PRESERVING THE TRACE OF THE KRONECKER SUM

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Abstract. The aim of this paper is to study linear preservers of the trace of Kronecker sums $A \oplus B$ and their connection with preservers of determinants of Kronecker products. The partial trace and partial determinant play a fundamental role in characterizing the preservers of the trace of Kronecker sums and preservers of the determinant of Kronecker products respectively.

1. Introduction

For positive integers $m, n > 2$, let $M_n$ be the algebra of all $n \times n$ matrices over some field $F$ and let $M_{mn} = M_m \otimes M_n = M_m(M_n)$. Here we consider $F = \mathbb{R}$ or $F = \mathbb{C}$ and define $H_n \subset M_n$ to be the Hermitian matrices in $M_n$. Linear maps preserving properties of Kronecker products of matrices have received considerable attention in recent years. Such maps are closely connected to quantum information science (see, e.g., [1]). More recently, Ding et. al. considered linear preservers of determinants of Kronecker products of Hermitian matrices [2], i.e., linear maps $\phi : H_{mn} \to H_{mn}$ satisfying

$$\det(\phi(A \otimes B)) = \det(A \otimes B)$$

where $A$ and $B$ are Hermitian. A few of the results in [2] are restricted to the case when $A$ and $B$ are positive or negative semidefinite matrices. In order to study this problem more generally, we make use of the identity

$$\det(e^A \otimes e^B) = e^{\text{tr}(A \oplus B)}$$

where $A \oplus B$ is the Kronecker sum, i.e.,

$$A \oplus B := A \otimes I_n + I_m \otimes B,$$

where $I_k$, $k = m, n$ denotes the $k \times k$ identity matrix. Under exponentiation of Hermitian matrices, the Kronecker sum arises naturally as the unique map $\oplus : H_m \times H_n \to H_m \otimes H_n$ satisfying

$$e^{A \oplus B} = e^A \otimes e^B, \quad A \in H_m, B \in H_n.$$ 

Moreover, $A \in M_n$ over $\mathbb{C}$ is non-singular if and only if $A = e^B$ for some $B \in M_n$ [3 Example 6.2.15]. Thus, in studying linear maps $\psi : M_{mn} \to M_{mn}$ preserving determinants of (non-singular) Kronecker products

$$\det(\psi(A \otimes B)) = \det(A \otimes B)$$

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of non-singular matrices $A$ and $B$, it suffices to study maps $\phi : M_{mn} \rightarrow M_{mn}$ preserving the trace of Kronecker sums

$$(1) \quad \text{tr}(\phi(A \oplus B)) = \text{tr}(A \oplus B).$$

In this article we will confine our attention to linear maps $\phi$ satisfying (1). We denote by $GL_n(\mathbb{F})$ and $SL_n(\mathbb{F})$ the general and special linear groups in $M_n$, respectively. Thus, we are also interested in preservers of the determinant of Kronecker product in $GL_{mn}(\mathbb{F})$ in terms of linear preservers of the Kronecker sum.

In what follows, $m, n$ are positive integers. For a positive integer $k$, $I_k$ denotes the $k \times k$ identity matrix, $0_k$ the $k \times k$ zero matrix, and $E_{ij}^{(k)}$, $1 \leq i, j \leq k$, the $k \times k$ matrix whose entries are all equal to zero except for the $(i, j)$-th entry which is equal to one. As usual, the symbol $\delta_{ij}$ denotes the Kronecker delta, i.e.,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j ; \\ 0, & \text{if } i \neq j . \end{cases}$$

The partial trace plays a central role in our investigation. Let $A \in M_{mn} = M_m(M_n) = M_m \otimes M_n$. Then $A$ can be written as a block matrix $A = (A_{ij})_{ij}$ where $A_{ij} \in M_n$ and $i, j = 1, \ldots, m$. In terms of the Kronecker product we write

$$A = \sum_{i,j=1}^{m} E_{i,j}^{(m)} \otimes A_{ij}.$$ 

The second partial trace ($\text{tr}_2$) maps each $n \times n$ block of $A$ to its trace, i.e., $\text{tr}_2 : (A_{ij})_{ij} \mapsto (\text{tr}(A_{ij}))_{ij}$, or equivalently

$$\text{tr}_2(A) = \sum_{i,j=1}^{m} E_{i,j}^{(m)} \otimes \text{tr}(A_{ij}) = \sum_{i,j=1}^{m} \text{tr}(A_{ij})E_{i,j}^{(m)}.$$ 

Like the trace, the partial trace is a linear operation. Furthermore, the partial trace preserves the trace

$$\text{tr}(A) = \text{tr}(\text{tr}_2(A)).$$

Similarly, we can define the first partial trace ($\text{tr}_1$). This definition provides

$$\text{tr}_1(A) = \sum_{j=1}^{m} A_{jj}.$$ 

Finally, we note that for any $B \in M_n$

$$\text{tr}_1(A(I_m \otimes B)) = \sum_{j=1}^{m} A_{jj}B = \text{tr}_1(A)B$$

and similarly for any $C \in M_m$

$$\text{tr}_2(A(C \otimes I_n)) = \text{tr}_2(A)C.$$ 

The transpose of the matrix $A \in M_n$ will be denoted by $A^T$. First, we define RT-symmetry, which plays a similar role in our analysis similar to that of symmetry in matrix analysis.
2. RT-Symmetry

Noting that $M_n$ is an $n^2$-dimensional space, in general we have

$$\phi(A) = \sum_{j,k,u,v=1}^{n} \alpha_{jk;uv} E_{jk}^{(n)} A E_{uv}^{(n)}$$

for some $\alpha_{jk;uv}$ in the underlying field. We define the linear transform $\phi'$ of $\phi$ by

$$\phi'(A) := \sum_{j,k,u,v=1}^{n} \alpha_{uv;jk} E_{jk}^{(n)} A E_{uv}^{(n)} = \sum_{j,k,u,v=1}^{n} \alpha_{jk;uv} E_{jk}^{(n)} A E_{jk}^{(n)}.$$

Clearly, $(\phi')' = \phi$. Let $Φ$ be the matrix representing the linear map $\phi$ in the standard basis, and $Φ'$ be the matrix representing $\phi'$. Since

$$\phi(E_{pq}^{(n)}) = \sum_{j,p,q,v=1}^{n} \alpha_{jp;qv} E_{jp}^{(n)} \otimes E_{qv}^{(n)}, \quad \phi'(E_{pq}^{(n)}) = \sum_{q,k,u,p=1}^{n} \alpha_{qk;up} E_{up}^{(n)} \otimes E_{qk}^{(n)}.$$

it follows that

$$Φ = \sum_{j,p,q,v=1}^{n} \alpha_{jp;qv} E_{jp}^{(n)} \otimes E_{qv}^{(n)}, \quad Φ' = \sum_{q,k,u,p=1}^{n} \alpha_{qk;up} E_{up}^{(n)} \otimes E_{qk}^{(n)}.$$

Let $P$ denote the perfect shuffle (also known as the vec-permutation matrix) on $\mathbb{F}^n \otimes \mathbb{F}^n$ [4], i.e. $P^T(A \otimes B)P = B \otimes A$. Then

$$Φ^T = P^T Φ' P.$$ 

Consequently, $\phi = \phi'$ if and only if $Φ^T = P^T Φ' P$. Equivalently, $\phi = \phi'$ if and only if $R(Φ)^T = R(Φ')$ where $R$ is the rearrangement operator [6]. The rearrangement operator $R$ is linear, and defined by $R(A \otimes B) = (\text{vec } A)(\text{vec } B)^T$ where vec is the vec operator [4].

**Definition 2.1.** A linear map $\phi : M_n \rightarrow M_n$ satisfying $\phi = \phi'$ is said to be **RT-symmetric**. If $\phi = -\phi'$ then $\phi$ is said to be **skew RT-symmetric**.

The following lemma follows immediately from $\phi = \frac{1}{2}(\phi + \phi') + \frac{1}{2}(\phi - \phi')$.

**Lemma 2.2.** If the underlying field has characteristic not equal to 2, then every linear map $\phi : M_n \rightarrow M_n$ is the sum of an RT-symmetric map and a skew RT-symmetric map.

**Definition 2.3.** A linear map $\phi : M_n \rightarrow M_n$, over $\mathbb{C}$, satisfying $\phi = \overline{\phi}^T$ is said to be **RT-Hermitian**. If $\phi = -\overline{\phi}^T$ then $\phi$ is said to be **skew RT-Hermitian**.

3. Linear trace preservers of Kronecker sums

We may write a linear map $\phi : M_{mn} \rightarrow M_{mn}$ in the operator-sum form

$$\phi(M) = \sum_{i=1}^{r} P_i M Q_i,$$

for some matrices $P_i, Q_i$, $i = 1, \ldots, r$, of the appropriate sizes. If $\phi$ preserves the trace of a Kronecker sum $M$, then the cyclic property of the trace yields

$$\text{tr}(M) = \text{tr}(\phi(M)) = \sum_{i=1}^{r} \text{tr}(Q_i P_i M).$$
and so we need only consider preservers of the form
\[ \phi(M) = \sum_{i=1}^{r} Q_i P_i M = PM, \]
where
\[ P := \sum_{i=1}^{r} Q_i P_i \]
and the remaining preservers are all obtained by representations (2) of \( P \). First we consider maps of the form \( \phi(M) = PM, M \in M_{mn} \).

**Theorem 3.1.** Let \( \phi : M_{mn} \to M_{mn} \) be a map given by \( \phi : M \mapsto PM \) for some \( P \in M_{mn} \). Then \( \text{tr}(\phi(A \oplus B)) = \text{tr}(A \oplus B) \) for all \( A \in M_m \) and \( B \in M_n \), if and only if
\[ \text{tr}_1(P) = \text{tr}_1(I_{mn}) \quad \text{and} \quad \text{tr}_2(P) = \text{tr}_2(I_{mn}). \]

**Proof.** First, let us write \( P \) in block matrix form,
\[ P = \sum_{k,l=1}^{m} E_{kl}^{(m)} \otimes P_{kl}. \]
Since \( \phi \) is linear, the map \( \phi \) preserves the trace of Kronecker sums if and only if \( \phi \) preserves traces of Kronecker products of the form \( E_{ij}^{(m)} \otimes I_n \) and of the form \( I_m \otimes E_{kl}^{(n)} \). Thus, we have
\[ n \delta_{ij} = \text{tr}(E_{ij}^{(m)} \otimes I_n) = \text{tr}(\phi(E_{ij}^{(m)} \otimes I_n)) \]
and
\[ \text{tr}(\phi(E_{ij}^{(m)} \otimes I_n)) = \sum_{k,l=1}^{m} \delta_{jk} \delta_{il} \text{tr}(P_{kl}) = \text{tr}(P_{ij}). \]
It follows that
\[ \text{tr}_2(P) = \sum_{k,l=1}^{m} \text{tr}(P_{kl}) E_{kl}^{(m)} = nI_m. \]

For Kronecker products of the form \( I_m \otimes E_{kl}^{(n)} \), we find
\[ m \delta_{kl} = \text{tr}(I_m \otimes E_{kl}^{(n)}) = \text{tr}(\phi(I_m \otimes E_{kl}^{(n)})), \]
where
\[ \text{tr}(\phi(I_m \otimes E_{kl}^{(n)})) = \sum_{i,j=1}^{m} \delta_{ij} \text{tr}(P_{ij} E_{kl}^{(n)}) = \sum_{j=1}^{m} (P_{jj})_k. \]
Consequently,
\[ \text{tr}_1(P) = \sum_{i,j=1}^{m} P_{ij} = mI_n. \]
Conversely, suppose that \( \text{tr}_1(P) = mI_n \) and \( \text{tr}_2(P) = nI_m \). Then
\[ \text{tr}(\phi(A \oplus B)) = \text{tr}(\text{tr}_2(P(A \oplus I_n))) + \text{tr}(\text{tr}_1(P(I_m \otimes B))) \]
\[ = \text{tr}(\text{tr}_2(P)A) + \text{tr}(\text{tr}_1(P)B) = n \text{tr}(A) + m \text{tr}(B) = \text{tr}(A \oplus B). \]
\[ \Box \]
Corollary 3.2. Let \( \phi : M_{mn} \to M_{mn} \) be a map given by \( \phi : M \mapsto PM \) for some \( P \in M_{mn} \), where

\[
P = I_{mn} + \sum_{j=1}^{r} A_j \otimes B_j.
\]

Here, \( r \) is the tensor rank of \( P - I_{mn} \) over \( M_m \otimes M_n \) and \( A_j \in M_m, B_j \in M_n \) for \( j = 1, \ldots, r \). Then \( \text{tr}(A \oplus B) = \text{tr}(A \oplus B) \), if and only if \( \text{tr}(A_j) = \text{tr}(B_j) = 0 \) for \( j = 1, \ldots, r \).

Proof. By theorem 3.1 we need only show that \( \text{tr}_1(P) = mI_n \) and \( \text{tr}_2(P) = nI_m \) if and only if \( \text{tr}(A_j) = \text{tr}(B_j) = 0 \) for \( j = 1, \ldots, r \). The proof of (\( \Leftarrow \)) is immediate. For (\( \Rightarrow \)), suppose \( \text{tr}_1(P) = mI_n \) and \( \text{tr}_2(P) = nI_m \). It follows that

\[
\sum_{j=1}^{r} \text{tr}(A_j)B_j = 0_n, \quad \sum_{j=1}^{r} \text{tr}(B_j)A_j = 0_m.
\]

Since \( r \) is the tensor rank of \( P - I_{mn} \), the set \( \{ B_1, \ldots, B_r \} \) is a linearly independent set and \( \text{tr}(A_j) = 0 \) for \( j = 1, \ldots, r \). Similarly, \( \text{tr}(B_j) = 0 \) for \( j = 1, \ldots, r \). \( \square \)

As a consequence of Theorem 3.1, we have that \( \phi : M \mapsto PM \) satisfies \( \text{tr}(A \oplus B) = \text{tr}(A \oplus B) \) if and only if \( \text{tr}_1(\phi(I_{mn})) = \text{tr}_1(I_{mn}) \) and \( \text{tr}_2(\phi(I_{mn})) = \text{tr}_2(I_{mn}) \).

In general, this statement is true modulo a traceless matrix. We note that any linear map \( \phi : M_{mn} \to M_{mn} \) can be written in the form

\[
\phi(M) = M + \sum_{j=1}^{r} (A_j \otimes C_j)M(B_j \otimes D_j),
\]

where \( A_j, B_j \in M_m \) and \( C_j, D_j \in M_n \) for \( j = 1, \ldots, r \). In the following we will use the commutation operation \( [A, B] = AB - BA \) corresponding to the Lie product of matrices \( A \) and \( B \) of the appropriate sizes.

Lemma 3.3. Let \( \phi : M_{mn} \to M_{mn} \) be a linear map given by

\[
\phi(M) = M + \sum_{j=1}^{r} (A_j \otimes C_j)M(B_j \otimes D_j).
\]

Then \( \text{tr}(A \oplus B) = \text{tr}(A \oplus B) \) for all \( A \in M_m \) and \( B \in M_n \), if and only if

\[
\text{tr}_1(\phi(I_{mn}) - I_{mn}) = \sum_{j=1}^{r} \text{tr}(A_jB_j)[C_j, D_j]
\]

and

\[
\text{tr}_2(\phi(I_{mn}) - I_{mn}) = \sum_{j=1}^{r} \text{tr}(C_jD_j)[A_j, B_j].
\]

Proof. The linear map \( \phi : M_{mn} \to M_{mn} \) can be written in the form

\[
\phi(M) = M + \sum_{j=1}^{r} (A_j \otimes C_j)M(B_j \otimes D_j)
\]

where \( A_j, B_j \in M_m \) and \( C_j, D_j \in M_n \) for \( j = 1, \ldots, r \). Since \( \text{tr}(A \oplus B) = \text{tr}(A \oplus B) \) if and only if \( \text{tr}(A \otimes I_n) = \text{tr}(A \otimes I_n) \) and \( \text{tr}(I_m \otimes B) = \text{tr}(I_m \otimes B) \) for all \( A \in M_m \).
and $B \in M_n$, we consider these two cases separately. In the first case we have

$$\text{tr} \phi(A \otimes I_n) = n \text{tr}(A) + \text{tr} \left( \sum_{j=1}^{r} \text{tr}(C_j D_j) B_j A_j \right) A = n \text{tr}(A)$$

for all $A \in M_m$. This equation holds if and only if

$$0_m = \sum_{j=1}^{r} \text{tr}(C_j D_j) B_j A_j = \sum_{j=1}^{r} \text{tr}(C_j D_j) ([B_j, A_j] + A_j B_j)$$

$$= -Q + \sum_{j=1}^{r} \text{tr}(C_j D_j) A_j B_j$$

$$= -Q + \text{tr}_2 \phi(I_{mn}) - \text{tr}_2 I_{mn}$$

where $[A, B] := AB - BA$ is the commutator and

$$Q := \sum_{j=1}^{r} \text{tr}(C_j D_j) [A_j, B_j] = \text{tr}_2 \phi(I_{mn}) - \text{tr}_2 I_{mn}$$

is traceless (i.e., $\text{tr}(Q) = 0$). Similarly, the second case yields that $\text{tr} \phi(I_m \otimes B) = \text{tr}(I_m \otimes B)$ if and only if

$$\sum_{j=1}^{r} \text{tr}(A_j B_j) [C_j, D_j] = \text{tr}_1 \phi(I_{mn}) - \text{tr}_1 I_{mn}.$$  

The commutators in this lemma highlight the traceless character. However, the anti-commutator plays a similar role. Here, the anti-commutator of matrices $A$ and $B$ is given by $[A, B]_+ := AB + BA$. We state the following lemma without proof, which is almost identical to the previous.

**Lemma 3.4.** Let $\phi : M_{mn} \to M_{mn}$ be a linear map given by

$$\phi(M) = M + \sum_{j=1}^{r} (A_j \otimes C_j) M (B_j \otimes D_j).$$

Then $\text{tr} \phi(A \oplus B) = \text{tr}(A \oplus B)$ for all $A \in M_m$ and $B \in M_n$, if and only if

$$\text{tr}_1(\phi(I_{mn}) - I_{mn}) = \sum_{j=1}^{r} \text{tr}(A_j B_j) [C_j, D_j]_+$$

and

$$\text{tr}_2(\phi(I_{mn}) - I_{mn}) = \sum_{j=1}^{r} \text{tr}(C_j D_j) [A_j, B_j]_+.$$  

Lemma 3.3 shows that the partial traces of the identity matrix must be preserved modulo a traceless matrix. However, this traceless matrix is not arbitrary but precisely defined in terms of $\phi$. The following theorem shows that $\phi'$ plays a fundamental role in the characterization of $\phi$, and provides an succinct characterization for RT-symmetric and skew RT-symmetric maps in the subsequent two corollaries.

**Theorem 3.5.** Let $\phi : M_{mn} \to M_{mn}$ be a linear map. Then $\text{tr} \phi(A \oplus B) = \text{tr}(A \oplus B)$ for all $A \in M_m$ and $B \in M_n$, if and only if

$$\text{tr}_1 \phi'(I_{mn}) = \text{tr}_1(I_{mn}) \quad \text{and} \quad \text{tr}_2 \phi'(I_{mn}) = \text{tr}_2(I_{mn}).$$
Proof. Using the representation of $\phi$ from Lemma 3.3 provides

$$\text{tr}_1(\phi(I_{mn})) = \text{tr}_1 I_{mn} + \sum_{j=1}^{r} \text{tr}(A_jB_j)C_jD_j,$$

$$\text{tr}_1(\phi'(I_{mn})) = \text{tr}_1 I_{mn} + \sum_{j=1}^{r} \text{tr}(A_jB_j)D_jC_j$$

and subtracting these two equations yields

$$\text{tr}_1((\phi - \phi')(I_{mn})) = \sum_{j=1}^{r} \text{tr}(A_jB_j)[C_j,D_j].$$

Similarly,

$$\text{tr}_2((\phi - \phi')(I_{mn})) = \sum_{j=1}^{r} \text{tr}(C_jD_j)[A_j,B_j].$$

From Lemma 3.3 $\text{tr}(A \oplus B) = \text{tr}(A \oplus B)$ for all $A \in M_m$ and $B \in M_n$ if and only if

$$\text{tr}_1(\phi(I_{mn}) - I_{mn}) = \text{tr}_1((\phi - \phi')(I_{mn})) \quad \text{and} \quad \text{tr}_2(\phi(I_{mn}) - I_{mn}) = \text{tr}_2((\phi - \phi')(I_{mn}))$$

if and only if

$$\text{tr}_1(\phi'(I_{mn})) = \text{tr}_1 I_{mn} \quad \text{and} \quad \text{tr}_2(\phi'(I_{mn})) = \text{tr}_2 I_{mn}. \quad \square$$

Corollary 3.6. Let $\phi : M_{mn} \to M_{mn}$ be an RT-symmetric map. Then $\text{tr}(\phi(A \oplus B) = \text{tr}(A \oplus B)$ for all $A \in M_m$ and $B \in M_n$ if and only if

$$\text{tr}_1(\phi(I_{mn})) = \text{tr}_1 I_{mn} \quad \text{and} \quad \text{tr}_2(\phi(I_{mn})) = \text{tr}_2 I_{mn}.$$  

Corollary 3.7. Let $\phi : M_{mn} \to M_{mn}$ be a skew RT-symmetric map. Then $\text{tr}(\phi(A \oplus B) = \text{tr}(A \oplus B)$ for all $A \in M_m$ and $B \in M_n$ if and only if

$$\text{tr}_1(\phi(I_{mn})) = -\text{tr}_1 I_{mn} \quad \text{and} \quad \text{tr}_2(\phi(I_{mn})) = -\text{tr}_2 I_{mn}.$$  

It is straightforward to extend Corollaries 3.6 and 3.7 to the RT-Hermitian and skew RT-Hermitian cases since $\text{tr}_1(\phi'(I_{mn})) = \text{tr}_1 I_{mn}$ if and only if $\text{tr}_1(\phi'(I_{mn})) = \text{tr}_1 I_{mn}.$

Now we are ready to consider the connection with the work in [2]. The connection is provided by the exponential map, i.e.,

$$\det(e^A \otimes e^B) = e^{\text{tr}(A \oplus B)}.$$  

4. Determinant preservers of Kronecker products

The condition given in Lemma 3.3 implies that we may characterize a class of determinant preservers of Kronecker products in terms of partial determinants. However, the relationship between the partial trace and the partial determinant is not straightforward. If we restrict our attention to matrices over the complex numbers, $M_{mn}(C),$ then we have [7]

$$\det(e^A \otimes e^B) = e^{\text{Tr}(A \oplus B)} R_{mn}$$

where $\det(A) := \sqrt[n]{\det(A)} R_n$ and $\text{Tr}(A) := \text{tr}(A)/n$ for $A \in M_n,$ and $R_n$ is the multiplicative group of $n$-th roots of unity in $C.$ Furthermore, [7] showed that

$$\det_1(e^A \otimes e^B) = e^{\text{Tr}_1(A \oplus B)} R_{mn} \quad \text{and} \quad \det_2(e^A \otimes e^B) = e^{\text{Tr}_2(A \oplus B)} R_n.$$
Let $\Omega_{mn} \subset M_{mn}(\mathbb{C})$ denote the set of matrices in $M_{mn}(\mathbb{C})$ with each eigenvalue $\lambda$ satisfying $\text{Im}(\lambda) \in (-\pi, \pi]$. Thus we associate with every non-singular matrix $A$ a unique matrix $M \in \Omega_{mn}$ such that $A = e^M$. Let $\phi : M_{mn}(\mathbb{C}) \to M_{mn}(\mathbb{C})$ be a linear map and let $\psi : GL_{mn}(\mathbb{C}) \to GL_{mn}(\mathbb{C})$ be the non-linear map

$$\psi(e^M) = e^{\phi(M)}.$$  

The map is well defined since $M \in \Omega_{mn}$ is uniquely determined for every matrix in $GL_{mn}(\mathbb{C})$. We have

$$\det \psi(e^M) = \det e^{\phi(M)} = e^{\text{tr} \phi(M)}$$

so that $\det \psi(e^M) = \det(e^M)$ if and only if $e^{\text{tr} \phi(M)} = e^{\text{tr} M}$. By linearity of the trace and $\phi$, this holds if and only if $\text{tr} \phi(M) = \text{tr} M$. Clearly, linear preservers of the trace of Kronecker sums also preserve the Tr of Kronecker sums. Thus, Lemma 3.3 provides the following corollary. We use the same form for $\phi$ as in Lemma 3.3.

**Corollary 4.1.** Let $\phi : \Omega_{mn}(\mathbb{C}) \to M_{mn}(\mathbb{C})$ be a linear map. The map $\psi : GL_{mn}(\mathbb{C}) \to GL_{mn}(\mathbb{C})$ given by

$$\psi(e^M) = e^{\phi(M)}$$

satisfies $\text{Det}(\psi(A \otimes B)) = \text{Det}(A \otimes B)$ if and only if $\text{Det}_1(\psi(I)) = e^{I_n}UR_m$ for $U \in SL_n(\mathbb{C})$ and $\text{Det}_2(\psi(I)) = e^{I_m}VR_n$ for $V \in SL_m(\mathbb{C})$, where

$$U = \exp \left( \frac{1}{m} \sum_{j=1}^{r} \text{tr}(A_jB_j)[C_j,D_j] \right), \quad V = \exp \left( \frac{1}{n} \sum_{j=1}^{r} \text{tr}(C_jD_j)[A_j,B_j] \right).$$

The matrices $U \in SL_n(\mathbb{C})$ and $V \in SL_m(\mathbb{C})$ are not arbitrary. Theorem 3.5 and Corollaries 3.6 and 3.7 provide a stronger condition, which we present as our final theorem.

**Theorem 4.2.** Let $\phi : \Omega_{mn}(\mathbb{C}) \to M_{mn}(\mathbb{C})$ be an RT-symmetric or RT-Hermitian map. The map $\psi : GL_{mn}(\mathbb{C}) \to GL_{mn}(\mathbb{C})$ given by

$$\psi(e^M) = e^{\phi(M)}$$

satisfies $\text{Det}(\psi(A \otimes B)) = \text{Det}(A \otimes B)$ if and only if

$$\text{Det}_1(\psi(e^{I_m})) = e^{I_n}R_m \quad \text{and} \quad \text{Det}_2(\psi(e^{I_m})) = e^{I_m}R_n.$$

**Theorem 4.3.** Let $\phi : \Omega_{mn}(\mathbb{C}) \to M_{mn}(\mathbb{C})$ be a skew RT-symmetric or skew RT-Hermitian map. The map $\psi : GL_{mn}(\mathbb{C}) \to GL_{mn}(\mathbb{C})$ given by

$$\psi(e^M) = e^{\phi(M)}$$

satisfies $\text{Det}(\psi(A \otimes B)) = \text{Det}(A \otimes B)$ if and only if

$$\text{Det}_1(\psi(e^{I_m})) = e^{-I_n}R_m \quad \text{and} \quad \text{Det}_2(\psi(e^{I_m})) = e^{-I_m}R_n.$$

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