Abstract. We study translation covers of several triply periodic polyhedral surfaces that are intrinsically platonic. We describe their affine symmetry groups and compute the quadratic asymptotics for counting saddle connections and cylinders, including the count of cylinders weighted by area.

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1. Introduction

In this paper, we study geodesics on a class of regular polyhedral surfaces. By geodesics, we mean straight-line trajectories and we are particularly interested in trajectories that connect vertices. We study these by considering the translation cover of each polyhedral surface. The surfaces of our interest are classified in Coxeter [3] and Lee [6] as intrinsically platonic polyhedral surfaces. By intrinsically platonic, we mean that the automorphism group of the underlying surface acts transitively on the vertices, edges, and faces of the polyhedral surface. We show that there are no closed geodesics on the specific case of polyhedral surfaces. That is, there are no closed straight-line trajectories from a vertex to itself.

1.1. Objects of our interest. Given any polyhedral surface, there exists a canonically defined cone metric with singularities only at the vertices. For example, the boundary of a cube is a polyhedral surface with cone angle $\frac{2\pi}{2} \times 3 = 2\pi(1 - \frac{1}{4})$ at the vertices and $2\pi$ elsewhere. In other words, the cone metric is a quartic differential on $\mathbb{CP}^1$ with simple poles at the eight vertices. In Athreya-Aulicino-Hooper [1], the authors take the fourfold translation cover (to be defined in Section 1.2) of the cube (and corresponding covers of other platonic solids) branched over the vertices and investigate the resulting translation surface. The translation cover of the cube is equipped with a cone metric with cone angle $\frac{2\pi}{2} \times 4$ at each vertex ($2\pi$ elsewhere) and hence is a genus-nine translation surface.

We will consider translation covers of a class of polyhedral surfaces. Namely, we are interested in infinite polyhedral surfaces $\Pi \subset \mathbb{R}^3$ that are tiled by Euclidean polygons with no self-intersection. We say $\Pi$ is triply periodic if there is a rank-three lattice $\Lambda \subset \mathbb{R}^3$ so that $\Lambda \Pi = \Pi$, where $\Lambda$ acts by translations. The objects of our interest are triply periodic polyhedral surfaces whose underlying curve $X (= \Pi/\Lambda$ with identification via translations) is a compact Riemann surface that have identifiable conformal structures. In [6], Lee classifies...
such surfaces that arise from a particular construction. We denote a polyhedral surface by \{p,q\} (using Schl"afli symbols) if it is tiled by regular Euclidean p-gons and all vertices are q-valent. A polyhedral surface is *platonic* if there are two types of symmetries (an order-\(p\) rotation about the center of a polygonal face and an order-\(q\) rotation about a vertex) so that group generated by these two symmetries acts transitively on the vertices, edges, and faces. To distinguish between Euclidean and hyperbolic isometries, we say that a surface is *intrinsically platonic* if the isometries of order-\(p\) and \(q\) are not euclidean. That is, we disregard the embedding of the surface in Euclidean space and consider only the uniformized hyperbolic metric on \(X\). Lee’s classification includes classical examples found by Coxeter and Petrie \[3\] such as Mucube \{4,6\}, Muoctahedron \{6,4\}, and Mutetrahedron \{6,6\} (Figure 1), and introduces examples that we call Octa-4 \{3,8\}, Octa-8 \{3,12\}, and Truncated Octa-8 \{4,5\} (Figure 2). The naming of these surfaces will be explained in Section 2.

![Figure 1. Mucube, Muoctahedron, and Mutetrahedron. Adapted from 6.](image1)

![Figure 2. Octa-4, Octa-8, and Truncated Octa-8. Adapted from 6.](image2)

### 1.2. Translation surfaces

A *translation surface* is a collection of Euclidean polygons in \(\mathbb{R}^2\) with parallel sides identified by translation. Equivalently, it is given by a pair \((X,\omega)\) where \(X\) is a compact Riemann surface and \(\omega\) is a holomorphic 1-form (a section of the canonical bundle). In local coordinates, we can write \(\omega = f(z)dz\). An order-\(k\) zero of \(\omega\) corresponds to a point with cone angle \(2\pi(k+1)\). We say that \((X,\omega)\) lies in a stratum \(H_1(k_1,\ldots,k_n)\) where \(k_i\) denotes the order of a zero of \(\omega\). Then \(k_1 + \cdots + k_n = 2g - 2\) where \(g\) is the genus of \(X\). Integrating \(\omega\) away from its zeros, we get an atlas of charts to \(\mathbb{C}\) whose transition maps are translations. The integral \(\frac{1}{2}\int_X \omega \wedge \bar{\omega}\) represents area(\(\omega\)), the area of the translation surface, and the group \(SL(2,\mathbb{R})\) acts on the moduli space \(\Omega^1 M_g\) of genus-\(g\) unit area translation surfaces via \(\mathbb{R}\)-linear post-composition with charts.

*Saddle connections and cylinders.* A *saddle connection* on a translation surface is a geodesic segment that connects two singular points (zeros of \(\omega\)) with no singular points in its interior. It is a *closed saddle connection* if it connects a singular point to itself. We will be interested in counting saddle connections (organized by length) and understanding the existence of closed saddle connections. Given a saddle connection \(\gamma\), its holonomy vector is given by

\[ z_\gamma = \int_\gamma \omega, \]

and we denote the set of all holonomy vectors by \(\Lambda_\omega\). This is a discrete subset of \(\mathbb{C}\), and it varies equivariantly under the \(SL(2,\mathbb{R})\)-action, so for \(h \in SL(2,\mathbb{R})\),

\[ h\Lambda_\omega = \Lambda_{h\omega}. \]
A closed geodesic $\eta$ not passing through a singular point is part of a *cylinder*, and we denote the set of holonomy vectors of cylinders as $\Lambda^\eta_{\text{cyl}}$. Associated to each cylinder is the *area* of the cylinder, $a(\eta)$.

With the aid of the *Sage* package [7] surface_dynamics [4], we prove that there are no closed saddle connections on any of our examples.

**Theorem 1.1.** There are no closed saddle connections on Mucube, Muoctahedron, Mutetrahedron, Octa-4, or Truncated Octa-8.

The case for Octa-8 is shown in [1]. We will show in Theorem 1.2 and Section 3 that Octa-8 is a translation surface that corresponds to the translation cover of the octahedron.

*Translation covers.* Many of the polyhedral surfaces we consider are not themselves translation surfaces, because the identifications used to build them consist of translations and rotations by $2\pi/k$ for some integer $k > 1$. These yield $k$-differentials on surfaces, that is, sections of the $k$-th power of the canonical bundle. Associated to a pair $(X, \sigma)$, where $\sigma$ is a $k$-differential, is the *translation cover* (also called *spectral curve* or *unfolding*), a translation surface $(Y, \omega)$ where $Y$ is a branched $k$-cover of $X$ with branching at the zeros of $\sigma$. Geometrically, this construction is given by taking $k$-copies of $(X, \sigma)$ and using these to give identifications by translation instead of translation and rotation. A key point in our work is precisely identifying these covers in a variety of situations, and carefully computing the affine symmetry groups of the resulting translation surfaces, using key ideas from Gutkin-Judge [5].

Moreover, we show that certain polyhedral surfaces and platonic solids have common translation covers.

**Theorem 1.2.**

1. The polyhedral metric on Octa-8 is an abelian differential (that is, a translation surface) that corresponds to the translation cover of the octahedron.
2. The translation cover of the cube and Mucube are identical translation surfaces.

*Question.* Given a polyhedral surface equipped with a $k$-differential and for $k'|k$, can one find a geometric realization of an (intermediate) $k'$-cover as a (quotient of an infinite) polyhedral surface in $\mathbb{R}^3$?

*Veech groups and counting results.* The translation surfaces we consider, arising from these covering constructions, are known as *Veech* or *lattice surfaces*: they have large affine symmetry groups – the stabilizers $SL(X, \omega)$ (known as *Veech groups*) of $(X, \omega)$ under the $SL(2, \mathbb{R})$-action of these surfaces are *lattices*. For these surfaces, Veech [8] showed that the existence of a saddle connection in a fixed direction implies that the surface can be decomposed into parallel cylinders in that direction. Veech also showed [9] that the number of saddle connections of length at most $R$ grows quadratically. We use these results to explicitly compute the asymptotic growth rate for our surfaces. We will also compute the asymptotic growth of the number of cylinders of length at most $R$, weighted by area, a quantity known as the *area Siegel-Veech constant*.

We record our main result on the Veech groups of the spectral curves of our polyhedral surfaces. Each surface results in a branched cover of the square torus (branched over 0), known as a *square-tiled surface* or *origami*. By [5], these surfaces have Veech groups which are finite index subgroups of the modular group $SL(2, \mathbb{Z})$, the Veech group of the torus.

**Theorem 1.3.** The indices of the Veech group, the number of cusps, and the cusp widths for the translation covers of Mucube, Muoctahedron, Mutetrahedron, Octa-4, and Truncated Octa-8 are given by:

| Surface       | Index | Cusps         | Cusp Widths  |
|---------------|-------|---------------|--------------|
| Mucube        | 9     | $\{\infty, 1/2, 1\}$ | $\{4, 2, 3\}$ |
| Muoctahedron  | 16    | $\{\infty, 1/5, 1/2, 1/7\}$ | $\{9, 1, 3, 3\}$ |
| Mutetrahedron | 4     | $\{\infty, 1\}$ | $\{3, 1\}$ |
| Octa-4        | 4     | $\{\infty, 1/2\}$ | $\{3, 1\}$ |
| Truncated Octa-8 | 15 | $\{\infty, 3, 1, 4\}$ | $\{6, 2, 3, 4\}$ |
Given a translation surface \((X, \omega)\), we denote
\[
N(R) = \#|\Lambda \omega \cap B(0, R)|, \\
N^{(1)}(R) = \#|\Lambda_\omega \cap B(0, R)|, \\
A(R) = \sum_{\eta \in \Lambda^{\omega} \cap B(0, R)} a(\eta), \\
A^{(1)}(R) = \sum_{\eta \in \Lambda_\omega \cap B(0, R)} a(\eta).
\]
where \([\omega] = \omega / \text{area}(\omega)\) is the area-1 normalization of \(\omega\). Veech [9] showed that for the surfaces that arise in our paper, there are rational constants \(c, c^{(1)}, a, a^{(1)}\) so that
\[
\lim_{R \to \infty} \zeta(2) \frac{N(R)}{\pi R^2} = c, \\
\lim_{R \to \infty} \zeta(2) \frac{N^{(1)}(R)}{\pi R^2} = c^{(1)}, \\
\lim_{R \to \infty} \zeta(2) \frac{A(R)}{\pi R^2} = a, \\
\lim_{R \to \infty} \zeta(2) \frac{A^{(1)}(R)}{\pi R^2} = a^{(1)}.
\]

Theorem 1.4. The (normalized) asymptotic growth rates for saddle connection, cylinder, and area weighted cylinder counts for our surfaces are given by:

| Surface           | \(c\) | \(c^{(1)}\) | \(a\) | \(a^{(1)}\) |
|-------------------|-------|-------------|-------|-------------|
| Mucube            | 1     | 24          | 6 \cdot 24 | 6 \cdot 24^2 |
| Muoctahedron      | \frac{27}{16} | \frac{171}{4} \cdot 72 | \frac{171}{4} | \frac{171}{4} \cdot 72 |
| Mutetrahedron     | \frac{19}{16} | \frac{19}{16} \cdot 24 | 15 | 15 \cdot 24 |
| Octa-4            | 1     | 48          | 6 \cdot 48 | 6 \cdot 48^2 |
| Truncated Octa-8  | 1     | 120         | 6 \cdot 120 | 6 \cdot 120^2 |

The organization of this paper is as follows. In Section 2, we refer to [6] and show how we achieve a particular class of triply periodic polyhedral surfaces. In Section 3, we discuss the translation covers of our examples and show that they share common covers with translation covers of platonic solids. In Section 4, we show how to compute the asymptotics of counting problems. Specifically, we will study the cusps of associated Teichmüller curves which describe affine equivalence classes of saddle connections.

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2. Construction of triply periodic polyhedral surfaces

We summarize Chapter 4 from [6] and describe carefully the class of examples of our interest. In [3], Coxeter and Petrie introduced three triply periodic regular polyhedral surfaces as an analogue to platonic solids. For example, \(\{4, 3\}\) forms a square-tiling of the cube, \(\{4, 4\}\) forms a square-tiling of the plane. When one increases the valency, the faces cannot bound a convex body and their construction forces the faces to “go up and down in a zig-zag formation” (Figure 1). These are named Mucube, Muoctahedron, and Mutetrahedron, as they bound polyhedra that are built from multiple cubes, octahedra, and tetrahedra, respectively. In [6], Lee broadens this classification by dropping the up-and-down condition on the faces, while still viewing the surfaces as the boundary of a polyhedron. In other words, each surface is the boundary of a tubular neighborhood of a graph in \(\mathbb{R}^3\). Given a graph in \(\mathbb{R}^3\), Lee builds a tubular neighborhood by replacing the 0- and 1-simplices with solids and formulates a gluing pattern of the solids. The following definition is given in [6].
Definition 2.1. Let $\Gamma = \{V,E\}$ be a skeletal graph embedded in $\mathbb{R}^3$ where $V$ is a set of vertices (0-simplices) and $E$ is a set of edges (1-simplices). An edge $e \in E$ is a 2-element subset of $V$ which we denote as an unordered pair $e = \{v_1,v_2\}$ for some $v_1,v_2 \in V$. A decoration of $\Gamma$ is defined as a polyhedron built by replacing the 0-simplices and 1-simplices of $\Gamma$ with convex polyhedral solids so that 1) $\Gamma$ is a deformation retract of the polyhedron and 2) the solids are identified only along faces. In essence, if a 0-simplex and a 1-simplex in $\Gamma$ are incident, then their corresponding replacement solids are identified along a common face. A regular decoration is a decoration whose boundary surface can be denoted by Schlüff symbols $\{p,q\}$. An Archimedean decoration is a decoration where the 0- and 1-simplices are replaced only by Platonic solids and Archimedean solids.

Remark. We include prisms, anti-prisms, and the empty solid to replace 1-simplices but not 0-simplices. By letting empty solids replace 1-simplices, we allow two adjacent solids to retract to 0-simplices. Moreover, we will only allow the solids to be identified along one type of polygon.

A skeletal graph $\Gamma$ is periodic if $\Gamma$ is invariant under $\Lambda$, a lattice of translations. Given a periodic skeletal graph $\Gamma$, we define its compact quotient graph by $\Gamma' = \{V',E'\} := \Gamma/\Lambda$. A graph is symmetric if given any two edges $\{v_1,v_2\}, \{v'_1,v'_2\}$, there is an automorphism $\varphi : V \rightarrow V$ such that $\varphi(v_1) = v'_1$ and $\varphi(v_2) = v'_2$. For our construction, we consider only triply periodic graphs whose quotients are symmetric. Given a quotient graph $\Gamma'$, one can define its genus by $\text{genus}(\Gamma') := e - v + 1$ where $v = |V'|$ and $e = |E'|$. For a given genus, there are finitely many graphs (whose quotients are symmetric) and due to the finiteness of Archimedean solids, there are only finitely many regular Archimedean decorations of that genus. We consider only those whose quotient surfaces are compact Riemann surfaces of genus-three and four that have identifiable structure, and are intrinsically platonic. The surfaces are Mucube, Muoctahedron, Mutetrahedron, Octa-4, Octa-8, and Truncated Octa-8. Details on decorations can be found in Chapter 4 of [6].

Mucube and Muoctahedron are regular Archimedean decorations of a graph whose quotient is a 6-valent graph with one 0-simplex. For Mucube, the 0-simplex is replaced with a cube and the 1-simplices are replaced with square prisms. For Muoctahedron, we replace the 0-simplex with a truncated octahedron and the 1-simplices with empty solids. Both underlying Riemann surfaces (quotient under $\Lambda$) are genus-three surfaces. Mutetrahedron and Octa-4 are decorations of a graph whose quotient is 4-valent with two 0-simplices. For Mutetrahedron, we replace one of the 0-simplices with a tetrahedron, the other with a truncated tetrahedron, and the 1-simplices with the empty solid. For Octa-4, we replace the 0-simplices with octahedra and the 1-simplices with triangular anti-prisms. The genus of both underlying surfaces is three.

Octa-8 and Truncated Octa-8 are decorations of a graph whose quotient is an 8-valent graph with one 0-simplex. Octa-8 is achieved by replacing the 0-simplex with an octahedron and the 1-simplices by triangular anti-prisms. Truncated Octa-8 is achieved by replacing the 0-simplex with a truncated octahedron and the 1-simplices with hexagonal prisms. Their underlying surfaces are genus-four surfaces.

The names Octa-4, Octa-8, and Truncated Octa-8 arise from the replacement solid of the 1-simplices and the valency of the graph. Adapting this notation, Mucube, Muoctahedron, and Mutetrahedron can be written as Cube-6, Truncated Octa-6, and Tetra-Truncated Tetra-4, respectively.

Remark. By the induced hyperbolic metric, Lee [Chapter 5 of [6]] identifies the underlying surfaces of Octa-4, Octa-8, and Truncated Octa-8 as Fermat’s quartic, Schoen’s minimal I-WP surface, and Bring’s curve, respectively.

3. Translation covers of Platonic surfaces

We devote this section to translation covers of polyhedral surfaces and prove Theorem 1.2.

A polyhedral surface denoted by $\{p,q\}$ is equipped with a cone metric whose cone angle is $\frac{q(p-2)\pi}{p}$ at every vertex. That is, the surface is a $k$-differential with singularity at the vertices, where $k$ is the smallest positive integer so that $q\frac{(p-2)\pi}{p} k = 0 \pmod{2\pi}$. We take $k$ copies of the polyhedral surface, rotate each copy by $2\pi/k$ from the previous copy, identify the edges by translations and achieve a translation surface.

The following table is a replication of Table 1 in [1].

Remark. Although the cone angle at the vertices of Mutetrahedron is an integral multiple of $2\pi$, the polyhedral metric on this surface is a quadratic differential (Figure 7). This is due to the fact that its holonomy lies in $\mathbb{Z}/2\mathbb{Z}$. Note that the unfolding is not a branched cover and hence produces additional branch points.
The cone metric on Octa-8 is an abelian differential. We use results from [6] and show that the cone metric on Octa-8 corresponds to the cone metric of the unfolding of the octahedron.

**Theorem 3.1.** The polyhedral metric on Octa-8 is an abelian differential that corresponds to the translation cover of the octahedron.

**Proof.** Octa-8 is a triply periodic polyhedral surface denoted by \(\{3,12\}\). Its underlying surface is a genus-four surface with identification of edges as described in Figure 3.

The surface is invariant under the order-twelve rotation \(\varphi\) about the center of the tessellation. The quotient \(X/\langle \varphi \rangle\) is a sphere where the covering is branched over three points. However, \(X/\langle \varphi^4 \rangle\) is also a sphere and the threefold map is branched over six points. In Section 5.2 of [6], Lee shows that the quotient sphere \(X/\langle \varphi^4 \rangle\) is conformally equivalent to a regular octahedron.

On the other hand, one can see that the threefold unfolding of the octahedron (Figure 4) is identical to the surface described in Figure 3. That is, by mapping the hyperbolic \(\frac{\pi}{6}\)-triangles to equilateral euclidean triangles, the identification of edges is preserved.

\[\square\]

Next we show that the cube (quartic differential) and Mucube (quadratic differential) share the same translation cover. That is, their translation covers are identical translation surfaces.

**Theorem 3.2.** The spectral curve of the cube and Mucube are identical translation surfaces.
Proof. We prove this theorem by picture. Figure 5 describes Mucube and the twofold cover of the cube where the edges are identified by translation or a half-translation. These two coverings are identical as half-translation surfaces, hence their twofold translation covers are identical.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{cube_twofold_translation.png}
\caption{Mucube (left) and the twofold cover of the cube (right).}
\end{figure}

4. Asymptotics of counting problems

In the following subsections we study the Teichmüller curve associated to the translation cover of each polyhedral surface in Table 3 and prove Theorem 1.4. To ease our computations, we describe each translation surface as a surface tiled by unit squares. The advantage of this tool is that a surface is completely defined by a horizontal permutation $\sigma_h$ and a vertical permutation $\sigma_v$ on the squares. The top of square $i$ is glued to the bottom of square $\sigma_v(i)$, and the right side of square $i$ is glued to the left side of square $\sigma_h(i)$. The polyhedral surfaces tiled by triangles cover the doubled triangle (a rhombus), which we map to a square via the $GL(2,\mathbb{R})$-action as shown in [1]. Our examples include surfaces tiled by hexagons (Muoctahedron and Mutetrahedron), which can also be described as square-tiled surfaces. We subdivide each hexagon into six triangles and then pair two triangles to form a rhombus. Figure 6 illustrates the subdivision of Mutetrahedron into rhombi. Note that the covering is not regular over rhombi.

The input to the Sage package $\texttt{surface_dynamics}$ is the two permutations that define a square-tiled surface. It yields Theorem 1.3 where it computes the Veech group of the Teichmüller curve and its cusps and cups widths. We will use Theorem 1.3 to study the Teichmüller curves associated to the polyhedral surfaces.
4.1. Mutetrahedron. The underlying surface (quotient of the infinite polyhedral surface by the lattice of translations) of Mutetrahedron is tiled by four hexagons. The cone angles at each vertex is $2\pi$, however the edges are identified by translation and a $180^\circ$-rotation. Hence its spectral curve is a 2-cover. By subdivision of hexagons into squares, the translation cover of Mutetrahedron is tiled by 24 squares, described by a horizontal permutation $\sigma_h$ and a vertical permutation $\sigma_v$.

$$\sigma_h = (1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12)(13, 14, 15, 16, 17, 18)(19, 20, 21, 22, 23, 24)$$

$$\sigma_v = (1, 7, 13, 11, 3, 21)(2, 20, 14, 12, 18, 22)(4, 10, 16, 8, 6, 24)(5, 23, 17, 9, 15, 19)$$

The translation cover is a genus five surface with eight simple zeros. Its Veech group $\Gamma$ is an index-4 subgroup of $SL(2, \mathbb{Z})$, which is described completely by the action of the generators of $SL(2, \mathbb{Z})$ on the right cosets $\Gamma\backslash SL(2, \mathbb{Z})$. Given the generators $s_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $s_3 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $l = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $r = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\Gamma$ has the following representation:

$$s_2 = (1, 4)(2, 3)$$

$$s_3 = (1, 3, 2)$$

$$l = (1, 3, 4)$$

$$r = (1, 2, 4)$$

The cusps are at $\infty$ and 1 with width 3 and 1, respectively. Each cusp corresponds to a rational direction in which the surface decomposes into parallel cylinders. We denote the saddle connections by $v_\infty = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v'_\infty = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, and $v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. However, neither of these are closed saddle connections as they do not connect vertices of the same color (Figure 7). To compute the quadratic asymptotics of saddle connections, we follow Section 16 of [9] and Appendix C of [2]. Veech [9] Theorem 16.1] showed that for any non-uniform lattice...
\( \Gamma \subset SL(2, \mathbb{R}) \) and any vector \( \mathbf{v} \) stabilized by a maximal parabolic subgroup \( \Lambda \subset \Gamma \), we have

\[
\lim_{R \to \infty} \frac{|g \Gamma \mathbf{v} \cap B(0, R)|}{\pi R^2} = c(\Gamma, \mathbf{v}).
\]

Our goal is to find this limit for all saddle connection vectors \( \mathbf{v} \in \Lambda_\omega \), that is,

\[
\lim_{R \to \infty} \frac{N(R)}{\pi R^2} = \sum_{\mathbf{v} \in \Lambda_\omega} c(\Gamma, \mathbf{v}).
\]

Given \( \mathbf{v} \), we compute \( c(\Gamma, \mathbf{v}) \) as follows. Let \( g_0 \in SL(2, \mathbb{R}) \) be such that \( g_0^{-1} \Lambda g_0 = \Lambda_0 \), with \( \Lambda_0 = \{(\frac{1}{n} 1) : n \in \mathbb{Z}\} \). Setting \( v_0 = 1 \) and \( g_0 v_0 = t \mathbf{v} \), we have \( c(\Gamma, \mathbf{v}) = t^2 \text{vol}(\mathbb{H}^2/\Gamma)^{-1} \). Here, \( t^2 \) corresponds to the weight of the cusp. For \( \mathbf{v} = \mathbf{v}_\infty \), we have

\[
g_0 = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \end{pmatrix}
\]

hence \( c(\Gamma, \mathbf{v}) = 3 \cdot \frac{6}{\pi^2} \cdot \frac{1}{4} = \frac{9}{2 \pi^2} \). Since \( \mathbf{v}'_\infty = 2 \mathbf{v}_\infty \), we have \( c(\Gamma, \mathbf{v}'_\infty) = (\frac{1}{2})^2 c(\Gamma, \mathbf{v}_\infty) \).

Following this computation, we have

\[
\lim_{R \to \infty} \frac{N(R)}{\pi R^2} = \left( \frac{3}{4} \left( 1 + \frac{1}{4} \right) + \frac{1}{4} \right) \frac{6}{\pi^2} = \frac{19}{16} \cdot \frac{6}{\pi^2}.
\]

We normalize the area of the surface to 1 and denote by \( N^{(1)}(R) \) the number of saddle connection vectors of length at most \( R \) on the unit-area surface. Then

\[
\lim_{R \to \infty} \frac{N^{(1)}(R)}{\pi R^2} = \frac{19}{16} \cdot \frac{6}{\pi^2} \cdot 24.
\]

Now we consider the cylinder decomposition of the surface arising from each saddle connection vector and compute the asymptotic growth of (weighted) cylinders whose core curves are at most length \( R \). In the vertical direction, the surface decomposes into sixteen cylinders: eight have \( \mathbf{v}_\infty \) as their core curves and eight have \( \mathbf{v}'_\infty \) as their core curves, hence the area of the cylinders are 1 and 2 respectively. Similarly, in the direction of \( \mathbf{v}_1 \), the surface decomposes into eight cylinders of unit area and eight cylinders of area 2. Hence

\[
\lim_{R \to \infty} \frac{A(R)}{\pi R^2} = \left( \frac{3}{4} \left( 8 + \frac{16}{4} \right) + \frac{1}{4} (8 + 16) \right) \frac{6}{\pi^2} = 15 \cdot \frac{6}{\pi^2}
\]

and on the unit-area surface, we have

\[
\lim_{R \to \infty} \frac{A^{(1)}(R)}{\pi R^2} = 15 \cdot \frac{6}{\pi^2} \cdot 24.
\]

4.2. Mucube. The translation cover of Mucube is defined by the following horizontal and vertical permutations on 24 squares. It is a genus nine surface with eight simple zeros.

\[
\sigma_h = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12)(13, 14, 15, 16)(17, 18, 19, 20)(21, 22, 23, 24)
\]

\[
\sigma_v = (1, 9, 14, 22)(2, 20, 13, 7)(3, 24, 16, 11)(4, 5, 15, 18)(6, 10, 17, 21)(8, 23, 19, 12)
\]

\[\text{Figure 8. Double cover of Mucube}\]
The Veech group is an index-9 subgroup of \( SL(2, \mathbb{Z}) \) which we describe by the action of the generators of \( SL(2, \mathbb{Z}) \) on the right cosets:

\[
s_2 = (2, 4)(3, 7)(5, 9)(6, 8) \\
s_3 = (1, 9, 6)(2, 5, 7)(3, 8, 4) \\
l = (1, 8, 7, 9)(2, 3)(4, 6, 5) \\
r = (1, 5, 3, 6)(2, 9, 8)(4, 7)
\]

The cusps are at \( \infty, 1/2, \) and 1, with cusp width 4, 2, and 3, respectively. These are associated to the vertical direction, direction of slope 1/2, and 1/3. Note that none of the saddle connections are closed as there is no vertex on both top and bottom of any cylinder in the cusp directions (Figure 8). Then we have the following quantities:

\[
\lim_{R \to \infty} \frac{N(R)}{\pi R^2} = \left( \frac{4}{9} + \frac{2}{9} + \frac{3}{9} \right) \frac{6}{\pi^2} = \frac{6}{\pi^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{N^{(1)}(R)}{\pi R^2} = \frac{6 \cdot 24}{\pi^2}.
\]

Since every cylinder has unit-area,

\[
\lim_{R \to \infty} \frac{A(R)}{\pi R^2} = \frac{6 \cdot 24}{\pi^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{A^{(1)}(R)}{\pi R^2} = \frac{6 \cdot 24^2}{\pi^2}.
\]

4.3. Muoctahedron. The translation cover of Muoctahedron is a genus-nineteen surface with twelve order-three zeros. The following vertical and horizontal permutations define the square-tiled surface.

\[
\sigma_v = (1, 29, 20, 36, 39, 54, 37, 3, 10)(2, 18, 46, 38, 53, 21, 28, 19, 11)(4, 60, 70, 58, 35, 23, 33, 6, 16) \\
(5, 15, 24, 34, 22, 71, 59, 69, 17)(7, 42, 51, 40, 57, 64, 55, 9, 13)(8, 14, 50, 41, 52, 72, 56, 63, 12) \\
(25, 31, 27, 66, 62, 68, 47, 44, 49)(26, 32, 48, 45, 61, 67, 63, 30)
\]

\[
\sigma_h = (1, 2, 3, 4, 5, 6, 7, 8, 9)(10, 11, 12, 13, 14, 15, 16, 17, 18)(19, 20, 21, 22, 23, 24, 25, 26, 27) \\
(28, 29, 30, 31, 32, 33, 34, 35, 36)(37, 38, 39, 40, 41, 42, 43, 44, 45)(46, 47, 48, 49, 50, 51, 52, 53, 54) \\
(55, 56, 57, 58, 59, 60, 61, 62, 63)(64, 65, 66, 67, 68, 69, 70, 71, 72)
\]

\[\text{Figure 9. Triple cover of Muoctahedron}\]

The Veech group is an index-16 subgroup of \( SL(2, \mathbb{Z}) \), which we describe by the action of generators on the right cosets:

\[
s_2 = (1, 14)(2, 9)(3, 11)(4, 10)(5, 6)(7, 15)(8, 12)(13, 16) \\
s_3 = (2, 16, 9)(3, 15, 6)(4, 14, 11)(5, 8, 10)(7, 13, 12) \\
l = (1, 14, 10, 12, 16, 9, 13, 15, 11)(3, 5, 4)(6, 7, 8) \\
r = (1, 14, 3, 7, 16, 2, 13, 8, 4)(5, 12, 15)(6, 11, 10)
\]

The cusps are at \( \infty, 1/5, 1/2, \) and 1/7, with width 9, 1, 3, and 3, respectively. Again, none of the saddle connections are closed. The number of saddle connection vectors in the disk of radius \( R \) grows quadratically
and we get
\[
\lim_{R \to \infty} \frac{N(R)}{\pi R^2} = \left( \frac{9}{16} \left( 1 + \frac{1}{4} \right) + \frac{1}{16} \left( 1 + \frac{1}{4} \right) + \frac{3}{16} \left( 1 + \frac{1}{4} \right) + \frac{3}{16} \right) \frac{6}{\pi^2} = \frac{77}{64} \cdot \frac{6}{\pi^2}
\]
and
\[
\lim_{R \to \infty} \frac{N^{(1)}(R)}{\pi R^2} = \frac{77}{64} \cdot \frac{6}{\pi^2} \cdot 72.
\]
For all directions, there are 24 cylinders of unit-area and 24 cylinders of area 2. For directions \(\infty, 1/5,\) and \(1/2,\) the area of the cylinder is proportional to the length of the core curve.

\[
\lim_{R \to \infty} \frac{A(R)}{\pi R^2} = \frac{9}{16} \left( 24 + \frac{48}{4} \right) + \frac{1}{16} \left( 24 + \frac{48}{4} \right) + \frac{3}{16} \left( 24 + \frac{48}{4} \right) + \frac{3}{16} \cdot (24 + 48) \cdot \frac{6}{\pi^2}
\]
and
\[
\lim_{R \to \infty} \frac{A^{(1)}(R)}{\pi R^2} = \frac{171}{4} \cdot \frac{6}{\pi^2} \cdot 72.
\]

4.4. **Octa-4.** The translation cover of Octa-4 is a genus-nineteen surface with twelve order-three zeros. The cover is defined by vertical and horizontal permutations as described in Figure 10.

\[
\sigma_v = (1, 6, 35)(2, 32, 41)(3, 48, 7)(4, 9, 10)(5, 38, 20)(8, 17, 26)(11, 25, 30)(12, 43, 39)
(13, 18, 47)(14, 44, 29)(15, 36, 19)(16, 21, 22)(23, 37, 42)(24, 31, 27)(28, 33, 34)(40, 45, 46)
\]

\[
\sigma_h = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15)(16, 17, 18)(19, 20, 21)(22, 23, 24)
(25, 26, 27)(28, 29, 30)(31, 32, 33)(34, 35, 36)(37, 38, 39)(40, 41, 42)(43, 44, 45)(46, 47, 48)
\]

![Figure 10. Triple cover of Octa-4](image)

Its Veech group is an index-four subgroup of \(SL(2, \mathbb{Z})\) with the following description on the right cosets:

\[
s_2 = (1, 2)(3, 4), \quad s_3 = (1, 3, 4), \quad l = (1, 3, 2), \quad r = (1, 4, 2).
\]

The two cusps are \(\infty\) and \(1/2\) with width 3 and 1, respectively. There are no closed saddle connections. We achieve

\[
\lim_{R \to \infty} \frac{N(R)}{R^2} = \left( \frac{3}{4} + \frac{1}{4} \right) \frac{6}{\pi^2} = \frac{6}{\pi^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{N^{(1)}(R)}{\pi R^2} = \frac{6 \cdot 48}{\pi^2}.
\]

Since all cylinders are unit-area, we get

\[
\lim_{R \to \infty} \frac{A(R)}{\pi R^2} = \frac{6 \cdot 48}{\pi^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{A^{(1)}(R)}{\pi R^2} = \frac{6 \cdot 48^2}{\pi^2}.
\]
4.5. Truncated Octa-8. The translation cover of Truncated Octa-8 is a genus-49 surface with 24 order-4 zeros. The square-tiled surface is defined by the following permutations on 120 squares.

\[
\sigma_v = (1, 34, 75, 81, 120, 7)(2, 20, 111, 80, 62, 25)(3, 40, 67, 79, 107, 15)(4, 10, 117, 84, 78, 31)
(5, 28, 65, 83, 114, 23)(6, 18, 104, 82, 70, 37)(8, 52, 71, 74, 96, 17)(9, 24, 86, 73, 64, 43)
(11, 14, 93, 77, 68, 49)(12, 46, 61, 76, 89, 21)(13, 22, 99, 69, 66, 55)(16, 58, 63, 72, 102, 19)
(26, 45, 115, 110, 90, 33)(27, 38, 98, 109, 103, 56)(29, 36, 87, 113, 118, 48)(30, 59, 106, 112, 101, 41)
(32, 92, 108, 116, 50, 39)(35, 42, 53, 119, 105, 95)(44, 57, 91, 85, 97, 51)(47, 54, 100, 88, 94, 60)
\]

\[
\sigma_h = (1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12)(13, 14, 15, 16, 17, 18)(19, 20, 21, 22, 23, 24)
(25, 26, 27, 28, 29, 30)(31, 32, 33, 34, 35, 36)(37, 38, 39, 40, 41, 42)(43, 44, 45, 46, 47, 48)
(49, 50, 51, 52, 53, 54)(55, 56, 57, 58, 59, 60)(61, 62, 63, 64, 65, 66)(67, 68, 69, 70, 71, 72)
(73, 74, 75, 76, 77, 78)(79, 80, 81, 82, 83, 84)(85, 86, 87, 88, 89, 90)(91, 92, 93, 94, 95, 96)
(97, 98, 99, 100, 101, 102)(103, 104, 105, 106, 107, 108)(109, 110, 111, 112, 113, 114)
(115, 116, 117, 118, 119, 120)
\]

![Quadruple Cover of Truncated Octa-8](image)

Figure 11. Quadruple cover of Truncated Octa-8

Its Veech group is an index-15 subgroup of SL(2, \( \mathbb{Z} \)) with the following description on the right cosets:

\[
s_2 = (2, 7)(3, 11)(4, 8)(5, 13)(6, 12)(9, 15)(10, 14)
\]
\[
s_3 = (1, 13, 6)(2, 9, 11)(3, 14, 7)(4, 10, 12)(5, 15, 8)
\]
\[
l = (1, 12, 14, 11, 15, 13)(2, 3)(4, 5, 7, 10, 8, 9)
\]
\[
r = (1, 5, 9, 3, 10, 6)(2, 15, 4, 14)(7, 11)(8, 13, 12)
\]

The cusps are at \( \infty \), 1/3, 1, and 1/4, and none of the saddle connections are closed. We get

\[
\lim_{R \to \infty} \frac{N(R)}{\pi R^2} = \left( \frac{6}{15} + \frac{2}{15} + \frac{3}{15} + \frac{4}{15} \right) \frac{6}{\pi^2} = \frac{6}{\pi^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{N^{(1)}(R)}{\pi R^2} = \frac{6 \cdot 120}{\pi^2}.
\]
Since every cylinder is unit-area,
\[
\lim_{R \to \infty} \frac{A(R)}{\pi R^2} = \frac{6 \cdot 120}{\pi^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{A^{(1)}(R)}{\pi R^2} = \frac{6 \cdot 120^2}{\pi^2}.
\]

4.6. **Platonic solids.** In this section, we present the quadratic asymptotics of saddle connection vectors on the translation cover of platonic solids represented as square-tiled surfaces. The following is Table 2 from [1].

| Cusp Widths | Cusps |
|-------------|-------|
| {∞}         |       |
| {1}         |       |
| {∞, 3}      | {1}   |
| {4, 2}      | {1, 3}|
| {4, 2, 3}   |       |
| {∞, 1/3, 1/2} | {5, 2, 3} |

**Table 2.** Cusp and cusp widths of Platonic solids

Then we have the following quantities for the spectral curve of the octahedron, cube, and icosahedron:

**3-cover of the octahedron.**
\[
\lim_{R \to \infty} \frac{N(R)}{\pi R^2} = \left( \frac{3}{4} + \frac{1}{4} \right) \frac{6}{\pi^2} = \frac{6}{\pi^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{N^{(1)}(R)}{\pi R^2} = \frac{6 \cdot 12}{\pi^2},
\]

and all cylinders are unit-area:
\[
\lim_{R \to \infty} \frac{A(R)}{\pi R^2} = \frac{6 \cdot 12}{\pi^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{A^{(1)}(R)}{\pi R^2} = \frac{6 \cdot 12}{\pi^2}.
\]

**4-cover of the cube (identical to the 2-cover of Mucube).**
\[
\lim_{R \to \infty} \frac{N(R)}{\pi R^2} = \left( \frac{4}{9} + \frac{2}{9} + \frac{3}{9} \right) \frac{6}{\pi^2} = \frac{6}{\pi^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{N^{(1)}(R)}{\pi R^2} = \frac{6 \cdot 24}{\pi^2},
\]

Similarly, all cylinders are unit-area, hence
\[
\lim_{R \to \infty} \frac{A(R)}{\pi R^2} = \frac{6 \cdot 24}{\pi^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{A^{(1)}(R)}{\pi R^2} = \frac{6 \cdot 24^2}{\pi^2}.
\]

**6-cover of the icosahedron.**
\[
\lim_{R \to \infty} \frac{N(R)}{\pi R^2} = \left( \frac{5}{10} + \frac{2}{10} + \frac{3}{10} \right) \frac{6}{\pi^2} = \frac{6}{\pi^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{N^{(1)}(R)}{\pi R^2} = \frac{6 \cdot 60}{\pi^2},
\]

Also, all cylinders are unit-area, hence
\[
\lim_{R \to \infty} \frac{A(R)}{\pi R^2} = \frac{6 \cdot 60}{\pi^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{A^{(1)}(R)}{\pi R^2} = \frac{6 \cdot 60^2}{\pi^2}.
\]

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