ON SPACES EXTREMAL FOR THE GOMORY-HU
INEQUALITY

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Abstract. Let \((X, d)\) be a finite ultrametric space. In 1961 E.C. Gomory and T.C. Hu proved the inequality \(|\text{Sp}(X)| \leq |X|\) where \(\text{Sp}(X) = \{d(x, y) : x, y \in X\}\). Using weighted Hamiltonian cycles and weighted Hamiltonian paths we give new necessary and sufficient conditions under which the Gomory-Hu inequality becomes an equality. We find the number of non-isometric \((X, d)\) satisfying the equality \(|\text{Sp}(X)| = |X|\) for given \(\text{Sp}(X)\). Moreover it is shown that every finite semimetric space \(Z\) is an image under a composition of mappings \(f: X \to Y\) and \(g: Y \to Z\) such that \(X\) and \(Y\) are finite ultrametric space, \(X\) satisfies the above equality, \(f\) is an \(\epsilon\)-isometry with an arbitrary \(\epsilon > 0\), and \(g\) is a ball-preserving map.

1. Introduction

Recall some necessary definitions from the theory of metric spaces. An ultrametric on a set \(X\) is a function \(d: X \times X \to \mathbb{R}^+, \mathbb{R}^+ = [0, \infty)\), such that for all \(x, y, z \in X\):

(i) \(d(x, y) = d(y, x)\),
(ii) \((d(x, y) = 0) \iff (x = y)\),
(iii) \(d(x, y) \leq \max\{d(x, z), d(z, y)\}\).

Inequality (iii) is often called the strong triangle inequality. By studying the flows in networks, R. Gomory and T. Hu [1], deduced an inequality that can be formulated, in the language of ultrametric spaces, as follows: if \((X, d)\) is a finite nonempty ultrametric space with the spectrum

\[ \text{Sp}(X) = \{d(x, y) : x, y \in X\}, \]

then

\[ |\text{Sp}(X)| \leq |X|. \]

Definition 1.1. Define by \(\mathcal{U}\) the class of finite ultrametric spaces \(X\) with \(|\text{Sp}(X)| = |X|\).
Two descriptions of $X \in \mathcal{U}$ were obtained in terms of the representing trees $\alpha_n$, respectively, so-called diametrical graphs of $X$ (see [2] theorems 2.3 and 3.1.). Our paper is also a contribution to this line of studies. We give a new criterium of $X \in \mathcal{U}$ in terms of weighted Hamiltonian cycles and weighted Hamiltonian paths (see Theorem 2.5) and find the number of non-isometric $X \in \mathcal{U}$ with given $\text{Sp}(X)$ (see Proposition 3.2). It is also shown that every finite semimetric $X$ is an image of a space $Y \in \mathcal{U}$, $X = g(f(Y))$, where $g$ is a ball-preserving map and $f$ is an $\varepsilon$-isometry (see Theorem 4.4 and Theorem 4.5).

Recall that a graph is a pair $(V,E)$ consisting of nonempty set $V$ and (probably empty) set $E$ elements of which are unordered pairs of different points from $V$. For the graph $G = (V,E)$, the set $V = V(G)$ and $E = E(G)$ are called the set of vertices and the set of edges, respectively. A graph $G$ is empty if $E(G) = \emptyset$. A graph is complete if $\{x,y\} \in E(G)$ for all distinct $x, y \in V(G)$. Recall that a path is a nonempty graph $P = (V,E)$ of the form

$$V = \{x_0, x_1, ..., x_k\}, \quad E = \{\{x_0, x_1\}, ..., \{x_{k-1}, x_k\}\},$$

where $x_i$ are all distinct. The number of edges of a path is the length. Note that the length of a path can be zero. A Hamiltonian path is a path in the graph that visits each vertex exactly once. A finite graph $C$ is a cycle if $|V(C)| \geq 3$ and there exists an enumeration $(v_1, v_2, ..., v_n)$ of its vertices such that

$$\{v_i, v_j\} \in E(C) \iff (|i - j| = 1 \quad \text{or} \quad |i - j| = n - 1).$$

For the graph $G = (V,E)$ a Hamiltonian cycle is a cycle which is a subgraph of $G$ that visits every vertex exactly once. A connected graph without cycles is called a tree. A tree $T$ may have a distinguished vertex called the root; in this case $T$ is called a rooted tree.

Generally we follow terminology used in [3]. A graph $G = (V,E)$ together with a function $w: E \rightarrow \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$, is called a weighted graph, and $w$ is called a weight or a weighting function. The weighted graphs we denote by $(G,w)$.

A nonempty graph $G$ is called complete $k$-partite if its vertices can be divided into $k$ disjoint nonempty subsets $X_1, ..., X_k$ so that there are no edges joining the vertices of the same subset $X_i$ and any two vertices from different $X_i, X_j$, $1 \leq i, j \leq k$ are adjacent. In this case we write $G = G[X_1, ..., X_k]$. 
2. Cycles in ultrametric spaces

In the following we identify a finite ultrametric space \((X,d)\) with a complete weighted graph \((G_X,w_d)\) such that \(V(G_X) = X\) and
\[
\forall x, y \in X, x \neq y: \quad w_d(\{x,y\}) = d(x,y).
\]

The following lemma was proved in [4].

**Lemma 2.1.** Let \((X,d)\) be an ultrametric space with \(|X| \geq 3\). Then for every cycle \(C \subseteq G_X\) there exist at least two distinct edges \(e_1, e_2 \in C\) such that
\[
w_d(e_1) = w_d(e_2) = \max_{e \in E(C)} w_d(e).
\]

We shall say that a weighted cycle \((C,w)\) is characteristic if the following conditions hold.

(i) There are exactly two distinct \(e_1, e_2 \in E(C)\) such that \((2.2)\) holds.

(ii) The restriction of \(w\) on the set \(E(C) \setminus \{e_1, e_2\}\) is strictly positive and injective.

**Remark 2.2.** Let us explain the choice of a name for such a type of cycles. It was proved in [4] that for every characteristic weighted cycle \((C,w)\) there is a unique ultrametric \(d: V(C) \times V(C) \to \mathbb{R}^+\) such that
\[
d(x,y) = w(\{x,y\})
\]
for all \(\{x,y\} \in E(C)\). In other words we can uniquely reconstruct whole the ultrametric space \((X,d)\) by characteristic cycle \((C,w_d) \subseteq (G_X,w_d)\) if \(|V(C)| = |X|\).

We need the following definition.

**Definition 2.3** ([1]). Let \((X,d)\) be a finite ultrametric space. Define the graph \(G^d_X\) as follows \(V(G^d_X) = X\) and
\[
(\{u,v\} \in E(G^d_X)) \iff (d(u,v) = \text{diam } X).
\]
We call \(G^d_X\) a diametrical graph of the space \((X,d)\).

**Lemma 2.4** ([1]). Let \((X,d)\) be a finite ultrametric space, \(|X| \geq 2\). If \((X,d) \in \mathcal{U}\), then \(G^d_X\) is a bipartite graph, \(G^d_X = G^d_X[X_1,X_2]\) and \(X_1 \in \mathcal{U}, X_2 \in \mathcal{U}\).

We shall say that a weighted path \((P,w)\) is characteristic if the weighting function \(w: E(P) \to \mathbb{R}^+\) is injective and strictly positive.

The next theorem is the main result of this section.

**Theorem 2.5.** Let \((X,d)\) be a finite ultrametric space with \(|X| \geq 3\). Then the following conditions are equivalent.
(i) \((X, d) \in \mathcal{U}\).

(ii) There exists a characteristic Hamiltonian path in \(G_X\).

(iii) There exists a characteristic Hamiltonian cycle in \(G_X\).

**Proof.** (i)\(\Rightarrow\)(ii). We shall prove the implication (i)\(\Rightarrow\)(ii) by induction on \(|X|\). Let \((X, d) \in \mathcal{U}\). If \(|X| = 3\), then the existence of a characteristic Hamiltonian path is evident. Suppose the implication (i)\(\Rightarrow\)(ii) holds for \(X\) with \(|X| \leq n - 1\). Let \(|X| = n\). Let us prove that there exists a characteristic Hamiltonian path in \(G_X\). According to Lemma 2.4 we have

\[
G^d_X = G^d_X[\mathcal{X}_1, \mathcal{X}_2], \quad |\mathcal{X}_1| \leq n - 1, \quad |\mathcal{X}_2| \leq n - 1
\]

and \(\mathcal{X}_1 \in \mathcal{U}, \mathcal{X}_2 \in \mathcal{U}\). By the induction supposition there exist characteristic Hamiltonian paths \(P_1 \subseteq G_{X_1}^d\) and \(P_2 \subseteq G_{X_2}^d\). Let \(V(P_1) = \{x_1, \ldots, x_m\}\) and \(V(P_2) = \{x_{m+1}, \ldots, x_n\}\), \(1 \leq m \leq n - 1\). Since \(G^d_X = G^d_X[\mathcal{X}_1, \mathcal{X}_2]\), we have

\[
\text{diam} \ X \notin \text{Sp}(\mathcal{X}_1) \quad \text{and} \quad \text{diam} \ X \notin \text{Sp}(\mathcal{X}_2).
\]

Moreover, the equality

\[
\text{Sp}(\mathcal{X}_1) \cap \text{Sp}(\mathcal{X}_2) = \{0\}
\]

holds. Indeed, it is clear that

\[
0 \in \text{Sp}(\mathcal{X}_1) \cap \text{Sp}(\mathcal{X}_2),
\]

but if \(\vert \text{Sp}(\mathcal{X}_1) \cap \text{Sp}(\mathcal{X}_2) \vert \geq 2\), then using the equality

\[
\text{Sp}(\mathcal{X}) = \text{Sp}(\mathcal{X}_1) \cup \text{Sp}(\mathcal{X}_2) \cup \{\text{diam} \ X\}
\]

and the Gomory–Hu inequality we obtain

\[
\vert \text{Sp}(\mathcal{X}) \vert \leq 1 + \vert \mathcal{X}_1 \vert + \vert \mathcal{X}_2 \vert - \vert \mathcal{X}_1 \cap \mathcal{X}_2 \vert < \vert \mathcal{X}_1 \vert + \vert \mathcal{X}_2 \vert = \vert \mathcal{X} \vert
\]

contrary to \((X, d) \in \mathcal{U}\). The equality \(d(x_m, x_{m+1}) = \text{diam} \ X\), (2.4) and (2.3) imply that the path \(P\) with \(V(P) = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_n\}\) is a characteristic Hamiltonian path in \(G_X\).

(ii)\(\Rightarrow\)(iii). Let \(P\) be a characteristic Hamiltonian path in \(G_X\) with \(V(P) = \{x_1, \ldots, x_n\}\). Consider the cycle \(C = (x_1, \ldots, x_n)\). It is clear that \(C\) is Hamiltonian. According to Lemma 2.1 the equality

\[
w_d(\{x_1, x_n\}) = \max_{e \in E(P)} w_d(e)
\]

holds. This means that \(C\) is characteristic.

(iii)\(\Rightarrow\)(i). Let \((X, d)\) be a finite ultrametric space and let \(C\) be a characteristic Hamiltonian cycle in \(G_X\). Using Lemma 2.1 with this \(C\) we easily show that \(\vert \text{Sp}(\mathcal{X}) \vert = \vert \mathcal{X} \vert\). Condition (i) follows.

\[\square\]
With every finite ultrametric space \((X, d)\), we can associate (see [2]) a labeled rooted \(m\)-ary tree \(T_X\) by the following rule. If \(X = \{x\}\) is a one-point set, then \(T_X\) is a tree consisting of one node \(x\) considered strictly binary by definition. Let \(|X| \geq 2\) and \(G^d_X = G^d_X[X_1, \ldots, X_k]\) be the diametrical graph of the space \((X, d)\). In this case the root of the tree \(T_X\) is labeled by \(\text{diam} X\) and, moreover, \(T_X\) has \(k\) nodes \(X_1, \ldots, X_k\) of the first level with the labels

\[
(2.6) \quad l_i = \begin{cases} 
\text{diam} X_i, & \text{if } |X_i| \geq 2, \\
x, & \text{if } X_i \text{ is a one-point set with the single element } x,
\end{cases}
\]

\(i = 1, \ldots, k\). The nodes of the first level indicated by labels \(x \in X\) are leaves, and those indicated by labels \(\text{diam} X_i\) are internal nodes of the tree \(T_X\). If the first level has no internal nodes, then the tree \(T_X\) is constructed. Otherwise, by repeating the above-described procedure with \(X_i \subset X\) corresponding to internal nodes of the first level, we obtain the nodes of the second level, etc. Since \(|X|\) is finite, and the cardinal numbers \(|Y|, Y \subseteq X\), decrease strictly at the motion along any path starting from the root, consequently all vertices on some level will be leaves, and the construction of \(T_X\) is completed. The above-constructed labeled tree \(T_X\) is called the representing tree of the space \((X, d)\). We note that every element \(x \in X\) is ascribed to some leaf, and all internal nodes are labeled as \(r \in \text{Sp}(X)\). In this case, different leaves correspond to different \(x \in X\), but different internal nodes can have coinciding labels.

Recall that a rooted tree is strictly binary if every internal node has exactly two children. Note that the correspondence between trees and ultrametric spaces is well known [5–7].

Define by \(L_T\) the set of leaves of the tree \(T\) and by \(l(v)\) the label of the vertex \(v\).

The proof of the following two lemmas is immediate.

**Lemma 2.6.** Let \((X, d)\) be a finite ultrametric space having a strictly binary tree \(T_X\). If \(v_0\) and \(v_1\) are interval nodes of \(T_X\) and \(v_1\) is a direct successor of \(v_0\) then the inequality \(l(v_1) < l(v_0)\) holds.

**Lemma 2.7.** Let \((X, d)\) be a finite ultrametric space with \(|X| \geq 3\) and let \(G^d_X = G^d_X[X_1, \ldots, X_k]\) be the diametrical graph of \((X, d)\). Then a tree \(T_X\) is strictly binary if and only if \(k = 2\) and \(T_{X_1}\) and \(T_{X_2}\) are strictly binary.

**Proposition 2.8.** Let \((X, d)\) be a finite ultrametric space with \(|X| \geq 3\). The following conditions are equivalent.
(i) $T_X$ is strictly binary.
(ii) If $X_1 \subseteq X$ and $|X_1| \geq 3$, then there exists a Hamiltonian cycle $C \subseteq G_{X_1}$ with exactly two edges of maximal weight.
(iii) There is no equilateral triangle in $(X,d)$.

Proof. (i)$\Rightarrow$(ii). Suppose $T_X$ is strictly binary. Let $X_1$ be a subset of $X$, $|X_1| \geq 3$. According to construction of $T_X$ all elements of $X_1$ are labels of leaves of $T_X$. Let $v_0$ be a smallest common predecessor for the leaves of $T_X$ labeled by elements of $X_1$. Let $v_0^1$ and $v_0^2$ be the two offsprings of $v_0$ (direct successors) and let $T_1$ and $T_2$ be the subtrees of the tree $T_X$ with the roots $v_0^1$ and $v_0^2$. Let $L_1 = L_{T_1} \cap X_1$ and $L_2 = L_{T_2} \cap X_1$ and let $P_1 = \{x_1, \ldots, x_m\}$ and $P_2 = \{x_{m+1}, \ldots, x_{|X_1|}\}$, $1 \leq m \leq |X_1| - 1$, be Hamiltonian paths in the spaces $(L_1,d)$ and $(L_2,d)$. By the property of representing trees of ultrametric spaces we have $d(x,y) = l(v_0)$ for all $x \in L_1$ and $y \in L_2$. Since $X_1 = L_1 \cup L_2$, we obtain that the Hamiltonian cycle $C = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{|X_1|})$ has exactly the two edges $\{x_1, x_{|X_1|}\}$ and $\{x_m, x_{m+1}\}$ of maximal weight.

(ii)$\Rightarrow$(iii). This implication is evident.

(iii)$\Rightarrow$(i). We will prove (i) by induction on $|X|$. The statement (i) evidently follows from (iii) if $|X| = 3$. Assume that (iii)$\Rightarrow$(i) is satisfied for all finite ultrametric spaces $(X,d)$ with $3 \leq |X| \leq n$, $n \in \mathbb{N}$. Let $G_X^d = G_X^d[X_1, \ldots, X_k]$ be the diametrical graph of $(X,d)$. Statement (i) holds if $k = 2$. Indeed, since the inequality $|X_i| < |X|$ holds, the induction assumption implies that for every $i = 1, \ldots, k$, $T_{X_i}$ is a strictly binary tree. Hence if $k = 2$, then $T_X$ is a strictly binary tree by Lemma 2.7. To complete the proof it suffices to note that if $k \geq 3$ and $x_i \in X_i$ for $i = 1, 2, 3$, then the points $x_1$, $x_2$, $x_3$ form an equilateral triangle with $d(x_1,x_2) = d(x_2,x_3) = d(x_3,x_1) = \text{diam } X$.

3. The number of non-isometric $X \in \mathcal{U}$ with given $\text{Sp}(X)$.

Let $n \in \mathbb{N}$ and $\mathcal{U}_n$ denote the class of ultrametric spaces $X \in \mathcal{U}$ such that $|X| = n$. In the present section we study the following question: how many non-isometric spaces having the same spectrum are in the class $\mathcal{U}_n$? Let us denote this number by $\kappa(\mathcal{U}_n)$.

Definition 3.1 (K). Let $(X,d_X)$, $(Y,d_Y)$ be metric spaces. A bijective mapping $\Phi: X \to Y$ is a weak similarity if there is a strictly increasing bijective function $f$: $\text{Sp}(Y) \to \text{Sp}(X)$ such that the equality

$$d_X(x,y) = f(d_Y(\Phi(x),\Phi(y)))$$

holds for all $x, y \in X$. Write $X \simeq Y$ if a weak similarity $\Phi: X \to Y$ exists.
It is clear that \(\simeq\) is an equivalence relation. It was proved in \([8]\) that if \(X\) and \(Y\) are compact ultrametric spaces with the same spectrum, then every week similarity \(\Phi: X \to Y\) is an isometry. So, the main question of this section can be reformulated as follows. How many spaces are there in \(\mathcal{U}_n\) up to weak similarity? 

**Proposition 3.2.** Let \(\mathcal{U}_n := \{X \in \mathcal{U} : |X| = n\}, n \in \mathbb{N}\), let \(\mathcal{U}_n/\simeq\) be the quotient set of \(\mathcal{U}_n\) by \(\simeq\) and let

\[
\kappa(\mathcal{U}_n) := \text{card}(\mathcal{U}_n/\simeq).
\]

Then the equality

\[
(3.2) \quad \kappa(\mathcal{U}_n) = \sum_{k=2}^{n-1} C_{n-3}^{k-2} \kappa(\mathcal{U}_k) \kappa(\mathcal{U}_{n-k})
\]

holds for every integer \(n \geq 3\) with \(\kappa(\mathcal{U}_1) = \kappa(\mathcal{U}_2) = 1\) and

\[
C_{n-3}^{k-2} = \frac{(n - 3)!}{(k - 2)!(n - k - 1)!}.
\]

**Proof.** Directly we can find the initial values

\(\kappa(\mathcal{U}_1) = \kappa(\mathcal{U}_2) = 1\).

Let \(n \geq 3\). The number \(\kappa(\mathcal{U}_n)\) coincides with the number of non-isometric \((X, d) \in \mathcal{U}_n\) having the spectrum \(\{0, 1, ..., n - 1\}\). For every such \((X, d) \in \mathcal{U}_n\) we write \(G_X^d[X_1, X_2]\) for the diametrical graph of \((X, d)\). The inequality \(n \geq 3\) implies that \(\text{diam} X = n - 1 > 1\). Since

\[
\text{Sp}(X) = \{n - 1\} \cup \text{Sp}(X_1) \cup \text{Sp}(X_2)
\]

and

\[
\text{Sp}(X_1) \cap \text{Sp}(X_2) = \{0\},
\]

we may assume without loss of generality that

\(1 \in \text{Sp}(X_1)\) and \(1 \notin \text{Sp}(X_2)\).

Let \(|X_1| = k\). It follows from \(1 \in \text{Sp}(X_1)\) that \(k \geq 2\). Moreover the statement \(X_2 \neq \emptyset\) implies that \(k \leq n - 1\). As was noted in the second section of the paper we have

\(X_1 \in \mathcal{U}_k\) and \(X_2 \in \mathcal{U}_{n-k}\).

Let \(\text{Sp}(X_1) = \{0, 1, n_1, ..., n_{k-2}\}\) where \(1 < n_1 < ... < n_k\) (if \(k \geq 3\)). The set \(\{n_1, ..., n_{k-2}\}\) can be selected from the set \(\{2, ..., n-2\}\) in \(C_{n-3}^{k-2}\) ways. It is clear that if \((X, d), (Y, \rho) \in \mathcal{U}_n\) and

\[
\text{Sp}(X) = \text{Sp}(Y) = \{0, 1, ..., n - 1\}
\]

Then the equality

\[
(3.2) \quad \kappa(\mathcal{U}_n) = \sum_{k=2}^{n-1} C_{n-3}^{k-2} \kappa(\mathcal{U}_k) \kappa(\mathcal{U}_{n-k})
\]

holds for every integer \(n \geq 3\) with \(\kappa(\mathcal{U}_1) = \kappa(\mathcal{U}_2) = 1\) and

\[
C_{n-3}^{k-2} = \frac{(n - 3)!}{(k - 2)!(n - k - 1)!}.
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**Proof.** Directly we can find the initial values

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Let \(n \geq 3\). The number \(\kappa(\mathcal{U}_n)\) coincides with the number of non-isometric \((X, d) \in \mathcal{U}_n\) having the spectrum \(\{0, 1, ..., n - 1\}\). For every such \((X, d) \in \mathcal{U}_n\) we write \(G_X^d[X_1, X_2]\) for the diametrical graph of \((X, d)\). The inequality \(n \geq 3\) implies that \(\text{diam} X = n - 1 > 1\). Since

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and

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we may assume without loss of generality that

\(1 \in \text{Sp}(X_1)\) and \(1 \notin \text{Sp}(X_2)\).

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\(X_1 \in \mathcal{U}_k\) and \(X_2 \in \mathcal{U}_{n-k}\).

Let \(\text{Sp}(X_1) = \{0, 1, n_1, ..., n_{k-2}\}\) where \(1 < n_1 < ... < n_k\) (if \(k \geq 3\)). The set \(\{n_1, ..., n_{k-2}\}\) can be selected from the set \(\{2, ..., n-2\}\) in \(C_{n-3}^{k-2}\) ways. It is clear that if \((X, d), (Y, \rho) \in \mathcal{U}_n\) and

\[
\text{Sp}(X) = \text{Sp}(Y) = \{0, 1, ..., n - 1\}
\]
and if for the diametrical graphs \( G'_X[X_1, X_2], G'_Y[Y_1, Y_2] \) we have
\[ 1 \in \text{Sp}(X_1) \text{ and } 1 \in \text{Sp}(Y_1), \]
the \( X \) and \( Y \) are isometric if and only if \( X_1 \) is isometric to \( Y_1 \) and \( X_2 \) is isometric to \( X_2 \). Now using the multiplication principle and additional principle we obtain (3.2). □

**Corollary 3.3.** The number of all non-isometric spaces \( X \in \mathcal{U}_n \) with given \( \text{Sp}(X) \) equals to
\[
\sum_{k=2}^{n-1} C_{n-k}^{k-2}\kappa(\mathcal{U}_k)\kappa(\mathcal{U}_{n-k}),
\]
where \( \kappa(\mathcal{U}_1) = \kappa(\mathcal{U}_2) = 1 \).

Using formula (3.2) we can find \( \kappa(\mathcal{U}_3) = 1, \kappa(\mathcal{U}_4) = 2, \kappa(\mathcal{U}_5) = 5, \kappa(\mathcal{U}_6) = 16, \kappa(\mathcal{U}_7) = 61 \) and so on.

**Remark 3.4.** As was shown in [2] there is an isomorphism between spaces from \( \mathcal{U} \) and strictly decreasing binary trees.

It is easy to see that there is also a bijection between the strictly decreasing binary trees and the ranked trees \( \mathcal{R}_n \). The definition of the ranked trees \( \mathcal{R}_n \) one can find in [9]. It was noted in [9] that numbers of \( \mathcal{R}_n \) correspond to sequence A000111 from [10].

4. **Ball-preserving mappings, \( \varepsilon \)-isometries and semimetric spaces**

Let \( X \) be a set. A **semimetric** on \( X \) is a function \( d: X \times X \rightarrow \mathbb{R}^+ \) such that \( d(x, y) = d(y, x) \) and \( (d(x, y) = 0) \Leftrightarrow (x = y) \) for all \( x, y \in X \).

A pair \((X, d)\), where \( d \) is a semimetric on \( X \), is called a **semimetric space** (see, for example, [11]).

A **directed graph** or **digraph** is a set of nodes connected by edges, where the edges have a direction associated with them. In formal terms a digraph is a pair \( G = (V, A) \) of
- a set \( V \), whose elements are called vertices or nodes,
- a set \( A \) of ordered pairs of vertices, called arcs, directed edges, or arrows.

An arc \( e = \langle x, y \rangle \) is considered to be directed from \( x \) to \( y \); \( y \) is said to be a **direct successor** of \( x \), and \( x \) is said to be a **direct predecessor** of \( y \). If a path made up of one or more successive arcs leads from \( x \) to \( y \), then \( y \) is said to be a **successor** of \( x \), and \( x \) is said to be a **predecessor** of \( y \).

A Hasse diagram for a partially ordered set \((X, \leq_X)\) is a digraph \((X, A_X)\), where \( X \) is the set of vertices and \( A_X \subseteq X \times X \) is the set of
directed edges such that the pair $\langle v_1, v_2 \rangle$ belongs to $A_X$ if and only if $v_1 \leq_X v_2$, $v_1 \neq v_2$, and implication
\[(v_1 \leq_X w \leq_X v_2) \Rightarrow (v_1 = w \lor v_2 = w)\]
holds for every $w \in X$.

Recall that a subset $B$ of a semimetric space $(X,d)$ is called a closed ball if it can be represented as follows:
\[B = B_r(t) = \{x \in X : d(x,t) \leq r\},\]
where $t \in X$ and $r \in [0, \infty)$. Denote by $B_X$ the set of all distinct balls of semimetric space $(X,d)$.

**Definition 4.1.** Let $X$ and $Y$ be semimetric spaces. A mapping $F: X \to Y$ is ball-preserving if
\[(4.1) \quad F(Z) \in B_Y,\]
for every $Z \in B_X$.

**Definition 4.2.** Let $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$ be directed graphs. A map $F: V_1 \to V_2$ is a graph homomorphism if the implication
\[(\langle u, v \rangle \in A_1) \Rightarrow (\langle F(u), F(v) \rangle \in A_2)\]
holds for all $u, v \in V_1$. A homomorphism $F: V_1 \to V_2$ is an isomorphism if $F$ bijective and the inverse map $F^{-1}$ is also a homomorphism.

According to [12] we shall say that a graph homomorphism $F: V_1 \to V_2$ from $G_1 = (V_1, A_1)$ to $G_2 = (V_2, A_2)$ is arc-surjective if for every $\langle x, y \rangle \in A_2$ there is $\langle u, v \rangle \in A_1$ such that $\langle x, y \rangle = (F(u), F(v))$.

It is evident that every isomorphism is arc-surjective. Furthermore, if $G_1$ and $G_2$ have no isolated points, then every injective arc-surjective homomorphism $F: V_1 \to V_2$ is an isomorphism. It was shown in [13] that if $X$ and $Y$ are finite ultrametric spaces, then the following conditions equivalent.

- There is a bijective ball-preserving mapping $F: X \to Y$ such that the inverse mapping $F^{-1}: Y \to X$ is also ball-preserving.
- The Hasse diagrams $(B_X, A_{B_X})$ and $(B_Y, A_{B_Y})$ of the posets $(B_X, \subseteq)$ and $(B_Y, \subseteq)$ are isomorphic as directed graphs.

**Definition 4.3.** Let $(X,d)$ and $(Y,\rho)$ be semimetric spaces and let $\varepsilon > 0$. A surjective mapping $F: X \to Y$ is an $\varepsilon$-isometry if the inequality
\[|d(x,y) - \rho(F(x), F(y))| \leq \varepsilon\]
holds for all $x, y \in X$.

The main result of the present section is the following two theorems.
Theorem 4.4. Let $X$ be a finite nonempty semimetric space. Then there is a finite ultrametric space $Y$ and a surjective ball-preserving function $F: Y \to X$ such that the mapping

$$B_Y \ni B \mapsto F(B) \in B_X$$

is an arc-surjective homomorphism from the Hasse diagram $(B_Y, \subseteq)$ of $(B_Y, \subseteq)$ to the Hasse diagram $(B_X, \subseteq)$ of $(B_X, \subseteq)$.

Theorem 4.5. Let $(Y, d)$ be a finite ultrametric space. Then for every $\varepsilon > 0$ there is a bijective $\varepsilon$-isometry $\Phi: W \to Y$ such that $W \in \mathcal{U}$.

Theorems 4.4 and 4.5 imply the following

Corollary 4.6. For every finite nonempty semimetric space $X$ and every $\varepsilon > 0$ there are mappings $F: Y \to X$ and $\Phi: Z \to Y$ such that $Y$ is finite and ultrametric, $Z \in \mathcal{U}$, $F$ is ball-preserving, $\Phi$ is an $\varepsilon$-isometry and

$$X = F(\Phi(Z)).$$

The next lemma will be used in the proof of Theorem 4.4.

Lemma 4.7. Let $X$ be a finite semimetric space. If $B \in B_X$ and $|B| \geq 2$, then the following statements hold.

(i) The ball $B$ has at least two direct predecessors in the Hasse diagram $(B_X, A_{B_X})$.

(ii) The union of all direct predecessors of $B$ coincides with $B$.

Proof. Let $B \in B_X$ and $|B| \geq 2$. The set of all direct predecessors of $B$ is simply the set of all maximal elements of the subset

$$(4.2) \quad S = \{S \in B_X : S \subseteq B \text{ and } S \neq B\}$$

of the poset $(B_X, \subseteq)$. The inequality $|B| \geq 2$ implies

$$B \subseteq \bigcup S, \quad S \in S,$$

because $\{x\} \in S$ for every $x \in B$. Since $X$ is finite, $S$ is also finite and consequently for every $x \in B$ there is a maximal element $S$ of $S$ such that $x \in S$. Statement (ii) follows. Now to finish the proof it suffices to note that if $B$ contains a unique direct predecessor $S$, then $B = S$ contrary to (4.2). \(\square\)

Proof of Theorem 4.4. Let $(B_X, A_{B_X})$ be a Hasse diagram of the poset $(B_X, \subseteq)$. To this diagram we assign $n$-ary rooted labeled tree $T$ by the following procedure. Let the root $v_0$ of $T$ be labeled by $X$. Let $B_1, \ldots, B_k$ be direct predecessors of $X$ in $(B_X, A_{B_X})$. Define $v_1, \ldots, v_k$ to be the children (nodes of the first level) of $v_0$ with the labels $B_1, \ldots, B_k$ respectively. Let us look at the nodes of the first level of the tree $T$. 


Define the children of the nodes \( v_i, i = 1, ..., k \), as follows: if there is no \( Y \) such that \( \langle Y, B_i \rangle \in A_{B_X} \) then \( v_i \) is a leaf of \( T \); if \( B_{i1}, B_{i2}, ..., B_{in} \) are direct predecessors of \( B_i \) in \( (B_X, A_{B_X}) \), then define \( v_{i1}, v_{i2}, ..., v_{in} \) to be the children of \( v_i \) (nodes of the second level) with labels \( B_{i1}, B_{i2}, ..., B_{in} \) respectively. Note that the nodes of the second level may have the identical labels in the case when \( B_{ij} \) is a direct predecessor both \( B_{k1} \) and \( B_{k2} \). Do the same procedure with the nodes of the second level and so on. By Lemma 4.7 \( T \) is \( n \)-ary tree with \( n \geq 2 \). Note also that the leaves of \( T \) are labeled with the balls \( \{x_i\}, x_i \in X \).

Let \( n \) be the number of leaves of \( T \). We define a new names \( y_i, i = 1, ..., n \), for the leaves of \( T \) in any order but save the labels of these leaves. Let \( Y \) be an ultrametric space with representing tree isomorphic to \( T, Y = \{y_1, ..., y_n\} \). Define \( F: Y \to X \) by the rule

\[
F(y_i) = x_i \text{ if the label of } y_i \text{ is } x_i.
\]

We claim that \( F \) is ball-preserving. Indeed, by Lemma 4 in [13] for every \( B \in B_Y \) there exists a node \( \tilde{v} \) of \( T \) such that \( \Gamma_T(\tilde{v}) = B \), where \( \Gamma_T(\tilde{v}) \) is the set of all leaves of subtree with the root \( \tilde{v} \). And let \( \tilde{B} \) be the label of \( \tilde{v} \). According to Lemma 4.7 and the construction of \( T \) the set \( F(B) \) coincides with \( \tilde{B} \). It suffice to note that \( \tilde{B} \) is a ball in \( B_X \) because all the nodes in \( T \) are labeled by balls of semimetric space \( X \). Furthermore, it is easily seen that the mapping

\[
B_X \ni B \mapsto F(B) \in B_Y
\]

is an arc-surjective homomorphism from \( (B_Y, A_Y) \) to \( (B_X, A_X) \) as required.

**Definition 4.8.** Let \( (Y, d_Y) \) and \( (W, d_W) \) be bounded metric spaces and let \( \Delta > 0 \). The Gromov–Hausdorff distance \( d_{GH}(Y, W) \) is less than \( \Delta \) if there exists a metric spaces \( (Z, d_Z) \) with subspaces \( Y' \) and \( W' \) such that

- \( Y \) and \( Y' \) are isometric;
- \( W \) and \( W' \) are isometric;
- We have the inclusions

\[
Y' \subseteq \bigcup_{w \in W'} O_\Delta(w) \text{ and } W' \subseteq \bigcup_{y \in Y'} O_\Delta(y)
\]

where for \( t \in Z, O_\Delta(t) = \{z \in Z : d_Z(t, z) < \Delta\} \) is an open ball from \( (Z, d_Z) \) that has the radius \( \Delta \).

The next lemma is a reformulation Proposition 4.1 from [2].
Lemma 4.9. Let $Y$ be a finite ultrametric space and let $\varepsilon > 0$. Then there is a finite ultrametric space $W \in \mathcal{U}$ such that $|Y| = |W|$ and $d_{GH}(Y, W) < \varepsilon$.

Now we are ready to prove Theorem 4.5.

Proof of Theorem 4.5. The theorem is trivial if $|Y| \leq 2$. Let $|Y| \geq 3$, let $\varepsilon > 0$ and let $
abla = \min\{d_Y(x, y) : x, y \in Y, x \neq y\}$.

Since $3 \leq |Y| < \infty$, we have $0 < \nabla < \infty$. By Lemma 4.9 for every $\Delta$ from the interval $(0, \min (\frac{1}{2}, \frac{1}{2}))$ there is $W \in \mathcal{U}$ such that $d_{GH}(Y, W) < \Delta$. Let $(Z, d_Z)$ be a metric space with contains isometric copies $Y'$ and $W'$ of $Y$ and $W$ respectively such that inclusions (4.3) hold. We claim that for every $w \in W'$ there is a unique $y \in Y'$ such that $y \in O_{\Delta}(w)$.

Let $\Phi : W \rightarrow Y$ be an isometry. For all $w \in W$ and $y \in Y$ the first part of the proof shows that this definition is correct and $\Phi$ is bijective. It remains to prove that $\Phi$ is an $\varepsilon$-isometry. For this purpose note that if $w_1, w_2 \in W$ and $y_1 = \Phi(w_1)$, $y_2 = \Phi(w_2)$ then
\[
d_W(w_1, w_2) = d_Z(\varphi(w_1), \varphi(w_2)),
\]
\[
d_Y(\Phi(w_1), \Phi(w_2)) = d_Z(\psi(\Phi(w_1)), \psi(\Phi(w_2)))
\]
and, by (4.4),
\[ d_Z(\varphi(w_i), \psi(\Phi(w_i))) < \Delta \]
for \( i = 1, 2 \). Now using the triangle inequality and the inequality \( \Delta < \frac{\varepsilon}{2} \) we obtain
\[
|d_W(w_1, w_2) - d_Y(\Phi(w_1), \Phi(w_2))| \\
= |d_Z(\varphi(w_1), \varphi(w_2)) - d_Z(\psi(\Phi(w_1)), \psi(\Phi(w_2)))| \\
\leq d_Z(\varphi(w_1), \psi(\Phi(w_1))) + d_Z(\varphi(w_2), \psi(\Phi(w_2))) < \varepsilon.
\]
Thus \( \Phi \) is an \( \varepsilon \)-isometry as required. \( \square \)

The class \( \mathcal{U} \) consisting of finite ultrametric spaces which are extremal for the Gomory-Hu inequality can be extended by the following way. If \( X \) is a compact ultrametric space, then we define \( X \in \mathcal{U}_C \) if \( Y \in \mathcal{U} \) for every finite \( Y \subseteq X \). It was shown in [2] that \( Y \in \mathcal{U} \) if \( Y \subset X \) and \( X \in \mathcal{U} \). Hence the class \( \mathcal{U} \) is a subclass of \( \mathcal{U}_C \). The following conjecture seems to be a natural generalization of theorems 4.4 and 4.5.

**Conjecture 4.10.** Let \( X \) be a compact nonempty semimetric space and let \( \varepsilon > 0 \). Then there are continuous mappings \( F: Y \to X \) and \( \Phi: W \to Y \) such that \( Y \) is compact ultrametric, \( W \in \mathcal{U}_C \), \( \Phi \) is an \( \varepsilon \)-isometry and \( F \) is ball-preserving and
\[
B_Y \ni B \mapsto F(B) \in B_X
\]
is an arc-surjective homomorphism from \( (B_Y, A_Y) \) to \( (B_X, A_X) \).

This statement can be considered as a variation of the following “universal” property of the Cantor set: “Any compact metric space is a continuous image of the Cantor set.”

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