THE STRONG SMALL INDEX PROPERTY FOR FREE HOMOGENEOUS STRUCTURES

GIANLUCA PAOLINI AND SAHARON SHELAH

Abstract. We show that in countable homogeneous structures with canonical amalgamation and locally finite algebraicity the small index property implies the strong small index property. We use this and the main result of [12] to deduce that countable free homogeneous structures in a locally finite irreflexive relational language have the strong small index property. As an application, we exhibit new continuum sized classes of \(\aleph_0\)-categorical structures with the strong small index property whose automorphism groups are pairwise non-isomorphic.

1. Introduction

The small index property (SIP) of a countable structure \(M\) asserts that any subgroup of \(\text{Aut}(M)\) of index less than \(2^{\omega}\) contains the pointwise stabilizer of a finite set \(A \subseteq M\). While \(M\) is said to have the strong small index property (SSIP) if every subgroup of \(\text{Aut}(M)\) of index less than \(2^{\omega}\) lies between the pointwise and the setwise stabilizer of a finite set \(A \subseteq M\). In [1] Cameron proves that the random graph \(R\) has the strong small index property and uses this to show that \(\text{Aut}(R)\) is not isomorphic to any group of automorphisms of a graph (other than \(R\)) or digraph which is transitive on vertices, ordered edges, and ordered non-edges.

In the present paper we generalize the first result of Cameron. First we prove:

Theorem 1. Let \(M\) be a countable homogeneous structure with canonical amalgamation and locally finite algebraicity. If \(M\) has the small index property, then \(M\) has the strong small index property.

Then we use the main result of [12] to infer:

Corollary 2. Let \(M\) be a countable free homogeneous structure in a locally finite irreflexive relational language. Then \(M\) has the strong small index property.

We then define what we call \(\eta\)-hypergraphs and random \(\zeta\)-free \(\eta\)-hypergraphs, for some \(\eta \in 2^{\omega}\) and \(\zeta \in \omega^{\omega}\), and use Corollary 2 and the main result of [8] to show:

Theorem 3. Let \(K\) be one of the following two classes of countable structures:

1. the random \(\eta\)-hypergraphs \(M(\eta)\), for some \(\eta \in 2^{\omega}\) (cf. Definition [12]);
2. the random \(\zeta\)-free \(\eta\)-hypergraphs \(M(\eta, \zeta)\), for some \(\eta \in 2^{\omega}\) and \(\zeta \in \omega^{\omega}\) (cf. Definition [14]).

If \(M = M(x), N = N(y) \in K\) and \(x \neq y\), then \(\text{Aut}(M) \ncong \text{Aut}(N)\).
In particular, we exhibit new continuum sized classes of homogeneous \( \aleph_0 \)-categorical structures with the strong small index property whose automorphism groups are pairwise non-isomorphic (as in the case of Henson digraphs \([4]\)).

The results of this paper pair with those of \([8]\), where the strong small index property is used to obtain new reconstructibility results, and to analyse outer automorphism groups of groups of automorphisms of countable structures.

2. Proofs

As a matter of notation, given a structure \( M \) and \( A \subseteq M \), and considering \( Aut(M) = G \) in its natural action on \( M \), we denote the pointwise (resp. setwise) stabilizer of \( A \) under this action by \( G_{(A)} \) (resp. \( G_{\{A\}} \)).

We say that a structure \( M \) is homogeneous if every isomorphism between finitely generated substructures of \( M \) extends to an automorphism of \( M \). As well known, an homogeneous structure \( M \) is the so-called Fraïssé of its age \( K = K(M) \), i.e. the collection of finitely generated substructures of \( M \) (for details on these notions see e.g. \([6\) Chapter 6]). Ages of homogeneous structures are characterized by the following three conditions:

(I) If \( B \in K \) and \( A \) is a substructure of \( B \) (denoted \( A \subseteq B \)), then \( A \in K \) (Hereditary Property).

(II) For every \( B_1, B_2 \in K \) there are \( C \in K \) and embeddings \( f_i : B_i \to C \) (\( i = 1, 2 \)) (Joint Embedding Property).

(III) For every \( A, B_1, B_2 \in K \) and embeddings \( f_i : A \to B_i \) (\( i = 1, 2 \)), there are \( C \in K \) and embeddings \( g_i : B_i \to C \) (\( i = 1, 2 \)) such that \( g_1 f_1 | A = g_2 f_2 | A \) (Amalgamation Property).

**Definition 4.** (1) We refer to classes of finitely generated structures satisfying conditions (I)-(III) above as amalgamation classes.

(2) The structure \( C \) in (III) above is called an amalgam of \( B_1 \) and \( B_2 \) over \( A \).

(3) We say that the amalgamation from (III) is disjoint if in addition the \( C \) and \( g_i \) can be chosen so that \( g_1 (B_1) \cap g_2 (B_2) = g_1 f_1 (A) \).

We will deal with homogeneous structures satisfying two additional conditions: locally finite algebraicity and canonical amalgamation, as defined below.

**Definition 5.** Let \( M \) be a structure and \( G = Aut(M) \).

(1) We say that \( a \) is algebraic over \( A \subseteq M \) in \( M \) if the orbit of \( a \) under \( G_{(A)} \) is finite.

(2) The algebraic closure of \( A \subseteq M \) in \( M \), denoted as \( acl_M(A) \), is the set of elements of \( M \) which are algebraic over \( A \).

(3) We say that \( M \) has locally finite algebraicity if for every finite \( A \subseteq M \), \( acl_M(A) \) is finite.

Notice that in homogeneous structures with disjoint amalgamation (cf. Definition 4) we have \( acl_M(A) = (A)_M \), i.e. the algebraic closure of \( A \) in \( M \) equals the substructure generated by \( A \) in \( M \) (see e.g. \([6\) Theorem 7.1.8]).

**Definition 6.** Let \( M \) be an homogeneous structure and \( K = K(M) \) its age. We say that \( M \) has canonical amalgamation if there exists an operation \( B_1 \oplus_A B_2 \) on \( K^3 \) satisfying the following conditions:

(a) \( B_1 \oplus_A B_2 \) is defined when \( A \subseteq B_i \) (\( i = 1, 2 \)) and \( B_1 \cap B_2 = A \);
(b) $B_1 \oplus_A B_2$ is an amalgam of $B_1$ and $B_2$ over $A$ (cf. Definition 4);
(c) if $B_1 \oplus_A B_2$ and $B'_1 \oplus_{A'} B'_2$ are defined, and there exist $f_i : B_i \cong B'_i \ (i = 1, 2)$ with $f_1 \upharpoonright A = f_2 \upharpoonright A$, then there is $f : B_1 \oplus_A B_2 \cong B'_1 \oplus_{A'} B'_2$ such that $f \upharpoonright B_1 = f_1$ and $f \upharpoonright B_2 = f_2$.

Given $A, B, C \subseteq M$ we write $A \equiv_B C$ to mean that there is an automorphism of $M$ fixing $B$ pointwise and mapping $A$ to $C$.

**Definition 7.** Let $M$ be an homogeneous structure with canonical amalgamation. We define a ternary relation between finite substructures of $M$ by setting $A \downarrow_C B$ if and only if $(A, B, C)_M \cong (A, C)_M \oplus_C (B, C)_M$.

In many cases the relation defined in Definition 7 satisfies several properties of interest (cf. the notion of stationary independence relation from [12]), but at this level of generality we only have the following three properties (which will suffice for our purposes):

**Proposition 8.** Let $M$ be an homogeneous structure with canonical amalgamation.

(A) (Invariance) If $A \downarrow_C B$ and $f \in \text{Aut}(M)$, then $f(A) \downarrow_{f(C)} f(B)$.

(B) (Existence) For every $A, B, C \subseteq M$, there exists $A' \equiv_B A$ such that $A' \downarrow_C B$.

(C) (Stationarity) If $A \equiv C A', A \downarrow_C B$ and $A' \downarrow_C B$, then $A \equiv (B C) A'$.

We will use these properties to prove Theorem 1; we first need a fact from [6].

**Fact 9 (6, Theorem 4.2.9).** Let $M$ be a countable homogeneous structure, $G = \text{Aut}(M)$, and suppose that for every finite algebraically closed $A, B \subseteq M$ we have $G_{(A \cup B)} = (G_{(A)} \cup G_{(B)})_G$. Let $H$ be a subgroup of $G$ such that there is some finite algebraically closed set $A$ with $G_{(A)} \leq H$. Then there is a unique smallest algebraically closed set $B \subseteq A$ such that $G_{(B)} \leq H$. Furthermore, for this set $B$ we have $H \leq G_{(B)}$.

**Proof of Theorem 7.** By Fact 9 it suffices to prove that for finite substructures $A, B \subseteq M$ we have:

$$G_{(A \cup B)} = (G_A \cup G_B)_G.$$ 

The containment from right to left is trivial, so let $g \in G_{(A \cup B)}$. Let $f : A' \cong g(A)$ be such that $A' \cap B = A \cap B$ and $A' \oplus_{A \cap B} B \cong (A' \cup B)_M$. Then $f \cup id_B$ extends to an automorphism of $(A' \cup B)_H$, and so to an automorphism $h \in G_{(B)}$. Thus, $hg(A) \downarrow_{A \cup B} B$. Similarly, we find $h' \in G_{(B)}$ such that $h'(A) \downarrow_{A \cap B} B$. Notice now that $hg(A) \equiv (A \cup B) h'(A)$, since $g \in G_{(A \cup B)}$ and $h, h' \in G_{(B)}$. Hence, by Stationarity, there exists $h'' \in G_{B}$ such that $h''hg \upharpoonright A = h' \upharpoonright A$. Thus, $h^* = (h')^{-1}h''hg \in G_{(A)}$, and so we have:

$$g = h^{-1}(h'')^{-1}h'h^* \in (G_A \cup G_B)_G,$$

since $h^* \in G_{(A)}$ and $h, h', h'' \in G_{(B)}$. 

Given a relational language $L$ and $L$-structures $A, B_1, B_2$, with $A \subseteq B_1, B_2$, there is a very natural way of amalgamating $B_1$ and $B_2$ over $A$: the structure with domain $B_1 \cup B_2$ where the only relations are the relation from $B_1$ and the relation from $B_2$. This way of amalgamating is known as free amalgamation, and it is an instance of canonical amalgamation as in Definition 6. We say that an homogeneous relational structure is free if its age is closed under free amalgamation.

Using the following fact from [12] it is now immediate to deduce Corollary 2. Recall that a relational language $L$ is said to be locally finite if for every $n < \omega$
there are only finitely many relations of arity \( n \) in \( L \). With some abuse of notation, we say that a language \( L \) is irreflexive if we only consider \( L \)-structures \( M \) such that if \( R \in L \) and \( M \models R(a_1, ..., a_n) \), then the \( a_i \) are distinct.

**Fact 10** ([12]). Let \( K \) be a free amalgamation class of finite structures in a finite relational language. Then \( K \) has the extension property for partial automorphisms.

**Proof of Corollary 2.** Of course Fact 10 can be used to show the extension property for partial automorphisms also for free amalgamation classes of finite irreflexive structures in a locally finite relational language. Thus, by Sections 6.1 and 6.2 of [7] we have the small index property. Hence, by Theorem 1 we are done. \( \blacksquare \)

**Corollary 11.** The following structures have the strong small index property:

1. the random graph [11];
2. the universal homogeneous \( K_n \)-free graph \((n \geq 3)\) [5];
3. the \( n \)-coloured random graph [14];
4. the random directed graph;
5. the continuum many Henson digraphs [4];
6. the \( k \)-uniform random hypergraph \((k \geq 2)\) [14];
7. the random \( m \)-free \( k \)-uniform hypergraph \((m > k \geq 2)\) [10].

The domain of applicability of Theorem 1 is well-beyond Corollary 2. In [9] Theorem 1 is in fact used outside of the relational context to infer the strong small index property of Hall’s universal locally finite group [3].

We now define the class of hypergraphs that appear in Theorem 3.

**Definition 12.** Let \( \eta \in 2^\omega \) and \( L(\eta) \) be the relational language which has exactly one relation symbol \( R_n \) of arity \( n \) if and only if \( \eta(n) = 1 \). For non-trivial reasons we restrict to the class of \( L(\eta) \) with \( \eta(0) = \eta(1) = 0 \). Let \( K(\eta) \) be the class of finite \( L(\eta) \)-structure such that the relations \( R_n \) (for \( \eta(n) = 1 \)) are symmetric and irreflexive, i.e. if \( K \models R(a_1, ..., a_n) \) then the \( a_i \) are distinct and \( R(a_1, ..., a_n) \) iff \( R(\sigma(a_1), ..., \sigma(a_n)) \) for every \( \sigma \in \text{Sym} \{1, ..., n\} \). The class \( K(\eta) \) is an amalgamation class. We call its Fraïssé limit \( M(\eta) \) the random \( \eta \)-hypergraph.

For \( \eta \) such that \( \eta(n) = 1 \) iff \( n = k \), for fixed \( k \geq 2 \), the structure \( M(\eta) \) is simply the random \( k \)-uniform hypergraph [14]. Notice that \( K(\eta) \) is closed under free amalgamation, and so by our Corollary 2 we have:

**Corollary 13.** For every \( \eta \) as above, the structure \( M(\eta) \) has the strong small index property.

**Definition 14.** Let \( \eta \) and \( L(\eta) \) be as in the previous paragraph. Let \( \zeta \in \omega^\omega \) be such that \( \zeta(n) \geq n \) if \( \eta(n) = 1 \), and \( \zeta(n) = 0 \) otherwise. Let now \( K(\eta, \zeta) \) be the class of structures \( K \in K(\eta) \) such that, for \( n < \omega \) such that \( \zeta(n) > n \), \( K \) does not embed the structure of size \( \zeta(n) \) such that every tuple of \( n \) distinct elements is in the relation \( R_n \). The class \( K(\eta, \zeta) \) is an amalgamation class. We call its Fraïssé limit \( M(\eta, \zeta) \) the random \( \zeta \)-free \( \eta \)-hypergraphs.

For fixed \( m > k \geq 2 \) and \( \eta \) such that \( \eta(n) = 1 \) iff \( n = k \), and \( \zeta \) such that \( \zeta(k) = m \) and \( \zeta(n) = 0 \) for \( n \neq k \), the structure \( M(\eta, \zeta) \) is simply the random \( m \)-free \( k \)-uniform hypergraph [10]. Notice that for \( \zeta \in \omega^\omega \) such that \( \zeta(n) = \eta(n)n \) we have \( M(\eta, \zeta) \cong M(\eta) \), and so this setting actually generalizes the previous. Finally, also in this case \( K(\eta, \zeta) \) is closed under free amalgamation, and so by our Corollary 2 we have:
Corollary 15. For every $\eta$ and $\zeta$ as above, the structure $M(\eta, \zeta)$ has the strong small index property.

We now state the main result of [8] and use it to prove Theorem 3.

Definition 16. We say that two structures $M$ and $N$ are bi-definable if there is a bijection $f : M \to N$ such that for every $A \subseteq M^n$, $A$ is $\emptyset$-definable in $M$ if and only if $f(A)$ is $\emptyset$-definable in $N$.

Fact 17 ([8]). Let $M$ and $N$ be countable $\aleph_0$-categorical structures with the strong small index property and no algebraicity. Then $\pi : \text{Aut}(M) \cong \text{Aut}(N)$ if and only if $M$ and $N$ are bi-definable. Furthermore, letting $f : M \to N$ be as in Definition 16, the isomorphism $\pi : \text{Aut}(M) \cong \text{Aut}(N)$ is induced by $f$.

Proof of Theorem 3. As noticed after Definition 14, every random $\eta$-hypergraph can be considered as a $\zeta$-free $\eta$-hypergraph for appropriate $\zeta$, and so it suffices to deal with the class of $\zeta$-free $\eta$-hypergraphs. Let $M_1 = M(\eta_1, \zeta_1)$ and $M_2 = M(\eta_2, \zeta_2)$ and suppose that $\pi : \text{Aut}(M_1) \cong \text{Aut}(M_2)$. We will show that $(\eta_1, \zeta_1) = (\eta_2, \zeta_2)$. By Fact 17, there is $f : M_1 \to M_2$ witnessing bi-definability and inducing $\pi$. Without loss of generality $f = \text{id}_{M_1}$, and so $\text{Aut}(M_1) = \text{Aut}(M_2) := H$. Denote the domain of $M_1$ as $A$ (which is the same as the domain of $M_2$). First of all we show that $\eta_1 = \eta_2$. Notice that for every $1 < n < \omega$ the following assertions are equivalent:

(a) $n \eta(n) = 1$ ($\ell = 1, 2$);
(b) there are $\bar{a} \neq \bar{b} \in A^n$ each with no repetitions such that:
   (i) for every $h \in H$, $h(\bar{a}) \neq h(\bar{b})$;
   (ii) for every $i < n$ there exists $h \in H$ such that $\bigwedge_{i \neq j < n} h(a_j) = b_j$.

Since the statement (b) does not depend on $\ell$ we conclude that $\eta_1 = \eta_2$. Suppose now that $\eta_1 = \eta_2$, we show that $\zeta_1 = \zeta_2$. Given $n \leq k < \omega$ we denote by $[k]^n$ the set of subsets of $\{1, \ldots, k\}$ of size $n$. Notice that for every $1 < n < k < \omega$ the following assertions are equivalent:

(a') for each $\{\bar{a}_x = (a_{x(1)}^1, \ldots, a_{x(n)}^1) : x = \{x(1) < \cdots < x(n)\} \in [k]^n\}$ each with no repetitions if (i') then (ii'), where:
   (i') for every $x \neq y \in [k]^n$ with $|x \cap y| > 0$ there exists $h_{x,y} \in H$ such that $\bigwedge_{p \in x \cap y} h_{x,y}(a_p^y) = a_p^y$;
   (ii') there exists $\bar{a}_* = (a_1^*, \ldots, a_k^*) \in A^k$ with no repetitions and $h_\omega \in H$,
   $x \in [k]^n$, such that $\bigwedge_{i < n} h_\omega(a_{x(i)}^*) = a_{x(i)}^*$.

Since the statement (b) does not depend on $\ell$ we conclude that $\zeta_1 = \zeta_2$.

REFERENCES

[1] Peter J. Cameron. The Random Graph Has the Strong Small Index Property. Discrete Math. 291 (2005), 41-43.
[2] Peter J. Cameron. The Random Graph. In: The Mathematics of Paul Erdos II, Ronald Graham, Jaroslav Nesetril and Steve Butler (eds.). Springer-Verlag New York, 2013.
[3] Philip Hall. Some Constructions for Locally Finite Groups. J. Lond. Math. Soc. 31-34 (1959), no. 3, 305-319.
[4] C. Ward Henson. Countable Homogeneous Relational Structures and $\aleph_0$-Categorical Theories. J. Symbolic Logic 37 (1972), no. 3, 494-500.
[5] C. Ward Henson. A Family of Countable Homogeneous Graphs. Pacific J. Math. 38 (1971), no. 1, 69-83.
[6] Wilfrid Hodges. Model Theory. Cambridge University Press, 1993.

[7] Alexander S. Kechris and Christian Rosendal. Turbulence, Amalgamation, and Generic Automorphisms of Homogeneous Structures. Proc. Lond. Math. Soc. 94 (2007), no. 2, 302-350.

[8] Gianluca Paolini and Saharon Shelah. Reconstructing Structures with the Strong Small Index Property up to Bi-Definability. Submitted, available on the arXiv.

[9] Gianluca Paolini and Saharon Shelah. The Automorphism Group of Hall’s Universal Group. Submitted, available on the arXiv.

[10] Ehud Hrushovski. Pseudo-Finite Fields and Related Structures. In: Model Theory and Applications (ed. L. Bélair et al.), pp. 151-212, Quaderni di Mathematica, Volume 11 (Seconda Universita di Napoli, 2002).

[11] Richard Rado. Universal Graphs and Universal Functions. Acta Arith. 9 (1964), no. 4, 331-340.

[12] Daoud Siniora and Slawomir Solecki. Coherent Extension of Partial Automorphisms, Free Amalgamation, and Dense Locally-Finite Subgroups. To appear.

[13] Katrin Tent and Martin Ziegler. On the Isometry Group of the Urysohn Space. J. London Math. Soc. 87 (2013), no. 1, 289-303.

[14] Simon Thomas. Reducts of Random Hypergraphs. Ann. Pure Appl. Logic 80 (1996), no. 2, 165-193.

[15] John K. Truss. The Group of the Countable Universal Graph. Math. Proc. Cambridge Philos. Soc. 98 (1985), no. 2, 213-245.

EINSTEIN INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, ISRAEL

EINSTEIN INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, ISRAEL AND DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, U.S.A.