\( \eta \)-Exponents in the one dimensional antiferromagnetic Heisenberg model with next to nearest neighbour coupling

C. Gerhardt, A. Fledderjohann, E. Aysal, K.-H. Mütter*
Physics Department, University of Wuppertal, 42097 Wuppertal, Germany

J.F. Audet, H. Kroger
Department of Physics, Université Laval, Quebec, Canada
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We investigate the critical exponents \( \eta_3(\alpha, M) \), \( \eta_1(\alpha, M) \) associated with the singularities in the longitudinal and transverse structure factors of the one dimensional antiferromagnetic Heisenberg model with nearest \((J_1)\) and next to nearest \((J_2)\) neighbour coupling of relative strength \( \alpha = \frac{J_2}{J_1} \) and an external field \( B \) with magnetization \( M(B) \).

I. INTRODUCTION

In this paper we continue the investigation of the one dimensional spin \( \frac{1}{2} \) Heisenberg model

\[
H = 2 \sum_{x=1}^{N} (\tilde{S}(x)\tilde{S}(x+1) + \alpha \tilde{S}(x)\tilde{S}(x+2)) + 2B \sum_{x=1}^{N} S_3(x) \tag{1.1}
\]

with next to nearest neighbour coupling parameter \( \alpha \) and external field \( B \).

Let us briefly summarize those results [1] - [9] relevant for our later investigation.

In the absence of an external field \((B = 0)\), the ground state of the model is a singlet \((S = 0)\) state with momentum \( p_0 \), where \( p_0 = 0 \) for \( N = 4, 8, 12, \ldots \) and \( p_0 = \pi \) for \( N = 6, 10, 14, \ldots \). This statement holds at least in the interval \(-\frac{1}{2} < \alpha < \frac{1}{2}\). At \( \alpha = \frac{1}{2} \) the model reduces to the Majumdar Ghosh model with degenerate dimer ground states.

The quantum numbers of the first excited state change with \( \alpha \): There is a triplet \((S = 1)\) state for \( 0 < \alpha < \alpha_c = 0.241 \ldots \) and a singlet \((S = 0)\) state for \( \alpha_c < \alpha < \frac{1}{2} \). The momentum of the first excited state is \( p_1 = p_0 + \pi \). Moreover there is no gap in the ‘spinfluid’ phase and a gap in the ‘dimer’ phase \( \alpha > \alpha_c \). The structure of these two phases can be exploited by means of the static and dynamical correlation functions of appropriate operators. In the spinfluid phase \( 0 < \alpha < \alpha_c \) the \( \Delta S = 1 \) operator

\[
S_3(p) = \frac{1}{\sqrt{N}} \sum_x e^{ipx} S_3(x) \tag{1.2}
\]

generates the transition from the singlet to the triplet excited state. To examine the dimer phase \( \alpha > \alpha_c \) we need a \( \Delta S = 0 \) operator. These transitions are generated by the dimer operator

\[
D(p) = \frac{1}{\sqrt{N}} \sum_x e^{ipx} (\tilde{S}(x)\tilde{S}(x+1) - \langle \tilde{S}(x)\tilde{S}(x+1) \rangle) \tag{1.3}
\]

The corresponding static structure factors

\[
S_3(\alpha, p, N) = \langle S_3^+(p)S_3(p) \rangle \quad \text{and} \quad D(\alpha, p, N) = \langle D^+(p)D(p) \rangle \tag{1.4}
\]

behave as follows for \( N \to \infty \):

\( S_3(\alpha, p = \pi, N) \) diverges logarithmically for \( \alpha \leq \alpha_c \) but stays finite for \( \alpha > \alpha_c \).

\( D(\alpha, p = \pi, N) \) diverges with a power depending on \( \alpha \) for \( \alpha > \alpha_c \) and stays finite for \( \alpha < \alpha_c \). The power behaviour degenerates to a logarithmic behaviour for \( \alpha = \alpha_c \).

In the presence of an external field \( B \) the ground state of the model has total spin \( S = M \cdot N \), where \( M = M(\alpha, B) \)

* e-mail:muetter@wpts0.physik.uni-wuppertal.de
is the magnetization. The behaviour of the magnetization curve $M(\alpha, B)$ near saturation $B \to B_s$, $M \to \frac{1}{2}$ changes with $\alpha$.

At $\alpha = 0$ it is known to develop a square root singularity:

$$M(\alpha = 0, B) \to \frac{1}{2} - \frac{1}{\pi} (B_s - B)^{1/2}, \text{ for } B \to B_s,$$

(1.5)

whereas the numerical data for $\alpha = \frac{1}{4}(N \leq 28)$ support a quartic root singularity for $\alpha = \frac{1}{4}$.

$$M(\alpha = \frac{1}{4}, B) \to \frac{1}{2} - \frac{1}{2\epsilon_4^2}(B_s - B)^{1/4} \quad \epsilon_4 = 1.70(5), \text{ for } B \to B_s.$$

(1.6)

The gap in the dimer phase $\alpha > \alpha_c$ appears in the low field behaviour of the magnetization curve:

$$M(\alpha, B) \to 0, \quad B < B_c(\alpha), \quad \alpha > \alpha_c.$$

(1.7)

Note that the model with $B > B_c(\alpha)$ is gapless, provided that there are no 'plateaus' ($M(\alpha, B) = \text{const}, B_{1c} \leq B \leq B_{2c}$) in the magnetization curve.

The momentum of the ground state $p_s$ follows Marshall’s sign rule:

$$p_s = 0 \text{ for } 2S + N = 4n, \quad p_s = \pi \text{ for } 2S + N = 4n + 2$$

(1.8)

for $0 < \alpha < \frac{1}{4}$.

The singularities in the static structure factors change, if we switch on an external field: The transverse structure factor at $p = \pi, \alpha = 0$

$$S_1(\alpha, p = \pi, M, N) \approx B_1(\alpha, M)N^{1-\eta_1(\alpha, M)} + A_1(\alpha, M)$$

(1.9)

diverges with a field dependent critical exponent $\eta_1(\alpha, M)$. $\eta_1(\alpha = 0, M)$ has been calculated in Ref. by means of the Bethe Ansatz. A second, weaker singularity, which moves with the external field appears at the softmode momentum $p = p_1(M) = 2\pi M$. To our knowledge, the positions of both singularities do not depend on $\alpha$. The critical exponent $\eta_1(M, \alpha)$ however does. It has been found to be:

$$\eta_1(\alpha = 0, M = \frac{1}{4}) = 0.65 \quad \text{and} \quad \eta_1(\alpha = \frac{1}{4}, M = \frac{1}{4}) = 1.16$$

(1.10)

The longitudinal structure factor at $p = \pi (M > 0)$ stays finite for $\alpha < \frac{1}{4}$ but develops a singularity at the field dependent softmode momentum $p = p_3(M) = \pi(1 - 2M)$:

$$S_3(\alpha, p = p_3(M), M, N) \approx B_3(\alpha, M)N^{1-\eta_3(\alpha, M)} + A_3(\alpha, M)$$

(1.11)

Again, the position of the singularity does not depend on $\alpha$, whereas the critical exponent $\eta_3(\alpha, M)$ changes drastically with $\alpha$:

$$\eta_3(\alpha = 0, M = \frac{1}{4}) = 1.50 \quad \text{and} \quad \eta_3(\alpha = \frac{1}{4}, M = \frac{1}{4}) = 0.84$$

(1.12)

The soft mode singularity at $p = p_3(M)$ is also found in the dimer structure factor $D(\alpha, p, N)$ defined in (1.3) and (1.4). A finite size analysis of the type (1.11) for $D(\alpha, p, N)$ yields for the critical exponents:

$$\eta_D(\alpha = 0, M = \frac{1}{4}) = 1.49 \quad \eta_D(\alpha = \frac{1}{4}, M = \frac{1}{4}) = 0.82$$

(1.13)

These values almost coincide with those of the longitudinal structure factor given in (1.12).

It is the purpose of this paper to determine the complete $\alpha$ dependence of the critical exponents $\eta_1(\alpha, M), \eta_3(\alpha, M)$. The paper is organized as follows:

In section 2 we exploit the range of validity of Marshall’s sign rule (1.8). In section 3 we study the impact of the next to nearest neighbour coupling on the lowlying excitations and on the static structure factors. The latter are computed numerically on systems up to $N = 32$. A finite size analysis of (1.13) and (1.11) yields the critical exponents $\eta_1(\alpha, M), \eta_3(\alpha, M)$. Section 4 is devoted to the study of an unexpected phenomenon, which we found for negative $\alpha$-values: For $\alpha < \alpha_-(M)$ the finite size behaviour of the longitudinal structure factor (1.11) changes systematically from a monotonic increase to a decrease.

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II. THE IMPACT OF FRUSTRATION ON THE GROUND STATES. LEVELCROSSINGS.

It was pointed out in the introduction that the momenta \( p_S(\alpha) \) of the ground state \( |S, S_z = S, p_S(\alpha) > \) in the sectors with total spin \( S \) follow Marshall’s sign rule for \( \alpha \leq \frac{1}{4} \). We found deviations from this rule for

\[
\alpha > \alpha_0(M = \frac{S}{N})
\]  

(2.1)

We computed the ground state energies \( E(\alpha, p_S(\alpha), M = \frac{S}{N}, N) \) on small systems with \( N = 10, \ldots, 20 \) sites. The dependence on the frustration parameter \( \alpha \) is shown in Fig. 1 for \( N = 12 \). Here we have marked the different ground state momenta by different symbols.

At \( M = \frac{S}{N} = 0 \) deviations from Marshall’s sign rule occur first at:

\[
\alpha_0(M = 0) = \frac{1}{2}
\]  

(2.2)

Here we meet the Majumdar Gosh model, which is known to have two degenerate ground states, namely dimer states with momenta \( p = 0 \) and \( p = \pi \), respectively.

A twofold degeneracy - with respect to the ground state momenta \( p_S^{(1)}(\alpha), p_S^{(2)}(\alpha) \) emerges along the whole curve \( \alpha = \alpha_0(M) \), which is plotted for \( N = 10,12,14,16,18,20 \) in Fig. 2.

The first momentum \( p_S^{(1)}(\alpha) \) follows Marshall’s sign rule (1.8). We have looked for an empirical rule for the second momentum \( p_S^{(2)}(\alpha) \) but we did not find such a rule which holds for all momenta and system sizes \( N \).

In the saturating field limit \( M \rightarrow \frac{1}{2} \) the curve \( \alpha_0(M) \) meets the point

\[
\alpha_0(M \rightarrow \frac{1}{2}) = \frac{1}{4}
\]  

(2.3)

Indeed, the eigenvalue problem can be solved analytically for \( S = \frac{N}{2} - 1 \) with the ansatz (1 magnon states)

\[
|p, S = \frac{N}{2} - 1 > = \frac{1}{\sqrt{N}} \sum_{x} e^{ipx} |x >
\]  

(2.4)

where \( |x > \) denotes a spin state with spin \(-\frac{1}{2}\) at site \( x \) and spin \( \frac{1}{2} \) at all other sites. The energy of the 1 magnon state is found to be:

\[
E(\alpha, p, M = \frac{1}{2} - \frac{1}{N}, N) = 2 \cos p + \alpha 2 \cos(2p) + (\frac{N}{2} - 2)(1 + \alpha)
\]  

(2.5)

The ground state energy and its momentum \( p = p_S(\alpha) \) follows by minimizing (2.5) with respect to \( p \). For \( \alpha < \frac{1}{4} \) the ground state momentum is found to be \( p_S^{(1)}(\alpha) = \pi, S = \frac{N}{2} - 1 \) in accord with Marshall’s sign rule. For \( \alpha > \frac{1}{4} \) however, the minimum is found for \( p = p_S^{(2)}(\alpha) \) where

\[
\cos p_S^{(2)}(\alpha) = \frac{1}{4\alpha} \quad S = \frac{N}{2} - 1 \quad N \rightarrow \infty
\]  

(2.6)

On finite lattices, the difference between the two momenta turns out to be:

\[
\Delta p_S(\alpha_0 = \frac{1}{4}) = |p_S^{(1)}(\alpha_0 = \frac{1}{4}) - p_S^{(2)}(\alpha_0 = \frac{1}{4})| = \frac{2\pi}{N} \quad S = \frac{N}{2} - 1
\]  

(2.7)

As a consequence of the levelcrossing at \( \alpha = \alpha_0(M = \frac{S}{N}) \), \( M \) fixed, the derivatives of the ground state energies

\[
\frac{\partial}{\partial \alpha} E(\alpha, p_S(\alpha), M = \frac{S}{N}, N)
\]  

(2.8)

change discontinuously, as can be seen in an amplification of Fig. 1.
III. SOFTMODES IN THE EXCITATION SPECTRUM AND THE ASSOCIATED $\eta$-EXPO-NENTS.

We have studied the finite-size dependence of the energy gaps

$$\omega_{\Delta S}(\alpha, p, M, N) = E(\alpha, p = p_S + p, M) - E(\alpha, p = p_S, M) = \frac{S + \Delta S}{N}$$  \tag{3.1}$$

for $\Delta S = 0$ and $\Delta S = 1$ in the domain $\alpha < \alpha_0(M)$ where the ground state momentum follows Marshall’s sign rule. In this regime the gap $\omega_{\Delta S=1}(\alpha, p = \pi, M, N)$ vanishes in the thermodynamical limit in such a way that the scaled quantity

$$\lim_{N \to \infty} N\omega_{\Delta S=1}(\alpha, p = \pi, M, N) = \Omega_1(\alpha, M) \quad \alpha < \alpha_0(M)$$  \tag{3.2}$$

converges to a finite non-vanishing limit. The same holds for the gap $\omega_{\Delta S=0}(\alpha, p = p_3(M), M, N)$, $p_3(M) = \pi(1-2M)$, if the next to nearest neighbour coupling $\alpha$ is positive:

$$\lim_{N \to \infty} N\omega_{\Delta S=0}(\alpha, p = p_3(M), M, N) = \Omega_3(\alpha, M)$$  \tag{3.3}$$

For negative $\alpha$-values ($\alpha < \alpha_-(M) < 0$) however, we observe a tendency in the numerical data, which at least hints to the emergence of a gap at the momentum $p = p_3(M)$:

$$\lim_{N \to \infty} \omega_{\Delta S=0}(\alpha, p = p_3(M), M, N) = \Delta_3(\alpha, M) \quad \alpha < \alpha_-(M) < 0$$  \tag{3.4}$$

as can be seen from Fig. 3. It is hard to decide from the finite system results ($N = 16, 20, 24$) the exact position $\alpha = \alpha_-(M)$ where the gap (3.4) opens.

In the gapless regimes, where (3.2) and (3.3) is valid, we expect that the critical behaviour of the system is properly described by conformal field theory. This means in particular that the ratios

$$2\theta_a(\alpha, M) = \frac{\Omega_a(\alpha, M)}{\pi v(\alpha, M)}, \quad a = 3, 1 \tag{3.5}$$

can be identified with the critical exponents $\eta_a(\alpha, M)$:

$$2\theta_a(\alpha, M) = \eta_a(\alpha, M), \quad a = 1, 3 \tag{3.6}$$

Here

$$v(\alpha, M) = \frac{1}{2\pi} \lim_{N \to \infty} N(E(\alpha, p = p_S + \frac{2\pi}{N}, M = \frac{S}{N}, N) - E(\alpha, p_S, M = \frac{S}{N}, N))$$  \tag{3.7}$$

is the spinwave velocity. For fixed values of $M$ ($M = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$) we have determined the $\alpha$-dependence of $2\theta_a(\alpha, M), \quad a = 1, 3$, from the energy differences (3.1), (3.3) as they enter in the ratios (3.5). The result can be seen from the solid curves in Figs 4a,b,c. The solid dots represent the determination of the critical exponents $\eta_1(\alpha, M), \eta_3(\alpha, M)$ as they follow from a fit of the form (3.9), (3.11) to the finite system results ($N \leq 32$). Comparing the two determinations we come to the following conclusions:

1. The identity $2\theta_1(\alpha, M) = \eta_1(\alpha, M)$ for the critical exponent in the transverse structure factor is well established for $-0.5 < \alpha < 0.25$.

The same holds for the identity $2\theta_3(\alpha, M) = \eta_3(\alpha, M)$ for the critical exponent in the longitudinal structure factor in the interval $\alpha_-(M) < \alpha < 0.25$. If we approach the curve $\alpha = \alpha_0(M)$ the convergence of the Lanczos algorithm slows down more and more, due to the emergence of the level-crossing discussed in section 2.

2. The two curves $2\theta_1(\alpha, M), 2\theta_3(\alpha, M)$ cross each other at $\alpha = \alpha_c(M)$

$$2\theta_1(\alpha_c(M), M) = 2\theta_3(\alpha_c(M), M) = 2\theta(M)$$  \tag{3.8}$$

where

$$\alpha_c(M) = \frac{1}{6} = 0.18 \quad \alpha_c(M) = \frac{1}{4} = 0.20 \quad \alpha_c(M) = \frac{1}{3} = 0.32$$  \tag{3.9}$$

4
and
\[
2\theta(M = \frac{1}{6}) = 1.01 \quad 2\theta(M = \frac{1}{4}) = 1.02 \quad 2\theta(M = \frac{1}{3}) = 1.02
\] (3.10)

The $\alpha$-values are quite close to the transition point $\alpha_c(M = 0) = 0.241$ from the spinfluid to the dimer phase. The same holds for the critical exponents $\eta(M)$, which deviate only slightly from $\eta(M = 0) = 1$.

3. The relation
\[
4\theta_1(\alpha, M)\theta_3(\alpha, M) = 1
\] (3.11)
appears to be satisfied within a few percent for $0 \leq \alpha \leq \frac{1}{4}$.

4. For negative values of $\alpha$, we observe in the data for $\eta_3(\alpha, M)$ a discontinuous structure (open symbols). Looking at the numerical data, which enter in the determination of $\eta_3(\alpha, M)$ via eq. (1.11), we found a systematic change in the finite size dependence. For $\alpha > \alpha_{-}(M)$
\[
\alpha_{-}(M = \frac{1}{6}) = -0.31 \quad \alpha_{-}(M = \frac{1}{4}) = -0.19 \quad \alpha_{-}(M = \frac{1}{3}) = -0.15
\] (3.12)
the longitudinal structure factor monotonically increases with $N$, whereas it decreases for $\alpha < \alpha_{-}(M)$. In the latter regime we expect the emergence of the gap (3.3).

IV. THE DISAPPEARANCE OF A FIELD DEPENDENT SOFTMODE

The change in the finite size dependence of the gap (3.3, 3.4) and of $S_3(\alpha, p_3(M), M, N)$ provides us with a first hint, that the field dependent softmode at $p = p_3(M) = \pi(1 - 2M)$ might disappear for $\alpha < \alpha_{-}(M)$. In this section, we are looking for further evidence for this hypothesis. In Figs 5a,b we compare the momentum distribution of $S_3(\alpha, p, M = \frac{1}{3}, N)$ for $\alpha = \alpha_{-}(M = \frac{1}{3}) = -0.19$ and $\alpha = -0.4$, respectively. At $\alpha = \alpha_{-}(M = \frac{1}{3}) = -0.19$ (Fig 5a) the momentum distribution is well approximated by two straight lines with different slopes for $p < p_3(M)$ and $p > p_3(M)$, respectively. This discontinuity is more and more washed out, if the next to nearest neighbour coupling decreases further. E.g., at $\alpha = -0.4$ (Fig 5b) the $p$-distribution of the longitudinal structure factor appears to be smooth in the thermodynamic limit. The approach to this limit is indicated by an arrow. $S_3(\alpha, p, M, N), \alpha \leq \alpha_{-}(M)$ is monotonically decreasing with $N$ for $p \leq p_3(M)$ but increasing for $p > p_3(M)$.

A more drastic effect can be seen in the dynamical structure factor:
\[
S_3(\alpha, \omega, p, M, N) = \sum_n \delta(\omega - (E_n - E_\omega))| < n|S_3(p)|s > |^2
\] (4.1)
which we computed by means of the recursion method\[8, 9\] for $M = \frac{1}{3}$ and $N = 28$. The excitation spectrum is plotted in Figs 6 a,b for $\alpha = -0.19$ and $\alpha = -0.4$, respectively. The numbers denote the corresponding relative spectral weight in percentage terms. The curves guide the eye to the excitations with dominant spectral weight. For $\alpha = -0.19$ (Fig 5 a) the spectral weight is distributed over a band of excitation energies which broadens in the vicinity of the momentum $p = p_3(M)$. For $\alpha = -0.4$ (Fig 5b), however, the spectral weight is more concentrated at higher excitation energies. In particular, the lowest excitation at $p = p_3(M)$ has a relative spectral weight less than 10% for $N = 28$.

V. DISCUSSION AND CONCLUSION

In this paper, we studied the impact of a next to nearest neighbour coupling $\alpha$ and an external field $B$ on the zero temperature properties of the one dimensional spin $\frac{1}{2}$ antiferromagnetic Heisenberg model. We found the following features:

1. The momentum of the ground state follows Marshall’s sign rule (1.8) for $\alpha \leq \alpha_0(M)$ where $\alpha_0(0) = \frac{1}{2}$ and $\alpha_0(\frac{1}{2}) = \frac{1}{4}$. The ground state is twofold degenerate with respect to its momentum for $\alpha = \alpha_0(M)$.
2. A study of the finite size dependence (1.9) and (1.11) yields the \( \alpha \)-dependence of the critical exponents \( \eta_1(\alpha, M), \eta_3(\alpha, M) \) associated with the softmode singularities at \( p = \pi \) and \( p = p_3(M) \) in the transverse and longitudinal structure factor, respectively. Good agreement is found with the prediction (3.5), (3.6) of conformal field theory for \( \eta_1(\alpha, M) < \frac{1}{2} < \alpha < \frac{1}{4} \) and for \( \eta_3(\alpha, M) \quad 0 < \alpha < \frac{1}{4} \) (Figs 4a-c). For these \( \alpha \)-values the spectral weight – entering into the definition of the corresponding dynamical structure factors (4.1) – is concentrated around the lower bound of the excitation spectrum. This seems to be a crucial condition in order that the critical behaviour is described correctly by conformal field theory. In the thermodynamical limit the dynamical structure factors \( S_1(\alpha, \omega, p = \pi, M) \) and \( S_3(\alpha, \omega, p = p_3(M), M) \) develop infrared singularities \( \omega^{-(2-\eta_2(\alpha))} \quad a = 1, 3 \), which can clearly be seen in a finite size scaling analysis. Such an analysis was performed in Ref. 14 for \( \alpha = 0 \).

3. Deviations from the relation (3.7) – predicted by conformal field theory – appear in the longitudinal case \( a = 3 \) (Figs 4a-c) for negative values of the next to nearest neighbour coupling and increasing \( M \)-values. This is accompanied by the fact that the spectral weight in (4.1) is distributed over a band of excitation energies, which broadens with decreasing values of \( \alpha \).

4. There are several indications that the field dependent soft mode at \( p = p_3(M) = \pi(1 - 2M) \) disappears for negative next to nearest neighbour couplings \( \alpha < \alpha_-(M) < 0 \): A gap (3.4) opens and the longitudinal structure factor (1.11) changes its finite size dependence from a monotonic increase to a decrease. Moreover the cusp-like singularity in the momentum dependence at \( p = p_3(M) \) is washed out and the spectral weight is shifted from low to higher excitation energies. Therefore, we find a further confirmation of the hypothesis – formulated in Ref. 22 – namely that field dependent soft modes only exist if the system is sufficiently frustrated. As was pointed out in Ref. 22 this condition is not satisfied in the twodimensional spin \( \frac{1}{2} \) antiferromagnetic Heisenberg model with nearest neighbour coupling.

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Figure Captions

FIG. 1. The ground state energies $E(\alpha, p_S, M = \frac{S}{N}, N)$ in the sector with total spin $S$ on a ring with $N = 12$ sites. The ground state momenta $p_S(\alpha)$ change with the next to nearest neighbour coupling $\alpha$ as indicated by the different symbols.

FIG. 2. The curve $\alpha = \alpha_0(M = \frac{S}{N})$ where the ground state $|S, p>$ in the sector with total spin $S$ is degenerate with respect to the momentum $p = p_S^{(1)}(\alpha), p_S^{(2)}(\alpha)$.

FIG. 3. Finite size dependence of the gap $\omega_{\Delta S=1}(\alpha, p_1(M), M, N)$ for $\alpha = 0.1, 0.0, -0.1, -0.2, -0.3, -0.4, -0.5$.

FIG. 4. Comparison of the ratio $2\theta_i(\alpha, M)$ for $i = 1, 3$ (solid curves) and the critical exponents $\eta_i(\alpha, M)$ for $i = 1, 3$ in the static structure factors (1.9)(1.11). a) $M = \frac{1}{6}$ b) $M = \frac{1}{4}$ c) $M = \frac{1}{3}$

FIG. 5. The momentum dependence of the longitudinal structure factor $S_3(\alpha, p, M = \frac{1}{4}, N = 28, 24, 20, \ldots$. a) $\alpha = \alpha_-(M = \frac{1}{4}) = -0.19$ b) $\alpha = -0.40$

FIG. 6. Excitation energies and relative spectral weights in the dynamical structure factor $S_3(\alpha, \omega, p, M = \frac{1}{4}, N = 28)$ in percentage terms. The lines connect the excitations with the dominant spectral weight. a) $\alpha = \alpha_-(M = \frac{1}{4}) = -0.19$ b) $\alpha = -0.4$
The diagram illustrates the energy $E(\alpha, p, S=M/S) = S/N, N=12$ as a function of $\alpha$ for different values of $S$. The energy is shown for $p=0, \pi/3, \pi/2, 2\pi/3, 5\pi/6, \pi$. Each line represents a different value of $S$. For $S=5$, the energy remains relatively constant with $\alpha$, while for lower values of $S$, the energy decreases as $\alpha$ increases. This suggests a trend where the energy is minimized at certain values of $\alpha$ for each $S$. The graph is labeled as Fig. 1.
Fig. 2
Fig. 3

\[ \omega_{\Delta S=0}(\alpha, p_3(M), M=1/4, N) \]

vs.

\[ \frac{1}{N} \]
Fig. 4
Fig. 5

\[ S_3(\alpha, p, M = 1/4, N) \]

\( \alpha = -0.19 \)

\( \alpha = -0.40 \)

\( 0 \leq p/\pi \leq 1 \)
Figure 6

$S_3(\alpha, \omega, p, M=1/4, N=28)$

$\alpha = -0.19$

$\alpha = -0.40$