Research Article

On a Diophantine equation related to the difference of two Pell numbers

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Abstract

In this paper, the Diophantine equation $P_n - P_m = 3^a$ is considered and all solutions for this equation are obtained. In the proof of the main theorem, lower bounds for the absolute value of linear combinations of logarithms and a version of the Baker-Davenport reduction method are used.

Keywords: Pell numbers; Diophantine equation; Baker’s theory.

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1. Introduction

In recent years, many researchers investigated the solutions of Diophantine equations of the form

$$u_n \pm u_m = p^a$$

where $(u_n)$ is a fixed linear recurrence sequence and $p$ is a prime. For example, Bravo and Luca [3,4] solved the equation $u_n + u_m = 2^a$ for the cases when $(u_n)$ is the Fibonacci sequence and when $(u_n)$ is the Lucas sequence. Also, Bitim and Keskin [1] found all the solutions of the equation $u_n - u_m = 3^a$ for the case when $(u_n)$ is the Fibonacci sequence. Also, many other researches on this topic, such as [6], have been carried out.

In this paper, we search all the solutions of the Diophantine equation

$$P_n - P_m = 3^a$$

(1)

where $P_n$ is the Pell sequence and $n$, $m$ and $a$ are nonnegative integers such that $n \geq m$. The main argument used for the solution of such problems is Baker’s theory (lower bound for the absolute value of linear combinations of logarithms of algebraic numbers) and a version of the Baker-Davenport reduction method.

2. Preliminaries

A linear recurrence sequence of order $k$ is a sequence whose general term is $(a_n) = L(a_{n-1}, a_{n-2}, \ldots, a_{n-k})$ where $k$ is a fixed positive integer and $L$ is a linear function. A linear recurrence sequence of order 2 is known as a binary recurrence sequence. Pell sequence, one of the most familiar binary recurrence sequence, is defined by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$. Some of the terms of the Pell sequence are given by 0, 1, 2, 5, 12, 29, 70, . . . . Its characteristic polynomial is of the form $x^2 - 2x - 1 = 0$ whose roots are $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Binet’s formula enables us to rewrite the Pell sequence by using the roots $\alpha$ and $\beta$ as

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}.$$  

(2)

Also, it is known that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1}.$$  

(3)

We give the definition of the logarithmic height of an algebraic number and some of its properties.

Definition 2.1. Let $\xi$ be an algebraic number of degree $d$ with minimal polynomial

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \cdot \prod_{i=1}^{d} (x - \xi_i)$$

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where the $a_i$'s are relatively prime integers with $a_0 > 0$ and the $\xi_i$'s are conjugates of $\xi$. Then

$$h(\xi) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^{d} \log \left( \max \{ |\xi_i|, 1 \} \right) \right)$$

is called the logarithmic height of $\xi$.

**Proposition 2.1.** Let $\xi, \xi_1, \xi_2, \ldots, \xi_t$ be elements of an algebraic closure of $\mathbb{Q}$ and $m \in \mathbb{Z}$. Then

1. $h(\xi_1 \cdots \xi_t) \leq \sum_{i=1}^{t} h(\xi_i)$,
2. $h(\xi_1 + \cdots + \xi_t) \leq (t-1) \log 2 + \sum_{i=1}^{t} h(\xi_i)$,
3. $h(\xi^m) = |m| h(\xi)$.

We will use the following theorem (see [8] or Theorem 9.4 in [5]) and lemma (see [2] which is a variation of the result due to [7]) for proving our results.

**Theorem 2.1 (Matveev’s theorem).** Let $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive elements of a number field $\mathbb{L}$ of degree $D$, and $b_1, b_2, \ldots, b_t$ be rational integers. Set

$$B := \max \{ |b_1|, \ldots, |b_t| \}$$

and

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1.$$ 

If $\Lambda$ is nonzero, then

$$\log |\Lambda| > -3 \cdot 30^{t+4} \cdot (t+1)^{5.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 \log (tB)) \cdot A_1 \cdots A_t$$

where

$$A_i \geq \max \{ D \cdot h(\gamma_i), |\log \gamma_i|, 0.16 \}$$

for all $1 \leq i \leq t$. If $\mathbb{L} \subset \mathbb{R}$, then

$$\log |\Lambda| > -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log B) \cdot A_1 \cdots A_t.$$ 

**Lemma 2.1.** Let $A, B, \mu$ be some real numbers with $A > 0$ and $B > 1$, and let $\gamma$ be an irrational number and $M$ be a positive integer. Take $p/q$ as a convergent of the continued fraction of $\gamma$ such that $q > 6M$. Set $\varepsilon := \|\mu q\| - M \|\gamma q\| > 0$ where $\|\cdot\|$ denotes the distance from the nearest integer. Then there is no solution to the inequality

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers $u, v$ and $w$ with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log \frac{\mu q}{\varepsilon}}{\log B}.$$ 

3. **Main result**

**Theorem 3.1.** The only triples of nonnegative integers $n, m, a$ with $n \geq m$ satisfying the Diophantine equation (1) are the following:

$$(n, m, a) \in \{(1, 0, 0), (2, 1, 0), (3, 2, 1), (5, 2, 3)\}.$$

**Proof.** In the case that $n = m$, it is obvious that there exists no solution for the Diophantine equation (1). So we consider the case that $n > m$ in the rest of the paper.

By a simple computation, we observe that all triples $(n, m, a)$ with $0 \leq m < n \leq 200$ satisfying the equation (1) form the set $\{(1, 0, 0), (2, 1, 0), (3, 2, 1), (5, 2, 3)\}$.

Assume that $n > 200$. From (1) and (3), we get

$$3^a = P_n - P_m \leq P_n \leq \alpha^{n-1} < 3^n$$

and so $a < n$. When we replace $P_n$ in the equation (1) with its closed form, we obtain

$$\frac{\alpha^n}{2\sqrt{2}} - 3^a = \frac{\beta^n}{2\sqrt{2}} + P_m.$$
By taking the absolute value of both sides of the above relation and using the upper bound in relation (3), it is yielded that

$$\left| \frac{\alpha^n}{2\sqrt{2}} - 3^n \right| \leq \frac{|\beta^n|}{2\sqrt{2}} + P_m < \frac{1}{6} + \alpha^{m-1}. $$

When we multiply both sides of the expression above by $\frac{2\sqrt{2}}{\alpha^n}$ to apply Matveev’s result in Theorem 2.1, we have

$$\left| 1 - 3^n \cdot \alpha^{-n} \cdot 2\sqrt{2} \right| < \frac{2\sqrt{2}}{\alpha^n} \left( \frac{1}{6} + \alpha^{m-1} \right)$$

$$= 2\sqrt{2} \alpha^{-n} \left( \frac{1}{6} \alpha^{-m} + \frac{1}{\alpha} \right)$$

$$< 2\sqrt{2} \alpha^{-n} \left( \frac{1}{6} + \frac{1}{2} \right)$$

$$= \frac{4\sqrt{2}}{3} \alpha^{-n}$$

$$< \frac{2}{\alpha^{n-m}}. \quad (4)$$

Let us take $t := 3$, $(\gamma_1,\gamma_2,\gamma_3) := (3,\alpha,2\sqrt{2})$ and $(b_1,b_2,b_3) := (a,-n,1)$. We have $D := 2$ since each $\gamma_i$ belongs to $\mathbb{Q}(\sqrt{2})$. Note that $1 - 3^n \cdot \alpha^{-n} \cdot 2\sqrt{2}$ is nonzero. Indeed, if it were zero, we could get

$$3^n = \frac{\alpha^n}{2\sqrt{2}} \Rightarrow \alpha^n = 3^n \cdot 2\sqrt{2} \Rightarrow \alpha^{2n} = 8 \cdot 3^{2n},$$

and so $\alpha^{2n} \in \mathbb{Z}$, which is a contradiction.

$A_1, A_2, A_3$ and $B$ can be chosen as follows:

$$A_1 := 2.2 > 2.1972 \simeq 2 \cdot \log 3 = D \cdot h (\gamma_1),$$

$$A_2 := 0.9 > 0.8813 \simeq \log \alpha = D \cdot h (\gamma_2),$$

$$A_3 := 2.1 > 2.079 \simeq 2 \cdot \log (2\sqrt{2}) = D \cdot h (\gamma_3),$$

$$B := n.$$

From Theorem 2.1, we obtain that

$$\left| 1 - 3^n \cdot \alpha^{-n} \cdot 2\sqrt{2} \right| > \exp (-C_1 \cdot (1 + \log n) \cdot 2.2 \cdot 0.9 \cdot 2.1)$$

$$\frac{2}{\alpha^{n-m}} > \exp (-C_1 \cdot (1 + \log n) \cdot 2.2 \cdot 0.9 \cdot 2.1) \quad \text{from (4)}$$

where $C_1 = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2 \cdot (1 + \log 2)$. Proceeding to appropriate operations, we have

$$\frac{2}{\alpha^{n-m}} > \exp (-C_1 \cdot (1 + \log n) \cdot 2.2 \cdot 0.9 \cdot 2.1)$$

$$(n - m) \log \alpha - \log 2 < C_1 \cdot (1 + \log n) \cdot 2.2 \cdot 0.9 \cdot 2.1.$$

Since $C_1 < 9.7 \cdot 10^{11}$ and $1 + \log n < 2 \log n$ for $n \geq 3$, we get

$$(n - m) \log \alpha - \log 2 < 9.7 \cdot 10^{11} \cdot (1 + \log n) \cdot 2.2 \cdot 0.9 \cdot 2.1$$

$$(n - m) \log \alpha < 8.2 \cdot 10^{12} \cdot \log n \quad (5)$$

To find an upper bound on $n$, let’s rewrite the equation (1) as a second linear form in logarithms and perform some operations as follows:

$$\frac{\alpha^n}{2\sqrt{2}} - \frac{\alpha^m}{2\sqrt{2}} - 3^n = \frac{\beta^n}{2\sqrt{2}} - \frac{\beta^m}{2\sqrt{2}}$$

and taking the absolute value of both sides, we have

$$\left| \frac{\alpha^n}{2\sqrt{2}} - \frac{\alpha^m}{2\sqrt{2}} - 3^n \right| = \left| \frac{\beta^n}{2\sqrt{2}} - \frac{\beta^m}{2\sqrt{2}} \right|.$$ 

It follows from the triangle inequality that

$$\left| \frac{\alpha^n}{2\sqrt{2}} (1 - \alpha^{-m}) - 3^n \right| \leq \frac{|\beta|^n + |\beta|^m}{2\sqrt{2}}.$$
Dividing both sides by \( \frac{\alpha^n}{\sqrt{2}} (1 - \alpha^{m-n}) \), we obtain
\[
\left| 1 - 3^a \alpha^{-n} 2 \sqrt{2} (1 - \alpha^{m-n})^{-1} \right| \leq \frac{|\beta|^n + |\beta|^m}{\alpha^n (1 - \alpha^{m-n})}.
\]
It follows from the fact that \( \frac{|\beta|^n + |\beta|^m}{(1-\alpha^{m-n})} < 0.59 \) for \( n \geq 3 \) and \( m \geq 1 \), that
\[
\left| 1 - 3^a \alpha^{-n} 2 \sqrt{2} (1 - \alpha^{m-n})^{-1} \right| < 0.59 \alpha^n.
\]
(6)

Let us apply the result of Matveev once more. We take \( t := 3 \), \( (\gamma_1, \gamma_2, \gamma_3) := \left( 3, \alpha, 2 \sqrt{2} (1 - \alpha^{m-n})^{-1} \right) \) and \( (b_1, b_2, b_3) := (a, -n, 1) \). We have \( D := 2 \) since each \( \gamma_i \) belongs to \( \mathbb{Q} (\sqrt{2}) \). Note that \( 1 - 3^a \cdot \alpha^{-n} \cdot 2 \sqrt{2} \cdot (1 - \alpha^{m-n})^{-1} \) is nonzero. Indeed, if it were zero, we could get
\[
\begin{align*}
3^a \cdot 2 \sqrt{2} &= \alpha^n (1 - \alpha^{m-n}) \\
3^a \cdot 2 \sqrt{2} &= \alpha^n - \alpha^m \\
-3^a \cdot 2 \sqrt{2} &= \beta^n - \beta^m
\end{align*}
\]
and the last two equations would imply that
\[
\alpha^n < \alpha^n + \alpha^m = |\beta^n - \beta^m| \leq |\beta|^n + |\beta|^m < 1,
\]
which contradicts that \( \alpha^n > 1 \) for positive integer \( n \).

\( A_1, A_2 \) and \( B \) can be chosen as follows:
\[
\begin{align*}
A_1 &:= 2.2 > 2.1972 \simeq 2 \cdot \log 3 = D \cdot h (\gamma_1), \\
A_2 &:= 0.9 > 0.8813 \simeq \log \alpha = D \cdot h (\gamma_2), \\
B &:= n.
\end{align*}
\]

Now, let’s find an appropriate value for \( A_3 \):
\[
\begin{align*}
h (\gamma_3) &= h \left( 2 \sqrt{2} \right) \left( \frac{1 - \alpha^{m-n}}{1 - \alpha^{m-n}} \right) \\
&\leq h (2 \sqrt{2}) + h (1 - \alpha^{m-n}) \quad \text{from Proposition 2.1(1)} \\
&\leq \log (2 \sqrt{2}) + h (1) + h (\alpha^{m-n}) + \log 2 \quad \text{from Proposition 2.1(2)} \\
&= \log (4 \sqrt{2}) + |m-n| \cdot h (\alpha) \quad \text{from Proposition 2.1(3)} \\
&= \log (4 \sqrt{2}) + (n-m) \cdot \log \alpha \quad \text{from Proposition 2.1(3)}
\end{align*}
\]
and so,
\[
A_3 := 3.47 + (n-m) \cdot \log \alpha > \log 32 + (n-m) \cdot \log \alpha = \max \{ 2 h (\gamma_3), |\log \gamma_3|, 0.16 \}.
\]

Now Theorem 2.1 implies that
\[
\begin{align*}
\frac{0.59}{\alpha^n} &> \left| 1 - 3^a \alpha^{-n} 2 \sqrt{2} (1 - \alpha^{m-n})^{-1} \right| \\
&> \exp (-C_2 \cdot (1 + \log n) \cdot 2.2 \cdot 0.9 \cdot (3.47 + (n-m) \log \alpha)) \\
&= \exp (-C_2 \cdot (1 + \log n) \cdot 1.98 \cdot (3.47 + (n-m) \log \alpha))
\end{align*}
\]
where \( C_2 := 1.4 \cdot 30^6 \cdot 3^{1.5} \cdot 2^2 \cdot (1 + \log 2) < 9.7 \cdot 10^{11} \). Taking the logarithm of both sides in the last inequality, considering that \( 1 + \log n < 2 \log n \) for \( n \geq 3 \) and using the inequality (5), one can see that
\[
\begin{align*}
\text{log } 0.59 - n \log \alpha &> -C_2 \cdot (1 + \log n) \cdot 1.98 \cdot (3.47 + (n-m) \log \alpha) \\
\text{nlog } \alpha &< \text{log } 0.59 + C_2 \cdot (1 + \log n) \cdot 1.98 \cdot (3.47 + (n-m) \log \alpha) \\
\text{nlog } \alpha &< 3.85 \cdot 10^{12} \cdot \log n \cdot (3.47 + (n-m) \log \alpha) \\
\text{nlog } \alpha &< 3.85 \cdot 10^{12} \cdot \log n \cdot (3.47 + 8.2 \cdot 10^{12} \log n).
\end{align*}
\]
Thus, we obtain
\[ n < 3.59 \cdot 10^{25} \log^2 n \]
and so,
\[ n < 1.63 \cdot 10^{29}. \]  
(8)

Now let’s improve the upper bound on \( n \) a little bit more. Set
\[ z_1 := a \log 3 - n \log \alpha + \log \left(2\sqrt{2}\right). \]

The inequality (4) can be also written as
\[ |1 - e^{z_1}| < \frac{2}{\alpha^{n-m}}. \]

By using (1) and (2), we get
\[ \frac{\alpha^n}{2\sqrt{2}} = P_n + \frac{\beta^n}{2\sqrt{2}} > P_n > P_n - P_m = 3^a. \]

Therefore, we have
\[ z_1 = \log \left(\frac{3^a 2\sqrt{2}}{\alpha^n}\right) < 0. \]

It is easy to see that \( \frac{2}{\alpha^{n-m}} < 0.829 \) for all \( n-m \geq 1 \). Therefore we have \( e^{|z_1|} < 5.85. \) Then we get
\[ 0 < |z_1| < e^{|z_1|} - 1 \leq e^{|z_1|} |1 - e^{z_1}| < \frac{12}{\alpha^{n-m}} \]
and so
\[ 0 < \left| a \log 3 - n \log \alpha + \log \left(2\sqrt{2}\right) \right| < \frac{12}{\alpha^{n-m}}. \]

Thus we have
\[ 0 < \left| a \frac{\log 3}{\log \alpha} - n + \frac{\log \left(2\sqrt{2}\right)}{\log \alpha} \right| < \frac{12}{\log \alpha} \cdot \alpha^{-(n-m)} \]
(9)

by dividing both sides of the inequality above by \( \log \alpha \). From Lemma 2.1, we have the irrational number \( \gamma = \frac{\log 3}{\log \alpha} \) with
\[ \mu = \frac{\log \left(2\sqrt{2}\right)}{\log \alpha}, \quad A = \frac{12}{\log \alpha}, \quad B = \alpha, \quad w = n - m. \]

On the other hand, we recall that \( \alpha < n < 1.63 \cdot 10^{29} \). From Lemma 2.1, we can set \( M := 1.63 \cdot 10^{29} \) and if we take the denominator of the 58th convergent of \( \gamma \), then we get \( q = 15.50 \cdot 10^{29} > 6M \). By using Mathematica Script Language, we obtain \( \varepsilon = \|\mu q\| - M \|\gamma q\| = 0.184766 > 0 \).

Applying Lemma 2.1 to the above parameters, we conclude that there is no solution to the inequality (9) for the values \( n - m \) with
\[ n - m < \frac{\log \left(Aq/\varepsilon\right)}{\log B} = 81.788. \]

Therefore, for the inequality (9) to be solvable, our upper limit for \( n - m \) must be at most 81. By substituting the upper bound value for \( n - m \) in the inequality (7), we get \( n < 1.211 \cdot 10^{16} \). Let us improve this upper bound value on \( n \) a little more. Put
\[ z_2 := a \log 3 - n \log \alpha + \log \left(2\sqrt{2} \left(1 - \alpha^{m-n}\right)^{-1}\right). \]

Therefore, (6) implies that
\[ |1 - e^{z_2}| < \frac{0.59}{\alpha^n}. \]
It is easy to see that \( \frac{0.59}{\alpha^n} < \frac{1}{2} \). Suppose that \( z_2 > 0 \). Then \( 0 < z_2 < e^{z_2} - 1 < \frac{0.59}{\alpha^n} \). If \( z_2 < 0 \), then \( 1 - e^{z_2} < \frac{0.59}{\alpha^n} < \frac{1}{2} \) and we obtain \( \frac{1}{2} < e^{z_2} \) so that again \( e^{|z_2|} < 2. \) Therefore, we have
\[ 0 < |z_2| < e^{|z_2|} - 1 \leq e^{|z_2|} |1 - e^{z_2}| < 2 \cdot \frac{0.59}{\alpha^n} \]
and
\[ 0 < \left| a \frac{\log 3}{\log \alpha} - n + \frac{\log \left(2\sqrt{2} \left(1 - \alpha^{m-n}\right)^{-1}\right)}{\log \alpha} \right| < 1.18 \cdot \alpha^{-n} \]  
(10)
by dividing both sides of the inequality above by $\log \alpha$. Now to apply Lemma 2.1 again, set
\[
\gamma = \log\frac{3}{\log \alpha}, \quad \mu = \frac{\log\left(2\sqrt{2}(1 - \alpha^{m-n})^{-1}\right)}{\log \alpha}, \quad A = \frac{1.18}{\log \alpha}, \quad B = \alpha, \quad w = n.
\]
Firstly, we can choose $M = 1.211 \cdot 10^{16}$. Since $6M = 7.266 \cdot 10^{18}$, in order to apply Lemma 2.1, we must choose $q = 8.27 \cdot 10^{18}$ which is the 33rd denominator of the continued fraction of $\gamma$. Therefore, with the aid of Mathematica, we get $\varepsilon \leq 0.49473$ for $n - m \in \{1, \ldots, 81\}$. From Lemma 2.1, there is no solution to the inequality (10) for
\[
n \geq \frac{\log(Aq/\varepsilon)}{\log B} = 50.551.
\]
Thus, $n$ must be less than or equal to 50 for a solution which contradicts our assumption. This completes the proof.

4. Conclusion

We obtain all solutions of the Diophantine equation $P_n - P_m = 3^a$. Linear forms in logarithms and Baker’s theory are the main tools used in our proofs. The method used in this paper may be applied to other Diophantine equations.

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