Market equilibrium in multi-tier supply chain networks

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Abstract

We model multistage competition in supply chains by a sequential game theoretical model in a network. We show the uniqueness of equilibrium for series parallel networks and provide a linear time algorithm to compute the equilibrium. We study the impact of the network structure on the firms’ payoffs, the total trade flow, and the overall social welfare.

KEYWORDS

competition, multistage, series parallel networks, supply chain networks

1 INTRODUCTION

Supply chain networks are typically multi-tier and heterogeneous. A firm’s decision influences not only other firms within the same tier but also across. The literature on game-theoretical models of supply chain networks, however, has largely focused on two extreme cases: heterogeneous two-tier or three-tier networks (Bimpikis, Candogan, & Ehsani, 2019; Bimpikis, Ehsani, & Ilkilç, 2019; Cachon & Lariviere, 2005; Kranton & Minehart, 2001) and a linear chain of n-tier firms (Corbett & Karmarkar, 2001; Wright & Wong, 2014). One main reason for this is that most models of sequential decision-making in multi-tier supply chain networks are intractable. Sequential decision making is a well-observed phenomenon in supply chains because firms at the top tier typically need to make decisions on the quantity and price to sell to firms in the next tier and the buying firms then decide how much to buy from which suppliers, and continue to pass on the goods by determining the quantity and price to firms at the next level.

To study such models, one needs to analyze multistage games and their subgame perfect equilibria, where each firm internalizes the decisions of the firms downstream and competes with all the firms of the same tier. For example, in a two-tier supply chain network consists of wholesalers and retailers, each wholesaler anticipates the reaction of the downstream retailers and considers the potential strategies of the other wholesalers for decision making. For general supply chain networks, one factor that further complicates the models is that even the basic concept of tiers is ambiguous because there are often multiple routes of different lengths that goods are traded from the original producers to the consumers. Our paper studies a model of sequential network game motivated by supply chain network applications. Our main goal is to understand the existence of an equilibrium and the effect of network structure on the efficiency of the system.

The length and the number of trading routes are the two main factors that impact the efficiency of a supply chain network. On one hand, a large variety of options to trade indicates a high degree of competition. On the other hand, a long trading path causes double, triple, and higher degree marginalization problems. To capture these ideas, we consider a sequential game-theoretical model for a special class of networks, series-parallel graphs. We focus our analysis on these networks because they are rich enough for studying the factors described above and simple enough for characterizing the equilibrium outcomes. In particular, series-parallel networks have two natural compositions. A parallel composition, which merges two different sub-networks at the source and the sink, can capture the increase in competition. A series composition, which attaches two sub-networks sequentially, corresponds to the increase in the length of trading. A serial supply chain, for example, is a special series–parallel network constructed by only series compositions. General series–parallel networks extend by employing parallel compositions to capture local firm competition in a network market.

We consider a sequential decision model where each firm makes a decision on the buying quantity from its sellers, and the selling quantity and price to its buyers, given that all its
sellers have made their decisions. The single-source producer of the network starts the decision-making with a fixed material cost. The single ending market of the network accepts all the goods offered by the firms in the last tier and the market price is an affine decreasing function of the total quantity of goods. Each firm strives to maximize its utility, assuming that its downstream buyers also strive to maximize their utility by reacting to the decision-making from the upstream. An equilibrium is the collection of firm decisions such that no firm has the incentive to change its decision by anticipating the reaction of the downstream buyers and assuming that the decisions of other firms remain unchanged. Our main results are listed as follows.

- We show a polynomial-time algorithm that finds the unique equilibrium in a series-parallel network. A crucial step is to derive a closed-form expression of the price at each firm in terms of quantity.
- We show a rich set of equilibrium comparative statics for series-parallel networks, including the firm location advantage of upstream firms and network-component-based efficiency analysis.
- We show that extending series-parallel networks makes the problem intractable.

Let us elaborate. First, to compute the equilibrium in series-parallel networks, we observe that the flow conservation property is satisfied at equilibrium, that is, inflow equals outflow for each intermediary firm. The main strategy is to formulate the price offered to each firm in terms of its inflow. Our algorithm starts the inductive price computation from the ending market in reverse topological order. In the computation of each firm, we take the partial derivative of the firm’s utility with respect to the buying quantity and obtain a closed-form expression. The challenge here is when a seller has multiple buyers. In this case, since the utility is a quadratic function of the buying quantity, we formulate a linear complementarity problem to compute the coefficients for each trade from this seller to her buyer. Once the price-quantity relation is obtained at the source, we compute the total flow needed for the network. Next, the algorithm computes the flow in a topological order starting from the source producer.

For the comparative analysis of firm utility based on the location in a network, we obtain a closed-form expression. By this expression, we conclude that an upstream firm that controls the flow of a downstream firm has at least twice the utility of the downstream firm. For the network-component-based efficiency analysis, we focus on analyzing the flow value and social welfare at equilibrium. We show that with the same production cost and ending market price, swapping the order of two components in a series composition does not change the flow value and the social welfare.

Finally, we consider two extensions with multiple source producers or end markets. When the generalized series-parallel graph has a single source and multiple markets, we consider a simple network and observe that the price function of the intermediary firm can be piecewise linear and discontinuous. This enforces the source to apply either the high price or low price strategy. There can be multiple pure strategy equilibria when both strategies give the source the same utility. When the generalized series-parallel graph has multiple sources and a single ending market, an equilibrium may not exist.

Our paper is organized as follows. In Section 2, we introduce the model and the parallel-serial networks together with the compositions. In Section 3, we provide the algorithm to compute the unique equilibrium. In Section 4, we analyze how firm location affects individual utility and how network structure influences efficiency. In Section 5, we discuss extensions to other classes of networks and show that pure strategy equilibrium might not exist in general networks.

1.1 Related work
Our paper is at the intersection of the literature on supply chain and network economics. Network consideration in supply chain literature is not new but expanding. One of the early related papers that combines competition with a network in the supply chain is Carr and Karmarkar (2005). Here, the authors considered an assembly tree network where agents make a sequential decision, and the competition is modeled by a “coordinated successive Cournot” mechanism. The analysis for a tree network is substantially simpler because each firm has a single downstream node that it can sell the products to. In our game, each firm needs to decide on the selling quantity and price to each of its buyers. As we show, some of the quantities on some of the links can be zero. Such “inactive” links make the analysis more complicated.

Other approaches that do not rely on tree networks often assume other special network assumptions such as two and three-tier networks. For example, Bimpikis, Candogan, and Ehsani (2019) and Bimpikis, Ehsani, and Ilkılıç (2019) and Pang et al. (2017), considered a Cournot game while Nakkas and Xu (2019) analyze bargaining game in two-sided markets. Several papers in the operation management literature assume three-tier networks, where retailers intermediate by purchasing goods from a single manufacturer and selling goods to the ending market have been extensively studied (e.g., Bernstein and Federgruen (2005); Cachon and Lariviere (2005); Netessine and Zhang (2005)). In addition to multiple retailers, Adida and DeMiguel (2011) further considered general three-tier networks with multiple manufacturers, where each manufacturer maximizes its profit by anticipating the reaction of risk-averse retailers. The main objective of this work is

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1Subgame perfect equilibria necessitate the specified trading strategies for all possible history. For simplicity, we consider the equilibrium decision where the source already makes the best decision and passes on the decision making to the downstream and the downstream reacts by the best response accordingly.
to study how simultaneous competition among manufacturers and retailers, risk aversion, and retailer differentiation affect the aggregate firm utility in decentralized and centralized supply chains. Recently, Nguyen and Kannan (2018) studied Cournot games in three-tier networks. However, the two-tier structure of the network in these papers, and the assumption that only the middle tiers make decisions. Thus, it assumes away the complex sequential decision making considered in our work.

At the other extreme, several papers study line network structures (Nguyen, 2017; Wright & Wong, 2014). These papers analyze sequential decisions in a game-theoretical framework, but they lack the heterogeneous nature of competition. Our model using a series–parallel network offers a natural combination of these two aspects. Corbett and Karmarkar (2001) also assumes a linear structure of the supply chain and firms simultaneously make decisions in the entry stage. The equilibrium analysis is on the network structure and stability.

Federgruen and Hu (2016) consider dynamic game in a more general framework, but focuses on price competition. The main comparative analysis of this paper is for a two-stage model, while our paper focuses on multi-stage Cournot competition. Kotowski and Leister (2019) studied a multipartite network where each transaction in the sequential trade is modeled by an efficient auction protocol. Each intermediary firm is inactive with some probability, in which case the firm does not participate. An active firm can resell the goods to the lower tiers or consume the goods by itself. When intermediary firms are speculators that solely earn profit by reselling instead of retaining goods, only a few speculators participate at equilibrium. Bimpikis, Candogan, and Elhansi (2019) and Bimpikis, Elhansi, and Ilkiliç (2019) also considered a sequential game with a similar structure to Kotowski and Leister (2019). They investigate how the number of firms in each tier, production cost, and disruption risk affects firm profits and a discussion similar to Kotowski and Leister (2019) about endogenous supply chain formulation where an equilibrium might cause inefficiency in terms of the number of engaged firms. Perakis and Roels (2007) studied the efficiency in supply chains that use price-only contracts where a buyer and a seller agree only on the transaction price, without specifying the quantity. The efficiency is measured by the profit ratio between a decentralized and a centralized supply chain and is characterized by various configurations including pull or push inventory positioning, two or multiple stages, etc. The focus of the papers described above is on the uncertainty of yields, which is different from the motivation in our paper.

More broadly, this work belongs to the growing literature on network games and their applications in supply chains (see Jackson (2010) and the citations therein). The literature is too large and an extensive summary is beyond the scope of this paper. Within this literature, our paper studies intermediation in networks, which is an active area of research (see Condorelli and Galeotti (2016) for a survey). We describe a few papers that is relatively close to ours, including Nava (2015; Nguyen (2017); Condorelli et al. (2017); Manea (2018). These papers, however, are different from ours in the main focus as well as the modeling approach. Nava (2015) studied a Cournot game in general networks where supply chains are endogenously determined based on the decision of the traders and firms make simultaneous decisions. Simultaneous games are easier to analyze but do not capture the essential elements of sequential decision-making of firms in supply chain networks as in our work. Nguyen (2017), analyzes bargaining games in networks with simpler structures. Condorelli et al. (2017); Manea (2018) analyzes network and intermediation and bargaining in a more general network but both papers assume a single unit of good. Our paper complements this line of research with a tractable framework on multistage Cournot competition.

2 | PRELIMINARIES

We introduce the sequential decision game and the definition of series–parallel graph.

Let \( G = (V \cup \{s, t\}, E) \) be a simple directed acyclic network that represents an economy where \( s \) is the producer firm at the source, \( t \) is the sink market, and the set of nodes \( V \) represents intermediary firms. The arcs of \( G \) represent the possibility of trade between two agents. The direction of an arc indicates the direction of trade. The outgoing end of the arc corresponds to the seller, and the incoming end is the buyer, while \( s \) has only outgoing arcs, and \( t \) has only incoming arcs. The remaining nodes \( i \in V \) representing intermediary firms have both incoming and outgoing arcs. For a node \( i \), \( B(i) \) and \( S(i) \) are the sets of agents that can be buyers and sellers in a trade with \( i \), respectively. Node \( k \) is a parent node or an upstream firm of \( i \) if there is a directed path from \( k \) to \( i \). The set of parent nodes of \( i \) is denoted as \( P(i) \). Similarly, we can define child nodes and the set of children \( C(i) \). Firms in \( C(i) \) are the downstream firms of \( i \). If a firm is neither an upstream nor a downstream firm of \( i \), then it is a competing firm of \( i \).

2.1 | The model

2.1.1 | The sequential decision game

Agents start making their decision after the output of their sellers is determined. Each firm \( i \) decides on how much to buy from each of its sellers, how much to sell to each of its buyer, and the price to sell to each of its buyers. Formally, \( i \)'s decision includes:

- The buying quantity \( x_{ki}^b \geq 0 \) of arc \( ki \in E \) for every \( k \in S(i) \).
- The selling quantity \( x_{ij}^s \geq 0 \) of arc \( ij \in E \) to every \( j \in B(i) \).
- The selling price \( p_{ij} \geq 0 \) of arc \( ij \in E \) to every \( j \in B(i) \).
The source producer does not buy any goods, so the decision of $s$ is the selling quantity $x_{ij}^\text{out} \geq 0$ and the price $p_{ij}$ of arc $sj$ to every $j \in B(s)$.

The marginal cost of production or the production cost per unit $p_s$ of the source $s$ is given and assumed to be a constant $a_s$.

$$p_s = a_s \text{ where } a_s > 0.$$ 

The sink node $t$ does not represent a firm, it corresponds to an end market. Firms supplying to the sink market do not choose the price. Instead, the price function $p_t$ at sink node $t$ is given and assumed to be an affine decreasing function on the total amount of goods, $X_t$, sold to the market $t$.

$$p_t = a_t - b_t X_t, \text{ where } X_t = \sum_{i \in S(i)} x_{ij}^\text{in} = \sum_{i \in S(i)} x_{ij}^\text{out},$$

$$a_t > a_s \text{ and } b_t > 0.$$ 

$a_t$ is the price intercept of the market $t$. We note that the market must accept all the goods, thus do not have a choice to reject them. That is, $x_{ij}^\text{in} = x_{ij}^\text{out}$ for each $i \in S(t)$. Generally, for a trade $ij \in E$, the buyer $j$ cannot obtain more than what the seller $i$ offers, thus $x_{ij}^\text{in} \leq x_{ij}^\text{out}$. We assume that each intermediary firm $i$ cannot get goods from any other source besides its sellers. Firms do not get any value from retaining the goods. The trade of each firm $i$ must be feasible, that is, the outflow value cannot exceed the inflow value.

$$\sum_{k \in S(i)} x_{ki}^\text{in} \geq \sum_{j \in B(i)} x_{ij}^\text{out}.$$ 

The utility of the source firm $s$ is

$$\Pi_s = \sum_{j \in B(s)} p_{sj} x_{ij}^\text{in} - p_s \sum_{j \in B(s)} x_{ij}^\text{out}. \quad (1)$$

The utility of an intermediary firm $i \in V$ is

$$\Pi_i = \sum_{j \in B(i)} p_{ij} x_{ij}^\text{in} - \sum_{k \in S(i)} p_{ki} x_{ik}^\text{in}. \quad (2)$$

The formula decomposes the utility function into two terms: the total revenue from $j \in B(i)$ and the total cost of materials from $k \in S(i)$.

The timing of the game is as follows. The decision-making is in topological order. The source producer makes its decision first. A firm makes its decision on the buying quantity from its sellers, and the selling quantity and price to all its buyers, once all of its sellers have made their decisions. A firm with multiple buyers cannot wait for the decision made by one of its buyers and then make the decision related to another buyer. When firms decide their accepting quantities to maximize their utilities, they also need to take into account the strategies of the non-upstream firms. Each firm knows the structure of the network, the source marginal cost of production, the price function of the sink, and the decisions made by the upstream firms, but not the decisions made by the competing firms.

2.1.2 Equilibrium characteristics

We focus on pure strategy equilibria throughout. In a topological order traversal, the firms that have already made their decisions form a subgraph of the entire network. The set of feasible decisions made by these firms is a history.

**Definition 2.1** (History) Let $V' \subseteq V \cup \{s\}$ where $V'$ is formed by a topological order traversal on $G$ from the source $s$, then a feasible set of decisions where

1. $x_{ij}^\text{out}$ and $p_{ij}$ for $j \in B(s)$, and
2. for each $i \in V' \setminus \{s\}$, $x_{ki}^\text{in}$ for $k \in S(i)$, and $x_{ij}^\text{out}$ and $p_{ij}$ for $j \in B(i)$

is a history. A history is empty when the topological order traversal does not visit any node.

Given a history $h$, a subgame denoted $SG(h)$ is a game which starts at $h$, and the firms that have not made their decisions in $h$ play the game $SG(h)$. The source plays the subgame $SG(\emptyset)$. A firm $i$ is a first player of the subgame if $i$ did not make a decision in $h$ and all the parents of $i$ have made their decisions in $h$. We note that a subgame can have multiple first players, and a first player does not have knowledge of its competing firm that has already played in $h$. A subgame perfect equilibrium is a set of best response strategies of each firm for all feasible histories $h$ where the firm is a first player in $SG(h)$.

**Definition 2.2** (Subgame Perfect Equilibrium) A subgame perfect equilibrium is a set of decisions for each firm $i \in V \cup \{s\}$ and each history $h$ where $i$ is a first player, such that $i$ makes the best decision for any subgame $SG(h)$, assuming that all the downstream firms of $i$ also make the best decision as a response to any history created based on $i$’s decision.

To illustrate the subgame perfect equilibrium concept, we present a line network as an example. The source producer controls the quantity and price, thus affecting the decision of the intermediary firm.

**Example 1** (Subgame perfect equilibrium in a line network)

Consider the following line network.

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 1 -- 2 -- 3
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$p_s = 1 \quad p_t = 9 - x_t \quad p_1 = 9 - x_{1t} = 9 - x_{1t}^\text{in}$

Suppose $s$ offers $v$ quantity $x_{sv}^\text{out}$ and price $p_{sv}$, $v$ has to make a decision on $x_{sv}^\text{in}$. We assume that $p_{sv} \in [1, 9]$ so that $s$ and $v$ can obtain non-negative utility.

$v$ cannot decide $p_s$ because it is already fixed as $9 - x_{sv}^\text{out}$. The sink market $t$ accepts all the goods, so the utility of $v$ is

$$\Pi_v = (9 - x_{sv}^\text{out}) x_{sv}^\text{in} - p_{sv} x_{sv}^\text{in} = (9 - x_{sv}^\text{out}) x_{sv}^\text{out} - p_{sv} x_{sv}^\text{in}.$$
If v is a rational player who maximizes $\Pi_v$, then v will only buy goods it intends to sell, so $x_{in}^v = x_{out}^v$ and $\Pi_v = (9 - x_{in}^v, x_{in}^v - p_{sv} x_{in}^v)$. By taking the derivative:

$$\frac{d\Pi_v}{dx_{in}^v} = 9 - 2x_{in}^v - p_{sv} = 0 \Rightarrow x_{in}^v = \frac{9 - p_{sv}}{2},$$

which implies that v is willing to buy $(9 - p_{sv})/2$ units of goods given price $p_{sv}$. Since $\Pi_v$ is an increasing function of $x_{in}^v$ when $x_{in}^v \leq (9 - p_{sv})/2$, given the history $p_{sv}$ and $x_{in}^v$ where the decision is only made by s, v will accept $x_{in}^s = \min\{x_{in}^s, (9 - p_{sv})/2\}$ unit of goods from s and sell $x_{out}^s = x_{in}^s$ of goods to t. For example, if the history is $p_{sv} = 7$ and $x_{in}^s = 1$, then v respond by accepting 1 unit of good offered by s and sell 1 unit of good to t, that is, $x_{in}^v = x_{out}^v = 1$.

If s intends to maximize its utility $\Pi_s = p_{sv} x_{in}^s - p_s x_{out}^s$, s will sell as many goods as possible to v without over-selling and over-buying (otherwise goods are wasted), so $x_{in}^s = x_{out}^s = (9 - p_{sv})/2$. This implies that if s is playing the best strategy, the utility is $$\Pi_s = p_{sv} \cdot \frac{9 - p_{sv}}{2} - 1 \cdot \frac{9 - p_{sv}}{2}$$

which is maximized when $p_{sv} = 5$. Given the empty history, the best strategy for s is $p_{sv} = 5$ and $x_{in}^s = (9 - p_{sv})/2 = 2$.

Interestingly, a subgame that starts at a feasible history does not guarantee pure strategy equilibria even in some simple networks.² Therefore, instead of considering all possible outcomes which include off-equilibrium history paths, we focus on pure strategy equilibria where all the firms make their best decision on an equilibrium history path. At an equilibrium, the source starts with a best strategy following the empty history. The second node gets an offer from the source and reacts to the history created by the source’s decision by playing the best strategy and pass on the decision making until the sink market is reached.

Definition 2.3 (Equilibrium) An equilibrium is a set of strategies with respect to a topological order such that

1. s plays a best strategy in subgame $SG(\emptyset)$, and
2. when node i is reached, i plays the best strategy in the subgame that starts at the history created by the decisions made by previous nodes in the topological order.

Since firms do not know the decisions made by competing firms that appear earlier in the topological order, the game is equivalent for any topological order traversal. Therefore, regardless of the configuration of the history, we can define an equilibrium as an assignment of good quantities and prices such that no firm is willing to change its decision after knowing the decision of the other competing firms, and assuming that all the downstream firms also make their best decisions. The equivalent definition of an equilibrium is as follows.

**Definition 2.4 (Equilibrium)** An equilibrium is a set of feasible decisions where each firm $i \in \{s\} \cup \mathcal{V}$ does not have the incentive to change its strategy for a better utility, assuming that each downstream firm of $i$ plays the best strategy that maximizes its utility, and the strategies of the non-downstream firms of $i$ remain the same.

We present an example with two competing intermediary firms to illustrate the equilibrium concept. The source producer controls the quantity and price, thus affecting the decisions of its two intermediary firms.

**Example 2** (The equilibrium with two intermediary firms)

Consider the following network.

![Network Diagram](image)

Suppose the decision of s is

| $p_{su}$ | $p_{sv}$ | $x_{out}^s$ | $x_{out}^v$ |
|---|---|---|---|
| 3 | 4 | 1 | 1 |

and u and v are rational firms, that is, they will sell all goods they buy to maximize their utility. For simplicity, let $x = x_{in}^u$ and $y = x_{in}^v$. The utilities of u and v are

$$\Pi_u = (7 - x - y) x - 3 x \quad \text{and} \quad \Pi_v = (7 - x - y) y - 4 y.$$  

By taking the derivative, we have that

$$\frac{\partial \Pi_u}{\partial x} = 4 - 2 x - y \quad \text{and} \quad \frac{\partial \Pi_v}{\partial y} = 3 - x - 2 y.$$

We claim that the best decision of u and v is $x = 1$ and $y = 1$. $\Pi_u$ is a concave function so it is maximized when $\frac{\partial \Pi_u}{\partial x} = 0$, which implies that the best response to $x = 1$ is $y = 1$. We observe that $\Pi_u$ is also concave. $\Pi_u$ is maximized when $\frac{\partial \Pi_u}{\partial x} = 0$. However, when $y = 1$, x cannot be $\frac{9}{2}$ since $x \leq x_{out}^u = 1$. While $y = 1$ and $x \in [0, 1]$, $\Pi_u$ is an increasing function, so the best response to $y = 1$ is $x = 1$.

The utility of s for this decision is

$$\Pi_s = p_{su} x + p_{sv} y - p_s (x_{out}^u + x_{out}^v) = 3 \times 1 + 4 \times 1 - 1 \times (1 + 1) = 5.$$
Given that $p_{uv} = 3$ and $p_{uv} = 4$, there is a better quantity decision for $s$. We recall that $\Pi_u$ and $\Pi_v$ are both concave, so it suffices to show that \( \frac{\partial \Pi_u}{\partial x} \) and \( \frac{\partial \Pi_v}{\partial x} \) are both zeros. This happens when $x = \frac{5}{3}$ and $y = \frac{2}{3}$. If $x_{iu}^{	ext{out}} = \frac{5}{3}$ and $x_{iv}^{	ext{out}} = \frac{2}{3}$, then $u$ and $v$ will buy all the goods from $s$ and the utility of $s$ is

$$\Pi_s = p_{su}x + p_{sv}y - p_s(x_{iu}^{	ext{out}} + x_{iv}^{	ext{out}}) = 3 \times \frac{5}{3} + 4 \times 2 - 1 \times \left( \frac{5}{3} + \frac{2}{3} \right) = \frac{16}{3}. $$

However, this is not the best decision of $s$. The equilibrium for this network is as follows.

| $p_{su}$ | $p_{sv}$ | $x_{iu}^{	ext{out}}$ | $x_{iv}^{	ext{out}}$ | $x$ | $y$ |
|---------|---------|----------------|----------------|-----|-----|
| 4       | 4       | 1              | 1              | 1   | 1   |

One can verify that $\Pi_u$ and $\Pi_v$ are concave and the derivatives are zeros. The utility of $s$ is

$$\Pi_s = p_{su}x + p_{sv}y - p_s(x_{iu}^{	ext{out}} + x_{iv}^{	ext{out}}) = 4 \times 1 + 4 \times 1 - 1 \times (1 + 1) = 6.$$ 

We observe that the best strategy for each firm $i \in V$ is to always sell as much as bought since it cannot benefit from paying more for those unsold goods. At the selling side, suppose firm $i$ is willing to offer $x_{ij}^{	ext{out}}$ quantity of goods to firm $j$, but part of the goods got rejected, that is, $x_{ij}^{	ext{in}} < x_{ij}^{	ext{out}}$. This can never happen at equilibrium because $i$ will be better off by rejecting $x_{ij}^{	ext{out}} - x_{ij}^{	ext{in}}$ amount of goods from its upstream before selling. The following observation lists the properties of the supplied quantities at an equilibrium.

**Observation 2.1** (Equilibrium flow conservation) An equilibrium satisfies:

1. $x_{ij}^{	ext{out}} = x_{ij}^{	ext{in}}$ for each $ij \in E$.
2. $\sum_{k \in S(i)} x_{ki}^{	ext{out}} = \sum_{j \in B(i)} x_{ij}^{	ext{out}}$, that is, inflow is equal to outflow for each firm $i \in V$.

For later notations, at an equilibrium, we denote $x_{ij}$ as the flow of arc $ij$, that is, $x_{ij} = x_{ij}^{	ext{out}} = x_{ij}^{	ext{in}}$, and no longer use $x_{ij}^{	ext{in}}$ and $x_{ij}^{	ext{out}}$. Meanwhile, since each firm accepts all the offers and sells everything they bought, we denote this sum of flow as $X_i := \sum_{k \in S(i)} x_{ki} = \sum_{j \in B(i)} x_{ij}$. The utility of firm $i$ in (2) becomes

$$\Pi_i = \sum_{j \in B(i)} p_{ij}x_{ij} - \sum_{k \in S(i)} p_{ki}x_{ki} \quad (3)$$

and the utility of source firm $s$ in (1) becomes

$$\Pi_s = \sum_{j \in B(s)} p_{sj}x_{sj} - \sum_{j \in B(s)} x_{sj} \quad (4)$$

For flow activities, an arc $ij \in E$ is active if $x_{ij} > 0$, and inactive if $x_{ij} = 0$. For each firm $i \in V$ and active arcs $ki \in E$ and $ij \in E$, $p_{ki} \leq p_{ij}$. That is, the buying price should not exceed the selling price. Otherwise, $i$ could have been better off by rejecting some goods from $k$ and choosing not to offer the same amount of goods to $j$.

**Observation 2.2** For every $ki \in E$ and $ij \in E$ that are active, the price at an equilibrium satisfies $p_{ki} \leq p_{ij}$.

To define the equilibrium uniqueness, we require the set of active arcs to be unique, as well as the flow and price of each active arc. The prices of inactive trades, on the other hand, can be arbitrary since they do not contribute to the seller revenue.

**Definition 2.5** An equilibrium of a network $G$ is unique if the set of active arcs is unique, as well as the flow and price of each active arc.

### 2.2 Series parallel graph

**2.2.1 General series–parallel graphs**

We consider the case when $G$ is a *series–parallel graph* (SPG). The networks in Examples 1 and 2 are both SPGs. Our main goal is to compute the equilibrium in networks that belong to this special graph family. This class of networks is well studied and has several applications in graph theory (e.g., Duffin (1965)). For completeness, we provide a formal definition as follows.

**Definition 2.6** (SPG) A single-source-and sink SPG (or SPG) is a graph that can be constructed by a sequence of series and parallel compositions starting from a set of copies of a single-arc graph. The compositions are defined as follows:

1. Series composition of $X$ and $Y$: given two SPGs $X$ with source $s_X$ and sink $t_X$, and $Y$ with source $s_Y$ and sink $t_Y$, form a new graph $G = S(X, Y)$ by identifying $s = s_X$, $t_X = s_Y$, and $t = t_Y$.
2. Parallel composition of $X$ and $Y$: given two SPGs $X$ with source $s_X$ and sink $t_X$, and $Y$ with source $s_Y$ and sink $t_Y$, form a new graph $G = P(X, Y)$ by identifying $s = s_X = s_Y$ and $t = t_X = t_Y$.

Both compositions do not affect the existing arcs.

**2.2.2 Shortcut-free series–parallel graphs**

We start with the definition of shortcuts.

**Definition 2.7** Given an SPG $G = (V, E)$, let $i, j \in V$. Consider a path $\ell_{ij} = (i, v_1, \ldots, v_k, j)$ from node $i$ to node $j$. If there is an arc $ij \in E$, then we say $ij$ is a shortcut of $\ell_{ij}$, or $ij$ dominates path $\ell_{ij}$. 
Definition 2.8  An SPG is shortcut-free if it has no shortcuts, that is, there is no path dominated by an arc.

Given a shortcut-free SPG, if we consider the relationship between a direct (or adjacent) parent and child $i$ and $j$, where $ij \in E$, there are three possibilities$^3$:

- Single seller and single buyer, $|S(j)| = |B(i)| = 1$. (SS)
- Multiple sellers and single buyer, $|S(j)| \geq 2$, $|B(i)| = 1$. (MS)
- Single seller and multiple buyers, $|S(j)| = 1$, $|B(i)| \geq 2$. (SM)

Sometimes there are multiple paths from a parent node to one of its children, we call these paths disjoint if they do not have any common intermediary nodes, that is, all nodes except the starting and the ending ones are different. Based on this definition, we can define the merging nodes with respect to node $i$.

Definition 2.9  (Self-merging Child Node) Node $j \in C(i)$ is a self-merging child node of $i$ if there are disjoint paths from $i$ to $j$. The set of such nodes $j$ is termed $CS(i)$.

Definition 2.10  (Parent-merging Child Node) Node $j \in C(i)$ is a parent-merging child node of $i$, if there exists a node $k \in P(i)$, such that there are disjoint paths from $k$ to $j$. The set of such nodes $j$ is denoted as $CP(i)$.

For $ij \in E$ in the SM case, we introduce the set of special self-merging child nodes of $i$ and its direct child $j$ as $CT(i,j) = CS(i) \cap C(j) \backslash CP(i)$. This notation helps us capture the “internal” merging nodes that are responsible for the selling price and quantity offered to $i$ in the later technical proofs.

Observation 2.3  An SPG has the following properties:

(1) $CP(s) = CP(t) = \emptyset$.
(2) In the SS case for $ij \in E$, $CP(j) = CP(i)$.

$^3$Multiple sellers and multiple buyers case does not exist in shortcut-free SPGs, we refer the proof to Appendix B.1.

3 | EQUILIBRIUM COMPUTATION AND UNIQUENESS

We start with shortcut-free SPGs and show that all arcs are active at equilibrium. We first derive a closed-form relation between the quantity and price offered to the firms at equilibrium via a backward computation. Then, we show that the unique optimal quantity and price offered to each firm can be solved following the decision sequence from the source to sink by the closed-form relation. For SPGs with shortcuts, we show that the trade on paths dominated by shortcuts is inactive. Thus, the equilibrium for an SPG with shortcuts can be found by the same algorithm after removing dominated paths.

3.1 | Shortcut-free series parallel graphs

3.1.1  Equilibrium computation and uniqueness

We compute the equilibrium price and quantity relation then proceed to the flow and price computation. A key characteristic of the equilibrium is that all edges are active when there are no shortcuts. The equilibrium price has a closed-form expression in terms of the equilibrium quantity based on the structure of the SPG.

Theorem 3.1  Given a shortcut-free SPG $G$, at an equilibrium, all arcs are active. For each firm $i \in V$, each seller $k \in S(i)$ offers $i$ the same price $p_i$ to $k$, and the following relation holds

$$p_i = a_i - b_i X_i - \sum_{\ell \in CP(i)} b_{\ell} X_{\ell}$$

where $b_i$ for each $i \in V \cup \{s\}$ is a positive constant that only depends on the structure of $G$.

Theorem 3.1 shows a concise way to present the price and quantity relation at equilibrium. We start with the algorithm that computes the price function in Theorem 3.1 followed by the flow computation and defer the proof after the algorithm description. The high-level strategy for deriving the closed-form expression is via a backward induction starting from the sink market $t$. We recall that the upstream firms make their decision based on the best decision of the downstream. Since the price of the sink market $t$ is an affine decreasing function, we can inductively show that the utility of each intermediary firm is a concave function of the quantity. We observe that the derivative of the utility with respect
to the quantity of trade cannot be positive since otherwise, the upstream firm can be better off by raising the price. On the other hand, the derivative with respect to the quantity of an active trade must be zero. This is because the upstream firms assume that the downstream firms make their decision to maximize their utility. By this observation, we derive a linear complementarity problem and show that the best price offered to the downstream is an affine decreasing function of the quantity. When the price and quantity relation of the source is computed, we obtain the flow starting from the source.

After having the source flow and the closed-form relation between the equilibrium price and quantity, we find the equilibrium flow and price. Consider the quantity decision from firm $i$ to its downstream buyers $j \in B(i)$. In the SS and MS case, firm $i$ only has one buyer, that is, $|B(i)| = 1$. By Observation 2.1, inflow equals outflow at firm $i$, and firm $j$ will accept all the offer from $i$, formally, $x_{ij} = X_i$. For the SM case, we distribute the flow proportionally by the convex coefficients and show that the optimal flow distributing strategy is unique, which implies that the equilibrium is unique.

Algorithm 1 has two phases. In the first phase that computes the price function in terms of quantity, we start from the sink market $t$. Given the downstream $b_j$ where $j \in B(i)$, we compute $b_t$ and this can be done in $O(\text{deg}^+(i))$ time where $\text{deg}^+(i)$ is the out-degree of $i$. The calculation is in a reverse topological order. Particularly, in the SM case, we store the convex coefficients of each downstream node $j \in B(i)$ which are used later in the second phase. The number of the $b_t$ computation is bounded by $O(V)$. Therefore, it takes linear time to compute the price function in terms of quantity by Algorithm 1. When $s$ is reached, we already have $p_s = a_t - b_sX_t = a_s$ since $C_p(s) = \emptyset$. We set $X_i = \sum_{j} b_jX_j$ so that the expected price of $s$ meets the given production cost.

In the second phase that computes the flow and price, we start from the source $s$ and distribute the flow to the downstream firms in a topological order. The price is set accordingly to the function computed in the first phase.

We show Example 5 in Appendix for the equilibrium computation by Algorithm 1.

### 3.1.2 Proof of the correctness and uniqueness

In this section, we show the proof of Theorem 3.1 and the uniqueness of the equilibrium.

**Proof of Theorem 3.1** We start from the sink $t$ and argue inductively via reverse topological traversal that the price proposed to $i$ must be an affine decreasing function of $X_i$. We note that the computation is always under the flow reservation condition by Observation 2.1.

**Starting from the sink $t$**

We start with the behavior of the direct upstream firms of sink $t$. For a firm $i \in S(t)$, if arc $it$ belongs to the SS case, then the utility of $i$ is

$$\Pi_i = (a_t - b_tX_i)x_{it} - \sum_{k \in S(i)} p_{ki}x_{ki} = \left( a_t - b_t \sum_{k \in S(i)} x_{ki} \right) \sum_{k \in S(i)} x_{ki} - \sum_{k \in S(i)} p_{ki}x_{ki}. $$

If $p_{ki}$ given by the selling firms are regarded as constants to $i$. $\Pi_i$ is a concave function. By taking the derivative of $\Pi_i$ with respect to $x_{ki}$, we have that

$$\frac{\partial \Pi_i}{\partial x_{ki}} = a_t - 2b_t \sum_{k \in S(i)} x_{ki} - p_{ki}. $$

The price and quantity at equilibrium is a solution of the following linear complementarity problem. Intuitively, $\frac{\partial \Pi_i}{\partial x_{ki}} > 0$ cannot happen since otherwise $k$ could have raised the price $p_{ki}$ such that $\frac{\partial \Pi_i}{\partial x_{ki}} = 0$. This makes $i$ accept all the goods from $k$ and $k$ would obtain a higher payoff, which contradicts the equilibrium condition. If $\frac{\partial \Pi_i}{\partial x_{ki}} < 0$, then $k$ will not offer any goods to $i$ so $x_{ki} = 0$ since the payoff of $i$ will decrease if $i$ accepts some goods from $k$.

\[
\begin{aligned}
\sum_{k \in S(i)} \frac{\partial \Pi_i}{\partial x_{ki}} x_{ki} &= 0, \\
\frac{\partial \Pi_i}{\partial x_{ki}} &\leq 0 \quad \forall k \in S(i), \\
x_{ki} &\geq 0 \quad \forall k \in S(i).
\end{aligned}
\] (LCP)

When $ki$ is active, $p_{ki}$ is such that $\frac{\partial \Pi_i}{\partial x_{ki}} = 0$. Therefore, for an active arc $ki$,

$$p_{ki} = p_i = a_t - 2b_t \sum_{k \in S(i)} x_{ki} = a_t - 2b_tX_t = a_t - b_tX_t - \sum_{\ell \in C_p(i)} b_{\ell}X_{\ell},$$

where $b_t = 2b_t$, $X_t = X_t$, and $C_p(i) = \emptyset$.

If arc $it$ belongs to the MS case, then the utility of $i$ is

$$\Pi_i = \left( a_t - b_t \sum_{j \in S(i)} x_{jt} \right) \sum_{k \in S(i)} x_{ki} - \sum_{k \in S(i)} p_{ki}x_{ki}. $$

$p_{ki}$ and $x_{jt}$ where $j \neq i$ are regarded as constants to $i$. $i$ is not willing to change its decision at equilibrium. $\Pi_i$ is a concave function. By taking the derivative of $\Pi_i$ with respect to $x_{ki}$, we have that

$$\frac{\partial \Pi_i}{\partial x_{ki}} = a_t - b_t - \sum_{k \in S(i)} x_{ki} - b_t \sum_{j \in S(i)} x_{jt} - p_{ki}. $$

By a similar argument as before, the price and quantity at equilibrium is a solution of (LCP). When $ki$ is active, $p_{ki}$ is
Algorithm 1. Equilibrium computation

**Input:** A shortcut-free SPG $G = (V \cup \{s, t\}, E)$, the price function $p_i = a_i - b_i X_i$, and the source production cost $p_s = a_s$.

**Output:** The equilibrium flow and price.

**Phase 1: backward equilibrium price function computation**

Starting from $t$, given the downstream buyer(s) coefficients $b$, compute the upstream seller(s) its price function case by case in a reverse topological order:

- For the SS case,
  
  $$b_i = 2b_j + \sum_{\ell \in C_p(j)} b_{\ell}. \tag{SS}$$

- For the MS case, for each seller $i$,
  
  $$b_i = b_j + \sum_{\ell \in C_p(j)} b_{\ell}. \tag{MS}$$

- For the simple SM case $(|C_3(i)| = 1)^*$, suppose $b_j$ was calculated for all $j \in B(i)$,
  
  $$b_i = \frac{2}{\sum_{j \in B(i)} b_j} + 2 \sum_{\ell \in C_p(j) \setminus C_r(i)} b_{\ell} + \sum_{\ell \in C_r(i)} b_{\ell}. \tag{Simple SM}$$

For each $j \in B(i)$, assign the convex coefficient $a_j = \frac{2}{\sum_{j \in B(i)} b_j}$ to arc $ij$.

Set the price function for seller $i$: $p_i = a_i - b_i X_i - \sum_{\ell \in C_p(j)} b_{\ell} X_{\ell}$.

**if** seller $i$ is the source $s$ **then**

Set $X_s = \frac{a_i - a_j}{b_i}$.

**Phase 2: forward equilibrium flow and price computation**

Assign quantity and price according to a topological order as the following:

- For the SS and MS case, the flow to the buyer is the sum of the upstream flow.
- For the SM case, assign the downstream flow proportionally to the convex coefficients.
- Set the price accordingly to the quantity by the price function for each case.

**if** the buyer is sink $t$ **then**

**return**

such that $\frac{\partial \Pi}{\partial x_{s_j}} = 0$. Therefore, for an active arc $ki$,

$$p_{ki} = p_i = a_i - b_i \sum_{k \in S(i)} x_{k} - b_i \sum_{j \in S(i)} x_{j} = a_i - b_i X_i - b_i X_{\ell}$$

$$= a_i - b_i X_i - \sum_{\ell \in C_p(i)} b_{\ell} X_{\ell}$$

where $b_i = b_j$ and $C_p(i) = \{i\}$.

**Before reaching the SM case**

The same procedure as before can be inductively repeated whenever we meet an MS or SS case by the reverse topological traversal from $t$. Given that the downstream price must be an affine decreasing function of the inflow of the parent merging child nodes, the derivative of the firm utility with respect to the quantity decision variables must always be zero whenever the quantity is positive.

Before reaching the SM case during the reverse topological traversal from $t$, consider firm $i \in V$. By inductive hypothesis, suppose for each arc $x_{j} > 0$,

$$p_{ij} = p_j = a_i - b_j X_j - \sum_{\ell \in C_p(j)} b_{\ell} X_{\ell}.$$
If \( ki \) is the MS case and \( x_{ki} > 0 \), then by (LCP), \( \frac{\partial L}{\partial x_{ki}} = 0 \), thus

\[
p_{ki} = p_j + \frac{\partial p_j}{\partial x_{ki}} \sum_{k' \in S(i)} x_{k'i}
\]

\[
= a_t - b_j X_j - \sum_{\ell \in C(j)} b_{\ell} X_{\ell} - \left( b_j + \sum_{\ell \in C(j)} b_{\ell} \right) \sum_{k'^{'} \in S(i)} x_{k'i}
\]

\[
= a_t - \left( b_j + \sum_{\ell \in C(j)} b_{\ell} \right) X_j - \sum_{\ell \in C(j)} b_{\ell} X_{\ell}
\]

(6)

where \( X_i = \sum_{k' \in S(i)} x_{k'i} \) and \( C_p(i) = C_p(j) \cup \{ j \} \) by Observation 2.3 (4).

### Reaching the SM case

Define the set of nodes \( N_G \) such that for any node \( i \in N_G \), \( i \) itself and all the children of \( i \) all have an empty self-merging child node set. Formally,

\[
N_G = \{ i | i \in V \cup \{ s \} \text{ such that } \forall j \in C(i) \cup \{ i \}, C_s(j) = \emptyset \}.
\]

\( N_G \) denotes the set of nodes starting from \( t \) via reverse topological traversal before we reach a set of nodes that are sellers in the SM case. These sellers can be defined as the set of nodes \( S_G \), such that for any node \( i \in S_G \), there exists a buyer of \( i \) that belongs to \( N_G \). Formally,

\[
S_G = \{ i | i \in V \cup \{ s \} \text{ such that } B(i) \cap N_G = \emptyset \}.
\]

In the following figure, \( N_G \) consists of the red nodes \( j_1, v_1, v_2, \ell, k \), and \( t \), while \( S_G \) consists of the black nodes \( s \) and \( j_2 \).

![Diagram of nodes and edges](image)

There must exist a node \( i \in S_G \) such that all its buyers \( j \in B(i) \) belong to \( N_G \), and \( i \) is the seller in the SM case. Given that \( p_j \) where \( j \in B(i) \) are affine decreasing functions, the utility of \( i \) is

\[
\Pi_i = \sum_{j \in B(i)} p_j x_j - \sum_{k \in S(i)} p_{ki} x_{ki}.
\]

(7)

Suppose the sellers of \( i \) make their best decision and offer \( i \) total inflow \( X_i = C \) such that \( i \) accepts everything. \( i \) does not have control over the buying cost \( \sum_{k \in S(i)} p_{ki} x_{ki} \) and the total inflow \( \sum_{k \in S(i)} x_{ki} \) at equilibrium. What \( i \) can decide is how to distribute \( C \) to its buyers in \( N_G \). Here \( p_{ki} \) are regarded as constants given to \( i \) and \( \Pi_i \) is a concave function. There is also a constraint \( \sum_{j \in B(i)} x_j = \sum_{k \in S(i)} x_{ki} = C \). Therefore, we can rewrite the problem of maximizing \( \Pi_i \) as the following convex quadratic program:

\[
\max \sum_{j \in B(i)} p_j x_j
\]

subject to \( \sum_{j \in B(i)} x_j = C \).

(CQP)

Consider the Lagrangian function:

\[
L(x, \lambda) = \sum_{j \in B(i)} p_j x_j - \lambda \left( \sum_{j \in B(i)} x_j - C \right).
\]

By taking the derivative of \( L(x, \lambda) \) with respect to \( x_j \), we have that

\[
\frac{\partial L(x, \lambda)}{\partial x_j} = p_j - \lambda = a_t - b_j x_j - \sum_{\ell \in C(j)} b_{\ell} X_{\ell} - b_j X_j - \sum_{\ell \in C(j)} b_{\ell} X_{\ell}
\]

\[
= a_t - 2b_j x_j - \sum_{\ell \in C(j)} b_{\ell} X_{\ell} - \lambda
\]

(8)

where the second equality follows by the inductive hypothesis on \( p_j \) and rearranging and summing the inflow value of the merging nodes, while the third equality follows by Observation 2.3 (3). We note that \( \sum_{j \in B(i)} p_j x_j \) is a constant since \( X_j \) and \( C \) are decided by the upstream sellers of \( i \).

\[
\sum_{j \in B(i)} p_j x_j \text{ is maximized when } \frac{\partial L(x, \lambda)}{\partial x_j} = 0 \text{ for each } j \in B(i).
\]

This indicates that

\[
a_t - 2b_j x_j - \sum_{\ell \in C(j)} b_{\ell} X_{\ell} = \sum_{\ell \in C(j)} b_{\ell} (X_{\ell} + C) + \lambda.
\]

By rearranging, this can be formulated as a system of linear equations

\[
\begin{cases}
2p_j x_j + 2 \sum_{\ell \in C(j)} b_{\ell} X_{\ell} = D & \forall j \in B(i), \\
\sum_{j \in B(i)} x_j = C,
\end{cases}
\]

where \( D := a_t - \sum_{\ell \in C(j)} b_{\ell} (X_{\ell} + C) - \lambda \). Since the right hand side is the same for each linear constraint \( j \in B(i) \), \( x_j = a_j C \) where \( \sum_{j \in B(i)} a_j = 1 \) is the solution of (CQP).

**Lemma 3.1** For the SM case, \( \Pi_i \) is maximized by distributing the flow to \( j \in B(i) \) proportionally to the convex coefficients \( a_j \).

**Finding the convex coefficients and the price**

Consider the SM case where \( j \in E \). We show the simple SM case and defer the proof for the general SM case to Appendix B.2.

**Simple SM**

\( |B(i)| \geq 2, \ |S(j)| = 1, \) and \( |C_S(i)| = 1 \):
In the simple SM case, \( C_T(i, j) = C_S(i) \setminus C_P(i) \) is the same for each \( j \in B(i) \), so \( 2 \sum_{\ell \in C_T(i, j)} b_{\ell} X_\ell \) are the same. By (9), it suffices to find \( a_i \) so that \( b_j x_j \) are the same for each \( j \in B(i) \). Let

\[
a_j = \frac{1}{b_j} \sum_{j' \in B(i)} \frac{1}{b_{j'}} \cdot \tag{10}
\]

then when \( x_j = a_j C \), \( 2 b_j x_j + 2 \sum_{\ell \in C_T(i, j)} b_{\ell} X_\ell \) are the same for \( j \in B(i) \) in (9). The best strategy for \( i \) is to assign \( a_j C \) on arc \( ij \). For all \( j' \in B(i) \), \( b_{j'} \) is positive so \( a_j \) is also positive. Therefore, if \( C \) is positive, then \( x_j \) are all active.

Price \( p_j \) is the same for each \( j \in B(i) \):

\[
p_j = a_i - b_j x_j - \sum_{\ell \in C_T(i, j)} b_{\ell} X_\ell
\]

\[
= a_i - \frac{1}{b_j} \left( -C - \sum_{\ell \in C_P(i)} b_{\ell} X_\ell - \sum_{\ell \in C_T(i)} b_{\ell} X_\ell \right)
\]

\[
= a_i - \frac{1}{b_j} \left( -X_i - \sum_{\ell \in C_T(i)} b_{\ell} X_\ell - \sum_{\ell \in C_P(i)} b_{\ell} X_\ell \right)
\]

where the last equality holds since \( X_\ell = X_i \) when \( \ell \in C_S(i) \setminus C_P(i) \) and \( |C_S(i)| = 1 \). The utility of \( i \) is

\[
\Pi_i = \left( a_i - \frac{1}{b_j} \left( -X_i - \sum_{\ell \in C_T(i)} b_{\ell} X_\ell - \sum_{\ell \in C_P(i)} b_{\ell} X_\ell \right) \right) X_i - \sum_{k \in S(i)} p_{ki} x_{ki}.
\]

By taking the derivative of \( \Pi_i \) with respect to \( x_{ki} \), we have that

\[
\frac{d\Pi_i}{dx_{ki}} = a_i - \frac{1}{b_j} \left( -X_i - \sum_{\ell \in C_T(i)} b_{\ell} X_\ell - \sum_{\ell \in C_P(i)} b_{\ell} X_\ell \right) - \left( \sum_{\ell \in C_T(i)} b_{\ell} + \sum_{\ell \in C_P(i)} b_{\ell} \right) X_i - p_{ki}
\]

\[
= a_i - \left( \sum_{\ell \in C_T(i)} b_{\ell} + \sum_{\ell \in C_P(i)} b_{\ell} \right) X_i - \sum_{\ell \in C_T(i)} b_{\ell} X_\ell - p_{ki}
\]

where \( \frac{dx_i}{dx_{ki}} = 1 \) and \( \frac{dx_j}{dx_{ki}} = 1 \) for \( \ell \in C_T(i) \cup C_S(i) \) since \( x_{ki} \) is part of the flow of \( X_i \) and \( X_\ell \).

By (LCP), if \( x_{ki} > 0 \), then \( \frac{d\Pi_i}{dx_{ki}} = 0 \), so

\[
p_{ki} = a_i - \left( \sum_{\ell \in C_T(i)} b_{\ell} + \sum_{\ell \in C_P(i)} b_{\ell} \right) X_i - \sum_{\ell \in C_T(i)} b_{\ell} X_\ell.
\]

\[
\tag{11}
\]

**General SM**

\[ [B(i)] \geq 3, |S(j)| = 1, \text{ and } |C_S(i)| \geq 2: \]

See Appendix B.2 for the proof. We note that all the convex coefficients \( a_{ij} \) are positive so each arc \( ij \) is active if the inflow \( X_i \) is positive.

**Reverse topological traversal until reaching the source**

We have shown that when the price function of \( i \) is an affine decreasing function of the inflow \( X_i \), then by induction, at node \( i \), whenever the SS, the MS, or the SM case is encountered, at equilibrium, the price of each active trade \( p_{ki} \) on arc \( ki \) is the same. This price can be written as \( p_i \), which is an affine decreasing function of the inflow \( X_i \). When the source \( s \) is reached, the price at \( s \) satisfies \( p_s = a_i - b_i X_i \). Since \( a_i > a_s \text{ and } b_i > 0 \), we have that \( X_i > 0 \). At equilibrium, by the flow conservation property, the fact that goods are distributed accordingly to the positive convex coefficients in the SM case, and the assumption that there are no shortcuts, all arcs in \( E \) are active, each seller \( k \in S(i) \) offers \( i \) the same price \( p_i \), and \( p_i = a_i - b_i X_i - \sum_{\ell \in C_T(i)} b_{\ell} X_\ell \).

For completeness, we list the closed-form of \( b_i \) and the convex coefficient \( a_j \). The closed-form expression is used in Algorithm 1.

**SS**

By (5),

\[ b_i = 2 b_j + \sum_{\ell \in C_T(i, j)} b_{\ell}. \]

**MS**

By (6),

\[ b_i = b_j + \sum_{\ell \in C_T(i, j)} b_{\ell}. \]

**Simple SM**

By (11) and (10),

\[ b_i = \frac{2}{\sum_{\ell \in B(i)} b_{\ell}} + 2 \sum_{\ell \in C_T(i)} b_{\ell} + \sum_{\ell \in C_P(i)} b_{\ell} \text{ and } a_j = \frac{1}{b_j}. \]

**General SM**

See Appendix B.2 (19) and (24).
Lemma 3.2  The equilibrium is unique for shortcut-free SPGs.

Proof  The flow value of the source \( s \) is a fixed quantity computed by Algorithm 1. Whenever we encounter the SS and MS case, the seller firm \( i \) always offers the buyer firm \( j \) quantity \( X_j \). For the SM case, \( i \) offers \( j \) the quantity proportional to the convex coefficient \( q_j \). Since (CQP) is strongly convex, this solution is unique, which implies that the equilibrium is unique. □

3.2  General series parallel graphs

Suppose the given SPG \( G = (V,E) \) has a shortcut \( ij \in E \) that dominates a path \( \ell_{ij} = (i, v_1, \ldots, v_k, j) \). When the price \( p_j \) is a decreasing function of \( X_j \), \( i \) always prefers selling to \( j \) directly than through the intermediary firms along the path \( \ell_{ij} \) to obtain better utility. We refer the proof details to Appendix B.3 and show Example 7 in Appendix that illustrates an equilibrium with inactive trades in an SPG with a shortcut.

Lemma 3.3  Given an SPG, at an equilibrium, if \( ij \in E \) is a shortcut of a path \( \ell_{ij} \) and the price \( p_j \) is a decreasing function of \( X_j \), then there is no trade on \( \ell_{ij} \), that is, all the arcs on the path \( \ell_{ij} \) are inactive.

In the price computation for shortcut-free SPGs, we show by induction that \( p_j \) is an affine decreasing function of \( X_j \). By Lemma 3.3, given a general SPG, we can remove the dominated paths and obtain a shortcut-free SPG. The equilibrium can be found by Algorithm 1 in linear time. The uniqueness follows by Lemma 3.2. We conclude by the following theorem.

Theorem 3.2  There exists a linear time algorithm to find the equilibrium quantity and price for SPGs, and the equilibrium is unique.

4  STRUCTURAL ANALYSIS

We present some structural analyses, including the relation between firm location and utility, and the influence and invariants of different SPG component compositions on the equilibrium. We consider shortcut-free SPGs throughout this section.

4.1  Firm location and individual utility

This section focuses on the firm’s utility at equilibrium. Specifically, how does the position of a firm in the network influence its utility at equilibrium? The following example is useful to address this question.

Example 3  (Firm utility in a line network)

\[ p_s = 1 \quad s \rightarrow x \rightarrow a \rightarrow t \quad p_t = 2 - x \]

The price at firm \( a \) is \( p_a = 2 - 2x \) and at producer \( s \) is \( p_s = 2 - 4x \). Therefore, the utility at firm \( a \) is \( \Pi_a = (p_t - p_a)x = x^2 \) and at producer \( s \) is \( \Pi_s = (p_s - p_a)x = 2x^2 = 2\Pi_a \).

The example above shows an intuition of the location advantage that the firm closer to the source may have higher utility than its downstream buyers. However, this is not always true, especially when there is strong competition among upstream buyers (i.e., the MS case). The upstream firm that controls all the flow of its downstream firm has a better utility at equilibrium. Therefore, we introduce the following definition.

Definition 4.1  (Dominating Parent) \( i \) is a dominating parent of \( j \) if all the path from source \( s \) to \( j \) must go through \( i \).

Before analyzing the utility relation between a dominating parent and a dominated child, let us first focus on individual utility. By using the coefficient relation between the buyer \( i \) and the seller \( j \in B(i) \) as in (SS), (MS), and (Simple SM), we show the closed-form expression of the utility in Lemma 4.1. The proof is provided in Appendix C.1.

Lemma 4.1  \( \Pi_i = \frac{1}{2} \left( b_i + \sum_{e \in E_{i(i)}} b_e \right) X_i^2 \)

\[ \forall i \in V \cup \{s\} \]

Lemma 4.1 presents a closed-form expression of the price offered by the source, which is irrelevant to the structure of the supply chain network. Then, we show the double utility rule of a dominating parent.

The utility of the source is

\[ \Pi_s = \frac{1}{2} b_s X_s^2 = \frac{1}{2} p_s X_s \frac{a_t - a_s}{b_i} = \frac{a_t - a_s}{2} X_s. \]

By Lemma 3.1, \( s \) offers its buyers the same price at equilibrium. Let \( p = p_j \) for \( j \in B(s) \), we have that

\[ \Pi_i = \frac{a_i - a_s}{2} X_i = (p - a_i)X_i p = \frac{a_t - a_s}{2}. \]

Proposition 4.1  At equilibrium, the source offers the price \( \frac{a_t - a_s}{2} \) to its buyers.

We prove the following propositions which show the location advantage of a dominating parent. We show that in the SS and SM cases, the seller benefits a lot from the competition among the buyer side, and the proof is provided in Appendix C.2.

\[ 5 \text{Also, (24) in the appendix.} \]
Proposition 4.2  In the SS and SM case, the utility of the seller is at least twice the utility of the buyers’ total utility.

If a firm controls all the flow of another child firm in the supply chain, then its utility is at least twice as much as that child. We refer the proof to Appendix C.3.

Proposition 4.3  If firm i is a dominating parent of firm j, then firm i has at least twice the utility of firm j.

To sum up, a dominating parent always has better utility and the double utility rule always holds, which demonstrates the great value of controlling the upstream trades.

4.2  Network efficiency and component composition

To measure how firms would benefit from the network, we may care about not only the flow value but also the social welfare. The flow value is the total amount of goods that the source sends, or equivalently, the total amount of goods that the sink receives. The social welfare is the total utility of the source and intermediary firms plus the consumer surplus.

\[ SW(G) = \sum_{i \in V \cup \{s\}} \Pi_i + \frac{1}{2} b_t X_t^2 \]

\[ = \frac{1}{2} \sum_{i \in V \cup \{s\}} \left( b_i + \sum_{k \in C(i)} b_k \right) X_t^2 + \frac{1}{2} b_t X_t^2. \tag{12} \]

The social welfare can also be interpreted as the product of the flow and the price difference between the sink and the source (the producer surplus), plus the consumer surplus:

\[ SW(G) = (a_t - a_s - b_t X_s) X_s + \frac{1}{2} b_t X_t^2 \]

\[ = \left( a_t - a_s - \frac{b_t}{2} X_s \right) X_s. \tag{13} \]

The criteria of interest are welfare efficiency and flow efficiency defined as follows.

Definition 4.2  (Welfare Efficiency) A supply chain network is more welfare efficient if it provides larger social welfare at equilibrium.

Definition 4.3  (Flow Efficiency) A supply chain network is more flow efficient if it provides a larger flow value at equilibrium.

We defer the component composition results and analysis to Appendix A.

5  EQUILIBRIUM IN GENERALIZED SERIES PARALLEL GRAPHS

We discuss the equilibrium properties in the extension cases when the series–parallel graph has multiple sources or sinks. In particular, we show that:

- Single-source-and-multiple-sinks SPG: Price function of a firm may be piecewise linear and discontinuous under simple settings. There may exist multiple equilibria.
- Multiple-sources-and-single-sink SPG: An equilibrium may not exist.

5.1  Single source and multiple sinks

A series–parallel graph with a single source and multiple sinks (SMSPG) is defined as follows.

Definition 5.1  (SMSPG) G is a single-source-and-multiple-sink SPG if it can be constructed by deleting the sink node of an SPG and setting the adjacent nodes of the sink as the new sink nodes. The set of sinks is denoted as T.

First, we consider a special case that all the markets have the same price intercept, then all markets are active at equilibrium, that is, every market has a positive incoming flow. The proof is similar to Theorem 3.2 and we provide the sketch in Appendix D.1.

Proposition 5.1  Given an SMSPG, if all markets have the same price intercept, then there exists a unique equilibrium that can be found in polynomial time.

With different price intercepts \( a_t \) for \( t \in T \), the markets may be inactive, that is, the incoming quantity is zero. The market behavior of SMSPG is usually intractable. In particular, we focus on the supply chain networks of the shape in Figure 1. Based on the activity status of the markets, we introduce two types of strategies for the upstream firm.

Definition 5.2  (Low Price Strategy) A firm processes a relatively large quantity of goods at a relatively low price, such that all markets are active.

Definition 5.3  (High Price Strategy) A firm processes a relatively small quantity of goods at a relatively high price, such that some markets are inactive.

The firm plays its strategy to maximize its utility. Because of the various choices of strategies, the price function might be piecewise linear, and discontinuous. Furthermore, some

\[ \text{The consumer surplus is the triangular area of size } b_t X_t^2 / 2, \text{ with width } X_t \text{ and height } b_t X_s \text{, under the price function and above the selling price at the sink } t. \]
Counterintuitive results will occur, that is, the increase of price intercept may result in the decrease of total flow and social welfare.\textsuperscript{7} To understand these differences, it is helpful to consider an example as in Figure 1, where the two supply chain networks have an identical structure with different price intercepts.

Intuitively, supply chain 2 with a higher price intercept should have a larger flow and social welfare. However, supply chain 1 is more flow and welfare efficient. The equilibrium price functions at $s$ and $v$ are shown in Figure 2. We note that the source firm $s$ has two strategies when $p_s = 7$, and both low and high price strategies are feasible. Interestingly, when $a_{t_1} = 20$, the utility of $s$ is maximized by choosing high price strategy and only market $t_1$ is active. However, when the price intercept at market $t_1$ drops, the low price strategy is preferred by $s$.

By fixing the price intercept at market $t_2$ and adjusting the price intercept at market $t_1 (a_{t_1})$, Figure 3 shows the results of the source utility, consumer surplus, total flow, and social welfare. The intersecting point at $a_{t_1} \approx 19.07$ shows that increasing price intercept at market $t_1$ hurts the supply chain efficiency. When $a_{t_1}$ is the intersecting point then there are multiple equilibria since $s$ has no preference between the high price and low price strategy. Besides, $a_{t_1}$ is feasible only in the interval $(12, 22]$. The calculation details are provided in Example 8 in Appendix.

**Proposition 5.2** An SMSPG may have multiple equilibria.

\textsuperscript{7}Compared with Proposition 8 in the appendix.
Assume that the source producers make their decision simultaneously, an equilibrium may not exist. A key factor of the equilibrium non-existence for MSSPGs is that the production quantities of the sources are not controlled by any upstream firms. Suppose a super source is added as an upstream of the sources, then the network is an SPG. At the equilibrium of this SPG, the super source sets a quantity upper bound for each source, which forces the sources to sell the goods at an optimal price to the downstream.

Without the quantity upper bound, producers potentially compete by setting lower prices, and firms with higher production costs drop out. The absence of high production firms affects the structure of the network. The optimal strategy of the attending firms with respect to the modified structure is not consistent with the strategy when the absent firms are also considered, thus causing the non-existence of the equilibrium.

We show Example 9 that illustrates this scenario in Appendix.

**Proposition 5.3** A pure strategy equilibrium in an MSSPG may not exist.

6 | CONCLUSION

We consider a model of sequential competition in supply chain networks. Our main contribution is that when the network is series–parallel, the model is tractable and allows a rich set of comparative analyses. In particular, we provide a linear-time algorithm to compute the equilibrium and the algorithm helps us study the influence of the network on the flow value and social welfare of the equilibrium.

To derive our results, several assumptions are needed, the main of which is the underlying series–parallel network. The series composition of such a network corresponds to the increase in the length of the sequential decisions, while the parallel composition represents the increase of competitiveness of the market. For this reason, this class of networks allows us to naturally study the combined effects of sequential decisions and competition in supply chain networks. We see our results as an initial theoretical step to understand this complex issue. Supply chain networks in practice might not have this special structure, but when they can be approximated with series and parallel compositions, our results provide valuable insights into their behavior.

From a technical perspective, generalizing series–parallel networks to a broader class is a main open question. To this end, we show that slightly extending the network structure beyond series–parallel graphs with a single source and multiple sinks (SMSPG) makes the model intractable. The first open problem is to design an efficient algorithm that verifies if an SMSPG has an equilibrium and finds one if it exists. The main challenge is the piecewise linearity and the discontinuity of the price function for intermediary firms. This problem may be computationally intractable, but it is unclear what a reasonable proof strategy would be.

Another open problem is to efficiently find an equilibrium in general DAGs with a single source and a single sink. The trades can be inactive for shortcut-free DAGs as shown in Example 10 in Appendix. We conjecture that there is always an equilibrium and the active trades form a shortcut-free SPG. A natural approach is to compute the price function in reverse topological order from the sink inductively. However, this enforces one to solve linear complementarity problems (LCP) that correspond to the firms, where the LCPs of the upstream firms are derived from the LCPs of the downstream firms. Solving the LCP system requires determining inactive trades where the number of combinations of active and inactive trades is exponential. Therefore, a potential strategy to show computational intractability is a reduction from an LCP based or a quadratic programming problem.

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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**SUPPORTING INFORMATION**

Additional supporting information may be found online in the Supporting Information section at the end of the article.

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