CONTROLLABILITY OF SEMILINEAR SCHROEDINGER EQUATION VIA LOW-DIMENSIONAL SOURCE TERM

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Abstract. We study controllability of 2D defocusing cubic Schroedinger equation under periodic boundary conditions and control applied via source term (additively). The source term is a linear combination of few complex exponentials (modes) with time-variant coefficients - controls. We manage to prove that controlling just 4 modes one can achieve controllability of this equation in any finite-dimensional projection of its evolution space $H^{1+\sigma}(\mathbb{T}^2)$, as well as approximate controllability in $H^{1+\sigma}(\mathbb{T}^2), \sigma > 0$. We also present negative result regarding exact controllability of cubic Schroedinger equation via a finite-dimensional source term.

Keywords: semilinear Schroedinger equation, approximate controllability, geometric control

1. Introduction

Lie algebraic approach of geometric control theory to nonlinear distributed systems has been initiated recently. An example of its implementation is study of 2D Navier-Stokes/Euler equations of fluid motion controlled by low-dimensional forcing in [1, 2], where for the mentioned equations one arranged sufficient criteria for approximate controllability and for controllability in finite-dimensional projections of evolution space.

Here we wish to develop similar approach to another class of distributed system - cubic defocusing Schroedinger equation (cubic NLS):

\begin{equation}
-i \partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + F(t,x), \quad u|_{t=0} = u^0,
\end{equation}

controlled via source term $F(t,x)$.

We restrict ourselves to 2-dimensional periodic case: space variable $x$ belongs to torus $\mathbb{T}^2$.

Our problem setting is distinguished by two features. First, control is introduced via source term, i.e. in additive form, on the contrast to bilinear form, characteristic for quantum control. More particular feature is finite-dimensionality of the range of the controlled source term:

\begin{equation}
F(t,x) = \sum_{k \in \hat{K}} v_k(t) e^{i k \cdot x}, \quad \hat{K} \subset \mathbb{Z}^2 - \text{finite},
\end{equation}

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which means that for each \( t \) the value \( F(t) \) belongs to a finite-dimensional subspace \( \mathcal{F}_{\hat{\mathcal{K}}} = \text{Span}\{e^{ik\cdot x}, \ k \in \hat{\mathcal{K}}\} \) of the evolution space for NLS.

The control functions \( v_k(t), \ t \in [0,T], \ k \in \hat{\mathcal{K}} \), which enter the source term, can be chosen freely in \( L^\infty[0,T] \), or in any functional space, which is dense in \( L^1[0,T] \).

By this choice of 'small-dimensional' control our problem setting differs from the studies of controllability of NLS (see end of Section 3 for few references to alternative settings and approaches), in which controls have infinite-dimensional range. In some of the studies controls are supported on a subdomain and one is interested in tracing propagation of the controlled energy to other parts of domain. On the contrast, in our case controls affect few directions - modes - in functional evolution space for NLS and we are interested in the way this controlled action spreads to other (higher) modes.

One could opt for more general finitely generated control \( \sum_{k \in \hat{\mathcal{K}}} v_k(t) F_k(x) \), but then representation of the NLS equation and in particular of its nonlinear term on \( \mathbb{T}^2 \) becomes much more intricate. Similar difficulties arise, when one studies NLS equation under general boundary conditions.

We will treat NLS equation (1) as an evolution equation in \( H^{1+\sigma}(\mathbb{T}^2) \), \( \sigma > 0 \). The 'high regularity' helps us to avoid certain analytic difficulties which are unrelated to the controllability issue.

Imposing the initial condition \( u(0) = u_0 \in H^{1+\sigma}(\mathbb{T}^2) \), we set problems of:

1. controllability in finite-dimensional projections, meaning that one can steer in time \( T > 0 \) the trajectory of the equation (1) from \( u_0 \) to a state \( \hat{u} \in H^{1+\sigma} \) with any preassigned orthogonal projection \( \PiL \hat{u} \) onto any given finite-dimensional subspace \( L \subset H^{1+\sigma} \);
2. approximate controllability meaning that attainable set of (1) from each \( u_0 \) is dense in \( H^{1+\sigma} \);
3. exact controllability in \( H^{1+\sigma} \).

Definitions of some types of controllability and exact problem setting are provided in the next Section together with the main results. First of the results asserts that controllability in projection on each finite-dimensional subspace of \( H^{1+\sigma} \) and approximate controllability in \( H^{1+\sigma} \) can be achieved by (universal family of) 4-dimensional controls (\( \#\hat{\mathcal{K}} = 4 \)). Corollary 6.5 describes a class of sets of controlled modes which suffice for achieving these types of controllability. The second main result asserts lack of exact controllability in \( H^{1+\sigma} \) by controlling any finite number of modes.

2. Cubic Schrödinger equation on \( \mathbb{T}^2 \); problem setting and main results

2.1. Controllability: definitions.

2.1.1. Global controllability. As we said evolution space of NLS equation will be Sobolev space \( H = H^{1+\sigma}(\mathbb{T}^2) \).
We say that control \( q \) steers the system \( (1) \) from \( u_0 \in H \) to \( \hat{u} \in H \) in time \( T > 0 \), if solution of \( (1) \) with initial condition \( u_{|t=0} = u^0 \) exists, is unique, belongs to \( C([0,T], H) \) and satisfies \( u(T) = \hat{u} \). The equation is \textit{globally time-} \( T \) (exactly) controllable from \( u_0 \), if it can be steered in time \( T \) from \( u_0 \) to any point of \( H \); it is globally (exactly) controllable from \( u_0 \), if for each \( \hat{u} \) the equation can be steered from \( u_0 \) to \( \hat{u} \) in some time \( T > 0 \).

2.1.2. Controllability in finite-dimensional projections and in finite-dimensional component. Let \( L \) be a closed linear subspace of \( H \), \( \Pi^L \) be orthogonal projection of \( H \) onto \( L \).

Equation \( (1) \)-\( (2) \) is (time-\( T \)) globally controllable from \( u_0 \) in projection onto \( L \), if for each \( \hat{q} \in L \) the system can be steered (in time \( T \)) from \( u_0 \) to some point \( \hat{u} \) with \( \Pi^L(\hat{u}) = \hat{q} \).

The NLS equation \( (1) \)-\( (2) \) is (time-\( T \)) globally controllable from \( u_0 \) in finite-dimensional projections if for each finite-dimensional subspace \( L \) it is (time-\( T \)) globally controllable from \( u_0 \) in projection onto \( L \); note that the set \( \hat{K} \) of controlled modes is assumed to be the same for all \( L \).

Whenever \( L \) is a 'coordinate subspace' \( L = \text{span}\{e^{ik \cdot x} \mid k \in K^o \} \), with \( K^o \subset \mathbb{Z}^2 \) being a finite set of observed modes, then controllability in projection on \( L \) is called controllability in observed \( K^o \)-component.

Remark 2.1. It is convenient to characterize time-\( T \) controllability in terms of surjectiveness of the end-point map \( E_T : v(\cdot) \mapsto F(v(\cdot)) \mapsto u(T) \) of the controlled NLS equation \( (1) \)-\( (2) \), which maps a control \( v(\cdot) = (v_k(t)), k \in \hat{K} \), into the 'final' point \( u(T) \) of the trajectory \( u(t) \) of this equation, driven by source term \( F = \sum_{k \in \hat{K}} v_k(t) e^{ik \cdot x} \) and starting at \( u(0) = u^0 \). Similarly controllability in projection on \( L \) means that the composition \( \Pi^L \circ E_T \) is onto (covers) \( L \). □

2.1.3. Approximate controllability. The NLS equation \( (1) \)-\( (2) \) is time-\( T \) approximately controllable from \( u_0 \) in \( H \), if it can be steered from \( u_0 \) to each point of a dense subset of \( H \). □

2.1.4. Solid controllability (cf. \( (2) \)). On the contrast to previous definitions the word 'solid' does not refer to a new type of controllability but means property of stability of controllability with respect to certain class of perturbations.

Let \( \Phi : M^1 \mapsto M^2 \) be a continuous map between two metric spaces, and \( S \subseteq M^2 \) be any subset. We say that \( \Phi \) covers \( S \) solidly, if \( S \subseteq \Phi(M^1) \) and the inclusion is stable with respect to \( C^0 \)-small perturbations of \( \Phi \), i.e. for some \( C^0 \)-neighborhood \( \Omega \) of \( \Phi \) and for each map \( \Psi \in \Omega \), there holds: \( S \subseteq \Psi(M^1) \).

Controllability in projection on finite-dimensional subspace \( L \) for the NLS equation \( (1) \)-\( (2) \) is solid, if for any bounded set \( S \subseteq L \) there exists a family of controls \( V_S = \{v(t,b) \mid b \in B \text{ - compact in } \mathbb{R}^d \} \), such that projected end-point map \( (\Pi^L \circ E_T) |_{V_S} \) (see Remark 2.1) covers \( S \) solidly. We will say that \( S \) is solidly attained by the controlled NLS equation.
2.2. Problem setting and main results. Our first goal is establishing sufficient criteria for controllability of cubic defocusing NLS in all finite-dimensional projections and approximate controllability in $H^{1+\sigma}$, $\sigma > 0$. Common criterion is formulated in terms of a set of controlled modes $\mathcal{K}$, which is fixed and the same for all projections and for approximate controllability.

Second objective is negative result regarding exact controllability of cubic NLS via finite-dimensional source term.

Main result 1 (criterion for controllability in finite-dimensional projections and approximate controllability). Given 2D periodic defocusing cubic Schrödinger equation (1), controlled via source term (2), one can find a 4-element set $\hat{\mathcal{K}} \subset \mathbb{Z}^2$ of controlled modes such that for any initial data $u^0 \in H^{1+\sigma}(T^2)$ and any $T > 0$: i) for each finite-dimensional subspace $\mathcal{L}$ of $H^{1+\sigma}(T^2)$ the equation (1)-(2) is time-$T$ controllable from $u^0$ in projection on $\mathcal{L}$; ii) the equation is approximately controllable from $u^0$ in $H^{1+\sigma}(T^2)$. □

Remark 2.2. An example of a set $\hat{\mathcal{K}}$ able to guarantee the controllability properties is $\hat{\mathcal{K}} = \{(0,0), (1,0), (0,1), (1,1)\}$. Corollary 6.5 introduces a class of sets $\hat{\mathcal{K}}$ of controlled modes, which suffice for the two types of controllability. □

Main result 2 (negative result on exact controllability). For 2D periodic defocusing cubic Schrödinger equation (1), controlled via source term (2) with arbitrary finite set $\hat{\mathcal{K}} \subset \mathbb{Z}^2$ of controlled modes, for each $T > 0$ and each initial data $u^0 \in H^{1+\sigma}(T^2)$, the time-$T$ attainable set $\mathcal{A}_{T,u^0}$ of (1)-(2) from $u^0$ is contained in a countable union of compact subsets of $H^{1+\sigma}(T^2)$ and therefore the complement $H^{1+\sigma}(T^2) \setminus \mathcal{A}_{T,u^0}$ is dense in $H^{1+\sigma}(T^2)$. □

3. Outline of the approach: Lie extensions, fast-oscillating controls, resonances. Other approaches

Study of controllability of NLS equation is based (as well as our earlier joint work with A.Agrachev on Navier-Stokes/Euler equation) on method of iterated Lie extensions. Lie extension of control system $\dot{x} = f(x,u)$, $u \in U$ is a way to add vector fields to the right-hand side of the system guaranteeing (almost) invariance of its controllability properties. The additional vector fields are expressed via Lie brackets of $f(\cdot,u)$ for various $u \in U$. If after a series of extensions one arrives to a controllable system, then the controllability of the original system will follow.

This approach can not be extended automatically onto infinite-dimensional setting due to the lack of adequate Lie algebraic tools. So far in the infinite-dimensional context Lie algebraic formulae are rather used as guiding tools, whose implementation has to be justified by analytic means. In the rest of this Section we provide geometric control sketch for the proof of main result.
When studying controllability we look at cubic NLS equation as at particular type of infinite-dimensional control-affine system:

\[-i\partial_t u = c(u, t) + \sum_{k \in \mathcal{K}} e_k v_k(t), \quad e_k = e^{ik \cdot x},\]

where \(c(u, t)\) is cubic drift vector field, \(e_k\) are constant controlled vector field in \(H^{1+\sigma}(\mathbb{T}^2)\) with values \(e^{ik \cdot x} \in H^{1+\sigma}(\mathbb{T}^2)\).

Lie extensions, we use, are implemented iteratively. At each iteration they involve two controlled vector fields \(e_m, e_n\) and outcome is fourth-order Lie bracket \([e_n, [e_m, c]]\), which appears as extending controlled vector field. The vector field is constant (as far as the vector field \(c\) is cubic) and is seen as direction of action of an extended control.

Different type of Lie brackets which makes its appearance for each Lie extension is third-order Lie bracket \([e_m, [e_m, c]]\), which can be seen as obstruction to controllability, along the vector field ‘unilateral drift’ of the system takes place. This drift can not be locally compensated but for NLS equation one can nullify average drift by imposing integral (isoperimetric) relations onto the controls involved.

To design needed motion in the extending direction \([e_n, [e_m, [e_m, c]]]\) and to oppress motion in the directions, not needed, we employ fast-oscillating controls. Use of such controls is traditional for geometric control theory and although a ‘general theory’ is hardly available, the approach can be effectively applied in particular cases (see, for example treatment of ‘single-bracket case’ in [16]).

In our study we feed fast-oscillating controls

\[v_m(t)e^{ia_m t/\varepsilon}e^{im \cdot x}, v_n(t)e^{ia_n t/\varepsilon}e^{in \cdot x}\]

into the right-hand side of the NLS equation at looks at interaction of the two controls via the cubic term. The idea is to design needed resonance in the course of such interaction, that is to choose oscillation frequencies and magnitudes in such a way that the interaction ‘in average’ influences dynamics of (few) certain modes. In our treatment we manage to limit the influence to unique basis mode \(e^{i(2m-n) \cdot x}\); the resonance term is seen as additional (extending) control along this mode. The procedure is interpreted as elementary extension of the set of controlled modes: for any \(m, n \in \mathcal{K}: \mathcal{K} \mapsto \mathcal{K} \cup \{2m - n\}\).

Final controllability result is obtained by (finite) iteration of the elementary extensions. If one seeks controllability in observed \(K^o\)-component with \(K^o \supset \mathcal{K}\), then one should look (when possible) for a series of elementary extensions \(\mathcal{K} = K^1 \subset K^2 \subset \cdots \subset K^N = K^o\). Getting extended controls available for each observed mode \(k \in K^o\) we conclude controllability of the extended system in \(K^o\)-component by an easy Lemma 5.2. On the contrast controllability of the original system in \(K^o\)-component will follow by virtue of rather technical Approximative Lemma 5.1 which formalizes the resonance design.
From controllability for each finite-dimensional component one derives controllability in projection on each finite-dimensional subspace as well as approximate controllability; this is proved in Section 7.

Note that the analysis of interaction of different terms via cubic nonlinearity in the case of periodic NLS equation is substantially simplified by choice of special basis of exponential modes.

Besides the design of proper resonances there are two analytic problems to be fixed. First problem consists of studying NLS with fast-oscillating right-hand side and of establishing the continuity, approximating properties and the limits of corresponding trajectories, as the frequency of oscillation tends to $+\infty$. Second problem is to cope with the fact that at each iteration we are only able to approximate the desired motion, therefore the controllability criteria need to be stable with respect to the approximation errors.

The second problem is fixed with the help of the notion of solid controllability (see previous Section), which guarantees stability of controllability property with respect to approximation error.

The solution to the first problem in finite-dimensional setting is provided by theory of relaxed controls. For general nonlinear PDE such theory is unavailable; although for semilinear infinite-dimensional control systems relaxation results have been obtained in [9, 8]. We provide formulations and proofs needed for our analysis in Subsection 5.5.

What regards negative result on exact controllability stated in Main result 2, then the key point for its proof is continuity of input-trajectory map in some weaker topology of the (functional) space of inputs (controls) in which the space is countable union of compacts and as a consequence attainable sets are meager. This kind of argument has been used in [3] for establishing noncontrollability of some bilinear distributed systems. Finer method, based on estimates of Kolmogorov’s entropy has been invoked in [15] for proving lack of exact controllability by finite-dimensional forcing for Euler equation of fluid motion.

At the end of the Section we wish to mention just few references to other approaches to controllability of linear and semilinear Schroedinger equation controlled via bilinear or additive control, this latter being "internal" or boundary.

First we address the readers to [18, 11] which provide nice surveys of the results on:

- exact controllability for linear Schroedinger equation with additive control in relation to observability of adjoint system and to geometric control condition ([13] and references in [18] on other results up to 2003);
- controllability of linear Schroedinger equation with control entering bilinearly; besides references in the above cited surveys there are notable results [4, 5] on local (exact) controllability in $H^T$ of 1-D equation; another interesting result is (obtained by geometric methods)
criterion of approximate controllability for the case in which 'drift Hamiltonian' has discrete non-resonant spectrum (see bibliographic references in [4, 5, 6] to preceding work);

- exact controllability of semilinear Schrödinger equation by means of internal additive control; in addition to references in [18, 11] we mention more recent publications [7, 14] where the property has been established for 2D and 1D cases. The key tool in the study of the semilinear case is 'linearization principle', going back to [12]. In contrast our approach makes direct and exclusive use of the nonlinear term.

4. Preliminaries on existence, uniqueness and continuous dependence of trajectories

Notions of controllability, introduced above, involve trajectories of cubic NLS equation with source term. The trajectories are sought in the space \( C\left([0,T]; H\right) \), \( H \) being Hilbert space of functions \( u(x) \) defined on \( \mathbb{T}^2 \). We opt for \( H = H^{1+\sigma}(\mathbb{T}^2) \).

In this Section we collect results on existence/uniqueness and on continuity in the right-hand side for solutions of semilinear equations

\[
(-i\partial_t + \Delta)\tilde{u} = G(t,\tilde{u}), \quad \tilde{u}(0) = \tilde{u}^0
\]

and of its 'perturbation':

\[
(-i\partial_t + \Delta)u = G(t,u) + \phi(t,u), \quad u(0) = u^0.
\]

Below we identify the equations (3),(4) with their integral forms (10),(11) obtained via applications of Duhamel formula.

We assume the nonlinear terms \( G(t,\cdot), \phi(t,\cdot) : H \to H \) to be continuous, and to satisfy the conditions

\[
G(t,0) = 0,
\]

\[
\forall b > 0, \exists \beta_b(t) \in L^1([0, T], \mathbb{R}_+), \text{ such that } \forall t \in [0, T], \forall \|u\| \leq b,
\]

\[
\|G(t,u)\|_H \leq \beta_b(t), \quad \|G(t,u') - G(t,u)\|_H \leq \beta_b(t)\|u'-u\|_H,
\]

\[
\|\phi(t,u)\|_H \leq \beta_b(t), \quad \|\phi(t,u') - \phi(t,u)\|_H \leq \beta_b(t)\|u'-u\|_H.
\]

Local existence of solutions under the assumptions could be established via fixed point argument for contracting map in \( C([0,T]; H) \).

**Proposition 4.1** (local existence and uniqueness of solutions). Let \( G \) satisfy conditions [9]. Then for each \( B > 0, \exists T_B > 0 \) such that for \( \|\tilde{u}^0\|_H \leq B \) there exists unique strong solution \( u(\cdot) \in C([0,T_B], H) \) of Cauchy problem [2]. □

We choose \( H = H^{1+\sigma}(\mathbb{T}^2) \), so that the cubic term of the NLS equation [11] would satisfy conditions [6],[7]. One can invoke the following technical result for verification.
Proof. As it is known the solution of equation (4) can be continued in time form
\[ \delta > 0 \text{ sufficiently small} \]
as long as \[ u \] then solution
\[ \| (8) \]
is unique and admits an upper bound
\[ \| (9) \]
Proposition 4.1 hold and let
\[ \| (10) \]
solution \[ u \] and respective global formulation for cubic defocusing NLS with source term.
\[ \| (11) \]
Let time-variant source term \[ t \rightarrow F(t, \cdot) \] belong to \( L^1([0, T], H^{1+\sigma}) \). Then for each initial condition \( u(0) = u^0 \in H^{1+\sigma} \) the Cauchy problem for the equation (10) has unique strong solution \( u(\cdot) \in C([0, T], H^{1+\sigma}) \).

No we provide few results on continuity of trajectories in the right-hand side of the NLS equation.

Proposition 4.3 (global existence and uniqueness). Let time-variant source term \( t \rightarrow F(t, \cdot) \) belong to \( L^1([0, T], H^{1+\sigma}) \). Then for each initial condition \( u(0) = u^0 \in H^{1+\sigma} \) the Cauchy problem for the equation (10) has unique strong solution \( u(\cdot) \in C([0, T], H^{1+\sigma}) \).

Lemma 4.2 ('Product Lemma'; [17]). For Sobolev spaces \( H^s(\mathbb{T}^d) \) of functions on \( d \)-dimensional torus there holds:
\[ \text{for } s \geq 0 : \| f g \|_{H^s} \leq C(s, d) (\| f \|_{H^s} \| g \|_{L^\infty} + \| f \|_{L^\infty} \| g \|_{H^s}); \]
\[ \text{for } s > d/2 : \| f g \|_{H^s} \leq (C(s, d) \| f \|_{H^s} \| g \|_{H^s}. \]

This Lemma allows verification of the conditions (8), (9) for more general Nemytskii-type operators \( u \mapsto G(t, u), \ u \mapsto \phi(t, u) \) of the form
\[ u(t, x) \mapsto F_0(t, x) + \sum_{j=1}^p P_j(u(t, x), \bar{u}(t, x); t), \]
where \( P_j : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \) are polynomials of degree \( j \) in \( u, \bar{u} \) with coefficients \( p_{j\alpha}(t) \in L^1([0, T], \mathbb{C}) \), while \( F_0(t, x) \) belongs to \( L^1([0, T], \mathbb{C}) \). Recall that the source term (2) is trigonometric polynomial in \( x \) and \( F(t, x) \in L^\infty([0, T], H^{1+\sigma}). \)

Global existence and uniqueness results for cubic defocusing NLS equation (1) are classical under assumptions we made; see, for example, [7] for respective global formulation for cubic defocusing NLS with source term.

Proposition 4.3 (global existence and uniqueness). Let time-variant source term \( t \rightarrow F(t, \cdot) \) belong to \( L^1([0, T], H^{1+\sigma}) \). Then for each initial condition \( u(0) = u^0 \in H^{1+\sigma} \) the Cauchy problem for the equation (10) has unique strong solution \( u(\cdot) \in C([0, T], H^{1+\sigma}) \).

Proof. As it is known the solution of equation (10) can be continued in time as long as \( H^{1+\sigma} \)-norm remains bounded. Therefore from the estimate (9) for sufficiently small \( \delta > 0 \) one gets extendibility of solution of (10) onto \([0, T]\).
Then
\[ u(t) - \tilde{u}(t) = e^{it\Delta} \left( (u^0 - \tilde{u}^0) + i \int_0^t e^{-i\tau \Delta} \phi(\tau, \tilde{u}(\tau)) d\tau \right) + \\
+ e^{it\Delta} i \int_0^t e^{-i\tau \Delta} (\langle G(\tau, u(\tau)) - G(\tau, \tilde{u}(\tau)) \rangle \phi(\tau, u(\tau)) - \phi(\tau, \tilde{u}(\tau))) d\tau. \]

Given that \( e^{it\Delta} \) is an isometry of \( H^{1+\sigma} \), we get
\[ \|u(t) - \tilde{u}(t)\|_H \leq \|u^0 - \tilde{u}^0\|_H + \left\| \int_0^t e^{-i\tau \Delta} \phi(\tau, \tilde{u}(\tau)) d\tau \right\|_H + \\
+ \int_0^t \|e^{-i\tau \Delta} (\langle G(\tau, u(\tau)) - G(\tau, \tilde{u}(\tau)) \rangle \phi(\tau, u(\tau)) - \phi(\tau, \tilde{u}(\tau))) \|_H d\tau \leq \\
\leq \|u^0 - \tilde{u}^0\|_H + \left\| \int_0^t e^{-i\tau \Delta} \phi(\tau, \tilde{u}(\tau)) d\tau \right\|_H + 2 \int_0^t \beta_b(\tau) \|u(\tau) - \tilde{u}(\tau)\|_H d\tau.
\]

By Gronwall inequality
\[ \|u(t) - \tilde{u}(t)\|_H \leq \left( \|u^0 - \tilde{u}^0\|_H + \left\| \int_0^t e^{-i\tau \Delta} \phi(\tau, \tilde{u}(\tau)) d\tau \right\|_H \right) C' e^{\int_0^t \beta_b(\tau) d\tau}, \]
for some \( C, C' > 0 \) and whenever (8) is satisfied, we get
\[ \|u(t) - \tilde{u}(t)\|_H \leq C'' \left( \|u^0 - \tilde{u}^0\| + \int_0^t \|\phi(\tau, \tilde{u}(\tau))\| d\tau \right) \leq C'' \delta. \]

\[ \square \]

Below we derive more general continuity result (Proposition 5.7) which incorporates perturbations \( \phi(t, x) \), fast-oscillating in time, and relaxation metric for the right-hand sides.

Similarly to the previous Proposition one gets

**Lemma 4.5.** Consider family of equations
\[ (-i\partial_t + \varepsilon \Delta) u^\varepsilon = \varepsilon G(t, u^\varepsilon) + \phi(t, u^\varepsilon), \quad u^\varepsilon(0) = u^0, \quad \varepsilon > 0, \]
depending on parameter \( \varepsilon > 0 \), with \( G, \phi \) satisfying (8), (7). Consider 'limit equation' for \( \varepsilon = 0 \):
\[ -i \partial_t \tilde{u} = \phi(t, \tilde{u}), \quad \tilde{u}|_{t=0} = u^0. \]

For solution \( \tilde{u}(\cdot) \in C([0, T], H) \) of (16) there exists \( \varepsilon_0 \) (depending on \( T \)), such that for \( \varepsilon \in [0, \varepsilon_0] \) solutions \( u^\varepsilon(t) \) of (15) exist on \([0, T]\) and
\[ \sup_{t \in [0, T]} \|u^\varepsilon(t) - \tilde{u}(t)\|_H = o(1), \quad \text{as } \varepsilon \to 0. \]
Proof. By Duhamel formula we get as in \((12)\)
\[
\|u^\varepsilon(t) - \tilde{u}(t)\|_H \leq \|e^{it\Delta}u^0 - u^0\| + \left\| \int_0^t e^{-i\varepsilon\tau\Delta} \varepsilon G(\tau, u(\tau)) d\tau \right\| + \\
+ \left\| \int_0^t e^{-i\varepsilon\tau\Delta} \phi(\tau, u^\varepsilon(\tau)) - \phi(\tau, \tilde{u}(\tau)) d\tau \right\|_H \leq \\
\leq \varepsilon \left\| \int_0^t e^{-i\varepsilon\tau\Delta} G(\tau, u(\tau)) d\tau \right\| + \left\| (e^{it\Delta} - I)u^0 \right\|_H + \\
+ \left\| \int_0^t (e^{-i\varepsilon\tau\Delta} - I) \phi(\tau, \tilde{u}(\tau)) d\tau \right\|_H + \int_0^t \|\phi(\tau, u^\varepsilon(\tau)) - \phi(\tau, \tilde{u}(\tau))\|_H d\tau.
\]
The last addend at the right-hand side is bounded by \(\int_0^t \beta(\|u^\varepsilon(\tau) - \tilde{u}(\tau)\|_H) d\tau\).

We will arrive to the needed conclusion by virtue of Gronwall inequality, when proving that the other three addends are \(o(1)\) as \(\varepsilon \to +0\).

We comment on the addend \(\left\| \int_0^t (e^{-i\varepsilon\tau\Delta} - I) \phi(\tau, \tilde{u}(\tau)) d\tau \right\|_H\), the other two assertions being obvious. For each \(\delta > 0\) one can approximate the function \(\tau \mapsto \phi(\tau, \tilde{u}(\tau)), \tau \in [0,T]\) by a piecewise constant function \(\psi^\delta(\tau) : [0,T] \to \mathbb{R}\) such that \(\|e^{-i\varepsilon\tau\Delta} - I\| \leq 2\) one gets
\[
\left\| \int_0^t (e^{-i\varepsilon\tau\Delta} - I) \phi(\tau, \tilde{u}(\tau)) d\tau \right\|_H \leq \left\| \int_0^t (e^{-i\varepsilon\tau\Delta} - I)\psi^\delta(\tau) d\tau \right\|_H + 2\delta.
\]
For a piecewise constant function \(\psi^\delta\) the first addend tends to 0, as \(\varepsilon \to 0\). \(\square\)

5. Extension of control

Here we introduce our main tool - extension of control. The outcome of the Section, to be employed later, is Proposition 5.3 which establishes sufficient criterion for controllability in finite-dimensional component, wherefrom one will derive in Section 7 controllability in projections and approximate controllability (Main Result 1). Proposition 5.3 is in its turn derived from rather technical Approximative Lemma 5.1 for extensions, accompanied by elementary Lemma 5.2 on controllability by full-dimensional control.

In what follows the metrics \(L^1([t_0,t_1],H^{1+\sigma}), L^1([t_0,t_1],C^\kappa), [t_0,t_1] \subset \mathbb{R}\) will be denoted both by \(L^1_t\) by abuse of notation.

5.1. Extensions: approximative lemma. Consider NLS equation \((1)-(2)\) with controls applied to the modes, indexed by a set \(\hat{\mathcal{K}} \subset \mathbb{Z}^2\), or the same with the controlled source term \(\sum_{k \in \hat{\mathcal{K}}} v_k(t)e^{ikx}\).

Pick two vectors \(r, s\) from the set \(\hat{\mathcal{K}}\) and call \(\mathcal{K} = \hat{\mathcal{K}} \cup \{2r - s\}\) an elementary extension of \(\hat{\mathcal{K}}\). Call \(\mathcal{K}\) proper extension of \(\hat{\mathcal{K}}\) if there exits a finite sequence of sets \(\hat{\mathcal{K}} = \mathcal{K}^1 \subset \mathcal{K}^2 \subset \cdots \subset \mathcal{K}^N = \mathcal{K}\), such that each \(\mathcal{K}^j\) is elementary extension of \(\mathcal{K}^{j-1}\), \(j \geq 2\).
The following Lemma states that controls (energy) fed into the modes, indexed by $\hat{K}$, can be cascaded to and moreover can approximately control larger set $K$ of modes, whenever $K$ is proper extension of $\hat{K}$.

**Lemma 5.1** (approximative lemma). Let $K$ be a proper extension of $\hat{K}$. Given a family of controls

\[ b \mapsto W(t; b) = \sum_{k \in K} w_k(t, b) e^{ik \cdot x}, \quad b \in B \text{ - compact in } \mathbb{R}^d, \]

parameterized by $b \in B$ continuously in $L_1$-metric, one can construct for each $\delta > 0$ another family of controls

\[ b \mapsto V^\delta(\cdot, b) = \sum_{k \in \hat{K}} v_k(t, b) e^{ik \cdot x}, \quad b \in B, \]

continuous in $L_1$-metric, such that for the respective end-point maps (see Remark 2.1) of the NLS equations,

\[ -i\partial_t u(t, x) + \Delta u(t, x) = |u(t, x)|^2 u(t, x) + W(t, b), \]

controlled via source terms $F = W$ and $F = V^\delta$, there holds

\[ \|E_T(V^\delta(b)) - E_T(W(b))\| \leq \delta, \quad \forall b \in B. \]

**Remark 5.1.** Note that controls (17) take their values in 'low-dimensional' space $F_{\hat{K}}$ in comparison with the 'high-dimensional' space $F_K$ - the range of controls (17). □

**Remark 5.2.** It suffices to prove the Lemma for $K$ being an elementary extension of $\hat{K}$, the rest being accomplished by induction. □

5.2. **Full-dimensional control.** Before proving that controllability can be achieved by means of low-dimensional controls we formulate general result for the case, where control is full-dimensional.

**Lemma 5.2** (full-dimensional control lemma). Controlled semi-linear equation

\[ -i\partial_t u(t, x) + \Delta u(t, x) = G(t, u) + \sum_{k \in \hat{K} = K^o} w_k(t) e^{ik \cdot x}, \quad u(0) = u^0, \]

with coinciding sets of controlled and observed modes $K^1 = K^o$, is time-$T$ solidly controllable for each $T > 0$ in observed $K^o$-component. □

**Proof of Lemma 5.2.** Without lack of generality assume the initial condition to be $u(0) = 0_H$. Take a ball $B$ in $F_{K^o} = \text{span}\{e^{ik \cdot x} | k \in K^o\}$. We will prove that $B$ is solidly attainable for the controlled equation (22).

Restrict (22) to an interval $[0, \varepsilon]$, where small $\varepsilon > 0$ will be specified later on. Proceed with time substitution $t = \varepsilon \tau$, $\tau \in [0, 1]$ under which (22) takes
form:

\[(23) \quad -i\partial_t u + \varepsilon \Delta u = \varepsilon G(t, u) + \varepsilon \sum_{k \in K^o} w_k(t)e^{ik\cdot x}, \quad u(0) = 0, \quad \tau \in [0, 1].\]

Fix $\gamma > 1$. For each $b \in \gamma B$, $b = (b_1, \ldots, b_N)$ consider control $w(\cdot; b) = -i\varepsilon^{-1}\sum_{k \in K^o} b_k e^{ik\cdot x}$. Substituting the control into (23) we get

\[-i\partial_t u + \varepsilon \Delta u = \varepsilon G(t, u) - i \sum_{k \in K^o} b_k e^{ik\cdot x}, \quad u(0) = 0, \quad \xi \in [0, 1].\]

For $\varepsilon = 0$ we get the 'limit equation'

\[(24) \quad \partial_t u = \sum_{k \in K^o} b_k e^{ik\cdot x}, \quad u(0) = 0, \quad \tau \in [0, 1].\]

Let $E^0_1$ be the time-1 end-point map of (24). In the basis $e^{ik\cdot x}$ of $H^{1+\sigma}$ it has form $(b_1, \ldots, b_N) \mapsto \sum_{k \in K^o} b_k e^{ik\cdot x}$.

Obviously the map $b \mapsto \Phi(b) = \Pi^0 \circ E^0_1(w(t; b))$, where $\Pi^0$ is orthogonal projection onto $\mathcal{L}^o$, coincides on $\gamma B \supset B$: with the identity map $\text{Id}_{\gamma B}$ and $(I - \Pi^0)E^0_1(w(t; b)) = 0$.

According to Lemma 4.5 for the continuous maps $\Phi^\varepsilon : b \mapsto E^\varepsilon_1(w(\cdot, b))$, where $E^\varepsilon_1$ are end-point maps of the control systems (23), there holds $\|\Phi^\varepsilon - \Phi^0\|_{C^0(B)} \to 0$ as $\varepsilon \to 0$.

By degree theory argument there exists $\varepsilon_0$ such that $\forall \varepsilon \leq \varepsilon_0$ the image of $(\Pi^0 \circ \Phi^\varepsilon)(\gamma B)$ covers $B$ solidly. □

**Remark 5.3.** In fact we only established controllability for small times $T > 0$. Still controllability in any time can be concluded by a standard trick of guiding the system from $u^0$ to the origin of $H^{1+\sigma}$ in small time $\delta > 0$, maintaining it at the origin under zero control for time length $T - 2\delta$ and then guiding it to preassigned $\dot{u}$ in time $\delta > 0$. □

**Remark 5.4.** From the proof of the Lemma it follows, that in addition to controllability one can arrange for each $\delta > 0$ a proper choice of controls, so that the estimate $\| (I - \Pi^0) (u(T) - u^0) \| \leq \delta$ will hold for the projection $I - \Pi^0 = \Pi^\perp$ onto orthogonal complement to $\mathcal{F}_{K^o}$. □

**Remark 5.5.** Without lack of generality we may assume, that $w(t, b)$ are smooth with respect to $t$ and that any finite number of derivatives $\frac{\partial^{k}w}{\partial t^{k}}(\cdot, b)$ depend continuously in $L^1$-metric on $b \in B$. Indeed smoothing $w(t, b)$ by convolution with a smooth $\varepsilon$-approximation $h_\varepsilon(t)$ of Dirac function $\delta(t)$, one gets a family of smooth controls $w^\varepsilon(t, b)$, which provides solid controllability, for small $\varepsilon > 0$. The continuous dependence in $L^1$-metric of $\frac{\partial^{k}w}{\partial t^{k}}(\cdot, b)$ on $b$ is verified directly. □

5.3. **Controllability in finite-dimensional component via extensions.**

The following result regarding controllability in observed component is a corollary of Lemmata 5.1,5.2.
Proposition 5.3. If a set of observed modes $\mathcal{K}^o$ is proper extension of a set of controlled modes $\mathcal{K}^1$, then NLS equation

$$-i\partial_t u + \Delta u = |u|^2 u + V(\cdot, b),$$

is solidly controllable in the observed $\mathcal{K}^o$-component. $\square$

Proof. Let $S$ be a compact subset of $\mathcal{F}_{\mathcal{K}^o} = \text{span}\{f_k | k \in \mathcal{K}^o\}$. According to Lemma 5.2 we can choose a family of $\mathcal{F}_{\mathcal{K}^o}$-valued controls $W(\cdot, b)$ which provides solid controllability. If $\delta > 0$ is small enough and family $V(t; b)$ satisfies conclusion of the Approximative Lemma 5.1, then $\Pi^o \circ E_T(V(t; b))$ covers $S$ solidly. $\square$

5.4. Proof of Approximative Lemma 5.1. According to the Remark 5.2 it suffices to treat the case where $\mathcal{K}$ is elementary extension of $\mathcal{K}$:

$$\mathcal{K} = \mathcal{K} \bigcup \{2r - s\}, \ r, s \in \mathcal{K}.$$

It is convenient to proceed with time-variant change of basis in $H^{1+\sigma}$, passing from the exponentials $e^{ik \cdot x}$ to the exponentials $f_k = e^{i(k \cdot x + |k|^2 t)}, \ k = (k_1, k_2) \in \mathbb{Z}^2$.

Therefore from now on we consider $\mathcal{F}_{\mathcal{K}}$-valued family of controls

$$b \mapsto W(t, b) = \sum_{k \in \mathcal{K}} w_k(t; b) f_k,$$

parameterized by $b \in B$ - compact in Euclidean space. We wish to construct family of controls $V(t; b) = \sum_{k \in \mathcal{K}} v_k(t; b) f_k$, whose range has one dimension less and which satisfy (21).

5.4.1. Substitution of variables. We will seek the family $b \mapsto V(t, b)$ in the form

$$V(t, b) = \tilde{W}(t, b) + \partial_t v_r(t, b) f_r + \partial_t v_s(t, b) f_s,$$

where $\tilde{W}(t, b)$, whose range is $\mathcal{F}_K$, and families of Lipschitzian functions $t \mapsto v_r(t, b), v_s(t, b)$ will be specified in the course of the proof. For some time we will omit dependence on $b$ in notation.

Feeding the controls (26) into the right-hand side of equation (11) we get

$$(-i\partial_t + \Delta) u = |u|^2 u + \tilde{W}(t) + \dot{v}_r(t) f_r + \dot{v}_s(t) f_s.$$

This equation can be given form

$$(-i\partial_t + \Delta) (u - iV_{rs}(t)) = |u|^2 u + \tilde{W}(t),$$

where $V_{rs}(t) = v_r(t) f_r + v_s(t) f_s$. We used the fact that $(-i\partial_t + \Delta) f_k = 0, \forall k \in \mathbb{Z}^2$.

By time-variant substitution

$$u^* = u - iV_{rs}(t),$$

**CONTROLLABILITY OF NLS VIA LOW-DIMENSIONAL SOURCE TERM**

13
we transform (28) into equation:

\[ (30) \quad (-i\partial_t + \Delta) u^* = |u^* + iV_{rs}(t)|^2(u^* + iV_{rs}(t)) + \tilde{W}(t) = |u^*|^2u^* - i(u^*)^2V_{rs} + 2i|u^*|^2V_{rs} - V_{rs}^2u^* + 2u^*|V_{rs}|^2 + i|V_{rs}|^2V_{rs} + \tilde{W}(t). \]

Imposing constraints

\[ (31) \quad v_r(0) = v_s(0) = 0, \quad v_r(T) = v_s(T) = 0, \]

we keep end-points unchanged under the substitution (29): \( u(0) = u^*(0), \) \( u(T) = u^*(T). \) Hence the end-point maps \( E_T \) for the controlled equations (27) and (30) coincide for those Lipschitzian controls \( v_r(t), v_s(t) \), which meet (31).

5.4.2. Fast oscillations and resonances. Now we put into game fast-oscillations, by choosing \( V_{rs}(t) \) in (29), (30) of the form

\[ (32) \quad V_{rs}(t) = v_r(t)f_r + v_s(t)f_s = e^{i(t/\varepsilon + \rho(t))}\tilde{v}_r(t)f_r + e^{i2t/\varepsilon}\tilde{v}_s(t)f_s, \]

where \( \tilde{v}_r(t), \tilde{v}_s(t), \rho(t) \) are Lipschitzian real-valued functions, which together with small \( \varepsilon > 0 \), will be specified in the course of the proof.

The terms at the right-hand side of (30), which contain \( V_{rs}, V_{rs}^r \), are to be classified as non-resonant and resonant with respect to the substitution (32). We call a term non-resonant if, after the substitution it results in a sum of fast-oscillating factors of the form \( p(u, V_{rs}, t)e^{i\beta t/\varepsilon}, \beta \neq 0 \), where \( p(u, V_{rs}, t) \) is polynomial in \( u, \tilde{u}, V_{rs}, V_{rs}^r \), with coefficients Lipschitzian in \( t \), independent of \( \varepsilon \). Otherwise, when no factor \( e^{i\beta t/\varepsilon} \) is present, the term is resonance. Crucial fact, which will be established below, is that influence of non-resonant (fast-oscillating) terms to the end-point map can be made arbitrarily small, when the frequency of the oscillating factor \( e^{i\beta t/\varepsilon} \) is sufficiently large.

Direct verification shows that the terms

\[ i(u^*)^2V_{rs}^r, \quad 2i|u^*|^2V_{rs}, \quad V_{rs}^2u^* \]

at the right-hand side of (30) are all non-resonant with respect to (32).

5.4.3. Resonance monomials in the quadratic term \( 2u^*|V_{rs}|^2 \): an obstruction. Consider the quadratic term \( 2u^*|V_{rs}|^2 \), which after the substitution (32) takes form

\[ 2u^*|V_{rs}|^2 = 2u^*(|\tilde{v}_r(t)|^2 + |\tilde{v}_s(t)|^2) + 4u^*\tilde{v}_r(t)\tilde{v}_s(t)Re \left( e^{-it/\varepsilon}e^{ip(t)}f_rf_s \right). \]

The last addend in the parenthesis is non-resonant, while the resonant term \( 2u^*(|\tilde{v}_r(t)|^2 + |\tilde{v}_s(t)|^2) \) is an example of so-called obstruction to controllability in terminology of geometric control.

We can not annihilate or compensate this term but, as far as the group \( e^{it\Delta} \) corresponding to linear Schrödinger equation is quasiperiodic, one can impose conditions on controls in such a way, that for a chosen \( T > 0 \) the influence of the obstructing term onto time-\( T \) end-point map \( E_T \) will be nullified.
Indeed, proceeding with time-variant substitution:

\[ u^* = u^* e^{-2i\Upsilon(t)}, \quad \Upsilon(t) = \int_0^t (|\ddot{v}_r(t)|^2 + |\ddot{v}_s(t)|^2) \, dt, \]

one gets for \( u^* \) the equality:

\[ (-i\partial_t + \Delta) u^* e^{2i\Upsilon(t)} = (-i\partial_t + \Delta) u^* - 2u^* (|\ddot{v}_r(t)|^2 + |\ddot{v}_s(t)|^2). \]

The equation (30) rewritten for \( u^* \) gets form

\[ (-i\partial_t + \Delta) u^* = |u^*|^2 u^* - i(u^*)^2 V_{rs} e^{2i\Upsilon(t)} + 2i|u^*|^2 V_{rs} e^{-2i\Upsilon(t)} - V_{rs}^2 u^* e^{-4i\Upsilon(t)} + 4u^* 2\text{Re} \left( e^{i(t/\varepsilon + \rho(t))} \dot{v}_r(t) \ddot{v}_s(t) \right) e^{-2i\Upsilon(t)} + \bar{W}(t) e^{-2i\Upsilon(t)} + i|V_{rs}|^2 V_{rs} e^{-2i\Upsilon(t)}. \]

For the sake of maintaining (for a given \( T > 0 \)) the time-\( T \) end-point map \( E_T \) unchanged, additional isoperimetric conditions on \( \ddot{v}_r(t), \ddot{v}_s(t) \)

\[ \int_0^T (|\ddot{v}_r(t)|^2 + |\ddot{v}_s(t)|^2) \, dt = \Upsilon(T) = \pi N, \quad N \in \mathbb{Z}, \]

could be imposed. The equality would imply \( u^*(0) = u^*(0), \; u^*(T) = u^*(T). \)

**Remark 5.6.** Although right-hand side of (34) gained ‘oscillating factors’ of the form \( e^{-4it\Upsilon(t)} \), the notion of resonant and resonant terms will not suffer changes, as long as \( e^{-2it\Upsilon(t)} \) is not ‘fast oscillating’; in further construction \( \Upsilon(t) \) will be chosen bounded uniformly in \( t \) and \( b \) with bounds independent of \( \varepsilon > 0. \)"
The equation (30) can be represented as

\[-i\partial_t + \Delta]u^* = |u^*|^2u^* + \tilde{W}(t)e^{-2i\Upsilon(t)} +
\]

\[+\tilde{v}_s^2(t)\tilde{v}_s(t)e^{2i(\rho(t) - |r-s|^2t - \Upsilon(t))}f_{2r-s} + \tilde{N}_\varepsilon(u, t).
\]

(39)

In Subsection 5.5 we will show that the influence of the fast-oscillating term \(\tilde{N}_\varepsilon(u, t)\) onto the end-point map can be made arbitrarily small by choice of small \(\varepsilon > 0\). By now we will take care of other addends at the right-hand side of (39). We wish to choose families of functions \(\tilde{W}(t; b), \tilde{v}_r(t; b), \tilde{v}_s(t; b)\) in such a way that

\[\tilde{W}(t)e^{-2i\Upsilon(t)} + \tilde{v}_s^2(t)\tilde{v}_s(t)e^{2i(\rho(t) - |r-s|^2t - \Upsilon(t))}f_{2r-s}
\]

approximates \(W(t; b)\) in \(L_1^1\)-metric uniformly in \(b \in B\).

Get family of controls \(\tilde{W}(t; b) = \sum_{k \in K} w_k(t; b)f_k\), by truncating the summand \(w_{2r-s}f_{2r-s}\) from \(W(t; b)\) (see (25)). We put \(\tilde{W}(t; b) = \tilde{W}(t; b)e^{2i\Upsilon(t; b)}\).

The controls \(\tilde{v}_r(t; b), \tilde{v}_s(t; b)\) will be constructed according to the

Lemma 5.4. For a continuous in \(L_1^1\)-metric family of controls \(b \mapsto w(t; b) \in L^\infty[0, T]\), and each \(\varepsilon' > 0\) one can construct continuous in \(L_1^1\)-metric families of real-valued functions

\[b \mapsto \tilde{v}_r(t; b, \varepsilon'), b \mapsto \tilde{v}_s(t; b, \varepsilon'),
\]

such that: i) they are Lipschitzian in \(t\); ii) their partial derivatives in \(t\) depend on \(b\) continuously in \(L_1^1\)-metric; iii) for each \(b, \varepsilon'\) the conditions (31), (35) hold for them; iv) their \(L_2^1\)-norms are equibounded for all \(\varepsilon' > 0, b \in B\); and v)

\[\|D_{\varepsilon'}\|_{L_1^1} = \int_0^T \|\tilde{v}_s^2(t; b, \varepsilon')\tilde{v}_s(t; b, \varepsilon') - |w_{2r-s}(t, b)|\| dt \leq \varepsilon'.
\]

(41)

uniformly in \(b \in B\).

The Lemma is proved in Appendix. Now we formulate a corollary, which defines the family \(b \mapsto \rho(t; b, \cdot)\).

Corollary 5.5. Given family (47), constructed in the Lemma, there exists a continuous in \(L_1^1\)-metric family of Lipschitzian functions \(b \mapsto \rho(\cdot; b)\) for which

\[\int_0^T |v_s^2(t; b, \varepsilon')\tilde{v}_s(t; b, \varepsilon')e^{2i(\rho(t) - |r-s|^2t - \Upsilon(t))} - w_{2r-s}(t, b)| dt \leq \varepsilon'.
\]

(42)

Recall that \(\Upsilon(t)\) is defined by (33).

To prove the Corollary we choose

\[\rho(t; b) = \frac{1}{2}{\rm Arg}(w_{2r-s}(t, b)) + |r - s|^2t + \Upsilon(t; b).
\]

(43)

According to Remark 5.5 we may think that \(w_{2r-s}(t, b)\) are smooth in \(t\) and hence \(\rho(t; b)\) is Lipschitzian in \(t\). Its dependence on \(b\) is continuous in \(L_1^1\)-metric. By (41), (43) we conclude (42).
Taking $\varepsilon' = \varepsilon$ and substituting the constructed controls $v_r, v_s, \tilde{W}$ into (36) we get the equation

$$(-i\partial_t + \Delta)u^* = |u^*|^2 u^* + W(t) + D^v_r(t) + \tilde{N}^\varepsilon(u^*, t, b).$$

By construction the end-point maps $\tilde{E}_T$ and $E_T$ of the systems (44) and (27) coincide on the set of controls, satisfying (31), (32), (35).

**Lemma 5.6.** The end-point map $E^\varepsilon_T(b)$ of the system (44) calculated for the family of controls, defined by Proposition 5.4, tends to the end-point map $E_{lim}$ of the ‘limit system’ (19) uniformly in $b$ as $\varepsilon \to 0$. □

Would the term $\tilde{N}^\varepsilon(u^*, t, b)$ be missing in (44) we could derive Lemma 5.6 from Proposition 4.4. The passage to limit, as $\varepsilon \to 0$, in the presence of fast-oscillating $\tilde{N}^\varepsilon(t, u)$ tends to 0, will be established in Proposition 5.7.

The proof of Approximative Lemma 5.1 is complete modulo proof of Lemmas 5.4, 5.6.

### 5.5. On continuity of solutions in the right-hand side with respect to relaxation metric.

The results, we are going to present briefly in this Section, regard continuous dependence of solutions of NLS equation on the perturbations of its right-hand side, which are small in so-called relaxation norm. This norm is suitable for treating fast oscillating terms. In finite-dimensional context the continuity results are part of theory of relaxed controls. A number of relaxation results for semilinear systems in Banach spaces can be found in [8, 9]. Below we provide version adapted for our goal - proof of Lemma 5.6.

Consider semilinear equation (3) and its perturbation (4).

We assume the perturbations $\phi : [0, T] \times H \to H$ to belong to a family $\Phi$. Elements of $\Phi$ are continuous; the family $\Phi$ is equibounded and equi-Lipschitzian meaning that each $\phi \in \Phi$ together with $G : [0, T] \times H \to H$ satisfy properties (5), (6), (7) with the same function $\beta_0(t)$.

Besides we admit complete boundedness assumption, which would guarantee the complete boundedness (precompactness) in $H$ of the set $\{\phi(t, u(t)) | t \in [0, T], \phi \in \Phi\}$ for each choice of $u(\cdot) \in C([0, T], H)$. To get the property it suffices, for example, to assume complete boundedness of the sets $\Phi(t, u) = \{\phi(t, u) | \phi \in \Phi\}$ for each fixed couple $(t, u)$ together with upper semicontinuity of the set valued map $(t, u) \mapsto \Phi(t, u)$.

We introduce relaxation seminorm $\| \cdot \|^{Rx}_{b}$ for the elements of $\Phi$ by the formula:

$$\| \phi \|^{Rx}_{b} = \sup_{t, t' \in [0, T], \|u\| \leq b} \left\| \int_{t}^{t'} \phi(\tau, u) d\tau \right\|_H.$$  

The seminorm is well adapted to the functions oscillating in time. The relaxation seminorms of fast-oscillating functions are small. For example $\|f(t) e^{it/\varepsilon}\|^{Rx}_{b} \to 0$, as $\varepsilon \to 0$ for each function $f \in L^1[0, T]$ (Lebesgue-Riemann lemma).
Now we formulate needed continuity result from which Lemma 5.6 will follow.

**Proposition 5.7.** Let solution $\tilde{u}(t)$ of the NLS equation (3) exist on $[0,T]$, belong to $C([0,T],H)$ and satisfy $\sup_{t \in [0,T]} \|u(t)\|_H < b$. Let family $\Phi$ of perturbations satisfy the continuity, equiboundedness, equi-Lipschitzianness and complete boundedness assumption, introduced above. Then $\forall \varepsilon > 0 \exists \delta > 0$ such that whenever $\phi \in \Phi$, $\|\phi\|^x + \|u^0 - \tilde{u}^0\|_H < \delta$, then the solution $u(t)$ of the perturbed equation (4) exists on the interval $[0,T]$, is unique and satisfies the bound $\sup_{t \in [0,T]} \|u(t) - \tilde{u}(t)\|_H < \varepsilon$. □

**Sketch of the proof.** Under the assumptions of the Proposition solutions of the equations (3), (4) exist locally and are unique (see Proposition 4.1). Global existence will follow from the bound on the $H^{1+\sigma}$-norm of the solution on $[0,T]$.

We start with the estimate (13) obtained in the course of the proof of Proposition 4.4

$$\|u(t) - \tilde{u}(t)\| \leq \left( \|u^0 - \tilde{u}^0\| + \left\| \int_0^t e^{-i\tau \Delta} \phi(\tau, \tilde{u}(\tau)) d\tau \right\| \right) C' e^{C \int_0^T \rho_b(\tau) d\tau}.$$ \( □ \)

The conclusion of Proposition 5.7 will follow from

**Lemma 5.8.** Let family $\Phi$ satisfy assumptions of the Proposition 5.7, and let $\tilde{u}(t)$ be solution of (3). Then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall \phi \in \Phi$: \( \|\phi\|^x < \delta \Rightarrow \left\| \int_0^t e^{-i\tau \Delta} \phi(\tau, \tilde{u}(\tau)) d\tau \right\| < \varepsilon. □ \)

Proof of this Lemma can be found in Appendix. We finish by remark on validity of conditions of Proposition 5.7 for NLS.

**Remark 5.7.** The nonlinear terms $\mathcal{N}^\varepsilon(u,t)$ at the right-hand side of (44) is Nemytskii-type operator of the form

$$\mathcal{N}^\varepsilon(u,t) = W^0(t,x) + uW^{11}(t,x) + uW^{12}(t,x) + u^2W^{21}(t,x) + |u|^2W^{22}(t,x),$$

where $W^{ij}(t,x)$ have form $w(t)e^{i(k visibly continued...
Definition 6.1. A finite set $\hat{K} \subset \mathbb{Z}^2$ of modes is called saturating if $K^\infty = \mathbb{Z}^2$. □

From Proposition 5.3 we conclude

Proposition 6.2. Let set $\hat{K}$ of controlled modes, involved in the source term (2), be saturating. Then for each $T > 0$ the controlled NLS equation (1)-(2) on $\mathbb{T}^2$ is time-$T$ solidly controllable in each finite-dimensional component. □

As we will see in the next section controllability in each finite-dimensional component (in projection on each coordinate subspace) implies controllability in projection on each finite-dimensional subspace and also approximate controllability.

Corollary 6.3. Let the set $\hat{K}$ of controlled modes be saturating. Then for any $T > 0$ the controlled defocusing NLS equation (1)-(2) on $\mathbb{T}^2$ is time-$T$ solidly controllable in each finite-dimensional projection and $H^{1+\sigma}$-approximately controllable. □

Now we introduce a class of saturating sets.

Proposition 6.4. Let vectors $k, \ell \in \mathbb{Z}^2$ be such that $k \wedge \ell = \pm 1$. Then the set $\{0, k, \ell, k + \ell\} \subset \mathbb{Z}^2$ is saturating. □

Proof. i) First note that if $z \in K^\infty$, then $-z = 2 \cdot 0 - z \in K^\infty$.

We prove that $K^\infty$ coincides with the set of all integer combinations $C = \{\alpha k + \beta \ell | \alpha, \beta \in \mathbb{Z}\}$.

ii) The set $C$ is obviously invariant with respect to the operation $(v, w) \mapsto v - 2w$. We will prove that $K^\infty \supset C$.

If $\pm z \in K^\infty$, then by induction $z + 2\omega - 2\beta \ell \in K^\infty, \forall \omega, \beta \in \mathbb{Z}$. In particular

$2\omega k + 2\beta \ell \in K^\infty, \forall \omega, \beta \in \mathbb{Z}$ and $k + 2\omega k + 2\beta \ell, \ell + 2\omega k + 2\beta \ell \in K^\infty$.

Thus $K^\infty$ contains all the combinations $mv + nw$ with at least one of the coefficients $m, n$ even. Note that the set of such combinations is invariant with respect to the operation $(x, y) \mapsto 2x - y$ involved in (25) and $0, k, \ell$ all are "combinations" of this type.

iii) "Invoking" $k + \ell \in \hat{K}$ we conclude by ii) that $\forall \alpha, \beta \in \mathbb{Z}$:

$$(2\alpha + 1)k + (2\beta + 1)\ell = (k + \ell) + 2\alpha k + 2\beta \ell \in K^\infty.$$  

2) Now we prove that whenever $k \wedge \ell = \pm 1$, then the set $\{\alpha k + \beta \ell | \alpha, \beta \in \mathbb{Z}\}$ coincides with $\mathbb{Z}^2$.

Assume $k \wedge \ell = 1$. Take any vector $y \in \mathbb{Z}^2$. Set $\alpha = y \wedge \ell, -\beta = y \wedge k$; obviously $\alpha, \beta$ are integer. We claim that $\alpha k + \beta \ell = y$.

By direct computation

$$(\alpha k + \beta \ell) \wedge \ell = \alpha (k \wedge \ell) = \alpha, (\alpha k + \beta \ell) \wedge k = \beta (\ell \wedge k) = -\beta.$$  

Then $(y - (\alpha k + \beta \ell)) \wedge \ell = 0$, $(y - (\alpha k + \beta \ell)) \wedge k = 0$. As far as $k, \ell$ are linearly independent, we conclude $y - (\alpha k + \beta \ell) = 0$. □
Corollary 6.5. Let vectors \( k, \ell \in \mathbb{Z}^2 \) be such that \( k \wedge \ell = \pm 1 \) and the controlled source term \( \Xi \) of the NLS equation \( (\mathfrak{1}) \) be of the form

\[
v_0(t) + v_k(t)e^{ikx} + v_\ell(t)e^{i\ell x} + v_{k+\ell}(t)e^{i(k+\ell)x}
\]

Then for any \( T > 0 \) the NLS equation \( (\mathfrak{1}) \) is time-\( T \) controllable in each finite-dimensional projection and \( H^{1+\sigma} \)-approximately controllable. \( \square \)

The space of controlled modes, introduced in Remark 2.2, satisfies hypotheses of the Corollary for \( k = (1,0), \ell = (0,1) \).

7. Controllability proofs (Main result 1)

7.1. Approximate controllability. We have established that whenever set of controlled modes is saturating, then NLS is solidly controllable in projection on any finite-dimensional coordinate subspace. Using this fact we will now prove \( H^{1+\sigma} \)-approximate controllability and controllability in each finite-dimensional projection.

Let us fix \( \tilde{\varphi}, \varphi \in H_2 \) and \( \varepsilon > 0 \) and assume that we want to steer the NLS equation from \( \tilde{\varphi} \) to the \( \varepsilon \)-neighborhood of \( \varphi \) in \( H_2 \)-metric.

Consider the Fourier expansions for \( \tilde{\varphi}, \varphi \) with respect to \( e^{ikx}, k \in \mathbb{Z}^2 \). Denote by \( \Pi_N \) the projection of \( \varphi \in H^{1+\sigma} \) onto the space of modes \( e^{ikx}, |k| \leq N \). Obviously \( \Pi_N(\tilde{\varphi}) \to \tilde{\varphi}, \Pi_N(\varphi) \to \varphi \) in \( H_0 \) as \( N \to \infty \).

Choose such \( N \) that the \( H^{1+\sigma} \)-norms of \( \Pi_N^\perp(\tilde{\varphi}) = -\Pi_N(\tilde{\varphi}) + \tilde{\varphi}, \Pi_N^\perp(\varphi) = -\Pi_N(\varphi) + \varphi \) are \( \leq \varepsilon/4 \).

By Lemma 5.2 there exists family of controls \( W(b) = \sum_{\|k\| \leq N} w_k(t; b)f_k \) such that \( \Pi_N(W(b)) \) covers \( \Pi_N(\tilde{\varphi}) \) solidly and besides \( \|\Pi_N^\perp E_T(W(b)) - \Pi_N^\perp(\tilde{\varphi})\| \leq \varepsilon/4 \). Then \( \|\Pi_N^\perp E_T(W(b))\| \leq \varepsilon/2 \).

If a set \( \mathcal{K} \) of controlled modes is saturating, then \( \{k| |k| \leq N\} \) is proper extension of \( \mathcal{K} \). By Approximative Lemma 5.3 there exists family of controls \( V(b) = \sum_{k \in \mathcal{K}} v_k(t; b)f_k \) such that

\[
\|E_T(V(b)) - E_T(W(b))\| \leq \varepsilon/4, \forall b \in B,
\]

and \( \Pi_N E_T(V(b)) \) covers the point \( \Pi_N(\tilde{\varphi}) \). Then \( \forall b : \|\Pi_N^\perp E_T(V(b))\| \leq 3\varepsilon/4 \) and for some \( \hat{b} \) : \( \Pi_N E_T(V(\hat{b})) = \Pi_N \tilde{\varphi} \). Then \( \|E_T(V(\hat{b})) - \tilde{\varphi}\| \leq \varepsilon \). \( \square \)

7.2. Controllability in finite-dimensional projections. Let \( \mathcal{L} \) be \( \ell \)-dimensional subspace of \( H^{1+\sigma} \) and \( \Pi^\mathcal{L} \) be orthogonal projection of \( H^{1+\sigma} \) onto \( \mathcal{L} \).

First we construct a finite-dimensional coordinate subspace which is projected by \( \Pi^\mathcal{L} \) onto \( \mathcal{L} \). Moreover for each \( \varepsilon > 0 \) one can find a finite-dimensional coordinate subspace \( \mathcal{L}_\varepsilon \) with its \( \ell \)-dimensional (non-coordinate) subspace \( \mathcal{L}_\varepsilon^\perp \), which is \( \varepsilon \)-close to \( \mathcal{L} \). The latter means that not only \( \Pi^\mathcal{L} \mathcal{L}_\varepsilon = \mathcal{L} \) but also the isomorphism \( \Pi_\varepsilon = \Pi^\mathcal{L}|_{\mathcal{L}_\varepsilon} \) is \( \varepsilon \)-close to the identity operator. It is an easy linear-algebraic computation; which can be found in [\( \text{II} \) Section 7].

Without lack of generality we may assume that \( \|\Pi_S(\tilde{\varphi}) - \tilde{\varphi}\|_0 \leq \varepsilon \).
As far as the set $\hat{K}$ of controlled modes is saturating, $\mathcal{S}$ is proper extension of $\hat{K}$ and the system is solidly controllable in the observed component $q^S$.

Let $B$ be a ball in $L$. Consider $B^\varepsilon = (\Pi_L)^{-1}B$; obviously $B^\varepsilon \subset L^\varepsilon \subset L^S$. We take a ball $B_S$ in $L^S$, which contains $B^\varepsilon$ and hence $\Pi_L(B_S) \supset B$.

Reasoning as in the previous Subsection one establishes existence of a family of controls $V(b) = \sum_{k \in \hat{K}} v_k(t;b)f_k$ such that $\Pi_S E_T(V(b))$ covers $B_S$ solidly and $\forall b: \|\Pi_S^\perp E_T(V(b))\| \leq 2\varepsilon$.

Then choosing $\varepsilon > 0$ sufficiently small we achieve that

$$\Pi^\varepsilon E_T(V(b)) = \Pi^\varepsilon \left( \Pi_S + \Pi_S^\perp \right) E_T(\hat{V}(b))$$

covers $B$.

8. LACK OF EXACT CONTROLLABILITY PROOF (MAIN RESULT 2)

Let us write cubic defocusing NLS equation (11)-(22) in the form

$$(-i\partial_t + \Delta)u = |u|^2u + \sum_{k \in \hat{K}} \hat{w}_k(t) f_k, \quad u|_{t=0} = u^0 \in H^{1+\sigma},$$

where $f_k = e^{i(k \cdot x + |k|^2t)}$, $\hat{K} \subset \mathbb{Z}^2$ is a finite set, $\hat{K} = \kappa$. Controls $\hat{w}_k(t)$ are taken from $L^1([0,T], \mathbb{C})$ and therefore are derivatives of absolutely continuous functions $w_k(t)$, $w_k(0) = 0$. In this Section $W^{1,1}([0,T], \mathbb{C}^\kappa)$ stays for the space of $\mathbb{C}^\kappa$-valued absolutely continuous functions, vanishing at $t = 0$.

Global existence and uniqueness results for solution of this equation in $C([0,T], H^{1+\sigma})$ is classical (Section 4).

Consider the end-point map $E_T: (\hat{w}_k(t)) \mapsto u|_{t=T}$ which maps the space of inputs $(\hat{w}_k(t)) \in L^1([0,T], \mathbb{C}^\kappa)$ into the state space $H^{1+\sigma}$. The image of $E_T$ is time-$T$ attainable set of the controlled equation (46). We wish to prove that this set is contained in a countable union of compacts and in particular has a dense complement in $H^{1+\sigma}$.

Introducing $W(t,x) = \sum_{k \in \hat{K}} w_k(t)f_k$, we rewrite (see Subsection 5.3) the equation (46) as $(-i\partial_t + \Delta)(u - iW(t,x)) = |u|^2u$, and after time-variant substitution $u - iW(t,x) = u^*(t)$ in the form

$$(47) \quad (-i\partial_t + \Delta)u^* = |u^*|^2u^* + iW(t,x)^2(u^* + iW(t,x)), \quad u|_{t=0} = u^0,$$

which we look at as semilinear control system with the input $W(t)$. Obviously for each absolutely continuous $W(t) = (w_k(t))$, $k \in \hat{K}$ solution of (47) exists and is unique on $[0,T]$.

Introduce input-trajectory map $E^*: W(\cdot) \mapsto u^*(\cdot)$ of (47). The following result is essentially a corollary of Proposition 4.4.

Lemma 8.1. Input-trajectory map $E^*$ is Lipschitzian on any ball $B_R = \{ W(\cdot) \in W^{1,1}([0,T], \mathbb{C}^\kappa) \mid \|W(\cdot)\|_{W^{1,1}} \leq R \}$, endowed with $L^1([0,T], \mathbb{C}^\kappa)$-metric, while the space of trajectories $u^*(\cdot)$ is endowed with $C([0,T], H^{1+\sigma})$-metric. In other words

$$\exists L_R > 0: \|u^*_2(t) - u^*_1(t)\|_H \leq L_R \int_0^T \|W_2(t) - W_1(t)\|_{\mathbb{C}^\kappa} dt, \quad \forall t \in [0,T],$$

where $\|u\|_H = \|u\|_{H^{1+\sigma}}$. 


From Lemma 5.4 proved in Appendix, Main Result 2 can be deduced easily.

Consider composition of maps

\[(\dot{w}_k)_{k \in K} \rightarrow W(\cdot) = (w_k)_{k \in K} \mapsto E_T^*(W) = E^*(W)|_{t = T};\]

\(E_T^*\) is the end-point map \(W(\cdot) \mapsto u|_{t = T}\) for the equation (47).

The relation between the end-point maps of the controlled equations (46) and (47) results \(E_T((\dot{w}) = E_T^*(W) + iW(T, x)\) and therefore the image of \(E_T\) (the attainable set) is contained in the image of the map

\[\Theta : (W(\cdot), \vartheta) \mapsto E_T^*(W(\cdot)) + \vartheta, \ (W(\cdot), \vartheta) \in W^{1,1}([0, T], \mathbb{C}^\kappa) \times \mathbb{C}^\kappa.\]

Represent \(L^1([0, T], \mathbb{C}^\kappa)\) as a union of balls \(\bigcup_{n \geq 1} B_n\) of radii \(n \in \mathbb{N}\). The image of each \(B_n\) under the map \(I : \dot{w}(\cdot) \mapsto (w(\cdot), w(T))\) is bounded in \(W^{1,1}([0, T], \mathbb{C}^\kappa) \times \mathbb{C}^\kappa\). If one endows \(W^{1,1}([0, T], \mathbb{C}^\kappa) \times \mathbb{C}^\kappa\) with the metric of \(L^1([0, T], \mathbb{C}^\kappa) \times \mathbb{C}^\kappa\) then \(I(B_n)\) is pre-compact (and completely bounded) in this metric.

By Lemma 5.1 the map \(E_T^*\) is Lipschitzian in the metric of \(L^1([0, T], \mathbb{C}^\kappa)\); hence \(\Theta\) is also Lipschitzian in the metric of \(L^1([0, T], \mathbb{C}^\kappa) \times \mathbb{C}^\kappa\) and therefore \(E_T(B_n)\) is contained in completely bounded image \(\Theta(I(B_n)) \subset H^{1+\sigma}\). Hence the attainable set of (46) is contained in a countable union of pre-compacts \(\bigcup_{n \geq 1} \Theta(I(B_n))\) and by Baire category theorem has a dense complement in \(H^{1+\sigma}\). \(\square\)

9. Appendix: Proofs of Lemmas 5.4, 5.8, 8.1

9.1. Proof of Lemma 5.4. First we choose \(\tilde{v}_\varepsilon^x(t)\) coinciding with real-valued nonnegative continuous piecewise-linear function, which vanishes at \(\{0, T\}\), is constant and equal \(\pi(T - \varepsilon^2)^{-1}\) on \([\varepsilon^2, T - \varepsilon^2]\) and is linear on \([0, \varepsilon^2]\) and \([T - \varepsilon^2, T]\). Evidently \(\int_0^T \tilde{v}_\varepsilon^x(t) dt = \pi\).

According to Remark 5.5 we may assume \(w_{2r-s}(t, b), \partial_t w_{2r-s}(t, b)\) to be smooth in \(t\) and depend on \(b\) continuously in \(L^1_t\)-metric. This implies that \(\|w_{2r-s}(t, b)\|_{L^\infty}\) are equibounded by \(C_w > 0\).

Denote \(I_\varepsilon = [0, \varepsilon^2] \bigcup [T - \varepsilon^2, T]\) and \(w^\varepsilon(t, b)\) the restrictions of \(w_{2r-s}(t, b)\) onto the interval \([0, T] \setminus I_\varepsilon\). Let

\[\int_{\varepsilon^2}^{T-\varepsilon^2} |w^\varepsilon(t, b)|^2 dt = A(b), \ A = \max_{b \in B} A(b);\]

the maximum is achieved. Put \(N = \lceil A/\pi \rceil + 1\) and extend \(|w^\varepsilon(t, b)|\) to a Lipschitzian function \(\tilde{v}_s(t, b)\) on \([0, T]\) in such a way that \(\tilde{v}_s(0, b) = \tilde{v}_s(T, b) = \)
0 and \( \int_0^T |\tilde{v}_s(t, b)|^2 dt = \pi N \)\footnote{One can take for example \( \tilde{v}_s(t, b) = \varepsilon^{-2} \sqrt{a_1(b)t} + a_2(b)\varepsilon^{-2}t^2 \) on \([0, \varepsilon^2]\). Parameters \( a_1(b), a_2(b) \) can be chosen continuously depending on \( b \). Similar construction can be arranged for the interval \([T - \varepsilon^2, T]\).} Then
\[
\int_0^T |\tilde{v}_r(t)|^2 + |\tilde{v}_s(t, b)|^2 dt = \pi(N + 1).
\]

Obviously \( \tilde{v}_r^2(t; b)|v_s(t; b) = |w_{2r-s}(t, b)| \) on \([\varepsilon^2, T - \varepsilon^2]\). Also
\[
\int_{I_s} |\tilde{v}_s(t)|^2 dt \leq \pi N,
\]
and by Cauchy-Schwarz inequality
\[
\int_{I_s} |\tilde{v}_s(t; b)| dt \leq \varepsilon \sqrt{2\pi N}.
\]

Then
\[
\int_{I_s} (|\tilde{v}_r^2(t)\tilde{v}_s(t, b)| + |w_{2r-s}(t; b)|) dt \leq \|\tilde{v}_r^2(t)\|_{L^\infty} \varepsilon \sqrt{2\pi N} + 2C \varepsilon^2. \Box
\]

9.2. **Proof of Lemma [5.8]**. Given that \( \tilde{u}(t) \) is continuous and \( \phi \) possesses Lipschitzian property, we can conclude that \( \forall \delta > 0, \exists \delta' > 0 \) such that \( \forall \phi \in \Phi \):

\[
(48) \sup_{t, t' \in [0, T], \|u\| \leq b} \left\| \int_t^{t'} \phi(\tau, \tilde{u}(\tau)) d\tau \right\| < \delta' \Leftrightarrow \sup_{t, t' \in [0, T]} \left\| \int_t^{t'} \phi(\tau, \tilde{u}(\tau)) d\tau \right\| < \delta.
\]

Indeed (compare with [10, Chap.4]) if \( \omega(\tau) \) is modulus of continuity for \( \tilde{u}(t) \) and \( \sup_{t \in [0, T]} \|\tilde{u}(t)\| \leq b \), then

\[
\sup_{t, t' \in [0, T]} \left\| \int_t^{t'} \phi(\tau, \tilde{u}(\tau)) d\tau \right\| = \left\| \int_t^{t'} \phi(\tau, \tilde{u}(\tau)) d\tau \right\| \leq \sum_{j=0}^{N-1} \left\| \int_{t_j}^{t_{j+1}} \phi(\tau, \tilde{u}(\tau)) d\tau \right\|,
\]

where \( t = t_0 < t_1 < \cdots < t_N = t' \) is a partition of \([t, t'] \subset [0, T]\) into \( N \leq T/\eta \) subintervals of length \( \eta \). Then

\[
\sup_{t, t' \in [0, T]} \left\| \int_t^{t'} \phi(\tau, \tilde{u}(\tau)) d\tau \right\| \leq \sum_{j=0}^{N-1} \left\| \int_{t_j}^{t_{j+1}} (\phi(\tau, \tilde{u}(\tau)) - \phi(\tau, \tilde{u}(t_j))) d\tau \right\| +
\]

\[
+ \sum_{j=0}^{N-1} \left\| \int_{t_j}^{t_{j+1}} \phi(\tau, \tilde{u}(t_j)) d\tau \right\| \leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \beta(\tau)\|\tilde{u}(\tau)\| d\tau + N\|\phi\|_{\text{rx}} \leq
\]

\[
\leq C \omega(\eta) + \frac{T}{\eta}\|\phi\|_{\text{rx}}.
\]
Choosing $\eta = \|\phi\|_{r_x}^{1/2}$ we get

$$
\sup_{t, t' \in [0, T]} \left\| \int_t^{t'} \phi(\tau, \tilde{u}(\tau)) d\tau \right\| \leq C\omega(\|\phi\|_{r_x}^{1/2}) + T\|\phi\|_{r_x}^{1/2}
$$

and conclude (48).

Introduce

$$\tilde{\Phi} = \{\phi(\tau, \tilde{u}(\tau)), \phi \in \Phi\}.$$

According to the aforesaid it suffices to prove the assertion

$$
(49) \quad \varphi \in \tilde{\Phi} \land \|\varphi\|_{r_x} < \delta \Rightarrow \left\| \int_0^t e^{-i\tau \Delta} \varphi(\tau) d\tau \right\| < \varepsilon.
$$

The set $R = \{\varphi(\tau) \mid \tau \in [0, T], \varphi \in \tilde{\Phi}\}$ is completely bounded by assumption.

Taking an orthonormal basis $h_1, h_2, \ldots, h_n, \ldots$ in $H$ and denoting by $\Pi_n$ the orthogonal projection onto $\text{Span}\{h_1, h_2, \ldots, h_n\}$, we conclude by standard compactness criterion that $\sup_{x \in F} \|x - \Pi_n x\| \to 0$, as $n \to \infty$.

Take a partition $0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$ of the interval $[0, T]$ into subintervals of lengths $\eta = T/N$. We represent the integral in (49) as a sum

$$
\int_0^t e^{-i\tau \Delta} \varphi(\tau) d\tau = \int_0^t e^{-i\tau \Delta} (\varphi(\tau) - \Pi_n \varphi(\tau)) d\tau +
\sum_{j=1}^{\omega} e^{-i\tau_j \Delta} \int_{\tau_{j-1}}^{\tau_j} \Pi_n \varphi(\tau) d\tau + \sum_{j=0}^{N-1} \int_{\tau_{j-1}}^{\tau_j} e^{-i\tau_j \Delta} (e^{-i(\tau - \tau_j) \Delta} - I) \Pi_n \varphi(\tau) d\tau.
$$

Recalling that:

i: $e^{-i\tau \Delta}$ is an isometry of $H$;

ii: $\left\| \int_{\tau_{j-1}}^{\tau_j} \Pi_n \varphi(\tau) d\tau \right\| \leq \|\varphi\|_{r_x}$;

iii: $\| (\varphi(\tau) - \Pi_n \varphi(\tau)) \| \leq \rho_n$, $\rho_n \xrightarrow{n \to \infty} 0$ uniformly for $\varphi \in \tilde{\Phi}$, $\tau \in [0, T]$;

iv: $\sup_{0 \leq \xi \leq \tau} (e^{-i\xi \Delta} - I) \circ \Pi_n = \gamma_n(\tau)$, $\forall n$ : $\lim_{\tau \to 0} \gamma_n(\tau) = 0$,

we conclude

$$
(50) \quad \left\| \int_0^t e^{-i\tau \Delta} \varphi(\tau) d\tau \right\| \leq T\rho_n + T\eta^{-1}\|\varphi\|_{r_x} + \gamma_n(\eta) \int_0^T \|\varphi(\tau)\| d\tau.
$$

Recall that $\int_0^T \|\varphi(\tau)\| d\tau$ are bounded by a constant $c_1$ for all $\varphi \in \tilde{\Phi}$.

Taking $n$ large enough so that $T\rho_n < \varepsilon/3$, we then choose $\eta > 0$ small enough so that $c_1\gamma_n(\eta) < \varepsilon/3$. If we impose $\|\varphi\|_{r_x} < \varepsilon\eta/3T$, then (50) will imply $\left\| \int_0^t e^{-i\tau \Delta} \varphi(\tau) d\tau \right\| < \varepsilon$. $\square$
9.3. Proof of Lemma 8.1. By the inequalities (13)-(14) we get
\[ \|u_2(t) - u_1(t)\|_H \leq C \int_0^T \|\Phi_{12}(\tau, u_1^*(\tau))\|_H d\tau e^{C' \int_0^T \beta_b(\tau)d\tau}, \]
where \( \Phi_{12}(\tau, u) \) and \( \beta_b(t) \) are defined by
\[
\Phi_{12}(\tau, u) = |u + iW_1(t, x)|^2 (u + iW_1(t, x)) - |u + iW_2(t, x)|^2 (u + iW_2(t, x)),
\]
\[
(51) |u' + iW(t, x)|^2 (u' + iW(t, x)) - |u + iW(t, x)|^2 (u + iW(t, x)) |H \leq \\
\leq \beta_b(t) \|u' - u\|_H, \forall W(\cdot) \in B_R, \|u\|_H \leq b.
\]

What regards \( \beta_b(t) \), then by Product Lemma 4.2 the left-hand side of (51) is bounded from above by \( C(1 + b^2 + \|W(t)\|^2_H) \). Hence \( \beta_b \) can be chosen constant, equal to \( C'(1 + b^2 + R^2) \), as far as \( W(t, x) \) are trigonometric polynomials in \( x \) with \( t \)-variant coefficients equibounded in \( W^{1,1}[0, T] \).

Similarly
\[ \|\Phi_{12}(\tau, u_1^*(\tau))\|_H \leq C_1 (1 + b^2 + R^2) \|W_2(\tau) - W_1(\tau)\|_{C^\infty}. \]

Then for \( L_R = CC_1 (1 + b^2 + R^2)e^{C'(1+b^2+R^2)T} \):
\[ \|u_2(t) - u_1^*(t)\|_H \leq L_R \int_0^T \|W_2(\tau) - W_1(\tau)\|d\tau. \]

\[ \square \]

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CONTROLLABILITY OF THE CUBIC SCHROEDINGER EQUATION VIA A LOW-DIMENSIONAL SOURCE TERM

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Abstract. We study controllability of \(d\)-dimensional defocusing cubic
Schroedinger equation under periodic boundary conditions. The control
is applied additively, via a source term, which is a linear combination of
few complex exponentials (modes) with time-variant coefficients - con-
trols. We manage to prove that controlling at most \(2^d\) modes one can
achieve controllability of the equation in any finite-dimensional projec-
tion of the evolution space \(H^s(\mathbb{T}^d)\), \(s > d/2\), as well as approximate con-
trollability in \(H^s(\mathbb{T}^d)\). We also present negative result regarding exact
controllability of cubic Schroedinger equation via a finite-dimensional
source term.

Keywords: semilinear Schroedinger equation, approximate controlla-
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1. Introduction

Lie algebraic approach of geometric control theory to the nonlinear dis-
tributed systems has been initiated recently. An example of its implemen-
tation is the study of 2-dimensional Navier-Stokes/Euler equations of fluid
motion controlled by low-dimensional forcing in \[1, 2\], where one arranged
sufficient criteria for approximate controllability and for controllability in
finite-dimensional projections of the evolution space.

Here we wish to develop similar approach for another class of distributed
system, which is cubic defocusing Schroedinger equation (cubic defocusing
NLS):

\[
- \partial_t u(t, x) + \Delta u(t, x) = |u(t, x)|^2 u(t, x) + V(t, x), \quad u|_{t=0} = u^0,
\]

controlled via a source term \(V(t, x)\).

We treat the periodic case: the space variable \(x\) belongs to the torus \(\mathbb{T}^d\).

Our problem setting is distinguished by two features. First, the control is
introduced via a source term, i.e. in additive form, on the contrast to the
bilinear form, characteristic for quantum control models. Second feature is
finite-dimensionality of the range of the controlled source term:

\[
V(t, x) = \sum_{k \in \mathcal{K}} v_k(t) e^{ik \cdot x},
\]
with $\hat{K} \subset \mathbb{Z}^d$ being a finite set. Obviously for each $t$ the value $V(t, \cdot)$ belongs to a finite-dimensional subspace $F_{\hat{K}} = \text{Span}\{e^{ik \cdot x}, \ k \in \hat{K}\}$ of the evolution space for NLS.

The control functions $v_k(\cdot), \ k \in \hat{K}$, which enter the source term, can be chosen freely in $L^\infty[0, T]$, or in any functional space, which is dense in $L^1[0, T]$.

By the choice of 'low-dimensional' control our problem setting differs from the studies of controllability of NLS (see the end of the Section 3 for few references to the alternative settings and approaches), in which controls have infinite-dimensional range. In some of the studies the controls have their support on a subdomain and one is interested in tracing the propagation of the controlled energy to other parts of the domain. On the contrast, in our case the controls affect few modes - 'directions' - in the functional evolution space for NLS and we are interested in the way this controlled action spreads to other (higher) modes.

We will treat NLS equation (1) as an evolution equation in Sobolev space $H^s(\mathbb{T}^d), \ s > d/2$. By opting for 'higher regularity' of solutions with respect to the space variables we seek to avoid analytic complications (see Remark 4.1), which are not directly related to the controllability issue under study.

Choosing initial condition $u(0) = u_0 \in H^s(\mathbb{T}^d)$, we set the problems of:

1. controllability in finite-dimensional projections, meaning that one can steer in time $T > 0$ the trajectory of the equation (1) from $u_0$ to a state $\hat{u} \in H^s$ with any preassigned orthogonal projection $\Pi^L \hat{u}$ onto a given finite-dimensional subspace $L \subset H^s$;
2. approximate controllability meaning that the attainable set of (1) from each $u_0$ is dense in $H^s$;
3. exact controllability in $H^s$.

Definitions of several types of controllability and exact problem setting are provided in the next Section together with the main results. First of the results asserts that controllability in projection on each finite-dimensional subspace of $H^s$ and approximate controllability in $H^s$ can be achieved by applying controls to at most $2^d$ modes, which can be chosen the same for all subspaces. Propositions 8.1, 8.2 describe classes of saturating sets of controlled modes which suffice for achieving these types of controllability. The second main result asserts lack of exact controllability in $H^s$ by controlling any finite number of modes.

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1In the formulation of results below and in the course of the proofs we speak of orthogonal projections on finite-dimensional subspaces. The formulations and proofs are valid for orthogonality coming from either $L^2$ inner product, or $H^s$ inner product. Note that the complex exponentials form orthogonal system with respect to both products.
2. Controllability: definitions.

2.1. Global controllability. As we mentioned before the evolution space of NLS equation is Sobolev space \( H = H^s(\mathbb{T}^d) \), \( s > d/2 \).

We say that the controlling source term \( (2) \) steers the system \( (1) \) from \( u_0 \in H \) to \( \hat{u} \in H \) in time \( T > 0 \), if mild (see Section 4) solution of \( (1) \) exists, is unique and satisfies \( u(T) = \hat{u} \). The set \( A_{T,u_0} \) of points to which the system can be steered to from \( u^0 \) in time \( T > 0 \) is called the attainable set.

The equation is \( \text{globally time-} T \) (exactly) controllable from \( u_0 \), if it can be steered in time \( T \) from \( u_0 \) to any point of \( H \), or, the same, \( A_{T,u_0} = H \).

2.1.2. Controllability in finite-dimensional projections and in finite-dimensional components. Let \( L \) be closed linear subspace of \( H \), \( \Pi_L \) be the orthogonal projection of \( H \) onto \( L \).

The equation \( (1)-(2) \) is time- \( T \) globally controllable in projection on \( L \), if for each \( \hat{q} \in L \) this equation can be steered in time \( T \) to some point \( \hat{u} \) with \( \Pi_L(\hat{u}) = \hat{q} \).

The NLS equation \( (1)-(2) \) is time- \( T \) globally controllable in finite-dimensional projections if for each finite-dimensional subspace \( L \) the equation is time- \( T \) globally controllable in projection on \( L \); the set \( \hat{K} \) of controlled modes is assumed to be the same for all \( L \).

Whenever \( L \) is a 'coordinate subspace' \( L = \mathcal{F}_{K_o} = \text{Span}\{ e^{ikx} | k \in K_o \} \), with a finite set \( K_o \subset \mathbb{Z}^d \) of observed modes, then controllability in projection on \( L \) is called controllability in observed \( K_o \)-component.

Remark 2.1. It is convenient to characterize the time- \( T \) controllability in terms of surjectiveness of the end-point map \( E_T : v(\cdot) \to u(T) \), of the controlled NLS equation \( (1)-(2) \). The map \( E_T \) maps a control \( v(\cdot) = \{ v_k(\cdot) | k \in \hat{K} \} \), into the end-point \( u(T) \) of the trajectory \( u(t) \) of \( (1) \), driven by the source term \( (2) \). Equivalently one can see the end-point map, as mapping the controlling source term \( V = \sum_{k \in \hat{K}} v_k(t) e^{ikx}, \ t \in [0,T] \) to \( u(T) \).

The time- \( T \) controllability in projection on \( L \) means surjectiveness of the composition \( \Pi_L \circ E_T \).

2.1.3. Approximate controllability. The NLS equation \( (1)-(2) \) is time- \( T \) approximately controllable from \( u_0 \) in \( H \), if it can be steered to each point of a dense subset of \( H \).

2.1.4. Solid controllability (cf. \( [2] \)). On the contrast to the previous definitions the word 'solid' does not refer to a new type of controllability but means stability of the controllability property with respect to certain classes of perturbations.

Let \( \Phi : \mathcal{M}^1 \to \mathcal{M}^2 \) be a continuous map between two metric spaces, and \( S \subseteq \mathcal{M}^2 \) be any subset. We say that \( \Phi \) covers \( S \) solidly, if \( \Phi(\mathcal{M}^1) \supseteq S \) and the inclusion is stable with respect to \( C^0 \)-small perturbations of \( \Phi \), i.e.
for some $C^0$-neighborhood $\Omega$ of $\Phi$ and for each map $\Psi \in \Omega$, the inclusion $\Psi(M) \supset S$ holds.

Given a finite-dimensional subspace $L \subset H^s$ and a bounded set $S \subset L$ we say that $S$ is solidly attainable for (1)-(2) in time $T$, if there exists a family of controls $\{v(\cdot,b) \mid b \in B_S\}$ parameterized by a compact set $B_S \subset \mathbb{R}^N$, such that the map $b \mapsto \left(\Pi_L \circ E_T\right)(v(\cdot,b))$ is continuous on $B_S$ and covers $S$ solidly.

Time-$T$ solid controllability in projection on finite-dimensional subspace $L$ for the NLS equation (1)-(2) means that any bounded subset $S \subset L$ is solidly attainable in time $T$.

2.2. Main results. Our first goal is establishing for the cubic defocusing NLS on $\mathbb{T}^d$ sufficient criteria of controllability in finite-dimensional projections and of approximate controllability in Sobolev space $H^s$, $s > d/2$. The common criterion is formulated in terms of the set of controlled modes $\hat{K}$, which is fixed and the same for all projections and for the approximate controllability.

Main result 1 (sufficient criterion for controllability in finite-dimensional projections and for approximate controllability). For each integer $d \geq 1$ there exists a $2^d$-element set $\hat{K}$ of controlled modes ($\hat{K} \subset \mathbb{Z}^d$) such that for each $s > d/2$, any initial data $u^0 \in H^s(\mathbb{T}^d)$ and any $T > 0$ the defocusing cubic Schroedinger equation (1) on $\mathbb{T}^d$, controlled via the source term (2) is: i) time-$T$ controllable in projection on each finite-dimensional subspace $L$ of $H^s(\mathbb{T}^d)$; ii) time-$T$ approximately controllable in $H^s(\mathbb{T}^d)$. □

Our second result is

Main result 2 (lack of exact controllability via a finite-dimensional additive control). Let the defocusing cubic Schroedinger equation (1) on $\mathbb{T}^d$, $d \geq 1$, be controlled via a source term (2), with the set $\hat{K} \subset \mathbb{Z}^d$ of controlled modes being arbitrary finite. Then for each $s > d/2$, each $T > 0$ and each initial data $u^0 \in H^s(\mathbb{T}^d)$, the time-$T$ attainable set $A_{T,u^0}$ of (1)-(2) is contained in a countable union of compact subsets of $H^s(\mathbb{T}^d)$ and hence its complement $H^s(\mathbb{T}^d) \setminus A_{T,u^0}$ is dense in $H^s(\mathbb{T}^d)$. □

3. Outline of the approach: Lie extensions, fast-oscillating controls, resonances. Other approaches

Study of controllability of the NLS equation is based (as well as our earlier work [1],[2] on Navier-Stokes/Euler equation) on the method of iterated Lie extensions. Lie extension of a control system $\dot{x} = f(x,u)$, $u \in U$ is a way to add vector fields to the right-hand side of the system, maintaining at the same time (almost) invariance of its controllability properties. The additional vector fields are expressed via the Lie brackets of the vector fields $f(\cdot,u)$ for various $u \in U$. If after a series of extensions one arrives to a
controllable (extended) system, then the controllability of the original system would follow.

This approach can not be propagated automatically to the infinite-dimensional setting due to the lack of adequate Lie algebraic tools. So far Lie algebraic viewpoint in the infinite-dimensional context is mainly used as a guiding tool; the implementation has to be justified by the analytic means. In the rest of this Section we provide geometric control sketch for the proof of main result.

When studying controllability we will look at the cubic NLS equation as at a particular type of (infinite-dimensional) control-affine system:

\[-i\partial_t u = c(u, t) + \sum_{k \in \hat{\mathcal{K}}} e_k v_k(t),\]

where \(c(u, t)\) is the cubic drift vector field, \(e_k\) are constant (independent of \(u\)) controlled vector field in \(H^s(\mathbb{T}^d)\) with the values \(e^{ik \cdot x}\).

The Lie extensions, we implement, involve two controlled vector fields \(e_m, e_n\) and the drift vector field \(c\). An outcome of each extension is the fourth-order Lie bracket \([e_n, [e_m, [e_m, c]]]\), which is again constant vector field (given that the vector field \(c\) is cubic). The Lie bracket is seen as a direction of the action of an extended control.

Another Lie bracket, which makes its appearance at each Lie extension, is the third-order Lie bracket \([e_m, [e_m, c]]\). The motion of the system along its direction is ‘unilateral’; hence it can be seen as obstruction to controllability. This motion can not be locally reverted nor compensated but for the NLS equation one can nullify it in average by imposing integral (isoperimetric) relations onto the controls involved in the extension.

To design the needed motion in the extending direction \([e_n, [e_m, [e_m, c]]]\) and to oppress the obstructing motions, we employ fast-oscillating controls. Use of such controls is traditional for geometric control theory. A ‘general theory’ of their application is hardly available, but particular cases can be effectively studied (see, for example [18]).

In our study we invoke for each extension a couple of fast-oscillating controls

\[v_m(t)e^{ia_m t/\varepsilon}e_m, v_n(t)e^{ia_n t/\varepsilon}e_n,\]

feed them into the right-hand side of the NLS equation and trace the interaction of the two controls via the cubic term of the equation. The idea is to design needed resonance in the course of the interaction, that is to choose oscillation frequencies and magnitudes of controls in such a way that the averaged interaction influences dynamics of specific modes.

By our design we manage to limit the influence of the resonance to a single mode \(e_{2m-n} = e^{i(2m-n) \cdot x}\); the resonance term will be seen as an (extending) control for this mode. The procedure can be interpreted as an elementary extension of the set of controlled modes: \(\hat{\mathcal{K}} \mapsto \hat{\mathcal{K}} \cup \{2m - n\}, m, n \in \hat{\mathcal{K}}.\)

\(^2\)Other choices of resonant modes are possible, being this one one of simplest
Final controllability result (Main result 1) is obtained by (finite) iteration of the elementary extensions. If one seeks controllability in observed $K^o$-component with $K^o \supset \hat{K}$, then one should find (when possible) a series of elementary extensions $\hat{K} = K^1 \subset K^2 \subset \cdots \subset K^N = K^o$. Getting extended controls available for each observed mode $k \in K^o$, we may conclude controllability of extended system in $K^o$-component by Lemma 5.3. On the contrast controllability of the original system in $K^o$-component will follow by virtue of technical Approximative Lemma 5.2 which formalizes the resonance design.

From the controllability in all finite-dimensional components one derives controllability in projection on each finite-dimensional subspace as well as the approximate controllability; this is proved in Section 6.

Note that in the case of periodic NLS equation the analysis of interaction of different terms via cubic nonlinearity is substantially simplified by our choice of a special basis of exponential modes.

Besides the design of proper resonances there are two analytic problems to be fixed. First problem consists of studying NLS with fast-oscillating right-hand sides and of establishing the continuity, the approximating properties and the limits of corresponding trajectories, as the frequency of oscillation tends to infinity. Second problem is coping with the fact that at each iteration we are only able to approximate the desired motion, therefore the controllability criteria need to be stable with respect to the approximation errors.

The second problem is fixed with the help of the notion of solid controllability (see previous Section). The solution to the first problem in finite-dimensional setting is provided by theory of relaxed controls (12). For general nonlinear PDE such theory is unavailable; while for semilinear infinite-dimensional control systems relaxation results have been obtained in 11, 10. We provide formulations and proofs needed for our analysis in Subsection 5.5.

What regards negative result on exact controllability (Main result 2), then the key point for its proof is endowing the (functional) space of controls with a weaker topology, in which this space is a countable union of compacts and the end-point map is Lipschitzian in the respective metric. Then the attainable sets - the images of the end point map - are meager. Similar kind of argument has been used in 3 for establishing noncontrollability of some bilinear distributed systems. Finer method, based on estimates of Kolmogorov’s entropy has been invoked in 17 for proving lack of exact controllability by finite-dimensional forcing for Euler equation of fluid dynamics.

At the end of the Section we wish to mention few references to other approaches to controllability of linear and semilinear Schroedinger equation controlled via bilinear or additive control, this latter being "internal" or boundary.

References 20, 13 provide nice surveys of the results on:
• exact controllability for linear Schroedinger equation with additive control in relation to observability of adjoint system and to geometric control condition (see also [15]);
• controllability of linear Schroedinger equation with control entering bilinearly; besides references in the above cited surveys there are notable results [4, 5] on local (exact) controllability in $H^7$ of 1-D equation; another interesting result is (obtained by geometric methods) criterion [8] of approximate controllability for the case in which ’drift Hamiltonian’ has discrete non-resonant spectrum (see bibliographic references in [4, 5, 8] to the preceding work);
• exact controllability of semilinear Schroedinger equation by means of internal additive control; in addition to references in [20, 13] we mention more recent publications [9, 16] where the property has been established for 2D and 1D cases. The key tool in the study of the semilinear case is ’linearization principle’, going back to [14]. On the contrast our approach makes exclusive use of nonlinear term.

4. Preliminaries on existence, uniqueness and continuous dependence of trajectories

In this Section we collect results on existence/uniqueness and continuity in the right-hand side for the mild solutions of a Cauchy problem for a class of semilinear equations:

$$\begin{align*}
(-i\partial_t + \Delta) u &= G(t, u), \quad u(0) = \tilde{u}^0.
\end{align*}$$

The mild solutions satisfy integral form of the equation (3), obtained via Duhamel formula:

$$u(t) = e^{it\Delta} \left( \tilde{u}^0 + i \int_0^t e^{-i\tau\Delta} G(\tau, u(\tau)) d\tau \right).$$

The mild solutions $u(\cdot)$ are sought in the space $C([0, T]; H)$, with $H$, being Hilbert space of functions $u(x)$ defined on $\mathbb{T}^d$. We opt for $H = H^s(\mathbb{T}^d)$, $s > d/2$.

Accordingly a ’perturbation’ of the Cauchy problem (3):

$$\begin{align*}
(-i\partial_t + \Delta) u &= G(t, u) + \phi(t, u), \quad u(0) = u^0,
\end{align*}$$

admits the integral form

$$u(t) = e^{it\Delta} \left( u^0 + i \int_0^t e^{-i\tau\Delta} (G(\tau, u(\tau)) + \phi(\tau, u(\tau))) d\tau \right),$$

whose solutions are the mild solutions of (4).
We assume the nonlinear terms $G(\cdot, \cdot), \phi(\cdot, \cdot) : [0, T] \times H \to H$ in (3)-(5) to satisfy the conditions:

(6) \hspace{1cm} G, \phi \text{ are continuous;}

(7) \hspace{1cm} \forall c > 0, \, \exists \beta_c(t) \in L^1([0, T], \mathbb{R}_+), \text{ such that } \forall t \in [0, T], \, \forall \|u\| \leq c, \|G(t, u')\|_H \leq \beta_c(t), \|G(t, u'') - G(t, u')\|_H \leq \beta_c(t)\|u' - u\|_H,

(8) \hspace{1cm} \|\phi(t, u')\|_H \leq \beta_c(t), \|\phi(t, u'') - \phi(t, u')\|_H \leq \beta_c(t)\|u' - u\|_H.

**Proposition 4.1** (local existence and uniqueness of mild solutions). Let $G$ satisfy the conditions (6), (7). Then for each $c > 0$, $\exists T_c > 0$ such that for $\|\tilde{u}^0\|_H \leq c$ there exists unique mild solution $u(\cdot) \in C([0, T_c], H)$ of the Cauchy problem (4). □

The result is proved via fixed point argument for contracting map in $C([0, T]; H)$.

Our choice of $H = H^s(\mathbb{T}^d), \, s > d/2$, guarantees that the cubic term of the NLS equation (1) would satisfy conditions (6), (7) according to the following version of embedding theorem.

**Lemma 4.2** (‘Product Lemma’; [19]). For Sobolev spaces $H^s(\mathbb{T}^d)$ of functions there holds:

for $s \geq 0$: $\|fg\|_{H^s} \leq C(s, d) (\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s});$

for $s > d/2$: $\|fg\|_{H^s} \leq C'(s, d)\|f\|_{H^s}\|g\|_{H^s}.$ □

This Lemma allows verification of the conditions (3), (4), (5) for more general classes of operators of the form

$u(x) \mapsto P(u(x), \bar{u}(x))$,

where $P$ is a polynomial in $u, \bar{u}$ with coefficients $p_{jk}(t, x)$ such that $p_{jk}(\cdot, x) \in L^1([0, T], \mathbb{C})$. Recall that the source term (2) is a trigonometric polynomial in $x$ and $F(\cdot, x) \in L^\infty([0, T], H)$.

**Remark 4.1.** Let us remark on the obstacles, which may arise, when one considers solutions of the NLS with lower regularity in space variables. For proving local existence, uniqueness and well-posedness of solutions of the NLS in $H^s$ with $s \leq d/2$, one invokes Strichartz inequalities. While for the NLS equation in $\mathbb{R}^n$ Strichartz inequalities are derived from dispersion estimates ([19]), this approach would fail on a compact manifold ([4]), since there the dispersion estimates are not available. Still kind of Strichartz inequalities with the loss of derivatives can be established on a compact Riemannian manifold; see [7] and references therein to the previous work. For flat torus $\mathbb{T}^d$ Strichartz estimates have been derived in [6] by methods of harmonic analysis.

For settling controllability issue in the low regularity setting one would need results on continuity with respect to the right-hand side, analogous to Propositions 4.4, 4.5, and more important analogues of the results of Subsection 5.3, which justify application of relaxed (and fast-oscillating) controls.
to the NLS equation. We trust that all the mentioned obstacles can be overcome and controllability criteria for more general setting will appear in future publications. □

Global existence results for a cubic defocusing NLS equation \( \Box \) are classical under assumptions we made.

**Proposition 4.3** (global existence and uniqueness of mild solutions). Let \( H = H^s(\mathbb{T}^d), \ s > d/2 \) and the time-variant source term \( t \mapsto F(t, \cdot) \) belong to \( L^1([0, T], H) \). Then for each initial condition \( u(0) = u^0 \in H \) the Cauchy problem \( \Box \) has a unique mild solution \( u(\cdot) \in C([0, T], H) \). □

One can consult [9], where such result is established for cubic defocusing NLS with source term and weaker regularity of the data.

Now we provide few results on the continuity of the solutions of the NLS equation in the right-hand side. The topology in the space of right-hand sides is introduced via the seminorms:

\[
\| \phi(t, u) \|_{T,c}^1 = \int_0^T \sup_{\| u \|_H \leq c} \| \phi(t, u) \|_H dt.
\]

**Proposition 4.4** (continuity of the solutions with respect to the right-hand side). Let \( G, \phi \) satisfy the conditions \( \Box \), \( \Box \), \( \Box \), \( \Box \) and \( \tilde{u}(t) \in C([0, T], H) \) be a mild solution of \( \Box \); assume \( \sup_{t \in [0, T]} \| \tilde{u}(t) \|_H < c \). Then \( \exists \delta > 0, C > 0, \) such that whenever

\[
\| u^0 - \tilde{u}^0 \| + \| \phi(t, u) \|_{T,c}^1 < \delta,
\]

then mild solution \( u(t) \) of the perturbed equation \( \Box \) exists on the interval \( [0, T], \) is unique and admits an estimate

\[
\sup_{t \in [0, T]} \| u(t) - \tilde{u}(t) \| < C \left( \| u^0 - \tilde{u}^0 \| + \| \phi(t, u) \|_{T,c}^1 \right). \ □
\]

**Proof.** A solution of the equation \( \Box \) can be continued in time as long as its norm in \( H \) remains bounded. Therefore one is able to conclude extendibility of the solution onto \( [0, T] \) from the estimate \( \Box \).

Estimating the difference

\[
u(t) - \tilde{u}(t) = e^{it\Delta} \left( (u^0 - \tilde{u}^0) + i \int_0^t e^{-i\tau\Delta} \phi(\tau, \tilde{u}(\tau)) d\tau \right) +
+ e^{it\Delta} \int_0^t e^{-i\tau\Delta} \left( (G(\tau, u(\tau)) - G(\tau, \tilde{u}(\tau))) + (\phi(\tau, u(\tau)) - \phi(\tau, \tilde{u}(\tau))) \right) d\tau,
\]
and noting that $e^{it\Delta}$ is an isometry of $H$, we get
\[
\|u(t) - \tilde{u}(t)\|_H \leq \|u^0 - \tilde{u}^0\|_H + \left\| \int_0^te^{-i\tau\Delta}\phi(\tau, \tilde{u}(\tau))d\tau \right\|_H + \\
+ \int_0^t \|e^{-i\tau\Delta}(G(\tau, u(\tau)) - G(\tau, \tilde{u}(\tau)) + \phi(\tau, u(\tau)) - \phi(\tau, \tilde{u}(\tau)))\|_{H}d\tau \leq \\
\leq \|u^0 - \tilde{u}^0\|_H + \left\| \int_0^te^{-i\tau\Delta}\phi(\tau, \tilde{u}(\tau))d\tau \right\|_H + 2\int_0^t \beta_c(\tau)\|u(\tau) - \tilde{u}(\tau)\|_{H}d\tau.
\]
By the Gronwall inequality
\[
\|u(t) - \tilde{u}(t)\|_H \leq \left( \|u^0 - \tilde{u}^0\|_H + \left\| \int_0^te^{-i\tau\Delta}\phi(\tau, \tilde{u}(\tau))d\tau \right\|_H \right) C' e^{C't} \int_0^t \beta_c(\tau)d\tau,
\]
for some $C' > 0$ and whenever $[\mathbb{D}]$ is satisfied, we get for some $C$
\[
\|u(t) - \tilde{u}(t)\|_H \leq C \left( \|u^0 - \tilde{u}^0\| + \int_0^t \|\phi(\tau, \tilde{u}(\tau))\|_{H}d\tau \right) \leq C\delta.
\]
One should choose $\delta > 0$ such that $C\delta < c - \sup_{t \in [0,T]} \|\tilde{u}(t)\|_H$. \hfill \QED

Next proposition is parametric reformulation of the previous result.

**Proposition 4.5** (continuous dependence of the solutions on parameter). Let a family of operators $G(t, u, b)$, parameterized by $b \in B \subset \mathbb{R}^N$: i) be continuous in $b$ with respect to each seminorm $\| \cdot \|_{T,c}$; ii) satisfy the conditions $[\mathbb{Z}]$ with the same $\beta_c(\cdot)$ for all $b \in B$. Then the mild solutions of the equations
\[
(-i\partial_t + \Delta)u = G(t, u, b), \quad u(0) = u^0,
\]
depend continuously on $b, u^0$ in the $C^0$-metric.

Besides if $\tilde{u}(t) \in C([0,T], H)$ is a mild solution of $[\mathbb{T}]$ for $b = \tilde{b}$, $u^0 = \tilde{u}^0$, and $\sup_{t \in [0,T]} \|u(t)\|_H dt < c$, then $\exists \delta > 0, C > 0$, such that whenever
\[
\|u^0 - \tilde{u}^0\| + \|G(t, u, b) - G(t, u, \tilde{b})\|_{T,c} < \delta,
\]
the solutions $u(t)$ of the equation $[\mathbb{T}]$ exist on the interval $[0,T]$, are unique and
\[
\sup_{t \in [0,T]} \|u(t) - \tilde{u}(t)\| \leq C \left( \|u^0 - \tilde{u}^0\| + \|G(t, u, b) - G(t, u, \tilde{b})\|_{T,c} \right). \hfill \QED
\]

Next Lemma treats the case in which a parameter also affects the linear term of the equation.

**Lemma 4.6.** Consider the family of equations
\[
(-i\partial_t + \varepsilon\Delta)u = \varepsilon G(t, u, b) + \phi(t, u, b), \quad u(0) = u^0, \quad \varepsilon > 0,
\]
depending on the parameters $\varepsilon > 0, b \in B$, where $B$ is a compact subset of $\mathbb{R}^N$. Let $G : [0,T] \times H \times B \to H$ be continuous in $b$ with respect to each
CONTROLLABILITY OF THE NLS VIA A LOW-DIMENSIONAL SOURCE TERM

seminorm \( \| \cdot \|_{L^1_{T,c}} \) and satisfy (7) with the functions \( \beta_c(\cdot) \), the same for all \( b \in B \). Let \( \phi : [0,T] \times H \times B \to H \) be continuous.

Consider the 'limit equation' for \( \varepsilon = 0 \):

\[
-i \partial_t u = \phi(t,u,b), \quad u|_{t=0} = u^0.
\]

Assume mild solutions \( \tilde{u} (\cdot, b) \in C([0,T], H) \) of (16) to be defined on \([0,T]\) for each \( b \in B \). Then there exists \( \varepsilon_0 \), such that for \( \varepsilon \in [0, \varepsilon_0) \) the solutions \( u^\varepsilon (t, b) \) of (15) exist on \([0,T]\) for each \( b \in B \), and

\[
\sup_{t \in [0,T]} \| u^\varepsilon (t, b) - \tilde{u}(t, b) \|_H \to 0, \quad \text{as} \ \varepsilon \to 0,
\]

uniformly in \( b \in B \). □

The proof of the Lemma can be found in Appendix.

Below we will formulate and employ more general continuity result (Proposition 5.9) which incorporates perturbations \( \phi(t,x) \), which are fast-oscillating in time, and with (9), substituted by a weaker estimate, based on relaxation metric for the right-hand sides.

5. Extension of control and controllability

In this Section we introduce extensions of control which are the main tools for establishing controllability. The outcome of the Section is Proposition 5.4 which establishes sufficient criterion for controllability in a finite-dimensional component, wherefrom one derives in Section 6 criteria for controllability in projections and approximate controllability (Main Result 1).

In what follows the metrics \( L^1 ([t_0, t_1], H^s), L^1 ([t_0, t_1], C^k), [t_0, t_1] \subset \mathbb{R} \) will be denoted both by \( L^1_t \) by abuse of notation.

5.1. Extensions: approximative lemma. Consider the NLS equation (1)-(2) with controls applied to the modes, indexed by a set \( \hat{K} \subset \mathbb{Z}^d \).

Definition 5.1. Given a finite set \( \hat{K} \subset \mathbb{Z}^d \), we define:

i) elementary extensions of \( \hat{K} \), being the sets \( K = \hat{K} \cup \{2r - s\} \), where \( r, s \in \hat{K} \) are arbitrary;

ii) proper extensions \( K \) of \( \hat{K} \), such that there exist finite sequences of sets \( \hat{K} = K^1 \subset K^2 \subset \cdots \subset K^N = K \), with \( K^j \) being elementary extensions of \( K^{j-1}, j = 2, \ldots; \)

iii) saturating sets \( \hat{K} \) such that, each finite subset \( K \subset \mathbb{Z}^d \) is proper extension of \( \hat{K} \). □

It turns out that saturating sets of modes are essential for controllability in projections and for the approximate controllability. Examples of saturating sets are provided in Section 8.

The following Lemma states that controls (energy) fed into the modes, indexed by \( \hat{K} \), can be cascaded to and moreover can approximately control a larger set \( K \) of modes, whenever \( K \) is proper extension of \( \hat{K} \).
Lemma 5.2 (approximative lemma). Let \( \mathcal{K} \) be a proper extension of \( \hat{\mathcal{K}} \). Consider a family of controls \( \{w_k(t,b)\} k \in \mathcal{K}, t \in [0,T]\), parameterized by \( b \) from a compact set \( B \subset \mathbb{R}^N \), and depending continuously in \( L^1_t \)-metric on \( b \). Then for each \( \delta > 0 \) one can construct a family of controls \( \{v_k(t,b)\} k \in \hat{\mathcal{K}}, t \in [0,T]\) continuously depending on \( b \) in \( L^1_t \)-metric, and such that

\[
\|E_T(v(\cdot,b)) - E_T(w(\cdot,b))\| \leq \delta, \ \forall b \in B.
\]

for the respective end-point map \( E_T \) (see Remark 5.1) of the NLS equation \( (\Pi) \). □

Remark 5.1. The controls, which appear in the formulation of the Lemma, correspond to the source terms

\[
W(t;b) = \sum_{k \in \mathcal{K}} w_k(t,b)e^{ikx}, \quad V(\cdot,b) = \sum_{k \in \mathcal{K}} v_k(t,b)e^{ikx}, \quad t \in [0,T], \ b \in B;
\]

\( V(t,b) \) take values in the 'low-dimensional' space \( \mathcal{F}_{\hat{\mathcal{K}}} \) on the contrast to the 'high-dimensional' space \( \mathcal{F}_{\mathcal{K}} \), which is the range of \( W(t,b) \). □

5.2. Full-dimensional control. Before proving that controllability can be achieved by means of low-dimensional controls we formulate a general controllability result for the case, where the control is 'full-dimensional'.

Lemma 5.3 (controllability by 'full-dimensional' control). Controlled semi-linear equation

\[
(i) - i\partial_t u(t,x) + \Delta u(t,x) = G(t,u) + \sum_{k \in \hat{\mathcal{K}} = \mathcal{K}^o} w_k(t)e^{ikx}, \quad u(0) = u^0,
\]

with coinciding sets of controlled and observed modes \( \hat{\mathcal{K}} = \mathcal{K}^o \), is time-\( T \) solidly controllable in observed \( \mathcal{K}^o \)-component for each \( T > 0 \).

In addition, for each \( \delta > 0 \), any bounded subset \( S \subset \mathcal{F}_{\mathcal{K}^o} \) is time-\( T \) solidly attainable for the equation \( (18) \) by means of controlled trajectories \( u(\cdot) \), which satisfy the estimate \( \|(I - \Pi^o)(u(T) - u^0)\| \leq \delta \), where \( \Pi^o, I - \Pi^o \) are the orthogonal projections onto \( \mathcal{F}_{\mathcal{K}^o} \) and its orthogonal complement. □

Proof of Lemma 5.3. Take a ball \( \mathcal{B} \) in \( \mathcal{F}_{\mathcal{K}^o} \). One can assume without lack of generality, that \( \mathcal{B} \) is centered at the origin and the initial condition is \( u(0) = 0_H \). We will prove that \( \mathcal{B} \) is solidly attainable for the controlled equation \( (18) \).

Restrict \( (18) \) to a small interval \( [0, \varepsilon] \) to be specified later. Proceed with the time substitution \( t = \varepsilon \tau, \ \tau \in [0,1] \) under which \( (18) \) takes form:

\[
(ii) - i\partial_\tau u + \varepsilon \Delta u = \varepsilon G(t,u) + \varepsilon \sum_{k \in \mathcal{K}^o} w_k(t)e^{ikx}, \quad u(0) = 0, \quad \tau \in [0,1].
\]

Fix \( \gamma > 1 \). For each \( b \in B_\gamma = \gamma \mathcal{B} \), consider the control

\[
w(\cdot;b) = -i\varepsilon^{-1} \sum_{k \in \mathcal{K}^o} b_k e^{ikx},
\]
which we substitute into (19) getting
\[-i\partial_t u + \varepsilon \Delta u = \varepsilon G(t, u) - i \sum_{k \in K^o} b_k e^{ik \cdot x}, \ u(0) = 0, \ \xi \in [0, 1].\]

For \(\varepsilon = 0\) we get the 'limit equation'
\[
(20) \quad \partial_t u = \sum_{k \in K^o} b_k e^{ik \cdot x}, \ u(0) = 0, \ \tau \in [0, 1].
\]

Let \(E_1^0\) be the time-1 end-point map of (20). In the basis \(e^{ik \cdot x}\) the map \(b \mapsto \Phi^0(b) = E_1^0(w(t; b))\) has form \(\{b_k | k \in K^o\} \mapsto u(1) = \sum_{k \in K^o} b_k e^{ik \cdot x}\).

Obviously the map \(b \mapsto \Phi^0(b) = \Pi^o \circ E_1^0(w(t; b))\) coincides with the identity map \(\text{Id}_{B_o}\) and \((I - \Pi^o)E_1^0(w(t; b)) = 0\).

According to the Lemmas 4.5,4.6 the maps \(\Phi^\varepsilon : b \mapsto E_1^\varepsilon(w(\cdot, b)), \) with \(E_1^\varepsilon\) being the end-point maps of the control systems (19), are continuous and \(\|\Phi^\varepsilon - \Phi^0\|_{C^0(B_o)} \to 0\) as \(\varepsilon \to 0\).

By the degree theory argument there exists \(\varepsilon_0\) such that \(\forall \varepsilon \leq \varepsilon_0\) the image of \((\Pi^0 \circ \Phi^\varepsilon)(B_o)\) covers \(B\) solidly.

We proved controllability in the observed component for small times \(T > 0\). Controllability in any time \(T > 0\) can be concluded by applying zero control on a time interval \(0, T - \delta\) (the trajectory is maintained in a bounded domain due to the conservation law) and then employing the previous reasoning on the interval \([T - \delta, T]\). \(\square\)

**Remark 5.2.** Without lack of generality we may assume, that the controls \(w(t, b)\), constructed in the Lemma are smooth with respect to \(t\) and that any finite number of derivatives \(\frac{\partial w}{\partial t}(\cdot, b)\) depend continuously on \(b \in B\) in \(L^1_t\)-metric. Indeed smoothing \(w(t, b)\) by the convolution with a smooth \(\varepsilon\)-approximation of Dirac function, one gets a family of smooth controls \(w^\varepsilon(t, b)\), which provides solid controllability, for small \(\varepsilon > 0\). The continuous dependence of \(\frac{\partial w^\varepsilon}{\partial t}(\cdot, b)\) on \(b\) in \(L^1_t\)-metric is verified directly. \(\square\)

### 5.3. Controllability in finite-dimensional component via extensions.

The following result is a corollary of Lemmas 5.2, 5.3.

**Proposition 5.4** (controllability in observed component). If the set of observed modes \(K^o\) is proper extension of the set of controlled modes \(K\), then the NLS equation (1)-(2) is solidly controllable in the observed \(K^o\)-component.

**Proof.** Let \(S\) be a compact subset of \(F_{K^o} = \text{span}\{e_k | k \in K^o\}\). According to the Lemma 5.3 we can choose a family of \(F_{K^o}\)-valued controlling source terms \(W(\cdot, b)\) by which \(S\) is solidly attainable. If a family \(V(t; b)\) in (2) satisfies the conclusion of the Approximative Lemma 5.2 and \(\delta > 0\) is small enough, then \(\Pi^o \circ E_T(V(t; b))\) covers \(S\) solidly. \(\square\)

**Corollary 5.5.** If the set of controlled modes \(K\) is saturating, then the NLS equation is solidly controllable in each finite-dimensional component. \(\square\)

 Examples of saturated sets are provided in Section 8.
5.4. Proof of Approximative Lemma 5.2. It suffices to prove the Approximative Lemma for $\mathcal{K}$ being an elementary extension of $\hat{\mathcal{K}}$, the rest being accomplished by induction. Let $\mathcal{K} = \hat{\mathcal{K}} \cup \{2r - s\}$, $r, s \in \hat{\mathcal{K}}$.

It is convenient to proceed with the time-variant change of basis in $H$, passing from the exponentials $e^{i k x}$ to the exponentials

$$f_k = e^{i(k x + |k|^2 t)}, \quad k \in \mathbb{Z}^d.$$

Note that $(-i \partial_t + \Delta) f_k = 0$, $\forall k \in \mathbb{Z}^d$.

We take a $\mathcal{F}_\mathcal{K}$-valued family of the controlling source terms

$$(21) \quad b \mapsto W(t, b) = \sum_{k \in \mathcal{K}} w_k(t; b) f_k,$$

parameterized by $b$ from a compact $B \subset \mathbb{R}^N$, and wish to construct a family of the controlling source terms $V(t; b) = \sum_{k \in \hat{\mathcal{K}}} v_k(t; b) f_k$, which satisfy (17) and whose range $\mathcal{F}_{\hat{\mathcal{K}}}$ has one dimension less.

5.4.1. Time-variant substitution. We will seek the family $b \mapsto V(t, b)$ in the form

$$(22) \quad V(t, b) = \tilde{V}(t, b) + \partial_t v_r(t, b) f_r + \partial_t v_s(t, b) f_s,$$

where $\tilde{V}(t, b) = \sum_{k \in \hat{\mathcal{K}}} \tilde{v}_k(t; b) f_k$, and the Lipschitzian functions $v_r(t, b), v_s(t, b)$ will be specified in the course of the proof. For some time we will omit dependence on $b$ in the notation.

Feeding the controls (22) into the right-hand side of equation (11) we get

$$(23) \quad (-i \partial_t + \Delta) u = |u|^2 u + \tilde{V}(t) + \dot{v}_r(t) f_r + \dot{v}_s(t) f_s.$$

This equation can be given form

$$(24) \quad (-i \partial_t + \Delta) (u - i V_{rs}(t)) = |u|^2 u + \tilde{V}(t),$$

where $V_{rs}(t) = v_r(t) f_r + v_s(t) f_s$.

By a time-variant substitution

$$u^* = u - i V_{rs}(t),$$

we transform (24) into the equation:

$$(25) \quad (-i \partial_t + \Delta) u^* = |u^* + i V_{rs}(t)|^2 (u^* + i V_{rs}(t)) + \tilde{V}(t) = |u^*|^2 u^* - i(u^*)^2 V_{rs} + 2i |u^*|^2 V_{rs} - V_{rs}^2 u^* + 2u^* |V_{rs}|^2 + i |V_{rs}|^2 V_{rs} + \tilde{V}(t).$$

Imposing the constraints

$$(26) \quad v_r(0) = v_s(0) = 0, \quad v_r(T) = v_s(T) = 0,$$

we achieve: $u(0) = u^*(0)$, $u(T) = u^*(T)$, and hence the end-point maps $E_T, E_T^*$ of the controlled equations (23) and (24) coincide provided (26) hold.
5.4.2. Fast oscillations and resonances. Now we put fast-oscillations, into the game, choosing $V_{rs}(t)$ in (25) of the form

$$V_{rs}(t) = v_r(t)f_r + s(t)f_s = e^{i(t/\varepsilon + \rho(t))v_r(t)}f_r + e^{i2t/\varepsilon}s(t)f_s,$$

where $v_r(t), s(t), \rho(t)$ are Lipschitzian real-valued functions, which together with small $\varepsilon > 0$ will be specified in the course of the proof.

We classify those terms at the right-hand side of (25), which contain $V_{rs}, V_{rs}, \text{as non-resonant and resonant}$ with respect to the substitution (27).

We call a term non-resonant if, after the substitution (27) the term results non-resonant, as $\varepsilon > 0$, in a sum of fast-oscillating factors of the form $p(u^*, V_{rs}, t)e^{i\beta t/\varepsilon}$, $\beta \neq 0$, where $p(u, V_{rs}, t)$ is polynomial in $u^*, u^*, V_{rs}, V_{rs}$, with coefficients, which are Lipschitzian in $t$ and independent of $\varepsilon$. Otherwise, i.e. $\beta = 0$, the term is resonant.

Crucial fact, to be established below, is that the influence of non-resonant (fast-oscillating) terms onto the end-point map can be made arbitrarily small, if the frequency $\beta/\varepsilon$ of the oscillating factor $e^{i\beta t/\varepsilon}$ is chosen sufficiently large.

Direct verification shows that the terms

$$i(u^*)^2V_{rs}, 2|u^*|^2V_{rs}, V_{rs}^2u^*$$

at the right-hand side of (25) are all non-resonant with respect to (27).

5.4.3. Resonant monomials in the quadratic term $2u^*|V_{rs}|^2$: an obstruction. Consider the quadratic term $2u^*|V_{rs}|^2$, which after the substitution (27) takes form

$$2u^*|V_{rs}|^2 = 2u^* (|v_r(t)|^2 + |s(t)|^2) + 4u^*v_r(t)s(t)\operatorname{Re} \left( e^{-it/\varepsilon}e^{i\rho(t)}f_r f_s \right).$$

The last addend in the parenthesis is non-resonant, while the resonant term $2u^* (|v_r(t)|^2 + |s(t)|^2)$ is an example of so-called obstruction to controllability in the terminology of geometric control.

We can not annihilate or compensate this term but, as far as the group $e^{it/\varepsilon}$ corresponding to the linear Schrödinger equation is quasiperiodic, one can impose conditions on the functions $v_r(\cdot), s(\cdot)$ in such a way, that for a chosen $T > 0$ the influence of the obstructing term onto the end-point map $E_T$ will be nullified.

Indeed, proceeding with the time-variant substitution:

$$u^* = u^*e^{-2i\Upsilon(t)}, \quad \Upsilon(t) = \int_0^t (|v_r(t)|^2 + |s(t)|^2)dt,$$

one gets for $u^*$ the equality:

$$(-i\partial_t + \Delta)u^* e^{2i\Upsilon(t)} = (-i\partial_t + \Delta)u^* - 2u^* (|v_r(t)|^2 + |s(t)|^2).$$

The equation (25) rewritten for $u^*$ becomes

$$(-i\partial_t + \Delta)u^* = |u^*|^2u^* - i(u^*)^2V_{rs}e^{2i\Upsilon(t)} + 2i|u^*|^2V_{rs}e^{-2i\Upsilon(t)} -$$

$$- V_{rs}^2u^*e^{-4i\Upsilon(t)} + 4u^*2\operatorname{Re} \left( e^{i(t/\varepsilon + \rho(t))}\bar{v}_r(t)\bar{s}(t) \right) e^{-2i\Upsilon(t)} +$$

$$+ \bar{V}(t)e^{-2i\Upsilon(t)} + i|V_{rs}|^2V_{rs}e^{-2i\Upsilon(t)}.$$
For the sake of maintaining the end-point map $E_T$ unchanged, one imposes isoperimetric conditions on $\tilde{v}_r(t), \tilde{v}_s(t)$

\begin{equation}
\int_0^T (|\tilde{v}_r(t)|^2 + |\tilde{v}_s(t)|^2)dt = \Upsilon(T) = \pi N, \quad N \in \mathbb{Z};
\end{equation}

this guarantees for the respective trajectories $u^*(0) = u^*(0), \ u^*(T) = u^*(T)$.

**Remark 5.3.** Although the right-hand side of (22) has gained the 'oscillating factors' of the form $e^{-i\mu T(t)}$, the notions of resonant and non-resonant terms will not suffer changes, as long as $e^{-i\mu T(t)}$ is not 'fast oscillating'; for this sake in the ongoing construction $\Upsilon(t)$ will be chosen bounded uniformly in $t, b$ and $\varepsilon > 0$. □

We introduce the notation $\tilde{N}^\varepsilon(u, t)$ for the sum of the non-resonant terms at the right-hand side of (29) and arrive to the equation

\begin{equation}
(-i\partial_t + \Delta)u^* = |u^*|^2 u^* + \tilde{V}(t)e^{-2i\Upsilon(t)} + i|V_{rs}|^2 V_{rs}e^{-2i\Upsilon(t)} + \tilde{N}^\varepsilon(u, t).
\end{equation}

**5.4.4. Extending the control via cubic resonance monomial.** The only resonant monomial in the cubic term $i|V_{rs}|^2 V_{rs}e^{-2i\Upsilon(t)} = i(V_{rs})^2 \bar{V}_{rs}e^{-2i\Upsilon(t)}$, with $V_{rs}, \Upsilon$, defined by (27), (28), is

\begin{equation}
e^{2i(\rho(t) - \Upsilon(t))} \tilde{v}_r^2(t)\tilde{v}_s(t) f_2^2 \hat{f}.
\end{equation}

We join all the non-resonant monomials of this term to $\tilde{N}^\varepsilon(u, t)$ in (31).

Recalling that $f_m = e^{i(m\cdot x + |m|^2 t)}$, we compute

\[ f_2^2 \hat{f}_s = e^{i(2r-s)\cdot x + (2|r|^2 - |s|^2) t} = f_{2r-s} e^{i((2|r|^2 - |s|^2 - |2r-s|^2) t) = f_{2r-s} e^{-2|r-s|^2 t},}\]

and rewrite (32) in the form

\[ \tilde{v}_r^2(t)\tilde{v}_s(t)e^{2i(\rho(t) - |r-s|^2 t - \Upsilon(t))} f_{2r-s},\]

seeing it as an extending control for the mode $f_{2r-s}$.

The equation (31) becomes now

\begin{equation}
(-i\partial_t + \Delta)u^* = |u^*|^2 u^* + \tilde{V}(t)e^{-2i\Upsilon(t)} +
\end{equation}

\[ + \tilde{v}_r^2(t)\tilde{v}_s(t)e^{2i(\rho(t) - |r-s|^2 t - \Upsilon(t))} f_{2r-s} + \tilde{N}^\varepsilon(u, t).\]

Now we take care of the addend

\begin{equation}
\tilde{V}(t)e^{-2i\Upsilon(t)} + \tilde{v}_r^2(t)\tilde{v}_s(t)e^{2i(\rho(t) - |r-s|^2 t - \Upsilon(t))} f_{2r-s}
\end{equation}

at the right-hand side of (33). We wish to choose families of functions $\tilde{V}(t; b), \tilde{v}_r(t; b), \tilde{v}_s(t; b)$ in such a way that (34) approximates $W(t; b)$ (see (21)) in $L_1$-metric uniformly in $b \in B$.

Let

\begin{equation}
\tilde{V}(t; b) = \hat{W}(t; b)e^{2i\Upsilon(t; b)},
\end{equation}

where $\hat{W}(t; b) = \sum_{k \in K, k \neq 2r-s} w_k(t; b)f_k$, is obtained by omission of the summand $w_{2r-s}f_{2r-s}$ in $W(t; b)$.

The controls $\tilde{v}_r(t; b), \tilde{v}_s(t; b)$ will be constructed according to the following
Lemma 5.6. For a family of controls \( b \mapsto w(t; b) \in L^\infty[0, T] \), continuous in \( L^1_t \)-metric, and any \( \varepsilon' > 0 \) one can construct families of real-valued functions
\[
(36) \quad b \mapsto \hat{v}_r(t; b, \varepsilon'), b \mapsto \hat{v}_s(t; b, \varepsilon'),
\]
such that: i) the functions (36) are Lipschitzian in \( t \);
ii) \( \hat{v}_r(0) = \hat{v}_s(0) = 0, \ \hat{v}_r(T) = \hat{v}_s(T) = 0 \);
iii) the functions (36) and their partial derivatives in \( t \) depend on \( b \) continuously in \( L^1_t \)-metric; iii) for each \( b, \varepsilon' \) the conditions (27), (29) hold for them; iv) the \( L^2_t \)-norms of the functions (36) are equibounded for all \( \varepsilon' > 0, b \in B \); and v) for
\[
D_{rs}^\varepsilon' = \hat{v}_r^2(t; b, \varepsilon')\hat{v}_s(t; b, \varepsilon') - |w_{2r-s}(t, b)|,
\]
the estimate
\[
(37) \quad \|D_{rs}^\varepsilon'\|_{L^1_t} = \int_0^T \|\hat{v}_r^2(t; b, \varepsilon')\hat{v}_s(t; b, \varepsilon') - |w_{2r-s}(t, b)|\, dt \leq \varepsilon'.
\]
holds uniformly in \( b \in B \). □

Lemma 5.6 is proved in the Appendix. Meanwhile we construct the family \( \rho(t; b) \).

Lemma 5.7. Given the family (29), constructed in the previous Lemma, there exists a family of Lipschitzian functions \( \rho(\cdot; b) \), such that \( b \mapsto \rho(\cdot; b) \) and \( b \mapsto \partial_t \rho(\cdot; b) \) are continuous in \( L^1_t \)-metric and
\[
(38) \quad \int_0^T \|\hat{v}_r^2(t; b, \varepsilon')\hat{v}_s(t; b, \varepsilon')e^{2i(\rho(t)-|r-s|^2t-Y(t))} - w_{2r-s}(t, b)\, dt \leq \varepsilon'. \Box
\]

Proof. We choose
\[
(39) \quad \rho(t; b) = \frac{1}{2} Arg \left( w_{2r-s}(t, b) \right) + |r-s|^2t + Y(t; b).
\]
As in the Remark 5.2 we may think that \( w_{2r-s}(t, b) \) are smooth in \( t \) and hence \( \rho(t; b) \) are Lipschitzian in \( t \). The dependence of \( \rho \) and \( \partial \rho/\partial t \) on \( b \) is continuous in \( L^1_t \)-metric. By (37), (39) we conclude (38). □

Remark 5.4. For each \( \varepsilon, \varepsilon' > 0 \) the functions \( v^r(t, b), v^s(t, b) \) defined by (22), (35), (27), as well as their derivatives in \( t \), depend continuously on \( b \) in \( L^1_t \)-metric.

By the construction the map \( b \mapsto E_T(v(t, b, \varepsilon, \varepsilon')) \), where \( E_T \) is the end-point map of the equation (??), coincides with the end-point map \( b \mapsto E_T^{\varepsilon, \varepsilon'}(b) \) of the equation
\[
(40) \quad (-i\partial_t + \Delta)u^* = |u^*|^2u^* + W(t, b) + D_{rs}^\varepsilon(t) + \hat{N}_{rs}^\varepsilon(u^*, t, b), \quad u^*(0) = u^0.
\]

The proof of Approximative Lemma 5.2 would be completed by the following
Lemma 5.8. The end-point map $E^{\varepsilon}_{T}(b)$ of the system (39) calculated for the family of controls, defined by the Lemmae 5.6, 5.7 converges to the end-point map $E_{T}^{\lim}$ of the 'limit system'\

\begin{equation}
(-i\partial_{t} + \Delta)u^{*} = |u^{*}|^{2}u^{*} + W(t, b),
\end{equation}

uniformly in $b$ as $\varepsilon + \varepsilon' \to 0$. □

Would the term $\tilde{N}^{\varepsilon}(u^{*}, t, b)$ be missing in (40) we could derive the conclusion of the Lemma from the Proposition 4.5. In the presence of the fast-oscillating term $\tilde{N}^{\varepsilon}(t, u)$ the needed passage to a limit system will be established by virtue of Proposition 5.9 of the next Subsection.

5.5. On continuity of solutions in the right-hand side with respect to the relaxation metric. The results presented in this Subsection, regard continuous dependence of the mild solutions of NLS equation with respect to the perturbations of its right-hand side, which are small in so-called relaxation seminorms.

The seminorms are suitable for treating fast oscillating terms. In finite-dimensional context the respective continuity results are part of theory of relaxed controls. Several relaxation results for semilinear systems in Banach spaces can be found in [10, 11]. Below we provide version adapted for the proof of Lemma 5.8.

Let us come back to the semilinear equation (3) and its perturbation (4). We assume the perturbations $\phi : [0, T] \times H \to H$ in (4) to belong to a family $\Phi$, which satisfies the following conditions:

(i) elements of $\Phi$ are continuous functions;
(ii) the family $\Phi$ is equibounded and equi-Lipschitzian, which means that each $\phi \in \Phi$ together with $G : [0, T] \times H \to H$ satisfy the properties (3), (4), (5) with the same functions $\beta_{c}(t)$.
(iii) the set $\{\phi(t, u(t)) | t \in [0, T], \phi \in \Phi\}$ is completely bounded in $H$ for each $u(\cdot) \in C([0, T], H)$.

We introduce the relaxation seminorm $||\cdot||_{c}^{rx}$ for the elements of $\Phi$ by the formula:

$$||\phi||_{c}^{rx} = \sup_{t, t' \in [0, T], ||u|| \leq c} \left|\int_{t}^{t'} \phi(\tau, u) d\tau\right|_{H}.$$  

As one can see the relaxation seminorms of fast-oscillating functions are small. For example $||f(t) e^{it/\varepsilon}||_{c}^{rx} \to 0$, as $\varepsilon \to 0$ for each function $f \in L^{1}[0, T]$ (and each $c$) according to Lebesgue-Riemann lemma.

Now we formulate the needed continuity result from which Lemma 5.8 will follow.

\footnote{The property of complete boundedness would follow, for example, from the complete boundedness of the sets $\Phi(t, u) = \{\phi(t, u) | \phi \in \Phi\}$ for each fixed couple $(t, u)$ together with upper semicontinuity of the set-valued map $(t, u) \mapsto \Phi(t, u)$.}
Proposition 5.9. Let a mild solution \( \tilde{u}(t) \in C([0,T], H) \) of the NLS equation (3) satisfy \( \sup_{t \in [0,T]} ||u(t)||_H < c \). Let the family \( \Phi \) of perturbations satisfy the conditions (i)-(iii) just introduced. Then \( \forall \varepsilon > 0 \exists \delta > 0 \) such that, whenever \( \phi \in \Phi, \|\phi\|^2 + \|u^0 - \tilde{u}\|_H < \delta \), then the mild solution \( u(t) \) of the perturbed equation (4) exists on the interval \([0,T]\), is unique and satisfies the estimate \( \sup_{t \in [0,T]} ||u(t) - \tilde{u}(t)||_H < \varepsilon. \)

Sketch of the proof of the Proposition 5.9. Under the assumptions of the Proposition the solution of the equation (4) exists locally and is unique (see Proposition 4.1). Global existence will follow from the boundedness of the estimate (12) by which:

\[
\|u(t) - \tilde{u}(t)\| \leq \left( \|u^0 - \tilde{u}\| + \left\| \int_0^t e^{-i\tau \Delta} \phi(\tau, \tilde{u}(\tau)) d\tau \right\| \right) C^t e^{C \int_0^t \beta_c(\tau) d\tau}.
\]

The conclusion of the Proposition 5.9 would follow from Lemma 5.10. Let the family \( \Phi \) satisfy the assumptions of the Proposition 5.9 and let \( \tilde{u}(t) \) be solution of (3). Then \( \forall \varepsilon > 0, \exists \delta > 0 \) such that, \( \forall \phi \in \Phi: \)

\[
\|\phi\|^2 < \delta \Rightarrow \left\| \int_0^t e^{-i\tau \Delta} \phi(\tau, \tilde{u}(\tau)) d\tau \right\| < \varepsilon. \]

Proof of the Lemma can be found in Appendix.

Let us remark on validity of conditions of the Proposition 5.9 for the limit equation (11) and for its perturbation (10).

The perturbation \( D\phi\varepsilon(t) + \hat{N}\varepsilon(u^*, t, b) \) at the right-hand side of (40) can be seen as an operator \( \phi : [0,T] \times H \rightarrow H: \)

\[
\forall u(\cdot) \in H : \phi \varepsilon : u(x) \mapsto W^0(t, x) + u(x)W^{11}(t, x) + \tilde{u}W^{12}(t, x) + u^2(x)W^{21}(t, x) + |u(x)|^2W^{22}(t, x),
\]

where \( W^{ij}(t, x) = w(t) e^{ik \cdot x} e^{ip(t) t + \varepsilon / \theta} \), and \( w(t), p(t) \) are Lipschitzian. The continuity, equiboundedness and equi-Lipschitzianness are concluded by application of 'Product Lemma' (1.2).

To confirm the complete boundedness assumption for \( \phi \varepsilon(t, \tilde{u}(t)) \), where \( \tilde{u}(t) \) is continuous on \([0,T]\), we substitute the factors \( e^{it \Delta / \varepsilon} \) in the coefficients \( W^{ij}(t, x) \) by \( e^{it \theta} \), \( \theta \in \mathbb{T}^1 \) and arrive to a function \( \psi(t, \theta) \), \( \theta \in \mathbb{T}^1 \), which is continuous on \([0,T] \times \mathbb{T}^1 \), and whose compact range contains the range of \( \phi \varepsilon(t, \tilde{u}(t)) \) for all \( \varepsilon > 0 \).

6. CONTROLLABILITY PROOFS (MAIN RESULT 1)

We have established above (Corollary 5.5) that whenever set \( \tilde{K} \) of the controlled modes is saturating, then the NLS equation (1) is solidly controllable in projection on any 'coordinate' subspace \( \mathcal{F}_K \) with finite \( K \subset \mathbb{Z}^d \).
In the Section we prove that this implies controllability in projection on any finite-dimensional subspace of $H^s$, $s > d/2$ and $H^s$-approximate controllability.

Together with the description of the classes of saturating sets, provided in Section 5 this would complete the proof of the Main result 1.

6.1. Approximate controllability. Let us fix $\tilde{\varphi}, \varphi \in H^s$ and $\varepsilon > 0$, we wish to steer the NLS equation from $u^0 = \tilde{\varphi}$ to the $\varepsilon$-neighborhood of $\varphi$ in the $H^s$-metric.

Consider the Fourier expansions for $\tilde{\varphi}, \varphi$ with respect to $e^{ik \cdot x}$, $k \in \mathbb{Z}^d$. Denote by $\Pi_N$ the orthogonal projection of $\varphi$ onto the space of modes $\mathcal{F}_N = \text{Span}\{e^{ik \cdot x}, |k| \leq \}$. Obviously $\Pi_N(\tilde{\varphi}) \to \tilde{\varphi}, \Pi_N(\varphi) \to \varphi$ in $H^s$ as $N \to \infty$.

Choose such $N$ that the $H^s$-norms of $(I - \Pi_N)\varphi$ and $(I - \Pi_N)\tilde{\varphi}$ are $\leq \varepsilon/4$.

By Lemma 5.3 there exists a family of controlling source terms $W(b) = \sum_{|k| \leq N} w_k(t; b)f_k$ such that $\Pi_N(W(b))$ covers $\Pi_N(\tilde{\varphi})$ solidly and in addition $\|(I - \Pi_N)(E_T(W(b)) - \tilde{\varphi})\| \leq \varepsilon/4$. Then $\|(I - \Pi_N)E_T(W(b))\| \leq \varepsilon/2$.

If a set $\mathcal{K}$ of controlled modes is saturating, then $\{k \mid |k| \leq N\}$ is a proper extension of $\mathcal{K}$. By Approximative Lemma 5.2 there exists family of controlling source terms $V(b) = \sum_{k \in \mathcal{K}} v_k(t; b)f_k$ such that

$$\|E_T(V(b)) - E_T(W(b))\| \leq \varepsilon/4, \forall b \in B,$$

and $\Pi_N E_T(V(b))$ covers the point $\Pi_N(\tilde{\varphi})$. Then $\forall b : \|(I - \Pi_N)E_T(V(b))\| \leq 3\varepsilon/4$ and for some $\tilde{b}$: $\Pi_N E_T(V(\tilde{b})) = \Pi_N \tilde{\varphi}$. Then $\|E_T(V(\tilde{b})) - \tilde{\varphi}\| \leq \varepsilon$. $\square$

6.2. Controllability in finite-dimensional projections. Let $\mathcal{L}$ be $\ell$-dimensional subspace of $H^s$ and $\Pi^\mathcal{L}$ be the orthogonal projection of $H^s$ onto $\mathcal{L}$.

First we construct a finite-dimensional coordinate subspace which is projected by $\Pi^\mathcal{L}$ onto $\mathcal{L}$. Moreover for each $\varepsilon > 0$ one can find a finite-dimensional coordinate subspace $\mathcal{L}^\varepsilon$ and its $\ell$-dimensional (non-coordinate) subspace $\mathcal{L}_\varepsilon$, which is $\varepsilon$-close to $\mathcal{L}$. The latter means that not only $\Pi^\mathcal{L} \mathcal{L}_\varepsilon = \mathcal{L}$ but also the isomorphism $\Pi_\varepsilon = \Pi^\mathcal{L}|_{\mathcal{L}_\varepsilon}$ is $\varepsilon$-close to the identity operator. It is an easy linear-algebraic construction; which can be found in [1, Section 7].

Without lack of generality we may assume that the orthogonal projection $\Pi_\varepsilon$ onto $\mathcal{L}^\varepsilon$ satisfies: $\|\Pi_\varepsilon(\varphi) - \tilde{\varphi}\|_{H^s} \leq \varepsilon$.

As far as the set $\mathcal{K}$ of controlled modes is saturating, $\mathcal{L}$ is proper extension of $\mathcal{K}$ and the system is solidly controllable in the observed component $q^\mathcal{L}$.

Let $\mathcal{B}$ be a ball in $\mathcal{L}$ centered at the origin. Consider $\mathcal{B}^\varepsilon = (\Pi_\varepsilon)^{-1}\mathcal{B}$; obviously $\mathcal{B}^\varepsilon \subset \mathcal{L}^\varepsilon \subset \mathcal{L}^\varepsilon$. We take a ball $\mathcal{B}_\varepsilon$ in $\mathcal{L}_\varepsilon$, which contains $\mathcal{B}^\varepsilon$ and hence $\Pi_\varepsilon(\mathcal{B}_\varepsilon) \supset \mathcal{B}$.

Reasoning as in the previous Subsection one establishes existence of a family of controls $V(b) = \sum_{k \in \mathcal{K}} v_k(t; b)f_k$ such that $\Pi_\varepsilon E_T(V(b))$ covers $\mathcal{B}_\varepsilon$ solidly and $\forall b : \|(I - \Pi_\varepsilon)E_T(V(b))\| \leq 2\varepsilon$.

Then choosing $\varepsilon > 0$ sufficiently small we achieve that

$$\Pi^\mathcal{L} E_T(V(b)) = \Pi^\mathcal{L} (\Pi_\varepsilon + (I - \Pi_\varepsilon)) E_T(\tilde{V}(b))$$

covers $\mathcal{B}$. $\square$
7. Lack of exact controllability proof (Main result 2)

Let us represent the controlled cubic defocusing NLS equation (1-2) in the form
\[(42)\]
\((-i\partial_t + \Delta)u = |u|^2u + \sum_{k \in \mathcal{K}} \dot{w}_k(t)f_k, \quad u|_{t=0} = u^0 \in H^s,\]

where \(f_k = e^{i(k \cdot x + |k|^2t)}\) and \(\mathcal{K} \subset \mathbb{Z}^d\) is a finite set. The controls \(\dot{w}_k(t) \in L^1([0, T], \mathbb{C})\) are time-derivatives of absolutely continuous functions \(w_k(t), w_k(0) = 0\). In this Section \(W^{1,1}([0, T], \mathcal{F}_\mathcal{K})\) stays for the space of \(\mathcal{F}_\mathcal{K}\)-valued absolutely continuous functions, vanishing at \(t = 0\).

Global existence and uniqueness results for solution of this equation in \(C([0, T], H^s)\) is classical (Section 4).

Consider the end-point map \(E_T : (\sum_{k \in \mathcal{K}} \dot{w}_k(t)f_k) \mapsto u|_{t=T}\) which maps the space of controlling source terms from \(L^1([0, T], \mathcal{F}_\mathcal{K})\) into the state space \(H^s\). The image of \(E_T\) is time-\(T\) attainable set \(A_{T,u^0}\) of the controlled equation (42).

Introducing \(W(t) = \sum_{k \in \mathcal{K}} w_k(t)f_k \in W^{1,1}([0, T], \mathcal{F}_\mathcal{K})\) we bring (as in Subsection 5.2) the equation (42) to the form \((-i\partial_t + \Delta)(u - iW(t)) = |u|^2u,\) and after another time-variant substitution \(u - iW(t, x) = u^*(t)\) to the form
\[(43)\]
\((-i\partial_t + \Delta)u^* = |u^* + iW(t)|^2(u^* + iW(t)), \quad u|_{t=0} = u^0,\]

which we look at as a semilinear control system with the inputs \(W(\cdot)\) belonging to \(W^{1,1}([0, T], \mathcal{F}_\mathcal{K})\). Obviously for each input \(W(\cdot)\) the solution of (43) exists and is unique on \([0, T]\).

Introduce the input-trajectory map \(E^* : W(\cdot) \mapsto u^*(\cdot)\) of the equation (43). The following result is essentially a corollary of Proposition 4.3.

Lemma 7.1. Consider a ball
\[\mathcal{B}_R = \{W(\cdot) \in W^{1,1}([0, T], \mathcal{F}_\mathcal{K}) \mid \|W(\cdot)\|_{W^{1,1}} \leq R\}.\]

Then
\[\exists L_R > 0 : \|u^*_2(t) - u^*_1(t)\|_H \leq L_R \int_0^T \|W_2(t) - W_1(t)\|_{\mathcal{F}_\mathcal{K}} dt, \quad \forall t \in [0, T],\]

\[\forall W_1(\cdot), W_2(\cdot) \in \mathcal{B}_R\] and the corresponding trajectories \(u^*_1(t), u^*_2(t)\) of (43). This means that the input-trajectory map \(E^*\) is Lipschitz on \(\mathcal{B}_R\), endowed with \(L^1([0, T], \mathcal{F}_\mathcal{K})\)-metric, the space of trajectories \(u^*(\cdot)\) endowed with \(C([0, T], H^s)\)-metric. \(\square\)

We postpone the proof of Lemma 7.1 to the Appendix, and now derive the Main Result 2.

Consider the composition of maps
\[(\dot{w}_k)_{k \in \mathcal{K}} \mapsto W(\cdot) \mapsto E^*_T(W) = E^*(W)|_{t=T},\]

where \(E^*_T\) is the end-point map \(W(\cdot) \mapsto u|_{t=T}\) for the equation (43).
The relation between the end-point maps of the controlled equations (12) and (13) provides the equality \( E_T(\dot{w}) = E_T^*(W(\cdot)) + iW(T) \) and therefore the image of \( E_T \) (the attainable set) is contained in the image of the map

\[
\Theta : (W(\cdot), \vartheta) \mapsto E_T^*(W(\cdot)) + \vartheta, \quad (W(\cdot), \vartheta) \in W^{1,1}([0, T], F_{\bar{K}})^2 \times C^\kappa.
\]

Representing \( L^1([0, T], C^\kappa) \) as a union of the balls \( \bigcup_{n \geq 1} B_n \) of radii \( n \in \mathbb{N} \) we note that the image \( \mathcal{I}(B_n) \) under the map \( \mathcal{I} : \dot{w}(\cdot) \mapsto (w(\cdot), w(T)) \) is bounded in \( W^{1,1}([0, T], F_{\bar{K}})^2 \times C^\kappa \) and pre-compact (completely bounded) in \( W^{1,1}([0, T], F_{\bar{K}})^2 \times C^\kappa \).

By the Lemma 7.1 the map \( E_T^* \) is Lipschitzian in the metric of \( L^1([0, T], F_{\bar{K}})^2 \); therefore \( \Theta \) is Lipschitzian in the metric of \( L^1([0, T], F_{\bar{K}})^2 \times C^\kappa \) and hence \( E_T(B_n) \) is contained in completely bounded image \( \Theta(\mathcal{I}(B_n)) \subset H^\kappa \). Then the attainable set of (12) is contained in a countable union of pre-.compacts \( \bigcup_{n \geq 1} \Theta(\mathcal{I}(B_n)) \) and by Baire category theorem has a dense complement in \( H^\kappa \). \( \square \)

8. Saturating sets of controlled modes and controllability

As we have established above the saturating sets of controlled modes suffice for providing the approximate controllability and controllability in projection on each finite-dimensional subspace for the NLS equation (11).

Here we describe some classes of saturating sets in \( \mathbb{Z}^d, d \geq 1 \).

Starting with a set \( \hat{K} \subset \mathbb{Z}^d \) and recalling the definition of proper extension we define the sequence of sets \( K^j \subset \mathbb{Z}^d \):

\[
K^1 = \hat{K}, K^j = \{2m - n \mid m, n \in K^{j-1} \}, \quad j = 2, \ldots ; \quad K^\infty = \bigcup_{j=1}^\infty K^j.
\]

Taking \( m = n \) in (44) we conclude that \( K^1 \subset \cdots \subset K^j \subset \cdots \subset K^\infty \) and for each pair \( j < j' \) the set \( K^{j'} \) is proper extension of \( K^j \). The set \( K^\infty \) is invariant with respect to the operation

\[
(k, \ell) \mapsto 2k - \ell, \quad k, \ell \in \mathbb{Z}^d.
\]

A finite set \( \hat{K} \subset \mathbb{Z}^d \) of modes is saturating if and only if \( K^\infty = \mathbb{Z}^d \).

**Proposition 8.1.** For each \( k \in \mathbb{Z} \) the two-element set \( \hat{K} = \{k, k + 1\} \) is saturating in \( \mathbb{Z} \). \( \square \)

**Proof.** Note that \( k - 1 = 2k - (k + 1) \in K^\infty \).

Assume that for some natural \( N \): \( k - N, k - N + 1, \ldots, k + N \in K^\infty \). Then

\[
2(k + 1) - (k - (N - 1)) = k + (N + 1) \in K^\infty,
\]

\[
2(k - 1) - (k + (N - 1)) = k - (N + 1) \in K^\infty,
\]

and the needed conclusion follows by induction. \( \square \)
Proposition 8.2. Let the vectors \( k_1, \ldots, k_d \in \mathbb{Z}^d \) be such that
\[
D = \det (k_1, k_2, \ldots, k_d) = \pm 1. \tag{46}
\]
Then the \( 2^d \)-element set
\[
\hat{C}^d = \left\{ \sum_{j=1}^p k_{i_j} \mid i_1 < \cdots < i_p, \ 0 \leq p \leq d \right\} \subset \mathbb{Z}^d \tag{47}
\]
is saturating. \( \square \)

Remark 8.1. In (17) \( p = 0 \) corresponds to the null vector \( 0 \in \mathbb{Z}^d \). \( \square \)

Proof. i) Note that if \( z \in \mathbb{C}^\infty \), then \( -z = 2 \cdot 0 - z \in \mathbb{C}^\infty \).

ii) We prove first that for any set \( \hat{C} = \{0, \ell_1, \ldots, \ell_N\} \) the set \( \mathbb{C}^\infty \), defined by (14) coincides with the set \( C_e^N \) of all integer linear combinations \( \sum_{s=1}^N \alpha_s \ell_s \) with at most one odd coefficient \( \alpha_s \).

Indeed \( C_e^N \supset \hat{C} \) and is invariant with respect to the operation (15).

Also \( C_e^N \supset -\hat{C} \) and if \( \pm z \in \mathbb{C}^\infty \), then \( \forall \ell_{s_0} \) : \( \pm z + 2 \ell_{s_0} \in \mathbb{C}^\infty \) and by induction \( \pm z + \sum_{s=1}^N 2\alpha_s \ell_s \in \mathbb{C}^\infty \), for any \( \alpha_1, \ldots, \alpha_N \in \mathbb{Z} \). In particular each linear combination \( \sum_{s=1}^N 2\alpha_s \ell_s \) with even coefficients belongs to \( \mathbb{C}^\infty \), together with \( \ell_{s_0} + \sum_{s=1}^N 2\alpha_s \ell_s \in \mathbb{C}^\infty \) for each \( s_0 \). Hence \( C_e^N = \mathbb{C}^\infty \).

iii) Now we prove that for \( \hat{C}^d \) defined by (17) the extension \( \mathbb{C}^\infty \) contains all integer linear combinations \( \sum_{j=1}^d \alpha_j k_j \).

Picking any combination \( \sum_{j=1}^d \alpha_j k_j \) we select its odd coefficients \( \alpha_{j_1}, \ldots, \alpha_{j_p}, \ j_1 < \cdots < j_p \). Then
\[
\sum_{j=1}^d \alpha_j k_j = (k_{j_1} + \cdots + k_{j_p}) + \sum_{j=1}^d \alpha'_{j} k_j,
\]
with all \( \alpha'_{j} \) even. Hence \( \sum_{j=1}^d \alpha_j k_j \) is representable as a linear combination of the elements of (17) with at most one odd coefficient. By ii) this proves that \( \sum_{j=1}^d \alpha_j k_j \in \mathbb{C}^\infty \).

iv) Finally we prove that whenever (16) holds, the set of all integer combinations \( \sum_{j=1}^d \alpha_j k_j \) coincides with \( \mathbb{Z}^d \).

Picking any \( \ell \in \mathbb{Z}^d \), we solve the equation \( \sum_{j=1}^d \alpha_j k_j = \ell \) by Kramer rule computing \( \alpha_j = D_j / D, \ j = 1, \ldots, d \), where \( D_j = \det(k_1, \ldots, \ell_j, \ldots, k_d) \) are integer, and \( D = \pm 1 \) according to (16). \( \square \)

Remark 8.2. An example of a saturating set \( \hat{C} \subset \mathbb{Z}^d \) is the set of all vectors with components equal either 0 or 1 or, in other words, the set of vertices of the unit cube \( [0, 1]^n \). \( \square \)
9. Appendix: some proofs

9.1. Proof of the Lemma 4.6

Proof. Similarly to [11] we get an estimate (all the norms are taken in $H$):

$$
\|u^\varepsilon(t) - \hat{u}(t)\| \leq \|e^{i\varepsilon t \Delta} u^0 - u^0\| + \left\| \int_0^t e^{-i\varepsilon \tau \Delta} \varepsilon G(\tau, u^\varepsilon(\tau), b) d\tau \right\| + \\
\leq \varepsilon \left\| \int_0^t e^{-i\varepsilon \tau \Delta} G(\tau, u^\varepsilon(\tau), b) d\tau \right\| + \left\| (e^{i\varepsilon \tau \Delta} - I) u^0 \right\| + \\
+ \left\| \int_0^t (e^{-i\varepsilon \tau \Delta} - I) \phi(\tau, \hat{u}(\tau), b) d\tau \right\| + \int_0^t \left\| \phi(\tau, u^\varepsilon(\tau), b) - \phi(\tau, \hat{u}(\tau), b) \right\| d\tau.
$$

The rightmost addend admits an upper bound $\int_0^t \beta_c(\tau) \|u^\varepsilon(\tau) - \hat{u}(\tau)\| d\tau$. We would arrive to the needed conclusion by virtue of Gronwall inequality, after proving that the other three addends tend to 0, as $\varepsilon \to 0$.

We only comment on the addend $\left\| \int_0^t (e^{-i\varepsilon \tau \Delta} - I) \phi(\tau, \hat{u}(\tau), b) d\tau \right\|$, the other two assertions being obvious.

By the assumptions of the Lemma the map $(\tau, b) \mapsto \phi(\tau, \tilde{u}(\tau), b)$ is continuous on $[0, T] \times B$ and hence its range $R$ is a compact subset of Hilbert space $H$. Being $\pi_N$ the orthogonal projection of $H$ on $F_N$ and $R_N = \pi_N(R)$, we assert that

$$
\forall \varepsilon > 0 \exists N, \forall n \geq N : \text{dist}(R, R_N) < \varepsilon/2.
$$

Also there exists $c_N > 0$ such that

$$
\forall y \in R_N, \forall t \in [0, T] : \left\| (e^{-i\varepsilon \tau \Delta} - I) y \right\| \leq c_N \varepsilon.
$$

Given that $\left\| e^{-i\varepsilon \tau \Delta} - I \right\| \leq 2$ one concludes

$$
\forall z \in R : \left\| (e^{-i\varepsilon \tau \Delta} - I) z \right\| \leq \left\| (e^{-i\varepsilon \tau \Delta} - I) (z - \pi_N z) \right\| + \\
+ \left\| (e^{-i\varepsilon \tau \Delta} - I) \pi_N z \right\| \leq 2(\varepsilon/2) + c_N \varepsilon = (c_N + 1)\varepsilon.
$$

and

$$
\left\| \int_0^t (e^{-i\varepsilon \tau \Delta} - I) \phi(\tau, \hat{u}(\tau), b) d\tau \right\| \leq (c_N + 1)\varepsilon T.
$$

\[\square\]

9.2. Proof of the Lemma 5.6

In the proof we use $\varepsilon$ in place of $\varepsilon'$.

As in the Remark 5.2 we can smoothen the family $w_{2r-s}(t, b)$ arriving to a family of smooth functions $\hat{w}(t, b)$ such that $\forall b : \|\hat{w}(t, b) - w_{2r-s}\|_{L^1} \leq \varepsilon/2$, and $\hat{w}(t, b), \partial_t \hat{w}(t, b)$ depend on $b$ continuously in $L^1$-metric. This implies that $\|\hat{w}(t, b)\|_{L^\infty}$ are equibounded by some $C_w > 0$, and for each $\tau \in [0, T]$ the values $\hat{w}(\tau, b)$ depend continuously on $b$.
First we choose real-valued nonnegative continuous function \( \tilde{v}_s(t) \) such that \( \tilde{v}_s^2(t) \) is: piecewise-linear, vanishing at 0, \( T \), and (constant) equal to \( \pi(T - \varepsilon^2)^{-1} \) on \( [\varepsilon^2, T - \varepsilon^2] \), while being linear on \( [0, \varepsilon^2] \) and \( [T - \varepsilon^2, T] \).

By the construction \( \int_0^T \tilde{v}_s^2(t)dt = \pi \). Assuming \( \varepsilon^2 < T/2 \), we get \( \|\tilde{v}_s^2\|_{L^{\infty}} \leq 2\pi/T \).

Denote \( I_\varepsilon = [0, \varepsilon^2] \cup [T - \varepsilon^2, T] \) and let \( w_\varepsilon(t, b) \) be the restriction of the function \( \pi^{-1}(T - \varepsilon^2)\tilde{w}(t, b) \) onto \( [0, T) \setminus I_\varepsilon \). Let

\[
\int_{\varepsilon^2}^{T-\varepsilon^2} |w_\varepsilon(t, b)|^2dt = A(b), \quad A = \max_{b \in B} A(b);
\]

\( A(b) \) is continuous and the maximum is achieved. Take \( N = \lceil A/\pi \rceil + 1 \). Extend \(^4\) the family \( |w_\varepsilon(t, b)| \) to a family of Lipschitzian functions \( \tilde{v}_s(t, b) \) on \( [0, T] \) such that \( \tilde{v}_s(0, b) = \tilde{v}_s(T, b) = 0, \int_0^T |\tilde{v}_s(t, b)|^2dt = \pi N \) and \( \tilde{v}_s(t, b), \partial_t\tilde{v}_s(t, b) \) depend continuously on \( b \) in \( L^1 \)-metric.

Then

\[
\int_0^T |\tilde{v}_r(t)|^2 + |\tilde{v}_s(t, b)|^2dt = \pi(N + 1).
\]

Obviously \( \tilde{v}_r^2(t; b)\tilde{v}_s(t; b) = |w_{2r-s}(t, b)| \) on \( [\varepsilon^2, T - \varepsilon^2] \) and

\[
\int_{I_\varepsilon} |\tilde{v}_s(t)|^2dt \leq \pi N;
\]

hence \( \int_{I_\varepsilon} |\tilde{v}_s(t; b)|dt \leq \varepsilon\sqrt{2\pi N} \) by Cauchy-Schwarz inequality.

Then

\[
\int_0^T |\tilde{v}_r^2(t)\tilde{v}_s(t, b) - |w_{2r-s}(t, b)|| dt \leq \frac{\varepsilon}{2} + \\
+ \int_0^T |\tilde{v}_r^2(t)\tilde{v}_s(t, b) - |\tilde{w}(t, b)|| dt \leq \varepsilon/2 + \int_{I_\varepsilon} (|\tilde{v}_r^2(t)\tilde{v}_s(t, b)| + |\tilde{w}(t; b)|) dt \leq \\
\leq \frac{\varepsilon}{2} + \|\tilde{v}_r^2(t)\|_{L^{\infty}}\sqrt{2\pi N} + 2C_we^2 \leq \frac{\varepsilon}{2} + (2\pi \varepsilon/T)\sqrt{2\pi N} + 2C_we^2. \quad \Box
\]

9.3. Proof of Lemma 5.10. For a continuous \( \tilde{u}(t) \) and \( \Phi \) possessing Lipschitzian property, we prove that \( \forall \delta > 0 \exists \delta' > 0 \) such that \( \forall \phi \in \Phi \):

\[
(48) \quad \sup_{t, t' \in [0, T], \|u\| \leq b} \left| \int_t^{t'} \phi(\tau, u)d\tau \right| < \delta' \Rightarrow \sup_{t, t' \in [0, T]} \left| \int_t^{t'} \phi(\tau, \tilde{u}(\tau))d\tau \right| < \delta.
\]

Indeed if \( \omega(\tau) \) is the modulus of continuity for \( \tilde{u}(t) \) and \( \sup_{t \in [0, T]} \|\tilde{u}(t)\| \leq c \), then

\[
\sup_{t, t' \in [0, T]} \left| \int_t^{t'} \phi(\tau, \tilde{u}(\tau))d\tau \right| = \left| \int_t^{t'} \phi(\tau, \tilde{u}(\tau))d\tau \right| \leq \sum_{j=0}^{N-1} \left| \int_{t_j}^{t_{j+1}} \phi(\tau, \tilde{u}(\tau))d\tau \right|)
\]

\(^4\)For example by polynomials of fixed degree, defined on \( [0, \varepsilon^2] \) and \( [T - \varepsilon^2, T] \), with coefficients depending continuously on \( b \).
where $t = t_0 < t_1 < \cdots < t_N = t'$ is a partition of $[t, t'] \subset [0, T]$ into $N \leq T/\eta$ subintervals of length $\eta$. Then
\[
\sup_{t, t': [0, T]} \left\| \int_t^{t'} \phi(\tau, \bar{u}(\tau)) d\tau \right\| \leq \sum_{j=0}^{N-1} \left\| \int_{t_j}^{t_{j+1}} (\phi(\tau, \bar{u}(\tau)) - \phi(\tau, \bar{u}(t_j))) d\tau \right\| + \\
+ \sum_{j=0}^{N-1} \left\| \int_{t_j}^{t_{j+1}} \phi(\tau, \bar{u}(t_j)) d\tau \right\| \leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \beta_c(\tau) \left\| \bar{u}(\tau) - \bar{u}(t_j) \right\| d\tau + \\
+ N\|\phi\|_{rx} \leq C\omega(\eta) + \frac{T}{\eta} \|\phi\|_{rx}.
\]
Choosing $\eta = \|\phi\|_{rx}^{1/2}$ we get
\[
\sup_{t, t': [0, T]} \left\| \int_t^{t'} \phi(\tau, \bar{u}(\tau)) d\tau \right\| \leq C\omega(\|\phi\|_{rx}^{1/2}) + T\|\phi\|_{rx}^{1/2}
\]
and conclude (48).

Let us introduce
\[
\tilde{\Phi} = \{ \varphi(\cdot) | \varphi(\cdot) = \phi(\cdot, \bar{u}(\cdot)), \phi \in \Phi \}.
\]
According to the aforesaid it suffices to prove that $\forall \varepsilon > 0 \exists \delta > 0$ such that $\forall \varphi \in \tilde{\Phi}$:
\begin{equation}
\|\varphi\|_{rx} < \delta \Rightarrow \left\| \int_0^t e^{-i\tau\Delta} \varphi(\tau) d\tau \right\| < \varepsilon.
\end{equation}

The set $R = \{ \varphi(\tau) | \tau \in [0, T], \varphi \in \tilde{\Phi} \}$ is completely bounded by the assumptions of the Lemma.

Taking for a natural $n$ the orthogonal projection $\Pi_n$ onto $\mathcal{F}_n = \text{Span}\{f_k | |k| \leq n\}$, we conclude by precompactness of $R$ that $\sup_{x \in R} \|x - \Pi_n x\| \to 0$, as $n \to \infty$.

Take a partition $0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$ of the interval $[0, T]$ into subintervals of lengths $\eta = T/N$. We represent the integral in (49) as a sum
\[
\int_0^t e^{-i\tau\Delta} \varphi(\tau) d\tau = \int_0^t e^{-i\tau\Delta} (\varphi(\tau) - \Pi_n \varphi(\tau)) d\tau + \\
+ \sum_{j=1}^\omega e^{-i\tau_j \Delta} \int_{\tau_j}^{\tau_j} \Pi_n \varphi(\tau) d\tau + \\
+ \sum_{j=0}^{N-1} \int_{\tau_{j+1}}^{\tau_j} e^{-i\tau_j \Delta} \left( e^{-i(\tau - \tau_j)\Delta} - I \right) \Pi_n \varphi(\tau) d\tau.
\]
Recalling that:
\begin{itemize}
\item $e^{-i\tau\Delta}$ is an isometry of $H$;
\end{itemize}
Then for $W_9.4$.

Proof of Lemma 7.1. Where $c$ implies $c$ while $eta$ we conclude

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