On the Existence of a Solution to a Spectral Estimation Problem à la Byrnes-Georgiou-Lindquist

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Abstract—A parametric spectral estimation problem in the style of Byrnes, Georgiou, and Lindquist was posed in [1], but the existence of a solution was only proved in a special case. Based on their results, we show that a solution indeed exists given an arbitrary matrix-valued prior density. The main tool in our proof is the topological degree theory.

Index Terms—Approximation of spectral densities, spectral estimation, generalized moment problems, topological degree theory, covariance extension.

I. INTRODUCTION

This note concerns a spectral estimation problem subjected to a generalized moment constraint. The setup of the problem (scalar version) was first introduced by Byrnes, Georgiou, and Lindquist in [2], and then further elaborated in [3] in order to allow for an a priori information. This formulation, known under the name of THREE-like spectral estimation, has now become nearly standard and includes as special cases some important problems in the field of systems and control such as covariance extension (cf. e.g., [4]–[9]) and Nevanlinna–Pick interpolation (cf. [10]–[12] and references therein).

It is worth pointing out that moment problems [13], [14] form a special class of inverse problems that are typically not well-posed in the sense of Hadamard. To remedy this, the mainstream approach today is to first define an entropy-like distance index \( d(\Phi, \Psi) \) between two bounded and coercive spectral densities, and then to find the “best” \( \Phi \) given the prior \( \Psi \) by minimizing the distance index subject to the generalized moment constraint. Still, it is not trivial to solve such an optimization problem. Indeed, although the dual problem is typically convex, the dual variable (i.e., the Lagrange multiplier) is a Hermitian matrix that lives in an open, unbounded domain and this usually gives rise to a number of numerical issues. With reference to the scalar case, results in this direction include the aforementioned [3], in which the chosen distance index is the Kullback–Leibler divergence (cf. also [15]–[18]), and [19], where a general family of divergences (the Alpha divergence family) is considered. In the multivariate case, the problem becomes much more challenging and its feasibility strongly depends on the selected distance. We mention, in particular, the papers [20], where a multivariate extension of the Kullback–Leibler divergence, the quantum relative entropy, is considered; [21], [22], which deal with a sensible generalization of the Hellinger distance; and [23], [24], where the selected distance index coincides with the multivariate Itakura–Saito distance. It is worth remarking that the latter two approaches lead to rational solutions with bounded McMillan degrees when the prior is rational. Finally, [25] and [26] introduce two more general frameworks based on the notion of Beta and Tau divergence families, wherein the multivariate Kullback–Leibler and Itakura–Saito distance can be recovered as particular cases.

There are a few attempts in directions different from optimization; see [1], [27], [28]. In particular, a parametric family of spectral densities was introduced in [1], and a certain map from the parameter space to the space of (generalized) moments was studied in the light of a Hadamard-type global inverse function theorem [29]. The proposed parametrization has been shown to be amenable for the implementation of a matricial version of an extremely simple and efficient fixed-point algorithm introduced in [15], whose convergence properties have been investigated in [16]–[18]. However, the result in [1] was not satisfactory because the authors only showed that a solution exists when the prior \( \Psi \) has a very special structure. In fact, this is the motivation of the current note. As a continuation of the work in [1], here we will show that a solution to the parametric spectral estimation problem exists given any fixed matrix-valued prior density that is bounded and coercive. Of course the problem is still open to a large extent since uniqueness of the solution (and, a fortiori, well-posedness of the problem) is not known, and the convergence properties of the algorithm proposed in [1] to compute a solution are yet to be examined.

The main machinery behind our existence proof is the topological degree theory from nonlinear analysis. As a historical remark, Georgiou was the first to apply the degree theory to rational covariance extension [5], [6], [10] to show existence of a solution, and it was further developed by Byrnes, Lindquist, and coworkers [7] to prove the uniqueness and well-posedness. These theories were established before the discovery of the cost function in the optimization framework [8], [9], [11], which was later called generalized entropy criterion.

The outline of this note is as follows. In Section III we first set up some notations before reviewing the problem formulation. The important special case of multivariate covariance extension is detailed for illustration. Our main result is presented in Section III. A part of the degree theory is
reviewed in order to carry out our proof. We conclude with some open questions on the uniqueness of the solution and convergence of an algorithm leading to such a solution.

List of symbols

- \( E \), mathematical expectation.
- \( \mathbb{Z} \), the set of integers.
- \( \mathbb{C} \), the complex plane.
- \( \mathbb{D} \), the open complex unit disk \( \{ z \in \mathbb{C} : |z| < 1 \} \).
- \( T \equiv \partial \mathbb{D} \), the unit circle, where \( \partial \) stands for the boundary.
- \( \text{GL}(n, \mathbb{C}) \), group of \( n \times n \) invertible complex matrices.
- \( \mathcal{H}_n \), the vector space of \( n \times n \) Hermitian matrices.
- \( \mathcal{H}_{+n} \), the subset of \( \mathcal{H}_n \) that contains positive definite matrices.
- \( C(T; \mathcal{H}_m) \), the space of \( \mathcal{H}_m \)-valued continuous functions on \( T \).
- \( \mathcal{M}_m \), the family of \( \mathcal{H}_{+m} \)-valued functions defined on \( T \) that are bounded and coercive.
- \( (\cdot)^* \), complex conjugate transpose. When considering a rational matrix-valued function \( G(z) \), \( G^*(z) \) stands for the analytic continuation of the function that for \( z \in T \) equals the complex conjugate transpose of \( G(z) \).
- \( (\cdot)^- \), shorthand for \( (\cdot)^* \).

II. A MULTIVARIATE SPECTRAL ESTIMATION PROBLEM

Consider a linear system with a state-space representation

\[
x(t + 1) = Ax(t) + By(t),
\]

where \( A \in \mathbb{C}^{n \times n} \) is Schur stable, i.e., has all its eigenvalues in \( \mathbb{D} \), \( B \in \mathbb{C}^{n \times m} \) is of full column rank \( n \geq m \). Moreover, the pair \((A, B)\) is assumed to be reachable. The input process \( y(t) \) is zero-mean wide-sense stationary with an unknown spectral density matrix \( \Phi(z) \). The transfer function of \((1)\) is just

\[
G(z) = (zI - A)^{-1}B,
\]

which can be interpreted as a bank of filters. An estimate of the steady-state covariance matrix \( \Sigma := \mathbb{E}[x(t)x(t)^*] \) of the state vector \( x(t) \) is assumed to be known. (For the problem of estimating covariance matrices in this setting, we refer to \([30]-[32]\).) Hence we have

\[
\int G\Phi G^* = \Sigma,
\]

where the function is integrated on \( T \) with respect to the normalized Lebesgue measure \( \frac{d\theta}{2\pi} \). This notation will be adopted throughout the note.

Given the matrix \( \Sigma \in \mathcal{H}_{+n} \), we want to estimate the spectral density \( \Phi \) such that the generalized moment constraint \((3)\) is satisfied. For example, consider the following choice of the matrix pair \((A, B)\):

\[
A = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.
\]

Here each block in \( A \) or \( B \) is of \( m \times m \) and \( A \) is a \((p + 1) \times (p + 1)\) block matrix while \( B \) is a \((p + 1)\)-block column vector. It is easy to verify that in this case

\[
G(z) = (zI - A)^{-1}B = \begin{bmatrix} z^{-p+1}I_m \\ \vdots \\ z^{-1}I_m \end{bmatrix},
\]

Symbolically, the steady state vector

\[
x(t) = G(z)y(t) = \begin{bmatrix} y(t - p+1) \\ \vdots \\ y(t - 1) \end{bmatrix},
\]

and the covariance matrix \( \Sigma \) has a block-Toeplitz structure, i.e.,

\[
\Sigma = \begin{bmatrix} \Sigma_0 & \Sigma_1 & \cdots & \Sigma_p \\ \Sigma_1 & \Sigma_0 & \cdots & \Sigma_{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_p & \Sigma_{p-1} & \cdots & \Sigma_0 \end{bmatrix},
\]

where \( \Sigma_k := \mathbb{E}\{y(t+k)y(t)^*\} \in \mathbb{C}^{m \times m} \) with a slight abuse of notation. In fact, the constraint \((3)\) is equivalent to the set of moment equations

\[
\int_{-\pi}^{\pi} e^{jk\theta} \Phi(e^{j\theta}) \frac{d\theta}{2\pi} = \Sigma_k, \quad k = 0, 1, \ldots, p.
\]

To find a spectral density \( \Phi \) satisfying \((8)\) is the classical covariance extension problem \([14]\).

In general, existence of \( \Phi \in \mathcal{M}_m \) satisfying \((3)\) is not trivial. Such feasibility problem was addressed in \([33], [34]\), see also \([1], [16], [21]-[23], [30], [31], [35]\). In order for \( \Sigma > 0 \) to be a state covariance, a certain Lyapunov-equation has to be solvable or an equivalent rank condition must hold.Interested readers can consult the references for details. Here we shall take the feasibility as a standing assumption. More precisely, let us define the linear operator \( \Gamma : C(T; \mathcal{H}_m) \to \mathcal{H}_n \) as

\[
\Gamma : \Phi \mapsto \int G\Phi G^*.
\]

Then we assume that the covariance matrix \( \Sigma \in \text{Range} \Gamma \). Various properties of the set \( \text{Range} \Gamma \) are elaborated in e.g., \([31]\) Sec. III. In particular, by Proposition 3.1 of that paper, \( \text{Range} \Gamma \subset \mathcal{H}_n \) is a linear space with real dimension \( m(2n - m) \).

Moreover, define the set

\[
\mathcal{L}_+ := \{ A \in \mathcal{H}_n : G^*(z)\Lambda G(z) > 0, \forall z \in T \}.
\]

By the continuous dependence of eigenvalues on the matrix entries, one can verify that \( \mathcal{L}_+ \) is an open subset of \( \mathcal{H}_n \). For \( \Lambda \in \mathcal{L}_+ \), take \( W_\Lambda \) as the unique stable and minimum phase (right) spectral factor of \( G^*\Lambda G \) \([1]\) Lemma 11.4.1, i.e.,

\[
G^*\Lambda G = W_\Lambda^*W_\Lambda.
\]

Our problem is formulated as follows.
Problem 1. Given the filter bank \( G(z) \) in (2), let \( \Sigma \in \text{Range}_+ \Gamma := \text{Range} \Gamma \cap \mathcal{S}_{+,n} \), and \( \Psi \in \mathcal{P}_m \). Find \( \Lambda \in \mathcal{L}_+ \) such that
\[
\Phi_\Lambda := W_{\Lambda}^{-1} \Psi W_{\Lambda}^* ,
\]
(12)
satisfies
\[
\int G\Phi_\Lambda G^* = \Sigma.
\]
(13)
Define \( \mathcal{L}_+^\Gamma := \mathcal{L}_+ \cap \text{Range} \Gamma \), and consider the map \( \omega: \mathcal{L}_+^\Gamma \rightarrow \text{Range}_+ \Gamma \) given by
\[
\omega: \Lambda \mapsto \int G\Phi_\Lambda G^*.
\]
(14)
As indicated in [1] and will be clear in Subsection III-B this is a continuous map between open subsets of the linear space \( \text{Range} \Gamma \), and Problem [1] is feasible if and only if \( \omega \) is surjective. Theorem 11.4.3 in [1] guarantees such surjectivity when the prior is a scalar density times a positive definite matrix. In the next section, we shall extend that result to accommodate an arbitrary matrix spectral density \( \Psi \).

III. Existence of a Solution

The proof of our main result relies on the notion of topological degree of a continuous map. The degree theory forms an important part of differential topology and is closely related to fixed-point theory, cf. [36, Ch. I] for a rather informative historical account. In particular, the degree theory is a powerful tool to prove existence of a solution to a system of nonlinear equations. There are several versions of the theory for different types of maps. Although the maps that we consider in this note are between open subsets of the Euclidean space, we shall use the more general degree theory for continuous maps between smooth, connected, boundary-less manifolds. Some main points of the theory are reviewed below.

A. A short review of the degree theory

We mainly follow the lines of [36, Ch. III]. Suppose \( U, V \subset \mathbb{R}^n \) are open and connected, and \( f: U \rightarrow V \) is a proper \( C^1 \) function. Recall that \( f \) is called proper if the preimage of every compact set in \( V \) is compact in \( U \). Our major concern is solvability of the equation
\[
f(x) = y.
\]
(15)
A point \( y \in V \) is called a regular value of \( f \) if either
(i) for any \( x \in f^{-1}(y) \), \( \det f'(x) \neq 0 \) or
(ii) \( f^{-1}(y) \) is empty.
Here \( f^{-1}(y) \) denotes the preimage of \( y \) under \( f \), i.e., the set
\[
\{ x \in U : f(x) = y \},
\]
and \( f'(x) \) denotes the Jacobian matrix of \( f \) evaluated at \( x \). Let \( y \) be a regular value of type (i), and the degree of \( f \) at \( y \) is defined as
\[
\deg(f, y) := \sum_{f(x) = y} \text{sign} \ |\det f'(x)| ,
\]
(16)
where the sign function
\[
\text{sign}(x) = \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0 
\end{cases}
\]
and not defined at 0.

Throughout this note, properness will be a crucial property of our function. Since \( f \) is proper, one can show that the preimage \( f^{-1}(y) \) is finite following the classical inverse function theorem, and hence the sum above is well defined. For regular values of type (ii), we set \( \deg(f, y) = 0 \). Moreover, the set of regular values is dense in \( V \) by Sard–Brown Theorem [36, p. 63]. Further properties of the degree related to our problem are listed below:

- The degree of \( f \) at \( y \) does not depend on the choice of regular value. Therefore, we can define the degree of \( f \) as
\[
\deg(f) = \deg(f, y)
\]
for any regular value \( y \).
- If \( \deg(f) \neq 0 \), then for any \( y \in V \), there exists \( x \in U \) such that \( f(x) = y \), that is, the map \( f \) is surjective. A proof of this fact can be found in [7, p. 1849].
- Homotopy invariance. If \( H: U \times [0, 1] \rightarrow V \), \( (x, t) \mapsto y \) is jointly continuous in \( (x, t) \) and proper, then \( \deg(H_t, y) \) is defined and independent of \( t \in [0, 1] \). Here \( H_t: U \rightarrow V \) is defined by \( H_t(x) = H(x, t) \).

One important point of theory is that degree can be defined for continuous functions through approximation by smooth functions [36 Proposition and Definition 3.1, p. 111], and [16] is just a way of computing it in the special case of \( C^1 \) [37, Remark p. 71]. In particular, the homotopy invariance of the degree holds in the continuous case [36 Proposition 3.4, p. 112].

B. Proof of existence

Our main theorem will be preceded by some lemmas. Take \( \Psi = I \) the identity matrix, and the map \( \omega \) would reduce to
\[
\tilde{\omega}: \mathcal{L}_+^\Gamma \rightarrow \text{Range}_+ \Gamma
\]
\[
\Lambda \mapsto \int G(G^* \Lambda G)^{-1} G^*.
\]
(17)

Lemma 1. The map \( \tilde{\omega} \) is continuously differentiable.

Proof. The map
\[
\text{GL}(n, \mathbb{C}) \rightarrow \text{GL}(n, \mathbb{C}) : X \mapsto X^{-1}
\]
(18)
is smooth, which follows from Cramer’s rule in linear algebra. Hence, the function \( \tilde{F}_\Lambda(e^{\theta}) := G(G^* \Lambda G)^{-1} G^* \) inside the integral of (17) is also smooth in \( \Lambda \). Moreover, since \( G \) is a rational function, all the partial derivatives of \( \tilde{F}_\Lambda(e^{\theta}) \) with respect to \( \Lambda \) are continuous in \( \theta \) (and \( \Lambda \)). Then by Leibniz’s rule for differentiation under the integral sign, partial derivatives of \( \tilde{\omega} \) of all orders exist.

Next, we show that the first order partial derivatives are continuous. For the time being, let us consider the map \( \tilde{\omega} \) defined on \( \mathcal{L}_+ \). (We made the domain restricted to the intersection with \( \text{Range} \Gamma \) out of the consideration of dimensionality.) From [18], the differential of the map (18) at \( X \) is given by
\[
\mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n} : V \mapsto -X^{-1} V X^{-1}.
\]
Moreover, there exists \( \mu > 0 \) such that \( G^*s(\theta)\Lambda \delta(\theta) > 0 \) for all \( \theta \in [-\pi, \pi] \) and \( k \). Hence, there exists \( \lambda_{\min} \) such that \( \mu \geq \lambda_{\min} \) for all \( k \). On the other hand, since \( \delta \Lambda \) is fixed, it must hold that \( G^*\Lambda \delta \Lambda \leq M_1 \), where

\[
M := \max \rho \left( G^*(\theta)\delta \Lambda(\theta) \right).
\]

Here \( \rho(\cdot) \) denotes the spectral radius of a matrix. Therefore, we have

\[
\delta \tilde{F}_{\Lambda, \delta} \leq M \mu^{-2} G^* \delta \Lambda, \quad \forall k.
\]

Moreover,

\[
|\delta \tilde{F}_{\Lambda, \delta}|_{\mathcal{H}} \leq M \mu^{-2} G_{\max}, \quad \forall k \geq 1, \quad \forall i, \ell,
\]

where \( G_{\max} := \max_{\theta, i, \ell} \|G^*\|_{\mathcal{H}} \) is finite since the entries of \( G^*(\theta)G^*(\theta) \) are continuous functions of \( \theta \), analytic in an open annulus containing the unit circle. Hence, by Lebesgue’s dominated convergence theorem, we have

\[
\lim_{k \to \infty} \delta \hat{W}_{\Lambda, k}(\delta \Lambda) = - \int \lim_{k \to \infty} \delta \tilde{F}_{\Lambda, \delta} \delta \Lambda = \delta \hat{W}_{\Lambda}(\delta \Lambda).
\]

Partial derivatives can then be recovered by the operation \( \delta \hat{W}_{\Lambda}(\delta \Lambda_1, \delta \Lambda_2) \) by choosing \( \delta \Lambda_k, \ k = 1, 2 \) to be the standard basis matrices of \( \delta \Lambda \), where the notation \( \langle M_1, M_2 \rangle := \text{tr}(M_1 M_2) \) denotes the standard inner product in \( \delta \Lambda \). In this way, one can see that every partial derivative of \( \hat{W} \) is continuous in \( \Lambda \).

**Lemma 2.** The map

\[
H : \mathcal{L}_+^{\mathbb{R}} \times [0, 1] \to \text{Range}_* \Phi
\]

\[
(\Lambda, t) \mapsto G\Phi_{\Lambda, t} G^*.
\]

is a proper continuous homotopy between \( \omega \) and \( \hat{\omega} \), where

\[
\Phi_{\Lambda, t} := W_{\Lambda}^{-1} \left[ t \Psi + (1 - t) I \right] W_{\Lambda}^*.
\]

**Proof.** By definition we need to show two things, namely that \( H \) is jointly continuous in \( \Lambda \) and \( t \) and that \( H \) is proper. In order to prove joint continuity, we first notice that the spectral factor \( W_\Lambda(z) \) can be written as [1], Lemma 11.4.1

\[
W_\Lambda(z) := L_\Lambda^* B^* P_\Lambda (z I - A)^{-1} B + L_\Lambda,
\]

where \( P_\Lambda \) is the unique stabilizing solution of the following Discrete-time Algebraic Riccati Equation (DARE)

\[
\Pi = A^* \Pi A - A^* \Pi B (B^* \Pi B)^{-1} B^* \Pi A + \Lambda,
\]

and \( L_\Lambda \) is the right Cholesky factor of \( B^* \Lambda P_\Lambda B \), i.e.,

\[
B^* \Lambda P_\Lambda B = L_\Lambda^* \Lambda L_\Lambda
\]

with \( L_\Lambda \) being lower triangular having real and positive diagonal entries. Next, let us introduce a change of variables by letting

\[
C_\Lambda := L_\Lambda^{-1} \Lambda B^* P_\Lambda.
\]

Then, it is not difficult to recover the relation \( L_\Lambda = C_\Lambda B \). In this way, the spectral factor (22) can be rewritten as

\[
W_\Lambda(z) = C_\Lambda A(z I - A)^{-1} B + C_\Lambda B.
\]

According to [40], Thm. A.5.5, the dependence of the \( m \times n \) matrix \( C_\Lambda \) defined above on \( \Lambda \in \mathcal{L}_+^{\mathbb{R}} \) turns out to be a homeomorphism. From this fact it follows that \( W_\Lambda(\theta) \) depends continuously on \( \Lambda \in \mathcal{L}_+^{\mathbb{R}} \), for all \( \theta \in [-\pi, \pi] \). Consider now

\[
\Phi_{\Lambda, t}(\theta) = W_{\Lambda}^{-1}(\theta) \left[ t \Psi(\theta) + (1 - t) I \right] W_{\Lambda}^*(\theta).
\]

As a linear combination in \( t \in [0, 1] \) of continuous functions of \( \Lambda \), \( \Phi_{\Lambda, t}(\theta) \) is jointly continuous w.r.t. \( t \in [0, 1] \) and \( \Lambda \in \mathcal{L}_+^{\mathbb{R}} \), for all \( \theta \in [-\pi, \pi] \).

Next we need to show the continuity together with the integral. Consider any sequence \( \{(\Lambda_k, t_k)\}_{k \geq 1} \subset \mathcal{L}_+^{\mathbb{R}} \times [0, 1] \) such that \( \lim_{k \to \infty} t_k = \ell \in [0, 1] \) and \( \lim_{k \to \infty} \Lambda_k = \Lambda \in \mathcal{L}_+^{\mathbb{R}} \). Following the same line of reasoning as in the proof of Lemma [1] there exists \( \mu > 0 \) such that \( G^* \Lambda_k \delta \Lambda \geq \mu I \), \( \forall k \). Therefore, it holds that

\[
G\Phi_{\Lambda_k, t_k} G^* \leq K G^* \Lambda_k G^* \leq K \mu^{-2} G^* \delta \Lambda, \quad \forall k \geq 1,
\]

where \( K \) is a positive real number such that

\[
t\Psi(\theta) + (1 - t) I \leq K I, \quad \forall t \in [0, 1], \theta \in [-\pi, \pi].
\]

Such \( K \) exists since \( \Psi \) is bounded. The rest argument is also similar. Given the joint continuity of \( \Phi_{\Lambda, t} \) in \( \Lambda \) and \( t \), one can show that the following limit holds

\[
\lim_{k \to \infty} \int G\Phi_{\Lambda_k, t_k} G^* = \int \lim_{k \to \infty} G\Phi_{\Lambda_k, t_k} G^* = \int G\Phi_{\Lambda, t} G^*.
\]

This proves joint continuity of \( H \) in \( \Lambda \) and \( t \).

Once we have joint continuity, the properness is not difficult to prove. In fact, let \( K \subset \text{Range}_* \Phi \) be a compact subset, and we next show that the set

\[
H^{-1}(K) := \{(\Lambda, t) \in \mathcal{L}_+^{\mathbb{R}} \times [0, 1] : H(\Lambda, t) \in K\}
\]

is compact. The argument is essentially the same as the proof of Theorem 11.4.1 of [1]. Since our setting is finite-dimensional, a set being compact is equivalent to being closed.
and bounded. If $H^{-1}(K)$ is unbounded, one can then find a sequence $\{(\Lambda_k, t_k)\} \subset H^{-1}(K)$ such that $\|\Lambda_k, t_k\| \to \infty$ as $k \to \infty$, which necessarily implies $\|\Lambda_k\| \to \infty$. However, in this case $H(\Lambda_k, t_k)$ would tend to be singular, which contradicts the premise of $K$ being compact. This proves the boundedness.

To prove the closedness, if a sequence $\{(\Lambda_k, t_k)\}$ in $H^{-1}(K)$ converges to $(\Lambda, t)$, then $\Lambda$ cannot be on the boundary of $L^+$, otherwise $\|H(\Lambda_k, t_k)\| \to \infty$, which again contradicts the compactness of $K$. To see the latter fact, notice that

$$H(\Lambda_k, t_k) = \int G\Phi_{\Lambda_k, t}G^*$$

$$= \int GW_{\Lambda_k}^{-1}[t\Psi + (1 - t)I]W_{\Lambda_k}^{-*}G^*$$

$$\geq K_{\min} \int G(G^*\Lambda_kG)^{-1}G^*,$$

where $K_{\min} := \min_{t, \theta} \lambda_{\min}(t\Psi(e^{i\theta}) + (1 - t)I) > 0$ since $\Psi$ is coercive. Now if $\{\Lambda_k\}$ approaches $\partial L^+$, then $G^*(e^{i\theta}\Lambda_kG(e^{i\theta}))$ tends to be singular for some $\theta$. Since $G$ has rank $m$ on $\mathbb{T}$, this in turn implies that $\|H(\Lambda_k, t_k)\| \to \infty$ as $k \to \infty$. Therefore, by the joint continuity of $H$, $(\Lambda, t) \in H^{-1}(K)$. This concludes the proof of properness.

Theorem 1. The map $\omega$ is surjective.

Proof. Given the second listed property of the degree, the claim follows directly if we can show that

$$\text{deg}(\omega) \neq 0.$$  

We notice first that $\omega$ is proper by Theorem 11.4.1 from [1], and thus the degree is well defined. By Lemma 2 and the homotopy invariance of the degree,

$$\text{deg}(\omega) = \text{deg}(\tilde{\omega}).$$

As a consequence of Sard–Brown theorem [36, p. 63], the codomain $\text{Range}_\Gamma \Gamma$ must contain a regular value of $\tilde{\omega}$ since it has positive $\text{Range}_\Gamma \Gamma$-Lebesgue measure. By Lemma 1 the $C^1$ degree of $\omega$ at a regular value is well-defined. Meanwhile, from Theorem 11.4.2 of [1], we know that $\tilde{\omega}$ is bijective. Therefore, we must have

$$\text{deg}(\tilde{\omega}) \neq 0,$$

and this concludes the proof.

C. The special case of covariance extension

Given $\Lambda \in L^+$ and $G(z)$ in [5], $G^*\Lambda G$ is now a matrix Laurent polynomial that takes positive definite values on the unit circle. Let us take

$$Q(z) := \sum_{k=-p}^{p} Q_k z^k \equiv G^*\Lambda G, \quad Q_{-k} = Q_k^* \in \mathbb{C}^{m \times m}.$$  

Then according e.g. to [41], $Q(z)$ admits a spectral factorization

$$Q(z) = D^*(z)D(z),$$

where $D(z) = \sum_{k=0}^{p} D_k z^{-k}$ is a $m \times m$ matrix polynomial (with negative powers) and the scalar polynomial $\det D(z)$ has all its roots strictly inside the unit circle. We shall call such $D(z)$ Schur. Therefore, the outer spectral factor in (11) is just

$$W_\Lambda(z) \equiv D(z).$$  

We have the following corollary of Theorem 1.

Corollary 1. Given a finite $m \times m$ matrix covariance sequence $\Sigma_0, \Sigma_1, \ldots, \Sigma_p$, for any $\Psi \in \mathbb{C}^p$, there exists a Schur polynomial $D(z)$ of degree $p$ such that the spectral density

$$\Phi := D^{-1}\Psi D^{-*}$$

satisfies the moment equations (8). The polynomial $D(z)$ is a right Schur spectral factor of $G^*\Lambda G$ for some $\Lambda \in L^+$.  

In particular, when taking $\Psi(z) = N(z)N^*(z)$ with $N(z) = \sum_{k=0}^{p} N_k z^{-k}$, $N_k \in \mathbb{C}^{m \times m}$, which is the spectral density of a moving-average process, the spectral density $\Phi$ in (20) would correspond to an $m$-dimensional vector ARMA process

$$\sum_{k=0}^{p} D_k y(t-k) = \sum_{k=0}^{p} N_k w(t-k), \quad t \in \mathbb{Z},$$

and we recover one of the main results of [5] Section V under a more general setting.

IV. Concluding remarks

We have shown that the multivariate spectral estimation problem posed in [1] admits a solution under an arbitrary prior matrix density. An immediate question is uniqueness of the solution and, more strongly, well-posedness of the problem. Following previous work on rational covariance extension [7], we intend to pursue the uniqueness problem in the frame of the global inverse function theorem now attributed to Hadamard.

The next theorem appears in [42]; see also [43, p. 127].

Theorem 2 (Hadamard). Let $M_1$ and $M_2$ be connected, oriented, boundary-less $n$-dimensional manifolds of class $C^1$, and suppose that $M_2$ is simply connected. Then a $C^1$ map $f : M_1 \to M_2$ is a diffeomorphism if and only if $f$ is proper and the Jacobian determinant of $f$ never vanishes.

The next proposition is simple.

Proposition 1. The set $\text{Range}_\Gamma \Gamma$ is simply connected.

Proof. By definition [43, p. 127], we need to show that: whenever $f : [0, 1] \to \text{Range}_\Gamma \Gamma$ is a closed curve, i.e., $f$ is continuous with $f(0) = f(1) = \Sigma$, there exists a continuous function $F : [0, 1] \times [0, 1] \to \text{Range}_\Gamma \Gamma$ such that

(i) $F(t, 0) = f(t)$, for all $t \in [0, 1]$,

(ii) $F(0, u) = F(1, u) = \Sigma$, for all $u \in [0, 1]$, and

(iii) $F(t, 1) = \Sigma$, for all $t \in [0, 1]$.

One can easily verify that $F(t, u) := (1 - u)f(t) + u\Sigma$ is the desired function. 

1Moreover, one can make such spectral factor unique if the constant matrix coefficient $D_0$ is required to be lower triangular with real and positive diagonal elements.
Given Theorem 2 and the above proposition, the open question becomes: Is our map $\omega$ continuously differentiable? If so, how to compute its Jacobian?

Another research direction concerns the computation of a solution to the problem. To accomplish this task, in [11] the following matricial fixed-point iteration was introduced

$$
\Lambda_{k+1} = \int \Lambda_k^{1/2} G(W^{-1}_k \Psi W_k^{-1}) G^* \Lambda_k^{1/2}, \tag{28}
$$

where the initialization is set to $\Lambda_0 = \frac{1}{n} I$. Iteration (28) can be seen as a multivariate generalization of the scalar algorithm proposed in [15] for the Kullback–Leibler estimation of spectral densities. The latter algorithm has proved to be extremely efficient and numerically robust, and its convergence properties have been thoroughly investigated in [16]–[18]. The extension of these convergence results to the multivariate case will be another subject of future investigation.

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