Learning Noisy Hedonic Games

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We consider the learning task of prediction of formation of core stable coalition structure in hedonic games based on agents’ noisy preferences. We have considered two cases: complete information (noisy preferences of all the agents are entirely known) and partial information (noisy preferences over some coalitions are only known). We introduce a noise model that probabilistically scales the valuations of coalitions. The performance metric is the probability of our prediction conditioned on all or few noisy preferences of the agents be correct. The nature of our results is that this prediction probability is relatively low, including being zero, and rarely it is one. In the complete information two-agent model, in which each agent ‘retains’ or ‘inflates’ the values of its coalitions, we identify the expressions of the prediction probabilities in terms of the noise probability. We identify the interval of the noise probability such that the prediction probability is at least a user-given threshold. It turned out that, for some noisy games, the noise probability interval does not exist for a threshold as low as 0.1481, thus demonstrating that the prediction probabilities are generally low even in this model.

In the partial information setup, we consider \( n \) agent games with \( l \) support of noise values, and such noisy preferences are available for some coalitions only. We obtain the bounds on the prediction probability of a partition to be \( \epsilon \)-PAC stable in the noise-free game in the cases when the realized noisy game has or hasn’t \( \epsilon \)-PAC stable outcome.

1 INTRODUCTION

Coalition formation games are of great interest to researchers because they model natural interactions among multi-agent societies. Individuals in social, economic, or political environments often form coalitions to carry out particular task rather than performing it alone. The coalition formation process can be formalized using the framework of hedonic games. In these games, each agent has a preference over the set of all possible coalitions it can form with other agents. The utility of an agent depends only on the composition of the coalition to which the agent belongs.

Transferable utility (TU) cooperative game theory assumes that binding agreements between the players are possible, and there is some medium of exchange between the players, for instance, money. TU cooperative game has two primary problems: which coalitions to form and how to allocate the worth of grand coalition to each agent. To this end, there are many allocation rules like Shapley value, core, nucleolus, etc. (for more details, see [Narahari, 2014, Peleg and Sudhölter, 2007]). Compared to the TU cooperative games, the hedonic games belong to non-transferable utility (NTU) games. In NTU games, there is no medium of exchange between players. Hedonic games primarily focus on each agent’s preferences over the coalitions, and hence addresses the first problem, i.e., which coalitions to form [Aziz and Savani, 2016]. The answer to this question is via various stability notions such as Core, Individual Rational, Perfect, Strict core, etc.

Since the coalition formation in a hedonic game depends on the agents’ preferences, any stability notion generally assumes complete information over all the preferences, which is one of the strong assumptions in coalition formation games. Often, it is tough to know what are the actual preferences of each agent. Nonetheless, under the full information setting, i.e., all the agents’ preferences are known, there is significant work in finding the stable partition of the agent set if it exists. We have started with the full information assumption to get some interesting insights, but later on, we have relaxed it to the partial information setup.

In [Sliwinski and Zick, 2017] it is assumed that the preferences over some coalitions are only known. They have done remarkable work by introducing the notion of \( \epsilon \)-Probably Approximately Correct (\( \epsilon \)-PAC) stability for learning the stable outcome in the hedonic games if it exists. Therefore, they bypassed the need to learn the unknown preferences to find the core stable outcome. For
several classes of hedonic games, authors have addressed the stabilizability and learnability problem of hedonic games.

Apart from the assumption on the preferences’ full information, we can have preferences after being corrupted by some noise, i.e., the exact preferences of agents are not available. Instead, the preferences with errors are observed. We call such observed erroneous preferences noisy preferences. The presence of noise in the preferences may hinder the actual information, and hence the partition of the set of agents will be noisy. For example, let us consider the game with two agents such that agent 1 prefers to stay alone, and agent 2 prefers to form the coalition with agent 1. Thus, a grand coalition is not a core stable outcome. Now suppose we have the simplest noise model, ‘retain or inflate,’ i.e., the value of each coalition is either retained or inflated by a fixed value $\alpha$ with some unknown probability $p$. With this noise, there are four possible noisy games. Out of these games, there is one game with grand coalition as core stable outcome. If this game is realized, then the noisy partition will be a grand coalition, and hence agent one will be ‘unhappy.’

Another motivation for this work is to assume that the agents have different preferences over the coalitions, and these preferences are their private information. An external agent has a mechanism for partitioning the agents set with the limited knowledge of the preferences available to it. Suppose there is a channel through which the agents will communicate this partial information to the external agent. If the channel has some inherent error, then the external agent at the receiver’s end will get the noisy preferences instead of the correct preferences. The external agent will partition the agents set based on this noisy information and obtain a noisy partition, if it exists.

In this paper, we consider a new learning task of prediction of formation of core stable coalition structure in hedonic cooperative games based on agents’ noisy preferences. We introduce a relevant noise model that probabilistically scales the valuations of coalitions. The performance metric is the probability of our prediction be correct conditioned on all or few noisy preferences of the agents.

1.1 Notations and Preliminaries

In this section, we will give some notations, definitions and other related background that we will use in rest of the paper. A hedonic game is the coalition formation game, where the objective of the agents is to form the coalitions with some desirable properties. In the entire paper we will use $N = \{1, 2, \ldots, n\}$ to denote the set of agents. Each agent is endowed with a reflexive, transitive and complete preference relation $\succeq$ over the set of coalitions containing agent $i$ denoted by $\mathcal{N}_i = \{S \subseteq N | i \in S\}$. Therefore, the hedonic game can be represented as a tuple $(N, \succeq)$, where $\succeq = (\succeq_1, \ldots, \succeq_n)$ are the preferences of each agent with the assumption that for any agent $i \in N$, any two coalitions $S, T \in \mathcal{N}_i$, $S \sim_i T$ iff $S \succeq_i T$ and $T \succeq_i S$. These preferences represent the agents’ willingness to form the coalitions with other agents. Since these preferences are assumed to be reflexive, transitive and complete and hence there exists a value function, $v$ such that $S \succeq_i T \iff v_i(S) \geq v_i(T)$ [Mas-Colell et al., 1995, Narahari, 2014]. Based on the representation of the preferences, hedonic games are classified into several classes [Aziz et al., 2019]. For example if the value $v_i(S) = \sum_{j \in S \setminus \{i\}} v_i(j)$, the game is called the Additively Separable Hedonic Games (ASHGs), whereas if the value is $v_i(S) = \frac{1}{|S|} \sum_{j \in S} v_i(j)$ it is called the Fractional Hedonic Games (FHGs) etc., where $v_i(j)$ is the valuation of agent $j$ in eyes of agent $i$. Throughout this paper we have assumed that $v_i(j)$’s are non-negative for all $i, j \in N$.

One of the major tasks in the coalition formation games or hedonic games, as its name suggests, is the formation of coalitions that are stable with respect to some stability criterion (for more details
on various stability criteria see [Aziz and Savani, 2016]). A coalition structure is a partition \( \pi \) of \( N \); we refer to the coalition containing \( i \) in partition \( \pi \) as \( \pi(i) \). We say that a coalition \( S \) core blocks a coalition structure \( \pi \), if every agent \( i \in S \) strictly prefers \( S \) to \( \pi(i) \), i.e., \( S \succ_i \pi(i) \forall i \in S \).

**Definition 1.1 (Core(\( N, \geq \)) [Aziz and Savani, 2016])**. A coalition structure \( \pi \) is said to be core stable if there is no coalition that core blocks \( \pi \), i.e., there is at least one agent \( i \in S \) such that \( \pi(i) \succeq S \). The set of all core stable coalition structures is called the core of \((N, \geq)\), denoted \( \text{Core}(N, \geq) \).

In this paper, we are looking at noisy hedonic games. We considered two settings: the full but noisy information is available for all the coalitions, and the second one where the noisy information is available for some coalitions only. In the full information setting, we have considered the two-agent games only, whereas, in the partial information setting, \( n \) agent games are considered. As the number of agents increases, the number of noisy games possible for a given game increases exponentially, and it is hard to enumerate all possible noisy games. In any case, for a given game, we are only given the noisy values for coalitions, and hence the noise-free game \((N, v)\) is not known.

Now, suppose there is a distribution \( N \) for noise over each coalition. Thus for each coalition \( S \) a value \( \alpha(S) > 0 \) is obtained according to \( N \). This \( \alpha(S) \) represents the noise it will impart to the value of each agent \( i \) in coalition \( S \), i.e., to \( v_i(S) \forall i \in S \). The distribution of noise \( N \) is also not known. We are assuming that for each agent in a given coalition, the noise is the same, i.e., \( \alpha_i(S) = \alpha(S) \forall i \in S \). Therefore, based on the noise and the true values for a coalition, we have the noisy values \( \tilde{o}_i(S) = \alpha(S) o_i(S) \), for all \( i \in S \). This noisy game is represented by \((N, \geq')\) or equivalently, \((N, \tilde{o})\), where \( \geq' = (\geq_1', \geq_2',..., \geq_n') \) are the noisy preferences of each agents. Our multiplicative noise structure is, of course, equivalent to an additive noise model via a logarithmic transformation.

For the full information case one can find a core stable partition for the noisy game \((N, \tilde{o})\), using a standard algorithm [Ballester, 2004, Sung and Dimitrov, 2010]. Whereas, in case of partial information setup we will use notion of \( \epsilon \)-PAC stability from [Sliwinski and Zick, 2017]. For the partial information setup we assume that we only have the noisy preferences over the sample \( S_{tr} = \{(S_1, \tilde{o}(S_1))..., (S_m, \tilde{o}(S_m))\} \), where coalitions \( S_1,...,S_m \) are coalitions drawn \( i.i.d. \) from a distribution \( D \) over \( 2^N \). Based on this information we can find the \( \epsilon \)-PAC stable partition \( \tilde{\pi} \) of the noisy game \((N, \tilde{o})\), if it exists, using [Sliwinski and Zick, 2017]. Our aim in both the cases (full information and partial information) is to check if \( \tilde{\pi} \) is core or \( \epsilon \)-PAC stable for the noise free game \((N, v)\), without knowing the actual values \( v \).

Next, we will describe the learning theoretic notion of core stability for the noisy game, based on the PAC framework of [Sliwinski and Zick, 2017]. An algorithm \( A \) PAC stabilizes a class \( \mathcal{H} \) of noisy hedonic games \((N, \tilde{o})\) if after seeing some examples, it can propose a partition \( \tilde{\pi} \) that is unlikely to be core blocked by a coalition sampled from \( D \). Formally, given \( \epsilon > 0 \), a partition \( \tilde{\pi} \) is \( \epsilon \)-PAC stable under \( D \) if
\[
\Pr_{T \sim D}[T \text{ core blocks } \tilde{\pi}] < \epsilon. \tag{1}
\]
Furthermore, the algorithm \( A \) PAC stabilizes \( \mathcal{H} \) if for any hedonic game \((N, \tilde{o}) \in \mathcal{H} \), any distribution \( D \) over \( 2^N \), and any pair of the error and the confidence parameters, \( \epsilon, \delta > 0 \), with probability at least \( 1 - \delta \), \( A \) outputs a \( \epsilon \)-PAC stable coalition structure, or reports that the core is empty. Also, \( m_{tr} \) is required to be polynomial in \( n, \frac{1}{\epsilon} \) and \( \log \frac{1}{\delta} \).

Throughout the paper, we have used \([I]\) to represent the set of the integers \( \{1,\ldots,l\} \). Moreover, we will use \((n, l)\) to represent the games with \( n \) agents and \( l \) support of the noise. We are interested in finding the noise intervals for which a certain prediction probability \( \eta(n,l)(\pi, \tilde{o}, G) \) defined below is achieved.
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Definition 1.2 (Prediction probability). In a $n$ agents $l$ support noise model, the prediction probability, $\eta_{n,l}(\pi, \tilde{v}, G)$, is the conditional probability of formation of partition $\pi$ in the noise-free game when the noisy values $\tilde{v}$ are according to a given noisy game $G$.

Specifically, our major contributions are as follows:

- In Section 2, we have considered two agents game with complete information of the noisy values. With 'retain or inflate' noise structure, we proved that the prediction probability of grand coalition formation is less than one for all but one case. In particular, the prediction probability is zero in many cases. This prediction probability is one for any noise probability when the inflation parameters are within some ball around the noisy values.
- For some user-given threshold $\zeta$, we have identified the noise probabilities for which the prediction probability is $\zeta$. It turned out that this $\zeta$ can be as low as 0.1481 for a specific noisy game in the 'retain or inflate' noise model, i.e., no matter what noise probability is, the prediction probability is very bad.
- The prediction probabilities are polynomials of noise probabilities in a suitable partition of noise level space. This insight leads to the above interesting results.
- Motivating the partial information noisy hedonic games, we have identified the lower bounds on the probability of formation of $\epsilon$-PAC stable outcome in the noise free game, given that the noisy game has $\epsilon$-PAC stable outcome, for $n$ agent with $2$ noise support and $n$ agent $l$ noise support models in Section 3.1, and Section 3.2 respectively.
- Relaxing the assumption of Section 3 we obtained a lower bound for the probability that noise-free game doesn’t have the $\epsilon$-PAC stable outcome, given that there is no $\epsilon$-PAC stable outcome in the noisy game, for $n$ agent with $2$ support and $n$ agent $l$ support models respectively in Section 4.

2 2 AGENT FULL INFORMATION NOISY HEDONIC GAMES

This section will begin with the simplest noise structure called the 'retain or inflate' noise structure.

2.1 Retain or Inflate – 2 Agent Full Information Model

As the name suggests, retain or inflate noise will mean that the value $v(S)$ of each coalition $S$ is either retained or inflated. Formally, the support of the noise is $\alpha(S) \in \{1, \alpha\}$ $\forall S \subseteq N$ with $\alpha > 1$ such that $P(\alpha(S) = \alpha) = p = 1 - P(\alpha(S) = 1)$ for any coalition $S$. In other words, the value of each coalition is either inflated by $\alpha > 1$ with some unknown probability $p$, or retained with probability $1 - p$. The aim is to find the conditions on noise $p$ such that the noise-free game $(N, v)$ (unknown) has grand coalition as the core stable outcome, i.e., $\pi = \{12\} = N$. Since for two-agent games, there can be four possible noisy hedonic games, and we will look at each of them individually. Moreover, we will assume that the value for each coalition in these noisy games is given (this we call full information noisy hedonic game).

**Game 1.** Let us consider a noisy game $(N, \tilde{v})$ with $N = \{1, 2\}$ such that $\tilde{v}$ are

\[
\tilde{v}_1(12) > \tilde{v}_1(1) \\
\tilde{v}_2(12) > \tilde{v}_2(2).
\]  

(game 1)

Clearly $\tilde{\pi} = \{12\} = N$ is the core stable outcome. To this end, we have the following lemma

**Lemma 2.1.** For 2 agent noisy game with 'retain or inflate' noise for full information of $\tilde{v}$'s as given in game 1, the conditional probability that the noise free game has grand coalition as core stable
outcome, i.e., \( P[\pi = N|\tilde{\nu}_1(\cdot), \tilde{\nu}_2(\cdot) \text{ as in game 1}] =: \eta_{(2,2)}(\pi, \tilde{\nu}, \text{game 1}) \) is given by:

\[
\eta_{(2,2)}(\pi, \tilde{\nu}, \text{game 1}) = \begin{cases} 
 p^3 + p^2(1 - p) + 2p(1 - p)^2 + (1 - p)^3, & \text{if } \alpha \geq \bar{r} \\
 1, & \text{if } \alpha < \bar{r}
\end{cases}
\]

where \( \bar{r} = \max \left\{ \hat{\tilde{\nu}}(1,1), \hat{\tilde{\nu}}(1,2), \hat{\tilde{\nu}}(2,1), \hat{\tilde{\nu}}(2,2) \right\} \), and \( \bar{r} = \min \left\{ \hat{\tilde{\nu}}(1,1), \hat{\tilde{\nu}}(1,2), \hat{\tilde{\nu}}(2,1), \hat{\tilde{\nu}}(2,2) \right\} \).

Also, this conditional probability \( \eta_{(2,2)}(\pi, \tilde{\nu}, \text{game 1}) \) is convex in \( p \). So, while the minimal value for \( \eta_{(2,2)}(\pi, \tilde{\nu}, \text{game 1}) \) occurs for noise probabilities around \( p = 0.5 \) (depending on the relative values of \( \{\hat{\tilde{\nu}}(\cdot)\}_{i=1,2} \) i.e., depending on \( \alpha, \bar{r} \) and \( \bar{r} \)), the maximal value of it is 1 at \( p = 0 \) and \( p = 1 \).

Proof. (Sketch only. The full proof is available in the supplementary material A.1). Firstly, we note that there are four possible hedonic games with two agents. Out of these four games, only one game has the grand coalition as a core stable outcome. Now, if game 1 is realized as a noisy game, then there are eight possible choices (because the value of each coalition is either retained or inflated, and there are three coalitions) for the noise combinations. For different intervals of \( \alpha \)'s, the above probability captures the event such that the noise-free game will have \( \pi = N \). The convexity follows as the second derivatives of \( \eta_{(2,2)}(\pi, \tilde{\nu}, \text{game 1}) \) with respect to \( p \) are \( 6p^2, 2 \), and 0 for \( \alpha \geq \bar{r}, \alpha \geq r, \) and \( \alpha < r \) respectively.

The lemma provides some interesting insights: the probability of grand coalition formation will depend on three factors, \( p, \alpha, \) and \( \tilde{\nu} \). But \( p \) is not known because the noise distribution is not known. Moreover, since \( \tilde{\nu} \)'s are known, we know \( r \), and \( \bar{r} \). From Figure 1 it is clear that if \( p \) is close to 0.5, (that is the noisy channel is randomly inflating or retaining the values of each coalition), then the probability \( \eta_{(2,2)}(\pi, \tilde{\nu}, \text{game 1}) \) is close to 0.62, for \( \alpha \geq \bar{r} \), and close to 0.75 for \( \alpha \geq \bar{r} \). Thus, whether the noise-free game has core stable outcome or not, given the noisy game 1, can be predicted only with probability approximately equal to 0.62, if \( \alpha \geq \bar{r} \), and equal to 0.75 for \( \alpha \geq \bar{r} \). Therefore, when the noise is random, the prediction is worst. Moreover, if \( p \approx 0 \) or \( p \approx 1 \), the probability of predicting grand coalition formation in unobserved game having observed game 1 is more than 0.9. It may seem counter-intuitive that for very high noise, i.e., \( p \approx 1 \), the probability of grand coalition is also high. It is because very high noise here means that the values of almost all the coalitions are inflated. Thus the preferences of each agent in the noise-free game and noisy game are mostly the same. It allows the formation of the grand coalition in both the noisy and noise-free game.

Now, suppose we relax the condition that we want \( \eta_{(2,2)}(\pi, \tilde{\nu}, \text{game 1}) = 1 \). If we allow the formation of the grand coalition in the noise-free game with probability more than a user-given threshold, say \( \zeta \). In that case, for different ranges of \( \alpha \), we can get the interval of the noise probability allowable for attaining the user given probability \( \zeta \) by drawing a horizontal line \( \eta_{(2,2)}(\pi, \tilde{\nu}, \text{game 1}) = \zeta \) in the graph shown in Figure 1. For example, if \( \zeta = 0.9 \), then the noise intervals are

\[
I^*(\zeta = 0.9) = \begin{cases}
[0, 0.101] \cup [0.946, 1], & \text{if } \alpha \geq \bar{r} \\
[0, 0.113] \cup [0.887, 1], & \text{if } \alpha \geq r \\
[0, 1], & \text{if } \alpha < r.
\end{cases}
\]

The aforementioned implies that for the grand coalition to be formed in the noise-free game with probability equal to \( \zeta = 0.9 \), the noise parameter \( p \) has to be in the interval \( I^*(\zeta = 0.9) \). For any other \( \zeta \), the intervals can be similarly obtained.

As mentioned above, with two agents, there are four possible noisy games. The only noisy game having the grand coalition is game 1. In Equation (2) we have obtained the conditional probability \( \eta_{(2,2)}(\pi, \tilde{\nu}, \text{game 1}) \). We will now state the results for this conditional probability when the other
Fig. 1. In two agent hedonic game 1 with ‘retain or inflate’ noise model, we plot the probability of formation of grand coalition in the noise free game \((N, v)\) given the noisy game 1, i.e., \(\eta_{(2, 2)}(\pi, \tilde{v}, \text{game } 1)\) for different ranges of \(\alpha\). Note that for \(\alpha \geq \bar{r}\) the probability \(\eta_{(2, 2)}(\pi, \tilde{v}, \text{game } 1)\) is as bad as 0.62, and it is as bad as 0.75 for \(\alpha \geq r\); and this occurs when the noise probability is \(p \approx 0.5\). Moreover, for the user given threshold \(\zeta = 0.9\) an interval of noise probability is identified for different ranges of \(\alpha\). This noise interval is \([0, 0.101] \cup [0.09456, 1]\) if \(\alpha \geq \bar{r}\), and \([0, 0.1127] \cup [0.8873, 1]\) if \(\alpha \geq r\), and \([0, 1]\) if \(\alpha < r\). Three noisy games are realized. Note that we are not providing the proofs of these results because they use the same idea as in the proof of Lemma 2.1.

**Game 2.** Let us consider a noisy game \((N, \tilde{v})\) with \(N = \{1, 2\}\) such that \(\tilde{v}\) are

\[
\tilde{v}_1(1) > \tilde{v}_1(12) \\
\tilde{v}_2(2) > \tilde{v}_2(12). 
\]  

Clearly \(\tilde{\pi} = \{\{1\}, \{2\}\} \neq N\) is the core stable outcome. To this end, we have following lemma

**Lemma 2.2.** For 2 agent noisy game with ‘retain or inflate’ noise for full information of \(\tilde{v}\)’s as given in game 2, the conditional probability that the noise free game has grand coalition as core stable outcome, i.e., \(\mathbb{P}[\pi = N|\tilde{v}_1(\cdot), \tilde{v}_2(\cdot)\text{ as in game } 2] =: \eta_{(2, 2)}(\pi, \tilde{v}, \text{game } 2)\) is given by:

\[
\eta_{(2, 2)}(\pi, \tilde{v}, \text{game } 2) = \begin{cases} 
p^2(1 - p) & \text{if } \frac{1}{\alpha} < r \\
0 & \text{if } \frac{1}{\alpha} \geq r
\end{cases}
\]  

Also, this conditional probability \(\eta_{(2, 2)}(\pi, \tilde{v}, \text{game } 2)\) is concave in \(p\). So, while the minimal value for \(\eta_{(2, 2)}(\pi, \tilde{v}, \text{game } 2)\) is for zero noise at \(p = 0\), the maximal value of it is 0.1481 at \(p = 2/3\).

**Game 3.** Let us consider a noisy game \((N, \tilde{v})\) with \(N = \{1, 2\}\) such that \(\tilde{v}\) are

\[
\tilde{v}_1(1) > \tilde{v}_1(12) \\
\tilde{v}_2(12) > \tilde{v}_2(2). 
\]  

Clearly \(\tilde{\pi} = \{\{1\}, \{2\}\} \neq N\) is the core stable outcome. To this end, we have following lemma
Lemma 2.3. For 2 agent noisy game with ‘retain or inflate’ noise for full information of \( \tilde{v} \)'s as given in game 3, the conditional probability that the noise free game has grand coalition as core stable outcome, i.e., \( \mathbb{P}[\pi = N|\tilde{v}_1(\cdot), \tilde{v}_2(\cdot) \text{ as in game 3}] =: \eta_{(2,2)}(\pi, \tilde{v}, \text{game 3}) \) is given by:

\[
\eta_{(2,2)}(\pi, \tilde{v}, \text{game 3}) = \begin{cases} 
(1 - p)^2 p + (1 - p)p^2, & \text{if } \frac{1}{\alpha} < \frac{\tilde{v}_1(12)}{\tilde{v}_1(1)} \\
0, & \text{if } \frac{1}{\alpha} \geq \frac{\tilde{v}_1(12)}{\tilde{v}_1(1)} \end{cases}
\]

Also, this conditional probability \( \eta_{(2,2)}(\pi, \tilde{v}, \text{game 3}) \) is concave in \( p \). So, while the minimal value for \( \eta_{(2,2)}(\pi, \tilde{v}, \text{game 3}) \) is for zero noise at \( p = 0 \), the maximal value of it is 0.25 at \( p = 0.5 \).

Game 4. Let us consider a noisy game \((N, \tilde{\nu})\) with \( N = \{1, 2\} \) such that \( \tilde{\nu} \) are

\[
\tilde{\nu}_1(12) > \tilde{\nu}_1(1) \\
\tilde{\nu}_2(2) > \tilde{\nu}_2(12).
\]

Clearly \( \tilde{\pi} = \{\{1\}, \{2\}\} \neq N \) is the core stable outcome. To this end, we have following lemma

Lemma 2.4. For 2 agent noisy game with ‘retain or inflate’ noise for full information of \( \tilde{v} \)'s as given in game 4, the conditional probability that the noise free game has grand coalition as core stable outcome, i.e., \( \mathbb{P}[\pi = N|\tilde{v}_1(\cdot), \tilde{v}_2(\cdot) \text{ as in game 4}] =: \eta_{(2,2)}(\pi, \tilde{v}, \text{game 4}) \) is given by:

\[
\eta_{(2,2)}(\pi, \tilde{v}, \text{game 4}) = \begin{cases} 
(1 - p)^2 p + (1 - p)p^2, & \text{if } \frac{1}{\alpha} < \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)} \\
0, & \text{if } \frac{1}{\alpha} \geq \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)} \end{cases}
\]

Also, this conditional probability \( \eta_{(2,2)}(\pi, \tilde{v}, \text{game 4}) \) is concave in \( p \). So, while the minimal value for \( \eta_{(2,2)}(\pi, \tilde{v}, \text{game 4}) \) is for zero noise at \( p = 0 \), the maximal value of it is 0.25 at \( p = 0.5 \).

Fig. 2. In 2 agent hedonic game with ‘retain or inflate’ noise model, the above plots show the probability of formation of grand coalition in the noise free game \((N, \nu)\) given that the noisy games are respectively game 2 in Fig 2(a), game 3, (or game 4) in Fig 2(b). For game 2, this conditional probability is always less than 0.1481, whereas for game 3, (or game 4) it is less than 0.25. Therefore, no matter what the noise probability is, the prediction probability is very bad for these 3 games.

We summarize the main observations of the two agents with ‘retain or inflate’ noise model in the Theorem below:
Thus, the value of coalitions in the noisy game are realized as $\tilde{\pi}_1(\cdot), \tilde{\pi}_2(\cdot)$ for some function $\tilde{\pi}_1(\cdot), \tilde{\pi}_2(\cdot)$ as in noisy game 1, $\tilde{\pi}_1(\cdot), \tilde{\pi}_2(\cdot)$ as in noisy game 1, for all $p, \tilde{\pi}_1(\cdot), \tilde{\pi}_2(\cdot), \alpha$, which is strictly less than 1, according to $\tilde{\pi}_1(\cdot), \tilde{\pi}_2(\cdot)$ and $\alpha$. In particular, $q(\tilde{\pi}_1(\cdot), \tilde{\pi}_2(\cdot), \alpha) = 0$ if

1. $\tilde{\pi}_1(\cdot), \tilde{\pi}_2(\cdot)$ are as in noisy game 2 and $\frac{1}{\alpha} \geq r$, or
2. $\tilde{\pi}_1(\cdot), \tilde{\pi}_2(\cdot)$ are as in noisy game 3 and $\frac{1}{\alpha} \geq \frac{\tilde{\pi}(12)}{\frac{\tilde{\pi}(12)}{\alpha}}$, or
3. $\tilde{\pi}_1(\cdot), \tilde{\pi}_2(\cdot)$ are as in noisy game 4 and $\frac{1}{\alpha} \geq \frac{\tilde{\pi}(12)}{\frac{\tilde{\pi}(12)}{\alpha}}$.

**Proof.** The proof of this theorem follows from Lemma 2.1, Lemma 2.2, Lemma 2.3, and Lemma 2.4, where $q(\tilde{\pi}_1(\cdot), \tilde{\pi}_2(\cdot), \alpha)$ is identified explicitly for the respective noisy games along with associated intervals of $\alpha$. The above theorem provides the fundamental limit to the formation of the grand coalition in the noise-free game for all but one specialized case where the noisy game is game 1, with the requirement $\alpha < r$. Therefore, even for a very stylized and optimistic game with two agents and two support noises, the prediction is not good.

**Remark 1.** For the noisy game 1, we have the noise regime in which the $P(\pi = N) = \zeta$, for any user given $\zeta$. Whereas in noisy game 2 this can’t be done for $\zeta > 0.1481$, and for noisy game 3, and game 4 this can’t be done for $\zeta > 0.25$. Therefore, there is no noise regime in which the prediction probability is more than 0.25 for 3 noisy games for which the grand coalition is not a stable outcome, but $\tilde{\pi} = \{\{1\}, \{2\}\}$.

**Remark 2.** The condition on the inflation value, $\alpha < r$ that is required for prediction probability of grand coalition formation for the clean/unobserved game is in fact 1 for any noise probability $p$ captures that values of coalitions of $(N, v)$ are within $r$ of the observed values $\tilde{\pi}_1(12), \tilde{\pi}_2(12)$ after these values are normalized by $\tilde{\pi}_1(1)$ and $\tilde{\pi}_2(2)$ respectively.

### 3 n AGENTS PARTIAL INFORMATION NOISY HEDONIC GAMES

For $n$ agents, we won’t further consider full information noise models because the number of games increases exponentially in the number of agents. Hence, computing the conditional probability $\eta_N(\pi, \tilde{\pi}, \text{game 1})$ is tedious. It motivates to study the games where agents’ preferences over all the coalitions are not known, i.e., noisy hedonic games with partial information.

We will now move to the partial information case where we have information only over a sample of coalitions $S_{ir} = \{(S_1, \tilde{\pi}(S_1), \ldots, S_{mi}, \tilde{\pi}(S_{mi}))\}$ sampled i.i.d from the distribution $\tilde{D}$ over $2^N$. Thus, the value of coalitions in the noisy game are realized as $\tilde{\pi}(S_1), \ldots, \tilde{\pi}(S_{mi})$. Since, the values of all the coalitions are not known, we will use the $\varepsilon$-PAC stable notion from [Sliwinski and Zick, 2017]. In this section, we will assume that the realized noisy game has $\varepsilon$-PAC stable outcome. However, we have relaxed this assumption in Section 4.

Let $\tilde{\pi}$ be the $\varepsilon$-PAC stable outcome of the realized noisy game (obtained using [Sliwinski and Zick, 2017]). Therefore $\forall \varepsilon > 0$ we have,

\[ P_{T-D}[\text{T core blocks } \tilde{\pi}] < \varepsilon, \quad (8) \]
\[ \text{or } P_{T-D}[\tilde{\pi}(i) > \tilde{\pi}(i)] \forall i \in T \] < \varepsilon, \quad (9)
\[ \text{or } P_{T-D}[\cup_{i \in T} \tilde{\pi}(i)] \geq \tilde{\pi}(i)] \geq 1 - \varepsilon, \quad (10) \]
with confidence at-least $1 - \delta$, that is,
\[
\forall \delta > 0, \quad \mathbb{P}_S\left[\mathbb{P}_{T\to\tilde{\mathcal{D}}} \left[ \bigcup_{i \in T} \tilde{v}_i(\tilde{\pi}(i)) \geq \tilde{v}_i(T) \right] \geq 1 - \varepsilon \right] \geq 1 - \delta.
\] (11)

Moreover, the number of samples $m_{\text{tr}}$ are required to be polynomial in $n, \frac{1}{\varepsilon}$ and $\log \frac{1}{\delta}$. Since the noisy sample is $S_{\text{tr}}$ and we are looking at $\varepsilon$-PAC stability in the noise-free game; thus, for notation simplicity, we will drop the outer probability term in the entire paper. This we are doing with the understanding that the noisy $\varepsilon$-PAC stable partition, $\tilde{\pi}$ is obtained using $S_{\text{tr}}$, but we are predicting the probability of the partition $\tilde{\pi}$ to be also $\varepsilon$-PAC stable for the noise-free game.

For this, we define the set $\mathcal{R}(T)$ for any coalition $T$ as
\[
\mathcal{R}(T) = \{ \tilde{\pi}(i) \in \tilde{\pi} \mid i \in T \}.
\] (12)

$\mathcal{R}(T)$ consists of set of all coalitions $\tilde{\pi}(i)$ for each agent $i$ in the coalition $T$. Moreover, let $E(T, \tilde{\pi})$ be the set of all pairs of scaling levels, $(\alpha(T), \alpha(\tilde{\pi}(i)))$ for coalitions $\tilde{\pi}(i) \in \mathcal{R}(T)$ which prefers coalition $\tilde{\pi}(i)$ over coalition $T$ for all agents $i \in T$ in both noisy and noise free games, i.e.,
\[
E(T, \tilde{\pi}) = \{ \alpha(T), \alpha(\tilde{\pi}(i)) : \forall \tilde{\pi}(i) \in \mathcal{R}(T), \bigcap_{i \in T} v_i(\tilde{\pi}(i)) \geq v_i(T) \cap \alpha(\tilde{\pi}(i)) \geq \alpha(T) v_i(T) \}
\] (13)

**Lemma 3.1.** Let $\tilde{\pi}$ be $\varepsilon$-PAC stable outcome for the noisy game $(N, \tilde{\varepsilon})$. Then the probability that any coalition $T \sim \tilde{\mathcal{D}}$ core blocks $\tilde{\pi}$ in the noise free game $(N, \varepsilon)$ is given by:
\[
\mathbb{P}_{T\to\tilde{\mathcal{D}}} \left[ \bigcup_{i \in T} \tilde{v}_i(\tilde{\pi}(i)) \geq \tilde{v}_i(T) \right] \geq (1 - \varepsilon) f_T(p, \alpha)^1
\]

**Proof.** Let us consider the following probability
\[
\mathbb{P}_{T\to\tilde{\mathcal{D}}} \left[ \bigcup_{i \in T} \tilde{v}_i(\tilde{\pi}(i)) \geq \tilde{v}_i(T) \right]
\]
\[
\geq \mathbb{P}_{T\to\tilde{\mathcal{D}}} \left[ \bigcup_{i \in T} \tilde{v}_i(\tilde{\pi}(i)) \geq \tilde{v}_i(T) \mid \bigcup_{j \in T} \tilde{v}_j(\tilde{\pi}(j)) \geq \tilde{v}_j(T) \right] \mathbb{P}_{T\to\tilde{\mathcal{D}}} \left[ \bigcup_{j \in T} \tilde{v}_j(\tilde{\pi}(j)) \geq \tilde{v}_j(T) \right]
\]
\[
\geq (1 - \varepsilon) \mathbb{P}_{T\to\tilde{\mathcal{D}}} \left[ \bigcup_{i \in T} \tilde{v}_i(\tilde{\pi}(i)) \geq \tilde{v}_i(T) \cap \bigcup_{j \in T} \tilde{v}_j(\tilde{\pi}(j)) \geq \tilde{v}_j(T) \right]
\]
\[
= (1 - \varepsilon) \mathbb{P}_{T\to\tilde{\mathcal{D}}} \left[ \bigcap_{i \in T} v_i(\tilde{\pi}(i)) \geq v_i(T) \cap \bigcup_{j \in T} \tilde{v}_j(\tilde{\pi}(j)) \geq \tilde{v}_j(T) \right]
\]
\[
\geq (1 - \varepsilon) \mathbb{P}_{T\to\tilde{\mathcal{D}}} \left[ \bigcap_{i \in T} v_i(\tilde{\pi}(i)) \geq v_i(T) \cap \tilde{v}_i(\tilde{\pi}(i)) \geq \tilde{v}_i(T) \right]
\]
\[
= (1 - \varepsilon) \mathbb{P}_{T\to\tilde{\mathcal{D}}} \left[ \mathcal{R}(T) \right]
\]
\[
= (1 - \varepsilon) f_T(p, \alpha)
\]

This ends the proof. \hfill \Box

Since, our objective is to check whether the noise free game $(N, \varepsilon)$ has $\tilde{\pi}$ as $\varepsilon$-PAC stable outcome or not, we want $\forall \varepsilon > 0,$
\[
\mathbb{P}_{T\to\tilde{\mathcal{D}}} \left[ \bigcup_{i \in T} \tilde{v}_i(\tilde{\pi}(i)) \geq \tilde{v}_i(T) \right] \geq 1 - \varepsilon.
\] (14)

But what we are getting instead of $(1 - \varepsilon)$ is $(1 - \varepsilon) f_T(p, \alpha)$ (Lemma 3.1). For Equation (14) to be satisfied we need $f_T(p, \alpha) = 1.$ But this will be possible only for certain noise probability. For example, as we see later, for two support noise model this is possible if and only if $p = 0$ or $p = 1$, i.e., the value of each coalition in the noisy game is either the same or, are scaled by $\alpha$. Thus, we relax the requirement of $f_T(p, \alpha) = 1$, and ask for a weaker condition, i.e., $(1 - \varepsilon) f_T(p, \alpha) \approx 1 - \varepsilon$, or

\[^1\text{Note that we have used } p, \alpha \text{ because the support of noise can possibly be a set, and each element in the support has some probability associated to it.}\]
equivalently, \( f_T(p, \alpha) \approx 1 \). Therefore, for a user given threshold, \( \zeta \) we want \( (1-\epsilon)f_T(p, \alpha) = \zeta(1-\epsilon) \). The \( \zeta \) captures, in some situations, the satisfaction level or the security level of an external agent who is trying to predict the partition of the noise free game.

**Remark 3.** Working with the user given threshold \( \zeta \), we will get a noise interval \( I^*(T, \zeta) \) on \( p \) for each coalition \( T \), that is the noise regime for which the coalition \( T \) can’t block \( \tilde{\pi} \) with error less than \( \epsilon \) for the user given \( \zeta \). So for each coalition in \( S_{te} = \{ T_1, T_2, \ldots, T_{m_{te}} \} \) sampled i.i.d from \( \tilde{D} \) we will have \( f_{T_1}(p, \alpha), f_{T_2}(p, \alpha), \ldots, f_{T_{m_{te}}}(p, \alpha) \). Therefore, the noise intervals \( I^*(T_1, \zeta), I^*(T_2, \zeta), \ldots, I^*(T_{m_{te}}, \zeta) \) are obtained. Hence for \( S_{te} \), the interval of the noise for which the noise free game has \( \epsilon \)-PAC stable outcome for the user given threshold \( \zeta \) is \( I^*(\zeta) = \cap_{i=1}^{m_{te}} I^*(T_i, \zeta) \). Thus, if noise values are from this interval then, the noise free game will have \( \tilde{\pi} \) as \( \epsilon \)-PAC stable outcome with user given threshold \( \zeta \).

To get some more insights, we will now consider the noise models with 2 support, and \( l \) support respectively.

### 3.1 \( n \) Agents 2 Support Partial Information Noise Model

In case of two support model, i.e., \( \alpha(S) \in \{ 1, \alpha \} \forall S \subseteq N \) with \( \alpha > 1 \), such that \( P(\alpha(S) = 1) = p = 1 - P(\alpha(S) = \alpha) \), the following lemma gives the form of \( f_T(p, \alpha) \).

**Lemma 3.2.** For \( n \) agents noisy hedonic games with \( \alpha(S) \in \{ 1, \alpha \} \forall S \subseteq N \) as support of noise and \( \tilde{\pi} \) as the \( \epsilon \)-PAC stable outcome of the noisy game, \( f_T(p, \alpha) \) is given by:

\[
f_T(p, \alpha) = \begin{cases} 
1, & \text{if } \tilde{\pi}(i) = T \forall i \in T \\
p + (1-p)|\mathcal{R}(T)|+1-[I(\alpha,T)], & \text{otherwise},
\end{cases}
\]

where \( I(\alpha,T) \) is defined as:

\[
I(\alpha,T) = \left\{ \tilde{\pi}(i) \in \mathcal{R}(T) \mid \frac{\tilde{v}_i(\tilde{\pi}(i))}{\tilde{v}_i(T)} \geq \alpha \right\}.
\]

**Proof.** (Sketch only. The proof of this lemma is deferred to the supplementary material A.2.)

Firstly, we note that \( f_T(p, \alpha) \) represents the probability of set of all pairs \( \{ (\alpha(T), \alpha(\tilde{\pi}(i))) \} \) \( \tilde{\pi}(i) \in \mathcal{R}(T) \) for which coalition \( \tilde{\pi}(i) \) is preferred over coalition \( T \) by all agents \( i \in T \) in both noisy and noise free games. If \( \tilde{\pi}(i) = T \) for all \( i \in T \), then \( f_T(p, \alpha) = p + (1-p) = 1 \) corresponding to the cases when values are scaled up or retained. If \( \tilde{\pi}(i) \neq T \) for at-least one \( i \in T \) we consider two cases, viz. \( I(\alpha,T) = \emptyset \), or \( I(\alpha,T) \neq \emptyset \). With respect to each case the probability of \( E(T, \tilde{\pi}) \) is obtained by looking at various choices of \( \alpha \).

Here are some more observations about \( f_T(p, \alpha) \).

**Remark 4.** If \( \tilde{\pi}(i) \neq T \) for at-least one \( i \in T \), then \( f_T(p, \alpha) = 1 \forall \alpha \) if and only if \( p = 0 \) or \( p = 1 \). That is, if the value of all the coalitions are retained, or if value of all of them are inflated by \( \alpha \), then for all \( i \in T \), and for all \( \tilde{\pi}(i) \in \mathcal{R}(T) \), one has \( \tilde{\pi}(i) \geq_1 T \), and \( \tilde{\pi}(i) \geq_1' T \). Thus, the noise free game has \( \epsilon \)-PAC stable outcome.

**Remark 5.** For the case when we allow for the \( f_T(p, \alpha) = \zeta \) for some user given threshold \( \zeta \), we will get an interval of the noise in accordance to the Remark 3. In this case, the noise interval will also depend on \( |\mathcal{R}(T)| \), and \( |I(\alpha,T)| \) for each \( T \in S_{te} \).

\footnote{Note that we are using \( f_T(p, \alpha) \) instead of \( f_T(p, \alpha) \) because for two support the only parameters are \( \alpha \), and \( p \).}
Remark 6. When \( \tilde{\pi} = N \), i.e., the grand coalition is \( \epsilon \)-PAC stable outcome in the noisy game, then \( R(T) = \{N\} \) for any coalition \( T \) by Equation (12). Thus, \( I(\alpha, T) = \emptyset \), or \( I(\alpha, T) = \{N\} \). Therefore \( f_T(p, \alpha) \) simplifies to

\[
f_T(p, \alpha) = \begin{cases} 1, & \text{if } I(\alpha, T) = N \\ (1 - p)^2 + p, & \text{if } I(\alpha, T) = \emptyset. \end{cases}
\]

(17)

Remark 7. Suppose we assume \( n = 2 \), with full information about the preferences of all the agents of all the coalitions in the noisy game, i.e., \( \epsilon = 0 \). In that case, we see that we will recover the results of Section 2.1 for various ranges of \( \alpha \).

3.2 \( n \) Agent / Support Partial Information Noise Model

Next, we will move to the \( l \geq 3 \) support case, i.e., \( \alpha(S) \in \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \forall S \subseteq N \) such that \( \alpha_i < \alpha_j \) if \( i < j \), with respective probabilities \( p_1, p_2, \ldots, p_l \), such that \( \sum_{j \in [l]} p_j = 1 \). Here \( \alpha_j > 0 \) for all \( j \in [l] \). For all \( r, s \) such that \( \alpha_r > \alpha_s \) let us define

\[
I(\alpha_r, \alpha_s, T) = \left\{ \tilde{\pi}(i) \in R(T) \left| \frac{\hat{v}_i(\tilde{\pi}(i))}{\bar{v}_i(T)} \geq \frac{\alpha_r}{\alpha_s} \right. \right\}.
\]

(18)

Following lemma provides the form of \( f_T(p, \alpha) \) in case of \( l \)-support noise.

Lemma 3.3. For \( n \) agents noisy hedonic game with \( \alpha(S) \in \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \forall S \subseteq N \) such that \( \alpha_i < \alpha_j \) if \( i < j \) with \( \mathbb{P}(a_j) = p_j \forall j \in [l] \), and if \( \tilde{\pi} \) is \( \epsilon \)-PAC stable outcome of the noisy game, then \( f_T(p, \alpha) \) is given by:

\[
f_T(p, \alpha) = \begin{cases} 1, & \text{if } \tilde{\pi}(i) = T \forall i \in T \\ \sum_{r,s \in [l]:\alpha_r > \alpha_s} p_s^{\left|R(T)|-|I(\alpha_r, \alpha_s, T)|+1\right|} (p_r + p_s) |\bar{v}_i(T)| - p_s \left|I(\alpha_r, \alpha_s, T)\right| \\ + \sum_{a=1}^l p_a \left(\sum_{b=1}^a p_b\right)^{|R(T)|}, & \text{otherwise}. \end{cases}
\]

(19)

Proof. (Sketch only. The full proof is deferred to supplementary material A.3). The proof follows from induction on \( l \), with the base case of two support as proved in Lemma 3.2.

Similar to the two support case, the expression \( f_T(p, \alpha) \) for \( \tilde{\pi}(i) \neq T \) for at least one \( i \in T \) has two terms, first term captures the probability of all such \( (\alpha(\tilde{\pi}(i)), \alpha(T)) \) such that \( I(\alpha_r, \alpha_s, T) = \emptyset \forall \alpha_r > \alpha_s \), whereas the second term captures the probability of all such \( (\alpha(\tilde{\pi}(i)), \alpha(T)) \) such that \( I(\alpha_r, \alpha_s, T) \neq \emptyset \) at-least for one \( \alpha_r > \alpha_s \), and hence the preferences will be preserved depending on the relative values of \( \hat{v}_i(\tilde{\pi}(i)), \bar{v}_i(T) \), and \( \alpha_r, \alpha_s \).

Remark 8. Similar to the 2 support noise model, if we allow \( f_T(p, \alpha) = \zeta \) for some user given threshold \( \zeta \), we will get an interval \( I^*(T, \zeta) \) of the noise in accordance to the Remark 3. In this case, the noise interval will also depend on \( |R(T)| \), and \( |I(\alpha_r, \alpha_s, T)| \forall \alpha_r > \alpha_s \) for each \( T \in S_{te} \).

Remark 9. For 2 agent model with support 2, i.e., when \( n = 2 \) and \( l = 2 \) with \( \epsilon = 0 \) and \( \tilde{\pi} = N \), we see that we have the results of Section 2.1, for various ranges of \( \alpha \) and \( (\alpha_1, \alpha_2) \) respectively.

4 \( n \) AGENTS PARTIAL INFORMATION NOISY HEDONIC GAMES WITH NO CORE

In this section, we will look at the case where the noisy game itself does not have the \( \epsilon \)-PAC stable outcome. This is either because only a few coalitions are observed, or the noise is such that the observed noisy game has no \( \epsilon \)-PAC stable outcome.

In particular, we are more interested in the case where there is no \( \epsilon \)-PAC stable partition in the noisy game \((N, \bar{v})\) as well as for the noise-free game.
Given a noisy game \((N, \bar{v})\) we say that it does not have any \(\epsilon\)-PAC stable outcome if, for any partition of the agent set, there is a coalition that \(\epsilon\) core blocks it with high probability. Formally, \(\forall \delta > 0, \forall \bar{\pi} \in \mathcal{P}, \exists \epsilon > 0\) and \(T \sim \mathcal{D}\), such that
\[
\Pr_{\mathcal{S}_T} [\Pr_{T,\mathcal{D}}[\cap_{i \in T} (\bar{v}_i(T) > \bar{v}_i(\bar{\pi}(i)))] \geq 1 - \epsilon] \geq 1 - \delta,
\]
where \(\mathcal{P}\) is the set of all partitions of agent set. Moreover, the number of samples \(m_T\) are required to be polynomial in \(n, \frac{1}{\epsilon}\) and \(\log \frac{1}{\delta}\). Similar to Section 3, for notation simplicity, we will drop the outer probability term. Let \(F(T, \bar{\pi})\) be the set of all pairs \((\alpha(T), \alpha(\bar{\pi}(i)))\) for coalitions \(\bar{\pi}(i) \in \mathcal{R}(T)\) which prefers coalition \(T\) over coalition \(\bar{\pi}(i)\) for all agents \(i \in T\) in both noisy and noise free games, i.e.,
\[
F(T, \bar{\pi}) = \{\alpha(T), \alpha(\bar{\pi}(i)) : \forall \bar{\pi}(i) \in \mathcal{R}(T), \cap_{i \in T} [v_i(T) \geq v_i(\bar{\pi}(i)) \cap \alpha(T)v_i(T) \geq \alpha(\bar{\pi}(i))v_i(\bar{\pi}(i))]\}
\]
(21)

We will first find the probability that a noise-free game also does not have a \(\epsilon\)-PAC stable outcome. Here is the lemma towards it.

**Lemma 4.1.** Suppose the noisy game doesn’t have the core stable partition, that is \(\forall \bar{\pi} \in \mathcal{P}, \exists \epsilon > 0, T \sim \mathcal{D}\) such that equation (20) is satisfied, then probability that the noise free game doesn’t have \(\epsilon\)-PAC stable outcome is given by:
\[
\Pr_{T,\mathcal{D}}[\cap_{i \in T} (v_i(T) > v_i(\bar{\pi}(i)))] \geq (1 - \epsilon)h_T(p, \alpha), \text{ where } h_T(p, \alpha) = \Pr_{T,\mathcal{D}}[F(T, \bar{\pi})].
\]
(22)

**Proof.** Consider the following:
\[
\begin{align*}
\Pr_{T,\mathcal{D}}[\cap_{i \in T} (v_i(T) > v_i(\bar{\pi}(i)))] & \geq \Pr_{T,\mathcal{D}}[\cap_{i \in T} (v_i(T) > v_i(\bar{\pi}(i))) | \cap_{i \in T} (\bar{v}_i(T) > \bar{v}_i(\bar{\pi}(i)))] \Pr_{T,\mathcal{D}}[\cap_{i \in T} (\bar{v}_i(T) > \bar{v}_i(\bar{\pi}(i)))] \\
& \geq (1 - \epsilon)\Pr_{T,\mathcal{D}}[\cap_{i \in T} (v_i(T) > v_i(\bar{\pi}(i))) | \cap_{i \in T} (\bar{v}_i(T) > \bar{v}_i(\bar{\pi}(i)))] \\
& \geq (1 - \epsilon)\Pr_{T,\mathcal{D}}[\cap_{i \in T} (v_i(T) > v_i(\bar{\pi}(i)) \cap \bar{v}_i(T) > \bar{v}_i(\bar{\pi}(i)))] \quad \therefore \Pr(A|B) \geq \Pr(A \cap B) \\
& = (1 - \epsilon)\Pr_{T,\mathcal{D}}[\cap_{i \in T} (v_i(T) > v_i(\bar{\pi}(i)) \cap \alpha(T)v_i(T) > \alpha(\bar{\pi}(i))v_i(\bar{\pi}(i)))] \\
& = (1 - \epsilon)\Pr_{T,\mathcal{D}}[F(T, \bar{\pi})] \\
& = (1 - \epsilon)h_T(p, \alpha).
\end{align*}
\]
\(\square\)

Our objective is to find the noise regime such that both noise-free and noisy game does not have a \(\epsilon\)-PAC stable outcome. Thus, we want \(\forall \bar{\pi} \in \mathcal{P}, \exists \epsilon > 0, T \in \mathcal{D}\) such that
\[
\Pr_{T,\mathcal{D}}[\cap_{i \in T} v_i(T) \geq v_i(\bar{\pi}(i))] \geq 1 - \epsilon.
\]
(23)

But what we are getting instead of \((1 - \epsilon)\) is \((1 - \epsilon)h_T(p, \alpha)\) (Lemma 4.1). For Equation (23) to be satisfied we need \(h_T(p, \alpha) = 1\). But this will be possible only for certain noise probability regimes. For example, as we see later, for two support noise model, this is possible if and only if \(p = 0\) or \(p = 1\), i.e., the value of all coalitions in the noise game is either the same or, are scaled by \(\alpha\) for all the coalitions.

Thus, we relax the requirement of \(h_T(p, \alpha) = 1\), and ask for a weaker condition, i.e., \((1 - \epsilon)h_T(p, \alpha) \approx 1 - \epsilon\), or equivalently, \(h_T(p, \alpha) \approx 1\). Therefore, for a user given threshold, \(\zeta\) we want \((1 - \epsilon)h_T(p, \alpha) = \zeta(1 - \epsilon)\).

**Remark 10.** Working with the user given threshold, \(\zeta\) we will get a noise interval \(I^*(T, \zeta)\) on \(p\) for the coalition \(T\), that is the noise regime in which the coalition \(T\) blocks \(\bar{\pi}\) with error less than \(\epsilon\) for the
user given $\zeta$. Moreover, this will hold for all $\tilde{\pi} \in \mathcal{P}$. Thus, we have a noise regime in which the $\varepsilon$-PAC stable outcome does not exist for the noise-free game when it does not exist in the noisy game.

To get some more insights, we will now consider the particular noise model with two support, and $l$ support, and provide the expression of $h_T(p, \alpha)$.

### 4.1 $n$ Agents 2 Support Partial Information Noisy Hedonic Games With No Core

In case of two support model, i.e., $\alpha(S) \in \{1, \alpha\} \forall S \subseteq N$, such that $\alpha > 1$, and $\mathbb{P}(\alpha(S) = \alpha) = p = 1 - \mathbb{P}(\alpha(S) = 1)$, the following lemma provides the expression of $h_T(p, \alpha)$.

**Lemma 4.2.** Consider the $n$ agent noisy hedonic game with partial information. Let the noise support $\alpha(S) \in \{1, \alpha\} \forall S \subseteq N$. Further assume that the noisy game doesn’t have the $\varepsilon$-PAC stable outcome, i.e., $\forall \tilde{\pi} \in \mathcal{P}$, $\exists \varepsilon > 0, T \sim \mathcal{D}$ such that equation (20) is satisfied, then the form of $h_T(p, \alpha)$ is

$$
    h_T(p, \alpha) = \begin{cases} 
    1, & \text{if } \tilde{\pi}(i) = T \forall i \in T \\
    (1 - p) + p |\mathcal{R}(T)| + 1 - |\mathcal{J}(\alpha, T)|, & \text{otherwise},
    \end{cases}
$$

where $\mathcal{J}(\alpha, T)$ is defined as:

$$
    \mathcal{J}(\alpha, T) = \left\{ \tilde{\pi}(i) \in \mathcal{R}(T) \left| \frac{\hat{b}_i(\tilde{\pi}(i))}{\hat{b}_i(T)} \geq \frac{1}{\alpha} \right. \right\}
$$

**Proof.** (Sketch only. The proof of this lemma is deferred to the supplementary material A.4). Firstly we note that $h_T(p, \alpha) \geq 1$ if and only if $p = 0$ or $p = 1$. That is, if the value of all the coalitions are retained, or if value of all of them are inflated by $\alpha$, then the coalition $\tilde{\pi}(i) \geq T$, and $\tilde{\pi}(i) \geq \alpha T$ for all $i \in T$, and for all $\tilde{\pi}(i) \in \mathcal{R}(T)$. Thus, neither the noise free game nor the noisy game will have the $\varepsilon$-PAC stable outcome.

**Remark 11.** If $\tilde{\pi}(i) \neq T$ for at least one $i \in T$, then $h_T(p, \alpha) = 1 \forall \alpha$ if and only if $p = 0$ or $p = 1$. That is, if the value of all the coalitions are retained, or if value of all of them are inflated by $\alpha$, then the coalition $\tilde{\pi}(i) \geq T$, and $\tilde{\pi}(i) \geq \alpha T$ for all $i \in T$, and for all $\tilde{\pi}(i) \in \mathcal{R}(T)$. Thus, neither the noise free game nor the noisy game will have the $\varepsilon$-PAC stable outcome.

**Remark 12.** For the case when we allow for the $h_T(p, \alpha) = \zeta$ for some user given threshold $\zeta$, we will get an interval of the noise in accordance to the Remark 10. In this case, the noise interval will also depend on $|\mathcal{R}(T)|$, and $|\mathcal{J}(\alpha, T)|$ for the coalition $T$ for all $\tilde{\pi} \in \mathcal{P}$.

### 4.2 $n$ Agents $l$ Support Partial Information Noisy Hedonic Games With No Core

Next, we will move to the $l \geq 3$ support case, i.e., $\alpha(S) \in \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \forall S \subseteq N$ such that $\alpha_i < \alpha_j$ if $i < j$, with respective probabilities $p_1, p_2, \ldots, p_l$, such that $\sum_{j \in [l]} p_j = 1$. Here $\alpha_j > 0$ for all $j \in [l]$. For all $r, s$ such that $\alpha_r > \alpha_s$ let us define

$$
    \mathcal{J}(\alpha_r, \alpha_s, T) = \left\{ \tilde{\pi}(i) \in \mathcal{R}(T) \left| \frac{\hat{b}_i(\tilde{\pi}(i))}{\hat{b}_i(T)} \geq \frac{\alpha_s}{\alpha_r} \right. \right\}
$$

Similar to the 2 support case, the set $\mathcal{J}(\alpha_r, \alpha_s, T)$ is the set of all coalition in the set $\mathcal{R}(T)$, such that $\alpha_r > \alpha_s$, and the relative value $\frac{\hat{b}_i(\tilde{\pi}(i))}{\hat{b}_i(T)} \geq \frac{\alpha_s}{\alpha_r}$. Note that, in the 2 support case, there is only one set because $\alpha > 1$. For $l$ support as defined above the following lemma provides the expression of $h_T(p, \alpha)$.

\(^3\)Note that we are using $h_T(p, \alpha)$ instead of $h_T(p, \alpha)$ because for two support the only parameters are $\alpha$, and $p$. 

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Lemma 4.3. For n agent noisy hedonic game with \( \alpha(S) \in \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \) \( \forall S \subseteq N \) such that \( \alpha_i < \alpha_j \forall i < j \) with \( \mathbb{P}(\alpha_j) = p_j \forall j \in [l] \), \( h_T(p, \alpha) \) is given by:

\[
h_T(p, \alpha) = \begin{cases} 
1, & \text{if } \tilde{\pi}(i) = T \forall i \in T \\
\sum_{r,s \in [l]: \alpha_r > \alpha_s} \frac{p_r^{\lvert R(T) \rvert} - \lvert J(\alpha_r, \alpha_s, T) \rvert}{|R(T)|} + \sum_{a=1}^{l} p_a \left( \sum_{b=1}^{l} p_b \right) \frac{1}{|R(T)|}, & \text{otherwise.}
\end{cases}
\]

(27)

**Proof.** Sketch only. The full proof is deferred to supplementary material A.5. The proof uses induction on \( l \) with the two support noise model as the base case proved in Lemma 4.2.

Similar to the two support case, the expression \( h_T(p, \alpha) \) for \( \tilde{\pi}(i) \neq T \) for at-least one \( i \in T \) has two terms, first term captures the probability of all such \((\alpha(\tilde{\pi}(i)), \alpha(T))\) such that \( J(\alpha_r, \alpha_s, T) = \emptyset \forall \alpha_r > \alpha_s \), whereas the second term capture the probability of all such \((\alpha(\tilde{\pi}(i)), \alpha(T))\) such that \( J(\alpha_r, \alpha_s, T) \neq \emptyset \) at-least for one \( \alpha_r > \alpha_s \), and hence the preferences will be preserved depending on the relative values of \( \bar{u}_i(\tilde{\pi}(i)), \bar{u}_i(T) \), and \( \alpha_r, \alpha_s \).

Remark 13. Similar to the two support noise model, if we allow for the \( h_T(p, \alpha) = \zeta \) for some user given threshold \( \zeta \), we will get an interval of the noise in accordance to the Remark 10. In this case, the noise interval will also depend on \( |R(T)| \), and \( |J(\alpha_r, \alpha_s, T)| \) for \( \alpha_r > \alpha_s \) for the coalition \( T \) for all \( \tilde{\pi} \in \mathcal{P} \).

Remark 14. The Lemma 4.1 provides the probability of non-existence of the \( \epsilon \)-PAC stable outcome in the noise-free game, given that the noisy game does not have the \( \epsilon \)-PAC stable outcome. Therefore, the probability that the noise-free game has a \( \epsilon \)-PAC stable outcome, given that the noisy game does not have one, is the complement of the probability in Equation (24).

### 5 RELATED WORK

Hedonic games have been extensively studied by researchers in the computational social choice community. Some of the early works done on hedonic coalition formation games describing the economic situations include that of [Dreze and Greenberg, 1980]. The agents act in collaboration and have personal preferences for belonging to a specific coalition. Based on these preferences, the agents get the partition of the grand coalition. However, which partition to form led to the notion of stability in the coalition formation game. There has been extensive work exploring the stability of various solution concepts for hedonic games by [Aziz and Brandl, 2012, Banerjee et al., 2001, Bogomolnaia and Jackson, 2002]. In particular, [Banerjee et al., 2001] has proposed the core for the simple hedonic games. There are quite a few stability criteria mainly based on the group deviation or the single-agent deviation, whereas in the case of group deviation, notions such as core, strong core, strict core, etc., defined via the notion of blocking a coalition by another coalition for all group members. The solutions based on single-agent deviations consist of perfect, individually stable, Nash stability, etc. Most of these solutions are defined in [Bogomolnaia and Jackson, 2002]. However, in the current work, we are working with core stability.

Apart from the various stability notions, there is much literature for representing the hedonic games. Various representations are needed because as the number of agents increases, the number of coalitions available to each agent increases exponentially. Some of the representations include individually rational lists of coalitions (IRLC) [Ballester, 2004], hedonic coalition nets (HCNs) [Elkind and Wooldridge, 2009], additively Separable games, fractional hedonic games, \( \mathcal{B} \)-games, \( \mathcal{W} \)-games, top-responsive games [Alcalde and Revilla, 2004] etc. A detailed survey of the hedonic games is available in [Aziz and Savani, 2016], and for more details, see [Aziz et al., 2019, Cechlárová and Hajduková, 2004]. However, we are not using any particular representation, and rather our analysis is free of any representation in both full information and partial information cases.
Another line of literature focuses on the algorithmic aspects of the solution concepts of the hedonic games. Majorly, for a given solution concept, such as core stability, and a partition \( \pi \) of the grand coalition, there are three questions to be asked: does \( \pi \) satisfy the solution concept’s condition; does there exists such a partition \( \pi \) satisfying the solution concepts properties; and if there is such a \( \pi \), find one. To this end, for various classes of hedonic games, there are various algorithms, and hardness results such as [Rahwan et al., 2009, Sung and Dimitrov, 2010, Woeginger, 2013].

Recently, there has been increasing interest from the learning theory perspective for finding the solution concepts in the cooperative games and hedonic games. Based on the additive and multiplicative noise structures, authors in [Li and Conitzer, 2015] introduced new solution concepts. They described the notion of least core, multiplicative nucleolus, and partial nucleolus. Uncertainty in the agents’ preferences in the cooperative games has been carefully studied by [Balcan et al., 2015]. Authors used the PAC learning model (for more elucidation, see, [Shalev-Shwartz and Ben-David, 2014]) to learn an underlying game. A new connection was established between PAC learnability and core stability for various classes of TU cooperative games. They mentioned that learning with the general structure of hedonic games is quite challenging. Therefore they took several classes of hedonic games with incomplete information. Finally, an algorithm for finding the stable and learnable partition is proposed. It turned out that only a few classes of hedonic games are learnable as well as stable. Expanding on this work [Balkanski et al., 2017] approximates the Shapley value of a transferable utility (TU) cooperative game, from its samples, under certain conditions on its cost function. They also introduced a new concept of a statistical Shapley value built upon statistical analogs of the standard Shapley. This Shapley value can be easily approximated from samples taken from any distribution and for any cost function. [Sliwinski and Zick, 2017] extended the PAC learning approach to the premise of hedonic coalition formation games, where full information about individual preferences is not available. We build upon this work for the partial information model to provide the noise model such that the stability is preserved.

6 DISCUSSION AND LOOKING AHEAD

In the current work we have considered a new class of learning tasks for noisy hedonic games. In many real life scenarios the true preferences of the agents are not available, and hence any solution concept will not be able to correctly partition the set of agents. We formally proved that this interesting problem persists even with the simplest noise structure in a two agent full information game. In the stylized two agent with ‘retain or inflate’ noise model, we prove that predicting probability of the grand coalition formation in noise free game is quite low, including being zero. We identified the noise interval where the prediction probability of grand coalition in the noise free game is at least a user given threshold \( \zeta \). Moreover, having incomplete/partial information about the preferences over the coalitions creates one more level of uncertainty. Nonetheless, we consider \( n \) agent hedonic games with \( l \geq 3 \) support of noise. Using the \( \epsilon \)-PAC stability notion in partial information setup, we obtain the bounds for the probability of \( \epsilon \)-PAC stable partition in the noise free game in terms of a user given threshold \( \zeta \).

Extending our analysis for the case of continuous distribution of the noise might be helpful in many more real world scenarios. For a given noise model, various stability notions may offer different coalition formations. Then, the prediction probabilities can change accordingly. One can see the impact of the noise model on various stability notions. An interesting model to consider, is when the noisy preferences of various coalitions are collected in a sequential manner, say when agents spread over a geographical region having possibly large noise support sets. Then, it would be interesting to investigate suitable adaption of our schemes for such situations.
REFERENCES

José Alcalde and Pablo Revilla. 2004. Researching with whom? Stability and manipulation. Journal of Mathematical Economics 40, 8 (2004), 869–887.

Haris Aziz and Florian Brandl. 2012. Existence of stability in hedonic coalition formation games. In Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 2. 763–770.

Haris Aziz, Florian Brandl, Felix Brandt, Paul Harrenstein, Martin Olsen, and Dominik Peters. 2019. Fractional hedonic games. ACM Transactions on Economics and Computation (TEAC) 7, 2 (2019), 1–29.

Haris Aziz and Rahul Savani. 2016. Hedonic Games (chapter 15). In Handbook of Computational Social Choice, F. Brandt, V. Conitzer, J. Lang U. Endriss, and A.D. Procaccia (Eds.). Cambridge University Press, Cambridge.

Maria-Florina Balcan, Ariel D Procaccia, and Yair Zick. 2015. Learning cooperative games. In Proceedings of the 24th International Conference on Artificial Intelligence. 475–481.

Eric Balkanski, Umar Syed, and Sergei Vassilvitskii. 2017. Statistical cost sharing. In Advances in Neural Information Processing Systems. 6221–6230.

Coralio Ballester. 2004. NP-completeness in hedonic games. Games and Economic Behavior 49, 1 (2004), 1–30.

Suryapratim Banerjee, Hideo Konishi, and Tayfun Sönmez. 2001. Core in a simple coalition formation game. Social Choice and Welfare 18, 1 (2001), 135–153.

Anna Bogomolnaia and Matthew O Jackson. 2002. The stability of hedonic coalition structures. Games and Economic Behavior 38, 2 (2002), 201–230.

Katarina Cechlárová and Jana Hajduková. 2004. Stability of Partitions Under $B^W$-Preferences and $WB^*$- Preferences. International Journal of Information Technology & Decision Making 3, 04 (2004), 605–618.

Jacques H Dreze and Joseph Greenberg. 1980. Hedonic coalitions: Optimality and stability. Econometrica (pre-1986) 48, 4 (1980), 987.

Edith Elkind and Michael J Wooldridge. 2009. Hedonic coalition nets.. In AAMAS (1). Citeseer, 417–424.

Yuqian Li and Vincent Conitzer. 2015. Cooperative game solution concepts that maximize stability under noise. In Twenty-Ninth AAAI Conference on Artificial Intelligence.

Andreu Mas-Colell, Michael Dennis Whinston, Jerry R Green, et al. 1995. Microeconomic theory. Vol. 1. Oxford university press New York.

Yadati Narahari. 2014. Game theory and mechanism design. Vol. 4. World Scientific.

Bezalel Peleg and Peter Sudhölter. 2007. Introduction to the theory of cooperative games. Vol. 34. Springer Science & Business Media.

Talal Rahwan, Sarvpali D Ramchurn, Nicholas R Jennings, and Andrea Giovannucci. 2009. An anytime algorithm for optimal coalition structure generation. Journal of artificial intelligence research 34 (2009), 521–567.

Shai Shalev-Shwartz and Shai Ben-David. 2014. Understanding machine learning: From theory to algorithms. Cambridge university press.

Jakub Sliwinski and Yair Zick. 2017. Learning Hedonic Games.. In IJCAI. 2730–2736.

Shao-Chin Sung and Dinko Dimitrov. 2010. Computational complexity in additive hedonic games. European Journal of Operational Research 203, 3 (2010), 635–639.

Gerhard J Woeginger. 2013. Core stability in hedonic coalition formation. In International Conference on Current Trends in Theory and Practice of Computer Science. Springer, 33–50.
Learning Noisy Hedonic Games – Supplementary Material

A PROOFS AND OTHER DETAILS

In this section, we will re-iterate the important results and give their proofs.

A.1 Proof of Lemma 2.1

For 2 agent noisy game with ‘retain or inflate’ noise for full information of \( \tilde{\nu} \)'s such that \( \tilde{\nu}_1(12) > \tilde{\nu}_1(1) \) and \( \tilde{\nu}_2(12) > \tilde{\nu}_2(2) \), the conditional probability that the noise free game has grand coalition as core stable outcome, i.e., \( \mathbb{P}[\pi = N | \tilde{\nu}_1(\cdot), \tilde{\nu}_2(\cdot) \text{ as in game 1}] = \eta_{(2,2)}(\pi, \tilde{\nu}, \text{game 1}) \) is given by:

\[
\eta_{(2,2)}(\pi, \tilde{\nu}, \text{game 1}) = \begin{cases} 
  p^3 + p^2(1-p) + 2p(1-p)^2 + (1-p)^3, & \text{if } \alpha \geq \tilde{r} \\
  1, & \text{if } \alpha < \tilde{r}
\end{cases}
\]

where \( \tilde{r} = \max \left\{ \frac{\tilde{\nu}_1(12)}{\tilde{\nu}_1(1)}, \frac{\tilde{\nu}_2(12)}{\tilde{\nu}_2(2)} \right\} \), and \( \tilde{r} = \min \left\{ \frac{\tilde{\nu}_1(12)}{\tilde{\nu}_1(1)}, \frac{\tilde{\nu}_2(12)}{\tilde{\nu}_2(2)} \right\} \).

**Proof.** Recall that the noisy game is given as:

\[
\begin{align*}
\tilde{\nu}_1(12) &> \tilde{\nu}_1(1) \\
\tilde{\nu}_2(12) &> \tilde{\nu}_2(2).
\end{align*}
\]

(game 1)

For this game \( \tilde{\pi} = N \). Now, consider the noise values \( \alpha(S) \in \{1, \alpha\} \), where \( \alpha > 1 \) such that \( \mathbb{P}(\alpha(S) = \alpha) = p, \) and \( \mathbb{P}(\alpha(S) = 1) = 1 - p, \) for some fixed and unknown \( p. \) Given noisy game 1 there are 8 possible combinations of \( \alpha \)'s (because each coalition has two option ‘retain or inflate’).

We will now enumerate all such possibilities:

1. \( \alpha(1) = 1; \alpha(2) = 1; \alpha(12) = 1 \). Probability of such alpha’s is \( (1-p)^3 \). Thus the noise free values are \( \nu_1(1) = \tilde{\nu}_1(1); \nu_2(2) = \tilde{\nu}_2(2), \nu_1(12) = \tilde{\nu}_1(12) \text{ and } \nu_2(12) = \tilde{\nu}_2(12). \) Therefore, the noise free game is

\[
\begin{align*}
\nu_1(12) &> \nu_1(1) \\
\nu_2(12) &> \nu_2(2).
\end{align*}
\]

2. \( \alpha(1) = 1; \alpha(2) = 1; \alpha(12) = \alpha \). Probability of such alpha’s is \( p(1-p)^2 \). Thus the actual values are \( \nu_1(1) = \tilde{\nu}_1(1); \nu_2(2) = \tilde{\nu}_2(2), \nu_1(12) = \frac{\tilde{\nu}_1(12)}{\alpha} \text{ and } \nu_2(12) = \frac{\tilde{\nu}_2(12)}{\alpha}. \) Therefore the actual preferences will depend on the relative values of \( \alpha \) and \( \tilde{\nu} \). If \( \alpha \) and \( \tilde{\nu} \)'s are such that \( \frac{\tilde{\nu}_1(12)}{\alpha} > \tilde{\nu}_1(1) \) and \( \frac{\tilde{\nu}_2(12)}{\alpha} > \tilde{\nu}_2(2), \) then \( \pi = N, \) otherwise \( \pi = \{\{1\}, \{2\}\}. \)

3. \( \alpha(1) = 1; \alpha(2) = \alpha; \alpha(12) = 1 \). Probability of such alpha’s is \( p(1-p)^2 \). Thus the actual values are \( \nu_1(1) = \tilde{\nu}_1(1); \nu_2(2) = \frac{\tilde{\nu}_2(2)}{\alpha}, \nu_1(12) = \tilde{\nu}_1(12) \text{ and } \nu_2(12) = \tilde{\nu}_2(12). \) Since, \( \tilde{\nu}_2(12) > \tilde{\nu}_2(2) > \frac{\tilde{\nu}_2(12)}{\alpha}. \) The noise free game is

\[
\begin{align*}
\nu_1(12) &> \nu_1(1) \\
\nu_2(12) &> \nu_2(2).
\end{align*}
\]

4. \( \alpha(1) = \alpha; \alpha(2) = 1; \alpha(12) = 1 \). Probability of such alpha’s is \( p(1-p)^2 \). Thus the actual values are \( \nu_1(1) = \frac{\tilde{\nu}_1(1)}{\alpha}; \nu_2(2) = \tilde{\nu}_2(2), \nu_1(12) = \tilde{\nu}_1(12) \text{ and } \nu_2(12) = \tilde{\nu}_2(12). \) Since, \( \tilde{\nu}_1(12) > \tilde{\nu}_1(1) > \frac{\tilde{\nu}_1(12)}{\alpha}. \) Therefore the noise free game is

\[
\begin{align*}
\nu_1(12) &> \nu_1(1) \\
\nu_2(12) &> \nu_2(2).
\end{align*}
\]
From this game it is clear that \( \pi = N \).

(5) \( \alpha(1) = 1; \alpha(2) = \alpha; \alpha(12) = \alpha. \) Probability of these alpha’s is \( p^2(1 - p) \). Thus, actual values are \( v_1(1) = \tilde{v}_1(1); v_2(2) = \frac{\tilde{v}_2(2)}{\alpha}, v_1(12) = \frac{\tilde{v}_1(12)}{\alpha} \) and \( v_2(12) = \frac{\tilde{v}_2(12)}{\alpha} \). The actual preferences will depend on the relative values of \( \alpha \) and \( \tilde{\alpha} \). If \( \alpha \) and \( \tilde{\alpha} \)’s are such that \( \frac{\tilde{v}_1(12)}{\alpha} > \tilde{v}_1(1) \), then \( \pi = N \), otherwise \( \pi = \{1, 2\} \).

(6) \( \alpha(1) = \alpha; \alpha(2) = 1; \alpha(12) = \alpha. \) Probability of such alpha’s is \( p^2(1 - p) \). Thus the actual values are \( v_1(1) = \frac{\tilde{v}_1(1)}{\alpha}, v_2(2) = \tilde{v}_2(2); v_1(12) = \frac{\tilde{v}_1(12)}{\alpha} \) and \( v_2(12) = \frac{\tilde{v}_2(12)}{\alpha} \). The actual preferences will depend on the relative values of \( \alpha \) and \( \tilde{\alpha} \). If \( \alpha \) and \( \tilde{\alpha} \)’s are such that \( \frac{\tilde{v}_1(12)}{\alpha} > \tilde{v}_2(2) \), then \( \pi = N \) otherwise \( \pi = \{1, 2\} \).

(7) \( \alpha(1) = \alpha; \alpha(2) = \alpha; \alpha(12) = 1. \) Probability of such alpha’s is \( p^2(1 - p) \). Thus, the actual values are \( v_1(1) = \frac{\tilde{v}_1(1)}{\alpha}, v_2(2) = \frac{\tilde{v}_2(2)}{\alpha}; v_1(12) = \frac{\tilde{v}_1(12)}{\alpha} \) and \( v_2(12) = \frac{\tilde{v}_2(12)}{\alpha} \). Therefore, the noise free game is

\[
\begin{align*}
v_1(12) &> v_1(1) \\
v_2(12) &> v_2(2).
\end{align*}
\]

From this game it is clear that \( \pi = N \).

(8) \( \alpha(1) = \alpha; \alpha(2) = \alpha; \alpha(12) = \alpha. \) Probability of such alpha’s is \( p^3. \) Thus the actual values are \( v_1(1) = \frac{\tilde{v}_1(1)}{\alpha}, v_2(2) = \frac{\tilde{v}_2(2)}{\alpha}; v_1(12) = \frac{\tilde{v}_1(12)}{\alpha} \) and \( v_2(12) = \frac{\tilde{v}_2(12)}{\alpha} \). Therefore, the noise free game is

\[
\begin{align*}
v_1(12) &> v_1(1) \\
v_2(12) &> v_2(2).
\end{align*}
\]

From this game it is clear that \( \pi = N \).

Since \( r = \max \left\{ \frac{\tilde{v}_1(12)}{\tilde{v}_1(1)}, \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)} \right\} \), and \( \tilde{r} = \min \left\{ \frac{\tilde{v}_1(12)}{\tilde{v}_1(1)}, \frac{\tilde{v}_2(12)}{\tilde{v}_2(2)} \right\} \). From above cases, we can see that there are 5 cases (Case 1,3,4,7,8) in which the grand coalition \( \pi = N \) is formed in noise-free game. In these conditions, the relative value of \( \tilde{v}_1(\cdot), \tilde{v}_2(\cdot) \) should satisfy \( \alpha \geq \tilde{r} \), and this constitute the first expression, \( p^3 + p^2(1 - p) + 2p(1 - p)^2 + (1 - p)^3 \) of \( \eta(\alpha \tilde{\alpha}, game 1) \). Apart from this, if case (6) above is also allowed, then \( p^2(1 - p) \) will be added to the above probability and \( \alpha \) can also be in the interval \( r \leq \alpha < \tilde{r} \), i.e., possible \( \alpha \)’s are now \( \alpha \geq r \), and hence we have \( \eta(\alpha \tilde{\alpha}, game 1) \) corresponding to it. And finally, if \( \alpha < r \), then for all cases the grand coalition will form. Thus, 

\[
\eta(\alpha \tilde{\alpha}, game 1) = \begin{cases} 
p^3 + p^2(1 - p) + 2p(1 - p)^2 + (1 - p)^3, & \text{if } \alpha \geq \tilde{r} \\
p^3 + 2p^2(1 - p) + 2p(1 - p)^2 + (1 - p)^3, & \text{if } \alpha \geq r \\
1, & \text{if } \alpha < r \end{cases}
\]

(29) This ends the proof. \( \Box \)

A.2 Proof of Lemma 3.2

For \( n \) agent noisy hedonic games with \( \alpha(S) \in \{1, \alpha\} \forall S \subseteq N \) as support of noise and \( \tilde{\pi} \) as the \( \varepsilon \)-PAC stable outcome of the noisy game, \( f_r(p, \alpha) \) is given by:

\[
f_r(p, \alpha) = \begin{cases} 
1, & \text{if } \tilde{\pi}(i) = T \forall i \in T \\
p + (1 - p)^{|R(T)| + 1 - |I(\alpha, T)|}, & \text{otherwise},
\end{cases}
\]

(30) where \( I(\alpha, T) \) are defined as:

\[
I(\alpha, T) = \left\{ \tilde{\pi}(i) \in R(T) \mid \frac{\tilde{v}_1(\tilde{\pi}(i))}{\tilde{v}_1(T)} \geq \alpha \right\}.
\] (31)
Recall from Lemma 3.1 we have the following
\[ P_{T- \emptyset}^C \left[ \bigcup_{i \in T} v_i(\pi(i)) \geq v_i(T) \right] \geq (1 - \epsilon) P_{T- \emptyset}^C \left[ \bigcap_{i \in T} \left( (v_i(\pi(i)) \geq v_i(T)) \cap (\tilde{v}_i(\pi(i)) \geq \tilde{v}_i(T)) \right) \right] \\
= (1 - \epsilon) f_T(p, \alpha) \tag{32} \]

For two support we will use \( f_T(p, \alpha) \) instead of \( f_T(p, \alpha) \).

Let us consider the following event:
\[ E(T, \pi) = \{ \alpha(T), \alpha(\pi(i)) : \forall \pi(i) \in R(T), \bigcap_{i \in T} \{ v_i(\pi(i)) \geq v_i(T) \cap \alpha(\pi(i)) \cap v_i(\pi(i)) \geq \alpha(T)v_i(T) \} \} \]

Therefore, \( f_T(p, \alpha) = \mathbb{P}[E(T, \pi)] \). Moreover, we have
\[ R(T) = \{ \pi(i) | i \in T \} \tag{33} \]
\[ I(\alpha, T) = \left\{ \pi(i) \in R(T) \left| \tilde{v}_i(\pi(i)) \geq \alpha \right. \right\} \tag{34} \]

The event \( E(T, \pi) \) will be true only if \( \alpha(\pi(i)) \cap \pi(i) \in R(T) \), and \( \alpha(T) \) will satisfy one of the following in two different cases \( I(\alpha, T) = \emptyset \), and \( I(\alpha, T) \neq \emptyset \) as given below:

- \( I(\alpha, T) = \emptyset \)
  - \( \alpha(\pi(i)) = 1 \forall \pi(i) \in R(T) \), and \( \alpha(T) = 1 \). Probability of such choice of \( \alpha 's \) is \( (1 - p)^{|R(T)|+1} \).
  - \( \alpha(\pi(i)) = 1 \forall \pi(i) \in R(T) \), and \( \alpha(T) = \alpha \). Probability of such choice of \( \alpha 's \) is \( p(1 - p)^{|R(T)|} \).
  - \( \alpha(\pi(i)) = \alpha \) for exactly one \( \pi(i) \in R(T) \), and \( \alpha(\pi(i)) = 1 \) for rest of the coalitions in \( R(T) \), and \( \alpha(T) = \alpha \). Probability of such choice of \( \alpha 's \) is \( p^2(1 - p)^{|R(T)|-1} \). And there are \( \frac{|R(T)|}{1} \) such possibilities for exactly one coalition \( \pi(i) \). Thus the overall probability is \( \left( \frac{|R(T)|}{1} \right) p^2(1 - p)^{|R(T)|-1} \).
  - \( \alpha(\pi(i)) = \alpha \) for exactly two \( \pi(i) \in R(T) \), and \( \alpha(\pi(i)) = 1 \) for rest of the coalitions in \( R(T) \), and \( \alpha(T) = \alpha \). Probability of such choice of \( \alpha 's \) is \( p^3(1 - p)^{|R(T)|-2} \). And there are \( \frac{|R(T)|}{2} \) such possibilities for exactly two coalitions \( \pi(i) \). Thus the overall probability is \( \left( \frac{|R(T)|}{2} \right) p^3(1 - p)^{|R(T)|-2} \).

- \( I(\alpha, T) \neq \emptyset \). Then in addition to above possible cases, we will have few other cases which are:
  - \( \alpha(\pi(i)) = \alpha \) for exactly one \( \pi(i) \in I(\alpha, T) \), \( \alpha(\pi(i)) = 1 \) for rest of the coalitions in \( R(T) \) and \( \alpha(T) = 1 \). Probability of such choice of \( \alpha 's \) is \( p(1 - p)^{|R(T)|-1}(1 - p) = p(1 - p)^{|R(T)|} \). And there are \( \left( \frac{|I(\alpha, T)|}{1} \right) p^{|R(T)|} \) such possibilities for exactly one coalition \( \pi(i) \in I(\alpha, T) \). Thus the overall probability is \( \left( \frac{|I(\alpha, T)|}{1} \right) p(1 - p)^{|R(T)|} \).
  - \( \alpha(\pi(i)) = \alpha \) for exactly two \( \pi(i) \in I(\alpha, T) \), \( \alpha(\pi(i)) = 1 \) for rest of the coalitions in \( R(T) \) and \( \alpha(T) = 1 \). Probability of such choice of \( \alpha 's \) is \( p^2(1 - p)^{|R(T)|-2}(1 - p) = p^2(1 - p)^{|R(T)|-2} \). And there are \( \left( \frac{|I(\alpha, T)|}{2} \right) \) such possibilities for exactly two coalitions \( \pi(i) \in I(\alpha, T) \). Thus the overall probability is \( \left( \frac{|I(\alpha, T)|}{2} \right) p^2(1 - p)^{|R(T)|-2} \).
\[\alpha(\pi(i)) = \alpha \text{ for all but one } \pi(i) \in \mathcal{I}(\alpha, T), \alpha(\pi(i)) = 1 \text{ for rest of the coalitions in } \mathcal{R}(T) \text{ and } \alpha(T) = 1. \] Probability of such choice of \( \alpha' \)'s is \( p^{I(\alpha(T)) - 1}(1 - p)^{|\mathcal{R}(T)| - |I(\alpha(T))| + 1}(1 - p) = p^{I(\alpha(T)) - 1}(1 - p)^{|\mathcal{R}(T)| - |I(\alpha(T))| + 2}. \) And there are \( \binom{|I(\alpha(T))|}{I(\alpha(T)) - 1} \) such possibilities for \textbf{all but one} coalition \( \pi(i) \in \mathcal{I}(\alpha, T). \) Thus the overall probability is \( \binom{|I(\alpha(T))|}{I(\alpha(T)) - 1} p^{I(\alpha(T)) - 1}(1 - p)^{|\mathcal{R}(T)| - |I(\alpha(T))| + 2}. \)

\[\alpha(\pi(i)) = \alpha \text{ for all } \pi(i) \in \mathcal{I}(\alpha, T), \alpha(\pi(i)) = 1 \text{ for rest of the coalitions in } \mathcal{R}(T) \text{ and } \alpha(T) = 1. \] Probability of such choice of \( \alpha' \)'s is \( p^{I(\alpha(T)) - 1}(1 - p)^{|\mathcal{R}(T)| - |I(\alpha(T))| + 1}(1 - p) = p^{I(\alpha(T)) - 1}(1 - p)^{|\mathcal{R}(T)| - |I(\alpha(T))| + 1}. \) And there are \( \binom{|I(\alpha(T))|}{I(\alpha(T))} \) such possibilities for all coalitions \( \pi(i) \in \mathcal{I}(\alpha, T). \) Thus the overall probability is \( \binom{|I(\alpha(T))|}{I(\alpha(T))} p^{I(\alpha(T)) - 1}(1 - p)^{|\mathcal{R}(T)| - |I(\alpha(T))| + 1}. \)

Therefore, the probability of the event \( E(T, \pi), \) i.e., \( \mathbb{P}[E(T, \pi)] \) is given by:

\[
\mathbb{P}[E(T, \pi)] = (1 - p)^{|\mathcal{R}(T)| + 1} + p(1 - p)^{|\mathcal{R}(T)}| + \binom{|\mathcal{R}(T)|}{1} p^2(1 - p)^{|\mathcal{R}(T)| - 1} + \binom{|\mathcal{R}(T)|}{2} p^3(1 - p)^{|\mathcal{R}(T)| - 2} + \ldots + \binom{|\mathcal{R}(T)|}{|\mathcal{R}(T)| - 1} p^{|\mathcal{R}(T)| - 1}(1 - p) + \binom{|\mathcal{R}(T)|}{|\mathcal{R}(T)|} p^{|\mathcal{R}(T)| + 1} + \binom{|I(\alpha(T))|}{1} (1 - p)^{|I(\alpha(T))| - 1} + \binom{|I(\alpha(T))|}{2} p^2(1 - p)^{|I(\alpha(T))| - 1} + \ldots + \binom{|I(\alpha(T))|}{|I(\alpha(T))| - 1} p^{|I(\alpha(T))| - 1}(1 - p) + \binom{|I(\alpha(T))|}{|I(\alpha(T))|} p^{|I(\alpha(T))| + 1} + \binom{|I(\alpha(T))|}{|I(\alpha(T))| + 1} (1 - p)^{|I(\alpha(T))| - 1} + \binom{|I(\alpha(T))|}{|I(\alpha(T))| + 2} p^2(1 - p)^{|I(\alpha(T))| - 2} + \ldots + \binom{|I(\alpha(T))|}{|I(\alpha(T))| - 1} p^{|I(\alpha(T))| - 1}(1 - p) + \binom{|I(\alpha(T))|}{|I(\alpha(T))|} p^{|I(\alpha(T))| + 1}
\]
This ends the proof. □

A.3 Proof of Lemma 3.3

For an agent noisy hedonic game with \( \alpha(S) = \{\alpha_1, \alpha_2, \ldots, \alpha_t\} \ \forall S \subseteq N \) such that \( \alpha_i < \alpha_j \ \forall i < j \) with \( \mathbb{P}(\alpha_j) = p_j \ \forall j \in [L] \), and \( \bar{\pi} \) as the \( \epsilon \)-PAC stable outcome of the noisy game, \( f_T(p, \alpha) \) is given by:

\[
  f_T(p, \alpha) = \begin{cases} 
  1, & \text{if } \bar{\pi}(i) = T \ \forall i \in T \\
  \sum_{r,s \in [L]} a_r a_s \sum_{a=1}^L p_a^{2a} |R(T)|^{-|I(\alpha_r, \alpha_s, T)|} + \sum_{r,s \in [L]} a_r a_s \sum_{a=1}^L p_a^{2a} |R(T)|^{-|I(\alpha_r, \alpha_s, T)|} \end{cases}
\]

\[
= (1 - p) |R(T)|^{-|I(\alpha_T)|} + p + (1 - p) |R(T)|^{-|I(\alpha_T)|} [1 - (1 - p) |I(\alpha_T)|]
\]

\[
= p + (1 - p) |R(T)|^{-|I(\alpha_T)|} + 1.
\]

PROOF. We will prove this via induction on \( l \). Clearly, this is true for \( l = 2 \) (from Lemma 3.2 above). Let us assume that it is true for \( l = k \), i.e., there are sets

\[ I(\alpha_r, \alpha_s, T) = \left\{ \bar{\pi}(i) \in \mathcal{R}(T) \mid \bar{\delta}_{l}(\bar{\pi}(i)) \geq \frac{\alpha_r}{\alpha_s} \right\}, \]

such that the support \( \alpha(S) = \{\alpha_1, \ldots, \alpha_k\} \) for all \( \alpha_s < \alpha_r, 1 \leq s < r \leq k \). Let us consider the following event

\[ E(T, \bar{\pi}) = \{ \alpha(T), \alpha(\bar{\pi}(i)) : \forall \bar{\pi}(i) \in \mathcal{R}(T), \cap_{i \in T} \{ v_i(\bar{\pi}(i)) \geq v_i(T) \cap \alpha(\bar{\pi}(i))v_i(\bar{\pi}(i)) \geq \alpha(T)v_i(T) \} \} \]

This event captures the set of all \( \alpha(\bar{\pi}(i)), \alpha(T) \ \forall \bar{\pi}(i) \in \mathcal{R}(T) \) such that \( \bar{\pi}(i) \geq T \) and \( \bar{\pi}(i) \geq T \) for all \( i \in T \). For this \( k \) we have \( f_T(p, \alpha; j \in [k]) =: f_T(p, \alpha) \) (by assumption)

\[
  f_T(p, \alpha) = \sum_{a=1}^k p_a \left( \sum_{b=1}^a p_b \right) |R(T)|^{-|I(\alpha_r, \alpha_s, T)|} + \sum_{r,s \in [k]} a_r a_s \sum_{a=1}^L p_a^{2a} |R(T)|^{-|I(\alpha_r, \alpha_s, T)|} \frac{\bar{\delta}_{l}(\bar{\pi}(i))}{\bar{\delta}_{l}(\bar{\pi}(i))} \alpha_{k+1} \alpha_s \]

\[
= \sum_{a=1}^k p_a \left( \sum_{b=1}^a p_b \right) |R(T)|^{-|I(\alpha_r, \alpha_s, T)|} + \sum_{r,s \in [k]} a_r a_s \sum_{a=1}^L p_a^{2a} |R(T)|^{-|I(\alpha_r, \alpha_s, T)|} \frac{\bar{\delta}_{l}(\bar{\pi}(i))}{\bar{\delta}_{l}(\bar{\pi}(i))} \alpha_{k+1} \alpha_s \]

\[
= (1 - p) |R(T)|^{-|I(\alpha_r, \alpha_s, T)|} + p + (1 - p) |R(T)|^{-|I(\alpha_r, \alpha_s, T)|} \frac{\bar{\delta}_{l}(\bar{\pi}(i))}{\bar{\delta}_{l}(\bar{\pi}(i))} \alpha_{k+1} \alpha_s \]

\[
= p + (1 - p) |R(T)|^{-|I(\alpha_r, \alpha_s, T)|} + 1.
\]

We will now show that this is true for \( l = k + 1 \). To this end define for all \( s \in [k] \) such that \( \alpha_{k+1} > \alpha_s \) the following sets:

\[ I(\alpha_{k+1}, \alpha_s, T) = \left\{ \bar{\pi}(i) \in \mathcal{R}(T) \mid \bar{\delta}_{l}(\bar{\pi}(i)) \geq \frac{\alpha_{k+1}}{\alpha_s} \right\}, \]

Now, there are two cases, \( I(\alpha_{k+1}, \alpha_s, T) = \emptyset \ \forall \alpha_s, s \in [k] \), or \( I(\alpha_{k+1}, \alpha_s, T) \neq \emptyset \) at least for one \( s \in [k] \).

**Case 01:** (\( I(\alpha_{k+1}, \alpha_s, T) = \emptyset \)). Clearly with one more element in the support, apart from the existing pairs of \( \alpha(\bar{\pi}(i)), \alpha(T) \) for \( k \) support case, it will also have \( \alpha(T) = \alpha_{k+1} \), and \( \alpha(\bar{\pi}(i)) \in \{\alpha_1, \alpha_2, \ldots, \alpha_{k+1}\} \ \forall \bar{\pi}(i) \in \mathcal{R}(T) \). The probability of such alpha’s is \( p_{k+1} \sum_{b=1}^{a-1} p_b \). Therefore, the overall probability is \( \sum_{a=1}^k p_a \left( \sum_{b=1}^a p_b \right) |R(T)|^{-|I(\alpha_{k+1}, \alpha_s, T)|} \sum_{a=1}^{k+1} p_a \left( \sum_{b=1}^{a-1} p_b \right) |R(T)|^{-|I(\alpha_{k+1}, \alpha_s, T)|} \frac{\bar{\delta}_{l}(\bar{\pi}(i))}{\bar{\delta}_{l}(\bar{\pi}(i))} \alpha_{k+1} \alpha_s \]

**Case 02:** (\( I(\alpha_{k+1}, \alpha_s, T) \neq \emptyset \)). In this case, apart from all such pairs of \( \alpha(\bar{\pi}(i)), \alpha(T) \) for all \( \bar{\pi}(i) \in I(\alpha_s, \alpha_s, T) \), we have pairs of \( \alpha(\bar{\pi}(i)), \alpha(T) \) for all \( \bar{\pi}(i) \in I(\alpha_{k+1}, \alpha_s, T) \). For this set, the possible pairs of \( \alpha(\bar{\pi}(i)), \alpha(T) \) are such \( \alpha(\bar{\pi}(i)) = \alpha_s \ \forall \bar{\pi}(i) \in \mathcal{R}(T) \setminus I(\alpha_{k+1}, \alpha_s, T) \), and \( \alpha(T) = \alpha_{k+1} \). Thus their combined probability will be \( p_s |R(T)|^{-|I(\alpha_{k+1}, \alpha_s, T)|} \frac{\bar{\delta}_{l}(\bar{\pi}(i))}{\bar{\delta}_{l}(\bar{\pi}(i))} \alpha_{k+1} \alpha_s \). Hence for \( k + 1 \)
support the probability is:
\[
\sum_{r,s \in [k] : \alpha_r > \alpha_s} p_s |R(T)\!-\!| J(\alpha_r, \alpha_s, T)\!+\!1| \left( (p_r + p_s) |J(\alpha_r, \alpha_s, T)| - p_s |J(\alpha_r, \alpha_s, T)| \right)
+ p_s |R(T)\!-\!| J(\alpha_{k+1}, \alpha_s, T)\!+\!1| \left( (p_{k+1} + p_s) |J(\alpha_{k+1}, \alpha_s, T)| - p_s |J(\alpha_{k+1}, \alpha_s, T)| \right).
\]
(39)

From case 01, and case 02 for \( k + 1 \) support we have
\[
f_T(p_j, \alpha; j \in [k + 1]) = \sum_{a=1}^{k+1} \left( \sum_{b=1}^{a} p_a \right) |R(T)|
+ \sum_{r,s \in [k+1] : \alpha_r > \alpha_s} p_s |R(T)\!-\!| J(\alpha_r, \alpha_s, T)\!+\!1| \left( (p_r + p_s) |J(\alpha_r, \alpha_s, T)| - p_s |J(\alpha_r, \alpha_s, T)| \right).
\]

And hence it is true for \( k + 1 \) support. Thus, from principle of mathematical induction this is true for any \( l \in \mathbb{N} \).

\( \square \)

A.4 Proof of Lemma 4.2

Consider the \( n \) agent noisy hedonic game with partial information. Let the noise support \( \alpha(S) \in \{1, \alpha\} \forall S \subseteq N \). Further assume that the noisy game doesn’t have the \( \varepsilon \)-PAC stable outcome, then the general form of \( h_T(p, \alpha) \) is given by:

\[
h_T(p, \alpha) = \begin{cases} 1, & \text{if } \tilde{\pi}(i) = T \forall i \in T \\ (1 - p) + p |R(T)| + 1 - |J(\alpha, T)|, & \text{otherwise}, \end{cases}
\]
(40)

where \( J(\alpha, T) \) is defined as:

\[
J(\alpha, T) = \left\{ \tilde{\pi}(i) \in R(T) \left| \frac{\tilde{v}_i(\tilde{\pi}(i))}{\tilde{v}_i(T)} \geq \frac{1}{\alpha} \right. \right\}
\]
(41)

Proof. Recall from Lemma 4.1 we have the following
\[
\mathbb{P}_{T \sim D}[\bigcup_{i \in T} v_i(\tilde{\pi}(i)) \geq v_i(T) ] \geq (1 - \varepsilon) \mathbb{P}_{T \sim D}[\bigcap_{i \in T} v_i(T) > v_i(\tilde{\pi}(i)) \cap \tilde{v}_i(T) > \tilde{\pi}(i))]
\]
(42)

For two support noise model we will use \( h_T(p, \alpha) \) instead of \( h_T(p, \alpha) \). Let us consider the following event:

\[
F(T, \tilde{\pi}) = \{ \alpha(T), \alpha(\tilde{\pi}(i)) : \forall \tilde{\pi}(i) \in R(T), \cap_{i \in T} \{ v_i(T) > v_i(\tilde{\pi}(i)) \cap \alpha(T)v_i(T) > \alpha(\tilde{\pi}(i))v_i(\tilde{\pi}(i)) \} \}
\]
(43)

Therefore, \( h_T(p, \alpha) = \mathbb{P}[F(T, \tilde{\pi})] \). Moreover, we have

\[
J(\alpha, T) = \left\{ \tilde{\pi}(i) \in R(T) \left| \frac{\tilde{v}_i(\tilde{\pi}(i))}{\tilde{v}_i(T)} \geq \frac{1}{\alpha} \right. \right\}
\]
(44)

The event \( F(T, \tilde{\pi}) \) will be true only if \( \alpha(\tilde{\pi}(i)) \forall \tilde{\pi}(i) \in R(T) \), and \( \alpha(T) \) will satisfy one of the following in two different cases (\( J(\alpha, T) = \emptyset \), and \( J(\alpha, T) \neq \emptyset \)) as given below:

- \( J(\alpha, T) = \emptyset \)
  - \( \alpha(\tilde{\pi}(i)) = \alpha \forall \tilde{\pi}(i) \in R(T) \), and \( \alpha(T) = \alpha \). Probability of such choice of \( \alpha \)'s is \( p^{|R(T)|+1} \).
  - \( \alpha(\tilde{\pi}(i)) = \alpha \forall \tilde{\pi}(i) \in R(T) \), and \( \alpha(T) = 1 \). Probability of such choice of \( \alpha \)'s is \( (1 - p)p^{|R(T)|} \).
  - \( \alpha(\tilde{\pi}(i)) = 1 \) for exactly one \( \tilde{\pi}(i) \in R(T) \), and \( \alpha(\tilde{\pi}(i)) = \alpha \) for rest of the coalitions in \( R(T) \), and \( \alpha(T) = 1 \). Probability of such choice of \( \alpha \)'s is \( (1 - p)^2 p^{|R(T)|-1} \). And there are \( \binom{|R(T)|}{1} \) such possibilities for exactly one coalition \( \tilde{\pi}(i) \). Thus the overall probability is \( \binom{|R(T)|}{1}(1 - p)^2 p^{|R(T)|-1} \).
\( \alpha(\tilde{\pi}(i)) = 1 \) for exactly two \( \tilde{\pi}(i) \in \mathcal{R}(T) \), and \( \alpha(\tilde{\pi}(i)) = \alpha \) for rest of the coalitions in \( \mathcal{R}(T) \), and \( \alpha(T) = 1 \). Probability of such choice of \( \alpha \)'s is \((1-p)^3p^{\mathcal{R}(T)\setminus 1}\). And there are \( \binom{2}{2}(1-p)^3p^{\mathcal{R}(T)\setminus 2} \) such possibilities for exactly two coalition \( \tilde{\pi}(i) \). Thus the overall probability is 
\[
\sum_{\mathcal{R}(T)\setminus 1} \binom{2}{2}(1-p)^3p^{\mathcal{R}(T)\setminus 2} = \frac{p^{\mathcal{R}(T)\setminus 1}}{1} \frac{p^{\mathcal{R}(T)\setminus 2}}{2}.
\]

\[\vdots\]

\( \alpha(\tilde{\pi}(i)) = 1 \) for all but one \( \tilde{\pi}(i) \in \mathcal{R}(T) \), and \( \alpha(\tilde{\pi}(i)) = \alpha \) for one of the coalitions in \( \mathcal{R}(T) \), and \( \alpha(T) = 1 \). Probability of such choice of \( \alpha \)'s is \((1-p)^\mathcal{R}(T)p \). And there are \( \binom{\mathcal{R}(T)\setminus 1}{1}(1-p)^\mathcal{R}(T)p \) such possibilities for all but one coalition \( \tilde{\pi}(i) \). Thus the overall probability is 
\[
\binom{\mathcal{R}(T)\setminus 1}{1}(1-p)^\mathcal{R}(T)p = \frac{p^{\mathcal{R}(T)\setminus 1}}{1} \frac{p^{\mathcal{R}(T)\setminus 2}}{2}.
\]

\( \alpha(\tilde{\pi}(i)) = 1 \) for all \( \tilde{\pi}(i) \in \mathcal{R}(T) \), and \( \alpha(T) = 1 \). Probability of such choice of \( \alpha \)'s is \((1-p)^\mathcal{R}(T)p \). And there are \( \binom{\mathcal{R}(T)\setminus 1}{1}(1-p)^\mathcal{R}(T)p \) such possibilities for all coalitions \( \tilde{\pi}(i) \). Thus the overall probability is 
\[
\binom{\mathcal{R}(T)\setminus 1}{1}(1-p)^\mathcal{R}(T)p = \frac{p^{\mathcal{R}(T)\setminus 1}}{1} \frac{p^{\mathcal{R}(T)\setminus 2}}{2}.
\]

\( \mathcal{F}(a, T) \neq \emptyset \). Then in addition to above possible cases, we will have few other cases which are:

\( \alpha(\tilde{\pi}(i)) = 1 \) for exactly one \( \tilde{\pi}(i) \in \mathcal{F}(a, T) \), \( \alpha(\tilde{\pi}(i)) = \alpha \) for rest of the coalitions in \( \mathcal{R}(T) \) and \( \alpha(T) = \alpha \). Probability of such choice of \( \alpha \)'s is \((1-p)p^\mathcal{F}(a,T)\setminus 1p = (1-p)p^\mathcal{F}(a,T)\setminus 2 \). And there are \( \binom{\mathcal{F}(a,T)\setminus 1}{1}(1-p)p^\mathcal{F}(a,T)\setminus 2 \) such possibilities for exactly one coalition \( \tilde{\pi}(i) \in \mathcal{F}(a, T) \). Thus the overall probability is 
\[
\binom{\mathcal{F}(a,T)\setminus 1}{1}(1-p)p^\mathcal{F}(a,T)\setminus 2 = \frac{p^{\mathcal{R}(T)\setminus 1}}{1} \frac{p^{\mathcal{R}(T)\setminus 2}}{2}.
\]

\( \alpha(\tilde{\pi}(i)) = 1 \) for exactly two \( \tilde{\pi}(i) \in \mathcal{F}(a, T) \), \( \alpha(\tilde{\pi}(i)) = \alpha \) for rest of the coalitions in \( \mathcal{R}(T) \) and \( \alpha(T) = \alpha \). Probability of such choice of \( \alpha \)'s is \((1-p)^2p^\mathcal{F}(a,T)\setminus 2 \). And there are \( \binom{\mathcal{F}(a,T)\setminus 1}{2}(1-p)^2p^\mathcal{F}(a,T)\setminus 2 \) such possibilities for exactly two coalitions \( \tilde{\pi}(i) \in \mathcal{F}(a, T) \). Thus the overall probability is 
\[
\binom{\mathcal{F}(a,T)\setminus 1}{2}(1-p)^2p^\mathcal{F}(a,T)\setminus 2 = \frac{p^{\mathcal{R}(T)\setminus 1}}{1} \frac{p^{\mathcal{R}(T)\setminus 2}}{2}.
\]

\( \mathcal{F}(a, T) \neq \emptyset \). Then in addition to above possible cases, we will have few other cases which are:

\( \alpha(\tilde{\pi}(i)) = 1 \) for exactly one \( \tilde{\pi}(i) \in \mathcal{F}(a, T) \), \( \alpha(\tilde{\pi}(i)) = \alpha \) for rest of the coalitions in \( \mathcal{R}(T) \) and \( \alpha(T) = \alpha \). Probability of such choice of \( \alpha \)'s is \((1-p)p^\mathcal{F}(a,T)\setminus 1p = (1-p)p^\mathcal{F}(a,T)\setminus 2 \). And there are \( \binom{\mathcal{F}(a,T)\setminus 1}{1}(1-p)p^\mathcal{F}(a,T)\setminus 2 \) such possibilities for all but one coalitions \( \tilde{\pi}(i) \in \mathcal{F}(a, T) \). Thus the overall probability is 
\[
\binom{\mathcal{F}(a,T)\setminus 1}{1}(1-p)p^\mathcal{F}(a,T)\setminus 2 = \frac{p^{\mathcal{F}(a,T)\setminus 1}}{1} \frac{p^{\mathcal{F}(a,T)\setminus 2}}{2}.
\]

\( \alpha(\tilde{\pi}(i)) = 1 \) for all \( \tilde{\pi}(i) \in \mathcal{F}(a, T) \), \( \alpha(\tilde{\pi}(i)) = \alpha \) for rest of the coalitions in \( \mathcal{R}(T) \) and \( \alpha(T) = \alpha \). Probability of such choice of \( \alpha \)'s is \((1-p)^2p^\mathcal{F}(a,T)\setminus 2 \). And there are \( \binom{\mathcal{F}(a,T)\setminus 1}{1}(1-p)^2p^\mathcal{F}(a,T)\setminus 2 \) such possibilities for all coalitions \( \tilde{\pi}(i) \in \mathcal{F}(a, T) \). Thus the overall probability is 
\[
\binom{\mathcal{F}(a,T)\setminus 1}{1}(1-p)^2p^\mathcal{F}(a,T)\setminus 2 = \frac{p^{\mathcal{F}(a,T)\setminus 1}}{1} \frac{p^{\mathcal{F}(a,T)\setminus 2}}{2}.
\]

Therefore, the probability of the event \( F(T, \tilde{\pi}) \), i.e., \( \mathbb{P}[F(T, \tilde{\pi})] \) is given by:

\[
\mathbb{P}[F(T, \tilde{\pi})] = p^{\mathcal{R}(T)\setminus 1} + (1-p)p^{\mathcal{R}(T)} + \frac{p^{\mathcal{R}(T)\setminus 1}}{1} \frac{p^{\mathcal{R}(T)\setminus 2}}{2} (1-p)^2p^{\mathcal{R}(T)\setminus 1} \]

\[
+ \binom{\mathcal{F}(a,T)\setminus 1}{1}(1-p)p^\mathcal{F}(a,T)\setminus 2 + \frac{p^{\mathcal{F}(a,T)\setminus 1}}{1} \frac{p^{\mathcal{F}(a,T)\setminus 2}}{2} (1-p)^2p^{\mathcal{F}(a,T)\setminus 2} + \ldots + \frac{p^{\mathcal{F}(a,T)\setminus 1}}{1} \frac{p^{\mathcal{F}(a,T)\setminus 2}}{2} (1-p)^2p^{\mathcal{F}(a,T)\setminus 2} + \ldots
\]

\[
+ \binom{\mathcal{F}(a,T)\setminus 1}{1}(1-p)p^\mathcal{F}(a,T)\setminus 2 + \frac{p^{\mathcal{F}(a,T)\setminus 1}}{1} \frac{p^{\mathcal{F}(a,T)\setminus 2}}{2} (1-p)^2p^{\mathcal{F}(a,T)\setminus 2} + \ldots
\]
\[
= p^{|R(T)|} \left( |R(T)| \left[ 1 - \left( |R(T)| - 1 \right) + \left( |R(T)| - 1 \right) p \right] |R(T)| \right)
+ \left( |R(T)| \right)^2 \left( |R(T)| - 1 \right) p |R(T)| + \left( |R(T)| - 1 \right) p^{2 |R(T)|}
+ \left( |R(T)| \right)^3 \left( |R(T)| - 1 \right) p |R(T)| + \left( |R(T)| - 1 \right) p^{3 |R(T)|}
+ \cdots
\]

This ends the proof. \(\Box\)

A.5 Proof of Lemma 4.3

For an agent noisy hedonic game with \(\alpha(S) \in \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \forall S \subseteq N\) such that \(\alpha_i < \alpha_j \forall i < j\) with \(\Pr(\alpha_j) = p_j \forall j \in [L]\), \(h_T(p, \alpha)\) is given by:

\[
h_T(p, \alpha) = \begin{cases} 
1, & \text{if } \tilde{\pi}(i) = T \forall i \in T \\
\sum_{r,s \in [L]: \alpha_r > \alpha_s} p_r |R(T)| - |J(\alpha_r, \alpha_s, T)| + p_s \left( p_r |J(\alpha_r, \alpha_s, T)| - p_s \left| J(\alpha_r, \alpha_s, T) \right| \right) \\
+ \sum_{a=1}^l p_a \left( \sum_{b=0}^l p_b \right)^{|R(T)|}, & \text{otherwise.}
\end{cases}
\]

Proof. We will prove this via induction on \(l\). Clearly, this is true for \(l = 2\) (from Lemma 4.2 above). Let us assume that it is true for \(l = k\), i.e., there are sets

\[
\mathcal{J}(\alpha_r, \alpha_s, T) = \left\{ \tilde{\pi}(i) \in R(T) \mid \frac{\tilde{v}(i)}{\tilde{v}(T)} \geq \frac{\alpha_s}{\alpha_r} \right\},
\]

such that the support \(\alpha(S) = \{\alpha_1, \ldots, \alpha_k\} \forall S \subseteq N\) for all \(\alpha_s < \alpha_r, 1 \leq s < r \leq k\). Let us consider the following event

\[
F(T, \tilde{\pi}) = \{\alpha(T), \alpha(\tilde{\pi}(i)) : \forall \tilde{\pi}(i) \in R(T), \cap_{i \in T} \left( (v_i(T) \geq v_i(\tilde{\pi}(i))) \cap \alpha(T)v_i(T) \geq \alpha(\tilde{\pi}(i))v_i(\tilde{\pi}(i)) \right) \}
\]
This event captures the set of all \( \alpha(\tilde{\pi}(i)), \alpha(T) \forall \tilde{\pi}(i) \in \mathcal{R}(T) \) such that \( T \geq_i \tilde{\pi}(i) \) and \( T \geq_i^j \tilde{\pi}(i) \) for all \( i \in T \). For this \( k \) we have \( f_T(p_j, \alpha_j : j \in [k]) =: h_T(p, \alpha) \) (by assumption)

\[
h_T(p, \alpha) = \sum_{a=1}^{k} p_a \left( \sum_{b=a}^{k} p_b \right)^{|\mathcal{R}(T)|} + \sum_{r,s \in [k], \alpha_r > \alpha_s} p_r^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha_r, \alpha_s, T)|+1} \left( (p_r + p_s) |\mathcal{J}(\alpha_r, \alpha_s, T)| - p_r^{|\mathcal{J}(\alpha_r, \alpha_s, T)|} \right) \tag{47}
\]

We will now show that this is true for \( l = k + 1 \). To this end define for all \( s \in [k] \) such that \( \alpha_{k+1} > \alpha_s \), the following sets:

\[
\mathcal{J}(\alpha_{k+1}, \alpha_s, T) = \left\{ \tilde{\pi}(i) \in \mathcal{R}(T) \mid \frac{\tilde{v}_i(\tilde{\pi}(i))}{\tilde{v}_i(T)} \geq \frac{\alpha_s}{\alpha_{k+1}} \right\}, \tag{48}
\]

Now, there are two cases, \( \mathcal{J}(\alpha_{k+1}, \alpha_s, T) = \emptyset \forall \alpha_s, s \in [k] \), or \( \mathcal{J}(\alpha_{k+1}, \alpha_s, T) \neq \emptyset \) at-least for one \( s \in [k] \).

**Case 01:** Clearly with one more element in the support, apart from the existing pairs of \( \alpha(\tilde{\pi}(i)), \alpha(T) \) for \( k \) support case, it will also have \( \alpha(T) = \alpha_{k+1} \), and \( \alpha(\tilde{\pi}(i)) = \alpha_{k+1} \) \( \forall \tilde{\pi}(i) \in \mathcal{R}(T) \). The probability of such alpha’s is \( p_{k+1} \left( \sum_{b=k+1}^{k} p_b \right)^{|\mathcal{R}(T)|} \). Therefore, the overall probability is

\[
\sum_{a=1}^{k} p_a \left( \sum_{b=a}^{k} p_b \right)^{|\mathcal{R}(T)|} + p_{k+1} \left( \sum_{b=k+1}^{k} p_b \right)^{|\mathcal{R}(T)|} = \sum_{a=1}^{k+1} p_a \left( \sum_{b=a}^{k} p_b \right)^{|\mathcal{R}(T)|}.
\]

**Case 02:** In this case, apart from all such pairs of \( \alpha(\tilde{\pi}(i)), \alpha(T) \) for all \( \tilde{\pi}(i) \in \mathcal{J}(\alpha_r, \alpha_s, T) \), we have pairs of \( \alpha(\tilde{\pi}(i)), \alpha(T) \) for all \( \tilde{\pi}(i) \in \mathcal{J}(\alpha_{k+1}, \alpha_s, T) \). For this set, the possible pairs of \( \alpha(\tilde{\pi}(i)), \alpha(T) \) are such \( \alpha(\tilde{\pi}(i)) = \alpha_s \) \( \forall \tilde{\pi}(i) \in \mathcal{R}(T) \setminus \mathcal{J}(\alpha_{k+1}, \alpha_s, T) \), and \( \alpha(T) = \alpha_{k+1} \). Thus their combined probability is \( p_{k+1}^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha_{k+1}, \alpha_s, T)|+1} \left( (p_{k+1} + p_s) |\mathcal{J}(\alpha_{k+1}, \alpha_s, T)| - p_{k+1}^{|\mathcal{J}(\alpha_{k+1}, \alpha_s, T)|} \right) \). Hence for \( k + 1 \) support the probability is

\[
\sum_{r,s \in [k], \alpha_r > \alpha_s} p_r^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha_r, \alpha_s, T)|+1} \left( (p_r + p_s) |\mathcal{J}(\alpha_r, \alpha_s, T)| - p_r^{|\mathcal{J}(\alpha_r, \alpha_s, T)|} \right)
\]

\[
+p_{k+1}^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha_{k+1}, \alpha_s, T)|+1} \left( (p_{k+1} + p_s) |\mathcal{J}(\alpha_{k+1}, \alpha_s, T)| - p_{k+1}^{|\mathcal{J}(\alpha_{k+1}, \alpha_s, T)|} \right) \tag{49}
\]

From, case 01, and case 02 for \( k + 1 \) support we have

\[
h_T(p_j, \alpha_j; j \in [k + 1]) = \sum_{a=1}^{k+1} p_a \left( \sum_{b=a}^{k+1} p_b \right)^{|\mathcal{R}(T)|} + \sum_{r,s \in [k+1], \alpha_r > \alpha_s} p_r^{|\mathcal{R}(T)|-|\mathcal{J}(\alpha_r, \alpha_s, T)|+1} \left( (p_r + p_s) |\mathcal{J}(\alpha_r, \alpha_s, T)| - p_r^{|\mathcal{J}(\alpha_r, \alpha_s, T)|} \right). \tag{49}
\]

And hence it is true for \( k + 1 \) support. Thus, from principle of mathematical induction this is true for any \( l \in \mathbb{N} \). \( \square \)