Constraints of mixing matrix elements in the sequential fourth generation model

Wu-Jun Huo
Institute of High Energy Physics, Academia Sinica, P.O. Box 918(4),
Beijing 100039, P.R. China

Abstract

We review our works on the sequential fourth generation model and focus on the constraints of $4 \times 4$ quark mixing matrix elements. We investigate the quark mixing matrix elements from the rare $K, B$ meson decays. We talk about the hierarchy of the $4 \times 4$ matrix and the existence of fourth generation.
1 Introduction

The Standard Model (SM) is a very successful theory of the elementary particles known today. But it must be incomplete because it has too many unpredicted parameters (nineteen!) to be put by hand. Most of these parameters are in the fermion part of the theory. We don’t know the source of the quarks and leptons, as well as how to determine their mass and number theoretically. We have to get their information all from experiment. There is still no successful theory which can be described them with a unified point, even if the Grand Unified Theory\cite{1} and Supersymmetry\cite{2}. Perhaps elementary particles have substructure and we need to progress more elementary theories. But this is beyond our current experimental level. On the other hand, the recent measurement of the muon anomalous magnetic moment by the experiment E821 \cite{3} disagrees with the SM expectations at more than \(2.6\sigma\) level. There are convincing evidences that neutrinos are massive and oscillate in flavor \cite{4}. It seems to indicate the presence of new physics.

From the point of phenomenology, for fermions, there is a realistic question is number of the fermions generation or weather there are other additional quarks or leptons. The present experiments can tell us there are only three generation fermions with light neutrinos which mass are less smaller than \(M_Z/2\)\cite{5} but the experiments don’t exclude the existence of other additional generation, such as the fourth generation, with a heavy neutrino, i.e. \(m_{\nu_4} \geq M_Z/2\)\cite{6}. Many refs. have studied models which extend the fermions part, such as vector-like quark models\cite{7}, sterile neutrino models\cite{8} and the sequential four generation standard model (SM4)\cite{9} which we talk in this note. We consider a sequential fourth generation non-SUSY model\cite{9}, which is added an up-like quark \(t'\), a down-like quark \(b'\), a lepton \(\tau'\), and a heavy neutrino \(\nu'\) in the SM. The properties of these new fermions are all the same as their corresponding counterparts of other three generations except their masses and CKM mixing, see tab.1,

|          | up-like quark | down-like quark | charged lepton | neutral lepton |
|----------|---------------|-----------------|----------------|----------------|
| SM fermions | \(u\)         | \(d\)           | \(e\)          | \(\nu_e\)     |
|          | \(c\)         | \(s\)           | \(\mu\)        | \(\nu_\mu\)   |
|          | \(t\)         | \(b\)           | \(\tau\)       | \(\nu_\tau\)  |
| new fermions | \(t'\)        | \(b'\)          | \(\tau'\)      | \(\nu'\)      |

Table 1: The elementary particle spectrum of SM4

In SM4, the quark mixing matrix can be written as,

\[
V = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} & V_{ub'} \\
V_{cd} & V_{cs} & V_{cb} & V_{cb'} \\
V_{td} & V_{ts} & V_{tb} & V_{tb'} \\
V_{t'd} & V_{t's} & V_{t'b} & V_{t'b'}
\end{pmatrix}
\] (1)

where \(V_{qb'}\) and \(V_{t'q}\) are the \(4 \times 4\) mixing matrix elements of the fourth generation SM and rest elements are the usual CKM matrix. In this note, we review our works on the SM4
and put the constraints of the fourth generation mixing matrix elements from rare meson and lepton decays.

## 2 Constraints of some 4th generation quark CKM elements

### 2.1 Constraints of \( V^*_{t's}V'_{tb} \) from \( B \rightarrow X_s\gamma \)

The rare decay \( B \rightarrow X_s\gamma \) plays an important role in present day phenomenology. The effective Hamiltonian for \( B \rightarrow X_s\gamma \) at scales \( \mu_b = \mathcal{O}(m_b) \) is

\[
H_{\text{eff}}(b \rightarrow s\gamma) = -\frac{G_F}{\sqrt{2}} V^*_{ts}V_{tb} \left[ \sum_{i=1}^{6} C_i(\mu_b) Q_i + C_7(\mu_b) Q_7 + C_8(\mu_b) Q_8 \right],
\]

where the magnetic–penguin operators

\[
Q_7 = \frac{e}{8\pi^2} m_b \bar{s}_a \sigma^{\mu\nu}(1 + \gamma_5) b_\mu F_{\mu\nu}, \quad Q_8 = \frac{g}{8\pi^2} m_b \bar{s}_a \sigma^{\mu\nu}(1 + \gamma_5) T^a_{\alpha\beta} b_\mu G^{a\mu\nu}
\]

The leading logarithmic calculations can be summarized in a compact form as follows [11]:

\[
R_{\text{quark}} = \frac{Br(B \rightarrow X_s\gamma)}{Br(B \rightarrow X_c e\bar{\nu}_e)} = \frac{|V^*_{ts}V_{tb}|^2}{|V_{cb}|^2} \frac{6\alpha}{\pi f(z)} |C_{7,8}\text{eff}(\mu_b)|^2,
\]

where

\[
f(z) = 1 - 8z + 8z^3 - z^4 - 12z^2 \ln z \quad \text{with} \quad z = \frac{m_{c,pole}^2}{m_{b,pole}^2}
\]

is the phase space factor in \( Br(B \rightarrow X_c e\bar{\nu}_e) \) and \( \alpha = e^2/4\pi \). In the case of four generation there is an additional contribution to \( B \rightarrow X_s\gamma \) from the virtual exchange of the fourth generation up quark \( t' \). The Wilson coefficients of the dipole operators are given by

\[
C_{7,8}\text{eff}(\mu_b) = C_{7,8}^{\text{(SM)eff}}(\mu_b) + \frac{V^*_{ts}V'_{tb}}{V^*_{ts}V_{tb}} C_{7,8}^{(4)\text{eff}}(\mu_b),
\]

where \( C_{7,8}^{(4)\text{eff}}(\mu_b) \) present the contributions of \( t' \) to the Wilson coefficients, and \( V^*_{ts} \) and \( V'_{tb} \) are two elements of the \( 4 \times 4 \) CKM matrix which now contains nine paremeters, i.e., six angles and three phases. We recall here that the CKM coefficient corresponding to the \( t \) quark contribution, i.e., \( V^*_{ts}V_{tb} \), is factorized in the effective Hamiltonian. The formulas for calculating the Wilson coefficients \( C_{7,8}(m_W) \) are same as their counterpares in the SM except exchanging \( t' \) quark not \( t \) quark and the corresponding Fenymann figurers are shown in fig. 1.
With these Wilson coefficients and the experiment results of the decays of $B \to X_s \gamma$ and 
$Br(B \to X_c e \bar{\nu}_e)$ [12], we obtain the results of the fourth generation CKM factor $V_{t's}^* V_{t'b}$. 
There exist two cases, a positive factor and a negative one:

$$V_{t's}^+ V_{t'b} = [C_7(0)_{\text{eff}}(\mu_b) - C_7(\text{SM})_{\text{eff}}(\mu_b)] \frac{V_{t's}^* V_{t'b}}{C_7(\text{SM})_{\text{eff}}(\mu_b)}$$

$$= [\sqrt{\frac{R_{\text{quark}}|V_{cb}|^2 \pi f(z)}{|V_{t's} V_{t'b}|^2 6\alpha}} - C_7(\text{SM})_{\text{eff}}(\mu_b)] \frac{V_{t's}^* V_{t'b}}{C_7(\text{SM})_{\text{eff}}(\mu_b)}$$

(7)

$$V_{t's}^- V_{t'b} = [-\sqrt{\frac{R_{\text{quark}}|V_{cb}|^2 \pi f(z)}{|V_{t's} V_{t'b}|^2 6\alpha}} - C_7(\text{SM})_{\text{eff}}(\mu_b)] \frac{V_{t's}^* V_{t'b}}{C_7(\text{SM})_{\text{eff}}(\mu_b)}$$

(8)

as in tab. 2,

| $m_{t'}$ (Gev) | 50   | 100  | 150  | 200  | 250  | 300  | 400  |
|----------------|------|------|------|------|------|------|------|
| $V_{t's}^+ V_{t'b} \times 10^{-2}$ | -11.591 | -9.259 | -8.126 | -7.501 | -7.116 | -6.861 | -6.548 |
| $V_{t's}^- V_{t'b} \times 10^{-3}$ | 3.5684  | 2.8503 | 2.5016 | 2.3092 | 2.191  | 2.113  | 2.016  |

Table 2: The values of $V_{t's}^* \cdot V_{t'b}$ due to masses of $t'$ for $Br(B \to X_s \gamma) = 2.66 \times 10^{-4}$

In the numerical calculations we set $\mu_b = m_b = 5.0 \text{GeV}$ and take the $t'$ mass value of 
50 GeV, 100 GeV, 150 GeV, 200 GeV, 250 GeV, 300 GeV, 400 GeV.

The CKM matrix elements obey unitarity constraints, which states that any pair of rows, 
or any pair of columns, of the CKM matrix are orthogonal. This leads to six orthogonality 
conditions [13]. The one relevant to $b \to s \gamma$ is

$$\sum_i V_{i's}^* V_{ib} = 0,$$

(9)

i.e.,

$$V_{u's} V_{ub} + V_{c's} V_{cb} + V_{t's} V_{tb} + V_{t's}^* V_{t'b} = 0.$$

(10)

We take the average values of the SM CKM matrix elements from Ref. [12]. The sum of 
the first three terms in eq. (10) is about $7.6 \times 10^{-2}$. If we take the value of $V_{t's}^+ V_{t'b}$ given 
in Table 2, the result of the left of (10) is much better and much more close to 0 than 
that in SM, because the value of $V_{t's}^+ V_{t'b}$ is very close to the sum but has the opposite 
sign. If we take $V_{t's}^- V_{t'b}$, the result would change little because the values of $V_{t's}^- V_{t'b}$ are about $10^{-3}$ order, ten times smaller than the sum of the first three ones in the left of 
(10). Considering that the data of CKM matrix is not very accurate, we can get the error 
range of the sum of these first three terms. It is about $\pm 0.6 \times 10^{-2}$, much larger than 
$V_{t's}^+ V_{t'b}$. Thus, the values of $V_{t's}^* V_{t'b}$ in the both cases satisfy the CKM matrix unitarity 
constraints.
2.2 Constraints on CKM Factor $V_{t's}^* V_{t'd}$ in SM4 [14]

The following three rare $K$ meson decays: two semi-leptonic decays $K^+ \to \pi^+ \nu\bar{\nu}$ and $K_L \to \pi^0 \nu\bar{\nu}$, and one leptonic decay $K_L \to \mu^+\mu^-$ [13] can provide certain constraints on the fourth generation CKM factors, $V_{t's}^* V_{t'd}$, $\text{Im}V_{t's}^* V_{t'd}$ and $\text{Re}V_{t's}^* V_{t'd}$ respectively.

|                | $\text{Br}(K^+ \to \pi^+ \nu\bar{\nu})$ | $\text{Br}(K_L \to \pi^0 \nu\bar{\nu})$ | $\text{Br}(K_L \to \mu^+\mu^-)$ |
|----------------|------------------------------------------|------------------------------------------|---------------------------------|
| **Experiment** | $< 2.4 \times 10^{-9}$ [16]              | $< 1.6 \times 10^{-9}$ [17]              | $(6.9 \pm 0.4) \times 10^{-9}$ [18] |
|                | $(4.2 + 9.7 - 3.5) \times 10^{-10}$ [23] | $(6.1 \pm 10^{-9}$ [19]                 | $(7.9 \pm 0.7) \times 10^{-9}$ [20] |
| **SM**         | $(8.2 \pm 3.2) \times 10^{-11}$ [22]     | $(3.1 \pm 1.3) \times 10^{-11}$ [22]    | $(1.3 \pm 0.6) \times 10^{-9}$ [21] |

Table 3: Comparison of $\text{Br}(K^+ \to \pi^+ \nu\bar{\nu})$, $\text{Br}(K_L \to \pi^0 \nu\bar{\nu})$ and $\text{Br}(K_L \to \pi^0 \nu\bar{\nu})$ among the experimental values and SM predictions with maximum mixing.

In the SM4, the branching ratios of the three decay modes mentioned above receive additional contributions from the up-type quark $t'$ [24]

$$\text{Br}(K^+ \to \pi^+ \nu\bar{\nu}) = \kappa_+ \left| \frac{V_{cd} V_{cs}^*}{\lambda} P_0 + \frac{V_{td} V_{ts}^*}{\lambda^5} \eta_t X_0(x_t) + \frac{V_{t'd} V_{t'd}^*}{\lambda^5} \eta_{t'} X_0(x_{t'}) \right|^2,$$  \hspace{1cm} (11)

$$\text{Br}(K_L \to \pi^0 \nu\bar{\nu}) = \kappa_L \left| \frac{\text{Im}V_{td} V_{ts}^*}{\lambda^5} \eta_t X_0(x_t) + \frac{\text{Im}V_{t'd} V_{t'd}^*}{\lambda^5} \eta_{t'} X_0(x_{t'}) \right|^2,$$  \hspace{1cm} (12)

$$\text{Br}(K_L \to \mu^+\mu^-)_{SD} = \kappa_{\mu} \left[ \frac{\text{Re}(V_{cd} V_{cs}^*)}{\lambda} P_0' + \frac{\text{Re}(V_{td} V_{ts}^*)}{\lambda^5} Y_0(x_t) + \frac{\text{Re}(V_{t'd} V_{t'd}^*)}{\lambda^5} Y_0(x_{t'}) \right]^2,$$  \hspace{1cm} (13)

where $\kappa_+, \kappa_L, \kappa_{\mu}, X_0(x_t), X_0(x_{t'}), Y_0(x_t), Y_0(x_{t'})$, $P_0, P_0'$ may be found in Refs [23, 24]. The QCD correction factors are taken to be $\eta_t = 0.985$ and $\eta_{t'} = 1.0$ [24].

To solve the constrains of the 4th generation CKM matrix factors $V_{t's}^* V_{t'd}$, $\text{Im}V_{t's}^* V_{t'd}$ and $\text{Re}V_{t's}^* V_{t'd}$, we must calculate the Wilson coefficients $X_0(x_{t'})$ and $Y_0(x_{t'})$. They are the functions of the mass of the 4th generation top quark, $m_{t'}$. Here we give their numerical results according to several values of $m_{t'}$, (see table 4) We found that the

| $m_{t'}$(GeV) | 50  | 100 | 150 | 200 | 250 | 300 | 400 | 500 | 600 |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $X_0(x_{t'})$ | 0.404 | 0.873 | 1.357 | 1.884 | 2.474 | 3.137 | 4.703 | 6.615 | 8.887 |
| $Y_0(x_{t'})$ | 0.144 | 0.443 | 0.833 | 1.303 | 1.856 | 2.499 | 4.027 | 5.919 | 8.179 |

Table 4: Wilson coefficients $X_0(x_{t'})$, $Y_0(x_{t'})$ to $m_{t'}$

Wilson coefficients $X_0(x_{t'})$ and $Y_0(x_{t'})$ increase with the $m_{t'}$. To get the largest constrain of the factors in eq. (11), (12) and (13), we must use the little value of $m_{t'}$. Considering
that the 4th generation particles must have the mass larger than $\frac{M_Z}{2}$, we take $m_t$ with 50 GeV to get our constrains of those three factors.

Then, from (11), (12) and (13), we arrive at the following constraints

\[ |V^*_{t's} V'_{t'd}| \leq 2 \times 10^{-4}, \tag{14} \]
\[ |\text{Im}V^*_{t's} V'_{t'd}| \leq 1.2 \times 10^{-4}, \tag{15} \]
\[ |\text{Re}V^*_{t's} V'_{t'd}| \leq 1.0 \times 10^{-4}. \tag{16} \]

For the numerical calculations, we will take $|\text{Im}V^*_{t's} V'_{t'd}| \leq 1.2 \times 10^{-4}$.

It is easy to check that the equation (14) obeys the CKM matrix unitarity constraint, which states that any pair of rows, or any pair of columns, of the CKM matrix are orthogonal.\[12\]. The relevant one to those decay channels is

\[ V^*_{us} V_{ud} + V^*_{cs} V_{cd} + V^*_{ts} V_{td} + V^*_{t's} V'_{t'd} = 0. \tag{17} \]

Here we have taken the average values of the SM CKM matrix elements from Ref.\[12\]. Considering the fact that the data of CKM matrix is not yet very accurate, there still exists a sizable error for the sum of the first three terms. Using the value of $V^*_{t's} V'_{t'd}$ obtained from eq. (14), the sum of the four terms in the left hand of (17) can still be close to 0, because the values of $V^*_{t's} V'_{t'd}$ are about $10^{-4}$ order, ten times smaller than the sum of the first three ones in the left of (17). Thus, the values of $V^*_{t's} V'_{t'd}$ remain satisfying the CKM matrix unitarity constraints in SM4 within the present uncertainties.

2.3 \( V^*_{t'b} V'_{t'd} \) from experimental measurements of \( \Delta M_{B_d} \)[27]

\( B^0_{d,s} - \bar{B}^0_{d,s} \) mixing proceeds to an excellent approximation only through box diagrams with internal top quark exchanges in SM. In SM, the effective Hamiltonian \( \mathcal{H}_{\text{eff}}(\Delta B = 2) \) for \( B^0_{d,s} - \bar{B}^0_{d,s} \) mixing, relevant for scales $\mu_b = \mathcal{O}(m_b)$ is given by\[11\]

\[ \mathcal{H}_{\text{eff}}^{\Delta B = 2} = \frac{C_F^2}{16\pi^2} M_W^2 (V^*_{t'b} V'_{t'd}) S_0(x_t) Q(\Delta B = 2) + \text{h.c.} \tag{18} \]

where $Q(\Delta B = 2) = (\bar{b}_q q_A)(\bar{b}_q q_A) - (\bar{b}_q q_A)(\bar{b}_q q_A)$, with $q = d, s$ for $B^0_{d,s} - \bar{B}^0_{d,s}$ respectively and $S_0(x_t)$ is the Wilson coefficient which is taken the form

\[ S_0(x_t) = \frac{4x_t - 11x_t^2 + x_t^3}{4(1-x_t)^2} - \frac{3}{2} \cdot \frac{x_t^3}{(1-x_t)^3} \cdot \ln x, \tag{19} \]

where $x_t = m_t^2/M_W^2$. The mass differences $\Delta M_{d,s}$ can be expressed in terms of the off-diagonal element in the neutral $B$-meson mass matrix

\[ \Delta M_{d,s} = 2 |M_{12}^{d,s}| \tag{20} \]
\[ 2m_{B_{d,s}} |M_{12}^{d,s}| = |\langle \bar{B}_{d,s}^0 | \mathcal{H}_{\text{eff}}(\Delta B = 2) | B_{d,s}^0 \rangle|. \]
If we add a fourth sequential fourth generation up-like quark $t'$, the above equations would have some modification. There exist other box diagrams contributed by $t'$ (see fig. 2), similar to the leading box diagrams in MSSM\[23\]. The mass differences $\Delta M_d$ in SM4 can be expressed

$$\Delta M_d = \frac{G_F^2}{6\pi^2} M_B^2 m_{B_d} (\hat{B}_{B_d} \hat{F}_{B_d}) [\eta_t (V_{tb}^* V_{td})^2 S_0(x_t) + \eta_{t'} (V_{t'b}^* V_{t'd})^2 S_0(x_{t'}) + \eta_{t'} (V_{t'b}^* V_{t'd}) \cdot (V_{tb}^* V_{td}) S_0(x_t, x_{t'})]$$

(21)

The new Wilson coefficients $S_0(x_{t'})$ present the contribution of $t'$, which like $S_0(x_t)$ in eq. (19) except exchanging $t'$ quark not $t$ quark. $S_0(x_t, x_{t'})$ present the contribution of a mixed $t - t'$, which is taken the form\[29\]

$$S_0(x, y) = x \cdot y \left[ -\frac{1}{y-x} \left( \frac{1}{4} + \frac{3}{2} \cdot \frac{1}{1-x} - \frac{3}{4} \cdot \frac{1}{(1-x)^2} \right) \ln x + \frac{1}{4} \cdot \frac{1}{(1-x)(1-y)} \right]$$

(22)

where $x = x_t = m_t^2/M_W^2$, $y = x_{t'} = m_{t'}^2/M_W^2$. The numerical results of $S_0(x_{t'})$ and $S_0(x_t, x_{t'})$ is shown on the tab. 5.

| $m_t$(GeV) | 50   | 100  | 150  | 200  | 250  | 300  | 350  | 400  | 450  | 500  |
|-----------|------|------|------|------|------|------|------|------|------|------|
| $S_0(x_{t'})$ | 0.33 | 1.07 | 2.03 | 3.16 | 4.44 | 5.87 | 7.49 | 9.23 | 11.15 | 13.25 |
| $S_0(x_t, x_{t'})$ | 0.48 | -7.03 | -4.94 | -5.09 | -5.39 | -5.87 | -5.99 | -6.25 | -6.49 | -6.72 |

| $m_{t'}$(GeV) | 550  | 600  | 650  | 700  | 750  | 800  | 850  | 900  | 950  | 1000 |
|---------------|------|------|------|------|------|------|------|------|------|------|
| $S_0(x_{t'})$ | 15.52 | 17.97 | 20.60 | 23.41 | 26.40 | 29.57 | 32.93 | 36.47 | 40.96 | 44.11 |
| $S_0(x_t, x_{t'})$ | -6.92 | -7.11 | -7.28 | -7.44 | -7.60 | -7.74 | -7.87 | -7.99 | -8.12 | -8.23 |

Table 5: The Wilson coefficients $S_0(x_{t'})$ and $S_0(x_t, x_{t'})$ to $m_{t'}$

The short-distance QCD correction factors $\eta_t$ and $\eta_{t'}$ can be calculated like $\eta_c$ and $\eta_{ct}$ in the mixing of $K^0 - \bar{K}^0$, which the NLO values are given in refs\[11, 30\], relevant for scale not $\mathcal{O}(\mu_t)$ but $\mathcal{O}(\mu_b)$. In leading-order, $\eta_t$ is calculated by

$$\eta_t^0 = [\alpha_s(\mu_t)]^{(6/23)}$$

$$\alpha_s(\mu_t) = \alpha_s(M_Z) [1 + \sum_{n=1}^{\infty} (\frac{\beta_0}{2\pi}) \frac{\alpha_s(M_Z)}{\mu_t} \ln \frac{M_Z}{\mu_t} ]$$

(23)

with its numerical value in tab. 6. The formulae of factor $\eta_{t'}$ is similar to the above equation except for exchanging $t$ by $t'$. For simplicity, we take $\eta_{t'} = \eta_{t'}$. We give the numerical results in tab. 7. In the last of this section, we give other input parameters necessary in this note. (See the following tab.).

Now, we can put the constraints of the fourth generation CKM factor $V_{t'b}^* V_{t'd}$ from the present experimental value of $\Delta M_{B_d}$. We change the form of eq. (21) as a quadratic equation about $V_{t'b}^* V_{t'd}$.

By solving it, we can get two analytical solution $V_{t'd}^* V_{t'b}^{(1)}$ (absolute
value is the large one) and $V_{td}^* V_{t'b}^{(2)}$ (absolute value is the small one). However, experimentally, it is not accurate for the measurement of CKM matrix element $V_{td}$\cite{11, 12}. So, we have to search other ways to solve this difficulty. Fortunately, the CKM unitarity triangle\cite{13}, i.e. the graphic representation of the unitarity relation for $d, b$ quarks, which come from the orthogonality condition on the first and third row of $V_{CKM}$,

$$V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0,$$

(24)

can be conveniently depicted as a triangle relation in the complex plane, as shown in the following figure. From the above equation, we can give the constraints of $V_{td} V_{tb}^*$\cite{32},

$$0.005 \leq |V_{td} V_{tb}^*| \leq 0.013$$

(25)

Then, we give the final results as shown in the figs. 3.

We must announce that figs. 3 only show the curves with $V_{td}^* V_{t'b}^{(2)}$ (absolute value is the small one) firstly. Because the absolute value of $V_{td}^* V_{t'b}^{(1)}$ is generally larger than 1. This is contradict to the unitarity of CKM matrix. So, we don’t think about this solution. From the figs. 3, we found all curves are in the range from $-1 \times 10^{-4}$ to $0.5 \times 10^{-4}$ when we considering the constraint of $V_{td} V_{tb}^*$. That is to say, the absolute value of $V_{td}^* V_{t'b}$ is about $\sim 10^{-4}$ order. This is a very interesting result.

These CKM matrix elements obey unitarity constraints. With the fourth generation quark $t'$, eq. (9) change to,

$$V_{ud}^* V_{ub} + V_{cd}^* V_{cb} + V_{td}^* V_{tb} + V_{t'd}^* V_{t'b} = 0.$$

(26)

We take the average values of the SM CKM matrix elements from Ref. \cite{12}. The sum of the first three terms in eq. (24) is about $\sim 10^{-2}$ order. If we take the value of $V_{t's} V_{t'b}^{(2)}$

\begin{table}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
$m_t$(GeV) & 50 & 100 & 150 & 200 & 250 & 300 & 350 & 400 & 450 & 500 \\
\hline
$\eta_t'$ & 0.968 & 0.556 & 0.499 & 0.472 & 0.455 & 0.443 & 0.433 & 0.426 & 0.420 & 0.416 \\
\hline
$m_t$(GeV) & 550 & 600 & 650 & 700 & 750 & 800 & 850 & 900 & 950 & 1000 \\
\hline
$\eta_t'$ & 0.412 & 0.408 & 0.405 & 0.401 & 0.399 & 0.396 & 0.395 & 0.393 & 0.391 & 0.389 \\
\hline
\end{tabular}
\caption{The short-distance QCD factors $\eta_t', \eta_t(= \eta_t')$ to $m_t'$}
\end{table}

\begin{table}
\begin{tabular}{|c|c|c|c|c|}
\hline
$m_c(m_c(pole))$ & $\bar{m}_c(m_c(pole))$ & $\Delta M_{B_d}$ & $\Delta M_{B_s}$ & $1.25 \pm 0.05$GeV & $175$GeV & $(0.473 \pm 0.016)(ps)^{-1}$ & $\hat{F}_{B_d} \sqrt{B_{B_d}}$ & $M_W$ & $80.2$GeV & $G_F$ & $1.14 \pm 0.06$ & $1.166 \times 10^{-5}$GeV$^{-2}$ \\
\hline
\end{tabular}
\caption{Neumerical values of the input parameters\cite{31}.}
\end{table}
the result of the left of (26) is better and more close to 0 than that in SM, when \( V_{t's}^* V_{t'b}^{(2)} \) takes negative values. Even if \( V_{t'd}^* V_{t'b}^{(2)} \) takes positive values, the sum of (26) would change very little because the values of \( V_{t'd}^* V_{t'b}^{(2)} \) are about \( 10^{-4} \) order, two orders smaller than the sum of the first three ones in the left of (24). Considering that the data of CKM matrix is not very accurate, we can get the error range of the sum of these first three terms. It is much larger than \( V_{t'd}^* V_{t'b}^{(2)} \). Thus, in the case the values of \( V_{t'd}^* V_{t'b}^{(2)} \) satisfy the CKM matrix unitarity constraints.

We can see the order of these 4th generation CKM matrix elements, such as \( V_{t'd}^* V_{t'b}^{(2)} \) doesn’t contradict to the hierarchy of the CKM matrix elements or the quarks mixing angles\(^{35, 33} \). Moreover, it seem to prove the hierarchy. The hierarchy in the quarks mixing angles is clearly presented in the Wolfenstein parameterization\(^{34} \) of the CKM matrix. Let’s see CKM matrix firstly,

\[
V_{\text{CKM}} = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} & \cdots \\
V_{cd} & V_{cs} & V_{cb} & \cdots \\
V_{td} & V_{ts} & V_{tb} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \sim \begin{pmatrix}
1 & \lambda & \lambda^3 & \cdots \\
-\lambda & 1 & \lambda^2 & \cdots \\
\lambda^3 & -\lambda^2 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\] (27)

with \( \lambda = \sin^2 \theta = 0.23 \). Now, the hierarchy can be expressed in powers of \( \lambda \). We found, the magnitudes of the mixing angles are about 1 among the same generations, \( V_{ud}, V_{cs} \) and \( V_{tb} \). For different generations, the magnitudes are about \( \lambda \) order between 1st and 2nd generation, \( V_{us} \) and \( V_{cd} \), as well as about \( \lambda^2 \) order between 2nd and 3rd generation, \( V_{cb} \) and \( V_{ts} \). The magnitudes are about \( \lambda^3 \) order between the 1st and third generation, \( V_{ub} \) and \( V_{td} \). Then, there should be an interesting problem: If the fourth generation quarks exist, how to choose the order do the magnitude of the mixing angles concern the fourth generation quarks? Because there is not direct experimental measurement of the fourth generation quark mixing angles, one have to look for other indirect methods to solve the problem. Many refs. have already talked about these additional CKM mixing angles\(^7, 8, 9, 36\), like the vector-like quark models\(^7\), the four neutrinos models\(^8\) and the sequential four generations models\(^9\). For simple, we give a guess for the magnitude of the fourth generation mixing angles. Similar to the general CKM matrix elements magnitude order, the fourth generation ones are about \( \lambda^4 \sim \lambda^5 \) order between the 1st and 4th generation, such as \( V_{t'd} \), as well as \( \lambda^2 \sim \lambda^3 \) between the 2nd and 4th generation, such as \( V_{t's} \). For the mixing between the 3rd and 4th generation quarks, such as \( V_{t'b} \), we take the magnitude as 1 because the mass of the fourth generation quark \( t' \) is the same order, \( 10^2 \), as the top quark \( t \). So \( V_{t'b} \) should take the order of \( V_{tb} \). Then, the magnitude order of the fourth generation CKM factor \( V_{t'd}^* V_{t'b}^{(2)} \) is about \( \lambda^4 \sim \lambda^5 \), i.e. \( < \lambda^4 \). From figs. 3, we found that the numerical results, \( V_{t'd}^* V_{t'b}^{(2)} \), satisfy this guess. At last, the factor \( V_{t'd}^* V_{t'b}^{(2)} \) constrained from \( \Delta M_{B_d} \) does not contradict to the CKM matrix texture. Moreover, it seem to support the existence of the fourth generation.
3 Conclusion

In summary, we study the constraints of some 4th generation quark mixing matrix from rare $K, B$ decays. We find they satisfy the unitarity conditions of the CKM matrix. We also talk about the texture of the fourth generation CKM matrix. All these constraints could provide a possible signal of new physics.

Acknowledgments

This research is supported by the the Chinese Postdoctoral Science Foundation and CAS K.C. Wong Postdoctoral Research Award Fund.

References

[1] P. Langacker, Phys. Rep. 72, No. 4, (1981) 185.

[2] M.F. Sohnius, Phys. Rep. 128, No. 2&3 (1985) 39.

[3] Muon g-2 Collaboration, H.N. Brown et al., Phys. Rev. Lett. 86, 2227 (2001).

[4] Y. Fukuda et al., Phys. Lett. B436 33 (1998); Phys. Rev. Lett. 81, 1562 (1998).

[5] G.S. Abrams et al., Mark II Collab., Phys. Rev. Lett. 63 (1989) 2173; B. Advera et al., L3 Collab., Phys. Lett. B 231 (1989) 509; L. Decamp et al., OPAL Collab., ibid., 231 (1989) 519; M.Z. Akrawy et al., DELPHI Collab., ibid., 231 (1989) 539; C. Caso et al., (Particle Data Group), Eur. Phys. J.C 3 (1998) 1.

[6] Z. Berezhiani and E. Nardi, Phys. Rev. D 52 (1995) 3087; C.T. Hill, E.A. Paschos, Phys. Lett. B 241 (1990) 96.

[7] Y. Nir and D. Silverman, Phys. Rev. D42 (1990) 1477; W-S. Choong and D. Silverman, Phys. Rev. D49 (1994) 2322; L.T. Handoko, hep-ph/9708447.

[8] V. Barger, Y.B. Dai, K. Whisnant and B.L. Young, hep-ph/9901380; R.N. Mohapatra, hep-ph/9702229; S. Mohanty, D.P. Roy and U. Sarkar, hep-ph/9810309; S.C. Gibbons, et al., Phys. lett. B430 (1998) 296; V. Barger, K. Whisnant and T.J. Weiler, Phys. lett. B427, (1998) 97; V. Barger, S. Pakvasa, T.J. Weiler and K. Whisnant, Phys. Rev. D58 (1998) 093016.

[9] J.F. Gunion, Douglas W. McKay, H. Pois, Phys. Lett. B 334 (1994) 339; Phys. Rev. D 51 (1995) 201.

[10] C.S. Huang, W.J. Huo and Y.L. Wu, Mod. Phys. Lett. A14, (1999) 2453.
[11] Andrzej J. Buras; hep-ph/9806471.

[12] C.Caso et al., (Particle Data Group), Eur. Phys. J. C3 (1998) 1; B. Grinstein, M.J. Savage, M.B. Wise, Nucl. Phys. B319 (1998) 271.

[13] A. Ali, hep-ph/9606324, hep-ph/9612262.

[14] C.S. Huang, W.J. Huo and Y.L. Wu, Phys.Rev. D64 (2001) 016009.

[15] R. D. Peccei, hep-ph/9909236; T. Hattori, T. Hasuike and S. Wakaizumi, hep-ph/9808412; A.J. Buras, hep-ph/9901409.

[16] S. Adler, et al., Phys. Rev. Lett. B76 (1996) 1421.

[17] J. Adams, et al., hep-ph/9806007.

[18] A.P. Heinson, et al., Phys. Rev. D51 (1995) 985.

[19] Y. Grossman and Y. Nir, Phys. Lett. B398 (1997) 163.

[20] T. Akagi, et al., Phys. Rev. D51 (1995) 2061.

[21] F. Gabbiani, hep-ph/9901262.

[22] G. Buchalla, A.J. Buras, hep-ph/9901288.

[23] S. Adler, et al., Phys. Rev. Lett. B79 (1997) 2204.

[24] T. Hattori, T. Hasuike, S. Wakaizumi, hep-ph/9804412.

[25] A.J. Buras, hep-ph/9806471.

[26] G. Buchalla, A.J. Buras, M.E. Lautenbacher, Rev. of Mod. Phys. 68 (1996) 1125 and references therein; E. A. Paschos, Y.L. Wu, Mod. Phys. Lett. A6 (1991) 93.

[27] C.S. Huang, W. J. Huo and Y.L. Wu, hep-ph/0006110.

[28] I. Hinchliife and N. Kersting, hep-ph/0003090.

[29] J.F. Donoghue, E. Golowich and B.R. Holstein, Dynamics of the Standard Model (Cambridge University Press, New York, 1992).

[30] S. Herrich and U. Nierste, hep-ph/9604330; hep-ph/9310311; S. Herrich, hep-ph/9609370.

[31] A. Ali and D. London, hep-ph/0002167.

[32] G. Barenboim, G. Eyal and Y. Nir, hep-ph/9905397.

[33] M. Leurer, Y. Nir, N. Seiberg, Nucl. Phys. B420 (1994) 468; P. Kielanowski, et al, hep-ph/0002062.
[34] L. Wolfenstein, Phys. Rev. Lett. **51** (1983) 1945.

[35] T. Hattori, T. Hasuike and S. Wakaizumi, Phys. Rev. **D60** (1999) 113008.

[36] DØ Collab., S. Abachi et al., Phys. Rev. Lett. **78** (1997) 3818.
Figure 1: Magnetic Photon (a) and Gluon (b) Penguins with $t'$. 

Figure 2: The Additional Box Diagrams to $B_{d,s}^0 - \bar{B}_{d,s}^0$ with the fourth up-like quark $t'$. 

13
Figure 3: Constraint of the 4th generation CKM factor $V_{td}^* V_{tb}^*$ to (a) $|V_{td} V_{tb}^*|$ with $m_{t'}$ range from 50GeV to 800GeV, (b) to $m_{t'}$ with $|V_{td} V_{tb}^*|$ range from 0.005 to 0.013.