\xi_i}^* E_i$, $P$ and $\Gamma$ are designed the same as in Theorem 1. By the way, each $\tilde{h}_i$ converges to a certain finite state value.

**Proof.** Consider a feasible Lyapunov function:

$$V_2 = \sum_{i=1}^{N} \left[ \frac{1}{2} \hat{b}_i (2\hat{b}_i + \hat{\phi}_i) \hat{\phi}_i + \frac{1}{4} \sum_{i=1}^{N} \hat{b}_i - \alpha_2 \right]^2$$

where $\tilde{G} \triangleq \text{diag}(\tilde{g}_1, ..., \tilde{g}_N) > 0$ satisfies $\tilde{G} L_{\omega_1} + L_{\omega_1}^T \tilde{G} > 0$, and $\alpha_2 > 0$ is an appropriate constant selected later. Since Assumptions 2 and 4 hold, based on the analysis in Proposition 2, we can see that such a positive definite matrix $\tilde{G}$ exists. It is not hard to see that $\hat{b}_i > 0$ for each $i > 0$, as $\hat{b}_i(0) > 0$ and $\hat{b}_i > 0$. Thus, $V_2$ is positive definite.

Similar to the derivation in Theorem 1, the time derivative of $V_2$ along the trajectory of (17) is described by

$$\dot{V}_2 \leq \tilde{\xi}^T [(\tilde{F} I + \tilde{\phi}) \tilde{G} \otimes (\tilde{A}^T P + PA + \Gamma)]$$

where $\tilde{\lambda}_0$ represents the smallest feature value of $\tilde{G} L_{\omega_1} + L_{\omega_1}^T \tilde{G}$. It is not difficult to see that

$$\tilde{\xi}^T [(\tilde{A} \tilde{\lambda}_0 (\tilde{F} I + \tilde{\phi})^2 + \alpha_2 \tilde{G}) \otimes \Gamma] \tilde{\xi} \geq \frac{9 \text{max}_{i=1}^{N} \hat{\theta}_i}{4 \hat{\lambda}_0}$$

Substituting (20) into (19) and choosing $\alpha_2 \geq \frac{9 \text{max}_{i=1}^{N} \hat{\theta}_i}{4 \hat{\lambda}_0}$, we have

$$\dot{V}_2 \leq \tilde{\xi}^T [(\tilde{F} I + \tilde{\phi}) \tilde{G} \otimes (\tilde{A}^T P + PA - 2\Gamma)] \tilde{\xi} \leq 0.$$  

Following the rest proof of Theorem 1, we can draw the conclusion that each $\tilde{h}_i$ converges to a certain finite value, and the bipartite tracking problem with input saturation is achieved. For brevity, here we omit the detailed proof.

**Remark 7.** It can be viewed from Theorem 2 that not only the protocol design, but also the Lyapunov function construction for the leader–follower bipartite consensus tracking problem is similar to those in leaderless bipartite consensus problem. However, it is worth noting that the achievement of bipartite consensus tracking depends on the proposition that $L_{\omega_1}$ is similar to a nonsingular $M$-matrix, while the leaderless bipartite consensus relies on the proposition of Laplacian matrix associated with strongly connected topology.

### 4 | Bipartite Consensus with Input Saturation Over Relative Output Feedback Adaptive Protocols

In Section 3, we have designed adaptive protocols to achieve bipartite consensus with input saturation of general linear MASs under leaderless or leader–follower signed digraph, where relative state information should be available. In practice, the state information may not be accessible under many circumstances, where only output information is available. In this section, we would discuss the case where only relative output information can be used to design adaptive control protocols.

Since only relative output information is available, the following assumption on agent dynamics should be introduced.

**Assumption 5.** $(A, B, C)$ is stabilizable and detectable.

### 4.1 | Leaderless Bipartite Consensus

Based on the relative output information among its neighbour agents, we propose a fully distributed adaptive protocol to each agent as follows:

$$\dot{x}_i = Ax_i + Fu_i,$$

$$\dot{y}_i = (A + BK)x_i - (\chi_i + \theta_i)FC_i,$$

$$\dot{u}_i = f(w_i) + K_i x_i$$

where $\hat{e}_i \in \mathbb{R}^n$ and $\hat{v}_i \in \mathbb{R}^n$ denote internal states of protocols with $\hat{e}_i(0) = 0$, $\hat{e}_i$ is a local observer used to estimate consensus error $e_i \triangleq \sum_{j=1}^{N} |a_{ij}|(x_i - \text{sgn}(a_{ij})x_j)$, $\hat{v}_i$ is a distributed observer used to realize consensus, $w_i$ is a distributed compensator, $\psi_i \triangleq \sum_{j=1}^{N} |a_{ij}|(w_i - \text{sgn}(a_{ij})(e_i + w_j))$, $\chi_i$ denotes the coupling time-varying weight of $i$th agent with $\chi_i(0) > 0$, $K$ is a feedback gain matrix and $A + BK$ is Hurwitz, $F = -QC^T$ and $e_i = (\hat{e}_i - \psi_i)^T Q^{-1} (\hat{e}_i - \psi_i)$, with $Q > 0$ satisfying the following ARI:

$$AQ + QA^T + 2QC^TCQ < 0.$$  

Let $e \triangleq [e_1^T, ..., e_N^T]^T$, $\hat{e} \triangleq [\hat{e}_1^T, ..., \hat{e}_N^T]^T$, $\psi \triangleq [\psi_1^T, ..., \psi_N^T]^T$, $x \triangleq [x_1^T, ..., x_N^T]^T$, $v \triangleq [v_1^T, ..., v_N^T]^T$ and $w \triangleq [w_1^T, ..., w_N^T]^T$. Then we have

$$e = (L_2 \otimes I_n)x$$

$$\psi = (L_3 \otimes I_n)(v + w)$$  

(24)
where $L_\sigma$ is the Laplacian matrix of signed topology with $N$ agents. From the first equality in (24), we know that the bipartite consensus can be achieved if and only if the consensus error $\epsilon$ asymptotically converges.

Combining (22) and (1), we can obtain

$$
\dot{x} = (IN \otimes A)x + (L_\sigma \otimes B)\delta(u)
$$

$$
\dot{\chi} = (IN \otimes A)\dot{\chi} + (L_\sigma \otimes B)\delta(u) - (IN \otimes FC)(\epsilon - \dot{\chi})
$$

$$
\dot{\psi} = [IN \otimes A]\psi + (L_\sigma \otimes B)\delta(u) - [L_\sigma(x + \theta) \otimes FC](\epsilon - \psi)
$$

$$
\dot{\chi}_i = (\chi_i - \psi_i)^T C^T C(\chi_i - \psi_i)
$$

where $X \triangleq \text{diag}(X_1, \ldots, X_N)$ and $\theta \triangleq \text{diag}(\theta_1, \ldots, \theta_N)$. Combining (25) with (24), we can get the closed-loop dynamics

$$
\dot{\epsilon} = (IN \otimes A)\epsilon + L_\sigma \otimes B)\delta(u)
$$

$$
\dot{\epsilon}_i = (IN \otimes A)\epsilon_i + L_\sigma \otimes B)\delta(u) - (IN \otimes FC)(\epsilon - \epsilon_i)
$$

$$
\dot{\psi}_i = (\epsilon_i - \psi_i)^T C^T C(\epsilon_i - \psi_i)
$$

where $\chi_i \triangleq \chi_i - \psi_i$. Then, we have

$$
\dot{\sigma} = [IN \otimes (A + FC)]\sigma
$$

$$
\dot{\phi} = [IN \otimes (A + L_\sigma(x + \Theta) \otimes FC)]\phi - (IN \otimes FC)\sigma
$$

$$
\dot{\chi}_i = \phi_i^T C^T C \phi_i.
$$

Based on previous discussion, main result is summarized as follows.

**Theorem 3.** Suppose that the signed topology $G$ satisfies Assumptions 1, 2, 3 and 5. The leaderless bipartite consensus with input saturation for linear MASs described in (1) can be achieved under the adaptive relative output feedback protocol (22). Furthermore, each coupling weight converges to a certain finite steady-state value.

**Proof.** Since the equation $(A + FC)Q + Q(A + FC)^T = QA^T + AQ - 2QC^T CQ < 0$ naturally holds, we have that $A + FC$ is Hurwitz, which indicates that $\sigma$ in (27) asymptotically converges. Then, we will illustrate that $\dot{\phi}$ in (27) converges.

Consider a feasible Lyapunov function candidate as follows:

$$
\dot{V}_3 = \frac{1}{2} \sum_{i=1}^{N} r_i(2\chi_i + \Theta)\delta_i + \frac{1}{2} \sum_{i=1}^{N} r_i(\chi_i - \beta_i)^2 + \gamma_1 \sigma^T(IN \otimes Q^{-1}) \sigma
$$

where $\beta_1$ and $\gamma_1$ are positive constants to be determined, and $r_i \triangleq [r_1, \ldots, r_N]^T$ represents the left feature vector of $L_\sigma$ mentioned in previous analysis. By Lemma 2, $R \triangleq \text{diag}(r_1, \ldots, r_N) > 0$. Obviously, we have $\Theta_i(t) \geq 0$ and $\chi_i(t) > 0$, so $V_3$ is positive definite.

The time derivative of $V_3$ along the trajectories of (27) is described by

$$
\dot{V}_3 = \phi^T [(X + \Theta)R \otimes (Q^{-1}A + ATQ^{-1} + CT)C] - ((X + \Theta)L_\sigma(x + \Theta) + \beta_1 R) \otimes CT \phi
$$

$$
+ \phi^T [(X + \Theta)R \otimes CT] \sigma - \chi_i \sigma (IN \otimes W) \sigma
$$

where $L_\sigma \triangleq RL_\sigma + L_\sigma R$ and $W = -(Q^{-1}A + ATQ^{-1} - 2CT)C > 0$. Denote $\lambda_2(L_\sigma)$ the smallest nonzero eigenvalue of $L_\sigma$. Similar to the discussions in Theorem 1 and by Proposition 1, we have

$$
\phi^T [(X + \Theta)L_\sigma(x + \Theta) \otimes I_\sigma] \phi
$$

$$
\geq \frac{\lambda_2(L_\sigma)}{N} \phi^T [(X + \Theta)^2 \otimes I_\sigma] \phi
$$

Notice that

$$
\phi^T [(X + \Theta)R \otimes CT] \sigma
$$

$$
\leq \frac{\lambda_1(L_\sigma)}{2N} \phi^T [(X + \Theta)^2 \otimes CT] \sigma + \frac{2N}{\lambda_1(L_\sigma)} \phi^T \sigma
$$

Choosing $\beta_1 \geq \frac{9N \max_{\Theta} \|A + FC\|}{2\lambda_1(L_\sigma)}$, we can obtain

$$
\frac{-\lambda_1(L_\sigma)}{2N} \phi^T [(X + \Theta)^2 \otimes CT] \phi - \phi^T [\beta_1 R \otimes CT] \phi
$$

$$\leq -3\phi^T [(X + \Theta)R \otimes CT] \phi
$$

Choosing $\gamma_1 \geq \frac{2N \lambda_2(L_\sigma) \|A + FC\|}{\lambda_1(L_\sigma) \lambda_{\text{min}}(W)}$ and substituting (30)–(32) into (29), we can get that

$$\dot{V}_3 \leq \phi^T [(X + \Theta)R \otimes (Q^{-1}A + ATQ^{-1} - 2CT)C] \phi
$$

$$\leq - \phi^T [X(0)R \otimes W] \phi
$$

$$\leq 0
$$

Hence, $V_3$ is bounded, which ensures the coupling weights $\chi_i$ and $\phi$ to be bounded. Note that $\dot{\chi}_i \geq 0$, we can draw the conclusion that $\chi_i$ converges to a certain finite value. Integrating the second inequity of system (33),

$$\int_{0}^{\infty} \phi^T [X(0)R \otimes W] \phi \leq V^{(0)} - V^{(\infty)}
$$

we can conclude that $\phi \in L_2$. Because of the boundedness of $(X + \Theta)$, the term $[(X + \Theta) \otimes FC] \xi_2$ is also bounded and $[(X + \Theta) \otimes FC] \xi_2 \in L_2$. Notice that $A + BK$ is Hurwitz, then
\( v_i \) is bounded and \( v_i \in L_2 \). In light of Lemma 4, \( w_i \) is bounded and \( w_i \in L_2 \). So are \( \Psi \) and \( \hat{\rho} \) by the completeness of \( L_2 \) space. It is not difficult to verify that \( \sigma \in L_2 \), then \( \epsilon \in L_2 \) and \( \epsilon \) is bounded. Then, the boundedness of \( \hat{\epsilon} \) can be ensured by the first equation of system (26). In light of the Barbab’s lemma, the bipartite consensus error \( \epsilon \) converges to 0. Hence, the bipartite consensus with input saturation is achieved.

Remark 8. The protocol (22) can achieve bipartite consensus with input saturation under strongly connected signed digraphs. It should be noticed that only relative output information is used to construct the protocol, making the distributed relative output feedback adaptive protocol (22) more practical in reality, compared with the relative state feedback adaptive protocols presented in previous section. The distributed observer \( v_i \) is needed to be transmitted among neighbouring agents due to communication topology.

4.2 Leader–follower bipartite consensus tracking

In this subsection, we further discuss the tracking bipartite consensus with input saturation for linear MASs containing one leader and \( N \) followers, whose dynamics are given in (1). Similar to Section 3.2, the leader is indexed as 0 with \( \eta_0 = 0 \) and followers are indexed as \( i = 1, \ldots, N \).

On relative output information among neighbour agents, two observers in this subsection are designed as

\[
\dot{\hat{\epsilon}}_i = A\hat{\epsilon}_i + F_i[C\hat{\epsilon}_i - \sum_{j=0}^{N} |a_{ij}|(\eta_i - \text{sgn}(a_{ij})\eta_j)] + B\sum_{j=0}^{N} |a_{ij}|(\hat{\sigma}_i - \text{sgn}(a_{ij})\hat{\sigma}_j)
\]

\[
\hat{\epsilon}_i = A\hat{\epsilon}_i + B\hat{\sigma}_i - K\hat{\epsilon}_i,
\]

\[
\dot{\hat{\epsilon}}_i = (\hat{\sigma}_i - \hat{\psi}_i)C^T(\hat{\sigma}_i - \hat{\psi}_i),
\]

where \( \hat{\epsilon}_i \) is a local observer designed to estimate consensus errors, so that only relative output information is needed. \( \hat{\sigma}_i \) is the distributed observer with \( \hat{\sigma}_0 = 0 \), designed to realize tracking consensus. \( \hat{\psi}_i \) is the designed compensation observer. \( \hat{\psi}_i \equiv \sum_{j=0}^{N} |a_{ij}|[(\eta_i + \hat{\eta}_i) - \text{sgn}(a_{ij})(\hat{\sigma}_j + \hat{\eta}_j)] \).

By such observer design structure, state observation and consensus realization can realize decoupling \( \chi_i \) is the coupling time-varying weight of \( i \)-th agent with \( \chi_i(0) > 0 \), and \( \Theta_i = (\hat{\epsilon}_i - \hat{\psi}_i)^TQ^{-1}(\hat{\epsilon}_i - \hat{\psi}_i) \). \( F, K \) and \( Q \) are designed as Theorem 3.

Based on above three observers, we provide the fully distributed bipartite consensus tracking protocol for each follower

\[
u_i = f(\hat{\eta}_i) + K\hat{\epsilon}_i
\]

It is noteworthy that no extra observer of the leader is needed, and the neighbour followers only need relative output information, which simplifies the complexity of calculation.

Let \( \hat{\tau} \equiv [\hat{\tau}_0, \ldots, \hat{\tau}_N]^T \), \( \hat{\tau} \equiv [\hat{\tau}_0, \ldots, \hat{\tau}_N]^T \), \( \hat{\psi} \equiv [\hat{\psi}_0, \ldots, \hat{\psi}_N]^T \), \( \chi \equiv [\chi_0, \ldots, \chi_N]^T \), \( \hat{\tau} \equiv [\hat{\tau}_0, \ldots, \hat{\tau}_N]^T \) and \( \hat{\psi} \equiv [\hat{\psi}_0, \ldots, \hat{\psi}_N]^T \).

Then, we have

\[
\hat{\epsilon} = (L_{\sigma} \otimes I_n)(x - d \otimes x_0),
\]

\[\quad \hat{\eta} = (L_{\sigma} \otimes I_n)(\hat{\epsilon} + \hat{\nu}). \quad (36)\]

The bipartite tracking consensus is achieved if and only if the bipartite consensus error \( \hat{\epsilon} \) asymptotically converges.

Combining (34) and (35) with (1), the closed-loop dynamics are described by

\[
\hat{\tau} = (L_N \otimes A)\hat{\eta} + (L_{\sigma} \otimes B)\hat{\psi}(u),
\]

\[
\hat{\tau} = (L_N \otimes A)\hat{\eta} + (L_{\sigma} \otimes B)\hat{\psi}(u) - (L_N \otimes FC)(\hat{\epsilon} - \hat{\tau}),
\]

\[
\hat{\psi} = (L_N \otimes A)\hat{\psi} - [L_{\sigma}(X + \Theta) \otimes FC](\hat{\epsilon} - \hat{\psi}),
\]

\[
\dot{\chi}_i = (\hat{\epsilon}_i - \hat{\psi}_i)^TC^T(\hat{\nu}_i - \hat{\psi}_i)^TC^T(\hat{\nu}_i - \hat{\psi}_i) \quad (37)
\]

Let \( \hat{\sigma} \equiv \hat{\epsilon} - \hat{\tau}, \) and \( \hat{\phi} \equiv [\hat{\phi}_0, \ldots, \hat{\phi}_N]^T \) be ensured by Proposition 2. It is obvious that \( \hat{\sigma} \equiv [\hat{\sigma}_0, \ldots, \hat{\sigma}_N]^T \)

\[
\hat{\sigma} = [L_N \otimes (A + FC)]\hat{\sigma}, \quad \hat{\phi} = [L_N \otimes (A + L_{\sigma}(X + \Theta) \otimes FC)]\hat{\phi} - (L_N \otimes FC)\hat{\sigma},
\]

\[
\hat{\tau} = (L_N \otimes A)\hat{\eta} + (L_{\sigma} \otimes B)\hat{\psi}(u) - (L_N \otimes FC)\hat{\sigma},
\]

\[
\hat{\chi}_i = (\hat{\psi}_i)^TC^T(\hat{\nu}_i - \hat{\psi}_i), \quad i = 1, \ldots, N. \quad (38)
\]

Theorem 4. Suppose that the signed topology \( G \) satisfies Assumptions 1, 2, 4 and 5. The tracking bipartite consensus with input saturation for MASs given in (1) can be achieved under the designed observers (34) and the adaptive relative output feedback control protocols (35). Moreover, each coupling weight converges to a certain finite steady-state value.

Proof. It is not difficult to see that \( \hat{\sigma} \) in (38) asymptotically converges. Then, we will prove that \( \hat{\phi} \) in (38) converges. Consider a feasible Lyapunov function candidate

\[
V_4 = \frac{1}{2} \sum_{i=1}^{N} \bar{\bar{\chi}}_i(2\chi_i + \Theta_i)\delta_i + \frac{1}{2} \sum_{i=1}^{N} \bar{\bar{\chi}}_i(\chi_i - \beta_2)^2
\]

\[
+ \gamma_2 \delta^T(I_N \otimes Q^{-1})\delta \quad (39)
\]

where \( \bar{\bar{G}} \equiv \text{diag}(\bar{\bar{G}}, \ldots, \bar{\bar{G}}) \) \( \bar{\bar{G}} > 0 \) satisfies \( L_{\sigma} = \bar{\bar{G}}L_{\sigma} + L_{\sigma}^T \bar{\bar{G}} > 0 \), and \( \bar{\bar{G}} \) and \( \gamma_2 \) are positive constants. The existence of \( \bar{\bar{G}} \) can be ensured by Proposition 2. It is obvious that \( V_4 \) is positive definite.
The time derivative of \( V_4 \) along the trajectory of (38) is described by
\[
\dot{V}_4 = \sum_{i=1}^{N} \dot{g}_i ((X_i + \Theta_i) \dot{\vartheta}_i + (X_i + \vartheta_i - \beta_2) X_i - \gamma_2 T (I_N \otimes W) \dot{\vartheta}_i) \\
= \dot{\phi}^T [(X + \Theta) \tilde{G} \otimes (\mathcal{Q}^{-1} A + A^T \mathcal{Q}^{-1} + C^T C)] \\
- \dot{\phi}^T [(X + \Theta) \tilde{L}_0 (X + \Theta) + \tilde{\beta}_2 \tilde{G} \otimes C^T C] \tilde{\vartheta} \\
+ \dot{\phi}^T [(X + \Theta) \tilde{G} \otimes 2C^T C] \dot{\vartheta} - \gamma_2 \tilde{T} (I_N \otimes W) \dot{\vartheta} \\
\leq \dot{\phi}^T [(X + \Theta) \tilde{G} \otimes (\mathcal{Q}^{-1} A + A^T \mathcal{Q}^{-1} + C^T C)] \\
- (\lambda_0 (X + \Theta)^2 + \tilde{\beta}_2 \tilde{G} \otimes C^T C) \dot{\vartheta} \\
+ \dot{\phi}^T [(X + \Theta) \tilde{G} \otimes 2C^T C] \dot{\vartheta} - \gamma_2 \tilde{T} (I_N \otimes W) \dot{\vartheta} \tag{40}
\]
where \( \lambda_0 \) represents the smallest eigenvalue of \( \tilde{L}_0 \).

It is easy to see that
\[
\dot{\phi}^T [(X + \Theta) \tilde{G} \otimes 2C^T C] \dot{\vartheta} \\
\leq \frac{\lambda_0}{2} \dot{\phi}^T [(X + \Theta)^2 \otimes C^T C] \dot{\vartheta} + \frac{2}{\lambda_0} \tilde{T} (G^2 \otimes C^T C) \dot{\vartheta} \tag{41}
\]
and by choosing \( \tilde{\beta}_2 \geq [9\lambda_{\text{max}}(\tilde{G})]/2\lambda_0 \), we obtain the following inequality:
\[
- \frac{\lambda_0}{2} \dot{\phi}^T [(X + \Theta)^2 \otimes C^T C] \dot{\vartheta} - \dot{\phi}^T (\tilde{\beta}_2 \tilde{G} \otimes C^T C) \dot{\vartheta} \\
\leq -3 \dot{\phi}^T [(X + \Theta) \tilde{G} \otimes C^T C] \dot{\vartheta} \tag{42}
\]
Substituting (41) and (42) into (40), and choosing \( \gamma_2 \geq [2\lambda_{\text{max}}(\tilde{G})\lambda_{\text{max}}(C^T C)]/[\lambda_0 \lambda_{\text{min}}(W)] \), \( V_4 \) can be scaled as
\[
\dot{V}_4 \leq \dot{\phi}^T (X + \Theta) \tilde{G} \otimes (\mathcal{Q}^{-1} A + A^T \mathcal{Q}^{-1} - 2C^T C) \dot{\vartheta} \\
\leq - \dot{\phi}^T (X(0) \tilde{G} \otimes W) \dot{\vartheta} \\
\leq 0 \tag{43}
\]
It is obvious that \( V_4 \) and coupling weights \( \chi_i \) are both bounded. Combining \( \chi_i \geq 0 \), we can get the conclusion that \( \chi_i \) converges to a certain finite value. Following the last subsection’s analyses, we can verify that the bipartite consensus error \( \tilde{\tau} \) is bounded and \( \tilde{\tau} \in \mathcal{L}_2 \). It is not difficult to show the boundedness of \( \tilde{\tau} \). Thus, the bipartite consensus error \( \tilde{\tau} \) converges to 0. That is, the bipartite consensus tracking with input saturation is solved.

Remark 9. The adaptive bipartite consensus problem for nonlinear MASs has been studied in [32] and [38]. The topology discussed in [32] is undirected and the adaptive protocols are used to estimate the linearization parameter, where global eigenvalue information is still needed. In the present study, the provided protocols are fully distributed protocol in real meaning. And the topology is directed, which is more practical. In [38], nonstrict feedback nonlinear MASs are considered with directed graph, where the nonlinear model is more practical. However, absolute output information is demanded to be available. Moreover, consensus errors and observer errors only converge to a neighbourhood around the origin. In this study, only relative output information is needed, and both the consensus errors and observer errors converge to zero, which is more convenient and accurate.

Remark 10. Recently, a novel dynamic model is presented in [39] to illustrate the patterns of amity and enmity, and it is revealed that a finite-time convergence with four factions or that with two factions can be achieved under different initial values. The results proposed in [39] mainly focus on the dynamical evolution with certain initial value, while this paper intends to realize bipartite consensus by designing fully distributed adaptive controller.

5 | NUMERICAL SIMULATIONS

In this section, we provide some simulation examples to illustrate the effectiveness of the designed distributed adaptive protocols. Consider the linear MAS described by (1), where the coefficient matrices are chosen as \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \) and \( C = \begin{bmatrix} 1 & 0 \end{bmatrix} \). Choose an appropriate positive definite solution of \( A P + (P A)^T = Q \), where \( Q = \begin{bmatrix} 2.5892 & 0.8680 \\ 0.8680 & 0.8631 \end{bmatrix} \). Then the feedback gain matrices can be determined as \( F = \begin{bmatrix} -2.5892 & -0.8680 \end{bmatrix} \) and \( K = \begin{bmatrix} -0.8680 & -2.5892 \end{bmatrix} \).

5.1 | Leaderless case

The topology for the leaderless case is displayed in Figure 1, which is a strongly connected signed digraph. The gauge transformation matrix is then given as \( D = \begin{bmatrix} I_{10} & 0 \\ 0 & -I_{10} \end{bmatrix} \).

Let \( \chi_i(0) = 1 \), \( i = 1, \ldots, 20 \), \( e_i = \sum_{j=1}^{N} \left| a_{ij} \right| (x_i - \text{sgn}(a_{ij})x_j) \) as the consensus error. Under the adaptive protocol (22), the tracking trajectory of state \( x_i \) is given in Figure 2.

Figure 3 shows the trajectories of consensus error \( e_i \), where \( \epsilon(1) \) is the consensus error under the designed protocol (22) in Figure 3. Figures 3 indicates that the bipartite consensus with input saturation under adaptive protocol (22) is achieved. The convergence trajectory of adaptive coupling weights \( \chi_i \) in (22), with initial values as \( 1 \), are depicted in the first subgraph of Figure 4 and the second subgraph shows the trajectory of control input \( u_i \) with dotted line and saturation input \( \delta(u_i) \) with solid line.

5.2 | Leader–follower case

The topology for the leader–follower case is shown as in Figure 5, which is a directed signed graph, containing a directed spanning tree, where the leader is the root node.
The gauge transformation matrix is then given as $D = \begin{bmatrix} I_{10} & 0 \\ 0 & -I_{10} \end{bmatrix}$, and $d = \left[ 1_{10}^T, -1_{10}^T \right]^T$. Choose initial value of coupling gain as $\chi_i(0) = 1$ and let $\xi = x_i - d_i x_0$.

The state trajectory under leader–follower graph is given in Figure 6, where the dotted line represents the leader’s trajectory while solid line represents the followers’ trajectory. And the bipartite consensus tracking error under adaptive saturated protocol (34) and (35) is shown in Figure 7, which indicates that the bipartite consensus tracking is realized. The first subgraph of Figure 8 shows the coupling weights’ trajectory in (35) and the second subgraph of Figure 8 shows the trajectories of input $u_i$ and saturation input $\delta(u_i)$, where the dotted line represents $u_i$ and solid line represents $\delta(u_i)$. The directed signed topology with a leader, where the solid line represents the weight as 1 and the dashed line represents the weight as $-1$. The trajectory of consensus errors $e_i$ under leaderless bipartite adaptive saturated protocol (22) is shown in Figure 3. The trajectory of coupling weights $\chi(1)$, control input $u_i$ and saturation input $\delta(u_i)$ under leaderless bipartite adaptive saturated protocol (22) is shown in Figure 4. The strongly connected signed topology with no leader, where the solid line represents the weight as 1 and the dashed line represents the weight as $-1$. The trajectory of state $x_i$ under leaderless bipartite saturated protocol (22) is shown in Figure 2. The directed signed topology with a leader, where the solid line represents the weight as 1 and the dashed line represents the weight as $-1$. The trajectory of state $x_i$ under leaderless bipartite saturated protocol (22) is shown in Figure 2.
6 | CONCLUSION

This paper has presented both state feedback and relative output feedback fully distributed adaptive protocols to achieve bipartite consensus with input saturation for linear MASs over signed digraph. The protocols given in this study only rely on local state or output information among neighbouring agents, without any use of global connectivity information of Laplacian matrix. Compared with the existing works, it appears to be first design of fully distributed adaptive protocols applicable for bipartite consensus with input saturation under signed digraphs. Future work can be done to further consider the leader-follower bipartite consensus problem with the leader of nonzero control input, or bipartite consensus problem of nonlinear MASs.

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REFERENCES

1. Ren, W., et al.: A survey of consensus problems in multiagent coordination. Proc. Am. Control Conf. 3, 1859–1864 (2005)
2. Olfati-Saber, R., et al.: Consensus and cooperation in networked multi-agent systems. Proc. IEEE 95, 215–233 (2007)
3. Feng, Y., et al.: Consensus of heterogeneous first- and second-order multi-agent systems with directed communication topologies. Int. J. Robust Nonlinear Control 25, 362–375 (2013)
4. Ren, W.: On consensus algorithms for double-integrator dynamics. IEEE Trans. Autom. Control 53, 1503–1509 (2008)
5. Wang, X., Hong, Y.: Finite-time consensus for multi-agent networks with second-order agent dynamics. IFAC Proc. Volumes 41, 15185–15190 (2008)
6. You, K., et al.: Consensus condition for linear multi-agent systems over randomly switching topologies. Automatica 49, 3125–3132 (2013)
7. Li, Z., et al.: Distributed consensus of linear multi-agent systems with adaptive dynamic protocols. Automatica 49, 1986–1995 (2013)
8. Lv, Y., et al.: Distributed adaptive consensus protocols for linear multi-agent systems over directed graphs with relative output information. IET Control Theory Appl. 12, 613–620 (2018)
9. Yu, X., et al.: Consensus of nonlinear multi-agent systems with exogenous disturbances. In: 12th International Conference on Informatics in Control, Automation and Robotics (ICINCO), pp. 281–286 IEEE, Colmar (2015)
10. Wang, Q., et al.: Lq synchronization of discrete-time multiagent systems: A distributed optimization approach. IEEE Trans. Autom. Control 64, 5183–5190 (2019)
11. Wang, Q., et al.: Distributed model predictive control for linear-quadratic performance and consensus state optimization of multiagent systems. IEEE Trans. Cybern. (2020). https://doi.org/10.1109/TCYB.2020.3001347
12. Duan, P., et al.: Distributed finite-horizon extended Kalman filtering for uncertain nonlinear systems. IEEE Trans. Cybern. (2019). https://doi.org/10.1109/TCYB.2019.2919919
13. Hong, H., et al.: Distributed robust fixed-time consensus for nonlinear and disturbed multiagent systems. IEEE Trans. Syst. Man Cybern. Syst. 47, 1464–1473 (2017)
14. Rezaee, H., Abdollahi, F.: Consensus problem in high-order multiagent systems with Lipschitz nonlinearities and jointly connected topologies. IEEE Trans. Syst. Man Cybern. Syst. 47, 741–748 (2017)
15. Agha, R., et al.: Adaptive distributed consensus control of one-sided Lipschitz nonlinear multiagents. IEEE Trans. Syst. Man Cybern. Syst. 49, 568–578 (2019)
16. Altafini, C.: Consensus problems on networks with antagonistic interactions. IEEE Trans. Autom. Control 58, 935–946 (2013)
17. Altafini, C., Lini, G.: Predictable dynamics of opinion forming for networks with antagonistic interactions. IEEE Trans. Autom. Control 60, 342–357 (2015)
18. Shang, Y.: Consensus in averager–copier–voter networks of moving dynamical agents. Chaos 27, 023116 (2017)
19. Meng, D., et al.: Uniform convergence for signed networks under directed switching topologies. Automatica 90, 8–15 (2018)
20. Meng, D., et al.: Finite-time consensus for multiagent systems with cooperative and antagonistic interactions. IEEE Trans. Neural Networks Learn. Syst. 27, 762–770 (2016)
21. Wang, H., et al.: Finite-time bipartite consensus for multi-agent systems on directed signed networks. IEEE Trans. Circuits Syst. I Regul. Pap. 65, 4336–4348 (2018)
22. Zhang, H.: Output feedback bipartite consensus and consensus of linear multi-agent systems. In: 54th Conference on Decision and Control (CDC), pp. 1731–1735 IEEE, Osaka (2015)
23. Zhang, H., Chen, J.: Bipartite consensus of multi-agent systems over signed graphs: State feedback and output feedback control approaches. Int. J. Robust Nonlinear Control 27, 3–14 (2016)
24. Hu, J., et al.: Consensus control of general linear multiagent systems with antagonistic interactions and communication noises. IEEE Trans. Autom. Control 64, 2122–2127 (2019)
25. Guo, X., et al.: Bipartite consensus for multi-agent systems with heterogeneous unknown inertias and control gains under a directed graph. IEEE Trans. Autom. Control 61, 2019–2034 (2016)
26. Teel, A.R.: On $\mathcal{L}_2$ performance induced by feedbacks with multiple saturations. ESAIM: Control Optim. Calculus Variations 1, 225–240 (1996)
27. Roy, S.: Scaled consensus. Automatica 51, 259–262 (2015)
28. Shahvali, M., et al.: Adaptive output-feedback bipartite consensus for nonstrict-feedback nonlinear multi-agent systems: A finite-time approach. Neurocomputing, 2018, 318 (NOV.27): 7–17
29. Shang, Y.: On the structural balance dynamics under perceived sentiment. Bull. Iran. Math. Soc. 46, 717–724 (2020)

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