Symmetry operators of the asymmetric two-photon quantum Rabi model

You-Fei Xie\textsuperscript{1,*} and Qing-Hu Chen\textsuperscript{1,2,†}

\textsuperscript{1} Zhejiang Province Key Laboratory of Quantum Technology and Device, School of Physics, Zhejiang University, Hangzhou 310027, China
\textsuperscript{2} Collaborative Innovation Center of Advanced Microstructures, Nanjing University, Nanjing 210093, China

E-mail: xieyoufei@zju.edu.cn and qhchen@zju.edu.cn

Abstract. The true level crossings in a subspace of the asymmetric two-photon quantum Rabi model (tpQRM) have been observed when the bias parameter of qubit is an even multiple of the renormalized cavity frequency. Generally, such level crossings imply the existence of the hidden symmetry because the bias term breaks the obvious symmetry exactly. In this work, we propose a Bogoliubov operator approach (BOA) for the asymmetric tpQRM to derive the symmetry operators associated with the hidden symmetry hierarchically. The explicit symmetry operators consisting of Lie algebra at low biases can be easily obtained in our general scheme. We believe the present approach can be extended for other asymmetric Rabi models to find the relevant hidden symmetry.

Keywords: asymmetric two-photon quantum Rabi model, hidden symmetry, Bogoliubov operator approach, symmetry operators
1. Introduction

The quantum Rabi model (QRM) [1], which describes the simplest interaction between a two-level system (qubit) and a single-mode cavity is a paradigmatic model in the quantum optics [2]. In the past decades, fascinating and inspiring research has proved on the QRM and other related generalized models, which has been highlighted in [3–6]. Besides, the exact solution of the QRM, which was firstly found by Braak in the Bargmann space representation [7] and later by Chen et al. [8] in a more familiar Hilbert space using Bogoliubov operator approach (BOA), further promotes the developments in the models. On the other hand, the QRM is ubiquitous in the modern advanced solid devices, such as circuit quantum electrodynamics (QED) system [9–11], trapped ions [12], and quantum dots [13], which can be described in the framework of an artificial qubit and a resonator coupling system [14,17].

In contrast to the cavity QED systems, the static bias of the qubit plays an essential role in the modern solid devices, resulting in the so-called asymmetric quantum Rabi model (AQRM). In the superconducting qubit-oscillator system [14,15], the bias could be manipulated by the external magnetic flux and the persistent current in the qubit loop. Recently, the AQRM has attracted a lot of attentions, since the level crossings in the spectrum have been surprisingly observed in this model [15,21] for certain bias values. In general, the additional bias term breaks the well known $\mathbb{Z}_2$-symmetry associated with the level crossings, the AQRM should not possess any obvious symmetry. Nevertheless, it exhibits the phenomenon of energy level crossings, i.e. double degeneracy [22–24], certainly due to the hidden symmetry beyond any known symmetry. To explore the hidden symmetry, a numerical study [25] was proposed that the symmetry operator commuting with the Hamiltonian must depend on the different system parameters unlike the symmetric case. It was quickly verified analytically in the literature [26–28] that the symmetry operators can be constructed one by one within Fock space. Most recently, the present authors also reproduced the symmetry operators for arbitrary integer biases within BOA in a more simple and general scheme [29]. In addition, the symmetry operators of other related asymmetric one-photon models have been investigated in the references [30,31].

Besides the proposals and experimental realizations of the various QRM models, the two-photon quantum Rabi model (tpQRM) also has been realized in superconducting circuits [32,33], or simulated in trapped ions [34,35] and cold atoms [36] to explore new quantum effects. In particular, the asymmetric tpQRM could be realized by the addition of an external bias current [33,37], namely

$$H_{tp} = \frac{\Delta}{2} \sigma_z + \frac{\epsilon}{2} \sigma_x + \omega a^\dagger a + g \left[ (a^\dagger)^2 + a^2 \right] \sigma_x,$$

(1)

where the first two terms fully describe a qubit with the energy splitting $\Delta$ and the static bias $\epsilon$, $\sigma_{x,z}$ are the Pauli matrices, $a^\dagger$ and $a$ are the creation and annihilation operators with the cavity frequency $\omega$, and $g$ is the qubit-cavity coupling strength.

For the symmetric case ($\epsilon = 0$), the exact solution of the tpQRM was first discovered...
Symmetry operators of the asymmetric two-photon quantum Rabi model

by Chen et al. [8] using BOA in 2012. Very interestingly, Braak reproduced this solution in the Bargmann space [35] after 10 years, and pointed out that only the G-function proposed in [8] exhibits an explicitly known pole structure which dictates the approach to the collapse point [39]. The novel behavior of the spectral collapse and dynamics for the tpQRM have attracted increasing attentions [34, 40–45]. It has been investigated that the tpQRM has $\mathbb{Z}_4$-symmetry generated by the parity $\hat{P} = \exp(i\pi a^\dagger a/2)\sigma_z$. The eigenvalues $\pm 1, \pm i$ of $\hat{P}$ are always independent of the system parameters, and thus the whole Hilbert space $\mathcal{H}$ separates into four invariant subspaces. The parity $\hat{P}^2 = \exp(i\pi a^\dagger a)$ associated with $\mathbb{Z}_2$-symmetry acts only in the bosonic part of $\mathcal{H}$ with eigenvalues $\pm 1$ corresponding to the even ($\mathcal{H}_{q=1/4}$) and odd ($\mathcal{H}_{q=3/4}$) bosonic subspaces [38, 39]. In the same $q$-subspace, there are degenerate level crossings called Juddian solutions in the spectrum, which is related to the symmetry group $\mathbb{Z}_2 \simeq \mathbb{Z}_4/\mathbb{Z}_2$ corresponding to the well known parity of the one-photon QRM [7].

While for the asymmetric case ($\epsilon \neq 0$), the presence of the bias term breaks the level crossings in the same $q$-subspace. As a result, the $\mathbb{Z}_2$-symmetry is broken in the subspace. Recently, the level crossings in the same $q$-subspace of the asymmetric tpQRM have been also uncovered by the present authors [46] if the bias value is an even multiple of the renormalized cavity frequency, i.e. $\epsilon/(2\beta) = N$, $N$ is an integer. To discuss the hidden symmetry responsible for this double degeneracy, we will follow the same BOA scheme in [29].

In this paper, we derive the symmetry operators of the asymmetric tpQRM with a BOA scheme. The paper is structured as follows: In section 2, we apply the BOA and $su(1,1)$ Lie algebra for the asymmetric tpQRM to derive the constraint relations for the symmetry operators. In section 3, the solutions of the symmetry operators for the asymmetric tpQRM at low biases are calculated specifically. Based on the symmetry operators, the characteristics of level crossings are discussed. A brief summary is given in the last section.

2. General scheme within BOA

To facilitate the BOA scheme, we rewrite the Hamiltonian after a unitary transformation $\exp(\frac{\pi}{4}\sigma_y)$ for the asymmetric tpQRM (1) in the matrix form (unit is taken of $\omega = 1$)

$$H_0 = \begin{pmatrix} a^\dagger a + g \left[ (a^\dagger)^2 + a^2 \right] + \frac{\epsilon}{2} & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & a^\dagger a - g \left[ (a^\dagger)^2 + a^2 \right] - \frac{\epsilon}{2} \end{pmatrix}. \quad (2)$$

The level crossings in $\mathcal{H}_{1/4}(\mathcal{H}_{3/4})$ of the asymmetric tpQRM has been demonstrated in [16] if the bias parameter of qubit is an even multiple of the renormalized cavity frequency, i.e. $\epsilon/(2\beta) = N$, $N$ is an integer. To discuss the hidden symmetry responsible for this double degeneracy, we will follow the same BOA scheme in [29].

To generate a simple quadratic form of the diagonal Hamiltonian matrix elements,
we first perform the following Bogoliubov transformation on the bosonic operators,

\[ a_+ = ua + va^\dagger, \quad a_+^\dagger = ua^\dagger + va, \]  

with

\[ u = \sqrt{\frac{1 + \beta}{2\beta}}, \quad v = \sqrt{\frac{1 - \beta}{2\beta}}, \]

and \( \beta = \sqrt{1 - 4g^2} \) whereby \( [a_+, a_+^\dagger] = 1 \) is still met. Then the upper diagonal element becomes

\[ H_{11} = a^\dagger a + g \left[ (a^\dagger)^2 + a^2 \right] + \frac{\epsilon}{2} = \beta a_+^\dagger a_+ - \frac{1 - \beta}{2} + \frac{\epsilon}{2}. \]

Analogously, another Bogoliubov operators are introduced as

\[ a_- = ua - va^\dagger, \quad a_-^\dagger = ua^\dagger - va, \]  

leading to

\[ H_{22} = \beta a_-^\dagger a_- - \frac{1 - \beta}{2} - \frac{\epsilon}{2}. \]

For convenience, we employ the operators in a representation of the non-compact \( su(1,1) \) Lie algebra,

\[ K_0^{a_i} = \frac{1}{2} \left( a_i^\dagger a_i + \frac{1}{2} \right), \quad K_+^{a_i} = \frac{1}{2} \left( a_i^\dagger \right)^2, \quad K_-^{a_i} = \frac{1}{2} a_i^2. \]

The \( su(1,1) \) generators obey spin-like commutation relations

\[ [K_0^{a_i}, K_+^{a_i}] = \pm K_+^{a_i}, \quad [K_-^{a_i}, K_+^{a_i}] = 2K_0^{a_i}, \]

where \( a_i = a, a_+ \) and \( a_- \), respectively.

In terms of the Lie algebra, the Hamiltonian can be expressed as

\[ H = \begin{pmatrix} 2\beta K_0^{a_+} - \frac{1}{2} + \frac{\epsilon}{2} & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & 2\beta K_0^{a_-} - \frac{1}{2} - \frac{\epsilon}{2} \end{pmatrix}. \]

Given the diagonalized Hamiltonian, we move to the symmetry operator \( J \) associated with the hidden symmetry. It should satisfy the commutation condition \([J, H] = 0\), as same as the one-photon AQRM in \[26,29\]. For the asymmetric tpQRM, we define \( J \) as

\[ J = e^{i\pi a_+ a_+^\dagger} Q. \]

Note that the form of \( J \) is not necessarily unique, here we only show one concise form for it. By the relations \( e^{i\pi a_+ a_+^\dagger} a = -ia e^{i\pi a_+ a_+^\dagger} \), \( e^{i\pi a_+ a_+^\dagger} a^\dagger = ia^\dagger e^{i\pi a_+ a_+^\dagger} \), we have

\[ QH = \tilde{H} Q, \]

where the Hamiltonian \( \tilde{H} \) reads

\[ \tilde{H} = \begin{pmatrix} 2\beta K_0^{a_+} - \frac{1}{2} + \frac{\epsilon}{2} & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & 2\beta K_0^{a_-} - \frac{1}{2} - \frac{\epsilon}{2} \end{pmatrix}. \]
Symmetry operators of the asymmetric two-photon quantum Rabi model

Similar to the one-photon AQRM case, the matrix $Q$ is described as

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$ 

Using the relation (8), the four elements in the matrix $Q$ result in the four equations below,

$$AK_0^{a+} - K_0^{a-} A + \frac{\Delta}{4\beta} (C - B) = 0, \quad (9)$$

$$K_0^{a+} D - DK_0^{a-} + \frac{\Delta}{4\beta} (C - B) = 0, \quad (10)$$

$$[B, K_0^{a-}] - \frac{\epsilon}{2\beta} B + \frac{\Delta}{4\beta} (D - A) = 0, \quad (11)$$

$$[K_0^{a+}, C] - \frac{\epsilon}{2\beta} C + \frac{\Delta}{4\beta} (D - A) = 0. \quad (12)$$

We will search for all the operators $A, B, C, D$ which satisfy the above equalities. The last two equations can be further reduced to

$$[B, K_0^{a-}] - \frac{\epsilon}{2\beta} B = [K_0^{a+}, C] - \frac{\epsilon}{2\beta} C. \quad (13)$$

Note the Lie algebra has the general commutation relations,

$$\left( (K_0^{a-})^N, K_0^{a-} \right) = N (K_0^{a-})^N, \quad \left[ K_0^{a+}, (K_0^{a+})^N \right] = N (K_0^{a+})^N, \quad (14)$$

where $N = \epsilon/(2\beta)$ is an integer for the level crossing case.

Comparing above equations (14) with (13), we speculate the four elements in the matrix $Q$ are

$$M = \sum_{n,m=0}^{n+m\leq 2N} M_{n,m} (K_0^{a+})^n (K_0^{a-})^m, \quad M = A, B, C, D, \quad (15)$$

where the coefficient $M_{n,m}$ depends on the parameters $\beta, g, \Delta, \epsilon$. Here the operator basis $(K_0^{a+})^n (K_0^{a-})^m$ is equivalent to $2^{-n-m} (a_+^\dagger)^n (a_-)^m$, while the commutation relation $[a_-, a_+^\dagger] = 1/\beta$. To reduce the power series of $M$, we choose the Lie algebra products of $K_0^{a+}$ and $K_0^{a-}$ instead of the normal Bogoliubov operators $a_+^\dagger$ and $a_-$ to expand the four elements. It would be also very cumbersome if we expand the four elements in terms of operators $a, a_\downarrow$ in the original Fock space. The index $n + m$ (the corresponding degree is $(n + m)/2$) in the general expression (15) is restricted to $2N$. The highest index more than $2N$ is also allowed, but the operator $M$ will be indeed complex and lengthy with additional polynomial terms of high degree. We deduce that $n + m = 2N$ is the minimum among the highest indices for the operator $M$ at present, as long as it satisfy the constraint conditions (9) (12). It should be noted that $n + m$ must be an even number due to the remaining $\mathbb{Z}_2$-symmetry, which means the operator $M$ must be located in the $q$-subspace. Based on (15), comparing (13) with (14), to eliminate the highest degree $N$-terms, the highest order in the operators $B$ and $C$ are given by

$$B = (K_0^{a-})^N + ..., \quad C = (K_0^{a+})^N + .... \quad (16)$$
Let us now consider the symmetry operator $J$ is self-adjoint, leading to
\[ e^{i\pi a\dagger a/2} Q e^{-i\pi a\dagger a/2} = Q^\dagger, \]
and thus the coefficients have
\[ A_{n,m} = (-1)^{\frac{n+m}{2}} A_{m,n}, \quad B_{n,m} = (-1)^{\frac{n+m}{2}} C_{m,n}, \quad D_{n,m} = (-1)^{\frac{n+m}{2}} D_{m,n}. \] (17)

Inserting (15) to (9–12), using (17) and comparing terms in $(K_+^a)^{\frac{n}{2}} (K_-^a)^{\frac{m}{2}}$ respectively, one can obtain the following recurrence equations for the coefficients,
\[ A_{n-2,m} + A_{n,m-2} + \frac{m+1}{\beta} A_{n-1,m+1} + \frac{m-n}{4g} A_{n,m} + \frac{n+1}{\beta} A_{n+1,m-1} \]
\[ + \frac{(m+1)(m+2)}{4\beta^2} A_{n,m+2} + \frac{(n+1)(n+2)}{4\beta^2} A_{n+2,m} = \frac{\Delta}{8g\beta} \left( B_{n,m} - (-1)^{\frac{n+m}{2}} B_{m,n} \right), \] (18)
\[ D_{n-2,m} + D_{n,m-2} + \frac{n-m}{4g} D_{n,m} = \frac{\Delta}{8g\beta} \left( B_{n,m} - (-1)^{\frac{n+m}{2}} B_{m,n} \right), \] (19)
\[ \left( \frac{m-n}{2} - N \right) B_{n,m} + \frac{2g}{\beta} (n+1) B_{n+1,m-1} + \frac{g}{2\beta^2} (n+1) (n+2) B_{n+2,m} \]
\[ = \frac{\Delta}{4\beta} (A_{n,m} - D_{n,m}), \] (20)
\[ \left( \frac{n-m}{2} - N \right) C_{n,m} - \frac{2g}{\beta} (m+1) C_{n-1,m+1} - \frac{g}{2\beta^2} (m+1) (m+2) C_{n,m+2} \]
\[ = \frac{\Delta}{4\beta} (A_{n,m} - D_{n,m}). \] (21)
The four constraint relations are the crucial ones to determine the symmetry operator $J$. In order to solve the four elements in (15), we will discuss the coefficient $M_{n,m}$ in detail one by one. For convenience, we focus on the derivation of the coefficients $A_{n,m}$, $B_{n,m}$ and $D_{n,m}$. The coefficient $C_{n,m}$ can be obtained from (17) straightforwardly.

3. Symmetry operators

We derive the symmetry operators of the asymmetric tpQRM within BOA scheme rigorously in this section. Recall the symmetry operator $J_N = e^{i\pi a\dagger a/2} Q_N$ for $N = \epsilon/(2\beta)$. Using the constraint relations (18–21) of the coefficients in the previous section, we first demonstrate the symmetry operators for $N = 0, 1, 2, 3$ in detail. In particular, considering (18) and (19) for $n + m = 2N + 2$, the two recurrence relations are reduced to
\[ A_{n-2,m} + A_{n,m-2} + \frac{m+1}{\beta} A_{n-1,m+1} = 0, \quad D_{n-2,m} + D_{n,m-2} = 0. \]

Given the initial values $A_{0,2N} = D_{0,2N} = 0$, increasing $n$ one by one until $2N + 2$, we can obtain
\[ A_{n,2N-n} = D_{n,2N-n} = 0. \] (22)
The symmetry operator for $N = 0$. In this case, we can immediately have $B = C = 1$ from (16), then (22) gives $A = D = 0$ easily. Interestingly, the solution is given by

$$J_0 = e^{i\frac{\pi a^+ a}{2}} \sigma_x,$$

which is just the parity operator in the symmetric tpQRM [39]. In the following, we discuss the symmetry operators in the asymmetric cases.

The symmetry operator for $N = 1$. According to (16), we infer that

$$B = K_a^-, \quad C = -K^+_a,$$

where the coefficient $B_{0,2} = 1$. Then we move to the operators $A$ and $D$. Equation (22) has given $A_{0,2} = A_{2,0} = 0$ and $D_{0,2} = D_{2,0} = 0$. Inserting $B_{0,2}$ to the relations (18) and (19) for $n = 0, m = 2$ respectively, one can get

$$A_{0,0} = \frac{\Delta}{8g\beta}, \quad D_{0,0} = \frac{\Delta}{8g\beta}.$$  

Therefore, the solution is given by

$$J_1 = e^{i\frac{\pi a^+ a}{2}} \begin{pmatrix} \frac{\Delta}{8g\beta} & K_a^- \\ -K_a^+ & \frac{\Delta}{8g\beta} \end{pmatrix}.$$  

One can see the symmetry operator still have a simple expression for $N = 1$, but for $N = 2$ and $3$, the expressions would be slightly more complicated with the increasing degree.

The symmetry operator for $N = 2$. In this case, we assume

$$B = (K_a^-)^2 + B_{0,0},$$

where $B_{0,4} = 1$ and $B_{0,0}$ is to be determined. For simplicity, here we set the coefficients $B_{n,m}$ for $n + m = 2$ are equivalent to zeros. Similar to the above case $N = 1$, we can find $A_{n,4-n} = D_{n,4-n} = 0$ by (22). Then for $n = 0, m = 4$, the relations (19) and (17) gives

$$D_{0,2} = -D_{2,0} = \frac{\Delta}{8g\beta},$$

for $n = 2, m = 0$, the relation (19) yields $D_{0,0} + \frac{\Delta}{2g} D_{2,0} = 0$, where

$$D_{0,0} = \frac{\Delta}{16g^2\beta}.$$  

On the basis of the above process to derive the operator $D$, we can get the operator $A$ similarly and results in $A_{2,0} = -A_{0,2} = -\frac{\Delta}{8g\beta}, A_{0,0} = -\frac{\Delta}{16g^2\beta}$. Now there only remains $B_{0,0}$ to solve, inserting the values of $A_{0,0}$ and $D_{0,0}$ to the relation (20) for $n = m = 0$ results in

$$B_{0,0} = \frac{\Delta^2}{64g^2\beta^2}.$$
Using the equation (17), the operator $C$ can be immediately arrived. Summarily, the symmetry operator for $N = 2$ is given by

$$J_2 = e^{\frac{\Delta a\dagger a}{2}}\left(\frac{\Delta}{8g\beta}(-K^a_+ + K^a_-) - \frac{\Delta^2}{16g^2\beta^2} (K^a_+)^2 + \frac{\Delta^2}{64g^2\beta^2} (-K^a_+ + K^a_-) + \frac{\Delta}{16g^2\beta}\right).$$  \hspace{1cm} (27)

One can find the symmetry operator still keeps a compact way.

The symmetry operator for $N = 3$. In this case, we can preliminarily set

$$B = (K^a_-)^3 + B_{2,0}K^a_+ + B_{0,2}K^a_- + B_{0,0}.$$ \hspace{1cm} (28)

Inserting $n = 0$ one by one in the condition of $n + m = 6$, by the equations (22) and (18) we see

$$A_{0,4} = A_{4,0} = -A_{2,2} = \frac{\Delta}{8g\beta}.$$ By the relation (19), we also have

$$D_{0,4} = D_{4,0} = -D_{2,2} = \frac{\Delta}{8g\beta}.$$ For $n + m = 4$, the relation (18) is reduced to

$$A_{n-2,m} + A_{n,m-2} + \frac{m + 1}{\beta}A_{n-1,m+1} + \frac{m - n}{4g}A_{n,m} + \frac{n + 1}{\beta}A_{n+1,m-1} = 0.$$ Starting from $m = 0$, we have $A_{2,0} = \frac{\Delta}{8g^2\beta}$, then increasing $m$ one by one, the coefficients are

$$A_{1,1} = -\frac{\Delta}{4g^2\beta^2}, \quad A_{0,2} = -\frac{\Delta}{8g^2\beta}.$$ To consider the operator $D$, for $n + m = 4$, the relation (19) becomes

$$D_{n-2,m} + D_{n,m-2} + \frac{n - m}{4g}D_{n,m} = 0.$$ When $n = 0, 2$ respectively, the above relation arrives

$$D_{0,2} = -D_{2,0} = \frac{\Delta}{8g^2\beta}.$$ For $n + m = 2$, if $n = 0, 2$, using the known coefficients, the relation (20) gives

$$B_{0,2} = \frac{\Delta^2}{32g^2\beta^2}, \quad B_{2,0} = -\frac{\Delta^2}{64g^2\beta^2}.$$ Then we refer to the relations (18) and (19), for $n = 2, m = 0$, inserting the obtained coefficients above results in

$$A_{0,0} = \frac{\Delta}{16g^3\beta} + \frac{\Delta^3}{512g^3\beta^3}, \quad D_{0,0} = \frac{\Delta}{16g^3\beta} + \frac{\Delta^3}{512g^3\beta^3}.$$
Therefore, the symmetry operator is shown as

\[ J_3 = e^{\frac{i\pi a_+ a_-}{2} \left( A_3 B_3 C_3 D_3 \right)} \]

with

\[ A_3 = \frac{\Delta}{8g\beta} \left[ (K_{+}^{a+})^2 - K_{+}^{a+} K_{-}^{a} + (K_{-}^{a-})^2 \right] + \frac{\Delta}{8g^2\beta} (K_{+}^{a+} - K_{-}^{a-}) \]

\[ - \frac{\Delta}{4g\beta} \left( K_{+}^{a+} \right)^2 \left( K_{-}^{a-} \right)^2 + \frac{\Delta^3}{16g^3\beta^3} - \frac{\Delta}{16g\beta^3}, \]

\[ B_3 = (K_{-}^{a-})^3 - \frac{\Delta^2}{32g^2\beta^2} \left( \frac{1}{2} K_{+}^{a+} - K_{-}^{a-} \right), \]

\[ C_3 = (-K_{+}^{a+})^3 - \frac{\Delta^2}{32g^2\beta^2} \left( K_{+}^{a+} - \frac{1}{2} K_{-}^{a-} \right), \]

\[ D_3 = \frac{\Delta}{8g\beta} \left[ (K_{+}^{a+})^2 - K_{+}^{a+} K_{-}^{a} + (K_{-}^{a-})^2 \right] - \frac{\Delta}{8g^2\beta} (K_{+}^{a+} - K_{-}^{a-}) \]

\[ + \frac{\Delta^3}{16g^3\beta^3} + \frac{\Delta^3}{512g^3\beta^3}. \]  

(29)

We have checked that all the coefficients of the symmetry operators satisfy the constraint relations (18–21), and they are the unique solutions for \( N = 0, 1, 2, 3 \) in the present scheme. In the solutions within BOA for the symmetry operator, one can see many terms of the same degree share the same coefficients, such as the dominant terms \((n + m = 2N - 2)\) in \( D \) (also in \( A \)). The terms for \( n + m = 2N - 2 \) in \( B \) (also in \( C \)) even vanish which simplify the symmetry operators effectively. If the symmetry operator within BOA is expanded in the original Fock space, for example, the term \((K_{+}^{a+})^N \sim (ua_+ + va)^{2N}\), the largest degree will rise to \( 2N \) and the power series will increase rapidly along with \( 2N \). In the present work, we focus on the low biases to demonstrate our BOA scheme. Since the constraint relations (18–21) have been given, one can continue to compute the symmetry operator for larger biases if interested, which should be feasible within our BOA scheme.

According to the definition of the symmetry operator \( J \) for the asymmetric tpQRM, we have

\[ J^2 = Q^1 Q. \]  

(30)

For \( N = 0 \), one can immediately obtain \( J_0^2 = I \) (\( I \) is the \( 2 \times 2 \) unit matrix), which results in the eigenvalues \( \pm 1 \) of \( J_0 \). It coincides with the usual definition of the parity operator for the symmetric case [39]. For the asymmetric tpQRM, we find \((J_1)^2\) is in a quadratic polynomial in \( H_1 \) following as

\[ (J_1)^2 = \frac{1}{4\beta^2} (H_1)^2 + \frac{\Delta^2}{64g^2} + \frac{g^2}{4\beta^2} I. \]  

(31)

Therefore, for the subspace, if an eigenvalue \( E \) of the Hamiltonian \( H_1 \) is given, the operator \( J_1 \) has two possible eigenvalues \( J_1(E) \), where

\[ J_1(E) = \pm \sqrt{\frac{1}{4\beta^2} E^2 + \frac{\Delta^2}{64g^2} + \frac{g^2}{4\beta^2}}. \]
Figure 1. (Color online) Energy spectrum as a function of the coupling strength $g$ for (a) $\epsilon/(2\beta) = 0$, (b) $\epsilon/(2\beta) = 0.5$, (c) $\epsilon/(2\beta) = 1$, and (d) $\epsilon/(2\beta) = 2$ in the 1/4-subspace. The values of $\Delta$ are randomly selected between 1 and 3. For clarity, the energies are rescaled with $(E + 1/2)/\beta$. The blue (red) lines correspond to the positive (negative) parity.

The subspace will be again divided into two sectors with positive and negative eigenvalues of $J_1$, which may still imply a $Z_4$-symmetry in the asymmetric tpQRM although the separation depends on the system parameters.

We also calculated $J$-square for $N = 2, 3$, resulting in

$$J_N^2 = \sum_{i=0}^{2N} y_i^{(N)}(g, \beta, \Delta) (H_N)^i,$$

(32)

where $y_i^{(N)}$ are the coefficients. Interestingly, the highest degree of the polynomial (32) is $2N$ in the asymmetry tpQRM rather than $N$ in the one-photon AQR at the same order level crossings. This is because the nonlinear two-photon interaction, similar to the asymmetric Rabi-Stark model case where the Stark term $\Delta a^\dagger a\sigma_z/2$ also induce the
2N-degree in the J-square [31], which need further research.

To define a parity operator $\Pi_N$ independent of the system parameters, we can rescale the symmetry operator, following as

$$\Pi_N = \frac{J_N}{\sqrt{\sum_{i=0}^{2N} y_i^{(N)} (H_N)^i}}.$$  

The eigenvalues of the parity $\Pi_N$ is $\pm 1$, analogous to the parity in the symmetric case, i.e. $J_0$. The energy spectra of the 1/4-subspace are illustrated numerically in figure 1 for different $\epsilon/(2\beta)$ and $\Delta$, which is similar to the figure 7 in reference [46]. In figure 1 (a),(c) and (d), for the integer $\epsilon/(2\beta) = 0, 1, 2$, the energy levels can be divided into two sectors with positive and negative parities. The positive (negative) parity is denoted by blue (red) lines, which can be achieved by solving the eigenvalues of equation (33). Obviously, the level crossings between the positive and negative parities are double degenerate which have been solved through polynomials in reference [46]. As a comparison, in figure 1 (b), where $\epsilon/(2\beta) = 0.5$ breaks the symmetry, the level crossings disappear in the spectrum and hereby the symmetry operator is absent. Note the existence of level crossings is $\Delta$ independent as long as $\Delta \neq 0$, shown in the figure 1 with random values of $\Delta$. These results indicate that only when $\epsilon/(2\beta) = N$ with integer $N$ for the asymmetric tpQRM, the level crossings can appear in the spectrum associated with the symmetry operator which can label the energy levels as positive and negative parities.

4. Conclusion

In this work, we have developed a BOA scheme to derive the symmetry operator responsible for the hidden symmetry of the asymmetric tpQRM systematically. We demonstrate explicit solutions of the symmetry operators at low biases. The solutions consist of the Lie algebra derived from the Bogoliubov operators in a compact way, and the derivation is very concise and accessible. For the asymmetric tpQRM, the Bogoliubov operator in our BOA scheme is the linear combination of the original operators in the Fock space, which means a few Bogoliubov operators could capture many more original operators. It would sharply decrease the power series of the solutions. Besides, the Lie algebra can reduce half degree of the original operators. Therefore, the BOA scheme by the combination of Bogoliubov operators and Lie algebra greatly increases the efficiency and clarity to deal with the symmetry operators than the previous approach [26][28] in the original Fock space. In addition, the square of the symmetry operator for the asymmetric tpQRM can be expanded in a polynomial of the Hamiltonian, showing twice order of that in the one-photon AQRM at the same order level crossings because of the nonlinear two-photon process. For the asymmetric tpQRM, the energy levels in the spectrum can be divided into positive and negative parities based on the corresponding symmetry operator, and the level crossings reappear, similar to the symmetric case. We believe the approach within BOA can be easily extended to the
other more complicated light-matter interaction systems to detect the hidden symmetry.

Finally, we would like to present some remarks. For the various one-photon AQRMs, because the Bogoliubov operators are actually expressed linearly in terms of the original operator, although the previous method based on the expansions of the elements in $2 \times 2$ matrix in the Fock space is rather complicated, it still works. In contrast, for the nonlinear coupling systems, such as the asymmetric two-photon QRM, it is unclear whether the nature of the hidden symmetry can be reached by the direct expansion in the Fock space. Interestingly, the BOA could be very effective both in the analytical solutions and the attempt to find the underlying symmetry operators.

Note added: The preliminary symmetry operators of the asymmetric tpQRM for low biases $\epsilon = 2\beta$ and $4\beta$ were given in our first version of this arXiv preprint [47]. Interestingly, the same idea was used to derive the lowest symmetry operator in the asymmetric two-mode quantum Rabi model most recently [48].

Acknowledgments

The authors thank Liwei Duan for helpful discussions. This work was supported by the National Science Foundation of China under Grant No. 11834005 and the National Key Research and Development Program of China under Grant No. 2017YFA0303002.

References

[1] Rabi I I 1937 Phys. Rev. 51 652
[2] Scully M O and Zubairy M S 1997 Quantum Optics (Cambridge: Cambridge University Press)
[3] Meystre P 2021 Quantum Optics (Berlin: Springer)
[4] Xie Q T, Zhong H H, Batchelor M T and Lee C H 2017 J. Phys. A: Math. Theor. 50 113001
[5] Braak D, Chen Q H, Batchelor M T and Solano E 2016 J. Phys. A: Math. Theor. 49 300301
[6] Choi M S, 2020 Adv. Quantum Technol. 3 1900140
[7] Braak D 2011 Phys. Rev. Lett. 107 100401
[8] Chen Q H, Wang C, He S, Liu T and Wang K L 2012 Phys. Rev. A 86 023822
[9] Niemczyk T et al 2010 Nat. Phys. 6 772
[10] Forn-Díaz P, García-Ripoll J J, Peropadre B, Orgiazzi J-L, Yurtalan M A, Belyansky R, Wilson C M and Lupascu A 2016 Nat. Phys. 13 39
[11] Blais A, Girvin S M and Wallraff A 2021 Rev. Mod. Phys. 93 025005
[12] Leibfried D, Blatt R, Monroe C and Wineland D 2003 Rev. Mod. Phys. 75 281
[13] Hennessy K et al. 2007 Nature 445 896
[14] Forn-Díaz P, Lisenfeld J, Marcos D et al. 2010 Phys. Rev. Lett. 105 237001
[15] Yoshihara F, Fuse T, Ashhab S, Kakuyanagi K, Saito S and Semba K, 2017 Nat. Phys. 13 44
[16] Forn-Díaz P, Lamata L, Rico E, Kono J and Solano E 2019 Rev. Mod. Phys. 91 025005
[17] Kockum A F, Miranowicz A, Liberato S D, Savasta S and Nori F 2019 Nat. Rev. Phys. 1 19
[18] Zhong H-H, Xie Q-T, Batchelor M and Lee C-H 2014 J. Phys. A: Math. Theor. 47 045301
[19] Li Z M and Batchelor M T 2015 J. Phys. A: Math. Theor. 48 454005 2016 J. Phys. A: Math. Theor. 49 369401
[20] Batchelor M T, Li Z M, Zhong H Q 2016 J. Phys. A: Math. Theor. 49 01LT01
[21] Wakayama M, 2017 J. Phys. A: Math. Theor. 50 174001 Kimoto K, Reyes-Bustos C and Wakayama M, 2020 Int. Math. Res. Not. 2021 9458-544
Symmetry operators of the asymmetric two-photon quantum Rabi model

[22] Maciejewski A J, Przybylska M and Stachowiak T 2014 Phys. Lett. A 378 3445
[23] Li Z M and Batchelor M T 2021 Phys. Rev. A 103 023719
[24] Li Z M, Ferri D, Tilbrook D and Batchelor M T 2021 J. Phys. A: Math. Theor. 54 405201
[25] Ashhab S, 2020 Phys. Rev. A 101 023808
[26] Mangazeev V, Batchelor M T and Bazhanov V V 2021 J. Phys. A: Math. Theor. 54 12LT01
[27] Reyes-Bustos C, Braak D and Wakayama M 2021 J. Phys. A: Math. Theor. 54 285202
[28] Reyes-Bustos C and Wakayama M arXiv: 2106.08916
[29] Xie Y F and Chen Q H 2022 J. Phys. A: Math. Theor. 55 225306
[30] Lu X L, Li Z M, Mangazeev V V and Batchelor M T 2021 J. Phys. A: Math. Theor. 54 325202
[31] Lu X L, Li Z M, Mangazeev V V and Batchelor M T 2022 Chin. Phys. B. 31 014210
[32] Bertet P, Chiorescu I, Burkard G, Semba K, Harmans C J P M, DiVincenzo D P and Mooij J E 2005 Phys. Rev. Lett. 95 257002
[33] Felicetti S, Rossatto D Z, Rico E, Solano E and Forn-Díaz P 2018 Phys. Rev. A 97 013851
[34] Felicetti S, Pedernales J S, Eguquisita I L, Romero G, Lamata L, Braak D and Solano E 2015 Phys. Rev. A 92 033817
[35] Puebla R, Hwang M-J, Casanova J and Plenio M B 2017 Phys. Rev. A 95 063844
[36] Schneeweiss P, Derau A and Sayrin C, 2018 Phys. Rev. A 98 021801(R)
[37] Burkard G, DiVincenzo D P, Bertet P, Chiorescu I and Mooij J E 2005 Phys. Rev. B 71 134504
[38] Braak D 2022 arXiv: 2206.02509
[39] Duan L W, Xie Y-F, Braak D and Chen Q-H 2016 J. Phys. A: Math. Theor. 49 464002
[40] Ng K M, Lo C F and Liu K L 1999 Eur. Phys. J. D 6 119-126
[41] Cong L, Sun X M, Liu M X, Ying Z J and Luo H G, 2019 Phys. Rev. A 99 013815
[42] Cui S, Cao J P, Fan H and Amico L 2017 J. Phys. A: Math. Theor. 50 204001
[43] Lii Z G et al 2017 J. Phys. A: Math. Theor. 50 074002
[44] Chan C K 2020 J. Phys. A: Math. Theor. 53 385303
[45] Lo C F 2020 Scientific Reports 10 14792
[46] Xie Y-F and Chen Q-H 2021 Phys. Rev. Res. 3 033057
[47] Xie Y-F and Chen Q-H 2021 unpublished (arXiv: 2106.05817v1)
[48] Yan Z Y et al 2022 J. Phys. A: Math. Theor. 55 155303