TYPICAL PROPERTIES OF PERIODIC TEICHMÜLLER GEODESICS

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Abstract. Call a property $P$ for periodic orbits of the Teichmüller flow acting on the moduli space $Q$ of area one abelian differentials on a surface of genus $g$ typical if the growth rate of orbits with property $P$ is maximal. We show that the following property is typical. The eigenvalues of the symplectic matrix defined by the orbit are arbitrarily close to the Lyapunov exponents of $Q$, and its the trace field is a totally real splitting field of degree $g$ over $\mathbb{Q}$. If $g \geq 3$ then orbits with full orbit closures under the action of $SL(2, \mathbb{R})$ are typical as well.

1. Introduction

The mapping class group $\text{Mod}(S)$ of a closed surface $S$ of genus $g \geq 2$ acts by precomposition of marking on the Teichmüller space $T(S)$ of marked complex structures on $S$. The action is properly discontinuous, with quotient the moduli space $\mathcal{M}_g$ of complex structures on $S$.

The goal of this paper is to describe properties of this action which are typical, i.e. which are invariant under conjugation and hold true for typical conjugacy classes of mapping classes in the following sense.

Let $Q \to \mathcal{M}_g$ be the moduli space of area one abelian differentials on $S$. There is a natural $SL(2, \mathbb{R})$-action on $Q$. The action of the diagonal subgroup is the Teichmüller flow $\Phi^t$.

Let $\gamma \subset Q$ be a periodic orbit for $\Phi^t$. Then $\gamma$ determines the conjugacy class of a pseudo-Anosov mapping class $\rho(\gamma) \in \text{Mod}(S)$. Each mapping class acts on the first homology group of the surface. This defines a natural homomorphism

$$\Psi : \text{Mod}(S) \to Sp(2g, \mathbb{Z}).$$

The conjugacy class of the pseudo-Anosov mapping class $\rho(\gamma)$ then determines the conjugacy class $[A(\gamma)]$ of a matrix $A(\gamma) \in Sp(2g, \mathbb{Z})$.

Let $\Gamma$ be the set of all periodic orbits for $\Phi^t$ on $Q$. The length of a periodic orbit $\gamma \in \Gamma$ is denoted by $\ell(\gamma)$. Let $h > 0$ be the entropy of the unique $\Phi^t$-invariant
Borel probability measure on $\mathcal{Q}$ in the Lebesgue measure class. As an application of [EMR12] (see also [EM11, H11]) we showed in [H13] that
\[
\sharp\{\gamma \in \Gamma \mid \ell(\gamma) \leq R\} \sim \frac{e^{hR}}{hR}.
\]

Call a subset $\mathcal{A}$ of $\Gamma$ typical if
\[
\sharp\{\gamma \in \mathcal{A} \mid \ell(\gamma) \leq R\} \sim \frac{e^{hR}}{hR}.
\]
Thus a subset of $\Gamma$ is typical if its growth rate is maximal. The intersection of two typical subsets is typical. The aim of this work is to describe some typical properties for $\Gamma$.

Our first result is valid for a surface of any genus $g \geq 2$. For its formulation, let
\[
1 = \lambda_1 > \lambda_2 > \cdots > \lambda_g > 0
\]
be the positive Lyapunov exponents of the Kontsevich Zorich cocycle with respect to the normalized Lebesgue measure on $\mathcal{Q}$. The fact that there are no multiplicities in the Lyapunov spectrum was shown in [AV07]. For $\gamma \in \Gamma$ let $\alpha_i(\gamma)$ be the $i$-th eigenvalue of the matrix $\frac{1}{\alpha_i(\gamma)}A(\gamma)^iA(\gamma)$ ordered in decreasing order. Thus
\[
1 = \alpha_1(\gamma) \geq \cdots \geq \alpha_g(\gamma) \geq 0 \geq -\alpha_g(\gamma) \geq \cdots \geq -\alpha_1(\gamma) = -1.
\]
As eigenvalues of matrices are invariant under conjugation, for $1 \leq i \leq g$ we obtain in this way a function $\alpha_i : \Gamma \to [-1, 1]$.

The characteristic polynomial of a symplectic matrix $A \in Sp(2g, \mathbb{Z})$ is a reciprocal polynomial of degree $2g$ with integral coefficients. Its roots define a number field $K$ of degree at most $2g$ over $\mathbb{Q}$ which is a quadratic extension of the so-called trace field of $A$. The Galois group of $K$ is isomorphic to a subgroup of the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^g \rtimes \Sigma_g$ where $\Sigma_g$ is the symmetric group in $g$ variables (see [VV02] for details). The trace field and the Galois group only depend on the conjugacy class of $A$.

For $\gamma \in \Gamma$ let $G(\gamma)$ be the Galois group of the number field defined by the conjugacy class $[A(\gamma)]$. We show

**Theorem 1.** The following two subsets of $\Gamma$ are typical.

1. For $\epsilon > 0$ the set $\{\gamma \in \Gamma \mid |\alpha_i(\gamma) - \lambda_i| < \epsilon\}$.
2. The set of all $\gamma \in \Gamma$ such that the trace field of $[A(\gamma)]$ is totally real, of degree $g$ over $\mathbb{Q}$, and $G(\gamma) = (\mathbb{Z}/2\mathbb{Z})^g \rtimes \Sigma_g$.

Call a matrix $A \in Sp(2g, \mathbb{Z})$ liftable if there is a periodic orbit $\gamma \in \Gamma$ so that $A(\gamma) = A$ (up to conjugation). The periodic orbit $\gamma$ is called a lift of $A$. Although the homomorphism $\Psi$ is surjective, there are non-liftable matrices whose preimages under $\Psi$ consist of pseudo-Anosov elements [McM13].

**Theorem 2.** For $g \geq 3$ the following two subsets of $\Gamma$ are typical.

1. The set of $\gamma \in \Gamma$ whose $SL(2, \mathbb{R})$-orbits are dense in $\mathcal{Q}$.
For each $k > 0$ the set \( \{ \gamma \in \Gamma \mid z(\gamma) \in \Gamma \mid [A(\gamma)] = [A(\gamma')] \} \geq k \} \).

As an immediate consequence, we obtain a property reminiscent of properties of hyperbolic surfaces: There are arbitrarily large multiplicities in the length spectrum of $Q$.

For $g = 2$, the first part of Theorem 2 is false in a very strong sense. Namely, McMullen [McM03] showed that in this case, the orbit closure of any periodic orbit with quadratic trace field is contained in the preimage under the Torelli map of the corresponding Hilbert modular surface in the moduli space $A_2$ of principally polarized abelian varieties of rank two. We do not know whether the second part of Theorem 2 holds true for $g = 2$.

**Organization:** The proof of the first part of Theorem 1 is contained in Section 2. Section 3 contains the proof of the second part of Theorem 2 and Section 4 is devoted to the proof of the second part of Theorem 2. The proof of Theorem 2 is completed in Section 5.

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## 2. Lyapunov Exponents

In this section we consider more generally any component $Q$ of a stratum of abelian differentials. In fact, all our results are equally valid for any affine invariant submanifold for the $SL(2,\mathbb{R})$-action [EMM13].

Denote by $\hat{Q}$ a component of the preimage of $Q$ in the Teichmüller space of marked area one abelian differentials. Let $h > 0$ be the entropy of the Teichmüller flow on $Q$ with respect to the invariant Lebesgue measure $\lambda$.

Let $\hat{S} \subset \hat{Q}$ be the set of all points $\hat{q}$ with the property that any element in $\text{Mod}(S)$ which fixes $\hat{q}$ fixes the entire component $\hat{Q}$ pointwise. Then $\hat{S}$ projects to an open dense $\Phi^t$-invariant submanifold $S \subset Q$ (Lemma 4.5 of [H13]). In particular, $S$ has full Lebesgue measure.

Call a point $q \in S$ *recurrent* if it is contained in its own $\alpha$- and $\omega$ limit set. The set of recurrent points in $S$ has full Lebesgue measure.

The following is Lemma 4.7 of [H13] together with the claim in the proof of Lemma 5.3 of [H13].

**Proposition 2.1.** Let $q \in S$ be a recurrent point and let $V$ be a neighborhood of $q$ in $Q$. Then for every $\delta > 0$ there are closed neighborhoods $Z_0 \subset Z_1 \subset Z_2 \subset Z_3 \subset V_0 \subset V$ of $g$ in $Q$ with dense interior, and there are numbers $t_0 > 0, R_0 > 0$ such that for $R > R_0$ the following properties are satisfied.

(i) $V_0$ is contractible.
(ii) $\lambda(Z_3) \leq \lambda(Z_0)(1 + \delta)$. 
(iii) Let \( z \in Z_1 \) and assume that \( \Phi^\tau z = z \) for some \( \tau \in (R-t_0, R+t_0) \). Let \( \bar{E} \) be the component containing \( z \) of the intersection \( \Phi^\tau V_0 \cap V_0 \) and let \( E = \bar{E} \cap \Phi^\tau Z_2 \cap Z_3 \). Then
\[
\lambda(E) \in [e^{-hR}\lambda(Z_1)/(1+\delta), e^{-hR}\lambda(Z_1)(1+\delta)],
\]
and the length of the connected orbit subsegment of \((\cup_{t \in \mathbb{R}} \Phi^t z) \cap Z_1\) containing \( z \) equals \( 2t_0 \). Moreover, the \( \Phi^t \)-orbit through \( z \) is the only periodic orbit for \( \Phi^t \) which intersects \( E \) and whose period is contained in the interval \((R-t_0, R+t_0)\).

(iv) Let \( u \in Z_0 \) be such that \( \Phi^R u \in Z_1 \). Let \( \gamma \) be the arc obtained by closing the orbit segment connecting \( u \) to \( \Phi^R u \) with an arc contained in \( V_0 \); then there is a periodic orbit passing through \( Z_3 \) of length contained in \([R-\delta, R+\delta]\) which contains \( \gamma \) in its \( \delta \)-neighborhood.

Let \( 1 = \lambda_1 \geq \cdots \geq \lambda_g \geq 0 \) be the nonnegative Lyapunov exponents of the Teichmüller flow on \( Q \). If \( Q \) is an entire stratum, then the exponents are all distinct \cite{AV07}, but for arbitrary affine invariant manifolds this need not be the case. We refer to \cite{A13} for a discussion and examples.

Let
\[
\Gamma \subset Q
\]
be the countable collection of all periodic orbits for \( \Phi^t \) in \( Q \). Let \( \epsilon > 0 \). For a periodic orbit \( \gamma \in \Gamma \) define \( \chi_\epsilon(\gamma) = 1 \) if \( |\alpha_i(\gamma) - \lambda_i| < \epsilon \) for all \( i \in \{1, \ldots, g\} \) and define \( \chi_\epsilon(\gamma) = 0 \) otherwise.

For \( R_1 < R_2 \) let \( \Gamma(R_1, R_2) \subset \Gamma \) be the set of all periodic orbits of prime period contained in the interval \((R_1, R_2)\). For an open or closed subset \( V \) of \( Q \) define
\[
H(V, R_1, R_2) = \sum_{\gamma \in \Gamma(R_1, R_2)} \int_{\gamma} \chi(V) \quad \text{and} \quad H_\epsilon(V, R_1, R_2) = \sum_{\gamma \in \Gamma(R_1, R_2)} \int_{\gamma} \chi(V)\chi_\epsilon(\gamma)
\]
where \( \chi(V) \) is the characteristic function of \( V \). In \cite{H13} (Corollary 4.8 and Proposition 5.4) we showed

**Proposition 2.2.** For every recurrent point \( q \in S \), for every neighborhood \( V \) of \( q \) in \( S \) and for every \( \delta > 0 \) there is a closed neighborhood \( Z_3 \subset V \) of \( q \) in \( S \) with the properties specified in Proposition \ref{h3} and a number \( t_0 > 0 \) such that
\[
H(Z_3, R-t, R+t)e^{-hR} \leq 2t\lambda(Z_3)(1-\delta, 1+\delta)
\]
for all \( t < t_0 \) and for all sufficiently large \( R \).

The first part of Theorem \ref{main} now follows from

**Proposition 2.3.** For every \( \epsilon > 0 \), for every recurrent point \( q \in S \), for every neighborhood \( V \) of \( q \) in \( S \) and for every \( \delta > 0 \) there is a closed neighborhood \( Z_3 \subset V \) of \( q \) in \( S \) and there are numbers \( t_0 > 0, R_0 > 10 \) with the properties stated in Proposition \ref{h3} such that for \( t \leq t_0 \) we have
\[
\lim_{R \to \infty} \inf H_\epsilon(Z_3, R-t-\delta, R+t+\delta)e^{-hR} \geq 2t\lambda(Z_3)(1-\delta).
\]
Proof. It suffices to show the following. For all $\sigma > 0$, $\varepsilon > 0$, every recurrent point $q \in \mathcal{S}$ admits neighborhoods $Z_0 \subset Z_3$ with the properties described in Proposition 2.1 such that

$$e^{-hR} \sum_{\gamma \in \Gamma(0,R)} \int_{\gamma} \chi(Z_3) \chi_{\varepsilon}(\gamma) \geq \lambda(Z_0)(1 - \sigma)$$

provided that $R > 0$ is sufficiently large.

Denote by $\mathcal{H}^1(\mathbb{R}) \to \mathbb{Q}$ the lift to $\mathbb{Q}$ of the flat bundle over moduli space whose fibre at $x$ is the real first cohomology group $H^1(S, \mathbb{R})$. Thus the fibre of $\mathcal{H}^1(\mathbb{R})$ is a symplectic vector space of real dimension $2g$.

Let $|||$ be the Hodge norm on the bundle $\mathcal{H}^1(\mathbb{R}) \to \mathbb{Q}$. The Teichmüller flow $\Phi^t$ has a natural lift to a symplectic flow $\Theta^t$ on $\mathcal{H}^1(\mathbb{R})$ determined by the Gauss Manin connection. This flow extension is called the Kontsevich Zorich cocycle for $\Phi^t$.

For $z \in \mathbb{Q}$ and $t > 0$ let $\zeta_i(t, z)$ be the minimum of the operator norms with respect to $||$ of the time-$t$-map of the Kontsevich Zorich cocycle at $z$ restricted to a symplectic subspace of $\mathcal{H}^1(\mathbb{R})_z$ of real dimension $2(g - i + 1)$. Define

$$\lambda_i(t, z) = \frac{1}{t} \log \zeta_i(t, z).$$

Let $\varepsilon > 0$, $\delta > 0$ and let $q \in \mathbb{Q}$ be a recurrent point. Since the Kontsevich Zorich cocycle is locally constant, there is a neighborhood $V_0$ of $q$ with the properties specified in Proposition 2.1 for $\delta$ and that moreover the following property holds true.

Let $Z_0 \subset Z_1 \subset Z_2 \subset Z_3 \subset V_0$ be the subsets as in Proposition 2.1. Let $z \in Z_0$ and let $R > 10$ be such that $\Phi^Rz \in Z_1$. Connect $\Phi^Rz$ to $z$ by an arc in $V_0$ and let $\eta$ be the resulting loop. By (iv) of Proposition 2.1 there is a unique periodic orbit $\gamma$ for $\Phi^t$ which passes through $Z_3$ and intersects $\Phi^RZ_2 \cap Z_3 \cap \hat{E}$ where $\hat{E}$ is the connected component of $\Phi^RV_0 \cap V_0$ containing $z$. This periodic orbit is contained in the $\delta$-neighborhood of $\eta$. We then have

$$|\lambda_i(R, z) - \alpha_i(\gamma)| \leq \varepsilon/2.$$

By Proposition 2.1 moreover the following holds true. Let $R_0 > 0$ be as in the proposition. Let again $z \in Z_0$ and let $R \geq R_0$ be such that $\Phi^Rz \in Z_1$. Let $\hat{E}$ be the connected component containing $z$ of the intersection $\Phi^RV_0 \cap V_0$. Then the Lebesgue measure of the intersection $\Phi^RZ_2 \cap Z_3 \cap \hat{E}$ is contained in the interval

$$[e^{-hR}\lambda(Z_1)/(1 + \delta), e^{-hR}\lambda(Z_1)(1 + \delta)].$$

The length of the connected orbit subsegment of $\bigcup_{t \in \mathbb{R}} \Phi^t z \cap Z_1$ containing $z$ equals $2t_0$.

On the other hand, since the Lebesgue measure is mixing under the Teichmüller flow, for sufficiently large $R > R_0$ we have

$$\lambda(\Phi^RZ_0 \cap Z_1) \geq \lambda(Z_1)^2/(1 + \delta).$$
Together this implies that the number of such intersection components is at least
\[ e^{hR} \lambda(Z_1)/(1 + \delta)^2. \]

By Oszeledec’s theorem and ergodicity, there is number \( R(\epsilon) > 0 \) and a subset \( B \) of \( Z_0 \) of measure \( \lambda(B) > \lambda(Z_0)(1 + \delta)^{-1} \) with the following property. Let \( u \in B \) and let \( R > R(\epsilon) \); then \( |\lambda_i(u, R) - \lambda_i| \leq \epsilon/2. \)

Since the Teichmüller flow is mixing, there is a number \( R > R_0 \) such that
\[ \lambda(\Phi_t B \cap B) \geq \lambda(B)^2(1 + \delta)^{-1} \geq \lambda(Z_0)^2(1 + \delta)^{-3} \]
for all \( t \geq R \). By the above estimate, this means that the number of intersection components containing points in \( B \) is at least \( e^{ht} \lambda(Z_0)(1 + \delta)^{-5} \). By construction, for the periodic orbit contained in each such component the required inequality holds true. This shows the proposition. \( \square \)

3. Galois groups

In this section we prove the second part of Theorem 1. Throughout, we denote by \( Q \) the moduli space of abelian differentials, and by \( \tilde{Q} \) the Teichmüller space of abelian differentials on \( S \).

We use a strategy suggested by [R08]. Let \( p \geq 3 \) be any odd prime and let \( F_p \) be the field with \( p \) elements. There is a natural reduction map
\[ \Lambda_p : Sp(2g, \mathbb{Z}) \to Sp(2g, F_p). \]
Let \( N(p) \) be the cardinality of the group \( Sp(2g, F_p) \).

For a periodic orbit \( \gamma \in \Gamma \) of length \( \ell(\gamma) > 0 \) let \( \delta_\gamma \) be the natural \( \Phi^t \)-invariant measure on \( \gamma \) of total mass \( \ell(\gamma) \). Denote by \( \lambda \) the normalized invariant Lebesgue measure on \( Q \).

A periodic orbit of \( \Phi^t \) on \( Q \) determines a conjugacy class in \( Sp(2g, \mathbb{Z}) \) and hence a conjugacy class in \( Sp(2g, F_p) \). As in Section 2 let \( \tilde{S} \subset \tilde{Q} \) be the open dense set of all points whose stabilizer in Mod(\( S \)) stabilizes \( \tilde{Q} \) pointwise. Let
\[ \Pi : \tilde{Q} \to Q \]
bethe canonical projection and write \( S = \Pi(\tilde{S}) \).

It will be convenient to look at actual elements of \( Sp(2g, F_p) \) rather than at conjugacy classes. To this end choose a recurrent point \( q \in S = \Pi(\tilde{S}) \). Let \( R_0 > 0, t_0 > 0, \delta > 0 \) and let \( Z_0 \subset Z_1 \subset Z_2 \subset Z_3 \subset V_0 \) be a family of neighborhoods of \( q \) in \( \tilde{S} \subset \tilde{Q} \) which satisfy the properties in Proposition 2.1 Use points from \( Z_3 \) as basepoints for loops in \( Q \). By this we mean the following.

Let \( z \in Z_3 \) and let \( T >> 0 \) be such that \( \Phi^T z \in Z_3 \). Connect \( \Phi^T z \) to \( z \) by an arc entirely contained in \( V_0 \). Call the resulting closed curve a characteristic curve. By Lemma 5.1 of [H13] and by Proposition 5.4 of [H13] and its proof, if \( z \in Z_0 \) and if \( \Phi^T z \in Z_1 \) then this curve determines uniquely a periodic orbit \( \gamma \) of \( \Phi^t \) which intersects \( Z_3 \) in an arc of length \( 2t_0 \) (here \( t_0 > 0 \) is determined by the construction of the sets \( Z_i \)). Choose the midpoint of this intersection arc \( \gamma \cap Z_3 \) as a basepoint.
for \( \gamma \) and as an initial point for a parametrization of \( \gamma \). Let \( \Gamma_0 \) be the set of all parametrized periodic orbits of this form. By the third part of Proposition 2.1 of \([H13]\), the map which associates to a component of \( \Phi^T V_0 \cap V_0 \) containing points in \( \Phi^T Z_0 \cap Z_1 \) the corresponding parametrized periodic orbit in \( \Gamma_0 \) is a bijection.

Fix once and for all a lift \( \tilde{V}_0 \) of \( V_0 \) to \( \tilde{Q} \). Recall that \( V_0 \) is contractible. A periodic orbit \( \gamma \in \Gamma_0 \) lifts to a subarc of a flow line of the Teichmüller flow on \( \tilde{Q} \) with starting point in \( \tilde{V}_0 \) whose endpoints are identified by a pseudo-Anosov element \( \Theta(\gamma) \in \text{Mod}(S) \).

The following was proved in \([H13]\).

**Lemma 3.1.** For \( \gamma_1, \ldots, \gamma_k \in \Gamma_0 \), there is a point \( z \in Z_3 \), and there are numbers \( 0 < t_1 < \cdots < t_k \) with the following properties.

1. \( \Phi^{t_i} z \in Z_3 \).
2. The characteristic curve \( \zeta \) of the orbit segment \( \{ \Phi^t \mid 0 \leq t \leq t_k \} \) is contained in the \( \delta \)-neighborhood of a periodic orbit \( \gamma \).
3. \( \Theta(\gamma) = \Theta(\gamma_k) \circ \cdots \circ \Theta(\gamma_1) \).

Recall the homomorphism \( \Psi : \text{Mod}(S) \to Sp(2g, \mathbb{Z}) \). We use Lemma 3.1 to show

**Proposition 3.2.** Let \( B \in Sp(2g, F_p) \) be arbitrary and define

\[
B(R, B) = \{ \gamma \in \Gamma_0 \mid \ell(\gamma) \leq R, \Lambda_p \circ \Psi \circ \Theta(\gamma) = B \}.
\]

Then

\[
\frac{N(p)}{e^{hR}} \sum_{\gamma \in B(R, B)} \delta_\gamma \to \lambda(V_0)^2 \lambda.
\]

**Proof.** We show first that there is a number \( c > 0 \) such that

\[
\#B(R, B) \geq e^{hR}
\]

for all \( B \in Sp(2g, F_p) \) and for all sufficiently large \( R \).

Let \( R_0 > 0, \delta > 0 \) and let \( Z_0 \subset Z_1 \subset Z_2 \subset Z_3 \subset V_0 \) be sets as in Proposition 2.1. By Lemma 3.1 the set \( \{ \Theta(\gamma) \mid \gamma \in \Gamma_0 \} \) generates a subsemigroup \( G \) of \( \text{Mod}(S) \). This semigroup consists of pseudo-Anosov elements corresponding to periodic orbits for \( \Phi^t \) passing through \( Z_3 \). The image of \( \Theta(G) \) under the map \( \Psi : \text{Mod}(S) \to Sp(2g, \mathbb{Z}) \) is a subsemigroup of \( Sp(2g, \mathbb{Z}) \) consisting of Perron Frobenius elements. This semigroup is mapped by \( \Lambda_p \) to a subsemigroup \( G_p \) of the finite group \( Sp(2g, F_p) \). As \( Sp(2g, F_p) \) is finite, for every \( A \in Sp(2g, F_p) \) there is some \( \ell \geq 1 \) such that \( A^\ell = A^{-1} \). As a consequence, for all \( x, y \in G_p \) we have \( xy^{-1} \in G_p \) as well and hence \( G_p < Sp(2g, F_p) \) is a group.

We claim that \( G_p = Sp(2g, F_p) \). To this end assume to the contrary that \( G_p \) is a proper subgroup of \( Sp(2g, F_p) \). Let

\[
\Sigma = \{ \beta \in \text{Mod}(S) \mid \Lambda_p \circ \Psi(\beta) \in G_p \};
\]

then \( \Sigma \) is a subgroup of \( \text{Mod}(S) \) of finite index which defines a finite orbifold covering of \( \mathcal{M}_g \) and a finite orbifold covering \( \tilde{Q} \) of \( Q \).
By construction, every periodic orbit of $\Phi^t$ which corresponds to a component of $\Phi^tV_0 \cap V_0$ containing points in $\Phi^t Z_0 \cap Z_1$ lifts isometrically to a periodic orbit on $\mathcal{Q}$. By the mixing properties of the Teichmüller flow and equidistribution of periodic orbits [H13], the closure of the set of all points on periodic orbits of this form is all of $\mathcal{Q}$. As each such periodic orbit lifts isometrically, we conclude that $\hat{\mathcal{Q}} = \mathcal{Q}$ and hence indeed $G_p = Sp(2g, F_p)$.

Since $Sp(2g, F_p)$ is a finite group, the above discussion shows that there is a number $\hat{R} > R_0$ such that for each $A \in Sp(2g, F_p)$ there is a periodic orbit $\gamma(A) \in \Gamma_0$ of period at most $\hat{R}$ which is determined in the above way by a point $z \in Z_0$ with $\Phi^t z \in Z_1$ and such that $\Lambda_p \Psi(\Theta(\gamma(A))) = A$.

Write $Z = Z_0$. Let $v \in Z$ be such that $\Phi^T v \in Z, \Phi^{T+U} v \in Z$ for some $T, U > 0$. By the version of the Anosov closing lemma from [H13] (see Proposition 2.1), the pseudo-orbits $\{\Phi^t v \mid 0 \leq t \leq T\}$ and $\{\Phi^t v \mid T \leq t \leq T+U\}$ determine two periodic orbits $\gamma_1, \gamma_2$ for $\Phi^t$ which in turn define elements $\Lambda_p(\Theta(\gamma_1)), \Lambda_p(\Theta(\gamma_2)) \in Sp(2g, F_p)$. Let $\gamma = \gamma_2 \gamma_1$ be the periodic orbit for $\{\Phi^t v \mid 0 \leq t \leq T + U\}$. The notation $\gamma = \gamma_2 \gamma_1$ indicates that $\gamma$ is the concatenation of two pseudo-orbits defining $\gamma_1$ and $\gamma_2$. We then have

$$\Lambda_p(\Theta(\gamma_2 \gamma_1)) = \Lambda_p(\Theta(\gamma_1)) \circ \Lambda_p(\Theta(\gamma_2)).$$

Fix an element $B \in Sp(2g, F_p)$. If $A \in Sp(2g, F_p)$ is arbitrary and if $\zeta \in \Gamma_0$ is such that $\Lambda_p(\Theta(\zeta)) = A$ then $\Lambda_p(\Theta(\gamma(\hat{BA}^{-1}) \zeta))) = B$. In particular, for sufficiently large $R > R_0$, the number of periodic orbits $\gamma \in \Gamma_0$ with $\ell(\gamma) \leq R + \hat{R}$ and $\Lambda_p(\Theta(\gamma))) = B$ is not smaller than $c e^{\frac{hR}{\rho}}$ where $\nu = \lambda(Z) > 0$ and where $c > 0$ does not depend on $B$.

As a consequence, up to passing to a subsequence, the measures

$$e^{-hR} \sum_{\gamma \in B(R, B)} \delta_{\gamma}$$

converge to a measure $\lambda_B$ of total volume contained in $[c, 1]$. Since periodic orbits are equidistributed for the Lebesgue measure, this measure is contained in the Lebesgue measure class. Moreover, it is invariant under the Teichmüller flow. Now the Lebesgue measure $\lambda$ is ergodic under $\Phi^t$ and hence $\lambda_B = c(B) \lambda$ for a number $c(B) \in [c, 1]$.

Our goal is to show that $c(B) = 1/N(p)$ independent of $B$. To this end recall that the Lebesgue measure $\lambda$ is mixing of all orders. In particular, for large enough numbers $R, S > 0$ we have

$$\lambda(Z \cap \Phi^R Z \cap \Phi^{R+S}(Z)) \sim \lambda(Z)^3.$$ 

Fix a number $\epsilon > 0$. Let $B \in Sp(2g, F_p)$ be such that $c(B)$ is minimal. Such an element exists since $Sp(2g, F_p)$ is finite. For $R > 0, T > 0$ let $\Gamma(T, R, B, Z)$ be the set of all periodic orbits $\gamma \in \Gamma_0$ for $\Phi^t$ with the following properties.

1. The length of $\gamma$ is contained in the interval $[T + R - 2\epsilon, T + R + 2\epsilon]$. 
(2) $\gamma$ is determined by some $v \in \mathbb{Z}$, and there is a number $U \in [R - \epsilon, R + \epsilon]$ such that $\Phi^U v \in \mathbb{Z}$.

(3) $\Lambda_p \Psi(\rho(\gamma)) = B$.

Define similarly a set $\Gamma'(R, B, Z)$ where we require only a single return to $Z$ within the interval $[R - \epsilon, R + \epsilon]$. It follows from the above discussion that

$$\sharp \Gamma'(R, B, Z) \sim c(B) \lambda(Z) e^{hR}$$

for large enough $R$.

Let $R_0 > 0$ be sufficiently large that the following holds true. Let $R \geq R_0$ and let

$$\mu(T, R, B, Z) = \sum_{\gamma \in \Gamma'(T, R, B, Z)} \delta_\gamma;$$

then $\mu(T, R, B, Z)(Z) \sim c(B) \lambda(Z)^3$.

Each orbit $\gamma \in \Gamma'(T, R, B, Z)$ can be represented in the form $\gamma = \gamma_1 \hat{\gamma}_2$ for some $\gamma_1 \in \Gamma'(T, A, Z)$ and some $\gamma_2 \in \Gamma'(R, C, Z)$. Moreover, if $\gamma_1 \in \Gamma'(T, A, Z)$ for some $A \in Sp(2g, \mathbb{F}_p)$ then for any $\gamma_2 \in \Gamma'(R, B \circ A^{-1})$ we have $\gamma_1 \hat{\gamma}_2 \in \Gamma'(T, R, B, Z)$. In particular, the orbit $\gamma_2 \hat{\gamma}_1$ contributes towards $\mu(T, R, B, Z)$. Since $c(B)$ is minimal, for sufficiently large $R > 0$ and as $T \to \infty$ we observe that

$$\sharp \Gamma'(T, R, B, Z) \sim \sum_{A \in Sp(2g, \mathbb{F}_p)} \# \Gamma'(T, BA^{-1}, Z) \Gamma'(R, A, Z) \geq \frac{e^{h(T+R)}}{h(T+R)} (c(B) - \epsilon) \lambda(Z) \sum_A \Gamma'(R, A, Z) \geq (c(B) - \epsilon) \lambda(Z)^3.$$ 

Assume that there is some $A \in Sp(2g, \mathbb{F}_p)$ such that $c(A) > c(B) + \rho$ for some $\rho > 2\epsilon$. By the choice of $B$ we know that $c(BA^{-1}) \geq c(B)$. As a consequence, in the above sum, the term

$$\Gamma'(T, BA^{-1}, Z) \Gamma'(R, A, Z)$$

contributes towards the sum with a contribution of at least $(c(B) + \rho)c(B)$, independent of $\epsilon$. For sufficiently small $\epsilon$, this is impossible. The proposition follows. \hfill \Box

Now we are ready to complete the proof of the second part of Theorem\[1\] To this end recall that the characteristic polynomial of a symplectic matrix $A \in Sp(2g, \mathbb{Z})$ is reciprocal of degree $2g$, and every reciprocal polynomial of degree $2g$ arises. The roots of such a polynomial come in pairs. The Galois group of the number field defined by the characteristic polynomial of $A$ is a subgroup of the semidirect product

$$(\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$$

where $S_g$ is the symmetric group in $g$ elements (see [VV02] for a nice account on this classical fact), and $S_g$ acts on $(\mathbb{Z}/2\mathbb{Z})^g$ by permutation of the factors. In the sequel we call the Galois group of the field defined by the characteristic polynomial of a matrix $A \in Sp(2g, \mathbb{Z})$ simply the Galois group of $A$. It only depends on the conjugacy class of $A$. We say that the Galois group of $A$ is full if it coincides with $(\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$. 

Having a full Galois group makes also sense for an element in $Sp(2g, F_p)$. We use this as in [R08] as follows.

Let $p \geq 5$ be a prime. By Proposition 3.2, for large enough $R$ and every $B \in Sp(2g, F_p)$, the number of periodic orbits $\gamma \in \Gamma_0$ of length at most $R$ with $\Lambda_p \circ \Psi \circ \Theta(\gamma) = B$ roughly equals $\frac{e^{\lambda R} N(p) \lambda(V_0)^2}{N(p) R^2}$ where $N(p)$ is the number of elements in $Sp(2g, F_p)$. On the other hand, if we denote by $R_p(2g)$ the subset of $Sp(2g, F_p)$ of elements with reducible characteristic polynomial then

$$|R_p(2g)| \ll N(p)^{3/3g}$$

(see Theorem 6.2 of [R08] for a reference for this classical result of Borel).

Now we follow the proof of Theorem 6.2 of [R08]. Namely, let $p_1, \ldots, p_k$ be $k$ distinct primes, and let $K = p_1 \cdots p_k$. Then the reduction $\Lambda_K(A)$ modulo $K$ of any element $A \in Sp(2g, \mathbb{Z})$ is defined, and we have

$$\Lambda_K(A) = \Lambda_{p_1}(A) \times \cdots \times \Lambda_{p_k}(A).$$

Thus by Proposition 3.2 for large enough $R$ the probability that the Galois group of the characteristic polynomial of $A(\gamma)$ for a periodic orbit of length at most $R$ is not full is at most of the order of $(1 - \frac{1}{3g})^k$. Now as $k \to \infty$, we conclude that the Galois group of a typical periodic orbit for $\Phi^t$ is full.

Let $\omega \in \tilde{Q}$ be a lift of a point on a typical periodic orbit for $\Phi^t$. The periods of $\omega$ define an abelian subgroup $\Lambda = \omega(H_1(S, \mathbb{Z}))$ of $\mathbb{C}$ of rank two. Let $e_1, e_2 \in \Lambda$ be two points which are linearly independent over $\mathbb{R}$. Let $K$ be the smallest subfield of $\mathbb{R}$ such that every element of $\Lambda$ can be written as $ae_1 + be_2$, with $a, b \in K$; then $\Lambda \otimes_K K = K^2$. If we write $T = \Psi(A(\gamma)) + \Psi(A(\gamma))^{-1}$, then the field $K$ also is the field of the characteristic polynomial of $T$. We call $K$ the trace field of $\beta$.

**Definition 3.3.** The pseudo-Anosov element $\beta$ is called algebraically primitive if the trace field $K$ of $\beta$ is a totally real number field of degree $g$ over $\mathbb{Q}$, with maximal Galois group.

If $\beta$ is algebraically primitive, then so is every conjugate of $\beta$. Thus it makes sense to talk about algebraically primitive periodic orbits in $\mathbb{Q}$. The length of an algebraically primitive periodic orbit $\gamma$ in $\mathbb{Q}$ is the logarithm of the Perron Frobenius eigenvalue of the corresponding matrix $A(\gamma)$.

The following corollary completes the proof of Theorem 2

**Corollary 3.4.** Algebraically primitive periodic orbits for $\Phi^t$ are typical.

**Proof.** Since the Lyapunov spectrum of $\mathbb{Q}$ is simple [AV07], by the first part of Theorem 2 we only have to show that for a symplectic matrix $A \in Sp(2g, \mathbb{R})$ with 2g distinct real eigenvalues $r_i, r_i^{-1}$ ($i \leq g, r_i > 1$) the field defined by $A + A^{-1}$ is totally real. However, this is immediate from the fact that the roots of the polynomial defining the trace field are of the form $r_i + r_i^{-1}$ where $r_i$ are the roots of the characteristic polynomial of $A$. \qed
4. Liftability

Call an element $A \in Sp(2g, \mathbb{Z})$ a candidate if the following conditions are satisfied.

1. $A$ is irreducible.
2. $A$ is Perron-Frobenius.
3. $A$ is not cyclotomic.

The preimage under the homomorphism $\Psi : \text{Mod}(S) \to Sp(2g, \mathbb{Z})$ of every candidate consists of pseudo-Anosov mapping classes [CBSS].

**Definition 4.1.** An element $A \in Sp(2g, \mathbb{Z})$ is called liftable if there is a periodic orbit $\gamma$ in $Q$ so that $A(\gamma) = A$.

If $A \in Sp(2g, \mathbb{Z})$ is liftable then $A$ is necessarily a candidate. However, although the canonical homomorphism $\Psi : \text{Mod}(S) \to Sp(2g, \mathbb{Z})$ is surjective, not every candidate $A \in Sp(2g, \mathbb{Z})$ is liftable. An explicit example was given by McMullen [McM13].

As periodic orbits in $Q$ define conjugacy classes in $\text{Mod}(S)$, if $A \in Sp(2g, \mathbb{Z})$ is liftable then so is every conjugate of $A$.

The following example is a variation of a construction which was communicated to me by Chris Leininger.

**Example:** Let $a, b$ be non-separating simple closed curves on a closed surface $S$ of genus 2. Assume that $S - (a \cup b)$ consists of a collection of topological discs and that for a choice of an orientation of $a$ and an orientation of $b$, all intersections between $a$ and $b$ are positive. Let $T_a, T_b$ be the positive Dehn twist about $a, b$. Then $\zeta = T_b^{-1}T_a$ is a pseudo-Anosov mapping class, moreover $\zeta$ preserves an abelian differential (see [L04]).

Let $\Sigma$ be the surface obtained from $S$ by a two-sheeted covering as follows. Cut $S$ open along $a$, take two copies of the cut open surface and cross-glue. The oriented curve $a$ lifts to two oriented curves $a_1, a_2$ in $\Sigma$ so that $a_1$ is homologous to $a_2^{-1}$. The preimage of the curve $b$ is a multicurve $\hat{b}$ in $\Sigma$ which consists of one or two components. Assume that $\hat{b}$ is connected; this is the case if $b$ intersects $a$ in an odd number of points.

Consider the curve system $a_1 \cup \hat{b}$ on $\Sigma$. It decomposes $\Sigma$ into topological discs. Similarly, the curve system $a_2 \cup \hat{b}$ decomposes $\Sigma$ into discs. As a consequence, the mapping classes $\psi_1 = T_b^{-1}T_{a_1}$ and $\psi_2 = T_b^{-1}T_{a_2^{-1}}$ of $\Sigma$ are pseudo-Anosov, moreover they preserve orientable foliations. As the curves $a_1, a_2$ are homologous, the actions of $\psi_1, \psi_2$ on $H_1(\Sigma, \mathbb{Z})$ coincide (compare [L04]).

Note that the flat structure of the abelian differential $\omega_i$ defined by the curve system $a_i \cup \hat{b}$ is square tiled [L04]. In particular, the $SL(2, \mathbb{R})$-orbit of the abelian differential $\omega_i$ is closed and arises from a branched cover of the two-torus. The two branched covers $\Sigma \to T^2$ are conjugated by the deck group of the covering $\Sigma \to S$. 
The Veech groups of the two square tiled surfaces are normalized by the mapping class \( \varphi \) but not centralized.

**Remark 4.2.** The construction in the above example generalizes to curve systems on \( S \) with more than one component and yields candidates with more than one lift in every genus \( g \geq 3 \). We do not know whether there are candidates with more than one lift for surfaces of genus 2.

We next show Theorem 4.2 from the introduction. It can be viewed as a generalization of the above example.

**Proposition 4.3.** For \( g \geq 3 \) and for every \( k > 0 \), the set of all orbits \( \gamma \in \Gamma \) so that \( A(\gamma) \) has at least \( k \) lifts is typical.

**Proof.** Choose a bounding pair \( c_1, c_2 \) for \( S \), oriented in such a way that \( c_1 \) and \( c_2 \) are homologous. Then \( S - (c_1 \cup c_2) \) is a disconnected surface whose components have positive genus.

There is a birecurrent oriented train track \( \tau \) on \( S \) which decomposes \( S \) into quadrangles and which contains \( c_1, c_2 \) as oriented embedded circles. We may assume that this train track is orientable and adapted to the principal stratum of abelian differentials as described in \([H11]\). Every orbit of \( \Phi^t \) on \( Q \) can be encoded by a splitting and shifting sequence of train tracks whose topological types form a finite alphabet \([H11]\). An orbit which is typical for the Lebesgue measure is encoded by a normal sequence containing every admissible finite subsequence infinitely often. In particular, a typical sufficiently long orbit segment contains a subword containing a multi-twist of length at least \( k \) about each of the two curves \( c_1, c_2 \). By the results in Section 2, this then also holds true for a train track expansion of a typical periodic orbit.

Let \( T_i \) be the positive Dehn twist about \( c_i \). Let \( \gamma \) be such a periodic orbit and choose the train track expansion in such a way that it ends with an \( \ell \)-multitwist about \( c_1 \) and \( c_2 \) for some \( \ell \geq k \). Let \( \varphi \) be the thus defined pseudo Anosov element. Then for each \( j \leq k \) there is another pseudo-Anosov element \( \zeta_j \) with a train track expansion which differs from the one for \( \gamma \) by decreasing the twist about \( c_1 \) by \( j \leq k \) and increasing the twist about \( c_2 \) by \( j \). Then \( \zeta_j = T_1^{-j}T_2^j \varphi \) and hence \( \varphi \zeta_j^{-1} \) is contained in the Torelli group for all \( j \). As \( \varphi \) and \( \zeta_j \) have train track expansions modeled on orientable train tracks, they give rise to distinct periodic orbits in \( Q \) with the same action on homology. This yields the proposition. \( \square \)

5. **Orbit closures**

The goal of this section is to show the first part of Theorem 4.2 from the introduction. Assume from now on that \( g \geq 3 \).

There is a natural action of the group \( SL(2, \mathbb{R}) \) on the Teichmüller space \( \hat{Q} \) of marked abelian differentials which commutes with the action of the mapping class group. For any \( \omega \in \hat{Q} \), the projection \( P(SL(2, \mathbb{R})(\omega)) \) to \( T(S) \) of the \( SL(2, \mathbb{R}) \)-orbit of \( \omega \) is a holomorphic disc in \( T(S) \) which is called an *abelian Teichmüller disc*. 
Let $U(g)$ be the unitary group of rank $g$. By a result of Kra [Kr81], the image of an abelian Teichmüller disc under the Torelli map

$$\mathcal{J} : \mathcal{T}(S) \to \mathcal{D}_g = Sp(2g, \mathbb{R})/U(g)$$

is a Kobayashi geodesic in the Siegel upper half-space $\mathcal{D}_g$. A point in $\mathcal{D}_g$ is a principally polarized abelian variety of dimension $g$. There is a natural holomorphic vector bundle $\tilde{\mathcal{V}} \to \mathcal{D}_g$ whose fibre over $y$ is just the complex vector space defining $y$. The polarization and the complex structure define a Hermitean metric $h$ on $\tilde{\mathcal{V}}$. The group $Sp(2g, \mathbb{R})$ acts from the left on this bundle preserving the Hermitean metric. Thus the bundle $\tilde{\mathcal{V}}$ projects to a holomorphic Hermitean vector bundle $\mathcal{V} \to Sp(2g, \mathbb{Z})/Sp(2g, \mathbb{R})/U(g) = \mathcal{A}_g$.

The Hermitean holomorphic vector bundle $\mathcal{V}$ admits a unique Chern connection $\nabla$ (see e.g. [GH78]). The Chern connection defines parallel transport of the fibres along smooth curves. As the Chern connection is Hermitian, parallel transport preserves the metric.

Let again $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus $g$. The Hodge bundle

$$\mathcal{P} : \mathcal{H} \to \mathcal{M}_g$$

is just the pullback of the tautological holomorphic bundle $\mathcal{V}$ on $\mathcal{A}_g$ under the Torelli map. Thus $\mathcal{H}$ is a $g$-dimensional holomorphic vector bundle on $\mathcal{M}_g$. The natural Hermitean inner product on $\mathcal{H}$ is given by

$$(\omega, \zeta) = \frac{i}{2} \int \omega \wedge \overline{\zeta}.$$ 

For $q \in \mathcal{H}$ let $q^\perp$ be the orthogonal complement of $q$ in $\mathcal{H}$ for this Hermitean metric. Note that $q^\perp$ is a $g - 1 \geq 2$-dimensional complex subspace of $\mathcal{H}$. The sphere bundle in $\mathcal{H}$ for the inner product $(,)$ equals the moduli space $\mathcal{Q}$ of area one abelian differentials.

Let $\mathcal{H}_+ \subset \mathcal{H}$ be the complement of the zero section in the Hodge bundle $\mathcal{H}$. This is a complex orbifold. The pull-back

$$\mathcal{L} \to \mathcal{H}_+$$

to $\mathcal{H}_+$ of the Hodge bundle on $\mathcal{M}_g$ is a holomorphic vector bundle over $\mathcal{H}_+$. The Hermitian structure on $\mathcal{H}$ defines a Hermitian structure on $\mathcal{L}$. The bundle $\mathcal{L}$ naturally splits as a direct sum

$$\mathcal{L} = \mathcal{T} \oplus \mathcal{L}^\perp$$

of holomorphic vector bundles. Here the fibre of $\mathcal{V}$ over a point $q$ is just the $\mathbb{C}$-span of $q$, and the fibre of $\mathcal{L}^\perp$ is the orthogonal complement of $\mathcal{V}$ with respect to the Hermitian metric. As a holomorphic vector bundle, it is equivalent to the quotient $\mathcal{L}/\mathcal{T}$.

Let

$$\mathcal{P} \to \mathcal{Q}$$

be the bundle over $\mathcal{Q}$ whose fibre at $q$ is the projectivization of the restriction of $\mathcal{L}^\perp$ to $q$. The fibre of $\mathcal{P}$ over $q$ is a complex projective space $\mathbb{C}P^{g-2}$. As $g \geq 3$, this fibre is non-trivial. There is a Borel probability measure $\lambda_P$ on $\mathcal{P}$ in the Lebesgue
measure class which projects onto the $\Phi^t$-invariant normalized Lebesgue measure $\lambda$ on $Q$. Conditional measures on the fibres are just the normalized Lebesgue measures induced by the Fubini Study metric.

The projection to $M_g$ of an $SL(2,\mathbb{R})$-orbit on $Q$ is an abelian Teichmüller disc $\mathcal{C}$. Its image under $J$ is a Kobayashi geodesic $[KIN]$ in $D_g$. Thus the complex curve $J(\mathcal{C})$ has no singular points, and the restriction to $J(\mathcal{C})$ of the holomorphic vector bundle $\hat{V}$ naturally splits as a sum of the holomorphic tangent bundle of $J(\mathcal{C})$ and a holomorphic vector bundle $\hat{V}^\perp$ which is just the orthogonal complement of $T_J(\mathcal{C})$ for the Hermitean metric (see [MV10] for details). By naturality, this decomposition is preserved by the Chern connection for $\hat{V}$ restricted to $J(\mathcal{C})$, i.e. it is invariant under parallel transport along curves in $J(\mathcal{C})$.

By naturality, the restriction of the Chern connection $\nabla$ for $L$ to the orbits of the $SL(2,\mathbb{R})$-action preserves the decomposition $L = T \oplus L^\perp$. Thus $\nabla$ defines parallel transport of the fibres of $P$ along curves in the leaves of the $SL(2,\mathbb{R})$-foliation of $Q$. This parallel transport preserves the Lebesgue measures on the fibres, i.e. it preserves the conditional measures of $\lambda_P$.

As a consequence, the Teichmüller flow and the horocycle flow to $P$ lift to flows on $P$ which we call frame flows. Since the Teichmüller flow and the horocycle flow preserve the Lebesgue measure $\lambda$ on $Q$, these frame flows preserve the measure $\lambda_P$. We observe as in [BG80] Lemma 5.1.

**Lemma 5.1.** The measure $\lambda_P$ is ergodic under the frame flows $\Psi^t, H^t$ covering the Teichmüller flow and the horocycle flow.

**Proof.** We show the lemma for the frame flow $\Psi^t$, the argument for the flow $H^t$ is completely analogous.

A point in $P$ can be viewed as a pair consisting of a unit vector $q$ in a fibre $S^{2g-1} \subset \mathbb{C}g$ of the projection $Q \to \mathcal{M}_g$ and a complex line $Z$ in the orthogonal complement of the complex line in $\mathbb{C}g$ spanned by $v$. Thus if we view $P$ as a bundle over $\mathcal{M}_g$, then its structure group is the group $U(g)$.

The action of the Teichmüller flow $\Phi^t$ on $Q$ with respect to the Lebesgue measure is ergodic. Thus the algebraic hull $L$ of the cocycle over $\Phi^t$ defining the action on the frame flow $\Psi^t$ on $P$ is defined. This hull defines a measurable reduction of the structure group of $P$ to $L$.

By Proposition 9.2.6 of [ZS4], the restriction of the cocycle to the measurable bundle with structure group $L$ is ergodic. Now if $g \equiv 1$ mod 2 then there is no proper reduction of the structure group $U(g)$ for the bundle over $S^{2g-1}$ whose fibre over $v$ is the complex projective space of complex lines in the orthogonal complement of $v$. Ergodicity of the frame flow $\Psi^t$ now follows as in [BG80].

If $g \equiv 0$ mod 2 then the only possible reductions of the structure group which are subgroups of $U(g)$ are the groups $Sp(g/2), Sp(g/2) \times U(1), Sp(g) \times Sp(1)$. This reduction then defines a $\Phi^t$-invariant quaternionic structure on $Q$. 
However, this would imply the existence of a \(\text{Mod}(\mathcal{S})\)-invariant quaternionic structure on the Hodge bundle over Teichmüller space. As the mapping class group surjects onto \(Sp(2g, \mathbb{Z})\), this is impossible. □

Now let \(\gamma\) be a periodic orbit for \(\Phi^t\). Then \(\gamma\) defines an orbit for the action of \(SL(2, \mathbb{R})\). Denote by \(C(\gamma)\) the closure of this orbit in \(\mathcal{Q}\). By [EMM13], \(C(\gamma)\) is an invariant affine submanifold of \(\mathcal{Q}\). Moreover, \(C(\gamma)\) is a projective variety [F13]. In particular, the affine structure is compatible with the Hodge structure and the holomorphic tangent bundle \(TC(\gamma)\) of \(C(\gamma)\) is defined. This holomorphic tangent bundle projects to a holomorphic subbundle \(P(TC(\gamma)) \subset \mathcal{H}|C(\gamma)\).

The next proposition is the main step towards a proof of the first part of Theorem 2.

**Proposition 5.2.** If \(\gamma\) is a typical periodic orbit then
\[
P(TC(\gamma)) = \mathcal{H}|C(\gamma).
\]

**Proof.** Let \(H^1(S, \mathbb{C})\) be the flat vector bundle over \(\mathcal{Q}\) whose fibre over \(z\) is the complex first cohomology group of \(S\). Let \(\gamma\) be a periodic orbit. As the affine invariant submanifold \(C(\gamma)\) is defined in period coordinates, given by relative periods, there is a natural forgetful map \(p : TC(\gamma) \to H^1(S, \mathbb{C})\). By [W14], there is a decomposition \(H^1(S, \mathbb{C}) = (\oplus \rho V_\rho) \oplus W\) where \(p(TC(\gamma))\) is a summand of \(\oplus \rho V_\rho\) (and the other summands are Galois conjugates in the sense worked out in [W14] which however will not be relevant for our purpose). This is a decomposition of flat vector bundles.

It follows from [PT13] that this decomposition is compatible with the Hodge decomposition. In particular, if \(p(TC(\gamma)) \neq H^1(S, \mathbb{C})\) then \(p(TC(\gamma))\) defines a non-trivial holomorphic subbundle of the restriction of \(\mathcal{H}\) to \(C(\gamma)\) (which is just the bundle \(P(TC)\) in the statement of the proposition).

Let \(O \subset C(\gamma)\) be the \(SL(2, \mathbb{R})\)-orbit of \(\gamma\). By construction, \(O\) is dense in \(C(\gamma)\). For every \(q \in O\), the tangent space of \(O\) at \(q\) defines a complex one-dimensional subspace \(T_q\) of \(p(TC(\gamma))_q\).

We first show that for orbit closures of typical periodic orbits, either \(T_q = p(TC(\gamma))_q\) for all \(q\) or \(p(TC(\gamma)) = \mathcal{H}\).

To this end choose a point \(q \in \mathcal{Q}\) which is typical for both the Teichmüller flow and the horocycle flow. Assume moreover that \(q\) is a smooth point of \(\mathcal{Q}\). Almost every \(q \in \mathcal{Q}\) will do. Let \(Z \in \mathcal{P}_q\) be a point in \(\mathcal{P}\) over \(q\) and assume that \(Z\) is typical for the extensions \(\Psi^t, H^t, \Phi^t, h^t\) to \(\mathcal{P}\). Once again, almost every point has this property.

Choose a neighborhood \(U\) of \(q\) in \(\mathcal{Q}\) diffeomorphic to a ball such that the restriction of \(\mathcal{P}\) to \(U\) admits a bundle trivialization. We use this fixed bundle trivialization to identify the fibres of \(\mathcal{P}\) over \(U\) with the fibre over \(q\).

The structure group \(U(q)\) of the bundle \(\mathcal{P}\) is compact and therefore every closed subgroup of \(U(q)\) is a compact Lie group. Moreover, there are open sets \(V_1, \ldots, V_k\)
in $U(g)$ so that any subgroup of $U(g)$ generated by elements $y_1, \ldots, y_k$ with $y_i \in V_i$ is dense.

By ergodicity of the action of $H^t$ on $\mathcal{P}$ we can find $s_k > \cdots > s_1$ such that the following holds true.

1. $h^{s_i}q \in U$ for all $i$.
2. The return map on the fibres of $\mathcal{P}$ induced by $H^{s_i}$ is contained in $V_i$.

Since the horocycle frame flow $H^t$ on $\mathcal{P}$ is smooth, there is a neighborhood $V \subset U$ of $q$ in $U$ with the following property. If $z \in V$ then $H^{s_i}z \in U$, moreover the above properties hold true as well. However, this means that over closures of orbits of the horocycle flow issuing from $V$, the structure group of $\mathcal{P}$ can not be reduced. By the description of typical periodic orbits for $\Phi^t$ in the previous sections, this implies that for a typical periodic orbit $\gamma$ for $\Phi^t$, we indeed either have $P(T\mathcal{C}(\gamma)) = \mathcal{H}$ as claimed, or the restriction to the $SL(2,\mathbb{R})$-orbit of the invariant subbundle coincides with the complex one-dimensional subbundle $T$.

We have to rule out the second case. To this end recall that the trace field of a typical periodic orbit is a number field of degree $g$ over $\mathbb{Q}$. By [W14], in the second case the field of definition for $\mathcal{C}(\gamma)$ in the sense of [W14] is this number field, and the bundle $H^1(S, \mathbb{C})$ splits over $\mathcal{C}(\gamma)$ as a sum of $g$ complex line bundles.

Using [F13], this decomposition is compatible with the Hodge decomposition and hence we obtain a decomposition of $\mathcal{H}$ over $\mathcal{C}(\gamma)$ as a sum of $g$ complex line bundles. This was ruled out by the above discussion. The proposition follows.

We can now complete the proof of the first part of Theorem 2 as follows. Let $\mathcal{C}(\gamma)$ be the orbit closure of a typical periodic orbit. It is contained in the principal stratum.

Theorem 4.1 of [Mo08] shows that $\mathcal{C}(\gamma)$ either is all of $\mathcal{Q}$, or it is the preimage of the hyperelliptic locus in $\mathcal{M}_g$, or it parametrizes curves with a Jacobian whose endomorphism ring is strictly larger than $\mathbb{Z}$. The second case can be ruled out for typical $\gamma$ since the hyperelliptic locus in $\mathcal{M}_g$ is a proper subvariety of positive codimension. Thus we have to rule out the third case.

However, in the third case, the projection $PT(\mathcal{C}(\gamma))$ is a proper holomorphic subbundle of $\mathcal{H}|\mathcal{C}(\gamma)$ which violates Proposition 5.2.

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