Abstract

In this paper, the two-dimensional Burgers’ equations are numerically analyzed by a meshfree numerical scheme, which is a combination of the implicit Euler method, the generalized finite difference method (GFDM) and the fictitious time integration method (FTIM). Since both of the convective and the diffusive terms simultaneously appear in the time-dependent quasi-linear Burgers’ equations, it is necessary and very difficult to develop a reliable numerical scheme to solve it. The GFDM, which can truly get rid of time-consuming mesh generation and numerical quadrature, and the implicit Euler method are used for spatial and temporal discretization, respectively. Then, the resultant system of nonlinear algebraic equations for every time step is resolved by the newly-developed FTIM. Since, in comparing with the Newton’s method, the calculation of the inverse of Jacobian matrix can be avoided in the FTIM, to adopt the FTIM for solving the system of nonlinear algebraic equations is very efficient and has great potential for large-scale engineering problems. Some numerical results and comparisons are provided to validate the accuracy and the simplicity of the proposed meshfree scheme.

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Keywords: Burgers’ equations; implicit Euler method; generalized finite difference method; fictitious time integration method.
1. Introduction

The Burgers’ equations are similar with the momentum equations of the well-known Navier-Stokes equations, which are the governing equations for viscous incompressible fluid flow. The Burgers’ equations are simpler than the Navier-Stokes equations, so many researches [1-2] paid attentions on accurately solving the Burgers’ equations to know the characteristic of both systems. In this paper, a meshfree numerical scheme, which is the combination of the implicit Euler method, the generalized finite difference method (GFDM) and the fictitious time integration method (FTIM), is adopted to accurately and efficiently analyze the two-dimensional quasi-linear Burgers’ equations.

Since the Burgers’ equations are a system of time- and space-dependent partial differential equations, the implicit Euler method and the GFDM are adopted for temporal and spatial discretizations, respectively. During the past decades, there are many promising meshfree methods proposed, such as the method of fundamental solutions (MFS) [3], the collocation Trefftz method (CTM) [4], the local RBF (radial basis function) collocation method [5-6], the GFDM [7-10], etc. In comparing with the MFS and the CTM, the concept of star in the GFDM can form the sparse matrix system, such that the GFDM has great potential to be extended to various engineering problems without knowing any fundamental solution or any set of T-complete functions. The GFDM, which is truly free from mesh generation and numerical quadrature, is evolved from classical finite difference method. Based on the moving-least-squares method, the expressions of the derivatives with respect to space coordinates can be acquired and are the linear combinations of nearest nodes with different weightings.

After the GFDM and the implicit Euler method are used for discretizations of the Burgers’ equations, a system of nonlinear algebraic equations will be formed for every time step. When a system of nonlinear algebraic equations is considered, the well-known and the most popular solver is the Newton’s method. Although the Newton’s method can quickly solve the system of nonlinear algebraic equations, the calculations of the inverse of Jacobian matrix will hugely increase the computational time and cost a lot of computer resource for large system. Besides, the Newton’s method is very sensitive to the initial guesses under some circumstances.

In order to efficiently analyze the system of nonlinear algebraic equations, the FTIM is proposed by Liu and Atluri [11] in 2008 by introducing a pseudo-time variable. The system of nonlinear algebraic equations will be transformed to a system of ordinary differential equations. Then, the explicit Euler method is used for numerically integrating the system of ordinary differential equations. In the FTIM, the calculation of the inverse of Jacobian matrix can be avoided, such that the computational efficiency can be greatly improved, especially for large-scale problems. In some numerical tests in [11], the FTIM outperforms the Newton’s method. Therefore, the FTIM is adopted in this paper to efficiently resolve the system of nonlinear algebraic equations for every time step.

2. Burger’s equations

The two-dimensional time- and space-dependent quasi-linear Burgers’ equations, which are considered in this paper, are shown as follows:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{R} \nabla^2 u
\]
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \frac{1}{R} \nabla^2 v
\] 

(2)

where \( u \) and \( v \) are \( x \)- and \( y \)-directional velocity components. \( \nabla^2 \) is the Laplacian operator. \( R \) is a non-dimensional parameter and is similar to the Reynolds number in the well-known Navier-Stokes equations. Furthermore, \( R \) can be seemed as the ratio between the magnitudes of convective term and the diffusive term, such that the Burgers’ equations could be either parabolic or hyperbolic. With suitable initial and boundary conditions, the above-mentioned Burger’s equations can be efficiently analyzed by the proposed meshfree numerical scheme in this paper.

3. Numerical methods

3.1 Implicit Euler method

Since the Burgers’ equations are a system of time-dependent partial differential equations, it is very important to adopt a stable and accurate numerical method for temporal discretization. In comparing with the explicit Euler method, to adopt the implicit Euler method can release strict limitation for the size of time step and have better accuracy of solutions,

\[
\frac{u^{n+1} - u^n}{\Delta t} + u^n \frac{\partial u^{n+1}}{\partial x} + v^n \frac{\partial u^{n+1}}{\partial y} = \frac{1}{R} \nabla^2 u^{n+1}
\] 

(3)

\[
\frac{v^{n+1} - v^n}{\Delta t} + u^n \frac{\partial v^{n+1}}{\partial x} + v^n \frac{\partial v^{n+1}}{\partial y} = \frac{1}{R} \nabla^2 v^{n+1}
\] 

(4)

where \( \Delta t \) is the time increment. The superscript \( n \) and \( n+1 \) denote the \( n^{th} \) and the \( (n+1)^{th} \) time steps. Since the implicit Euler method is used for temporal discretization of the Burgers’ equations, there are only the derivatives for the space coordinates in Eqs.(3)-(4). Then, the GFDM will be adopted for the spatial discretization of the above equations.

3.2 Generalized finite difference method

The idea of the GFDM is based on the moving-least-squares method to approximate the derivatives at every node in computational domain by linear summation of nodal values. For a given \( i^{th} \) node which is centered at the star, the \( n_i \) nearest nodes around the \( i^{th} \) node will be found. Since the \( i^{th} \) node and the \( n_i \) nearest nodes are used to form a star, the shape of the star in this paper is circular.

When the star for the \( i^{th} \) node is formed, the Taylor series is adopted to expand the function in the star and definite a new functional, \( B(u) \). Let \( u_i \) be the function value at the central node of the star with coordinates \( (x_i, y_i) \) and \( u_j \) \( (j = 1, 2, 3, \ldots, n_i) \) denote the values of the rest nodes with coordinates \( (x_j', y_j') \). Defining a functional as [7-10]:

\[
B(u) = \sum_{j=1}^{n_i} \left[ u_i - u_j + h_{ij} \frac{\partial u_i}{\partial x} + k_{ij} \frac{\partial u_i}{\partial y} + \frac{1}{2} \left( h_{ij}^2 \frac{\partial^2 u_i}{\partial x^2} + k_{ij}^2 \frac{\partial^2 u_i}{\partial y^2} + 2h_{ij}k_{ij} \frac{\partial^2 u_i}{\partial x \partial y} \right) \right] w(h_i, k_i)
\] 

(5)

where \( j \) is the local index in the star. \( h_{ij} = x_i - x_j \) and \( k_{ij} = y_i - y_j \). \( w(h_i, k_i) \) is the weighting function at \( (x_j', y_j') \).

After some mathematical derivations [7-10], the explicit expressions for the partial derivatives at the central node by linear combination of \( n_i \) approximate values at the nodes of star can be acquired.

\[
\frac{\partial u}{\partial x} = w_{x_i} u_i + \sum_{j=1}^{n_i} w_{x_j} u_j
\] 

(6)

\[
\frac{\partial u}{\partial y} = w_{y_i} u_i + \sum_{j=1}^{n_i} w_{y_j} u_j
\] 

(7)
\[
\frac{\partial^2 u}{\partial x^2} = w_{xx}u_i + \sum_{j=1}^{n} w_{xxj}u_j
\]  
(8)

\[
\frac{\partial^2 u}{\partial y^2} = w_{yy}u_i + \sum_{j=1}^{n} w_{yyj}u_j
\]  
(9)

\[
\frac{\partial^2 u}{\partial x \partial y} = w_{xy}u_i + \sum_{j=1}^{n} w_{xyj}u_j
\]  
(10)

where \( \{w_{x_i}\}_{j=0}^{n}, \{w_{y_j}\}_{j=0}^{n}, \{w_{xx_j}\}_{j=0}^{n}, \{w_{yy_j}\}_{j=0}^{n}, \{w_{xy_j}\}_{j=0}^{n} \) are weighting coefficients corresponding to the \( i^{th} \) node and are obtained via the above numerical procedures. This procedure should be implemented at every node inside the computational domain, and the linear combination of the approximate nodal values will be used to replace the derivatives at every node. A system of linear algebraic equations will be formed by enforcing the satisfactions of the boundary conditions at boundary nodes and the governing equations at interior nodes. Finally, the numerical solutions can be obtained by solving the linear matrix system.

3.3 Fictitious time integration method

The FTIM, which is proposed by Liu and Atluri [11] in 2008, is a newly-developed solver for nonlinear algebraic equations. The considered system of nonlinear algebraic equations can be shown as,

\[
F(x) = 0
\]  
(11)

where \( F \in \mathbb{R}^m \) and \( x \in \mathbb{R}^m \), \( m \) is the number of equations and is also the number of unknowns. By introducing a time-like variable, \( \tau \), the system of nonlinear algebraic equations can be converted to a system of ordinary differential equations,

\[
\frac{dx}{d\tau} = F(x).
\]  
(12)

For simplicity, the explicit Euler method is used to numerically integrate the above system of ordinary differential equations and obtain the convergent solutions. From previous study [10-11], the FTIM outperforms the well-known Newton’s method, especially for large-scale problems. In this paper, we used the FTIM to numerically investigate the system of nonlinear algebraic equations, acquired by the GFDM and the implicit Euler method.

4. Numerical results and comparisons

4.1 Example 1

In order to examine the accuracy and the efficiency of the proposed meshfree scheme, the computational domain in this example is a unit square and the following analytical solutions [1] are used,

\[
u(x,y,t) = \frac{3}{4} - \frac{1}{4\left[1 + \exp\left((R/32)(-4x + 4y - t)\right)\right]} 
\]  
(13)

\[
u(x,y,t) = \frac{3}{4} + \frac{1}{4\left[1 + \exp\left((R/32)(-4x + 4y - t)\right)\right]} 
\]  
(14)

The initial and boundary conditions are directly derived from the above analytical solutions. In this example, 441 nodes are uniformly distributed along the boundary and inside the computational domain. \( R=100, n_s=12 \) and \( d\tau = 0.0005 \) are adopted in these numerical experiments.
Fig. 1. Distributions of numerical solutions for $u$ at (a) $t=0.01$, (b) $t=0.5$ and (c) $t=2$.

Fig. 2. Distributions of numerical solutions for $v$ at (a) $t=0.01$, (b) $t=0.5$ and (c) $t=2$.

The distributions of x- and y-directional velocity components, $u$ and $v$, at different time are demonstrated in figures 1 and 2. From these two figures, it is obvious that a sharp font appears in both of solutions for $u$ and $v$. In the meantime, the sharp fronts in $u$ and $v$ will move toward the same direction. The figures for the analytical solutions are omitted since these numerical solutions in figures 1 and 2 are almost identical to the analytical solutions.

Fig. 3. Distributions of absolute errors for $u$ at (a) $t=0.01$, (b) $t=0.5$ and (c) $t=2$.

Fig. 4. Distributions of absolute errors for $v$ at (a) $t=0.01$, (b) $t=0.5$ and (c) $t=2$.

In addition, the distributions of maximum absolute errors for $u$ and $v$ at $t=0.01$, $t=0.5$ and $t=2$ are depicted in figures 3 and 4. It is very surprising that the numerical errors are very small even by using such few nodes. Besides, the errors will appear at the sharp fronts and move with the fronts. In order to examine the efficiency and the stability of the proposed meshfree numerical scheme, the evolutionary profiles of maximum absolute errors by using
different sizes of time step for numerically integrating in the FTIM are revealed in figure 5. It can be found that it is very stable for the FTIM by using different stepsizes for the time-like variable. From the numerical results and the comparisons, the accuracy, the simplicity and the efficiency of the proposed meshfree numerical scheme is validated.

![Profiles of maximum absolute errors](image)

Fig. 5. Profiles of maximum absolute errors for (a)\(u\) and (b)\(v\) by using different \(W'\) for numerically integrating at the FTIM.

5. Conclusions

In this paper, the time-dependent quasi-linear two-dimensional Burgers’ equations are numerically investigated by a proposed meshfree numerical, which is a combination of the implicit Euler method, the GFDM and the FTIM. The GFDM, which is truly free from mesh generation and numerical quadrature, and the implicit Euler method, which can release strict limitation of size of time step, are responsible for the spatial and the temporal discretizations, respectively, while the FTIM is used for efficiently solving the system of nonlinear algebraic equations at every time step. A numerical example is provided in this paper, and the distributions of solutions and absolute errors are shown to validate the accuracy of the proposed method. Besides, solutions by using different stepsizes in the FTIM are given to demonstrate the stability of the proposed method. Therefore, the simplicity, the accuracy and the stability of the proposed meshfree numerical scheme, are verified. More numerical examples will be used in the future for further testing the ability of the proposed meshfree scheme for solving the Burgers’ equations.

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