Paley-Wiener theorems for a $p$-adic spherical variety

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Abstract. Let $\mathcal{S}(X)$ be the Schwartz space of compactly supported smooth functions on the $p$-adic points of a spherical variety $X$, and let $\mathcal{C}(X)$ be the topological space of Harish-Chandra Schwartz functions. Under assumptions on the spherical variety, which are satisfied at least when it is symmetric, we prove Paley-Wiener theorems for the two spaces, characterizing them in terms of their spectral transforms. As a corollary, we get a relative analog of the (smooth or tempered) Bernstein center – rings of multipliers for $\mathcal{S}(X)$ and $\mathcal{C}(X)$. When $X$ is a reductive group, our theorem for $\mathcal{C}(X)$ specializes to the well-known theorem of Harish-Chandra, and our theorem for $\mathcal{S}(X)$ corresponds to a first step – enough to recover the structure of the Bernstein center – towards the well-known theorem of Bernstein and Heiermann.

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1. Introduction

The goal of this paper is to characterize the spectral transform of the spaces of Schwartz (i.e. smooth, compactly supported) and Harish-Chandra Schwartz functions on the points of a spherical variety over a $p$-adic field. We do it under some assumptions on the variety, which essentially restrict us to the case of a symmetric space (possibly a few others, but we cannot name any). These assumptions are explained at the beginning of section 2.

Let $X$ be a spherical variety for a group $G$ over a non-archimedean local field $F$, satisfying those assumptions. We will be denoting $X(F)$ simply by $X$ (and similarly for other varieties), when this causes no confusion. We assume that $X = X(F)$ is endowed with a $G$-eigenmeasure, and normalize the action of $G$ on $L^2(X)$ (and other spaces of functions on $X$) so that it is unitary. To $X$ one associates some "simpler" spherical $G$-spaces $X_{\Theta}$ with more symmetries, called the boundary degenerations, parametrized by standard Levi subgroups in the "dual group" of $X$, whose Weyl group we denote by $W_X$. When $G$ is not split then we demand that $X$ is symmetric, and these symbols refer to their "relative" versions, cf. §2.3.

1.1. Paley-Wiener for the Harish-Chandra Schwartz space. The definition of the Harish-Chandra Schwartz space $\mathcal{E}(X_{\Theta})$ (including the case $X_{\Theta} = X$) is recalled in §2.4. In §5 it is shown that it has a direct summand, the intersection with the "discrete modulo-center" part $L^2(X_{\Theta})_{\text{disc}}$ of $L^2(X_{\Theta})$, which we will denote by $\mathcal{E}(X_{\Theta})_{\text{disc}}$. That carries an action of a ring of multipliers $\mathfrak{z}^\text{disc}(X_{\Theta})$, the "discrete center" of $X_{\Theta}$, which is isomorphic to the ring of $C^\infty$ functions on the "discrete spectrum" $\widehat{X_{\Theta}}_{\text{disc}}$ of $X_{\Theta}$ (to be explained below):

$$\mathfrak{z}^\text{disc}(X_{\Theta}) := C^\infty(\widehat{X_{\Theta}}_{\text{disc}}).$$

By [SV, Del], for each $\Theta$ one has a canonical "Bernstein map":

$$\iota_\Theta : L^2(X_{\Theta}) \to L^2(X).$$

Moreover, for each $w \in W_X(\Omega, \Theta)$, i.e. each element of $W_X$ which takes the set of a standard Levi $\Theta$ of the dual group to a standard Levi $\Omega$, there is a canonical "scattering map"

$$S_w : L^2(X_{\Theta}) \sim L^2(X_{\Omega}),$$
which is \( w \)-equivariant with respect to the “centers” (i.e. \( G \)-automorphism groups) of these spaces and such that we have a decomposition:

\[
t^*_{\Omega, \Theta} = \sum_{w \in W_X(\Omega, \Theta)} S_w. \tag{1.1}
\]

Notice that, despite the notation, the scattering operators are not parametrized by elements of \( W_X \), but by triples \((\Theta, \Omega, w \in W_X(\Omega, \Theta))\).

The main theorem [SV, Theorem 7.3.1], [Del, Theorem 6] on the Plancherel decomposition states:

1.2. Theorem. Let \( t^*_{\Theta, \text{disc}} \) denote the map \( t^*_\Theta \) composed with projection to the discrete spectrum. The sum:

\[
t^* := \sum_{\Theta} \left( \frac{t^*_{\Theta, \text{disc}}}{c(\Theta)} \right) : L^2(X) \to \bigoplus_{\Theta} L^2(X_{\Theta})_{\text{disc}}, \tag{1.2}
\]

where \( c(\Theta) \) is the number of “Weyl chambers” associated to \( \Theta \) \((= \#\{w \in W_X | w\Theta \subset \Delta_X\})\), is an isometric isomorphism of \( L^2(X) \) onto \( \bigoplus_{\Theta} L^2(X_{\Theta})_{\text{disc}} \) the subspace consisting of collections \((f_\Theta)_{\Theta}\) such that for all triples \((\Theta, \Omega, w \in W_X(\Omega, \Theta))\) we have: \( S_w f_\Theta = f_\Omega \).

Our first version of the Paley-Wiener theorem for the Harish-Chandra Schwartz space reads:

1.3. Theorem (cf. Theorem 13.5). The scattering maps \( S_w \) restrict to \( \mathfrak{g}^{\text{disc}}(X_{\Theta}) \)-equivariant isomorphisms (of LF-spaces) on the discrete part of the Harish-Chandra Schwartz spaces:

\[
S_w : \mathfrak{g}(X_{\Theta})_{\text{disc}} \xrightarrow{\sim} \mathfrak{g}(X_{\Omega})_{\text{disc}},
\]

where \( \mathfrak{g}^{\text{disc}}(X_{\Theta}) \) acts on \( \mathfrak{g}(X_{\Omega})_{\text{disc}} \) via the isomorphism:

\[
\mathfrak{g}^{\text{disc}}(X_{\Theta}) \xrightarrow{\sim} \mathfrak{g}^{\text{disc}}(X_{\Omega}) \tag{1.3}
\]

induced by \( w \).

The sum (1.2) restricts to a topological isomorphism:

\[
t^* : \mathfrak{g}(X) \xrightarrow{\sim} \bigoplus_{\Theta} (\mathfrak{g}(X_{\Theta})_{\text{disc}})^{\text{inv}}. \tag{1.4}
\]

There is also a more explicit version of this theorem, in terms of “normalized Eisenstein integrals” and “normalized constant terms”. Let \( \pi \) be an irreducible smooth representation of \( G \). One defines the space of coinvariants:

\[
\mathcal{S}(X)_\pi = \text{Hom}_G(\mathcal{S}(X), \pi)^* \otimes \pi,
\]

\[\text{In Theorem 13.8 we extend this statement to the whole Harish-Chandra Schwartz space, but this is not necessary for formulating the Paley-Wiener theorem and only comes as a corollary of it.}\]
which is a canonical quotient of $\mathcal{S}(X)$. Its smooth dual can be identified with a canonical submodule $C^\infty(X)^\pi$ of $C^\infty(X)$. One defines various subspaces $C^\infty_{\text{disc}}(X)^\pi$, $C^\infty_{\text{cusp}}(X)^\pi$ corresponding to the condition of square integrability, resp. compact support modulo center, and denotes the sets of unitary representations which appear discretely, resp. cuspidally, by $\hat{\mathcal{X}}_{\text{disc}}$, resp. $\hat{\mathcal{X}}_{\text{cusp}}$. Dually, we have the correspond quotients:

$$S(X)^\pi \twoheadrightarrow S(X)^\pi_{\text{disc}} \twoheadrightarrow S(X)^\pi_{\text{cusp}}.$$  

The same definitions can be given for any boundary degeneration $X\Theta$, but taking into account that this space is “parabolically induced” from a “Levi spherical variety” $X\Theta^L$ for a Levi subgroup $L\Theta$, i.e.:

$$X\Theta \simeq X\Theta^L \times_{\Theta^G} \tilde{G},$$  

the corresponding coinvariants are also parabolically induced, and indexed by representations of $L\Theta$.

As $\sigma$ varies over the set $X\Theta^L_{\text{disc}}$ of those representations which appear discretely-mod-center, the spaces $\mathcal{L}_\Theta,\sigma := S(X\Theta)^\pi_{\text{disc}}$ are the fibers of a complex algebraic vector bundle (actually, a countable direct limit of such) $\mathcal{L}_\Theta$ over the complexification of $X\Theta^L_{\text{disc}}$, and the canonical quotient maps give rise to a surjective morphism:

$$S(X\Theta) \twoheadrightarrow \mathbb{C}[X\Theta^L_{\text{disc}}, \mathcal{L}_\Theta].$$  

In Theorem 5.2 we show that this extends continuously to an isomorphism (in the literature sometimes referred to as the partial Fourier transform) of LF spaces:

$$\mathcal{C}(X\Theta)^\text{disc} := L^2(X\Theta)^\text{disc} \cap \mathcal{C}(X\Theta) \overset{\sim}{\twoheadrightarrow} C^\infty(X\Theta^L_{\text{disc}}, \mathcal{L}_\Theta).$$  

The aforementioned action of the discrete center $\mathfrak{z}_{\text{disc}}(X\Theta^L)$ on the left is, by definition, the action of $C^\infty(X\Theta^L_{\text{disc}})$ on the right.

It follows from their $\mathfrak{z}_{\text{disc}}(X\Theta^L)$-equivariance that the operators $S_w$ act fiberwise on these vector bundles; more precisely, it turns out that there are elements:

$$\mathcal{J}_w \in \Gamma \left( \frac{X\Theta^L_{\text{disc}}}{\mathfrak{z}_{\text{disc}}(X\Theta^L)}, \text{Hom}_G(\mathcal{L}_\Theta, w^* \mathcal{L}_\Theta) \right),$$  

where $\Gamma$ denotes rational sections whose poles do not meet the unitary set (cf. §3.2), such that the following diagram of isomorphisms commutes:

$$\begin{array}{ccc}
\mathcal{C}(X\Theta)^\text{disc} & \sim & C^\infty(X\Theta^L_{\text{disc}}, \mathcal{L}_\Theta) \\
\mathcal{S}_w \downarrow & & \downarrow \mathcal{J}_w \\
\mathcal{C}(X\Theta)^\text{disc} \sim & C^\infty(X\Theta^L_{\text{disc}}, \mathcal{L}_\Theta). 
\end{array}$$
Similarly, the Bernstein maps $\iota_{\Theta}$ are explicitly given by normalized Eisenstein integrals associated to discrete data, which are explicitly defined maps:

$$E_{\Theta,\sigma,\text{disc}} : \mathcal{L}_{\Theta,\sigma} = C^{\infty}(X_{\Theta})_{\text{disc}}^{\sigma} \rightarrow C^{\infty}(X)$$

(1.7)

where $\hat{\cdot}$ denotes smooth dual, varying rationally with $\sigma$. If $f \in L^2(X_{\Theta})_{\text{disc}}^{\sigma}$ admits the decomposition:

$$f = \int_{X_{\Theta}^{\text{disc}}} f^{\sigma}(x) d\sigma$$

with $f^{\sigma} \in C^{\infty}(X_{\Theta})_{\text{disc}}^{\sigma}$, then its image under the Bernstein map is the wave packet:

$$\iota_{\Theta} f = \int_{X_{\Theta}^{\text{disc}}} E_{\Theta,\sigma,\text{disc}} f^{\sigma} d\sigma,$$

If $f^{\sigma} \in C^{\infty}(X_{\Theta})_{\text{disc}}^{\sigma}$, then its image under the Bernstein map is the wave packet:

$$\iota_{\Theta} f = \int_{X_{\Theta}^{\text{disc}}} E_{\Theta,\sigma,\text{disc}} f^{\sigma} d\sigma,$$

(1.7)

cf. [SV, Theorem 15.6.1], [Del, Theorem 7]. We use this to prove that the normalized Eisenstein integrals (which are a priori rational in $\sigma$) have no poles on the imaginary axis, thus dually we get normalized constant terms (often called normalized Fourier transforms in the literature on symmetric spaces):

$$E_{\Theta,\text{disc}}^{\sigma} : S(X) \rightarrow \Gamma(X_{\Theta}^{\text{disc}}, \mathcal{L}_{\Theta}),$$

(1.8)

representing $\iota_{\Theta,\text{disc}}^{\sigma}$, where by $\Gamma(\bullet)$ we denote rational sections whose poles do not meet the unitary set. Combining all of this with Theorem 1.3 we get the following explicit Paley-Wiener theorem for the Harish-Chandra Schwartz space:

1.4. Theorem (cf. Theorem 13.6). The normalized constant terms (1.8) extend to isomorphisms of LF-spaces:

$$C(X) \xrightarrow{\sim} \left( \bigoplus_{\Theta} C^{\infty}(X_{\Theta}^{\text{disc}}, \mathcal{L}_{\Theta}) \right)^{\text{inv}},$$

where $\text{inv}$ here denotes $\mathcal{S}_{\text{w}}$-invariants, i.e. collections of sections $(f_{\Theta})_{\Theta}$ such that for all triples $(\Theta, \Omega, w \in W_{\chi}(\Omega, \Theta))$ we have: $\mathcal{S}_{\text{w}} f_{\Theta} = f_{\Omega}.$

In the group case, this theorem is part of the Plancherel formula of Harish-Chandra, appearing in Waldspurger [Wal03].

We remark that (1.7), in combination with the fact that $\iota_{\Theta}^{\ast} \iota_{\Theta}$ is the identity on $\mathcal{S}_{\text{w}}$-invariants, provide an explicit way to invert this map by means of normalized Eisenstein integrals. Notice that we do not explicitly identify the scattering maps; this can be the object of further research, with a number-theoretic flavor since their poles are often related to $L$-functions. We only describe their relation to normalized Eisenstein integrals in (10.16).

A corollary of this theorem (or its previous version 1.3) is the existence of a ring of multipliers on $C(X)$. Notice that each $w \in W_{\chi}(\Omega, \Theta)$ induces the
isomorphism (1.3) between discrete centers. Let:

$$\mathfrak{z}^{\text{temp}}(X) = \left( \bigoplus_{\Theta} \mathfrak{z}^{\text{disc}}(X^L_{\Theta}) \right)^{\text{inv}}$$

denote the invariants of these isomorphisms, for all triples $(\Theta, \Omega, w \in W_X(\Theta, \Theta))$. One can call this ring the tempered center of $X$ – it is the relative analog of the tempered center of Schneider and Zink [SZ08] (whose structure can also be inferred directly from the Plancherel theorem of Harish-Chandra [Wal03]).

1.5. **Corollary** (s. Corollary 13.7). There is a canonical action of $\mathfrak{z}^{\text{temp}}(X)$ by continuous $G$-endomorphisms on $\mathcal{C}(X)$, characterized by the property that it is compatible with the isomorphisms of Theorems 1.3, 1.4.

1.6. **Paley-Wiener for the Schwartz space.** We now come to the Paley-Wiener theorem for the Schwartz space $\mathcal{S}(X)$ of compactly supported smooth functions on $X$. In analogy with the previous case, this has a distinguished direct summand $\mathcal{S}(X)_{\text{cusp}}$, its “cuspidal part”; it is characterized by the fact that its orthogonal complement is orthogonal to the image of any of the spaces $\mathcal{C}^\infty_{\text{cusp}}(X)^\pi$ introduced before.

The same definitions hold for the boundary degenerations $X_\Theta$, and the space $\mathcal{S}(X_\Theta)_{\text{cusp}}$ comes equipped with the action of a “cuspidal center” $\mathfrak{z}^{\text{cusp}}(X^L_{\Theta})$, identified with the ring of polynomial functions on the subset $\widehat{X^L_{\Theta}}_{\text{cusp}} \subset \widehat{X^L_{\Theta}}_{\text{disc}}$:

$$\mathfrak{z}^{\text{cusp}}(X) := \mathbb{C}[\widehat{X^L_{\Theta}}_{\text{cusp}}].$$ (1.9)

Here we have the “equivariant exponential maps”:

$$e_{\Theta} : \mathcal{S}(X_\Theta) \to \mathcal{S}(X),$$ (1.10)

which are a convenient way to generalize the classical theory of asymptotics of matrix coefficients (see [SV, §5]). The space $\mathcal{S}(X)$ is the sum of all $e_{\Theta}\mathcal{S}(X_\Theta)_{\text{cusp}}$:

1.7. **Proposition** (s. Proposition 14.1). We have:

$$\mathcal{S}(X) = \sum_{\Theta \in \Delta_X} e_{\Theta}\mathcal{S}(X_\Theta)_{\text{cusp}}.$$ (1.11)

We note that this fails to be true without the assumption that $X$ is strongly factorizable; interesting phenomena await the researcher who will work on the general case!

A basic element in our analysis is a similar to the unitary case decomposition into “smooth scattering maps” when $\Theta$ and $\Omega$ are conjugate:

$$e_{\Omega}^* e_{\Theta} = \sum_{w \in W_X(\Omega, \Theta)} \mathcal{G}_w,$$ (1.11)
where the maps \( \mathcal{S}_w : \mathcal{S}(X_\Theta) \to C^\infty(X_\Theta) \) are \( \delta_{\text{cusp}}(X_L^{\Theta}) \)-equivariant when this ring acts on \( C^\infty(X_\Theta) \) via the isomorphism:

\[
\delta_{\text{cusp}}(X_L^{\Theta}) \xrightarrow{\sim} \delta_{\text{cusp}}(X_L^{\Omega})
\]

induced by \( w \). Note that in this case neither the scattering maps nor the isomorphisms between cuspidal spectra of \( X_L^{\Theta} \) and \( X_L^{\Omega} \) are provided by the \( L^2 \)-theory: all these are results that we need to establish.  

Notice that the adjoint \( e_\Theta^* \) ("smooth asymptotics map") of \( e_\Theta \) does not preserve compact support, therefore the maps \( \mathcal{S}_w \) have image in some subspace of \( C^\infty(X_\Theta) \). If we let \( \mathcal{S}_\mathcal{S}^{\ast} \) denote the space generated by the images of those \( \mathcal{S}_w \) (for all associates \( \Theta \) of \( \Omega \) and all \( w \in W_X(\Omega, \Theta) \)), then (cf. Theorem 9.2) each scattering map \( \mathcal{S}_w \) extends canonically to an isomorphism:

\[
\mathcal{S}_w : \mathcal{S}_\mathcal{S}^{\ast} \xrightarrow{\sim} \mathcal{S}^{\ast}.
\]

The first version of our Paley-Wiener theorem for the Schwartz space reads:

1.8. **Theorem** (cf. Theorem 14.3). Let \( \epsilon_{\Theta, \text{cusp}}^* \) denote the map \( \epsilon_{\Theta}^* \) composed with projection to the cuspidal summand. The sum:

\[
e^* := \sum_{\Theta} \epsilon_{\Theta, \text{cusp}}^* : \mathcal{S}(X) \to \bigoplus_{\Theta} \mathcal{S}^+(X_\Theta)_{\text{cusp}}
\]

is an isomorphism into the \( (\mathcal{S}_w)_{\omega} \)-invariants of the space on the right, i.e. the subspace consisting of collections \( (f_\Theta)_{\Theta} \) such that for all triples \( (\Theta, \Omega, w \in W_X(\Omega, \Theta)) \) we have: \( \mathcal{S}_w f_\Theta = f_\Omega \).

Again there is a more explicit version of this theorem. Consider the bundle \( \mathcal{L}_\Theta \) whose fibers are the cuspidal coinvariants \( \mathcal{L}_{\Theta, \sigma} := \mathcal{S}(X_\Theta)_{\sigma, \text{cusp}}; \) it is a (countable direct limit of) complex algebraic vector bundle(s) over the complexification of the subset \( \widetilde{X}_L^{\text{cusp}} \subset \widetilde{X}_L^{\text{disc}} \) where these spaces are nonzero.

In analogy to (1.6), the canonical quotient maps give rise to isomorphisms:

\[
\mathcal{S}(X_\Theta)_{\text{cusp}} \xrightarrow{\sim} \mathbb{C}[\widetilde{X}_L^{\text{cusp}}, \mathcal{L}_\Theta].
\]

The action of the cuspidal center \( \delta_{\text{cusp}}(X_L^{\Theta}) \) is nothing but the action of \( \mathbb{C}[\widetilde{X}_L^{\text{cusp}}] \) on the right hand side.

For the space \( \mathcal{S}^+(X_\Theta)_{\text{cusp}} \) this extends to an identification with a “fractional ideal” (i.e. a subspace of the space of rational sections which, when multiplied by a suitable element of \( \mathbb{C}[\widetilde{X}_L^{\text{cusp}}] \), becomes regular):

\[
\mathcal{S}^+(X_\Theta)_{\text{cusp}} \xrightarrow{\sim} \mathbb{C}^+[\widetilde{X}_L^{\text{cusp}}, \mathcal{L}_\Theta] \subset \mathbb{C}([\widetilde{X}_L^{\text{cusp}}, \mathcal{L}_\Theta]),
\]

\[2\text{Again, only the restriction of the above statements to cuspidal summands is of interest for the formulation of the Paley-Wiener theorem, and the extension to the whole space follows a posteriori, s. Theorem 14.6.}\]
but this identification needs some explanation. The fractional ideal will not be identified (except in specific examples); this seems to be a number-theoretic question, as in all known examples it involves $L$-functions. We only know that it is obtained by inverting “linear polynomials” (see §3.4 for the definition of “linear”). Despite the notation, it does not only depend on the isomorphism class of $X_{\Theta}$, but it actually depends on $X$ itself. They can in principle be computed by (10.16) whenever the normalized Eisenstein integrals can.

Using the isomorphism (1.14), the smooth scattering maps $S_w$ can be expressed in terms of the same fiberwise scattering maps $F_w$ as before (but restricted, of course, to the subbundle $L_{\Theta}$ of $L_{\tilde{\Theta}}$ which they turn out to preserve). Namely, the isomorphism (1.14) fits into a commuting diagram:

$$
\begin{array}{ccc}
S^+(X_{\Theta})_{\text{cusp}} & \overset{\sim}{\longrightarrow} & \mathbb{C}^+[X_{\Theta}^L_{\text{cusp}}, L_{\Theta}] \\
S_w & \downarrow & \\
S^+(X_{\Omega})_{\text{cusp}} & \overset{\sim}{\longrightarrow} & \mathbb{C}^+[X_{\Omega}^L_{\text{cusp}}, L_{\Omega}].
\end{array}
$$

Although the fiberwise scattering maps $F_w$ are the same as before, the inversion of (1.14) is not, as in the Harish-Chandra case, by continuous extension of the map (1.5) – the elements on the left cannot be reconstructed by elements on the right via an integral over the unitary set $X_{\Theta}^L_{\text{cusp}}$ – and hence the smooth scattering maps $S_w$ do not coincide with the unitary scattering maps $S_w$.

Similarly, the explicit version of the equivariant exponential map $e_\Theta$ is again given by normalized Eisenstein integrals, but using shifted wave packets this time. More precisely, if we fix a Haar-Plancherel measure $d\sigma$ on $X_{\Theta}^L_{\text{cusp}}$ and use it to write $f \in S(X_{\Theta})_{\text{cusp}}$ as:

$$
f = \int_{X_{\Theta}^L_{\text{cusp}}} f^\tilde{\sigma} (x) d\sigma
$$

with $f^\tilde{\sigma} \in C^\infty(X_{\Theta})_{\tilde{\sigma}} = \tilde{L}_{\Theta,\sigma}$, then by (1.13) $f^\tilde{\sigma}$ extends polynomially to non-unitary $\sigma$’s and we have:

$$
e_\Theta f(x) = \int_{\omega^{-1}X_{\Theta}^L_{\text{cusp}}} E_{\Theta,\sigma,\text{cusp}} f^\tilde{\sigma} (x) d\sigma. \quad (1.15)
$$

for every “sufficiently positive” character $\omega$, cf. Theorem 7.3. (For symmetric spaces, the fact that shifted wave packets are compactly supported can also be proved using the results of [CD14] and a technique due to Heiermann in the group case [Hei01].)

Dually, this gives an expression of $e^\tilde{\sigma}_{\Theta,\text{cusp}}$ as a normalized constant term:

$$
S(X) \rightarrow \Gamma(X_{\Theta}^L_{\text{cusp}}, L_{\Theta}), \quad (1.16)
$$
which is the same map as (1.8) composed with the natural quotient \( \mathcal{L}_\Theta \to \mathcal{L}_\Theta \).

Our explicit version of the Paley-Wiener theorem for the Schwartz space reads:

1.9. **Theorem** (cf. Theorem 14.4). The morphisms (1.16) give an isomorphism:

\[
S(X) \simeq \left( \bigoplus_\Theta \mathbb{C}^+ [X^\text{cusp}_\Theta, \mathcal{L}_\Theta] \right)^{\text{inv}},
\]

where \( \text{inv} \) here denotes \( \mathcal{S}_w \)-invariants.

A corollary of this theorem (or its previous version (1.8)) is the existence of a ring of multipliers on \( S(X) \). Notice that each \( w \in W_X(\Omega, \Theta) \) induces the isomorphism (1.12) between cuspidal centers. Let:

\[
\mathfrak{z}^{\text{sm}}(X) = \left( \bigoplus_\Theta \mathfrak{z}^\text{cusp}(X^L_\Theta) \right)^{\text{inv}}
\]

denote the invariants of these isomorphisms, for all triples \( (\Theta, \Omega, w \in W_X(\Omega, \Theta)) \). One can call this ring the smooth center of \( X \) – it is the relative analog of the Bernstein center (cf. §15.1). Then:

1.10. **Corollary** (s. Corollary 14.5). There is a canonical action of \( \mathfrak{z}^{\text{sm}}(X) \) on \( S(X) \), characterized by the property that it is compatible with the isomorphisms of Theorems 1.8, 1.9.

In section 15 we discuss the example of \( X = \) a reductive group \( H \) under the \( G = H \times H \)-action by left and right multiplication. We show that the multiplier ring \( \mathfrak{z}^{\text{sm}}(X) \) that we described above provides an alternative proof for the structure of the Bernstein center as the algebra of polynomials on the “space” of cuspidal supports. We also discuss the relationship of our Paley-Wiener theorem with that of Bernstein [Ber] and Heiermann [Hei01]: in this case, our work is analogous to part A of [Hei01], and one needs to apply part B, which is the hardest part of this paper, to obtain the usual Paley-Wiener theorem. This is a good point to reflect on what our theorem really represents: It represents a reduction of the study of smooth functions on \( X \) to (relatively) cuspidal spectra plus the study of scattering operators; it does not, however, reveal much about the nature of these operators, which can be the object of further research.

1.11. **Proofs.** We outline the basic steps in the proofs of the aforementioned theorems.

After introducing the necessary structure theory of spherical varieties in section 2 and the bundles of discrete and cuspidal coinvariants in sections 3 and 4, the first step is to show that the discrete, resp. cuspidal summand of \( \mathcal{G}(X) \), resp. \( S(X) \), is a direct summand. This is relatively easy to do, and is done in sections 5 and 6.
The spectral characterization of \( \mathcal{C}(X)_{\text{disc}} \) (1.6) and \( \mathcal{S}(X)_{\text{cusp}} \) (1.13) is the next step, and the basis for those is the surjection (1.5); this follows from the definition of the bundle \( \mathcal{L}_\Theta \), and an application of Nakayama’s lemma (Proposition 5.3). After this, (1.6) follows from the analogous statement for abelian groups (we use here the assumption that \( X \), and later \( X_{\Omega} \), are all factorizable, cf. §2), and (1.13) is immediate by projection from discrete to cuspidal.

The unitary scattering operators \( S_w \) were introduced in [SV], but here we need to prove that they preserve Harish-Chandra Schwartz spaces (at least their discrete summands). The explicit expression (1.7) for \( \iota_\Theta \) allows us to relate the fiberwise versions \( \mathcal{S}_w \) of the scattering maps to the asymptotics of normalized Eisenstein integrals and normalized constant terms, hence deducing their rationality in the parameter by a linear algebra argument, Proposition 10.12. Essentially, the operator \( \mathcal{S}_w \), for \( w \in W_X(\Omega, \Theta) \), is the “\( w \)-equivariant part” of the asymptotics \( e^*_\Omega f \) of the normalized Eisenstein integral \( E_{\Theta, \text{disc}} \). A priori knowledge of \( L^2 \)-boundedness of the operators \( S_w \), together with the “linear” form of the poles of Eisenstein integrals 8.2, cf. also [BD08], allow us to show that their poles do not meet the unitary spectrum, Theorem 9.3, and since they are unitary it follows from (1.6) that \( S_w \), for \( w \in W_X(\Omega, \Theta) \), maps \( \mathcal{C}(X_{\Theta})_{\text{disc}} \) isomorphically onto \( \mathcal{C}(X_{\Omega})_{\text{disc}} \).

Using this fact, and a characterization of the Bernstein maps \( \iota_{\Theta} \) from [SV], we are able to prove that \( \iota_{\Theta} \) maps \( \mathcal{C}(X_{\Theta})_{\text{disc}} \) into \( \mathcal{C}(X) \) (Proposition 13.1). This is essentially the fact that some wave packets are in the Harish-Chandra Schwartz space (cf. [DH14] for a result of this type for symmetric spaces). Vice versa, the description of \( \iota_{\Theta} \) in terms of normalized Eisenstein integrals (1.7), together with the regularity of normalized Eisenstein integrals on the unitary spectrum, proves that \( \iota_{\Theta, \text{disc}} \) continuously sends the space \( \mathcal{C}(X) \) into \( \mathcal{C}(X_{\Theta})_{\text{disc}} \) (Proposition 13.2), and this is enough to prove the Paley-Wiener theorems 1.3, 1.4 for the Harish-Chandra Schwartz space.

To construct the smooth scattering operators \( \mathfrak{S}_w \) one needs to study properties of the compositions \( e^*_\Omega e_{\Theta} \) restricted to cuspidal summands, and more precisely that the restriction of this composition to \( \mathcal{S}(X_{\Theta})_{\text{cusp}} \) is zero if \( \Omega \) does not contain an associate of \( \Theta \), has cuspidal image if \( \Omega \) is associate to \( \Theta \) and has image in the orthogonal complement of the cuspidal summand if \( \Omega \) strictly contains an associate of \( \Theta \), Theorem 9.2. The proofs of these facts are accomplished in section 12 using the explicit description (1.15) of these maps in terms of normalized Eisenstein integrals. The proof relies in a crucial way on a theorem in [SV] (which in turn was based on a theorem of Bezrukavnikov and Kazhdan [BK]) which says that the support of \( e^*_\Omega f \) for \( f \in \mathcal{S}(X) \), is bounded, i.e. of compact closure in a (fixed) affine embedding of \( X_{\Theta} \); this allows to prove the vanishing of certain “exponents” of the Eisenstein integrals. By totally different methods these results were obtained by Carmona-Delorme [CD14] for symmetric spaces, via an explicit description
of the constant term of Eisenstein integrals, starting from cuspidal data, in terms of “C-functions”.

As was mentioned in Proposition 1.7, the space $S(X)$ is the sum of all “shifted cuspidal wave packets”, i.e. the sum of all $e_\Theta S(X_\Theta)_{\text{cusp}}$. Then (1.11) can be understood as the cuspidal constant term of shifted wave packets. The proof of Theorem 1.9 rests mainly on (1.11).

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Part 1. Structure, notation and preliminaries

2. Boundary degenerations, exponents, Schwartz and Harish-Chandra Schwartz spaces

2.1. Assumptions. We let $X$ be a homogeneous spherical variety for a reductive group $G$ over a non-archimedean local field $F$ in characteristic zero. The assumption on characteristic is in order to use the results of [SV] which freely applied the structure theory of spherical varieties in characteristic zero, but with minor modifications these results, and hence our methods, should work in arbitrary characteristic. We notice that for symmetric spaces, [Del] only required that the characteristic of the field be different than 2.

We will make the following assumptions on $X$:

1. If $G$ is not split, $X$ is symmetric. The symmetric condition (whether $G$ is split or not) subsumes all the conditions that follow, but should be considered as a provisional assumption in order to use the non-split analogs of spherical root systems used in [Del] (cf. §2.3). Our methods do not depend heavily on the structure of spherical varieties. If $G$ is split, we assume:

2. $X$ is wavefront;

3. $X$ is strongly factorizable (cf. below for both of these notions);

4. $X$ satisfies a strong version of the “generic injectivity” condition (cf. §10.5).

Up to now, our assumptions guarantee the validity of the full Plancherel decomposition of [SV, Theorem 7.3.1], [Del, Theorem 6]. Finally, we require the validity of the explicit Plancherel formula in terms of normalized Eisenstein integrals:

5. The explicit Plancherel formula of [SV, Theorem 15.6.2], [Del, Theorem 8] holds; this is the case, for instance, if the “small Mackey restriction” of [SV, §15.5] is generically injective.
Again, all these conditions are satisfied if $X$ is symmetric; for the strong version of the generic injectivity assumption in the symmetric case, which was not used in the aforementioned references, we prove this in §10.5.

2.2. The split case. We start by giving definitions when the group $G$ is split. We will then modify them for non-split $G$, when the space $X$ is symmetric (following [Del]).

Given a spherical variety $X$ for a group $G$, we define the center of $X$ as the connected component of its $G$-automorphism group:

\[ \mathcal{Z}(X) := \text{Aut}_G(X)^0. \]

It is known to be a torus, and we assume throughout (as we may, without loss of generality, by enlarging $G$ if necessary, that the natural map is surjective:

\[ \mathcal{Z}(G)^0 \twoheadrightarrow \mathcal{Z}(X). \] (2.1)

For any fixed Borel subgroup, we denote by $X$ the open Borel orbit on $X$.

Our varieties will be homogeneous, $X = H \backslash G$, and we let $X^\text{ab}$ be the homogeneous variety under the abelianization $G^\text{ab}$ of $G$ which is obtained by dividing $X$ by the action of the commutator group $[G, G]$. If we choose a point $x \in X$ with stabilizer $H$ and let $\overline{H^\text{ab}}$ be the image of $H$ in $G^\text{ab}$, then as algebraic varieties: $X^\text{ab} = G^\text{ab}/\overline{H^\text{ab}}$.

$X$ is called factorizable if $\dim X^\text{ab} = \dim \mathcal{Z}(X)$; all symmetric varieties have this property. In the wavefront case, it is called strongly factorizable if all of its Levi varieties are also factorizable; the notions of “wavefront” and “Levi variety” are defined below. Symmetric varieties are strongly factorizable [SV, Proposition 9.4.2]. If $X$ is factorizable then as algebraic varieties (but not necessarily in terms of their $F$-points):

\[ X \simeq \mathcal{Z}(X) \cdot X', \]

where $X' = H^0 \cap [G, G]\backslash[G, G]$. This, of course, depends on the choice of base point defining the isomorphism $X \simeq H \backslash G$, and if we choose different such points $x_1, x_2, \ldots$ we get different subvarieties $X'_1, X'_2, \ldots$.

Then, at the level of $F$-points, there are a finite number of points $x_i$ such that $X$ is the disjoint union of open-closed subsets:

\[ X(F) = \bigcup_{i=1}^n \mathcal{Z}(X)(F) \cdot X'_i(F). \] (2.2)

The non-canonical subvarieties $X'_i$ will never appear in the statements, but will sometimes be used in the proofs.

We will generally denote the points of a variety $Y$ over our fixed nonarchimedean field $F$ simply by $Y$, when this creates no confusion. The group of unitary complex characters of the $F$-points of the torus $X^\text{ab}$ will be denoted by $\hat{X}^\text{ab}$, and its complexification (which can be identified with the
group of not-necessarily-unitary characters) by $\hat{X}_{\text{ab}}$. The connected component of $X_{\text{ab}}$, i.e. the group of unramified unitary characters, will be denoted by $\hat{X}_{\text{unr}}$, and this notation ($\text{unr}$) will be used more generally to denote groups of unramified characters.

To every spherical variety $X$ one associates its set of (simple) spherical roots $\Delta_X$ and the “little Weyl group” $W_X$, cf. [SV, §2]. The spherical roots live in the lattice $\mathcal{X}(X)$ of characters of a Borel subgroup which are trivial on stabilizers of generic points, and $W_X$ acts by automorphisms on $\mathcal{X}(X)$. They are actually part of the root data of the “dual group” $\hat{G}_X$ of $X$, although this will not be of particular concern to us, and the same holds for the precise choice of lengths of roots. (We recall from loc.cit. that there are the “unnormalized” and “normalized” spherical roots, but this plays no role here.)

What is important for us is that one has the following set of data:

- **Boundary degenerations:** For every subset $\Theta \subset \Delta_X$, a spherical $G$-variety $X_\Theta$ of the same dimension, with the property that $\dim (Z(X_\Theta)) = \dim (Z(X)) + |\Delta_X \setminus \Theta|$. We alternatively denote:

  $$A_{X,\Theta} := Z(X_\Theta).$$

Under the convention that $Z(X) = Z(G)^0$ that we are using, $X$ is called *wavefront* if for every $\Theta$ the variety $X_\Theta$ is parabolically induced from a spherical variety $X^{L_\Theta}$ (called *Levi variety*) for the Levi quotient $L_\Theta$ of a parabolic $P^{-}_\Theta$:

$$X_\Theta \simeq X^{L_\Theta} \times^{P^{-}_\Theta} G$$

(2.3)

such that the action of $Z(X_\Theta)$ is induced from the action of the connected center of $L_\Theta$ on $X^{L_\Theta}$. Only the conjugacy class of $P^{-}_\Theta$ is canonically defined (and then $X^{L_\Theta}$ is the fixed point set of its unipotent radical on $X_\Theta$); thus, whenever we use those Levi varieties we will be careful that the non-canonical choice of a representative for $P^{-}_\Theta$ does not affect our results. The isomorphism (2.3) shows that $X^{L_\Theta}$ can also be identified as the quotient of the open $P_\Theta$-orbit on $X_\Theta$ (where $P_\Theta$ is in the class of parabolics opposite to $P^{-}_\Theta$) by the (free) action of the unipotent radical $U_\Theta$ of $P_\Theta$. Since $X P_\Theta / U_\Theta \simeq X_\Theta P_\Theta / U_\Theta$ canonically [SV, Lemma 2.8.1], the Levi variety is also identified with the analogous quotient for $X$.

For $\Theta = \Delta_X$ we have $X_\Theta = X$, and for $\Theta = \emptyset$ the variety $X_\Theta$ is *horospherical*, i.e. stabilizers contain maximal unipotent subgroups of $G$. We denote $A_{X,\emptyset}$ simply by $A_X$; its character group has a canonical identification with $\mathcal{X}(X)$. For every $\Theta$, $A_{X,\Theta}$ is canonically identified with the connected kernel of $\Theta$ in $A_X$, and we denote by $A^+_{X,\Theta}$ the monoid of elements $a \in A_{X,\Theta}(F)$ with the property that $|e^{\gamma}(a)| \leq 1$ for all $\gamma \in \Delta_X$, and by $\hat{A^+}_{X,\Theta}$ the subset of those elements
with $|e^\gamma(a)| < 1$ for all $\gamma \in \Delta_X \setminus \Theta$. (We use the exponential symbol in order to use additive notation for the group $X(X)$).

- **Exponential map:** For every open compact subgroup $J$ of $G$, a system of $J$-stable subsets $N_\Theta$ of $X = X(F)$, with $N_\Theta \subset N_\Omega$ if $\Theta \subset \Omega$, and for each $\Theta$ a $J \times A_{X,\Theta}^+$-stable subset $N_\Theta^+$ of $X_\Theta$, which generates all $X_\Theta$ under the action of $A_{X,\Theta}$, together with identifications:

$$N_\Theta/J = N_\Theta^+/J,$$

and such that a certain property on asymptotics of representations (to be explained in §7) is satisfied. Such a set $N_\Theta$ will be called a "$J$-good neighborhood of $\Theta$-infinity", and from now on we will not distinguish in notation between $N_\Theta$ and $N_\Theta^+$, i.e. we will be denoting the latter also by $N_\Theta$. The above identifications clearly also identify $N_\Omega/J$, for all $\Omega \subset \Theta$, with subsets of $X_\Theta/J$, and the set:

$$N_\Theta \setminus \bigcup_{\Omega \subset \Theta} N_\Omega$$

has compact image in $X_\Theta/A_{X,\Theta}$. We have $N_{\Delta_X} = X$, hence the complement of $\bigcup_{\Theta \subset \Delta_X} N_\Theta$ is compact modulo the action of $Z(X)$.

The modular character of $P_\Theta$, i.e. the inverse of the modular character of $P_\Theta^+$, will be denoted by $\delta_\Theta$. (Our convention is that a modular character is the quotient of right by left Haar measure.) The functor of *normalized* induction from $P_\Theta$, resp. $P_\Theta^-$, will be denoted by $I_\Theta$, resp. $I_\Theta^-$:

$$I_\Theta V := \{ f : G \to V \text{ smooth} | f(pg) = \delta_\Theta^{1/2} f(g) \text{ for all } p \in P_\Theta \},$$

$$I_\Theta^- V := \{ f : G \to V \text{ smooth} | f(pg) = \delta_\Theta^{-1/2} f(g) \text{ for all } p \in P_\Theta^- \}.$$  

We similarly denote, for every representation $\pi$ of $G$, the *normalized* Jacquet modules with respect to $P_\Theta$, resp. $P_\Theta^-$, by $\pi_\Theta$, resp. $\pi_\Theta^-$. These are, by definition, the coinvariants of the corresponding unipotent radicals, tensored by the inverse square root of the corresponding modular character, so that we have canonical $L_\Theta$-morphisms:

$$(I_\Theta V)_\Theta \to V, \quad (I_\Theta^- V)_\Theta^- \to V.$$ 

Actions of Weyl groups will always be defined to be *left* actions. We define the Weyl group $W$ of $G$ as an automorphism group of its *universal Cartan* $A = B/N$ (where $B$ is any Borel subgroup, with unipotent radical $N$, so that the universal Cartan is a unique torus up to unique isomorphism). For subset $S$ of the positive simple roots of $A$ in $G$, corresponding to a class of parabolics $P_S$, any element $w$ which maps $S$ into the positive simple roots gives rise to an isomorphism between the Levi quotients $L_S$ and $L_{wS}$ of the corresponding parabolics, unique up to inner conjugacy. In particular, this is true for the Levi quotients $L_\Theta$, $L_\Omega$ (where $\Theta, \Omega \subset \Delta_X$) and an element $w \in W_X(\Omega, \Theta) \subset W_X \subset W$, where $W_X(\Omega, \Theta)$ denotes the set of elements of $W_X$ mapping $\Theta$ to $\Omega$. Finally, the notation $\Theta \sim \Omega$ will mean that $\Theta$ and $\Omega$ are associates, i.e. $W_X(\Omega, \Theta) \neq \emptyset$. 


2.3. The general symmetric case. In the general symmetric case (when $G$ is not necessarily split), the boundary degenerations $X_{\Theta}$ are defined in [Del, §3.1]. They are denoted there by $X_{P}$, while the Levi varieties $X_{L}^{\Theta}$ are denoted by $X_{M}$.

The center $Z(X)$ is defined in §7.3 of loc.cit., denoted by $A_{G}$, as the maximal split torus in the center of $G$ made by anti-invariant elements of the given involution; similarly for the tori $A_{X,\Theta} = Z(X_{\Theta})$ (denoted by $A_{M}$ there).

These tori correspond to the maximal split tori of what, over the algebraic closure, is $Z_{p}$ under the definitions of the previous subsection.

While it is not very good to have notation which is not stable under base change, it is convenient here that the emphasis is not on geometry but on harmonic analysis, and we will adopt it.

2.4. Normalized action and the various Schwartz spaces. We assume that $X_{\Theta}$ carries a $G$-eigenmeasure\(^\text{3}\) with eigencharacter $\eta$, and any choice of such measure endows all the spaces $X_{\Theta}(F)$ with $G$-eigenmeasures with the same eigencharacter which make the identifications (2.4) of neighborhoods of the form $N_{\Theta}/J$ measure-preserving, cf. [SV, §4.1], [Del, Theorem 2]. This measure on $X_{\Theta}$ is also an $A_{X,\Theta}$-eigenmeasure, and whenever a group acts on a space $Y$ endowed with an eigenmeasure with eigencharacter $\chi$, we normalize the action of the group on functions on $Y$ so that it is an $L^{2}$-isometry:

$$ (g \cdot f)(y) = \sqrt{\chi(g)} f(yg). \quad (2.5) $$

This also identifies the space $C^{\infty}(Y)$ (uniformly constant functions on $Y$) with the smooth dual of $\mathcal{S}(Y) = C_{c}^{\infty}(Y)$.

On the Levi varieties $X_{L}^{\Theta} = \hat{X}P_{\Theta}/U_{\Theta}$ the measure on $X$ gives rise to an $L_{\Theta}$-eigenmeasure for which the following is true:

$$ \int_{X_{P_{\Theta}}} f(x)dx = \int_{X_{L}^{\Theta}} \int_{U_{\Theta}} f(ux)dudx. $$

This depends on the choice of Haar measure on $U_{\Theta}$. The character by which $L_{\Theta}$ acts on this measure is $\delta_{\Theta}\eta$ (recall that $\eta$ is the eigencharacter of the measure on $X$). Thus, we need to twist the unnormalized action of $L_{\Theta}$ on functions by $(\eta \delta_{\Theta})^{\frac{1}{2}}(l)$ in order to obtain a unitary representation.

Another way to describe this twisting is the following: if we identify $X_{L}^{\Theta}$ as a subvariety of $X_{\Theta}$ fixed by the parabolic $P_{\Theta}^{-}$, and $g \in P_{\Theta}^{-}$ with image $l \in L_{\Theta}^{-}$, then for a function $f$ a function on $X_{\Theta}$ we have:

$$ l \cdot (f|_{X_{L}^{\Theta}}) := \delta_{\Theta}^{\frac{1}{2}}(l)(g \cdot f)|_{X_{L}^{\Theta}}. \quad (2.6) $$

\(^3\)In fact, under our assumption of factorizability it is possible to twist such a measure and make it invariant; however, even if we do this for $X$ it will not be the natural choice for the Levi varieties $X_{L}^{\Theta}$, as we will see, so one ends up working with eigenmeasures anyway.
(The twist by $\sqrt{\pi}$ is already contained in the $G$-action on $X_{\Theta}$.) An important observation is that, by introducing this twisted action for $L_{\Theta}$, the action of the connected center of $L_{\Theta}$ on $f|_{X_{\Theta}}$ coincides with the action of $\mathcal{Z}(X_{\Theta})$ on $C^\infty(X_{\Theta})$, under the identification of $\mathcal{Z}(X_{\Theta}) = A_{X,\Theta}$ as a quotient of $\mathcal{Z}(L_{\Theta})^0$. Indeed, the twist by $\delta_{\Theta}^2$ is contained in (2.5), by taking into account the eigencharacter of the measure under the action of $\mathcal{Z}(X_{\Theta})$.

We caution the reader that this may not be the most natural-looking action; for instance, if $X$ has a $G$-invariant measure and we consider the Levi variety $X_{\Theta}^L \simeq A_X$, the usual action of $A$ on $C^\infty(A_X)$ is twisted by the square root of the modular character of $P(X)$. However, this definition is such that the space of $L^2$-functions on $X_{\Theta}$ is \textit{unitarily induced} from the analogous space on $X_{\Theta}^L$:

$$L^2(X_{\Theta}) = I_{\Theta^{-1}} L^2(X_{\Theta}^L), \tag{2.7}$$

The Schwartz space $S(X)$ is, by definition, the space $C_c^\infty(X)$ of compactly supported smooth functions on $X$ (and similarly for any homogeneous space). The twist (2.6) on functions on $X_{\Theta}^L$ allows us to write, using again the functor of normalized induction from $P_{\Theta}^*$:

$$S(X_{\Theta}) = I_{\Theta^{-1}} S(X_{\Theta}^L), \tag{2.8}$$

Moreover, if $X$ is a direct product:

$$X = Z(X) \times X',$$

where $X'$ is a $[G, G]$-spherical variety, we clearly have a decomposition:

$$S(X) = S(Z(X)) \otimes S(X').$$

In the general factorizable case, using a decomposition such as (2.2) and pulling back functions by the action map:

$$Z(X) \times X'_i \to Z(X) \cdot X'_i,$$

it is immediate to identify $S(X)$ with:

$$\bigoplus_i (S(Z(X)) \otimes S(X'_i))^{(Z(X)) \cap [G, G]^{\text{diag}}}, \tag{2.9}$$

i.e. invariants under the simultaneous action of the subgroup $Z(X) \cap [G, G]$ on both factors. (Recall that $Z(G)^0 \to Z(X)$ under our conventions.)

For any function on $X_{\Theta}$ which is $A_{X,\Theta}$-finite (i.e. its translates under the normalized action (2.5) of $A_{X,\Theta}$ span a finite-dimensional space) we call \textit{exponents} its generalized $A_{X,\Theta}$-characters, considered as a multiset (i.e. each character appears with a certain multiplicity).

We say that a function $f \in C^\infty(X)$ (invariant, say, by an open compact subgroup $J$) is \textit{tempered} if for every $\Theta \subset \Delta_X$ there is a $J$-good neighborhood of $\Theta$-infinity where $|f|$ is bounded by an $A_{X,\Theta}$-finite function with \textit{trivial exponents} (equivalently: by the absolute value of an $A_{X,\Theta}$-finite function with \textit{unitary} exponents).
The Harish-Chandra Schwartz space \( \mathcal{C}(X) \) is the space of those functions \( f \in C^\infty(X) \) such that for every tempered function \( F \) we have:

\[
\rho_F(f) := \int_X |f \cdot F| dx < \infty.
\]

For example, in the abelian case \( X = \mathbb{Z}(X) \) (by choosing a base point), any smooth function descends to a function on a finitely generated abelian group \( \cong R \) (torsion) \( \times \mathbb{Z}^r \), and it is in the Harish-Chandra Schwartz space iff its restriction to any \( \mathbb{Z}^r \)-orbit is bounded by the multiple of the inverse of any polynomial in the coordinates \( n_1, n_2, \ldots, n_r \).

We similarly define this notion for the spaces \( X_\Theta \). Again, the twisted action (2.6) allows us to write the Harish-Chandra Schwartz space of \( X_\Theta \) as the normalized induction of the Harish-Chandra Schwartz space of \( X^L_\Theta \):

\[
\mathcal{C}(X_\Theta) = I_{\Theta^*} \mathcal{C}(X^L_\Theta), \tag{2.10}
\]

Indeed, the action of \( G \) is clearly the correct one as was the case for \( L^2(X_\Theta) \) and \( \mathcal{S}(X_\Theta) \); and the notion of “unitary exponents” used to define tempered functions and, by duality, the Harish-Chandra Schwartz space coincides for the action of \( A_{X_\Theta} \) on functions on \( X_\Theta \) and \( X^L_\Theta \).

The \( J \)-invariants of each of those Harish-Chandra Schwartz spaces have a natural Fréchet space structure, defined by a system of seminorms \( \rho_F \) as above for \( F \) belonging in a sequence \( (F_n)_n \) of tempered, \( J \)-invariant functions with the property: for every tempered function \( F' \) there is an \( n \) and a positive scalar \( c \) such that \( |F'| \leq c \cdot |F_n| \). (In fact, this is a nuclear Fréchet space.) Thus, the space \( \mathcal{C}(X) \) is an LF-space, i.e. a countable strict inductive limit of Fréchet spaces.

In case \( X \) is a direct product:

\[
X = \mathbb{Z}(X) \times X',
\]

where \( X' \) is a \([G, G]\)-spherical variety, we have a decomposition:

\[
\mathcal{C}(X) = \mathcal{C}(\mathbb{Z}(X)) \hat{\otimes} \mathcal{C}(X'),
\]

where the completed tensor product is defined as a strict inductive limit over the corresponding spaces of invariants under compact open subgroups, and for each such subgroup it is uniquely defined by nuclearity. In simple terms, this means the following: We may choose the sequence as above of tempered functions \( F_n \) to consist of product functions: \( F_{ij} = F_{i}^{(1)} \otimes F_{j}^{(2)} \), where \( F_{i}^{(1)} \) and \( F_{j}^{(2)} \) denote, respectively, similar sequences on \( \mathbb{Z}(X) \) and \( X' \). Then \( \mathcal{C}(X)^J \) is the completion of \( \mathcal{S}(X)^J = \mathcal{S}(\mathbb{Z}(X))^{J,\mathbb{Z}(X)} \otimes \mathcal{S}(X')^{J,\mathbb{Z}(X')^J} \) with respect to the corresponding seminorms.

In the general factorizable case, using a decomposition such as (2.2) and pulling back functions by the action map:

\[
\mathbb{Z}(X) \times X'_i \rightarrow \mathbb{Z}(X) \cdot X'_i,
\]
it is immediate to identify:
\[ \mathcal{C}(X) \simeq \bigoplus_i \left( \mathcal{C}(Z(X)) \otimes \mathcal{C}(X_i) \right) (Z(X) \cap [G,G])^{\text{diag}}. \]  
(2.11)

Notice that we also have:
\[ L^2(X) \simeq \bigoplus_i \left( L^2(Z(X)) \otimes L^2(X_i) \right) (Z(X) \cap [G,G])^{\text{diag}}, \]  
(2.12)
where the completed tensor product here is the Hilbert space tensor product.

2.4.1. Comparison with alternative definitions. Since the definition of Harish-Chandra Schwartz space is sometimes phrased differently in the literature, we would like to verify that the one we gave coincides with other versions. We start from the general definition given in [Ber88, §3.5]; according to it, the Harish-Chandra Schwartz subspace of \( C^\infty(X) \) is the one defined by the norms of \( L^2(X, (1 + r)^d \mu_X)^d \), \( d \geq 1 \). Here \( r \) is a radial function on \( X \), and the measure \( \mu_X \) is a \( G \)-invariant measure. (We leave the case of an eigenmeasure to the reader – cf. §3.7 of loc.cit.)

We remind that a radial function \( r : X \to \mathbb{R}^+ \) is a locally bounded proper function such that for every compact subset \( B \subset G \) there is a constant \( C > 0 \) with \( |r(gx) - r(x)| < C \) for all \( g \in B, x \in X \). The definition of the Harish-Chandra Schwartz space using radial functions generally depends on the radial function chosen up to the equivalence relation:

\[ r \sim r' \iff \exists C > 0 \text{ s.t. } C^{-1}(1 + r) \leq 1 + r' \leq C(1 + r). \]

However, it is possible to see that in our case there is a "maximal" such equivalence class, in the sense that is is dominating every other, i.e. if \( r' \) is any radial function then there is an \( r \) in that class such that:

\[ 1 + r' \leq 1 + r, \]

and this is the class that gives the Harish-Chandra Schwartz space that we defined. This class makes \( X \) into a space of polynomial growth in the language of Bernstein, and an element of this class can be described as follows:

We will use the existence of a (weak) Cartan decomposition for \( X \), namely a subvariety \( A_X \subset G \), which is an orbit of a Cartan subgroup \( A \) of \( G \), such that:

\[ X = A_X^+ U \]  
(2.13)
for some large enough compact subset \( U \) of \( G \), where \( A_X^+ \) denotes a certain notion of “dominant” elements of \( A_X \), cf. [BO07, DS11] for the symmetric case and [SV, Lemma 5.3.1] for the general split case. This decomposition can be used to cover \( X \) by Siegel domains in the sense of [Ber88, §4.6]. In particular, if we fix a radial function \( R \) on \( A_X \), the following is a radial function on \( X \):

\[ r(x) := \min \{ R(a) | a \in A_X^+, x \in aU \}. \]
In fact, as the proof of [SV, Lemma 5.3.1] shows, the subset $A_X$ in the above decomposition can be identified with the torus $A_{X,\Theta} = \mathcal{Z}(X_\Theta)$ of $G$-automorphisms of the most degenerate boundary degeneration, of which the other tori $A_{X,\Theta}$ are subtori. Then the above Cartan decomposition can be taken to be compatible with the $A_{X,\Theta}$-actions on good neighborhoods of infinity, in the following sense: For a fixed open compact subgroup $J$ (such that $U$ is $J$-invariant), there is a disjoint union: $X = \sqcup_{\Theta} N_\Theta$, where $N_\Theta$ belongs to a $J$-good neighborhood of $\Theta$-infinity, is $A_{X,\Theta}^+$-invariant and compact modulo the $A_{X,\Theta}$-action. Then, in a neighborhood of any point of "$\Theta$-infinity", the action of $A_{X,\Theta}^+$ on $J$-orbits in $N_\Theta$ is compatible with its action on $A_X$:

$$(a \cdot a_X)\cdot J = a \cdot (a_X\cdot J).$$

Here the action on the left is the action on $A_X$ when $A_{X,\Theta}$ is identified as a subtorus, and the action on the right is the action of $A_{X,\Theta} = \mathcal{Z}(X_\Theta)$ on $X_\Theta$ (where we have identified $J$-orbits on $N_\Theta$ with $J$-orbits on $X_\Theta$).

Thus, the above radial function is equivalent to the following one: Fix, for every $\Theta$, a $J$-invariant compact subset $M_\Theta \subset N_\Theta$ such that $A_{X,\Theta}^+ M_\Theta = N_\Theta$, and let, for each $x \in N_\Theta$:

$$r'(x) = \min \{ R(a) | a \in A_{X,\Theta}^+ \},$$

where $R(a)$ is the fixed radial function on $A_{X,\Theta} \subset A_X$.

Now observe that the functions $F_d(x) = (1 + r'(x))^{\frac{d}{2}} \mu_X(x) J^{-\frac{1}{2}}$ form a basis of tempered functions such as the ones used to define the Fréchet structure on the Harish-Chandra Schwartz space above. Thus, the system of norms of the spaces $L^2(X, (1 + r)^d \mu_X) J$ is equivalent to the system of norms $\rho_{F_d}$ defined using those functions. This shows the equivalence of our definition with Bernstein’s.

On the other hand, we have [DH14, Definition 3] in the case of a symmetric space, which defines the Harish-Chandra Schwartz space in terms of bounds of the form:

$$|f(x)| \ll \Theta_G(x)(N_d(x))^{-1},$$

$d > 0$. The function $N_d$ is of the form $(1 + r)^d$, for an algebraic radial function $r$; namely, $X$ is realized as a closed subvariety of affine space, and the function $r$ is the maximum of the absolute values of the coordinates. Such a radial function can easily be seen to be equivalent to the ones used above, cf. also [Ber88, §4.5]. The function $\Theta_G$ is a nonvanishing positive smooth function, which up to a power of $(1 + r)$ and a constant coincides with the volume of $xJ$, by a result of Lagier [DH14, (2.27)] and an estimate of the volumes in [KT10, Proposition 2.6]. Thus, this definition of $\mathcal{C}(X)$ also coincides with the above ones.

3. Bundles over tori
3.1. Bundles with flat connections over complex tori. Let $T$ be a complex algebraic torus, and let $V$ be a finite-dimensional complex vector space. Let $\Gamma \subset T$ be a finite subgroup, and let $\rho : \Gamma \to \text{GL}(V)$ be a representation. Thus, $\Gamma$ acts on the total space of the vector bundle $T \times V$, and the quotient is a vector bundle over the quotient torus $Y = T/\Gamma$. In algebro-geometric language, one defines this vector bundle as the sheaf of $\Gamma$-invariant sections of $T \times V$, which a priori is just an étale sheaf because it is not immediately clear that there are enough such sections on Zariski open subsets.

This will actually not be an issue: the representation $\rho$ always extends to a complex algebraic homomorphism $\tilde{\rho} : T \to \text{GL}(V)$: it decomposes into a finite sum of characters of $\Gamma$, and every character of $\Gamma$ extends to a complex algebraic character of $T$ (one can choose coordinates $T = (\mathbb{C}^\times)^n$ and then induce the character to a character of a finite subgroup generated by “coordinate subgroups” – the induced representation decomposes again to a finite sum of characters). Then, the quotient vector bundle with total space $(T \times V)/\Gamma$ is trivializable: once we choose a basis $(v_1, \ldots, v_n)$ of $V$, the trivialization is given by the sections: $(t, v_i(t))$, where $v_i(t) = \tilde{\rho}(t)v_i$.

In our setting we will apply this to $T = \check{X}^\text{ab}_\mathbb{C}, T/\Gamma = \text{a connected component of } \check{X}^\text{disc}_\mathbb{C}$ (and the corresponding tori for “boundary degenerations” – see §4 for the definitions). The vector bundle will come from certain spaces of coinvariants of $S(X)$.

Now we want to endow this quotient vector bundle with a flat connection, hence an action (on its sections) of the ring $\mathcal{D}(Y)$ of differential operators on $Y = T/\Gamma$. There are two obvious choices for doing that: One is to choose a trivialization by sections $v_i(t)$ as before and require that the $v_i(t)$’s are flat sections;

this is not the action that we will use. Rather, we consider the natural connection on the trivial vector bundle $T \times V$: $D\left(\sum_i c_i(t)v_i\right) = \sum_i (Dc_i(t))v_i \quad (D \in \mathcal{D}(T))$.

This descends to a connection on the quotient vector bundle $(T \times V)/\Gamma$ over $Y$. Indeed, differential operators on $Y$ are the $\Gamma$-invariant differential operators on $T$, i.e. those whose action on functions commutes with the action of $\Gamma$; the action of all differential operators also commutes with the action of $\text{GL}(V)$ on sections, hence preserves $\Gamma$-invariant sections. This is the action that we will be using.

For convenience we introduce a notion of flat functional on the vector bundle with total space $E = (T \times V)/\Gamma$. A flat functional will be an element of the dual vector space $V^*$, thought of as a flat section of the dual of the pull-back of $E$ to $T$. Thus, it is by abuse of language that we call it a “flat functional on $E$” since it is really a flat section of the dual vector bundle over an étale cover of $Y = T/\Gamma$, and not over $Y$. Any section $y \mapsto f_y$ of $\mathscr{L}$, together with a flat functional $v^*$, give rise to a function $F(f_y, v^*) : t \mapsto \langle f_t, v^* \rangle$ on $T$ (not on $Y = T/\Gamma$). The action of differential operators was
defined in such a way that for every section \( f_y \), every flat section \( v^* \) and every differential operator \( D \in \mathcal{D}(Y) \) we have:

\[
F(Df_y, v^*) = DF(f_y, v^*).
\]

3.2. **Various spaces of sections.** Let \( T \) now denote the compact real subtorus of a complex torus \( T_\mathbb{C} \), or a torsor thereof. Let \( L \) be a (finite dimensional) complex algebraic vector bundle over \( T_\mathbb{C} \). We introduce the following notation, for sections of \( L \):

- We denote by \( \mathbb{C}[T, L] \) the regular sections of \( L \) over \( T \) – that is, over \( T_\mathbb{C} \).
- We denote by \( \mathbb{C}(T, L) \) the rational sections.
- We denote by \( \Gamma(T, L) \) the rational sections which are regular on the real subset \( T \); by “regular” we mean that their polar divisors do not intersect \( T \); however, with an extra restriction on the poles which we will introduce, this will turn out to be equivalent to the weaker condition that they extend to \( C^\infty \), or even \( L^2 \), sections (see Lemma 3.5).
- Thinking of (the set of real points) \( T \) as a smooth manifold and of \( L \) as a smooth vector bundle over \( T \), we denote by \( C^\infty(T, L) \) the smooth sections over \( T \); it carries a canonical structure of a Fréchet space.

If \( T \) is trivializable, we have a canonical isomorphism:

\[
C^\infty(T, L) = \mathbb{C}[T, L] \otimes_{\mathbb{C}[T]} C^\infty(T).
\]

- If we choose a Haar measure on \( T \) and a global section of \( L^* \otimes \bar{L}^* \) (the bundle of hermitian forms on \( L \), where \( \bar{\cdot} \) denotes the same module of sections as a set but with the coordinate ring of \( T_\mathbb{C} \) acting via complex conjugation induced from the compact real form \( T \)) then we have a notion of \( L^2 \)-sections of \( L \) over \( T \). All such choices with the chosen hermitian form positive definite over \( T \) give isomorphic topological vector spaces of \( L^2 \)-sections, although of course the Hilbert norm will depend on the choices.

In this paper, all \( L^2 \)-structures on such bundles will be such, i.e. positive definite on each fiber over \( T \). In fact, it will usually be the case that \( L^* \otimes L^* \) has a canonical trivialization, and that the hermitian form is taken to be constant with respect to this trivialization.

The following easy lemma will be useful:

3.3. **Lemma.** Let \( T \) be a real torus and \( \Gamma \) a finite subgroup. The natural map:

\[
\mathbb{C}[T] \otimes_{\mathbb{C}[T/\Gamma]} C^\infty(T/\Gamma) \to C^\infty(T)
\]

is an isomorphism.
Proof. We can extend each complex character $\chi$ of $\Gamma$ to a character $\tilde{\chi}$ of $T$ ($\tilde{\chi} \in \mathbb{C}[T]$), then write each $f \in C^\infty(T)$ as a linear combination of its $\chi$-equivariant parts:

$$f = \sum_{\chi} f_{\chi}, \text{ where } f_{\chi}(t) = \frac{1}{|\Gamma|} \sum_{\gamma} \chi^{-1}(\gamma) f(\gamma t),$$

and finally $f_{\chi} = \tilde{\chi} \cdot \frac{L_{\chi}}{\bar{\chi}}$, with the last factor an element of $C^\infty(T/\Gamma)$. This shows surjectivity, and injectivity is easy. □

3.4. Linear poles. We continue assuming that $L$ is a trivializable vector bundle over a complex torus or torso thereof $T_C$, whose compact real form we denote by $T$. A linear divisor on $T$ will be the scheme-theoretic zero set of a polynomial of the form:

$$\prod_i (\chi_i - r_i)$$

(3.1)

where:

- the $r_i$’s are nonzero scalars;
- the $\chi_i$’s are “characters” of $T$ – more precisely: nonzero eigenfunctions for the torus acting on $T$.

In particular, a linear divisor is always principal. The word “linear” stems from the fact that under an exponential map: $t \to T$ (and its complexification $t_C \to T_C$) their preimages are unions of linear hyperplanes – in fact, linear hyperplanes associated to real functionals.

We say that a rational section $f \in \mathbb{C}(T, L)$ has linear poles if:

$$\prod_i (\chi_i - r_i) f \in \mathbb{C}[T, L] \text{ (regular sections)}$$

for a finite set of $\chi_i$’s and $r_i$’s as above. A very crucial lemma will be the following:

3.5. Lemma. If $f \in \mathbb{C}(T)$ has linear poles and belongs to $L^1(T)$, then it belongs to $\Gamma(T)$, i.e. its poles do not meet the unitary set $T$.

The notion of $L^1(T)$ is defined with respect to any Haar measure on $T$.

Proof. Using the exponential map, we can pull back the function to a holomorphic function $F$ on the complexification $t_C$ of the Lie algebra, with poles along complex hyperplanes and locally integrable on the real subspace $t$. Thus, locally around a point on $t$ which without loss of generality we may assume to be zero, the pullback looks like:

$$h \cdot \prod_i l_i^{-r_i},$$

where the $l_i$’s are real linear functionals, the $r_i$’s are positive integers, and $h$ is holomorphic which does not identically vanish on the zero set of any of the $l_i$’s. If such a function is locally integrable, the same is true a fortiori.
when the denominator is replaced by a single linear functional \( l_1 \), thus we may assume that \( F = h \cdot l_1^{-1} \), where \( h \) is not divisible by \( l_1 \).

There is a (real) point of \( t \) in any neighborhood of zero which is in the kernel of \( l_1 \) but not on the zero set of \( h \) (otherwise, \( h \) would be divisible by \( l_1 \)). Thus, in a neighborhood of that point the function is bounded by a constant times \( l_1^{-1} \), and cannot be integrable.

\[ \Box \]

Finally, if \( L_1, L_2 \) are two vector bundles as above, then \( \text{Hom}(L_1, L_2) \) is also such a vector bundle, and we can talk about “rational sections” of that and “linear poles” for those sections. In particular, we have the following easy corollary of the previous lemma:

3.6. **Corollary.** Suppose that \( L_1, L_2 \) are endowed with hermitian structures as described in §3.2 (i.e., positive definite on each fiber over \( T \)), and \( M \in \mathbb{C}(T, \text{Hom}(L_1, L_2)) \) has linear poles and induces a bounded map:

\[ L^2(T, L_1) \to L^2(T, L_2). \]

Then \( M \in \Gamma(T, \text{Hom}(L_1, L_2)) \), i.e. its poles do not meet the unitary set.

**Proof.** By taking fiberwise operator norms, we easily see that:

\[ T \ni t \mapsto \|M_t\|^2 \in \mathbb{C}(T) \]

has only linear poles, no fewer than those of \( M \). The norm of \( M \) as a bounded map: \( L^2(T, L_1) \to L^2(T, L_2) \) is the \( L^1 \)-norm of this function. \( \Box \)

4. **Coinvariants and the bundles of \( X \)-discrete and \( X \)-cuspidal representations**

4.1. **Coinvariants.** For an irreducible representation \( \pi \) of \( G \), the \( \pi \)-coinvariants of \( S(X) \) is the quotient of \( S(X) \) by the common kernel of all morphisms: \( S(X) \to \pi \). They can be canonically identified with:

\[ S(X)_\pi = (\text{Hom}_G(S(X), \pi))^* \otimes \pi. \]

A subspace of \( \text{Hom}_G(S(X), \pi) \) corresponds to a quotient of the space \( S(X)_\pi \) of \( X \)-coinvariants. Let \( \pi \) have unitary central character \( \chi_\pi \); recall here that by (2.1) we assume that the connected center of \( G \) surjects onto the “center of \( X \)”. We call an element of \( \text{Hom}_G(S(X), \pi) \) “cuspidal” if \( \pi \) has unitary central character and the dual:

\[ \tilde{\pi} \to C^\infty(X) \]

has image in the space of compactly supported functions modulo the center. We call it “discrete” if the dual has image in \( L^2(X/\mathcal{Z}(X), \chi_{\tilde{\pi}}) \), where \( \chi_{\tilde{\pi}} \) is the central character. We call it “tempered” the dual has image in the space of tempered functions or, equivalently, if the morphism extends continuously to the Harish-Chandra Schwartz space \( \mathcal{C}(X) \). The “continuous” assumption will be implicit whenever we write homomorphisms from \( \mathcal{C}(X) \).
Thus, we have natural surjections:

\[ S(X) \pi \to S(X)_{\pi, \text{temp}} \to S(X)_{\pi, \text{disc}} \to S(X)_{\pi, \text{cusp}}, \tag{4.1} \]

where the second corresponds to tempered morphisms. The second, third and fourth are also coinvariants for the Harish-Chandra Schwartz space, i.e. the canonical quotient map from \( S(X) \) extends continuously to:

\[ \mathcal{E}(X) \to S(X)_{\pi, \text{temp}}. \tag{4.2} \]

If \( \pi \) does not have unitary central character, we will still be using the notation \( S(X)_{\pi, \text{temp}}, S(X)_{\pi, \text{disc}} \) and \( S(X)_{\pi, \text{cusp}} \) for quotients of \( S(X) \) such that the corresponding morphisms:

\[ S(X) \to \pi \text{ or } \bar{\pi} \to C^\infty(X) \]

have the aforementioned properties up to a twist by a character of the group.

4.2. \( X \)-discrete and \( X \)-cuspidal components. We let \( \hat{X}_{\text{cusp}} \) denote the set of irreducible representations \( \pi \) with unitary central character such that \( S(X)_{\pi, \text{cusp}} \neq 0 \); we let \( \hat{X}_{\text{disc}} \) denote the set of irreducible representations \( \pi \) with unitary central character such that \( S(X)_{\pi, \text{disc}} \neq 0 \). Thus, \( \hat{X}_{\text{cusp}} \subset \hat{X}_{\text{disc}} \).

Both sets have a natural topology, and split into disjoint, possibly infinite, unions of compact components which can naturally be identified with the real points of real algebraic varieties of the same dimension, each of which is a principal homogeneous space for a torus. This structure arises as follows:

Recall from §2 that \( X_{\text{ab}} \) is a quotient variety of \( X \), which is a torsor for the torus \( G^\text{ab}/H^\text{ab} \); we denote by \( \hat{X}_{\text{ab}} \) the real torus of unitary unramified characters of this torus. It acts with finite stabilizers on \( \hat{X}_{\text{disc}} \) because a morphism \( S(X) \to \pi \) can be “twisted” by any element of \( \hat{X}_{\text{ab}} \); the following is [SV, Theorem 9.2.1], [Del, Theorem 4]:

4.3. Proposition. For each open compact subgroup \( \mathcal{J} \) of \( G \), the set of \( \hat{X}_{\text{ab}} \)-orbits on elements of \( \hat{X}_{\text{disc}} \) with nonzero \( \mathcal{J} \)-fixed vectors is finite.

In particular, the set of \( \hat{X}_{\text{ab}} \)-orbits on \( \hat{X}_{\text{disc}} \) is countable, and the action endows the latter with a real algebraic structure. We denote by \( \hat{X}_{\text{C}}^{\text{cusp}}, \hat{X}_{\text{C}}^{\text{disc}}, \hat{X}^{\text{ab}}_{\text{C}} \) the complex points of these varieties.

4.4. The bundles \( \mathcal{L}_\pi, \mathcal{L}'_\pi \). Consider the associations:

\[ \hat{X}_{\text{C}}^{\text{cusp}} \ni \pi \mapsto \mathcal{L}_\pi := S(X)_{\pi, \text{cusp}}, \]

\[ \hat{X}_{\text{C}}^{\text{disc}} \ni \pi \mapsto \mathcal{L}'_\pi := S(X)_{\pi, \text{disc}}. \]

We will discuss in more detail the process of twisting by unramified characters, in order to use the formalism of §3.1 to understand these associations as fibers of complex algebraic vector bundles \( \mathcal{L}, \mathcal{L}' \) over \( \hat{X}_{\text{C}}^{\text{cusp}} \), resp. \( \hat{X}_{\text{C}}^{\text{disc}} \).
endowed with (slightly noncanonical) flat connections. More precisely, we will do that for the \( J \)-coinvariants \( L^J, \mathcal{L}^J \), which will be vector bundles supported over a finite number of connected components by Proposition 4.3 and with finite-dimensional fibers, and then we will define the space of sections of \( L, \mathcal{L} \) to be direct limits over all \( J \) of \( J \)-invariant sections.

For notational simplicity, we only discuss the case of \( L^J(X) \)-discrete; the other is identical and, after all, it is just a quotient of \( L^J \) (and, as we shall see later, also a direct summand).

To begin, we notice again that \( S(X)_{\pi,\text{disc}} \) is a canonical quotient of \( S(X) \) which depends only on the isomorphism class of \( \pi \) and not on its realization. In particular, if \( \pi \cong \pi \otimes \omega \) for some character \( \omega \) then we canonically have:

\[
S(X)_{\pi,\text{disc}} \cong S(X)_{\pi \otimes \omega,\text{disc}}.
\]  

(4.3)

To explain this in terms of the isomorphism:

\[
S(X)_{\pi,\text{disc}} = (\text{Hom}_G(S(X), \pi)_{\text{disc}})^* \otimes \pi,
\]  

(4.4)

we notice that the difference between any two choices of isomorphisms: \( \pi \cong \pi \otimes \omega \) is a scalar which gets cancelled when we tensor \( \pi \) with the linear dual of \( \text{Hom}_G(S(X), \pi)_{\text{disc}} \).

On the other hand, the realization (4.4) will allow us to identify the spaces \( S(X)_{\pi \otimes \omega,\text{disc}} \) as \( \omega \) varies along all elements of \( \hat{X}^\text{ab}_C \) – of course, such an identification will only be a vector space identification, not \( G \)-equivariant. Also, this identification will not coincide with the canonical \( G \)-equivariant identification when \( \pi \cong \pi \otimes \omega \).

The way to perform this linear identification is by fixing a point \( x_0 \in X \) (or, at least, its image in \( X^\text{ab} \)) and hence identifying \( X^\text{ab} \) with the abelian quotient of \( G \) of which it is a torsor. For each \( \omega \in \hat{X}^\text{ab}_C \) we define linear isomorphisms between the finite-dimensional vector spaces (4.4) as follows:

1. We identify the space of the representation \( \pi \otimes \omega \) with the vector space \( W \) of \( \pi \), simply by tensoring any element with \( \omega \) (here \( \omega \) is considered, by restriction, a character of \( G(F) \)); notice that this identification does not necessarily respect isomorphisms of the form: \( \pi \cong \pi \otimes \omega \).

2. We define a linear isomorphism:

\[
\text{Hom}_G(S(X), \pi)_{\text{disc}} \ni M \mapsto M_\omega \in \text{Hom}_G(S(X), \pi \otimes \omega)_{\text{disc}}
\]

by:

\[
M_\omega(\Phi) = M(\Phi \cdot \omega).
\]  

(4.5)

This trivializes the topological vector bundle:

\[
\hat{X}^\text{ab}_C \ni \omega \mapsto S(X)^J_{\pi \otimes \omega,\text{disc}},
\]  

(4.6)

and thus gives it the structure of an algebraic line bundle over \( \hat{X}^\text{ab}_C \). Explicitly, a section \( \omega \mapsto f_\omega \in S(X)_{\pi \otimes \omega} \) is regular iff for every \( w^* \) in the dual
vector space \( W^* \) we have: \( \langle (M_\omega)(f_\omega), w^* \rangle \in \mathbb{C}[\hat{X}^{ab}_C] \), where \( M_\omega \) was defined above.

The algebraic structure on (4.6) does not depend on the choice of base point \( \pi \) for our representations, or base point \( x_0 \) on \( X \). However, the corresponding flat connection, and hence the notion of flat functionals of \( \Sect \) depends on the choice of base point \( x_0 \) up to a character of the torus \( \hat{X}^{ab}_C \). More precisely, the “flat functionals” are the expressions \( f_\omega \mapsto \langle (M_\omega)(f_\omega), w^* \rangle \) that we encountered above.

Now we define the complex vector bundle \( \mathcal{L}^J \) over \( \hat{X}^{\text{disc}}_C \) so that the bundle (4.6) is simply its pull-back under the orbit map \( \omega \mapsto \pi \otimes \omega \). In the notation of \( \Sect \), \( V \) is the space \( S(X)^J_{\pi, \text{disc}} \), \( T = \hat{X}^{ab}_C \), \( \Gamma \) is the subgroup of those \( \omega \) such that:

\[
\pi \otimes \omega \simeq \pi,
\]
and \( \rho(\omega) \) is the canonical \( G \)-equivariant isomorphism (4.3) of the corresponding spaces of coinvariants.

Hence, \( \mathcal{L}^J \) is the vector bundle over \( Y = T/\Gamma \) (and, by repeating the same process for each connected component, over \( \hat{X}^{\text{disc}}_C \)) with total space \( (T \times V)/\Gamma \). We have:

4.5. **Lemma.** The vector bundle \( \mathcal{L}^J \) over \( \hat{X}^{\text{disc}}_C \) is trivializable (over each connected component).

This follows from the generalities discussed in \( \Sect \), but it can also be seen explicitly, since \( \mathcal{L}^J \) can be trivialized by sections coming from \( S(X)^J \). Indeed, it is clear from the definitions that the natural map:

\[
S(X)^J \to S(X)^J_{\pi, \text{disc}}
\]
gives rise to regular sections of \( \mathcal{L}^J \), as \( \pi \) varies, i.e. it gives a canonical map:

\[
S(X) \to \mathbb{C}[\hat{X}^{\text{disc}}_C, \mathcal{L}].
\]

(4.8) (Similarly, by composing with the quotient map \( \mathcal{L} \to \mathcal{L} \) we get a canonical map:

\[
S(X) \to \mathbb{C}[\hat{X}^{\text{cusp}}_C, \mathcal{L}].
\]

(4.9)

Secondly, the map \( S(X)^J \to S(X)^J_{\pi, \text{disc}} \) is surjective for any irreducible \( \pi \), hence (4.8), composed with evaluation at each fiber, is surjective. Thirdly, choose a finite number \( f_i \) of characteristic functions on \( J \)-orbits \( x_i, J \) on \( X \) such that their images form a basis of \( S(X)_{\pi} \), for some fixed \( \pi \). If \( \hat{f}_i(\omega) \) denotes the image of \( f_i \) in \( S(X)_{\pi \otimes \omega} \) with all those vector spaces identified with each other as above, it is immediate from the above definitions that, in this common vector space, \( \hat{f}_i(\omega) = \omega(x_i) \cdot \hat{f}_i(1) \). Hence, the images of the \( f_i \)'s form a basis for every fiber of (4.6). But the images of the \( f_i \)'s are well-defined sections of \( \mathcal{L}^J \), and thus this line bundle is also trivializable. We will see in Proposition 5.3 that elements of \( S(X) \) actually provide all global sections of \( \mathcal{L} \).
We let:
\[
\mathcal{L} = \lim_\to \mathcal{L}^J,
\]
\[
\hat{\mathcal{L}} = \lim_\to \hat{\mathcal{L}}^J
\]
as direct limits of sheaves, i.e. the corresponding sections will be, by definition, sections of the finite-dimensional vector bundle of $J$-invariants for some open compact subgroup $J$.

Fixing our $G$-invariant measure (or eigenmeasure) on $X$, the discrete part of the Plancherel decomposition for $L^2(X)$ endows the space $\hat{X}^{\text{disc}}$ with a canonical measure valued in the space of hermitian forms on $\mathcal{E}(X)$; more precisely, it is a measure valued in the space of invariant (and, in fact, non-degenerate) hermitian forms on the vector bundle $\mathcal{E}$, i.e. it is valued in the vector bundle $(\mathcal{L}^* \otimes \mathcal{E}^*)^G$. (Since our “vector bundles” are really direct limits of finite-dimensional ones, their duals will be by definition the inverse limits of the duals – however, the fibers of $G$-invariants in this case are just finite-dimensional. We point the reader to [SV, §6.1] for a general discussion of the formalism of Plancherel formulas.)

We will be denoting this measure by:
\[
\langle \ , \ \rangle_\pi d\pi.
\]
(4.10)

Notice that the bundle $(\mathcal{L}^* \otimes \mathcal{E}^*)^G$ is canonically trivializable, because for every unitary character $\omega$ and unitary representation $\pi$ of $G$, the unitary structure on $\pi$ induces one on $\pi \otimes \omega$ (by identifying the vector spaces $\pi$ and $\pi \otimes \omega$). It is then trivial to see from the definitions that:

4.6. Lemma. The quotient of the measure $\langle \ , \ \rangle_\pi d\pi$ by a Haar measure on $\hat{X}^{\text{disc}}$ is constant with respect to the canonical trivialization of $(\mathcal{L}^* \otimes \mathcal{E}^*)^G$.

In other words, the vector bundles $\mathcal{L}^J$ are endowed with an $L^2$-structure like the one considered in §3.2.

Dividing by a Haar measure on $\hat{X}^{\text{disc}}$, we get canonical – up to a scalar – hermitian forms on the fibers $\mathcal{L}_\pi$, which allow us to split the canonical quotient from discrete to cuspidal: $\mathcal{L} \to \hat{\mathcal{L}}$. This is, of course, just the orthogonal projection to the cuspidal subspace of $L^2(X/\mathcal{Z}(X), \chi)_{\text{disc}}$, for every unitary character $\chi$. The above lemma shows that there are trivializations compatible with this splitting, i.e. $\mathcal{L}$ is a direct summand of $\mathcal{E}$ as algebraic vector bundles over $X^{\text{disc}}_G$.

Similarly, the Plancherel decomposition for $L^2(X_\Theta)_{\text{disc}}$ gives a canonical $(\mathcal{L}_\Theta^* \otimes \mathcal{E}_\Theta^*)^G$-valued measure:
\[
\langle \ , \ \rangle_\sigma d\sigma.
\]
(4.11)
on $X^{\text{disc}}_\Theta$. 


4.7. The case of boundary degenerations. We have a finer decomposition for the analogous spaces of $X_{\Theta}$, not in terms of representations of $G$ but in terms of representations of a Levi subgroup. Recall the isomorphisms (2.8), (2.10):

$$S(X_\Theta) = I_\Theta \cdot S(X^L_\Theta),$$

$$E(X_\Theta) = I_\Theta \cdot E(X^L_\Theta).$$

For each irreducible representation $\sigma$ of $L_\Theta$, by inducing the quotient $S(X^L_\Theta)_{\sigma,\text{disc}}$ of $S(X^L_\Theta)$ we get a representation:

$$S(X_\Theta)_{\sigma,\text{disc}} := \left( \text{Hom}_{L_\Theta}(\tilde{\sigma}, C^\infty(X^L_\Theta)) \right) ^* \otimes I_\Theta \cdot \sigma,$$

together with a canonical map:

$$S(X_\Theta) \rightarrow S(X_\Theta)_{\sigma,\text{disc}}.$$

(This is the “discrete” quotient of the space $S(X_\Theta)$ defined in §15.2.6 of [SV].) Similarly, we define the cuspidal $\sigma$-coinvariants: $S(X_\Theta)_{\sigma,\text{cusp}}$. In other words, the spaces of discrete and cuspidal $\sigma$-coinvariants are the quotients corresponding to morphisms (of the corresponding type) from $\tilde{\sigma}$ to $C^\infty(X^L_\Theta)$, induced by Frobenius reciprocity to morphisms: $I_\Theta \cdot \tilde{\sigma} \rightarrow C^\infty(X^L_\Theta)$.

The spaces $S(X_\Theta)_{\sigma,\text{disc}}$ form a trivializable complex vector bundle over $X^L_\Theta$, which we will denote by $\mathcal{L}_\Theta$ (again as a direct limit over $J$-invariants, to be precise). The spaces $S(X_\Theta)_{\sigma,\text{cusp}}$ form a trivializable complex vector bundle over $X^L_\Theta$, which we will denote by $\mathcal{L}_\Theta$. Again, the definition of the algebraic structure of these vector bundles is obtained by pulling back to $X^L_\Theta$, and they are endowed with the flat connections described in §3 (depending on the choice of a base point on $X^L_\Theta$).

Although the isomorphism (2.3), and the subsequent isomorphisms (2.10), (2.8), depend on the choice of parabolic $P^-_\Theta$ in its class, it is clear that the spaces $S(X_\Theta)_{\sigma,\text{disc}}$, $S(X_\Theta)_{\sigma,\text{cusp}}$ can be considered as canonical quotients of $S(X_\Theta)$, and hence the vector bundles $\mathcal{L}_\Theta$, $\mathcal{L}_\Theta$ do not depend on choices. Indeed, the kernels of the maps $S(X_\Theta) \rightarrow S(X_\Theta)_{\sigma,\text{disc}}$, $S(X_\Theta) \rightarrow S(X_\Theta)_{\sigma,\text{cusp}}$ do not depend on the choice of parabolic.

Part 2. Discrete and cuspidal parts

5. Discrete part of the Harish-Chandra Schwartz space

Recall that the Hilbert space $L^2(X)$ has an orthogonal direct sum decomposition $L^2(X) = L^2(X)_{\text{disc}} \oplus L^2(X)_{\text{cont}}$, where $L^2(X)_{\text{disc}}$ has a Plancherel decomposition in terms of discrete morphisms from irreducible representations: $\pi \rightarrow C^\infty(X)$, in the sense of §4.1, i.e. with unitary central characters and in $L^2$ modulo the center. We let $\mathcal{E}(X)_{\text{disc}} = \mathcal{E}(X) \cap L^2(X)_{\text{disc}}$, and similarly for the spaces $X_\Theta$. 
5.1. **Proposition.** Let \( Y \) be a connected component of \( \hat{X}^{\text{disc}} \); it corresponds to a direct summand \( L^2(X)_Y \) of \( L^2(X)_{\text{disc}} \) by restriction of the Plancherel measure to \( Y \). The orthogonal projection of an element of \( \mathcal{E}(X) \) to \( L^2(X)_Y \) lies in \( \mathcal{E}(X) \). In particular, the orthogonal projection of an element of \( \mathcal{E}(X) \) to \( L^2(X)^{\text{disc}} \) lies in \( \mathcal{E}(X) \), and we have a direct sum decomposition:

\[
\mathcal{E}(X) = \mathcal{E}(X)_{\text{disc}} \oplus \mathcal{E}(X)_{\text{cont}},
\]

where \( \mathcal{E}(X)_{\text{disc}} = \mathcal{E}(X) \cap L^2(X)_{\text{disc}} \) and \( \mathcal{E}(X)_{\text{cont}} = \mathcal{E}(X) \cap L^2(X)_{\text{cont}} \).

Since for every open compact subgroup \( J \) there is only a finite number of connected components \( Y \) with \( L^2(X)_Y \neq 0 \), the proposition actually gives a finer decomposition of \( \mathcal{E}(X)_{\text{disc}} \):

\[
\mathcal{E}(X)_{\text{disc}} = \bigoplus_Y \mathcal{E}(X)_Y,
\]

(5.1)

where \( Y \) ranges over all connected components of \( \hat{X}^{\text{disc}} \).

**Proof.** Since \( X \) is assumed to be factorizable, we may represent \( \mathcal{E}(X) \) as in (2.11). Clearly, “projection to discrete” can be defined only with respect to the action of \([G,G]\), which reduces the statement to the spaces \( \mathcal{E}(X)' \) in the notation of (2.11), i.e. reduces the problem to the case: \( Z(X) = 1 \).

In this case, the finiteness theorem 9.2.1 of [SV] for discrete series with \( J \)-invariant vectors the usual criterion of Casselman characterizing discrete series as those representations which appear with subunitary exponents in all directions (s. Kato-Takano [KT10] for the symmetric case) imply that the projection of any smooth element of \( L^2 \) to the discrete part lies in the Harish-Chandra Schwartz space. The first statement follows, and it easily implies the second. \( \square \)

Similar decompositions hold for all the boundary degenerations \( X^\Theta \); this is seen simply by inducing from the Levi varieties \( X^\Theta_L \), i.e. it follows from (2.10). The spectral description of \( \mathcal{E}(X^\Theta)_{\text{disc}} \) is as follows:

5.2. **Theorem.** For every \( \Theta \), the canonical quotient maps: \( \mathcal{E}(X^\Theta) \to \mathcal{E}(X^\Theta)_{\sigma,\text{disc}} \) gives rise to a canonical isomorphism:

\[
\mathcal{E}(X^\Theta)_{\text{disc}} \cong C^\infty(X^\Theta_{\text{disc}})^{\text{cont}},(\mathcal{L}^\Theta).
\]

(5.2)

The “isomorphism”, here and throughout the paper, is in the category of \( G \)-representations on LF-spaces (countable strict inductive limits of Fréchet spaces).

In preparation for the proof of Theorem 5.2, recall the vector bundle of \( X \)-discrete series \( \mathcal{L} \). Essentially as a corollary of the definition, the image of elements of \( S(X) \) is in \( \mathbb{C}[\hat{X}^{\text{disc}}, \mathcal{L}] \). It turns out that we get all global sections this way:

5.3. **Proposition.** The global (regular) sections of \( \mathcal{L}^\Theta \) over \( \hat{X}^{\text{disc}} \) are precisely the images of elements of \( S(X)^\Theta \).
Proof. If $Z$ denotes the center of $X$ and $\hat{Z}$ its character group, we have a “homomorphism” with finite kernels: $\hat{X}^{\text{disc}} \to \hat{Z}$ (central character). Here “homomorphism” stands for an equivariant map with respect to the natural action of $\hat{X}^{\text{ab}}$ on both, and “finite kernels” means finite fibers when restricted to each connected component of $X^{\text{disc}}$.

We may push $\mathcal{L}^J$ forward to $\hat{Z}$ – by definition, global sections of the push-forward are global sections of the original bundle. The fiber of the pushforward over a point $\chi \in \hat{Z}_C$ is equal to:

$$\bigoplus_{\pi \mapsto \chi} \mathcal{S}(X)^J_{\pi, \text{disc}};$$

this is a finite sum, since there are finitely many $X$-discrete series with given central character and nonzero $J$-fixed vectors.

By Nakayama’s lemma, it suffices to show that the image of $\mathcal{S}(X)^J$ is a $\mathbb{C}[\hat{Z}]$-module and surjects onto the fiber at each point. It is clearly a $\mathbb{C}[\hat{Z}]$-module because this corresponds to the action of $\hat{Z}$. And we already know that $\mathcal{S}(X)$ surjects on each summand of (5.3). But the representations $\pi$ indexing the sum are irreducible and nonisomorphic, so it has to surject to the sum.

Similarly, the Plancherel decomposition gives rise to a canonical isomorphism:

$$L^2(X)_{\text{disc}} \cong L^2(\hat{X}^{\text{disc}}, \mathcal{L}).$$

The hermitian structure on the right hand side was discussed in §4, (4.10).

Finally, we are ready to prove Theorem 5.2:

**Proof of Theorem 5.2.** The proof relies on the Payley-Wiener theorem for the Harish-Chandra Schwartz space of, essentially, finitely generated abelian groups.

By the isomorphism (2.10), it is enough to prove the theorem for $\mathcal{C}(X^{\Theta})_{\text{disc}}$, hence we are reduced to the case of $X^{\Theta} = X$.

First of all, we claim that the image of:

$$\mathcal{C}(X) \to L^2(\hat{X}^{\text{disc}}, \mathcal{L})$$

lies in $C^\infty$, and that the resulting map:

$$\mathcal{C}(X) \to C^\infty(\hat{X}^{\text{disc}}, \mathcal{L})$$

(5.5)

is continuous with respect to the finer topologies of these spaces. This is easy to see using the fact that $X$ is factorizable, the decomposition (2.9), and the Paley-Wiener theorem for finitely generated abelian groups.

Next, we claim that the map (5.5) is onto. For this, we let $Z \subset Z(X)$ be a free abelian subgroup such that $Z(X)/Z$ is compact. Notice that the group ring of $Z$ is canonically isomorphic to the (complexification of the) coordinate ring of $\hat{Z}$, the torus of unitary characters of $Z$. Moreover, this extends to an isomorphism:

$$\mathcal{C}(Z) \cong C^\infty(\hat{Z}).$$

(5.6)
By restriction of central characters we get embeddings:
\[ \mathbb{C}[\hat{Z}] \to \mathbb{C}[\hat{X}_{ab}] \]
and:
\[ C^\infty(\hat{Z}) \to C^\infty(\hat{X}_{ab}) \]

Recall the surjection of Proposition 5.3:
\[ S(X) \to \mathbb{C}[\hat{X}_{\text{disc}}, \mathcal{L}] \quad (5.7) \]

The action of the Harish-Chandra Schwartz algebra of \( \mathbb{Z} \) on \( S(X) \):
\[ \mathcal{C}(Z) \otimes S(X) \to \mathcal{C}(X) \]
translates on the right hand side of (5.7) as multiplication by \( C^\infty(\hat{Z}) \). Finally, by Lemma 3.3 the multiplication map is surjective:
\[ C^\infty(\hat{Z}) \otimes_{\mathbb{C}[\hat{Z}]} \mathbb{C}[\hat{X}_{\text{disc}}, \mathcal{L}] \to C^\infty(\hat{X}_{\text{disc}}, \mathcal{L}) \]

This shows surjectivity of (5.5). The kernel is, essentially by definition, the subspace \( \mathcal{C}(X)_{\text{cont}} \), and thus the map induces an isomorphism of \( \mathcal{C}(X)_{\text{disc}} \) with \( C^\infty(\hat{X}_{\text{disc}}, \mathcal{L}) \).

5.4. The discrete center of \( X \). From Theorem 5.2 we deduce that the ring \( C^\infty(\hat{X}_\Theta^{L_{\text{disc}}}) \) of smooth functions on \( \hat{X}_\Theta^{L_{\text{disc}}} \) acts \( G \)-equivariantly on the Harish-Chandra Schwartz space \( \mathcal{C}(X_\Theta)_{\text{disc}} \). We extend this action to the whole space \( \mathcal{C}(X_\Theta) \) by demanding that it acts as zero on \( \mathcal{C}(X_\Theta)_{\text{cont}} \). We will call this ring the discrete center of \( X_\Theta^{L_{\text{disc}}} \), and denote it by:
\[ C^\infty(\hat{X}_\Theta^{L_{\text{disc}}}) : = \mathcal{C}_{\text{disc}}(X_\Theta^{L_{\text{disc}}}) \]

In the case of \( X_\Theta = X \), we can think of this as the relative analog of the discrete part of the center of the Harish-Chandra Schwartz algebra, i.e. the discrete part of the “tempered Bernstein center” of Schneider and Zink [SZ08].

Remark. Maybe from the point of view of the “relative Langlands program” this is not quite the full “center”. Notice that if, for \( \pi \in \hat{X}_{\text{disc}} \), the space \( S(X)_\pi_{\text{disc}} \) has multiplicity \( n > 1 \) as a \( G \)-representation, then there is a larger ring acting \( G \)-equivariantly on the connected component (call it \( Y \subset \hat{X}_{\text{disc}} \)) of \( \pi \) — but it is non-commutative: It is, noncanonically, the ring \( C^\infty(Y, \text{Mat}_n) \), i.e. \( \text{Mat}_n \)-valued smooth functions. The philosophy of the relative Langlands program proposed in [SV] suggests that this multiplicity should be related to the number of lifts of the Langlands parameter of \( \pi \) to a suitable “\( X \)-distinguished parameter” into the \( L \)-group \( ^L \! G_X \) of \( X \); a more precise statement involves Arthur parameters and packets, and we won’t get into that. That suggests that there might be a distinguished decomposition of \( S(X)_{\pi,\text{disc}} \) into a direct sum of multiplicity-free spaces, which would canonically (up to indexing the subspaces) identify this larger ring with
Then we would have the commutative ring of functions valued in the diagonal of $\text{Mat}_{n}$, as a larger ring than the $\mathcal{F}^{\text{disc}}(X)$ that we defined, but smaller than the full non-commutative ring acting – and it would have an interpretation in terms of smooth $\mathcal{L}G_{X}$-equivariant smooth functions on the set of $X$-distinguished parameters. This is not important for our analysis, but we mention it in order to relate the version of “center” that we are using here with that suggested by the Langlands picture.

6. CUSPIDAL PART OF THE SCHWARTZ SPACE

6.1. Main result. We have the following analog of Proposition 5.1 and Theorem 5.2: First of all, notice that the canonical quotients: $\mathcal{S}(X_{\Theta})_{\sigma,\text{disc}} \rightarrow \mathcal{S}(X_{\Theta})_{\sigma,\text{cusp}}$ give, for every connected component $Y$ of $\mathcal{X}^{L}_{\Theta}$, a direct summand $\mathcal{L}^{2}(X_{\Theta})_{Y,\text{cusp}}$ of $\mathcal{L}^{2}(X_{\Theta})_{Y}$; it is nonzero if and only if $Y \subset \mathcal{X}^{L}_{\Theta}$, as well. The sum of those is the cuspidal subspace $\mathcal{L}^{2}(X_{\Theta})_{\text{cusp}}$.

6.2. Theorem. For every connected component $Y$ of $\mathcal{X}^{L}_{\Theta}$, orthogonal projection of an element of $\mathcal{S}(X_{\Theta})$ to $\mathcal{L}^{2}(X_{\Theta})_{Y,\text{cusp}}$ lies in $\mathcal{S}(X_{\Theta})$. In particular, the orthogonal projection of an element of $\mathcal{S}(X_{\Theta})$ to $\mathcal{L}^{2}(X_{\Theta})_{\text{disc}}$ lies in $\mathcal{S}(X_{\Theta})$, and we have a direct sum decomposition:

$$\mathcal{S}(X_{\Theta}) = \mathcal{S}(X_{\Theta})_{\text{cusp}} \oplus \mathcal{S}(X_{\Theta})_{\text{noncusp}},$$

where $\mathcal{S}(X_{\Theta})_{\text{cusp}} = \mathcal{S}(X_{\Theta}) \cap \mathcal{L}^{2}(X_{\Theta})_{\text{cusp}}$ and $\mathcal{S}(X_{\Theta})_{\text{noncusp}} = \mathcal{S}(X_{\Theta}) \cap \mathcal{L}^{2}(X_{\Theta})_{\text{disc}}^{\perp}$.

Finally, the natural map (4.9) from $\mathcal{S}(X_{\Theta})$ to sections of $\mathcal{L}$ over $\mathcal{X}^{\text{cusp}}_{\Theta}$ is the composition of an isomorphism:

$$\mathcal{S}(X_{\Theta})_{\text{cusp}} \cong \mathbb{C}[\mathcal{X}^{\text{cusp}}_{\Theta}, \mathcal{L}_{\Theta}],$$

with the orthogonal projection from $\mathcal{S}(X_{\Theta})$ to $\mathcal{S}(X_{\Theta})_{\text{cusp}}$.

Remark. This is probably not the most conceptual definition of the summand $\mathcal{S}(X_{\Theta})_{\text{cusp}}$. For a different characterization, see Proposition 7.1.

Proof. First of all, by (2.8) and the analogous isomorphism for the bundle $\mathcal{L}_{\Theta}$, the theorem is reduced to the case $\mathcal{X}^{L}_{\Theta} = X$.

Since our space is assumed to be factorizable, by (2.9) the problem is reduced to the case $\mathcal{Z}(X) = 1$, in which case cuspidal morphisms: $\pi \rightarrow C^{\infty}(X)$ have image in $C^{\infty}_{\pi}(X)$. Given that for every such $\pi$ the space $\mathcal{S}(X)_{\pi,\text{cusp}}^{\perp}$ is finite-dimensional, it is immediate to see that orthogonal projection will preserve compact support.

By the fact that $\mathcal{L}$ is a direct summand of $\mathcal{L}$ (both trivializable vector bundles), and by Proposition 5.3, the image of the map $\mathcal{S}(X)$ into sections of $\mathcal{L}$ over $\mathcal{X}^{\text{cusp}}_{\Theta}$ is equal to $\mathbb{C}[\mathcal{X}^{\text{cusp}}_{\Theta}, \mathcal{L}]$, and the kernel is the space $\mathcal{S}(X)_{\text{noncusp}} = \mathcal{S}(X) \cap \mathcal{L}^{2}(X)_{\text{cusp}}^{\perp}$. This proves the last claim. □
For future reference, we note that since $C^\infty(X_\Theta)$ is the smooth dual of $S(X_\Theta)$ using the eigenmeasure that we have fixed (§2.4), there is a corresponding direct sum decomposition:

$$C^\infty(X_\Theta) = C^\infty(X_\Theta)_{\text{cusp}} \oplus C^\infty(X_\Theta)_{\text{noncusp}},$$

where $C^\infty(X_\Theta)_{\text{noncusp}}$ is defined as the orthogonal complement of $S(X_\Theta)_{\text{cusp}}$ and vice versa. Of course, $S(X_\Theta)_{\text{cusp}}$ belongs to $C^\infty(X_\Theta)_{\text{cusp}}$.

6.3. **The cuspidal center of $X$.** The cuspidal center of $X$ (and similarly for $X_{\Theta}^L$) is the ring:

$$\mathfrak{z}^{\text{cusp}}(X) := \mathbb{C}[X^{\text{cusp}}].$$

It acts on $S(X)$, as zero on $S(X)_{\text{noncusp}}$ and via the isomorphism (6.1) on $S(X)_{\text{cusp}}$.

Again, as in Remark 5.4, we could have a larger, noncommutative ring acting on $S(X)_{\text{cusp}}$ by $G$-automorphisms, if we wanted to take into account the multiplicity of the spaces $S(X)_{\pi, \text{disc}}$, but we know no good way of doing that.

**Part 3. Eisenstein integrals**

7. **Smooth and unitary asymptotics**

The theory of asymptotics of smooth representations [SV, §4] provides us with canonical morphisms:

$$\epsilon_\Theta : S(X_\Theta) \to S(X),$$

characterized by the property that for a $J$-good neighborhood $N_\Theta \subset X$ of $\Theta$-infinity (s. (2.4)) the map $\epsilon_\Theta$ respects the identification: $N_\Theta/J = N'_\Theta/J$, where $N'_\Theta$ is the corresponding subset of $X_\Theta$, in the sense that for every $J$-orbit on $N'_\Theta$ the map $\epsilon_\Theta$ takes its characteristic function to the characteristic function of the corresponding $J$-orbit on $N_\Theta$. (Throughout the paper, we allow ourselves to identify notationally the subsets $N_\Theta$ and $N'_\Theta$, when statements are made which in reality apply to the quotients $N_\Theta/J$ and $N'_\Theta/J$.)

On the other hand, the theory of unitary asymptotics [SV, §11] provides us with canonical morphisms (the “Bernstein maps”):

$$\iota_\Theta : L^2(X_\Theta) \to L^2(X),$$

characterized by the fact that they are “asymptotically equal to $\epsilon_\Theta$” close to $\Theta$-infinity (cf. loc.cit. for details).

We can characterize the spaces $L^2(X)_{\text{disc}}$, $S(X)_{\text{cusp}}$ using these maps:

**7.1. Proposition.** We have:

$$S(X)_{\text{cusp}} = \bigcap_{\Theta \neq \Delta_X} \ker \epsilon_\Theta^*|_{S(X)},$$

$$L^2(X)_{\text{disc}} = \bigcap_{\Theta \neq \Delta_X} \ker \iota_\Theta^*.$$
Proof. For $L^2(X)_{\text{disc}}$ this is part of the Plancherel formula of [SV], [Del].

Let $f \in S(X)$, and let $f = \int_{\mathcal{Z}(X)} f^\chi d\chi$ be its pointwise Plancherel decomposition with respect to the action of $\mathcal{Z}(X)$, i.e. $f^\chi \in L^2(X/\mathcal{Z}(X), \chi)$ and $f(x) = \int_{\mathcal{Z}(X)} f^\chi(x) d\chi$ for every $x \in X$. Then all $f^\chi$ are compactly supported-mod-center.

Essentially by definition, we have $f \in S(X)_{\text{cusp}}$ if and only if for almost every $\chi$, the subrepresentation $\mathcal{V}_\chi \subset L^2(X/\mathcal{Z}(X), \chi)^{\perp}$ generated by $f^\chi$ is admissible (and hence finite-length, necessarily semisimple). This is the case if and only if (for almost all $\chi$) $V^J_\chi$ is supported in a compact-mod-center subset of $X$, which is equivalent to the statement that $e_\Theta^*f^\chi = 0$ for all $\Theta \neq \Delta_X$.

On the other hand, we have that $e_\Theta^*f = 0$ iff $e_\Theta^*f^\chi = 0$ for almost all $\chi$. Indeed, by applying the Plancherel formula with respect to $\mathcal{Z}(X)$ to the inner product $\langle f, e_\Theta \Phi \rangle$ we get:

$$\langle f, e_\Theta \Phi \rangle = \int_{\mathcal{Z}(X)} \langle f^\chi, e_\Theta \Phi \rangle d\chi = \int_{\mathcal{Z}(X)} \langle e_\Theta^*f^\chi, \Phi \rangle d\chi,$$

which is zero for all $\Phi \in S(X_\Theta)$ iff $e_\Theta^*f^\chi$ for almost all $\chi$.

This proves the proposition. \hfill \square

Under the assumptions of the present paper (in particular, in the case of symmetric varieties), and conjecturally always, these smooth and unitary asymptotics have spectral expansions in terms of normalized Eisenstein integrals.

Recall the spaces of discrete and cuspidal $\sigma$-coinvariants defined in section 4; the normalized constant terms (restricted, here, to discrete and cuspidal spectra) are explicitly defined morphisms:

$$E^*_\Theta,\sigma,\text{disc} : S(X) \to \mathcal{L}_\Theta,\sigma = S(X_\Theta)_{\sigma,\text{disc}},$$

$$E^*_\Theta,\sigma,\text{cusp} : S(X) \to \mathcal{L}_\Theta,\sigma = S(X_\Theta)_{\sigma,\text{cusp}},$$

the latter obtained from the former via the natural quotient maps: $\mathcal{L}_\Theta,\sigma \to \mathcal{L}_\Theta,\sigma$, which vary rationally in $\sigma$, i.e. they are really:

$$E^*_\Theta,\text{disc} \in \mathbb{C} \left( \tilde{X}_\Theta^\text{disc}, \text{Hom}_G(S(X), \mathcal{L}_\Theta) \right),$$

$$E^*_\Theta,\text{cusp} \in \mathbb{C} \left( \tilde{X}_\Theta^\text{cusp}, \text{Hom}_G(S(X), \mathcal{L}_\Theta) \right).$$

Here $\text{Hom}_G(S(X), \mathcal{L}_\Theta)$ denotes the sheaf over $\tilde{X}_\Theta^\text{disc}$ whose sections over an open subset $U$ is the space of $G$-morphisms: $S(X) \to \mathbb{C}[U, \mathcal{L}_\Theta]$ (and similarly for $\mathcal{L}_\Theta$ over $\tilde{X}_\Theta^\text{cusp}$). A priori this sheaf could be infinite-dimensional, but we claim:

7.2. Lemma. $\text{Hom}_G(S(X), \mathcal{L}_\Theta)$ is a coherent, torsion-free sheaf over $\tilde{X}_\Theta^\text{disc}$. 


**Proof.** Indeed, for every open compact subgroup $J$ the space $S(X)^J$ is a finitely generated module for the Hecke algebra $\mathcal{H}(G, J)$ [AAG12, Theorem A], [SV, Remark 5.1.7]. Therefore,

$$\text{Hom}_G(S(X), \mathcal{L}_\Theta) = \lim_{J} \text{Hom}_{\mathcal{H}(G, J)}(S(X)^J, \mathcal{L}_\Theta^J),$$

and the individual Hom-spaces on the right are coherent sheaves over $X_\Theta^{\text{disc}}$. Moreover, for every connected component $Y$ of $X_\Theta^{\text{disc}}$ there is a compact open subgroup $J$ such that

$$\text{Hom}_G(S(X), \mathcal{L}_\Theta|_Y) = \text{Hom}_{\mathcal{H}(G, J)}(S(X)^J, \mathcal{L}_\Theta^J|_Y),$$

therefore $\text{Hom}_G(S(X), \mathcal{L}_\Theta)$ is a coherent sheaf over $X_\Theta^{\text{disc}}$ (and similarly for $\mathcal{L}_\Theta$ over $X_\Theta^{\text{cusp}}$). Moreover, it is a subsheaf of the locally free sheaf $\text{Hom}(S, \mathcal{L}_\Theta^J|_J)$, where $S$ is a finite set of generators of $S(X)^J$, and hence it is torsion-free. □

The definition of normalized constant terms will be recalled in the next subsection, where we will also prove the important property of regularity on the unitary set. We will also recall there the notion of a character $\omega \in X_\Theta^{\text{unit}}$ being large, denoted $\omega \gg 0$.

The normalized constant terms are adjoint to “normalized Eisenstein integrals”, which can be described as morphisms:

$$E_{\Theta, \sigma, \text{disc}} : \mathcal{L}_{\Theta, \sigma} \rightarrow C^\infty(X),$$

$$E_{\Theta, \sigma, \text{cusp}} : \mathcal{L}_{\Theta, \sigma} \rightarrow C^\infty(X),$$

varying rationally in $\sigma$. (By $\tilde{\cdot}$ we denote smooth duals.)

Now we recall the way in which Eisenstein integrals can be used to explicate smooth and unitary asymptotics. To formulate it, start from the Plancherel formula for $X_\Theta$, which canonically attaches to every $f \in L^2(X_\Theta)^{\text{disc}}$ a $C^\infty(X_\Theta)$-valued measure $\tilde{f} \, d\sigma$ on $X_\Theta^{\text{disc}}$. Explicitly, $\tilde{f}$ belongs to the “discrete $\sigma$-equivariant eigenspace of $C^\infty(X_\Theta)$” (i.e. the dual of $\mathcal{L}_{\Theta, \sigma}$), and is characterized by the property that for every $\Phi \in S(X_\Theta)$ we have:

$$\langle f, \tilde{\Phi} \rangle_{L^2(X_\Theta)} = \int_{X_\Theta^{\text{disc}}} \int_{X_\Theta} \tilde{f}(x)\Phi(x) \, dx \, d\sigma. \quad (7.3)$$

When $f \in S(X_\Theta)$ the measure $\tilde{f} \, d\sigma$ extends to a $C^\infty(X_\Theta)$-valued differential form on $X_\Theta^{\text{cusp}}$, and another way to describe it is as follows. Recall the canonical map:

$$S(X_\Theta) \rightarrow \mathbb{C}[X_\Theta^{\text{disc}}, \mathcal{L}_\Theta],$$

$$f \mapsto (\sigma \mapsto f_{\sigma, \text{disc}})$$

(where $f_{\sigma, \text{disc}}$ denotes the image of $f$ in the discrete $\sigma$-coinvariants), and the canonical differential form: $(\cdot, \cdot)_{\sigma} \, d\sigma$, valued in hermitian forms on $\mathcal{L}_\Theta$.
obtained from the discrete part of the Plancherel formula of $X_\Theta$ (see (4.10)). Then:

$$f^\sigma d\sigma = \langle \cdot, \tilde{f}_{\sigma,\text{disc}} \rangle d\sigma$$

as differential forms valued in the smooth dual of $S(X_\Theta)$.

We are particularly interested in the case when $f \in S(X_\Theta)_{\text{cusp}}$, in which the form $f^\sigma d\sigma$ is valued in the dual of $L_{\Theta,\sigma}$ and supported on $X_{\Theta,cusp}^L$.

Since the integrand in (7.3) is entire and supported on $X_{\Theta,cusp}^L$, we can shift the contour of integration and write:

$$\langle f, \bar{\Phi} \rangle_{L^2(X_\Theta)_{\text{disc}}} = \int_{\omega^{-1}X_{\Theta,cusp}^L} \int_{X_\Theta} f^\sigma(x)\Phi(x) dx d\sigma \quad (7.4)$$

for any character $\omega$ of $X_{\Theta,cusp}^L$.

7.3. **Theorem** ([SV, Theorem 15.4.2]). For any $\omega \gg 0$, if $f \in S(X_\Theta)_{\text{cusp}}$ admits the decomposition (7.4) then:

$$e_\Theta f(x) = \int_{\omega^{-1}X_{\Theta,cusp}^L} E_{\Theta,\sigma,\text{cusp}} f^\sigma(x) d\sigma. \quad (7.5)$$

7.4. **Theorem** ([SV, Theorem 15.6.1], [Del, Theorem 7]). If $f \in L^2(X_\Theta)_{\text{disc}}^\infty$ admits the decomposition (7.3), then:

$$\iota_\Theta f(x) = \int_{X_{\Theta,\text{disc}}} E_{\Theta,\sigma,\text{disc}} f^\sigma(x) d\sigma. \quad (7.6)$$

We need to extend the validity of Theorem 7.3 to the cases considered in [Del]. The proof of Theorem 15.4.2 in [SV] carries over verbatim, up to Proposition 5.4.5 which we need to prove in the setting of [Del]:

7.5. **Proposition.** There is an affine embedding $X_\Omega \hookrightarrow Y$ such that for every $\Phi \in S(X)$, the support of $e_\Omega^\ast \Phi$ has compact closure in $Y$.

**Proof.** We choose a finite extension $E$ of our field over which $G$ splits, and we let $\tilde{X}$, $\tilde{G}$ etc. denote points over $E$. Then, in [SV, §5.5] there is a filtration of $\tilde{X}$ defined by certain subsets $\tilde{X}_{\geq \mu}$ indexed by points $\mu$ in a rational vector space $\tilde{a}_X$, and similarly for the spaces $\tilde{X}_\Theta$. By taking intersections with $X$, $X_\Theta$, we have obtain filtrations for this space.

Similarly, there is a filtration $\tilde{H}_{\geq \lambda}$ of the full Hecke algebra of $\tilde{G}$ determined by the support of its elements, where $\lambda$ lies in a rational vector space $\tilde{a}$ endowed with a surjective map: $\tilde{a} \to \tilde{a}_X$. We may analogously define a filtration of the full Hecke algebra of $G$, by imposing the same conditions on the support of its elements (considered as a subset of $\tilde{G}$).

The rest of the argument of [SV], which was due to [BK], hence [SV, Lemma 5.5.5] and Proposition 7.5, now follow verbatim. □

We complement this with a statement of moderate growth, that will be used later:
7.6. Proposition. For any open compact subgroup $J$ the image of $\mathcal{S}(X)^I$ under $e_\Theta^*$ is a space of functions of uniformly moderate growth on $X_\Theta$: i.e. there is a finite number of rational functions $F_i$, whose sets of definition cover $X_\Theta$, such that each $f \in e_\Theta^* (\mathcal{S}(X)^I)$ satisfies:

$$|f| \leq C_f \cdot \min_i (1 + |F_i|)$$
onumber

on $X_\Theta$ (for some constant $C_f$ depending on $f$).

This is [SV, Proposition 15.4.3], whose proof holds in the general case.

8. Definition and regularity of Eisenstein integrals

The normalized constant terms:

$$E_{\Theta,\sigma}^0 : \mathcal{S}(X) \to \mathcal{S}(X_\Theta)_\sigma$$

are defined as the composition: $T_{\Theta,\sigma}^{-1} \circ R_{\Theta,\sigma}$, where $R_{\Theta,\sigma}$ and $T_{\Theta,\sigma}$ are operators fitting in a diagram (to be explained):

$$\begin{array}{ccc}
\mathcal{S}(X) & \xrightarrow{R_{\Theta,\sigma}} & \mathcal{S}(X^h_\Theta, \delta_\Theta)_\sigma \\
\downarrow T_{\Theta,\sigma} & & \uparrow T_{\Theta,\sigma} \\
\mathcal{S}(X_\Theta)_\sigma & & \\
\end{array}$$

The space $X^h_\Theta$ is the space of (generic) $\Theta$-horocycles of $X$, classifying pairs $(Q, \mathcal{O})$, where $Q$ is a parabolic in the conjugacy class opposite to that of $P_\Theta$ (defined in §3) and $\mathcal{O}$ is an orbit for its unipotent radical $U_Q$ on the open $Q$-orbit on $X$. Saying the same words about $X_\Theta$ would produce a canonically isomorphic variety [SV, Lemma 2.8.1], and the operators $R_{\Theta}$ and $T_{\Theta}$ are defined in completely analogous ways as operators from $\mathcal{S}(X)$, resp. $\mathcal{S}(X_\Theta)$, to $\mathcal{S}(X^h_\Theta, \delta_\Theta)_\sigma$, so for notational simplicity we only describe below the definition of the former. (From the definition it will be clear that $T_{\Theta}$ factors through the quotient $\mathcal{S}(X_\Theta)_\sigma$, which we noted in the diagram above in order to make sense of the inverse of $T_{\Theta}$. In some sense, $T_{\Theta}$ is the standard intertwining operator between induction from two opposite parabolics, and $R_{\Theta}$ is its $X$-analog, showing up in the spectral decomposition of Radon transform.)

Let $\Lambda \in \text{Hom}_{L_\Theta}(\mathcal{S}(X^L_\Theta), \sigma)$, where $\sigma$ is an irreducible representation of $L_\Theta$. Recall from §3 that the Levi variety $X^L_\Theta$ can be identified with the quotient of the open $P_\Theta$-orbit on $X$ by its unipotent radical $U_{\Theta}$. We define a $P_\Theta$-morphism:

$$\tilde{\Lambda} : \mathcal{S}(X) \to \sigma \otimes \delta_\Theta$$

by integrating over $U_{\Theta}$-orbits in the open $P_\Theta$-orbit and then applying the operator $\Lambda$. There are two difficulties here: First, integrating over $U_{\Theta}$-orbits
requires fixing a measure on them; secondly the result of this integration will not be compactly supported on $X^L_{\Theta}$.

Without fixing measures on $U_{\Theta}$-orbits, the operation (Radon transform) of integrating over them canonically takes values in a line bundle over $X^L_{\Theta}$ whose smooth sections we denote by $C^\infty(X^L_{\Theta}, \delta_{\Theta})$, and is noncanonically isomorphic to $C^\infty(X^L_{\Theta}) \otimes \delta_{\Theta}$, cf. [SV, §5.4.1]. The image of “integration over generic $U_{\Theta}$-orbits” will be denoted by:

$$S(X) \mapsto C^\infty(X^L_{\Theta}, \delta_{\Theta}) \subset C^\infty(X^L_{\Theta}, \delta_{\Theta}).$$  \hspace{1cm} (8.3)

Inducing this $P_{\Theta}$-functional to $G$, we get a $G$-morphism (denoted by the same symbol):

$$S(X) \mapsto C^\infty(X^h_{\Theta}, \delta_{\Theta})_X,$$

where this notation stands for the corresponding line bundle over $X^h_{\Theta}$.

For ease of presentation, let us now fix an isomorphism:

$$C^\infty(X^L_{\Theta}, \delta_{\Theta})_X \simeq C^\infty(X^h_{\Theta})_X \otimes \delta_{\Theta}. \hspace{1cm} (8.4)$$

Let us now describe how to extend the morphism $\Lambda$ to $C^\infty(X^h_{\Theta})_X$. Let $\tilde{v} \in \tilde{\sigma}$, and consider the following distribution on $X^L_{\Theta}$:

$$S(X^L_{\Theta}) \ni \Phi \mapsto \langle \Lambda(\Phi), \tilde{v} \rangle.$$

Consider also the invariant-theoretic quotient $X_{\Theta} \sslash U_P$. It can be shown that it contains $X^L_{\Theta}$ as an open orbit, whose preimage is precisely the open $P_{\Theta}$-orbit in $X$. For a character $\omega \in \widehat{X^L_{\Theta} \cap \Theta}$, considered as a function on $X^L_{\Theta}$ (this requires fixing a base point), we write $\omega \gg 0$ if it vanishes sufficiently fast around the complement of the open orbit; such characters contain an open subset of the whole character group.

Twisting by $\omega$ we get from $\Lambda$ and $\tilde{v}$ functionals:

$$S(X^L_{\Theta}) \ni \Phi \mapsto \langle \Lambda(\Phi \cdot \omega), \tilde{v} \rangle \hspace{1cm} (8.5)$$

factoring through $\sigma \otimes \omega^{-1}$-coinvariants.

Then, for $\omega \gg 0$ (not depending on the choice of $\tilde{v}$) and any $f \in C^\infty(X^L_{\Theta})_X$, the distributions (8.5) are in $L^1(X, f)$ (i.e. when divided by $dx$ they are in $L^1(X, f dx)$). That gives a natural way to extend them to $C^\infty(X^L_{\Theta})_X$, and it turns out that their pairing against any such $f$ is rational in the variable $\omega$; more precisely:

8.1. Lemma. For any $f \in C^\infty(X^L_{\Theta})_X$ and $\tilde{v} \in \tilde{\sigma}$ the integral:

$$\langle \Lambda(f \cdot \omega), \tilde{v} \rangle$$

is rational in the variable $\omega \in \hat{X^L_{\Theta}}^{ab}$, with linear poles.
Proof. This follows from the theory of Igusa integrals and the proof of [SV, Proposition 15.3.5], where this integral is, locally over a blow-up of $X$, represented by an Igusa integral of the form:

$$I(\omega) := \int F \cdot |\Omega| \cdot \prod_i |f_i|^{s_i(\omega)},$$

where:
- $F$ is a $D$-finite function, for some divisor $D$;
- $\Omega$ an algebraic volume form, with polar divisor contained in $D$;
- $f_i$ rational functions with polar divisor contained in $D$;
- The exponents $s_i(\omega)$ vary linearly in $\omega$.

We point the reader to loc.cit. for an explanation of the notation. □

Remark. For symmetric spaces, an alternative proof of rationality and linearity of the poles was given by Blanc-Delorme in [BD08, Theorem 2.8(iv) and Theorem 2.7(i)].

Composing this with the map $R_{\Theta}$ of (8.3) we now get, for every $\Lambda \in \text{Hom}_{L_{\Theta}}(S(X_{\Theta}^h), \sigma)$, a rational family of $P_{\Theta}$-morphisms:

$$\tilde{\Lambda}_\omega : S(X) \to \sigma \otimes \omega^{-1}\delta_{\Theta}.$$  

If we let $\Lambda$ vary, this defines a rational family of $P_{\Theta}$-morphisms from $S(X)$ to the coinvariant space $S(X_{\Theta}^h)_{\sigma \otimes \omega^{-1}\delta_{\Theta}}$ and inducing from $P_{\Theta}$ (and forgetting the isomorphism (8.4)) we land in a certain coinvariant space:

$$S(X_{\Theta}^h, \delta_{\Theta})_{\sigma}.$$  

This completes the definition of the map $R_{\Theta}$, and the definition of $T_{\Theta}$ is completely analogous. This gives the definition of normalized constant terms, and normalized Eisenstein integrals are by definition their adjoints:

$$E_{\Theta, \sigma} : S(X_{\Theta})_{\sigma} \to C^\infty(X).$$  

Notice that $S(X_{\Theta})_{\sigma}$ can be identified under the duality between $S(X_{\Theta})$ and $C^\infty(X_{\Theta})$ with a certain subspace of $C^\infty(X_{\Theta})^\theta$, which we may denote by $C^\infty(X_{\Theta})^\theta$. The functions $f^\theta$ of Theorems 7.3 and 7.4 live there (and, more precisely, in the “cuspidal”, resp. “discrete” subspaces).

Now we project normalized constant terms to discrete and cuspidal quotients of $S(X_{\Theta})_{\sigma}$, to obtain the morphisms used in the previous section:

$$E_{\Theta, \text{disc}}^\sigma \in \mathbb{C} \left( \widehat{X_{\Theta}^L, \text{disc}}, \text{Hom}_G(S(X), L_{\Theta}) \right),$$  

$$E_{\Theta, \text{cusp}}^\sigma \in \mathbb{C} \left( \widehat{X_{\Theta}^L, \text{cusp}}, \text{Hom}_G(S(X), L_{\Theta}) \right).$$  

Linearity of the poles and their role in the Plancherel formula imply that the poles actually do not meet the unitary set:
8.2. **Proposition.** Normalized Eisenstein integrals are regular on the subsets of unitary representations, i.e.:

\[
E_{\Theta, \text{disc}}^* \in \Gamma \left( \widetilde{X}_{\Theta}^{\text{disc}}, \text{Hom}_G(S(X), L_{\Theta}) \right),
\]

and:

\[
E_{\Theta, \text{cusp}}^* \in \Gamma \left( \widetilde{X}_{\Theta}^{\text{cusp}}, \text{Hom}_G(S(X), L_{\Theta}) \right).
\]

**Proof.** Theorem 7.4 shows that for every \( \Phi \in S(X) \), and every \( L^2 \)-section \( \sigma \mapsto v_\sigma \) of \( L_{\Theta} \), the inner product:

\[
\langle E_{\Theta, \text{disc}}^*, \Phi, v \rangle_\sigma \ d\sigma
\]

(see (4.11) for the unitary structure on \( L_{\Theta} \)) is integrable over \( \widetilde{X}_{\Theta}^L \).

Dividing by a Haar measure \( d\sigma \), and assuming that \( v \) is actually regular, we get a function \( \sigma \mapsto \langle E_{\Theta, \text{disc}}^*, \Phi, v \rangle_\sigma \) which is known a priori to have linear poles. By Lemma 3.5, these poles cannot meet the unitary spectrum \( \widetilde{X}_{\Theta}^L \). \( \square \)

**Part 4. Scattering**

9. **Goals**

This part is technically at the heart of our proof of Paley-Wiener theorems. The goal here is to prove Theorems 9.1, 9.2; before we formulate them, let us explain what we will mean by saying that for associates \( \Theta, \Omega \subset \Delta_X \), and \( w \in W_X(\Omega, \Theta) \), a morphism:

\[
S(X_{\Theta})_{\text{cusp}} \rightarrow C^\infty(X_{\Omega})_{\text{cusp}} \tag{9.1}
\]

is “\( w \)-equivariant with respect to the cuspidal center” \( \tilde{\chi}^{\text{cusp}}(X_{\Theta}) \) (\S 6.3), and similarly that a morphism:

\[
C(X_{\Theta})_{\text{disc}} \rightarrow C(X_{\Omega})_{\text{disc}}
\]

is “\( w \)-equivariant with respect to the discrete center” \( \tilde{\chi}^{\text{disc}}(X_{\Theta}) \) (\S 5.4).

Conjugation by \( w \) induces an isomorphism between the Levi \( L_{\Theta} \) and the Levi \( L_{\Omega} \) (unique up to conjugacy), and hence between their unitary duals. It is not a priori clear that this preserves the discrete and cuspidal subsets:

\[
\widetilde{X}_{\Theta}^L_{\text{disc}} \rightarrow \widetilde{X}_{\Omega}^L_{\text{disc}},
\]

\[
\widetilde{X}_{\Theta}^L_{\text{cusp}} \rightarrow \widetilde{X}_{\Omega}^L_{\text{cusp}},
\]

however this will be implicit (and hence will be proven) whenever we say that a morphism of the form (9.1) is “\( w \)-equivariant”. Recall from \S 6.3, 5.4 that the cuspidal, resp. discrete center of \( X_{\Theta} \) can be identified with regular, resp. smooth functions on \( \widetilde{X}_{\Theta}^L_{\text{cusp}} \), resp. \( \widetilde{X}_{\Theta}^L_{\text{disc}} \). Thus, \( w \) induces isomorphisms:

\[
\tilde{\chi}^{\text{cusp}}(X_{\Theta}) \rightarrow \tilde{\chi}^{\text{cusp}}(X_{\Omega})
\]

\[
\tilde{\chi}^{\text{disc}}(X_{\Theta}) \rightarrow \tilde{\chi}^{\text{disc}}(X_{\Omega})
\]
and:

$$\delta^\text{disc}(X^L_\Theta) \cong \delta^\text{disc}(X^L_\Omega),$$

and by saying that the map is “$w$-equivariant” we mean with respect to this isomorphism. Notice that, by duality to $S(X_\Omega)$, the space $C^\infty(X_\Omega)$ decomposes as a direct sum:

$$C^\infty(X_\Omega) = C^\infty(X_\Omega)_{\text{cusp}} \oplus C^\infty(X_\Omega)_{\text{noncusp}},$$

and the action of $\delta^\text{cusp}(X^L_\Omega)$ on $C^\infty(X_\Omega)_{\text{cusp}}$ is defined by duality in such a way that it extends the action on $S(X_\Omega)_{\text{cusp}}$: for $Z \in \mathbb{C}[X^L_\Omega_{\text{cusp}}]$ we let $Z^\vee$ denote the dual element: $Z^\vee(\pi) = Z(\bar{\pi})$ and we define $Z \cdot f$, for each $f \in C^\infty(X_\Omega)_{\text{cusp}}$ by the property:

$$\int_{X_\Omega} \Phi \cdot (Z \cdot f) = \int_{X_\Omega} (Z^\vee \Phi) f$$

for all $\Phi \in S(X_\Omega)_{\text{cusp}}$.

Notice that $\Omega$ could be equal to $\Theta$, but $w \neq 1$, in which case the isomorphism between centers is not the identity map.

9.1. Theorem. Consider the composition $i^*_\Omega \circ i_\Theta$, restricted to $L^2(X_\Theta)_{\text{disc}}$. It is zero unless $\Omega$ contains a $W_X$-translate of $\Theta$, and it has image in $L^2(X_\Omega)_{\text{cont}}$ unless $\Omega$ is equal to a $W_X$-translate of $\Theta$, in which case it has image in $L^2(X_\Omega)_{\text{disc}}$. In the last case, it admits a decomposition:

$$i^*_\Omega \circ i_\Theta|_{L^2(X_\Theta)_{\text{disc}}} = \sum_{w \in W_X(\Omega, \Theta)} S_w,$$

where the morphism:

$$S_w : L^2(X_\Theta)_{\text{disc}} \to L^2(X_\Omega)_{\text{disc}}$$

is $w$-equivariant with respect to $\delta^\text{disc}(X^L_\Omega)$, and is an isometry.

The operators $S_w$ restrict to continuous morphisms between Harish-Chandra Schwartz spaces:

$$S_w : C'(X_\Theta)_{\text{disc}} \to C'(X_\Omega)_{\text{disc}},$$

and they satisfy the natural associativity conditions:

$$S_{w'} \circ S_w = S_{w'w} \text{ for } w \in W_X(\Omega, \Theta), w' \in W_X(\Xi, \Omega). \quad (9.2)$$

(In particular, since $S_1 = 1$, they are topological isomorphisms.)

The theorem is part of the main $L^2$-scattering theorem [SV, Theorem 7.3.1], [Del, Theorem 6], except for two assertions: First, the condition on equivariance with respect to $\delta^\text{disc}(X^L_\Theta)$; indeed, the condition used in loc. cit. to characterize the scattering maps $S_w$ was $w$-equivariance with respect to the action of $A'_{X, \Theta}$ (via $w : A_{X, \Theta} \cong A_{X, \Omega}$ on $L^2(X_\Omega)$), where $A'_{X, \Theta}$ denotes the image of the $F$-points of $Z(L_\Theta)$ via the quotient map: $Z(L_\Theta) \to A_{X, \Theta}$. This condition is slightly weaker than $\delta^\text{disc}(X^L_\Theta)$-equivariance. Secondly and most importantly, the fact that the scattering maps (continuously) preserve Harish-Chandra Schwartz spaces.
9.2. **Theorem.** Consider the composition $e^*_\Omega \circ e_\Theta$, restricted to $\mathcal{S}(X_\Theta)_{\text{cusp}}$. It is zero unless $\Omega$ contains a $W_X$-translate of $\Theta$, and it has image in $C^\infty(X_\Omega)_{\text{noncusp}}$ unless $\Omega$ is equal to a $W_X$-translate of $\Theta$, in which case it has image in $C^\infty(X_\Omega)_{\text{cusp}}$. In the last case, it admits a decomposition:

$$e^*_\Omega \circ e_\Theta|_{\mathcal{S}(X_\Theta)_{\text{cusp}}} = \sum_{w \in W_X(\Omega, \Theta)} \mathcal{S}_w,$$

where the morphism:

$$\mathcal{S}_w : \mathcal{S}(X_\Theta)_{\text{cusp}} \to C^\infty(X_\Omega)_{\text{cusp}}$$

is $w$-equivariant with respect to $S^\infty(X_\Omega)_{\text{cusp}}$.

If we denote the subspace of $C^\infty(X_\Omega)_{\text{cusp}}$ spanned by the images of all operators $\mathcal{S}_w$ by $S^+(X_\Omega)_{\text{cusp}}$, as $\Theta$ varies and $w \in W_X(\Omega, \Theta)$, then there is a unique extension of the operators $\mathcal{S}_w$ to the spaces $S^+$, i.e.:

$$\mathcal{S}_w : S^+(X_\Theta)_{\text{cusp}} \to S^+(X_\Omega)_{\text{cusp}}$$

satisfying the natural associativity conditions:

$$\mathcal{S}_{w'} \circ \mathcal{S}_w = \mathcal{S}_{w'w} \text{ for } w \in W_X(\Omega, \Theta), \; w' \in W_X(\Xi, \Omega). \quad (9.3)$$

Both types of scattering operators have spectral expressions, which should be seen as the analogs of Theorems 9.3 and 7.4; to formulate them, let $w \in W_X(\Omega, \Theta)$ and denote by $w^* \mathcal{L}_\Omega$ the pullback of the vector bundle $\mathcal{L}_\Omega$ to $X^\text{disc}_\Theta$ under the isomorphisms afforded by $w: X^\text{disc}_\Theta \to X^\text{disc}_\Pi$. We will not distinguish between sections of $\text{Hom}_G(\mathcal{L}_\Theta, w^* \mathcal{L}_\Omega)$ over $X^\text{disc}_\Theta$ and sections of $\text{Hom}_G((w^{-1})^* \mathcal{L}_\Theta, \mathcal{L}_\Omega)$ over $X^\text{disc}_\Pi$; this allows us to compose such sections for a sequence of maps:

$$X^\text{disc}_\Theta \to X^\text{disc}_\Pi \to X^\text{disc}_\Xi.$$

9.3. **Theorem.** For each $w \in W_X(\Omega, \Theta)$ there is a rational family of operators, with linear poles which do not meet the unitary set:

$$\mathcal{S}_w \in \Gamma \left( X^\text{disc}_\Theta, \text{Hom}_G(\mathcal{L}_\Theta, w^* \mathcal{L}_\Omega) \right),$$

preserving the cuspidal direct summands $\mathcal{L}_\Theta, \mathcal{L}_\Omega$, such that the scattering operators of Theorems 9.1 and 9.2 admit the following decompositions:

- For any $\omega \gg 0$, if $f \in \mathcal{S}(X_\Theta)_{\text{cusp}}$ admits the decomposition (7.4) then:

$$\mathcal{S}_w f(x) = \int_{\omega^{-1} X^\text{disc}_\Theta} \mathcal{S}_w^* \omega^{-1} f^\beta(x) d\sigma. \quad (9.4)$$

- If $f \in L^2(X_\Theta)_{\text{disc}}$ admits the decomposition (7.3), then:

$$S_w f(x) = \int_{X^\text{disc}_\Theta} S_w^* \omega^{-1} f^\beta(x) d\sigma. \quad (9.5)$$
The operators $\mathcal{S}_w$ satisfy the natural associativity conditions:

$$\mathcal{S}_{w'} \circ \mathcal{S}_w = \mathcal{S}_{w'w} \text{ for } w \in W_X(\Omega, \Theta), w' \in W_X(\Xi, \Omega).$$

(9.6)

Remark. The operator $\mathcal{S}_{w^{-1}} = \mathcal{S}_w^{-1}$ is a rational section of morphisms:

$$L_{\Theta, \sigma} \rightarrow L_{\Theta, \sigma} \text{ as } \sigma \text{ varies in } X^L_{\Theta}$$

and hence its adjoint $\mathcal{S}^*_{w^{-1}}$ is a rational section of morphisms:

$$L_{\Theta, \sigma} \rightarrow L_{\Theta, \sigma}.$$

The spaces $L_{\Theta, \sigma} \rightarrow L_{\Theta, \sigma}$ are considered as subspaces of $C^p_{\sigma}$ and $C^p_{\sigma}$, respectively, by duality with $S(X_{\Theta})$, resp. $S(X_{\Theta})$ – recall that $f_{\sigma} \in L_{\Theta, \sigma}$.

10. EIGENSPACE DECOMPOSITION OF EISENSTEIN INTEGRALS

Recall that the (discrete part of the) normalized constant term gives the morphisms (1.8):

$$E^\ast_{\Theta, \text{disc}} : \mathcal{S}(X) \rightarrow \mathcal{C}(X^L_{\Theta}, \mathcal{L}_{\Theta}).$$

If we compose with the equivariant exponential map $e_{\Omega}$ (for some $\Omega \subset \Delta_X$), we get morphisms which we will denote by $E^*_{\Theta, \text{disc}}$:

$$E^*_{\Theta, \text{disc}} := E^*_{\Theta, \text{disc}} \circ e_{\Omega} : \mathcal{S}(X_{\Theta}) \rightarrow \mathcal{C}(X^L_{\Theta}, \mathcal{L}_{\Theta}).$$

(10.1)

These morphisms express the asymptotics, in the $\Omega$-direction, of Eisenstein integrals. Their projection to the cuspidal part will be denoted by $E^*_{\Theta, \text{cusp}}$.

We notice that we have an action of a torus $A_{X, \Omega}$ on $\mathcal{S}(X_{\Theta})$. If we fix $\sigma \in X^L_{\Theta}$, any map:

$$\mathcal{S}(X_{\Theta}) \rightarrow L_{\Theta, \sigma}$$

(= the fiber $\mathcal{S}(X_{\Theta})_{\sigma, \text{disc}}$ of $L_{\Theta}$ over $\sigma$) is finite under the $A_{X, \Omega}$-action, but we will prove a stronger statement which takes into account the variation of $\sigma$. This has nothing to do with the Eisenstein integrals per se, and in fact we want to apply it to their derivatives as well (in order to prove that the normalized constant term of the Harish-Chandra space gives smooth sections over the spectrum), so we will discuss it in a more general setting.

We have already explained in §7 how $\text{Hom}_G(\mathcal{S}(X), L_{\Theta})$ denotes a coherent, torsion-free sheaf over $X^L_{\Theta}$; the same applies to:

$$\mathfrak{M} := \text{Hom}_G (\mathcal{S}(X_{\Theta}), L_{\Theta}).$$

Let us fix a connected component $Y \subset X^L_{\Theta}$. We let:

$$\mathcal{M}_Y := \mathcal{C}(Y, \text{Hom}_G (\mathcal{S}(X_{\Theta}), L_{\Theta})) = \text{Hom}_G (\mathcal{S}(X_{\Theta}), \mathcal{C}(Y, L_{\Theta}))$$

(10.3)

denote the rational global sections of this sheaf over $Y$ – they form a finite-dimensional vector space over the field $K_Y := \mathcal{C}(Y)$.

This vector space carries a smooth $A_{X, \Omega}$-action via the action of this torus on $\mathcal{S}(X_{\Theta})$. Thus, over a finite extension of $K_Y$, it splits into a direct sum of generalized eigenspaces. We will describe the eigencharacters.
Let \( \hat{A} \) denote the unitary dual of the maximal reductive quotient of the minimal parabolic of \( G \). In what follows, characters of a Levi \( L_\Theta \) are considered as characters of \( A \) via the embedding of the minimal parabolic into the parabolic \( P_\Theta \) (not \( P_6 \)). Thus, we have restriction maps:

\[
\tilde{X}^\text{unr}_{\Theta C} \rightarrow \hat{L}_\Theta \rightarrow \hat{A}_C, \tag{10.4}
\]

and recall that there is also a quotient map:

\[
\mathcal{Z}(L_\Omega)^0 \rightarrow A_{X,\Omega}, \tag{10.5}
\]

whose image at the level of \( F \)-points we are denoting by \( A'_{X,\Omega} \).

For a character \( \chi \in \hat{A}_C \) (not necessarily unitary) and an element \( w \in W \), it may happen that for all \( \omega \in X^\text{unr}_{\Theta C} \) the restrictions:

\[
w(\chi\omega)|_{\mathcal{Z}(L_\Omega)^0}
\]

factor through the quotient map (10.5). For example, this is the case if \( \chi \) arises as the restriction of a character in (10.4), and \( w \in W_X(\Omega, \Theta) \). We will then write:

\[
w(\chi\omega)|_{A'_{X,\Omega}},
\]

and whenever we use this notation we will implicitly mean that the characters do factor through \( A'_{X,\Omega} \). This is also implicitly assumed for the characters that appear in the following:

10.1. Proposition. Choose a base point \( \sigma \in Y \), and use it to construct the finite cover:

\[
X^\text{unr}_{\Theta} \ni \omega \mapsto \sigma \otimes \omega \in Y. \quad \text{Let } \tilde{K}_Y = \mathbb{C}(\tilde{X}^\text{unr}_{\Theta}) \quad \text{be the corresponding finite extension. Then all eigencharacters of } A'_{X,\Omega} \text{ on } \mathcal{M}_Y \text{ are defined over } \tilde{K}_Y.
\]

More precisely, if \( \tilde{\tau} : L_\Theta \rightarrow \mathbb{C}[X^\text{unr}_{\Theta}]^\times \subset \tilde{K}_Y^\times \) denotes the tautological character \( a \mapsto (\omega \mapsto \omega(a)) \), there is a character \( \chi \in \hat{A}_C \), with restriction to \( \mathcal{Z}(L_\Theta)^0 \) equal to the central character of \( \sigma \), and a subset \( W_1 \subset W_{L_\Omega} \setminus W \) such that the operator:

\[
\prod_{w \in W_1} (z - w(\chi\tilde{\tau})(z)) \tag{10.6}
\]

annihilates \( \mathcal{M}_Y \otimes_{\tilde{K}_Y} \tilde{K}_Y \), for every \( z \in A'_{X,\Omega} \).

We used \( \tilde{\tau} \) for the tautological character of \( L_\Theta \) into \( \tilde{K}_Y^\times \) in order to reserve the symbol \( \tau \) for the tautological character:

\[
t : A'_{X,\Theta} \rightarrow \mathbb{C}[Y]^\times \subset \mathbb{K}_Y^\times,
\]

\( a \mapsto (\sigma \mapsto \chi_\sigma(a)) \), where \( \chi_\sigma \) denotes the central character of \( \sigma \).

Proof. Since \( \mathcal{L}_\Theta \) is a direct sum of a finite number of copies of \( I_{\Theta^{-}}(\sigma') \) (considered as a sheaf over \( Y \) as \( \sigma' \in Y \) varies), it is enough to prove the proposition for the module:

\[
\mathcal{M}_Y := \text{Hom}_{G}((\mathcal{S}(X_\Omega), \mathbb{C}(Y, \sigma' \mapsto I_{\Theta^{-}}(\sigma'))).
\]
We have:
\[ M'_Y \otimes_{K_Y} \tilde{K}_Y \subset M''_Y := \text{Hom}_G \left( S(X_\Omega), \mathbb{C}(\mathbb{X}_\theta^L \cup \mathbb{X}_0^L), \omega \mapsto I_{\Theta^-}(\sigma \otimes \omega) \right), \]
therefore it is enough to show that \( M''_Y \) decomposes into a direct sum of generalized eigenspaces as in the statement of the proposition.

We can represent \( \sigma \) as a subrepresentation of a representation parabolically induced from a supercuspidal \( \tau \), and then \( I_{\Theta^-}(\sigma \otimes \omega) \) becomes a subrepresentation of \( I_P(\tau \otimes \omega) \), where \( P \) is a suitable parabolic.

Notice that:
\[ \text{Hom}_G \left( S(X_\Omega), I_P(\tau \otimes \omega) \right) \overset{(*)}{\cong} \text{Hom}_G \left( \bar{I}_P(\tau \otimes \omega^{-1}), C^\infty(X_\Omega) \right) \cong \text{Hom}_{L_\Omega} \left( \bar{I}_P(\tau \otimes \omega^{-1})_{\Omega^-}, C^\infty(X_\Omega^L) \right), \]
where we recall that the index \( \Omega^- \) denotes normalized Jacquet module with respect to the parabolic \( P_{\Omega^-} \).

The Jacquet module \( I_P(\tau \otimes \omega^{-1})_{\Omega^-} \) is \( \mathcal{Z}(L_\Omega) \)-finite, and it is annihilated by the product:
\[ \prod_{w \in (W_L \backslash W/W_{L_\Omega})} \left( z - w(\chi_{\tau \omega})^{-1}(z) \right), \]
(corresponding to its canonical filtration in terms of \( P \setminus G/P_{\Omega^-} \)-orbits) for every \( z \in \mathcal{Z}(L_\Omega) \), where \( \chi_{\tau \omega} \) is the central character of \( \tau \). Notice that at the step \((*)\) we have used a duality which inverts characters of \( A'_{X,\Omega}\) therefore the module \( \text{Hom}_G \left( S(X_\Omega), I_P(\tau \otimes \omega) \right) \), for \( \omega \) in general position, will be annihilated by:
\[ \prod_{w \in (W_L \backslash W/W_{L_\Omega})^\ast} \left( z - w(\chi_{\tau \omega})(z) \right), \]
where by \((W_L \backslash W/W_{L_\Omega})^\ast\) we denote the subset of those cosets for which the restriction of elements of \( w(\chi_{\tau \omega}) \) to \( \mathcal{Z}(L_\Omega) \) factors through the quotient \((10.5)\). (We could alternatively have used second adjunction from the beginning, to analyze the \( A'_{X,\Omega} \)-action in terms of the Jacquet module \( I_P(\tau \otimes \omega)_{\Omega^-} \).)

After choosing representatives for this subset in \( W_{L_\Omega} \backslash W \), and an extension \( \chi_\tau \) of \( \chi_{\tau \omega} \) (which is a character of the center of the Levi of \( P \)) to \( A \), we arrive at the statement of the proposition. Notice that there is no reason to choose this extension \( \chi \) of \( \chi_{\tau \omega} \), but it allowed us to formulate the proposition without reference to the parabolic \( P \).

As is obvious from the last sentence of the proof, there is some choice involved in the subset \( W_1 \subset W_{L_\Omega} \backslash W \). Notice, however, that for cosets represented by elements \( w_X \in W_X(\Omega, \Theta) \) there is no choice involved, since \( W_{L_\Omega}w_XW_{L_\Omega} = W_{L_\Omega}w_X \).

We can think of the eigencharacters described in the last proposition as correspondences \( Y_\mathbb{C} \longrightarrow A'_{X,\Omega} \mathbb{C} \).
10.2. **Derivatives.** Now recall that \( \mathcal{L}_\sigma \) carries a flat connection, which depends (in a very mild way) on choosing a base point \( x \in X \). The resulting action of \( D(Y) \) (the ring of differential operators on \( Y \)) on elements of the space:

\[
\mathcal{M}_Y = \text{Hom}_G(\mathcal{S}(X_\Omega), \mathcal{C}(Y, \mathcal{L}_\sigma))
\]

does not preserve \( G \)-equivariance, but it does preserve eigencharacters up to multiplicity:

**Lemma.** Let \( E \in \mathcal{M}_Y \) and let \( W_E \subset W_1 \), in the notation of (10.6), be any subset such that the corresponding operator:

\[
P_E(z) := \prod_{w \in W_E} (z - w(\chi))(z)
\]

annihilates \( E \), for every \( z \in \mathfrak{A}_{X,\Omega} \).

Let \( D \in D(Y) \), so \( DE \in \text{Hom}(\mathcal{S}(X_\Omega), \mathcal{C}(Y, \mathcal{L}_\sigma)) \). Then a fixed power of \( P_E(z) \) annihilates \( DE \), for every \( z \in \mathfrak{A}_{X,\Omega} \).

**Proof.** For every fixed \( z \), the polynomial:

\[
P_{E,z}(x) = \prod_{w \in W_E} (x - w(\chi))(z)
\]

is divided by the minimal polynomial of the operator \( z \) acting on \( E \). The ring \( D(Y) \) acts on polynomials with coefficients in \( K_Y \), simply by acting on the coefficients. If \( D \in D(Y) \) is of degree \( n \), then the commutator:

\[
[D, P_{E,z}^{n+1}]
\]

lies in the ideal generated by \( P_{E,z} \); therefore, \( P_{E,z}^{n+1}(z) = P_E^{n+1}(z) \) annihilates \( DE \).

10.3. **Weak tangent space of a family.** Recall that an irreducible representation \( \sigma \) of \( G \) is a subquotient of a parabolically induced supercuspidal representation \( \tau \) of a Levi subgroup \( L \), and the pair \((\tau, L)\) is called the supercuspidal support of \( \sigma \). It is well defined modulo \( G \)-conjugacy, and the set of \( G \)-conjugacy classes of such pairs has a natural orbifold structure. Notationally, we can also write \((\tau, P)\) when \( P \) is a parabolic with Levi subgroup \( L \).

Let us denote by \( SP_G \) the space of supercuspidal pairs \((\tau, L)\) and by \( SC_G \) the orbifold of their equivalence classes. The tangent space of the fiber of \( SP_G \) over a fixed \( L \) can be canonically identified with the Lie algebra of
the unramified character group $\hat{L}_{\text{unr}}^*$. Notice that $\hat{L}_{\text{unr}}^* \subset \hat{A}_{\text{unr}}^*$. For a pair $x = (\tau, L)$ we will call weak tangent space $WT_{G,x}$ the image of the Lie algebra $\mathfrak{l}^*_C$ of $\hat{L}_{\text{unr}}^*$ in the set-theoretic quotient $\mathfrak{a}^*_C/W$, where

$$\mathfrak{a}^*_C = \text{Hom}(A, \mathbb{G}_m) \otimes \mathbb{C}$$

is the Lie algebra of $\hat{A}_C$.

The weak tangent space descends to the orbifold $SC_G$, i.e. it makes sense for any point of $SC_G$; in fact, if by tangent space $T_x$ of a point $x$ on an orbifold we mean the quotient of the tangent space of a preimage on the covering manifold by the finite stabilizer, there is a well defined map:

$$T_x \to WT_{G,x}$$

at every $x \in SC_G$.

Now, consider a set $I$ of finite-length representations of $G$. Let $I'$ be the set of isomorphism classes of irreducible subquotients of elements of $I$, and let $SC_G(I) \subset SC_G$ denote the set of supercuspidal supports of elements of $I'$. Then at every point $x \in SC_G(I)$ we have a well-defined subset:

$$WT_{G,x}(I) \subset WT_{G,x} \subset \mathfrak{a}^*_C/W,$$

defined as the union of the tangent spaces at $x$ of all embedded suborbifolds: $S \subset SC_G(I) \subset SC_G$. We use “embedded suborbifold” to refer to the image in $SC_G$ of a smooth embedded submanifold of a finite (smooth manifold) cover of $SC_G$.

The union of the spaces $WT_{G,x}(I)$ over all $x \in SC_G(I)$ will be called, for brevity, the “weak tangent space of $I$” instead of “weak tangent space of the supercuspidal support of $I$”, and denoted $WT_G(I)$.

In what follows, we will apply these notions to Levi subgroups, instead of the group $G$. Notice that for a Levi subgroup $L$, the corresponding notion of weak tangent space for $SC_L$ gives a subset of $\mathfrak{a}^*_L/W_L$, where $W_L$ is the Weyl group of $L$. If $L = L_\Omega$ for some $\Omega \subset \Delta_X$ we will be using the index $\Omega$ instead of $L_\Omega$.

Let $\Omega \subset \Delta_X$. Let $J$ be a family of $G$-morphisms $\{S(X_\Omega) \to \pi\}_\pi$, where $\pi$ varies over a set $J_0$ of finite length $G$-representations. Each such morphism is equivalent, by second adjunction, to a morphism:

$$S(X_\Omega^I) \to \pi_\Omega.$$

where $\pi_\Omega$ denotes the Jacquet module of $\pi$ with respect to $P_\Omega$ – also of finite length. Let $I$ denote the union, over all $\pi \in J_0$, of the images of the maps (10.8). We define:

$$SC_\Omega(J) := SC_\Omega(I)$$

and

$$WT_\Omega(J) := WT_\Omega(I),$$

the latter whenever $SC_\Omega(J)$ is a suborbifold of $SC_\Omega$. 
These definitions can also be given without appealing to second adjunction, of course; it suffices to dualize the morphisms:

$$\tilde{\pi} \rightarrow C^{\infty}(X_{\Omega})$$

and to use Frobenius reciprocity, as in the proof of the preceding proposition.

In the notation of the previous subsection \((Y \subset \tilde{X}^L_{\Theta} \text{ disc})\), an element \(E \in \text{Hom}_G(S(X_{\Omega}), \mathbb{C}(Y, \mathcal{L}_{\Theta}))\) will be considered as a family \(\mathcal{F}\) as above by considering the evaluations of its points wherever they are defined, and we will also be using \(SC(E), WT(E)\) to denote the supercuspidal support, resp. weak tangent space, of this family.

In this language, the proof of Proposition 10.1 shows:

10.4. Corollary. Let \((Y \subset \tilde{X}^L_{\Theta} \text{ disc})\) be a connected component and let \(E \in \mathcal{M}_Y = \text{Hom}_G(S(X_{\Omega}), \mathbb{C}(Y, \mathcal{L}_{\Theta})).\)

Then:

$$SC_{\Omega}(E) \subset \bigcup_{g \in G(x, L_{\Omega})} \left\{ g \cdot \tilde{X}^L_{\Theta} \right\}, \quad (10.9)$$

$$WT_{\Omega}(E) = \bigcup_{w \in W \subset W_L \backslash W} \left\{ w \left( a^*_x, \Theta, C \right) \right\}, \quad (10.10)$$

where:

- \(x\) is a supercuspidal pair for \(L_{\Theta}\) (i.e. \(x \in SP_{\Theta}\));
- \(G(x, L_{\Omega})\) denotes the set of elements in \(G\) carrying the Levi \(L\) of \(x\) into \(L_{\Omega}\);
- \(a^*_x, \Theta, C\) denotes the Lie algebra of \(\tilde{X}^L_{\Theta} \text{ disc}\), which can also be identified with the Lie algebra of \(\tilde{A}^L_{\Theta} \text{ disc}\) (hence the notation), inside of \(a^*_C\);
- \([\bullet]\) denotes classes in \(SC_{\Omega}\), resp. \(WT_{\Omega}\);
- \(W^\prime\) denotes some subset of the given set.

Proof. The proof of the corollary is essentially identical to that of Proposition 10.1, if we replace the action of \(Z(L_{\Omega})\) by that of the Bernstein center \(\mathcal{Z}(L_{\Omega})\). The reader should notice here that the present corollary will only be used in the proof of Proposition 12.1 when \(\Omega \neq \Delta_X\); thus, we are not applying a circular argument when reproving the structure of the Bernstein center in §15.1, since one can establish it inductively on the size of the group.

In the notation of the proof of Proposition 10.1, by choosing a basis for \(\text{Hom}_{L_{\Theta}}(S(X_{\Omega}), \sigma_{\text{disc}})\), we can identify the bundle \(\omega \mapsto \mathcal{L}_{\Theta, \sigma \otimes \omega}\) over \(\sigma \otimes \omega\) with a subbundle of \(\omega \mapsto I_P(\tau \otimes \omega)^r\), for some \(r\), and hence \(E\) with an element of:

$$\text{Hom}_G(S(X_{\Omega}), \omega \mapsto I_P(\tau \otimes \omega)^r).$$

Its dual, \(E^*\), will be a rational section (in the variable \(\omega\)) of:

$$\text{Hom}_G(I_P(\hat{\tau} \otimes \omega^{-1})^r, C^{\infty}(X_{\Omega})) = \text{Hom}_{L_{\Omega}}(I_P(\hat{\tau} \otimes \omega^{-1})^r_{\hat{\Omega}}, C^{\infty}(X_{\Omega})).$$
The specialization, thought of as an element of the latter space, will be denoted by $E^*_{\Omega, \omega}$.

The $L_\Omega$-supercuspidal support of the Jacquet module $I_P(\hat{\tau} \otimes \omega^{-1})_{\Gamma^\omega}$ consists of the classes of pairs $(w(\hat{\tau} \otimes \omega^{-1}), wL)$, where $w$ ranges in $W_{L_\Omega} \setminus W/W_L$. We can partition the set $W_{L_\Omega} \setminus W/W_L$ into equivalence classes $W_i$ so that the classes of $(w_1(\hat{\tau} \otimes \omega^{-1}), w_1L), (w_2(\hat{\tau} \otimes \omega^{-1}), w_2L)$ are equal (for all $\omega$) when $w_1, w_2 \in W_i$ and distinct, for generic $\omega \in \widehat{X^L_{\Omega C}}$, when $w_1 \in W_i, w_2 \in W_j, i \neq j$.

Denote $\tilde{K}_Y := \mathbb{C}(\widehat{X^L_{\Omega, \omega}})$ as in Proposition 10.1, and let:

$$\mathfrak{z}_{\Omega, K} := \mathfrak{z}(L_\Omega) \otimes_{\mathbb{C}} \tilde{K}_Y,$$

where $\mathfrak{z}(L_\Omega)$ is the Bernstein center of the category of smooth representations of $L_\Omega$.

As in the proof of Proposition 10.1, generically distinct supercuspidal supports means distinct generalized eigencharacters for the action of the algebra $\mathfrak{z}_{\Omega, K}$ on its module generated by $E^*$, and these generalized eigencharacters are defined over $\tilde{K}_Y$. Thus, we can find a partition of unity:

$$1 = \sum z_i, \quad z_i \in \mathfrak{z}_{\Omega, K},$$

such that the composition:

$$I_P(\hat{\tau} \otimes \omega^{-1})_{\Gamma^\omega} \xrightarrow{E^*_{\Omega, \omega}} C^\infty(X^L_{\Omega}) \xrightarrow{z_i} C^\infty(X^L_{\Omega}),$$

which varies rationally in $\omega$, if nonzero factors through a quotient of $I_P(\hat{\tau} \otimes \omega^{-1})_{\Gamma^\omega}$, with supercuspidal support equal to $(w(\hat{\tau} \otimes \omega^{-1}), w_iL)$, for $w_i \in W_i$.

Thus, there is a finite subset $R$ of indices $i$, and a (nonempty) Zariski open subset $U \subset \widehat{X^L_{\Omega C}}$ such that for $\omega \in U$ the supercuspidal support of $E_{\Omega, \omega} : S(X^L_{\Omega}) \to I_P(\hat{\tau} \otimes \omega^{-1})_{\Gamma^\omega}$ is precisely equal to the set of classes of the pairs $(w(\hat{\tau} \otimes \omega), w_iL)$, for $i \in R$ and $w_i \in W_i$. It is not difficult to see from this that the supercuspidal support of $E_{\Omega, \omega}$ at all points where it is defined is contained in this set of classes, and this proves the corollary.

\[\square\]

10.5. Generic injectivity. We recall the notion of "generic injectivity of the map: $a^*_X/W_X \to a^*/W$" in the language of [SV, §14.2] (for brevity we will just say: "generic injectivity"), where $a^*_X = \text{Hom}(A_X, \mathbb{G}_m) \otimes \mathbb{Q} \subset a^* = \text{Hom}(B, \mathbb{G}_m) \otimes \mathbb{Q}$. We will also introduce a stronger version of this notion, to be termed "strong generic injectivity", and will show that it holds for symmetric spaces.

For each $\Theta \subset \Delta_X$ we let:

$$a^*_{X, \Theta} := \text{Hom}(A_{X, \Theta}, \mathbb{G}_m) \otimes \mathbb{Q} \simeq \text{Hom}((X^L_{\Theta})^{ab}, \mathbb{G}_m) \otimes \mathbb{Q},$$
which is embedded into \( a^*_X = a^*_{X,\emptyset} \) by the map induced from:

\[
(X^L_{\emptyset})^{ab} \rightarrow (X^L_{\emptyset})^{ab}.
\]

We say that \( X \) satisfies the condition of **generic injectivity** if the following holds:

Whenever the action of an element \( w \) of the **full** Weyl group \( W \) on \( a^* \) restricts to an isomorphism:

\[
a^*_{X,\theta} \overset{\sim}{\rightarrow} a^*_{X,\Omega}
\]

(for any \( \Theta, \Omega \subset \Delta_X \), obviously of the same order, and possibly equal), there is an element of the **little** Weyl group \( W_X \) which induces the same isomorphism.

The condition holds for all symmetric varieties, by [Del, Lemma 15]. Together with the wavefront and strong factorizability assumptions (both of which hold for symmetric spaces), it guarantees the validity of the full Plancherel decomposition [SV, Theorem 7.3.1], [Del, Theorem 6].

We will say that \( X \) satisfies the **strong generic injectivity** condition if the following holds:

Whenever an element \( w \) of the **full** Weyl group \( W \) on \( a^* \) restricts to an injection:

\[
a^*_{X,\Theta} \hookrightarrow a^*_{X,\Omega}
\]

(for any \( \Theta, \Omega \subset \Delta_X \), obviously with \( |\Omega| \leq |\Theta| \)) there is an element \( w_X \in W_X \) such that:

\[
w_X|_{a^*_{X,\Theta}} = w|_{a^*_{X,\emptyset}}.
\]

10.6. **Lemma.**

(1) If \( X \) is a symmetric variety, then it satisfies the strong generic injectivity assumption.

(2) If \( X \) satisfies the strong generic injectivity condition, the element \( w_X^{-1} \) in the definition of this condition can be taken to map the set of simple spherical roots \( \Omega \) into \( \Theta \).

**Proof.** By [Del, Lemma 15(iv)], for every \( Z \in a^*_{X,\emptyset} \) there is an element \( w_Z \in W_X \) such that:

\[
w_Z(Z) = w(Z).
\]

Since \( W_X \) is finite, an element \( w_X \in W_X \) will be equal to \( w_Z \) for a Zariski dense set of elements of \( a^*_{X,\emptyset} \) and the first claim follows from continuity of the \( w_X \)-action.

The second claim follows from known root system combinatorics: Among elements in \( W_X \) which take the subspace \( a^*_{X,\emptyset} \) into \( a^*_{X,\emptyset} \) if we take \( w_X \) to be of minimal length in \( W_X,\emptyset \), \( W_X,\emptyset \) (where \( W_X,\emptyset \) denote the Weyl groups of the corresponding Levis, i.e. those generated by the simple reflections corresponding to \( \bullet \)) then:

\[
w_X \Theta > 0 \text{ and } w_X^{-1} \Omega > 0.
\]
The second statement implies that \( w_X^{-1} \Omega \) belongs to the positive span of \( \Theta \), and the first that it actually belongs to \( \Theta \). □

**Remark.** The strong version of the generic injectivity condition will only be used to prove that “scattering maps preserve cuspidality”, cf. Proposition 12.1. This result has been proven in a different way for symmetric varieties by [CD14], relying heavily on the structure of these varieties. In each specific case, the strong generic injectivity condition is easy to check once one knows the dual group of the spherical variety; of course, it would be desirable to have a proof of this property in some more general setting.

For the following lemma we identify \( a^*_{X,\Omega} \), as we did before, with a subspace of \( a^*_{\Omega} \):

\[
\text{Hom}(P_{\Omega}^{-}, \mathbb{G}_m) \otimes \mathbb{Q} \subset a^* \Omega
\]
on the other hand, we have a restriction map from characters of the Borel \( B \) to characters of the center \( Z(L_{\Omega}) \) of the Levi of \( P_{\Omega}^{-} \); we will write:

\[
\text{Cent}_{\Omega} : a^* \rightarrow a^*_{\Omega} \simeq \text{Hom}(Z(L_{\Omega}), \mathbb{G}_m) \otimes \mathbb{Q}
\]

for the corresponding map.

**10.7. Lemma.** Assume the strong generic injectivity condition for \( X \). Then:

1. For \( w \in W \) we have \( w \left( a^*_{X,\Theta,\mathcal{C}} \right) \supset a^*_{X,\Theta,\mathcal{C}} \) iff \( w \) is equivalent in \( W_{L_{\Omega}} \setminus W \) to an element \( w_X \in W_X \) with \( w_X^{-1} \Omega \subset \Theta \).

2. For \( w_X \in W_X(\Omega, \Theta) \) and \( w \in W \) we have:

\[
\text{Cent}_{\Omega} \circ w_X | a^*_{X,\Theta,\mathcal{C}} = \text{Cent}_{\Omega} \circ w | a^*_{X,\Theta,\mathcal{C}}
\]

iff \( w \equiv w_X \) in \( W_{L_{\Omega}} \setminus W \).

We prove the lemma below. If the meaning of the lemma is not immediately obvious, the following corollary has a representation-theoretic content related to supercuspidal supports, more precisely their “weak tangent spaces”. Recall that we denoted by \( t : A'_{X,\Theta} \rightarrow K_{Y}^{\times} \) the tautological character.

**10.8. Corollary.**

1. Let \( Y \subset X_{\Theta}^{\text{disc}} \) be a connected component, and define \( \mathcal{M}_Y \) as before. Then the only components on the right hand side of (10.10) which belong to \( a^*_{X,\Theta,\mathcal{C}} \) are those indexed by classes of elements \( w_X \in W_X \) with \( w_X^{-1} \Omega \subset \Theta \).

2. The eigencharacters \( w_t, w \in W_X(\Omega, \Theta) \), appear in (10.6) with multiplicity one. The same holds if we replace \( A'_{X,\Theta} \) by any subgroup of finite index.

Strictly speaking, the discussion up to this point implies that the eigencharacters corresponding to elements of \( W_X(\Omega, \Theta) \) appear with multiplicity at most one. However, scattering theory implies that they appear – already in the asymptotics of Eisenstein integrals. We omit the details, since we will encounter this point later.
Proof of Lemma 10.7. For the first statement, we notice that if \( w \left( a_{X,\Theta}^* \right) = a_{X,\Theta}^* \), then, by generic injectivity, there is a \( w_X \in W_X(\Omega, \Theta) \) such that \( w_X^{-1} \cdot w \) fixes all points of \( a_{X,\Theta}^* \).

However, it is known that \( a_{X,\Theta}^* \) contains strictly \( P^-_\Theta \)-dominant elements \([SV, \text{Proof of Corollary 15.3.2}]\), i.e. elements that are positive on each coroot corresponding to the unipotent radical of \( P^-_\Theta \). Therefore the only elements of \( W \) which act trivially on it are the elements of \( W_{L_\Theta} \). Hence, \( w \in w_X W_{L_\Theta} \) or, equivalently, \( w \in W_{L_\Theta} w_X \).

For the second statement we notice that in terms of an orthogonal \( W \)-invariant inner product on \( a_{X,\Theta}^* \), the operator \( \text{Cent}_\Theta \) represents the orthogonal projection onto \( a_{X,\Theta}^* \). On the other hand, \( w_X |_{a_{X,\Theta}^*} \) already has image into \( a_{X,\Omega}^* \subset a_{X,\Theta}^* \), therefore the only way that \( \text{Cent}_\Theta \circ w_X |_{a_{X,\Theta}^*} \) is that \( w |_{a_{X,\Theta}^*} \) also has image in \( a_{X,\Omega}^* \). By the first statement, this implies that \( w \) is equivalent to \( w_X \) in \( W_{L_\Theta} \backslash W \). \( \square \)

Combining Proposition 10.1 with Corollary 10.8, and observing that the eigencharacters \( \nu^t \), \( w \in W_X(\Omega, \Theta) \), are already defined over \( K_Y \), we arrive at the following:

10.9. Proposition. The module \( \mathcal{M}_Y \) admits a decomposition:

\[
\mathcal{M}_Y = \bigoplus_{w \in W_X(\Omega, \Theta)} \mathcal{M}_Y^w \oplus \mathcal{M}_Y^\text{rest},
\]

(10.12)

where \( \mathcal{M}_Y^w \) is the eigenspace with eigencharacter \( \nu^t \), and the space \( \mathcal{M}_Y^\text{rest} \) contains none of these eigencharacters.

Of course, this proposition is vacuous unless \( \Theta \sim \Omega \).

10.10. Polynomial decomposition of morphisms. We now return to the torsion-free sheaf

\[
\mathcal{M} := \text{Hom}_{\mathcal{L}}(\mathcal{S}(X_\Theta), \mathcal{L}_\Theta),
\]

whose rational sections over a connected component \( Y \subset X^\text{disc}_\Theta \) we denoted before by \( \mathcal{M}_Y \).

Let us discuss to what extent the decomposition of Proposition 10.9 extends to a decomposition of this sheaf – the goal being to determine the poles that might get introduced when decomposing an element of \( \mathcal{M}_Y \) as in (10.12). Our approach is similar to [DH14, Proposition 2], based on the theory of the resultant, which in turn was inspired by the proof of Lemma VI.2.1 in Waldspurger [Wal03], except that we use a more algebro-geometric language to avoid choosing elements of \( A_{X,\Theta}^* \).
10.11. Proposition. Let \( U \subset Y \) denote the complement of the images of the divisors\(^4\) given by equations:

\[
w_X(\chi_\mathcal{Y})|_{A'_{X,\Omega}} = w(\chi_\mathcal{Y})|_{A'_{X,\Omega}},
\]

(10.13)

for those pairs \( w_X \in W_X(\Omega, \Theta), w \in W_1 \) in the notation of (10.6) for which this equality represents a divisor in \( X^L_{\Theta} \).

The restriction \( \mathfrak{M}_U \) of the sheaf \( \mathfrak{M} = \text{Hom}_G(S(X_\Omega), \mathcal{Z}_\Theta) \) over \( U \) decomposes as a direct sum of subsheaves:

\[
\mathfrak{M}_U = \bigoplus_{w \in W_X(\Omega, \Theta)} \mathfrak{M}_U^w \oplus \mathfrak{M}_U^{\text{rest}},
\]

where \( \mathfrak{M}_U^w \) denotes the sheaf of \( A'_{X,\Theta} \)-equivariant morphisms with respect to the map \( w : A'_{X,\Theta} \to A'_{X,\Omega} \).

Remark. Not all pairs \((w_X, w)\) as above represent a divisor. For example, if \( \Theta = \Omega = \emptyset \) so that \( W_X(\Omega, \Theta) = W_X \), only the reflections corresponding to roots represent a divisor.

Proof. Clearly, by the previous section, the sheaf \( \mathfrak{M}_Y \) admits a direct sum decomposition into a finite number of eigenspaces for the maximal compact subgroup of \( A'_{X,\Omega} \). Each eigencharacter defines a connected component of \( A'_{X,\Omega} \). We fix such a component \( V \) and consider \( \mathfrak{M}_Y \) as a sheaf over \( V_\mathbb{C} \times Y_\mathbb{C} \) (Elements of \( A'_{X,\Omega} \) now restrict to polynomials over \( V_\mathbb{C} \)).

Set \( \mathcal{R} := \mathbb{C}[V \times Y] \). The annihilator of \( \mathfrak{M}_Y \) in \( \mathcal{R} \) is the ideal \( \mathcal{I} \) generated by the “minimal polynomials” (10.6), where \( z \) ranges over all elements of \( A'_{X,\Omega} \). (Clearly, a finite set of elements generating \( A'_{X,\Omega} \) modulo its maximal compact subgroup suffices.)

The spectrum of the ring \( \overline{\mathcal{R}} = \mathcal{R}/\mathcal{I} \) has a finite number of irreducible components, parametrized by \( (X^L_{\Theta}/Y) \)-Galois orbits of the distinct factors of (10.6). We denote by \( Z_w \) the components corresponding to \( w \in W_X(\Omega, \Theta) \), and by \( Z_{\text{rest}} \) the rest of the components; we use \( P_w, P_{\text{rest}} \) for the corresponding prime ideals.

Let \( Y^\text{sing}_\mathbb{C} \subset Y_\mathbb{C} \) denote the union of the images of all subvarieties given by equations (10.13), whether these equations represent divisors or subvarieties of larger codimension. For any \( f \in \overline{\mathcal{R}} \) which is not a zero divisor and vanishes on \( Y^\text{sing}_\mathbb{C} \), consider the localization:

\[
\mathfrak{M}_Y[f^{-1}]
\]

which is a sheaf over the spectrum of \( \overline{\mathcal{R}}[f^{-1}] \).

Notice that the components \( Z_\sigma \) have no intersection over the complement of \( Y^\text{sing}_\mathbb{C} \); therefore, \( \overline{\mathcal{R}}[f^{-1}] \) is a direct sum of integral domains, and we have

\(^4\)Recall that these equations are on \( X^L_{\Theta}/Y \); “images” refers to the map \( \omega \mapsto \sigma \otimes \omega \in Y \).
a corresponding decomposition of the unit element:

\[ 1 = \sum_w 1_w + 1_{\text{rest}}, \quad (10.14) \]

where \( 1_w \in \mathbb{R}[f^{-1}] \). This gives a decomposition of \( M_Y \) over the complement of the zero set of \( f \). Since the only requirement on \( f \) was that it vanishes on \( Y^{\text{sing}}_C \), we get a decomposition of \( M_{Y^C \setminus Y^{\text{sing}}_C} \):

\[ M_{Y^C \setminus Y^{\text{sing}}_C} = \bigoplus_{w \in W_X(\Omega, \Theta)} M^w_{Y^C \setminus Y^{\text{sing}}_C} \oplus M^{\text{rest}}_{Y^C \setminus Y^{\text{sing}}_C}. \]

Finally, recall that \( M = \text{Hom}_G(S(X_\Theta), \mathcal{L}_\Theta) \). A section of \( \mathcal{L}_\Theta \) defined in a neighborhood of a subvariety of codimension \( \geq 2 \) extends uniquely to this subvariety. Therefore, the above decomposition of \( M_{Y^C \setminus Y^{\text{sing}}_C} \) extends to the complement of all divisors contained in \( Y^{\text{sing}}_C \), i.e. to \( U \). \( \square \)

Remark. The statement is not true without removing the divisors of the form \( (10.13) \). Indeed, while it is true that the direct sum of sheaves \( M^w_{X^C} \), \( w \in W_X(\Omega, \Theta) \), embeds into \( M_Y \), it is easy to construct sections of these sheaves with poles over such a divisor, whose sum does not have any pole. We will soon find out that our scattering maps are not of this form, at least not on the divisors of the form \( (10.13) \) where \( w \) is also in \( W_X(\Omega, \Theta) \).

We return to the asymptotics of the normalized constant terms introduced in (10.1):

\[ E^{*, \Omega}_{\Theta, \text{disc}} = E^{*, \text{disc}}_{\Theta, \Omega} \circ e_{\Omega} : S(X_\Omega) \to \mathbb{C}(X^L_{\Theta} \text{ disc}, \mathcal{L}_\Theta). \]

10.12. Corollary. Let \( \Omega \sim \Theta \). There is a decomposition:

\[ E^{*, \Omega}_{\Theta, \text{disc}} = \sum_{w \in W_X(\Theta, \Omega)} \mathcal{S}_w + \mathcal{S}_{\text{Subunit}}, \quad (10.15) \]

where all summands are elements of:

\[ \mathcal{M} = \text{Hom}_G \left( S(X_\Omega), \mathbb{C}(X^L_{\Theta} \text{ disc}, \mathcal{L}_\Theta) \right) \]

with the following properties:

1. The operator \( \mathcal{S}_w \) is an eigenvector of \( A'_{X, \Omega} \) on \( \mathcal{M} \); more precisely, it is \( w \)-equivariant with respect to the action of \( A'_{X, \Omega} \).
2. The operator \( \mathcal{S}_{\text{Subunit}} \) has no \( w \)-equivariant direct summand, for any \( w \in W_X(\Omega, \Theta) \).
3. The poles of all summands are linear; for each component \( Y \subset X^L_{\Theta} \), they are contained in the union of the poles of \( E^{*, \Omega}_{\Theta, \text{disc}} \) and the images of divisors given by equations:

\[ w_X(\chi \tilde{t})|_{A'_{X, \Omega}} = w(\chi \tilde{t})|_{A'_{X, \Omega}}. \]
for those pairs $w_X \in W_X(\Omega, \Theta), w \in W_1$ in the notation of (10.6) for which this equality represents a divisor in $X^\text{unr}_{\Theta}$.

The statements of this corollary will be strengthened in the next couple of sections, in order to arrive at the results of §9. The notation $\mathcal{S}_{\text{Subunit}}$ is due to the fact that, as we will see in the next section using $L^2$-theory, the exponents of these morphisms over the unitary subset $X^\text{disc}_\Theta$ are “subunitary”.

10.13. **Explication of the fiberwise scattering maps.** Here we would like to emphasize here that the fiberwise scattering maps $\mathcal{S}_w$ play the role of “functional equations” between the normalized Eisenstein integrals. We use the fact that these maps are equivariant with respect to discrete centers, which is a yet-unproven statement of Theorem 9.3, because we are not going to use the following result anywhere.

10.14. **Proposition.** Let $\Theta, \Omega$ be associates, and $w \in W_X(\Omega, \Theta)$. The corresponding fiberwise scattering map $\mathcal{S}_w \in \mathbb{C} \left( \tilde{X}_\Theta^\text{disc}, \text{Hom}_G(\mathcal{L}_\Theta, w^* \mathcal{L}_\Omega) \right)$ is the unique rational family of maps making the following diagram commute (for almost all $\sigma \in X^\text{disc}_\Theta$):

$$
\begin{array}{ccc}
\mathbb{S}(X) & \mathcal{S}(X)_{\Theta, \text{disc}} \\
\mathcal{S}_w & \downarrow \\
\mathcal{S}(X)_{\Omega, \text{disc}} & \mathcal{S}(X)^w_{\sigma, \text{disc}}
\end{array}
$$

(10.16)

**Proof.** We have $\iota_{\Theta} f = \iota_{\Omega} S_w f$. By Theorem 7.4 and (9.5) this becomes:

$$
\int_{\tilde{X}_\Theta^\text{disc}} E_{\Theta, \sigma, \text{disc}} f^\sigma d\sigma = \int_{\tilde{X}_\Theta^\text{disc}} E_{\Omega, w^* \sigma, \text{disc}} \mathcal{S}_w^{-1} f^\sigma d\sigma,
$$

and disintegrating over $\sigma$ we get that $E_{\Theta, \sigma, \text{disc}}$ and $E_{\Omega, w^* \sigma, \text{disc}} \circ \mathcal{S}_w$ must be equal for almost every $\sigma$, hence equal as rational functions of $\sigma$. □

This proof is actually rather indirect, to avoid the discussion of “small Mackey restriction” of [SV, §15.5]; it can be inferred directly from this discussion, when “injectivity of small Mackey restriction” is known (such as in the case of symmetric varieties).

This result is essentially equivalent to the description of the constant term of Eisenstein integrals in terms of “$B$-matrices” and intertwining integrals in [CD14, Theorem 8.4]; that work can be considered as a qualitative study of these functional equations in the case of symmetric spaces, which
among other things gives some results that we prove here without relying so much on the structure of symmetric varieties, such as the fact that “scattering maps preserve cuspidal summands” (Theorem 9.3).

11. SCATTERING: THE UNITARY CASE

The unitary asymptotics (adjoints of Bernstein maps) in [SV, §11.4] were obtained by filtering out the unitary exponents of the Plancherel decomposition. More precisely, given a (smooth, say) function \( \Phi \in L^2(X) \) with Plancherel decomposition:

\[
\Phi(x) = \int_G \Phi^\tau(x) \mu(\pi),
\]

then it is known that \( e^*_\Omega \Phi^\pi \) is \( A_{X,\Omega} \)-finite with only unitary and subunitary exponents (generalized eigencharacters) for \( \mu \)-almost all \( \pi \). We recall the notion of subunitary exponents for a morphism from \( S(X_\Omega) \) to a smooth representation \( V \): it means that the morphism is \( A_{X,\Omega} \)-finite, and the image of its dual: \( V \to C^\infty(X_\Omega) \) has subunitary exponents under the action of \( A_{X,\Omega} \), i.e. generalized characters which are \( < 1 \) on \( A^+_X \). (For the definition of \( A^+_X \) see §2.)

By construction we have:

\[
e^*_\Omega(\Phi) = \int_G (e^*_\Omega \Phi^\pi)^{\text{unit}} \mu(\pi), \tag{11.1}
\]

where \( (e^*_\Omega \Phi^\pi)^{\text{unit}} \) refers to isolating the part of \( (e^*_\Omega \Phi^\pi) \) with unitary generalized exponents, cf. [SV, Proposition 11.4.2].

Moreover, [SV], [Del] have proven Theorem 9.1 restricted to \( L^2(X_\Theta)_{\text{disc}} \) and with the modification that the condition of \( w \)-equivariance with respect to \( \varrho_{\text{disc}}(X_\Theta) \) be replaced by the weaker condition of \( w \)-equivariance with respect to \( A^+_X \). In other words,

\[
I^*_\Omega \circ I_{\Theta} |_{L^2(X_\Theta)_{\text{disc}}} = \sum_{w \in W X(\Omega, \Theta)} S_w, \tag{11.2}
\]

with \( S_w \) being \( w \)-equivariant with respect to \( A^+_X \).

Combining all the above with Theorem 7.4 and Corollary 10.12 we obtain:

11.1. Proposition. Let \( \Theta, \Omega \subset \Delta_X \). For every \( \sigma \in X^L_{\Theta, \text{disc}} \) (hence unitary), the \( A_{X,\Omega} \)-exponents of \( E^*_{\Theta, \text{disc}} \) are unitary or subunitary.

Let \( \Theta \sim \Omega \). Consider the operator \( \mathcal{S}_{\text{Subunit}} \) of Corollary 10.12. For every \( \sigma \in X^L_{\Theta} \) (hence unitary) where this operator is defined (regular), the resulting morphism:

\[
S(X_\Omega) \to \mathcal{L}_{\Theta, \sigma}
\]

has subunitary exponents.
We have:

\[ i_\Omega^* \iota_\Theta f(x) = \sum_{w \in W X(\Omega, \Theta)} \int_{X^\text{disc}_\Theta}^{\wedge} \mathcal{F}^{w-1} f^\sigma(x) d\sigma. \tag{11.3} \]

Proof. In the notation of Theorem (7.4):

\[ e^*_\Omega \iota_\Theta f(x) = \int_{X^\text{disc}_\Theta}^{\wedge} E^*_{\Theta, \sigma, \text{disc}} f^\sigma(x) d\sigma, \]

and therefore, by the above, \( e^*_{\Theta, \sigma, \text{disc}} \) can only have unitary or subunitary \( A_{X, \Omega} \)-exponents (for almost all \( \sigma \)); hence, the same holds for \( \mathcal{F}_{\text{Subunit}} \).

On the other hand, (11.1) together with the property of \( w \)-equivariance with respect to \( A_{X, \Theta} \) of the maps \( S_w \) of (11.2) implies that all unitary \( A_{X, \Omega} \)-exponents of \( e^*_{\Theta, \sigma, \text{disc}} \) are contained among the exponents of the \( \mathcal{F}_{w} \)'s.

This proves (11.3), and it shows that \( \mathcal{F}_{\text{Subunit}} \) only has subunitary exponents. □

This proves assertion (9.5) of Theorem 9.3: the scattering operator \( S_w : L^2(X_\Theta) \to L^2(X_{\Omega}) \), i.e. the \( w \)-equivariant part of \( i_\Omega^* \iota_\Theta f \) with respect to the action of \( A_{X, \Theta} \), is given by:

\[ S_w f(x) = \int_{X^\text{disc}_\Theta}^{\wedge} \mathcal{F}^{w-1} f^\sigma(x) d\sigma. \]

The next proposition includes, among others, the regularity property of the fiberwise scattering maps, i.e. the fact that the operators \( \mathcal{F}_w \), and hence also \( \mathcal{F}_{\text{Subunit}} \) by Proposition 8.2, are actually regular on the unitary set; hence, the condition “where its specialization is defined” turns out to be superfluous in the last statement.

The following proves the assertion on \( z^\text{disc}(X^L_\Theta) \)-equivariance of Theorem 9.1; assertion (9.6) of Theorem 9.3; and the regularity statement of Theorem 9.3:

11.2. Proposition. Let \( \Theta \sim \Omega, w \in W X(\Theta, \Omega) \). For every \( \sigma \in X^L_{\Theta, \text{disc}} \) where the operator \( \mathcal{F}_w \) is defined, the resulting morphism:

\[ S(X_{\Omega}) \to \mathcal{L}_{\Theta, \sigma} \]

factors through the discrete \( w\sigma \)-coinvariants \( S(X_{\Omega})_{w\sigma, \text{disc}} = \mathcal{L}_{\Omega, w\sigma} \) and is generically an isomorphism between \( \mathcal{L}_{\Omega, \sigma} \) and \( \mathcal{L}_{\Theta, \sigma} \). In particular, \( w \) induces an isomorphism:

\[ X^L_{\Theta, \text{disc}} \to X^L_{\Omega, \text{disc}}. \]

Thus, \( \mathcal{F}_w \) is a rational section of the sheaf \( \text{Hom}_G(w^* \mathcal{L}_\Omega, \mathcal{L}_\Theta) \) over \( X^L_{\Theta, \text{disc}} \). Its poles do not meet the unitary set, i.e.:

\[ \mathcal{F}_w \in \Gamma \left( X^L_{\Theta, \text{disc}}, \text{Hom}_G(w^* \mathcal{L}_\Omega, \mathcal{L}_\Theta) \right). \tag{11.4} \]

The operators \( \mathcal{F}_w \) satisfy the natural associativity conditions:

\[ \mathcal{F}_{w'} \circ \mathcal{F}_w = \mathcal{F}_{w'w} \text{ for } w \in W X(\Omega, \Theta), w' \in W X(\Xi, \Omega). \]
The scattering map $S_w$ is $w$-equivariant with respect to the discrete center $y_{\text{disc}}(X^L_\Omega)$.

**Proof.** By $w$-equivariance with respect to $A'_{X,\Theta}$, it follows that the specialization of $S_w$ at $\sigma$ factors through the $A'_{X,\Omega}$-coinvariant space:

$$S(X_\Omega)^w_{\chi_\sigma},$$

where $\chi_\sigma$ is the central character of $\sigma$. Moreover, since $S_w$ is a morphism: $L^2(X_\Theta)_{\text{disc}} \to L^2(X_\Omega)_{\text{disc}}$, it follows that $S_w$ is zero on the kernel of the map:

$$S(X_\Omega)^w_{\chi_\sigma} \to L^2(X_\Omega/A'_{X,\Omega}, w_{\chi_\sigma})_{\text{disc}}$$

(for almost all, and hence for all $\sigma$ where it is defined), and hence factors through the discrete coinvariants $L_{\Omega,\sigma} = S(X_\Omega)^w_{\chi_\sigma, \text{disc}}$. By the fact that $S_w$ is an isometry, we get that $\mathcal{S}_w$ is nonzero on every connected component of $X^L_\Omega$.

By definition, the space $S(X_\Omega)^w_{\chi_\sigma, \text{disc}}$ is equal to:

$$\bigoplus_\tau L_{\Omega,\tau},$$

where $\tau$ ranges over the fiber of the map: $\tilde{X}^L_{\Omega,\tau} \to \tilde{A}_{X,\Omega,\tau}$ (central character) over $w_{\chi_\sigma}$.

We claim that for $\sigma$ in general position, the only such $\tau$ with the property that $I_\Theta(\sigma)$ and $I_\Omega(\tau)$ have a common subquotient is $\tau = w_{\chi_\sigma}$. Indeed, this follows from the following lemma:

**11.3. Lemma.** Suppose that $\tau_1, \tau_2$ are non-isomorphic, irreducible unitary representations of the Levi quotient of a parabolic $P$, and that $X$ is a subtorus of the unramified characters of $P$ containing strictly $P$-dominant characters (i.e. those which are positive on all coroots corresponding to the unipotent radical of $P$).

Then, for $\omega \in X$ in general position, $I_P(\tau_1 \otimes \omega)$ and $I_P(\tau_2 \otimes \omega)$ are irreducible and non-isomorphic.

The irreducibility statement follows from [Cas, Theorem 6.6.1] and the fact that it is an open condition, and the fact that the induced representations are non-isomorphic can be deduced, for example, from the Langlands quotient theorem: For strictly $P$-dominant characters $\omega$, the representations $I_P(\tau_1 \otimes \omega)$ and $I_P(\tau_2 \otimes \omega)$ are Langlands quotients from different inducing data, since this is the case for $\tau_1$ and $\tau_2$.

We will apply this to $P = P_{\Omega}$, $\tau_1 \otimes \omega = \tau$, $\tau_2 \otimes \omega = w_{\chi_\sigma}$, $X = \tilde{X}^L_{\Omega,\tau}$, for $\sigma$ in general position. Notice that irreducibility implies that the standard intertwining operator is an isomorphism:

$$I_\Omega(w_{\chi_\sigma}) \simeq I_\Theta(\sigma),$$

hence any nonzero morphism:

$$I_\Omega(\tau) \to I_\Theta(\sigma)$$
gives a nonzero morphism: $I_\Omega(\tau) \to I_\Omega(\sigma^w)$. By the lemma, we cannot have a nonzero morphism $I_\Omega(\tau) \to I_\Omega(\sigma^w)$ unless $\tau \simeq w\sigma$.

Thus, the specialization of $\mathcal{S}_w$ at $\sigma$ factors through $\mathcal{L}_{\Omega,w^\sigma}$. Using (9.5), this proves that the scattering map $S_w$ is $w$-equivariant with respect to the discrete center $z^\text{disc}(X^\text{l}_\Theta)$. The fact that $S_w$ is an isometry now proves that the resulting map: $\mathcal{L}_{\Omega,w^\sigma} \to \mathcal{L}_{\varnothing,\sigma}$ is an isomorphism for generic $\sigma$.

For the regularity statement, we will proceed as in the proof of Proposition 8.2, where a priori knowledge of the integrability of Eisenstein integrals gave us their regularity on the unitary spectrum. Here we will use the a priori knowledge (Theorem 9.1) that the scattering operators are bounded operators between $L^2$-spaces (in fact, isometries, but we will not use that):

$$S_w : L^2(X^\Theta)^\text{disc} \to L^2(X^\Omega)^\text{disc}.$$ 

In terms of Theorem 5.2, this can be written as a map:

$$L^2(X^\text{l}_\Theta)^\text{disc} \to L^2(X^\text{l}_\Omega)^\text{disc},$$

which, we now know, is induced by some element:

$$\mathcal{S}_w \in \mathcal{C}(X^\text{l}_\Theta^\text{disc}, \text{Hom}_G(\mathcal{L}_\Theta, w^*\mathcal{L}_\varnothing)).$$

By Corollary 10.12, $\mathcal{S}_w$ has linear poles. Corollary 3.6 now implies that it is regular on the unitary spectrum.

Finally, the associativity conditions on $\mathcal{S}_w$ follow from those of the unitary scattering maps $S_w$. Indeed, the only way that the composition of the following maps:

$$L^2(X^\text{l}_\Theta^\text{disc}, \mathcal{L}_\Theta) \to L^2(X^\text{l}_\Omega^\text{disc}, \mathcal{L}_\Omega) \to L^2(X^\text{l}_\Xi^\text{disc}, \mathcal{L}_\Xi),$$

given by fiberwise application of $\mathcal{S}_w$ and $\mathcal{S}_w'$, is equal to the fiberwise application of $\mathcal{S}_{w'w}$ is that $\mathcal{S}_{w'w}|_{\mathcal{L}_\Theta,\sigma} = \mathcal{S}_w'|_{\mathcal{L}_\Theta,\sigma} \circ \mathcal{S}_w|_{\mathcal{L}_\Theta,\sigma}$ for almost all $\sigma \in X^\text{l}_\Theta^\text{disc}$, and thus for all.

Finally, we prove the continuous preservation of Harish-Chandra Schwartz spaces under the scattering maps, thus completing the proof of Theorem 9.1:

**Proof that $S_w, w \in W_X(\Omega, \Theta)$, restricts to a continuous map: $\mathcal{C}(X^\Theta)^\text{disc} \to \mathcal{C}(X^\Omega)^\text{disc}$.**

By (9.5) the following diagram commutes:

$$\begin{array}{ccc}
L^2(X^\Theta)^\text{disc} & \xrightarrow{S_w} & L^2(X^\Omega)^\text{disc} \\
\downarrow & & \downarrow \\
L^2(X^\text{l}_\Theta^\text{disc}, \mathcal{L}_\Theta) & \xrightarrow{\mathcal{S}_w} & L^2(X^\text{l}_\Omega^\text{disc}, \mathcal{L}_\Omega),
\end{array}$$

where the vertical arrows are the isomorphisms of the Plancherel formula (5.4).
By the regularity statement of Proposition 11.2, the restriction of the bottom arrow to smooth sections gives an isomorphism:

\[ C^\infty(X_\Theta^{-\text{disc}}, L_\Theta) \xrightarrow{\mathcal{S}_w} C^\infty(X_\Omega^{-\text{disc}}, L_\Omega), \]

which by Theorem 5.2 corresponds to an isomorphism between discrete summands of the corresponding Harish-Chandra Schwartz spaces.

\[ \square \]

12. Scattering: the smooth case

We now turn to the smooth case, in order to prove Theorem 9.2 and the remainder of Theorem 9.3. As in the unitary case, the smooth scattering maps \( \mathcal{S}_w \) will be given by integrating the fiberwise scattering maps \( \mathcal{S}_w \), but now over a shift of the unitary set in analogy to Theorem 7.3. However, there is an important result that needs to be proven first: that “cuspidal scatters to cuspidal”. This is the analog of “discrete scatters to discrete” in the unitary case, which was proven in the course of the development of the Plancherel theorem. Similarly, here, “cuspidal scatters to cuspidal” will be proven using a priori knowledge about smooth asymptotics, and more precisely the support theorem 7.5.

We notice that both the statement “discrete scatters to discrete” and “cuspidal scatters to cuspidal” have been proven by Carmona-Delorme [CD14] in the symmetric case. The proofs there heavily use the structure of symmetric varieties. Here we present a different argument which may be possible to generalize to all spherical varieties.

Recall again the asymptotics of normalized constant terms, defined in section 10:

\[ E^*_\Theta,\Omega = E^*_\Theta,\text{disc} \circ e_\Omega : \mathcal{S}(X_\Omega) \to \mathcal{C}(X_\Theta^{-\text{disc}}, L_\Theta). \]

We may project those to the cuspidal quotient (and summand) \( L_\Theta \) of \( \mathcal{L}_\Theta \), in which case we will denote them by:

\[ E^*_\Theta,\text{cusp} : \mathcal{S}(X_\Omega) \to \mathcal{C}(X_\Theta^{-\text{cusp}}, L_\Theta). \]

12.1. Proposition. If \( \Omega \) does not contain a conjugate of \( \Theta \), then \( E^*_\Theta,\text{cusp} \) is zero.

If \( \Omega \sim \Theta \) then \( E^*_\Theta,\text{cusp} \) factors through \( \mathcal{S}(X_\Omega)_{\text{cusp}} \):

\[ E^*_\Theta,\text{cusp} \in \text{Hom}_G \left( \mathcal{S}(X_\Omega), \mathcal{C}(X_\Theta^{-\text{cusp}}, L_\Theta) \right). \]

The summands \( \mathcal{S}_w \) of (11.4) restrict to elements of:

\[ \Gamma \left( X_\Theta^{-\text{cusp}}, \text{Hom}_G(w^*L_\Omega, \mathcal{L}_\Theta) \right) \]
(i.e. preserve the cuspidal summands of the bundles $\mathcal{L}_*$), and the projection of $\mathcal{S}_{\text{Subunit}}$ of (10.15) to $L_{\Theta}$ is zero, hence we have a decomposition:

$$E_{\Theta, \text{cusp}}^* = \sum_{w \in W_X(\Theta, \Omega)} \mathcal{S}_w |_L,$$

where $|_L$ denotes the restriction of $\mathcal{S}_w$ to the subbundle $L_*$.

We will prove this proposition below; let us first see how it implies Theorems 9.2 and 9.3.

First of all, the decomposition 12.1, combined with Theorem 7.3, allow us to express the composition $e_{\Omega}^* e_{\Theta}$, restricted to $S(X_{\Theta})_{\text{cusp}}$ as a sum:

$$e_{\Omega}^* e_{\Theta} |_{S(X_{\Theta})_{\text{cusp}}} = \sum_{w \in W_X(\Omega, \Theta)} \mathcal{S}_w,$$

as claimed in Theorem 9.2, where $\mathcal{S}_w$ is defined as in (9.4). Notice that (9.4) is independent of $\omega$ as long as $\omega \neq 0$; this follows from Corollary 10.15, according to which the $\mathcal{S}_w$ are rational with linear poles (hence no poles for $\omega \neq 0$). Moreover, by Proposition 12.1,

$$\mathcal{S}_w(S(X_{\Theta})_{\text{cusp}}) \subset C^\infty(X_{\Omega})_{\text{cusp}}.$$

Now define, as in Theorem 9.2 the space $\mathcal{S}_w S(X_{\Omega})_{\text{cusp}}$ for $\Omega \sim \Theta$ and $w \in W_X(\Theta, \Omega)$. This includes the case $w = 1$ where $\mathcal{S}_w = 1$, so $S^+(X_{\Theta})_{\text{cusp}} \supset S(X_{\Theta})_{\text{cusp}}$. Let us prove that the scattering maps (and, incidentally, the Bernstein maps) extend to the spaces $S^+(X_{\Theta})_{\text{cusp}}$, and that they satisfy the associativity properties asserted in Theorem 9.2:

**12.2. Lemma.** For $f \in S^+(X_{\Theta})_{\text{cusp}}$ and $w \in W_X(\Omega, \Theta)$, define $\mathcal{S}_w f \in S^+(X_{\Omega})_{\text{cusp}}$ and $e_{\Theta} f \in C^\infty(X)$ as follows: If

$$f = \sum_{(\Theta', w')} \mathcal{S}_{w'} f_{(\Theta', w')}$$

with $w' \in W_X(\Theta, \Theta')$ and $f_{(\Theta', w')} \in S(X_{\Theta'})_{\text{cusp}}$, we set:

$$\mathcal{S}_w f = \sum_{(\Theta', w')} \mathcal{S}_{ww'} f_{(\Theta', w')}.$$

and:

$$e_{\Theta} f = \sum_{(\Theta', w')} e_{\Theta'} f_{(\Theta', w')}.$$

Then the resulting maps are well-defined (do not depend on the decomposition of $f$ chosen), and $\mathcal{S}_w$ is an isomorphism:

$$\mathcal{S}_w : S^+(X_{\Theta})_{\text{cusp}} \rightarrow S^+(X_{\Omega})_{\text{cusp}}.$$
Moreover, $\mathcal{S}_w$ is $w$-equivariant with respect to the actions of $S^\ast(u(X))$ on $C^\infty(X_{\Theta})_{\text{cusp}}$, $C^\infty(X_{\Omega})_{\text{cusp}}$, and the maps $\mathcal{S}_w$ satisfy the associativity properties of Theorem 9.2.

**Proof.** First of all, (9.4) implies that every element of $S^+(X_{\Theta})_{\text{cusp}}$ is of moderate growth, as was the case for elements of $\mathcal{S}_\Theta$ of moderate growth, cf. Proposition 7.6. Hence, every element of $S^+(X_{\Theta})_{\text{cusp}}$ admits a unique spectral decomposition of the form (7.4), with the only difference from (7.4) being that the forms $f^\sigma d\sigma$ are not polynomial, but rational with linear poles, given by (9.4). We point the reader to [SV,§15.4.4] for details on the spectral decomposition of functions of moderate growth.

If $f = \sum_{(\Theta', w')} \mathcal{S}_w' f(\Theta', w')$ as in the statement of the lemma then, using (9.4) and the associativity property (9.6) of the fiberwise scattering maps $\mathcal{S}_w$ we conclude that the operator $\mathcal{S}_w$ described in the lemma also admits the expression (9.4), which proves that it is well-defined.

Similarly for $\mathcal{S}_\Theta$.

Moreover, (9.3) now follows from (9.6), and the fact that $\mathcal{S}_1 = \text{Id}$ shows that these maps are isomorphisms. The extension of the action of the cuspidal center with the given properties is obvious.

The associativity relations of the operators $\mathcal{S}_w$ follow from those of the operators $\mathcal{S}_w$, which were proven in the previous section. □

This completes the proof of Theorem 9.3, and of Theorem 9.2 for the case $\Omega \sim \Theta$.

If $\Omega$ does not contain a conjugate of $\Theta$ then the same calculation and Proposition 12.1 show that the projection of $\mathcal{E}_\Theta^\ast e_{\Omega}$ to $C^\infty(X_{\Theta})_{\text{cusp}}$ is zero or, equivalently, $\mathcal{E}_\Theta^\ast e_{\Theta}$, when restricted to $S(X_{\Theta})_{\text{cusp}}$, is zero.

Finally, if $\Omega$ contains, but is not equal to, a conjugate of $\Theta$ then by switching the roles of $\Theta$ and $\Omega$ in the above argument, since $\Theta$ does not contain a conjugate of $\Omega$ we have:

$$e_{\Theta,\text{cusp}}^\ast e_{\Omega}|_{S(X_{\Omega})_{\text{cusp}}} = 0,$$

which means that $e_{\Omega}^\ast e_{\Theta}$, when restricted to $S(X_{\Theta})_{\text{cusp}}$, has image in $C^\infty(X_{\Omega})_{\text{noncusp}}$.

This completes the proof of Theorem 9.2.

Now let us come to the proof of Proposition 12.1.

**Proof of Proposition 12.1.** The proof is based on the same result as Theorem 7.3, namely Proposition 7.5 on the support of elements of $\mathcal{E}_\Theta^\ast(S(X))$. This proposition implies, in particular, that for every $f \in S(X_{\Theta})_{\text{cusp}}$, the support of $e_{\Omega}^\ast e_{\Theta} f$ has compact closure in an affine embedding $Y$ of $X_{\Omega}$. Moreover, Proposition 7.6 states that this function is of moderate growth.

By Theorem 7.3,

$$e_{\Omega}^\ast e_{\Theta} f(x) = \int_{\omega^{-1}X_{\Theta}^\ast_{\text{cusp}}} E_{\Theta,\sigma,\text{cusp}}^\Omega f^\sigma(x) d\sigma. \quad (12.2)$$

We first claim that if $|\Omega| < |\Theta|$, i.e. $\dim A_{X,\Omega} > \dim A_{X,\Theta}$, then $e_{\Omega}^\ast e_{\Theta} f$ has to be zero. Since we may translate $f$ by the action of $G$, it is enough to fix
an $A_{X,\Omega}$-orbit $Z$ and show that
\[ e_{\Omega}^* e_{\Theta} f|_Z \equiv 0. \]

We identify $Z$ with $A_{X,\Omega}$ by fixing a base point. Let $Y$ be an affine embedding of $X_\Omega$ as above, and let $\psi$ be an algebraic character of $A_{X,\Omega}$ which vanishes on the complement of the open orbit, and this set (monoid) of characters has to generate the character group; in particular, such a character $\psi$ exists. The function $e_{\Omega}^* e_{\Theta} f$ is of moderate growth; since $\bar{Z} \setminus Z$ is a divisor, this is equivalent to saying that there is an open cover $\bar{Z} = \cup_i U_i$ and for every $i$ a function $F_i$ which is regular on $U_i \cap Z$ such that $|e_{\Omega}^* e_{\Theta} f| \leq |F_i|$ on $U_i \cap Z$. Multiplied by a high enough power of $\psi$, $F$ becomes regular on the whole $U$, and therefore for a large enough $n$ we have that $|\psi|^n \cdot e_{\Omega}^* e_{\Theta} f|_Z \in L^2(\bar{Z})$. In particular, if we restrict it to an $A_{X,\Omega}$-orbit, its abelian Fourier/Mellin transform will be in $L^2(\hat{A}_{X,\Omega})$.

On the other hand, let us revisit Proposition 10.1: As $\omega$ varies in $X^{\text{unr}}_{\Theta,\C}$, each factor of (10.6) varies over a set of characters of $A_{X,\Omega}$ of positive codimension in $\hat{A}_{X,\Omega,C}$. More precisely, the support of the $\C[a_{X,\Omega}]$-module generated by $E_{\Theta,\text{cusp}}^*$ is contained in a subscheme $S$ of $\hat{A}_{X,\Omega,C}$ whose reduction is the union:
\[ \bigcup_{\omega \in W_1} \omega(\chi\hat{A}_{X,\Theta,C})|_{A_{X,\Omega}} \subset \hat{A}_{X,\Omega,C}, \]
in the notation of (10.6). (To be sure, our notation here, in particular $\chi$ and $W_1$, concern the restriction of $E_{\Theta,\text{cusp}}^*$ to a single component of $X^{\text{cusp}}_{\Theta}$, and one should consider the union over all components.)

In particular, since the dimension of $S$ is smaller than that of $\hat{A}_{X,\Omega,C}$, there is $P \in \C[\hat{A}_{X,\Omega}]$, nonzero on every connected component, which vanishes on $S$ (i.e. vanishes with the appropriate multiplicity, since $S$ is not necessarily reduced). Explicitly, if for every factor of (10.6) of the form $(z - w(\chi\hat{r})(z))$ we choose an element $z_{\chi,w} \in A_{X,\Omega}$ on which $w(\chi\omega)(z_{\chi,w})$ is equal to some constant $a_{\chi,w}$ for all $\omega \in X^{\text{unr}}_{\Theta}$ (such an element exists for dimension reasons), then the restriction of $P$ to a connected component of $\hat{A}_{X,\Omega}$ can be taken to be the product, over all pairs $(\chi,w)$ such that $w\chi$ belongs to that component, of the terms $(z_{\chi,w} - a_{\chi,w})$ (where $z_{\chi,w}$ is by evaluation a polynomial on this component and $a_{\chi,w}$ we repeat, is a constant complex number).

The restriction of $P$ to any finite number of components is the Mellin transform of a compactly supported smooth measure $h \in \mathcal{M}^c (A_{X,\Omega})$. This measure has, by construction, the property that $h \ast E_{\Theta,\sigma,\text{cusp}}^* = 0$ for all $\sigma \in X^{\text{cusp}}_{\Theta}$, and hence $h \ast e_{\Omega}^* e_{\Theta} f = 0$ for all $f \in S(X_\Theta)_{\text{cusp}}$ by (12.2). Similarly, if we translate the argument of $P$ by $|\psi|^{-n}$ and restrict again to a
finite number of components we get the Mellin transform of a measure 
\( h_n \in \mathcal{M}_\infty^c(A_{X,\Omega}) \) which annihilates \( |\psi|^{n}e_{\Theta}^{*}f|_{Z} \) for all \( f \in \mathcal{S}(X_{\Theta})_{\text{cusp}} \).

But the Mellin transform of the multiplicative convolution \( h_{n} \ast |\psi|^{n}e_{\Theta}^{*}f|_{Z} = \) is a translate of \( P \), restricted to a finite number of components, times the Mellin transform of \( |\psi|^{n}e_{\Theta}^{*}f|_{Z} \), which by the above should be in \( L^2(\tilde{A}_{X,\Omega}) \). Since \( P \) is nonzero on a Zariski dense set of \( \tilde{A}_{X,\Omega} \), it follows that \( e_{\Theta}^{*}f|_{Z} = 0 \).

This proves that \( e_{\Theta}^{*}f = 0 \) when \( |\Omega| < |\Theta| \).

Now assume that \( \Omega \) does not contain a conjugate of \( \Theta \). By induction on \( |\Omega| \), we may assume that \( e_{\Theta}^{*}f \in C^{\infty}(X_{\Omega})_{\text{cusp}} \) for all \( f \in \mathcal{S}(X_{\Theta})_{\text{cusp}} \). That means that \( \mathcal{E}_{0,\text{cusp}}^{*} \) factors through \( \mathcal{S}(X_{\Theta})_{\text{cusp}} \), so by evaluating at points of regularity we get a family \( \mathcal{I} \) of morphisms:

\[
\mathcal{S}(X_{\Omega})_{\text{cusp}} \to \mathcal{L}_{\Theta,e}.
\]

In the language of §10.3, we will examine the weak tangent space of this family which, we recall, has to do with the set of morphisms obtained by second adjunction:

\[
\mathcal{S}(X_{\Omega})_{\text{cusp}} \to (\mathcal{L}_{\Theta,e})_{\Omega}.
\]

By Corollary 10.4 we have that \( \text{WT}_{\Omega}(\mathcal{I}) \), if nonempty, is a union of sets of the form:

\[
[w(\mathfrak{a}_{X_{\Theta},C}^{*})],
\]

where \([ \bullet ]\) denotes image in \( \mathfrak{a}^{*}/W_{L_{\Omega}} \). On the other hand, we can twist the morphisms (12.3) by elements of \( \mathcal{X}^{L}_{\Omega,\text{cusp}} \), thus obtaining a possibly larger family \( \mathcal{J} \) whose weak tangent space will be a union of components of the form:

\[
[w(\mathfrak{a}_{X_{\Theta},C}^{*})] + \mathfrak{a}_{X_{\Theta},C}^{*}.
\]

By Lemma 10.7, the dimension of this is strictly larger than the dimension of \( \mathfrak{a}_{X_{\Theta},C}^{*} \). This is a contradiction: we cannot have a family of finite-length quotients of \( \mathcal{S}(X_{\Omega})_{\text{cusp}} \) whose weak tangent space has dimension larger than that of \( A_{X,\Omega} \). This shows that \( \mathcal{E}_{\Theta,\text{cusp}}^{*} \) is zero.

Finally, consider the case \( \Omega \sim \Theta \). First of all:

12.3. \textbf{Lemma.} Let \( Y \subset X_{\Omega}^{\text{cusp}} \) be a connected component, and \( E \in \mathcal{M}_{Y}^{\text{cusp}} := \text{Hom}_{G}(\mathcal{S}(X_{\Theta}),\mathbb{C}(Y,\mathcal{L}_{\Theta})) \). Let \( K_{Y} = \mathbb{C}(Y) \), and let \( E = \sum_{i}E_{i} \) be the decomposition of \( E \) into elements of (distinct) generalized eigenspaces for the action of \( A_{X,\Omega} \) on the finite-dimensional \( K_{Y} \)-vector space \( \mathcal{M}_{Y}^{\text{cusp}} \). If \( E \) factors through \( \mathcal{S}(X_{\Omega})_{\text{cusp}} \), so does each of the \( E_{i} \)s.

The validity of the lemma is obvious, since \( \text{Hom}_{G}(\mathcal{S}(X_{\Theta})_{\text{cusp}},\mathbb{C}(Y,\mathcal{L}_{\Theta})) \) is an \( A_{X,\Omega} \)-stable subspace of \( \mathcal{M}_{Y}^{\text{cusp}} \). Thus, the projections to \( \mathcal{L}_{\Theta} \) of all summands \( \mathcal{J}_{w} \) of (10.15) all factor through \( \mathcal{S}(X_{\Theta})_{\text{cusp}} \), hence through \( \mathcal{L}_{\Theta} \). The unitarity of the operators \( S_{w} \) now shows that the image of sections of \( \mathcal{L}_{\Theta} \) under \( \mathcal{J}_{w} \) lies in sections of \( \mathcal{L}_{\Theta} \).
Finally, we claim that the projection of $S_{\text{Subunit}}$ to $L_\Theta$ is zero. The argument here is identical to the one above using the weak tangent space of the resulting family of maps:

$$S(X^I_{\Omega})_{\text{cusp}} \rightarrow (L_{\Theta,\sigma})_{\Omega}$$

and Lemma 10.7.

\[\square\]

Remark. Regarding the last step of the proof: in the discrete case, there is no contradiction to the existence of subunitary exponents. The reason is that the “subunitary parts” of Eisenstein integrals do not need to be “discrete modulo center” (while the cuspidal parts were necessarily cuspidal modulo center). Indeed, they could be non-discrete, but with a central character that makes them decay “towards infinity”.

Part 5. Paley-Wiener theorems

13. The Harish-Chandra Schwartz space

We start by proving the following two results:

13.1. Proposition. $\iota_{\Theta}$ takes $C(X_{\Theta})_{\text{disc}}$ continuously into $C(X)$.

And in the other direction:

13.2. Proposition. $\iota^*_\Theta$ takes $C(X)$ continuously into $C(X_{\Theta})_{\text{disc}}$.

We first reduce both statements to the case $Z(X) = 1$. This is achieved by using (2.11) and (2.12), which identify $C(X)$ and $L^2(X)$ as closed subspaces of a direct sum of spaces of the form:

$$C(Z(X) \times Y) \cong C(Z(X)) \hat{\otimes} C(Y),$$

respectively:

$$L^2(Z(X) \times Y) \cong L^2(Z(X)) \hat{\otimes} L^2(Y),$$

where $Y$ is a spherical $[G, G]$-variety with $Z(Y) = 1$.

It is obvious from the definitions that the Bernstein maps on those spaces are induced by Bernstein maps for the second factor:

$$\iota_{\Theta} : L^2(Y_{\Theta}) \rightarrow L^2(Y),$$

which reduces both problems to the case $Z(X) = 1$. We will assume this for the two proofs.

The proof of Proposition 13.1 will require a lemma: Fix an open compact subgroup $J \subset G$ and a collection $(\tilde{N}_\Theta)_{\Theta}$ of $J$-good neighborhoods of infinity. We may, and will, assume that this collection is determined by the neighborhoods $\tilde{N}_{\alpha}$, where $\alpha$ runs over all simple spherical roots and $\tilde{\alpha} := \Delta_X \setminus \{\alpha\}$, in the following sense:

$$\tilde{N}_\Theta = \bigcap_{\alpha \neq \Theta} \tilde{N}_{\tilde{\alpha}}. \quad (13.1)$$
We will also be denoting:

\[ N_\Theta := \bar{N}_\Theta \setminus \bigcup_{\Omega \in \Theta} \bar{N}_\Omega, \]

remembering that the image of \( N_\Theta \) in \( X_\Theta/A_{X,\Theta} \) is compact.

By a decaying function on \( X \) we will mean a positive, smooth function whose restriction to each \( \bar{N}_\Theta \) is \( A_{X,\Theta} \)-finite function with subunitary exponents. Notice that, by our definition, a subunitary exponent on \( \bar{A}_{X,\Theta}^+ \) is allowed to be unitary on a “wall” of \( A_{X,\Theta}^+ \) (it only has to be \( \approx 1 \) on \( \bar{A}_{X,\Theta}^- \)). However, by demanding that the exponents of our function are \( A_{X,\Theta}^+ \)-subunitary on every \( \bar{N}_\Theta \), this possibility is ruled out, and together with our assumption that \( Z(X) = 1 \) a decaying function is automatically in \( C^p_{\bar{N}_\Theta} \).

This definition is compatible with the way the notion of “decaying function” was used in [SV] (and will be used in the proof below): Indeed, for each \( \Theta \subset \Delta_X \), a decaying function on \( A_{X,\Theta}^+ \) was defined in [SV] to be the restriction of a positive, \( A_{X,\Theta} \)-finite function with \( A_{X,\Theta}^+ \)-subunitary exponents. Hence, a decaying (\( J \)-invariant) function \( f \) in our present sense has the property that for every \( x \in \bar{N}_\Theta \) the function:

\[ A_{X,\Theta}^+ \ni a \mapsto | \langle f, a^{-1} \cdot 1_{xJ} \rangle |, \tag{13.2} \]

where \( 1_{xJ} \) denotes the characteristic function of \( xJ \), is bounded by a decaying function on \( A_{X,\Theta}^+ \). (Notice that this is stronger than saying that the naive restriction of the function to the \( A_{X,\Theta}^+ \)-orbit is a decaying function; our definition of the action of \( A_{X,\Theta}^+ \) is normalized by the square root of the volume, s. §2.4, so the above bound is equivalent to a bound of \( f(ax) \) by \( \text{Vol}(axJ)^{-\frac{1}{2}} \) times a decaying function on \( A_{X,\Theta}^+ \).

Let \( \tilde{\tau}_\Theta \) denote, for each \( \Theta \), the map of restriction to \( \bar{N}_\Theta \).

13.3. Lemma. When \( Z(X) = 1 \), for any \( \Phi \in L^2(X)^J \) the alternating sum:

\[ \text{Alt}(\Phi) := \sum_{\Theta \subset \Delta_X} (-1)^{|\Theta|} \tilde{\tau}_\Theta \ast \Phi \]

\[ \tag{13.3} \]

is bounded in absolute value by \( \| \Phi \|_{L^2(X)} \) times a decaying function which depends only on \( J \).

Proof. We claim that for every \( \alpha \in \Delta_X \), and every \( x \in N_\alpha \), the restriction of (13.3) to \( A_{X,\alpha}^+ \cdot x \) satisfies the bound:

\[ | \text{Alt}(\Phi)(a \cdot x) | \leq \| \Phi \|_{L^2(X)} \text{Vol}(axJ)^{-\frac{1}{2}} R_\alpha(a), \tag{13.4} \]

where \( R_\alpha \) is a decaying function on \( \bar{A}_{X,\alpha}^+ \) that only depends on \( J \).
This will prove the Lemma: Indeed, for an arbitrary $\Theta$, $x \in N_\Theta$ and $a \in A_{X,H}^+$ we get a bound:

$$|\text{Alt}(\Phi)(a \cdot x)| \leq \|\Phi\|_{L^2(X)} \text{Vol}(axJ)^{\frac{-1}{2}} R_\Theta(a),$$

where $R_\Theta$ is the decaying function on $A_{X,H}^+$ defined as:

$$R_\Theta(a) = \min_{a' \in A_{X,H}} \left( \min_{a'' \in A_{X,H}} R_\alpha(b) \right).$$

To prove (13.4), we notice that $\text{Alt}(\Phi)$ is equal to the sum over all $\Theta$ containing $\alpha$ of the terms:

$$(-1)^{|\Theta|}(\tau_\Theta \iota_{\Theta^*}^{\alpha} \Phi - \tau_{\Theta \setminus \{\alpha\}} \iota_{\Theta \setminus \{\alpha\}}^{\ast} \Phi).$$

By the transitivity property of Bernstein maps:

$$\iota_{\Theta}^{\alpha} \Phi = \iota_{\Theta'}^{\alpha'} \circ \iota_{\Theta}^{\alpha} \Phi$$

for $\Theta \subset \Theta'$, where $\iota_{\Theta'}^{\alpha'}$ is the corresponding adjoint Bernstein map for the variety $X_{\Theta'}$, and the fact that the norm of $\iota_{\Theta}^{\alpha} \Phi$ for some $\Theta' \subset \Delta_X$ with $\alpha \in \Theta'$, is bounded by a fixed multiple of $\|\Phi\|_{L^2(X)}$, it is enough to prove the statement when (13.3) is replaced by $\Phi - \iota_{\alpha}^{\ast} \Phi$ (the rest of the terms being similar, with $\Phi$ replaced by , whose norm is bounded by a constant times $\|\Phi\|_{L^2(X)}$).

Thus, we need to prove that for every $x \in N_{\hat{\alpha}}$ the restriction of $\Phi - \iota_{\alpha}^{\ast} \Phi$ to $A_{X,\hat{\alpha}}^+ \cdot x$ is bounded by $\|\Phi\|_{L^2(X)} \text{Vol}(axJ)^{\frac{-1}{2}} \cdot R_\alpha$, where $R_\alpha$ is a decaying function on $A_{X,\hat{\alpha}}^+$ that only depends on $J$.

This follows from [SV, Lemma 11.5.1] (and its proof): Indeed, if $\Psi = 1_{xJ}$, the characteristic function of some $J$-orbit on $N_{\hat{\alpha}}$, and $a \in A_{X,\hat{\alpha}}^+$, then we have:

$$|\langle \Phi - \iota_{\alpha}^{\ast} \Phi, a^{-1} \cdot \Psi \rangle| \leq \|\Phi\|_{L^2(X)} \cdot \|e_{\hat{\alpha}} - \iota_{\alpha}^{-1} a\| \cdot \Psi\|_{L^2(X)} \leq \|\Phi\|_{L^2(X)} C_\Psi Q_\gamma(a)$$

in the notation of loc.cit. so we can set $R_\alpha(a) = C_\Psi Q_\gamma(a)$, where $Q_\gamma(a)$ is a decaying function on $A_{X,\hat{\alpha}}^+$ which is independent of $\Phi, \Psi$. The proof of [SV, Lemma 11.5.1] shows, actually, that we can choose $C_\Psi$ to be proportional to the $L^2$ norm of $\Psi$ – and of the norm of $e_{\Theta}$ applied to a finite number of its translates, which in the case that $\Psi$ is the characteristic function of a $J$-orbit on $N_{\hat{\alpha}}$ will just be equal to the $L^2$ norm of $\Psi$. Therefore, for every $x \in N_{\hat{\alpha}}$ and $a \in A_{X,\hat{\alpha}}^+$ we get:

$$|\langle \Phi - \iota_{\alpha}^{\ast} \Phi, a^{-1} \cdot 1_{xJ} \rangle| \leq \|\Phi\|_{L^2(X)} \cdot \text{Vol}(xJ)^{\frac{1}{2}} Q_\gamma(a).$$
where the implicit constant only depends on $J$. Taking into account that the measure on $N_\Theta$ is an $A_{X,\Theta}$-eigenmeasure, say with character $\delta_\Theta$, we have by definition:

$$\langle \Phi - \iota_\alpha^* \Phi, a^{-1} \cdot 1_{xJ} \rangle = \left\langle \Phi - \iota_\alpha^* \Phi, \delta_\Theta^{-\frac{1}{2}}(a) 1_{axJ} \right\rangle =$$

$$= (\Phi - \iota_\alpha^* \Phi)(ax) \delta_\Theta^{-\frac{1}{2}}(a) \operatorname{Vol}(axJ) = (\Phi - \iota_\alpha^* \Phi)(ax) \delta_\Theta^\frac{1}{2}(a) \operatorname{Vol}(xJ),$$

and hence the above inequality becomes:

$$|\Phi - \iota_\alpha^* \Phi|(ax) \ll \|\Phi\|_{L^2(X)} \cdot \operatorname{Vol}(a \cdot xJ)^{-\frac{1}{2}} Q^J(a).$$

This proves the lemma.

\[\square\]

**Proof of Proposition 13.1.** Fix a $\Theta$, as in a statement of the proposition. We will prove the proposition inductively on $|\Delta_X| - |\Theta|$, the base case $\Theta = \Delta_X$ being trivial. Assume that it has been proven for all orders of $|\Delta_X| - |\Theta|$ smaller than the given one.

We have already reduced the proposition to the case $Z(X) = 1$, which we will henceforth assume. (Notice, however, that by the inductive assumption we are free to assume it for any smaller value of $|\Delta_X| - |\Theta|$ without this assumption.) We fix an open compact subgroup $J$ and $J$-good neighborhoods $\tilde{N}_\Theta$ of $\Omega$-infinity as in the setup of Lemma 13.3, and use the notation $N_\Theta$ as before.

By Lemma 13.3, it is enough to prove:

For all $\Omega \subseteq \Delta_X$, the composition of $\iota_{\Omega\Theta}^* \iota_\Theta : L^2(X_\Theta) \to L^2(X_\Omega)$ with restriction to $N_\Omega$ takes $\mathcal{C}(X_\Theta)_{\text{disc}}$ continuously into $\mathcal{C}(N_\Omega)$.

Indeed, if the above was saying “continuously into $\mathcal{C}(\tilde{N}_\Theta)$” then by Lemma 13.3 the difference of $\iota_\Theta \Phi$ from an alternating sum of terms of the form $\tilde{\iota}_\Omega \iota_{\Omega\Theta}^* \iota_\Theta \Phi$ is bounded by a decaying function times $\|\iota_\Theta \Phi\| \ll \|\Phi\|$, and we would be done. But this argument can be used, inductively on $|\Omega|$ to extend the above claim from $N_\Omega$ to $\tilde{N}_\Theta$, so indeed the claim above will suffice to prove the proposition.

The claim is trivially true when $\Omega = \Delta_X$, since we are assuming that $Z(X) = 1$, so $N_{\Delta_X}$ is compact. We now want to reduce any other case to the case $\Omega = \Theta$ and the map $\iota_{\Omega}^* : L^2(X_\Theta) \to L^2(X_\Omega)$, where is it true by the inductive assumption.

First of all, if $\Omega$ does not contain a conjugate of $\Theta$ then $\iota_{\Omega\Theta}^* \iota_\Theta = 0$ and there is nothing to prove. Thus, let $\Omega_1 \subset \Omega$ with $\Omega_1 \sim \Theta$ and denote by $\iota_{\Omega_1}^* : L^2(X_{\Omega_1}) \to L^2(X_{\Omega})$ the pertinent Bernstein map for the variety $X_{\Omega}$. Then there is a nonzero integer $c(\Omega_1, \Omega)$ such that:

$$\iota_{\Omega_1}^* \circ \iota_\Theta = c(\Omega_1, \Omega)^{-1} \iota_{\Omega_1}^* \circ \iota_{\Omega_1}^* \circ \iota_\Theta. \quad (13.5)$$
Indeed, we first write $i_{\Omega_1}^* = i_{\Omega_1}^{*\Theta} \circ i_{\Theta}^*$ by the transitivity property of Bernstein maps and then we observe that the map:

$$c(\Omega_1, \Omega)^{-1} i_{\Omega_1}^{*\Theta} \circ i_{\Theta}^{*\Omega} : L^2(X_{\Omega}) \rightarrow L^2(X_{\Theta})$$

is the identity on the part of the spectrum where $i_{\Omega_1}^{*\Theta}$ lives (namely, the direct summand $L^2(X_{\Omega_1})_{\Omega_1}$ equal to the image of $L^2(X_{\Omega_1})_{\text{disc}}$ under $i_{\Omega_1}^{*\Theta}$). Here $c(\Omega_1, \Omega)$ is the number of “chambers” on the “wall” corresponding to $\Omega_1$ in the root system of $\Omega$, i.e. the analog of $c(\bullet)$ of (1.2) when we work with $X_{\Omega}$ instead of $X$.

The map $i_{\Omega_1}^{*\Theta} \circ i_{\Theta}$ is a continuous map (actually, isomorphism):

$$\mathcal{C}(X_{\Theta})_{\text{disc}} \rightarrow \mathcal{C}(X_{\Omega_1})_{\text{disc}}$$

by Theorem 9.1, so by replacing $\Theta$ by $\Omega_1$ we have reduced the statement to the case $\Theta \subset \Omega$, and have replaced $i_{\Theta}$ by $i_{\Omega_1}^{\Theta}$. This completes the proof of the proposition. □

Now we come to proving the other direction. We keep assuming that $\mathcal{Z}(X) = 1$, having reduced the problem to this case.

Proof of Proposition 13.2. Let $f \in \mathcal{C}(X)^J$, and let $X = \sqcup \Theta N_{\Theta}$ be a decomposition as above; then $f_{\Theta} := f|_{N_{\Theta}} \in \mathcal{C}(N_{\Theta}) \subset \mathcal{C}(X_{\Theta})$. By Theorem 5.2, we need to show that the image of $i_{\Theta,\text{disc}}^* f$ in $L^2(X_{\Theta}^L_{\Theta}, \mathcal{L}_{\Theta})$ actually lies in $C^\infty(X_{\Theta}^L_{\Theta}, \mathcal{L}_{\Theta})$ (smooth sections), and that the resulting map:

$$\mathcal{C}(X)^J \rightarrow C^\infty(X_{\Theta}^L_{\Theta}, \mathcal{L}_{\Theta})$$

is continuous.

First of all, for each $\Theta$ and $\Omega$ consider the composition of maps:

$$S(X_{\Omega}) \xrightarrow{E_{\Theta,\text{disc}}} S(X) \xrightarrow{i_{\Theta,\text{disc}}^*} L^2(X_{\Theta})_{\text{disc}} \rightarrow L^2(X_{\Theta}^L_{\Theta}, \mathcal{L}_{\Theta}). \quad (13.6)$$

By (7.4), this composition is given by the restriction of the maps $E_{\Theta,\text{disc}}^*$ of (10.1) ($\Theta$-asymptotics of normalized constant terms). Recall that by Proposition 8.2, the image of $E_{\Theta,\text{disc}}^*$ lies in $\Gamma(X_{\Theta}^L_{\Theta}, \mathcal{L}_{\Theta})$ (i.e. rational sections whose poles do not meet the unitary set).

Our goal is show that the maps $E_{\Theta,\text{disc}}^*$ extend continuously to the bottom horizontal row of the following diagram, where the vertical arrows are the natural inclusions:

$$\mathcal{C}(N_{\Theta}) \xrightarrow{\rightarrow} \Gamma(X_{\Theta}^L_{\Theta}, \mathcal{L}_{\Theta})$$

This will prove the proposition, once we know it for all $\Omega$. 
Fix a connected component $Y$ of $X_L^{\text{disc}}$, and recall that $\Gamma(Y, L_\Theta)$ is actually a $D(Y)$-module (module for the ring of differential operators on $Y$). Fix any $D \in D(Y)$ and apply it to the operator $E^{*, \Omega}_{\Theta, \text{disc}}$. As we have seen in Lemma 10.2, the resulting element:

$$DE^{*, \Omega}_{\Theta, \text{disc}} \in \text{Hom} \left( S(X_\Omega), \Gamma(Y, L_\Theta) \right)$$

(not $G$-equivariant) has the same exponents, possibly with higher multiplicity, as $E_{\Theta, \text{disc}}^{*, \Omega}$, and by Proposition 11.1, these are either unitary or subunitary with respect to $A^+_{X, \Theta}$.

We will use the following lemma of linear algebra:

13.4. **Lemma.** Suppose $S$ is a finitely generated abelian group together with a finitely generated submonoid $S^+ \subset S$ that generates $S$. If $S$ has a locally finite action on a finite-dimensional complex vector space $V$, with uniformly bounded degree and generalized eigencharacters which are unitary or subunitary with respect to $S^+$, and if $\| \cdot \|$ is any norm on $V$, there exist a tempered function $T$ on $S$ and a finite subset $S_0 \subset S$, depending only on the rank of $S$ and the maximal degree, with the property that:

$$\| s \cdot v \| \leq T(s) \max_{s' \in S_0} \| s' \cdot v \|$$

for all $s \in S^+$.

Compare this with [SV, Lemma 10.2.5]. The proof easily reduces to the case $S = \mathbb{Z}, S^+ = \mathbb{N}$, in which case it follows by induction on the degree, i.e. writing the minimal polynomial of the generator $\lambda' \in \mathbb{N}$ on an element $v \in V$ as $P(x) = (x - \zeta)Q(x)$ and applying an induction hypothesis on the vector $(\lambda' - \zeta)v$.

We fix a Haar measure $d\sigma$ on $X_L^{\text{disc}}$, which determines norms $\| \cdot \|_\sigma$ on the fibers of $L_\Theta$ over the unitary set, cf. (4.11). By regularity, for any $F \in S(X_\Omega)$ the numbers

$$\left\| DE^{*, \Omega}_{\Theta, \text{disc}, \sigma}(F) \right\|_{\sigma}$$

are uniformly bounded in $\sigma$. If we now fix a set of $J$-orbits on $N_\Omega$ whose $A^+_{X, \Omega}$-translates cover $N_\Omega$, and denote by $F_i$ their characteristic functions, there is, by the above lemma, a finite set $S_0$ of elements of $A_{X, \Omega}$ and a tempered function $T$ on $A_{X, \Omega}$ such that:

$$\left\| DE^{*, \Omega}_{\Theta, \text{disc}, \sigma}(aF_i) \right\|_{\sigma} \leq T(a) \max_{s \in S_0} \left\| DE^{*, \Omega}_{\Theta, \text{disc}, \sigma}(sF_i) \right\|_{\sigma}$$

(13.7)

for all $a \in A^+_{X, \Theta}$.

For an arbitrary element $\Phi \in L^2(N_\Omega)^J$, writing it as a series in $A^+_{X, \Omega}$-translates of the $F_i$: $\Phi = \sum_{i,j} c_{ij} a_j \cdot F_i$,
its image in $L^2(\mathbf{X}^L_{\Theta}, \mathcal{L}_\Theta)$ is given by the corresponding series:

$$\sum_{i,j} c_{ij} E_{\Theta, \text{disc}}^*(a_j \cdot F_i).$$

If, in particular, $\Phi \in \mathcal{C}(N_\Theta)$, by (13.7) we deduce that the corresponding series for $DE_{\Theta, \text{disc}}(\Phi)$ converges in $L^2(\mathbf{X}^L_{\Theta}, \mathcal{L}_\Theta)$, and is bounded by continuous seminorms on $\mathcal{C}(N_\Theta)$.

Since the seminorms:

$$f \mapsto \|Df\|_{L^2(\mathbf{X}^L_{\Theta}, \mathcal{L}_\Theta)}, \quad D \in D(Y),$$

form a complete system of seminorms for $C^\infty(\mathbf{X}^L_{\Theta}, \mathcal{L}_\Theta)$, we deduce that the maps $E_{\Theta, \text{disc}}^*$, restricted to $\mathcal{S}(N_\Theta)^J$, extend continuously to:

$$\mathcal{C}(N_\Theta)^J \to C^\infty(\mathbf{X}^L_{\Theta}, \mathcal{L}_\Theta).$$

We are now ready to prove our main result:

13.5. **Theorem.** For each $\Theta$, orthogonal projection to $L^2(\mathbf{X}_\Theta)_{\text{disc}}$ gives a topological direct sum decomposition:

$$\mathcal{C}(X_\Theta) = \mathcal{C}(X_{\Theta, \text{disc}}) \oplus \mathcal{C}(X_{\Theta, \text{cont}}).$$

For each $w \in W_X(\Omega, \Theta)$ the scattering map $S_w$ restricts to a topological isomorphism:

$$\mathcal{C}(X_{\Theta, \text{disc}}) \xrightarrow{\sim} \mathcal{C}(X_{\Omega, \text{disc}}).$$

The map $i^*$ of (1.2) restricts to a topological isomorphism:

$$\mathcal{C}(X) \xrightarrow{\sim} \left( \bigoplus_{\Theta \in \Delta_X} \mathcal{C}(X_{\Theta, \text{disc}}) \right)^{\text{inv}}. \quad (13.8)$$

**Proof.** By Theorem 9.2, the space $\left( \bigoplus_{\Theta \in \Delta_X} \mathcal{C}(X_{\Theta, \text{disc}}) \right)^{\text{inv}}$ makes sense, and by Proposition 13.2 and Theorem 1.2 the space $\mathcal{C}(X)$ maps continuously into it. Finally, since $\sum_\Theta (i^*_\Theta \circ i_\Theta)$ is a multiple of the identity on $\left( \bigoplus_{\Theta \in \Delta_X} L^2(\mathbf{X}_{\Theta, \text{disc}}) \right)^{\text{inv}}$, it follows from Proposition 13.1 that the map from $\mathcal{C}(X)$ to $\left( \bigoplus_{\Theta \in \Delta_X} \mathcal{C}(X_{\Theta, \text{disc}}) \right)^{\text{inv}}$ is onto. \[ \Box \]

The combination of Theorems 7.4 and 13.5 gives Theorem 1.4, which we repeat for convenience of the reader:
13.6. **Theorem.** The normalized discrete constant terms $E_{\Theta, \text{disc}}$ composed with the natural isomorphisms (5.2) give an isomorphism:

$$
\mathcal{C}(X) \overset{\sim}{\rightarrow} \left( \bigoplus_{\Theta} C^\infty(X^L_{\Theta}, \mathcal{L}_{\Theta}) \right)^{\text{inv}},
$$

where $\text{inv}$ here denotes $S_w$-invariants.

In particular, the existence of a ring $\mathfrak{z}_{\text{temp}}(X)$ of multipliers on $\mathcal{C}(X)$, as described in Corollary 1.5, immediately follows from either of the above two versions of the Paley-Wiener theorem for the Harish-Chandra Schwartz space:

13.7. **Corollary.** Let

$$
\mathfrak{z}_{\text{temp}}(X) = \left( \bigoplus_{\Theta} \mathfrak{z}^\text{disc}(X^L_{\Theta}) \right)^{\text{inv}},
$$

where the exponent $\text{inv}$ denotes invariants of all the isomorphisms induced by triples $(\Theta, \Omega, w \in W_X(\Omega, \Theta))$.

There is a canonical action of $\mathfrak{z}_{\text{temp}}(X)$ by continuous $G$-endomorphisms on $\mathcal{C}(X)$, characterized by the property that for every $\Theta$, considering the map:

$$
\iota_{\Theta, \text{disc}}^* : \mathcal{C}(X) \rightarrow \mathcal{C}(X^L_{\Theta})
$$

we have:

$$
\iota_{\Theta, \text{disc}}^*(z \cdot f) = z_{\Theta}(\iota_{\Theta, \text{disc}}^* f)
$$

for all $z \in \mathfrak{z}_{\text{temp}}(X)$, where $z_{\Theta}$ denotes the $\Theta$-coordinate of $z$.

**Proof.** Indeed, $\mathfrak{z}_{\text{temp}}(X)$ acts by continuous $G$-automorphisms on the right hand side of (13.8) or (13.9), and the action is characterized by the stated property. \(\square\)

We complete this section by formulating an extension of the properties of Bernstein and scattering maps from the discrete components of Harish-Chandra Schwartz spaces to the whole space. The proof is straightforward given Theorem 13.5 and the associativity/composition properties of these maps, and is left to the reader:

13.8. **Theorem.** For every triple $(\Theta, \Omega, w \in W_X(\Omega, \Theta))$ the scattering map:

$$
S_w : L^2(X_{\Theta}) \rightarrow L^2(X_{\Omega})
$$

restricts to a topological isomorphism:

$$
\mathcal{C}(X_{\Theta}) \rightarrow \mathcal{C}(X_{\Omega})
$$

which is $\mathfrak{z}_{\text{temp}}(X^L_{\Theta})$-equivariant with respect to the obvious isomorphism:

$$
\mathfrak{z}_{\text{temp}}(X^L_{\Theta}) \overset{\sim}{\rightarrow} \mathfrak{z}_{\text{temp}}(X^L_{\Omega})
$$

induced by $w$.

The Bernstein maps $\iota_{\Theta}$ and their adjoints $\iota_{\Theta}^*$ map $\mathcal{C}(X_{\Theta})$ continuously into $\mathcal{C}(X)$ and vice versa.
14. The Schwartz space

We now come to the Paley-Wiener theorem for the Schwartz space of compactly supported, smooth functions on $X$. Besides the properties of the scattering operators $S_w$ of §9, we will use the following basic result:

14.1. Proposition. Let $[\Theta]$ run over all associate classes of subsets of $\Delta_X$, and for each such class let $S(X)_{[\Theta]}$ denote the space generated by all $e_\Omega S(X_\Omega)_{\text{cusp}}$, $\Omega \in [\Theta]$. Then:

$$S(X) = \bigoplus_{[\Theta]} S(X)_{[\Theta]}.$$  \hspace{1cm} (14.1)

Proof. The sum is direct by Theorem 9.2. We need to show that the map:

$$\sum_{\Theta} e_\Theta : \bigoplus_{[\Theta]} S(X)_{\text{cusp}} \to S(X)$$

is surjective.

We will use induction on the size of $\Delta_X$, the case $\Delta_X = \emptyset$ being tautologically satisfied (because then $S(X) = S(X)_{\text{cusp}}$). Assume that the proposition has been proven when $X$ is replaced by $X_\Omega$, for all $\Omega \subseteq \Delta_X$. Let us denote by $e_\Omega^\Theta : S(X_\Theta) \to S(X_\Omega)$ ($\Theta \subset \Omega$) the corresponding maps for the variety $X_\Omega$. Recall the transitivity property:

$$e_\Theta \circ e_\Omega^\Theta = e_\Theta.$$

Therefore,

$$\sum_{\Theta} e_\Theta (S(X)_{\text{cusp}}) = S(X)_{\text{cusp}} + \sum_{\Theta \neq \Delta_X} e_\Theta (S(X_\Theta)).$$

Assume that $S(X) \neq \bigoplus_{[\Theta]} S(X)_{[\Theta]}$, then there would be a nonzero subspace $V$ of the smooth dual (i.e. $C^\infty(X)$) which would vanish on all the spaces on the right hand side of the last equation. In particular, $e_\emptyset^\emptyset V = 0$ for all $\emptyset \neq \Delta_X$, hence the elements of $V$ are compactly supported modulo the center of $X$. But then they cannot be orthogonal to the cuspidal part $S(X)_{\text{cusp}} = S(X|_{\Delta_X})$.

\hfill $\square$

Remark. This proposition seems to be false in the non-factorizable case. For example, if $X = \text{PGL}_2$ under the $G = \mathbb{G}_m \times \text{PGL}_2$ action (with $\mathbb{G}_m$ acting as a split subtorus by multiplication on the left, and $\text{PGL}_2$ acting by multiplication on the right) then it is known that the tensor product of the trivial character of $F^\times$ by the Steinberg representation of $\text{PGL}_2$ is relatively cuspidal on $X$, while this is not the case for non-trivial characters of $F^\times$. What this means is that the subspace of $C^\infty(X)$ corresponding to the image of the Steinberg representation will be orthogonal to $e_\emptyset(S(X_\emptyset))$, but also orthogonal to $S(X)_{\text{cusp}}$. 

Recall that for each $\Theta$ we have defined $S^+(X_\Theta)_{\text{cusp}}$ as the subspace of $C^\infty(X_\Theta)$ generated by all spaces of the form:

$$S_wS(X_\Omega)_{\text{cusp}}$$

where $\Omega$ is an associate of $\Theta$ and $w \in W_X(\Omega, \Theta)$, and in Theorem 9.2 (and Lemma 12.2) we extended the scattering operators $S_w$ to isomorphisms between these spaces.

We are now ready to prove the Paley-Wiener theorem, reminding first that the exponent $^{\text{inv}}$ in:

$$(\bigoplus_{\Theta} S^+(X_\Theta)_{\text{cusp}})^{\text{inv}}$$

denotes invariants of these maps. Notice that, as follows easily from the definitions, any element of \eqref{14.2} can be obtained by averaging elements of the spaces $S(X_\Theta)_{\text{cusp}}$ via the operators $S_w$, i.e.:

14.2. Lemma. For any element $f = (f_\Theta)_{\Theta}$ of \eqref{14.2} there is a (non-unique) element $(f'_\Theta)_{\Theta} \in \bigoplus_{\Theta} S(X_\Theta)_{\text{cusp}}$

such that:

$$f_\Theta = \sum_{\Omega; w \in W_X(\Theta, \Omega)} S_wf'_\Omega.$$  

14.3. Theorem. The sum of the morphisms $e^*_\Theta S_{\Theta, \text{cusp}}$ defines an isomorphism:

$$S(X) \cong (\bigoplus_{\Theta} S^+(X_\Theta)_{\text{cusp}})^{\text{inv}}.$$  

Proof. It is an immediate corollary of Proposition 14.1 and Theorem 9.2 that the image of $\bigoplus_{\Theta} e^*_\Theta S_{\Theta, \text{cusp}}$ lies in $(\bigoplus_{\Theta} S^+(X_\Theta)_{\text{cusp}})^{\text{inv}}$.

Lemma 14.2 shows that the map is surjective, and injectivity follows from Proposition 7.1.

It is easy from this to deduce the fiberwise version in terms of normalized constant terms. First of all, for every $\Theta \subset \Delta_X$ let:

$$\mathbb{C}^+ \left[ \hat{X}_\Theta^{\text{cusp}}, \mathcal{L}_\Theta \right] \subset \mathbb{C} \left( \hat{X}_\Theta^{\text{cusp}}, \mathcal{L}_\Theta \right)$$

be the subspace generated by the images of all fiberwise scattering maps $\mathcal{I}_w$, for $\Omega$ and associate of $\Theta$ and $w \in W_X(\Theta, \Omega)$. Notice that by the regularity of scattering maps on the unitary spectrum (Theorem 9.3), we might as well have written $\hat{\Gamma}(\cdot)$ instead of $\mathbb{C}(\cdot)$. Then it is clear that such an $\mathcal{I}_w$ induces an isomorphism:

$$\mathbb{C}^+ \left[ \hat{X}_\Theta^{\text{cusp}}, \mathcal{L}_\Theta \right] \cong \mathbb{C}^+ \left[ \hat{X}_\Omega^{\text{cusp}}, \mathcal{L}_\Omega \right],$$

and the combination of Theorems 7.3 and 14.3 gives Theorem 1.9, which we repeat for convenience of the reader:
14.4. Theorem. The normalized cuspidal constant terms $E^{\ast}_{\Theta, \text{cusp}}$ composed with the natural isomorphisms (6.1) give an isomorphism:

$$S(X) \sim \left( \bigoplus_{\Theta} \mathbb{C}^+ \left[ X^{L}_{\Theta} \right], \mathcal{L}_{\Theta} \right)^{\text{inv}},$$

where $\text{inv}$ here denotes $\mathcal{S}_w$-invariants.

In particular, the existence of a ring $\mathfrak{s}_\text{sm}(X)$ of multipliers on $S(X)$, as described in Corollary 1.10, immediately follows from either of the above two versions of the Paley-Wiener theorem for the Schwartz space:

14.5. Corollary. Let

$$\mathfrak{s}_\text{sm}(X) = \left( \bigoplus_{\Theta} \mathfrak{s}_\text{cusp}(X^{L}_{\Theta}) \right)^{\text{inv}},$$

where the exponent $\text{inv}$ denotes invariants of all the isomorphisms induced by triples $(\Theta, \Omega, w \in W_X(\Omega, \Theta))$.

There is a canonical action of $\mathfrak{s}_\text{sm}(X)$ by continuous $G$-endomorphisms on $S(X)$, characterized by the property that for every $\Theta$, considering the map:

$$e^{\ast}_{\Theta, \text{cusp}} : S(X) \to S^{+}(X^{L}_{\Theta})_{\text{cusp}}$$

we have:

$$e^{\ast}_{\Theta, \text{cusp}}(z \cdot f) = z_{\Theta}(e^{\ast}_{\Theta, \text{cusp}}f)$$

for all $z \in \mathfrak{s}_\text{sm}(X)$, where $z_{\Theta}$ denotes the $\Theta$-coordinate of $z$.

Proof. Indeed, $\mathfrak{s}_\text{sm}(X)$ acts by continuous $G$-automorphisms on the right hand side of (14.3) or (14.4), and the action is characterized by the stated property. \hfill \Box

We complete this section by formulating an extension of the properties of the asymptotics and smooth scattering maps from the cuspidal components of Schwartz spaces to the whole space. The proof is straightforward given Theorem 14.3 and the associativity/composition properties of these maps, and is left to the reader:

14.6. Theorem. There are unique extensions of the smooth scattering maps, for all triples $(\Theta, \Omega, w \in W_X(\Omega, \Theta))$:

$$\mathcal{S}_w : S(X_\Theta) \to C^\infty(X_\Omega),$$

such that for all $\Theta' \subset \Theta$, setting $\Omega' = w\Theta$:

$$e^{\Omega}_{\Theta'} \circ \mathcal{S}_w|_{S(X_\Theta)_{\text{cusp}}} = \mathcal{S}_w \circ e^{\Theta}_{\Omega'}|_{S(X_\Theta)_{\text{cusp}}},$$

where as usual we denote by $e^{\Theta}_{\Theta'}$, $e^{\Theta'}_{\Omega'}$ the analogous equivariant exponential maps for the varieties $X_\Theta$, $X_\Omega$, respectively.
These maps satisfy the same associativity relations as their restrictions to cuspidal spectra (s. Theorem 9.2), and $S_w$ is $\hat{\mathfrak{s}}^{\text{sm}}(X^L_{\Theta})$-equivariant with respect to the obvious isomorphism:

$$\hat{\mathfrak{s}}^{\text{sm}}(X^L_{\Theta}) \overset{\sim}{\rightarrow} \hat{\mathfrak{s}}^{\text{sm}}(X^L_{\Omega})$$

induced by $w$.

Finally, for any $\Theta \sim \Omega$ we have:

$$e^*_{\Omega} e_\Theta = \sum_{w \in W_X(\Omega, \Theta)} S_w.$$

15. The Bernstein center and the group Paley-Wiener theorem

15.1. The Bernstein center. We will now see how our Paley-Wiener theorem, and in particular the description of multipliers (Corollary 1.10), implies the well-known theorem on the structure of the Bernstein center in the case of the group, $X = H, G = H \times H$. The argument is inductive in the size of $H$; in particular, we have used the structure of the Bernstein center for its proper Levi subgroups in Corollary 10.4 and hence Proposition 12.1 in order to deduce our Paley-Wiener theorem and the existence of the multiplier ring $\hat{\mathfrak{s}}^{\text{sm}}(H)$ on $S(H)$.

Recall that the Bernstein center $\hat{\mathfrak{s}}(H)$ is, by definition, the center of the category $\mathcal{M}(H)$ of smooth representations of $H$, i.e. the algebra of natural transformations of the identity functor of $\mathcal{M}(H)$. When $X = H$ the boundary degenerations $X_\Theta, \Theta \subset \Delta_X$ are parametrized by classes of parabolics in $H$, where for a given parabolic $P$ corresponding to $\Theta \subset \Delta_X$ we have:

$$X_P := X_\Theta \simeq L_P^{\text{diag}} \backslash (U_P \backslash H \times U_P \backslash H) \simeq L_P \times P^\times P^- (H \times H).$$

Here $P^-$ is an opposite parabolic, $L_P = P \cap P^-$ a Levi subgroup and $U_P, U_P^-$ the corresponding unipotent radicals.

For all $H \times H$-representations that appear below, if not specified otherwise, we let the Bernstein center of $H$ act via the embedding $H \overset{\text{Id} \times 1}{\longrightarrow} H \times H$.

15.2. Theorem. (1) The canonical morphism:

$$\hat{\mathfrak{s}}(H) \rightarrow \text{End}_{H \times H}(S(H))$$

is an isomorphism.

(2) For every class of parabolics $P$ in $H$ (corresponding to $\Theta \subset \Delta_X$) the Bernstein center acts fibrewise, i.e. $\hat{\mathfrak{s}}^{\text{cusp}}(X^L_{\Theta}) = \mathbb{C}[\hat{\mathcal{L}}_{\Theta}]$-equivariantly, on $S(X_\Theta)_{\text{cusp}} \simeq \mathbb{C}[\hat{\mathcal{L}}_{P^{\text{cusp}}}]$.

(3) The action of any element of $\hat{\mathfrak{s}}(H)$ on each fiber of $\mathcal{L}_\Theta$ is scalar; this scalar varies polynomially on $\hat{\mathcal{L}}_{P^{\text{cusp}}}$, i.e. we get a canonical morphism:

$$\hat{\mathfrak{s}}(H) \rightarrow \bigoplus_P \mathbb{C}[\hat{\mathcal{L}}_{P^{\text{cusp}}}]$$

(15.2)
(4) The above map gives rise to an isomorphism:

\[ \mathfrak{z}(H) \sim \left( \bigoplus_P \mathbb{C}[\hat{L}_P^{\text{cusp}}] \right)^{\text{inv}} = \mathfrak{z}^{\text{sm}}(H), \]

(15.3)

where the exponent \( \text{inv} \) denotes invariants with respect to the isomorphisms:

\[ \hat{L}_P^{\text{cusp}} \cong \hat{L}_Q^{\text{cusp}} \]

induced by all \( w \in W_H(P, Q) \).

Proof.  

(1) Choose a Haar measure \( dh \) on \( H \), and let \( z \mapsto \alpha(z) \) denote the morphism (15.1). We can construct an inverse to \( \alpha \) as follows: Let \( (\pi, V) \) be a smooth representation of \( H \) and let \( J \) be an open compact subgroup. For \( Z \in \text{End}_{H \times H}(S(H)) \) we define an endomorphism \( \beta(Z) \) of \( V^J \) by:

\[ \beta(Z)(v) = \pi(Z(1_J/\text{Vol}(J))dh)(v), \]

where \( 1_J \) is the characteristic function of \( J \). It is easy to see that this defines an endomorphism \( \beta(Z) \) of \( V \), and that the collection of these endomorphisms is an element of the Bernstein center (also to be denoted by \( \beta \)). Finally, the fact that \( \beta \) is inverse to \( \alpha \) follows from applying any \( z \in \mathfrak{z}(H) \) to the morphism of smooth \( H \)-representations:

\[ S(H) \otimes \pi \ni f \otimes \pi \mapsto \pi(fdh)(v) \in \pi, \]

where the left hand side is considered as an \( H \)-module only via the action on \( S(H) \) by left multiplication.

(2) This is obvious from the definition of the Bernstein center and the fact that the \( \mathfrak{z}^{\text{cusp}}(X^H_{\Theta}) \)-action commutes with the \( G = H \times H \)-action.

(3) The action is generically scalar because for \( \sigma \in \hat{L}^{\text{cusp}}_{\mathbb{C}} \) in general position the representations \( I^H_{P, \infty}(\sigma) \) and \( I^H_{\text{f}}(\sigma) \) are irreducible (s. [Cas, Theorem 6.6.1]). On the other hand, it has to preserve the space \( S(X_{\Theta})_{\text{cusp}} \cong \mathbb{C}[\hat{L}^{\text{cusp}}_P, L_{\Theta}] \) of regular sections of \( L_{\Theta} \), so it has to be polynomial in \( \sigma \).

(4) From the Paley-Wiener theorem (e.g. in the form of Theorem 14.4) and the \( \mathfrak{z}^{\text{cusp}}(X^H_{\Theta}) \)-equivariance properties of the scattering maps, it follows that the image of (15.2) has to lie in the invariants. On the other hand, by the inverse of (15.1) and the fact that \( \mathfrak{z}^{\text{sm}}(H) \subset \text{End}_{H \times H}(S(H)) \), every invariant induces an \( H \times H \)-equivariant endomorphism of \( S(H) \), thus by the first assertion of this proposition we get the desired isomorphism. 

\( \square \)
15.3. **Paley-Wiener theorem.** In the case of the group, \( X = H, G = H \times H \), we would like to explain the relation of our theorem to the well-known Paley-Wiener theorem of Bernstein [Ber] and Heiermann [Hei01]. We clarify that our theorem goes only half-way towards this result; for the other half, one needs to appeal to Proposition (0.2) of [Hei01], which is probably also the hardest part of that paper. This is because the usual version of the Paley-Wiener theorem for the group is not the one that generalizes to spherical varieties; and there is a non-trivial way to cover in order to obtain one from the other, accomplished through the aforementioned proposition of Heiermann. In fact, the steps taken in part A of [Hei01] can be recast in the setting of our general proof; thus, our work provides a generalization, but not a new proof of the Paley-Wiener theorem for reductive groups. We find it important, nevertheless, to explain the connection.

To state the Paley-Wiener theorem of Bernstein and Heiermann we will use the language of bundles, as in §3, 4; we will not explicitly detail the structure of the bundles that we will encounter, since the process is identical to the one we have used thus far. Instead of formulating it in terms of parabolics with semi-standard Levi as in [Hei01], we prefer to fix a class of standard parabolic subgroups (i.e. fix a minimal parabolic) and use only those.

15.4. **Theorem** (Bernstein [Ber], Heiermann [Hei01]). For every parabolic \( P \) of \( H \), denoting its Levi quotient by \( L \), consider the bundle \( \sigma \mapsto \text{End} (I_H^L (\sigma)) \) over \( \hat{L}_{\text{cusp}} \). Fixing a Haar measure \( dh \), for every smooth representation \( \pi \) we have the canonical map:

\[
S(H) \ni f \mapsto \pi(f \, dh) \in \text{End} (\pi).
\]

Then this map gives rise to an isomorphism:

\[
S(H) \xrightarrow{\sim} \left( \bigoplus_P \mathbb{C} \left[ \hat{L}_{\text{cusp}}, \text{End} (I_P^H (\bullet)) \right] \right)^{\text{inv}},
\]

where:

- \( P \) ranges over all standard parabolics;
- the exponent \( \text{inv} \) refers to sections of the bundle of endomorphisms which commute with all standard intertwining operators.

The standard intertwining operators between representations induced from parabolics \( P, Q \) which share a common Levi subgroup will be denoted:

\[
T_{Q|P} : I_P^H (\sigma) \to I_Q^H (\sigma).
\]

They are rational in \( \sigma \); the definition of \( T_{Q|P} \) depends on choosing a Haar measure on the quotient of \( U_Q \) (the unipotent radical of \( Q \)) by \( U_Q \cap P \); the exact choice does not matter, as long as it makes \( T_{Q|P} \) (applied to \( I_P^H (\sigma) \)) and \( T_{P|Q} \) (applied to \( I_Q^H (\tilde{\sigma}) \)) adjoints, having also fixed the duality between
$I_P(\sigma)$ and $I_P(\tilde{\sigma})$ etc. When $P$ and $Q$ are opposite (denoted: $P = Q^-$ or $Q = P^-$), we will also denote $T_{Q|P}$ or $T_{P|Q}$ by $T_0$.

To relate the above theorem to our result, we will explicitly describe our scattering maps in terms of standard intertwining operators. Notice that for $X = H$ the boundary degenerations are in bijection with conjugacy classes of parabolics in $H$, hence with standard parabolic subgroups. We will write $\Theta \leftrightarrow P$ for this correspondence. The space $X_{\text{cusp}}$ is the space $L_{\text{cusp}}$ of supercuspidal representations of the corresponding Levi quotient, and in a slightly non-canonical way we may identify the bundle $\mathcal{L}_\Theta$ with the bundle whose fiber over $\sigma \in L_{\text{cusp}}$ is the representation $I_P^H(\sigma) \otimes I_P^H(\tilde{\sigma})$.

On the other hand, the bundle $\text{End}(I_P^H(\bullet))$ of Theorem 15.4 can (again slightly non-canonically) be identified with the bundle $\sigma \mapsto I_P^H(\sigma) \otimes I_P^H(\tilde{\sigma})$. The non-canonicity of both isomorphisms depends simply on certain choices of measures (which in the diagram below will be counterbalanced by the choice of measure for the definition of $T_0$), and we will leave the details about choices of measures to the reader.

Thus, the morphism $f \mapsto \pi(fdh)$ can be understood as a morphism:

$$M^* : \mathcal{S}(X) \rightarrow \mathbb{C} \left[ \hat{L}^\text{cusp}, I_P^H(\sigma) \otimes I_P^H(\tilde{\sigma}) \right] \quad (15.4)$$

where the notation $M^*$ is due to the fact that this is dual to the operation of taking matrix coefficients. On the other hand, the condition of invariance under standard intertwining operators in Theorem 15.4 can be translated to the condition of invariance under the operators:

$$\mathbb{C} \left( \hat{L}^\text{cusp}, \sigma \mapsto I_P^H(\sigma) \otimes I_P^H(\tilde{\sigma}) \right) \xrightarrow{T_{Q|P} \otimes T_{P|Q}^{-1}} \mathbb{C} \left( \hat{L}^\text{cusp}, \sigma \mapsto I_Q^H(\sigma) \otimes I_Q^H(\tilde{\sigma}) \right).$$

By [SV, §15.7], one obtains the normalized cuspidal constant terms $E_{\Theta, \text{cusp}}^*$ out of this by composing with the inverse of the standard intertwining operator $T_0 : I_P^H(\tilde{\sigma}) \rightarrow I_P^{H^\perp}(\tilde{\sigma})$ in the second variable:

$$\mathcal{S}(X) \xrightarrow{M^*} \mathbb{C} \left[ I_P^H(\sigma) \otimes I_P^H(\tilde{\sigma}) \right] \xrightarrow{1 \otimes T_0^{-1}} \mathbb{C} \left[ I_Q^H(\sigma) \otimes I_Q^H(\tilde{\sigma}) \right].$$

Thus, we have a commutative diagram:

$$\begin{align*}
\mathbb{C} \left[ I_P^H(\sigma) \otimes I_P^H(\tilde{\sigma}) \right] & \xrightarrow{T_{Q|P} \otimes T_{P|Q}^{-1}} \mathbb{C} \left[ I_Q^H(\sigma) \otimes I_Q^H(\tilde{\sigma}) \right] \\
\simeq & \xrightarrow{1 \otimes T_0^{-1}} \mathbb{C} \left( I_P^H(\sigma) \otimes I_P^{H^\perp}(\tilde{\sigma}) \right) \\
\simeq & \xrightarrow{1 \otimes T_0^{-1}} \mathbb{C} \left( I_Q^H(\sigma) \otimes I_Q^{H^\perp}(\tilde{\sigma}) \right)
\end{align*}$$

where the compositions of slanted and vertical arrows are the normalized constant terms, and we have for brevity omitted $\hat{L}^\text{cusp}$ from the notation.
From this diagram one sees that for $\Theta \sim \Omega$ and $w \in W_X(\Omega, \Theta)$ (we can represent those data by parabolics $P$ and $Q$ of $H$ having a common Levi subgroup), the fiberwise scattering operators $\mathcal{S}_w$ are given by:

$$\mathcal{S}_w = T_{Q|P} \otimes T_{P^{-1}|Q^{-}} : I^H_{P}(\sigma) \otimes I^H_{P^{-}}(\tilde{\sigma}) \rightarrow I^H_{Q}(\sigma) \otimes I^H_{Q^{-}}(\tilde{\sigma}).$$

(Again, $T_{P/Q}$ etc. denote standard intertwining operators; notice that there is no lack of symmetry here, because $T_{Q|P} \otimes T_{P^{-1}|Q^{-}} = T_{P|Q} \otimes T_{Q^{-}|P^{-}}$.)

Our theorem, 14.3 states that the sum of normalized constant terms induces an isomorphism:

$$S(H) \xrightarrow{\sim} \left( \bigoplus_P \mathbb{C}^+ \left[ I^H_{Q}(\sigma) \otimes I^H_{Q^{-}}(\tilde{\sigma}) \right] \right)^{\text{inv}},$$

(15.6)

where $\text{inv}$ denotes invariants of the fiberwise scattering maps $\mathcal{S}_w$. The space $\mathbb{C}^+ \left[ I^H_{Q}(\sigma) \otimes I^H_{Q^{-}}(\tilde{\sigma}) \right]$ is generated by applying these scattering maps to regular sections.

To see that this implies Theorem 15.4, the only non-trivial statement to prove is that every element of:

$$\left( \bigoplus_Q \mathbb{C} \left[ I^H_{Q}(\sigma) \otimes I^H_{Q^{-}}(\tilde{\sigma}) \right] \right)^{\text{inv}}$$

(15.7)

corresponds to an element of the right hand side of (15.6) under the diagram (15.5), but this is [Hei01, Proposition 0.2] which, in our language, states:

15.5. **Proposition.** For every element $\varphi = (\varphi_Q)_Q$ of (15.7) there is an element $\xi = (\xi_P)_P \in \bigoplus_P \mathbb{C} \left[ I^H_{Q}(\sigma) \otimes I^H_{Q^{-}}(\tilde{\sigma}) \right]$ such that:

$$\varphi = \left( \bigoplus_{P \sim Q, w \in W_H(P,Q)} T_{Q|P} \otimes T_{Q^{-}|P^{-}} \xi_P \right)_Q.$$

Notice that $T_{Q|P} \otimes T_{Q^{-}|P^{-}} = T_{Q|P} \otimes (T_0 \circ T_{P^{-1}|Q^{-}})$. Thus, under the vertical arrows of diagram (15.5), the element $\varphi$ corresponds to the element:

$$\left( \bigoplus_{P \sim Q, w \in W_H(P,Q)} T_{Q|P} \otimes T_{P^{-1}|Q^{-}} \xi_P \right)_Q$$

of $\left( \bigoplus_Q \mathbb{C}^+ \left[ I^H_{Q}(\sigma) \otimes I^H_{Q^{-}}(\tilde{\sigma}) \right] \right)^{\text{inv}}$. This recovers Theorem 15.4.

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