On Cauchy dual operator and duality for Banach spaces of analytic functions

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Abstract. In this paper, two related types of dualities are investigated. The first is the duality between left-invertible operators and the second is the duality between Banach spaces of vector-valued analytic functions. We will examine a pair \((\mathcal{B}, \Psi)\) consisting of a reflexive Banach spaces \(\mathcal{B}\) of vector-valued analytic functions on which a left-invertible multiplication operator acts and an operator-valued holomorphic function \(\Psi\) on an open subset of complex plane \(\Omega\). We prove that there exist a dual pair \((\mathcal{B}', \Psi')\) such that the space \(\mathcal{B}'\) is unitarily equivalent to the space \(\mathcal{B}^*\) and the following intertwining relations hold

\[ \mathcal{L}U = U\mathcal{M}_z^* \quad \text{and} \quad \mathcal{M}_z U = U\mathcal{L}^*, \]

where \(U\) is the unitary operator between \(\mathcal{B}'\) and \(\mathcal{B}^*\). In addition we show that \(\Psi\) and \(\Psi'\) are connected through the relation

\[ \langle (\Psi'(z)e_1)\lambda, e_2 \rangle = \langle e_1, (\Psi(\lambda)e_2)(z) \rangle \]

for every \(e_1, e_2 \in E, z \in \Omega, \lambda \in \Omega'.\)

If a left-invertible operator \(T \in \mathcal{B}(\mathcal{H})\) satisfies certain conditions, then both \(T\) and the Cauchy dual operator \(T'\) can be modelled as a multiplication operator on reproducing kernel Hilbert spaces of vector-valued analytic functions \(\mathcal{H}\) and \(\mathcal{H}'\), respectively. We prove that Hilbert space of the dual pair of \((\mathcal{H}, \Psi)\) coincide with \(\mathcal{H}'\), where \(\Psi\) is a certain operator-valued holomorphic function. Moreover, we characterize when the duality between spaces \(\mathcal{H}\) and \(\mathcal{H}'\) obtained by identifying them with \(\mathcal{H}\) is the same as the duality obtained from the Cauchy pairing.

1. Introduction

Duality is one of the mathematical principles that allows one to look at the same object from two points of view. This is its great advantage and one of the reasons why it attracts the attention of researchers. Different types of dualities appear in many branches of mathematics and physics. We refer the reader to a nice survey article by M. Atiyah \([5]\) concerning this topic. In this paper, we consider two related types of dualities. The first is the duality between left-invertible operators and the second is the duality between Banach spaces of vector-valued analytic functions.

Let \(\mathcal{B}\) be a Banach space of analytic functions on the unit disc \(\mathbb{D}\) continuously contained in the space \(\text{Hol}(\mathbb{D})\) with the topology given by uniform convergence on

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compact sets. Assume that $\mathcal{B}$ contains the space $\text{Hol}(\overline{D})$ of analytic functions in a neighbourhood of $D$ as a dense subset. Define an operator $U: \mathcal{B}^* \to \text{Hol}(\overline{D})$ by

\begin{equation}
U\varphi(\lambda) := \varphi\left(\frac{1}{1 - \lambda z}\right), \quad \varphi \in \mathcal{B}^*.
\end{equation}

As shown in [1, p. 616] $U$ is injective. Now let $\mathcal{B}'$ be the image of $\mathcal{B}^*$ by $U$ with the norm induced from $\mathcal{B}^*$, so that $U$ is unitary. With this norm the space $\mathcal{B}'$ is called the Cauchy dual of $\mathcal{B}$. The name is justified by the fact that the dual of $\mathcal{B}$ is represented by $\mathcal{B}'$ via the Cauchy pairing

\begin{equation}
\varphi(f) = \lim_{r \to 1^-} \int_0^{2\pi} f(r e^{it}) U\varphi(r e^{it}) \frac{dt}{2\pi}, \quad f \in \text{Hol}(\overline{D}), \varphi \in \mathcal{B}^*.
\end{equation}

In Banach spaces $\mathcal{B}$ of analytic functions where the dilations $f_r, 0 < r < 1$ of a function $f \in \mathcal{B}$ converge to $f$ in $\mathcal{B}$ as $r \to 1^+$ the above holds for all $f \in \mathcal{B}$ (see [1, p. 616]). This notion is well-known in the theory of spaces of analytic functions. For example, the dual of Bergman space $\mathcal{B}$ of analytic functions on disc can be identified of course as $\mathcal{B}$ itself, via the usual Hilbert-space duality, defined using the inner product. However, in many applications it is more appropriate to identify the dual of $\mathcal{B}$ with the Dirichlet space of analytic functions on disc via the Cauchy pairing (see [2, Example 1.4]). We note that the notion of Cauchy duality is close to the notion of triplet of Hilbert spaces when the middle space is the Hardy space $H^2$, which is why it is also called the $H^2$-duality. For more examples and information on Cauchy duals we refer the reader to [2, 22, 1].

In [24] S. Shimorin constructed an analytic model for a left-invertible analytic operator $T \in \mathcal{B}(\mathcal{H})$. Namely, he showed that operator $T$ is unitarily equivalent to multiplication operator acting in some reproducing kernel Hilbert space $\mathcal{H}$ of vector-valued holomorphic functions defined on a disc. The construction of this analytic model is based on the following unitary isomorphism:

$$U: \mathcal{H} \ni x \to \sum_{n=0}^{\infty} (P_n T^* x) z^n \in \mathcal{H}, \quad z \in \mathbb{D}(r(T')^{-1}),$$

where $E := \mathcal{N}(T^*)$ and $T'$ is the Cauchy dual operator of $T$. The Cauchy dual operator $T'$ of a left-invertible $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$T' := T(T^* T)^{-1}$$

and was introduced and studied by S. Shimorin in [24]. The Cauchy dual operator of a left-invertible analytic operator $T$ is itself left-invertible. Moreover, if $T'$ is also analytic, then for both operators $T$ and $T'$ one can construct Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ of vector-valued holomorphic functions defined on a disc. S. Shimorin observed that the duality between $\mathcal{H}$ and $\mathcal{H}'$ obtained by identifying them with $\mathcal{H}$ is the same as the duality obtained from the Cauchy pairing, that is,

\begin{equation}
\langle U^{-1} f, U'^{-1} g \rangle_\mathcal{H} = \sum_{n=0}^{\infty} \langle \hat{f}(n), \hat{g}(n) \rangle_E
\end{equation}

\textsuperscript{1}The dilation $f_r: \mathbb{D} \to \mathbb{C}$ is defined by $f_r(z) := f(rz), z \in \mathbb{C}$.
for $E$-valued polynomials
\[ f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in \mathcal{H} \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n \in \mathcal{H}', \]

In the recent paper [19, Section 3], the author provided a new analytic model on an annulus for left-invertible operators, which are not necessarily analytic operators. The construction of the analytic model on an annulus for a left-invertible operator $T \in B(\mathcal{H})$ is based on the following unitary isomorphism:
\[ U : \mathcal{H} \ni x \rightarrow \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^n x) z^n \in \mathcal{H}, \]
where $E$ is a closed subspace of $\mathcal{H}$ satisfying some certain condition (see (♣)). The model extends both Shimorin’s analytic model for left-invertible analytic operators (see [19, Theorem 3.3]) and Gellar’s model for a bilateral weighted shift (see [19, Example 5.2]). As shown in [19, Theorem 3.2 and 3.8] a left-invertible operator $T$, which satisfies certain conditions can be modelled as a multiplication operator on a reproducing kernel Hilbert space of vector-valued analytic functions on an annulus or a disc. We refer the reader to [24, 19] for more information on analytic model for left-invertible operator. For other results related to the analytic model see, for example [12, 14, 20, 13].

It is worth noting that the notion of the Cauchy dual operator is also interesting because the map $T \rightarrow T'$ sets up the correspondence between some classes of operators:
\[
\begin{array}{ccc}
T & \rightarrow & T' \\
\text{expansion} & \rightarrow & \text{contraction} \\
\text{2-hyperexpansive operator} & \rightarrow & \text{hyponormal contraction} \\
\text{completely hyperexpansive} & \rightarrow & \text{contractive subnormal} \\
\text{weighted shift} & \rightarrow & \text{weighted shift} \\
\cdots & \rightarrow & \cdots
\end{array}
\]

(see [9, pp. 639/640]). Recently the Cauchy dual subnormality problem, which asks whether the Cauchy dual operator of a 2-isometry is subnormal was solved negatively in the class of 2-isometric operators (see [3]). The topics related to the Cauchy dual operator are currently being studied intensively from several points of view (see e.g. [3, 4, 6, 9, 10, 11, 15]).

Let $T$ be a left-invertible operator such that both operators $T$ and $T'$ can be modelled as a multiplication operator on a reproducing kernel Hilbert space of vector-valued analytic functions on an annulus. In view of the above, one may ask whether the duality between $\mathcal{H}$ and $\mathcal{H}'$ obtained by identifying them with $\mathcal{H}$ is the same as the duality obtained from the Cauchy pairing? Can the Hilbert space $\mathcal{H}'$ be constructed in a similar way to (1.1)?

Research on these problems led us to investigate a pair $(\mathcal{B}, \Psi)$ consisting of a reflexive Banach spaces $\mathcal{B}$ of vector-valued analytic functions on which a left-invertible multiplication operator $\mathcal{M}_z$ acts and an operator-valued holomorphic function $\Psi : (\Omega')^* \rightarrow B(E, \mathcal{B})$ which satisfies the following conditions:

(A1) $\mathcal{B} \ni \text{Hol}(\Omega, E)$ the inclusion map is both injective and continuous (the space Hol$(\Omega, E)$ with the topology of uniform convergence on compact sets),

(A2) the subspace $\text{lin}\{\Psi(\lambda)e : e \in E, \lambda \in \Omega'\}$ is dense in $\mathcal{B}$,
(A3) $\mathcal{L}\Psi(\lambda) = \bar{\lambda}\Psi(\lambda)$ for every $\lambda \in \Omega'$,  
where $E$ is a Hilbert space and $\Omega, \Omega' \subset \mathbb{C}$ are open sets.

It turns out that this type of Banach spaces include the classical Banach spaces of holomorphic functions in the unit disc: the Hardy space, the Bergman space and the Dirichlet space (see Example 5.1) as well as the Hilbert spaces of vector-valued analytic functions on an annulus $\mathcal{H}$ associated with the analytic models for left-invertible operators (see Examples 5.2 and 5.3). We prove that there exist a dual pair $(B', \Psi')$ such that the space $B'$ is unitarily equivalent to the space $B^*$ and the following intertwining relations hold

$$\mathcal{L}U = \mathcal{U} \mathcal{L}^* \quad \text{and} \quad U\mathcal{L} = \mathcal{U} \mathcal{L}^*_U,$$

where $\mathcal{U}$ is the unitary operator between $B'$ and $B^*$. In addition we show that $\Psi$ and $\Psi'$ are connected through the relation

$$\langle (\Psi'(z)e_1)(\lambda), e_2 \rangle = \langle e_1, (\Psi(\bar{\lambda})e_2)(z) \rangle$$

for every $e_1, e_2 \in E$, $z \in \Omega$, $\lambda \in \Omega'$. We define the dual space $B'$ as the image of $B^*$ by $\mathcal{U} : B^* \to \text{Hol}(\Omega', E)$ with the norm induced from $B^*$, where $\mathcal{U}$ is given by

$$\varphi(\Psi(\bar{\lambda})e) = (e, (\varphi(\lambda))(z)), \quad e \in E, \lambda \in \Omega'.$$

We describe the relationship between the analytic model for $T$ and analytic model for the Cauchy dual operator $T'$. Namely, we prove that Hilbert space associated with the analytic model for $T'$ coincides with the Hilbert space obtained in the above construction with $\mathcal{H}$ in place of $B$ and $\Psi : (\Omega')^* \to B(E, \mathcal{H})$ defined by

$$\Psi(\lambda) := \sum_{n=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\lambda^n} \mathcal{L}^n + \sum_{n=0}^{\infty} \lambda^n \mathcal{M}^n_z, \quad \lambda \in (\Omega')^*.$$  

Moreover, we characterize left-invertible operators, for which the duality between $\mathcal{H}$ and $\mathcal{H}'$ obtained by identifying them with $\mathcal{H}$ is the same as the duality obtained from the Cauchy pairing.

2. Preliminaries

In this paper, we use the following notation. The field of complex numbers is denoted by $\mathbb{C}$. The symbols $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ stand for the sets of integers, positive integers and nonnegative integers, respectively. Set $\mathbb{D}(r) = \{ z \in \mathbb{C} : |z| < r \}$ and $\mathbb{A}(r, r') = \{ z \in \mathbb{C} : r^- < |z| < r^+ \}$ for $r, r^- < r^+ \in [0, \infty)$. If $\Omega \subset \mathbb{C}$ then $\Omega^*$ is the set $\{ z : z \in \Omega \}$. We denote by card($X$) the cardinal number of a set $X$.

All Hilbert spaces considered in this paper are assumed to be complex. Let $T$ be a linear operator in a complex Hilbert space $\mathcal{H}$. Denote by $T^*$ the adjoint of $T$. We write $B(\mathcal{H})$ for the $C^*$-algebra of all bounded operators in $\mathcal{H}$. Let $T \in B(\mathcal{H})$. The spectrum and spectral radius of $T \in B(\mathcal{H})$ is denoted by $\sigma(T)$ and $r(T)$ respectively.

We say that $T$ is left-invertible if there exists $S \in B(\mathcal{H})$ such that $ST = I$. We call $T$ analytic if $\mathcal{H}_\infty = \bigcap_{n=1}^{\infty} T^n \mathcal{H} = \{ 0 \}$.

Let $X$ be a countable set and $\varphi : X \to X$ be a selfmap. If $n \in \mathbb{Z}_+$, then the $n$-th iterate of $\varphi$ is given by $\varphi^{(n)} = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n \text{ times}}, \varphi$ composed with itself $n$-times and $\varphi^{(0)}$ is identity function. For $x \in X$ the set

$$[x]_\varphi = \{ y \in X : \text{there exist } i, j \in \mathbb{N} \text{ such that } \varphi^{(i)}(x) = \varphi^{(j)}(y) \}$$
is called the orbit of \( \varphi \) containing \( x \). If \( x \in X \) and \( \varphi^{(i)}(x) = x \) for some \( i \in \mathbb{Z}_+ \), then the cycle of \( \varphi \) containing \( x \) is the set

\[
\mathcal{C}_\varphi = \{ \varphi^{(i)}(x) : i \in \mathbb{N} \}.
\]

Define the function \([\varphi] : X \to \mathbb{Z}\) by

\[
\begin{align*}
(i) & \quad [\varphi](x) = 0 \text{ if } x \text{ is in the cycle of } \varphi, \\
(ii) & \quad [\varphi](x^*) = 0, \text{ where } x^* \text{ is a fixed element of orbit } F \text{ of } \varphi \text{ not containing a cycle,} \\
(iii) & \quad [\varphi](\varphi(x)) = [\varphi](x) - 1 \text{ if } x \text{ is not in a cycle of } \varphi.
\end{align*}
\]

We set

\[
\text{Gen}_\varphi(m, n) := \{ x \in X : m \leq [\varphi](x) \leq n \}
\]

for \( m, n \in \mathbb{Z} \). We say that \( \varphi \) has finite branching index if

\[
\sup \{ [[\varphi](x)] : \text{card}(\varphi^{-1}(x)) \geq 2, \ x \in X \} < \infty.
\]

Let \( w : X \to \mathbb{C} \) be a complex function on \( X \). By a weighted composition operator \( C_{\varphi, w} \) in \( \ell^2(X) \) we mean a mapping

\[
\begin{align*}
\mathcal{D}(C_{\varphi, w}) &= \{ f \in \ell^2(X) : w(f \circ \varphi) \in \ell^2(X) \}, \\
C_{\varphi, w}f &= w(f \circ \varphi), \quad f \in \mathcal{D}(C_{\varphi, w}).
\end{align*}
\]

We call \( \varphi \) and \( w \) the symbol and the weight of \( C_{\varphi, w} \) respectively. Let us recall some useful properties of composition operator we need in this paper:

**Lemma 2.1** ([19, Lemma 2.1]). Let \( X \) be a countable set, \( \varphi : X \to X \) be a selfmap and \( w : X \to \mathbb{C} \) be a complex function. If \( C_{\varphi, w} \in \mathcal{B}(\ell^2(X)) \), then for any \( x \in X \) and \( n \in \mathbb{N} \)

\[
\begin{align*}
(i) & \quad C_{\varphi, w}^*e_x = w(x)e_{\varphi(x)}, \\
(ii) & \quad C_{\varphi, w}e_x = \sum_{y \in \varphi^{-1}(x)} w(y)e_y, \\
(iii) & \quad C_{\varphi, w}^n e_x = w(x)w(\varphi(x)) \cdots w(\varphi^{(n-1)}(x))e_{\varphi^{(n)}(x)}, \\
(iv) & \quad C_{\varphi, w}^* e_x = \sum_{y \in \varphi^{-1}(x)} w(y)w(\varphi(y)) \cdots w(\varphi^{(n-1)}(y))e_y, \\
v) & \quad C_{\varphi, w}^* C_{\varphi, w} e_x = \left( \sum_{y \in \varphi^{-1}(x)} |w(y)|^2 \right) e_x.
\end{align*}
\]

We now describe the Cauchy dual of a weighted composition operator.

**Lemma 2.2** ([19, Lemma 2.2]). Let \( X \) be a countable set, \( \varphi : X \to X \) be a selfmap and \( w : X \to \mathbb{C} \) be a complex function. If \( C_{\varphi, w} \in \mathcal{B}(\ell^2(X)) \), then the Cauchy dual \( C_{\varphi, w}^* \) of \( C_{\varphi, w} \) is also a weighted composition operator \( C_{\varphi', w} \) with the same symbol \( \varphi : X \to X \) and weight \( w' : X \to \mathbb{C} \) defined by

\[
w'(x) := \frac{w(x)}{\sum_{y \in \varphi^{-1}(\varphi(x))} |w(y)|^2}.
\]

### 3. Analytic model

Since the analytic model for left-invertible operator introduced in the recent paper [19] by the author plays a major role in this paper, we outline it in the following discussion. Let \( T \in \mathcal{B}(\mathcal{H}) \) be a left-invertible operator and \( E \) be a closed subspace of \( \mathcal{H} \) denote by \([E]_{T^* \cdot T} \) the following subspace of \( \mathcal{H} \):

\[
[E]_{T^* \cdot T} := \bigvee \left( \{T'^*x : x \in E, n \in \mathbb{N} \} \cup \{T'^nx : x \in E, n \in \mathbb{N} \} \right)
\]
where $T'$ is the Cauchy dual of $T$.

To avoid repetition, we state the following assumption which will be used frequently in this paper.

\[ \text{(♣) The operator } T \in B(H) \text{ is left-invertible and } E \text{ is a closed subspace of } H \text{ such that } [E]_{T^*, T'} = H. \]

Suppose (♣) holds. In this case we may construct a Hilbert $\mathcal{H}$ associated with $T$, of formal Laurent series with vector coefficients. We proceed as follows. For each $x \in H$, define a formal Laurent series $U_x$ with vector coefficients as

\[ U_x(z) = \sum_{n=1}^{\infty} (P_E T^{*n} x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^{*n} x) z^n. \]

Let $\mathcal{H}$ denote the vector space of formal Laurent series with vector coefficients of the form $U_x$, $x \in H$. Consider the map $U : H \to \mathcal{H}$ defined by $Ux := U_x$. As shown in [19, Lemma 3.1] $U$ is injective. In particular, we may equip the space $\mathcal{H}$ with the norm induced from $H$, so that $U$ is unitary. Observe that every $f \in \mathcal{H}$ can be represented as follows

\[ f = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n, \]

where

\[ \hat{f}(n) = \begin{cases} P_E T^{*n} U^* f & \text{if } n \in \mathbb{N}, \\ P_E T'^{-n} U^* f & \text{if } n \in \mathbb{Z} \setminus \mathbb{N}. \end{cases} \]

By [19, Theorem 3.2] the operator $T$ is unitarily equivalent to the operator $M_z : H \to H$ of multiplication by $z$ on $H$ given by

\[ (M_z f)(z) = zf(z), \quad f \in \mathcal{H} \]

and operator $T'^*$ is unitarily equivalent to the operator $L : \mathcal{H} \to \mathcal{H}$ given by

\[ (L f)(z) = \frac{f(z) - (P_{N(H)} f)(z)}{z}, \quad f \in \mathcal{H}. \]

Following [24], the reproducing kernel for $\mathcal{H}$ is an $B(E)$-valued function of two variables $\kappa_{\mathcal{H}} : \Omega \times \Omega \to B(E)$ that

(i) for any $e \in E$ and $\lambda \in \Omega$

\[ \kappa_{\mathcal{H}}(\cdot, \lambda)e \in \mathcal{H}, \]

(ii) for any $e \in E$, $f \in \mathcal{H}$ and $\lambda \in \Omega$

\[ (f(\lambda), e)_E = (f, \kappa_{\mathcal{H}}(\cdot, \lambda)e)_E. \]

It turns out that if the series (3.1) is convergent in $E$ on $\Omega \subset \mathbb{C}$ for every $x \in H$, then $\mathcal{H}$ is a reproducing kernel Hilbert space of vector-valued holomorphic functions on $\Omega$ (see [19, Theorem 3.8]).

For left-invertible operator $T \in B(H)$, among all subspaces satisfying condition (♣) we will distinguish those subspaces $E$ which satisfy the following condition

\[ \text{(♠) } E \perp T^n E \quad \text{and} \quad E \perp T'^n E, \quad n \in \mathbb{Z}_+. \]
4. Duality

In this section, we will consider a quintuple $(B, \Psi, \mathcal{L}, \mathcal{M}_z)$ consisting of: a reflexive Banach space $B$ of $E$-valued analytic functions on which a left-invertible multiplication operator $\mathcal{M}_z : B \rightarrow B$, defined by

$$(\mathcal{M}_z f)(z) = z f(z), \quad f \in B,$$

acts, $\mathcal{L} : B \rightarrow B$ is a left inverse of $\mathcal{M}_z$ and $\Psi : (\Omega')^* \rightarrow \mathcal{B}(E, B)$ is an operator-valued holomorphic function, where $E$ is a Hilbert space and $\Omega, \Omega' \subset \mathbb{C}$ are open sets. We assume that the following conditions hold:

(A1) $B \hookrightarrow \text{Hol}(\Omega, E)$ the inclusion map is both injective and continuous (the space $\text{Hol}(\Omega, E)$ with the topology of uniform convergence on compact sets),

(A2) the subspace $\text{lin}\{\Psi(\lambda)e : e \in E, \lambda \in \Omega'\}$ is dense in $B$,

(A3) $\mathcal{L}\Psi(\lambda) = \lambda \Psi(\lambda)$ for every $\lambda \in \Omega'$.

We prove that there exist a dual quintuple $(B', \Psi', \mathcal{L}', \mathcal{M}_z')$ such that the space $B'$ is unitarily equivalent to the space $B^*$ and the unitary operator $\mathcal{U} : B^* \rightarrow B'$ between these spaces intertwines $\mathcal{M}_z^*$ and $\mathcal{L}^*$ on $B^*$ with the $\mathcal{L}$ and $\mathcal{M}_z$ on $B'$, respectively, that is:

$$\mathcal{L}\mathcal{U} = \mathcal{U}\mathcal{M}_z^* \quad \text{and} \quad \mathcal{M}_z\mathcal{U} = \mathcal{U}\mathcal{L}^*.$$

In addition, we show that $\Psi$ and $\Psi'$ are connected through the relation

$$(\Psi(\bar{z})e_1)(\lambda), e_2) = \langle e_1, (\Psi(\bar{\lambda})e_2)(z) \rangle$$

for every $e_1, e_2 \in E, z \in \Omega, \lambda \in \Omega'$.

Now we provide a construction of the dual quintuple $(B', \Psi', \mathcal{L}', \mathcal{M}_z')$. Define an operator $\mathcal{U} : B^* \rightarrow \text{Hol}(\Omega', E)$ such that for every $\varphi \in B^*$ the following equation holds

$$(\Psi(\bar{z})e)(\lambda) = \langle e, (\mathcal{U}\varphi)(\lambda) \rangle, \quad e \in E, \lambda \in \Omega'.$$

Now let $B'$ be the image of $B^*$ by $\mathcal{U}$ and equip this space with the norm induced from $B^*$, so that $\mathcal{U}$ is unitary. Define an operator $\mathcal{L}' := \mathcal{U}\mathcal{M}_z\mathcal{U}^*$ and an operator-valued holomorphic function $\Psi' : \Omega' \rightarrow B(E, B')$ by

$$\Psi'(z)e = \mathcal{U}\varphi_{z,e}, \quad z \in \Omega', e \in E,$$

where a functional $\varphi_{z,e} : B \rightarrow \mathbb{C}$ for $e \in E, z \in \Omega$ is given by

$$\varphi_{z,e}(f) = \langle f(z), e \rangle, \quad f \in B.$$ 

Below, we show that $\mathcal{U}$ and $\Psi'$ are well-defined.

**Lemma 4.1.** Suppose that the quintuple $(B, \Psi, \mathcal{L}, \mathcal{M}_z)$ is as above. Then $\mathcal{U}$ and $\Psi'$ are well-defined. Moreover, the operator $\mathcal{U}$ is injective.

**Proof.** First we prove that $\mathcal{U}$ is well-defined. Fix $\varphi \in B^*$. Note that $E \ni e \rightarrow \varphi(\Psi(\bar{\lambda})e) \in \mathbb{C}$ for $\lambda \in \Omega'$ is a continuous linear functional thus by Riesz-Fréchet representation theorem (see [23, Theorem 12.5]) and (4.2) the function $\mathcal{U}\varphi$ is uniquely determined on the set $\Omega'$. Since $\Psi$ is an operator-valued holomorphic function we infer from (4.2) that $\mathcal{U}\varphi$ is a weakly holomorphic function. Using the fact that weakly holomorphic functions are strongly holomorphic (see [23, Theorem 3.31]), we see that $\mathcal{U}\varphi$ is strongly holomorphic, which yields $\mathcal{U}\varphi \in \text{Hol}(\Omega', E)$.
Since the set \( \{ \psi(\bar{\lambda})e : e \in E, \lambda \in \Omega' \} \) is dense in \( B \) the map \( U : B^* \ni \varphi \to U\varphi \in \text{Hol}(\Omega', E) \) is injective.

Now we justify that \( \Psi' \) is well-defined. Fix \( z \in \Omega \). It is easy to see that \( E \ni e \to \varphi_{z,e} \in B^* \) is a linear functional. This and (4.3) implies that \( \Psi'(z) \) is linear. By (A1) there exist a constant \( C > 0 \) such that
\[
\|f(z)\|_E \leq C\|f\|_B, \quad f \in B.
\]
This combined with (4.4) shows that
\[
(4.5) \quad |\varphi_{z,e}(f)| = |\langle f(z), e \rangle| \leq \|f(z)\|_E\|e\|_E \leq C\|f\|_B\|e\|_E.
\]
for \( f \in B \) and \( e \in E \). Hence, \( \varphi_{z,e} \) is bounded and
\[
\|\varphi_{z,e}\|_{B^*} \leq C\|e\|_E, \quad e \in E.
\]

Thus, by (4.3), we see that
\[
(4.6) \quad \|\Psi'(z)e\|_{B^*} = \|U\varphi_{z,e}\|_{B^*} = \|\varphi_{z,e}\|_{B^*} \leq C\|e\|_E
\]
for all \( e \in E \). Therefore \( \Psi'(z) \) is a bounded operator for every \( z \in \Omega \).

□

Now we prove that \( \Psi' \) is actually an operator-valued holomorphic function. In fact, we give two proofs of this theorem. The first appeals to the generalization of Hartogs’ theorem (see [18, Theorem 36.1]) which states that separately vector-valued holomorphic functions are strongly holomorphic. The second utilizes the Cauchy Integral Formula.

**Theorem 4.2.** Suppose that the quintuple \((B, \Psi, \mathcal{L}, \mathcal{M}_z)\) is as above. Then the following conditions hold:

(i) \( \Psi' \) is an operator-valued holomorphic function,

(ii) \( \Psi \) and \( \Psi' \) are connected through the relation (4.1) for every \( e_1, e_2 \in E, z \in \Omega, \lambda \in \Omega' \).

**Proof.** (ii) Combining (4.2), (4.3) with (4.4), we get
\[
\langle e_2, (\Psi'(\bar{z})e_1)(\lambda) \rangle = \langle e_2, (U\varphi_{z,e_1})(\lambda) \rangle = \varphi_{z,e_1}(\Psi(\bar{\lambda})e_2)
\]
for \( z \in \Omega, \lambda \in \Omega', e_1, e_2 \in E \).

(i) Fix \( e \in E \). Let \( \vartheta : \Omega^* \times \Omega' \to E \) be a two variable function defined by
\[
\vartheta(z, \lambda) = (\Psi'(z)e)(\lambda), \quad z \in \Omega^*, \lambda \in \Omega'.
\]
Combining (4.1) with the fact that \( \Psi \) is an operator-valued holomorphic function and \( B \) is a space of holomorphic functions, we deduce that \( \vartheta \) is a separately weakly holomorphic function. More precisely, \( \vartheta \) is a weakly holomorphic function in each variable \( z \) and \( \lambda \), while the other variable is held constant. Since weak holomorphic functions are strongly holomorphic (see [23, Theorem 3.31]), we deduce that \( \vartheta \) is separately strongly holomorphic function. By Hartogs’ theorem for vector-valued holomorphic functions (see [18, Theorem 36.1, p.265]), each separately vector-valued holomorphic function is strongly holomorphic thus \( \vartheta \) is strongly holomorphic.

We will now give an alternative proof of holomorphicity of \( \vartheta \) without using the generalized Hartogs’ theorem. We claim that \( \vartheta \) is a continuous function. Let
Let \( z_0 \in \Omega^* \) and choose \( r > 0 \) such that \( \overline{D(z_0, r)} \subset \Omega^* \). Put \( K := \overline{D(z_0, r)} \). By (A1) there exist \( C > 0 \) such that
\[
(4.7) \quad \|f(z)\|_E \leq C\|f\|_B, \quad z \in K, f \in B.
\]
Fix \( f \in B \) and \( e \in E \). We define a function \( g_{f,e} : \Omega \rightarrow \mathbb{C} \) by
\[
g_{f,e}(z) = \langle f(z), e \rangle, \quad z \in \Omega.
\]
Then, by (4.7)
\[
|g_{f,e}(z)| = |\langle f(z), e \rangle| \leq \|f(z)\|_E \|e\|_E \leq C\|f\|_B \|e\|_E, \quad z \in K,
\]
which gives
\[
(4.8) \quad \sup_{z \in K} |g_{f,e}(z)| \leq C\|f\|_B \|e\|_E.
\]
The Cauchy Integral Formula yields
\[
g_{f,e}(z) - g_{f,e}(z_0) = \frac{1}{2\pi i} \int_{\partial K} \left( \frac{g_{f,e}(\xi)}{\xi - z} - \frac{g_{f,e}(\xi)}{\xi - z_0} \right) d\xi
\]
\[
= \frac{z - z_0}{2\pi i} \int_{\partial K} \frac{g_{f,e}(\xi)}{(\xi - z)(\xi - z_0)} d\xi
\]
for \( z \in \mathbb{D}(z_0, r) \). By (4.8) and the standard integral estimate we obtain,
\[
|g_{f,e}(z) - g_{f,e}(z_0)| \leq \frac{|z - z_0|}{2\pi} \int_{\partial K} \left| \frac{g_{f,e}(\xi)}{(\xi - z)(\xi - z_0)} \right| d\xi \leq \frac{2}{r} |z - z_0| \sup_{\xi \in K} |g_{f,e}(\xi)|
\]
\[
\leq \frac{2}{r} |z - z_0| C\|f\|_B \|e\|_E
\]
for \( z \in \mathbb{D}(z_0, \frac{r}{2}) \). It follows from the above that
\[
|\varphi_{z,e} - \varphi_{z_0,e}||(f)\| \overset{(4.3)}{=} |\langle f(z), e \rangle - \langle f(z_0), e \rangle| = |g_{f,e}(z) - g_{f,e}(z_0)|
\]
\[
\leq \frac{2}{r} |z - z_0| C\|f\|_B \|e\|_E
\]
for \( z \in \mathbb{D}(z_0, \frac{r}{2}) \). Since \( f \) was arbitrarily chosen, we get
\[
(4.9) \quad \|\varphi_{z,e} - \varphi_{z_0,e}\|_{B^*} \leq |z - z_0| C\|e\|_E \frac{2}{r}, \quad z \in \mathbb{D}(z_0, \frac{r}{2}).
\]
Let \( \lambda_0 \in \Omega' \) and choose \( \rho > 0 \) such that \( \overline{\mathbb{D}(\lambda_0, \rho)} \subset \Omega' \). By continuity of \( \Psi \), there exist a constant \( D > 0 \) such that \( \|\Psi(\lambda)\|_{B(E,B)} < D \) for \( \lambda \in \overline{\mathbb{D}(\lambda_0, \rho)} \). Thus, by (4.2)
\[
|\langle e, (U\varphi)(\lambda) \rangle| \leq \|\varphi\|_{B^*} \|\Psi(\lambda)e\|_B \leq D\|\varphi\|_{B^*} \|e\|_E
\]
for \( e \in E, \lambda \in \overline{\mathbb{D}(\lambda_0, \rho)} \) and \( \varphi \in B^* \), which implies
\[
\|(U\varphi)(\lambda)\|_E \leq D\|\varphi\|_{B^*}, \quad \lambda \in \overline{\mathbb{D}(\lambda_0, \rho)}, \varphi \in B^*.
\]
This combined with (4.9) gives
\[
\|U\varphi_{z,e}(\lambda) - U\varphi_{z_0,e}(\lambda)\|_E \leq D\|U\varphi_{z,e} - U\varphi_{z_0,e}\|_{B^*} \leq D\|\varphi_{z,e} - \varphi_{z_0,e}\|_{B^*}
\]
\[
\leq CD|z - z_0|\|e\|_E \frac{2}{r}.
\]
for $z \in \mathbb{D}(z_0, \frac{r}{2})$ and $\lambda \in \overline{\mathbb{D}(\lambda_0, \rho)}$. Hence, we have

$$
\|\vartheta(z, \lambda) - \vartheta(z_0, \lambda_0)\|_E = \|(\Psi'(z)e)(\lambda) - (\Psi'(z_0)e)(\lambda_0)\|_E \\
\overset{(4.3)}{=} \|U\varphi_{z,e}(\lambda) - U\varphi_{z_0,e}(\lambda_0)\|_E \\
\leq \|U\varphi_{z,e}(\lambda) - U\varphi_{z_0,e}(\lambda)\|_E + \|U\varphi_{z_0,e}(\lambda) - U\varphi_{z_0,e}(\lambda_0)\|_E \\
\leq CD\|z - z_0\|_E^{2} + \|U\varphi_{z_0,e}(\lambda) - U\varphi_{z_0,e}(\lambda_0)\|_E
$$

for $z \in \mathbb{D}(z_0, \frac{r}{2})$ and $\lambda \in \overline{\mathbb{D}(\lambda_0, \rho)}$, which yields $\vartheta$ is continuous. By [18, Lemma 8.9] a function is holomorphic if and only if it is separately holomorphic and continuous, which implies that $\vartheta$ is strongly holomorphic.

Since $\vartheta$ is strongly holomorphic, we infer from [16, Corollary 15.3.3] that a function

$$
\Omega \ni z \mapsto (\Omega' \ni \lambda \mapsto \vartheta(\lambda, z) \in E) \in \mathcal{B}
$$

is a $\mathcal{B}$-valued holomorphic function. Therefore $\Omega \ni z \mapsto \Psi'(z)e \in \mathcal{B}$ for all $e \in \mathcal{E}$ is also strongly holomorphic. By criterion for the holomorphy of operator-valued functions (see [17, Theorem 1.7.1]), $\Psi'$ is an operator-valued holomorphic function.

Our next goal is to show that the quintuple $(\mathcal{B}', \Psi', \mathcal{L}', \mathcal{M}_z)$ satisfies the conditions (A1)-(A3). The next theorem is inspired by [2, Proposition 5.2] (cf. Example 5.1).

**Theorem 4.3.** Suppose that the quintuple $(\mathcal{B}, \Psi, \mathcal{L}, \mathcal{M}_z)$ is as above. Then the quintuple $(\mathcal{B}', \Psi', \mathcal{L}', \mathcal{M}_z)$ satisfies the conditions (A1)-(A3). Moreover,

(i) the following intertwining relations hold

$$
\mathcal{L}U = U\mathcal{M}_z^* \quad \text{and} \quad \mathcal{M}_z\mathcal{U} = U\mathcal{L}^*,
$$

(ii) $\Psi$ and $\Psi'$ are connected through the relation

$$
\langle (\Psi'(z)e_1)(\lambda), e_2 \rangle = \langle e_1, (\Psi(\bar{\lambda})e_2)(z) \rangle
$$

for every $e_1, e_2 \in \mathcal{E}$, $z \in \Omega$, $\lambda \in \Omega'$

**Proof.** Condition (ii) follows from Theorem 4.2.

Let $K$ be a compact subset of $\Omega'$. Since $\Psi$ is continuous, there exist a constant $C_K > 0$ such that $\|\Psi(\lambda)\|_{\mathcal{B}(\mathcal{E}, \mathcal{E})} < C_K$ for $\lambda \in K$. By (4.2), we have

$$
|\langle e, (U\varphi)(\lambda) \rangle| = |\varphi(\bar{\lambda})e| \leq \|\varphi\|_{\mathcal{B}'} \cdot \|\bar{\lambda}\|_{\mathcal{B}}\|e\|_E \leq C_K\|U\varphi\|_{\mathcal{B}'L_2} \|e\|_E,
$$

for $e \in \mathcal{E}, \varphi \in \mathcal{B}'$, $\lambda \in K$, which yields

$$
\|\langle U\varphi(\lambda) \rangle \| \leq C_K\|U\varphi\|, \quad \varphi \in \mathcal{B}', \lambda \in K.
$$

This proves condition (A1). We show that the subspace $\text{lin}\{\Psi'(\bar{z})e : e \in \mathcal{E}, z \in \Omega\}$ is dense in $\mathcal{B}'$. Let $V := \text{lin}\{\varphi_{z,e} : e \in \mathcal{E}, z \in \Omega\}$. Suppose that the subspace $V$ is not dense in $\mathcal{B}'$. Then there exist $F \in \mathcal{B}''$ such that $F \neq 0$ and $F|_V = 0$. By reflexivity there exist $f \in \mathcal{B}$ such that

$$
F(\varphi) = \varphi(f), \quad \varphi \in \mathcal{B}'.
$$

This implies that

$$
F(\varphi_{z,e}) = \langle f(z), e \rangle, \quad e \in \mathcal{E}, z \in \Omega.
$$
Since $F|_V = 0$, we deduce that $(f(z),e) = 0$ for every $e \in E$ and $z \in \Omega$. By Identity theorem $f = 0$. This shows that $F = 0$ and thus $V$ is dense in $B^*$. An application of (4.3) completes the proof of property (A2).

We show that $L\Psi'(\bar{z}) = \bar{z}\Psi'(\bar{z})$, for every $z \in \Omega$. Since $L = U^{\ast}MzU$, by (4.1) the following equalities hold

$$\langle e_1,(L\Psi'(\bar{z})e_2)(\lambda) \rangle = \langle e_1,(UM^\ast\varphi_{z,e_2}z)(\lambda)e_1 \rangle = \varphi_{z,e_2}(z)(\lambda)e_1 = \langle e_1,z(\Psi(\bar{z}))e_2(\lambda) \rangle$$

for every $e_1,e_2 \in E$, $z \in \Omega$ and $\lambda \in \Omega'$. This shows property (A3).

It remains to prove that $MzU = U^{\ast}L$. Combining (4.2) with (A3), we get

$$\langle e,UL^{\ast}\varphi(\lambda) \rangle = \varphi(L\Psi(\bar{z}))e_1 = \varphi(\bar{z}\Psi(\bar{z}))e_1 = \langle e,LM\varphi(\lambda) \rangle = \langle e,MzU\varphi(\lambda) \rangle$$

for $e \in E$, $\varphi \in B^*$.

\[\square\]

### 5. Examples

In this section, we collect a variety of examples of Banach spaces that satisfy the conditions (A1)-(A3). These examples include the classical Banach spaces of holomorphic functions in the unit disc: the Hardy space, the Bergman space and the Dirichlet space as well as the Hilbert spaces of vector-valued analytic functions on an annulus associated with analytic models for left-invertible operators.

In [2, Sec. 5] A. Aleman, S. Richter and W. T. Ross studied the Banach space $B$ of analytic functions on $\mathbb{D}$ which satisfies the following properties:

- (B1) $MzB \subset B$,
- (B2) $B \to \text{Hol}(\mathbb{D})$ the inclusion map is both injective and continuous (the space Hol($\mathbb{D}$) with the topology of uniform convergence on compact sets),
- (B3) $1 \in B$,
- (B4) $L\lambda B \subset B$,
- (B5) $\sigma(Mz) = \overline{\mathbb{D}}$,
- (B6) the polynomials are dense in $B$,
- (B7) $B$ is reflexive,

where for $\lambda \in \mathbb{D}$ an operator $L\lambda : B \to B$ is given by

$$(L\lambda f)(z) = \frac{f(z) - f(\lambda)}{z - \lambda}, \quad z \in \mathbb{D}.$$  

This example include the classical Banach spaces of holomorphic functions in the unit disc: the Hardy spaces, Bergman spaces and the Dirichlet spaces (see [2, Examples 1.3 and 1.4]).

**Example 5.1.** Let $B$ be a Banach space of analytic functions on $\mathbb{D}$ which satisfies the conditions (B1)-(B7). The multiplication operator $Mz$ is left-invertible and operator $L : B \to B$ given by

$$(L f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in B$$
is its left inverse. Let $\Psi : \mathbb{D} \to \mathcal{B}(\mathbb{C}, \text{Hol}(\mathbb{C}))$ be a linear map defined by

$$\Psi(\lambda)\omega := \omega k_{\lambda}, \quad \omega \in \mathbb{C}, \lambda \in \mathbb{D},$$

where $k_{\lambda} : \mathbb{C} \to \mathbb{C}, \lambda \in \mathbb{D}$ is a holomorphic function defined by

$$k_{\lambda}(z) = \frac{1}{1 - \lambda z}, \quad z \in \mathbb{D}.$$  

It is trivial that the quintuple $(\mathcal{B}, \Psi, L_z, M_z)$ satisfies conditions (A1) and (A3). The fact that quintuple satisfies condition (A2) follows from [2, Proposition 2.2].

Following [24, Definition 2.4], we say that $T \in \mathcal{B}(\mathcal{H})$ possesses the wandering subspace property, if

$$[\mathcal{N}(T^*)]_T = \bigvee\{T^n\mathcal{N}(T^*) : n \in \mathbb{N}\} = \mathcal{H}.$$  

It turns out that for a left-invertible operator $T$, $T$ is analytic if and only if the Cauchy dual $T'$ of $T$ possesses wandering subspace property (see [24, Proposition 2.7]). The next two examples are related to the Shimorin’s analytic model and the model constructed in [19, Section 3]. For the sake of completeness, we only provide definitions of the quintuple $(\mathcal{H}, \Psi, L_z, M_z)$ here, the justification is given in the next section (see Theorem 6.2).

**Example 5.2.** Let $T \in \mathcal{B}(\mathcal{H})$ be a left-invertible operator with the wandering subspace property and $E := \mathcal{N}(T^*)$. Let $\mathcal{H}$ be a Hilbert space of vector-valued analytic functions associated with $T$. The multiplication operator $M_z$ is left-invertible and $L_z : \mathcal{H} \to \mathcal{H}$ given by

$$(L_z f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in \mathcal{H},$$

is its left inverse. Let $\Psi : \mathbb{D} \to \mathcal{B}(E, \mathcal{B})$ be an operator-valued holomorphic function defined by

$$\Psi(\lambda) := \sum_{n=0}^{\infty} \lambda^n M_z^n.$$  

It turns out that the quintuple $(\mathcal{H}, \Psi, L_z, M_z)$ satisfies conditions (A1)-(A3).

**Example 5.3.** Let $T \in \mathcal{B}(\mathcal{H})$ be a left-invertible operator and $E$ be a closed subspace of $\mathcal{H}$ such that condition (6.2) below holds. Let $\mathcal{H}$ be a Hilbert space of vector-valued analytic functions associated with $T$, $L_z : \mathcal{H} \to \mathcal{H}$ be a left inverse of $M_z$ given by

$$(L_z f)(z) = \frac{f(z) - (P_{\mathcal{N}(\mathcal{M}_z^*)}f)(z)}{z}, \quad f \in \mathcal{H}.$$  

and $\Psi : (\mathcal{O}')^* \to \mathcal{B}(E, \mathcal{B})$ be an operator-valued holomorphic function defined by

$$\Psi(\lambda) := \sum_{n=1}^{\infty} \frac{1}{\lambda^n} L_z^n + \sum_{n=0}^{\infty} \lambda^n M_z^n,$$

where $\mathcal{O}'$ is as in (6.2). In the next section we show that the quintuple $(\mathcal{H}, \Psi, L_z, M_z)$ satisfies conditions (A1)-(A3).
6. Duality for analytic model

In this section, we show that the analytic model for a left-invertible operator $T$ is a natural example of a Banach space of vector-valued analytic functions considered in Section 4. We will describe the relationship between the analytic model for $T$ and the analytic model for the Cauchy dual operator $T'$.

The Cauchy dual operator $T'$ of a left-invertible operator is itself left-invertible. Assume now that there exist a closed subspace $E \subset \mathcal{H}$ such that $[E]_{T^*, T'} = \mathcal{H}$ and $[E]_{T, T'} = \mathcal{H}$ hold. Then for both operators $T$ and $T'$ one can construct Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ of $E$-valued Laurent series. Therefore, by (3.1) $\mathcal{H}'$ is the space of Laurent series of the form $U'_x$, $x \in \mathcal{H}$, where

$$U'_x(z) := \sum_{n=1}^{\infty} (P_E T^m x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T^{m+n} x) z^n. \tag{6.1}$$

To avoid repetition, we state the following assumption which will be used frequently in this section.

(6.2)

The operator $T \in \mathcal{B}(\mathcal{H})$ is left-invertible and $E$ is a closed subspace of $\mathcal{H}$ such that $[E]_{T^*, T'} = \mathcal{H}$ and $[E]_{T^*, T} = \mathcal{H}$. Suppose that the series (3.1) and (6.1) are convergent in $E$ on an annulus $\Omega := \mathbb{A}(r^-, r^+)$ and $\Omega' := \mathbb{A}(r'^-, r'^+)$ respectively, where $0 \leq r^- < r^+$ and $0 \leq r'^- < r'^+$.

We will be consider the quintuple $(\mathcal{H}, \Psi, \mathcal{L}, \mathcal{M}_z)$, where

- $\mathcal{H}$ is a Hilbert space of vector-valued analytic functions associated with $T$,
- $\mathcal{M}_z : \mathcal{H} \to \mathcal{H}$ is a multiplication operator,
- $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ is a left inverse of $\mathcal{M}_z$ given by
  $$\mathcal{L}(f)(z) = \frac{f(z) - (P_N(M_z^*) f)(z)}{z}, \quad f \in \mathcal{H},$$
- $\Psi : (\Omega')^* \to \mathcal{B}(E, B)$ is an operator-valued holomorphic function defined by
  $$\Psi(\lambda) := \sum_{n=1}^{\infty} \frac{1}{\lambda^n} \mathcal{L}^n + \sum_{n=0}^{\infty} \lambda^n \mathcal{M}_z^n. \tag{6.3}$$

The following lemma shows that $\Psi$ is well-defined.

**Lemma 6.1.** Suppose (6.2) holds. Then the following conditions hold:

(i) The series in (6.3) converges absolutely and uniformly in operator norm on any compact subset contained in $\mathbb{A}(r'^-, r'^+)$.  
(ii) The function $\Psi$ is well-defined and holomorphic on $\mathbb{A}(r'^-, r'^+)$.

**Proof.** (i) By [19, Theorem 3.8] with $T'$ in place of $T$ the series

$$\sum_{n=1}^{\infty} (P_E T'^n) \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} (P_E T'^{n+1}) \lambda^n \in \mathcal{B}(\mathcal{H}, E) \tag{6.4}$$

converges absolutely and uniformly in operator norm on any compact subset contained in $\mathbb{A}(r'^-, r'^+)$. Therefore, the series

$$\sum_{n=1}^{\infty} (T'^n P_E) \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} (T^n P_E) \lambda^n \in \mathcal{B}(E, \mathcal{H})$$
also converges absolutely and uniformly in operator norm on any compact subset contained in $A(r^-, r^+)$. This combined with the fact that the operators $T, T^*$ are unitarily equivalent to the operators $\mathcal{M}, \mathcal{L}$ respectively, completes the proof.

(ii) This is a direct consequence of (i). $\square$

We now show that the quintuple $(\mathcal{H}, \Psi, \mathcal{L}, \mathcal{M}_z)$ satisfies properties (A1)-(A3).

**Theorem 6.2.** Suppose (6.2) holds. Then the quintuple $(\mathcal{H}, \Psi, \mathcal{L}, \mathcal{M}_z)$ satisfies properties (A1)-(A3), that is

(i) the inclusion map

$$i : \mathcal{H} \hookrightarrow \text{Hol}(A(r^-, r^+), E),$$

is both injective and continuous, where $\text{Hol}(A(r^-, r^+), E)$ is with the topology of uniform convergence on compact sets.

(ii) the subspace $\text{lin}\{\Psi(\lambda)e : e \in E, \lambda \in \Omega\}$ is dense in $\mathcal{B}$.

(iii) $L\Psi(\lambda) = \lambda \Psi(\lambda)$.

**Proof.** (i) Since by [19, Theorem 3.8] the series

$$\sum_{n=1}^{\infty} (P_E T^n) \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} (P_E T^* n) \lambda^n \in \mathcal{B}(\mathcal{H}, E)$$

converges absolutely and uniformly in operator norm on any compact set contained in $A(r^-, r^+)$ there exist constant $C_K > 0$ for every compact subset $K \subset A(r^-, r^+)$ such that

$$\| \sum_{n=1}^{\infty} (P_E T^n) \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} (P_E T^* n) \lambda^n \| \leq C_K, \quad \lambda \in K.$$ 

This implies that

$$\| Ux(\lambda) \|_E = \| \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} (P_E T^* n x) \lambda^n \|_E$$

$$\leq C_K \| x \|_\mathcal{H} = C_K \| Ux \|_\mathcal{H}$$

for $x \in \mathcal{H}$ and $\lambda \in K$. Therefore, the inclusion map $i$ is continuous in the topology of uniform convergence on compact sets.

(ii) Suppose that there exist $f \in \mathcal{H}$ such that

$$\langle \Psi(\lambda)e, f \rangle = 0, \quad e \in E, \lambda \in A(r^-, r^+).$$

This is equivalent to

$$\sum_{n=1}^{\infty} (P_E T^n U f) \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} (P_E T^* n U f) \lambda^n = 0, \quad \lambda \in A(r^-, r^+).$$

An application of [19, Lemma 3.1] with $T'$ in place of $T$ completes the proof of assertion (ii).

(iii) By [19, Theorem 3.2], we have

$$\mathcal{L} = UT^*,$$

$$\Psi(\lambda) = U(\sum_{n=1}^{\infty} \frac{1}{\lambda^n} T^n + \sum_{n=0}^{\infty} \lambda^n T^n).$$

An easy calculation shows that (iii) holds.
Now, we show that both Hilbert space constructed for the Cauchy dual operator in (3.1) and the Cauchy dual space obtained in construction (4.2) coincides.

**Theorem 6.3.** Suppose that (6.2) holds. Then the Hilbert space constructed for the Cauchy dual operator $T'$ in (3.1) coincide with the Cauchy dual space obtained in construction (4.2) for the quintuple $(\mathcal{H}, \Psi, \mathcal{L}, \mathcal{M}_z)$. Moreover, if $\varphi \in \mathcal{H}^*$ is represented by $g \in \mathcal{H}$, that is,

$$\varphi(f) = (f, g), \quad f \in \mathcal{H},$$

then $U\varphi = U'T^*g$.

**Proof.** Note that

$$\varphi(\bar{\Psi}(\lambda)e) = \langle \bar{\Psi}(\lambda)e, g \rangle_{\mathcal{H}} = \langle U(\sum_{n=1}^{\infty} \frac{1}{\lambda^n} T^{*n}e + \sum_{n=0}^{\infty} \bar{\lambda}^n T^n e), g \rangle_{\mathcal{H}}$$

$$= \langle \sum_{n=1}^{\infty} \frac{1}{\lambda^n} T^{*n}e + \sum_{n=0}^{\infty} \bar{\lambda}^n T^n e, U^*g \rangle_{\mathcal{H}}$$

$$= \langle e, \sum_{n=1}^{\infty} (P_E T^n U^*g) \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} (P_E T^n U^*g) \bar{\lambda}^n \rangle_{\mathcal{H}}.$$

for every $e \in E$, $\lambda \in \mathbb{A}(r^-, r^+)$. Therefore

$$U\varphi(\lambda) = \sum_{n=1}^{\infty} (P_E T^n U^*g) \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} (P_E T^n U^*g) \bar{\lambda}^n$$

$$= (U'U^*g)(\lambda),$$

for $\lambda \in \mathbb{A}(r^-, r^+)$. This completes the proof.

Our next aim is to characterise when the duality between $\mathcal{H}$ and $\mathcal{H}'$ obtained by identifying them with $\mathcal{H}$ is the same as the duality obtained from the Cauchy pairing. Let us point out that the Cauchy pairing in (1.3) is between two $E$-valued polynomials. Note that if left invertible operator possesses the wandering subspace property, then $E$-valued polynomials are dense in $\mathcal{H}$. Therefore, in order to obtain an analogue of (1.3) we replace $E$-valued polynomials with a dense subspace $\mathcal{V} := \mathcal{H} \cap \text{Hol}(\mathbb{A}(r^-, \infty), E)$ of $\mathcal{H}$, which includes polynomials. First, we prove the following auxilliary lemma.

**Lemma 6.4.** Suppose that (6.2) holds, $r^- < 1$ and $r'^- \leq 1 \leq r'^+$. Then there exist an open neighbourhood $W \subset (0, \infty)$ of 1 such that the series below is convergent absolutely for every $r \in W$, $f \in \mathcal{V} := \mathcal{H} \cap \text{Hol}(\mathbb{A}(r^-, \infty), E)$ and

$$\lim_{r \to 1^+} \int_0^{2\pi} (\Psi(re^{it}))(f(re^{it})) \frac{dt}{2\pi} = \lim_{r \to 1^+} U \left( \sum_{n=1}^{\infty} \frac{1}{r^{2n}} T^{*n} P_E T^n U^* f + \sum_{n=0}^{\infty} r^{2n} T^n P_E T^{*n} U^* f \right)$$

$$= U \left( \sum_{n=1}^{\infty} T^{*n} P_E T^n U^* f + \sum_{n=0}^{\infty} T^n P_E T^{*n} U^* f \right).$$
PROOF. Take $\rho_1 \in (r^-, \infty)$ and $\rho_2 \in (r', r^+)$ such that $\rho_1 \rho_2 > 1$. By \cite[Theorem 3.8]{19} the series (6.4) is absolutely convergent in $E$ on an annulus $A(r^-, r^+)$ thus there exists a constant $M_1 > 0$ such that

$$\|P_E T'^s f\rho^2_n\| \leq M_1, \quad n \in \mathbb{N}.$$ 

Since the series (3.1) with $U^* f$ in place of $x$ is convergent there exist a constant $D_1 > 0$, such that

$$\|P_E T'^s f\rho^2_n\| \leq D_1, \quad n \in \mathbb{N}.$$ 

This implies that

$$\|T^n P_E T'^s f\| \leq \|T^n P_E\| \|P_E T'^s f\| \leq \frac{M_1 D_1}{(\rho_1 \rho_2)^n}.$$ 

Therefore, the series $\sum_{n=0}^{\infty} T^n P_E T'^s f z^n$ is absolutely convergent on $D(\rho_1 \rho_2)$ and we see that

$$\lim_{r \to 1} \sum_{n=0}^{\infty} r^{2n} T^n P_E T'^s f = \sum_{n=0}^{\infty} T^n P_E T'^s f.$$ 

Take $\rho_3 \in (r^-, \infty)$ and $\rho_4 \in (r', r^+)$ such that $\rho_3 \rho_4 < 1$. Similarly, we see that there exist constants $M_2, D_2 > 0$, such that

$$\|P_E T'^s f\| \|\frac{1}{\rho_3^2}\| \leq D_2, \quad \|P_E T'^s f\| \|\frac{1}{\rho_4^2}\| \leq M_2, \quad n \in \mathbb{N}.$$ 

As a consequence, we have

$$\|T'^s P_E T'^s f\| \|\frac{1}{\rho_3^2}\| \|P_E T'^s f\| \leq C_2 D_2 (\rho_3 \rho_4)^n.$$ 

We see that the series $\sum_{n=1}^{\infty} T'^s P_E T'^s f \frac{1}{\rho_3^2}$ converges in $A(\rho_3 \rho_4, \infty)$ and

$$\lim_{r \to 1} \sum_{n=1}^{\infty} \frac{1}{r^{2n}} T'^s P_E T'^s f = \sum_{n=1}^{\infty} T'^s P_E T'^s f.$$ 

This combined with (6.6) gives the second equality in (6.5).

Since $T'$ is unitarily equivalent to $T'$ (see Section 3), we have

$$\frac{1}{\lambda^m} \mathcal{L}^m(f(\lambda)) = \frac{1}{\lambda^m} U T'^s m \sum_{n=1}^{\infty} (P_E T'^s f) \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} (P_E T'^s f) \lambda^n)$$ 

for $\lambda \in A(r^-, \infty), m \in \mathbb{N}$. As a consequence, we get

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r e^{it}} \mathcal{L}^m(f(re^{it})) dt = \frac{1}{r^{2m}} U T'^s m P_E T'^s f,$$

for $r \in (r^-, \infty)$ and $m \in \mathbb{N}$. Similarly, we obtain that

$$\tilde{\lambda} m \tilde{z}^m f(\lambda) = \tilde{\lambda} m U \tilde{T} m (\sum_{n=1}^{\infty} (P_{\tilde{T}} T'^s f) \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} (P_{\tilde{T}} T'^s f) \lambda^n)$$ 

for $\lambda \in A(r^-, \infty), m \in \mathbb{N}$. Hence, we have

$$\int_0^{2\pi} r^{2m} e^{it} \tilde{z}^m f(re^{it}) dt \frac{2\pi}{2\pi} = r^{2m} U T'^s m P_E T'^s f.$$
for \( r \in (r_-, \infty) \) and \( m \in \mathbb{N} \). This, combined with (6.7), Lemma 6.1 and changing order of summation and integration yields

\[
\int_0^{2\pi} (\Psi(re^{it}))(f(re^{it})) \frac{dt}{2\pi} = \int_0^{2\pi} \left( \sum_{n=1}^{\infty} \frac{1}{r^n e^{it\theta}} \mathcal{L}^n + \sum_{n=0}^{\infty} r^n e^{it\theta} z^n \right) f(re^{it}) \frac{dt}{2\pi} = U \left( \sum_{n=1}^{\infty} \frac{1}{r^{2n}} T'^*n P_E T^n U^* f \right) + \sum_{n=0}^{\infty} r^{2n} T^n P_E T'^*n U^* f).
\]

This gives the first equality in (6.5) and completes the proof.

We are now in a position to prove the main theorem of this section.

**Theorem 6.5.** Suppose that (6.2) holds, \( r^- < 1 \) and \( r^- \leq r^+ \). Then the subspace \( \mathcal{V} := \mathcal{H} \cap \text{Hol}(\mathbb{A}(r^-, \infty), E) \) is dense in \( \mathcal{H} \), the limits in (i) and (ii) exist, the series in (iii) and (iv) converges and the following conditions are equivalent:

(i) \[ f = \lim_{r \to 1^-} \int_0^{2\pi} (\Psi(re^{it}))(f(re^{it})) \frac{dt}{2\pi}, \quad f \in \mathcal{V}, \]

(ii) \[ \varphi(f) = \lim_{r \to 1^-} \int_0^{2\pi} (f(re^{it}), U\varphi(re^{it})) \frac{dt}{2\pi}, \quad \varphi \in \mathcal{H}^*, f \in \mathcal{V}, \]

(iii) \[ \langle U^*f, U'^*g \rangle_{\mathcal{H}} = \sum_{n=-\infty}^{\infty} \langle \hat{f}(n), \hat{g}(n) \rangle_E, \quad g \in \mathcal{H}', f \in \mathcal{V}, \]

(iv) \[ \sum_{n=1}^{\infty} T'^*n P_E T^n U^* f + \sum_{n=0}^{\infty} T^n P_E T'^*n U^* f = U^* f, \quad f \in \mathcal{V}. \]

**Proof.** First, we prove that the subspace \( \mathcal{V} \) is dense in \( \mathcal{H} \). Note that the series (3.1) is convergent in \( E \) on an annulus \( \mathbb{A}(r^-, r^+) \). Hence, we see that the series

\[
UT'^*m e = \sum_{n=1}^{\infty} (P_E T'^*n T'^*m e) \frac{1}{2^n}, \quad m \in \mathbb{N}, e \in E
\]

is convergent on an annulus \( \mathbb{A}(r^-, \infty) \). Thus \( UT'^*m e \in \mathcal{V} \). Observe that

\[
UT^m e = z^m e, \quad m \in \mathbb{N}
\]

and \( UT^m e \in \mathcal{V} \). Since \([E]_{T^*, T} = \mathcal{H}\), this and (6.8) shows that \( \mathcal{V} \) is dense in \( \mathcal{H} \).

It follows from Lemma 6.4 that the limit in (i) exist and the series in (iv) converges. Fix any \( \varphi \in \mathcal{H}^* \) and \( f \in \mathcal{H} \). Note that by (4.2), we have

\[
\varphi(\int_0^{2\pi} (\Psi(re^{it}))(f(re^{it})) \frac{dt}{2\pi}) = \int_0^{2\pi} \varphi((\Psi(re^{it}))(f(re^{it}))) \frac{dt}{2\pi} = \int_0^{2\pi} (f(re^{it}), U\varphi(re^{it})) \frac{dt}{2\pi}.
\]

This combined with Lemma 6.4 shows that the limit in (ii) exist.
It follows from (3.2) that
\[
(6.10) \quad \sum_{n=-\infty}^{\infty} r^{2n} \langle \hat{f}(n), \hat{g}(n) \rangle = \sum_{n=1}^{\infty} \frac{1}{r^{2n}} \langle T^{*n} P_{E} T^n U^* f, g \rangle + \sum_{n=0}^{\infty} r^{2n} \langle T^n P_{E} T^{*n} U^* f, g \rangle
\]
for \( f \in \mathcal{V}, g \in \mathcal{H}' \) and \( r \in W \), where \( W \) is as in Lemma 6.4. By Lemma 6.4, we see that that the series in (iii) converges.

(i) \Rightarrow (ii) It follows from (6.9).

(ii) \Rightarrow (iii) Combining (6.10) with Lemma 6.4, we deduce that the following limit exists and
\[
(6.11) \quad \lim_{r \to 1} \sum_{n=-\infty}^{\infty} r^{2n} \langle \hat{f}(n), \hat{g}(n) \rangle = \sum_{n=-\infty}^{\infty} \langle \hat{f}(n), \hat{g}(n) \rangle, \quad f \in \mathcal{V}, g \in \mathcal{H}'.
\]
Fix \( g \in \mathcal{H}' \). Let \( \varphi \in \mathcal{H}^* \) be defined by
\[
\varphi(f) = \langle f, U U^* g \rangle, \quad f \in \mathcal{H}.
\]
Then (ii) combined with (6.11), Lemma 6.4 and Theorem 6.3 implies that
\[
(6.12) \quad \langle U^* f, U U^* g \rangle = \langle f, U U^* g \rangle = \varphi(f) = \lim_{r \to 1} \int_{0}^{2\pi} \langle f(re^{it}), U \varphi(re^{it}) \rangle \frac{dt}{2\pi}
\]
\[
= \lim_{r \to 1} \int_{0}^{2\pi} \langle f(re^{it}), g(re^{it}) \rangle \frac{dt}{2\pi}
\]
\[
= \lim_{r \to 1} \int_{0}^{2\pi} \langle \sum_{n=-\infty}^{\infty} \hat{f}(n)(re^{it})^n, \sum_{n=-\infty}^{\infty} \hat{g}(n)(re^{it})^n \rangle \frac{dt}{2\pi}
\]
\[
= \lim_{r \to 1} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_{0}^{2\pi} \langle \hat{f}(n)(re^{it})^n, \hat{g}(m)(re^{it})^m \rangle \frac{dt}{2\pi}
\]
\[
= \lim_{r \to 1} \sum_{n=-\infty}^{\infty} r^{2n} \langle \hat{f}(n), \hat{g}(n) \rangle = \sum_{n=-\infty}^{\infty} \langle \hat{f}(n), \hat{g}(n) \rangle.
\]

(iii)\Rightarrow(iv) Combining (6.10) with the fact that both series in (iii) and (iv) are convergent completes the proof of equivalence (iii)\Rightarrow(iv).

(iv) \Rightarrow (i) Using Lemma 6.4, we obtain
\[
\lim_{r \to 1} \int_{0}^{2\pi} \langle \Psi(re^{it}) f(re^{it}) \rangle \frac{dt}{2\pi} = U \left( \sum_{n=1}^{\infty} T^{*n} P_{E} T^n U^* f + \sum_{n=0}^{\infty} T^n P_{E} T^{*n} U^* f \right) = f
\]
for \( f \in \mathcal{V} \), which completes the proof.

\[\square\]

7. Weighted composition operators

In this section, we illustrate Theorem 6.5 by considering examples of composition operators. Since the analytic structure of composition operators plays a major role in this section, we outline it in the following discussion. Let \( X \) be a countable set, \( w : X \to \mathbb{C} \) be a complex function, \( \varphi : X \to X \) be a transformation of \( X \), which has finite branching index and \( C_{\varphi, w} \in B(\ell^2(X)) \) be a weighted composition operator. We only consider composition functions with one orbit, since an orbit induces a reducing subspace to which the restriction of the weighted composition operator
induces a directed graph \((X, E)\) given by
\[
E^\varphi := \{(x, y) \in X \times X : x = \varphi(y)\}.
\]
Perhaps it is appropriate at this point to note that a self-map with one orbit can have at most one cycle. The directed graph \((X, E^\varphi)\) is a directed three in the case of when \(\varphi\) has one orbit and does not have a cycle. The next lemma shows that in the case of rootless directed tree with finite branching index there exist some special vertex.

**Lemma 7.1** ([12, Lemma 6.1]). Let \(\mathcal{T} = (V, E)\) be a rootless directed tree with finite branching index \(m\). Then there exist a vertex \(\Omega \in V_{<}\) such that
\[
\text{card}(\text{Chi}(\text{par}(n)(\Omega))) = 1, \quad n \in \mathbb{Z}_+.
\]
Moreover, if \(V_{<}\) is non-empty, then there exists a unique \(\Omega \in V_{<}\) satisfying (7.2).

The vertex \(\Omega \in V_{<}\) appearing in the statement of Lemma 7.1 is called generalized root. We put \(x^* := \text{par}(\Omega)\) in the definition of function \([\varphi] : X \to \mathbb{Z}\) for orbit \(F\) of \(\varphi\) not containing a cycle (see Section 2).

The following lemma describes a subspace \(E \subset \ell^2(X)\), which satisfies condition (\(\spadesuit\)) with \(C_{\varphi,w}\) in place of \(T\).

**Lemma 7.2** ([19, Lemma 4.2]). Let \(X\) be a countable set, \(w : X \to \mathbb{C}\) be a complex function on \(X\) and \(\varphi : X \to X\) be a transformation of \(X\), which has finite branching index. Let \(C_{\varphi,w}\) be a weighted composition operator in \(\ell^2(X)\) and
\[
E := \left\{ \bigoplus_{x \in \text{Gen}_x(1,1)} \langle e_x \rangle \oplus N((C_{\varphi,w}|_{\ell^2(\text{Des}(x))})^*) \text{ when } \varphi \text{ has a cycle,} \right. \\
\left. \langle e_\Omega \rangle \oplus N(C_{\varphi,w}^*) \quad \text{otherwise}, \right.
\]
where \(\text{Des}(x) := \bigcup_{n=0}^\infty \phi(-n)(x)\) and \(\Omega\) is a generalized root of the tree. Then the subspace \(E\) has the following properties:
(i) \([E]_{C_{\varphi,w},C_{\varphi,w}^*} = \mathcal{H}\) and \([E]_{C_{\varphi,w},C_{\varphi,w}^*} = \mathcal{H}\),
(ii) \(E \perp C_{\varphi,w}^n E\) and \(E \perp C_{\varphi,w}^n E, n \in \mathbb{Z}_+\).

Suppose that the series (3.1) with \(C_{\varphi,w}\) in place of \(T\) is convergent in \(E\) on an annulus \(\mathbb{A}(r^-, r^+)\) with \(r^- < r^+\) and \(r^-, r^+ \in [0, \infty)\) for every \(x \in \mathcal{H}\). In [19, see (4.7) and (4.8)] the inner and outer radius of convergence for weighted composition operator was described only in terms of its weights. In this case (see [19, Theorem 4.3]), there exist a \(z\)-invariant reproducing kernel Hilbert space \(\mathcal{H}\) of \(E\)-valued holomorphic functions defined on the annulus \(\mathbb{A}(r^-, r^+)\) and a unitary mapping \(U : \ell^2(V) \to \mathcal{H}\) such that \(\mathcal{M}_z U = UC_{\varphi,w}\), where \(\mathcal{M}_z\) denotes the operator of multiplication by \(z\) on \(\mathcal{H}\). Moreover, in the case when \(\varphi\) does not have a cycle the linear subspace generated by \(E\)-valued polynomials in \(z\) and \(E\)-valued polynomials involving only negative powers of \(z\) is dense in \(\mathcal{H}\), that is
\[
\bigvee \{(z^n E : n \in \mathbb{N}) \cup \{ \frac{1}{z^n} \tilde{E} \oplus n \in \mathbb{Z}_+ \}) = \mathcal{H},
\]
where \(\tilde{E} := \bigvee \{e_x : x \in \text{Gen}_x(1,1)\}\). If \(\varphi\) has a cycle \(C_{\varphi}\), then there exist \(\tau\) functions \(f_1, \ldots, f_\tau\) on \(\mathbb{A}(r^-, r^+)\) given by the following Laurent series
\[
f_i(z) := \sum_{k=0}^\infty \sum_{j=1}^\tau \Lambda^k A^i_j \frac{1}{z^{kr+j}}, \quad i = 1, \ldots, \tau.
\]
where \( \tau := \text{card} \mathcal{G}_\varphi \), \( \Lambda := \prod_{x \in \mathcal{G}_\varphi} w(x) \) and \( A^i_j \in \tilde{E} \), \( i, j = 1, \ldots, \tau \) such that the linear subspace generated by \( E \)-valued polynomials in \( z \) and the above functions is dense in \( \mathcal{H} \), that is
\[
\bigvee \{ \{ z^n E : n \in \mathbb{N} \} \cup \{ f_i : i \in \{ 1, \ldots, \tau \} \} \} = \mathcal{H}.
\]

Recently, the analytic structure of weighted composition operators and related operators, like weighted shifts on directed trees was studied by several authors (see [8, 7, 12, 19]).

We begin by proving that in the case of left-invertible weighted composition operators on \( \ell^2(X) \) the duality between \( \mathcal{H} \) and \( \mathcal{H}' \) obtained by identifying them with \( \ell^2(X) \) is the same as the duality obtained from the Cauchy pairing for dense subspace \( \mathcal{H} \cap \text{Hol}(\mathbb{A}(r^-, \infty), E) \), which contain all vector-valued polynomials.

**Theorem 7.3.** Let \( X \) be a countable set, \( w : X \to \mathbb{C} \) be a complex function on \( X \) and \( \varphi : X \to X \) is a transformation of \( X \), which has finite branching index. Let \( C_{\varphi,w} \) be a weighted composition operator in \( \ell^2(X) \) such that (6.2) holds with \( C_{\varphi,w} \) in place of \( T \) and \( E \) as in (7.3). Suppose that \( r^- < 1 \) and \( r^+ \leq r^+ \). Then the duality between \( \mathcal{H} \) and \( \mathcal{H}' \) obtained by identifying them with \( \ell^2(X) \) is the same as the duality obtained from the Cauchy pairing
\[
\langle U^{-1} f, U^{-1} g \rangle_{\ell^2(X)} = \sum_{n = -\infty}^{\infty} \langle \hat{f}(n), \hat{g}(n) \rangle_E
\]
for \( f \in \mathcal{H} \cap \text{Hol}(\mathbb{A}(r^-, \infty), E) \) and \( g \in \mathcal{H}' \).

**Proof.** Set \( T := C_{\varphi,w} \). Suppose that \( \varphi \) has a cycle. Let \( s : \mathcal{G}_\varphi \to [0, \infty) \) be a function defined by
\[
s(x) := \sum_{y \in \varphi^{-1}(\varphi(x))} |w(y)|^2.
\]
Fix any \( x \in \mathcal{G}_\varphi \). Let \( h : \{ 0, 1, \ldots, \tau \} \to [0, \infty) \) be a function given by
\[
h(m) := \begin{cases} \prod_{k=m}^{\tau-1} (w(\varphi(k)(x)))w'(\varphi(k)(x)) & \text{if } m < \tau, \\ \prod_{k=m}^{\tau-1} \frac{|w(\varphi(k)(x))|^2}{s(\varphi(k)(x))} & \text{if } m = \tau, \end{cases}
\]
where \( w' \) is as in Lemma 2.2. Let \( m \in \mathbb{N} \) be such that \( 0 \leq m < \tau \). We claim that
\[
\text{(7.4)}
\]
\[
T^{*n}p_{(\varphi_k)} T^n e_x = \begin{cases} h(0)^l \frac{|w(y)|^2}{s(\varphi^{m_l}(x))} h(m + 1) e_x & \text{if } n = (l + 1)\tau - m, \ l \in \mathbb{N}, \\ 0 & \text{otherwise}, \end{cases}
\]
for \( y \in \varphi^{-1}(\varphi^{m+1}(x)) \setminus \{ \varphi^{(m)}(x) \} \). Indeed, by Lemma 2.1 if \( v, y \in \text{Gen}_\varphi(1,1) \) and \( n \in \mathbb{Z}_+ \) then \( T^n e_v \in \lim\{ e_u : u \in \text{Gen}_\varphi(n + 1, n + 1) \} \) and, consequently, \( P_{(\varphi_k)} T^n e_v = 0 \). Combining this fact with Lemma 2.1, we deduce that
\[
\text{(7.5)}
\]
\[
P_{(\varphi_k)} T^n e_{\varphi^k(x)} = P_{(\varphi_k)} T^{n-1} \sum_{u \in \varphi^{-1}(\varphi^k(x))} w(u) e_u
\]
\[
= P_{(\varphi_k)} T^{n-1} (w(\varphi^{k-1}(x))) e_{\varphi^{k-1}(x)}
\]
\[
+ \sum_{u \in \varphi^{-1}(x) \setminus \{ \varphi^{-1}(x) \}} w(u) e_u
\]
\[
= w(\varphi^{k-1}(x)) P_{(\varphi_k)} T^{n-1} e_{\varphi^{k-1}(x)}
\]
for $y \in \text{Gen}_{\varphi}(1, 1)$, $n \geq 2$ and $k \geq 1$. Similarly, we obtain

$$P_{\text{lin}(e_y)}T^n e_x = w(\varphi^{-1}(x))P_{\text{lin}(e_y)}T^{n-1}e_{\varphi^{-1}(x)}$$

for $y \in \text{Gen}_{\varphi}(1, 1)$ and $n \geq 2$.

(7.7) \[ P_{\text{lin}(e_y)}T^n e_x = \begin{cases} w(y)e_y & \text{if } y \in \varphi^{-1}(x), \\ 0 & \text{otherwise}, \end{cases} \]

for $y \in \text{Gen}_{\varphi}(1, 1)$. Combining (7.5), (7.6) and (7.7), we deduce that

$$P_{\text{lin}(e_y)}T^n e_x = \begin{cases} w(y)e_y & \text{if } y \in \varphi^{-1}(x), \\ 0 & \text{otherwise}, \end{cases}$$

for $y \in \varphi^{-1}(\varphi^{m+1}(x)) \setminus \{\varphi^m(x)\}$, $n = (l + 1)\tau - m$, $l \in \mathbb{N}$ and $P_{\text{lin}(e_y)}T^n e_x = 0$ in the other case. This and Lemma 2.2 gives (7.4).

Our next goal is to show that the following equality holds

(7.8) \[ \sum_{n=1}^{\infty} T^{\tau n} P_E T^n e_x + \sum_{n=0}^{\infty} T^n P_E T^{\tau n} e_x = e_x, \quad x \in Y, \]

where

$$Y := \begin{cases} \mathcal{C}_{\varphi} & \text{when } \varphi \text{ has a cycle} \\ \{x : [\varphi](x) \leq 0\} & \text{otherwise.} \end{cases}$$

We now consider two disjunctive cases which cover all possibilities. First we consider the case when $\varphi$ does not have a cycle. Fix $[\varphi](x) < 0$. Using Lemmas 2.2 and 7.2, one can verify that $T^n P_E T^{\tau n} e_x = 0$ for every $n \in \mathbb{N}$ and

(7.9) \[ T^{\tau n} P_E T^n e_x = \begin{cases} e_x & \text{if } n = -[\varphi](x) + 1, \\ 0 & \text{otherwise}, \end{cases} \]

which completes the proof of the case when $\varphi$ does not have a cycle.

It remains to consider the other case when $\varphi$ has a cycle. Fix any $x \in \mathcal{C}_{\varphi}$. Define the subspace $E_m := \bigvee \{e_y : y \in \varphi^{-1}(\varphi^{m+1}(x)) \setminus \{\varphi^m(x)\}\}$ for every $0 \leq m < \tau$. It follows from (7.4) that

(7.10) \[ T^{\tau n} P_{E_m} T^n e_x = \sum_{y \in \varphi^{-1}(\varphi^{m+1}(x)) \setminus \{\varphi^m(x)\}} T^{\tau n} P_{\text{lin}(e_y)} T^n e_x \]

$$= \begin{cases} h(0)^l s(\varphi^m(x)) - |w(\varphi^m(x))|^2 \sum_{l=0}^{m+1} h(l) e_x & \text{if } n = (l + 1)\tau - m, l \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}$$

Summing over all $n \in \mathbb{Z}_+$, we get

(7.11) \[ \sum_{n=1}^{\infty} T^{\tau n} P_{E_m} T^n e_x = \frac{s(\varphi^m(x)) - |w(\varphi^m(x))|^2}{s(\varphi^m(x))} h(m + 1) e_x \sum_{l=0}^{\infty} h(0)^l e_x \]

$$= \frac{h(m + 1) - h(m)}{1 - h(0)} e_x. \]

Looking at the formula (7.3), we deduce that

(7.12) \[ T^s e = \begin{cases} w^*(x)e_{\varphi(x)} & \text{if } e = e_x, x \in \text{Gen}_{\varphi}(1, 1) \\ 0 & \text{if } e \in \bigoplus_{x \in \text{Gen}_{\varphi}(1, 1)} \mathcal{N}(\mathcal{C}_{\varphi,w|\mathcal{E}(\text{Des}(x))}). \end{cases} \]
Since \( \bigvee \{ e_x : x \in \text{Gen}_x(1,1) \} = \bigoplus_{k=0}^{r-1} E_k \) by (7.12) we get
\[
(7.13) \quad T^{s n} P E T^n e_x = \sum_{k=0}^{r-1} T^{s n} P_{E_k} T^n e_x
\]
which yields
\[
(7.14) \quad \sum_{n=0}^{\infty} T^{s n} P E T^n e_x = \sum_{k=0}^{r-1} \sum_{n=1}^{\infty} T^{s n} P_{E_k} T^n e_x
\]
Combining (7.11) with (7.14), we deduce that (observe that \( h(0) < 1 \))
\[
\sum_{n=0}^{\infty} T^{s n} P E T^n e_x = \frac{1}{1 - h(0)} \sum_{k=0}^{r-1} [h(k+1) - h(k)] e_x = e_x.
\]
This proves our claim.

Our next goal is to prove that
\[
(7.15) \quad \sum_{n=1}^{\infty} T^{s n} P E T^n U^* f + \sum_{n=0}^{\infty} T^n P E T^{s n} U^* f = U^* f
\]
for \( f \in \mathcal{H} \cap \text{Hol}(A(r^-, \infty), E) \). Let \( f \in \mathcal{H} \cap \text{Hol}(A(r^-, \infty), E) \) and \( \{ a_x \}_{x \in X} \subset \mathbb{C} \)
be such that
\[
U^* f = \sum_{x \in X} a_x e_x \in \ell^2(X).
\]
We define \( f_1, f_2 \in \mathcal{H} \) by
\[
f_1 := U \left( \sum_{x \in X \setminus Y} a_x e_x \right) \quad \text{and} \quad f_2 := U \left( \sum_{x \in Y} a_x e_x \right).
\]
It follows from Lemmas 2.1, 2.2 and 7.2 that \( P E T^{s n} e_x = 0, \ x \in Y, \ n \in \mathbb{N} \). This implies
\[
(7.16) \quad \hat{f}_1(n) = P E T^{s n} \left( \sum_{x \in X \setminus Y} a_x e_x \right) = P E T^{s n} \left( \sum_{x \in X} a_x e_x \right) = \hat{f}(n), \ n \in \mathbb{N}.
\]
By kernel-range decomposition and (7.3), we get
\[
\hat{f}_1(n) = P E T^{-n} \left( \sum_{x \in X \setminus Y} a_x e_x \right) = P \{ e_x : x \in \text{Gen}_x(1,1) \} T^{-n} \left( \sum_{x \in X \setminus Y} a_x e_x \right) = 0
\]
for \( n \in \mathbb{Z} \setminus \mathbb{N} \), which yields
\[
f_1 = \sum_{n=0}^{\infty} \hat{f}(n) z^n.
\]
Combined (7.16) with the fact that \( f \in \text{Hol}(A(r^-, \infty), E) \) we obtain
\[
\sum_{n=0}^{\infty} \hat{f}(n) z^n \in \text{Hol}(\mathbb{C}, E)
\]
and
\[
(7.17) \quad \limsup_{n \to \infty} \sqrt[n]{\| \hat{f}(n) \|} = 0.
\]
Now we show that \( \sum_{n=0}^{\infty} T^n \hat{f}(n) \) converges absolutely. Indeed, applying the root test [23, page 199] and (7.17) we get
\[
\limsup_{n \to \infty} \sqrt[n]{\|T^n \hat{f}(n)\|} \leq \|T\| \limsup_{n \to \infty} \sqrt[n]{\|\hat{f}(n)\|} = 0.
\]
This and (♠) in turn implies that the following double sum converges
\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \|T^n P_E T^{s_n} T^m \hat{f}(m)\| = \sum_{n=0}^{\infty} \|T^n \hat{f}(n)\| < \infty.
\]
By (♠) again and changing the order of summation we have the following equalities
\[
(7.18) \quad \sum_{n=0}^{\infty} T^n P_E T^{s_n} U^* f_1 = \sum_{n=0}^{\infty} T^n P_E T^{s_n} \left( \sum_{m=0}^{\infty} T^m \hat{f}(m) \right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} T^n P_E T^{s_n} T^m \hat{f}(m) = \sum_{m=0}^{\infty} T^m \hat{f}(m) = U^* f_1.
\]
By the same kind of reasoning we see that
\[
(7.19) \quad \sum_{n=0}^{\infty} T^{s_n} P_E T^n U^* f_1 = 0.
\]
If \( \varphi \) does not have a cycle, then by (7.9) we have
\[
(7.20) \quad \sum_{n=0}^{\infty} \sum_{x \in Y} \|T^{s_n} P_E T^n a_x e_x\| = \sum_{x \in Y} \|a_x\|^2 < \infty.
\]
Let us pass to the other case when \( \varphi \) has a cycle. Since in this case \( Y \) is finite, by (7.10), (7.11) and (7.13) the following double series converges
\[
(7.21) \quad \sum_{n=0}^{\infty} \sum_{x \in Y} \|T^{s_n} P_E T^n a_x e_x\| = \sum_{n=0}^{\infty} \sum_{x \in Y} \sum_{k=0}^{\tau-1} \|T^{s_n} P_E T^n a_x e_x\| \leq \sum_{n=0}^{\infty} \sum_{x \in Y} \sum_{k=0}^{\tau-1} \|T^{s_n} P_E T^n a_x e_x\| = \sum_{x \in Y} \sum_{k=0}^{\tau-1} \|T^{s_n} P_E T^n a_x e_x\| = \sum_{x \in Y} \sum_{k=0}^{\tau-1} \frac{h(k+1) - h(k)}{1 - h(0)} < \infty.
\]
Using (7.8), (7.20), (7.21) and changing the order of summation we get

\[
\sum_{n=0}^{\infty} T^* T^n U^* f_2 = \sum_{n=0}^{\infty} T'^* T^n \left( \sum_{x \in Y} a_x e_x \right) = \sum_{n=0}^{\infty} \sum_{x \in Y} a_x T'^* T^n e_x = \sum_{x \in Y} a_x e_x = U^* f_2
\]

Similarly we see that

\[
\sum_{n=0}^{\infty} T^n P E T^* P E U^* f_2 = 0.
\]

This combined with (7.18), (7.19) and (7.22) gives (7.15). An application of Theorem 6.5 completes the proof. □

Now we give an example of left-invertible composition operator satisfying the conditions of Theorem 7.3.

**Example 7.4.** Set \( m \in \mathbb{N}, \lambda, \lambda_1, \lambda_2, \ldots, \lambda_m \in (0, 1) \) and \( X = \{0, 1, \ldots, m\} \cup \{(0, i): i \in \mathbb{N}\} \). Let \( \varphi : X \to X \) be a transformation of \( X \) defined by

\[
\varphi(x) = \begin{cases} 
(0, i - 1) & \text{for } x = (0, i), \ i \in \mathbb{N} \setminus \{0\}, \\
m & \text{for } x = (0, 0), \\
i - 1 & \text{for } x = i \text{ and } i \in \{1, \ldots, m\}, \\
m & \text{for } x = 0,
\end{cases}
\]

(see Figure 1) and \( w : X \to \mathbb{C} \) be a function defined by

\[
w(x) = \begin{cases} 
1 & \text{for } x = 0, \\
\lambda_i & \text{for } x = i, \ i \in \{1, \ldots, m\}, \\
\sqrt{\frac{1}{X^{m+1}} \prod_{i=1}^{m} \lambda_i} & \text{for } x = (0, 0), \\
\frac{1}{X} & \text{for } x = (0, i), \ i \in \mathbb{Z}_+.
\end{cases}
\]

Then \( C_{\varphi, w} : \ell^2(X) \to \ell^2(X) \) is a left-invertible composition operator. It is easily seen that

\[
C_{\varphi, w} e_0 = \begin{cases} 
w((0, i + 1)) e_{(0, i+1)} & \text{for } x = (0, i), \ i \in \mathbb{N} \setminus \{0\}, \\
w(i + 1) e_{i+1} & \text{for } x = i \text{ and } i \in \{0, 1, \ldots, m\}, \\
w(0) e_0 + w((0, 0)) e_{(0,0)} & \text{for } x = m.
\end{cases}
\]

It is routine to verify that \( \mathcal{N}(C_{\varphi, w}) = \overline{\{w((0,0)) e_0 - w(0) e_{(0,0)}\}} \). Let \( E := \overline{\{e_{(0,0)}\}} \). One can check that this one-dimensional subspace satisfies (\( \heartsuit \)). The formulas for the inner and outer radius of convergence take the following form

\[
r^+ = \liminf_{n \to \infty} \sqrt[n]{\prod_{i=1}^{n} |w((0, i))|}
\]

and

\[
r^- = \prod_{i=0}^{m+1} |w(i)|
\]
(see [19, Example 5.3]). Therefore the inner and outer radius of convergence of (3.1) with $C_{\phi,w}$ in place of $T$ are

$$r^- = m+1 \prod_{i=1}^{m} \lambda_i \quad \text{and} \quad r^+ = \frac{1}{\lambda}.$$

Using Lemma 2.2 we see that

$$w'(x) = \begin{cases} 
1 & \text{for } x = 0, \\
\lambda^{-m-1} \prod_{i=1}^{m} \lambda_i & \text{for } x = i \text{ and } i \in \{1, \ldots, m\}, \\
\lambda^{m+1} \prod_{i=1}^{m} \lambda_i & \text{for } x = (0,0), \\
\lambda & \text{for } x = (0,i), i \in \mathbb{N} \setminus \{0\}. 
\end{cases}$$

As a consequence, we obtain the inner and outer radius of convergence of (3.1) with $C_{\phi,w}$ in place of $T$

$$r'^- = \lambda^{m+1} \sqrt{\prod_{i=1}^{m} \lambda_i} \quad \text{and} \quad r'^+ = \lambda.$$

Note that $r^- < 1 < r^+$ and $r'^- < r'^+ < 1$ therefore $C_{\phi,w}$ satisfy assumption of Theorem 7.3.

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