Isoperimetric solution to problem of prismatic bar torsion

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Abstract. In the article the simple torsion problem of an elastic prismatic bar is presented in the isoperimetric form. It proves that equal sections with a convex contour have a geometrical rigidity of torsion, depending only on one parameter – the form factor. The main isoperimetric properties of the form factor are stated, and on their basis, isoperimetric properties of the given geometrical rigidity of torsion are formulated. Elliptic sections, rectangular, and triangular sections in the form of the regular figures are considered. For these subsets of bars, the approximating functions satisfying known decisions for the corresponding forms of sections with an accuracy of 2%, are constructed.

1. Introduction

Development of analytical methods for solving the problem of cores torsion of non-circular cross section is one of the most important problems of mechanics. The cores, which perceive torsion deformations, are widespread in mechanical engineering. Despite the century path, passed by the scientists, who were actively dealing with this problem, there are a few exact solutions of the task. For cores of complicate types, generally approximate methods (variation and numerical) are used. In the last decades, the final element method is very effectively used.

For the first time it was proposed to solve the torsion problem of prismatic cores of non-circular cross section in movements by Sen-Venan [1]. He assumed that u and v shifts in the section plane are defined by the same relationships as that for a core of round section, and using the Lame’s general equations, consolidated the solution of a problem to the movement definition of w in the twisted core. Only half a century later, in 1948 G. Polia confirmed Sen-Venan's conclusion and provided the strict proof of the theorem asserting that from all prismatic cores with equal section the core with round section has the greatest rigidity [2].

A large number of works were devoted to the research of problems of torsion by experimental methods with the use of membrane analogy. Piotr Gorzelaczyk and Jan A. Kolodziej [3] carried out numerical experiments related with the elastic torsion of prismatic rods.

Many scientists in Russia and the Soviet Union were also engaged in the solution of this problem. It is necessary to outline N.I. Muskhelishvili, N.H. Arutyunyan and B.L. Abramyan, A.D. Chernyshov [4], E.V. Lomakin [5], L.M. Zubov [6] and others.

2. Solution

At the research of the strain-stress state of twisted cores, it is necessary to define geometrical torsion rigidity $I_k$. In the present article the isoperimetric method, theoretical foundation of which was laid in the article [7], is applied for a calculation of mentioned parameter.

In the work [7], geometrical torsion rigidity $I_k$ for any cross section is presented by a ratio:
\begin{equation}
I_k = \max 4 \left( \iint_A f dA \right)^2 \iint_A \left( \frac{f_x^2}{r^2} + \frac{f_y^2}{r^2} \right) dA.
\end{equation}

Here \( f(x, y) \) is a sufficiently smooth function (stress function – the Prandtl function), which satisfies an equality condition to zero on a contour and to an inequality \( 0 < f < 1 \); the integral in a denominator is called the Dirichlet integral.

Let use the modified Rayleigh - Ritz method and transform expression (1), having replaced its two-parametrical single tension function of \( f(x, y) \) with one-parametrical function \( g(\rho) \) which level lines are similar to a contour of section and are located as follows:

\begin{equation}
f(x, y) = g\left[ \frac{t}{r(\phi)} \right] = g(\rho),
\end{equation}

where \( \rho = \frac{t}{r(\phi)} \) is a dimensionless polar coordinate; \( t \) and \( \phi \) are polar coordinates; \( r(\phi) \) - the section contour equation. Let us substitute function (2) in expression (1) and carry out necessary transformations of a numerator and a denominator separately:

\begin{equation}
\iint_A f(x, y) dA = \int_0^2 \int_0^r g^2(\rho) t d\phi = \int_0^2 r^2 d\phi \int_0^r g(\rho) \frac{t}{r} \frac{dt}{r} = 2A \int_0^1 g(\rho) \rho d\rho.
\end{equation}

Finally,

\begin{equation}
\iint_A f(x, y) dA = 2A \int_0^1 g(\rho) \rho d\rho,
\end{equation}

\begin{equation}
4 \left( \iint_A f dA \right)^2 = 16A^2 \left( \int_0^1 g(\rho) \rho d\rho \right)^2.
\end{equation}

The Dirichlet integral can be represented in polar coordinates:

\begin{equation}
\iint_A \left( f_x^2 + f_y^2 \right) dA = \int_0^2 \int_0^r \left( f_x^2 + \frac{1}{t^2} f_y^2 \right) t d\phi.
\end{equation}

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\end{equation}

Let us find derivative of function \( f(t, \varphi) \):

\begin{equation}
f_t = \left[ g\left( \frac{t}{r} \right) \right]' = g'\left( \frac{t}{r} \right) \frac{1}{r} = g'(\rho) \frac{1}{r};
\end{equation}

\begin{equation}
f_\varphi = \left[ g\left( \frac{t}{r(\varphi)} \right) \right]' = -g'\left[ \frac{t}{r(\varphi)} \right] \frac{r'(\varphi)}{r^2} = -g'(\rho) \frac{r' t}{r^2}.
\end{equation}

Let us substitute these derivatives in the Dirichlet integral:

\begin{equation}
\int_0^2 \int_0^r \left( f_x^2 + \frac{1}{t^2} f_y^2 \right) t d\phi = \int_0^2 \int_0^r \left( g'^2(\rho) \frac{1}{r^2} + \frac{1}{t^2} g''^2(\rho) \frac{r'^2}{r^2} \right) t d\phi =
\end{equation}

\begin{equation}
= \int_0^2 \left( \frac{1}{t^2} \frac{r'^2}{r^2} \right) d\phi \int_0^r g''(\rho) \frac{t}{r} \frac{dt}{r} = K \int_0^1 g''(\rho) \rho d\rho.
\end{equation}
Here \( K_f \) is the coefficient of a section form (figure 1).

![Figure 1](image)

**Figure 1.** The coefficient of a section form is as follows.

For the curved area (Figure 1, a), it is:

\[
K_f = \min \int \left( 1 + \frac{r'^2}{r^2} \right) d\varphi.
\]

For the polygon area (Figure 1, b), it is:

\[
K_f = \sum_{i=1}^{n} \frac{l_i}{h_i} = \sum_{i=1}^{n} (\text{ctg} \alpha_i + \text{ctg} \beta_i).
\]

Detailed researches of its properties are given in articles [8-10]. Finally,

\[
\int \int f_s^2 + f_r^2 dA = K_f \int_0^1 g r'(\rho) \rho d\rho.
\]

Substituting expressions (4) and (8) in (1), it is possible to obtain:

\[
I_k \geq \frac{16A^2}{K_f} \left( \int_0^1 g \rho d\rho \right)^2 \left/ \int_0^1 g r'(\rho) \rho d\rho \right. .
\]

The sense of inequality is because of the using the Rayleigh - Ritz variation method. The second fraction in this expression represents the number depending on the accuracy of the Prandtl function choice. This fraction can be attributed to the \( K_f \) proportionality factor. Then,

\[
I_k \geq K_f A^2 / K_f .
\]

In this expression the coefficient of proportionality \( K_f \) depends on the section form (triangular, rectangular, trapezoid, etc.), or a type of geometrical transformation. In other words, this coefficient is function. Therefore, instead of expression (10), it is reasonable to use inequalities [8-10]:

\[
I_k \geq K_f A^2 / K_f^p , \text{ or } I_k \geq A^2 \left( K_f + B / K_f \right) , \text{ or } I_k \geq A^2 \left( K_f + B \times f( K_f ) \right) .
\]

where there are two unknown parameters. If some subset of the sections united by one continuous or discrete geometrical transformation is considered and in that transformation there are two sections with known decisions («reference» solutions), then unknown parameters in (11) expressions can be found.
Henceforth it is more advantageous to define the normalized geometrical rigidity of section \( i_k = I_k / A^2 \):

\[
i_k \geq K_f / K_f^* , \quad \text{or} \quad i_k \geq I_f + B / K_f , \quad \text{or} \quad i_k \geq K_f + B \times f( K_f ).
\]  

(12)

Since the form factor has isoperimetric properties, normalized geometrical rigidity of torsion \( i_k \) has the same properties.

Expression (9) shows that for any sections the geometrical rigidity of torsion \( I_k \) depends only on \( K_f \), therefore it can be considered as geometrical analogue of \( i_k \). That is it; all isoperimetric properties of the form factor are associated to the geometrical rigidity of sections:

- from all set of sections with a convex contour a circle has the greatest \( i_k \) value;
- from all set of sections in the form of a convex quadrangle a square section has the smallest \( i_k \) value;
- from all set of sections in the form of a triangle a section in the form of an equilateral triangle has the smallest \( i_k \) value;
- from all set of sections in the form of a convex n-gon a section in the form of the regular n-gon has the smallest \( i_k \) value;
- from two sections in the form of the regular n-gons a section with the bigger number of the sides has a smaller \( i_k \) value;
- all set of \( i_k \) values for sections with a convex contour, presented in coordinate \( i_k - K_f \) axes, is limited from below by \( i_k \) values for polygons, all sides of which contact an inscribed circle (including sections in the form of the regular n-gons and isosceles triangles), and from above by \( i_k \) values for sections in the form of ellipses;
- all set of values for sections in the form of convex quadrangles and triangles is limited from above by \( i_k \) values for sections in the form of a rectangle (figure 2).

![Figure 2. Graphs \( i_k - 1/K_f \)](image)

Two last properties of bilateral values limitation of the given geometrical rigidity for all set of figures with a convex contour are represented in figure 2, where on the abscissa axis, the form factor is postponed. In figure 1, the I curve describes \( i_k \) values for sections in the form of polygons (including regular), all parties of which contact an inscribed circle. Points 0, 8, 6, 5, 4, 3 correspond to \( i_k \) values
for a circle to $i_{k0}$ and sections in the form of the regular figures ($i_{k8}$ for an octagon, $i_{k6}$ for a hexagon, $i_{k5}$ for a pentagon, $i_{k4}$ for a square, $i_{k3}$ for an equilateral triangle). The II curve corresponds to $i_k$ values for sections in the form of any triangles (including isosceles); the III curve corresponds to $i_k$ values for sections in the form of an ellipse; the IV curve corresponds to $i_k$ values for sections in the form of a rectangle.

3. Conclusions

1. The normalized geometrical rigidity of sections torsion depends only on one parameter – the area form factor.

2. On the basis of isoperimetric properties of the form factor, isoperimetric properties of the normalized geometrical rigidity are formulated.

3. A set of values of the normalized geometrical rigidity of sections with a convex contour is limited from two sides: elliptic sections give the lower bound, and sections in the form of the regular figures and isosceles triangles give the top bound; for quadrangular sections, the lower bound is formed by rectangular sections.

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