The Hadronic Spectrum and Confined Phase in (1+1)-Dimensional Massive Yang-Mills Theory

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Including joint work with Peter Orland
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The Principal Chiral Sigma Model (PCSM)

Action: \( S = \frac{N}{2g^2} \int d^2x \, \text{Tr} \partial_\mu U^\dagger(x) \partial^\mu U(x), \)

\( U(x) \in SU(N) : \)

\( SU(N) \times SU(N) \) symmetry: \( U(x) \rightarrow V_L U(x) V_R, \, V_{L,R} \in SU(N). \)

Associated Noether currents:

\[ j^L_\mu(x)_c = \frac{-iN}{2g^2} \partial_\mu U_{ab}(x) U^\dagger_{bc}(x), \]

\[ j^R_\mu(x)_d = \frac{-iN}{2g^2} U^\dagger_{da}(x) \partial_\mu U_{ab}(x) \]

Asymptotically free theory of massive particles, with left and right color.
The S-Matrix

Particles and antiparticles have two color charges (color dipoles).

\[
\langle P, \theta', c_1, d_1; P, \theta'_2, c_2, d_2 | P, \theta_1, a_1, b_1; P, \theta_2, a_2, b_2 \rangle_{\text{out}} \langle \theta'_{\text{out}} | \theta_{\text{in}} \rangle_{\text{in}} = \frac{\sinh(\theta/2 - \frac{\pi i}{N})}{\sinh(\theta/2 + \frac{\pi i}{N})} \left[ \frac{\Gamma(i\theta/2\pi + 1)\Gamma(-i\theta/2\pi - \frac{1}{N})}{\Gamma(i\theta/2\pi + 1 - \frac{1}{N})\Gamma(-i\theta/2\pi)} \right]^2 \times \left( \delta_{a_1}^{c_1} \delta_{a_2}^{c_2} - \frac{2\pi i}{N\theta}\delta_{a_2}^{c_2}\delta_{a_2}^{c_1} \right) \times \left( \delta_{b_1}^{d_1} \delta_{b_2}^{d_2} - \frac{2\pi i}{N\theta}\delta_{b_2}^{d_2}\delta_{b_2}^{d_2} \right) \langle \theta'_1 | \theta_1 \rangle \langle \theta'_2 | \theta_2 \rangle
\]

\[\theta_j = \text{rapidity} : E_j = m \cosh \theta_j, \quad p_j = m \sinh \theta_j, \quad E^2 = p^2 + m^2\]

rapidity difference \(\theta = \theta_1 - \theta_2\)

P. B. Wiegmann, Phys. Lett. 142 B (1984)
Two-particle form factor

For \( N > 2 \)

\[
\langle 0 | j^L_{\mu}(0)_{a_0c_0} | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2 \rangle = (p_1 - p_2)_\mu \left( \delta_{a_0a_2} \delta_{c_0a_1} - \frac{1}{N} \delta_{a_0c_0} \delta_{a_1a_2} \delta_{b_1b_2} \right) \times \frac{2\pi i}{(\theta + \pi i)} \exp \int_0^\infty \frac{dx}{x} \left[ \frac{-2 \sinh \left( \frac{2x}{N} \right)}{\sinh x} + \frac{4e^{-x} \left( e^{2x/N} - 1 \right)}{1 - e^{-2x}} \right] \sin^2 \left[ x(\pi i - \theta)/2\pi \right] \frac{\sinh x}{\sinh x}
\]

A. C. C., Phys. Rev. D 86 (2012) 025025

For \( N = 2 \), the form factors are known from the \( O(4) \) sigma model, by

\[
SU(2) \times SU(2) \simeq O(4),
\]

M. Karowski and P. Weisz, Nucl. Phys. B 139 (1978) 455
Gauged PCSM
Not integrable anymore

\[ S \int d^2 x \left( -\frac{1}{4} \text{Tr} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2 g_0^2} \text{Tr} D_\mu U^\dagger D^\mu U \right) \]

with

\[ D_\mu = \partial_\mu + ie A^L_\mu \]

The left $SU(N)$ symmetry is now a local gauge symmetry.

There is a “Gauss Law” that requires the left color indices of sigma-model particles to contract into singlets.

What is the mass spectrum?
Unitary gauge $U = 1$

$$S = \int d^2 x - \frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{e^2}{2g_0^2} \text{Tr} A_\mu A^\mu$$

In unitary gauge, the PCSM works as a Higgs field, giving mass $e/g_0$ to the gluon.

Asymptotic freedom forces $g_0 \to 0$! The gluon would have a huge mass, not visible at low energies.

Is there more to life than this?
Hamiltonian in the (completely fixed) Axial gauge

Find Hamiltonian in the axial gauge $A_0 = 0, \ A_1(t = 0) = 0$.

$$H = H_{\text{PCSM}} - \frac{e^2}{2g_0^4} \int dx^1 \int dy^1 |x^1 - y^1| j_0^L(x^1) j_0^L(y^1).$$

The system is in a confined phase. The physical particles are hadron-like bound states of sigma model particles

Mesons: one sigma model particle and one antiparticle, with string tension $\sigma = e^2 C_N$.

The meson spectrum is of the form $M = 2m + E$. 
Nonrelativistic meson wave function

\( (x = x^1 - y^1) \)

\[-\frac{1}{m} \frac{d^2}{dx^2} \Psi(x) + \sigma |x| \Psi(x) = E \Psi(x) \]

The wave function for particles confined by the potential \( V(x^1, y^1) = \sigma |x^1 - y^1| \) is

\[ \Psi(x) = CAi \left\{ (m\sigma)^{\frac{1}{3}} \left[ |x| - \frac{E}{\sigma} \right] \right\} \]

The \( N \)-particle hadron spectrum can be computed in principle by solving the \( N \)-body problem with potential

\[ V(x_1, \ldots, x_N) = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \sigma |x_i^1 - x_j^1| \]
The particle-antiparticle wave function

For a free sigma model particle and antiparticle, the wave function is

\[ \Psi(x^1, y^1) = \begin{cases} 
  e^{ip_1 x^1 + ip_2 y^1}, & \text{for } x^1 < y^1 \\
  e^{ip_2 x^1 + ip_1 y^1} S(\theta), & \text{for } x^1 > y^1 
\end{cases} \]

The free and confined wave functions must agree at \( x^1 \approx y^1 \). Quantization condition for the binding energies \( E \)!
The meson spectrum

\[ M_n = 2m + E_n, \quad n = 0, 1, 2, \ldots \]

\[ E_n = \left\{ \left[ \epsilon_n + \left( \epsilon_n^2 + \beta_N^3 \right)^{\frac{1}{2}} \right]^{\frac{1}{3}} + \left[ \epsilon_n - \left( \epsilon_n^2 + \beta_N^3 \right)^{\frac{1}{2}} \right]^{\frac{1}{3}} \right\}^\frac{1}{2}, \]

where

\[ \epsilon_n = \frac{3\pi}{4} \left( \frac{\sigma}{m} \right)^{\frac{1}{2}} \left( n + \frac{1}{2} \pm \frac{1}{4} \right), \]

\[ \beta_N = \frac{\sigma^{\frac{1}{2}}}{2\pi m} \int_0^\infty \frac{d\xi}{\sinh \xi} \left[ 2(e^{2\xi/N} - 1) - \sinh(2\xi/N) \right], \]

where \( \pm = + \) for the \( SU(N)^R \) \((N^2 - 1)\)-plet, and \( \pm = - \) for the singlet.
Form factors and correlation functions
Two-quark approximation
P. Fonseca, A. Zamolodchikov, RUNHETC-2001-37

\[ |B, \phi, n\rangle \approx e^{ix^{1}M_{n}\sinh \phi} \frac{1}{\sqrt{m}} \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \Psi_{n}(\theta) |A, \theta, a_{1}, b_{1}; P, -\theta, a_{1}, b_{1}\rangle \]

Bound state form factor

\[ \langle 0|A(x^{1}, x^{2})|B, \phi, n\rangle = e^{s\phi} e^{ix^{1}M_{n}\sinh \phi} \int dz \int \frac{d\theta}{4\pi} e^{izm\sinh \theta} \frac{1}{\sqrt{m}} \left( \frac{E_{n}}{\sigma^{H}} \right)^{\frac{1}{4}} \text{Ai}\left[(m\sigma^{H})^{\frac{1}{3}} \left(|z| - \frac{E_{n}}{\sigma^{H}}\right)\right] \times \langle 0|A(0, x^{2})|A, \theta, a_{1}, b_{1}; P, -\theta, a_{1}, b_{1}\rangle \]

Correlation functions

\[ \langle 0|A(x^{1}, x^{2})A(0, x^{2})|0\rangle = \sum_{\Psi} \langle 0|A(x^{1}, x^{2})|\Psi\rangle \langle \Psi|A(0, x^{2})|0\rangle \]
Lattice results by Gongyo and Zwanziger \((SU(2))\)
Order parameter: \(\Psi = \frac{1}{2} \text{Tr}[\bar{U}^\dagger U]\)

S. Gongyo and D. Zwanziger, arXiv:1402.7124

Also quark-antiquark potential from Wilson loop, and W-boson propagator suggest a confined phase, and a Higgs-like (Kosterlitz-Thouless) phase that seems to go away at large volume.
Where are the W bosons, hiding at finite volume?

Look at the action in axial gauge $A_1 = 0$

\[ S = \int d^2x \left[ \frac{1}{2} \text{Tr} (\partial_1 A_0)^2 + \frac{1}{2g_0^2} \text{Tr} (\partial_0 U^\dagger + ieU^\dagger A_0)(\partial_0 U - ieA_0 U) \right. \\
\left. - \frac{1}{2g_0^2} \text{Tr} \partial_1 U^\dagger \partial_1 U \right] \]

Integrate out $A_0$:

\[ S = \int d^2x \left( \frac{1}{2g_0^2} \text{Tr} \partial_\mu U^\dagger \partial^\mu U + \frac{1}{2} j^L_{0\ a} \frac{1}{2} \partial_1^2 + e^2 / g_0^2 U^\dagger U j^L_{0\ a} \right) \]

Problem! The renormalized field, $\Phi(x) \sim Z(g_0, \Lambda)^{-1/2} U(x)$, is not unitary, and in fact $Z^{-1/2} \to \infty$ as $\Lambda \to \infty$. 
Screening at finite volume?
Expectation values at finite volume (Mussardo-LeClair formula):

\[ \langle \Phi^\dagger(0) \Phi(0) \rangle_V = Z_V^{-1} \]

\[ = \sum_n \frac{1}{n! (2\pi)^n} \int \left[ \prod_{i=1}^n d\theta_i \frac{e^{-\epsilon(\theta_i)}}{1 + e^{-\epsilon(\theta_i)}} \right] \text{F.P.} \langle \theta_n, \ldots, \theta_1 | \Phi^\dagger(0) \Phi(0) | \theta_1, \ldots, \theta_n \rangle \]

where the pseudo energies, \( \epsilon \) are obtained from the thermodynamic Bethe ansatz

\[ \epsilon(\theta) = mL \cosh(\theta) - \int \frac{d\theta'}{2\pi} \left[ i \log \frac{d}{dx} S(x) \right] \big|_{x=\theta-\theta'} \log(1 + e^{-\epsilon(\theta')}) \]
An anticlimactic Lattice ’14 picture for an anticlimactic Lattice ’14 talk