Perfect matchings and Hamiltonicity in the Cartesian product of cycles

John Baptist Gauci
Department of Mathematics, University of Malta, Malta.

Jean Paul Zerafa
Dipartimento di Scienze Fisiche, Informatiche e Matematiche,
Università di Modena e Reggio Emilia, Via Campi 213/B, Modena, Italy

Abstract

A pairing of a graph \( G \) is a perfect matching of the complete graph having the same vertex set as \( G \). If every pairing of \( G \) can be extended to a Hamiltonian cycle of the underlying complete graph using only edges from \( G \), then \( G \) has the PH–property. A somewhat weaker property is the PMH–property, whereby every perfect matching of \( G \) can be extended to a Hamiltonian cycle of \( G \). In an attempt to characterise all 4–regular graphs having the PH–property, we answer a question made in 2015 by Alahmadi et al. by showing that the Cartesian product \( C_p □ C_q \) of two cycles on \( p \) and \( q \) vertices does not have the PMH–property, except for \( C_4 □ C_4 \) which is known to have the PH–property.

Keywords: Cartesian product of cycles, Hamiltonian cycle, perfect matching.

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1 Introduction

All graphs considered are finite, simple (without loops or multiple edges) and connected. A perfect matching of a graph \( G \) is a set of independent edges of \( G \) which cover the vertex set \( V(G) \) of \( G \). If for a given perfect matching \( M \) of \( G \) there exists another perfect matching \( N \) of \( G \) such that \( M \cup N \) is a Hamiltonian cycle of \( G \), then we say that \( M \) can be extended to a Hamiltonian cycle. A graph admitting a perfect matching has the Perfect–Matching–Hamiltonian property (for short the PH–property) if each of its perfect matchings can be extended to a Hamiltonian cycle. In this case we also say that \( G \) is PHM. Graphs having this property and other similar concepts have been studied by various authors such as in [1, 2, 3, 5, 6, 7, 8, 9, 10]. For a more detailed introduction to the subject we suggest the reader to [1].

The path graph, cycle graph and complete graph on \( n \) vertices are denoted by \( P_n, C_n \) and \( K_n \), respectively. A vertex of degree one is called an end vertex. For any graph \( G \), \( K_G \) denotes the complete graph on the same vertex set \( V(G) \) of \( G \). Let \( G \) be of even order. A perfect matching of \( K_G \) is said to be a pairing of \( G \). In [2], the authors say that a graph \( G \) has the Pairing–Hamiltonian property (for short the PH–property) if every pairing \( M \) of \( G \) can be extended to a Hamiltonian cycle \( H \) of \( K_G \) in which \( E(H) - M \subseteq E(G) \). Clearly, this is a stronger property than the PMH–property and if a graph has the PH–property then

E-mail addresses: john-baptist.gauci@um.edu.mt (John Baptist Gauci), jeanpaul.zerafa@unimore.it (Jean Paul Zerafa)
it is also PMH. Amongst other results, the authors characterise which cubic graphs have the PH–property: $K_4$, the complete bipartite graph $K_{3,3}$ and the 3–dimensional hypercube $Q_3$. Most of the notation and terminology that we use in the sequel is standard, and we refer the reader to [4] for definitions and notation not explicitly stated.

Having a complete characterisation of cubic graphs that have the PH–property, a natural pursuit would be to characterise 4–regular graphs having the same property, as also suggested by the authors in [2]. Although Seongmin Ok and Thomas Perrett privately communicated to the authors of [2] the existence of an infinite family of 4–regular graphs having the PH–property, it was suggested to tackle this characterisation problem by looking at the Cartesian product of two cycles $C_p \square C_q$ (Open Problem 3 in [2]). In particular, the authors ask for which values of $p$ and $q$ does $C_p \square C_q$ have the PH–property.

In this work we show that $C_p \square C_q$ has the PH–property only when both $p$ and $q$ are equal to 4. In fact, the graph $C_4 \square C_4$ is isomorphic to the 4–dimensional hypercube $Q_4$, which was proved to have the PH–property in [5] together with all other $n$–dimensional hypercubes. More precisely, we show that except for $Q_4$, $C_p \square C_q$ is not PMH.

2 Main Result

**Definition 2.1.** The Cartesian product $G \square H$ of two graphs $G$ and $H$ is a graph whose vertex set is the Cartesian product $V(G) \times V(H)$ of $V(G)$ and $V(H)$. Two vertices $(u_i, v_j)$ and $(u_k, v_l)$ are adjacent precisely if $u_i = u_k$ and $v_jv_l \in E(H)$ or $u_iu_k \in E(G)$ and $v_j = v_l$. Thus,

$$V(G \square H) = \{(u_r, v_s) : u_r \in V(G) \text{ and } v_s \in V(H)\},$$

and

$$E(G \square H) = \{(u_i, v_j)(u_k, v_l) : u_i = u_k, v_jv_l \in E(H) \text{ or } u_iu_k \in E(G), v_j = v_l\}.$$

For simplicity, we shall refer to the vertex $(u_r, v_s)$ as $\omega_{r,s}$. In this work we restrict our attention to the Cartesian product of a cycle graph and a path graph and to that of two cycle graphs, noting that the latter is also referred to in literature as a torus grid graph. In the sequel we tacitly assume that operations (including addition and subtraction) in the indices of the vertices of a cycle $C_n$ are carried out in a “cyclic sense”, that is, going to 1 upon reaching $n$, and vice-versa.

We first prove the following result.

**Lemma 2.2.** The graph $C_p \square P_q$ is not PMH, for every $p, q \geq 3$.

**Proof.** Label the vertices of $C_p$ and $P_q$ consecutively as $u_1, u_2, \ldots, u_p$ and $v_1, v_2, \ldots, v_q$, respectively, such that $v_1$ and $v_q$ are the two end vertices of $P_q$. If $p$ is even (and so $q$ is even, otherwise $C_p \square P_q$ does not have a perfect matching), then there exists a perfect matching of $C_p \square P_q$ containing an odd cut, say $\{\omega_{1,q-1}, \omega_{1,q}, \ldots, \omega_{p,q-1}, \omega_{p,q}\}$. Clearly, this perfect matching cannot be extended to a Hamiltonian cycle. Thus, we can assume that $p$ is even. Let $M$ be a perfect matching of $C_p \square P_q$ containing $\omega_{i,q-1} \omega_{i+1,q-1}$ and $\omega_{i-1,q} \omega_{i,q}$, for every odd $i \in [p]$, where $[p] = \{1, \ldots, p\}$. For contradiction, suppose that $N$ is a perfect matching of $C_p \square P_q$ such that $M \cup N$ is a Hamiltonian cycle. Then, for every odd $i \in [p]$, $N$ contains either $\omega_{i,q} \omega_{i+1,q}$ or the two edges $\omega_{i,q-1} \omega_{i,q}$ and $\omega_{i+1,q-1} \omega_{i+1,q}$. Therefore, $M \cup N$ contains a cycle with vertices belonging to $\{\omega_{1,q-1}, \omega_{p,q-1}, \omega_{1,q}, \ldots, \omega_{p,q}\}$. Since $q > 2$, $M \cup N$ is not a Hamiltonian cycle, a contradiction. Consequently, $C_p \square P_q$ is not PMH. \qed
Now, we prove our main result.

**Theorem 2.3.** Let \( p, q \geq 3 \). The graph \( C_p \Box C_q \) is PMH only when \( p = 4 \) and \( q = 4 \).

**Proof.** The 4–dimensional hypercube \( Q_4 = C_4 \Box C_4 \) has the PH–property by Fink’s result in [5]. Moreover, the authors in [2] showed that \( C_4 \Box C_q \) is not PMH when \( q \neq 4 \). Thus, in what follows we shall assume that \( p \) is even and at least 6 and that \( q \) is not equal to 4.

Let the consecutive vertices of \( C_p \) and \( C_q \) be labelled \( u_1, u_2, \ldots, u_p \) and \( v_1, v_2, \ldots, v_q \), respectively.

We first consider the case when \( q = 3 \). For simplicity, let the vertices \( \omega_i,1, \omega_i,2, \omega_i,3 \) be referred to as \( a_i, b_i, c_i \), for each \( i \in [p] \), and let \( M \) be a perfect of \( C_p \Box C_3 \) containing the following nine edges: \( a_1a_2, b_1b_2, c_1c_2, a_3a_4, b_3b_4, a_5a_6, b_5b_6, a_4b_6, c_4c_6 \), as shown in Figure 1. Since \( p \) is even, such a perfect matching \( M \) clearly exists.

![Figure 1: Edges belonging to the perfect matching \( M \) in \( C_p \Box C_3 \)](image)

We claim that \( M \) cannot be extended to a Hamiltonian cycle. For, suppose not, and let \( N \) be a perfect matching of \( C_p \Box C_3 \) such that \( M \cup N \) is a Hamiltonian cycle. Each of the two sets \( X_1 = \{ a_3a_4, c_3c_4 \} \) and \( X_2 = \{ a_5a_6, c_5c_6 \} \) is a 2–edge-cut of the cubic graph \( C_p \Box C_3 - M \), and so \( |X_1 \cap N| \) is even for each \( i = 1, 2 \). Moreover, the edge \( b_4b_5 \) is a bridge of the graph \( C_p \Box C_3 - M \), and consequently, \( M \cup N \) contains a cycle of length 4, 6 or 8 with vertices belonging to \( \{ a_3, a_4, a_5, a_6, c_3, c_4, c_5, c_6 \} \), a contradiction. Therefore, \( q \geq 5 \).

Similar to above, for each \( i \in [p] \), let the vertices \( \omega_i,1, \omega_i,2, \ldots, \omega_i,6 \) be referred to as \( a_i, b_i, \ldots, f_i \) as in Figure 2, with \( f_i \) being equal to \( a_i \) if \( q = 5 \). For each \( i \in [p] \), let \( L_i \) and \( R_i \) represent \( b_i \cdots c_i \) and \( d_i \cdots e_i \), respectively, whilst \( L := \{ L_i : i \in [p] \} \) and \( R := \{ R_i : i \in [p] \} \). Let \( M \) be a perfect matching of \( C_p \Box C_q \) containing the following edges:

(i) \( a_i a_{i+1} \) and \( f_i f_{i+1} \), for every even \( i \in [p] \),

(ii) \( b_i b_{i+1} \) and \( e_i e_{i+1} \), for every odd \( i \in [p] \), and

(iii) \( c_i d_i \), for every \( i \in [p] \).
Once again, since $p$ is even, such a perfect matching $M$ exists. For contradiction, suppose that $N$ is a perfect matching of $C_p \Box C_q$ such that $M \cup N$ is a Hamiltonian cycle $H$ of $C_p \Box C_q$. The set of edges $L$ (and similarly $R$) is an even cut of order $p$ in the cubic graph $C_p \Box C_q - M$. Consequently, both $|L \cap N|$ and $|R \cap N|$ are even. We claim that both sets $L$ and $R$ must be intersected by $N$. For, suppose that $R \cap N$ is empty, without loss of generality. In this case, $M \cup N$ forms a Hamiltonian cycle of $C_p \Box C_q - R$, which is isomorphic to $C_p \Box P_q$. By a similar reasoning to that used in the proof of Lemma 2.2, this leads to a contradiction, and so $M$ cannot be extended to a Hamiltonian cycle. Therefore, both $L \cap N$ and $R \cap N$ are non-empty.

Next, we claim that a maximal sequence of consecutive edges belonging to $L - N$ (or $R - N$) is of even length, whereby “consecutive edges” we mean that the indices of these edges are consecutive integers in a cyclic sense. For, suppose there exists such a sequence made up of an odd number of edges. Without loss of generality, let $L_s$ and $L_{s+2t}$ be the first and last edges of this sequence, for some $s \in [p]$ and $0 \leq t < p/2$. Thus, $L_{s-1}$ and $L_{s+2t+1}$ are in $N$. In order for $N$ to cover all the vertices of the graph it must induce a perfect matching of the path $c_s c_{s+1} \ldots c_{s+2t}$, which has an odd number of vertices. This is not possible, and so our claim holds. Consequently, there exists $L_\gamma \in N$, for some odd $\gamma \in [p]$. We pair the edge $L_\gamma$ with the edge $L_{\gamma'}$, where $\gamma'$ is the least integer greater than $\gamma$ in a cyclic sense such that $L_{\gamma'} \in N$. More formally,

$$
\gamma' = \begin{cases} 
\min\{j \in \{\gamma + 1, \ldots, p\} : L_j \in N\} & \text{if such a minimum exists,} \\
\min\{j \in \{1, \ldots, \gamma - 1\} : L_j \in N\} & \text{otherwise.}
\end{cases}
$$

By the last claim we note that $\gamma'$ is even and that the next integer $\beta > \gamma'$ in a cyclic sense (if any) for which $L_\beta$ is in $N$ must be odd. Repeating this procedure on all the edges in $L \cap N$ we get a partition of $L \cap N$ into pairs of edges $\{L_\gamma, L_{\gamma'}\}$ where $\gamma$ is odd and $\gamma'$ is even. The edges in $R \cap N$ are partitioned into pairs in a similar way.

We remark that if we start tracing the Hamiltonian cycle $H$ from $c_\gamma$ going towards $b_\gamma$,
then $H$ contains a path with edges alternating in $N$ and $M$, starting from $c_\gamma$ and ending at $c_{\gamma'}$. More precisely, if $\gamma' = \gamma + 1$, then $H$ contains the path $c_\alpha b_i b_j c_\gamma$. Otherwise, if $\gamma' \neq \gamma + 1$, then, for every even $j \in \{\gamma + 1, \ldots, \gamma' - 2\}$, $N$ contains either $b_j b_{j+1}$ or the two edges $a_j b_j$ and $a_{j+1} b_{j+1}$. Consequently, the internal vertices on this path belong to the set $\{b_{\gamma}, a_{\gamma+1}, b_{\gamma+1}, \ldots, a_{\gamma'-1}, b_{\gamma'-1}, b_{\gamma'}\}$. In each of these two cases we shall refer to such a path between $c_\alpha$ and $c_{\gamma'}$ as an $L_\gamma L_{\gamma'}$-bracket, or just a left–bracket, with $L_\gamma$ and $L_{\gamma'}$ being the upper and lower edges of the bracket, respectively.

Having arrived at $c_{\gamma'}$, and noting that $c_{\gamma'} d_{\gamma'} \in M$, $H$ also traverses this edge to arrive at vertex $d_{\gamma'}$. At this point we can potentially take one of three directions, depending on whether $R_{\gamma'}$ is in $N$ or otherwise. If $R_{\gamma'} \in N$, then there exists an $R_\alpha R_{\gamma'}$-bracket for some odd $\alpha \in [p]$, where $\alpha$ is the greatest integer smaller than $\gamma'$ in a cyclic sense such that $R_\alpha \in N$. As above, this bracket consists of a path with edges alternating in $N$ and $M$, starting from $d_{\gamma'}$ and ending at $d_\alpha$, such that the other vertices of this path belong to:

$$\{e_{\gamma'}, f_{\gamma'-1}, e_{\gamma'-1}, \ldots, f_{\alpha+1}, e_{\alpha+1}, e_\alpha\} \text{ if } \alpha \neq \gamma' - 1,$$

$$\{e_{\gamma'}, e_\alpha\} \text{ if } \alpha = \gamma' - 1.$$ 

Otherwise, if $R_{\gamma'} \notin N$, we either have $d_{\gamma'-1} d_{\gamma'} \in N$ or $d_{\gamma'} d_{\gamma'+1} \in N$. Continuing this process, the Hamiltonian cycle $H$ must eventually reach the vertex $c_{\gamma}$. Thus, $H$ contains only vertices in the set $\{a_i, b_i, c_i, d_i, e_i, f_i : i \in [p]\}$, giving a contradiction if $q \geq 7$. Henceforth, we can assume that $5 \leq q \leq 6$. Notwithstanding whether or not $R_{\gamma'}$ is in $N$, if $q = 6$, then there is no instance in the above procedure which leads to $H$ passing through the vertices $a_{\gamma}$ and $a_{\gamma'}$, a contradiction. Hence, we can further assume that $q = 5$.

We now note that for the vertices in the set $\{a_i, b_i, e_i : i \in [p]\}$ to be in $H$, they must belong either to a left–bracket or to a right–bracket. Thus, if $R_i \in N$ is a lower edge of a right–bracket, for some even $i \in [p]$, then, $R_{i+1}$ must be an upper edge of another right–bracket (that is, $R_{i+1} \in N$), otherwise, the vertex $e_{i+1}$ is not contained in any bracket. This observation, together with the fact that a maximal sequence of consecutive edges belonging to $R - N$ is of even length, implies that if $R_i \notin N$, for some even $i \in [p]$, then $d_i d_{i+1} \in N$.

We revert back to the last remaining case, that is, when $q = 5$. The only way how the Hamiltonian cycle $H$ can contain the vertices $a_{\gamma}$ and $a_{\gamma'}$ is when both $R_\gamma$ and $R_{\gamma'}$ do not belong to $N$, in which case $a_{\gamma}$ and $a_{\gamma'}$ can be reached by some right–bracket (or right–brackets). Therefore, suppose that $R_\gamma$ and $R_{\gamma'}$ do not belong to $N$.

Consequently, tracing $H$ starting from $c_\gamma$ and going in the direction of $b_\gamma$, after traversing the $L_\gamma L_{\gamma'}$-bracket, $H$ must then contain the path $c_\gamma d_{\gamma'} d_{\gamma'+1} e_{\gamma'+1}$. First assume that $\gamma' + 1 \neq \gamma$. By the same reasoning used for the edges in $R \cap N$, the lower edge $L_{\gamma'}$ must be followed by an upper edge, and thus $L_{\gamma'+1} \in N$. We trace the Hamiltonian cycle through an $L_{\gamma'+1} L_{\gamma''}$-bracket, noting in particular that for $a_{\gamma''}$ to be in $H$, $R_{\gamma''}$ does not belong to $N$, and hence $d_{\gamma''} d_{\gamma''+1} \in N$, since $\gamma''$ is even. Continuing this procedure, $H$ must eventually reach again the vertex $c_\gamma$, without having traversed any right–bracket. The same conclusion can be obtained if $\gamma' + 1 = \gamma$. In either case, the vertices $a_{\gamma}$ and $a_{\gamma'}$, together with several other vertices of $C_9 \square C_9$, are untouched by $H$, a contradiction. As a result $M$ cannot be extended to a Hamiltonian cycle, proving our theorem.

\[ \square \]

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