Spatially self-similar locally rotationally symmetric perfect fluid models

Ulf Nilsson *
Department of Physics, Stockholm University,
Box 6730, S-113 85 Stockholm, Sweden
and
Claes Uggla † ‡
Department of Physics, Stockholm University,
Box 6730, S-113 85 Stockholm, Sweden
and
Department of Physics, Luleå University of Technology
S–951 87 Luleå, Sweden

December 29, 2021

Abstract

Einstein’s field equations for spatially self-similar locally rotationally symmetric perfect fluid models are investigated. The field equations are rewritten as a first order system of autonomous ordinary differential equations. Dimensionless variables are chosen in such a way that the number of equations in the coupled system of differential equations is reduced as far as possible. The system is subsequently analyzed qualitatively for some of the models. The nature of the singularities occurring in the models is discussed.

PACS numbers: 0420, 9530S, 9880H.

Short title: Spatially self-similar perfect fluid models

* e-mail: ulfn@vanosf.physto.se
† Supported by the Swedish Natural Science Research Council
‡ e-mail: uggla@vanosf.physto.se


1 Introduction

Self-similar models, with a group acting transitively on 3-dimensional hypersurfaces, has been discussed in the literature for several decades. Within this family of models one finds a number of different interesting physical phenomena; shock waves and violation of cosmic censorship being perhaps the most prominent ones. Self-similar models are also interesting because they constitute asymptotic states for more general models and that they thus act as building blocks when it comes to understanding wider classes of models. They also generalize the spatially homogeneous (SH) models which admit two spacelike commuting Killing vectors, and in this context they can be seen as the first step towards the construction of inhomogeneous cosmological models. They might also shed light on the generality of properties found in the SH subclass of models. For example, in this article we discuss if whimper singularities, found in some of the SH models, exist in the self-similar case.

One finds two approaches toward the self-similar perfect fluid models. One is the “fluid adapted” approach in which one chooses a timelike coordinate along the fluid lines (see e.g., [1]–[4]). The second is the “homothetic” approach in which one chooses a coordinate along the orbits of the homothetic Killing vector (see e.g., [5]–[9]). In general, the symmetry surface changes causality.

One often considers models that are diagonalizable. In this case the homothetic approach, taken together with the diagonal gauge requirement, has the drawback that one has to cover the spacetime with several coordinate patches. On the other hand, the homothetic approach reveals that there is a considerable structural similarity between the field equations for the self-similar models and the hypersurface-homogeneous models (e.g., SH models). There is a considerable literature about on how one deals with hypersurface-homogeneous models (much larger than the one on self-similar models). Thus the homothetic approach makes it possible to transfer ideas from the hypersurface-homogeneous arena to the self-similar one. Because of this advantage we will use the homothetic approach. However, note that results obtained in this picture can be transferred to the fluid adapted formulation by a coordinate transformation and vice versa.

In this article we will focus on the “spatial part” of self-similar models exhibiting a locally rotationally symmetric (LRS) isometry group (in addition to the self-similar symmetry). The line elements for the spatially-self-similar (SSS) LRS models have been given by Wu [9]. Collectively they can be written as

\[ ds^2 = e^{-2f} \left[ -dt^2 + D_1(t)^2 dx^2 + D_2(t)^2 e^{-2ax} \left( dy^2 + k^{-1} \sin (\sqrt{k}y) dz^2 \right) \right] \]  

(1)

where \( f, a, k \) are parameters describing the symmetry groups of the various models. Canonical values for these parameters are given in Table 1. Note that \( ak = 0 \) and that, for the sake of brevity, we have denoted the LRS type \( fI \) (\( fV \)) models with \( fI \) (\( fV \)) even though they also contain a type \( fVII_0 \) (\( fVII_h \)) group (see [9]). The SSS models with spherical symmetry are denoted by \(*KS\) where KS stands for Kantowski-Sachs in analogy with the corresponding SH case.

We will consider perfect fluid models. The energy momentum tensor, \( T_{ab} \), is thus given by

\[ \tilde{T}_{ab} = \tilde{\mu} u_a u_b + \tilde{p}(u_a u_b + g_{ab}) \]  

(2)

where \( \tilde{\mu} \) is the energy density; \( \tilde{p} \) is the pressure; and \( u^a \) the 4-velocity of the fluid. We will assume

\[ \tilde{p} = (\gamma - 1) \tilde{\mu} \]  

(3)

as an equation of state where the parameter \( \gamma \) takes values in the interval \( 1 \leq \gamma < 2 \), which includes dust (\( \gamma = 1 \)) and radiation (\( \gamma = 4/3 \)). Thus we have excluded the value \( \gamma = 2 \), which corresponds to
Table 1: Canonical choices of the symmetry parameters $a, f$ and $k$.

|        | SSS           | SH            |
|--------|---------------|---------------|
| $f$    | $f$           | 0             |
| $a$    | 1             | 1             |
| $k$    | 0             | -1            |

A stiff fluid. The reason for this is that the corresponding models behave quite differently compared to those in the interval $1 \leq \gamma < 2$, and thus need special treatment. Note that all dust models are known \[10\]. Note also that all self-similar LRS models are of Petrov type D (or 0) and that the magnetic part of the Weyl tensor is zero.

The outline of the article is the following: In section 2 we rewrite the field equations in two steps. We first express the field equations in a set of variables associated with the normal congruence of the symmetry surface. We then introduce a dimensionless set of variables in order to obtain a maximal reduction of the coupled system of ordinary differential equations.

In section 3 the reduced phase spaces and invariant submanifolds of the various models are discussed. The dimensionality of the fully reduced phase spaces together with the relation between the models, in terms of Lie contractions, are given in a diagram. Similarities and differences between the various models are discussed. The relation between the field equations for the SSS models and the timelike-self-similar (TSS) models and the possibility of extending the SSS models to the TSS sector is commented on.

In section 4 an equilibrium (critical, singular) point analysis of the SSS type $1_I$ and $f V$ LRS models is carried out. Asymptotic expressions for the line element and the kinematic fluid quantities are given. The remaining SSS and TSS LRS models need special treatment, and will be discussed elsewhere.

In section 5 complete phase portraits are given for a number of type $1_I$ and $f V$ models. Some global issues are also discussed. In section 6 we discuss the nature of the singularities occurring in the SSS type $1_I$ and $f V$ models. Appendix A describes the properties of the fluid congruence and the condition for the spacetimes to belong to Petrov type 0. Appendix B gives the relation between the presently used coordinates and those of the fluid approach.

2 Derivation of the autonomous DE

Misner has introduced a useful metric parametrization in the context of SH cosmology \[11\]. In the present case an analogous parametrization can be introduced

$$D_1 = e^{\beta^0 - 2\beta^+}, \quad D_2 = e^{\beta^0 + \beta^+}.$$  \hspace{1cm} (4)

This parametrization is closely related to the kinematic properties of the normal congruence of the $ds^2$ geometry (which is conformally related to the SSS geometry). The expansion, $\theta$, and the shear,
which can be described by a quantity $\sigma_+$, of this congruence, are related to $\beta^{0+}$ by

$$\theta = 3\beta^0, \quad \sigma_+ = 3\beta^+,$$

where the dot stands for $d/dt$. The tetrad components of the fluid velocity are conveniently parametrized by $(1,v,0,0)/\sqrt{1-v^2}$ where $v$ is just the 3-velocity with respect to the symmetry surfaces.

Einstein’s equations, $\bar{G}_{\alpha\beta} = \bar{T}_{\alpha\beta}$, and $\bar{T}^{\alpha\beta}_{\ ;\beta} = 0$ lead to:

**Evolution equations**

\[
\begin{align*}
\dot{\theta} &= -\frac{1}{3}\theta^2 - \frac{2}{3}\sigma_+^2 + 2(a+f) f B_1^2 - \frac{1}{2} (3\gamma - 2) + (2 - \gamma) v^2 \mu_n, \\
\dot{\sigma}_+ &= -\theta \sigma_+ + 2 \left[ (a + \frac{2}{3} f) \sigma_+ - \frac{1}{3} f \theta \right] v B_1 + 2 (a + f) f B_1^2 - k B_2^2, \\
\dot{B}_1 &= \frac{1}{3} (-\theta + 2\sigma_+) B_1, \\
\dot{B}_2 &= -\frac{1}{3} (\theta + \sigma_+) B_2, \\
\dot{v} &= \frac{1 - v^2}{3\gamma (1 - (\gamma - 1)v^2)} \left[ \gamma ((3\gamma - 4) \theta + 2\sigma_+) v + 3 \left( (\gamma - 1) (2f - (2a + 3f) \gamma) v^2 + (2 - \gamma) f \right) B_1 \right],
\end{align*}
\]

where

$$B_1 = e^{-\beta^{0+}2\beta^+} = D_1^{-1}, \quad B_2 = e^{-\beta^{0+}2\beta^+} = D_2^{-1}.$$  

The quantity $\mu_n$ stands for the energy density of the fluid, measured by an observer associated with the normal congruence of the symmetry surfaces, multiplied with the factor $e^{-2fx}$. It is related to $\tilde{\mu}$, the energy density of the fluid, by

$$\mu_n = \frac{1 + (\gamma - 1)v^2}{1 - v^2} \tilde{\mu} e^{-2fx}.$$  

The above evolution equations are not independent. Instead they are related by

**Constraint equation**

$$\gamma v \mu_n + 2 \left[ 1 + (\gamma - 1)v^2 \right] \left[ (a + \frac{2}{3} f) \sigma_+ - \frac{1}{3} f \theta \right] B_1 = 0.$$  

**Defining equation for $\mu_n$**

$$\mu_n = \frac{1}{3} \left( \theta^2 - \sigma_+^2 - 3(a + f) (a + f) B_1^2 + 3kB_2^2 \right).$$  

Of interest is also:

**Auxiliary equation**
\[\begin{align*}
\mu_n &= \frac{\gamma \mu_n}{1 + (\gamma - 1) v^2} \left(-\theta + 2(a + f)B_1 v + \frac{1}{3} (2\sigma_+ - \theta) v^2\right). \\
\end{align*}\] (11)

Note the close relationship between the above presentation of the field equations and the one given by Hewitt and Wainwright for the SH type V case [12]. This close connection will allow us to transfer some of the ideas used by Hewitt and Wainwright to the present class of models. In particular, we will follow their treatment of the constraint and the introduction of \(\theta\)-normalized dimensionless variables:

\[\begin{align*}
\Sigma_+ &= \frac{\sigma_+}{\theta}, \quad A = \frac{B_1}{\theta}, \quad K = \left(\frac{B_2}{\theta}\right)^2. \\
\end{align*}\] (12)

The density \(\mu_n\) is replaced by the density parameter \(\Omega_n\), which is defined by

\[\begin{align*}
\Omega_n &= \frac{3 \mu_n}{\theta^2}.
\end{align*}\] (13)

An introduction of a dimensionless time variable \(\tau\),

\[\begin{align*}
\frac{dt}{d\tau} &= \frac{3}{\theta},
\end{align*}\] (14)

leads to a decoupling of the \(\theta\)-equation

\[\begin{align*}
\frac{d\theta}{d\tau} &= -(1 + q) \theta,
\end{align*}\] (15)

where the deceleration parameter \(q\), associated with the normal congruence of the SH \(ds^2\) geometry, is given by

\[\begin{align*}
q &= 2\Sigma_+^2 - 6(a + f) f A^2 + \frac{1}{2} \frac{(3\gamma - 2) + (2 - \gamma) v^2}{1 + (\gamma - 1) v^2} \Omega_n.
\end{align*}\] (16)

Alternatively one can write \(q\) as

\[\begin{align*}
q &= 2 - 6(a + f)(3a + 2f) A^2 + 6kK - \frac{1}{2} \frac{(3(2 - \gamma) + (5\gamma - 6)v^2)}{1 + (\gamma - 1)v^2} \Omega_n,
\end{align*}\] (17)

by using the definition of \(\Omega_n\). The remaining evolution equations can now be written in dimensionless form:

**Evolution equations**

\[\begin{align*}
\Sigma'_+ &= [-2 + q + 2v A (3a + 2f)] \Sigma_+ - 2f v A + 6f A^2 (a + f) - 3kK, \\
A' &= (2\Sigma_+ + q) A, \\
K' &= 2 (-\Sigma_+ + q) K, \\
v' &= \frac{1 - v^2}{\gamma (1 - (\gamma - 1) v^2)} \left[\gamma ((3\gamma - 4) + 2\Sigma_+) v + 3 \left((\gamma - 1)(2f - (2a + 3f) \gamma) v^2 + (2 - \gamma) f\right) A\right],
\end{align*}\] (18)

where a prime denotes \(d/d\tau\).

**Constraint equation**
\[ \gamma v \Omega_n + 2 \left[ 1 + (\gamma - 1) v^2 \right] [(3a + 2f) \Sigma_+ - f] A = 0 . \]  
\[ \text{Defining equation for } \Omega_n \] 
\[ \Omega_n = 1 - \Sigma_+^2 - 3(3a + f)(a + f) A^2 + 3kK . \] 
\[ \text{Auxiliary equation} \]
\[ \Omega'_n = \frac{\Omega_n}{1 + (\gamma - 1)v^2} \left[ -(3\gamma - 2) + 2q + 6\gamma(a + f)vA + (2\gamma \Sigma_+ + 2q(\gamma - 1) - (2 - \gamma)) v^2 \right] . \]  
The above equation system exhibits a discrete symmetry. The field equations are invariant under the transformations 
\[ (\Sigma_+, A, K, v) \rightarrow (\Sigma_+, -A, K, -v) , \]  
The line element can be obtained when \( K, A \) and \( \theta \) has been found through the relations 
\[ D_1^2 = (\theta A)^{-2} , \quad D_2^2 = (\theta^2 K)^{-1} , \quad t = 3 \int \frac{d\tau}{\theta} . \]  

The relationship between the kinematic fluid quantities and the \( \theta, K, A, \Sigma_+, v \) variables is given in Appendix A.

3 Reduced phase spaces and invariant submanifold structure

The \( K \)-equation decouples for the type I and \( fV \) models leaving a reduced system of equations for the \( \Sigma_+, A, v \) variables, related by the constraint in eq. (19). The relationship between the various models is given in terms of Lie contractions in Diagram 1 together with the dimension of the reduced phase space.

We have chosen to present the equation in expansion normalized variables since this leads to relatively simple equations. However, a very desirable property is compactness of the reduced phase space. In this context eq. (19) plays an essential role. One can use the fact that \( \mu_n \) is nonnegative to produce inequalities which can be used in order to find “dominant” quantities. The expansion, \( \theta \), is such a quantity for some of the models (those for which \( k \leq 0, (3a + f)(a + f) \geq 0 \); as seen by eq. (19)). Note that \( \theta \) cannot change sign for the matter filled “\( \theta \)-dominated” models because of eq. (19). Furthermore, because of the discrete symmetry in eq. (22) one can therefore, without loss of generality, assume \( A \geq 0 \) (this just corresponds to looking at models with positive expansion). Unfortunately one cannot make this assumption for the other models where \( \theta \), and thus also \( A \), may change sign.

The boundary consists of a number of invariant subsets which plays an important role when determining the qualitative properties of the orbits in the reduced phase space. The boundary of the reduced phase space will thus be included. We have the following invariant submanifolds on the boundary: (i) \( A = 0 \), (ii) \( v = 1 \), (iii) \( v = -1 \), (iv) the vacuum submanifold \( 1 - \Sigma_+^2 - 3(3a + f)(a + f) A^2 = 0 \). For the \( k \neq 0 \) models we also have (v) \( K = 0 \) as an invariant submanifold of the reduced phase space.

The \( v = \pm 1 \) submanifolds has a physical interpretation in terms of models with directed fluxes of neutrinos (see [9]). In the vacuum submanifold case one can interpret \( v \) as the velocity of a test fluid (see [2]). The constraint leads to \( v \Omega_n = 0 \) when \( A = 0 \). For \( v = 0 \) we obtain the reduced equations for the orthogonal SH LRS type I, III and KS models. When \( \Omega_n = 0 \) one obtains the
Diagram 1: Lie contractions for self similar and spatially homogeneous LRS models. The abbreviations Caus., HH, HSS stand for causal character, hypersurface homogeneous and hypersurface self-similar respectively.

reduced equations for the vacuum models for the same models but with a test fluid. For the $k \neq 0$ models, the $K = 0$ manifold yields the same equations as the reduced type $1I$ equations. Note that $v = 0$, in contrast to the SH models, is not an invariant submanifold (except for $\gamma = 2$). Thus one naturally obtains, for the present class of LRS models, the well known result that SSS models are tilted if $\gamma \neq 2$ (i.e., there are no SSS models with a fluid flow orthogonal with respect to the symmetry surfaces) [9].

3.1 The reduced phase space of type $fV$ and $1I$ models

For these models $k = 0$. This results in a decoupling of the equation for $K$. Since $K$ decouples the boundary value $K = 0$ is of no interest (it is enough to analyze the reduced system of equations). Thus $K$ satisfy the inequality $K > 0$. This leaves a coupled system of equations for the remaining variables ($\Sigma_+, A, v$), related by the constraint equation. As mentioned above, the $(3a+f)(a+f) = 0$ and the $(3a+f)(a+f) < 0$ models, do not have a compact reduced phase space. Moreover, the $3a+f = 0$ and $a+f = 0$ models behaves drastically different compared to the other models. This is seen in the equilibrium point analysis below. Shikin has been able to find the general solution for the $3a+f$ and $a+f = 0$ cases [2, 3]. Shikin has also qualitatively investigated the $(3a+f)(a+f) < 0$ models [4]. Below we will focus on the “$\theta$-dominant” $(3a+f)(a+f) > 0$ case which contains the SH type $V$ models and the SSS type $1I$ models. Recall that one can assume that $A \geq 0$ for the $\theta$-dominant models. However, we will also list the equilibrium points for the remaining models since this gives an indication of how different the three types of models.
There are four invariant submanifolds describing the boundary of the reduced equation system of the \( \theta \)-dominant models: The \( v = \pm 1 \) submanifolds; the vacuum submanifold \( 1 - \Sigma_+^2 - 3(3a + f)(a + f)A^2 = 0 \), and the \( A = 0 \) submanifold which can be divided into three parts depending on \( v < 0, v = 0, v > 0 \).

3.2 The reduced phase space of type \( ^*_{\text{III}} \) and \( ^*_{\text{KS}} \) models

It follows from eq. (10) that one obtains a compact reduced phase space in the (SSS) type \( ^*_{\text{III}} \) models. However, the \( \theta \) normalized variables are not compact for the \( ^*_{\text{KS}} \) models. In a forthcoming article we will show how one can use eq. (10) to produce compact reduced phase space variables in the \( ^*_{\text{KS}} \) case. The type \( ^*_{\text{III}} \) models are characterized by the choices \( k = -1, a = 0, f = -1 \). The reduced system consists of equations for the \( \Sigma_+, A, K, v \) variables, related by the constraint in eq. (19). Here the \( K = 0 \) value is included. This leads to the inequality \( K \geq 0 \). The boundary is described by five pieces: The \( v = \pm 1 \) submanifolds; the vacuum submanifold \( 1 - \Sigma_+^2 - 3A^2 - 3K = 0 \); the \( A = 0 \) submanifold, and the \( K = 0 \) submanifold.

3.3 The relation between the SSS and TSS field equations

It is important to note that the above equation system (where \( \mu_n \) (\( \Omega_n \)) is assumed to be eliminated by use of eq. (10) (eq. (20))) also describes the field equations in the TSS region if one makes the change \( K \rightarrow -K \) in the \( k \neq 0 \) case. However, in the TSS case \( v^2 \geq 1 \) instead of, as presently, \( v^2 \leq 1 \).

It’s natural to extend solutions to the TSS sector when the symmetry surface changes causality and when one does not run into a (non-coordinate) singularity (compare with the LRS type V discussion by Collins and Ellis [13]). A number of phenomena occurs in the TSS region which do not exist in the SSS case. Hence, we will investigate the TSS region separately in a forthcoming article. This will allow us to obtain a more global picture of the self-similar LRS models than in the present article which will, from now on, deal exclusively with the SSS case.

4 Equilibrium points and asymptotic analysis

The reduced phase space is determined by a coupled system \( dX/d\tau = F(X) \) subject to a constraint \( G(X) = 0 \) given by eq. (13) (\( X \) constitute the reduced phase space variables). Of central importance to the investigation of the dynamical system are the equilibrium points which are determined by the equations \( F(X) = 0, G(X) = 0 \). Below the equilibrium points are given together with \( \Omega_n \), which shows if an equilibrium point is located on a vacuum submanifold or not. The gradient of \( G(X) \), which is used to locally solve the constraint to linear order, and the eigenvalues and eigenvectors of the remaining locally unconstrained system are then listed.

A local analysis of the equilibrium points is then used to express the fluid energy density (\( \tilde{\mu} \)), expansion (\( \tilde{\theta} \)), shear scalar (\( \tilde{\sigma} \)), acceleration (\( \tilde{a} \)), divergence of the acceleration (\( \tilde{a}^\mu_{\mu} \)); the line element (\( D_{1,2} \)); and the Weyl scalar (\( C \)), for the various asymptotes in the interior matter filled phase space. The tables describing asymptotic properties differ from the ones given by Collins and Ellis for the SH type V models [13]. Their tables relate the initial and final state of a given orbit (or orbits). Such tables will be given in the next section when we discuss global behaviour. Our tables also differ from those in reference [13] in that we express the behaviour in terms of the independent
variable $\tau$ instead of the proper time associated with the comoving fluid frame. The reason for this is that the SSS and SH models need separate treatment if one wants to use proper fluid time. The results in this section and those by [13, 15] for SH type V models are easily related by using, e.g., $\tilde{\theta}$, as independent variable. We only give the $\tau$-dependence; the self-similar dependence on $e^{f\tau}$ is given in Appendix A. The acceleration scalar, $\tilde{a}$, and the divergence of the acceleration, $\tilde{a}^\mu_{\mu}$, are of course zero for the dust, $\gamma = 1$, models.

4.1 Type $f$V and $I$ models

The reduced phase space consists of the constraint surface in the $(\Sigma_+, A, v)$ space, subject to the inequalities $\Omega_n \geq 0, v^2 \leq 1$. For the $\theta$-dominated models we also have $A \geq 0$.

4.1.1 Equilibrium points with zero tilt, $v = 0$

The equilibrium point $F$:

$$\Sigma_+ = 0, \quad A = 0; \quad \Omega_n = 1;$$

$$\nabla G = (0, -2f, \gamma), \quad (v \text{ eliminated}) ;$$

$$\quad -\frac{3}{2}(2 - \gamma), \quad (1, 0); \quad \frac{1}{2}(3\gamma - 2), \quad (0, 1),$$

for $1 \leq \gamma < 2$, where the constraint has been used to solve for the $v$ variable and where the eigenvalues of the resulting unconstrained system are grouped together with the corresponding eigenvectors. The equilibrium point is a saddle point. In Table 2 we give asymptotic expressions for the single initial asymptote entering the interior of the reduced physical phase space of the type $f$V and $I$ models.

| As.St. | $\tilde{\mu}$ | $\tilde{\theta}$ | $\tilde{\sigma}$ | $\tilde{a}$ | $\tilde{a}^\mu_{\mu}$ | $D_1$ | $D_2$ | $C$ |
|--------|---------------|-----------------|-----------------|-------------|-------------------|------|------|-----|
| i      | $e^{-3\gamma\tau}$ | $e^{-3\gamma\tau/2}$ | $e^{(3\gamma-4)\tau/2}$ | $f e^{(\gamma-1)\tau}$ | $f e^{(3\gamma-4)\tau}$ | $e^{\tau}$ | $e^{\tau}$ | $f e^{-2\tau}$ |

Table 2: Asymptotic expressions for the single initial asymptote coming from the equilibrium point $F$. Here, and below As. St. is an abbreviation for asymptotic state. Initial states are denoted by i and final states by f.

The equilibrium points $K^0_\pm$:

$$\Sigma_+ = \pm 1, \quad A = 0; \quad \Omega_n = 0;$$

$$\nabla G = (0, 2 (3a + f), 0), \quad \text{for} \quad \Sigma_+ = +1, \quad (A \text{eliminated}) ;$$

$$\nabla G = (0, -6 (a + f), 0), \quad \text{for} \quad \Sigma_+ = -1;$$

$$3(2 - \gamma), \quad (1, 0); \quad 2\Sigma_+ + (3\gamma - 4), \quad (0, 1),$$

for $1 \leq \gamma < 2$. Different equilibrium points are often closely related through sign changes or zero values of $v$ and $\Sigma_+$. In such cases we denote the equilibrium points with a common kernel and
indicate the value of $v$ in the upper position and of $\Sigma_+$ in the lower (if there only exists points with a given value of $v$ or $\Sigma_+$ we do not indicate the corresponding value). Thus in the present case of $K_0^0$, $0$ refers to $v = 0$ and $\pm$ refers to the sign of $\Sigma_+$. (Note that this leads to a somewhat different notation than that used by Hewitt and Wainwright for the SH type V case [12].) The point $K_0^0$ is a saddle point on the boundary and does not give rise to any asymptotes in the interior phase space. The point $K_0^0$ on the other hand is a stable source giving rise to a 1-parameter set of asymptotes described in Table 3.

| As. St. | $\bar{\mu}$ | $\bar{\theta}$ | $\bar{\sigma}$ | $\bar{\tilde{a}}$ | $\bar{a}_{\mu}^\mu$ | $D_1$ | $D_2$ | $C$ |
|---------|-------------|-------------|-------------|-------------|-------------|-----|-----|-----|
| $i, \Sigma_+ = +1$ | $e^{-3\gamma\tau}$ | $e^{-3\tau}$ | $e^{(3\gamma-5)\tau}$ | $e^{2(3\gamma-5)\tau}$ | $e^{-\tau}$ | $e^{2\tau}$ | $e^{-6\tau}$ |

Table 3: Asymptotic expressions for the fluid congruence around the $K_+^0$-point.

The equilibrium point $M^0$:

This point only exists for $f = 0$ and therefore we can choose $A > 0$. $\Sigma_+ = 0$, $A = \frac{1}{3a}$; $\Omega_n = 0$;
\[
\nabla G = (2, 0, 0) ; \quad (\Sigma_+ \text{ eliminated}) ;
- (3\gamma - 2), \quad (1, 0) ; \quad 3\gamma - 4, \quad (0, 1).
\]

There exists a bifurcation for $\gamma = 4/3$, which needs special consideration; we refer to [12]. For $1 \leq \gamma < 4/3$ the point is stable leading to a 1-parameter set of final asymptotes, among them the Friedmann asymptote. For $4/3 \leq \gamma < 2$ the point is a saddle and has no asymptotes into the physical phase space, except for the special Friedmann asymptote. The asymptotic behaviour is described in Table 4.

| As. St. | $\bar{\mu}$ | $\bar{\theta}$ | $\bar{\sigma}$ | $\bar{\tilde{a}}$ | $\bar{a}_{\mu}^\mu$ | $D_1$ | $D_2$ | $C$ |
|---------|-------------|-------------|-------------|-------------|-------------|-----|-----|-----|
| $f, \gamma > 4/3$ | $e^{-3\gamma\tau}$ | $e^{-\tau}$ | $0$ | $0$ | $e^{\tau}$ | $e^{\tau}$ | $0$ |
| $f, \gamma < 4/3$ | $e^{-3\gamma\tau}$ | $e^{-\tau}$ | $e^{(3\gamma-5)\tau}$ | $e^{(3\gamma-5)\tau}$ | $e^{-3(2-\gamma)\tau}$ | $e^{\tau}$ | $e^{\tau}$ | $e^{-4\tau}$ |

Table 4: Asymptotic expressions for the fluid congruence around the $M^0$-point.
4.1.2 Points with intermediate tilt, $0 < v^2 < 1$

The equilibrium points $M^{v\pm}$: $1 < \gamma < 2$; $\gamma \neq 2f/(2a+3f)$.

\[
\Sigma_+ = \frac{f}{3a+2f}, \quad A = \frac{\epsilon}{3a+2f}, \quad v = \frac{\epsilon_1 c_2}{2c_1(\gamma-1)} \pm \sqrt{\frac{\gamma^2 c_2^2 + 4(2-\gamma)(\gamma-1)f c_1}{4c_1^2(\gamma-1)^2}}; \quad \Omega_n = 0;
\]

\[
c_1 := (3\gamma - 2)f + 2a\gamma, \quad c_2 := 2(\gamma - 1)f + (3\gamma - 4)a;
\]

\[
\nabla G = \left( \frac{2\epsilon_1 v (3a+2f)(\gamma-1) - 2f \gamma v + 2\epsilon (3a+2f)}{3a+2f}, -\frac{\epsilon_1 v (3a+f)(a+f)}{3a+2f}, 0 \right);
\]

\[
(A \text{ eliminated}); \quad -\frac{3(1-\epsilon_1_v)(c_1 v - \epsilon(2-\gamma) f)}{3a+2f}\gamma v; \quad \frac{\epsilon_1 v + c_2}{(\gamma-1)^2 c_1}; \quad (0,1)
\]

\[
e_1 := -3\epsilon((2\gamma - 1)(2 - \gamma) f^2 - c_2 f^3)(\gamma - 1)c_1^2 + 2(4(\gamma - 1)^2 - (2 - \gamma)(\gamma - 1)^2 c_1^2 f
\]

\[
+ 4(\gamma - 1)^2 a_1 c_1^2 - 4(\gamma - 1)(\gamma - 2)c_1 c_2 f \gamma^2 + c_2^2 \gamma^4),
\]

\[
e_2 := -3((3\gamma - 2)f + c_1)(\gamma - 1)(\gamma - 2)c_1 f - (\gamma - 1)^2 c_1^2 - (\gamma - 2)c_2 f \gamma^2) c_2 \gamma.
\]

The $\pm$ in $M^{v\pm}$ refers to the different signs of the root in the expression for $v$. The condition $\gamma \neq 2f/(2a+3f)$ corresponds to $c_1 \neq 0$. The eigenvector of the second eigenvalue above points in the direction of the vacuum submanifold, while the eigenvector of the first eigenvalue points into the physical phase space. The expression for the eigenvector associated with the first eigenvalue is quite cumbersome and will not be given. Note that there are two sets of points for the non-$\theta$-dominated models since $A$ can be both positive and negative in that case. For the $\theta$-dominated models there is only a single pair of points. The requirement of $A \geq 0$ taken together with the sign of $3a + f$ determines the sign of the parameter $e$ which can take the values $\pm 1$. The first eigenvalue is negative for the $\theta$-dominated models. Hence it follows that, in the $\theta$-dominated cases, possible initial asymptotes lies in the vacuum submanifold. Thus the only interior asymptote is a final one. Its generic behaviour is given in Table 5.

| As. St. | $\tilde{\mu}$ | $\tilde{\theta}$ | $\tilde{\sigma}$ | $\tilde{a}$ | $\tilde{a}_{\mu}$ | $D_1$ | $D_2$ | $C$
|---|---|---|---|---|---|---|---|---
| $f$ | $e^{-\lambda^2 \tau}$ | $e^{-\Gamma_1 \tau}$ | $e^{-\Gamma_1 \tau}$ | $e^{-2\Gamma_1 \tau}$ | $e^{\Gamma_1 \tau}$ | $e^{\Gamma_2 \tau}$ | $e^{-\lambda^2 \tau}$

Table 5: Asymptotic expressions for the fluid congruence around the $M^{v\pm}$-points. The constants $\Gamma_{1,2}$ are given by $\Gamma_1 = \frac{3a}{3a + 2f}$ and $\Gamma_2 = \frac{3(a + f)}{3a + 2f}$. Here and below $\lambda^2$ will stand for a constant which will not be given because it is quite complicated (the actual value of $\lambda^2$ is different for different tables).

The equilibrium points $M^{v\pm}$: $\gamma = 1$. 

11
\[ \Sigma_+ = \frac{f}{3a + 2f}, \quad A = \pm \frac{1}{3a + 2f}, \quad v = \pm \frac{f}{a}; \quad \Omega_n = 0; \]

\[ \nabla G = \left( \pm \frac{2(3a - f)(a + f)}{a(3a + 2f)}, -\frac{6f(3a + f)(a + f)}{a(3a + 2f)}, 0 \right); \]

(A eliminated),

The local analysis yields a single multiple valued eigenvalue \( m \) below and a linear solution of the following form:

\[ \Sigma_+ = \frac{f}{3a + 2f} + k_2 e^{m \tau}, \quad v = \pm \frac{f}{a} + (k_1 + k_2 p \tau) e^{m \tau}; \]

\[ m := -\frac{3(a + f)(a - f)}{(3a + 2f)a}, \quad p := \frac{3f(3a + 2f)(a - f)(a + f)^2}{(3a - f)a^3}, \]

where \( k_1 \) and \( k_2 \) are constants. For \( \theta \)-dominated models with \( (3a + f) < 0 \) and models with \( (a + f) > 0 \) and \( a - f < 0 \) none of these points exist in the physical phase space. However, for models with \( (a + f) > 0 \) and \( a - f > 0 \) the point \( M^{v+} \) exists and is a sink. For non-\( \theta \)-dominated models it can be a sink or a source depending on the values of \( a \) and \( f \). Asymptotic behaviour is given in Table 6.

| As, St. | \( \hat{\mu} \) | \( \hat{\theta} \) | \( \hat{\sigma} \) | \( \hat{\sigma}_\mu \) | \( D_1 \) | \( D_2 \) | \( C \) |
|--------|----------------|----------------|----------------|----------------|----------|----------|--------|
| \( f \) | \( e^{-\lambda_1^2 \tau} \) | \( e^{-\Gamma_1 \tau} \) | \( e^{-\Gamma_1 \tau} \) | 0 | \( e^\Gamma_1 \tau \) | \( e^\Gamma_2 \tau \) | \( e^{-\lambda_1 \tau} \) |

Table 6: Asymptotic expressions for the fluid congruence around the \( M^{v\pm} \)-points for \( \gamma = 1 \). The constants \( \Gamma_{1,2} \) are given by \( \Gamma_1 = \frac{3a}{3a+2f} \) and \( \Gamma_2 = \frac{3(a+f)}{3a+2f} \) while \( \lambda_1^2 = \frac{3(2a^2-f^2)}{a(3a+2f)} \).

The equilibrium points \( M^{v\pm} \): \( \gamma = 2f/(2a + 3f) \).

\[ \Sigma_+ = \frac{f}{3a + 2f}, \quad A = \pm \frac{1}{3a + 2f}, \quad v = \pm \frac{2a + 3f}{4a + f}; \quad \Omega_n = 0; \]

\[ \nabla G = \left( \pm \frac{4(6a^2 + 2af - 3f^2)(3a + f)}{(4a + f)^2(3a + 2f)}, -\frac{12f(3a + f)(a + f)}{(4a + f)(3a + 2f)}, 0 \right); \]

(A eliminated); \[ \frac{12(a + f)(a - f)}{(3a + 2f)(2a + 3f)}, -\frac{6(4a + f)(a + f)(a - f)}{(5a^2 + 4af + f^2)(2a + 3f)}. \]

This point only exists for models which are non-\( \theta \)-dominated, and therefore we have only given the eigenvalues and not the eigenvectors. We have also refrained from giving a table with the asymptotic properties.
4.1.3 Points with extreme tilt, \( v^2 = 1 \)

The equilibrium points \( M^\pm \):

\[
\Sigma_\pm = \frac{f}{3a + 2f}, \quad A = \mp \frac{1}{3a + 2f}, \quad v = \pm 1; \quad \Omega_n = 0;
\]

\[
\nabla G = \left( \pm \frac{6\gamma(a + f)}{3a + 2f}, \frac{6\gamma(a + f)(3a + f)}{3a + 2f}, 0 \right); \quad (\Sigma_+ \text{ eliminated});
\]

\[
= \frac{12(a + f)}{(3a + 2f)}, \quad (1, 0) ; \quad -\frac{6(a + f)(5\gamma - 6)}{(3a + 2f)(2 - \gamma)}, \quad (0, 1).
\]

For \( \theta \)-dominated models with \((3a + f) < 0 \) only \( M^+ \) exists, while for models with \((a + f) > 0 \) only \( M^- \) exists. The quotient \((a + f)/(3a + 2f)\), appearing in both eigenvalues, is positive for all \( \theta \)-dominated models. However, for non-\( \theta \)-dominated models different signs are possible and thus this factor will affect the stability of the point in this case. For \( \theta \)-dominated models, the only bifurcation is for \( \gamma = 6/5 \). For \( \gamma < 6/5 \) the existing point is a saddle and there is no asymptote from or into the interior region of the phase space, while for \( \gamma > 6/5 \), the points are always sinks. Note that the fluid quantities require separate treatment when \( \gamma = 4/3 \), even though there is no bifurcation (note the asymptotic behaviour of \( \bar{\sigma} \) in Table 7).

For \( \gamma > 6/5 \), the sink associated with the existing point does not correspond to a curvature singularity. For \( \theta \)-dominated models with \((3a + f) < 0 \) there is no final singular behaviour at all. This is also true for the case \((a + f) > 0 \) and \( f \leq -2a(3\gamma - 4)/(5\gamma - 6) \); but if \( f < -2a(3\gamma - 4)/(5\gamma - 6) \), then the point corresponds to a crushing singularity (i.e., the kinematic quantities blow up while the curvature stays finite (it goes to zero)).

In the above discussion about singularities, we should really only say that the curvature and/or kinematic quantities blow up asymptotically since we have not investigated if one really reaches the asymptotic stage in a finite amount of proper fluid time or in a finite affine distance along a geodesic. Such an investigation will not be undertaken in the initial and final asymptotic analysis below either. To prove that we really have singularities we would need to do an additional analysis involving asymptotic coordinate variable transformations for each case. However, for the closely related SH case and for the unphysical SH geometry associated with the SSS geometry, the initial and final states corresponding to possible singularities are reached in a finite time. Hence we believe that these states correspond to singularities and we will thus refer to them as such.

| As. St. | \( \tilde{\mu} \) | \( \tilde{\theta} \) | \( \tilde{\sigma} \) | \( \tilde{a} \) | \( \tilde{a}_{\mu} \) | \( D_1 \) | \( D_2 \) | \( C \) |
|---------|-----------------|-----------------|-----------------|-------------|-----------------|-----------|-----------|-------|
| \( f, \gamma > 6/5, \gamma \neq 4/3 \) | \( e^{\Gamma_1 \tau} \) | \( e^{\Gamma_1 \tau} \) | \( e^{\Gamma_1 \tau} \) | \( e^{\Gamma_1 \tau} \) | \( e^{3(\alpha + f)/(3a + 2f)} \) | \( e^{3(\alpha + f)/(3a + 2f)} \) | \( e^{-6\tau} \) |
| \( f, \gamma > 6/5, \gamma = 4/3 \) | \( e^{\Gamma_2 \tau} \) | \( e^{\Gamma_1 \tau} \) | \( e^{\Gamma_1 \tau} \) | \( e^{\Gamma_1 \tau} \) | \( e^{3(\alpha + f)/(3a + 2f)} \) | \( e^{3(\alpha + f)/(3a + 2f)} \) | \( e^{-6\tau} \) |

Table 7: Asymptotic expressions for the fluid congruence around the \( M^\pm \)-point. The constants \( \Gamma_{1,2,3} \) are given by \( \Gamma_1 = \frac{3((\alpha + f)(5\gamma - 6)/(2 - \gamma)a)}{(3a + 2f)(2 - \gamma)} \), \( \Gamma_2 = \frac{6((3\gamma - 2)f + 2\alpha \gamma)}{(3a + 2f)(2 - \gamma)} \) and \( \Gamma_3 = -\frac{2a + f}{3a + 2f} \).
The equilibrium points $\mathcal{H}^\pm$:

$$\Sigma_+ = -1 \pm 3(a + f)A, \quad v = \pm 1; \quad \Omega_n = \frac{2(1 + \Sigma_+)(f - (3a + 2f)\Sigma_+)}{3(a + f)};$$

$$\nabla G = \left(\pm \frac{2\gamma(3a + 2f - f\Sigma_+)}{3(a + f)}, \pm 2\gamma(3a + 2f - f\Sigma_+), \frac{2(\gamma - 2)(1 + \Sigma_+)((3a + 2f)\Sigma_+ - f)}{3(a + f)}\right);$$

$$e_1 := \frac{-2(2a + 3f)\Sigma_+ - a}{a + f}, \quad (1 + \Sigma_+)((3a + 2f)\Sigma_+ - f)e_1, \quad (2(a + 3f)\Sigma_+ - a)e_2,$$

$$e_2 := 3a + 2f - f\Sigma_+.$$ \hspace{1cm} (32)

For $\theta$-dominated models only one of the lines $\mathcal{H}^\pm$ exists in the physical part of the phase space. For models with $(3a + f) < 0$, only $\mathcal{H}^-$ exists, while for models with $(a + f) > 0$ only $\mathcal{H}^+$ is present. The condition that $\Omega_n \geq 0$, limits the possible values of $\Sigma_+$ to lie in the interval $-1 \leq \Sigma_+ \leq f/(3a + 2f)$. We also note that for the models with $(3a + f) < 0$ the existing line of equilibrium points splits into two parts. The part where $\Sigma_+$ attain the values $-1 \leq \Sigma_+ < 2a/(2a + 3f)$ constitutes a source while the part where $2a/(2a + 3f) < \Sigma_+ \leq f/(3a + 2f)$ is a sink. For models with $(a + f) > 0$ and $(f - a) < 0$, the entire line $\mathcal{H}^+$ is a source. However, for models with $f \geq a$ the line splits into two parts where the stability of the two parts is governed by the same inequalities as for the $(3a + f) < 0$ case. The asymptotic behaviour is given in Table 8.

All $\theta$-dominated models with an initial point on one of the lines $\mathcal{H}^\pm$ have an initial crushing singularity. This crushing singularity is a curvature singularity if $\Gamma_1$ in Table 8 is negative. Thus for models with $f > 0$ there is an initial curvature singularity.

The part of $\mathcal{H}^\pm$ corresponding to a sink is never associated with a final curvature singularity when it comes to $\theta$-dominated models. If $(3a + f) < 0$ and $2a/(2a + 3f) \leq \Sigma_+ \leq (2a + f)/(4a + 5f)$ there are no final singularities at all, but for $(3a + f) < 0$ and $(2a + f)/(4a + 5f) \leq \Sigma_+ \leq f/(3a + 2f)$ there is a crushing singularity. For $\theta$-dominated models with $(a + f) > 0$ and $a < f \leq 2a$, there is no final singular behaviour at all. For $2a < f$, $\mathcal{H}^+$ again splits into two parts whose behaviour is governed by the same inequalities as in the “splitted” $\mathcal{H}^-$ case. For non-$\theta$-dominated models the situation, which will not be discussed further, is more complicated.

| As. St. | $\tilde{\mu}$ | $\tilde{\sigma}$ | $\tilde{\phi}$ | $\tilde{\theta}$ | $\tilde{\sigma}$ | $\tilde{\phi}$ | $\tilde{\mu}$ | $D_1$ | $D_2$ | $C$ |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|------|------|------|
| i, f  | $e^{\Gamma_1\tau}$ | $e^{\Gamma_2\tau}$ | $e^{\Gamma_1\tau}$ | $e^{\Gamma_2\tau}$ | $e^{\Gamma_1\tau}$ | $e^{\Gamma_2\tau}$ | $e^{\Gamma_1\tau}$ | $e^{(1 + \Sigma_+)^\tau}$ | $e^{(1 + \Sigma_+)^\tau}$ | $e^{\Gamma_1\tau}$ |

Table 8: Asymptotic expressions for the fluid congruence around the $\mathcal{H}^\pm$-points. The constants $\Gamma_{1,2}$ are given by $\Gamma_1 = \frac{-2f(1 + \Sigma_+)}{(a + f)}$, $\Gamma_2 = \frac{(5\Sigma_+ - 1)f - 2a(1 - 2\Sigma_+)}{(a + f)}$. 

14
The equilibrium points $K_{\pm}^\pm$:

$$\Sigma_+ = \pm 1, \quad A = 0, \quad v^2 = 1; \quad \Omega_n = 0;$$

$$\nabla G = (-2\gamma \text{sgn} v, 2\gamma (3a + f), 0), \quad \text{for } \Sigma_+ = 1,$$

$$\nabla G = (-2\gamma \text{sgn} v, -6\gamma (a + f), 0), \quad \text{for } \Sigma_+ = -1; \quad (A \text{ eliminated})$$

(33)

The points $K_{\pm}^\pm$ constitute one of the boundaries of the physical part of the lines $H^\pm$. They describe the boundary asymptotes which play an important role in the dynamical system analysis. This motivates their special treatment. The points $K_{\pm}^\pm$ have no asymptotes entering the interior region of the phase space.

The equilibrium point $\bar{M}_{\pm}$:

$$\bar{M}_{\pm} = \frac{f}{3a + 2f}, \quad A = \pm \frac{1}{3a + 2f}, \quad v = \pm 1; \quad \Omega_n = 0;$$

$$\nabla G = \left(\mp \frac{2\gamma(3a + f)}{3a + 2f}, -\frac{6\gamma(3a + f)(a + f)}{3a + 2f}, 0\right), \quad (\Sigma_+ \text{ eliminated})$$

(34)

$$0, \quad (1, 0); \quad \frac{6(a - f)}{3a + 2f}, \quad (0, 1).$$

These points are at the other end (compared to $K_{\pm}^\pm$) of the lines $H^\pm$. The points have no asymptotes entering the interior region of the phase space.

5 Global SSS behaviour

5.1 Type I models

The type I models are defined by $a = 0$ and $f = -1$ (see Table 1). The model is $\theta$-dominated which implies that our variables leads to a compact reduced phase space. Only the line $H^-$ of the two lines $H^\pm$ lies in the physical phase space. The stability of $H^-$ is governed by the sign of $\Sigma_+$: for $\Sigma_+ < 0$ it is a source, and for $\Sigma_+ > 0$ it is a sink. Note that the requirement $\Omega_n \geq 0$ limits the range of $\Sigma_+$ to $-1 \leq \Sigma_+ \leq 1/2$. The only bifurcation appears for $\gamma = 6/5$, when the equilibrium point $M^{v_+}$ passes through $M^+$ and enters the physical phase space, stabilizing $M^+$. As $\gamma$ approaches 2, the point $M^{v_+}$ moves along the line $M^+ - M^{v_+} - M^-$, while $v$ approaches 0. Thus there are only two cases: $1 \leq \gamma \leq 6/5$ and $6/5 < \gamma < 2$. In both cases there are no interior equilibrium points, and hence no limit cycles. In addition, consideration of the orbits on the boundary leads to the conclusion that there are no heteroclinic cycles (compare with the SH type V case discussed in [12]).

5.1.1 Models with $1 \leq \gamma \leq 6/5$

The different orbits can be written symbolically as $H^- \to H^-, F \to H^-$ and $K_0^+ \to H^-$. The intermediate evolution of orbits which define one-parameter families of solutions can be approximated by the following heteroclinic sequences:
(1) $K_0^+ \rightarrow K_+^+ \rightarrow M^+ \rightarrow \mathcal{H}^-$

(2) $K_0^+ \rightarrow F \rightarrow \mathcal{H}^-$

(3) $\mathcal{H}^- \rightarrow K_0^0 \rightarrow F \rightarrow \mathcal{H}^-$. 

From Table 9, which describes the initial and final states of all asymptotes in the interior phase space (note the difference with the tables in the previous section where the tables referred to a single equilibrium point/line), it follows that all models begin with a curvature singularity and approach $\mathcal{H}^-$, with $0 \leq \Sigma_+ \leq 1/2$, at late times (see Diagram 2a). None of these endpoints correspond to curvature singularities, although for $\Sigma_+ > 1/5$ they are crushing singularities.

| As. St. | $\tilde{\mu}$ | $\tilde{\theta}$ | $\tilde{\sigma}$ | $\tilde{a}$ | $\tilde{a}_{\mu}$ | $D_1$ | $D_2$ | $C$ |
|---------|---------------|-----------------|-----------------|-------------|------------------|-------|-------|-----|
| $F$     | i             | $e^{-3\gamma \tau}$ | $e^{-3\gamma \tau}/2$ | $e^{(\gamma-1)\tau}$ | $e^{3(\gamma-4)\tau}$ | $e^\tau$ | $e^\tau$ | $e^{-2\tau}$ |
| $\mathcal{H}^-$ | f             | $e^{\Gamma_1 \tau}$ | $e^{\Gamma_2 \tau}$ | $e^{\Gamma_3 \tau}$ | $e^{2\Gamma_2 \tau}$ | $e^{(2\Sigma_+-1)\tau}$ | $e^{-\Gamma_1 \tau}/2$ | $e^{\Gamma_1 \tau}$ |
| $\mathcal{H}^-$ | i             | $e^{\Gamma_1 \tau}$ | $e^{\Gamma_2 \tau}$ | $e^{\Gamma_3 \tau}$ | $e^{2\Gamma_2 \tau}$ | $e^{(2\Sigma_+-1)\tau}$ | $e^{-\Gamma_1 \tau}/2$ | $e^{\Gamma_1 \tau}$ |
| $\mathcal{H}^-$ | f             | $e^{\Gamma_1 \tau}$ | $e^{\Gamma_2 \tau}$ | $e^{\Gamma_3 \tau}$ | $e^{2\Gamma_2 \tau}$ | $e^{(2\Sigma_+-1)\tau}$ | $e^{-\Gamma_1 \tau}/2$ | $e^{\Gamma_1 \tau}$ |
| $K_0^+$  | i             | $e^{-3\gamma \tau}$ | $e^{-3\gamma \tau}$ | $e^{(3\gamma-5)\tau}$ | $e^{2(3\gamma-5)\tau}$ | $e^{-\tau}$ | $e^{2\tau}$ | $e^{-6\tau}$ |
| $\mathcal{H}^-$ | f             | $e^{\Gamma_1 \tau}$ | $e^{\Gamma_2 \tau}$ | $e^{\Gamma_3 \tau}$ | $e^{2\Gamma_2 \tau}$ | $e^{(2\Sigma_+-1)\tau}$ | $e^{-\Gamma_1 \tau}/2$ | $e^{\Gamma_1 \tau}$ |

Table 9: Initial and final points for the type $I$-models with $1 \leq \gamma \leq 6/5$. The constants $\Gamma_{1,2}$ are given by $

\Gamma_1 = -2(1 + \Sigma_+)$ and $\Gamma_2 = (5\Sigma_+-1)$.

5.1.2 Models with $6/5 < \gamma < 2$

The orbits are described by $\mathcal{H}^- \rightarrow \mathcal{H}^-, F \rightarrow \mathcal{H}^-, K_0^0 \rightarrow \mathcal{H}^-, K_0^+ \rightarrow M^{v+}$ and $K_0^0 \rightarrow M^+$. It follows from Table 10 that all models start with a curvature singularity. None of the end-points correspond to a curvature singularity. Points on $\mathcal{H}^-$ with $\Sigma_+ > 1/5$ and $M^+$ correspond to crushing singularities. The remaining final points are not associated with any singular behaviour at all. The intermediate evolution of orbits can be approximated by the following heteroclinic sequences:

(1) $K_0^+ \rightarrow K_+^+ \rightarrow M^+$

(2) $K_0^+ \rightarrow M^{v+} \rightarrow M^+$

(3) $K_0^0 \rightarrow F \rightarrow \mathcal{H}^-$

(4) $\mathcal{H}^- \rightarrow K_0^0 \rightarrow F \rightarrow \mathcal{H}^-$. 

16
Table 10: Initial and final points for the type I models with $6/5 < \gamma < 2$. The constants $\Gamma, \mu = 1 - 5$, are given by $\Gamma_1 = 2(1 + \Sigma_+), \Gamma_2 = 5(\Sigma_+ - 1), \Gamma_3 = 3(5\gamma - 6)/(2(2 - \gamma))$, $\Gamma_4 = 3(3\gamma - 2)\tau/(2 - \gamma)$ and $\Gamma_5 = 3(a + f)\tau/(3a + 2f)$. Here and below fin. is an abbreviation for finite constant.

5.2 Some comments on LRS spatially homogeneous type V models

The SH LRS type V models have been extensively discussed in the literature \cite{12, 13, 14, 15}. However, we will here make some remarks about the existence of monotonic functions. The SH type V models differ from the SSS type f V and 1 I models in that there exists an exact solution corresponding to an invariant submanifold described by $\nu = 0, \Sigma_+ = 0$ (this is just the open Friedmann-Robertson-Walker (FRW) model). This separatrix divides the reduced 2-dimensional phase space of the SH type V models into two regions. For the type f V and 1 I models one also has a separatrix leaving the equilibrium point F. However, at what equilibrium point this separatrix ends up at depends on $\gamma$ and $f$, and so far one has not been able to find the corresponding exact solution. The simple division of the phase space created by the FRW separatrix in the SH type V models allows one to find monotonic functions valid in the $v > 0$ and $v < 0$ regions respectively.

The conditions $A, \Omega_n > 0$ (for the interior matter part of the type V phase space) and the constraint, lead to $\nu \Sigma_+ < 0$. Let us first consider $v < 0$. Then $\Sigma_+ (A)$ is a monotonically decreasing (increasing) function for $1 \leq \gamma < 2$. The variable $v$ is monotonically decreasing when $4/3 \leq \gamma < 2$. If $v > 0$ then $v$ is monotonically decreasing if $1 \leq \gamma \leq 4/3$. Thus one can obtain considerable qualitative dynamical information because of the simple separatrix structure. Unfortunately this structure is not available for the SSS models.
5.3 Models with radiation, $\gamma = \frac{4}{3}$

The general structure of the type $fV$ models is complicated by the number of parameters. To avoid being lost in details we here therefore restrict ourselves to the physically interesting radiation case, $\gamma = \frac{4}{3}$. For $\gamma = \frac{4}{3}$ we obtain a number of bifurcations for different values of the parameters $a$ and $f$.

5.3.1 Type $fV$ models with $3a + f < 0$

The phase space has the same structure as for the type $1I$ models with $\gamma = \frac{4}{3}$. It follows from Diagram 3a that the initial and final end points of the orbits are described by $\mathcal{H}^- \rightarrow \mathcal{H}^-, F \rightarrow \mathcal{H}^-$, $K_+^0 \rightarrow \mathcal{H}^-$, $K_+^0 \rightarrow M_{v+}$ and $K_+^0 \rightarrow M^+$. From Table 11 it is seen that all models start with a curvature singularity and that the point $M^+$ corresponds to a crushing singularity. If the final points on $\mathcal{H}^-$ are crushing singularities or not depends on which of the inequalities given in section 4 the final value of $\Sigma_+$ satisfies. It is clear though, that the line has points corresponding to crushing singularities or no singularities at all. We have the following heteroclinic sequences:

1. $K_+^0 \rightarrow K_+^{v} \rightarrow M^+$
2. $K_+^0 \rightarrow M^{v+} \rightarrow M^+$
3. $K_+^0 \rightarrow F \rightarrow \mathcal{H}^-$
(4) $\mathcal{H}^- \to K^0 \to F \to \mathcal{H}^-$. 

| As. St. | $\tilde{\mu}$ | $\tilde{\theta}$ | $\tilde{\sigma}$ | $\tilde{a}$ | $\tilde{a}^\mu$ | $D_1$ | $D_2$ | $C$ |
|---------|----------------|-----------------|-----------------|-------------|--------------|------|------|-----|
| $F$     | $e^{-4\tau}$  | $e^{-2\tau}$    | fin.            | $e^{\tau/3}$ | $e^{-8\tau}$ | $e^\tau$ | $e^\tau$ | $e^{-2\tau}$ |
| $\mathcal{H}^-$ | $e^{\Gamma_1 \tau}$ | $e^{\Gamma_2 \tau}$ | $e^{\Gamma_2 \tau}$ | $e^{2\Gamma_2 \tau}$ | $e^{(2\Sigma_+ - 1)\tau}$ | $e^{-\Gamma_1 \tau/2}$ | $e^{\Gamma_1 \tau}$ | $e^{\Gamma_1 \tau}$ |
| $K^0_+$ | $e^{-4\tau}$  | $e^{-4\tau}$    | $e^{-3\tau}$    | $e^{-\tau}$  | $e^{-2\tau}$  | $e^\tau$  | $e^\tau$  | $e^{-6\tau}$ |
| $M^+$   | $e^{\Gamma_4 \tau}$ | $e^{\Gamma_3 \tau}$ | $e^{\Gamma_3 \tau}$ | $e^{2\Gamma_3 \tau}$ | $e^{3\alpha/(3a+2f)}$ | $e^{3(a+f)/(3a+2f)}$ | $e^{-6\tau}$ |

Table 11: Initial and final points for the type $fV$-models with $\gamma = 4/3$ and $(3a + f) < 0$. The constants $\Gamma_{1,2,3,4}$ are given by $\Gamma_1 = -2f(1 + \Sigma_+)/(a + f)$, $\Gamma_2 = ((5\Sigma_+ - 1)f - 2a(1 - 2\Sigma_+))/(a + f)$, $\Gamma_3 = -2a + f$ and $\Gamma_4 = -6(4a + 3f)/(3a + 2f)$.

### 5.3.2 Type $fV$ models with $a + f > 0$ and $f < 0$

The orbits are described by $\mathcal{H}^+ \to M^-$, $F \to M^-$, and $K^0_+ \to M^-$ (see Diagram 3b). It follows from Table 12 that the orbits starting from $\mathcal{H}^+$ do not have an initial curvature singularity. The other two types of orbits have an initial curvature singularity. The heteroclinic sequences are given by:

1. $\mathcal{H}^+ \to M^-$
2. $\mathcal{H}^+ \to K^0_- \to F \to M^-$
3. $K^0_+ \to F \to M^-$
4. $K^0_+ \to K^0_- \to M^-$. 

As seen in Diagram 3b, all orbits approach $M^-$ for late times, and as can be seen from Table 12, this point corresponds to no singular behaviour at all.

### 5.3.3 Type V models

For $f = 0$, $v = 0$ is an invariant submanifold, characterized by the separatrix $F \to M^0$. The other orbits are characterized by $\mathcal{H}^+ \to M^0$ and $K^0_+ \to M^-$ (see Diagram 3c). Orbits starting from the
Table 12: Initial and final points for the type \( f \)V-models with \( \gamma = 4/3 \) and \( (a + f) >, f < 0 \). The constants \( \Gamma_{1,2,3,4} \) are given by \( \Gamma_1 = -2 f (1 + \Sigma_+) / (a + f) \), \( \Gamma_2 = ((5 \Sigma_+ - 1) f - 2 a (1 - 2 \Sigma_+) ) / (a + f) \), \( \Gamma_3 = -2 a + f / (3 a + 2 f) \) and \( \Gamma_4 = -6 (4 a + 3 f) / (3 a + 2 f) \).

points \( K_0^+ \) and \( F \) have an initial curvature singularity while orbits starting from \( H^+ \) do not. The point \( M^0 \), which only exists for these models, has a zero eigenvalue for \( \gamma = 4/3 \) and needs special treatment. We therefore refrain from giving a table of initial and final asymptotic expressions and instead refer to [13]. It follows from Table 7 that \( M^- \) is associated with no singular behaviour at all while Table 8 shows that the line \( H^+ \) corresponds to a crushing singularity. The intermediate evolution of orbits can be approximated by the following heteroclinic sequences:

1. \( H^+ \rightarrow K_0^- \rightarrow F \rightarrow M^0 \)
2. \( K_0^+ \rightarrow F \rightarrow M^0 \rightarrow M^- \)
3. \( K_0^- \rightarrow K_+^- \rightarrow M^- \).

5.3.4 Type \( f \)V models with \( a + f > 0 \) and \( 0 < f < a \)

In this case the SH type V point \( M^0 \) splits into the two points \( M^{v\pm} \). These points move apart as \( f \) increases. When \( f \) approaches \( a \), \( M^{v+} \) approaches the point \( \bar{M}^+ \). The initial and final stages of the orbits are given by \( H^+ \rightarrow M^{v+} \), \( F \rightarrow M^{v+} \), \( K_0^+ \rightarrow M^{v+} \), \( K_0^+ \rightarrow M^{v-} \) and \( K_0^+ \rightarrow M^- \) (see Diagram 3d). It follows from Table 13 that all orbits have an initial curvature singularity. Solutions ending in \( M^- \) have a final crushing singularity. The intermediate evolution of orbits can be approximated by the following heteroclinic sequences:

1. \( H^+ \rightarrow K_0^- \rightarrow F \rightarrow M^{v+} \)
2. \( K_0^+ \rightarrow F \rightarrow M^{v+} \)
3. \( K_0^+ \rightarrow M^{v-} \rightarrow M^{v+} \)
4. \( K_0^+ \rightarrow M^{v-} \rightarrow M^- \)
5. \( K_0^+ \rightarrow K_+^- \rightarrow M^- \).
Table 13: Initial and final points for the type $fV$-models with $\gamma = 4/3$ and $(a + f) > 0, 0 < f < a$. The constants $\Gamma_{1,2,3,4}$ are given by $\Gamma_1 = -2f(1+\Sigma_+)//(a+f)$, $\Gamma_2 = ((5\Sigma_+ -1)f-2a(1-2\Sigma_+))/(a+f)$, $\Gamma_3 = -2a+f/3a+2f$, and $\Gamma_4 = -6(4a+3f)/(3a+2f)$. The constants $\lambda_{1,2}^2$ have long expressions which will not be given.

### 5.3.5 Type $fV$ models with $a + f > 0$ and $f \geq a$

The asymptotic properties of the orbits are described by $\mathcal{H}^+ \rightarrow \mathcal{H}^+$, $F \rightarrow \mathcal{H}^+$, $K_0^+ \rightarrow \mathcal{H}^+$, $K_0^+ \rightarrow M_-$, and $K_0^- \rightarrow M^-$ (see Diagram 3e). From Table 14 it is seen that all models have an initial curvature singularity. From Table 14 it also follows that orbits ending in the point $M^-$ have a crushing singularity. A solution ending on the line $\mathcal{H}^+$ may have a final crushing singularity or no singularity at all. The existence of a final crushing singularity depends on which of the previously given inequalities the final value of $\Sigma_+$ satisfies (see section 4). We have the following heteroclinic sequences:

1. $K_0^+ \rightarrow K_+^- \rightarrow M^-
2. K_+^0 \rightarrow M^+ \rightarrow M^-
3. K_+^0 \rightarrow F \rightarrow \mathcal{H}^+
4. K_+^0 \rightarrow M^+ \rightarrow \mathcal{H}^+
5. \mathcal{H}^+ \rightarrow K_-^0 \rightarrow F \rightarrow \mathcal{H}^+.

### 5.4 Some remarks

All models starting from points with non-extreme tilt start from the points $F$ or $K_+^0$ and are characterized by curvature singularities. Models belonging to the class $a + f > 0$ and $f \leq 0$, ...
|   | \( \dot{\rho} \) | \( \dot{\theta} \) | \( \dot{\sigma} \) | \( \ddot{\rho} \) | \( \ddot{\theta} \) | \( \tau \) | \( \mu \) | \( C \) | \( D_1 \) | \( D_2 \) |
|---|---|---|---|---|---|---|---|---|---|---|
| \( F \) | \( e^{-4\tau} \) | \( e^{-2\tau} \) | \( \text{fin.} \) | \( e^{\tau/3} \) | \( e^{-8\tau} \) | \( e^{\tau} \) | \( e^{\tau} \) | \( e^{-2\tau} \) | \( \text{fin.} \) | \( \text{fin.} \) |
| \( \mathcal{H}^+ \) | \( e^{\Gamma_1 \tau} \) | \( e^{\Gamma_2 \tau} \) | \( e^{\Gamma_2 \tau} \) | \( e^{2\Gamma_2 \tau} \) | \( e^{(2\Sigma_+ - 1)\tau} \) | \( e^{-\Gamma_1 \tau/2} \) | \( e^{\Gamma_1 \tau} \) | \( \text{fin.} \) | \( \text{fin.} \) | \( \text{fin.} \) |
| \( K^0_+ \) | \( e^{-4\tau} \) | \( e^{-4\tau} \) | \( e^{-3\tau} \) | \( e^{-\tau} \) | \( e^{-2\tau} \) | \( e^{\tau} \) | \( e^{2\tau} \) | \( e^{-6\tau} \) | \( \text{fin.} \) | \( \text{fin.} \) |
| \( M^\sigma_+ \) | \( e^{-\lambda^2 \tau} \) | \( \text{fin.} \) | \( \text{fin.} \) | \( \text{fin.} \) | \( \text{fin.} \) | \( e^{3\tau/2} \) | \( e^{-\lambda^2 \tau} \) | \( \text{fin.} \) | \( \text{fin.} \) | \( \text{fin.} \) |
| \( K^0_+ \) | \( e^{-4\tau} \) | \( e^{-3\tau} \) | \( e^{-3\tau} \) | \( e^{-\tau} \) | \( e^{-2\tau} \) | \( e^{\tau} \) | \( e^{2\tau} \) | \( e^{-6\tau} \) | \( \text{fin.} \) | \( \text{fin.} \) |
| \( H^+ \) | \( e^{\Gamma_1 \tau} \) | \( e^{\Gamma_2 \tau} \) | \( e^{\Gamma_2 \tau} \) | \( e^{2\Gamma_2 \tau} \) | \( e^{(2\Sigma_+ - 1)\tau} \) | \( e^{-\Gamma_1 \tau/2} \) | \( e^{\Gamma_1 \tau} \) | \( \text{fin.} \) | \( \text{fin.} \) | \( \text{fin.} \) |
| \( H^+ \) | \( e^{\Gamma_1 \tau} \) | \( e^{\Gamma_2 \tau} \) | \( e^{\Gamma_2 \tau} \) | \( e^{2\Gamma_2 \tau} \) | \( e^{(2\Sigma_+ - 1)\tau} \) | \( e^{-\Gamma_1 \tau/2} \) | \( e^{\Gamma_1 \tau} \) | \( \text{fin.} \) | \( \text{fin.} \) | \( \text{fin.} \) |
| \( K^0_+ \) | \( e^{-4\tau} \) | \( e^{-3\tau} \) | \( e^{-3\tau} \) | \( e^{-\tau} \) | \( e^{-2\tau} \) | \( e^{\tau} \) | \( e^{2\tau} \) | \( e^{-6\tau} \) | \( \text{fin.} \) | \( \text{fin.} \) |
| \( M^- \) | \( e^{\Gamma_4 \tau} \) | \( e^{\Gamma_3 \tau} \) | \( e^{\Gamma_3 \tau} \) | \( e^{2\Gamma_3 \tau} \) | \( e^{3\sigma/(3a+2f)} \) | \( e^{3(a+f)/3a+2f} \) | \( e^{-6\tau} \) | \( \text{fin.} \) | \( \text{fin.} \) | \( \text{fin.} \) |

Table 14: Initial and final points for the type \( fV \)-models with \( \gamma = 4/3 \) and \((a + f) > 0, f \geq a\). The constants \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \) are given by \( \Gamma_1 = -2f(1 + \Sigma_+)//(a + f) \), \( \Gamma_2 = ((5\Sigma_+ - 1)f - 2a(1 - 2\Sigma_+))/(a + f) \), \( \Gamma_3 = -\frac{2a+f}{3a+2f} \) and \( \Gamma_4 = -6(4a + 3f)/(3a + 2f) \).

which initially have extreme tilt, may be extended so that they come from the TSS region. The remaining models on the other hand have an initial curvature singularity. A solution which ends at a point corresponding to extreme tilt can be extended to the TSS region since these points do not correspond to a curvature singularity (they only correspond to a crushing singularity or no singularity at all). There are no interior equilibrium points in any of the \( \theta \)-dominated type \( fV \) models, and hence no limit cycles. Furthermore, consideration of the orbits on the boundary leads to the conclusion that there are no heteroclinic cycles either.
Diagram 3: The reduced phase space of the $fV$ models with (a) $3a + f < 0$, (b) $a + f > 0$ and $f < 0$, (c) $a > 0$ and $f = 0$, (d) $a + f > 0$ and $0 < f < a$ and (e) $a + f > 0$ and $a \geq 0$. 
6 Discussion

Note that we do not obtain any final conformal singularities (i.e., curvature singularities for which only the Weyl tensor blows up) at the Cauchy horizon since $C \to 0$ whenever $\mu \to 0, \text{const.}$ This is in contrast to [13] where it is claimed that one has a conformal singularity at the horizon for the SH type V models with $4/3 < \gamma < 2$ (see the lower part of figure 7 in [13]). We only obtain a crushing singularity in this case and thus one should be able to extend the space time beyond the Cauchy horizon. This result is supported by our asymptotic analysis (done with help of the computer algebra systems REDUCE and the SHEEP package CLASSI) as well as numerical calculations for the orbits throughout their entire evolution. Thus our result supports conjecture 2 in [16] (p. 147) which states that matter flow lines do not end at a conformal curvature singularity.

Note that for the type $fV$ models it is only in the SH type V model one has $\mu = \text{const.} \neq 0$ on the horizon when $1 \leq \gamma < 2$ (this follows from Table 8 where we have $\mu \propto e^{-2f(1+\Sigma_{+})/(a+f)}$). Collins has shown that only stiff perfect fluids ($\gamma = 2$) allow $\mu = 0$ on the horizon [15] in the SH case (this implies that there cannot be any fluid flow over the horizon). This lead him to the conclusion that this behaviour probably was associated with a rather esoteric equation of state. However, for the self-similar models presently considered one always has $\mu = 0$ or $\mu = \infty$ at the horizon. When the SH type V models are seen as a special case of the present class of models it is rather the case $\mu = \text{const.} \neq 0$ which is exceptional. Moreover, the existence of whimper singularities is intimately connected with $\mu = \text{const.} \neq 0$ (see [16] p. 136). Hence we believe that no whimper singularities occur in the present self-similar models (a proof would require a more subtle comparison of the decrease in $\mu$ and the increase associated with the growth of the affine parameter; see [16]). The present analysis (exemplified by the result $\mu \propto e^{-2f(1+\Sigma_{+})/(a+f)}$) thus reinforce the impression that whimper singularities are highly special.

A Properties of the fluid congruence and Petrov type conditions

The fluid expansion is given by

$$\dot{\theta} = \frac{e^{fx}}{(1 - v^2)^{3/2}} \left\{ vv + (1 - v^2) \left( \theta - (2a + 3f)B_1v \right) \right\} =$$

$$= \frac{e^{fx}}{(1 - v^2)^{3/2}} \left\{ \frac{vv'}{3} + (1 - v^2) (1 - (2a + 3f)Av) \right\} \theta .$$

(35)

The fluid shear scalar is

$$\tilde{\sigma} = \sqrt{\frac{1}{2} \tilde{\sigma}_{ab} \tilde{\sigma}^{ab}} = \frac{e^{fx}}{\sqrt{3(1 - v^2)^{3/2}}} \left| vv - \left( 1 - v^2 \right) \left( \sigma_{+} - aB_1v \right) \right| =$$

$$= \frac{e^{fx}}{\sqrt{3(1 - v^2)^{3/2}}} \left\{ \frac{vv'}{3} - (1 - v^2) (\Sigma_{+} - aAv) \right\} \theta .$$

(36)
The fluid acceleration scalar is
\[ \ddot{a} = \sqrt{\ddot{a}^\mu \ddot{a}_\mu} = \frac{e^{fx}}{3(1 - v^2)^{3/2}} \left| 3\ddot{v} + (1 - v^2) (\theta v - 2\sigma_+ v - 3fB_1) \right| = \]
\[ = \frac{e^{fx}}{3(1 - v^2)^{3/2}} \left| \left\{ v' + (1 - v^2)(v - 2\Sigma_+ v - 3fA) \right\} \theta \right|. \tag{37} \]

The divergence of the acceleration vector is given by
\[ \ddot{u}^a = \frac{e^{2fx}}{3(1 - v^2)^{3/2}} (3v(1 - v^2)\ddot{v} - 3(1 + 3v^2)\dot{v}^2 + (1 - v^2)(5v\theta - 4v\sigma_+ - 3fv^2B_1 - 3(2a + 3f)B_1)\dot{v} - (1 - v^2)(3fv\dot{B}_1 + (2\dot{\sigma}_+ - \dot{\theta} - \theta^2 + 2\theta\sigma_+ + v^2 - 2B_1(a + f)(2v\sigma_+ - 3fB_1))) \]
\[ = \frac{e^{2fx}}{3(1 - v^2)^3} \left( \frac{1}{3} v (1 - v^2) v'' + \frac{1}{3} (1 + 3v^2) (v')^2 \right) \]
\[ - \frac{1}{3} (1 - v^2) \left( 3A_1(2a + 3f + fv^2) + 4v\Sigma_+ - 5v \right) v' \]
\[ + \frac{1}{3} (1 + q) (1 - v^2) \left( \left( 1 - v^2 \right) (3fvA_1 + 2v^2\Sigma_+ - v) - vv' \right) \]
\[ - \left( 1 - v^2 \right) (2A_1(a + f) - v) \left( 3fvA_1 + 2v^2\Sigma_+ - v \right) + \frac{1}{3} \left( 3fvA_1' + 2v^2\Sigma_+ ' \right) \theta^2 \tag{38} \]

The Weyl scalar is
\[ C = \sqrt{C_{abcd}C^{abcd}} = \frac{2e^{2fx}}{3\sqrt{3}} \left| -3\dot{\sigma}_+ + 3kB_2^2 - \theta\sigma_+ + 2\sigma_+^2 \right| = \]
\[ = \frac{2e^{2fx}}{3\sqrt{3}} \left| \left\{ \Sigma_+ ' + (2\Sigma_+ + q)\Sigma_+ + 3kK \right\} \theta^2 \right|. \tag{39} \]

Note that the magnetic part of the Weyl tensor is identically zero for all models. The space-time is of Petrov type 0 if
\[ -3\dot{\sigma}_+ - \sigma_+ \theta + 2\sigma_+^2 + 3kB_2^2 = 0 \tag{40}, \]
which can be written as
\[ (\Sigma_+ ' + (2\Sigma_+ + q)\Sigma_+ + 3kK)\theta^2 = 0 \tag{41}. \]
otherwise it is of type D.

B Relation to the fluid approach

The fluid approach, where one uses a coordinate system adapted to the fluid velocity, is more common than the “homothetic” approach used in this article. The line element, of the presently
considered LRS models, takes the following form in the fluid approach \[2\]:

\[
d\tilde{s}^2 = -e^{\Psi(\lambda)}dT^2 + e^{\Lambda(\lambda)}dX^2 + Y^2(\lambda)X^2 \left(dy^2 + k^{-1} \sin(\sqrt{k}y)dz^2\right),
\]

where \( \lambda = X/T \) and \( \delta = (a + f)/f \). The relation \( ak = 0 \) thus takes the form \( (\delta - 1)k = 0 \). The fluid velocity, \( u^a \), is given by \( e^{-\Psi/2}(1,0,0,0) \).

The following transformation:

\[
X = e^{-f(x - F(\bar{t}))}, \quad \lambda = X/T = e^{\bar{f}},
\]

leads to the “homothetic” adapted SSS line element

\[
d\tilde{s}^2 = e^{-2fx}ds^2 = e^{-2fx} \left[-N^2d\bar{t}^2 + D_1^2dx^2 + D_2^2e^{-2ax} \left(dy^2 + k^{-1} \sin(\sqrt{k}y)dz^2\right)\right],
\]

where

\[
N^2 = f^2e^{-2fF(\bar{t})+\Lambda+\Psi-2f\bar{t}}(e^\Lambda - e^{\Psi-2f\bar{t}})^{-1},
\]

\[
D_1^2 = f^2e^{-2fF(\bar{t})}(e^\Lambda - e^{\Psi-2f\bar{t}}), \quad D_2^2 = e^{-2f\delta F(\bar{t})}Y^2.
\]

The fluid velocity \( v \) is given by \( v^2 = e^{-\Lambda+\Psi-2f\bar{t}} \).
References

[1] M.E. Cahill and A.H. Taub, Commun. Math. Phys. 21 (1971), 1.

[2] I.S. Shikin, Sov. Phys. JETP 54 (1981), 427.

[3] I.S. Shikin, Gen. Relativ. Grav. 11 (1979), 433.

[4] A. Ori and T. Piran, Phys. Rev. D 42 (1990), 1068.

[5] D.M. Eardley, Commun. Math. Phys. 37 (1974), 289.

[6] J.P. Luminet, Gen. Relativ. Grav. 9 (1978), 673.

[7] O.I. Bogoyavlensky, Sov. Phys. JETP 46 (1977), 633.

[8] O.I. Bogoyavlensky, Methods in the Qualitative Theory of Dynamical Systems in Astrophysics and Gas Dynamics (Springer-Verlag), (1985).

[9] C. Wu, Gen. Relativ. Grav. 13 (1981), 625.

[10] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, Exact Solutions of Einstein's Field Equations (Cambridge University Press, Cambridge), 1980.

[11] C.W. Misner, Phys. Rev. 186 (1969), 1319.

[12] C.G. Hewitt and J. Wainwright, Phys. Rev. D 46 (1992), 4242.

[13] C.B. Collins and G.F.R. Ellis, Phys. Rep. 56 (1979), 65.

[14] I.S. Shikin, Sov. Phys. JETP 41 (1976), 794.

[15] C.B. Collins, Commun. Math. Phys. 39 (1974), 131.

[16] G.F.R. Ellis and A.R. King, Commun. Math. Phys. 38 (1974), 119.