THE ABSOLUTE CONTINUITY OF CONVOLUTIONS OF
ORBITAL MEASURES IN SYMMETRIC SPACES

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Abstract. We characterize the absolute continuity of convolution products of orbital measures on the classical, irreducible Riemannian symmetric spaces $G/K$ of Cartan type $III$, where $G$ is a non-compact, connected Lie group and $K$ is a compact, connected subgroup. By the orbital measures, we mean the uniform measures supported on the double cosets, $KzK$, in $G$. The characterization can be expressed in terms of dimensions of eigenspaces or combinatorial properties of the annihilating roots of the elements $z$.

A consequence of our work is to show that the convolution product of any rank $G/K$, continuous, $K$-bi-invariant measures is absolutely continuous in any of these symmetric spaces, other than those whose restricted root system is type $A_n$ or $D_3$, when rank $G/K +1$ is needed.

1. Introduction

In this paper, we study the smoothness properties of $K$-bi-invariant measures on the irreducible Riemannian symmetric spaces $G/K$, where $G$ is a connected Lie group and $K$ is a compact, connected subgroup fixed by a Cartan involution of $G$. Inspired by earlier work of Dunkl [6] on zonal measures on spheres, Ragozin in [27] and [28] showed that the convolution of any dim $G/K$, continuous, $K$-bi-invariant measures on $G$ was absolutely continuous with respect to the Haar measure on $G$. This was improved by Graczyk and Sawyer, who in [10] showed that if $G$ was non-compact and $n = \text{rank } G/K$, then any $n+1$ convolutions of such measures is absolutely continuous and that this is sharp for the symmetric spaces whose restricted root system was type $A_n$. They conjectured that $n+1$ was always sharp. One consequence of our work is to show this conjecture is false. In fact, for all the classical, non-compact symmetric spaces of rank $n$, other than those whose restricted root system is type $A_n$, the convolution product of any $n$ $K$-bi-invariant, continuous measures is absolutely continuous and this is sharp.

We obtain this result by studying a particular class of examples of $K$-bi-invariant, continuous measures, the so-called orbital measures $\nu_z = m_K * \delta_z * m_K$, where $m_K$ is the Haar measure on $K$. These are the uniform measures supported on the double cosets $KzK$ in $G$. They are purely singular, probability measures. The main objective of this paper is to characterize the $L$-tuples $(z_1, ..., z_L)$ such that $\nu_{z_1} * \cdots * \nu_{z_L}$ is absolutely continuous for the classical symmetric spaces of Cartan type $III$.

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The convolution product $\nu_{z_1} \ast \cdots \ast \nu_{z_L}$ is supported on the product of double cosets $K z_1 K z_2 \cdots K z_L K$ and hence if the convolution product is absolutely continuous than the product of double cosets has positive Haar measure in $G$. In fact, if the convolution product is absolutely continuous, the product of double cosets has non-empty interior and the converse is true, as well, thus we also characterize which products of double cosets have non-empty interior.

Given any $z_j \in G$ there is some $Z_j \in \mathfrak{g}$, the Lie algebra of $G$, such that $z_j = \exp Z_j$. Our characterization is in terms of combinatorial properties of the set of annihilating roots of the elements $Z_j$. It also can be expressed in terms of the dimensions of the largest eigenspaces when we view the $Z_j$ as matrices in the classical Lie algebras.

In a series of papers, (see [11] - [13] and the examples cited therein), Graczyk and Sawyer found a characterization for the absolute continuity of $\nu_{z_1} \ast \nu_{y}$ for certain of the type $\text{III}$ (mainly) classical symmetric spaces. Our approach was inspired by their work, but is more abstract and relies heavily upon combinatorial properties of the root systems and root spaces of Lie algebras.

The Cartan involution also gives rise to a decomposition of the Lie algebra as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and put $A = \exp \mathfrak{a}$. Then $G = KAK$, thus we can always assume $z \in A$ when studying orbital measures. Closely related to the $K$-bi-invariant orbital measures supported on double cosets in $G$ are the $K$-invariant, uniform measures, $\mu_Z$, supported on the orbits under the $\text{Ad}(K)$ action of elements $Z \in \mathfrak{a}$. It is known (11) that $\mu_{Z_1} \ast \cdots \ast \mu_{Z_L}$ is absolutely continuous with respect to Lebesgue measure on $\mathfrak{p}$ if and only if $\nu_{z_1} \ast \cdots \ast \nu_{z_L}$ is absolutely continuous on $G$ when $z_j = \exp Z_j$, and this is also equivalent to the sum of the $\text{Ad}(K)$ orbits generated by the $Z_j$ having non-empty interior.

The absolute continuity of convolution products of orbital measures is connected with questions about spherical functions $\phi_{\lambda}(\exp X)$ where $\lambda$ is a complex-valued linear form on $\mathfrak{a}$ and $X \in \mathfrak{a}$. This is the spherical Fourier transform of the orbital measure $\nu_{\exp X}$. The product formula states that

$$\phi_{\lambda}(e^{Z_1}) \cdots \phi_{\lambda}(e^{Z_L}) = \int_G \phi_{\lambda} d\nu_{z_1} \ast \cdots \ast \nu_{z_L},$$

hence the absolute continuity of convolutions of orbital measures gives a formula for a product of spherical functions.

An example of a compact, symmetric space is $(G \times G)/\Delta(G)$ where $G$ is a compact, connected, simple Lie group and $\Delta(G) = \{(g, g) : g \in G\} \simeq G$. These are the symmetric spaces of Cartan type $\text{II}$. In this case, the orbital measure $\nu_z$ for $z = (g, g)$, supported on the double coset $\Delta(G) z \Delta(G)$, can be identified with the uniform measure supported on the conjugacy class in $G$ containing the element $g$. The $\Delta(G)$-invariant measure, $\mu_Z$ for $Z \in \mathfrak{g}$, can be identified with the measure on the compact Lie algebra $\mathfrak{g}$ that is uniformly distributed on the adjoint orbit in $\mathfrak{g}$ containing $Z$. These symmetric spaces are dual to the non-compact, symmetric spaces of Cartan type $\text{IV}$, $G^C/G$, where $G^C$ is the non-compact Lie group corresponding to the Lie algebra $\mathfrak{g} \oplus \mathfrak{i} \mathfrak{g}$ (over $\mathbb{R}$) (see [1], [20]). It can be seen from [1] that the absolute continuity problem for Cartan type $\text{IV}$ symmetric spaces can be deduced from the analogous problem for the corresponding $G$-invariant orbital measures $\mu_Z$, for $Z \in \mathfrak{g}$, from the (dual) Cartan type $\text{II}$ spaces.
In a series of papers, the authors (with various coauthors) studied the absolute continuity problem for orbital measures in the compact setting. The sharp exponent \( n = n(Z) \) (or \( n(z) \)) with the property that the \( n \)-fold convolution product of \( \mu_Z \) (or \( \nu_z \)) was absolutely continuous was determined for the classical compact Lie algebras \( g \) in [18], for the classical compact Lie groups in [15], and for the exceptional compact Lie groups and algebras in [19]. Sufficient conditions for these problems were found using harmonic analysis methods not generally available in the symmetric space setting.

In [31], Wright used geometric methods to extend this result to the convolution of different orbital measures on the Lie algebras of type \( A_n \). Later, the authors in [17] obtained a (almost complete) characterization for the absolute continuity of convolution products of arbitrary orbital measures in all the classical compact Lie algebras and hence also those for the classical symmetric spaces of Cartan type \( IV \). This was done by mainly algebraic/combinatorial methods. These ideas are key to this paper, particularly for the symmetric spaces with one-dimensional, restricted root spaces, but many additional technical complications arise in the more general symmetric space setting.

Earlier, Ricci and Stein in [29] and [30] studied the smoothness properties of convolutions of measures supported on manifolds whose product has non-empty interior. They proved, for example, that if the surface measure of a compact manifold has an absolutely continuous convolution product, then the density function of that convolution product is actually in \( L^{1+\varepsilon} \) for some \( \varepsilon > 0 \). A number of authors have attempted to compute the density function (in some special cases), but this is very hard. We refer the reader to [3], [7] and [9], for example. Sums of adjoint orbits have also been studied, such as in [25] where the sum of two adjoint orbits in \( su(n) \) is described. The smoothing properties of convolution is also of interest in the study of random walks on groups and hypergroups; c.f., [2], [22], [26].

1.1. Organization of the paper. We begin in the second section by introducing terminology, including the definition of orbital measures and the very important notion of annihilating roots. We also explain the connection between the absolute continuity problem for the two classes of orbital measures and the connection with questions about sums of orbits / products of double cosets. In section three we explain what is meant by the type of an element, and what is meant by eligible and exceptional tuples. These ideas come from [17]. We also give the formal statement of our main theorem, that absolute continuity is characterized by eligibility and non-exceptionality. An immediate corollary is that except when the restricted root space is type \( A_n \), any convolution product of \( n = \text{rank} G/K \) continuous bi-invariant measures on \( G \) is absolutely continuous. Moreover, we can describe which \( (n-1) \)-fold products of orbital measures are not absolutely continuous. The proof of sufficiency of our characterization is the content of section four and occupies the majority of the paper. Finally, in section five we prove that the non-eligible and the exceptional tuples do not give rise to absolutely continuous convolution products. Useful basic facts about the symmetric spaces of Cartan type \( III \) are summarized in the appendix.

2. Set Up and Preliminary results

2.1. Cartan decomposition. Let \( G \) be a non-compact, connected, Lie group with Lie algebra \( g \) and suppose \( \theta \) is an involution of \( G \). Let \( K = \{ g \in G : \theta(g) = g \} \)
and assume $K$ is compact and connected. The quotient space, $G/K$, is called a symmetric space. We let $W$ denote its Weyl group.

The map $\theta$ induces an involution of $\mathfrak{g}$, also denoted $\theta$. We put

$$\mathfrak{k} = \{ X \in \mathfrak{g} \mid \theta(X) = X \} \quad \text{and} \quad \mathfrak{p} = \{ X \in \mathfrak{g} \mid \theta(X) = -X \},$$

the $\pm 1$ eigenspaces of $\theta$, respectively. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the Cartan decomposition of the Lie algebra $\mathfrak{g}$. The subspaces $\mathfrak{k}$ and $\mathfrak{p}$ satisfy the following rules:

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

We fix a maximal abelian (as a subalgebra of $\mathfrak{g}$) subspace $\mathfrak{a}$ of $\mathfrak{p}$ and let $\mathfrak{a}^*$ be the dual of $\mathfrak{a}$.

As is standard, we will let $\text{ad}(X)(Y) = [X, Y]$, denote by $\text{Ad}(\cdot)$ the adjoint action of $G$ on $\mathfrak{g}$ and write $\exp$ for the exponential map from $\mathfrak{g}$ to $G$.

It is known that $\text{Ad}(k) : \mathfrak{p} \rightarrow \mathfrak{p}$ whenever $k \in K$. Moreover, $\exp \mathfrak{k} = K$ and if we put $A = \exp \mathfrak{a}$, then $G = KAK$ (24, p. 459).

By $\mathfrak{a}^+$ we mean the subset

$$\mathfrak{a}^+ = \{ H \in \mathfrak{a} : \alpha(H) > 0 \ \text{for all} \ \alpha \in \mathfrak{a}^* \}.$$

The sets $w(\mathfrak{a}^+)$ are disjoint for distinct $w \in W$ and $\mathfrak{a} = \bigcup_{w \in W} w(\mathfrak{a}^*)$.

Put $A^+ = \exp \mathfrak{a}^+$. It is known (20, p. 402) that $G = K A^+ K$. Indeed, given any $g \in G$, there is a pair $k_1, k_2 \in K$ and a unique $H \in \mathfrak{a}^*$ such that $g = k_1(\exp H)k_2$. We define a map $A : G \rightarrow \mathfrak{a}^+ \subseteq \mathfrak{a}$ by $A(g) = H$. We also speak of $A$ as a map from $G \rightarrow A^+$ by taking $A(g) = \exp H$. It will be clear from the context which we mean.

For non-zero $\alpha \in \mathfrak{a}^*$ we consider the set

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} : [H, X] = \alpha(H) X \ \text{for all} \ H \in \mathfrak{a} \}.$$

The set of restricted roots, $\Phi$, is defined by

$$\Phi = \{ \alpha \in \mathfrak{a}^* : \mathfrak{g}_\alpha \neq 0 \}$$

and the subset of positive roots is denoted by $\Phi^+$. The set $\Phi$ is a root system, although not necessarily reduced because it is possible for both $\alpha$ and $2\alpha$ to be a root.

The vector spaces $\mathfrak{g}_\alpha$ corresponding to $\alpha \in \Phi$ are known as the restricted root spaces and need not be one-dimensional. It is well known that $\theta \mathfrak{g}_\alpha = -\mathfrak{g}_\alpha$. The Lie algebra $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha, \ \text{where} \ \mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$$

with $\mathfrak{m} = \{ X \in \mathfrak{k} : [X, \mathfrak{a}] = 0 \}$.

The $\pm 1$ eigenspaces of $\theta$ can also be described as

$$\mathfrak{e} = sp \left\{ X + \theta X : X \in \mathfrak{g}_\alpha, \ \alpha \in \Phi^+ \cup \{ 0 \} \right\} = sp \left\{ X + \theta X : X \in \mathfrak{g}_\alpha, \ \alpha \in \Phi^+ \right\} \oplus \mathfrak{m}$$

and

$$\mathfrak{p} = sp \left\{ X - \theta X : X \in \mathfrak{g}_\alpha, \ \alpha \in \Phi^+ \right\} \oplus \mathfrak{a}$$

where by $sp$ we will mean the real span.

We will put

$$X^+ = X + \theta X \ \text{and} \ \ X^- = X - \theta X.$$
We will write $g_n$, $t_n$ etc if we want to emphasize that the rank of the symmetric space is $n$.

Throughout this paper we will assume $G/K$ is an irreducible, Riemannian, globally symmetric space of Type III for which the root system of the Lie group $G$ is not exceptional. These have Cartan classifications $A_I, A_{III}, A_{III}, B_I, C_I, C_{II}$, and $D_{III}$. Their restricted root systems have Lie types $A_n, B_n, C_n, D_n$ or $BC_n$. In the appendix, we summarize basic information about these symmetric spaces, including the choices of $G, K$, rank and the dimension of the restricted root spaces. This information is taken from [3], [4], [20] and [24]. For general facts about root systems, we refer the reader to [21] and [23].

2.2. Orbital measures, orbits and double cosets. A measure $\mu$ on $p$ will be said to be $K$-invariant if $\mu(E) = \mu(\text{Ad}(k)E)$ for all $k \in K$ and Borel sets $E \subseteq p$. Corresponding to each $Z \in p$ is a $K$-invariant probability measure called the orbital measure, $\mu_Z$, defined by

$$\int_p f d\mu_Z = \int_K f(\text{Ad}(k)Z) dm_K(k)$$

for any continuous, compactly supported function $f$ on $p$. Here $m_K$ denotes the Haar measure on the compact group $K$. This measure is supported on the $\text{Ad}_K$-orbit of $Z$, meaning the orbit of $Z$ under the action of $K$ on $p$. We denote this set by $O_Z$, thus

$$O_Z = \{ \text{Ad}(k)Z : k \in K \}.$$ 

Every $\text{Ad}_K$-orbit contains an element of $a$ since $p = \bigcup_{k \in K} \text{Ad}(k)a$ ([24], p. 455)), hence in studying orbital measures, $\mu_Z$, there is no loss in assuming $Z \in a$.

$\text{Ad}_K$-orbits are manifolds of proper dimension in $p$ and hence have Haar measure zero and empty interior. Thus the orbital measures are singular with respect to Haar measure.

A measure $\mu$ on $G$ will be said to be $K$-bi-invariant if $\mu(E) = \mu(k_1Ek_2)$ for all $k_1, k_2 \in K$ and Borel sets $E \subseteq G$. These measures can be naturally identified with the left $K$-invariant measures on the symmetric space $G/K$. An example is the measure we will denote by $\nu_z$ for $z \in G$, defined by

$$\int_G f d\nu_z = \int_K \int_K f(k_1zk_2) dm_K(k_1) dm_K(k_2)$$

for any compactly supported, continuous function $f$ on $G$. This is the measure supported on the double coset $KzK$ in $G$ and is also called an orbital measure. It is easy to see that $\nu_z = m_K \ast \delta_z \ast m_K$ where $\delta_z$ is the point mass measure at $z$. Since $G = KAK$, every double coset contains an element of $A$ and hence there is no loss of generality in assuming $z \in A$.

Double cosets are also manifolds of proper dimension, hence have Haar measure zero in $G$ and empty interior. It follows that the measures $\nu_z$ are singular with respect to Haar measure.

The $K$-bi-invariant measures are also known as zonal measures.

2.3. Annihilating roots and Tangent spaces. Given $Z \in a$, we let

$$\Phi_Z = \{ \alpha \in \Phi : \alpha(Z) = 0 \}$$
be the set of annihilating roots of $Z$ and let
\[ N_Z = sp\{X_\alpha - \theta X_\alpha : X_\alpha \in g_\alpha, \alpha \notin \Phi_Z\} \subseteq p. \]
The set $\Phi_Z$ is itself a root system and is a proper root subsystem provided $Z \neq 0.$
As we will see, these root subsystems, $\Phi_Z,$ and the associated spaces, $N_Z,$ are of fundamental importance in studying orbits and orbital measures.

A very useful fact is that if $Z \in \mathfrak{a}$ and $X_\alpha \in g_\alpha,$ then it holds for almost all $\{Z, X, Y\} : Z, X, Y \in \mathfrak{g}_{\alpha},$ then
\[ [Z, X^+] = [Z, X_\alpha + \theta X_\alpha] = \alpha(Z)(X_\alpha - \theta X_\alpha) = \alpha(Z)X^-_. \]

In particular,
\[ N_Z = sp\{[Z, X] : X \in \mathfrak{k}\} = \text{Imad}(Z)|_t. \]

It is well known that the tangent space to the $Ad_K$-orbit of $Z$ is $T_Z(O_Z) = \{[Z, Y] : Y \in \mathfrak{t}\}.$ Hence
\[ T_Z(O_Z) = N_Z \]
and, in particular,
\[ \dim O_Z = \dim N_Z = \sum_{\alpha \in \Phi^+ \setminus \Phi_Z} \dim g_\alpha. \]

More generally, if $X \in O_Z,$ say $X = Ad(k)Z,$ then $T_X(O_Z) = Ad(k)T_Z(O_Z) = Ad(k)N_Z.$

Ragozin [27] proved that questions about the absolute continuity of convolution products of orbital measures are related to geometric questions about orbits and tangent spaces. Here are some key ideas.

**Proposition 1.** Let $Z_1, \ldots, Z_t \in \mathfrak{a}$ and $z_j = \exp Z_j \in A.$ The following are equivalent:

1. The convolution product of orbital measures $\mu_{Z_1} * \cdots * \mu_{Z_t}$ is absolutely continuous with respect to Lebesgue measure on $\mathfrak{p}.$
2. The sum $O_{Z_1} + \cdots + O_{Z_t}$ has non-empty interior (equivalently, positive Lebesgue measure) in $\mathfrak{p}.$
3. There is some $k_1 = Id, k_2, \ldots, k_t \in K$ such that
   \[ sp\{Ad(k_j)N_{Z_j} : j = 1, \ldots, t\} = \mathfrak{p}. \]
4. The convolution product of orbital measures $\nu_{z_1} * \cdots * \nu_{z_t}$ is absolutely continuous with respect to Haar measure on $G.$
5. The product of the double cosets $Kz_1Kz_2K \cdots Kz_tK$ has non-empty interior (equivalently, positive Haar measure) in $G.$
6. There is some $k_2, \ldots, k_t \in K$ such that
   \[ \{X_1 + Ad(z_1)X_2 + \cdots + Ad(z_1k_2z_2 \cdots k_tz_t)X_{t+1} : X_j \in \mathfrak{t}\} = \mathfrak{g}. \]

Furthermore, in the case that (3) or (6) holds for some $(t-1)$-tuple, $(k_2, \ldots, k_t),$ then it holds for almost all $(k_2, \ldots, k_t) \in K^{t-1}.$

**Proof.** This is an amalgamation of ideas that can mainly be found in [1] and [27].

Consider the maps
\[ F = F_{Z_1, \ldots, Z_t} : O_{Z_1} \times \cdots \times O_{Z_t} \to \mathfrak{p} \]
\[ F(X_1, \ldots, X_t) = X_1 + \cdots + X_t \]
and
\[ f = f_{z_1, \ldots, z_t} : K^{t+1} \to G \]
is of type $A$. A root system is type $A$ can be identified with the $n$-dimensional vector space.

3.1. Type and eligibility. Here we modify those definitions for the symmetric space scenario.

### Corollary 1.

$(Z_1, ..., Z_m)$ is an absolutely continuous tuple if and only if the orbital measure $\nu_{z_1} \ast \cdots \ast \nu_{z_m}$ is absolutely continuous on $G$ for $z_j = \exp Z_j$.

3. Statement of the Characterization Theorem

#### 3.1. Type and eligibility.

As in [17], our theorem will depend upon what we call type and eligibility. Here we modify those definitions for the symmetric space scenario.

### Type of an element.

When the restricted root system of the symmetric space is of type $A_{n-1}$ (we also call this type $SU(n)$ as this is the classical Lie group whose root system is type $A_{n-1}$), after applying a suitable Weyl conjugate any $Z \in \mathfrak{a}_n$ can be identified with the $n$-vector

$$Z = \begin{pmatrix} (a_1, \ldots, a_{s_1}, a_2, \ldots, a_{s_2}, \ldots, a_{s_m}, \ldots, a_m) \end{pmatrix},$$

where the $a_j \in \mathbb{R}$ are distinct and $\sum_{j=1}^{m} s_j a_j = 0$. The set of annihilating roots $\Phi_Z = \Psi_1 \cup \cdots \cup \Psi_m$ where

- $\Psi_1^+ = \{e_i - e_j : 1 \leq i < j \leq s_1\}$ and
- $\Psi_l^+ = \{e_i - e_j : s_1 + \cdots + s_{l-1} < i < j \leq s_1 + \cdots + s_l\}$ for $l > 1$. 

But the same argument works for general $Z_j$. As $Ad(k_j)N_{Z_j} \subseteq \mathfrak{p}$ for any $k_j \in K$, (2.3) holds if and only if property (3) holds, i.e., $sp\{Ad(k_j)N_{Z_j} : j = 1, ..., t\} = \mathfrak{p}$.

To show that a convolution product of orbital measures is absolutely continuous, we will typically establish that property (3) of the proposition holds.

### Notation 1.

We will call $(Z_1, ..., Z_m)$ an absolutely continuous tuple if any of these equivalent conditions are satisfied.

For emphasis, we highlight:

### Corollary 1.

$(Z_1, ..., Z_m)$ is an absolutely continuous tuple if and only if the orbital measure $\nu_{z_1} \ast \cdots \ast \nu_{z_m}$ is absolutely continuous on $G$ for $z_j = \exp Z_j$. 

$$f(k_1, ..., k_{l+1}) = k_1 z_1 k_2 \cdots k_l z_l k_{l+1}.$$
Following [13], we say that $Z$ is type $SU(s_1) \times \cdots \times SU(s_m)$ as this is the Lie type of its set of annihilating roots.

If the restricted root system is type $B_n$, $C_n$, $D_n$ or $BC_n$, then up to a Weyl conjugate, $Z \in a_n$ can be identified with the $n$-vector

$$Z = (0, \ldots, 0, a_1, \ldots, a_m, \ldots, ( \pm )a_m)$$

where the $a_j > 0$ are distinct. We remark that the minus sign is needed only in type $D_n$ and only if $J = 0$. (This is because the Weyl group in type $D_n$ changes only an even number of signs.)

The set of annihilating roots of $Z$ can be written as $\Phi_Z = \Psi_0 \cup \Psi_1 \cdots \cup \Psi_m$ where

$$\Psi_0^+ = \begin{cases} 
\{ e_k, e_i \pm e_j : 1 \leq i, j, k \leq J, i < j \} & \text{if } \Phi \text{ type } B_n \\
\{ 2e_k, e_i \pm e_j : 1 \leq i, j, k \leq J, i < j \} & \text{if } \Phi \text{ type } C_n \\
\{ e_i \pm e_j : 1 \leq i < j \leq J \} & \text{if } \Phi \text{ type } D_n \\
\{ e_k, 2e_k, e_i \pm e_j : 1 \leq i, j, k \leq J, i < j \} & \text{if } \Phi \text{ type } BC_n 
\end{cases}$$

and for $l \geq 1$,

$$\Psi_l^+ = \{ e_i - e_j : J + s_1 + \cdots + s_{l-1} < i < j \leq J + s_1 + \cdots + s_l \},$$

except if $Z = (a_1, \ldots, a_1, \ldots, a_m, \ldots, -a_m)$ in $D_n$ when

$$\Psi_m^+ = \{ e_i - e_j, e_i + e_n : n - s_m < i < j \leq n - 1 \}.$$ 

In the case that $\Phi$ is type $B_n$, we will say that such an element $Z$ is type $B_J \times SU(s_1) \times \cdots \times SU(s_m)$ as this is the Lie type of $\Phi_Z$. We make a similar definition if $\Phi$ is type $C_n$, $D_n$ or $BC_n$. We understand $SU(1)$ and $B_0$ to be empty, $B_1$ to be the subsystem $\{ e_1 \}$ and define $C_0$, $BC_0$, $C_1$ and $BC_1$ similarly. In the case of type $D_n$, we understand both $D_0$ and $D_1$ to be empty and $D_2$ to be $\{ e_i \pm e_j \}$. We often omit the writing of the empty root systems in our descriptions.

Note that there are two distinct subsystems (up to Weyl conjugacy) of annihilating roots of elements of type $SU(n)$ in $D_n$.

3.1.2. Dominant type. Suppose the symmetric space has restricted root system of type $B_n$ and $Z \in a_n$ is type $B_J \times SU(s_1) \times \cdots \times SU(s_m)$. We will say $Z$ is dominant $B$ type if $2J \geq \max s_j$, and is dominant $SU$ type otherwise. We define dominant $C$, $D$ and $BC$ type similarly for $Z$ in a symmetric space with restricted root system of type $C_n$, $D_n$ or $BC_n$.

3.1.3. Eligible and Exceptional Tuples.

**Notation 2.** If $Z$ is of type $SU(s_1) \times \cdots \times SU(s_m)$ in a symmetric space with restricted root system of type $A_n$, put $S_X = \max s_j$.

If $Z$ is type $B_J \times SU(s_1) \times \cdots \times SU(s_m)$ in a symmetric space with restricted root system of type $B_n$, put

$$S_X = \begin{cases} 
2J & \text{if } X \text{ is dominant } B \text{ type} \\
\max s_j & \text{else} 
\end{cases}$$

Define $S_X$ similarly when $Z$ is in a symmetric space with restricted root system of type $C_n$, $D_n$ or $BC_n$. 
Definition 1. (1) We will say that the \( L \)-tuple \((Z_1, Z_2, \ldots, Z_L) \in a^L \) in a symmetric space with restricted root system of type \( A_n \) is \textit{eligible} if

\[
\sum_{i=1}^{L} S_{X_i} \leq (L - 1)(n + 1).
\]

(2) We will say that the \( L \)-tuple \((Z_1, Z_2, \ldots, Z_L) \in a^L \) in a symmetric space with restricted root system of type \( B_n, C_n, D_n \) or \( BC_n \) is \textit{eligible} if

\[
\sum_{i=1}^{L} S_{X_i} \leq (L - 1)2n.
\]

Definition 2. We will say that \((Z_1, Z_2, \ldots, Z_L) \in a^L \) is an \textit{exceptional tuple} in any of the following situations:

1. The symmetric space has restricted root system of type \( A_{2n-1} \), \( L = 2, n \geq 2 \) and \( Z_1 \) and \( Z_2 \) are both of type \( SU(n) \times SU(n) \);
2. The symmetric space has restricted root system of type \( D_n \), \( L = 2, Z_1 \) is type \( SU(n) \) and \( Z_2 \) is either type \( SU(n) \) or type \( SU(n-1) \);
3. The symmetric space has restricted root system of type \( D_4 \), \( L = 2, Z_1 \) is type \( SU(4) \) and \( Z_2 \) is either type \( SU(2) \times SU(2) \) and \( \Phi_{Z_2} \) is Weyl conjugate to a subset of \( \Phi_{Z_1} \), or \( Z_2 \) is type \( SU(2) \times D_2 \);
4. The symmetric space has restricted root system of type \( D_n, n = 3 \) or \( 4, L = 3 \) and \( Z_1, Z_2, Z_3 \) are all of type \( SU(n) \) with Weyl conjugate sets of annihilators in the case of \( n = 4 \).

3.2. Main Result. Our main result is that other than for the exceptional tuples, eligibility characterizes absolute continuity of the convolution product. The proof of this theorem will occupy most of the remainder of the paper. Here is the formal statement of the theorem.

Theorem 1. Let \( G/K \) by a symmetric space of type \( III \) and suppose \( Z_j \in a, Z_j \neq 0 \) for \( j = 1, 2, \ldots, L \) and \( L \geq 2 \). The orbital measure \( \mu_{Z_1} \ast \mu_{Z_2} \ast \cdots \ast \mu_{Z_L} \) is absolutely continuous with respect to Lebesgue measure on \( p \) if and only if \((Z_1, Z_2, \ldots, Z_L) \) is eligible and not exceptional.

Corollary 2. Let \( Z_j \in a, Z_j \neq 0 \) for \( j = 1, 2, \ldots, L \) and \( L \geq 2 \), and let \( z_j = \exp Z_j \). The orbital measure on \( G, \nu_{z_1} \ast \cdots \ast \nu_{z_L} \), is absolutely continuous with respect to Haar measure on \( G \) if and only if \((Z_1, Z_2, \ldots, Z_L) \) is eligible and not exceptional.

Proof. The proof is immediate from the Theorem and Cor. [1] \( \square \)

Remark 1. The characterization of absolute continuity for pairs of orbital measures, \( \nu_{z_1} \ast \nu_{y_2} \), was established by Graczyk and Sawyer for the Type III symmetric spaces of Cartan types \( AI \) and \( AII \) in [11] and for the Cartan types \( AIII, CII \) and \( BDI \) in [12] and [13]. They use an induction argument, but how it is applied depends upon the particular symmetric space.

We will give a complete proof of sufficiency for all Cartan types and all \( L \geq 2 \). As with Graczyk and Sawyer, we also use an induction argument, but it relies upon the Lie type of the restricted root space rather than the symmetric space itself. In fact, it is the combinatorial structure of the root systems and root vectors that is key to our approach.
Corollary 3. (1) If $G/K$ is a symmetric space of Cartan type $A_I$ or $A_{II}$, of rank $n$ (hence the restricted root system is type $A_n$), then the convolution of any $n + 1$ orbital measures (on $G$ or $p$) is absolutely continuous. Moreover, this is sharp since any $n$-tuple of elements all of type $SU(n)$ is not absolutely continuous. Furthermore, these are the only $n$-tuples that fail to be absolutely continuous.

(2) If $G/K$ is a symmetric space of rank $n$ whose restricted root system is not type $A_n$ or type $D_3$, then the convolution of any $n$ orbital measures (on $G$ or $p$) is absolutely continuous. This is sharp since any $(n-1)$-tuple of elements of type $B_{n-1}$ (or $C_{n-1}$, $D_{n-1}$, $BC_{n-1}$ depending on the restricted root system) is not absolutely continuous. Furthermore, except in type $D_4$, these are the only $(n-1)$-tuples that fail to be absolutely continuous.

We remind the reader that when we speak of an $L$-tuple of elements being absolutely continuous, we mean that the convolution of their corresponding orbital measures is absolutely continuous.

Proof. In both cases, just check the eligibility and non-exceptionality criterion. □

Remark 2. We remark that this corollary partially improves upon [10] where it was shown that in any symmetric space the convolution of rank $+ 1$ orbital measures is absolutely continuous and that in the symmetric space with restricted root system of type $A_n$, the $n$-fold convolution of the orbital measure $\mu_X$, where $X$ is type $SU(n)$ is not absolutely continuous.

This corollary also answers Conjecture 10 of [10] negatively.

A $K$-bi-invariant measure $\mu$ on $G$ is said to be continuous if $\mu(gK) = 0$ for all $g \in G$. Ragozin in [27] proved that the convolution of any $\dim G/K$ continuous $K$-bi-invariant measures is absolutely continuous. This too can be improved.

Corollary 4. If $G/K$ is a symmetric space of rank $n$, then the convolution of any $n$ (resp., $n + 1$) continuous $K$-bi-invariant measures on $G$ is absolutely continuous if the restricted root system is not type $A_n$ or type $D_3$ (resp., if the restricted root system is type $A_n$ or $D_3$).

Proof. In [27] it was actually shown that if for each $Z_1, ..., Z_m \in a_n, sp\{Ad(k)N_{Z_j} : j = 1, ..., m\} = p$ for almost all $k_j \in K$, then any $m$ continuous $K$-bi-invariant measures on $G$ is absolutely continuous. From the Theorem and Prop. [11] we know this holds with $m = n$. □

Most of the remainder of the paper will be aimed at proving this theorem. The proof is organized as follows. We focus first on sufficiency. We begin by showing that the problem can largely be reduced to the study of the problem on the symmetric spaces whose restricted root spaces are all of dimension one. For these spaces we give an induction argument; this is the key combinatorial idea that was also used in the study of the analogous problem for convolutions of orbital measures in the classical Lie algebras (see [17]). We will apply this first to the problem of convolving two orbital measures and then will show how to handle more than two convolutions.

Of course, an induction argument can only be used if we can establish the base case(s). Some of these cases are non-trivial and for those we prove another sufficient combinatorial condition that was motivated by a result in [31].
4. Reduction to multiplicities one problems.

Lemma 1. Let $G/K$ be a symmetric space and $x_1, \ldots, x_m \in A$. Then $\nu_{x_1} \cdots \nu_{x_m}$ is absolutely continuous on $G$ if and only if $\mathcal{A}(x_1 K x_2 \cdots K x_m)$ has non-empty interior in $A$.

Proof. Suppose $V$ is an open subset of $A$ contained in $\mathcal{A}(x_1 K x_2 \cdots K x_m)$. Let $A^+ = \exp A^+$. Then $V \setminus \text{bdy}(A^+) \subseteq A^+$ is open in $A$. Since one can easily check that $\text{bdy}(A^+)$ has $A$-Haar measure zero, $V \setminus \text{bdy}(A^+)$ is non-empty.

It is known that the map $\mathcal{A}$ restricted to $K A^+ K$ is a smooth map onto $A^+$ (\cite{8}), thus the preimage of $V \setminus \text{bdy}(A^+)$ is open in $K A^+ K$ and hence also in $G$ since $K A^+ K$ is open in $G$. But this non-empty open set is a subset of $K x_1 K \cdots K x_m K$ and therefore by Prop. 1 $\nu_{x_1} * \cdots * \nu_{x_m}$ is absolutely continuous.

Conversely, suppose $\nu_{x_1} * \cdots * \nu_{x_m}$ is absolutely continuous on $G$. Applying Prop. 1 we can find an open, non-empty subset $V$ of $G$ contained in $K x_1 K \cdots K x_m K$. But then also $K V K$ is an open set in $G$ contained in $K x_1 K \cdots K x_m K$ and hence $K V K \cap A$ is open in $A$ (the topology on $A$ being the relative topology). It is non-empty since every double coset admits elements of $A$.

Now

$$K V K \cap A = \bigcup_{w \in W} \left( K V K \cap w(A^+) \right) \bigcup_{w \in W} \left( K V K \cap \text{bdy}(w(A^+)) \right).$$

The sets $K V K \cap w(A^+)$ are all open in $A$. If for some $w \in W$, $K V K \cap w(A^+)$ is non-empty, then since $w^{-1}(K V K) = K V K$, it would follow that $K V K \cap A^+ \subseteq K x_1 K \cdots K x_m K$ is open and non-empty. But then $\mathcal{A}(K V K \cap A^+)$ is also open and non-empty, hence $\mathcal{A}(K x_1 K \cdots K x_m K) = \mathcal{A}(x_1 K \cdots K x_m)$ has non-empty interior, as we desired to show.

Otherwise, $K V K \cap A = \bigcup_{w \in W} \left( K V K \cap \text{bdy}(w(A^+)) \right)$. But the set on the right has Haar measure zero, while the set on the left is open and non-empty, so this is impossible.

Terminology: Let $G_1/K_1$ and $G_2/K_2$ be two symmetric spaces. We say that $G_1/K_1$ is embedded into $G_2/K_2$ if there is a mapping $\mathcal{I} : G_1 \to G_2$ satisfying the following properties.

Definition 3. (1) $\mathcal{I}$ is a group isomorphism into $G_2$.
(2) $\mathcal{I}$ restricted to $A_1$ is a topological group isomorphism onto $A_2$.
(3) $\mathcal{I}$ maps $K_1$ into $K_2$. 

In passing from the symmetric spaces with one dimensional, restricted root spaces to the other symmetric spaces, there are a few special cases of $L$-tuples that we will also need to handle using this other sufficient condition.

We then turn to necessity. The necessity of eligibility will be seen to follow from elementary linear algebra arguments. For the exceptional tuples, we need other reasoning. The simple fact that the dimension of the underlying orbits are simply not large enough to have a chance to satisfy Prop. 1(3) can often be used.

4. Proof of Sufficiency

4.1. Reduction to multiplicities one problems. We begin the proof of sufficiency by showing we can focus our attention primarily on the symmetric spaces whose restricted root spaces are all of dimension one. The first lemma seems to be known, but we could not find a proof in the literature.

Lemma 1. Let $G/K$ be a symmetric space and $x_1, \ldots, x_m \in A$. Then $\nu_{x_1} \cdots \nu_{x_m}$ is absolutely continuous on $G$ if and only if $\mathcal{A}(x_1 K x_2 \cdots K x_m)$ has non-empty interior in $A$.

Proof. Suppose $V$ is an open subset of $A$ contained in $\mathcal{A}(x_1 K x_2 \cdots K x_m)$. Let $A^+ = \exp A^+$. Then $V \setminus \text{bdy}(A^+) \subseteq A^+$ is open in $A$. Since one can easily check that $\text{bdy}(A^+)$ has $A$-Haar measure zero, $V \setminus \text{bdy}(A^+)$ is non-empty.

It is known that the map $\mathcal{A}$ restricted to $K A^+ K$ is a smooth map onto $A^+$ (\cite{8}), thus the preimage of $V \setminus \text{bdy}(A^+)$ is open in $K A^+ K$ and hence also in $G$ since $K A^+ K$ is open in $G$. But this non-empty open set is a subset of $K x_1 K \cdots K x_m K$ and therefore by Prop. 1 $\nu_{x_1} \cdots \nu_{x_m}$ is absolutely continuous.

Conversely, suppose $\nu_{x_1} \cdots \nu_{x_m}$ is absolutely continuous on $G$. Applying Prop. 1 we can find an open, non-empty subset $V$ of $G$ contained in $K x_1 K \cdots K x_m K$. But then also $K V K$ is an open set in $G$ contained in $K x_1 K \cdots K x_m K$ and hence $K V K \cap A$ is open in $A$ (the topology on $A$ being the relative topology). It is non-empty since every double coset admits elements of $A$.

Now

$$K V K \cap A = \bigcup_{w \in W} \left( K V K \cap w(A^+) \right) \bigcup_{w \in W} \left( K V K \cap \text{bdy}(w(A^+)) \right).$$

The sets $K V K \cap w(A^+)$ are all open in $A$. If for some $w \in W$, $K V K \cap w(A^+)$ is non-empty, then since $w^{-1}(K V K) = K V K$, it would follow that $K V K \cap A^+ \subseteq K x_1 K \cdots K x_m K$ is open and non-empty. But then $\mathcal{A}(K V K \cap A^+)$ is also open and non-empty, hence $\mathcal{A}(K x_1 K \cdots K x_m K) = \mathcal{A}(x_1 K \cdots K x_m)$ has non-empty interior, as we desired to show.

Otherwise, $K V K \cap A = \bigcup_{w \in W} \left( K V K \cap \text{bdy}(w(A^+)) \right)$. But the set on the right has Haar measure zero, while the set on the left is open and non-empty, so this is impossible. 

Terminology: Let $G_1/K_1$ and $G_2/K_2$ be two symmetric spaces. We say that $G_1/K_1$ is embedded into $G_2/K_2$ if there is a mapping $\mathcal{I} : G_1 \to G_2$ satisfying the following properties.

Definition 3. (1) $\mathcal{I}$ is a group isomorphism into $G_2$.
(2) $\mathcal{I}$ restricted to $A_1$ is a topological group isomorphism onto $A_2$.
(3) $\mathcal{I}$ maps $K_1$ into $K_2$. 

In passing from the symmetric spaces with one dimensional, restricted root spaces to the other symmetric spaces, there are a few special cases of $L$-tuples that we will also need to handle using this other sufficient condition.

We then turn to necessity. The necessity of eligibility will be seen to follow from elementary linear algebra arguments. For the exceptional tuples, we need other reasoning. The simple fact that the dimension of the underlying orbits are simply not large enough to have a chance to satisfy Prop. 1(3) can often be used.
Property (2) ensures that the symmetric spaces have the same rank. Here are some examples of embeddings.

**Lemma 2.** In the following cases $G_1/K_1$ embeds into $G_2/K_2$:

| Cartan class | $G_1$ | $K_1$ | $G_2$ | $K_2$ |
|--------------|-------|-------|-------|-------|
| $A I$        | $SL(n, \mathbb{R})$ | $SO(n)$ | $SL(n, \mathbb{R})$ | $Sp(n)$ |
| $B D I$      | $SO_0(p, q), q \geq p$ | $SO(p) \times SO(p)$ | $SO_0(p, q + 1), q \geq p$ | $SO(p) \times SO(q)$ |
| $B D I$      | $SO_0(p, q), q \geq p$ | $SO(p) \times SO(q)$ | $SU(p, q), q \geq p$ | $SU(p) \times SU(q)$ |
| $A H I I$    | $SU(p, q), q \geq p$ | $SU(p) \times SU(q)$ | $Sp(p, q), q \geq p$ | $Sp(p) \times Sp(q)$ |
| $T y p e I V$ | $SO(n, \mathbb{C})$ | $SO(n)$ | $SO^*(2n)$ | $U(n)$ |

| Cartan class | $G_1$ | $K_1$ | $G_2$ | $K_2$ |
|--------------|-------|-------|-------|-------|
| $A I I$      | $SO_0(p, q), q \geq p$ | $SO(p) \times SO(p)$ | $SO_0(p, q + 1), q \geq p$ | $SO(p) \times SO(q)$ |
| $B D I$      | $SO_0(p, q), q \geq p$ | $SO(p) \times SO(q)$ | $SU(p, q), q \geq p$ | $SU(p) \times SU(q)$ |
| $A H I I$    | $SU(p, q), q \geq p$ | $SU(p) \times SU(q)$ | $Sp(p, q), q \geq p$ | $Sp(p) \times Sp(q)$ |
| $C I I$      | $SO^*(2n)$ | $U(n)$ | $SO^*(2n)$ | $U(n)$ |

**Proof.** In fact, it is obvious that in all but the last case that the embedding map $\mathcal{I}$ is the identity and $A_1 = A_2$.

For the final case, we remind the reader that $SO(n, \mathbb{C})$ is the set of $n \times n$ complex matrices $g$ satisfying $g' g = Id$ and $SO^*(2n)$ are the matrices in $SO(2n, \mathbb{C})$ with the additional requirement that $g' J_n g = J_n$ where $J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. The subgroup $SO(n)$ embeds into $SO(n, \mathbb{C})$ in the natural way and $U(n)$ embeds into $SO^*(2n)$ as follows: The matrix $X + iY \in U(n)$, where $X,Y$ are real, maps to $\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$.

Here $A_1 = \{ \exp iX : X \in \mathfrak{t} \}$ where $\mathfrak{t}$ is the maximal torus of the Lie algebra of $SO(n)$.

If we define $\mathcal{I} : SO(n, \mathbb{C}) \to SO^*(2n)$ by $\mathcal{I}(g) = \begin{bmatrix} g & 0 \\ 0 & 7 \end{bmatrix}$, then $\mathcal{I}(A_1) = A_2$ and the other conditions of the embedding lemma are also satisfied.

The embedding property is important because we can deduce absolute continuity of certain convolution products of orbital measures in the ‘larger’ space $G_2/K_2$ from the property in the ‘smaller’ symmetric space $G_1/K_1$.

**Proposition 2.** Suppose $G_1/K_1$ is embedded into $G_2/K_2$ with the mapping $\mathcal{I}$.

1. Let $x_1, \ldots, x_m \in A_1$. If $\nu_{x_1} \ast \cdots \ast \nu_{x_m}$ is absolutely continuous on $G_1$ and $z_j = \mathcal{I}(x_j)$, then $\nu_{z_1} \ast \cdots \ast \nu_{z_m}$ is absolutely continuous on $G_2$.
2. Let $X_1, \ldots, X_m \in A_1$. If $(X_1, \ldots, X_m)$ is an absolutely continuous tuple on $p_1$ and $\exp(Z_j) = \mathcal{I}(\exp X_j)$, then $(Z_1, \ldots, Z_m)$ is absolutely continuous on $p_2$.

**Proof.** Put $A_j : G_j \to A_j$ for $j = 1, 2$. One can check from the definitions that $\mathcal{I} \circ A_j = A_2 \circ \mathcal{I}$. As $\mathcal{I}$ is a group isomorphism, for all $k_1, \ldots, k_{m-1} \in K_1$ we have

$$\mathcal{I}(A_1(x_1 k_1 \cdots k_{m-1} x_m)) = A_2(\mathcal{I}(x_1) \mathcal{I}(k_1) \mathcal{I}(x_2) \cdots \mathcal{I}(k_{m-1}) \mathcal{I}(x_m)) \subseteq A_2(z_1 K_{2, z_2} \cdots K_{2, z_m}).$$

Hence $\mathcal{I}(A_1(x_1 K_1 \cdots K_1 x_m)) \subseteq A_2(z_1 K_{2, z_2} \cdots K_{2, z_m})$.

As $\nu_{x_1} \ast \cdots \ast \nu_{x_m}$ is absolutely continuous on $G_1$, Lemma 1 implies there is an open set $V \subseteq A_1$ with $V \subseteq A_1(x_1 K_{2, z_2} \cdots K_{2, z_m})$. Since $\mathcal{I}(V)$ is open in $A_2$ and contained in $A_2(z_1 K_{2, z_2} \cdots K_{2, z_m})$, it follows by another application of the lemma that $\nu_{z_1} \ast \cdots \ast \nu_{z_m}$ is absolutely continuous on $G_2$.

Part (2) follows from (1) and Corollary 1.

**Remark 3.** This idea is implicit in the work of Graczyk and Sawyer, in the special case of the embedding map being the identity.
4.2. Induction argument. Let \( G_n/K_n \) be a symmetric space of rank \( n \), with restricted root system of type \( B_n, C_n, D_n \) or \( BC_n \). Let \( Z \in \mathfrak{a}_n \), say
\[
Z = \left( 0, \ldots, 0, a_1, \ldots, a_m, (\pm) a_m \right) \in \mathfrak{a}_n,
\]
where \( s_1 = \max s_j \). We denote by \( Z' \) the element of \( \mathfrak{a}_{n-1} \) given by
\[
Z' = \begin{cases} 
(0, \ldots, 0, a_1, \ldots, a_m, (\pm) a_m) & \text{if } 2J \geq s_1 \\
(0, \ldots, 0, a_1, \ldots, a_m, (\pm) a_m) & \text{if } 2J < s_1.
\end{cases}
\]

(4.1)

Define \( Z' \) similarly when the restricted root system of \( G_n/K_n \) is type \( A_n \).

We embed \( \mathfrak{a}_{n-1} \) into \( \mathfrak{a}_n \) by taking the standard basis vectors \( e_1, \ldots, e_n \) in \( \mathbb{R}^n \) (or \( e_1 - e_{n+1}, \ldots, e_n - e_{n+1} \) in the case of type \( A_n \)) as the basis for \( \mathfrak{a}_n \) and taking the vectors \( e_2, \ldots, e_n \) (resp., \( e_2 - e_{n+1}, \ldots, e_n - e_{n+1} \)) as the basis for \( \mathfrak{a}_{n-1} \). This also gives a natural embedding of \( \Phi_{n-1} \) into \( \Phi_n \) and together these give an embedding \( \mathfrak{g}_{n-1} \supset \mathfrak{p}_{n-1} \) and \( \mathfrak{t}_{n-1} \) into \( \mathfrak{g}_n \), \( \mathfrak{p}_n \) and \( \mathfrak{t}_n \) respectively, an embedding of \( G_{n-1} \) into \( G_n \), and an embedding of \( K_{n-1} \) into \( K_n \). We will also view \( Z' \) as an element of \( \mathfrak{a}_n \) in the natural way.

With this understanding, put
\[
\Omega_Z = \mathcal{N}_Z \setminus \mathcal{N}_{Z'} \subseteq \mathfrak{p}_n.
\]

**Lemma 3.** If \((X, Y)\) is an eligible pair in \( \mathfrak{a}_n \), and \( X, Y \) are not both of type \( SU(m) \times SU(n) \) in a symmetric space with restricted root system of type \( A_n \) where \( n = 2m - 1 \), then the reduced pair, \((X', Y')\), is eligible in \( \mathfrak{a}_{n-1} \).

**Proof.** The proof is a straightforward calculation. The details are worked out for the compact Lie algebra case in [17, Lemma 3]. \( \square \)

We next adapt the general strategy used in [17, Prop. 2] for the corresponding problem in the classical compact Lie algebra setting.

**Proposition 3. (General Strategy)** Let \( G_n/K_n \) be a symmetric space of rank \( n \), with associated Lie algebra \( \mathfrak{g}_n = \mathfrak{t}_n \oplus \mathfrak{p}_n \) and maximal abelian subspace \( \mathfrak{a}_n \). Let \( X_i \in \mathfrak{a}_n \), \( i = 1, \ldots, L \) for \( L \geq 2 \) and assume \((X_1', \ldots, X_L')\) is an absolutely continuous tuple in \( \mathfrak{p}_{n-1} \).

Let \( \mathcal{V}_j = \mathfrak{p}_j \oplus \mathfrak{a}_j \) for \( j = n - 1, n \). Suppose \( \Omega \) is a subset of \( \mathcal{V}_n \cap \mathcal{V}_{n-1} \) that contains all \( \Omega_{X_i} \) and has the property that \( \text{ad}(H)(\Omega) \subseteq \text{sp}(\Omega) \) whenever \( H \in \mathfrak{t}_{n-1} \). Fix \( \Omega_0 \subseteq \Omega_{X_i} \) and assume there exists \( k_1, \ldots, k_{L-1} \in K_{n-1} \) and \( M \in \mathfrak{t}_n \) such that
(i) \( \text{sp}\{\text{Ad}(k_i)(\Omega_{X_i}), \Omega_{X_i} \setminus \Omega_0 : i = 1, \ldots, L - 1\} = \text{sp}(\Omega) \);
(ii) \( \text{ad}^k(M) : \mathcal{N}_{X_i} \setminus \Omega_0 \rightarrow \text{sp}(\Omega, \mathfrak{p}_{n-1}) \) for all positive integers \( k \); and
(iii) The span of the projection of \( \text{Ad}^s(M)(\Omega_0) \) onto the orthogonal complement of \( \text{sp}(\mathfrak{p}_{n-1}, \Omega) \) in \( \mathfrak{p}_n \) is a surjection for all small \( s > 0 \).

Then \((X_1, \ldots, X_L)\) is an absolutely continuous tuple in \( \mathfrak{p}_n \).

**Proof.** As \((X_1', \ldots, X_L')\) is an absolutely continuous tuple, Prop. [1] tells us that
\[
\text{sp}\{\text{Ad}(h_i)(\Omega_{X_i}), \Omega_{X_i} \setminus \Omega_0 : i = 1, \ldots, L - 1\} = \mathfrak{p}_{n-1}
\]
for a dense set of \((h_1, \ldots, h_{L-1}) \in K_{n-1}^{L-1} \). Given \( \varepsilon > 0 \), choose such \( h_i = h_i(\varepsilon) \in K_{n-1} \) with \( \| \text{Ad}(h_i) - \text{Ad}(k_i) \| < \varepsilon \), where the elements \( k_i \in K_{n-1} \) are the ones given in the hypothesis of the proposition and the norm is the operator norm.
An elementary linear algebra argument, together with assumption (i), shows that for sufficiently small \( \varepsilon > 0 \),

\[
\dim(sp\Omega) = \dim(sp\{Ad(h_i)(\Omega_{X_i}), \Omega_{X_i}\setminus\Omega_0 : i = 1, \ldots, L - 1\}).
\]

Since \( ad(H)(\Omega) \subseteq sp(\Omega) \) for all \( H \in \mathfrak{e}_{n-1} \) and \( h_i = \exp H_i \) for some \( H_i \in \mathfrak{e}_{n-1} \) we have \( Ad(h_i)(\Omega) = \exp(ad(H_i)(\Omega)) \subseteq sp(\Omega) \) for all \( h_i \in K_{n-1} \). Thus for sufficiently small \( \varepsilon > 0 \),

\[
sp\{Ad(h_i)(\Omega_{X_i}), \Omega_{X_i}\setminus\Omega_0 : i = 1, \ldots, L - 1\} = sp\Omega.
\]

For such a choice of \( \varepsilon \) (now fixed) we have

\[
sp\{Ad(h_i)(\mathcal{N}_{X_i}), \mathcal{N}_{X_i}\setminus\Omega_0 : i = 1, \ldots, L - 1\} = sp\{\Omega, p_{n-1}\}.
\]

Assumption (ii) and the fact that \( \mathcal{N}_{X_i}\setminus\Omega_0 \subseteq sp\{\Omega, p_{n-1}\} \) implies that for any \( s > 0 \), \( exp(s \cdot adM) = Ad(exp sM) \) maps \( \mathcal{N}_{X_i}\setminus\Omega_0 \) to \( sp\{\Omega, p_{n-1}\} \). Moreover, \( \|Id - Ad(exp sM)\| \to 0 \) as \( s \to 0 \), thus similar reasoning to that above shows that for all small enough \( s > 0 \),

\[
sp\{\Omega, p_{n-1}\} = sp\{Ad(h_i)(\mathcal{N}_{X_i}), Ad(exp sM)(\mathcal{N}_{X_i}\setminus\Omega_0) : i = 1, \ldots, L - 1\} = p_n,
\]

with the final equality coming from (iii). Another application of Prop. [1] proves that \( \mu_{X_1} \cdots \mu_{X_L} \) is absolutely continuous. \( \Box \)

We will now focus on the symmetric spaces all of whose restricted root spaces have dimension one. These are the symmetric spaces of Cartan type \( AI, CI, DI \) and \( BI \), the latter two in the cases when the symmetric space is \( SO_0(p+q)/SO(p) \times SO(q) \) with \( q = p \) and \( q = p + 1 \) respectively.

For such spaces we will introduce the notation \( E_\alpha \) for a (fixed choice of) basis vector of the restricted root space \( \mathfrak{g}_\alpha \), \( \alpha \in \Phi^+ \), and put

\[
E_\alpha^+ = E_\alpha + \theta E_\alpha, \quad E_\alpha^- = E_\alpha - \theta E_\alpha.
\]

The following is well known and very important for us:

\[
[E_\alpha^+, E_\beta^-] = cE_{\alpha + \beta}^- + dE_{\alpha - \beta}^-
\]

where \( c \) (or \( d \)) \( \neq 0 \) if \( \alpha + \beta \) (respectively, \( \alpha - \beta \)) is a restricted root and \( E_{\alpha + \beta}^- \) (or \( E_{\alpha - \beta}^- \)) should be understood to be the zero vector if \( \alpha + \beta \) (resp. \( \alpha - \beta \)) is not a restricted root. Furthermore,

\[
0 \neq [E_\alpha^+, E_\alpha^-] \in \mathfrak{a}
\]

and if for some subset of roots \( I \), \( \{\alpha : \alpha \in I\} \) spans \( sp \Phi \), then \( \{ [E_\alpha^+, E_\alpha^-] : \alpha \in I \} \) spans \( \mathfrak{a} \).

Here is the key induction argument, the most significant step in the proof of sufficiency.

**Theorem 2.** Assume \( G_n/K_n \) is a Type III symmetric space of rank \( n \), whose restricted root spaces all have dimension one. Suppose \( (X, Y) \) is an eligible pair in \( \mathfrak{a}_n \). Assume that \( X \) and \( Y \) are not both of type \( SU(n) \) when the restricted root system is type \( D_n \) and are not both of type \( SU(m) \times SU(m) \) for \( n = 2m - 1 \) when the restricted root system is type \( A_{2m-1} \). Assume, also, that the reduced pair, \( (X', Y') \), is an absolutely continuous pair in \( p_{n-1} \). Then \( (X, Y) \) is an absolutely continuous pair in \( p_n \).
Proof. As mentioned above, these are the symmetric spaces of Cartan type $AI, CI$, $DI$ ($q = p$) and $BI$ ($q = p + 1$) and hence their restricted root systems are types $A_n, C_n, D_n$ (with $n = p$) and $B_n$ (with $n = p$), respectively.

The proof of the theorem is essentially the same as that given in [12] Prop. 3 for orbital measures in the classical, compact Lie algebras, with an appropriate change of notation. But as the ideas are so important for this paper we will present a condensed overview of the arguments for the restricted root spaces of type $B_n, C_n$ or $D_n$. Type $A_n$ is similar to case 1(a) below, but easier, and is left for the reader.

Different arguments will be needed depending on the dominant type of $X$ and $Y$.

Case 1: Neither $X$ nor $Y$ are dominant $SU$ type.

Suppose $S_X = 2J$ and $S_Y = 2N$. Applying a Weyl conjugate if necessary (which corresponds to the $Ad$-action of an element in $K$) we can assume, without loss of generality, that

$$
\Omega_X = \{E_{e_1}^+ : j > J\} \text{ and } \Omega_Y = \{E_{e_1}^- : j > N\}.
$$

Case 1(a): The restricted root system is type $D_n$. Put

$$
\Omega = \{E_{e_1}^+ : j = 2, ..., n\} \text{ and } \Omega_0 = \{E_{e_1}^-\}.
$$

Property (4.2) ensures that $ad(H)(\Omega) \subseteq sp\Omega$ whenever $H \in \mathfrak{t}_{n-1}$.

Take $k \in K_{n-1}$ the Weyl conjugate that permutes the letters $1 + j$ with $N + j$ for $j = 1, ..., J - 1$. The eligibility assumption ensures $\{Ad(k)(\Omega_Y), \Omega_X \setminus \Omega_0\} = \Omega$, thus Prop. (3) is satisfied.

Set $M = E_{e_1}^+ \in \mathfrak{t}_n$ and note that Prop. (3)(ii) is also met.

The projection of $ad(M)(E_{e_1}^-)$ maps onto the orthogonal complement of $sp\{\Omega, p_{n-1}\}$ in $\mathfrak{p}_n$ since $sp\{\Omega, p_{n-1}\}$ is of co-dimension one, thus (iii) is also fulfilled with any $s > 0$. Applying Prop. (3) we conclude that $\mu_X * \mu_Y$ is absolutely continuous.

Case 1(b): The restricted root space is type $B_n$. The arguments are similar. Take

$$
\Omega = \{E_{e_1}^+ : j = 2, ..., n\} \text{ and } \Omega_0 = \{E_{e_1}^-\}.
$$

Let $k \in K_{n-1}$ be the Weyl conjugate that permutes the letters $1 + j$ with $N + j$ for $j = 1, ..., J - 1$, as in the previous case, and let $k_t = (\exp t E_{e_1}^+) k \in K_{n-1}$ for small $t > 0$. Since

$$
Ad(k_t)(E_{e_1}^-) = a(t)E_{e_1}^- + tb(t)E_{e_1}^- + t^2 c(t)E_{e_1}^- + \mathfrak{a}_n
$$

where $a(t) \to 1$ as $t \to 0$, and $b(t), c(t)$ converge to non-zero scalars, one can deduce that

$$
sp\{Ad(k)(\Omega_Y), \Omega_X \setminus \Omega_0\} = sp\Omega.
$$

Now take $M = E_{e_1}^+$ and apply the general strategy.

Case 1(c): The restricted root space is type $C_n$. Here we begin with

$$
\Omega = \{E_{e_1}^+ : j = 2, ..., n\} \text{ and } \Omega_0 = \{E_{e_1}^-\},
$$

and let $k \in K_{n-1}$ be the Weyl conjugate that permutes the letters $1 + j$ with $N + j$ for $j = 1, ..., J - 1$. As in the proof of the general strategy, the induction assumption implies there is some $h \in K_{n-1}$ such that

$$
sp\{Ad(h)\Omega_Y, \Omega_X \setminus \Omega_0\} = sp\{\Omega, p_{n-1}\}.
$$
We again take \( M = E_{e_1}^+ + e_n \in \mathfrak{g}_n \), but in this case cannot call directly upon the general strategy as it is not true that \( ad^k(M)(H) \in sp\{\Omega, p_{a-1}\} \) for all \( H \in \mathcal{N}_X \setminus \Omega_0 \). However, one can check that for small \( s > 0 \),
\[
sp\{E_{e_1}^-, E_{e_2}^-, \ldots, E_{e_n}^-\} \cap \Omega_0 \subseteq \mathcal{N}_X \setminus \Omega_0
\]
and using this fact it can be shown that
\[
sp\{Ad(h)\mathcal{N}_Y, Ad(\exp sM)(\mathcal{N}_X \setminus \Omega_0)\} = p_n \cap sp(a_1)
\]
where \( a_1 \) is the standard basis vector of \( a_n \oplus a_{n-1} \). Standard arguments then allow one to deduce that for small enough \( s > 0 \),
\[
sp\{Ad(h)\mathcal{N}_Y, Ad(\exp sM)(\mathcal{N}_X)\} = p_n.
\]
For the details of this technical argument we refer the reader to the proof of Prop. 3 Case 1(c) in [17].

Case 2: Both \( X, Y \) are dominant type \( SU \). First, assume the restricted root space is either \( B_n \) or \( C_n \). Let
\[
\Omega = \{E_{e_1}^\pm, E_{(2)e_1}^\pm : j = 2, \ldots, n\} \text{ and } \Omega_0 = \{E_{(2)e_1}^\pm\}
\]
(with the choice \( E_{(2)e_1}^\pm \) depending on whether the underlying root system is type \( C_n \) or \( B_n \)). Applying a Weyl conjugate change of sign, as needed, there is no loss of generality in assuming \( \Omega_X \) contains all \( E_{e_1}^\pm, E_{j}^\pm \) for \( j \geq 2 \) and \( E_{(2)e_1}^\pm \), and \( \Omega_Y \) contains all \( E_{e_1}^\pm, E_{j}^\pm \) for \( j \geq 2 \) and (again) \( E_{(2)e_1}^\pm \). Hence \( \{\Omega_Y, \Omega_X \setminus \Omega_0\} = \Omega \). Now take \( M = E_{(2)e_1}^\pm \).

If, instead, the restricted root space is \( D_n \), then we take \( \Omega = \{E_{e_1}^\pm, E_{j}^\pm : j = 2, \ldots, n\} \). Since \( X \) and \( Y \) are not both of type \( SU(n) \), we can assume \( \Omega_X \) contains all the roots \( E_{e_1}^\pm, E_{j}^\pm \) for \( j \geq 2 \) and both \( E_{e_1}^\pm, E_{n+1}^\pm \), and \( \Omega_Y \) contains all \( E_{e_1}^\pm, E_{j}^\pm \) for \( 2 \leq j \leq n-1 \) and at least one of \( E_{n+1}^\pm \). Take \( \Omega_0 \) to be the one of \( E_{e_1}^\pm \) that belongs to \( \Omega_Y \) and argue as above.

Case 3: \( X, Y \) are of different dominant type, say \( X \) is dominant \( SU \) type with \( S_X = m \) and \( S_Y = 2J \). Take \( \Omega = \mathcal{V}_n \oplus \mathcal{V}_{n-1} \). Applying suitable Weyl conjugates, we can assume
\[
\Omega_X = \{E_{e_1, e_j}^-, E_{e_1-e_n, (2)e_1}^-, E_{(2)e_1}^\pm : j \geq 2, k > m\} \text{ and } \Omega_0 = \{E_{e_1, e_j}^-, E_{e_1-e_n, (2)e_1}^-, E_{(2)e_1}^\pm : 2 \leq j \leq n-J+1\}
\]
(with appropriate modifications in type \( D_n \)). Put
\[
\Omega_0 = \{E_{e_1+e_n-j+1}^\pm\} \subseteq \Omega_X \cap \Omega_Y.
\]
If \( n-J+1 \geq m \), then \( \{\Omega_Y, \Omega_X \setminus \Omega_0\} = \Omega \) and the rest of the argument is easy. Otherwise, let
\[
\Omega_1 = \{E_{e_1+e_k}^- : 2 \leq k \leq n-J\} \subseteq (\Omega_Y \cap \Omega_X) \setminus \Omega_0.
\]
Define
\[
H = \sum_{j=2}^{m+n-J} E_{e_j+e_n-j}^+.
\]
The eligibility condition implies \( \Omega_1 \supset \Omega_Y \cap \Omega_0 \), \( H \subseteq \Omega_Y \cap \Omega_0 \). It follows that
\[
sp\{ad(H)(\Omega_1), \Omega_Y \setminus \Omega_1, \Omega_X \setminus \Omega_0\} = sp\Omega.
\]
A linear algebra argument implies there is some \( k \in K_{n-1} \) (namely, \( k = \exp tH \) for sufficiently small \( t > 0 \)) such that

\[
sp\{Ad(k)(\Omega_Y), \Omega_X \setminus \Omega_0\} = sp\Omega.
\]

Now, take \( M = E_{-e_1+e_{n-J+1}}^+ \) and apply the general strategy to complete the argument. \( \Box \)

### 4.3. Another sufficient condition

To use the induction argument outlined in the previous subsection we will, of course, need to do the base case s. A sufficient condition for absolute continuity that will be helpful to us for in doing this (and also for dealing with some special tuples when the restricted root spaces are higher dimensional) is the following variant of a result of [31].

By the rank of a root subsystem we mean the dimension of the Euclidean space it spans. By the dimension of a root subsystem \( \Phi_0 \), we mean

\[
dim \Phi_0 := \dim sp\{X_{-\alpha}: X_\alpha \in g_\alpha, \alpha \in \Phi_0\}.
\]

This is the cardinality of \( \Phi_0^+ \) counted by multiplicity of the corresponding restricted root spaces. When the multiplicities of all the restricted root spaces coincide, say are equal to \( r \), then \( \dim \Phi_0 = r \cdot card(\Phi_0^+) \).

**Theorem 3.** Assume \( G/K \) is a symmetric space with restricted root system \( \Phi \) and Weyl group \( W \). Suppose \( Z_1, ..., Z_m \in a \). Assume

\[
(m - 1)(\dim \Phi - \dim \Psi) - 1 \geq \sum_{i=1}^m \left( \dim \Phi_{Z_i} - \min_{\sigma \in W} \dim(\Phi_{Z_i} \cap \sigma(\Psi)) \right)
\]

for all root subsystems \( \Psi \subseteq \Phi \) of co-rank 1 and having the property that \( sp(\Psi) \cap \Phi = \Phi \). Then \( \mu_{Z_1} \ast \cdots \ast \mu_{Z_m} \) is absolutely continuous.

We first prove several lemmas. Throughout, \( Z_1, ..., Z_m \in a \) will be fixed. For the proof, denote by

\[
n_X = ker(adX)|_p = \{Y \in p: [X, Y] = 0\}.
\]

**Lemma 4.** The sum \( O_{Z_1} + \cdots + O_{Z_m} \) has non-empty interior in \( p \) if and only if there exist \( k_1, ..., k_m \in K \) such that

\[
\bigcap_{j=1}^m \text{Ad}(k_j)n_{Z_j} = \{0\}.
\]

**Proof.** This is a Hilbert space argument taking the inner product given by the Killing form. From Prop. \( \Box \) \( O_{Z_1} + \cdots + O_{Z_m} \) has non-empty interior if and only if there is some \( k_1 = Id, k_2, ..., k_m \in K \) such that

\[
sp\{Ad(k_j)n_{Z_j} : j = 1, ..., m\} = p.
\]

We note that \( spN_{Z_j} = \text{Im ad}(Z_j)|_t \). Thus (4.4) holds if and only if

\[
p = \sum_{j=1}^m \text{Ad}(k_j)\text{Im ad}(Z_j)|_t = \sum_{j=1}^m \text{Ad}(k_j) (\ker(ad(Z_j)|_p))^\perp
\]

\[
= \sum_{j=1}^m \text{Ad}(k_j) (n_{Z_j})^\perp = \left( \bigcap_{j=1}^m \text{Ad}(k_j)n_{Z_j} \right)^\perp,
\]

where the orthogonal complements are all understood to be in \( p \). \( \Box \)
We call $Z \in \mathfrak{p}$ maximally singular if whenever $W = \text{Ad}(k)Z \in \mathfrak{a}$ for $k \in K$, then $\Phi_W$ is of co-rank one. This is equivalent to saying $O_Z$ contains an element in $\mathfrak{a}$ whose set of annihilating roots is a co-rank one root subsystem.

**Lemma 5.** If the intersection
\[
\bigcap_{j=1}^{m} \text{Ad}(k_j) n_{Z_j} \neq \{0\}
\]
for some $k_j \in K$, then the intersection contains a maximally singular element.

**Proof.** Suppose $Z \in \bigcap_{j=1}^{m} \text{Ad}(k_j) n_{Z_j}$ for some $Z \neq 0$. Choose a maximal abelian subalgebra $\mathfrak{a}'$ of $\mathfrak{p}$ that contains $Z$ and let
\[
\mathfrak{a}'_Z = \{ H \in \mathfrak{a}' : \alpha(H) = 0 \text{ for all } \alpha \in \Phi_Z \},
\]
where we understand the root system $\Phi$ to be with respect to this subalgebra $\mathfrak{a}'$.

For each $\alpha \in \Phi^+$, choose bases $\{E^{(i)}_\alpha : i \in I_\alpha\}$ for the restricted root spaces $\mathfrak{g}_\alpha$.

Temporarily fix an index $j$. Since $\text{Ad}(k_j)Z_j \in \mathfrak{p}$, we can find $H \in \mathfrak{a}'$ and coefficients $c^{(i)}_\alpha$ (depending on $j$) such that
\[
\text{Ad}(k_j)Z_j = H + \sum_{\alpha \in \Phi^+} \sum_{i \in I_\alpha} c^{(i)}_\alpha E^{(i)}_\alpha.
\]

Now, $Z \in \text{Ad}(k_j) n_{Z_j}$, hence there is some $Y_j \in n_{Z_j}$ such that $Z = \text{Ad}(k_j)Y_j$. Thus
\[
[Z, \text{Ad}(k_j)Z_j] = [\text{Ad}(k_j)Y_j, \text{Ad}(k_j)Z_j] = \text{Ad}(k_j)[Y_j, Z_j] = 0.
\]

But we also have
\[
[Z, \text{Ad}(k_j)Z_j] = [Z, H + \sum_{\alpha \in \Phi^+} \sum_{i \in I_\alpha} c^{(i)}_\alpha E^{(i)}_\alpha] = \sum_{\alpha \in \Phi^+} \sum_{i \in I_\alpha} \alpha(Z)c^{(i)}_\alpha E^{(i)}_\alpha.
\]

It follows that $c^{(i)}_\alpha = 0$ for all $\alpha \in \Phi^+$ such that $\alpha(Z) \neq 0$, i.e., for all $\alpha \notin \Phi_Z$. Hence
\[
\text{Ad}(k_j)Z_j = H + \sum_{\alpha \in \Phi^+} \sum_{i \in I_\alpha} c^{(i)}_\alpha E^{(i)}_\alpha.
\]

Pick $H' \in \mathfrak{a}'_Z$. Since $\alpha(H') = 0$ for all $\alpha \in \Phi_Z$, we have
\[
[H', \text{Ad}(k_j)Z_j] = [H', H + \sum_{\alpha \in \Phi^+} \sum_{i \in I_\alpha} c^{(i)}_\alpha E^{(i)}_\alpha] = 0.
\]

Thus $\mathfrak{a}'_Z \subseteq \text{Ad}(k_j) n_{Z_j}$ for all $j$ and hence is contained in $\bigcap_{j=1}^{m} \text{Ad}(k_j) n_{Z_j}$. To complete the proof, simply choose a maximally singular element in $\mathfrak{a}'_Z$. \hfill $\square$

**Remark 4.** Note that the proof actually shows that the intersection contains all the elements of $\mathfrak{a}'_Z$ whose set of annihilating roots is co-rank one in $\mathfrak{a}'$.

There are only finitely many co-rank one root subsystems of the (original) restricted root system $\Phi$, so we may choose a finite set $S \subseteq \mathfrak{a}$ such that $\{ \Phi_Z : Z \in S \}$ is the complete set. For each $Z \in S$, consider the map $f_Z : O_Z \times K^m \to \mathfrak{t}^m$ defined by
\[
f_Z(Z', k_1, ..., k_m) = ([Z', \text{Ad}(k_1)Z_1], ..., [Z', \text{Ad}(k_m)Z_m]).
\]
Note that \( f_Z'(Z', k_1, \ldots, k_m) = 0 \) if and only if \( Z' \in \bigcap_{j=1}^m \text{Ad}(k_j)n_{Z_j} \).

For \( Z \in \mathfrak{a} \), set
\[
G_Z = \{ g \in G : \text{Ad}(g)Z = Z \} \quad \text{and} \quad K_Z = \{ k \in K : \text{Ad}(k)Z = Z \}.
\]

The associated Lie algebras are given by:
\[
\mathfrak{g}_Z = \{ X \in \mathfrak{g} : [X, Z] = 0 \} \quad \mathfrak{k}_Z = \{ X \in \mathfrak{k} : [X, Z] = 0 \} \quad \mathfrak{p}_Z = \{ X \in \mathfrak{p} : [X, Z] = 0 \}.
\]

Let \((G_Z)_0\) and \((K_Z)_0\) be the connected components containing the identity of \( G \). Their Lie algebras are also \( \mathfrak{g}_Z \) and \( \mathfrak{k}_Z \), respectively.

**Lemma 6.** The pair \((G_Z)_0/(K_Z)_0\) is a symmetric space whose rank is equal to that of the dimension of \( \mathfrak{a} \). Moreover, \( \mathfrak{g}_Z = \mathfrak{k}_Z \oplus \mathfrak{p}_Z \).

**Proof.** Let \( h \in (G_Z)_0 \) and pick \( H \in (\mathfrak{g}_Z)_0 \) such that \( h = \exp H \). Standard facts imply
\[
\text{Ad}(\theta h)Z = \text{Ad}(\theta \exp H)Z = \exp(\text{ad}(\theta)_{\mathfrak{e}} H)Z = Z
\]
since \([d\theta]_{\mathfrak{e}} H, Z] = 0\). Thus \( \theta h \in (G_Z)_0 \).

Since \( Z \in \mathfrak{a} \), \([X, Z] = 0 \) for all \( X \in \mathfrak{a} \). Thus \( \mathfrak{a} \subseteq \mathfrak{p}_Z \) and hence must be a maximal abelian subalgebra. \( \square \)

**Lemma 7.** The rank of \( Df_Z \) at \((Z', k_1, \ldots, k_m) \in f_Z^{-1}(0)\) is at least
\[
\sum_{j=1}^m \min_{\sigma \in \mathcal{W}} \dim \left( N_Z \cap N_{\sigma(Z_j)} \right).
\]

**Proof.** Fix \((Z', k_1, \ldots, k_m) \in f_Z^{-1}(0)\). Consider the \( j \)’th inclusion map: \( K \to O_Z \times K^m \) given by \( k \mapsto (Z', k_1, \ldots, k_{j-1}, k, k_{j+1}, \ldots, k_m) \) and let \( f_Z^{(j)} : K \to \mathfrak{t}^m \) be the composition of this inclusion with \( f_Z \).

As the derivative of \( f_Z^{(j)} \) lies in the \( j \)’th coordinate of \( \mathfrak{t}^m \),
\[
\text{rank} Df_Z \geq \sum_{j=1}^m \text{rank} Df_Z^{(j)}.
\]

Suppressing the unused components of the domain of \( f_Z^{(j)} \), we can write \( f_Z^{(j)}(k) = [Z', \text{Ad}(k)Z_j] \). We will compute the rank of \( f_Z^{(j)} \) at \( k_j \).

We claim that there is a \( k \in K \) such that \( \text{Ad}(k)Z' = Z \) and \( \text{Ad}(kk_j)Z_j = \sigma(Z_j) \) for some \( \sigma \in W \). To see this, choose \( h_1 \in K \) such that \( \text{Ad}(h_1)Z' = Z \) and consider \([Z, \text{Ad}(h_1)k_j]Z_j \). As \( \text{Ad}(h_1k_j)Z_j \in \mathfrak{p}_Z \), there is some \( h_2 \in (K_Z)_0 \) such that \( \text{Ad}(h_2)\text{Ad}(h_1k_j)Z_j \in \mathfrak{a} \). Set \( k = h_2h_1 \). By construction, \( \text{Ad}(k)Z' = Z \). Since \( kk_j \in K \) and \( \text{Ad}(kk_j)Z_j \in \mathfrak{a} \), the claim now follows from the fact that if two elements, \( Q, Q' \), of \( \mathfrak{a} \) are \( \text{Ad}(K) \) related, then there is an element \( \sigma \in W \) with \( \sigma(Q) = Q' \).

Since the rank of \( f_Z^{(j)} \) is \( \text{Ad} \)-invariant, we can assume that \( Z' = Z \) and \( \text{Ad}(k_j)Z_j = \sigma(Z_j) \).
For $X \in \mathfrak{k}$, we have
\[(Df_Z^{(j)})_{k_j}(X) = [Z, \text{Ad}(k_j)[X, Z_j]] = [Z, [\text{Ad}(k_j)X, \sigma(Z_j)]]\]
Thus
\[
\text{Im}(Df_Z^{(j)})_{k_j} = [Z, N_{\sigma(Z_j)}] = \text{sp}\left\{X_{\alpha}^+ : X_{\alpha} \in \mathfrak{g}_{\alpha}, \alpha \in \Phi_Z \cap \Phi_{\sigma(Z_j)}^c\right\}
\]
and this has the same dimension as $N_Z \cap N_{\sigma(Z_j)}$. \hfill \Box

Proof. [of Theorem] The proof of the theorem can now be completed as in [31]. The hypothesis of the theorem implies that
\[
\sum_{j=1}^m \min_{\sigma \in W} \dim (N_Z \cap N_{\sigma(Z_j)}) > \dim N_Z
\]
for all maximally singular elements $Z$, thus the rank of $f_Z$ at any element of $f_Z^{-1}(0)$ is greater than the dimension of $O_Z$. Consequently, $f_Z^{-1}(0)$ has dimension less than $K^m$. If we let $\pi_Z : O_Z \times K^m \to K^m$ be the projection, then $\pi_Z(f_Z^{-1}(0))$, and thus also $\bigcup_{Z \in S} \pi_Z(f_Z^{-1}(0))$, has measure zero in $K^m$. If $(k_1, \ldots, k_m) \notin \bigcup_{Z \in S} \pi_Z(f_Z^{-1}(0))$, then $\bigcap_{j=1}^m \text{Ad}(k_j)n_{Z_j} = \{0\}$ and hence $O_{Z_1} + \cdots + O_{Z_m}$ has non-empty interior. \hfill \Box

4.4. Completion of the proof of sufficiency for symmetric spaces with all one-dimensional, restricted root spaces. In this subsection we will apply the previous results to prove that eligible, non-exceptional $L$-tuples in symmetric spaces with (all) one-dimensional, restricted root spaces are absolutely continuous. We begin with two lemmas that will allow us to establish the base cases.

We call $Z \in \mathfrak{a}$ a regular if $\Phi_Z$ is empty.

Lemma 8. If $Z$ is a regular element and $Y \in \mathfrak{a}$ is non-zero, then $\mu_Z \ast \mu_Y$ is absolutely continuous.

Proof. The proof given in [14] can be easily adapted or see [8]. \hfill \Box

Lemma 9. Any eligible, non-exceptional pair, $(X, Y)$, in the symmetric space $G_n/K_n$ where $G_n = \text{SO}_0(n, n)$, $K_n = \text{SO}(n) \times \text{SO}(n)$ for $n = 3, 4, 5$, is absolutely continuous.

Proof. When $n = 3$, the only eligible, non-exceptional pairs where neither $X$ nor $Y$ are regular are the pairs of type $(D_2, \text{SU}(2))$ and $(\text{SU}(2), \text{SU}(2))$. Since the annihilators of any element of type $\text{SU}(2)$ are contained in the annihilators of an element of type $D_2$, it suffices to check that the former pair is absolutely continuous. For this one can easily verify (4.3).

The induction argument, Theorem [2], can then be called upon to see that all eligible pairs $(X, Y)$ in $G_n/K_n$ with $n = 4$ are absolutely continuous, except those for which $X'$ is type $\text{SU}(3)$ and $Y'$ is either $\text{SU}(3)$) or $\text{SU}(2)$. These are the pairs $(X, Y)$ of types $(\text{SU}(4), \text{SU}(4))$, $(\text{SU}(4), \text{SU}(3))$, $(\text{SU}(4), D_2 \times \text{SU}(2))$, or $(\text{SU}(4), \text{SU}(2) \times \text{SU}(2))$. Notice that all these are exceptional pairs, except for the pairing $X$ of type $\text{SU}(4)$ and $Y$ of type $\text{SU}(2) \times \text{SU}(2)$ with $\Phi_Y$ not Weyl conjugate to a subset of $\Phi_X$. For the last pair we can easily check (4.3) is satisfied.

The arguments are similar for $n = 5$; it suffices to check the Wright criterion for a pair of type $(\text{SU}(5), D_2 \times \text{SU}(3))$ and this can be done as in [17, Lemma 6]. \hfill \Box
Theorem 4. Suppose $G/K$ is a symmetric space whose restricted root spaces are all of dimension one. If $(X,Y)$ is an eligible, non-exceptional pair of non-zero elements in $\mathfrak{a}$, then $(X,Y)$ is an absolutely continuous pair.

Proof. This will be an induction argument based on the rank of the symmetric space. The previous lemma establishes the result for the symmetric spaces with $G_n = SO_0(n,n)$ and $K_n = SO(n) \times SO(n)$ (restricted root space type $D_n$) for $n = 3, 4, 5$. Hence for this family of symmetric spaces we can start the induction argument with rank $n = 6$, taking $n = 5$ as the base case. All the other symmetric spaces under consideration have restricted root spaces of types $A_n$, $B_n$ or $C_n$. For all of these we can take $n = 1$ as the base case and there the result trivially holds by Lemma 8 since any non-zero element is regular.

We now assume inductively that the result holds for symmetric spaces of rank $n - 1$ and proceed to consider the problem for rank $n$. Assume $(X,Y)$ is an eligible, non-exceptional pair of non-zero elements in $\mathfrak{a}_n$. From Lemma 3 we know the reduced pair, $(X',Y')$, is also eligible. If the restricted root system is type $B_n$ or $C_n$, then clearly $(X',Y')$ is not exceptional. If the restricted root system is type $D_n$ (with $n \geq 6$) then $(X',Y')$ can only be exceptional if $X'$ is type $SU(n-1)$ and $Y'$ is either that type or type $SU(n-2)$. But this happens only if $X$ is of type $SU(n)$ and $Y$ is either type $SU(n)$ or $SU(n-1)$ which is not true as the pair $(X,Y)$ is not exceptional. Lastly, we remark that in the case of type $A_n$, we can be sure the pair $(X',Y')$ is not exceptional because if $X',Y'$ were both of type $SU(n/2) \times SU(n/2)$ in the symmetric space with restricted root system of type $A_{n-1}$, then $X,Y$ would both be type $SU(n/2 + 1) \times SU(n/2)$ and that’s not an eligible pair in the original symmetric space.

The induction hypothesis thus implies that $(X',Y')$ is an absolutely continuous pair. Appealing to Theorem 9 we conclude that the same is true for $(X,Y)$.   

We now turn to the problem of $L \geq 3$ where the results are new for all types. Type $D_n$ is the most complicated because of the exceptional cases. (The exceptional pairs present difficulties even for dealing with $L \geq 3$.) These problems were also addressed for the Lie algebra case in [17] (see especially Lemmas 6, 7 in that paper), but in [17] some of the arguments relied upon $L^2$ density results for convolutions of orbital measures and such results are generally unknown in the symmetric space setting.

We begin the argument with several technical lemmas which will enable us to address these complications.

Lemma 10. Consider the symmetric space of Cartan type $DI$, $SO_0(n,n)/SO(n) \times SO(n)$ with $n \geq 3$. Let $X,Y \in \mathfrak{a}_n$ be of dominant SU type and $Z \in \mathfrak{a}_n$ be non-zero.

1. When $n = 4$, the triple $(X,Y,Z)$ is absolutely continuous if $X,Y,Z$ are all type $SU(4)$, but their annihilating root systems are not Weyl conjugates.
2. Suppose $n \geq 4$. If $(X',Y',Z')$ is an absolutely continuous triple, then $(X,Y,Z)$ is also absolutely continuous.
3. If $n \geq 4$ and $Z$ is also of dominant SU type, then $(X,Y,Z)$ is an absolutely continuous triple, except if $n = 4$ and all three of $X,Y,Z$ are type $SU(4)$ with Weyl conjugate sets of annihilating roots.
4. If $n \geq 4$ and $Z$ is not of dominant SU type, then $(X,Y,Z)$ is absolutely continuous.
Proof: (1) We use the criterion of [13] to prove this. The root subsystems of rank 3 in $D_4$ are of type $D_2 \times SU(2)$, $D_3$ and $SU(4)$. The key points to observe are:

(i) the intersection of any positive root systems of type $SU(4)$ with one of type $D_1 \times SU(4-J)$ has cardinality at least 1 if $J = 2$ and at least 3 if $J = 3$;

(ii) the intersection of any two Weyl conjugate positive root systems of type $SU(4)$ has cardinality at least four; the intersection of any two non-Weyl conjugate positive root systems of type $SU(4)$ has cardinality at least six.

(2) We will use the notation of Theorem 2. Let $\Omega = \{E_\alpha : \alpha = e_1 \pm e_j, 2 \leq j \leq n\}$. Without loss of generality we can assume $\Omega_X \supseteq \{E_\alpha : \alpha = e_1 + e_j, 2 \leq j \leq n\}$ and $\Omega_Y$ either contains the same set again or $\Omega_Y \supseteq \{E_\alpha, E_{e_1-e_n} : \alpha = e_1 + e_j, 2 \leq j \leq n-1\}$. Let $k \in K_n$ be a Weyl conjugate that changes the signs of $2, ..., n-1$ (and $n$ if necessary, to be an even sign change). Then $\Omega_X \cup \Omega_Y$ contains all of $\Omega$, except possibly $E_{e_1-e_n}$.

If $Z$ is not type $SU(n)$, applying a Weyl conjugate, if needed, we can assume $\Omega_Z \supseteq \{E_\alpha : \alpha = e_1 \pm e_n\}$. We take $\Omega_0 = \{E_{\pm e_1} \pm e_n\}$ and $M = E_{e_1+e_n}^\pm$. By assumption, $(X', Y', Z')$ is absolutely continuous, hence we can appeal to the general strategy, Prop. 3 to see that $(X, Y, Z)$ is absolutely continuous.

If $Z$ is of type $SU(n)$, after applying a Weyl conjugate we can assume $\Omega_Z$ contains $E_{e_1-e_n}^-$ and $E_{e_2}^-$ for $\beta$ one of $e_1 \pm e_2$. Take $\Omega_0 = \{E_{e_1}^\pm\}$ and $M = E_{e_2}^\pm$, and appeal to the same proposition again.

(3) It is useful to note that if $\mu_X * \mu_Y$ is absolutely continuous, then so is $\mu_X * \mu_Y * \mu_Z$ for any $Z$.

Assume, first, that $n \geq 5$. Since all pairs of dominant $SU$ type are absolutely continuous except the pairs $(SU(n), SU(n))$ and $(SU(n), SU(n-1))$, and the annihilating root system of any element of type $SU(n-1)$ is contained in one of type $SU(n)$, it suffices to check that the triple $(SU(n), SU(n), SU(n))$ is absolutely continuous. To do this, we will verify the result holds for $n = 5$ and then appeal to the induction argument established in the previous part of this lemma.

The most efficient way to prove a triple $(X, Y, Z)$, each of type $SU(5)$ with $n = 5$, is absolutely continuous is to establish that the reduced triple $(X', Y', Z')$ is absolutely continuous and again appeal to the previous part of the lemma. The reduced elements are each of type $SU(4)$, but by applying an even number of sign changes as needed, to the original triple, we can assume the three $SU(4)$ annihilating root subsystems are not Weyl conjugate. Thus part (1) of the lemma establishes the absolute continuity of $(X', Y', Z')$.

When $n = 4$, the pairs $(X, Y)$ that are each of dominant $SU$ type that are not absolutely continuous are those where (without loss of generality) $X$ is type $SU(4)$ and $Y$ is type $SU(4), SU(3)$ or $SU(2) \times SU(2)$ where, in the latter case, $\Phi_Y \subseteq \Phi_X$. Thus it will be enough to check that the triples $(SU(4), SU(4), SU(3))$ and $(SU(4), SU(4), SU(2) \times SU(2))$ are absolutely continuous. The first follows from part (1) since any root system of type $SU(3)$ can be viewed as a subset of either of the two Weyl conjugacy classes of root systems of type $SU(4)$. The second can be deduced from [13].

(4) First assume $n = 4$. The triples with precisely two terms that are dominant $SU$ type, that cannot be seen to be absolutely continuous by arguing that some pair in the triple is absolutely continuous, are the types $(SU(4), SU(4), D_3)$, $(SU(4), SU(3), D_3)$ and $(SU(4), SU(4), D_2 \times SU(2))$. Actually, it is enough to check
Theorem 5. Suppose \( L \) have dimension one. Let \( X, Y \) hypotheses, the pair \( (X, Y) \) is eligible and not exceptional.

Proof. Suppose an element \( L \) of \( SU(n) \) and \( Z \) is not dominant \( SU \) type. We proceed by induction using the comment of the previous paragraph to start the base case, \( n = 4 \), noting that \( X' \) and \( Y' \), but not \( Z' \), are type \( SU(n - 1) \). If \( Z' \) is not dominant \( SU \) type, then by the induction hypothesis the triple \( (X', Y', Z') \) is absolutely continuous and we appeal to part (2) to conclude that \( (X, Y, Z) \) is absolutely continuous. If \( Z' \) is dominant \( SU \) type, we appeal to (3) to see that the reduced triple is absolutely continuous and then call again upon (2). \( \square \)

Here is a useful and immediate corollary.

Corollary 5. Suppose \( n \geq 5 \), non-zero \( X_j \in a_\alpha \) for \( j = 1, ..., L \geq 3 \) and at least two \( X_j \) are dominant \( SU \) type. Then \( (X_1, ..., X_L) \) is an absolutely continuous tuple.

Lemma 11. Consider the symmetric space \( SO_0(n, n)/SO(n) \times SO(n) \) with \( n \geq 5 \).

1. Suppose \( X \) is of dominant \( SU \) type, \( Y \) is of dominant \( D \) type and \( Y' \) is of dominant \( SU \) type. Then the pair \( (X, Y') \) is absolutely continuous.

2. Suppose \( X, Y \) are both of dominant \( D \) type, but \( X', Y' \) are both of dominant \( SU \) type. Then the pair \( (X, Y) \) is absolutely continuous.

Proof. Suppose an element \( Z \) is type \( D_j \times SU(s_1) \times \cdots \times SU(s_m) \) and is of dominant \( D \) type, but \( Z' \) is dominant \( SU \) type. Then \( 2J > s_1 \geq 2J - 1 \) and consequently, \( J \leq (n + 1)/3 \). Using this fact it is easy to check that under either of the two hypotheses, the pair \( (X, Y) \) is eligible and not exceptional. \( \square \)

Theorem 5. Suppose \( G/K \) is a symmetric space whose restricted root spaces all have dimension one. Let \( L \geq 3 \). If \( (X_1, X_2, ..., X_L) \) is an eligible, non-exceptional \( L \)-tuple of non-zero elements in \( a \), then \( (X_1, X_2, ..., X_L) \) is absolutely continuous.

Proof. We will first give the argument when the restricted root system is type \( D_n \); the other cases are easier, as we indicate below. The key idea is again an induction argument that is similar to one used in [17] and builds upon the \( L = 2 \) result.

The base cases \( D_3 \) and \( D_4 \) will be discussed at the conclusion of the proof, so assume that the result is true for \( n = 3, 4 \) and that now \( n \geq 5 \). We will let

\[
\Omega = \{ E_{\alpha}^+ : \alpha = e_1 \pm e_j, 2 \leq j \leq n \}.
\]

By appealing to Cor. 5 and Lemma 11 we can assume that for at least \( L - 1 \) indices \( j \), (say all but \( j = L \)) both \( X_j \) and \( X'_j \) are dominant \( D \) type for otherwise there is some pair, \( (X_k, X_\ell) \), which is already absolutely continuous. For these \( L - 1 \) indices we have \( S_{X_j} = S_{X'_j} - 2 \) and from this it is easy to see that the reduced tuple, \( (X'_1, ..., X'_L) \), is eligible. It is clearly not exceptional. By the induction assumption, the reduced tuple is absolutely continuous.

For \( j \neq L \), we have \( \Omega_{X_j} = \{ E_{\alpha_k}^+ : k > J_j \} \) where \( 2J_j = S_{X_j} \). By choosing suitable Weyl conjugates, \( k_i \in K \), we can arrange for \( \bigcup_{i=1}^{L-1} Ad(k_i)(\Omega_{X_i}) \) to contain
\( \Omega_Y \) for some choice of \( Y \) which is of type \( D_m \) for \( m = n - (L - 1)n + \sum_{i=1}^{L-1} J_i \) (or regular \( Y \) if \( m < 2 \)); for more details on how to do this, we refer the reader to [17].

The eligibility assumption ensures that the pair \((Y, X_L)\) is eligible and not exceptional. The arguments given in the proof of Theorem 2 Case 1 or Case 3, depending on whether \( X_L \) is dominant \( D \) or dominant \( SU \) type, can then be used to show that there is some \( k \in K_{n-1}, M \in \xi_n \) and \( \Omega_0 \subseteq \Omega_{X_L} \) such that

\[
sp \Omega = sp \{Ad(k)(\Omega_Y), \Omega_{X_L} \setminus \Omega_0\} = sp \{Ad(\Omega_Y), \Omega_{X_L} \setminus \Omega_0 : i = 1, ..., L - 1\};
\]

\( ad^k(M) \) maps \( N_{X_L} \setminus \Omega_0 \to sp \{\Omega, \Omega_{\xi_n} \} \) for all positive integers \( k \); and the span of the projection of \( Ad(\exp sM)(\Omega_0) \) onto the orthogonal complement of \( sp \{\Omega_{p_n-1}, \Omega_0\} \) in \( \mathfrak{p}_n \) is a surjection for all small \( s > 0 \).

Calling upon the general strategy, Prop. 3 with \( k_i \) replaced there by \( kk_i \), we deduce that \((X_1, ..., X_L)\) is an absolutely continuous tuple.

This completes the induction argument and we now turn to the base cases.

When the restricted root system is type \( D_3 \), we want to show all four-tuples are absolutely continuous and all triples are absolutely continuous, except when all three elements are of type \( SU(3) \). This reduces to checking that the triples \((SU(3), SU(3), D_2), (SU(3), D_2, D_2)\) and \((D_2, D_2, D_2)\) are absolutely continuous, and that the four-tuple consisting of all elements of type \( SU(3) \) is absolutely continuous. These are easily checked using the criteria [13], noting that any two root systems of type \( SU(3) \) will intersect non-trivially.

For type \( D_4 \), we have already seen in Lemma 10 that any triple with at least two terms that are dominant \( SU \) type is absolutely continuous, other than the exceptional triple. Thus we only need to check the triples with two or three terms that are dominant \( D \) type. But in any of these cases the reduced triple in type \( D_3 \) will be absolutely continuous and we can use the induction argument given earlier in this proof, taking \( D_3 \) as the base case. This finishes the argument for restricted root systems of type \( D_n \).

When the restricted root system is type \( B_n \) or \( C_n \) the base case argument is trivial because when \( n = 1 \) the convolution of any two non-zero orbital measures is absolutely continuous. To begin the induction argument, we note that if two or more \( X_i \) are dominant \( SU \) type, then that pair is itself eligible and hence absolutely continuous. Similarly, if two or more \( X_i' \) are dominant \( SU \) type, it is easy to see that the corresponding \( X_i \) are an eligible pair and hence are absolutely continuous.

Thus, again we can assume \( S_{X_i'} = S_{X_i} - 2 \) for all but at most one \( i \) and that ensures the reduced tuple is eligible. Now apply the induction argument as done for type \( D_n \) above, (starting with \( \Omega = \{E_{\alpha}, E_{(2)e_1} : \alpha = e_1 \pm e_j, 2 \leq j \leq n\} \)) obtaining a \( Y \) that is either type \( B_m \) (or \( C_m \)) if \( m = n - (L - 1)n + \sum_{i=1}^{L-1} J_i \geq 2 \) or \( B_1 \) (or \( C_1 \)) otherwise.

The argument is easier, still, for type \( A_{n-1} \). If two or more \( X_i \) have \( S_{X_i'} = S_{X_i} \), then these satisfy \( S_{X_i} \leq n/2 \) and this fact implies the reduced \( L \)-tuple is eligible. Otherwise, at most one \( X_i \) has \( S_{X_i'} = S_{X_i} \) and again we deduce that the reduced tuple is eligible. Now apply the induction argument in a similar manner.

\[ \square \]

4.5. **Proof of sufficiency for symmetric spaces with higher dimensional, restricted root spaces.** We prove two more technical lemmas before completing the proof of sufficiency for symmetric spaces with higher dimensional, restricted root spaces.
Lemma 12. In the symmetric spaces whose restricted root systems are type $B_n$, $C_n$ or $BC_n$, the pairs of type $(SU(n), SU(n))$ are absolutely continuous.

Proof. This will be an induction argument on $n$, similar to the argument given in Case 2 of Theorem 2. The base case, $n = 1$, is trivial, so assume the result holds for $n - 1$, $n \geq 2$. We will write the proof for type $C_n$; only notational changes are needed for the other types. Let

$$\Omega = \{E^{(u)}_{e_1 \pm e_2}, E^{(v)}_{2e_1} : j = 2, \ldots, n; u, v\},$$

where $\{E^{(u)}_{e_1 \pm e_2} : u\}$ is a basis for the restricted root space $\mathfrak{g}_{e_1 \pm e_2}$ and $\{E^{(v)}_{2e_1} : v\}$ a basis for $\mathfrak{g}_{2e_1}$. Let $\Omega_0$ be any one of the vectors $E^{(v)}_{2e_1}$. Applying a Weyl conjugate, if necessary, there is no loss of generality in assuming

$$\Omega_X = \Omega_Y = \{E^{(u)}_{e_1 \pm e_2}, E^{(v)}_{2e_1} : j = 2, \ldots, n; u, v\}.$$

Taking the Weyl conjugate $k \in K_{n-1}$ that changes the sign of the letters $2, \ldots, n$, we have

$$\{Ad(k)(\Omega_Y), \Omega_X \setminus \Omega_0\} = \Omega.$$

As $X', Y'$ are both type $SU(n-1)$, they are an absolutely continuous pair in $\mathfrak{p}_{n-1}$. Now take $M = E_{2e_1}^\pm$. Since the complement of $sp\{\mathfrak{p}_{n-1}, \Omega\}$ in $\mathfrak{p}_n$ is spanned by any one basis vector in $\mathfrak{a}_n \ominus \mathfrak{a}_{n-1}$, an application of the general strategy, Prop. 3, completes the argument. \hfill \square

Lemma 13. The pair $(C_{k-2} \times SU(2), SU(k))$ is absolutely continuous in any symmetric space whose restricted root system is type $C_k$, $k = 3, 4$.

Proof. We will give the proof for $C_4$ and leave $C_3$ as an exercise. Suppose the multiplicities of the long roots are $m_L$ and the multiplicities of the short roots are $m_S$. In terms of this notation, $\dim \Phi = 12m_S + 4m_L$. The co-rank one root subsystems are types $C_3, C_2 \times SU(2), C_1 \times SU(3)$ and $SU(4)$. The chart below summarizes the pertinent information. When we write $\min \dim(\Psi \cap \Psi')$ we mean the minimal dimension of the span of $\{X^{±}_\alpha : \alpha \in \sigma(\Psi) \cap \Psi'\}$ where $\Phi'$ is any root subsystem of type $\Psi'$ and $\sigma$ is any Weyl conjugate.

| $\Psi$ | $C_3$ | $C_2 \times SU(2)$ | $C_1 \times SU(3)$ | $SU(4)$ |
|--------|-------|---------------------|---------------------|---------|
| $\dim \Psi$ | $6m_S + 3m_L$ | $3m_S + 2m_L$ | $m_L + 3m_S$ | $6m_S$ |
| $\min \dim(\Psi \cap SU(4))$ | $3m_S$ | $m_S$ | $m_S$ | $2m_S$ |
| $\min \dim(\Psi \cap C_2 \times SU(2))$ | $m_L + m_S$ | $m_S$ | $0$ | $m_S$ |

With these facts it is easy to check that the criterion is satisfied. \hfill \square

Completion of the Proof of Sufficiency: Theorems 4 and 5 establish the absolute continuity of all eligible, non-exceptional $L$-tuples in symmetric spaces all of whose restricted root spaces have dimension one. That means the sufficiency result is proven for the Cartan classes $AI$, $CI$, $BI$ (with $q = p + 1$) and $DI$ (with $p = q \geq 3$).

Lemma 2 shows that Cartan class $AI$ embeds into $AII$ (of the same rank) with the identity map. Since the eligible, non-exceptional tuples in type $AII$ are the same as those in type $AI$, sufficiency follows for $AII$ directly from Prop. 2 (the embedding proposition).

Similarly, $BI$ with $q = p + 1$ embeds into $BDI$ with $q > p$ and also into Cartan types $AIII$ and $CII$ with $q > p$, in all cases with the identity map. There are no
exceptional tuples in restricted root systems of type $B_p$ (the restricted root system of type $BDI$ when $q > p$) and the eligible tuples are the same in all these cases, hence we again obtain the result for those types from the embedding proposition.

The Cartan type $DI$ with $p = q \geq 3$ has restricted root system type $D_p$ and embeds into type $AIII$ $(q = p)$ with restricted root system of type $C_p$. This in turn embeds into type $CII$ also with restricted root system of type $C_p$. Both embeddings are given by the identity map. Thus any eligible $L$-tuple in type $AIII$ or $CII$ with $p \geq 3$, that is not identified with an exceptional tuple in type $DI$, is absolutely continuous. By Lemma 12, the eligible pair of type $(CII)$ in either $AIII$ or $CII$ of rank $p$ is an absolutely continuous pair, and this implies the same conclusion for the pair $(SU(p), SU(p-1))$ and the triple $(SU(p), SU(p), SU(p))$ when $p = 3, 4$. By Lemma 13, the pairs $(C_{p-2} \times SU(2), SU(p))$ for $p = 3, 4$, and hence also the pair $(SU(2) \times SU(2), SU(4))$ when $p = 4$, are absolutely continuous in types $AIII$ and $CII$. This shows that the eligible, but exceptional tuples in $DI$ are absolutely continuous tuples in types $AIII$ and $CII$, when $p = q \geq 3$.

It was noted in the appendix that for Cartan type $AIII$ with $p = q$ we can assume $p \geq 3$ and for type $CII$ with $p = q$ we can assume $p \geq 2$. For $CII$ with $p = q = 2$ all pairs are eligible and we can check absolute continuity by verifying the criterion (4.3). This is very easy as the only non-regular elements are types $C_1$ or $SU(2)$, so it remains only to check the pairs $(C_1, C_1)$ and $(C_1, SU(2))$. We leave the details for the reader.

Lastly, consider the symmetric $SO^\ast(2n)/U(n)$, the space of Cartan type $DIII$ where, as noted in the appendix, $n \geq 6$ when $n$ is even and $n \geq 3$ when $n$ is odd.

In Lemma 2 we saw that the Cartan type $IV$ symmetric space $SO(n, \mathbb{C})/SO(n)$ embeds into $SO^\ast(2n)/U(n)$ with the canonical map. The symmetric spaces $SO(n, \mathbb{C})/SO(n)$ are dual to the compact Lie groups $SO(n)$, considered as symmetric spaces, and have root systems of type $D_{n/2}$ when $n \geq 6$ is even, or $B_{[n/2]}$ when $n \geq 3$ is odd. In [17] it was shown that all eligible, non-exceptional tuples in these settings were absolutely continuous. In type $B_{[n/2]}$ there are no exceptional tuples, so all eligible tuples in Cartan type $DIII$ with $n$ odd are absolutely continuous. When $n$ is even, the restricted root system of Cartan type $DIII$ is type $C_{n/2}$, so again images of the exceptional tuples from type $D_{n/2}$ must be shown to be absolutely continuous. This is done in the same manner as for $AIII$ above.

5. Proof of Necessity

Finally, to complete the proof of the characterization theorem, we turn to proving the necessity of eligibility and non-exceptionality.

5.1. Eligibility is necessary.

**Proposition 4.** If $(X_1, \ldots, X_L) \in \mathfrak{a}^L$ is not eligible, then $\mu_{X_1} \ast \ldots \ast \mu_{X_L}$ is a singular measure.

**Proof.** We will prove necessity using properties of the underlying symmetric spaces, but the core idea is elementary linear algebra.

Case: Cartan type $AI$ and $AII$ of rank $n - 1$. Restricted root space is type $A_{n-1}$.

Here $\mathfrak{g} = sl(n, F)$, the $n \times n$ matrices over $F$ with the real part of their trace 0, where $F$ is $\mathbb{R}$ for type $AI$ and $F$ is the Quaternions for type $AII$. The space $p$
consists of the Hermitian members of \( \mathfrak{g} \) and \( \mathfrak{a} \) can be taken to be the real diagonal matrices in \( \mathfrak{p} \) (see [24 p.371]).

For \( X \in \mathfrak{a} \), \( S_X \) is the dimension of the largest eigenspace of matrix \( X \).

Given non-eligible \( L \)-tuple, \( (X_1, \ldots, X_L) \in \mathfrak{a}^L \), let \( \alpha_j \) be the eigenvalue of \( X_j \) (viewed as an element of \( \mathfrak{sl}(n,F) \)) with greatest multiplicity. Let \( k_j \in K \) and denote by \( V_j \) the eigenspace of \( \text{Ad}(k_j)X_j \) corresponding to \( \alpha_j \). Then

\[
\dim \bigcap_{j=1}^L V_j \geq \sum_{j=1}^L \dim V_j - n(L-1) = \sum_{j=1}^L S_{X_j} - n(L-1) \geq 1.
\]

Now, for any \( v \in \bigcap_{j=1}^L V_j \) we have \( \text{Ad}(k_j)X_j)v = \alpha_j v \), hence the matrix \( \sum_{j=1}^L \text{Ad}(k_j)X_j \)

has eigenvalue \( \sum \alpha_j \). This proves every element of \( \sum_{j=1}^L O_{X_j} \) has eigenvalue \( \sum \alpha_j \) and that implies \( \sum_{j=1}^L O_{X_j} \) must have empty interior. Prop. 1 tells us \( \mu_{X_1 \cdots X_L} \) is a singular measure.

Case: Cartan type \( CI \) of rank \( n \). Restricted root space is type \( C_n \).

In this symmetric space \( \mathfrak{p} \) is the space of \( 2n \times 2n \) matrices of the form

\[
\begin{bmatrix}
Z_1 & Z_2 \\
Z_2 & -Z_1
\end{bmatrix},
\]

where \( Z_1, Z_2 \) are \( n \times n \) real matrices with \( Z_1 \) symmetric and \( Z_2 \) skew-symmetric. Take for \( \mathfrak{a} \) the diagonal matrices in \( \mathfrak{p} \) (see [20 p.454]). We identify the diagonal matrix \( \text{diag}(b_1, \ldots, b_n, -b_1, \ldots, -b_n) \) with the \( n \)-vector \( (b_1, \ldots, b_n) \).

If \( X \in \mathfrak{a} \) is type \( C_1 \times SU(s_1) \times \cdots \times SU(s_l) \), then \( 0 \) is an eigenvalue of \( X \) with multiplicity \( 2J \) and \( X \) has pairs of non-zero eigenvalues with multiplicity \( s_j \).

It follows that the dimension of the largest eigenspace of \( \text{Ad}(k)X \) is \( S_X \) for any \( k \in K \).

As above, we argue that if \( (X_1, \ldots, X_L) \) is not eligible, then every element of \( \sum_{j=1}^L O_{X_j} \) has a common eigenvalue and hence the sum must have empty interior.

Case: Cartan types \( BI \) and \( DI \) - Symmetric space \( SO_0(p,q)/SO(p) \times SO(q) \) with \( q \geq p \). Restricted root space is type \( C_p \) (if \( q > p \)) or type \( D_p \) if \( q = p \).

Here \( \mathfrak{p} \) consists of the \( (p+q) \times (p+q) \) matrices

\[
\begin{bmatrix}
0 & Z \\
Z^t & 0
\end{bmatrix}
\]

where \( Z \) is a real \( p \times q \) matrix. The space \( \mathfrak{a} \) can be taken to be the set of matrices of the form

\[
(5.1) \quad X = \begin{bmatrix}
0_{p\times p} & H & 0_{p\times (q-p)} \\
H & 0_{p\times p} & 0_{p\times (q-p)} \\
0_{(q-p)\times p} & 0_{(q-p)\times p} & 0_{(q-p)\times (q-p)}
\end{bmatrix}
\]

where \( H \) is a real diagonal \( p \times p \) matrix (see [12]). One can see from this description that a subset of \( \mathfrak{p} \) in which every element has \( 0 \) as an eigenvalue with multiplicity greater than \( q-p \), or contains only elements which have a common non-zero eigenvalue, cannot be open.

We identify the matrix \( X \in \mathfrak{a} \) with the \( p \)-vector whose entries are the diagonal entries of \( H \),

\[
(5.2) \quad X = (0, \ldots, 0, a_1, \ldots, a_1, \ldots, a_m, \ldots, a_m).
\]

Then \( 0 \) is an eigenvalue of \( X \) with multiplicity \( 2J + q - p \) and each \( \pm ia_j \) is an eigenvalue of multiplicity \( s_j \).
Suppose \((X_1, \ldots, X_L)\) is not eligible and \(k_j \in K\). If all \(X_j\) are dominant \(C\) or \(D\) type and \(V_j\) is the eigenspace of \(Ad(k_j)X_j\) corresponding to the eigenvalue 0, then

\[
\sum_{j=1}^{L} \dim V_j = \sum_{j=1}^{L} S_{X_j} + L(q - p) \geq (L - 1)2p + 1 + L(q - p).
\]

Thus

\[
\dim \bigcap_{j=1}^{L} V_j \geq (L - 1)2p + 1 + L(q - p) - (L - 1)(p + q) \geq q - p + 1.
\]

Consequently, 0 is an eigenvalue of every element of \(\sum_{j=1}^{L} O_{X_j}\) with multiplicity greater than \(q - p\) and hence this sum must have empty interior.

If, instead, one \(X_j\) is dominant \(SU\) type, then a similar argument shows every element of \(\sum_{j=1}^{L} O_{X_j}\) has a non-zero eigenvalue with multiplicity at least one. Again it follows that \(\sum_{j=1}^{L} O_{X_j}\) has empty interior.

If two or more \(X_j\) are dominant \(SU\) type, then \((X_1, \ldots, X_L)\) is eligible so we do not need to consider this case.

Case: Cartan types \(A_{III}\) and \(C_{II}\)

The arguments are the same as for \(BDI\) as the only difference is that \(p\) consists of complex or quaternion valued matrices.

Case: Cartan type \(D_{III}\) with rank \(m = \lfloor n/2 \rfloor\). Restricted root space is \(C_m\) or \(BC_m\) depending on whether \(n\) is even or odd.

Here \(p\) consists of the purely imaginary, trace zero matrices of the form

\[
\begin{bmatrix}
Z_1 & Z_2 \\
Z_2 & -Z_1
\end{bmatrix},
\]

where \(Z_1, Z_2\) are \(n \times n\) symmetric matrices. The space \(a\) consists of the matrices in \(p\) of the form \(X = \begin{bmatrix} H & 0 \\ 0 & -H \end{bmatrix}\) where \(H\) is block diagonal with \(m\) \(2 \times 2\) blocks

\[
\begin{bmatrix}
0 & b_j \\
b_j & 0
\end{bmatrix}
\]

when \(n\) is even and an additional entry of 0 in the \((n, n)\) position if \(n\) is odd (see [20, p.454-5]). We identify \(X\) with the \(m\)-vector \((b_1, \ldots, b_m)\).

If \(X\) is type \((B)C_I \times SU(s_1) \times \cdots \times SU(s_t)\) (depending on whether \(n\) is even or odd), then 0 is an eigenvalue with eigenspace of dimension \(4J\) if \(n\) is even and \(4J + 2\) if \(n\) is odd. The non-zero eigenvalues have multiplicities \(2s_j\).

If \(n\) is even, the dimension of the largest eigenspace is \(2S_X\). As the matrices in \(p\) are size \(4m \times 4m\), the argument for the necessity of eligibility is similar to type \(CI\).

If \(n\) is odd, then we require a slight variant on the argument. If all \(X_j\) are dominant \(BC\) type, then each has 0 as its eigenvalue of greatest multiplicity \(2S_{X_j} + 2\). If \(V_j\) is the corresponding eigenspace of \(Ad(k_j)X_j\), then

\[
\dim \bigcap_{j=1}^{L} V_j \geq \sum_{j=1}^{L} (2S_{X_j} + 2) - (L - 1)2n \\
\geq 2((L - 1)2m + 1) + 2L - 2n(L - 1) \geq 4.
\]

Since the generic element of \(p\) has 0 as an eigenvalue with multiplicity 2, this implies \(\sum_{j=1}^{L} O_{X_j}\) has empty interior.
If precisely one $X_j$ is of dominant $SU$ type, then as above we argue that each element of $\sum_{j=1}^L O_{X_j}$ has a common non-zero eigenvalue with multiplicity at least one. If two or more $X_j$ are dominant $SU$ type, then $(X_1, \ldots, X_L)$ is eligible.  

5.2. Exceptional tuples are not absolutely continuous.

**Proposition 5.** If $(X_1, \ldots, X_L) \in a_n^L$ is exceptional, then $\mu_{X_1} \ast \cdots \ast \mu_{X_L}$ is a singular measure.

*Proof.* Different arguments will be needed for the different exceptional tuples.

1. Cartan type $DI$ of rank $n$

Case: $X$ is type $SU(n)$, $Y$ is either type $SU(n)$ or $SU(n-1)$; restricted root system of type $D_n$.

It suffices to check the family of pairs $(SU(n), SU(n-1))$ is singular. One can see that $\dim N_X = \binom{n}{2}$ and $\dim N_Y = \binom{n-1}{2} + 2(n-1)$. Since $\dim p_n = n^2$, it is impossible for $\text{Ad}(k_1)N_X + \text{Ad}(k_2)N_Y$ to equal $p_n$ for any choice of $k_1, k_2$ and hence the pair is singular.

Case: Other exceptional tuples in $D_4$.

Consider the isomorphism $\pi$ that identifies a root system of type $D_3$ with one of type $A_3$. With this identification, the exceptional tuples are all identified with tuples that fail to be eligible in $D_4$ and hence are not absolutely continuous. For example, a triple of Weyl conjugate root subsystems of type $SU(4)$ is identified with a triple of root subsystems of type $D_3$ in $D_4$ and such a triple is not eligible. The isomorphism $\pi$ lifts to an isomorphism of the symmetric space that preserves $p_4$ and $\xi_4$ and hence the failure of the absolute continuity of the original tuples follows.

We remark that a different argument was given in [13] for the exceptional pairs in type $D_4$.

Case: $X, Y, Z$ all type $SU(3)$ in $D_3$.

If such a triple was absolutely continuous, the induction argument, Lemma 10(2), would imply any triple of elements of type $SU(4)$ in $D_4$ would be absolutely continuous.

2. Cartan type $AI$ or $AII$ of rank $n-1$

Case: $X, Y$ both of type $SU(n/2) \times SU(n/2)$; restricted root system of type $A_{n-1}$.

These pairs were proven to be singular in [11]. We note that for the Cartan type $AI$, a dimension argument, as was given in the first case above, would also establish the failure of absolute continuity.  

6. Appendix

In the chart below we list the irreducible, Riemannian globally symmetric spaces of Type III. For each, we give the non-compact group $G$, the compact subgroup $K$, its Cartan class, the Lie type of its restricted root system and the dimensions of the restricted root spaces $g_\alpha$. The rank is the subscript on the label of the restricted root system.

These details can be found in [20, ch.X], [24, VI.4], [3, p.219] and [4, p.72].
We have omitted some from the list as they are isomorphic to others.

- **AIH** with $p = q = 1$ is isomorphic to $AI$ with $n = 2$
- **AIH** with $p = q = 2$ is isomorphic to $BI$ with $q = 4, p = 2$
- **CII** with $p = q = 1$ is isomorphic to $BI$ with $q = 4, p = 1$
- **DIH** with $n = 4$ is isomorphic to $DI$ with $q = 6, p = 2$

Type **DI** with $p = 2 = q$ and type **DIH** with $n = 2$ are not irreducible.

We also describe below the restricted root systems of types $A_n, B_n, C_n, D_n$ and $BC_n$.

| Cartan class | $G/K$ | Restricted root system | $dim g_\alpha$ for $\alpha = e_i, \alpha = 2e_i$ |
|--------------|-------|------------------------|-----------------------------------------------|
| **AI**       | $SL(n, \mathbb{R})/SO(n), n \geq 2$ | $A_{n-1}$ | 1  |
| **II**       | $SL(n, \mathbb{H})/SO(n), n \geq 2$ | $A_{n-1}$ | 4  |
| **AIH**      | $SU(p, q)/SU(p) \times SU(q)$, $p = q \geq 3, q > p \geq 1$ | $C_p$ (if $p = q$) | 2 $(q-p)$ 1 |
| **CI**       | $Sp(n, \mathbb{R})/SU(n), n \geq 1$ | $C_n$ | 1  |
| **CII**      | $Sp(p, q)/Sp(p) \times Sp(q)$, $p = q \geq 2, q > p \geq 1$ | $C_p$ (if $p = q$) | 3 $(q-p)$ 4 |
| **DII(even)** | $SO^*(2n)/U(n), n \geq 6$ | $C_{n/2}$ | 4  |
| **DII(odd)** | $SO^*(2n)/U(n), n \geq 3$ | $BC_{[n/2]}$ | 4  |
| **BI(p + q odd)** | $SO_0(p, q)/SO(p) \times SO(q), q > p \geq 1$ | $B_p$ | 1  |
| **DI(p + q even)** | $SO_0(p, q)/SO(p) \times SO(q), p \geq 3$ | $D_p$ | 1  |

We also describe below the restricted root systems of types $\Phi^+_n$.

| Root system type | Restricted root system $\Phi^+_n$ |
|-----------------|----------------------------------|
| $A_n$           | $\{e_i - e_j : 1 \leq i < j \leq n + 1\}$ |
| $B_n$           | $\{e_i, e_i \pm e_j : 1 \leq i \neq j \leq n\}$ |
| $C_n$           | $\{2e_i, e_i \pm e_j : 1 \leq i \neq j \leq n\}$ |
| $D_n$           | $\{e_i \pm e_j : 1 \leq i \neq j \leq n\}$ |
| $BC_n$          | $\{e_i, 2e_i, e_i \pm e_j : 1 \leq i \neq j \leq n\}$ |

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