Cylindrically Symmetric Solution in Teleparallel Theory of Gravitation

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The field equations of a special class of teleparallel theory of gravitation and electromagnetic fields have been applied to tetrad space having cylindrical symmetry with four unknown functions of radial coordinate $r$ and azimuth angle $\theta$. The vacuum stress-energy momentum tensor with one assumption concerning its specific form generates one non-trivial exact analytic solution. This solution is characterized by a constant magnetic field parameter $B_0$. If $B_0 = 0$ then, the solution will reduces to the flat spacetime. The energy content is calculated using the superpotential given in the framework of teleparallel geometry. The energy contained in a sphere is found to be different from the pervious results.
1. Introduction

Theories of gravity based on the geometry of distance parallelism [1]∼[8] are commonly considered as the closest alternative to the general relativity theory. Teleparallel gravity models possess a number of attractive features both from the geometrical and physical viewpoints. Teleparallelism is naturally formulated by gauging external (spacetime) translation and underlain the Weitzenböck spacetime characterized by the metricity condition and by the vanishing of the curvature tensor. Translations are closely related to the group of general coordinate transformations which underlies general relativity. Therefore, the energy-momentum tensor represents the matter source in the field equations of tetradic theories of gravity like in general relativity.

An important point of teleparallel gravity is that it corresponds to a gauge theory for the translation group. As a consequence of translations, any gauge theory including these transformations will differ from the usual internal gauge models in many ways, the most significance being the presence of the tetrad field. The tetrad field can be used to define a linear Weitzenböck connection, from which torsion can be defined but no curvature. Also tetrad field can be used to define a Riemannian metric, in terms of which Lie-Civita connection is constructed. It is important to keep in mind that torsion and curvature are properties of a connection and many different connection can be define on the same manifold.

Teleparallel theories of gravity have been considered long time ago in connection with attempts to define the energy of gravitational field [9, 10]. It is clear from the properties of the solutions of Einstein field equation of an isolated system that a consistent expression for the energy density of the gravitational field would be given in terms of second order derivatives of the metric tensor. It is well known that there exists no covariant, nontrivial expression constructed out of the metric tensor, both in three and four dimensions that contain such derivatives. However, covariant expressions that contain second order derivatives of the tetrad fields are feasible. Thus it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy-momentum is related to the geometrical description of the gravitational field rather than being an intrinsic drawback of the theory [11].

The problem of defining a consistence and unequivocal expression for the energy of the gravitational and electromagnetic fields is still open and an important question in general relativity. It is well know that the principle of equivalence has led to the belief that the gravitational energy cannot be localized [12, 13]. However, the argument based on this principle regarding the nonlocalizibility of gravitational energy is controversial and not generally accepted [11]. Therefore, it is legitimate to conjecture that the difficulties associated to the problem of defining the gravitational energy is related to the geometrical description of the gravitational field, rather being an intrinsic nuisance of the theory [11]. The first attempts to define the energy of the gravitational field were based on pseudotensors [14], which make use of coordinate dependent expressions. More recently the idea of quasi-local energy, i.e., energy associated to a closed spacelike two surface, in the context of the Hilbert-Einstein action integral, has emerged as a tentative description of the gravitational energy [15].

Teleparallel Theories of gravity, whose basic entities are the tetrad field $e_{a\mu}$ (a and $\mu$ are $SO(3,1)$ and space-time indices, respectively) have been considered long time ago by
Møller [10] in connection also with attempts to define the energy of the gravitational field. Teleparallel theories of gravity are defined on the Weitzenböck spacetime [16], which is endowed with the affine connection. The curvature tensor constructed out of this connection vanishes identically. This connection defines a space-time with an absolute parallelism or teleparallelism of vector fields [3]. In this geometrical framework the gravitational effects are due to the torsion tensor corresponding to the above mentioned connection.

As stated above that the calculations of energy within the framework of general relativity theory have some problems [10]. It is the aim of the present work to study a tetrad having cylindrical symmetry and apply it to the field equation of gravitation and electromagnetic. Solving the resulting non linear differential equation an exact solution is obtained. We then, calculate the energy using the superpotential of Mikhail et al. [18].

In section 2, we give a brief review of the gravitational and electromagnetic theory. The tetrad having cylindrical symmetry is applied to the field equations of the gravitational and electromagnetic theory in section 3. The solution of the resulting field equations is also given in section 3. The singularities of this solution are given in section 4. In section 5, the calculations of energy using the energy-momentum complex derived by Mikhail et al. are given. Section 6 is devoted for the discussion and a summary of the results. A comparison between the energy calculated and that of general relativity is also given in section 6.

2. The tetrad theory of gravitation and electromagnetism

In teleparallel theory of gravity and electromagnetism the spacetime is represented by the Weitzenböck manifold $W^4$ of distance parallelism. This theory naturally arises within the gauge approach based on the group of the spacetime translations. Accordingly, at each point of this manifold, a gauge transformation is defined as a local translation of the tangent-space coordinate [19],

$$x^a \rightarrow x'^a = x^a + b^a,$$

where $b^a = b^a(x^\mu)$ are the transformation parameters. For an infinitesimal transformation we have

$$\delta x^a = \delta b^a P_a x^a,$$

with $\delta b^a$ the infinitesimal parameters, and $P_a = \partial_a$ the generators of translations. Denoting the translational gauge potential by $\Lambda^a_{\mu}$, the gauge covariant derivative for a scalar field $\Phi(x^\mu)$ reads [20]

$$D_\mu \Phi = e^a_\mu \partial_a \Phi,$$  \hspace{1cm} (1)

where

$$e^a_\mu = \partial_\mu x^a + \Lambda^a_{\mu},$$  \hspace{1cm} (2)

is the tetrad field which satisfies the orthogonality condition

$$e^a_\mu e^\nu_a = \delta^\nu_\mu.$$  \hspace{1cm} (3)
This nontrivial tetrad field induces a teleparallel structure on spacetime which is directly related to the presence of the gravitational field, and the Riemannian metric arises as

\[ g^{\mu\nu} \overset{\text{def.}}{=} e_i^\mu e_i^\nu. \]  

(4)

The gravitational Lagrangian \( L \) is an invariant constructed from \( \gamma_{\mu\nu\rho} \) and \( g^{\mu\nu} \), where \( \gamma_{\mu\nu\rho} \) is the contorsion tensor given by

\[ \gamma_{\mu\nu\rho} \overset{\text{def.}}{=} e_i^\mu e_i^\nu \rho, \]  

(5)

where the semicolon denotes covariant differentiation with respect to Christoffel symbols. The most general Lagrangian density invariant under parity operation is given by the form [21]

\[ L \overset{\text{def.}}{=} (-g)^{1/2} \left( \alpha_1 \Phi^\mu \Phi^\mu + \alpha_2 \gamma^{\mu\nu\rho} \gamma_{\mu\nu\rho} + \alpha_3 \gamma^{\mu\nu\rho} \gamma_{\rho\nu\mu} \right), \]  

(6)

where

\[ g \overset{\text{def.}}{=} \det(g_{\mu\nu}), \]  

(7)

and \( \Phi^\mu \) is the basic vector field defined by

\[ \Phi^\mu \overset{\text{def.}}{=} \gamma^\rho_{\mu\rho}. \]  

(8)

Here \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) are constants determined by Møller such that the theory coincides with general relativity in the weak fields [21]:

\[ \alpha_1 = -\frac{1}{\kappa}, \quad \alpha_2 = \frac{\lambda}{\kappa}, \quad \alpha_3 = \frac{1}{\kappa}(1 - \lambda), \]  

(9)

where \( \kappa \) is the Einstein constant and \( \lambda \) is a free dimensionless parameter*. The same choice of the parameters was also obtained by Hayashi and Nakano [2].

The electromagnetic Lagrangian density \( L_{\text{e.m.}} \) is given by [22]

\[ L_{\text{e.m.}} = -\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, \]  

(10)

where \( F_{\rho\sigma} \) is given by†

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]  

(11)

where \( A_\mu \) is the electromagnetic potential.

The gravitational and electromagnetic field equations for the system described by \( L_G + L_{\text{e.m.}} \) are the following

\[ G_{\mu\nu} + H_{\mu\nu} = -\kappa T_{\mu\nu}, \]  

(12)

\[ K_{\mu\nu} = 0, \]  

(13)

*Throughout this paper we use the relativistic units, \( c = G = 1 \) and \( \kappa = 8\pi \).
†Heaviside Lorentz rationalized units will be used throughout this paper.
\[ \partial_{\nu} \left( \sqrt{-g} F^{\mu\nu} \right) = 0 \]  
(14)

where the Einstein tensor \( G_{\mu\nu} \) is defined by

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \]  
(15)

Here \( H_{\mu\nu} \) and \( K_{\mu\nu} \) are given by

\[ H_{\mu\nu} \overset{\text{def.}}{=} \lambda \left[ \gamma_{\rho\sigma\mu} \gamma_{\rho\sigma\nu} + \gamma_{\rho\sigma\mu} \gamma_{\nu\rho\sigma} + \gamma_{\rho\sigma\nu} \gamma_{\mu\rho\sigma} + g_{\mu\nu} \left( \gamma_{\rho\sigma\lambda} \gamma^{\lambda\sigma\rho} - \frac{1}{2} \gamma_{\rho\sigma\lambda} \gamma^{\rho\sigma\lambda} \right) \right], \]  
(16)

and

\[ K_{\mu\nu} \overset{\text{def.}}{=} \lambda \left[ \Phi_{\mu,\nu} - \Phi_{\nu,\mu} - \Phi_{\rho} \left( \gamma_{\mu\nu}^{\rho} - \gamma_{\nu\mu}^{\rho} \right) + \gamma_{\mu\nu}^{\rho} \right], \]  
(17)

and they are symmetric and skew symmetric tensors, respectively. The energy-momentum tensor \( T^{\mu\nu} \) is given by

\[ T^{\mu\nu} = g_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} - \frac{1}{4} g^{\mu\nu} g^{\lambda\rho} g^{\sigma\delta} F_{\lambda\rho} F_{\sigma\delta} \]  
(18)

### 3. Cylindrically symmetric solution

The tetrad space having cylindrical symmetry is given by

\[ (e^\mu_i) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & iB \sin \theta \cos \phi & \frac{iB1}{r} \cos \theta \cos \phi & -\frac{iB2 \sin \phi}{r \sin \theta} \\ 0 & iB \sin \theta \sin \phi & \frac{iB1}{r} \cos \theta \sin \phi & \frac{iB2 \cos \phi}{r \sin \theta} \\ 0 & iB \cos \theta & -\frac{iB1}{r} \sin \theta & 0 \end{pmatrix} \]  
(19)

where \( i = \sqrt{-1} \) to preserve Lorentz signature and \( A, B, B1 \) and \( B2 \) are unknown functions in \( r \) and \( \theta \).

Applying (19) to the field equations (12)~(14) we note that the two tensors \( H_{\mu\nu} \) and \( K_{\mu\nu} \) are vanishing identically regardless of the value of the functions \( A, B, B1 \) and \( B2 \). Thus Møller field equations reduce for the tetrad (19) to Einstein’s equations. Applying the following transformation

\[ R = \frac{r}{B}, \]  
(20)
(e_\mu) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & i(1 - rB') \sin \theta \cos \phi & \frac{iB1}{rB} \cos \theta \cos \phi & -iB2 \sin \phi \\ 0 & i(1 - rB') \sin \theta \sin \phi & \frac{iB1}{rB} \cos \theta \sin \phi & \frac{iB2 \cos \phi}{rB \sin \theta} \\ 0 & i(1 - rB') \cos \theta & -\frac{iB1}{rB} \sin \theta & 0 \end{pmatrix}, \quad (21)

with \( B' = \frac{dB(r)}{r} \). Then the field equations (12)~(14) take the form

\[-\kappa T_{00} = G_{00}, \quad -\kappa T_{11} = G_{11}, \quad -\kappa T_{12} = G_{11}, \quad -\kappa T_{21} = G_{21}, \quad -\kappa T_{22} = G_{22}, \quad -\kappa T_{33} = G_{33}, \]

where \( G_{\mu\nu} \) is defined by (15).

Now we are going to find a special solution to the non linear partial differential equations (22), when

\[ A = \frac{1}{\Lambda}, \quad B = \frac{1}{2} \ln(4\sqrt{\Lambda}), \quad B1 = \frac{B}{\Lambda}, \quad B2 = B\Lambda, \]

\[ \Lambda \] is defined as

\[ \Lambda = 1 + \frac{1}{4} B_0^2 r^2 \sin^2 \theta. \]

The form of the vector potential \( A_\mu \), the antisymmetric electromagnetic tensor field \( F_{\mu\nu} \) and the stress energy-momentum tensor \( T_{\mu\nu} \) are given by

\[ A_t(r) = B_0 r \cos \theta, \quad A_t(\theta) = B_0 r \cos \theta, \quad F = B_0 (\cos \theta dt \wedge dr + r \sin \theta d\theta \wedge dt) , \]

\[ T_0^0 = -T_3^3 = \frac{B_0^2}{8\pi \Lambda^4}, \quad T_1^1 = -T_2^2 = \frac{B_0^2 (1 - 2 \sin^2 \theta)}{8\pi \Lambda^4}, \]

\[ T_1^2 = r^2 T_2^1 = -\frac{B_0^2 \sin \theta \cos \theta}{8\pi r \Lambda^4}, \]

and the tetrad (21) takes the form

\[ (e_\mu) = \begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & \frac{i \sin \theta \cos \phi}{\Lambda} & \frac{i \cos \theta \cos \phi}{r \Lambda} & -\frac{i \Lambda \sin \phi}{r \sin \theta} \\ 0 & \frac{i \sin \theta \sin \phi}{\Lambda} & \frac{i \cos \theta \sin \phi}{r \Lambda} & \frac{i \Lambda \cos \phi}{r \sin \theta} \\ 0 & \frac{i \cos \theta}{\Lambda} & -\frac{i \sin \theta}{r \Lambda} & 0 \end{pmatrix}, \quad (26) \]

where \( \Lambda \) is given by (24) with the associated Riemannian metric

\[ ds^2 = \Lambda^2 \left( dt^2 - dr^2 - r^2 d\theta^2 \right) - \frac{r^2 \sin^2 \theta}{\Lambda^2} d\phi^2, \]

\[ (27) \]
which is the line-element given before by Melvin [17].

Thus we have an exact cylindrically symmetric solution satisfy the field equations (12)~(14) and leads to the metric given by Melvin in spherical polar coordinate.

In what follows we will examine the singularities of solution (23) and then, calculated the energy using the superpotential derived form Møller’s theory by Mikhail et al. [18].

4. Singularities

In teleparallel theories we mean by singularity of spacetime [22] the singularity of the scalar concomitants of the torsion and curvature tensors.

Using the definitions of the Riemann-Christoffel curvature tensor, Ricci tensor, Ricci scalar, torsion tensor, basic vector, traceless part and the axial vector part [23], we obtain the scalars of (23) in the form

\[
R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma} = \frac{B_0^4}{4\Lambda^8} \left[ 80 - 24B_0^2 r^2 \sin^2 \theta + 3B_0^4 r^4 \sin^4 \theta \right],
\]

\[
R^{\mu\nu} R_{\mu\nu} = \frac{4B_0^4}{\Lambda^8},
\]

\[
R = 0,
\]

\[
T^{\mu\nu\lambda} T_{\mu\nu\lambda} = \frac{B_0^4 r^2 \sin^2 \theta}{2048\Lambda^4} \left[ r^8 B_0^8 \sin^8 \theta + 24r^6 B_0^6 \sin^6 \theta + 272r^4 B_0^4 \sin^4 \theta + 1536r^2 B_0^2 \sin^2 \theta + 6144 \right],
\]

\[
\Phi^{\mu} \Phi_{\mu} = \frac{B_0^8 r^6}{4096\Lambda^4} \left[ -r^4 B_0^4 \sin^{10} \theta - 24r^2 B_0^2 \sin^8 \theta - 144 \sin^6 \theta \right],
\]

\[
t^{\mu\nu\lambda} t_{\mu\nu\lambda} = \frac{B_0^4 r^2}{4096\Lambda^4} \left[ -r^8 B_0^8 \sin^{10} \theta - 24r^6 B_0^6 \sin^8 \theta - 336r^4 B_0^4 \sin^6 \theta - 2304r^2 B_0^2 \sin^4 \theta - 9216 \sin^2 \theta \right],
\]

\[
a^{\mu} a_{\mu} = 0.
\]

As is clear from (28) that all the above scalars have a singularity when \( \Lambda = 0 \). Also the scalars of Riemann-Christoffel curvature tensor and Ricci tensor will approach infinity rapidly than that of torsion tensor, basic vector and traceless part when \( \Lambda \to 0 \).

5. The Energy content using Møller definition

The superpotential of the Møller theory was given by Mikhail et al. [18] as

\[
U_{\mu}^{\nu\lambda} = \frac{(-g)^{1/2}}{2\kappa} P_{\chi\rho\sigma}^{\nu\lambda} \left[ \Phi^{\rho} g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu} \gamma^{\chi\rho\sigma} - (1 - 2\lambda) g_{\tau\mu} \gamma^{\sigma\rho\chi} \right],
\]
where $U_{\mu}^{\nu \lambda}$ is the superpotential derived from Møller’s theory and $P_{\chi \rho \sigma}^{\tau \nu \lambda}$ is defined as

$$P_{\chi \rho \sigma}^{\tau \nu \lambda} \overset{\text{def}}{=} \delta_{\chi}^{\tau} g_{\rho \sigma}^{\nu \lambda} + \delta_{\tau}^{\rho} g_{\sigma \chi}^{\nu \lambda} - \delta_{\sigma}^{\tau} g_{\chi \rho}^{\nu \lambda}$$

(30)

with $g_{\rho \sigma}^{\nu \lambda}$ being a tensor defined by

$$g_{\rho \sigma}^{\nu \lambda} \overset{\text{def}}{=} \delta_{\rho}^{\nu} \delta_{\sigma}^{\lambda} - \delta_{\sigma}^{\nu} \delta_{\rho}^{\lambda}.$$  

(31)

The energy is expressed by the surface integral [24]

$$E = \lim_{r \to \infty} \int_{r=\text{constant}} U_{0}^{0 \alpha} n_{\alpha} dS,$$

(32)

where $n_{\alpha}$ is the unit 3-vector normal to the surface element $dS$.

Now we are in a position to calculate the energy associated with solution (23) using superpotential (29). As is clear from (32), the only components which contributes to the energy is $U_{0}^{0 \alpha}$. Thus substituting from (23) into (29) we obtain the following non-vanishing value

$$U_{0}^{01} = \frac{B_{0}^{2} x \left[ B_{0}^{2} (x^{2} + y^{2}) + 8 \right]}{128 \pi}, \quad U_{0}^{02} = \frac{B_{0}^{2} y \left[ B_{0}^{2} (x^{2} + y^{2}) + 8 \right]}{128 \pi}, \quad U_{0}^{03} = 0.$$  

(33)

Substituting from (33) into (32) we get

$$E(r) = \frac{1}{6} B_{0}^{2} r^{3} + \frac{1}{60} B_{0}^{4} r^{5}.$$  

(34)

This result depends on the constant $B_{0}$ as to be expected.

### 6. Main results and Discussion

In this paper we have studied a cylindrically symmetric solution in the teleparallel theory of gravitation and electromagnetic fields. The axial vector part of the torsion, $a^{\mu}$ of this solution is identically vanishing.

The solution gives rise to the same Riemannian metric given before by Melvin [17, 25]. Melvin presented a rigorous solution of Einstein-Maxwell equations which correspond to a configuration of parallel magnetic lines of force in equilibrium under their mutual gravitational attraction. This solution had been obtained earlier by Misra and Radhakrishna [26]. This spacetime is invariant under rotation about and translation along an axis of symmetry. This is also invariant under reflection in planes comprising that axis or perpendicular to it. Wheeler [27] demonstrated that a magnetic universe could also be obtained in Newton’s
theory of gravitation and showed that it is unstable according to the elementary Newtonian analysis. Further Melvin [25] showed that his universe to be stable against small radial perturbation and Thorne [28] proved the stability of the magnetic universe against arbitrary large perturbation. Further Thorne [28] pointed out that the Melvin magnetic universe might be of great value in understanding the nature of extragalactic sources of radio waves and thus the Melvin solution to the Einstein-Maxwell equations is of immense astrophysical interest. Virbhadra and Prasanna [29] studied spin dynamics of charged massive test particles in this spacetime. Energy distribution in Melvin’s universe computed by many authors [31, 12] using different definitions of the energy momentum complex within the framework of general relativity theory.

It was shown by Møller [32] that a tetrad description of a gravitational field equation allows a more satisfactory treatment of the energy-momentum complex than does general relativity. Therefore, we have applied the superpotential method given by Mikhail et al. [18] to calculate the energy. As is clear from equation (34) that energy depends on the constant magnetic $B_0$ and if this constant equal zero then, $E = 0$ since solution (23) will reduce to a flat space time i.e.

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 - \sin^2 \theta d\phi^2).$$

It is of interest to mention that Radinschi [30] has calculated the energy distribution of Melvin metric using Møller energy momentum complex within the framework of general relativity and obtained

$$E(r) = \frac{1}{3}B_0^2 r^3 - \frac{1}{15}B_0^4 r^5 + \frac{1}{70}B_0^6 r^7. \quad (35)$$

Also Xulu [31] has calculated the energy of the same metric using the energy momentum complex given by Landau and Lifshitz and the energy momentum complex given by Papapetrou and obtained

$$E(r) = \frac{1}{6}B_0^2 r^3 + \frac{1}{20}B_0^4 r^5 + \frac{1}{140}B_0^6 r^7 + \frac{1}{2520}B_0^8 r^9. \quad (36)$$

As is clear that there is a difference between the results obtained by Radinschi, Xulu and the results obtained here. This is expected because as we stated above that general relativity theory has a problem in the calculation of the quasi local energy [33, 34]. This problem is not exists in the teleparallel theories because it has only one definition of the energy-momentum complex [21].
A comparison between the results of the energy using different energy momentum complexes are given in the following table

Table (I) comparison between the results of energy using different energy momentum complexes

| Energy Momentum Complex | Energy | Equation |
|-------------------------|--------|----------|
| Møller                  | \[\frac{1}{6}B_0^2 r^3 + \frac{1}{60}B_0^4 r^5\] | (34) |
| Within the Framework of |        |          |
| Tetrad Spacetime        |        |          |
| Møller                  | \[\frac{1}{3}B_0^2 r^3 - \frac{1}{15}B_0^4 r^5 + \frac{1}{70}B_0^6 r^7\] | (35) |
| Within the Framework of |        |          |
| General Relativity Spacetime |        |          |
| Landau and Lifshitz, Papapetrou | \[\frac{1}{6}B_0^2 r^3 + \frac{1}{20}B_0^4 r^5 + \frac{1}{140}B_0^6 r^7 + \frac{1}{2520}B_0^8 r^9\] | (36) |
| Within the Framework of |        |          |
| General Relativity Spacetime |        |          |
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