SUBCRITICAL MULTITYPE BRANCHING PROCESS IN RANDOM ENVIRONMENT

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Abstract. We study the asymptotic behaviour of the survival probability of a multitype branching process in random environment. The class of processes we consider here corresponds, in the one-dimensional situation, to the strongly subcritical case. We also prove a conditional limit theorem describing the distribution of the number of particles in the process given its survival for a long time.

1. Introduction and statement of results

We study in this note asymptotic properties of multitype branching processes in random environments (BPRE’s). Informally speaking, these processes serve as a stochastic model for the evolution of a population with \( p \) different types of particles, where each particle may have children of all types, the number of descendants is random with distribution changing from generation to generation in a random fashion.

In order to give a rigorous definition of multitype BPRE’s and to formulate our results we need to introduce some notation. First, \( p \)-dimensional deterministic vectors with non-negative coordinates and \( p \times p \) deterministic matrices with non-negative entries will be denoted by bold lower case symbols. In particular, \( \mathbf{0} := (0, 0, \ldots, 0) \) and \( \mathbf{1} := (1, 1, \ldots, 1) \). The standard basis vectors will be denoted by \( \mathbf{e}_i, i = 1, 2, \ldots, p \). For \( p \)-dimensional vectors \( \mathbf{x} = (x^1, \ldots, x^p) \) and \( \mathbf{y} = (y^1, \ldots, y^p) \) we set

\[
(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{p} x^i y^i, \quad |\mathbf{x}| := \sum_{i=1}^{p} |x^i| \quad \text{and} \quad \mathbf{x}^\mathbf{y} := \prod_{i=1}^{p} (x^i)^{y^i}.
\]

For every \( p \)-tuple \( (\mu^1, \mu^2, \ldots, \mu^p) \) of measures on \( \mathbb{Z}_+^p \), we define its multidimensional generating function \( f = (f^1, f^2, \ldots, f^p) \) by the relations

\[
f^i(s) := \sum_{\mathbf{z} \in \mathbb{Z}_+^p} s^{\mathbf{z}} \mu^i(\mathbf{z}), \quad s \in [0, 1]^p, \quad i = 1, 2, \ldots, p.
\]

Any sequence \( \{f_n, n \geq 1\} \) of multidimensional generating functions will be called a varying environment. The corresponding measures \( \mu^i_n \) will be interpreted as the offspring law for a particle of type \( i \) in generation \( n - 1 \). We now define a \( p \)-type branching process \( \mathbf{Z}_n = (Z^1_n, Z^2_n, \ldots, Z^p_n) \), \( n \geq 0 \), where \( Z^i_n \) is the number of type \( i \) particles in the process at moment \( n \). This process has a deterministic starting point \( \mathbf{Z}_0 \) and the population sizes of the subsequent generations of the process are

1991 Mathematics Subject Classification. Primary 60J80; Secondary 60F99.

Key words and phrases. Branching process, random environment, non-extinction probability.
specified by the following standard recursion:

\[ Z_n^i = \sum_{j=1}^{p} \sum_{k=1}^{Z_{n-1}^j} X_{n,j,k}^i, \quad i = 1, 2, \ldots, p, \quad n \geq 1, \quad (1) \]

where \( X_{n,j,k} = (X_{n,j,k}^1, X_{n,j,k}^2, \ldots, X_{n,j,k}^p), \quad k \geq 1 \) are independent random vectors distributed according to \( \mu_{n,j} \). Here and in what follows we denote random objects by upper case symbols using the respective bold symbols if the objects are vectors or matrices.

Equipping the set of all tuples \((\mu_1, \mu_2, \ldots, \mu_p)\) with the total variation distance we obtain a metric space. Therefore, we may consider probability measures on this space. Due to the one-to-one correspondence between \( p \)-tuples of measures and \( p \)-dimensional generating functions, we may assume that we are given a probability measure on the space of generating functions. With this agreement in hands we call a sequence of independent, identically distributed \( p \)-dimensional generating functions \( \{F_n = (F_{n,1}, F_{n,2}, \ldots, F_{n,p})\} \) a random environment. We say that \( \{Z_n, n \geq 0\} \) is a \( p \)-type BPRE, if its conditional distribution is determined by (1) for every fixed realization of the environmental sequence.

Branching processes in random environment with one type of particles have been intensively investigated during the last two decades and their properties are well understood and described in the literature. The reader may find a modern and unified presentation of the corresponding results in a recent book by Kersting and Vatutin [12]. The multi-dimensional case is much less studied and many basic questions, e.g. a detailed classification, asymptotics of the survival probability and the corresponding conditional limit theorems, are still not answered in full generality. For instance, only recently an asymptotic representation for the survival probability in the critical multitype BPRE’s was found under relatively general conditions, see Le Page, Peigne and Pham [13] and Vatutin and Dyakonova [14].

The purpose of this note is to study asymptotic properties of a class of subcritical multi-type BPRE’s, which correspond to the so-called strongly subcritical BPRE’s with one type of particles.

It is a general phenomenon that the asymptotic behaviour of BPRE’s is mainly specified by the properties of certain basic characteristics of the environment. In the case of multi-type processes the crucial role is played by the (random) mean matrices

\[ M_n = (M_{n,i,j})_{i,j=1}^p := \left( \frac{\partial F_n}{\partial s^j}(1) \right)_{i,j=1}^p, \quad n \geq 1. \]

If \( F_n \) are independent and distributed as a generating function \( F = (F_1, \ldots, F_p) \), then, obviously, \( M_n \) are independent probabilistic copies of the random matrix

\[ M = (M_{i,j})_{i,j=1}^p := \left( \frac{\partial F^i}{\partial s^j}(1) \right)_{i,j=1}^p. \]

We assume that the distribution of \( M \) satisfies the following assumptions:

- **Condition H0.** \( P(\|M\| > 0) = 1 \), where \( \|M\| \) is the operator norm of \( M \).
- **Condition H1.** The set \( \Theta := \{ \theta > 0 : \mathbb{E} \left[ \|M\|^\theta \right] < \infty \} \) is non-empty.
- **Condition H2.** The support of the distribution of \( M \) acts strongly irreducibly on the semi-group of matrices with non-negative entries, i.e. no
proport finite union of subspaces of \( \mathbb{R}^p \) is invariant with respect to all elements of the multiplicative semi-group generated by the support of \( \mathbf{M} \).

- **Condition H3.** There exists a positive number \( \gamma > 1 \) such that

\[
1 \leq \max_{i,j} M_{i,j} \leq \gamma.
\]

We also need to consider the so-called Hessian matrices

\[
\mathbf{B}(k) := \left( \frac{\partial^2 F_k}{\partial s^i \partial s^j}(1) \right)_{i,j=1}^p, \quad k = 1, 2, \ldots, p
\]

and the random variables

\[
\mathbf{T} := \frac{1}{\|\mathbf{M}\|} \sum_{k=1}^p \|\mathbf{B}(k)\| \quad \text{and} \quad \mathbf{T}_n := \frac{1}{\|\mathbf{M}_{n}\|} \sum_{k=1}^p \|\mathbf{B}_n(k)\|, \quad n = 1, 2, \ldots.
\]

Thus, \( \mathbf{T}_n \) are independent probabilistic copies of \( \mathbf{T} \). We shall impose, along with Conditions H0 – H3 the following restriction on the distribution of \( \mathbf{T} \):

- **Condition H4.** There exists an \( \varepsilon > 0 \) such that

\[
\mathbb{E} \left[ \|\mathbf{M}\| \log \mathbf{T} \right]^{1+\varepsilon} < \infty.
\]

Using the standard subadditivity arguments, one can easily infer that for every \( \theta \in \Theta \) the limit

\[
\lambda(\theta) := \lim_{n \to \infty} \left( \frac{\mathbb{E} \left[ \|\mathbf{M}_n \cdots \mathbf{M}_1\|^{\theta} \right]}{\|\mathbf{M}_n\|^2} \right)^{1/n} < \infty
\]

is well defined. This function is an analogue of the moment generating function for the associated random walk in the case of BPRE’s with single type of particles.

Set

\[
\Lambda(\theta) := \log \lambda(\theta), \quad \theta \in \Theta.
\]

**Theorem 1.** Assume that Conditions H0 – H4 are valid, the point \( \theta = 1 \) belongs to the interior of the set \( \Theta \) and \( \Lambda'(1) < 0 \). Then

(a) there exists a vector \( \mathbf{c} = (c^1, \ldots, c^p) \) with strictly positive components such that

\[
\mathbb{P} \left( |\mathbf{Z}_n| > 0 \mid \mathbf{Z}_0 = \mathbf{e}_i \right) \sim c^i \Lambda^n(1), \quad n \to \infty.
\]

(b) for each \( \mathbf{s} \in [0, 1]^p, \mathbf{s} \neq \mathbf{1} \)

\[
\lim_{n \to \infty} \mathbb{E} \left[ \mathbf{s}^{\mathbf{Z}_n} \mid |\mathbf{Z}_n| > 0; \mathbf{Z}_0 = \mathbf{e}_i \right] = \Phi_i(\mathbf{s}),
\]

where \( \Phi_i(\mathbf{s}) \) is the probability generating function of a proper distribution on \( \mathbb{Z}_+^p \).

If \( p = 1 \) then the assumption \( \Lambda'(1) < 0 \) reduces to \( \mathbb{E}[\mathbf{M} \log \mathbf{M}] < 0 \). The one-type BPRE’s satisfying this condition are usually called strongly subcritical. The asymptotic behaviour of the survival probability for one-type strongly subcritical processes has been studied by Guivarch and Liu [10]. The corresponding conditional limit theorem for the distribution of the number of particles given survival has been proved by Geiger, Kersting and Vatutin [9].

Dyakonova [6] has obtained (2) under the assumption that all possible realizations of \( \mathbf{M} \) have a common deterministic right eigen-vector corresponding to the Perron root \( \phi(\mathbf{M}) \) of \( \mathbf{M} \). In this special case the condition \( \Lambda'(1) < 0 \) reduces to the inequality \( \mathbb{E}[\phi(\mathbf{M}) \log \phi(\mathbf{M})] < 0 \). The case \( \mathbb{E}[\phi(\mathbf{M}) \log \phi(\mathbf{M})] = 0 \) has been
studied by Dyakonova [7] under the assumption that there exists a common left eigen-vector corresponding to the Perron root of $M$.

2. Proof of Theorem [1]

2.1. Representation for the generating function of the process. For every fixed realization of the environmental sequence $F_n$ and $k < n$ we define

$$ F_{k,n}(s) := F_{k+1} \circ F_{k+2} \circ \cdots \circ F_n(s), $$

$$ F_{n,k}(s) := F_n \circ F_{n-1} \circ \cdots \circ F_{k+1}(s), $$

and set

$$ F_{n,n}(s) := s. $$

It is immediate from the definition of the process $\{Z_n, n \geq 0\}$ that

$$ \mathbb{E} \left[ s^{Z_n} \mid Z_0 = e_i, F_1, F_2, \ldots, F_n \right] = F_{i,n}^i(s). $$

Therefore,

$$ \mathbb{E}[s^{Z_n} \mid Z_0 = e_i] = \mathbb{E}[F_{i,n}^i(s)]. \quad (3) $$

Setting here $s = 0$, we get

$$ \mathbb{P} (|Z_n| > 0 \mid Z_0 = e_i) = 1 - \mathbb{E}[F_{0,n}^i(0)] = \mathbb{E}[1 - F_{0,n}^i(0)]. \quad (4) $$

Let $f$ be the generating function of a $p$-tuple $(\mu^1, \mu^2, \ldots, \mu^p)$, and let $m$ be the corresponding mean matrix, i.e.,

$$ m = \left( \frac{\partial f_i}{\partial s^j}(1) \right)_{i,j=1}^p. $$

For a generating function $f$ and a matrix $a$ define

$$ \psi_{f,a}(s) := \frac{|a|}{|a(1 - f(s))|} - \frac{|a|}{|am(1 - s)|}, $$

where, by a slight abuse of notation, $|\cdot|$ is used to denote the $L_1$-norm of matrices.

Let $a_i$ be the matrix with $a_{i,i} = 1$ and $a_{k,l} = 0$ for all $(k,l) \neq (i,i)$. Then, clearly,

$$ 1 - F_{i,n}^i(s) = |a_i(1 - F_{0,n}(s))|. $$

Using now the definition of $\psi$, we have

$$ \frac{1}{1 - F_{i,n}^i(s)} = \frac{|a_i|}{|a_i(1 - F_{0,n}(s))|} = \frac{|a_i|}{|a_iM(1 - F_{1,n}(s))|} + \psi_{F, a_i}(F_{1,n}(s)) $$

$$ = \frac{1}{|a_iM(1 - F_{1,n}(s))|} + \psi_{F, a_i}(F_{1,n}(s)). $$

Iterating this procedure, we obtain

$$ \frac{1}{1 - F_{i,n}^i(s)} = \frac{1}{|a_iR_n(1 - s)|} + \sum_{k=1}^{n} \frac{1}{|a_iR_k|} \psi_{F, a_i R_{k-1}}(F_{k,n}(s)), $$

where

$$ R_0 := \text{Id} \quad \text{and} \quad R_k := M_1M_2\cdots M_k, \quad k \geq 1. $$
Now, recalling that $F_k$ are i.i.d. random elements, we may use the substitution $F_k \leftrightarrow F_{n-k+1}$. As a result we get

$$
\mathbb{E}[1 - F^i_{0,n}(s)] = \mathbb{E}[1 - F^i_{n,0}(s)]
$$

$$
= \mathbb{E}
\begin{pmatrix}
\frac{1}{a_i L_{n,1}(1-s)} + \sum_{k=1}^{n} \frac{1}{|a_i L_{n,n-k+1}|} \psi_{F_{n-k+1}, a_i L_{n,n-k+2}}(F_{n-k,0}(s)) \psi_{F_{k-1}, 0}(F_{k-1,0}(s))^{-1}
\end{pmatrix}
$$

where

$$
L_{n,n+1} := \text{Id} \quad \text{and} \quad L_{n,k} := M_n M_{n-1} \cdots M_k, \quad 1 \leq k \leq n.
$$

We shall see later, that the asymptotic, as $n \to \infty$, behaviour of $1 - F^i_{0,n}(s)$ will be determined by the summands corresponding to fixed values $k$. To control $\psi_{F_k, a_i, R_k}$ we shall use Lemma 5 in [14]: under the assumption H3

$$
0 \leq \psi_{F_k, a_i L_{n,k+1}}(F_{k-1,0}(s)) \leq \gamma p^2 T_k
$$

for all $s \in [0,1]^p$, where the value of $\psi_{F_k, a_i L_{n,k+1}}(s)$ at point $s = 1$ is specified by continuity.

2.2. Exponential change of measure. Define

$$
\mathbb{S}_+ := \{ x \in \mathbb{R}_+^p : |x| = 1 \}.
$$

Then, for every matrix $m$ with positive entries the vector

$$
m \cdot x := \frac{mx}{|mx|}
$$

is the projection of $mx$ on the set $\mathbb{S}_+$. Denote by $C(\mathbb{S}_+)$ the set of all continuous functions on $\mathbb{S}_+$. For $\theta \in \Theta$, $g \in C(\mathbb{S}_+)$, and $x \in \mathbb{S}_+$ define the transition operator

$$
P_\theta g(x) := \mathbb{E}
\begin{pmatrix}
|Mx|^{\theta} g(M \cdot x)
\end{pmatrix}.
$$

If Conditions H1 – H3 hold, then, according to Proposition 3.1 in [1], $\lambda(\theta)$ is the spectral radius of $P_\theta$ and there exist a unique strictly positive function $r_\theta \in C(\mathbb{S}_+)$ and a unique probability measure $l_\theta$ meeting the scaling

$$
\int_{\mathbb{S}_+} r_\theta(x) dl_\theta(x) = 1
$$

and such that

$$
l_\theta P_\theta = \lambda(\theta) l_\theta, \quad P_\theta r_\theta = \lambda(\theta) r_\theta.
$$

Following [2], we introduce the functions

$$
p^\theta_n(x, m) := \frac{|mx|^{\theta} r_\theta(M \cdot x)}{\lambda^n(\theta) r_\theta(x)}, \quad x \in \mathbb{S}_+.
$$

It is easy to see that, for every $n \geq 1$, every $x \in \mathbb{S}_+$ and every matrix $m$,

$$
\mathbb{E} p^\theta_{n+1}(x, Mm) = p^\theta_n(x, m)
$$

and, in particular,

$$
\mathbb{E} p^\theta_n(x, L_{n,1}) = 1.
$$
For each \( n \geq 1 \), denote by \( \mathcal{F}_n \) the \( \sigma \)-algebra generated by random elements \( Z_1, Z_2, \ldots, Z_n \) and \( F_1, F_2, \ldots, F_n \). Let \( I_A \) be the indicator of the event \( A \). It follows from (10) that
\[
P_\theta^n(A) := \mathbb{E} \left[ p_\theta^n(x, L_{n,1}) I_A \right]
\]
is a probability measure on \( \mathcal{F}_n \). Furthermore, (9) implies that \( \{P_\theta^n, n \geq 1\} \) is a sequence of consistent probability measures. It can be extended to a probability measure \( P_\theta \) on our original probability space \((\Omega, \mathcal{F})\).

We now take \( \theta = 1 \) and apply the corresponding change of measure to the representation (3). Since \( 1 - F_{n,0}^i(s) \) is measurable with respect to \( \mathcal{F}_n \), we have
\[
\mathbb{E}[1 - F_{n,0}^i(s)] = \lambda^v(1) r_1(e_i) \mathbb{E} \left[ \frac{(1 - F_{n,0}^i(s))}{L_n,1 e_i | r_1(L_{n,1} \cdot e_i)} \right]
= \lambda^v(1) r_1(e_i) \mathbb{E} \left[ \frac{1 - F_{n,0}^i(s)}{L_n,1 e_i | r_1(L_{n,1} \cdot e_i)} \right].
\]

Applying now (3), recalling the definition of \( a_i \) and using the equality \( |a_i L_{n,k}| = |a_i L_{n,k}^\perp| \) we obtain
\[
\mathbb{E}[1 - F_{n,0}^i(s)] = \lambda^v(1) r_1(e_i) \mathbb{E} \left[ \frac{1}{r_1(L_{n,1} \cdot e_i)} \frac{|e_i L_{n,1}(1 - s)|}{|L_{n,1} e_i|} \Xi_n(s) \right],
\]
where
\[
\Xi_n(s) := \left( 1 + \sum_{k=1}^n \frac{|e_i L_{n,1}(1 - s)|}{|a_i L_{n,k}|} \psi_{F_k, a_i L_{n,k+1}}(F_{k-1,0}(s)) \right)^{-1}.
\]

Set, for brevity,
\[
\tilde{\psi}_{n,k}(s) := \psi_{F_k, a_i L_{n,k+1}}(F_{k-1,0}(s)).
\]

We fix some \( N \geq 1 \) and study the asymptotic behaviour of the expectation
\[
K_n(N; s) := \mathbb{E} \left[ \frac{1}{r_1(L_{n,1} \cdot e_i)} \frac{|e_i L_{n,1}(1 - s)|}{|L_{n,1} e_i|} \Xi_{n,N}(s) \right],
\]
where
\[
\Xi_{n,N}(s) := \left( 1 + \sum_{k=1}^N \frac{|e_i L_{n,1}(1 - s)|}{|a_i L_{n,k}|} \tilde{\psi}_{n,k}(s) \right)^{-1}.
\]

For each \( k \leq N \) we have the equality
\[
\frac{|e_i L_{n,k}|}{|e_i L_{n,1}(1 - s)|} = \frac{|e_i L_{n,N} L_{N-1,k}|}{|e_i L_{n,N} L_{N-1,1}(1 - s)|}.
\]

By Theorem 1 in Hennion [11], there exist a sequence of random numbers \( \{\lambda_n(N) > 0, n \geq 1\} \) and a tuple of random vectors \( \{U_n^{(N)}, V_n^{(N)}, n \geq 1\} \) such that, as \( n \to \infty \)
\[
\frac{L_{n,N}}{\lambda_n(N)} - V_n^{(N)} \otimes U_n^{(N)} \to 0 \quad \text{a.s.}
\]
(Here and in what follows we agree to associate with vectors \( v = (v^1, \ldots, v^p) \) and \( u = (u^1, \ldots, u^p) \) the matrix \( v \otimes u = (v_i u_j)_{i,j=1}^p \).) Besides, the sequence of random
vectors \( \{ (U_n^{(N)}, V_n^{(N)})/|V_n^{(N)}|, n \geq 1 \} \) weakly converges, as \( n \to \infty \), to a vector \((U^{(N)}, V^{(N)})\). As a result, the sequence of ratios

\[
\frac{|e_i L_{n,N} L_{N-1,k}|}{|e_i L_{n,N} L_{N-1,1}(1 - s)|} = \frac{|e_i L_{n,N} L_{N-1,k}|}{|e_i L_{n,N} L_{N-1,1}(1 - s)|}, \quad k \leq N
\]

weakly converges, as \( n \to \infty \), to

\[
\frac{|e_i (V^{(N)} \otimes U^{(N)})) L_{N-1,k}|}{|e_i (V^{(N)} \otimes U^{(N)})) L_{N-1,1}(1 - s)|}, \quad k \leq N.
\]

By the same arguments, the vectors \((\tilde{\psi}_{n,1}(s), \ldots, \tilde{\psi}_{n,N}(s))\) weakly converge, as \( n \to \infty \) to a vector \((\tilde{\psi}_{1}(s), \ldots, \tilde{\psi}_{N}(s))\), and \( r_1(L_{n,1} \cdot e_i) \) weakly converges, as \( n \to \infty \) to a bounded random variable. Therefore, there exists

\[
\lim_{n \to \infty} K_n(N; s) =: K^i(N; s)
\]

and the sequence \( K^i(N; s) \) is decreasing in \( N \) for each fixed \( s = (s^1, \ldots, s^p) \).

Let \( A(s) = \{ 1 \leq j \leq p : s^j < 1 \} \). Setting \( \Delta(s) := \min_{j \in A(s)} (1 - s^j) \) and using Lemma 2 in Furstenberg and Kesten [8] it is not difficult to check that

\[
\frac{\Delta(s)}{p^2 \gamma^2} \leq \frac{|e_i L_{n,1}(1 - s)|}{|L_{n,1} e_i|} \leq \frac{|e_i L_{n,1}|}{|L_{n,1} e_i|} \leq \gamma^2 p^2, \quad n \geq 1.
\]

Combining this estimate with the fact that \( r_1 \) is bounded away from zero, we obtain for some absolute constant \( C = C(s) \) the estimates

\[
0 \leq K_n(N; s) - \mathbb{E}^1 \left[ \frac{1}{r_1(L_{n,1} \cdot e_i)} \frac{|e_i L_{n,1}(1 - s)|}{|L_{n,1} e_i|} \Xi_n(s) \right] \leq C \mathbb{E}^1 \left[ \frac{\sum_{k=N+1}^n \frac{|e_i L_{n,k+1}(1 - s)|}{|e_i L_{n,k+1}|} \tilde{\psi}_k(s)}{1 + \sum_{k=N+1}^n \frac{|e_i L_{n,k+1}(1 - s)|}{|e_i L_{n,k+1}|} \tilde{\psi}_k(s)} \right].
\]

Obviously,

\[
|e_i L_{n,1}(1 - s)| = |e_i L_{n,k} L_{k-1,1}(1 - s)| \leq |e_i L_{n,k}| \cdot \|L_{k-1,1}\|, \quad k \leq n.
\]

From this bound and (10), we get for some constant \( C_1 \)

\[
0 \leq K_n(N; s) - \mathbb{E}^1 \left[ \frac{1}{r_1(L_{n,1} \cdot e_i)} \frac{|e_i L_{n,1}(1 - s)|}{|L_{n,1} e_i|} \Xi_n(s) \right] \leq C_1 \mathbb{E}^1 \left[ \frac{\sum_{k=N+1}^n \|L_{k-1,1}\| T_k}{1 + \sum_{k=N+1}^n \|L_{k-1,1}\| T_k} \right] := K(N).
\]

Assume that we can show that the series

\[
\Psi := 1 + \sum_{k=1}^\infty \|L_{k-1,1}\| T_k < \infty \quad \mathbb{P}^1 - a.s.
\]

Then, clearly,

\[
K(N) \to 0 \quad \text{as} \ N \to \infty.
\]

Combining this with (11), we see that there exists

\[
\lim_{n \to \infty} \mathbb{E}^1 \left[ \frac{1}{r_1(L_{n,1} \cdot e_i)} \frac{|e_i L_{n,1}(1 - s)|}{|L_{n,1} e_i|} \Xi_n(s) \right] = \lim_{N \to \infty} K^i(N; s) =: \phi_i(s).
\]
The limit at the right hand side exists, since the sequence $K^i(N; s)$ is decreasing in $N$. Taking into account (13) and the estimate $\Xi_n(s) \geq \Psi^{-1}$ we conclude that this limit is strictly positive for each $s \in [0, 1]^p$, $s \neq 1$.

Having this result in hands we deduce that, as $n \to \infty$
\[
\mathbb{P}(|Z_n| > 0; Z_0 = e_i) = \mathbb{E}[1 - F_{0,n}^i(0)] \sim \lambda^\nu(1)r_1(e_i)\phi_i(0)
\]
and
\[
\lim_{n \to \infty} \mathbb{E}[s^{Z_n}||Z_n| > 0; Z_0 = e_i] = 1 - \lim_{n \to \infty} \frac{\mathbb{E}[1 - F_{0,n}^i(s)]}{\mathbb{E}[1 - F_{0,n}^i(0)]} = 1 - \frac{\phi_i(s)}{\phi_i(0)} =: \Phi_i(s).
\]

Therefore, to complete the proof of the theorem it remains to check the validity of (13).

We first note that our assumption that 1 belongs to the interior of $\Theta$ provides the finiteness of $\mathbb{E}^1[\log \|M\|]$. Moreover, by Condition $H3$
\[
\frac{\min_{x \in S_+} \frac{|xM|}{|x|}}{\gamma^{-1}} \geq \|M\|.
\]
Thus, all the conditions of Theorem 2 in [11] are valid and, therefore,
\[
\lim_{k \to \infty} \frac{\log \|L_{k,1}\|}{k} = \Lambda'(1) = 0 \quad \mathbb{P}^1 - a.s.
\]
In particular, for every $\varepsilon > 0$,
\[
\|L_{k,1}\| = O\left(e^{-k^{1-\varepsilon}/2}\right) \quad \mathbb{P}^1 - a.s. \tag{14}
\]
By the Markov inequality,
\[
\mathbb{P}^1(\log T_k > k^{(1+\delta)/(1+\varepsilon)}) \leq \mathbb{E}^1\left[\frac{\log T_k}{k^{1+\delta}}\right].
\]
Using now the definition of the measure $\mathbb{P}^1$, we obtain
\[
\mathbb{P}^1(\log T_k > k^{(1+\delta)/(1+\varepsilon)}) \leq C_2 \mathbb{E}[\|M\| \log T_k^{1+\varepsilon}] \cdot \frac{1}{k^{1+\delta}}
\]
for some constant $C_2$. Taking into account Condition $H4$ and applying the Borel-Cantelli lemma, we conclude that
\[
T_k = O\left(e^{k^{(1+\delta)/(1+\varepsilon)}}\right) \quad \mathbb{P}^1 - a.s.
\]
Combining this estimate with (14) and choosing $\delta$ sufficiently small, we obtain (13).

Acknowledgement. This work was supported by the Russian Science Foundation under the grant 17-11-01173 and was fulfilled in the Novosibirsk state university.

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