Entanglement Wedge Cross Section from the Dual Density Matrix

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We generalize the entanglement entropy so that we can extract the entanglement wedge cross section from a given mixed state dual to the entanglement wedge. The entanglement wedge cross section has been introduced as minimal cross section of the entanglement wedge, a natural generalization of the Ryu-Takayanagi surface. By using the replica trick, we explicitly compute the generalized entropy for two-dimensional CFT with the bulk dual (AdS$_{3}$ and planner BTZ blackhole) and see agreement with the entanglement wedge cross section. We conjecture this relation will hold in the generic dimension.

I. INTRODUCTION AND SUMMARY

The entanglement entropy (EE) characterizes the quantum entanglement for a pure state. It is defined by the von Neumann entropy of a reduced density matrix $\rho_{A}$ on a subsystem $A$, $S(\rho_{A}) = -\text{Tr}_{A}\rho_{A} \log \rho_{A}$. If one considers states on the conformal field theories (CFT) with the gravity dual[1], the EE tells us the area of minimal surface anchored on the boundary of asymptotically AdS$_{2}$[2,3]. This strongly suggests that the bulk gravity is encoded into the structure of quantum entanglement in the boundary. Since the EE cannot tell us all structure of the entanglement, the minimal surface also cannot tell us the whole structure of geometry in the same manner. Therefore, finding the generalization on both sides is quite important in order to decode the profound connection between the entanglement and the geometry[4,6].

Recently, the entanglement wedge cross section, a generalization of the minimal surface, has been introduced[7,8]. It is defined by $E_{W}(\rho_{AB}) = \sigma_{min}/AG_{N}$, where $\sigma_{min}$ is the area of minimal cross section of the entanglement wedge and $G_{N}$ is the Newtonian constant. The $\rho_{AB}$ is supposed to be the state dual to the entanglement wedge[9,11]. The cross section has been conjectured to be the entanglement of purification (EoP) [12], which is a correlation measure for mixed states (for recent progress, refer to [13-18]). The EoP is also a generalization of the EE and has many nice properties consistent with the entanglement wedge cross section. However, computing the EoP in QFT is a really hard task because we need to find the minimized value from all possible purifications.

Can we then extract the entanglement wedge cross section directly from a given mixed state in QFT? In this paper, we answer yes to this question and demonstrate it explicitly; however, from another (rather “odd”) generalization of the EE.

We first summarize the main result of the present paper. Let $\rho_{AB}$ be a mixed state acting on bi-partite Hilbert space $\mathcal{H} = \mathcal{H}_{A} \otimes \mathcal{H}_{B}$. Then we define a quantity

$$S_{o}(n_{o}) \left( \rho_{AB} \right) \equiv \frac{1}{1-n_{o}} \left[ \text{Tr}_{B} \left( T_{B}^{n_{o}} \rho_{AB} T_{B}^{n_{o}} \right) - 1 \right],$$

where $T_{B}$ is the partial transposition[19] with respect to the subsystem $B$. Namely, we will consider the Tsallis entropy[20] for the partially transposed $\rho_{AB}$. We are especially interested in the limit $n_{o} \to 1$,

$$S_{o}(\rho_{AB}) = \lim_{n_{o} \to 1} S_{o}(n_{o}) \left( \rho_{AB} \right),$$

where $n_{o}$ is analytic continuation of an odd integer. Since the odd integer analytic continuation is crucial in the later discussion, we will call $S_{o}$ as “odd entanglement entropy” or OEE in short. Loosely speaking, the OEE is the von Neumann entropy with respect to $\rho_{AB}$; however, $\rho_{AB}$ potentially contains negative eigenvalues. In section III we will be more precise on that point. Moreover, we will show especially the following three statement: First, $S_{o}(\rho_{AB})$ reduces to the EE $S(\rho_{A})$ if $\rho_{AB}$ is a pure state. Second, $S_{o}(\rho_{AB})$ reduces to the von Neumann entropy $S(\rho_{AB})$ if $\rho_{AB}$ is a product state. Third, if one considers two-dimensional holographic CFT, direct calculation indeed agrees with

$$E_{W}(\rho_{AB}) = S_{o}(\rho_{AB}) - S(\rho_{AB}) = E_{W}(\rho_{AB}).$$

In particular, we consider subregion of the vacuum state (section III) and the thermal state (section IV). We conjecture this relation will hold even for the higher dimensional cases. From our viewpoint, the $E_{W}$ is similar to the coherent information[21,22], which can take even negative value. We conclude with a discussion on this point in section IV. We also leave a example of two-qubit system in appendix A.

II. DEFINITION AND PROPERTIES OF $S_{o}$ & $E_{W}$

II.1. Partial transposition and a generalized EE

We first introduce the partial transposition which is relevant to the definition of $\rho_{AB}$. Let $\rho_{AB}$ be a state acting on Hilbert space $\mathcal{H} = \mathcal{H}_{A} \otimes \mathcal{H}_{B}$ and let $|e_{i}^{(A,B)}\rangle_{S}$ ($i = 1,2,\cdots, \dim\mathcal{H}_{A,B}$) be a complete set thereof. Using these basis, we can expand a given density matrix,

$$\rho_{AB} = \sum_{ik} \sum_{j\ell} |e_{i}^{(A)}\rangle |e_{j}^{(B)}\rangle \rho_{AB} |e_{k}^{(A)}\rangle |e_{\ell}^{(B)}\rangle |e_{k}^{(A)}\rangle |e_{\ell}^{(B)}\rangle = \sum_{i} |e_{i}^{(A)}\rangle \rho_{i}^{(A)} |e_{i}^{(A)}\rangle + \sum_{j} |e_{j}^{(B)}\rangle \rho_{j}^{(B)} |e_{j}^{(B)}\rangle.$$
We define the partial transposition of the \( \rho_{AB} \) with respect to \( \mathcal{H}_{A,B} \) as

\[
\langle e_i^{(A)} e_j^{(B)} | \rho_{AB} | e_{k}^{(A)} e_{\ell}^{(B)} \rangle = \langle e_k^{(A)} e_{\ell}^{(B)} | \rho_{AB} | e_i^{(A)} e_j^{(B)} \rangle ,
\]
(5)

\[
\langle e_i^{(A)} e_j^{(B)} | \rho_{AB} | e_{k}^{(A)} e_{\ell}^{(B)} \rangle = \langle e_k^{(A)} e_{\ell}^{(B)} | \rho_{AB} | e_i^{(A)} e_j^{(B)} \rangle .
\]
(6)

Note that the partial transposition does not change its normalization \( \text{Tr}_{H} \rho_{AB}^T = \text{Tr}_{H} \rho_{AB} = \text{Tr}_{H} \rho_{AB} = 1 \), whereas it changes the eigenvalues. Since the partial transposition is not the completely positive map, the \( \rho_{AB} \) can include negative eigenvalues. This negative property is actually a sign of conformal blocks and its partial transposition in terms of the correlation functions.

The \( n \)-th power of the \( \rho_{AB}^T \) depends on the parity of \( n \):

\[
\text{Tr}_{H} \rho_{AB}^T \rho_{AB}^T \cdots \rho_{AB}^T = \left\{ \begin{array}{c}
\sum_{\lambda_{i} > 0} |\lambda_{i}|^{n} - \sum_{\lambda_{j} < 0} |\lambda_{j}|^{n} \quad (n : \text{odd}), \\
\sum_{\lambda_{i} > 0} |\lambda_{i}|^{n} + \sum_{\lambda_{j} < 0} |\lambda_{j}|^{n} \quad (n : \text{even}),
\end{array} \right.
\]

(7)

where \( \lambda_{i} \)'s are the eigenvalues of the \( \rho_{AB}^T \). This argument is completely the same as the negativity using the replica trick [23, 26]. The main difference in the present paper is that we are just choosing the odd integer. Therefore, OEE can be formally written as

\[
S_o(\rho_{AB}) = -\sum_{\lambda_{i} > 0} |\lambda_{i}| \log |\lambda_{i}| + \sum_{\lambda_{j} < 0} |\lambda_{j}| \log |\lambda_{j}| .
\]

II.2. Pure states

Let \( |\Psi_{AB}\rangle \) be a pure state in bi-partite Hilbert space \( \mathcal{H} = \mathcal{H}_{A} \otimes \mathcal{H}_{B} \). Using the Schmidt decomposition, we can write the \( |\Psi_{AB}\rangle \) as a simple form,

\[
|\Psi_{AB}\rangle = \sum_{n=1}^{N} \sqrt{p_{n}} |n_{A}\rangle |n_{B}\rangle ,
\]

(9)

where \( 0 \leq p_{n} \leq 1 \), \( \sum_{n} p_{n} = 1 \). The \( N \) can be taken as \( \min(\dim \mathcal{H}_{A} , \dim \mathcal{H}_{B} ) \). One can show that the corresponding density matrix \( \rho_{AB} = |\Psi_{AB}\rangle \langle \Psi_{AB}| \) and its partial transposition \( \rho_{AB}^{T} \) have the eigenvalues,

\[
\text{Spec}(\rho_{AB}) = \{1, 0, \cdots , 0\} ,
\]

(10)

\[
\text{Spec}(\rho_{AB}^{T}) = \{p_i , \cdots , p_{N} , +\sqrt{p_{1}p_{2} - \sqrt{p_{1}p_{2}}} \},
\]

\[
\cdots , +\sqrt{p_{N-1}p_{N} - \sqrt{p_{N-1}p_{N}}} \} .
\]

(11)

Here each \( \pm \sqrt{p_{i}p_{j}} \) \( (i \neq j) \) in (11) appears just once respectively. In particular, from the definition of the (8), these contributions completely cancel out. Thus, one can conclude that

\[
\mathcal{E}_{W}(\rho_{AB}) = S_o(\rho_{AB}) = S(\rho_{A}) \quad \text{(for pure states)},
\]

(12)

where the \( S(\rho_{A}) \) is the EE for \( |\Psi_{AB}\rangle \).

II.3. Product states

Let \( \rho_{A_1B_1} \otimes \sigma_{A_2B_2} \) be a product state with respect to the bi-partition \( \mathcal{H}_{A_1B_1} \otimes \mathcal{H}_{B_2A_2} \). Then the \( \mathcal{E}_{W} \) is additive,

\[
\mathcal{E}_{W}(\rho_{A_1B_1} \otimes \sigma_{A_2B_2}) = \mathcal{E}_{W}(\rho_{A_1B_1}) + \mathcal{E}_{W}(\sigma_{A_2B_2}) .
\]

(13)

In particular, if

\[
\tau_{AB} = \tau_{A} \otimes \tau_{B} ,
\]

(14)

we have \( \text{Tr}_{H} \tau_{AB}^n = \text{Tr}_{H} (\tau_{AB}^T)^n \). This fact immediately leads \( S_o(\tau_{AB}) = S(\tau_{AB}) \). Thus, we also obtain \( \mathcal{E}_{W}(\tau_{AB}) = 0 \).

Note that all properties discussed the above are consistent with the entanglement wedge cross section.

III. VACUUM STATE IN HOLOGRAPHIC CFT

In this section, we compute the \( S_o \) and the \( \mathcal{E}_{W} \) for mixed states in CFT on \( \mathbb{R}^2 \). We divide the total Hilbert space of CFT into \( \mathcal{H}_{A} \otimes \mathcal{H}_{A'} \), where the corresponding subregion \( A \) and its complement \( A' \) are not necessarily to be connected. Then we can prepare a mixed state \( \rho_{A_1A_2} \equiv \text{Tr}_{A'} |00\rangle \langle 00| \), where \( |0\rangle \) is the vacuum state in CFT. Here we further divided the remaining subspace \( \mathcal{H}_{A} \) into two pieces, \( \mathcal{H}_{A_1} \) and \( \mathcal{H}_{A_2} \). We will focus on the holographic CFT2.

III.1. Two disjoint intervals

First, we consider disjoint interval \( A_1 = [u_1, v_1], A_2 = [u_2, v_2] \) on a time slice \( \tau = 0 \). In order to compute the \( S_o \) and the \( \mathcal{E}_{W} \), we can apply the replica trick as usual [23, 26]. In particular, one can write \( n \)-th power of the density matrix and its partial transposition in terms of the correlation functions for a cyclic orbifold theory \( \mathcal{CFT}^n/\mathbb{Z}_n \),

\[
\text{Tr}_{H_\mathcal{A}} (\rho_{A_1A_2}^T)^n = \langle \sigma_n(u_1)\bar{\sigma}_n(v_1)\sigma_n(u_2)\bar{\sigma}_n(v_2) \rangle_{\mathcal{CFT}^n/\mathbb{Z}_n} ,
\]

(15)

\[
\text{Tr}_{H_\mathcal{A}} (\rho_{A_1A_2}^T)^n = \langle \sigma_n(u_1)\bar{\sigma}_n(v_1)\sigma_n(u_2)\bar{\sigma}_n(v_2) \rangle_{\mathcal{CFT}^n/\mathbb{Z}_n} ,
\]

(16)

where \( \sigma_n(\bar{\sigma}_n) \) is the (anti)twist operator with scaling dimension \( h_{\sigma_n} = h_{\bar{\sigma}_n} = \frac{c}{24}(n - \frac{1}{2}) \). Hereafter, we will omit the suffix of the correlation function, \( \mathcal{CFT}^n/\mathbb{Z}_n \), for brevity. Since (15) is studied in [27], we focus on the latter one. Let us expand (16) into the conformal blocks in t-channel,

\[
\langle \sigma_n(u_1)\bar{\sigma}_n(v_1)\sigma_n(u_2)\bar{\sigma}_n(v_2) \rangle / (|u_1 - v_2||v_1 - u_2|)^{-\frac{c}{24}(n - \frac{1}{2})} = \sum_{p} b_{p} F(c, \bar{h}, h, p, 1 - x) F(c, \bar{h}, h, p, 1 - \bar{x}) ,
\]

(17)

where \( F(c, h, p, x) \) and \( F(c, h, p, x) \) are the Virasoro conformal blocks and \( b_p \)'s are the OPE coefficients. We defined the cross ratio,

\[
x = \frac{(u_1 - v_1)(u_2 - v_2)}{(u_1 - u_2)(v_1 - v_2)} ,
\]

(18)
and impose $x = \bar{x}$ since we are interested in the time slice $\tau = 0$. The dominant contribution at the large-$c$ limit will come from a conformal family with the lowest scaling dimension in the channel\cite{27}. This approach should be valid only for some specific region $x_c < x < 1$. We do not specify the lower bound $x_c$, but just expect $x_c \sim 1/2$. In this channel, the dominant one is universally $\sigma_n^2$ (and $\bar{\sigma}_n^2$) due to the twist number conservation,

$$\langle \sigma_n(u_1) \bar{\sigma}_n(v_1) \bar{\sigma}_n(v_2) \rangle / (|u_1 - v_2| |u_1 - u_2|) \sim b_{\sigma_n^2} F(c, \bar{h}_{\sigma_n}, \bar{h}_{\sigma_n^2}, 1 - x).$$

Next we would like to specify the analytic form of the above conformal blocks. First, the scaling dimension of the $\sigma_n^2$ depends on the parity of $n$\cite{25, 26, 27}.

\begin{equation}
\bar{h}_{\sigma_n^2} = h_{\sigma_n^2} = \begin{cases}
\frac{c}{24} (n - 1) \quad (n: \text{odd}), \\
\frac{c}{12} n + \frac{c}{2} \quad (n: \text{even}).
\end{cases}
\end{equation}

Since we are interested in the odd integer case, this coincides with $h_{\sigma_n}$. Second, this contribution of the conformal block consists only of light operators in the heavy-light limit\cite{28}. In this case, these analytic forms are known in the literatures\cite{28, 29, 31}. In our situation, the block for $\sigma_n^2$ has a simple form,

$$\log F(c, h_{\sigma_n}, h_{\sigma_n^2}, 1 - x) = -h_{\sigma_n} \log \left[ \frac{1 + \sqrt{x}}{1 - \sqrt{x}} \right],$$

where we assumed analytic continuation of odd integer $n = n_0$, and the light limit $c \gg 1$ with fixed $h_i/c, h_\sigma/c \ll 1$. Here we took the normalization in\cite{29}. Therefore, we have obtained

$$S_o(\rho_{A_1 A_2}) = S(\rho_{A_1 A_2}) + \frac{c}{6} \log \left[ \frac{1 + \sqrt{x}}{1 - \sqrt{x}} \right] + \text{const.}, \quad (22)$$

where

$$S(\rho_{A_1 A_2}) = \frac{c}{3} \log \frac{|u_1 - v_2|}{\epsilon} + \frac{c}{3} \log \frac{|v_1 - u_2|}{\epsilon}. \quad (23)$$

Here we introduced UV cutoff $\epsilon$. The constant terms do not depend on the position. For a while, we just assume the contribution from $b_{\sigma_n^2}$ is negligible at the large-$c$ limit. This assumption will be justified when we consider the pure state limit discussed in the next subsection.

In the same way, we can compute the s-channel limit $x \to 0$. In this case, the dominant contribution will be the vacuum block as like the EE. Hence, we obtain

$$S_o(\rho_{A_1 A_2}) = \frac{c}{3} \log \frac{|u_1 - v_2|}{\epsilon} + \frac{c}{3} \log \frac{|v_1 - u_2|}{\epsilon} = S(\rho_{A_1 A_2}). \quad (24)$$

Therefore, we have confirmed

$$E_W(\rho_{A_1 A_2}) = \begin{cases}
\frac{1}{4 G_N} \log \left[ \frac{1 + \sqrt{x}}{1 - \sqrt{x}} \right] \quad (\text{t-channel, } x \sim 1), \\
0 \quad (\text{s-channel, } x \sim 0).
\end{cases} \quad (26)$$

in the two disjoint interval case. Here we used the relation between the central charge and the three-dimensional Newtonian constant $c = \frac{3}{2G_N}(\bar{2})$. The (26) precisely matches the minimal entanglement wedge cross section for AdS$_3$ (see FIG 1).

III.2. Pure state limit

Let us consider the single interval limit $u_2 \to v_1$ and $v_2 \to u_1$. This corresponds to the pure state limit for the initial mixed state. In this case, our calculation reduces to two point function of the twist operators. Hence, we can get the usual EE with single interval $A = [u_1, v_1]$ in this limit. This is generic statement for any CFT$_2$, but let us see this behavior from (22). If one takes the distance $|u_2 - v_1|$ and $|v_2 - u_1|$ to the cutoff scale $\epsilon$, the second term of r.h.s. of (22) reduces to the length of the geodesics anchored on the boundary points $u_1$ and $v_1$. Moreover, this argument guarantees the constant terms from $b_{\sigma_n^2}$ is irrelevant at the large-$c$ limit because of the position independence.

III.3. Multiple disjoint intervals

We briefly illustrate the case of more than two disjoint intervals. In general, the previous $A_1$ and $A_2$ themselves also have disconnected pieces within their subregions. To compute $\text{Tr}_{\rho_{A_1 A_2}^S}$, we assign twist operators $\sigma_n$ and $\bar{\sigma}_n$ for each boundary of subregions. This was considered in\cite{27} and reproduces the usual Ryu-Takayanagi formula. We can also compute the $\text{Tr}_{\rho_{A_1 A_2}^{T_n}}$ from the twist operators’ correlation function by just flipping the order of $\sigma_n$ and $\bar{\sigma}_n$ belonging to the $A_2$. Let us consider

$$\langle \sigma_n(z_1) \bar{\sigma}_n(z_2) \bar{\sigma}_n(z_3) \sigma_n(z_4) \sigma_n(z_5) \bar{\sigma}_n(z_6) \sigma_n(z_7) \bar{\sigma}_n(z_8) \rangle,$$

which corresponds to $\text{Tr}_{\rho_{A_1 A_2}^{T_n}}$ for left panel of FIG 2 for concreteness. If two points belonging to $A_1$ and $A_2$ respectively are sufficiently close, the OPE channel in which twist operators $\sigma_n S$ (or $\bar{\sigma}_n S$) on these points fuse into $\sigma_n^2 (\bar{\sigma}_n^2 S)$ will be favored. Here we just assume the lightest conformal
block approximation could still work. Eventually, we can pick the OPE channel in right panel of FIG. 2. The corresponding eight point conformal blocks factorizes into four point ones due to the identity exchange with the identity exchange with the identity exchange with the identity exchange with the identity exchange with the identity exchange with the identity exchange with the identity exchange.

IV. THERMAL STATE IN HOLOGRAPHIC CFT₂

In this section, we consider the thermal state in CFT₂ with the bulk dual, which is genuinely mixed state and is dual to the static (planner) BTZ blackhole. Namely, we will consider the CFT₂ on cylinder $S^1_β \times \mathbb{R}$, with single interval on the time slice, $A = [−\ell/2, \ell/2]$. The $A^c$ denotes its complement on the slice.

To compute $\text{Tr}(\rho_{AA^c})_{n_o}$ by using the replica trick, one needs to take care about the location of branch cut, which cannot be realized as the naive conformal map from the plane $z$ (previous results in section III) to the cylinder $w = σ + iτ$. The correct prescription is given by

\[
\text{Tr}(\rho_{AA^c})_{n_o} = \langle σ_{n_o} (−L/2) σ_{n_o}^2 (−\ell/2) σ_{n_o}^2 (\ell/2) σ_{n_o} (L/2) \rangle_β
\]

where we introduced a finite but large cutoff $L$ so that the conformal map can work. Thus, our “complement” $A^c$ is now $[−L/2, −\ell/2] \cup [\ell/2, L/2]$, although the true time slice is the infinite line. After taking the limit $n_o \rightarrow 1$, we let $L \rightarrow \infty$. Here the suffix of correlation function $β$ denotes the inverse temperature. Then the corresponding $\text{Tr}(\rho_{AA^c})_{n_o}$ should be

\[
\text{Tr}(\rho_{AA^c})_{n_o} = \langle σ_{n_o} (−L/2) σ_{n_o} (L/2) \rangle_β
\]

By using the conformal map $z = e^{2\pi w/β}$, one can write the above correlation function as

\[
\text{Tr}(\rho_{AA^c})_{n_o} = \left( \frac{2\pi}{β} \right)^{8h_{σ_{n_o}}} \langle σ_{n_o} (e^{−\frac{L}{2β}}) σ_{n_o}^2 (e^{−\frac{\ell}{2β}}) σ_{n_o}^2 (e^{\frac{\ell}{2β}}) σ_{n_o} (e^{\frac{L}{2β}}) \rangle
\]

(30)

\[
\text{Tr}(\rho_{AA^c})_{n_o} = \left( \frac{2\pi}{β} \right)^{4h_{σ_{n_o}}} \langle σ_{n_o} (e^{−\frac{L}{2β}}) σ_{n_o} (e^{\frac{L}{2β}}) \rangle
\]

(31)

Then one can expand the (30) by using the conformal blocks. The dominant contribution can be again approximated by the single conformal block contribution which depends on the value of the cross ratio. Here the cross ratio is $x = e^{−\frac{2π}{β} \ell}$ for sufficiently large $L$.

First, we consider the t-channel ($x \rightarrow 1$) limit, $\ell \ll β$. Then the dominant contribution from the channel is the vacuum block; hence, the (30) reduces to the product of two point functions. After simple calculation, we obtain

\[
\mathcal{E}_W = \frac{c}{3} \log \frac{β}{π\epsilon} \left(\sinh \frac{π\ell}{β}\right) + \text{const.} \quad (x \sim 1),
\]

(32)

where we introduced the UV cutoff $\epsilon$ from the dimensional analysis. The constant term comes from the normalization of two-point functions. This precisely matches the $E_W$ for the planner BTZ black hole (see FIG. 3).

Next, we consider the s-channel ($x \rightarrow 0$) limit, $\ell \gg β$. The dominant contribution in the channel is now the twist operator $σ_{n_o} (σ_{n_o})$ because of the twist number conservation. Then we have obtained

\[
\mathcal{E}_W = \frac{c}{3} \log \frac{β}{π\epsilon} + \text{const.} \quad (x \sim 0),
\]

(33)

where the constant terms come from the normalization of two-point functions and the OPE coefficients. This again agrees with the $E_W$; however, it is important to note that this result is exact at the leading order of small $x$ expansion. There is the position dependent deviation of order $O(x^1)$. 

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FIG. 2. Left: Calculation of (27) in the bulk side. Each line and region represent the same one as FIG. 1. At the boundary, we have disconnected subregion for both $A_1$ and $A_2$. In the boundary calculation, we flip the order of twist operators on $A_2$ in contrast to the entanglement entropy. Right: the OPE channel we took in the calculation of the left panel. The $1, \ldots, 8$ corresponds to operators sitting on $z_1, \ldots, z_8$. In particular, this conformal block factorizes into two pieces due to the identity exchange.

FIG. 3. Calculation of $E_W$ for the static planner BTZ black hole. The inverse temperature $β$ is determined by the radius of the horizon. If the subsystem $A$ is sufficiently small $\ell \ll β$, the $E_W$ computes the geodesics anchored on the boundary of $A$ (black curve) which agrees with the $\mathcal{E}_W$. For $\ell \gg β$, the $E_W$ does the disconnected surfaces (dotted vertical lines) which is consistent with the $\mathcal{E}_W$. 

FIG. 4. An example of computation of $\mathcal{E}_W$ for the static planner BTZ black hole. The inverse temperature $β$ is determined by the radius of the horizon. If the subsystem $A$ is sufficiently small $\ell \ll β$, the $E_W$ computes the geodesics anchored on the boundary of $A$ (black curve) which agrees with the $\mathcal{E}_W$. For $\ell \gg β$, the $E_W$ does the disconnected surfaces (dotted vertical lines) which is consistent with the $\mathcal{E}_W$. 

FIG. 5. An example of computation of $\mathcal{E}_W$ for the static planner BTZ black hole. The inverse temperature $β$ is determined by the radius of the horizon. If the subsystem $A$ is sufficiently small $\ell \ll β$, the $E_W$ computes the geodesics anchored on the boundary of $A$ (black curve) which agrees with the $\mathcal{E}_W$. For $\ell \gg β$, the $E_W$ does the disconnected surfaces (dotted vertical lines) which is consistent with the $\mathcal{E}_W$. 

FIG. 6. An example of computation of $\mathcal{E}_W$ for the static planner BTZ black hole. The inverse temperature $β$ is determined by the radius of the horizon. If the subsystem $A$ is sufficiently small $\ell \ll β$, the $E_W$ computes the geodesics anchored on the boundary of $A$ (black curve) which agrees with the $\mathcal{E}_W$. For $\ell \gg β$, the $E_W$ does the disconnected surfaces (dotted vertical lines) which is consistent with the $\mathcal{E}_W$. 

FIG. 7. An example of computation of $\mathcal{E}_W$ for the static planner BTZ black hole. The inverse temperature $β$ is determined by the radius of the horizon. If the subsystem $A$ is sufficiently small $\ell \ll β$, the $E_W$ computes the geodesics anchored on the boundary of $A$ (black curve) which agrees with the $\mathcal{E}_W$. For $\ell \gg β$, the $E_W$ does the disconnected surfaces (dotted vertical lines) which is consistent with the $\mathcal{E}_W$.
V. DISCUSSION

We firstly make comments on the possible connection of $\mathcal{E}_W$ to the entanglement of purification (EoP). In [15], the calculation of the EoP has been identified with the conformal blocks with internal twist operators with the aid of the holographic code model[35]. In this case, the corresponding correlation function consists of twist operators with the twist number $\pm \frac{n+1}{2}$ where $n$ is an odd integer. From the path integral perspective, these operators and $\sigma_n(\bar{\sigma}_n)$s would play the same role effectively. The $\mathcal{E}_W$ is not the EoP in general. In particular, the $\mathcal{E}_W$ can be negative (see appendix A); thus, the $\mathcal{E}_W$ is farther from the entanglement measure than the EoP.

The $\mathcal{E}_W(\rho_{AB})$ is rather similar to the coherent information $I(A|B) \equiv S(\rho_B) - S(\rho_{AB})$ [21,22], or equivalently, the conditional entropy with the minus sign $S(A|B) \equiv -I(A|B)$. Remarkably, these quantities can have either positive and negative values. The conditional entropy has already been discussed in the context of the differential entropy from which one can draw the bulk convex surfaces [36,37]. In particular, these were defined together with its orientation (with ± sign) [38,39]. For the differential entropy, one needs infinite series of density matrices associated with each infinitesimal subregion. On the other hand, our present result has been derived from a single density matrix $\rho_{AB}$ dual to the entanglement wedge. This is a crucial difference compared with the differential entropy. It is very interesting to study operational interpretation of $S_o$ as like the differential entropy [40]. Further study on generic properties of the $S_o$ is also important.

From the viewpoint of the $\mathcal{E}_W$, positive semi-definiteness of the $\mathcal{E}_W$ might be understood as the specific nature of holographic states which are globally entangled enough [35]. Derivation of the $\mathcal{E}_W$ using the on-shell gravity action [21,42] would test our conjecture in the generic dimension. Another obvious extension is to study the time-dependent setup on both sides. We would like to report these issues in near future.

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Appendix A: A two-qubit example

Let us consider the following state for two-qubit system as a simple example,

$$\rho_{AB} = q|\text{Bell}_{AB}\rangle\langle\text{Bell}_{AB}| + \frac{(1-q)}{4} I_A \otimes I_B,$$  \hspace{1cm} (A1)

where the $I_{A(B)}$ is the identity operator acting on $A(B)$ and the $|\text{Bell}_{AB}\rangle$ is the singlet Bell state, $2^{-\frac{1}{2}}(|0A1B\rangle - |1A0B\rangle)$. The eigenvalues of this density operator and its partial transposition are given by

$$\text{Spec}(\rho_{AB}) = \left\{ \frac{1+3q}{4}, \frac{1-q}{4}, \frac{1-q}{4}, \frac{1-q}{4} \right\},$$  \hspace{1cm} (A2)

$$\text{Spec}(\rho_{AB}^T) = \left\{ \frac{1-3q}{4}, \frac{1+q}{4}, \frac{1+q}{4}, \frac{1+q}{4} \right\}. \hspace{1cm} (A3)$$

Then one can easily compute the $S_o(\rho_{AB})$, the $\mathcal{E}_W(\rho_{AB})$ and the mutual information respectively. See FIG. 4. It is known that, in the two-qubit cases (and two-qutrit cases), the PPT criterion [19,43] is necessarily and sufficient condition for the separability. Namely, if all eigenvalues for the $\rho_{AB}^T$ is non-negative, our given state $\rho_{AB}$ is separable. This is just the case for $0 \leq q \leq 1/3$. In other words, if $q \leq 1/3$, the $\rho_{AB}$ can be constructed from the product state via the LOCC process. To summarize, the $S_o(\rho_{AB})$ and the $\mathcal{E}_W(\rho_{AB})$ do not take the lowest value for separable states; hence, we can not say these are measure of the quantum entanglement. Rather interestingly, the $S_o(\rho_{AB})$ takes the lowest value for the pure state $|\text{Bell}_{AB}\rangle$ and is bounded below by its EE $S(\rho_A)$.

![FIG. 4. Plot of the $S_o(\rho_{AB})$ (blue), the $\mathcal{E}_W(\rho_{AB})$ (orange) and half of the mutual information (green) for the $|\text{Bell}_{AB}\rangle$. The horizontal axis represents the value of $q$. For sufficiently large mixture of the entangled state and classical state, we have $\mathcal{E}_W < 0$. The inflection point appears at $q = 1/3$. Under the LOCC classification, the states with $q \leq 1/3$ are separable.](image)

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