GENERALIZING THE KANTOROVICH METRIC TO PROJECTION VALUED MEASURES

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ABSTRACT. Given a compact metric space X, the collection of Borel probability measures on X can be made into a complete metric space via the Kantorovich metric (see [5]). We generalize this well known result to projection valued measures. In particular, given a Hilbert space \( \mathcal{H} \), consider the collection of projection valued measures from X into the projections on \( \mathcal{H} \). We show that this collection can be made into complete metric space via a generalized Kantorovich metric. As an application, we use the Contraction Mapping Theorem on this complete metric space of projection valued measures to provide an alternative method for proving a fixed point result due to P. Jorgensen (see [7] and [8]). This fixed point, which is a projection valued measure, arises from an iterated function system on X, and is related to Cuntz Algebras. We also show that the space of positive operator valued measures from X into the positive operators on \( \mathcal{H} \) can be made into a complete metric space, via the same generalized Kantorovich metric.

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1. PRELIMINARIES:

Let \((X, d)\) be a compact metric space, and define \(M(X)\) to be the collection of Borel probability measures on X. It is well known (see [5]) that \(M(X)\) can be equipped with the Kantorovich metric, \(H\), given by:

\[
H(\mu, \nu) = \sup_{\phi \in \text{Lip}_1(X)} \left\{ \left| \int_X \phi d\mu - \int_X \phi d\nu \right| \right\},
\]

where \(\mu\) and \(\nu\) are elements of \(M(X)\), and where,
\( \text{Lip}_1(X) = \{ \phi : X \to \mathbb{R} : |\phi(x) - \phi(y)| \leq d(x, y) \text{ for all } x, y \in X \}. \)

**Definition 1.1.** A sequence measures \( \{ \mu_n \}_{n=1}^{\infty} \subseteq M(X) \) converges weakly to a measure \( \mu \in M(X) \), written \( \mu_n \Rightarrow \mu \), if for all \( f \in C(\mathbb{R}) \), \( \int_X f \, d\mu_n \to \int_X f \, d\mu \), where \( C(\mathbb{R}) \) is the collection of continuous real valued functions on \( X \).

The motivation behind the above definition is the following. For each \( \mu \in M(X) \), define a mapping \( \hat{\mu} : M(X) \to \mathbb{R} \) by \( \mu \mapsto \int_X f \, d\mu \). For each \( \nu \in M(X) \), for each \( \epsilon > 0 \), and for any finite collection of functions \( \{f_1, \ldots, f_k\} \subseteq M(X) \), consider the subset, \( \{ \mu \in M(X) : |\hat{f}_j(\mu) - \hat{f}_j(\nu)| < \epsilon \text{ for all } 1 \leq j \leq k \} \), of \( M(X) \). The collection of all finite intersections of such subsets defines a basis for a topology on \( M(X) \) which is called the weak topology on \( M(X) \). Note that since \( C(\mathbb{R}) \) is a separable metric space, the weak topology is first countable, and hence, can be characterized by sequences.

**Remark 1.2.** A sequence \( \{ \mu_n \} \) of measures in \( M(X) \) converges to a measure \( \mu \) in the weak topology on \( M(X) \) if and only if \( \hat{\mu}(\mu_n) \to \hat{\mu}(\mu) \) for all \( f \in C(\mathbb{R}) \).

This leads us to the following two well known facts (presented in [9]).

**Proposition 1.3.**

1. \((M(X), H)\) is compact.
2. The topology induced by the metric \( H \) on \( M(X) \) coincides with the weak topology on \( M(X) \).

**Corollary 1.4.** \((M(X), H)\) is a complete metric space.

We continue with some additional preliminaries. Let \( S = \{ \sigma_0, \ldots, \sigma_{N-1} \} \) be an iterated function system (IFS) on \((X, d)\). That is, for all \( 0 \leq i \leq N - 1 \), \( \sigma_i : X \to X \) such that for all \( x, y \in X \),

\[
d(\sigma_i(x), \sigma_i(y)) \leq r_i d(x, y),
\]

where \( 0 < r_i < 1 \). Indeed, each \( \sigma_i \) is a Lipschitz contraction on \( X \), and \( r_i \) is the Lipschitz constant associated to \( \sigma_i \). Let \( \sigma : X \to X \) be a Borel measurable function such that \( \sigma \circ \sigma_i = \text{id}_X \) for all \( 0 \leq i \leq N - 1 \).

Assume further that,

\[
X = \bigcup_{i=0}^{N-1} \sigma_i(X), \quad (1.2)
\]

where the above union is disjoint. We have the following important result due to Hutchinson.

**Theorem 1.5.** [5] The map \( T : M(X) \to M(X) \) by

\[
\nu(\cdot) \mapsto \sum_{k=0}^{N-1} \frac{1}{N} \nu(\sigma_k^{-1}(\cdot)),
\]

is a Lipschitz contraction in the \((M(X), H)\) metric space, with Lipschitz constant \( r := \max_{0 \leq i \leq N-1} \{ r_i \} \).
By applying the Contraction Mapping Theorem to the Lipschitz contraction $T$, there exists a unique measure, $\mu \in M(X)$, such that $T(\mu) = \mu$. That is,

$$\mu(\cdot) = \frac{1}{N}\sum_{k=0}^{N-1}\mu(\sigma_k^{-1}(\cdot)).$$

This unique invariant measure, $\mu$, is called the Hutchinson measure associated to $S$.

Consider the Hilbert space $L^2(X, \mu)$. Define,

$$S_i : L^2(X, \mu) \to L^2(X, \mu) \text{ by } \phi \mapsto (\phi \circ \sigma)\sqrt{N}\delta_{\sigma_i(X)},$$

for all $i = 0, \ldots, N-1$, and it’s adjoint,

$$S_i^* : L^2(X, \mu) \to L^2(X, \mu) \text{ by } \phi \mapsto \frac{1}{\sqrt{N}}(\phi \circ \sigma_i),$$

for all $i = 0, \ldots, N-1$. This leads to the following result due to Jorgensen.

**Theorem 1.6.** [6] The maps $\{S_i : 0 \leq i \leq N-1\}$ are isometries, and the maps $\{S_i^* : 0 \leq i \leq N-1\}$ are their adjoints. Moreover, these maps and their adjoints satisfy the Cuntz relations:

1. $\sum_{i=0}^{N-1} S_i S_i^* = 1_H$
2. $S_i^* S_j = \delta_{i,j} 1_H$ where $0 \leq i, j \leq N-1$.

**Corollary 1.7.** [6] The Hilbert space $L^2(X, \mu)$ admits a representation of the Cuntz algebra, $\mathcal{O}_N$, on $N$ generators.

Let $\Gamma_N = \{0, \ldots, N-1\}$. For $k \in \mathbb{Z}_+$, let $\Gamma_N^k = \Gamma_N \times \ldots \times \Gamma_N$, where the product is $k$ times. If $a = (a_1, \ldots, a_k) \in \Gamma_N^k$, where $a_j \in \{0, 1, \ldots, N-1\}$ for $1 \leq j \leq k$, define

$$A_k(a) = \sigma_{a_1} \circ \ldots \circ \sigma_{a_k}(X).$$

Using that (1.2) is a disjoint union, we conclude that $\{A_k(a)\}_{a \in \Gamma_N^k}$ partitions $X$ for all $k \in \mathbb{Z}_+$. Indeed,

$$X = \bigcup_{i=0}^{N-1}\sigma_i(K) = \bigcup_{i=0}^{N-1}\sigma_i(\bigcup_{j=1}^{N-1}\sigma_j(X)) = \bigcup_{i,j}\sigma_i \circ \sigma_j(X) = \bigcup_{a \in \Gamma_N^k} A_k(a) = \ldots = \bigcup_{a \in \Gamma_N^k} A_k(a).$$

For $k \in \mathbb{Z}_+$ and $a = (a_1, \ldots, a_k) \in \Gamma_N^k$, define,

$$P_k(a) = S_a S_a^*,$$

where $S_a = S_{a_1} \circ \ldots \circ S_{a_k}$. The Cuntz relations suggest that $P_k(a)$ is a projection on the Hilbert space $L^2(X, \mu)$.

**Definition 1.8.** If $T$ is a set, $\Omega$ is a $\sigma$-algebra of subsets of $T$, and $\mathcal{H}$ is a Hilbert space, a projection valued measure for $(T, \Omega, \mathcal{H})$ is a function $F : \Omega \to B(\mathcal{H})$ (where $B(\mathcal{H})$ denotes the $C^*$ algebra of bounded operators on the Hilbert space $\mathcal{H}$) such that,
(1) For each $\Delta \in \Omega$, $F(\Delta)$ is a projection on $\mathcal{H}$;
(2) $F(\emptyset) = 0$ and $F(\Omega) = id_{\mathcal{H}}$ (where $id_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$);
(3) $F(\Delta_1 \cap \Delta_2) = F(\Delta_1)F(\Delta_2)$ for $\Delta_1, \Delta_2 \in \Omega$;
(4) If $\{\Delta_n\}_{n=1}^{\infty}$ are pairwise disjoint sets in $\Omega$, then for any $h \in \mathcal{H}$,
\[ F\left(\bigcup_{n=1}^{\infty} \Delta_n\right)(h) = \sum_{n=1}^{\infty} (F(\Delta_n)h). \]

We now state an important result due to Jorgensen.

**Theorem 1.9.** \([7] [8]\) There exists a unique projection valued measure $E(\cdot)$ defined on the Borel subsets of $X$ taking values in the projections on $L^2(X, \mu)$ such that,

1. $E(\cdot) = \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\cdot)) S_i^*$, and
2. $E(A_k(a)) = P_k(a)$ for all $k \in \mathbb{Z}^+$ and $a \in \Gamma_k N$.

Jorgensen’s proof of Theorem 1.9 is achieved by showing that the map $A_k(a) \mapsto P_k(a)$ extends uniquely to a projection valued measure from the Borel subsets of $X$, denoted $B(X)$, into the projections on $L^2(X, \mu)$. The main goal of this paper is to provide an alternative proof of this theorem. In particular, we will realize the map,

\[ F(\cdot) \mapsto \sum_{i=0}^{N-1} S_i F(\sigma_i^{-1}(\cdot)) S_i^*, \]

as a Lipschitz contraction on a complete metric space of projection valued measures from $B(X)$ into the projections on $L^2(X, \mu)$. The Contraction Mapping Theorem will then guarantee the existence of a unique projection valued measure $E$ satisfying part (1) of Theorem 1.9 Part (2) of Theorem 1.9 will follow as a consequence.

2. A Metric Space of Projection Valued Measures on $X$:

Let $(X, d)$ be the compact metric space defined above. Let $\mathcal{H}$ be an arbitrary Hilbert space. In Section 3, we will restrict to the situation that $\mathcal{H} = L^2(X, \mu)$.

2.1. A Generalized Kantorovich Metric:

**Lemma 2.1.** \([3]\) Let $F$ be a projection valued measure from $B(X)$ into the projections on $\mathcal{H}$. Let $g, h \in \mathcal{H}$. For all $\Delta \in \mathcal{B}(X)$ define,

\[ F_{g,h}(\Delta) = \langle F(\Delta)g, h \rangle, \]

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$. Then $F_{g,h}(\cdot)$ defines a countably additive measure on $\mathcal{B}(X)$ with total variation less than or equal to $||g|| \cdot ||h||$. Moreover, $F_{g,h}(\cdot) = F_{h,g}(\cdot)$.

**Remark 2.2.** If $h \in \mathcal{H}$, $F_{h,h}(\cdot)$ is a positive measure with total mass equal to $||h||^2$.

**Proposition 2.3.** \([3]\) Let $F$ be a projection valued measure from $B(X)$ into the projections on $\mathcal{H}$. Let $\psi : X \to \mathbb{C}$ be a bounded Borel measurable function. Then there exists a unique bounded operator, which we denote by $\int \psi dF$, that satisfies,

\[ \langle \left( \int \psi dF \right) g, h \rangle = \int_X \psi dF_{g,h}, \]
for all $g, h \in \mathcal{H}$. Moreover, $|\int \psi dF| \leq ||\psi||_\infty$, where $|| \cdot ||$ denotes the operator norm, and $|| \cdot ||_\infty$ denotes the supremum norm.

Let $P(X)$ be the collection of all projection valued measures from $\mathcal{B}(X)$ into the projections on $\mathcal{H}$. Define a metric $\rho$ on $P(X)$ by,

$$\rho(E, F) = \sup_{\phi \in \text{Lip}_1(X)} \left\{ \left\| \int \phi dE - \int \phi dF \right\| \right\}, \quad (2.1)$$

where $|| \cdot ||$ denotes the operator norm in $\mathcal{B}(\mathcal{H})$, and $E$ and $F$ are arbitrary members of $P(X)$. We show that $\rho$ is a metric on $P(X)$. We begin with several facts.

Claim 2.4. For $h \in \mathcal{H}$, and $E \in P(X)$, the positive measure $E_{h,h}(\cdot)$ is regular on $\mathcal{B}(X)$.

Proof. This follows from the fact that positive Borel measures are regular on metric spaces (see [2]).

Claim 2.5. If $\phi : X \to \mathbb{R}$ is continuous and $E \in P(X)$, then $\int \phi dE$ is a self-adjoint operator on $\mathcal{B}(L^2(X, \mu))$.

Proof. Note that since $\phi$ is continuous on the compact space $X$, it is a bounded Borel measurable function. Therefore, one can define $\int \phi dE$. If $g, h \in \mathcal{H}$, then,

$$\left\langle \left( \int \phi dE \right) g, h \right\rangle = \int_X \phi dE_{g,h} = \int_X \overline{\phi dE_{h,g}} = \int_X \phi dE_{h,g} = \left\langle \left( \int \phi dE \right) h, g \right\rangle = \left\langle g, \left( \int \phi dE \right) h \right\rangle.$$

Claim 2.6. If $\lambda \in \mathbb{C}$ is a constant, then,

$$\left\langle \left( \int \lambda dE \right) g, h \right\rangle = \lambda \text{id}_\mathcal{H}.$$

Proof. Let $g, h \in \mathcal{H}$. Then,
\[
\langle \left( \int \lambda dE \right) g, h \rangle = \int_X \lambda dE_{g,h}(x)
\]
\[
= \lambda \int_X dE_{g,h}(x)
\]
\[
= \lambda \langle E(X)g, h \rangle
\]
\[
= \lambda \langle \text{id}_H g, h \rangle
\]
\[
= \langle \lambda \text{id}_H g, h \rangle.
\]

\[
\square
\]

**Theorem 2.7.** $\rho$ is a metric on $P(X)$.

**Proof.**

(1) Let $E, F \in P(X)$. We show $\rho(E, F) < \infty$. Let $\phi \in \text{Lip}_1(X)$ and $x_0 \in X$. Then, by Claim 2.6 ($\lambda = \phi(x_0)$),

\[
\left| \int \phi dE - \int \phi dF \right| = \left| \int \phi dE - \phi(x_0)\text{id}_H + \phi(x_0)\text{id}_H - \int \phi dF \right|
\]
\[
= \left| \int \phi dE - \int \phi(x_0) dE - \left( \int \phi dF - \int \phi(x_0) dF \right) \right|
\]
\[
\leq \left| \int (\phi - \phi(x_0)) dE \right| + \left| \int (\phi - \phi(x_0)) dF \right|
\]

(2.2)

By Claim 2.3 since $\phi - \phi(x_0)$ is a real-valued continuous function on $X$, $\int (\phi - \phi(x_0)) dE$ and $\int (\phi - \phi(x_0)) dF$ are self-adjoint operators and therefore,

\[
\left| \int (\phi - \phi(x_0)) dE \right| = \sup_{h \in H, ||h|| = 1} \left\{ \left| \langle \left( \int (\phi(x) - \phi(x_0)) dE \right) h, h \rangle \right| \right\}
\]

Let $h \in H$ with $||h|| = 1$. Then,
\begin{align*}
\left| \left( \int (\phi(x) - \phi(x_0))dE \right) h, h \right| &= \left| \int_X (\phi(x) - \phi(x_0))dE_{h,h}(x) \right| \\
&\leq \int_X |\phi(x) - \phi(x_0)|dE_{h,h}(x) \\
&\leq \int_X d(x,x_0)dE_{h,h}(x) \\
&\leq \text{diam}(X) \int_X dE_{h,h}(x) \\
&= \text{diam}(X) \langle E(X)h,h \rangle \\
&= \text{diam}(X)||h||^2 \\
&= \text{diam}(X) < \infty,
\end{align*}

where \( \text{diam}(X) \) denotes the diameter of the metric space \( X \). This quantity is finite because \( X \) is compact. Hence,

\[ \left| \int (\phi - \phi(x_0))dE \right| \leq \text{diam}(X) < \infty, \]

and similarly,

\[ \left| \int (\phi - \phi(x_0))dF \right| \leq \text{diam}(X) < \infty, \]

which implies that the last line of (2.2) is less than or equal to \( 2 \text{diam}(X) < \infty \). Since \( \text{diam}(X) \) is independent of the choice of \( \phi \in \text{Lip}_1(X) \), \( \rho(E,F) \leq 2 \text{diam}(X) < \infty \).

(2) Let \( E,F \in P(X) \). It is clear from the definition of \( \rho \) that \( \rho(E,F) = \rho(F,E) \).

(3) Let \( E,F \in P(X) \). We show that \( \rho(E,F) = 0 \) if and only if \( E = F \). The backwards direction is clear from the definition of \( \rho \). For the forwards direction, suppose that \( \rho(E,F) = 0 \). We need to show that \( E = F \). That is, for all \( \Delta \in \mathcal{B}(X) \), we need to show that \( E(\Delta) = F(\Delta) \). Choose a closed subset \( C \subseteq X \).

Define \( f_n : X \to \mathbb{R} \) for \( n = 1, \ldots, \infty \) by \( f_n(x) = \max \{1 - nd(x,C)\} \). Note that \( f_n \in \text{Lip}_1(X) = \{ f : X \to \mathbb{R} : |f(x) - f(y)| \leq nd(x,y) \text{ for all } x,y \in X \} \). Therefore, \( \frac{1}{n} f_n \in \text{Lip}_1(X) \). Since \( \rho(E,F) = 0 \),

\[ \frac{1}{n} \int f_n dE = \frac{1}{n} \int f_n dF, \]

for all \( n \), which implies,

\[ \int f_n dE = \int f_n dF, \]

for all \( n \). Note that \( f_n \downarrow 1_C \) pointwise. Choose \( h \in \mathcal{H} \) with \( ||h|| = 1 \). By the Dominated Convergence Theorem,

\[ E_{h,h}(C) = \int_X 1_C dE_{h,h} = \lim_{n \to \infty} \int_X f_n dE_{h,h}. \]
and,
\[ F_{h,h}(C) = \int_X 1_C dF_{h,h} = \lim_{n \to \infty} \int_X f_n dF_{h,h}. \]

By (2.3),
\[ \int_X f_n dE_{h,h} = \int_X f_n dF_{h,h}, \]
for all \( n \), and hence,
\[ E_{h,h}(C) = F_{h,h}(C) \]
for all closed sets \( C \subseteq X \). Since \( E_{h,h}(\cdot) \) and \( F_{h,h}(\cdot) \) are regular measures, \( E_{h,h}(\Delta) = F_{h,h}(\Delta) \), or equivalently,
\[ \langle (E(\Delta) - F(\Delta))h, h \rangle = 0 \]
for all \( \Delta \in \mathcal{B}(X) \). Since \( E(\Delta) - F(\Delta) \) is a self-adjoint operator (being the difference of two projections),
\[ ||E(\Delta) - F(\Delta)|| = \sup_{h \in \mathcal{H}, ||h|| = 1} |\langle (E(\Delta) - F(\Delta))h, h \rangle| = 0. \]

Therefore, \( E(\Delta) = F(\Delta) \) for all \( \Delta \in \mathcal{B}(X) \).

(4) Let \( E, F, G \in P(X) \). We need to show that \( \rho \) satisfies:
\[ \rho(E, G) \leq \rho(E, F) + \rho(F, G). \]

Choose \( \phi \in \text{Lip}_1(X) \). Then,
\[ \left| \left| \int \phi dE - \int \phi dG \right| \right| \leq \left| \left| \int \phi dE - \int \phi dF \right| \right| + \left| \left| \int \phi dF - \int \phi dG \right| \right|. \]

By taking the supremum of both sides over all \( \text{Lip}_1(X) \) functions (2.4) follows.

**Corollary 2.8.** The metric space \( (P(X), \rho) \) is bounded.

**Proof.** In (1) of the above proof, we showed that for any \( E, F \in P(X) \), \( \rho(E, F) \leq 2 \text{diam}(X) < \infty \). \( \square \)

2.2. \( (P(X), \rho) \) is Complete: We show that the metric space \( (P(X), \rho) \) is complete. We begin with several facts.

**Definition 2.9.** Let \( C(X) \) denote the \( C^* \)-algebra of continuous functions functions from \( X \) to \( \mathbb{C} \), and \( B(H) \) denote the \( C^* \)-algebra of bounded operators on \( H \). A representation \( \pi : C(K) \rightarrow B(H) \) is a \( * \)-homomorphism that preserves the identity.

**Theorem 2.10.** [3] Let \( E : \mathcal{B}(X) \rightarrow B(H) \) be a projection valued measure. The map \( \pi : C(X) \rightarrow B(H) \) given by,
\[ f \mapsto \int f dE, \]
is a representation.

**Theorem 2.11.** [3] Let \( \pi : C(X) \rightarrow B(H) \) be a representation. There exists a unique projection valued measure \( E : \mathcal{B}(X) \rightarrow B(H) \) such that,
\[ \pi(f) = \int f dE, \]
for all \( f \in C(X) \).
Lemma 2.12. \( \text{Lip}(X) \) is dense in \( C_\mathbb{R}(X) \), where \( \text{Lip}(X) \) is the collection of real valued Lipschitz functions on \( X \).

Theorem 2.13. The metric space \( (P(X), \rho) \) is complete.

**Proof.** Let \( \{E_n\}_{n=1}^\infty \subseteq P(X) \) be a Cauchy sequence of projection valued measures in the \( \rho \) metric. For each \( n = 1, 2, \ldots \), use Theorem 2.10 to define a representation \( \pi_n : C(X) \to \mathcal{B}(\mathcal{H}) \) by,

\[
f \mapsto \int f \, dE_n.
\]

**Claim 2.14.** Let \( f \in C(X) \). The sequence of operators \( \{\pi_n(f)\}_{n=1}^\infty \) is Cauchy in the operator norm.

**Proof of claim:** Let \( \epsilon > 0 \). Let \( f = f_1 + if_2 \), where \( f_1, f_2 \in C_\mathbb{R}(X) \). By Lemma 2.12, choose \( g_1, g_2 \in \text{Lip}(X) \) such that \( ||f_1 - g_1||_\infty < \frac{\epsilon}{6} \) and \( ||f_2 - g_2||_\infty < \frac{\epsilon}{6} \).

There is a \( K > 0 \) such that \( \frac{1}{K}g_1 \in \text{Lip}_1(X) \) and \( \frac{1}{K}g_2 \in \text{Lip}_1(X) \). Since \( \{E_n\}_{n=1}^\infty \) is a Cauchy sequence in the \( \rho \) metric, the sequence \( \{\pi_n(\frac{1}{K}g_1)\}_{n=1}^\infty \) is Cauchy in the operator norm, and hence, \( \{\pi_n(g_1)\}_{n=1}^\infty \) is Cauchy in the operator norm. Similarly, \( \{\pi_n(g_2)\}_{n=1}^\infty \) is Cauchy in the operator norm. Therefore, choose \( N \) such that for \( n, m \geq N \),

\[
||\pi_n(g_1) - \pi_m(g_1)|| < \frac{\epsilon}{6} \quad \text{and} \quad ||\pi_n(g_2) - \pi_m(g_2)|| < \frac{\epsilon}{6}.
\]

If \( m, n \geq N \),

\[
||\pi_n(f_1) - \pi_m(f_1)|| \leq ||\pi_n(f_1) - \pi_n(g_1)|| + ||\pi_n(g_1) - \pi_m(g_1)|| + ||\pi_m(g_1) - \pi_m(f)||
\]

\[
\leq ||\pi_n(f_1 - g_1)|| + \frac{\epsilon}{6} + ||\pi_m(f_1 - g_1)||
\]

\[
\leq \frac{\epsilon}{2},
\]

where the third inequality is because \( ||\pi_n(f_1 - g_1)|| \leq ||f_1 - g_1||_\infty \) and \( ||\pi_m(f_1 - g_1)|| \leq ||f_1 - g_1||_\infty \). Similarly, \( ||\pi_n(f_2) - \pi_m(f_2)|| \leq \frac{\epsilon}{2} \). Then, if \( n, m \geq N \),

\[
||\pi_n(f) - \pi_m(f)|| = ||\pi_n(f_1 + if_2) - \pi_m(f_1 + if_2)||
\]

\[
= ||(\pi_n(f_1) - \pi_m(f_1)) + i(\pi_n(f_2) - \pi_m(f_2))||
\]

\[
\leq ||\pi_n(f_1) - \pi_m(f_1)|| + ||\pi_n(f_2) - \pi_m(f_2)||
\]

\[
\leq \epsilon.
\]

This proves the claim.

Define \( \pi : C(X) \to \mathcal{B}(\mathcal{H}) \) by \( f \mapsto \lim_{n \to \infty} \pi_n(f) \). This map is well defined by Claim 2.14 and the fact that \( \mathcal{B}(\mathcal{H}) \) is complete in the operator norm. We show that \( \pi \) is a representation.

1. \( \pi \) is linear:

Let \( f, g \in C(X) \) and \( \alpha \in \mathbb{C} \). Then,
\[ \pi(\alpha f + g) = \lim_{n \to \infty} \pi_n(\alpha f + g) \]
\[ = \lim_{n \to \infty} (\alpha \pi_n(f) + \pi_n(g)) \]
\[ = \alpha \lim_{n \to \infty} \pi_n(f) + \lim_{n \to \infty} \pi_n(g) \]
\[ = \alpha \pi(f) + \pi(g). \]

(2) \( \pi \) is an algebra homomorphism:

Let \( f, g \in C(X) \). Then,

\[ \pi(fg) = \lim_{n \to \infty} \pi_n(fg) \]
\[ = \lim_{n \to \infty} \pi_n(f) \lim_{n \to \infty} \pi_n(g) \]
\[ = \pi(f)\pi(g). \]

(3) \( \pi \) is a \( * \)-homomorphism:

Let \( f \in C(X) \). Then,

\[ \pi(f^*) = \lim_{n \to \infty} \pi_n(f^*) \]
\[ = \lim_{n \to \infty} \pi_n(f)^* \]
\[ = \pi(f)^*, \]

where the last equality is because \( ||\pi_n(f) - \pi(f)|| = ||(\pi_n(f) - \pi(f))^*|| = ||\pi_n(f)^* - \pi(f)^*||. \)

(4) \( \pi \) preserves the identity:

\[ \pi(1) = \lim_{n \to \infty} \pi_n(1) \]
\[ = \lim_{n \to \infty} 1_{\mathcal{H}} \]
\[ = 1_{\mathcal{H}}. \]

By Theorem 2.11 there exists a unique projection valued measure \( E : \mathcal{B}(X) \to \mathcal{B}(\mathcal{H}) \) such that,

\[ \pi(f) = \int f \, dE, \]

for all \( f \in C(X) \). We show that \( E_n \to E \) in the \( \rho \) metric as \( n \to \infty \). Let \( \epsilon > 0 \). Choose \( N \) such that for \( n, m \geq N \),

\[ \rho(E_n, E_m) < \epsilon. \]

Let \( n, m \geq N \) and \( \phi \in \text{Lip}_1(X) \). Observe,
\[ \left\| \int \phi dE_n - \int \phi dE \right\| = \lim_{m \to \infty} \left\| \int \phi dE_n - \int \phi dE_m \right\| \leq \epsilon, \]

where the equality is because \( \lim_{m \to \infty} \int \phi dE_m = \lim_{m \to \infty} \pi_m(\phi) = \pi(\phi) = \int \phi dE \)
and the inequality is because \( \rho(E_n, E_m) < \epsilon \). Since the choice of \( N \) is independent of the choice of \( \phi \), we have for \( n, m \geq N \),

\[ \rho(E_n, E) = \sup_{\phi \in \text{Lip}(X)} \left\{ \left\| \int \phi dE_n - \int \phi dE \right\| \right\} \leq \epsilon. \]

Hence, \( E_n \to E \) in the \( \rho \) metric as \( n \to \infty \) and the metric space \( (P(X), \rho) \) is complete.

2.3. Defining the Weak Topology on \( P(X) \):

**Definition 2.15.** A sequence of projection valued measures \( \{F_n\}_{n=1}^{\infty} \subseteq P(X) \) converges weakly to a projection valued measure \( F \in P(X) \), written \( F_n \Rightarrow F \), if for all \( f \in C_{\mathbb{K}}(X) \), \( \int f dF_n \to \int f dF \), where convergence is in the operator norm on \( \mathcal{B}(\mathcal{H}) \).

**Theorem 2.16.** The topology induced by the \( \rho \) metric on \( P(X) \) coincides with the weak topology on \( P(X) \).

**Proof.** Suppose that \( \{E_n\}_{n=1}^{\infty} \subseteq P(X) \) converges to a projection valued measure, \( E \in P(X) \), in the \( \rho \) metric. We will show that \( E_n \Rightarrow E \). Toward this end, choose \( f \in C_{\mathbb{K}}(X) \) and let \( \epsilon > 0 \). Using Lemma 2.12, choose a function \( g \in \text{Lip}(X) \), with Lipschitz constant \( K > 0 \), such that \( \|f - g\|_{\infty} \leq \frac{\epsilon}{3K} \). Note that \( \frac{g}{K} \in \text{Lip}_1(X) \).

Since \( E_n \to E \) in the \( \rho \) metric, we know there exists an \( N \) such that for \( n \geq N \),

\[ \rho(E_n, E) \leq \frac{\epsilon}{3K}. \]

In particular, for \( n \geq N \),

\[ \left\| \int_X \frac{g}{K} dE_n - \int_X \frac{g}{K} dE \right\| \leq \rho(E_n, E) \leq \frac{\epsilon}{3K}, \]

which implies that,

\[ \left\| \int_X g dE_n - \int_X g dE \right\| \leq \frac{\epsilon}{3}. \]

Combining this information, we get that for \( n \geq N \),

\[ \left\| \int_X f dE_n - \int_X f dE \right\| \leq \left\| \int_X f dE_n - \int_X g dE \right\| + \left\| \int_X g dE_n - \int_X g dE \right\| \]

\[ + \left\| \int_X g dE - \int_X f dE \right\| \]

\[ \leq \|f - g\|_{\infty} + \frac{\epsilon}{3} + \|f - g\|_{\infty} \]

\[ = \epsilon, \]

which implies that \( E_n \Rightarrow E \).
Next, suppose that $E_n \Rightarrow E$. We show that $E_n$ converges to $E$ in the $\rho$ metric. Choose $x_0 \in X$. Consider the set $B = \{ f \in C_{\mathbb{R}}(X) : f \in \text{Lip}_1(X) \text{ and } f(x_0) = 0 \}$.

- $B$ is closed in the supremum norm in $C_{\mathbb{R}}(X)$.
- $B$ is pointwise bounded: If $x \in X$, and $f \in B$, then $|f(x)| = |f(x) - f(x_0)| \leq d(x, x_0) \leq \text{diam}(X) < \infty$.
- $B$ is equicontinuous: Let $x \in X$ and $\epsilon > 0$. Then, if $y \in X$ such that $d(x, y) < \epsilon$,

$$|f(x) - f(y)| \leq d(x, y) < \epsilon,$$

for all $f \in B$.

Therefore, by Ascoli’s Theorem, see [10], $B$ is compact in the supremum norm. Accordingly, choose $\{f_1, \ldots, f_k\} \subseteq B$ such that $B \subseteq \bigcup_{j=1}^{k} \mathcal{O}_{\epsilon_j}(f_j)$. Since $E_n \Rightarrow E$, and $f_j \in C_{\mathbb{R}}(X)$ for all $1 \leq j \leq k$, there exists an $N$ such that for $n \geq N$,

$$\left| \int_X f_j dE_n - \int_X f_j dE \right| \leq \frac{\epsilon}{3},$$

for all $1 \leq j \leq k$. Let $\phi \in \text{Lip}_1(X)$. Define $f(x) = \phi(x) - \phi(x_0)$, and note that $f \in B$. There exists an $f_j$ such that $\|f - f_j\|_{\infty} \leq \frac{\epsilon}{3}$. Observe that if $n \geq N$,

$$\left| \int_X \phi dE_n - \int_X \phi dE \right| = \left| \int_X f dE_n - \int_X f dE \right| \leq \left| \int_X f dE_n - \int_X f_j dE_n \right| + \left| \int_X f_j dE_n - \int_X f_j dE \right| + \left| \int_X f_j dE - \int_X f dE \right| \leq \|f - f_j\|_{\infty} + \frac{\epsilon}{3} + \|f - f_j\|_{\infty} = \epsilon.$$

Since $N$ does not depend on the choice of $\phi$, $\rho(E_n, E) \leq \epsilon$ if $n \geq N$.

2.4. Isomorphism of Metric Spaces:

**Theorem 2.17.** Suppose the $\mathcal{H}_1$ and $\mathcal{H}_2$ are two isomorphic Hilbert spaces with isomorphism $S : \mathcal{H}_1 \to \mathcal{H}_2$. Consider the two associated complete metric spaces $(P_{\mathcal{H}_1}(X), \rho)$ and $(P_{\mathcal{H}_2}(X), \rho)$. Define $\Theta : (P_{\mathcal{H}_1}(X), \rho) \to (P_{\mathcal{H}_2}(X), \rho)$, by

$$E(\cdot) \mapsto SE(\cdot)S^*.$$

Then, $\Theta$ is a bijective isometry of metric spaces.

**Proof.** We first show that $\Theta$ is well defined. Choose some $E \in (P_{\mathcal{H}_1}(X), \rho)$ and show that $\Theta(E)$ is a projection valued measure in $(P_{\mathcal{H}_2}(X), \rho)$. By construction, $\Theta(E)(\Delta)$ is a bounded operator in $B(\mathcal{H}_2)$ for all Borel subsets $\Delta \subseteq X$.

- $\Theta(E)(\emptyset) = SE(\emptyset)S^* = 0$.
- $\Theta(E)(X) = SE(X)S^* = SS^* = 1_{\mathcal{H}_2}$. 

• \( \Theta(E)(\Delta_1 \cap \Delta_2) = SE(\Delta_1 \cap \Delta_2)S^* = SE(\Delta_1)E(\Delta_2)S^* = SE(\Delta_1)SE(\Delta_2)S^* = \Theta(E)(\Delta_1)\Theta(E)(\Delta_2) \) for all Borel subsets \( \Delta_1, \Delta_2 \subseteq X \).

• Let \( \{\Delta_n\}_{n=1}^{\infty} \) be a sequence of pairwise disjoint Borel subsets of \( X \) and let \( h \in H_2 \). Then,

\[
\Theta(E)(\bigcup_{n=1}^{\infty} \Delta_n)(h) = SE(\bigcup_{n=1}^{\infty} \Delta_n)S^*h = S \left( \sum_{n=1}^{\infty} E(\Delta_n)S^*h \right) = \sum_{n=1}^{\infty} SE(\Delta_n)S^*h = \sum_{n=1}^{\infty} \Theta(E)(\Delta_n)h,
\]

where the third equality is because \( S \) is continuous. Hence, \( \Theta(E) \) is a projection valued measure. Now we show that \( \Theta \) preserves the metric \( \rho \). In particular, let \( E, F \in (P_{H_1}(X), \rho) \). We want to show that

\[
\rho(\Theta(E), \Theta(F)) = \rho(E, F).
\]

To this end, choose \( \phi \in \text{Lip}_1(X) \) and suppose \( h \in H_2 \) with ||h|| = 1. Observe,

\[
\left| \left\langle \left( \int \phi d\Theta(E) - \int \phi d\Theta(F) \right) h, h \right\rangle \right| =
\]

\[
\left| \left\langle \left( \int \phi dSE(\cdot)S^* - \int \phi dSF(\cdot)S^* \right) h, h \right\rangle \right| =
\]

\[
\left| \left\langle \left( \int \phi dE - \int \phi dF \right) S^*h, S^*h \right\rangle \right|.
\]

Since \( S^* \) is a surjective isometry,

\[
\{k \in H_1 : ||k|| = 1\} = \{S^*h : h \in H_2, ||h|| = 1\}.
\]

Hence,
\[
\left\| \int \phi d\Theta(E) - \int \phi d\Theta(F) \right\| = \sup_{h \in \mathcal{H}_2, ||h||=1} \left\langle \left( \int \phi dE - \int \phi dF \right) h, h \right\rangle
\]
\[
= \sup_{h \in \mathcal{H}_2, ||h||=1} \left\langle \left( \int \phi dE - \int \phi dF \right) S^* h, S^* h \right\rangle
\]
\[
= \sup_{k \in \mathcal{H}_1, ||k||=1} \left\langle \left( \int \phi dE - \int \phi dF \right) k, k \right\rangle
\]
\[
= \left\| \int \phi dE - \int \phi dF \right\|
\]

By taking the supremum over all Lip\(_1\)(\(X\)) functions we get that, \(\rho(\Theta(E), \Theta(F)) = \rho(E, F)\).

Next, we show that \(\Theta\) is surjective. Choose \(E \in (P_{\mathcal{H}_2}(X), \rho)\). Consider \(S^* E(\cdot) S \in (P_{\mathcal{H}_1}(X), \rho)\). Then, \(\Theta(S^* E(\cdot) S) = SS^* E(\cdot) SS^* = E(\cdot)\), and \(\Theta\) is surjective (where here we are using the fact that \(SS^* = \text{id}_{\mathcal{H}_2}\)). To show \(\Theta\) is injective, suppose \(E, F \in P_{\mathcal{H}_1}(X)\) are such that \(SE(\cdot)S^* = SF(\cdot)S^*\). By using the fact that \(S^* S = \text{id}_{\mathcal{H}_1}\), we get that \(E = F\).

\[\square\]

### 3. An Application for the Metric Space \((P(X), \rho)\):

We now restrict to the situation that \(\mathcal{H} = L^2(X, \mu)\), and we consider the associated complete metric space \((P(X), \rho)\).

**Theorem 3.1.** The map \(\Phi : P(X) \to P(X)\) given by,

\[
E(\cdot) \mapsto \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\cdot)) S_i^*
\]

is a Lipschitz contraction in the \(\rho\) metric.

**Proof.** We begin by showing that the map \(\Phi\) is well defined. That is, we show that if \(E \in P(X)\), then \(\Phi(E)\) is a projection valued measure.

- Let \(\Delta \in \mathcal{B}(X)\). Then,
\[(\Phi(E)(\Delta))^2 = \left( \sum_{i=0}^{N-1} S_i E(\sigma^{-1}_i(\Delta)) S_i^* \right)^2 \]
\[= \sum_{i=0}^{N-1} S_i E(\sigma^{-1}_i(\Delta)) S_i^* \sum_{j=0}^{N-1} S_j E(\sigma^{-1}_j(\Delta)) S_j^* \]
\[= \sum_{i=0}^{N-1} S_i E(\sigma^{-1}_i(\Delta))^2 S_i^* \]
\[= \sum_{i=0}^{N-1} S_i E(\sigma^{-1}_i(\Delta)) S_i^* \]
\[= \Phi(E)(\Delta), \]

where the third equality is because \(S_i^* S_j = \delta_{i,j} \text{id}_H\), and the fourth equality is because \(E(\sigma^{-1}_i(\Delta))\) is a projection (in particular an idempotent) for all \(0 \leq i \leq N - 1\).

- \(\Phi(E)(\emptyset) = \sum_{i=0}^{N-1} S_i E(\sigma^{-1}_i(\emptyset)) S_i^* = \sum_{i=0}^{N-1} S_i E(\emptyset) S_i^* = 0.\)

- \(\Phi(E)(X) = \sum_{i=0}^{N-1} S_i E(\sigma^{-1}_i(X)) S_i^* = \sum_{i=0}^{N-1} S_i E(X) S_i^* = \sum_{i=0}^{N-1} S_i S_i^* = \text{id}_H.\)

- Let \(\Delta_1, \Delta_2 \in \mathcal{B}(X)\). Then,

\[
\Phi(E)(\Delta_1 \cap \Delta_2) = \sum_{i=0}^{N-1} S_i E(\sigma^{-1}_i(\Delta_1 \cap \Delta_2)) S_i^* \\
= \sum_{i=0}^{N-1} S_i E(\sigma^{-1}_i(\Delta_1) \cap \sigma^{-1}_i(\Delta_2)) S_i^* \\
= \sum_{i=0}^{N-1} S_i E(\sigma^{-1}_i(\Delta_1)) E(\sigma^{-1}_i(\Delta_2)) S_i^* \\
= \sum_{i=0}^{N-1} S_i E(\sigma^{-1}_i(\Delta_1)) S_i^* S_i E(\sigma^{-1}_i(\Delta_2)) S_i^* \\
= \sum_{i=0}^{N-1} S_i E(\sigma^{-1}_i(\Delta_1)) S_i^* \sum_{j=0}^{N-1} S_j E(\sigma^{-1}_j(\Delta_2)) S_j^* \\
= \Phi(E)(\Delta_1) \Phi(E)(\Delta_2),
\]

where the third equality is because \(E\) is a projection valued measure, and the fourth and fifth equalities are because \(S_i^* S_j = \delta_{i,j} \text{id}_H\).
Let \( \{\Delta_n\}_{n=1}^\infty \) be a sequence of disjoint subsets in \( \mathcal{B}(X) \). Let \( h \in \mathcal{H} \). Note that,

\[
\Phi(E)(\bigcup_{n=1}^\infty \Delta_n)(h) = \left( \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\bigcup_{n=1}^\infty \Delta_n)) S_i^* \right) (h) = \\
\sum_{i=0}^{N-1} \left( S_i E(\sigma_i^{-1}(\bigcup_{n=1}^\infty \Delta_n)) S_i^* \right) h.
\]  

(3.1)

For each \( 0 \leq i \leq N - 1 \), since \( E \) is a projection valued measure,

\[
E(\sigma_i^{-1}(\bigcup_{n=1}^\infty \Delta_n)) S_i^* h = E(\bigcup_{n=1}^\infty \sigma_i^{-1}(\Delta_n)) S_i^* h = \sum_{n=1}^\infty E(\sigma_i^{-1}(\Delta_n)) S_i^* h.
\]

Since \( S_i \) is continuous,

\[
S_i \left( \sum_{n=1}^\infty E(\sigma_i^{-1}(\Delta_n)) S_i^* h \right) = \sum_{n=1}^\infty S_i E(\sigma_i^{-1}(\Delta_n)) S_i^* h,
\]

which implies that (3.1) is equal to,

\[
\sum_{i=0}^{N-1} \sum_{n=1}^\infty S_i E(\sigma_i^{-1}(\Delta_n)) S_i^* h = \sum_{n=1}^\infty \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\Delta_n)) S_i^* h = \sum_{n=1}^\infty (\Phi(E)(\Delta_n) h).
\]

Claim 3.2. Let \( h \in \mathcal{H} \). Then,

\[
\Phi(E)_{h,h}(\Delta) = \sum_{i=0}^{N-1} E S_i^* h, S_i^* h(\sigma_i^{-1}(\Delta)),
\]

for all \( \Delta \in \mathcal{B}(X) \).

Proof of claim: Let \( \Delta \in \mathcal{B}(X) \). Then,

\[
\Phi(E)_{h,h}(\Delta) = \left( \Phi(E)(\Delta) h, h \right) = \left( \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\Delta)) S_i^* \right) h, h = \\
\sum_{i=0}^{N-1} \left( S_i E(\sigma_i^{-1}(\Delta)) S_i^* h, h \right) = \\
\sum_{i=0}^{N-1} \sum_{i=0}^{N-1} \left( E(\sigma_i^{-1}(\Delta)) S_i^* h, S_i^* h \right) = \\
\sum_{i=0}^{N-1} \sum_{i=0}^{N-1} E S_i^* h, S_i^* h(\sigma_i^{-1}(\Delta)),
\]

which completes the proof of the claim.
We now show that \( \Phi \) is a Lipschitz contraction in the \( \rho \) metric. Accordingly, choose \( E, F \in P(X) \). We show that,

\[
\rho(\Phi(E), \Phi(F)) \leq r \rho(E, F),
\]

where \( r = \max_{0 \leq i \leq N-1} \{r_i\} \) (\( r_i \) is the Lipschitz constant associated to \( \sigma_i \)). Choose \( \phi \in \text{Lip}_1(X) \), \( h \in \mathcal{H} \) with \( ||h|| = 1 \), and \( x_0 \in X \). By Claim 2.5,

\[
\int \phi d\Phi(E) - \int \phi d\Phi(F),
\]

is self adjoint operator, and hence,

\[
\left\langle \left( \int \phi d\Phi(E) - \int \phi d\Phi(F) \right), h, h \right\rangle,
\]

is a real number. Suppose without loss of generality,

\[
\left| \left\langle \left( \int \phi d\Phi(E) - \int \phi d\Phi(F) \right), h, h \right\rangle \right| = \left\langle \left( \int \phi d\Phi(E) - \int \phi d\Phi(F) \right), h, h \right\rangle.
\]

Then,

\[
\left\langle \left( \int \phi d\Phi(E) - \int \phi d\Phi(F) \right), h, h \right\rangle = \left\langle \left( \int \phi d\Phi(F) \right), h, h \right\rangle = \int_X \phi d\Phi(E)_{h,h} - \int_X \phi d\Phi(F)_{h,h} =
\]

\[
\sum_{i=0}^{N-1} \int_X \phi dE_{S_i^* h, S_i^* h}(\sigma_i^{-1}(\cdot)) - \sum_{i=0}^{N-1} \int_X \phi dF_{S_i^* h, S_i^* h}(\sigma_i^{-1}(\cdot)) =
\]

\[
\sum_{i=0}^{N-1} \int_X (\phi \circ \sigma_i) dE_{S_i^* h, S_i^* h} - \sum_{i=0}^{N-1} \int_X (\phi \circ \sigma_i) dF_{S_i^* h, S_i^* h} =
\]

\[
\sum_{i=0}^{N-1} \int_X (\phi(\sigma_i(x)) - \phi(\sigma_i(x_0))) dE_{S_i^* h, S_i^* h}(x) + \sum_{i=0}^{N-1} \int_X \phi(x_0) dE_{S_i^* h, S_i^* h}(x) -
\]

\[
\sum_{i=0}^{N-1} \int_X (\phi(\sigma_i(x)) - \phi(\sigma_i(x_0))) dF_{S_i^* h, S_i^* h}(x) - \sum_{i=0}^{N-1} \int_X \phi(x_0) dF_{S_i^* h, S_i^* h}(x) =
\]

\[
\sum_{i=0}^{N-1} \int_X (\phi(\sigma_i(x)) - \phi(\sigma_i(x_0))) dE_{S_i^* h, S_i^* h}(x) -
\]

\[
\sum_{i=0}^{N-1} \int_X (\phi(\sigma_i(x)) - \phi(\sigma_i(x_0))) dF_{S_i^* h, S_i^* h}(x) \leq
\]

\[
\sum_{i=0}^{N-1} \int_X r d(x, x_0) dE_{S_i^* h, S_i^* h}(x) - \sum_{i=0}^{N-1} r \int_X \phi(\sigma_i(x)) - \phi(\sigma_i(x_0)) \frac{dE_{S_i^* h, S_i^* h}(x)}{r} =
\]
\[ r \left( \sum_{i=0}^{N-1} \left\langle \left( \int d(x, x_0) dE \right) S_i^* h, S_i^* h \right\rangle - \left\langle \left( \int g(x) dF \right) S_i^* h, S_i^* h \right\rangle \right) = \]
\[ r \left( \sum_{i=0}^{N-1} \left\langle \left( \int d(x, x_0) dE - \int g(x) dF \right) S_i^* h, S_i^* h \right\rangle \right) \]

where \( g(x) = \frac{\phi(\sigma_i(x)) - \phi(\sigma_i(x_0))}{r} \in \text{Lip}_1(X) \). Also, note that \( d(x, x_0) \in \text{Lip}_1(X) \).

Continuing,
\[ r \left( \sum_{i=0}^{N-1} \left\langle \left( \int d(x, x_0) dE - \int g(x) dF \right) S_i^* h, S_i^* h \right\rangle \right) \leq \]
\[ r \left( \sum_{i=0}^{N-1} \left\| \int d(x, x_0) dE - \int g(x) dF \right\| \left\| S_i^* h \right\|^2 \right) \leq \]
\[ r \rho(E, F) \left( \sum_{i=0}^{N-1} \left\langle S_i^* h, S_i^* h \right\rangle \right) = r \rho(E, F) \left( \sum_{i=0}^{N-1} \left\langle S_i S_i^* h, h \right\rangle \right) \]
\[ = r \rho(E, F) \left( \sum_{i=0}^{N-1} \left\langle S_i S_i^* h, h \right\rangle \right) \]
\[ = r \rho(E, F) \left\langle S_i S_i^* h, h \right\rangle \]
\[ = r \rho(E, F) \left\langle h, h \right\rangle \]
\[ = r \rho(E, F). \]

Hence,
\[ \left\| \int \phi d\Phi(E) - \int \phi d\Phi(F) \right\| \leq r \rho(E, F), \]
and since \( \phi \) is an arbitrary element of \( \text{Lip}_1(X) \),
\[ \rho(\Phi(E), \Phi(F)) \leq r \rho(E, F), \]
which proves that \( \Phi \) is a Lipschitz contraction on \( (P(X), \rho) \).

\[ \square \]

3.1. **An Alternative Proof of Theorem 1.9**: By Theorem 2.13 and Theorem 3.1, we know that \( \Phi \) is contraction on the complete metric space \( (P(X), \rho) \). By the Contraction Mapping Theorem, there exists a unique projection valued measure \( E \in P(X) \) such that,

\[ E(\cdot) = \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\cdot)) S_i^* . \quad (3.2) \]
It remains to show that \( E(A_k(a)) = P_k(a) \) for all \( k \in \mathbb{Z}_+ \) and \( a \in \Gamma_N^k \). This will be done by induction on \( k \). Indeed, suppose that \( k = 1 \), and consider \( A_1(j) = \sigma_j(X) \) for some \( j \in \Gamma_N \). Then, by (3.2),

\[
E(\sigma_j(X)) = \sum_{i=0}^{N-1} S_i E(\sigma_j^{-1}(\sigma_j(X))) S_i^*
\]

\[
= S_j E(X) S_j^*
\]

\[
= S_j S_j^*
\]

\[
= P_1(j).
\]

This proves the base case. Suppose that \( E(A_{k-1}(b)) = P_{k-1}(b) \) for all \( b \in \Gamma_{N-1}^k \) where \( k \in \mathbb{Z}_+ \) with \( k > 1 \). We will show that \( E(A_k(a)) = P_k(a) \). Choose some \( a \in \Gamma_N^k \). Suppose that \( a = (a_1, \ldots, a_k) \) and \( b = (a_2, \ldots, a_k) \). Then,

\[
E(A_k(a)) = \sum_{i=0}^{N-1} S_i E(\sigma^{-1}_i(A_k(a))) S_i^*
\]

\[
= \sum_{i=0}^{N-1} S_i E(\sigma^{-1}(\sigma_{a_1}(A_{k-1}(b)))) S_i^*
\]

\[
= S_{a_1} E(A_{k-1}(b)) S_{a_1}^*
\]

\[
= S_{a_1} P_{k-1}(b) S_{a_1}^*
\]

\[
= P_k(a).
\]

Hence, an alternative proof of Theorem \text{[1.9]} is complete.

**Remark 3.3.** The alternative proof that we have presented depends on the fact that the subsets \( A_k(a) \), for \( k \in \mathbb{Z}_+ \) and \( a = (a_1, \ldots, a_k) \in \Gamma_N^k \), satisfy

\[
A_k(a) = \sigma_{a_1} \circ \cdots \circ \sigma_{a_k}(X),
\]

where \( \{\sigma_i\}_{i=0}^{N-1} \) is an iterated function system of Lipschitz contractions on \( X \). The proof of Jorgensen does not require this assumption, and hence, it is more general. Indeed, it only requires that for each \( k \in \mathbb{Z}_+ \), there is a sequence of subsets \( \{A_k(a)\}_{a \in \Gamma_N^k} \) which partitions \( X \), such that

\[
\lim_{k \to \infty} \text{diam}(A_k(a)) = O(N^{-ck}) \text{ for } c > 0.
\]

At this juncture, we think it is useful to briefly review a result due to Jorgensen which identifies the relationship between the measure \( \mu \) and the projection valued measure \( E \). We first introduce some notation. Let \( C \) be the abelian \( C^*-\)subalgebra of \( B(L^2(X, \mu)) \) generated by the family of projections \( S_a S_a^* \) for \( a \in \Gamma_N^k \) and \( k \in \mathbb{Z}_+ \). For \( h \in L^2(X, \mu) \), let \( Ch = \{Ch : C \in C\} \), and denote the closure of \( Ch \) by \( H_h \).

**Theorem 3.4.** [7] There is a set \( h_1, h_2, \ldots \in L^2(X, \mu) \) (possibly finite) with \( ||h_i|| = 1 \) for all \( i = 1, 2, \ldots \) such that the following hold:
(1) The measures $E_{h_i, h_i}$ for $i = 1, 2, \ldots$ are mutually singular.
(2) $L^2(X, \mu) = \oplus_i \mathcal{H}_{h_i}$.
(3) For each $i = 1, 2, \ldots$, there exists a unique isometry,

$$V_i : L^2(X, E_{h_i, h_i}) \rightarrow L^2(X, \mu),$$

satisfying the following:

- $V_i(1_{A_k(a)}) = S_a S^*_a h_i$ for all $a \in \Gamma^k_N$ and $k \in \mathbb{Z}_+$,
- $V_i^* S_a S^*_a V_i = M_{A_k(a)}$ for all $a \in \Gamma^k_N$ and $k \in \mathbb{Z}_+$, where $M_{A_k(a)}$ is the operator on $L^2(X, E_{h_i, h_i})$ given by multiplication by $1_{A_k(a)}$, and
- $V_i(L^2(X, E_{h_i, h_i})) = \mathcal{H}_{h_i}$.

We note that Theorem 3.4 is presented in a more general form in [7]; namely, the Hilbert space $L^2(X, \mu)$ is replaced with an arbitrary Hilbert space $\mathcal{H}$ which admits a representation of the Cuntz algebra on $N$ generators.

4. Generalizing to Positive Operator Valued Measures:

**Definition 4.1.** Let $\mathcal{H}$ be an arbitrary Hilbert space. A positive operator valued measure for $(X, B(X), \mathcal{H})$ is a function $A : B(X) \rightarrow B(\mathcal{H})$ such that,

1. For each $\Delta \in B(X)$, $A(\Delta)$ is a positive operator on $\mathcal{H}$;
2. $A(\emptyset) = 0$ and $A(X) = id_{\mathcal{H}}$;
3. If $\{\Delta_n\}_{n=1}^\infty$ are pairwise disjoint sets in $B(X)$, then for all $h, k \in \mathcal{H}$,

$$\left\langle B \left( \bigcup_{n=1}^\infty \Delta_n \right), h, k \right\rangle = \sum_{n=1}^\infty \langle B(\Delta_n) h, k \rangle.$$

**Remark 4.2.** Lemma 2.1 and Proposition 2.3 are true for positive operator valued measures.

Let $S(X)$ be the collection of all positive operator valued measures from $X$ into $B(\mathcal{H})$, and consider the $\rho$ metric on $S(X)$. Using the same argument as before, we can show that $\rho$ is a metric on $S(X)$.

**Theorem 4.3.** The metric space $(S(X), \rho)$ is complete.

The proof of this theorem will be achieved by combining several results.

**Lemma 4.4.** Let $\{A_n\}_{n=1}^\infty \subseteq S(X)$ be a Cauchy sequence in the $\rho$ metric. For any $g, h \in \mathcal{H}$, there exists a unique complex valued measure $\mu_{g, h}$ such that $A_n \Rightarrow \mu_{g, h}$; that is, for any $f \in C_\mathbb{R}(X)$,

$$\int_X f dA_n \rightarrow \int_X f d\mu_{g, h}.$$

**Proof.** Choose some $h \in \mathcal{H}$ and consider the sequence of positive measures $\{A_{n_h, h}\}_{n=1}^\infty$.

**Claim 4.5.** For $f \in C_\mathbb{R}(X)$, the sequence $\{\int_X f dA_{n_h, h}\}_{n=1}^\infty$ is a Cauchy sequence of real numbers.
Proof of claim: Let $\epsilon > 0$. Since $\text{Lip}(X)$ is dense in $C_\mathbb{R}(X)$, choose a $g \in \text{Lip}(X)$, with Lipschitz constant $K > 0$, such that $\|f - g\|_\infty \leq \frac{\epsilon}{3||h||^2}$. Next, choose an $N$ such that for $n, m \geq N$, $\rho(A_n, A_m) \leq \frac{\epsilon}{3K||h||^2}$. Note that since $gK \in \text{Lip}_1(X)$, if $n, m \geq N$,

$$\left| \int_X \frac{g}{K} dA_{n,h} - \int_X \frac{g}{K} dA_{m,h} \right| = \left| \left( \int_X \frac{g}{K} dA_n - \int_X \frac{g}{K} dA_m \right) h, h \right|$$

$$\leq \left\| \int_X \frac{g}{K} dA_n - \int_X \frac{g}{K} dA_m \right\| ||h||^2$$

$$\leq \rho(A_n, A_m)||h||^2$$

$$\leq \frac{\epsilon}{3K},$$

or equivalently,

$$\left| \int_X g dA_{n,h} - \int_X g dA_{m,h} \right| \leq \frac{\epsilon}{3}.$$  

For $n, m \geq N$,

$$\left| \int_X f dA_{n,h} - \int_X f dA_{m,h} \right| = \left| \int_X f dA_{n,h} - \int_X g dA_{n,h} \right|$$

$$+ \left| \int_X g dA_{n,h} - \int_X g dA_{m,h} \right|$$

$$+ \left| \int_X g dA_{m,h} - \int_X f dA_{m,h} \right|$$

$$\leq ||f - g||_\infty ||h||^2 + \frac{\epsilon}{3} + ||f - g||_\infty ||h||^2$$

$$\leq \epsilon,$$

and the claim is proven.

Define $\mu_{h,h} : C_\mathbb{R}(X) \to \mathbb{R}$ by $f \mapsto \lim_{n \to \infty} \int_X f dA_{n,h}$. This map is well defined by the above claim, and the fact that $\mathbb{R}$ is complete. Since $\mu_{h,h}$ is a positive linear functional on $C(X)$, $\mu_{h,h}$ is the unique measure which satisfies,

$$\int_X f d\mu_{h,h} = \mu_{h,h}(f),$$

for all $f \in C_\mathbb{R}(X)$. Moreover, by construction, $A_{n,h} \Rightarrow \mu_{h,h}$.

Remark 4.6. Let $A$ be a positive operator (or projection) valued measure on $X$. Since the inner product associated to the Hilbert space $\mathcal{H}$ is sesquilinear, the map,

$$[g, h] \mapsto A_{g,h},$$

is sesquilinear. Also, for $g, h \in \mathcal{H}$, $A_{g,h} = A_{h,g}^\ast$ (this property is also inherited from the inner product).
Let \( g, h \in \mathcal{H} \). Using the above remark, observe that for all \( n \),

\[
A_{n g+h,g+h} = A_{n g,g} + A_{n g,h} + A_{n h,g} + A_{n h,h} \\
= A_{n g,g} + A_{n g,h} + A_{n g,h} + A_{n h,h} \\
= A_{n g,g} + 2\Re A_{n g,h} + A_{n h,h},
\]

and therefore, \( \Re A_{n g,h} = \frac{1}{2} (A_{n g+h,g+h} - A_{n g,g} - A_{n h,h}) \). Since \( A_{n g+h,g+h} \Rightarrow \mu_{g+h,g+h} \), \( A_{n g,g} \Rightarrow \mu_{g,g} \), and \( A_{n h,h} \Rightarrow \mu_{h,h} \),

\[
\Re A_{n g,h} = \frac{1}{2} (\mu_{g+h,g+h} - \mu_{g,g} - \mu_{h,h}) := \Re \mu_{g,h}.
\]

Similarly,

\[
A_{n i g+h,i g+h} = A_{n g,g} + iA_{n g,h} - iA_{n h,g} + A_{n h,h} \\
= A_{n g,g} + i(\Re A_{n g,h} + iA_{n g,h}) - i\overline{A_{n g,h}} + A_{n h,h} \\
= A_{n g,g} + i\Re A_{n g,h} - \Im A_{n g,h} - i(\Re A_{n g,h} - i\Im A_{n g,h}) + A_{n h,h} \\
= A_{n g,g} + i\Re A_{n g,h} - \Im A_{n g,h} - i\Re A_{n g,h} - \Im A_{n g,h} + A_{n h,h} \\
= A_{n g,g} - 2\Im A_{n g,h} + A_{n h,h},
\]

and therefore, \( \Im A_{n g,h} = -\frac{1}{2} (A_{n i g+h,i g+h} - A_{n g,g} - A_{n h,h}) \). Since \( A_{n i g+h,i g+h} \Rightarrow \mu_{i g+h,i g+h} \), \( A_{n g,g} \Rightarrow \mu_{g,g} \), and \( A_{n h,h} \Rightarrow \mu_{h,h} \),

\[
\Im A_{n g,h} = -\frac{1}{2} (\mu_{i g+h,i g+h} - \mu_{g,g} - \mu_{h,h}) := \Im \mu_{g,h}.
\]

Define \( \mu_{g,h} = \Re \mu_{g,h} + i\Im \mu_{g,h} \). By construction, \( A_{n g,h} \Rightarrow \mu_{g,h} \).  

\[\square\]

**Lemma 4.7.** The map \( [g, h] \mapsto \mu_{g,h} \) is sesquilinear.

**Proof.** We will show that \( [g, h] \mapsto \mu_{g,h} \) is linear in the first coordinate. The remaining properties of sesquilinearity are proved with a similar approach.

Let \( g, h, k \in \mathcal{H} \). We will show that,

\[
\mu_{g+h,k}(\Delta) = \mu_{g,k}(\Delta) + \mu_{h,k}(\Delta),
\]

for all Borel subsets \( \Delta \in \mathcal{B}(X) \). Let \( f \in C_{r}(X) \). Then,

\[
\int_X f d\mu_{g+h,k} = \lim_{n \to \infty} \int_X f dE_{n g+h,k} \\
= \lim_{n \to \infty} \left( \int_X f dE_{n g,k} + \int_X f dE_{n h,k} \right) \\
= \int_X f d\mu_{g,k} + \int_X f d\mu_{h,k}.
\]
Consider a closed subset $C \subseteq X$, and choose a sequence of functions $\{f_m\}_{m=1}^{\infty} \subseteq C_\mathbb{R}(X)$ such that $f_m \downarrow 1_C$ pointwise. By the Dominated Convergence Theorem,

$$
\int_X 1_C d\mu_{g+h,k} = \lim_{m \to \infty} \int_X f_m d\mu_{g+h,k}
$$

$$
= \lim_{m \to \infty} \left( \int_X f_m d\mu_{g,k} + \int_X f_m d\mu_{h,k} \right)
$$

$$
= \int_X 1_C d\mu_{g,k} + \int_X 1_C d\mu_{h,k}.
$$

Hence, we have shown that for any closed $C \subseteq X$,

$$
\mu_{g+h,k}(C) = \mu_{g,k}(C) + \mu_{h,k}(C). \tag{4.1}
$$

By decomposing the measures $\mu_{g+h,k}, \mu_{g,k}, \mu_{h,k}$ into their real and imaginary parts, we can show that (4.1) is equivalent to the following equations:

$$
\text{Re}\mu_{g+h,k}(C) = \text{Re}\mu_{g,k}(C) + \text{Re}\mu_{h,k}(C), \tag{4.2}
$$

and,

$$
\text{Im}\mu_{g+h,k}(C) = \text{Im}\mu_{g,k}(C) + \text{Im}\mu_{h,k}(C). \tag{4.3}
$$

Using the definition of the real part of the measures $\mu_{g+h,k}, \mu_{g,k}, \mu_{h,k}$, we can show, by rearranging terms, that (4.2) is equivalent to:

$$
M_1(C) = M_2(C), \tag{4.4}
$$

where $M_1$ is the positive Borel measure,

$$
\frac{1}{2}(\mu_{g+h+k,g+h+k} + \mu_{g,g} + 2\mu_{k,k}),
$$

and $M_2$ is the positive Borel measure,

$$
\frac{1}{2}(\mu_{g+k,g+k} + \mu_{g+h,g+h} + \mu_{h+k,h+k} + \mu_{k,k} + \mu_{h,h}).
$$

Since $M_1$ and $M_2$ are positive Borel measures on a metric space, $M_1$ and $M_2$ regular. That is, we can conclude that $M_1(\Delta) = M_2(\Delta)$ for any Borel subset $\Delta \in B(X)$. By invoking the equivalence of (4.2) and (4.4), we have that (4.2) is true for any Borel subset $\Delta \in B(X)$. A similar approach, will yield that (4.3) is true for any Borel subset $\Delta \in B(X)$. Hence, (4.1) is true for any Borel subset $\Delta \in B(X)$. This shows linearity in the first coordinate.

As mentioned above, the following additional properties listed below are proved similarly:

- Let $g, h, k \in \mathcal{H}$. Then $\mu_{g,h+k} = \mu_{g,h} + \mu_{g,k}$.
- Let $\alpha \in \mathbb{C}$ and $g, h \in \mathcal{H}$. Then $\mu_{\alpha g,h} = \alpha \mu_{g,h}$.
- Let $\beta \in \mathbb{C}$ and $g, h \in \mathcal{H}$. Then $\mu_{g,\beta h} = \beta \mu_{g,h}$.
Hence, the map \([g, h] \mapsto \mu_{g,h}\) is sesquilinear.

**Lemma 4.8.** For all \(g, h \in \mathcal{H}\), \(\mu_{g,h}(X) = \langle g, h \rangle\).

**Proof.** \(\mu_{g,h}(X) = \int_X 1d\mu_{g,h} = \lim_{n \to \infty} \int_X 1dA_{n_{g,h}} = \lim_{n \to \infty} \langle A_n(X)g, h \rangle = \langle g, h \rangle\), where the second equality is because \(1 \in C_\mathcal{R}(X)\), and \(A_{n_{g,h}} \Rightarrow \mu_{g,h}\). □

We now begin the proof of Theorem 4.3.

**Proof.** Let \(\{A_n\}_{n=1}^\infty \subseteq S(X)\) be a Cauchy sequence in the \(\rho\) metric. Our goal is to find an element \(A \in S(X)\) such that \(A_n \to A\) in the \(\rho\) metric. By Lemma 4.4 and Lemma 4.7, there exists a sesquilinear family of complex measures \(\{\mu_{g,h} : g, h \in \mathcal{H}\}\) such that for all \(f \in C_\mathcal{R}(X)\),

\[
\int_X f dA_{n_{g,h}} \to \int_X f d\mu_{g,h}.
\]

Let \(\Delta \in B(X)\). The map \([g, h] \mapsto \int_X 1\Delta d\mu_{g,h}\) is a bounded sesquilinear form with bound 1. Hence, by the Riesz Representation Theorem, there exists a unique bounded operator, \(A(\Delta) \in B(\mathcal{H})\), such that for all \(g, h \in \mathcal{H}\),

\[
\langle A(\Delta)g, h \rangle = \int_X 1\Delta d\mu_{g,h}.
\]

Accordingly, define \(A : B(X) \to B(\mathcal{H})\) by \(\Delta \mapsto A(\Delta)\).

**Claim 4.9.** \(A\) is a positive operator valued measure.

**Proof of claim:**

1. Let \(\Delta \in B(X)\), and \(h \in \mathcal{H}\). Then,

\[
\langle A(\Delta)h, h \rangle = \int_X 1\Delta d\mu_{h,h} \geq 0.
\]

Hence, \(A(\Delta)\) is a positive operator.

2. Let \(h \in \mathcal{H}\). Then,

\[
\langle A(X)h, h \rangle = \int_X d\mu_{h,h} = \langle h, h \rangle,
\]

and

\[
\langle A(\emptyset)h, h \rangle = \int_X 1_\emptyset d\mu_{h,h} = 0.
\]

Hence, \(A(X) = \text{id}_\mathcal{H}\) and \(A(\emptyset) = 0\).

3. If \(\{\Delta_n\}_{n=1}^\infty\) are pairwise disjoint sets in \(B(X)\), then for all \(h, k \in \mathcal{H}\),

\[
\left\langle A \left( \bigcup_{n=1}^\infty \Delta_n \right) h, k \right\rangle = \int_X 1_{\bigcup_{n=1}^\infty \Delta_n} d\mu_{h,k} =
\]
\[
\sum_{n=1}^{\infty} \mu_{h,k}(\Delta_n) = \\
\sum_{n=1}^{\infty} \int_X 1_{\Delta_n} d\mu_{h,k} = \\
\sum_{n=1}^{\infty} \langle A(\Delta_n) h, k \rangle.
\]

This completes the proof of the claim.

We now show that \(A_n \to A\) in the \(\rho\) metric. Let \(\epsilon > 0\). Choose an \(N\) such that for \(n, m \geq N\),

\[
\rho(A_n, A_m) < \epsilon.
\]

Next, choose \(\phi \in \text{Lip}_1(X)\). If \(n \geq N\), and \(h \in \mathcal{H}\) with \(||h|| = 1\),

\[
\left| \left\langle \left( \int \phi dA_n - \int \phi dA_m \right) h, h \right\rangle \right| = \left| \int X \phi dA_{n,h,h} - \int X \phi dA_{h,h} \right| \\
= \lim_{m \to \infty} \left| \int X \phi dA_{n,h,h} - \int X \phi dA_{m,h,h} \right| \\
= \lim_{m \to \infty} \left| \left\langle \left( \int \phi dA_n - \int \phi dA_m \right) h, h \right\rangle \right|,
\]

where the second equality is because \(A_{n,h,h} \Rightarrow \mu_{h,h} = A_{h,h}\). Now,

\[
\left| \left\langle \left( \int \phi dA_n - \int \phi dA_m \right) h, h \right\rangle \right| \leq \left\| \int \phi dA_n - \int \phi dA_m \right\| \left| h \right|^2 \\
= \left\| \int \phi dA_n - \int \phi dA_m \right\| \\
\leq \rho(A_n, A_m) \\
\leq \epsilon.
\]

Hence,

\[
\lim_{m \to \infty} \left| \left\langle \left( \int \phi dA_n - \int \phi dA_m \right) h, h \right\rangle \right| \leq \epsilon,
\]

and therefore,

\[
\left\| \int \phi dA_n - \int \phi dA \right\| \leq \epsilon.
\]

Since the choice of \(N\) is independent of \(\phi \in \text{Lip}_1(X)\), \(\rho(A_n, A) \leq \epsilon\), which shows that the metric space \((S(X), \rho)\) is complete.

\(\square\)
Corollary 4.10. The map $\Phi : S(X) \to S(X)$ given by,

$$A(\cdot) \mapsto \sum_{i=0}^{N-1} S_i A(\sigma_i^{-1}(\cdot)) S_i^*,$$

is a Lipschitz contraction in the $\rho$ metric.

Proof. The proof of this corollary is exactly the same as the proof of Theorem 3.1. □

Remark 4.11. Since we have previously shown that $(P(X), \rho)$ is a complete metric space, and $P(X) \subseteq S(X)$, where $(S(X), \rho)$ is also complete, we can conclude that $P(X)$ is a closed subset of $S(X)$ in the $\rho$ metric. Also, note that, by uniqueness, the fixed point for the map $\Phi : S(X) \to S(X)$ is the same as the fixed point for $\Phi : P(X) \to P(X)$.

Remark 4.12. We can also consider the weak topology on $S(X)$. Using the same argument as before, one can show that the weak topology on $S(X)$ coincides with the topology induced by the $\rho$ metric.

5. Conclusion:

Given a compact metric space $(X, d)$, we showed that it is possible to generalize the Kantorovich metric to the space of projection (positive operator) valued measures from $B(X)$ into the bounded operators on an arbitrary fixed Hilbert space $\mathcal{H}$. We showed that this metric space was complete. We used the Contraction Mapping Theorem as an alternative approach to show the existence and uniqueness of a projection valued measure that satisfies a fixed point result (this was first identified by P. Jorgensen using a different and slightly more general approach [7] [8]). We now identify a list of further related topics. We have considered the first four topics, but we plan to present these additional findings in a future paper. We have not considered the fifth topic listed below, but we look forward to investigating it.

(1) Finding a topology on $P(X)$ (or $S(X)$) which is compact.

Remark 5.1. The proof that the Kantorovich metric space of Borel probability measures on a compact space is compact (see Theorem 1.3) does not directly generalize to the $(P(X), \rho)$ (or $(S(X), \rho)$) metric space, because $B(\mathcal{H})$ does not have a compact unit ball in the operator norm.

(2) Generalizing $(X, d)$ to a non-compact metric space.
(3) Finding other Lipschitz contractions on the metric space $(S(X), \rho)$.
(4) Looking at the map,

$$E(\cdot) \mapsto \sum_{i=0}^{N-1} S_i E(\sigma_i^{-1}(\cdot)) S_i^*,$$

when the maps $\{\sigma_i\}$ constitute a weakly hyperbolic iterated function system on a compact metric space $(X, d)$ (see [1] and [4]). This is a weaker notion than when each $\sigma_i$ is a Lipschitz contraction on $X$.

(5) Considering an iterated function system $\{\sigma_i\}$ that has overlap (i.e. [1,2] is not a disjoint union).
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