ALGEBRAIC INDEPENDENCE RESULTS FOR VALUES OF THETA-CONSTANTS

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Abstract: Let $\theta(q) = 1 + 2 \sum_{\nu=1}^{\infty} q^{\nu^2}$ denote the Thetanullwert of the Jacobi Zeta function

$$\theta(z|\tau) = \sum_{\nu=-\infty}^{\infty} e^{\pi i \nu^2 \tau + 2 \pi i \nu z}.$$ 

For algebraic numbers $q$ with $0 < |q| < 1$ we prove the algebraic independence over $\mathbb{Q}$ of the numbers $\theta(q^n)$ and $\theta(q)$ for $n = 2, 3, \ldots, 12$ and furthermore for all $n \geq 16$ which are powers of two. An application for $n = 5$ proves the transcendence of the number

$$\sum_{j=1}^{\infty} (-1)^j \left( \frac{j}{5} \right) \frac{jq^j}{1 - q^j}.$$ 

Similar results are obtained for numbers related to modular equations of degree 3, 5, and 7.

Keywords: algebraic independence, theta-constants, Nesterenko’s theorem, independence criterion, modular equations.

1. Introduction and statement of results

Let $\tau$ with $\Im(\tau) > 0$ denote a complex variable. The series

$$\vartheta_2(\tau) = 2 \sum_{\nu=0}^{\infty} q^{(\nu+1/2)^2}, \quad \vartheta_3(\tau) = 1 + 2 \sum_{\nu=1}^{\infty} q^{\nu^2}, \quad \vartheta_4(\tau) = 1 + 2 \sum_{\nu=1}^{\infty} (-1)^\nu q^{\nu^2}$$

are known as theta-constants or Thetanullwerte, where $q = e^{\pi i \tau}$. Sometimes it is useful to write $\vartheta_2(q), \vartheta_3(q), \vartheta_4(q)$ instead of $\vartheta_2(\tau), \vartheta_3(\tau), \vartheta_4(\tau)$, respectively, where $q$ belongs to the unit circle around 0 of the complex plane. The theta-constants are modular forms of weight 1/2 for the principal congruence subgroup of level 2. In particular, $\theta(q) := \vartheta_3(q)$ is the Thetanullwert of the Jacobi zeta

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function \( \theta(z|\tau) = \sum_{\nu=-\infty}^{\infty} e^{\pi i \nu^2 \tau + 2 \pi i \nu z} \). Let \( n \geq 3 \) denote an odd positive integer. Set

\[
h_j(\tau) := n^2 \frac{\vartheta_j^4(n\tau)}{\vartheta_j^4(\tau)} \quad (j = 2, 3, 4), \quad \lambda = \lambda(\tau) := \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}, \quad \psi(n) := n \prod_{p | n} \left(1 + \frac{1}{p}\right),
\]

where \( p \) runs through all primes dividing \( n \). Also the function

\[
j(\tau) := 256 \left(\frac{\lambda^2 - \lambda + 1}{\lambda^2 - 1}\right)^3
\]

is a modular function with respect to the group \( SL(2, \mathbb{Z}) \) (cf. [5, ch.3,18]), for which identities of the form \( \Phi_n(j(\tau), n(\tau)) \) with polynomials \( \Phi_n(X, Y) \in \mathbb{Z}[X, Y] \) are known (cf. [5, ch.5]). Yu.V.Nesterenko [8] proved the existence of integer polynomials \( P_n(X, Y) \in \mathbb{Z}[X, Y] \) such that \( P_n(h_j(\tau), R_j(\lambda(\tau))) = 0 \) holds for \( j = 2, 3, 4 \), odd integers \( n \geq 3 \), and a suitable rational function \( R_2, R_3, \) or \( R_4 \), respectively:

**Theorem A ([8, Theorem 1.1, Corollary 3]).** For any odd integer \( n \geq 3 \) there exists a polynomial \( P_n(X, Y) \in \mathbb{Z}[X, Y] \), \( \deg_X P = \psi(n) \), such that

\[
P_n\left(h_2(\tau), 16 \frac{\lambda(\tau) - 1}{\lambda(\tau)}\right) = 0,
\]

\[
P_n\left(h_3(\tau), 16 \lambda(\tau)\right) = 0,
\]

\[
P_n\left(h_4(\tau), 16 \frac{\lambda(\tau)}{\lambda(\tau) - 1}\right) = 0.
\]

The polynomials \( P_3, P_5, P_7, P_9, \) and \( P_{11} \) are listed in the appendix. \( P_3 \) and \( P_5 \) are already given in [8], \( P_7, P_9, \) and \( P_{11} \) are the results of computer-assisted computations of the author.

There are various algebraic relationships between the theta-constants and arithmetic functions like Ramanujan’s Eisenstein series \( P(q), Q(q), R(q) \) (cf. [6]), the Dedekind eta-function \( \eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2 \pi i n \tau}) \), and others. For instance, it follows from Jacobi’s triple product identity that \( \theta(-q) = \eta^2(\tau)/\eta(2\tau) \) for \( \Im(\tau) > 0 \) and \( q = e^{2 \pi i \tau} \). Therefore, under suitable circumstances, an algebraic independence result for values of theta-constants can be transformed into an algebraic independence result for functions which are expressed in terms of theta-constants. For example, see [3] and Corollary1.1 below.

In this paper we focus on the problem to decide on the algebraic independence of \( \theta(q) \) and \( \theta(q^n) \) over \( \mathbb{Q} \) for algebraic numbers \( q \) and integers \( n > 1 \). We shall use Theorem A in connection with an algebraic independence criterion (Lemma 2.1) to settle the problem for the odd integers \( n = 3, 5, 7, 9, 11 \) and for three even numbers \( n = 6, n = 10, \) and \( n = 12 \). The central point of the algebraic independence criterion is the non-vanishing of a Jacobian determinant, which is hard to decide when the involved polynomials are not given explicitly. Using the double-argument formulae (3.1) for the theta-constants we construct suitable polynomials \( P_{2m}(X, Y) \)
(Lemma 3.1). In this case the polynomials $P_{2m}(X,Y)$ are given recursively such that we can solve the problem of the algebraic independence of $\theta(q)$ and $\theta(q^{2m})$ for arbitrary integers $m \geq 1$. But this method cannot be extended to decide on the algebraic independence of $\theta(q)$ and $\theta(q^n)$ for arbitrary odd integers $n$ by Theorem A. So the main results of this paper are given by the following theorem.

**Theorem 1.1.** Let $q$ be an algebraic number with $0 < |q| < 1$. Let $m \geq 1$ be an integer. Then, the two numbers $\theta(q)$ and $\theta(q^{2m})$ are algebraically independent over $\mathbb{Q}$ as well as the two numbers $\theta(q)$ and $\theta(q^n)$ for $n = 3, 5, 6, 7, 9, 10, 11, 12$.

Let $n \geq 3$ be any odd integer. If the polynomial $P_n(X,Y)$ from Theorem A is given explicitly, then by Theorem 4.1 in Section 4 one can decide on the algebraic independence of $\theta(q)$ and $\theta(q^n)$ over $\mathbb{Q}$ for any algebraic number $q$ satisfying the condition of Theorem 1.1.

The following identities are originally due to Ramanujan (cf. [1, §19, Entries 8 and 17]):

$$1 + S_1(q) = 1 + \sum_{j=1}^{\infty} (-1)^j \left(\frac{j}{5}\right) \frac{jq^j}{1-q^j} = \frac{1}{4}(5\theta(-q)\theta^3(-q^5) - \theta^3(-q)\theta(-q^5)),$$

$$24 + 40S_2(q) = 24 + 40 \sum_{j=1}^{\infty} \left(\frac{j}{5}\right) \frac{jq^j}{1+q^j} = 25\theta(q)\theta^3(q^5) - \theta^5(q)\theta(q^5),$$

$$1 + 2S_3(q) = 1 + 2 \sum_{j=1}^{\infty} \varepsilon_j \frac{q^j}{1-q^j} = \theta(q)\theta(q^7),$$

where $\left(\frac{j}{5}\right)$ denotes the Legendre symbol, and the cycle of coefficients $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{27})$ of length 28 is given by

$$(0,1,-1,-1,1,-1,1,0,1,1,1,1,-1,-1,0,1,1,-1,-1,-1,0,-1,1,-1,1,1,-1).$$

**Corollary 1.1.** Let $q$ be an algebraic number with $0 < |q| < 1$. Then the numbers $S_1(q), S_2(q),$ and $S_3(q)$ are transcendental.

From Entry 3 and Entry 4 in [1, §19] analogous results can be obtained for modular equations of degree 3.

2. Auxiliary results

A detailed discussion of theta-functions and theta-constants can be found in [4, part2, ch.2] and [9, ch.10]. At first we point out some properties of the functions

$$X_0(\tau) \in \left\{ n^2 \frac{\vartheta^4_3(n\tau)}{\vartheta^4_3(\tau)}, \frac{\vartheta^3_3(n\tau)}{\vartheta^3_3(\tau)} \right\} \quad \text{and} \quad Y_0(\tau) \in \left\{ 16 \frac{\vartheta^4_3(\tau)}{\vartheta^4_3(\tau)} \frac{\vartheta^4_3(\tau)}{\vartheta^4_3(\tau)}, \frac{\vartheta^3_3(\tau)}{\vartheta^3_3(\tau)} \right\},$$
From the theory of modular forms it is well known that in the upper half plane \( \mathbb{H} = \{ z \in \mathbb{C} | \Im(z) > 0 \} \) the theta-constants \( \vartheta_2(\tau), \vartheta_3(\tau) \) and \( \vartheta_4(\tau) \) are regular functions for \( \tau \in \mathbb{H} \). Moreover, \( \vartheta_3(\tau) \) does not vanish in \( \mathbb{H} \). Therefore, \( X_0(\tau) \) and \( Y_0(\tau) \) are regular functions in \( \mathbb{H} \).

The most important tool to transfer the algebraic independence of a set of \( m \) numbers to another set of \( m \) numbers, which all satisfy a system of algebraic identities, is given by the following lemma. We call it an algebraic independence criterion (AIC).

**Lemma 2.1 ([2, Lemma 3.1]).** Let \( x_1, \ldots, x_m \in \mathbb{C} \) be algebraically independent over \( \mathbb{Q} \) and let \( y_1, \ldots, y_m \in \mathbb{C} \) satisfy the system of equations

\[
 f_j(x_1, \ldots, x_m, y_1, \ldots, y_m) = 0 \quad (1 \leq j \leq m),
\]

where \( f_j(t_1, \ldots, t_m, u_1, \ldots, u_m) \in \mathbb{Q}[t_1, \ldots, t_m, u_1, \ldots, u_m] \) \((1 \leq j \leq m)\). Assume that

\[
 \det \left( \frac{\partial f_j}{\partial t_i}(x_1, \ldots, x_m, y_1, \ldots, y_m) \right) \neq 0.
\]

Then the numbers \( y_1, \ldots, y_m \) are algebraically independent over \( \mathbb{Q} \).

We shall apply the AIC to the sets \( \{x_1, x_2\} = \{\vartheta_2(\tau), \vartheta_3(\tau)\} \) and \( \{x_1, x_2\} = \{\vartheta_3(\tau), \vartheta_4(\tau)\} \). For this purpose we have to know that these pairs of numbers are algebraically independent.

**Lemma 2.2 ([3, Lemma 4]).** Let \( q \) be an algebraic number with \( q = e^{\pi i \tau} \) and \( \Im(\tau) > 0 \). Then, the numbers in each of the sets

\[
 \{\vartheta_2(\tau), \vartheta_3(\tau)\}, \quad \{\vartheta_2(\tau), \vartheta_4(\tau)\}, \quad \{\vartheta_3(\tau), \vartheta_4(\tau)\}
\]

are algebraically independent over \( \mathbb{Q} \).

This result can be derived from Yu.V.Nesterenko’s theorem [7] on the algebraic independence of the values \( P(q), Q(q), R(q) \) of the Ramanujan functions \( P, Q, R \) at a nonvanishing algebraic point \( q \).

### 3. Preparation of the proof of Theorem 1.1

In this section, we prove the following lemmas which are required to prove Theorem 1.1 when \( n \) is a power of two.

**Lemma 3.1.** For every integer \( m \geq 1 \) let \( n = 2^m \). There exists a polynomial \( P_n(X, Y) \in \mathbb{Z}[X, Y] \) such that

\[
 P_n\left( \frac{\vartheta_2^2(\tau)}{\vartheta_3^2(\tau)}, \vartheta_4(\tau) \right) = 0
\]

with \( \deg_X P_2(X, Y) = 1 \), and \( \deg_X P_n(X, Y) = 2^{m-2} \) for \( m \geq 2 \).
Proof. For simplicity we introduce the notation \( \vartheta_3 := \vartheta_3(\tau) \) and \( \vartheta_4 := \vartheta_4(\tau) \). Then
\[
\begin{align*}
2\vartheta_3^2(2\tau) &= \vartheta_3^2 - \vartheta_4^2, \\
2\vartheta_3^2(2\tau) &= \vartheta_3^2 + \vartheta_4^2, \\
\vartheta_4^2(2\tau) &= \vartheta_3 \vartheta_4.
\end{align*}
\] (3.1)

For every integer \( m \geq 1 \) let
\[
\begin{align*}
z_1 &= \vartheta_3^2(n\tau), \\
z_2 &= (\vartheta_3 + \vartheta_4)^2, \\
z_3 &= \vartheta_3 \vartheta_4.
\end{align*}
\]

Note that \( z_1 \) depends on \( n = 2^m \), while \( z_2 \) and \( z_3 \) do not depend on \( n \). First, we compute the polynomials \( P_n(X,Y) \) for \( n = 2, 4, 8 \).

\( n = 2 \): From (3.1) we have
\[
2\vartheta_3^2(2\tau) - (\vartheta_3 + \vartheta_4)^2 + 2\vartheta_3 \vartheta_4 = 2z_1 - z_2 + 2z_3 = 0. \] (3.2)

Dividing by \( \vartheta_3^2 \), it follows that
\[
2\left( \frac{\vartheta_3(2\tau)}{\vartheta_3} \right)^2 - \left( 1 + \frac{\vartheta_4}{\vartheta_3} \right)^2 + 2 \frac{\vartheta_4}{\vartheta_3} = 0.
\]

Hence, \( P_2(X,Y) = 2X - (1 + Y)^2 + 2Y \).

\( n = 4 \): In the second identity of (3.1) we replace \( \tau \) by \( 2\tau \) and then express \( \vartheta_3^2(2\tau) \) and \( \vartheta_4^2(2\tau) \) on the right-hand side again by (3.1) in terms of \( \vartheta_3 \) and \( \vartheta_4 \):
\[
4\vartheta_3^2(4\tau) - (\vartheta_3 + \vartheta_4)^2 = 4z_1 - z_2 = 0. \] (3.3)

Dividing by \( \vartheta_3^2 \), it follows that
\[
4\left( \frac{\vartheta_3(4\tau)}{\vartheta_3} \right)^2 - \left( 1 + \frac{\vartheta_4}{\vartheta_3} \right)^2 = 0.
\]

Hence, \( P_4(X,Y) = 4X - (1 + Y)^2 \).

\( n = 8 \): In (3.3) we replace \( \tau \) by \( 2\tau \). In order to express \( \vartheta_3(2\tau) \) and \( \vartheta_4(2\tau) \) in terms of \( \vartheta_3 \) and \( \vartheta_4 \), it becomes necessary to solve the identity for \( \vartheta_3 \vartheta_4 \) and square the equation. After some straightforward computations it turns out that
\[
0 = (8\vartheta_3^2(8\tau) - (\vartheta_3 + \vartheta_4)^2)^2 - 8((\vartheta_3 + \vartheta_4)^2 - 2\vartheta_3 \vartheta_4)\vartheta_3 \vartheta_4
\]
\[
= (8z_1 - z_2)^2 - 8(z_2 - 2z_3)z_3. \] (3.4)

Dividing by \( \vartheta_3^4 \), we find that
\[
P_8(X,Y) = (8X - (1 + Y)^2)^2 - 8((1 + Y)^2 - 2Y)Y.
\]
The polynomials in terms of $z_1, z_2, z_3$ in (3.2 - 3.4) are homogeneous of degrees 1, 1, and 2 respectively. Therefore, we try to prove the following statement by induction with respect to $m$:

For every $m \geq 1$ there is a homogeneous polynomial $T_n(t_1, t_2, t_3) \in \mathbb{Z}[t_1, t_2, t_3]$ of total degree $\lambda$ such that $T_n(z_1, z_2, z_3) = 0$ with $\lambda = \deg_t T_n(t_1, t_2, t_3) = 2^{m-2}$ for $m \geq 2$ and $\lambda = 1$ when $m = 1$.

We have already shown the existence of $T_2, T_4,$ and $T_8$. For $T_8$ we have $\lambda = 2$ by (3.4). So, let us assume that for some $m \geq 3$ such a homogeneous polynomial $T_{2m}$ with $\lambda = 2^{m-2}$ do exist. Then,

$$T_{2m} \left( \vartheta_3^2(2^m \tau), (\vartheta_3 + \vartheta_4)^2, \vartheta_3 \vartheta_4 \right) = 0, \quad (3.5)$$

where

$$T_{2m}(t_1, t_2, t_3) = \sum_\nu a_\nu t_1^{\nu_1} t_2^{\nu_2} t_3^{\nu_3}, \quad (3.6)$$

say, with $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{N}^3$, $a_\nu \in \mathbb{Z}$, and $\nu_1 + \nu_2 + \nu_3 = \lambda = \deg_t T_{2m}(t_1, t_2, t_3)$. Here, $\mathbb{N}$ denotes the set of nonnegative integers. The leading term with respect to $t_1$ occurs once only for $\nu = (\lambda, 0, 0)$. Next, in (3.5) we replace $\tau$ by $2\tau$:

$$T_{2m} \left( \vartheta_3^2(2^{m+1} \tau), (\vartheta_3(2\tau) + \vartheta_4(2\tau))^2, \vartheta_3(2\tau) \vartheta_4(2\tau) \right) = 0. \quad (3.7)$$

Setting $w := \vartheta_3(2\tau) \vartheta_4(2\tau)$, we have, using (3.1),

$$(\vartheta_3(2\tau) + \vartheta_4(2\tau))^2 = \frac{z_2}{2} + 2w.$$  

For $m + 1$ we set $z_1 := \vartheta_3^2(2^{m+1} \tau) = \vartheta_3^2((n + 1)\tau)$. Then, (3.7) and (3.6) can be expressed in terms of $z_1, z_2,$ and $w$:

$$0 = T_{2m} \left( z_1, \frac{z_2}{2} + 2w, w \right) = \sum_\nu a_\nu z_1^{\nu_1} \left( \frac{z_2}{2} + 2w \right)^{\nu_2} w^{\nu_3} = \sum_\mu b_\mu z_1^{\mu_1} z_2^{\mu_2} w^{\mu_3}$$

with $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{N}^3$, $b_\mu \in \mathbb{Q}$, and $\mu_1 + \mu_2 + \mu_3 = \lambda$. We separate the sum on $\mu = (\mu_1, \mu_2, \mu_3)$ into two parts according to the parity of $\mu_3$:

$$0 = \sum_{\mu = (\mu_1, \mu_2, \mu_3)} b_\mu z_1^{\mu_1} z_2^{\mu_2} w^{\mu_3} + \sum_{\mu = (\mu_1, \mu_2, \mu_3)} b_\mu z_1^{\mu_1} z_2^{\mu_2} w^{\mu_3}, \quad \mu_3 \equiv 0 \ (\text{mod} \ 2) \quad \mu_3 \equiv 1 \ (\text{mod} \ 2)$$
where the leading term with respect to \( z_1 \) is \( b_{(\lambda,0,0)} z_1^{2\lambda} \neq 0 \) occurring in the left-hand sum. It follows that

\[
0 = \left( \sum_{\mu = (\mu_1, \mu_2, \mu_3) \atop \mu_3 \equiv 0 \pmod{2}} b_{\mu} z_1^{\mu_1} z_2^{\mu_2} w^{\mu_3} \right)^2 - w^2 \left( \sum_{\mu = (\mu_1, \mu_2, \mu_3) \atop \mu_3 \equiv 1 \pmod{2}} b_{\mu} z_1^{\mu_1} z_2^{\mu_2} w^{\mu_3 - 1} \right)^2 = 0. \tag{3.8}
\]

Using (3.1) we express \( w^2 \) in terms of \( z_2 \) and \( z_3 \):

\[
w^2 = \frac{1}{2} (z_2 - 2z_3) z_3.
\]

Substituting this expression into (3.8), we obtain

\[
0 = \left( \sum_{\mu = (\mu_1, \mu_2, \mu_3) \atop \mu_3 \equiv 0 \pmod{2}} b_{\mu} z_1^{\mu_1} z_2^{\mu_2} 2^{-\mu_3/2} (z_2 - 2z_3)^{\mu_3/2} z_3^{\mu_3/2} \right)^2
- \frac{1}{2} (z_2 - 2z_3) z_3 \left( \sum_{\mu = (\mu_1, \mu_2, \mu_3) \atop \mu_3 \equiv 1 \pmod{2}} b_{\mu} z_1^{\mu_1} z_2^{\mu_2} 2^{-(\mu_3-1)/2} (z_2 - 2z_3)^{(\mu_3-1)/2} z_3^{(\mu_3-1)/2} \right)^2
= \sum_{\kappa} c_{\kappa} z_1^{\kappa_1} z_2^{\kappa_2} z_3^{\kappa_3},
\]

where \( \kappa = (\kappa_1, \kappa_2, \kappa_3) \in \mathbb{N}^3 \), \( c_{\kappa} \in \mathbb{Q} \), and \( \kappa_1 + \kappa_2 + \kappa_3 = 2\lambda \). The leading term with respect to \( z_1 \) is \( c_{(2\lambda,0,0)} z_1^{2\lambda} \neq 0 \). The homogeneous polynomial \( T_{2m+1} \in \mathbb{Z}[t_1, t_2, t_3] \setminus \{0\} \) can be chosen by

\[
T_{2m+1}(t_1, t_2, t_3) := 2^{2\lambda} \sum_{\kappa} c_{\kappa} t_1^{\kappa_1} t_2^{\kappa_2} t_3^{\kappa_3}.
\]

For this polynomial we have \( 2\lambda = 2^{m-1} \). This completes the proof of the existence of the homogeneous polynomials \( T_n(t_1, t_2, t_3) \) for every integer \( m \geq 1 \) with \( n = 2^m \) satisfying \( T_n(z_1, z_2, z_3) = 0 \). Let us consider a monomial of such a homogeneous polynomial \( T_n \) of degree \( \lambda \) given by (3.6). Then we have \( \nu_1 + \nu_2 + \nu_3 = \lambda \). After dividing \( T_n \) by \( \vartheta_3^\lambda \), the monomial takes the form

\[
\frac{a_\nu}{\vartheta_3^\lambda} \cdot z_1^{\nu_1} z_2^{\nu_2} z_3^{\nu_3} = \frac{a_\nu}{\vartheta_3^\lambda} \cdot \left( \vartheta_3(2^m \tau) \right)^{2\nu_1} \left( \vartheta_3 + \vartheta_4 \right)^{2\nu_2} \left( \vartheta_3 \vartheta_4 \right)^{\nu_3}
= a_\nu \left( \frac{\vartheta_3(2^m \tau)}{\vartheta_3} \right)^{2\nu_1} \left( 1 + \frac{\vartheta_4}{\vartheta_3} \right)^{2\nu_2} \left( \frac{\vartheta_4}{\vartheta_3} \right)^{\nu_3}
= a_\nu X^{\nu_1} (1 + Y)^{2\nu_2} Y^{\nu_3}
\]

with

\[
X := \frac{\vartheta_3(2^m \tau)}{\vartheta_3^2} \quad \text{and} \quad Y := \frac{\vartheta_4}{\vartheta_3}.
\]

Introducing the polynomial

\[
P_n(X, Y) := \sum_{\nu} a_\nu X^{\nu_1} (1 + Y)^{2\nu_2} Y^{\nu_3} = T_n(X, (1 + Y)^2, Y),
\]

we finish the proof of Lemma 3.1. \( \blacksquare \)
The polynomials $P_2, P_4, P_8, P_{16}$, and $P_{32}$ are listed in the appendix. The proof of the algebraic independence of $\vartheta_3(q^{2^m})$ and $\vartheta_3(q)$ over $\mathbb{Q}$ will require some more information on the polynomials $T_n(t_1, t_2, t_3)$ introduced in the proof of the preceding lemma.

**Lemma 3.2.** For every integer $m \geq 3$ let $n = 2^m$. Then there is a polynomial $U_n(t_1, t_2, t_3) \in \mathbb{Q}[t_1, t_2, t_3]$ such that the polynomial $T_n(t_1, t_2, t_3)$ from (3.5) can be written as

$$T_n(t_1, t_2, t_3) = (nt_1 - t_2)^{2m-2} + t_3 U_n(t_1, t_2, t_3)$$

with

$$U_n\left(\frac{1}{n}, 1, 0\right) = -2^{2m-1-1}. \quad (3.10)$$

**Proof.** Lemma 3.2 is true for $m = 3$ and $m = 4$. We have

$$T_8(t_1, t_2, t_3) = (8t_1 - t_2)^2 - 8(t_2 - 2t_3)t_3,$$

$$U_8(t_1, t_2, t_3) = -8(t_2 - 2t_3);$$

$$T_{16}(t_1, t_2, t_3) = (16t_1 - t_2)^4 + t_3 \left(16(t_2 - 2t_3)(16t_1 - t_2)^2 + 64(t_2 - 2t_3)^2t_3 - 128(t_2 - 2t_3)(8t_1 + \frac{t_2}{2})^2\right), \quad (3.11)$$

$$U_{16}(t_1, t_2, t_3) = 16(t_2 - 2t_3)(16t_1 - t_2)^2 + 64(t_2 - 2t_3)^2t_3 - 128(t_2 - 2t_3)(8t_1 + \frac{t_2}{2}).$$

For $m \geq 4$ we prove a more precise result on the particular shape of the polynomials $T_{2^m}$. We shall show the following. For every integer $m \geq 4$ we have

$$T_{2^m}(t_1, t_2, t_3) = (2^m t_1 - t_2)^{2m-2}
+ t_3 \sum_{\nu \in \mathbb{N}^5 \atop \nu \geq 2 \wedge \nu \geq 1} a_\nu (2^m t_1 - t_2)^{\nu_1} (t_2 - 2t_3)^{\nu_2} t_1^{\nu_3} t_2^{\nu_4} t_3^{\nu_5}
- 2^{2m-1-1} t_2^{2m-3-2} (t_2 - 2t_3)(2^{m-1} t_1 + \frac{t_2}{2})^{2m-3} t_3. \quad (3.12)$$

Here, $\nu = (\nu_1, \ldots, \nu_5) \in \mathbb{N}^5$, and the numbers $a_\nu$ are rationals. Only finitely many $a_\nu$ do not vanish. One can show that $T_{2^m}(t_1, t_2, t_3)$ is a polynomial with integer coefficients, but we do not need this fact. We point out that the conditions on the summation variables $\nu_1, \ldots, \nu_5$ read as follows: it is either $\nu_1 \geq 2$ or $\nu_5 \geq 1$ (or both), and we always have $\nu_2 \geq 1$. The second and third term on the right-hand side of (3.12) form $t_3 U_{2^m}(t_1, t_2, t_3)$, which implies (3.9). In particular, for $t_1 = 1/2^m$, $t_2 = 1$, and $t_3 = 0$, we have

$$2^m t_1 - t_2 = 0, \quad t_2 - 2t_3 = 1, \quad 2^{m-1} t_1 + \frac{t_2}{2} = 1,$$
such that

$$U_{2m} \left( \frac{1}{2^m}, 1, 0 \right) = -2^{2m-1}$$

proves (3.10) in Lemma 3.2.

**Proof of (3.12).** We proceed by induction on $m$. For $m = 4$ see (3.11). Next let us assume that (3.12) holds for some integer $m \geq 4$. Following the lines in the proof of Lemma 3.1, we construct step by step the new polynomial $T_{2^{m+1}}(t_1, t_2, t_3)$ from (3.12).

**Step 1:** After substituting the new expressions

$$t_1 \rightarrow t_1, \quad t_2 \rightarrow \frac{t_2}{2} + 2w, \quad t_3 \rightarrow w$$

into (3.12), we see that the resulting term equals to zero. Hence,

$$0 = \left( 2^m t_1 - \frac{t_2}{2} - 2w \right)^{2m-2} + w \sum_{\nu_1, \ldots, \nu_5} a_{\nu} \left( \begin{array}{c} 2^m t_1 - \frac{t_2}{2} - 2w \\ \nu_1 \\ \nu_2 \end{array} \right)^{\nu_1} \left( \frac{t_2}{2} \right)^{\nu_2} t_1^{\nu_3} \left( \frac{t_2}{2} + 2w \right)^{\nu_4} w^{\nu_5}$$

$$- 2^{2m-1-1} \left( \frac{t_2}{2} + 2w \right)^{2m-3-2} \frac{t_2}{2} \left( 2^{m-1} t_1 + \frac{t_2}{4} + w \right)^{2m-3} w.$$

Using four times the binomial theorem, the above expression becomes

$$0 = \left( 2^m t_1 - \frac{t_2}{2} - 2w \right)^{2m-2} + \sum_{\mu_1 = 1}^{2m-2} \left( \begin{array}{c} 2^m t_1 - \frac{t_2}{2} \\ \mu_1 \end{array} \right)^{(2m-2-\mu_1)} \left( \frac{t_2}{2} \right)^{(2m-2-\mu_1)} (2w)^{\mu_1}$$

$$+ w \sum_{\nu_1, \ldots, \nu_5} a_{\nu} \left( \sum_{\mu_2 = 0}^{\nu_1} \left( \begin{array}{c} \nu_1 \\ \mu_2 \end{array} \right)^{-1} \mu_2 \left( \begin{array}{c} 2^m t_1 - \frac{t_2}{2} \end{array} \right)^{\nu_1-\mu_2} (2w)^{\mu_2} \right) \left( \frac{t_2}{2} \right)^{\nu_2} t_1^{\nu_3}$$

$$\times \left( \sum_{\mu_3 = 0}^{\nu_4} \left( \begin{array}{c} \nu_4 \\ \mu_3 \end{array} \right)^{-\mu_3} (2w)^{\mu_3} \right) w^{\nu_5}$$

$$- 2^{2m-1-1} \left( \frac{t_2}{2} \right)^{2m-3-2} \frac{t_2}{2} \left( 2^{m-1} t_1 + \frac{t_2}{4} + w \right)^{2m-3} w$$

$$- 2^{2m-1-1} \left( \sum_{\mu_4 = 1}^{2m-3-2} \left( 2^{m-3-2} \frac{t_2}{2} \right)^{2m-3-2-\mu_4} (2w)^{\mu_4} \right) \frac{t_2}{2}$$

$$\times \left( 2^{m-1} t_1 + \frac{t_2}{4} + w \right)^{2m-3} w.$$

(3.13)
The last but one term on the right-hand side of (3.13) can be expanded by

\[
2^{2m-1-1} \left( \frac{t_2}{2} \right)^{2m-2} \frac{t_2}{2} \cdot \frac{1}{2^{2m-3}} \left( 2^m t_1 + \frac{t_2}{2} + 2w \right)^{2m-3} w
\]
\[
= 2^{2m-2} 2^{m-3} 2^{-m-3-1} 1 \cdot t_2^{m-3-1} \left( 2^m t_1 + \frac{t_2}{2} + 2w \right)^{2m-3} w
\]
\[
= 2^{2m-2} t_2^{m-3-1} \left( 2^m t_1 + \frac{t_2}{2} \right)^{2m-3} w
\]
\[
+ 2^{2m-2} t_2^{m-3-1} \left( \sum_{\mu_5=1}^{2m-3} \left( \frac{2^{m-3}}{\mu_5} \right) \left( 2^m t_1 + \frac{t_2}{2} \right)^{2m-3-\mu_5} (2w)^{\mu_5} \right) w.
\]

Substituting (3.14) for the last but one term into (3.13), we summarize the terms as follows.

\[
0 = \left( 2^m t_1 - \frac{t_2}{2} \right)^{2m-2} + \sum_{\substack{\mu_6, \ldots, \mu_9, \\
\mu_6, \mu_9 \geq 2, \\
\mu_9 \equiv 1 \pmod{2} \mu_9 \geq 1 \mu_9 \leq 2 \mu_9 \geq 1}} b_\mu (2^m + 1 t_1 - t_2)^{\mu_6} t_1^{\mu_7} t_2^{\mu_8} w^{\mu_9}
\]
\[
= \left( 2^m t_1 - \frac{t_2}{2} \right)^{2m-2} + \sum_{\mu_6, \ldots, \mu_9, \\
\mu_6, \mu_9 \geq 2, \\
\mu_9 \equiv 1 \pmod{2} \mu_9 \geq 1 \mu_9 \leq 2 \mu_9 \geq 1}} b_\mu (2^m + 1 t_1 - t_2)^{\mu_6} t_1^{\mu_7} t_2^{\mu_8} w^{\mu_9}
\]
\[
+ 2^{2m-2} t_2^{m-3-1} \left( 2^m t_1 + \frac{t_2}{2} \right)^{2m-3} w.
\]

Here, we abbreviate by \( \mu = (\mu_6, \ldots, \mu_9) \), and the coefficients \( b_\mu \) are again rational numbers.

**Step 2:** In (3.15) we separate the terms with an even power of \( w \) from those with an odd power of \( w \). This gives

\[
\left( 2^m t_1 - \frac{t_2}{2} \right)^{2m-2} + \sum_{\mu_6, \ldots, \mu_9, \\
\mu_6, \mu_9 \geq 2, \\
\mu_9 \equiv 0 \pmod{2} \mu_9 \geq 1}} b_\mu (2^m + 1 t_1 - t_2)^{\mu_6} t_1^{\mu_7} t_2^{\mu_8} w^{\mu_9}
\]
\[
= - \sum_{\mu_6, \ldots, \mu_9, \\
\mu_6, \mu_9 \geq 2, \\
\mu_9 \equiv 1 \pmod{2} \mu_9 \geq 1}} b_\mu (2^m + 1 t_1 - t_2)^{\mu_6} t_1^{\mu_7} t_2^{\mu_8} w^{\mu_9}
\]
\[
- 2^{2m-2} t_2^{m-3-1} \left( 2^m t_1 + \frac{t_2}{2} \right)^{2m-3} w.
\]
Step 3: Squaring (3.16), we obtain

\[
\left(2^m t_1 - \frac{t_2}{2}\right)^{2^{m-1}} + \left( \sum_{\mu_6, \ldots, \mu_9} b_\mu (2^{m+1} t_1 - t_2)^{\mu_6} t_1^{\mu_7} t_2^{\mu_8} w^{\mu_9} \right)^2
\]

\[
+ 2 \left(2^m t_1 - \frac{t_2}{2}\right)^{2^{m-2}} \sum_{\mu_6, \ldots, \mu_9} b_\mu (2^{m+1} t_1 - t_2)^{\mu_6} t_1^{\mu_7} t_2^{\mu_8} w^{\mu_9}
\]

\[
= \left( \sum_{\mu_6, \ldots, \mu_9} b_\mu (2^{m+1} t_1 - t_2)^{\mu_6} t_1^{\mu_7} t_2^{\mu_8} w^{\mu_9} \right)^2
\]

\[
+ 2^{2^{m-2}+1} t_2^{2^{m-3}-1} \left(2^m t_1 + \frac{t_2}{2}\right)^{2^{m-3}} \sum_{\mu_6, \ldots, \mu_9} b_\mu (2^{m+1} t_1 - t_2)^{\mu_6} t_1^{\mu_7} t_2^{\mu_8} w^{1+\mu_9}
\]

\[
+ 2^{2^{m-1}} t_2^{2^{m-2}-2} \left(2^m t_1 + \frac{t_2}{2}\right)^{2^{m-2}} w^2.
\]

This identity can be summarized as follows.

\[
0 = \left(2^m t_1 - \frac{t_2}{2}\right)^{2^{m-1}} + \sum_{\nu_6, \ldots, \nu_9} c_\nu \left(2^m t_1 - \frac{t_2}{2}\right)^{\nu_6} w^{2^\nu_7 t_1^{\nu_8} t_2^{\nu_9}}
\]

\[
- 2^{2^{m-1}} t_2^{2^{m-2}-2} \left(2^m t_1 + \frac{t_2}{2}\right)^{2^{m-2}} w^2,
\]

where \(\nu = (\nu_6, \ldots, \nu_9)\) and \(c_\nu \in \mathbb{Q}\).

Step 4: Multiplying (3.17) by \(2^{2^{m-1}}\) and replacing \(w^2\) by \(\frac{1}{3}(t_2 - 2t_3) t_3\), the right-hand side of (3.17) becomes the polynomial \(T_{2^{m+1}}(t_1, t_2, t_3)\). Thus we obtain
\[ T_{2^{m+1}}(t_1, t_2, t_3) = \left(2^{m+1}t_1 - t_2\right)^{2^{m-1}} + \sum_{\nu \in \mathbb{Z}} 2^{m-1} c_{\nu} \left(2^m t_1 - \frac{t_2}{2}\right)^{\nu} \frac{1}{2^{\nu^2}} (t_2 - 2t_3)^{\nu^2} t_3^{\nu^2} t_1^{2^m} \]

\[ - 2^{m-1} + 2^{m-1} t_2^{m-2} - 2^{m-2} \left(2^m t_1 + \frac{t_2}{2}\right)^{2^{m-2}} \frac{1}{2} t_2^{2^{m-1}} t_3 \]

where \( \nu = (\nu_6, \ldots, \nu_{10}) \), and \( d_{\nu} \in \mathbb{Q} \). Hence, (3.18) corresponds to (3.12) with \( m \) replaced by \( m + 1 \). This proves (3.12). \( \blacksquare \)

4. Proof of Theorem 1.1

By \( \text{Res}_t(f(t), g(t)) \) we denote the resultant of two polynomials \( f(t), g(t) \) with respect to the variable \( t \). It is consistent with the notation of theta-constants to write \( \theta_3(q) \) and \( \theta_3(\tau) \) instead of \( \theta(q) \) and \( \theta(\tau) \), respectively.

We divide the proof of Theorem 1.1 into several steps. The first step is an interim result given by the following theorem.

Theorem 4.1. Let \( n \) be either an odd integer \( \geq 3 \) or \( n = 2^m \) with \( m \geq 1 \). Let \( q \) be an algebraic number with \( q = e^{\pi i \tau} \) and \( \Im(\tau) > 0 \). If the polynomial

\[ \text{Res}_X \left( P_n(X, Y), \frac{\partial}{\partial Y} P_n(X, Y) \right) \]

does not vanish identically, then the numbers \( \theta_3(n\tau) \) and \( \theta_3(\tau) \) are algebraically independent over \( \mathbb{Q} \).

Proof. For any given odd integer \( n \geq 3 \) let

\[ X_0 := n^2 \frac{\partial_3^3(n\tau)}{\partial_3^4(\tau)}, \quad Y_0 := 16 \frac{\partial_3^4(\tau)}{\partial_3^3(\tau)}; \]

\[ x_1 := \theta_2(\tau), \quad x_2 := \theta_3(\tau), \]

\[ y_1 := \theta_3(n\tau), \quad y_2 := x_2 = \theta_3(\tau). \]

We know by Theorem A that \( P_n(X_0, Y_0) = 0 \), and by Lemma 2.2 and the conditions of Theorem 4.1 that \( x_1 \) and \( x_2 \) are algebraically independent over \( \mathbb{Q} \). Let

\[ P_n(X, Y) = \sum_{\nu=0}^{N} \sum_{\mu=0}^{M} a_{\nu, \mu} X^{\nu} Y^{\mu}, \quad (4.1) \]
where \(a_{\nu,\mu}\) are the integer coefficients of the polynomial \(P_n\). Consider the polynomials

\[
f_1 := f_1(t_1, t_2, u_1, u_2) := t_2^{4M} u_2^{4N} P_n \left( \frac{n^2 u_1^4}{u_2^4}, \frac{16t_1^4}{t_2^4} \right)
= \sum_{\nu=0}^{N} \sum_{\mu=0}^{M} a_{\nu,\mu} t_2^{4M} u_2^{4N} \left( \frac{n^2 u_1^4}{u_2^4} \right)^{\nu} \left( \frac{16t_1^4}{t_2^4} \right)^{\mu}
= \sum_{\nu=0}^{N} \sum_{\mu=0}^{M} 16^\nu n^{2\nu} a_{\nu,\mu} t_2^{4M} u_2^{4N} \left( \frac{n^2 u_1^4}{u_2^4} \right)^{\nu} \left( \frac{16t_1^4}{t_2^4} \right)^{\mu}.
\]

\[
f_2 := f_2(t_1, t_2, u_1, u_2) := u_2 - t_2.
\]

Note that \(f_j(x_1, x_2, y_1, y_2) = 0\) for \(j = 1, 2\). To prove the algebraic independence of \(y_1\) and \(y_2\) using the AIC (Lemma 2.1) we have to show that the determinant

\[
\Delta := \det \begin{pmatrix}
\frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2} \\
\frac{\partial f_2}{\partial t_1} & \frac{\partial f_2}{\partial t_2}
\end{pmatrix}
\]
does not vanish at \((x_1, x_2, y_1, y_2)\).

Since

\[
\frac{\partial f_2}{\partial t_1} = 0 \quad \text{and} \quad \frac{\partial f_2}{\partial t_2} = -1,
\]

the condition \(\Delta \neq 0\) is equivalent with the nonvanishing of the number

\[
\frac{\partial f_1}{\partial t_1} (x_1, x_2, y_1, y_2) := \frac{\partial f_1(t_1, t_2, u_1, u_2)}{\partial t_1} \bigg|_{(t_1=x_1, t_2=x_2, u_1=y_1, u_2=y_2)}.
\]

We have

\[
\frac{\partial f_1}{\partial t_1} (x_1, x_2, y_1, y_2) = \sum_{\nu=0}^{N} \sum_{\mu=0}^{M} 16^\nu n^{2\nu} a_{\nu,\mu} x_1^{4\mu-1} x_2^{4(M-\mu)} y_1^{4\nu} y_2^{4(N-\nu)}
= x_2^{4M} y_2^{4N} \sum_{\nu=0}^{N} \sum_{\mu=1}^{M} a_{\nu,\mu} \left( \frac{n^2 y_1^4}{y_2^4} \right)^{\nu} \left( \frac{16x_1^4}{x_2^4} \right)^{\mu-1} \left( \frac{64 x_1^3}{x_2^3} \right)
= 64 x_1^3 x_2^{4M-4} y_2^{4N} \frac{\partial P_n}{\partial Y} \left( \frac{n^2 y_1^4}{y_2^4}, \frac{16x_1^4}{x_2^4} \right).
\]

Since both, \(x_1\) and \(x_2\) do not vanish, it is clear that

\[
\Delta \neq 0 \iff \frac{\partial f_1}{\partial t_1} (x_1, x_2, y_1, y_2) \neq 0 \iff \frac{\partial P_n}{\partial Y} \left( \frac{n^2 y_1^4}{y_2^4}, \frac{16x_1^4}{x_2^4} \right) \neq 0. \quad (4.2)
\]
By the hypothesis of Theorem 4.1 the polynomial

\[ R = \text{Res}_X \left( P_n(X, Y), \frac{\partial}{\partial Y} P_n(X, Y) \right) \in \mathbb{Z}[Y] \]

does not vanish identically. For \( Y = Y_0 = 16x_1^4/x_2^4 \) we have \( R \in \mathbb{Q}(x_1, x_2) \), so that the algebraic independence of \( x_1, x_2 \) proves \( R \neq 0 \). In particular, \( P_n(X, Y) \) and \( \frac{\partial}{\partial Y} P_n(X, Y) \) as polynomials in \( X \) have no common root for fixed \( Y = Y_0 = 16x_1^4/x_2^4. \) Since \( P_n(X, Y) \) vanishes at \((X_0, Y_0) = (n^2y_1^4/y_2^4, 16x_1^4/x_2^4)\), it follows that

\[ \frac{\partial P_n}{\partial Y} \left( \frac{n^2y_1^4}{y_2^4}, \frac{16x_1^4}{x_2^4} \right) \neq 0. \]

Thus, Theorem 4.1 for odd integers \( n \geq 3 \) follows from (4.2) and the AIC (Lemma 2.1).

In the case \( n = 2^m \) (\( m \geq 1 \)) we introduce the quantities

\[ X_0 := \frac{\vartheta_3^2(n\tau)}{\vartheta_3^2(\tau)}, \quad Y_0 := \frac{\vartheta_3(\tau)}{\vartheta_3(\tau)}; \]

\[ x_1 := \vartheta_3(\tau), \quad x_2 := \vartheta_3(\tau), \]

\[ y_1 := \vartheta_3(n\tau), \quad y_2 := x_2 = \vartheta_3(\tau). \]

Here, we have \( P_n(X_0, Y_0) = 0 \) by Lemma 3.1, and

\[ f_1(t_1, t_2, u_1, u_2) := t_2^M u_2^{2N} P_n \left( \frac{u_1^2}{u_2^2}, \frac{t_1}{t_2} \right) = \sum_{\nu=0}^N \sum_{\mu=0}^M a_{\nu, \mu} t_1^{M-\mu} u_1^{2\nu} u_2^{2(N-\nu)}, \]

\[ \frac{\partial f_1}{\partial t_1}(x_1, x_2, y_1, y_2) = x_2^{M-1} y_2^{2N} \frac{\partial P_n}{\partial Y} \left( \frac{y_1^2}{y_2^2}, \frac{x_1}{x_2} \right), \]

\[ f_2(t_1, t_2, u_1, u_2) := u_2 - t_2. \]

Then,

\[ \Delta \neq 0 \iff \frac{\partial P_n}{\partial Y} \left( \frac{y_1^2}{y_2^2}, \frac{x_1}{x_2} \right) \neq 0. \]

Using similar arguments as above by considering the particular point \((X_0, Y_0) = (y_1^2/y_2^2, x_1/x_2)\), the algebraic independence of \( \vartheta_3(q^n) \) and \( \vartheta_3(q) \) for \( n = 2^m \) can be derived from the AIC.

First, using Theorem 4.1 we prove the algebraic independence of \( \vartheta_3(q^n) \) and \( \vartheta_3(q^m) \) for \( n = 2, 3, 4, 5, 7, 8, 9, 11 \) by computing the resultant of the polynomials \( P_n(X, Y) \) and \( \frac{\partial P_n(X, Y)}{\partial Y} \). We have to show that these resultants do not vanish identically. So, it suffices to compute the values of the resultants at the point \( Y = 0 \) for \( n = 3, 4, 5, 7, 8, 11 \) and at the point \( Y = 2 \) for \( n = 2, 9 \). Note that
Res\(_X(P_n(X, Y), \frac{\partial P_n(X, Y)}{\partial Y})\) vanishes at \(Y = 0\) for \(n = 2, 9\).

\[
\begin{align*}
\text{Res}_X(P_2(X, 2), \frac{\partial P_2}{\partial Y}(X, 2)) &= -2^2, \\
\text{Res}_X(P_3(X, 0), \frac{\partial P_3}{\partial Y}(X, 0)) &= 2^{16} \cdot 3^2, \\
\text{Res}_X(P_4(X, 0), \frac{\partial P_4}{\partial Y}(X, 0)) &= -2, \\
\text{Res}_X(P_5(X, 0), \frac{\partial P_5}{\partial Y}(X, 0)) &= 2^{60} \cdot 3^{10} \cdot 5^2, \\
\text{Res}_X(P_7(X, 0), \frac{\partial P_7}{\partial Y}(X, 0)) &= 2^{142} \cdot 3^{14} \cdot 7^2, \\
\text{Res}_X(P_8(X, 0), \frac{\partial P_8}{\partial Y}(X, 0)) &= 2^{12}, \\
\text{Res}_X(P_9(X, 0), \frac{\partial P_9}{\partial Y}(X, 0)) &= -2, \\
\text{Res}_X(P_{11}(X, 0), \frac{\partial P_{11}}{\partial Y}(X, 0)) &= 2^{236} \cdot 3^{22} \cdot 5^{22} \cdot 11^2. \\
\end{align*}
\]

Next we prove the algebraic independence of the numbers in each of the sets

\[
\{ \vartheta_3(6\tau), \vartheta_3(\tau) \} \quad \text{and} \quad \{ \vartheta_3(10\tau), \vartheta_3(\tau) \}.
\]

We shall not treat these two problems by the method shown before, but again the AIC will play an important role. We first consider the numbers \(\vartheta_3(6\tau)\) and \(\vartheta_3(\tau)\). Given any odd integer \(n \geq 3\) one can deduce the algebraic independence of \(\vartheta_3(2n\tau)\) and \(\vartheta_3(\tau)\) as follows. First we replace \(\tau\) by \(2\tau\) in Theorem A. Then,

\[
P_n(X, Y) = 0
\]

holds for

\[
X_0 := n^2 \frac{\vartheta_3^4(2n\tau)}{\vartheta_3^2(2\tau)} \quad \text{and} \quad Y_0 := 16 \frac{\vartheta_3^4(2\tau)}{\vartheta_3^2(2\tau)}.
\]

Next we express \(\vartheta_3^4(2\tau)\) and \(\vartheta_3^4(2\tau)\) in terms of \(\vartheta_3(\tau)\) and \(\vartheta_4(\tau)\):

\[
\begin{align*}
\vartheta_3^2(2\tau) &= \frac{1}{4} (\vartheta_3^2(\tau) - \vartheta_4^2(\tau))^2, \\
\vartheta_3^4(2\tau) &= \frac{1}{4} (\vartheta_3^2(\tau) + \vartheta_4^2(\tau))^2.
\end{align*}
\]

Hence (4.3) holds for

\[
X_0 = \frac{4n^2 \vartheta_3^4(2n\tau)}{(\vartheta_3^2(\tau) + \vartheta_4^2(\tau))^2} \quad \text{and} \quad Y_0 = \frac{16(\vartheta_3^2(\tau) - \vartheta_4^2(\tau))^2}{(\vartheta_3^2(\tau) + \vartheta_4^2(\tau))^2}.
\]
Setting
\[ x_1 := \partial_3(\tau), \quad x_2 := \partial_4(\tau), \]
\[ y_1 := \partial_3(2n\tau), \quad y_2 := x_1 = \partial_3(\tau), \]
we know that (4.3) holds for
\[ X_0 = \frac{4n^2y_1^4}{(y_2^2 + x_2^2)^2} \quad \text{and} \quad Y_0 = \frac{16(x_1^2 - x_2^2)^2}{(x_1^2 + x_2^2)^2}. \quad (4.4) \]
Beside (4.3) we have the identity \( y_2 - x_1 = 0 \), and the numbers \( x_1, x_2 \) are known to be algebraically independent over \( \mathbb{Q} \) for any algebraic number \( q = e^{\pi i \tau} \) with \( \Im(\tau) > 0 \) by Lemma 2.2. Using
\[ P_n(X, Y) = \sum_{\nu=1}^{N} \sum_{\mu=1}^{M} a_{\nu, \mu} X^{\nu} Y^{\mu}, \]
we now introduce the polynomials
\[ f_1(t_1, t_2, u_1, u_2) := (t_2 + u_2^2)^{2N}(t_1^2 + t_2^2)^{2M} P_n \left( \frac{4n^2u_1^4}{(t_1^2 + u_2^2)^2}, \frac{16(t_1^2 - t_2^2)^2}{(t_1^2 + t_2^2)^2} \right), \quad (4.5) \]
\[ f_2(t_1, t_2, u_1, u_2) := u_2 - t_1. \]
Using the AIC we have to show the nonvanishing of
\[ \Delta := \det \begin{pmatrix} \frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2} \\ \frac{\partial f_2}{\partial t_1} & \frac{\partial f_2}{\partial t_2} \end{pmatrix} = \frac{\partial f_1}{\partial t_2} \]
at \( (x_1, x_2, y_1, y_2) \). From now on we restrict the investigation to particular cases. First, let \( n = 3 \). For \( P_3(X, Y) \) we have \( N = 4 \) and \( M = 2 \) (cf. Appendix). We now compute \( \Delta = \frac{\partial f_1}{\partial t_2}(x_1, x_2, y_1, y_2) \), where \( f_1 \) is as in (4.5). Setting \( y_2 = x_1 \), we get
\[ \Delta = 72x_2(x_2^2 + x_1^2)^3(3440x_2^2y_1^4x_1^{10} + 7536y_1^4x_2^{10}x_1^2 - 19936x_1^6y_1^4x_2^6 \\
+ 34560x_1^8y_1^4x_2^2 - 10344x_1^8y_1^4x_2^2 + 186624x_1^2y_1^8x_2^2 + 920x_1^4y_1^4x_2^8 \\
+ 51840x_1^4y_1^4x_2^4 + 34560x_1^2y_1^8x_2^6 + 8640y_1^8x_2^8 - 93312x_1^4y_1^{12} + 210x_1^8x_2^8 \\
+ 168x_1^6x_2^{10} + 84x_1^4x_2^{12} + 168x_1^{10}x_2^6 + 84x_1^{12}x_2^4 + 24x_1^{14}x_2^2 + 744x_1^{12}y_1^4 \\
+ 24x_1^2x_2^{14} - 93312y_1^{12}x_2^4 + 3x_2^{16} + 186624y_1^{16} + 8640y_1^8x_2^8 + 3x_1^{16} - 280y_1^4x_2^{12}). \]
To prove that \( \Delta \neq 0 \) it suffices to consider the polynomial within the second parentheses, since \( 72x_2(x_2^2 + x_1^2)^3 \) does not vanish by the algebraic independence
of $x_1$ and $x_2$:

$$
\begin{align*}
& h_1(x_1, x_2, y_1) := 3440x_2^4y_1^4x_1^{10} + 7536y_1^4x_1^2x_1^2 - 19936x_1y_1^4x_2^6 \\
& + 34560x_1^2y_1^4x_2^2 - 10344x_1^4y_1^4x_2^2 - 186624x_1^{12}x_2^2 + 920x_1^4y_1^4x_2^8 \\
& + 51840x_1^2y_1^4x_2^2 + 34560x_1^2y_1^8x_2^2 + 8640y_1^{12}x_1^4y_1^{12} \\
& + 210x_1^8x_2^2 + 168x_1^6x_2^2 + 84x_1^4x_2^2 + 168x_1^6x_2^2 + 84x_1^4x_2^2 \\
& + 24x_1^{14}x_2^2 + 744x_1^{12}y_1^4 + 24x_1^{14}x_2^2 - 93312y_1^{12}x_1^2 + 3x_1^6 \\
& + 186624y_1^{16} + 8640x_1^8y_1^8 + 3x_1^6 - 280y_1^4x_2^4.
\end{align*}
$$

Let us assume that $\Delta = 0$, hence $h_1(x_1, x_2, y_1) = 0$. From (4.5) we have $(x_1^2 + x_2^2)^{2(N+M)}P_3(X_0, Y_0) = 0$. Using $y_2 = x_1$, it follows that

$$
0 = 9(x_2^2 + x_1^2)^4(-x_2^8 + 80y_1^2x_2^2x_1^2 - 72x_1^4x_1^4 - 4x_2^2x_1^6 + 432y_1^8 - x_2^8 \\
- 6x_1^2x_2^4 - 144y_1^4x_2^2x_2^2 + 80x_1^4x_2^2x_2^2 - 4x_2^6x_1^2 - 72x_1^4x_2^4 - 16y_1^2x_2^6 \\
- 16x_1^6y_1^2)(-x_2^8 - 80y_1^2x_2^2x_1^2 - 72x_1^4x_1^4 - 4x_2^2x_1^6 + 432y_1^8 - x_2^8 - 6x_2^4x_1^4 \\
- 144y_1^4x_2^2x_2^2 - 80x_1^4x_2^2x_2^2 - 4x_2^6x_1^2 - 72y_1^4x_2^2 + 16y_1^6x_2^4 + 16x_1^6y_1^2).
$$

The algebraic independence of $x_1, x_2$ over $\mathbb{Q}$ shows that $9(x_2^2 + x_1^2)^4 \neq 0$, hence the number

$$
\begin{align*}
& h_2(x_1, x_2, y_1) := (-x_2^8 + 80y_1^2x_2^2x_1^2 - 72x_1^4x_1^4 - 4x_2^2x_1^6 + 432y_1^8 - x_1^8 - 6x_2^4x_1^4 \\
& - 144y_1^4x_2^2x_2^2 + 80x_1^4x_2^2x_2^2 - 4x_2^6x_1^2 - 72x_1^4x_2^4 - 16y_1^2x_2^6 - 16x_1^6y_1^2)
\end{align*}
$$

vanishes. By the assumption $h_1 = 0$ it follows that $\text{Res}_{y_1}(h_1(x_1, x_2, y_1), h_2(x_1, x_2, y_1)) = 0$. We obtain

$$
0 = 2^{240}3^{72}x_1^{16}x_2^{8}(8x_1^4 + 29x_2^2x_1^2 + 27x_2^4)^4(x_2 - x_1)^{12}(x_1 + x_2)^{12} \\
\times (x_2^2 - 2x_1x_2 - x_1^2)^{16}(x_2^2 + 2x_1x_2 - x_1^2)^{16}(x_2^2 + x_1^2)^{64},
$$

a contradiction to the algebraic independence of $x_1, x_2$ over $\mathbb{Q}$. Thus the AIC proves the algebraic independence of $\vartheta_3(6\tau)$ and $\vartheta_3(\tau)$ over $\mathbb{Q}$.

Next, let $n = 5$. With $N = 6$, $M = 4$, and the polynomial $P_5(X, Y)$ listed in the appendix, an analogous computation finally gives the identity

$$
0 = 2^{592}5^{200}x_1^{32}x_2^8(128x_1^{12} - 816x_1^4x_2^4 + 603x_1^6x_2^6 + 5775x_1^4x_2^8 + 7569x_1^2x_2^{10} \\
+ 3125x_2^{12})^4(243x_1^4 - 3580x_2^2x_2^{12} - 315034x_1^4x_2^{20} + 1780x_1^6x_2^{18} + 1040093x_1^8x_2^{16} \\
+ 774920x_1^{10}x_2^{14} - 2001561x_2^{12}x_2^{12} + 774920x_1^{14}x_2^{10} + 1040093x_1^{16}x_2^8 \\
+ 1780x_1^{18}x_2^6 - 315034x_1^2x_2^{20} + 3580x_1^2x_2^{22} + 243x_1^{24})^8(x_1^2 - 2x_1x_2 - x_2^2)^{16} \\
\times (x_1^2 + 2x_1x_2 - x_2^2)^{16}(x_1 - x_2)^{20}(x_1 + x_2)^{20}(x_1^2 + x_2^2)^{96}.
$$
The contradiction proves the algebraic independence of \( \vartheta_3(10\tau) \) and \( \vartheta_3(\tau) \) over \( \mathbb{Q} \). For the proof of the algebraic independence of \( \vartheta_3(12\tau) \) and \( \vartheta_3(\tau) \) over \( \mathbb{Q} \) we have to modify the above formulae. From the double-argument formulae (3.1) we obtain

\[
\vartheta_4^4(4\tau) = 16(\vartheta_3 - \vartheta_4)^4,
\]
\[
\vartheta_3^4(4\tau) = 16(\vartheta_3 + \vartheta_4)^4.
\]

In Theorem A we replace \( \tau \) by \( 4\tau \) such that (4.3) holds with

\[
X_0 = \frac{n^2 \vartheta_3^4(4n\tau)}{\vartheta_3^4(4\tau)} = \frac{16n^2 y_1^4}{(y_2 + x_2)^4},
\]
\[
Y_0 = \frac{16 \vartheta_3^4(4\tau)}{\vartheta_3^4(4\tau)} = \frac{16(x_1 - x_2)^4}{(x_1 + x_2)^4},
\]

where \( y_1 = \vartheta_3(4n\tau) \). Finally, we replace (4.5) by

\[
f_1(t_1, t_2, u_1, u_2) = (t_2 + u_2)^4N (t_1 + t_2)^4M P_n\left(\frac{16n^2 u_1^4}{(t_2 + u_2)^4}, \frac{16(t_1 - t_2)^4}{(t_1 + t_2)^4}\right).
\]

Setting \( n = 3, N = 4, M = 2 \), and following the above lines of computations, we deduce the following identity:

\[
0 = 2^{376} 3^7 2^8 x_1^8 x_2^8 (x_1^4 - 12x_1^3 x_2 - 12x_1 x_2^3 + x_2^4 + 6x_1^2 x_2^2)^{16}
\]
\[
\times (3x_1^4 + 16x_1^3 x_2 + 30x_1^2 x_2^2 + 32x_1 x_2^3 + 27x_2^4)^4 (x_1 - x_2)^{32}(x_1 + x_2)^{128}.
\]

Here the contradiction proves the algebraic independence of \( \vartheta_3(12\tau) \) and \( \vartheta_3(\tau) \) over \( \mathbb{Q} \).

Finally, for Theorem 1.1 it remains to prove the algebraic independence of \( \vartheta_3(4^m \tau) \) and \( \vartheta_3(q) \) over \( \mathbb{Q} \) for any \( m \geq 3 \). Let \( n = 2^m \). By Theorem 4.1 it suffices to show that the polynomial

\[
\text{Res}_X\left(P_n(X,Y), \frac{\partial}{\partial Y} P_n(X,Y)\right) \in \mathbb{Z}[Y]
\]

does not vanish identically. We know from (3.9) in Lemma 3.2 that

\[
P_n(X,Y) = T_n(X, (1 + Y)^2, Y) = (nX - (1 + Y)^2)^{2m-2} + Y U_n(X, (1 + Y)^2, Y).
\]

Hence we obtain

\[
P_n(X, 0) = T_n(X, 1, 0) = (2^m X - 1)^{2m-2},
\]
\[
\frac{\partial P_n}{\partial Y}(X, 0) = -2^{m-1}(2^m X - 1)^{2m-2-1} + U_n(X, 1, 0).
\]
On the one hand, the polynomial \( P_n(X, 0) \) in (4.6) has a \( 2m^{-2} \)-fold root \( X_0 \) at \( X_0 = 1/2^m \). On the other hand, we know by (4.7) and (3.10) in Lemma 3.2 that

\[
\frac{\partial P_n}{\partial Y}(X_0, 0) = U_n\left(\frac{1}{n}, 1, 0\right) = -2^{-m-1 - 1} \neq 0.
\]

This shows that for \( Y = 0 \) the polynomials \( P_n(X, Y) \) and \( \partial P_n(X, Y)/\partial Y \) have no common root. Therefore, the resultant of both polynomials with respect to \( X \) does not vanish identically. This completes the proof of Theorem 1.1. ■

5. Appendix

The polynomials \( P_3, P_5, P_7, P_9, \) and \( P_{11} \) listed below were derived from the proof of Theorem 1.1 in [8].

\[
P_3 = 9 - (28 - 16Y + Y^2)X + 30X^2 - 12X^3 + X^4,
\]

\[
P_5 = 25 - (126 - 832Y + 308Y^2 - 32Y^3 + Y^4)X + (255 + 1920Y - 120Y^2)X^2
\]
\[+ (-260 + 320Y - 20Y^2)X^3 + 135X^4 - 30X^5 + X^6,
\]

\[
P_7 = 49 - (344 - 17568Y + 20554Y^2 - 6528Y^3 + 844Y^4 - 48Y^5 + Y^6)X
\]
\[+ (1036 + 156800Y + 88760Y^2 - 12320Y^3 + 385Y^4)X^2
\]
\[+ (-1736 - 185024Y + 18732Y^2 - 896Y^3 + 28Y^4)X^3
\]
\[+ (1750 + 31360Y - 1960Y^2)X^4 - (1064 - 2464Y + 154Y^2)X^5
\]
\[+ 364X^6 - 56X^7 + X^8,
\]

\[
P_9 = 6561 - (60588 - 18652032Y + 56033208Y^2 - 40036032Y^3 + 11743542Y^4
\]
\[+ 1715904Y^5 + 1325161Y^6 - 5184Y^7 + 81Y^8)X
\]
\[+ (250146 + 427613184Y + 2083563072Y^2 + 86274432Y^3 - 57982860Y^4
\]
\[+ 4249728Y^5 - 99288Y^6 + 576Y^7 - 9Y^8)X^2
\]
\[+ (607420 - 1418904064Y + 2511615520Y^2 - 353755456Y^3 + 19071754Y^4
\]
\[+ 612736Y^5 + 139601Y^6 - 64Y^7 + Y^8)X^3
\]
\[+ (959535 + 856286208Y + 8468928Y^2 - 2145024Y^3 - 80848Y^4
\]
\[+ 65664Y^5 - 1368Y^6)X^4
\]
\[+ (1028952 + 22899456Y + 1430352Y^2 - 505152Y^3 + 38826Y^4
\]
\[+ 1728Y^5 + 36Y^6)X^5
\]
\[+ (757596 - 13138944Y + 4160448Y^2 - 417408Y^3 + 13044Y^4)X^6
\]
\[+ (378072 + 1138176Y + 16416Y^2 - 10944Y^3 + 342Y^4)X^7
\]
\[+ (122895 + 64512Y - 4032Y^2)X^8 - (24060 - 11136Y + 696Y^2)X^9
\]
\[+ 2466X^{10} - 108X^{11} + X^{12},
\]
The polynomials $P_2$, $P_4$, $P_8$, $P_{16}$, and $P_{32}$ listed below were derived from the proof of Lemma 3.1:

\[ P_2 = 2X - Y^2 - 1, \]
\[ P_4 = 4X - (1 + Y)^2, \]
\[ P_8 = 64X^2 - 16(1 + Y)^2X + (1 - Y)^4, \]
\[ P_{16} = 65536X^4 - 16384(1 + Y)^2X^3 + 512(3Y^4 + 4Y^3 + 18Y^2 + 4Y + 3)X^2 - 64(1 + Y)^2(Y^4 + 28Y^3 + 6Y^2 + 28Y + 1)X + (1 - Y)^8, \]
\[ P_{32} = 2^{40}X^8 - 2^{38}(1 + Y)^2X^7 + 2^{32}(7Y^4 + 20Y^3 + 42Y^2 + 20Y + 7)X^6 - 2^{28}(1 + Y)^2(7Y^4 + 164Y^3 + 42Y^2 + 164Y + 7)X^5 + 2^{21}(35Y^8 + 552Y^7 + 2260Y^6 + 3864Y^5 + 5010Y^4 + 3864Y^3 + 2260Y^2 + 552Y + 35)X^4 - 2^{18}(1 + Y)^2(7Y^8 + 424Y^7 + 7492Y^6 + 2968Y^5 + 15082Y^4 + 2968Y^3 + 7492Y^2 + 424Y + 7)X^3 + 2^{12}(7Y^{12} - 5924Y^{11} + 4174Y^{10} + 33900Y^9 + 33161Y^8 + 36536Y^7 + 58436Y^6 + 36536Y^5 + 33161Y^4 + 33900Y^3 + 4174Y^2 - 5924Y + 7)X^2 - 2^8(1 + Y)^2(Y^{12} + 660Y^{11} + 15170Y^{10} + 68420Y^9 + 121327Y^8 + 212520Y^7 + 212380Y^6 + 212520Y^5 + 121327Y^4 + 68420Y^3 + 15170Y^2 + 660Y + 1)X + (1 - Y)^{16}. \]
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