Continuity in law with respect to the Hurst parameter of the local time of the fractional Brownian motion

Maria Jolis  Noèlia Viles
Departament de Matemàtiques, Universitat Autònoma de Barcelona
08193-Bellaterra, Barcelona, Spain.
E-mail addresses: mjolis@mat.uab.es  nviles@mat.uab.es

Abstract

We give a result of stability in law of the local time of the fractional Brownian motion with respect to small perturbations of the Hurst parameter. Concretely, we prove that the law (in the space of continuous functions) of the local time of the fractional Brownian motion with Hurst parameter $H$ converges weakly to that of the local time of $B^{H_0}$, when $H$ tends to $H_0$.

Keywords: Convergence in law; Fractional Brownian motion; Local time.

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1 Introduction

In this work we will prove the continuity in law with respect to the Hurst parameter of the family of laws of the local times of the fractional Brownian motions.

Recall that a fractional Brownian motion $B^H = \{B^H_t, \ t \geq 0\}$ with Hurst parameter $H$ is a centered Gaussian process whose covariance function is given by

$$R_H(s,t) = E[B^H_s B^H_t] = \frac{1}{2} (s^{2H} + t^{2H} - |t-s|^{2H}).$$

It is an easy exercise to see that, for any $T > 0$, the family of laws of the fractional Brownian motions $\{B^H, \ H \in (0,1)\}$ converges in law in $C([0,T])$ to that of $B^{H_0}$, as $H$ tends to $H_0 \in (0,1)$. In fact, from the equality

$$E(B^H_t - B^H_s)^{2m} = \frac{(2m)!}{2^m m!} |t-s|^{2Hm},$$

one can see, by using Billingsley criterion (see Theorem 12.3 of [4]), that, for any $H_1 \in (0,1)$, the family of laws of $\{B^H, \ H \in [H_1,1)\}$ is tight. On the other hand, it is clear that the covariance function $R_H(s,t)$ converges...
to $R_{H_0}(s,t)$, for any $s, t \in [0,T]$ as $H \to H_0$, and from this we obtain the convergence of the finite dimensional distributions.

It seems interesting to study similar results of convergence for some functionals of the fractional Brownian motion. This kind of results justifies the use of $B^H$ as a model in applications, when the actual value of the Hurst parameter is unknown and $\hat{H}$ is some estimation of it.

Concretely, in this paper we consider, for any $T, D > 0$, the family of laws in $\mathcal{C}([-D,D] \times [0,T])$ of the family $\{L^H, H \in (0,1)\}$, where $L^H = \{L^H_{x,t}, t \geq 0, x \in \mathbb{R}\}$ is the local time of $B^H$. We prove, that this family of laws converges weakly, as $H$ tends to $H_0 \in (0,1)$, to the law of $L^{H_0}$.

We point out that the existence of a continuous version of the local time for the fractional Brownian motion was first proved by Berman (see [2]). In fact, for the proof of tightness we will mainly use the techniques based on Fourier transforms developed by this author, jointly with a study of the correlation of the increments of $B^H$, when $H$ belongs to a neighborhood of $H_0$.

We have organized the paper as follows. In the first section of preliminaries we give the main definitions and results on the existence and continuity of the local time for Gaussian processes with stationary increments. In Section 2 we prove the tightness of the laws of $\{L^H\}$ with $H$ belonging to a certain neighborhood of $H_0$. In Section 3 we prove some general results on the convergence in law of local times and obtain as a corollary the desired convergence of the laws of the local times of the fractional Brownian motions. Finally, we have added an appendix with the proof of a result used in order to assure the existence of the local time as a limit in quadratic mean.

2 Preliminaries

**Definition 2.1.** Given a measurable stochastic process $X = \{X_t, t \in [0,T]\}$, the occupation measure of $X$ until the instant $t \in [0,T]$ is defined as the following finite measure

$$\mu^t(A) = \int_0^t 1_{\{X_s \in A\}} ds, \quad \forall A \in \mathcal{B}(\mathbb{R}).$$

**Definition 2.2.** A local time of the process $X$ will be a two-parameter process $L = \{L^t_x, x \in \mathbb{R}, t \in [0,T]\}$ such that for any $t \in [0,T]$ $\omega$-a.s. $L^t(\omega)$ is a version of the density (with respect to the Lebesgue measure) of the occupation measure $\mu^t$, in the case in which this density exists.
Remark 1. Notice that $L^0$ can be taken identically equal to 0 and that the existence of a density for $\mu^T$ implies the existence of a density for $\mu^t$ for any $t \in [0,T]$.

One of the main properties of the occupation measure is given by the following result, known as occupation formula (see for instance [1]).

**Proposition 2.1.** Let $X = \{X_t, \ t \in [0,T]\}$ be a measurable process. If $g : \mathbb{R} \to \mathbb{C}$ is a Borel measurable function, then

$$\int_{\mathbb{R}} g(u)d\mu^t(u) = \int_{0}^{t} g(X_s)ds,$$

and both integrals are well defined or not at the same time.

We will use the following version of Plancherel's theorem.

**Theorem 2.1.** If the Fourier transform $\phi$ of a measure $\mu$ belongs to $L^2(\mathbb{R})$, then $\mu$ is absolutely continuous and its density is also square integrable. Moreover the density, $f$, is the limit in $L^2(\mathbb{R})$ of

$$f_N(x) = \frac{1}{2\pi} \int_{-N}^{N} e^{-iux} \phi(u)du,$$

when $N \to \infty$.

**Remark 2.** We denote by $\phi^t$ the Fourier transform of the occupation measure $\mu^t$. Then, by using the occupation formula (Proposition 2.1), $\phi^t \in L^2(\mathbb{R})$ almost surely, if and only if

$$\int_{\mathbb{R}} \left| \int_{0}^{t} e^{iuX_s}ds \right|^2du < \infty \ a.s.,$$

or equivalently, if

$$\int_{\mathbb{R}} \left( \int_{0}^{t} e^{iux}du \right)^2 \left( \int_{0}^{t} e^{-iux}X_sdrds \right) < \infty \ a.s.$$

The fact that the limit in (2.1) is in $L^2(\mathbb{R})$ has the inconvenience that, in this case, the density $f(x)$ is defined for all $x$-a.e. By this reason we will need some results that allow to ensure the existence of $L = \{L^t_x, \ x \in \mathbb{R}, \ t \in [0,T]\}$ as a stochastic process. The following result is given in Theorem 4.1 of [1]:

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Theorem 2.2. Suppose that
\[ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{0}^{T} \int_{0}^{T} |E[e^{iuX_s + ivX_t}]| dsdrdudv < +\infty. \]
Define
\[ \psi_N(x,t,\omega) = \frac{1}{2\pi} \int_{-N}^{N} e^{-iux} \int_{0}^{t} e^{iuX_r(\omega)} drdu. \]
Then, for any \((x,t) \in \mathbb{R} \times [0,T]\) there exists a random variable \(L^t_x\) such that
\[ \lim_{N \to \infty} \sup_{(x,t) \in \mathbb{R} \times [0,T]} E|\psi_N(x,t) - L^t_x|^2 = 0. \]

The following theorem allows us to obtain the local time as a limit in quadratic mean. Its proof, that uses Theorem 2.2 and Plancherel’s theorem, is given in the Appendix.

Theorem 2.3. Let \(X = \{X_t, t \in [0,T]\}\) be a measurable stochastic process verifying the following conditions:

(i) \[ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{0}^{T} \int_{0}^{T} |E[e^{iuX_s + ivX_t}]| dsdrdudv < \infty. \]

(ii) For each \(t \in [0,T]\),
\[ \int_{\mathbb{R}} \int_{0}^{t} e^{iu(X_s - X_r)} drd\tau < \infty, \quad \text{a.s.} \]

Consider \(L = \{L^t_x, (x,t) \in \mathbb{R} \times [0,T]\}\) the process defined for any \((x,t)\) as the random variable \(L^t_x\) appearing in Theorem 2.2. Then, this process \(L\) is a local time of \(X\).

The next lemma (see Example 3.2 of [1]) gives a sufficient condition in order that a Gaussian process with stationary increments satisfies hypothesis (ii) of Theorem 2.3.

Lemma 2.1. Let \(X = \{X_t, t \in [0,T]\}\) be a measurable centered Gaussian process null at 0 with stationary increments. Denote by \(\sigma^2(t)\) the variance function of \(X\). If
\[ \int_{0}^{T} \int_{0}^{t} \sigma(t - s)^{-1} ds dt < \infty, \quad (2.2) \]
then, for any \(t \in [0,T]\),
\[ E \left( \int_{\mathbb{R}} \int_{0}^{t} e^{iu(X_s - X_r)} drd\tau \right) < \infty. \]
We will also need the following lemma.

**Lemma 2.2.** *(see Lema 8.1 of [3]) Let $Y_1, \ldots, Y_m$ be non-constant square integrable random variables. The following inequality is satisfied for all $v_1, \ldots, v_m \in \mathbb{R}$:

$$\text{Var} \left( \sum_{i=1}^{m} v_i Y_i \right) \geq \frac{\det \Gamma}{\prod_{i=1}^{m} \Gamma_{ii}} \frac{1}{m} \sum_{i=1}^{m} v_i^2 \Gamma_{ii},$$

where $\Gamma$ is the covariance matrix of $Y_1, \ldots, Y_m$.

We will also use the equality given in the following lemma.

**Lemma 2.3.** For any $a > 0$ and $0 < \alpha < 2$,

$$\int_{\mathbb{R}} |x|^{\alpha} e^{-ax^2} dx = a^{-\frac{\alpha + 1}{2}} \Gamma \left( \frac{\alpha + 1}{2} \right).$$

Given a function $F$, defined on $\mathbb{R}^2$, and $(s, t), (s', t') \in \mathbb{R}^2$ such that $s \leq s'$ and $t \leq t'$, we will denote by $\Delta_{s,t}F(s', t')$ the increment of $F$ over the rectangle $((s, t), (s', t')]$, that is,

$$\Delta_{s,t}F(s', t') = F(s', t') - F(s', t) - F(s, t') + F(s, t).$$

The next theorem is the main result of this section where sufficient conditions are given for a Gaussian process with stationary increments to have a local time possessing a continuous version. This theorem is an adaptation of Theorem 8.1 of [3]. We will give its proof because we will need a precise evaluation of the constants appearing in it.

**Theorem 2.4.** Let $X = \{X_t, t \in [0, T]\}$ be a centered Gaussian measurable process null at 0 with stationary increments such that its variance function $\sigma^2(t)$ is bounded by a constant $C_\sigma$. Suppose that

(i) There exists $m_0$, even natural number, such that for any $m \geq m_0$ even, the determinant of the covariances of the normalized increments

$$\frac{X_{t_j} - X_{t_{j-1}}}{\sigma(t_j - t_{j-1})}, \quad j = 1, \ldots, m,$$

is bounded from below by a constant $A_m > 0$ on the set\n
$$\{(t_1, \ldots, t_m) \in [0, T]^m : 0 = t_0 < t_1 < \cdots < t_m < T\}.$$
There exist \( \delta > 0 \) and \( \alpha > 0 \) such that

\[
\sup_{t \in [0,T]} \int_t^{t+h} [\sigma(s)]^{-(1+2\delta)} ds \leq C_{\alpha,\delta} h^\alpha.
\]  

(2.3)

Then,

a) For each \((x, t)\), there exists the local time \(L_t^x\) as a limit (uniform in \((x, t)\)) in quadratic mean.

b) For any even \(m \geq m_0\), there exists a positive constant \(C_1\) depending on \(m, A_m, \alpha\) and \(\delta\) such that

\[
E|\Delta_{0,t}L(0, t + h)|^m \leq C_1|h|^{\alpha m}.
\]

We can take \(C_1 = C_m A_m^{-m/2} (C_{\alpha,\delta})^m\), with \(C_m\) only depending on \(m\).

c) If \(m \geq m_0\) and even, there exists a positive constant \(C_2\) that depends on \(m, A_m, \delta, \alpha\) and \(\sigma\) such that

\[
E|\Delta_{x,t}L(x + k, t + h)|^m \leq C_2|h|^{\alpha m} |k|^{\delta m}.
\]

We can take

\[
C_2 = C_m \max(1, A_m^{-m/2})(\max(1, C_{\alpha,\delta}^2))^{m}C_{\alpha,\delta}^m,
\]

with \(C_m\) only depending on \(m\).

As a consequence of b) and c), by using Kolmogorov-Chentsov’s criterion, we obtain the existence of a version of the local time of \(X_\cdot\), \(L = \{L_t^x, (x, t) \in \mathbb{R} \times [0, T]\}\) that is jointly continuous in \((x, t)\).

Proof. First of all, we check that the hypotheses of this theorem imply those of Theorem 2.3. This will give (a).

Indeed, it is easy to see that condition (ii) implies inequality (2.2) of Lemma 2.1 and, as a consequence, hypothesis (ii) of Theorem 2.3 is satisfied.

On the other hand, it is not difficult to check that for \(m \geq 2\) even,

\[
\int_0^T \int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[e^{iuX_s + ivX_t}] ds \, dt \, du \, dv
\]

\[
\leq \left( \int_0^T \int_0^T \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} E[e^{\sum_{j=1}^{m} iu_j X_s}] \prod_{j=1}^{m} du_j \prod_{j=1}^{m} ds_j \right)^{2/m}.
\]
This last integral is finite for \( m \geq m_0 \), by the arguments that we will give below. This provides us condition (i) of Theorem 2.3.

Both (b) and (c) are proved in a similar way, we will only give the proof of (c) that is the more complicated one.

Taking into account (a), for \( m \) even, we can express the \( m \)-th moment of the 2-dimensional increment of the local time of \( X \) as

\[
E[\Delta_{x,t}L(x+k,t+h)] = E[\Delta_{x,t}L(x+k,t+h)]^m
\]

\[
= (2\pi)^{-m} E \left( \lim_{N \to \infty} \int_{-N}^{N} \cdots \int_{-N}^{N} \int_{t}^{t+h} \prod_{j=1}^{m} (e^{-iu_j(x+k)} - e^{-iu_jx}) \right.
\]

\[
\times \prod_{j=1}^{m} e^{iu_j X_{s_j}} \prod_{j=1}^{m} ds_j \prod_{j=1}^{m} du_j \left. \right). \]

It can be checked that the above expression can be bounded by

\[
(2\pi)^{-m} \int_{t}^{t+h} \cdots \int_{t}^{t+h} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{m} |e^{-iu_j(x+k)} - e^{-iu_jx}| E[e^{\sum_{j=1}^{m} iu_j X_{s_j}}] \prod_{j=1}^{m} du_j \prod_{j=1}^{m} ds_j.
\]

Using that \(|e^{ix} - e^{iy}| \leq 2|x - y|\), for any \( x, y \in \mathbb{R} \) and all \( \delta \in (0,1) \), we have that

\[
\int_{t}^{t+h} \cdots \int_{t}^{t+h} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{m} |e^{-iu_j(x+k)} - e^{-iu_jx}| E[e^{\sum_{j=1}^{m} iu_j X_{s_j}}] \prod_{j=1}^{m} du_j \prod_{j=1}^{m} ds_j
\]

\[
\leq 2^m |k|^m \delta \int_{t}^{t+h} \cdots \int_{t}^{t+h} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{m} |u_j| E[e^{\sum_{j=1}^{m} iu_j X_{s_j}}] \prod_{j=1}^{m} du_j \prod_{j=1}^{m} ds_j.
\]

Given that the integrand on the last expression is symmetric in \( s_1, \ldots, s_m \), we can change the domain of integration, \([t, t+h]^m\), by its subset

\[
\{(s_1, \ldots, s_m) : t \leq s_1 < \cdots < s_m \leq t + h\}.
\]

Making the following change of variables

\[
\begin{cases}
u_j = v_j - v_{j+1}, & \forall j = 1, \ldots, m - 1, \\
u_m = v_m,
\end{cases}
\]

and defining \( s_0 = 0 \), we have that

\[
\sum_{j=1}^{m} u_j X_{s_j} = \sum_{j=1}^{m} v_j (X_{s_j} - X_{s_{j-1}}).
\]
and this entails that
\[ E[e^{\sum_{j=1}^{m} i u_j X_{s_j}}] = e^{-\frac{1}{2} \text{Var}[\sum_{j=1}^{m} \sigma_j^2 (X_{s_j} - X_{s_{j-1}})]}. \]

By Lemma 2.2, we can majorize this expression as follows
\[ E[e^{\sum_{j=1}^{m} i v_j (X_{s_j} - X_{s_{j-1}})}] \leq \exp \left( -\frac{1}{2} \frac{R}{\prod_{j=1}^{m} \sigma_j^2 (s_j - s_{j-1})} \sum_{j=1}^{m} v_j^2 \sigma_j^2 (s_j - s_{j-1}) \right), \]
where \( R \) is the determinant of the covariance matrix of the increments \( X_{s_j} - X_{s_{j-1}} \) for \( j = 1, \ldots, m - 1 \).

Since the constant \( A_m \) is a lower bound of the determinant of the correlation matrix of the increments of \( X \) that coincides with \( \prod_{j=1}^{m} \sigma_j^2 (s_j - s_{j-1}) \), we have that
\[ E[e^{i \sum_{j=1}^{m} v_j (X_{s_j} - X_{s_{j-1}})}] \leq e^{-B_m \sum_{j=1}^{m} v_j^2 \sigma_j^2 (s_j - s_{j-1})}, \]
with \( B_m = \frac{A_m}{2^m} \).

On the other hand,
\[ \prod_{j=1}^{m} |u_j|^\delta = \left( \prod_{j=1}^{m-1} |v_j - v_{j+1}|^\delta \right) |v_0|^\delta \leq \left( \prod_{j=1}^{m-1} (|v_j|^\delta + |v_{j+1}|^\delta) \right) |v_m|^\delta. \]

This last product equals to the sum of \( 2^{m-1} \) terms, each of them containing at most \( m \) factors \( |v_1|^\delta, \ldots, |v_m|^\delta \) with exponents 0, 1 or 2.

Using this, we obtain
\[
E|\Delta_{x,t} L(x + k, t + h)|^m \\
\leq |k|^{\delta m} \pi^{-m \cdot m!} \sum_{\theta_j \in \{0, 1, 2\}} \int_{\{t \leq s_1 \leq \cdots \leq s_m \leq t + h\} \times \mathbb{R}^m} |v_1^{\theta_1} \cdots v_m^{\theta_m}|^\delta e^{-B_m \sum_{j=1}^{m} v_j^2 \sigma_j^2 (s_j - s_{j-1})} \prod_{j=1}^{m} dv_j \prod_{j=1}^{m} ds_j.
\]

By Fubini’s theorem and Lemma 2.3
\[
E|\Delta_{x,t} L(x + k, t + h)|^m \\
\leq |k|^{\delta m} \pi^{-m \cdot m!} \sum_{\theta_j \in \{0, 1, 2\}} \int \cdots \int_{\{t \leq s_1 \leq \cdots \leq s_m \leq t + h\}} \prod_{j=1}^{m} (B_m \sigma_j^2 (s_j - s_{j-1}))^{-\frac{(\theta_j + 1)}{2}} \\
\times \prod_{j=1}^{m} \Gamma \left( \frac{\theta_j + 1}{2} \right) ds_1 \cdots ds_m. \tag{2.4}
\]
Taking into account that $\max_{r \in [\frac{1}{2}, \frac{3}{2}]} \Gamma(r) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$, we can bound expression (2.4) in the following way

$$E|\Delta_{x,t}L(x + k, t + h)|^m \leq |k|^{m\delta} \pi^{-m/2} m! \max(1, B_m^{-m/2})$$

$$\times \sum_{\theta_j \in \{0, 1, 2\}} \int \cdots \int_{\{t \leq s_1 < \cdots < s_m \leq t + h\}} \prod_{j=1}^{m} [\sigma(s_j - s_{j-1})]^{-(\theta_j\delta + 1)} ds_1 \cdots ds_m.$$ 

$$\leq |k|^{m\delta} \pi^{-m/2} m! \max(1, B_m^{-m/2}) \left(\max(1, C_\sigma^{2\delta})\right)^m$$

$$\times \int \cdots \int_{\{t \leq s_1 < \cdots < s_m \leq t + h\}} \prod_{j=1}^{m} [\sigma(s_j - s_{j-1})]^{-(1+2\delta)} ds_1 \cdots ds_m.$$ 

Finally, using that

$$\int \cdots \int_{\{t \leq s_1 < \cdots < s_m \leq t + h\}} \prod_{j=1}^{m} [\sigma(s_j - s_{j-1})]^{-(1+2\delta)} ds_1 \cdots ds_m$$

$$\leq \left(\int_t^{t+h} \sigma(x)^{-(1+2\delta)} dx\right) \left(\int_0^h \sigma(x)^{-(1+2\delta)} dx\right)^{m-1},$$

denoting by $C_m$ the product of all the constants depending only on $m$ and using (2.3), we have that

$$E|\Delta_{x,t}L(x + k, t + h)|^m \leq C_m \max(1, A_m^{-m/2}) \left(\max(1, C_\sigma^{2\delta})\right)^m (C_{\alpha,\delta})^{m h \max |k|^{\max}}.$$

This ends the proof. \hfill \square

### 3 Existence and continuity of the local time of the fractional Brownian motions. Tightness of the family of their laws

Recall that the fractional Brownian motion of Hurst parameter $H \in (0, 1)$, denoted by $B^H$, is a centered Gaussian process with stationary increments, and taking a continuous version we ensure that it is also a measurable process.

Along this section we will check that the fractional Brownian motion satisfies the conditions of Theorem 2.4. And we will see that all the constants appearing in these conditions can be taken independent of the parameter $H$, at least in a neighborhood of $H_0$, for any $H_0 \in (0, 1)$. 

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First of all, notice that the variance function of the fractional Brownian motion of parameter $H$, $\sigma^2_H(t) = t^{2H}$ is bounded by $C_T = \max(1, T^2)$ for $t \in [0, T]$. Then, the constant $C_\sigma$ appearing in Theorem 2.4 equals to $C_T$.

We state in the next lemma, whose proof is a simple computation, that the variance function $\sigma^2_H$ also satisfies condition (2.3) of Theorem 2.4.

**Lemma 3.1.** Let $H \in (H_0 - \eta, H_0 + \eta) \subset (0, 1)$. Then, for any $\delta > 0$ satisfying $(H_0 + \eta)(1 + 2\delta) < 1$,

$$
\int_t^{t+h} [\sigma(s)]^{-(1+2\delta)} ds \leq C_{T,H_0,\eta,\delta} h^{1-(H_0+\eta)(1+2\delta)},
$$

where

$$
C_{T,H_0,\eta,\delta} = \max(1, T^{2\eta(1+2\delta)}) \left( 1 - (H_0 + \eta)(1 + 2\delta) \right)^{-1 - (H_0 + \eta)(1 + 2\delta)}.
$$

Now, we will prove that for any $m \geq 2$, the determinant of the covariance matrix of the normalized increments of the process is bounded from below by a positive constant $A_H$ over the set $\{(t_1, \ldots, t_m) \in [0, T]^m : 0 = t_0 < t_1 < \cdots < t_m < T\}$. Moreover, we will show that this constant can be taken independently of $H$, at least in a neighborhood of a point $H_0 \in (0, 1)$.

We will need to distinguish the cases $H_0 < \frac{1}{2}$ and $H_0 \geq \frac{1}{2}$. For $H_0 \in (0, \frac{1}{2})$, we will use the following lemma.

**Lemma 3.2.** (see [2]) Let $X = \{X_t, t \in [0, T]\}$ be a Gaussian process with stationary increments and concave variance function. Let $0 \leq t_0 < t_1 < \cdots < t_n$. Then the following inequality is satisfied

$$
\det(E[(X_{t_i} - X_{t_{i-1}})(X_{t_j} - X_{t_{j-1}})])_{1 \leq i,j \leq n} \geq 2^{-3n},
$$

for $n$ positive integer.

Since the variance function of the fractional Brownian motion with parameter $H \in (0, \frac{1}{2})$ is concave, we can apply this lemma and obtain that the constant $A_H$ equals to $2^{-3n}$ (notice that, in fact, for $H = \frac{1}{2}$ we can take $A_H = 1$).

When $H \in (\frac{1}{2}, 1)$, the above result does not apply. In fact, we will need to analyze the behaviour of the elements of the covariance matrix of the normalized increments of $B^H$ in a small neighborhood of $H_0 \in [\frac{1}{2}, 1)$.

The correlation of two disjoint increments of $B^H$ is given by

$$
\text{Corr}(B^H_t - B^H_s, B^H_v - B^H_u) = \frac{1}{2} \frac{(u-t)^{2H} - (u-s)^{2H} - (v-t)^{2H} + (v-s)^{2H}}{(t-s)^H(v-u)^H},
$$

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with $0 \leq s < t < u < v \leq T$. We know that, for $H > \frac{1}{2}$, this correlation is always positive.

If we write

$$v - u = \gamma(t - s),$$
$$u - t = \beta(t - s),$$

we will have

$$\text{Corr}(B^H_t - B^H_s, B^H_v - B^H_u) = \frac{1}{2} \frac{\beta^{2H} - (1 + \beta)^{2H} - (\beta + \gamma)^{2H} + (1 + \beta + \gamma)^{2H}}{\gamma^H}.$$ 

If we consider consecutive increments $B^H_t - B^H_s$ and $B^H_w - B^H_t$, therefore $u = t$ (this entails that $\beta = 0$) and we have

$$\text{Corr}(B^H_t - B^H_s, B^H_w - B^H_t) = \frac{1}{2} \frac{(1 + \gamma)^{2H} - \gamma^{2H} - 1}{\gamma^H}.$$ 

**Lemma 3.3.** Let $H \in (\frac{1}{2}, 1)$. For any $\gamma, \beta$ positive real numbers, the following inequality is satisfied

$$\frac{\beta^{2H} - (1 + \beta)^{2H} - (\beta + \gamma)^{2H} + (1 + \beta + \gamma)^{2H}}{\gamma^H} \leq \frac{(1 + \gamma)^{2H} - \gamma^{2H} - 1}{\gamma^H}.$$ 

**Proof.** The proof is a simple argument of convexity. \hfill \(\square\)

In the following proposition we will prove that condition (i) of Theorem 2.4 is satisfied and that the constant $A^H_m$ can be taken independently of $H$ for $H$ belonging to a neighborhood of $H_0 \geq \frac{1}{2}$.

**Proposition 3.1.** Let $H_0 \in [\frac{1}{2}, 1)$. For any $m \geq 2$ there exist $\eta > 0$ and a constant $A_{m,\eta,H_0} > 0$ depending only on $m$, $\eta$ and $H_0$, such that the determinant of the correlation matrix of the increments $B^H_{t_j} - B^H_{t_{j-1}}$, $j = 1, \ldots, m$, on the set $\{(t_1, \ldots, t_m) \in [0, T]^m : 0 = t_0 < \ldots < t_m < T\}$, is greater or equal than $A_{m,\eta,H_0}$, for any $H \in (H_0 - \eta, H_0 + \eta)$.

**Proof.** The existence, for any $H \in (\frac{1}{2}, 1)$, of a constant $A_{m,H} > 0$ such that it is a lower bound for the determinant of the correlation matrix of the increments of $B^H$ is proved in Theorem 6.2 of [2].

In order to prove the proposition, we will show that in a small enough neighborhood of $H_0$ the determinant of the correlation matrix corresponding to $B^{H_0}$ is near to the corresponding to $B^{H_0}$. 

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Notice that if $H_0 = \frac{1}{2}$, Lemma 3.2 says us that the determinant of the correlation matrix of the increments of $B^H$ is bounded from below by $2^{-3m}$ for $H \in (H_0 - \eta, H_0]$, for any $\eta < \frac{1}{2}$. The arguments that we will use from now on will provide us a neighborhood of the form $(H_0, H_0 + \eta)$ in which the uniform lower bound of the determinant also exists.

Since the determinant of a matrix is a sum of products of its components, and taking into account that, for any $H \in (0, 1)$,

$$\sup_{0 \leq s < t \leq u < v \leq T} |\text{Corr}(B^H_t - B^H_s, B^H_v - B^H_u)| \leq 1,$$

the proof will be concluded if we see that for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any $H \in (H_0 - \rho, H_0 + \rho)$ we have that

$$\sup_{0 \leq s < t \leq u < v \leq T} |\text{Corr}(B^H_t - B^H_s, B^H_v - B^H_u) - \text{Corr}(B^{H_0}_t - B^{H_0}_s, B^{H_0}_v - B^{H_0}_u)| < \varepsilon. \quad (3.1)$$

As we have seen above, we can express the correlation between two disjoint increments $B^H_t - B^H_s$ and $B^H_v - B^H_u$ in terms of the parameters $\beta$ and $\gamma$, with

$$v - u = \gamma(t - s),$$
$$u - t = \beta(t - s),$$

and $0 \leq s < t \leq u < v \leq T$. So, (3.1) is equivalent to

$$\sup_{0 < \gamma < +\infty} \sup_{0 < \beta < +\infty} \left| \frac{\beta^{2H} - (1 + \beta)^{2H} - (\beta + \gamma)^{2H} + (1 + \beta + \gamma)^{2H}}{\gamma^H} \right| < \varepsilon, \quad (3.2)$$

for $|H - H_0|$ small enough.

Now, we will show (3.2). Taking into account the different possible values of the parameters $\gamma$ and $\beta$, we will prove the following assertions.

(i) For any $\varepsilon > 0$ and any $0 < \delta < M, L > 0$ there exists $\rho_1$ (depending on $\varepsilon, M, \delta$ and $L$) such that, for $|H - H_0| < \rho_1$, we have

$$\sup_{\delta \leq \gamma \leq M} \sup_{0 \leq \beta \leq L} \left| \frac{\beta^{2H} - (1 + \beta)^{2H} - (\beta + \gamma)^{2H} + (1 + \beta + \gamma)^{2H}}{\gamma^H} \right| < \varepsilon.$$

(iii) For any $\varepsilon > 0$ and any $0 < \delta < M, L > 0$ there exists $\rho_1$ (depending on $\varepsilon, M, \delta$ and $L$) such that, for $|H - H_0| < \rho_1$, we have

$$\sup_{\delta \leq \gamma \leq M} \sup_{0 \leq \beta \leq L} \left| \frac{\beta^{2H} - (1 + \beta)^{2H} - (\beta + \gamma)^{2H} + (1 + \beta + \gamma)^{2H}}{\gamma^H} \right| < \varepsilon.$$
This is a consequence of the uniform continuity of the function \( f(x, y) = x^y \) on any compact that does not contain the point \((0, 0)\).

(ii) Given \( \varepsilon > 0 \), there exists \( \delta > 0 \) and \( \rho_2 > 0 \) such that for any \( |H - H_0| < \rho_2 \), we have

\[
\sup_{0 \leq \gamma < \delta} \frac{\beta^{2H} - (1 + \beta)^{2H} - (\beta + \gamma)^{2H} + (1 + \beta + \gamma)^{2H}}{\gamma^H} < \varepsilon. \tag{3.4}
\]

Notice that (3.4) gives (3.2) when we take the supremum on \( 0 < \gamma < \delta \), \( 0 \leq \beta < +\infty \), because (3.4) is also valid for \( H = H_0 \).

From Lemma 3.3 we know that

\[
\frac{\beta^{2H} - (1 + \beta)^{2H} - (\beta + \gamma)^{2H} + (1 + \beta + \gamma)^{2H}}{\gamma^H} \leq \frac{(1 + \gamma)^{2H} - \gamma^{2H} - 1}{\gamma^H}, \tag{3.5}
\]

Moreover, the two numerators of the above expressions are positive.

By the Mean Value theorem, we have that

\[
(1 + \gamma)^{2H} - \gamma^{2H} - 1 = 2H((1 + \xi)^{2H-1} - \xi^{2H-1}) \gamma, \quad \xi \in (0, \gamma).
\]

Using this inequality and taking into account that \( 0 < 2H - 1 < 1 \), we can bound the right-hand side of the inequality (3.5) in the following way

\[
\frac{(1 + \gamma)^{2H} - \gamma^{2H} - 1}{\gamma^H} \leq 2 (H_0 + \rho_2) \delta^{H_0 - \rho_2} < \varepsilon, \tag{3.6}
\]

by taking \( \rho_2 \) verifying \( 1 - H_0 - \rho_2 > 0 \) and

\[
0 < \delta < \left( \frac{\varepsilon}{2(H_0 + \rho_2)} \right)^{1/(H_0 - \rho_2)}.
\]

This gives (3.4).

(iii) Given \( \varepsilon > 0 \), there exists \( M > 0 \) and \( \rho_3 > 0 \) such that for \( |H - H_0| < \rho_3 \), we have

\[
\sup_{\gamma > M} \frac{\beta^{2H} - (1 + \beta)^{2H} - (\beta + \gamma)^{2H} + (1 + \beta + \gamma)^{2H}}{\gamma^H} < \varepsilon. \tag{3.7}
\]

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Indeed, using again Lemma 3.3
\[
\sup_{\gamma > M} \frac{\beta^{2H} - (1 + \beta)^{2H} - (\beta + \gamma)^{2H} + (1 + \beta + \gamma)^{2H}}{\gamma^H} \\
\leq \sup_{\gamma > M} \frac{(1 + \gamma)^{2H} - \gamma^{2H} - 1}{\gamma^H} \\
= \sup_{0 < y < \frac{1}{L}} \frac{\left(1 + \frac{1}{y}\right)^{2H} - \left(\frac{1}{y}\right)^{2H} - 1}{y^H} = \sup_{0 < y < \frac{1}{L}} \frac{(1 + y)^{2H} - (1 + y^{2H})}{y^H}.
\]
So, (3.7) is a consequence of (3.6).

(iv) Finally, given \(\varepsilon > 0\) and \(0 < \delta < M\), there exists \(L > 0\) and \(\rho_4 > 0\) such that for
\[
|H - H_0| < L < \rho_4, \quad |H| < \rho_4, \quad |H| < \rho_4 - 1
\]
Indeed, on one hand \(\gamma\) belongs to the compact, \([\delta, M]\), far away from 0.
So, it suffices to study the numerator of the above expression. Applying twice the Mean Value Theorem we obtain
\[
0 \leq (1 + \beta + \gamma)^{2H} - (\beta + \gamma)^{2H} - ((1 + \beta)^{2H} - \beta^{2H}) = 2H(\xi^{2H-1} - \eta^{2H-1}) \\
= 2H(2H - 1)v^{2H-2}(\xi - \eta),
\]
where \(\xi \in (\beta + \gamma, \beta + \gamma + 1), \eta \in (\beta, \beta + 1), v \in (\eta, \xi)\), from which we deduce that \(\xi \geq \eta\).

Taking into account that \(\xi \in (\beta + \gamma, \beta + \gamma + 1), \eta \in (\beta, \beta + 1)\) and \(\delta \leq \gamma \leq M\), we have that \(0 \leq \xi - \eta < M + 1\).
On the other hand, since \(L < v < \beta + M + 1\) and \(2H - 2 < 0\), we obtain
\[
(1 + \beta + \gamma)^{2H} - (\beta + \gamma)^{2H} - ((1 + \beta)^{2H} - \beta^{2H}) \leq 2H(2H - 1) \left(\frac{1}{L}\right)^{2-2H} (M+1).
\]
Finally, since we can take \(L > 1\), we obtain
\[
(1 + \beta + \gamma)^{2H} - (\beta + \gamma)^{2H} - ((1 + \beta)^{2H} - \beta^{2H}) \leq 2(H_0 + \rho_4)(2(H_0 + \rho_4) - 1) \\
\times \left(\frac{1}{L}\right)^{2-2(H_0+\rho_4)} (M + 1) < \varepsilon,
\]
if $|H - H_0| < \rho_4$, with $\rho_4 > 0$ satisfying $2 - 2(H_0 + \rho_4) > 0$ (or equivalently, $0 < \rho_4 < 1 - H_0$) and $L$ big enough.

This finishes the proof of (3.2). \hfill \Box

As a consequence of the previous results of this section and Theorem 2.4, we can state the following proposition.

**Proposition 3.2.** Let $H_0 \in (0,1)$. Then, there exists $\eta > 0$ such that the family $\{B^H, H \in (H_0 - \eta, H_0 + \eta)\}$ satisfies

(i) For any $(x,t)$ and each $H \in (H_0 - \eta, H_0 + \eta)$, there exists the local time $L_{x}^{H}$ as a limit (uniform in $(x,t)$) in quadratic mean.

(ii) There exist positive constants $C_1$ (depending on $m, \eta$ and $H_0$) and $\alpha$ (depending on $\eta$ and $H_0$), such that

$$E|\Delta_{0,t}L^{H}(0,t+h)|^m \leq C_1|h|^{m\alpha}.$$  

(iii) For all $m$ even and $m \geq 2$, there exist $\delta > 0$ and $\alpha > 0$ depending on $H_0$ and $\eta$ and also $C_2 > 0$ depending on $m$, $H_0$ and $\eta$ such that

$$E|\Delta_{x,t}L^{H}(x+k,t+h)|^m \leq C_2|h|^{m\alpha}|k|^{m\delta}.$$  

As a consequence of (ii) and (iii), by using the tightness criterion of [5], we obtain the tightness of the laws of $\{L^{H}, H \in (H_0-\eta, H_0+\eta)\}$ in $C([-D, D] \times [0,T])$, for any $D > 0$ and $T > 0$.

### 4 Identification of the limit law

The identification of the limit of any weakly converging sequence of laws of local times of fractional Brownian motions will be a consequence of some general results. The first one is inspired by the occupation formula.

**Proposition 4.1.** Let $\{X^n\}_{n \in \mathbb{N}}$ be a family of stochastic processes verifying

(a) $\{X^n\}_{n \in \mathbb{N}}$ converges in law to $X$ in $C([0,T])$, when $n \to +\infty$.

(b) Both the family $\{X^n\}_{n \in \mathbb{N}}$ and $X$ have local times $L^n$ and $L$ respectively, jointly continuous in $x$ and $t$.  

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(c) The family of local times $L^n$ converges in law to a process $Y$ in $\mathcal{C}([-D, D] \times [0, T])$, when $n \to \infty$.

Let $g : \mathbb{R} \times [-D, D] \to \mathbb{R}$ continuous with compact support such that there exist $\alpha \in (0, 1]$ and $C > 0$ for which

$$\sup_{x \in [-D, D]} \left| \frac{g(y, x) - g(z, x)}{y - z} \right|^\alpha < C.$$  \hspace{1cm} (4.1)

Then

$$\int_{\mathbb{R}} g(u, x) Y(u, t) du \overset{\mathcal{L}}{=} \int_0^t g(X_s, x) ds,$$

in $\mathcal{C}([-D, D] \times [0, T])$.

**Proof.** Fix $g$ continuous with compact support and satisfying condition (4.1). Consider the maps $T_g$ and $U_g$, defined in the following way

$$T_g : \mathcal{C}(\mathbb{R} \times [0, T]) \to \mathcal{C}([-D, D] \times [0, T])$$

$$y \mapsto T_g(y)(x, t) = \int_{\mathbb{R}} g(u, x)y(u, t) du,$$

$$U_g : \mathcal{C}([0, T]) \to \mathcal{C}([-D, D] \times [0, T])$$

$$f \mapsto U_g(f)(x, t) = \int_0^t g(f(s), x) ds.$$

It is easily checked that $T_g$ and $U_g$ are continuous maps with respect to the usual topologies.

Proposition 2.1 implies that, for any $(x, t)$ and $n \in \mathbb{N},$

$$\int_{\mathbb{R}} g(u, x)L^{t, n}_u du = \int_0^t g(X^n_s, x) ds.$$  \hspace{1cm} (4.2)

Due to the continuity of $T_g$ and $U_g$ and the convergence in law of the families $\{X^n\}_{n \in \mathbb{N}}$ and $\{L^n\}_{n \in \mathbb{N}}$ to $X$ and $Y$ in the spaces $\mathcal{C}([0, T])$ and $\mathcal{C}([-D, D] \times [0, T])$, respectively, we have

$$\int_{\mathbb{R}} g(u, x)L^{t, n}_x du \to \int_{\mathbb{R}} g(u, x)Y(u, t) du$$

and

$$\int_0^t g(X^n_s, x) ds \to \int_0^t g(X_s, x) ds,$$
in \( C([-D, D] \times [0, T]) \), as \( n \to \infty \).

Taking into account this convergence and using (4.2) we obtain

\[
\int_{\mathbb{R}} g(u, x)Y(u, t)\,du \xrightarrow{\mathcal{L}} \int_{0}^{t} g(X_{s}, x)\,ds.
\]

This concludes the proof. \( \square \)

In the next proposition, we will prove that, under the hypotheses of this last result, the finite dimensional distributions of \( Y \) coincide with those of \( L \).

**Proposition 4.2.** Let \( \{X_{n}\}_{n \in \mathbb{N}} \) be a family of processes satisfying (a), (b) and (c) of the above proposition. Then, for any \( (x_1, t_1), \ldots, (x_k, t_k) \in [-D, D] \times [0, T] \)

\[
(\int_{\mathbb{R}} g_{\varepsilon}(u, x_1)Y(u, t_1)\,du, \ldots, \int_{\mathbb{R}} g_{\varepsilon}(u, x_k)Y(u, t_k)\,du) \xrightarrow{\mathcal{L}} (L_{x_1}^{t_1}, \ldots, L_{x_k}^{t_k}).
\]

**Proof.** Let \( \varphi \in C^1 \) with compact support contained in \([-1, 1]\) and such that \( \int_{\mathbb{R}} \varphi(x)\,dx = 1 \). Define, for any \( \varepsilon > 0 \) and any \( (u, x) \in \mathbb{R} \times [-D, D] \), \( g_{\varepsilon}(u, x) = \frac{1}{\varepsilon} \varphi \left( \frac{u-x}{\varepsilon} \right) \).

By Proposition 4.1 we have that the following random vectors are equal in law in \( \mathbb{R}^k \)

\[
\left( \int_{\mathbb{R}} g_{\varepsilon}(u, x_1)Y(t_1, u)\,du, \ldots, \int_{\mathbb{R}} g_{\varepsilon}(u, x_k)Y(t_k, u)\,du \right) \xrightarrow{\mathcal{L}} \left( \int_{0}^{t_1} g_{\varepsilon}(X_{s}, x_1)\,ds, \ldots, \int_{0}^{t_k} g_{\varepsilon}(X_{s}, x_k)\,ds \right).
\]

Since \( \{g_{\varepsilon}\} \) is an approximation of the identity, for any fixed \( x \in \mathbb{R} \), \( t \in [0, T] \) and any \( \omega \in \Omega \) we have that

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}} g_{\varepsilon}(u, x)Y(u, t)\,du = Y(x, t),
\]

and this implies that

\[
\left( \int_{\mathbb{R}} g_{\varepsilon}(u, x_1)Y(t_1, u)\,du, \ldots, \int_{\mathbb{R}} g_{\varepsilon}(u, x_k)Y(t_k, u)\,du \right) \xrightarrow{\mathcal{L}} (Y(x_1, t_1), \ldots, Y(x_k, t_k)),
\]

as \( \varepsilon \) tends to 0.

Using again that \( g_{\varepsilon} \) is an approximation of the identity we obtain

\[
\left( \int_{\mathbb{R}} g_{\varepsilon}(u, x_1)L_{u}^{t_1}\,du, \ldots, \int_{\mathbb{R}} g_{\varepsilon}(u, x_k)L_{u}^{t_k}\,du \right) \xrightarrow{\mathcal{L}} (L_{x_1}^{t_1}, \ldots, L_{x_k}^{t_k}),
\]

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or equivalently, by using Proposition 2.1
\[
\left( \int_0^t g_\varepsilon(X_s, x_1) ds, \ldots, \int_0^t g_\varepsilon(X_s, x_k) ds \right) \xrightarrow{L} \left( L_{x_1}^{H_1}, \ldots, L_{x_k}^{H_k} \right),
\]
when \( \varepsilon \to 0 \).

From this, we conclude
\[
(Y(x_1, t_1), \ldots, Y(x_k, t_k)) \overset{L}{=} (L_{x_1}^{H_1}, \ldots, L_{x_k}^{H_k}).
\]

The above general result can be applied to the fractional Brownian motions. Using also the results of the preceding sections we obtain the desired convergence in law of the family of local times.

**Corollary 4.1.** Given \( H_0 \in (0, 1) \), the family \( \{L^H \}_{H \in (0,1)} \) of local times of the fractional Brownian motions converges in law to the local time \( L^{H_0} \) of \( B^{H_0} \) in \( C([-D, D] \times [0, T]) \), for any \( D, T > 0 \), when \( H \) tends to \( H_0 \).

**Proof.** By Proposition 3.2 we have the tightness of the laws of the family \( \{L^H \}_{H \in (H_0-\eta, H_0+\eta)} \), for certain \( \eta > 0 \), in \( C([-D, D] \times [0, T]) \).

Take a sequence \( \{H_n\}_{n \in \mathbb{N}} \) converging to \( H_0 \) as \( n \to \infty \) such that
\[
L_{x_1}^{H_{n}} \xrightarrow{L} Y(x, t), \tag{4.3}
\]
in \( C([-D, D] \times [0, T]) \), as \( n \to \infty \).

Proposition 4.2 gives that for any fixed \( (x_1, t_1), \ldots, (x_k, t_k) \) we have that
\[
(Y(x_1, t_1), \ldots, Y(x_k, t_k)) \overset{L}{=} (L_{x_1}^{H_0-t_1}, \ldots, L_{x_k}^{H_0-t_k}).
\]
So, \( \mathcal{L}(Y) = \mathcal{L}(L^{H_0}) \) in \( C([-D, D] \times [0, T]) \).

\[\square\]

### 5 Appendix

**Proof of Theorem 2.3.**

Fix \( t \in [0, T] \). By condition (ii), the Fourier transform of the occupation measure belongs to \( L^2(\mathbb{R}) \), \( \omega \)-a.s.

By applying Theorem 2.1 we have that \( \omega \)-a.s. the measure \( \mu^t \) is absolutely continuous and its density \( f^t \in L^2(\mathbb{R}) \) is square integrable in \( x \). Moreover, defining
\[
f_N^t(x) = \frac{1}{2\pi} \int_{-N}^N e^{-iux} \phi^t(u) du = \frac{1}{2\pi} \int_0^t \int_{-N}^N e^{-iux} e^{iuX_r} dr du,
\]

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we have
\[ f_t^N \xrightarrow{L^2(\mathbb{R})} f_t, \quad \omega - a.s. \] (5.1)

By Theorem 2.2 for any \((x,t)\) there exists a random variable, \(L^t_x\), such that
\[ \sup_{(x,t) \in \mathbb{R} \times [0,T]} \mathbb{E} |\psi_N(x,t) - L^t_x|^2 \xrightarrow{N \to \infty} 0, \]
where \(\psi_N(x,t) = \frac{1}{2\pi} \int_{-N}^N e^{-iux} \int_0^t e^{iux} \, dr \, du = f_t^N(x)\).

We will see that \(\omega\)-a.s. \(L^t_x\) is a version of the density \(f_t\) of \(\mu^t\). Due to the above uniform convergence, in any \([-A,A]\), with \(A > 0\), the following convergence follows
\[ \mathbb{E} \left( \int_{-A}^A |\psi_N(x,t) - L^t_x|^2 \, dx \right) \xrightarrow{N \to \infty} 0. \]

That is,
\[ \int_{-A}^A |\psi_N(x,t) - L^t_x|^2 \, dx \xrightarrow{L^1(\Omega)} 0, \]
when \(N \to \infty\). So, there exists a subsequence \(\{N_k\}_{k \in \mathbb{N}}\) and \(\Omega' \subset \Omega\) with \(P(\Omega') = 1\) such that for any \(\omega \in \Omega'\), the integral
\[ \int_{-A}^A |\psi_{N_k}(x,t) - L^t_x|^2 \, dx \xrightarrow{N_k \to \infty} 0, \] (5.2)
converges to 0, as \(N_k \to \infty\).

On the other hand, by (5.1) we have, with probability 1, that
\[ \int_{-\infty}^{\infty} |\psi_N(x,t) - f^t(x)|^2 \, dx \xrightarrow{N \to \infty} 0, \] (5.3)
when \(N \to \infty\).

Finally, from (5.2) and (5.3), we can deduce that \(L^t_x = f^t(x)\) \(x\text{-a.e.}\), with probability 1. So, \(\{L^t_x\}\) is a local time for \(X\).

\[ \square \]

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